Tsallis distribution from minimally selected order statistics

G. Wilk∗ and Z. Włodarczyk†

∗The Andrzej Sołtan Institute for Nuclear Studies; Hoża 69; 00-681 Warsaw, Poland
†Institute of Physics, Świętokrzyska Academy, Świętokrzyska 15; 25-406 Kielce, Poland and University of Arts and Sciences (WSU), Wesoła 52, 25-353 Kielce, Poland

Abstract. We demonstrate that selection of the minimal value of ordered variables leads in a natural way to its distribution being given by the Tsallis distribution, the same as that resulting from Tsallis nonextensive statistics. The possible application of this result to the multiparticle production processes is indicated.

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Tsallis distribution $p(E)$ of some variable $E$ is defined as

\[ p_q(E) = \frac{2 - q}{T} \left[ 1 - (1 - q) \frac{E}{T} \right]^{\frac{1}{1 - q}}, \]  

(with $E \in (0, \infty)$ for $q \geq 1$ and $E \in [0, T/(1 - q)]$ for $q < 1$) where $T$ is some scale parameter (for example, if $E$ is energy then $T$ is temperature). The mean value of variable $E$ is $\langle E \rangle = T/(3 - 2q)$. In the limit $q \to 1$ Tsallis distribution (1) becomes the usual exponential (Boltzmann) distribution,

\[ p_{q=1}(E) = \frac{1}{T} \exp \left( -\frac{E}{T} \right). \]  

Distributions of type (1) are ubiquitous in all fields of research [1]. Their origin is rooted in the notion of nonextensive statistics introduced by Tsallis [3] (see also [4, 1] and references therein), which for $q \to 1$ becomes the usual Boltzmann-Gibbs one. There are numerous ways to obtain Tsallis distribution (1) which are discussed in the literature [4] and from which we would like to mention here only two. The first is based on the observation that some specific intrinsic fluctuations of the parameter $T$ in the distribution (2) result in the eq. (1); in this case $(q - 1)$ measures the strength of these fluctuations [5, 6]. This can be contrasted with the second approach proposed in [7] where suitable choice of particles in the phase space (introducing effectively some specific correlations among them) also results in eq. (1) [6]. In this note we shall

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1 This includes also applications to multiparticle production processes, which are of special interest to us, see review [2].
2 Similar to this is the approach based on the assumed fractality of phase space proposed in [8].
follow similar way of reasoning and demonstrate that distribution of the minimal values
of some specific choices of variable $E$ (known in the literature as order statistics [9]) also results in Tsallis distribution (1) but this time without necessity of correlating the corresponding variables.

Let us imagine therefore that we have to our disposal a number of $n$ "ghost-particles" with energies $\varepsilon_i, i = 1, \ldots, n$, with $\varepsilon_i$ following some distribution $f(\varepsilon)$. Let us now order the values of $\varepsilon_i$, i.e., let us introduce order statistics in this set. As result we are getting the ordered set of energies, $\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n$, out of which we shall now choose the actually observed ("real") particle defined as particle with the lowest energy, $E = \varepsilon_1 = \min\{\{\varepsilon_i\}\}$. Probability to find a particle with such energy $E$ among $n$ elements is equal to $n f(\varepsilon)$ whereas probability to find particle with energy exceeding $E$ is equal to $1 - F(E)$, where $F(E) = \int_0^E d\varepsilon f(\varepsilon)$ is distribuant of $f(\varepsilon)$. If particle of energy $E$ is already that of the minimal energy it means that the remain $n-1$ particles have to posses higher energies. Probability of such event is equal $[1 - F(E)]^{n-1}$. It means therefore that distribution of the minimal value in the sample of $n$ elements is

$$g(E) = n f(E)[1 - F(E)]^{n-1}. \quad (3)$$

For some specific forms of distribution $f(\varepsilon)$ (3) can be converted exactly into Tsallis distribution (1) with parameter $T$ being independent of $q$. We shall in what follows discuss examples of such distributions, separately for $q < 1$ and $q > 1$ cases.

For $q < 1$ energies $\varepsilon$ of all $n$ particles considered must be limited to the interval $0 < \varepsilon < (n-1)T$ ($T$ is scale parameter mentioned before). Let us now assume that they are distributed uniformly according to

$$f(\varepsilon) = \frac{1}{(n-1)T}. \quad (4)$$

Distribuant of this distribution is $F(E) = E/[(n-1)T]$, therefore distribution of the minimal value (3) takes the following form

$$g(E) = \frac{n}{n-1} \left(1 - \frac{\varepsilon}{n-1} T\right)^{n-1} \left[1 - \left(1 - q\right) \frac{E}{T}\right]^{\frac{1}{1-q}}, \quad (5)$$

i.e., it coincides with Tsallis distribution with $q = (n-2)/(n-1) < 1$ and $E < T/(1-q)$. Notice that increasing the range of allowed $\varepsilon$, i.e., for $n \to \infty$, one obtains exponential distribution (2).

For $q > 1$ let us consider $n$ particles with unlimited energies $\varepsilon$, $0 \leq \varepsilon < \infty$, assumed to be distributed according to

$$f(\varepsilon) = \frac{c}{(1+c\varepsilon)^2}, \quad c = \frac{1}{(n+1)T}. \quad (6)$$

Notice that, because $f(E) = dF(E)$, distribution $g(E)$ is properly normalized if $f(E)$ is normalized.
FIGURE 1. Examples of $T f(\varepsilon)$ versus $\varepsilon/T$ used here (left panel) and corresponding to them $T g(E)$ versus $E/T$ (middle panel) calculated for $n = 20$. Continuous line is for $q < 1$, dotted line for $q = 1$ and dashed line for $q > 1$. The right panel displays the $q$ as function of $n$ corresponding to (5) (continuous line) and (7) (dashed line). They both converge to dotted line for $q = 1$ for $n \to \infty$.

Distribuant of this distribution is $F(E) = 1 - 1/(1 + cE)$ and distribution of the minimal value (3) takes the form

$$g(E) = \frac{n}{n+1} \frac{1}{T} \left(1 + \frac{1}{n+1} \frac{E}{T}\right)^{-(n+1)} = \frac{2 - q}{T} \left[1 - (1-q) \frac{E}{T}\right]^{-\frac{1}{q}} ,$$

(7)
i.e., again, the form of Tsallis distribution with $q = (n + 2)/(n + 1) > 1$ and with no limit on $E$. Again, for $n \to \infty$ one recovers the exponential distribution (2).

Notice that in both cases for $n \to \infty$ one gets distribution with $q = 1$ (approached, respectively, from below or from above). Actually, one can have the $q = 1$ distribution independently on the value of $n$ only for the exponential form of the initial distribution of $\varepsilon$,

$$f(\varepsilon) = \frac{1}{nT} \exp \left(-\frac{\varepsilon}{nT}\right).$$

(8)
In this case distribuant is $F(E) = 1 - \exp \left(-E/(nT)\right)$ and distribution of the minimal value, $g(E)$, has form of eq. (2), independent of $n$. The above results are illustrated in Fig. I.

Let us now calculate the changes of entropy $S = - \int p(x) \ln[p(x)] dx$ caused by the selection of the minimal value,

$$\Delta S_q = S_f - (\langle S_f' \rangle + S_g).$$

(9)
Here $S_f$ is entropy of the initial distribution $f(\varepsilon)$, which changes to $S_f'(E)$ after choosing the minimal value of $\varepsilon$. It depends on $E$ because the remaining values of $\varepsilon$ are all above $E$. Its average over $g(E)$ is denoted by $\langle S_f' \rangle$, whereas the entropy of the selected particle is denoted by $S_g$. One has

$$\Delta S_q = \ln(n) - \frac{n}{n+1} \ln(T) - \frac{n}{n+1} \ln(n+\xi) - \frac{(n-\xi)(n+2)}{(n+1)^2} + \frac{1}{2} \xi (\xi - 1) \left(\frac{1}{n} + \frac{1}{n+1}\right),$$

(10)
where $\xi = (−1, 0, +1)$ for $q < 1$, $q = 1$ and $q > 1$, respectively. Notice that entropy always increases and that its increase is, in the limit of $n \to \infty$, the same and equal $\Delta S_q = −[1 + \ln(T)]$. Because $S_f \neq (S_f') + S_g$ the process of selection of the minimal value is nonextensive.

To summarize: the order statistics is per se the important branch of science [9] but we are in no position to discuss it here. On the other hand, as shown in [10, 2], the selection of energies, very much alike to the one applied here, when applied to hadronic physics results in the power-like distributions observed in experiments. This fact has been interpreted as indication of the need of some kind of new statistical physics (Tsallis statistics) being at work here [11]. In fact, one can easily invent a non-thermal scenario leading to thermal-like form of the observed spectra, see, for example, recent work [12]. In such approach the resultant distribution emerges not because of the equilibrating of energies due to some collisions (i.e., because of the kinematic thermalization) but rather because of the process of erasing of memory of the initial state and is the result of the approaching to a state of maximal entropy (called in [12] stochastic thermalization). This seems to be very promising idea which needs further investigations.

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