Towards a q-series for $osp(2|2n)$

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Abstract

A series invariant for a certain class of closed 3-manifolds associated with a type I Lie superalgebra $sl(m|n)$ was introduced recently. We find a q-series for the other Lie superalgebra of the same type of the minimum rank.

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1 Introduction

Motivated by a prediction for a categorification of the Witten-Reshitikhin-Turaev invariant of a closed oriented 3-manifold [19, 17, 18] in [11, 10], a topological invariant q-series for graph 3-manifolds associated with a type I Lie superalgebra $sl(m|n)$ was introduced recently in [4]. This q-series denoted as $\hat{Z}_{ab}$ is labeled by a pair of $Spin^c$ structures of the manifolds as opposed to one label for a q-series ($\hat{Z}_b$) corresponding to a classical Lie algebra [16]. Another core difference is that the q-series invariant $\hat{Z}_{ab}$ decomposes an extension of the WRT invariant, namely, the CGP invariant based on a Lie superalgebra [2]. In the most general topological setting, the CGP invariant is a topological invariant of a triple consisting of a closed oriented 3-manifold, a link and a certain
The construction of the CGP invariant associated to a Lie (super)algebra involves a new facet: the modified quantum dimension, which was first introduced in [8] for the quantum groups of Lie superalgebra of type I and then for the quantum groups of $\text{sl}(2)$ at roots of unity [9]. In general, for a complex Lie superalgebra, the standard quantum dimension vanishes, which makes the WRT invariant of links and 3-manifolds trivial. The modified quantum dimension overcomes the obstacle. In the case of a complex simple Lie algebra, the modified quantum dimension enables to extend the WRT invariant to the above mentioned triplet. Furthermore, for a fixed Lie (super)algebra, the modified dimension can be defined for semisimple and nonsemisimple ribbon categories. Specific examples of the former type were given in [9] and [5]. The latter type, which is relevant to this paper, was first dealt with in [6]. And then it was expanded into a Lie superalgebra in [1] and [12] in which finite dimensional irreducible representations of the (unrolled) quantum groups of $\text{sl}(m|n) (m \neq n)$ at roots of unity was analyzed (the latter paper focuses on $\text{sl}(2|1)$). The CGP invariant contains a variety of information such as the multivariable Alexander polynomial and the ADO polynomial of a link. The relation between the CGP invariant and the WRT invariant for $\text{sl}(2;\mathbb{C})$ was conjectured in [3] and was proved for certain classes of 3-manifolds.

In this paper we analyze $\hat{Z}_{ab}$ associated with $\text{osp}(2|2)$ for plumbed 3-manifolds and observe that it is either even or odd power series in the examples. The rest of the paper is organized as follows. In Section 2 we review the ingredients involved in the CGP invariant and in $\hat{Z}_{ab}$ for $\text{sl}(m|n)$. In Section 3 we present the formula for homological blocks $\hat{Z}_{ab}$ for $\text{osp}(2|2)$. And then in Section 4, we apply the formula to a few examples.

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2 Background

The twist $\theta$, the S-matrix and the modified quantum dimension $d$ for the quantum group $\mathcal{U}_h(g)$ of $g = \text{type I} = \text{sl}(m|n), \text{osp}(2|2n)$ are given in [8], where $h$ is a formal variable related to $q = e^{h/2}$:

$$\theta_V(\lambda) = q^{(\lambda, \lambda + 2\rho)} 1_V,$$

where $\lambda$ is the highest weight of a highest weight $\mathcal{U}_h(g)$-module $V$ coloring the a link component and $\rho = \rho_0 - \rho_1$ and $\langle - , - \rangle$ is the bilinear form (see the appendix A for the details of representation theoretic concepts).

$$S(V, V') = \varphi_{\mu + \rho}(\text{sch}(V(\lambda))),$$

where $\lambda$ and $\mu$ are weights of (irreducible) $\mathcal{U}_h(g)$-modules $V$ and $V'$, respectively, coloring each link component. Moreover, $\text{sch}(V(\lambda))$ is a supercharacter of $V$ and $\varphi_\beta$ is a map from a group ring $\mathbb{Z}[\Lambda]$ to $\mathbb{C}[[h]]$. The modified quantum dimension $d$ is

$$d(\mu)S(\lambda, \mu) = d(\lambda)S(\mu, \lambda).$$

In the case of $\text{sl}(m|n)$, for the unrolled quantum group at roots of unity $\mathcal{U}_1^H(\text{sl}(m|n)) (m \neq n)$,

It is denoted $\omega \in H^1(Y; \mathbb{Z}/2\mathbb{Z})$, which induces coloring on a surgery link; this link is not part of the triplet [2].

$2^e a^{\alpha} \mapsto q^{2(\alpha, \beta)}$ for any weight $\beta$.
the above formulas modify \[1\]:

\[
\theta_V(\lambda; l) = \xi^{(\lambda, \lambda + \pi)} 1_V \quad \xi = e^{i \frac{2\pi}{\rho}} \quad l \geq m + n - 1
\]

\[
S(V, V'; l) = \varphi_{\lambda + \frac{\pi}{2}}(\text{sch}(V(\lambda))), \quad \pi = 2 \rho - 2 l \rho_0 \in \Lambda_R
\]

\[
= \xi^{2(\lambda + \frac{\pi}{2} \mu + \frac{\pi}{2})} \prod_{\alpha \in \Delta_0^+} \frac{\{l (\lambda + \frac{\pi}{2}, \alpha)\} \xi}{\{\langle \lambda + \frac{\pi}{2}, \alpha \rangle\} \xi} \prod_{\alpha \in \Delta_1^+} \{\langle \lambda + \frac{\pi}{2}, \alpha \rangle\} \xi
\]

\[
d(\mu; l)S(\lambda, \mu; l) = d(\lambda; l)S(\mu, \lambda; l).
\]

The second equality in the S-matrix element assumes \(V\) is a simple \(U_{H \lambda}^H(sl(m|n))\)-module that is typical having dimension \(D\). The changes of the formula are due to the different pivotal structure of \(U_{H \lambda}^H(sl(m|n))\). Since the modifications are on the representation theory level and don’t seem to involve unique features of \(sl(m|n)\), we assume that the same modifications occur for \(osp(2|2n)\). We next summarize the concepts involved in the homological blocks \(Z_{ab}\) associated with a Lie superalgebra \[4\].

**Generic graph** In type I Lie superalgebra case, plumbing graph used in practice needs to satisfy certain conditions due to the functional form of the edge factor in the integrand of \(\hat{Z}\); a plumbing graph is called generic, if

1. at least one vertex of a graph has degree \(> 2\)
2. \(V|_{\text{deg} > 2} \neq U \sqcup W\) such that \(B_{IJ}^{-1} = 0\) for some \(I \in V, J \in W,\)

where \(V\) is a set of vertices of a graph.

**Good Chamber** For \(\hat{Z}\) associated with type I Lie superalgebra, there is a notion of a good chamber introduced in \[4\]. Existence of a good chamber guarantees that \(\hat{Z}\) is a power series with integer coefficients. Specifically, an adjacency matrix \(B\) of a generic plumbing graph needs to admit a good chamber. It turns out that \(osp(2|2)\) requires positive definite (generic) plumbing graphs, its criteria for good chamber existence shown below are opposite of \(sl(2|1)\) criteria in \[4\]. Good chamber exists, if there exists a vector \(\alpha\) whose components are

\[
\alpha_I = \pm 1, \quad I \in V|_{\text{deg} \neq 2}
\]

such that

\[
B_{IJ}^{-1} \alpha_I \alpha_J \geq 0 \quad \forall I \in V|_{\text{deg} = 1}, \quad J \in V|_{\text{deg} \neq 2}
\]

\[
B_{IJ}^{-1} \alpha_I \alpha_J > 0 \quad \forall I, J \in V|_{\text{deg} = 1}, \quad I \neq J
\]

\(X_{IJ}\) is copositive \(X_{IJ} = B_{IJ}^{-1} \alpha_I \alpha_J \quad I, J \in V|_{\text{deg} > 2}\) are satisfied. Furthermore, a good chamber corresponding to such \(\alpha\) is

\[
deg(I) = 1 : \left\{ \begin{array}{ll}
|y_I|^{\alpha_I} < 1 \\
|z_I|^{\alpha_I} > 1
\end{array} \right.
\]

\[
deg(I) > 2 : \left| \frac{y_I}{z_I} \right|^{\alpha_I} < 1.
\]
Chamber Expansion The series expansions for the good chambers for $osp(2|2)$ turn out to be the same as that of $sl(2|1)$ in [4]. We record here the expansions.

- degree(I) = 2 + $K > 2$
\[
\left(\frac{1-z_I}{y_I-z_I}\right)^K = \begin{cases} 
(1-y_I)^K (1-z_I^{-1}) \sum_{r_I=0}^{\infty} \frac{(r_I+1)(r_I+2)\cdots(r_I+K-1)}{(K-1)!} (z_I/y_I)^{r_I}, & |z_I| < |y_I| \\
(1-z_I)^K (1-y_I^{-1}) \sum_{r_I=0}^{\infty} \frac{(r_I+1)(r_I+2)\cdots(r_I+K-1)}{(K-1)!} (y_I/z_I)^{r_I}, & |z_I| > |y_I|
\end{cases}
\]

- degree(I) = 1
\[
\frac{y_I-z_I}{(1-z_I)(1-y_I)} = \begin{cases} 
\sum_{r_I=1}^{\infty} y_I^{r_I} + \sum_{r_I=0}^{\infty} z_I^{-r_I}, & |y_I| < 1, |z_I| > 1 \\
- \sum_{r_I=0}^{\infty} y_I^{-r_I} - \sum_{r_I=1}^{\infty} z_I^{r_I}, & |y_I| > 1, |z_I| < 1
\end{cases}
\]

3 Result

For closed oriented plumbed 3-manifolds $Y(\Gamma)$ having $b_1(Y) = 0$ and $a, b \in Spin^c(Y) \cong H_1(Y)$, $\hat{Z}_{ab}[Y; q]$ associated with $osp(2|2)$ is
\[
\hat{Z}_{ab}^{osp(2|2)}[Y; q] = (-1)^{\varphi} \int_{\Omega} \prod_{J \in V} \frac{dz_I}{i2\pi z_I} \frac{dy_I}{i2\pi y_I} \left(\frac{y_I-z_I}{(1-z_I)(1-y_I)}\right)^{2-\text{deg}(I)} \sum_{n \in BZ^+_n} q^{-2nB^{-1}m} \prod_{J \in V} z_J^{m_j} y_J^{n_j}
\]
\[ \in \mathbb{Q} + q^{\Delta_{ab}} \mathbb{Z}[[q]], \quad \Delta_{ab} \in \mathbb{Q} \quad H_1(Y) = \mathbb{Z}^L/B\mathbb{Z}^L \quad |q| < 1\]

where $B = B(\Gamma)$ is an adjacency matrix of a generic plumbing graph $\Gamma$. Moreover, the convergent q-series in a complex unit disc requires $\Gamma$ to be positive definite plumbing graphs inside a complex unit disc. This is opposite of the $sl(m|n)$ case (and for $su(n)$). The edge factor in the integrand turns out to be the same as that of $sl(2|1)$ [3]. The integration prescription states that one chooses a chamber for an expansion of the integrand using the chamber expansion in the previous section and then picks constant terms in $y_I$ and $z_I$.

4 Examples

We apply the above formula to $ZHS^3$ and $QHS^3$. Each $\hat{Z}$ is either even or odd power series and the regularized constants are given by the zeta function $\zeta(s)$ or the Hurwitz zeta function $\zeta(s, x)$ (see [4] for details).

- $Y = S^3$ The adjacency matrix of a generic plumbing graph of $S^3$ (Figure 1 in appendix B) is

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[3] This seems to be no longer true in higher rank cases
\[
B(\Gamma) = \begin{pmatrix} 4 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \end{pmatrix} \quad \text{Det}B = 1 \quad |H_1(Y)| = |\text{Det}B|
\]

The two chambers are
\[
\alpha = \pm (1, -1, -1, -1).
\]

\[
\hat{Z}^{osp(2|2)}_{00}[Y; q] = 1 + 2\zeta(-1) + 2\zeta(0) - 2q^2 - 4q^4 - 4q^6 - 6q^8 - 4q^{10} - 8q^{12} - 4q^{14} - 8q^{16} - 6q^{18} + \cdots
\]

- \text{Y} = \Sigma(2, 3, 7) \quad \text{The adjacency matrix of a (generic) plumbing graph (Figure 2 in appendix B) is}

\[
B(\Gamma) = \begin{pmatrix} 1 & 1 & 1 & 1 \\
1 & 2 & 0 & 0 \\
1 & 0 & 3 & 0 \\
1 & 0 & 0 & 7 \end{pmatrix} \quad \text{Det}B = 1
\]

The two chambers are
\[
\alpha = \pm (1, -1, -1, -1).
\]

\[
\hat{Z}^{osp(2|2)}_{00} = 1 + 2\zeta(-1) + 2\zeta(0) - 2q^2 - 4q^4 - 4q^6 - 6q^8 - 4q^{10} - 6q^{12} - 4q^{14} - 6q^{16} - 6q^{18} - 8q^{20} - 4q^{22} - 10q^{24} - 6q^{26} - 8q^{28} + \cdots
\]

- \text{Y} = M(-1|\frac{1}{2}, \frac{1}{3}; \frac{1}{8}) \quad \text{The adjacency matrix of a (generic) plumbing graph (Figure 3 in appendix B) is}

\[
B(\Gamma) = \begin{pmatrix} 1 & 1 & 1 & 1 \\
1 & 2 & 0 & 0 \\
1 & 0 & 3 & 0 \\
1 & 0 & 0 & 8 \end{pmatrix} \quad \text{Det}B = 2
\]

The two chambers are
\[
\alpha = \pm (1, -1, -1, -1).
\]

The independent \(\hat{Z}_{ab}\) are

\[
\hat{Z}_{00} = 1 + 2\zeta(-1) + 2\zeta(0) - 2q^2 - 4q^4 - 4q^6 - 6q^8 - 4q^{10} - 2q^{12} - 6q^{16} - 2q^{18} - 6q^{20} - 2q^{22} - 8q^{24} - 2q^{26} - 4q^{28} + \cdots
\]

\[
\hat{Z}_{11} = -2q^3 - 2q^5 - 2q^7 - 2q^9 - 4q^{11} - 4q^{13} - 6q^{15} - 4q^{17} - 6q^{19} - 4q^{21} - 4q^{23} - 6q^{25} + \cdots
\]

\[
\hat{Z}_{01} = 2\zeta(-1, 1/2) + \zeta(0, 1/2) - q^2 - q^4 - 3q^6 - 2q^8 - 3q^{10} - 4q^{12} - 4q^{14} - 3q^{16} - 5q^{18} - 5q^{20} - 4q^{22} - 5q^{24} - 4q^{26} - 5q^{28} + \cdots
\]

where the labels denote the last components of elements of \(H_1(Y)\).
\(Y = M(-1|\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\) The adjacency matrix of a (generic) plumbing graph (Figure 4 in appendix B) is
\[
B(\Gamma) = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & 3 \\
1 & 0 & 0 & 9
\end{pmatrix}
\]
\[DetB = 3\]

The independent \(\hat{Z}^{\text{osp}(2|2)}_a[Y]\) are
\[
\hat{Z}_{00} = 1 + 2\zeta(-1) + 2\zeta(0) - 2q^6 - 2q^8 - 4q^{12} - 2q^{14} - 2q^{16} - 4q^{18} - 2q^{20} - 2q^{22} - 6q^{24} - 2q^{26} - 2q^{28} + \cdots
\]
\[
\hat{Z}_{11} = q^\frac{1}{2} (-q - q^3 - 3q^5 - 2q^7 - 3q^9 - 2q^{11} - 5q^{13} - 2q^{15} - 4q^{17} - 2q^{19} - 5q^{21} - 4q^{23} - 2q^{25} - q^{27} - 3q^{29} + \cdots)
\]
\[
\hat{Z}_{22} = q^\frac{1}{3} (-q - q^3 - 3q^5 - 2q^7 - 3q^9 - 2q^{11} - 5q^{13} - 2q^{15} - 4q^{17} - 2q^{19} - 5q^{21} - 4q^{23} - 2q^{25} - q^{27} - 3q^{29} + \cdots)
\]
\[
\hat{Z}_{01} = 3\zeta(-1, 1/2) + \zeta(0, 1/2) - 2q^4 - q^6 - 2q^8 - 3q^{10} - 3q^{12} - 2q^{14} - 4q^{16} - 2q^{18} - 5q^{20} - 3q^{22} - 2q^{24} - 2q^{26} - 3q^{28} + \cdots
\]
\[
\hat{Z}_{02} = 3\zeta(-1, 1/2) + \zeta(0, 1/2) - q^2 - q^4 - q^6 - 3q^8 - 2q^{10} - 2q^{12} - 3q^{14} - 4q^{16} - 2q^{18} - 4q^{20} - 2q^{22} - 3q^{24} - 2q^{26} - 4q^{28} + \cdots
\]
\[
\hat{Z}_{12} = q^{-\frac{1}{3}} (-2q^3 - 2q^5 - 2q^7 - 2q^9 - 4q^{11} - 2q^{13} - 4q^{15} - 2q^{17} - 6q^{19} - 2q^{21} - 4q^{23} - 2q^{25} - 4q^{27} + \cdots)
\]

Appendix

A Type I Lie superalgebra and its quantization

We give a summary of the representation theory of \(osp(2|2n)\) and the quantum group \(U_h(\text{type I})\) in [8]4 For \(osp(2|2n)\), the set of positive roots is \(\Delta^+ = \Delta_0^+ \cup \Delta_1^+, \) with
\[
\Delta_0^+ = \{\delta_i \pm \delta_j | 1 \leq i < j \leq n\} \cup \{2\delta_i\} \quad \text{and} \quad \Delta_1^+ = \{\epsilon \pm \delta_i\}, \quad n \in \mathbb{Z}_+
\]
where \(\{\epsilon, \delta_1, \cdots, \delta_n\}\) form the dual basis of the Cartan subalgebra. Their inner products are
\[
\langle \epsilon, \epsilon \rangle = 1 \quad \langle \delta_i, \delta_j \rangle = -\delta_{ij} \quad \langle \epsilon, \delta_j \rangle = 0
\]
The half sums of positive roots are
\[
\rho_0 = \sum_i (n + 1 - i)\delta_i, \quad \rho_1 = n\epsilon \quad \text{and} \quad \rho = \rho_0 - \rho_1.
\]

4For reviews on Lie superalgebras, see [13] [14] [15]
The fundamental weights are
\[ w_1 = \epsilon, \quad w_{k+1} = \epsilon + \sum_{i=1}^{k} \delta_i \quad \text{and} \quad k = 1, \ldots, n. \]

The weight \( \lambda^c_a \) decomposes as
\[ \lambda^c_a = aw_1 + c_1w_2 + \cdots + c_nw_{n+1}, \]
where \( c \in \mathbb{N}^{r-1}, a \in \mathbb{C} \).

Let \( g \) be a Lie superalgebra of type I, \( sl(m|n) \) or \( osp(2|2n) \) \((m \neq n)\). \( U_h(g) \) is the \( \mathbb{C}[[h]] \)-Hopf superalgebra generated by \( E_i, F_i, h_i, i = 1, \ldots, r = m + n - 1 \) or \( n + 1 \) satisfying the relations
\[ [h_i, h_j] = 0, \quad [h_i, E_j] = A_{ij}E_j, \quad [h_i, F_j] = A_{ij}F_j, \]
\[ [E_i, F_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad E_s^2 = F_s^2 = 0 \]
and the quantum Serre-type relations; they are quadratic, cubic and quartic relations among \( E_i \) or \( F_i \) (see [20] Definition 4.2.1). \( A_{ij} \) is \( r \times r \) Cartan matrix and \( \{s\} = \tau \subset \{1, \ldots, r\} \) determining the parity of the generators. \( E_s, F_s \) are the only odd generators. Moreover, the (anti)commutator is \([x, y] := xy - (-1)^{\bar{x}\bar{y}}yx\). The Hopf algebra structure is
\[ \Delta(E_i) = \Delta(E_i) \otimes 1 + q^{-h_i} \otimes \Delta(E_i), \quad \epsilon(\Delta(E_i)) = 0, \quad S(\Delta(E_i)) = -q^{h_i} \Delta(E_i) \]
\[ \Delta(F_i) = \Delta(F_i) \otimes 1 + q^{-h_i} \otimes \Delta(F_i), \quad \epsilon(\Delta(F_i)) = 0, \quad S(\Delta(E_i)) = -\Delta(F_i)q^{h_i} \]
\[ \Delta(h_i) = \Delta(F_i) \otimes 1 + 1 \otimes \Delta(h_i), \quad \epsilon(\Delta(h_i)) = 0, \quad S(\Delta(h_i)) = -h_i \]

### B Invariance checks

We display the weighted positive definite generic plumbing graphs and their equivalent graphs of the examples in Section 4. Each vertex corresponds to a \( S^1 \)-bundle over \( S^2 \); a positive integer is the Euler number of a bundle. The graphs are related by the Kirby-Neumann moves:

The invariance of \( \hat{Z}_{ab} \) under the Kirby-Neumann moves for the exhibited graphs and the graphs mentioned in the captions of Figure 3 and 4 were verified.
Figure 1: A generic plumbing graph of $S^3$ (left most) and its equivalent graphs

Figure 2: A generic plumbing graph of $\Sigma(2, 3, 7)$ (the first graph) and its equivalent graphs
C Derivations

We setup for the derivations of the ingredients in the CGP invariant formula using the appendix A and outline the derivation of \( \hat{Z}^{osp(2|2)} \) in Section 3.

The root lattice \( \Lambda_R \) of \( osp(2|2) \) are generated by \( 2\delta \) and \( \epsilon - \delta \), hence, two dimensional. The pivotal element \( \pi \)

\[
\pi = 2(\rho_0 - \rho_1) - 2l\rho_0 \in \Lambda_R \\
= -2(\epsilon - \delta) - l(2\delta)
\]

This implies that

\[
K_\pi = K_1^{-l}K_2^{-2} \in U_\xi^H(osp(2|2))
\]

The weight \( \lambda \) of \( V \) is

\[
\lambda = \mu_1 w_1 + \mu_2 w_2 \\
= (\mu_1 + \mu_2)\epsilon + \mu_2\delta,
\]

where \( w_1 = \epsilon \) and \( w_2 = \epsilon + \delta \). Under the assumption mentioned in Section 2, we arrive at

\[
\theta_V(\vec{\mu}; l) = \xi^{2l\mu_2}x^{\mu_1^2 + 2\mu_1 \mu_2 - 4\mu_2 - 2\mu_1} 1_V \\
l = \text{odd and } \geq 3.
\]

\[
S(\vec{\mu}, \vec{\mu}'; l) = \xi^{2l(\mu_2+\mu_2')}x^{2(\mu_1\mu_1' + \mu_1\mu_2' + \mu_2\mu_2') - 2(\mu_1 + \mu_1' + 2\mu_2 + 2\mu_2')} \\
\times \left\{ \frac{2l(\mu_2' + 1 - l)}{2(\mu_2' + 1 - l)} \right\} \{\mu_1' - 2 + l\} \{\mu_1' + 2\mu_2' - l\}
\]

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\[ d(\vec{\mu}; l) = \frac{\{2(\mu_2 + 1 - l)\}}{\{2(\mu_2 + 1 - l)\} \{\mu_1 - 2 + l\} \{\mu_1 + 2\mu_2 - l\}} \cdot \frac{1}{\{x\}} \xi := \xi^x - \xi^{-x} \quad \xi = q^{1/2} = e^{i2\pi/l}, \]

where \((\mu_1, \mu_2)\) and \((\mu'_1, \mu'_2)\) are the coefficients in the weight decompositions of \(\lambda\) and \(\mu\) for \(V\) and \(V'\), respectively (see appendix A). For the S-matrix, notations for \(V\) and \(V'\) are switched.

In order to apply the derivation of the superalgebra \(\hat{Z}\) given in [4], we need to adapt the above three expressions into a plumbing graph setup. We first let \(\vec{\alpha} = (\alpha_1 = \mu_1 - 2 + l, \alpha_2 = 2\mu_2 + 2 - 2l) \in \mathbb{C}^2\)

Then

\[ \theta_V = \xi^{\alpha_1^2 + \alpha_1\alpha_2} \]

\[ S' = \xi^{2\alpha_1\alpha_1' + \alpha_1\alpha_2' + \alpha_2\alpha_1'} \frac{\{l\alpha_2'\}}{\{\alpha_2'\}} \xi \left( \frac{\{\alpha_1'\}}{\{\alpha_2'\}} \right) \xi \]

We next shift \(\alpha_1\) and \(\alpha_2\) by \(s\) and \(t\), respectively. And then we associate \(\theta_V\) to each vertex. This leads to

\[ \theta_{V_{\alpha_I^s\alpha_J^t}} = \xi^{(\mu_1 - 2 + s)^I (\mu_1 + 2\mu_2 + s + t)^J}. \]

Similarly, the S-matrix elements between to vertices \(I\) and \(J\) of graphs contain

\[ \prod_{(I,J)\in\text{Edges}} S'(\alpha_{s_I^J}^{I}, \alpha_{s_J^I}^{J}) \supset \xi^{\sum_{I,J} B_{I,J} (\mu_1 - 2 + s)^I (\mu_1 + 2\mu_2 + s + t)^J}, \]

which is the relevant part for the derivation. For the modified quantum dimension \(d(\vec{\alpha})\)

\[ d(\vec{\alpha}) = \frac{\{\alpha_2\}}{\{l\alpha_2\} \{\alpha_1\} \{\alpha_1 + \alpha_2\} \xi}, \]

after shifting by \(s\) and \(t\) as above and some manipulations, the roots of unity dependent factor becomes

\[ d(\vec{\alpha}) \supset \frac{\xi^{2(\mu_1 + 2\mu_2 + s + t)} - \xi^{2(\mu_1 + s - 2)}}{(1 - \xi^{2(\mu_1 + 2\mu_2 + s + t)}) (1 - \xi^{2(\mu_1 + s - 2)})}. \]

We define coordinates of the 2D Cartan subalgebra of \(osp(2|2)\) to be

\[ y = \xi^{2(\mu_1 + 2\mu_2 + s + t)} \quad z = \xi^{2(\mu_1 + s - 2)} \]

Then the modified quantum dimension for a graph contains

\[ d(y, z) \supset \frac{yz - zI}{(1 - yI)(1 - zI)}. \]
We next sketch the derivation of \( \hat{Z}_{\text{osp}(2|2)} \) in Section 3 by applying the procedure in the appendix D of [4]. For a closed oriented graph 3-manifold \( Y \) equipped with \( \omega \in H^1(Y; \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}) \), the CGP invariant for a pair \((Y, \omega)\) in which \( Y \) is presented by Dehn surgery is

\[
N_l(Y, \omega) = \sum_{s^I, t^I = 0}^{l-1} \prod_{l \in \text{Vert}} d(a^I_{s^I, t^I}) \prod_{l \in \text{Edges}} d(\theta_{a^I_{s^I, t^I}}) \left( \theta_{a^I_{s^I, t^I}} \right)^{B_{l^I}} \prod_{(l, j) \in \text{Edges}} S'(\alpha^I_{s^I, t^I}, \alpha^J_{s^I, t^I})
\]

\( \omega \in H^1(Y; \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}) \cong B^{-1} \mathbb{Z}^L / \mathbb{Z} L \times B^{-1} \mathbb{Z}^L / \mathbb{Z} L \)

\( \omega([m_I]) = \mu^I = (\mu_1^I, \mu_2^I) \in \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}, \quad \sum_J B_{l^I} \mu^I_J = 0 \mod \mathbb{Z} \times \mathbb{Z}, \)

where \( m_I \) is a meridian of I-th component \( L_I \) of a surgery link \( L \) and \([m_I]\) is its homology class and \( B \) is a linking matrix of \( L \). A color of \( L_I \) is set by \( \omega([m_I]) \). The origin of \( \omega \) can be found in the footnote in Section 1.3 of [2]. After substituting the ingredients, the right hand side becomes

\[
N_l(Y, \omega) = \frac{1}{l^L+1} \prod_{l \in V} \left( e^{2\pi \mu^I_l} - e^{-2\pi \mu^I_l} \right)^{\deg(I)-2} \times \sum_{a^I, b^I \in \mathbb{Z}^L / \mathbb{I} \mathbb{Z}^L} F \left( \left\{ \xi^2(\mu_1+2\mu_2+a), \xi^2(\mu_1+b) \right\} \right) \xi \sum_{IJ} B_{l^I l^J} (\mu_1+2\mu_2+a)^I (\mu_1+b)^J,
\]

where \( a^I = s^I + t^I \), \( b^I = s^I - 2 \) and

\[
F(y, z) = \prod_{l \in V} \left( \frac{y_l - z_l}{1 - y_l}(1 - z_l) \right)^{2 - \deg(l)}.
\]

We next expand \( F(y, z) \), which modifies the summand as

\[
\sum_{a^I, b^I \in \mathbb{Z}^L / \mathbb{I} \mathbb{Z}^L} \xi \sum_{IJ} B_{l^I l^J} (\mu_1+2\mu_2+a)^I (\mu_1+b)^J + 2 \sum_{I} n_I (\mu_1+2\mu_2+a)^I + m_I (\mu_1+b)^I
\]

And then we recast it in terms of matrices by forming

\[
\mathcal{M} = \frac{1}{2} \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, \quad r = \begin{pmatrix} b \\ a \end{pmatrix}, \quad p = \begin{pmatrix} B(\mu_1+2\mu_2)+2m \\ B\mu_1+2n \end{pmatrix},
\]

This enables us to apply the appropriate Gauss reciprocity formula

\[
\sum_{r \in \mathbb{Z}^L / \mathbb{I} \mathbb{Z}^L} \exp \left( \frac{i2\pi}{l} r^T \mathcal{M} r + \frac{i2\pi}{l} p^T r \right) = \frac{\exp \left( \frac{i\pi \sigma(\mathcal{M})}{4}(l/2)^{N/2} \right)}{|\text{Det}\mathcal{M}|^{1/2}} \sum_{\delta \in \mathbb{Z}^L / 2M\mathbb{Z}^L} \exp \left( -\frac{i\pi l}{2} \left( \delta + \frac{p}{l} \right)^T \mathcal{M}^{-1} \left( \delta + \frac{p}{l} \right) \right),
\]

where \( \mathcal{M} \) is a non-degenerate symmetric \( 2L \times 2L \) matrix over \( \mathbb{Z} \) and \( \sigma(\mathcal{M}) \) is its signature. From the \( p \)-quadratic term in the exponential we read off

\[
\text{RHS} \supset \xi^{-4m^2B^{-1}n},
\]

which becomes the \( q \)-term in the summand in Section 3 after we analytically continue from a complex unit circle into an interior of an unit disc coordinatized by \( q \).
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