FOLIATIONS ON DOUBLE-TWISTED PRODUCTS

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Abstract. In this paper we present a certain class of geodesic vector fields of the double-twisted product $\mathbb{R} \times (f, g) \mathbb{R}$. Some examples of totally geodesic foliations are given.

1. Introduction

The concept of double-twisted product was introduced by Ponge and Reckziegel (see [5]) in the context of the semi-Riemannian Geometry.

Definition 1. Let $(M_1, g_1)$ and $(M_2, g_2)$ be (pseudo)Riemannian manifolds. Let $\lambda_i : M_1 \times M_2 \to \mathbb{R}$ $(i = 1, 2)$ be a positive and differentiable function. Consider the canonical projections $\pi_i : M_1 \times M_2 \to M_i$ for $i = 1, 2$. Then the double-twisted product $M_1 \times (\lambda_1, \lambda_2) M_2$ of $(M_1, g_1)$ and $(M_2, g_2)$ is the differentiable manifold $M_1 \times M_2$ equipped with the (pseudo)Riemannian metric $g$ defined by

$$g(X, Y) = \lambda_1 g_1(d\pi_1(X), d\pi_1(Y)) + \lambda_2 g_2(d\pi_2(X), d\pi_2(Y))$$

for all vector fields $X$ and $Y$ of $M_1 \times M_2$.

This definition generalizes the R. L. Bishop’s notion of an umbilic product (denoted by $M_1 \times_\lambda M_2$), which B. Y. Chen calls a twisted product and which is the double twisted product $M_1 \times (1, \lambda) M_2$ (see [1] and [2]). If in this situation $\lambda$ only depends on the points of $M_1$, then $M_1 \times_\lambda M_2$ is a warped product by definition.

If $M$ and $N$ are complete Riemannian manifolds, then the double-twisted product $M \times (\lambda_1, \lambda_2) N$ is not, in general, a complete Riemannian manifold. But, for our purposes, the following Lemma is enough (the proof is similar of that of Lemma 40 in [4])

Lemma 1. Suppose that $M$ and $N$ are complete Riemannian manifolds. Given two differentiable functions $\lambda_1, \lambda_2 : M \times N \to [1, \infty)$, then the double-twisted product $M \times (\lambda_1, \lambda_2) N$ is a complete Riemannian manifold.

Proof. Denote by $\pi : M \times N \to M$ and $\sigma : M \times N \to N$ the natural projections. Denote by $\mathcal{L}$ and $d$ the arc length and the distance in $M \times (\lambda_1, \lambda_2) N$, respectively. We use the metric completeness criterion from the Hopf-Rinow theorem. Note first that if $v$ is tangent to $M \times (\lambda_1, \lambda_2) N$ then, since $\lambda_1 \geq 1$ and $\lambda_2 > 0$, we have $\langle v, v \rangle \geq \langle d\pi(v), d\pi(v) \rangle$. Hence $\mathcal{L}(\alpha) \geq \mathcal{L}(\pi \circ \alpha)$ for any curve segment $\alpha$ in $M \times (\lambda_1, \lambda_2) N$. Analogously, $\mathcal{L}(\alpha) \geq \mathcal{L}(\sigma \circ \alpha)$. Then we have the inequalities

$$d(x, y) \geq d(\pi(x), \pi(y)) \quad \text{and} \quad d(x, y) \geq d(\sigma(x), \sigma(y)),$$

$\forall x, y \in M \times (\lambda_1, \lambda_2) N$

Both inequalities implies that if $((p_k, q_k))_{k \in \mathbb{N}}$ is a Cauchy sequence in the double-twisted product $M \times (\lambda_1, \lambda_2) N$, then $(p_k)_{k \in \mathbb{N}}$ and $(q_k)_{k \in \mathbb{N}}$ are Cauchy sequences, respectively, in $(M, g_1)$ and $(N, g_2)$. As $M$ and $N$ are complete, $(p_k)_{k \in \mathbb{N}}$ and $(q_k)_{k \in \mathbb{N}}$ are convergent and then $((p_k, q_k))_{k \in \mathbb{N}}$ is convergent in $M \times (\lambda_1, \lambda_2) N$. $\square$
2. Preliminaries

Some computations are necessary in order to find the examples. We denote by \( \partial_x \) and \( \partial_y \) the canonical vector fields of \( \mathbb{R}^2 \). The new metric in \( \mathbb{R}^2 \) is given by

\[
\begin{aligned}
\langle \partial_x, \partial_x \rangle_{(x,y)} &= [f(x,y)]^2 \\
\langle \partial_x, \partial_y \rangle_{(x,y)} &= 0 \\
\langle \partial_y, \partial_y \rangle_{(x,y)} &= [g(x,y)]^2
\end{aligned}
\]

where \((x,y) \in \mathbb{R}^2\) and \(f, g : \mathbb{R}^2 \to (0, +\infty)\) are functions of class \( C^\infty \). The real plane equipped with this (Riemannian) metric is the double twisted product of \( \mathbb{R} \) and \( \mathbb{R} \) and denoted by \( \mathbb{R} \times (f,g) \mathbb{R} \). We find now its connection \( \nabla \) and curvature \( \mathbf{R} \).

By metric compatibility we know that

\[
\begin{aligned}
2f f_x &= \partial_x \langle \partial_x, \partial_x \rangle = 2 \langle \partial_x, \nabla_{\partial_x} \partial_x \rangle \\
2f f_y &= \partial_y \langle \partial_x, \partial_x \rangle = 2 \langle \partial_x, \nabla_{\partial_x} \partial_y \rangle \\
2g g_x &= \partial_x \langle \partial_y, \partial_y \rangle = 2 \langle \partial_y, \nabla_{\partial_x} \partial_y \rangle \\
2g g_y &= \partial_y \langle \partial_y, \partial_y \rangle = 2 \langle \partial_y, \nabla_{\partial_y} \partial_y \rangle \\
0 &= \partial_x \langle \partial_x, \partial_y \rangle = \langle \nabla_{\partial_x} \partial_x, \partial_y \rangle + \langle \partial_x, \nabla_{\partial_y} \partial_y \rangle \\
0 &= \partial_y \langle \partial_x, \partial_y \rangle = \langle \nabla_{\partial_y} \partial_x, \partial_y \rangle + \langle \partial_x, \nabla_{\partial_y} \partial_y \rangle \\
\nabla_{\partial_x} \partial_y &= \nabla_{\partial_y} \partial_x
\end{aligned}
\]

where in the last equation we used \([\partial_x, \partial_y] = 0\).

Using the equations above, we find the Riemannian connection \( \nabla \) of \( \mathbb{R} \times (f,g) \mathbb{R} \):

\[
\begin{aligned}
\nabla_{\partial_x} \partial_x &= \frac{1}{f^2} \langle \nabla_{\partial_x} \partial_x, \partial_x \rangle \partial_x + \frac{1}{g^2} \langle \nabla_{\partial_x} \partial_x, \partial_y \rangle \partial_y \\
&= \frac{f_x}{f} \partial_x - \frac{f f_y}{g^2} \partial_y \\
\nabla_{\partial_y} \partial_y &= \frac{1}{f^2} \langle \nabla_{\partial_y} \partial_y, \partial_x \rangle \partial_x + \frac{1}{g^2} \langle \nabla_{\partial_y} \partial_y, \partial_y \rangle \partial_y \\
&= -\frac{g_x g}{f^2} \partial_x + \frac{g_y}{g} \partial_y \\
\nabla_{\partial_y} \partial_x &= \frac{1}{f^2} \langle \nabla_{\partial_y} \partial_x, \partial_x \rangle \partial_x + \frac{1}{g^2} \langle \nabla_{\partial_y} \partial_x, \partial_y \rangle \partial_y \\
&= \frac{f_y}{f} \partial_x + \frac{g_x}{g} \partial_y
\end{aligned}
\]
Moreover, by definition we have
\[
R(\partial_x, \partial_y) \partial_x = \nabla_{\partial_x} \nabla_{\partial_y} \partial_x - \nabla_{\partial_y} \nabla_{\partial_x} \partial_x - \nabla_{[\partial_x, \partial_y]} \partial_x
\]
\[
= \nabla_{\partial_x} \nabla_{\partial_y} \partial_x - \nabla_{\partial_y} \nabla_{\partial_x} \partial_x
\]
\[
= \nabla_{\partial_x} \left( \frac{f_x}{f} \partial_x - \frac{ff_y}{g^2} \partial_y \right) - \nabla_{\partial_y} \left( \frac{f_y}{f} \partial_x + \frac{gg_x}{g} \partial_y \right)
\]
\[
= \frac{f_x}{f} \nabla_{\partial_x} \partial_x + \left( \frac{f_x}{f} \right)_y \partial_y - \frac{ff_y}{g^2} \nabla_{\partial_y} \partial_y - \left( \frac{ff_y}{g^2} \right)_y \partial_y
\]
\[
- \left( \frac{f_y}{f} \nabla_{\partial_x} \partial_x + \left( \frac{f_y}{f} \right)_y \partial_y + \frac{gg_x}{g} \nabla_{\partial_y} \partial_y + \left( \frac{gg_x}{g} \right)_y \partial_y \right)
\]
\[
= \left[ \frac{f_x}{f} \frac{g^2}{g^2} + \frac{f_x g_x}{f g} - \frac{ff_y g_y}{g^2} - \left( \frac{ff_y}{g^2} \right)_y \right] \partial_y
\]
and then
\[
\langle R(\partial_x, \partial_y) \partial_x, \partial_y \rangle = g^2 \left[ \frac{f_x}{f} \frac{g^2}{g^2} + \frac{f_x g_x}{f g} - \frac{ff_y g_y}{g^2} - \left( \frac{ff_y}{g^2} \right)_y - \left( \frac{gg_x}{g} \right)_y \right]
\]

Analogously, we have
\[
R(\partial_x, \partial_y) \partial_y = \nabla_{\partial_y} \nabla_{\partial_x} \partial_y - \nabla_{\partial_x} \nabla_{\partial_y} \partial_y - \nabla_{[\partial_x, \partial_y]} \partial_y
\]
\[
= \nabla_{\partial_y} \nabla_{\partial_x} \partial_y - \nabla_{\partial_x} \nabla_{\partial_y} \partial_y
\]
\[
= \nabla_{\partial_y} \left( \frac{f_y}{f} \partial_x + \frac{gg_x}{g} \partial_y \right) - \nabla_{\partial_x} \left( \frac{f_x}{f} \partial_y + \frac{gg_y}{g} \partial_x \right)
\]
\[
= \frac{f_y}{f} \nabla_{\partial_y} \partial_x + \left( \frac{f_y}{f} \right)_x \partial_x + \frac{gg_x}{g} \nabla_{\partial_y} \partial_y + \left( \frac{gg_x}{g} \right)_y \partial_y
\]
\[
+ \frac{gg_x}{f^2} \nabla_{\partial_y} \partial_x + \left( \frac{gg_x}{f^2} \right)_x \partial_x - \frac{gg_y}{g} \nabla_{\partial_x} \partial_y - \left( \frac{gg_y}{g} \right)_y \partial_y
\]
\[
= \frac{f_y}{f} \left[ \frac{f_x}{f} \partial_x + \frac{gg_x}{g} \partial_y \right] + \left( \frac{f_y}{f} \right)_y \partial_y + \frac{gg_x}{f} \partial_x + \frac{gg_y}{f} \partial_y + \left( \frac{gg_y}{g} \right)_y \partial_y
\]
\[
+ \left[ \frac{gg_x}{f^2} \left( \frac{f_x}{f} \partial_x - \left( \frac{gg_x}{f^2} \right)_x \partial_x - \frac{gg_y}{f^2} \partial_y \right) - \left( \frac{gg_y}{g} \right)_y \partial_y \right]
\]
\[
= \left[ \frac{f_y}{f} \left( \frac{f_x}{f} \frac{g^2}{g^2} + \frac{gg_x}{f^2} \right) - \left( \frac{gg_x}{g} \right)_y \right] \partial_y
\]

3. Geodesic vector fields

A codimension-one foliation $\mathcal{F}$ on a Riemannian manifold $M$ is totally geodesic if their leaves are totally geodesic submanifolds of $M$. If an unit vector field
\[
U_{(x,y)} = a(x,y) \partial_x + b(x,y) \partial_y
\]
defined on $\mathbb{R} \times (f,g)$ $\mathbb{R}$ is geodesic, then its integral curves determines a totally geodesic foliation on $\mathbb{R} \times (f,g) \mathbb{R}$. The vector field $U$ is geodesic $\iff \nabla_U U = 0$. But

$$\nabla_U U = a \nabla_{\partial_x} U + b \nabla_{\partial_y} U$$

$$= a \left( a \partial_x (a \partial_x + b \partial_y) \right) + b \left( a \partial_y (a \partial_x + b \partial_y) \right)$$

$$= a \left( a \left( \frac{f_x}{f} \partial_x - \frac{f_y}{g^2} \partial_y \right) + a_x \partial_x + b \left( \frac{f_x}{f} \partial_x + \frac{g_x}{g} \partial_y \right) + b_x \partial_y \right)$$

$$+ b \left( a \left( \frac{f_y}{g^2} \partial_x + \frac{g_x}{g} \partial_y \right) + a_y \partial_x + b \left( - \frac{g_y}{f^2} \partial_x + \frac{g_y}{g} \partial_y \right) + b_y \partial_y \right)$$

$$= \left( a^2 \frac{f_x}{f} - b^2 \frac{g_x}{g} + 2ab \frac{f_y}{f} + a a_x + b a_y \right) \partial_x$$

$$+ \left( -a^2 \frac{f_y}{g^2} + b^2 \frac{g_y}{g} + 2ab \frac{g_x}{g} + a b_x + b b_y \right) \partial_y$$

Therefore, the condition $\nabla_U U = 0$ is equivalent to the following system

$$\begin{cases} 
    a^2 \frac{f_x}{f} - b^2 \frac{g_x}{g} + 2ab \frac{f_y}{f} + a a_x + b a_y = 0 \\
    -a^2 \frac{f_y}{g^2} + b^2 \frac{g_y}{g} + 2ab \frac{g_x}{g} + a b_x + b b_y = 0
\end{cases}$$

**Theorem 1.** Consider the following unit vector field defined on $\mathbb{R} \times (f,g) \mathbb{R}$

$$U(x,y) := \frac{1}{\sqrt{2f(x,y)}} \partial_x + \frac{1}{\sqrt{2g(x,y)}} \partial_y$$

Then the vector field $U$ above is an unit geodesic vector field on $\mathbb{R} \times (f,g) \mathbb{R}$ iff

$$f_y = g_x$$

**Proof.** In this case we have $a = 1/\sqrt{2f}$ and $b = 1/\sqrt{2g}$. Substituting this in the above system we obtain

$$\begin{cases} 
    \frac{1}{2f^2} \cdot \frac{f_x}{f} - \frac{1}{2g^2} \cdot \frac{g_x}{g} + \frac{1}{f} \cdot \frac{f_y}{f} + \frac{1}{\sqrt{2f}} \cdot \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{2f}} \right) + \frac{1}{\sqrt{2g}} \cdot \frac{\partial}{\partial y} \left( \frac{1}{\sqrt{2g}} \right) = 0 \\
    -\frac{1}{2f^2} \cdot \frac{f_y}{g^2} + \frac{1}{2g^2} \cdot \frac{g_x}{g} + \frac{1}{f} \cdot \frac{f_y}{g} + \frac{1}{\sqrt{2f}} \cdot \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{2f}} \right) + \frac{1}{\sqrt{2g}} \cdot \frac{\partial}{\partial y} \left( \frac{1}{\sqrt{2g}} \right) = 0
\end{cases}$$

After simplifications, both equations are equivalent to the equation $f_y = g_x$. □

**Example.** If $f(x,y) := e^{x+y} + \sin^2(x) + 1$ and $g(x,y) := e^{x+y} + 1$, then $f_y = g_x$ and the following unit vector field of $\mathbb{R} \times (f,g) \mathbb{R}$

$$U(x,y) := \frac{1}{\sqrt{2(e^{x+y} + \sin^2(x) + 1)}} \partial_x + \frac{1}{\sqrt{2(e^{x+y} + 1)}} \partial_y$$

is a geodesic normal vector field.

**4. Foliations with geodesic normal vector field**

We recall some results about totally geodesic foliations. Let $M^{n+1}$ be an orientable Riemannian manifold and $\mathcal{F}$ be a codimension-one $C^\infty$-foliation on $M$. Suppose that $\mathcal{F}$ is transversely orientable, i.e., we may choose a differentiable unit vector field $N \in \mathfrak{X}(M)$ normal to the leaves of $\mathcal{F}$.
Given a point \( p \in M \) we may always choose an orthonormal frame field
\[
\{ e_1, e_2, \ldots, e_n, e_{n+1} \}
\]
defined in a neighborhood of \( p \) and such that the vectors \( e_1, \ldots, e_n \) are tangent to the leaves of \( \mathcal{F} \) and \( e_{n+1} = N \). The Ricci curvature in the direction \( e_{n+1} = N \) is
\[
\text{Ric}(N) := \frac{1}{n} \sum_{i=1}^{n} \langle R(e_i, N)e_i, N \rangle
\]

The divergence of a vector field \( V \in \mathfrak{X}(M) \) is locally defined as
\[
\text{div}(V) := \sum_{k=1}^{n+1} \langle \nabla e_k V, e_k \rangle
\]

In [3] the Author proved the following

**Theorem 2.** Let \( \mathcal{F} \) be a codimension-one foliation of a complete Riemannian manifold \( M^n \) and let \( N \) be an unit vector field normal to the leaves of \( \mathcal{F} \). Then
\[
\mathcal{F} \text{ is totally geodesic } \iff \text{Ric}(N) - \frac{1}{n} \text{div}(\nabla N N) = 0
\]

As a consequence we obtain

**Corollary 1.** Let \( \mathcal{F} \) be a codimension-one foliation of a complete Riemannian manifold \( M \) and let \( N \) be a geodesic unit vector field normal to the leaves of \( \mathcal{F} \). Then \( \mathcal{F} \) is a totally geodesic foliation if, and only if, \( \text{Ric}(N) = 0 \).

The following theorem gives us some examples of such foliations

**Theorem 3.** Let \( f = f(x) \geq 1 \) and \( g = g(y) \geq 1 \) be differentiable functions. Consider the following unit vector field defined on \( \mathbb{R} \times (f,g) \mathbb{R} \)
\[
N(x,y) := \frac{1}{\sqrt{2f(x)}} \partial_x + \frac{1}{\sqrt{2g(y)}} \partial_y
\]
Then \( N \) is a normal vector field of a totally geodesic foliation of \( \mathbb{R} \times (f,g) \mathbb{R} \).

**Proof.** We have \( f_y = g_x = 0 \) and then \( N \) is a geodesic vector field. Moreover, for this choice of functions, the double-twisted product \( \mathbb{R} \times (f,g) \mathbb{R} \) is flat (see the expressions of curvature \( R \) in the §2) and complete (by Lemma 1). Therefore, \( \text{Ric}(N) = 0 \) and (by Corollary 1) \( N \) is a normal vector field of a totally geodesic foliation of \( \mathbb{R} \times (f,g) \mathbb{R} \).

**Example.** If \( f(x) := e^x + 1 \) and \( g(y) := \sin^2(y) + 1 \), the following vector field
\[
N(x,y) := \frac{1}{\sqrt{2(e^x + 1)}} \partial_x + \frac{1}{\sqrt{2(\sin^2(y) + 1)}} \partial_y
\]
is a geodesic vector field and normal to the leaves of a totally geodesic foliation of \( \mathbb{R} \times (f,g) \mathbb{R} \).
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