Fault Tolerant QR Factorization for General Matrices

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Abstract—This paper presents a fault-tolerant algorithm for the QR factorization of general matrices. It relies on the communication-avoiding algorithm, and uses the structure of the reduction of each part of the computation to introduce redundancies that are sufficient to recover the state of a failed process. After a process has failed, its state can be recovered based on the data held by one process only. Besides, it does not add any significant operation in the critical path during failure-free execution.

I. INTRODUCTION

Fault tolerance for high performance distributed applications can be achieved at system-level or application-level. System-level fault tolerance is transparent for the application and requires a specific middleware that can restart the failed processes and ensure coherent state of the application [BCH+08]. [BLKC04].

Application-level fault tolerance requires the application itself to handle the failures and adapt to them. Of course, it implies that the middleware that supports the distributed execution must be robust enough to survive the failures and provide the application with primitives to handle them [FD00]. Moreover, it requires that the application uses fault-tolerant algorithms that can deal with process failures [BDDDL09].

Recent efforts in the MPI-3 standardization process [For12a] defined an interface for a mechanism called User-Level Failure Mitigation (ULFM) [BBH+13] and Run-Through Stabilization [HGB+11].

This paper deals with the QR factorization of general matrices. After a quick overview of techniques for fault tolerance (section II.A), we describe the communication-avoiding QR factorization algorithm we are relying on in this paper in section II-B. Then we give the full fault-tolerant algorithm in sections II-B for the panel and III-C for the trailing matrix.

II. ALGORITHM-BASED FAULT TOLERANCE

FT-MPI [FD00], [FGB+04] defined four error-handling semantics that can be defined on a communicator. SHRINK consists in reducing the size of the communicator in order to leave no hole in it after a process of this communicator died. As a consequence, if one process \( p \) which is part of a communicator of size \( N \) dies, after the failure the communicator has \( N - 1 \) processes numbered in \([0, N - 2]\). On the opposite, BLANK leaves a hole in the communicator: the rank of the dead process is considered as invalid (communications return that the destination rank is invalid), and surviving processes keep their original ranks in \([0, N - 1]\). While these two semantics survive failures with a reduced number of processes, REBUILD spawns a new process to replace the dead one, giving it the place of the dead process in the communicators it was part of, including giving it the rank of the dead process. Last, the ABORT semantics corresponds to the usual behavior of non-fault-tolerant applications: the surviving processes are terminated and the application exits.

Using the first three semantics, programmers can integrate failure-recovery strategies directly as part of the algorithm that performs the computation. For instance, diskless checkpointing [PLP95] uses the memory of other processes to save the state of each process. Arithmetic on the state of the processes can be used to store the checksum of a set of processes [CFG+05]. When a process fails, its state can be recovered from the checkpoint and the states of the surviving processes. This approach is particularly interesting for iterative processes. Some matrix operations exhibit some properties on this checkpoint, such as checkpoint invariant for LU factorization [DBB+12].

A proposal for run-through stabilization introduced new constructs to handle failures at communicator-level [HGB+11]. Other mechanisms, at process-level, have been integrated as a proposal in the MPI 3.1 standard draft [For12b]. It is called user-level failure mitigation [BBH+13]. Failures are detected when an operation involving a failed process fails and returns an error. As a consequence, operations that do not involve any failed process can proceed unknowingly.

III. FAULT-TOlarant communication-Avoiding QR Factorization

In this section, we first recall how communication-avoiding QR works in section III-A Then we give the fault-tolerant algorithm in two parts: for the processes involved in the panel factorization in section III-B and for the processes involved in the update of the trailing matrix in section III-C.

A. CAQR algorithm

Communication-avoiding algorithms were introduced in [DGHL08] [DGHL12]. They minimize the number of communications, at the cost of some extra computations. Given the
relative computation vs communication speeds of the current architectures, these algorithms are faster than traditional algorithms that maximize the parallelism between the processing elements and involve more communications on a wide range of architectures, from multicores [DGG10] to grids [ACD+10] and GPUs [BDD+12].

CAQR relies on two operations: a panel factorization and an update of the trailing matrix. A set of columns on the left of the matrix is used as a panel. The panel is factorized and, using the result of the factorization, the part of the matrix on the right of this panel, called the trailing matrix, is updated. This organization is represented in Figure 1.

The algorithm can be decomposed as follows on a matrix $A$ that can be represented by blocks:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = Q_1 \begin{pmatrix} R_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

1) Panel factorization: $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = Q_1 \begin{pmatrix} R_{11} \\ 0 \end{pmatrix}$

2) Compact representation: $Q_1 I = Y_1 T_1 Y_1^T$

3) Update the trailing matrix:

$$\begin{pmatrix} I - Y_1 T_1 Y_1^T \end{pmatrix} \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} - Y_1 \begin{pmatrix} T_1 \\ Y_1^T \end{pmatrix} \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = \begin{pmatrix} R_{12} \\ A_{22} \end{pmatrix}$$

4) Continue recursively on the submatrix $A_{22}$

The panel factorization (step 1) is a specific kind of QR factorization. Since it factorizes a matrix with a particular shape (called tall and skinny), a dedicated algorithm is used: TSQR [BDG+14] [Lan10].

B. Fault-tolerant TSQR

In [Cot16], we have presented a set of algorithms to achieve fault tolerance in the TSQR panel factorization. The idea was to exploit the idle processes along the reduction tree in order to integrate redundancy with a very low overhead. Instead of just having odd-number (modulo the step number) processes sending their intermediate $\tilde{R}$ factor to an even-numbered (modulo the step number) process and stop computing, the two processes exchange their intermediate $\tilde{R}$ factors and both compute the same new intermediate $\tilde{R}$ factor. In other words, the reduction turns into an all-reduce operation, where the number of processes that own the same data (and therefore, the resilience of the computation) doubles at each step (see Figure 2).

This process has shown to have little overhead during fault-free execution and potentially no overhead or just the time for the MPI middleware to detect the failure and start a new process to recover from a failure.

C. Fault-tolerant QR factorization of 2D matrices

TSQR is a basic block of the QR factorization. It is sufficient for tall and skinny matrices, but achieving fault-tolerance in general matrices requires to be also able to tolerate failures in the trailing matrix. The purpose of this paper is to present how it can be achieved in order to implement a fault-tolerant QR factorization for 2D general matrices.

As stated in Section III-A, the update of the trailing matrix is made by applying it the transpose of the current panel’s $Q$ factor. If we denote the current matrix after the factorization by $C_i'$, the update consists of computing the $\hat{C}_i'$ factors on the right side of the panel:

$$\begin{pmatrix} R_0 & C_0' \\ R_1 & C_1' \end{pmatrix} = QR \begin{pmatrix} C_0' \\ C_1' \end{pmatrix}$$

The update consists of computing the $\hat{C}_i'$ factors on the right side of the panel:

$$A = Q \begin{pmatrix} R & \hat{C}_i' \\ \hat{C}_i' & C_1' \end{pmatrix}$$

The blocs of the left side of the matrix are decomposed into two parts: the top part contains as many lines as the number of columns of each block, the bottom part contains the rest of the lines. If the width of a block is denoted by $N$ and $C_i[: N - 1]$ denotes the first $N$ lines of matrix $C_i$:

$$C_i = \begin{pmatrix} C_i' \\ C_i'' \end{pmatrix} = \begin{pmatrix} C_i[: N - 1] \\ C_i[N :] \end{pmatrix}$$

The compact representation of the matrix is computed, as stated in section III-A, as follows:

$$\begin{pmatrix} C_i' \\ C_i'' \end{pmatrix} = \begin{pmatrix} I - (Y_0) \end{pmatrix}^T \begin{pmatrix} I \end{pmatrix}^T \begin{pmatrix} Y_1 \end{pmatrix}^T \begin{pmatrix} C_i' \\ C_i'' \end{pmatrix}$$

FIG. 1: Panel/update organization of the QR factorization.

FIG. 2: Computing the $R$ of a matrix using a TSQR factorization on 4 processes with redundant $R$ factors.
An algorithm for computing this in parallel is given in [DGHL08]. A graphical representation of this algorithm in a pair of processes is given in Figure 3, corresponding to Algorithm 1. As noticed by [DGHL08], the $T$ factors can be computed on either process: it is on the critical path anyway.

**Algorithm 1: Parallel trailing matrix update algorithm.**

```plaintext
Data: Trailing submatrix $A$
1 step = 0;
2 while ! done() do
3   if isOdd( step ) then
4      /* I am a sender - I am odd-numbered */
5         $C_0 = \text{topOfMatrix}( A );$
6         $Y_0 = \text{computeY}( ) ;$
7         $b = \text{myBuddy}( \text{step} ) ;$
8         send( $C_0, b$ );
9         recv( $W, b$ );
10        $\hat{C}_0 = C_0^I - Y_0 W ;$
11        return; /* done with my part of the update */
12   else
13      /* I am even-numbered */
14        $C_1 = \text{topOfMatrix}( A );$
15        $T = \text{computeT}( ) ;$
16        $Y_1 = \text{computeY}( ) ;$
17        $b = \text{myBuddy}( \text{step} ) ;$
18        recv( $C_0, b$ );
19        $W = T^T( C_0^I + Y_1^T C_1^I );$
20        send( $W, b$ );
21        $\hat{C}_1 = C_1^I - Y_1 W ;$
22        step++;
23
```

**Fig. 3:** Update of the trailing matrix in parallel on two processes.

The algorithm follows a binary tree by pairs, as represented by Figure 4. We can see that, in a similar way as with TSQR, processes exchange data and compute by pair and one of them is done with the update. As a consequence, at each step, half of the working processes become idle.

The idea of the fault-tolerant algorithm is to use these processes that become idle and, instead, introduce some redundancy with them. Hence, they keep computing and the data they keep can be used to recover the state of the computation after a process has failed and has been restarted.

A graphical representation of this algorithm is given in Figure 5 in order to give the reader the intuition behind this algorithm. The idea is that since both processes can compute the $T$ factors, all they need to compute their $\hat{C}_i$ update is the other processes’ $C_j$. With this $C_j$, they can compute the $W$ and then their own $\hat{C}_i$.

**Fig. 4:** Tree formed by the parallel update of the trailing matrix.

**Fig. 5:** Fault-tolerant update of the trailing matrix in parallel on two processes.

The algorithm itself is given by Algorithm 2. We can see that, instead of having two one-way communications in each direction between the two processes, we have an exchange. Implemented on dual-channel communication hardware, the latter is faster than the former, because the two communications made by the exchange overlap. Besides, it does not increase the length of the critical path. On the other hand, this algorithm requires both processes to compute while of of them could be idle: it is less energy-efficient.

At the end of the execution of each step, between processes $i$ and $j$:

- $P_i$ has $W$, $T$, $C_j^I$, $C_j^I$ and $\hat{C}_i^I$; therefore, if $P_j$ fails, $P_i$ can provide the required data to recalculate $\hat{C}_i^I = C_j^I - Y_j W$ on $P_j$ (or any process that has $Y_j$)
- $P_j$ has $W$, $T$, $C_j^I$, $C_i^I$, $Y_i$ and $\hat{C}_j^I$; therefore, if $P_i$ fails, $P_j$ can recalculate $\hat{C}_j^I = C_i^I - Y_i W$ on $P_i$ (or any process that has $Y_i$)

Therefore, the state of a failed process can be recovered using its subpart of the initial matrix and some data kept by (at least) one process. However, although several processes may have this data, retrieving from only one of them is necessary.

One minor modification would require that, instead of having $P_i$ sending $C_j^I$ and $P_j$ sending $C_i^I$ and $Y_j$, they both exchange their $C_j^I$ and $Y_j^I$: hence, the reconstruction would be symmetric.
Algorithm 2: Fault-tolerant parallel trailing matrix update algorithm.

Data: Trailing submatrix A

1: step = 0;
2: while !done() do
3:   if isOdd(step) then
4:     /* I am a sender - I am odd-numbered */
5:     C0 = topOfMatrix(A);
6:     T = computeT();
7:     Y0 = computeY();
8:     b = myBuddy(step);
9:     sendrecv(C0, C1 + Y1, b);
10:    W = T^T(C0^T + Y1^T C1^T);
11:    C0 = C0 - Y0 W;
12:   return; /* done with my part of the update */
13: else /* I am even-numbered */
14:    C1 = topOfMatrix(A);
15:    T = computeT();
16:    Y1 = computeY();
17:    b = myBuddy(step);
18:    sendrecv(C1^T + Y1, C1^T, b);
19:    W = T^T(C0^T + Y1^T C1^T);
20:    send(W, b);
21:    C1 = C1 - Y1 W;
22:    step++;

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