FINITE CHAINS INSIDE THIN SUBSETS OF $\mathbb{R}^d$

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Abstract. In a recent paper, Chan, Laba, and Pramanik investigated geometric configurations inside thin subsets of the Euclidean set possessing measures with Fourier decay properties. In this paper we ask which configurations can be found inside thin sets of a given Hausdorff dimension without any additional assumptions on the structure. We prove that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d+1}{2}$, then there exists a non-empty interval $I$ such that given any sequence $\{t_1, t_2, \ldots, t_k; t_j \in I\}$, there exists a sequence $\{x_j\}_{j=1}^{k+1}$, such that $x_j \in E$ and $|x_j+1 - x_i| = t_j$, $1 \leq i \leq k$. In other words, $E$ contains vertices of a chain of arbitrary length with prescribed gaps.

1. Introduction

The problem of determining of which geometric configurations one can find inside various subsets of Euclidean space is a classical subject matter. The basic problem is to understand how large a subset of the Euclidean space needs to be in order to contains the vertices of a congruent and possibly scaled copy of a given polyhedron or another geometric shape. In the case of a finite set, “large” refers to the number of points, while in infinite sets, it refers to the Hausdorff dimension or Lebesgue density. The resulting class of problems has been attacked by a variety of authors using combinatorial, number theoretic, ergodic and Fourier analytic techniques, creating a rich set of ideas and interactions.

We begin with a comprehensive result due to Tamar Ziegler, ([19]) which generalizes an earlier result due to Furstenberg, Katznelson and Weiss ([7]). See also ([4]).

Theorem 1.1. [Ziegler] Let $E \subset \mathbb{R}^d$, of positive upper Lebesgue density in the sense that

$$\limsup_{R \to \infty} \frac{\mathcal{L}^d(E \cap [-R, R]^d)}{(2R)^d} > 0,$$

where $\mathcal{L}^d$ denotes the $d$-dimensional Lebesgue measure. Let $E_\delta$ denote the $\delta$-neighborhood of $E$. Let $V = \{0, v^1, v^2, \ldots, v^{k-1}\} \subset \mathbb{R}^d$, where $k \geq 2$ is a positive integer.

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Then there exists $l_0 > 0$ such that for any $l > l_0$ and any $\delta > 0$ there exists \( \{x^1, \ldots, x^k\} \subset E_\delta \) congruent to \( lV = \{0, lv^1, \ldots, lv^{k-1}\} \).

In particular, this result shows that we can recover every simplex similarity type inside a subset of \( \mathbb{R}^d \) of positive upper Lebesgue density. It is reasonable to wonder whether the assumptions of Theorem 1.1 can be weakened, but the following result due to Maga ([10]) shows that conclusion may fail even if we replace the upper Lebesgue density condition with the assumption that the set is of dimension \( d \).

**Theorem 1.2.** [Maga] For any \( d \geq 2 \) there exists a full dimensional compact set \( A \subset \mathbb{R}^d \) such that \( A \) does not contain the vertices of any parallelogram. If \( d = 2 \), then given any triple of points \( x^1, x^2, x^3, x^j \in A \), there exists a full dimensional compact set \( A \subset \mathbb{R}^2 \) such that \( A \) does not contain the vertices of any triangle similar to \( \triangle x^1x^2x^3 \).

In view of Maga's result, it is reasonable to ask whether interesting point configurations can be found inside thin sets under additional structural hypotheses. This question was recently addressed by Chan, Laba and Pramanik in [1]. Before stating their result, we provide two relevant definitions.

**Definition 1.3.** Fix integers \( n \geq 2, p \geq 3 \), and \( m = n\lceil \frac{p+1}{2} \rceil \). Suppose \( B_1, \ldots, B_p \) are \( n \times (m-n) \) matrices.

(a) We say that \( E \) contains a \( p \)-point \( B \)-configuration if there exists vectors \( z \in \mathbb{R}^n \) and \( w \in \mathbb{R}^{m-n} \setminus \{0\} \) such that

\[
\{z + B_jw\}_{j=1}^p \subset E.
\]

(b) Moreover, given any finite collection of subspaces \( V_1, \ldots, V_q \subset \mathbb{R}^{m-n} \) with \( \dim(V_i) < m-n \), we say that \( E \) contains a non-trivial \( p \)-point \( B \)-configuration with respect to \( (V_1, \ldots, V_q) \) if there exists vectors \( z \in \mathbb{R}^n \) and \( w \in \mathbb{R}^{m-n} \setminus \bigcup_{i=1}^q V_i \) such that

\[
\{z + B_jw\}_{j=1}^p \subset E.
\]

**Definition 1.4.** Fix integers \( n \geq 2, p \geq 3 \), and \( m = n\lceil \frac{p+1}{2} \rceil \). We say that a set of \( n \times (m-n) \) matrices \( \{B_1, \ldots, B_p\} \) is non-degenerate if

\[
\text{rank} \begin{pmatrix} B_{i_2} - B_{i_1} \\ \vdots \\ B_{i_{m/n}} - B_{i_1} \end{pmatrix} = m-n
\]

for any distinct indices \( i_1, \ldots, i_{m/n} \in \{1, \ldots, p\} \).
Theorem 1.5. [Chan, Laba and Pramanik] Fix integers $n \geq 2$, $p \geq 3$, and $m = n\left[\frac{p+1}{2}\right]$. Let $\{B_1, \ldots, B_p\}$ be a collection of $n \times (m-n)$ non-degenerate matrices in the sense of Definition 1.4. Then for any constant $C$, there exists a positive number $\epsilon_0 = \epsilon_0(C, n, p, B_1, \ldots, B_p) << 1$ with the following property: Suppose the set $E \subset \mathbb{R}^p$ with $|E| = 0$ supports a positive, finite, Radon measure $\mu$ with two conditions: (a) (ball condition) $\sup_{x \in E} \frac{\mu(B(x, r))}{r^\alpha} \leq C$ if $n - \epsilon_0 < \alpha < n$, (b) (Fourier decay) $\sup_{\xi \in \mathbb{R}^n} |\hat{\mu}(\xi)|(1 + |\xi|)^{\beta/2} \leq C$.

Then

(i) $E$ contains a $p$–point $\mathcal{B}$–configuration in the sense of Definition 1.3 (a).

(ii) Moreover, for any finite collection of subspaces $V_1, \ldots, V_q \subset \mathbb{R}^{m-n}$ with $\dim(V_i) < m-n$, $E$ contains a non-trivial $p$–point $\mathcal{B}$–configuration with respect to $(V_1, \ldots, V_q)$ in the sense of Definition 1.3 (b).

One can check that the Chan-Laba-Pramanik result covers some geometric configurations but not others. For example, their non-degeneracy condition allows them to consider triangles in the plane, but not simplexes in $\mathbb{R}^3$ where three faces meet at one of the vertices at right angles, forming a three-dimensional corner. Most relevant to this paper is the fact that the conditions under which Theorem 1.5 holds are satisfied for chains (see Definition 1.6 below), but the conclusion requires decay properties for the Fourier transform of a measure supported on the underlying set. We shall see that in the case of chains, such an assumption is not needed and the existence of a wide variety of chains can be established under an explicit dimensional condition alone.

1.1. **Focus of this article.** In this paper we establish that a set of sufficiently large Hausdorff dimension, with no additional assumptions, contains an arbitrarily long chain with vertices in the set and preassigned admissible gaps.

Definition 1.6. (See Figure 1 above) A $k$-chain in $E \subset \mathbb{R}^d$ with gaps $\{t_i\}_{i=1}^k$ is a sequence

$$\{x^1, x^2, \ldots, x^{k+1} : x^j \in E; |x^{i+1} - x^i| = t_i; 1 \leq i \leq k\}.$$  

We say that the chain is non-degenerate if all the $x^i$’s are distinct.

Our main result is the following.

Theorem 1.7. Suppose that the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d+1}{2}$. Then for any $k \geq 1$, there exists an open interval $I$, such that for any $\{t_i\}_{i=1}^k \subset I$ there exists a non-degenerate $k$-chain in $E$ with gaps $\{t_i\}_{i=1}^k$. 

In the course of establishing Theorem 1.7 we shall prove the following result which is interesting in its own right and has a number of consequences for Falconer type problems. See [5], [2] and [18] for the background and the latest results pertaining to Falconer distance problem.

**Theorem 1.8.** Suppose that $\mu$ is a compactly supported Borel measure such that
\begin{equation}
\mu(B(x,r)) \leq Cr^s, \tag{1.1}
\end{equation}
where $B(x,r)$ is the ball of radius $r$ centered at $x \in \mathbb{R}^d$, for some $s > \frac{d+1}{2}$. Then for any $t_1, \ldots, t_k > 0$,
\begin{equation}
\mu \times \mu \times \cdots \times \mu\{(x^1, x^2, \ldots, x^{k+1}) : t_i \leq |x^{i+1} - x^i| \leq t_i + \epsilon; \ i = 1, 2, \ldots, k\} \leq C\epsilon^k. \tag{1.2}
\end{equation}

**Corollary 1.9.** Given a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, define
\[ \Delta_k(E) = \{|x^1 - x^2|, |x^2 - x^3|, \ldots, |x^k - x^{k+1}| : x^i \in E\}. \]
Suppose that the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$. Then
\[ \mathcal{L}^k(\Delta_k(E)) > 0. \]

**Definition 1.10.** We say that a compact subset $E$ of $\mathbb{R}^d$ is Ahlfors-David regular, then there exists $C > 0$ such that
\[ C^{-1}r^s \leq \mu(B(x,r)) \leq Cr^s \]
for every $x \in E$, where $E$ has Hausdorff dimension $s$ and $\mu$ is the restriction of the $s$-dimensional Hausdorff measure to $E$.

Using the techniques in [3] and Theorem 1.8 one can recover the following result which gives an upper bound on the dimension of the set of chains of a given type.

**Corollary 1.11.** Let $E$ be an Ahlfors-David regular subset of $\mathbb{R}^d$ of Hausdorff dimension $> \frac{d+1}{2}$. Then for any sequence of positive real numbers $t_1, t_2, \ldots, t_k$, the
upper Minkowski dimension of
\[ \{(x^1, x^2, \ldots, x^{k+1}) \in E^{k+1} : |x^{j+1} - x^j| = t_j, 1 \leq j \leq k\} \]
does not exceed \((k + 1) \dim_H(E) - k\).

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2. Proof of Theorem 1.7 and Theorem 1.8

We shall repeatedly use the following result due to Iosevich, Sawyer, Taylor and Uriarte-Tuero ([9]).

**Theorem 2.1.** Let \( T_\lambda f(x) = \lambda * (f\mu)(x) \), where \( \lambda, \mu \) are compactly supported Borel measures on \( \mathbb{R}^d \). Suppose that \( \mu \) satisfies (1.1) and for some \( \alpha > 0 \)
\[ |\hat{\lambda}(\xi)| \leq C|\xi|^{-\alpha}. \]
Suppose that \( \nu \) is a compactly supported Borel measure supported on \( \mathbb{R}^d \) satisfying (1.1) with \( s_\mu \) replaced by \( s_\nu \) and suppose that \( \alpha > d - s \), where \( s = \frac{s_\mu + s_\nu}{2} \). Then
\[ ||T_\lambda f||_{L^2(\nu)} \leq C||f||_{L^2(\mu)}. \]

Since the proof of Theorem 2.1 is short, we give the argument below for the sake of keeping the presentation as self-contained as possible. It is enough to show that
\[ \langle T_{K^*} f, g\nu \rangle \leq C||f||_{L^2(\mu)} \cdot ||g||_{L^2(\nu)}. \]
The left hand side equals
\[ \int \hat{\lambda}(\xi) \hat{f}(\xi) \hat{g}\nu(\xi) d\xi. \]
By the assumptions of Theorem 2.1, the modulus of this quantity is bounded by
\[ C \int |\xi|^{-\alpha} |\hat{f}(\xi)||\hat{g}\nu(\xi)| d\xi, \]
and applying Cauchy-Schwarz bounds this quantity by
\[ C \left( \int |\hat{f}(\xi)|^2 |\xi|^{-\alpha_\mu} d\xi \right)^{\frac{1}{2}} \cdot \left( \int |\hat{g}\nu(\xi)|^2 |\xi|^{-\alpha_\nu} d\xi \right)^{\frac{1}{2}} \]
for any \( \gamma_\mu, \gamma_\nu > 0 \) such that \( \alpha = \frac{\alpha_\mu + \alpha_\nu}{2} \).
By Lemma (2.3) below, the quantity (2.1) is bounded by \( C \| f \|_{L^2(\mu)} \cdot \| g \|_{L^2(\nu)} \) after choosing, as we may, \( \alpha_\mu > d - s_\mu \) and \( \alpha_\nu > d - s_\nu \). This completes the proof of Theorem 2.1.

A rough outline of this section is the following. We shall first prove Theorem 1.8 using Theorem 2.1. This will give us an upper bound on the quantity

(2.2) \[ \limsup_{\epsilon \to 0} \epsilon^{-k} \mu \times \cdots \times \{(x^1, \ldots, x^{k+1}) : t_i \leq |x^{i+1} - x^i| \leq t_i + \epsilon; 1 \leq i \leq k\} \]

We shall then use a variant of the technique from [8] (see also [12]) and the Cauchy-Schwartz inequality to obtain a lower bound for (2.2) with \( \limsup \) replaced by \( \liminf \). Using both the upper and the lower bound, combined with an inductive argument, we shall complete the proof of Theorem 1.7.

2.1. Proof of Theorem 1.8 and Corollary 1.9. We will fix a set \( \{t_1, \ldots, t_n\} \in \mathbb{R}^n \) with \( n \) the desired chain length and with the values \( t_i \) in the valid interval. Divide the left hand side of (1.2) by the right hand side and note that it suffices to establish the estimate

\[ C_k^\epsilon(\mu) = \int \left( \prod_{i=1}^{k} \sigma_i^\epsilon(x^{i+1} - x^i) d\mu(x^i) \right) d\mu(x^{k+1}) \leq A_k, \]

with \( A_k \) independent of \( \epsilon \). Here \( \sigma_i^\epsilon(x) = \sigma_r \ast \rho_\epsilon(x) \), with \( \sigma_r \) the Lebesgue measure on the sphere of radius \( r \), \( \rho \) a smooth cut-off function with \( \int \rho = 1 \) and \( \rho_\epsilon(x) = \epsilon^{-d} \rho\left( \frac{x}{\epsilon} \right) \).

Define

(2.3) \[ d\mu_0(x) = d\mu(x) \equiv f_0(x)d\mu(x); \quad d\mu_k^\epsilon(x) = \sigma_{t_k}^\epsilon \ast \mu_{k-1}^\epsilon(x)d\mu(x) \equiv f_k^\epsilon(x)d\mu(x). \]

Observe that

\[ \int d\mu_k^\epsilon(x) = C_k^\epsilon(\mu). \]

Another way of putting this is that

(2.4) \[ \| f_k^\epsilon \|_{L^1(\mu)} = C_k^\epsilon(\mu). \]

Moreover, if we let \( T_k^\epsilon = T_{\sigma_{t_{k+1}}}^\epsilon \), with \( T_{\sigma_{t_k}}^\epsilon \) as in Theorem 2.1, then

(2.5) \[ \| T_k^\epsilon f_k^\epsilon \|_{L^1(\mu)} = C_{k+1}^\epsilon(\mu). \]

Using Cauchy-Schwartz (keeping in mind that \( \int d\mu(x) = 1 \)), we have

\[ C_{k+1}^\epsilon(\mu) \leq \left( \int [(T_k^\epsilon f_k^\epsilon)(x)]^2 d\mu(x) \right)^{1/2}. \]
Next, we will induct on the chain length \( k \) to show that \( \|T_\epsilon^k f_\epsilon^k\|_{L^2(\mu)} \) is always bounded by a constant depending on \( n \) but not \( \epsilon \). Certainly, we can bound \( \|f_0\|_{L^2(\mu)} \) with an absolute constant \( A_0 \), since \( \mu \) is a finite measure.

Now suppose that

\[
\|f_\epsilon^k\|_{L^2(\mu)} \leq A_k.
\]

By definition,

\[
\|f_\epsilon^{k+1}\|_{L^2(\mu)} = \|T_\epsilon^k f_\epsilon^k\|_{L^2(\mu)}.
\]

Then, by Theorem 2.1, \( \|T_\epsilon^k f_\epsilon^k\|_{L^2(\mu)} \leq c_k \|f_\epsilon^k\|_{L^2(\mu)} \).

Thus \( C_{k+1}(\mu) \leq c_{k+1}A_k \) and the proof of Theorem 1.8 is complete.

We now recover Corollary 1.9. Cover \( \Delta_k(E) \) with cubes of the form

\[
\bigcup_i \prod_{j=1}^d (t_{ij}, t_{ij} + \epsilon_i),
\]

where \( \prod \) denotes the Cartesian product. We have

\[
1 = \mu^{(k+1)}(E^{k+1}) \leq \sum_i \mu^{(k+1)}\{(x_1, \ldots, x_{k+1}) : t_{ij} \leq |x^{j+1} - x^j| \leq t_{ij} + \epsilon_i; 1 \leq j \leq k\}.
\]

By Theorem 1.8, the right hand side of (2.6) is bounded by

\[
(2.7) \quad C \sum_i \epsilon_i^k
\]

and we conclude that (2.7) is bounded from below by \( \frac{1}{C} > 0 \). It follows that \( \Delta_k(E) \) cannot have measure 0 and the proof of Corollary 1.9 is complete.

We now continue with the proof of Theorem 1.7.

2.2. Lower bound on \( C_k(\mu) \).

**Theorem 2.2.** With the notation above, suppose that the Hausdorff dimension of \( E \) is greater than \( \frac{d+1}{2} \). Then there exists a non-empty open interval \( I \) such that if all the gaps \( t_i \) belong to \( I \), then

\[
(2.8) \quad \liminf_{\epsilon \to 0} C_k(\mu) > 0.
\]
Here we will make use of the measures $\mu_k$ defined in (2.3) above. By Frostman’s lemma (see e.g. [6], [17] or [11] ) we may take $\mu$ to satisfy the same properties as in the statement of Theorem 1.8 above with $\frac{d+1}{2} < s_\mu < dim_H(E)$. We will show that, for all $n$,

$$\liminf_{\epsilon \to 0} \int d\mu^\epsilon_n(x) > 0.$$  

Again, we use induction. First, $\int d\mu^\epsilon_0(x) = \int d\mu(x) = 1 > 0$. Now suppose that

$$\liminf_{\epsilon \to 0} \int d\mu^\epsilon_k(x) > 0.$$  

Combined with the upper bound for this quantity obtained above, we see $\lim_{\epsilon \to 0} \mu^\epsilon_k$ is well-defined non-zero Borel measure supported on $E$. This allows us to consider $\mu_k$ in place of $\mu^\epsilon_k$ in (2.3) above.

We now employ a variant of the argument in ([8]). Consider the quantity

$$\int d\mu^\epsilon_{k+1}(x) = \int \int \sigma^\epsilon_{t_{k+1}}(x-y)d\mu(x)d\mu_k(y)$$

$$= \int \int (\sigma_{t_{k+1}} * \rho_\epsilon)(x-y)d\mu(x)d\mu_k(y).$$

Using Fourier inversion and properties of the Fourier transform, this is equal to

$$\int \int \int e^{2\pi i (x-y) \cdot \xi} \hat{\sigma}_{t_{k+1}}(\xi) \hat{\rho}_\epsilon(\xi)d\mu(x)d\mu_k(y)d\xi.$$  

Simplifying further, we get

$$\int \hat{\sigma}(t_{k+1} \xi) \hat{\rho}(\epsilon \xi) \hat{\mu}(\xi) \hat{\mu}_k(-\xi)d\xi.$$  

We substitute $t$ for $t_{k+1}$ and divide this quantity into $M_k(t) + R_k^\epsilon(t)$, where

$$M_k(t) = \int \hat{\sigma}(t \xi) \hat{\mu}(\xi) \hat{\mu}_k(-\xi)d\xi$$  

and

$$R_k^\epsilon(t) = \int \hat{\sigma}(t \xi) (\hat{\rho}(\epsilon \xi) - 1) \hat{\mu}(\xi) \hat{\mu}_k(-\xi)d\xi.$$
We will prove that $M_k(t)$ is continuous away from $t = 0$ and that 
\[ \lim_{\epsilon \to 0} R_k^\epsilon(t) = 0. \]

Once this is established, we will show that $M_k(t)$ is not identically 0. It will then follow by continuity that $M_k(t)$ is bounded from below on a closed interval.

We shall verify the continuity of $M_k(t)$ using the Dominated Convergence Theorem. Because $\sigma$ is the surface measure on the $d-1$ sphere, it follows by the method of stationary phase (see e.g. [14], [15]) that
\[ |\hat{\sigma}(\xi)| \leq C|\xi|^{-\frac{d-1}{2}} \]
and thus
\[ |\hat{\sigma}(t\xi)| \leq Ct^{-\frac{d-1}{2}}|\xi|^{-\frac{d-1}{2}} \text{ if } t \neq 0. \]

It follows that if $t \neq 0$, then
\[ |M_k(t)| \leq t^{-\frac{d-1}{2}} \int |\xi|^{-\frac{d-1}{2}}|\hat{\mu}(\xi)||\hat{\mu}_k(-\xi)|d\xi. \]

Using Cauchy-Schwarz, this quantity is bounded above by
\[ t^{-\frac{d-1}{2}} \left( \int |\xi|^{-\frac{d-1}{2}}|\hat{\mu}(\xi)|^2d\xi \right)^{1/2} \left( \int |\xi|^{-\frac{d-1}{2}}|\hat{\mu}_k(\xi)|^2d\xi \right)^{1/2}. \]

Now we will verify that these two integrals are, in fact, finite. The first is the $\frac{d+1}{2}$-energy integral of $\mu$ and is finite since the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$ by assumption. To treat the second integral in the case that $k > 0$, we will need the following calculation.

**Lemma 2.3.** Let $\mu$ be a compactly supported Borel measure such that $\mu(B(x,r)) \leq Cr^s$ for some $s \in (0, d)$. Suppose that $\alpha > d - s$. Then for $f \in L^2(\mu)$,
\[ \int |\hat{\mu}(\xi)|^2 |\xi|^{-\alpha}d\xi \leq C''|f|^2_{L^2(\mu)}. \]

To prove Lemma 2.3, observe that
\[ \int |\hat{\mu}(\xi)|^2 |\xi|^{-\alpha}d\xi = C \int \int f(x)f(y)|x-y|^{-d+\alpha}d\mu(x)d\mu(y) = \langle Tf, f \rangle, \]
where
\[ Tf(x) = \int |x-y|^{-d+\alpha}f(y)d\mu(y) \]
and the inner product above is with respect to $L^2(\mu)$. Observe that
\[ \int |x-y|^{-d+\alpha}d\mu(y) \approx \sum_{j>0} 2^{j(d-\alpha)} \int_{|x-y|\approx 2^{-j}} d\mu(y) \leq C \sum_{j>0} 2^{j(d-\alpha-s)} \leq C'' \]
since $\alpha > d - s$.

By symmetry, $\int |x - y|^{d + \alpha} d\mu(x) \leq C'$ and Schur’s test ([13], see also Lemma 7.5 in [17]) implies at once that

$$||Tf||_{L^2(\mu)} \leq C'||f||_{L^2(\mu)},$$

which implies that conclusion of Lemma 2.3 in view of (2.10) and the Cauchy-Schwarz inequality. This completes the proof of Lemma 2.3. We note that Lemma 2.3 can also be recovered from the fractal Plancherel estimate due to R. Strichartz ([16]). See also Theorem 7.4 in [17] where a similar statement is proved by the same method as above.

Taking $f = f_k$ and applying the lemma, we see that the second factor is indeed bounded. Recall that we verified that $f_k \in L^2(\mu)$ in the proof of Theorem 1.8.

Next, we wish to show that $\lim_{\epsilon \to 0} R_k(t) = 0$. Recall that $R_k(t)$ is equal to

$$\int (\hat{\rho}(\epsilon\xi) - 1)\hat{\sigma}(t\xi)\hat{\mu}(\xi)\hat{\mu}_k(-\xi)d\xi.$$

Notice that the size of the integrand is negligibly small for $|\xi| \leq 1/\epsilon$, and so it suffices to estimate the quantity

$$\int_{|\xi| > 1/\epsilon} |\hat{\sigma}(t\xi)||\hat{\mu}(\xi)||\hat{\mu}_k(-\xi)|d\xi.$$

Proceeding as in the estimation of $M_k(t)$ above, we bound this integral above with

$$Ct^{-d-1/2} \int_{|\xi| > 1/\epsilon} |\xi|^{-d-1/2} |\hat{\mu}(\xi)||\hat{\mu}_k(-\xi)|d\xi$$

and then use Cauchy Schwarz to bound it further with

$$Ct^{-d-1} \left( \int_{|\xi| > 1/\epsilon} |\xi|^{-d-1} |\hat{\mu}(\xi)|^2 d\xi \right)^{1/2} \left( \int_{|\xi| > 1/\epsilon} |\xi|^{-d-1} |\hat{\mu}_k(\xi)|^2 d\xi \right)^{1/2}.$$

We have already shown that the second integral is finite. The first integral is bounded by

$$\sum_{j > \log_2(1/\epsilon)} 2^{-j(d+1)} \int_{2^j \leq |\xi| < 2^{j+1}} |\hat{\mu}(\xi)|^2 d\xi.$$

We may choose a smooth cut-off function $\psi$ such that the inner integral is bounded by
\[
\int |\hat{\mu}(\xi)|^2 \hat{\psi}(2^{-j}\xi) d\xi.
\]

By Fourier inversion his integral is equal to
\[
2^dj \int \int \psi(2^j(x-y))d\mu(x)d\mu(y) \leq C2^{j(d-s)}.
\]

Returning to the sum, we now have the estimate
\[
C \sum_{j > \log_2(1/\epsilon)} 2^{-j(d+1)} \cdot 2^{j(d-s)} \leq C \sum_{j > \log_2(1/\epsilon)} 2^{j(d+1-s)}.
\]

As long as \(s > \frac{d+1}{2}\), this is \(< < \epsilon^{s-d+1}\). Thus \(R_k^\epsilon(t)\) tends to 0 with \(\epsilon\) as long as \(\dim_H(E) > \frac{d+1}{2}\).

In order to complete the proof of the lower bound, we must show that \(M_k(t)\) is not identically 0. To see this, observe that by construction, \(M_k(t)\) is the Radon-Nikodym derivative of the measure \(\nu_k\) given by the relation
\[
\int f(t) d\nu_k(t) = \int \int f(|x-y|) d\mu(x) d\mu_{k-1}(y).
\]

Since \(\int d\mu(x) = 1\) and \(\int d\mu_{k-1}(y) > 0\) by the induction hypothesis, we see that \(\int d\nu_k(t) > 0\) and this \(M_k(t)\) is not identically 0. This completes the proof of the lower bound.

2.3. The non-degeneracy argument and completion of the proof of Theorem 1.7. We shall need a robust definition of a non-degenerate chain.

**Definition 2.4.** Fix \(\epsilon > \) very small. We say that a \(k\)-chain \(\{x^1, \ldots, x^{k+1} : x^j \in E\}\) is \((\epsilon, N)\) non-degenerate if \(|x^i - x^j| \geq N\epsilon\) for all \(i \neq j\), \(1 \leq i, j \leq k\).

We know that \(\liminf_{\epsilon \to 0} C_k^\epsilon(\mu) > 0\) and \(C_k^\epsilon(\mu) \leq A\) for all \(k\). It is enough to show that the lower bound cannot result from degenerate chains alone.

Fix \(\epsilon > 0\) small. Consider

\[
(2.11) \int \ldots \int_{\{(x^1, \ldots, x^{k+1}) \in E^{k+1} : |x^i - x^j| \leq N\epsilon\ \text{for some} \ i \neq j\}} \left( \prod_{j=1}^{k} \sigma_{t_j}^\epsilon(x^{i+1} - x^i)d\mu(x^i) \right) d\mu(x^{k+1}).
\]

Let \(i_0\) denote the smallest index such that \(|x^{i_0} - x^j| \leq N\epsilon\) for some \(j\).
FINITE CHAINS INSIDE THIN SUBSETS OF $\mathbb{R}^d$

Figure 2. On the left we have a 6-chain where $x^1$ and $x^5$ are within $N\epsilon$. The picture on the right depicts the chain that results after we integrate in $x^1$ and cut the link between $x^1$ and $x^2$.

2.3.1. The case $i_0 = 1$. Estimating $\sigma^*(x^1 - x^2) \leq C\epsilon^{-1}$ and integrating in $d\mu(x^1)$, we see that the expression in (2.11) with the additional restriction that $x^1$ is within $N\epsilon$ of another vertex $x^j$ is bounded by

$$C \cdot k \cdot (N\epsilon)^s \cdot \epsilon^{-1} \cdot C_k^{\epsilon}(\mu) \leq C'kN^s\epsilon^{s-1}.$$ 

Note that the factor of $k$ above results from the fact that there are $k$ choices for $x^j$, the factor $\epsilon^{-1}$ is there because we eliminated the link connecting $x^1$ and $x^2$, and $C_k^\epsilon(\mu)$ is what is left after $x^1$ and the link connecting it to $x^2$ is eliminated from consideration. See Figure 2 where this argument is depicted in the case $k = 6, i_0 = 1$.

2.3.2. The case $2 \leq i_0 \leq k + 1$. Then we simply integrate in $x^1, x^2, \ldots, x^{i_0-1}$. This gives us a $(k - i_0 - 1)$-chain, except that the measure on $x^{i_0}$ is $d\mu_{i_0}$ instead of $d\mu$. We would be able to repeat the argument in the previous paragraph if it were true that $\mu_{i_0}(B(x, r)) \leq Cr^s$ for some $s > \frac{d+1}{2}$, given our assumption that the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$. While this is the case for $\mu$, the property does not necessarily extend to $d\mu_{i_0}$. We get around this problem using a simple pigeon-holing approach.

We have shown above that $\int d\mu_k^*(x)$ is bounded above with constants independent of $\epsilon$. Recall that

$$\int d\mu_k^*(x) = \int \int \sigma^*_i(x - y)d\mu_{k-1}(y)d\mu(x) = \int \sigma^*_i \ast \mu_{k-1}(x)d\mu(x).$$

By Chebyshev’s inequality,

$$\mu\{x: \sigma^*_i \ast \mu_{k-1}(x) > R\} \leq \frac{1}{R} \int \sigma^*_i \ast \mu_{k-1}(x)d\mu(x)$$

$$= \frac{1}{R} \int d\mu_k^*(x) \leq R^{-1}C_k$$

independently of $\epsilon$. 


Choose \( R = R_k \) to be large enough so that \( R^{-1} C_k < \frac{1}{2} \). Then there exists \( E' \subset E \) with \( \mu(E') > \frac{1}{2} \) such that \( \sigma_t^* \mu_{k-1}(x) \leq R_k \) for \( x \in E' \). This implies, in particular, that

\[
\mu_k(B(x, r) \cap E') \leq R_k \mu(B(x, r)) \leq R_k r^s,
\]

where \( \frac{d+1}{2} < s < \text{dim}_H(E) \) is in accordance with the Frostman property of \( \mu \).

We deduce that for each \( k \), we can find \( E' \subset E \) such that \( \mu(E') > 0 \) and

\[
\mu_j(B(x, r) \cap E') \leq R_k r^s \quad \forall \, j \leq k - 1.
\]

In view of this, replacing \( E \) with \( E' \) reduces the case \( 2 \leq i_0 \leq k \) to the case \( i_0 = 1 \). We note that replacing \( E \) with \( E' \) does not affect the conclusion of Theorem 1.7 because any non-degenerate \( k \)-chain in \( E' \) is also a non-degenerate \( k \)-chain in \( E \).

In conclusion, we have

\[
\int \cdots \int_{\{ (x^1, \ldots, x^{k+1}) \in E^{k+1} : |x^i - x^j| > N \epsilon \text{ for all } i \neq j \}} \left( \prod_{j=1}^{k} \sigma_{ij}^*(x^{i+1} - x^i) d\mu(x^i) \right) d\mu(x^{k+1}) > c_k > 0
\]

as long as, say, \( N < C \epsilon^{-1+\frac{1}{s}+\delta} \) for some \( \delta > 0 \). If \( \delta > 0 \) is chosen small enough, \( \epsilon^{-1+\frac{1}{s}+\delta} \to \infty \) as \( \epsilon \to 0 \). Taking a liminf as \( \epsilon \to 0 \) we see that there exists a non-degenerate \( k \)-chain with the prescribed gaps.

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