Local bilinear operators on the lattice and their perturbative renormalisation including $O(a)$ effects

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Abstract

Some basic concepts are discussed to derive renormalisation factors of local lattice operators relevant to deep inelastic structure functions and to other measurable quantities. These $Z$ factors can be used to relate matrix elements measured by lattice techniques to their continuum counterparts. We discuss the $O(a)$ improvement of point and one-link lattice quark operators. Suitable bases of improved operators are derived. Tadpole improvement is applied to get more reliable perturbative results.

1 Introduction and some basic definitions of DIS and OPE

In deep inelastic lepton scattering (DIS, see e.g. [1]) (with 4-momenta $k$ and $k'$) on hadron ($p$ with $p^2 = M^2$) the inclusive differential cross section for $eP \rightarrow e'X$ in the hadron rest frame is given by

$$\frac{d^3\sigma}{dx\,dy\,d\phi} = \frac{e^4}{16\pi^2 Q^4} y \, l^{\mu\nu}(k, q, s_l) \, W_{\mu\nu}(p, q)_{\lambda\lambda}$$

with the standard notations

$$x = \frac{Q^2}{2p\cdot q}, \quad y = \frac{p\cdot q}{p\cdot k}.$$ 

$q = k - k'$ (with $-q^2 = Q^2$) is the momentum transfer in the scattering process, $\phi$ the azimuthal scattering angle of the outgoing lepton, $s_l$ (with $s_l^2 = -m_l^2$) the initial lepton polarisation vector and $\lambda$ denotes the initial hadron polarisation ($\pm 1/2$ for spin 1/2 target). All information about

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the cross section is contained in the leptonic and hadronic tensors $l_{\mu\nu}$ and $W_{\mu\nu}$. While $l_{\mu\nu}$ is known, the hadronic tensor

$$W_{\mu\nu}^{\mu\nu}(p, q)_{\lambda\lambda} = \frac{1}{4\pi} \int d^4x \, e^{iq\cdot x} \langle p, \lambda' | [j^\mu(x), j^\nu(0)] | p, \lambda \rangle$$

with the electromagnetic hadronic currents $j^\mu(x)$ contains the strong interaction effects which are not completely accessible to perturbative QCD.

The most general hadronic tensor for polarised DIS from spin 1/2 targets is usually written in the form

$$W_{\mu\nu}(p, q, s) = F_1 \left( -g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2} \right) + \frac{F_2}{p\cdot q} \left( p_{\mu} - \frac{p\cdot q}{q^2} q_{\mu} \right) \left( p_{\nu} - \frac{p\cdot q}{q^2} q_{\nu} \right) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho}{p\cdot q} \left( (g_1 + g_2) s^\sigma - g_2 \frac{s\cdot q}{p\cdot q} p^\sigma \right).$$

Here $F_{1,2}$ and $g_{1,2}$ are the structure functions and $s_\mu$ the polarisation vector of the nucleon with $s^2 = -M^2$. In a parton model interpretation the structure functions $F_{1,2}$ contain information about the overall density of quarks (and gluons) in the nucleon and $g_1$ probes the distribution of quarks of given helicity in a longitudinally polarised nucleon ($Q_i$ is the quark charge, $q_\pm(x) = (\bar{q}_\pm(x)$) denotes the distribution function of quark (antiquark) with momentum fraction $x$ and helicity parallel/antiparallel to its parent nucleon)

$$F_1(x, Q^2) = \sum_i \frac{Q_i^2}{2} \left( q_+(x) + q_-(x) + \bar{q}_+(x) + \bar{q}_-(x) \right),$$

$$F_2(x, Q^2) = \sum_i \frac{Q_i^2}{2} \left( q_+(x) + q_-(x) + \bar{q}_+(x) + \bar{q}_-(x) \right),$$

$$g_1(x, Q^2) = \sum_i \frac{Q_i^2}{2} \left( q_+(x) - q_-(x) + \bar{q}_+(x) - \bar{q}_-(x) \right).$$

The structure function $g_2$ has no simple interpretation in the parton model.

One can derive sum rules (moments) for the structure functions directly from QCD using the operator product expansion (OPE). The starting point is the time–ordered product of two hadronic currents

$$t_{\mu\nu} = i \int d^4z \, e^{iq\cdot z} \, T(j_\mu(z) j_\nu(0))$$

the matrix element of which gives the well–known Compton amplitude. This amplitude is related to the hadronic tensor $W_{\mu\nu}$ via the optical theorem

$$2\pi W_{\mu\nu}(p, q)_{\lambda\lambda} = \text{Im} \langle p, \lambda' | t_{\mu\nu} | p, \lambda \rangle.$$

To calculate the Compton amplitude in QCD one relies on OPE which allows to write the product of two local operators $O_a(z)$ and $O_b(0)$ for vanishing distance $z$ as an expansion in local operators. In the Fourier transform version of OPE one has

$$\lim_{q \to \infty} \int d^4z \, e^{iq\cdot z} \, O_a(z) O_b(0) = \sum_d c_{abcd}(q) O_d(0).$$

The Wilson coefficients $c_{abcd}(q)$ (in general singular at $q \to \infty$) are independent of the matrix elements, provided $q$ is much larger than the characteristic momentum in any of the external states.
The local operators in OPE for QCD are quark and gluon operators with arbitrary dimension $d$ and spin $n$. It can be shown that the contribution of any operator to $l^{\mu
u}W_{\mu
u}$ is of the order

$$c_{\mu_1...\mu_n}O_{d,n}^{\mu_1...\mu_n} \propto x^{-n} \left( \frac{Q^2}{M^2} \right)^{(2-t)/2}$$

with the twist introduced as $t = d - n$. Therefore, the most important operators in OPE at $Q^2 \to \infty$ are those with twist two contributing to $F_{1,2}$ and $g_1$. The structure function $g_2$ involves twist three operators allowing a direct measurement of higher twist matrix elements.

For unpolarised DIS we use as conventional basis for twist two quark and gluon operators

$$O^{(u,d)}_{\mu_1...\mu_n} = \left( \frac{i}{2} \right)^n \bar{\psi}^{(u,d)}(\gamma_\mu_1 \vec{D}_{\mu_2} \cdots \vec{D}_{\mu_n} \psi^{(u,d)}) - \text{traces},$$

$$O^{(g)}_{\mu_1...\mu_n} = i^{n-2} \text{tr}F_{\mu_1}D_{\mu_2} \cdots D_{\mu_{n-1}}F_{\alpha\mu_n} - \text{traces}$$

where $\psi^{(u,d)}$ denote the quark fields, $F_{\alpha\beta}$ is the gluon field strength tensor and the covariant derivatives $D_\mu$ and $\vec{D}_\mu = \vec{D}_\mu - D_\mu$. Taking into account polarisation, the following towers of operators have to be added

$$O^{5(u,d)}_{\sigma\mu_1...\mu_n} = \left( \frac{i}{2} \right)^n \bar{\psi}^{(u,d)}(\gamma_\sigma \gamma_5 \vec{D}_{\mu_1} \cdots \vec{D}_{\mu_n} \psi^{(u,d)}) - \text{traces},$$

$$O^{5(g)}_{\sigma\mu_1...\mu_n} = i^{n-1} \text{tr}F_{\alpha\sigma}D_{\mu_1} \cdots D_{\mu_{n-1}}F_{\alpha\mu_n} - \text{traces}$$

with the dual field strength tensor $\tilde{F}_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$.

The moments of nucleon structure functions are then written at large $Q^2$ in the form

$$2 \int_0^1 dx \ x^{n-1}F_1(x,Q^2) = \sum_{f=u,d,g} \ c_{1,n}^{(f)} \left( \frac{\mu^2}{Q^2}, g(\mu) \right) v_{n}^{(f)}(\mu),$$

$$\int_0^1 dx \ x^{n-2}F_2(x,Q^2) = \sum_{f} c_{2,n}^{(f)} \left( \frac{\mu^2}{Q^2}, g(\mu) \right) v_{n}^{(f)}(\mu)$$

and

$$2 \int_0^1 dx \ x^n g_1(x,Q^2) = \frac{1}{2} \sum_{f=u,d,g} e_{1,n}^{(f)} \left( \frac{\mu^2}{Q^2}, g(\mu) \right) a_{n}^{(f)}(\mu),$$

$$2 \int_0^1 dx \ x^n g_2(x,Q^2) = \frac{1}{2n+1} \sum_{f=u,d,g} \left( e_{2,n}^{(f)} \left( \frac{\mu^2}{Q^2}, g(\mu) \right) d_{n}^{(f)}(\mu) - e_{1,n}^{(f)} \left( \frac{\mu^2}{Q^2}, g(\mu) \right) a_{n}^{(f)}(\mu) \right).$$

Due to the symmetry of the structure functions under charge conjugation only even $n$ contribute to $F_{1,2}$ and $g_{1,2}$. In the so called quenched approximation of lattice QCD, however, also operators with odd $n$ are relevant. The coefficient functions $c_{1,n}$, $c_{2,n}$, $e_{1,n}$ and $e_{2,n}$ are calculable in QCD perturbation theory, the measured scaling violations are usually described by their $Q^2$ evolution.

On the other hand the computation of structure functions themselves (at a given low momentum scale $\mu$) requires nonperturbative methods ab initio. Note that $\mu$–dependence in the matrix elements $v_{n}^{(f)}(\mu)$, $a_{n}^{(f)}(\mu)$ and $d_{n}^{(f)}(\mu)$ arising from the $\mu$–dependence of the renormalisation constants defined below should be cancelled by the corresponding $\mu$–dependence in the coefficient functions. The matrix elements are defined from the operators as follows

$$\frac{1}{2} \sum_{\vec{p}, \vec{s}} \langle \vec{p}, \vec{s} | O_{\mu_1...\mu_n}^{(f)} | \vec{p}, \vec{s} \rangle = 2v_{n}^{(f)}(p_{\mu_1} \cdots p_{\mu_n} - \text{traces}),$$

$$\frac{1}{2} \sum_{\vec{p}, \vec{s}} \langle \vec{p}, \vec{s} | O_{\mu_1...\mu_n}^{(f)} | \vec{p}, \vec{s} \rangle = 2a_{n}^{(f)}(p_{\mu_1} \cdots p_{\mu_n} - \text{traces}),$$

$$\frac{1}{2} \sum_{\vec{p}, \vec{s}} \langle \vec{p}, \vec{s} | O_{\mu_1...\mu_n}^{(f)} | \vec{p}, \vec{s} \rangle = 2d_{n}^{(f)}(p_{\mu_1} \cdots p_{\mu_n} - \text{traces}).$$
Here \( \{ \ldots \} \) denotes symmetrisation and \( \ldots \) antisymmetrisation.

The traceless and symmetric operators \( O^{(f)}_{\{a_{\mu_1} \ldots a_{\mu_n}\}} \) and \( O^{(f)}_{\{a_{\mu_1} \ldots a_{\mu_{n+1}}\}} \) transform irredicably under the Lorentz group. The r.h.s. of eqs. (16) and (17) are the only traceless, symmetric tensors of maximum spin – i.e. \( n \) and \( n+1 \), respectively – one can build from a single momentum vector and the polarisation vector \( s_\mu \). Both operators have twist two. The operator \( O^{(f)}_{\{a_{\mu_1} \ldots a_{\mu_{n+1}}\}} \), which is also traceless but of mixed symmetry, transforms irredicably as well and has spin \( n \) and twist three.

Besides the operators (10–13) used in DIS also point quark operators are of interest to calculate e.g. masses or decay constants:

\[
O_T = \bar{\psi} \Gamma \psi
\]  

with

\[
\Gamma = 1, \, \gamma_\mu, \gamma_5, \, \gamma_\mu \gamma_5, \, \sigma_{\mu\nu}, \, \sigma_{\mu\nu} \gamma_5.
\]  

The matrix elements for the lowest spins are calculable on the theoretical basis of lattice QCD using numerical simulation techniques which allow in principle to define the structure functions as physical observables on one common theoretical basis QCD. A lot of results for the matrix elements has been obtained by our QCDSF–collaboration, part of them are discussed in the talk of G. Schierholz [3] at this workshop.

2 Action and operator improvements

To reduce systematic discretisation errors in realistic lattice calculations to \( O(a^2) \), an improved action, proposed by Sheikholeslami and Wohlert [3] can be used. In their approach a higher dimensional operator is added to the Wilson action \( S_W \) which is restricted by the same symmetries as the original unimproved action (\( a \) is the lattice spacing, \( r \) the Wilson coefficient)

\[
S_{\text{imp}} = S_W + c_{sw} g i \left( a^4 \sum_{x,\mu\nu} \frac{ar}{4} \bar{\psi}(x) \sigma_{\mu\nu} F_{\mu\nu}^{\text{clover}}(x) \psi(x) \right).
\]  

\( F_{\mu\nu}^{\text{clover}} \) denotes the clover leaf form of the lattice field strength, \( \sigma_{\mu\nu} = (i/2)(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \). The constant \( c_{sw} \) which gives the strength of the higher dimensional operator is given in a perturbative expansion as \( c_{sw} = 1 + 0.2659 g^2 \) [4, 5]. The coefficient can be tuned to obtain on–shell improved Green’s functions. The extra action piece adds a vertex contribution to the lattice Feynman rules.

In Euclidean space-time the Lorentz group is replaced by the orthogonal group \( O(4) \), which on the lattice reduces to the hypercubic group \( H(4) \subset O(4) \). Accordingly, the lattice operators are classified by their transformation properties under the hypercubic group and charge conjugation. In ref. [3] we have identified all irreducible representations of the operators \( \mathcal{O} \) and \( \mathcal{O}^5 \) up to rank four. In the (Wick rotated) operators the covariant derivatives are replaced by the lattice covariant derivatives (with the link matrix \( U_{x,\mu} \))

\[
\tilde{D}_{\mu} \psi(x) = \frac{1}{2a} \left( U_{x,\mu} \psi(x + \mu) - U_{x-\mu,\mu} \psi(x - \mu) \right),
\]

\[
\bar{\psi}(x) \tilde{D}_{\mu} = \frac{1}{2a} \left( \bar{\psi}(x + \mu) U_{x,\mu}^\dagger - \bar{\psi}(x - \mu) U_{x-\mu,\mu} \right).
\]  

\[
\langle \bar{p}, \bar{s} | O^{(f)}_{\{a_{\mu_1} \ldots a_{\mu_n}\}} | \bar{p}, \bar{s} \rangle = \frac{1}{n+1} a_n^{(f)} \left( s_\sigma p_{\mu_1} \ldots p_{\mu_n} + \ldots - \text{traces} \right), \tag{17}
\]

\[
\langle \bar{p}, \bar{s} | O^{(f)}_{\{a_{\mu_1} \ldots a_{\mu_{n+1}}\}} | \bar{p}, \bar{s} \rangle = \frac{1}{n+1} d_n^{(f)} \left( s_\sigma p_{\mu_1} p_{\mu_2} \ldots p_{\mu_n} + \ldots - \text{traces} \right). \tag{18}
\]
To be consistent in an improvement program, besides of an improved action, the operators under discussion have to be improved, too. We have constructed the fundamental bases necessary to achieve full $O(a)$ improvement of point and one–link quark operators. This is achieved by adding higher dimensional operators with the same symmetry properties (parity, charge conjugation) as the original unimproved ones. The bases for point and one–link operators are listed in the following:

- **$S = \bar{\psi}\psi$**
  
  $$\left(\bar{\psi}\psi\right)^{\text{imp}} = (1 + a b m) \bar{\psi}\psi - \frac{1}{2} a c_1 \bar{\psi} \not{D} \psi, \quad (23)$$

- **$A_5 = \bar{\psi}\gamma_5\psi$**
  
  $$\left(\bar{\psi}\gamma_5\psi\right)^{\text{imp}} = (1 + a b m) \bar{\psi}\gamma_5\psi + \frac{1}{2} a c_1 \partial_\mu \left(\bar{\psi}\gamma_\mu\gamma_5\psi\right), \quad (24)$$

- **$V_\mu = \bar{\psi}\gamma_\mu\psi$**
  
  $$\left(\bar{\psi}\gamma_\mu\psi\right)^{\text{imp}} = (1 + a b m) \bar{\psi}\gamma_\mu\psi - \frac{1}{2} a c_1 \bar{\psi} \not{D}_\mu \psi + i \frac{1}{2} a c_2 \partial_\lambda \left(\bar{\psi}\sigma_{\mu\lambda}\psi\right), \quad (25)$$

- **$A_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$**
  
  $$\left(\bar{\psi}\gamma_\mu\gamma_5\psi\right)^{\text{imp}} = (1 + a b m) \bar{\psi}\gamma_\mu\gamma_5\psi - i \frac{1}{2} a c_1 \bar{\psi}\sigma_{\mu\lambda}\gamma_5 \not{D}_\lambda \psi + \frac{1}{2} a c_2 \partial_\mu \left(\bar{\psi}\gamma_5\psi\right), \quad (26)$$

- **$t_{\mu\nu} = \bar{\psi}\sigma_{\mu\nu}\psi$**
  
  $$\left(\bar{\psi}\sigma_{\mu\nu}\psi\right)^{\text{imp}} = (1 + a b m) \bar{\psi}\sigma_{\mu\nu}\psi + i \frac{1}{2} a c_1 \epsilon_{\mu\lambda\nu\sigma} \bar{\psi}\gamma_\sigma\gamma_5 \not{D}_\lambda \psi + i \frac{1}{2} a c_2 \left(\partial_\mu \left(\bar{\psi}\gamma_\nu\psi\right) - \partial_\nu \left(\bar{\psi}\gamma_\mu\psi\right)\right), \quad (27)$$

- **$(h_1)_{\mu\nu} = \bar{\psi}\sigma_{\mu\nu}\gamma_5\psi$ for the Drell-Yan process**
  
  $$\left(\bar{\psi}\sigma_{\mu\nu}\gamma_5\psi\right)^{\text{imp}} = (1 + a b m) \bar{\psi}\sigma_{\mu\nu}\gamma_5\psi + i \frac{1}{2} a c_1 \psi \left(\gamma_\mu \not{D}_\nu - \gamma_\nu \not{D}_\mu\right)\gamma_5\psi + i \frac{1}{2} a c_2 \epsilon_{\mu\nu\lambda\tau} \partial_\tau \left(\bar{\psi}\gamma_\lambda\psi\right), \quad (28)$$

- **$O_{\mu\nu} = \bar{\psi}\gamma_\mu \not{D}_\nu \psi$**
  
  $$O_{\mu\nu}^{\text{imp,1}} = (1 + a b m) \bar{\psi}\gamma_\mu \not{D}_\nu \psi - \frac{1}{4} a c_2 \psi \left(\gamma_\mu \not{D}_\nu + \gamma_\nu \not{D}_\mu\right) \psi - a i c_1 g \bar{\psi}\sigma_{\mu\lambda} F_{\nu\lambda}^{\text{clover}} \psi$$

  $$- \frac{1}{4} a c_2 \psi \left(\gamma_\mu \not{D}_\nu + \gamma_\nu \not{D}_\mu\right) \psi + \frac{1}{2} a i c_3 \partial_\lambda \left(\bar{\psi}\sigma_{\mu\lambda} \not{D}_\nu \psi\right), \quad (29)$$

- **$O_{\mu\nu}^{\text{imp,2}} = (1 + a b m) \bar{\psi}\gamma_\mu \not{D}_\nu \psi + \frac{1}{4} a i c_1 \bar{\psi}\sigma_{\mu\lambda} \not{D}_\nu D_\lambda \psi - \frac{1}{4} a c_2 \psi \left(\gamma_\mu \not{D}_\nu + \gamma_\nu \not{D}_\mu\right) \psi + \frac{1}{2} a i c_3 \partial_\lambda \left(\bar{\psi}\sigma_{\mu\lambda} \not{D}_\nu \psi\right), \quad (30)$$

- **$O_{\mu\nu}^5 = \bar{\psi}\gamma_\mu \not{D}_\nu \psi$**
  
  $$O_{\mu\nu}^{5,\text{imp,1}} = (1 + a b m) \bar{\psi}\gamma_\mu \gamma_5 \not{D}_\nu \psi - a i c_1 g \bar{\psi}\gamma_5 F_{\mu\nu}^{\text{clover}} \psi$$

  $$- \frac{1}{4} a i c_2 \bar{\psi}\sigma_{\mu\lambda}\gamma_5 \left(D_\lambda, \not{D}_\nu\right) \psi + \frac{1}{2} a i c_3 \partial_\lambda \left(\bar{\psi}\gamma_5 \not{D}_\nu \psi\right), \quad (31)$$

- **$O_{\mu\nu}^{5,\text{imp,2}} = (1 + a b m) \bar{\psi}\gamma_\mu \gamma_5 \not{D}_\nu \psi - \frac{1}{4} a i c_1 g \bar{\psi}\gamma_5 \left(D_\mu, \not{D}_\nu\right) \psi$$

  $$- \frac{1}{4} a i c_2 \bar{\psi}\sigma_{\mu\lambda}\gamma_5 \left(D_\lambda, \not{D}_\nu\right) \psi + \frac{1}{2} a i c_3 \partial_\lambda \left(\bar{\psi}\gamma_5 \not{D}_\nu \psi\right). \quad (32)$$
Due to $[\vec{D}_\mu, \vec{D}_\nu]^{\text{lattice}} = 4i g F_{\mu\nu}^{\text{clove}} + O(a^2)$ two prescriptions for the improved one–link lattice operators are possible.

Inserting the improved operators into forward matrix elements the surface term $\partial_\mu (\bar{\psi} O \psi)$ (where $O$ is any operator) vanishes due to momentum conservation. Using the equation of motion it is possible for each improved operator to eliminate one base operator. For the coefficients $c_i = 1 + O(g^2)$ the perturbative expansion is not known, $c_i$ and $b$ are different quantities for every operator considered.

3 Renormalisation

3.1 Renormalisation conditions

In order to relate the matrix elements computed on the lattice to continuum matrix elements the so called $Z$ factors have to be calculated. A consistent way would be to do this also non-perturbatively, e.g. on the lattice. Here we present one–loop perturbative calculations (using totally anticommuting $\gamma_5$) which can be used as a first step to control the nonperturbative result. We present the $Z$ factors with coefficients $c_i$ and $c_{sw}$ kept arbitrary. This allows to define the perturbative contributions of the various terms and their relative magnitudes. Moreover, this will allow to implement tadpole improved perturbation theory.

There are several possibilities to determine renormalised quantities which can be compared to data obtained in experiments. We use the projection onto the tree structure (cf. e.g. \cite{8}). The finite quark operators renormalised at finite scale $\mu$ with their multiplicative renormalisation factors are given by the relations

\begin{align}
O(\mu) &= Z_O O(a), \\
\langle q(p)|O(\mu)|q(p)\rangle &= Z_O Z_\psi^{-1} \langle q(p)|O(a)|q(p)\rangle, \\
\langle q(p)|O(\mu)|q(p)\rangle|_{p^2=\mu^2} &= \langle q(p)|O(a)|q(p)\rangle|_{p^2=\mu^2}. \quad (33)
\end{align}

$\langle q(p)|O(a)|q(p)\rangle$ is the amputated Green’s function. $Z_\psi$ is the wave function renormalisation factor determined either via the quark propagator or via the conserved vector current \cite{7}.

\begin{equation}
Z_\psi = \frac{1}{48} \text{tr} \left( \Lambda_{V_\mu}(pa) \right)|_{p^2=\mu^2},
\end{equation}

where $\Lambda_{V_\mu}$ is the amputated Green’s function of the conserved vector current. The definition (33) corresponds to the momentum subtraction scheme.

3.2 About the calculation and program code

The calculation basically amounts to the computation of integrals of the form

\begin{equation}
\mathcal{I}_{\mu_1\ldots\mu_n}(a, p) = \int \frac{d^4 k}{(2\pi)^4} \mathcal{K}_{\mu_1\ldots\mu_n}(a, p, k), \quad (34)
\end{equation}

where $\mathcal{K}$ contains lattice quark and gluon propagators and sin, cos of lattice momenta $p_\mu$ and $k_\mu$, the integration is over the first Brillouin zone $-\pi/a \leq k_\mu < \pi/a$. The calculation of the loop integrals $\mathcal{I}$ is performed in two parts. We decompose (34) \cite{1, 8}

\begin{equation}
\mathcal{I} = \tilde{\mathcal{I}} + (\mathcal{I} - \tilde{\mathcal{I}}), \quad (35)
\end{equation}
where
\[
\tilde{I}(a, p) = \sum_{n=0}^{N} \frac{p_{\alpha_1} \cdots p_{\alpha_n}}{n!} \frac{\partial^n}{\partial p_{\alpha_1} \cdots \partial p_{\alpha_n}} I(a, p)
\]
and the order of the expansion \( N \) is determined by the degree of ultraviolet (UV) divergences of \( I(a, P) \) in the limit \( a \to 0 \). Therefore, \( I - \tilde{I} \) is rendered UV finite and is computed in the continuum. The Taylor expansion of the lattice integral (the first term) will in general create an infrared (IR) divergence. To regularise the integrals dimensional regularisation is used. The IR poles of \( \tilde{I} \) cancel those of \( I - \tilde{I} \). UV divergent contributions (\( \propto 1/a^n \)) of the lattice integrals will cancel in the operator representations which we are interested in.

Let us summarise some of the basic features of the developed program code.

- The program package written in Mathematica.
- Symbolic Feynman rules used as input, the program computes one–loop forward matrix elements of bilinear quark and gluon operators on a hyper-cubic lattice including \( O(a) \) contributions and in the continuum in symbolic form.
- Special features of the program:
  - Dimensional regularisation
  - Symmetrisation tables are used to accelerate the momentum integration over the Brillouin zone
  - General index handling in the complicated case of hyper-cubic H(4) symmetry (non-Lorentz covariant structures)
  - Algebraic isolation of the infrared poles which leads to an exact cancellation of the divergences
  - Results given with general index structure what allows an easy generation of all group representations
  - All finite lattice integrals represented by symbols which are accurately calculated numerically

A part of the results has been checked in a completely independent calculation based on a code in Form.

### 3.3 Quark self energy to order \( O(a) \)

First we discuss the one–loop self energy for quarks which contributes to almost all matrix elements of the operators discussed below. To transform between various renormalisation schemes both lattice and continuum contributions are listed.

The bare fermion propagator is given by
\[
S^{-1} = i \not{p} + m + arp^2/2 - \Sigma^{\text{latt}}
\]
where we present the self energy in the standard form
\[
\Sigma^{\text{latt}} = \frac{g^2}{16\pi^2} C_F \left( i \not{p} \Sigma_1^{\text{latt}} + m \Sigma_2^{\text{latt}} + ar \left( p^2 \Sigma_3^{} + m i \not{p} \Sigma_4^{} + m^2 \Sigma_5^{} \right) + O(a^2) \right)
\]
\[
\Sigma^{\text{cont}} = \frac{g^2}{16\pi^2} C_F \left( i \not{p} \Sigma_1^{\text{cont}} + m \Sigma_2^{\text{cont}} \right)
\]
with $C_F = 4/3$ for SU(3). The bare mass $m$ is defined by

$$ma \equiv \frac{1}{2\kappa} - \frac{1}{2\kappa_c} = \frac{1}{2\kappa} - 4 - \frac{g^2}{16\pi^2}C_F\Sigma_0.$$ (40)

The perturbative calculation is performed by expanding the massive fermion propagator in the mass parameter up to order $\mathcal{O}(m^2)$ for $m^2 \ll p^2$. For $r = 1$ we obtain in covariant gauge

$$\Sigma_0 = -51.4347 + 13.7331 c_{sw} + 5.7151 c_{sw}^2,$$

$$\Sigma_1^{\text{latt}} = +16.6444 - 2.2489 c_{sw} - 1.3973 c_{sw}^2 + \alpha L(ap) - \alpha,$$

$$\Sigma_1^{\text{cont}} = -\alpha K(\epsilon, p/\mu) - \alpha,$$

$$\Sigma_2^{\text{latt}} = +11.0680 - 9.9868 c_{sw} - 0.0169 c_{sw}^2 + (3 + \alpha)L(ap) - 2\alpha,$$

$$\Sigma_2^{\text{cont}} = -(3 + \alpha)K(\epsilon, p/\mu) - 4 - 2\alpha,$$

$$\Sigma_3 = +7.1389 + 0.4857 c_{sw} - 0.0817 c_{sw}^2 - 0.0719\alpha + (-3 + 3 c_{sw} + 2\alpha)\frac{1}{2}L(ap),$$

$$\Sigma_4 = -6.3466 - 1.4850 c_{sw} + 1.2860 c_{sw}^2 + 0.1437\alpha + (-3 - 3 c_{sw} - 2\alpha)\frac{1}{2}L(ap),$$

$$\Sigma_5 = -14.9857 + 16.9857 c_{sw} - 1.5234 c_{sw}^2 + 2.0719\alpha + (-9 + 6 c_{sw} - \alpha)\frac{1}{2}L(ap)$$

with

$$L(x) = \gamma_E - F_0 + \ln x^2, \quad K(\epsilon, x) = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln x^2. \quad (41)$$

$\alpha$ is the gauge parameter ($\alpha = 1(0)$ for Feynman (Landau) gauge), $F_0 = 4.369225$, $\gamma_E = 0.5772\ldots$

### 3.4 Operator renormalisation

The renormalisation factors $Z_O$ in the momentum subtraction scheme are given in the form

$$Z_O(a\mu, g) = 1 - \frac{g^2}{16\pi^2}C_F (\gamma_O \ln(a\mu) + B_O) + O(g^4), \quad (42)$$

where $\gamma_O$ is the anomalous dimension and $B_O$ the finite part of $Z_O$. As can be seen from (33) $B_O$ receives contributions from the one–loop amputated Green’s function and the self energy diagrams (wave function renormalisation)

$$B_O = B_O^{\text{amputated}} + B_O^{\text{self}} \quad (43)$$

with

$$B_O^{\text{self}} = \Sigma_1^{\text{latt}} - \alpha L(ap).$$

Since the Wilson coefficients are usually computed in the MS or $\overline{\text{MS}}$ scheme, one has to present the renormalisation constants in these schemes, too. The transformations between the different schemes are

$$B_O^{\text{MS}} = B_O - B_O^{\text{cont}}, \quad B_O^{\overline{\text{MS}}} = B_O^{\text{MS}} + \frac{\gamma_O}{2}(\gamma_E - \ln 4\pi), \quad (44)$$

where $\gamma_O$ and $B_O^{\text{cont}}$ are given in Table 1.
\[ 4 + \frac{\gamma_0}{2} (\gamma_E - \ln 4\pi) + \alpha \]

\[ 0 \]

\[ \frac{\gamma_0}{2} (\gamma_E - \ln 4\pi) - \alpha \]

\[ -\frac{40}{9} + \frac{\gamma_0}{2} (\gamma_E - \ln 4\pi) + 1 - \alpha \]

Table 1: Anomalous dimensions and finite contributions of continuum integrals to \( B_O \)

### 3.5 Examples for renormalisation of operators \( A_\mu \) and \( O_{\mu\nu}^5 \)

The matrix element of point quark operator \( A_\mu \) to order \( O(g^2) \) up to \( O(a) \) is given by the form

\[ \langle A_\mu \rangle_{g^2} = \frac{g^2}{16\pi^2} C_F \left( \langle A_\mu^{(0)} \rangle + a \langle A_\mu^{(1)} \rangle \right) \]  

As result for the amputated Green’s functions we find

\[ \langle A_\mu^{(0)} \rangle = \gamma_\mu \gamma_5 \left( -0.8481 + 2.4967 c_{sw} - 0.8541 c_{sw}^2 - c_1 (19.3723 - 10.3167 c_{sw} + 0.8846 c_{sw}^2) \right) + \alpha - \alpha L(ap) \]  

\[ + \frac{2 \alpha \not p \gamma_5 P_\mu}{p^2} = \gamma_\mu \gamma_5 \left( B_{\gamma_\mu \gamma_5}^{\text{amputated}} - \alpha L(ap) \right) - 2 \alpha \frac{\not p \gamma_5 P_\mu}{p^2}, \]  

\[ \langle A_\mu^{(1)} \rangle = \frac{i}{2} \left( \not p \gamma_\mu \gamma_5 + \gamma_\mu \gamma_5 \not p \right) \left( 0.6760 + 4.7905 c_1 - 1.7181 c_{sw} + 0.5430 c_1 c_{sw} \right) + 0.1302 c_{sw}^2 + 0.0537 c_1 c_{sw} + \alpha \left( 0.8563 - 4.0583 c_1 \right) + \alpha(1 + c_1) L(ap), \]  

\[ \langle A_\mu^{(0)\text{cont}} \rangle = \gamma_\mu \gamma_5 \alpha \left( 1 + K(\epsilon, p/\mu) \right) - 2 \alpha \frac{\not p \gamma_5 P_\mu}{p^2}. \]

Note that \( O(a) \) terms do not contribute to the tree-level structure, what is valid in general. Therefore, no dangerous \( O(g^2 a) \) or \( O(g^2 a \log a) \) terms are present for massless quarks in \( Z \) factors defined in this momentum subtraction scheme. The finite contribution to the \( Z \) factor in \( \overline{\text{MS}} \) is then found as

\[ B_{\gamma_\mu \gamma_5}^{\text{MS}} = 15.7963 + 0.2478 c_{sw} - 2.2514 c_{sw}^2 - c_1 \left( 19.3723 - 10.3167 c_{sw} + 0.8846 c_{sw}^2 \right). \]

The calculation in the case of link operators including improved action and operators is much more cumbersome. The \( O(a) \) contributions are still not known yet, however, they will also not contribute to the tree level structure. The \( Z \) factors have to be calculated in a special representation. Choosing the representation \( O_{\{14\}}^5 \), only terms \( \propto c_2 \) contribute, and the two prescriptions [12][32] coincide

\[ B_{\gamma_\mu \gamma_5}^{\text{NS}} = -4.0988 - 1.3593 c_{sw} - 1.8926 c_{sw}^2 - c_2 \left( 27.5719 - 16.1193 c_{sw} + 0.7570 c_{sw}^2 \right). \]

### 4 Tadpole improvement

It is known that many results from (naive) lattice perturbation theory are in poor agreement with their numerically determined counterparts. One of the reasons is that tadpoles (lattice artifacts) spoil the expansion. Lepage and Mackenzie [10] proposed a rearrangement in the perturbative series in order to get rid of the numerically large tadpole contributions. These contributions are included in an essentially nonperturbative way using e.g. the measured value of the plaquette.
In lattice theory tadpole corrections renormalise the link operator so that the vev is considerably smaller than one. As a recipe ones scales the link variables with $u_0(g^2)$ measured in Monte Carlo

$$u_0 = \left( \frac{1}{3} \text{Tr} U_{\text{Plaq}} \right)^{\frac{1}{4}}.$$ (51)

This leads to the consequences ($g^*^2$ renormalised at some physical scale) of rescaling the variables

$$g^2 \rightarrow g^*^2 = u_0^{-4} g^2, \quad c_{sw} \rightarrow c_{sw}^* = u_0^3 c_{sw}, \quad c_i \rightarrow c_i^* = u_0^{n_i-n_D} c_i.$$ (52)

Here $n_i$ denotes the number of covariant derivatives in the higher dimensional operator part (proportional to $c_i$) and $n_D$ the number of covariant derivatives (links) of the original operator. In that case one obtains a tadpole improved perturbative expansion of $Z$

$$Z_O \equiv \left( \frac{u_0}{u_0} \right)^{n_D-1} Z_O = u_0^{1-n_D} \left( 1 - \frac{g^*^2}{16 \pi^2} C_F \left( \gamma_O \ln(a\mu) + B_O^* \right) \right) + O(g^*^4) = Z_O^* + O(g^*^4)$$ (53)

with the improved perturbative expansion

$$u_0 \approx 1 - \frac{g^*^2}{16 \pi^2} C_F \pi^2.$$ (54)

which implies

$$B_O^* = B_O(c_{sw}^*, c_i^*) + (n_D - 1) \pi^2.$$ (55)

For the representation $O^5_{\{14\}}$ we get

$$B_O^*^{\text{MS}} = 0.3456 - 1.3593 u_0^3 c_{sw} - 1.8926 u_0^6 c_{sw}^2 - c_2 \left( 27.5719 u_0 - 16.1193 u_0^4 c_{sw} + 0.7570 u_0^7 c_{sw}^2 \right)$$ (56)

To demonstrate the influence of tadpole improvement and addition of higher dimensional operators we show in Fig. 1 the dependence of the critical hopping parameter $\kappa_c$ on $c_{sw}$ and the gauge coupling. The nonperturbative determination of $c_{sw}$ is taken from the ALPHA–collaboration [11]. The tadpole improved perturbation theory describes the Monte Carlo data (crosses, circles) significantly better, however a nonperturbatively determined improvement coefficient $c_{sw}$ is favoured.

In Figs. 2 a and b perturbative predictions in $\overline{\text{MS}}$ scheme without and with tadpole improvement are shown for the renormalisation factor of the operator $O^5_{\{14\}}$ as function of $c_2$ and $g^2$. Note the significant difference in the predictions.
Figure 2: One-loop $Z_{\text{MS}}^F (O_5^{(14)})$ representation) (a) vs. $c_2 \ (g^2 = 1)$ and (b) vs. $g^2 \ (c_2 = 1)$ in naive and tadpole improved perturbation theory, $c_{sw}(g^2)$ is taken from the ALPHA–collaboration

5 Summary

1. We have developed an algebraic computer package to perform one-loop calculations in lattice QCD perturbation theory based on *Mathematica*. Part of the results have been checked by a completely independent code in *Form*.
2. The fundamental bases are constructed which are necessary to remove completely $O(a)$ effects for all bilinear operators up to spin 2.
3. The $Z$ factors are calculated in one-loop for arbitrary coefficients of the counterterms to the operators and to the action.
4. The contributions including $O(a)$ have been calculated for operators without derivatives.
5. It is planned to determine all renormalisation constants nonperturbatively.

Finally, Table 2 gives an overview over the calculated renormalisation factors of lattice bilinear operators. We would like to mention that in [5] and [22] the coefficients $b_{PS,V,A}$ and $c_i$ (for another basis of local point quark operators) have been obtained using the Schrödinger functional approach.

| $O$ | $S_{w}$ | $S_{\text{imp}} \ c_{sw}, c_i = 1$ | $S_{\text{imp}} \ c_{sw}, c_i \neq 1$ | $S_{\text{imp}} \ O(a)$ |
|-----|--------|-------------------------------|-------------------------------|---------------------|
| $\bar{\psi}\psi$ | 12 | 13 | 14 | 15, 16 |
| $\bar{\psi}\gamma_5\psi$ | 12 | 13 | 14 | 15, 16 |
| $\bar{\psi}\gamma_\sigma\psi$ | 12 | 13 | 14 | 15, 16 |
| $\bar{\psi}\gamma_\sigma\gamma_5\psi$ | 12 | 13 | 14 | 15, 16 |
| $\bar{\psi}\sigma_\tau\psi$ | 12 | 13 | 14 | 15, 16 |
| $\bar{\psi}\gamma_\sigma\leftrightarrow D_\mu\psi$ | 17, 18 | 17 | 16 | in preparation |
| $\bar{\psi}\gamma_\sigma\gamma_5\leftrightarrow D_\mu\psi$ | 18 | 17 | 16 | in preparation |
| $\bar{\psi}\gamma_\sigma\leftrightarrow D_\mu D_\nu\psi$ | 18, 19 | 18 | 16 | 16 |
| $\bar{\psi}\gamma_\sigma\gamma_5\leftrightarrow D_\mu D_\nu\psi$ | 18 | 18 | 16 | 16 |
| $\bar{\psi}\sigma_\sigma\leftrightarrow D_\mu D_\nu D_\rho\psi$ | 18 | 18 | 16 | 16 |
| $F_{\mu\nu}F_{\rho\sigma}$ | 21, 17, 21, 14 | | | |

Table 2: Overview on published works on renormalisation factors of lattice bilinear operators.
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