Lorentz invariance violation and charge (non-)conservation:
A general theoretical frame for extensions of the Maxwell equations

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All quantum gravity approaches lead to small modifications in the standard laws of physics which lead to violations of Lorentz invariance. One particular example is the extended standard model (SME). Here, a general phenomenological approach for extensions of the Maxwell equations is presented which turns out to be more general than the SME and which covers charge non-conservation (CNC), too. The new Lorentz invariance violating terms cannot be probed by optical experiments but need, instead, the exploration of the electromagnetic field created by a point charge or a magnetic dipole. Some scalar–tensor theories and higher dimensional brane theories predict CNC in four dimensions and some models violating Special Relativity have been shown to be connected with CNC and its relation to the Einstein Equivalence Principle has been discussed. Due to this upcoming interest, the experimental status of electric charge conservation is reviewed. Up to now there seem to exist no unique tests of charge conservation. CNC is related to the precession of polarization, to a modification of the $1/r$–Coulomb potential, and to a time-dependence of the fine structure constant. This gives the opportunity to describe a dedicated search for CNC.

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I. INTRODUCTION AND MOTIVATION

The dynamics of the electromagnetic field is usually described by the homogeneous and inhomogeneous Maxwell equations

$$\partial_{\mu} F_{\nu\rho} = 0, \quad \partial_{\nu} F^{\mu\nu} = 4\pi j^\mu, \quad (1)$$

where $F_{\mu\nu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu}$. These equations can be based on charge conservation

$$\partial_{\mu} j^\mu = \rho + \nabla \cdot j = 0, \quad (2)$$

and on the conservation of the magnetic flux together with the constitutive relation (see [? ] for discussion of formal structure of ordinary Maxwell equations). From (1) it follows that light rays are characterized by $\eta^{\mu\nu} k_{\mu} k_{\nu} = 0$ and propagate along geodesics, and that the charge is given by

$$Q = \int_{\Sigma} j^0 d\Sigma = \int \rho d^3 x, \quad (3)$$

where $\Sigma$ is some space–like hypersurface, is identified with the total charge related to the 4–current density $j^\mu$. (As usual, we assume that the charge and current density fall off faster than $1/r^2$ for large space–like distances.) From (2) we get $dQ/dt = 0$, where $t$ is the parameter connected to the space–like hypersurfaces. Charge non–conservation (CNC), on the other hand, would imply

$$\frac{d}{dt} Q \neq 0, \quad (4)$$

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This is directly related to $\partial_\mu j^\mu \neq 0$.

The homogeneous Maxwell equations can be related to a conservation of the magnetic flux [?] and serve as reason for defining the Maxwell potential $A_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. (For another relation to physical phenomena, see below.) With the help of the homogeneous equations, the inhomogeneous equations can be derived from a Lagrangian $L = \frac{1}{2} \eta^{\mu\nu} F_{\mu\nu} F_{\rho\sigma} + j^\mu A_\mu$ where $\eta^{\mu\nu}$ is the Minkowski space–time metric. Charge conservation is a consequence of the variational principle.

In the approach of Kostelecký and coworkers [?], it has been suggested that modifications of the effective Maxwell equations arising from string theory are described by a replacement of

$$\eta^{\rho\mu} j^{\rho\sigma} \rightarrow \eta^{\rho\mu} j^{\rho\sigma} + k^{\mu\nu\rho\sigma}$$

in the ordinary Maxwell Lagrangian. This leads to a violation of the Lorentz invariance of the physics of the SME which is related to the coefficients encoded in the tensor $k^{\mu\nu\rho\sigma}$. The question we want to address is whether this is the most general extension of the ordinary Maxwell theory for the inclusion of Lorentz violating terms. In the answer to this question we do not start from a Lagrangian but, instead, use the field equations only. Indeed, using this approach we are able to introduce further Lorentz violating terms and, thereby, are able to include charge non-conservation.

The new Lorentz-violating extension cannot be tested by optical experiments. We also think that, being such an important part of present scheme of theoretical physics, charge conservation needs also to be questioned and should be subject to experimental tests in the same way as Lorentz invariance.

Charge conservation is a very important feature of ordinary Maxwell theory. It is

- basic for an interpretation of Maxwell–theory as $U(1)$ gauge theory, and
- it is necessary for the compatibility with standard quantum theory in the sense that it is related to the conservation of probability.

The more important a particular feature of physics is, the more firmly this feature should be based on experimental facts. Therefore, in this paper we are going to address the questions

- How good is charge conservation experimentally verified?
- How to describe theoretically in a consistent way charge no-conservation?
- Are there further consequences of charge non-conservation?
- Do these consequences provide new tests of charge conservation?

It is also clear that, apart from its theoretical importance, Lorentz invariance as well as charge conservation are central for metrology, that is, for our current system of physical units and the systems of fundamental constants. Any dependence of the velocity of light from the velocity and from the orientation of the laboratory as well as charge non-conservation will abolish the uniqueness and universality of the definition of the second: Each clock then will depend in its own way from the velocity and orientation and, through the time–dependence of the fine structure constant, from the used atomic transition. Related to that is the definition of the meter and, of course, the definition of all electrical units like the resistance and the voltage, both based today on quantum phenomena, namely the von Klitzing effect and the Josephson effect. Furthermore, as has been shown by Ni [?] that any modification of the Maxwell equations (except the one described by some axion field) will violate the validity of the Universality of Free Fall, too, and is thus deeply connected with the geometrization of the gravitational interaction, and, consequently, important for the frame of General Relativity. Since the ordinary Maxwell equations leads to Lorentz invariant effects and to charge conservation, any violation of Lorentz invariance and CNC has necessarily to be described by a modification of this set of Maxwell equations.

II. EXPERIMENTAL FACTS

Since the experimental facts concerning the search for Lorentz violation are well documented in every textbook we will restrict ourselves to a few aspects of the experimental facts related to charge conservation. There seem to be only three classes of experiments related to charge conservation:

1. Electron disappearing: One aspect of charge conservation is the spontaneous electron disappearing. This is related to elementary particle decay processes like $e \to \nu_e + \gamma$ or, more general, to $e \to \gamma$ any particles. Decays of this kind have been searched for in processes in high energy storage rings but nothing has been observed [? ?].

   For the general process, the probability for such a process has been estimated to be $2 \cdot 10^{-22}$ y$^{-1}$ [?] and for two specific processes the probability can be as less as $3 \cdot 10^{-26}$ y$^{-1}$ [? ].
We note that even for a strict non–disappearing of electrons, the charge of electron may vary in time and thus may give rise to CNC. Therefore, while charge–conservation implies a non–disappearing of electrons, electron non–disappearing does not imply charge conservation.

2. Equality of electron and proton charge: Another aspect of charge conservation is the equality of the absolute value of the charge of all separable elementary particles like electrons and protons (we let aside fractional charges of quarks because these particles cannot be observed in isolated states). Tests of the equality of $q_e$ and $q_p$ through the neutrality of atoms [?] yield very precise estimates. The reason for that is that macroscopic numbers of atoms can be observed. The experiment consists of the observation of sound waves in gas induced by an externally applied time–dependent electric field which should lead to a characteristic frequency of these sound waves if there is an excess charge. The result is $|(q_e - q_p)/q_e| \leq 10^{-19}$.

3. Time–variation of α: The most direct test of charge conservation is implied by searches for a time–dependence of the fine structure constant $\alpha = q_e q_p / \hbar c$. Since different hyperfine transitions depend in a different way on the fine structure constant, a comparison of various transitions is sensitive to a variation of $\alpha$. Recent comparisons of different hyperfine transitions [?] lead to $|\dot{\alpha}/\alpha| \leq 7.2 \cdot 10^{-16}$ $\text{y}^{-1}$. This may be translated into an estimate for charge conservation $|q_e/q_e| \leq 3.6 \cdot 10^{-16}$ $\text{y}^{-1}$, provided $\hbar$ and $c$ are constant and $q_p = q_e$. However, this cannot be done within, e.g., the frame of varying $c$ theories.

Although there are individual experiments which have been used to set limits on charge conservation, such as $e \rightarrow \nu_e + \gamma$ [? ? ], a general framework is needed to allow the limits from different experiments to be compared.

III. MODELS WITH VIOLATION OF LORENTZ INVARIANCE AND CNC

Models describing the violation of Lorentz invariance in the context of the theory of electromagnetic fields have been discussed since the early seventies. Ni [?] considered a Lorentz invariance violation in the Maxwell Lagrangian given by [?] and derived conditions on the coefficient $K^{\mu\nu\rho\sigma}$ from the requirement that no birefringence and no anisotropic speed of light should come out. Using this approach he found a non–metric extension of Maxwell’s theory which is still compatible with the Weak Equivalence Principle which, thus, constituted a counterexample to Schiff’s conjecture. Lateron, Haugan and Kauffmann [?] used the same model in order to analyze astrophysical observations related to birefringence. Recently, Kostelecký and coworkers set up a general scheme, the so–called Standard Model Extension (SME) including the Maxwell as well as the fermion sector of particles in order to describe violations of Lorentz invariance [? ]. For the Maxwell sector, again a modification on the level of the Lagrangian has been used, and has been confronted with astrophysical observations related to birefringence and to laboratory experiments related to the isotropy of the velocity of light [? ? ] which, in the context of a broad class of gravity theories, also lead to general time– and position–dependent effects and violations of the Weak Equivalence Principle [? ]. A simple model including some mass vector for the photon and which already leads to CNC has been introduced in [? ].

Recently, some models which allow for a violation of charge conservation have been discussed: Within higher dimensional brane theories it has been argued that charge may escape into other dimensions [? ? ] thus leading to CNC in four–dimensional space–time. Also in connection with variable–speed–of–light theories CNC may occur [? ]. A very important aspect of CNC is its relation to the Einstein Equivalence Principle which is lying at the basis of General Relativity [? ]. This again emphasizes the fact that the structure of Maxwell’s equations is deeply connected with the structure of space–time. CNC also necessarily appears if one introduces phenomenologically a mass of the photon into the Maxwell equations in a gauge–independent way [? ]. This particular approach is also the starting point for the considerations in this paper.

In this paper we do not want to proceed along the line of a particular model but, instead, want to set up a general frame for a theoretical description of CNC. The idea behind that is that any violation of charge conservation must show up in a modification of the Maxwell equations, and any modification of the Maxwell equations should lead to a variety of effects which should be accessible to various experimental tests. Such effects are, e.g., birefringence, dispersion, damping, anisotropy of the speed of light, etc.

IV. HOW TO MEASURE CHARGE? – THE DEFINITION OF THE ELECTROMAGNETIC FIELD

As a first step in the treatment of general Lorentz invariance violations and CNC one first should clear how the electromagnetic field and charges can be defined uniquely in an operational way.
A. Charge in classical mechanics

In the frame of classical mechanics the electric and magnetic field as well as the charge is measured using Lorentz’ law \[ F = qE + \frac{q}{c}v \times B. \] \((6)\)

However, if one allows for accelerating observers it is not possible even to characterize uniquely what is a vanishing charge. Furthermore, in this frame forces are probes of charge. If charges, e.g., vary in time, then the force between charges varies with time. This can be probed using e.g. springs. However, since the physics of spring also heavily depends on the electromagnetic field and the properties of the charges of the constituents, no unique identification of charge conservation can be made.

![Diagram of forces between increasing charges](image)

**FIG. 1:** Forces between increasing charges can be probed with a spring.

B. Charges in quantum mechanics

A more fundamental approach is the physics of the hydrogen atom, which can be used to probe the electromagnetic interaction between charged particles on the quantum level. Here, charge comes in via

\[ E = -\text{Ry} \frac{1}{n^2}, \quad \text{Ry} = \frac{\mu q_e^2 q_p^2}{\hbar^2}, \quad \mu = \frac{m_e m_p}{m_e + m_p}, \] \((7)\)

where \(q_e, q_p, m_e,\) and \(m_p\) are the charges and masses of the electron and the proton, respectively. For other levels, like hyperfine levels, the expressions are more complicated but still depend on the same set of constants.

![Diagram of atomic energy levels](image)

**FIG. 2:** Atomic energy levels (here the Balmer series, for example) are sensitive to increasing charges.

The important point in both cases is that one needs a reference "system", either a spring (depends on electric properties, too), or quantum mechanics. Since also the spring basically is determined by quantum properties, both reference systems depend, beside the charge, on other parameters like \(\hbar, c, m_e, m_p\). Therefore, these methods are not applicable for a unique and independent definition of charge.

We may ask the following question: If we start from the ordinary Schrödinger equation minimally coupled to the electromagnetic field, what parameter is effectively probed by experiments described by the Schrödinger equation? If we rewrite the Schrödinger equation in the form

\[ i \frac{\partial}{\partial t} \psi = -\frac{1}{2m/\hbar} \left( \nabla - \frac{i q_e}{\hbar c} A \right)^2 - \frac{m}{\hbar} \mu \psi + \frac{q_e}{\hbar} \phi \psi, \] \((8)\)

then it is clear that there are two effective parameters, \(m/\hbar\) and \(q_e/\hbar\) where \(m/\hbar\) can be determined with high precision with neutron or atom interferometry. In order to determine the charge or the mass of the quantum particle, one has
to know $\hbar$. This quantity today can be measured best using the Watt balance [? ]. However, this is based on the primary standard kilogram, on the measurement of the gravitational acceleration as well as on the units of electrical resistance and voltage defined through the von–Klitzing effect and the Josephson effect which, in turn, again depends on the charge. Since the thorough analysis of all the dependence between the various definitions is beyond this paper, we use $m/\hbar$ and $q_e/\hbar$ as (effective) mass and charge.

Inspired by the structure of the Schrödinger equation we may proceed with an operational definition of charge and of the electromagnetic field using the Aharonov–Bohm effect.

C. Definition of charge and of the electromagnetic field

In order to develop a general frame for describing in a consistent way the dynamics of the electromagnetic field and, thus, violations of Lorentz invariance and CNC, we first define the electromagnetic field strength $F_{\mu\nu}$ and the electric charge. This will be done through the Aharonov–Bohm effect for charged quantum particles.

If we perform an interference experiment for charged particles in an electromagnetic field, then we make the basic experience that the phase shift depends in a linear way on the area enclosed by the particles’ path. For doing this observation we do not need any description of the electromagnetic field. We just need to have some apparatus which produces some electromagnetic field which we also should be able to vary. This basic experience can be encoded in the general formula ($\mu, \nu, \ldots$ are indices ranging from $0, \ldots, 3$)

$$\Delta \phi = \frac{q}{\hbar} \int F_{\mu\nu} d\sigma^{\mu\nu},$$

(9)

where, $\sigma^{\mu\nu}$ is the area element which is an antisymmetric tensor. For given area and measured phase, this equation defines the electromagnetic field $F_{\mu\nu}$. This phase shift, of course, coincides with the phase shift for charged particle when using (8). However, here we use (9) independent from (8).

By observing this phase shift for different particles in the same experimental situation, we recognize that the linear dependence of the phase shift on the area which differs only by a proportionality factor $q/\hbar$. This proportionality factor we call the charge of the particle. This charge can indeed be defined uniquely. The reason for that is that in nature there are neutral particles which, by means of our approach, can be identified uniquely: Neutral particles are defined as those particles whose interference pattern does not vary in the case of a varying electromagnetic field (though the Aharonov–Bohm effect is used for the definition of the electromagnetic field, this is no logical circle because for the identification of neutral particles we do not need the quantitative definition of the electromagnetic field, we just need to vary it in some way). The fact that here we are able to uniquely identify neutral particles while this is not possible by using the Lorentz force equation only shows that the Aharonov-Bohm effect is of superior importance.

After having identified neutral particles, we are in the position to identify particles with different charges $q$ by merely comparing the phases for different particles travelling along the same geometrical path: $q_1/q_2 = \Delta \phi_1/\Delta \phi_2$. By consideration of all kinds of charged elementary particles, we can identify that one with the smallest charge, and it is also a basic experience, again from the Aharonov–Bohm effect, that all particles come with charges which is a multiple of the elementary charge $e$, that is, particles have a discrete spectrum of charges only. However, the fact that all particles have charges $q/\hbar = ne/\hbar$, $n \in \mathbb{N}$, does not mean that charge is conserved. Indeed, it is still possible that the elementary charge in $e/\hbar$ varies with time. In this case, all the observations connected with charged particle interferometry and derived notions are still consistent. However, as we shall see later, a varying charge is not consistent with the ordinary Maxwell equations.

As a first result, the uniqueness of the phase shift (9) requires (for simplicity, we exclude any non–trivial topology of the underlying space–time)

$$\partial_{[\mu} F_{\rho\sigma]} = 0.$$  

(10)

what establishes the homogenous Maxwell equations.

V. A GENERAL FRAME FOR THE DYNAMICS OF ELECTROMAGNETIC FIELDS

A. The general ansatz

The homogeneous equations (10) are dynamically incomplete because they provide only three dynamical equations for the six components encoded in $F_{\mu\nu}$. Therefore (10) has to be completed by another set containing three further
dynamical equations. In a pragmatic approach, we start from the ordinary inhomogenous Maxwell equations which are experimentally verified to a very high degree of accuracy [? ]. Therefore, if in nature there is some violation of Lorentz invariance or CNC, then this must be a tiny correction to the ordinary equations. The most general modification of the inhomogeneous Maxwell equations for field strength $F$ which still is linear in $F$ and first order in derivative is given by

$$\eta^{\mu\nu} \psi^\sigma + \chi^{\mu\nu\rho\sigma} \partial_\rho F_{\sigma\nu} + \chi^{\mu\rho\sigma} F_{\rho\nu} = 4\pi j^\mu.$$  \hspace{1cm} (11)

Here, $\eta^{\mu\nu}$ plays the role of a generalized constitutive or material tensor and $\chi^{\mu\nu\rho\sigma}$ possess properties of an anisotropic mass of the photon. $\eta^{\mu\nu}$ is the Minkowski metric with signature $(+, -, -, -)$. We assume the $\chi^{\mu\nu\rho\sigma}$ and $\chi^{\mu\rho\sigma}$ to be constant. It is no problem to generalize this approach to the case of a Riemannian metric $\eta^{\mu\nu}$ instead of the Minkowski metric and to consider position dependent tensors $\chi^{\mu\nu\rho\sigma}$ and $\chi^{\mu\rho\sigma}$. We restrict ourselves to this more simple case since, as far as Lorentz invariance is concerned, all conclusions will be the same. — The charge underlying the 4–current on the right hand side is identified with the charge which has been defined using the Aharonov–Bohm effect. That means that we identify active and passive electromagnetic charges.

Eq. (11) is our model describing violations of Lorentz invariance and CNC. Before discussing Lorentz non–invariance and CNC, we make a few general comments:

- The generalized constitutive tensor possess the symmetry

$$\chi^{\mu\nu\rho\sigma} = \chi^{\mu[\nu\rho\sigma]}$$ \hspace{1cm} (12)

and, thus, possesses 96 components. Below we impose two more conditions on this constitutive tensor: first $\chi^{\mu[\nu\rho\sigma]} = 0$ which eliminates the degrees of freedom which will drop out from the inhomogenous Maxwell equations (11) due to the homogeneous ones (10). Furthermore, the uniqueness of the Cauchy–problem requires $\chi^{(\alpha\beta\mu\nu)} = 0$. This reduces the number of independent coefficients to 35. One of these components can be absorbed by a redefinition of the charge. This generalized constitutive tensor implies birefringence, an anisotropic velocity of light, an anisotropic Coulomb potential and a modified magnetic field.

- In equations which use field strengths only it is not possible to assign a scalar mass to the photon. Masses have to be tensor valued. This is in contrast to the Proca equation which is used as model for describing a scalar photon mass.

- The mass tensor possesses the symmetry $\chi^{\mu\rho\sigma} = \chi^{\mu[\rho\sigma]}$ and, thus, 24 independent components. It leads an anisotropy and velocity dependence of the velocity of light, to birefringence, dispersion, a damping of the intensity of radiation, and to an anisotropy and a Yukawa modification of the Coulomb potential.

- Our model, in general, cannot be derived from a Lagrangian. In fact, in order to get the second term, the corresponding term in the Lagrangian has to be linear in $\chi^{\mu\rho\sigma}$ and quadratic in the potential $A_\mu$. The only combination is $\chi^{\mu\rho\sigma} A_\mu F_{\rho\sigma}$. This term is gauge invariant only, if $\chi^{\mu\rho\sigma} \partial_\mu F_{\rho\sigma}$ vanishes. This is the case only for $\chi^{\mu\rho\sigma} = \chi^{[\mu\rho\sigma]}$.

### B. Decomposition of the constitutive tensor

First, we introduce the decomposition of the general constitutive tensor into irreducible parts (see Appendix for the definitions)

$$\chi^{\alpha\beta\mu\nu} = \frac{1}{(1)} W_{\alpha \beta \mu \nu} + \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \eta^{\rho [\alpha} \phi_{\beta]} \sigma + \frac{1}{12} \chi \epsilon^{\alpha \beta \mu \nu} - \eta^{\rho [\alpha} \phi_{\beta] \nu} + \eta^{\nu [\alpha} \phi_{\beta] \mu} + \eta^{\mu [\alpha} \phi_{\beta] \nu} - \eta^{\nu [\alpha} \phi_{\beta] \mu} + \eta^{\nu [\alpha} \phi_{\beta] \mu} + \frac{1}{3} \epsilon^{\mu \rho \sigma} \eta^{\rho [\alpha} \phi_{\beta] \nu} - \frac{1}{2} \eta^{\nu [\alpha} \phi_{\beta] \mu} + \eta^{\rho [\alpha} \phi_{\beta] \nu} - \frac{1}{2} \eta^{\nu [\alpha} \phi_{\beta] \mu} + \eta^{\nu [\alpha} \phi_{\beta] \mu} + \frac{1}{4} \eta^{\rho [\alpha} \phi_{\beta] \nu} - \frac{1}{2} \eta^{\nu [\alpha} \phi_{\beta] \mu}$$ \hspace{1cm} (13)

The generalized constitutive tensor is more general than the ordinary constitutive tensor which possesses the symmetries of the Riemann tensor, namely

$$k^{\mu \nu \rho \sigma} = k^{[\mu \nu \rho \sigma]} = k^{\mu \nu [\rho \sigma]} = k^{\rho \sigma \mu \nu}, \quad k^{\mu [\nu \rho \sigma]} = 0$$ \hspace{1cm} (14)
and, thus, consists of 20 components. Within a Lagrange–ansatz $\mathcal{L} = \frac{1}{2\mu} (\eta^{\mu\nu} \eta^{\rho\sigma} + k^{\mu\rho\nu\sigma}) F_{\mu\nu} F_{\rho\sigma}$, the totally antisymmetric part results in a divergence and the scalar part can be absorbed into a redefinition of the coupling to matter, so that $k^{\mu\rho\nu\sigma}$ effectively possesses 19 components only. The implications of $k^{\mu\rho\nu\sigma}$ have been described by Ni [? ? ? ? ], Haugan and Kauffmann [? ] and recently by Kostelecký, Mewes and others [? ? ? ? ].

The "mass" term can be decomposed into three irreducible parts, the totally antisymmetric part, a trace and the rest

$$\chi^{\alpha\mu\nu} = (1) \chi^{\alpha\mu\nu} + \epsilon^{\alpha\beta\mu\nu} a_\beta + \eta^{\alpha [\mu \nu]}$$

(15)

where $a_\beta = \frac{1}{6} \epsilon_{\alpha\beta\mu\nu} \chi^{\alpha\mu\nu}$, $t^\nu = \frac{2}{3} \eta_{\alpha\mu} \chi^{\alpha\mu\nu}$, and $(1) \chi^{\alpha\mu\nu} = \chi^{\alpha\mu\nu} - \epsilon^{\alpha\beta\mu\nu} a_\beta - \eta^{\alpha [\mu \nu]}$. Both vectors $a_\mu$ and $t^\mu$ possess four and $(1) \chi^{\alpha\mu\nu}$ possesses 16 components.

C. Unique time evolution

A 3 + 1–decomposition of the generalized Maxwell equations (11) gives

$$4 \pi \rho = \nabla \cdot E + \chi^{00i} \dot{E}_i + \chi^{0ij} \dot{B}_{ij} + \chi^{0i\rho\sigma} \partial_\rho F_{\sigma\rho} + \chi^{0\rho\sigma} F_{\rho\sigma}$$

(16)

$$4 \pi j^i = - (\nabla \times B)^i + \dot{E}_i + \chi^{0ij} \dot{E}_j + \chi^{0j\rho} \partial_\rho B_{ij} + \chi^{ij\rho\sigma} \partial_\rho F_{\rho\sigma} + \chi^{i\rho\sigma} F_{\rho\sigma},$$

(17)

where $E_i = F_{0i}$, $B_{ij} = F_{ij}$ and $B_1 = \frac{1}{b} \epsilon_{ijk} B_{ijk}$. The time derivative on $B$ can be replaced by a spatial derivative of the electric field with the help of the homogenous equations. The important point is, that, owing to the term $(18)$, both equations are dynamical equations for the electric field. In order that the generalized Maxwell equation leads to a consistent dynamical description, we have to require the vanishing of the coefficient $\chi^{00i}$ (since we want to recover the ordinary Maxwell equations for vanishing $\chi^{\mu\rho\sigma}$ we cannot choose $\chi^{00j} = - \delta^j_3$ though this, in principle, would also solve the problem). Since this should be true for any chosen frame of reference, we have to require

$$\chi^{(\mu\nu\rho)\sigma} = 0.$$  

(18)

This consistency requirement can also be read off directly from the following observation: Solving the second equation for $\dot{E}_i$ and inserting this into the first equation (only keeping terms first order in the $\chi$’s), gives the condition

$$4 \pi \rho = \nabla \cdot E + \chi^{00i} (j^i - (\nabla \times B)^i) + \chi^{0ij} \partial_\rho E_j + \chi^{0i\rho\sigma} \partial_\rho F_{\sigma\rho} + \chi^{0\rho\sigma} F_{\rho\sigma}.$$  

(19)

For given charge density $\rho$ and current $j^i$ and given initial conditions on $E_i$ and $B_i$, this represents an additional condition on the sources implying that one is not free to choose arbitrary sources. Since this is non–physical, we have to require (18).

The requirement (18) imposes conditions on the general constitutive tensor. These conditions are readily shown to be

$$\chi^{(\alpha\beta\mu\nu)} = 0 \implies (1) Z_{(\alpha\beta\mu\nu)} = 0, \quad \Xi_{\mu\nu} = 0, \quad \Delta_{\mu\nu} = - \frac{3}{4} Z_{\mu\nu}.$$  

(20)

The decomposition of the constitutive tensor then simplifies to be

$$\chi^{\alpha\beta\mu\nu} = (1) W^{\alpha\beta\mu\nu} + \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} \eta^{\alpha\beta \rho\sigma} + \frac{1}{12} X \epsilon^{\alpha\beta\mu\nu} - \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} + \eta^{\alpha\beta \mu\nu}$$

$$+ \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} - \frac{1}{6} W \eta^{\alpha\beta \rho\sigma} + \frac{1}{3} \epsilon^{\mu\nu\rho\sigma} \eta^{\alpha\beta \rho\sigma} - \frac{1}{2} \left( \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} - \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} + \frac{1}{2} \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} \right)$$

(21)

and the corresponding Maxwell equations then read

$$\chi^{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu} = \partial_\mu F_{\nu\sigma} + (1) W^{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu} - 2 \eta^{\mu\rho\nu} \eta^{\alpha\beta \rho\sigma} \partial_\beta F_{\mu\nu}$$

$$+ \left( \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} - \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} \right) \partial_\beta F_{\mu\nu} - \frac{1}{6} W \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} \partial_\beta F_{\mu\nu}$$

$$+ \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \left( \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} + \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} \right) \partial_\beta F_{\mu\nu} - \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} \partial_\beta F_{\mu\nu} + \frac{1}{2} \eta^{\alpha\beta \rho\sigma} \eta^{\mu\rho\nu} \partial_\beta F_{\mu\nu}.$$  

(22)

It is easy to see that neither the $Y$– nor the $Z$–terms contribute to a time derivative of the electric field.
D. Charge (non–)conservation

Within our model (11) the divergence of the 4–current in general is not assumed to vanish

\[ 4\pi \partial_\alpha j^\alpha = \chi^{\alpha \mu \nu} \partial_\mu F_{\nu \rho} + \chi^{\mu \rho \sigma} \partial_\mu F_{\rho \sigma} . \]  

(23)

However, it is amazing that already at this stage, that is using (18) only, the principal part of the Maxwell equations respect charge conservation: Employing the above decomposition we get for the divergence of the anomalous part of the principal part

\[ \partial_\alpha (\chi^{\alpha \beta \mu \nu} \partial_\beta F_{\mu \nu}) = \frac{3}{8} \epsilon^{\alpha \beta \mu \nu} \partial_\alpha \partial_\beta F_{\mu \nu} - \frac{1}{2} \eta^{\alpha \beta} Z_{\mu \nu} \partial_\alpha \partial_\beta F_{\mu \nu} \]

\[ + \frac{3}{2} \eta^{\beta \mu} Z^{\alpha \nu} \partial_\alpha \partial_\beta F_{\mu \nu} + \frac{1}{2} \eta^{\alpha \beta} Z_{\mu \nu} \partial_\alpha \partial_\beta F_{\mu \nu} + \chi^{\mu \rho \sigma} \partial_\mu F_{\rho \sigma} , \]  

(24)

where we showed those terms only which do not vanish trivially due to the antisymmetry in \( \alpha \) and \( \beta \). Here, the \( \Psi \) vanishes because of its total antisymmetry in \( \alpha, \mu, \nu \). The \( Z \)–terms can be treated as follows

\[ Z \text{–terms} = \eta^{\alpha \beta} Z_{\mu \nu} \partial_\alpha (\partial_\mu F_{\beta \nu} + \partial_\beta F_{\mu \nu}) = 0 \]  

(25)

due to the homogenous Maxwell equations. Therefore, even for the very general ansatz (11) with (13) the dynamical consistency automatically ensures charge conservation for the principal part of the generalized Maxwell equations. Thus, charge non–conservation can arise from the \( \chi^{\alpha \mu \nu} \) only

\[ 4\pi \partial_\alpha j^\alpha = \chi^{\alpha \mu \nu} \partial_\alpha F_{\mu \nu} = (1) \chi^{\alpha \mu \nu} \partial_\alpha F_{\mu \nu} + \eta^{\alpha \mu \nu} \partial_\alpha F_{\mu \nu} . \]  

(26)

The totally antisymmetric part of \( \chi^{\alpha \mu \nu} = e^{\alpha \beta \mu \nu} a_\beta \) still is compatible to charge conservation[1]. The quantity \( a_\beta \) is the axion introduced first by Ni [?]. All other parts of \( \chi^{\alpha \mu \nu} \) lead to CNC.

The change of the total charge can be expressed in terms of the tensor \( \chi^{\mu \rho \sigma} \),

\[ \frac{dQ}{dt} = \lim_{\delta t \to 0} \frac{1}{\delta t} \left( \int_{t}^{t+\delta t} \rho d^3x - \int_{t}^{t+\delta t} \rho d^3x \right) \]

\[ = \int \partial_\mu j^\mu d^3x \]

\[ = \int \left( (1) \chi^{\mu \rho \sigma} \partial_\mu F_{\rho \sigma} + \eta^{\rho \mu} t^\sigma \partial_\mu F_{\rho \sigma} \right) d^3x . \]  

(27)

The variation of the charge depends on the actual solution \( F_{\mu \nu} \) of the generalized Maxwell equations. In the case \( (1) \chi^{\mu \rho \sigma} = 0 \) this simplifies considerably

\[ \frac{dQ}{dt} = t_0 Q + \int t \cdot j d^3x , \]  

(28)

where we assumed that the charge density falls off fast enough at spatial infinity. If there are no currents present, then the change of the charge can be given directly in terms of the charge solely, \( \frac{dQ}{dt} = t_0 Q \) [?].

It is obvious that already in this simple case there is no static solution for the electric field of a point charge. That means, in particular, that we do not have energy conservation in the sense of an invariance of the solution with respect to time translations. One then might think of testing CNC parameters by analyzing energy conservation in high energy experiments. This will not lead to good results since in scattering processes the time scale is too short for probing the time dependence of the energy or the charge. For our purposes, long term experiments are needed as, e.g., searches for a time–dependence of the fine structure constant.

E. Use of the homogeneous Maxwell equations

Because of (10) the corresponding part in the inhomogenous Maxwell equations, \( \chi^{\alpha [\beta \rho \sigma]} \partial_\beta F_{\rho \sigma} \), will vanish identically and drops out of the inhomogeneous Maxwell equations. The corresponding parts of the irreducible decomposition
play no role and can be assumed to vanish, \( \chi^{\alpha[\beta\mu\nu]} = 0 \) or, equivalently, \( \epsilon_{\delta\beta\mu\nu} \chi^{\alpha[\beta\mu\nu]} = 0 \). From the irreducible decomposition we get

\[
\epsilon_{\delta\beta\mu\nu} \chi^{\alpha[\beta\mu\nu]} = -\frac{1}{2} X_{\delta} - \epsilon_{\delta i\mu\nu} (Z - W_{\delta})_{\mu\nu} + (2\Psi - \Psi)^{\alpha}_{\delta}. \tag{29}
\]

That means

\[
\chi^{\alpha[\beta\mu\nu]} = 0 \quad \Rightarrow \quad X = 0, \quad Z_{\mu\nu} = W_{\delta} \mu\nu, \quad \Upsilon_{\mu\nu} = \frac{1}{2} \Psi_{\mu\nu}. \tag{30}
\]

With this result the constitutive tensor reduces to

\[
\chi^{\alpha\beta\mu\nu} = (1) W^{\alpha\beta\mu\nu} + \frac{1}{8} \epsilon_{\rho\sigma}^{\alpha\beta\mu\nu} \left( 3\eta^{\rho\alpha} \Phi^{\beta\sigma} - \eta^{\rho\beta} \Phi^{\alpha\sigma} \right) - \eta^{\mu[\alpha} \Phi_{\beta]} + \frac{1}{2} \eta^{\alpha\beta} Z^{\mu\nu} - \frac{1}{6} W \eta^{\alpha[\mu} \eta_{\beta]}^{\nu} \tag{31}
\]

Now the effective Maxwell equations are

\[
4\pi j^\alpha = \partial_{\nu} F_{\mu\nu} + \chi^{\alpha\beta\mu\nu} \partial_{\beta} F_{\mu\nu}
= \partial_{\nu} F_{\mu\nu} + (1) W^{\alpha\beta\mu\nu} \partial_{\beta} F_{\mu\nu} + \frac{3}{8} \epsilon_{\rho\sigma}^{\alpha\beta\mu\nu} \Psi^{\beta\sigma} \partial_{\beta} F_{\mu\nu} - 2\eta^{\rho[\alpha} \Phi_{\beta]} \partial_{\beta} F_{\mu\nu}
- \frac{1}{2} \eta^{\alpha\beta} Z^{\mu\nu} \partial_{\beta} F_{\mu\nu} + \frac{3}{2} \eta^{\beta[\mu} Z^{\nu]} \partial_{\beta} F_{\mu\nu} + \frac{1}{2} \eta^{\alpha\beta} Z^{\mu\nu} \partial_{\beta} F_{\mu\nu} - \frac{1}{6} W \eta^{\alpha[\mu} \eta_{\beta]}^{\nu} \partial_{\beta} F_{\mu\nu}. \tag{32}
\]

Since these Maxwell equations are more general than those in the extended standard model we now have to look anew for ways how to confront these equations with the experiment in order to give unique experimental criteria for the Lorentz invariance of the theory. It has been shown by Ni [2] and Kostelecký and Mewes [3] that for the SME the requirement of vanishing birefringence and isotropic speed of light leads to a Lorentz invariant theory. The question now is whether this remains true for the present framework. Compared with the Lagrangian based approach by Ni, Haugan, Kostelecký and others, we are more general by the terms \( \Psi^{\mu\nu} \) and \( Z^{\mu\nu} \).

### VI. PROPAGATION OF LIGHT

Now we want to analyze some physical consequences of the generalized Maxwell equations [42]. A first step is to determine the wave equation for the electromagnetic field, to calculate the dispersion relation and to discuss consequences like birefringence and anisotropic speed of light. Also the propagation of the polarization is sensitive to effects connected with the violation of Lorentz invariance. We will see, that not all Lorentz invariance violating terms are accessible by radiation effects, that is, by optical experiments: Therefore, there is a need to discuss also the static electromagnetic fields of point charges and magnetic moments (Sec VII).

#### A. The wave equation

We derive the wave equation in vacuum in the usual way by differentiating Eq.(32) and substituting the time derivative of the magnetic field using the homogeneous equation

\[
0 = \dot{E}_i - \Delta E_i + (\nabla (\nabla \cdot E))^i_i + \chi^{\mu\nu\rho\sigma} \partial_{\rho} \partial_{\sigma} E_j + 2\chi^{i0j} \dot{E}_i + \chi^{i0j} \partial_{k} E_j \tag{33}
\]

In order to determine the propagation of the waves we insert the plane wave ansatz \( E = E^0 e^{ik \cdot x - \omega t} \) into wave equation and take the derivatives of the amplitude into account

\[
0 = \dot{E}_i^0 - 2i\omega \dot{E}_i^0 - \omega^2 E_i^0 - (\Delta E^0_i + 2ik \cdot \nabla E^0_i - k^2 E^0_i) + \partial_i \partial_j E_j^0 + ik_0 \partial_0 E_j^0 + ik_j \partial_j E_j^0 - k_j E_j^0 k_i + 2\chi^{i0j} (\partial_i \partial_j E_j^0 + ik_k \partial_k E_j^0 + ik_0 \partial_0 E_j^0 - k_j k_0) + 2i\omega \chi^{i0j} E_j^0 - ik_0 \chi^{i0j} E_j^0 \tag{34}
\]

To first order the amplitude is constant, \( \partial E_i^0 = 0 \). The corresponding real part gives the relation

\[
0 = ((\omega^2 - k^2) \delta_{ij} + k_i k_j + 2\chi^{i0j} k_k k_0) E_j^0. \tag{35}
\]
The next order is related to the derivatives of the amplitude and gives, due to the real and imaginary part, two equations

\[
0 = -2\omega \dot{E}_i^0 - 2k \cdot \nabla E_i^0 + k_j \partial_i E_j^0 + k_i \partial_j E_j^0 + 2\chi^{i\mu j} (k_\mu \partial_i E_j^0 + k_\nu \partial_j E_j^0) + 2\omega \chi^{0i} E_j^0 - k_k \chi^{kj} E_j^0
\]

\[
0 = \dot{E}_i^0 - \Delta E_i^0 + \partial_i \partial_j E_j^0 + 2\chi^{i\mu j} \partial_j \partial_k E_j^0 .
\]

B. The dispersion relation

The existence of a solution \( E_j^0 \) for Eq. \((35)\) requires that the determinant of the coefficient matrix vanishes

\[
0 = \det ((\omega^2 - k^2)\delta_{ij} + k_i k_j + 2\chi^{i\mu j} k_\mu k_\nu) ,
\]

which establishes a relation between the frequency \( \omega \) and the wave vector \( k \). To first order in the modifications, the frequency is given by

\[
\omega = \left( 1 + \rho(k) \pm \sqrt{\sigma^2(k) - \rho^2(k)} \right) |k|
\]

with

\[
\rho = \frac{1}{2} (\eta_{\sigma\sigma} \chi^{\mu\sigma\nu} - \chi^{\mu\nu0i} n_i) n_\mu n_\nu
\]

\[
\sigma^2 = \frac{1}{2} \left( \chi^{\mu0\nu} \chi^{\rho0\sigma} n_i n_j + 2\chi^{\mu0\nu} \chi^{\nu0\sigma} n_j + \chi^{\mu\nu0} \chi^{\rho\sigma} n_\mu n_\nu - \chi^{0\nu0} \chi^{0\rho0} n_\mu n_\nu \right) n_\mu n_\nu n_\rho n_\sigma ,
\]

with \( n_\mu = k_\mu / \omega = (1, k/|k|) \). This generalizes the results in [? ?]. As mentioned in [? ], the velocity of light is, to leading order in the anomalous terms, given by \( v = 1 + \rho(k) \pm \sqrt{\sigma^2 - \rho^2} \).

It is possible to simplify further the above quantities \( \rho \) and \( \sigma \). Inserting the irreducible decomposition, we get for \( \rho \)

\[
\rho = -\frac{1}{2} \Phi^{\mu\nu} n_\mu n_\nu .
\]

The \( \sigma \)-term can be considerably simplified by using the condition \( \chi^{(\alpha\beta\mu)\nu} = 0 \) from the uniqueness of the Cauchy problem. For doing so, we complete the summation over \( i, j \), etc. to a summation over all indices, e.g., \( \chi^{\mu\nu0} n_i n_\mu n_\nu = \chi^{\alpha\nu0} n_\alpha n_\nu n_\nu - \chi^{0\nu0} n_\mu n_\nu = -\chi^{0\nu0} n_\mu n_\nu . \) Finally we get

\[
\sigma^2 = \frac{1}{2} \eta_{\alpha\gamma} \eta_{\beta\delta} \chi^{\alpha\beta\mu} \chi^{\rho\sigma} n_\mu n_\nu n_\rho n_\sigma ,
\]

where we set \( \eta^{\mu\nu} n_\mu n_\nu = 0 \) since this produces higher order corrections only.

C. No birefringence

Kostelecký and Mewes [? ?] analyzed very carefully the light from distant galaxies and inferred that to very high precision there is no birefringence. That means \( 0 = \sigma^2 - \rho^2 \), that is

\[
0 = -\frac{1}{2} \left( \chi^{\alpha\mu\nu} \chi^{\beta\rho\sigma} - \frac{1}{2} \Phi^{\mu\nu} \Phi^{\rho\sigma} \right) n_\mu n_\nu n_\rho n_\sigma .
\]

This means that the totally symmetric part of \( \chi^{\alpha\mu\nu} \chi^{\beta\rho\sigma} - \frac{1}{2} \Phi^{\mu\nu} \Phi^{\rho\sigma} \) has to be proportional to the unperturbed metric \( \eta^{\mu\nu} \):

\[
\chi^{(\alpha\mu\nu} \chi^{\beta\rho\sigma)} - \frac{1}{2} \Phi^{(\mu\nu} \Phi^{\rho\sigma)} = \eta^{(\mu\nu} \rho\sigma)}
\]

where \( \mu \) is some symmetric tensor. Using again the decomposition \((31)\) we get after some lengthy calculations

\[
(1) \Phi^{\mu\nu\rho\sigma} = 0 , \quad \Phi^{\mu\nu} = 0 .
\]
With this result, the constitutive tensor reduces to
\[
\chi^{\alpha\beta\mu\nu} = -\eta^{\mu[\alpha} \Phi^{\beta]\nu} + \eta^{\nu[\alpha} \Phi^{\beta]\mu} - \frac{1}{2} \eta^{\mu[\alpha} Z^{\nu]\beta} + \frac{3}{2} \eta^{\beta[\mu} Z^\nu\alpha] + \frac{1}{2} \eta^{\alpha\beta} Z^{\mu\nu} - \frac{1}{6} W \eta^{\alpha[\mu} \eta^{\beta]_\nu} \tag{47}
\]
and the Maxwell equations are
\[
4\pi j^\alpha = \partial_\nu F^{\mu\nu} - 2\eta^{\mu[\alpha} \Phi^{\beta]_\nu} \partial_\beta F_{\mu\nu} \]
\[
= -\frac{1}{2} \eta^{\mu\beta} Z^{\nu\alpha} \partial_\alpha F_{\mu\nu} + \frac{3}{2} \eta^{\beta\mu} Z^{\nu\alpha} \partial_\beta F_{\mu\nu} + \frac{1}{2} \eta^{\alpha\beta} Z^{\mu\nu} - \frac{1}{6} W \eta^{\alpha[\mu} \eta^{\beta]_\nu} \partial_\beta F_{\mu\nu} . \tag{48}
\]
These are the most general Maxwell equations which do not lead to birefringence.

\[\text{D. Isotropy of speed of light}\]

If we add experiments on the isotropy of light propagation, that is, Michelson–Morley type experiments either based on interferometers or cavities (see, e.g., [?]), then we have access to the additional tensor \( \Phi^{\mu\nu} \) only: If the speed of light is required to be isotropic, then we get the condition
\[
\Phi^{\mu\nu} = 0 \tag{49}
\]
leading to the constitutive tensor
\[
\chi^{\alpha\beta\mu\nu} = -\frac{1}{2} \eta^{\alpha[\mu} Z^{\nu]\beta} + \frac{3}{2} \eta^{\beta[\mu} Z^\nu\alpha] + \frac{1}{2} \eta^{\alpha\beta} Z^{\mu\nu} - \frac{1}{6} W \eta^{\alpha[\mu} \eta^{\beta]_\nu} \tag{50}
\]
and the Maxwell equations
\[
4\pi j^\alpha = (1 - \frac{1}{6} W) \partial_\nu F^{\mu\nu} - \frac{1}{2} \eta^{\mu\beta} Z^{\nu\alpha} \partial_\alpha F_{\mu\nu} + \frac{3}{2} \eta^{\beta\mu} Z^{\nu\alpha} \partial_\beta F_{\mu\nu} + \frac{1}{2} \eta^{\alpha\beta} Z^{\mu\nu} - \frac{1}{6} W \eta^{\alpha[\mu} \eta^{\beta]_\nu} \partial_\beta F_{\mu\nu} . \tag{51}
\]
That means, vanishing birefringence and isotropy of light propagation is not enough to establish the Lorentz invariance of the theory – in contrast to the Lagrangian based SME where the constitutive tensor is of ordinary form only. Our frame is much more general than a Lagrangian based theory. Since the \( Z^{\mu\nu} \) cannot be probed by radiation phenomena we later analyze the field created by point charges and magnetic moments. But first we consider the propagation of the polarization of the radiation field.

\[\text{E. Propagation of polarization states}\]

In order to determine the propagation of the amplitude from the wave equation we reformulate \(33\) and get
\[
0 = \left( \delta^j_i - 2\chi^{i00j} - 2\chi^{i(k)j} n_k \right) E^0_j + \left( n_i \delta^j_i - \frac{1}{2} n_j \delta^i_i - \frac{1}{2} n_i \delta^j_i - 2\chi^{i(0)j} - 2\chi^{i(k)j} n_k \right) \partial_\nu E^0_j + (2\omega \chi^{0j} - k_k \chi^{kj}) E^0_j . \tag{52}
\]
We now apply the theorem from matrix theory [?] that for any matrix \( A \) there is a minor \( M_A \) so that \( M_A A = \det A \). (In the case that \( \det A \) possesses zeros of higher order, then \( M_A \) can be chosen so that on the right hand side there appears a first order zero only: \( M_A^A = \det' A \). In the unperturbed case \( \det A = \omega^2 (\omega^2 - k^2)^2 \), so that \( \det' A = \omega^2 (\omega^2 - k^2) \). In our case we have \( \det A = \omega^2 (\omega^2 - k^2) (\omega^2 + k^2) \) with first order zeros only. Differentiation of \( M_A A = \det A \) with respect to the wave vector gives
\[
\left( \frac{\partial (M_A)_{ij}}{\partial k_\mu} A^j_i + (M_A)_{ij} \frac{\partial A^j_i}{\partial k_\mu} \right) \delta^j_i = \nu^\mu \delta^j_i \tag{53}
\]
where \( \nu^\mu \) is the group velocity.

We apply this result to the amplitude. On–shell, the first term will vanish due to \( A^j_i E^0_j = 0 \). Therefore, we have on–shell
\[
(M_A)_{ij} \frac{\partial A^j_i}{\partial k_\mu} \delta^j_i = \nu^\mu \delta^j_i \partial_\mu E^0_j . \tag{54}
\]
In our case

\[
(M_A)_{ij} \partial A_i^j \partial \mu E_0^0 = (M_A')_{ij} \partial \partial \mu \left( \omega_2 - k^2 \right) \delta_i^j + k^l k_j - 2 \chi^{(\sigma \tau)} k_i k_\sigma \right) \partial_\mu E_0^0
\]

where on the right hand side the matrix in (12) shows up. Therefore, the amplitude precesses during the transport along the light rays

\[
v^\mu \partial_\mu E_0^0 = - \left( 2 \omega \chi^{0j} - k_k \chi^{ikj} \right) E_0^0.
\]

The polarization is sensitive only to the mass tensor \( \chi^{\mu \sigma} \), no additional information for the constitutive tensor can be deduced. The axion part of \( \chi^{\alpha \mu \nu} \) is an example leading to a non–vanishing evolution of \( E_1^0 \) which has been analyzed in [? ?].

In terms of the irreducible decompositions \( \epsilon \), we get

\[
v^\mu \partial_\mu E_0^0 = - \omega \left( 2 \chi^{(1)} \chi^{0j} + \epsilon^{0jk} a_k + \eta^{0(0j)} - n_k \chi^{(1)} \chi^{ikj} + \epsilon^{ikj} a_0 + \eta^{0(0j)} \right) E_0^0.
\]

To first order in the parameters encoded in \( \chi^{\mu \sigma} \), we get for the two characteristic projections

\[
n_i v^\mu \partial_\mu E_0^0 = - \omega \chi^{(1)} n_i E_0^0 + \epsilon^{0jk} n_k a_j - \left( \chi^{(1)} \chi^{ikj} n_i n_k E_0^0 - \frac{1}{2} \epsilon^j E_0^0 \right)
\]

and

\[
E_0^i v^\mu \partial_\mu E_0^0 = - \omega \left( 2 \left( \chi^{(1)} \chi^{0j} E_0^0 + \epsilon^{0jk} n_k E_0^0 \right) + \frac{1}{2} \epsilon^j (E_0^0)^2 \right) - \left( \chi^{ikj} n_k E_0^0 E_j^0 + \frac{1}{2} n_k n_j (E_0^0)^2 \right)
\]

where we used \( k_i E_i = \mathcal{O}(\chi) \). Each part can be isolated by varying independently the polarization \( E_0^0 \) and the direction of propagation. If we assume that no precession of the polarization will be observed (what is well confirmed by observations, see below), then \( \chi^{\mu \sigma} \) has to vanish. Since \( \chi^{\mu \sigma} \) is related to CNC, observations of the precession of the polarization of electromagnetic radiation can be used for testing the validity of charge conservation.

## VII. 3+1 DECOMPOSITION

For a further analysis of our generalized Maxwell equation we perform a 3+1 decomposition. With the two 3–vectors \( \zeta^i := \frac{3}{2} Z^0 \) and \( \tilde{\zeta}^i := \frac{3}{2} \epsilon_{ijk} Z^j \) we get from (51) in SI units

\[
\rho = \frac{1}{\epsilon_0} \left( 1 - \frac{1}{2} W \right) \nabla \cdot E - \tilde{\zeta} \cdot (\nabla \times E) - c \tilde{\zeta} \cdot (\nabla \times B) - \frac{1}{\epsilon_0} \nabla \cdot \left( \frac{1}{c} \nabla \cdot E \right) - B \cdot \nabla \times \left( \frac{1}{c} \nabla \phi \right) - \frac{1}{c} \zeta (\nabla \times E),
\]

\[
\mu_0 j = \frac{1}{c} \left( 1 - \frac{1}{2} W \right) E - \frac{1}{c} \zeta \times E - \tilde{\zeta} \cdot \nabla B + \left( 1 - \frac{1}{2} W \right) \nabla \times B - \frac{1}{c} \nabla (\zeta \cdot E) + \frac{1}{c} \zeta (\nabla \cdot E),
\]

where \( c^2 = 1/(\epsilon_0 \mu_0) \). This kind of violation of Lorentz invariance in the Maxwell theory encoded in the two parameters \( \zeta \) and \( \tilde{\zeta} \) has not been treated hitherto. It can be shown explicitly that this is compatible with all the requirements stated until now. The remaining Lorentz–invariance violating terms \( \zeta \) and \( \tilde{\zeta} \) can be probed by studying the fields of point charges and magnetic moments only. These coefficients cannot be probed by radiation phenomena. Here we restrict ourselves to the principal part of the Maxwell equations only. Effects due to a particular choice for the mass term have been discussed in [? ?] where an anisotropic speed of light and CNC has been derived; similar effects will occur for the general case.

### A. Solution for a point charge

The generalized Maxwell equations for a point charge at the origin are given by (56) with \( \rho = q \delta(r) \) and \( j = 0 \). Since we have a static problem, we neglect the time derivatives. We furthermore chose \( E = \nabla \phi \) and \( B = \nabla \times A \) and the gauge \( \nabla \cdot A = 0 \). Then the generalized Maxwell equations are

\[
\frac{q}{\epsilon_0} \delta(r) = - \left( 1 - \frac{1}{2} W \right) \Delta \phi + c \zeta \cdot \Delta A
\]

and

\[
0 = - (\tilde{\zeta} \cdot \nabla) \nabla \times A - \left( 1 - \frac{1}{2} W \right) \Delta A - \frac{1}{c} \nabla \nabla \cdot \nabla \phi + \frac{1}{c} \zeta \Delta \phi.
\]
FIG. 3: The magnetic field of a point charge located at the origin. The vector field $\zeta$ is assumed to point in the $z$–direction.

The solution should be of the form

$$\phi = \frac{1}{4\pi \varepsilon_0} \frac{q}{r} + \delta \phi, \quad A = \delta A,$$

(64)

where $\delta \phi$ and $\delta A$ are small quantities, at most of the order of $\zeta$. Therefore, modifications of the static electrical potential will be of second order only, $\delta \phi = O(\zeta^2)$. However, inserting the unperturbed solution $\phi = q/(4\pi \varepsilon_0 r)$ into the second equation (63), we get to first order in the perturbations

$$\Delta A = \frac{1}{c} \frac{q}{4\pi \varepsilon_0 c r} \zeta \Delta \phi = -\frac{1}{c} \frac{q}{\varepsilon_0} \frac{\delta}{\delta} \frac{\zeta}{r} (r)$$

(65)

with the solution

$$A = \frac{q \zeta}{4\pi \varepsilon_0 c r}.$$  

(66)

This gives a magnetic field

$$B = \frac{q}{4\pi \varepsilon_0 c} \frac{\zeta \times r}{r^3}.$$  

(67)

Therefore, our model includes the feature that a point charge also creates a magnetic field. This field is different from a field of a magnetic moment. If the point charge is at the origin, and if we take the coordinate system such that $\zeta$ points in $e_z$ direction, then the magnetic field lines are circles in the $x$–$y$–plane, similar to the magnetic field lines around a wire, see Fig. 3. The strength, however, varies with $1/r^2$ where $r$ is the distance from the origin.

If we take a charged line with line–density $\lambda$ in direction $n$, then the magnetic field is

$$B = -\frac{\lambda}{2\pi \varepsilon_0 c} \frac{\zeta \cdot n}{\rho} e_\varphi - \frac{\lambda}{2\pi \varepsilon_0 c} \frac{\zeta \times e_\rho}{\rho} \cdot n,$$

(68)

where $\rho$ is the distance from the charged line, $e_\rho$ the radial unit vector orthogonal to $n$, and $e_\varphi$ the unit tangent vector of a circle around that line. For $n \sim \zeta$ the magnetic field is

$$B = -\frac{\lambda}{2\pi \varepsilon_0 c} \frac{e_\varphi}{\rho},$$

(69)
This has the form of the magnetic field of a wire carrying the current \(-\lambda \zeta/(2\pi c_0 c)\). For \(n \perp \zeta\)

\[
B = -\frac{\lambda}{2\pi c_0 c} \frac{n \cdot (\zeta \times e_\rho)}{\rho} n.
\]

(70)

### B. Solution for a magnetic moment

If the source of the Maxwell equations is a magnetic moment \(m\) localized at the origin, then the Maxwell equations are (60,61) with \(\rho = 0\) and \(j = m \times \nabla \delta(r)\). We assume again a static situation

\[
0 = -\left(1 - \frac{1}{\pi} W\right) \Delta \phi + c \zeta \cdot \Delta A
\]

(71)

\[
\mu_0 m \times \nabla \delta(r) = -(\zeta \cdot \nabla) \nabla \times A - \left(1 - \frac{1}{\pi} W\right) \Delta A - \frac{1}{c} \nabla (\zeta \cdot \nabla \phi) + \frac{1}{c} \zeta \Delta \phi
\]

(72)

and make the ansatz

\[
A = \frac{\mu_0}{4\pi} \frac{m \times r}{r^3} + \delta A, \quad \phi = \delta \phi.
\]

(73)

Similar to the previous case, insertion of the unperturbed solution for \(A\) into the first equation leads, in first order of the perturbations, to

\[
\Delta \phi = c \zeta \cdot (\mu_0 m \times \nabla \delta(r)) = \mu_0 c (\zeta \times m) \nabla \delta(r).
\]

(74)

This is the equation for an electrical dipole with dipole moment \(d = \mu_0 c_0 \zeta \times m\). The solution is

\[
\phi = \frac{\mu_0 c (\zeta \times m) \cdot r}{4\pi} r^3.
\]

(75)

Therefore, a magnetic moment also creates an electric field. This feature is "dual" to the previous case.

The parameter \(\zeta\) gives rise to small deviations from the unperturbed quantities only, that is, it induces a small additional term \(\delta \phi\) for a given \(\phi = q/(4\pi c_0 r)\) and a small additional \(\delta A\) for a large \(A = \mu_0 m \times r/(4\pi r^3)\). This cannot be measured such precisely. On the contrary, measurements of the parameter \(\zeta\) always amount to a null-test and are, thus, much more precise.

### VIII. CONFRONTATION WITH EXPERIMENT

Using astrophysical observations and laboratory experiments, the possibility of birefringence and an anisotropy of the velocity of light has been estimated to very hight precision. Kostelecký and Mewes gave an upper limit of \(|\phi| = 10^{-42}\) (\[\]\)). The today’s most precise Michelson–Morely experiment by Müller and coworkers (\[\]\)) using optical resonators restricted a possible anisotropy to \(|\phi| \leq 10^{-15}\). Here, all estimates are valid for each component in the actual laboratory frame. See (\[\]\) for a refined description of that experiments.

A possible precession of the polarization has also been estimated from astrophysical observations to very high precision. In a model with a totally antisymmetric \(\chi^{\mu\rho}\) Carroll, Field and Jackiw analyzed the polarization from distant galaxies (\[\]\) and obtained, since no precession of the polarization has been found, the estimate \(\chi^{\mu\rho} \leq 10^{-42}\) GeV which is equivalent to \(|\chi^{\mu\rho}| \leq 3 \cdot 10^{-17}\) s\(^{-1}\). Since the other irreducible parts of \(\chi^{\mu\rho}\) lead to a precession of the polarization, too, we extend this result to the other parts: \(|\chi^{\mu\rho}| \leq 3 \cdot 10^{-17}\) s\(^{-1}\). This in particular also means that the charge is conserved to that order: \(|\hat{Q}/Q| \leq 3 \cdot 10^{-17}\) s\(^{-1}\). Since this result is not connected with the choice of a time dependence of any other "constant", it represents a clear and dedicated statement about charge conservation. This result on the conservation of the electric charge is one order better than what one gets from tests of the time–dependence of the fine structure constant (\[\]\) by assuming a constant \(c\) and \(h\).

As far as the new Lorentz invariance violating parameters \(\zeta\) and \(\zeta\) are concerned, we are, unfortunately, not aware of any dedicated experiment searching for, e.g., a magnetic field which is created by a point charge. In order to get some feeling for the accuracy of the validity of the ordinary Maxwell equations, that is for \(\zeta = 0\) and \(\zeta = 0\), we discuss the accuracy of some possible high precision measurements of magnetic fields. Magnetic field can be measured with the help of SQUIDs (measurements based on the Hall effect are not such precise).

With SQUIDs weak magnetic fields of down to \(10^{-14}\) T can be measured. We assume that even a dedicated search for a magnetic field from a point charge does not lead to any magnetic field larger than the SQUID sensitivity. Then,
from $|\lambda \zeta/(2\pi \epsilon_0 e \rho)| \leq 10^{-14}$ T for a line charge density $\lambda = 0.01$ C/m at a distance of 1 cm, we get the estimate $|\zeta| \leq 2.7 \cdot 10^{-17}$. However, this is just the estimate which would result if such a kind of experiments yields a null–result; a dedicated experiment of this kind has not yet been carried through.

Another method to search for this kind of effects is to use atomic spectroscopy. Since the charge of the proton leads to a magnetic field, a hyperfine splitting, additional to the usual one, should occur. Due to the different radial structure of the magnetic field, the result also should be different from the ordinary hyperfine splitting. With obvious notations, we get for the interaction Hamiltonian of an electron in the magnetic field $B$ of the nucleus

$$H_{\zeta} = \mu_1 \cdot \frac{q \zeta \times r}{4\pi \epsilon_0 c r^3}.$$  \hspace{1cm} (76)

If we choose the $z$–axis in direction of $\zeta$, then the corresponding energy shift $\Delta E_{nlm} = \langle \psi_{nlm} | H_{\zeta} | \psi_{nlm} \rangle$ is

$$\Delta E_{nlm} = -\frac{q \lambda \zeta \mu_x}{4\pi \epsilon_0 c} \int \psi_{nlm}^* \cos \vartheta \psi_{nlm} dr \sin \vartheta d\vartheta d\varphi.$$ \hspace{1cm} (77)

This does not vanish for, e.g., $\psi_{210} = R_{21} Y_{10}$ where $R_{21} = \frac{1}{\sqrt{2} \sqrt{2\pi a}} a^{-15/2} e^{-r/(2a)}$, $Y_{10} = \sqrt{\frac{3}{2\pi}} \cos \vartheta$ where $a$ in the Bohr radius (contrary to the ordinary hyperfine splitting, there is no shift for the $s$ states). In this case we get

$$\Delta E_{210} = -\frac{q \lambda \zeta \mu_x}{48\pi \epsilon_0 c a^2}.$$ \hspace{1cm} (78)

With $\mu_z = e \hbar / m_e$ this yields $\Delta E_{210} = \zeta \cdot 1.8 \cdot 10^{-2}$ eV. The state of the art of high precision measurements of energy levels is of the order $\Delta E/E \approx 10^{-15}$. Since the measured energy levels are still well described within the standard theory one gets for energies of about 10 eV at best an estimate $|\zeta| \leq 10^{-14}$ which, however, is not as good as a direct measurement discussed above might yield.

\section{IX. CONCLUSION}

We discussed the most general model for the dynamics of the electromagnetic field which is linear and of first order in the derivative of the field strength. This is tantamount to the most general ansatz linear in the field strength leading to violation of Lorentz invariance in the Maxwell theory. It is shown that the condition of dynamical consistency is responsible for the charge conservation of the principal part of the differential equations: CNC can be induced only by an additional term without derivative in generalized Maxwell equations. Our model cannot be derived from a Lagrangian. Due to this very general and systematic approach we were able to identify CNC terms and, furthermore, Lorentz violating terms encoded in the constitutive tensor which are beyond the SME. Beside the usual tests of the time–dependence of the fine structure constant (which interpretation depends on assumptions on the time–dependence of the velocity of light and the Planck’s constant), the CNC parameters can be tested through its effects on the polarization of electromagnetic radiation. Using a previous analysis of astrophysical observations on the polarization of the light of distant galaxies, charge conservation can be confirmed at a slightly better level than from laboratory $\alpha$–experiments.

If we describe all experiments related to CNC in terms of the parameters $\chi^{\mu \nu}$ and $t^\mu$, then the results from electron disappearing experiments, from searches for a time–dependence of the fine structure constant and from astrophysical observations on a precession of the polarization of light from distant galaxies can be compared, see Table I. Being a discrete process, the electron disappearing in principle may have a different cause than a continuous variation of the elementary charge. Therefore, the first experiment in Table I has a slightly different status than the other two.

The Lorentz invariance violating terms in the constitutive tensor which are beyond the SME cannot be probed by radiation phenomena. This means, optical experiments (in particular Michelson–Morley experiments together
with observations on birefringence) are not sufficient to prove or to establish uniquely the Lorentz invariance of the theory. These additional Lorentz violating terms lead to the effects that an electrical point charge, beside the ordinary Coulomb potential, also creates a magnetic field and a magnetic moment also an electric field. A rough discussion of what might be expected by carrying through dedicated experiments using current technology leads to estimates of $|\zeta| \leq 2 \cdot 10^{-17}$. Therefore experiments which explore the fields of point charges and point–like magnetic moments are needed and we strongly suggest experimentalists to carry out such experiments.

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**APPENDIX A: DECOMPOSITION OF GENERALIZED CONSTITUTIVE TENSOR**

We shortly describe the irreducible decomposition of generalized constitutive tensor $\chi^{\alpha\beta\mu\nu} = \chi^{\alpha\beta[\mu\nu]}$ after Hehl et al. [?]. First we split the tensor into its symmetric and antisymmetric pieces

$$\chi^{\alpha\beta\mu\nu} = W^{\alpha\beta\mu\nu} + Z^{\alpha\beta\mu\nu} \quad (A1)$$

with

$$W^{\alpha\beta\mu\nu} := \chi^{[\alpha\beta][\mu\nu]} \quad Z^{\alpha\beta\mu\nu} := \chi^{(\alpha\beta)\mu\nu}. \quad (A2)$$

Here, $R$ possesses 36 independent components and $Z$ 60 components. In the following, indices are raised and lowered with the metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. We define

$$W := \chi_{\mu\nu}^{\mu\nu} \quad (A3)$$

$$W^\alpha_\mu := W^{\alpha\beta}_\beta\mu \quad (A4)$$

$$X^{\alpha}_\mu := -\frac{1}{6} \epsilon^{\rho\sigma} W^{\alpha\rho\sigma}_\mu \quad (A5)$$

$$X := -\frac{1}{6} \epsilon^{\mu\rho\sigma} W_{\mu\rho\sigma} \quad (A6)$$

$$\Psi_{\alpha\mu} := X_{\alpha\mu} - \frac{1}{4} \eta_{\alpha\mu} X - X_{[\alpha\mu]} \quad (A7)$$

$$\Phi_{\alpha\mu} := W_{\alpha\mu} - \frac{1}{4} \eta_{\alpha\mu} W - W_{[\alpha\mu]} \quad (A8)$$

$$Z^{\mu\nu} := \eta_{\alpha\beta} Z^{\alpha\beta\mu\nu} \quad (A9)$$

$$Z^{\alpha\beta\mu\nu}_{\text{tracefree}} := Z^{\alpha\beta\mu\nu} - \frac{1}{4} \eta_{\alpha\beta} Z^{\mu\nu} \quad (A10)$$

$$Z^{\alpha\mu}_{\text{tracefree}} := \eta_{\beta\nu} Z^{\alpha\beta\mu\nu}_{\text{tracefree}} \quad (A11)$$

$$\Delta^{\mu\nu} := Z^{\mu\nu}_{\text{tracefree}} \quad (A12)$$

$$Y^{\alpha\mu} := \frac{1}{6} \epsilon^{\rho\sigma\tau} Z^{\alpha\rho\sigma\tau}_{\text{tracefree}} \quad (A13)$$

$$\Xi^{\alpha\mu} := Z^{\alpha\mu}_{\text{tracefree}} - Z^{[\alpha\mu]}_{\text{tracefree}} \quad (A14)$$

$$\Upsilon^{\alpha\mu} := Y^{\alpha\mu} - Y^{[\alpha\mu]} \quad (A15)$$

These tensors have the following interpretation
\[ W_{\alpha\beta\mu\nu} \quad \text{Weyl tensor} \]

\[ W \quad \text{scalar of antisymmetric part} \]

\[ W^{\alpha}_{\mu} \quad \text{Ricci–tensor} \]

\[ X^{\alpha}_{\mu} \quad \text{trace of the right–dual } \epsilon^{\mu\rho\sigma} W_{\alpha\beta\rho\sigma} \]

\[ X \quad \text{pseudoscalar} \]

\[ \Psi^{\alpha}_{\mu} \quad \text{symmetric tracefree part of trace of right–dual} \]

\[ \Phi^{\alpha}_{\mu} \quad \text{symmetric tracefree part of Ricci–tensor} \]

\[ Z_{\alpha\beta\mu\nu} \quad \text{Ricci tensor of symmetric tracefree part, is tracefree} \]

\[ Z_{\alpha\mu} \quad \text{genuine trace of symmetric part (trace with respect to first two indices)} \]

\[ Y^{\alpha}_{\mu} \quad \text{trace of right–dual, is tracefree} \]

\[ \Xi^{\alpha}_{\mu} \quad \text{symmetric part of } Y^{\alpha}_{\mu} \text{, tracefree} \]

\[ \Delta^{\mu\nu} \quad \text{antisymmetric part of Ricci tensor of symmetric tracefree part} \]

The irreducible decomposition of \( \chi^{\alpha\beta\mu\nu} \) is then given by

\[ \chi^{\alpha\beta\mu\nu} = \sum_{i=1}^{6} (i) \, W^{(i)}_{\alpha\beta\mu\nu} + \sum_{i=1}^{5} (i) \, Z^{(i)}_{\alpha\beta\mu\nu} \] (A16)

with

\[ (1) \, W^{\alpha\beta\mu\nu} = W^{\alpha\beta\mu\nu} - \sum_{i=2}^{6} (i) \, W^{(i)}_{\alpha\beta\mu\nu} \] (A17)

\[ (2) \, W^{\alpha\beta\mu\nu} = \frac{1}{4} \epsilon^{\mu\rho\sigma} (\eta^{\alpha\rho} \Psi^{\beta\sigma} - \eta^{\beta\rho} \Psi^{\alpha\sigma}) \] (A18)

\[ (3) \, W^{\alpha\beta\mu\nu} = \frac{1}{12} X^{\alpha\beta\mu\nu} \] (A19)

\[ (4) \, W^{\alpha\beta\mu\nu} = -2 \eta^{[\alpha} \Phi^{\beta]\mu} + \eta^{[\rho} \Phi^{\beta]\mu} \] (A20)

\[ (5) \, W^{\alpha\beta\mu\nu} = \frac{1}{2} \left( \eta^{\alpha\mu} W_{\alpha}^{\beta\nu} - \eta^{\alpha\nu} W_{\alpha}^{\beta\mu} - \eta^{\beta\mu} W_{\alpha}^{\alpha\nu} + \eta^{\beta\nu} W_{\alpha}^{\alpha\mu} \right) \] (A21)

\[ (6) \, W^{\alpha\beta\mu\nu} = -\frac{1}{6} W^{[\alpha[\mu} \eta^{\beta]\nu]} \] (A22)

\[ (1) \, Z^{\alpha\beta\mu\nu} = Z^{\alpha\beta\mu\nu} - \sum_{i=2}^{5} (i) \, Z^{(i)}_{\alpha\beta\mu\nu} \] (A23)

\[ (2) \, Z^{\alpha\beta\mu\nu} = \frac{1}{4} \epsilon^{\mu\rho\sigma} (\eta^{\alpha\rho} \Psi^{\beta\sigma} + \eta^{\beta\rho} \Psi^{\alpha\sigma}) \] (A24)

\[ (3) \, Z^{\alpha\beta\mu\nu} = \frac{1}{3} \left( \eta^{\alpha\mu} \Delta^{\beta\nu} - \eta^{\alpha\nu} \Delta^{\beta\mu} + \eta^{\beta\mu} \Delta^{\alpha\nu} - \eta^{\beta\nu} \Delta^{\alpha\mu} - \eta^{\alpha\beta} \Delta^{\mu\nu} \right) \] (A25)

\[ (4) \, Z^{\alpha\beta\mu\nu} = \frac{1}{4} \eta^{\alpha\beta} Z^{\mu\nu} \] (A26)

\[ (5) \, Z^{\alpha\beta\mu\nu} = \frac{1}{4} \left( \eta^{\alpha\mu} \Xi^{\beta\nu} - \eta^{\alpha\nu} \Xi^{\beta\mu} + \eta^{\beta\mu} \Xi^{\alpha\nu} - \eta^{\beta\nu} \Xi^{\alpha\mu} \right) \] (A27)

where we defined \( W^{\alpha\mu}_{\nu} = W^{[\alpha\mu]} \) as the antisymmetric part of the Ricci tensor. From this the decomposition follows.

[1] In a Lagrangian formulation such a term comes from the totally antisymmetric part of the constitutive tensor which then, however, has to be position dependent: Indeed, taking as part of the Lagrangian \( \theta F_{\mu\nu} \epsilon^{\mu\rho\sigma\nu} F_{\rho\sigma} \), then this leads to a term \( \partial_{\nu} \theta^{\alpha\rho\sigma\nu} F_{\rho\sigma} \) in the inhomogeneous Maxwell equations. This is the axion as introduced by Ni [7] establishing a counterexample to Schiff’s conjecture. On the level of the field equations the axion is not part of the constitutive tensor but, instead, part of the "mass" tensor.

[2] For negative charges, this line charge density is the principal limit for a wire of 1 mm diameter when taking into account that field emission starts at approx. \( 10^{11} \) V/m at the surface of the wire. Also the possibilities to create a sufficient high voltages limits the charge line density. Therefore we extend this limit to positive charges, too.