WELL-POSEDNESS AND AVERAGING PRINCIPLE OF MCKEAN-VLASOV SPDES DRIVEN BY CYLINDRICAL $\alpha$-STABLE PROCESS

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Abstract. In this paper, we first study the well-posedness of a class of McKean-Vlasov stochastic partial differential equations driven by cylindrical $\alpha$-stable process, where $\alpha \in (1, 2)$. Then by the method of the Khasminskii’s time discretization, we prove the averaging principle of a class of multiscale McKean-Vlasov stochastic partial differential equations driven by cylindrical $\alpha$-stable processes. Meanwhile, we obtain a specific strong convergence rate.

1. Introduction

Let $H$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $| \cdot |$. Let $\mathcal{P}$ be the set of all probability measures on $(H, \mathcal{B}(H))$. For any $p \geq 1$, define

$$\mathcal{P}_p := \left\{ \mu \in \mathcal{P} : \mu(| \cdot |^p) := \int_H |x|^p \mu(dx) < \infty \right\}.$$ 

Then space $(\mathcal{P}_p, \mathbb{W}_p)$ is a complete metric space, where $\mathbb{W}_p$ is the $L^p$-Wasserstein distance, i.e.,

$$\mathbb{W}_p(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}_{\mu_1, \mu_2}} \left[ \int_{H \times H} |x - y|^p \pi(dx, dy) \right]^{1/p}, \quad \mu_1, \mu_2 \in \mathcal{P}_p,$$

where $\mathcal{C}_{\mu_1, \mu_2}$ is the set of all couplings for $\mu_1$ and $\mu_2$.

We first consider the following McKean-Vlasov stochastic partial differential equations (SPDEs for short) in $H$:

$$\begin{cases} 
    dX_t = AX_t dt + B(X_t, \mathcal{L}_{X_t}) dt + dL_t, \\
    X_0 = \xi \in H,
\end{cases} \tag{1.1}$$

where $\mathcal{L}_{X_t}$ is the law of random variable $X_t$, $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator, which is the infinitesimal generator of a linear strongly continuous semigroup $(e^{tA})_{t \geq 0}$. The process $L = (L_t)_{t \geq 0}$ is a cylindrical $\alpha$-stable process with $\alpha \in (1, 2)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t \geq 0}$. $\xi$ is an $H$-valued $\mathcal{F}_0$-measurable random variable, map $B : H \times \mathcal{P}_p \to H$ satisfies proper condition.

The McKean-Vlasov stochastic differential equations (SDEs for short), also called distribution dependent SDEs, describe stochastic systems whose evolution is determined by both the microcosmic location and the macrocosmic distribution of the particle, see [21]. When the noise is the classical Brownian motion, the well-posedness of such kind of stochastic equations have been studied intensively (see e.g. [16, 17, 22, 31, 37] for finite dimension and
which also have been investigated in many references (see e.g. deviations, ergodicity, Harnack inequality and the Bismut formula for the Lions Derivative, \(\alpha\)-stable process with \(\eta\) and some appropriate conditions, and \(\{X_t\}\) component the so-called averaged equation in various senses. In this paper we intend to prove the slow \(X_t\) component of singular SPDEs driven by \(\alpha\) has been drawn much attentions, see e.g. \(\alpha\) of singular SPDEs driven by \(\alpha\) by \(\alpha\) has been drawn much attentions, see e.g. \(\alpha\) and \(\eta\) are mutually independent cylindrical \(\alpha\)-stable process with \(\alpha \in (1, 2)\) defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). \(\xi\) and \(\eta\) are two \(H\)-valued \(\mathcal{F}_0\)-measurable random variables.

The averaging principle of stochastic system (1.2) is to describe the asymptotic behavior of the slow component \(X_t^\varepsilon\) as \(\varepsilon \to 0\), which says that the slow component will converge to the so-called averaged equation in various senses. In this paper we intend to prove the slow component \(X_t^\varepsilon\) convergent to \(\bar{X}\) in the strong sense, i.e., for any initial values \(\xi\) and \(\eta\) which are two \(H\)-valued random variables having finite \(k\)-th moment for \(m \in [1, \alpha)\), one tries to find a constant \(r > 0\) such that for \(T > 0\)

\[
\left[ \mathbb{E} \left( \sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t|^m \right) \right]^{1/m} \leq C \varepsilon^r, \tag{1.3}
\]

where \(C\) is a positive constant only depends on \(T, \xi\) and \(\eta\), and \(\bar{X}\) is the solution of the corresponding averaged equation (see equation (3.21) below).

The theory of averaging principle was first developed for the ordinary differential equations by Bogoliubov and Mitropolsky [4], and extended to SDEs by Khasminskii [18] and SPDEs by Cerrai and Freidlin [7]. Nowadays, the averaging principle for slow-fast stochastic system has been drawn much attentions, see e.g. [5, 6, 9, 10, 11, 12, 20, 23, 24, 30, 36, 38, 39] and the references therein.

In the distribution-independent case, the averaging principle for multiscale SDEs driven by \(\alpha\)-stable processes has been studied in a number of papers. For instance, Bao et al. [3] prove the strong averaging principle for two-time scale SPDEs driven by \(\alpha\)-stable processes. Chen et al. [8] prove the strong averaging principle for stochastic Burgers equations. Sun et al. [33] prove the strong and weak convergence rates for a class of multiscale SDEs driven by \(\alpha\)-stable processes. Sun and Zhai [34] prove the strong averaging principle for stochastic Ginzburg-Landau equation. Sun et al. [35] prove the strong averaging principle for a class of singular SPDEs driven by \(\alpha\)-stable processes.

In the distribution dependent case, it seems there are few results. For example, Röckner et al. [29] study the averaging principle for a class of slow-fast McKean-Vlasov stochastic
differential equations driven by Wiener noise. However unlike the Wiener noise, the cylindrical $\alpha$-stable process only has finite $p$-th moment for $p \in (0, \alpha)$, thus some methods developed in [29] are not suitable to treat the cylindrical $\alpha$-stable noises, therefore we require new and different techniques to deal with the cylindrical $\alpha$-stable noise.

Note that the exponential ergodicity of the transition semigroup of the corresponding frozen equation plays an important role in the proof of strong averaging convergence. However it does not hold if the coefficient $G$ in the fast component depends on the law of the fast component. In this situation, the corresponding frozen equation is also a McKean-Vlasov SPDEs. As a result, the corresponding Markov operator of the frozen equation $P_t f(y) := \mathbb{E} f(Y^{x,\mu,y}_t)$ is not a semigroup anymore (see [37, (1.11)]), where $\{Y^{x,\mu,y}_t\}_{t \geq 0}$ denotes the unique solution of the frozen equation by fixed $x \in H$, $\mu \in \mathcal{P}_p$. Then it will brings some essential difficulties. Consequence, we focus on the coefficients depending only on the law of the slow component here.

The paper is organized as follows. In section 2, we study the well-posedness of a class of McKean-Vlasov SPDEs driven by $\alpha$-stable processes. In sections 3, we study the averaging principle for a class of multiscale Mckean-Vlasov SPDEs. Section 4 is the appendix.

2. Well-posedness of McKean-Vlasov SPDEs driven by $\alpha$-stable process

In this section, we first give some notations and assumptions. Then we prove the existence and uniqueness of the mild solution of a class of McKean-Vlasov SPDEs driven by $\alpha$-stable processes.

2.1. Notations and assumptions. We recall the main equation

$$\left\{ \begin{array}{l}
   dX_t = AX_t dt + B(X_t, \mathcal{L}_{X_t}) dt + dL_t, \\
   X_0 = \xi \in H,
\end{array} \right. \quad \tag{2.1}$$

where $\{L_t\}_{t \geq 0}$ is a cylindrical $\alpha$-stable process with $\alpha \in (1, 2)$, which is given by

$$L_t = \sum_{k=1}^{\infty} \beta_k L^k_t e_k, \quad t \geq 0$$

where $\{e_k\}_{k=1}^{\infty}$ is a complete orthonormal basis of $H$, $\{\beta_k\}_{k=1}^{\infty}$ is a given sequence of positive numbers and $\{L^k_t\}_{k=1}^{\infty}$ is a sequence of independent one dimensional symmetric $\alpha$-stable processes satisfies for any $k \geq 1$ and $t \geq 0$,

$$\mathbb{E}[e^{iL^k_th}] = e^{-t|h|^\alpha}, \quad h \in \mathbb{R}.$$
A3. Assume that \( \sum_{k=1}^\infty \frac{\beta_k^2}{\alpha_k} < \infty \).

For any \( s \in \mathbb{R} \), we define

\[
H^s := \mathcal{D}((−A)^{s/2}) := \left\{ u = \sum_{k=1}^\infty u_k \epsilon_k : u_k \in \mathbb{R}, \sum_{k=1}^\infty \lambda_k^s u_k^2 < \infty \right\}
\]

and

\[
(−A)^{s/2} u := \sum_{k=1}^\infty \lambda_k^{s/2} u_k \epsilon_k, \quad u \in \mathcal{D}((−A)^{s/2}),
\]

with the associated norm \( \|u\|_s := \|−(A)^{s/2}u\| = (\sum_{k=1}^\infty \lambda_k^s u_k^2)^{1/2} \). It is easy to see \( \|\cdot\|_0 = |\cdot| \).

The following smoothing properties of the semigroup \( e^{tA} \) (see [5, Proposition 2.4]) will be used quite often later in this paper:

\[
|e^{tA}x| \leq e^{-\lambda t}|x|, \quad t \geq 0, \quad (2.2)
\]

\[
\|e^{tA}x\|_{\sigma_2} \leq C_{\sigma_1, \sigma_2} t^{\frac{s-\sigma_1}{2}} e^{-\lambda t} \|x\|_{\sigma_1}, \quad x \in H^{\sigma_2}, \sigma_1 \leq \sigma_2, t > 0, \quad (2.3)
\]

\[
|e^{tA}x - x| \leq C_{\sigma,t} t^{\frac{s}{2}} \|x\|_{\sigma}, \quad x \in H^s, \sigma > 0, t \geq 0. \quad (2.4)
\]

2.2. The proof of well-posedness of equation (2.1). In this subsection, we shall prove the well-posedness of equation (2.1). To do this, we first give the definition of the solution.

**Definition 2.1.** We call a predictable \( H \)-valued stochastic process \( \{X_t\}_{t \geq 0} \) defined on the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with the initial value \( \xi \in H \) a mild solution of equation (2.1) if the following statements are satisfied:

i) \( \{X_t\}_{t \geq 0} \) is \( \{\mathcal{F}_t\}_{t \geq 0} \) adapted.

ii) \( \mathbb{P} \left( \int_0^\infty |e^{(t-s)A}B(X_s, \mathcal{L}_s)| ds < \infty \right) = 1, \quad \forall t \geq 0. \)

iii) For any \( t \geq 0 \), \( \{X_t\}_{t \geq 0} \) satisfies

\[
X_t = e^{tA} \xi + \int_0^t e^{(t-s)A}B(X_s, \mathcal{L}_s) ds + \int_0^t e^{(t-s)A}dL_s, \quad \mathbb{P} \text{ a.s..} \quad (2.5)
\]

Next, we present our first main result.

**Theorem 2.2.** Suppose that assumptions **A1-A3** hold. Then for any initial value \( \xi \in H \) satisfying \( \mathbb{E}[|\xi|^p] < \infty \), equation (2.1) has a unique mild solution. Furthermore, if \( \mathbb{E}[|\xi|^m] < \infty \) for some \( m \in [p, \alpha) \), then for any \( T > 0 \), there exists \( C_{T,m} > 0 \) such that

\[
\sup_{t \in [0,T]} (\mathbb{E}[|X_t|^m]^{1/m}) \leq C_T \left[ 1 + (\mathbb{E}[|\xi|^m]^{1/m}) \right]. \quad (2.6)
\]

**Proof.** The detailed proof is divided into three steps.

**Step 1:** For the fixed \( p \in [1, \alpha) \) in assumption **A2**, let \( C([0,T], L^p(\Omega, \mathbb{P}; H)) \) be the Banach space of continuous maps \( \{Z_t\}_{t \geq 0} \) from \([0,T]\) to \( L^p(\Omega, \mathbb{P}; H) \) satisfying \( \sup_{t \in [0,T]} \mathbb{E}[|Z_t|^p] < \infty \). Let \( \Lambda_p \) be the closed subspace of \( C([0,T], L^p(\Omega, \mathbb{P}; H)) \) consisting of measurable and \( \{\mathcal{F}_t\}_{t \geq 0} \) adapted process \( \{X_t\}_{t \geq 0} \). Then \( \Lambda_p \) is a Banach space with the norm topology given by

\[
\|Z\|_{\Lambda_p} := \sup_{t \in [0,T]} e^{-\lambda t} \left[ \mathbb{E}[|Z_t|^p] \right]^{1/p},
\]

where \( \lambda > 0 \) is a large enough constant. We define by \( C([0,T], \mathcal{P}_p) \) the complete metric space of continuous functions from \([0,T]\) to \((\mathcal{P}_p, \mathcal{W}_p)\) with the metric:

\[
D_T(\mu, \nu) := \sup_{t \in [0,T]} e^{-\lambda t} \mathcal{W}_p(\mu_t, \nu_t), \quad \mu, \nu \in C([0,T], \mathcal{P}_p).
\]
For a fixed $\mu \in C([0, T], \mathcal{P}_p)$, we first study the well-posedness of the following equation:

$$dX_t = AX_t dt + B(X_t, \mu_t) dt + dL_t, \quad X_0 = \xi.$$  

(2.7)

Let us define the operator on $\Lambda_p$ as follows:

$$(\Phi_\mu X)(t) = e^{tA} \xi + \int_0^t e^{(t-s)A} B(X_s, \mu_s) ds + \int_0^t e^{(t-s)A} dL_s.$$

On one hand, refer to [25, (4.12)], the assumption A3 implies that

$$\sup_{t \geq 0} \mathbb{E} \left| \int_0^t e^{(t-s)A} dL_s \right|^p \leq C_{\alpha, p} \left( \sum_{k=1}^{\infty} \frac{\beta_k}{\lambda_k^\alpha} \right)^{p/\alpha} < \infty, \quad 0 < p < \alpha. $$

(2.8)

Note that by assumption A2, it is easy to see that

$$|B(x, \mu)| \leq C \left[ 1 + |x| + \mu(| \cdot |^{p/\alpha}) \right], \quad \forall x \in H, \mu \in \mathcal{P}_p.$$  

(2.9)

Then by (2.2), (2.8), (2.9) and Minkowski’s inequality, it follows for any $t \in [0, T]$

$$\mathbb{E} |(\Phi_\mu X)(t)|^p \leq \mathbb{E} \left( e^{tA} \xi \right)^p + \int_0^t \mathbb{E} \left| e^{(t-s)A} B(X_s, \mu_s) \right|^p ds + \left( \mathbb{E} \int_0^t e^{(t-s)A} dL_s \right)^{p/\alpha}$$

$$\leq \mathbb{E} \xi^p + C \int_0^t \mathbb{E} (X_s)^p + [\mu_s(| \cdot |^{p/\alpha})]^{p/\alpha} ds + \left( \mathbb{E} \int_0^t e^{(t-s)A} dL_s \right)^{p/\alpha}$$

$$\leq C_{T, p},$$

which implies that $\|\Phi_\mu X\|_{\Lambda_p} < \infty$. It is easy to see that $(\Phi_\mu X)(t)$ is $\{F_t\}_{t \geq 0}$ adapted process if $\{X_t\}_{t \geq 0}$ is adapted, thus

$$\Phi_\mu : \Lambda_p \to \Lambda_p.$$  

(2.10)

On the other hand, for any $X, Y \in \Lambda_p$ with $X_0 = Y_0$, we have for any $t \in [0, T]$,

$$e^{-\lambda t} \mathbb{E} |(\Phi_\mu X)(t) - (\Phi_\mu Y)(t)|^p \leq e^{-\lambda t} \left[ \mathbb{E} \left| \int_0^t e^{(t-s)A} [B(X_s, \mu_s) - B(Y_s, \mu_s)] ds \right|^p \right]^{1/p}$$

$$\leq e^{-\lambda t} \int_0^t C (\mathbb{E} (X_s - Y_s)^p)^{1/p} ds$$

$$\leq C \int_0^t e^{-\lambda (t-s)} ds \|X - Y\|_{\Lambda_p}$$

$$\leq \frac{C}{\lambda} \|X - Y\|_{\Lambda_p}.$$  

Let us choose $\lambda$ large enough such that $c_0 := \frac{C}{\lambda} < 1$, thus it follows

$$\|\Phi_\mu X - \Phi_\mu Y\|_{\Lambda_p} \leq c_0 \|X - Y\|_{\Lambda_p}. $$

(2.11)

Hence (2.10) and (2.11) yield that $\Phi_\mu$ is a contraction map on $\Lambda_p$. As a result, $\Phi_\mu$ has a unique fixed point $X_\mu$ in $\Lambda_p$, which is the unique solution of equation (2.7), i.e.,

$$X_\mu(t) = e^{tA} \xi + \int_0^t e^{(t-s)A} B(X_\mu(s), \mu_s) ds + \int_0^t e^{(t-s)A} dL_s.$$

(2.12)

**Step 2:** We define an operator $\Psi$ on $\mu \in C([0, T], \mathcal{P}_p)$ by

$$\Psi : \mu \to \mathcal{L}_{X_\mu},$$

where $\mathcal{L}_{X_\mu} = \{\mathcal{L}_{X_\mu(t)}, t \in [0, T]\}$ is the law of process $\{X_\mu(t)\}_{t \geq 0}$. Obviously, if $X$ is a mild solution of equation (2.1). Then its Law $\{\mu_t := \mathcal{L}_{X_\mu(t)}\}_{t \geq 0}$ is a fixed point of $\Psi$. Thus in
order to complete the proof, it is sufficient to show that the operator $\Psi$ has a unique fixed point.

For any $\mu \in C([0, T], \mathcal{P}_p)$, we first prove that $\Psi(\mu) \in C([0, T], \mathcal{P}_p)$. By Step 1, it is easy to see that $\sup_{t \in [0, T]} \mathbb{E}|X_\mu(t)|^p < \infty$. Thus it remains to show that $t \to L_{X_\mu(t)}$ is continuous in $(\mathcal{P}_p, \mathbb{W}_p)$.

In fact, note that for any $0 \leq t \leq t + h \leq T$,

$$X_\mu(t + h) - X_\mu(t) = (e^{hA} - I)X_\mu(t) + \int_t^{t+h} e^{(t-h-s)A} B(X_\mu(s), L_{X_\mu(s)}) ds + \int_t^{t+h} e^{(t-h-s)A} dL_s.$$ 

By dominated convergence theorem, we have

$$\lim_{h \to 0} \left[ \mathbb{E}\left| (e^{hA} - I)X_\mu(t) \right|^p \right]^{1/p} = 0. \quad (2.13)$$

By (2.2) and (2.9), it is easy to see

$$\lim_{h \to 0} \left[ \mathbb{E}\left| \int_t^{t+h} e^{(t-h-s)A} B(X_s, L_{X_\mu(s)}) ds \right|^p \right]^{1/p} \leq \lim_{h \to 0} Ch \left( 1 + \sup_{t \in [0, T]} \mathbb{E}|X_\mu(t)|^p \right)^{1/p} = 0. \quad (2.14)$$

Define $\{ \tilde{L}_t \}_{t \geq 0} := \{ L_{t+h} - L_h \}_{t \geq 0}$, which is also a cylindrical $\alpha$-stable process. Then refer to [25, (4.12)], we have

$$\mathbb{E}\left| \int_t^{t+h} e^{(t-h-s)A} dL_s \right|^p = \mathbb{E}\left| \int_0^h e^{(h-s)A} d\tilde{L}_s \right|^p \leq \left[ \sum_{k=1}^{\infty} \frac{(1 - e^{-\alpha \lambda_k h})^\beta_k}{\alpha \lambda_k} \right]^{p/\alpha}.$$ 

Then by dominated convergence theorem and assumption $A_3$, we get

$$\lim_{h \to 0} \left[ \mathbb{E}\left| \int_t^{t+h} e^{(t-h-s)A} dL_s \right|^p \right]^{1/p} \leq \lim_{h \to 0} C \left[ \sum_{k=1}^{\infty} \frac{(1 - e^{-\alpha \lambda_k h})^\beta_k}{\alpha \lambda_k} \right]^{1/\alpha} = 0. \quad (2.15)$$

Thus by (2.14)-(2.15), it is easy to see that

$$\lim_{h \to 0} \mathbb{W}_p(\mathcal{L}_{X_\mu(t+h)}, \mathcal{L}_{X_\mu(t)}) \leq \lim_{h \to 0} \left[ \mathbb{E}|X_\mu(t + h) - X_\mu(t)|^p \right]^{1/p}$$

$$\leq \lim_{h \to 0} \left\{ \mathbb{E}\left| (e^{hA} - I)X_\mu(t) \right|^p \right\}^{1/p} + \mathbb{E}\left[ \int_t^{t+h} e^{(t-h-s)A} dL_s \right]^p$$

$$+ \left[ \mathbb{E}\left| \int_t^{t+h} e^{(t-h-s)A} B(X_\mu(s), L_{X_\mu(s)}) ds \right|^p \right]^{1/p} = 0.$$ 

Next, we shall prove that $\Psi$ is a contraction operator. Let $X_\mu$ and $X_\nu$ be the corresponding solutions of equation of (2.7) for $\mu, \nu \in C([0, T], \mathcal{P}_p)$ respectively. Then it follows

$$\left[ \mathbb{E}|X_\mu(t) - X_\nu(t)|^p \right]^{1/p} \leq \left[ \mathbb{E}\left| \int_0^t e^{(s-t)A} (B(X_\mu(s), \mu_s) - B(X_\nu(s), \nu_s)) ds \right|^p \right]^{1/p}$$

$$\leq C \int_0^t \left[ \mathbb{E}|X_\mu(s) - X_\nu(s)|^p \right]^{1/p} + \mathbb{W}_p(\mu_s, \nu_s) ds,$$

which implies

$$\sup_{t \in [0, T]} e^{-\lambda t} \left[ \mathbb{E}|X_\mu(t) - X_\nu(t)|^p \right]^{1/p} \leq C \sup_{s \in [0, T]} e^{-\lambda s} \left[ \mathbb{E}|X_\mu(s) - X_\nu(s)|^p \right]^{1/p} + C \sup_{s \in [0, T]} e^{-\lambda s} \mathbb{W}_p(\mu_s, \nu_s).$$
Choose \( \lambda > 0 \) large enough such that \( \frac{C}{\lambda} < 1/2 \), then we get
\[
\sup_{t \in [0,T]} e^{-\lambda t} \left[ \mathbb{E}[X_\mu(t) - X_\nu(t)]^p \right]^{1/p} \leq \frac{2C}{\lambda} \sup_{t \in [0,T]} e^{-\lambda t} \mathbb{W}_p(\mu_t, \nu_t).
\]

Note that \( \mathbb{W}_p(\mathcal{L}_{X_\mu(t)}, \mathcal{L}_{X_\nu(t)}) \leq \left[ \mathbb{E}[X_\mu(t) - X_\nu(t)]^p \right]^{1/p} \), it follows
\[
D_T(\Psi(\mu), \Psi(\nu)) \leq \frac{2C}{\lambda} D_T(\mu, \nu).
\]

Hence \( \Psi \) is a contraction operator in \( C([0,T], \mathscr{P}_p) \).

**Step 3:** Let \( X_t \) be the unique solution of equation (2.1). By [25, (4.12)], it follows for any \( m \in [p, \alpha) \) and \( t \in [0,T] \), we have
\[
\left( \mathbb{E}|X_t|^m \right)^{1/m} \leq \left( \mathbb{E}|e^{A\xi}|^m \right)^{1/m} + \int_0^t \left[ \mathbb{E}|e^{(t-s)A}B(X_s, \mathcal{L}_{X_s})|^m \right]^{1/m} ds + \left[ \mathbb{E} \left| \int_0^t e^{(t-s)A}dL_s \right|^m \right]^{1/m} \]
\[
\leq \left( \mathbb{E}|\xi|^m \right)^{1/m} + \int_0^t C \left( \mathbb{E}|X_s|^m \right)^{1/m} ds + C \left( \mathbb{E}|X_s|^p \right)^{1/p} ds + \left[ \mathbb{E} \left| \int_0^t e^{(t-s)A}dL_s \right|^m \right]^{1/m} \]
\[
\leq C \left[ 1 + \left( \mathbb{E}|\xi|^m \right)^{1/m} \right] + \int_0^t C \left( \mathbb{E}|X_s|^m \right)^{1/m} ds.
\]

Then by Gronwall’s inequality, it is easy to see
\[
\left( \mathbb{E}|X_t|^m \right)^{1/m} \leq Ce^{Ct} \left[ 1 + \left( \mathbb{E}|\xi|^m \right)^{1/m} \right].
\]

The proof is complete. \( \square \)

3. Averaging principle for multiscale Mckean-Vlasov SPDEs driven by \( \alpha \)-stable processes

In this section, we further study a class of multiscale Mckean-Vlasov SPDEs driven by \( \alpha \)-stable processes. Under some proper assumptions, we will prove the strong averaging principle holds with some order \( r > 0 \). The main technique is based on the classical Khasminskii’s time discretization method.

3.1. Assumptions and main result. We recall the multiscale Mckean-Vlasov SPDEs driven by \( \alpha \)-stable processes in the Hilbert space \( H \):
\[
\begin{align*}
\left\{ \begin{array}{l}
dX_t^\xi = \left[ AX_t^\xi + F(X_t^\xi, \mathcal{L}_{X_t^\xi}, Y_t^\xi) \right] dt + dL_t, \quad X_0^\xi = \xi \in H, \\
\quad dY_t^\xi = \frac{1}{\alpha} [AY_t^\xi + G(X_t^\xi, \mathcal{L}_{X_t^\xi}, Y_t^\xi)] dt + \frac{1}{\epsilon^{1/\alpha}} dZ_t, \quad Y_0^\xi = \eta \in H,
\end{array} \right.
\end{align*}
\]
where operator \( A \) satisfies assumption A1. \( \xi, \eta \) are two \( H \)-valued random variable. \( \{L_t\}_{t \geq 0} \) and \( \{Z_t\}_{t \geq 0} \) are two mutually independent cylindrical \( \alpha \)-stable processes with \( \alpha \in (1,2) \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with natural filtration \( \{\mathcal{F}_t := \sigma(\xi, \eta, L_s, Z_s, s \leq t)\}_{t \geq 0} \), i.e.,
\[
L_t = \sum_{k=1}^{\infty} \beta_k L_t^k e_k, \quad Z_t = \sum_{k=1}^{\infty} \gamma_k Z_t^k e_k, \quad t \geq 0,
\]
where \( \{\beta_k\}_{k \in \mathbb{N}_+} \) and \( \{\gamma_k\}_{k \in \mathbb{N}_+} \) are two given sequence of positive numbers, \( \{L_t^k\}_{k \in \mathbb{N}_+} \) and \( \{Z_t^k\}_{k \in \mathbb{N}_+} \) are two sequences of independent one dimensional symmetric \( \alpha \)-stable processes satisfying for any \( k \in \mathbb{N}_+ \) and \( t \geq 0 \),
\[
\mathbb{E}[e^{tL_t^k}] = \mathbb{E}[e^{tZ_t^k}] = e^{-t|h|^\alpha}, \quad h \in \mathbb{R}.
\]

Now, we suppose the following conditions hold throughout this section:
B1. Suppose that there exists a constant \( p \in [1, \alpha) \) such that
\[
F(\cdot, \cdot, \cdot) : H \times \mathcal{P}_p \times H \to H,
\]
\[
G(\cdot, \cdot, \cdot) : H \times \mathcal{P}_p \times H \to H.
\]
Furthermore, there exist constants \( C, L_G > 0 \) such that for any \( x_i, y_i \in H \) and \( \mu_i \in \mathcal{P}_p, i = 1, 2, \)
\[
|F(x_1, \mu_1, y_1) - F(x_2, \mu_2, y_2)| \leq C \left[ |x_1 - x_2| + \mathbb{W}_p(\mu_1, \mu_2) + |y_1 - y_2| \right],
\]
\[
|G(x_1, \mu_1, y_1) - G(x_2, \mu_2, y_2)| \leq C \left[ |x_1 - x_2| + \mathbb{W}_p(\mu_1, \mu_2) + L_G|y_1 - y_2| \right].
\]

B2. There exists \( \theta \in (0, 2/\alpha] \) such that \( \sum_{k=1}^{\infty} \frac{\beta_k^\theta}{\lambda_k^{1-\theta/2}} < \infty \) and \( \sum_{k=1}^{\infty} \frac{\gamma_k^\theta}{\lambda_k^{1-\theta/2}} < \infty \).

B3. Suppose that \( \lambda_1 - L_G > 0 \) and there exists \( C > 0 \) such that
\[
\sup_{x, y \in H} |F(x, \mu, y)| \leq C \left[ 1 + (\mu(\cdot | \cdot)^{1/p}) \right].
\]

The following are some comments on the assumptions above:

Remark 3.1. Suppose that assumptions A1, B1 and B2 holds. By Theorem 2.2, for any given \( \varepsilon > 0 \) and initial value \((\xi, \eta) \in H \times H\) satisfying \( \mathbb{E}|\xi|^p < \infty \) and \( \mathbb{E}|\eta|^p < \infty \), equation (3.1) admits a unique mild solution \((X^\varepsilon_t, Y^\varepsilon_t)\), i.e., \( \mathbb{P}\)-a.s.,
\[
\begin{cases}
X^\varepsilon_t = e^{tA}\xi + \int_0^t e^{(t-s)A}F(X^\varepsilon_s, \mathcal{L}_t, Y^\varepsilon_s)ds + \int_0^t e^{(t-s)A}dL_s, \\
Y^\varepsilon_t = e^{tA/\varepsilon}\eta + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}G(X^\varepsilon_s, \mathcal{L}_t, Y^\varepsilon_s)ds + \frac{1}{\varepsilon^{2\alpha}} \int_0^t e^{(t-s)A/\varepsilon}dZ_s.
\end{cases}
\]

Remark 3.2. The condition \( \lambda_1 - L_G > 0 \) in assumption B3 is called the strong dissipative condition, which is used to prove the existence and uniqueness of the invariant measures and the exponential ergodicity of the transition semigroup of the frozen equation.

Remark 3.3. The method used here is the classical Khasminskii’s time discretization, which highly depends on the square calculation in the proof, hence the finite second moment of the solution \( X^\varepsilon_t \) is required usually. But the solution \( X^\varepsilon_t \) for system (1.1) only has finite \( p \)-th moment \( 0 < p < \alpha \), thus the condition (3.2) in assumption B3 is used to weaken the required finite second moment to finite first moment. However by the technique of Poisson equation (see e.g. [29, 33]), the condition (3.2) could be removed, which is left to our further work.

Remark 3.4. Refer to [26, Lemma 4.1], if \( \sum_{k=1}^{\infty} \frac{\beta_k^\theta}{\lambda_k^{1-\theta/2}} < \infty \) holds for some \( \theta \geq 0 \) then for any \( 0 < m < \alpha \), we have
\[
\sup_{t \geq 0} \mathbb{E} \left\| \int_0^t e^{(t-s)A}dL_s \right\|^m \leq C_{\alpha, p} \left( \sum_{k=1}^{\infty} \frac{\beta_k^\theta}{\lambda_k^{1-\theta/2}} \right)^{m/\alpha}.
\]

Now, we state our second result in this paper, whose proof is left in subsection 3.3 below.

Theorem 3.5. Suppose that assumptions A1 and B1-B3 hold. Then for any \( T > 0, \xi, \eta \in H \) satisfying \( \mathbb{E}|\xi|^m < \infty \) for some \( m \in [p, \alpha) \) and \( \mathbb{E}|\eta|^p < \infty \), then we have
\[
\left[ \mathbb{E} \left( \sup_{t \in [0, T]} |X^\varepsilon_t - \bar{X}_t|^m \right)^{1/m} \right] \leq C_T \left[ 1 + (\mathbb{E}|\xi|^m)^{1/m} + (\mathbb{E}|\eta|^p)^{1/p} \right] \varepsilon^{\frac{\theta}{2(1+\theta)}}.
\]
Remark 3.6. In contrast to the model in [3], the coefficients \( F \) and \( G \) in equation (3.1) all depend on the distribution. Meanwhile, the main result (3.5) implies that the strong convergence order is \( \frac{\theta}{2(1+\theta)} \), which depends on the regular assumption on the noise in the slow component.

3.2. Some a priori estimates. In this subsection, we will prove some a priori estimates of the solutions \( (X^\varepsilon_t, Y^\varepsilon_t) \) and an auxiliary process \( \tilde{Y}^\varepsilon_t \). Note that we always assume that the initial value \( \xi, \eta \) satisfying \( \mathbb{E}|\xi|^m < \infty \) for some \( m \in [p, \alpha) \) and \( \mathbb{E}|\eta|^p < \infty \).

Lemma 3.7. For any \( T > 0 \), there exists a constant \( C_T > 0 \) such that

\[
\sup_{\varepsilon \in (0,1), t \in [0,T]} (\mathbb{E}|X^\varepsilon_t|^m)^{1/m} \leq C_T \left[ 1 + (\mathbb{E}|\xi|^m)^{1/m} \right] \tag{3.6}
\]

and

\[
\sup_{\varepsilon \in (0,1), t \in [0,T]} (\mathbb{E}|Y^\varepsilon_t|^p)^{1/p} \leq C_T \left[ 1 + (\mathbb{E}|\xi|^p)^{1/p} + (\mathbb{E}|\eta|^p)^{1/p} \right]. \tag{3.7}
\]

Proof. Recall that

\[ X^\varepsilon_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(X^\varepsilon_s, \mathcal{L}X^\varepsilon_s, Y^\varepsilon_s) ds + \int_0^t e^{(t-s)A} dL_s. \]

Then by (3.2), (3.4) and Minkowski's inequality, we have

\[
\sup_{t \in [0,T]} (\mathbb{E}|X^\varepsilon_t|^m)^{1/m} \leq (\mathbb{E}|\xi|^m)^{1/m} + C \int_0^T (\mathbb{E}|X^\varepsilon_s|^p)^{1/p} ds + \sup_{t \in [0,T]} \left[ \mathbb{E} \left| \int_0^t e^{(t-s)A} dL_s \right|^m \right]^{1/m}
\]

\[
\leq C \left[ 1 + (\mathbb{E}|\xi|^m)^{1/m} \right] + C \int_0^T (\mathbb{E}|X^\varepsilon_t|^m)^{1/m} dt + C \left( \sum_{k=1}^{\infty} \frac{\beta_k^p}{\lambda_k^{1-a\theta/2}} \right)^{m/\alpha}. \tag{3.6}
\]

Then the Gronwall’s inequality implies that (3.6) holds.

Now, we proceed to show estimate (3.7). Recall that

\[ Y^\varepsilon_t = e^{tA/\varepsilon}\eta + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon} G(X^\varepsilon_s, \mathcal{L}X^\varepsilon_s, Y^\varepsilon_s) ds + \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon} dZ_s. \]

Note that by assumption B1, it is easy to see that

\[ |G(x, \mu, y)| \leq C \left[ 1 + |x| + \mu(|\cdot|^p)^{1/p} \right] + L_G|y|, \quad \forall x, y \in H, \mu \in \mathcal{P}_p. \tag{3.8}
\]

Then by (2.2) and (3.8), we have for any \( t \geq 0 \),

\[ |Y^\varepsilon_t| \leq \left| \eta \right| + \frac{1}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} (C + C|X^\varepsilon_s| + C (\mathbb{E}|X^\varepsilon_s|^p)^{1/p} + L_G|Y^\varepsilon_s|) ds + \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon} dZ_s. \]

Define \( \tilde{Z}_t := \frac{1}{\varepsilon^{1/\alpha}} Z_{t\varepsilon} \), which is also a cylindrical \( \alpha \)-stable process. Then by [25, (4.12)],

\[
\mathbb{E} \left| \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon} dZ_s \right|^p \leq C \left( \sum_{k=1}^{\infty} \frac{\gamma_k^\alpha}{\alpha \lambda_k} \right)^{p/\alpha} \leq C \left( \sum_{k=1}^{\infty} \frac{\gamma_k^\alpha}{\alpha \lambda_k} \right)^{p/\alpha}.
\]
which together with (3.6), by Minkowski’s inequality, we have for any \( t \leq T \),
\[
(\mathbb{E}|Y_t^\varepsilon|^p)^{1/p} \leq (\mathbb{E}|\eta|^p)^{1/p} + \frac{C}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} (\mathbb{E}|X_s^\varepsilon|^p)^{1/p} \, ds \\
+ \frac{L_G}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} (\mathbb{E}|Y_s^\varepsilon|^p)^{1/p} \, ds + \left[ \mathbb{E}\left[ \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)\lambda_1/\varepsilon} dZ_s \right] \right]^{1/p} \\
\leq C_T \left[ 1 + (\mathbb{E}|\eta|^p)^{1/p} + (\mathbb{E}|\eta|^p)^{1/p} \right] + \frac{L_G}{\lambda_1} \sup_{0 \leq t \leq T} (\mathbb{E}|Y_t^\varepsilon|^p)^{1/p}.
\]

Hence by the condition \( L_G < \lambda_1 \) in assumption \textbf{B3}, it is easy to see that (3.7) holds. The proof is complete. \( \square \)

**Remark 3.8.** Note that if without the condition (3.2). Then by (2.9), we can obtain following a prior estimate:
\[
\sup_{\varepsilon \in (0,1), t \in (0,T]} \left[ (\mathbb{E}|X_t^\varepsilon|^m)^{1/m} + (\mathbb{E}|Y_t^\varepsilon|^m)^{1/m} \right] \leq C_T \left[ 1 + (\mathbb{E}|\eta|^m)^{1/m} + (\mathbb{E}|\eta|^m)^{1/m} \right].
\]

**Lemma 3.9.** For any \( x, y \in H \) and \( T > 0 \), there exists a constant \( C_T > 0 \) such that for any \( \varepsilon \in (0,1) \), we have
\[
\int_0^T \left[ \mathbb{E}|X_t^\varepsilon - X_{\delta t}^\varepsilon|^m \right]^{1/m} \, dt \leq C_T \delta^{\frac{\theta}{2}} \left[ 1 + (\mathbb{E}|\xi|^m)^{1/m} \right], \tag{3.9}
\]
where \( \theta \) is the one in assumption \textbf{B2} and \( t(\delta) := \lfloor \frac{t}{\delta} \rfloor \delta \) with \( \lfloor \frac{t}{\delta} \rfloor \) is the integer part of \( \frac{t}{\delta} \).

**Proof.** By (3.6) and Minkowski’s inequality, it is easy to check that
\[
\int_0^T \left[ \mathbb{E}|X_t^\varepsilon - X_{\delta t}^\varepsilon|^m \right]^{1/m} \, dt = \int_0^\delta \left[ \mathbb{E}|X_t^\varepsilon - \xi|^m \right]^{1/m} \, dt + \int_\delta^T \left[ \mathbb{E}|X_t^\varepsilon - X_{\delta t}^\varepsilon|^m \right]^{1/m} \, dt \\
\leq C_T \delta \left[ 1 + (\mathbb{E}|\xi|^m)^{1/m} \right] + \int_\delta^T \left[ \mathbb{E}|X_t^\varepsilon - X_{t-\delta}^\varepsilon|^m \right]^{1/m} \, dt \\
+ \int_\delta^T \left[ \mathbb{E}|X_t^\varepsilon - X_{t-\delta}^\varepsilon|^m \right]^{1/m} \, dt. \tag{3.10}
\]

Recall that the mild solution \( X_t^\varepsilon \) in (3.3). Then by (2.3), (3.2) and Remark 3.4, it is easy to see that for any \( \theta \in (0, 2/\alpha), t \in (0, T] \) and \( m \in [p, \alpha) \),
\[
(\mathbb{E}|X_t^\varepsilon|^m)_\theta^{1/m} \leq (\mathbb{E}|e^{(t-s)A}X_s|^m)_\theta^{1/m} + \left[ \mathbb{E}\left[ \int_0^t (-A)^{\theta/2} e^{(t-s)A} F(X_s^\varepsilon, L_{X_s^\varepsilon}, Y_s^\varepsilon) \, ds \right]^m \right]^{1/m} \\
+ \left[ \mathbb{E}\left[ \int_0^t e^{(t-s)A} dL_s \right]^m \right]^{1/m} \\
\leq C t^{-\theta/2} (\mathbb{E}|\xi|^m)^{1/m} + C \int_0^t (t-s)^{-\theta/2} (\mathbb{E}|X_s^\varepsilon|^p)^{1/p} \, ds \\
+ C \left( \sum_{k=1}^{\infty} \frac{\beta_k^\alpha}{\lambda_k^{1-\alpha \theta/2}} \right)^{1/\alpha} \\
\leq C t^{-\theta/2} (\mathbb{E}|\xi|^m)^{1/m} + C_T. \tag{3.11}
\]

Note that
\[
X_t^\varepsilon - X_{t-\delta}^\varepsilon = (e^{A\delta} - I)X_{t-\delta}^\varepsilon + \int_{t-\delta}^t e^{(t-s)A} F(X_s^\varepsilon, L_{X_s^\varepsilon}, Y_s^\varepsilon) \, ds + \int_{t-\delta}^t e^{(t-s)A} dL_s.
\]
Then by \((2.4)\) and Minkowski’s inequality, we have
\[
\int_{\delta}^{T} \left[ \mathbb{E}[|e^{A\delta} - I|X_{t-\delta}^\varepsilon|^m] \right]^{1/m} dt \leq C\delta^{\frac{2}{\alpha}} \int_{\delta}^{T} \left( \mathbb{E}[|X_{t-\delta}^\varepsilon|^m] \right)^{1/m} dt
\]
\[
\leq C\delta^{\frac{2}{\alpha}} \left[ \int_{\delta}^{T} (t-\delta)^{-\theta/2} dt \left( \mathbb{E}[|\xi|^m] \right)^{1/m} + C_T \right]
\]
\[
\leq C_T \delta^{\frac{2}{\alpha}} \left[ 1 + \left( \mathbb{E}[|\xi|^m] \right)^{1/m} \right] . \tag{3.12}
\]
By \((3.2)\), it follows
\[
\int_{\delta}^{T} \left[ \mathbb{E} \left[ \left| \int_{t-\delta}^{t} e^{(t-s)A} F(X_{s}^\varepsilon, \mathcal{L}X_{s}^\varepsilon, Y_{s}^\varepsilon) ds \right|^m \right] \right]^{1/m} dt \leq C_T \delta \left[ 1 + \left( \mathbb{E}[|\xi|^m] \right)^{1/m} \right] . \tag{3.13}
\]
By \((3.4)\) and assumption \textbf{B2}, we have
\[
\int_{\delta}^{T} \left[ \mathbb{E} \left[ \left| \int_{t-\delta}^{t} e^{(t-s)A} dL_{s} \right|^m \right] \right]^{1/m} dt \leq C \int_{\delta}^{T} \left[ \sum_{k=1}^{\infty} \beta_k^\alpha \frac{(1 - e^{-\lambda_k \delta})^\alpha}{\lambda_k} \right]^{1/\alpha} dt
\]
\[
\leq C \delta^{\frac{2}{\alpha}} \int_{\delta}^{T} \left( \sum_{k=1}^{\infty} \frac{\beta_k^\alpha}{\lambda_k^{1-\theta/2}} \right)^{1/\alpha} dt \leq C_T \delta^{\frac{2}{\alpha}} , \tag{3.14}
\]
where we use the fact that \(1 - e^{-x} \leq Cx^{\alpha\theta/2}\) for any \(x > 0\).

Combining \((3.12)-(3.14)\), we obtain
\[
\int_{\delta}^{T} \left( \mathbb{E}[|X_{t}^\varepsilon - X_{t-\delta}^\varepsilon|^m] \right)^{1/m} dt \leq C_T \delta^{\frac{2}{\alpha}} \left[ 1 + \left( \mathbb{E}[|\xi|^m] \right)^{1/m} \right] . \tag{3.15}
\]
Similar as the argument above, we also have
\[
\int_{\delta}^{T} \left[ \mathbb{E}[|X_{(t-\delta)}^\varepsilon - X_{t-\delta}^\varepsilon|^m] \right]^{1/m} dt \leq C_T \delta^{\frac{2}{\alpha}} \left[ 1 + \left( \mathbb{E}[|\xi|^m] \right)^{1/m} \right] . \tag{3.16}
\]
Finally, \((3.10), (3.15)\) and \((3.16)\) imply \((3.9)\) holds. The proof is complete. \qed

Inspired from the idea introduced by Khasminskii in \cite{18}, we construct an auxiliary process \(\hat{Y}_{t}^\varepsilon \in H\), i.e., we split \([0, T]\) into some subintervals of size \(\delta > 0\), where \(\delta > 0\) depends on \(\varepsilon\) and will be chosen later. With the initial value \(\hat{Y}_{0}^\varepsilon = Y_{0}^\varepsilon = \xi\), we construct the process \(\hat{Y}_{t}^\varepsilon\) on each time interval \([l\delta, (l + 1)\delta \wedge T], l \in \mathbb{N}\),
\[
d\hat{Y}_{t}^\varepsilon = \frac{1}{\varepsilon} \left[ A\hat{Y}_{t}^\varepsilon + G(X_{l\delta}^\varepsilon, \mathcal{L}X_{l\delta}^\varepsilon, \hat{Y}_{l\delta}^\varepsilon) \right] dt + \frac{1}{\varepsilon^{1/\alpha}} dZ_{t}, \quad t \in [l\delta, (l + 1)\delta \wedge T],
\]
i.e.,
\[
d\hat{Y}_{t}^\varepsilon = \frac{1}{\varepsilon} \left[ A\hat{Y}_{t}^\varepsilon + G(X_{t\delta}^\varepsilon, \mathcal{L}X_{t\delta}^\varepsilon, \hat{Y}_{t\delta}^\varepsilon) \right] dt + \frac{1}{\varepsilon^{1/\alpha}} dZ_{t},
\]
which satisfies for any \(t \in [0, T]\)
\[
\hat{Y}_{t}^\varepsilon = e^{tA/\varepsilon}\xi + \frac{1}{\varepsilon} \int_{0}^{t} e^{(t-s)A/\varepsilon} G(X_{s\delta}^\varepsilon, \mathcal{L}X_{s\delta}^\varepsilon, \hat{Y}_{s\delta}^\varepsilon) ds + \frac{1}{\varepsilon^{1/\alpha}} \int_{0}^{t} e^{(t-s)A/\varepsilon} dZ_{s}. \tag{3.17}
\]
By following the same argument as in the proof of \((3.7)\), we have
\[
\sup_{\varepsilon \in (0, 1), t \in [0, T]} \left( \mathbb{E}[|\hat{Y}_{t}^\varepsilon|^p] \right)^{1/p} \leq C_T \left[ 1 + \left( \mathbb{E}[|\xi|^p] \right)^{1/p} + \left( \mathbb{E}[|\eta|^p] \right)^{1/p} \right] . \tag{3.18}
\]
Lemma 3.10. For any $T > 0$, there exists a constant $C_T > 0$ such that
$$
\int_0^T \left( \mathbb{E}|Y_{t\varepsilon} - \hat{Y}_{t\varepsilon}|^m \right)^{1/m} dt \leq C_T \delta^{\frac{m}{2}} \left[ 1 + (\mathbb{E}|\xi|^m)^{1/m} \right].
$$

Proof. By the construction of $Y_{t\varepsilon}$ and $\hat{Y}_{t\varepsilon}$, we have
$$
Y_{t\varepsilon} - \hat{Y}_{t\varepsilon} = \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon} \left[ G(X_{s\varepsilon}, \mathcal{L}_{X_{s\varepsilon}, Y_{s\varepsilon}}) - G(X_{s\varepsilon}, \mathcal{L}_{X_{s\varepsilon}, \hat{Y}_{s\varepsilon}}) \right] ds.
$$

Then for any $t > 0$,
$$
|Y_{t\varepsilon} - \hat{Y}_{t\varepsilon}| \leq \frac{1}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} \left[ C|X_{s\varepsilon} - X_{s(\delta)}\varepsilon| + \left( \mathbb{E}|X_{s\varepsilon} - X_{s(\delta)}\varepsilon|^p \right)^{1/p} \right] ds
$$
$$
+ \frac{1}{\varepsilon} \int_0^t e^{-\lambda_1(t-s)/\varepsilon} L_G|Y_{s\varepsilon} - \hat{Y}_{s\varepsilon}| ds.
$$

By Fubini’s theorem, we have
$$
\int_0^T |Y_{t\varepsilon} - \hat{Y}_{t\varepsilon}| dt \leq \frac{1}{\varepsilon} \int_0^T \int_0^t e^{-\lambda_1(t-s)/\varepsilon} \left[ C|X_{s\varepsilon} - X_{s(\delta)}\varepsilon| + \left( \mathbb{E}|X_{s\varepsilon} - X_{s(\delta)}\varepsilon|^p \right)^{1/p} \right] ds dt
$$
$$
+ \frac{1}{\varepsilon} \int_0^T \int_0^t e^{-\lambda_1(t-s)/\varepsilon} L_G|Y_{s\varepsilon} - \hat{Y}_{s\varepsilon}| ds dt
$$
$$
= \frac{C}{\varepsilon} \int_0^T \left( \int_s^T e^{-\lambda_1(t-s)/\varepsilon} dt \right) |X_{s\varepsilon} - X_{s(\delta)}\varepsilon| ds
$$
$$
+ \frac{C}{\varepsilon} \int_0^T \left( \int_s^T e^{-\lambda_1(t-s)/\varepsilon} dt \right) \left( \mathbb{E}|X_{s\varepsilon} - X_{s(\delta)}\varepsilon|^p \right)^{1/p} ds
$$
$$
+ \frac{L_G}{\varepsilon} \int_0^T \left( \int_s^T e^{-\lambda_1(t-s)/\varepsilon} dt \right) |Y_{s\varepsilon} - \hat{Y}_{s\varepsilon}| ds
$$
$$
\leq \frac{C}{\lambda_1} \int_0^T |X_{s\varepsilon} - X_{s(\delta)}\varepsilon| ds + \frac{C}{\lambda_1} \int_0^T \left( \mathbb{E}|X_{s\varepsilon} - X_{s(\delta)}\varepsilon|^p \right)^{1/p} ds + \frac{L_G}{\lambda_1} \int_0^T |Y_{s\varepsilon} - \hat{Y}_{s\varepsilon}| ds.
$$

By Minkowski’s inequality, Lemma 3.9 and $L_G < \lambda_1$, it is easy to see
$$
\int_0^T \left( \mathbb{E}|Y_{t\varepsilon} - \hat{Y}_{t\varepsilon}|^m \right)^{1/m} dt \leq C_T \delta^{\frac{m}{2}} \left[ 1 + (\mathbb{E}|\xi|^m)^{1/m} \right].
$$

The proof is complete. \qed

3.3. The frozen and averaged equation. For any fixed $x \in H$ and $\mu \in \mathcal{P}_p$, we consider the following frozen equation:
$$
\begin{cases}
    dY_t = [AY_t + G(x, \mu, Y_t)] dt + dZ_t, \\
    Y_0 = y \in H.
\end{cases}
$$

(3.19)

Since $G(x, \mu, \cdot)$ is Lipschitz continuous, it is easy to prove that equation (3.19) has a unique mild solution denoted by $Y_{t\varepsilon, \mu, y}$, which is a time homogeneous Markovian process. By a straightforward computation, it is easy to prove
$$
\sup_{t \geq 0} \left( \mathbb{E}|Y_{t\varepsilon, \mu, y}|^p \right)^{1/p} \leq C \left[ 1 + |x| + |y| + (\mu \cdot |y|)^{1/p} \right].
$$

(3.20)

Let $P_{t\varepsilon, \mu}$ be the transition semigroup of $Y_{t\varepsilon, \mu, y}$, that is, for any bounded measurable function $\varphi$ on $H$ and $t \geq 0$,
$$
P_{t\varepsilon, \mu} \varphi(y) = \mathbb{E}\varphi(Y_{t\varepsilon, \mu, y}), \quad y \in H.
$$
The asymptotic behavior of $P^x_t$ has been studied in many literatures, by a minor revision in [3, Lemma 3.3], we have the following result:

**Proposition 3.11.** For any $x \in H$ and $\mu \in \mathcal{P}_p$, $\{P^x_t\}_{t \geq 0}$ admits a unique invariant measure $\nu^{x,\mu}$. Moreover, for any $t > 0$,

$$\left| \mathbb{E}F(x, \mu, Y^x_t) - \int_H F(x, \mu, z) \nu^{x,\mu}(dz) \right| \leq C \left[ 1 + |x| + (\mu(| \cdot |))^{1/p} + |y| \right] e^{- (\lambda_1 - L_F)t},$$

where $C$ is a positive constant which is independent of $t$.

Now, we define $\tilde{F}(x, \mu) := \int_H F(x, \mu, y) \nu^{x,\mu}(dy)$. Let $\tilde{X}$ be the solution of the corresponding averaged equation:

$$\begin{cases}
  d\tilde{X}_t = \left[ A\tilde{X}_t + \tilde{F}(\tilde{X}_t, \mathcal{L}\tilde{X}_t) \right] dt + dL_t, \\
  \tilde{X}_0 = \xi \in H.
\end{cases} \tag{3.21}$$

The well-posedness of equation (3.21) is the following:

**Theorem 3.12.** Equation (3.21) exists a unique mild solution $\tilde{X}_t$ which satisfies

$$\tilde{X}_t = e^{ tA} \xi + \int_0^t e^{(t-s)A} \tilde{F}(\tilde{X}_s, \mathcal{L}\tilde{X}_s) ds + \int_0^t e^{(t-s)A} dL_s. \tag{3.22}$$

**Proof.** It is sufficient to check that the $\tilde{F}(\cdot, \cdot)$ is Lipschitz continuous, then the well-posedness can be easily obtained by following the procedures in Theorem 2.2.

Note that it is easy to prove that $x_1, x_2 \in H$, $\mu_1, \mu_2 \in \mathcal{P}_p$, we have

$$\sup_{t \geq 0, y \in H} |Y^{x_1,\mu_1}_t - Y^{x_2,\mu_2}_t| \leq C|x_1 - x_2| + C\mathbb{W}_p(\mu_1, \mu_2), \tag{3.23}$$

Then by (3.23) and Proposition 3.11, we get

$$|\tilde{F}(x_1, \mu_1) - \tilde{F}(x_2, \mu_2)| = \left| \int_H F(x_1, \mu_1, z) \nu^{x_1,\mu_1}(dz) - \int_H F(x_2, \mu_2, z) \nu^{x_2,\mu_2}(dz) \right|$$

$$\leq \left| \int_H F(x_1, \mu_1, z) \nu^{x_1,\mu_1}(dz) - \mathbb{E}F(x_1, \mu_1, Y^{x_1,\mu_1}_t) \right|$$

$$+ \left| \mathbb{E}F(x_1, \mu_1, Y^{x_1,\mu_1}_t) - \mathbb{E}F(x_2, \mu_2, Y^{x_2,\mu_2}_t) \right|$$

$$+ \left| \mathbb{E}F(x_2, \mu_2, Y^{x_2,\mu_2}_t) - \int_H F(x_2, \mu_2, z) \nu^{x_2,\mu_2}(dz) \right|$$

$$\leq C \left[ 1 + |x_1| + |x_2| + (\mu_1(| \cdot |))^{1/p} + (\mu_2(| \cdot |))^{1/p} + |y| \right] e^{- (\lambda_1 - L_F)t}$$

$$+ C (|x_1 - x_2| + \mathbb{W}_p(\mu_1, \mu_2) + \mathbb{E}|Y^{x_1,\mu_1}_t - Y^{x_2,\mu_2}_t|)$$

$$\leq C \left[ 1 + |x_1| + |x_2| + (\mu_1(| \cdot |))^{1/p} + (\mu_2(| \cdot |))^{1/p} + |y| \right] e^{- (\lambda_1 - L_F)t}$$

$$+ C|x_1 - x_2| + C\mathbb{W}_p(\mu_1, \mu_2).$$

Hence by letting $t \to \infty$, it follows

$$|\tilde{F}(x_1, \mu_1) - \tilde{F}(x_2, \mu_2)| \leq C|x_1 - x_2| + C\mathbb{W}_p(\mu_1, \mu_2).$$

The proof is complete. \qed

**Remark 3.13.** Under the condition (3.2), it is easy to check that

$$\sup_{x \in H} |\tilde{F}(x, \mu)| \leq C \left[ 1 + (\mu(| \cdot |))^{1/p} \right]. \tag{3.24}$$
3.4. The proof of Theorem 3.5. In this subsection, we will give the detailed proof of Theorem 3.5.

Proof. It is easy to see that

\[
X_t^\varepsilon - \bar{X}_t = \int_0^t e^{(t-s)A} \left[ F(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon) - F(\bar{X}_s, \mathcal{L}_{\bar{X}_s}) \right] ds
\]

\[
= \int_0^t e^{(t-s)A} \left[ F(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon) - F(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_{s(\delta)}^\varepsilon}, \hat{Y}_s^\varepsilon) \right] ds
\]

\[
+ \int_0^t e^{(t-s)A} \left[ F(\bar{X}_s^\varepsilon, \mathcal{L}_{\bar{X}_s}) - F(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_{s(\delta)}^\varepsilon}) \right] ds
\]

Then by Lemmas 3.9 and 3.10, we have

\[
\left[ \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon - \bar{X}_t|^m \right) \right]^{1/m} \leq \left[ \mathbb{E} \left( \int_0^T \left| F(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon) - F(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_{s(\delta)}^\varepsilon}, \hat{Y}_s^\varepsilon) \right| ds \right)^m \right]^{1/m}
\]

\[
+ \left[ \mathbb{E} \left( \int_0^T \left| F(\bar{X}_s^\varepsilon, \mathcal{L}_{\bar{X}_s}) - F(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_{s(\delta)}^\varepsilon}) \right| ds \right)^m \right]^{1/m}
\]

\[
+ \left[ \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^t e^{(t-s)A} \left[ F(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, \hat{Y}_s^\varepsilon) - F(\bar{X}_s^\varepsilon, \mathcal{L}_{\bar{X}_s}) \right] ds \right)^m \right]^{1/m}
\]

\[
\leq C \int_0^T \left[ \mathbb{E} \left( |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^m \right) \right]^{1/m} + \left[ \mathbb{E} \left( |Y_s^\varepsilon - \hat{Y}_s^\varepsilon|^m \right) \right]^{1/m} + \left[ \mathbb{E} \left( |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^p \right) \right]^{1/p} ds
\]

\[
+ C \int_0^T \left[ \mathbb{E} \left( |\varepsilon_s^\varepsilon - \bar{X}_s|^m \right) \right]^{1/m} + \left[ \mathbb{E} \left( |\varepsilon_s^\varepsilon - \bar{X}_s|^p \right) \right]^{1/p} ds
\]

\[
+ \left[ \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^t e^{(t-s)A} \left[ F(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, \hat{Y}_s^\varepsilon) - F(\bar{X}_s^\varepsilon, \mathcal{L}_{\bar{X}_s}) \right] ds \right)^m \right]^{1/m}
\]

\[
\leq C \int_0^T \left[ \mathbb{E} \left( |X_t^\varepsilon - \bar{X}_t|^m \right) \right]^{1/m} dt + C_T \left[ \left( \mathbb{E} \left( |\varepsilon|^m \right) \right)^{1/m} + 1 \right] \delta^\theta + C_T J(T, m, \varepsilon, \delta),
\]

Then by Gronwall’s inequality we obtain that

\[
\left[ \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon - \bar{X}_t|^m \right) \right]^{1/m} \leq C_T \left[ \left( \mathbb{E} \left( |\varepsilon|^m \right) \right)^{1/m} + 1 \right] \delta^\theta + C_T J(T, m, \varepsilon, \delta),
\]
where
\[
J(T, m, \varepsilon, \delta) := \left\{ \begin{array}{c} E \sup_{t \in [0, T]} \int_0^t e^{(t-s)A} \left[ F(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{F}(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon) \right] ds \end{array} \right\}^{1/m}.
\]

Then by (3.2) and (3.24), we have
\[
J(T, m, \varepsilon, \delta) \leq \left\{ E \sup_{t \in [0, T]} \int_0^t e^{(t-s)A} \left[ F(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{F}(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon) \right] ds \right\}^{1/2} + C_T \delta
\]
\[
\leq C \left\{ E \sup_{t \in [0, T]} \left[ \int_0^t e^{(t-s)A} \left[ F(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{F}(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon) \right] ds \right] \right\}^{1/2} + C_T \delta
\]
\[
\leq C \left\{ \int_0^T \psi_t(s, r) dsdr \right\}^{1/2} + C_T \delta,
\]
where for any \( 0 \leq r \leq s \leq \frac{T}{\delta} \) and \( l = 0, 1, \ldots, [T/\delta] - 1 \),
\[
\psi_t(s, r) := \mathbb{E} \left\{ e^{(s-r)A} \left[ F(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{F}(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon) \right] \right\} + e^{(s-r)A} \left[ F(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{F}(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon) \right].
\]

Refer to Appendix, the following estimation holds:
\[
\psi_t(s, r) \leq C_T \left[ 1 + (\mathbb{E}[\xi]^p)^{2/p} + (\mathbb{E}[\eta]^p)^{2/p} \right] e^{-(\lambda_1 - L_C)(s-r)}, \quad \forall t = 0, 1, \ldots, [T/\delta] - 1, \quad (3.26)
\]

As a result, we obtain
\[
J(T, m, \varepsilon, \delta) \leq C_T \left[ 1 + (\mathbb{E}[\xi]^m)^{1/m} + (\mathbb{E}[\eta]^m)^{1/m} \right] \left( \frac{\varepsilon^{1/2}}{\delta^{1/2}} + \delta \right).
\]

By (3.25) and (3.27), it is easy to see
\[
\left[ \mathbb{E} \left( \sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t| \right)^m \right] \leq C_T \left[ 1 + (\mathbb{E}[\xi]^m)^{1/m} + (\mathbb{E}[\eta]^m)^{1/m} \right] \left( \delta^{\frac{\varepsilon}{m}} + \frac{\varepsilon^{1/2}}{\delta^{1/2}} + \delta \right).
\]

Finally, taking \( \delta = \varepsilon^{1/m} \), then we obtain that
\[
\left[ \mathbb{E} \left( \sup_{t \in [0, T]} |X_t^\varepsilon - \bar{X}_t| \right)^m \right] \leq C_T \left[ 1 + (\mathbb{E}[\xi]^m)^{1/m} + (\mathbb{E}[\eta]^m)^{1/m} \right] \varepsilon^{\frac{\delta}{m(1+7)}}.
\]

The proof is complete.
Remark 3.14. Note that under the condition (3.2), it follows
\[
\sup_{t \in [0,T]} |X_t^\varepsilon - \bar{X}_t| \leq \int_0^T |F(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon, Y_t^\varepsilon) - \bar{F}(\bar{X}_t, \mathcal{L}\bar{X}_t)| dt \leq C_T.
\]
Thus for any $T > 0$ and $k \geq 1$, we have
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{t \in [0,T]} |X_t^\varepsilon - \bar{X}_t|^k \right] = 0.
\]

4. Appendix

In this section, we give the detailed proof of (3.26).

The proof of (3.26): For any $\mu \in \mathcal{P}_p$ and random variables $\tilde{\xi}, \tilde{\eta} \in \mathcal{F}_s$, we consider the following equation
\[
d\tilde{Y}_{t}^{\varepsilon,s,\xi,\mu,\eta} = \frac{1}{\varepsilon} A\tilde{Y}_{t}^{\varepsilon,s,\xi,\mu,\eta} dt + \frac{1}{\varepsilon} G(\tilde{\xi}, \tilde{\eta}, \tilde{Y}_{t}^{\varepsilon,s,\xi,\mu,\eta}) dt + \frac{1}{\varepsilon^{1/\alpha}} dZ_t, \quad t \geq s.
\]
with $\tilde{Y}_{t}^{\varepsilon,s,\xi,\mu,\eta} = \tilde{\eta}$. Then by the construction of $\hat{Y}_{t}^{\varepsilon}$, for any $l \in \mathbb{N}_+$, we have
\[
\hat{Y}_{t}^{\varepsilon} = \tilde{Y}_{t}^{\varepsilon,l,\delta,X_{l,\delta}^\varepsilon,\mathcal{L}X_{l,\delta}^\varepsilon,Y_{l,\delta}^\varepsilon}, \quad t \in [l\delta, (l + 1)\delta],
\]
which implies
\[
\Psi_t(s, r) = \mathbb{E} \left\{ e^{(\delta - se)A} \left[ F(X_{l,\delta}^\varepsilon, \mathcal{L}X_{l,\delta}^\varepsilon, Y_{l,\delta}^\varepsilon) - \bar{F}(X_{l,\delta}^\varepsilon, \mathcal{L}X_{l,\delta}^\varepsilon) \right] \right\}.
\]

Note that for any fixed $x, y \in H$ and $\mu \in \mathcal{P}_p$, $\tilde{Y}_{s+\delta,l,\delta}^{\varepsilon,l,\delta,x,\mu,y}$ is independent of $\mathcal{F}_{l,\delta}$. $X_{l,\delta}^\varepsilon$ and $\tilde{Y}_{l,\delta}^\varepsilon$ are $\mathcal{F}_{l,\delta}$-measurable, we have
\[
\Psi_t(s, r) = \mathbb{E} \left\{ \mathbb{E} \left[ e^{(\delta - se)A} \left( F \left( X_{l,\delta}^\varepsilon, \mathcal{L}X_{l,\delta}^\varepsilon, Y_{l,\delta}^\varepsilon \right) - \bar{F} \left( X_{l,\delta}^\varepsilon, \mathcal{L}X_{l,\delta}^\varepsilon \right) \right) \right] \right\}.
\]

By the construction of the process $\{\tilde{Y}_{t}^{\varepsilon,s,\xi,\mu,\eta} \}_{t \geq s}$, which has the integrated form:
\[
\tilde{Y}_{s+\delta,l,\delta}^{\varepsilon,l,\delta,x,\mu,y} = y + \int_{\delta}^{s+\delta} A\tilde{Y}_{r}^{\varepsilon,l,\delta,x,\mu,y} dr + \frac{1}{\varepsilon} \int_{\delta}^{s+\delta} G \left( x, \mu, \tilde{Y}_{r}^{\varepsilon,l,\delta,x,\mu,y} \right) dr + \frac{1}{\varepsilon^{1/\alpha}} \int_{\delta}^{s+\delta} dZ_r
\]
\[
= y + \int_{0}^{s} A\tilde{Y}_{r}^{\varepsilon,l,\delta,x,\mu,y} dr + \int_{0}^{s} G \left( x, \mu, \tilde{Y}_{r}^{\varepsilon,l,\delta,x,\mu,y} \right) dr + \int_{0}^{s} d\tilde{Z}_r, \quad (4.1)
\]
where \( \{ \tilde{Z}_t := \frac{1}{\sqrt{\delta}}(Z_{\varepsilon t + \delta t} - Z_{\varepsilon t}) \}_{t \geq 0} \), which is also a cylindrical \( \alpha \)-stable process. Recall the solution of the frozen equation satisfies

\[
Y^{x,\mu,y}_s = y + \int_0^s AY^{x,\mu,y}_r dr + \int_0^s G(x, \mu, Y^{x,\mu,y}_r) dr + \int_0^s dZ_r. \tag{4.2}
\]

The uniqueness of the solutions of equation (4.1) and equation (4.2) implies that the distribution of \( \{ Y^{x,\mu,y}_{s \in [\varepsilon t, \delta t]} \}_{0 \leq s \leq \delta / \varepsilon} \) coincides with the distribution of \( \{ Y^{x,\mu,y} \}_{0 \leq s \leq \delta / \varepsilon} \).

Hence by Markov property, Proposition 3.11, (3.2), (3.18) and (3.20), we have

\[
\Psi_I(s, r) = \mathbb{E} \left[ e^{(\delta - \varepsilon)A} \left[ F(x, \mu, Y^{x,\mu,y}_s - \tilde{F}(x, \mu) \right] \right |
\begin{align*}
&= \mathbb{E} \left[ e^{(\delta - \varepsilon)A} \mathbb{E} \left[ F(x, \mu, Y^{x,\mu,y}_s - \tilde{F}(x, \mu) \right] \bigg | \mathcal{F}_r \right] \\
&= \mathbb{C} \mathbb{E} \left[ e^{(\delta - \varepsilon)L_F(s - r)} \right] \\
&\leq \mathbb{C} \mathbb{E} \left[ e^{(\delta - \varepsilon)L_F(s - r)} \right] \\
&\leq C T \left[ 1 + (\mathbb{E}[\xi^p]^{1/p})^2 + (\mathbb{E}[\eta^p]^{1/p})^2 \right] e^{-(\lambda_1 - L_F)(s - r)},
\end{align*}
\]

which completes the proof.

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