Is there still a strong CP problem?

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The rôle of a chiral U(1) phase in the quark mass in QCD is analysed from first principles. In operator formulation, there is a parity symmetry and the phase can be removed by a change in the representation of the Dirac \(\gamma\) matrices. Moreover, these properties are also realized in a Pauli-Villars regularized version of the theory. In the functional integral scenario, attempts to remove the chiral phase by a chiral transformation are thought to be obstructed by a nontrivial Jacobian arising from the fermion measure and the chiral phase may therefore seem to break parity. But if one starts from the regularized action with the chiral phase also present in the regulator mass term, the Jacobian for a combined chiral rotation of quarks and regulators is seen to be trivial and the phase can be removed by a combined chiral rotation. This amounts to a taming of the strong CP problem.

I. INTRODUCTION

The strong interactions do not violate parity in the manner of the weak interactions: the gauge interactions in quantum chromodynamics involve vector currents instead of chiral currents. Experiments too do not indicate CP violation in the QCD sector. However, if one writes the Lagrangian density as

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - me^{i\theta'\gamma_5})\psi - \frac{1}{4} \text{tr} F_{\mu\nu}F^{\mu\nu} - m f \frac{g^2}{32\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu},
\]

(1)

it contains the topological term with the QCD vacuum angle \(\theta\), which violates CP and, if nonzero, may lead to experimentally detectable CP violating effects. Furthermore, the quark mass term has a chiral phase \(\theta'\), which comes from symmetry breaking in the electroweak sector and may be large (\(\approx 1\)). This phase is expected to violate CP by mixing a scalar and a pseudoscalar. One then has to require that the CP violation due to \(\theta'\) should be cancelled by that due to the \(\theta\) term. In the popular functional integral scenario, the phase \(\theta'\) is supposed to be convertible into a \(\theta\)-like term through anomalous chiral rotations of the quarks, so that there is an effective parameter \(\bar{\theta}\) and according to popular methods of calculation the CP-violating electric dipole moment of the neutron is thought to be about \(10^{-16} \bar{\theta}\) e-cm, to be compared with the experimental upper bound of \(10^{-26}\) e-cm. This is interpreted to mean that \(|\bar{\theta}| < 10^{-10}\), requiring an unbelievable amount of fine-tuning between \(\theta'\) and \(\theta\). This is referred to as the strong CP problem [1]. Various modifications of QCD continue to be proposed to avoid this supposed need for fine tuning. But none of them has been experimentally verified.

In such a situation it may be profitable to reanalyse the problem from first principles. There have been several attempts to prove that the \(\theta\) parameter does not really lead to any CP violation in QCD [2–4]. As the nonperturbative dynamics of QCD with the topological term is difficult, we find it useful to postpone the consideration of \(\theta\) and concentrate on CP violation generated from the fermionic \(\theta'\) term. We use a single flavour for simplicity. This means that \(m\) in (1) is a (real) number instead of a matrix. An extension can be made to more quark flavours without difficulty.

In first order, \(\theta'\) is involved in an apparently pseudoscalar term, and its matrix element has therefore been thought to be a measure of the CP violation produced by the term. However, it is not so simple. In theories like QCD with chirally invariant interactions, perturbation theory cannot find any effect of such phases as they cancel at each vertex [5]. Thus the quark propagator can be written as

\[
(\gamma^\mu p_\mu - me^{i\theta'\gamma_5})^{-1} = e^{-i\frac{\theta'}{2}\gamma_5}(\gamma^\mu p_\mu - m)^{-1} e^{-i\frac{\theta'}{2}\gamma_5},
\]

(2)

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and the chiral phases formally cancel at each gauge vertex because
\[ e^{-i \frac{\theta'}{2} \gamma_5} e^{-i \frac{\theta'}{2} \gamma_5} = \gamma^\mu. \] (3)

Moreover, in spite of the presence of the chiral phase in the quark mass term, parity can be defined so as to be conserved at the classical level. This parity transformation, which leaves the fermionic part of the Lagrangian \( \int d^3 x \mathcal{L}_\psi \) invariant [5], involves the usual parity operation for gauge fields, while the operation for fermions includes a chiral rotation:
\[ \bar{\psi}(x_0, \vec{x}) \rightarrow \bar{\psi}(x_0, -\vec{x}) e^{i \theta' \gamma_5 \gamma_0}, \]
\[ \psi(x_0, \vec{x}) \rightarrow \gamma^0 e^{i \theta' \gamma_5} \psi(x_0, -\vec{x}). \] (4)

The full mass term, with the chiral phase, is a scalar under this parity.

In spite of these direct and transparent arguments, the view that the chiral U(1) phase \( \theta' \) in the quark mass term gives rise to CP violating effects prevails in the literature. The raison d’être of this view is the perception that there is an anomaly in the axial U(1) current:
\[ \partial_\mu (\bar{\psi} \gamma_\mu \gamma_5 \psi) \propto \text{tr}[F_{\mu \nu} \tilde{F}_{\mu \nu}] \] (5)

in euclidean metric. But it must be pointed out that this anomaly does not translate \( \theta' \) into a physically relevant parameter unless supplemented by a nontrivial topology of the gauge field:
\[ \int d^4 x \frac{g^2}{16\pi^2} \text{tr}[F_{\mu \nu} \tilde{F}_{\mu \nu}] = 2\nu \neq 0. \] (6)

The fundamental point to be observed is that these two relations are contradictory: the anomaly has the simple form only when \( \nu = 0 \). In the presence of a fermion mass, one must have
\[ \int d^4 x \langle \partial_\mu (\bar{\psi} \gamma_\mu \gamma_5 \psi) \rangle = 0 \] (7)
for consistency with a nontrivial topological charge because the anomaly equation also contains a mass term, whose integral is
\[ \int d^4 x [2m \langle \bar{\psi} \gamma_5 \psi \rangle] = 2(n_+ - n_-) = 2\nu, \] (8)
by the index theorem. Here \( n_\pm \) count the zero modes of the Dirac operator with positive and negative chiralities. The theory becomes undefined unless the infrared divergences arising from these zero modes are suitably regularized. In the infrared regularized theory, the \( m \rightarrow 0 \) limit of the LHS is nonvanishing, so that the anomaly equation for massless fermions contains an extra piece involving the zero modes [6] in the form \( \sum_i \epsilon_i \phi_0^i(x) \phi_0^i(x) \), where \( \phi_0^i \) is the \( i \)th zero mode eigenfunction of the euclidean Dirac operator and \( \epsilon_i \) its chirality. By virtue of (7) which continues to hold on infrared regularization, \( \theta' \) becomes unphysical.

It has therefore to be examined carefully whether the abovementioned formal parity properties of the theory survive regularization and anomalies. We shall first review in the next section how the chiral anomaly formally appears to lead to the breaking of parity by \( \theta' \). We shall thereafter introduce a (Pauli-Villars) regularization and demonstrate that it is compatible with the parity defined in (4). In other words, \( \theta' \) does not cause any violation of parity in the regularized quantum theory. Then all that is needed for removing CP violation is to set \( \theta = 0 \). This is natural in the technical sense à la ’t Hooft, because it increases the CP symmetry of the action and does not involve fine tuning between two quantities.

II. ANOMALY AND CHIRAL ROTATION IN UNREGULARIZED THEORY

The popular belief in the physicality of the \( \theta' \) term and its equivalence with a vacuum angle like \( \theta \) arises mainly because of a nontrivial Jacobian produced by a spacetime independent chiral transformation in the euclidean functional integral. For a chiral transformation
\[ \bar{\psi} \rightarrow \bar{\psi} e^{i\alpha(x) \gamma_5}, \quad \psi \rightarrow e^{i\alpha(x) \gamma_5} \psi, \] (9)
where $\alpha$ may depend on $x$, the Jacobian reads

$$J = e^{i \int d^4 x \alpha(x) X(x)},$$

(10)

where $X(x) = \sum_n \phi_n^\dagger(x) \gamma_5 \phi_n(x)$, the functions $\phi_n$ being eigenfunctions of the Dirac operator in euclidean metric. A regularized calculation yields [7]

$$X(x) = \frac{g^2}{16 \pi^2} \text{tr} F_{\mu \nu} \tilde{F}_{\mu \nu}.$$  

(11)

$X(x)$ is identified as the chiral anomaly [7] from the anomalous Ward identity

$$\langle \partial_\mu (\bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x)) \rangle = -2 \langle \bar{\psi} m \gamma_5 \psi \rangle + X(x),$$  

(12)

which follows from the rule that the functional integral over $\psi, \bar{\psi}$ cannot change under a chiral rotation of integration variables.

If $\alpha(x) = -\theta'/2$, the chiral phase gets removed from the mass term, but the Jacobian causes a parity violating vacuum angle term to be added to the action:

$$Z[\theta'] \equiv \int d\psi d\bar{\psi} e^{i \int d^4 x \bar{\psi} (\gamma_5 D_\mu - m e^{i \theta'} \gamma_5) \psi}$$

$$= Z[0] e^{i \theta' \frac{g^2}{16 \pi^2} \int d^4 x \text{tr} F_{\mu \nu} \tilde{F}_{\mu \nu}}.$$  

(13)

Unlike the perturbative argument or the parity constructed earlier, this functional integral argument takes the anomaly into account, and hence it may appear to be more robust. However, a regularization is used here only in the evaluation of $X$, while the remaining pieces of (12) are unregularized. We shall consider a fully regularized theory to resolve the apparent discrepancy between the arguments of the previous section and the present one. In the euclidean functional integral of the regularized theory, it will turn out that the Jacobian corresponding to the physical fermions can be cancelled by that corresponding to the regulators although both are separately nontrivial because of the existence of the chiral anomaly.

### III. REGULARIZED QCD

In the generalized Pauli-Villars regularization, the Lagrangian density has to be augmented to include some extra species:

$$\mathcal{L}^{[0]}_{\psi, \text{reg}} = \bar{\psi} (i \gamma^\mu D_\mu - m) \psi + \sum_j |c_j| \sum_k \bar{\chi}_{jk} (i \gamma^\mu D_\mu - M_j) \chi_{jk}.$$  

(14)

Here the $\chi_{jk}$ are regulator spinor fields with fermionic or bosonic statistics, which determines the signs, positive or negative, of the integers $c_j$ [8]; the $c_j$-s have to satisfy relations

$$1 + \sum_j c_j = 0, \quad m^2 + \sum_j c_j M_j^2 = 0$$  

(15)

to cancel divergences. The masses $M_j$ are to be taken to infinity at the end of calculations.

#### A. Parity in regularized QCD in presence of chiral phase

The easiest proof of the unphysicality of $\theta'$ is in the operator formulation of quantum chromodynamics. Instead of an anomalous chiral transformation of the quark fields, a change of $\gamma$-matrices can also be used to remove the chiral phase $\theta'$. Using the relation

$$\bar{\psi} = \psi^\dagger \gamma^0,$$

(16)

one can write the spinor part of Eq. (1) as
\[ L_\psi^{[\theta']} = \bar{\psi} (i \gamma^0 \gamma^\mu D_\mu - m \gamma^0 e^{i \theta' \gamma_5}) \psi, \]  

(17)

where \( D_\mu, m, \theta' \) carry no spinorial indices. It can also be rewritten as

\[ L_\psi^{[\theta']} = \bar{\psi} (i \hat{\gamma}^0 \gamma^\mu D_\mu - m \hat{\gamma}^0) \psi, \]  

(18)

where

\[ \hat{\gamma}^\mu \equiv e^{-i \theta' \gamma_5 / 2} \gamma^\mu e^{i \theta' \gamma_5 / 2}. \]  

(19)

It is easy to see that the new matrices satisfy the Dirac algebra

\[ \hat{\gamma}^\mu \hat{\gamma}^\nu + \hat{\gamma}^\nu \hat{\gamma}^\mu = 2 g^{\mu\nu} \]  

(20)

and also have the same hermiticity properties as their parent matrices. Thus the chiral phase can be absorbed in a simple redefinition of \( \gamma \)-matrices, and can have no physical effect. Note that the parity transformation (4) is the usual parity in terms of the \( \tilde{\gamma} \) representation.

This argument goes through as long as there are no additional chirally noninvariant interactions besides the one generating fermion mass. It is also to be noted that this argument does not go through directly in euclidean field theory, where \( \bar{\psi} \) is taken to be independent of \( \psi \). One can however modify the argument so that it holds in euclidean spacetime. As it is Minkowski spacetime that one is finally interested in, we shall not go into this complication here.

The argument skirts the issue of the chiral anomaly arising from short distance singularities at the quantum level. We therefore need to examine whether or not it is jeopardized in the regularized theory. The regularization (14) is standard, but in the presence of the chiral phase \( \theta' \) in the physical fermion mass term, it is convenient to use the freedom to provide the same chiral phase in the regulator mass terms as well:

\[ L_{\psi, \text{reg}}^{[\theta']} = \bar{\psi} (i \gamma^\mu D_\mu - m e^{i \theta' \gamma_5}) \psi + \sum_{j,k} \chi_{jk} \bar{\chi}_{jk} D_\mu (i \gamma^\mu D_\mu - M_j e^{i \theta' \gamma_5}) \chi_{jk}. \]  

(21)

This is invariant under the parity transformation (4) extended to the regulator fields besides the physical fermion. Moreover, all the \( \gamma_5 \)-s in (21) can be removed by changing over to the matrices \( \hat{\gamma}^\mu \) because the same chiral phase has been chosen for all \( j, k \):

\[ L_{\psi, \text{reg}}^{[\theta']} = \bar{\psi} (i \hat{\gamma}^0 \gamma^\mu D_\mu - m \gamma^0) \psi + \sum_{j,k} \chi_{jk} \bar{\chi}_{jk} (i \hat{\gamma}^\mu D_\mu - M_j) \chi_{jk}. \]  

(22)

This is no more parity violating than (14). Thus parity is conserved and \( \theta' \) is unphysical in the regularized theory.

If (21) were modified by changing the chiral phases in the regulator sector, it would no longer respect the naive extension of (4) to the regulators mentioned above. Such a choice of phases would be equivalent to choosing nonzero phases for the regulators in the absence of a phase in the physical mass term. However, it would still be possible to define a parity respected by the regularized action by using different phases for the physical and regulator masses.

**B. Jacobian for chiral rotation in regularized functional integral**

The preceding arguments for the unphysicality of \( \theta' \) suggest that something is amiss in the anomaly-based argument leading to (13). It is easy to guess what it may be: the functional integral in (13) is not properly regularized. The term \( X(x) \), representing the anomaly, is calculated [7] through an *a posteriori* regularization. In this section, we start instead from the regularized Lagrangian density (21). The measure of integration now includes the Pauli-Villars fields:

\[ d\mu = d\psi d\bar{\psi} \prod_{jk} d\chi_{jk} d\bar{\chi}_{jk}. \]  

(23)

The fermionic functional integral is defined by

\[ Z_{\text{reg}}^{[\theta']} = \int d\mu e^{-\int d^4 x L_{\psi, \text{reg}}^{[\theta']}}, \]  

(24)
One can apply a chiral transformation purely to the physical fermion fields and obtain formulas similar to (13). However, this removes only the phase in the physical fermion fields in $Z_{\text{reg}}^{[\theta']}$, leaving behind some $\theta'$-dependence through the regularization. To remove both, one has to extend the chiral transformation (9) to the regulators. The Jacobian factors corresponding to the different $\chi_{jk}, \bar{\chi}_{jk}$ come with powers $c_j/|c_j|$ by virtue of the fermionic or bosonic statistics, while $\psi, \bar{\psi}$ obey fermionic statistics. Altogether,

$$J_{\text{reg}} = e^{i(1+\sum_j c_j) \int d^4x \alpha(x) x(x) = 1},$$

(25)

because $1 + \sum_j c_j = 0$ [7]. Thus in the regularized framework the Jacobian for a combined chiral transformation on the physical fermion fields and the regulators is trivial.

Because of the trivial Jacobian associated with a combined chiral rotation, we see that the fermionic functional integral $Z_{\text{reg}}^{[\theta']}$ can be simplified by rotating $\theta'$ away from both the physical fermion fields and the regulators:

$$Z_{\text{reg}}^{[\theta']} = Z_{\text{reg}}^{[0]},$$

(26)

i.e., the regularized theories with and without $\theta'$ are equivalent. In other words, the chiral phase $\theta'$ is completely unphysical in the regularized theory. The unphysicality of the chiral phase $\theta'$ proved earlier in operator formulation thus stands vindicated in the framework of regularized functional integrals in the euclidean metric.

The lesson of the above exercise in the Pauli-Villars scheme is that a regularized theory signals a trivial Jacobian and conversely, a non-trivial Jacobian signals an unregularized theory. What happens to the chiral anomaly in this regularized approach is an interesting question, which we now briefly discuss.

C. Anomaly in regularized functional integral

The regularized axial Ward identity is obtained from the combined chiral transformation acting on physical fermions and regulators:

$$\langle \partial_\mu J_{\mu 5 \text{ reg}} \rangle = -2m \langle \bar{\psi} \gamma_5 \psi \rangle - 2 \sum_{jk} M_j \langle \bar{\chi}_{jk} \gamma_5 \chi_{jk} \rangle,$$

(27)

where

$$J_{\mu 5 \text{ reg}} = \bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x) + \sum_{jk} \bar{\chi}_{jk}(x) \gamma_\mu \gamma_5 \chi_{jk}(x).$$

(28)

Both sides of (27) are regularized and well-defined. One may wonder how the anomaly is manifested in this regularized approach where the Jacobian for the combined chiral rotation is trivial and the Ward identity does not look like an anomalous one. In the regularized Ward identity (27), divergent pieces in the physical mass term are cancelled exactly by those in the regulator terms. As the regularization is removed [9],

$$-2 \sum_{jk} \lim_{M_j \to \infty} M_j \langle \bar{\chi}_{jk} \gamma_5 \chi_{jk} \rangle_{\text{reg}} = \frac{g^2}{16 \pi^2} \text{tr} F_{\mu \nu} \tilde{F}_{\mu \nu},$$

(29)

yielding the chiral anomaly and the regularized version of (12):

$$\langle \partial_\mu J_{\mu 5 \text{ reg}} \rangle = -2m \langle \bar{\psi} \gamma_5 \psi \rangle_{\text{reg}} + \frac{g^2}{16 \pi^2} \text{tr} F_{\mu \nu} \tilde{F}_{\mu \nu},$$

(30)

IV. DISCUSSION

We have presented detailed arguments in this paper for the unphysical nature of the phase $\theta'$. One proof works in Minkowski spacetime and involves a change of $\gamma$-matrices. As is well known, no representation of $\gamma$-matrices is more sacred than others, and as the phase can be removed by a mere change of representation, it cannot be physically observable. Another way of understanding this is to see that while the usual parity operation involves $\gamma^0$, the presence of a chiral phase in the mass term does not destroy the symmetry but merely changes this matrix factor to $\gamma^0$. This
new symmetry transformation involves a chiral transformation of the fermion fields, which alerts us to the possibility of a parity anomaly. We have used a generalized Pauli-Villars regularization to explore this possibility. It is consistent with the symmetry, showing that there is no parity anomaly. An historically important approach involves a chiral transformation in which the chiral phase in the quark term is rotated away. It is convenient for this purpose to make the same choice of chiral phase in the regulator sector as in the physical mass term. In the regularized theory, the regulator fields are also chirally rotated and the chiral phase is fully rotated away: the measure of the regularized euclidean functional integral does not change in this combined transformation, and no $F\tilde{F}$ term gets generated, so that the theories with and without $\theta'$ are equivalent. While the present paper uses an explicit regularization, the result can alternatively be understood by considering the parity symmetry of the fermion functional measure [10]. It is not difficult to understand why $\theta'$ was so long thought to lead to CP violation. The formal functional integral formulation chooses to ignore the regularization of the action and goes for a regularization of only the change in the measure, which leads to a $\theta'$-dependent Jacobian modifying the $F\tilde{F}$ term. A regularized action ensures that the functional integrals are well-defined and yields a trivial Jacobian for the combined chiral transformation while the chiral anomaly is of course unchanged.

CP violation has been usually believed to get a contribution from the chiral phase $\theta'$ of the quark mass term as well as the $\theta$ term in the gauge sector. We have shown that $\theta'$ does not really have any such effect. What can one say about the $\theta$ term? This term changes sign under the standard parity transformation of gauge fields, and there does not appear to be any new definition of parity which can restore the symmetry. This term is known to be an exact divergence, and does not have any effect in perturbation theory. Its effect has sometimes been estimated by conversion to a $\theta'$ term. Since such a conversion is now seen to be impossible in a regularized theory, this method has to be abandoned as invalid. There have been arguments in the literature against CP violation by the $\theta$ term, but one can still look for new ways of estimating CP violating quantities on the basis of the $\theta$ term.

Meanwhile, the unphysicality of $\theta'$ already has a bearing on the strong CP problem. It used to be thought that an unnaturally fine tuned cancellation occurs between the $\theta'$ and $\theta$ terms, and this was precisely the strong CP problem [1]. The present work demonstrates that the $\theta'$ term produces no CP violation. Thus $\theta'$ becomes irrelevant and there is no need for fine tuning $\theta$ to balance its effects. Even without knowing how parity-violating $\theta$ is, one can avoid all CP violation by having $\theta = 0$ in the $\theta F\tilde{F}$ term. This is no doubt a special value, but this choice increases the symmetry of the action, because $\theta'$ does not break the symmetry. Consequently it is a natural choice according to 't Hooft's criterion. The situation is similar to working with Lorentz-invariant theories, setting possible noncovariant terms to zero by hand, exploiting naturalness. Regularization of QCD does not force $\theta$ or the CP violation to vanish, but certainly makes it natural to avoid CP violation by setting $\theta$ equal to zero.

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