Gauge theory in Riem(M)

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Abstract

In this paper we restrict our attention to an open subset $\mathcal{M}'$ of $\text{Riem}(M) = \mathcal{M}$, consisting solely of metrics with no global symmetry beyond the identity. Therein we have a natural principal fiber bundle (PFB) structure $\text{Diff}(M) \hookrightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{M}/\mathcal{D}(M)$ in which we input the connection form $\omega$ and try to construct elements of gauge theory over the associated bundle $\mathcal{M}' \times_\rho F$, with $F = \Gamma(E)$, with $E$ being constrained by elements of the theory to be taken as a tensorial bundle over $M$. In spite of the infinite dimensional setting, we succeed in obtaining an explicitly local curvature form and Yang-Mills equations for gauge fields in superspace. We are also able to obtain a wave equation for a global section of the associated bundle $\mathcal{M}' \times_{\text{Ad}} \Gamma(TM)$.

1 Introduction

In the ADM formulation of General Relativity [1] space-time is assumed to be the topological product $M \times \mathbb{R}$ and the dynamics are cast in a constrained Hamiltonian system with (unconstrained) configuration space $\text{Riem}(M) = \mathcal{M}$: the space of all Riemannian metrics over $M$. Two isometric metrics over $M$ are physically indistinguishable, since there is no operational labeling of a point which does not involve the metric thereof.

Hence to get rid of redundancies we must quotient out the isometries, leaving only the intrinsic geometries. That means identifying the metric $h = f^*g$ with $g$, where $f \in \mathcal{D}(M) := \text{Diff}(M)$ acts by pull-back. The resulting orbit space, $\mathcal{S}$ is called superspace, and is the proper configuration space in which momentary spatial geometries evolve. Classical space-time is a curve in $\mathcal{S}$, and each point can be seen as an instantaneous spatial configuration.

Unfortunately, $\mathcal{S}$ is not actually a manifold, since as we will see geometries which possess symmetry beyond the identity (i.e. $\text{SYM}_g := \{f^*g = g \mid f \neq \text{Id}\} \neq \emptyset$) don’t have neighborhoods homeomorphic to neighborhoods of less symmetric geometries. The metrics that do allow isometries impede the quotient space $\mathcal{M}/\mathcal{D}(M)$ to have a manifold structure. These metrics are singular points in superspace (they have “less” dimension than generic geometries) [2].

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1For now we are considering metrical fields only.

2Not to be confused with supersymmetry terminology.
However, for any $M$, the metrics that do not admit isometries form a generic, (open\footnote{In an yet to be defined topology.}) subset of $\mathcal{M}$ \footnote{For suppose that $\text{SYM}_g = \emptyset$, that $h = f_0^*g$ and that $f_1^*h = h$. Now, $f_1^*h = f_1^*f_0^*g = h = f_0^*g$, hence $(f_0^{-1})^*f_1^*f_0^*g = g$ but since $\text{SYM}_g = \emptyset$ that means $f_0 \circ f_1 \circ (f_0^{-1}) = \text{Id}$ hence $f_1 = \text{Id}$.}. The same is true of the respective projected subset, consisting of unsymmetrical geometries of $\mathcal{S}$. To be explicit, any metric that lies in the same orbit of a metric without non-trivial symmetries is unsymmetrical as well\footnote{For suppose that $\text{SYM}_g = \emptyset$, that $h = f_0^*g$ and that $f_1^*h = h$. Now, $f_1^*h = f_1^*f_0^*g = h = f_0^*g$, hence $(f_0^{-1})^*f_1^*f_0^*g = g$ but since $\text{SYM}_g = \emptyset$ that means $f_0 \circ f_1 \circ (f_0^{-1}) = \text{Id}$ hence $f_1 = \text{Id}$.}, so an unsymmetrical geometry is well defined as the equivalence class of unsymmetrical metrics. On the other hand, it is easy to construct paths between any two unsymmetrical metrics avoiding symmetrical singularities by adding unsymmetrical bumps to the metric. Therefore the space of generic geometries forms a connected subset of $\mathcal{S}$.

In this work, we will consider $\mathcal{M}'$ to be the open connected set of unsymmetric metrics in $\text{Riem}(\mathcal{M})$, which clearly contains all of its orbits.

We will describe earlier work showing that it is possible to construct a principal fiber bundle structure

$$\mathcal{D}(\mathcal{M}) \hookrightarrow \mathcal{M}' \xrightarrow{\pi} \mathcal{M}'/\mathcal{D}(\mathcal{M})$$

Thence, we will try to prescribe a gauge connection over $\mathcal{M}'$ which is completely equivalent to Barbour’s et al construction of General Relativity via a “best-matching” procedure for an equilocality principle \cite{4},\cite{5}.

Sections 1 and 2 contain no new results, they present merely a setting for our problem and a brief introduction to the theory of superspace. Section 3 solely restates in this setting many of the usual results and machinery of the usual gauge theory, for the sake of familiarity with the notation and clarity of correspondence with finite-dimensional gauge theory. It is only an extension of the constructions made on the first two sections, but it will be the bread and butter of the last section, where we attempt to set up the ingredients for a gauge theory over an associated bundle with typical fiber $\Gamma(E = TM \otimes TM \otimes ... \otimes TM^* \otimes ... \otimes TM^*)$.

We find that most of the usual constructions work, paving the way for an application in quantum gravity in a different perspective of Wheeler’s geometrodynamics \cite{6}\cite{7}, and for another route to a wave equation for superspace. For many of the results, we use the proofs of \cite{8}, since they are usually done without coordinates, with no dimension input.

Even though there are many similarities with the finite dimensional case, many modifications have to be made for the theory to make sense, such as a different point of view regarding the canonical inner product of the associated tensor bundle. We will be forced to regard the fibers of the associated bundle to be non-isometric amongst themselves, but we do this while retaining a canonical inner product defined for each one of them. These inner products are naturally dependent on the base point geometry $\tilde{g} \in \mathcal{S}'$. The question of transporting this procedure to Kaluza-Klein theories naturally arises.

The connection form is seen as describing the vertical projection of any metric change. As we will see it yields the part of the infinitesimal metric change that can be done away with by a change of “coordinates” (more precisely, identification of points over $\mathcal{M}$). This is exactly the description of what a best-matching field is supposed to do \cite{4},\cite{5}. Utilizing this more intuitive interpretation of best-matching fields as the values of the connection form, we give a natural interpretation of the curvature form using an analogy of holonomies in this setting.
Although we construct what we regard to be the necessary machinery to pursue a theory with (the analogous) “matter fields” and “pure fields”, we only get around to exploring the latter, coming up with a Yang-Mills field equation. It can be seen as somewhat surprising that the Yang-Mills equation and the equations defining the curvature form are explicitly local in $M$. This is an aftermath of the orthogonality condition between vectors in $\mathcal{M}'$, which has a local dependence on the fields.

2 PFB Structure for $\mathcal{M}'$

Let $M$ be a ($C^\infty$) compact, oriented, connected, 3-dimensional manifold without boundary. We will call $L^2_S(TM)$ the tensor bundle of continuous symmetric bilinear forms and $L^2_S^+(TM)$ the subspace consisting of the positive definite elements. $L^2_S(TM)$ is a $C^\infty$ tensor bundle over $M$, and since the subset $L^2_S^+(TM)$ is defined by an open condition it inherits the manifold structure.

The vector space $\Gamma^r(L^2_S(TM))$ of $r$-differentiable cross sections of $L^2_S(TM)$ is clearly infinite dimensional. But of course infinite dimensional calculus can be constructed over an infinite dimensional vector space in the same way ordinary calculus can be constructed over the usual linear space $\mathbb{R}^n$. In fact, a substantial portion of the definitions and theorems valid in ordinary calculus can be proven substituting $\mathbb{R}^n$ by a complete locally convex topological vector space (CLCTV), such as Banach and Hilbert spaces.

For each $r$, $0 < r < \infty$, $\Gamma^r(L^2_S(TM))$ is a second countable Banach space in the topology of uniform convergence in $r$ derivatives, or $C^r$ topology. We have that

$$\Gamma^\infty(L^2_S(TM)) = \bigcap_{r=0}^\infty \Gamma^r(L^2_S(TM))$$

is a Fréchet space that is dense in each $\Gamma^r(L^2_S(TM))$ and is paracompact. We will call $\Gamma^\infty(L^2_S(TM)) := S_2(M)$, and a few times we will abuse notation and use $\bigcap_{r=0}^s \Gamma^r(L^2_S^+(TM)) = \mathcal{M}^s$, a notation which we will (mildly) elucidate later.

Due to the lack of completeness, for general Fréchet spaces no existence theorems for ordinary differential equations exist, neither do Implicit function Theorems. However, $S_2(M)$ is in fact a separable ILB-space (or even a ILH space), where we do have integral curves and the usual Implicit Function Theorems defined for certain types of mappings. Namely, for mappings that are ILH-normal (and obey the usual conditions for IFTs). Since for each $s$, $\bigcap_{r=0}^s \Gamma^r(L^2_S(TM))$ is a separable Hilbert space, it has partitions of unity, and with this condition it is possible to demonstrate the existence and uniqueness of integral curves and of a connection (and hence of the exponential mapping) thereat.

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5 A topological locally convex vector space is a Hausdorff vector space with a topology such that the operations of addition and scalar ($\mathbb{R}$) multiplication are continuous, and such that any neighborhood of 0 contains an open neighborhood $U$ of 0 such that for $x, y \in U$, the line segment $x\bar{y} \in U$. An alternative definition can be given using semi-norms and a condition for separating points.

6 A Fréchet space is a metrizable, complete, locally convex vector space. The metrizability condition may be replaced by stating that 0 has a countable neighborhood base or that its topology is generated by a countable family of semi-norms which separates points. The main thing to notice is that these spaces are not necessarily complete.
Now, $\mathcal{M}$ is not a linear space since for $g, h \in \mathcal{M}$ it can happen that $g - h \notin \mathcal{M}$. But $g + \lambda h \in \mathcal{M}$, for any $\lambda \in C^\infty(M)$ such that $\lambda > 0$, so it is a positive cone in $S_2(M)$. Hence it is a ILB-manifold\footnote{In fact, it can just as easily be given a ILH structure, so wherever we mention inverse limit Banach, inverse limit Hilbert can be read instead.} modeled on the ILB-space $\Gamma^\infty(L^2_2(TM))$, and since $\mathcal{M}$ is open in $S_2(M)$, the tangent space of $\mathcal{M}$ at $g$ is

$$T_g \mathcal{M} = \{g\} \times S_2(M) \simeq S_2(M)$$

i.e. its tangent bundle is trivial and is also an ILH manifold, modeled on the ILH space $S_2(M) \oplus S_2(M)$.

We will assume the following facts \footnote{Fischer, in his Stratification Theorem \cite{2], has shown that this allows $\mathcal{S}$ to be partitioned into manifolds of geometries, such that the geometries with higher symmetry are completely}{3}:

- The set $\mathcal{D}(M) := \text{Diff}(M)$ of smooth diffeomorphisms of $M$ is an ILH Lie group, and it acts on $\text{Riem}(M)$ on the right as a group of transformations by pulling back metrics:

  $$\Psi : \mathcal{M} \times \mathcal{D}(M) \rightarrow \mathcal{M}$$

  $$(g,f) \mapsto f^* g$$

  an action which is ILH-smooth with respect to the ILH structures of $\mathcal{M}$ and $\mathcal{D}(M)$. The natural action is on the right since of course $(f_1 f_2)^* g = f_2^* f_1^* g$. It is clear that two metrics are isometric if and only if they lie in the same orbit.

- The derivative of the orbit map $\Psi_g : \mathcal{D}(M) \rightarrow \mathcal{M}$ at the identity

  $$T_e \Psi_g : \Gamma(TM) \rightarrow T_g \mathcal{M}$$

  $$X \mapsto L_X g$$

  where $X$ is the infinitesimal generator of a given curve of diffeomorphisms of $M$, forms a closed linear space with closed complement in $S_2(M)$ (it splits). Hence $\Psi_g : \mathcal{D}(M) \rightarrow \mathcal{M}$ embeds $\mathcal{D}(M)$ in $\mathcal{M}$, that is $\mathcal{O}_g := \Psi_g(\mathcal{D}(M))$ is a smooth closed submanifold of $\mathcal{M}$ with tangent space at $h \in \Psi_g(\mathcal{D}(M))$ given by the range of $T_e \Psi_h$.

  The diffeomorphisms that fix $g \in \mathcal{M}$, i.e. $\{f \in \mathcal{D}(M) \mid f^* g = g\}$, form a closed subgroup of $\mathcal{D}(M)$, the isometry or symmetry group of $g$ which we will call $\mathcal{I}_g(M)$. As an isometry group of $(M, g)$, it is a compact Lie group whose Lie algebra $\mathcal{I}_g(M)$ is composed of the vector fields whose flux are isometries, i.e. it is the Lie algebra of Killing fields $\{X \in \Gamma(TM) \mid L_X g = 0\}$, with the usual Lie bracket between sections of $TM$. Clearly since $L_{[X,Y]} = L_X L_Y - L_Y L_X$ this is well defined.

  Now, since the isometry groups are compact, $\mathcal{I}_g(M) = \text{Ker}(T_e \Psi_g)$ is a finite dimensional subspace of $\Gamma(TM)$. By the closed graph theorem it has a closed complementary subspace such that $\Gamma(TM)$ is top linearly isomorphic to their product (it splits in $\Gamma(TM)$). Since $\Psi_g$ is a smooth embedding, $T_e \Psi_g$ maps $\Gamma(TM)/\mathcal{I}_g(M)$ injectively into a closed splitting subspace of $S_2(M)$.\footnotetext{In fact, it can just as easily be given a ILH structure, so wherever we mention inverse limit Banach, inverse limit Hilbert can be read instead.}
contained in the boundary of geometries with lower symmetry. \( S \) is stratified, and the manifolds that consist the strata are indexed by the conjugacy classes in \( D(M) \) of the isometry groups \( I_g(M) \). In the present work however we will very few times mention the occurrences of these boundaries, remaining safe within one of the given strata.

We will now describe a part of the Ebin-Palais slice theorem \([12]\) which is analogous to the usual slice theorem, and which allows us to construct and visualize a PFB structure in \( S' \).

Now, clearly the map \( \Psi : M' \times D(M) \rightarrow M' \) acts freely since if \( f^*g = g \) for \( g \in M' \), \( f = Id \) by construction of \( M' \). It can be shown that the action is also proper \([13]\).

**Theorem 1 (Slice)** If \( M' \) is a smooth manifold and \( D(M) \) acts smoothly, freely and properly on \( M' \) on the right, than through any point \( g \in M' \) there exists a submanifold \( \Sigma \) such that \( \Sigma \) is transversal to the orbits and intersects each one of them at a single point. Such a submanifold is called a section through \( g \).

**Proof** (sketch):

Given \( g \in M \), it induces a pointwise metric in all tensor bundles over \( M \), which for \( S_2(M) \) we will call \( G_0(\cdot, \cdot)_g \). Being explicit, for elements of \( S_2(M) \) of the form

\[
w = \frac{1}{2}(w_1 \otimes w_2 + w_2 \otimes w_1), \quad \lambda = \frac{1}{2}(\lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_1)
\]

denoting by \( g^* \) the metric on \( TM^* \) dual\(^8\) to \( g \), and since the bundle metric is of the form:

\[
G_0(\lambda_1 \otimes \lambda_2, w_1 \otimes w_2) = g^*(\lambda_1, w_1)g^*(\lambda_2, w_2)
\]

we have:

\[
G_0(\lambda, w)_g = \frac{1}{2}G_0(\lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_1, (w_1 \otimes w_2 + w_2 \otimes w_1))_g
= \frac{1}{2}(g^*(\lambda_1, w_1)g^*(\lambda_2, w_2) + g^*(\lambda_1, w_2)g^*(\lambda_2, w_1))
\]

(2)

In coordinates this would be \( G^{ijkl}_0 = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) \). Hence we define the weak \( L_2 \) Riemannian metric \( \mathcal{G}_0 \) of \( M \): for \( v_1, v_2 \in S_2(M) \) and \( g \in M \)

\[
\mathcal{G}_0(v_1, v_2)_g = \int_M G_0(v_1, v_2)_g d\mu_g
\]

(3)

where \( d\mu_g \) is the volume element associated to \( g \) (it is an \( n \)-form only if \( M \) is orientable) locally given by \( \sqrt{\det g} dx \), and \( \mathcal{G}_0 \) is invariant by \( D(M) \), since, by linearity, \( Tf^* = f^* \)

\[
((f^*)\mathcal{G})_0(v_1, v_2)_g = \int_M G_0(Tf^*v_1, Tf^*v_2)_{f^*g} d\mu_{f^*g}
= \int_M (G_0(v_1, v_2)_g \circ f)(f^*(d\mu_g))
= \int_M f^*(G_0(v_1, v_2)_gd\mu_g)
= \mathcal{G}_0(v_1, v_2)_g
\]

\(^8\)A submanifold in the infinite dimensional case is given by an injective immersion, where \( f : X \rightarrow Y \) is an immersion at \( x \in X \) if \( T_xf \) is injective and splits.

\(^9\)I.e. since \( \xi : TM \rightarrow TM^* \) is an isomorphism, \( g^*\langle v^*, w^* \rangle_x = g(v, w)_x, v, w \in T_xM \)
Clearly this is a positive definite inner product, and hence $\mathcal{G}_0$ defines a linear injective mapping from the tangent space $T_g\mathcal{M}$ to its dual, $T_g\mathcal{M}^*$, the space of distributional densities with values in $TM \times_s TM$. This linear mapping is not surjective, and that is why $\mathcal{G}_0$ is only a weak metric, leaving all objects defined implicitly by the metric (such as the Levi-Civita connection) a priori existing only on the Sobolev completion of the tangent spaces. Even though existence cannot be guaranteed for implicit objects, the definiteness of the metric indeed guarantees uniqueness of such objects. Ebin [31], explicitly constructs the covariant derivative, and hence, implicitly, the unique exponential mapping with all the standard properties.

Now, as we mentioned, $T_e\Psi_g$ maps $\Gamma(TM)/\mathcal{I}_g(M)$ injectively into a closed subspace of $S_2(M)$ (in our case $\mathcal{I}_g(M) = \mathrm{Id}(M)$) which splits. Denoting the orbit $\Psi_g(\mathcal{D}(M))$ by $\mathcal{O}_g$, we shall now consider the normal bundle $\nu\mathcal{O}_g$, whose smoothness we will have to assume: we only possess a weak Riemannian metric the usual construction does not work.

We note however, that employing the individual separable Hilbert manifolds $S^0\mathcal{M}^s$ and $\mathcal{O}_g^s$, since we have assumed $\mathcal{O}_g$ to be a smooth submanifold, the inclusion $\iota^s : \mathcal{O}_g^s \hookrightarrow \mathcal{M}^s$ induces a splitting injection $T_h\iota^s : T_h(\mathcal{O}_g^s) \rightarrow T_h\mathcal{M}^s$. And so over an open neighborhood $V^s$ of $g \in \mathcal{M}^s$ we may assume an isometric chart of the form $\eta^s : V^s \rightarrow H^1 \oplus H^2$, where $H^1, H^2$ are separable Hilbert spaces, $H^1 \oplus H^2$ has the orthogonal inner product and $\eta^s(V^s \cap \mathcal{O}_g^s) = H^1 \times \{0\}$.

So $(\nu\mathcal{O}_g^s) \cap V^s = \mathrm{Ker}(\mathrm{pr}_2 \circ T\eta^s)$, where $\mathrm{pr}_2 : H^1 \oplus H^2 \rightarrow H^2$ is the usual projection. Since manifolds modeled on separable Hilbert spaces (separable Hilbert manifolds) have partitions of unity, we can smoothly glue together the various charts, respecting the orthogonal decomposition, and hence directly obtain the normal bundle. We will not prove that this holds true in the inverse limit. Now we derive the conditions for orthogonality with respect to the $L_2$ metric.

Since $M$ is compact, every $X \in \Gamma(TM)$ is complete and $\Gamma(TM)$ forms a Lie algebra under the usual Lie bracket. We define $X$’s integral flow, as $\phi(tX) \in \mathcal{D}(M)$, which has the following trivial properties\(^{10}\)

$$\phi(tX) \circ \phi(sX) = \phi((t+s)X) \quad \text{and} \quad \phi(0) = \mathrm{Id}$$

and for which the usual existence and uniqueness theorems are valid. We may then define

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\(^{10}\) See [14] or [3] for definitions of $\mathcal{M}^s$. We give the gist of the explanation: Let us call $J^s(L^2_0(TM))$ the sth jet bundle of $L^2_0(TM)$. If $J^s(L^2_0(TM))$ is endowed with a Riemannian structure $(\langle \cdot , \cdot \rangle_s)$, given a volume element $d\mu$ on $M$, it induces an inner product in $C^\infty(J^s(L^2_0(TM)))$ and hence in $C^\infty(L^2_0(TM))$ as well. $\mathcal{M}^s$ is the enlarged Hilbert manifold made from the Sobolev completion of $C^\infty(L^2_0(TM))$ with respect to $(\langle \cdot , \cdot \rangle_s)$: its tangent space at any point is the space of sections of $L^2_0(TM)$ which have square integrable partial derivatives up to order $s$.

\(^{11}\) We note that $\mathcal{D}(M)$ qualifies as a strong ILH-Lie group, but the image of the exponential mapping does not cover a neighborhood of the identity, since any (metric) open set containing the identity has non-null intersection with the set of “large diffeomorphisms”, i.e. those not contained in the identity component $\mathcal{D}(M)_0$ of $\mathcal{D}(M)$. However, the subgroup generated by the flow of vector fields is a non-trivial normal subgroup of the simple group $\mathcal{D}(M)_0$, hence they are equivalent. There is an interesting debate about the role of the large diffeomorphisms in physical theories, as to whether or not they are observable symmetries or should be treated as the usual gauge transformations [15]. This debate arises, because, as we will mention later, the supermomentum constraint requires only invariance with respect to $\mathcal{D}(M)_0$, not $\mathcal{D}(M)$. In [14] the alternative spaces $S_0 := \mathcal{M}/\mathcal{D}(M)_0$, $C := \frac{\mathcal{M}/P}{\mathcal{D}(M)_0}$ and $C_0 := \frac{\mathcal{M}/P}{\mathcal{D}(M)_0}$, where $P = \mathrm{Pos}(M)$ is the abelian group of smooth positive functions on $M$, are considered.
\( j_g \) to be the isomorphism

\[
j_g := T_e \Psi_g : \Gamma(TM) \to V_g
\]

\[
X \mapsto L_X g
\]

which is well defined since we know \( V_g := T_g(\mathcal{O}_g) = \{L_X g \mid X \in \Gamma(TM)\} \). Denoting by \( \nabla \) the Levi-Civita connection on \( TM \), we take the formula

\[
L_X g = g(\cdot, \nabla X) + g(\nabla X, \cdot) = (X_{ij} + X_{ji}) dx^i \otimes dx^j
\]

which is easily derivable from the product rule for derivatives (both the Lie and \( \nabla \)) and the zero torsion condition of \( \nabla \) (and thence converting it to local coordinates, where \( ; \) denotes covariant derivative), and input it in our weak metric, for \( v \in T_g \mathcal{M}' \):

\[
G_0(j_g(X), v)_g = \int_M \frac{1}{2} (g^{ik}g^{jl} + g^{il}g^{jk})(X_{ij} + X_{ji})v_{kl}(d\mu_g)
\]

\[
= \int_M (X_{ij} + X_{ji})v_{ij}(d\mu_g)
\]

\[
= -2 \int_M X_i v_{ij}^i (d\mu_g)
\]

where we have used Stoke’s Theorem and discarded divergencies. Thus, \( \text{Im}(T_e \Psi_g) = T_g \mathcal{O}_g = \{L_X g \mid X \in \Gamma(TM)\} \) has a closed \( \mathcal{G} \)-orthogonal complement in \( S_2(M) \), where the condition for orthogonality is:

\[
H_g := \{v \in S_2(M) \mid \delta_g v = 0\}
\]

which in abstract tensor notation\(^{12}\) becomes \( \nabla^a v_{ab} = 0 \), the divergence-free two covariant tensor fields in \(^{13}\) \( M \). So we have the direct sum decomposition \( T_g \mathcal{M} = T_g \mathcal{O}_g \oplus H_g \), which, by the equivariance of the metric and equivariance of \( \mathcal{O}_g \) is easily seen to be equivariant.

Now, we will denote the exponential map of the \( \mathcal{D}(M) \)-invariant \( L_2 \) Riemannian metric on \( \mathcal{M} \) by \( \text{Exp} \), it is a \( C^{\infty} \), \( \infty \) ILH-normal mapping\(^{11}\). It trivially induces an exponential map on the normal bundle, which we will denote by \( \text{Exp} \) as well.

We define the submanifold \( \Sigma_g \) by exponentiating \( H^0_g \), a (sufficiently small) open neighborhood of the origin in \( H_g \). But is the exponential of \( H^0_g \) really a submanifold? We have to check it is an immersion (in the infinite dimensional sense).

The tangent space at a zero normal vector over any point can be given the direct sum decomposition\(^\text{14}\)

\[
T_{(g,0)}(\nu \mathcal{O}_g) \simeq T_g \mathcal{O}_g \oplus \nu_g \mathcal{O}_g = T_g \mathcal{O}_g \oplus H_g = T_g \mathcal{M}
\]

\(^{12}\)In this notation, the indices merely indicate the type of tensor field, and repeated indices indicate contraction.

\(^{13}\)In \( \mathcal{M}' \), the construction of the normal bundle is made through the following argument: for \( \mathcal{M}' \) we have that \( \text{Ker}_{\mathcal{O}_g}(\mathcal{O}_g) = 0 \), and since the operator \( (\mathcal{O}_g)^* \) is elliptic, its inverse is a continuous (with respect to the \( L_2 \) metric) map between smooth vector fields and we have that

\[
\Gamma(TM) = \text{Im}(\mathcal{O}_g) \oplus \text{Ker}(\mathcal{O}_g)^*
\]

orthogonally and continuously with respect to the \( L_2 \) metric.

\(^{14}\) See theorem 19 of \( \mathcal{M} \).
Over a fixed fiber of $T\mathcal{M}$, i.e. for $v \in T_g\mathcal{M}$, $\text{Exp}(g,v) = \text{Exp}_g(v)$ and we have, taking $(w,u) = \xi \in T_g\mathcal{M}$:

$$T_{(g,0)}\text{Exp}(\xi) = \frac{d}{dt}|_{t=0}\left(\text{Exp}(\gamma(t),0)\right) + \frac{d}{dt}|_{t=0}\left(\text{Exp}(g,tu)\right)$$

$$= \frac{d}{dt}|_{t=0}\left(\gamma(t),0\right) + \frac{d}{dt}|_{t=0}\left(\text{Exp}_g(tu)\right)$$

$$= (w,0) + (0,u) = \xi$$

So utilizing the identification (5), we have shown that

$$T_{(g,0)}(\text{Exp}) = \text{Id}|_{T_g\mathcal{M}}$$

(6)

which by the Inverse Function Theorem\(^{15}\) makes the normal exponential a diffeomorphism which respects the normal decomposition. So $T_{(g,0)}\text{Exp}|_g^\mathcal{H}_g$ splits and $\Sigma_g$ is indeed a submanifold, which by construction is transversal to the orbits at $g \in \mathcal{M}$. By the openness of the trasversality condition, it is transversal over an open neighborhood of $g$ in $\Sigma_g$, which we may replace for $\Sigma_g$ itself.

The other condition, that $\Sigma_g$ only intersect orbits at a single point, is much more delicate, and we won’t go into the more complicated details.

We have in $\mathcal{M}$ a $\mathcal{D}(M)$-invariant metric, therefore, $\mathcal{D}(M)$ takes geodesics to geodesics. But in our case, if $f \neq e$ then $f^*g \neq g$ and for $u \in H^0_g$, $f^*\text{Exp}(g, tu)$ is a geodesic that passes through $f^*g$ with tangent

$$\frac{d}{dt}|_{t=0}(f^*\text{Exp}(g, tu)) = f^* \circ \frac{d}{dt}|_{t=0}(\text{Exp}(g, tu)) = f^*(u)$$

the same as $\text{Exp}(f^*g, tf^*(u))$, thence, by uniqueness

$$f^*\text{Exp}(g, u) = \text{Exp}(f^*g, f^*(u))$$

(7)

Moreover, as we mentioned, since $\mathcal{G}$ is $\mathcal{D}(M)$ equivariant, the action of $\mathcal{D}(M)$ preserves the orthogonality of the normal fibers and $\mathcal{O}_g$, and preserves also the length of $u \in H_g$. Hence

$$f^*(u) \in f^*H^0_g \subset H_{f^*g} = \nu_{f^*g}\mathcal{O}_g$$

(8)

So by (8) and (7) we have the equivariance $f^*\Sigma_g = \Sigma_{f^*g}$.

But this does not yet mean that $f^*\Sigma_g \cap \Sigma_g = \emptyset$ for $f \neq \text{Id}$, which is what we wish to show and is the complicated part which we will really just mention. Using the fact that the normal exponential is a local diffeomorphism between a neighborhood of the null section in $\nu\mathcal{O}_g$ and an open set in $\mathcal{M}$ containing $\mathcal{O}_g$, and the properness of the action of $\mathcal{D}(M)$, remembering that proper maps between metrizable spaces are closed, it is possible to show that the normal exponential is a global diffeomorphism between a neighborhood of the null section $W \subset \nu\mathcal{O}_g$ and an open set in $\mathcal{M}$ containing $\mathcal{O}_g$.

Therefore, the correspondent family of submanifolds $\{\Sigma_h \mid h \in \mathcal{O}_g^*\}$ is such that for $g \neq h \in \mathcal{O}_g$ we have $\Sigma_g \cap \Sigma_h = \emptyset$. And so by the equivariance above $f^*\Sigma_g \cap \Sigma_g = \emptyset$ for $f \neq \text{Id}$ which proves the second part. $\square$

---

\(^{15}\)The Inverse function Theorem is not usually valid in Frechét manifolds, however it is valid for ILH-normal mappings in ILH manifolds [11], and the exponential mapping is indeed ILH-normal [3].
2.1 Sections in $\pi : \mathcal{M}' \to \mathcal{S}'$

We will call $\mathcal{M}'/\mathcal{D}(M) := \mathcal{S}'$ (local) superspace and the (continuous open) projection by

$$\pi : \mathcal{M}' \to \mathcal{S}'$$

$$g \mapsto \tilde{g}$$

Now, since $\text{Exp} : \mathcal{W} \to \mathcal{M}$ is a diffeomorphism onto an open subset of $\mathcal{M}$, we have that for any $\xi \in \text{Exp}(\mathcal{W}) \subset \mathcal{M}$, there exist unique $(h, w) \in W \subset \nu\mathcal{O}_g$, $f \in \mathcal{D}(M)$ and $u \in \nu_g \mathcal{O}_g$ such that

$$\xi = \text{Exp}(h, w) = \text{Exp}(f^*g, f^*u) = f^*\text{Exp}(g, u) = \Psi(\text{Exp}(g, u), f)$$

And since $\text{Exp}(g, u) \in \Sigma_g$, the section we have constructed guarantees that for every $\tilde{g} \in \mathcal{S}'$ there exists an open set $\tilde{U} \subset \mathcal{S}'$ such that we have a diffeomorphism

$$\varphi : \Sigma_g \times \mathcal{D}(M) \to \pi^{-1}(\tilde{U})$$

$$(h, f) \mapsto f^*h$$

So now $\mathcal{S}'$ has a proper (ILH) smooth manifold structure induced by the sections, that is, by defining the bijection $\pi|_{\Sigma_g}$ to be a diffeomorphism. This is well defined, since general sections over the same open (in the quotient topology) set $\theta \subset \mathcal{M}'/\mathcal{D}(M)$, are diffeomorphic amongst themselves (just by being smooth submanifolds with single intersections with the orbits and to them transversal\[16\]. Then $\pi|_{\Sigma}$ is a diffeomorphism for any section $\Sigma$, and we may associate to $\Sigma$ a diffeomorphism playing the correspondent role as $\varphi$ above.

Now, we may rename as a ‘section’ the map $s : \tilde{U} \to \mathcal{M}'$, that has as its range the submanifold we previously called ‘section’. I.e. given the associated diffeomorphism $\varphi$, it would be defined as

$$\zeta(\tilde{g}) := \varphi(\pi^{-1}\tilde{g}, \text{Id})$$

(9)

If $\tau : \tilde{U} \subset \mathcal{S}' \to \mathcal{M}'$ is any other section over $\tilde{U}$ we have $\tau(\tilde{g}) = \varphi(\tilde{g}, f_\tau^*)$ for an appropriate smooth map $f_\tau^* : \tilde{U} \to \mathcal{D}(M)$, for which it is a trivial exercise to check cocycle\[17\] conditions:

$$f_\tau^*(\tilde{g}) = (f_\tau^*(\tilde{g}))^{-1} \quad \text{and} \quad f_\tau^*(\tilde{g})f_\tau^*(\tilde{g})f_\tau^*(\tilde{g}) = \text{Id}$$

(10)

Even if we were not considering $\mathcal{M}'$, but $\mathcal{M}$ in its totality, Fischer\[13\] devised an ingenious scheme to unfold these singularities through the use of additional structure over $M$. He considered the action of $\mathcal{D}(M)$ on the product space $\mathcal{M} \times F(M)$, where $F(M)$ is the frame bundle of $M$, over which there also exists a natural right action of $\mathcal{D}(M)$:

$$\chi : F(M) \times \mathcal{D}(M) \to F(M)$$

$$(u, f) \mapsto f^*(u) := f_*^{-1}(u)$$

\[16\]One can easily see this by observing the smooth image of this more general section (i.e. not constructed via the exponential mapping) by the diffeomorphism $\varphi^{-1}$ induced by the section $\Sigma_g$ (constructed via the exponential mapping)\[8\]. This entails the existence of smooth map between points in $\Sigma$ and points in $\Sigma_g$. Single intersections mean this is an injective map, and transversality then yields the same decomposition of total space, meaning that the tangent to this map is an isomorphism.

\[17\]We will generalize this transformation to gauge transformations in (Sec. 3.1), when we discuss the meaning of such transformations.
The point revolves around the fact that the combined action:

$$\Phi : (\mathcal{M} \times F(\mathcal{M})) \times \mathcal{D}(\mathcal{M}) \rightarrow \mathcal{M} \times F(\mathcal{M})$$

$$(g, u, f) \mapsto (f^*g, f^*(u))$$

is free.

That is, if $f$ is an isometry that fixes a frame, it is the identity. To see this, consider two isometries $f, h \in I_g(\mathcal{M}) \subset \mathcal{D}(\mathcal{M})$ such that $T_pf = T_p h$, hence $f(p) = h(p)$. Now the set $A = \{ x \in M \mid T_x f = T_x h \}$ is closed in $M$ by continuity. Now, suppose $x \in A$, and consider a normal neighborhood $U$ of $x$ (where the exponential is injective). For $q \in U$ there exists $v \in T_x M$ such that $\exp_x v = q$, where $\exp$ is the exponential map of $(\mathcal{M}, g)$. Now,

$$f(q) = f(\exp_x v) = \exp f(x) T_x f(v) = \exp h(x) T_x h(v) = h(\exp_x v) = h(q)$$

So $A$ is open as well, hence, since we are assuming $M$ connected, $f = h$. Putting $h = \text{Id}$, we have that $f = \text{Id}$ as well.

Fischer is able to show that $\mathcal{S}_{FM} := (\mathcal{M} \times F(\mathcal{M})) / \mathcal{D}(\mathcal{M})$ is a smooth manifold and indeed the base space for the $\mathcal{D}(\mathcal{M})$ PFB $\pi_1 : \mathcal{M} \times F(\mathcal{M}) \rightarrow \mathcal{S}_{FM}$. Furthermore, we have a covering:

$$\pi_2 : \mathcal{S}_{FM} \rightarrow \mathcal{S}$$

$$[(g, u)] \mapsto \tilde{g}$$

and in fact the (non-isomorphic) fibers $\pi_2^{-1}(\tilde{g})$ index the isometry group of the metric $g$. We note that in all the cases treated here, the metrics we used are only auxiliary, and of course the PFB-smooth structure constructed is independent thereof.

3 Connections in $\mathcal{S}'$

3.1 Interpretation

Smoothness of “Coordinates”

How can we interpret a section in the PFB $\pi : \mathcal{M}' \rightarrow \mathcal{S}'$? We hold that it should be seen as an identification of points over a smooth substratum, diffeomorphic to $\mathcal{M}$, where there exists no a priori notion of location. With respect to such an identification, a metric is described. To facilitate comprehension we will call from now on an “identification of points”, or a “notion of location”, a “choice of coordinates”. We note that as usually understood, a choice of coordinates is only local in $\mathcal{M}$, and conveys a much more passive note than the active meaning of an identification of points.

Anyhow, in this way we parameterize the elements of $\pi^{-1}(\tilde{g})$ by choices of coordinates alone. According to these notions of locations, a geometry is described. We consider the action of the group of diffeomorphisms as giving another identification of points (coordinates) over $\mathcal{M}$.

To better appreciate this instance of active diffeomorphisms, we may forego of the particular manifold $\mathcal{M}$ and consider the points of $\mathcal{O}_g$ as being truly different manifolds endowed
with compatible metrics. That is:

\[ \mathcal{W}_g = \{(N,h) \mid h \in \text{Riem}(N), h = f^*g \text{ where } f : N \to M \text{ is a diffeomorphism}\} \]
\[ = \{f^*(M,g) \mid f : N \to M \text{ is a diffeomorphism} \} \]

The generalization from single orbits to \( \mathcal{M}' \) is made by dropping the assumption above that \( h = f^*g \). That is, to take the following as the elements of our considered space: an identification of points over a substratum and a metric therein.

This is done as a heuristic instrument, to visualize a section in \( \pi : \mathcal{M}' \to \mathcal{S}' \) as a smooth selection of manifolds endowed with metrics. Smooth selection of manifolds only makes sense with respect to the metrics themselves:

**Definition 1** A one parameter family of manifolds is said to be \( C^s \) if and only if it comes along with a one parameter family of metrics, i.e. \( \{M_t, g_t\} \), and there exists a one parameter family of diffeomorphisms \( f_t \) connecting them such that, as a curve of metrics over \( M_0 \),

\[ \frac{d}{dt}f_t^*(g_t) \text{ is pointwise continuous in the topology induced by } g_0 \text{ on } TM^* \otimes_S TM^* \].

Clearly, in classical general relativity we are only worried about “coordinates” along a curve of metrics. However, it is not hard to see that an appropriate way to choose “coordinates” over an open set of geometries in \( \mathcal{S}' \) consists of choice of submanifold of \( \mathcal{M}' \) transversal to the orbits, i.e. a section.

Using the notion of smoothness presented in the above definition, we can then define a generalization of a time dependent change of “coordinates”:

**Definition 2** A gauge transformation is a diffeomorphism \( \Upsilon : \mathcal{M}' \to \mathcal{M}' \) such that for any \( f \in \mathcal{D}(M) \), \( g \in \mathcal{M}' \), we have \( \Upsilon(f^*g) = f^*\Upsilon(g) \) (i.e. it is a bundle automorphism) and \( \pi(\Upsilon) = \pi \) (i.e. it covers the identity).

For any gauge transformation \( \Upsilon \) we can define a function \( \gamma : P \to \mathcal{D}(M) \), called the \( \mathcal{D}(M) \)-form of \( \Upsilon \), such that \( \Upsilon(g) = \gamma(g)^*g \). We have that

\[ \Upsilon(f^*g) = f^*\Upsilon(g) = f^*\gamma(g)^*g = (\gamma(g) \circ f)^*g \]
\[ = \gamma(f^*g) = f^*g = (f \circ \gamma(f^*g))^*g \]
\[ \therefore \gamma(g) \circ f = f \circ \gamma(f^*g) \]
\[ \therefore \gamma(f^*g) = f^{-1} \circ \gamma(g) \circ f \]

where in the third line we used that \( g \) has no symmetries, i.e. if \( f^*g = g \) then \( f = \text{Id} \). So we can identify the space of gauge transformations with the space of mappings \( \gamma : \mathcal{M}' \to \mathcal{D}(M) \) satisfying \( \gamma(f^*g) = f^{-1} \circ \gamma(g) \circ f \).

**Connections and “Coordinates”**

Now, given two identification of points over the substratum and metrics over them, we can canonically quantify any change in “coordinates” only if we are sitting over a single geometry, or \( O_g \), since there exists a single diffeomorphism taking each “coordinate” to each other “coordinate”. The same happens with a frame bundle over \( TM \): while over a single fiber, there exists a unique canonical element of \( \text{GL}(F) \), where \( F \simeq T_zM \), that connects two given frames.
But when you move from fiber to fiber, you don’t have a canonical backdrop against which you can measure “intrinsic changes of coordinate”, since there is no canonical identification between neighboring fibers. This distinction can be seen as a consequence (or cause) of the fact that there is a canonical projection $\pi : M' \to S'$ and therefore a canonical vertical subspace at any $g \in M'$ given by $V_g := \text{Ker}(d\pi_g)$. However there is no canonical complement to the subbundle $V \subset S_2(M)$.

Of course, in the case of the frame bundle over $TM$ a connection provides such an identification along paths. The exact same thing goes on here: when we move from fiber to fiber along a smooth path, we have no canonical way of quantifying “how much” of the metric is changing due to the changing “coordinates” on the substratum, and how much is “intrinsically” changing. A connection form over a bundle of bases, $\omega$ allows us to determine the share of a given infinitesimal dislocation in $P$ due to a change of basis. The kernel of $\omega$ determines the directions in $P$ in which bases are identified, i.e. remain “constant”. Changing the word “basis” to “identification of points over $M$” accomplishes this analogy.

As a matter of fact, many of the usual results for a general principal fiber bundle (see Sec. 2.2 in [8]) are valid for this infinite-dimensional case, and coordinate and dimension free proofs can usually be transposed without further complications.

### 3.2 Formalism

If our connection is going to measure the infinitesimal intrinsic change of “coordinates”, i.e. if it is going to be the flux of a curve of diffeomorphisms, it has to be a $\Gamma(TM)$-valued one form $\omega : S_2(M) \to \Gamma(TM)$, or loosely $\omega \in \Omega^1(M', \Gamma(TM))$.

A connection form is basically a projection on the vertical subspaces, hence it is equivalent to a choice of horizontal distribution, i.e $H_g = \text{Ker}(\omega_g)$, or, given the decomposition $TM = V \oplus H$ and denoting the respective linear projections by $\hat{V}, \hat{H}$, we have

$$\omega_g = j_g^{-1} \circ \hat{V}_g$$

Both the horizontal and the vertical distributions should be invariant by the group action. This means they respect the canonical identification between elements of the bundle $TM|_{O_g}$ by the group action (i.e. by $f^* \in \mathcal{D}(M)$)[8]. Thus our connection form should obey some equivariance condition. To find out what it is we have to explore some of the group structure of $\mathcal{D}(M)$, for which we follow the nomenclature and results of [11].

### The Adjoint Representation

The natural representation of any Lie group on itself, $\rho : G \to \text{Aut}(G)$, must be a Lie automorphism, which restricts its form naturally to $\rho(g)(h) = ghg^{-1}$. The derivative of $\rho(g) : G \to G$, given by $\rho(g) : \mathfrak{g} \to \mathfrak{g}$, is called the adjoint representation.

In our infinite-dimensional case[12] we have the analogous notion of an ILH-representation. Besides a smoothness condition on which we will not touch, a ILH representation must obey linearity with respect to $v$ and $\rho(\rho(f)(v), h) = \rho(f \circ h)(v)$, for $f, h \in \mathcal{D}(M), v \in F$.

---

18 Or integer powers thereof.
19 We will give further justification for the adjoint representation at Sec. 4.1.
Furthermore, the space of bundle automorphisms of sections over a vector bundle, Aut(Γ(E)) will be defined to be the one induced by Aut(E). More explicitly for \( f \in \text{Aut}(E) \) we define \( \hat{f} \in \text{Aut}(\Gamma(E)) \) to act pointwise as \( f \), i.e. \( \hat{f}(s)(x) = f(s(x)) \).

For \( G = D(M) \), we define the adjoint representation as the continuous linear mapping:

\[
\text{Ad}(f) : \Gamma(TM) \rightarrow \Gamma(TM) \\
X \mapsto \frac{d}{dt}_{|t=0} (f \circ \phi(tX) \circ f^{-1})
\]

By uniqueness of integral curves (passing through Id),

\[ \phi(t\text{Ad}(f)(X)) = f \circ (\phi(tX)) \circ f^{-1} \quad (11) \]

In fact, by (Lemma 4.3):

**Lemma 1** We have the following properties of the Ad map:

(i) \( \text{Ad}(f)\text{Ad}(h) = \text{Ad}(fh) \) for \( f, h \in D(M) \).

(ii) For \( X, Y \in \Gamma(TM) \),

\[
[X, Y] = \frac{d}{dt}_{|t=0} \text{Ad}(\phi(tX))Y
\]

is an element of \( \Gamma(TM) \) and \([,] : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)\) is a smooth skew symmetric bilinear mapping.

(iii) \( \text{Ad}(f)[X,Y] = [\text{Ad}(f)X,\text{Ad}(f)Y] \)

(iv) \([X[Y,Z]] + [Y[Z,X]] + [Z[X,Y]] = 0 \)

It is easily checked that \( \text{Ad} : D(M) \rightarrow \text{Aut}(\Gamma(TM)) \) is indeed a ILH-representation (besides the smoothness condition, that we will assume Prop. 4.2): clearly \( \text{Ad}(f)(cX) = c\text{Ad}(f)X \), and from property (i) of Lem. it is ILH-representation. Furthermore, calling \( \text{ad} := T_{\text{Id}} \rho \) we have, by (ii) of Lem. it \( \text{ad}_X Y = \frac{d}{dt}_{|t=0} \text{Ad}(\phi(tX))Y = [X,Y] \), i.e.

\[
\text{ad} : \Gamma(TM) \rightarrow \text{End}(\Gamma(TM)) \\
X \mapsto [X,\cdot]
\]

Now, to find the promised equivariance condition on our connection form, since

\[
\mathcal{J}_g(X) = \frac{d}{dt}(\phi(tX)^*(g))
\]

and \( Tf^* = f^* \), by (11) we have that

\[
\mathcal{J}_g(\text{Ad}(f)(X)) = \frac{d}{dt}_{|t=0} (f \circ \phi(tX) \circ f^{-1})^*g 
\] \quad (12)

\[
\therefore f^*(\mathcal{J}_g(\text{Ad}(f)(X))) = \frac{d}{dt}_{|t=0} (f \circ \phi(tX)f^{-1} \circ f)^*g 
\] \quad (13)

\[
= \frac{d}{dt}_{|t=0} (\phi(tX))^*(f^*g) = f^*_g(X) 
\] \quad (14)
Hence, \( j_{f^*g}^{-1} = \text{Ad}(f^{-1})j_g^{-1} \circ (f^{-1})^* \) and utilizing the equivariance of \( \hat{\nabla} \)

\[
(f^*\omega)_g = j_{f^*g}^{-1} \circ \hat{\nabla}_{f^*g} = j_{f^*g}^{-1} \circ f^* \circ \hat{\nabla}_g
\]

\[
f^*\omega = \text{Ad}(f^{-1})j_g^{-1} \circ \hat{\nabla}_g
\]

Which matches the usual equivariance condition over a connection form.

**The Exterior Covariant Derivative**

We know that ultralocally (i.e. over a single point \( g \in \mathcal{M}' \)), a \( p \)-form \( \eta \in \Lambda^p(\mathcal{M}') \) is described by a section in \( \Lambda^p(TM^* \otimes_S TM^*) \). Since vectors in \( T_g\mathcal{M}' \) are sections in \( S_2(M) = \Gamma(TM^* \otimes_S TM^*) \), the action of forms over fields can be defined pointwise in \( \mathcal{M}' \) as the integration of the usual action of forms over fields in \( M \).

Let \( \eta \in \Lambda^1(\mathcal{M}') \) and \( v \in T_g\mathcal{M}' \), then, since we prioritize the ultralocal form, we will abuse notation and denote by \( \eta_g(v) \in C^\infty(M) \) the function which when integrated yields the real number

\[
\eta_g[v] = \int_M \eta_g(v) d\mu_g
\]

The bracket here denotes functional dependence on the section (over \( M \)) given by \( v \). In this way we can define

**Definition 3** *The exterior derivative* \( \tilde{\text{d}} : \Lambda^1(\mathcal{M}') \to \Lambda^2(\mathcal{M}') \) *is pointwise given by*

\[
(\tilde{\text{d}}\eta)_g[v_1, v_2] := \int_M d\eta_g(v_1, v_2) d\mu_g
\]

where \( d : \Lambda^p(TM^* \otimes_S TM^*) \to \Lambda^{p+1}(TM^* \otimes_S TM^*) \). The same definition can be extended for \( p \neq 1 \) and more general operators.

As we have being doing so far, we will continue restricting our attention to the (ultra)local, non-functional, expressions, and will henceforth omit the distinction between \( \tilde{\text{d}} \) and \( d \).

If \( \eta \in \Omega^p(\mathcal{M}', \Gamma(E)) \), i.e. a \( p \) alternating form on \( \mathcal{M}' \) taking values in \( \Gamma(E) \), for \( E \) a vector bundle over \( M \), is of the form

\[
\sum_i \eta^i \otimes X_i \text{ where } \eta^i \in \Lambda^p(\mathcal{M}') \text{ and } X_i \in \Gamma(E)
\]

we can define the exterior derivative applied to \( \eta \) simply as \( d\eta := \sum_i d\eta^i \otimes X_i \).

Since pointwise we can uniquely decompose any element of \( \Omega^p(\mathcal{M}', \Gamma(E)) \) in terms of a countable basis of such tensor products (these spaces are modeled on separable Hilbert spaces), we define the exterior covariant derivative for a \( \Gamma(E) \)-valued \( p \)-form :

\[
d^H \eta = d\eta \circ \hat{H}
\]

and the curvature form is defined as usual:

\[
\Omega := d^H \omega = d\omega\left(\hat{H}(\cdot), \hat{H}(\cdot)\right) \in \Omega^2(\mathcal{M}', \Gamma(TM))
\]
for which the usual Bianchi Identity holds \(11\): \(d^H \Omega = 0\). Once again, the reason these formulas work, even in this infinite-dimensional setting, is that operators such as the exterior derivative and constraints such as horizontality are enforced (ultra)locally on the fields defined in \(\mathcal{M}'\).

There is an easier to handle formula for the curvature form

\[
\Omega = \omega([\hat{H}, \hat{H}]) = \omega([\text{Id} - J^{-1} \circ \omega, \text{Id} - J^{-1} \circ \omega])
\]  

(17)

the proof is identical to the finite-dimensional case, and follows by applying the curvature form to horizontal and vertical fields (see Theo. 23 \[8\]). It is well to keep in mind that the curvature is non-null strictly for horizontal fields, and clearly it is the lack of integrability of the horizontal subbundle.

However, this is not yet the most common visualization for the curvature form. To get it, we introduce a general formulation for the wedge (or exterior) product on forms taking values in \(\Gamma(TM)\), pointwise, by

\[
\wedge : \Omega^p(\mathcal{M}', \Gamma(TM)) \times \Omega^q(\mathcal{M}', \Gamma(TM)) \rightarrow \Omega^{p+q}(\mathcal{M}', \Gamma(TM) \otimes \Gamma(TM))
\]

\[
\lambda^1 \wedge \lambda^2(v_1, \ldots, v_{p+q}) = \frac{p!q!}{(p+q)!} \sum_{\sigma} \tau(\sigma) \lambda^1(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \otimes \lambda^2(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})
\]

(18)

where \(\sigma \in P(p+q)\) the group of permutations of \(p+q\) elements. For \(p = q = 1\) and \(\lambda^1 = \lambda^2 = \lambda\), regarding vector fields as derivations on \(C^\infty(M)\) and substituting the tensor product above by composition of derivatives we get

\[
\lambda \wedge \lambda(v_1, v_2) = \frac{1}{2} \lambda(v_1)\lambda(v_2) - \lambda(v_2)\lambda(v_1) = \frac{1}{2} [\lambda(v_1), \lambda(v_2)] =: \frac{1}{2} [\lambda, \lambda](v_1, v_2)
\]

(19)

At which point it is possible to derive a simple formula for the exterior derivative: given \(\eta \in \Omega^p(\mathcal{M}', \Gamma(TM))\) and a representation \(\rho : \mathcal{D}(M) \rightarrow \text{Aut}(\Gamma(E))\),

\[
d^H \eta = d\eta + \frac{1}{2} T_{\text{Id}} \rho(\omega) \wedge \eta
\]

(20)

where in this instance the wedge product takes into account the natural action of \(\text{End} \times \text{End} \rightarrow \text{End}\) by contraction:

\[
\tilde{\wedge} : \Omega^1(\mathcal{M}', \text{End}(\Gamma(TM))) \times \Omega^1(\mathcal{M}', \Gamma(TM)) \rightarrow \Omega^2(\mathcal{M}', \Gamma(TM))
\]

Formula (20) is derived by explicitly computing both sides again for horizontal and vertical vectors.

For the substitution \(g \rightarrow \Gamma(TM)\), i.e. for \(E = TM\), \(\rho = \text{Ad}\) and \(u, v \in T_g \mathcal{M}'\) we have:

\[
ad_{\omega} \tilde{\omega}(u, v) = \frac{1}{2} ([\omega(u), \omega(v)] - [\omega(v), \omega(u)]) = [\omega(u), \omega(v)]
\]

(21)

Hence by (19) and (20):

\[
d^H \omega = d\omega + \frac{1}{2} \text{ad}_\omega \wedge \omega = d\omega + \omega \wedge \omega
\]

(22)
There is a more geometrical way in which we can visualize the effects of curvature for our case which we will see at the end of the next section.

We shall utilize the section $\zeta : \tilde{U} \subset S' \rightarrow \mathcal{M}'$, which we defined in (9), to describe the connection in terms of some choice of “coordinates”. As we have shown before, there exists a trivialization over $\tilde{U}$, i.e. a ILH-diffeomorphism $\varphi : \tilde{U} \times \mathcal{D}(M) \rightarrow \pi^{-1}(\tilde{U})$ such that $\zeta(\tilde{g}) = \varphi(\tilde{g}, \text{Id})$. We again note that all our descriptions of physical phenomena should be done with respect to such a section. The following will be useful when we attempt to work with quantities living in superspace.

Now we describe the analogous gauge field and gauge potential. As usual, given the connection form $\omega$, we have that

$$ A^{\zeta} = T_{\text{Id}} \rho(\zeta^* \omega) = \text{ad}(\zeta^* \omega)_{\tilde{g}} : T_{\tilde{g}} S' \rightarrow \text{End}(\Gamma(TM)) $$

$$ \tilde{u} \mapsto [\omega_{\zeta(\tilde{g})}(T_{\tilde{g}} \zeta(\tilde{u})), \cdot] $$

(and similarly with $F^{\zeta} = \text{ad}(\zeta^* \Omega)$) where we still use “∼” to denote superspace sets as opposed to sets in $\text{Riem}(M)$.

Similarly to finite dimensional PFB gauge theory, due to the representation being the “adjoint” one and the orthogonality of the metric, all usual theorems relating the principal fiber bundle connection to a local connection are valid (utilizing coordinate and dimension-free proofs, as in [8]).

For instance, letting $\tau$ be another section over $\tilde{U}$ and $\gamma : \mathcal{M}' \rightarrow \mathcal{D}(M)$ the $\mathcal{D}(M)$-form of the gauge transformation relating $\zeta$ and $\tau$ defined by $(f_{\tau}^\zeta) \circ \zeta := \gamma \circ \zeta = \tau$, since $\rho$ is an ILH representation (using the same proof as Prop. 15 [8]) we have

$$ A^{\tau} = \text{Ad}(f_{\tau}^\zeta)^{-1} A^{\zeta} + \text{ad} \left( (f_{\tau}^\zeta)^{-1} T f_{\tau}^\zeta \right) $$

$$ F^{\tau} = \text{Ad}(f_{\tau}^\zeta)^{-1} F^{\zeta} $$

but by property $(iii)$ of the adjoint map we have $\text{Ad}(f)\text{ad}_X Y = \text{ad}_{\text{Ad}(f)X} \text{Ad}(f) Y$, and since $\text{ad}(\omega^\zeta) = A^{\zeta}$ we have

$$ A^{\tau} = \text{Ad}(f_{\tau}^\zeta)^{-1} A^{\zeta} + \text{ad} \left( (f_{\tau}^\zeta)^{-1} T f_{\tau}^\zeta \right) $$

$$ F^{\tau} = \text{Ad}(f_{\tau}^\zeta)^{-1} F^{\zeta} $$

Which are the usual forms of these equations, and will be useful whenever we want to change the choice of “coordinates”.

### 3.3 Barbour’s Approach

Mach’s point particle principles can be loosely stated as:

1. In particle dynamics, only the relative distances between the particles are physically-relevant. I.e. motion is relative.

20It is a trivial exercise to verify that these transition functions defined by a $\mathcal{D}(M)$-form of some gauge transformation satisfy cocycle conditions.
2. Time is nothing but an arbitrary monotonic parameter \( \lambda \) used to label the sequence of relative configurations that the universe passes through. I.e. time is derived from change.

Barbour, in [4], attempted to implement these principles to a study of space-time through successive configurations of three dimensional Riemannian manifolds. To implement the first principle heuristically, for any two metrics over manifolds \( M_1 \) and \( M_2 \), we keep the coordinates in \( M_1 \) fixed while shuffling around the coordinate of \( M_2 \) until their metrics are as “close as possible”. The infinitesimal process of finding the “best-matched” coordinates is called best-matching (B.M.), and it is a correction of velocities of the type \( v \mapsto v - L_X g \) where \( X \in \Gamma(TM) \) is the vector field that infinitesimally accomplishes the best-matching.

To implement the second principle, one must find an action on superspace (the configuration space of the theory) that is reparametrization invariant with respect to the time parameter \( \lambda \).

His aim was to construct a fully Machian (since external reference frames are to be eliminated), and Jacobi type formulation. That is, it must yield only paths in configuration space, and not their velocities traversing it, which would have no meaning. They would have no meaning because an “instant” is seen as one configuration of the entire system, and any velocity whatsoever is measured by changes in relative configurations of objects within the system itself. A path in configuration space would contain the motion of all objects in the system, so if we speed up the velocity with which we traverse the path in configuration space, the clocks used to measure speed will be speeded up as much as the motion it measures.

By enforcing these principles, Barbour arrived at a BSW-type action [16], which is the lapse eliminated ADM action of 3 + 1 General Relativity:

\[
S[g, \frac{dg}{d\lambda}] = \int dt \int \sqrt{R \sqrt{T}} d\mu_g \tag{27}
\]

where

\[
T = G_{abcd}^1 \frac{dg_{ab}}{d\lambda} \frac{dg_{cd}}{d\lambda} = G_{abcd}^1 \left( \frac{\partial g_{ab}}{\partial \lambda} - L_X g_{ab} \right) \left( \frac{\partial g_{cd}}{\partial \lambda} - L_X g_{cd} \right)
\]

\( R \) is the curvature scalar for the three metric \( g \) and

\[
G_{abcd}^1 = \frac{\sqrt{g}}{2} \left( g^{ac} g^{bd} + g^{ad} g^{bc} - 2 g^{ab} g^{cd} \right)
\]

is the DeWitt metric. We will explain the significance of the lower index “1” shortly.

If \( G_1 \) induced a positive-definite inner product \( \mathcal{M}' \), like our \( G_0 \), composed with the horizontal projection it would induce an inner product in \( S' \) by projection, since given a connection there is a canonical isomorphism between \( H_g \) and \( T_g S' \). In this way \( T \) in (27) would be the squared norm of vectors in \( S' \).

One of Barbour’s conceptual ideas is that actions which are \( \lambda \)-reparametrization invariant could be considered as geodesic principles on configuration space: the reduction of the physical problem of motion to the geometrical problem of finding the geodesics of the configuration space geometry. This followed from an insight regarding homogeneous-quadratic Newtonian mechanics, according to which the use of Jacobi’s principle reduces the problem of motion to a problem (conformal to) the well-defined, well-studied problem of finding the
geodesics of the Riemannian configuration space geometry \cite{17}. In our case, this configuration space geometry would be implicitly defined by (27). However, as we will shortly see, there are some problems with this elegant interpretation.

For constant lapse and the shift vector $X \equiv 0$, Einstein’s equations without cosmological constant decompose into the constraints (denoting differentiation with respect to $\lambda$ by overhead dots)

\begin{equation}
G_{abcd}^1 \dot{g}_{ab} \dot{g}_{cd} - 4\sqrt{R} = 0 \quad \text{Hamiltonian Constraint} \tag{28}
\end{equation}

\begin{equation}
\nabla_b(G_{abcd}^1 \dot{g}_{cd}) = \nabla_b \dot{g}^{ab} \quad \text{Momentum Constraint} \tag{29}
\end{equation}

(where we raised indices by the DeWitt metric) and the dynamical part, which we write here only for the sake of exposition:

\begin{equation}
\ddot{g}_{ab} + \Gamma_{ijkl}^a \dot{g}_{ij} \dot{g}_{kl} = -2(R_{ab} - \frac{1}{4}g_{ab}R) \tag{30}
\end{equation}

where $R_{ab}$ is the Ricci tensor of $g_{ab}$ and $\Gamma$ are the Christoffel symbols for the DeWitt metric.

For constant lapse, we may utilize (28), by which $\sqrt{R} = 1/4G_{abcd}^1 \dot{g}_{ab} \dot{g}_{cd}$, and define a metric for $u, v \in T_g \mathcal{M}$ by restricting the (27) action to the horizontal bundle \cite{21} as:

\[
\mathcal{G}_1[u, v]_g := \int G_1(u, v)_g d\mu_g
\]

$\mathcal{G}_1$ is called the Wheeler-deWitt (WDW) metric \cite{22}. We call attention to the fact that in this instance the bracket denotes functional dependence.

The geometry defined by (27) generalizes the WDW supermetric and provides meaning to the formal definition of a distance function given in \cite{18}. However, there are serious problems regarding the BSW-action as providing a metric structure, since it apparently does not result in even Finsler geometries \cite{19}, and even the WDW metric does not possess a constant signature \cite{20}, resulting in non-trivial intersections of the vertical and horizontal spaces (if they are to be implicitly defined by the metric).

Giulini, in \cite{20} studied a one parameter family of ultralocal metrics:

\[
G_{\beta}^{abcd} = \frac{\sqrt{\beta}}{2}(g^{ac}g^{bd} + g^{ad}g^{bc} - 2\beta g^{ab}g^{cd}) \tag{31}
\]

For $\beta = 1$ above, we have the DeWitt metric. We will call $G_1^{\beta}$ the inverse of (31). The reason for the index “1” is that further on we will have to induce a metric on spaces of $p$-alternating forms over $\mathcal{M}'$ - the metric induced by $G_{\beta}^{0}$ - which we will call $G_{\beta}^{0}$.

Going back, the metrics $\mathcal{G}_\beta$ are non-degenerate for $\beta \neq 1/3$: of mixed signature for $\beta > 1/3$ and positive definite for $\beta < 1/3$. Denoting by $\mathcal{H}_\beta$ the horizontal subbundle defined implicitly by the metric, we have that for a positive definite metric in $\text{Riem}(\mathcal{M})$, i.e. $\beta = a < 1/3$, $V_g \cap \mathcal{H}_\beta = \{0\}$, but for $\beta = b > 1/3$ we generally have $V_g \cap \mathcal{H}_\beta \neq \{0\}$. However, for many cases of $\beta > 1/3$ we can still maintain $V_g \cap \mathcal{H}_\beta = \{0\}$, such as for Einstein

\footnote{And as, we have mentioned, there is a unique metric in $S'$ given by the projection of a “horizontal” metric.}

\footnote{We note that (29) is the condition for horizontal vector fields (with respect to $\mathcal{G}_1$).}

18
metrics and for $\beta \leq 1$ and for $\text{Ric} < 0$, Ricci-negative geometries ($\mathbb{R}^n_\beta$ with strictly negative eigenvalues). In this last case $\mathcal{G}_\beta|_V$ is positive definite, thus $V_g \cap H_g = \{0\}$.

In (Sec. 4), we will completely sidestep these issues, imposing orthogonality of $H, V$ with respect to the metric in $\mathcal{M}'$ by giving $\mathcal{M}'$ a bundle metric.

**Best-Matching as the action of the connection form**

Now, suppose given a vector $v \in T_g \mathcal{M} \simeq S_2(M)$, such that $v \notin V_g$. We consider the space of all the vectors $u \in T_g \mathcal{M} \simeq S_2(M)$ such that $T\pi(u) = T\pi(v)$. We have that

$$u = u^H + u^V \in H_g \oplus V_g$$

where $T\pi(u^H) = T\pi(v)$ and since $T\pi|_{H_g}$ is an isomorphism $u^H$ is fixed.

Now, suppose that the vector $u_0$ is the one that “extremizes the metric change” (with respect to $\mathcal{G}$) between the two infinitesimally neighboring geometries $\hat{g}, \hat{g} + \delta d\pi(v)$. We have that, since the decomposition $V \oplus H$ is orthogonal with respect to $\mathcal{G}$ and $u_0^H$ is fixed, to minimize $||u||$ we must have $||u_0^V|| = 0$. If from this we could assert that $u_0^V = 0$ we would have:

$$u_0 = \hat{H}_g(v)$$

However, for a weak or a non-positive-definite metric, we cannot conclude that to be the case right away, but using a (strong) positive definite metric the conclusion is indeed valid. In any case, any theory that implements this “best-matching principle” must have dynamical orbits orthogonal to the diffeomorphism orbits. In the case we expounded earlier, $\beta = 0$, the velocity field in $M$ correspondent to a dynamical path must be a divergence-free two covariant tensor field, i.e. it obeys $\nabla^a v_{ab} = 0$ as shown before.

In practice, the implementation of this principle can be achieved by projection of the velocity $v \in T_g \mathcal{M}$ in the horizontal subspace.

$$v \mapsto v - L_{\omega(v)}g$$

Where $\omega(v) \in \Gamma(TM)$ is such that $v - L_{\omega(v)}g$ is divergence-free. Equation (32) is exactly the formula representing B.M. in [3].

Since $D(M)$ acts here as a group of isometries of $\mathcal{M}'$ (not of $M'$!), as in the finite dimensional case the geodesics in $\mathcal{M}'$ have constant vertical projection. That is, let $\gamma : I \rightarrow \mathcal{M}'$ be a geodesic and $\xi \in \Gamma(TM')$ a vertical field. Since $\xi$ is also a Killing field with respect to $G$, we have $G(\nabla\xi, \cdot) + G(\cdot, \nabla\xi) = 0$ and so:

$$\frac{d}{dt}G(\xi, \gamma'(t)) = G(\nabla_{\gamma'(t)}\xi, \gamma'(t)) = 0$$

This lemma gives rise to constraint propagation on dynamical orbits, such as the momentum constraint, since geodesics that start off horizontally will obligatory remain horizontal. Of course what really has any physical meaning are geodesics in $S'$, since we have complete freedom on how to lift them, i.e. according to which “coordinates” we want to use.
Curvature

Now, let us present a re-interpretation of the curvature for \( m \in M' \), recalling the analogy between “identification of points” over the geometries of \( S' \) on the one hand, and the bundle of basis of a given finite-dimensional vector bundle \( E \) on the other.

Imagine any two paths \( \tilde{\gamma}_1, \tilde{\gamma}_2 : I \subset \mathbb{R} \to S' \) connecting the geometries \( \tilde{g}, \tilde{h} \in S' \), such that \( \tilde{\gamma}_1 \) and \( \tilde{\gamma}_2 \) are homotopic. The effect of the curvature can be seen as follows: the horizontal lifts \( \gamma_{H_1}, \gamma_{H_2} \) of \( \tilde{\gamma}_1, \tilde{\gamma}_2 \), starting from the same point \( g \in \pi^{-1}(\tilde{g}) \), will end up at the same \( h \in \pi^{-1}(\tilde{h}) \) if and only if the curvature is null.

Now, let us assume we have non-null curvature. A lift \( \gamma \) of \( \tilde{\gamma} \), not necessarily horizontal, is the only way any observer can describe the metric change over time (since we don’t have a way of describing the metric in terms of equivalence relations). Following the “best-matching” identification of points over time (i.e. following the horizontal lifts), means identifying points according to the time-dependent integral curves of \( \omega(\dot{\gamma}(t)) \in \Gamma(TM) \).

I.e, given an identification of points over \( M \) along time, which we denote by \( x \), we redefine \( x \)’s “equilocal” points through time in the following way:

\[
x_{\gamma}(t) = x - \left( \int_0^t (\omega(\dot{\gamma}(s))(x_{\gamma}(s))ds \right)
\]

where \( \omega(\dot{\gamma}) : I \times M \to TM \) is a time-dependent vector field and we are applying the usual time-dependent integral curves theorem.

The null curvature condition is equivalent to the statement that

\[
x_{\gamma_1}(1) = x_{\gamma_2}(1)
\]

That is, if we identify points along time (or using the muddier expression, “coordinatize” space throughout time) in such a way, the two sequence of geometries will arrive at the geometry \( \tilde{h} \) with the same localization of space points if and only if the curvature form is null. Otherwise, even if two hypothetical observers that started out together have chosen the spatial “coordinates” in such a way that the metric change over time is minimal, they will arrive at the second geometry not agreeing over which point is \( x \).

So curvature can be said to be the lack of existence of a well defined (path independent) equilocality relation over a simply connected open set in \( S' \). This translates to our case what in the usual terminology is meant by saying that there is no integral submanifold of the horizontal distribution.

Example: \( H_g := \{ \delta_g v = 0 | v \in T_g M' \} \)

For better visualization, we give a brief account on what would the curvature and connections look like when we take the horizontal bundle to be defined by the orthogonal space of the vertical bundle with respect to the metric \( G_0 \).

- We always raise and lower indices by the usual ultralocal metric \( g \). For elements of \( S_2(M) \) this coincides with raising double indices by the metric \( G_0 \). The covariant derivative of vector fields is the Levi-Civita one (for the point \( g \in M' \)). Denoting covariant derivatives by semi-colons:
• A vertical vector at \( g \in \mathcal{M}' \), denoted by \( \xi = \xi_{ij} dx^i \otimes_S dx^j \in T_g \mathcal{M}' \simeq S_2(M) \), satisfies, for some smooth curve \( g : I \to \mathcal{M}' \)
\[
\xi_{ij} = \dot{g}_{ij} := \frac{dg_{ij}(t)}{dt} \bigg|_{t=0} = LX(g_{ij}(0)) = X_{i;j} - X_{j;i}
\]
for some \( X \in \Gamma(TM) \) (\( X = j^{-1}_g(\xi) \)). For a general vector \( \dot{g}_{ij} \in T_g \mathcal{M}' \) the vertical projection is given by
\[
(\dot{g}_{ij})^V = L_{\omega(g)} g_{ij}(0)
\]
that is \( X = \omega_g(\dot{g}) \)

• In the same way, a horizontal vector satisfies: \( \dot{g}_{ij}^H = 0 \) and the horizontal projection is given by
\[
(\dot{g}_{ij})^H = \dot{g}_{ij} - X_{i;j} + X_{j;i}
\]
where, in abstract index notation, given \( v_{ab} \in S_2(M) \), \( X \) is the unique solution of:
\[
\nabla^a(v_{ab} - \nabla_a X_b + \nabla_b X_a) = 0
\]

• The curvature form \( \Omega \in \Omega^2(\mathcal{M}', \Gamma(TM)) \) will be given (ultra)locally by:
\[
\Omega = \Omega_i^b e^i \otimes \frac{\partial}{\partial x^k}
\]
where \( e^i = e^j \wedge e^j \), \( i > j \). For clearer visualization of what a collection of \( e^i \)'s look like, we give the (ultra)local example:
\[
e^1 = \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} \quad ; \quad e^2 = \frac{\partial}{\partial x^2} \otimes \frac{\partial}{\partial x^2} \quad ; \quad e^3 = \frac{\partial}{\partial x^3} \otimes \frac{\partial}{\partial x^3} \quad ; \quad e^4 = \frac{1}{2} \left( \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^2} \right) \quad ; \quad e^5 = \frac{\partial}{\partial x^2} \otimes_S \frac{\partial}{\partial x^3} \quad ; \quad e^6 = \frac{\partial}{\partial x^1} \otimes_S \frac{\partial}{\partial x^3}
\]

Considering the formula
\[
\Omega(u, v) = \hat{V}([\hat{H}(u), \hat{H}(v)])
\]
and equation (36), a perhaps more intrinsic way of describing the curvature as applied to \( u, v \in T_g \mathcal{M}' \simeq S_2(M) \) is simply
\[
\Omega(u, v) = L_Z g \quad \text{or in abstract index} \quad \Omega(u, v)_{ab} = Z_{a;b} - Z_{b;a}
\]
where, \( Z \in \Gamma(TM) \), as a vertical projection, is the unique solution to
\[
\nabla^e ([u_{ab} - X_{a;b} + X_{b;a}, v_{cd} - Y_{c;d} + Y_{d;c}]_{ef} - Z_{e;j} + Z_{j;e}) = 0
\]
where by their turn \( X \in \Gamma(TM) \) and \( Y \in \Gamma(TM) \) are the unique solutions to
\[
\nabla^a(u_{ab} - X_{a;b} + X_{b;a}) = 0
\]
\[
\nabla^c(v_{cd} - Y_{c;d} + Y_{d;c}) = 0
\]
the terms inside the parenthesis are always the horizontal projection of the first term inside the parenthesis. This sets up the proper tools for any coordinatized evaluation in the setting of pure fields we have considered.
4 Associated Bundles and Gauge theory in S’

We now cite the basic ingredients for the description of the interaction between a gauge field and a particle (see [8] for comments and explanations), while parenthetically remarking the changes of description necessary to our approach. Without the risk of confusion, since we will concentrate on the infinite-dimensional case, we will denote the usual maps utilized in PFB theory by the same names.

1. **A smooth semi-Riemannian manifold** $M$ (resp. $S'$). - $M$ is simply the space particles inhabit (resp. the space of possible geometries).

2. **A finite dimensional (resp. complete locally convex topological) vector space** $F$ equipped with an inner product $\langle \cdot, \cdot \rangle$. - In the orthodox interpretation, this is the space where wave functions take values. This space is usually determined by the internal structure of the particle (phase, isospin, etc) and is called internal space. Typical examples are $\mathbb{C}, \mathbb{C}^2, \mathbb{C}^4$ or the Lie algebras $\mathfrak{u}(1), \mathfrak{su}(2)$. By the inner product one computes the function norm and therefore quantum probabilities. In our case, $F$ will be the space of sections $\Gamma(E)$ of a tensor bundle $\pi_E : E = TM \otimes TM \otimes ... \otimes TM^* \otimes ... \otimes TM^* \to M$. Even though in our case $F$ doesn’t come with a single pre assigned inner product, we can still recover all aspects of the theory.

3. **A Lie group** $G$ (resp. Frechét Lie group $\mathcal{D}(M)$) and a representation $\rho : G$ (resp. ILH-representation $\rho : \mathcal{D}(M) \to \text{Aut}(F)$) orthogonal with respect to $\langle \cdot, \cdot \rangle$. - $G$ then acts over the frames of the internal states over each point. If $P$ is seen as the $G$-frame bundle for $E$, orthogonality of the representation in necessary so that the inner product is frame independent (resp. “coordinate” independent).

4. **A $G$ (resp. $\mathcal{D}(M)$)-principal fiber bundle over $M$ (resp. $S'$)** $(P, \pi, M, G)$ (resp. $(\mathcal{M}', \pi, S', \mathcal{D}(M))$). This bundle may be identified with the bundle of $G$-admissible frames over $M$ (resp. identifications of $M$ over $S'$). The fiber over each point is a copy of $G$ seen as the $G$-admissible orthonormal frames of the internal states. A section of $P$ here would be a reference frame, made up of $G$-admissible frames, according to which we describe internal states (resp. according to which we would describe the geometry of the Universe).

5. **A connection form** $\omega$ in $P$ (resp. $\mathcal{M}'$), with curvature $\Omega$. - This connection form provides us with the intrinsic variation of the frames (resp. intrinsic variation of “coordinates”). Applied over a local section $\zeta$ we obtain the local potential, $\mathcal{A} = T_{\text{Id}}\rho(\zeta^*\omega)$. Similarly we obtain the local gauge field $\mathcal{F} = T_{\text{Id}}\rho(\zeta^*\Omega)$.

6. **A global section** $\Psi$ of the associated bundle $P$ (resp. $\mathcal{M}'$) $\times_\rho F$. - Matter fields will be associated to such sections that satisfy the Euler-Lagrange equations of some action functional that involves the local gauge potentials $\mathcal{A}$ (resp. since $S'$ is not compact, we cannot integrate over it, hence this prescription has to be somewhat

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\(^{23}\text{We will explore the case of a general bundle } E \text{ in the end of the paper.}\)
modified in superspace). Given a local reference frame we can locally associate these sections to $G$ (resp. $D(M)$) -equivariant functions $\psi : \theta \to F$, which provide us with the so called wave functions.

7. A non-negative, smooth, real valued function $U : F \to \mathbb{R}$- which is invariant with respect to the action of $G : U(g \cdot v) = U(v)$, and which is to be regarded as the self interaction energy of the matter field. Again, we will not regard $U$ as a function of the typical fiber, nonetheless we are able to recover all fundamental results.

8. An action $S(\Psi, \omega)$ whose singular points are to be considered the physically relevant field configurations. - Typically, this functional is of the following form:

$$S[\Psi, \omega] = c \int_M \|\Omega\|^2 + c_1 \|d^H \Psi\|^2 + c_2 U \circ \Psi$$

where again here the bracket denotes functional dependence. Here $d^H$ is the exterior covariant derivative determined by $\omega$, $c$ is called the normalization constant and $c_1, c_2$ are coupling constants.

4.1 The case $F = \Gamma(TM)$

$D(M)$-action and Smooth Structure

Let us at first take $F = \Gamma(E)$, where $E$ is some as yet unidentified vector bundle over $M$. One might suppose the action of $D(M)$ on $\Gamma(E)$ to be of the form $\rho : (f, s) \mapsto s \circ f$. Of course, this would not have image in $\text{Aut}(\Gamma(E))$ since $s \circ f \notin \Gamma(E)$ as can be promptly noticed projecting it onto $\Gamma' : \pi_E(s \circ f)(x) = f(x)$.

So we must somehow have an action that brings the section back to $E_x$. We only have a canonical way to do this if $f$ has a lift\(^{24}\) from $M$ to $E$. That is, a smooth $\tilde{f} : E \to E$ such that $\pi_E \circ \tilde{f} = f \circ \pi_E$. Then the action could be $\rho : (f, s) \mapsto \tilde{f}^{-1} \circ s \circ f$ which is easily verified to be a section, applying $\pi_E$:

$$(\pi_E \circ \tilde{f}^{-1}) \circ s \circ f = f^{-1} \circ (\pi_E \circ s) \circ f = f^{-1} \circ f = \text{Id}$$

So it must be an adjoint action. Taking $E$ not to be trivial, the simplest case where we can find a canonical lift for $f$ is for $E$ being a general tensor bundle. The canonical lift $\tilde{f}$ will be a tensor product of pull-backs and push-forwards of $f$. We believe all, or most, of the forthcoming results can be applied to the more general tensor bundle with the adjoint action.

\(^{24}\)We briefly comment on a construction which might allow us to remove this constraint on the typical fiber, and regard any vector bundle over $M$, i.e. any $(\pi_E : E \to M) \in \text{Vect}(M)$. We denote by $\text{Vect}_E(M)$ the space $\text{Vect}_E(M) := \{F \in \text{Vect}(M) \mid \exists f_F \in D(M) \text{ such that } F = f^* E\}$, and $\mathcal{V}_E := \{s \in \Gamma(F) \mid F \in \text{Vect}_E(M)\}$. Now, $\mathcal{V}_E$ has a linear structure defined, for $s_F \in \Gamma(F), s_B \in \Gamma(E)$, and $x \in M$, by $(s_F + s_B)(x) := s_F \circ f_F^{-1}(x) + s_B \circ f_B^{-1}(x)$, which is clearly a section of $\Gamma(E)$. Hence we can induce a linear structure on $\mathcal{V}_E$ by the linear structure of $\Gamma(E)$. We can define the action of $D(M)$ on $\mathcal{V}_E$ naturally as $f^* s := s \circ f$ which, for $s \in \Gamma(E)$, we have $f^* s \in \Gamma(f^*(E))$. 

23
The simplest example of such tensor bundles is of course $\Gamma(TM)$, on which we expand now. The natural representation of $D(M)$ is the adjoint, for which we find an Ad-invariant metric $^{25}$.

The associated bundle we wish to construct is $(\mathcal{M}' \times_{\text{Ad}} \Gamma(TM))$ with the quotient topology corresponding to the equivalence relation $(f^*g, \text{Ad}(f^{-1})X) \sim (g, X)$. We shall denote the equivalence class of $(g, X)$ as $\{g, X\}$ (since we will be heavily utilizing $[\cdot, \cdot]$ as the commutator of vector fields, and would like to avoid confusion). Clearly $\{f^*g, X\} = \{g, \text{Ad}(f)X\}$.

The projection on $\mathcal{S}'$ is given by

$$\pi_E : (\mathcal{M}' \times_{\text{Ad}} \Gamma(TM)) \rightarrow \mathcal{S}'$$

$$(g, X) \mapsto \pi(g) = \tilde{g}$$

each fiber $\pi_{E}^{-1}(\tilde{g})$ is diffeomorphic to $\Gamma(TM)$ (see $^{22}$). Now, we again shall utilize the section $\zeta : \tilde{U} \subset \mathcal{S}' \rightarrow \mathcal{M}'$, which we defined in $^{19}$. Remembering that over $\tilde{U}$, we have an ILH-diffeomorphism $\varphi : \tilde{U} \times D(M) \rightarrow \pi^{-1}(\tilde{U})$ such that $\zeta(\tilde{g}) = \varphi(\tilde{g}, \text{Id})$.

So we have

$$\pi_{E}^{-1}(\tilde{U}) \simeq (\pi^{-1}(\tilde{U}) \times \Gamma(TM))/D(M) \simeq (\tilde{U} \times D(M) \times \Gamma(TM))/D(M) \simeq \tilde{U} \times \Gamma(TM)$$

and a topological structure is induced on $\mathcal{M}' \times_{\text{Ad}} \Gamma(TM)$. The smooth structure is induced by the maps corresponding to the smooth sections of $\mathcal{M}'$:

$$\Phi^{\zeta} : \tilde{U} \times \Gamma(TM) \rightarrow \pi_{E}^{-1}(\tilde{U})$$

$$(\tilde{g}, X) \mapsto \{\zeta(\tilde{g}), X\}$$

Clearly if $\tau : \tilde{U} \subset \mathcal{S}' \rightarrow \mathcal{M}'$ is another principal section over $\tilde{U}$, utilizing the transition map $f_\tau : \tilde{U} \rightarrow D(M)$ defined by $\tau(\tilde{g}) = \varphi(\tilde{g}, f_\tau)$ (see $^{11}$) then

$$(\Phi^{\tau})^{-1} \circ \Phi^{\zeta}(\tilde{g}, X) = (\Phi^{\tau})^{-1}\{\zeta(\tilde{g}), X\}$$

$$= (\Phi^{\tau})^{-1}\{(f_\tau^*(\tilde{g}))^*\zeta(\tilde{g}), \text{Ad}((f_\tau^*(\tilde{g}))^{-1})X\}$$

$$= (\Phi^{\tau})^{-1}\{\tau(\tilde{g}), \text{Ad}((f_\tau^*(\tilde{g}))^{-1})X\}$$

$$= (\tilde{g}, \text{Ad}((f_\tau^*(\tilde{g}))^{-1})X)$$

and so it possesses smooth transition functions that satisfy the cocycle conditions and so on and so forth. A perhaps more succinct way of phrasing this is to define an action of the group of gauge transformations of $\mathcal{M}'$ on $\mathcal{M}' \times_{\text{Ad}} \Gamma(TM)$ as

$$\Upsilon_{\Gamma(TM)}(\{g, X\}) = \{\Upsilon(g, X) = \{\gamma(g)^*g, X\} = \{g, \text{Ad}(\gamma(g))X\}\}$$

(42)

where $\Upsilon(g) = \gamma(g)^*g$ is that given by $^{(\text{Def. 2})}$. Clearly $\Upsilon_{\Gamma(TM)}(\{g, X\}) = \Upsilon_{\Gamma(TM)}(\{f^*g, \text{Ad}(f^{-1})X\}$ and is hence well defined.

$^{25}$As a matter of fact, we note that in this setting, as in classical gauge electrodynamics, $\rho^n = \text{Ad}^n$ would also be a valid representation, and the forthcoming proofs could be repeated almost "ipsis literis". We have not investigated questions of irreducibility or of what meaning such a representation would have, an aspect we leave to further work.
Matter Fields

A section $\chi$ of $\mathcal{M}' \times_{\text{Ad}} \Gamma(TM)$ will sometimes be alternatively called matter field. Given $\chi$ and $g \in U$, there is a unique element $\Psi_{\chi}(g) \in \Gamma(TM)$ such that

$$\chi(\tilde{g}) = \{g, \Psi_{\chi}(g)\}$$

In fact, there exists a one-to-one correspondence between sections $\chi$ of $\mathcal{M}' \times_{\text{Ad}} \Gamma(TM)$ and maps $\Psi_{\chi} : \mathcal{M}' \to \Gamma(TM)$, such that $\Psi_{\chi}(f^*g) = \text{Ad}(f^{-1})\Psi_{\chi}(g)$. Indeed,

$$\{\zeta(\tilde{g}), \Psi_{\chi}(\zeta(\tilde{g}))\} = \{f^*_\zeta(\tilde{g})^*\tau(\tilde{g}), \text{Ad}((f^*_\zeta(\tilde{g}))^{-1})\Psi_{\chi}(\tau(\tilde{g}))\} = \{\tau(\tilde{g}), \Psi_{\chi}(\tau(\tilde{g}))\}$$

that is, they fit together over the different trivializations, so we have a globally defined equivariant map

$$\Psi_{\chi} : \mathcal{M}' \to \Gamma(TM)$$

As $\Gamma(TM)$-valued maps on $S'$, the wavefunctions are only locally defined, but all of these local wavefunctions piece together and can be represented by a globally defined $\Gamma(TM)$-valued map (0-form) on $\mathcal{M}'$.

It is a trivial step to now define the exterior covariant derivative for a section $\chi$ of $\mathcal{M}' \times_{\text{Ad}} \Gamma(TM)$ by utilizing the exterior derivative for $\Gamma(E)$-valued $p$-forms over $M'$ we defined on Sec. 3.2 (for the particular case $p = 0$):

$$d^H \chi := d\Psi_{\chi} \circ \hat{H}$$

which is a $\Gamma(TM)$-valued one form on $\mathcal{M}'$.

We will momentarily revert to the general fiber $\Gamma(E)$ and representation $\rho : \mathcal{D}(M) \to \text{Aut}(\Gamma(E))$, so that no confusion arises about dependence upon our choice of typical fiber $\Gamma(E) = \Gamma(TM)$ (since the connection form is a $\Gamma(TM)$-valued one form over $\mathcal{M}'$ irrespectively of $E = TM$). Given the connection form $\omega$ in $\mathcal{M}'$, we construct a $\Gamma(E)$ valued one form from $\Psi_{\chi}$ as follows:

$$(\omega \cdot \Psi_{\chi})_g(v) := T_{\text{Id}}\rho(\omega_g(v))(\Psi_{\chi}(g))$$

for $v \in T_g\mathcal{M}'$.

Taking $E = TM$ and $\rho(f) = \text{Ad}(f)$, omitting the subscript $\chi$, this translates to

$$T_{\text{Id}}\rho(\omega_g(v))(\Psi(g)) = \text{ad}_{\omega_g(v)}(\Psi(g)) = [\omega_g(v), (\Psi(g))] \in \Gamma(TM)$$

Now, we assert that

$$d\Psi \circ \hat{H} = d\Psi + \omega \cdot \Psi$$

To prove it, noting that both sides are linear, we must merely check the values it takes separately for vertical and horizontal fields. The result for horizontal fields is obvious, since it annihilates $\omega$. For a vertical vector

$$v = j(X) = j\left(\frac{d}{dt}\big|_{t=0} \phi(tX)\right) = \frac{d}{dt}\big|_{t=0} (\phi(tX)^*g)$$
we have \( d\Psi \circ \dot{H} = 0 \). Remembering \( \Psi(f^*g) = \text{Ad}(f^{-1})\Psi(g) \), and \( \omega_g(v) = X \),

\[
\begin{align*}
    d\Psi(v) &= \frac{d}{dt}_{|t=0} (\Psi(\phi(tX)^*g)) \\
              &= \frac{d}{dt}_{|t=0} (\text{Ad}(\phi(-tX))\Psi(g)) \\
              &= -\text{ad}_X \Psi(g)
\end{align*}
\]

as stated. We note that \( \omega \cdot \Psi(v) = [X, \Psi(g)] \in \Gamma(TM) \) explicitly contains the “best-matched” vector field \( X \in \Gamma(TM) \).

Now, taking the global wavefunction \( \Psi : \mathcal{M}' \to \Gamma(TM) \), we get a local wavefunction simply by composing with \( \zeta \), i.e. \( \psi^\zeta := \Psi \circ \zeta : \tilde{U} \to \Gamma(TM) \). Going back to equation (44), applying \( \zeta^* \) to both sides we get:

\[
\zeta^*(d^H\Psi) = \zeta^*(d\Psi) + \zeta^*(T_{Id}\rho(\omega)(\Psi)) \\
= d(\Psi \circ \zeta) + T_{Id}\rho(\zeta^*\omega)(\Psi \circ \zeta) \\
= d\psi^\zeta + A^\zeta(\psi^\zeta)
\]

We will denote the covariant exterior derivative for local wave functions as \( D^\omega \) and, more specifically to our case, we have:

\[
D^\omega \psi^\zeta = d\psi^\zeta + [\zeta^*\omega, \psi^\zeta]
\]  

(45)

4.2 Inner Product on \( \Gamma(TM) \) valued forms on \( \mathcal{M}' \)

\( \Gamma(TM) \) Norm

In standard finite-dimensional gauge theory, if we are dealing with a semi-simple compact Lie group \( G \), the \( \text{Ad} \)-invariant inner product used for \( \mathfrak{g} \)-valued forms is the negative of the Killing form \( K(u,v) = \text{tr}(\text{ad}_u \text{ad}_v) \) where \( u,v \in \mathfrak{g} \). This could be seen as resulting from taking the inner product on the (image of) the adjoint representation of \( \mathfrak{g} \) on the endomorphisms of \( \mathfrak{g} \) [8]. However, usually the representation of \( G \) on the \( k \)-dimensional vector space \( F \), \( \rho : G \to \text{End}(F) \cong \text{GL}(k) \), is seen as the inclusion \( G \hookrightarrow \text{GL}(k) \) [21], and one foregoes of the Killing form, replacing it simply with \( (u,v) \mapsto -\text{tr}(uv) \). This inner product can be shown to differ from the Killing form only by a positive scalar multiple [23].

We cannot in our case however take the norm relative to the “trace”, even though \( \text{ad}_\omega \) is a continuous linear operator (and hence bounded) it is not a trace class operator. We will not dwell on the possibility of finding an inner product analogous to the (negative of the) Killing form.

To avoid confusion between the brackets of functional dependence and that denoting the commutator of vector fields, we will discontinue the usage for functional dependence, since in most cases this dependence is evident from the context.

Now, in a finite-dimensional associated bundle \( P \times_{\rho} F \), we have an inner product over the typical fiber \( F \), \( \langle \cdot, \cdot \rangle \), with regards to which the representation is necessarily orthogonal,
so that \( \langle \{ p \cdot h, \rho(h)v \}, \{ p \cdot h, \rho(h)v' \} \rangle = \langle \{ p, v \}, \{ p, v' \} \rangle \), i.e. it doesn’t depend on what \( p \in \pi^{-1}(\tilde{p}) \) we choose. We usually use this inner product on \( F \) to induce an inner product on the associated bundle, i.e. for \( \pi : P \rightarrow M, p \in \pi^{-1}(x) \subset P \) and \( v, v' \in F \)

\[
\langle \{ p, v \}, \{ p, v' \} \rangle_x := \langle v, v' \rangle
\]  

(46)

In our case, for whatever \( g \in \mathcal{M}' \) we choose, and for given elements \( X, Y \) of our typical fiber \( \Gamma(TM) \)

\[
\alpha_g(X, Y) := \int_M g(X, Y)d\mu_g
\]

is \( \rho \)-invariant. So we do not take a standard inner product on \( \Gamma(TM) \), but take advantage of the fact that there is a unique, natural, \( \text{Ad} \) invariant inner product over each fiber. Instead of (46), we write

\[
\langle \{ g, X \}, \{ g, Y \} \rangle_{\tilde{g}} := \int_M g(X, Y)d\mu_g
\]

(47)

which is the inner product over each fiber of \( \mathcal{M}' \times_{\text{Ad}} \Gamma(TM) \). Let us check this is indeed consistent for any \( h \in \pi^{-1}(\tilde{g}) \), that is for \( h = f^*g \):

\[
\langle \{ f^*g, \text{Ad}(f^{-1})X \}, \{ f^*g, \text{Ad}(f^{-1})Y \} \rangle_{\tilde{g}} = \int_M f^*g(T f^{-1}(X \circ f), T f^{-1}(Y \circ f))d\mu_{f^*g}
\]

\[
= \int_M g(X \circ f, Y \circ f)f^*(d\mu_g)
\]

\[
= \int_M g(X, Y)d\mu_g
\]

That takes care of one parcel of the required inner product, it gives us a prototype for all \( \Gamma(TM) \) valued forms.

However, taken for the values of \( \omega \), this inner product is not identical over all of \( \mathcal{M}' \), unlike the analogous Killing form, but will depend on the base point \( g \in \mathcal{M}' \). This dependence is better represented if we regard \( \omega \) as just the vertical projection. I.e. since for \( \langle \cdot, \cdot \rangle \), an \( \text{Ad} \)-invariant inner product on \( \Gamma(TM) \), \( \langle \text{Ad}(f)X, \text{Ad}(h)v \rangle = \langle X, Y \rangle \) for all \( X, Y \in \Gamma(TM), f, h \in \mathcal{D}(M) \), utilizing the isomorphism given by \( j_g : \Gamma(TM) \simeq V_g \), we obtain a \( \mathcal{D}(M) \)-invariant metric over the vertical bundle. That is, given \( j_g(X), j_g(Y) \in V_g \) denoting in the same way the inner product over \( V \) we define:

\[
\langle j_g(X), j_g(Y) \rangle_g := \langle \{ g, X \}, \{ g, Y \} \rangle_{\tilde{g}}
\]

(48)

Substituting \( f \) by \( f^{-1} \) and \( g \) for \( f^*g \) in (13) we get

\[
(f^{-1})^*(j_{f^*g}(\text{Ad}(f^{-1}))) = j_g
\]

Hence, for \( f \in \mathcal{D}(M) \):

\[
\langle f^*(j_g(X)), f^*(j_g(Y)) \rangle_{f^*g} = \langle j_{f^*g}(\text{Ad}(f^{-1})X), j_{f^*g}(\text{Ad}(f^{-1})Y) \rangle_{f^*g}
\]

\[
= \langle \{ f^*g, \text{Ad}(f^{-1})X \}, \{ f^*g, \text{Ad}(f^{-1})Y \} \rangle_{\tilde{g}}
\]

\[
= \langle \{ g, X \}, \{ g, Y \} \rangle_{\tilde{g}}
\]

\[
= \int_M g(X, Y)d\mu_g
\]
and so no inconsistency arises by taking $\alpha$ as the inner product for connection and curvature forms valued in the vertical subbundle $V \subset TM'$. This prompts us to consider bundle metrics in $\mathcal{M}'$ which we will define in the next topic.

As an aside, we mention that one of the most vexing issues with Kaluza-Klein theories is the fixed contribution of the structural group $G$ to the total scalar curvature of the principal bundle $P$. By explicitly demanding the Lie algebra inner product to depend on the base point in a consistent way, as we have done here, we feel perhaps this problem can be amended.

$\mathcal{M}'$ Norm

Now we have to approach the $\mathcal{M}'$ part. As we recall $T'_g \mathcal{M}' \simeq S_2(M)$, which is the space of symmetric two covariant sections over $M$. An element of $S_2(M)$, assigns to each point in $M$ an element of the 6-dimensional space $T_xM^* \otimes S T_xM^*$. Hence we let $\{e_i\}_{i=1}^6$ be a basis for the possible values of an element of $S_2(M)$ at $x \in M$, a basis orthonormal with respect to $G_0$.

So we define the Hodge star operator for $\Lambda^p(S_2(M))$ pointwise as taking each element $e_i$ of a basis of $\Lambda^p(T_xM^* \otimes S T_xM^*)$ to the corresponding oriented complement $\tau(I)e_I C$. Here we are using the multi-index notation, i.e. $I$ is an increasing subset of $p$ elements of $\{1, \ldots, 6\}$, $I^C$ is the increasing complement of $I$, of $6-p$ elements, and $\tau(I)$ is the sign of the permutation $I$. Clearly

$$e_I \wedge *e_I = e_I \wedge \tau(I)e_{I^C} = \tau(I)\tau(I)e_I \wedge e_{I^C} = d\mu_g(x)$$

By the above, if we call $G^0_\theta$ the inner product in $\Lambda^p(T_xM^* \otimes S T_xM^*)$ induced by $G_0$, we have: $e_I \wedge *e_I = G^0_\theta(e_I, e_I) d\mu_g(x)$, hence for $\lambda, \gamma \in \Lambda^p(T_xM^* \otimes S T_xM^*)$

$$G^0_\theta(\lambda, \gamma) d\mu_g = \lambda \wedge *\gamma$$

where $* : \Lambda^p(T_xM^* \otimes S T_xM^*) \rightarrow \Lambda^{6-p}(T_xM^* \otimes S T_xM^*)$ is a linear operator (acts solely on the basis elements).

Now, taking $\mathcal{G}_0$ as the pointwise inner product in $\mathcal{M}'$, we have a pointwise inner product in $\Lambda^2(\mathcal{M}')$, denoted by $\mathcal{G}^0_2(\cdot, \cdot)_g$. For $\Omega_1, \Omega_2 \in \Lambda^2(\mathcal{M}')$:

$$\mathcal{G}^0_2(\Omega_1, \Omega_2) := \int_M \Omega_1 \wedge *\Omega_2 = \int_M G^0_2(\Omega_1, \Omega_2) d\mu_g$$  \hspace{1cm} (49)

Bundle Inner Product in $\mathcal{M}'$

This is all well and fine, but as stated in the topic regarding $\Gamma(TM)$ Norm, we would like to vary the connection, which means either that we will keep the metric $\mathcal{G}_0$ but the horizontal and vertical subbundles will lose orthogonality, or that we will keep in some way the orthogonality and modify the original metric $\mathcal{G}_0$. We will opt for the second, since all of our proofs involving the metric (except (44)), require orthogonality and/or invariance with respect to the action of $\mathcal{D}(M)$.

The inner product $\mathcal{G}_0$ is well defined for the vertical subbundle and invariant by the group action. We also know that $\mathcal{G}_0$ is invariant by the group action as well. Hence given the connection form $\omega \in \Omega^1(\mathcal{M}', \Gamma(TM))$ we define the bundle metric:

$$(\mathcal{B}_0)_g = (\mathcal{G}_0)_g \circ \hat{H}_g + \alpha_g \circ \omega_g$$  \hspace{1cm} (50)

The horizontal projection is then defined as $\hat{H} := \text{Id} - \gamma^{-1} \circ \omega$. 

26 The horizontal projection is then defined as $\hat{H}$.
Clearly \( B_0 \) is \( \mathcal{D}(M) \)-invariant and keeps the horizontal and vertical subspaces orthogonal at each point. In the finite dimensional theory of PFB’s, the bundle metric is induced by a connection form \( \omega \in \Omega^1(P, g) \), a (semi)riemannian metric \( g \) over the base space \( M \) and an \( \text{Ad} \)-invariant inner product \( K \) for \( g \) in the following way:

\[
B_p = \pi^* g + K \circ \omega_p
\]

In our case however, we cannot pull-back a metric from \( S' \) since metrics thereat are only defined by projecting metrics from \( \mathcal{M}' \). Suppose then that we have such an inner product. Then the projection of \( G_0 \) applied to \( \tilde{u}, \tilde{v} \in T_{\tilde{g}} S' \) is defined by \( \tilde{G}_0(\tilde{u}, \tilde{v}) := G_0(u^H, v^H) \), where \( u^H \) denotes the horizontal lifting of \( \tilde{u} \). Then the pull-back of the projection applied to \( u, v \in T_g \mathcal{M}' \) is:

\[
(\pi^* \tilde{G}_0)(u, v)_g = \tilde{G}_0(T_g \pi(u), T_g \pi(v))_\tilde{g} = G_0(u^H, v^H)_g = (G_0)_g \circ \tilde{H}_g(u, v)
\]

which trivially establishes their equivalence.

Everything we have proved involving the metric \( G_0 \) is valid for the metric \( B_0 \), since as we have proved, it maintains orthogonality of \( H \oplus V \) and is also \( \mathcal{D}(M) \)-invariant. The Hodge star operator we have defined above is now taken to be the one relative to the inner product \( B_0 \). Nonetheless it is worthwhile to mention that, by (17) the curvature form is non-null only over horizontal vectors, hence the Hodge operator of the self-action will be the same as the one described in the topic above, since for horizontal vectors \( B_0 = G_0 \), and hence \( B_2 = G_2 \).

\( \Omega^p(\mathcal{M}', \Gamma(TM)) \) Norm

It we take (18) to incorporate the composition of vector fields (as derivatives) as in (19), \( \Omega \wedge \ast \Omega \) would not be again a vector field, so we have to discard this version of exterior product for \( \Omega^p(\mathcal{M}', \Gamma(TM)) \). If we consider as in (21) the exterior product to take into account the representation of the Frechét Lie algebra on itself:

\[
(T \text{id}_\rho(\Omega)) \ast \Omega(v_1, v_2, v_3, v_4, v_5, v_6) = \frac{1}{3} \sum_\sigma \tau(\sigma)[\Omega(v_{\sigma(1)}, v_{\sigma(2)}), \ast \Omega(v_{\sigma(3)}, v_{\sigma(4)}, v_{\sigma(5)}, v_{\sigma(6)})] \equiv 0
\]

(51)

because each element of the sum vanishes, as can be readily seen by straight computation in each term.

And so we are left with the simplest norm, substituting the tensor product on (18) by the metric, i.e.

\[
\lambda_1 \wedge \lambda_2(v_1, \ldots, v_{p_1+p_2}) = \frac{p!q!}{(p+q)!} \sum_\sigma \tau(\sigma)g \left( \lambda_1(v_{\sigma(1)}, \ldots, v_{\sigma(p_1)}), \lambda_2(v_{\sigma(p_1+1)}, \ldots, v_{\sigma(p_1+p_2)}) \right)
\]

denoting \( \langle \cdot, \cdot \rangle_g \) the pointwise inner product for \( \Omega_1, \Omega_2 \in \Omega^p(\mathcal{M}', \Gamma(TM)) \) we get

\[
\langle \Omega_1, \Omega_2 \rangle_g := \int_M g(\Omega_1 \wedge \ast \Omega_2)
\]

(53)

or, for elements of the form \( \Omega_1 \otimes \xi_1, \Omega_2 \otimes \xi_2 \in \Gamma(\Lambda^2(\mathcal{M}') \otimes V) \):

\[
\langle \Omega_1 \otimes \xi_1, \Omega_2 \otimes \xi_2 \rangle_g := \int_M \langle \Omega_1 \wedge \ast \Omega_2(g_j(\xi_1), j_g(\xi_2)) \rangle
\]

(54)
where we have used the notation $\Gamma(\Lambda^2(\mathcal{M}') \otimes V)$ to denote the sections of the tensor bundle over $\mathcal{M}'$ given by $\Lambda^2(T_g \mathcal{M}') \otimes V_g$ for each $g \in \mathcal{M}'$. For sections in $\Omega^2(\mathcal{M}', \Gamma(TM))$, we have the entirely compatible

$$\langle \langle \cdot, \cdot \rangle \rangle_g := B^0_2(\cdot, \cdot)_g \otimes \alpha_g$$

(55)

where

$$(B^0_2)_g = \int_M \left( (G^0_2)_g \circ \hat{H}_g \right) d\mu_g + \alpha_g \circ \omega_g \quad \text{and} \quad \alpha_g = \int_M g d\mu_g$$

and where $G^0_1$ is the inverse of the metric defined for $\beta = 0$ (see (31), $G^0_p$ being the induced inner product on the $p$-forms over $S_2(M)$).

Now, we have the necessary elements to approach items (7) and (8). The potential function $U : \Gamma(TM) \to \mathbb{R}$ can be taken for example to be some multiple of

$$U(\Psi(g)) = \langle \sqrt{R_0} \Psi(g), \sqrt{R_0} \Psi(g) \rangle = \int_M R_0 g(\Psi(g), \Psi(g)) d\mu_g$$

where $R_0$ is the Riemannian scalar curvature of $(M, g)$. Summing up we have two important results so far:

- The inner product for $\Gamma(TM)$-valued one forms in $\mathcal{M}'$ is (55)

$$\langle \langle \cdot, \cdot \rangle \rangle_g := B^0_1(\cdot, \cdot)_g \otimes \alpha_g$$

where $\alpha$ is a smoothly $g$-dependent, $\mathcal{D}(M)$-invariant, well defined inner product, both for the vertical bundle and for sections of the associated bundle $\mathcal{M}' \times_{\mathbb{A}_q} \Gamma(TM)$ over $\mathcal{S}'$. Here $G^0_1$ is simply the inverse metric defined in (51) for $\beta = 0$. We remind the reader that at any point (metric) $g$, tensor fields over $\mathcal{M}'$ are reduced to sections of tensor bundles over $M$, with which we know how to deal.

- For $\Psi \in \Omega^p(\mathcal{M}', \Gamma(TM))$, the exterior covariant derivative is (44):

$$d\Psi \circ \hat{H} = d\Psi + \Psi \wedge \omega$$

where the wedge product implies an action of $\omega$ on $\Gamma(TM)$ through $\text{ad}_\omega$. Of course, for a matter field, i.e. for $p = 0$, $\Psi \wedge \omega_g(v) = [\Psi(g), \omega_g(v)]$. The usual localized formulas apply on $\mathcal{S}'$.

### 4.3 Item (8)

**Pure Fields**

As a simplified first example, let us take the “pure field” case, i.e. $\Psi \equiv 0$.

Of course, we cannot integrate over $\mathcal{S}'$ or over $\mathcal{M}'$, since they are not even locally compact. Nonetheless, the minimization of the action we are seeking is not subject to any boundary condition, we have to minimize the action \textit{pointwise} in $\mathcal{M}'$ (or $\mathcal{S}'$), and this we know how to do since at each point the inner product is an integral over the compact 3-manifold $M$.
Now, if $\kappa \in \Omega^1(M', \Gamma(TM))$, by (20)

$$d(\omega + \kappa) + (\omega + \kappa) \wedge (\omega + \kappa) = d^H \omega + d^H \kappa + \kappa \wedge \kappa$$

where the horizontal bundle is determined by $\omega$. Making use of the affine structure of connection space, we take a linear variation of $\omega$ given by $\omega + tk$. We have

$$\frac{d}{dt_{|t=0}} \langle \langle \Omega_{(\omega+tk)}, \Omega_{(\omega+tk)} \rangle \rangle = \langle \langle \Omega, d^H \kappa \rangle \rangle$$

We get as expected $(d^H)^* \Omega = 0$ where the adjoint is taken with respect to $\langle \langle \cdot, \cdot \rangle \rangle_g$ (see (55)). Since we are now working over a single metric $g \in M'$, we are dealing with sections of tensor bundles over the finite-dimensional manifold $M$. Hence using geodesic coordinates at $x \in M$ we have $d^H = d$, and it is easy to find the adjoint of $d$, using Stoke’s Theorem and the linearity of $d$:

$$\int_M g(d\Omega_1 \wedge *\Omega_2) = -\int_M g(\Omega_1 \wedge d^* \Omega_2)$$

$$= \text{sign} \int_M g(\Omega_1 \wedge (*d \Omega_2))$$

$$= \text{sign} \langle \langle \Omega_1, *d \Omega_2 \rangle \rangle_g$$

where here $\Omega_1$ is a $p$-form, $\Omega_2$ is a $p+1$ form, and $\text{sign} = (-1)^{6(p+1)+1}$. And so, pointwise in $M'$, for $p = 2$, we obtain

$$(d^H)^* \Omega_g = - * d^H * d^H \omega_g = 0 \quad (56)$$

We will leave the questions concerning the all important variation of the metric itself (and with it the variation of $B_0^2$ and $*$ (see (55)) to the future.

If we can solve (56), it will give an (ultra)local equilocality condition. Using the procedure represented by (33) for comparing best-matching coordinates along two separate paths in superspace, we may compare the values of $\omega$ with suitable single particle diffraction experiments (since in classical mechanics we cannot traverse two independent paths in $S'$).

**Matter Fields**

Now, widening our attention to the full action given in item 8, for extremization we must independently vary the action with respect to the connection $\omega \in \Omega^1(M', \Gamma(TM))$ and to the matter field $\Psi \in \Gamma(M' \times \text{Ad} \Gamma(TM)) \subset \Omega^0(M', \Gamma(TM))$. The lagrangian we are considering is

$$\mathcal{L}(\omega, \Psi)_g = \int_M \left( g(\Omega \wedge *_2 \Omega) + g(d^H \Psi \wedge *_1 d^H \Psi) + R_g g(\Psi(g), \Psi(g)) \right) d\mu_g \quad (57)$$

Where we have denoted $*_2$ as the Hodge operator with respect to the metric given in (55), while $*_1$ is from it obtained by substitution of $G^0_2$ by $G^0_1$.

Now, $d^H \Psi = d\Psi + \omega \cdot \Psi$, which besides the self action of the pure field is the only other term that depends on the connection form. So, since the space of connections is an affine space,
we can again take a general variation to be of the form \( \omega + \kappa \), where \( \kappa \in \Omega^1(\mathcal{M}', \Gamma(TM)) \). Hence we will get \( d^H \Psi = d \Psi + \omega \cdot \Psi + \kappa \cdot \Psi = d^H \Psi + \kappa \cdot \Psi \), and

\[
\frac{\delta \| d^H \Psi \|^2}{\delta \omega} = \frac{d}{dt|_{t=0}} \int_M g \left( d^H \Psi + t(\kappa \cdot \Psi(g)) \right) \wedge \left( d^H \Psi + t(\kappa \cdot \Psi(g)) \right) d\mu_g
\]

\[
= \int_M g \left( d^H \Psi \wedge \kappa \cdot \Psi(g) \right) d\mu_g
\]

\[
= - \int_M g \left( d^H \Psi \wedge L_{\Psi(g)}(\kappa) \right) d\mu_g
\]

\[
= - \int_M g \left( L_{\Psi(g)}^*(d^H \Psi) \wedge \kappa \right) d\mu_g
\]

where we used \( \kappa \cdot \Psi(g) = [\kappa, \Psi(g)] = -L_{\Psi(g)} \kappa \). We denoted the transpose of \( L_{\Psi(g)} \), by the inner product \((\mathcal{B}^1)g \otimes \alpha_g \) (see (55)), as \( L_{\Psi(g)}^* \). Let us calculate \( L_{\Psi(g)}^* \). For simplicity, we can consider the usual Lie derivative of vector fields over a Riemannian manifold \( M \). Then, for \( X, Y, Z \in \Gamma(TM) \):

\[
\langle [X, Y], Z \rangle := \int_M g([X, Y], Z) d\mu_g
\]

\[
= \int_M (g(\nabla_X Y, Z) - g(\nabla_Y X, Z)) d\mu_g
\]

\[
= \int_M (-g(Y, \nabla_X Z) - g((\nabla_X(Y)), Z)) d\mu_g
\]

\[
= \int_M (-g(Y, \nabla_X Z) - g(Y, (\nabla_X)^t(Z))) d\mu_g
\]

\[
= \int_M g(Y, -(\nabla_X + (\nabla_X)^t)(Z)) d\mu_g
\]

Where \( \nabla \in \mathcal{C}(M) \) is the usual Levi-Civita connection of \( g \) over \( M \), \( \nabla X \in \Gamma(L(TM, TM)) \simeq \Gamma(TM^* \otimes TM) \) is a \((1, 1)\)-type tensor field, and \((\nabla X)^t\) is simply the transpose of this linear transformation.

Since we have already calculated how the self-action varies through a change of connection, we get the inhomogeneous field equation:

\[
- *d^H \Omega_g = L_{\Psi(g)}^*(d^H \Psi(g)) \tag{58}
\]

or

\[
* d^H \Omega_g = \nabla_{\Psi(g)}(d^H \Psi(g)) + (\nabla \Psi(g))^t(d^H \Psi(g)) \tag{59}
\]

We have also the secondary, conservation equation:

\[
* d^H \left( L_{\Psi(g)}^*(d^H \Psi(g)) \right) = 0 \tag{60}
\]
Now we proceed to the variation with respect to $\Psi$. From (57):

$$\frac{d}{dt}|_{t=0} L(\omega, \Psi + t\Phi) = \int_M \left( g(d^H \Psi \wedge *d^H \Phi) + R_g g(\Psi, \Phi) \right) d\mu_g$$

$$= \int_M \left( g(*d^H *d^H \Psi, \Phi) + R_g g(\Psi, \Phi) \right) d\mu_g$$

$$= \int_M g(*d^H *d^H \Psi - R_g \psi(g), \Phi(g)) d\mu_g$$

Hence, for the extreme we have:

$$*d^H *d^H \psi(g) = R_g \psi(g) \quad (61)$$

Which tops the three equations we have uncovered (Bianchi, (61) and (58)), and might be considered our tentative “wave equation” of the Universe. There are of course several problems with extracting information from these equations, but that is a problem we will deal with in the near future, in further work.

5 Concluding Remarks

For our construction of gauge theory in $\mathcal{M}' \subset \text{Riem}(M)$ we had to make several departures from the usual structures in gauge theory. Amongst them, since we do not have the equivalent of the Killing form for non-compact groups, was the utilization of an inner product on the Frechet Lie algebra $\Gamma(TM)$ dependent on the base point $\tilde{g} \in S'$. It is a device which we plan to apply on Kaluza-Klein theory, so as to give the contribution of the scalar curvature due to the group more freedom over $M$.

We also had to constrain the typical fiber of any given associated bundle over $S'$ to be that of sections of a tensor bundle over $M$, so that we could have a natural action of the diffeomorphism group on the automorphisms of the fiber.

In spite of some of these natural deviations, the majority of the structures present in gauge theory can be suitably transplanted to this infinite-dimensional setting, a fact due mostly to the existence of a cross-section of $\mathcal{M}'$ relative to $D(M)$.

As we mentioned, we cannot properly define the action for the whole of $\mathcal{M}'$. Nonetheless, due to the absence of boundary conditions for the curvature forms, minimization of this hypothetical action over all of $\mathcal{M}'$ is equivalent to minimization pointwise in $\mathcal{M}'$, which produces as usual the Yang-Mills equation for the curvature form.

We note that even though we are considering an infinite-dimensional manifold $\mathcal{M}'$, we were able to derive explicitly (ultra)local equations for the curvature form, that is, we do not need to know what is happening over the entire Universe to solve our equations. Of course, in classical physics we cannot choose two metric velocities $u, v \in S_2(M)$ to input into the curvature form.

As we have mentioned, if we can find solutions for (56), we may infer what will be the best-matching “coordinates” along two separate paths in $\mathcal{M}'$ (see (53). The connection form will yield a measure of disagreement between these “coordinates”. Hence it may be possible

\footnote{See footnote 24 for a possible way out of this restriction.}
to test these solutions with simple experiments of single particle diffractions. Equilocality relations already embody relativity of motion [4], and in our view provide a true meaning of space throughout time. Even if the pure Yang-Mills equation concerns solely the identification of points of $M$ over different geometries, it may relate to properties of space, and should not be considered as simply gauge fixing.

If the curvature form indicates the lack of existence of a path independent notion of (best-matching) equilocality over an open set of $S'$, what do the Yang-Mills equation and the Bianchi Identity mean in this context? Do they concern solely the best-matching vector field $X$ as we said? What modifications arise from considering $M_0'$ and $C_0$ [14]? All of these questions remain to be answered in future work, as well as exploring results for different $\beta < 1/3$ (see 31), and different (tensor) bundles as typical fibers.

Another interesting topic left for future research is that of exploring more fully the bundle metric (55). For instance, what is its scalar curvature, and what do we get if we, as in Kaluza-Klein theory, minimize it with respect to $(g, \omega)$?

Lastly, we have to try to answer the most important, obvious question regarding an interpretation of equations (61) and (58). Can $\Psi$ be considered a wave function for the Universe?

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28Actually, the concept of scalar curvature itself is not obvious in $\mathcal{M}$, since if its Ricci curvature is not zero it induces a topolinar isomorphism between infinite-dimensional subspaces of $T_p\mathcal{M}$, and hence is never of trace-class. However, Gil-Medrano and Michor have done a lot to develop the Riemannian geometry of $\mathcal{M}$ in [24]. There, they use the concept of scalar-like curvature and Ricci-like curvature, which considers the pointwise trace of the linear operator $\xi \mapsto R_g(\zeta, \xi)\eta$, $\xi, \zeta, \eta \in \Gamma(T\mathcal{M})$. This device can be used in trying to find the horizontal contribution to the scalar curvature of the bundle metric (since the vertical scalar curvature is pointwise (in $M$) simply $R_g$; the usual scalar curvature for $g$).
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35