Skew Randić Matrix and Skew Randić Energy*

Ran Gu, Fei Huang, Xueliang Li
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China
Email: guran323@163.com, huangfei06@126.com, lxl@nankai.edu.cn

Abstract

Let \( G \) be a simple graph with an orientation \( \sigma \), which assigns to each edge a direction so that \( G^\sigma \) becomes a directed graph. \( G \) is said to be the underlying graph of the directed graph \( G^\sigma \). In this paper we define skew Randić matrix \( R_{S}(G^\sigma) \) as the real skew symmetric matrix \( [ (r_{s})_{ij} ] \) where \( (r_{s})_{ij} = (d_{i}d_{j})^{-\frac{1}{2}} \) if \( v_{i} \rightarrow v_{j} \) is an arc of \( G^\sigma \), otherwise \( (r_{s})_{ij} = (r_{s})_{ji} = 0 \). In this paper, we obtain some properties of the skew Randić energy of a digraph. And we find that the skew Randić energy of a directed tree is independent of its orientation, moreover, it is equal to the Randić energy of the underlying tree. Also, we study the relationship between the spectrum of Randić matrix and skew Randić matrix of a bipartite graph.

Keywords: Skew Randić matrix, Skew Randić energy

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1 Introduction

In this paper we are concerned with simple finite graphs. Undefined notation and terminology can be found in [3]. Let \( G \) be a simple graph with vertex set \( V(G) = \{v_{1}, v_{2}, \ldots, v_{n}\} \), and let \( d_{i} \) be the degree of vertex \( v_{i}, i = 1, 2, \ldots, n. \)

The Randić index [6] of \( G \) is defined as the sum of \( \frac{1}{\sqrt{d_{i}d_{j}}} \) over all edges \( v_{i}v_{j} \) of \( G \). This topological index was first proposed by Randić [6] in 1975 under the name “branching index”. In 1998, Bollobás and Erdős [2] generalized this index as \( R_{\alpha} = R_{\alpha}(G) = \sum_{i \sim j}(d_{i}d_{j})^{\alpha} \), called general Randić index.

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Let $A(G)$ be the $(0, 1)$-adjacency matrix of $G$. The spectrum $Sp(G)$ of $G$ is defined as the spectrum of $A(G)$. The Randić matrix $R = R(G)$ of order $n$ can be viewed as a weighted adjacency matrix, whose $(i, j)$-entry is defined as

$$r_{ij} = \begin{cases} 
0 & \text{if } i = j, \\
(d_id_j)^{-\frac{1}{2}} & \text{if the vertices } v_i \text{ and } v_j \text{ of } G \text{ are adjacent,} \\
0 & \text{if the vertices } v_i \text{ and } v_j \text{ of } G \text{ are not adjacent.}
\end{cases}$$

The polynomial $\varphi_R(G, \lambda) = \det(\lambda I_n - R)$ will be referred to as the $R$–characteristic polynomial of $G$. Here and later by $I_n$ is denoted the unit matrix of order $n$.

The Randić spectrum $Sp_R(G)$ of $G$ is defined as the spectrum of $R(G)$. The Randić spectrum $Sp_R(G)$ of $G$ is defined as the spectrum of $R(G)$. Denote the spectrum $Sp_R(G)$ of $G$ by $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and label them in non-increasing order. The energy of the Randić matrix is defined as $RE = RE(G) = \sum_{i=1}^n |\lambda_i|$ and is called Randić energy.

Let $G$ be a simple graph with an orientation $\sigma$, which assigns to each edge a direction so that $G^\sigma$ becomes a directed graph. $G$ is said to be the underlying graph of the directed graph $G^\sigma$. With respect to a labeling, the skew-adjacency matrix $S(G^\sigma)$ is the real skew symmetric matrix $[s_{ij}]$ where $s_{ij} = 1$ and $s_{ji} = -1$ if $v_i \rightarrow v_j$ is an arc of $G^\sigma$, otherwise $s_{ij} = s_{ji} = 0$.

Similarly, we define a skew Randić matrix $R_s(G^\sigma)$ of order $n$, whose $(i, j)$-entry is

$$s_{ij} = \begin{cases} 
(d_id_j)^{-\frac{1}{2}} & \text{if } v_i \rightarrow v_j, \\
-(d_id_j)^{-\frac{1}{2}} & \text{if } v_j \rightarrow v_i, \\
0 & \text{Otherwise.}
\end{cases}$$

If $G$ does not possess isolated vertices, and $\sigma$ is an orientation of $G$, then it is easy to check that

$$R_s(G^\sigma) = D^{-\frac{1}{2}}S(G^\sigma)D^{-\frac{1}{2}},$$

where $D$ is the diagonal matrix of vertex degrees.

The polynomial $\varphi_{R_s}(G, \lambda) = \det(\lambda I_n - R_s)$ will be referred to as the $R_s$-characteristic polynomial of $G^\sigma$. It is obvious that $R_s(G^\sigma)$ is a real skew symmetric matrix. Hence the eigenvalues $\{\rho_1, \rho_2, \ldots, \rho_n\}$ of $R_s(G^\sigma)$ are all purely imaginary numbers. The skew Randić spectrum $Sp_{R_s}(G^\sigma)$ of $G^\sigma$ is defined as the spectrum of $R_s(G^\sigma)$.

The energy of $R_s(G^\sigma)$, called skew Randić energy can be defined as the sum of its singular values, and equals to the sum of the absolute values of its eigenvalues. If we denote the skew Randić energy of $G^\sigma$ by $RE_s(G^\sigma)$, then $RE_s(G^\sigma) = \sum_{i=1}^n |\rho_i|$.
2 Basic properties of skew Randić energy

Theorem 2.1. Let \( \varphi_{R_s}(G, \lambda) = \det(\lambda I_n - R_s) = c_0 \lambda^n + c_1 \lambda^{n-1} + \ldots + c_{n-1} \lambda + c_n \) be the \( R_s \)-characteristic polynomial of \( G^\sigma \). Then (i) \( c_0 = 1 \), (ii) \( c_1 = 0 \), (iii) \( c_2 = R_{-1}(G) \), the general Randić index with \( \alpha = -1 \) and (iv) \( c_i = 0 \) for all odd \( i \).

Proof. We can obtain Theorem 2.1.

Theorem 2.2.

Proof. We can obtain Theorem 2.2.

Note that the determinant of a skew-symmetric matrix of odd order is zero. For odd \( i \), \( c_i \) is equal to the sum of determinants of all the \( i \times i \) principal submatrices of \( R_s \). Since a principal submatrix of a skew-symmetric matrix is also skew-symmetric, \( c_i = 0 \) clearly follows.

Theorem 2.2. \( \sqrt{2R_{-1}(G) + n(n - 1)p^\frac{2}{n}} \leq RE_s(G^\sigma) \leq \sqrt{2R_{-1}(G)n} \), where \( p = |\det R_s| = \prod_{i=1}^{n} |\rho_i| \).

Proof. \( \sum_{i=1}^{n} \rho_i^2 = tr(R_s^2) = \sum_{i=1}^{n} \sum_{k=1}^{n} (r_s)_{ik}(r_s)_{ki} = - \sum_{i=1}^{n} \sum_{k=1}^{n} (r_s)_{ik}^2 = -2R_{-1}(G) \)

Since \( R_s \) is skew-symmetric, \( \rho_i^2 = -|\rho_i|^2 \) and so \( \sum_{i=1}^{n} |\rho_i|^2 = 2R_{-1}(G) \). By applying the Cauchy-Schwarz inequality we have that

\[
RE_s(G^\sigma) = \sum_{i=1}^{n} |\rho_i| \leq \sqrt{ \sum_{i=1}^{n} |\rho_i|^2 \sqrt{n} } \leq \sqrt{2nR_{-1}(G)}. \tag{1}
\]

By using arithmetic geometric average inequality, one can get that

\[
\sum_{i \neq j} |\rho_i||\rho_j| \geq n(n - 1)(\prod_{i=1}^{n} |\rho_i|)^2 = n(n - 1)p^\frac{2}{n}.
\]

Then

\[
[RE_s(G^\sigma)]^2 = [\sum_{i=1}^{n} |\rho_i|^2]^2 = \sum_{i=1}^{n} |\rho_i|^2 + \sum_{i \neq j} |\rho_i||\rho_j| \geq 2R_{-1}(G) + n(n - 1)p^\frac{2}{n}.
\]

Therefore we can obatian the lower bound of skew Randić energy,

\[
RE_s(G^\sigma) \geq \sqrt{2R_{-1}(G) + n(n - 1)p^\frac{2}{n}}. \tag{2}
\]
3 Skew Randić energies of trees

It is well known that the skew energy of a directed tree is independent of its orientation[1]. In this section, we investigate the skew Randić energy for trees. Similarly, we present a basic lemma. The proof is also similar to the proof given in [1].

Lemma 3.1. Let $D$ be a digraph, and let $D'$ be the digraph obtained from $D$ by reversing the orientations of all the arcs incident with a particular vertex of $D$. Then $RE_s(D) = RE_s(D')$.

Proof. Let $R_s(D)$ be the skew Randić matrix of $D$ of order $n$ with respect to a labeling of its vertex set. Suppose the orientations of all the arcs incident at vertex $v_i$ of $D$ are reversed. Let the resulting digraph be $D'$. Then $R_s(D') = P_i R_s(D) P_i$, where $P_i$ is the diagonal matrix obtained from the identity matrix of order $n$ by changing the $i$–th diagonal entry to $-1$. Hence $R_s(D)$ and $R_s(D')$ are orthogonally similar, and so have the same eigenvalues, and hence $D$ and $D'$ have the same skew Randić energy.

Let $\sigma$ be an orientation of a graph $G$. Let $W$ be a subset of $V(G)$ and $\overline{W} = V(G) \setminus W$. The orientation $\tau$ of $G$ obtained from $\sigma$ by reversing the orientations of all arcs between $W$ and $\overline{W}$ is said to be obtained from $G^{\sigma}$ by a switching with respect to $W$. Moreover, two oriented graphs $G^{\tau}$ and $G^{\sigma}$ of $G$ are said to be switching-equivalent if $G^{\tau}$ can be obtained from $G^{\sigma}$ by a switching.

From lemma 3.1 we know that if $G^{\tau}$ and $G^{\sigma}$ are switching-equivalent, they have the same skew Randić energy.

Lemma 3.2. [1] Let $T$ be a labeled directed tree rooted at vertex $v$. It is possible, through reversing the orientations of all arcs incident at some vertices other than $v$, to transform $T$ to a directed tree $T'$ in which the orientations of all the arcs go from low labels to high labels.

We can also show that the skew energy of a directed tree is independent of its orientation by using Lemma 3.1 and Lemma 3.2.

Theorem 3.3. The skew Randić energy of a directed tree is independent of its orientation.

Corollary 3.4. The skew Randić energy of a directed tree is the same as the Randić energy of its underlying tree.
4 The relationship between $Sp_{R_s}(G^\sigma)$ and $iSp_R(G)$

The relationship between $Sp_s(G^\sigma)$ and $iSp(R)$ has been concerned in [7]. Similarly, we concentrate on the relationship between $Sp_{R_s}(G^\sigma)$ and $iSp_R(G)$, and we obtain some analogous results. The following two lemmas given in [7] will be used.

**Lemma 4.1.** [7] Let $A = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix}$ be two real matrices. Then $Sp(B) = iSp(A)$.

Let $|X|$ denote the matrix whose entries are the absolute values of the corresponding entries in $X$. For real matrices $X$ and $Y$, $X \leq Y$ means that $Y - X$ has nonnegative entries. $\rho(X)$ denotes the spectral radius of a square matrix $X$.

**Lemma 4.2.** [7] Let $A$ be an irreducible nonnegative matrix and $B$ be a real positive semi-definite matrix such that $|B| \leq A$ (entry-wise) and $\rho(A) = \rho(B)$. Then $A = DBD$ for some real matrix $D$ such that $|D| = I$, the identity matrix.

**Theorem 4.3.** $G$ is a bipartite graph if and only if there is an orientation $\sigma$ such that $Sp_{R_s}(G^\sigma) = iSp_R(G)$.

**Proof.** (Necessity) If $G$ is bipartite, then there is a labeling such that the Randić matrix of $G$ is of the form

$$R(G) = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}.$$ 

Let $\sigma$ be the orientation such that the skew Randić matrix of $G^\sigma$ is of the form

$$R_s(G^\sigma) = \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix}.$$ 

By Lemma 4.1, $Sp_{R_s}(G^\sigma) = iSp_R(G)$.

(Sufficiency) Suppose that $Sp_{R_s}(G^\sigma) = iSp_R(G)$, for some orientation $\sigma$. Since $R_s(G^\sigma)$ is a real skew symmetric matrix, $Sp_{R_s}(G^\sigma)$ has only pure imaginary eigenvalues and so is symmetric about the real axis. Then $Sp_R(G) = -iSp_{R_s}(G^\sigma)$ is symmetric about the imaginary axis. Hence $G$ is bipartite.

Trees are special bipartite graphs, actually, we will prove that $G$ is a tree if and only if for any orientation $\sigma$, $Sp_{R_s}(G^\sigma) = iSp_R(G)$. The next lemma plays an important role in the proof of above statement.
Lemma 4.4. Let \( X = \begin{pmatrix} C & \ast \\ \ast & \ast \end{pmatrix} \) be a nonnegative matrix, where \( C = (c_{ij}) \) is a \( k \times k \) \((k > 2)\) matrix whose nonzero entries are \( c_{i,i-1} \) and \( c_{i,i} \) with the subscripts modulo \( k \), for \( 1 \leq i \leq k \). Let \( Y \) be obtained from \( X \) by changing the \((1,1)\) entry to \(-c_{1,1}\). If \( X^T X \) is irreducible then \( \rho(X^T X) > \rho(Y^T Y) \).

Proof. Note that \( |Y^T Y| \leq X^T X \) (entry-wise), and so \( \rho(X^T X) \geq \rho(Y^T Y) \) by Perron-Frobenius theory [5]. Now suppose that \( \rho(X^T X) = \rho(Y^T Y) \). Since \( X^T X \) is irreducible, by Lemma 4.2 there exists a signature matrix \( D = \text{Diag}(d_1, d_2, \ldots, d_n) \) such that \( X^T X = DY^T YD \). Therefore \( [X^T X]_{ij} = d_id_j[Y^T Y]_{ij} \) for all \( i, j \). Now, for \( i = 1, \cdots, k-1 \), \( [X^T X]_{i,i+1} = [Y^T Y]_{i,i+1} \neq 0 \). Using \( [X^T X]_{ij} = d_id_j[Y^T Y]_{ij} \), we have \( d_id_j = 1 \) for \( i = 1, \cdots, k-1 \). Hence \( d_id_k = 1 \). On the other hand, let \( [X^T X]_{1k} = c_{1,1}c_{1,k} + M \), we have \(-c_{1,1}c_{1,k} + M = d_id_k[Y^T Y]_{1k} = [X^T X]_{1k} = c_{1,1}c_{1,k} + M \), which is impossible. \( \square \)

Theorem 4.5. \( G \) is a tree if and only if for any orientation \( \sigma \), \( \text{Sp}_{R_\sigma}(G^\sigma) = i \text{Sp}_{R}(G) \).

Proof. (Necessity) See the proof of Theorem 3.3.

(Sufficiency) Suppose that \( \text{Sp}_{R_\sigma}(G^\sigma) = i \text{Sp}_{R}(G) \), for any orientation \( \sigma \). By Theorem 4.3 \( G \) is a bipartite graph. So there is a labeling such that the Randić matrix of \( G \) is of the form

\[
R(G) = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix},
\]

where \( X \) is a nonnegative matrix. Since \( G \) is connected, \( X^T X \) is indeed a positive matrix and so irreducible. Now assume that \( G \) is not a tree. Then \( G \) has at least an even cycle because \( G \) is bipartite. W.L.O.G. \( X \) has the form \( \begin{pmatrix} C & \ast \\ \ast & \ast \end{pmatrix} \) where \( C = (c_{ij}) \) is a \( k \times k \) \((k > 2)\) matrix whose nonzero entries are \( c_{i,i-1} \) and \( c_{i,i} \) with the subscripts modulo \( k \), for \( 1 \leq i \leq k \). Let \( Y \) be obtained from \( X \) by changing the \((1,1)\) entry to \(-c_{1,1}\). Consider the orientation \( \sigma \) of \( G \) such that

\[
R_\sigma(G^\sigma) = \begin{pmatrix} 0 & Y \\ -Y^T & 0 \end{pmatrix}.
\]

By hypothesis, \( \text{Sp}_{R_\sigma}(G^\sigma) = i \text{Sp}_{R}(G) \) and hence \( X \) and \( Y \) have the same singular values. It follows that \( \rho(X^T X) = \rho(Y^T Y) \), which contradicts Lemma 4.4. \( \square \)

Corollary 4.6. \( G \) is a forest if and only if for any orientation \( \sigma \), \( \text{Sp}_{R_\sigma}(G^\sigma) = i \text{Sp}_{R}(G) \).

Proof. (Necessity) Let \( G = G_1 \cup \ldots \cup G_r \), where \( G_j \)'s are trees. Then \( G^\sigma = G_1^\sigma \cup \ldots \cup G_r^\sigma \).

By Theorem 4.5 \( \text{Sp}_{R_\sigma}(G_j^\sigma) = i \text{Sp}_{R}(G_j) \) for all \( j = 1, 2, \cdots, r \). Hence \( \text{Sp}_{R_\sigma}(G^\sigma) = \text{Sp}_{R_\sigma}(G_1^\sigma) \cup \cdots \cup \text{Sp}_{R_\sigma}(G_r^\sigma) = i \text{Sp}_{R}(G_1) \cup \cdots \cup i \text{Sp}_{R}(G_r) = i \text{Sp}_{R}(G_1 \cup \ldots \cup G_r) = i \text{Sp}_{R}(G) \).
(Sufficiency) Suppose that $G$ is not a forest. Then $G = G_1 \cup \ldots \cup G_r$ where $G_1 \ldots G_t$ are connected, but not trees, and $G_{t+1} \ldots G_r$ are trees. By Theorem 4.3, $G$ is a bipartite graph. So there is a labeling such that the Randić matrix of $G$ is of the form

$$ R(G) = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}, $$

where $X = X_1 \oplus \ldots \oplus X_r$. Let $Y_j$ be obtained from $X_j$ by changing the $(1,1)$ entry to its negative. Consider the orientation $\sigma$ of $G$ such that

$$ R_\sigma(G^\sigma) = \begin{pmatrix} 0 & Y \\ -Y^T & 0 \end{pmatrix}. $$

where $Y = Y_1 \oplus \ldots \oplus Y_r$. By Lemma 4.1, $\text{Sp}_{R_\sigma}(G^\sigma) = \text{iSp}_R(G)$ implies that the singular values of $X$ coincide with the singular values of $Y$. Since $G_{t+1} \ldots G_r$ are trees, the singular values of $X_j$ coincide with the singular values of $Y_j$ for $j = t+1, \ldots, r$. Hence the singular values of $X_1 \oplus \ldots \oplus X_t$ coincide with the singular values of $Y_1 \oplus \ldots \oplus Y_t$. Since $G_1 \ldots G_t$ are not trees, we have $\rho(X_j^T X_j) > \rho(Y_j^T Y_j)$ for $j = 1, \ldots, t$. Consequently, for some $j_0$,

$$ \max_{1 \leq j \leq t} \rho(X_j^T X_j) = \max_{1 \leq j \leq t} \rho(Y_j^T Y_j) = \rho(Y_{j_0}^T Y_{j_0}) < \rho(X_{j_0}^T X_{j_0}) \leq \max_{1 \leq j \leq t} \rho(X_j^T X_j). $$

a contradiction.

\[ \square \]

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