HYPERBOLIC GEOMETRY AND DISTANCE FUNCTIONS ON DISCRETE GROUPS

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February 2008
Acknowledgements

Professor Michael Cowling has been a mentor for a number of years. He suggested the specific direction taken in this work, always answered my questions clearly, and spent much time away from our meetings preparing material. Professor Cowling’s thorough editing of the entire thesis is much appreciated.

Dr James Franklin’s sincere commitment to this project, and his willingness to challenge my thinking, have been extremely valuable. As well as supervising the historical part of this work, Dr Franklin read the mathematical chapters, and helped with the preparation of my seminar. His suggestions led to important improvements.

I would like to thank Dr Ian Doust, our honours coordinator, for always being available to provide both clear guidance on the preparation of this project, and reassurance. The School of Mathematics, as a whole, has been an exciting and supportive environment in which to study.

I could not have completed this thesis without help from my friends. In particular, I would like to thank David for discussions which clarified several mathematical issues, and Claire, Edwina, Lisa, Shaun and Stephanie for, at the appropriate times, giving me a break from my work, and leaving me alone to get on with it.

My family have been patient and supportive throughout my honours year. I am especially grateful for the way they have cared for me over the last few months.

Anne Thomas, 12 June 2002.
Introduction

This thesis reflects my interest in the history of mathematics, and in areas of mathematics which combine geometry and algebra. The historical and mathematical sections of this work can be read independently. One of the main themes of the history, though, is how in the nineteenth century a new understanding of mathematical space developed, in which groups became vital to geometry. The idea of using algebraic concepts to investigate geometric structures is present throughout the work.

Chapter 1 is a short history of non-Euclidean geometry. This chapter is a synthesis of my readings of mainly secondary sources. Historians of mathematics have traditionally adopted an ‘internalist’ approach, explaining changes to the content of mathematics only in terms of factors such as the imperative to generalise, or the desire to remove inconsistencies. An internalist narrative is certainly necessary for a satisfactory history of mathematics, but it is not sufficient. First, the history of mathematics is part of broader intellectual history. Euclidean geometry held great philosophical prestige, and non-Euclidean geometry challenged fundamental assumptions about the nature of space and the truth value of mathematics. Second, there are social factors in the history of mathematics. Mathematicians themselves are part of a community, the mathematical community. The structure of this community in the nineteenth century helps to explain the reception and dissemination of non-Euclidean geometry. The sources consulted for this chapter each adopted one or more of the approaches to the history of mathematics outlined here, but none of them seemed to discuss all the relevant issues.

Chapter 2 begins the strictly mathematical part of this thesis. One aim of this chapter is to provide a deeper understanding of some of the mathematics discussed in Chapter 1. To this end, each of the main models of hyperbolic geometry is presented, and important results for each model proved. The second aim of Chapter 2 is to describe thoroughly the way in which the action of the group $PSL(2, \mathbb{Z})$ induces a tessellation of the upper half-plane. Most of the material in Chapter 2 is selected and adapted from Chapters 3–6 of Ratcliffe [21].

Chapter 3 poses a question about the upper half-plane, and then provides the theory needed to frame the question precisely and answer it in a variety of settings. Suppose $z$ is a point in the upper half-plane, and $\gamma z$ the result of the action of an element $\gamma \in PSL(2, \mathbb{Z})$ on the point $z$. The geodesic segment joining $z$ and $\gamma z$ has a finite length, and crosses a finite number of tiles in the tessellation of the upper half-plane. The question is whether or not there is any relationship between the length of the geodesic segment, and the number of tiles it crosses. We begin by defining a symmetric space. Then, if $X$ is a symmetric space, and $\Gamma$ a discrete group of isometries of $X$, we define two distance functions on the group $\Gamma$. The first is the geometric distance function, and is induced by the distance function on the space.
The second distance function on $\Gamma$ is the word distance function. The word distance function is induced by the generators of $\Gamma$, and corresponds to the number of tiles in the tessellation crossed by a geodesic segment. We now ask whether or not these two distance functions are equivalent. We prove that, when the tiles of the induced tessellation are compact, the two distance functions are equivalent. Then, in order to generalise the action of $PSL(2, \mathbb{Z})$ on the upper half-plane, we describe a symmetric space of matrices on which the group $PSL(n, \mathbb{Z})$ acts.

Chapter 4 is devoted to the proof of Theorem 4.1.1. This theorem states that, when $n = 2$, the geometric and word distance functions on $PSL(n, \mathbb{Z})$ are not equivalent, but that for all $n \geq 3$, these two distances on $PSL(n, \mathbb{Z})$ are equivalent.

The content of Chapter 3 and Chapter 4 is based upon two papers by Lubotzky, Mozes and Raghunathan, \[13\] and \[14\]. These papers are not easy to read. There is sometimes ambiguity about the hypotheses on the space $X$ and the discrete group $\Gamma$ acting on $X$. We have, here, assumed $X$ to be a symmetric space. This assumption ensures that geometric distance functions and word distance functions exist, and that structures such as geodesics and fundamental domains behave nicely. The authors do not prove the ‘well-known’ result that the geometric and word distance functions are well-defined up to Lipschitz equivalence, so we fill this gap. They do not discuss the interesting geometric interpretation of the two distance functions, in terms of a geodesic crossing tiles in the space. Also, these papers inaccurately refer to what we have called the geometric distance function as a Riemannian metric. The more recent article, \[14\], was used only for the proof of Theorem 3.4.1. By assuming that $X$ is a symmetric space rather than a ‘coarse path metric space’, we have simplified the proof a little, as we may choose the sequence of points $\{p_i\}_{i=0}^m$ to lie on the geodesic. Besides, it is not obvious that a ‘coarse path metric space’ even contains a point with a trivial stabiliser subgroup in $\Gamma$. The proofs in Chapter 4 are based on those in the earlier paper, \[13\], which concerns Lipschitz equivalence for the group $PSL(d, \mathbb{Z})$. This paper does not describe the symmetric space on which $PSL(d, \mathbb{Z})$ acts, so we have added this material. Also, we have, throughout Chapter 4, rewritten statements of lemmas, propositions and so on, in order to clarify what each of the many constants depends upon. In the paper, for the case $d = 2$, the claim that the word length of $u^n$ grows linearly in $n$ is not proved. For the cases $d \geq 3$, there is no explicit statement of Theorem 4.4.1 and no indication of how to combine the results of Corollary 4.4.10 and Corollary 4.4.14 to prove the result of Theorem 4.4.1. In the proof of Proposition 4.4.4 results 4.4.5–4.4.8 are stated without proof. In Proposition 4.4.10 the construction suggested by the authors does not actually satisfy the statement of the proposition, and there is no proof of the claims of Lemmas 4.4.11 and 4.4.12. Finally, the way in which the result of Theorem 4.4.1 implies that the word distance function is bounded above by the geometric distance function is not proved.
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Chapter 1
A History of Non-Euclidean Geometry

I entreat you, leave the doctrine of parallel lines alone; you should fear it like a sensual passion; it will deprive you of health, leisure and peace—it will destroy all joy in your life.

from a letter by Farkas Bolyai to his son János[1]

I have now resolved to publish a work on parallels ... the goal is not yet reached, but I have made such wonderful discoveries that I have been almost overwhelmed by them ... I have created a new universe from nothing.

from a letter by János Bolyai to his father Farkas[2]

1.1 Introduction
This chapter is a short history of non-Euclidean geometry. The intellectual history of developments internal to mathematics is presented, together with discussion of relevant social and philosophical trends within the mathematical community and in wider contexts. Chapter 2 will provide a rigorous and modern treatment of many of the mathematical terms and results referred to here. In order to properly acknowledge sources, the referencing style adopted for Chapter 1 is different from that in the rest of this thesis.

The discovery and acceptance of non-Euclidean geometry forms a crucial strand of the history of mathematics. For thousands of years, Euclidean geometry held great prestige. However, there were small doubts about the parallel axiom. After numerous attempts to validate this axiom, in the nineteenth century there was a shift to advancing alternative geometries. The coming-together of developments in many areas of mathematics aided the acceptance of non-Euclidean geometry by the mathematical community. In this process mathematicians were forced to reconsider and alter their ideas about the nature of mathematics and its relationship with the real world.

1.2 Euclid’s Elements
Although Egyptian and Babylonian mathematicians had posed and solved geometric problems, geometry only acquired logical structure with the Greeks[3].

[1]Quoted in B. A. Rosenfeld. A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space. Springer-Verlag, 1988, p. 108.
[2]Quoted in Jeremy Gray. Ideas of Space: Euclidean, Non-Euclidean and Relativistic. Clarendon Press, 1989, p. 107.
[3]Carl B. Boyer. A History of Mathematics. John Wiley & Sons, Inc., 1991, p. 47.
geometers were the first to write proofs as deductive sequences of statements. Euclid’s thirteen-volume *Stoichia (Elements)*, which appeared in about 300 B.C.E., was an exposition of the fundamentals of classical Greek mathematics. The first six books covered plane geometry, and the last three solid geometry.

The *Elements* opens with a list of 23 definitions, followed by ten axioms. Propositions and their proofs fill the remainder of the work. This logical structure reflected the efforts by mathematicians of the classical period to establish geometry as a deductive system dependent on as few assumptions as possible.

Euclid’s fifth axiom, which is also known as the parallel axiom, states:

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

To understand this, suppose that the two straight lines are \( l \) and \( m \), and that the transversal falling on them is line \( t \). The axiom is saying that if the interior angles on the same side of \( t \) sum to less than \( \pi \), the three lines \( l, m \) and \( t \) form a triangle. This triangle has two vertices on the line \( t \). Its third vertex lies in the half-plane bounded by the line \( t \), and on the same side of \( t \) as the interior angles which sum to less than \( \pi \).

![Diagram of Euclid's parallel axiom](image)

The parallel axiom does not say anything about the possibility that the pairs of interior angles on both sides of the transversal sum to \( \pi \). With the word parallel defined as meaning ‘never intersecting’, Euclid proved, without using the fifth axiom, that if a transversal falling on two straight lines makes the interior angles on the same side add to \( \pi \), then the two lines are parallel. To prove the converse of this statement, Euclid did need to use the parallel axiom. The parallel axiom was also essential to the proof of the key theorem that the angle sum of a triangle is \( \pi \).

The *Elements* made little use of the idea of motion in definitions or methods. In the sixth century B.C.E., Pythagoreans had defined a line as the trace of a moving point, and a surface as the trace of a moving line. But Euclid largely conformed to the views of Aristotle (384–322 B.C.E.), who rejected motion in geometry because he considered mathematical objects to be abstractions of physical objects. So, in the *Elements*, a line is a “breadthless length” rather than the successive positions of a moving point.

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4. Roberto Torretti. *Philosophy of Geometry from Riemann to Poincaré*. D. Reidel Publishing Company, 1984, p. 2.

5. The ten axioms are sometimes called five postulates and five common notions; surviving manuscripts of the *Elements* are not consistent. See Boyer, *History of Mathematics*, p. 105.

6. Torretti, *Philosophy of Geometry*, p. 5.

7. Quoted in Boyer, *History of Mathematics*, p. 106.

8. Rosenfeld, *History of Non-Euclidean Geometry*, pp. 110–112.

9. Gray, *Ideas of Space*, p. 28.
of Propositions 4 and 8, but chose to prove Proposition 26, where he could also have used superposition, with a longer method instead. This suggests that he was unhappy with the method of superposition.10

Euclidean geometry contained many ambiguities and hidden assumptions. The opening definitions, such as “a surface is that which has length and breadth only”, served no logical purpose, since Euclid provided no prior set of undefined concepts in terms of which definitions could be framed.11 Euclid’s second and third axioms, 2. [It is possible] to produce a finite straight line continuously in a straight line,
3. [It is possible] to describe a circle with any centre and radius 12 embody the assumption that geometric space is infinite. Greek geometers also assumed that space was homogeneous, so that any construction could be performed at any point with the same results.13 The congruence of figures depends on this property of space. The existence of parallel lines was also taken for granted.14

Despite these logical weaknesses, the Greeks and many later thinkers considered the Elements to be the model for philosophical inquiry. After the fall of Rome, the survival and development of Greek mathematics, including Euclidean geometry, depended on Arabic-speaking mathematicians in Islamic Africa and Asia. In the twelfth century, Arabic translations of the Elements were translated in turn into Latin. The Elements were first printed in 1482, and versions in various European vernaculars appeared in the sixteenth century. Euclid’s Elements achieved huge influence in Europe, with more than one thousand editions published.15 Until about 1800, Euclidean geometry enjoyed secure philosophical and mathematical status.

1.3 Doubts about the parallel axiom

From early on there were, though, questions raised about the parallel axiom. A lot of basic geometry, such as the angle sum of a triangle, depended on it, so its soundness was very important. Yet the parallel axiom appeared much less self-evident than the others. Euclid’s formulation of the axiom seemed long-winded and complicated, and it was not clear that assuming the existence of infinite straight lines was reasonable.16 Mathematicians did not seriously doubt the truthfulness of the axiom until the nineteenth century; they just wanted to make sure that it was formally unassailable. Proclus (410–485 C.E.), for example, wrote that the parallel axiom

...ought to be struck from the axioms altogether. For it is a theorem — one that invites many questions.17

10Morris Kline. Mathematical Thought from Ancient to Modern Times. Oxford University Press, 1972, p. 60.
11Boyer, History of Mathematics, p. 105.
12Quoted in Kline, Mathematical Thought, p. 59.
13Gray, Ideas of Space, p. 26.
14Ibid, p. 29.
15Boyer, History of Mathematics, p. 119.
16Kline, Mathematical Thought, p. 863.
17Proclus. A Commentary on the First Book of Euclid’s Elements. Princeton University Press, 1969, p. 150.
Many mathematicians tried to settle uncertainty about the parallel axiom by proving it from the other nine axioms. They were all unsuccessful, as they relied on implicit and unsupported assumptions, some of them actually equivalent to the parallel axiom. In his ‘proof’, Ptolemy (d. 168 C.E.) made tacit assumptions about parallel lines. Proclus found the flaws in Ptolemy’s work, but then based his own efforts on a rather dubious Aristotelian axiom about the finiteness of the universe, which was inconsistent with Euclidean assumptions of the infinitude of space. Thabit ibn Qurra (836–901) wrote two treatises trying to prove the parallel axiom. In his first, he assumed that if two lines diverge on one side of a transversal, they converge on the other side. In the second, he used kinematic arguments, asserting that

If any solid is imagined to move as a whole in one direction with one simple and straight movement, then every point in it will have a straight movement and will thus draw a straight line on which it will pass.

Thus, the curve at a fixed distance from a straight line is itself straight. Ibn al-Haytham (965–1041) gave a false proof of this last statement, then used it to prove the parallel axiom. The first substantial European effort on parallels was that of John Wallis (1616–1703), who realised that many of Euclid’s proofs depended on unstated assumptions. He derived the parallel axiom by assuming the existence of triangles similar to a given triangle. Other European attempts included those of Giordano Vitale (1633–1711), and A. M. Legendre (1752–1833).

A different approach was to attempt to replace or reformulate the parallel axiom. Ptolemy, Proclus, and the Arabian editor of Euclid, Nazir-Eddin (1201–1274), as well as the Europeans Wallis, Joseph Fenn, John Playfair (1748–1819) and Legendre, all attempted substitute axioms. The version generally given in modern textbooks is due to Playfair: “Through a given point P not on a line l there exists only one line in the plane of P and l which does not meet l.” All these efforts, though, were unsatisfactory, relying on assertions about infinity which were no more self-evident than those in Euclid’s version.

A third strategy was to investigate the consequences of denying the parallel axiom. Aristotle connected negating the axiom with triangles having an angle sum greater than \(\pi\). The resulting geometry is actually spherical geometry. From a modern point of view, there are no parallels in spherical geometry, because all ‘lines’ (great circles) intersect. However, despite its antiquity, and its many applications in navigation, spherical geometry was never perceived as a challenge to Euclidean geometry. This was probably because spherical geometry was not considered to be a geometry of the plane. For example, in the first work in the area, the Sphaerica

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18 Kline, Mathematical Thought, pp. 863–864.
19 Rosenfeld, History of Non-Euclidean Geometry, pp. 49–56.
20 Quoted in ibid., p. 52.
21 Gray, Ideas of Space, p. 57.
22 Quoted in Kline, Mathematical Thought, p. 865.
23 Kline, Mathematical Thought, p. 866.
24 Gray, Ideas of Space, pp. 33–34.
25 Rosenfeld, History of Non-Euclidean Geometry, pp. 1–33.
26 Gray, Ideas of Space, p. 71.
of Theodosius (c. 20 B.C.E.), propositions were formulated mostly in terms of planes intersecting the sphere, rather than in terms of the sphere’s intrinsic geometry.

Many mathematicians who considered negating the parallel axiom found that their conclusions were counter-intuitive. Common sense and everyday experience favoured Euclidean geometry. Some abandoned their efforts. For instance, Omar Khayyam (1048?–1122) entertained the possibility of a quadrilateral with angle sum different from $2\pi$, but straight away dismissed the idea as inconsistent.

Girolamo Saccheri (1667–1733), though, investigated in some depth the geometry which resulted from denying the parallel axiom. He was the first to try to prove that a contradiction would follow from supposing the parallel axiom false. Saccheri began by establishing that $\angle ACD = \angle BDC$ in the following quadrilateral.

He then considered three hypotheses about angles $ACD$ and $BDC$: they could both be obtuse, or both right angles, or both acute. He showed that these three hypotheses implied that the angle sum of a triangle was respectively more than, equal to, or less than $\pi$. Implicitly assuming that straight lines could be extended indefinitely, and so ruling out spherical geometry, Saccheri then rejected the obtuse angle hypothesis, by showing that it implied the parallel axiom. This was a contradiction, since the parallel axiom implies that $ACD$ and $BDC$ are right angles. He then considered the acute angle hypothesis. Assuming that it held, Saccheri proved several theorems, and found that two straight lines might diverge in one direction and come asymptotically close together in the other, so that they had a common point at infinity. Although he found this idea repugnant, he only rejected the acute angle hypothesis after deriving a contradiction from an error he made in computing arc length.

Because there had been no rigorous invalidation of the results of negating the parallel axiom, a few mathematicians began to speculate that the alternative geometry might hold somewhere outside everyday experience.

J. H. Lambert (1728–1777) considered the same hypotheses as Saccheri, and like Saccheri obtained from the obtuse angle hypothesis a contradiction. In Euclidean geometry, he observed, there is an absolute measure of angles, but length varies according to the unit of length selected. Under the acute angle hypothesis Lambert found he could actually define an absolute unit of length, by associating one line segment with one angle. He also saw that the area of a triangle was proportional to its angle sum. Counter-intuitive though such consequences seemed, Lambert was unable to reject the acute angle hypothesis. He supposed that it might hold on some “imaginary sphere”, a concept he never explained clearly.

Ferdinand Karl Schweikart (1780–1859) and his nephew Franz Adolf Taurinus (1794–1874) were influenced by Saccheri and Lambert’s work. Assuming the angle sum of a triangle was not two right angles, they obtained what they called “astral

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27 Rosenfeld, *History of Non-Euclidean Geometry*, pp. 98–99.
28 Ibid., p. 99.
29 Ibid., p. 101.
geometry”, since it might occur in the region of the stars. Schweikart and Taurinus were the first to use analysis in non-Euclidean geometry, and this enabled progress past the classical approach of Saccheri and Lambert. Their analytic results included trigonometric formulae, and the area of a triangle with a vertex at infinity, in terms of a constant $K$. They derived their trigonometric formulae from those of spherical trigonometry, by replacing ordinary trigonometric functions with hyperbolic trigonometric functions. The radius of the sphere concerned was $K$.

Carl Friedrich Gauß (1777–1855) did a little work on non-Euclidean geometry at the same time as Schweikart and Taurinus. He never published anything in the area. According to his letters and rough notes, Gauß realised that rejecting the parallel axiom implied the existence of an absolute measure of length, and doubted whether Euclidean geometry could be proved.

1.4 Systems of non-Euclidean geometry

Nikolai Ivanovich Lobachevsky (1792–1856) and János Bolyai (1802–1860) were the first people to publish, without any reservations, research on a system of non-Euclidean geometry. They worked independently. Lobachevsky initially set out his ideas on what he called “imaginary geometry” in a presentation at the University of Kazan in 1826. (The city of Kazan lies on the Volga River in central Russia.) Lobachevsky published papers in Kazan journals in 1829 and 1835–37, and in 1837 a third paper appeared in *Journal für Mathematik*. Lobachevsky also published books on his theories, in 1840 and 1855. As for János Bolyai, until 1821 he agreed with his father Farkas that the parallel axiom must be true. Over the next two years, though, János investigated the alternative geometry, arriving at many of the same results as Lobachevsky. In 1832 János published his work on what he called “absolute geometry”, in an appendix to his father’s book on the theory of parallels.

In their systems, Lobachevsky and Bolyai rejected the parallel axiom but regarded all Euclidean propositions proved without using this axiom as still being valid. In Lobachevsky’s account, if $C$ is a point at a perpendicular distance $a$ from the line $AB$, there exists an angle $\pi(a)$ such that all lines through $C$ which make an angle less than $\pi(a)$ with the perpendicular $CD$ will intersect $AB$; all other lines through $C$ do not intersect $AB$. In the Euclidean sense of the word parallel, there are infinitely many parallels through $C$.

Other results of Lobachevsky and Bolyai included trigonometry, arc length $ds$, the area of a circle, and theorems on the area of plane regions and volumes of solids.

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30 Gray, *Ideas of Space*, p. 97.
31 Rosenfeld, *History of Non-Euclidean Geometry*, pp. 214–215.
32 Torretti, *Philosophy of Geometry*, p. 53.
33 For this reason I will refer to their geometry, in which the angle sum of a triangle is less than $\pi$, as hyperbolic geometry rather than using one or both of their names.
34 Torretti, *Philosophy of Geometry*, p. 40.
35 discussed in Kline, *Mathematical Thought*, pp. 874–875.
The founders of non-Euclidean geometry were not formalists, and did not consider their work to be an investigation of an axiomatic system with only one difference from that of Euclid. This is shown by their willingness to consider the possibility that their geometry was the geometry of real physical space. When Lobachevsky wrote ‘straight line’, he meant straight in the ordinary sense of the word, showing that he regarded his geometry as (at least potentially) real. Indeed, the discovery of these different geometries made it seem possible that Euclidean geometry might be proved inexact by experiment, at some point. Bolyai commented that if non-Euclidean geometry were real, a measurement would determine the size of the constant $K$, the radius of the imaginary sphere. Since Euclidean geometry had been used so successfully in physics, if non-Euclidean geometry were indeed the geometry of the real world, then any differences would only be detectable on a very large scale. Lobachevsky thus suggested using the parallax of stars to decide which geometry held. His measurements were inconclusive. According to some sources, Gauß measured distances between mountain peaks for the purpose of deciding between geometries. It seems more likely that Gauß realised that the distances involved were too small, and he actually made these measurements for his research in geodesy. This is certainly suggested by his writing about the constant $K$: “in the light of our astronomical experience, the constant must be enormously larger than the radius of the earth”.

1.5 Reception of non-Euclidean geometry

Until the 1860s, the European mathematical community did not regard investigations into non-Euclidean geometry as respectable, and non-Euclidean geometry was separate from the rest of mathematics. Only about 1870 was the work of Lobachevsky and Bolyai recognised. There were social reasons for this reluctant reception. As well, there were mathematical issues which had to be resolved, and a system of alternative geometry raised many philosophical controversies.

1.5.1 Social factors

During the nineteenth century mathematics became professionalised, and important mathematical communities developed at centres including Paris, Berlin and Göttingen. Bolyai, as a Hungarian army officer, was socially and geographically a long way from these mathematical hubs. Lobachevsky’s 1829 paper was published in a very obscure journal, got bad reviews and was regarded as incomprehensible. His article in the mainstream Journal für Mathematik was virtually impossible to read, and depended on results in the 1829 paper. Gauß, at Göttingen, approved of Lobachevsky’s 1829 paper and Bolyai’s 1832 article, but only praised them in letters to friends.

Because of Gauß’s prestige, when his correspondence was published in the early 1860s, these comments made a strong impression on the European mathematical community.

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36 Torretti, Philosophy of Geometry, p. 61.
37 Ibid., pp. 62–63.
38 Ibid., p. 254.
39 Ibid., p. 63.
40 For example, Kline, Mathematical Thought, pp. 872–873.
41 Gray, Ideas of Space, p. 123.
42 Quoted in Torretti, Philosophy of Geometry, p. 63.
community, and led to intensive discussion of new geometrical ideas. In 1865, Arthur Cayley (1821–1895) published a commentary on Lobachevsky, which did much to disseminate his work. In the years 1866 and 1867, French, German, Italian and Russian translations and publications of Lobachevsky’s work appeared, so that by 1870 Lobachevsky and Bolyai’s research was known to geometers all across Europe. From the 1890s, there were university courses and textbooks on non-Euclidean geometry, showing that it was well-accepted by the mathematical community.

1.5.2 Mathematical developments

Internal developments in mathematics were needed before non-Euclidean geometry could be recognised, because in the 1830s there were sound mathematical reasons for holding reservations about the new geometry. Neither Lobachevsky nor Bolyai proved that their geometry was consistent, that is, that it did not contain any contradictions. Lobachevsky was convinced that he had established consistency, since he had obtained his trigonometric formulae from those of spherical trigonometry by multiplying the sides of the triangle by the complex number \( i \). His argument was in fact insufficient. It only showed that his trigonometric formulae followed from the assumptions he had made in setting up hyperbolic geometry.

Models of hyperbolic geometry were critical to overcoming these mathematical reservations. The models found by the mathematicians Beltrami, Klein and Poincaré were structured collections of objects satisfying a set of mathematical statements, given a suitable, if unusual, interpretation of the key terms such as ‘straight line’. The (relative) consistency of non-Euclidean geometry was proved using a model, and models aided the acceptance of the new geometry in other ways. They provided a method of visualising concepts contrary to intuition, such as the existence of infinitely many parallels through a point. Models also made it possible to apply non-Euclidean geometry to solving problems in other areas of mathematics. This demonstrated its usefulness.

In the 1830s, Ferdinand Minding (1806–85) had investigated surfaces of constant negative curvature, such as the pseudosphere, which is obtained by rotating a tractrix about a vertical axis. For such surfaces, Minding obtained the same trigonometric formulae as in hyperbolic geometry, and published them in 1840. Minding’s work was largely forgotten until Eugenio Beltrami (1835–1900) noticed that Minding’s trigonometric formulae were identical to Lobachevsky’s. He thus realised that non-Euclidean plane geometry could be regarded as the geometry of a surface of constant negative curvature. In his model of 1868, Beltrami identified the hyperbolic plane with the interior of a fixed circle in the Euclidean plane. Hyperbolic straight lines were identified with open chords of this circle, and points at infinity with the circle itself. By constructing this first model of the hyperbolic plane, Beltrami established the relative consistency of non-Euclidean geometry. This was because his model was defined wholly in terms of Euclidean geometry, and he proved, using differential geometry, that it satisfied all the axioms of hyperbolic geometry. Thus, the new geometry was consistent if Euclidean geometry was consistent.

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43 Rosenfeld, History of Non-Euclidean Geometry, p. 227.
44 Torretti, Philosophy of Geometry, p. 132.
45 John G. Ratcliffe. Foundations of Hyperbolic Manifolds. Springer-Verlag, 1994, p. 7.
Felix Klein (1849–1925) interpreted Beltrami’s model in terms of projective geometry in 1871. He modelled hyperbolic geometry as metric projective geometry in the interior of some conic. Beltrami’s model was the special case where the conic was a circle.

In 1882, Henri Poincaré (1854–1912) showed that linear fractional transformations with real coefficients preserve the complex upper half-plane. Poincaré’s model is this upper half-plane with arc length defined as $ds = \sqrt{dx^2 + dy^2}/y$, and geodesics as vertical lines or semicircles orthogonal to the real axis. Distance is then arc length along the geodesic joining two points. Poincaré’s model allows groups of linear fractional transformations to be represented as discrete groups of orientation-preserving isometries of the hyperbolic plane. The model can be mapped to the unit disc, and in later works Poincaré extended his model to three dimensions, and discussed a two-sheeted hyperboloid model as well.

Using these models, Poincaré found many applications for hyperbolic geometry, which helped the new geometry to seem useful and familiar. Indeed, Poincaré’s work showed that hyperbolic geometry was already part of mainstream mathematics. For example, he used hyperbolic geometry to prove the existence of linear fractional transformations. He then studied the group of such transformations, using the tessellation of the upper half-plane pictured here.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{tessellation.png}
\caption{Tessellation of the upper half-plane.}
\end{figure}

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46 Scott Walter. The non-Euclidean style of Minkowskian relativity. In Jeremy Gray, editor, *The Symbolic Universe: Geometry and Physics 1890–1930*, pp. 91–127. Oxford University Press, 1999, p. 92.

47 John Stillwell. Introduction to Poincaré’s ‘Theory of Fuchsian groups’, ‘Memoir on Kleinian groups’, ‘On the applications of non-Euclidean geometry to the theory of quadratic forms’. In John Stillwell, editor, *Sources of Hyperbolic Geometry*, pp. 113–122. American Mathematical Society, 1996, p. 113
Other applications included complex analysis, differential equations and number theory.\[48\]

The models and applications of hyperbolic geometry which were so important to its acceptance could not have developed without advances in many different areas of mathematics in the nineteenth century. These areas included differential geometry, group theory, geometric transformations and projective geometry.

Gauß’s theory of curved surfaces (1827) was essential to the eventual recognition of hyperbolic geometry.\[49\] The language established by the theory of curvature could be used to describe or analyse surfaces in terms of their intrinsic properties, that is, without consideration of the space in which they were embedded.\[50\] Geodesics in surfaces could be defined by analogy with straight lines in the plane.\[51\] These techniques were employed in modelling hyperbolic geometry, as all the models were surfaces of constant negative curvature with geodesics defined intrinsically.

Riemannian geometry was also crucial to the reception of hyperbolic geometry. At Göttingen in 1854, G. F. B. Riemann (1826–1866) gave a lecture entitled “Über die Hypothesen, welche der Geometrie zu Grunde liegen” (“On the Hypotheses which lie at the Foundations of Geometry”). It was published in 1867. In this lecture, Riemann described a very general notion, that of an \(n\)-dimensional manifold. A manifold is characterised by a surface, and a metric chosen from among infinitely many possible alternatives. The theory of manifolds included Euclidean geometry as a special case, and hyperbolic geometry as an infinite family of cases where the surface has constant negative curvature.\[52\] By showing that infinitely many geometries were possible, and allowing the design of geometries very different from Euclid’s, Riemann’s work helped get hyperbolic geometry admitted to mathematical respectability.\[53\]

Many of the applications of hyperbolic geometry used group theory. The idea of a group began with concrete cases, such as permutations of the roots of algebraic equations, which were studied by J. H. Lagrange (1736–1813) and Paolo Ruffini (1765–1822) in the eighteenth century. In 1826, Niels Henrik Abel (1802–1829) used such groups to prove that it was impossible to obtain a solution in radicals of the general equation of degree greater than or equal to 5. Augustin Louis Cauchy (1789–1857) wrote many papers on the theory of groups of substitutions. The term ‘group’ was first used in an article by Evariste Galois (1811–1832), who used it only for groups of substitutions. Cayley, in 1854, was the first to formulate the concept of an abstract group, by giving an axiomatic definition of a group. Klein brought group theory into geometry, by noticing that geometric transformations can form groups under composition.

Although Euclid had rejected the use of motion in geometry, other mathematicians employed many different kinds of geometric transformations.\[54\]

\[48\]John Stillwell, “Introduction”, pp. 113–114.

\[49\]John Milnor. Hyperbolic Geometry: the first 150 years. In Bulletin of the American Mathematical Society, 6:9–24, 1982, p. 10.

\[50\]Torretti, Philosophy of Geometry, p. 78.

\[51\]Gray, Ideas of Space, pp. 136–137.

\[52\]Torretti, Philosophy of Geometry, p. 40.

\[53\]Gray, Ideas of Space, pp. 141–145.

\[54\]Rosenfeld, History of Non-Euclidean Geometry, p. 112–151.
(b. 287? B.C.E.) used reflections, and Apollonius (262?–190? B.C.E.) used dilations and inversions. Greek and Arabian astronomers applied stereographic projection to construct astrolabes and maps. The geometric transformations employed by Isaac Newton (1642–1727) included central projection and projective transformations. Alexis Claude Clairaut (1713–1765) defined general affine transformations in words in 1733, and Leonhard Euler (1707–1783) gave general formulae for affine and similarity transformations in 1748, and classified plane motions. Euler also used conformal mappings of the plane and linear fractional transformations, which define transformations generated by similarities and inversions in circles. A. F. Möbius (1790–1860) introduced the general concept of a geometric transformation as a one-to-one correspondence between figures.

In his *Erlangen Program* of 1872, Klein described geometry as the study of those properties of figures which are preserved by a particular group of geometric transformations. Geometries with non-trivial symmetries, such as projective geometries and metric spaces with constant curvature, could now be characterised by classifying groups of transformations. Two geometries were equivalent if their characteristic groups were isomorphic. For example, Euclidean geometry in the plane is the study of those properties of figures, such as angles and the ratios between lengths, which remain invariant under the group of similarities of the plane, which includes all translations, dilations and rotations. Klein’s *Erlangen Program* was at first little noticed, but achieved international influence by the end of the nineteenth century. As a result of this program, the study of some geometries became the study of transformation groups. Sophus Lie (1842–1899), for example, used this idea to characterise all three-dimensional geometries by determining all subgroups of motions in three-dimensional space. Hyperbolic geometry was now just one of many alternatives which could be investigated using algebraic techniques, and was a source of applications of group theory.

Projective geometry, with its roots in antiquity, was largely developed in the nineteenth century. Depending on the projective metric chosen, the requirements of spherical, Euclidean or hyperbolic geometry can be met. So, as Klein discovered, projective metric spaces can model hyperbolic geometry.

### 1.5.3 Philosophical disputes

The philosophical controversy about non-Euclidean geometry was extensive. Since the Greeks, Euclidean geometry had been considered the correct representation of space, which accorded with intuition and possessed admirable logical structure. The discovery of non-Euclidean geometry led to debates on issues including the relationship between mathematics and the real world, and the source of geometric notions. Changes to the ideas held in these areas were necessary for non-Euclidean geometry to be accepted, while non-Euclidean geometry itself inspired investigations into the foundations of mathematics.

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55 John D. Norton. Geometries in collision: Einstein, Klein and Riemann. In Gray, editor, *The Symbolic Universe*, pp. 128–144, p. 129.

56 Boyer, *History of Mathematics*, p. 549.

57 Jeremy Gray. Geometry — formalisms and intuitions. In Gray, editor, *The Symbolic Universe*, pp. 58–83, p. 62.

58 Torretti, *Philosophy of Geometry*, p. 110.
One of the basic issues raised by non-Euclidean geometry was the connection between geometry and physical space. In classical physics, Euclidean geometry was held to be the science of space, a view which implied that this geometry really existed. The ontological status of geometry was also linked to debates over the source of geometric knowledge. If our geometric notions are inborn, \textit{a priori}, then geometry really exists. Further, our intuition must be obeyed, and so the counter-intuitive nature of many non-Euclidean results is a sound argument against them. If, on the other hand, geometry is an abstraction of the physical world, then Euclidean geometry is just one of many possible human constructs. It may well be the idealisation which is favoured by experience, but this does not imply that it really exists.

Disputes about the source of mathematical knowledge went back to the Greeks. In Plato’s philosophy, mathematical knowledge was \textit{a priori}. Geometric concepts were particular cases of the Forms, those eternal perfect objects which are the template for physical things, and to which we have access before we are born. To prove these ideas, Plato describes in \textit{Meno} a conversation between Socrates and a slave-boy. After some leading questions, the boy proves a result about the square of the hypotenuse of a right-angled triangle. In the dialogue, Socrates concludes

\begin{quote}
\ldots now these opinions have been newly aroused in this boy as if in a dream, but if someone asks him these same things many times and in many ways, you can be sure that in the end he will come to have exact knowledge of these things as well as anyone else does \ldots he will come to have knowledge without having been taught by anyone, but only having been asked questions, having recovered this knowledge himself, from himself.
\end{quote}

The opposing view, put by Aristotle, was that geometric concepts are abstractions of real world objects: a mathematician “effects an abstraction, for in thought it is possible to separate figures from motions”. Whether they regarded it as inborn knowledge or an abstraction, the Greeks considered Euclidean geometry to correctly represent the physical world.

The idea of space as an empty receptacle which is then occupied by matter was not familiar to Greek philosophers. The classical Greek words \textit{topos} (place) and \textit{kenon} (void) do not correspond to the word space, since place is the place of a body, which is determined by its relationship with other nearby bodies, while a void is always filled and is finite. The modern concept, though, was current in Europe by the fifteenth century. Natural philosophers such as Giordano Bruno (1548–1600) held that space was infinite and existed independent of matter. Bruno wrote

\begin{quote}
Space is a continuous three-dimensional natural quantity, in which the magnitude of bodies is contained, which is prior by nature to all bodies and subsists without them but indifferently receives them all, and is free from the conditions of action and passion, unmixible, impenetrable,
\end{quote}

\footnotesize
\begin{itemize}
\item \textsuperscript{59} Torretti, \textit{Philosophy of Geometry}, p. 25.
\item \textsuperscript{60} Rosenfeld, \textit{History of Non-Euclidean Geometry}, pp. 186–187.
\item \textsuperscript{61} Plato. \textit{Meno}. Aris & Phillips Publishers, 1985, p. 79.
\item \textsuperscript{62} Quoted in Rosenfeld, \textit{History of Non-Euclidean Geometry}, p. 186.
\item \textsuperscript{63} Torretti, \textit{Philosophy of Geometry}, pp. 25–26.
\end{itemize}
unshapeable, non-locatable, outside all bodies yet encompassing and incomprehensibly containing them all.64

Once this concept of space was part of philosophical discourse, the dispute about the source of geometry embraced the origin of our ideas of space as well as of more specific geometrical notions. Immanuel Kant (1724–1804) is important for his association with the doctrine that our concept of space is *a priori*. His major work *Kritik der reinen Vernunft (Critique of Pure Reason)* (1781) presented his theories of time and space. Here, Kant wrote that

1. Space is not a conception which has been derived from outward experiences. For, in order that certain sensations may relate to something without me . . . the representation of space must already exist as a foundation. . . .

2. Space then is a necessary representation *a priori*, which serves for the foundation of all external intuitions.65

As for geometry, this was discussed by Kant in the *Kritik*, and in a chapter on pure mathematics in the *Prolegomena* (1783), a work written as a survey of the main points in the *Kritik*. Kant argued that geometry was the true science of space, that is, of space the *a priori* representation, and that geometry was revealed by pure intuition in full agreement with Euclid’s *Elements*.66 Because of these doctrines, followers of Kantian orthodoxy in the nineteenth century opposed non-Euclidean geometry, dismissing it as an intellectual exercise unrelated to the real world.67

Much of classical physics depended on Euclidean geometry.68 Johann Kepler (1571–1630) regarded geometry as the foundation of both celestial and terrestrial physics, writing

> We see that the motions [of the planets] occur in time and place and that the force [that binds them to the sun] emanates from its source and diffuses through the spaces of the world. All these are geometrical things. Must not that force be subject also to other geometrical things?69

The coordinate geometry of René Descartes (1596–1650) was crucial to the status of geometry in physics.70 With the Cartesian coordinate system, space could be equated with a structured set of points and, importantly, motion could be analysed.

Gottfried Wilhelm Leibniz (1646–1716) and Isaac Newton are considered the founders of mathematical physics. They had differing views of space. Leibniz (like Descartes) considered it impossible for space to be empty. In *Nouveaux essais sur l’entendement humain (New Essays on Human Understanding)*, Leibniz wrote that “it is necessary rather to conceive space as full of a matter originally fluid, susceptible of all the divisions”.71 Newton, however, thought that space was independent from matter. In the *Principia* (1687), he defined absolute space as:

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64 Quoted in ibid., p. 28.
65 Quoted in Rosenfeld, *History of Non-Euclidean Geometry*, p. 187.
66 Torretti, *Philosophy of Geometry*, p. 33.
67 Ibid.
68 Kline, *Mathematical Thought*, pp. 880–881.
69 Quoted in Torretti, *Philosophy of Geometry*, p. 23.
70 Torretti, *Philosophy of Geometry*, p. 34.
71 Quoted in Rosenfeld, *History of Non-Euclidean Geometry*, p. 184.
Absolute space, in its own nature, without relation to anything external, remains always similar and immovable.\textsuperscript{72} Absolute space was essential to the results of the \textit{Principia}, and it was Newton’s ideas on the nature of space which prevailed as theoretical mechanics was developed in the eighteenth century. The implicit assumptions about space made by Newton and later mechanists included that it was continuous, infinite, three-dimensional and homogeneous, and that it satisfied Euclidean theorems.\textsuperscript{73}

Because physics depended on Euclidean geometry, undermining the parallel axiom was a real physical problem.\textsuperscript{74} There were sound reasons to think that physical space was Euclidean. How then could an alternative geometry exist? The word ‘existence’ here was taken to mean true physical existence, and this is certainly the interpretation of ‘existence’ held by Bolyai and Lobachevsky. Apart from the parallel axiom, they never questioned Euclidean geometry, which Bolyai described as “the absolutely true science of space”.\textsuperscript{75} Concrete notions of existence definitely hampered the acceptance of non-Euclidean geometry: contemporaries could discount Lobachevsky and Bolyai’s work because neither author succeeded in establishing the physical existence of their geometry.\textsuperscript{76}

The development of Riemannian geometry was very important in changing mathematicians’ ideas about the nature of mathematical existence. Riemann began his 1854 lecture on the foundations of geometry by observing that all previous presentations of geometry had presupposed the nature of space. He then described space, meaning real physical space, as just one instance of a general concept which he called a \textit{mehrfach ausgedehnte Grö"osse} (multiply extended quantity).\textsuperscript{77} Riemann showed that such an extended quantity (in modern terminology, a manifold) may have arbitrarily many “metric relations”; therefore “space constitutes only a special case of a threefold extended quantity”.\textsuperscript{78} The implication of Riemann’s geometry for the philosophy of space was that, since there can be only one geometry of real space among infinitely many mathematical possibilities, the science of space could not be determined by mathematical processes alone. As Riemann said, “those properties which distinguish space from other conceivable threefold extended quantities can be gathered only from experience”.\textsuperscript{79} Riemann acknowledged that Euclidean geometry was empirically successful within “the limits of observation”\textsuperscript{80} but his work meant that mathematical existence could no longer be equated with physical existence. This was a very significant change in mathematical discourse.

Because each possible manifold had equal mathematical validity, Euclidean and non-Euclidean geometries could now exist side by side. Indeed, Beltrami’s model showed that the consistency of Euclidean geometry implied the consistency of hyperbolic geometry. The models of hyperbolic geometry, such as Beltrami’s, also

\textsuperscript{72}Quoted in ibid., p. 185. \\
\textsuperscript{73}Torretti, \textit{Philosophy of Geometry}, p. 25. \\
\textsuperscript{74}Kline, \textit{Mathematical Thought}, pp. 880–881. \\
\textsuperscript{75}Quoted in Torretti, \textit{Philosophy of Geometry}, p. 62. \\
\textsuperscript{76}Gray, \textit{Ideas of Space}, p. 170. \\
\textsuperscript{77}Torretti, \textit{Philosophy of Geometry}, p. 83. \\
\textsuperscript{78}Quoted in ibid., p. 83. \\
\textsuperscript{79}Quoted in ibid., p. 84. \\
\textsuperscript{80}Quoted in ibid., p. 104.
represented a move towards semantic neutrality. With words such as ‘straight’ being interpreted in ways differing from their everyday meaning, physical existence became irrelevant. Poincaré, in the spirit of the Erlangen Program, characterised geometries using groups, and then explained that alternative geometries can exist because the existence of one group is not incompatible with the existence of another.\footnote{Arthur Miller. Einstein, Poincaré, and the testability of geometry. In Gray, editor, The Symbolic Universe, pp. 47–57, p. 54.}

The development of projective geometry further aided the acceptance of hyperbolic geometry. Since projective geometry was so counter-intuitive, it was quickly characterised by axioms, a project which was completed in 1882 by Moritz Pasch (1843–1930).\footnote{Torretti, Philosophy of Geometry, p. 190.} Objections to non-Euclidean geometry based on intuition had little weight once mathematics accepted systems as strange as projective geometry.

By the late nineteenth century, mathematicians made no claims that their work represented the physical truth. For example, in his popularising work of 1912, La science et l’hypothèse, Poincaré wrote that the question of whether Euclidean geometry was true had no meaning. He dismissed the argument that geometric ideas are \textit{a priori} by pointing out that alternative systems, such as projective geometry, are conceivable.\footnote{Henri Poincaré. Science and Hypothesis. Dover Publications Inc., 1952, p. 48.} Also, he argued, geometry could not be a body of experimental truths, since if it were experimental it would not be exact, and would be subject to continuous revision.\footnote{Ibid., pp. 49–50.} Thus the choice of which geometry to use in representing physical space was a matter of convention, with experience guiding but not determining our choice of the most convenient system.\footnote{Ibid., p. 50.} Poincaré concluded: “One geometry cannot be more true than another; it can only be more convenient.”\footnote{Ibid.}

This attenuation of mathematical truth-claims meant that the mathematical notion of existence was very much weaker than that of physics. In fact, over the course of the nineteenth century, it became for the first time possible, then necessary, to distinguish between the subjects of mathematics and physics, as they developed quite distinct institutions and aims.\footnote{Jeremy Gray. Introduction. In Gray, editor, The Symbolic Universe, pp. 1–21, p. 3.} The leading physicist Ludwig Boltzmann (1844–1906), for instance, considered mathematics to be unrelated to the real world, and to have only a service role in physics.\footnote{Ibid., p. 4.} The writings of Ernst Mach (1838–1916) were very influential in physics. His notion of existence was stronger than that prevailing in mathematics, and he wrote of nineteenth-century geometry:

*Analogothe geometry we are familiar with, are constructed on broader and more general assumptions for any number of dimensions, with no pretension to being regarded as more than intellectual scientific experiments and with no idea of being applied to reality. \ldots Seldom have thinkers become so steeped in reverie, or so far estranged from reality, as to imagine for our space a number of dimensions exceeding the three of the given space of sense, or to conceive of representing that*
space by any geometry that appreciably departs from the Euclidean.
Gauss, Lobachevsky, Bolyai and Riemann were perfectly clear on this
point, and certainly cannot be held responsible for the grotesque fictions
subsequently stated in this field\[89\]

It is interesting, then, that in twentieth-century physics, space is Riemannian and
not Euclidean.\[90\] Einstein’s theories of special and general relativity depend on ge-
ometry quite different from the three-dimensional Euclidean geometry of classical
physics.\[91\] Indeed, from 1908 to 1916, the laws of special relativity were reformu-
lated and reinterpreted in terms of hyperbolic geometry.\[92\] Even so, non-relativistic
physics still employs only one geometry to represent physical space, while mathem-
aticians consider a whole variety of geometries.

Non-Euclidean geometry stimulated discussion of the foundations of mathemat-
ics. Works responding to the new geometries began to appear in the 1860s. Rigorous
axiomatisation was their aim, and they embodied the new, weaker notion of exis-
tence, and a lesser role for intuition. Lie, for example, published two papers on the
foundations of geometry in 1890. In these, he asked what properties were necessary
and sufficient to define Euclidean geometry and the non-Euclidean geometries of
constant non-zero curvature. Thus Lie considered the problem of space as an issue
of pure mathematics only, showing the divide between mathematics and ideas about
physical space.\[93\]

The investigations into foundations culminated in 1899 with the publication of
Grundlagen der Geometrie by David Hilbert (1862–1943). This was a complete
axiomatisation of geometry, which rectified the logical inadequacies of Euclid’s El-
ements. For Hilbert, a geometry was defined by a set of unproved axioms, any of
which might be negated to obtain a different theory. The axioms contained unde-
fined primitive terms, which could be modelled as referring to particular objects
such as points, lines and planes, restricted only by the implications of the axioms.
Such a level of formalism further weakened claims as to the truth of geometry.\[94\] At
the end of the nineteenth century, Euclidean and hyperbolic geometry had become
two abstract axiomatic systems, differing in only one axiom, and equally consistent.
This philosophy was very different from that of Bolyai and Lobachevsky, but their
work contributed to it.\[95\]

1.6 Conclusion

This history shows how the development and reception of non-Euclidean geometry
were associated with fundamental changes in the way mathematics is done and
thought about. The recognition and study of abstract mathematical spaces, and
the realisation of the connection between geometry and groups, are stages in this
history which are of particular relevance for subsequent chapters. Indeed, the style

\[89\] Quoted in Rosenfeld, History of Non-Euclidean Geometry, p. 203.
\[90\] Gray, ‘Geometry’, p. 80.
\[91\] Miller, ‘Einstein, Poincaré’, p. 55.
\[92\] Walter, ‘Non-Euclidean style’, p. 91.
\[93\] Torretti, Philosophy of Geometry, p. 154.
\[94\] Gray, ‘Geometry’, p. 69.
\[95\] Torretti, Philosophy of Geometry, p. 61.
of exposition adopted for the rest of this thesis, with explicit definitions, acknowledgment of assumptions and so forth, illustrates well the abstraction and logical precision of modern mathematics. The shift in conceptions of geometry which resulted in this modern approach was stimulated by the discovery of non-Euclidean geometry, and also aided the acceptance of the new geometry by mathematicians.
Chapter 2
Hyperbolic Geometry

2.1 Introduction
In this chapter we survey hyperbolic geometry. We provide a modern treatment of much of the geometry discussed in Chapter 1, and lay the foundations for Chapters 3 and 4. After a section of preliminary definitions and results, we consider four models of hyperbolic geometry. Hyperbolic geometry is actually a continuum of geometries, each distinguished by its Gaussian curvature. For convenience, the models of hyperbolic geometry presented here all have constant curvature of $-1$. The first model discussed is called the hyperboloid model, because the hyperbolic plane in this model is one sheet of a two-sheeted hyperboloid. In general, the hyperboloid model is the sphere of radius $i$ in a non-Euclidean space called Lorentzian $n$-space. Lorentzian 4-space is the model of space-time in special relativity. Next is a brief treatment of Beltrami’s model, which was reinterpreted by Klein, and is also known as the projective disc model. The model introduced by Poincaré was the upper half-plane, and he showed how this model could be mapped to the unit disc. Poincaré’s models have now been generalised to higher dimensions, and are known respectively as the upper half-space model, and the conformal ball model. We will prove that, in the upper half-space and the conformal ball, hyperbolic isometries may be identified with compositions of finitely many reflections in planes and spheres. In the upper half-plane, we go on to consider how tessellation by triangles is related to the action of the discrete group $SL(2, \mathbb{Z})$. Most of the material in this chapter is selected and adapted from the treatment of hyperbolic geometry in Chapters 3–6 of [21]. The proofs of relevant exercises in [21] are provided. Some additional results which will be needed in later chapters are established.

2.2 Geometric preliminaries
This section comprises useful definitions and results relating to Euclidean $n$-space and to general metric spaces. The geometric transformations discussed include reflections in planes and spheres. Compositions of finitely many such reflections are called Möbius transformations, and Möbius transformations turn out to be very important for understanding hyperbolic geometry.

2.2.1 Euclidean $n$-space
We define Euclidean $n$-space, $E^n$, to be the metric space consisting of $\mathbb{R}^n$ together with the distance function
$$d_E(x, y) = |x - y|.$$
Here, the right-hand side is the Euclidean norm, $|x| = (x \cdot x)^{1/2}$. The Euclidean inner product is the usual dot product given by
\[
x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n.
\]
We are referring to $d_E$ as a distance function rather than as a metric in order to avoid confusion with Riemannian metrics in Chapter 3. The Cauchy–Schwartz inequality for $x, y \in E^n$,
\[
|x \cdot y| \leq |x||y|
\]
with equality if and only if $x$ and $y$ are linearly dependent, allows us to define $\theta(x, y)$, the Euclidean angle between non-zero vectors $x$ and $y$, by
\[
x \cdot y = |x||y| \cos \theta(x, y).
\]

2.2.2 Geometric transformations and group actions

An isometry from a metric space $(X, d_X)$ to a metric space $(Y, d_Y)$ is a bijection $\phi : X \to Y$ such that
\[
d_X(x_1, x_2) = d_Y(\phi(x_1), \phi(x_2))
\]
for all $x_1, x_2 \in X$. The set of isometries from a metric space $X$ to itself, denoted by $I(X)$, forms a group under composition. Isometries of $E^n$ are known as Euclidean isometries.

A function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is called an orthogonal transformation if
\[
\phi(x) \cdot \phi(y) = x \cdot y
\]
for all $x, y \in \mathbb{R}^n$. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Then, it can be proved that a map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation if and only if $\phi$ is linear and $\{\phi(e_1), \phi(e_2), \ldots, \phi(e_n)\}$ is an orthonormal basis of $\mathbb{R}^n$. It follows that every orthogonal transformation is a Euclidean isometry, and that orthogonal transformations may be identified with orthogonal $n \times n$ matrices. The set of all orthogonal $n \times n$ matrices, denoted by $O(n)$, forms a group under matrix multiplication.

A group $G$ is said to act on a set $X$ if there exists a function from $G \times X$ to $X$, written $(g, x) \mapsto gx$, such that for all $g, h \in G$, and all $x \in X$, we have
\[
1x = x \quad \text{and} \quad g(hx) = (gh)x.
\]
For example, the group $I(X)$ acts on $X$, and the group $O(n)$ acts on the set of $m$-dimensional vector subspaces of $\mathbb{R}^n$, for $1 \leq m \leq n$. The action of $G$ on $X$ is said to be transitive if for each $x, y \in X$ there is a $g \in G$ such that $gx = y$.

**Proposition 2.2.1.** For each dimension $m$, the action of the group $O(n)$ on the set of $m$-dimensional vector subspaces of $\mathbb{R}^n$ is transitive.

**Proof.** Let $V$ be any $m$-dimensional subspace of $\mathbb{R}^n$. Identify $\mathbb{R}^m$ with the span of the set $\{e_1, e_2, \ldots, e_m\}$ in $\mathbb{R}^n$. Since $O(n)$ is a group, it suffices to show there is an $A \in O(n)$ such that $A(\mathbb{R}^m) = V$. 


Choose an orthonormal basis \( \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \} \) for \( V \). Then, using the Gram–Schmidt process, extend it to an orthonormal basis \( \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \} \) for \( \mathbb{R}^n \). Let \( A \) be the \( n \times n \) matrix which has \( \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \) as its columns. Then \( A \) is orthogonal, and \( A(\mathbb{R}^m) = V \).

An \( m \)-plane of \( E^n \) is defined to be a coset \( \mathbf{a} + V \), where \( \mathbf{a} \in E^n \) and \( V \) is an \( m \)-dimensional subspace of \( E^n \).

**Corollary 2.2.2.** For each dimension \( m \), the group \( I(E^n) \) acts transitively on the set of \( m \)-planes of \( E^n \).

**Proof.** Let \( \mathbf{a} + V \) and \( \mathbf{b} + W \) be \( m \)-planes of \( E^n \). By Proposition 2.2.1, there is an \( A \in O(n) \) such that \( A(V) = W \). Define the map \( \phi : E^n \rightarrow E^n \) by

\[
\phi(\mathbf{x}) = (\mathbf{b} - A\mathbf{a}) + A\mathbf{x}.
\]

Then \( \phi(\mathbf{a} + V) = \mathbf{b} + W \), and

\[
|\phi(\mathbf{x}) - \phi(\mathbf{y})| = |A\mathbf{x} - A\mathbf{y}| = |\mathbf{x} - \mathbf{y}|,
\]

since \( A \) is an isometry. Hence \( \phi \) is an isometry.

The following proposition shows that every Euclidean isometry is the composition of an orthogonal transformation and a translation.

**Proposition 2.2.3.** The function \( \phi : E^n \rightarrow E^n \) is an isometry if and only if \( \phi \) is of the form

\[
\phi(\mathbf{x}) = \mathbf{a} + A\mathbf{x},
\]

where \( A \) is an orthogonal matrix and \( \mathbf{a} = \phi(\mathbf{0}) \).

**Proof.** Suppose \( \phi \) is an isometry. Define a map \( A \) by \( A(\mathbf{x}) = \phi(\mathbf{x}) - \phi(\mathbf{0}) \). Then \( A(\mathbf{0}) = \mathbf{0} \), and, since \( \phi \) preserves Euclidean norms,

\[
|A(\mathbf{x}) - A(\mathbf{y})| = |\phi(\mathbf{x}) - \phi(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|
\]

for all \( \mathbf{x}, \mathbf{y} \in E^n \). Therefore, if \( \mathbf{x} \in E^n \),

\[
|A(\mathbf{x})| = |A(\mathbf{x}) - A(\mathbf{0})| = |\mathbf{x} - \mathbf{0}| = |\mathbf{x}|,
\]

so \( A \) preserves Euclidean norms. It follows that \( A \) is orthogonal, since

\[
2(A\mathbf{x} \cdot A\mathbf{y}) = |A\mathbf{x}|^2 + |A\mathbf{y}|^2 - |A\mathbf{x} - A\mathbf{y}|^2
\]

\[
= |\mathbf{x}|^2 + |\mathbf{y}|^2 - |\mathbf{x} - \mathbf{y}|^2
\]

\[
= 2(\mathbf{x} \cdot \mathbf{y}).
\]

So we have an orthogonal matrix \( A \) such that \( \phi(\mathbf{x}) = \phi(\mathbf{0}) + A\mathbf{x} \).

Conversely, suppose \( \phi \) is of the form \( \phi(\mathbf{x}) = \phi(\mathbf{0}) + A\mathbf{x} \). Then \( \phi \) is the composition of an orthogonal transformation and a translation, both of which are isometries. Therefore \( \phi \) is an isometry.
A function \( \phi : X \to Y \) between metric spaces \((X, d_X)\) and \((Y, d_Y)\) is called a similarity if it is a bijection, and there is a scale factor \( k > 0 \) such that

\[
d_X(x_1, x_2) = kd_Y(\phi(x_1), \phi(x_2))
\]

for all \( x_1, x_2 \in X \). The set of similarities from a metric space \( X \) to itself, denoted by \( S(X) \), forms a group under composition. The isometries of \( X \) are a subgroup of \( S(X) \). Under transformation by elements of \( S(E^n) \), all the theorems of Euclidean geometry remain true. Thus the similarities of \( E^n \) are its characteristic group, in the sense used by Klein in his Erlangen Program. The following characterisation of similarities follows from Proposition 2.2.3.

**Proposition 2.2.4.** The function \( \phi : E^n \to E^n \) is a similarity if and only if \( \phi \) is of the form

\[
\phi(x) = a + kAx,
\]

where \( A \) is an orthogonal matrix, \( k \) is a positive constant and \( a = \phi(0) \).

### 2.2.3 Geodesics

Let \((X, d)\) be a metric space, and let \( \alpha : [a, b] \to X \) be a continuous injection, where \( a < b \) in \( \mathbb{R} \). We say \( \alpha \) is a geodesic curve if, for all \( s, t \in [a, b] \),

\[
d(\alpha(s), \alpha(t)) = |s - t|.
\]

For points \( x \) and \( y \) in \( X \), we denote by \([x, y]\) the image in \( X \) of a geodesic curve \( \alpha : [a, b] \to X \) such that \( \alpha(a) = x \) and \( \alpha(b) = y \). We call \([x, y]\) a geodesic segment. Then \([x, y] \cup [y, z]\) is a geodesic segment joining \( x \) to \( z \) if and only if

\[
d(x, y) + d(y, z) = d(x, z).
\]

A geodesic line is defined to be a function \( \lambda : \mathbb{R} \to X \) which is locally distance-preserving. This means that for any \( x \in \lambda(\mathbb{R}) \), with \( x = \lambda(t) \), there is a set \([a, b] \subseteq \mathbb{R}\) containing \( t \) such that \( \lambda \) restricted to \([a, b]\) is a geodesic curve. A geodesic is the image in \( X \) of a geodesic line \( \lambda : \mathbb{R} \to X \).

It turns out that, if the metric space \((X, d)\) satisfies the axioms of Euclidean or hyperbolic geometry, then the shortest path from \( x \) to \( y \) in \( X \) is along the unique geodesic segment joining \( x \) and \( y \), and the length of this shortest path is \( d(x, y) \).

### 2.2.4 Reflections

Let \( u \) be a unit vector in \( E^n \), and let \( t \) be a real number. The hyperplane of \( E^n \) with unit normal vector \( u \) passing through the point \( tu \) is the set

\[
P(u, t) = \{ x \in E^n : u \cdot x = t \}.
\]

We will sometimes refer to hyperplanes as just planes. The reflection \( \rho \) of \( E^n \) in the plane \( P(u, t) \) is given by

\[
\rho(x) = x + 2(t - u \cdot x)u.
\]
Then \( \rho(x) = x \) if and only if \( x \in P(u,t) \), and \( \rho^2(x) = x \) for all \( x \in \mathbb{R}^n \). We now show that every Euclidean isometry is a composition of finitely many reflections in hyperplanes.

**Lemma 2.2.5.** Let \( x, y \) be in \( \mathbb{R}^n \) and \( \rho \) be the reflection in the plane \( P(u,t) \). Then

\[
|\rho(x) - \rho(y)| = |x - y|.
\]

That is, \( \rho \) is a Euclidean isometry.

**Proposition 2.2.6.** Every isometry of \( \mathbb{R}^n \) is a composition of at most \( n + 1 \) reflections in hyperplanes.

**Proof.** Let \( \phi : \mathbb{R}^n \to \mathbb{R}^n \) be an isometry. We construct reflections \( \rho_0, \rho_1, \ldots, \rho_n \) such that \( \phi = \rho_0 \rho_1 \cdots \rho_n \).

Set \( v_0 = \phi(0) \). If \( v_0 = 0 \) let \( \rho_0 \) be the identity, and otherwise let \( \rho_0 \) be the reflection in the plane \( P(v_0/|v_0|,|v_0|/2) \). Then

\[
\rho_0(v_0) = v_0 + 2 \left( \frac{|v_0|}{2} - \frac{v_0 \cdot v_0}{|v_0|} \right) = 0,
\]

and so \( \rho_0 \phi(0) = 0 \). Let \( \phi_0 = \rho_0 \phi \). Then the composition \( \phi_0 \) is an isometry, and \( \phi_0(x) = \rho_0 \phi(0) + \rho_0 \phi(x) \), so by Proposition 2.2.3, \( \phi_0 \) is orthogonal.

For \( k = 1, 2, \ldots, n \), assume inductively that \( \phi_{k-1} \) is an orthogonal transformation of \( \mathbb{R}^n \) which fixes \( 0 \) and the basis vectors \( e_1, e_2, \ldots, e_{k-1} \). Let \( v_k \) be the vector \( \phi_{k-1}(e_k) - e_k \). If \( v_k = 0 \) let \( \rho_k \) be the identity, and otherwise let \( \rho_k \) be the reflection in the plane \( P(v_k/|v_k|,0) \). Then, by using the assumption that \( \phi_{k-1} \) is orthogonal to expand out \( \rho_k \phi_{k-1}(e_k) \), we find that \( \rho_k \phi_{k-1} \) fixes \( e_k \). Also, for \( 1 \leq j \leq k-1 \),

\[
v_k \cdot e_j = (\phi_{k-1}(e_k) - e_k) \cdot e_j = \phi_{k-1}(e_k) \cdot e_j = e_k \cdot e_j = 0.
\]

Therefore, for \( 1 \leq j \leq k-1 \), the vector \( e_j \) is in the plane \( P(v_k/|v_k|,0) \), hence is fixed by the reflection \( \rho_k \). Consequently, \( \phi_k = \rho_k \phi_{k-1} \) fixes the basis vectors \( e_1, e_2, \ldots, e_k \). Also, since \( 0 \) is in the plane \( P(v_k/|v_k|,0) \), \( \rho_k \) fixes \( 0 \). By induction, then, the maps \( \rho_0, \rho_1, \ldots, \rho_n \) are each either the identity or a reflection, and the composition \( \rho_n \cdots \rho_1 \rho_0 \phi \) fixes \( 0, e_1, e_2, \ldots, e_n \). Using Proposition 2.2.3 again, we infer that \( \rho_n \cdots \rho_1 \rho_0 \phi \) is an orthogonal, hence linear, map which fixes all the basis vectors, and so is the identity. Since each reflection is its own inverse, this means that \( \rho_0 \rho_1 \cdots \rho_n = \phi \).

We now discuss reflections in spheres. These maps are also known as inversions. For \( a \in \mathbb{R}^n \) and \( r > 0 \), we denote by \( S(a,r) \) the sphere with centre \( a \) and radius \( r \). The reflection \( \sigma \) of the set \( \mathbb{R}^n \setminus \{a\} \) in the sphere \( S(a,r) \) is given by

\[
\sigma(x) = a + r^2 \frac{x - a}{|x - a|^2}.
\]
Then $\sigma(x) = x$ if and only if $x \in S(a, r)$, and $\sigma^2(x) = x$ for all $x \neq a$. Abusing notation, we will say that $\sigma$ is the reflection of $E^n$ in $S(a, r)$. Although $\sigma$ is not an isometry, we do have the following identity.

**Lemma 2.2.7.** Let $\sigma$ be the reflection of $E^n$ in the sphere $S(a, r)$. For all $x, y \neq a$,

$$|\sigma(x) - \sigma(y)| = \frac{r^2|x - y|}{|x - a||y - a|}.$$ 

We will sometimes denote by $S^{n-1}$ the sphere $S(0, 1)$ in $E^n$.

Reflections in planes and spheres are continuous maps with respect to the usual metric topology of $\mathbb{R}^n$, which is that induced by the distance function $d_E$. They are also differentiable functions. The following result shows that reflections in hyperplanes and spheres preserve angles. Let $\Omega$ be an open subset of $E^n$ and let $\phi: \Omega \to E^n$ be a differentiable function. We say $\phi$ is conformal if $\phi$ preserves angles between differentiable curves in $\Omega$. An important class of conformal maps is the set of orthogonal matrices. These are conformal because they preserve the Euclidean inner product.

**Proposition 2.2.8.** Every reflection of $E^n$ in a hyperplane or sphere is conformal.

*Proof.* Let $\rho$ be the reflection of $E^n$ in the plane $P(u, t)$. As $\rho(x) = x + 2(t - u \cdot x)u$, the $j$th component of $\rho(x)$ is $x_j + 2(t - (u_1x_1 + u_2x_2 + \cdots + u_nx_n))u_j$. Thus

$$D\rho(x) = (\delta_{ij} - 2u_iu_j) = I - 2U,$$

where $D\rho(x)$ is the matrix of partial derivatives of $\rho(x)$, and $U$ is the matrix $(u_iu_j)$. As $D\rho(x)$ is independent of $t$, we may assume without loss of generality that $t = 0$. Then $\rho(x) = (I - 2U)x$, and

$$\rho(x) \cdot \rho(y) = (x - 2(u \cdot x)u) \cdot (y - 2(u \cdot y)u)$$

$$= x \cdot y - 4(u \cdot x)(u \cdot y) + 4(u \cdot x)(u \cdot y)(u \cdot u)$$

$$= x \cdot y,$$

so $\rho$ is an orthogonal transformation. Therefore $I - 2U$ is an orthogonal matrix, and so the reflection $\rho$ is conformal.

For reflections in spheres, we consider first the reflection $\sigma_r$ in the sphere $S(0, r)$. We have $\sigma_r(x) = r^2x/|x|^2$, so the $j$th component of $\sigma_r(x)$ is $r^2x_j/(x_1^2 + x_2^2 + \cdots + x_n^2)$. Hence,

$$D\sigma_r(x) = r^2 \left( \frac{\delta_{ij} |x|^2 - 2x_i x_j}{|x|^4} \right) = \frac{r^2}{|x|^2}(I - 2U),$$

where $U$ is the matrix $(x_i x_j/|x|^2)$. From above, $I - 2U$ is orthogonal, as $x/|x|$ is a unit vector. Thus $\sigma_r$ is conformal.

Now let $\sigma$ be the reflection in the sphere $S(a, r)$ and let $\tau$ be translation by $a$. Then $\sigma = \tau \sigma_r \tau^{-1}$. Since $\tau$ and $\sigma_r$ are conformal, this means $\sigma$ is conformal. □

### 2.2.5 Möbius transformations

We now consider the one-point compactification of $E^n$, $E^n \cup \{\infty\}$, which we denote by $\hat{E}^n$. The reflections $\rho$ and $\sigma$ above may be extended to be maps from $\hat{E}^n$ to
\(\hat{E}^n\), by setting \(\rho(\infty) = \infty\), \(\sigma(a) = \infty\) and \(\sigma(\infty) = a\). A sphere of \(\hat{E}^n\) is defined to be either a Euclidean sphere \(S(a, r)\), or an extended plane \(P(u, t) \cup \{\infty\}\). For simplicity of notation, we will also use \(P(u, t)\) to denote extended planes. In the case \(n = 2\), we may identify \(\hat{E}^2\) with the Riemann sphere \(\mathbb{C}\).

A Möbius transformation of \(\hat{E}^n\) is a finite composition of reflections in spheres of \(\hat{E}^n\). The set of all Möbius transformations of \(\hat{E}^n\), denoted by \(M(\hat{E}^n)\), forms a group under composition. By Proposition 2.2.6, any Euclidean isometry may be written as a finite composition of reflections, so we may by extending these reflections regard the group of Euclidean isometries to be a subgroup of \(M(\hat{E}^n)\). From the properties of reflections, Möbius transformations are continuous, differentiable and conformal.

We now discuss the action of \(M(\hat{E}^n)\) on spheres. A Möbius transformation of particular importance is the reflection of \(\hat{E}^n\) in the sphere \(S(\mathbf{e}_n, \sqrt{2})\). Identify \(\hat{E}^{n-1} \times \{0\}\) in \(\hat{E}^n\). The stereographic projection \(\pi\) of \(\hat{E}^{n-1}\) onto the sphere \(S^{n-1}\) is defined to be the projection of \(x \in \hat{E}^{n-1}\) towards (or away from) the vector \(\mathbf{e}_n\) until it meets the sphere \(S^{n-1}\). Let \(\sigma\) be the reflection of \(\hat{E}^n\) in the sphere \(S(\mathbf{e}_n, \sqrt{2})\). Then, by comparing the explicit formulae for \(\pi\) and \(\sigma\), it can be shown that the restriction of \(\sigma\) to \(\hat{E}^{n-1}\) is stereographic projection \(\pi : \hat{E}^{n-1} \to S^{n-1}\).

**Lemma 2.2.9.** Let \(\sigma\) be the reflection of \(\hat{E}^n\) in the sphere \(S(a, r)\), and let \(\sigma_1\) be the reflection of \(\hat{E}^n\) in the sphere \(S(0, 1)\). Define a map \(\tau : \hat{E}^n \to \hat{E}^n\) by \(\tau(x) = a + rx\).

Then \(\sigma = \tau \sigma_1 \tau^{-1}\).

**Proposition 2.2.10.** The group \(M(\hat{E}^n)\) acts transitively on the set of spheres of \(\hat{E}^n\).

**Proof.** We first need to show that \(M(\hat{E}^n)\) maps spheres to spheres. Let \(\phi\) be a Möbius transformation and let \(\Sigma\) be a sphere of \(\hat{E}^n\). As \(\phi\) is a composition of reflections, we may assume that \(\phi\) is a reflection. Suppose first that \(\phi\) is a reflection in an extended plane. Since \(\phi\) is an isometry, by Corollary 2.2.2, \(\phi\) maps hyperplanes to hyperplanes, and by the definition of a sphere \(\phi\) maps spheres to spheres. Now suppose that \(\phi\) is the reflection in the Euclidean sphere \(S(a, r)\), and let \(\tau\) be the same map as in Lemma 2.2.9. By Proposition 2.2.4 \(\tau\) is a Euclidean similarity, and so \(\tau\) maps spheres to spheres. We may thus, by Lemma 2.2.9 assume without loss of generality that \(\phi\) is the reflection in the sphere \(S(0, 1)\), that is, \(\phi(x) = x/|x|^2\).

We may characterize the spheres of \(\hat{E}^n\) by a vector \((a_0, a_1, \ldots, a_{n+1})\) in \(\mathbb{R}^{n+2}\), called a coefficient vector, which is unique up to multiplication by a non-zero scalar. The equations for the sphere \(S(a, r)\) and the extended plane \(P(a, t)\) in \(\hat{E}^n\) are respectively

\[
|\mathbf{x}|^2 - 2\mathbf{a} \cdot \mathbf{x} + |\mathbf{a}|^2 - r^2 = 0 \quad \text{and} \quad -2\mathbf{a} \cdot \mathbf{x} + 2t = 0.
\]

These may be written in the common form

\[a_0|\mathbf{x}|^2 - 2\mathbf{a} \cdot \mathbf{x} + a_{n+1} = 0, \quad (2.1)\]

with \(|\mathbf{a}|^2 > a_0a_{n+1}\). Conversely, any vector \((a_0, a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+2}\) such that \(|\mathbf{a}|^2 > a_0a_{n+1}\) determines a sphere of \(\hat{E}^n\) satisfying equation (2.1).
Now let \((a_0, a_1, \ldots, a_{n+1})\) be a coefficient vector for \(\Sigma\). Then \(x \in \Sigma\) satisfies equation (2.1), and so if \(y = \phi(x)\), by the formula for \(\phi\), the vector \(y\) satisfies
\[
a_0 - 2a \cdot y + a_{n+1}|y|^2 = 0.
\]
This is the equation of another sphere, say \(\Sigma'\). Hence \(\phi\) maps \(\Sigma\) into \(\Sigma'\). Since \(\phi\) is its own inverse, the same argument shows that \(\phi\) maps \(\Sigma'\) into \(\Sigma\). We have shown that \(M(\hat{E}^n)\) acts on the set of spheres of \(\hat{E}^n\).

To show that this action is transitive, let \(\Sigma\) be a sphere of \(\hat{E}^n\). It suffices to prove there is a Möbius transformation \(\phi\) such that \(\phi(\Sigma) = \hat{E}^{n-1}\). By Corollary 2.2.2, the group of Euclidean isometries acts transitively on the set of hyperplanes of \(E^n\). Since Euclidean isometries are Möbius transformations, this completes the proof if \(\Sigma\) is a hyperplane. We may thus assume that \(\Sigma\) is a Euclidean sphere. Now, the group of Euclidean similarities acts transitively on the set of spheres of \(E^n\). Let \(\psi\) be the similarity which maps \(\Sigma\) to \(S^{n-1}\). Then by Proposition 2.2.4, \(\psi\) has the form \(\psi(x) = \psi(0) + kA x\), for some orthogonal matrix \(A\) and some \(k > 0\). The map \(x \mapsto kx\) is the composition of the reflection in \(S(0, 1)\) followed by the reflection in \(S(0, \sqrt{k})\), and so is a Möbius transformation. Since \(A\) is orthogonal, \(x \mapsto Ax\) is an isometry, and so is also a Möbius transformation. Hence \(\psi\) is the composition of an isometry (translation) and a Möbius transformation. Therefore \(\psi\) is a Möbius transformation as well. We may now assume that \(\Sigma = S^{n-1}\). Let \(\sigma\) be the reflection in the sphere \(S(e_n, \sqrt{2})\). Then \(\sigma\) is stereographic projection, so we have \(\sigma(S^{n-1}) = \hat{E}^{n-1}\), and the proof is complete.

The proof of the following important lemma is omitted for reasons of brevity. See [21].

**Lemma 2.2.11.** Let \(\phi\) be a Möbius transformation which fixes every point of a sphere \(\Sigma\) of \(\hat{E}^n\). Then \(\phi\) is either the identity or the reflection in \(\Sigma\).

### 2.3 The hyperboloid model

We first establish useful definitions and results in Lorentzian \(n\)-space. Then, we define the hyperboloid model and show that it is a metric space. A discussion of its geodesics and trigonometry completes this section.

#### 2.3.1 Lorentzian \(n\)-space

Let \(x\) and \(y\) be vectors in \(\mathbb{R}^n\), where \(n > 1\), and consider the symmetric bilinear form
\[
x \circ y = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.
\]
This form is known as the Lorentzian inner product, although it is not positive definite. The space of \(\mathbb{R}^n\) together with \(\circ\) is called Lorentzian \(n\)-space. The Lorentzian norm derived from the Lorentzian inner product is \(\|x\| = (x \circ x)^{1/2}\). Then, \(\|x\|\) may be positive, zero, or positive imaginary. If \(\|x\|\) is imaginary, we denote its modulus by \(\|x\|\).

Using the Lorentzian norm, the vectors in \(\mathbb{R}^n\) may be partitioned into space-, light- and time-like sets (these labels derive from the use of Lorentzian 4-space as a model for special relativity). Space-like vectors are defined to be those with positive
Lorentzian norm, and light-like vectors those with Lorentzian norm zero. A vector $x$ is time-like if $\|x\|$ is imaginary, and is positive or negative time-like as $x_1 > 0$ or $x_1 < 0$. In $\mathbb{R}^3$, the light-like vectors are the cone $x_1^2 = x_2^2 + x_3^2$, the space-like vectors are the exterior of this cone, and the time-like vectors are the two disjoint convex sets forming the cone’s interior.

We now concentrate on time-like vectors and prove an analogue of the Cauchy–Schwartz inequality. Two vectors $x$ and $y$ are said to be Lorentz orthogonal if $x \circ y = 0$. If vectors $x$ and $y$ with non-zero Lorentzian norm are Lorentz orthogonal then they are linearly independent.

**Proposition 2.3.1.** Let $x$ and $y$ be non-zero Lorentz orthogonal vectors in $\mathbb{R}^n$. If $x$ is time-like then $y$ is space-like.

**Proof.** Since $x$ is time-like, $x_1^2 > x_2^2 + \cdots + x_n^2$. Therefore,

$$1 > \left( \sum_{i=2}^{n} x_i^2 \right) x_1^{-2}.$$  

As $x \circ y = 0$, we have $x_1 y_1 = x_2 y_2 + \cdots + x_n y_n$. Then, using the Cauchy–Schwartz inequality for the Euclidean inner product,

$$\|y\|^2 = y_1^2 + y_2^2 + \cdots + y_n^2$$

$$= -\left[ \left( \sum_{i=2}^{n} x_i y_i \right) x_1^{-1} \right]^2 + \sum_{i=2}^{n} y_i^2$$

$$\geq - \left( \sum_{i=2}^{n} x_i^2 \right) \left( \sum_{i=2}^{n} y_i^2 \right) x_1^{-2} + \sum_{i=2}^{n} y_i^2$$

$$= \left( \sum_{i=2}^{n} y_i^2 \right) \left[ 1 - \left( \sum_{i=2}^{n} x_i^2 \right) x_1^{-2} \right]$$

$$\geq 0.$$  

From the second last line, if $\|y\|^2 = 0$ then $\sum_{i=2}^{n} y_i^2 = 0$, implying $y_i = 0$ for $2 \leq i \leq n$. Then $y_1 = (\sum_{i=2}^{n} x_i y_i) x_1^{-1} = 0$, and so $y = 0$. But $y$ is non-zero. Thus, $\|y\| > 0$, that is, $y$ is space-like. \[\Box\]

An $n \times n$ matrix $A$ such that $A x \circ A y = x \circ y$ for all vectors $x$ and $y$ in $\mathbb{R}^n$ is said to be Lorentzian. The set of all Lorentzian matrices, denoted by $O(1, n - 1)$, forms a group under matrix multiplication. The subset of matrices $A \in O(1, n - 1)$ which transform positive time-like vectors into positive time-like vectors is in fact a subgroup of index two of $O(1, n - 1)$, and is denoted by $PO(1, n - 1)$. A vector subspace of $\mathbb{R}^n$ is said to be time-like if it contains a time-like vector.

**Proposition 2.3.2.** For each dimension $m$, the action of $PO(1, n - 1)$ on the set of $m$-dimensional time-like vector subspaces of $\mathbb{R}^n$ is transitive.

**Proof.** The method is a modification of the Gram–Schmidt process. Let $V$ be an $m$-dimensional time-like vector subspace of $\mathbb{R}^n$. Choose a basis $\{u_1, u_2, \ldots, u_m\}$.
Proposition 2.3.1. The vector $v$ and let $w$, then, $PO$ serves the Lorentzian inner product and is thus an element of $A$. By Proposition 2.3.2, there is a Lorentzian matrix $A$.

Proof. Consider some $x > y$ with equality if and only if $x = y$. Proposition 2.3.3. Suppose $x$ and $y$ are positive time-like vectors in $\mathbb{R}^n$. Then $x \circ y \leq ||x|| ||y||$, with equality if and only if $x$ and $y$ are linearly dependent.

Proof. By Proposition 2.3.2 there is a Lorentzian matrix $A$ so that $Ax = te_1$ for some $t > 0$. We may thus replace $x$ and $y$ by $Ax$ and $Ay$ without affecting the inner product or norms. So we may assume, without loss of generality, that $x = x_1 e_1$. Then,

$$||x||^2 ||y||^2 = -x_1^2 (-y_1^2 + y_2^2 + \cdots + y_n^2)$$

$$= x_1^2 y_1^2 - x_1^2 (y_2^2 + \cdots + y_n^2)$$

$$\leq x_1^2 y_1^2$$

$$= (x \circ y)^2,$$

with equality if and only if $y_2^2 + \cdots + y_n^2 = 0$, that is, $y = y_1 e_1$. Now, $||x||$ and $||y||$ are both positive imaginary, so $||x|| ||y|| < 0$. Also,

$$x \circ y = -x_1 y_1 < 0.$$  

So $x \circ y \leq ||x|| ||y||$, with equality if and only if $x$ and $y$ are linearly dependent.

The following corollary uses this inequality to define the Lorentzian time-like angle between positive time-like vectors $x$ and $y$. The Lorentzian angle is denoted $\eta(x, y)$, and will be used to define a distance function for the hyperboloid model of hyperbolic $n$-space.
Lemma 2.3.5. Facts about the Lorentzian cross product.

We have

\[ x \circ y = \|x\|\|y\| \cosh \eta(x, y). \]

Moreover, \( \eta(x, y) = 0 \) if and only if \( x \) and \( y \) are positive scalar multiples of each other.

To conclude our discussion of Lorentzian \( n \)-space, we define a Lorentzian cross product between vectors in \( \mathbb{R}^3 \). Let \( x \) and \( y \) be in \( \mathbb{R}^3 \), and let \( J \) be the matrix

\[
J = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

The Lorentzian cross product of \( x \) and \( y \) is defined to be \( x \otimes y = J(x \times y) \), where \( \times \) is the usual cross product in \( \mathbb{R}^3 \). It can be calculated that \( x \otimes y = Jy \times Jx \), and also that \( x \circ y = x \cdot Jy \). These identities can then be used to prove the following facts about the Lorentzian cross product.

Lemma 2.3.5. If \( w, x, y \) and \( z \) are vectors in \( \mathbb{R}^3 \), then

1. \( x \circ (x \otimes y) = y \circ (x \otimes y) = 0 \),
2. \( x \otimes y = -y \otimes x \),
3. \( (x \otimes y) \circ z = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \),
4. \( x \circ (y \otimes z) = (x \otimes y) \circ z \),
5. \( x \otimes (y \otimes z) = (x \circ y)z - (z \circ x)y \),
6. \( (x \otimes y) \circ (z \otimes w) = \det \begin{pmatrix} x \circ w & x \circ z \\ y \circ w & y \circ z \end{pmatrix} \).

The following corollaries define the Lorentzian angle \( \eta(x, y) \) between pairs of space-like vectors \( x \) and \( y \), and provide some useful identities involving Lorentzian angles and trigonometric or hyperbolic trigonometric functions.

Corollary 2.3.6. Let \( x \) and \( y \) be linearly independent positive time-like vectors in \( \mathbb{R}^3 \). Then \( x \otimes y \) is space-like, and \( \|x \otimes y\| = -\|x\|\|y\| \sinh \eta(x, y). \)

Proof. We have

\[
\|x \otimes y\|^2 = (x \otimes y) \circ (x \otimes y) \\
= (x \circ y)^2 - \|x\|^2\|y\|^2 \\
= \|x\|^2\|y\|^2 \cosh^2 \eta(x, y) - \|x\|^2\|y\|^2 \\
= \|x\|^2\|y\|^2 \sinh^2 \eta(x, y).
\]

Since \( \eta(x, y) > 0 \), and \( \|x\| \) and \( \|y\| \) are positive imaginary, on taking square roots of both sides we get

\[
\|x \otimes y\| = -\|x\|\|y\| \sinh \eta(x, y).
\]

The right-hand side is real and positive, so \( x \otimes y \) is space-like. \( \square \)
Corollary 2.3.7. Let \( \mathbf{x} \) and \( \mathbf{y} \) be space-like vectors in \( \mathbb{R}^3 \). If \( \mathbf{x} \otimes \mathbf{y} \) is time-like then \( |\mathbf{x} \odot \mathbf{y}| < \|\mathbf{x}\|\|\mathbf{y}\| \), and so there is a unique real number \( \eta(\mathbf{x}, \mathbf{y}) \in (0, \pi) \) such that

\[
\mathbf{x} \odot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos \eta(\mathbf{x}, \mathbf{y}).
\]

Proof. Follows immediately from the equality \( \|\mathbf{x} \otimes \mathbf{y}\|^2 = (\mathbf{x} \odot \mathbf{y})^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \).

Corollary 2.3.8. Let \( \mathbf{x} \) and \( \mathbf{y} \) be space-like vectors in \( \mathbb{R}^3 \). If \( \mathbf{x} \otimes \mathbf{y} \) is time-like, then

\[
\|\mathbf{x} \otimes \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\| \sin \eta(\mathbf{x}, \mathbf{y}).
\]

Proof. By Corollary 2.3.7 since \( \mathbf{x} \otimes \mathbf{y} \) is time-like, \( \mathbf{x} \odot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos \eta(\mathbf{x}, \mathbf{y}) \). Then by Lemma 2.3.5

\[
\|\mathbf{x} \otimes \mathbf{y}\|^2 = (\mathbf{x} \odot \mathbf{y})^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \cos^2 \eta(\mathbf{x}, \mathbf{y}) - \|\mathbf{x}\|^2\|\mathbf{y}\|^2 = -\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \sin^2 \eta(\mathbf{x}, \mathbf{y}),
\]

and the conclusion follows from \( \mathbf{x} \otimes \mathbf{y} \) being time-like.

2.3.2 The hyperboloid model

We define \( H^n \), the hyperboloid model of hyperbolic \( n \)-space, to be the set of positive time-like vectors of norm \( i \) in Lorentzian \( (n + 1) \)-space:

\[
H^n = \{ \mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\|^2 = -1 \text{ and } x_1 > 0 \}.
\]

We are disregarding the negative time-like vectors of norm \( i \) so as to obtain a connected set. The hyperbolic plane \( H^2 \) is thus one sheet of the two-sheeted hyperboloid \( x_1^2 = x_2^2 + x_3^2 + 1 \). Let \( \mathbf{x} \) and \( \mathbf{y} \) be vectors in \( H^n \) and let \( \eta(\mathbf{x}, \mathbf{y}) \) be the Lorentzian time-like angle between them. The hyperbolic distance between \( \mathbf{x} \) and \( \mathbf{y} \) is defined to be

\[
d_H(\mathbf{x}, \mathbf{y}) = \eta(\mathbf{x}, \mathbf{y}).
\]

Since \( \mathbf{x} \odot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cosh \eta(\mathbf{x}, \mathbf{y}) \), and the norms of \( \mathbf{x} \) and \( \mathbf{y} \) are equal to \( i \), we have the explicit formula

\[
d_H(\mathbf{x}, \mathbf{y}) = \cosh^{-1}(-\mathbf{x} \odot \mathbf{y}).
\]

Proposition 2.3.9. The function \( d_H \) is a distance function.

Proof. Since \( \mathbf{x} \odot \mathbf{y} = \mathbf{y} \odot \mathbf{x} \), \( d_H \) is symmetric. By the definition of \( \eta(\mathbf{x}, \mathbf{y}) \) in Corollary 2.3.4, \( d_H \) is non-negative. If \( \mathbf{x} = \mathbf{y} \), then \( \cosh^{-1}(-\mathbf{x} \odot \mathbf{y}) = \cosh^{-1}(1) = 0 \), so \( d_H(\mathbf{x}, \mathbf{y}) = 0 \). Conversely, if \( d_H(\mathbf{x}, \mathbf{y}) = 0 \), then by the definition of \( \eta(\mathbf{x}, \mathbf{y}) \), the vectors \( \mathbf{x} \) and \( \mathbf{y} \) must be positive scalar multiples of each other. But \( \mathbf{x} \) and \( \mathbf{y} \) have the same norm, which implies \( \mathbf{x} = \mathbf{y} \).

For the triangle inequality, we need to show that for all \( \mathbf{x}, \mathbf{y} \) and \( \mathbf{z} \) in \( H^n \),

\[
d_H(\mathbf{x}, \mathbf{z}) \leq d_H(\mathbf{x}, \mathbf{y}) + d_H(\mathbf{y}, \mathbf{z}).
\]

By Proposition 2.3.2 there exists a Lorentzian matrix \( A \) such that \( A\mathbf{x}, A\mathbf{y} \) and \( A\mathbf{z} \) are in the subspace of \( \mathbb{R}^{n+1} \) spanned by \( \mathbf{e}_1, \mathbf{e}_2 \) and \( \mathbf{e}_3 \). So we may, without loss of
generality, assume that \( n = 2 \). This allows us to use results about the Lorentzian cross product. By Corollary 2.3.6 \( x \otimes y \) and \( y \otimes z \) are space-like, and
\[
\|x \otimes y\| = \sinh \eta(x, y), \quad \|y \otimes z\| = \sinh \eta(y, z).
\]

We have, also,
\[
(x \otimes y) \otimes (y \otimes z) = ((x \otimes y) \circ y)z - (z \circ (x \otimes y))y = -(z \circ (x \otimes y))y,
\]
so the vectors \( y \) and \( (x \otimes y) \otimes (y \otimes z) \) are linearly dependent. Thus, \( (x \otimes y) \otimes (y \otimes z) \) is either zero or time-like, so we have by Corollary 2.3.7 that
\[
|(x \otimes y) \circ (y \otimes z)| \leq \|x \otimes y\| \|y \otimes z\|.
\]

Combining all these facts, we obtain
\[
cosh(\eta(x, y) + \eta(y, z)) = \cosh \eta(x, y) \cosh \eta(y, z) + \sinh \eta(x, y) \sinh \eta(y, z)
\]
\[
= (-x \circ y)(-y \circ z) + \|x \otimes y\| \|y \otimes z\|
\]
\[
\geq (x \circ y)(y \circ z) + (x \circ y) \circ (y \otimes z)
\]
\[
= (x \circ y)(y \circ z) + ((x \circ z)(y \circ y) - (x \circ y)(y \circ z))
\]
\[
= -x \circ z
\]
\[
= \cosh \eta(x, z).
\]

Since \( \cosh \) is monotonic increasing, we infer that \( \eta(x, z) \leq \eta(x, y) + \eta(y, z) \), as required.

The topology induced on \( H^n \) by the distance function \( d_H \) is the same as the subspace topology of \( H^n \), where \( H^n \) is regarded as a subspace of \( \mathbb{R}^{n+1} \) with its usual metric topology.

**Hyperbolic geodesics**

We now define hyperbolic lines, and show that the hyperbolic lines of \( H^n \) are its geodesics. A hyperbolic line of \( H^n \) is the non-empty intersection of \( H^n \) with a two-dimensional vector subspace of \( \mathbb{R}^{n+1} \). If \( x \) and \( y \) are distinct vectors in \( H^n \) then \( L(x, y) = H^n \cap \text{span}\{x, y\} \) is the unique hyperbolic line containing \( x \) and \( y \). Observe that \( L(x, y) \) is one branch of a hyperbola. We first establish a sufficient condition for points of \( H^n \) to lie on the same hyperbolic line.

**Lemma 2.3.10.** If \( x, y \) and \( z \) are points of \( H^n \) and
\[
\eta(x, y) + \eta(y, z) = \eta(x, y),
\]
then \( x, y \) and \( z \) are hyperbolically collinear, that is, there is a hyperbolic line of \( H^n \) containing \( x, y \) and \( z \).

**Proof.** Since \( \text{span}\{x, y, z\} \) has dimension at most 3, we may assume that \( n = 2 \) and use Lorentzian cross product results. From the proof that \( d_H \) is a distance function, \( \eta(x, y) + \eta(y, z) = \eta(x, y) \) if and only if
\[
(x \otimes y) \circ (y \otimes z) = \|x \otimes y\| \|y \otimes z\|. \tag{2.2}
\]
Also from that proof, the vector \((x \otimes y) \otimes (y \otimes z)\) is either zero or time-like. But if it were time-like, equation (2.2) would contradict Corollary 2.3.7. So \((x \otimes y) \otimes (y \otimes z)\) is 0. Now, since
\[
(x \otimes y) \otimes (y \otimes z) = -((x \otimes y) \circ z)y
\]
and y is time-like, we have that \((x \otimes y) \circ z = 0\). Therefore, by Lemma 2.3.5, the vectors \(x, y\) and \(z\) are linearly dependent. So \(x, y\) and \(z\) lie in a two-dimensional subspace of \(\mathbb{R}^{n+1}\), and are thus hyperbolically collinear.

The next proposition relates a geodesic curve \(\alpha\) to a parametrisation using Lorentz orthonormal vectors, and to a differential equation. Two vectors \(x, y\) in \(\mathbb{R}^{n+1}\) are said to be Lorentz orthonormal if \(\|x\|^2 = -1\), \(x \circ y = 0\), and \(\|y\|^2 = 1.\) This definition means that \(x\) must be time-like and \(y\) space-like. Note that \(e_1\) and \(e_2\) are Lorentz orthonormal.

**Proposition 2.3.11.** Let \(\alpha : [a, b] \to H^n\) be a curve. Then the following are equivalent:

1. The curve \(\alpha\) is a geodesic curve.
2. There are Lorentz orthonormal vectors \(x, y\) in \(\mathbb{R}^{n+1}\) such that
   \[
   \alpha(t) = (\cosh(t - a))x + (\sinh(t - a))y.
   \]
3. The curve \(\alpha\) satisfies the differential equation \(\alpha'' - \alpha = 0\).

**Proof.** If \(A\) is a Lorentzian matrix then \((A\alpha)' = A\alpha'\), so \(\alpha\) satisfies the differential equation if and only if \(A\alpha\) does. We may thus transform \(\alpha\) by a Lorentzian matrix.

Suppose \(\alpha\) is a geodesic curve, and \(t \in [a, b]\). We prove the second statement holds. Now,

\[
\eta(\alpha(a), \alpha(b)) = b - a = (t - a) + (b - t) = \eta(\alpha(a), \alpha(t)) + \eta(\alpha(t), \alpha(b)).
\]

By Lemma 2.3.10, \(\alpha(a), \alpha(t)\) and \(\alpha(b)\) are hyperbolically collinear. So \(\alpha([a, b])\) is contained in a hyperbolic line of \(H^n\). Thus we may assume that \(n = 1\), and so consider \(H^1\) as the branch of the hyperbola \(x_1^2 = 1 + x_2^2\) on which \(x_1 > 0\). By Proposition 2.3.2, we may apply a Lorentz transformation to map \(\alpha(a)\) to \(e_1\), and so we may assume that \(\alpha(a) = e_1\). Then,

\[
e_1 \cdot \alpha(t) = (-\alpha(a)) \circ \alpha(t) = \cosh \eta(\alpha(a), \alpha(t)) = \cosh(t - a).
\]

From the equation \(x_1^2 = 1 + x_2^2\), we have \(e_2 \cdot \alpha(t) = \pm \sinh(t - a)\). Since \(\alpha\) is continuous, either \(e_2 \cdot \alpha(t) = \sinh(t - a)\) for all \(t\), or \(e_2 \cdot \alpha(t) = -\sinh(t - a)\) for all \(t\). In the second case, we may reflect in the \(x_2\)-axis, and so we may assume that \(\alpha\) has the form

\[
\alpha(t) = (\cosh(t - a))e_1 + (\sinh(t - a))e_2.
\]

Now suppose there are Lorentz orthonormal vectors \(x, y\) such that

\[
\alpha(t) = (\cosh(t - a))x + (\sinh(t - a))y.
\]
Let \( s \) and \( t \) be such that \( a \leq s \leq t \leq b \). Then

\[
\cosh \eta(\alpha(s), \alpha(t)) = -\alpha(s) \circ \alpha(t)
\]
\[
= -(-\cosh(s-a)\cosh(t-a) + \sinh(s-a)\sinh(t-a))
\]
\[
= \cosh((t-a)-(s-a))
\]
\[
= \cosh(t-s),
\]
so \( \eta(\alpha(s), \alpha(t)) = t - s \). Therefore \( \alpha \) is a geodesic curve.

If the second statement holds then just differentiate to find that \( \alpha'' - \alpha = 0 \).

Finally, suppose that \( \alpha'' - \alpha = 0 \). Then

\[
\alpha(t) = (\cosh(t-a))\alpha(a) + (\sinh(t-a))\alpha'(a)
\] (2.3)
is a solution of this differential equation which satisfies the initial value conditions at \( t = a \). By uniqueness of such solutions, \( \alpha(t) \) must have the form (2.3). We now just need to prove that \( \alpha(a) \) and \( \alpha'(a) \) are Lorentz orthonormal. By differentiating the equation \( \alpha(t) \circ \alpha(t) = -1 \) we obtain \( \alpha(t) \circ \alpha'(t) = 0 \) for all \( t \). In particular, \( \alpha(a) \circ \alpha'(a) = 0 \). Also,

\[
\|\alpha(t)\|^2 = \alpha(t) \circ \alpha(t) = \cosh^2(t-a)\|\alpha(a)\|^2 + \sinh^2(t-a)\|\alpha'(a)\|^2.
\]

Since \( \alpha \) maps into \( H^2 \), \( \|\alpha(a)\|^2 = \|\alpha(t)\|^2 = -1 \), and so \( \|\alpha'(a)\|^2 = 1 \). This completes the proof.

The next proposition uses these results to characterise the geodesic lines of \( H^n \).

**Proposition 2.3.12.** A function \( \lambda : \mathbb{R} \to H^n \) is a geodesic line if and only if there are Lorentz orthonormal vectors \( x \) and \( y \) such that

\[
\lambda(t) = (\cosh t)x + (\sinh t)y.
\]

**Proof.** Suppose such vectors \( x \) and \( y \) exist. Then \( \lambda \) satisfies the differential equation \( \lambda'' - \lambda = 0 \). By Proposition 2.3.11 the restriction of \( \lambda \) to any interval \([a, b]\) is a geodesic curve, and so \( \lambda \) is a geodesic line.

For the converse, suppose \( \lambda \) is a geodesic line. Then \( \lambda'' - \lambda = 0 \) so, as in the proof of Proposition 2.3.11

\[
\lambda(t) = (\cosh t)\lambda(0) + (\sinh t)\lambda'(0).
\]
The vectors \( \lambda(0) \) and \( \lambda'(0) \) are Lorentz orthonormal by the same argument as in the proof of Proposition 2.3.11.

**Corollary 2.3.13.** The geodesics of \( H^n \) are its hyperbolic lines.

**Proof.** Suppose \( \lambda \) is a geodesic line of \( H^n \). By Proposition 2.3.12 there exist vectors \( x \) and \( y \), one of which is time-like, such that

\[
\lambda(t) = (\cosh t)x + (\sinh t)y.
\]
Then the image of $\lambda$ is the intersection of $H^n$ and $\text{span}\{x, y\}$, which is a hyperbolic line.

Conversely, let $L$ be a hyperbolic line of $H^n$. As in the proof of Proposition 2.3.11 we may assume that $n = 1$. Then $L = H^1$. Define $\lambda : \mathbb{R} \to H^1$ by

$$\lambda(t) = (\cosh t)e_1 + (\sinh t)e_2.$$  

Then by Proposition 2.3.12 $\lambda$ is a geodesic line which is onto $H^1 = L$. So $L$ is a geodesic.

**Hyperbolic trigonometry**

We now restrict our attention to the hyperbolic plane $H^2$, and discuss hyperbolic trigonometry. For reasons of space, the treatment here is cursory; again, see [21] for the details. In particular, the area of regions in the hyperbolic plane is not needed at all in later chapters, so we do not define area.

The definition of a hyperbolic triangle is as follows. Suppose $x, y$ and $z$ in $H^2$ are hyperbolically non-collinear. Let $L(x, y)$ be the unique hyperbolic line of $H^2$ containing $x$ and $y$, and let $H(x, y, z)$ be the closed half-plane of $H^2$ with $z$ in its interior and $L(x, y)$ as its boundary. The hyperbolic triangle with vertices at $x, y$ and $z$ is defined to be

$$T(x, y, z) = H(x, y, z) \cap H(y, z, x) \cap H(z, x, y).$$

The sides of this triangle are the geodesic segments $[x, y]$, $[y, z]$ and $[z, x]$. To define the angles of this triangle, let $a = d_H(y, z)$, $b = d_H(z, x)$ and $c = d_H(x, y)$. Let $f : [0, a] \to H^2$, $g : [0, b] \to H^2$ and $h : [0, c] \to H^2$ be the geodesic curves joining $y$ to $z$, $z$ to $x$, and $x$ to $y$ respectively. Then, the hyperbolic angle $\alpha$ at the vertex $x$ is defined to be the Lorentzian angle between the vectors $-g'(b)$ and $h'(0)$. Angles $\beta$ and $\gamma$ at vertices $y$ and $z$ are defined similarly. The following sketch shows the relationships between vectors, sides and angles.

The trigonometric identities below may be proved using Lemma 2.3.5 and Corollaries 2.3.6, 2.3.8. In each of these identities, $\alpha$, $\beta$ and $\gamma$ are the angles of a hyperbolic triangle, and $a$, $b$ and $c$ are the lengths of the opposite sides.
Theorem 2.3.14 (Sine Rule).
\[
\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}.
\]

Theorem 2.3.15 (First Cosine Rule).
\[
\cos \gamma = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}.
\]

Theorem 2.3.16 (Second Cosine Rule).
\[
\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}.
\]

We now generalise the definition of hyperbolic triangle to include the possibility of vertices at infinity. Suppose \(L_1\) and \(L_2\) are hyperbolic lines, with \(L_1\) the intersection of \(H^2\) with the plane \(\text{span}\{x_1, y_1\}\), and \(L_2\) the intersection of \(H^2\) with the plane \(\text{span}\{x_2, y_2\}\). If \(\text{span}\{x_1, y_1\} \cap \text{span}\{x_2, y_2\}\) is a one-dimensional subspace of \(\mathbb{R}^{n+1}\) consisting of light-like vectors, then \(L_1\) and \(L_2\) are said to meet at infinity. Hyperbolic lines which meet at infinity are disjoint, but become asymptotically close in one direction. A generalised hyperbolic triangle, then, is defined in the same way as a hyperbolic triangle, except that the hyperbolic lines defining adjacent sides may meet at infinity. The angle at a vertex at infinity is defined to be zero.

The Second Cosine Rule may be extended to a generalised hyperbolic triangle with one vertex at infinity. If such a triangle has angles \(\alpha, \beta = \frac{\pi}{2}\) and \(\gamma = 0\), and finite side of length \(c\), we obtain
\[
c = \cosh^{-1}\left(\frac{1}{\sin \alpha}\right).
\]

This shows that there is an absolute unit of length in hyperbolic geometry, as the length \(c\) depends only on the angle \(\alpha\).

Theorem 2.3.17. Let \(T\) be a generalised hyperbolic triangle. If the angles of \(T\) are \(\alpha, \beta\) and \(\gamma\), then the area of \(T\) is \(\pi - \alpha - \beta - \gamma\).

Corollary 2.3.18. The angle sum of a generalised hyperbolic triangle is less than \(\pi\).

2.4 The projective disc model

The projective disc model is mapped to \(H^n\) by a projection known as gnomonic projection. Its chief feature is that its hyperbolic lines are the same as its Euclidean lines.

The open unit disc in \(\mathbb{R}^n\) is the set \(D^n = \{x \in \mathbb{R}^n : |x| < 1\}\). Identify \(\mathbb{R}^n\) with \(\mathbb{R}^n \times \{0\}\) in \(\mathbb{R}^{n+1}\). Let \(\mu\) be the translation of \(D^n\) vertically by \(e_{n+1}\), followed by radial projection to \(H^n\). The map \(\mu\) is known as gnomonic projection, is a bijection, and has the explicit formula
\[
\mu(x) = \frac{x + e_{n+1}}{\|x + e_{n+1}\|}.
\]
We now define a distance function $d_D$ on $D^n$ by

$$d_D(x, y) = d_H(\mu(x), \mu(y)).$$

The projective disc model of hyperbolic $n$-space is the metric space consisting of $D^n$ together with the distance function $d_D$.

A subset $L$ of $D^n$ is called a hyperbolic line of $D^n$ if $\mu(L)$ is a hyperbolic line of $H^n$. Since $\mu : D^n \to H^n$ is an isometry, and the geodesics of $H^n$ are its hyperbolic lines, the geodesics of $D^n$ are its hyperbolic lines. The following characterisation of the hyperbolic lines of $D^2$ shows that in this model, hyperbolic lines are the same as Euclidean lines.

**Proposition 2.4.1.** A subset $L$ of $D^2$ is a hyperbolic line of $D^2$ if and only if $L$ is an open chord of $D^2$.

**Proof.** Let $L$ be a hyperbolic line of $H^2$. Then $L$ is the intersection of $H^2$ with the plane $P(u, 0)$, where $u$ is some unit vector in $\mathbb{R}^3$. Radial projection maps $L$ onto the line of intersection of the plane $P(u, 0)$ and the plane $P(\mathbf{e}_3, 1)$. The translation by the vector $-\mathbf{e}_3$ which follows ensures that $\mu^{-1}(L)$ is an open chord of $D^2$. This argument can be reversed for the converse. \qed

### 2.5 The conformal ball model

The conformal ball model is mapped to $H^n$ by a projection known as stereographic projection. We show that the isometries of this model are M"obius transformations, and that the hyperbolic angles of this model are the same as Euclidean angles.

First, we redefine the Lorentzian inner product on $\mathbb{R}^{n+1}$ to be

$$\mathbf{x} \circ \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$ 

All results of Section 2.3 are still true, once the order of coordinates in $\mathbb{R}^{n+1}$ is reversed. Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. Let $B^n$ be the open unit ball in $\mathbb{R}^n$ (this is the same set as $D^n$, but we will soon be defining a different distance function for $B^n$, and so wish to distinguish the two metric spaces). The stereographic projection $\zeta$ of $B^n$ onto $H^n$ is the projection of $\mathbf{x} \in B^n$ away from the vector $-\mathbf{e}_{n+1}$ until it meets $H^n$ in the unique point $\zeta(\mathbf{x})$. The explicit formula for this map is

$$\zeta(\mathbf{x}) = \left(\frac{2x_1}{1 - |\mathbf{x}|^2}, \ldots, \frac{2x_n}{1 - |\mathbf{x}|^2}, \frac{1 + |\mathbf{x}|^2}{1 - |\mathbf{x}|^2}\right). \quad (2.4)$$

The projection $\zeta$ is a bijection from $B^n$ to $H^n$.

The conformal ball model of hyperbolic $n$-space is defined to be the set $B^n$ together with the distance function $d_B$ given by

$$d_B(x, y) = d_H(\zeta(x), \zeta(y)).$$
By the explicit formula for $d_H$, and (2.4),
\[
\cosh d_H(\zeta(x), \zeta(y)) = -\zeta(x) \circ \zeta(y) = \frac{-4x \cdot y + (1 + |x|^2)(1 + |y|^2)}{(1 - |x|^2)(1 - |y|^2)} = 1 + \frac{2|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}.
\]
Thus an explicit formula for $d_B$ is
\[
d_B(x, y) = \cosh^{-1}\left(1 + \frac{2|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}\right).
\]
The topology induced on $B^n$ by $d_B$ is the same as the subspace topology of $B^n$ as a subspace of $\mathbb{R}^n$.

A subset $L$ of $B^n$ is defined to be a hyperbolic line of $B^n$ if $\zeta(L)$ is a hyperbolic line of $H^n$. By Corollary 2.3.13 and the fact that $\zeta$ is an isometry, the geodesics of $B^n$ are its hyperbolic lines. We now characterise the hyperbolic lines of $B^2$.

**Proposition 2.5.1.** A subset $L$ is a hyperbolic line of $B^2$ if and only if $L$ is either an open diameter of $B^2$, or the intersection of $B^2$ with a circle orthogonal to $S^1$.

**Proof.** Let $L$ be an open diameter of $B^2$. Then $L$ is the intersection of $B^2$ with the one-dimensional subspace of $\mathbb{R}^3$ spanned by, say, the vector $v$. The projection $\zeta$ maps $L$ onto the hyperbolic line of $H^2$ obtained by intersecting $H^2$ with the two-dimensional subspace spanned by $\{v, e_3\}$. Thus $L$ is a hyperbolic line of $B^2$. This argument can be reversed to show that if $\zeta(L)$ is a hyperbolic line of $H^2$ which passes through the point $e_3$, then $L$ is an open diameter of $B^2$.

Now, if $x = (x_1, x_2, x_3) \in H^2$ then $x_1^2 + x_2^2 - x_3^2 = -1$. The inverse of the projection $\zeta$ is thus the map
\[
\zeta^{-1}(x_1, x_2, x_3) = \frac{1}{x_3 + 1}(x_1, x_2).
\]
Let $R$ be a rotation of $\mathbb{R}^3$ about the $x_3$ axis, and identify $R$ with a rotation of $\mathbb{R}^2$. Then $\zeta^{-1}(R(x)) = R(\zeta^{-1}(x))$ for any $x \in H^2$. So we may consider, without loss of generality, a hyperbolic line of $H^2$ which is of the form $H^2 \cap P(w, 0)$, where $w = (-\sin \theta, 0, \cos \theta)$ for some $\theta \in (\frac{\pi}{2}, \frac{\pi}{3})$. Let $L = \zeta^{-1}(H^2 \cap P(w, 0))$. Then $L$ is a hyperbolic line of $B^2$. Now, $x \in P(w, 0)$ if and only if $x_3 = x_1 \tan \theta$. This means that $x \in H^2 \cap P(w, 0)$ if and only if
\[
x_2^2 = x_1^2 \tan^2 \theta - x_1^2 - 1.
\]
Abbreviate $\tan \theta$ to $t$. Then
\[
\zeta^{-1}(x) = \frac{1}{x_3 + 1}(x_1, x_2) = \frac{1}{x_1 t + 1}\left(x_1, \sqrt{x_1^2 t^2 - x_1^2 - 1}\right).
\]
The circle in $\mathbb{R}^2$ with centre $(t,0)$ and radius $\sqrt{t^2 - 1}$ meets $S^1$ and is orthogonal to it. See the diagram in the proof of Lemma 2.5.2 below. We have

$$|\zeta^{-1}(x) - (t,0)|^2 = \left(\frac{x_1}{x_1 t + 1} - t\right)^2 + \left(\frac{\sqrt{x_1^2 t^2 - x_1^2 - 1}}{x_1 t + 1}\right)^2 = \cdots = t^2 - 1.$$ 

So $L$, the image of $H^2 \cap P(w, 0)$ under $\zeta^{-1}$, lies in the circle centre $(t,0)$ and radius $\sqrt{t^2 - 1}$. Since hyperbolic lines of $H^2$ are connected, and $\zeta^{-1}$ is continuous, $L$ is a connected arc. As $x_1 \to \infty$, the arc $L$ approaches the boundary of $B^2$. Thus $L$ is the intersection of $B^2$ with a circle orthogonal to $S^1$. This argument can be reversed to prove the converse.

### 2.5.1 Möbius transformations and isometries

A map $\phi : \hat{E}^n \to \hat{E}^n$ is said to leave $B^n$ invariant if it maps $B^n$ bijectively onto itself. (If, in addition, $\phi(x) = x$ for all $x$ in $B^n$, we say that $\phi$ fixes each point of $B^n$.) We define a Möbius transformation of $B^n$ to be a Möbius transformation of $\hat{E}^n$ which leaves $B^n$ invariant. Our aim is to show that $I(B^n)$, the group of isometries of $B^n$, is isomorphic to $M(B^n)$, the group of Möbius transformations of $B^n$.

The following lemma gives sufficient and necessary conditions for a reflection in a sphere to leave $B^n$ invariant. Two spheres are said to be orthogonal if they intersect in $E^n$, and at each point of intersection their normal lines are orthogonal.

**Lemma 2.5.2.** Let $\sigma$ be the reflection of $\hat{E}^n$ in the sphere $S(a, r)$. Then $\sigma$ leaves $B^n$ invariant if and only if $S(a, r)$ is orthogonal to $S^{n-1}$.

**Proof.** By considering the intersection of $S(a, r)$, $B^n$ and $S^{n-1}$ with two-dimensional subspaces of $\hat{E}^n$ which contain the vector $a$, we may assume that $n = 2$. The circles $S(a, r)$ and $S^1$ are orthogonal if and only if, at their points of intersection, their respective radii are orthogonal. Thus, $S(a, r)$ is orthogonal to $S^1$ if and only if $|a|^2 = r^2 + 1$. 

---

![Diagram](image-url)
Suppose \( S(a, r) \) is orthogonal to \( S^1 \). Then for \( x \in B^2 \),

\[
|\sigma(x)|^2 = \left| a + r^2 \frac{x - a}{|x - a|^2} \right|^2 \\
= \left| a + \frac{|a|^2 - 1}{|x - a|^2} (x - a) \right|^2 \\
= \ldots \\
= \frac{|x - a|^2 + (1 - |x|^2)(1 - |a|^2)}{|x - a|^2} < 1,
\]

since \( |x| < 1 \) and \( |a| > 1 \). Thus \( \sigma(B^2) \) is contained in \( B^2 \). To show that \( \sigma \) is onto \( B^2 \), let \( y \) be in \( B^2 \). Then, since \( \sigma \) is its own inverse, \( \sigma(x) = y \) if and only if \( x = \sigma(y) \). But \( \sigma \) maps \( B^2 \) into \( B^2 \), so \( x \in B^2 \), hence \( \sigma \) is onto. Therefore \( \sigma \) leaves \( B^2 \) invariant.

For the converse, we identify \( B^2 \) with the open unit disc in \( \mathbb{C} \) and \( S^1 \) with the unit circle in \( \mathbb{C} \). Without loss of generality, we may consider the centre of the circle \( S(a, r) \) to lie on the real axis at a point \( a \geq 0 \). We prove the contrapositive. Suppose \( S(a, r) \) is not orthogonal to \( S^1 \). We show that there exists a point of \( B^2 \) which is mapped outside \( B^2 \) by \( \sigma \). If \( a < 1 \), let \( \delta > 0 \) be such that \( a + \delta < 1 \). Then \( a + \delta \in B^2 \), but for sufficiently small \( \delta \),

\[
\sigma(a + \delta) = a + \frac{r^2}{\delta} > 1.
\]

If \( a \geq 1 \) and \( a^2 < r^2 + 1 \), then for small enough \( \delta > 0 \), we have \( a^2 + \delta(a + 1) < r^2 + 1 \) and \( 1 - \delta \in B^2 \). However,

\[
\sigma(1 - \delta) = a - \frac{r^2}{a - (1 - \delta)} < -1.
\]

Finally, if \( a > 1 \) and \( a^2 > 1 + r^2 \), then choose \( \delta > 0 \) so that \( a^2 - \delta(a - 1) > 1 + r^2 \) and \( -1 + \delta \in B^2 \). We have \( \sigma(-1 + \delta) > 1 \). So, if \( S(a, r) \) is not orthogonal to \( S^1 \), the reflection \( \sigma \) does not leave \( B^2 \) invariant.

**Lemma 2.5.3.** If \( \phi \) is a M"{o}bius transformation of \( \hat{E}^n \) leaving \( B^n \) invariant, and \( x, y \) are in \( B^n \), then

\[
\frac{|\phi(x) - \phi(y)|^2}{(1 - |\phi(x)|^2)(1 - |\phi(y)|^2)} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}.
\]

**Proof.** Since \( \phi \) is a composition of reflections, we may assume that \( \phi \) is a reflection. If \( \phi \) is a reflection in a plane then, in order that \( \phi \) leave \( B^n \) invariant, \( \phi \) must be a reflection in a plane passing through the origin. Then \( \phi \) preserves Euclidean distances, and \( \phi(0) = 0 \), so \(|\phi(x)| = |x|, |\phi(y)| = |y| \) and \(|\phi(x) - \phi(y)| = |x - y| \). The result follows immediately.
We are left with the possibility that \( \phi \) is a reflection in a sphere \( S(a, r) \). By Lemma 2.2.7
\[
\frac{|\phi(x) - \phi(y)|}{|x - y|} = \frac{r^2}{|x - a||y - a|}.
\]
By Lemma 2.5.2, \( S(a, r) \) must be orthogonal to \( S^{n-1} \), and from its proof, this means that \( |a|^2 = r^2 + 1 \). The formula for the reflection \( \phi \) is
\[
\phi(x) = a + \frac{r^2}{|x - a|^2}(x - a),
\]
hence
\[
|\phi(x)|^2 - 1 = |a|^2 - 1 + \frac{2r^2}{|x - a|^2}a \cdot (x - a) + \frac{r^4}{|x - a|^2}
= \frac{r^2|x - a|^2 + 2r^2a \cdot (x - a) + r^4}{|x - a|^2}
= \ldots
= \frac{r^2(|x|^2 - 1)}{|x - a|^2}.
\]
Putting all this together, we obtain
\[
\frac{|\phi(x) - \phi(y)|^2}{|x - y|^2} = \frac{r^4}{|x - a|^2|y - a|^2} = \frac{(1 - |\phi(x)|^2)(1 - |\phi(y)|^2)}{(1 - |x|^2)(1 - |y|^2)}.
\]
Rearrange this equation to finish the proof.
\[\Box\]

**Theorem 2.5.4.** Every M"obius transformation of \( \hat{E}^n \) which leaves \( B^n \) invariant restricts to an isometry of \( B^n \), and every isometry of \( B^n \) extends to a unique M"obius transformation of \( \hat{E}^n \) which leaves \( B^n \) invariant.

**Proof.** The claim that M"obius transformations restrict to isometries follows immediately from the explicit formula for \( d_B \) together with Lemma 2.5.3

Conversely, let \( \phi : B^n \to B^n \) be an isometry. We construct an extension for \( \phi \) as follows. First, for any \( b \in B^n \), we construct a M"obius transformation \( \tau_b \) such that \( \tau_b \) leaves \( B^n \) invariant, and \( \tau_b(0) = b \). Next, we consider the map \( \tau_{\phi(0)}^{-1} \phi \), which fixes \( 0 \), and show that this map is orthogonal. It follows that \( \tau_{\phi(0)}^{-1} \phi \) extends to a M"obius transformation of \( \hat{E}^n \). We then prove that this extension is unique.

Let \( S(a, r) \) be a sphere of \( E^n \) orthogonal to \( S^{n-1} \), and let \( \sigma_a \) be the reflection in this sphere (since \( r^2 = |a|^2 - 1 \), the radius \( r \) is a function of \( a \)). Let \( \rho_a \) be the reflection in the hyperplane through \( 0 \) normal to \( a \). Then by Lemma 2.5.2 \( \sigma_a \) and \( \rho_a \) leave \( B^n \) invariant, so the composition \( \rho_a \sigma_a \) leaves \( B^n \) invariant. A calculation shows that \( \rho_a \sigma_a(0) = -a/|a|^2 \). For \( b \neq 0 \) in \( B^n \), set \( b' = -b/|b|^2 \). If \( r = (|b|^2 - 1)^{1/2} \) then \( S(b', r) \) is orthogonal to \( S^{n-1} \), so we may define a M"obius transformation of \( B^n \) by the formula \( \tau_b = \rho_{b'} \sigma_{b'} \). Define \( \tau_0 \) to be the identity map. Then for all \( b \) in \( B^n \), \( \tau_b(0) = b \).
We now define a map $\psi : B^n \to B^n$ by $\psi = \tau_{\phi(0)}^{-1}$. We have $\psi(0) = 0$. And, since $\psi$ is a composition of an isometry and a Möbius transformation, by the first part of this theorem, $\psi$ is an isometry of $B^n$. Let $x$ and $y$ be in $B^n$. Since $d_B(\psi(x), 0) = d_B(x, 0)$, we have

$$\frac{|\psi(x)|^2}{1 - |\psi(x)|^2} = \frac{|x|^2}{1 - |x|^2},$$

hence $|\psi(x)| = |x|$. Also, since $d_B(\psi(x), \psi(y)) = d_B(x, y)$, we have

$$\frac{|\psi(x) - \psi(y)|^2}{(1 - |\psi(x)|^2)(1 - |\psi(y)|^2)} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)},$$

and so $|\psi(x) - \psi(y)| = |x - y|$. Thus $\psi$ preserves Euclidean distances in $B^n$. It is known that any mapping $\psi$ of the open ball $B^n$ which preserves Euclidean distances and fixes the point $0$ is the restriction to $B^n$ of an orthogonal transformation, say $A$, of $E^n$. Hence, $\tau_{\phi(0)}A$ is an extension of $\phi$. Now, $A$ is a Euclidean isometry, so is also a Möbius transformation of $E^n$. Thus $\tau_{\phi(0)}A$ is a Möbius transformation of $\hat{E}^n$ which leaves $B^n$ invariant and extends $\phi$.

To show that this extension is unique, suppose that $v$ is another Möbius transformation of $B^n$ which extends $\phi$, and let $\xi$ be the composition $v^{-1}\tau_{\phi(0)}A$. By the continuity of $\xi$, we have $\xi(x) = x$ for all $x \in B^n$. So by Lemma 2.2.11, $\xi$ is the identity. Therefore $\tau_{\phi(0)}A$ is the unique Möbius extension of $\phi$. \qed

**Corollary 2.5.5.** The groups $I(B^n)$ and $M(B^n)$ are isomorphic.

We can now explain why the hyperbolic angles of $B^n$ are the same as its Euclidean angles. From the geometric definition of the map $\zeta : B^n \to H^n$, this projection preserves the Euclidean angle between any two hyperbolic lines of $B^n$ which intersect at the origin. By considering the tangent space to $H^n$ at $\zeta(0) = e_{n+1}$, it can be proved that the hyperbolic angle between two hyperbolic lines of $H^n$ intersecting at $\zeta(0)$ is the same as the Euclidean angle. So, the hyperbolic angle between two hyperbolic lines of $B^n$ which intersect at the origin is the same as the Euclidean angle. Moreover, the isometries of $B^n$ are Möbius transformations, which are conformal and act transitively on the set of hyperbolic lines of $B^n$. Therefore, the hyperbolic angle between any two intersecting hyperbolic lines in $B^n$ is the same as the Euclidean angle between these lines. This is why $B^n$ is called the conformal ball model of hyperbolic $n$-space.

### 2.6 The upper half-space model

The upper half-space model is mapped to $B^n$ by a Möbius transformation. As in the conformal ball model, we may identify isometries and Möbius transformations. In the two-dimensional case, there are many interesting relationships between isometries, maps known as linear fractional transformations, and groups of matrices.

Consider the upper half-space $U^n = \{x \in \hat{E}^n : x_n > 0\}$. There is a standard transformation $\eta$ from $U^n$ onto the open unit ball $B^n$. It is given by $\eta = \sigma \rho$, where $\rho$ is the reflection in the boundary $\partial \hat{E}^{n-1}$, and $\sigma$ is the reflection in the sphere
The upper half-space model of hyperbolic space is the set $U^n$ together with the distance function $d_U$ defined by

$$d_U(x, y) = d_B(\eta(x), \eta(y)).$$

The topology induced on $U^n$ by $d_U$ is the subspace topology of $U^n$ as a subspace of $\mathbb{R}^n$. We will mainly be considering the upper half-plane $U^2$, which may be identified with the set of complex numbers

$$\{ z \in \mathbb{C} : \text{Im}(z) > 0 \}.$$

**Lemma 2.6.1.** The explicit formula for $d_U$ is

$$d_U(x, y) = \cosh^{-1} \left( 1 + \frac{|x - y|^2}{2x_n y_n} \right).$$

**Proof.** From the explicit formula for $d_B$, we have

$$\cosh d_U(x, y) = \cosh d_B(\eta(x), \eta(y)) = 1 + \frac{2|\sigma \rho(x) - \sigma \rho(y)|^2}{(1 - |\sigma \rho(x)|^2)(1 - |\sigma \rho(y)|^2)}.$$

Then, by Lemma 2.2.7 applied to $\sigma$, and $\rho$ being a Euclidean isometry, we have

$$|\sigma \rho(x) - \sigma \rho(y)| = \frac{2|\rho(x) - \rho(y)|}{|\rho(x) - e_n||\rho(y) - e_n|} = \frac{2|x - y|}{|x + e_n||y + e_n|}.$$

Also,

$$|\sigma \rho(x)|^2 = \left| e_n + \frac{2(\rho(x) - e_n)}{|\rho(x) - e_n|^2} \right|^2$$

$$= 1 + \frac{4e_n \cdot (\rho(x) - e_n)}{|\rho(x) - e_n|^2} + \frac{4}{|\rho(x) - e_n|^2}$$

$$= 1 + \frac{4[\rho(x)]_n}{|x + e_n|^2},$$

hence

$$1 - |\sigma \rho(x)|^2 = \frac{4x_n}{|x + e_n|^2}.$$

These results may be combined to complete the proof. \qed

**Corollary 2.6.2.** Let $ia$ and $ib$, where $b \geq a > 0$, be points on the imaginary axis in the upper half-plane $U^2$. Then the hyperbolic distance between $ia$ and $ib$ is $\log \frac{b}{a}$.

A subset $L$ of $U^n$ is, by definition, a hyperbolic line of $U^n$ if $\eta(L)$ is a hyperbolic line of $B^n$. Thus the geodesics of $U^n$ are its hyperbolic lines. Now, the boundary of $U^n$ is mapped onto the boundary of $B^n$ by $\eta$ which is a Möbius transformation. So $\eta$ is conformal and maps spheres to spheres. By Proposition 2.5.1, a hyperbolic line of $B^2$ is the intersection of $B^2$ with either a Euclidean line or a Euclidean circle which is orthogonal to its boundary $S^1$. Hence, $L$ is a hyperbolic line of $U^2$ if and
only if $L$ is the intersection of $U^2$ with either a Euclidean line or a Euclidean circle which is orthogonal to the real axis.

Another consequence of $\eta$ being conformal is that, since $B^n$ is a conformal model of hyperbolic $n$-space, $U^n$ is also a conformal model.

2.6.1 Möbius transformations, isometries and linear fractional transformations

As in the conformal ball model, we may identify the Möbius transformations and isometries of $U^n$. A Möbius transformation of $U^n$ is a Möbius transformation of $\mathring{E}^n$ which maps $U^n$ bijectively onto itself.

**Theorem 2.6.3.** Every Möbius transformation of $\mathring{E}^n$ which leaves $U^n$ invariant restricts to an isometry of $U^n$, and every isometry of $U^n$ extends to a unique Möbius transformation of $\mathring{E}^n$ which leaves $U^n$ invariant.

**Proof.** Follows immediately from Theorem 2.5.4 and the fact that $\eta$ is an isometry. 

**Corollary 2.6.4.** The groups $I(U^n)$ and $M(U^n)$ are isomorphic.

We now consider just the two-dimensional case, where $U^2$ is the complex upper half-plane. An important class of maps on $\mathring{C}$ is the linear fractional transformations. A linear fractional transformation is a continuous map $\phi : \mathring{C} \to \mathring{C}$ of the form

$$\phi(z) = \frac{az + b}{cz + d},$$

where $a, b, c$ and $d$ are in $\mathbb{C}$ and $ad - bc \neq 0$. Let $LF(\mathring{C})$ be the set of all linear transformations of $\mathring{C}$. Let $B(\mathring{C})$ be the set of all bijections of $\mathring{C}$. Then $B(\mathring{C})$ is a group under composition. Note that a linear fractional transformation is a bijection, so $LF(\mathring{C}) \subseteq B(\mathring{C})$.

We will need the definitions of several groups of matrices. First, $GL(n, \mathbb{C})$ is the group of invertible $n \times n$ complex matrices. Then, $GL(n, \mathbb{C})$ contains the subgroup $SL(n, \mathbb{C})$, which is the group of $n \times n$ complex matrices which have determinant 1. The groups $SL(n, \mathbb{R})$ and $SL(n, \mathbb{Z})$ are defined similarly. The group $SO(n, \mathbb{R})$ is the group of real orthogonal matrices with determinant 1. So $SO(n, \mathbb{R})$ is a subgroup of $SL(n, \mathbb{R})$.

We discuss the relationships between $LF(\mathring{C})$, Möbius transformations, groups of matrices and the isometries of $U^2$. Let $\Xi : GL(2, \mathbb{C}) \to LF(\mathring{C})$ be the map

$$\left( \Xi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)(z) = \frac{az + b}{cz + d}.$$

**Lemma 2.6.5.** The set $LF(\mathring{C})$ is a group under composition.

**Proof.** The map $\Xi$ is into the group $B(\mathring{C})$. For $g, h \in GL(2, \mathbb{C})$, it can be checked that $\Xi(gh)$, the image of the matrix product $gh$, is the same map as the composition $\Xi(g)\Xi(h)$. Thus, $\Xi$ is a group homomorphism. Therefore, the image of $\Xi$ is a subgroup of $B(\mathring{C})$. Since $\Xi$ maps onto $LF(\mathring{C})$, we conclude that $LF(\mathring{C})$ is a group. 

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Lemma 2.6.5 permits us to define an action of the group $GL(2, \mathbb{C})$ on $\hat{\mathbb{C}}$. Let $g$ be the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$. The action is defined by setting $gz = (\Xi(g))(z)$, that is,
\[
 gz = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.
\tag{2.5}
\]

We now define some quotient groups. Let $\mathbb{C}^*$ be $\mathbb{C} \setminus \{0\}$. The group $PGL(n, \mathbb{C})$ is the quotient of the group $GL(n, \mathbb{C})$ by the normal subgroup $\{\lambda I : \lambda \in \mathbb{C}^*\}$. The group $PSL(n, \mathbb{C})$ is then the quotient $SL(n, \mathbb{C})/\{\lambda I : \lambda \in \mathbb{C}^*\}$. The groups $PSL(n, \mathbb{R})$ and $PSL(n, \mathbb{Z})$ are defined similarly. Then, if $n$ is even, $PSL(n, \mathbb{R}) = SL(n, \mathbb{R})/\{\pm I\}$, and $PSL(n, \mathbb{Z}) = SL(n, \mathbb{Z})$. The fact that the groups $PGL(2, \mathbb{C})$ are $PSL(2, \mathbb{C})$ isomorphic is used to prove the following corollary.

**Corollary 2.6.6.** The group $LF(\hat{\mathbb{C}})$ is isomorphic to the group $PSL(2, \mathbb{C})$.

**Proof.** The homomorphism $\Xi : GL(2, \mathbb{C}) \to LF(\hat{\mathbb{C}})$ is onto, so

$$LF(\hat{\mathbb{C}}) \cong GL(2, \mathbb{C})/\ker \Xi.$$ 

Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the kernel of $\Xi$. Then, for all $z \in \hat{\mathbb{C}},$
\[
 az + b = cz + d = z.
\]
Rearranging, we obtain the equation $cz^2 + (d - a)z + b = 0$. Since this equation holds for all $z$, we have $c = 0$, $d = a$ and $b = 0$. Hence, the kernel of $\Xi$ is the subgroup $\{\lambda I : \lambda \in \mathbb{C}^*\}$. Therefore $LF(\hat{\mathbb{C}}) \cong PSL(2, \mathbb{C})$. \qed

**Proposition 2.6.7.** Every linear fractional transformation of $\hat{\mathbb{C}}$ is a M"{o}bius transformation of $\hat{\mathbb{C}}$.

**Proof.** We first show that the group $SL(2, \mathbb{C})$ is generated by matrices of the form
\[
u(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
where $m$ is a complex number. For complex numbers $m$, $n$ and $p$, consider the product
\[
u(m)vu(n)vu(p) = \begin{pmatrix} mn - 1 & mnp - p - m \\ n & np - 1 \end{pmatrix} \in SL(2, \mathbb{C}).
\]
Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$. Suppose first that $c \neq 0$. Set $n = c$, and then choose $m$ and $p$ so that $mn - 1 = a$ and $np - 1 = d$. Since both $g$ and the product
$u(m) v u(n) v u(p)$ have determinant 1, we must have $b = mnp - p - m$. Thus $g = u(m) v u(n) v u(p)$. Now suppose that $c = 0$. Since $\det(g) = 1$, $a \neq 0$. We have

$$v^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix},$$

which may be written $u(m) v u(n) v u(p)$ for some complex numbers $m$, $n$ and $p$, since $a \neq 0$. Thus $g = v u(m) v u(n) v u(p)$. Therefore $SL(2, \mathbb{C})$ may be generated by the matrix $v$, and matrices of the form $u(m)$, where $m \in \mathbb{C}$.

Let $\phi_u(m) = \Xi(u(m))$ and $\phi_v = \Xi(v)$. Then $\phi_u(m)(z) = z + m$ and $\phi_v(z) = -1/z$. We have $v^2 = -I$, and, from Corollary 2.6.6, $LF(\hat{\mathbb{C}}) \cong PSL(2, \mathbb{C})$. Hence $LF(\hat{\mathbb{C}})$ is generated by the maps $\phi_u(m)$ and $\phi_v$. So it now suffices to prove that $\phi_u(m)$ and $\phi_v$ are Möbius transformations. First, $\phi_u(m)$ is a translation, and translations are isometries. Therefore $\phi_u(m)$ is a Möbius transformation. For $\phi_v$, let $\sigma$ be the reflection in the unit circle $S^1$, and let $\tau$ be the reflection in the imaginary axis, that is, $\tau(z) = -\overline{z}$. Then

$$\tau \sigma(z) = \tau \left( \frac{z}{|z|^2} \right) = \tau \left( \frac{1}{\overline{z}} \right) = -\frac{1}{z},$$

hence $\phi_v = \tau \sigma$. Thus $\phi_v$ is a Möbius transformation.

**Proposition 2.6.8.** Let $\rho : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be complex conjugation, that is, $\rho(z) = \overline{z}$. Then $M(\hat{\mathbb{C}}) = LF(\hat{\mathbb{C}}) \cup LF(\hat{\mathbb{C}}) \rho$.

**Proof.** Since $\rho$ is a reflection (in the real axis), and Proposition 2.6.7 shows that $LF(\hat{\mathbb{C}}) \subseteq M(\hat{\mathbb{C}})$, we have the inclusion

$$LF(\hat{\mathbb{C}}) \cup LF(\hat{\mathbb{C}}) \rho \subseteq M(\hat{\mathbb{C}})$$

immediately.

For the other inclusion, let $\phi$ be a Möbius transformation of $\hat{\mathbb{C}}$. Since $\phi$ is a composition of reflections in spheres of $\hat{\mathbb{C}}$, we first consider the cases where $\phi$ is a single reflection. Suppose that $\phi$ is the reflection in the circle $S(a, r)$. Then

$$\phi(z) = a + r^2 \frac{z - a}{|z - a|^2} = a + \frac{r^2}{z - a} = \frac{a \overline{z} + r^2 - \overline{a}}{\overline{z - a}} = \psi \rho(z),$$

where $\psi$ is the map given by

$$\psi(z) = \frac{az + (r^2 - |a|^2)}{z - \overline{a}}.$$ 

Since

$$a(-\overline{a}) - (r^2 - |a|^2)(1) = -|a|^2 - r^2 + |a|^2 = -r^2 \neq 0,$$

the map $\psi$ is a linear fractional transformation. Hence, $\phi \in LF(\hat{\mathbb{C}}) \rho$. 

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Now suppose that \( \phi \) is the reflection in the line \( P(u,t) \). Let \( \phi_0 \) be the reflection in the line through the origin \( P(u,0) \). As \( P(u,t) \) is the image of \( P(u,0) \) under translation by \( tu \), we have

\[
\phi(z) = \phi_0(z - tu) + tu.
\]  

(2.6)

Next, the line \( P(u,0) \) is the image of the real axis under rotation through an angle of \( \theta + \frac{\pi}{2} \), where \( u = e^{i\theta} \). Since \( \rho \) is reflection in the real axis,

\[
\phi_0(z) = e^{i(\theta + \frac{\pi}{2})} \rho(e^{-i(\theta + \frac{\pi}{2})}z) = iuiz = -u^2z.
\]  

(2.7)

Combining (2.6) and (2.7), we obtain

\[
\phi(z) = -u^2z + tu^2z + tu = \psi \rho(z),
\]

where \( \psi \) is the linear fractional transformation

\[
\psi(z) = \frac{-u^2z + tu^2z + tu}{0z + 1}.
\]

Thus \( \phi \in LF(\hat{C})\rho \).

We now consider the effect of composing reflections. Let \( \phi \) and \( \phi' \) be in \( LF(\hat{C}) \). By expanding out, it can be shown that the composition \( \phi \rho \phi' \) is in \( LF(\hat{C}) \). The composition \( \phi \phi' \) is a linear fractional transformation, as \( LF(\hat{C}) \) is a group. Also, both of the compositions \( \phi \rho \phi' \) and \( \phi' \rho \phi \) are in \( LF(\hat{C}) \rho \). Thus every Möbius transformation is either in \( LF(\hat{C}) \), or in \( LF(\hat{C}) \rho \), depending on whether it contains an even or odd number of reflections respectively. Hence \( M(\hat{C}) \subseteq LF(\hat{C}) \cup LF(\hat{C})\rho \).

The Möbius transformations belonging to \( LF(\hat{C}) \) are usually called orientation preserving.

**Lemma 2.6.9.** A linear fractional transformation \( \phi \) of \( \hat{C} \) leaves \( U^2 \) invariant if and only if there exist real numbers \( a, b, c, d \) with \( ad - bc = 1 \) such that

\[
\phi(z) = \frac{az + b}{cz + d}.
\]

**Proof.** Suppose there exist real numbers \( a, b, c, d \) with \( ad - bc = 1 \) such that \( \phi(z) = (az + b)/(cz + d) \). Recall that \( \phi \) is a bijection. Let \( z \in U^2 \). Then

\[
\text{Im}(\phi(z)) = \frac{\text{Im}(z)}{|cz + d|^2} > 0.
\]

Thus \( \phi \) maps into \( U^2 \). To show that \( \phi \) is onto \( U^2 \), let \( w \) be any point of \( U^2 \). Then \( \phi(z) = w \) if and only if

\[
z = \frac{b - dw}{cw - a}.
\]

Here, \( \text{Im}(z) = \text{Im}(w)/|cw - a|^2 > 0 \), so \( \phi \) is onto \( U^2 \). Thus \( \phi \) leaves \( U^2 \) invariant.
Conversely, suppose $\phi$ leaves $U^2$ invariant. Since $\phi$ is a homeomorphism, $\phi$ maps the boundary of $U^2$, which is $\mathbb{E}^1 = \mathbb{R} \cup \{\infty\}$, bijectively onto itself. We know that for some complex numbers $a$, $b$, $c$ and $d$ with $ad - bc \neq 0$,

$$\phi(z) = \frac{az + b}{cz + d}.$$

We consider two cases, the first being if $b = 0$. Since $ad - bc \neq 0$, it follows that $a \neq 0$. Then, we may divide $a$, $b$, $c$ and $d$ by $a$ without changing the function $\phi$, so we may assume that $a = 1$. We have $\phi(1) = 1/(c + d)$ and $\phi(-1) = -1/(-c + d)$ in $\mathbb{E}^1$, so $c + d$ and $-c + d$ are real. Thus $c$ and $d$ are real.

The second case is when $b \neq 0$. By dividing each of $a$, $b$, $c$ and $d$ by $b$, we may assume that $b = 1$. Then $\phi(0) = 1/d \in \mathbb{E}^1$, so $d \in \mathbb{R}$. If $a = 0$ then $c \neq 0$, and $\phi(1) = 1/(c + d) \in \mathbb{E}^1$, so $c + d \in \mathbb{R}$, implying $c \in \mathbb{R}$. If $a \neq 0$, we have that $\phi(-1/a) = 0$, hence $a \in \mathbb{R}$. Then,

$$\phi(1) = \frac{a + 1}{c + d} \in \mathbb{E}^1,$$

implying $c \in \mathbb{R}$.

Since $\phi$ maps the upper half-plane onto itself, $\text{Im}(\phi(i)) > 0$. We find that $\text{Im}(\phi(i)) = (ad - bc)/(c^2 + d^2)$. Therefore, the determinant $ad - bc$ must be positive. Let $A = a/\sqrt{ad - bc}$, $B = b/\sqrt{ad - bc}$, $C = c/\sqrt{ad - bc}$ and $D = d/\sqrt{ad - bc}$. Then $A$, $B$, $C$ and $D$ are real, $AD - BC = 1$, and $\phi(z) = (Az + B)/(Cz + D)$.

**Theorem 2.6.10.** Let $-\rho$ be the map $(-\rho)(z) = -\overline{z}$. Then

$$I(U^2) = \Xi(SL(2, \mathbb{R})) \cup \Xi(SL(2, \mathbb{R}))(-\rho).$$

**Proof.** Let $\phi$ be in $\Xi(SL(2, \mathbb{R}))$. By Proposition 2.6.7, $\phi$ is a Möbius transformation of $\mathbb{C}$. By Proposition 2.6.9, $\phi$ leaves $U^2$ invariant, so $\phi$ is a Möbius transformation of $U^2$. Thus, by Theorem 2.6.3, $\phi$ restricts to an isometry of $U^2$. Since $-\rho$ is a reflection, which moreover leaves $U^2$ invariant, $\phi(-\rho)$ is also a Möbius transformation of $U^2$. Hence, the composition $\phi(-\rho)$ is an isometry of $U^2$ as well.

For the other inclusion, let $\phi$ be an isometry of $U^2$. By Theorem 2.6.3, $\phi$ extends to a unique Möbius transformation of $U^2$. So by Proposition 2.6.8, either $\phi \in LF(\mathbb{C})$, or $\phi \in LF(\mathbb{C})\rho$.

If $\phi \in LF(\mathbb{C})$, then by Lemma 2.6.9, since $\phi$ leaves $U^2$ invariant, $\phi$ is the image of some matrix in $SL(2, \mathbb{R})$ under the map $\Xi$. If $\phi \in LF(\mathbb{C})\rho$, let $\tau$ be the linear fractional transformation $\tau(z) = (-z + 0)/(0z + 1) = -z$. Then $(-\rho)(z) = \tau\rho(z)$, and so $LF(\mathbb{C})\rho = LF(\mathbb{C})\tau\rho = LF(\mathbb{C})(-\rho)$. Now, the map $-\rho$ leaves $U^2$ invariant, and $\phi$ leaves $U^2$ invariant, so $\phi$ has the form

$$\phi(z) = \frac{a(-\overline{z}) + b}{c(-\overline{z}) + d},$$

where $a$, $b$, $c$ and $d$ are real and $ad - bc = 1$. Hence $\phi \in \Xi(SL(2, \mathbb{R}))(-\rho)$. \qed
2.6.2 Topological groups and discrete subgroups

A topological group is a group $G$ which is also a topological space, such that the multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$ are continuous functions in $G$. Many of the groups discussed in this chapter are topological groups. Groups of matrices such as $GL(n, \mathbb{C})$ and $SL(n, \mathbb{R})$ are topological groups, and their topology is the metric topology induced by the distance function $d(A, B) = |A - B|$, where the right-hand side is the matrix norm defined by

$$|A| = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$ 

Then, quotient groups such as $PSL(2, \mathbb{R})$ are topological groups as well, equipped with the quotient topology. The group of isometries of $\mathbb{B}^n$ or $\mathbb{U}^n$, and the group of linear fractional transformations of $\mathbb{H}$, are also topological groups. They are topologised with the subspace topology inherited from the spaces $C(\mathbb{B}^n)$, $C(\mathbb{U}^n)$ or $C(\mathbb{H})$, respectively, of continuous maps with the compact-open topology.

An isomorphism of topological groups $G$ and $H$ is a map $\phi : G \to H$ which is both a group isomorphism and a homeomorphism. The isomorphism of $LF(\mathbb{H})$ and $PSL(2, \mathbb{C})$ established in Proposition 2.6.6 is an isomorphism of topological groups.

Let $\Gamma$ be a subgroup of a topological group $G$. Then $\Gamma$ is said to be a discrete subgroup of $G$ if for all $\gamma \in \Gamma$ there exists an open neighbourhood $\Omega$ of $\gamma$ in $G$ such that $\Omega \cap \Gamma = \{\gamma\}$. Recall that the groups $SL(2, \mathbb{R})$ and $PSL(2, \mathbb{R})$ act on $\mathbb{U}^2$ by isometries. We now show that $SL(n, \mathbb{Z})$ is a discrete subgroup of $SL(n, \mathbb{R})$, and that $PSL(n, \mathbb{Z})$ is a discrete subgroup of $PSL(n, \mathbb{R})$.

**Lemma 2.6.11.** Let $G$ be a topological group which is also a metric space, and let $\Gamma$ be a subgroup of $G$. Suppose every convergent subsequence in $\Gamma$ is eventually constant. Then $\Gamma$ is discrete.

**Proof.** Suppose that every convergent subsequence in $\Gamma$ is eventually constant, but that $\Gamma$ is not discrete. Then there is a point $\gamma \in \Gamma$ such that for all $n \geq 1$, the ball $B(\gamma, 1/n)$ contains a point $\gamma_n$ of $\Gamma$ which is different from $\gamma$. Then $\gamma_n \to \gamma$, but the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is not eventually constant, which is a contradiction. We conclude that $\Gamma$ must be discrete.

**Proposition 2.6.12.** A subgroup $\Gamma$ of $SL(n, \mathbb{R})$ is discrete if for all $r > 0$ the set $\{A \in \Gamma : |A| < r\}$ is finite.

**Proof.** Suppose that $\{A \in \Gamma : |A| \leq r\}$ is finite for each $r > 0$. Let $B_j \to B$ in $\Gamma$. As the norm function is continuous, $|B_j| \to |B|$. Thus, there exists an integer $j_0$ such that for all $j \geq j_0$, $|B_j - |B|| \leq 1$. If $|B_j - |B|| \leq 1$ then $|B_j| \leq 1 + |B|$. Fix $r = 1 + |B|$, then the set $\{A \in \Gamma : |A| \leq 1 + |B|\}$ is finite. Hence, the sequence $\{B_j\}$ is eventually constant. Since $SL(n, \mathbb{R})$ is a metric space, we may apply Lemma 2.6.11 to conclude that $\Gamma$ is discrete.

**Corollary 2.6.13.** The group $SL(n, \mathbb{Z})$ is a discrete subgroup of $SL(n, \mathbb{R})$. 

Corollary 2.6.14. The group $PSL(n, \mathbb{Z})$ is a discrete subgroup of $PSL(n, \mathbb{R})$.

We now discuss group actions in greater detail. Let $G$ be a group acting on a set $X$ and let $x$ be an element of $X$. The subset of $X$ given by $Gx = \{gx : g \in G\}$ is called the $G$-orbit of $x$, or the orbit of $x$ under $G$. The $G$-orbits partition $X$.

Lemma 2.6.15. If $\Gamma$ is a discrete group of isometries of a metric space $X$, and $K$ is compact in $X$, then $K$ contains only finitely many points of each orbit $\Gamma x$.

Proof. Let $K$ be compact in $X$. Suppose $K$ contains infinitely many points of an orbit $\Gamma x$. Then $K$ contains a convergent subsequence $\{\gamma_n x\}$ of $\Gamma x$, and there are infinitely many distinct points $\gamma_n x$. Since $\Gamma$ is topologised with the compact open topology, $\gamma_n x \rightarrow \gamma x$ in $K$ if and only if $\gamma_n \rightarrow \gamma$ in $\Gamma$. As $\Gamma$ is discrete, the sequence $\{\gamma_n\}$ is eventually constant, so the sequence $\{\gamma_n x\}$ is eventually constant, that is, it contains only finitely many distinct elements. This is a contradiction. \qed

The stabiliser subgroup $G_x$ of $x$ in $G$ is the subgroup $\{g \in G : gx = x\}$. As a useful application of the ideas of orbits and stabiliser subgroups, we have the following decomposition of $SL(2, \mathbb{Z})$.

Lemma 2.6.16. Let $\gamma \in SL(2, \mathbb{Z})$. Then $\gamma$ may be expressed in the form $k_1 a k_2$, where $k_1$ and $k_2$ are in $SO(2, \mathbb{R})$, and $a$ is in $SL(2, \mathbb{R})$ with the form \[
\begin{pmatrix}
s & 0 \\
0 & s^{-1}
\end{pmatrix}
\] for some $s \geq 1$.

Proof. We first show that $SO(2, \mathbb{R})$ is the stabiliser subgroup of $z = i$ in $SL(2, \mathbb{R})$. Let $k \in SO(2, \mathbb{R})$. Then $k$ has the form

$$
k_\theta = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
$$

for some $\theta \in [0, 2\pi)$. It can be calculated that $k_\theta$ fixes the point $i$. Now suppose that $k$ is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \mathbb{R})$, and that $ki = i$. Then $ai + b = di - c$, hence $a = d$ and $b = -c$. Since $ad - bc = 1$, we have $a^2 + c^2 = 1$, and so the matrix

$$
k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}
$$

is an element of $SO(2, \mathbb{R})$. Therefore, if $k \in SL(2, \mathbb{R})$, $ki = i$ if and only if $k$ is in $SO(2, \mathbb{R})$.

We next show that for each $r \geq 0$, the orbit of the point $e^r i$ under the group $SO(2, \mathbb{R})$ is the Euclidean circle with Cartesian equation

$$
x^2 + (y - \cosh r)^2 = \sinh^2 r.
$$

We will denote this circle by $C_r$. When $r = 0$, the point $e^r i$ is just $i$. Since $SO(2, \mathbb{R})$ is the stabiliser subgroup of $i$, the orbit is then the single point $i$. For $r > 0$, a tedious calculation shows that for any $k_\theta \in SO(2, \mathbb{R})$, the point $k_\theta e^r i$ lies in the circle $C_r$. We now argue that the orbit is onto the circle $C_r$. The action of $SO(2, \mathbb{R})$ on the point $e^{-r}i$ may be regarded as a continuous function of $\theta$. Thus, the image of the interval $[0, \pi/2]$ is a connected subset of $C_r$. When $\theta = 0$, we get
Since \( e^k = e^i \). When \( \theta = \pi/2 \), we get \( ke^\theta = e^{-\theta}i \). For \( 0 < \theta < \pi/2 \), the real part of \( ke^\theta \) is negative. Hence, the image of \([0, \pi/2]\) is those points of the circle \( C_r \) with non-positive real part. Similarly, the image of the interval \([\pi/2, \pi]\) is those points of \( C_r \) with non-negative real part. Therefore, the orbit of the point \( e^\theta i \) under \( SO(2, \mathbb{R}) \) is the whole circle \( C_r \).

We now apply these results to complete the proof. Let \( p \) be the point \( \gamma \). Then, since \( p \) lies in the orbit \( C_r \) for some \( r \geq 0 \), there exists a \( k \in SO(2, \mathbb{R}) \) such that \( kp = e^\theta i \). Let \( \alpha \) be the matrix \( \begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix} \). Then \( \alpha e^\theta i = i \). Let \( k' = \alpha k \gamma \). Then \( k'i = i \), so \( k' \in SO(2, \mathbb{R}) \). Making \( \gamma \) the subject and then relabelling, we obtain \( \gamma = k^{-1} \alpha^{-1} k' = k_1 a k_2 \).

2.6.3 Tessellation by fundamental domains

In this subsection, we provide general definitions of concepts needed to describe tesselations of metric spaces which are induced by the action of discrete groups of isometries. Most of the proofs are for just the upper half-plane. However, with a few exceptions, the proofs for the general case are not much different. See Chapter 6 of [24] for a full treatment of the general case.

Let \((X, d)\) be a metric space and let \( \Gamma \) be a non-trivial group of isometries of \( X \). A subset \( R \) of \( X \) is said to be a fundamental region for the group \( \Gamma \) if

1. the set \( R \) is open in \( X \),
2. the members of \( \{ \gamma R : \gamma \in \Gamma \} \) are mutually disjoint, and
3. \( X = \cup \{ \gamma R : \gamma \in \Gamma \} \).

A fundamental domain is a connected fundamental region. Now, if \( S \) is a collection of subsets of \( X \), then \( S \) is described as locally finite if for each point \( x \in X \), there is an open neighbourhood of \( x \) which meets only finitely many members of \( S \). We say that a fundamental domain \( D \) is a locally finite fundamental domain if the collection \( \{ \gamma D : \gamma \in \Gamma \} \) is a locally finite collection of sets.

A subset \( F \) of \( X \) is a fundamental set for the group \( \Gamma \) if \( F \) contains exactly one point from each \( \Gamma \)-orbit in \( X \). Note that a fundamental domain is not a fundamental set. As the orbits of \( \Gamma \) on the space \( X \) partition \( X \), if \( F \) is a fundamental set, then we have \( X = \cup \{ \gamma F : \gamma \in \Gamma \} \).

We consider the metric space \( U^2 \). One feature of the upper half-plane is that any isometry of \( U^2 \) which fixes every point of a non-empty open subset of \( U^2 \) is the identity map. This fact is used to prove the following lemma.

**Lemma 2.6.17.** An open subset \( R \) of the metric space \( U^2 \) is a fundamental region for a group \( \Gamma \) of isometries of \( U^2 \) if there is a fundamental set \( F \) for \( \Gamma \) such that \( R \subseteq F \subseteq \overline{R} \).

**Proof.** Suppose that there exists a fundamental set \( F \) for the group \( \Gamma \) such that \( R \subseteq F \subseteq \overline{R} \). We first show that the elements of the set \( \{ \gamma R : \gamma \in \Gamma \} \) are mutually disjoint. Suppose that \( \gamma \) and \( \delta \) are elements of \( \Gamma \) such that \( \gamma R \cap \delta R \) is non-empty. Then, there are points \( z \) and \( w \) in \( R \) such that \( \gamma z = \delta w \). Hence, \( \delta^{-1} \gamma z = w \). As \( R \subseteq F \), the points \( z \) and \( w \) are in \( F \). Since \( F \) contains only one point from the orbit \( \Gamma z \), we have \( \delta^{-1} \gamma z = z \). Thus the isometry \( \delta^{-1} \gamma \) fixes each point of the open set \( R \cap \gamma^{-1} \delta R \). Therefore, \( \delta^{-1} \gamma \) is the identity, and so \( \delta = \gamma \).
Next, since $F \subseteq R$, we have
\[ X = \bigcup_{\gamma \in \Gamma} \gamma F = \bigcup_{\gamma \in \Gamma} \gamma R. \]
Therefore $R$ is a fundamental region for $\Gamma$. \qed

We now describe a general method for constructing a fundamental domain.

Suppose there is a point $p_0$ in $X$ which has a trivial stabiliser subgroup $\Gamma_{p_0}$. For each $\gamma \neq 1$ in $\Gamma$, we define the open half-space $H_\gamma(p_0)$ by
\[ H_\gamma(p_0) = \{ x \in X : d(x, p_0) < d(x, \gamma p_0) \}. \]
Then, the Dirichlet domain $D(p_0)$ with centre $p_0$ is defined to be the intersection of these half-spaces:
\[ D(p_0) = \bigcap \{ H_\gamma(p_0) : \gamma \neq 1 \text{ in } \Gamma \}. \]

If $\Gamma = \text{PSL}(2, \mathbb{Z})$, we may regard $\Gamma$ as a subgroup of the group of isometries of $U^2$. We will, abusing notation, write $\gamma$ for both a matrix $\gamma \in \text{SL}(2, \mathbb{Z})$, and the coset of this matrix in $\text{PSL}(2, \mathbb{Z})$. There are uncountably many points of $U^2$ which have trivial stabilisers in $\Gamma$. This is because, if
\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) z = z, \]
then by the quadratic formula $z = ((a - d) \pm \sqrt{(d - a)^2 + 4bc})/2c$, and $\mathbb{Z}$ is countable. Some points which do have trivial stabilisers are all the points $t_i$, where $t > 1$. We now show that in the upper half-plane, a Dirichlet domain with centre at $p_0$ is a locally finite fundamental domain. To show that $D(p_0)$ is open, we will show that its complement is closed in $U^2$. In general, a union of closed sets need not be closed. However, if a collection $\mathcal{S}$ of closed subsets is locally finite, then the union of the members of $\mathcal{S}$ is closed.

**Proposition 2.6.18.** Let $\Gamma = \text{PSL}(2, \mathbb{Z})$ and let $p_0$ be a point of $U^2$ such that $\Gamma_{p_0}$ is trivial. Then the Dirichlet domain $D(p_0)$ is a locally finite fundamental domain for the action of $\Gamma$ on $U^2$.

**Proof.** For each $\gamma \neq 1$ in $\text{PSL}(2, \mathbb{Z})$, let $K_\gamma = U^2 \setminus H_\gamma(p_0)$. Then $K_\gamma$ is closed. We show that $\mathcal{K} = \{ K_\gamma : \gamma \neq 1 \text{ in } \Gamma \}$ is a locally finite collection of sets. Let $z$ be any point of $U^2$, and let $r > 0$ be such that $B(p_0, r)$ contains $z$. We will show that $B(p_0, r)$ meets only finitely many of the sets in $\mathcal{K}$. If $B(p_0, r)$ meets none of the sets in $\mathcal{K}$ then we are done, so suppose $B(p_0, r)$ intersects the set $K_\gamma$ in a point $w$. Then we have
\[ d_U(p_0, \gamma p_0) \leq d_U(p_0, w) + d_U(w, \gamma p_0) \leq d_U(p_0, w) + d_U(w, p_0) < 2r. \]
Hence, the ball $B(p_0, 2r)$ contains $\gamma p_0$. Now, the closure of this ball is compact, and $\text{PSL}(2, \mathbb{Z})$ is discrete, so by Lemma 2.6.15, $\overline{B(p_0, 2r)}$ contains only finitely many points of the orbit $\Gamma p_0$. Thus $B(p_0, r) \subseteq \overline{B(p_0, 2r)}$ meets only finitely many of the sets $K_\gamma$. Therefore $\mathcal{K}$ is a locally finite family of closed sets in $U^2$. This means
that the union of the sets in $\mathcal{K}$ is closed. So $D(p_0)$, which is the complement of this union, is open.

We now construct a fundamental set $F$ and show that $D(p_0) \subseteq F \subseteq \overline{D(p_0)}$. From each orbit $\Gamma z$, we may select a point which is the minimum distance from the point $p_0$ (if there is more than one closest point, choose any). Let $F$ be the set of these points. Then $F$ is a fundamental set for $\Gamma$.

Let $z$ be in $D(p_0)$ and $\gamma \neq 1$ in $\Gamma$. Then, since $\gamma$ is an isometry,

$$d_U(z, p_0) < d_U(z, \gamma p_0) = d(\gamma^{-1}z, p_0),$$

so $z$ is the unique closest point of the orbit $\Gamma z$ to $p_0$. Hence $D(p_0) \subseteq F$.

To show $F \subseteq \overline{D(p_0)}$, let $z$ be in $F$. If $z = p_0$ then $z \in \overline{D(p_0)}$ immediately, so we assume that $z \neq p_0$. Let $\gamma \neq 1$ be in $\Gamma$. If $d_U(z, p_0) > d_U(z, \gamma p_0)$, then $d_U(z, p_0) > d(\gamma^{-1}z, p_0)$, which contradicts our construction of $F$. Therefore, $d_U(z, p_0) \leq d_U(z, \gamma p_0)$. Let $[p_0, z]$ be the geodesic segment in $U^2$ joining the points $p_0$ and $z$, and let $w$ be a point in the interior of this segment. Then

$$d_U(z, w) + d_U(w, p_0) = d_U(z, p_0),$$

so

$$d_U(w, p_0) = d_U(z, p_0) - d_U(z, w) \leq d_U(z, \gamma p_0) - d_U(z, w) \leq d_U(w, \gamma p_0),$$

with equality only if $d_U(z, p_0) = d_U(z, \gamma p_0)$ and $d_U(z, \gamma p_0) = d_U(z, w) + d_U(w, \gamma p_0)$. Suppose we have equality. Then $[z, w] \cup [w, \gamma p_0]$ is the geodesic segment $[z, \gamma p_0]$. The segments $[z, p_0]$ and $[z, \gamma p_0]$ have the same length, and both extend $[z, w]$, so $p_0 = \gamma p_0$. This contradicts the stabiliser subgroup $\Gamma_{p_0}$ being trivial. Thus we have the strict inequality

$$d_U(w, p_0) < d_U(w, \gamma p_0)$$

for all $\gamma \neq 1$ in $\Gamma$. Hence, $w \in D(p_0)$. This implies that the half-open segment $[p_0, z]$ is contained in $D(p_0)$, and so $z \in \overline{D(p_0)}$. Therefore, $F \subseteq \overline{D(p_0)}$. By Lemma 2.6.17, $D(p_0)$ is a fundamental region.

We have shown that, for any $w \in D(p_0)$, the geodesic segment $[p_0, w]$ is in $D(p_0)$. This means $D(p_0)$ is connected. Therefore $D(p_0)$ is a fundamental domain.

Finally we show that $D(p_0)$ is a locally finite fundamental domain. Write $D$ for $D(p_0)$. Let $r > 0$, and suppose the ball $B(p_0, r)$ meets $\gamma D$ for some $\gamma \in \Gamma$. Then there is a $z$ in $D$ such that $d_U(p_0, \gamma z) < r$. Moreover,

$$d_U(p_0, \gamma p_0) \leq d_U(p_0, \gamma z) + d(\gamma z, \gamma p_0)$$

$$< r + d_U(z, p_0)$$

$$\leq r + d_U(z, \gamma^{-1}p_0)$$

$$= r + d_U(\gamma z, p_0)$$

$$< 2r.$$

Since $\Gamma$ is discrete, $d_U(p, \gamma p_0) < 2r$ for only finitely many elements $\gamma$. Hence, the ball $B(p_0, r)$ meets only finitely many sets $\gamma D$. \qed
Let $T$ be the generalised hyperbolic triangle with vertices at $\pm \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\infty$, and let $T^\circ$ be the interior of the triangle $T$. This triangle is pictured overleaf.

**Proposition 2.6.19.** Let $p_0 = ti$ for some $t > 1$. The Dirichlet domain $D(p_0)$ is equal to $T^\circ$.

**Proof.** Let $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $u$ and $v$ are in $\text{PSL}(2, \mathbb{Z})$, so

$$D(p_0) \subseteq (H_u(p_0) \cap H_v(p_0) \cap H_{u^{-1}}(p_0)).$$

We have $up_0 = ti + 1$, $vp_0 = i/t$ and $u^{-1}p_0 = ti - 1$. Thus the geodesic $\text{Re}(z) = 1/2$ is the boundary of $H_u(p_0)$, and $H_u(p_0) = \{z \in U^2 : \text{Re}(z) < 1/2\}$. The boundary of $H_{u^{-1}}(p_0)$ is the geodesic $\text{Re}(z) = -1/2$, and $H_{u^{-1}}(p_0) = \{z \in U^2 : \text{Re}(z) > -1/2\}$. For $H_v(p_0)$, all points in the intersection of the circle $|z| = 1$ with $U^2$ are the same hyperbolic distance from $p_0$ as from $vp_0$. Thus, this circle, which is a geodesic since it is orthogonal to the real axis, is the boundary of $H_v(p_0)$, and $H_v(p_0)$ is the set $\{z \in U^2 : |z| > 1\}$. The three boundary geodesics form the sides of a generalised hyperbolic triangle with one vertex at infinity. The finite vertices are at $\pm \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Hence, $H_u(p_0) \cap H_v(p_0) \cap H_{u^{-1}}(p_0)$ is the interior of the triangle $T$. Thus $D(p_0) \subseteq T^\circ$.

Suppose $D(p_0) \nsubseteq T^\circ$, and write $D$ for $D(p_0)$. Then there is a point $z \in T^\circ$ such that $z \notin D$. Since $D$ is a fundamental domain, $U^2 = \{\gamma D : \gamma \in \Gamma\}$, hence there is a $\gamma \in \Gamma$, $\gamma \neq 1$, such that $z \in \gamma D$. Now, the action of $\Gamma$ on $U^2$ is by homeomorphisms, and $D$ and $T^\circ$ are both open sets, with $D \nsubseteq T^\circ$. Thus $T^\circ$ must properly contain the boundary of $D$. So we may, in fact, find some $\gamma \in \Gamma$ such that $z \in \gamma D$. Let
w = γ^{-1}z ∈ D. Then, since D ⊆ T^0, w is in T^0. We show that this implies a contradiction.

Let δ be γ^{-1}, and suppose that δ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. Then, since z is in T^0, |z| > 1 and Re(z) < 1/2, so

\[
|cz + d|^2 = c^2|z|^2 + 2cd\text{Re}(z) + d^2 \\
> c^2 - |cd| + d^2 \\
= (|c| - |d|)^2 + |cd|.
\]

This lower bound is a non-negative integer. It is zero if and only if c = d = 0; since \(ad - bc = 1\) this is impossible. Therefore, \(|cz + d|^2 > 1\). Thus,

\[
\text{Im}(\delta z) = \frac{\text{Im}(z)}{|cz + d|^2} < \text{Im}(z),
\]

that is, \(\text{Im}(w) < \text{Im}(z)\). Now replace \(z\) by \(w\) and \(\delta\) by \(\gamma\) in the above argument. We find that \(\text{Im}(\gamma w) < \text{Im}(w)\), hence \(\text{Im}(z) < \text{Im}(w)\). This is a contradiction. Therefore, \(D = T^0\).

We conclude this discussion of fundamental domains by showing how they are related to tessellations, and using this relationship to find generators for the groups \(PSL(2, \mathbb{Z})\) and \(SL(2, \mathbb{Z})\). A convex polyhedron in a metric space \(X\) is a non-empty, closed, convex subset of \(X\) with finitely many sides and a non-empty interior. A tessellation of \(X\) is a collection \(\mathcal{P}\) of convex polyhedra in \(X\) such that

1. the interiors of the polyhedra in \(\mathcal{P}\) are mutually disjoint, and
2. the union of the polyhedra in \(\mathcal{P}\) is equal to \(X\).

By comparing this definition with that of a fundamental domain, it can be seen that, since \(T^0\) is a locally finite fundamental domain, \(\mathcal{T} = \{\gamma T : \gamma ∈ PSL(2, \mathbb{Z})\}\) is a locally finite tessellation of \(U^2\). Moreover, it is an exact tessellation, meaning that each side of a triangle \(\gamma T\) is a side of exactly two triangles \(\gamma T\) and \(δ T\) in \(\mathcal{T}\). An exact tessellation \(\mathcal{P}\) is said to be connected if for each \(P, Q ∈ \mathcal{P}\) there is a finite sequence \(P_0, P_1, \ldots, P_m\) in \(\mathcal{P}\) such that \(P = P_0, Q = P_m\), and for each \(1 ≤ i ≤ m\), the polyhedra \(P_{i−1}\) and \(P_i\) have a common side. The tessellation \(\mathcal{T}\) is pictured in Chapter 1 on page 53.

**Lemma 2.6.20.** The tessellation \(\mathcal{T}\) of \(U^2\) is connected.

**Proof.** Let \(δ T\) be a triangle in \(\mathcal{T}\), and let \(V\) be the union of all the triangles \(\gamma T\) where there exists a finite sequence \(γ_0 T, γ_1 T, \ldots, γ_m T\) such that \(δ T = γ_0 T\), \(γ T = γ_m T\), and the triangles \(γ_{i−1} T\) and \(γ_i T\) share a common side for \(1 ≤ i ≤ m\). Since \(V\) contains \(δ T\), \(V\) is non-empty. The tessellation \(\mathcal{T}\) is a locally finite collection of closed subsets of \(U^2\). Hence, \(V\) is closed.

We now show that the set \(V\) is also open. Let \(z\) be in \(V\). If \(z\) is in the interior of some triangle \(γ T\), then since an interior is open, there is an \(ε > 0\) such that the ball \(B(z, ε)\) is contained in the interior, and so this ball is contained in \(V\). The other possibility is that \(z\) lies on the side of some triangle in \(V\). Choose \(ε\) so that \(\overline{B(z, ε)}\) meets only the triangles in \(\mathcal{T}\) which have \(z\) on one of their sides. Let \(S(z, ε)\) be the hyperbolic circle centred at \(z\) of radius \(ε\). Then \(\mathcal{T}\) restricts to an exact tessellation
of $S(z, \varepsilon)$. This restriction is a connected tesselation, so any triangle in $T$ which has $z$ on a side is in $V$. Thus $V$ contains $B(z, \varepsilon)$. We have shown that $V$ is both open and closed, and is non-empty. Since $U^2$ is connected, this means $V = U^2$. □

**Proposition 2.6.21.** The group $\Gamma = PSL(2, \mathbb{Z})$ is generated by the set

$$\Sigma = \{ \gamma \in \Gamma : T \cap \gamma T \text{ is a side of } T \}.$$ 

**Proof.** Let $\gamma$ be any element of $\Gamma$. Then, since $T$ is connected, there is a finite sequence of elements $\gamma_0, \gamma_1, \ldots, \gamma_m$ of $\Gamma$ such that $T = \gamma_0T$, $\gamma_m T = \gamma T$, and $\gamma_{i-1}T$ and $\gamma_i T$ share a common side for $1 \leq i \leq m$. Note that $\gamma_0 = I$ and $\gamma_m = g$. Since $\gamma_{i-1}^{-1}\gamma_{i-1}T = T$, the triangles $T$ and $\gamma_{i-1}^{-1}\gamma_i T$ share a common side for each $i$. We may assume that for each $i$, $\gamma_{i-1} \neq \gamma_i$. Then $\gamma_{i-1}^{-1}\gamma_i \in \Sigma$. The product

$$\gamma_0(\gamma_0^{-1}\gamma_1)(\gamma_1^{-1}\gamma_2) \cdots (\gamma_{m-1}^{-1}\gamma_m) = \gamma$$

is thus a product of the identity $\gamma_0$ and elements of $\Sigma$, hence $\Sigma$ generates $\Gamma$. □

**Corollary 2.6.22.** The group $SL(2, \mathbb{Z})$ is generated by the matrices $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. 

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Chapter 3
Symmetric Spaces and Distance Functions

3.1 Introduction
The upper half-plane $U^2$ is a metric space with distance function $d_U$. The group $PSL(2, \mathbb{R})$ acts on $U^2$ by isometries, and a tesselation of $U^2$ is induced by the action of the discrete subgroup $PSL(2, \mathbb{Z})$. It is natural to ask if, for $\gamma \in PSL(2, \mathbb{Z})$ and $z \in U^2$, there is any relationship between the hyperbolic distance $d_U(z, \gamma z)$, and the number of tiles of this tesselation which are crossed by the geodesic segment $[z, \gamma z]$. Now, the tiles in this tesselation of $U^2$ are the images under elements of $PSL(2, \mathbb{Z})$ of the triangle $T$ with vertices at $\pm \frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $\infty$. Moreover, the group $PSL(2, \mathbb{Z})$ may be generated by those elements $u$ and $v$ which map $T$ to a triangle adjacent to $T$. Thus, it seems there is some relationship between the number of tiles crossed by the geodesic segment $[z, \gamma z]$, and the expression for $\gamma$ in terms of $u$ and $v$.

In this chapter, we present the theory needed to more precisely frame questions about, and analyse, these kinds of relationships between distances, tesselations and generators. The upper half-plane is an example of a very general concept called a
symmetric space. Where \((X, d)\), a symmetric space, is acted on by a discrete group of isometries \(\Gamma\), we define two distance functions on \(\Gamma\), and introduce a notion of equivalence of distance functions. The geometric distance function on \(\Gamma\) is induced by the distance function \(d\) on \(X\). The word distance function on \(\Gamma\), an algebraic concept, is induced by the generators of \(\Gamma\). The remainder of this thesis investigates the circumstances in which the geometric and word distance functions are equivalent. We prove that, in spaces where the tiles of the tesselation are compact, these two distance functions are indeed equivalent. Next, so as to generalise the action of \(PSL(2, \mathbb{R})\) on \(U^2\), we describe a space \(P^{(n)}\) on which \(PSL(n, \mathbb{R})\), \(n \geq 2\), acts by isometries. Chapter 4 comprises a proof that the geometric and word distances on \(PSL(n, \mathbb{Z})\) are not equivalent for \(n = 2\), but they are equivalent for all \(n \geq 3\).

### 3.2 Symmetric Spaces

Let \(X\) be a differentiable manifold which is also a metric space, and let \(d\) be a Riemannian distance function on \(X\) (see [8] for a discussion of Riemannian metrics and Riemannian distance functions). Suppose there exists a group \(G\) which acts transitively and by isometries on \(X\). Also suppose there exists a special point, say \(0\), in \(X\), and an element \(k \in G\), such that \(k \circ 0 = 0\) and \(Dk(0) = -I\). Then, \((X, d)\) is said to be a symmetric space. (The element \(k\) is actually a reflection in the point 0.)

The upper half-plane is a symmetric space. Considered as an open subset of \(\mathbb{R}^2\), it is a two-dimensional differentiable manifold, and it is a metric space with distance function \(d_U\). Let \(G\) be the group \(SL(2, \mathbb{R})\). We know that \(G\) acts by isometries on \(X\). For transitivity, it suffices to show that for all \(z \in U^2\), there exists a \(g \in G\) so that \(gi = z\). Let \(z = x + iy\) be in \(U^2\), and let \(g \in G\) be the matrix

\[
g = \begin{pmatrix}
  \sqrt{y} & \zbar{y} \\
  0 & \sqrt{y}
\end{pmatrix}.
\]

Then \(gi = z\). If \(k\) is the element \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) in \(G\), then \(ki = i\) and \(Dk(i) = -I\).

Other examples of symmetric spaces are hyperbolic \(n\)-space \(H^n\), Euclidean \(n\)-space \(E^n\), and spherical \(n\)-space \(S^n\). There are also symmetric spaces of matrices, one of which is described in Section 3.3 below. A space which is not a symmetric space is an ellipsoid. There is no transitive group of isometries for this space. For example, reflections which are not in the axes of the ellipsoid do not preserve the space.

It turns out that symmetric spaces have the features needed to construct tesselations like that of the upper half-plane, as described in Subsection 2.6.3. More precisely, let \(X\) be a symmetric space which is acted on by a group of isometries \(G\). The group \(G\) has a normal subgroup, say \(N\), consisting of all elements of \(G\) which fix every point of the space \(X\). (If \(X\) is the upper half-plane and \(G = SL(2, \mathbb{R})\), this normal subgroup is \(\{\pm I\}\).) Then, the quotient group \(G/N\) has a discrete subgroup \(\Gamma\), such that for some point \(p_0\) in \(X\), the stabiliser subgroup \(\Gamma_{p_0}\) is trivial. The Dirichlet domain \(D\) with centre \(p_0\) is a locally finite fundamental domain for the action of \(\Gamma\) on \(X\). The closure of \(D\) is a polyhedron which, acted on by \(\Gamma\), yields a
tessellation of $X$. This tessellation is locally finite, and any compact set in $X$ meets only finitely many tiles of the tessellation. Finally, the group $\Gamma$ may be generated by those finitely many elements of $\Gamma$ which map $\overline{D}$ to a tile which shares a side with $\overline{D}$. For the details of the considerable amount of theory needed to establish these results, see Chapter 4 of [8] and Chapter 6 of [21].

3.3 Distance functions on discrete groups

Let $\Gamma$ be a discrete group of isometries of a symmetric space $X$ and let $d$ and $d'$ be distance functions on $\Gamma$. Then $d$ and $d'$ are said to be Lipschitz equivalent if there exist constants $c, C > 0$ such that, for all $\gamma_1, \gamma_2 \in \Gamma$,

$$cd(\gamma_1, \gamma_2) \leq d'(\gamma_1, \gamma_2) \leq Cd(\gamma_1, \gamma_2).$$

Lipschitz equivalence is an equivalence relation. We write $d \approx d'$ if distance functions $d$ and $d'$ are in the same equivalence class.

Lipschitz equivalence allows distance functions on $\Gamma$ to be compared. We now define the two distance functions that we wish to compare.

3.3.1 The geometric distance function

Let $p_0$ be a point in $(X, d)$ such that the stabiliser subgroup $\Gamma_{p_0}$ is trivial. A geometric distance function $d_R$ on $\Gamma$ is defined by

$$d_R(\gamma_1, \gamma_2) = d(\gamma_1 p_0, \gamma_2 p_0).$$

The letter $R$ is used since $d$ is a Riemannian distance function. From the properties of the distance function $d$ on $X$, it follows immediately that $d_R$ is non-negative and symmetric, that $d_R$ satisfies the triangle inequality, and that $\gamma_1 = \gamma_2$ implies $d_R(\gamma_1, \gamma_2) = 0$. Now suppose $d_R(\gamma_1, \gamma_2) = 0$. Then $d(\gamma_1 p_0, \gamma_2 p_0) = 0$, and so $\gamma_1 p_0 = \gamma_2 p_0$. Therefore, $p_0 = \gamma_1^{-1} \gamma_2 p_0$, meaning that $\gamma_1^{-1} \gamma_2$ is in the stabiliser subgroup of $p_0$. But this subgroup is trivial, hence $\gamma_1 = \gamma_2$. Thus $d_R$ really is a distance function.

The point $p_0$ used to define the geometric distance function is not, in general, the only point of $X$ whose stabiliser subgroup in $\Gamma$ is trivial. For instance, in the upper half-plane, all points $ti$, where $t > 1$, have trivial stabiliser subgroups in $PSL(2, \mathbb{Z})$. The geometric distance functions which are then defined using, say, the points $2i$ and $3i$ will not be the same. For our purposes, this difference does not matter. We are interested in Lipschitz equivalence classes of distance functions, and the following result shows that all geometric distance functions on $\Gamma$ acting on the space $(X, d)$ are Lipschitz equivalent.

Lemma 3.3.1. Let $\Gamma$ be a discrete group of isometries of the symmetric space $(X, d)$. The geometric distance function on $\Gamma$ depends on the choice of point in $X$ with trivial stabiliser subgroup, but only up to Lipschitz equivalence.

Proof. Let $d_R^p$ and $d_R^q$ be two geometric distance functions defined with respect to the points $p$ and $q$ in $(X, d)$. Let $\gamma_1$ and $\gamma_2$ be elements of $\Gamma$. By the triangle
inequality, and the fact that $\Gamma$ acts by isometries,
\[
\begin{align*}
d_R^p(\gamma_1, \gamma_2) &= d(\gamma_1 p, \gamma_2 p) \\
&\leq d(\gamma_1 p, \gamma_1 q) + d(\gamma_1 q, \gamma_2 q) + d(\gamma_2 q, \gamma_2 p) \\
&= 2d(p, q) + d_R^q(\gamma_1, \gamma_2).
\end{align*}
\]
As the group $\Gamma$ is discrete, there is an $r_q > 0$ such that $d_R^p(\gamma_1, \gamma_2) \geq r_q$ for all distinct $\gamma_1$ and $\gamma_2$. Choose the constant $C_1$ so that $2d(p, q) \leq (C_1 - 1)r_q$. Then, for all $\gamma_1 \neq \gamma_2$,
\[
2d(p, q) + d_R^q(\gamma_1, \gamma_2) \leq (C_1 - 1)r_q + d_R^q(\gamma_1, \gamma_2) \\
\leq (C_1 - 1)d_R^p(\gamma_1, \gamma_2) + d_R^q(\gamma_1, \gamma_2) \\
= C_1 d_R^q(\gamma_1, \gamma_2).
\]
Thus $d_R^p(\gamma_1, \gamma_2) \leq C_1 d_R^q(\gamma_1, \gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$. Similarly, there exists a constant $C_2 > 0$ such that
\[
d_R^p(\gamma_1, \gamma_2) \leq C_2 d_R^p(\gamma_1, \gamma_2)
\]
for all $\gamma_1$ and $\gamma_2$ in $\Gamma$. Combining these results,
\[
\frac{1}{C_1} d_R^p(\gamma_1, \gamma_2) \leq d_R^q(\gamma_1, \gamma_2) \leq C_2 d_R^p(\gamma_1, \gamma_2),
\]
which completes the proof.

This lemma shows that changing the point $p_0$ in the definition of a geometric distance function $d_R$ does not affect Lipschitz equivalence. Now, the definition of $d_R$ also depends on the distance function $d$ on $X$. It is reasonable to wonder whether the geometric distance is well-defined, at least up to Lipschitz equivalence, if the distance function on the space $X$ is changed. It turns out that, because $X$ is a symmetric space, any two Riemannian distance functions $d$ and $d'$ on $X$ are scalar multiples of each other. Hence, the respective induced geometric distance functions $d_R$ and $d'_R$ are indeed Lipschitz equivalent.

We will from now on abuse notation and refer to $d_R$ as the geometric distance function on $\Gamma$.

If $x$ is in $X$ and $\gamma_1$ in $\Gamma$, the distance in $X$ along the geodesic joining $x$ and $\gamma_1 x$ is $d(x, \gamma_1 x)$. Since $\Gamma$ acts transitively on $X$, there is a $\gamma_2 \in \Gamma$ such that $\gamma_2 x = p_0$. Then,
\[
d(x, \gamma_1 x) = d(\gamma_2 x, \gamma_2 \gamma_1 x) = d(p_0, \gamma_2 \gamma_1 \gamma_2^{-1} p_0) = d_R(1, \gamma_2 \gamma_1 \gamma_2^{-1}).
\]
So the geometric distance provides a way of relating a distance in the space $X$ between points $x$ and $\gamma x$ to a distance in the group $\Gamma$. In particular, the distance along the geodesic joining the points $p_0$ and $\gamma p_0$ is Lipschitz equivalent to $d_R(1, \gamma)$.

### 3.3.2 The word distance function

To define the second distance function on $\Gamma$, fix a finite set of generators for $\Gamma$, denote this set of generators by $\Sigma$, and let $\Sigma^{-1}$ be the set $\{\sigma^{-1} : \sigma \in \Sigma\}$. A word distance function $d_\Sigma$ on $\Gamma$ is induced by $\Sigma$ as follows. For $\gamma_1$ and $\gamma_2$ in $\Gamma$, $d_\Sigma(\gamma_1, \gamma_2) = n$, where $n$ is the smallest integer such that $\gamma_1^{-1} \gamma_2$ may be written as a
word of length $n$ in terms of elements of $\Sigma \cup \Sigma^{-1}$. Let $l_\Sigma(\gamma)$ denote the length of a shortest word for $\gamma$ in terms of elements of $\Sigma \cup \Sigma^{-1}$. Then $d_\Sigma(\gamma_1, \gamma_2) = l_\Sigma(\gamma_1^{-1}\gamma_2)$.

The word distance function corresponds to distance in the Cayley graph for $\Gamma$ with respect to $\Sigma$.

We now show that $d_\Sigma$ does satisfy the definition of a distance function. Since the identity is the only element of $\Gamma$ which may be written as a word of length zero, $d_\Sigma(\gamma_1, \gamma_2) = 0$ if and only if $\gamma_1 = \gamma_2$. All words for elements other than the identity have positive length, so $d_\Sigma(\gamma_1, \gamma_2) > 0$ for all $\gamma_1, \gamma_2 \in \Gamma$. Also, for any $\gamma_1, \gamma_2 \in \Gamma$, by taking the inverse of a shortest word for $\gamma_1^{-1}\gamma_2$, we see that $d_\Sigma(\gamma_1, \gamma_2) = d_\Sigma(\gamma_2, \gamma_1)$.

Finally, for any $\gamma_1, \gamma_2$ and $\gamma_3$ in $\Gamma$,

$$d_\Sigma(\gamma_1, \gamma_3) = l_\Sigma(\gamma_1^{-1}\gamma_3)$$
$$= l_\Sigma(\gamma_1^{-1}\gamma_2\gamma_2^{-1}\gamma_3)$$
$$\leq l_\Sigma(\gamma_1^{-1}\gamma_2) + l_\Sigma(\gamma_2^{-1}\gamma_3)$$
$$= d_\Sigma(\gamma_1, \gamma_2) + d_\Sigma(\gamma_2, \gamma_3),$$

proving the triangle inequality.

**Lemma 3.3.2.** The word distance function on $\Gamma$ depends on the choice of generators, but only up to Lipschitz equivalence.

**Proof.** Let $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ and $\Sigma' = \{\sigma'_1, \sigma'_2, \ldots, \sigma'_m\}$ be two finite sets of generators for $\Gamma$, with $d_\Sigma$ and $d'_{\Sigma'}$ the respective induced word distance functions. For $\gamma_1, \gamma_2 \in \Gamma$, we have $d_\Sigma(\gamma_1, \gamma_2) = d_\Sigma(1, \gamma_1^{-1}\gamma_2)$, and $d'_{\Sigma'}(\gamma_1, \gamma_2) = d'_{\Sigma'}(1, \gamma_1^{-1}\gamma_2)$.

Thus, to prove that $d_\Sigma \approx d'_{\Sigma'}$, it suffices to prove that for all $\gamma \in \Gamma$, there exists a constant $C > 0$ such that

$$d_\Sigma(1, \gamma) \leq C d'_{\Sigma'}(1, \gamma), \quad \text{and} \quad d'_{\Sigma'}(1, \gamma) \leq C d_\Sigma(1, \gamma).$$

As $\Sigma$ and $\Sigma'$ are sets of generators, if $\sigma_i \in \Sigma$ then both $\sigma_i$ and $\sigma_i^{-1}$ may be expressed in terms of elements of $\Sigma' \cup (\Sigma')^{-1}$, and similarly if $\sigma'_j \in \Sigma'$ then $\sigma'_j$ and $(\sigma'_j)^{-1}$ may be expressed in terms of elements of $\Sigma \cup \Sigma^{-1}$.

For all $\gamma \in \Gamma$, by taking the inverse of a shortest word for $\gamma$, we find that $d_\Sigma(1, \gamma) = d_\Sigma(1, \gamma^{-1})$, and $d'_{\Sigma'}(1, \gamma) = d'_{\Sigma'}(1, \gamma^{-1})$. So let

$$c_1 = \max_{1 \leq i \leq n} \{d_\Sigma(1, \sigma_i)\},$$
$$c_2 = \max_{1 \leq j \leq m} \{d_{\Sigma'}(1, \sigma'_j)\},$$
$$C = \max\{c_1, c_2\}.$$

Then for any $\gamma \in \Gamma$,

$$d'_{\Sigma'}(1, \gamma) \leq c_1 d_\Sigma(1, \gamma) \leq C d_\Sigma(1, \gamma),$$

and

$$d_\Sigma(1, \gamma) \leq c_2 d'_{\Sigma'}(1, \gamma) \leq C d'_{\Sigma'}(1, \gamma).$$

Therefore $d_\Sigma \approx d'_{\Sigma'}$. \qed
As with the geometric distance function, this result allows us to abuse notation. If we are interested only in Lipschitz equivalence of distance functions, we will now write $d_W$ for any word distance function, and refer to $d_W$ as the word distance function on $\Gamma$. We will continue to denote by $d_\Sigma$ the word distance function induced by a particular set of generators $\Sigma$.

The action of $\Gamma$ on $X$ induces a tesselation of $X$. For $\gamma \in \Gamma$, the geodesic segment $[p_0, \gamma p_0]$ crosses a sequence of tiles in this tesselation. Let $T$ be the tile which has the point $p_0$ in its interior. Then $[p_0, \gamma p_0]$ intersects a finite sequence of tiles $\{\gamma_i(T)\}_{i=0}^n$, say, where $\gamma_0 = 1$ and $\gamma_n = \gamma$. There is a slight complication if the geodesic segment $[p_0, \gamma p_0]$ passes through the vertex of some tile in the tesselation. Since the tesselation is locally finite, if this happens we can always choose a finite set of successively adjacent tiles meeting at the vertex. Then, we may assume that for $1 \leq i \leq n$, the tile $\gamma_{i-1}T$ is adjacent to the tile $\gamma_i T$. Thus, the tile $T$ is adjacent to the tile $\gamma_{i-1}\gamma_i T$. As in the upper half-plane, the set of elements of $\Gamma$ which map $T$ to a tile adjacent to $T$ generate $\Gamma$. This means that

$$
\gamma = (\gamma_0^{-1}\gamma_1)(\gamma_1^{-1}\gamma_2) \cdots (\gamma_{n-1}^{-1}\gamma_n)
$$

is a word for $\gamma$ with respect to a finite set of generators for $\Gamma$. Its length is the same as the number of tiles of the tesselation which are crossed by the geodesic segment $[p_0, \gamma p_0]$. There is no shorter word for $\gamma$ with respect to this set of generators (this is proved for a case when $\Gamma = PSL(2, \mathbb{Z})$ in Section 4.3, and the proof given there generalises). Therefore, $d_W(1, \gamma)$ has a geometric significance as well as its algebraic meaning: it is Lipschitz equivalent to the number of tiles in the tesselation which are crossed by the geodesic segment $[p_0, \gamma p_0]$.

### 3.4 Fundamental domains of compact closure

A straightforward example of a symmetric space is the Euclidean plane, acted on by the group of Euclidean isometries. Let $s$ and $t$ be the orthogonal translations $s(x) = x + (0, 1)$ and $t(x) = x + (1, 0)$. Let $\Gamma$ be the discrete subgroup of isometries generated by $s$ and $t$, and let $p_0$ be the point $(1/2, 1/2)$. Then $p_0$ has trivial stabiliser subgroup in $\Gamma$, and the Dirichlet domain with centre $p_0$ is the open square $(0, 1) \times (0, 1)$. It seems intuitively reasonable that in the tesselation induced, which is a grid of unit squares, the Euclidean distance between $x$ and $\gamma x$ is roughly the same as the number of tiles crossed by the line segment $[x, \gamma x]$. 
The following theorem shows that this intuition is correct. In spaces, like this one, where the induced tessellation has compact tiles, the geometric and word distance functions are Lipschitz equivalent.

**Theorem 3.4.1.** [14] Let $D$ be a fundamental domain for the action of $\Gamma$ on $(X, d)$. Suppose $D$ has compact closure. Then the geometric distance function $d_R$ on $\Gamma$ is Lipschitz equivalent to the word distance function $d_W$ on $\Gamma$.

**Proof.** Let $\Sigma$ be a finite set of generators for $\Gamma$ such that $\Sigma = \Sigma^{-1}$, and let $d_\Sigma$ be the word distance function induced by the set $\Sigma$. Let $\gamma$ be in $\Gamma$. Choose $C_0$ to be the constant

$$C_0 = \max\{d_R(1, \sigma) : \sigma \in \Sigma\}.$$ 

We prove that $d_R(1, \gamma) \leq C_0 d_\Sigma(1, \gamma)$ by induction on $m = d_\Sigma(1, \gamma)$. For the inductive step, we may write $\gamma = \sigma_1 \sigma_2 \cdots \sigma_m$. Then,

$$d_R(1, \gamma) = d(p_0, \sigma_1 \sigma_2 \cdots \sigma_m p_0)$$
$$= d(\sigma_1^{-1} p_0, \sigma_2 \cdots \sigma_m p_0)$$
$$\leq d(\sigma_1^{-1} p_0, p_0) + d(p_0, \sigma_2 \cdots \sigma_m p_0)$$
$$= d_R(\sigma_1^{-1}, 1) + d_R(1, \sigma_2 \cdots \sigma_m)$$
$$\leq C_0 d_\Sigma(\sigma_1^{-1}, 1) + C_0 d_\Sigma(1, \sigma_2 \cdots \sigma_m)$$
$$\leq C_0 + C_0 (m - 1)$$
$$= C_0 m$$
$$= C_0 d_\Sigma(1, \gamma).$$

Thus there exists a constant $C_1 > 0$ such that, for all $\gamma \in \Gamma$, $d_R(1, \gamma) \leq C_1 d_W(1, \gamma)$.

To show that $d_W(1, \gamma) \leq C_2 d_R(1, \gamma)$, consider the geodesic segment $[p_0, \gamma p_0]$. Since $\Gamma$ is discrete, the constant $K_0$ defined by

$$K_0 = \inf\{d(p_0, \theta p_0) : \theta \in \Gamma\}$$

would be equal to $C_1$.
is strictly positive. By the division algorithm, there exists a positive integer \( q \) and real number \( r \), where \( 0 \leq r < \frac{K_0}{2} \), such that

\[
d(p_0, \gamma p_0) = \frac{K_0}{2} q + r.
\]

Thus, there exists a finite sequence of points \( p_0, p_1, p_2, \ldots, p_n = \gamma p_0 \) lying on the geodesic segment \([p_0, \gamma p_0] \) such that, for \( 1 \leq i \leq n \),

\[
\frac{K_0}{2} \leq d(p_{i-1}, p_i) < K_0.
\]

Note that

\[
d_R(1, \gamma) = d(p_0, \gamma p_0) = \sum_{i=1}^{n} d(p_{i-1}, p_i) \geq n \frac{K_0}{2}.
\]

Now, the tiles of the tesselation induced by \( \Gamma \) are the images of \( \overline{D} \) under elements of \( \Gamma \). So each point \( p_i \) belongs to (at least one) image of \( \overline{D} \). Thus, for \( 0 \leq i \leq n \),

\[
d(\gamma^{-1} p_{i-1}, \gamma^{-1} p_i) = d(x_{i-1}, \gamma^{-1} \gamma_i x_i) = d(x_{i-1}, r_i x_i) < K_0,
\]

so the point \( r_i x_i \) is in the ball \( B(x_{i-1}, K_0) \).

Consider the set

\[
B = \bigcup_{x \in \overline{D}} B(x, K_0).
\]

Since \( \overline{D} \) is compact, there is an \( R_0 > 0 \) such that

\[
\sup\{d(x, y) : x, y \in \overline{D} \} \leq R_0.
\]

That is, the distance between points in \( \overline{D} \) is bounded above by the constant \( R_0 \).

Let \( \Theta \) be the following set of elements in \( \Gamma \):

\[
\Theta = \{ \theta \in \Gamma : \theta \overline{D} \cap B \neq \emptyset \}.
\]

Then, using (3.1), and the fact that \( p_0 \) is in \( \overline{D} \),

\[
B \subseteq \bigcup_{\theta \in \Theta} \theta \overline{D} \subseteq \overline{B(p_0, K_0 + R_0)}.
\]

Since the closed ball \( \overline{B(p_0, K_0 + R_0)} \) is compact, it meets only finitely many images of \( \overline{D} \). Thus \( \Theta \) is a finite set. Therefore, if \( x \) and \( y \) are points in \( \overline{D} \), and the point \( \theta y \) is in the ball \( B(x, K_0) \), the element \( \theta \) belongs to the finite set \( \Theta \).
Now, for $1 \leq i \leq n$, the points $x_{i-1}$ and $x_i$ are in $\overline{D}$, and the point $r_i x_i$ is in the ball $B(x_{i-1}, K_0)$. This means the element $r_i$ belongs to the set $\Theta$. Moreover, since $r_1 r_2 \cdots r_n$ is a word for $\gamma$, and $\Theta$ is finite and independent of the element $\gamma$, the set $\Theta$ is a finite set of generators for the group $\Gamma$. Let $d_\Theta$ be the word distance function induced by the set $\Theta$. Then, $d_\Theta(1, \gamma) \leq n$. Thus

$$d_R(1, \gamma) \geq \frac{K_0}{2} n \geq \frac{K_0}{2} d_\Theta(1, \gamma).$$

Therefore, there is a constant $C_2 > 0$ such that $d_W(1, \gamma) \leq C_2 d_R(1, \gamma)$. \hfill \qed

Note that we did not need the fact that $\overline{D}$ is compact to prove the inequality $d_R(1, \gamma) \leq C_1 d_W(1, \gamma)$.

3.5 A group action of $PSL(n, \mathbb{R})$

As described in Subsection 2.6.1, the group $PSL(2, \mathbb{R})$ acts on the upper half-plane by isometries. We now describe how the group $PSL(n, \mathbb{R})$, where $n \geq 2$, acts by isometries on a symmetric space $P^{(n)}$ of $n \times n$ matrices. First, we define $P^{(n)}$, and show that it is a metric space. Then, we define an action of the group $SL(n, \mathbb{R})$ on $P^{(n)}$, and show that this action is by isometries and is transitive. Following this, we conclude that $P^{(n)}$ is a symmetric space. Next, we show that the upper half-plane is isometric to $P^{(2)}$, once the distance function on $U^2$ is multiplied by a constant. This shows that the action of $SL(n, \mathbb{R})$ on $P^{(n)}$ is a logical generalisation of the action of $SL(2, \mathbb{R})$ on $U^2$. Finally, we define the action of the quotient group $PSL(n, \mathbb{R})$ on $P^{(n)}$. The proofs of many statements in this section depend on elementary results of matrix theory. See, for example, [12].

A linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be positive definite if $T \mathbf{x} \cdot \mathbf{x} \geq 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$, and $T \mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$. If $T$ is positive definite and $T \mathbf{x} = \mathbf{0}$, then $T \mathbf{x} \cdot \mathbf{x} = 0$, so $\mathbf{x} = \mathbf{0}$. Thus positive definite matrices are invertible.

Let $P^{(n)}$ be the set of symmetric, positive definite $n \times n$ real matrices of determinant 1. In other words, $P^{(n)}$ is the subset of symmetric positive definite matrices in $SL(n, \mathbb{R})$. Note that the identity matrix $I$ is in $P^{(n)}$. The space $P^{(n)}$ is a differentiable manifold of dimension $\frac{n}{2}(n+1)$.

We now show that $P^{(n)}$ is a metric space. From now on, we denote the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$ by $\|\mathbf{x}\|$. The operator norm of a matrix $A$ is then, by definition,

$$\|A\| = \sup_{\|\mathbf{x}\| = 1} \|A \mathbf{x}\|.$$ 

Note that this matrix norm is different from the norm used to equip the group $SL(n, \mathbb{R})$ with a topology. We define the distance function $d_P$ on $P^{(n)}$ by

$$d_P(S_1, S_2) = \log \|S_1^{-1} S_2\| + \log \|S_2^{-1} S_1\|.$$ 

Clearly, $d_P(S_1, S_2) = d_P(S_2, S_1)$. For any two $n \times n$ matrices $A$ and $B$, we have $\|AB\| \leq \|A\| \|B\|$ (this is a standard result for bounded operators on a Hilbert space; see [11]). Since $I = S_1^{-1} S_2 S_2^{-1} S_1$, it follows that

$$1 = \|I\| \leq \|S_1^{-1} S_2\| \|S_2^{-1} S_1\|.$$
Thus

\[ d_P(S_1, S_2) = \log \|S_1^{-1}S_2\|\|S_2^{-1}S_1\| \geq 0. \]

If \( S_1 = S_2 \) then \( d_P(S_1, S_2) = \log \|I\|^2 = \log 1 = 0 \). If \( d_P(S_1, S_2) = 0 \), then

\[ \|S_1^{-1}S_2\|\|S_2^{-1}S_1\| = 1. \]

Write \( M = S_1^{-1}S_2 \), so that \( \|M\|\|M^{-1}\| = 1 \). Since \( M \) is in \( SL(n, \mathbb{R}) \), there are matrices \( K_1 \) and \( K_2 \) in \( SO(n, \mathbb{R}) \), and a diagonal matrix \( A \in SL(n, \mathbb{R}) \) with positive diagonal entries, such that \( M = K_1AK_2 \). (This decomposition is established for the case \( n = 2 \) in Lemma \ref{Lemma 2.6.16}. The general case is proved below in Lemma \ref{Lemma 3.5.2}, so as not to interrupt the proof that \( P^{(n)} \) is a metric space.) Then, since \( \|Kx\| = \|x\| \) for all \( K \in SO(n, \mathbb{R}) \) and all \( x \in \mathbb{R}^n \),

\[ \|M\| = \|K_1AK_2\| = \|A\|, \quad \text{and} \quad \|M^{-1}\| = \|K_2^{-1}A^{-1}K_1^{-1}\| = \|A^{-1}\|. \]

Also, \( \det(A) = 1 \). So we have a diagonal matrix \( A \) with determinant 1 and \( \|A\|\|A^{-1}\| = 1 \). We now show that \( A \) must be the identity.

**Lemma 3.5.1.** Let \( D \) be a diagonal matrix. Suppose \( D \) has positive diagonal entries \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \). Then the operator norm of \( D \) is

\[ \|D\| = \max_{1 \leq i \leq n} \{\lambda_i\}. \]

**Proof.** We have

\[ \|D\|^2 = \sup_{\|x\| = 1} \|Dx\|^2 = \sup\{\lambda_1^2x_1^2 + \lambda_2^2x_2^2 + \cdots + \lambda_n^2x_n^2 : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}. \]

If \( \lambda_I \) is the maximum of the set \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), then for \( x \) such that \( \|x\| = 1 \),

\[ \lambda_1^2x_1^2 + \lambda_2^2x_2^2 + \cdots + \lambda_n^2x_n^2 \leq \lambda_I^2(x_1^2 + x_2^2 + \cdots + x_n^2) = \lambda_I^2. \]

So \( \|D\|^2 \leq \lambda_I^2 \). If \( x = e_I \), then

\[ \lambda_1^2x_1^2 + \lambda_2^2x_2^2 + \cdots + \lambda_n^2x_n^2 = \lambda_I^2, \]

so equality is attained. \( \square \)

Let the diagonal entries of \( A \) be \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \). We have \( \det(A) = 1 \), so the product \( \prod_i \lambda_i = 1 \). By Lemma \ref{Lemma 3.5.1} the equation \( \|A\|\|A^{-1}\| = 1 \) means that \( (\max \lambda_i)(\max \lambda_i^{-1}) = 1 \). Hence,

\[ (\max \lambda_i)(\min \lambda_i)^{-1} = 1. \quad (3.2) \]

Suppose first that \( \max \lambda_i > 1 \). Then \( \min \lambda_i < 1 \), as otherwise \( \det(A) = \prod_i \lambda_i > 1 \). It follows that

\[ (\max \lambda_i)(\min \lambda_i)^{-1} > 1, \]

which contradicts \((3.2)\). Therefore \( \max \lambda_i \leq 1 \). But if \( \max \lambda_i < 1 \), \( \det(A) < 1 \). Hence, \( \max \lambda_i = 1 \). For the minimum diagonal entry, we now have \( \min \lambda_i \leq 1 \). If
min \lambda_i < 1 \text{ then } \det(A) < 1, \text{ so } \min \lambda_i = 1. \text{ Therefore, } \lambda_i = 1 \text{ for all } i, \text{ and } A \text{ is the identity matrix.}

Since } A \text{ is the identity, } M = K_1 K_2 \in SO(n, \mathbb{R}). \text{ So } S_2^{-1} S_2 \text{ is in } SO(n, \mathbb{R}). \text{ Hence, } S_2 = S_1 U \text{ for some } U \text{ in } SO(n, \mathbb{R}). \text{ As the matrices } S_1 \text{ and } S_2 \text{ are symmetric, we have } S_i = (S_i S_i^T)^{1/2} \text{ for } i = 1, 2. \text{ Then, }

\[
S_2 = (S_2 S_2^T)^{1/2} = (S_1 U U^T S_1^T)^{1/2} = (S_1 S_1^T)^{1/2} = S_1.
\]

This proves that } d_P(S_1, S_2) = 0 \text{ if and only if } S_1 = S_2.

For the triangle inequality,

\[
d_P(S_1, S_3) = \log \| S_1^{-1} S_2 \| \| S_3^{-1} S_1 \|
\]

\[
= \log \| S_1^{-1} S_2 S_2^{-1} S_3 \| \| S_3^{-1} S_3 S_1 \|
\]

\[
\leq \log \| S_1^{-1} S_2 \| \| S_2^{-1} S_3 \| \| S_3^{-1} S_2 \| \| S_2^{-1} S_1 \|
\]

\[
= \log \| S_1^{-1} S_2 \| \| S_2^{-1} S_1 \| + \log \| S_2^{-1} S_3 \| \| S_3^{-1} S_2 \|
\]

\[
= d_P(S_1, S_2) + d_P(S_2, S_3).
\]

We conclude that } P^{(n)} \text{ with the distance function } d_P \text{ is a metric space.}

We now define an action of the group } SL(n, \mathbb{R}) \text{ on } P^{(n)}. \text{ For } M \in SL(n, \mathbb{R}) \text{ and } S \in P^{(n)}, \text{ the action of } M \text{ on } S \text{ is }

\[
M \circ S = MSM^T.
\]

Since } S \text{ is symmetric, } MSM^T = (MSM^T)^T. \text{ Since } S \text{ is positive definite, }

\[
MSM^T \mathbf{x} \cdot \mathbf{x} = S(M^T \mathbf{x}) \cdot (M^T \mathbf{x}) \geq 0,
\]

with equality if and only if } M^T \mathbf{x} = \mathbf{0}. \text{ But } M^T \in SL(n, \mathbb{R}), \text{ so } M^T \mathbf{x} = \mathbf{0} \text{ if and only if } \mathbf{x} = \mathbf{0}. \text{ Hence } M \circ S \in P^{(n)}. \text{ Since } ISI^T = S, \text{ and }

\[
(M_1 M_2) S(M_1 M_2)^T M_1 (M_2 S M_2^T) M_1^T,
\]

this is indeed a group action.

The stabiliser subgroup of } I \text{ in } SL(n, \mathbb{R}) \text{ is }

\[
\{ M \in SL(n, \mathbb{R}) : M \circ I = I \} = \{ M \in SL(n, \mathbb{R}) : MM^T = I \},
\]

which is the special orthogonal group } SO(n, \mathbb{R}). \text{ This fact is used to prove the following decomposition of } SL(n, \mathbb{R}).

**Lemma 3.5.2.** Let } M \in SL(n, \mathbb{R}). \text{ Then there exist matrices } K_1 \text{ and } K_2 \text{ in } SO(n, \mathbb{R}), \text{ and a diagonal matrix } A \text{ in } SL(n, \mathbb{R}) \text{ with positive diagonal entries, such that } M = K_1 A K_2.

**Proof.** First, } M \circ I = MIM^T \text{ is a symmetric positive definite matrix. Since it is symmetric, there exists an orthogonal basis of eigenvectors of } MIM^T. \text{ If } K \text{ is the orthogonal matrix which has these eigenvectors as columns, then the matrix } KMIM^T K^{-1} = KMIM^T K^T \text{ is diagonal, and the diagonal entries of } KMIM^T K^T \text{ are the eigenvalues of } MIM^T. \text{ By multiplying the columns of the matrix } K \text{ by
a suitable constant, we may assume that \( \det(K) = 1 \), so that \( K \in SO(n, \mathbb{R}) \). As the matrix \( MIM^T \) is positive definite, its eigenvalues are real and positive. So there exists a diagonal matrix \( D \) such that \( KIMM^T K^T = D^2 \). Note that \( \det(D) = \sqrt{\det(D^2)} = 1 \), so \( D \in SL(n, \mathbb{R}) \). Then

\[
D^{-1}KIMM^T K^T D^{-1} = (D^{-1}KM)I(D^{-1}KM)^T = (D^{-1}KM) \circ I = I.
\]

As the group \( SO(n, \mathbb{R}) \) is the stabiliser subgroup of \( I \) in \( P^{(n)} \), we have \( D^{-1}KM = K' \) for some \( K' \in SO(n, \mathbb{R}) \). Making \( M \) the subject and then relabelling, we obtain \( M = K^{-1}DK' = K_1AK_2 \).

We now further investigate the action of \( SL(n, \mathbb{R}) \) on \( P^{(n)} \). For \( M \in SL(n, \mathbb{R}) \) and \( S_1, S_2 \in P^{(n)} \),

\[
d_P(M \circ S_1, M \circ S_2) = \log \| S_1^{-1}M^{-1}MS_2 \| \| S_2^{-1}M^{-1}MS_1 \|
\]

\[
= \log \| S_1^{-1}S_2 \| \| S_2^{-1}S_1 \|
\]

\[
= d_P(S_1, S_2).
\]

Thus \( SL(n, \mathbb{R}) \) acts by isometries on \( P^{(n)} \).

The action of \( SL(n, \mathbb{R}) \) on \( P^{(n)} \) is also transitive. Let \( S \) be in \( P^{(n)} \). As in the proof of Lemma 3.5.2, since \( S \) is symmetric and positive definite, there exists \( K \in SO(n, \mathbb{R}) \), and a diagonal matrix \( D \in SL(n, \mathbb{R}) \) with positive diagonal entries, such that \( KSK^T = D^2 \). Then

\[
S = K^TDDK = (K^T DK')(K^T D)^T.
\]

Setting \( M = K^TD \), we have \( M \in SL(n, \mathbb{R}) \) and \( M \circ I = S \).

The only thing now required to conclude that \( P^{(n)} \) is a symmetric space is the point 0 in \( P^{(n)} \) and element \( k \in SL(n, \mathbb{R}) \) such that \( k \) fixes 0 and \( Dk(0) = -I \). For any \( n \), we take the point 0 to be the identity matrix. Then, for \( n = 2 \), take \( k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). For \( n \geq 3 \), we may take as \( k \) a matrix which has all zeros except for the diagonal running from bottom left to top right. On this diagonal, each entry is either 1 or \(-1\), arranged so that \( \det(k) = 1 \).

The following lemma motivates our investigation of the space \( P^{(n)} \), as it shows that the upper half-plane and the space \( P^{(2)} \) are isometric after adjusting the distance function on \( U^2 \) (or \( P^{(2)} \)) by a constant. Lipschitz equivalence is not affected by multiplying a distance function by a constant.

**Lemma 3.5.3.** There is a bijection \( \varphi : U^2 \to P^{(2)} \) so that for some constant \( C > 0 \),

\[
d_U(z, w) = Cd_P(\varphi(z), \varphi(w))
\]

for all \( z, w \in U^2 \). That is, \( \varphi \) is a similarity.

**Proof.** Let \( G = SL(2, \mathbb{R}) \) and \( K = SO(2, \mathbb{R}) \). Recall that \( K \) is the stabiliser subgroup of \( i \) when \( G \) is acting on \( U^2 \) (see the proof of Lemma 2.7.16), and of \( I \) when \( G \) is acting on \( P^{(2)} \).
Define a map \( \varphi : U^2 \to P^{(2)} \) by \( \varphi(g i) = g \circ I \). The domain of \( \varphi \) is \( U^2 \) because \( U^2 = G i \), and \( \varphi \) is onto \( P^{(2)} \) since \( P^{(2)} = G \circ I \). To show \( \varphi \) is injective, suppose \( \varphi(g_1 i) = \varphi(g_2 i) \). Then \( g_1 \circ I = g_2 \circ I \), and so \( I = g_2^{-1} g_1 \circ I \), hence \( g_2^{-1} g_1 \in K \). Thus \( g_2^{-1} g_1 i = i \), and so \( g_1 i = g_2 i \). Therefore \( \varphi \) is a bijection.

Now, for all \( g_1, g_2 \in G \),

\[
d_U(g_1 i, g_2 i) = d_U(i, g_1^{-1} g_2 i), \quad \text{and} \quad d_P(g_1 \circ I, g_2 \circ I) = d_P(I, g_1^{-1} g_2 \circ I).
\]

So, to show that \( d_U(g_1 i, g_2 i) = C d_P(\varphi(g_1 i), \varphi(g_2 i)) \), it suffices to prove that for all \( g \in G \),

\[
d_U(i, g i) = C d_P(I, g \circ I).
\]

By Lemma 2.6.16 we may write \( g \) as \( k_1 a k_2 \), where \( k_1, k_2 \in K \) and \( a = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \) for some \( s \geq 1 \). Since \( K \) stabilises \( i \) in \( U^2 \) and \( I \) in \( P^{(2)} \), \( d_U(i, g i) = d_U(i, ai) \) and \( d_P(I, g \circ I) = d_P(I, a \circ I) \). In \( U^2 \), \( ai = s^2 i \), and \( d_U(s^2 i, i) = 2 \log s \). In \( P^{(2)} \),

\[
a \circ I = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \circ I = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}^2 = \begin{pmatrix} s^2 & 0 \\ 0 & s^{-2} \end{pmatrix}.
\]

Then, by Lemma 3.5.1 \( \|a \circ I\| = \|(a \circ I)^{-1}\| = s^2 \), so \( d_P(a \circ I, I) = 2 \log s^2 = 4 \log s \). Thus, with \( C = 2 \), \( d_U(g_1 i, g_2 i) = C d_P(g_1 \circ I, g_2 \circ I) \).

We now, finally, define the action of the group \( PSL(n, \mathbb{R}) \) on \( P^{(n)} \). Let \( M \) be in \( SL(n, \mathbb{R}) \) and \( S \) in \( P^{(n)} \). Then \( MSM^T = (-M) S (-M)^T \), so \( M \circ S = (-M) \circ S \). Write \([M]\) for the coset of \( M \) in the group \( PSL(n, \mathbb{R}) \). Then \([M]\circ S = MSM^T \) is a well-defined group action of \( PSL(n, \mathbb{R}) \) on the space \( P^{(n)} \). So, the group \( PSL(n, \mathbb{R}) \) acts transitively and by isometries on the metric space \( P^{(n)} \).
4.1 Introduction

We have, in Chapter 3, described the action of the group $PSL(n, \mathbb{R})$ on the symmetric space $P^{(n)}$, and shown that this action is a generalisation of the action of the group $PSL(2, \mathbb{R})$ on the upper half-plane. In Chapter 4, we consider the Lipschitz equivalence of distance functions on the discrete subgroup $PSL(n, \mathbb{Z})$ of $PSL(n, \mathbb{R})$. The content of this chapter is an elaboration of the paper [13] by Lubotzky, Mozes and Raghunathan.

A fundamental domain for the action of $PSL(n, \mathbb{Z})$ on $P^{(n)}$ does not have compact closure [10]. So Theorem 3.4.1 does not apply to $PSL(n, \mathbb{Z})$. Our aim in this chapter is to prove the following theorem.

**Theorem 4.1.1.** Let $d$ be an integer greater than or equal to 2. Let $\Gamma$ be the group $PSL(d, \mathbb{Z})$ acting on the space $P^{(d)}$. When $d = 2$, the geometric distance function $d_R$ on $\Gamma$ is not Lipschitz equivalent to the word distance function $d_W$ on $\Gamma$. For all $d \geq 3$, the distance function $d_R$ on $\Gamma$ is Lipschitz equivalent to the distance function $d_W$ on $\Gamma$.

(The switch to the letter $d$ is so that the letter $n$ is available to be used in proofs.)

Before proving Theorem 4.1.1 we will discuss briefly how the distance functions $d_R$ and $d_W$ are actually defined for the group $PSL(d, \mathbb{Z})$. Then, we prove Theorem 4.1.1 for the case where $\Gamma = PSL(2, \mathbb{Z})$. To prove Theorem 4.1.1 for the case $\Gamma = PSL(d, \mathbb{Z})$, where $d \geq 3$, it suffices to prove that there exist constants $C_1, C_2 > 0$ such that, for all $\gamma \in \Gamma$,

$$d_R(1, \gamma) \leq C_1 d_W(1, \gamma), \quad \text{and} \quad d_W(1, \gamma) \leq C_2 d_R(1, \gamma).$$

The proof is then broken into three main steps.

1. The inequality $d_R(1, \gamma) \leq C_1 d_W(1, \gamma)$ can be established in exactly the same way as in Theorem 3.4.1.
2. The element $\gamma \in PSL(d, \mathbb{Z})$ is the coset of some matrix $g \in SL(d, \mathbb{Z})$. There exists a constant $K > 0$ such that $d_W(1, \gamma) \leq K \log \|g\|$, where $\|g\|$ is the operator norm of the matrix $g$. This step is Theorem 4.4.1.
3. We conclude, using the definition of the distance function $d_P$ on $P^{(n)}$, that $d_W(1, \gamma) \leq C_2 d_R(1, \gamma)$.

The proof of Theorem 4.1.1 forms the bulk of this chapter.

We will often need to compare the growth of functions, and so state here a useful definition. Let $f$ and $g$ be functions on a discrete set $X$. We say that the function $f$
is $O(g)$, or $f = O(g)$, if there exists a constant $C > 0$, and a finite subset $X_0 \subseteq X$, such that $f(x) \leq Cg(x)$ for all $x \in X \setminus X_0$. The following lemma is also useful.

**Lemma 4.1.2.** Let $f$ and $g$ be functions $\mathbb{R} \to \mathbb{R}$. Then $f(x) \leq g(x)$ for all $x \geq x_0$ if $f(x_0) \leq g(x_0)$, and $f'(x) \leq g'(x)$ for all $x \geq x_0$.

### 4.2 Distance functions on $PSL(d, \mathbb{Z})$

Here we discuss briefly how the geometric and word distance functions on the discrete group $PSL(d, \mathbb{Z})$ are defined.

The definition of the geometric distance function is with respect to some matrix $S_0 \in P(d)$ which has a trivial stabiliser subgroup in $PSL(d, \mathbb{Z})$. Setting $S_0$ to be the identity $I$ does not suffice, because if $K$ is any orthogonal matrix then $K \circ I = I$. However, there does exist a suitable $S_0$ arbitrarily close to $I$ in $P(d)$. For example, when $d = 2$, for small $\varepsilon$ the matrix

$$
\begin{pmatrix}
1 + \varepsilon^2 & \varepsilon \\
\varepsilon & 1
\end{pmatrix} \in P(2)
$$

has a trivial stabiliser subgroup in $PSL(2, \mathbb{Z})$.

For the word distance function, it will often be convenient to consider words in the group $SL(d, \mathbb{Z})$ rather than in $PSL(d, \mathbb{Z})$. For $g \in SL(d, \mathbb{Z})$, write $[g]$ for the coset of $g$ in $PSL(d, \mathbb{Z})$. Then if $g_1 g_2 \cdots g_n$ is a word for $g$ in terms of elements of $SL(d, \mathbb{Z})$, a word for the coset $[g]$ is $[g_1][g_2] \cdots [g_n]$. Going in the other direction, if $[g] = [g_1][g_2] \cdots [g_n]$ is a word in $PSL(d, \mathbb{Z})$ then, in $SL(d, \mathbb{Z})$, either $g = g_1 g_2 \cdots g_n$, or $g = (-I)g_1 g_2 \cdots g_n$. Multiplying by the matrix $-I$ adds 1 to the word length, which will not affect Lipschitz equivalence. Thus, it is reasonable to move back and forth between words in $SL(d, \mathbb{Z})$ and words in $PSL(d, \mathbb{Z})$ when constructing words and proving results about the word distance function $d_W$. We will usually write $\gamma$ both for the coset $\gamma \in PSL(d, \mathbb{Z})$, and for an element $g \in SL(d, \mathbb{Z})$ such that $[g] = \gamma$; we may describe $\gamma$ as a matrix rather than a coset.

We now describe a particular finite set $\Sigma_d$ of generators of $PSL(d, \mathbb{Z})$. This set contains the (cosets of the) following matrices. First, it contains the upper-triangular matrices which have all diagonal entries equal to 1, and one 1 above the diagonal. The set $\Sigma_d$ also contains all permutation matrices which have determinant 1. Finally, $\Sigma_d$ contains all other permutation matrices, but with 1 changed to $-1$ wherever this will ensure a determinant of $+1$. So, generators for $PSL(2, \mathbb{Z})$ are

$$
\Sigma_2 = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},
$$
and generators for \( PSL(3, \mathbb{Z}) \) are

\[
\Sigma_3 = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\} 
\]

\[
\cup \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\} 
\]

\[
\cup \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \right\},
\]

where the signs in the last set of matrices are arranged so that the determinant in each case is 1. The proof that the sets \( \Sigma_d \) do generate the groups \( PSL(d, \mathbb{Z}) \) is by induction on \( d \). For the inductive step, it is possible by multiplying by elements of \( \Sigma_d \cup \Sigma_d^{-1} \) to transform a matrix \( \gamma \) in \( PSL(d, \mathbb{Z}) \) into a matrix \( \theta \) which has first column \((1, 0, \ldots, 0)^T\) and first row \((1, 0, \ldots, 0)\). Then, by the inductive assumption, the \((d - 1) \times (d - 1)\) matrix in the lower right-hand corner of \( \theta \) may be written in terms of elements of \( \Sigma_{d-1} \cup \Sigma_{d-1}^{-1} \). The set of generators \( \Sigma_{d-1} \) may then be considered a subset of \( \Sigma_d \), by embedding \( PSL(d - 1, \mathbb{Z}) \) in the lower right-hand corner of \( PSL(d, \mathbb{Z}) \).

### 4.3 Proof of Theorem 4.1.1 where \( \Gamma = PSL(2, \mathbb{Z}) \)

The group \( PSL(2, \mathbb{Z}) \) may be generated by \( u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Let \( \Sigma = \{ u, v \} \) and let \( d_\Sigma \) be the word distance function induced by the set of generators \( \Sigma \). We show that there is no \( C_1 > 0 \) such that \( d_W(1, u^n) \leq C_1 d_R(1, u^n) \) for all \( n \geq 2 \).

Let \( n \geq 2 \). Then \( d_\Sigma(1, u^n) \leq n \). To show that \( d_\Sigma(1, u^n) \geq n \), we consider the tessellation of the upper half-plane induced by the action of \( \Gamma = PSL(2, \mathbb{Z}) \). Recall that the point \( p_0 = 2i \) lies in the tile \( T \), where \( T \) is the hyperbolic triangle with vertices at \( \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \) and \( \infty \). Let \( w_1 w_2 \ldots w_l \) be any word for \( u^n \) in terms of the elements of \( \Sigma \cup \Sigma^{-1} \). Let \( W_0 \) be the identity in \( \Gamma \). For \( 1 \leq j \leq l \), let \( W_j \) be the partial word \( w_1 w_2 \ldots w_j \). Then the sequence of partial words, \( \{W_j\}_{j=0}^{l} \), corresponds to a sequence of tiles, \( \{W_j T\}_{j=0}^{l} \), such that \( W_0 T = T \) and \( W_l T = u^n T \). In this sequence of tiles, for each \( j \) such that \( 1 \leq j \leq l \), the tiles \( W_{j-1} T \) and \( W_j T \) share a boundary. For \( k \in \mathbb{Z} \), let \( L_k \) be the geodesic of \( U^2 \) which is given by \( \text{Re}(z) = k + 1/2 \). Then \( L_k \) is a boundary between pairs of tiles. See the picture of the tessellation on page 9.

Since \( u^n p_0 = n + 2i \), the \( n \) boundaries \( L_0, L_1, \ldots, L_{n-1} \) all lie between the points \( p_0 \) and \( u^n p_0 \). Thus, the sequence \( \{W_j T\}_{j=0}^{l} \) must contain at least \( n \) successive pairs of tiles which have a geodesic \( L_k \) as their common boundary. Therefore, \( l \geq n \). So \( d_\Sigma(1, u^n) \geq n \); hence \( d_\Sigma(1, u^n) = n \). This implies that the word distance \( d_W(1, u^n) \) grows linearly in \( n \).

For the geometric distance function, we use Lemma 3.5.3 which states that the upper half-plane is isometric to \( P(2) \) once the distance function on one of these
spaces is multiplied by a scalar. Multiplying a distance function by a scalar does not affect Lipschitz equivalence, so there is some constant $C > 0$ such that

\[ d_R(1, u^n) = d_P(S_0, u^n \circ S_0) = C d_U(p_0, u^n p_0) = C \cosh^{-1} \left( 1 + \frac{n^2}{8} \right). \]

The function $\cosh^{-1}(1 + n^2/8)$ is $O(\log n)$, so $d_R(1, u^n)$ is $O(\log n)$. But there is no constant $C_1 > 0$ such that $n \leq C_1 \log n$ for all $n \geq 2$. Hence, there is no $C_1 > 0$ such that $d_W(1, u^n) \leq C_1 d_R(1, u^n)$ for all $n \geq 2$. So, in this case, the geometric distance function $d_R$ is not Lipschitz equivalent to the word distance function $d_W$.

### 4.4 Proof of Theorem 4.4.1

**where $\Gamma = PSL(d, \mathbb{Z})$ and $d \geq 3$**

The first step in this proof is to show that there exists a constant $C_1 > 0$ such that, for all $\gamma \in \Gamma$,

\[ d_R(1, \gamma) \leq C_1 d_W(1, \gamma). \tag{4.1} \]

As claimed in the introduction to this chapter, the proof in Theorem 3.4.1 suffices. This is because the proof of inequality (4.1) in Theorem 3.4.1 did not depend on the compactness of tiles in the tesselation induced by $\Gamma$. Indeed, inequality (4.1) holds in the case $\Gamma = PSL(2, \mathbb{Z})$.

The second step in the proof of Theorem 4.4.1 for the case $d \geq 3$ is the following theorem. We denote by $\|g\|$ the operator norm of a matrix $g \in SL(d, \mathbb{Z})$, that is, $\|g\| = \sup_{\|x\|=1} \|gx\|$. Now, if $g \in SL(d, \mathbb{Z})$, then $\|g\| = \|g^{-1}\|$. So, if $\gamma \in PSL(d, \mathbb{Z})$ is the coset $\gamma = [g]$, then it is well-defined to declare the norm of $\gamma$ to be $\|\gamma\| = \|g\|$.

**Theorem 4.4.1.** Let $d$ be an integer greater than or equal to 3. Then there exists a constant $K > 0$ such that, for all $\gamma \in PSL(d, \mathbb{Z})$, $d_W(1, \gamma) \leq K \log \|\gamma\|$.

Before we prove Theorem 4.4.1, we show how it implies that there exists a constant $C_2 > 0$ such that, for all $\gamma \in \Gamma$,

\[ d_W(1, \gamma) \leq C_2 d_R(1, \gamma). \tag{4.2} \]

Inequality (4.2) is the final step in the proof of Theorem 4.4.1. Applying Theorem 4.4.1 to first $\gamma$ then $\gamma^{-1}$, we obtain

\[ d_W(1, \gamma) \leq K \log \|\gamma\|, \quad \text{and} \quad d_W(1, \gamma^{-1}) \leq K \log \|\gamma^{-1}\|. \]

Now, $d_W(1, \gamma) = d_W(1, \gamma^{-1})$, which gives us

\[ d_W(1, \gamma) \leq \frac{K}{2} \log \|\gamma\||\gamma^{-1}|. \tag{4.3} \]

Next, we consider distances in the space $P^{(d)}$. The distance between the identity $I$ and the matrix $\gamma \circ I$ is

\[ d_P(I, \gamma \circ I) = d_P(I, \gamma \gamma^T) = \log \|\gamma \gamma^T\| \|(\gamma \gamma^T)^{-1}\| = \log \|\gamma \gamma^T\| \|(\gamma^{-1})^T \gamma^{-1}\|. \]

It is a standard result that if $A$ is a bounded operator on a Hilbert space then $\|A\|^2 = \|A^*\|^2 = \|AA^*\|$, where $A^*$ is the adjoint of $A$; see [4]. Applying this to the
matrix $\gamma$, which has adjoint $\gamma^T$, we obtain

$$d_P(I, \gamma \circ I) = \log \|\gamma\|^2 \|\gamma^{-1}\|^2 = 2 \log \|\gamma\| \|\gamma^{-1}\|.$$  

Combining this with (4.3) yields

$$d_W(1, \gamma) \leq \frac{K}{4} d_P(I, \gamma \circ I). \quad (4.4)$$

The geometric distance function on $\Gamma$ is $d_R(1, \gamma) = d_P(S_0, \gamma \circ S_0)$. Since $\Gamma$ is discrete, there is a constant $r_0 > 0$ such that for all $\gamma \in \Gamma$, with $\gamma \neq 1$, we have $d_R(1, \gamma) \geq r_0$. Choose the constant $C$ so that $2d_P(I, S_0) \leq (C - 1)r_0$. Then, by the triangle inequality, for $\gamma \neq 1$,

$$d_P(I, \gamma \circ I) \leq d_P(I, S_0) + d_P(S_0, \gamma \circ S_0) + d_P(\gamma \circ S_0, \gamma \circ I)
= 2d_P(I, S_0) + d_R(1, \gamma)
\leq (C - 1)r_0 + d_R(1, \gamma)
\leq (C - 1)d_R(1, \gamma) + d_R(1, \gamma)
= Cd_R(1, \gamma).$$

Hence, using (4.4), we conclude that there exists a constant $C_2 > 0$ such that, for all $\gamma \in \Gamma$,

$$d_W(1, \gamma) \leq C_2 d_R(1, \gamma).$$

4.4.1 Outline of the proof of Theorem 4.4.1

Since Theorem 4.4.1 seeks to bound the word length of $\gamma \in \Gamma$, we begin by, in Subsection 4.4.2, constructing a special word

$$\gamma = \delta_1 \delta_2 \cdots \delta_d.$$  

Each of the terms $\delta_i$ in this factorisation belongs to some copy of $PSL(2, \mathbb{Z})$ in $PSL(d, \mathbb{Z})$. We want to relate the word for $\gamma$ to the norm of $\gamma$. Since the norm of $\gamma$ depends on the entries of $\gamma$, we show that, in fact, the elements $\delta_i$ may be chosen so that the entries of each $\delta_i$ are bounded by a fixed polynomial in the entries of $\gamma$.

Next, in Subsection 4.4.3, we find a bound on the word length of any $\delta$ belonging to some copy of $PSL(2, \mathbb{Z})$ in $PSL(d, \mathbb{Z})$. Let $d_W$ be the word distance function on $PSL(d, \mathbb{Z})$. We show that there exists a constant $K_1 > 0$ such that, for all such $\delta$,

$$d_W(1, \delta) \leq K_1 \log \|\delta\|. \quad (4.5)$$

To establish this inequality, we first consider the elementary matrices which have one above-diagonal entry 1, all diagonal entries 1, and all other entries 0. We show that if $\delta$ is such a matrix, then, roughly, $d_W(1, \delta^n) = O(\log |n|)$. It is here that $d \geq 3$ is necessary, for if $d = 2$ and $\delta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $d_W(1, \delta^n) \neq O(\log |n|)$.

Next, we restrict our attention to $PSL(2, \mathbb{Z})$. We construct a special word for any $\delta \in PSL(2, \mathbb{Z})$ in terms of only the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \in \mathbb{Z}$.
Moreover, there is a fixed polynomial \( p \leq 1 \leq 1 \gamma \) the first component of \( \gamma \) just put \( \gamma \). If \( x \) we will denote by \( E \)

Proof. Since \( SL \) diagonal entries 1, and all other entries 0. Another definition needed is that of the \( \{1 \). Some elements of the set of integers \( \gamma \) which fixes the basis vectors \( e \)

\( \text{Lemma 4.4.2.} \)

\( \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \in SL(2,\mathbb{Z}) \), then

\[
SL^{1,2}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad SL^{1,3}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a_{11} & 0 & a_{12} \\ 0 & 1 & 0 \\ a_{21} & 0 & a_{22} \end{pmatrix} \middle| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}.
\]

\textbf{Lemma 4.4.2.} Let \( x = (x_1, x_2, \ldots, x_d)^T \in \mathbb{Z}^d \) be a unimodular vector. Then there is an element \( \gamma_1 \in \{ E_{1i}(1) \mid i \neq 1 \} \cup \{ I \} \), and elements \( \gamma_i \in SL^{1,i}(2,\mathbb{Z}) \) for \( 2 \leq i \leq d \), such that

\[
\gamma_d \cdots \gamma_2 \gamma_1 x = (1, 0, \ldots, 0)^T.
\]

Moreover, there is a fixed polynomial \( p \) in the components of \( x \), such that, for \( 1 \leq i \leq d \), and \( 1 \leq k, l \leq d \), the \( (k, l) \) entry of \( \gamma_i \) satisfies

\[
| (\gamma_i)_{kl} | \leq |p(x_1, x_2, \ldots, x_d)|.
\]

\textbf{Proof.} Since \( x \) is unimodular, \( x \neq 0 \). If the first component \( x_1 \) is non-zero then just put \( \gamma_1 = I \). If \( x_1 = 0 \), then \( x_j \neq 0 \) for some \( 2 \leq j \leq d \); put \( \gamma_1 = E_{1j}(1) \). Then the first component of \( \gamma_1 x \) is non-zero.

For \( i = 2, 3, \ldots, d \) in turn, we construct \( \gamma_i \). Let \( y^i \) be the vector

\[
y^i = (y_1^i, y_2^i, \ldots, y_d^i)^T = \gamma_{i-1} \cdots \gamma_2 \gamma_1 x.
\]
By the construction which follows, we have $y'_j \neq 0$, and $y'_1 = 0$ for $1 < j < i$. Let $a_i$ be the greatest common divisor of $y'_i$ and $y'_i$. Then, by the Euclidean algorithm, there exist integers $u$ and $v$ such that $uy'_i + vy'_i = a_i$.

We claim that $u$ and $v$ may be chosen so that $|u|$ and $|v|$ are both less than or equal to the polynomial $(y'_i)^2 + (y'_i)^2$. If $y'_i = 0$, then with $u = 1$ and $v = 0$ the claim follows immediately. If $|y'_i| = |y'_i| \neq 0$ then, again, put $u = 1$ and $v = 0$. Otherwise, we may suppose without loss of generality that $|y'_1| > |y'_i|$. Now, since $uy'_i + vy'_i = a_i$, it follows that

$$(u + my'_i)y'_i + (v - my'_i)y'_i = a_i$$

for all $m \in \mathbb{Z}$ as well. We may now choose $m$ so that $|u + my'_i| \leq |y'_i|/2$. Then, relabel $u + my'_i$ as $u$ and $v - my'_i$ as $v$, so that $|u| \leq |y'_i|/2 \leq |y'_i|$. Because $|y'_i| \neq |y'_i|$, the greatest common divisor $a_i$ is not equal to $y'_i$ or to $-y'_i$, thus $|a_i| \leq |y'_i|/2$. So, for $|v|$, we have

$$|v| = \frac{|a_i| - u}{y'_i} \leq \frac{|a_i|}{|y'_i|} + \frac{|u|}{|y'_i|} |y'_i| \leq |a_i| + \frac{1}{2} |y'_i| \leq \frac{1}{2} |y'_i| + \frac{1}{2} |y'_i| = |y'_i|.$$  

And thus, since $|u| \leq |y'_i|$ and $|v| \leq |y'_i|$, it follows that $|u|, |v| \leq (y'_i)^2 + (y'_i)^2$.

Since $a_i$ is a factor of $y'_i$ and $y'_i$, there exist integers $b_1$ and $b_i$ such that $y'_i = a_i b_i$ and $y'_i = a_i b_i$. Put $z = b_1$ and $w = -b_i$, then

$$a_i = uy'_i + vy'_i = (ub_1 + vb_i) = a_i(uz - vw).$$

Therefore the matrix $\begin{pmatrix} u & v \\ w & z \end{pmatrix}$ is in $SL(2, \mathbb{Z})$. Then,

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} \begin{pmatrix} y'_i \\ y'_i \end{pmatrix} = \begin{pmatrix} uy'_i + vy'_i \\ -b_i y'_i + b_i y'_i \end{pmatrix}.$$  

Note that $-b_i y'_i + b_i y'_i = -b_i a_i b_1 + b_i a_i b_i = 0$. It follows that

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} \begin{pmatrix} y'_i \\ y'_i \end{pmatrix} = \begin{pmatrix} a_i \\ 0 \end{pmatrix}.$$  

The integers $|z|$ and $|w|$ are bounded by the same polynomial as $|u|$ and $|v|$. We have

$$|z| = \frac{|y'_i|}{|a_i|} \leq |y'_i| \leq (y'_i)^2 + (y'_i)^2,$$

and, similarly, $|w| \leq (y'_i)^2 + (y'_i)^2$.

Let $\gamma_i$ be the matrix $\begin{pmatrix} u & v \\ w & z \end{pmatrix} \in SL^i(d, \mathbb{Z})$. Then

$$\gamma_i y'^i = \gamma_i \gamma_{i-1} \cdots \gamma_2 \gamma_1 x = (a_i, 0, \ldots, 0, y'_{i+1}, \ldots, y'_d)^T.$$
By our construction, $y_i^t = x_i$, so

\[ a_i = \gcd(y_i^t, y_i^0) = \gcd(\gcd(y_{i-1}^t, y_{i-1}^0), x_i) = \gcd(y_{i-1}^t, x_{i-1}, x_i) = \cdots = \gcd(x'_1, x_2, \ldots, x_i), \]

where $x'_1$ is the first component of $\gamma_1 x$. In particular,

\[ a_d = \gcd(x'_1, x_2, \ldots, x_d) = \gcd(x_1, x_2, \ldots, x_d) = 1. \]

Thus we obtain $\gamma_1, \gamma_2, \ldots, \gamma_d$ such that $\gamma_d \cdots \gamma_2 \gamma_1 x = (1, 0, \ldots, 0)^T$.

We have shown that the entries of $\gamma_i$, for $2 \leq i \leq d$, are bounded by the polynomial $(y_i^1)^2 + (y_i^0)^2 = (\gcd(y_{i-1}^1, y_{i-1}^0))^2 + x_i^2$. Now,

\[
(\gcd(y_{i-1}^1, y_{i-1}^0))^2 + x_i^2 = (\gcd(x'_1, x_2, \ldots, x_{i-1}))^2 + x_i^2 \
\leq ((x_1^2 + x_2^2 + \cdots + x_{i-1}^2))^2 + x_i^2 \
\leq (2(x_1^2 + x_2^2 + \cdots + x_{i-1}^2)) + x_i^2 \
\leq 5(x_1^2 + x_2^2 + \cdots + x_d^2). \]

Let $p$ be the polynomial

\[ p(x_1, x_2, \ldots, x_d) = 5(x_1^2 + x_2^2 + \cdots + x_d^2)^2. \]

Then $p$ is a fixed polynomial in the components of $x$, and the entries of each $\gamma_i$, for $2 \leq i \leq d$, are bounded by $p$. The matrix $\gamma_1$ is either the identity, or $E_{1j}(1)$ for some $j$, so in either case the entries of $\gamma_1$ are bounded by 1. The entries of $\gamma_i$ are thus bounded by the polynomial $p$ as well.

**Corollary 4.4.3.** Let $\gamma \in SL(d, \mathbb{Z})$. Then $\gamma$ may be written as a product

\[ \gamma = \delta_1 \delta_2 \cdots \delta_d, \]

where each $\delta_i$ belongs to some $SL^{\alpha}(2, \mathbb{Z}) \subseteq SL(d, \mathbb{Z})$. Moreover, there is a fixed polynomial $P$ in the entries of $\gamma$, such that, for $1 \leq i \leq d^2$, and for $1 \leq k, l \leq d$, the $(k, l)$ entry of $\delta_i$ satisfies

\[ |(\delta_i)_{kl}| \leq |P(\gamma_{11}, \gamma_{12}, \ldots, \gamma_{dd})|. \]

**Proof.** We prove this corollary by induction on $d$. The idea is to multiply $\gamma$ by appropriate matrices on the left to obtain a matrix $\gamma'$ with first column $(1, 0, \ldots, 0)^T$. Then, we multiply $\gamma'$ by appropriate matrices on the right to obtain a matrix $\gamma''$ with first column $(1, 0, \ldots, 0)^T$ and first row $(1, 0, \ldots, 0)$. The inductive assumption is then applied to $\gamma''$.

If $d = 2$ then we may let $\delta_1 = \gamma$, and for $2 \leq i \leq d^2$, let $\delta_i$ be the identity matrix. For $d \geq 3$, let $x = (x_1, x_2, \ldots, x_d)^T \in \mathbb{Z}^d$ be the first column of the matrix...
\( \gamma \). Since \( \mathbf{x} \) is a column of a matrix in \( SL(d, \mathbb{Z}) \), \( \mathbf{x} \) is unimodular. This is because the determinant of \( \gamma \) may be calculated by expanding along its first column, so we may obtain the equation

\[
m_1x_1 + m_2x_2 + \cdots + m_dx_d = 1,
\]

where the \( m_i \), for \( 1 \leq i \leq d \), are integers. Thus, by Lemma 4.4.2, there exist matrices \( \gamma_1, \gamma_2, \ldots, \gamma_d \) belonging to various copies of \( SL(2, \mathbb{Z}) \) in \( SL(d, \mathbb{Z}) \) such that

\[
\gamma_d \cdots \gamma_2 \gamma_1 \mathbf{x} = (1, 0, \ldots, 0)^T.
\]

Hence,

\[
\gamma' = \gamma_d \cdots \gamma_2 \gamma_1 \gamma
\]

is an element of \( SL(d, \mathbb{Z}) \) which has as its first column \((1, 0, \ldots, 0)^T\). Also by Lemma 4.4.2, each of the matrices \( \gamma_i \) has entries bounded by the fixed polynomial \( p \) in the components \( x_j \), where \( 1 \leq j \leq d \). Since \( x_j = \gamma_1 \), which is an entry in the first column of the matrix \( \gamma \), we may now consider \( p \) to be a fixed polynomial in the entries of \( \gamma \). Therefore, each matrix \( \gamma_i \) has entries bounded by a fixed polynomial \( p \) in the entries of \( \gamma \).

Let the first row of \( \gamma' \) be \((1, y_2, y_3, \ldots, y_d)\). Multiply \( \gamma' \) on the right by \( \gamma_2 \gamma_3' \cdots \gamma_d' \), where for \( 2 \leq j \leq d \), the matrix \( \gamma_j' \) is \( \begin{pmatrix} 1 & -y_j \\ 0 & 1 \end{pmatrix} \) in \( SL^{1-j}(2, \mathbb{Z}) \). This results in a matrix

\[
\gamma'' = \gamma_d \cdots \gamma_2 \gamma_1 \gamma_2' \gamma_3' \cdots \gamma_d'
\]

with first column \((1, 0, \ldots, 0)^T\) and first row \((1, 0, \ldots, 0)\). We may thus consider \( \gamma'' \) to belong to \( SL(d-1, \mathbb{Z}) \) embedded in the lower right-hand corner of \( SL(d, \mathbb{Z}) \).

The entries in the matrices of the form \( \gamma_j' \), for \( 2 \leq j \leq d \), are bounded by the polynomial \( y_j^2 + 1 \). Now, if \( \gamma_1 \) is the identity, and \( \gamma_j = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \in SL^{1-j}(d, \mathbb{Z}) \), then

\[
y_j = u \gamma_{1j} + v \gamma_{jj}.
\]

If \( \gamma_1 = E_{1k}(1) \) for some \( k \neq 1 \), and \( \gamma_j = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \in SL^{1-j}(d, \mathbb{Z}) \), then

\[
y_j = u(\gamma_{1j} + \gamma_{kj}) + v \gamma_{jj}.
\]

In either case, the entries in the matrices of the form \( \gamma_j' \), for \( 2 \leq j \leq d \), are bounded by

\[
y_j^2 + 1 \leq (|u|(|\gamma_{1j}| + |\gamma_{kj}|) + |v||\gamma_{jj}|)^2 + 1
\]

\[
\leq (p(\gamma)(\gamma_{1j}^2 + \gamma_{kj}^2 + \gamma_{jj}^2))^2 + 1
\]

\[
\leq (p(\gamma)q(\gamma))^2 + 1,
\]

where \( p(\gamma) \) is the fixed polynomial \( p \) in the entries of \( \gamma \), and \( q(\gamma) \) is the polynomial \( q \) in the entries of \( \gamma \) given by

\[
q(\gamma_{11}, \gamma_{12}, \ldots, \gamma_{dd}) = 2(\gamma_{12}^2 + \gamma_{13}^2 + \cdots + \gamma_{1d}^2 + \gamma_{22}^2 + \gamma_{33}^2 + \cdots + \gamma_{dd}^2).
\]

Thus the entries of the matrices of the form \( \gamma_j' \), for \( 2 \leq j \leq d \), are bounded by a fixed polynomial \((p(\gamma)q(\gamma))^2 + 1\) in the entries of \( \gamma \).
Applying the inductive hypothesis to the matrix $\gamma''$, we may write $\gamma''$ as a product $\gamma'' = \delta_1\delta_2 \cdots \delta_{(d-1)^2}$, where for $1 \leq k \leq (d-1)^2$, each $\delta_k$ belongs to some $SL^{s,t}(2,\mathbb{Z})$, and each $\delta_k$ has entries bounded by a fixed polynomial in the entries of the $(d-1) \times (d-1)$ submatrix in the lower right-hand corner of $\gamma''$. So we may consider each $\delta_k$ to have entries bounded by a fixed polynomial in the entries of $\gamma''$. Now, $\gamma''$ is the product of the matrix $\gamma$, and matrices of the form $\gamma_i$ and $\gamma'_j$. So, each entry of $\gamma''$ is some polynomial in the entries of $\gamma$, the $\gamma_i$ and the $\gamma'_j$. But the matrices $\gamma$, $\gamma_i$ and $\gamma'_j$ all have entries bounded by fixed polynomials in the entries of $\gamma$. This means the entries of $\gamma''$ are bounded by a (higher degree) fixed polynomial in the entries of $\gamma$. So we can find a fixed polynomial $P$ in the entries of $\gamma$ which is sufficiently large to bound the entries of the $\gamma_i$, the $\gamma'_j$ and the $\delta_k$.

Rearranging to make $\gamma$ the subject, we now have
\[
\gamma = \gamma_1^{-1}\gamma_2^{-1} \cdots \gamma_d^{-1}\delta_2 \cdots \delta_{(d-1)^2}(\gamma_d')^{-1} \cdots (\gamma_3')^{-1}(\gamma_2')^{-1}.
\]
The total number of matrices on the right-hand side of this expression is $d^2$. The inverses $\gamma_i^{-1}$ and $(\gamma'_j)^{-1}$ have entries bounded by the same fixed polynomial as for $\gamma_i$ and $\gamma'_j$. This is because the inverse of $\gamma_i = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$ is $\begin{pmatrix} z & -v \\ -w & u \end{pmatrix}$, and the inverse of $\gamma'_j = \begin{pmatrix} 1 & -y_j \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & y_j \\ 0 & 1 \end{pmatrix}$. In both cases, the bounds on the entries are unchanged. We conclude that $\gamma$ may be expressed as a product of the required form.

\[\Box\]

4.4.3 Factorisation of $\delta \in PSL(d,\mathbb{Z})$

As in Subsection 4.4.2, the results established here are in terms of the group $SL(d,\mathbb{Z})$, rather than $PSL(d,\mathbb{Z})$. Propositions 4.4.4 and 4.4.10, and Lemma 4.4.13 are all used to prove Corollary 4.4.14. Corollary 4.4.14 says that there exists a constant $K_6 > 0$ such that, for all elements $\delta \in SL^{s,t}(2,\mathbb{Z}) \subseteq SL(d,\mathbb{Z})$, we have $d_W(1,\delta) \leq K_6 \log \|\delta\|$. The proofs of Propositions 4.4.4 and 4.4.10 are long, each involving several lemmas.

We say that $\gamma \in SL(d,\mathbb{Z})$ is a U1-element of $SL(d,\mathbb{Z})$, or just $\gamma$ is U1, if there exists a constant $C_\gamma > 0$ such that, for all integers $n$,
\[
d_W(1,\gamma^n) \leq C_\gamma \log(|n| + 1).
\]

Proposition 4.4.4. The matrix $\gamma = E_{ij}(1)$ (where $1 \leq i \neq j \leq d$) is U1-element of $SL(d,\mathbb{Z})$ for $d \geq 3$.

Proof. First, we show why it suffices to prove this result for just the case where $\gamma$ is $E_{13}(1) \in SL(3,\mathbb{Z})$. Then, we identify a subgroup $V$ of $SL(3,\mathbb{R})$ with the vector space $\mathbb{R}^2$, and identify a discrete subgroup of $V$ with the lattice $\mathbb{Z}^2$. The matrix $E_{13}(1)$ is in this discrete subgroup. We construct a set $S \subseteq \mathbb{Z}^2$ such that every point of $\mathbb{Z}^2$ is at a bounded distance from some point of $S$. For a vector $v \in S$, we find a bound on the norm of $v$, and a bound on the word length of the element of $SL(3,\mathbb{R})$ which corresponds to $v$. This allows us to bound the word length of those elements of $SL(3,\mathbb{R})$ which are identified with the lattice $\mathbb{Z}^2$. To make the proof of this proposition more digestible, most of these steps are presented as lemmas.
To show why it suffices to prove this proposition for just the case where $\gamma$ is $E_{13}(1) \in SL(3, \mathbb{Z})$, let $\gamma$ and $\delta$ be elements of $SL(d, \mathbb{Z})$, and suppose that $\gamma$ is U1. Then, there exist constants $C_\gamma$ and $C_\delta$ such that, for all non-zero integers $n$,

$$d_W(1, (\gamma \delta \delta^{-1})^n) = d_W(1, \gamma^n \delta^{-1})$$

$$\leq d_W(1, \gamma^n) + 2d_W(1, \delta)$$

$$\leq C_\gamma \log(|n| + 1) + 2d_W(1, \delta)$$

$$\leq 2(C_\gamma + 2d_W(1, \delta)) \log(|n| + 1)$$

$$\leq C_\delta \log(|n| + 1).$$

If $n = 0$ then both sides are equal to 0. This shows that if $\gamma$ is U1, so are all conjugates of $\gamma$.

We now show that all elements $E_{ij}(1)$, where $1 \leq i \neq j \leq d$, are conjugate to $E_{13}(1)$ in $SL(d, \mathbb{Z})$. We proceed by induction on $d$. If $d = 3$, then each matrix $E_{ij}(1)$, for $1 \leq i \neq j \leq 3$, may be written as $\delta E_{13}(1) \delta^{-1}$ for some $\delta \in SL(3, \mathbb{Z})$:

- For $E_{12}(1)$, $\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$;
- For $E_{21}(1)$, $\delta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$;
- For $E_{23}(1)$, $\delta = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;
- For $E_{31}(1)$, $\delta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$;
- For $E_{32}(1)$, $\delta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

The inductive assumption is that for some $d \geq 3$, all of the elements $E_{ij}(1)$, where $1 \leq i \neq j \leq d$, are conjugate to $E_{13}(1)$ in $SL(d, \mathbb{Z})$. Consider $E_{ij}(1) \in SL(d+1, \mathbb{Z})$. If $i$ and $j$ are both less than or equal to $d$, $E_{ij}(1)$ may be regarded as an element of $SL(d, \mathbb{Z})$ embedded in the upper left-hand corner of $SL(d+1, \mathbb{Z})$, so by assumption $E_{ij}(1)$ is conjugate to $E_{13}(1)$. Otherwise, one of $i$ or $j$ must be equal to $d + 1$; we show that $E_{ij}(1)$ is conjugate to some matrix $E_{st}(1)$ with $1 \leq s \neq t \leq d$, hence is conjugate to $E_{13}(1)$. For $i = d + 1$ and $1 \leq j \leq d - 1$, and for $j = d + 1$ and $1 \leq i \leq d - 1$, the matrix by which we conjugate is

$$\begin{pmatrix} I_{d-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $I_{d-1}$ is the $(d-1) \times (d-1)$ identity matrix. For $(i, j) = (d + 1, d)$ and $(i, j) = (d, d + 1)$, we conjugate by

$$\begin{pmatrix} I_{d-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

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It remains to show that if the matrix \( E_{13}(1) \) is \( \text{U}1 \) in \( SL(3, \mathbb{Z}) \), then it is \( \text{U}1 \) in \( SL(d, \mathbb{Z}) \). For this, let \( \Sigma_3 \) be a finite set of generators for \( SL(3, \mathbb{Z}) \). By embedding \( SL(3, \mathbb{Z}) \) in the top left-hand corner of \( SL(d, \mathbb{Z}) \), we may extend \( \Sigma_3 \) to \( \Sigma \), a finite set of generators for \( SL(d, \mathbb{Z}) \). Let \( \gamma \) be a \( \text{U}1 \) element of \( SL(3, \mathbb{Z}) \), embedded in \( SL(d, \mathbb{Z}) \). The sets of generators \( \Sigma_3 \) and \( \Sigma \) induce word distance functions which we will denote by \( d^3_{\Sigma} \) and \( d_{\Sigma} \), respectively. Then, there exists a constant \( C_\gamma > 0 \) such that, for all integers \( n \),

\[
d_\Sigma(1, \gamma^n) \leq d^3_\Sigma(1, \gamma^n) \leq C_\gamma \log(|n| + 1).
\]

Therefore \( d_W(1, \gamma^n) \leq C_\gamma \log(|n| + 1) \), and \( \gamma \) is \( \text{U}1 \) in \( SL(d, \mathbb{Z}) \).

We have now shown that it suffices to prove that the matrix \( E_{13}(1) \) is \( \text{U}1 \) in \( SL(3, \mathbb{Z}) \). In the next part of the proof, we consider a special element and some subgroups of \( SL(3, \mathbb{Z}) \). Let

\[
A = \begin{pmatrix}
2 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
V = \left\{ v(x, y) = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R} \right\},
\]

\[
L = V \cap SL(3, \mathbb{Z}) = \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \\ 0 & 0 \\ 1 \end{pmatrix} : k, l \in \mathbb{Z} \right\}.
\]

Multiplication in \( V \) is given by \( v(x, y)v(x', y') = v(x + x', y + y') \), so \( V \) is an Abelian group. The group \( V \) is isomorphic to the vector space \( \mathbb{R}^2 \), where the latter is considered as an Abelian group under addition of vectors. We may, then, identify \( \mathbb{L} \subseteq V \) with the integer lattice \( \mathbb{Z}^2 \subseteq \mathbb{R}^2 \). The action of \( A \) on \( V \) by conjugation takes \( v(x, y) \) to \( v(2x + y, x + y) \), which corresponds under this isomorphism to the vector in \( \mathbb{R}^2 \) given by

\[
\begin{pmatrix} 2x + y \\ x + y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

Hence, the action of \( A \) on \( V \) by conjugation corresponds to the linear action of \( A' = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) on \( \mathbb{R}^2 \). Moreover, this action preserves \( L \), and so preserves the integer lattice \( \mathbb{Z}^2 \).

We will now consider \( V \) as the vector space \( \mathbb{R}^2 \) and write the multiplication in it as addition. We also identify \( \mathbb{L} \) with \( \mathbb{Z}^2 \). For \( v \in V \) we will write \( Av \) meaning the conjugation of \( v \) by \( A \), that is, multiplication by the matrix \( A' \), which we will write as \( A \). The eigenvectors of the \( 2 \times 2 \) matrix \( A \) are then \( \lambda = (3 + \sqrt{5})/2 \) and \( \lambda^{-1} = (3 - \sqrt{5})/2 \). Note that \( 2 < \lambda < 3 \) and \( \lambda^{-1} < 1 \). We will denote by \( v_1 \) and \( v_2 \) fixed eigenvectors corresponding to the eigenvalues \( \lambda \) and \( \lambda^{-1} \) respectively. For convenience, we may choose \( v_1 \) and \( v_2 \) so that \( ||v_1|| = ||v_2|| = 1 \). We denote by \( W_1 \) and \( W_2 \) the corresponding eigenspaces of \( V \). Let \( y_0 \) be the fixed vector \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V \).
We define the following subsets of $L$:

$$
S_1 = \left\{ \pm \sum_{i=1}^{m} a_i A^i y_0 : a_i \in \{0, 1, 2\}, a_m \neq 0 \right\}
$$

$$
S_2 = \left\{ \pm \sum_{j=1}^{n} b_j A^{-j} y_0 : b_j \in \{0, 1, 2\}, b_n \neq 0 \right\}.
$$

The relationship between the sets $S_1$ and $S_2$ and the eigenspaces $W_1$ and $W_2$ is described in the following lemma.

**Lemma 4.4.5.** For $k = 1, 2$, each point of the eigenspace $W_k$ is within a bounded distance of some point of the set $S_k$.

**Proof.** Let 

$$
v = \pm \sum_{i=1}^{m} a_i A^i y_0
$$

be a point of $S_1$. Since $v_1$ and $v_2$ span $\mathbb{R}^2$, we have $y_0 = \alpha v_1 + \beta v_2$ for fixed real numbers $\alpha$ and $\beta$. Then

$$
A^i y_0 = \alpha \lambda^i v_1 + \beta \lambda^{-i} v_2,
$$

and so,

$$
v = \pm \left( \alpha \sum_{i=1}^{m} a_i \lambda^i v_1 + \beta \sum_{i=1}^{m} a_i \lambda^{-i} v_2 \right).
$$

The sum $\sum_{i=1}^{m} a_i \lambda^{-i}$ is bounded. This is because $a_i < \lambda$ and $\lambda^{-1} < 1$, so

$$
\sum_{i=1}^{m} a_i \lambda^{-i} < \frac{1}{1 - \lambda^{-1}}.
$$

Let $u$ belong to $W_1$. Then $u = \alpha' v_1$ for some $\alpha' \in \mathbb{R}$. Suppose first that $\alpha' \geq 0$, and consider an element $v$ in $S_1$ with the form

$$
v = \alpha \sum_{i=1}^{m} a_i \lambda^i v_1 + \beta \sum_{i=1}^{m} a_i \lambda^{-i} v_2.
$$

Note here that $\alpha > 0$. The distance from $v$ to $u$ is

$$
\|v - u\| = \left\| \left( \alpha \sum_{i=1}^{m} a_i \lambda^i - \alpha' \right) v_1 + \left( \beta \sum_{i=1}^{m} a_i \lambda^{-i} \right) v_2 \right\|
$$

$$
\leq \alpha \left| \sum_{i=1}^{m} a_i \lambda^i - \frac{\alpha'}{\alpha} \right| + |\beta| \sum_{i=1}^{m} a_i \lambda^{-i}.
$$

Since $\alpha$ and $\beta$ are constant, and $\sum_{i=1}^{m} a_i \lambda^{-i}$ is bounded, it now suffices to prove that for all $a \geq 0$ there exists an integer $m \geq 1$, and a set of integers $\{a_i\}_{i=1}^{m}$, with
\(a_i \in \{0, 1, 2\}\) and \(a_m \neq 0\), such that
\[
\left| \sum_{i=1}^{m} a_i \lambda^i - a \right| \leq \lambda. \tag{4.6}
\]

If \(0 \leq a < 1\) then \(m = 1\) and \(a_1 = 1\) will do. For \(a \geq 1\), the proof is by induction on \(k\), where
\[\lambda^k \leq a < \lambda^{k+1}.
\]
When \(k = 0\), we have \(1 \leq a < \lambda\), so take \(m = 1\) and \(a_1 = 1\). Now assume inductively that, for all \(l\) such that \(0 \leq l < k\) and all \(a\) such that \(\lambda^l \leq a < \lambda^{l+1}\), there exist integers \(\{a_i\}_{i=1}^l\), with \(a_i \in \{0, 1, 2\}\) and \(a_i \neq 0\), such that (4.6) holds. Suppose that \(\lambda^k \leq a < \lambda^{k+1}\). Then, since \(2 < \lambda < 3\), either \(\lambda^k \leq a < 2\lambda^k\), or \(2\lambda^k \leq a < 3\lambda^k\). Put \(a_k = 1\) in the first case and \(a_k = 2\) in the second case. Then
\[a - a_k \lambda^k < \lambda^k,
\]
so we have
\[\lambda^l \leq a - a_k \lambda^k < \lambda^{l+1}
\]
for some \(l < k\). By the inductive assumption, there exists a set of integers \(\{a_i\}_{i=1}^l\), with \(a_i \in \{0, 1, 2\}\) and \(a_i \neq 0\), such that
\[
\left| \sum_{i=1}^{l} a_i \lambda^i - (a - a_k \lambda^k) \right| \leq \lambda.
\]

So
\[
\left| \sum_{i=1}^{l} a_i \lambda^i + a_k \lambda^k - a \right| \leq \lambda,
\]
and if we put \(a_i = 0\) for \(l < i < k\), we obtain a set \(\{a_i\}_{i=1}^k\), with \(a_i \in \{0, 1, 2\}\) and \(a_k \neq 0\), satisfying (4.6). Thus \(\|v - u\|\) is bounded.

When \(\alpha' < 0\) the proof is almost the same; consider
\[
v = -\alpha \sum_{i=1}^m a_i \lambda^i v_1 - \beta \sum_{i=1}^m a_i \lambda^{-1} v_2.
\]

To show that every point of \(W_2\) is within a bounded distance of some point of \(S_2\), the proof is similar.

Let \(S\) be the set \(S_1 + S_2\). Note that \(S\) is a subset of the integer lattice \(L\).

**Corollary 4.4.6.** Each point of \(L\) is within a bounded distance of some point in \(S\).

**Proof.** The eigenvectors \(v_1\) and \(v_2\) are linearly independent, so \(\mathbb{R}^2 = W_1 + W_2\). Each point of \(L\), then, can be written as \(w_1 + w_2\) for some \(w_1 \in W_1\) and \(w_2 \in W_2\). By Lemma 4.4.5 \(w_1\) is within a bounded distance of some point of \(S_1\), and \(w_2\) is within a bounded distance of some point of \(S_2\). Thus \(w_1 + w_2\) is within a bounded distance of some point of \(S_1 + S_2\). \(\square\)
We now establish a lower bound on the norm of vectors in $S$. Let $v$ be in $S$. Then $v$ may be expressed in the form

$$v = \pm_1 \sum_{i=1}^{m} a_i A^i y_0 \pm_2 \sum_{j=1}^{n} b_j A^{-j} y_0,$$

where $\pm_1$ and $\pm_2$ are independent of each other. We first prove a general result about linearly independent vectors.

**Lemma 4.4.7.** Let $w_1$ and $w_2$ be linearly independent vectors in the vector space $\mathbb{R}^m$, where $m \geq 2$. Then there exists an $M > 0$ such that, for all $\mu_1, \mu_2 \in \mathbb{R}$,

$$\|\mu_1 w_1 + \mu_2 w_2\| \geq M \max\{\|\mu_1 w_1\|, \|\mu_2 w_2\|\}.$$

**Proof.** The two-dimensional vector space spanned by $w_1$ and $w_2$ may be identified with the complex plane $\mathbb{C}$. Indeed we may, without loss of generality, identify $w_1$ with the point $1e^{i\theta} = 1$, and $w_2$ with the point $re^{i\theta}$ where $r > 0$ and $0 < \theta < \pi$.

Let $M = \sin \theta$. Then, if $\mu_1 = 0$, there is nothing more to prove. Otherwise, we may assume without loss of generality that $\mu_1 > 0$, and so we are trying to prove that

$$|\mu_1 + \mu_2 re^{i\theta}| \geq M \max\{\mu_1, |\mu_2 r|\}$$

for all $\mu_1 > 0$ and all $\mu_2 \in \mathbb{R}$. Divide through this inequality by $\mu_1$ and put $R = \mu_2 r / \mu_1$. To establish (4.8), it now suffices to prove that for all $R \in \mathbb{R}$,

$$1 + 2R \cos \theta \geq M^2 \max\{1, |R|\}.$$ 

Now, $|1 + Re^{i\theta}|^2 = 1 + 2R \cos \theta + R^2$. We have

$$(\cos \theta + R)^2 \geq 0$$

$$\Rightarrow \cos^2 \theta + 2R \cos \theta + R^2 \geq 0$$

$$\Rightarrow 1 + 2R \cos \theta + R^2 \geq \sin^2 \theta.$$ 

Also,

$$(1 + R \cos \theta)^2 \geq 0$$

$$\Rightarrow 1 + 2R \cos \theta + R^2 \cos^2 \theta \geq 0$$

$$\Rightarrow 1 + 2R \cos \theta + R^2 \geq R^2 \sin^2 \theta.$$ 

It follows that

$$1 + 2R \cos \theta + R^2 \geq M^2 \max\{1, R^2\}.$$ 

Take square roots of both sides to complete the proof. 

Note that the value of $M$ in Lemma 4.4.7 depends only on the angle between the vectors $w_1$ and $w_2$. When we use Lemma 4.4.7 in the next proof, the value of $M$ will thus depend only on the angle between the fixed eigenvectors $v_1$ and $v_2$. So $M$, as far as it is used here, is a constant.
Lemma 4.4.8. There exists a constant $c_1 > 0$ such that, for any $v$ in $S$, if $k$ is the maximum of $m$ and $n$ where $v$ has the form (4.7), then $c_1 \lambda^k \leq \|v\|$.

Proof. As in Lemma 4.4.5, $y_0 = \alpha v_1 + \beta v_2$ for fixed $\alpha$ and $\beta$. Then, we get

$$v = \pm \sum_{i=1}^{m} a_i (\alpha \lambda^{i} v_1 + \beta \lambda^{-i} v_2) \pm \sum_{j=1}^{n} b_j (\alpha \lambda^{-j} v_1 + \beta \lambda^{j} v_2)$$

$$= \left( \pm \sum_{i=1}^{m} a_i \lambda^{i} \pm \sum_{j=1}^{n} b_j \lambda^{-j} \right) \alpha v_1 + \left( \pm \sum_{i=1}^{m} a_i \lambda^{-i} \pm \sum_{j=1}^{n} b_j \lambda^{j} \right) \beta v_2.$$

Let

$$\mu_1 = \left( \pm \sum_{i=1}^{m} a_i \lambda^{i} \pm \sum_{j=1}^{n} b_j \lambda^{-j} \right) \alpha, \quad \text{and} \quad \mu_2 = \left( \pm \sum_{i=1}^{m} a_i \lambda^{-i} \pm \sum_{j=1}^{n} b_j \lambda^{j} \right) \beta.$$

By Lemma 4.4.7, there exists a constant $M > 0$ such that

$$\|v\| \geq M \max\{\|\mu_1 v_1\|, \|\mu_2 v_2\|\} = M \max\{|\mu_1|, |\mu_2|\}.$$

Suppose first that $m \geq n$, and consider

$$\frac{|\mu_1|}{\alpha} = \left| \pm \sum_{i=1}^{m} a_i \lambda^{i} \pm \sum_{j=1}^{n} b_j \lambda^{-j} \right|.$$

Since $a_m \neq 0$, $\sum_{i=1}^{m} a_i \lambda^{i} \geq \lambda^m$. For the other sum, we have

$$0 < \sum_{j=1}^{n} b_j \lambda^{-j} < \sum_{j=0}^{\infty} \lambda^{-j} = \frac{1}{1 - \lambda} = \frac{\lambda}{\lambda - 1} < 2.$$

It follows that

$$\left| \pm \sum_{i=1}^{m} a_i \lambda^{i} \pm \sum_{j=1}^{n} b_j \lambda^{-j} \right| \geq \left| \sum_{i=1}^{m} a_i \lambda^{i} \right| - \left| \sum_{j=1}^{n} b_j \lambda^{-j} \right|$$

$$= \sum_{i=1}^{m} a_i \lambda^{i} - \sum_{j=1}^{n} b_j \lambda^{-j}$$

$$\geq \lambda^m - 2$$

$$\geq \frac{1}{5} \lambda^m.$$

Thus, $\|v\| \geq M|\mu_1| \geq c_1 \lambda^m$ where $c_1 > 0$ is constant. In the case $n > m$, the proof is similar, with a lower bound being established for $|\mu_2|$. \hfill \Box

Corollary 4.4.6 and Lemma 4.4.8 will be used in the conclusion of the proof of Proposition 4.4.4. Before this, we return to the interpretation of $V$ as a subgroup of $SL(3,\mathbb{Z})$. If $v \in S$ is considered as an element of $SL(3,\mathbb{Z})$, then its word length is bounded as follows.
Lemma 4.4.9. There exists a constant $c_2 > 0$ such that, for any $v$ in $S$, if $k$ is the maximum of $m$ and $n$ where $v$ has the form \((4.4)\), then $v$ may be written as a word in the elements $A$ and $Y_0$ of $SL(3, \mathbb{Z})$ of length at most $c_2 k$.

Proof. The expression

$$v = \pm_1 \sum_{i=1}^{m} a_i A^i y_0 \pm_2 \sum_{j=1}^{n} b_j A^{-j} y_0$$

may be rewritten as

$$v = \pm_1 A(a_1 y_0 + A(a_2 y_0 + \cdots + A(a_{m-1} y_0 + A a_m y_0) \cdots))$$

$$\pm_2 A^{-1}(b_1 y_0 + A^{-1}(b_2 y_0 + \cdots + A^{-1}(b_{n-1} y_0 + A^{-1} b_n y_0) \cdots)).$$

Using the correspondence between $\mathbb{R}^2$ and $SL(3, \mathbb{Z})$, we may translate this expression for $v$ into a word written multiplicatively in the elements

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

of $SL(3, \mathbb{Z})$. Terms of the form $a_i y_0$ have length at most 2, since $0 \leq a_i \leq 2$. There are $m$ terms of this form. Each multiplication by $A$ in $\mathbb{R}^2$ corresponds to conjugation by $A$ in $SL(3, \mathbb{Z})$, so we add 2 to the length for each of the $m$ multiplications by $A$. Similar calculations for the second half of the expression show that $v$ may be written as a word of length at most $2m + 2m + 2n + 2n = 4(m + n) \leq 4(2k) = C_2 k$. \qed

We are now in a position to conclude the proof of Proposition 4.4.4. Let $\Sigma$ be a finite set of generators of $SL(3, \mathbb{Z})$ which includes the elements $A$, $Y_0$, $E_{13}(1)$ and $E_{23}(1)$, and let $d_\Sigma$ be the word distance function induced by the set $\Sigma$. Let $v$ be in the set $S$. Using logarithm laws to rearrange the inequality $c_1 \lambda^k \leq \|v\|$ from Lemma 4.4.8, we obtain

$$k \leq \frac{\log \|v\| - \log c_1}{\log \lambda}.$$

By Lemma 4.4.9 and this inequality, the word distance function $d_\Sigma$ satisfies

$$d_\Sigma(1, v) \leq c_2 k \leq \frac{c_2}{\log \lambda} (\log \|v\| - \log c_1) \leq B \log \|v\|,$$

for some constant $B > 0$.

Next, by Corollary 4.4.6, there exists some constant $b_1 > 0$ such that, for all $w \in L$, we can find a vector $v \in S$ so that $\|w - v\| \leq b_1$. Let $w$ be any element of $L$, and write $v' = w - v$, so that $\|v'\| \leq b_1$. We now prove that

$$d_\Sigma(1, w) \leq B' \log(\|w\| + 1),$$

for some constant $B' > 0$. Since $w$ and $v$ are elements of the lattice $L$, so is $v'$. We may then translate the vector $w = v + v' \in \mathbb{R}^2$ into a word, written multiplicatively, in terms of elements of $SL(3, \mathbb{Z})$. We use the elements $A$ and $Y_0$ for $v$, and $E_{13}(1)$.
and $E_{23}(1)$ for $v'$. Since the value of $\|v'\|$ is bounded, this multiplicative expression for $v'$ is bounded in length, by say $b_2 > 0$. Thus

$$d_{\Sigma}(1, w) \leq d_{\Sigma}(1, v) + d_{\Sigma}(1, v') \leq B \log \|v\| + b_2.$$ 

By the triangle inequality, $\|v\| = \|w - v'\| \leq \|w\| + \|v'\|$, so

$$B \log \|v\| + b_2 \leq B \log(\|w\| + \|v'\|) + b_2 \leq B \log(\|w\| + b_1) + b_2 \leq B' \log(\|w\| + 1),$$

for some constant $B' > 0$.

Let $n$ be any integer. Then the matrix $(E_{13}(1))^n$ is in the lattice $L$. It corresponds to the vector $w = \begin{pmatrix} n \\ 0 \end{pmatrix}$, which has norm equal to $|n|$. So, for some $C > 0$, and all integers $n$,

$$d_{U}(1, (E_{13}(1))^n) \leq C \log(|n| + 1).$$

Thus, $E_{13}(1)$ is U1.

**Proposition 4.4.10.** There exist constants $c_3, c_4 > 0$ so that any $\delta \in SL(2, \mathbb{Z})$ may be written as a word of the form $\delta = r_1 r_2 \cdots r_m$, where

1. each $r_j$, for $1 \leq j \leq m$, is either $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 1 & n_j \\ 0 & 1 \end{pmatrix}$ for some non-zero integer $n_j$,
2. we have

$$c_3 d_{U}(p_0, \delta p_0) \leq \sum_{j=1}^{m} f(r_j) \leq c_4 d_{U}(p_0, \delta p_0),$$

where

$$f(r) = \begin{cases} 1 & \text{if } r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \log(|n| + 1) & \text{if } r = \begin{pmatrix} 1 & n_j \\ 0 & 1 \end{pmatrix}, \end{cases}$$

the point $p_0$ is $2i$ in $U^2$ the upper half-plane, and $d_{U}(\cdot, \cdot)$ is the hyperbolic distance function on $U^2$.

**Proof.** Let $T$ be the hyperbolic triangle with vertices at $\pm \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\infty$. Let $u(n)$ denote the matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \in \mathbb{Z}$, and let $v$ be the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Given $\delta \in SL(2, \mathbb{Z})$, we consider the geodesic segment $[p_0, \delta p_0]$ connecting the points $p_0$ and $\delta p_0$ in $U^2$. This geodesic segment passes through a sequence of tiles $\{\theta_k T\}_{k=0}^{n}$, such that $\theta_0 = 1$ and $\theta_n = \delta$, and for $1 \leq k \leq n$, tile $\theta_{k-1} T$ is adjacent to tile $\theta_k T$. Let $s_k = \theta^{-1}_{k-1} \theta_k$. Then the product of the elements $s_k$ is

$$s_1 s_2 \cdots s_n = \theta^{-1}_0 \theta_1^{-1} \theta_2 \cdots \theta^{-1}_{n-1} \theta_n = \delta.$$
Since successive tiles in the sequence \(\{\theta_k T\}_{k=0}^n\) are adjacent, we have, for \(1 \leq k \leq n\),
\[
s_k = \theta_{k-1}^{-1} \theta_k \in \{u(1), u(-1), v\}.
\]

Notice also that since \(v^2\) is the identity in \(\text{PSL}(d, \mathbb{Z})\), there are no two consecutive matrices \(s_{k-1}\) and \(s_k\) which are both equal to \(v\).

We now, by reducing the product of the \(s_k\), construct a new product \(r_1 r_2 \cdots r_m\) which is also equal to \(\delta\), a corresponding subsequence of tiles \(\{\theta_j T\}_{j=0}^m\), and a set of points \(\{p_j\}_{j=0}^m\) in \(U^2\). The basic idea is to multiply together all adjacent matrices of the form \(u(1)\) or \(u(-1)\). Let \(s_i\) be the first matrix in the product \(s_1 s_2 \cdots s_n\) which has the form \(s_i = u(1)\) or \(s_i = u(-1)\). For all \(k\) such that \(1 \leq k < i\), let \(r_k = s_k\), and let the point \(p_k = \theta_k p_0\). There is an integer \(l > 0\) such that \(s_i = u(n_i), s_{i+1} = u(n_{i+1}), \ldots, s_{i+l} = u(n_{i+l})\), but \(s_{i+l+1} = v\). Let \(r_i\) be the matrix \(r_i = u(n_i')\), where \(n_i' = n_i + n_{i+1} + \cdots + n_{i+l}\). Then,
\[
r_i = s_i s_{i+1} \cdots s_{i+l} = \theta_{i-1}^{-1} \theta_i \theta_{i+1}^{-1} \theta_i \cdots \theta_{i+l-1}^{-1} \theta_{i+l} = \theta_{i-1}^{-1} \theta_{i+l}.
\]

Let \(p_i\) be the point \(\theta_{i+l} p_0\). Relabel \(\theta_{i+l}\) as \(\theta_i\), so that \(r_i = \theta_{i-1}^{-1} \theta_i\) and \(p_i = \theta_i p_0\). We now have a product
\[
r_1 \cdots r_i s_{i+l+1} \cdots s_n = \delta,
\]
a sequence of tiles \(\{\theta_j T\}_{j=0}^i\), and a set of points \(\{p_j\}_{j=0}^i\), such that, for \(1 \leq j \leq i\), \(r_j = \theta_{j-1}^{-1} \theta_j\) and \(p_j = \theta_j p_0\). Iterate the above reduction process on the product \(s_{i+l+1} \cdots s_n\) until a product
\[
r_1 r_2 \cdots r_m = \delta,
\]
a sequence \(\{\theta_j T\}_{j=0}^m\) and a set \(\{p_j\}_{j=0}^m\) is obtained, where, for \(1 \leq j \leq m\), \(r_j = \theta_{j-1}^{-1} \theta_j\) and \(p_j = \theta_j p_0\). In this product, no two consecutive matrices \(r_{j-1}\) and \(r_j\) have the respective forms \(r_{j-1} = u(n_{j-1})\) and \(r_j = u(n_j)\), where \(n_{j-1}\) and \(n_j\) are non-zero integers.

We now consider the path \(\alpha\) in \(U^2\) which is the following union of geodesic segments:
\[
\alpha = [p_0, p_1] \cup [p_1, p_2] \cup \cdots \cup [p_{m-1}, p_m].
\]
The length of this path is
\[
\text{length}(\alpha) = \sum_{j=1}^m d_U(p_0, r_j p_0), \quad (4.9)
\]
since
\[
\sum_{j=1}^m d_U(p_{j-1}, p_j) = \sum_{j=1}^m d_U(\theta_{j-1} p_0, \theta_j p_0) = \sum_{j=1}^m d_U(p_0, \theta_{j-1}^{-1} \theta_j p_0) = \sum_{j=1}^m d_U(p_0, r_j p_0).
\]
The following lemma shows that, roughly speaking, the length of \(\alpha\) is not too different from the length of the geodesic joining the points \(p_0\) and \(\delta p_0\).
Lemma 4.4.11. There exist constants $K_1, K_2 > 0$ such that

$$K_1 d_U(p_0, \delta p_0) \leq \text{length}(\alpha) \leq K_2 d_U(p_0, \delta p_0).$$

Proof. Since the geodesic segment $[p_0, \delta p_0]$ is the shortest path from $p_0$ to $\delta p_0$, we have immediately that

$$d_U(p_0, \delta p_0) \leq \text{length}(\alpha).$$

To prove the other required inequality, for $0 \leq j \leq m$ let $q_j$ be the point on the geodesic segment $[p_1, \delta p_0]$ which is closest to the point $p_j$. Note that $q_0 = p_0$ and $q_m = p_m$, and also that the point $q_j$ need not be in the same tile of the tesselation as $p_j$. We claim that, for $1 \leq j \leq m$,

$$d_U(p_0, q_{j-1}) < d_U(p_0, q_j). \quad (4.10)$$

This claim implies that the geodesic segments $[q_{j-1}, q_j]$ are disjoint except for their endpoints, and so

$$d_U(p_0, \delta p_0) = \sum_{j=1}^{m} d_U(q_{j-1}, q_j). \quad (4.11)$$

In proving this claim, we will establish that the sequence $\{q_j\}_{j=0}^m$ is well-defined.

Let $L$ be the geodesic which contains the geodesic segment $[p_0, \delta p_0]$. As $\theta_{j-1}^{-1}$ is an isometry, $(4.10)$ holds if and only if $d_U(\theta_{j-1}^{-1} p_0, \theta_{j-1}^{-1} q_{j-1}) < d_U(\theta_{j-1}^{-1} p_0, \theta_{j-1}^{-1} q_j)$. Since $q_{j-1}$ is the closest point on $L$ to $p_{j-1}$, the point $\theta_{j-1}^{-1} q_{j-1}$ is the closest point on the geodesic $\theta_{j-1}^{-1} L$ to $\theta_{j-1}^{-1} p_{j-1} = p_0$. Similarly, $\theta_{j-1}^{-1} q_j$ is the closest point on the geodesic
\( \theta_{j-1}^{-1}L \) to the point \( \theta_{j-1}^{-1}p_j \). Now, the geodesic \( L \) intersects the tiles \( \theta_jT \) and \( \theta_{j-1}T \), and the tiles \( \theta_jT \) and \( \theta_{j-1}T \) are adjacent. So, the geodesic \( \theta_{j-1}^{-1}L \) intersects the tiles \( \theta_{j-1}^{-1}\theta_jT \) and \( T \), and these tiles are also adjacent. By symmetry in the imaginary axis, we need only consider the cases \( \theta_{j-1}^{-1}\theta_jT = u(1)T \) and \( \theta_{j-1}^{-1}\theta_jT = vT \). Thus the point \( \theta_{j-1}^{-1}p_j \) is either, respectively, \( 1 + 2i \), or \( i/2 \). To simplify notation, we now write \( L \) for \( \theta_{j-1}^{-1}L \), \( p \) for \( p_0 \), \( p' \) for \( \theta_{j-1}^{-1}p_j \), \( q \) for \( \theta_{j-1}^{-1}p_j-1 \) and \( q' \) for \( \theta_{j-1}^{-1}q_j \).

The first case is when \( p' = 1 + 2i \). Here, the geodesic \( L \) is a semicircle orthogonal to the real axis, which intersects the tiles \( T \) and \( u(1)T \). We may parametrise \( z \in L \) by

\[
z = a + re^{i\phi},
\]

where \( a \in \mathbb{R}, r > 0 \) and \( \phi \in (0, \pi) \). Since \( L \) meets \( T \) and \( u(1)T \),

\[
r \geq \left| a - \left( \frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \right| \quad \Rightarrow \quad r^2 \geq a^2 - a + 1. \tag{4.12}
\]

The distance from the point \( n + 2i \), where \( n \in \mathbb{Z} \), to a point \( z = a + re^{i\phi} \in L \) is

\[
d_U(n + 2i, z) = \cosh^{-1} \left( 1 + \frac{(n - a - r \cos \phi)^2 + (2 - r \sin \phi)^2}{4r \sin \phi} \right). \tag{4.13}
\]

For fixed \( n \), \( a \) and \( r \), this function has a unique minimum when

\[
\cos \phi = \frac{2(n - a)r}{(n - a)^2 + r^2 + 4}. \tag{4.14}
\]

Thus, the points \( q \) and \( q' \) are the unique closest points on \( L \) to \( p \) and \( p' \) respectively. So, in this first case, the choice of the \( q_j \) is well-defined. If \( q = a + re^{i\phi} \) and \( q' = a + re^{i\phi'} \), to prove (4.10) it now suffices to prove that \( \phi' < \phi \). By (4.14),

\[
\cos \phi = -\frac{2ar}{a^2 + r^2 + 4} \quad \text{and} \quad \cos \phi' = \frac{2(1-a)r}{(1-a)^2 + r^2 + 4}.
\]

If \( 0 \leq a \leq 1 \), then by considering the signs of \( \cos \phi' \) and \( \cos \phi \), it follows that \( \phi' < \phi \). If \( a > 1 \) or \( a < 0 \) then \( \cos \phi \) and \( \cos \phi' \) have the same sign. Since cosine is decreasing on \((0, \pi)\), it suffices to prove that \( \cos \phi < \cos \phi' \). The inequality

\[
\frac{-2ar}{a^2 + r^2 + 4} < \frac{2(1-a)r}{(1-a)^2 + r^2 + 4}
\]

holds if and only if \( a^2 < a + r^2 + 4 \). By (4.12), we have

\[
a + r^2 + 4 > a + r^2 - 1 \geq a^2,
\]

and so \( \cos \phi < \cos \phi' \). This completes the proof of the case \( p' = 1 + 2i \).

The second case is when \( p' = i/2 \). Here, \( L \) intersects the tiles \( T \) and \( vT \), so is either a vertical line, or a semicircle orthogonal to the real axis. If \( L \) is a vertical line, to prove (4.10) it suffices to prove that \( \text{Im}(q') < \text{Im}(q) \). We may parametrise \( z \in L \) by \( z = a + iy \) for some \( a \in [-1/2, 1/2] \). Then the unique closest point on
\[ L \text{ to } p = 2i \text{ is } q = a + i\sqrt{a^2 + 4}, \text{ and the unique closest point on } L \text{ to } p' = i/2 \text{ is } q' = a + i(\sqrt{4a^2 + 1}/2). \text{ Therefore the } q_j \text{ are well-defined here. It is straightforward to prove that } \sqrt{4a^2 + 1}/2 < \sqrt{a^2 + 4}. \text{ If } L \text{ is a semicircle, we again parametrise } z \in L \text{ by } z = a + re^{i\phi}. \text{ Without loss of generality, } a > 0 \text{ (if } a = 0 \text{ then } L \text{ cannot intersect both } T \text{ and } vT). \text{ Again, the closest points to } p \text{ and } p' \text{ are unique; we find that the unique minimum distances from } p \text{ and } p' \text{ are achieved when, respectively,}

\[
\cos \phi = \frac{-2ar}{a^2 + r^2 + 4}, \text{ and } \cos \phi' = \frac{-8ar}{4a^2 + 4r^2 + 1}.
\]

So \( q = a + re^{i\phi} \) and \( q' = a + re^{i\phi'} \). To establish (4.10), it suffices to prove that \( \phi' > \phi \). Since \( \cos \phi < 0 \) and \( \cos \phi' < 0 \), we wish to show that \( \cos \phi > \cos \phi' \); this is again a straightforward inequality.

We have now proved (4.10), and so (4.11) holds. The plan for the rest of the proof of Lemma 4.4.11 is as follows. After considering two special cases, we show that when \( r_j = u(n_j) \), there exists a constant \( K'_2 \) such that \( d_U(p_{j-1}, p_j) \leq K'_2 d_U(q_{j-1}, q_j) \). This involves parametrisation of a geodesic which is a semicircle with centre \( a \) and radius \( r \). The case \( j = 1 \) is treated first, then for \( j > 1 \) we find the distance between \( q_{j-1} \) and \( q_j \) in terms of \( a \), \( r \) and \( n_j \), and thus show that there exists a constant \( K \) so that \( \cosh^{-1}((1 + n_j^2)/K) \leq d_U(q_{j-1}, q_j) \). From the explicit formula for \( d_U \), it follows that \( d_U(p_{j-1}, p_j) \leq K'_2 d_U(q_{j-1}, q_j) \). Then, we consider the \( r_j \) which are equal to \( v \). We show that

\[
d_U(p_{j-1}, p_j) + d_U(p_j, p_{j+1}) \leq 4K'_2 (d_U(q_{j-1}, q_j) + d_U(q_j, q_{j+1})).
\]

Adding together these kinds of inequalities completes the proof.

In the special case where \( \delta = 0 \), Lemma 4.4.11 is trivial. The other special case is when \( \delta = v \). Here, we have \( d_U(p_0, p_1) = d_U(q_0, q_1) \), so the path \( \alpha \) is identical to the geodesic segment \([p_0, \delta p_0]\). Provided \( K_2 \geq 1 \), there is nothing more to prove in this case. So if, later, we arrive at a value of \( K_2 \) which is less than 1, we will take \( K_2 = 1 \) instead.

For all other \( \delta \in SL(2, \mathbb{Z}) \), at least one of the \( r_j \) must have the form \( u(n_j) \), where \( n_j \) is a non-zero integer. We now show that when \( r_j = u(n_j) \), there exists a constant \( K'_2 \) such that

\[
d_U(p_{j-1}, p_j) \leq K'_2 d_U(q_{j-1}, q_j).
\]

We have

\[
d_U(p_{j-1}, p_j) = d_U(\theta_j p_0, \theta_j p_0) = d_U(p_0, \theta_j^{-1} \theta_j p_0) = d_U(p_0, r_j p_0).
\]

Also,

\[
d_U(q_{j-1}, q_j) = d_U(\theta_j^{-1} q_{j-1}, \theta_j^{-1} q_j).
\]

Since in both cases we have applied the isometry \( \theta_j^{-1} \), we may now, without loss of generality, compare the distances between the points \( p_0 \) and \( r_j p_0 \), and \( \theta_j^{-1} q_{j-1} \) and \( \theta_j^{-1} q_j \). By symmetry in the imaginary axis, we may in addition assume that \( n_j \geq 1 \).
We are considering geodesics \( L \) which contain the image of \([p_0, \delta p_0]\) under \( \theta_{j-1}^{-1} \), and which, since \( p_0 \in T \) and \( r_j p_0 \in u(n_j)T \), intersect the tiles \( T \) and \( u(n_j)T \). We may, as above, parametrise \( z \in L \) by \( z = a + re^{i\phi} \), where \( a \in \mathbb{R}, r > 0 \) and \( 0 < \phi < \pi \).

If \( j = 1 \), then the isometry \( \theta_{j-1}^{-1} \) is just the identity. Thus, as \( p_0 = q_0 = 2i \), the geodesic \( L \) passes through the point \( 2i \) in the tile \( T \). Now, \( L \) either intersects the arc of the circle \( |z| = 1 \) between the points \( z = \frac{1}{2} + \frac{\sqrt{3}}{2}i \) and \( z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \), or \( L \) intersects the ray \( R \) given by

\[
R = \left\{ z \in U^2 : \text{Re}(z) = -\frac{1}{2} \text{ and } \text{Im}(z) \geq \frac{\sqrt{3}}{2} \right\}.
\]

The case where \( L \) intersects the arc may be dealt with as below, where \( \theta_{j-1}^{-1} \) is not the identity. We now suppose \( L \) intersects the ray \( R \) in the point \( w \). Then, the point \( a \), which is the centre of the semicircle \( L \), is where the perpendicular bisector of the points \( w \) and \( 2i \) meets the real axis. The value of \( a \) thus has a maximum, when \( w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \). It follows that there are only finitely many positive integers \( n_1 \) such that \( p_1 = n_1 + 2i \). Let \( N \) be the maximum possible value of \( n_1 \). By the reductions carried out above to form the product \( r_1 r_2 \cdots r_m \), as the angle \( \phi \) increases, the geodesic \( L \) must pass straight from a tile not of the form \( u(m_1)T \), where \( m_1 \in \mathbb{Z} \), into the tile \( u(n_1)T \). Now, for \( 1 \leq n_1 \leq N \), the smallest value of \( d_U(2i, q_1) \) occurs when \( L \) passes through the vertex \( \left(n_1 - \frac{1}{2}\right) + \frac{\sqrt{3}}{2}i \) of the tile \( u(n_1)T \). Since there are only finitely many cases to consider, we can thus find a \( K'_2 \) sufficiently large so that, for each instance \( 1 \leq n_1 \leq N \),

\[
d_U(p_0, p_1) = d_U(2i, n_1 + 2i) \leq K'_2 d_U(2i, q_1) = d_U(q_0, q_1).
\]

For all \( j > 1 \), to simplify notation, write \( n = n_j, p = p_0, p' = n + p_0, q = \theta_{j-1}^{-1} q_{j-1} \) and \( q' = \theta_{j-1}^{-1} q_j \). We first establish bounds on the values of \( a \) and \( r \) in terms of \( n \). By the reductions carried out above to form the product \( r_1 r_2 \cdots r_m \), as the angle \( \phi \) increases, the geodesic \( L \) must pass straight from a tile not of the form \( u(m_1)T \) into \( u(n)T \), and later on from \( T \) straight into a tile not of the form \( u(m_2)T \), where \( m_1, m_2 \in \mathbb{Z} \).
Thus, the radius of $L$ is bounded by the inequalities

$$|a - \left( n - 1/2 + i\sqrt{3}/2 \right)| \leq r \leq |a - \left( n + 1/2 + i\sqrt{3}/2 \right)|$$  \hspace{1cm} (4.15)$$

and

$$|a - \left( 1/2 + i\sqrt{3}/2 \right)| \leq r \leq |a - \left( -1/2 + i\sqrt{3}/2 \right)|.$$  \hspace{1cm} (4.16)

By squaring, expanding and combining (4.15) and (4.16), we get

$$\frac{n - 1}{2} \leq a \leq \frac{n + 1}{2}.$$  \hspace{1cm} (4.17)

Then, using (4.16) and (4.17), we obtain

$$\frac{n^2 - 4n + 7}{4} \leq r^2 \leq \frac{n^2 + 4n + 7}{4}.$$  \hspace{1cm} (4.18)

Now, we find the hyperbolic distance between $q$ and $q'$ in terms of $a$, $r$ and $n$. Using (4.13), the closest point to $p = 2i$ on the geodesic $L$ is

$$q = a + \frac{-2ar^2}{a^2 + r^2 + 4} + ir \frac{\sqrt{(a^2 + r^2 + 4)^2 - 4a^2 r^2}}{a^2 + r^2 + 4},$$

and the closest point to $p' = n + 2i$ on the geodesic $L$ is

$$q' = a + \frac{2(n - a)r^2}{(n - a)^2 + r^2 + 4} + ir \frac{\sqrt{(n - a)^2 + r^2 + 4)^2 - 4(n - a)^2r^2}}{(n - a)^2 + r^2 + 4}.$$

Then, the distance between $q$ and $q'$ is, for fixed $n$,

$$d_U(q, q') = \cosh^{-1} \left( 1 + \frac{F(a, r)}{G(a, r)} \right),$$

where $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the function

$$F(a, r) = |q - q'|^2 = (\text{Re}(q) - \text{Re}(q'))^2 + (\text{Im}(q) - \text{Im}(q'))^2$$

$$= \left( \frac{2(n - a)r^2}{(n - a)^2 + r^2 + 4} + \frac{2ar^2}{a^2 + r^2 + 4} \right)^2 + \frac{r^2 \left( \frac{\sqrt{(a^2 + r^2 + 4)^2 - 4a^2 r^2}}{a^2 + r^2 + 4} - \frac{\sqrt{(n - a)^2 + r^2 + 4)^2 - 4(n - a)^2r^2}}{(n - a)^2 + r^2 + 4} \right)^2},$$

and $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ is

$$G(a, r) = 2 \text{Im}(q) \text{Im}(q')$$

$$= 2r^2 \frac{\sqrt{((a^2 + r^2 + 4)^2 - 4a^2 r^2)((n - a)^2 + r^2 + 4)^2 - 4(n - a)^2}(a^2 + r^2 + 4)((n - a)^2 + r^2 + 4)}{(a^2 + r^2 + 4)((n - a)^2 + r^2 + 4)}.$$
From the picture of the geodesic \( L \) above, it is clear that \( \text{Im}(q) \) may equal \( \text{Im}(q') \). To show that \( F(a, r) \) is bounded below, we thus consider the difference between \( \text{Re}(q) \) and \( \text{Re}(q') \). We then show that \( G(a, r) \) is bounded above. These facts together will prove that for all \( n \geq 1 \), and all \( a \) and \( r \) satisfying (4.17) and (4.18) respectively,

\[
\frac{n^2}{K} \leq \frac{F(a, r)}{G(a, r)}, \tag{4.19}
\]

where \( K > 0 \) is a constant. This implies a lower bound on \( d_U(q, q') \).

From (4.17), we have \( a \leq (n+1)/2 \), and \( n-a \leq (n+1)/2 \). Using these bounds and (4.18),

\[
\frac{2(n-a)r^2}{(n-a)^2 + r^2 + 4} + \frac{2ar^2}{a^2 + r^2 + 4} \geq \frac{2(n-a)r^2 + 2ar^2}{(n+1)^2/4 + (n^2 + 4n + 7)/4 + 4} = \frac{2nr^2}{n^2/2 + 3n/2 + 6} \geq \frac{n(n^2 - 4n + 7)}{n^2 + 3n + 12}.
\]

The right-hand side of the final line goes to \( n \) as \( n \to \infty \), so

\[
F(a, r) \geq \left( \frac{2(n-a)r^2}{(n-a)^2 + r^2 + 4} + \frac{2ar^2}{a^2 + r^2 + 4} \right)^2 \geq \frac{n^2}{K^2},
\]

for some positive constant \( k \). For \( G \), by a similar process using (4.17) and (4.18) repeatedly, we obtain

\[
G(a, r) \leq \frac{(n^2 + 4n + 7)/2}{(n^2/2 - 3n/2 + 6)^2} \frac{25n^2/4 + 33n/2 + 137/4}{n^2/2 + 3n/2 + 6}.
\]

The function of \( n \) on the right-hand side converges to a constant as \( n \to \infty \), so \( G(a, r) \) is bounded above, by say \( K/k \). The lower bound on \( F(a, r)/G(a, r) \) at (4.19) follows. Therefore, the distance between the points \( q \) and \( q' \) is bounded below:

\[
\cosh^{-1} \left( 1 + \frac{n^2}{K} \right) \leq \cosh^{-1} \left( 1 + \frac{F(a, r)}{G(a, r)} \right) = d_U(q, q').
\]

We can find a constant \( K'_2 > 0 \) such that for all \( n \geq 1 \),

\[
\cosh^{-1} \left( 1 + \frac{n^2}{8} \right) \leq K'_2 \cosh^{-1} \left( 1 + \frac{n^2}{K} \right).
\]

Therefore, \( d_U(p, p') \leq K'_2 d_U(q, q') \). We have established (4.14).

We next consider the \( r_j \) of the form \( r_j = v \), where \( j \) is strictly less than \( m \). Rather than comparing the distances \( d_U(q_{j-1}, q_j) \) and \( d_U(p_{j-1}, p_j) \), we show instead that

\[
d_U(p_{j-1}, p_j) + d_U(p_j, p_{j+1}) \leq 4K'_2 (d_U(q_{j-1}, q_j) + d_U(q_j, q_{j+1})). \tag{4.20}
\]
When \( r_j = v \), we have
\[
d_U(p_{j-1}, p_j) = d_U(p_0, r_j p_0) = d_U(2i, i/2).
\]

By the construction of the product \( r_1 r_2 \cdots r_m \), and the assumption that \( j < m \), if \( r_j = v \) then \( r_{j+1} = u(n_{j+1}) \) for some non-zero integer \( n_{j+1} \). So, by (4.14), we have
\[
d_U(p_j, p_{j+1}) \leq K'_2 d_U(q_j, q_{j+1}).
\]

Then, the following proves (4.20):
\[
d_U(p_{j-1}, p_j) + d_U(p_j, p_{j+1}) = d_U(2i, i/2) + d_U(p_j, p_{j+1})
\]
\[
= \log 4 + d_U(p_j, p_{j+1})
\]
\[
\leq 3 \cosh^{-1}(1 + 1/8) + d_U(p_j, p_{j+1})
\]
\[
\leq 3 \cosh^{-1}(1 + n_{j+1}^2/8) + d_U(p_j, p_{j+1})
\]
\[
= 3d_U(p_0, p_0 + n_{j+1}) + d_U(p_j, p_{j+1})
\]
\[
= 3d_U(p_0, r_{j+1} p_0) + d_U(p_j, p_{j+1})
\]
\[
= 3d_U(p_j, p_{j+1}) + d_U(p_j, p_{j+1})
\]
\[
= 4d_U(p_j, p_{j+1})
\]
\[
\leq 4K'_2 d_U(q_j, q_{j+1})
\]
\[
\leq 4K'_2 (d_U(q_{j-1}, q_j) + d_U(q_j, q_{j+1})).
\]

The final case is when \( j = m \) and \( r_m = v \). Then \( r_{m-1} \) must be \( u(n_{m-1}) \) for some non-zero integer \( n_{m-1} \). It can be proved, as for (4.20), that
\[
d_U(p_{m-2}, p_{m-1}) + d_U(p_{m-1}, p_m) \leq 4K'_2 (d_U(q_{m-2}, q_{m-1}) + d_U(q_{m-1}, q_m)).
\]

Let the constant \( K_2 \) be \( K_2 = 2 \times 4K'_2 \). (Multiplication by 2 is needed to cater for the possibility of counting \( d_U(q_{m-2}, q_{m-1}) \) twice.) Then,
\[
\sum_{j=1}^{m} d_U(p_{j-1}, p_j) \leq K_2 \sum_{j=1}^{m} d_U(q_{j-1}, q_j),
\]
and so by (4.9) and (4.11), \( \text{length}(\alpha) \leq K_2 d_U(p_0, \delta p_0) \).

The next lemma shows that the length of the path \( \alpha \) is also roughly the same as the sums of the values of \( f \) at each of the terms \( r_j \).

**Lemma 4.4.12.** There exist constants \( K_3, K_4 > 0 \) such that
\[
K_3 \text{length}(\alpha) \leq \sum_{j=1}^{m} f(r_j) \leq K_4 \text{length}(\alpha).
\]
Proof. If \( r_j = v \), then \( d_U(p_0, r_j p_0) = \log 4 \) and \( f(r_j) = 1 \). If \( r_j = u(n_j) \) for \( n_j \neq 0 \), then \( d_U(p_0, r_j p_0) = \cosh^{-1} \left( 1 + n_j^2 / 8 \right) \) and \( f(r_j) = \log(|n_j| + 1) \). Let the set \( J \) be

\[
J = \{ j \in \mathbb{Z} : 1 \leq j \leq m \text{ and } r_j = v \},
\]

and denote its cardinality by \( |J| \). Then

\[
\text{length}(\alpha) = \sum_{j=1}^{m} d_U(p_0, r_j p_0) = |J| \log 4 + \sum_{j \notin J} \cosh^{-1} \left( 1 + n_j^2 / 8 \right),
\]

and

\[
\sum_{j=1}^{m} f(r_j) = |J| + \sum_{j \notin J} \log(|n_j| + 1).
\]

Now, for all \( n_j \neq 0 \),

\[
\cosh^{-1}(1 + n_j^2 / 8) \leq 3 \log(|n_j| + 1)
\]

and, also,

\[
\log(|n_j| + 1) \leq 3 \cosh^{-1}(1 + n_j^2 / 8).
\]

It follows that

\[
\frac{1}{3} \text{length}(\alpha) \leq \sum_{j=1}^{m} f(r_j) \leq 3 \text{length}(\alpha).
\]

We may now conclude the proof of Proposition 4.4.10. By Lemmas 4.4.11 and 4.4.12 there exist constants \( K_1, K_2, K_3 \) and \( K_4 \) such that

\[
K_3 K_1 d_U(p_0, \delta p_0) \leq K_3 \text{length}(\alpha) \leq \sum_{j=1}^{m} f(r_j) \leq K_4 \text{length}(\alpha) \leq K_4 K_2 d_U(p_0, \delta p_0).
\]

\[ \square \]

Lemma 4.4.13. Let \( p_0 \) be the point \( 2i \) in \( U^2 \). Then there is a constant \( K_5 > 0 \) such that for any \( \delta \in SL(2, \mathbb{Z}) \),

\[
d_U(p_0, \delta p_0) \leq K_5 \log \| \delta \|.
\]

Proof. Let \( \delta \) be in \( SL(2, \mathbb{Z}) \). As shown in Lemma 2.6.16 there exist matrices \( k_1 \) and \( k_2 \) in \( SO(2, \mathbb{R}) \), and a matrix \( a \) in \( SL(2, \mathbb{R}) \) of the form

\[
\begin{pmatrix}
s & 0 \\
0 & s^{-1}
\end{pmatrix}
\]

where \( s \geq 1 \),
such that $\delta = k_1ak_2$. Then, by Lemma 3.5.1, $\|\delta\| = \|a\| = s$. Since the group $SO(2, \mathbb{R})$ stabilises $i$, we have $d_U(i, \delta i) = d_U(i, at) = d_U(i, s^2i) = 2\log s$. Thus,

$$d_U(p_0, \delta p_0) \leq d_U(2i, i) + d_U(i, \delta i) + d_U(\delta i, \delta(2i))$$

$$= 2d_U(2i, i) + d_U(i, \delta i)$$

$$= 2\log 2 + 2\log s$$

$$= \log 4 + 2\log \|\delta\|$$

$$\leq K_5 \log \|\delta\|,$$

for some constant $K_5 > 0$. \hfill \Box

**Corollary 4.4.14.** There exists a constant $K_6 > 0$ such that, for all $s$ and $t$ where $1 \leq s \neq t \leq d$, and all $\delta \in SL^{st}(2, \mathbb{Z}) \subseteq SL(d, \mathbb{Z})$,

$$d_W(1, \delta) \leq K_6 \log \|\delta\|.$$

**Proof.** Let $r_1r_2 \cdots r_m$ be a word for $\delta$ as in Proposition 4.4.10 where each $r_j$ belongs to $SL^{st}(2, \mathbb{Z})$. Let $\Sigma$ be a fixed finite set of generators of $SL(d, \mathbb{Z})$, and let $d_\Sigma$ be the word distance function on $SL(d, \mathbb{Z})$ induced by $\Sigma$.

First, we consider the terms $r_j$ which are of the form $\begin{pmatrix} 1 & n_j \\ 0 & 1 \end{pmatrix}$ for some non-zero integer $n_j$. Such elements may be written as $r_j = (E_{st}(1))^{n_j}$. By Proposition 4.4.4 there exists a constant $C_{st}$ such that

$$d_\Sigma(1, r_j) \leq C_{st} \log(|n_j| + 1) = C_{st}f(r_j),$$

where $f$ is the function defined in the statement of Proposition 4.4.10. Let $C > 0$ be the maximum of the constants $C_{st}$ over the finitely many values of $s$ and $t$ such that $1 \leq s \neq t \leq d$. Then $d_\Sigma(1, r_j) \leq Cf(r_j)$.

If the element $r_j$ is of the form $r_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $r_j$ may be written as a shortest word of fixed length $L_{st}$ in our generators $\Sigma$. For such $r_j$, we have $f(r_j) = 1$. Let $L > 0$ be the maximum of the constants $L_{st}$ over the finitely many values of $s$ and $t$ such that $1 \leq s \neq t \leq d$. Then $d_\Sigma(1, r_j) \leq Lf(r_j)$.

Let $M$ be the maximum of $L$ and $C$. By Proposition 4.4.10 and Lemma 4.4.13 the word length of $\delta$ with respect to $\Sigma$ is bounded as follows:

$$d_\Sigma(1, \delta) \leq \sum_{j=1}^{m} d_\Sigma(1, r_j) \leq M \sum_{j=1}^{m} f(r_j) \leq Mc_4d_U(p_0, \delta p_0) \leq Mc_4K_5 \log \|\delta\|.$$

Thus there exists a constant $K_6 > 0$ such that $d_W(1, \delta) \leq K_6 \log \|\delta\|$. \hfill \Box

### 4.4.4 Bounds on the operator norm of a matrix

**Lemma 4.4.15.** There exist constants $c_5$, $c_6 > 0$ such that, for any $d \times d$ real matrix $A = (a_{ij})$,

$$c_5 \max_{i,j} |a_{ij}| \leq \|A\| \leq c_6 \max_{i,j} |a_{ij}|.$$
Proof. Suppose the entry of $A$ which has maximum absolute value occurs in column $k$. Then
\[
\|A\| = \sup_{\|x\|=1} \|Ax\| \geq \|Ae_k\| = \left( \sum_{i=1}^d |a_{ik}|^2 \right)^{1/2} \geq \left( \max_i |a_{ik}|^2 \right)^{1/2} = \max_{i,j} |a_{ij}|. 
\]

For the other inequality, let $x$ be any vector such that $\|x\| = 1$, and write $y = Ax$. For $1 \leq j \leq d$, we have $|x_j| \leq 1$, so
\[
\|Ax\| = \|y\| = \left( \sum_{i=1}^d |y_i|^2 \right)^{1/2} \leq \sqrt{d} \max_i |y_i| = \sqrt{d} \max_{i,j} |a_{ij} x_j| \leq \sqrt{dd} \max_{i,j} |a_{ij} x_j| \leq d^{3/2} \max_{i,j} |a_{ij}|. 
\]

Thus,
\[
\|A\| = \sup_{\|x\|=1} \|Ax\| \leq d^{3/2} \max_{i,j} |a_{ij}|. 
\]

Note that we may take the constant $c_5$ to be 1. \qed

4.4.5 Conclusion of the proof of Theorem 4.4.1
Let $\gamma$ be in $SL(d, \mathbb{Z})$. By Corollary 4.4.3, $\gamma$ may be written as a word
\[
\gamma = \delta_1 \delta_2 \cdots \delta_{d^2},
\]
where each $\delta_i \in SL^{s,t}(\mathbb{Z})$ for some $1 \leq s \neq t \leq d$. Moreover, the entries of these $\delta_i$ are bounded by a fixed polynomial $P$ in the entries of $\gamma$; we will write $P(\gamma)$ for $P(\gamma_{11}, \gamma_{12}, \ldots, \gamma_{dd})$. Then for $1 \leq i \leq d^2$, the $(k, l)$ entry of the matrix $\delta_i$ satisfies
\[
|\delta_{i,k,l}| \leq |P(\gamma)|. 
\]

Let $\delta$ be the matrix in the expression $\delta_1 \delta_2 \cdots \delta_{d^2}$ which has maximum norm. That is, $\delta = \delta_i$ for some $1 \leq i \leq d^2$, and $\|\delta\| \geq \|\delta_i\|$ for all $1 \leq i \leq d^2$. Then, by Corollary 4.4.14 the word distance from 1 to $\gamma$ is bounded as follows:
\[
d_W(1, \gamma) \leq \sum_{i=1}^{d^2} d_W(1, \delta_i) \leq \sum_{i=1}^{d^2} K_6 \log \|\delta_i\| \leq K_7 \log \|\delta\|,
\]
where $K_6$ and $K_7$ are constants.
where $K_7$ is some positive constant. Using Lemma 4.4.15 and (4.21), and writing $\delta = (\delta_{kl})$, we then obtain

$$K_7 \log \|\delta\| \leq K_7 \log \left( c_6 \max_{k,l} |\delta_{kl}| \right) \leq K_7 \log \left( c_6 |P(\gamma)| \right).$$

Let the degree of the polynomial $P$ be $\deg(P)$, and let the sum of the the absolute values of the coefficients of $P$ be $S$. Then for all $\gamma$,

$$|P(\gamma)| \leq S \left( \max_{i,j} |\gamma_{ij}| \right)^{\deg(P)},$$

so

$$K_7 \log(C_6 |P(\gamma)|) \leq K_7 \log \left( c_6 S \left( \max_{i,j} |\gamma_{ij}| \right)^{\deg(P)} \right)$$

$$\leq K_7 \log \left( c_6 S \max_{i,j} |\gamma_{ij}|^{\deg(P)} \right)$$

$$= K_7 \deg(P) \log \left( c_6 S \max_{i,j} |\gamma_{ij}| \right)$$

$$\leq K \log \left( \max_{i,j} |\gamma_{ij}| \right)$$

for some constant $K > 0$. Finally, using Lemma 4.4.15 again,

$$K \log \left( \max_{i,j} |\gamma_{ij}| \right) \leq K \log \|\gamma\|.$$ 

Thus

$$d_W(1, \gamma) \leq K \log \|\gamma\|,$$

and the proof of Theorem 4.3.1 is complete.
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