We introduce analogs of the Hopf algebra of Free quasi-symmetric functions with bases labelled by colored permutations. When the color set is a semigroup, an internal product can be introduced. This leads to the construction of generalized descent algebras associated with wreath products $\Gamma \wr S_n$ and to the corresponding generalizations of quasi-symmetric functions. The associated Hopf algebras appear as natural analogs of McMahon’s multisymmetric functions. As a consequence, we obtain an internal product on ordinary multi-symmetric functions. We extend these constructions to Hopf algebras of colored parking functions, colored non-crossing partitions and parking functions of type $B$.

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1. INTRODUCTION

The Hopf algebra of Free Quasi-Symmetric Functions FQSym \[4\] is an algebra of noncommutative polynomials associated with the sequence \((\mathfrak{S}_n)_{n \geq 0}\) of all symmetric groups. It is connected by Hopf homomorphisms to several other important algebras associated with the same sequence of groups: Free symmetric functions (or coplactic algebra) FSym \[36\ \[4\], Non-commutative symmetric functions (or descent algebras) Sym \[7\], Quasi-Symmetric functions QSym \[9\], Symmetric functions Sym, and also, Planar binary trees PBT \[22\ \[12\], Matrix quasi-symmetric functions MQSym \[4\ \[11\], Parking functions PQSym \[19\ \[29\], and so on.

Most of these Hopf algebras are endowed with an internal product, generalizing the one of ordinary symmetric functions. The basic example is provided by noncommutative symmetric functions, whose homogeneous components can be identified with the Solomon descent algebras of symmetric groups \[7\].

Symmetric groups are the Coxeter groups of type \(A\), and there are descent algebras for other types as well. However, the direct sums of the descent algebras of types B or D are not Hopf algebras in any natural way. But there are Hopf algebras associated with wreath products \(\mathbb{Z}_\ell \wr \mathfrak{S}_n\), the Mantaci-Reutenauer algebras \[27\], which admit internal products, and contain the Solomon algebras of type B for \(\ell = 2\).
From the point of view of symmetric functions, MR(\(\ell\)), the Mantaci-Reutenauer algebra of level \(\ell\) is the free product of \(\ell\) copies of \(\text{Sym}\). It is therefore the natural noncommutative analog of \((\text{Sym})^{\otimes \ell} \simeq \text{Sym}(X_0; \ldots; X_{\ell-1})\), the algebra of symmetric functions in \(\ell\) independent alphabets, which is also the Grothendieck ring of the tower of algebras \((\mathbb{C}[\mathbb{Z}_\ell \wr \mathfrak{S}_n])\). And indeed, it has been shown in [14] that MR(\(\ell\)) was the Grothendieck ring of projective modules over the 0-Ariki-Koike-Shoji algebras, a degeneracy of the Hecke algebras associated with \(\mathbb{Z}_\ell \wr \mathfrak{S}_n\).

However, with \(\ell\) independent alphabets, one can build a larger Hopf algebras. In the commutative case, it is the algebra of multi-symmetric functions, first introduced by McMahon [26], and briefly investigated by Gessel from a modern point of view in [10]. It is defined as follows. Setting \(X_i = \{x_{i,j} | j = 1, \ldots, n\}\), the multi-symmetric polynomials are the invariants of \(\mathfrak{S}_n\) in \(\mathbb{C}[X_0, \ldots, X_{\ell-1}]\) for the diagonal action (by the automorphisms \(\sigma(x_{i,j}) = x_{i,\sigma(j)}\)). This is an algebra, which, as usual, acquires a Hopf algebra structure in the limit \(n \to \infty\).

In the following, we will start with a level \(\ell\) analogue of \(\text{FQSym}\), whose bases are labelled by \(\ell\)-colored permutations. Imitating the embedding of \(\text{Sym}\) in \(\text{FQSym}\), we obtain a Hopf subalgebra of level \(\ell\) called \(\text{Sym}(\ell)\), which is a natural noncommutative analog of McMahon’s algebra of multi-symmetric functions, and turns out to be dual to Poirier’s quasi-symmetric functions. Its homogenous components can be endowed with an internal product, thus providing an analog of Solomon’s descent algebras for wreath products, bigger than the Mantaci-Reutenauer algebras, and in which most useful properties such as the splitting formula remain valid. By commutative image, this yields an internal product on multi-symmetric functions.

The Mantaci-Reutenauer descent algebra MR(\(\ell\)) arises as a natural Hopf subalgebra of \(\text{Sym}(\ell)\) and its dual is computed in a straightforward way by means of an appropriate Cauchy formula.

Finally, we introduce a Hopf algebra of colored parking functions, and use it to define Hopf algebras structures on parking functions and non-crossing partitions of type \(B\).

The main results of this paper have been announced in the draft [32]. Since then, some of these results, in particular the construction of \(\text{Sym}(\ell)\), have been used by Baumann and Hohlweg [2], whose paper provide detailed proofs. Hence, we shall only include the proofs which cannot be found in their paper. In particular, we propose an alternative approach to the internal product, which is introduced by a duality argument, and derive its main properties from those of the dual coproduct.

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2. Background and notations

We first explain how to adapt the classical definitions and operations to the \(\ell\)-colored case.
2.1. Colored alphabets. We shall start with an $\ell$-colored alphabet
\[ A = A^0 \sqcup A^1 \sqcup \cdots \sqcup A^{\ell-1}, \]
such that all $A^i$ are of the same cardinality $N$, which will be assumed to be infinite in the sequel. Let $C$ be the alphabet $\{c_0, \ldots, c_{\ell-1}\}$ and $A$ be the auxiliary ordered alphabet $\{1, 2, \ldots\}$ (the letter $C$ stands for colors and $A$ for alphabet) so that $A$ can be identified with the cartesian product $A \times C$:
\[ A \simeq A \times C = \{(a, c) : a \in A, c \in C\}. \]
A colored letter $(i, c)$ will be denoted in bold type $i$. Given two colored words, their concatenation is obtained by concatenating separately the elements coming from $A$ and from $C$. We will sometimes allow $\ell = \infty$.

2.2. Colored standardization. Let $w$ be a word in $A$, represented as $(v, u)$ with $v \in A^*$ and $u \in C^*$. Then the colored standardized word $\text{Std}(w)$ of $w$ is
\[ \text{Std}(w) := (\text{Std}(v), u), \]
where $\text{Std}(v)$ is the usual standardization on words.

Recall that the standardization process sends a word $v$ of length $n$ to a permutation $\text{Std}(v) \in \mathfrak{S}_n$, called the standardized of $v$, defined as the permutation obtained by iteratively scanning $v$ from left to right, and labelling 1, 2, ... the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\text{Std}(v)$ is the permutation having the same inversions as $v$.

For example, $\text{Std}(abcadbdaa) = 157286934$:
\[
\begin{array}{cccccccccc}
& a & b & c & a & d & b & d & a & a \\
\text{a1} & b_5 & c_7 & a_2 & d_8 & b_6 & d_9 & a_3 & a_4 \\
1 & 5 & 7 & 2 & 8 & 6 & 9 & 3 & 4
\end{array}
\]
so that
\[ \text{Std}(abcdadbdaa, 144120100) = (157286934, 144120100). \]

2.3. Colored shifted operations. For an element $v = (v_1, v_2, \ldots)$ of $A$ and an integer $k$, denote by $v[k]$ the shifted word $(v_1 + k) \cdot \cdots \cdot (v_n + k)$, e.g., $312[4] = 756$. Given a colored word $\alpha = (\alpha, u)$, we set $\alpha[k] = (\alpha[k], u)$.

The shifted concatenation of two words $v$ and $v'$ is defined by
\[ v \cdot v' := v \cdot v'[k] \]
where $k$ is the length of $v$.

The shifted concatenation of two colored words $(v, c)$, $(v', c')$ is defined by
\[ (v, c) \cdot (v', c') := (v \cdot v'[k], c \cdot c'). \]
For example,
\[ (13241, 00322) \cdot (12, 23) = (1324167, 0032223). \]

Finally, recall that the shuffle product of two words $au$ and $bv$ is defined by
\[ au \shuffle bv = a(au \shuffle bv) + b(au \shuffle v), \]
where $a$ and $b$ are letters and $u$ and $v$ are words, with the initial conditions
\[(10) \quad u \epsilon v = \epsilon \epsilon u = u, \quad \epsilon \text{ being the empty word.}\]

This extends to the colored case, considering colored words as the concatenation of biletters.

The shifted shuffle product is
\[(11) \quad u \Psi v := u \epsilon v[k],\]
where $k$ is the size of $u$.

3. Free quasi-symmetric functions of level $\ell$

3.1. $\text{FQSym}^{(\ell)}$ and $\text{FQSym}^{(\Gamma)}$. A colored permutation is a pair $(\sigma, u)$, with $\sigma \in S_n$ and $u \in C^n$, the integer $n$ being the size of this permutation.

**Definition 3.1.** The dual free colored quasi-ribbon $G_{\sigma, u}$ labelled by a colored permutation $(\sigma, u)$ of size $n$ is the noncommutative polynomial
\[(12) \quad G_{\sigma, u} := \sum_{w \in A^n; \text{Std}(w) = (\sigma, u)} w \in \mathbb{Z} \langle A \rangle.\]

Recall that the convolution of two permutations $\sigma$ and $\mu$ is the set $\sigma * \mu$ (identified with the formal sum of its elements) of permutations $\tau$ such that the standardized word of the $|\sigma|$ first letters of $\tau$ is $\sigma$ and the standardized word of the remaining letters of $\tau$ is $\mu$ (see [39]). We then have:

**Theorem 3.2.** Let $(\sigma', u')$ and $(\sigma'', u'')$ be colored permutations. Then
\[(13) \quad G_{\sigma', u'} G_{\sigma'', u''} = \sum_{\sigma \in \sigma' * \sigma''} G_{\sigma, u' \cdot u''}.\]

Therefore, the dual free colored quasi-ribbons span a $\mathbb{Z}$-subalgebra $\text{FQSym}^{(\ell)}$ of the free associative algebra.

**Proof –** This is immediate from the product of the usual free quasi-symmetric functions $G_{\sigma}$:
\[(14) \quad G_{\sigma'} G_{\sigma''} = \sum_{\sigma \in \sigma' * \sigma''} G_{\sigma}.\]

Note that all colored permutations indexing a product of $G$ have given colors at the same places. For example,
\[(15) \quad G_{(21,41)} G_{(12,31)} = G_{(2134,4131)} + G_{(3124,4131)} + G_{(4123,4131)} + G_{(3214,4131)} + G_{(4213,4131)} + G_{(4312,4131)}.\]

One can define a coproduct by the usual trick of sums of alphabets: observe that we only need a total order on $A$ to define the colored standardisation, so that taking two isomorphic copies $A'$ and $A''$ of $A$, we define $A' \oplus A''$ as $(A' \oplus A'') \times C$, where
\( A' \oplus A'' \) denotes the ordered sum. Assuming furthermore that \( A' \) and \( A'' \) commute, we identify \( f(A')g(A'') \) with \( f \otimes g \) and define a coproduct by:

\[
\Delta f(A) = f(A' \oplus A'').
\]

By construction, this is an algebra morphism from \( \text{FQSym}^{(\ell)} \) to \( \text{FQSym}^{(\ell)} \otimes \text{FQSym}^{(\ell)} \), so that

**Theorem 3.3.** \( \text{FQSym}^{(\ell)} \) is a graded connected bialgebra. Hence, it is a Hopf algebra. The coproduct is given by

\[
\Delta G_{\sigma,u} := \sum_{(\sigma',u',u'') \in (\sigma,u) \uplus (\sigma'',u'')} G_{\sigma',u'} \otimes G_{\sigma'',u''}.
\]

For example,

\[
\Delta G_{3142,2412} = 1 \otimes G_{3142,2412} + G_{1,4} \otimes G_{231,212} + G_{12,42} \otimes G_{12,21}
\]

\[
+ G_{312,242} \otimes G_{1,1} + G_{3142,2412} \otimes 1.
\]

**Proof** – This is again immediate from the coproduct of the usual free quasi-symmetric functions \( G_{\sigma} \):

\[
\Delta G_{\sigma} = \sum_{\sigma \in \sigma' \uplus \sigma''} G_{\sigma'} \otimes G_{\sigma''}.
\]

\( \Box \)

3.2. Duality in \( \text{FQSym}^{(\ell)} \).

**Definition 3.4.** The free \( \ell \)-quasi-ribbon \( F_{\sigma,u} \) labelled by a colored permutation \( (\sigma,u) \) is the noncommutative polynomial

\[
F_{\sigma,u} := G_{\sigma^{-1},u^{-1}},
\]

where the action of a permutation on the right of a word permutes the positions of the letters of the word.

For example,

\[
F_{3142,2142} = G_{2413,142}.
\]

The product and coproduct of the \( F_{\sigma,u} \) can be easily described in terms of shifted shuffle and deconcatenation of colored permutations.

**Theorem 3.5.** Let \( \sigma' \) and \( \sigma'' \) be two colored permutations. Then

\[
F_{\sigma} F_{\sigma'} = \sum_{\sigma \subset \sigma' \uplus \sigma''} F_{\sigma},
\]

and

\[
\Delta F_{\sigma} = \sum_{w',w''} F_{\text{Std}(w')} \otimes F_{\text{Std}(w'')},
\]
Proof – Without colors, these formulas are the usual product of coproduct formulas of the \( F \) in \( \text{FQSym} \). With colors, one just has to observe that colors follow the letter to which they are attached.

Note that all colored permutations indexing a product of \( F \) have given colors associated with the same values, which is consistent with the corresponding remark on the \( G \) since places and values are exchanged when taking the inverse of a permutation.

For example, compare the following with Equation (15):

\[
\begin{align*}
F_{(21,14)}F_{(12,31)} &= F_{(2134,1341)} + F_{(2314,1341)} + F_{(2341,1314)} + F_{(3214,3141)} + F_{(3241,3114)} + F_{(3421,3114)}.
\end{align*}
\]

(24)

Here is an example of coproduct on the \( F \) basis:

\[
\begin{align*}
\Delta F_{(23514,1212)} &= 1 \otimes F_{(23514,1212)} + F_{(1,1)} \otimes F_{(241,312)} + F_{(12,14)} \otimes F_{(312,212)} \\
&+ F_{(123,14)} \otimes F_{(12,14)} + F_{(2341,1212)} \otimes F_{(1,12)} + F_{(23514,1212)} \otimes 1.
\end{align*}
\]

(25)

Let us define a scalar product on \( \text{FQSym}^{(\ell)} \) by

\[
\langle F_{\sigma,u}, G_{\sigma',u'} \rangle := \delta_{\sigma,\sigma'}\delta_{u,u'},
\]

where \( \delta \) is the Kronecker symbol.

**Theorem 3.6.** For any \( U, V, W \in \text{FQSym}^{(\ell)} \),

\[
\langle \Delta U, V \otimes W \rangle = \langle U, V W \rangle,
\]

(27)

so that, \( \text{FQSym}^{(\ell)} \) is self-dual: the map \( F_{\sigma,u} \mapsto G_{\sigma,u}^* \) is an isomorphism from \( \text{FQSym}^{(\ell)} \) to its graded dual.

**Proof –** Straightforward from Theorem 3.5.

**Note 3.7.** Let \( \phi \) be any bijection from \( C \) to \( C \), extended to words by concatenation. Then if one defines the free \( \ell \)-quasi-ribbon as

\[
F_{\sigma,u} := G_{\sigma^{-1},\phi(u)\cdot\sigma^{-1}},
\]

the previous theorems remain valid since one only permutes the labels of the basis \( (F_{\sigma,u}) \). Moreover, if \( C \) has a semigroup structure, the colored permutations \( (\sigma,u) \in \mathfrak{S}_n \times C^n \) can be interpreted as elements of the semi-direct product \( H_n := \mathfrak{S}_n \ltimes C^n \) with multiplication rule

\[
(\sigma;c_1,\ldots,c_n) \cdot (\tau;d_1,\ldots,d_n) := (\sigma\tau;c_{\tau(1)}d_1,\ldots,c_{\tau(n)}d_n).
\]

(29)

In furthermore \( C \) is a group, one can choose \( \phi(\gamma) := \gamma^{-1} \) and define the scalar product as before, so that the adjoint basis of the \( (G_h) \) becomes \( F_h := G_{h^{-1}} \). In the sequel, we will be mainly interested in the cases \( C := \mathbb{Z}/\ell\mathbb{Z} \), and we will indeed make that choice for \( \phi \) whenever \( C \) is a group.
3.3. Algebraic structure of FQSym\(ℓ\). Recall that a permutation \(σ\) of size \(n\) is connected \([27, 4]\) if, for any \(i < n\), the set \(\{σ_1, \ldots, σ_i\}\) is different from \(\{1, \ldots, i\}\).

We denote by \(C\) the set of connected permutations, and by \(c_n := |C_n|\) the number of such permutations in \(S_n\). For later reference, we recall that the generating series of \(c_n\) is Sequence A003319 of \([41]\):

\[
(30) \quad c(t) := \sum_{n \geq 1} c_n t^n = 1 - \left( \sum_{n \geq 0} n! t^n \right)^{-1} = t + t^2 + 3 t^3 + 13 t^4 + 71 t^5 + 461 t^6 + O(t^7).
\]

Let the connected colored permutations be the \((σ, u)\) with \(σ\) connected and \(u\) arbitrary. Their generating series is given by \(c(\ell t)\).

From \([4]\), we immediately get

**Proposition 3.8.** FQSym\(ℓ\) is free over the set \(F_{σ,u}\) (or \(G_{σ,u}\)), where \((σ, u)\) is connected.

For example, the generating series of the algebraic generators of FQSym\(2\) is

\[
(31) \quad 2 t + 4 t^2 + 24 t^3 + 208 t^4 + 2272 t^5 + 29504 t^6 + 441216 t^7 + \ldots
\]

3.4. Primitive elements of FQSym\(ℓ\). Let \(L(ℓ)\) be the primitive Lie algebra of FQSym\(ℓ\). Since \(Δ\) is not cocommutative, FQSym\(ℓ\) cannot be the universal enveloping algebra of \(L(ℓ)\). But since it is cofree, it is, according to \([23]\), the universal enveloping dipterous algebra of its primitive part \(L(ℓ)\).

Let \(G_{σ,u}\) be the multiplicative basis defined by \(G_{σ,u} = G_{σ_1,u_1} \cdots G_{σ_r,u_r}\) where \((σ, u) = (σ_1, u_1) \cdots (σ_r, u_r)\) is the unique maximal factorization of \((σ, u) ∈ S_n × C^n\) into connected colored permutations.

**Proposition 3.9.** Let \(V_{σ,u}\) be the adjoint basis of \(G_{σ,u}\). Then, the family \((V_{α,u})_{α ∈ C}\) is a basis of \(L(ℓ)\). In particular, we have \(\dim L_{n(ℓ)} = ℓ^n c_n\). Moreover, \(L(ℓ)\) is free.

**Proof** – The first part of the statement follows from \([4]\). The second part comes from the fact that FQSym\(ℓ\) is bidendriform (Theorem 3.10 below).

For example, since \(L(ℓ)\) is free, the generating series by degree of its generators is (with \(ℓ = 2\)):

\[
(32) \quad 1 - \prod_{n \geq 1} (1 - t^n)^{ℓ^n c_n} = 1 - (1 - t)^2(1 - t^2)^4(1 - t^3)^{24} \ldots
\]

\[
\quad = 2 t + 3 t^2 + 16 t^3 + 158 t^4 + 1796 t^5 + 24 250 t^6 + 372 656 t^7 + \ldots
\]

and the Hilbert series of the universal enveloping algebra of FQSym\(ℓ\) (its domain of cocommutativity) is, again with \(ℓ = 2\),

\[
(33) \quad \prod_{n \geq 1} (1 - t^n)^{-ℓ^n c_n} = 1 + 2 t + 7 t^2 + 36 t^3 + 283 t^4 + 2 898 t^5 + 36 169 t^6
\]

\[
\quad + 524 976 t^7 + \ldots
\]
3.5. **Dendriform structure of $\text{FQSym}^{(\ell)}$.** Foissy introduced the notion of bidendriform bialgebras [6], generalizing the notion of dendriform algebras (cf. [24]) and proved some conjectures about $\text{FQSym}$, presented in [4]. We shall adapt this technology to the colored case. We shall not recall all the theory, since complete details can be found in [6].

Recall that the generators of $\text{FQSym}^{(\ell)}$ as a dendriform algebra are called *totally primitive elements* [6] and that their generating series is given by

$$TP := \frac{PQ - 1}{PQ^2},$$

where $PQ$ is the generating series of $\text{FQSym}^{(\ell)}$.

**Theorem 3.10.** The algebra $\text{FQSym}^{(\ell)}$ has a structure of bidendriform bialgebra [6], hence is free as a Hopf algebra and as a dendriform algebra, cofree, self-dual, and its primitive Lie algebra is free.

Moreover, the totally primitive elements of $\text{FQSym}^{(\ell)}$ are the totally primitive elements of $\text{FQSym}$ with any coloring.

**Proof** – It has been done by Foissy in [6] in the case of $\text{FQSym}$. But since the dendriform and codendriform structure do not involve the color alphabet $C$, the property is true in this case as well.

Similarly, colors do not play any role in determining if a given element is (totally) primitive or not.

For example, the dendriform generators of $\text{FQSym}$ have as degree generating series

$$\sum_i \text{dg}_i t^i = t + t^3 + 6 t^4 + 39 t^5 + 284 t^6 + 2305 t^7 + \ldots$$

so that the dendriform generators of $\text{FQSym}^{(2)}$ have as degree generating series $2^i \text{dg}_i$:

$$2 t + 8 t^3 + 96 t^4 + 1248 t^5 + 18176 t^6 + 295040 t^7 + \ldots,$$

Note that there cannot be any relation, even dendriform relations, among the elements $F_{1,c}$ where $c \in C$, so that $\text{FQSym}^{(\ell)}$ contains the free dendriform algebra $\text{PBT}^{(\ell)}$ on $\ell$ generators.

3.6. **Internal product of $\text{FQSym}^{(\ell)}$.** When $C$ is a semigroup, an internal product can be defined on $\text{FQSym}^{(\ell)}$ by

$$F_{\sigma,u} \ast F_{\tau,v} = F_{\mu,w},$$

where $(\mu, w)$ is the product $(\sigma, u) \cdot (\tau, v)$ in the wreath product, defined by formula (29), that is

$$F_{(\sigma,u) \ast (\tau,v)} = F_{(\sigma, (u \tau) \cdot v)}.$$
For example, if the color group is $\mathbb{Z}$
\begin{equation}
F_{(1324,1011)} \ast F_{(2413,3200)} = F_{(3412,3311)}.
\end{equation}

This can be reduced to any $\mathbb{Z}/\ell\mathbb{Z}$, e.g., with $\ell = 3$,
\begin{equation}
F_{(165324,102011)} \ast F_{(625413,322011)} = F_{(462315,423023)}.
\end{equation}

In the $G$ basis, one has
\begin{equation}
G_{(\sigma,u)} \ast G_{(\tau,v)} = G_{(\tau \sigma,u \cdot (v \sigma))}.
\end{equation}

4. Noncommutative symmetric functions of level $\ell$

4.1. $\ell$-partite numbers. Following McMahon \[26\], we define an $\ell$-partite number $\mathbf{n} = (n_1, \ldots, n_\ell)$ as a column vector in $\mathbb{N}^\ell$, and a vector composition of $\mathbf{n}$ of weight $|\mathbf{n}| := \sum_i n_i$ and length $m$ as a $\ell \times m$ matrix $\mathbf{I}$ of nonnegative integers, with row sums vector $\mathbf{n}$ and no zero column.

For example,
\begin{equation}
\mathbf{I} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 1 \\ 4 & 2 & 1 & 3 \end{pmatrix}
\end{equation}
is a vector composition (or a 3-composition, for short) of the 3-partite number \(\binom{4}{5} \binom{5}{10}\) of weight 19 and length 4.

For each $\mathbf{n} \in \mathbb{N}^\ell$ of weight $|\mathbf{n}| = n$, we define a level $\ell$ complete homogeneous noncommutative symmetric function as
\begin{equation}
S_n := \sum_{u;|u_i| = n_i} G_{1 \cdots n_{u_i}}.
\end{equation}

It is the sum of all possible colorings of the identity permutation with $n_i$ occurrences of color $i$ for each $i$.

4.2. The Hopf algebra $\text{Sym}^{(\ell)}$. Let $\text{Sym}^{(\ell)}$ be the subalgebra of $\text{FQSym}^{(\ell)}$ generated by the $S_n$ (with the convention $S_0 = 1$). The Hilbert series of $\text{Sym}^{(\ell)}$ is easily found to be
\begin{equation}
S_\ell(t) := \sum_n \dim \text{Sym}^{(\ell)}_n t^n = \frac{(1-t)^\ell}{2(1-t)^\ell - 1}.
\end{equation}

For example, with $\ell = 2$, one has
\begin{equation}
S_2(t) := 1 + 2t + 7t^2 + 24t^3 + 82t^4 + 280t^5 + 956t^6 + 3264t^7 + \ldots
\end{equation}
which is Sequence A003480 of \[41\].
For general $\ell$, it is well-known in the combinatorial folklore (and easy to prove by means of generating series expansions) that, for all $n \geq 1$,

$$\text{ncs}_n(\ell) := n! \dim(\text{Sym}_n^{(\ell)}) = \sum_{k=1}^{n} S(n, k) p_k \ell^k$$

where $S(n, k)$ is the sequence of absolute values of Stirling numbers of the first kind (sequence A130534 of [41]) and $p_k$ is the sequence of ordered Bell numbers (also known as packed words or preferential arrangements, Sequence A000670 of [41]).

**Theorem 4.1.** $\text{Sym}^{(\ell)}$ is free over the set $\{S_n, |n| > 0\}$, so that a linear basis is given by

$$S^I = S_{i_1} \cdots S_{i_m},$$

where $i_1, \ldots, i_m$ are the columns of $I$.

Moreover, $\text{Sym}^{(\ell)}$ is a Hopf subalgebra of $\text{FQSym}^{(\ell)}$ and the coproduct of the generators is given by

$$\Delta S_n = \sum_{i+j=n} S_i \otimes S_j,$$

where the sum $i + j$ is taken in the space $\mathbb{N}^\ell$. In particular, $\text{Sym}^{(\ell)}$ is cocommutative.

**Proof** – Consider a linear relation between the $S^I$. Since without colors the algebra is free (it is $\text{Sym}$), the linear relation splits into many linear relations involving terms of the form $S^{i_1 \cdots i_r}$ with $|i_1|, \ldots, |i_r|$ fixed. But there are no relations of this form, thanks to the definition of the $G_{\sigma}$ as a sum of colored words.

Given the coproduct of the $G_{\sigma}$ of $\text{FQSym}^{(\ell)}$ and since the $S_n$ are sums of $G$, the coproduct of an $S_n$ amounts to unshuffling the color words, whence their coproduct formula. 

For example,

$$\Delta S^2 = S^{(1)} \otimes S^{(0)} + S^{(0)} \otimes S^{(1)} + S^{(1)} \otimes S^{(0)} + S^{(0)} \otimes S^{(1)} + S^{(1)} \otimes S^{(1)}.$$

**4.3. Algebraic structure of $\text{Sym}^{(\ell)}$.** The number of generators of $\text{Sym}^{(\ell)}$ of degree $n$ is given by the number of $\ell$-partite numbers of total sum $n$. So its generating series is

$$(1 - t)^{-\ell} - 1 = \sum_{n \geq 1} \binom{\ell + n - 1}{n} t^n.$$

We shall denote by $G(\ell)$ the set of nonzero $\ell$-partite numbers.
4.4. **Primitive elements of** $\text{Sym}^{(\ell)}$. $\text{Sym}^{(\ell)}$ being a graded connected cocommutative Hopf algebra, it follows from the Cartier-Milnor-Moore theorem that it is the universal enveloping algebra of $L^{(\ell)}$:

$$\text{Sym}^{(\ell)} = U(L^{(\ell)}),$$

where $L^{(\ell)}$ is the Lie algebra of its primitive elements. Let us now prove

**Theorem 4.2.** As a graded Lie algebra, the primitive Lie algebra $L^{(\ell)}$ of $\text{Sym}^{(\ell)}$ is free over a set indexed by $G(\ell)$.

**Proof** – If $L^{(\ell)}$ is free, by standard arguments on generating series, the number of generators of $L^{(\ell)}$ in degree $n$ must be the number of algebraic generators of $\text{Sym}^{(\ell)}$ in degree $n$, parametrized for example by $G(\ell)$. We will now show that $L^{(\ell)}$ has at least this number of generators and that those generators are algebraically independent, determining completely the dimensions of the homogeneous components $L_n^{(\ell)}$ of $L^{(\ell)}$ whose generating series begins by

$$t + \left(\frac{\ell + 1}{2}\right)t^2 + \left(\left(\frac{\ell + 1}{2}\right) + \left(\frac{\ell + 2}{3}\right)\right)t^3 + \ldots$$

Following Reutenauer [39] p. 58, denote by $\pi_1$ the Eulerian idempotent, that is, the endomorphism of $\text{Sym}^{(\ell)}$ defined by $\pi_1 = \log^*(Id)$. It is obvious, thanks to the definition of $S_p$ that

$$\pi_1(S_p) = S_p + \ldots,$$

where the dots stand for terms $S_I$ such that $I$ are vector compositions with strictly more than one column. Since the $S_p$ where $p$ is a $\ell$-partite number are algebraically independent, the dimension of $L_n^{(\ell)}$ is at least equal to the cardinality of the elements of $G(\ell)$ of size $n$. So $L^{(\ell)}$ is indeed free over a set of primitive elements parametrized by $G(\ell)$.

For example, with $\ell = 2$, the generating series of the dimensions of $L^{(\ell)}$ is

$$\sum_{n=0}^{\infty} S_n(A) x^n = \sum_{n=0}^{\infty} S_n(\ell^{(\ell-1)}) x^n \text{ where } x^n = (x^{(0)})^{n_0} \ldots (x^{(\ell-1)})^{n_{\ell-1}}.$$
This realization gives rise to a Cauchy formula (see [20] for the \( l = 1 \) case), which in turn allows one to identify the dual of \( \text{Sym}^{(\ell)} \) with an algebra introduced by S. Poirier in [35]. It is detailed in the following section.

Note that \( \text{Sym}^{(\ell)} \) is the natural noncommutative analog of McMahon’s algebra of multisymmetric functions [26, 10].

5. Quasi-symmetric functions of level \( \ell \)

5.1. Cauchy formula of level \( \ell \). Let \( X = X^0 \sqcup \cdots \sqcup X^{\ell-1} \), where \( X^i = \{ x^{(i)}_j, j \geq 1 \} \), be an \( \ell \)-colored alphabet of commutative variables, also commuting with \( A \). Imitating the level 1 case (see [4]), we define the Cauchy kernel

\[
K(X, A) = \prod_{j \geq 1} \sigma(x^{(0)}_{j_1}, \ldots, x^{(\ell-1)}_{j_{\ell-1}})(A).
\]

Expanding on the basis \( S^I \) of \( \text{Sym}^{(\ell)} \), we get as coefficients what can be called the level \( \ell \) monomial quasi-symmetric functions \( M_I(X) \)

\[
K(X, A) = \sum_I M_I(X) S^I(A),
\]

defined by

\[
M_I(X) = \sum_{j_1 < \cdots < j_m} x^{i_1}_{j_1} \cdots x^{i_m}_{j_m},
\]

with \( I = (i_1, \ldots, i_m) \).

These functions form a basis of a subalgebra \( QSym^{(\ell)} \) of \( \mathbb{K}[X] \), which we shall call the algebra of quasi-symmetric functions of level \( \ell \).

5.2. Poirier’s Quasi-symmetric functions. The functions \( M_I(X) \) can be recognized as a basis of one of the algebras introduced by Poirier: the \( M_I \) coincide with the \( M_{C,v} \) defined in [35], p. 324, formula (1), up to indexation.

Following Poirier, we introduce the level \( \ell \) quasi-ribbon functions by summing over an order on \( \ell \)-compositions: an \( \ell \)-composition \( C \) is finer than \( C' \), and we write \( C \preceq C' \), if \( C' \) can be obtained by repeatedly summing up two consecutive columns of \( C \) such that no nonzero element of the left one is strictly below a nonzero element of the right one.

This order can be described in a much easier and natural way if one recodes an \( \ell \)-composition \( I \) as a pair of words, the first one \( d(I) \) being the set of sums of the elements of the first \( k \) columns of \( I \), the second one \( c(I) \) being obtained by concatenating the words \( i^{k}_{\ell} \) while reading of \( I \) by columns, from top to bottom and from left to right.

For example, the 3-composition of Equation (43) satisfies

\[
d(I) = \{5, 10, 14, 19\} \quad \text{and} \quad c(I) = 13333 22233 1123 12333.
\]
Moreover, this recoding is a bijection if the two words \(d(I)\) and \(c(I)\) are such that the descent set of \(c(I)\) is a subset of \(d(I)\). The order previously defined on \(\ell\)-compositions is in this context the inclusion order on sets \(d: (d', c) \leq (d, c)\) iff \(d' \subseteq d\).

It allows us to define the level \(\ell\) quasi-ribbon functions \(F_I\) by

\[
F_I = \sum_{I' \leq I} M_{I'}.
\]

Notice that this last description of the order \(\leq\) is reminiscent of the order \(\leq'\) on descent sets used in the context of quasi-symmetric functions and non-commutative symmetric functions: more precisely, since it does not modify the word \(c(I)\), the order \(\leq\) restricted to \(\ell\)-compositions of weight \(n\) amounts for \(\ell^n\) copies of the order \(\leq'\). The computation of its Möbius function is therefore straightforward.

Moreover, one can obtain the \(F_I\) as the commutative image of certain \(F_{\sigma,u}\): any pair \((\sigma, u)\) such that \(\sigma\) has descent set \(d(I)\) and \(u = c(I)\) will do.

5.3. Coproducts and alphabets.

5.3.1. Recall that to define the \(G_{\sigma,u}(A)\) of an \(\ell\)-colored alphabet \(A = A \times C\), we only need a total order on \(A\). Hence, if \(B\) is another copy of \(A\) commuting with \(A\), we can define \(A + B\) as \((A + B) \times C\) where \(A + B\) is the ordinal sum, and thus make sense of \(G_{\sigma,u}(A + B)\).

As usual, we identify \(F(A)G(B)\) with \(F \otimes G\).

**Lemma 5.1.** For any \(F \in \text{FQSym}^{(\ell)}\), \(F(A + B) \in \text{FQSym}^{(\ell)} \otimes \text{FQSym}^{(\ell)}\), and

\[
F(A + B) = \Delta F,
\]

where \(\Delta\) is the coproduct defined by (17).

**Proof –** It is sufficient to show this for \(\ell = 1\), which is done in [4].

Let us observe that on this picture, it is clear that the restriction of \(\Delta\) to \(\text{Sym}^{(\ell)}\) is dual to the product of \(\text{QSym}^{(\ell)}\). By definition of the Cauchy kernel (58), we have

\[
K(X, A + B) = K(X, A)K(X, B),
\]

and by (59), this implies that \(\Delta\) is dual of the multiplication of \(\text{QSym}^{(\ell)}\).

5.3.2. The same description can be applied to the quasi-symmetric side. Let \(X = X \times C\) and \(Y = Y \times C\), where \(Y\) is a copy of \(X\). Again, define \(X + Y\) as the ordinal sum of \(X\) and \(Y\), and \(X + Y = (X + Y) \times C\).

**Lemma 5.2.** The map \(\nabla : F \mapsto F(X + Y)\) defines a coproduct on \(\text{QSym}^{(\ell)}\), which is dual to the product of \(\text{Sym}^{(\ell)}\).

**Proof –** This follows from the identity

\[
K(X + Y, A) = K(X, A)K(Y, A).
\]
5.3.3. The internal coproduct. From now on, we assume that the color set $C$ is an additive semigroup, such that every element $\gamma \in C$ has a finite number of decompositions $\gamma = \alpha + \beta$.

We define the $C$-product $T = \mathbb{X} \times_C \mathbb{Y}$ of two $C$-colored alphabets by
\begin{equation}
T(\gamma) = \left\{ t_{rs}^{(\gamma)} = \sum_{\alpha + \beta = \gamma} x_r^{(\alpha)} y_s^{(\beta)} \right\},
\end{equation}
with the pairs $(r, s)$ ordered lexicographically.

**Proposition 5.3.** The map $\delta : F \mapsto F(\mathbb{X} \times_C \mathbb{Y})$ defines a coassociative coproduct on $QSym(\ell)$.

We define the internal product $\ast$ of $\text{Sym}(\ell)$ as the dual product of the map $\delta$.

**Proof.** The coassociativity condition
\begin{equation}
(\delta \otimes \text{Id}) \circ \delta = (\text{Id} \otimes \delta) \circ \delta
\end{equation}
translates as the associativity of the $C$-product
\begin{equation}
(\mathbb{X} \times_C \mathbb{Y}) \times_C \mathbb{Z} = \mathbb{X} \times_C (\mathbb{Y} \times_C \mathbb{Z})
\end{equation}
which is clear since both sides are by definition
\begin{equation}
\left\{ t_{pqr}^{\mu} = \sum_{\alpha + \beta + \gamma = \mu} x_p^{(\alpha)} y_q^{(\beta)} z_r^{(\gamma)} \right\}
\end{equation}
with the lexicographic order on triples $(p, q, r)$. 

**Example 5.4.** With $\ell = 2$ and $C = \mathbb{Z}/2\mathbb{Z}$,
\begin{align*}
S\left(\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right) * S\left(\begin{array}{c} 0 \\ 0 \\ 2 \end{array}\right) &= \mu_2 \left[ \left( \left( S\left(\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right) \otimes S\left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right) \right) *_2 \Delta S\left(\begin{array}{c} 0 \\ 0 \\ 2 \end{array}\right) \right] \\
&= \left( S\left(\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right) * S\left(\begin{array}{c} 0 \\ 0 \\ 2 \end{array}\right) \right) \left( S\left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right) * S\left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right) \right) + \left( S\left(\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right) * S\left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right) \right) \left( S\left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right) * S\left(\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right) \right) \\
&= S\left(\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right) + S\left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right) + S\left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right) + S\left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right).
\end{align*}

5.3.4. The splitting formula. The definition of $\ast$ by duality with the $C$-product implies that it satisfies the splitting formula. We just need to check a trivial property:

**Lemma 5.5.** The $C$-product is distributive over the colored ordinal sum:
\begin{equation}
(\mathbb{X}' + \mathbb{X}'') \times_C \mathbb{Y} = \mathbb{X}' \times_C \mathbb{Y} + \mathbb{X}'' \times_C \mathbb{Y}.
\end{equation}
Proposition 5.6. Let $\mu_r : (\text{Sym}^{(r)})^{\otimes r} \to \text{Sym}^{(r)}$ be the product map. Let $\Delta^{(r)} : (\text{Sym}^{(r)}) \to (\text{Sym}^{(r)})^{\otimes r}$ be the $r$-fold coproduct, and $\star_r$ be the extension of the internal product to $(\text{Sym}^{(r)})^{\otimes r}$. Then, for $F_1, \ldots, F_r$, and $G \in \text{Sym}^{(r)}$,

$$
(F_1 \cdots F_r) \star G = \mu_r[(F_1 \otimes \cdots \otimes F_r) \star_r \Delta^{(r)} G].
$$

Proof – It is enough to consider the case $r = 2$. Let $F \in Q\text{Sym}^{(r)}$ and $U, W, W' \in \text{Sym}^{(r)}$. We have, writing $XY = X \times_C Y$ for short, and assuming duality between $X$ and $A$, $Y$ and $B$ and so on,

$$
\langle F, (UV) \star W \rangle = \langle F(XX''), (UV)(A')W(A'') \rangle \\
= \langle F(X' + Y')X'', U(A')V(B')W(A'') \rangle \\
= \langle F(XX'' + Y''Y''), U(A')V(B')W(A'' + B'') \rangle \\
= \langle F(X + Y'), U(A')V(B')W(A' + B') \rangle \\
= \langle \nabla F, (U \otimes V) \star_2 \Delta W \rangle = \langle F, \mu[(U \otimes V) \star_2 \Delta W] \rangle.
$$

5.3.5. Evaluation of internal products. Let us start with the simplest case, $S_i \star S_j$. The coefficient of $S^K$ in this product is the coefficient of $M_i \otimes M_j$ in $\delta M_K$, which is also the coefficient of $x^i y^j$ in $M_K(x \times_C y)$. This is zero if $K$ has more than one column, so that

$$
S_i \star S_j = \sum_n d^n_{ij} S_n
$$

contains only one-part vector compositions.

Lemma 5.7. If the color semigroup is $C = \mathbb{N}$, then the coefficient $d^n_{ij}$ of $S_n$ in $S_i \star S_j$ is equal to the coefficient of the monomial symmetric function $m_\mu$ in the product $m_\alpha m_\beta$, where $\mu$ is the partition $(0^a1^b \cdots)$, $\alpha = (0^a1^b \cdots)$, $\beta = (0^a1^b \cdots)$, the monomial functions being taken over an alphabet of $n$ letters, where $n = |i|$.

Proof – From (72), we only need to compute the coproducts $\delta M_n$. For this, it is sufficient to use alphabets of the form $X = \{x\} \otimes C$, $Y = \{y\} \otimes C$. Then,

$$
M_n(x \times_C y) = \prod_{\ell \geq 0} \left( \sum_{i+j=k} x^{(i)} y^{(j)} \right)^{n_k},
$$

and we see that the coefficient of $M_i(x)M_j(y) = x^i y^j$ is the same as the coefficient of $h_\alpha \otimes h_\beta$ in

$$
\prod_{\ell \geq 0} \left( \sum_{i+j=k} h_i \otimes h_j \right)^{n_k} = \delta (h_0^n h_1^n \cdots) = \Delta h_\mu.
$$

\[\square\]
For example,

\[ S^{\binom{2}{2}} \ast S^{\binom{3}{1}} = 3 S^{\binom{1}{3}} + S^{\binom{2}{1}}. \]

This result is compatible with the fact that

\[ m_{11}m_1 = 3m_{111} + m_{21}. \]

As another example,

\[ S^{\binom{0}{0}} \ast S^{\binom{1}{1}} = S^{\binom{0}{0}} + 2 S^{\binom{1}{0}} + 2 S^{\binom{2}{1}}. \]

One can then check that the previous result amounts to selecting the partitions of size at most 3 in

\[ m_{211}m_{21} = m_{421} + 2m_{331} + 2m_{322} + \ldots \]

Together with the splitting formula (71), this result determines all the products \( S^I \ast S^J \), since one also has

\[ S^I \ast S^J = S^J \ast S^I \]

thanks to the isomorphism of ordered colored alphabets

\[ x \times_C Y \simeq Y \times_C x, \]

where \( x = \{x\} \otimes C \).

When the color group is \( \mathbb{Z}/\ell \mathbb{Z} \), the result is obtained by reduction modulo \( \ell \), e.g., with \( l = 3 \), we get from example (77)

\[ S^{\binom{0}{0}} \ast S^{\binom{1}{1}} = 2 S^{\binom{0}{0}} + 2 S^{\binom{1}{0}} + 2 S^{\binom{2}{1}}. \]

Note that the coefficient of an \( S^I \) can change when computing modulo \( \ell \): for all pairs of parts \( k \) and \( k' \) added together, a factor \( \binom{k+k'}{k} \) appears.

5.4. **Generalized descent algebras.** In a preliminary draft of this work [32], we introduced the internal product in a different way. Assuming that \( C \) has a semigroup structure, we regard colored permutations as elements of the wreath product \( H = C \wr \mathfrak{S}_n \), and for \( h', h'' \in H \), we set

\[ G_{h'} \ast' G_{h''} = G_{h''h'}. \]

(opposite law, as in the classical case of \( \text{Sym} \)).

**Theorem 5.8.** Let \( C \) be a commutative semigroup.

1. \( \text{Sym}_n^{(C)} \) is a subalgebra of \( \text{FQSym}_n^{(C)} \), for the operation \( \ast' \) defined previously.
2. The restriction of \( \ast' \) to \( \text{Sym}_n^{(C)} \) satisfies the splitting formula (71).
3. \( S^I \ast' S^J \) is given by Lemma 5.7.
4. \( S^I \ast' S^J = S^J \ast' S^I \).
Proof – The proofs of (1) and (2) can be found in [2]. (3) and (4) follow from the definition in (44) and the internal product on $\text{FQSym}^{(C)}$ given by (38).

This provides an analogue of Solomon’s descent algebra for the wreath product $C \wr \mathfrak{S}_n$. Note that the definition remains valid for $C = \mathbb{Z}$, so that we get a descent algebra for the (extended) affine Weyl groups of type $A$, $\hat{\mathfrak{S}}_n = \mathbb{Z} \wr \mathfrak{S}_n$.

5.5. Ordinary multi-symmetric functions. A consequence of the above results is that the algebra $\text{Sym}^{(\ell)}$ of ordinary (commutative) multi-symmetric functions admits an internal product. If we denote by $F \in \text{Sym}^{(\ell)}$ the commutative image of $F \in \text{Sym}^{(\ell)}$, we have

\[(83) \quad F \ast G = F' \ast G' \quad \text{as soon as} \quad F = F' \quad \text{and} \quad G = G'.\]

For example,

\[
S^{\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}} \ast S^{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + S^{\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}} + S^{\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}} + S^{\begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \\
+ 2S^{\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}} + 2S^{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}} \\
+ S^{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} + S^{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} + S^{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} + S^{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \\
+ 2S^{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}} + 2S^{\begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}} + 2S^{\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}.
\]

\[
S^{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} + S^{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} + S^{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} + S^{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \\
+ 2S^{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} + 2S^{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \\
+ S^{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} + S^{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} + S^{\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} + S^{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \\
+ 2S^{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} + 2S^{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} + 2S^{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}.
\]
\[
S^{(3\ 0)}_{(1\ 1)} * S^{(2\ 1)}_{(1\ 1)} = S^{(0\ 0\ 2)}_{(0\ 1\ 0)} + S^{(0\ 0\ 2)}_{(1\ 0\ 1)} + S^{(0\ 0\ 0)}_{(1\ 0\ 0)} + S^{(0\ 0\ 1)}_{(1\ 1\ 0)} + 2S^{(0\ 0\ 1)}_{(2\ 0\ 0)} + 2S^{(0\ 0\ 0)}_{(0\ 0\ 0)}
\]

(86)

\[
S^{(3\ 0)}_{(1\ 1)} * S^{(2\ 1)}_{(1\ 1)} = S^{(2\ 0\ 0)}_{(1\ 0\ 1)} + S^{(2\ 0\ 0)}_{(0\ 1\ 0)} + S^{(2\ 0\ 0)}_{(1\ 0\ 0)} + 2S^{(2\ 0\ 0)}_{(2\ 1\ 1)} + 2S^{(2\ 0\ 0)}_{(0\ 1\ 0)} + 2S^{(2\ 0\ 0)}_{(0\ 0\ 1)} + 2S^{(2\ 0\ 0)}_{(0\ 0\ 0)}
\]

(87)

If one denotes by \( h \) the commutative image of \( S \), one easily checks that

\[
h^{(3\ 0)}_{(1\ 1)} h^{(2\ 1)}_{(1\ 1)} = 2h^{(2\ 1)}_{(1\ 0\ 1)} h^{(0\ 0\ 1)}_{(1\ 0\ 0)} + h^{(2\ 0\ 0)}_{(0\ 1\ 0)} h^{(0\ 0\ 1)}_{(1\ 0\ 0)} + h^{(2\ 0\ 0)}_{(1\ 0\ 0)} h^{(0\ 0\ 0)}_{(1\ 0\ 0)} + 2h^{(2\ 0\ 0)}_{(2\ 1\ 1)} h^{(0\ 0\ 1)}_{(1\ 0\ 0)} + 2h^{(2\ 0\ 0)}_{(0\ 1\ 0)} h^{(0\ 0\ 1)}_{(1\ 0\ 0)} + 2h^{(2\ 0\ 0)}_{(0\ 0\ 1)} h^{(0\ 0\ 0)}_{(1\ 0\ 0)} + 4h^{(2\ 0\ 0)}_{(0\ 0\ 0)} h^{(0\ 0\ 0)}_{(1\ 0\ 0)}
\]

(88)

6. The Mantaci-Reutenauer algebra

6.1. Monochromatic complete functions. Let \( e_i \) be the canonical basis of \( \mathbb{N}^r \).
For \( n \geq 1 \), let

\[
\sigma_n^{(i)} = S_{n-e_i} \in \text{Sym}^{(i)}
\]

(89)
be the monochromatic complete symmetric functions.

**Proposition 6.1.** The $S_n^{(i)}$ generate a Hopf subalgebra $\text{MR}^{(\ell)}$ of $\text{Sym}^{(\ell)}$, which is isomorphic to the Mantaci-Reutenauer descent algebra of the wreath products $\mathfrak{s}_n^{(i)} = (\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{s}_n$ if $C = \mathbb{Z}/\ell\mathbb{Z}$.

**Proof** – $\text{MR}^{(\ell)}$ is obviously stable by the product and coproduct coming from $\text{Sym}^{(\ell)}$, hence is a Hopf subalgebra of $\text{Sym}^{(\ell)}$.

As a Hopf algebra, it is clearly isomorphic to the Mantaci-Reutenauer algebra, having the same number of independent generators in each degree, and behaving in the same way (divided powers) under the coproduct. The isomorphism for the internal product comes from the splitting formula, which gives explicitly $S(I,u) * S(J,v)$. ■

Since $\text{MR}^{(\ell)}$ has $\ell$ generators in each degree, its dimensions are given by

$$1 - \sum_{k \geq 1} t^k \ell^k = 1 + \ell \sum_{n \geq 1} (\ell + 1)^{n-1} t^n.$$

The bases of $\text{MR}^{(\ell)}$ are labelled by colored compositions, that is, pairs formed by a composition and a color vector of the same length:

$$(I, u) = ((i_1, \ldots, i_m), (u_1, \ldots, u_m)).$$

6.2. **Primitive elements of $\text{MR}^{(\ell)}$.** $\text{MR}^{(\ell)}$ being a subalgebra of a graded connected cocommutative Hopf algebra, is itself a graded connected cocommutative Hopf algebra, so that, thanks to the Cartier-Milnor-Moore theorem, it is the universal enveloping algebra of $L^{(\ell)}$:

$$\text{MR}^{(\ell)} = U(L^{(\ell)}),$$

where $L^{(\ell)}$ is the Lie algebra of its primitive elements. The following property is obtained in the same way as Theorem 4.2.

**Theorem 6.2.** As a graded Lie algebra, the primitive Lie algebra $L^{(\ell)}$ of $\text{MR}^{(\ell)}$ is free over a set indexed by colored compositions.

For example, with $\ell = 2$, the generating series of the dimensions of $L^{(\ell)}$ is

$$1 + 2t + 3t^2 + 8t^3 + 18t^4 + 48t^5 + 116t^6 + 312t^7 + \ldots$$

With $\ell = 3$, one finds

$$1 + 3t + 6t^2 + 20t^3 + 60t^4 + 204t^5 + 670t^6 + 2340t^7 + \ldots$$

More generally, the dimension of $L_n^{(\ell)}$ is given by the Witt polynomials

$$q_n(\ell) := \begin{cases} \frac{1}{n} \sum_{d \mid n} \mu(d)(\ell + 1)^{n/d} & \text{if } n = 1, \\ \frac{1}{\ell} \sum_{d \mid n} \mu(d)(\ell + 1)^{n/d} & \text{if } n > 1, \end{cases}$$

so that, for $n \geq 2$, the dimension of $L_n^{(\ell)}$ coincide with those of a free Lie algebra on $\ell + 1$ generators of degree 1.
6.3. **Duality.** The duality is easily worked out by means of the appropriate Cauchy kernel. The generating function of the complete functions is

\[(96)\quad \sigma_{\text{MR}}^x(A) := 1 + \sum_{j=0}^{\ell-1} \sum_{n \geq 1} \sum_{\text{S}(j)}(x^{(j)})^n,\]

and the Cauchy kernel is as usual

\[(97)\quad K_{\text{MR}}(X, A) = \prod_{i \geq 1} \sigma_{\text{MR}}^{x_i}(A) = \sum_{(I, u)} M_{(I, u)}(X) S^{(I, u)}(A),\]

where \((I, u)\) runs over colored compositions \(I = (i_1, \ldots, i_m), (u_1, \ldots, u_m)\). The \(M_{(I, u)}\) are called the **monochromatic monomial quasi-symmetric functions** and satisfy

\[(98)\quad M_{(I, u)}(X) = \sum_{j_1 < \cdots < j_m} (x_{j_1}^{(u_1)})^{i_1} \cdots (x_{j_m}^{(u_m)})^{i_m}.\]

**Proposition 6.3.** The \(M_{(I, u)}\) span a subalgebra of \(C[X]\) which can be identified with the graded dual of \(\text{MR}^{(\ell)}\) through the pairing

\[(99)\quad \langle M_{(I, u)}, S^{(I', v)} \rangle = \delta_{I, I'} \delta_{u, v},\]

where \(\delta\) is the Kronecker symbol.

Note that this algebra can also be obtained by imposing the relations

\[(100)\quad x_i^{(p)} x_i^{(q)} = 0, \text{ for } p \neq q\]

on the variables of \(Q\text{Sym}^{(\ell)}\).

Baumann and Hohlweg [2] have another construction of the dual of \(\text{MR}^{(\ell)}\) (implicitly defined in [35], Lemma 11).

7. **Level \(\ell\) parking quasi-symmetric functions**

7.1. **Usual parking functions.**

The combinatorial objects. Recall that a **parking function** on \([n] = \{1, 2, \ldots, n\}\) is a word \(a = a_1 a_2 \cdots a_n\) of length \(n\) on \([n]\) whose nondecreasing rearrangement \(a^\dagger = a'_1 a'_2 \cdots a'_n\) satisfies \(a'_i \leq i\) for all \(i\). Let \(PF_n\) be the set of such words. It is well-known that \(|PF_n| = (n + 1)^{n-1}\).

One says that \(a\) has a **breakpoint** at \(b\) if \(|\{a_i \leq b\}| = b\). The set of all breakpoints of \(a\) is denoted by \(BP(a)\). Then, \(a \in PF_n\) is said to be **prime** if \(BP(a) = \{n\}\) (see [12]).

Let \(PPF_n \subset PF_n\) be the set of prime parking functions on \([n]\). It can easily be shown that \(|PPF_n| = (n - 1)^{n-1}\) (see [13]).

Finally, one says that \(a\) has a **match** at \(b\) if \(|\{a_i < b\}| = b - 1\) and \(|\{a_i \leq b\}| \geq b\). The set of all matches of \(a\) is denoted by \(Ma(a)\).
7.1.2. Algebraic structure on parking functions. The algebra \( \text{PQSym} \) of parking functions \([29, 31]\) is very similar to the algebra \( \text{FQSym} \) of permutations.

Since parking functions are closed under the shifted shuffle, one defines a product on the vector space with basis \((F_a)\) by \[
F_a F_{a'} = \sum_{a'' \in [a' \shuffle a'']} F_{a''}.
\]
The coproduct on this basis is given by the parkization algorithm \([29]\): for \( w = w_1 w_2 \cdots w_n \) on \( \{1, 2, \ldots\} \), let us define \[
d(w) := \min\{i | \#\{w_j \leq i\} < i\}.
\]
If \( d(w) = n + 1 \), then \( w \) is a parking function and the algorithm terminates, returning \( w \). Otherwise, let \( w' \) be the word obtained by decrementing all the elements of \( w \) greater than \( d(w) \). Then \( \text{Park}(w') := \text{Park}(w') \). Since \( w' \) is smaller than \( w \) in the lexicographic order, the algorithm terminates and always returns a parking function.

For example, let \( w = (3, 5, 1, 11, 8, 8, 2) \). Then \( d(w) = 6 \) and the word \( w' = (3, 5, 1, 1, 10, 7, 7, 2) \). Then \( d(w') = 6 \) and \( w'' = (3, 5, 1, 1, 9, 6, 6, 2) \). Finally, \( d(w'') = 8 \) and \( w''' = (3, 5, 1, 1, 8, 6, 6, 2) \), that is a parking function. Thus, \( \text{Park}(w) = (3, 5, 1, 1, 8, 6, 6, 2) \).

We then have \[
\Delta F_a = \sum_{u, v; u \bowtie v} F_{\text{Park}(u)} \otimes F_{\text{Park}(v)}.
\]

7.1.3. Duality. Let \( G_a = F_a^* \in \text{PQSym}^* \) be the dual basis of \((F_a)\). If \( \langle , \rangle \) denotes the duality bracket, the product on \( \text{PQSym}^* \) is given by \[
G_a G_{a'} = \sum_{a''} \langle G_{a''} \otimes G_{a''}, \Delta F_a \rangle G_a = \sum_{a''} G_{a''},
\]
where the convolution \( a' \ast a'' \) of two parking functions is defined as \[
a' \ast a'' = \sum_{u, v; u \bowtie v, \text{Park}(u) = a', \text{Park}(v) = a''} a.
\]
By duality, one easily gets the formula for the coproduct of \( G_a \) as \[
\Delta G_a := \sum_{u, v; a \bowtie u \bowtie v} G_u \otimes G_v.
\]

7.2. Colored parking functions. Let \( \ell \) be an integer, representing the number of allowed colors. A colored parking function of level \( \ell \) and size \( n \) is a pair composed of a parking function of length \( n \) and a word on \([\ell]\) of length \( n \).

Since there is no restriction on the coloring, it is obvious that there are \( \ell^n(n+1)^{n-1} \) colored parking functions of level \( \ell \) and size \( n \).

With two colors, one finds the sequence \( a_i = 2(2i+2)^{i-1} \), known as A097629 in \([31]\): \[
1 + 2t + 12t^2 + 128t^3 + 2000t^4 + 41472t^5 + 1075648t^6 + \ldots
\]
Since the convolution of two parking functions contains only parking functions, one easily builds as in [29] an algebra $\text{PQSym}^{(\ell)}$ on colored parking functions:

\begin{equation}
G(a',u') G(a'',u'') = \sum_{a \in a' \ast a''} G(a,u') \cdot G(a'',u'').
\end{equation}

We can define a coproduct using sums of alphabets: again, we only need a total order on $A$ to define the colored parkization, so that taking two isomorphic copies $A'$ and $A''$ of $A$, we define $A' \oplus A''$ as $(A' \oplus A'') \times C$, where $A' \oplus A''$ denotes the ordered sum. Assuming furthermore that $A'$ and $A''$ commute, we identify $f(A') g(A'')$ with $f \otimes g$ and define a coproduct by:

\begin{equation}
\Delta f(a,u) = f(A' \oplus A'').
\end{equation}

By construction, this is an algebra morphism from $\text{PQSym}^{(\ell)}$ to $\text{PQSym}^{(\ell)} \otimes \text{PQSym}^{(\ell)}$, so that

**Theorem 7.1.** $\text{PQSym}^{(\ell)}$ is a graded connected bialgebra, hence a Hopf algebra. More precisely, the coproduct can be computed in the following way:

\begin{equation}
\Delta G(a,u) = \sum_{(\alpha',\alpha'',u',u'') \in (a',u') \oplus (a'',u'')} G(\alpha',u') \otimes G(\alpha'',u'').
\end{equation}

**Proof** – Straightforward from the previous definitions.

For example,

\begin{equation}
\Delta G(41142,22115) = 1 \otimes G(41142,22115) + G(112,215) \otimes G(11,21) + G(41142,22115) \otimes 1.
\end{equation}

**7.3. Duality.** Let $F(a,u) = G^*_a \in \text{PQSym}^{(\ell)*}$ be the dual basis of $(G_a)$. If $\langle , \rangle$ denotes the duality bracket, the product on $\text{PQSym}^{(\ell)*}$ is given by

\begin{equation}
F(a',u') F(a'',u'') = \sum_{(a,u) \in (a',u') \oplus (a'',u'')} F(a,u),
\end{equation}

where the shifted shuffle of two colored parking functions is such that colors follow their letters.

Using the duality bracket once more, one easily gets the formula for the coproduct of $F(a,u)$ as

\begin{equation}
\Delta F(a,u) = \sum_{(p',u'),(p'',u'') \in (p',u') \oplus (p'',u'')} F(\text{Park}(p'),u') \otimes F(\text{Park}(p''),u'').
\end{equation}

**7.4. Algebraic structure of PQSym*^{(\ell)}.** Recall that a word $w$ over $\mathbb{N}^*$ is connected if it cannot be written as a shifted concatenation $w = u \cdot v$, and anti-connected if its mirror image $\overline{w}$ is connected.
We denote by $\mathcal{CP}$ the set of connected parking functions, and by $p_n := |\mathcal{CP}_n|$ the number of such parking functions of size $n$. For later reference, we recall that the generating series of $p_n$ is Sequence A122708 of [41]:

$$p(t) := \sum_{n \geq 1} p_n t^n = 1 - \left( \sum_{n \geq 0} (n + 1)^{n-1} t^n \right)^{-1}$$

$$= t + 2t^2 + 11t^3 + 92t^4 + 1014t^5 + 13795t^6 + 223061t^7 + \ldots$$

Let the connected colored parking functions be the $(a, u)$ with $a$ connected and $u$ arbitrary. Their generating series is given by $p(\ell t)$.

From [31], we immediately get

**Proposition 7.2.** $\text{PQSym}^{(\ell)}$ is free over the set $\mathbf{F}_{\sigma,u}$ (or $\mathbf{G}_{\sigma,u}$), where $(\sigma,u)$ is connected.

For example, one gets the following generating series for the algebraic generators (connected parking functions with $\ell = 2$):

$$2t + 8t^2 + 88t^3 + 1472t^4 + 32448t^5 + 882880t^6 + 28551808t^7 + \ldots$$

**7.5. Primitive elements of $\text{PQSym}^{(\ell)}$.** Let $\mathcal{L}^{(\ell)}$ be the primitive Lie algebra of $\text{PQSym}^{(\ell)}$. Since $\Delta$ is not cocommutative, $\text{PQSym}^{(\ell)}$ cannot be the universal enveloping algebra of $\mathcal{L}^{(\ell)}$. But since it is cofree, it is, according to [23], the universal enveloping dipterous algebra of its primitive part $\mathcal{L}^{(\ell)}$.

Let $\mathbf{G}^{a,u}$ be the multiplicative basis defined by $\mathbf{G}^{a,u} = \mathbf{G}_{a_1,u_1} \cdots \mathbf{G}_{a_r,u_r}$ where $(a,u) = (a_1,u_1) \cdots (a_r,u_r)$ is the unique maximal factorization of $(a,u) \in \mathcal{S}_n \times C^n$ into connected colored parking functions.

**Proposition 7.3.** Let $\mathbf{V}_{a,u}$ be the adjoint basis of $\mathbf{G}^{a,u}$. Then, the family $(\mathbf{V}_{a,u})_{(a,u) \in \mathcal{CP}}$ is a basis of $\mathcal{L}^{(\ell)}$. In particular, we have $\dim \mathcal{L}^{(\ell)} = \ell^n p_n$. Moreover, $\mathcal{L}^{(\ell)}$ is free.

**Proof** – The first part of the statement follows from [4]. The second part comes from the fact that $\text{PQSym}^{(\ell)}$ is bidendriform (Theorem 7.4 below).

For example, since $\mathcal{L}^{(\ell)}$ is free, the generating series of the degree of its generators is (with $\ell = 2$):

$$1 - \prod_{n \geq 1} (1 - t^n)^{\ell^n p_n} = 1 - (1 - t)^2(1 - t^2)^8(1 - t^3)^{88} \cdots$$

$$= 2t + 7t^2 + 72t^3 + 1276t^4 + 28944t^5 + 805288t^6 + 26462232t^7 + \ldots$$

and the Hilbert series of the universal enveloping algebra of $\text{PQSym}^{(\ell)}$ (its domain of cocommutativity) is, again with $\ell = 2$:

$$\prod_{n \geq 1} (1 - t^n)^{-\ell^n c_n} = 1 + 2t + 11t^2 + 108t^3 + 1713t^4 + 36470t^5 + 969919t^6 + 30847464t^7 + \ldots$$
7.6. Bidendriform and tridendriform structure.

**Theorem 7.4.** The algebra $\text{PQSym}^{(\ell)}$ has a structure of bidendriform bialgebra, hence is free as a Hopf algebra and as a dendriform algebra, cofree, self-dual, and its primitive Lie algebra is free.

Moreover, the totally primitive elements of $\text{PQSym}^{(\ell)}$ are the totally primitive elements of $\text{PQSym}$ with any coloring.

**Proof –** It has been done in [31] in the case of $\text{FQSym}$. But since the dendriform and codendriform structure do not involve the color alphabet $C$, the property is true in this case as well.

Also, colors do not play any role in determining whether a given element is (totally) primitive or not, so the last statement holds. ■

For example, the dendriform generators of $\text{PQSym}$ have as degree generating series

$$\sum_i \text{dgp}_i t^i = t + t^2 + 7 t^3 + 66 t^4 + 786 t^5 + 11 378 t^6 + 189 391 t^7 + \ldots$$

so that the dendriform generators of $\text{PQSym}^{(2)}$ have as degree generating series $2^i \text{dgp}_i$:

$$2 t + 4 t^2 + 56 t^3 + 1 056 t^4 + 25 152 t^5 + 721 792 t^6 + 24 242 048 t^7 + \ldots$$

7.6.1. Tridendriform structure.

**Conjecture 7.5.** As in the case of $\text{PQSym}$, we conjecture that $\text{PQSym}^{(\ell)}$ is a free dendriform trialgebra.

Note that there cannot be any relations, even tridendriform relations, among the elements $F_{1,c}$ where $c \in C$, so that $\text{PQSym}^{(\ell)}$ contains the free tridendriform algebra $\mathfrak{T}D^{(\ell)}$ on $\ell$ generators.

Recall that, if $\text{PQSym}^{(\ell)}$ is free as a tridendriform algebra, its generators have as generating series

$$\text{TD} := \frac{\text{PQ} - 1}{2\text{PQ}^2 - \text{PQ}},$$

where $\text{PQ}$ is the generating series of $\text{PQSym}^{(\ell)}$.

For example, the tridendriform generators of $\text{PQSym}$ have as degree generating series

$$\sum_i \text{tgp}_i t^i = t + 5 t^3 + 50 t^4 + 634 t^5 + 9 475 t^6 + 163 843 t^7 + \ldots$$

so that the tridendriform generators of $\text{PQSym}^{(2)}$ have as degree generating series $2^i \text{tgp}_i$:

$$2 t + 40 t^3 + 800 t^4 + 20 288 t^5 + 606 400 t^6 + 20 971 904 t^7 + \ldots$$
8. Type B Algebras

8.1. Parking functions of type B. In \[38\], Reiner defined non-crossing partitions of type B by analogy to the classical case. In our context, he defined the level 2 case. This allowed him to derive, by analogy with a simple representation theoretical result, a definition of parking functions of type B as the words on \([n]\) of size \(n\).

We shall build another set of words, also enumerated by \(n\) that sheds light on the relation between type-A and type-B parking functions and provides a natural Hopf algebra structure on the latter.

First, fix two colors \(0 < 1\). We say that a pair of words \((a, u)\) composed of a parking function and a binary colored word is a level 2 parking function if

- the only elements of \(a\) that can have color 1 are the matches of \(a\).
- for all element of \(a\) of color 1, all letters equal to it and to its left also have color 1,
- all elements of \(a\) have at least once the color 0.

For example, there are 27 level 2 parking functions of size 3: there are the 16 usual ones all with full color 0, and the eleven new elements

\[
(111, 100), (111, 110), (112, 100), (121, 100), (211, 010), (113, 100), (131, 100), (311, 010), (122, 010), (212, 100), (221, 100).
\]

The first time the first rule applies is with \(n = 4\), where one has to discard the words \((1122, 0010)\) and \((1122, 1010)\) since 2 is not a match of 1122. On the other hand, both words \((1133, 0010)\) and \((1133, 1010)\) are \(B_4\)-parking functions since 1 and 3 are matches of 1133.

**Theorem 8.1.** The restriction of \(\text{PQSym}^{(2)}\) to the \(F\) elements indexed by level 2 parking functions is a subalgebra of \(\text{PQSym}^{(2)}\). The restriction of \(\text{PQSym}^{(2)}\) to the \(G\) elements indexed by level 2 parking functions is a subcoalgebra of \(\text{PQSym}^{(2)}\).

**Proof** – The shifted shuffle of two \(F\) elements indexed by level 2 parking functions only consists in level 2 parking functions since the definition only involves matches (preserved by shifted shuffle) and positions of colors 0 and 1 on a given letter inside a word, also preserved by shifted shuffle. The same property holds for the coproduct on the \(G\) side: a match on either side of the tensor product comes from a match of the original word and all equal letters go to the same side of the tensor product in the same order.

8.2. Non-crossing partitions of type B. Remark that in the level 1 case, non-crossing partitions are in bijection with non-decreasing parking functions. To extend this correspondence to type B, let us start with a non-decreasing parking function \(b\) (with no color). We factor it into the maximal shifted concatenation of prime non-decreasing parking functions, and we choose a color, here 0 or 1, for each factor. We obtain in this way \(\binom{2n}{n}\) words \(\pi\), which can be identified with type B non-crossing partitions.
Let
\[ P^\pi = \sum_{a^\uparrow = \pi} F_a \]
where \( w^\uparrow \) denotes the nondecreasing rearrangement of the letters of \( w \). Then,

**Theorem 8.2.** The \( P^\pi \), where \( \pi \) runs over the above set of non-decreasing signed parking functions, form the basis of a cocommutative Hopf subalgebra \( \text{NCPQSym}^{(2)} \) of \( \text{PQSym}^{(2)} \).

**Proof** – The subalgebra part comes from the fact that the shifted shuffle does not mix prime factors. The coalgebra part selects pieces of each factor, hence satisfies that each letter of a (new) factor has identical color. The cocommutativity part comes from the fact that all rearrangements of a given word are considered at the same time.

All this can be extended to higher levels in a straightforward way: allow each prime non-decreasing parking function to choose any color among \( \ell \) and use the factorization as above. Since non-decreasing parking functions are in bijection with Dyck words, the choice can be described as: each block of a Dyck word with no return-to-zero, chooses one color among \( \ell \). In this version, the generating series is obviously given by
\[ \frac{1}{1 - \ell \frac{1 - \sqrt{1 - 4t}}{2}}. \]
For \( \ell = 3 \), we obtain Sequence A007854 of [41].

9. Colored analogs of planar binary trees: \( \text{PBT}^{(\ell)} \)

9.1. Definition of \( \text{PBT}^{(\ell)} \). In the case with one color, the Hopf algebra \( \text{PBT} \) of Planar binary trees initially defined by Loday and Ronco [22] can be embedded in \( \text{FQSym} \) in the following way [12, 13]:
\[ P_T = \sum_{\sigma; \text{shape}(P(\sigma))=T} F_\sigma, \]
where \( P \) is a simple algorithm: it is the well-known binary search tree insertion, such as presented, for example, by Knuth in [18].

In the colored case, the definition is almost the same:
\[ P_{T, u} = \sum_{(\sigma, u); \text{shape}(P(\sigma))=T} F_{(\sigma, u)}, \]
where \( u \) is a color word whose length is equal to the number of leaves of \( T \). Note that this algebra (without its realization on words) has been previously studied by Maria Ronco [40].

Given the definitions, the generating series of the dimensions of \( \text{PBT}^{(\ell)} \) is
\[ 1 + \ell t + 2\ell^2 t^2 + 5\ell^3 t^3 + 14\ell^4 t^4 + \ldots \]
that is, the generating series of Catalan numbers multiplied by \( \ell^n \).
9.2. Algebraic structure of $\text{PBT}^{(\ell)}$. Since $\text{PBT}$ is generated by the trees with no right branch (starting from the root), the same holds in $\text{PBT}^{(\ell)}$:

**Proposition 9.1.** $\text{PBT}^{(\ell)}$ is free over the set $\mathcal{P}_T$, where $T$ is a tree with no right branch.

For example, the generating series of the algebraic generators of $\text{PBT}^{(\ell)}$ is

\begin{equation}
\ell t + \ell^2 t^2 + 2\ell^3 t^3 + 5\ell^4 t^4 + 14\ell^5 t^5 + \ldots
\end{equation}

that is, the generating series of shifted Catalan numbers $C_{n-1}$ multiplied by $\ell^n$.

9.2.1. Primitive elements of $\text{PBT}^{(\ell)}$. Let $\mathcal{L}^{(\ell)}$ be the primitive Lie algebra of the algebra $\text{PBT}^{(\ell)}$. Since $\Delta$ is not cocommutative, $\text{PBT}^{(\ell)}$ cannot be the universal enveloping algebra of $\mathcal{L}^{(\ell)}$. But since it is cofree, it is, according to [23], the universal enveloping dipterous algebra of its primitive part $\mathcal{L}^{(\ell)}$.

Using the same arguments as in the case of $\text{FQSym}^{(\ell)}$, one then proves

**Proposition 9.2.** The Lie algebra $\mathcal{L}^{(\ell)}$ is free. Moreover

\begin{equation}
\dim \mathcal{L}_n^{(\ell)} = \ell^n C_{n-1},
\end{equation}

For example, since $\mathcal{L}^{(\ell)}$ is free, the generating series of the degree of its generators is (with $\ell = 2$):

\begin{equation}
1 - \prod_{n \geq 1} (1 - t^n)^{\ell^n C_{n-1}} = 1 - (1 - t)^2 (1 - t^2)^4 (1 - t^3)^{16} (1 - t^4)^{80} \ldots
\end{equation}

\begin{align*}
&= 2t + 3t^2 + 8t^3 + 46t^4 + 252t^5 \\
&\quad + 1558t^6 + 9800t^7 + \ldots
\end{align*}

and the Hilbert series of the universal enveloping algebra of $\text{PBT}^{(\ell)}$ (its domain of cocommutativity) is, again with $\ell = 2$,

\begin{equation}
\prod_{n \geq 1} (1 - t^n)^{-\ell^n C_{n-1}} = 1 + 2t + 7t^2 + 28t^3 + 139t^4 + 762t^5 + 4549t^6 \\
&\quad + 28464t^7 + \ldots
\end{equation}

9.2.2. Dendriform structure of $\text{PBT}^{(\ell)}$. Recall that $\text{PBT}$ is the free dendriform algebra on one generator. Since colors do not play any role in determining if a given element is (totally) primitive or not, the same holds for $\text{PBT}^{(\ell)}$:

**Proposition 9.3.** The algebra $\text{PBT}^{(\ell)}$ is the free dendriform algebra on $\ell$ generators. It has also the structure of bidendriform bialgebra.

**Note 9.4.** It is also possible to define a colored analog of $\text{CQSym}$ the Catalan Quasi-symmetric algebra, but the natural definition leads to a non-cocommutative algebra, hence not sharing the basic property of $\text{CQSym}$ itself.
10. Examples

10.1. Multigraded coinvariants and colored Klyachko idempotents. One of the very first applications of the theory of noncommutative symmetric functions was to provide an explanation for the following coincidence. On the one hand, consider the representation of $\mathfrak{S}_n$ in the coinvariant algebra

$$\mathcal{H}_n = \mathbb{C}[x_1, \ldots, x_n]_{\mathfrak{S}_n} = \mathbb{C}[x_1, \ldots, x_n]/\mathcal{J},$$

where $\mathcal{J}$ is the ideal generated by symmetric polynomials without constant term. It is known [25] that the graded Frobenius characteristic of the action of $\mathfrak{S}_n$ on $\mathcal{H}_n$ is

$$\text{ch}_q(\mathcal{H}_n) = \sum_k q^k \text{ch}(\mathcal{H}_n)^{(k)} = (q)_n h_n \left( \frac{X}{1 - q} \right) = \sum_{l=n} q^{\text{maj}(l)} r_I(X),$$

where $r_I(X)$ are the ribbon Schur functions.

On the other hand, Klyachko [17] introduced a remarkable Lie idempotent in $\mathbb{C} \mathfrak{S}_n$

$$\kappa_n = \sum_{\sigma \in \mathfrak{S}_n} \zeta^{\text{maj}(\sigma)} \sigma$$

where $\zeta$ is a primitive $n$-th root of unity.

In terms of noncommutative symmetric functions, both expressions can be interpreted as specializations of

$$K_n(q) := (q)_n s_n \left( \frac{A}{1 - q} \right) = \sum_{l=n} q^{\text{maj}(l)} R_I,$$

This is the graded noncommutative characteristic of an action of $H_n(0)$ on coinvariants. This is a projective module, hence also an $\mathfrak{S}_n$-module, and taking the commutative image, $(A = X)$, one obtains the characteristic of $\mathfrak{S}_n$. But $K_n(q)$ can also be interpreted as an element of the descent algebra of $\mathfrak{S}_n$. A simple computation shows that for $q = \zeta$, a primitive $n$-th root of unity, $K_n(\zeta)$ is a primitive element of $\text{Sym}$ of commutative image $p_n/n$, hence is a Lie idempotent (see [20]). Actually, $K_n(q)$ is the noncommutative Hall-Littlewood function $\tilde{H}_n(A; q)$, and this specialization property is a special case of a noncommutative version [11] of the classical property [11, 21, 25].

There is a similar phenomenon here. Let $q_1, \ldots, q_n$ be independent variables, and consider the noncommutative symmetric function

$$K_n(A; q_1, \ldots, q_n) := \sum_{l=n} q^\text{MAJ}(l) R_I,$$

where

$$q^\text{MAJ}(i_1, \ldots, i_r) := (q_{i_1} \ldots q_{i_r})^r (q_{i_1+i_2} \ldots q_{i_1+i_2+i_3} \ldots (q_{i_1+\cdots+i_{r-2}+\cdots+i_{r-1}}).$$

For example,

$$K_3 = R_3 + q_1 q_2 R_{21} + q_1 R_{12} + q_1^2 q_2 R_{111}.$$
\[ K_4 = R_4 + q_1 q_2 q_3 R_{31} + q_1 q_2 R_{22} + q_1^2 q_2^3 q_3 R_{211} + q_1 R_{13} + q_1^2 q_2 q_3 R_{121} + q_1^2 q_2 R_{112} + q_1^3 q_2^2 q_3 R_{1111}. \]

Its commutative image is the multigraded characteristic of \( H_n \) with respect to the partition degree (cf. [3]).

One may also regard \( K_n \) as an element of
\[ \text{Sym}_n^{(Z)} \subset \text{FQSym}_n^{(Z)} \]
by means of the identification
\[ \text{Sym}_n \cong \text{FQSym}_n \]
and writing it as
\[ K_n(A; q_1, \ldots, q_n) = \sum_{I \vdash n} R_I(A)q^{\text{MAJ}(I)} \]

Note that this element lives in the (descent) algebra of the extended affine Weyl group of type \( A \)
\[ \tilde{S}_n = \mathbb{Z}^n \rtimes S_n = P \rtimes S_n, \]
where \( P \) is the weight lattice. One can also interpret it as an element of the usual affine Weyl group
\[ \tilde{S}_n = Q \rtimes S_n \]
where \( Q \) is the root lattice
\[ Q = \{ \alpha \in P | \sum_{i=1}^n \alpha_i = 0 \}. \]

This amounts to imposing the relation
\[ q_1 \ldots q_n = 1, \]
which replaces the root of unity condition \( q^n = 1 \) in the one-parameter case.

It has been proved by McNamara and Reutenauer [28] that under condition (146), \( K_n(A; q_1, \ldots, q_n) \) was a Lie idempotent in \( \mathbb{C} \tilde{S}_n \). Within the formalism of the present paper, this can be seen as follows: the authors of [28] introduce a twisted product on \( A_n = \mathbb{C}(x_1, \ldots, x_n)[\tilde{S}_n] \) by the formula
\[ f(x)\sigma \cdot g(x)\tau = f(x)\sigma[g(x)]\sigma\tau, \]
where permutations act on functions as automorphisms, i.e., \( \sigma(x_i) = x_{\sigma(i)} \), and in particular, on monomials by \( \sigma[x^c] = x^{c\sigma^{-1}} \). Hence,
\[ (\sigma x^c) \cdot (\tau x^d) = \sigma \tau x^{c\sigma+d}, \]
which is the same as Formula (29) so that \( A_n \) can be identified with the homogeneous component of degree \( n \) of \( \text{FQSym}_n^{(Z)} \), by setting
\[ \sigma x^c = G_{\sigma} x^c = G_{\sigma,c}. \]
McNamara and Reutenauer then introduce the formal series in $\mathbf{FQSym}^{(\mathbb{Z})}$

\[(150) \Theta(x) = \sum_{n \geq 0} \sum_{\sigma \in S_n} \prod_{j \in \text{Des}(\sigma)} x_{\sigma(1)} \cdots x_{\sigma(j)} \sigma \]

which, applying (148), and under the identification (149), can be rewritten as

\[(151) \Theta(x) = \prod_{l \geq 0} \sum_{n \geq 0} G_{ld_{n,l}} = \prod_{l \geq 0} \sigma_1(A^{(l)}). \]

Indeed, introducing the new variables

\[(152) y_j = x_1 x_2 \cdots x_j, \]

and applying (148) we can write

\[(153) \Theta(x) = \sum_{n \geq 0} \sum_{\sigma \in S_n} G_{\sigma} \prod_{d \in \text{Des}(\sigma)} y_d \]

\[= \sum_{n \geq 0} K_n(y_1, \ldots, y_{n-1}) \left(\frac{1}{(y)_n}\right), \]

where $((y))_n = (1 - y_1)(1 - y_2) \cdots (1 - y_n)$ and

\[(154) K_n(y_1, \ldots, y_{n-1}) = \sum_{\sigma \in S_n} G_{\sigma} \prod_{d \in \text{Des}(\sigma)} y_d \]

is the twisted version of the multiparameter Klyachko element introduced in [20, (11)]. By Moebius inversion over the lattice of compositions of $n$, we have

\[(155) K_n(y_1, \ldots, y_{n-1}) = \sum_{I{:}n} R_I \cdot \prod_{d \in \text{Des}(I)} y_d \]

\[(156) K_n(y_1, \ldots, y_{n-1}) \left(\frac{1}{(y)_n}\right) = \sum_{j=1}^{n-j} S_j \cdot \prod_{d \in \text{Des}(J)} y_d \]

\[= \sum_{i \geq 0} S_i y_i \otimes \sum_{j \geq 0} S_j y_j, \]

which implies the expression

\[(157) \Theta(x) = \prod_{l \geq 0} \left(\sum_{n \geq 0} S_n y_n^l\right). \]

Each factor of the right-hand side is grouplike (for the coproduct of $\mathbf{FQSym}^{(l)}$):

\[(158) \Delta \sum_{n \geq 0} S_n y_n^l = \sum_{n \geq 0} \sum_{i+j=n} S_i y_i^{l} \otimes S_j y_j^{l} = \sum_{i \geq 0} S_i y_i \otimes \sum_{j \geq 0} S_j y_j^l \]

so that also

\[(159) \Delta \Theta(x) = \Theta(x) \otimes \Theta(x). \]

Extracting the term of degree $n$, we find

\[(160) \Delta K_n = \sum_{i+j=n} \left(K_i \frac{1}{((y)_i)} \otimes K_j \frac{1}{((y)_j)}\right) ((y)_n) \]
so that if we send \( y_n \) to 1, all terms vanish except the extreme ones, and we get a primitive element. This is the main result of [28].

10.2. A formula of Raney. Raney [37] gave a combinatorial interpretation of the coefficients of the unique solution \( g(t) \in \mathbb{Q}[Y, Z][[t]] \) of the functional equation

\[
  g(t) = t \sum_{k=1}^{\ell} y_k e^{z_k g(t)},
\]

with \( g(t) = \sum_{n \geq 0} \frac{g_n}{n!} t^{n+1} \). This defining equation is of the form

\[
  g(t) = t \phi(g(t)),
\]

with \( \phi(u) = \sum_{k=1}^{\ell} y_k e^{z_k u} \). Hence, the coefficient \( g_n \) of \( t^{n+1} \) in \( g(t) \) is

\[
  g_n = \frac{1}{n+1} \sum_{n_1 + \cdots + n_\ell = n+1} \binom{n+1}{n_1, \ldots, n_\ell} \binom{n}{q_1, \ldots, q_\ell} y_1^{n_1} \cdots y_\ell^{n_\ell} (n_1 z_1)^{q_1} \cdots (n_\ell z_\ell)^{q_\ell}.
\]

Thus, \( g_n \in \mathbb{N}[Y, Z] \). Its combinatorial interpretation can be mechanically derived by means of a colored version of the noncommutative Lagrange inversion formula as formulated in [33, 38]. Consider the functional equation

\[
  g = \sum_{k,n} b_k S_n^{(k)} g^n,
\]

where \( S_n^{(k)} = S_n(A^{(k)}) \) is a colored complete function and \( b_k \) are noncommuting letters. We can set

\[
  S_n = \sum_k b_k S_n^{(k)},
\]

so that (164) can be rewritten as

\[
  g = \sum_{n \geq 0} S_n g^n,
\]

and the solution of [33] reads

\[
  g = S^0 + S^{10} + (S^{200} + S^{110}) + \ldots
\]

\[
  = \sum_{\pi \in \text{NDPF}} S^{\text{Ev}(\pi)}.0,
\]

where NDPF is the set of nondecreasing parking functions. Note that \( S^0 = \sum b_k \) is a priori different from 1, and does not commute with the other \( S^i \). Each term \( S^{\text{Ev}(\pi)}.0 \) represents an ordered tree \( T \) in Polish notation, so that, for example

\[
  \text{\textbullet} \quad \text{\textbullet} \quad \text{\textbullet} \quad \text{\textbullet}
\]
Replacing each $S^i$ by $\sum_{k=1}^{\ell} b_k S_i^{(k)}$, the expression $S_{\text{Ev}(\pi)}^{(0)}$ becomes a sum over all $\ell$-colorings of the tree $T$, so that, one recovers the combinatorial interpretation of Raney (proved in a different way by Foata \cite{Foata2}):

let $n = (n_1, \ldots, n_\ell)$ and $q = (q_1, \ldots, q_\ell) \in \mathbb{N}^\ell$ and let $B(n, q)$ be the set of $\ell$-colored trees on $n = |n|$ vertices with $n_k$ vertices of color $k$ and such that the sum of the arities of vertices of color $k$ is $q_k$. Then

\begin{equation}
(169) \quad g = \sum_{n \geq 0} \frac{1}{n!} \sum |B(n, q)| y^n z^q.
\end{equation}

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