RESEARCH ARTICLE

Global and local asymptotic stability of an epidemic reaction-diffusion model with a nonlinear incidence

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1 | INTRODUCTION

In this manuscript, we consider the reaction-diffusion epidemic phenomenon proposed in Djebara et al.\textsuperscript{1} which is an extended version of the SIS epidemic model with the nonlinear incidence $u\varphi(v)$. The system is described

\begin{align}
\frac{\partial u}{\partial t} - d_1 \Delta u &= \Lambda - \mu u - \lambda u \varphi(v) =: F(u, v) \quad \text{in} \quad (0, \infty) \times \Omega, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= -\sigma v + \lambda u \varphi(v) =: G(u, v) \quad \text{in} \quad (0, \infty) \times \Omega.
\end{align}

Throughout this paper, the notation $\Delta$ denotes the Laplacian operator on $\Omega$, where $\Omega$ is an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. The constant parameters $d_1, d_2 > 0$ are the diffusion coefficients. We assume the initial conditions

\begin{equation}
\begin{align*}
&u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \quad \text{in} \quad \Omega,
\end{align*}
\end{equation}

where $u_0, v_0 \in C(\overline{\Omega})$ and impose homogeneous Neumann boundary conditions

\begin{equation}
\begin{align*}
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad (0, \infty) \times \partial \Omega.
\end{align*}
\end{equation}
with $v$ being the unit outer normal to $\partial \Omega$. We will also assume that the initial conditions $u_0(x), v_0(x) \in \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0} := [0, +\infty)$.

The considered system (1.1)–(1.3) may describe the transmission of a communicable disease between individuals such as HIV/AIDS. The population assumed in the model is divided into two classes, susceptible and infective. Functions $u(t, x)$ and $v(t, x)$ denote the non-dimensional population densities of the susceptible and infective individuals at time $t$ and location $x$, respectively. The constant parameter $\Lambda > 0$ represents the average growth of different categories of susceptibles whether through birth or migration, sometimes referred to as the recruitment rate of the population. In addition, $\mu$ denotes the natural death rate, $\lambda$ is the rate at which susceptibles turn into infectives, and $\sigma$ is the rate at which infectives recover from the disease. For the purpose of this study, we will assume that $\mu, \sigma, \lambda > 0$. The incidence function $\varphi(v)$ introduces a nonlinear relation between the two classes of individuals. We assume $\varphi$ to be a continuously differentiable function on $\mathbb{R}^+$ satisfying

$$\varphi(0) = 0, \quad (1.4)$$

and

$$0 < v \varphi'(v) \leq \varphi(v) \text{ for all } v > 0. \quad (1.5)$$

Infectious diseases are the leading cause of death in all living organisms. The study of epidemiology has attracted the attention of a vast number of researchers with the aim of improving the treatment of these diseases through planning and predictions of the spread of the disease thereby reducing mortality rates. Hethcote$^3$ studied a simple model resulting in the existence, and uniqueness of the system solution as well as the asymptotic stability of the equilibrium points using an Lyapunov function

$$V(t) = \int_{\Omega} \left[ \frac{1}{2} \left( S - \frac{\Lambda}{\mu} \right)^2 + \frac{\Lambda I}{\mu} \right] \, dx$$

was used. A similar investigation was carried out in a previous study.$^{15}$

A criss-cross infection model describing the spread of FIV (Feline immunodeficiency virus) was proposed and studied in Fitzgibbon et al.$^{16}$ This model falls under (1.1) with $n \geq 1$, $d_1 > 0$, $\Lambda = \mu = 0$, and bilinear incidence $u \varphi(v) = \lambda uv$. Another important study is Alexander and Moghadas.$^8$ where the local and global asymptotic stability of the simple epidemiological model with $\varphi(v) = \beta \left[ 1 + \varphi_p(v) \right] v$ were investigated by means of the Poincaré index theory. The study showed that the basic reproductive number is independent of the functional $\varphi_p(v)$. Lyapunov functions were proposed in Korobeinikov and Maini$^{17}$ for ODE SIS and SEIR models with incidence rate $\varphi(v)g(u)$ and $\Lambda = \mu \geq 0$. The same authors generalized the incidence rate to the very general form $f(u, v)$ and $\Lambda = \mu \geq 0$ in Korobeinikov.$^{18}$ Using extended Lyapunov functions based on those constructed in Korobeinikov and Maini,$^{17}$ they established the global asymptotic stability.
More sophisticated nonlinear incidences of the form \( k \frac{uv}{1+u} \) were studied in previous works\(^{19,20}\), where the authors considered an Avian influenza model and proved that the global asymptotic stability depends on the basic reproductive number. A more general Lyapunov function was developed in Li et al\(^{12}\) and used to establish the existence of an endemic equilibrium for ODE SIS models with the nonlinear incidence \( \phi(v) = \lambda u \phi(v) \). Furthermore, a spatially diffusive SIR epidemic model with two scenarios \((d_1 = 0, d_2 > 0)\) and \((d_1 > 0, d_2 = 0)\) was assumed in Kuniya and Wang\(^{21}\) with all parameters being spatially heterogeneous, where \( \phi(v) = \lambda(x) v \) and \( \sigma, \mu, \gamma, \lambda \in C(\Omega; \mathbb{R}) \) are all strictly positive.

The global asymptotic stability of \( E_0 = (b(x) \lambda(x), 0) \) and \( E^* = (S^*(x), I^*(x)) \) was studied, and the authors concluded that the standard threshold dynamical behaviors depend on the basic reproductive number. More recently, Jia and Qin\(^{11}\) investigated the stability of the disease-free equilibrium and the unique endemic equilibrium for an ODE SITAR epidemiological model with general nonlinear incidence rate \( u \phi(v) \) using the geometric approach.

In the present study, we study the existence of equilibria and their asymptotic stability conditions for the diffusive epidemic model considered in Djebara et al\(^1\), which is an extension of that proposed in Li et al\(^{12}\). We recall that in Djebara et al\(^1\), we established the global existence of solutions to problem (1.1)–(1.3). We have now established the system model and parameter descriptions. Section 2 will define the basic reproductive number \( R_0 \) of the proposed model and establish the existence of two equilibria. The local asymptotic stability and instability of the disease-free equilibrium and the endemic equilibrium are investigated. Section 3 will prove that the two steady states of the model are globally asymptotically stable using an appropriate Lyapunov functional. Finally, Section 4 will present some numerical examples to validate the theoretical analysis presented throughout the paper.

## 2 | PRELIMINARY PROPERTIES OF THE MODEL

Let us assume that the initial conditions \((u_0, v_0) \in \mathbb{R}^2_{\geq 0}\). Note that for \((u, v) \in \mathbb{R}^2_{\geq 0}\), we have

\[
\begin{align*}
F(0, v) &= \Lambda > 0, \\
G(u, 0) &= 0,
\end{align*}
\]

which makes the function \((F, G)^T\) essentially non-negative. Hence, the non-negative quadrant \(\mathbb{R}^2_{\geq 0}\) is an invariant set, see Haddad et al\(^{22}\) and Proposition 2.1 of Quittner and Souplet\(^{23}\). By dropping the spatial variable, the proposed system reduces to the following system of ordinary differential equations (ODE):

\[
\begin{align*}
\frac{du}{dt} &= F(u, v) \quad \text{in} \quad (0, \infty), \\
\frac{dv}{dt} &= G(u, v) \quad \text{in} \quad (0, \infty),
\end{align*}
\]

with initial conditions

\[ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \]

In the following subsections, we define an invariant region for the system, identify the system's equilibria and their relation to the basic reproduction number \( R_0 \), establish the global existence of solutions in time, and investigate the local stability of the system in the ODE and PDE scenarios.

### 2.1 | Invariant regions

Throughout this paper, we let \( N = u + v \) and \( \sigma_0 = \min(\sigma, \mu) \). We also define the region

\[ D = \left\{ (u, v) : u, v \geq 0 \quad \text{and} \quad u + v \leq \frac{\Lambda}{\sigma_0} \right\}. \]

The following proposition shows that \( D \) is an invariant region of system (2.1)–(2.2).

**Proposition 1.** The region \( D \) is non-empty, attracting and positively invariant.
Proof. We start by summing the equations of system (2.1)–(2.2), which yields

\[
\frac{d}{dt} N(t) = u_t + v_t \leq \Lambda - \sigma_0 N.
\]

This implies that

\[
N(t) \leq \frac{\Lambda}{\sigma_0} \left(1 - e^{-\sigma_0 t}\right) + N_0 e^{-\sigma_0 t}.
\]

Substituting the value of \(N\) yields

\[
(u + v)(t) \leq \frac{\Lambda}{\sigma_0} \left(1 - e^{-\sigma_0 t}\right) + (u + v)(0)e^{-\sigma_0 t},
\]

for \(t \geq 0\). If the initial states satisfy \((u + v)(0) \leq \frac{\Lambda}{\sigma_0}\), then \((u + v)(t) \leq \frac{\Lambda}{\sigma_0}\) and

\[
\limsup_{t \to \infty} N(t) \leq \frac{\Lambda}{\sigma_0}.
\]

As a result, region \(D\) is positively invariant and attracting within \(\mathbb{R}^2_{\geq 0}\). Therefore, it is sufficient to consider the dynamics of the model within \(D\) as \(D\) is the biologically feasible region of the system where the existence and uniqueness results hold for the system.

\[\square\]

2.2 Existence of equilibria and the basic reproduction number \(R_0\)

In this section, we aim to show the existence of equilibrium solutions for (2.1)–(2.2) and calculate the basic reproduction number.

**Theorem 1.** Subject to conditions (1.4)-(1.5), system (2.1)–(2.2) admits always a disease-free equilibrium \(E_0 = \left(\frac{\Lambda}{\mu}, 0\right)\).

If \(R_0 = \frac{\Lambda \lambda}{\mu \sigma} \varphi'(0) > 1\), then system (2.1)–(2.2) has an endemic equilibrium \(E^* = (u^*, v^*)\).

**Proof.** The positive equilibria of model (2.1)–(2.2) satisfy

\[
\begin{cases}
F(u, v) = \Lambda - \lambda uv (v) - \mu u = 0, \\
G(u, v) = \lambda uv (v) - \sigma v = 0.
\end{cases}
\]

(2.3)

If \(u = 0\), it is easy to see that the system has no equilibrium. On the other hand, only equilibrium is found for \(v = 0\) and that is \(E_0 = \left(\frac{\Lambda}{\mu}, 0\right)\).

Next, we study the existence conditions of an endemic steady state in the case \(v > 0\). From the second part of (2.3), and because \(\lambda > 0\) and \(\varphi(v) > 0\), we obtain

\[
u = \frac{\sigma v}{\lambda \varphi (v)}.
\]

Substituting this into the first equation yields

\[
h(v) = 0 \quad \text{for any} \quad v > 0,
\]

where

\[
h(v) = \frac{\Lambda \lambda}{\mu \sigma} \varphi (v) - \frac{\sigma \lambda}{\mu \sigma} \varphi (v) - 1
\]

is continuous for any \(v > 0\). It is clear that

\[\lim_{v \to 0} h(v) = \lim_{v \to 0} \frac{\Lambda \lambda}{\mu \sigma} \varphi' (v) - 1 = R_0 - 1.
\]
where $R_0$ is the basic reproduction number to be identified next. Using $\sigma_0 = \min(\sigma, \mu)$, we have

$$\lim_{v \to \frac{\Lambda}{\sigma_0}} h(v) = h \left( \frac{\Lambda}{\sigma_0} \right) = \frac{\lambda(\sigma_0 - \sigma)}{\mu \sigma} v \phi \left( \frac{\Lambda}{\sigma_0} \right) - 1 < 0.$$ 

Hence, for $R_0 > 1$,

$$h \left( \frac{\Lambda}{\sigma_0} \right) \lim_{v \to 0} h(v) = h \left( \frac{\Lambda}{\sigma_0} \right) (R_0 - 1) < 0.$$ 

By applying the intermediate value theorem, there exists a real $v^* \in \left( 0, \frac{\Lambda}{\sigma_0} \right)$ such that (2.4) holds. Using (1.5), we can show that the function $h$ is monotonically decreasing for all $v > 0$ as

$$\frac{dh}{dv}(v) = \frac{\Lambda \lambda [v \phi'(v) - \phi(v)] - \sigma \lambda v^2 \phi'(v)}{\mu \sigma v^2} < 0.$$ 

Then, there exists a unique real $v^* \in \left( 0, \frac{\Lambda}{\sigma_0} \right)$ such that $h(v^*) = 0$, which implies the existence of a unique $u^* = \frac{\sigma v^*}{\lambda \phi(v^*)}$.

Note that in $\left( \frac{\Lambda}{\sigma_0}, +\infty \right)$, the second equation of (2.3) has no solution because

$$\max_{v \in \left( \frac{\Lambda}{\sigma_0}, +\infty \right)} h(v) \leq h \left( \frac{\Lambda}{\sigma_0} \right) < 0.$$ 

This concludes the proof.

In the previous proof, we used the reproduction number $R_0$ of system (2.1)–(2.2). It is now time to identify its value by means of the next generation matrix method formulated in Al-Sheikh and Musali\textsuperscript{24} and described further in Lemma 1 of Van den Driessche and Watmough\textsuperscript{25} (page 32). System (2.1)–(2.2) may be rewritten in vector form as

$$\begin{pmatrix} v_t \\ u_t \end{pmatrix} = \begin{pmatrix} \lambda u \phi(v) - \sigma v \\ \Lambda - \lambda u \phi(v) - \mu u \end{pmatrix} = \begin{pmatrix} \lambda u \phi(v) \\ 0 \end{pmatrix} - \begin{pmatrix} -\Lambda + \lambda u \phi(v) + \mu u \end{pmatrix}.$$ 

The Jacobian matrices corresponding to vectors $\begin{pmatrix} \lambda u \phi(v) \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -\Lambda + \lambda u \phi(v) + \mu u \end{pmatrix}$ at the disease-free equilibrium $E_0 = \left( \frac{\Lambda}{\mu}, 0 \right)$ are given, respectively, by

$$\begin{pmatrix} \frac{\lambda \Lambda}{\mu} \phi'(0) \\ 0 \end{pmatrix} = \begin{pmatrix} S \\ 0 \end{pmatrix}$$ 

and

$$\begin{pmatrix} \frac{\lambda \sigma}{\mu} \phi'(0) \\ 0 \end{pmatrix} = \begin{pmatrix} V \\ 0 \end{pmatrix}.$$ 

The basic reproduction number $R_0$ is simply the spectral radius of the next generation matrix

$$K = SV^{-1} = \left( \frac{\lambda \Lambda}{\mu} \phi'(0) \right) (\sigma)^{-1} = \frac{\lambda \Lambda}{\mu \sigma} \phi'(0),$$

which is given by

$$R_0 = \rho \left( SV^{-1} \right) = \frac{\lambda \Lambda}{\mu \sigma} \phi'(0).$$
2.3 | The local ODE stability

Now that we have identified two constant steady states for system (2.1)–(2.2), namely, \( E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \) and \( E^* = (u^*, v^*) \), we move to study their local asymptotic stability as described in the following theorem.

**Theorem 2.** Assuming that the incidence function \( \varphi \) satisfies (1.4)–(1.5), the following statements hold for system (2.1)–(2.2):

(i) If \( R_0 < 1 \), the disease-free equilibrium solution \( E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \) is the only steady state of the system and is locally asymptotically stable.

(ii) If \( R_0 > 1 \), \( E_0 \) is unstable and the positive constant endemic steady state \( E^* = (u^*, v^*) \) is locally asymptotically stable.

**Proof.** To prove the local asymptotic stability of the constant steady states, we make advantage of the Jacobian matrix, which may be given by

\[
J(u, v) = \begin{pmatrix}
F_u(u, v) & F_v(u, v) \\
G_u(u, v) & G_v(u, v)
\end{pmatrix},
\]

where \( F_u, F_v, G_u, \) and \( G_v \) are the first partial derivatives. Evaluating the derivatives yields

\[
J(u, v) = \begin{pmatrix}
-\lambda \varphi (v) - \mu - \lambda u^* \varphi' (v) & \lambda u^* \varphi' (v) - \sigma \\
\lambda \varphi (v) & \lambda \varphi' (v) - \sigma
\end{pmatrix}.
\]  

Evaluating \( J(u, v) \) at \( E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \) with (1.4) in mind yields

\[
J(E_0) = \begin{pmatrix}
-\mu - \lambda \varphi'(0) & 0 \\
0 & \lambda \varphi'(0) - \sigma
\end{pmatrix}.
\]

The local stability of \( E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \) rests on the nature of the eigenvalues corresponding to \( J(E_0) \). The eigenvalues can be easily shown to be \( \lambda_1 = -\mu < 0 \) and \( \lambda_2 = \lambda \varphi'(0) - \sigma \). It is easy to see that \( \lambda_2 < 0 \) if \( R_0 < 1 \), leading to the asymptotic stability of \( E_0 \).

The second case is where \( R_0 > 1 \). The equilibrium \( E_0 \) is clearly unstable, but the system possesses a positive endemic equilibrium \( E^* = (u^*, v^*) \) where \( u^*, v^* > 0 \). Since \( E^* \) satisfies (2.3), we have

\[
\begin{align*}
\Lambda &= \lambda u^* \varphi(v^*) + \mu u^*, \\
\sigma &= \frac{\mu}{v^*} \varphi'(v^*).
\end{align*}
\]

Evaluating the Jacobian matrix (2.5) at \( E^* = (u^*, v^*) \) yields

\[
J(u^*, v^*) = \begin{pmatrix}
-\lambda \varphi (v^*) - \mu - \lambda u^* \varphi' (v^*) & \lambda u^* \varphi' (v^*) - \sigma \\
\lambda \varphi (v^*) & \lambda \varphi' (v^*) - \sigma
\end{pmatrix},
\]

which has trace

\[
\text{tr}(J(u^*, v^*)) = -\left[ \lambda \varphi (v^*) + \mu \right] - \sigma + \lambda u^* \varphi' (v^*).
\]

Using (2.6) and (1.5), this can be rewritten as

\[
\text{tr}(J(u^*, v^*)) = -\left( \frac{\Lambda}{u^*} - \lambda u^* \right) \frac{\varphi(v^*)}{v^*} - \varphi'(v^*) < 0.
\]

The determinant of the Jacobian may be given by

\[
det J(u^*, v^*) = \lambda \sigma \varphi (v^*) + \mu \sigma - \mu \lambda u^* \varphi' (v^*).\]
Using (2.6), we obtain
\[
\det(J(u^*, v^*)) = \lambda \frac{\lambda u^* v^*}{v^*} \frac{\varphi(v^*)}{v^*} + \mu \frac{\lambda u^* v^*}{v^*} - \mu \lambda u^* \varphi'(v^*)
\]
\[
= \frac{\lambda^2 u^* (\varphi(v^*))^2}{v^*} + \mu \lambda u^* \left[ \frac{\varphi(v^*)}{v^*} - \varphi'(v^*) \right],
\]
and from (1.5), we obtain \(\det(J(u^*, v^*)) > 0\). Hence, the equilibrium \(E^* = (u^*, v^*)\) is locally asymptotically stable. 

\[\square\]

2.4 | Global existence of solutions

Let us first state a lemma presented in Abdelmalek and Bendoukha,\(^2^6\) which will become useful later on.

**Lemma 1.** Condition (1.5) implies that
\[
0 < \frac{\varphi(v)}{v} \leq \varphi'(0) \text{ for all } v > 0. \tag{2.7}
\]

Now, assuming that the function \(\varphi\) satisfies conditions (1.4)–(1.5), the following proposition establishes that the solution of system (1.1)–(1.3) exists globally in time and is bounded by a parameter dependent constant.

**Proposition 2.** For any non-negative initial data \((u_0, v_0) \in C(\Omega) \times C(\Omega)\), the non-negative solution \((u(t, x), v(t, x))\) of system (1.1)–(1.3) exists uniquely and globally in time. In addition, there exists a positive constant \(C(u_0, v_0, \Lambda, \mu, \lambda, \sigma)\) such that \(\forall t > 0\),

\[
\|u(t, \cdot)\|_{L^\infty(\Omega)} + \|v(t, \cdot)\|_{L^\infty(\Omega)} \leq C. \tag{2.8}
\]

Furthermore, there exists a positive constant \(\bar{C}(\Lambda, \mu, \lambda, \sigma)\) such that \(\forall t \geq T\) for some large \(T > 0\),

\[
\|u(t, \cdot)\|_{L^\infty(\Omega)} + \|v(t, \cdot)\|_{L^\infty(\Omega)} \leq \bar{C}. \tag{2.9}
\]

**Proof.** Let us define \(\mathcal{X} = \mathcal{Y} \times \mathcal{Y}\) with \(\mathcal{Y} = C(\Omega)\). Given
\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\]
the norm on \(\mathcal{X}\) is defined as \(\|\psi\| = \|\psi_1\|_Y + \|\psi_2\|_Y\). Hence, \((\mathcal{X}, \|\cdot\|)\) is a Banach space. Let us also define the unbounded operator
\[
\bar{A} : D(\bar{A}) \subset \mathcal{X} \rightarrow \mathcal{X}
\]
such that
\[
\bar{A} \begin{pmatrix} u \\ v \end{pmatrix} = (\bar{A}_1 u, \bar{A}_2 v)^T = (d_1 \Delta, d_2 \Delta)^T,
\]
and
\[
D(\bar{A}) = D(\bar{A}_1) \times D(\bar{A}_2),
\]
with
\[
D(\bar{A}_i) = \left\{ \varphi \in C^2(\Omega) \cap C^1(\Omega) : \bar{A}_i \varphi \in C(\Omega), \frac{\partial \varphi}{\partial v} = 0 \text{ on } \partial \Omega \right\}, \text{ for } i = 1, 2.
\]
We may write system (1.1)–(1.3) in abstract form in \(\mathcal{X}\) as
\[
\begin{cases}
\frac{dU}{dt}(t) = AU(t) + Y(U(t)), \ t > 0, \\
\bar{U}(0) = U_0 \in \mathcal{X},
\end{cases}
\]
where \(U(t) = (u(t, \cdot), v(t, \cdot))^T, Y(U(t)) = (F(U(t)), G(U(t)))^T\) and \(A = \text{diag}(A_1, A_2)\), in which \(A_i\) is the closure of \(\bar{A}_i\) in \(\mathcal{Y}\) for \(i = 1, 2\). The operator \(A\) is the infinitesimal generator of an analytical semigroup of bounded linear operators \(\{T(t)\}_{t \geq 0}\) on \(\mathcal{X}\).\(^2^7\) Since \(\mathcal{Y}\) is locally Lipschitz in \(\mathcal{X}\), it follows that system (1.1)–(1.3) admits a unique local solution \((u(t, x), v(t, x))\) for \(t \in [0, T_{max})\) and \(x \in \Omega\) where \(T_{max}\) is the maximal existence time.\(^2^8\)
Let us now consider the case \( u(t, x) \in (0, T_{\text{max}}) \times \Omega \), which can be formulated as

\[
\begin{aligned}
\frac{\partial u}{\partial t} - d_1 \Delta u &= \Lambda - \mu u - \lambda u \varphi (v), \quad \text{in} \quad (0, T_{\text{max}}) \times \Omega, \\
u(0, x) &= u_0(x), \quad \text{on} \quad \Omega, \\
\frac{\partial u}{\partial v} &= 0, \quad \text{on} \quad (0, T_{\text{max}}) \times \partial \Omega.
\end{aligned}
\] (2.10)

We can easily observe that an upper solution exists for (2.10) for any positive functions \( u \) and \( v \). This upper solution is given by

\[ N_1 := \max \left\{ \frac{\Lambda}{\mu} , \| u_0 \|_{C(\overline{\Omega})} \right\}. \]

Using the comparison principle from Protter and Weinberger, we obtain \( u(t, x) \leq N_1 \) in \([0, T_{\text{max}}) \times \Omega\), and consequently, \( u \) is uniformly bounded in \([0, T_{\text{max}}) \times \Omega\). Integrating the equations of (1.1) over \( \Omega \) and taking the sum of the resulting two identities yields

\[
\frac{d}{dt} \int_{\Omega} (u(t, x) + v(t, x)) \, dx = |\Omega| \Lambda - \int_{\Omega} (\mu u(t, x) + \sigma v(t, x)) \, dx,
\]

\[
\leq |\Omega| \Lambda - \sigma_0 \int_{\Omega} (u(t, x) + v(t, x)) \, dx,
\] (2.11)

where \( \sigma_0 = \min \{ \mu, \sigma \} \). From the well-known Gronwall’s inequality, we have for \( t \in (0, T_{\text{max}}) \),

\[
\int_{\Omega} (u(t, x) + v(t, x)) \, dx \leq N_2,
\] (2.12)

where \( N_2 > 0 \). Hence, for \( t \in (0, T_{\text{max}}) \),

\[
\int_{\Omega} v(t, x) \, dx \leq N_2.
\] (2.13)

Following the footsteps of Alikakos (Theorem 3.1) and using the \( v \)-equation, we conclude \( \exists N_3 > 0 \) depending on \( N_2 \) such that \( \nu(t, x) \leq N_3 \) over \([0, T_{\text{max}}) \times \Omega\). Therefore, \( v \) is also uniformly bounded in \([0, T_{\text{max}}) \times \Omega\). Using the standard theory of semilinear parabolic systems described in Henry, we deduce \( T_{\text{max}} = \infty \).

When \( T_{\text{max}} = \infty \), system (2.10) becomes

\[
\begin{aligned}
\frac{\partial u}{\partial t} - d_1 \Delta u &= \Lambda - \mu u - \lambda u \varphi (v) \leq \Lambda - \mu u, \quad \text{in} \quad (0, \infty) \times \Omega, \\
u(0, x) &= u_0(x) \leq \| u_0 \|_{C(\overline{\Omega})}, \quad \text{on} \quad \Omega, \\
\frac{\partial u}{\partial v} &= 0, \quad \text{on} \quad (0, \infty) \times \partial \Omega.
\end{aligned}
\] (2.14)

By means of the comparison principle, we get \( u(t, x) \leq \omega(t) \) for \( t \in [0, \infty) \), where \( \omega(t) = \| u_0 \|_{C(\overline{\Omega})} e^{-\mu t} + \left( \frac{\Lambda}{\mu} \right) (1 - e^{-\mu t}) \) is the unique solution of the problem

\[
\begin{aligned}
\frac{d\omega}{dt} &= \Lambda - \mu \omega, \quad t > 0, \\
\omega(0) &= \| u_0 \|_{C(\overline{\Omega})}.
\end{aligned}
\] (2.15)

Consequently, for \( x \in \overline{\Omega} \), we have

\[ u(t, x) \leq \omega(t) \xrightarrow{t \to \infty} \frac{\Lambda}{\mu}. \]

Therefore, we have an upper bound for \( \| u(t, \cdot) \|_{L^\infty(\Omega)} \) independent of the initial conditions given a sufficiently large \( t \). By applying Peng and Zhao (Lemma 3.1), we find that \( \| v(t, \cdot) \|_{L^\infty(\Omega)} \) is also bounded by a positive constant independent of the initial conditions for a large enough \( t \).
2.5 | The local PDE stability

We have already established sufficient conditions for the local asymptotic stability of system (2.1)–(2.2) in the ODE scenario. Let us now examine the more general PDE case (1.1)–(1.3).

**Theorem 3.** Assuming that the incidence function \( \psi \) satisfies (1.4)–(1.5), the following statements hold for system (1.1)–(1.3):

(i) If \( R_0 < 1 \), the disease-free equilibrium \( E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \) is locally asymptotically stable.

(ii) If \( R_0 > 1 \), the positive constant endemic steady equilibrium \( E^* = (u^*, v^*) \) is locally asymptotically stable.

**Proof.** (i) In the presence of diffusion, the equilibrium point \( E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \) satisfies

\[
\begin{align*}
d_1 \Delta u + \Lambda - \lambda u^* \varphi (v^*) - \mu u^* &= 0, \\
d_2 \Delta v + \lambda u^* \varphi (v^*) - \sigma v^* &= 0,
\end{align*}
\]

with Neumann boundaries

\[
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \mathbb{R}^+ \times \partial \Omega.
\]

We denote the indefinite sequence of positive eigenvalues for the Laplacian operator \( \Delta \) over \( \Omega \) with Neumann boundary conditions by \( \lambda_0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \rightarrow +\infty \). Note that the first eigenfunction is a constant, which is why the corresponding eigenvalue is equal to zero. The corresponding sequence of eigenfunctions is denoted by \( (\Phi_{ij})_{j=1,m_i} \), where \( m_i \geq 1 \) is the algebraic multiplicity of \( \lambda_i \). These functions are the solutions of

\[
\begin{align*}
-\Delta \Phi_{ij} &= \lambda \Phi_{ij} \text{ in } \Omega, \\
\frac{\partial \Phi_{ij}}{\partial \nu} &= 0 \text{ on } \partial \Omega.
\end{align*}
\]

The eigenfunctions are normalized according to

\[
\left\| \Phi_{ij} \right\|_2 = \int_{\Omega} \Phi_{ij}^2 (x) \, dx = 1.
\]

The set of eigenfunctions \( \{ \Phi_{ij} : i \geq 0, j = 1, m_i \} \) forms a complete orthonormal basis in \( L^2(\Omega) \). In order to establish the local asymptotic stability of the steady states, we must examine all the eigenvalues of the linearizing operator, and if they all have negative real parts, then the solution is locally asymptotically stable. The linearizing operator may be given by

\[
L(E_0) = \begin{pmatrix}
d_1 \Delta - \mu - \lambda \mu \varphi'(0) & 0 \\
0 & d_2 \Delta + \lambda \mu \varphi'(0) - \sigma
\end{pmatrix}.
\]

Using the same method from Abdelmalek and Bendoukha,\(^{33}\) the stability of \( E_0 \) reduces to examining the eigenvalues of the matrices

\[
J_i(E_0) = \begin{pmatrix}
-d_1 \lambda_i - \mu - \lambda \mu \varphi'(0) \\
0 & -d_2 \lambda_i + \lambda \mu \varphi'(0) - \sigma
\end{pmatrix}
\]

for all \( i \geq 0 \), which are given for all \( i \geq 0 \) by

\[
\begin{align*}
\lambda_{i1} &= -d_1 \lambda_i - \mu, \\
\lambda_{i2} &= -d_2 \lambda_i + \lambda \mu \varphi'(0) - \sigma.
\end{align*}
\]

Since the Laplacian eigenvalues are positive and in ascending order, both \( \lambda_{i1} \) and \( \lambda_{i2} \) clearly have negative real parts for \( R_0 < 1 \) leading to the local stability of \( E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \).

(ii) The second steady state \( E^* = (u^*, v^*) \) satisfies

\[
\begin{align*}
d_1 \Delta u + \Lambda - \lambda u^* \varphi (v^*) - \mu u^* &= 0, \\
d_2 \Delta v + \lambda u^* \varphi (v^*) - \sigma v^* &= 0,
\end{align*}
\]

with Neumann boundaries

\[
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \mathbb{R}^+ \times \partial \Omega.
\]
The corresponding linearization operator is
\[
\mathcal{L}(E^*) = \begin{pmatrix} d_1\Delta - \lambda \varphi(v^*) - \mu - \lambda u^* \varphi'(v^*) & \lambda \varphi(v^*) \\ \lambda \varphi(v^*) & d_2\Delta + \lambda u^* \varphi'(v^*) - \sigma \end{pmatrix}.
\]

Hence, the stability of \(E^*\) rests on the negativity of the real parts of the eigenvalues of matrices
\[
J_i(E^*) = \begin{pmatrix} -d_1\lambda_i - \lambda \varphi(v^*) - \mu - \lambda u^* \varphi'(v^*) & \lambda \varphi(v^*) \\ \lambda \varphi(v^*) & -d_2\lambda_i + \lambda u^* \varphi'(v^*) - \sigma \end{pmatrix}, \quad \text{for all } i \geq 0,
\]
which is guaranteed if the trace and determinant of \(J_i(E^*)\) satisfies the conditions \(\text{tr}(J_i(E^*)) < 0\) and \(\text{det}(J_i(E^*)) > 0\), for all \(i \geq 0\). The trace of \(J_i(E^*)\) is given by
\[
\text{tr}(J_i(E^*)) = -d_1\lambda_i - d_2\lambda_i - \lambda \varphi(v^*) - \mu + \lambda u^* \varphi'(v^*) - \sigma,
\]
\[
= - \lambda_i (d_1 + d_2) + \text{tr}(J(u^*, v^*)).
\]

Since \(\text{tr}(J(u^*, v^*)) < 0\), it follows that \(\text{tr}(J_i(E^*)) < 0\) for all \(i \geq 0\). The determinant is given by
\[
\text{det}(J_i(E^*)) = d_1d_2\lambda_i^2 + \frac{\lambda \varphi(v^*)}{v^*} (-d_1\lambda_i u^* + d_1\sigma + \lambda \varphi(v^*) d_2 + \mu d_2)
\]
\[
+ \lambda \sigma \varphi(v^*) + \mu \sigma - \mu \lambda u^* \varphi'(v^*),
\]
\[
= d_1d_2\lambda_i^2 + \lambda_i H_0 + \text{det}(J(u^*, v^*)) \quad \text{for all } i \geq 0,
\]
where
\[
H_0 = -d_1\lambda u^* \varphi'(v^*) + d_1\sigma + \lambda \varphi(v^*) d_2 + \mu d_2.
\]
Using (1.5) and (2.6), it holds that
\[
H_0 \geq -d_1\lambda u^* \frac{\varphi(v^*)}{v^*} + d_1\frac{\lambda u^* \varphi(v^*)}{v^*} + \lambda \varphi(v^*) d_2 + \mu d_2,
\]
\[
= d_2 (\lambda \varphi(v^*) + \mu),
\]
\[
= d_2 \frac{\Lambda}{u^*} > 0,
\]
which leads to
\[
\text{det}(J_i(E^*)) = d_1d_2\lambda_i^2 + \lambda_i H_0 + \text{det}(J(u^*, v^*)) > 0 \quad \text{for all } i \geq 0.
\]
Hence, \(E^*\) is locally asymptotically stable.

3 | GLOBAL ASYMPTOTIC STABILITY

Next, we study the global asymptotic stability of the two steady states \(E_0\) and \(E^*\). The global stability depends on the reproduction number \(R_0\), which is why we have decided to treat the scenarios \(R_0 < 1\) and \(R_0 > 1\) separately. First, however, let us state a necessary lemma taken from Sigdel\textsuperscript{14} that will aid with the proofs to come.

**Lemma 2.** Given that \(\varphi\) satisfies criterion (1.5) and
\[
L(x) = x - 1 - \ln(x), \text{ for all } x > 0,
\]
the inequality
\[
L \left( \frac{\varphi(v)}{\varphi(v^*)} \right) \leq L \left( \frac{v}{v^*} \right),
\]
where \(v^*\) is the second component of the equilibrium point \(E^*\), holds.

Proof. Condition (1.5) guarantees that $\frac{\varphi(v)}{v}$ is a decreasing function for all $v > 0$. We may separate the proof into two regions:

1. The first region is $v \geq v^*$. Since $\frac{\varphi(v)}{v}$ is a decreasing function, we have

$$\frac{\varphi(v)}{v} \leq \frac{v}{v^*}.$$  

Note that (1.5) implies that $\varphi$ is non-decreasing, which leads to

$$\varphi(v) \geq \varphi(v^*),$$

and, consequently,

$$1 \leq \frac{\varphi(v)}{\varphi(v^*)} \leq \frac{v}{v^*}.$$  

Since $L$ is increasing for all $x > 1$, (3.2) follows.

$$L \left( \frac{\varphi(v)}{\varphi(v^*)} \right) \leq L \left( \frac{v}{v^*} \right)$$

for all $v \geq v^*$.

2. The second region is $0 < v < v^*$. Again, since $\frac{\varphi(v)}{v}$ is a decreasing function, we have

$$\frac{\varphi(v)}{\varphi(v^*)} > \frac{v}{v^*},$$

and given the non-decreasing nature of $\varphi$, we end up with

$$\varphi(v) < \varphi(v^*).$$

As a result, we get

$$1 > \frac{\varphi(v)}{\varphi(v^*)} > \frac{v}{v^*} > 0.$$  

Since $L$ is decreasing for $0 < x < 1$, (3.2) holds.

3.1 | Global asymptotic stability with $R_0 < 1$

To establish the global asymptotic stability of the disease-free equilibrium $E_0$, we consider

$$V_\theta(t) = \int_\Omega \left[ uv + \frac{\theta}{2} \left( u - \frac{\Lambda}{\mu} \right)^2 + \frac{1}{2} v^2 + \frac{\Lambda}{\sigma} v \right] dx,$$

where $\theta > 0$,

as our candidate Lyapunov function. The aim of this subsection is to obtain a weaker condition than that of Djebara et al.$^1$

**Theorem 4.** Assuming that (1.5) holds, if $R_0 < 1$, then $E_0$ is a globally asymptotically stable disease-free steady state for system (1.1)–(1.3) under the assumption

$$\varphi'(0) \leq \frac{\mu + \sigma}{\lambda \left( \frac{\sigma}{\mu} + \frac{\Lambda}{\sigma} \right)},$$

(3.3)

with

$$\theta \geq \frac{(d_1 + d_2)^2}{4d_1d_2}.$$  

(3.4)
Proof. To prove that \( E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \) is globally asymptotically stable, we must show that \( V_0(t) \) is a Lyapunov function. The positive definiteness of \( V_0(t) \) is evident. Evaluating its derivative with respect to time gives

\[
\frac{d}{dt} V_0(t) = \int_\Omega \left( \frac{\partial u}{\partial t} + u \frac{\partial v}{\partial t} \right) dx + \theta \int_\Omega \left( u - \frac{\Lambda}{\mu} \right) \frac{\partial u}{\partial t} dx + \int_\Omega v \frac{\partial v}{\partial t} dx + \frac{\Lambda}{\sigma} \int_\Omega \frac{\partial v}{\partial t} dx.
\]

Substituting the partial derivatives \( \frac{\partial u}{\partial t} \) and \( \frac{\partial v}{\partial t} \) with their respective values from (1.1) and applying Green’s formula with the assumed Neumann boundaries leads to

\[
\frac{d}{dt} V_0(t) = I_1 + I_2 + I_3,
\]

where

\[
I_1 = - (d_1 + d_2) \int_\Omega \nabla u \nabla v dx + \Lambda \int_\Omega v dx - \lambda \int_\Omega uv \varphi (v) dx + \lambda \int_\Omega u^2 \varphi (v) dx - (\sigma + \mu) \int_\Omega u v dx,
\]

\[
I_2 = -d_1 \theta \int_\Omega |\nabla u|^2 dx - \mu \theta \int_\Omega \left( u - \frac{\Lambda}{\mu} \right) dx - \lambda \theta \int_\Omega u^2 \varphi (v) dx + \lambda \theta \frac{\Lambda}{\mu} \int_\Omega u \varphi (v) dx,
\]

and

\[
I_3 = -d_2 \int_\Omega |\nabla v|^2 dx + \lambda \int_\Omega uv \varphi (v) dx - \sigma \int_\Omega v^2 dx + \lambda \int_\Omega u \varphi (v) dx - \Lambda \int_\Omega v dx.
\]

Taking the sum of terms \( I_1, I_2, \) and \( I_3 \) yields

\[
\frac{d}{dt} V_0(t) = - d_1 \theta \int_\Omega |\nabla u|^2 dx - (d_1 + d_2) \int_\Omega \nabla u \nabla v dx - d_2 \int_\Omega |\nabla v|^2 dx
\]

\[
+ \lambda (1 - \theta) \int_\Omega u^2 \varphi (v) dx - (\sigma + \mu) \int_\Omega u v dx - \mu \theta \int_\Omega \left( u - \frac{\Lambda}{\mu} \right)^2 dx
\]

\[
- \sigma \int_\Omega v^2 dx + \lambda \left( \theta \frac{\Lambda}{\mu} + \frac{\Lambda}{\sigma} \right) \int_\Omega u \varphi (v) dx.
\]

\[
= I + J,
\]

where

\[
I = - d_1 \theta \int_\Omega |\nabla u|^2 dx - (d_1 + d_2) \int_\Omega \nabla u \nabla v dx - d_2 \int_\Omega |\nabla v|^2 dx,
\]

and

\[
J = \lambda (1 - \theta) \int_\Omega u^2 \varphi (v) dx + \lambda \left( \theta \frac{\Lambda}{\mu} + \frac{\Lambda}{\sigma} \right) \int_\Omega u \varphi (v) dx
\]

\[
- (\sigma + \mu) \int_\Omega u v dx - \mu \theta \int_\Omega \left( u - \frac{\Lambda}{\mu} \right)^2 dx - \sigma \int_\Omega v^2 dx.
\]

The first term may be written as

\[
I = - \int_\Omega Q (\nabla u, \nabla v) dx,
\]

where \( Q \) is a quadratic form with respect to \( \nabla u \) and \( \nabla v \), that is,

\[
Q (\nabla u, \nabla v) = d_1 \theta |\nabla u|^2 + (d_1 + d_2) \nabla u \nabla v + d_2 |\nabla v|^2.
\]

It is easy to see that \( Q (\nabla u, \nabla v) \) is non-negative as \( \theta, d_1, \) and \( d_2 \) satisfy the conditions \( d_1 \theta > 0 \) and \( \theta \geq \frac{(d_1 + d_2)^2}{4d_1 d_2} \), from which we obtain \( I \leq 0 \).

On the other hand, using the inequality \( \theta \geq \frac{(d_1 + d_2)^2}{4d_1 d_2} \geq 1 \), we have

\[
J \leq \lambda \left( \theta \frac{\Lambda}{\mu} + \frac{\Lambda}{\sigma} \right) \int_\Omega u \varphi (v) dx - (\sigma + \mu) \int_\Omega u v dx - \mu \theta \int_\Omega \left( u - \frac{\Lambda}{\mu} \right)^2 dx - \sigma \int_\Omega v^2 dx.
\]
Applying Lemma 1 yields

\[ J \leq \int_{\Omega} \left[ \lambda \left( \frac{\Lambda}{\mu} + \frac{\Lambda}{\sigma} \right) \varphi'(0) - (\mu + \sigma) \right] uvdx - \mu \theta \int_{\Omega} \left( u - \frac{\Lambda}{\mu} \right)^2 dx - \sigma \int_{\Omega} \nu^2 dx. \]  

Assuming (3.3) holds, the derivative \( \frac{d}{dt} \mathcal{V}_0(t) \leq 0 \) for all \( t \geq 0 \) with \( \mathcal{V}_0(t) = 0 \) being true when \( (u, v) = \left( \frac{\Lambda}{\mu}, 0 \right) \). Finally, by Lyapunov's direct method, \( E_0 \) is globally asymptotically stable.

3.2 | Global asymptotic stability with \( R_0 > 1 \)

**Theorem 5.** Assuming that \( u_0, v_0 > 0 \) and (1.5) holds, if \( R_0 > 1 \), \( E^* \) is a globally asymptotically stable endemic steady state for system (1.1)–(1.3).

**Proof.** For this proof, we consider the candidate Lyapunov function

\[ \mathcal{V}(t) = \int_{\Omega} \left[ u^* L \left( \frac{u}{u^*} \right) + v^* L \left( \frac{v}{v^*} \right) \right] dx, \]  

which is a positive definite and continuously differentiable function. First, note that

\[ \frac{d}{dt} L \left( \frac{u}{u^*} \right) = \frac{1}{u^*} \left( 1 - \frac{u^*}{u} \right) \frac{du}{dt}. \]

Differentiating \( \mathcal{V}(t) \) with respect to time yields

\[ \frac{d}{dt} \mathcal{V}(t) = \int_{\Omega} \left( 1 - \frac{u^*}{u} \right) (d_1 \Delta u + \Lambda - \lambda u \varphi (v) - \mu u) dx + \int_{\Omega} \left( 1 - \frac{v^*}{v} \right) (d_2 \Delta v + \lambda u \varphi (v) - \sigma v) dx. \]

Similar to the previous scenario, we apply Green’s formula with Neumann boundaries to expand the derivative to

\[ \frac{d}{dt} \mathcal{V}(t) = - d_1 \int_{\Omega} \nabla \left( 1 - \frac{u^*}{u} \right) \nabla u dx + \int_{\Omega} \left( 1 - \frac{u^*}{u} \right) dx - \lambda \int_{\Omega} \left( 1 - \frac{u^*}{u} \right) u \varphi(v) dx \\
- \mu \int_{\Omega} \left( 1 - \frac{u^*}{u} \right) u dx - d_2 \int_{\Omega} \nabla \left( 1 - \frac{v^*}{v} \right) \nabla v dx + \lambda \int_{\Omega} \left( 1 - \frac{v^*}{v} \right) u \varphi(v) dx \\
- \sigma \int_{\Omega} \left( 1 - \frac{v^*}{v} \right) v dx = I + J, \]

where

\[ I = - \int_{\Omega} \left[ d_1 \frac{u}{u^*} |\nabla u|^2 + d_2 \frac{v}{v^*} |\nabla v|^2 \right] dx \leq 0, \]  

and

\[ J = \int_{\Omega} \left( 1 - \frac{u^*}{u} \right) [\Lambda - \lambda u \varphi(v) - \mu u] dx + \int_{\Omega} \left( 1 - \frac{v^*}{v} \right) [\lambda u \varphi(v) - \sigma v] dx. \]  

Using (2.6) and simplifying the resulting equation leads to

\[ J = \int_{\Omega} \mu u^* \left( 1 - \frac{u}{u^*} \right) \left( 1 - \frac{u^*}{u} \right) dx + \lambda u^* \varphi(v^*) \int_{\Omega} \left( \frac{u \varphi(v)}{u^* \varphi(v^*)} - \frac{v}{v^*} \right) \left( 1 - \frac{v^*}{v} \right) dx \\
+ \lambda u^* \varphi(v^*) \int_{\Omega} \left( 1 - \frac{u \varphi(v)}{u^* \varphi(v^*)} \right) \left( 1 - \frac{u^*}{u} \right) dx. \]  

Using (3.8) and simplifying further gives

\[ J = \int_{\Omega} \left[ \mu u^* J_1 + \lambda u^* \varphi(v^*) \right] dx, \]  

where
FIGURE 1 Numerical solutions of system (4.1) (ODE case) subject to the first and second sets of parameters [Colour figure can be viewed at wileyonlinelibrary.com]

where

\[ J_1 = \left( 1 - \frac{u^*}{u} \right) \left( 1 - \frac{u}{u^*} \right), \]

and

\[ J_2 = \frac{\varphi(v)}{\varphi(v^*)} - \frac{v}{v^*} - \frac{u\varphi(v)v^*}{u^*\varphi(v^*)v} - \frac{u^*}{u} + 2. \]

Since \( J_1 \) and \( J_2 \) can be rewritten in the forms

\[ J_1 = -L\left( \frac{u}{u^*} \right) - L\left( \frac{u^*}{u} \right), \tag{3.10} \]

and

\[ J_2 = -L\left( \frac{u}{u^*} \right) + L\left( \frac{\varphi(v)}{\varphi(v^*)} \right) - L\left( \frac{v}{v^*} \right) - L\left( \frac{u\varphi(v)v^*}{u^*\varphi(v^*)v} \right), \tag{3.11} \]

it follows that

\[ J = -\mu u^* \int_{\Omega} \left[ L\left( \frac{u}{u^*} \right) + L\left( \frac{u^*}{u} \right) \right] dx - \lambda u^* \varphi(v^*) \int_{\Omega} \left[ L\left( \frac{u}{u^*} \right) + L\left( \frac{u\varphi(v)v^*}{u^*\varphi(v^*)v} \right) \right] dx \]

\[ + \lambda u^* \varphi(v^*) \int_{\Omega} \left[ L\left( \frac{\varphi(v)}{\varphi(v^*)} \right) - L\left( \frac{v}{v^*} \right) \right] dx. \]

Taking advantage of the established positivity of \( L \) and applying Lemma 2, we end up with

\[ J \leq 0. \]

Hence, \( \frac{d}{dt} \mathcal{V}(t) \leq 0 \), and consequently, \( \mathcal{V} \) is non-increasing in time with \( \mathcal{V}(t) = 0 \) only at the steady state \( E^* \). The global asymptotic stability of \( E^* \) follows from Lyapunov’s direct methods.

3.3 Important remarks

It is important to note that, for system (1.1)–(1.2) in ODE case, we have the following results:

(i) For \( R_0 < 1 \), the disease-free equilibrium \( E_0 \) is globally asymptotically stable. This easy to establish as \( d_1 = d_2 = 0 \) and, thus, it suffices to choose \( \theta = 1 \) in \( \mathcal{V}_0(t) \).

(ii) For \( R_0 > 1 \) and provided that condition (1.5) is satisfied, the endemic equilibrium \( E^* = (u^*, v^*) \) is globally asymptotically stable.
APPLICATIONS AND NUMERICAL EXAMPLES

Although the analytical methods we used to obtain the main results of this study are theoretically sound, it is always beneficial to use numerical solutions to illustrate and confirm their validity. We consider an incidence function of the form $u \phi (v)$ satisfying (1.5) and assess the local and global asymptotic stability of the disease-free equilibrium $E_0$ when $R_0 < 1$ and the endemic equilibrium $E^*$ when $R_0 > 1$. Focus will be on the main results of the paper as reported in Theorems 4 (condition 3.3) and 5. In the following, we examine three different numerical examples.
4.1 First example

In this first example, we consider the function

\[ \varphi(v) = \alpha v, \quad \text{for all } \alpha > 0. \]

The resulting problem is given by

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= -\lambda \alpha uv + \Lambda - \mu u \quad \text{in } (0, \infty) \times \Omega, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= \lambda \alpha uv - \sigma v \quad \text{in } (0, \infty) \times \Omega, \\
u(0, x) &= u_0(x), \quad v(0, x) = v_0(x) \quad \text{on } \Omega, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad \text{on } (0, \infty) \times \partial \Omega, \\
\end{align*}
\]

\[ (4.1) \]
which is a special case of system (1.1)–(1.3). In fact, system (4.1) is identical to the bird system proposed in Section 3 of Kim et al\textsuperscript{14} and in a previous study\textsuperscript{15} but with $d_1 = d_2$. The ODE scenario of this model was treated earlier in previous works.\textsuperscript{6,35} Conditions (1.4) and (1.5) are clearly satisfied as

$$\begin{align*}
\varphi(0) &= 0, \\
\varphi'(v) &= \alpha > 0, \\
\varphi''(0) &= \alpha, \\
\alpha v &= v\varphi'(v) \leq \varphi(v) = \alpha v.
\end{align*}$$
TABLE 1  Simulation parameters for example 1: System (4.1)

| Set   | Figure | $u_0$ | $v_0$ | $d_1$ | $d_2$ | $\lambda$ | $\sigma$ | $\sigma$ | $\Lambda$ | $\mu$ | $R_0$ |
|-------|--------|-------|-------|-------|-------|------------|----------|----------|------------|-------|-------|
| ODE   | Set 1A | 1     | 2     | 3     | 4     | 5          | 6        | 7        | 8          | 9     | 10    |
|       |        | 1     | 2     | 3     | 4     | 5          | 6        | 7        | 8          | 9     | 10    |
| PDE   | Set 1B | 1     | 2     | 3     | 4     | 5          | 6        | 7        | 8          | 9     | 10    |
|       |        | 1     | 2     | 3     | 4     | 5          | 6        | 7        | 8          | 9     | 10    |

FIGURE 8  Numerical solutions of system (4.2) subject to the third set of parameters [Colour figure can be viewed at wileyonlinelibrary.com]

System (4.1) possesses two constant steady states

$$E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \quad \text{and} \quad E^* = \left( \frac{\sigma}{\lambda \alpha}, \frac{\mu}{\alpha \Lambda} \left( R_0 - 1 \right) \right).$$

Note that the second steady state $E^*$ exists only when the reproduction number $R_0 = \frac{\lambda \Lambda}{\mu \sigma} \phi'(0) = \frac{\lambda \Lambda}{\mu \sigma} \alpha > 1$ and is globally asymptotically stable. Also, note that the first steady state $E_0$ is globally asymptotically stable unconditionally in the ODE case and subject to

$$\frac{(d_1 + d_2)^2}{4d_1d_2} \leq \theta \leq \frac{\mu}{\Lambda} \left( \frac{\mu + \sigma}{\lambda \alpha} - \frac{\Lambda}{\sigma} \right)$$

when $d_1 \neq d_2$, and to $\theta = 1$ when $d_1 = d_2$ in the PDE case. As detailed in Table 1, we use different sets of parameters to obtain numerical solutions in the ODE and PDE. Note that throughout the PDE simulations, we assume a single spatial dimension with $\Omega = (0, 10)$.

The following is a description of the results:

- Figure 1 shows the solutions in the ODE case subject to sets 1 and 2, with $R_0 = 3.7333$ and $R_0 = 0.9524$, respectively. In the first case, as $R_0 > 1$, $E^* = (4.2857, 20.5)$ is globally asymptotically stable. In the second case, $R_0 < 1$. $E_0 = (6.8571, 0)$ is globally asymptotically stable.

- Figure 2 depicts the solution in the PDE case subject to parameter set 1, where $R_0 = 3.7333 > 1$, which by Theorem 5 means that $E^* = (4.2857, 20.5)$ is globally asymptotically stable.

- Figure 3 depicts the solution in the PDE case subject to parameter set 2, where $R_0 = 0.9524 < 1$. By Theorem 4 and given $\theta \in \left[ \frac{196}{180}, \frac{1805}{1643} \right]$, $E_0 = (6.8571, 0)$ is globally asymptotically stable.
4.2 | Second example

The second numerical example which we are interested in is the PDE extension of the ODE SIR model studied in Lahrouz et al.,\textsuperscript{20} which is a special case of (1.1) with $\varphi(v) = \frac{av}{1+kv}$, $a > 0$ and $k \geq 0$. The resulting system is described by

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= -\lambda \frac{av}{1+kv} + \Lambda - \mu u \quad \text{in} \ (0, \infty) \times \Omega, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= \lambda \frac{av}{1+kv} - \sigma v \quad \text{in} \ (0, \infty) \times \Omega, \\
u(0, x) &= u_0(x), \ v(0, x) = v_0(x) \quad \text{on} \ \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0, \quad \text{on} \ (0, \infty) \times \partial \Omega.
\end{align*}
\] (4.2)
The imposed conditions (1.4) and (1.5) can be easily verified. It is evident that

\[ \phi(0) = 0 \quad \text{and} \quad \phi(v) > 0 \quad \text{for all} \quad v > 0. \]

Also, the derivative of \( \phi(v) \) is given by

\[
\phi'(v) = \left( \frac{av}{1 + kv} \right)'
= \frac{\alpha}{(1 + kv)^2} > 0 \quad \text{and} \quad \phi'(0) = \alpha.
\]
In addition, we have

\[ v \frac{\varphi'(v)}{\varphi(v)} = v \frac{\alpha}{(1 + kv)^2} \leq \frac{av}{1 + kv} = \varphi(v). \]

The constant steady states of system (4.2) are

\[ E_0 = \left( \frac{\Lambda}{\mu}, 0 \right) \quad \text{and} \quad E^* = \left( \frac{\sigma (1 + kv^*)}{\lambda \alpha}, \mu \left( \frac{R_0 - 1}{\lambda \alpha + k \mu} \right) \right). \]
Again, $E^*$ exists and is globally asymptotically stable provided that the reproduction number $R_0 = \frac{\frac{\mu}{\lambda} \sigma}{\mu^2} > 1$. On the other hand, $E_0$ is globally asymptotically stable in the ODE case with no conditions and in the PDE case if $(\frac{d_1+d_2}{d_2})^\frac{\sigma}{\alpha} \leq \theta \leq \mu \left( \frac{\frac{\mu}{\lambda} \sigma}{\mu^2} - \frac{\lambda}{\alpha} \right)$ when $d_1 \neq d_2$, and $\theta = 1$ when $d_1 = d_2$. Table 2 shows the parameter sets used in the numerical simulations.

The following is a description of the results:

- Figures 4 and 5 depict the numerical solutions obtained using the four parameter sets in the ODE case with equilibrium points $E^* = (8.2688, 2.4524)$, $E_0 = (8.75, 0)$, $E^* = (9.8534, 0.5509)$, and $E_0 = (8.75, 0)$, respectively. In all four scenarios, both the analytical and numerical simulations agree that the equilibria are asymptotically stable.

- Figure 6 shows the PDE solution obtained using parameter set 1 with $E^* = (8.2688, 2.4524)$. In this case, $R_0 = 2.7764 > 1$ and by Theorem 5, $E^*$ is globally asymptotically stable.

- Figure 7 shows the PDE solution obtained using parameter set 2 with $E^* = (9.8534, 0.5509)$. Since $R_0 = 2.7764 > 1$, $E^*$ is globally asymptotically stable.

- Figure 8 shows the PDE solution obtained using parameter set 3 with $E_0 = (8.75, 0)$. In this case, $R_0 = 0.8750 < 1$ and using Theorem 4 with $\theta \in \left[ \frac{25}{24}, 1.2449 \right]$, $E_0$ is globally asymptotically stable.

- Figure 9 shows the PDE solution obtained using parameter set 4 with $E_0 = (8.75, 0)$. In this scenario, we again have $R_0 = 0.8750 < 1$ and with $\theta \in \left[ \frac{529}{504}, 1.2449 \right]$, $E_0$ is globally asymptotically stable.

### 4.3 Third example

The last example is obtained by substituting $\phi(v) = \frac{k_v}{1 + (\frac{v}{\sigma})}$, which yields the same system studied in previous studies, but with $d_1 = d_2 = 0$. The resulting system is given by

\[\begin{align*}
\frac{du}{dt} - d_1 \Delta u &= -\delta k \frac{v}{1 + (\frac{v}{\sigma})} u + \Lambda - \mu u \quad \text{in} \ (0, \infty) \times \Omega, \\
\frac{dv}{dt} - d_2 \Delta v &= \delta k \frac{v}{1 + (\frac{v}{\sigma})} u - \sigma v \quad \text{in} \ (0, \infty) \times \Omega, \\
u(0, x) &= u_0(x), \quad v(0, x) = v_0(x) \quad \text{on} \ \Omega, \\
\frac{du}{\partial v} &= \frac{du}{\partial v} = 0, \quad \text{on} \ (0, \infty) \times \partial \Omega,
\end{align*}\]

for $\alpha > 0$ and $k > 0$. The imposed conditions may be verified as

\[\begin{align*}
\phi (0) &= 0, \\
\phi' (v) &= \frac{k}{1 + (\frac{v}{\sigma})} > 0 \quad \text{for all} \ v \geq 0, \\
\nu \phi' (v) &= \frac{v}{1 + (\frac{v}{\sigma})} \leq \frac{k_v}{1 + (\frac{v}{\sigma})} = \phi (v).
\end{align*}\]

The steady states of system (4.3) are given by $E_0 = \left( \frac{\Lambda}{\mu}, 0 \right)$ and $E^* = \left( \frac{\sigma (\sigma + \nu)}{\delta k}, \mu a \frac{(R^*_0 - 1)}{\delta k + \mu} \right)$ with the reproduction number $R_0 = \frac{\frac{\Lambda k}{\mu}}{k} > 1$. Note that if $E^*$ exists than it is globally asymptotically stable and that $E_0$ is globally asymptotically stable.
if \( \frac{(d_1+d_2)^2}{4d_1d_2} \leq \theta \leq \frac{\mu}{\Lambda} \left( \frac{\mu+\sigma}{\sigma} - \frac{\Lambda}{\sigma} \right) \) when \( d_1 \neq d_2 \), and if \( \theta = 1 \) when \( d_1 = d_2 \) or in the ODE case. Table 3 details the sets of parameters used in the numerical simulations.

The following is a description of the results:

- Figures 10 and 11 show the numerical solutions of system (4.3) resulting the four parameter sets in the ODE case with equilibrium points \( E^* = (11.1857, 5.3), E_0 = (1.75, 0), E^* = (10.9548, 1.1613), \) and \( E_0 = (1.4, 0) \), respectively. In all four scenarios, both the analytical and numerical simulations agree that the equilibria are asymptotically stable.
- Figure 12 shows the PDE solution obtained using parameter set 1 with \( E^* = (11.1857, 5.3) \). In this case, \( R_0 = 18.6667 > 1 \) and by Theorem 5, \( E^* \) is globally asymptotically stable.
- Figure 13 shows the PDE solution obtained using parameter set 2 with \( E^* = (11.1857, 5.3) \). In this case, \( R_0 = 18.6667 > 1 \) and \( E^* \) is globally asymptotically stable.
- Figure 14 shows the PDE solution obtained using parameter set 3 with \( E^* = (11.1857, 5.3) \). In this case, \( R_0 = 10 > 1 \) and \( E^* \) is globally asymptotically stable.
- Figure 15 shows the PDE solution obtained using parameter set 4 with \( E_0 = (1.75, 0) \). In this case, \( R_0 = 0.1544 < 1 \) and by Theorem 4 with \( \theta \in \left[ \frac{280}{240}, 11.1701 \right] \), \( E_0 \) is globally asymptotically stable.
- Figure 16 shows the PDE solution obtained using parameter set 5 with \( E_0 = (1.4, 0) \). In this case, \( R_0 = 0.4941 < 1 \) and by Theorem 4 with \( \theta \in \left[ \frac{441}{416}, 2.9014 \right] \), \( E_0 \) is globally asymptotically stable.

![Figure 15](https://example.com/figure15.png)

**FIGURE 15** Numerical solutions of system (4.3) subject to the forth set of parameters [Colour figure can be viewed at wileyonlinelibrary.com]
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CONFLICT OF INTERESTS

This work does not have any conflict of interests.

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