Spin in the \(q\)-Deformed Poincaré Algebra

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Abstract
We investigate spin as algebraic structure within the \(q\)-deformed Poincaré algebra, proceeding in the same manner as in the undeformed case. The \(q\)-Pauli-Lubanski vector, the \(q\)-spin Casimir, and the \(q\)-little algebras for the massless and the massive case are constructed explicitly.

1 Introduction

From the beginnings of quantum field theory it has been argued that the pathological ultraviolet divergences should be remedied by limiting the precision of position measurements by a fundamental length [1–3]. In view of how position-momentum uncertainty enters into quantum mechanics, a natural way to integrate such a position uncertainty in quantum theory would have been to replace the commutative algebra of space observables with a non-commutative one [4]. However, deforming the space alone will in general break the symmetry of spacetime. In order to preserve a background symmetry the symmetry group must be deformed together with the space it acts on. This reasoning led to the discovery of quantum groups [5], that is, generic methods to continuously deform Lie algebras [6–8] and matrix groups [8–10] within the category of Hopf algebras. Starting from the non-commutative plane [11], the \(q\)-deformations of a series of objects, differential calculi on non-commutative spaces [12], Euclidean space [9], Minkowski space [13], the Lorentz group and the Lorentz algebra [14–17], led to the \(q\)-deformed Poincaré algebra [18,19].

Describing the symmetry of flat spacetime, the Poincaré group or, equivalently, its enveloping algebra is sufficient to construct special relativity and even a considerable part of relativistic quantum theory. More precisely, Wigner has
shown that a free elementary particle can be identified with an irreducible Hilbert space representation of the Poincaré group [20]. These representations are constructed using the method of induced representations, which reduces the representation theoretic problem to a structural analysis of the Poincaré algebra. In mathematical terms: We start from a representation of the inhomogeneous part of the algebra, determine the stabilizer (little algebra) of this representation, construct the irreducible representations of the stabilizer, and, finally, induce these representations to representations of the entire algebra, yielding all irreducible representations of the Poincaré algebra.

Seemingly abstract, each step in this construction has a clear physical interpretation: The representation of the inhomogeneous part is the description of a momentum eigenspace. The stabilizer is the spin symmetry lifting a possible momentum degeneracy. The representation of the stabilizer defines the transformations of the spin degrees of freedom, the canonical example being a massive spin particle at rest carrying a representation of SU(2). Finally, the induction is the boosting of a rest state to arbitrary momentum. We see that this procedure is not only the mathematical means to construct the wanted representations, but provides insight in the physical nature of spin. It tells us that there is spin, because in general momentum is not sufficient to characterize a particle uniquely. It tells us what the symmetry structure of the spin degrees of freedom is, SU(2) in the massive case but ISO(2) in the massless case. And it tells us, that momentum and spin are all possible exterior degrees of freedom of a particle.

The physical line of thought described in the last paragraph relies on the sole assumption that the Poincaré algebra describes the basic symmetry of spacetime. The \( q \)-deformed Poincaré algebra has been constructed to describe the basic symmetry of \( q \)-deformed space time. Therefore, we can proceed in exactly the same manner to find out what \( q \)-deformed spin is.

In Sec. 2 we review the \( q \)-Poincaré algebra with focus on its general structure. In Sec. 3 we define the key properties of a useful \( q \)-deformed Pauli-Lubanski vector and present such a vector in theorem 1. Its square yields the spin Casimir. Sec. 4 uses this \( q \)-Pauli-Lubanski vector to compute the \( q \)-little algebras for both the massive and the massless case.

Throughout this article, it is assumed that \( q \) is a real number \( q > 1 \). We will frequently use the abbreviations \( \lambda = q - q^{-1} \) and \([j] = \frac{q^j - q^{-j}}{q - q^{-1}}\) for a real number \( j \), in particular \([2] = q + q^{-1}\). The lower case Greek letters \( \mu, \nu, \sigma, \tau \) denote 4-vector indices running through \( \{0, -, +, 3\} \). The upper case Roman letters \( A, B, C \) denote 3-vector indices running through \( \{-1, 0, +1\} = \{-, 3, +\} \).

2 The \( q \)-Deformed Poincaré Algebra

The \( q \)-Poincaré algebra can be defined very explicitly by listing its generators and the commutation relations between them. This has been done in Appendix A.
Here, we give an overview of the more general algebraic structure.

The $q$-Lorentz algebra $\mathcal{H} = \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ is a Hopf-* algebra, with coproduct $\Delta$, counit $\varepsilon$, and antipode $S$. We will also use the Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$. Several forms of the $q$-Lorentz algebra can be found in the literature, which are essentially equivalent. Here, it is natural to use the form, where $\mathcal{H}$ is described as Drinfeld double of $\mathcal{U}_q(\mathfrak{su}_2)$ with its dual $SU_q(2)^{\text{op}}$ \cite{[14]},

$$\mathcal{H} = \mathcal{U}_q(\mathfrak{su}_2) \bowtie SU_q(2)^{\text{op}},$$

that is, the Hopf-* algebra generated by the algebra of rotations $\mathcal{U}_q(\mathfrak{su}_2)$ and the algebra of boosts $SU_q(2)^{\text{op}}$ with cross commutation relations

$$bl = \langle l_{(1)}, b_{(1)} \rangle l_{(2)} b_{(2)} \langle S(l_{(3)}), b_{(3)} \rangle$$

for all $l \in \mathcal{U}_q(\mathfrak{su}_2)$, $b \in SU_q(2)^{\text{op}}$, where $\langle l, b \rangle$ denotes the dual pairing. In addition to the Drinfeld-Jimbo generators $E, F, K = q^H$ of $\mathcal{U}_q(\mathfrak{su}_2)$ we will also use the the Casimir operator $W$ and the 3-vector $\{J_A\} = \{J_-, J_+, J_3\}$ of angular momentum. The generators $a, b, c, d$ of boosts form a multiplicative quantum matrix $(a \ b)$. \mathcal{H} possesses two universal $\mathcal{R}$-matrices, $\mathcal{R}_I$ and $\mathcal{R}_{II}$, the first of which is antireal $\mathcal{R}_I^{* \otimes *} = \mathcal{R}_I^{-1}$, the second is real $\mathcal{R}_{II}^{* \otimes *} = \mathcal{R}_{II21}$. We often write in a Sweedler like notation $\mathcal{R} = \mathcal{R}_{[1]} \otimes \mathcal{R}_{[2]}$.

The $q$-Minkowski space algebra $\mathcal{X} = \mathbb{R}^{1,3}_q$ is generated by the 4-momentum vector $\{P_{\mu}\} = \{P_0, P_-, P_+, P_3\}$ with relations

$$P_{\mu} P_{\nu} R_{I}^{\mu \sigma \tau} = P_\nu P_\sigma \leftrightarrow P_{\mu} P_{\nu} (R_{I}^{-1})^{\mu \nu \sigma \tau} = P_\tau P_\sigma ,$$

where the $R$-matrix $R_{I}^{\mu \nu \sigma \tau} = \Lambda((\mathcal{R}_{I[1]})^{\mu \nu}) \Lambda((\mathcal{R}_{I[2]})^{\nu \tau})$ is the 4-vector representation of $\mathcal{R}_I$. The 4-momentum vector is the basis of this 4-vector representation of $\mathcal{H}$, $h \triangleright P_\nu \equiv P_\mu \Lambda(h)^{\mu \nu}$, where $\Lambda$ is the representation map. Relations (3) are the only homogeneous commutation relations of $\mathcal{X}$, which are consistent with this representation and which have the right commutative limit. Consistency means that $\mathcal{X}$ is a left $\mathcal{H}$-module *-algebra, that is,

$$h \triangleright x x' = (h_{(1)} \triangleright x)(h_{(2)} \triangleright x'), \quad (h \triangleright x)^* = (Sh)^* \triangleright x^*$$

for all $h \in \mathcal{H}$, $x \in \mathcal{X}$.

The $q$-Poincaré algebra $\mathcal{A}$ is the Hopf semidirect product

$$\mathcal{A} = \mathcal{X} \rtimes \mathcal{H},$$

the *-algebra generated by the *-algebras $\mathcal{X}$ and $\mathcal{H}$ with cross commutation relations $h x = (h_{(1)} \triangleright x) h_{(2)}$. More accurately, we have the following

**Definition 1.** Let $\mathcal{H}$ be a Hopf-* algebra and $\mathcal{X}$ a left $\mathcal{H}$-module *-algebra. The semidirect product $\mathcal{X} \rtimes \mathcal{H}$ is the *-algebra defined as the vector space $\mathcal{X} \otimes \mathcal{H}$ with multiplication

$$(x \otimes h)(x' \otimes h') := x(h_{(1)} \triangleright x') \otimes h_{(2)} h'$$
and \( \ast\)-structure \((x \otimes h)^\ast = (1 \otimes h^\ast)(x^\ast \otimes 1)\). We often abbreviate \(x \equiv x \otimes 1\) and \(h \equiv 1 \otimes h\).

There is a left and a right Hopf adjoint action of \(\mathcal{H}\) on \(\mathcal{A}\) defined as

\[
\text{ad}_L h \triangleright a := h_{(1)} a S(h_{(2)}), \quad a \triangleleft \text{ad}_R h := S(h_{(1)}) a h_{(2)}. \tag{7}
\]

The commutation relations (6) are precisely such that the left Hopf adjoint action of \(\mathcal{H}\) on \(\mathcal{X}\) equals the module action \(\text{ad}_L h \triangleright x = h \triangleright x\). Let \(\rho\) be a finite representation of the \(q\)-Lorentz algebra \(\mathcal{H}\). We call a set of operators \(\{T_i\}\) a left or a right \(\rho\)-tensor operator if

\[
\text{ad}_L h \triangleright T_j = T_i \rho(h)^{i}_{j} \quad \text{or} \quad T_j \triangleleft \text{ad}_L h = T_i \rho(S^{-1}h)^{i}_{j} \tag{8}
\]

holds, respectively, for all \(h \in \mathcal{H}\). By definition of the \(q\)-Poincaré algebra, the momenta \(P_\mu\) form a left \(\Lambda\)-tensor operator, that is, a left 4-vector operator.

\section{The \(q\)-Pauli-Lubanski Vector and the Spin Casimir}

\subsection{Defining Properties of the Pauli-Lubanski Vector}

In the undeformed case one defines the Pauli-Lubanski (pseudo) 4-vector operator

\[
W^{q=1}_\mu := -\frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} L^{\nu\sigma} P^\tau, \tag{9}
\]

where \(\varepsilon\) is the totally antisymmetric tensor, \(L^{\nu\sigma}\) the matrix of Lorentz generators, and \(P^\tau\) the momentum 4-vector. It is useful because each component of \(W^{q=1}_\mu\) commutes with each component of \(P^\nu\), from which follows that the 4-vector square \(W^2 = \eta^{\mu\nu} W^{q=1}_\mu W^{q=1}_\nu\) is a Casimir operator. The eigenvalues of this Casimir operator are \(-m^2 s(s+1)\) where \(s\) is the spin. Therefore, \(W^{q=1}_\mu\) can be viewed as square root of the spin Casimir.

In the \(q\)-deformed case we can try to define \(W_\mu\) by Eq. (9), as well, with the \(q\)-deformed versions of the epsilon tensor, of the matrix of Lorentz generators, and of the momenta. By construction, this definition yields a left 4-vector operator. But the square of this 4-vector does not commute with the momenta and, hence, it is not the searched-for spin Casimir.

In general, the assumption that \(W_\mu\) and \(P_\nu\) commute is not consistent with both, \(W_\mu\) and \(P_\nu\), being left 4-vector operators. Otherwise, the expression \(\text{ad}_L h \triangleright [W_\mu, P_\nu]\) would have to vanish for all \(h \in \mathcal{H}\), that is,

\[
W_\mu' P_\nu' (\Lambda(h_{(1)})^{\mu'} \Lambda(h_{(2)})^{\nu'} - \Lambda(h_{(2)})^{\mu'} \Lambda(h_{(1)})^{\nu'}) \overset{1}{=} 0. \tag{10}
\]
Since the coproduct is not cocommutative as in the undeformed case, this seems only possible for degenerate forms of $W_{\mu}$. We can avoid this problem if we assume that the $q$-Pauli-Lubanski vector is a right 4-vector operator, making the following general observation:

**Proposition 1.** Let $a \in \mathcal{A} = \mathcal{X} \rtimes \mathcal{H}$ commute with $\mathcal{X}$, $[a, x] = 0$ for all $x \in \mathcal{X}$. Then $a \triangleleft \text{ad}_R h$ also commutes with $\mathcal{X}$ for any $h \in \mathcal{H}$. In other words, the centralizer of $\mathcal{X}$ is invariant under the right Hopf adjoint action of $\mathcal{H}$.

**Proof.** Let $x \in \mathcal{X}$ be any element of the quantum space. Then

\[
(a \triangleleft \text{ad}_R h) x = S(h(1))a h(2) x = S(h(1))a (h(2) \triangleright x) h(3)
= S(h(1))(h(2) \triangleright x) a h(3)
= (S(h(1))(h(2) \triangleright x) S(h(1))(2) a h(3)
= (S(h(2))h(3) \triangleright x) S(h(1)) a h(4)
= x (a \triangleleft \text{ad}_R h)
\]

for any $h \in \mathcal{H}$. \qed

This means, that if a single component of a right vector operator commutes with all momenta, then the other components commute with all momenta, as well. Hence, the requirement that $P_{\mu}$ and $W_{\nu}$ commute does no longer generate linear dependencies of type (10). We come to the following

**Definition 2.** A set of operators $W_{\mu} \in \mathcal{A}$ with the properties

- (PL1) $W_{\mu}$ is a right 4-vector operator,
- (PL2) each component $W_{\mu}$ commutes with all translations $P_{\nu}$,
- (PL3) $\lim_{q \rightarrow 1} W_{\mu} = W_{\mu}^{q=1}$ as defined in (9),

is called a $q$-Pauli-Lubanski vector.

Obviously, (PL1)-(PL3) do not determine $W_{\mu}$ uniquely. For example, we could multiply it by any $q$-polynomial which evaluates to 1 at $q = 1$. Property (PL1) tells us that the square of $W_{\mu}$ is a $q$-Lorentz scalar, that is, commutes with all $h \in \mathcal{H}$. As a consequence of (PL2) this square commutes with all momenta. Therefore, (PL1) and (PL2) together guarantee that the square of a $q$-Pauli-Lubanski vector is a Casimir operator. The additional property (PL3) is the obvious requirement that $W_{\mu}$ be a $q$-deformation of the undeformed Pauli-Lubanski vector.
3.2 Constructing the $q$-Pauli-Lubanski Vector

The only 4-vector operator we know so far is the 4-momentum $P_\mu$. By construction, it is a left 4-vector operator. Being given a universal $\mathcal{R}$-matrix, there is a generic way to construct a right tensor operator from of a given left tensor operator:

**Proposition 2.** Let $T_j$ be a left $\rho$-tensor operator, that is, $\text{ad}_L h \triangleright T_j = T_i \rho(h)^i_j$ for all $h \in \mathcal{H}$, where $\rho$ is a finite representation of $\mathcal{H}$. Let $\mathcal{R}$ be a universal $\mathcal{R}$-matrix of $\mathcal{H}$. The set of operators

$$\Sigma_\mathcal{R}(T_j) := S^2(\mathcal{R}_{[1]}^T)T_i \rho(\mathcal{R}_{[2]}^T)^i_j$$

(12)

is a right $\rho$-tensor operator.

**Proof.** Abbreviating $\text{ad}_L h \triangleright T_j \equiv h \triangleright T_j$, we have

$$\Sigma_\mathcal{R}(T_j) \triangleleft \text{ad}_R h = S(h(1))S^2(\mathcal{R}_{[1]}^T)h(1)S^{-1}(h(2))h(2) \triangleright T_j = S(h(1))S^2(\mathcal{R}_{[1]}^T)h(3)S^{-1}(h(2))h(2) \triangleright T_j$$

$$= S(h(1))S(h(2))h(3)S^{-1}(h(1))S(\mathcal{R}_{[2]}^T)h(2) \triangleright T_j = S^2(\mathcal{R}_{[1]}^T)S(h(2))h(3)S^{-1}(h(1))S^{-1}(h(1)) \triangleright T_j$$

$$= S^2(\mathcal{R}_{[1]}^T)S^{-1}h \triangleright T_j = S^2(\mathcal{R}_{[1]}^T)(\mathcal{R}_{[2]}^T)^{-1}h \triangleright T_j$$

$$= \Sigma_\mathcal{R}(T_i) \rho(S^{-1}h)^i_j.$$  

(13)

According to Eq. (13), $\Sigma_\mathcal{R}(T_j)$ is indeed a right $\rho$-tensor operator. $\square$

This proposition tells us in particular, that $\Sigma_\mathcal{R}(P_\mu)$ satisfies (PL1). The next proposition takes care of (PL2).

**Proposition 3.** Let $P_\mu$ be the momentum 4-vector, $\mathcal{R}_1$ the antireal universal $\mathcal{R}$-matrix of the $q$-Lorentz algebra, and $\Sigma$ be defined as in Proposition 2. Then

$$[\Sigma_{\mathcal{R}_1}(P_\mu), P_\nu] = 0, \quad [\Sigma_{\mathcal{R}_1^{-1}}(P_\mu), P_\nu] = 0$$

(14)

for all $\mu, \nu$.

**Proof.** We denote the 4-vector representation by $\text{ad}_L h \triangleright P_\nu = h \triangleright P_\nu = P_\mu \Lambda(h)^\mu_\nu$. Recall, that the commutation relations of the momenta can be written as

$$P_\mu P_\nu R_1^\mu_\sigma = P_\sigma P_\tau \iff P_\mu P_\nu (R_1^{-1})^\mu_\sigma = P_\tau P_\sigma,$$

(15)

where the $R$-matrix $R_1^\mu_\sigma = \Lambda(\mathcal{R}_{[1]}^\mu_\sigma)\Lambda(\mathcal{R}_{[2]}^\nu_\tau)$ is the 4-vector representation of the antireal universal $\mathcal{R}$-matrix $\mathcal{R}_1$. Using the commutation relations between

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tensor operators and the Hopf algebra, we find

\[
P_\sigma \Sigma_{R_1}(P_\tau) = P_\sigma S^2(R_{[1]}|P_\nu)\Lambda(R_{[2]})^{\nu}\tau = S^2(R_{[1]}|S(R_{[1]}) \triangleright P_\sigma)\Lambda(R_{[2]})^{\nu}\tau = S^2(R_{[1]}|P_\mu \Lambda(R_{[1]}^{-1})^{\mu}\sigma P_\nu \Lambda(R_{[2]})^{\nu}\sigma')\Lambda(R_{[2]})^{\nu'}\tau
\]

\[
= S^2(R_{[1]}|[P_\mu P_\nu (R_{[1]}^{-1})^{\mu}\sigma]\Lambda(R_{[2]})^{\nu}\tau
\]

\[
= S^2(R_{[1]}|P_\tau P_\sigma \Lambda(R_{[2]})^{\nu}\tau
\]

\[
= \Sigma_{R_1}(P_\tau) P_\sigma.
\]

(16)

On the second an third line we have used \([26]\), \(\Delta(R_{[1]} \otimes R_{[2]} = R_{[1]} \otimes R_{[1]} \otimes R_{[2]} R_{[2]'}\), and \(S(R_{[1]} \otimes R_{[2]} = R_{[1]}^{-1} \otimes R_{[2]}^{-1}\). The calculations for \(R_1 \rightarrow R_{121}^{-1}\) are completely analogous. \(\square\)

Propositions 2 and 3 tell us, that both \(\Sigma_{R_1}(P_\mu)\) and \(\Sigma_{R_{121}^{-1}}(P_\mu)\) satisfy properties (PL1) and (PL2), respectively. In order to check (PL3) we must find explicit expressions for \(\Sigma_{R_1}(P_\mu)\) and \(\Sigma_{R_{121}^{-1}}(P_\mu)\). This amounts to calculating the \(L\)-matrices

\[
(L_{1+}^A)^\mu_\nu := R_{[1]}\Lambda(R_{[2]})^{\mu}\nu, \quad (L_{1-}^A)^\mu_\nu := R_{[2]}^{-1}\Lambda(R_{[1]})^{\mu}\nu.
\]

(17)

For the 4-vector of these \(L\)-matrices we find

\[
(L_{1+}^A)^\mu_\nu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a^2 & b^2 & \frac{q^2}{2}ab \\
0 & c^2 & d^2 & \frac{q^2}{2}cd \\
0 & \frac{q^2}{2} ac & \frac{q^2}{2}bd & (1 + [2]bc)
\end{pmatrix}
\]

(18a)

\[
(L_{1-}^A)^\mu_\nu = \begin{pmatrix}
W & \lambda K^{-1} J_+ & \lambda K^{-1} J_+ & \lambda J_3 \\
-q^{-1}\lambda J_+ & 1 & 0 & -q^{-1}\lambda J_+ \\
-q\lambda J_+ & 0 & 1 & -q\lambda J_+ \\
\lambda J_3 & -\lambda K^{-1} J_+ & -\lambda K^{-1} J_+ & \lambda J_3 + K^{-1}
\end{pmatrix}
\]

(18b)

with respect to the basis \(\{0, -, +, 3\}\). Observe, that \((L_{1+}^A)^A_B\) is the 3-dimensional corepresentation matrix of \(SU_q(2)^{\text{op}}\). With a linear combination of these two \(L\)-matrices we can satisfy (PL3).

**Theorem 1.** The set of operators

\[
W_\nu := \lambda^{-1}[\Sigma_{R_{121}^{-1}}(P_\nu) - \Sigma_{R_1}(P_\nu)] = \lambda^{-1} S^2((L_{1-}^A)^\mu_\nu - (L_{1+}^A)^\mu_\nu) P_\mu
\]

is a \(q\)-Pauli-Lubanski vector in the sense of Definition 3.

**Proof.** Properties (PL1) and (PL2) have been shown in Propositions 2 and 3, respectively. It remains to show (PL3). We note that the undeformed limit of the (left and right) Hopf adjoint action is the ordinary adjoint action. Hence, the
limit $q \to 1$ preserves tensor operators. Since a 4-vector is an irreducible tensor operator it is sufficient to examine the limit of one component only. The limits of the other components follow by application of the adjoint action. We choose the zero component for which we have to show that

$$W_0 = \lambda^{-1}(W - 1)P_0 + J_A P_B g^{AB} \xrightarrow{q \to 1} W_0^{q=1} = J_A P_B g^{AB}.$$  \hspace{1cm} (20)

All there is to show is that

$$\lambda^{-1}(W - 1) = \lambda^{-1}([2]^{-1}[q^{-1}K + qK^{-1} + \lambda^2 EF] - 1)
= \lambda^{-1}[2]^{-1}(q^{-1}K + qK^{-1} - [2]) + \lambda[2]^{-1}EF \hspace{1cm} (21)$$

vanishes for $q \to 1$. Clearly, the $\lambda[2]^{-1}EF$ term of the last line vanishes. Using $K = q^H$ we get for the other terms

$$\lambda^{-1}(q^{-1}K + qK^{-1} - [2]) = \sum_{n=1}^{\infty} \frac{(q + (-1)^nq^{-1})(\ln q)^n}{\lambda n!} H^n
= \ln q H + \frac{2}{\lambda 2!} (\ln q)^2 H^2 + \frac{1}{\lambda 3!} (\ln q)^3 H^3 + \frac{2}{\lambda 4!} (\ln q)^4 H^4 + \ldots , \hspace{1cm} (22)$$

which vanishes for $q \to 1$, since $\lim_{q \to 1} (\ln q)^n = 0$ for $n \geq 1$.

### 3.3 The Spin Casimir

We proceed to calculate the spin Casimir

$$W^\tau W_\tau = \eta^{\tau\nu} W_\nu W_\tau
= \lambda^{-2} \eta^{\tau\nu} S^2 [(L_{1-}^\Lambda)^\mu_\nu - (L_{1+}^\Lambda)^\mu_\nu] P_\mu S^2 [(L_{1-}^\Lambda)^\sigma_\tau - (L_{1+}^\Lambda)^\sigma_\tau] P_\sigma
= \lambda^{-2} \eta^{\tau\nu} S^2 \left\{ [(L_{1-}^\Lambda)^\mu_\nu - (L_{1+}^\Lambda)^\mu_\nu][((L_{1-}^\Lambda)^\sigma_\tau - (L_{1+}^\Lambda)^\sigma_\tau)] P_\mu P_\sigma
= \lambda^{-2} \eta^{\tau\nu} S^2 \left\{ [(L_{1-}^\Lambda)^\mu_\nu (L_{1+}^\Lambda)^\sigma_\tau + (L_{1+}^\Lambda)^\mu_\nu (L_{1-}^\Lambda)^\sigma_\tau
- (L_{1-}^\Lambda)^\mu_\nu (L_{1+}^\Lambda)^\sigma_\tau] P_\mu P_\sigma \right\}. \hspace{1cm} (23)$$

This can be further simplified. We first note that the commutation relations of the $L$-matrices are such that

$$\eta^{\tau\nu}(L_{1-}^\Lambda)^\mu_\nu (L_{1+}^\Lambda)^\sigma_\tau P_\sigma P_\mu = \eta^{\tau\nu}(R_{1-}^1)^{\tau\nu} (R_{1+}^1)^{\rho\sigma} (L_{1-}^\Lambda)^\mu_\nu (L_{1+}^\Lambda)^{\rho\sigma} \tau P_1^\rho \sigma_\mu_\nu P_\sigma P_\mu,
= \eta^{\tau\nu}(L_{1+}^\Lambda)^\mu_\nu (L_{1-}^\Lambda)^\sigma_\tau P_\sigma P_\mu \hspace{1cm} (24)$$

where in the second step we have used the commutation relations \cite{3} of the momenta and that $(R_{1-}^{-1})^{\tau\nu} \nu \tau \eta^{\tau\nu} = \eta^{\tau\nu}$. Moreover, using Eq. \cite{55} one can see that

$$\eta^{\tau\nu}(L_{1+}^\Lambda)^\mu_\nu (L_{1-}^\Lambda)^\sigma_\tau = \eta^{\mu\sigma} \hspace{1cm} (25)$$

With the last two results Eq. \cite{3} becomes

$$W^\tau W_\tau = 2\lambda^{-2} S^2 [\eta^{\mu\sigma} - \eta^{\tau\nu}(L_{1+}^\Lambda)^\mu_\nu (L_{1-}^\Lambda)^\sigma_\tau] P_\sigma P_\mu \hspace{1cm} (26)$$
## 4 The Little Algebras

### 4.1 Little Algebras in the q-Deformed Setting

In classical relativistic mechanics the state of motion of a free particle is completely determined by its 4-momentum. In quantum mechanics particles can have an additional degree of freedom called spin: Let us assume we have a free relativistic particle described by an irreducible representation of the Poincaré algebra. We pick all states with a given momentum,

$$L_{\vec{p}} := \{ |\psi\rangle \in L : P_\mu |\psi\rangle = p_\mu |\psi\rangle \},$$  \hspace{1cm} (27)

where $L$ is the Hilbert space of the particle and $\vec{p} = (p_\mu)$ is the 4-vector of momentum eigenvalues. If the state of the particle is not uniquely determined by the eigenvalues of the momentum, then the eigenspace $L_{\vec{p}}$ will be degenerate. In that case we need, besides the momentum eigenvalues, an additional quantity to label the basis of our Hilbert space uniquely. This additional degree of freedom is spin. The spin symmetry is then the set of Lorentz transformations that leave the momentum eigenvalues invariant and, hence, act on the spin degrees of freedom only,

$$\mathcal{K}_{\vec{p}} := \{ h \in \mathcal{H} : P_\mu h |\psi\rangle = p_\mu h |\psi\rangle \ \text{for all} \ |\psi\rangle \in L_{\vec{p}} \},$$  \hspace{1cm} (28)

where $\mathcal{H}$ is the enveloping Lorentz algebra. In mathematical terms, $\mathcal{K}_{\vec{p}}$ is the stabilizer of $L_{\vec{p}}$. Clearly, $\mathcal{K}_{\vec{p}}$ is an algebra, called the little algebra.

A priori, there are a lot of different little algebras for each representation and each vector $p$ of momentum eigenvalues. In the undeformed case it turns out that for the physically relevant representations (real mass) there are (up to isomorphism) only two little algebras, depending on the mass being either positive or zero \[^{[20]}\]. For positive mass we get the algebra of rotations, $\mathcal{U}(su_2)$, for zero mass an algebra which is isomorphic to the algebra of rotations and translations of the 2-dimensional plane denoted by $\mathcal{U}(iso_2)$. The proof that $\mathcal{K}_{\vec{p}}$ does not depend on the particular representation but on the mass does not generalize to the $q$-deformed case: If we define for representations of the $q$-Poincaré algebra the little algebra as in Eq. (28), $\mathcal{K}_{\vec{p}}$ for a spin-$\frac{1}{2}$ particle will not be the same as for spin-1. We will therefore define the $q$-little algebras differently.

In the undeformed case there is an alternative but equivalent definition of the little algebras. $\mathcal{K}_{\vec{p}}$ is the algebra generated by the components of the $q$-Pauli-Lubanski vector as defined in Eq. (1) with the momentum generators replaced by their eigenvalues. Let us formalize this to see why this definition works and how it is generalized to the $q$-deformed case.

Let $\chi_{\vec{p}}$ be the map that maps the momentum generators to the eigenvalues, $\chi_{\vec{p}}(P_\mu) = p_\mu$. Being the restriction of a representation, $\chi_{\vec{p}}$ must extend to a one dimensional $\ast$-representation of the momentum algebra $\chi_{\vec{p}} : \mathcal{X} \rightarrow \mathbb{C}$, a non-trivial
condition only in the $q$-deformed case. Noting that every $a \in \mathcal{A} = \mathcal{X} \rtimes \mathcal{H}$ can be uniquely written as $a = \sum_i h_i x_i$, where $h_i \in \mathcal{H}$ and $x_i \in \mathcal{X}$, we can extend $\chi_\vec{p}$ to a linear map on all of $\mathcal{A}$ by defining $\hat{\chi}_\vec{p} : \mathcal{A} \to \mathcal{H}$ as

$$\hat{\chi}_\vec{p}(\sum_i h_i x_i) := \sum_i h_i \chi_\vec{p}(x_i).$$

(29)

The little algebra can now be alternatively defined as the unital algebra generated by the images of the $q$-Pauli-Lubanski vector under $\hat{\chi}_\vec{p}$,

$$\mathcal{K}_\vec{p} := \mathbb{C}\langle \hat{\chi}_\vec{p}(W_\mu) \rangle.$$

(30)

Why is this a reasonable definition? By construction the action of every element of $\mathcal{A}$ on $\mathcal{L}_\vec{p}$ is the same as of its image under $\hat{\chi}_\vec{p}$. For any $|\psi\rangle \in \mathcal{L}_\vec{p}$ this means

$$P_\mu \hat{\chi}_\vec{p}(W_\nu)|\psi\rangle = \hat{\chi}_\vec{p}(P_\mu W_\nu)|\psi\rangle = \hat{\chi}_\vec{p}(W_\nu P_\mu)|\psi\rangle = p_\mu \hat{\chi}_\vec{p}(W_\nu)|\psi\rangle,$$

(31)

which shows that $\mathcal{K}_\vec{p} \subset \mathcal{K}_\vec{p}'$. It still could happen, that $\mathcal{K}_\vec{p}$ is strictly smaller than $\mathcal{K}_\vec{p}'$. In the undeformed case there are theorems [21, 22] telling us that this cannot happen, so we really have $\mathcal{K}_\vec{p} = \mathcal{K}_\vec{p}'$. For the $q$-deformed case no such theorem is known [23]. However, if there were more generators in the stabilizer of some momentum eigenspace they would have to vanish for $q \to 1$. In this sense Eq. (30) with the $q$-deformed Pauli-Lubanski vector can be considered to define the $q$-deformed little algebras.

### 4.2 Computation of the $q$-Little Algebras

To begin the explicit calculation of the $q$-deformed little algebras, we need to figure out if there are eigenstates of $q$-momentum at all. That is, we want to determine the one-dimensional $\ast$-representations of $\mathcal{X} = \mathbb{R}^{1,3}_q$, that is the homomorphisms of $\ast$-Algebras $\chi : \mathcal{X} \rightarrow \mathbb{C}$. Let us again denote the eigenvalues of the generators by lower case letters $p_\mu := \chi(P_\mu)$. According to Eq. (51), we must have $p_0, p_3$ real and $p_\pm = -q p_\mp$ for $\chi$ to be a $\ast$-map. To find the conditions for $\chi$ to be a homomorphism of algebras, we apply $\chi$ to the relations (50) of $\mathcal{X} = \mathbb{R}^{1,3}_q$, yielding $p_A(p_0 - p_3) = 0$. There are two cases. The first is $p_0 \neq p_3$, which immediately leads to $p_A = 0$, and $p_0 = \pm m$. The second case is $p_0 = p_3$, leading to $m_0^2 = -|p_-|^2 - |p_+|^2$, where, if the mass $m$ is to be real, we must have $p_\pm = 0$.

In summary, for real mass $m$ we have a massive and a massless type of momentum eigenstate with eigenvalues given by

$$(p_0, p_-, p_+, p_3) = \begin{cases} (\pm m, 0, 0, 0), & m > 0 \\ (k, 0, 0, k), & m = 0, k \in \mathbb{R} \end{cases}$$

(32)
According to Eq. (30) we now have to replace the momenta in the definition (19) of the \( q \)-Pauli-Lubanski vector with these eigenvalues.

For the **massive case** we get

\[
\hat{\chi}_p(W_0) = \lambda^{-1}(W - 1)m \\
\hat{\chi}_p(W_-) = J_- K^{-1}m \\
\hat{\chi}_p(W_+) = J_+ K^{-1}m \\
\hat{\chi}_p(W_3) = \lambda^{-1}(W - K^{-1})m ,
\]

so the set of generators of the little algebra is essentially \( \{ W, K^{-1}, J_\pm K^{-1} \} \). Since \( K^{-1} \) stabilizes the momentum eigenspace, so does its inverse \( K \). Hence, it is safe to add \( K \) to the little algebra which would exist, anyway, as operator within a representation. We thus get

\[
\mathcal{K}_{(m,0,0,0)} = \mathcal{U}_q(\text{su}_2) ,
\]

completely analogous to the undeformed case.

The **massless case** is more interesting. Replacing the momentum generators with \( (P_0, P_-, P_+, P_3) \to (k, 0, 0, k) \) we get

\[
\hat{\chi}_p(W_0) = \lambda^{-1}(K - 1)k \\
\hat{\chi}_p(W_-) = -\lambda^{-1}q^{-\frac{3}{2}}[2]^{\frac{1}{2}}ac k \\
\hat{\chi}_p(W_+) = -\lambda^{-1}q^{\frac{1}{2}}[2]^{\frac{1}{2}}bd k \\
\hat{\chi}_p(W_3) = \lambda^{-1}(K - (1 + [2]bc))k .
\]

The set of generators of this little algebra is essentially \( \{ K, ac, bd, bc \} \). The commutation relations of these generators can be written more conveniently in terms of \( K \) and \( N_A := (L^A_{+})^3_A \), that is

\[
N_- = q^{\frac{1}{2}}[2]^{\frac{1}{2}}ac , \quad N_+ = q^{\frac{1}{2}}[2]^{\frac{1}{2}}bd , \quad N_3 = 1 + [2]bc .
\]

The commutation relations are

\[
N_B N_A \varepsilon^{AB} C = -\lambda N_C , \quad N_A N_B g^{BA} = 1 , \quad K N_A = q^{-2A} N_A K ,
\]

with conjugation \( N_A^* = N_B g^{BA} \), \( K^* = K \). In words: The \( N_A \) generate the opposite algebra of a unit quantum sphere, \( S_{q^{\infty}}^{\text{op}} \). \( K \), the generator of \( \mathcal{U}_q(\text{u}_1) \), acts on \( N_A \) as on a right 3-vector operator. In total we have

\[
\mathcal{K}_{(k,0,0,k)} = \mathcal{U}_q(\text{u}_1) \ltimes S_{q^{\infty}}^{\text{op}} .
\]

As opposed to the massive case, this is no Hopf algebra. However, since \( L \)-matrices are multiplicative, that is, \( \Delta([L^A_+]^{\mu} \sigma) = (L^A_+)^{\mu} \nu \otimes (L^A_+)^{\nu} \sigma \), we have

\[
\Delta(N_B) = N_A \otimes (L^A_+)^A_B ,
\]

hence, \( \mathcal{K}_{(k,0,0,k)} \) is a right coideal.
Appendix: The $q$-Poincaré Algebra

The Hopf-$\ast$ algebra generated by $E$, $F$, $K$, and $K^{-1}$ with relations

\begin{align}
KK^{-1} &= 1 = K^{-1}K, \quad KEK^{-1} = q^2 E, \\
KFK^{-1} &= q^{-2}F, \quad [E,F] = \lambda^{-1}(K-K^{-1}),
\end{align}

Hopf structure

\begin{align}
\Delta(E) &= E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \\
\Delta(K) &= K \otimes K, \quad \varepsilon(E) = 0 = \varepsilon(F), \quad \varepsilon(K) = 1,
\end{align}

and $\ast$-structure

\begin{align}
E^\ast &= FK, \quad F^\ast = K^{-1}E, \quad K^\ast = K
\end{align}

is called $\mathcal{U}_q(\mathfrak{su}_2)$, the $q$-deformation of the enveloping algebra $\mathcal{U}(\mathfrak{su}_2)$.

The set of generators $\{J_A\} = \{J_-, J_3, J_+\}$ of $\mathcal{U}_q(\mathfrak{su}_2)$ defined as

\begin{align}
J_- &:= q[2]^{-\frac{1}{2}}KF \\
J_3 &:= [2]^{-1}(q^{-1}EF - qFE) \\
J_+ &:= -[2]^{-\frac{1}{2}}E
\end{align}

is the left 3-vector operator of angular momentum. The center of $\mathcal{U}_q(\mathfrak{su}_2)$ is generated by

\begin{align}
W := K - \lambda J_3 = K - \lambda[2]^{-1}(q^{-1}EF - qFE),
\end{align}

the Casimir operator of angular momentum. $W$ is related to $J_A$ by

\begin{align}
W^2 - 1 = \lambda^2(J_3^2 - q^{-1}J_-J_+ - qJ_+J_-) = \lambda^2 J_A J_B g^{AB},
\end{align}

thus defining the 3-metric $g^{AB}$, by which we raise 3-vector indices $J^A = g^{AB}J_B$.

It is also useful to define an $\varepsilon$-tensor

\begin{align}
\varepsilon^{-3} &= q^{-1} & \varepsilon^3 &= -q \\
\varepsilon^{-+} &= 1 & \varepsilon^{+-} &= -1 & \varepsilon^{33} &= -\lambda \\
\varepsilon^{+-} &= q^{-1} & \varepsilon^+ &= -q
\end{align}

The Hopf-$\ast$ algebra generated by the $2 \times 2$-matrix of generators $B^i_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with relations

\begin{align}
ba &= qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd \\
b^c &= cb, \quad da - ad = (q - q^{-1})bc, \quad da - qbc = 1,
\end{align}
coproduct $\Delta(B^i_k) = B^i_j \otimes B^j_k$ (summation over $j$), counit $\varepsilon(B^i_j) = \delta^i_j$, antipode and $*$-structure

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix},$$

is $SU_q(2)^{op}$, the opposite algebra of the quantum group $SU_q(2)$.

The Hopf-$*$ algebra generated by the Hopf-$*$ sub-algebras $U_q(su_2)$ and $SU_q(2)^{op}$ with cross commutation relations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}E = \begin{pmatrix} qEa - q^2b \\ qEc + q^2Ka - q^2d \\ q^{-1}Ed + q^{-2}Kb \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}F = \begin{pmatrix} qFa + q^{-2}c \\ q^{-1}Fc \\ q^{-1}Fd - q^{-2}K^{-1}c \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}K = K\begin{pmatrix} a & q^{-2}b \\ q^2c & d \end{pmatrix},$$

which is the Drinfeld double of $U_q(su_2)$ and $SU_q(2)^{op}$, is the $q$-Lorentz algebra $H = U_q(sl_2(\mathbb{C}))$ [14].

The $*$-algebra generated by $P_0, P_-, P_+, P_3$ with commutation relations

$$P_0P_A = P_AP_0, \quad P_AP_B \varepsilon^{AB}C = -\lambda P_0P_C$$

and $*$-structure

$$P_0^* = P_0, \quad P_-^* = -q^{-1}P_+, \quad P_+^* = -q, \quad P_-^* = P_3$$

is the $q$-Minkowski space algebra $X = \mathbb{R}^{1,3}_q$. The center of $\mathbb{R}^{1,3}_q$ is generated by the mass Casimir

$$m^2 := P_\mu P_\nu \eta^{\mu\nu} = P_0^2 + q^{-1}P_-P_+ + qP_+P_- - P_3^2,$$

thus defining the 4-metric $\eta^{\mu\nu}$. It is related to the 3-metric by $\eta^{AB} = -g^{AB}$ for $A, B \in \{-, +, 3\}$.

The commutation relations of $X = \mathbb{R}^{1,3}_q$ are consistent with the 4-vector action $h \triangleright P_\nu = P_\nu \Lambda(h)^{\nu}_{\mu}$ of $H$ on $X$. $\Lambda$ is defined on the generators of rotations as

$$\Lambda(J_A) = \begin{pmatrix} \rho^0(J_A) & 0 \\ 0 & \rho^1(J_A) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_A B \end{pmatrix},$$

where $\rho^0$ and $\rho^1$ are the spin-0 and the spin-1 representations of $U_q(su_2)$, respectively. On the boost generators $\Lambda$ is given by

$$\Lambda(a) = \begin{pmatrix} [4][2]^{-2} & 0 & 0 & q\lambda[2]^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q^{-1}\lambda[2]^{-1} & 0 & 0 & 2[2]^{-1} \end{pmatrix}$$

(54a)
\[ \Lambda(b) = q^{-\frac{1}{2}} \lambda[2]^{-\frac{1}{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]  
(54b)

\[ \Lambda(c) = -q^{\frac{1}{2}} \lambda[2]^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]  
(54c)

\[ \Lambda(d) = \begin{pmatrix} [4][2]^{-2} & 0 & 0 & -q^{-1}\lambda[2]^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -q\lambda[2]^{-1} & 0 & 0 & 2[2]^{-1} \end{pmatrix} \]  
(54d)

with respect to the \( \{0, -, +, 3\} \) basis. It has the property

\[ \eta^{\nu\gamma} \Lambda(h)^{\mu'\nu'} \eta_{\mu'\mu} = \Lambda(S h)^{\nu}_{\mu} \]  
(55)

for all \( h \in \mathcal{H} \).

Finally, the \( q \)-Poincaré algebra is the *-algebra generated by the \( q \)-Lorentz algebra \( \mathcal{H} = U_q(sl_2(\mathbb{C})) \) and the \( q \)-Minkowski algebra \( \mathcal{X} = \mathbb{R}^1,3_q \) with cross-commutation relations

\[ h P_{\nu} = P_{\mu} \Lambda(h^{(1)})^{\mu}_{\nu}, h^{(2)} \quad \leftrightarrow \quad P_{\nu} h = h^{(2)} P_{\mu} \Lambda(S^{-1}(h^{(1)}))^{\mu}_{\nu}. \]  
(56)

More details and mathematical background information has been compiled in [25].

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