One-Instanton Tests of the Exact Results in $N = 2$ Supersymmetric QCD

Matthew J. Slater

Physics Department, Centre for Particle Theory
University of Durham, Durham DH1 3LE, UK
E-mail: m.j.slater@durham.ac.uk

Abstract

We use the microscopic instanton calculus to determine the one-instanton contribution to the quantum modulus $u_3 = \langle \text{Tr}(\phi^3) \rangle$ in $N = 2$ $SU(N_c)$ supersymmetric QCD with $N_f < 2N_c$ fundamental flavors. This is compared with the corresponding prediction of the hyperelliptic curves which are expected to give exact solutions in this theory. The results agree up to certain regular terms which appear when $N_f \geq 2N_c - 3$. The curve prediction for these terms depends upon the curve parameterization which is generically ambiguous when $N_f \geq N_c$. In $SU(3)$ theory our instanton calculation of the regular terms is found to disagree with the predictions of all of the suggested curves. For this theory we employ our results as input to improve the curve parameterization for $N_f = 3, 4, 5$. 
In their seminal work [1], Seiberg and Witten applied ideas of duality to $N = 2$ supersymmetric QCD (SQCD) with gauge group $SU(2)$ and $N_f = 0, 1, 2, 3, 4$ flavors of matter hypermultiplets, and were able to predict exact results, valid at both strong and weak coupling. This was achieved by identifying the quantum moduli spaces of these models with the moduli spaces of certain families of elliptic curves. Exact solutions for the holomorphic prepotential describing the low energy dynamics were then obtained via periods of a meromorphic one-form on the curves.

This analysis has subsequently been extended to $SU(N_c)$ models with $N_c > 2$ and $N_f \leq 2N_c$ fundamental flavors [2–5]. In the general case, the quantum moduli space is described by a family of genus $N_c - 1$ hyperelliptic curves. These are parameterized by the gauge invariant quantum moduli $u_n = \langle \text{Tr}(\phi^n) \rangle$ ($n = 2, 3, \ldots, N_c$), where $\phi$ is the adjoint Higgs. The proposed curves predict exact solutions for these objects as well as for the holomorphic prepotential.

The exact solutions can be explicitly expanded in the semiclassical regime [8] to give the expected one-loop perturbative contribution plus predictions for $k$-instanton corrections. These take the form of rational functions of the vacuum expectation values (VEV’s). Non-trivial tests of the curves can be performed by applying the microscopic instanton calculus to directly evaluate these non-perturbative contributions.

In $SU(2)$, instanton calculations have been carried out at the one-instanton [9, 10] and two-instanton [11–15] levels. The results completely agree with the curves for $N_f = 0, 1, 2$ fundamental flavors. However discrepancies have emerged for $N_f = 3$ [13] and $N_f = 4$ [14] fundamental flavors at the two-instanton level (as well as at the one-instanton level for one adjoint flavor in [10]). For general $N_c$, only the singular part of the one-instanton contribution to $u_2$ has been calculated, in [10, 11]. When $N_f < 2N_c - 2$ there are no additional regular terms and the result is in full agreement with the curves. It was further claimed in [17] that the results of an evaluation of the regular terms that appear when $N_f \geq 4$ in the $SU(3)$ model are in conflict with the predictions of the proposed curves.

In [18, 17, 10] it was shown that the $SU(2)$ discrepancies can be resolved in a way consistent with the analysis of Seiberg and Witten, by reinterpreting the parameters of the original curves. In fact it is a generic feature of curve construction that the curve parameterization may not be uniquely fixed when $N_f \geq N_c$. In [2–5] possible curve parameterizations are suggested on the basis of various assumed criteria. The results of [17] imply that none of the $SU(3)$ parameterizations are correct and therefore question the validity of these criteria.

In this letter, we continue the program of comparing the curve predictions with the results of first-principles instanton calculations, by evaluating the one-instanton contribution to the quantum modulus $u_3$ in $N = 2$ SQCD with $N_c > 2$ colors and $N_f < 2N_c$ fundamental flavors.

\footnote{By virtue of the Matone relation [3] (see [7] for instanton based derivations) between the prepotential and $u_2$, it actually suffices to consider only the solutions for the quantum moduli as independent predictions of the curves when $N_f < 2N_c$.}
We determine the most singular part of the answer, which for \( N_f < 2N_c - 3 \) is the complete answer and agrees exactly with the prediction which we extract from the curves. Our analysis also gives the coefficients of the regular terms which arise in the \( SU(3) \) theory when \( N_f \geq 3 \), and we find disagreement with the numbers obtained from the curves, for all of the suggested curve parameterizations. We then employ the set of microscopic instanton calculations in the \( SU(3) \) theory to improve the parameterizations of the \( N_f = 3, 4, 5 \) curves. For \( N_f = 3, 4 \) flavors the curves should be completely fixed, so that no discrepancies can appear at higher order instanton levels.

The field content of \( SU(N_c) \) \( N = 2 \) SQCD is as follows. There is an \( N = 2 \) hypermultiplet comprised of two \( N = 1 \) superfields, \( \Phi \) and \( W_\alpha \), which transforms under the adjoint representation of the gauge group. Components of the chiral superfield \( \Phi \) are a complex scalar Higgs \( \phi \) and its fermionic superpartner, the Higgsino \( \psi \). The vector superfield \( W_\alpha \) contains the gauge boson \( A_\mu \) and its superpartner, the gaugino \( \lambda \). The additional \( N_f \) matter hypermultiplets consist of chiral superfields \( Q_f \) and \( \tilde{Q}_f \) \((f = 1, 2, \ldots, N_f)\), which transform under the fundamental and its conjugate representation respectively. The associated component fields are the squarks \( q_f \) and quarks \( \chi_f \) along with their conjugate representation counterparts, \( \tilde{q}_f \) and \( \tilde{\chi}_f \).

In this paper we adopt the leading-order short-distance constrained instanton \([19]\) approach to semiclassical analysis as reviewed in Sections 3 and 4 of \([11]\). The constrained Euclideanized Euler-Lagrange equations in the short-distance region \(|x| < 1/M\), where \( M \) represents a typical W-boson mass, can be solved perturbatively in the coupling constant \( g \). Only the leading-order terms contribute to the holomorphic prepotential and quantum moduli \( u_n \). The defining equations of the instanton configuration are consequently \([11–15]\)

\[
F_{\mu\nu} = \tilde{F}_{\mu\nu},
\]

\[
\tilde{\nabla} \lambda = 0, \quad \tilde{\nabla} \psi = 0, \quad \tilde{\nabla} \chi = 0, \quad \tilde{\nabla} \tilde{\chi} = 0,
\]

\[
D^2 \phi = \sqrt{2}ig[\lambda, \psi], \quad D^2 \phi^a = \sqrt{2}ig\chi T^a \chi,
\]

\[
D^2 q = \sqrt{2}ig\lambda \chi, \quad D^2 \tilde{q} = -\sqrt{2}ig\tilde{\chi} \lambda, \quad D^2 q^\dagger = \sqrt{2}ig\tilde{\chi} \psi, \quad D^2 \tilde{q}^\dagger = \sqrt{2}ig\psi \chi.
\]

We use the convention \( \tilde{\nabla}^\alpha = \bar{e}_\mu^\alpha D_\mu \) where \( \bar{e}_\mu \) is the Hermitian conjugate of \( e_\mu = (i\bar{\sigma}, 1) \). For notational clarity we have dropped the flavor indices on the quark and squark fields. In the Coulomb branch of the theory, the moduli space of vacua results from a potential term \( V(\phi) \sim \text{Tr}([\phi, \phi^\dagger]^2) \) in the Lagrangian. Up to gauge transformations, the Higgs field acquires the matrix of vacuum expectation values

\[
\langle \phi \rangle = \text{diag}(a_1, a_2, \ldots, a_{N_c}).
\]

The \( a_i \) are complex parameters satisfying the constraint \( \sum_{i=1}^{N_c} a_i = 0 \) which ensures that \( \langle \phi \rangle \) lives in the Lie algebra of the group \( SU(N_c) \). This imposes a boundary condition on the instanton solution for the Higgs field, since it must approach its matrix of VEV’s at large distances.
The required self-dual solution to Eq. (1) of unit topological charge is given by the standard $SU(2)$ pure gauge field (BPST) instanton \[20\] ‘minimally embedded’ in the $SU(N_c)$ Lie algebra \[21\]. In singular gauge this is

$$A_\mu = \frac{2\rho^2}{g} \frac{y_\nu \bar{y}_\mu}{y^2 (y^2 + \rho^2)} T^a,$$

(6)

where $y_\mu = (x - x_0)_\mu$, and $x_0$ and $\rho$ give the location and size of the instanton respectively. We make use of the usual ‘t Hooft $\eta$-symbol \[22\] and choose a basis of generators such that $T^{1,2,3}_{ij} = \frac{1}{2} \sigma^{1,2,3}_{ij}$ (ie. these are normalized Pauli matrices in the ‘upper left corner’).

The above configuration is subject to global gauge transformations which rotate it into the space of the $SU(N_c)$ Lie algebra. However for the purposes of the instanton calculation we can choose to preserve the upper left embedding of the BPST instanton, and perform global gauge transformations of the matrix of VEV’s (5) instead \[23\]. In this case the boundary condition on the Higgs becomes

$$\lim_{|y| \to \infty} \phi = \Omega^\dagger \langle \phi \rangle \Omega = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

(7)

where $\Omega \in SU(N_c)$, and the second equality indicates a convenient partitioning of the rotated VEV matrix; $A_1$ and $A_4$ are $2 \times 2$ and $(N_c - 2) \times (N_c - 2)$ matrix blocks respectively.

The action corresponding to the leading-order instanton can be simplified by integrating by parts and using Eqs. (1)–(4). We are left with

$$S_0 = \frac{8\pi^2}{g^2} + \int d^4x \partial_\mu \left\{ 2 \text{Tr}(\phi^\dagger D_\mu \phi) + (D_\mu \tilde{q})^\dagger \tilde{q} + (D_\mu \tilde{q})^\dagger \tilde{q} \right\} + \sqrt{2} \text{im} \int d^4x \chi \tilde{\chi}$$

$$+ \sqrt{2}i g \int d^4x \left( \tilde{\chi} \phi \chi + q^\dagger \lambda \chi + \tilde{q} \psi \chi \right).$$

(8)

The $\sqrt{2}$ prefactor of the quark mass term allies us with the usual curve convention.

We now present the remaining singular gauge solutions to the defining equations, which we shall use to evaluate the above action. The normalized gaugino ‘zero-mode’ solutions are listed in \[23, 24\],

$$\lambda_{SC\alpha} = \frac{i \rho \ y_\nu \bar{y}_\mu \epsilon_{\mu \nu}}{\pi (y^2 + \rho^2)^2} \xi_{SC}\alpha T^a,$$

(9)

$$\lambda_{SS\alpha} = \frac{\sqrt{2} \rho^2 \ y_\nu \bar{y}_\mu \eta^{\nu \mu}_{\lambda \sigma} (\sigma^b \xi_{SS})_{\alpha}}{\pi y^2 (y^2 + \rho^2)^{3/2}} T^a,$$

(10)

$$(\lambda_{M\alpha})_{ij} = -\frac{\rho}{\sqrt{2} \pi} \ y_\mu (\epsilon_{\mu \nu})_{ai} \frac{y_{\nu}}{\sqrt{y^2 (y^2 + \rho^2)^{3/2}}} \xi_{Mj} \quad (j = 3, 4, \ldots, N_c),$$

(11)

$$(\lambda_{N\alpha})_{ij} = \frac{\rho}{\sqrt{2} \pi} \ y_\mu (\epsilon_{\mu \nu})_{aj} \frac{y_{\nu}}{\sqrt{y^2 (y^2 + \rho^2)^{3/2}}} \xi_{Ni} \quad (i = 3, 4, \ldots, N_c).$$

(12)

Here $\epsilon$ is the antisymmetric tensor satisfying $\epsilon^{12} = 1$. In addition to the two ‘superconformal’ (SC) and two ‘supersymmetric’ (SS) modes there are an additional $2(N_c - 2)$ modes which we
have chosen to partition such that the ‘M’ modes live in the upper right and the ‘N’ modes live
in the lower left parts of the matrix representation of the $SU(N_c)$ Lie algebra. The analogous
solutions for $\psi$ are obtained by switching the Grassmannian collective coordinates $\xi \rightarrow \zeta$.

The normalized solution for a quark flavor is \[ \chi_{\alpha i} = \frac{-\rho}{\pi \sqrt{y^2(y^2 + \rho^2)^{3/2}}} \eta. \] (13)

The conjugate quark solution satisfies $\bar{\chi}_{\alpha i} = \epsilon^{ij} \chi_{\alpha j}$ provided we exchange the Grassmannian
collective coordinate $\eta \rightarrow \bar{\eta}$.

Turning to the scalar fields, we separate the solution for $\phi$ into a part satisfying the homo-
genous equation, $\phi_h$, and a particular solution $\phi_p$ which arises in the presence of the Yukawa
source term. The homogeneous solution was found in \[16\] to be

\[ \phi_h = \left( \frac{y^2}{y^2 + \rho^2} A_{1(tl)} + \frac{1}{2} \text{Tr}(A_1) I_2 \right) \sqrt{\frac{y^2}{y^2 + \rho^2}} A_3 \sqrt{\frac{y^2}{y^2 + \rho^2}} A_4, \] (14)

where $A_{1(tl)} = A_1 - \frac{1}{2} \text{Tr}(A_1) I_2$ and $I_2$ is the $2 \times 2$ identity matrix. This solution manifestly
satisfies the boundary condition (7).

Linearity enables $\phi_p$ to be decomposed further. If we define $\phi_{A/B}$ as the particular solution
with fermionic modes $\lambda_A$ and $\psi_B$ inserted into the source term, then

\[ \phi_p = \sum_{A,B=SC,SS,M,N} \phi_{A/B}. \] (15)

We obtain the following list of independent solutions which enter the right hand side of this
equation:

$\phi_{SC/SC} = \frac{i g}{4 \sqrt{2} \pi^2} \frac{y^2 (\xi_{SC} \sigma^a \bar{\xi}_{SC})}{(y^2 + \rho^2)^2} T^a, $ (16)

$\phi_{SC/SS} = -\frac{i g \rho}{4 \pi^2} \frac{y \mu \bar{\eta}^a_{\mu \nu} \epsilon_{\nu \lambda} \eta_{\mu} (\xi_{SS} \sigma^b \bar{\xi}_{SS})}{(y^2 + \rho^2)^2} T^a, $ (17)

$\phi_{SS/SS} = -\frac{i g \rho^2}{2 \sqrt{2} \pi^2} \frac{y \mu \eta^a_{\mu \nu} \eta^b_{\nu \lambda} (\xi_{SS} \sigma^b \bar{\xi}_{SS})}{y^2 (y^2 + \rho^2)^2} T^a, $ (18)

$(\phi_{SC/M})_{ij} = \frac{i g}{8 \pi^2} \frac{\sqrt{y^2}}{(y^2 + \rho^2)^{3/2}} \xi_{SC} \epsilon_{ij} \bar{\zeta}_{MC}, $ (19)

$(\phi_{SC/N})_{ij} = \frac{i g}{8 \pi^2} \frac{\sqrt{y^2}}{(y^2 + \rho^2)^{3/2}} \zeta_{Ni} (\xi_{SC})_j, $ (20)

$(\phi_{SS/M})_{ij} = -\frac{i g \rho}{4 \sqrt{2} \pi^2} \frac{y \mu}{\sqrt{y^2 (y^2 + \rho^2)^{3/2}}} (\epsilon_{\mu \nu} \xi_{SS})^i \bar{\zeta}_{MC}, $ (21)

$(\phi_{SS/N})_{ij} = -\frac{i g \rho}{4 \sqrt{2} \pi^2} \frac{y \mu}{\sqrt{y^2 (y^2 + \rho^2)^{3/2}}} \zeta_{Ni} (\epsilon_{\mu \nu} \xi_{SS})^j, $ (22)
\[(\phi_{M/N})_{ij} = \frac{ig}{8\sqrt{2}\pi^{2}} \frac{1}{(y^{2} + \rho^{2})} \left\{ \delta_{ij} \delta_{i,j} \leq 2 \sum_{k=3}^{N_{c}} \zeta_{Nk} \xi_{Mk} - 2 \zeta_{Ni} \xi_{Mj} \delta_{i,j} \geq 3 \right\}, \quad (23)\]

\[\phi_{M/M} = \phi_{N/N} = 0. \quad (24)\]

The solution for \(\phi_{A/B}\) is deduced from the solution for \(\phi_{B/A}\) by changing the sign and making the exchange \(\xi \leftrightarrow \zeta\).

The conjugate Higgs also consists of a homogeneous and a particular solution. The homogeneous solution is simply the Hermitian conjugate of (14) whilst the particular solution is

\[(\phi_{p})_{ij} = \frac{ig}{4\sqrt{2}\pi^{2}} \frac{1}{(y^{2} + \rho^{2})} \left\{ \left( \frac{N_{c} - 2}{2N_{c}} \right) \delta_{ij} \delta_{i,j} \leq 2 - \frac{1}{N_{c}} \delta_{ij} \delta_{i,j} \geq 3 \right\} \tilde{\eta}. \quad (25)\]

Finally, each squark solution is a sum of particular solutions \([24]\),

\[q_{SCi} = \frac{ig}{4\sqrt{2}\pi^{2}} \frac{\sqrt{y^{2}}}{(y^{2} + \rho^{2})^{3/2}} \tilde{\zeta}_{SC} \eta, \quad (26)\]

\[q_{SSi} = -\frac{ig\rho}{4\pi^{2}} \frac{y_{\mu}}{\sqrt{y^{2}(y^{2} + \rho^{2})^{3/2}}} (\tilde{e}_{\mu} \xi_{SS}) \eta, \quad (27)\]

\[q_{Ni} = \frac{ig}{4\pi^{2}} \frac{1}{(y^{2} + \rho^{2})} \xi_{Ni} \eta, \quad (28)\]

where \(q_{A}\) represents the solution with \(\lambda_{A}\) inserted in the source term. The solutions for \(q_{\dagger}\) and the conjugate representation squarks may be obtained by straightforward manipulations of these configurations.

By plugging the above solutions into Eq. (8) we are immediately able to evaluate the leading-order instanton action. Ignoring supersymmetric zero-modes which are not lifted and give no contribution, we find

\[2 \int d^{4}x \partial_{\mu} \text{Tr}(\phi_{\dagger} D_{\mu} \phi) = 8\pi^{2} \rho^{2} F + g(\tilde{\zeta}_{SC}, \xi_{M}, \zeta_{N}) M(\tilde{\xi}_{SC}, \xi_{M}, \xi_{N})\dagger, \quad (29)\]

\[\int d^{4}x \partial_{\mu}((D_{\mu} q_{\dagger}) q) = \int d^{4}x \partial_{\mu}((D_{\mu} \tilde{q}) \dagger \tilde{q}) = 0, \quad (30)\]

\[\sqrt{2}i m \int d^{4}x \tilde{\chi} \chi = -i \sqrt{2} m \tilde{\eta}, \quad (31)\]

\[\sqrt{2}i g \int d^{4}x \tilde{\phi} \chi = -\frac{ig}{\sqrt{2}} \text{Tr}(A_{1}) \tilde{\eta} - \frac{g^{2}}{24\pi^{2}\rho^{2}} \sum_{k=3}^{N_{c}} (\xi_{MK} \zeta_{NK} + \zeta_{NK} \xi_{MK}) \tilde{\eta}, \quad (32)\]

\[\sqrt{2}i g \int d^{4}x (q_{\dagger} \lambda \chi + \tilde{q} \psi \chi) = -\frac{g^{2}}{12\pi^{2}\rho^{2}} \sum_{k=3}^{N_{c}} (\xi_{MK} \zeta_{NK} + \zeta_{NK} \xi_{MK}) \tilde{\eta}. \quad (33)\]

In Eq. (29) \(F\) and \(M\) are the same as in [16], namely

\[F = \text{Tr}(A_{1(\dagger)} A_{1(\dagger)} + \frac{1}{2} (A_{3} A_{2}^{\dagger} + A_{2} A_{3}^{\dagger})), \quad (34)\]
and

\[
M = i \begin{pmatrix}
\sqrt{2} \epsilon A^1_{1(t)} & (A^1_2)^t & 0 \\
A^1_2 & 0 & -\frac{\text{Tr} A^1_3}{\sqrt{2}} I_{N_c-2} + \sqrt{2} A^1_4 \\
(\epsilon A^1_3)^t & -\frac{\text{Tr} A^1_4}{\sqrt{2}} I_{N_c-2} + \sqrt{2} (A^1_4)^t & 0
\end{pmatrix},
\]

(35)

where \( I_{N_c-2} \) is the \((N_c - 2) \times (N_c - 2)\) identity matrix. The leading-order instanton action obtained by substituting Eqs. (29)-(33) into Eq. (8) is in full agreement with the calculation of Ito and Sasakura [17]. These authors treated the Yukawa terms in the action perturbatively, whereas we explicitly included these terms as sources in the defining equations (3) and (4) in the spirit of [11–14].

Our next consideration is the measure associated with integration over the collective coordinates which appear as free parameters in the instanton solutions. For the 1-instanton situation the relevant Jacobian factors are well known [22, 25, 24] and combine to give the measure

\[
\int d\omega = 2^{10} \pi^{2 N_c + 2} \mu^{2 N_c - N_f} \int d\Omega \int_0^\infty \rho^{4 N_c - 5} d\rho \int d^{2 N_c} \zeta d^{2 N_c} \xi \int d^4 x_0 \int d^{N_f} \eta d^{N_f} \bar{\eta}.
\]

(36)

Since the is action invariant under the subgroup \( H = U(1) \times SU(2) \times SU(N_c - 2) \), the \( d\Omega \) integration is to be taken over the group submanifold \( SU(N_c)/H \). The Pauli-Villars regularization mass \( \mu \) is eliminated by defining the RG-invariant dynamical scale

\[
\Lambda_{PV}^{2 N_c - N_f} = \mu^{2 N_c - N_f} e^{-\frac{8 \pi^2}{g^2}}
\]

(37)

It is convenient to switch from the Pauli-Villars scale to the dynamical scale used in the hyper-elliptic curves. In [17] it was shown using renormalization group matching arguments that the two scales are related by

\[
\Lambda_{PV}^{2 N_c - N_f} = 2^{2 - N_c + N_f/2} N_f \Lambda_{PV}^{2 N_c - N_f}.
\]

(38)

In supersymmetric theories an important simplification of the instanton calculus occurs in connection with the functional integration of the quadratic quantum fluctuations about the instanton action. Namely the resulting determinant factors due to fermionic and bosonic field fluctuations exactly cancel each other in the background gauge [26]. We are therefore now in a position to write down the 1-instanton contribution to \( u_n \). After assembling the relevant factors and performing the integration over the quark zero-modes we have

\[
u_{\sqrt{2} N_c - N_f} = \pi^{2 + 2 N_c} \Lambda^{2 N_c - N_f} \sum_{p=0}^{N_f} \int d\bar{\omega} \int d^2 \xi_{SS} d^2 \xi_{SS} \int d^4 x_0 \text{Tr}(\phi^n)
\]

\[
\times t_p \left( \text{Tr}(A_1) - \frac{ig}{4 \sqrt{2} \pi^2 \rho^2} \sum_{k=3}^{N_c} (\xi_{MK} \zeta_{NK} + \xi_{NK} \zeta_{MK}) \right)^{N_f-p} \exp(-S_H),
\]

(39)

where the \( x_0 \) and SS mode integrations have been separated from \( d\omega \), leaving

\[
\int d\bar{\omega} = \int d\Omega \int_0^\infty \rho^{4 N_c - 5} d\rho \int d^{2 N_c - 2} \zeta d^{2 N_c - 2} \xi.
\]

(40)
\[ S_H \] is just the contribution of the Higgs kinetic term to the action as given by (29) and the \( t_p \)
are symmetric polynomials in the quark masses,
\[ t_p = \sum_{i_1 < i_2 < \cdots < i_p} m_{i_1} m_{i_2} \cdots m_{i_p}. \]  

The Higgs field insertions into the integrand are to be evaluated using the short-distance configurations
listed above. These insertions saturate the integration over the collective coordinates corresponding
to the exact supersymmetric zero-modes. It follows that only the part of \( \text{Tr}(\phi^n) \) which contains precisely four SS Grassmann variables can give a non-zero contribution. When \( n = 2 \) this is just \( \text{Tr}(\phi_{SS/SS}^2) \) and using Eq. (18) we can perform the integration of the field operator over \( x_0 \) and the SS modes,
\[ \int d^2 \zeta_{SS} d^2 \xi_{SS} \int d^4 x_0 \text{Tr}(\phi^2) = -\frac{g^2}{2^4 \pi^2}. \]  

In [16, 17], the authors considered the integral expression (39) when \( n = 2 \). Since the group integration was not generally tractable they studied the particular case of two VEV’s being infinitesimally close. In this limit they found that the group integration linearized and could be carried out. Their answer exhibited a singularity structure associated with the infra-red divergence caused by the restoration of a non-Abelian subgroup when any two VEV’s coincide. Taking this to represent the only instance where the instanton integration diverges, and by considerations of dimensional analysis, gauge invariance and holomorphy, Ito and Sasakura deduced the full result
\[ u_2^{1I} = \frac{\Lambda^{2N_c-N_f}}{2} \sum_{p=0}^{N_f} t_p \left( \sum_{k=1}^{N_c} \frac{a_{k-p}}{\prod_{l \neq k} (a_l - a_k)^2} + \alpha_{N_c} \delta_{N_f-p,2N_c-2} + \beta_{N_c} \delta_{N_f-p,2N_c} \sum_{k=1}^{N_c} a_k^2 \right). \]  
The analysis fails to determine the constant coefficients of the regular terms, \( \alpha_{N_c} \) and \( \beta_{N_c} \). However in the specific case of \( SU(3) \) it was claimed in [17] that the integral expression for \( u_2^{1I} \) may be directly computed and gives \( (\alpha_3, \beta_3) = (-3/8, -15/64) \). For a range of input values for the VEV’s we have numerically verified these results.

Now we use our explicit solutions for \( \phi \) to evaluate \( u_3^{1I} \) along similar lines. This will provide the only remaining independent test for the \( SU(3) \) curves at the 1-instanton level. For insertion into the integrand, we require the part of \( \text{Tr}(\phi^3) \) which has the necessary quadrilinear dependence on the SS Grassmannian variables. This is
\[ 3 \text{Tr} \left\{ \phi_{SS/SS}^2 \left( \phi_h + \sum_{A,B \neq SS} \phi_{A/B} \right) \right\} + 3 \text{Tr} \left\{ \phi_{SS/SS} \left( \sum_{A \neq SS} \phi_{A/SS} + \sum_{B \neq SS} \phi_{SS/B} \right)^2 \right\}. \]  

\[ \]This is to be contrasted with the case where a Green’s function such as \( \langle \bar{\psi}(x_1)\psi(x_2)\bar{\lambda}(x_3)\lambda(x_4) \rangle \) is to be evaluated. Here it is the limit \( |x_i - x_j| \to \infty \) which is important (see eg. [8, 11]) and long-distance instanton solutions must be obtained.
Since $\phi_{SS/SS}^2$ is proportional to the $2 \times 2$ identity matrix in the upper left block of the matrix representation, the first term reduces to two distinct non-zero components,

$$3\text{Tr}(\phi_{SS/SS}^2 \phi_h) = \frac{3}{2} \text{Tr}(\phi_{SS/SS}^2) \text{Tr}(A_1),$$

$$3\text{Tr}(\phi_{SS/SS}^2 (\phi_{M/N} + \phi_{N/M})) = -\frac{3ig}{8\sqrt{2\pi^2}} \frac{\text{Tr}(\phi_{SS/SS}^2)}{y^2 + \rho^2} \sum_{k=3}^{N_c} (\xi_{Mk} \zeta_{Nk} + \xi_{Nk} \zeta_{Mk}).$$

The second term simplifies because $\phi_{SS/SS}$ is composed of Pauli matrices living in the upper left corner of the matrix representation. Closer inspection shows that the only contributing component is

$$3\text{Tr}(\phi_{SS/SS}^2 (\phi_{M/SS} \phi_{SS/N} + \phi_{SS/M} \phi_{N/SS})) = -\frac{3ig}{8\sqrt{2\pi^2}} \frac{\text{Tr}(\phi_{SS/SS}^2)}{y^2 + \rho^2} \sum_{k=3}^{N_c} (\xi_{Mk} \zeta_{Nk} + \xi_{Nk} \zeta_{Mk}).$$

Upon integrating over $x_0$ and the SS modes, we get

$$\int d^2 \zeta_{SS} d^2 \xi_{SS} \int d^4 x_0 \text{Tr}(\phi^3) = \frac{3}{2} \left( -\frac{g^2}{24\pi^2} \right) \left( \text{Tr}(A_1) - \frac{ig}{4\sqrt{2\pi^2}\rho^2} \sum_{k=3}^{N_c} (\xi_{Mk} \zeta_{Nk} + \xi_{Nk} \zeta_{Mk}) \right).$$

The first factor in brackets is just the corresponding result (12) for the $\text{Tr}(\phi^2)$ insertion whilst the second factor precisely matches the part of the instanton action which is pulled down by the integration over the quark Grassmannians.

This is a remarkable result since it allows us to immediately determine $u_{3I}^f$ from knowledge of $u_{2I}^f$. Using Eq. (33) and Eq. (43) and after accounting for a rescaling of the Higgs field, we find that for $N_f < 2N_c$,

$$u_{3I}^f = \frac{3\Lambda_{2N_c-N_f}}{2} \sum_{p=0}^{N_f} t_p \left( \sum_{k=1}^{N_c} a_k^{N_f-p+1} + \sum_{k \neq k'} (a_k - a_{k'})^2 \right) + \tilde{\alpha}_{N_c} \delta_{N_f-2N_c-3} + \tilde{\beta}_{N_c} \sum_{k=1}^{N_c} a_k^2,$$

where $(\tilde{\alpha}_{N_c}, \tilde{\beta}_{N_c}) \equiv (\alpha_{N_c}, \beta_{N_c})$. This equation is the main result of this letter, and constitutes a non-trivial independent prediction of the microscopic instanton calculus.

We now consider the exact results predicted by the hyperelliptic curves which have been proposed for $N_f < 2N_c$ in [2–5]. By making use of the freedom to shift the $x$-variable, we can write all of the suggested curves in the following form,

$$y^2 = P(x)^2 - Q(x),$$

where

$$Q(x) = \Lambda_{2N_c-N_f} \sum_{p=0}^{N_f} t_p x^{N_f-p} \quad \text{and} \quad P(x) = \prod_{k=1}^{N_c} (x - e_k) + \Lambda_{2N_c-N_f} T(x).$$
The moduli space parameters $e_k$ satisfy $\sum_{k=1}^{N_c} e_k = 0$ and are related to the moduli of the physical theory through the formula
\[ u_n = \sum_{k=1}^{N_c} e_k^n. \] (52)

The function $T(x)$ satisfies
\[ T(x) = \sum_{p=0}^{N_f} t_p T^{(N_f - p - N_c)}(x) \delta_{N_f - p \geq N_c}, \] (53)
where the $T^{(N_f - p - N_c)}(x)$ are polynomials of degree $(N_f - p - N_c)$ in $x$, with possible dependence on the dynamical scale and also on the moduli space parameters.

It is apparent that the considerations used in constructing the curves are insufficient to uniquely determine the function $T(x)$. Nonetheless the curve prediction for the holomorphic prepotential in terms of the $a_k$ is independent of this function [8]. In this respect $T(x)$ represents a superfluous degree of freedom in the curve parameterization. However the curve predictions for the quantum moduli $u_n$ are certainly affected by $T(x)$. For the physical correspondence to be complete, a definite form for $T(x)$ must exist which specifies curves whose predictions consistently agree with the results of instanton calculus. Furthermore, the authors of [2], [3], [4] and [5] all use different criteria to propose definite forms for $T(x)$. The validity of these criteria is open to testing by instanton calculations of the $u_n$.

Solutions for the $u_n$ are obtained from the curves through the periods
\[ a_k = \frac{1}{2\pi i} \oint_{A_k} x(P' - \frac{PQ}{2y}) dx, \] (54)
where the $A_k$ are a set of one-cycles enclosing branch cuts of the curves. These integrals can be expanded in powers of $\Lambda^{2N_c - N_f}$ in the semiclassical regime, and the result is [8]
\[ a_k = e_k + \sum_{m,n \geq 0; m + n \neq 0} \frac{(-1)^n (\Lambda^{2N_c - N_f})^{m+n} \partial^{2m+n-1}}{(m!)^2 n! 2^m} \partial e_k (S_k(e)^m R_k(e)^n), \] (55)
where
\[ S_k(e) = \frac{\sum_{p=0}^{N_f} t_p e_k^{N_f - p}}{\prod_{i \neq k} (e_k - e_i)^2} \quad \text{and} \quad R_k(e) = \frac{T(e_k)}{\prod_{i \neq k} (e_k - e_i)}. \] (56)

At the 1-instanton level it is a simple matter to invert this series and use the defining expression (54) to get the curve prediction for $u_n^{1\text{f}}$. The answer may be written in the form
\[ u_n^{1\text{f}} = \frac{n(n-1)\Lambda^{2N_c - N_f}}{4} \sum_{p=0}^{N_f} t_p \left( \sum_{k=1}^{N_c} a_k^{N_f - p + n - 2} \prod_{l \neq k}^{N_c} (a_k - a_l) + \frac{1}{n-1} \delta_{n=1} \right), \] (57)

3Excepting the requirement that for $N_f = 2N_c - 1$ theories the $x^{N_c - 1}$ term in $T^{(N_f - 1)}(x)$ has coefficient $\frac{1}{4}$ (this ensures that the meromorphic one-form has no residue at infinity when the bare masses are zero).
where $r_n^{(N_f-p)}$ is a regular function of the VEV’s given by

$$
r_n^{(N_f-p)} = \sum_{k=1}^{N_c} \frac{a_k^{N_f-p+n-2}}{\prod_{j \neq k}^N_c (a_k - a_j)^2} \left( 2a_k \sum_{l \neq k}^N_c \frac{1}{(a_k - a_l)} - (N_f - p + n - 1) \right) + 4\delta_{N_f-p \geq N_c} \sum_{k=1}^{N_c} a_k^{n-1} T^{(N_f-p-N_r)}(a_k) \bigg|_{\Lambda=0}.
$$

(58)

The non-singular nature of $r_n^{(N_f-p)}$ can be verified by expanding it in powers of the separation between two VEV’s.

We observe that when $N_f - p < 2N_c - n$, the regular function $r_n^{(N_f-p)}$ vanishes and the full answer is unambiguously given by the singular term in Eq. (57). So when $n = 3$ we find complete agreement with the prediction of the instanton analysis, Eq. (49), for $N_f < 2N_c - 3$. For $n = 2$ we confirm the similar observation made in (43), i.e. the agreement of all the proposed curves with the instanton prediction (43) when $N_f < 2N_c - 2$. When $N_f \geq 2N_c - 3$, the regular functions $r_2^{(2N_c-2)}$, $r_3^{(2N_c-3)}$ and $r_3^{(2N_c-1)}$ simplify to give the expected regular terms of Eqs. (43) and (49), but with multiplying constants which depend on the function $T(x)$. In Table 1 we summarize the curve predictions for the coefficients $\alpha_3$, $\tilde{\alpha}_3$ and $\tilde{\beta}_3$ pertinent to the $SU(3)$ theory with $N_f < 6$ flavors, according to the various suggestions for $T(x)$ in [2–5]. Our results confirm the curve predictions for $\alpha_3$ extracted in (17), and we see that none of the proposed curves give the numbers predicted by instanton calculus.

We can use Eq. (57) and the set of instanton calculations in $SU(3)$ theory to fix the curve parameterization at the one-instanton level for $N_f = 3, 4, 5$ fundamental flavors (see Table 1). Dimensional considerations imply that for $N_f = 3, 4$ the curves are completely fixed, so that no discrepancies should occur at higher order instanton levels. For $N_f = 5$ however, there may corrections up to the 3-instanton level. It would be interesting if an a priori criterion for curve construction could be found which predicts the parameterization required by the instanton calculus.

| Source of prediction | $T^{(0)}(x)$ | $T^{(1)}(x)$ | $T^{(2)}(x)$ | $(\alpha_3, \tilde{\alpha}_3, \tilde{\beta}_3)$ |
|----------------------|-------------|-------------|-------------|------------------------------------------|
| Ref. [2]             | $\frac{1}{4}$ | $\frac{1}{4}x$ | $\frac{1}{4}x^2$ | $(0,0, -1/4)$ |
| Refs. [4] and [3]    | 0           | 0           | $\frac{1}{4}x^2 + \frac{1}{45}x + \frac{5}{1728} - \frac{1}{24}u_2$ | $(-1, -1/2, -1/3)$ |
| Ref. [4]             | $\frac{1}{4}$ | $\frac{1}{4}x$ | $\frac{1}{4}x^2 + \frac{1}{8}u_2$ | $(0, 0, 0)$ |
| Instanton calculus   | $\frac{1}{16}$ | $\frac{5}{32}x$ | $\frac{1}{4}x^2 + \frac{1}{128}u_2$ | $(-3/8, -3/8, -15/64)$ |

Table 1: Predictions for the coefficients of the regular terms appearing in the one-instanton contributions to the moduli $u_2$ and $u_3$ in $SU(3)$ SQCD, according to suggested forms for $T(x)$ (defined by the polynomials $T^{(0)}(x)$, $T^{(1)}(x)$ and $T^{(2)}(x)$).
It is clearly desirable to extend the $SU(3)$ analysis to the two-instanton level to check that no further discrepancies appear for $N_f = 3, 4$ and to see if further curve fixing at this level is required for $N_f = 5$. It would also be interesting to investigate the model which has $N_f = 6$ fundamental flavors, particularly in the light of the recent discoveries for the similar $N_f = 4$ model in $SU(2)$ [18]. There is also the wider problem of fully evaluating one-instanton contributions to all the quantum moduli $u_n$, for general $N_c$. We hope to address these issues in future work.

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