Ultracold gases and multi-Josephson junctions as simulators of out-of-equilibrium phase transformations in superfluids and superconductors

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Abstract. The experiments on strongly nonequilibrium symmetry-breaking phase transformations in superfluids and superconductors revealed that the topological defects (e.g. vortices) are produced most efficiently in the systems of microscopic size or low dimensionality (D = 1), while in the macroscopic two-dimensional (2D) and 3D samples the efficiency of their formation was substantially suppressed (by a few orders of magnitude) as compared to theoretical predictions. A reasonable explanation for this behaviour is based on the specific thermal correlations between the phases of Bose–Einstein condensates formed in the spatial subregions disconnected during the phase transformation. Such correlations were initially revealed in the multi-Josephson-junction loop experiment (Carmi R et al 2000 Phys. Rev. Lett. 84 4966) and were confirmed recently by the experiments with ultracold atoms in periodic potentials (Hadzibabic Z et al 2006 Nature 441 1118). We begin our theoretical consideration from a phase transformation in the simplest $\phi^4$-model of the real scalar field and show that, under the presence of the above-mentioned correlations, the final symmetry-broken states are described by the effective Ising model. Its behaviour changes dramatically in passing from finite to infinite size of the system and from the low (D = 1) to higher (D $\geq$ 2) dimensionality, which is in qualitative agreement with the experimental results.

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1. Introduction

1.1. The concept of topological defects

Formation of topological defects by the strongly out-of-equilibrium symmetry-breaking phase transformations is the subject of interest both in condensed-matter and elementary-particle physics. This is because of a close similarity between the Lagrangian of Landau–Ginzburg theory, widely used to describe phase transitions in condensed matter (superconductors, superfluids, liquid crystals etc), and the Lagrangians of the modern field theories (such as the standard electroweak model or various kinds of Grand Unification Theories), which are also substantially based on the concept of spontaneous symmetry breaking. After the phase transformations in all the above-mentioned cases, stable topological defects of the order parameter can arise, depending on the symmetry group involved such as the monopoles, strings (vortices) and domain walls.

Qualitative prediction of the defect formation, due to the independent establishment of the symmetry-broken states in spatially separated subregions, was done by Bogoliubov [1] soon after the appearance of the first field models based on the concept of spontaneous symmetry breaking. Later, a detailed quantitative theory of this phenomenon was developed by Kibble [2] and Zurek [3], and it is usually called the Kibble–Zurek (KZ) mechanism.

The KZ scenario is based on simple causality arguments. Namely, if during a phase transformation the information about an order parameter can spread over the distance \( \xi_{\text{eff}} \), then phases of the order parameter should be established independently in the regions of characteristic size \( \xi_{\text{eff}} \).\(^2\) As a result, after some relaxation following the phase transformation, stable defects of the order parameter can be formed at the typical separation \( \xi_{\text{eff}} \) from each other. Therefore, their concentration, for example, in a three-dimensional (3D) system can be roughly estimated as \( n \approx 1/\xi_{\text{eff}}^d \), where \( d = 3 \), 2 and 1 for the monopoles, strings (vortices) and domain walls, respectively, while the effective correlation length \( \xi_{\text{eff}} \) depends on the particular substance under consideration.

\(^2\) The effective correlation length \( \xi_{\text{eff}} \) appearing in the KZ mechanism should not be confused with the coherence length commonly introduced in Landau–Ginzburg theory for systems in thermodynamic equilibrium.

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Table 1. Summary of experiments on the defect formation by strongly nonequilibrium symmetry-breaking phase transformations. A positive result (+) implies an agreement with the KZ estimate within a factor of about unity; and a negative result (−), implies a strong disagreement (by a few times to a few orders of magnitude).

| Experimental object | Initiation of phase transition | Method of detection | Dimension | Size | Result | References |
|---------------------|-------------------------------|---------------------|-----------|------|--------|------------|
| Superfluids         | Expansion of sample           | Second sound absorption Calorimetry | 3         | Macro | −      | [6, 7]     |
| $^4$He              | Neutron irradiation          |                     |           |      |        |            |
| $^3$He              |                               | Nuclear magnetic resonance |           |      |        |            |
| Superconductors     | Heating–cooling cycles       | SQUID               | 2         | Macro | −      | [8, 9]     |
| Thin films          |                               | SQUID               | 2         | Macro | −      | [12]       |
| Thin films          |                               | SQUID               | 1         | Macro | +      | [13]       |
| Multi-Josephson-junction loop |               | SQUID               |           |      |        |            |
| Annular Josephson tunnel junctions |               | Voltage measurement |           |      |        |            |

For example, in the symmetry breaking of Higgs fields by the cosmological phase transitions (i.e. generation of mass of the elementary particles) $\xi_{\text{eff}}$ is commonly taken to be $\xi_{\text{eff}} \lesssim c/H_{\text{PT}}$ [4], where $c$ is the speed of light and $H_{\text{PT}}$ is the Hubble constant at the instant of phase transition. In the consideration of vortex generation by a superfluid phase transformation, the corresponding quantity is defined as $\xi_{\text{eff}} \approx c_2 \tau_Q$, where $c_2$ is the speed of the second sound (which is a characteristic rate of propagation of information about the phase of the order parameter), and $\tau_Q$ is the so-called quench time (a characteristic time of the phase transformation), which can be determined as $1/\tau_Q = (1/T)(dT/dt)_{T=T_c}$. Similar definitions of $\xi_{\text{eff}}$ are used also for other condensed-matter systems (superconductors, liquid crystals etc).

1.2. Review of experimental data

Although the KZ mechanism was proposed initially in the context of elementary-particle models admitting the symmetry breaking, much work has been undertaken in the last 15 years to study the same phenomenon in laboratory experiments with condensed matter. These works were originated by Chuang et al [5], dealt with liquid crystals, and about a dozen of subsequent experiments were performed with various superfluid and superconducting systems. They are listed in table 1, which outlines the design

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of each experiment, cites the first publications by each group and summarizes their results.

Analysis of the table reveals a quite interesting tendency: the topological defects are formed most efficiently in the systems of small size or low dimensionality, for example, quasi-1D multi-Josephson-junction loops (MJJL) \[13\] and annular Josephson tunnel junctions \[14\], as well as in microscopic hot bubbles of $^3$He produced by neutron irradiation \[10, 11\]. On the other hand, the concentration of defects in the macroscopic systems of higher dimensionality, e.g. 2D superconductor films \[8, 9, 12\] and 3D volume samples of $^4$He \[6, 7\], was found to be considerably less than theoretical predictions \[4\].

The aim of the present work is to show that the above-mentioned features are natural consequences of the corrections which should be introduced into the standard KZ mechanism in view of the recent experiments on thermal dynamics of Bose–Einstein condensates (BECs) in periodic potentials \[13, 15\]. As will be seen in the next sections, the corresponding thermal corrections result in a quite universal behaviour of the efficiency of defect formation as a function of size and dimensionality of the system.

2. The model of defect formation

2.1. Initial assumptions and equations

Let us consider the simplest $\phi^4$-model of a real scalar field (the order parameter) whose Lagrangian $L = \int L \, dV$ with density

$$L = \frac{1}{2} \left[ \left( \partial_\mu \phi \right)^2 - \left( \nabla \phi \right)^2 \right] - \frac{\lambda}{4} \left[ \phi^2 - \left( \mu^2 / \lambda \right) \right]^2,$$

admits the discrete $\mathbb{Z}_2$ symmetry breaking. (See also discussion of the same model in \[16\].)

As is known, two stable vacuum states of this field (which, for the sake of convenience, will be marked by oppositely directed arrows) are

$$\phi_{\pm} = \pm \phi_0, \quad \text{where} \quad \phi_0 = \mu / \sqrt{\lambda}. $$

The structure of a domain wall (kink) between them, located at $x = x_0$, is described as

$$\phi(x) = \pm \phi_0 \tanh \left( \frac{\mu}{\sqrt{2}} (x - x_0) \right).$$

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3 This table does not show a large number of experiments with liquid crystals, aimed at the detailed study of the inner structure of the defects without measuring their formation rates. The preliminary results \[6\] were subsequently corrected in \[7\], and the initial results \[8\] were improved in \[9\]. It should be mentioned also that the experiments with $^3$He heated by neutron absorption \[10, 11\] involve much uncertainty because of the spatial nonuniformity of the resulting hot bubbles and poor understanding of the processes of conversion of the released nuclear energy into heat.

4 The small-size system is implied here to be a system not much larger than the characteristic size of the defect, so that only a few defects can be accommodated in it. The low-dimensional system is a system which is so thin in one or more directions that the defect covers the entire thickness of the sample, i.e. no multiple defects can be formed in this (these) direction(s).
and the specific energy concentrated in this wall (per unit length or area, depending on the dimensionality) equals

\[ E = \frac{2\sqrt{2}}{3} \mu^3 \lambda. \]  

(4)

Let a domain structure formed after a strongly nonequilibrium phase transformation be approximated by a regular (square, cubic etc) grid with a cell size of about the effective correlation length \( \xi_{\text{eff}} \), whose definition in the KZ scenario was already discussed in section 1.1. The particular value of \( \xi_{\text{eff}} \) is not of importance here, but we shall assume that it is sufficiently large in comparison with a characteristic thickness (\( \sim 1/\mu \)) of the domain wall. As a result, the final pattern of vacuum states (equation (2)) after the phase transformation will look like a distribution of spins on the regular grid.

The key assumption of the KZ mechanism is that the final symmetry-broken states of the field \( \phi \) in two neighbouring cells of the size \( \xi_{\text{eff}} \) are essentially independent of each other. Then, the probability of formation of a domain wall between them is given by the ratio of the number of statistical configurations involving the domain wall to the total number of configurations, \( P_{\text{KZ}} = 2/4 = 1/2 \); and the resulting concentration of the defects (domain walls) will be

\[ n_{\text{KZ}} = \frac{1}{2} \frac{D}{\xi_{\text{eff}}^D}, \]  

(5)

where \( D \) is the effective dimensionality of the system.

However, formula (5) is not sufficiently accurate, because it is based entirely on classical field dynamics and does not take into account the specific coherent effects of quantum fields. Namely, as follows from the latest experimental studies of BECs formed in isolated potential wells, there are pronounced residual correlations between their phases, i.e. a kind of ‘footprint’ of the initial thermal state of the entire system.

### 2.2. Overview of the experiments exhibiting the residual thermal correlations

The residual thermal correlations were observed for the first time in the MJJL experiment [13], whose sketch is presented in figure 1. A thin quasi-1D winding strip was engraved at the boundary between two crystalline grains of \( \text{YBa}_2\text{Cu}_3\text{O}_{7-\delta} \) high-temperature superconductor film, thereby forming a loop of 214 superconductor segments separated by the grain-boundary Josephson junctions. This system experienced multiple heating–cooling cycles in the temperature range 77 K to \( \sim 100 \) K, which covers both the critical temperature of superconducting phase transformation in the segments of the loop (\( T_c = 90 \) K) and in the junctions between them (\( T_{cJ} = 83–85 \) K).

There is evidently no order parameter in the entire loop as long as \( T > T_c \). Next, when the temperature drops below \( T_c \) but remains above \( T_{cJ} \) (i.e. \( T_{cJ} < T < T_c \)), some value of the order parameter should be established in each segment, as is schematically shown by arrows in the right-hand part of figure 1. Since these segments are separated by nonconducting Josephson junctions, it is reasonable to assume that the phase jumps between them are random (i.e. uncorrelated to each other). Finally, when the temperature drops below \( T_{cJ} \), the entire loop becomes superconducting and, due to the above-mentioned jumps, a phase integral along the loop, in general, should be nonzero. As a result the electric current \( I \) circulating along the loop, and the corresponding magnetic flux \( \Phi \), penetrating the loop, will be spontaneously generated.
So, if the phase jumps in the intermediate state $T_{cJ} < T < T_c$ were absolutely uncorrelated, then distribution of spontaneously trapped magnetic flux in the particular experimental setup [13] would be given by the normal (Gaussian) law with a characteristic width of $3.6\phi_0$ (where $\phi_0$ is the magnetic-flux quantum). On the other hand, the actual experimental distribution was found to be over twice as wide; and this anomaly was explained by authors of the experiment assuming that the phase jumps in the intermediate state were not random but correlated to each other, so that the probability $P(\delta_i)$ of the phase difference $\delta_i$, in the $i$th junction, was given just by the Boltzmann law:

$$P(\delta_i) \propto \exp\left[-\frac{E_J(\delta_i)}{T_c}\right],$$

(6)

where $E_J$ is the energy concentrated in the Josephson junction, and $T_c$ is the phase transformation temperature, measured in energy units.

Therefore, MJJL experiment provided the first evidence that a system experiencing the symmetry-breaking phase transformation ‘remembers’ its initial thermal state even if its separate parts cannot communicate with each other immediately during phase transformation. Unfortunately, this experiment did not verify the particular form of the functional dependence (6), because $T_c$ and $E_J$ were fixed by the material properties of the superconductor.

This obstacle was overcome a few years later by studying the interference between BECs of ultracold atoms. In particular, in the experiment of [15] two independent condensate clouds were formed in the wells of an optical potential. Then, the potential was abruptly switched off, the clouds began to expand and interfere with each other, and the experimentalists measured the number of sharp phase jumps (dislocations) in the interference pattern as a function of the initial temperature of the condensates (see figure 4 of [15]). Although the error bars were quite large, it was clearly seen that the probability of defect formation qualitatively followed the same Boltzmann dependence (6), i.e. quickly decreased with a decrease in temperatures.\(^5\)

We will not discuss further the problem of thermal influences on the occurrence of phase defects, but shall assume that formula (6) is typical for all BECs formed by the symmetry-breaking phase transformations. Next, we shall try to answer the question: what will be the resulting effect in the systems of variable size and dimensionality?

\(^5\)To avoid misunderstanding, it should be noted that the fraction of images with dislocations is plotted in figure 4(c) [15] as function of average central contract of the interference pattern, which is, roughly speaking, inversely proportional to temperature. (For more details, see figure 2 of [15].)
Table 2. Formal correspondence between the properties of $\varphi^4$ and Ising models.

| Item                                           | $\varphi^4$-model | Ising model                      |
|------------------------------------------------|-------------------|---------------------------------|
| 1. Elementary domains of the symmetry-broken field | Spins             |                                 |
| 2. Energy of an elementary domain wall $E$       | Effective spin–spin interaction constant $E$ | Variable temperature $T_c$ of a fixed system |
| 3. Critical temperatures $T_c$ for various $\varphi^4$-field systems |                      | Thermodynamical statistical sum $Z^{(D)}$ for spin distribution |
| 4. Statistical sum $Z^{(D)}$ for distribution of the domains with various vacuum states |                      | Thermodynamically equilibrium state |
| 5. State formed after a strongly out-of-equilibrium phase transformation | Ordering phase transition in the spin system |
| 6. Suppression of the domain wall formation      |                   |                                 |

2.3. Improvement of the classical estimates

Following from the above section, the probabilities of various field configurations after the phase transformation in our $\varphi^4$ lattice model should be calculated by taking into account the Boltzmann factors (6). As a result, instead of a random distribution of the domain phases (spins), we obtain a distribution exactly equivalent to the Ising model at some temperature $T_c$, which formally coincides with a critical temperature $T_c$ of the initial $\varphi^4$-model, while the energy of the elementary domain wall $E$ plays the role of an effective spin–spin interaction constant. All aspects of this formal correspondence are summarized in table 2. (The item 5 about the equivalence between the state formed after a strongly nonequilibrium phase transformation in $\varphi^4$-model and a thermodynamically equilibrium state of the Ising model, implies a mathematical identity between statistical sums for distribution of the domains of the symmetry-broken phase in $\varphi^4$-model, on one hand, and for the spin distribution in the Ising model, on the other hand.)

Then, the probability of a domain wall formation can be calculated just as an average energy of the system with domain walls divided by the energy of the elementary domain wall $E$ (i.e. at one boundary between two neighbouring cells) and by the total number of the sites $D N^D$, where these domain walls can be located:

$$P = \frac{T_c^2}{E D N^D} \frac{\partial}{\partial T_c} \ln Z^{(D)}. \quad (7)$$

Here, $N$ is the number of cells along each side of the lattice, and

$$Z^{(D)} = \sum_i \exp(-\varepsilon_i/T_c) \quad (8)$$

is the usual statistical sum over all possible spin configurations of the Ising model, where $\varepsilon_i$ is the total energy of the $i$th configuration.

This fact has a simple physical interpretation: the inner volume of the symmetry-broken cells does not contribute to the energy of the system because the vacuum field values (2) correspond to zero energy. Therefore, in the ‘thin-wall approximation’ all contributions come from the integration over the boundaries (transition regions) between the cells. This looks exactly like the ‘nearest-neighbour interaction’ between spins in the Ising model.
Particularly, for a 1D system with periodic boundary conditions,

\[ Z^{(1)} = \sum_{j=1}^{N} \sum_{s_j = \pm 1} \exp \left\{ - \frac{E}{T_c} \sum_{k=1}^{N} \frac{1}{2} (1 - s_k s_{k+1}) \right\}; \]  

for a 2D system,

\[ Z^{(2)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{s_{ij} = \pm 1} \exp \left\{ - \frac{E}{T_c} \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{1}{2} (2 - s_{kl} s_{k,l+1} - s_{kl} s_{k,l-1}) \right\}, \]

and so on. Here, \( s_k \) and \( s_{kl} \) are the spin-like variables describing the symmetry-broken states in the \( k \)th and \( (kl) \)th cells, respectively.

As is known [17], the Ising model for 1D as well as the finite-size higher-dimensional systems does not experience a phase transition to the ordered state at any value of the ratio \( E/T_c \). From the viewpoint of domain wall formation by strongly out-of-equilibrium phase transformation in the original \( \varphi^4 \)-model (table 2), this means that concentration of the defects will not differ considerably from the standard KZ estimate, because the probability of defect formation \( P \) at the scale of effective correlation length \( \xi_{\text{eff}} \) will not deviate substantially from \( P_{\text{KZ}} = 1/2 \sim 1 \).

On the other hand, the Ising model for the sufficiently large (infinite-size) 2D and 3D systems does experience a phase transition to the ordered state at some value of \( E/T_c \sim 1 \). As a result, the concentration of domain walls in the corresponding \( \varphi^4 \)-model at large ratios \( E/T_c \) should be suppressed dramatically due to formation of macroscopic regions with the same value of the order parameter, covering a great number of cells of the effective correlation length \( \xi_{\text{eff}} \). (To avoid misunderstanding, let us emphasize again that the different values of \( T_c \) should be understood here as critical temperatures of various physical systems described by the \( \varphi^4 \)-model, and they formally correspond to a variable temperature of the fixed Ising system.)

2.4. Particular example

The general conclusions formulated above can be illustrated by the particular example in figure 2, which represents the refined concentration of domain walls \( n \) normalized to standard KZ value, \( n_{\text{KZ}} \), as a function of \( E/T_c \) for the following three cases:

1. 1D infinite-size Ising model, which admits the exact solution (see, for example [18]);
2. 2D 6 \( \times \) 6-cell Ising model with periodic boundary conditions, which simulates quite well the system of infinite size; and
3. 2D 6 \( \times \) 6-cell Ising model with free boundaries (where no energy is concentrated), which is an example of a microscopic system.

(The particular size of 6 cells along each side of the lattice was taken arbitrarily, just as the value at which a numerical computation of the statistical sum (10) is not too cumbersome; for details, see the appendix.)

As is seen in figure 2, at \( E/T_c \sim 1 \) the concentration of defects in the 1D system (dashed green curve) differs from the standard KZ value by less than a factor of two; in microscopic 2D system (dotted blue curve), by three times; while in the macroscopic 2D system (solid red curve) it is suppressed by order of magnitude. Such suppression becomes much stronger when
the ratio $E/T_c$ increases: for example, at $E/T_c \sim 2$ the difference between each of the three values is over an order of magnitude.

3. Conclusions

Following from the above consideration, specific thermal correlations between the phases of BECs in separated spatial subregions revealed for the first time in the MJJL experiment and confirmed later by the experiments with ultracold atoms, can be a promising method for the explanation of the data presented in table 1. The effective Ising model discussed in sections 2.3 and 2.4 predicts a strong suppression of the defect formation in the macroscopic systems with dimensionality $D \geq 2$, as it was observed by studying the strongly out-of-equilibrium phase transformations of superfluids and superconductors (see rows marked by the minus sign in the table).

Unfortunately, results of the very simplified model (1) with a real field $\varphi$ cannot be compared quantitatively with the works cited in table 1, because superfluids and superconductors possess more complex order parameters and Lagrangians than (1). So, consideration of the more realistic models with complex order parameters (possessing $U(1)$ symmetry group) is necessary.

Although a general treatment of the complex case is not easy, it can be expected that the above-mentioned properties will remain valid. For example, a strongly out-of-equilibrium phase transformation in the $\varphi^4$-model, with a complex field $\varphi$ in 2D, should be approximately reduced to the XY-model (where spins are in the plane of the lattice). The presence of the well-known Berezinskii–Kosterlitz–Thouless phase transition in this model suggests that conclusions derived for the Ising model may be still valid.

Next, it is important to emphasize that the refined concentration of defects, obtained in the previous section, $n = f(E/T_c) n_{KZ}(\xi_{\text{eff}}(\tau_Q))$ possesses exactly the same dependence on the quench time $\tau_Q$ as in the classical KZ scenario. So, this dependence, often measured in the

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**Figure 2.** Concentration of the defects $n$ normalized to the standard KZ value $n_{KZ}$ as function of the ratio of the domain wall energy $E$ to the phase transformation temperature $T_c$ for the infinite-size 1D Ising model (dashed green curve), $6 \times 6$-cell 2D Ising model with periodic boundary conditions (solid red curve) and the same model with free boundaries (dotted blue curve).
experiments, cannot serve by itself for a discrimination between the mechanisms; the absolute values of the defect concentration are always necessary.

Finally, let us mention that the ideas described in the present work can also be applied to solving the problem of excessive concentration of topological defects predicted after the cosmological phase transitions of Higgs fields \([19, 20]\).

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**Appendix. Computation of the statistical sums**

As is known, all characteristics of the 1D Ising model can be calculated analytically (for a detailed description see, for example \([18]\)). The resulting formula for the defect (spin reversal) concentration in a chain of length \(N\) with periodic boundary conditions is

\[
\frac{n}{n_{KZ}} = 2 \exp\left(-\frac{E}{T_c}\right) \left(1 + \exp\left(-\frac{E}{T_c}\right)\right)^{N-1} - \left(1 - \exp\left(-\frac{E}{T_c}\right)\right)^{N-1} \left(1 + \exp\left(-\frac{E}{T_c}\right)\right)^N + \left(1 - \exp\left(-\frac{E}{T_c}\right)\right)^N.
\]  

(A.1)

In the case of infinite chain, this expression will take the form:

\[
\frac{n}{n_{KZ}} = \frac{2 \exp\left(-\frac{E}{T_c}\right)}{1 + \exp\left(-\frac{E}{T_c}\right)},
\]  

(A.2)

which is the formula used for plotting the respective curve in figure 2.

For the 2D infinite lattice, statistical sum \((8)\) can be estimated analytically just around the point of phase transition of the Ising model (e.g. monograph \([21]\)). Unfortunately, this result is of little value for the present work, since we need to know the behaviour of the statistical sum over a wide range of temperature. Besides, no analytical results are available for the finite-size lattices, which are also of considerable interest for us. So we used a more straightforward approach.

In general, statistical sum \((8)\) can be rewritten as

\[
Z^{(D)}(k) = \sum_k C_k \exp(-k E / T_c),
\]  

(A.3)

where \(C_k\) are the statistical weights of configurations involving \(k\) elementary domain walls each with energy \(E\). For the particular case of the 2D \(6 \times 6\)-cell Ising model, we have calculated the coefficients \(C_k\) by the methods of computer algebra, both for the lattice with periodic boundary conditions (in which every domain wall carries the energy \(E\)) and with the free boundaries (where only the inner domain walls contribute to the total energy of the configuration). The corresponding values of \(C_k^{(per)}\) and \(C_k^{(free)}\) are listed in tables A.1 and A.2.
Table A.1. Nonzero statistical weight factors for the $6 \times 6$-cell Ising model with periodic boundary conditions.

| $k$ | $C_k^{\text{(per)}}$ and $C_{72-k}^{\text{(per)}}$ | $k$ | $C_k^{\text{(per)}}$ and $C_{72-k}^{\text{(per)}}$ |
|-----|---------------------------------|-----|---------------------------------|
| 0   | 2                               | 17  | 569 080                        |
| 4   | 72                              | 22  | 71 789 328                     |
| 6   | 144                             | 24  | 260 434 986                    |
| 8   | 1 620                           | 26  | 808 871 328                    |
| 10  | 6 048                           | 28  | 2 122 173 684                  |
| 12  | 35 148                          | 30  | 4 616 013 408                  |
| 14  | 159 840                         | 32  | 8 196 905 106                  |
| 16  | 804 078                         | 34  | 11 674 988 208                 |
| 18  | 3 846 576                       | 36  | 13 172 279 424                 |

Table A.2. Nonzero statistical weight factors for the $6 \times 6$-cell Ising model with free boundaries.

| $k$ | $C_k^{\text{(free)}}$ and $C_{60-k}^{\text{(free)}}$ | $k$ | $C_k^{\text{(free)}}$ and $C_{60-k}^{\text{(free)}}$ |
|-----|------------------------------------------------------|-----|------------------------------------------------------|
| 0   | 2                                                   | 16  | 15 444 302                                           |
| 2   | 8                                                   | 17  | 33 435 520                                           |
| 3   | 48                                                  | 18  | 69 487 240                                           |
| 4   | 100                                                 | 19  | 138 380 976                                          |
| 5   | 288                                                 | 20  | 263 185 168                                          |
| 6   | 1 132                                               | 21  | 476 852 512                                          |
| 7   | 3 168                                               | 22  | 821 190 292                                          |
| 8   | 8 824                                               | 23  | 1 340 056 928                                        |
| 9   | 25 744                                              | 24  | 2 065 952 532                                        |
| 10  | 71 064                                              | 25  | 3 000 507 536                                        |
| 11  | 186 624                                             | 26  | 4 093 604 824                                        |
| 12  | 484 210                                             | 27  | 5 230 849 920                                        |
| 13  | 1 214 336                                           | 28  | 6 244 335 166                                        |
| 14  | 2 931 560                                           | 29  | 6 951 501 824                                        |
| 15  | 6 853 760                                           | 30  | 7 206 345 520                                        |

Next, the relative concentration of domain walls is obtained by substitution of (A.3) into (7), and the final result takes the form:

$$n/n_{KZ} = \frac{2}{DN^D} \sum_{k} k C_k \exp(-kE/T_c).$$  \hspace{1cm} (A.4)

In the particular cases plotted in figure 2, we used $D = 2$ and $N = 6$. 

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References

[1] Bogoliubov N N 1966 Suppl. Nuovo Cimento 4 346
[2] Kibble T W B 1976 J. Phys. A: Math. Gen. 9 1387
[3] Zurek W H 1985 Nature 317 505
[4] Klapdor-Kleingrothaus H V and Zuber K 1997 Particle Astrophysics (Bristol: IOP Publishing)
[5] Chuang I, Durrer R, Turok N and Yurke B 1991 Science 251 1336
[6] Hendry P C, Lawson N S, Lee R A M, McClintock P V E and Williams C D H 1994 Nature 368 315
[7] Dodd M E, Hendry P C, Lawson N S, McClintock P V E and Williams C D H 1998 Phys. Rev. Lett. 81 3703
[8] Carmi R and Polturak E 1999 Phys. Rev. B 60 7595
[9] Maniv A, Polturak E and Koren G 2003 Phys. Rev. Lett. 91 197001
[10] Bäuerle C, Bunkov Yu M, Fisher S N, Godfrin H and Pickett G R 1996 Nature 382 332
[11] Ruutu V M H, Eltsov V B, Gill A J, Kibble T W B, Krusius M, Makhlin Yu G, Plaçais B, Volovik G E and Wen Xu 1996 Nature 382 334
[12] Kirtley J R, Tsuei C C and Tafuri F 2003 Phys. Rev. Lett. 90 257001
[13] Carmi R, Polturak E and Koren G 2000 Phys. Rev. Lett. 84 4966
[14] Monaco R, Mygind J and Rivers R J 2002 Phys. Rev. Lett. 89 080603
[15] Hadzibabic Z, Krüger P, Cheneau M, Battelier B and Dalibard J 2006 Nature 441 1118
[16] Lombardo F C, Mazzitelli F D and Rivers R J 2001 Phys. Lett. B 523 317
[17] Rumer Yu B and Ryvkin M Sh 1980 Thermodynamics, Statistical Physics, and Kinetics (Moscow: Mir)
[18] Isihara A 1971 Statistical Physics (New York: Academic)
[19] Dumin Yu V 2000 Hot Points in Astrophysics (Dubna: Joint Institute of Nuclear Research) p 114
[20] Dumin Yu V 2003 Frontiers of Particle Physics (Singapore: World Scientific) p 289
[21] Landau L D and Lifshitz E M 1969 Statistical Physics (Oxford: Pergamon)