Multiple addition theorem for discrete and continuous nonlinear problems

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Abstract. The addition relation for the Riemann theta functions and for its limits, which lead to the appearance of exponential functions in soliton type equations is discussed. The presented form of addition property resolves itself to the factorization of \( N \)-tuple product of the shifted functions and it seems to be useful for analysis of soliton type continuous and discrete processes in the \( N + 1 \) space-time. A close relation with the natural generalization of bi- and tri-linear operators into multiple linear operators concludes the paper.

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1 Introduction

The main goal of this note is the presentation of the role of the addition property (AP), its relation to the famous bilinear operator formalism and its universality, since using AP, the quasi-periodic processes and soliton ones can be considered in an identical manner. An important step is however the generalization of the leading to the bilinear operator standard version of the AP to the version which can be linked with multilinear operators. It seems that this generalized version can be useful in a case of multidimensional soliton type problems.

There is opinion that a huge success of the bilinear formalism in the soliton theory can be related to the AP for \( \tau \)-functions which appear in the majority of soliton equations

\[
\tau(z + w)\tau(z - w) = \sum_\varepsilon W_\varepsilon(w)Z_\varepsilon(z),
\]

where \( z = \kappa x + \omega t \in C^g, \varepsilon \in Z_2^g; \tau: C^g \to C \) and \( W_\varepsilon(w), Z_\varepsilon(z): C^g \times Z^g \to C \).

The essential point here is the factorization of the right hand side of (1), in which functions \( W_\varepsilon \) and \( Z_\varepsilon \) depend on \( w \) and on \( z \), respectively and exclusively. There are a few version of AP, according to (1) scheme, \[1\], \[2\], \[3\], and the factorization appears in each one. In applications to the soliton type equations, the argument \( z \) usually depends on space and time, while \( w \) plays a role of a fixed constant parameter. In a few papers \[4\] it was shown that (1) allows in a straightforward manner to determine derivatives of logarithms of \( \tau \)-function, which are useful in differential version of soliton type equations. For the discrete soliton type equations the form (1) has a direct and immediate application.

As it was shown in the cited references, a class containing exponential functions as well as the Riemann theta functions has just the AP according to (1).

In order to illustrate an application of AP, we present below two examples: the discrete Hirota equation and doubly discrete sine-Gordon equation (dd-sGe). In the limit, when step tends to zero, the first - Hirota equation has a trivial limit, while the second one (dd-sGe) becomes a standard sGe.

2 The Hirota equation

As an elementary example of the addition property we present the system of dispersion equations for the functional (discrete) equations

\[
a\tau(x + h, y, t)\tau(x - h, y, t) + b\tau(x, y + h, t)\tau(x, y - h, t) + c\tau(x, y, t + h)\tau(x, y, t - h) = C\tau^2(x, y, t),
\]

where \( h \) is the step and \( a, b, c, C \) are constant. When \( C = 0 \) this equation is known as the Hirota equation and then its soliton solutions one can be found in \[3\]. Quasiperiodic solutions
are reported in [4] and another class of solutions in [5]. In the language of bilinear operator (2) can be written as
\[ [a \exp(hD_x) + b \exp(hD_y) + c \exp(hD_z)] (\tau \circ \tau) = C \tau^2, \] (3)
where bilinear operator \( D_x \) (in scalar version) is defined as
\[ (D_x)^n (\tau \circ \tau) := (\partial_x - \partial_{x_1})^n \tau (x, y, t) \tau (x_1, y, t) |_{x_1 = x} = (\partial_x)^n [\tau (x + s, y, t) \tau (x - s, y, t)] |_{s = 0}. \] (4)

Assuming that \( \tau \)–function argument is \( z = kx + ly + wt \in C^9 \), equation (4) can be rewritten as
\[ a \tau (z + kh) \tau (z - kh) + b \tau (z + lh) \tau (z - lh) + c \tau (z + wh) \tau (z - wh) = C \tau^2 (z), \] (5)
which is just suitable for the application of the addition property. We obtain functional equation
\[ \sum_{\varepsilon \in \mathbb{Z}_2^q} [a W_\varepsilon (kh) + b W_\varepsilon (lh) + c W_\varepsilon (wh) - CW_\varepsilon (0)] Z_\varepsilon (z) = 0, \] (6)
which in case of independent functions \( Z_\varepsilon (z) \), \( \varepsilon \in \mathbb{Z}_2^q \) leads to the system of algebraic equations
\[ a W_\varepsilon (kh) + b W_\varepsilon (lh) + c W_\varepsilon (wh) - CW_\varepsilon (0) = 0; \text{ for each } \varepsilon \in \mathbb{Z}_2^q. \] (7)

For the fixed type of solution (which determines the class of functions \( W_\varepsilon \) ) and for a fixed step \( h \), equations (7) determine the relations between \( a, b, c, C \) and \( k, l, w \). Thus the dispersion equations (7) are valid both for soliton and for quasiperiodic processes, and even for processes in the form of solitons on the periodic background.

### 3 Discrete sine - Gordon equation.

We consider the functional (discrete-discrete) version of sine - Gordon equation (dd-sGe) in the form:
\[ \frac{1}{h^2} \sin \left( \frac{u^{++} + u^{+-} - u^{+-} - u^{--}}{4} \right) = \sin \left( \frac{u^{++} + u^{+-} + u^{+-} + u^{--}}{4} \right), \] (8)
where \( u^{\pm \pm} := u (\xi \pm h, \tau \pm h) \). It is obvious that in the limit \( h \to 0 \) equation (8) becomes the traditional sGe in light cone coordinates
\[ u_{\xi \tau} = \sin u. \] (9)

We look for the quasi-periodic solutions (8) in the form, which is identical (up to constant parameters) with solutions of (4)
\[ u (\xi, \tau) = 2i \ln \frac{\theta (z + \frac{\alpha}{2} B)}{\theta (z + \frac{\beta}{2} B)}, \] (10)
where \( \theta (z|B) \) denotes the Riemann theta function of argument \( z = k\xi + u\tau \in C^9 \), and parametrized by the Riemannian matrix \( B \in C^{9 \times 9} \); \( e = [1, ..., 1] \in \mathbb{Z}^9 \), see e.g. [5], [6]. Moreover, in order to obtain the real solutions we require \( \theta (z + \frac{\alpha}{2} B) = \overline{\theta (z|B)} \).

Since
\[ u (\xi + h, \tau \pm h) = 2i \ln \frac{\theta (z + (k \pm u)_h + \frac{\alpha}{2} B)}{\theta (z + (k \pm u)_h|B)} = 2i \ln \frac{\theta (z + w \pm \frac{\alpha}{2} B)}{\theta (z + w|B)}, \] (11)
and similar relations hold for \( u (\xi - h, \tau \pm h) \), after simple manipulations we can rewrite (13) in the form

\[
\frac{\theta (z + w_+) \theta (z - w_+) - h^2 \theta (z + w_+ + e/2) \theta (z - w_+ + e/2)}{\theta (z + w_+) \theta (z - w_+)} = (C + 1) \theta (z + w_+) \theta (z - w_+),
\]

where \( w_+ = (k + u) h, \ w_- = (k - u) h \). Let us assume that they are constant (with respect to \( \xi \) and \( \tau \)). Both lead to the same result

\[
\theta (z + w_-) \theta (z - w_-) - h^2 \theta (z + w_- + e/2) \theta (z - w_- + e/2) = (C + 1) \theta (z + w_+) \theta (z - w_+).
\]

(13)

Now the addition property can be applied. The Riemann theta functions do have the addition property:

\[
\theta (z + w|B) \theta (z - w|B) = \sum_{\varepsilon \in Z_g^2} W_\varepsilon (w) \theta^2 \left( z + \frac{\varepsilon}{2} |B \right),
\]

(14)

for \( z, w \in C^g \). \( W_\varepsilon \) coefficients can be expressed by theta-constants \( \theta \), but - and it is important - do not depend on \( \xi \) and \( \tau \). Then equation (13) can be written as

\[
\sum_{\varepsilon \in Z_g^2} W_\varepsilon (w_-) \theta^2 \left( z + \frac{\varepsilon}{2} |B \right) - h^2 \sum_{\varepsilon \in Z_g^2} W_\varepsilon (w_-) \theta^2 \left( z + \left( \frac{\varepsilon + e}{2} \right) |B \right) = (C + 1) \sum_{\varepsilon \in Z_g^2} W_\varepsilon (w_+) \theta^2 \left( z + \frac{\varepsilon}{2} |B \right),
\]

(15)

or as a simple functional equation

\[
\sum_{\varepsilon \in Z_g^2} [W_\varepsilon (w_-) - h^2 W_{e-\varepsilon} (w_-) - (C + 1) W_\varepsilon (w_+)] \theta^2 \left( z + \frac{\varepsilon}{2} |B \right) = 0.
\]

(16)

Since \( \theta^2 \left( z + \frac{\varepsilon}{2} |B \right) \) labeled by \( \varepsilon \in Z_g^2 \) form a set of linearly independent functions, finally we arrive at the requirement that for any \( \varepsilon \in Z_g^2 \)

\[
W_\varepsilon (w_-) - h^2 W_{e-\varepsilon} (w_-) - (C + 1) W_\varepsilon (w_+) = 0.
\]

(17)

Equations (17) determine of \( k \) and \( u \), (and \( C \)) and these are linearly related to the propagation vectors \( \kappa \) and angular frequencies \( \omega \) in laboratory coordinate system \( (z = \kappa x + \omega t) \). Therefore these equations represent a system of dispersion equations for the discussed dd-sGe. Nontrivial solutions of (17) determines the solutions of starting equation (6), but for the higher \( g \), since then the system is overdetermined, also some and even all elements of the matrix \( B \).

As \((dd\text{-sGe})\rightarrow(s\text{Ge})\), with \( h \to 0 \), (17) also tends to the dispersion equation for standard sGe

\[
\sum_{i,j} k_i u_j W_{\varepsilon,ij} + \frac{1}{2} (\delta_{e,\varepsilon} - c \delta_{e,0}) = 0.
\]

(18)

where \( W_{\varepsilon,ij} := \frac{\partial^2}{\partial u_i \partial u_j} W_\varepsilon (w) |_{w=0} \). In order to prove this statement one can substitute a new constant \( c = -C/h^2 \) and use the relations \( W_\varepsilon (z) = W_\varepsilon (-z), W_\varepsilon (0) = \delta_{e,0} \), where \( \delta \) represents the Kronecker symbol.
4 Tri-linear operator

In order to extend direct methods in the spirit of bilinear operator formalism on the a broader class of equations, the trilinear operators $T$ and $T^*$ were introduced by R. Hirota, (c.f. [5], [7])

\[
(T)^n (\tau \circ \tau \circ \tau) := \left( \partial_z + j\partial_{w_1} + j^2\partial_{w_2} \right) \tau (z) \tau (w_1) \tau (w_2) \bigg|_{w_2 = w_1 = z}, \tag{19}
\]

\[
(T^*)^n (\tau \circ \tau \circ \tau) := \left( \partial_z + j^2\partial_{w_1} + j\partial_{w_2} \right) \tau (z) \tau (w_1) \tau (w_2) \bigg|_{w_2 = w_1 = z}, \tag{20}
\]

where $j = \exp \left( i2\pi / 3 \right)$.

In this language e.g. the 5-th order equation of the Lax hierarchy

\[
u_{5x} + 10uu_{3x} + 20uxux + 30u^2ux + ut = 0, \tag{21}
\]

although lacking a bilinear representation, can be written in trilinear form [8]

\[
(7T_x^6 + 20T_x^3T_x^3 + 2T_xT_x) F \circ F \circ F = 0, \tag{22}
\]

where $u = 2 \left( \ln F \right)_{xx}$. We will return to this equation in the last paragraph of this note. However, first let us try to generalize the concept of bi- and tri-linear operators.

5 Multiple addition property

One can introduce multilinear (J-linear) operator by the relation

\[
(T)^n (\tau \circ \tau \circ \cdots \circ \tau) := \left[ \partial_{z_0} + j\partial_{z_1} + \ldots + j^{J-1}\partial_{z_{J-1}} \right] \prod_{i=0}^{J-1} \tau (z_i) \bigg|_{z_{J-1} = \ldots = z_1 = z_0 = z}, \tag{23}
\]

where $j = \exp \left( i2\pi / J \right)$.

We are convinced that the effectivity of the bi- and tri-linear operators formalism depends on the relevant addition property of the functions to which this formalism is applied. Therefore, the fundamental question is which class of functions has the multiple AP. Instead of the class of exponential functions appearing in the solutions of soliton type equations we focus our attention on the Riemann theta functions constituting a more general class of functions and expressing the quasi-periodic solutions. Obviously, exponential functions can be considered as the limiting case of the theta functions.

Following [3], [2], [4], [5], we adopt the definition of the Riemann theta function as

\[
\theta (z|B) = \sum_{n \in \mathbb{Z}^g} \exp \left[ i\pi \left( 2 \langle z, n \rangle + \langle n, Bn \rangle \right) \right], \tag{24}
\]

where $z \in C^g$, $B \in C^{g \times g}$ is the Riemann matrix, (i.e. symmetric with positively defined imaginary part) and $\langle z, n \rangle := \sum_{j=1}^{g} z_jn_j$.

If $z, u^{(k)} \in C^g$, $k = 0, \ldots, J-1$, and

\[
\sum_{k=0}^{J-1} u^{(k)} = 0, \tag{25}
\]

one can prove [10] that

\[
\theta \left( z + u^{(0)}|B \right) \theta \left( z + u^{(1)}|B \right) \theta \left( z + u^{(2)}|B \right) \ldots \theta \left( z + u^{(J-1)}|B \right) = \sum_{\epsilon \in \mathbb{Z}^J} W_\epsilon \left( u^{(0)}, \ldots, u^{(J-1)} \right) Z_\epsilon (z), \tag{26}
\]

\[
\theta \left( z + u^{(0)}|B \right) \theta \left( z + u^{(1)}|B \right) \theta \left( z + u^{(2)}|B \right) \ldots \theta \left( z + u^{(J-1)}|B \right) = \sum_{\epsilon \in \mathbb{Z}^J} W_\epsilon \left( u^{(0)}, \ldots, u^{(J-1)} \right) Z_\epsilon (z), \tag{26}
\]
In the Table 1 below we present relations between \( \delta \) where

\[
W_\varepsilon \left( u^{(0)}, ..., u^{(J-1)} \right) = \exp \left( i2\pi <u^{(0)}, \varepsilon > \right) \theta \left( \begin{bmatrix}
0 & 0 & B \\
0 & 0 & B \\
B & 2B & B \\
B & B & 2B \\
\end{bmatrix}
\right).
\]

All \( \theta \)– functions are of order \( g \), except the appearing in \( (28) \), which is of order \( (J-1)g \). The sum in \( (28) \) is over \( \varepsilon \in \mathbb{Z}_J \) i.e. over \( g \)–dimensional vectors whose components are 0, 1, ..., \( J-1 \), and therefore the sum contains \( J^g \) elements. Equation \( (28) \), written here for theta functions, is a natural generalization of AP \( (3) \). The same form has the generalized AP for exponential functions defined by

\[
E \left( z|\tilde{B} \right) = \sum_{n \in \mathbb{Z}_g} \exp \left[ i\pi \left( 2 \langle z, n \rangle + \langle n, \tilde{B}n \rangle \right) \right],
\]

which appear in the solutions of standard soliton equations. (Observe that the difference between \( (24) \) and \( (29) \) is only in the number of elements in the sum.) It is convenient to assume that diagonal elements of \( \tilde{B} \in C^{g \times g} \) are real. The constraint \( (28) \) can be eliminated easily by introducing new parameters \( w^{(1)}, ..., w^{(J-1)} \) instead of \( u^{(0)}, u^{(1)}, ..., u^{(J-1)} \) according to the relation

\[
w^{(k)} = \frac{1}{j-1} \left( u^{(k)} - j u^{(k+1)} (1 - \delta_{k,J-1}) - j u^{(1)} \delta_{k,J-1} \right), \quad k = 1, ..., J-1,
\]

where \( \delta_{k,J-1} \) is the standard Kronecker symbol. Inversely

\[
u^{(k)} = j^{1-k} \left( j^{J-1} \sum_{m=1}^{k-1} j^m w^{(m)} + \sum_{m=1}^{J-1} j^m w^{(m)} \right) \quad k = 1, ..., J-1,
\]

\[
u^{(0)} = \sum_{m=1}^{J-1} w^{(m)}.
\]

In the Table 1 below we present relations between \( u^{(p)} \) and \( w^{(q)} \) for \( J = 2 - 5 \).

| \( J \) | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|
| \( j \) | 2 | 3 | 4 | 5 |
| \( u^{(0)} = w^{(1)} \) | \( w^{(1)} + w^{(2)} \) | \( w^{(1)} + w^{(2)} + w^{(3)} \) | \( w^{(1)} + w^{(2)} + w^{(3)} + w^{(4)} \) | \( \exp(i2\pi/3) \) |
| \( u^{(1)} = jw^{(1)} \) | \( jw^{(1)} + j2w^{(2)} \) | \( jw^{(1)} + j2w^{(2)} + j3w^{(3)} \) | \( jw^{(1)} + j2w^{(2)} + j3w^{(3)} + j4w^{(4)} \) | \( \exp(i2\pi/5) \) |
| \( u^{(2)} = j^2w^{(1)} \) | \( j^2w^{(1)} + jw^{(2)} \) | \( j^2w^{(1)} + jw^{(2)} + j2w^{(3)} \) | \( j^2w^{(1)} + jw^{(2)} + j2w^{(3)} + j3w^{(4)} \) |
| \( u^{(3)} = j^3w^{(1)} \) | \( j^3w^{(1)} + j^2w^{(2)} \) | \( j^3w^{(1)} + j^2w^{(2)} + j^3w^{(3)} \) | \( j^3w^{(1)} + j^2w^{(2)} + j^3w^{(3)} + j^4w^{(4)} \) |
| \( u^{(4)} = j^4w^{(1)} \) | \( j^4w^{(1)} \) | \( j^4w^{(1)} \) | \( j^4w^{(1)} \) | \( j^4w^{(1)} \) |

Table 1.

Note that the choice of \( w \) parameters is not unique. The set adopted here, gives a full correspondence with trilinear operators introduced earlier in soliton theory \( \mathbb{3} \). Since for arbitrary integer, \( J \) the sum \( \sum_{k=0}^{J-1} j^k = 0 \), it is seen that the requirement \( (28) \) is fulfilled for any set \( w^{(q)} \).

As it was already mentioned, there exist several versions of the addition theorem for theta functions. To our knowledge only one form \( (11) \) leads deals to the product of an arbitrary number of shifted theta functions as in \( (26) \), but the r.h.s. is essentially different and unusable for our purposes. For the fixed \( J \) \( (28) \) can be rewritten as

\[
\exp \sum_{j=1}^{J-1} \left[ \ln \theta \left( z + u^{(j)} \right) - \ln \theta \left( z \right) \right] = \theta \left( z \right)^{-J} \sum_{\varepsilon} W_\varepsilon \left( u^{(1)}, ..., u^{(J-1)} \right) Z_\varepsilon \left( z \right).
\]
Differentiating \[33\] with respect to different components of vectors \(u^{(k)}\), \((k = 1, \ldots, J - 1)\) we obtain

\[
\frac{\partial^{\rho+\ldots+q}}{(\partial u_{\alpha_1}^{(1)})^p \ldots (\partial u_{\beta_k}^{(q)})^q} \exp \left[ \sum_{k=0}^{J-1} \ln \theta \left( z + u^{(k)} \right) - J \ln \theta (z) \right] = \left[ \theta (z) \right]^{-J} \sum_{\varepsilon} \left[ \frac{\partial^{\rho+\ldots+q}}{(\partial u_{\alpha_1}^{(1)})^p \ldots (\partial u_{\beta_k}^{(q)})^q} W_\varepsilon \left( u^{(1)}, \ldots, u^{(J-1)} \right) \right] Z_\varepsilon (z),
\]

where \(j = \exp(2\pi/J)\) and \(u^{(0)} = \sum_{k=1}^{J} u^{(k)}\) Changing now the derivatives with respect \(u^{(k)}\) on l.h.s. into derivatives with respect to \(z\), we can find easily the relationship between the derivatives of logarithms of theta functions (with respect to \(z\)) and the derivatives of \(W_\varepsilon\) functions (with respect to \(u^{(k)}\)). All relations become simplier if \(u^{(k)}\) parameters are chosen to be zero.

As an example, the lowest nontrivial derivatives of \(W_\varepsilon \left( u^{(1)} \right)\) and \(W_\varepsilon \left( u^{(1)}, u^{(2)} \right)\), respectively, (at zero) up to fifth order are reported for \(J = 2, 3\) below in Table 2 and Table 3. In the Appendix we report the lowest nontrivial derivatives (up to 6-th order) also for \(J = 6\).

| \(J=2\) |
| --- |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}}\) | \(L_{\alpha\beta}\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(4 \left( 3 \times L_{\alpha\beta} L_{\gamma\delta} + 2L_{\alpha\beta\gamma\delta} \right)\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(8 \left( 15 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} \right) + 4 \left( 15 \times L_{\alpha\beta\gamma\delta} L_{\zeta\mu} \right) + 2L_{\alpha\beta\gamma\delta\zeta\mu}\) |

Table 2.

| \(J=3\) |
| --- |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}}\) | \(2L_{\alpha\beta}\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}}\) | \(L_{\alpha\beta}\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}}\) | \(-L_{\alpha\beta\gamma}\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(4 \left( 3 \times L_{\alpha\beta} L_{\gamma\delta} + 2L_{\alpha\beta\gamma\delta} \right)\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(2 \left( 3 \times L_{\alpha\beta} L_{\gamma\delta} + 2L_{\alpha\beta\gamma\delta} \right)\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(4L_{\alpha\beta} L_{\gamma\delta} + 2 \times L_{\alpha\beta\gamma\delta} + 2L_{\alpha\beta\gamma\delta}\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(-2 \left( 6 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} \right) - L_{\alpha\beta\gamma\delta}\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(8 \left( 15 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} \right) + 4 \left( 15 \times L_{\alpha\beta\gamma\delta} L_{\zeta\mu} \right) + 2L_{\alpha\beta\gamma\delta\zeta\mu}\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(4 \left( 15 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} \right)\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(++2 \left( 5 \times L_{\alpha\beta\gamma\delta} L_{\zeta\mu} + 10 \times L_{\alpha\beta\gamma\delta} L_{\delta\zeta\mu} + L_{\alpha\beta\gamma\delta\zeta\mu}\right)\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(8 \left( 3 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} \right) + 12 \left( 2 \times L_{\alpha\beta} L_{\gamma\delta} L_{\delta\zeta\mu} \right) +\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(4 \left( L_{\alpha\beta\gamma\delta} L_{\zeta\mu} + 4 \times L_{\alpha\beta\gamma\delta} L_{\delta\zeta\mu} + 4L_{\alpha\beta\gamma\delta} L_{\zeta\mu} \right) + i \neq j\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(2 \left( 6 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} \right) + \left( L_{\alpha\beta\gamma\delta} L_{\zeta\mu} \right) + 4 \left( 9 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} \right) + i \neq j\) |
| \((W_\varepsilon)_{u_\alpha^{(i)}u_\beta^{(i)}u_\gamma^{(i)}u_\delta^{(i)}}\) | \(9 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} + 2 \left( 6 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} \right) + 9 \times L_{\alpha\beta} L_{\gamma\delta} L_{\zeta\mu} i \neq j\) |

Table 3.
The remaining derivatives of the order less than 6 vanish. $L_{\alpha\beta} := \partial_{z_{\alpha}z_{\beta}} \ln \theta(z|B) \big|_{z=0}$ etc., and we adopted here the shorthand notation which includes all possible permutations with respect identically underlined indices: e.g. $3 \times L_{\alpha\beta}L_{\gamma\delta} := L_{\alpha\beta}L_{\gamma\delta} + L_{\alpha\gamma}L_{\beta\delta} + L_{\alpha\delta}L_{\beta\gamma}$.

Some introductory applications of the above results can be found in Ref. [10].

6 Addition property versus multilinear operators

The correspondence between the reported here system of dispersion equations and bilinear operators is quite obvious. This affinity can be extended even further. For fixed $J$, let us introduce a hierarchy of operators $T^{(n)}$ labeled by $n = 0, 1, \ldots, J - 1$

$$\left( T^{(n)} \right)^m (\tau)^{(J-1)} := \left( T^{(n)} \right)^m \tau \circ \tau \circ \ldots \circ \tau = \left( T^{(n)} \right)^m [\tau (z + u_0) \ldots \tau (z + u_{J-1})] \big|_{u_0 = \ldots = u_{J-1} = 0} ,$$

where

$$T^{(0)} = \sum_{m=0}^{J-1} \partial u^{(m)} ,$$

$$T^{(n)} = \partial u^{(0)} + \sum_{m=J-n+1}^{J-1} j^m \partial u^{(m-n)} + \sum_{m=1}^{J-n} j^m \partial u^{(m-n-1)} , \quad 0 \neq n < J ,$$

and differentiation relates of course to the indicated components of $u^{(m)}$ vectors i.e. $u^{(m)}_\alpha$. For $J = 2, 3, 4$ we have

| J   | 2   | 3   | 4   |
|-----|-----|-----|-----|
| $T^{(0)}$ | $\partial u^{(0)} + \partial u^{(1)}$ | $\partial u^{(0)} + \partial u^{(1)} + \partial u^{(2)}$ | $\partial u^{(0)} + \partial u^{(1)} + \partial u^{(2)} + \partial u^{(3)}$ |
| $T^{(1)}$ | $\partial u^{(0)} + j^3 \partial u^{(1)}$ | $\partial u^{(0)} + j^3 \partial u^{(1)} + j^2 \partial u^{(2)}$ | $\partial u^{(0)} + j^3 \partial u^{(1)} + j^2 \partial u^{(2)} + j^1 \partial u^{(3)}$ |
| $T^{(2)}$ | $\partial u^{(0)} + j^2 \partial u^{(1)}$ | $\partial u^{(0)} + j^2 \partial u^{(1)} + j \partial u^{(2)}$ | $\partial u^{(0)} + j^2 \partial u^{(1)} + j \partial u^{(2)} + j^1 \partial u^{(3)}$ |
| $T^{(3)}$ | $\partial u^{(0)}$ | $\partial u^{(0)} + j \partial u^{(1)}$ | $\partial u^{(0)} + j \partial u^{(1)} + j^1 \partial u^{(2)}$ |

It is seen that for $J = 2$ and $J = 3$ we have the standard two- and tri-linear operators, respectively. However, for $J > 3$, the operator $T^{(2)} \neq (T^{(1)})^*$ i.e. $T^{(2)}$ is not a complex conjugate to $T^{(1)}$. For this reason the operators $T^{(n)} (n = 1, \ldots, J - 1)$ for fixed $J$, will called associated operators. Operator $T^{(0)}$ is introduced here only for completeness.

Now, if $\tau-$ function from [33] has AP, the question arises how it reflects on the $W_\varepsilon (w^{(1)}, \ldots, w^{(J-1)})$ functions? Using (31) and (32) we have $\partial u^{(s)}_\alpha / \partial u^{(p)}_\alpha = j^{1-s+p} \left( 1 + (j^{J-1} - 1) \delta_{0<s}\right)$ and therefore

$$\partial u^{(p)}_\alpha = \sum_{s=0}^{J-1} j^{s} \partial u^{(s)}_\alpha \partial u^{(p)}_\alpha = \partial u^{(0)}_\alpha + j^{1+p} \sum_{s=0}^{J-1} \left(j^s \left( 1 + (j^{J-1} - 1) \delta_{0<s}\right) \right) \partial u^{(s)}_\alpha$$

$$= \partial u^{(0)}_\alpha + \sum_{m=J-n+1}^{J-1} j^m \partial u^{(m-p)}_\alpha + \sum_{m=1}^{J-p} j^m \partial u^{(m-p-1)} = T^{(p)}_\alpha .$$

This means that if $\tau-$function has the AP, multilinear operators according to (33)-(37) reduce to the simple differentiation of the $W_\varepsilon (w^{(1)}, \ldots, w^{(J-1)})$ functions with respect to their arguments. In the simplest cases of bi- and tri-linear operators this assertion allows immediately to write the system of dispersion equation on the basis of bi- or tri-linear approximation.

As an example, let us note the bi-linear form of Korteweg de Vries equation and tri-linear form of the reduction of self-dual Yang Mills equation

$$\left( D_x D_t + D^2_x \right) \tau \circ \tau = 0,$$

$$\left(T^2_x T^* + 8T^2_x T^* + 9T^2_x T^* \right) \tau \circ \tau = 0,$$
coincide with the relevant dispersion equation systems

\[
\sum_{ij} \kappa_i \omega_j W_{\varepsilon,w_iw_j} + \sum_{ijkl} \kappa_i \kappa_j \kappa_k \kappa_l W_{\varepsilon,w_iw_jw_kw_l} = CW_\varepsilon, \tag{41}
\]

\[
\sum_{ijklm} (\kappa_i \kappa_j \kappa_k \kappa_l \lambda_m - 8\kappa_i \kappa_j \kappa_l \lambda_m) W_{\varepsilon,w_iw_jw_kw_lw_m} + 9 \sum_{ijk} \kappa_i \kappa_j \omega_k W_{\varepsilon,w_iw_jw_k} = CW_\varepsilon, \tag{42}
\]

where we assumed that the arguments of \(\tau\)-functions depend linearly on space and time:

\[z_i = \kappa_i x + \omega_i t\]

and

\[z_i = \kappa_i x + \lambda_i y + \omega_i t\], respectively. Moreover, in the second equation \(W_\varepsilon = W_\varepsilon\left(w^{(1)}, w^{(2)}\right)\big|_{w^{(1)}=w^{(2)}=0}\), depends on two vectors, designed for typographic reasons as \(w\) and \(v\). In both cases, \(C\) appears as the integration constant and for soliton solutions it vanishes, while for quasi-periodic solution it has to be determined as an additional parameter.

Finally equations (41) and (42) should hold for any \(\varepsilon \in \mathbb{Z}_g\), i.e. the first one for \(\varepsilon \in \mathbb{Z}_g^2\), and the second - for \(\varepsilon \in \mathbb{Z}_g^3\).

In conclusion, we expect that the reported here generalized addition property can be useful for the analysis of multidimensional soliton equations.

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7 Appendix

The lowest nontrivial derivatives (up to 6-th order) of of \( W_c (w^{(1)}, ..., w^{(5)}) \) for \( J = 6 \).

| \((W_c)_{a_i} (u_j^{(i)})\) | \(2L_{\alpha\beta}\) |
|-----------------------------|----------------|
| \((W_c)_{a_i} u_{\gamma}^{(i)}\) | \(L_{\alpha\beta}\) |
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)}\) | \(-L_{\alpha\beta\gamma}\) |
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)}\) | \(-L_{\alpha\beta\gamma}\) |
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)} u_{\epsilon}^{(i)}\) | \(|i, j - different|\)
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)} u_{\epsilon}^{(i)}\) | \(|i, j, k, l - different|\)
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)} u_{\epsilon}^{(i)} u_{\mu}^{(i)}\) | \(-2 (6 L_{\alpha\beta\gamma\delta\epsilon\mu}) - L_{\alpha\beta\gamma\delta\epsilon\mu}\) |
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)} u_{\epsilon}^{(i)} u_{\mu}^{(i)}\) | \(|j \neq i, k \neq i|\)
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)} u_{\epsilon}^{(i)} u_{\mu}^{(i)}\) | \(|j \neq i, k \neq i|\)
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)} u_{\epsilon}^{(i)} u_{\mu}^{(i)}\) | \(|i, j, k - different|\)
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)} u_{\epsilon}^{(i)} u_{\mu}^{(i)}\) | \(|i, j, k, l - different|\)
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)} u_{\epsilon}^{(i)} u_{\mu}^{(i)}\) | \(|i, j, k, l, m - different|\)
| \((W_c)_{a_i} (u_j^{(i)}) u_{\gamma}^{(i)} u_{\delta}^{(i)} u_{\epsilon}^{(i)} u_{\mu}^{(i)}\) | \(|i, j, k, l, m - different|\)