ON DELOCALIZATION OF EIGENVECTORS OF RANDOM NON-HERMITIAN MATRICES

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Abstract. We study delocalization of null vectors and eigenvectors of random matrices with i.i.d entries. Let $A$ be an $n \times n$ random matrix with i.i.d real subgaussian entries of zero mean and unit variance. We show that with probability at least $1 - e^{-\log^2 n}$

$$\min_{I \subset [n], |I| = m} \|v_I\| \geq \frac{m^{3/2}}{n^{3/2} \log C n} \|v\|$$

for any real eigenvector $v$ and any $m \in [\log^C n, n]$, where $v_I$ denotes the restriction of $v$ to $I$.

Further, when the entries of $A$ are complex, with i.i.d real and imaginary parts, we show that with probability at least $1 - e^{-\log^2 n}$ all eigenvectors of $A$ are delocalized in the sense that

$$\min_{I \subset [n], |I| = m} \|v_I\| \geq \frac{m}{n \log C n} \|v\|$$

for all $m \in [\log^C n, n]$.

Comparing with related results, in the range $m \in [\log^C n, n/ \log C n]$ in the i.i.d setting and with weaker probability estimates, our lower bounds on $\|v_I\|$ strengthen an earlier estimate $\min_{|I|=m} \|v_I\| \geq c(m/n)^6 \|v\|$ obtained in [M. Rudelson, R. Vershynin, Geom. Func. Anal., 2016], and bounds $\min_{|I|=m} \|v_I\| \geq c(m/n)^2 \|v\|$ (in the real setting) and $\min_{|I|=m} \|v_I\| \geq c(m/n)^{3/2} \|v\|$ (in the complex setting) established in [K. Luh, S. O’Rourke, arXiv:1810.00489].

As the case of real and complex Gaussian matrices shows, our bounds are optimal up to the polylogarithmic multiples. We derive stronger estimates without the polylogarithmic error multiples for null vectors of real $(n-1) \times n$ random matrices.

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1. Introduction

We say that a random variable $\xi$ is subgaussian if $(E|\xi|^p)^{1/p} \leq K \sqrt{p}$ for all $p \geq 1$ and some fixed $K > 0$. For any real or complex vector $X = (x_1, x_2, \ldots, x_n)$, we define its non-increasing

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Delocalization of eigenvectors of random matrices has been actively studied, especially in the setting of Wigner (and generalized Wigner) matrices. The term delocalization usually refers to sup-norm ($\ell_\infty$–norm) delocalization or, more generally, to upper bounds for inner products $|\langle Y, v \rangle|$ for any fixed unit vector $Y$ and normalized eigenvectors $v$. With the $\ell_\infty$–delocalization, no single coordinate of the eigenvector carries a significant mass. In comparison, the no-gaps delocalization is the property of having few small coordinates, so that every subset of vector components carries a non-negligible mass. Both notions provide a way to measure how close the distribution of an eigenvector is to the uniform distribution on the sphere. A related concept of quantum unique ergodicity provides another viewpoint to this phenomenon. Among many others, we refer to papers [6, 7, 8, 17, 18, 24] for eigenvectors of Wigner and band random matrices, as well as to surveys [34, 35, 5] for overview of some of existing results on delocalization for various models of randomness and for further references.

No-gaps delocalization has found applications in both Hermitian and non-Hermitian settings. The Braess paradox on random networks asserts that under certain assumptions removing an edge in the random network can decrease the traffic congestion (see, in particular, [54, 11]). A closely related phenomenon is a decrease in the spectral gap of the normalized Laplacian of a random graph when adding a new edge (see [15, 38]). The probability of the decrease can be bounded using no-gaps delocalization estimates for the second eigenvector of the Laplacian [15, 38]. In particular, the no-gaps delocalization estimates obtained in [46] allowed to strengthen the main result of [15] and to show that randomly adding a new edge to a graph decreases the spectral gap with probability at least $1/2 - O(n^{-c})$ [35]. Further, in the non-symmetric setting, a weak form of no-gaps delocalization for unit normals to linear spans of columns of $d$–regular random matrices has been recently used in [28, 29] to establish the circular law for the limiting spectral distribution of sparse random $d$–regular directed graphs (see also [12, 2]).

In this paper, we consider non-Hermitian random matrices. Before stating the main theorems, let us discuss several existing results on delocalization of eigenvectors of such matrices, including no-gaps and sup-norm delocalization.

In the non-Hermitian setting, the sup-norm delocalization for the eigenvectors was studied, in particular, in [45], where it was shown that for an $n \times n$ random matrix $A$ with i.i.d subgaussian entries, with probability close to one every unit eigenvector $v$ of $A$ satisfies $\|v\|_\infty \leq C \log^9 n/n$. Further, in [33] the bound $\|v\|_\infty \leq C \sqrt{\log n/n}$ w.h.p. was derived for unit eigenvectors of $A$ corresponding to eigenvalues of small absolute value (of order $O(1)$). The primary object of interest in [33] was a unit vector $v$ in the kernel of an $(n-1) \times n$ random matrix $B$ with i.i.d subgaussian entries. It was shown that $\mathbb{P}\{|\|v\|_\infty \geq t/\sqrt{n}\| \leq e^{-ct^2}$ for all $t \geq C \sqrt{\log n}$, with $C, c > 0$ depending only on the subgaussian moment. It is not difficult to see, by considering the Gaussian matrix, that, disregarding the values of $C, c$, this latter result is optimal.

No-gaps delocalization was considered in [40] in a very general setting, including symmetric, skew-symmetric matrices and matrices with i.i.d entries. In particular, the main result of [40] implies that, given an $n \times n$ random matrix $A$ with i.i.d $K$–subgaussian entries of zero mean and
unit absolute second moment, with probability at least 1 − e−c(n−i), for every unit eigenvector \( \mathbf{v} \) of \( A \) and every \( i \in [n/2, n − Cn^{−7/2}] \), one has \( v_i^* \geq c(n − i)^{1/2}/n^5 \). The results of [46] easily imply no-gaps delocalization of null vectors of an \((n−1) \times n\) matrix \( B \) with i.i.d entries. Recently, developing the approach from [46], in work [32] a stronger bound for the order statistics was established in the non-Hermitian case; namely, it was shown in [32] that \( v_i^* \geq c(n − i)^{3/2}/n^2 \) (in the real case) and \( v_i^* \geq c(n − i)/n^{3/2} \) (in the complex case) for every \( i \in [1, n − \log^2 n] \) and every unit eigenvector of \( A \) with high probability. Moreover, the authors of [32] obtained lower bounds for \( v_i^* \) in the regime \( i \in [n − \log^2 n, n] \), i.e. for smallest order statistics. We note here that a variant of the no-gaps delocalization was considered in [33] where the magnitude of the largest/smallest singular values of respective matrices. This immediately implies that a variant of the no-gaps delocalization was considered in [33] where the magnitude of the smallest component of the null unit vector was estimated.

If \( G \) is an \( n \times n \) Gaussian matrix with i.i.d real standard normal entries, strong estimates for components of the non-increasing rearrangements of the real eigenvectors are immediately implied by the orthogonal invariance of the matrix distribution. Indeed, conditioned on the event that \( G \) has at least \( j \) real eigenvalues, the unit eigenvector \( \mathbf{v} \) corresponding to the \( j \)-th largest real eigenvalue is uniformly distributed on the unit sphere \( S^{n−1}(\mathbb{R}) \). In particular, with probability close to one all unit real eigenvectors \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) of \( G \) satisfy \( c(n − i)/n^{3/2} \leq v_i^* \leq C(n − i)/n^{3/2} \) for all \( n/2 \leq i < n − C \log n \), for some universal constants \( C, c > 0 \) (see e.g. [13]). A similar argument for a complex \( n \times n \) matrix with i.i.d standard complex Gaussian entries implies that with probability close to one every unit eigenvector \( \mathbf{v} \) of such matrix satisfies \( c\sqrt{n−i}/n \leq v_i^* \leq C\sqrt{n−i}/n \) for all \( n/2 \leq i < n − C \log n \).

Having in mind the Gaussian case, it is natural to suggest that for an \( n \times n \) matrix \( A \) with i.i.d subgaussian entries of zero mean and unit second absolute moment,

- In the real case, all real unit eigenvectors \( \mathbf{v} \) of \( A \) should satisfy
  \[
  v_i^* \geq \frac{c(n − i)}{n^{3/2}}, \quad i \in [n/2, n − C \log n]
  \]
  with probability close to one;

- In the complex case, all unit eigenvectors \( \mathbf{v} \) of \( A \) should satisfy
  \[
  v_i^* \geq \frac{c\sqrt{n−i}}{n}, \quad i \in [n/2, n − C \log n]
  \]
  with high probability.

Thus, the estimates in [46] [32] cited above are suboptimal in the considered range of \( i \). Moreover, without significant new ingredients, the approach of [46] [32] cannot produce the sharp bounds in that range for reasons explained further.

Consider a simpler setting of no-gaps delocalization estimates for a unit null vector \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) of \((n−1) \times n\) random matrix \( B \) with i.i.d real subgaussian entries of zero mean and unit variance. Given \( i \in [n/2, n − 1] \) and any \((n−i)\)-element subset \( I \subset [n] \), one has

\[
s_{\max}(B_I) \left\| \sum_{j \in I} u_j \mathbf{e}_j \right\|_2 \geq \left\| \sum_{j \in I} u_j \text{Col}_j(B) \right\|_2 = \left\| \sum_{j \in I^c} u_j \text{Col}_j(B) \right\|_2 \geq s_{\min}(B_{I^c}) \left\| \sum_{j \in I^c} u_j \mathbf{e}_j \right\|_2,
\]

where \( B_J \) is the \((n−1) \times |J|\) matrix with columns \( \text{Col}_j(B), j \in J \), and \( s_{\max}, s_{\min} \) denote the largest/smallest singular values of respective matrices. This immediately implies

\[
\max_{I \subset [n], |I| = n-i} s_{\max}(B_I) \sqrt{n−i} u_{i+1}^* \geq c \min_{I \subset [n], |I| = n-i} s_{\min}(B_{I^c}),
\]
whence for any $t > 0$, we have
\[
\mathbb{P}\{ u_{i+1}^* \geq t \} \geq 1 - \mathbb{P}\left\{ \max_{I \subset [n], |I| = n - i} s_{\max}(B_{1^c}) \leq \frac{t}{2} \min_{I \subset [n], |I| = n - i} s_{\min}(B_{1^c}) \sqrt{n - i} \right\}
\geq 1 - \sum_{I \subset [n], |I| = n - i} \mathbb{P}\{ c s_{\min}(B_{1^c}) \leq t s_{\max}(B) \sqrt{n - i} \}
= 1 - \left( \sum_{n-i} \mathbb{P}\{ c s_{\min}(B_{[i]}) \leq t s_{\max}(B) \sqrt{n - i} \} \right)
\]
where $[i] = \{1, 2, \ldots, i\}$. The relation (1) is the starting point of the argument of [46] (and also [32]) and is used to derive the main results of those papers (see [46] Proposition 4.1). But, while $s_{\max}(B) = O(\sqrt{n})$ with probability close to one, the small ball probabilities for $s_{\min}(B_{[i]})$ are too weak to derive the optimal bound for $u_{i+1}^*$ from the last relation. Indeed, it can be verified that when $B$ is Gaussian, we have for all $i$ in $[n/2, n - 1]$ and $\tau > 0$: $\mathbb{P}\{ s_{\min}(B_{[i]}) \leq \tau(n - i)/\sqrt{n} \} \geq (\tau n - i)^{n-i}$. But this implies that the probability bound in (1) is non-trivial only when $t \leq c'(n - i)^{3/2}/n^2$, i.e. (1) can only be used to show that $u_{i+1}^* \geq c'(n - i)^{3/2}/n^2$ with high probability, which falls short of the optimal lower bound $c'(n - i)/n^{3/2}$. In a similar way, it can be argued that lower bounds for order statistics of eigenvectors of square random matrices produced using this approach, are suboptimal.

An alternative approach to delocalization is considered in [33]. Let, as before, $u$ be a null unit vector of an $(n - 1) \times n$ random matrix $B$ with i.i.d entries of zero mean and unit absolute second moment. In [33], it is shown that there is a coupling of $u$ and a random vector $w = (w_1, \ldots, w_n)$ uniformly distributed on the unit sphere, such that with probability at least $1 - n^{-c}$, $|u_i - w_i^*| \leq n^{-1/2}c'$, where $i$ is in the range $[n^{-c}, n - n^{-c}]$ for a small constant $c > 0$ (see formula (36) in [33]). The proof of the result is based on the Berry–Esseen theorem for frames used earlier in [48]. This result of [33], provides much sharper estimates for $u_i^*$ compared to [46], though with much weaker probability bounds. However, it is not clear how the approach can be adapted to studying delocalization of eigenvectors rather than null vectors, and how to extend the range $[n^{-c}, n - n^{-c}]$ for the components of the non-increasing rearrangement for which the estimates are available. We also note that a development of this approach to delocalization is considered in [32], where sharp asymptotics for $\min_{I \subset [n], |I| = \delta n} \| \mathbf{v}_I \|$ and $\max_{I \subset [n], |I| = \delta n} \| \mathbf{v}_I \|$ is derived (for any constant $\delta$) for the null unit vectors.

In this paper, we follow a geometric approach to random matrices in our study of the eigenvectors. Our main motivation is to establish the optimal no-gaps delocalization estimates for real and complex non-Hermitian random matrices, as well as for null vectors of real rectangular matrices. The treatment of null vectors is significantly simpler, and we start by discussing the corresponding result first. The following theorem is a simplified version of Theorem 4.3 in the text.

**Theorem A.** For any $K \geq 1$ there are $C, c > 0$ depending only on $K$ with the following property. Let $n \geq C$, let $B$ be an $(n - 1) \times n$ random matrix with i.i.d $K$-subgaussian real entries with zero mean and unit variance. Further, let $u = (u_1, \ldots, u_n)$ be a random unit vector in ker($B$). Then for any $i \in [n - cn, n - C \log n]$ and $t \geq e^{-c(n-i)}$ we have
\[
\mathbb{P}\{ u_{i+1}^* \leq \frac{(n-i)t}{n^{3/2}} \} \leq (Ct)^{n-i} e^{-cn}.
\]

Note that a unit random vector $w$ uniformly distributed on the sphere $S^{n-1}(\mathbb{R})$ (i.e. the null vector of a standard $(n - 1) \times n$ Gaussian real matrix), satisfies two-sided estimates $(ct)^{n-i} \leq \mathbb{P}\{ w_{i+1}^* \leq \frac{(n-i)t}{n^{3/2}} \} \leq (Ct)^{n-i}$ for all $t \in (0, 1]$ and $i \geq n/2$. Thus, our result recovers both the correct order of magnitude for the $(i+1)$–st largest component of the null vector of $B$ and
Theorem B can be restated as follows: with probability at least 1 the optimal probability estimates (in a restricted range for $t$). Moreover, considering discrete distributions with atoms, one can easily see that the additive term $e^{-cn}$ in our estimate is in general not removable; for example, the $(n - 1) \times n$ Bernoulli matrix admits null vector $(1, 1, 0, 0, \ldots, 0)$ with probability $2^{1-n}$.

The generalized version of the above theorem — Theorem 4.3 — deals with almost null vectors. As simple corollaries of Theorem 4.3 we obtain no-gaps delocalization bounds on eigenvectors corresponding to small real eigenvalues and on singular vectors corresponding to small singular values of square random matrices; see Corollaries 4.4 and 4.5.

Let us remark on the approaches to delocalization used earlier in [33, 35]. As we noted before, [33] further develops an argument from [48] based on the Berry-esseen theorem for frames. This gives sharp estimates for the components of the null vector but in a limited range for the index $i$ and with a relatively weak probability. Another argument in [33], dealing with the sup-norm delocalization and estimates for the smallest component of the null vector, is based on using test projections — projections onto orthogonal complements to spans of some matrix columns. Test projections were employed earlier in [48] and [45] in a related context. Our proof of Theorem A uses test projections as a basic tool. Additionally, we employ an averaging procedure which, in a different form, was used earlier in [33], as well as in [33]. The third ingredient of the proof of Theorem A is a small ball probability estimate for the smallest singular value of certain auxiliary matrices. Those estimates are given in Section 3.

In the second part of the paper, we consider eigenvectors of non-Hermitian square random matrices. The main results of this paper are the following theorems corresponding to real and complex settings.

**Theorem B.** Let $n > 2$, let $A$ be an $n \times n$ random matrix with i.i.d $K$-subgaussian real entries with zero mean and unit variance. Then with probability at least $1 - e^{-\log^2 n}$ we have

$$v_i^* \geq \frac{n - i}{n^{3/2} \log C n} \quad \text{for every real unit eigenvector } v = (v_1, v_2, \ldots, v_n)$$

for every $i \in [n/2, n - \log C n]$, where $C > 0$ may only depend on $K$.

**Theorem C.** Let $n > 2$, let $A$ be an $n \times n$ random matrix with i.i.d $K$-subgaussian complex entries with zero mean, unit second absolute moment and i.i.d real and imaginary parts. Then with probability at least $1 - e^{-\log^2 n}$ we have

$$v_i^* \geq \frac{\sqrt{n - i}}{n \log C n} \quad \text{for every unit eigenvector } v = (v_1, v_2, \ldots, v_n)$$

for every $i \in [n/2, n - \log C n]$, where $C > 0$ may only depend on $K$.

In view of the above discussion, the estimates are optimal up to the polylogarithmic multiple $\log C n$. The probability estimate can be strengthened to $1 - e^{-\log^2 C' n}$ for any constant $C' > 2$, at expense of increasing the constant $C$ in the theorems (we would like to note here that the weaker lower bounds on $v_i^*$ obtained in [16, 32] hold with higher probability than in our paper). Theorem B can be restated as follows: with probability at least $1 - e^{-\log^2 n}$ any real eigenvector satisfies

$$\min_{I \subset [n], |I| = m} \|v_I\|_2 \geq \frac{m^{3/2}}{n^{3/2} \log C n} \|v\|_2 \quad \text{for every } m \geq \log C n,$$

where $v_I = (v_i)_{i \in I}$ denotes the restriction of $v$ to $I$. Similarly, Theorem C implies that with probability $1 - e^{-\log^2 n}$ we have

$$\min_{I \subset [n], |I| = m} \|v_I\|_2 \geq \frac{m}{n \log C n} \|v\|_2 \quad \text{for every } m \geq \log C n.$$
Note that Theorem B provides delocalization bounds for real eigenvectors only. Existence of real eigenvalues, hence real eigenvectors, of random real non-symmetric matrices with i.i.d entries was established in [45] under the assumption that the first four moments of the matrix entries match those of the standard Gaussian variable; statistics of real eigenvalues of Gaussian matrices were studied earlier in [14]. Delocalization of complex eigenvectors of real non-symmetric matrices is briefly discussed at the end of Subsection 5.4 of our paper.

The study of eigenvectors of random matrices with help of test projections, compared to the null vectors, is significantly more involved. The main difficulty consists in estimating the magnitude of projections of the coordinate vectors \( -e_j \) which arise when considering matrices \( A - z \). A strategy for estimating the projections was proposed earlier in [45], and was based on studying certain biorthogonal systems of random vectors and a special choice of the kernel of the projection operator. While some elements of that argument turn out extremely useful in our context, certain aspects of the proof, in particular, selection of the projection operator. While some elements of that argument turn out extremely useful in our context, certain aspects of the proof, in particular, selection of the spectral window [45], Section 5], do not seem applicable. In this paper, we develop a geometric approach based on studying dimensions of the ellipsoids generated by projected columns of the matrix \( A - z \), as well as the dual ellipsoid. We conclude the introduction with a brief description of the method.

For any two parameters \( \beta > \theta > 0 \), the condition

\[
A \text{ has a unit eigenvector } v = (v_1, \ldots, v_n) \text{ with } |v_i| \geq \beta \text{ and } |v_{j_1}|, |v_{j_2}|, \ldots, |v_{j_k}| \leq \theta
\]
deterministically implies that orthogonal projection \( P_F(\text{Col}_i(A - z)) \) (where \( z \) is the eigenvalue corresponding to \( v \)) is contained in the ellipsoid

\[
\left\{ \sum_{\ell=1}^k a_\ell P_F(\text{Col}_{j_\ell}(A - z)) : \left\| (a_1, \ldots, a_k) \right\|_2 \leq \frac{\theta \sqrt{k}}{\beta} \right\},
\]

where \( F \) is the orthogonal complement to the linear span of columns \( \text{Col}_q(A - z) \), \( q \in [n] \setminus \{i; j_1, j_2, \ldots, j_k\} \). This elementary observation reduces the no-gaps delocalization estimates to bounding probabilities of events of the form \( \{ X \in E \} \), where \( X \) is an appropriate random vector and \( E \) is an appropriate random ellipsoid. The actual reduction procedure that we use replaces no-gaps delocalization with two conditions of that form:

(a) \( P_F(-\theta e_i) \in \left\{ \sum_{\ell=1}^k a_\ell P_F(-\theta e_{j_\ell}) : \left\| a_\ell \right\|_2 \leq \frac{\theta \sqrt{k}}{\beta} \right\} + B \) and;

(b) \( P_F(\text{Col}_i(A - z)) \in \left\{ \sum_{\ell=1}^k a_\ell P_F(-\theta e_{j_\ell}) : \left\| a_\ell \right\|_2 \leq \frac{\theta \sqrt{k}}{\beta} \right\} + B' \),

where \( B, B' \) are appropriate dilations of the Euclidean ball in \( F \) (see Lemma 5.1). The two conditions are treated using different techniques. The first one can be restated in terms of the dual basis for \( P_F(-\theta e_i), P_F(-\theta e_{j_\ell}), \ell = 1, 2, \ldots, k \) in \( F \) as a statement about magnitudes of inner products of the dual basis vectors with a specially chosen vector in \( F \) (see Lemma 5.1), giving a “raw” deterministic statement. That relation is one of basic elements of the proof and essentially taken from [45], although not stated there explicitly. Bounding probability of (a) is then reduced to estimating probability of the event of the form

\[
\{ \exists Y \in F : \left\| Y \right\|_2 \leq T, \left\langle Y, Y_i \right\rangle = 1 \text{ and } \left| \left\langle Y, Y_{j_1} \right\rangle \right| \leq \delta \},
\]

where \( Y_i, Y_{j_1}, \ldots, Y_{j_k} \) is the dual basis for \( P_F(-\theta e_i), P_F(-\theta e_{j_1}), \ldots, P_F(-\theta e_{j_k}) \), and \( T, \delta > 0 \) are appropriate parameters. In fact, for technical reasons, we work with perturbations of the dual basis, but here we consider a simplified scheme. Estimating probability of the last event is challenging, first, because the random vectors \( Y_i, Y_{j_1}, \ldots, Y_{j_k} \) do not have to be isotropic (i.e. the covariance structure may be not a multiple of identity) and, second, since existence of a vector \( Y \) satisfying the conditions is not easy to restate in terms of anti-concentration properties of \( Y_i, Y_{j_1}'s \). The first difficulty — treatment of anisotropic random vectors — was addressed in [45], and we reuse some of the estimates from [45], while adding new relations.
(see Subsection 5.3). The second difficulty — a reduction of the condition for $X, Y_{j_k}$ and $Y$ to anti-concentration estimates for $X, Y_{j_k}$'s — is resolved by constructing a special discretization of the set of admissible vectors $Y$, and taking the union bound over the set. The principal issue in this part of the proof is to find a decoupling that would allow to resolve probabilistic dependencies between the admissible vectors $Y$ and vectors $X, Y_{j_k}$'s (otherwise, applying small ball probability estimates for $|\langle Y_{j_k}, Y \rangle|$ would be impossible in a direct way). This is the central part of our argument (see Lemmas 5.14 and 5.19).

To bound the (conditional) probability of the event (b), given a realization of the ellipsoid $E = \{ \sum_{k=1}^{n} a_k P_F(-z e_{j_k}) : \|a\|_2 \leq \frac{\sqrt{k}}{n} \}$, we utilize independence of $\text{Col}_i(A)$ from $F$ and $E$. A straightforward argument estimating the probability in terms of the volume of $E$ turns out too rough for our purposes. The volume of $E$ may be large because of few long semi-axes, while the probability of $P_F(\text{Col}_i(A - z)) \in E + B'$ is still small. In our estimate, we use that $P_F(\text{Col}_i(A))$ has relatively small Euclidean norm (of order $\sqrt{k}$ or slightly greater) with large probability, and replace the ellipsoid $E$ with its “truncation” obtained by intersecting with a Euclidean ball centered at $P_F(-z e_i)$ (see Lemma 5.21).

Our proof uses several discretizations — of the family of ellipsoids; of basic sequences; of vectors in a given linear subspace (see Definitions 5.3, 5.8, 5.13 of classes $C(R, b)$, $D(r, p)$, and $T(W_N, \delta, T)$). Some other aspects of the proof not mentioned here are discussed at the beginning of Section 5.

2. Preliminaries

Let us start by introducing notation. Given a finite set $I$, we denote by $|I|$ its cardinality. For a positive integer $k$, we write $[k]$ for the set $\{1, 2, \ldots, k\}$. By $1_A$ we denote the indicator of an event or a subset $A$. Everywhere in the text, $\Theta$ denotes either the field of real numbers $\mathbb{R}$, or complex numbers $\mathbb{C}$. For a complex number $z$, we denote by $\bar{z}$ the conjugate of $z$; $i$ stands for imaginary unit. The canonical inner product in $\Theta^n$ is denoted by $\langle \cdot, \cdot \rangle$, and the standard Euclidean norm in $\Theta^n$ — by $\| \cdot \|_2$. We denote the sup-norm (or $\ell_\infty$-norm) by $\| \cdot \|_\infty$. For an $m \times n$ matrix $B$, its columns are denoted by $\text{Col}_i(B) \in \Theta^m$, $i \in [n]$. When the matrix is clear from the context, we simply write $\text{Col}_i$ instead of $\text{Col}_i(B)$. The spectral norm of $B$ is denoted by $\|B\|_\text{HS}$, and its Hilbert–Schmidt norm — by $\|B\|_{\text{HS}}$. The conjugate transpose of a matrix $B$ will be denoted by $B^*$. For a linear subspace $E$ of $\Theta^k$, we write $E^\perp$ for its orthogonal complement. We will write $P_E : \Theta^k \to \Theta^k$ for the orthogonal projection operator onto $E$. Given two subsets $S_1, S_2$ of $\Theta^n$, we write $\text{dist}(S_1, S_2)$ for the Euclidean distance between $S_1$ and $S_2$. Further, we write $S_1 + S_2$ for the Minkowski sum of $S_1$ and $S_2$ defined as $S_1 + S_2 := \{x + y : x \in S_1, y \in S_2\}$. Let $B^k_2(\Theta)$ (resp., $S^{k-1}(\Theta)$) be the unit Euclidean ball (resp., Euclidean sphere) in $\Theta^k$. We also write $B^E_2 := B_2 \cap E$ for the unit Euclidean ball in a linear subspace $E$.

Given a real random variable $\xi$, let $\mathcal{L}(\xi, \cdot)$ be its Lévy concentration function defined by

$$\mathcal{L}(\xi, t) := \sup_{a \in \Theta} \mathbb{P}\{|\xi - a| \leq t\}, \quad t > 0.$$ 

More generally, for a random vector $X$ in $\Theta^m$ let

$$\mathcal{L}(X, t) := \sup_{W \in \Theta^m} \mathbb{P}\{|X - W|_2 \leq t\}, \quad t > 0.$$ 

Following [46] for any complex vector $W \in \mathbb{C}^m$ we define $\text{real}(W)$ as a vector in $\mathbb{R}^{2m}$ of the form $\mathbb{R}(W) \oplus \mathbb{I}(W)$, where $\mathbb{R}(W)$ and $\mathbb{I}(W)$ is the real and imaginary part of $W$, respectively, and $\oplus$ denotes vector concatenation. Further, for any complex subspace $E \subset \mathbb{C}^m$ let $\text{real}(E) \subset \mathbb{R}^{2m}$ be the subspace defined as $\text{real}(E) := \{\text{real}(W) : W \in E\}$. The following relation, taken from [46], holds:

$$\text{P}_{\text{real}(E)}(\text{real}(Y)) = \text{P}_{\text{real}(E)}(\text{real}(Y))$$

for any complex subspace $E \subset \mathbb{C}^m$, $Y \in \mathbb{C}^m$. 

The next lemma will be useful.

**Lemma 2.1.** Let \( E \) be a \( k \)-dimensional ellipsoid in \( \mathbb{C}^n \) defined as
\[
E := \left\{ \sum_{i=1}^{k} a_i X_i : \| (a_1, \ldots, a_k) \|_2 \leq 1 \right\},
\]
where \( \{X_1, X_2, \ldots, X_k\} \) is a collection of pairwise orthogonal vectors in \( \mathbb{C}^n \). Then \( \text{real}(E) \) is a \( 2k \)-dimensional ellipsoid in \( \mathbb{R}^{2n} \) with semi-axes \( \text{real}(X_1), \text{real}(iX_1), \ldots, \text{real}(X_k), \text{real}(iX_k) \).

Given a random vector \( X \) in \( \Theta^m \), we say that \( X \) is isotropic if any one-dimensional projection \( \langle X, w \rangle \) (\( w \in S^{m-1}(\Theta) \)) has zero mean and \( E(\langle X, w \rangle)^2 = 1 \). In particular, a complex vector with independent coordinates of zero mean and unit absolute second moment (with i.i.d real and imaginary parts) is isotropic. A variable \( \xi \) in \( \Theta \) is \( K \)-subgaussian if \( (E|\xi|^p)^{1/p} \leq K \sqrt[p]{p} \) for all \( p \geq 1 \). The definition of a \( K \)-subgaussian variable admits several equivalent formulations; see, for example, [55, Section 5.2.3].

Throughout the text, we deal with two models of random vectors \( X \):

- \((\star)\) \( X \) is real isotropic with i.i.d. \( K \)-subgaussian coordinates, if \( \Theta = \mathbb{R} \), and
- \((\star\star)\) \( X \) is complex isotropic with i.i.d. \( K \)-subgaussian coordinates,
  
each coordinate with i.i.d. real and imaginary parts, if \( \Theta = \mathbb{C} \).

Following [40] [42], for any \( \delta, \rho > 0 \), we define the set of incompressible unit vectors
\[
\text{Incomp}_n(\delta, \rho) := \left\{ X \in S^{n-1}(\Theta) : \text{dist}(X, \{ Y \in \Theta^n : |\text{supp} Y| \leq \delta n \}) > \rho \right\}.
\]
The incompressible vectors naturally appear in kernels of “almost square” random matrices with independent entries. The next standard lemma can be verified with help of an \( \varepsilon \)-net argument (see [30] [40] [42]) and elementary properties of incompressible vectors [40] Lemma 3.4):

**Lemma 2.2.** For any \( M \geq 1 \) and \( K \geq 1 \) there are \( \zeta > 0, c > 0 \) depending only on \( M \) and \( K \) with the following property. Let \( m \leq n \leq 2m \) and let \( W \) be an \( m \times n \) random matrix with i.i.d. rows satisfying \((\dagger)\) or \((\dagger\dagger)\). Further, let \( M \in \Theta^{n \times n} \) be any fixed matrix of spectral norm at most \( M \sqrt{n} \). Then with probability at least \( 1 - 2e^{-cm} \)
\[
\| (W + M)Y \|_2 \geq c\sqrt{n} \quad \text{for any} \quad Y \in S^{n-1}(\Theta) \setminus \text{Incomp}_n(\zeta, \zeta),
\]
and, in particular,
\[
\| (W + M)Y \|_2 \geq c\sqrt{n} \quad \text{for any} \quad Y \in S^{n-1}(\Theta) \quad \text{with} \quad \left| \left\{ i \leq n : |Y_i| \geq \zeta / \sqrt{n} \right\} \right| < \zeta n.
\]

The next notion and two theorems are taken from [40] [42]. Given a real vector \( X \in \mathbb{R}^n \), and parameters \( \alpha > 0 \) and \( \gamma \in (0,1) \), the least common denominator (LCD) of \( X \) is given by
\[
\text{LCD}_{\alpha,\gamma}(X) := \inf \{ \theta > 0 : \text{dist}(\theta X, \mathbb{Z}^n) < \min(\gamma \|\theta X\|_2, \alpha) \}.
\]
Moreover, if \( E \) is a linear subspace of \( \mathbb{R}^n \), then the LCD of \( E \) is defined as follows:
\[
\text{LCD}_{\alpha,\gamma}(E) := \inf \{ \text{LCD}_{\alpha,\gamma}(X) : X \in E \cap S^{n-1}(\mathbb{R}) \}
\]
\[
= \inf \{ \|Y\|_2 : Y \in E, \text{dist}(Y, \mathbb{Z}^n) < \min(\gamma \|\|Y\|_2, \alpha) \}.
\]

**Theorem 2.3** (Distance to a subspace, [42] Theorem 4.2). Let \( X \) be a random vector in \( \mathbb{R}^n \) satisfying \((\dagger)\) with a parameter \( K \). Let \( E \) be a subset of \( \mathbb{R}^n \) with \( m := \text{codim} E < n \). Then for every \( Y \in \mathbb{R}^n \), any \( \alpha > 0 \), \( \gamma \in (0,1) \) and for all \( t \geq m/\text{LCD}_{\alpha,\gamma}(E^\perp) \), we have
\[
\mathbb{P}\{\text{dist}(X, E + Y) \leq t\} \leq \left( \frac{Ct}{\gamma \sqrt{m}} \right)^m + C^m e^{-c\alpha^2},
\]
where \( c, C > 0 \) depend only on \( K \).
Theorem 2.4 (Structure of a random subspace. [42, Theorem 4.3]). For any $K, M \geq 1$ there exist $C_{[42]} > 0$ depending only on $K$ and $M$ with the following property. Let $n \geq C_{[42]}$ $n > m \geq n - C_{[42]}$. Let $X_1, \ldots, X_m$ be mutually independent random vectors in $\mathbb{R}^n$ satisfying $F$ with parameter $K$. Further, let $y_1, y_2, \ldots, y_m$ be fixed vectors in $\mathbb{R}^n$ such that the spectral norm of the $n \times m$ matrix with columns $y_1, \ldots, y_m$ is bounded above by $M\sqrt{n}$. Then for $E := \text{span} \{X_1 + y_1, X_2 + y_2, \ldots, X_m + y_m\}$ we have

$$P\{\text{LCD}_{\mathbb{R}^m}(E) \leq \sqrt{n}e^{\frac{2}{3}m/(n-m)}\} \leq e^{-\frac{2}{3}m}.$$

The next result is essentially proved for centered matrices in [40]; see [40, Lemma 5.8] and [42, Lemma 4.8]. For non-centered matrices, the statement follows by a straightforward modification of the argument.

Theorem 2.5. For any $K, M > 0$ there are $C_{[42]}, C_{[42]} > 0$ depending only on $K, M$ with the following property. Let $m \geq C_{[42]}$ and let $B$ be an $m \times m$ random matrix with i.i.d $K$-subgaussian real entries of zero mean and unit variance. Further, let $M$ be any fixed $m \times m$ real matrix with $\|M\| \leq M\sqrt{m}$. Then with probability at least $1 - e^{-\frac{2}{3}m}$ we have

$$\|(B + M)Y\|_2 \geq \frac{\max(\sqrt{m}, \text{LCD}_{\mathbb{R}^m}(Y))}{\text{LCD}_{\mathbb{R}^m}(E)} \text{ for any } Y \in S^{m-1}(\mathbb{R}).$$

The next theorem provides small ball probability estimates for projections of complex random vectors in the absence of strong structural assumptions on the range of the projection. In one-dimensional setting, those are reformulations of Erdős–Littlewood–Offord and Lévy–Kolmogorov–Rogozin inequalities [27, 16, 25, 37, 19, 22]; high-dimensional versions are studied, in particular, in [11, 31, 46]. The statement below can be proved by a reduction to the real setting via the relation $L(P_F(X), \sqrt{kt}) = L(P_{\text{real}(F)}(\text{real}(X)), \sqrt{kt})$ (see [46]) and applying the above anti-concentration estimates from [42].

Theorem 2.6. For any $\zeta > 0$ there is $C > 0$ depending only on $\zeta$ with the following property. Let $X$ be a random vector in $\mathbb{C}^m$ satisfying assumption $\mathbb{X}$. Further, let $Y$ be a fixed vector in $\text{Incomp}_m(\zeta, \zeta$. Then

$$L((X, Y), t) \leq (Ct + C\zeta^{-1/2})^2 \text{ for all } t > 0.$$

More generally, if $F$ is a fixed subspace in $\mathbb{C}^m$ of dimension $k$ such that $F \cap S^{m-1}(\mathbb{C}) \subset \text{Incomp}_m(\zeta, \zeta$ then

$$L(P_F(X), \sqrt{kt}) \leq (Ct + C\zeta^{-1/2})^{2k} \text{ for all } t > 0.$$

3. An estimate for the smallest singular value of real rectangular matrices

Let $M$ be a $d \times r$ real random matrix with independent isotropic columns. In this section, we are concerned with estimating the smallest singular value of $M$ (assuming certain small ball probability estimates and concentration for individual matrix columns). The problem of estimating $s_{\min}(M)$ is a standard question within the random matrix theory, due to its relevance to questions in statistics, convex geometric analysis, computer science. We refer, among others, to surveys and books [1, 43, 55, 34, 56] for more information and further references. For a tall matrix (for example, with $d \geq 2r$) with i.i.d subgaussian entries, a basic $\varepsilon$–net argument gives $s_{\min}(M) \geq c\sqrt{d}$ with probability at least $1 - e^{-cd}$, with $c > 0$ depending only on the subgaussian moment; with more elaborate arguments, similar estimates are available for “almost square” matrices and under more general assumptions on the matrix columns (see, in particular, [30, 42, 26, 52, 60]). This estimate is close to optimal in general: for example, the random Bernoulli matrix is of deficient rank with probability $e^{-c' d}$. However, with some additional assumptions on anti-concentration of the matrix columns, the singularity probability is much smaller.
To obtain satisfactory quantitative estimates for $s_{\min}$, we modify the basic $\varepsilon$-net argument by replacing the spectral norm of the matrix with the Hilbert–Schmidt norm: due to much better concentration properties of the latter, we get strong small ball probability estimates for $s_{\min}$, provided that such estimates exist for arbitrary linear combinations of the matrix columns. We start with a simple lemma obtained as a consequence of the Hanson–Wright inequality [21] [13].

**Lemma 3.1.** Let $m, r, d \in \mathbb{N}$, $K > 0$, and let $X_1, \ldots, X_r$ be i.i.d vectors in $\mathbb{R}^m$ satisfying [23]. Further, let $M$ be an $m \times r$ random matrix such that $\text{Col}_\ell(M) = P_F(X_\ell)$, $\ell = 1, \ldots, r$, with $F$ being a $d$-dimensional fixed subspace of $\mathbb{R}^m$. Then
\[
P\{\|M\|_{\text{HS}} \geq C_{mr} \sqrt{rd}\} \leq \exp(-c_{mr} d),
\]
for some $C_{mr} > 1$, $c_{mr} > 0$ depending only on the subgaussian moment $K$.

**Remark 3.2.** The above lemma can be viewed as a special case of [23] Theorem 2.1, in which we take $A$ as the $mr \times mr$ block-diagonal matrix with $r$ identical blocks representing orthogonal projection onto $F$, and consider a concatenated $mr$-dimensional random vector composed of $X_\ell$’s.

In the following statement we construct the required net inside the Euclidean ball. The construction is a direct application of the probabilistic method, and was previously used in other contexts in high-dimensional convex and discrete geometry. As a classical illustration of this method we refer to Rogers’ paper [36]. Among recent applications, see [20] for randomized coverings of convex sets and paper [23] where a statement similar to the one below is proved.

**Lemma 3.3.** Let $r \geq C_{mr}$ and let $m$ satisfy $m \leq 2^r$. For every $t \in (2^{-2^r}, 1]$, there exists a non-random subset $\mathcal{N} \subset \frac{1}{2}B_2^r(\mathbb{R}) \setminus \frac{1}{2}B_2^r(\mathbb{R})$ of cardinality at most $(C_{mr} t)^r$ with the following property: For every real $m \times r$ matrix $A$ with the Hilbert–Schmidt norm at most $\sqrt{r}$, and every $X \in S^{r-1}(\mathbb{R})$ there is $X' \in \mathcal{N}$ satisfying $\|AX - X'\|_2 \leq t$. Here, $C_{mr} > 0$ is a universal constant.

**Proof.** For brevity, we write $S^{r-1}, B_2^r$ instead of $S^{r-1}(\mathbb{R}), B_2^r(\mathbb{R})$. The proof involves a covering of the set of matrices with uniformly bounded Hilbert–Schmidt norms, and a standard covering of $S^{r-1}$.

We start with constructing a net of matrices. Let HS be the set of all real-valued $m \times r$ matrices having the Hilbert–Schmidt norm at most $\sqrt{r}$. Since the absolute values of coordinates of matrices from HS are bounded by $\sqrt{r}$, any $\sqrt{mr}$ net in the parallelootope $[-\sqrt{r}, \sqrt{r}]^{m \times r}$ (with respect to the $\ell_\infty$-metric) is also a 1–net for HS (with respect to the Hilbert–Schmidt norm). Thus, there is a 1–net $\mathcal{N}_{\text{HS}} \subset \text{HS}$ of cardinality at most $(2\sqrt{mr})^{mr}$.

Further, let $\mathcal{N}$ be a Euclidean $\frac{1}{2}2^{1/r}$-net on $S^{r-1}$ of cardinality at most $(6\sqrt{r}/t)^r$.

Given positive real numbers $t_1, t_2$ and a random $r$-dimensional vector $X = (x_1, \ldots, x_r)$ uniformly distributed in the Euclidean ball $t_1B_2^r$, for any fixed matrix $A \in \text{HS}$ we have
\[
P\{\|AX\|_2 > t_2\} \leq (t_1/t_2)^2.
\]
Indeed, we have $\mathbb{E}x_\ell^2 = t_1^2/(r + 2)$ for all $\ell \leq r$, and, applying the singular value decomposition to $A$, obtain $\mathbb{P}\{\|AX\|_2 > t_2\} = \mathbb{P}\{\sum_{\ell=1}^r s_\ell^2 x_\ell^2 > t_2^2\}$, where $s_1, \ldots, s_r$ are the singular values of $A$. Since $\sum_{\ell=1}^r s_\ell^2 \leq r$, Markov’s inequality implies
\[
P\{\|AX\|_2 > t_2\} \leq t_2^2 \sum_{\ell=1}^r s_\ell^2 \mathbb{E}x_\ell^2 \leq (t_1/t_2)^2.
\]

Now, define $S$ as a collection of mutually independent random vectors uniformly distributed in the shell $\frac{1}{2}B_2^r \setminus \frac{1}{2}B_2^r$ of cardinality $\lceil C^{-1} \rceil$, where the constant $C > 1$ is to be chosen later. Note that in order to prove the lemma, it is enough to show that
\[
P\{\forall A \in \text{HS} \ \forall X \in S^{r-1} \ \exists X' \in S : \|A(X - X')\|_2 \leq t\} > 0,
\]
and then take \( \mathcal{N} \) as an appropriate realization of \( S \). Note that for any \( X \in S^{r-1} \) there is \( X'' \in \tilde{\mathcal{N}} \) with \( \|X - X''\|_2 \leq \frac{1}{2\sqrt{r}} \), so that for any matrix \( A \in HS \) we have \( \|A(X - X'')\|_2 \leq \frac{t}{4} \). Thus, the above estimate holds if

\[
1 - p_0 := \mathbb{P}\{ \forall A \in HS \ \forall X \in \tilde{\mathcal{N}} \ \exists X' \in S : \|A(X - X')\|_2 \leq t/2 \} > 0.
\]

By the union bound,

\[
p_0 \leq \sum_{X \in \tilde{\mathcal{N}}} \sum_{A \in \mathcal{N}_{HS}} \mathbb{P}\{ \exists A \in HS \ \forall X' \in S : \|A(X - X')\|_2 > t/2 \}
\]

\[
\leq \sum_{X \in \tilde{\mathcal{N}}} \sum_{A \in \mathcal{N}_{HS}} \mathbb{P}\{ \forall X' \in S \cap \left( \frac{t}{8} B_2^r + X \right) : \|A(X - X')\|_2 > t/4 \}.
\]

Fix for a moment any \( A \in \mathcal{N}_{HS} \) and \( X \in \tilde{\mathcal{N}} \), and denote by \( \mathcal{E} \) the event that the Euclidean ball \( \frac{t}{8} B_2^r + X \) contains at least \((C/48)^r\) points from \( S \). Observe that \( \frac{t}{8} B_2^r + X \) is entirely contained in \( \frac{3}{2} B_2^r \setminus \frac{1}{2} B_2^r \); thus, the probability that a random vector uniformly distributed on \( \frac{3}{2} B_2^r \setminus \frac{1}{2} B_2^r \) falls into \( \frac{t}{8} B_2^r + X \) is greater than \((t/12)^r\). Hence, using the definition of \( S \), we get

\[
\mathbb{P}(\mathcal{E}) \leq \left( \frac{|S|}{(C/48)^r} \right) \left( 1 - \left( \frac{t}{12} \right)^r \right)^{|S|} \leq e^{(C/48)^r} \left( \frac{t}{12} \right)^r e^{-((C/48)^r)} \leq \exp(-e^r),
\]

as long as the constant \( C \) is chosen sufficiently large. Further, conditioned on \( \mathcal{E} \), we can estimate the probability \( \mathbb{P}\{ \forall X' \in S \cap \left( \frac{t}{8} B_2^r + X \right) : \|A(X - X')\|_2 > t/4 \} \) using relation \( \|\| \) and conditional independence; specifically,

\[
\mathbb{P}\{ \forall X' \in S \cap \left( \frac{t}{8} B_2^r + X \right) : \|A(X - X')\|_2 > t/4 \mid \mathcal{E} \} \leq 2^{-2(C/48)^r}.
\]

Finally, combining the estimates and taking the union bound, we obtain

\[
p_0 \leq |\tilde{\mathcal{N}}| |\mathcal{N}_{HS}| \left( 2^{-2(C/48)^r} + \exp(-e^r) \right) < 1,
\]

and the result follows.

As a consequence of the above lemmas, we obtain the main statement of the section.

**Proposition 3.4.** For any \( M, K \geq 1 \) there is \( C_{r, \delta} > 1 \) depending only on \( K, M \), with the following property. Let \( r \geq C_{r, \delta} m \leq 2^r, d \geq r \). Further, let \( M \) be an \( m \times r \) random matrix, where each column \( \text{Col}_i(M) \) is the orthogonal projection of a random isotropic vector \( X_i \in \mathbb{R}^m \) with i.i.d \( K \)-subgaussian coordinates, onto a \( d \)-dimensional fixed subspace \( F \); \( X_1, X_2, \ldots, X_r \) are mutually independent. Assume that for any unit vector \( a = (a_1, \ldots, a_r) \in \mathbb{R}^r \), the linear combination \( \sum_{i=1}^r a_i \text{Col}_i(M) \) satisfies

\[
\mathcal{L}\left( \sum_{i=1}^r a_i \text{Col}_i(M), \sqrt{dt} \right) \leq (\delta t)^d + \nu, \quad t > 0,
\]

for some numbers \( \delta \geq 1 \) and \( \nu > 0 \). Then for all \( t \geq (e^{-\nu/2} + \nu^{1/d})/\delta \) and any fixed \( m \times r \) matrix \( W \) with \( \|W\|_{HS} \leq M \sqrt{d} \), we have

\[
\mathbb{P}\{ \text{s}_{\min}(M + W) \leq C_{r, \delta} \sqrt{dt} \} \leq C_{r, \delta}^{r+d} \delta d t^{d-r}.
\]

**Proof.** Fix any \( t \in (2^{-2^r}, 1] \) and let \( \mathcal{N} \) be a discrete subset of \( \frac{3}{2} B_2^r \setminus \frac{1}{2} B_2^r \) constructed in Lemma 3.3, i.e a subset of cardinality at most \((C/48)^r\) such that for any \( m \times r \) matrix \( A \) of Hilbert–Schmidt norm at most \( \sqrt{t} \), and any \( X \in S^{r-1} \) there is \( X' \in \mathcal{N} \) such that \( \|A(X - X')\|_2 \leq t \).
For any vector $X' = (x'_1, x'_2, \ldots, x'_r) \in \mathcal{N}$ we have, by the assumption of the Proposition and since $\|X'\|_2 \geq 1/2$,\[
L \left( \sum_{\ell=1}^{r} x'_\ell \text{Col}(M), (2C_{\text{3.1}} + M) \sqrt{dt} \right) \leq ((4C_{\text{3.1}} + 2M) \delta t)^d + \nu.
\]
On the other hand, the modified $\varepsilon$-argument gives\[
s_{\min}(M + W) \geq \min_{X' \in \mathcal{N}} \| (M + W) X' \|_2 - t \| M + W \|_{HS} / \sqrt{r}
\]
deterministically. Taking the union bound over all $X' \in \mathcal{N}$ in the previous formula, and applying Lemma 3.1 we get\[
\mathbb{P} \{ s_{\min}(M + W) \leq C_{\text{3.1}} \sqrt{dt} \} \
\leq \mathbb{P} \left\{ \min_{X' \in \mathcal{N}} \| (M + W) X' \|_2 \leq (2C_{\text{3.1}} + M) \sqrt{dt} | \| M \|_{HS} \leq C_{\text{3.1}} \sqrt{r} \right\} \
+ \mathbb{P} \{ \| M \|_{HS} \geq C_{\text{3.1}} \sqrt{r} \} \
\leq (C_{\text{3.1}}/t)^r \left[ ((4C_{\text{3.1}} + 2M) \delta t)^d + \nu \right] + e^{-C_{\text{3.1}} d}.
\]
The result follows. \hfill \Box

**Remark 3.5.** Let us compare the above estimate with the standard $\varepsilon$-net argument, involving the spectral norm of the matrix. Assume for concreteness that $d = 2r$ and $\nu = 0$. The standard $\varepsilon$-net argument then gives\[
\mathbb{P} \{ s_{\min}(M) \leq C't \sqrt{d} \} \leq \left( \frac{3}{t} \right)^r \sup_{X \in S^{r-1}} \mathbb{P} \{ \| MX \|_2 \leq 2C't \sqrt{d} \} + \mathbb{P} \{ \| M \| \geq C't \sqrt{d}/t' \},
\]
where we optimize over $t' \in (0, t]$ — the parameter of the net on $S^{r-1}$. Under the assumptions on $M$, the best possible estimate for $\| M \|$ is $\mathbb{P} \{ \| M \| \geq \tau \} \leq e^{-cr^2}$ for all $\tau \geq C \sqrt{d}$ (it is not difficult to see that the reverse estimate holds with a different constant, say, when $X_1, \ldots, X_r$ are Gaussians). For all $t \leq \frac{1}{Cd}$, the optimization over $t'$ then gives\[
\mathbb{P} \{ s_{\min}(M) \leq C't \sqrt{d} \} \leq \bar{C}d \delta^d t^d \left( \sqrt{\log(t^{-1} \delta^{-2})} \right)^r,
\]
producing the extra logarithmic factor.

## 4. No-gaps delocalization of null vectors in the real setting

In this section, we consider delocalization of almost null vectors for rectangular matrices with i.i.d subgaussian entries. As we have discussed in the introduction, our goal is not only to get optimal bounds for the smallest coordinates of the vectors but also to derive optimal deviation estimates. We combine the well known technique of test projections used in [33] with additional ingredients: an efficient averaging procedure and small ball probability estimates for the smallest singular values of matrices of projections, which were considered in the previous section.

**Lemma 4.1.** Given $\tau \geq 0$ and $m \leq n$, let $B$ be an $m \times n$ matrix with entries in $\Theta$, and let $\mathbf{u}$ be a unit vector in $\Theta^n$ such that $\| B \mathbf{u} \|_2 \leq \tau$. Given $\beta > \theta > 0$, define two sets\[
I_\theta := \{ i \leq n : |u_i| \leq \theta \} \quad \text{and} \quad J_\beta := \{ i \leq n : |u_i| \geq \beta \},
\]
and assume that $k := |I_\theta| \neq 0$ and $r := |J_\beta| \neq 0$. Let\[
F := \text{span} \{ \text{Col}_i(B) : i \in (I_\theta \cup J_\beta)^c \}^\perp.
\]
Define an $m \times k$ matrix $M$ and an $m \times r$ matrix $M'$ as\[
M := (P_F(\text{Col}_i(B)))_{i \in I_\theta} \quad \text{and} \quad M' := (P_F(\text{Col}_j(B)))_{j \in J_\beta}.
\]
Then we have
\[ \beta \sqrt{r} s_{\min}(M') \leq \theta \sqrt{k} s_{\max}(M) + \tau. \]

Proof. Denote by \( E_1 \) the ellipsoid
\[ E_1 := \left\{ \sum_{i \in I_0} \alpha_i P_F(\text{Col}_i(B)) : \|\alpha_i\|_2 \leq \theta \sqrt{k} \right\}, \]
and by \( E_2 \) the ellipsoid (hypersurface)
\[ E_2 := \left\{ \sum_{i \in J_0} \alpha_i P_F(\text{Col}_i(B)) : \|\alpha_i\|_2 = \beta \sqrt{r} \right\}. \]

Then \( E_2 \cap (E_1 + \tau B_2^m(\Theta) \cap F) \) is non-empty. Indeed, setting
\[ v := \frac{\beta \sqrt{r}}{\|u_j\|_2} \sum_{j \in J_0} u_j P_F(\text{Col}_j(B)), \]
by construction we get \( v \in E_2 \). On the other hand, since \( \beta \sqrt{r}/\|u_j\|_2 \leq 1 \) and
\[ \sum_{j \in J_0} u_j P_F(\text{Col}_j(B)) + \sum_{i \in I_0} u_i P_F(\text{Col}_i(B)) = P_F Bu, \]
where \( P_F Bu \in \tau B_2^m(\Theta) \cap F \) and \( \sum u_i P_F(\text{Col}_i(B)) \in E_1 \), we have \( v \in (E_1 + \tau B_2^m(\Theta) \cap F) \).

Now, since \( E_2 \cap (E_1 + \tau B_2^m(\Theta) \cap F) \) is non-empty, there exist vectors
\[ a \in \theta \tau B_2^K(\Theta) \text{ and } a' \in \beta \tau S_r^{-1}(\Theta) \]
such that \( \|M'a' - Ma\|_2 \leq \tau \), and it remains to note that the l.h.s. is at least \( \beta \sqrt{r} s_{\min}(M') - \theta \sqrt{k} s_{\max}(M). \)

In what follows, this lemma allows to reduce the problem of estimating coordinates of \( u \) to comparing largest and smallest singular values of auxiliary random matrices. The probabilistic estimate on the singular values is obtained as a combination of Proposition 3.4 and structural results of [40] [42] stated in Section 2.

Proposition 4.2. Let \( m, d \in \mathbb{N} \) and let \( F \) be a subspace of \( \mathbb{R}^m \) of dimension \( d \). Further, let \( k, r < d \), and let \( X_1, X_2, \ldots, X_r, Y_1, Y_2, \ldots, Y_k \) be i.i.d random vectors in \( \mathbb{R}^m \) satisfying (4) with a parameter \( K \). Let \( M \) and \( M' \) be \( m \times r \) and \( m \times k \) random matrices with columns \( P_F(X_\ell) \), \( \ell \leq r \) and \( P_F(Y_\ell) \), \( \ell \leq k \), respectively. Finally, assume that \( W \) and \( W' \) are fixed \( m \times r \) and \( m \times k \) matrices with spectral norms at most \( M \sqrt{d} \). Then for any \( \varepsilon_1, \varepsilon_2 > 0, \tau \in [0, \varepsilon_2 \sqrt{d}] \) and \( \alpha > 0, \gamma \in (0, 1) \) we have
\[ \mathbb{P}\left\{ \varepsilon_1 s_{\min}(M + W) \leq \varepsilon_2 s_{\max}(M' + W') + \tau \right\} \leq (G_4^{\gamma})^d \left( \frac{\sqrt{d}}{\text{LCD}_{\alpha, \gamma}(F)} + \gamma e^{-\frac{\varepsilon_1}{\varepsilon_2}} + \varepsilon_2 + \frac{\varepsilon_2}{\varepsilon_1} \right)^{d-r}, \]
where \( G_4^{\gamma} > 0 \) may depend only on \( K, M \). In particular, if \( \text{LCD}_{\gamma \sqrt{m}, \gamma}(F) \geq \sqrt{m e^{m/d}} \), then the r.h.s. is less than \( (G_4^{\gamma / \sqrt{m}} (e^{-\frac{\varepsilon_1}{\varepsilon_2}} + e^{-\frac{\varepsilon_2}{\varepsilon_1}} + \varepsilon_2 / \varepsilon_1)^{d-r} \), where \( G_4^{\gamma / \sqrt{m}} > 0 \) depend only on \( K, M \) and \( \gamma \).

Proof. Fix for a moment any \( a = (a_1, \ldots, a_r) \in S_r^{-1}(\mathbb{R}) \), and let
\[ Z := \sum_{\ell=1}^r a_\ell P_F(X_\ell) = P_F \left( \sum_{\ell=1}^r a_\ell X_\ell \right). \]
Clearly, \( \sum_{i=1}^{\ell} a_i X_i \) is an isotropic random vector in \( \mathbb{R}^{n_0} \) with i.i.d coordinates. Moreover, each coordinate is \( C_0 \)-subgaussian, with \( C_0 > 0 \) being a universal constant (see, for example, [55], Lemma 5.9). Now, by Theorem 2.3 we have for any \( \alpha > 0, \gamma \in (0, 1) \) and for all \( t \geq d \log \alpha, \gamma \frac{C_0}{\ell} \): 
\[
\mathcal{L}(Z, t) \leq \left( \frac{\widetilde{C}_t}{\gamma \sqrt{d}} \right)^d + \widetilde{C}_0 e^{-\alpha c^2},
\]
where \( \widetilde{C}, c > 0 \) depend only on the subgaussian moment \( K \). Now, setting \( \delta = \frac{\widetilde{C}}{\gamma} \) and \( \nu = C d e^{-\alpha c^2} + \left( \frac{\widetilde{C}_t}{\gamma \sqrt{d}} \right)^d \), and applying Proposition 3.4 we get 
\[
P \{ \| s_{\min}(M + W) \| \leq C_\mathcal{L} \sqrt{d} \} \leq C_\mathcal{L}^{r+1} d^{t-\nu}, \quad t \geq (e^{-\frac{c_\mathcal{L}}{d}} + \nu^{1/d})/\delta.
\]
Further, it follows from the Hanson–Wright inequality (see Theorem 3.2 of [44]) that 
\[
P \{ \| s_{\max}(M' + W') \| \geq C'' h \sqrt{d} + k \} \leq e^{-h^2(d+k)} , \quad h \geq 1,
\]
where \( C'' \) may only depend on \( K, \mathcal{M} \). Finally, observe that, setting \( t_0 = (e^{-\frac{c_\mathcal{L}}{d}} + \nu^{1/d})/\delta \), we have 
\[
P \{ \| s_{\min}(M + W) \| \leq C_\mathcal{L} \sqrt{d} \} \leq (e^{-\frac{c_\mathcal{L}}{d}} + \nu^{1/d})/\delta.
\]
Then application of the above estimates gives the result. \( \square \)

Now, we can prove the main result of this section, which gives Theorem A from the introduction for an appropriate choice of parameters.

**Theorem 4.3.** For any \( K \geq 1 \) there are \( C_\mathcal{L}, C_n > 0 \) depending only on \( K \) with the following property. Let \( n \geq C_\mathcal{L} \), let \( 1 \leq n - m \leq n/\log n \), and let \( B \) be an \( m \times n \) random matrix with independent columns satisfying (\*). Then for any 
\[
G_{\mathcal{L}, m} (n-m) \log n \leq k \leq C_m n, \quad e^{-\frac{c_\mathcal{L}}{d}(n-m)} \leq t/1, \quad 0 \leq \tau \leq k^2 t/n^{3/2}
\]
and any \( m \times n \) fixed matrix \( W \) with \( \| W \| \leq \sqrt{k} \) we have 
\[
P \left\{ \exists u \in S^{n-1}(\mathbb{R}) : \| (B + W)u \|_2 \leq \tau \quad \text{and} \quad u^*_{n-k+1} \leq \frac{kt}{n^{3/2}} \right\} \leq (C_\mathcal{L} k)^k e^{-\frac{c_\mathcal{L}}{d} t}.
\]

**Proof.** The constants \( C_\mathcal{L}, C_m, C_{\mathcal{L}, m} \) can be recovered from the proof below. Given \( n \geq C_\mathcal{L} \) and \( 1 \leq n - m \leq n/\log n \), take any \( k, t, \tau \) satisfying 
\[
k \in [C_\mathcal{L}(n-m) \log n, C_{\mathcal{L}, m}], \quad t \leq 1, \quad \tau \leq k^2 t/n^{3/2} \leq \sqrt{\frac{c_\mathcal{L} n}{d}}
\]
and a fixed matrix \( W \) with \( \| W \| \leq \sqrt{k} \). Let \( u = (u_1, \ldots, u_n) \) be a unit random vector such that \( \| (B + W)u \|_2 \leq \tau \) everywhere on the probability space. It follows from [12], Lemma 2.6, that there is \( \tilde{c} = \tilde{c}(K) > 0 \) such that if \( \tilde{c} \leq \tilde{c} \),
\[
\beta := \tilde{c}/\sqrt{n} \quad \text{and} \quad J_\beta := \{ i \leq n : |u_i| \geq \beta \}
\]
then \( \mathbb{P} \{ |J_\beta| < \tilde{c} n \} < e^{-\tilde{c} n} \). Set 
\[
\theta := kt/n^{3/2} \quad \text{and} \quad I_\theta := \{ i \leq n : |u_i| \leq \theta \}
\]
and note that 
\[
u_{n-k+1} \leq \theta \quad \text{if and only if} \quad |I_\theta| \geq k.
\]
Thus we need to estimate probability of the event $\{ |I_\theta| \geq k \}$. Using Markov’s inequality, we get

$$
\mathbb{P}\{ |I_\theta| \geq k \} \leq k^{-c} \mathbb{E}( |I_\theta|^k 1_{\{|I_\theta| \geq k\}}) 
\leq (\tilde{c} n)^c \mathbb{E}( |I_\theta|^k |J_\beta|^k 1_{\{|I_\theta| \geq \tilde{c} n\}} 1_{\{|J_\beta| \geq \tilde{c} n\}}) + (n/k)^c e^{-\tilde{c} n},
$$

(6)

and we need to get a bound for $\mathbb{E}( |I_\theta|^k |J_\beta|^k 1_{\{|I_\theta| \geq \tilde{c} n\}} 1_{\{|J_\beta| \geq \tilde{c} n\}})$.

Let $\mathcal{E}_{k}$ be the event that all $m \times m$ submatrices of $B + W$ are of full rank. The main result of [40] implies that, as long as $n - m \leq c'n$ for a sufficiently small $c' > 0$ (depending on $K$), we have

$$
\mathbb{P}(\mathcal{E}_{k}) \geq 1 - e^{-c'n}
$$

(as we already mentioned, [40] deals with centered random matrices, however, adding a non-random shift with an appropriately bounded spectral norm does not in any way change the argument). Given any two distinct subsets of indices $I := \{i_1, \ldots, i_k\}$, $J := \{j_1, \ldots, j_k\} \subset [n]$, denote

$$
F = F(I, J) := \text{span} \{ \text{Col}_i(B + W), i \in (I \cup J)^c \}^\perp.
$$

Conditioned on $\mathcal{E}_{k}$, we have

$$
\dim F = m - (n - 2k) =: d \in [k, 2k].
$$

Further, assuming that $\epsilon_1 \leq \epsilon_2 / 2$, by Theorem 2.4 we get

$$
\mathbb{P}\{ \text{LCD}(\epsilon_1, \epsilon_2, F) \leq \sqrt{m} e^{2\epsilon_2 n/d} \} \leq e^{-\epsilon_2 m}.
$$

Let $M, M'$ be the $m \times k$ matrices with columns $P_F(\text{Col}_j(B + W))$, $j \in J$ and $P_F(\text{Col}_i(B + W))$, $i \in I$, respectively. Let $\epsilon_1 > \epsilon_2 \geq \tau / \sqrt{d}$. By Proposition 4.2 we have

$$
\mathbb{P}\{ \epsilon_1 s_{\min}(M) \leq \epsilon_2 s_{\max}(M') + \tau \mid \mathcal{E}_{k} \cap \{ \text{LCD}(\epsilon_1, \epsilon_2, F) \geq \sqrt{m} e^{2\epsilon_2 n/d} \} \}
\leq C d^{\epsilon_2 \epsilon_1} (e^{-cn/k} + e^{-ck} + \epsilon_2 / \epsilon_1)^{d-k},
$$

where $C$ and $c = \min\{\epsilon_1, \epsilon_2, \epsilon_3, c'\}$ depend on $K$ and $\epsilon_2$, and we used that $m/d \geq n/(2k)$.

Let $\chi_{i_1, \ldots, i_k}(\epsilon_1, \epsilon_2)$ be the indicator of the event

$$
\epsilon_1 s_{\min}(M) \leq \epsilon_2 s_{\max}(M') + \tau.
$$

It follows from the above that for all $\epsilon_2 / \epsilon_1 \geq e^{-cn/k} + e^{-ck}$ we have

$$
\mathbb{E}(\chi_{i_1, \ldots, i_k}(\epsilon_1, \epsilon_2)) \leq C d^{\epsilon_2 \epsilon_1} (2\epsilon_2 / \epsilon_1)^{d-k} + e^{-c'n} + e^{-\epsilon_2 \epsilon_1} \leq C d^{\epsilon_2 / \epsilon_1} \epsilon_1^{d-k}
$$

with $C > 2C_{4.2}$, so that

$$
\mathbb{E}\left( \sum \chi_{i_1, \ldots, i_k}(\epsilon_1, \epsilon_2) \right) \leq C d^{\epsilon_2 / \epsilon_1} n^{2k} (\epsilon_2 / \epsilon_1)^{d-k},
$$

(7)

where the sum is taken over all ordered $2k$-tuples $(j_1, \ldots, j_k, i_1, \ldots, i_k)$ with distinct components.

Now for any distinct $i_1, \ldots, i_k \in I_\theta$ and $j_1, \ldots, j_k \in J_\beta$ (whenever $|I_\theta|, |J_\beta| \geq k$) by Lemma 4.1 we have $\chi_{i_1, \ldots, i_k}(\epsilon_1, \epsilon_2) = 1$, hence deterministically

$$
\sum \chi_{i_1, \ldots, i_k}(\epsilon_1, \epsilon_2) \geq |I_\theta|( |I_\theta| - 1) \ldots (|I_\theta| - k + 1) |J_\beta|( |J_\beta| - 1) \ldots (|J_\beta| - k + 1) 1_{\{|I_\theta| \geq k\}} 1_{\{|J_\beta| \geq \tilde{c} n\}}
\geq 9^{-k} |I_\theta|^k |J_\beta|^k 1_{\{|I_\theta| \geq k\}} 1_{\{|J_\beta| \geq \tilde{c} n\}}.
$$

Set

$$
\epsilon_1 := \beta \sqrt{k}, \quad \epsilon_2 := \theta \sqrt{k}.
$$

Note that with this choice, $\tau \leq \sqrt{d} \epsilon_2$. Let

$$
t \geq t_0 := \tilde{c} n k^{-1} (e^{-cn/k} + e^{-ck/(n-m)}) \geq \tilde{c} n k^{-1} (e^{-cn/k} + e^{-ck}).
$$
Note that $t_0 < 1$ provided that $c_1^{-1} C_{4.3} > c^{-1}$ are big enough. Then necessarily
\[
\frac{\varepsilon_2}{\varepsilon_1} = \frac{\theta}{\beta} = \frac{kt}{cn} \geq e^{-cn/k} + e^{-e^k},
\]
which together with the above estimate and (17), yield
\[
\mathbb{E}(|I_\theta|^k | J_\theta|^k 1_{(|I_\theta| \geq k)} 1_{(|J_\theta| \geq \tilde{c} n)}) \leq (9C^2/\tilde{c})^k n^{3k-d} (kt)^d - k,
\]
hence, using (16) and the fact that $t \geq t_0$, we get
\[
\mathbb{P}\{ |I_\theta| \geq k \} \leq (3C/\tilde{c})^{2k(n/k)^{2k-d} t^d - k} + (n/k)^k e^{-\tilde{c} n}
\leq \left((3C/\tilde{c})^{2k} (n/(kt))^{n-m} + (n/(kt))^k e^{-\tilde{c} n}\right) t^k \leq (\tilde{C} t)^k,
\]
for some $\tilde{C} > 1$. Thus, for any $t \geq t_0$ and $\tau \leq k^2 t/n^{3/2}$, any unit random vector $u$ with
\[
\mathbb{P}\{|(B + W)u|_2 \leq \tau\} = 1
\]
satisfies
\[
\mathbb{P}\{ u_{n-k+1}^* \leq kt/n^{3/2} \} \leq (\tilde{C} t)^k.
\]
Also, $\mathbb{P}\{ u_{n-k+1}^* \leq kt/n^{3/2} \} \leq (\tilde{C} t_0)^k$ for $t \leq t_0$. Note that there exists $\tau > 0$ such that
\[
t_0 \leq \max(e^{-\tilde{c} n/k}, e^{-\tilde{c} k/(n-m)}).
\]
Hence, for all $t \geq e^{-\tilde{c} k/(n-m)}$ we get from the above
\[
\mathbb{P}\{ u_{n-k+1}^* \leq kt/n^{3/2} \} \leq (\tilde{C} \max(t, t_0))^k \leq (\tilde{C} \max(t, e^{-\tilde{c} n/k}))^k \leq (\tilde{C} t)^k + \tilde{C} e^{-\tilde{c} n}.
\]
The result follows. \qed

As an illustration of the above result, we consider no-gaps delocalization of eigenvectors of non-Hermitian random matrices corresponding to real eigenvalues of small absolute value. The statement follows by combining Theorem 4.3 with a simple discretization procedure for the interval $[-k, k]$; see proof of Theorem C in the last part of the paper for related details.

**Corollary 4.4.** For any $K \geq 1$ there are $C_{4.4} > 0$ depending only on $K$ with the following property. Let $n \geq C_{4.4} K$ and let $B$ be an $n \times n$ random matrix with independent columns satisfying (9). Then for any integer $k \in [K, \log n, \sqrt{n}]$ we have
\[
\mathbb{P}\{ \exists \mathbf{v} \in S^{n-1}(\mathbb{R}) : B\mathbf{v} = \lambda \mathbf{v} \text{ for some } -\sqrt{k} \leq \lambda \leq \sqrt{k} \text{ and } v_{n-k+1}^* \leq \frac{C_{4.4} K}{n^{3/2}} \} \leq e^{-k}.
\]

As another simple corollary, let us consider no-gaps delocalization for singular vectors corresponding to small singular values. We note that sup-norm delocalization of singular vectors is well studied in literature; see, in particular, section 8 of survey [34] and references therein, as well as papers [19, 9, 35, 14, 11, 57, 58].

For a square $n \times n$ matrix $B$ with i.i.d $K$-subgaussian entries, Wei [59] showed that the $\ell$–th smallest singular value $s_{n-\ell+1}(B)$ satisfies $s_{n-\ell+1}(B) \leq \frac{c}{\sqrt{n}}$ with probability $1 - e^{-e^{c \ell t}}$ for all $t \geq 1$, where $C, c > 0$ may only depend on $K$ (we refer to [33, 39, 51] for upper bounds for $s_{\min}(B)$). Combining this result with the above theorem, we immediately get

**Corollary 4.5.** For any $K \geq 1$ there are $C_{4.5} > 0$ depending only on $K$ with the following property. Let $n \geq C_{4.5} K$ and let $B$ be an $n \times n$ random matrix with independent columns satisfying (9). Then for any $\ell \in [n]$ and any $C_{4.5} \sqrt{n} \max(\ell, \log n) \leq k \leq C_{4.5} n$ we have
\[
\mathbb{P}\{ v_{n-k+1}^* \leq \frac{C_{4.5} k}{n^{3/2}}, \text{ for a unit vector } \mathbf{v} \text{ satisfying } B^T B \mathbf{v} = s_{n-\ell+1}(B)^2 \mathbf{v} \} \leq n^{-1}.
\]

**Remark 4.6.** The estimates in the above corollary are non-trivial for $\ell = o(n)$. 

5. Eigenvectors of non-Hermitian matrices

In this section, we study no-gaps delocalization of eigenvectors of non-Hermitian random matrices with i.i.d entries. The basic tool, as in the case of almost null vectors, is a test projection onto the orthogonal complement $F$ of $n - N$ columns of the random matrix (where $N$ is chosen polylogarithmic in $n$). The major difficulty in working with eigenvectors rather than null vectors of the matrix is the necessity to control magnitudes of projections of rescaled coordinate vectors $-ze_i$. When $z$ is small by absolute value, a trivial upper bound $\|P_F(-ze_i)\|_2 \leq |z|$ for the norm of the projection is already sufficient (see Corollary 4.4). However, when $|z|$ is of order $n^{1/2}$, such trivial estimate becomes useless.

A similar problem was considered earlier in [15] where the sup-norm delocalization was studied. The authors of [15] have developed a strategy based on comparing the magnitudes of projections of $-ze_i$ with each other rather than estimating their “absolute” magnitudes. The bound on the $\ell_{\infty}$-norm of an eigenvector was reduced to estimating probabilities of the form

$$P\{\|P(Col_N(A) - ze_N)\|_2 \leq \varepsilon \max_{i \leq N-1} \|P(Col_i(A) - ze_i)\|_2\},$$

where $P$ was a specially constructed test projection with $\ker(P) \supset F$. It was shown that, for $N$ polylogarithmic in $n$ and for $\varepsilon = \log^{O(n)}(n)$ for a sufficiently large constant $C > 0$, the above probability is very close to zero implying the upper bound $\frac{\text{polylog}(n)}{\sqrt{n}}$ for the $\ell_{\infty}$-norms of unit eigenvectors of $A$ [15, Section 5].

The upper bound for [5] obtained in [15] with the above choice of $\varepsilon$ is exp$(-ck/\log n)$ [15, Theorem 5.1], which is sufficient for $\ell_{\infty}$-delocalization. However, studying no-gaps delocalization with the same method requires much stronger estimates, as they need to be able to survive the union bound over a very large number of combinations of coordinates. In particular, in the real case, for any $\beta > \theta > 0$, the probability of the event

$$\{\exists \mathbf{v} \in S^{n-1}(\Theta) : \|v - z\|_2 \leq n^{-2} \text{ and } v_i \leq \theta, i < N, |v_N| \geq \beta\}$$

has to be bounded from above by $(N^C\theta/\beta)^N$, so that the averaging argument, similar to that in the proof of Theorem 4.3, would imply the desired lower bound for the order statistics of eigenvectors. The procedure of selecting spectral window developed in [15] and applied to bound [5], does not seem applicable to get the stronger bounds needed in our context.

5.1. Test projections and ellipsoids. An alternative procedure developed in this paper is based on a careful analysis of the ellipsoid

$$E' := \left\{ \sum_{i=1}^{N-1} a_i P_F(-ze_i) : \|(a_1, \ldots, a_{N-1})\|_2 \leq 1 \right\}$$

as well as the dual ellipsoid. Rather than comparing the lengths of $P_F(-ze_i)$, we consider the geometric problem of estimating probability of events of the form

$$P_F(-ze_N) \in \varepsilon E' + \tau B^F_2.$$

A simple but important observation is the following deterministic lemma:

**Lemma 5.1.** Given $n \in \mathbb{N}$, let $A$ be an $n \times n$ matrix with entries in $\Theta$, and let $z \in \Theta$. For $N < n$, let

$$F := \text{span} \{\text{Col}_i(A - z) : i \in [n] \setminus [N]\}.$$

Let $\mathbf{v} = (v_1, \ldots, v_n) \in S^{n-1}(\Theta)$ and let $\tau > 0$, $\beta > \theta > 0$ be parameters such that

$$\|(A - z)\mathbf{v}\|_2 \leq \tau, \quad |v_i| \leq \theta, i \leq N - 1, \quad \text{and} \quad |v_N| \geq \beta.$$
Assume also that for some $T > 0$ we have $\|P_{F}\text{Col}_\ell(A)\|_2 \leq T$, $\ell \in [N]$. Then
\[
P_{F}\text{Col}_N(A - z) \in \frac{\sqrt{N}}{\beta} E' + \left(\frac{\tau}{\beta} + \frac{T N \theta}{\beta} \right) B_2^F \quad \text{and}
\]
\[
P_{F}(-ze_N) \in \frac{\sqrt{N}}{\beta} E' + \left(\frac{\tau}{\beta} + \frac{T N \theta}{\beta} + T \right) B_2^F.
\]

Proof. Let $X := P_{F}\text{Col}_N(A - z)$. Applying the argument from the proof of Lemma 5.1 with $B = A - z$, $J_\beta = \{N\}$, $I_\theta = [N - 1]$, we obtain
\[
X \in \frac{\sqrt{N}}{\beta} \left\{ \sum_{\ell=1}^{N-1} a_\ell P_{F}\text{Col}_\ell(A - z) : \|(a_\ell)\|_2 \leq 1 \right\} \leq \frac{\tau}{\beta} B_2^F.
\]

On the other hand, by the conditions on vectors $\text{Col}_\ell(A)$, we have
\[
\left\{ \sum_{\ell=1}^{N-1} a_\ell P_{F}\text{Col}_\ell(A - z) : \|(a_\ell)\|_2 \leq 1 \right\} \subset \left\{ \sum_{\ell=1}^{N-1} a_\ell P_{F}(-ze_\ell) : \|(a_\ell)\|_2 \leq 1 \right\} + T \sqrt{N} B_2^F.
\]
Together with the previous inclusion, this gives
\[
X \leq \frac{\sqrt{N}}{\beta} E' + \left(\frac{T}{\beta} + \frac{T N \theta}{\beta} \right) B_2^F,
\]
proving the first assertion of the lemma. For the second assertion, it remains to note (again) that $\|P_{F}\text{Col}_N(A)\|_2 \leq T$. \(\square\)

The above lemma reduces the proof of Theorems B and C to computing probabilities of the two conditions on test projections. In what follows, rather than working with the sequence $(P_{F}(-ze_1), \ldots, P_{F}(-ze_N))$ we will consider a dual basis in $F$. We will need some definitions.

For brevity, we use capital calligraphic letters to denote sequences of vectors of a given length, for example, $X_N = (X_1, X_2, \ldots, X_N)$. Correspondingly, span $(X_N) = \text{span} \{X_1, \ldots, X_N\}$ is the linear space of $X_1, \ldots, X_N$. Further, given a sequence of vectors $X_N$ in a Euclidean space $E$ (over $\Theta$) we use notation
\[
E(X_N) := \left\{ \sum_{i=1}^{N} a_i X_i : a_1, \ldots, a_N \in \Theta, \|(a_1, \ldots, a_N)\|_2 \leq 1 \right\}
\]
for the ellipsoid generated by $X_N$.

We say that a pair $X_N, Y_N \in E^N$ forms a biorthogonal system in $E$ if
\[
\text{span} (X_N) = \text{span} (Y_N) \quad \text{and} \quad \langle X_i, Y_j \rangle = \delta_{ij}, \ 1 \leq i, j \leq N.
\]
The next observation is essentially taken from [45]:

**Lemma 5.2** ([45]). Let $A = (a_{ij})_{i,j}$, $z$, and subspace $F$ be as in Lemma 5.1. Assume that the $(n - N) \times (n - N)$ principal submatrix $\tilde{A}$ of $A$ obtained by crossing out first $N$ rows and $N$ columns, is invertible. Define $(n - N)$-dimensional vectors
\[
q_i = q_i(A, N) := (a_{ij})_{\ell \in [n]\setminus[N]}, \quad i \leq N,
\]
and $(n - N) \times (n - N)$ matrix $D := \tilde{z}^{-1}(\tilde{A}^* - \mathbf{I})^{-1}$. Finally, construct $n$-dimensional vectors
\[
V_i = V_i(A, z, N) := (-\tilde{z}^{-1}e_1) \oplus Dq_i, \quad i \in [N],
\]
where the direct sum “$\oplus$” should be understood as a concatenation of $N$-dimensional vectors $-\tilde{z}^{-1}e_1$ with $(n - N)$-dimensional vectors $Dq_i$'s. Then $V_i, i \in [N]$ form a basis of the space $F$ and, moreover, the sequences $P_{F}(-ze_1), \ldots, P_{F}(-ze_N)$ and $V_1, \ldots, V_N$ form a biorthogonal system in $F$. 

The rest of this section is split into three parts.

In Subsection 5.2 we derive some deterministic relations for biorthogonal systems and dual ellipsoids. This is a crucial step which allows to relate properties of the projections $P_F(-ze_i)$'s to certain estimates for the dual basis $V_1, \ldots, V_N$ (and its perturbations).

In Subsection 5.3 we consider probabilistic relations for anisotropic random vectors of the form $DX$ where $D$ is a fixed matrix and $X$ is a random vector with i.i.d. components with certain assumptions on the distribution. A systematic treatment of anti-concentration properties of anisotropic random vectors was given earlier in [45]. We reuse some of the estimates from [45] while adding some new ones. The central technical element of the subsection is a decoupling argument in the proof of Lemma 5.19.

Finally, in Subsection 5.4 we put the results together and, with help of argument similar to that in the proof of Theorem 4.3, obtain the main statements of this paper.

5.2. Biorthogonal systems and dual ellipsoids. Throughout this subsection we use notation $\mathcal{X}_N = (X_1, \ldots, X_N)$ for a sequence of linearly independent vectors in a Euclidean space $E$ over $\Theta$ and $\mathcal{Y}_N = (Y_1, \ldots, Y_N)$ for the dual sequence.

Let $E(\mathcal{X}_N)$ be the ellipsoid generated by $\mathcal{X}_N$ (see [9]). Clearly, the ellipsoid is the linear image of the unit Euclidean ball in $\Theta^N$ under the action of the linear operator $L : \ell_2^N(\Theta) \to E$ with $L(e_i) = X_i, i = 1, 2, \ldots, N$. Denote by $s_1(\mathcal{X}_N) \geq s_2(\mathcal{X}_N) \geq \cdots \geq s_N(\mathcal{X}_N)$ the singular values of $L$. Note that by the definition of a biorthogonal system we have $s_i(\mathcal{X}_N) = s_{N-i+1}(\mathcal{Y}_N), \ i \leq N$.

**Definition 5.3.** Given any number $R > 1$ and a non-increasing sequence of integers

\begin{equation} (12) \quad b = (b_1, \ldots, b_N) \in \mathbb{Z}^N, \ b_1 \geq b_2 \geq \cdots \geq b_N, \text{ such that } 1/2 \leq 2^{b_i} \leq R, \ i \leq N, \end{equation}

we say that $E(\mathcal{X}_N)$ belongs to the class $C(R, b)$ if

$$\min\left(\max(s_i(\mathcal{X}_N), 1), R\right) \in [2^{b_i}, 2^{b_i+1}), \ i \leq N,$$

i.e. if $\max(s_i(\mathcal{X}_N), 1) \geq 2^{b_i}$ and $\min(s_i(\mathcal{X}_N), R) < 2^{b_i+1}$.

Given $R > 1$, the classes $C(R, b)$ for all $b$ satisfying (12) form a partition of the set of ellipsoids $E(\mathcal{X}_N), \mathcal{X}_N \in E^N$. The next lemma is immediate:

**Lemma 5.4.** For any $R > 1$, the total number of classes $C(R, b)$ (for all admissible $b$) is bounded above by $(\log_2 R + 2)^N$.

The classes $C(R, b)$ provide a discretization of the set of ellipsoids of not-too-large complexity, and allow us (in a probabilistic context) condition on the event that a random ellipsoid generated by the test projections of vectors $-ze_i$, belongs to a given class, without affecting the probability estimates. The definition of the classes “truncates” large semi-axes of the ellipsoid (those exceeding $R$) and does not record information about the magnitude of small semi-axes (of length less than 1): the probabilistic argument proceeds in such a way that both large and very small semi-axes do not significantly affect the estimates.

Let $T, \delta > 0$ be some parameters. In what follows, we are interested in describing the condition

\begin{equation} (13) \quad E(\mathcal{X}_N) \in C(R, b) \quad \text{and} \quad X_N \in \delta E(\mathcal{X}_{N-1}) + T B_2^E \end{equation}

in terms of the sequence $\mathcal{Y}_N$, where $\mathcal{X}_{N-1}$ denotes the sequence $(X_1, X_2, \ldots, X_{N-1})$.

For technical reasons, we will work with perturbations of $\mathcal{Y}_N$. Let $\delta > 0$. We say that a sequence of vectors $W_N$ is a $\delta$-perturbation of $\mathcal{Y}_N$ if $\|Y_i - W_i\|_2 \leq \delta$ for all $i \leq N$. Note that we do not require $W_1, \ldots, W_N$ to be contained in span $\mathcal{Y}_N$. We have the following elementary consequence of perturbation inequalities for singular values (see, for example, [10] Theorem 1.3):

**Lemma 5.5.** Let $W_N$ be a $\delta$-perturbation of $\mathcal{Y}_N$, and let $s_1(\mathcal{Y}_N) \geq \cdots \geq s_N(\mathcal{Y}_N)$ and $s_1(W_N) \geq \cdots \geq s_N(W_N)$ be as above. Then $s_i(W_N) \leq s_i(\mathcal{Y}_N) + \delta \sqrt{N}, i \leq N$. 

In the probabilistic setting considered later, we will not have a “direct access” to the singular values $s_i(W_N)$ but instead will be able to measure the distances from $W_j$’s to spans of some other $W_i$’s. The ordering of the vectors $W_j$’s will be quite important. Given $W_N \in E^N$, we define $\sigma_W$ as a permutation of $[N]$ such that for all $1 \leq i \leq N$,

$$d_i := \text{dist}(W_{\sigma_W(i)}, \text{span}\{W_{\sigma_W(j)} : j < i\}) = \max_{\ell \leq N} \text{dist}(W_\ell, \text{span}\{W_{\sigma_W(j)} : j < i\});$$

in particular $d_1 = \|W_{\sigma_W(1)}\|_2 = \max_{i \leq N} \|W_i\|_2$.

Note that the permutation may be not unique; in what follows for any $W_N$ we fix some $\sigma_W$ satisfying (14). It is easy to check that $d_1 \geq s_1(W_N)/\sqrt{N}$ and $d_1 \geq d_2 \geq \cdots \geq d_N$. Moreover, we have

**Lemma 5.6.** Let $W_N \in E^N$ and let $s_1(W_N) \geq s_2(W_N) \geq \cdots \geq s_N(W_N)$ be as above. Then, with $d_i$’s defined by formula (14), we have $d_i \geq s_i(W_N)/\sqrt{N - i + 1}$, $i \leq N$.

**Proof.** It is convenient to regard $s_i(W_N)$’s as the singular values of the $m \times N$ matrix $Q$ with columns $W_i$, $i \leq N$. Fix any $2 \leq i \leq N$, and set $F := \text{span}\{W_{\sigma_W(j)} : j < i\}$. The min-max formula for singular values of $Q$ (see, for example, [10, Theorem 1.2]) implies that

$$s_i(W_N) \leq \max_{QX \in F, \|X\|_2 = 1} \|QX\|_2 \leq \|P_F Q\|,$$

where $P_F$ denotes the orthogonal projection onto $F$. On the other hand, by the definition of $\sigma_W$ be have

$$\|P_F W_\ell\|_2 = \text{dist}(W_\ell, \text{span}\{W_{\sigma_W(j)} : j < i\}) \leq d_i$$

for all $\ell \leq N$. Hence, $\|P_F Q\| \leq d_i \sqrt{N - i + 1}$, and the result follows.

It follows from the definition of $C(R, b)$ that for any $R > 1$, $b$ satisfying (12), and any sequence $X_N$, if $E(X_N) \in C(R, b)$ then

$$2^{-\sum_{i=1}^N b_i} \leq \prod_{i=1}^N \text{min} \left( \max(s_i(X_N), 1), R \right) < 2^{N + \sum_{i=1}^N b_i},$$

where the quantity in the middle serves as a measure (“truncated volume”) of $E(X_N)$. Lemma 5.6 allows to relate this measure to characteristics of a perturbation of a dual basis:

**Proposition 5.7.** Let $\delta > 0$, $N \geq 1$, and $1 < R \leq \delta^{-1} N^{-1/2}$. Let $X_N, Y_N$ be a biorthogonal system in $E$, $W_N$ be a $\delta$-perturbation of $Y_N$, and $d_i$, $i \leq N$, be defined by (14). Then

$$\prod_{i=1}^N \text{min}(d_i, 1) \leq (4N)^{N/2} 2^{-\sum_{i=1}^N b_i}.$$

**Proof.** Fix for a moment any $t > 0$, and let $I \subset [N]$ be the set of all indices $i \in [N]$ with $s_i(W_N) \leq t$. Then for every $i \in I^c := [N] \setminus I$ we have $\min(d_i, t) \leq t \leq \min(s_i(W_N), t)$, so that

$$\prod_{i \in I^c} \text{min}(d_i, t) \leq \prod_{i \in I} d_i \prod_{i \in I^c} \min(s_i(W_N), t).$$

Next, since $d_i \geq s_i(W_N)/\sqrt{N}$ by Lemma 5.6 and in view of the standard identity

$$\prod_{i \leq N} d_i = \prod_{i \leq N} s_i(W_N)$$

(see, for example, formula (3) in [10]), we get

$$\prod_{i \in I} d_i = \prod_{i \leq N} s_i(W_N) \prod_{i \in I^c} d_i^{-1} \leq N^{N/2} \prod_{i \in I} s_i(W_N) = N^{N/2} \prod_{i \in I} \min(s_i(W_N), t).$$
Hence
\[ \prod_{i=1}^{N} \min(d_i, t) \leq N^{N/2} \prod_{i=1}^{N} \min(s_i(W_N), t). \]

Further, in view of Lemma 5.5 and the duality relation between \( \mathcal{X}_N \) and \( \mathcal{Y}_N \), we have
\[ \min(s_i(W_N), t) \leq \min\left( s_{N-i+1}^{-1}(\mathcal{X}_N), \delta \sqrt{N}, t \right) \]
\[ \leq 2 \max\left( \min(s_{N-i+1}^{-1}(\mathcal{X}_N), t), \min(\delta \sqrt{N}, t) \right), \]

where, as before, \( s_1(\mathcal{X}_N) \geq \cdots \geq s_N(\mathcal{X}_N) \) are the singular values of the linear operator \( L : \Theta^N \to E \) with \( L(e_i) = X_i, i = 1, 2, \ldots, N. \)

Now, choosing \( t = 1 \), we get from the above and our assumption on \( R \)
\[ \min(s_i(W_N), 1) \leq 2 \max\left( \min(s_{N-i+1}^{-1}(\mathcal{X}_N), 1), \delta \sqrt{N} \right) \leq 2/ \min\left( \max(s_{N-i+1}(\mathcal{X}_N), 1), R \right), \]

and by (15),
\[ \prod_{i=1}^{N} \min(d_i, 1) \leq N^{N/2} \prod_{i=1}^{N} \min(s_i(W_N), 1) \]
\[ \leq (4N)^{N/2} \prod_{i=1}^{N} \min(\max(s_i(\mathcal{X}_N), 1), R)^{-1} \leq (4N)^{N/2} 2^{-\sum_{i=1}^{N} b_i}. \]

This finishes the proof. \( \square \)

Returning to formula (13), we can now describe the condition \( E(\mathcal{X}_N) \in C(R, b) \) in terms of a perturbation of \( \mathcal{Y}_N \). It will be convenient to introduce a classification of sequences of vectors based on statistics of distances.

**Definition 5.8.** Given \( 0 < r < 1 \), let \( p := (p_1, \ldots, p_N) \in \mathbb{Z}^n \) satisfy
\[ p_1 \geq \cdots \geq p_N \quad \text{and} \quad r/2 \leq 2^{p_i} \leq 1, \quad i \leq N. \]

We say that a sequence of vectors \( W_N \) in a Euclidean space \( E \) belongs to the class \( \mathcal{D}(r, p) \) if
\[ \min(\max(d_i, r), 1) \in [2^{p_i}, 2^{p_i+1}) \quad \forall i \leq N, \]

where the numbers \( d_i, i \leq N, \) are defined by (14).

The following observation is immediate:

**Lemma 5.9.** For any \( 0 < r < 1 \), the total number of classes \( \mathcal{D}(r, p) \) for all \( p \) satisfying (16) does not exceed \( (2 - \log_2 r)^N \).

Note that, for every \( 0 < r < 1 \), we have \( r \leq \min(\max(d_i, r), 1) \leq i \leq N. \) Thus, for any \( W_N \) there exists a sequence \( p \) satisfying (16) such that \( W_N \) belongs to \( \mathcal{D}(r, p) \), i.e. the classes form a partition of the \( N \)-sequences of vectors from \( E \). We have the following formal consequence of Proposition 5.7 describing the condition \( E(\mathcal{X}_N) \in C(R, b) \) in (13):

**Corollary 5.10.** Given \( \delta > 0, N \geq 1 \), let \( \mathcal{X}_N, \mathcal{Y}_N \) be a biorthogonal system in \( E \) and \( W_N \) be a \( \delta \)-perturbation of \( \mathcal{Y}_N \). Then given \( 1 < R \leq \delta^{-1} N^{-1/2}, b \) satisfying (12), and \( r < 1 \), if
\[ E(\mathcal{X}_N) \in C(R, b) \quad \text{and} \quad W_N \quad \text{is such that} \quad d_i(W_N) \geq r, \quad i \leq N, \]

then
\[ W_N \in \mathcal{D}(r, p) \quad \text{for some} \quad p \quad \text{satisfying} \quad 2^{\sum_{i=1}^{N} p_i} \leq (4N)^{N/2} 2^{-\sum_{i=1}^{N} b_i}. \]

In the second part of this subsection, we investigate the second condition in (13).
Lemma 5.11. (Inclusion into ellipsoid via biorthogonal system). Let $X_N$, $Y_N$ be a biorthogonal system in $E$. Assume further that for some $T, \delta > 0$ we have

$$X_N \in \delta E(X_{N-1}) + TB_2^E.$$ 

Then there is $Y \in E$ with $\|Y\|_2 \leq T$ such that $\sum_{t=1}^{N-1} |\langle Y_t, Y \rangle|^2 \leq \delta^2$, and $|\langle Y_N, Y \rangle| = 1$.

Proof. Let $B$ be a $N \times N$ matrix with columns $X_1, \ldots, X_N$. Then $Y_1, \ldots, Y_N$ are the rows of the complex conjugate of the inverse $B^{-1}$. Conditions of the lemma can be rewritten as $Ba = Y$, for some $a = (a_1, \ldots, a_N)$ with $a_N = -1$ and $\|(a_1, \ldots, a_{N-1})\|_2 \leq \delta$ and $\|Y\|_2 \leq T$. Equivalently, $a = B^{-1}Y$, and the result follows.

\[ \Box \]

Corollary 5.12. Let $T, \delta, X_N, Y_N$ be as in the lemma above, and let $W_N$ be a $\delta$-perturbation of $Y_N$. Assume additionally that $T \leq \frac{1}{2}$. Then there is a vector $Y' \in E$ with $\|Y'\|_2 \leq 2T$ such that $|\langle W_\ell, Y' \rangle| \leq 2\delta + 2\delta T$, $\ell \leq N - 1$, and $|\langle W_N, Y' \rangle| = 1$.

Proof. It follows from Lemma 5.11 that there exists $Y \in E$ with $\|Y\|_2 \leq T$ such that $|\langle W_\ell, Y \rangle| \leq \delta + \delta T$, and $|\langle W_N, Y \rangle| \geq 1 - \delta T \geq 1/2$. Hence, there is a number $0 < c \leq 2$ such that $Y' = cY$ satisfies the conclusion of the corollary. \[ \Box \]

As it follows from Corollary 5.12 in probabilistic setting, in order to estimate probability of the event $X_N \in \delta E(X_{N-1}) + TB_2$ it is sufficient to bound probability of the event of the form

\[ \{ \exists Y : \|Y\|_2 \leq T' \text{ and } |\langle W_\ell, Y \rangle| \leq \delta', \ell \leq N - 1; |\langle W_N, Y \rangle| = 1 \} \]

Note that taking here the union bound over all choices of $Y$ would lead to summation over a continuum. To resolve this issue we use Lemma 5.11 below, which allows to associate with a sequence $W_N$ a finite collection $\Upsilon(W_N, \delta', T)$ of vectors in $E$ which can be reconstructed from $W_N$ using certain relations for scalar products.

Definition 5.13. Given a sequence of linearly independent vectors $W_N$ in $E$ (over $\Theta = \mathbb{C}$) and parameters $\delta', T > 0$, define $\Upsilon(W_N, \delta', T)$ as the set of all vectors $Z \in \text{span}(W_N)$ such that

- $\|Z\| \leq T$;
- for any $i \in [N]$, $|\langle Z, W_{\sigma_W(i)} \rangle - 1_{\{\sigma_W(i)=N\}}| \leq \sqrt{2}(N + 1)\delta'$, and either 
\[ (\langle Z, W_{\sigma_W(i)} \rangle - 1_{\{\sigma_W(i)=N\}})/\delta' \in \mathbb{Z} + i\mathbb{Z} \]

or
\[ \langle Z, W_{\sigma_W(i)} \rangle = \langle Z, P_{\text{span}\{W_{\sigma_W(j), j<i}\}}(W_{\sigma_W(i)}) \rangle. \]

Here, $\sigma_W$ is the permutation of $[N]$ defined via relations $\{ 1 \}$. When $\Theta = \mathbb{R}$, the definition of the vector collection is modified by replacing the lattice $\mathbb{Z} + i\mathbb{Z}$ with $\mathbb{Z}$.

For example, in the simplest case when $W_1 = e_1, \ldots, W_N = e_N$, and $\delta' = (N + 1)^{-2}$, we have

$$\Upsilon(W_N, \delta', T) = \{ y \in TB_2^N(\mathbb{C}) : |y_i - 1_{\{i=N\}}| \leq \sqrt{2}(N + 1)^{-1} \text{ } \& \text{ } y_i \in \mathbb{Z} + i\mathbb{Z} \text{ } \forall i \in [N] \}.$$

The collection $\Upsilon(W_N, \delta', T)$ can be viewed as a discretization of the set of vectors $Y$ from Corollary 5.12 up to some adjustment of parameters. In particular, we allow the scalar products $\langle Z, W_{\sigma_W(i)} \rangle$ (for $Z \in \Upsilon(W_N, \delta', T)$) to take only discrete values from a rescaled lattice $\mathbb{Z} + i\mathbb{Z}$, unless $Z$ is orthogonal to the vector $W_{\sigma_W(i)} - P_{\text{span}\{W_{\sigma_W(j), j<i}\}}(W_{\sigma_W(i)})$.

Note that given a sequence of linearly independent vectors $W_N$, a subset $J \subset [N]$, and numbers $(k_j)_{j \in J}$ from $\Theta$, conditions

$$Z \in \text{span}(W_N), \quad \langle Z, W_{\sigma_W(i)} \rangle = \begin{cases} 1_{\{\sigma_W(i)=N\}} + k_i \delta', & \text{if } i \in J, \\
\langle Z, P_{\text{span}\{W_{\sigma_W(j), j<i}\}}(W_{\sigma_W(i)}) \rangle, & \text{if } i \in [N] \setminus J, \end{cases} \quad (18)$$
uniquely determine a vector $Z = Z_{W_N,J(k_j)}$. Thus we can say that if $T(W_N,\delta', T)$ is not empty then there exist a subset $J \subset [N]$ and complex numbers with integer real and imaginary parts, 

$$k_i \in \{z \in \mathbb{C} : |z| \leq \sqrt{2}(N + 1)\} \cap (\mathbb{Z} + i\mathbb{Z}), \quad i \in J,$$

such that vector $Z_{W_N,J(k_j)}$ defined in (13) satisfies 

$$\langle Z_{W_N,J(k_j)}, W_{\sigma_{W(i)}(i) = N} \rangle - 1_{\{\sigma_{W(i)} = N\}} \leq \sqrt{2}(N + 1) \delta', \quad i \in [N] \quad \text{and} \quad \|Z_{W_N,J(k_j)}\|_2 \leq T.$$

(If $\Theta = \mathbb{R}$ then $k_i \in (-\sqrt{2}(N + 1), \sqrt{2}(N + 1)) \cap \mathbb{Z}$.)

The key statement below shows that every vector satisfying relations from (17) can be perturbed into a vector satisfying conditions (18), without increasing its length.

**Lemma 5.14.** Let $W_N$ be a sequence of linearly independent vectors in $E$. Assume that $\delta' > 0$ and $Y \in E$ are such that 

$$|\langle Y, W_N \rangle| = 1 \quad \text{and} \quad |\langle Y, W_i \rangle| \leq \delta', \quad 1 \leq i \leq N - 1.$$

Then the set $T(W_N, \delta', \|Y\|_2)$ defined above is non-empty.

**Proof.** We will prove the statement when $\Theta = \mathbb{C}$; the real case follows by a straightforward adaptation of the argument. Without loss of generality, $\langle Y, W_N \rangle = 1$. For brevity, we will use notation $W'_i := W_{\sigma_{W}(i)}$, $i \leq N$. Further, set $q := \sigma^{-1}_{W}(N)$. Assuming conditions of the lemma, we will construct a vector $\tilde{V} \in T(W_N, \delta', \|Y\|_2)$ in $N$ steps via an inductive argument. Set $Y_0 := Y$. Fix any $m \in \{1, \ldots, N\}$ and denote $F_{m-1} := \text{span}\{W'_1, \ldots, W'_{m-1}\}$. Assume that there exists a vector $Y_{m-1} \in E$ satisfying the following conditions:

- a) $\|Y_{m-1}\|_2 \leq \|Y\|_2$,
- b) for any $i \in [N]$, $|\langle Y_{m-1}, W'_i \rangle - 1_{\{i = q\}}| \leq \sqrt{2}m\delta'$, and
- c) for any $i \in [m - 1]$, either 

$$\langle Y_{m-1}, W'_i \rangle - 1_{\{i = q\}} / \delta' \in \mathbb{Z} + i\mathbb{Z} \quad \text{or} \quad \langle Y_{m-1}, W'_i \rangle = \langle Y_{m-1}, P_{\text{span}\{W'_1, \ldots, W'_{m-1}\}} W'_i \rangle.$$

Note that $Y_0$ satisfies all of the above conditions with $m = 1$. We represent $Y_{m-1}$ in the form 

$$Y_{m-1} = Y''_{m-1} + Y''_{m-1} + Y'''_{m-1}, \quad \text{where} \quad Y'''_{m-1} = \langle Y_{m-1}, P_{F_{m-1}^\perp} W'_m \rangle \frac{P_{F_{m-1}^\perp} W'_m}{\|P_{F_{m-1}^\perp} W'_m\|_2};$$

$Y''_{m-1} \in F_{m-1}$, and $Y'''_{m-1} \in F_{m-1}^\perp$ (observe that this representation is unique, and the vectors $Y''_{m-1}, Y'''_{m-1}$, $Y'''_{m-1}$ are pairwise orthogonal). Now, define a parametric family of vectors 

$$Z(t) := Y''_{m-1} + t Y'''_{m-1} + Y'''_{m-1}, \quad t \in \mathbb{C}.$$

Clearly, $\|Z(t)\|_2 \leq \|Y_{m-1}\|_2$ for every $t$ with $|t| \leq 1$. Also we have 

$$\langle Z(t), W'_i \rangle = \langle Y_{m-1}, W'_i \rangle \quad \text{for every} \quad i \leq m - 1, \quad \text{and} \quad \langle Z(t), W'_m \rangle = \langle Y_{m-1}, W'_m \rangle + t \langle Y_{m-1}, P_{F_{m-1}^\perp} W'_m \rangle = \langle Y''_{m-1}, W'_m \rangle + t \langle Y'''_{m-1}, W'_m \rangle,$$

hence, in particular, $Z(t)$ satisfies (b) for every $t$ and every $i \in [m - 1]$. Now let 

$$\Gamma := \{t \in \mathbb{C} : \langle Z(t), W'_m \rangle - 1_{\{m = q\}} \in \delta' \mathbb{Z} + \delta' i\mathbb{Z}\}.$$

If $\Gamma = \emptyset$, then $\langle Y_{m-1}, P_{F_{m-1}^\perp} W'_m \rangle = 0$. In that case we set $Y_m := Y(1) = Y_{m-1}$ and observe that $Y_m$ satisfies the above conditions (a)–(c) with $m + 1$ in place of $m$.

Further, assume that $\Gamma \neq \emptyset$ and $|\langle Y_{m-1}, W'_m \rangle| \leq 2\delta'$. In this case we set $Y_m := Y(0)$. We then have 

$$\langle Y_m, W'_m \rangle = \langle Y''_{m-1}, W'_m \rangle = \langle Y''_{m-1}, P_{F_{m-1}^\perp} W'_m \rangle = \langle Y_m, P_{F_{m-1}} W'_m \rangle.$$
Further, by the definition of the permutation $\sigma_W$ (see (14)) and the sequence $W'_1, \ldots, W'_N$, for all $j > m$ we have
\[
\| \langle P_{F_{m-1}^\perp} W'_m, W'_j \rangle \| \leq \| P_{F_{m-1}^\perp} W'_m \|_2 \| P_{F_{m-1}^\perp} W'_j \|_2 \leq d_m^2 = \| \langle P_{F_{m-1}^\perp} W'_m, W'_m \rangle \|.
\]
Hence,
\[
(23) \quad \| \langle Y'_{m-1}, W'_j \rangle \| \leq \| \langle Y'_{m-1}, W'_m \rangle \|, \quad \forall j \geq m,
\]
and
\[
\| \langle Y_m, W'_j \rangle \| \leq \| \langle Y_{m-1}, W'_j \rangle \| + \| \langle Y''_{m-1}, W'_j \rangle \| \leq \| \langle Y_{m-1}, W'_j \rangle \| + \sqrt{2}\delta' \leq \sqrt{2}(m+1)\delta', \quad j > m.
\]
Combining this with (21) and (22), we get that the vector $Y_m$ satisfies conditions (a)–(c) with $m$ replaced with $m+1$.

Now assume that $\Gamma \neq \emptyset$ and $\| \langle Y''_{m-1}, W'_m \rangle \| > \sqrt{2}\delta'$. Note that for every $t_1, t_2 \in \Gamma$ we have
\[
\tau_i := \frac{\langle Z(t_1), W'_m \rangle - 1_{\{m=q\}}}{\| \langle Y''_{m-1}, W'_m \rangle \|} \in \frac{\delta'}{\| \langle Y''_{m-1}, W'_m \rangle \|}(\mathbb{Z} + i\mathbb{Z}), \quad i = 1, 2,
\]
and $|t_1 - t_2| = |\tau_1 - \tau_2|$. Hence, $\Gamma$ is an affine distance-preserving transformation of the lattice $(\delta'/(\| \langle Y''_{m-1}, W'_m \rangle \|))(\mathbb{Z} + i\mathbb{Z})$, and, in particular, any disc in $\mathbb{C}$ of radius greater or equal to $\delta'/(\sqrt{2}(\| \langle Y''_{m-1}, W'_m \rangle \|))$ contains at least one point from $\Gamma$. Thus, condition $\| \langle Y''_{m-1}, W'_m \rangle \| > \sqrt{2}\delta'$ guarantees that the intersection of the discs $\{t : |t| \leq 1\}$ and $\{t : |1-t| \leq \sqrt{2}\delta'/\| \langle Y''_{m-1}, W'_m \rangle \|}\}$ contains at least one point, say $t_0$, from $\Gamma$. Set $Y_m := Z(t_0)$. We need only to check (b) for $i \geq m$. Since $Z(1) = Y_{m-1}$, we have $Z(t_0) - Y_m = (t_0 - 1)Y_{m-1}$. Hence, for every $j \geq m$, the conditions on $t_0$ and (23) imply
\[
\| \langle Z(t_0), W'_j \rangle - 1_{\{j=q\}} \| \leq \| \langle Y_{m-1}, W'_j \rangle - 1_{\{j=q\}} \| + |t_0 - 1| \| \langle Y''_{m-1}, W'_j \rangle \| \leq \| \langle Y_{m-1}, W'_j \rangle - 1_{\{j=q\}} \| + \sqrt{2}\delta',
\]
where by the induction hypothesis $Y_{m-1}$ satisfies $\| \langle Y_{m-1}, W'_j \rangle - 1_{\{j=q\}} \| \leq \sqrt{2}m\delta'$. Thus, the vector $Y_m$ satisfies (a)–(c) with $m$ replaced with $m+1$.

The above procedure produces a sequence of vectors $Y_0, Y_1, \ldots, Y_N$ satisfying (a)–(c). Clearly, $Y_N \in \mathcal{Y}(W_N, \delta', \|Y\|_2)$. This finishes the proof. \qed

Summarising results of this section and combining Lemma 5.14 with Corollaries 5.10 and 5.12 we get:

**Proposition 5.15.** Let $X_N$, $\mathcal{Y}_N$ be a biorthogonal system in $E$ over $\mathbb{C}$ and $W_N$ be a $\delta$-perturbation of $Y_N$ (for some $\delta \in (0,1]$). Assume that for some $b$ satisfying (14) and some $R \geq 1$, $T > 0$ with $T\delta \leq 1/2$ and $R \leq \delta^{-1}N^{-1/2}$ we have
\[
E(X_N) \in \mathcal{C}(R, b) \quad \text{and} \quad X_N \in \delta E(X_N-1) + TB_F^2.
\]
Let $r < 1$ be such that $d_i(W_N) \geq r$, $i \leq N$, where $d_i$'s are defined in (14). Then there exists $p$ satisfying (16) and $2\sum_{i=1}^N p_i \leq (4N)^{N/2} 2^{-\sum_{i=1}^N b_i}$ such that
\[
(24) \quad W_N \in \mathcal{D}(r, p) \quad \text{and} \quad \mathcal{Y}(W_N, 2\delta(T+1), 2T) \neq \emptyset.
\]
In particular, the second condition in (24) implies that there exist a subset $J \subset [N]$ and numbers
\[
k_i \in \{z \in \mathbb{C} : |z| \leq \sqrt{2}(N+1)\} \cap (\mathbb{Z} + i\mathbb{Z}), \quad i \in J,
\]
such that vector $Z_{W_N, J_i(k_j)}$ defined in (18) with $\delta' = 2\delta(T+1)$ satisfies
\[
\| Z_{W_N, J_i(k_j)} W_{\sigma_W(i)} - 1_{\{\sigma_W(i) = N\}} \| \leq \sqrt{2}(N+1)\delta', \quad i \in [N] \quad \text{and} \quad \| Z_{W_N, J_i(k_j)} \|_2 \leq 2T.
\]
Note that if $\Theta = \mathbb{R}$, then the statement remains valid with $k_i \in (-\sqrt{2}(N+1), \sqrt{2}(N+1)) \cap \mathbb{Z}$. 


5.3. Anti-concentration for anisotropic vectors. In this subsection, we study anti-concentration properties of random vectors of the form $DX$, where $D$ is a fixed $m \times m$ matrix and $X$ is an $m$-dimensional isotropic random vector with i.i.d $K$-subgaussian coordinates. We distinguish two types of behavior: first, when many singular values of $D$ are relatively large, in which case $DX$ is well spread, and, second, when $D$ has only a few large singular values in which case $DX$ is essentially contained in a fixed low-dimensional subspace.

Let us start with the following corollary of the Hanson–Wright inequality proved in [45] which will be applied to matrices $D$ of the first type.

**Lemma 5.16** ([45] Theorem 4.1(i)). Let $D$ be an $m \times m$ fixed matrix, let $m \geq h > p \geq 1$, and let $X_1, \ldots, X_p$ be i.i.d random vectors in $\Theta^m$ satisfying $[\text{**}]$ or $[\text{***}]$ with a parameter $K$. Then with probability at least $1 - 2pe^{-c(h-p)}$ we have

$$\text{dist}(DX_i, \text{span}\{DX_j : j \in [p] \setminus \{i\}\}) \geq c\left(\sum_{\ell=h}^m s_\ell(D)^2\right)^{1/2} \quad \text{for all } i \in [p].$$

Here, $c > 0$ may only depend on $K$.

The next lemma will be used while dealing with matrices of the second type (with few large singular values).

**Lemma 5.17.** Let $D$ be an $m \times m$ fixed matrix, and let $X$ be a random vector in $\Theta^m$ satisfying $[\text{**}]$ or $[\text{***}]$ with a parameter $K$. Let $R > 0$, $m \geq h \geq 1$, and assume that $(\sum_{\ell=h}^m s_\ell(D)^2)^{1/2} \leq \psi$ for some $\psi > 0$. Then with probability at least $1 - 2me^{-c\psi^{-2}R^{-2}}$ we have

$$\|D^*Y\|_2 \geq \psi R/h \quad \text{for all vectors } Y \in \Theta^m \text{ with } \|Y\|_2 \leq R \text{ and } \langle DX, Y \rangle = 1.$$

Here, $c > 0$ may only depend on $K$.

**Proof.** Let $D = \sum_{i=1}^m s_i(D)u_i, v_i^*$ be the singular value decomposition of $D$ (with $u_i$ and $v_i$ being the normalized left and right singular vectors, respectively). Applying a concentration inequality for sums of subgaussian random variables (see, for example, [55] Proposition 5.10), we get

$$\mathbb{P}\{|\langle X, v_i \rangle| \geq t\} \leq 2e^{-c_1 t^2}, \quad t > 0,$

for some $c_1 = c_1(K) > 0$. Taking the union bound over all $i$, we obtain that for any $t > 0$ the event

$$\mathcal{E}_t := \{|\langle X, v_i \rangle| \leq t \text{ for all } i \leq m\}$$

has probability at least $1 - 2me^{-c_1 t^2}$. If we denote by $P$ the orthogonal projection onto the span of vectors $v_h, v_{h+1}, \ldots, v_m$ then, on the event $\mathcal{E}_t$, we have $\|DPX\|_2 \leq \psi t$.

Let $Y$ be any vector in $\Theta^m$ with $\|Y\|_2 \leq R$. Then

$$|\langle DX, Y \rangle| \leq R\|DPX\|_2 + |\langle (D(I_m - P)X, Y \rangle|.$$

Setting $t := 1/(2\psi R)$, we obtain from the above that everywhere on $\mathcal{E}_t$ we have

$$|\langle (D(I_m - P)X, Y \rangle| \geq 1/2 \quad \text{for all vectors } Y \in \Theta^m \text{ with } \|Y\|_2 \leq R \text{ and } |\langle DX, Y \rangle| = 1.$$

Note that condition $|\langle (D(I_m - P)X, Y \rangle| \geq 1/2$ immediately implies

$$s_i(D)|\langle X, v_i \rangle (u_i, Y) \rangle \geq 1/(2h) \quad \text{for some } i = i(Y) < h.$$

On the other hand, everywhere on $\mathcal{E}_t$ we have $|\langle X, v_i \rangle| \leq t$ for all $i < h$. Thus, we get that, conditioned on $\mathcal{E}_t$ (with the above choice of $t$), we have

$$s_i(D)|\langle u_i, Y \rangle| \geq \psi R/h \quad \text{for some } i = i(Y) < h \text{ whenever } \|Y\|_2 \leq R \text{ and } |\langle DX, Y \rangle| = 1.$$

But the leftmost condition immediately gives $\|D^*Y\|_2 \geq \psi R/h$. The result follows. $\square$
The above lemma emphasizes an important property of anisotropic vectors with few principal components. Consider first the opposite situation, when all singular values of $D$ are roughly comparable. Say, if $D = I_m$ then the relation $\langle DX, Y \rangle = \langle X, Y \rangle = 1$ holds for vector $Y = \frac{X}{\|X\|_2}$ which has Euclidean norm of order $O(m^{-1/2})$ with large probability. Thus, in this case there is no strong lower bound for $\|D^*Y\|_2$. On the other hand, when $DX$ has few principal components, that is, when most singular values of $D$ are small, the condition $\langle DX, Y \rangle = \langle X, D^*Y \rangle = 1$ does guarantee (with large probability) that $\|D^*Y\|_2$ is large. This happens because $DX$ is almost contained in a fixed low-dimensional subspace, and therefore no dependence on $m$ appears in the lower bound for $\|D^*Y\|_2$.

Let $m, n, N$ satisfy $m = n - N$. Let $X_1, \ldots, X_N$ be i.i.d. random vectors in $\Theta^m$ satisfying (**) or (***) with a parameter $K$. Given an $m \times m$ matrix $D$ and a parameter $\kappa \in \Theta \setminus \{0\}$, define $n$-dimensional random vectors

$$\tilde{V}_i := \kappa e_i \oplus DX_i, \quad i \leq N,$$

(see (**)). In what follows we will take $\tilde{V}_i = V_i - (\tilde{z}^{-1}/|z^{-1}|) \delta e_i$, so that $\tilde{V}_N$ is a $\delta$-perturbation of $V_N$. We apply the above lemmas to estimate probability of the event that $\tilde{V}_N$ falls into a given class $D(r, p)$ and that there exists a vector $\tilde{x}$ with prescribed scalar products with $\tilde{V}_i$’s. The proof is based on a decoupling argument which allows to deal with dependencies between $\tilde{x}$ and $\tilde{V}_N$.

**Lemma 5.18.** Let $m, n, N \in \mathbb{N}$, let $D$ be an $m \times m$ matrix with $\sum_{\ell = N^2}^m s_\ell(D)^2 \geq (NT')^{-2}$, $T' \geq 1$, and let $\tilde{V}_N$ be defined by (25) for some $\kappa \in \Theta \setminus \{0\}$. Fix any $0 < r < 1$ and any $p$ satisfying (10). Then

$$\mathbb{P}\{\tilde{V}_N \in D(r, p)\} \leq (CN)^{CN} \prod_{i=1}^N (2^pT')^U + 2e^{-cN^2},$$

with $U = 1$ for $\Theta = \mathbb{R}$ and $U = 2$ for $\Theta = \mathbb{C}$, for some $C, c > 0$ depending only on $K$.

**Proof.** In view of the assumptions on the singular values of $D$ and in view of Lemma 5.16 we get that with probability at least $1 - 2e^{-c_1N^2}$

$$(26) \quad \text{dist}(\tilde{V}_i, \text{span} \{\tilde{V}_j : j \neq i\}) \geq \text{dist}(DX_i, \text{span} \{DX_j : j \neq i\}) \geq c_1/(NT')$$

for all $i \leq N$ for some constant $c_1 \in (0, 1]$ which may only depend on $K$. Note that if $2^p_{i+1} \geq c_1/(NT')$ for all $i \leq N$ then, for sufficiently large $C'$, we get $(C'N)^{C'N} \prod_{i=1}^N (2^pT')^U \geq 1$, and the required probability estimate is trivial. On the other hand, if $2^p_{i+1} < c_1/(NT')$ for some $i \leq N$ then, conditioned on (25), we get $\min(\max(d_i, r), 1) > 2^p_{i+1}$ (where $d_i$’s are defined for the sequence $\tilde{V}_N$ by (14)). Hence, $\tilde{V}_N \notin D(r, p)$, and probability of the corresponding event is bounded from above by probability of the complement of the event (25), i.e. by $2e^{-c_1N^2}$. The result follows.

**Lemma 5.19.** Let $n \in \mathbb{N}$, let $N \geq \log n$, $m = n - N$; let $D$ be an $m \times m$ matrix with $\sum_{\ell = N^2}^m s_\ell(D)^2 < (NT')^{-2}$, and let $\kappa \in \Theta, |\kappa|, \delta' \geq n^{-1}, T' \geq 1$ be some parameters such that

$$\sqrt{2}(N + 1)\delta' + |\kappa|T' \leq 1/2.$$ 

Fix any $0 < r < 1$ and any $p$ satisfying (10) and consider the event

$$\mathcal{E} := \{\tilde{V}_N \in D(r, p) \quad \text{AND} \quad \Upsilon(\tilde{V}_N, \delta', T') \neq \emptyset\}.$$
Then for some $C, c > 0$ depending only on $K$ we have

$$\mathbb{P}(\mathcal{E}) \leq 2e^{-cN^2} + (CN)^C \prod_{i=1}^{N-1} \sup Y_i \mathbb{P}\{ \min(\langle X_1, D^* Y_i \rangle, T') \leq \eta_i \},$$

where $\eta_i := \sqrt{2}(N + 1) \delta' + 2^{p_i+1} T' + |\kappa| T'$, and for each $i \leq N - 1$ the supremum is taken over all vectors $Y_i \in \Theta^m$ with $\|Y_i\|_2 \leq T'$ and $\|D^* Y_i\|_2 \geq N^{-3}$.

Proof. Set $h := N^2$. It follows from the definition of $\Upsilon(\tilde{\nu}_N, \delta', T')$ that there exists a universal constant $\tilde{C} > 0$ such that

$$\mathbb{P}(\mathcal{E}) \leq (\tilde{C} N)^\tilde{C} \max_{J \subset [N]} \max_{(k_j)_{j \in J}} \mathbb{P}\{ \tilde{V}_N \in D(r, p) \text{ AND } Z_{\tilde{V}_N, J, (k_j)} \text{ satisfies } (20) \},$$

where $(k_j)_{j \in J}$ are sequences of numbers satisfying (19), and vectors $Z_{\tilde{V}_N, J, (k_j)}$ are defined by (18), with $W_N$ replaced with $\tilde{V}_N$. From now on we fix any $J \subset [N]$ and any admissible sequence $(k_j)_{j \in J}$. We have:

$$\mathbb{P}\{ \tilde{V}_N \in D(r, p) \text{ AND } Z_{\tilde{V}_N, J, (k_j)} \text{ satisfies } (20) \} \leq N! \max_{\sigma} \mathbb{P}(\mathcal{E}_\sigma),$$

where

$$\mathcal{E}_\sigma := \{ \tilde{V}_N \in D(r, p) \text{ and } \|\tilde{X}\|_2 \leq T' \text{ and } |\langle \tilde{X}, \tilde{V}_{\sigma(i)} \rangle - 1_{\{\sigma(i) = N\}}| \leq \sqrt{2}(N + 1) \delta', \ i \in [N], \text{ and } \sigma_{\tilde{V}} = \sigma \},$$

and $\tilde{X} = \tilde{X}_{\tilde{V}_N, J, (k_j)}$ is a random vector uniquely defined by the conditions

$$\tilde{X} \in \text{span}(\tilde{V}_N), \ (\tilde{X}, \tilde{V}_{\sigma(i)}) = \begin{cases} 1_{\{\sigma(i) = N\}} + k_i \delta', & \text{if } i \in J, \\ (\tilde{X}, P_{\text{span}(\tilde{V}_{\sigma(j), J \setminus i})}(\tilde{V}_{\sigma(i)})), & \text{if } i \in [N] \setminus J. \end{cases}$$

Note that the definition of $\tilde{X}$ does not involve the permutation $\sigma_{\tilde{V}}$, which is crucial for the claim below. Now to prove the lemma it is enough to show that

$$\mathbb{P}(\mathcal{E}_\sigma) \leq (C' N)^{C'} \prod_{i=1}^{N-1} \sup Y_i \mathbb{P}\{ \min(\langle X_1, D^* Y_i \rangle, T') \leq \eta_i \} + 2e^{-cN^2}.$$

Let $P : \Theta^n \to \Theta^m$ be the coordinate projection onto the last $m$ coordinates and for each $i \leq N$ let $P_i : \Theta^n \to \Theta^n$ be the orthogonal projection onto span $\{\tilde{V}_N; \tilde{V}_{\sigma(j), J \setminus i} \}$. Define the events

$$\mathcal{E}' := \{ \|\tilde{X}\|_2 > T' \text{ OR } |\langle \tilde{X}, \tilde{V}_{\sigma(i)} \rangle - 1_{\{\sigma(i) = N\}}| > \sqrt{2}(N + 1) \delta' \text{ for some } i \in [N] \} \quad \text{and} \quad \mathcal{E}'' := \{ \|D^* PP_i \tilde{X}\|_2 \geq N^{-3} \text{ for all } i \neq \sigma^{-1}(N) \};$$

so that on $(\mathcal{E}' \cup \mathcal{E}'')^c$ we have $\|\tilde{X}\|_2 \leq T'$, $|\langle \tilde{X}, \tilde{V}_{\sigma(i)} \rangle - 1_{\{\sigma(i) = N\}}| \leq \sqrt{2}(N + 1) \delta'$ for all $i \in [N]$ and there exists a (random) index $i_0 \leq N$ such that $\|D^* PP_{i_0} \tilde{X}\|_2 < N^{-3}$. Since $P\tilde{V}_i = DX_i$ and $\langle P_{i_0} \tilde{X}, \tilde{V}_N \rangle = \langle \tilde{X}, \tilde{V}_N \rangle$, we have

$$\langle PP_{i_0} \tilde{X}, DX_N \rangle = \langle \tilde{X}, \tilde{V}_N \rangle - \kappa(P_{i_0} \tilde{X}) N,$$

and by the above conditions, everywhere on $(\mathcal{E}')^c$ we have

$$\langle PP_{i_0} \tilde{X}, DX_N \rangle \geq |\langle \tilde{X}, \tilde{V}_N \rangle| - |\kappa| \|\tilde{X}\|_2 \geq 1 - \sqrt{2}(N + 1) \delta' - |\kappa| \|\tilde{X}\|_2,$$

By the choice of parameters and (27), $|\langle PP_{i_0} \tilde{X}, DX_N \rangle| \geq 1/2$ everywhere on $(\mathcal{E}')^c$. Hence, everywhere on $(\mathcal{E}' \cup \mathcal{E}'')^c$ the vector $y := PP_{i_0} \tilde{X}/\langle PP_{i_0} \tilde{X}, DX_N \rangle$, satisfies

$$\|y\|_2 \leq 2T', \quad |\langle y, DX_N \rangle| = 1, \quad \text{and } \|D^* y\| < 2N^{-3}.$$
Now Lemma 5.17 with \( \psi = 1/(NT') \) and \( R = 2T' \) implies
\[
\mathbb{P}((\mathcal{E}' \cup \mathcal{E}'')^c) \leq 2e^{-c_2 N^2}
\]
for some \( c_2 > 0 \) depending only on the subgaussian moment. Thus,
\[
\mathbb{P}(\mathcal{E}_\sigma) \leq \mathbb{P}(\mathcal{E}_\sigma \cap \mathcal{E}') + \mathbb{P}(\mathcal{E}_\sigma \cap \mathcal{E}'') + \mathbb{P}((\mathcal{E}' \cup \mathcal{E}'')^c) \leq \mathbb{P}(\mathcal{E}_\sigma \cap \mathcal{E}'') + 2e^{-c_2 N^2}.
\]
On \( \mathcal{E}_\sigma \cap \mathcal{E}'' \), for any \( i \leq N \) such that \( i \neq \sigma^{-1}(N) \) by our assumptions we have
\[
|\langle \bar{V}_{\sigma(i)}, \bar{X} \rangle| \leq \sqrt{2}(N+1)\delta', \quad \min\left(\text{dist}(\bar{V}_{\sigma(i)}, \text{span}\{\bar{V}_{\sigma(j)} : j < i\}), 1\right) \leq 2^{\rho_i+1},
\]
whence
\[
\min\left(|\langle DX_{\sigma(i)}, PP_i \bar{X} \rangle|, T'\right) \leq \min\left(|\langle \bar{V}_{\sigma(i)}, P_i \bar{X} \rangle|, T'\right) + |\kappa| T'
\]
\[
\leq |\langle \bar{V}_{\sigma(i)}, \bar{X} \rangle| + \min\left(|\langle \bar{V}_{\sigma(i)}, \bar{X} - P_i \bar{X} \rangle|, T'\right) + |\kappa| T' \leq \eta_i
\]
for all \( i \neq \sigma^{-1}(N) \). Introduce events
\[
\mathcal{E}_i := \left\{ \min\left(|\langle X_{\sigma(i)}, D^* PP_i \bar{X} \rangle|, T'\right) \leq \eta_i \quad \text{and} \quad \|D^* PP_i \bar{X}\|_2 \geq N^{-3} \quad \text{and} \quad \|P_i \bar{X}\|_2 \leq T' \right\}.
\]
From the above, we get
\[
\mathcal{E}_\sigma \cap \mathcal{E}'' \subset \bigcap_{i \neq \sigma^{-1}(N)} \mathcal{E}_i.
\]
The following observation is crucial.

Claim. For every \( i \in [N] \setminus \{\sigma^{-1}(N)\} \), the event \( \mathcal{E}_i \) is measurable with respect to the algebra generated by \( \{X_N; X_{\sigma(j)}, j \leq i\} \). Indeed, it is obvious that \( \mathcal{E}_i \) is measurable with respect to the algebra generated by \( X_{\sigma(i)} \) and \( P_i \bar{X} \). Further, the vector \( P_i \bar{X} \) satisfies \( \langle \bar{V}_{\sigma(j)}, P_i \bar{X} \rangle = \langle \bar{V}_{\sigma(j)}, \bar{X} \rangle, j \in \{1, \ldots, i-1; \sigma^{-1}(N)\} \), whence, by the definition of \( \bar{X} \),
\[
\langle P_i \bar{X}, \bar{V}_{\sigma(i)} \rangle = \begin{cases} 1_{\{\sigma(j)=N\}} + k_i \delta', & \text{if } j \in J, \\ \langle P_i \bar{X}, P_{\text{span}\{\bar{V}_{\sigma(i)}, \ell < j\}}(\bar{V}_{\sigma(j)}) \rangle, & \text{if } j \in [N] \setminus J. \end{cases}
\]
These conditions, together with the linear independence of \( \bar{V}_j \)'s, imply that \( P_i \bar{X} \) is uniquely determined by \( \{X_N; X_{\sigma(j)}, j < i\} \), and the claim follows.

Applying the above claim, we get
\[
\mathbb{P}(\mathcal{E}_\sigma \cap \mathcal{E}'') \leq \mathbb{P}\left(\bigcap_{i \neq \sigma^{-1}(N)} \mathcal{E}_i\right) \leq \prod_{i \neq \sigma^{-1}(N)} \text{ess sup} \mathbb{P}(\mathcal{E}_i | X_{N}; X_{\sigma(1)}, \ldots, X_{\sigma(i-1)}).
\]
Finally, we estimate
\[
\text{ess sup} \mathbb{P}(\mathcal{E}_i | X_{N}; X_{\sigma(1)}, \ldots, X_{\sigma(i-1)}).
\]
Fix any realization of \( X_N; X_{\sigma(1)}, \ldots, X_{\sigma(i-1)} \). If \( \|P_i \bar{X}\|_2 > T' \) then the conditional probability of \( \mathcal{E}_i \) given this realization of \( X_N; X_{\sigma(1)}, \ldots, X_{\sigma(i-1)} \), is equal to zero. Otherwise, the conditional probability can be bounded from above by
\[
\sup_{Y_i} \mathbb{P}\{ \min(|\langle X_{\sigma(i)}, D^* Y_i \rangle|, T') \leq \eta_i \},
\]
where the supremum is taken over all vectors \( Y_i \in \Theta'' \) with \( \|Y_i\|_2 \leq T' \) and \( \|D^* Y_i\|_2 \geq N^{-3} \). The result follows. \( \square \)

The next proposition is the main result of the subsection, obtained by combining the above Lemmas 5.18 and 5.19 with anti-concentration statements for random vectors with independent components.
Proposition 5.20. Let $n \in \mathbb{N}$, let $N \geq \log n$, $m = n - N$; let $D$ be an $m \times m$ matrix, and let $\kappa \in \Theta \setminus \{0\}$, $\delta' \geq n^{-1}$, $T' \geq 1$ be some parameters such that
\[
\sqrt{2}(N + 1)\delta' + |\kappa|T' \leq 1/2.
\]
Assume that for any vector $Y \in \Theta^m$ with $\|Y\|_2 \leq T'$ and $\|D^*Y\|_2 \geq N^{-3}$, we have
\[
\text{LCD}_{\min\{\eta \}} \left( \frac{D^*Y}{\|D^*Y\|_2} \right) \geq \frac{\eta}{NT'}, \quad \text{if } \Theta = \mathbb{R};
\]
\[
\frac{D^*Y}{\|D^*Y\|_2} \in \text{Incomp}_m(\zeta, \zeta) \text{ for some } \zeta > 0 \quad \text{if } \Theta = \mathbb{C}.
\]
Fix any $0 < r < 1$ and any $p$ satisfying \(16\) and consider the event
\[
\mathcal{E} := \{ \tilde{V}_N \in D(r, p) \text{ AND } T(\tilde{V}_N, \delta', T') \neq \emptyset \}.
\]
Then
\[
\mathbb{P}(\mathcal{E}) \leq (CN)^{CN} \prod_{i=1}^N \left( \delta' + |\kappa|T' + 2p_iT' + m^{-1}/U \right)^U + 2e^{-cN^2},
\]
where $U = 1$ for $\Theta = \mathbb{R}$ and $U = 2$ for $\Theta = \mathbb{C}$, and $C, c > 0$ depend only on the subgaussian moment $K$ and (in the complex case) the parameter $\zeta$.

Proof. We consider two cases. First, if $\sum_{\ell=1}^m s_\ell(D)^2 \geq (NT')^{-2}$ then the statement immediately follows from Lemma 5.18.

Otherwise, $\sum_{\ell=1}^m s_\ell(D)^2 < (NT')^{-2}$. Then, applying Lemma 5.19 we get
\[
\mathbb{P}(\mathcal{E}) \leq (CN)^{CN} \prod_{i=1}^{N-1} \sup_{Y_i} \mathbb{P}\left\{ \min(|\langle X_1, D^*Y_i \rangle|, T') \leq \eta_i \right\} + 2e^{-cN^2},
\]
where $\eta_i := \sqrt{2}(N + 1)\delta' + 2p_iT' + m^{-1}/U$, and each supremum is taken over all $Y_i \in \Theta^m$ satisfying $\|Y_i\|_2 \leq T'$ and $\|D^*Y_i\|_2 \geq N^{-3}$.

Fix any $i \leq N - 1$. If $T' \leq \eta_i$ then we have
\[
\sup_{Y_i} \mathbb{P}\left\{ \min(|\langle X_1, D^*Y_i \rangle|, T') \leq \eta_i \right\} \leq 1 \leq (\eta_i/T')^U \leq (N\eta_i)^U.
\]
If $T' \geq \eta_i$ then for every admissible $Y_i$ we have
\[
\mathbb{P}\left\{ \min(|\langle X_1, D^*Y_i \rangle|, T') \leq \eta_i \right\} \leq \mathbb{P}\left\{ |\langle X_1, D^*Y_i \rangle| \leq \eta_i \right\}.
\]
We will bound the latter using anti-concentration estimates for isotropic vectors.

In the case $\Theta = \mathbb{R}$, by the assumptions of the proposition, for every admissible $Y_i$ the vector $\frac{D^*Y_i}{\|D^*Y_i\|_2}$ has LCD of order at least $\frac{m}{NT'}$. Hence, applying Theorem 2.3
\[
\mathbb{P}\left\{ |\langle X_1, D^*Y_i \rangle| \leq \eta_i \right\} \leq \tilde{C} \frac{\eta_i}{\|D^*Y_i\|_2} + \tilde{C} NT'm^{-1}.
\]

In the case $\Theta = \mathbb{C}$, the vector $\frac{D^*Y_i}{\|D^*Y_i\|_2}$ is incompressible. Hence, applying Theorem 2.6 we get
\[
\mathbb{P}\left\{ |\langle X_1, D^*Y_i \rangle| \leq \eta_i \right\} \leq \tilde{C} \left( \frac{\eta_i}{\|D^*Y_i\|_2} \right)^2 + \tilde{C}m^{-1}
\]
for some $\tilde{C} > 0$ depending only on $\zeta$ and the subgaussian moment $K$.

It remains to recall in both cases that $\|D^*Y_i\|_2 \geq N^{-3}$. The statement follows. \qed
5.4. Comparison of coordinates, proofs of Theorems B and C. Using the results of the previous two subsections, here we estimate probabilities of the geometric relations given in Lemma 5.1 and then prove the main results of this paper.

**Lemma 5.21.** Given \( n \in \mathbb{N} \) and \( \log^2 n \leq N \), let \( F \) be a fixed \( N \)-dimensional subspace of \( \Theta^n \), and \( X_N \) be a fixed sequence of vectors in \( F \). Let \( n \geq R \geq 1 \). Let the \( N \)-dimensional ellipsoid \( E = E(X_N) \subset F \) be defined by (3). Assume that \( E \in C(R,b) \) for some admissible sequence of integers \( b \). Further, let \( Z \) be a random vector in \( \Theta^n \) satisfying (\ref{WT}) or (\ref{WT1}). Then for any \( \delta > 0 \), \( \tau > 0 \), \( t > 0 \) and any fixed vector \( Y \) we have,

\[
\mathbb{P}\{P_F(Z + Y) \in \delta E + t B_2^F \} \leq N^{C N \left( \frac{t + \delta}{\delta} \right)^{U} 2^U \sum_{i=1}^{N} \beta_i \mathcal{L}(P_F(Z), \delta) + 2e^{-c N^2}
\]

for some \( C, c > 0 \) depending only on \( K \), where \( U = 1 \) for \( \Theta = \mathbb{R} \) and \( U = 2 \) for \( \Theta = \mathbb{C} \).

**Proof.** Denote by \( B \) the Euclidean ball in \( F \) of radius \( N \) centered at \( P_F(Y) \). Applying the Hanson–Wright inequality \cite{21, 44}, we get that

\[
\mathbb{P}\{P_F(X + Y) \in B \} \geq 1 - 2e^{-c_1 N^2}
\]

for some \( c_1 > 0 \) depending only on \( K \), whence

\[
\mathbb{P}\{P_F(Z + Y) \in \delta E + t b B^F_2 \} \leq \mathbb{P}\{P_F(Z + Y) \in (\delta E + t B^F_2) \cap B \} + 2e^{-c_1 N^2}.
\]

To estimate \( \mathbb{P}\{P_F(Z + Y) \in (\delta E + t B^F_2) \cap B \} \), note that for any \( \tau > 0 \) and any covering of \( (\delta E + t B^F_2) \cap B \) by \( L \) translates of \( \tau B_2^F(\Theta) \), we have

\[
\mathbb{P}\{P_F(Z + Y) \in (\delta E + t B^F_2) \cap B \} \leq L \mathcal{L}(P_F(Z), \tau).
\]

The cardinality of a minimal covering of \( (\delta E + t B^F_2) \cap B \) in the real case \( \Theta = \mathbb{R} \) can be estimated from above with help of a standard volumetric argument by the ratio

\[
\frac{\text{Vol}_N(\delta E \cap B + (t + \frac{\tau}{2}) B_2^F(\mathbb{R}))}{\text{Vol}_N(\frac{\tau}{2} B_2^F(\mathbb{R}))},
\]

which in turn is bounded above by \( (CN)^{N/2} \tau^{-N} \prod_{i=1}^{N} \left( \min(\delta s_i(X_N), N) + t + \frac{\tau}{2} \right) \). In the complex case \( \Theta = \mathbb{C} \), applying Lemma 2.1, we reduce the problem to estimating cardinality of a minimal covering of \( (\text{real}(E) + \text{real}(B_2^F(\mathbb{C})) \cap \text{real}(B) \) by translates of \( \tau \text{real}(B_2^F(\mathbb{C})) \). Repeating the above argument, we get at upper bound

\[
(CN)^N \tau^{-2N} \prod_{i=1}^{N} \left( \min(\delta s_i(X_N), N) + t + \frac{\tau}{2} \right)^2.
\]

Thus, for any \( \tau > 0 \)

\[
\mathbb{P}\{P_F(Z + Y) \in (\delta E + t B_2^F(\Theta)) \cap B \} \leq (C' \sqrt{N}/\tau)^{U} \mathcal{L}(P_F(Z), \tau) \prod_{i=1}^{N} \max(\tau + t, \min(\delta s_i(X_N), N))^U,
\]

where \( U = 1 \) for \( \Theta = \mathbb{R} \) and \( U = 2 \) for \( \Theta = \mathbb{C} \). Taking \( \tau := \delta \), we get

\[
\mathbb{P}\{P_F(Z + Y) \in (\delta E + t B_2^F(\Theta)) \cap B \} \leq N^{C N \left( \frac{t + \delta}{\delta} \right)^{U} 2^U \sum_{i=1}^{N} \beta_i \mathcal{L}(P_F(Z), \delta)}.
\]

The result follows. \( \Box \)
Lemma 5.22. Let $M \geq 1$, $n \in \mathbb{N}$, $\log^2 n \leq N \leq \sqrt{n}$, $z \in \Theta$ with $N^2 \leq |z| \leq M \sqrt{n}$, and let $\beta, \theta > 0$ be such that
\[ n^{-1/2} \leq \theta/\beta \leq 1/N \quad \text{and} \quad \beta \geq 1/n. \]
Let $A$ be an $n \times n$ random matrix with i.i.d columns satisfying (**) with a parameter $K$ and with a bounded distribution density of the entries, and let $F$ be the orthogonal complement of the linear span of last $n - N$ columns of $A - z$. Denote $\lambda_N := (P_F(-ze_1), \ldots, P_F(-ze_N))$, and let ellipsoids $E = E(\lambda_N)$ and $E' = E(\lambda_{N-1})$ be defined by (10). Set $R := \beta/(\theta N)$. Then for any fixed (non-random) integer sequence $p$ satisfying (**), the event
\[ \mathcal{E} := \left\{ P_F(-ze_N) \in (\theta \sqrt{N}/\beta)E' + 3NB^2_2(\Theta) \quad \text{AND} \quad E \in C(R, b) \right\} \]
has probability at most $N^{CN} 2^{-U \sum_{i=1}^N b_i} + 2e^{-cN^2}$, where $C,c > 0$ may only depend on $M$ and $K$, and $U = 2$ for $\Theta = \mathbb{C}$ and $U = 1$ for $\Theta = \mathbb{R}$.

Proof. Without loss of generality, we can assume that $n$ is large and that $\theta/\beta \leq N^{-2}$; otherwise the assertion is trivial. Let $T := 3N$, $\delta := \theta \sqrt{N}/\beta$. Note that the definition of $\mathcal{E}$ matches condition (13) which was the object of study in Subsection 5.2.

Define a collection of vectors $\tilde{V}_i$, $i = 1, \ldots, N$ in $\Theta^a$, by the formula
\[ \tilde{V}_i := (-\tilde{z}^{-1}e_i - \frac{\bar{z}^{-1}}{|z|} \bar{z}^{-1}e_i) \oplus Dq_i, \quad i \in [N], \]
where the $(n - N) \times (n - N)$ matrix $D$ is defined in the same way as in (11). Note that $D$ is well defined with probability 1 since all square submatrices of $A - z$ are non-singular almost everywhere, in view of the assumption on the distribution density of the entries. It is obvious that $\tilde{V}_N = (\tilde{V}_i)_i$ is a $\delta$-perturbation of the sequence $(\tilde{V}_i)$ given by (11). Applying Proposition 5.15, we get that everywhere on $\mathcal{E}$ the set $\mathcal{Y}(\tilde{V}_N, 2\delta(T + 1), 2T)$ is non-empty.

Set $r := \delta$, and observe that with this choice of parameters we have $1 < R = \delta^{-1}N^{-1/2}$ and $0 < R < 1$. Moreover, $\text{dist}(\tilde{V}_i, \text{span}(\tilde{V}_j : j \neq i)) \geq \delta = r$ for all $i \leq N$ everywhere on the probability space. Further, by Proposition 5.15 we get that everywhere on $\mathcal{E}$ there is a (random) non-increasing sequence of integers $p$ satisfying (16) such that $\tilde{V}_N \in \mathcal{D}(\delta, p)$ and
\[ 2\sum_{i=1}^N p_i \leq (4N)^{N/2} 2^{-\sum_{i=1}^N b_i}. \]
By Lemma 5.9 the total number of possible realizations of $p$ is bounded above by $(2 - \log_2 \delta)^N$. Hence, to prove the lemma, it is enough to show that for every choice of a non-random sequence $p$ satisfying (28), the event
\[ \mathcal{E}_p := \{ \tilde{V}_N \in \mathcal{D}(\delta, p) \quad \text{AND} \quad \mathcal{Y}(\tilde{V}_N, 2\delta(T + 1), 2T) \neq \emptyset \} \]
has probability at most $N^{CN} 2^{U \sum_{i=1}^N p_i} + 2e^{-cN^2}$. Set
\[ T' := 2T, \quad \delta' := 2\delta T + 2\delta, \quad \kappa := -\tilde{z}^{-1} - \frac{\bar{z}^{-1}}{|z|} \delta. \]
Observe that $|\kappa|, \delta' \geq 1/n$ and that $(N + 1)\delta' + |\kappa|T' \leq 1/2$. Further, applying the definition of the matrix $D = \tilde{z}^{-1}(A^*)^{-1}$, we get that for any vector $Y \in \Theta^m$ such that $\|D^*Y\|_2 \geq N^{-3}$ and $\|Y\|_2 \leq T'$, we necessarily have
\[ \|\tilde{A}(z^{-1}(A)^{-1}Y)\|_2 \leq \frac{N^3T'}{|z|} \|z^{-1}(A)^{-1}Y\|_2, \]
where $\tilde{A}$ denotes the $(n - N) \times (n - N)$ submatrix of $A - z$ obtained by crossing out the first $N$ rows and columns.
If $\Theta = \mathbb{R}$ then, in view of the conditions on $|z|$ and Theorem 2.3, for any $Y$ with $\|D^*Y\|_2 \geq N^{-3}$ and $\|Y\|_2 \leq T'$ with probability at least $1 - 2e^{-cn}$ we have

$$\text{LCD}_{\text{bad}, \text{bad}}(D^*Y/\|D^*Y\|_2) \geq c_2(n - N)/(NT').$$

Otherwise, if $\Theta = \mathbb{C}$ then, in view of Lemma 2.2 for any $Y$ with $\|D^*Y\|_2 \geq N^{-3}$ and $\|Y\|_2 \leq T'$ with probability at least $1 - 2e^{-cn}$ we have $Y \in \text{Incomp}_n(\zeta, \zeta)$, where $\zeta > 0$ depends only on $K$ and $\mathcal{M}$.

In view of the above remarks and Proposition 5.20, we get

$$\mathbb{P}(\mathcal{E}_p) \leq (C_3N)^{C_4N} \prod_{i=1}^{N} (\delta' + |\kappa|T' + 2p_0T' + n^{-1/U}) + 2e^{-c_3N^2},$$

for some $C_3, C_4 > 0$ depending only on $K$ and $\mathcal{M}$. Note that for all $i \leq N$ we have $2p_i \geq |\kappa|$ just by the construction of $\tilde{V}_i$’s. Putting everything together, we get that $\mathbb{P}(\mathcal{E}_p)$ is bounded above by $C''N^{2U} + 2e^{-c_4N^2}$ for some $C'', c_4 > 0$ depending only on $K, \mathcal{M}$. The result follows. □

**Proposition 5.23.** Let $\mathcal{M} \geq 1$, $n \in \mathbb{N}$, and consider an $n \times n$ random matrix $A$ with i.i.d columns satisfying $\mathbb{1}$ or $\mathbb{2}$ with a parameter $K$. Let $\log^2 n \leq N \leq \sqrt{n}$, $z \in \Theta$ with $N^2 \leq |z| \leq \mathcal{M}\sqrt{n}$, and let $\beta > 0$ be such that

$$n^{-1/U} \leq \theta/\beta \leq 1 \quad \text{and} \quad \beta \geq 1/n.$$

Consider the event

$$\mathcal{E} := \{ \exists v \in S^{n-1}(\Theta) : \|(A - z)v\|_2 \leq n^{-2} \text{ and } |v_\ell| \leq \theta, \ell \leq N - 1; |v_N| \geq \beta \}.$$

Then $\mathbb{P}(\mathcal{E}) \leq (C\theta/\beta)^{UN}$ for some $C > 0$ depending only on $K$ and $\mathcal{M}$, where $U = 1$ for $\Theta = \mathbb{R}$ and $U = 2$ for $\Theta = \mathbb{C}$.

**Proof.** Without loss of generality, by adding an infinitely small Gaussian perturbation to the matrix entries, we can assume that the distribution density of the entries is bounded. We can also assume that $\theta/\beta \leq 1/N^4$.

Let $\mathcal{N} := (P_F(-ze_1), \ldots, P_F(-ze_N))$, and $E := E(\mathcal{N}), E' := E(\mathcal{N}_{-1})$. Denote by $F$ the orthogonal complement to the linear span of last $n - N$ columns of $A - z$. Set $T := N$, and observe that, by the Hanson–Wright inequality [21], [44],

$$\mathbb{P}\{\|P_F Col(A)\|_2 \leq T, \ell \in [N]\} \geq 1 - 2e^{-c_1N^2}$$

for some $c_1 > 0$ depending only on $K$. Hence, in view of Lemma 5.1, it is sufficient to show that the event

$$\mathcal{E}' := \{ P_F(-ze_N) \in \delta E' + 3TB_2^F \text{ and } P_F Col(A - z) \in \delta E' + 2T\sqrt{N}\delta B_2^F \},$$

with $\delta = \theta\sqrt{N}/\beta$, has probability at most $(NC\theta/\beta)^{UN}$, for some $C' > 0$ depending only on $K, \mathcal{M}$.

Set $R := \beta/(\theta N) = (\delta\sqrt{N})^{-1}$. Let $Q$ be a subset of $N$–dimensional linear subspaces of $\Theta^n$ which will be defined later. We have

$$\mathbb{P}(\mathcal{E}') \leq \sum_b \mathbb{P}(\mathcal{E}' \cap \{ E \in C(R, b) \} \mid F \in Q) \mathbb{P}\{ F \in Q \} + \mathbb{P}\{ F \notin Q \},$$

where the summation is taken over all integer sequences $b$ satisfying (12). Further, applying Lemma 5.4 we obtain

$$\mathbb{P}(\mathcal{E}') \leq (2 - \log_2(\sqrt{N}\delta)) N \sup_b \mathbb{P}(\mathcal{E}'_b \cap \mathcal{E}'_2(b) \mid F \in Q) + \mathbb{P}\{ F \notin Q \},$$

for some $c_2 > 0$ depending only on $K, \mathcal{M}$.
where
\[ E'_1(b) := \{ P_F(-ze_N) \in \delta E' + 3TB_2^F \} \cap \{ E \in \mathcal{C}(R, b) \}, \]
\[ E'_2(b) := \{ P_F\text{Col}_N(A - z) \in \delta E' + 2\sqrt{\delta}B_2^F \} \cap \{ E \in \mathcal{C}(R, b) \}. \]

Let us fix for a moment any admissible sequence \( b \). Note that the event \( E'_1(b) \) is measurable with respect to the sigma-algebra generated by \( F \). Hence,
\[
\mathbb{P}(E'_1(b) \cap E'_2(b) \mid F \in Q) = \mathbb{E}(1_{E'_1(b)} 1_{E'_2(b)} \mid F \in Q) = \mathbb{E}(1_{E'_1(b)} \mathbb{E}(1_{E'_2(b)} \mid F) \mid F \in Q) \leq \mathbb{P}(E'_1(b) \mid F \in Q) \sup_{F_0 \in Q} \mathbb{P}(E'_2(b) \mid F = F_0).
\]

By Lemma 5.21 there exists some \( \tilde{C} = \tilde{C}(K, \mathcal{M}) > 0 \) such that conditioned on any realization \( F_0 \) of \( F \), we have
\[
\mathbb{P}(E'_2(b) \mid F = F_0) \leq N\tilde{C}^{-1} \sup_{Y \in F_0} \mathbb{P}\left\{ \|P_{F_0}(Z) - Y\|_2 \leq \delta \right\} 2^U \sum_{i=1}^{N} b_i,
\]
where \( Z \) is a random vector equidistributed with columns of \( A \). On the other hand, in view of Lemma 5.22
\[
\mathbb{P}(E'_1(b)) \leq N^{C_{1}N} 2^{-U} \sum_{i=1}^{N} b_i + 2e^{-c_2N^2} \leq N^{C_{1}N} 2^{-U} \sum_{i=1}^{N} b_i,
\]
where we have used that \( 2^{-U} \sum_{i=1}^{N} b_i \geq R^{-U}N \gg e^{-c_2N^2} \), by our assumptions on parameters. Summarising, we get for some \( C'' > 0 \),
\[
\mathbb{P}(E') \leq N^{C''N} \sup_{F_0 \in Q} \sup_{Y \in F_0} \mathbb{P}\left\{ \|P_{F_0}(Z) - Y\|_2 \leq \delta \right\} + \mathbb{P}\{ F \notin Q \},
\]
assuming that, say, \( \mathbb{P}\{ F \in Q \} \geq 1/2 \).

It remains to define the subset \( Q \) depending on whether we are in the real or the complex setting.

If \( \Theta = \mathbb{R} \) then we set
\[
Q := \{ F_0 \subset \mathbb{R}^n : \text{dim } F_0 = N \quad \text{and LCD} \geq (\mathbb{E} v^T P_{F_0} v) \geq \sqrt{n}e^{2\pi^2 N^4} \}.
\]

Note that, by the definition of \( F \) and by Theorem 2.4, we have
\[
\mathbb{P}\{ F \in Q \} \geq 1 - e^{-2\pi^2 N^4}.
\]

On the other hand, for every \( F_0 \in Q \) we get, by Theorem 2.3,
\[
\sup_{Y \in F_0} \mathbb{P}\left\{ \|P_{F_0}(Z) - Y\|_2 \leq \delta \right\} \leq (C_4 \delta/\sqrt{N})^N + C_4^N e^{-c_4n},
\]
whence
\[
\mathbb{P}(E') \leq N^{C_5N}(\theta/\beta)^N.
\]

In the case \( \Theta = \mathbb{C} \), in view of Lemma 2.2 with probability at least \( 1 - 2e^{-c''n} \) all unit vectors in \( F \) are \((\zeta, \zeta')\)-incompressible, for some \( c', \zeta > 0 \) depending only on \( \mathcal{M}, K \). Define
\[
Q := \{ F_0 : F_0 \cap S^{n-1}(\mathbb{C}) \subset \text{Incomp}_n(\zeta, \zeta) \}.
\]

Applying Theorem 2.6 we get that whenever \( F_0 \in Q \), we have
\[
\sup_{Y \in F_0} \mathbb{P}\left\{ \|P_{F_0}(Z) - Y\|_2 \leq \delta \right\} \leq C_6^N (\delta + n^{-1/2})^{2N}.
\]

Hence,
\[
\mathbb{P}(E') \leq N^{C_1N}(\theta/\beta)^{2N}.
\]

The result follows. \( \square \)
For complex matrices, the following lemma supplements Proposition 5.23 covering the case of small eigenvalues. We have:

**Lemma 5.24.** Let $\Theta = C$, and let $n$, $N$, $\beta$, $\theta$, $\mathcal{M}$ be as in Proposition 5.23. Then for $z \in \mathbb{C}$ with $|z| \leq N^2$, the probability of the event $\mathcal{E}$ defined in Proposition 5.23 is less than $(N^C \theta / \beta)^{2N}$ for some $C > 0$ depending only on $K$ and $\mathcal{M}$.

**Proof.** Everywhere on $\mathcal{E}$, by applying Lemma 4.1 with $k = N - 1$, $r = 1$, $B = A - z$, $\tau = n^{-2}$, $I_\theta = [N - 1]$, $J_\beta = \{N\}$, $F = \text{span} \{\text{Col}_i(B) : i > N\}^\perp$, $M := (P_F \text{Col}_i(B))_{i \in [N-1]}$, $M' := (P_F \text{Col}_N(B))$, we get

$$\beta \|P_F \text{Col}_N(B)\|_2 = \beta s_{\min}(M') \leq \theta \sqrt{N - 1} s_{\max}(M) + n^{-2}.$$ 

Further, the Hanson–Wright inequality (see (5) with $h$ (see, for example, $\varepsilon$)) implies that

$$\mathbb{P}\{s_{\max}(M) \geq C' N\} \leq e^{-N^2}$$

for some $C'$ depending only on $K, \mathcal{M}$. Hence,

$$\mathbb{P}(\mathcal{E}) \leq \mathbb{P}\{\|P_F \text{Col}_N(B)\|_2 \leq C' \theta N^{3/2} / \beta + n^{-1}\} + e^{-N^2}.$$ 

Further, Lemma 2.2 implies that with probability at least $1 - e^{-cn}$ we have $F \cap S^{n-1}(\zeta, \zeta)$ for some $\zeta, c > 0$ depending only on $K, \mathcal{M}$. On the other hand, for any $N$-dimensional subspace $F_0$ such that $F_0 \cap S^{n-1}(\zeta, \zeta) \subset \text{Incomp}_n(\zeta, \zeta)$, we have, by Theorem 2.6, that

$$\mathbb{P}\{\|P_{F_0} \text{Col}_N(B)\|_2 \leq 2C' \theta N^{3/2} / \beta\} \leq C^N (2C' \theta N^{3/2} / \beta + n^{-1/2})^{2N},$$ 

and the result follows. \hfill \Box

**Proof of Theorems B and C.** Here we prove Theorem C, the proof of Theorem B follows the same lines with appropriate modifications (see Remark 5.25 below). Without loss of generality, we can assume that $n$ is large. For a random $n \times n$ matrix $A$ with i.i.d columns satisfying (**), there is $\mathcal{M}$ depending only on $K$ such that $\|A\| \leq M \sqrt{n}$ with probability at least $2^{-n}$ (see, for example, $\varepsilon$-net argument in [30]). Thus, it is sufficient to consider only eigenvectors corresponding to eigenvalues $z$ with $|z| \leq M \sqrt{n}$. It is enough to prove the following statement:

There exists constant $C > 0$ depending only on $K$ and $\mathcal{M}$ such that for any $k \in [\log 2C n, n/2]$ the event

$$\mathcal{E}_k := \left\{ \exists z \in \mathbb{C} \text{ with } |z| \leq M \sqrt{n} \text{ and } v \in S^{n-1}(\mathbb{C}) \text{ s.t. } Av = zv \text{ and } v^*_{n-k+1} < \frac{\sqrt{k}}{n \log C n} \right\},$$

has probability at most $n^{-1} e^{-\log^2 n}$.

A standard $\varepsilon$-net argument (a discretization of the disc $\{w : |w| \leq M \sqrt{n}\}$ in the complex plane) implies that $\mathbb{P}(\mathcal{E}_k) \leq 8M^2 n^5 \max_{|z| \leq M \sqrt{n}} \mathbb{P}(\mathcal{E}_k(z))$, where

$$\mathcal{E}_k(z) := \left\{ \exists v \in S^{n-1}(\mathbb{C}) : \|(A - z)v\|_2 \leq n^{-2} \text{ and } v^*_{n-k+1} < \frac{\sqrt{k}}{n \log C n} \right\}.$$ 

In what follows, we fix any $z$ with $|z| \leq M \sqrt{n}$. Let $v = (v_1, \ldots, v_n) \in S^{n-1}(\mathbb{C})$ be any random vector. We will estimate probability of the event $\{|I_\theta| \geq k \text{ and } \|(A - z)v\|_2 \leq n^{-2}\}$, where, similarly to the argument in the proof of Theorem 4.3, we define

$$\theta := \frac{\sqrt{k}}{n \log C n} \text{ and } I_\theta := \{i \leq n : |v_i| \leq \theta\}.$$
It follows from Lemma 5.22 that there is $\zeta > 0$ depending only on $K$ and $\mathcal{M}$ such that, setting
\[
\beta := \zeta / \sqrt{n} \quad \text{and} \quad J_\beta := \{i \leq n : |v_i| \geq \beta\},
\]
we get $\mathbb{P}\{\|A\| < \zeta n\} \leq e^{-n/2}$. Fix $N = \lceil \log^2 n \rceil$. Let $\mathcal{E}' := \{(A - z)v_2 \leq n^{-2}\}$. Using Markov’s inequality we obtain
\[
P\{\|A\| \geq k \quad \text{and} \quad (A - z)v_2 \leq n^{-2}\} = \mathbb{E}(1_{\|A\| \geq k} \mathcal{E}') \\
\leq k^{-N} \mathbb{E}(1_{\|A\| \geq k} \mathcal{E}'(\|A\| \geq \zeta n)) + (n/k)^N e^{-n/2}
\]
(29)
so we need to estimate $\mathbb{E}(1_{\|A\| \geq k} \mathcal{E}'(\|A\| \geq \zeta n))$.

Given a set of distinct indices $\{i_1, \ldots, i_N\}$, let $\chi_{i_1, \ldots, i_N}$ be the indicator of the event
\[
|v_{i_1}| \leq \theta, \ldots, |v_{i_{N-1}}| \leq \theta \quad \text{and} \quad |v_{i_N}| \geq \beta.
\]
Since for any distinct $i_1, \ldots, i_{N-1} \in I_\theta$ and $i_N \in J_\beta$ we have $\chi_{i_1, \ldots, i_N} = 1$, we get a deterministic relation valid on the entire probability space:
\[
\sum_{\chi_{i_1, \ldots, i_N}} \chi_{i_1, \ldots, i_N} \geq |I_\theta|(|I_\theta| - 1) \cdots (|I_\theta| - N + 1)|J_\beta|1_{\|A\| \geq N} \geq 3^{-N}|I_\theta|^N|J_\beta|1_{\|A\| \geq k},
\]
where the sum is taken over all ordered $N$-tuples $(i_1, \ldots, i_N)$ with distinct components. Hence,
\[
\mathbb{E}(1_{\|A\| \geq k} \mathcal{E}'(\|A\| \geq \zeta n)) \leq 3^N \sum \mathbb{E}(\chi_{i_1, \ldots, i_N} \mathcal{E}')
\]
and, by (29),
\[
P\{\|A\| \geq k \quad \text{and} \quad (A - z)v_2 \leq n^{-2}\} \leq (\zeta nk)^{-1} 3^N \sum \mathbb{E}(\chi_{i_1, \ldots, i_N} \mathcal{E}') + (n/k)^N e^{-n/2}.
\]

Consider two cases. First, assume that $N^2 \leq |z| \leq \mathcal{M}\sqrt{n}$. If $k \geq \zeta^2 \log^2 n$ then $\theta / \beta \geq n^{-1/2}$, and Proposition 5.23 implies that for any sequence $(i_1, \ldots, i_N)$ of distinct indices
\[
\mathbb{E}(\chi_{i_1, \ldots, i_N} \mathcal{E}') \leq (N^{2C'}) \theta / \beta \leq 2^N = \left(\frac{N^{2C'}}{\zeta^2 \log^2 n}\right)^N,
\]
where $C'$ depends only on $K$ and $\mathcal{M}$. Hence,
\[
P\{\|A\| \geq k \quad \text{and} \quad (A - z)v_2 \leq n^{-2}\} \leq (\zeta n)^{-1} (3/\zeta^2)^N N^{2C'} (\log n)^{-2CN} + (n/k)^N e^{-n/2}.
\]
Taking $C > 2C'$ big enough we get that the right hand side is less than $e^{-2\log^2 n}$. Since $v$ was arbitrary, this implies $\mathbb{E}(\mathcal{E}_k(z)) \leq e^{-2\log^2 n}$, and the result follows.

Now, assume that $|z| \leq N^2$. Then we repeat the above argument, replacing Proposition 5.23 with Lemma 5.23. This finishes the proof of Theorem C.

To strengthen the result and get probability estimate $1 - e^{-\log^2 C} n$ for any constant $C' > 2$, one can repeat the argument above taking $N := \lceil \log^2 n \rceil$ with sufficiently large $C''$.

\begin{remark}
The proof of Theorem B in the case of real eigenvalues $\lambda$ such that $\log^4 n \leq |\lambda| \leq \mathcal{M}\sqrt{n}$ follows the same scheme and is based on Proposition 5.23 with $U = 1$. For $|\lambda| \leq \log^4 n$ the statement follows from Corollary 4.4.
\end{remark}

\begin{remark}
Theorem B provides no-gaps delocalization estimates only for real eigenvectors. By considering the real Ginibre ensemble one can easily verify that the obtained bounds are optimal up the polylogarithmic multiple $\log^C n$. For complex eigenvectors of real matrices with i.i.d entries, the situation is not clear to us. Analysis of simple cases, in particular, eigenvectors of real Gaussian matrices corresponding to eigenvalues with big imaginary and small real parts, shows that such an eigenvector $v$ is more delocalized in our sense, so that a much stronger lower bound for $v_i^2$ than $c(n - i)/n^{3/2}$ can be verified. It is reasonable to expect similar phenomenon
\end{remark}
for any random non-symmetric matrix with normalized i.i.d real entries, that is, sharp no-gaps delocalization estimates should depend on the magnitudes of the real and imaginary parts of the eigenvalues. It is not clear if the method of this paper can be adapted to catch this property.

Remark 5.27. It is natural to expect that the method of this paper can be developed to treat the case of matrices with non-identically distributed entries and without a bounded subgaussian moment. We have not explored that direction.

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