Quasi-distributions for arbitrary non-commuting operators

J. S. Ben-Benjamin\textsuperscript{1}, L. Cohen\textsuperscript{2}

\textsuperscript{1} Institute for Quantum Science and Engineering, Texas A\&M University, Texas, USA
\textsuperscript{2} Hunter College and Graduate Center, City University of New York, NY, USA

Abstract

We present a new approach for obtaining quantum quasi-probability distributions, $P(\alpha, \beta)$, for two arbitrary operators, $a$ and $b$, where $\alpha$ and $\beta$ are the corresponding c-variables. We show that the quantum expectation value of an arbitrary operator can always be expressed as a phase space integral over $\alpha$ and $\beta$, where the integrand is a product of two terms: One dependent only on the quantum state, and the other only on the operator. In this formulation, the concepts of quasi-probability and correspondence rule arise naturally in that simultaneously with the derivation of the quasi-distribution, one obtains the generalization of the concept of correspondence rule for arbitrary operators.

1 Introduction

We present a new approach for obtaining quasi-probability distributions, $P(\alpha, \beta)$, for two arbitrary operators, $a$ and $b$, where $\alpha$ and $\beta$ are the corresponding c-variables. The method also yields the correspondence rule relating a quantum operator, $g$, to the corresponding c-function, $g(\alpha, \beta)$. We assume total generality for the operators.

Historically, quasi-probability distributions have been studied for the position-momentum case (qp case). Parallel to this development was the development aimed at finding the quantum operator for a given classical function. Such rules are called correspondence rules or rules of association. Moyal \cite{27} showed that the Wigner distribution \cite{37} can be derived using the Weyl correspondence rule \cite{36} and Cohen showed \cite{7} how to derive an infinite number of quasi-distributions and the associated correspondence rules. The methods used in those studies have been by way of the characteristic function operator. Scully \cite{33, 4, 5} emphasized a different approach for obtaining quasi-distributions of position and momentum, namely to start with quantum mechanics; One starts with the quantum expectation value of an arbitrary operator, $\langle g \rangle = \text{Tr}[g \rho]$, where $g$ is the operator and $\rho$ is the density matrix, and shows that $\text{Tr}[g \rho]$ can always be expressed as a phase space integral over position and momentum. Crucially the integrand factorizes into two terms: One dependent only on the density matrix, and the other only on the quantum operator. In this way, one not only derives the quasi-probability distributions, but also what have been called the inverse correspondence rules \cite{19, 15}. In this formulation, the concepts of quasi-probability and correspondence arise naturally and together.

To illustrate the basic idea, we give the original derivation of Scully. The expectation value of an operator $g$ is\footnote{We often represent multiple integrals by a single integration symbol, the differentials indicating the number of integrals. All integrals go from $-\infty$ to $\infty$.}

\begin{equation}
\langle g \rangle = \text{Tr}[g \rho] = \int \langle q | g \rho | q \rangle dq \tag{1}
\end{equation}

Inserting the identity operator three times, we have

\begin{equation}
\langle g \rangle = \iiint \langle p_1 | g | p_2 \rangle \langle p_2 | q_1 \rangle \langle q_1 | \rho | q_2 \rangle \langle q_2 | p_1 \rangle dq_1 dp_1 dq_2 dp_2 \tag{2}
\end{equation}
where bras and kets of \( p_1 \) and \( p_2 \) are momentum eigenstates, and those of \( q_1 \) and \( q_2 \) correspond to position eigenstates. We change variables in Eq. (2), defining the difference between the positions and between the momenta as \( \bar{q} = q_1 - q_2 \) and \( \bar{p} = p_2 - p_1 \), and the average position and momentum as \( q = \frac{1}{2}(q_1 + q_2) \) and \( p = \frac{1}{2}(p_1 + p_2) \), respectively. Eq. (2) then becomes

\[
\langle g \rangle = \int \int \left( \int \int \langle p - \frac{1}{2}\bar{p} \mid \rho \mid p + \frac{1}{2}\bar{p} \rangle e^{-i\bar{p}q/\hbar} d\bar{p} \right) \left( \int \int \langle q + \frac{1}{2}\bar{q} \mid \rho \mid q - \frac{1}{2}\bar{q} \rangle e^{-i\bar{q}p/\hbar} dq \right) d\bar{p}d\bar{q} \tag{3}
\]

which shows that the quantum expectation value can be written as a phase space integral; moreover, it shows that the integrand has been factorized as the product of two c-functions. The first factor is the classical function, \( g(q,p) \), that corresponds to the operator \( g \) by way of the inverse-Weyl rule \([19]\) and the second factor depends only on the density matrix \( \rho \), which in this case is the Wigner distribution. Recently the above program has been carried out for the general class of quasi-distributions of position and momentum \([5]\).

In this paper we generalize the case of position and momentum to arbitrary operators. We start with the expectation value of the operator \( g \), as per Eq. (1)

\[
\langle g \rangle = \text{Tr}[g \rho] \tag{4}
\]

and show that this can always be written in the form

\[
\langle g \rangle = \int \int \left\{ \begin{array}{c}
\text{a c-function of } \alpha \text{ and } \beta \\
\text{that depends only on } g
\end{array} \right\} \left\{ \begin{array}{c}
\text{a c-function of } \alpha \text{ and } \beta \\
\text{that depends only on } \rho
\end{array} \right\} d\alpha d\beta \tag{5}
\]

The left hand factor is the c-function, \( g(\alpha, \beta) \), corresponding to the operator and the right hand factor of the integrand is the quasi-distribution, \( P(\alpha, \beta) \). What Eq. (5) shows is that we can calculate the expectation value of a quantum operator by way of phase space averaging in the phase space of \( \alpha \) and \( \beta \),

\[
\langle g(\alpha, \beta) \rangle = \int \int g(\alpha, \beta) P(\alpha, \beta) d\alpha d\beta \tag{6}
\]

and that it equals \( \langle g \rangle \) calculated quantum mechanically as per Eq. (4).

There has been some previous work on obtaining quasi-probability distributions for quantities other than position and momentum. In particular, the case of spin components \([14, 16]\), local energy and local kinetic energy \([1, 2, 8, 28, 25]\), and local spread of an operator (the conditional standard deviation) have been considered \([29, 31, 23]\). Scully and M. S. Zubairy \([34]\) and Schleich \([32]\) used a similar approach to the one presented here to derive the Q distribution. Also, similar considerations arise in the field of time-frequency analysis \([3, 12, 9, 22]\). Previous attempts to generalize to arbitrary operators have been approached by way of the characteristic function operator method, but the general difficulty with that method is the disentanglement of the exponential \([35, 13, 10, 11]\).

We present our results by first deriving two special cases and then deriving a general form that generates an infinite number of distributions characterized by a kernel function. We deal with the continuous case, and the results for the discrete case follow naturally.

## 2 Notation, marginals, density matrix, and transformation matrix

The measurable quantities of the operators \( a \) and \( b \) are denoted by \( \alpha \) and \( \beta \), which are the eigenvalues of the respective operators obtained by solving the eigenvalue problem for each operator

\[
a u_\alpha(q) = \alpha u_\alpha(q) \quad ; \quad b v_\beta(q) = \beta v_\beta(q) \tag{7}
\]
where $u_\alpha(q)$ and $v_\beta(q)$ are the corresponding eigenfunctions in the position representation. The quantum probabilities for $\alpha$ and $\beta$ are respectively

$$P(\alpha) = |A(\alpha)|^2 = \left| \int \psi(q)u_\alpha^*(q) \, dq \right|^2 ; \quad P(\beta) = |B(\beta)|^2 = \left| \int \psi(q)v_\beta^*(q) \, dq \right|^2$$

where the wave function is expanded in terms of the eigenfunctions of $a$ or of $b$

$$\psi(q) = \int A(\alpha)u_\alpha(q) \, d\alpha ; \quad \psi(q) = \int B(\beta)v_\beta(q) \, d\beta$$

with $A(\alpha)$ and $B(\beta)$, respectively,

$$A(\alpha) = \int \psi(q)u_\alpha^*(q) \, dq ; \quad B(\beta) = \int \psi(q)v_\beta^*(q) \, dq$$

In wave function terminology, $A(\alpha)$ is the wave function in the $\alpha$ representation and $B(\beta)$ is the wave function in the $\beta$ representation [6].

For the quasi-distribution, $P(\alpha, \beta)$, the marginal conditions are,

$$\int P(\alpha, \beta) \, d\beta = |A(\alpha)|^2 = \left| \int \psi(q)u_\alpha^*(q) \, dq \right|^2$$

$$\int P(\alpha, \beta) \, d\alpha = |B(\beta)|^2 = \left| \int \psi(q)v_\beta^*(q) \, dq \right|^2$$

**Density matrix.** In the $\alpha$ representation, the pure state density matrix is

$$\rho(\alpha'', \alpha') = A^*(\alpha')A(\alpha'')$$

and the matrix elements $g_{\alpha''\alpha'}$ of the operator $g(x, p_x)$ in the $\alpha$ basis are

$$g_{\alpha''\alpha'} = \int \delta(\alpha' - x)g(x, p_x)\delta(\alpha'' - x) \, dx$$

In general, the expectation value of $g$ is

$$\langle g \rangle = \iint \rho(\alpha'', \alpha')g_{\alpha''\alpha'} \, d\alpha' d\alpha''$$

Explicitly,

$$\langle g \rangle = \iint A^*(\alpha')A(\alpha'')\delta(\alpha' - x)g(\alpha'' - x) \, d\alpha' d\alpha'' \, dx$$

Eq. (16) will be our starting point.

**Transformation matrix.** We define the transformation matrix, $T(\beta, \alpha)$, from the $\alpha$ to the $\beta$ representation by [6]

$$T(\beta, \alpha) = \int v_\beta^*(q)u_\alpha(q) \, dq$$

in which case, the eigenfunctions are related by

$$u_\alpha(q) = \int T(\beta, \alpha)v_\beta(q) \, d\beta$$

$$v_\beta(q) = \int T^*(\beta, \alpha)u_\alpha(q) \, d\alpha = \int T^\dagger(\alpha, \beta)u_\alpha(q) \, d\alpha$$
and the wave functions \( A(\alpha) \) and \( B(\beta) \) are related by

\[
A(\alpha) = \int B(\beta) T^*(\beta, \alpha) d\beta \tag{20}
\]

\[
B(\beta) = \int A(\alpha) T(\beta, \alpha) d\alpha \tag{21}
\]

The transformation matrix satisfies

\[
\int T^\dagger(\alpha, \beta) T(\beta, \alpha') d\beta = \delta(\alpha - \alpha') \tag{22}
\]

and

\[
T^\dagger(\beta, \alpha) = T^*(\alpha, \beta) \tag{23}
\]

3 Wigner type case

We start with the quantum expectation value, Eq. (16)

\[
\langle g \rangle = \int A^*(\alpha') A(\alpha'') \delta(\alpha' - x) g \delta(\alpha'' - x) dx d\alpha' d\alpha'' \tag{24}
\]

The aim is to express Eq. (24) in integrand factorized form as per Eq. (5). We insert the following three expressions in the right hand side of Eq. (24).

\[
\delta(\alpha'' - x) = \int T(\beta', \alpha'') T^\dagger(\beta', \beta') d\beta' \tag{25}
\]

\[
T^\dagger(\alpha', \beta') \frac{1}{T^\dagger(\alpha', \beta')} = 1 \tag{26}
\]

\[
\int \delta(\alpha' - \bar{\alpha}) \delta(\beta' - \bar{\beta}) d\bar{\alpha} d\bar{\beta} = 1 \tag{27}
\]

to obtain

\[
\langle g \rangle = \int A^*(\alpha') A(\alpha'') T(\beta', \alpha'') T^\dagger(\alpha', \beta') \frac{1}{T^\dagger(\alpha', \beta')} \delta(\alpha' - \bar{\alpha}) \delta(\beta' - \bar{\beta})
\]

\[
\delta(\bar{\alpha} - x) g T^\dagger(\bar{\beta} \beta) dx d\alpha' d\alpha'' d\bar{\alpha} d\bar{\beta} d\beta' \tag{28}
\]

We complete the integrand factorization by separating the delta functions into a product of two terms, one depending only on \( \alpha' \) and \( \beta' \), and the other only on \( \bar{\alpha} \) and \( \bar{\beta} \),

\[
\delta(\alpha' - \bar{\alpha}) \delta(\beta' - \bar{\beta}) = \left( \frac{1}{\pi \hbar} \right)^2 \int e^{-2i(\alpha - \alpha')(\beta - \beta')/\hbar} e^{2i(\alpha - \bar{\alpha})(\beta - \bar{\beta})/\hbar} d\alpha d\beta \tag{29}
\]

In addition, this is how the \( \alpha \) and \( \beta \) of Eq. 5 are introduced.

Inserting Eq. (29) into Eq. (28), we obtain

\[
\langle g \rangle = \left( \frac{1}{\pi \hbar} \right)^2 \int A^*(\alpha') A(\alpha'') T(\beta', \alpha'') T^\dagger(\alpha', \beta') e^{-2i(\alpha - \alpha')(\beta - \beta')/\hbar} \frac{1}{T^\dagger(\alpha', \beta')} e^{2i(\alpha - \bar{\alpha})(\beta - \bar{\beta})/\hbar}
\]

\[
\delta(\bar{\alpha} - x) g T^\dagger(\bar{\beta} \beta) dx d\alpha' d\alpha'' d\bar{\alpha} d\bar{\beta} d\beta' d\beta d\alpha \tag{30}
\]
This achieves the factorization of the integrand because we can write Eq. (30) as
\[
\langle g \rangle = \int \left\{ \frac{1}{\pi \hbar} \int \mathcal{A}^*(\alpha') \mathcal{A}(\alpha'') T(\beta', \alpha'') T(\alpha', \beta') e^{-2i(\alpha - \alpha')(\beta - \beta')/\hbar} d\alpha' d\beta' d\alpha'' \right\}
\[
\left\{ \frac{1}{\pi \hbar} \int \mathcal{T}^\dagger(\alpha, \beta) e^{2i(\alpha - \alpha')(\beta - \beta')/\hbar} \delta(\alpha - x) g T^\dagger(x, \beta) dxd\beta \right\} d\alpha d\beta
\]
where the first term depends only on the wave function and the second only on the operator \( g \). Therefore
\[
\langle g \rangle = \int g_W(\alpha, \beta) P_W(\alpha, \beta) d\alpha d\beta
\]
where the quasi-probability distribution is
\[
P_W(\alpha, \beta) = \frac{1}{\pi \hbar} \int \int \mathcal{A}^*(\alpha') \mathcal{A}(\alpha'') T^\dagger(\alpha', \beta') T(\beta', \alpha'') e^{-2i(\alpha - \alpha')(\beta - \beta')/\hbar} d\alpha' d\beta' d\alpha''
\]
and the corresponding c-function is given by
\[
g_W(\alpha, \beta) = \frac{1}{\pi \hbar} \int e^{2i(\alpha - \alpha')(\beta - \beta')/\hbar} \frac{1}{T^\dagger(\alpha, \beta)} \delta(\alpha - x) g T^\dagger(x, \beta) d\beta d\alpha dx
\]

Alternate forms. In Eq. (33) note that
\[
\int \mathcal{A}(\alpha'') T(\beta', \alpha'') d\alpha'' = \mathcal{B}(\beta')
\]
and hence we can write
\[
P_W(\alpha, \beta) = \frac{1}{\pi \hbar} \int \int \mathcal{A}^*(\alpha') \mathcal{B}(\beta') T^\dagger(\alpha', \beta') e^{-2i(\alpha - \alpha')(\beta - \beta')/\hbar} d\alpha' d\beta'
\]
Also, making the transformation in Eq. (33)
\[
\alpha' = u - \hbar \tau / 2 \quad ; \quad \alpha'' = u + \hbar \tau / 2
\]
with \( d\alpha' d\alpha'' = \hbar du d\tau \), we obtain
\[
P_W(\alpha, \beta) = \frac{1}{\pi} \int \int \mathcal{A}^*(u - \hbar \tau / 2) A(u + \hbar \tau / 2) T^\dagger(u - \hbar \tau / 2, \beta') T(\beta', u + \hbar \tau / 2) e^{-i2(\alpha - \alpha')(\beta - \beta')/\hbar} e^{-i(\beta - \beta') \tau} d\beta' d\tau du
\]

Marginals. To show that the marginals are satisfied, we integrate Eq. (33) with respect to \( \beta \),
\[
\int P_W(\alpha, \beta) d\beta = \int \int \mathcal{A}^*(\alpha) \mathcal{A}(\alpha'') T^\dagger(\alpha, \beta') T(\beta', \alpha'') d\alpha'' d\beta'
\]
Using Eq. (22) we obtain
\[
\int P_W(\alpha, \beta) d\beta = |\mathcal{A}(\alpha)|^2
\]
which is the marginal for the \( \alpha \) variable. To obtain the \( \beta \) marginal we integrate \( P_W(\alpha, \beta) \) with respect to \( \alpha \) obtain
\[
\int P_W(\alpha, \beta) d\alpha = \int \int \mathcal{A}^*(\alpha') \mathcal{A}(\alpha'') T^*(\beta, \alpha') T(\beta, \alpha'') d\alpha' d\alpha''
\]
\[
= \int |\mathcal{A}(\alpha) T(\beta, \alpha) d\alpha|^2 = |\mathcal{B}(\beta)|^2
\]
which is the marginal for the $\beta$ variable.

**Position-momentum case.** To specialize to the $qp$ case, we take

$$\alpha \to q = \text{position}$$  
$$\beta \to p = \text{momentum}$$

in which case the transformation matrix is

$$T(p,q) = \frac{1}{\sqrt{2\pi\hbar}} e^{-iqp/\hbar}$$

Inserting these into Eq. (38), we obtain that

$$P_W(\alpha,\beta) = \frac{1}{2\pi} \int e^{-2i(\alpha-\bar{\alpha})(\beta-\bar{\beta})/\hbar} e^{2i(\alpha-\bar{\alpha})(\beta-\bar{\beta})/\hbar} d\alpha d\beta$$

which is the Wigner distribution. This is the reason we have called Eq. (38) Wigner type for arbitrary operators.

For the correspondence rule, inserting Eqs. (43)–(46) into Eq. (34), one obtains that

$$g_W(q,p) = \hbar \int e^{i\tau p} \delta(q - \hbar \tau/2 - x) g(x, p_x) \delta(q + \hbar \tau/2 - x) dx d\tau$$

It has been previously shown [19] that Eq. (48) is the inverse Weyl, that is, it is the $c$-function obtained from the quantum operator.

### 4 Dirac notation derivation

It is of interest to give a derivation of Eqs. (33) and (34) in terms of Dirac notation and method. Starting with

$$\text{Tr} \{g\rho\} = \int \int \langle \alpha'|g|\beta' \rangle \langle \beta'|\rho|\alpha' \rangle d\alpha' d\beta'$$

Inserting

$$\langle \alpha'|\beta' \rangle \frac{1}{\langle \alpha'|\beta' \rangle} = 1$$

and

$$\delta(\alpha' - \bar{\alpha})\delta(\beta' - \bar{\beta}) = \frac{1}{(\pi\hbar)^2} \int e^{-2i(\alpha-\bar{\alpha})(\beta-\bar{\beta})/\hbar} e^{2i(\alpha-\bar{\alpha})(\beta-\bar{\beta})/\hbar} d\alpha d\beta$$

into Eq. (49) we obtain

$$\text{Tr} \{g\rho\} = \frac{1}{(\pi\hbar)^2} \int d\alpha d\beta \int d\alpha' d\beta' \langle \beta'|\rho|\alpha' \rangle \langle \alpha'|\beta' \rangle e^{-2i(\alpha-\bar{\alpha})(\beta-\bar{\beta})/\hbar}$$

$$\times \int d\bar{\alpha}d\bar{\beta} \frac{1}{\langle \bar{\alpha}|\beta \rangle} \langle \bar{\alpha}|g|\bar{\beta} \rangle e^{2i(\alpha-\bar{\alpha})(\beta-\bar{\beta})/\hbar}$$

which achieves the factorization of the integrand. Therefore the phase space distribution is

$$P_W(\alpha,\beta) = \frac{1}{\pi\hbar} \int d\bar{\alpha}d\bar{\beta} \langle \beta'|\rho|\alpha' \rangle \langle \alpha'|\beta' \rangle e^{-2i(\beta-\bar{\beta})(\alpha-\bar{\alpha})/\hbar}$$

and the corresponding $c$-number is

$$g_W(\alpha,\beta) = \frac{1}{\pi\hbar} \int d\bar{\alpha}d\bar{\beta} \frac{1}{\langle \bar{\alpha}|\beta \rangle} \langle \bar{\alpha}|g|\bar{\beta} \rangle e^{2i(\beta-\bar{\beta})(\alpha-\bar{\alpha})/\hbar}$$

Equivalence of Eqs. (53) and (54) with Eqs. (33) and (34) is readily shown.
5 Margenau-Hill type case

We start with Eq. (16)

\[ \langle g \rangle = \iiint A^*(\alpha) A(\alpha') \delta(\alpha - x) g(\alpha' - x) d\alpha d\alpha' dx \]  

(55)

and we see that replacing

\[ \delta(\alpha' - x) = \int T^\dagger(x, \beta) T(\beta, \alpha') d\beta \]  

(56)

and inserting Eq. (26) immediately achieves the factorization of the integrand,

\[ \langle g \rangle = \int \left\{ A^*(\alpha) T^\dagger(\alpha, \beta) \int A(\alpha') T(\beta, \alpha') d\alpha' \right\} \left\{ (T^\dagger(\alpha, \beta))^{-1} \int \delta(\alpha - x) g T^\dagger(x, \beta) dx \right\} d\beta d\alpha \]  

(57)

Therefore

\[ \langle g \rangle = \int g(\alpha, \beta) P(\alpha, \beta) d\alpha d\beta \]  

(58)

with

\[ P_{MH}(\alpha, \beta) = A^*(\alpha) T^\dagger(\alpha, \beta) \int A(\alpha') T(\beta, \alpha') d\alpha' \]  

(59)

\[ g_{MH}(\alpha, \beta) = \left( T^\dagger(\alpha, \beta) \right)^{-1} \int \delta(\alpha - x) g T^\dagger(x, \beta) dx \]  

(60)

Alternate forms. Also, since

\[ B(\beta) = \int A(\alpha') T(\beta, \alpha') d\alpha' \]  

(61)

we may also write

\[ P_{MH}(\alpha, \beta) = A^*(\alpha) T^\dagger(\alpha, \beta) B(\beta) \]  

(62)

\[ g_{MH}(\alpha, \beta) = \left( T^\dagger(\alpha, \beta) \right)^{-1} \int \delta(\alpha - x) g T^\dagger(x, \beta) dx \]  

(63)

We note that \( P_{MH}^*(\alpha, \beta) \) is also a quasi-distribution.

Position-momentum case. To specialize to the \(qp\) case, as per Eqs. (43)–(46) we have

\[ B(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(q') e^{-iq'p/\hbar} dq' = \varphi(p) \]  

(64)

which is the momentum wave function. Hence, substituting into Eq. (62) we obtain

\[ P_{MH}(q, p) = \frac{1}{\sqrt{2\pi\hbar}} \psi^*(q) e^{iqp/\hbar} \varphi(p) \]  

(65)

which is a distribution first mentioned by Kirkwood [20] and derived by Margenau and Hill [24] and studied by Mehta [26] and others, and is known in the field of time-frequency analysis as the Rihaczek distribution [30].

Using Eq. (63), we obtain that

\[ g_{MH}(q, p) = e^{-iqp/\hbar} g(q, p) e^{iqp/\hbar} \]  

(66)
6 General class

Cohen obtained the general class of quasi-distributions for the \(qp\) case by introducing a kernel function that characterizes each quasi-distribution and its correspondence rule \[21\]. By taking all possible kernel functions, one obtains the totality of bilinear quasi-distributions and rules of association \[21, 17, 18, 39\].

We now show how to generate an infinite number of quasi-distributions for arbitrary operators. Again, we start with the quantum expectation value

\[
\langle g \rangle = \int \mathcal{A}^*(\alpha')\mathcal{A}(\alpha'')\delta(\alpha' - x)g\delta(\alpha'' - x)\,dx\,d\alpha'\,d\alpha''
\]  

\(67\)

Inserting the three expressions Eqs. (25)-(27), which we repeat here

\[
\delta(\alpha'' - x) = \int T(\beta', \alpha'')T^\dagger(x, \beta')d\beta'
\]  

\(68\)

\[
T^\dagger(\alpha', \beta') \frac{1}{T^\dagger(\alpha', \beta')} = 1
\]  

\(69\)

\[
\int \delta(\alpha' - \bar{\alpha})\delta(\beta' - \bar{\beta})d\bar{\alpha}d\bar{\beta}
\]  

\(70\)

we obtain

\[
\langle g \rangle = \int \mathcal{A}^*(\alpha')\mathcal{A}(\alpha'')T(\beta', \alpha'')T^\dagger(\alpha', \beta') \frac{1}{T^\dagger(\alpha', \beta')}\delta(\alpha' - \bar{\alpha})\delta(\beta' - \bar{\beta})
\]  

\[
\delta(\alpha - x)gT^\dagger(x, \beta)d\alpha'\,d\alpha''\,d\bar{\alpha}d\bar{\beta}d\beta'
\]  

\(71\)

To introduce the phase space variables \(\alpha\) and \(\beta\), we insert

\[
\delta(\bar{\alpha} - \alpha')\delta(\bar{\beta} - \beta') = \frac{1}{(2\pi)^4} \int \frac{\Phi(\theta, \tau)e^{i\theta(\alpha'-\bar{\alpha})}e^{ir\theta h/2}e^{ir(\beta'-\bar{\beta})}}{\Phi(\theta', \tau')e^{i\theta(\bar{\alpha}-\alpha')}e^{ir\theta h/2}e^{ir(\bar{\beta}-\bar{\beta})}} \,d\theta d\theta' d\tau' d\alpha d\bar{\beta}
\]  

\(72\)

where \(\Phi(\theta, \tau)\) is the kernel function that characterizes the distribution. We obtain

\[
\langle g \rangle = \int \left\{ \frac{1}{4\pi^2} \int \mathcal{A}^*(\alpha')\mathcal{A}(\alpha'')T(\beta', \alpha'')T^\dagger(\alpha', \beta')\Phi(\theta, \tau)e^{i\theta(\alpha'-\bar{\alpha})}e^{ir\theta h/2}e^{ir(\beta'-\bar{\beta})} \,d\alpha' \,d\beta' \,d\alpha'' \,d\theta d\tau \right\}
\]

\[
\left\{ \int \frac{1}{4\pi^2} \frac{1}{\Phi(\theta', \tau')e^{i\theta(\bar{\alpha}-\alpha')}e^{ir\theta h/2}e^{ir(\bar{\beta}-\bar{\beta})}} \frac{1}{T^\dagger(\alpha', \beta')}\delta(\bar{\alpha} - x)gT^\dagger(x, \bar{\beta})dx \,d\bar{\alpha}d\bar{\beta}d\theta' d\tau' \right\} d\alpha d\bar{\beta}
\]  

\(73\)

which achieves the factorization of the integrand. Therefore

\[
\langle g \rangle = \int g_{\Phi}(\alpha, \beta)P_{\Phi}(\alpha, \beta)\,d\alpha d\beta
\]  

\(74\)

with the quasi-distribution given by

\[
P_{\Phi}(\alpha, \beta) = \frac{1}{4\pi^2} \int \Phi(\theta, \tau)\mathcal{A}^*(\alpha')\mathcal{A}(\alpha'')T(\beta', \alpha'')T^\dagger(\alpha', \beta')e^{i\theta(\alpha'-\bar{\alpha})}e^{ir\theta h/2}e^{ir(\beta'-\bar{\beta})} \,d\alpha' \,d\beta' \,d\alpha'' \,d\theta d\tau
\]  

\(75\)

and the corresponding \(c\)-function is given by

\[
g_{\Phi}(\alpha, \beta) = \frac{1}{4\pi^2} \int e^{-i\theta(\bar{\alpha}-\alpha)}e^{-ir\theta h/2}e^{-ir(\bar{\beta}-\bar{\beta})} \frac{1}{T^\dagger(\alpha, \beta)\Phi(\theta', \tau')} \delta(\bar{\alpha} - x)gT^\dagger(x, \bar{\beta})dx \,d\bar{\alpha}d\bar{\beta}d\theta' d\tau'
\]  

\(76\)
If we take $\Phi(\theta, \tau) = 1$, then we obtain the Wigner-type case above, and if one takes $\Phi(\theta, \tau) = e^{i\theta\hbar/2}$ we obtain Eq. (62).

**Alternate form.** Making the transformation in Eq. (75)

\[
\alpha' = u - \hbar\tau/2 \quad ; \quad \alpha'' = u + \hbar\tau/2
\]

One obtains

\[
P_{\Phi}(\alpha, \beta) = \frac{\hbar}{4\pi^2} \int \Phi(\theta, \tau) \mathcal{A}^\dagger(u - \hbar\tau/2) \mathcal{A}(u + \hbar\tau/2) T^\dagger(u - \hbar\tau/2, \beta') T(\beta', u + \hbar\tau/2) e^{i\theta[u - \alpha - \hbar(\tau - \tau')/2]} e^{i\tau'(\beta' - \beta)} d\beta' d\theta d\tau dud\tau
\]

(78)

**Position-momentum case.** To specialize to the position-momentum case we use Eqs. (43)–(46); inserting them into Eq. (78) we obtain

\[
P_{\Phi}(q, p) = \frac{1}{4\pi^2} \int \int \Phi(\theta, \tau) \psi^\dagger(u - \tau\hbar/2) \psi(u + \tau\hbar/2) e^{i\theta(u - q - i\tau p)} du d\theta d\tau
\]

(79)

which is the generalized quasi-probability distribution obtained by Cohen [7]. All bilinear position-momentum distributions are generated by Eq. (79) by taking different functions for $\Phi(\theta, \tau)$. Similarly all bilinear distributions for arbitrary operators are generated by Eq. (78).

For the correspondence rule we have

\[
g_{\Phi}(q, p) = \frac{1}{(2\pi)^2} \int \frac{1}{\Phi(\theta, \tau)} e^{i\tau p} e^{-i\theta q} \delta(\bar{\alpha} + q - \hbar\tau/2 - x) g(x, p_x) \delta(\bar{\alpha} + q + \hbar\tau/2 - x) d\bar{\alpha} dx d\theta d\tau
\]

(80)

In the original formulation for generating all correspondence rules, the operator is given by [7]

\[
g_{\Phi}(x, p_x) = \int \int g_{\Phi}(\theta, \tau) \Phi(\theta, \tau) e^{i\theta x + i\tau p_x} d\theta d\tau
\]

(81)

That Eq. (80) obtains the classical function appearing in Eq. (81) in terms of the operator can be readily verified by substituting Eq. (81) into Eq. (80).

**Commuting operators.** Commuting operators have common eigenfunctions and we may write

\[
a u_{\alpha}(q) = \alpha u_{\alpha}(q)
\]

(82)

\[
b u_{\alpha}(q) = \gamma(\alpha) u_{\alpha}(q)
\]

(83)

Cohen has shown, using the characteristic function operator method, that for all distributions

\[
P(\alpha, \beta) = \delta(\beta - \gamma(\alpha)) |\mathcal{A}(\alpha)|^2
\]

(84)

This results may also be derived from Eq. (75).

### 7 Characteristic function

For two random variables, $\alpha$ and $\beta$, the characteristic function, $M(\theta, \tau)$, and distribution, $P(\alpha, \beta)$, are respectively

\[
M(\theta, \tau) = \langle e^{i\theta\alpha + i\tau\beta} \rangle = \int \int e^{i\theta\alpha + i\tau\beta} P(\alpha, \beta) d\alpha d\beta
\]

(85)
\[ P(\alpha, \beta) = \frac{1}{4\pi^2} \int \int e^{-i\theta \alpha - i\tau \beta} M(\theta, \tau) d\alpha d\beta \]  

Consider now the characteristic function for the Wigner type, Eq. (33), and of the general class, Eq. (75),

\[ M_W(\theta, \tau) = \frac{1}{\pi \hbar} \int e^{i\theta \alpha + i\tau \beta} A^*(\alpha') A(\alpha'') T^\dagger(\alpha', \beta') T(\beta', \alpha'') e^{-2i(\beta - \beta')(\alpha - \alpha')/\hbar} d\alpha' d\beta' d\alpha'' d\beta \]  

and

\[ M_\Phi(\theta, \tau) = \frac{1}{4\pi^2} \int \Phi(\theta', \tau') A^*(\alpha') A(\alpha'') T^\dagger(\alpha', \beta') T(\beta', \alpha'') e^{i\theta'(\alpha' - \alpha)/\hbar} e^{i\tau'(\beta' - \beta)/\hbar} d\alpha' d\beta' d\alpha'' d\beta \]

Straight-forward manipulation leads to the fact that

\[ M_\Phi(\theta, \tau) = \Phi(\theta, \tau) M_W(\theta, \tau) \]

This is identical in form to the \( qp \) case [7, 21].

8 Example

We consider the Hamiltonian of a particle acted on by constant force, \( f \),

\[ H = fq + \frac{p^2}{2m} \]

and we seek a joint quasi-distribution of position and the Hamiltonian. We take

\[ a = q, \quad b = H \]

The eigenvalue problem for \( H \) can be readily solved in the momentum representation

\[ b \hat{v}_\beta(p) = \left\{ fq + \frac{p^2}{2m} \right\} \hat{v}_\beta(p) = \beta \hat{v}_\beta(p) \]

or

\[ \left\{ i\hbar f \frac{d}{dp} + \frac{p^2}{2m} \right\} \hat{v}_\beta(p) = \beta \hat{v}_\beta(p) \]

The delta function normalized solution is

\[ \hat{v}_\beta(p) = \frac{1}{\sqrt{2\pi \hbar f}} e^{-i(\beta p - p^3/6m)/\hbar f} \]

In the spatial domain

\[ b v_\beta(q) = \beta v_\beta(q) \]

the eigenfunctions are

\[ v_\beta(q) = \frac{1}{2\pi \sqrt{f}} \int e^{-i(\beta p - p^3/6m)/\hbar f} e^{iqp/\hbar} dp \]

and the transformation matrix,

\[ T(\beta, \alpha) = \int v_\beta^*(q) u_\alpha(q) dq \]
is

\[ T(\beta, \alpha) = \frac{1}{2\pi \hbar} \frac{1}{\sqrt{f}} \int \int e^{i(p_3 - p_3')/6m} e^{-i q \alpha / \hbar} \delta(q - \alpha) \, dq dp \]  

(98)

\[ = \frac{1}{2\pi \hbar} \frac{1}{\sqrt{f}} \int e^{-i q^2/6m\hbar} e^{i(\beta / f - \alpha) p / \hbar} dp \]  

(99)

Since \( A \) is now the position wave function, \( \psi \), we can write

\[ P(\alpha, \beta) = \frac{1}{\pi \hbar} \int \int \psi^* (\alpha') \psi (\alpha'') T^\dagger (\alpha', \beta') T (\beta', \alpha'') e^{-2i(q - \alpha'(\beta' - \beta'))/\hbar} \, d\alpha' d\beta' d\alpha'' \]  

(100)

with

\[ T^\dagger (q', \beta') T (\beta', q'') = \frac{1}{(2\pi \hbar)^2} \frac{1}{\sqrt{f}} \int \int dp dp' e^{i(p - p')^2 / 6m\hbar} e^{-i \beta''/\hbar} e^{i(\alpha' - \alpha'') p / \hbar} \]  

(101)

This could be simplified further by using the wave function in the \( b \) basis, \( B(\beta) \),

\[ P(\alpha, \beta) = \frac{1}{(2\pi \hbar)^2} \frac{1}{\pi} \int \int \psi^* (\alpha') B(\beta') T^\dagger (\alpha', \beta') e^{-2i(\alpha - \alpha')(\beta - \beta')/\hbar} \, d\alpha' d\beta' \]  

(102)

where

\[ B(\beta) = \int T(\beta, \alpha) \psi (\alpha) d\alpha \]  

(103)

**Momentum and energy.** We take

\[ a = p, \quad b = H \]  

(104)

The eigenfunctions for momentum are then,

\[ u_\alpha (q) = \frac{1}{\sqrt{2\pi \hbar}} \psi^{i q / \hbar} \]  

(105)

and the transformation matrix is

\[ T(\beta, \alpha) = \int v_\beta^* (q) u_\alpha (q) \, dq \]  

(106)

\[ = \frac{1}{2\pi \hbar} \int e^{i(\beta - \alpha^2 / 6m) / \hbar} \]  

(107)

The quasi-distribution of momentum and energy is

\[ P(\alpha, \beta) = \frac{1}{(2\pi \hbar)^2} \frac{1}{\pi} \int \int \varphi^* (\alpha') B(\beta') T^\dagger (\alpha', \beta') e^{-2i(\alpha - \alpha')(\beta - \beta')/\hbar} \, d\alpha' d\beta' \]  

(108)

where \( \varphi \) is the momentum wave function and

\[ A(\alpha) = \varphi (\alpha) = \frac{1}{\sqrt{2\pi \hbar}} \int e^{-i q \alpha / \hbar} \psi (q) \, dq \]  

(109)

\[ B(\beta) = \int T(\beta, \alpha) \varphi (\alpha) d\alpha \]  

(110)
9 Conclusion

We have obtained joint quasi-distributions for arbitrary operators. Starting with the quantum mechanical expression for the expectation value of an operator, we have expressed it as phase space integral where the integrand is factorized into two factors: One depending only on the density matrix and other only on the operator. Simultaneously with the derivation of the quasi-distribution, one obtains the generalization of the concept of correspondence rule for arbitrary operators. An advantage of our approach is that it shows straightforwardly the coupling of the quasi-distribution with its correspondence rule. Although we have presented our results for the pure case, generalization to the density matrix follows straightforwardly.

In the classic paper by Moyal, he derived the Wigner distribution by defining the characteristic function by way of

\[ M(\theta, \tau) = \int \psi^*(q)e^{iq\theta + ip\tau}\psi(q) \, dq \]  

(111)

where \( q \) and \( p \) are the position and momentum operators. For the case of position and momentum, the operator \( e^{iq\theta + ip\tau} \) can be simplified and \( M(\theta, \tau) \) calculated explicitly.

Scully and Cohen [35] generalized the method by defining the characteristic function for two arbitrary operators

\[ M(\theta, \tau) = \int \psi^*(q)e^{i\alpha q + i\beta p}\psi(q) \, dq \]  

(112)

and this has been carried out for a number of cases where \( e^{i\alpha q + i\beta p} \) can be simplified. However, the simplification of \( e^{i\alpha q + i\beta p} \) is generally difficult [38]. Of course there is a relation between the two approaches and this will be discussed in a future paper.

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