A geometric interpretation of the multiplication of complex numbers

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Abstract. In complex analysis courses, it is common to use physical interpretations as a didactic tool for teaching complex numbers. In the case of operations between complex numbers, the geometric interpretation of addition and subtraction is well known; however, many authors avoid the interpretation of the multiplication of complex numbers. In this paper, using the physical concepts of rotation and scaling, we will explain the multiplication of complex numbers through visualization in the Argand plane. In addition, we use visual representations in order to obtain proofs without words for some identities.

1. Introduction

Teaching mathematics and physics in engineering represents an opportunity to improve didactic methods; mathematical competencies must be developed for the student to perform applications in their study, keeping the mathematical rigor. There are many methods to facilitate teaching and learning of advanced mathematics courses, history as a didactic resource and didactic modeling are essential for the understanding of the origin of mathematical and physical concepts and its relevance in real life [1]; in the case of complex analysis, geometric interpretations of complex numbers are the main didactic method, but specific contents are not fully utilized [2]. Here we use visualization as a didactic tool, to explain the physical steps of the multiplication of complex numbers.

Following [3], visualization is the product and the process of creation, interpretation and reflection upon pictures, images and diagrams. We verify that in the course of advanced mathematics or complex analysis, in the majority of the cases, the study texts present complex numbers as objects that have an algebraic and geometric representation [4], next they present the geometric interpretation for sum and subtract of complex numbers. However, when defining the complex number product, they focus on numerical aspects and avoid the geometric interpretation [5,6].

Comparing the geometry of the real number line and the Argand diagram creates conceptual connections across various mathematical objects; however, researches that study geometrical representations for teaching complex numbers avoid the multiplication of complex numbers [7]. For this reason, it is important to review the texts that we usually find as sources of associated research, which show a disarticulation of the product of complex numbers and their geometric representation [8].
2. Mathematical background

Equations without real solutions, such as \( x^2 + 1 = 0 \), were observed early in history, Girolamo Cardano (1501-1576) was the first to use complex numbers for solving equations [9]; in 1732, Leonard Euler (1707-1783) introduce the symbol \( i \), defined as \( i = \sqrt{-1} \), years later Euler introduce the formula \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \), known as Euler’s formula. As consequence, we have \( e^{i\pi} + 1 = 0 \), currently known as the most beautiful theorem in mathematics [10]. Independently, Caspar Wessel (1745-1818), Carl Friedrich Gauss (1777-1855) and Jean-Robert Argand (1768-1822), introduce a geometric representation for complex numbers as points, or vectors, in the plane [11]. William Hamilton (1805-1865) present an algebraic definition as ordered pairs of real numbers, the Equation (1) shows the definition of sum and substraction of complex numbers.

\[
(a, b) \pm (c, d) = (a \pm c, b \pm d).
\]

Similarly, considering complex numbers as pairs of real numbers, the Equation (2) shows the multiplication of complex numbers.

\[
(a, b)(c, d) = (ac - bd, ad + bc).
\]

For any couple \((a, b)\) it is equivalent to the complex number \(a + bi\), cf. [12]. Complex numbers can be represented as points on the complex plane, also called Argand diagram [13], where the complex number \(z = a + bi\) is associated with the point \((a, b)\). Thus, \(Re(z) = a\) is associated with points on the \(x\)-axis and \(Im(z) = b\) correspond to points on the \(y\)-axis; in this context, \(x\)-axis and \(y\)-axis are called real and imaginary axis respectively. The Figure 1, shows the geometric interpretation of the sum of complex numbers, the Figure 2 shows the geometric interpretation of substraction. Substraction of complex numbers can be expressed in terms of a sum, \(z_1 - z_2 = z_1 + (-z_2)\).

![Figure 1. Geometric interpretation for the sum of complex numbers.](image1)

![Figure 2. Geometric interpretation substraction of complex numbers.](image2)

In this paper, we present a didactic proposal for teaching the multiplication of complex numbers through rotations and scaling on the complex plane; also, we present proofs without words for some identities, using Geogebra as a visualization software tool.
and $\overline{z}$, it is easy to check that $Arg(\overline{z}) = -Arg(z)$. From Figure 1 and Figure 2 we may deduce the parallelogram law.

3. Multiplication of complex numbers through Argand diagram
Since the product of complex numbers is not related to scalar multiplication, it is not easy to obtain a geometric interpretation. Here we present the following process for multiplication through the geometrical representation of complex numbers.

(i) Perform a rotation of the complex plane, such that the complex number $z_1$ is on the $x$-axis of the rotate plane.
(ii) The rotated plane is scaled, such that $z_1$ is $(1,0)$ in that plane.
(iii) Locate $z_2$ in the rotated complex plane and mark this point.
(iv) The marked point, considering from the original complex plane, represents $z_1z_2$.

For instance, we calculate $z_1z_2$ with $z_1 = 1+i$ and $z_2 = 1-3i$ following the described process. We start performing a rotation of $arg(z_1)$ in the complex plane. We perform a scaling in the rotated complex plane in such way that $z_1 = 1 + i$ represents $(1,0)$. In Figure 3, we see the rotated and scaled complex plane in blue.

Starting from the origin, in Figure 4 we locate $z_2 = 1 - 3i$ on the rotated complex plane. In Figure 5, we locate the obtained point in the initial complex plane; therefore, we conclude that $(1+i)(1-3i) = 4 - 2i$.

Some proofs without words via the geometric interpretation of the multiplication of complex numbers can be obtained; for instance, $z_1 = 1$ does not perform a rotation on the complex
plane, thus $1w = w$ for each $w \in \mathbb{C}$; multiplying $z$ by $i^n$, with $n$ a positive integer, means a counterclockwise rotation of $\frac{\pi}{2}n$. Let’s verify the identity $z\overline{z} = |z|^2$ considering $z = 1 + i$, therefore in Figure 6 we use the rotated plane presented in Figure 3.

Let $z$ and $w$ be complex numbers with $z = 1 + i$ and $w = 2 - 2i$, since $w = 2\overline{z}$ we get $zw = 2z\overline{z} = 2|z|^2$, from the above observation we may deduce $|\alpha z| = |\alpha||z|$.

In Figure 7 we perform the rotation and the scaling of the complex plane by the complex $w = 1 + i$; in Figure 8 we locate $w$ in the rotated plane; thus, by Figure 9 we conclude that $(1 + i)(2 - 2i) = 4$.

In Figure 10 we perform the rotation and the scaling of the complex plane by the complex $w = 2 - 2i$; in Figure 11 we locate $z$ in the rotated plane, thus obtaining the result of $wz$ in Figure 12. Since Figure 9 and Figure 12 show the same result we verify the commutative law $zw = wz$. Since $zw$ is calculated by rotating the complex plane $arg(z)$ and locating $w$ in the rotated plane we may deduce that $arg(zw) = arg(z) + arg(w)$ and $|zw| = |z||w|$, hence we may introduce the polar form of complex numbers.
4. Conclusions
The use of physical concepts facilitates the visualization of multiplication of complex numbers, thus turning an abstract object into one with a representation in the complex plane. To explain the visualization process, we follow steps described through physical concepts as rotation, scaling, and locating. Proofs without words for some identities through visualization were presented, they can be considered less elegant than the formal proofs; however, for students, they are easier to understand than formal proofs due to the chance to perform rotations and scaling by software tools.

The didactic proposal that we have presented is a useful method for teaching complex numbers through visual representations of their operations. It was performed at Universidad ECCI, generating better results than the formal proofs; students use smartphones and technological devices for interacting with an app developed in Geogebra, they investigate the identity 

\[ z(w + v) = zw + zv \]

for \( v \), \( w \), \( z \in \mathbb{C} \).

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