Performance Analysis for Data Compression Based Signal Classification Methods

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Abstract—In this paper, we present an information theoretic analysis of the blind signal classification algorithm. We show that the algorithm is equivalent to a Maximum A Posteriori (MAP) estimator based on estimated parametric probability models. We prove a lower bound on the error exponents of the parametric model estimation. It is shown that the estimated model parameters converge in probability to the true model parameters except some small bias terms.

I. INTRODUCTION

In this paper, we consider the blind signal classification problems. In the considered scenarios, a sequence of random signal samples $X_1, X_2, \ldots, X_N$ is observed, where each signal sample is a real number or a vector in a finite dimensional space. It is assumed that the $N$ signal samples are generated by $J$ information sources with different statistical properties. However, it is unknown from which information source each signal sample $X_n$ is emitted. The blind signal classification problems denote the problems of estimating the membership of each signal sample to the information sources. The signal classification problems find applications in many areas of image processing, computer vision and machine learning, for example, in image segmentation, and cluster analysis. For background information in these applications, we refer interested readers to [1] [2] and references therein.

In [3], a novel algorithm for the signal classification problems is proposed based on data compression. The algorithm is based on the intuitive idea that optimal classification induces optimal adaptive data compression. Therefore, the signal classification problems can be formulated as optimization problems. An analysis in an algorithmic viewpoint was also presented. It was shown in the paper [3] that a soft membership relaxation can be used to reduce the computational complexity with asymptotic vanishing optimality loss. Simulation results show that the algorithm has nice performance.

It is well known that there exist close connections between information theory and statistical inference. Especially, source coding and data compression have been used in statistical inference problems, such as prediction, estimation and modeling, see for instance [4], [5], [6], [7]. However, there exists no discussion on using data compression for classification and clustering until very recent. In [8], [9], a “clustering by compression” algorithm has been proposed. The approach in [8], [9] is different from the approach in [3] in terms of their ways of using data compression. In [8], [9], the data compression methods are used to compute distances between data items. The clustering results are then obtained by using conventional methods based on the computed distances.

In this paper, we present an information theoretical analysis to justify the intuitive idea of the blind signal classification algorithm in [3]. It is shown that the blind signal classification algorithm is equivalent to a Maximum A Posteriori (MAP) estimator based on estimated parametric probability models. We also discuss the error exponents of the model parameter estimation. It is shown that the estimated model parameters converge to the true model parameters in probability. These theoretical discussions suggest that the blind signal classification algorithm has nice performance.

The discussions in this paper focus on the cases that the information sources are independent and identically distributed (i.i.d.) Gaussian, and there are two information sources. Even though, more sophisticated cases are not covered in this paper, the discussions presented here can provide useful insights into these more general cases. The discussions can be easily generalized to the cases of non-Gaussian, Markov, stationary or ergodic information sources.

Notation: we use $\lfloor \cdot \rfloor$ to denote the floor function, that is, $\lfloor x \rfloor$ is the largest integer smaller than $x$. We use $\ln(\cdot)$ to denote the logarithmic function with base $e$. We use $H(P)$ and $D(P||Q)$ to denote the entropy function and information divergence respectively. If $P$ and $Q$ are discrete probability mass functions, then

$$H(P) = \sum_{x \in \mathcal{X}} -P(x) \ln(P(x)),$$

$$D(P||Q) = \sum_{x \in \mathcal{X}} P(x) \ln \frac{P(x)}{Q(x)},$$

(1)

where $\mathcal{X}$ is the discrete alphabet. If $P$ and $Q$ are probability density functions, then

$$H(P) = \int_{-\infty}^{\infty} -P(x) \ln(P(x))dx,$$

$$D(P||Q) = \int_{-\infty}^{\infty} P(x) \ln \frac{P(x)}{Q(x)}dx.$$  (2)

If $f(N)$ and $g(N)$ are two functions of the number $N$, we use $f(N) \sim g(N)$ to denote that

$$\lim_{N \to \infty} \frac{1}{N} \ln \left( \frac{f(N)}{g(N)} \right) = 0.$$  (3)
For a sequence $x_1 x_2 \ldots x_N$ over a discrete alphabet $\mathcal{X}$, the type of the sequence is defined as the corresponding empirical distribution $P$ over $\mathcal{X}$, that is, $P(a)$ is equal to the fraction of $x_i$ taking value $a$. For a type $P$, the type class $T(P, N)$ is the set of sequences with length $N$ and type $P$. We use $\mathbb{P}(T(P, N))$ to denote the probability that the type of the random sequence $x_1 x_2 \ldots x_N$ is $P$.

The rest of this paper is organized as follows. We review the data compression based signal classification algorithm in Section II. We discuss the necessary conditions for the optimal solutions of the blind signal classification algorithm in Section III. We discuss the error exponents of the parameter estimation in Section IV. The concluding remarks are presented in Section V.

II. BLIND SIGNAL CLASSIFICATION ALGORITHM

In this paper, we consider the scenario, where a sequence of random real-valued signal samples $X_1, X_2, \ldots, X_N$ is observed. Each signal sample $X_n$ is independently drawn from one of the several i.i.d. Gaussian information sources. The probability density function is

$$P(X_n) = \sum_{i=1}^{J} \frac{\alpha_i}{\sqrt{2\pi\sigma_i^2}} \exp\left(\frac{-(x_n - \mu_i)^2}{2\sigma_i^2}\right),$$

where $J$ is the total number of information sources, $\alpha_i$ is the probability that $X_n$ is drawn from the $i$-th information source, and $\mu_i, \sigma_i$ are the Gaussian distribution parameters of the $i$-th information source. The goal is to estimate the membership of each signal sample $x_n$ to the $J$ information sources.

The blind signal classification algorithm in [3] is based on a data compression argument that accurate signal classification results in efficient signal modeling and good data compression. Therefore, the signal samples should be classified, so that the coding efficiency is maximized. Let $m_{ni}$ denote the membership variable for the $n$-th signal sample with respect to the $i$-th class,

$$m_{ni} = \begin{cases} 1, & \text{if } x_n \text{ is classified into the } i-th \text{ class} \\ 0, & \text{otherwise} \end{cases}$$

The algorithm searches for the optimal values $m_{ni}$ such that the following objective function is minimized,

$$G = \exp\left(2H(\alpha_1, \ldots, \alpha_J)\right) \prod_{i=1}^{J} (\sigma_i^2)^{\alpha_i}$$

where $\alpha_i$ is the fraction of signal samples that are classified into the $i$-th class, $\sigma_i^2$ is the variance of signal samples in the $i$-th class, $H(\alpha_1, \ldots, \alpha_J)$ is the entropy function in nats,

$$\alpha_i = \frac{\sum_n m_{ni}}{N}, \quad \mu_i = \frac{\sum_n m_{ni} x_n}{\alpha_i N}, \quad \sigma_i^2 = \frac{\sum_n m_{ni} (x_n - \mu_i)^2}{\alpha_i N}.$$  

The objective function $G$ relates to the so-called classification gain and adaptive coding efficiency [3].

In [3], the hard membership variables $m_{ni}$ are relaxed into soft membership variables. That is, $m_{ni}$ can take real values, such that $\sum_i m_{ni} = 1$, $0 \leq m_{ni} \leq 1$. It is proven in [3] that the optimality loss due to the relaxation vanishes asymptotically.

III. NECESSARY CONDITION FOR OPTIMIZATION SOLUTIONS

In this section, we show a necessary condition for the solution in the above blind signal classification method with soft membership variables. It turns out that useful insights can be gained from the necessary condition.

Let us assume that the probability density function of $X_n$ is $P^*(X_n) = \alpha_1 f_1^*(x) + \alpha_2 f_2^*(x)$, where

$$f_i^*(x) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x - \mu_i)^2}{2\sigma_i^2}\right).$$

Let $\hat{m}_{ni}$ denote one global minimizer of the optimization programming in the blind signal classification method. Let $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2$ denote the distribution parameters corresponding to the optimization solution $\hat{m}_{ni}$.

**Theorem 3.1:** The optimal solution $\hat{m}_{ni}$ satisfies the following condition.

$$\hat{m}_{ni} = \begin{cases} 1, & \text{if } \hat{\alpha}_1 \hat{f}_1(x_n) > \hat{\alpha}_2 \hat{f}_2(x_n) \\ 0, & \text{if } \hat{\alpha}_1 \hat{f}_1(x_n) < \hat{\alpha}_2 \hat{f}_2(x_n) \end{cases}$$

where $\hat{f}_1$ and $\hat{f}_2$ are the Gaussian probability density functions corresponding to the parameters $\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2$.

**Proof:** (sketch) Consider $G$ as a function solely determined by $m_{ni}$. Taking a derivative, we have

$$\frac{\partial G}{\partial m_{ni}} = \frac{1}{N} \left[ \ln (\hat{\sigma}_1^2) - 2 \ln (\hat{\alpha}_1) + \frac{(x_n - \hat{\mu}_1)^2}{\hat{\sigma}_1^2} \right] - \frac{1}{N} \left[ \ln (\hat{\sigma}_2^2) - 2 \ln (\hat{\alpha}_2) + \frac{(x_n - \hat{\mu}_2)^2}{\hat{\sigma}_2^2} \right] = \frac{2}{N} \ln \left[ \sqrt{2\pi\hat{\sigma}_2} \hat{f}_2(x_n) \right] - \frac{2}{N} \ln \left[ \sqrt{2\pi\hat{\sigma}_1} \hat{f}_1(x_n) \right]$$

Therefore,

$$\frac{\partial G}{\partial m_{ni}} < 0, \quad \text{if } \hat{\alpha}_1 \hat{f}_1(x_n) > \hat{\alpha}_2 \hat{f}_2(x_n)$$

$$\frac{\partial G}{\partial m_{ni}} > 0, \quad \text{if } \hat{\alpha}_1 \hat{f}_1(x_n) < \hat{\alpha}_2 \hat{f}_2(x_n)$$

The theorem then follows from the KKT condition [10].

**Corollary 3.2:** Define three subsets $A, B, C$ of real numbers as follows, $A = \{ x|\hat{\alpha}_1 \hat{f}_1(x) > \hat{\alpha}_2 \hat{f}_2(x) \}$, $B = \{ x|\hat{\alpha}_1 \hat{f}_1(x) = \hat{\alpha}_2 \hat{f}_2(x) \}$, $C = \{ x|\hat{\alpha}_1 \hat{f}_1(x) < \hat{\alpha}_2 \hat{f}_2(x) \}$. We write $n \in A$ ($n \in B, n \in C$), if $x_n \in A$ ($x_n \in B, x_n \in C$). Then, the following statements hold.

$$\hat{\alpha}_1 = \frac{\sum_{n \in A} 1 + \sum_{n \in B} \hat{m}_{ni}}{N}, \quad \hat{\alpha}_2 = \frac{\sum_{n \in C} 1 + \sum_{n \in B} \hat{m}_{ni}}{N}.$$  

(14)
\[
\hat{\mu}_1 = \frac{\sum_{n \in A} x_n + \sum_{n \in B} \hat{m}_{n1} x_n}{\hat{a}_1 N},
\]
\[
\hat{\mu}_2 = \frac{\sum_{n \in A} x_n + \sum_{n \in B} \hat{m}_{n2} x_n}{\hat{a}_2 N},
\]
\[
\hat{\sigma}_1 = \frac{\sum_{n \in A}(x_n - \hat{\mu}_1)^2 + \sum_{n \in B} \hat{m}_{n1}(x_n - \hat{\mu}_1)^2}{\hat{a}_1 N},
\]
\[
\hat{\sigma}_2 = \frac{\sum_{n \in C}(x_n - \hat{\mu}_2)^2 + \sum_{n \in B} \hat{m}_{n2}(x_n - \hat{\mu}_2)^2}{\hat{a}_2 N}.
\]

Remark 1: Theorem 3.1 shows that the data compression based signal classification method is equivalent to the MAP estimation based on the estimated parametric probability models. Even though the probability model estimation is just a by-product of the classification algorithm, the accuracy of model estimation is critical to the performance of the algorithm.

IV. ERROR EXPONENT OF PARAMETER ESTIMATION

In this section, we investigate the accuracy of the parametric probability model estimation in the proposed signal classification method by using the method of types [11]. We need to introduce several auxiliary discrete probability distributions. Let \(M_N, L_N, W_N\) be numbers, which only depend on the number of signal samples \(N\),
\[
M_N = c(N)^{\frac{1}{2} + \zeta}, \quad L_N = 2 \left[ \frac{N^{1 - \eta}}{\log N} \right] + 1, \quad W_N = \frac{2M_N}{L_N},
\]
where \(c, \zeta, \eta\) are positive constants, and \(\zeta + \eta < 1/2\). Let \(\alpha_k\) denote the number of signal samples, which fall in the interval \([(k - 1/2)W_N, (k + 1/2)W_N]\). Let \(\mathcal{O}_N\) denote the random event that \(|x_n| \geq M_N\) for some \(n, 1 \leq n \leq N\). If \(\mathcal{O}_N\) does not occur, then \(\{\ldots, \alpha_k/N, \ldots\}\) is a well defined empirical probability mass function, where
\[
-\frac{L_N - 1}{2} \leq k \leq \frac{L_N - 1}{2}.
\]

We write \(k \in \mathcal{A}\), if the interval \([(k - 1/2)W_N, (k + 1/2)W_N]\) \(\subset \mathcal{A}\). We write \(k \in \mathcal{C}\), if \([(k - 1/2)W_N, (k + 1/2)W_N]\) \(\subset \mathcal{C}\). We write \(k \in \mathcal{B}\), if \([(k - 1/2)W_N, (k + 1/2)W_N]\) \(\cap \mathcal{B} \neq \emptyset\). If \(k \in \mathcal{B}\), we define \(c_k\) as,
\[
c_k = \frac{\sum_{n=1}^{N} m_{n1} I(x_n \in [(k - 1/2)W_N, (k + 1/2)W_N])}{\sum_{n=1}^{N} I(x_n \in [(k - 1/2)W_N, (k + 1/2)W_N])},
\]
where \(I(\cdot)\) is the indicator function.

Let \(\Theta\) denote the 6-tuple \(\{\alpha_1, \alpha_2, \mu_1, \mu_2, \sigma_1, \sigma_2\}\). We use \(P(x; \Theta)\) to denote the mixture Gaussian distribution,
\[
\left( \sum_{i=1}^{2} \alpha_i \frac{\exp \left( -\frac{(x - \mu_i)^2}{2\sigma_i^2} \right)}{\sqrt{2\pi\sigma_i}} \right).
\]

We use \(P_N(k; \Theta)\) to denote the following discrete probability distribution over the same alphabet of \(\{\ldots, \alpha_k/N, \ldots\}\),
\[
P_N(k; \Theta) = c_P \int_{(k-1/2)W_N}^{(k+1/2)W_N} \left( \sum_{i=1}^{2} \frac{\alpha_i}{\sqrt{2\pi\sigma_i}} \exp \left( -\frac{(x - \mu_i)^2}{2\sigma_i^2} \right) \right) dx,
\]
where \(c_P\) is a normalization constant, \(c_P \to 1, \) as \(N \to \infty\).

Lemma 4.1:
\[
P(\mathcal{O}_N) \leq \sum_{i=1}^{2} \exp \left( -\frac{(c(N)\frac{1}{2} + \zeta) + \mu_i^*}{2\sigma_i^2} \right) + \ln N
\]
\[
+ \sum_{i=1}^{2} \exp \left( -\frac{(c(N)\frac{1}{2} + \zeta) - \mu_i^*}{2\sigma_i^2} \right) + \ln N
\]

Proof:
\[
P(\mathcal{O}_N) \leq \sum_{n=1}^{N} \mathbb{P}(|x_n| \geq M_N|x_n| \text{is of class } i) \mathbb{P}(x_n \text{ is of class } i)
\]
\[
\leq \sum_{n=1}^{N} \sum_{i=1}^{2} \mathbb{P}(|x_n| \geq M_N|x_n| \text{is of class } i)
\]
\[
\leq N \sum_{i=1}^{2} Q \left( \frac{M_N + \mu_i^*}{\sigma_i^2} \right) + N \sum_{i=1}^{2} Q \left( \frac{M_N - \mu_i^*}{\sigma_i^2} \right)
\]
\[
\leq \sum_{i=1}^{2} \exp \left( -\frac{(c(N)\frac{1}{2} + \zeta) + \mu_i^*}{2\sigma_i^2} \right) + \ln N
\]
\[
+ \sum_{i=1}^{2} \exp \left( -\frac{(c(N)\frac{1}{2} + \zeta) - \mu_i^*}{2\sigma_i^2} \right) + \ln N
\]
\]
where, \(Q(\cdot)\) denotes the well known Gaussian tail function, (a) follows from the union bound, and (b) follows from the well known Chernoff bound \(Q(x) \leq \exp(-x^2/2)\) (see for example [12] Section 2.1-5).}

Lemma 4.2: Let \(\hat{\Theta} = \{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2\}\) be the estimated model parameters. Assume that the random event \(\mathcal{O}_N\) does not occur. Then, the following bound holds, which relates the type \(\{\alpha_k/N\}\) to the estimated probability model parameters.
\[
D \left( \frac{\alpha_k}{N} \parallel P_N(k; \hat{\Theta}) \right) + H \left( \frac{\alpha_k}{N} \right)
\]
\[
\leq \frac{\hat{\alpha}_1}{2} \ln(2\pi e \hat{\sigma}_1^2) + \frac{\hat{\alpha}_2}{2} \ln(2\pi e \hat{\sigma}_2^2) + H(\hat{\alpha}_1, \hat{\alpha}_2)
\]
\[
+ \left( \frac{\hat{\alpha}_1}{2\hat{\sigma}_1^2} + \frac{\hat{\alpha}_2}{2\hat{\sigma}_2^2} \right) W_N^2 - \ln W_N - \ln c_P.
\]
Proof: (sketch) The bound is proved in Eq. 27, where (a) follows from the fact that \(\ln(\cdot)\) is an increasing function, (b) follows from the mean-value theorem, and (c) follows from Eqs. 17, 18.

\[ \text{Theorem 4.3: Let } D \text{ denote a set of mixture Gaussian distributions with parameters } \{\alpha_1, \alpha_2, \mu_1, \mu_2, \sigma_1, \sigma_2\}, \text{ where } \sigma_1^2 \text{ are lower bounded by a positive constant } B_2. \text{ Assume that the true model distribution } \Theta^* \notin D, \Theta^* = \{\alpha_1^*, \alpha_2^*, \mu_1^*, \mu_2^*, \sigma_1^*, \sigma_2^*\}. \text{ Define the error exponent } E_r(D) = \lim_{N \to \infty} \frac{-1}{N} \ln P(\hat{P} \in D). \text{ Let } F(D) \text{ denote the set of probability distribution with well-defined probability density function } P(x), \text{ such that, there exists a } P(x; \Theta), \Theta \in D, \Theta = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1, \tilde{\sigma}_2\} \text{ and}
\]

\[
D(P(x); P(x; \tilde{\Theta})) + \frac{1}{2} \ln(2\pi e\tilde{\sigma}_1^2) + \frac{1}{2} \ln(2\pi e\tilde{\sigma}_2^2) + H(\tilde{\alpha}_1, \tilde{\alpha}_2).
\]

Then \(E_r(D) \geq E_b, \text{ where}
\]

\[
E_b(D) = \min_{P \in F(D)} D(P(x)||P(x; \Theta^*))
\]

Proof: (sketch) According to Lemma 4.1, the exponent of the random event \(P(O_N)\) is infinity. Therefore,

\[
P(\hat{\Theta} \in D) = \mathbb{P}(\hat{\Theta} \in D, O_N^c) + \mathbb{P}(\hat{\Theta} \in D|O_N)\mathbb{P}(O_N)
\]

\[
\sim \mathbb{P}(\hat{\Theta} \in D, O_N).
\]

Let \(F(D, \epsilon)\) denote the set of probability distribution with probability density function \(P(x)\), such that, there exists \(P(x; \Theta), \Theta \in D, \Theta = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1, \tilde{\sigma}_2\}, \text{ and}
\]

\[
D(P(x)||P(x; \tilde{\Theta})) + H(P(x))
\]

\[
\leq \frac{\tilde{\alpha}_1}{2} \ln(2\pi e\tilde{\sigma}_1^2) + \frac{\tilde{\alpha}_2}{2} \ln(2\pi e\tilde{\sigma}_2^2) + H(\tilde{\alpha}_1, \tilde{\alpha}_2) + \epsilon.
\]

Let \(G(D, N, \epsilon)\) denote the set of type \(P\) of sequences with length \(N\), such that, there exists \(P(x; \Theta), \Theta \in D, \Theta = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1, \tilde{\sigma}_2\}, \text{ and}
\]

\[
D(P(x)||P_N(k; \tilde{\Theta})) + H(P(x))
\]

\[
\leq \sum_{i=1}^{2} \frac{\tilde{\alpha}_i}{2} \ln(2\pi e\tilde{\sigma}_i^2) + H(\tilde{\alpha}_1, \tilde{\alpha}_2) - \ln N + \epsilon.
\]

According to Lemma 4.1 if \(\hat{\Theta} \in D, \text{ and } O_N \text{ does not occur, then the type } \{a_k/N\} \in G(D, N, \epsilon). \text{ Therefore,}
\]

\[
P(\hat{\Theta} \in D, O_N^c) \leq \sum_{P \in G(D, N, \epsilon)} \mathbb{P}(T(P, N))\]

\[
\approx \max_{P \in G(D, N, \epsilon)} \mathbb{P}(T(P, N))
\]

\[
\approx \exp \left( -\min_{P \in G(D, N, \epsilon)} D(P||P_N(k; \Theta^*))N \right),
\]

where, (a) follows from the fact that the number of type class is upper bounded by

\[
(N + 1)^{LN} \sim 1,
\]

and (b) follows from first principles in the method of types 11. Let \(\{b_k/N\} \text{ denote the above type, which minimizes } D(P||P_N(k; \Theta^*)). \text{ We can construct a probability distribution with probability density function } Q(x; N, b_k/N) \text{ as follows.}
\]

\[
Q(x; N, b_k/N) = \begin{cases} \frac{b_k}{W_N N^*}, & \text{if } |x| \leq M_N, |x - kW_N| < \frac{W_N}{2} \\ 0, & \text{otherwise} \end{cases}
\]

It can be checked that \(Q(x; N, b_k/N) \in F(D, \epsilon_1), \text{ and}
\]

\[
D(b_k/N||P_N(k; \Theta^*)) \geq D(Q(x; N, b_k/N)||P(x; \Theta^*)) - \epsilon_2,
\]

where \(\epsilon_1, \epsilon_2\) are some small positive numbers, \(\epsilon_1, \epsilon_2 \to 0, \text{ as } N \to \infty.
\]

As a consequence,

\[
\min_{P \in G(D, N, \epsilon)} D(P||P_N(k; \Theta^*))
\]

\[
\geq \min_{P \in F(D, \epsilon_1)} D(P(x)||P(x; \Theta^*)) - \epsilon_2.
\]

Finally, the theorem follows from the fact that all information divergence and entropy functions are continuous. 

\textbf{Theorem 4.4:} For sufficiently large \(N\), with probability close to one,

\[
D(P(x; \Theta^*))||P(x; \tilde{\Theta})) \leq H(\alpha_1^*, \alpha_2^*) + \epsilon,
\]

where \(\epsilon\) is a small positive number, \(\epsilon \to 0, \text{ as } N \to \infty.
\]

Proof: (sketch) We define \(a_k/N\) and \(Q(x; N, a_k/N)\) similarly as in the above. With probability close to one, \(O_N\) does not occur, and

\[
D(P(x; \Theta^*))||P(x; \tilde{\Theta})) - D(Q(x; N, a_k/N)||P(x; \tilde{\Theta})) \leq \epsilon_1.
\]

Note that

\[
H(P(x; \Theta^*)) \geq \frac{1}{2} \ln \left(2\pi e(\sigma_1^*)^2 \right) + \frac{1}{2} \ln \left(2\pi e(\sigma_2^*)^2 \right).
\]

By Lemma 4.1 and 4.2, we have with probability close to one

\[
D(P(x; \Theta^*))||P(x; \tilde{\Theta}))
\]

\[
\leq \frac{1}{2} \ln \left((\sigma_1^*)^2 \right) + \frac{1}{2} \ln \left((\sigma_2^*)^2 \right) + H(\tilde{\alpha}_1, \tilde{\alpha}_2)
\]

\[
- \frac{1}{2} \ln \left((\sigma_1^*)^2 \right) - \frac{1}{2} \ln \left((\sigma_2^*)^2 \right) + \epsilon_2.
\]

The theorem then follows from the fact that

\[
\frac{1}{2} \ln \left((\sigma_1^*)^2 \right) + \frac{1}{2} \ln \left((\sigma_2^*)^2 \right) + H(\tilde{\alpha}_1, \tilde{\alpha}_2) = \min \frac{\ln(G)}{2};
\]

with probability close to one

\[
\frac{1}{2} \ln \left((\sigma_1^*)^2 \right) + \frac{1}{2} \ln \left((\sigma_2^*)^2 \right) + H(\alpha_1^*, \alpha_2^*) + \epsilon_3.
\]
\[ D \left( a_k/N \| P_N(k; \hat{\Theta}) \right) + H(a_k/N) = \sum_k \frac{-a_k}{N} \ln \left( P_N(k; \hat{\Theta}) \right) \]

\[
= \sum_k \frac{-a_k}{N} \ln \left[ \int_{(k-1)/2}^{(k+1)/2} \frac{\alpha_1}{\sqrt{2\pi}\sigma_1} \exp \left( -\frac{(x - \hat{\mu}_1)^2}{2\sigma_1^2} \right) \, dx \right] + \frac{\alpha_2}{\sqrt{2\pi}\sigma_2} \exp \left( -\frac{(x - \hat{\mu}_2)^2}{2\sigma_2^2} \right) \, dx \]  
\[ \leq \sum_{k \in A} \frac{-a_k}{N} \ln \left[ \int_{(k-1)/2}^{(k+1)/2} \frac{\alpha_1}{\sqrt{2\pi}\sigma_1} \exp \left( -\frac{(x - \hat{\mu}_1)^2}{2\sigma_1^2} \right) \, dx \right] + \sum_{k \in B} \frac{-c_k a_k}{N} \ln \left[ \int_{(k-1)/2}^{(k+1)/2} \frac{\alpha_1}{\sqrt{2\pi}\sigma_1} \exp \left( -\frac{(x - \hat{\mu}_1)^2}{2\sigma_1^2} \right) \, dx \right] \\
+ \frac{\alpha_1}{2} \ln \left( 2\pi \sigma_1^2 \right) + \frac{\alpha_2}{2} \ln \left( 2\pi \sigma_2^2 \right) + H(\hat{\alpha}_1, \hat{\alpha}_2) - \ln(c_P) \\
\leq \sum_{n \in A} \frac{(2x_n - 2\hat{\mu}_1 + W_N)W_N}{2\sigma_1^2 N} + \sum_{n \in B} \frac{\tilde{m}_n(x_n - \hat{\mu}_1)^2}{2\sigma_1^2 N} \\
+ \sum_{n \in C} \frac{(2x_n - 2\hat{\mu}_2 + W_N)W_N}{2\sigma_2^2 N} + \sum_{n \in B} \frac{\tilde{m}_n(x_n - \hat{\mu}_2)^2}{2\sigma_2^2 N} \\
+ \frac{\alpha_1}{2} \ln \left( 2\pi \sigma_1^2 \right) + \frac{\alpha_2}{2} \ln \left( 2\pi \sigma_2^2 \right) + H(\hat{\alpha}_1, \hat{\alpha}_2) - \ln(W_N) - \ln(c_P) \\
\leq \frac{\alpha_1}{2} \ln \left( 2\pi \sigma_1^2 \right) + \frac{\alpha_2}{2} \ln \left( 2\pi \sigma_2^2 \right) + H(\hat{\alpha}_1, \hat{\alpha}_2) + \left( \frac{\alpha_1}{2\sigma_1^2} + \frac{\alpha_2}{2\sigma_2^2} \right) W_N^2 - \ln(W_N) - \ln(c_P) \\
\]

In the above, \( \epsilon_1, \epsilon_2, \epsilon_3 \) are all small positive real numbers, \( \epsilon_1, \epsilon_2, \epsilon_3 \to 0 \), as \( N \to \infty \).

Remark 2: Theorem 4.3 and 4.4 show that the estimated model parameters converge to the true model parameters in probability except some bias terms. The bound in Theorem 4.3 can be further improved. However, because the discussion is much involved, we leave it for future research.

V. CONCLUSION

In this paper, we present an information theoretic performance analysis of the blind signal classification algorithm proposed in [3]. We show that the obtained classification results in the algorithm is equivalent to a MAP estimator using the estimated parametric probability models. We further show that the by-product model parameter estimation is accurate. These theoretical analysis suggests that the algorithm has nice performance.

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