The effect of kick velocities on the spatial distribution of millisecond pulsars and implications for the Galactic center excess

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Abstract. Recently it has become apparent that the Galactic center excess (GCE) is spatially correlated with the stellar distribution in the Galactic bulge. This has given extra motivation for the unresolved population of millisecond pulsars (MSPs) explanation for the GCE. However, in the “recycling” channel the neutron star forms from a core collapse supernovae that undergoes a random “kick” due to the asymmetry of the explosion. This would imply a smoothing out of the spatial distribution of the MSPs. We use $N$-body simulations to model how the MSP spatial distribution changes. We estimate the probability distribution of natal kick velocities using the resolved gamma-ray MSP proper motions, where MSPs have random velocities relative to the circular motion with a scale parameter of $77 \pm 6$ km/s. We find that, due to the natal kicks, there is an approximately 10% increase in each of the bulge MSP spatial distribution dimensions and also the bulge MSP distribution becomes less boxy but is still far from being spherical.

Keywords: dark matter simulations, galaxy morphology, gamma ray theory, neutron stars

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1 Introduction

The Galactic Center Excess (GCE) is an extended source of gamma radiation in the central region of the Galaxy found in the Fermi Large Area Telescope (Fermi-LAT) data. When first discovered, its apparently spherically symmetric profile and spectrum peaking at a few GeV suggested that it may be evidence of self-annihilating weakly interacting massive particles (WIMPs) distributed according to a Navarro-Frenk-White (NFW) profile [1–4]. An alternative scenario that was proposed was one in which the GCE was produced by a population of unresolved millisecond pulsars (MSPs) [5]. More recently, evidence has suggested that in fact the GCE is not spherically symmetric but actually is correlated with the stellar mass in the Galactic bulge and so has a more boxy morphology [6–11]. Although one recent study, using different methods, still argues for a spherically symmetric GCE [12]. If the GCE does trace the stellar mass of the bulge this would favor the MSP (or other dim unresolved astrophysical point source) explanation. Additionally, several studies have claimed to find a non-Poissonian component to the GCE [13, 14] which may be further evidence for the MSP explanation. However, there is some controversy regarding the level of systematics in this approach [e.g., 11, 15–19].

In the “recycling” model of MSP formation a neutron star is spun up to millisecond periods through the transfer of mass from a binary companion. This requires that the binary system survives the kick produced by any asymmetry in the core collapse supernova explosion [20]. However, an alternative to the recycling channel is accretion induced collapse of white dwarfs into neutron stars which may produce more than half of all observed MSPs [21–24]. In this case the system does not receive a significant natal kick [25, 26]. This would imply that the MSPs have much smaller peculiar velocities in comparison to the recycling model [27–29].

Ploeg et al. [30] (hereafter referred to as P20) modelled Fermi-LAT detected MSPs as having a Maxwell distributed peculiar velocity with the scale parameter $\sigma_v$ found to be $77 \pm 6$ km s$^{-1}$ where we quote error bars at the 68% confidence interval throughout this article. This velocity applies for disk MSPs and we assume it will not be significantly different for bulge MSPs. Although the star formation histories are very different in the bulge and disk, as can be seen from figure 6 of P20 the probability distribution of luminosities in the bulge and disk only differ by a few percent. Also, as can be seen from figure 11 of P20, the bulge

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and disk have a ratio of number of MSPs formed per solar mass which is within one error
bar of each other. Based on this, we assume that the bulge and disk have the same mix of
MSP formation channels and thus the same probability distribution of natal kick velocities.

If the source of the GCE is a population of unresolved MSPs, then the spatial distribu-
tion may be smoothed to some degree relative to the stellar mass in the bulge. Eckner et
al. [31] used the virial theorem to estimate the “smoothing length” of MSPs as 700 – 900 pc
for kicks \( \lesssim 70 \text{ km s}^{-1} \). However, they assumed a spherically symmetric spatial distribution
for the MSPs.

In this article we use N-body simulations to estimate what are the effects of MSP kicks
for a boxy bulge like distribution. In section 2 we explain our method. Our results are given
in section 3, and our conclusions in section 4.

2 Method

For this work we use the code of Bedorf et al. [32] to run N-body simulations\(^1\) in order
to model the Milky Way. We use parameters corresponding to initial conditions labelled
MWa, MWb and MWc0.8 as denoted by Fujii et al. [33] as they were the initial conditions
that led to the best fitting simulations to Galaxy observations that Fujii et al. found. The
initial conditions consist of a parametric description of the dark matter halo, Galactic disk
and bulge. Comparing to bulge kinematics, bar length, and pattern speed observations Fujii
et al. found MWa was the initial conditions which gave the best fit to the data, followed
by MWb, and then MWc0.8. For each initial condition model we generated a total of 30
million disk, bulge and dark matter halo particles. These initial populations are generated
using the methods of Kuijken and Dubinski [34], Widrow and Dubinski [35], and Widrow
et al. [36].\(^2\) As in Fujii et al. [33], we use time-steps of \( \sim 0.6 \text{ Myr} \), an opening angle of
0.4 and ran the simulation for 10 Gyr. However, we use a softening length of 30 pc. Also,
our dark-matter halo particles have a mass 8 times larger than the disk and bulge particles.
Taking into account the masses of the various components, this implies that, out of the 30
million particles, approximately 10 million particles represent stellar mass and the remainder
represent dark matter.

In order to model the density of MSPs, we additionally include massless (so they do
not affect the simulation) disk and bulge particles that are given a normally distributed per-
turbation to each component of their velocity vector with mean zero and standard deviation
\( \sigma_k \). The kick velocity magnitude is therefore Maxwell distributed. The probability density
function of a Maxwell distribution can be written as:

\[
p(x) = \sqrt{\frac{2}{\pi}} \frac{x^2 \exp(-x^2/2\sigma^2)}{\sigma^3}
\]  

(2.1)

where \( x \) is the magnitude of a three dimensional vector with components sampled from the
normal distribution \( \mathcal{N}(0, \sigma^2) \). For each model we try a case where the kicks occurred at
the beginning of the N-body simulations and a case where the kicks occur randomly with
a uniform rate over the course of the 10 Gyr. These cases approximate scenarios in which
MSPs are largely born early or at a relatively steady rate. From these two extremes, we can
estimate the sensitivity of our results to the MSP age distribution.

\(^1\)Available at: https://github.com/treecode/Bonsai.
\(^2\)We used the implementation at: https://github.com/treecode/Galactics.parallel.
The first step is to estimate the kick velocity scale required to produce a peculiar velocity distribution consistent with P20 where for the best model the peculiar velocity scale parameter was \( \sigma_v = 77 \pm 6 \text{ km s}^{-1} \). We do this by running each initial condition model with 41 populations of \( 10^5 \) kicked particles with \( \sigma_k \) between 70 and 110 km s\(^{-1}\). We separate the velocity of each particle into two components:

\[
\mathbf{v} = \mathbf{v}_c + \mathbf{v}_p
\]

where \( \mathbf{v}_c \) is the velocity of a particle on a circular orbit around the center of the galaxy and \( \mathbf{v}_p \) is the peculiar velocity. The magnitude of \( \mathbf{v}_c \) for a particle with coordinates \( x, y, \) and \( z \) can be evaluated using the centripetal force:

\[
\| \mathbf{v}_c \| = \sqrt{\| \mathbf{a}_c(x, y, 0) \| R}
\]

where \( \mathbf{a}_c(x, y, z) \) is the acceleration toward the center of the galaxy which can be obtained from a small modification to the \( N \)-body code and \( R^2 = x^2 + y^2 \). For \( R \) outside the bulge region, and for small peculiar velocity, we are therefore assuming that particles are rotating with the disk, with \( \mathbf{v}_c \) the rotation velocity of the disk at \( R \) [37].

We use a maximum likelihood estimate of the final \( \sigma_v \) for each initial \( \sigma_k \). For a set of \( N \) particles with peculiar velocities \( v_1, \ldots, v_N \), the log-likelihood is obtained by assuming velocities have a Maxwell distribution:

\[
\log(L) = \frac{N}{2} \log \left( \frac{2}{\pi} \right) - 3N \log(\sigma_v) + \sum_{i=1}^{N} 2\log(v_i) - \frac{v_i^2}{2\sigma_v^2}
\]

and therefore

\[
\frac{d\log(L)}{d\sigma_v} = -\frac{3N}{\sigma_v} + \sum_{i=1}^{N} \frac{v_i^2}{\sigma_v^3}.
\]

Then solving for \( \sigma_v \) where \( d\log(L)/d\sigma_v = 0 \), we find the maximum likelihood estimate for \( \sigma_v \) is:

\[
\hat{\sigma}_v = \sqrt{\frac{\sum_{i=1}^{N} v_i^2}{3N}}
\]

This is done for particles where \( 4 \text{ kpc} \leq R \leq 12 \text{ kpc} \) and \( |z| \leq 2 \text{ kpc} \), ensuring we are estimating the peculiar velocity distribution scale parameter for particles in the disk region from which gamma-ray MSPs are most likely to be resolved and where \( \mathbf{v}_c \) approximately represents disk rotation.

Once we have a best fitting \( \sigma_k \) we use that \( \sigma_k \) to generate an additional population of kicked particles. We rerun the \( N \)-body simulations for each set of initial conditions (MWa, MWb, MWc0.8) including in each initial condition case two new populations of massless particles where each new population has \( 2 \times 10^6 \) particles. These new populations are made massless so as not to influence the gravitational potential of the \( N \)-body simulation. The first massless populations is sampled from the initial condition model. For the second population we duplicate the first population and then add a Maxwell distributed kick with scale parameter \( \sigma_k \) at the beginning of the simulation to each particle in the second population. The simulations are then run for 10 Gy to see what the final state of the two massless populations is for each set of initial conditions. We then repeat this process but instead of the second population having a Maxwell distributed kick added at the beginning of the simulation we add it randomly to members of the second population at a uniform rate throughout.
the simulation. So in the end for each initial condition we have two massless populations of unkicked particles, one massless population of particles kicked at the beginning and one for particles kicked at a uniform rate throughout the simulation. We used two populations of massless unkicked particles to make the kicked at beginning cases and uniform kick rate cases as independent as possible from each other.

In order to understand the effect of MSP kicks on the final spatial distribution of MSPs we found it clearer to have a parametric description of the spatial distribution. Then by comparing the parameters for the kick and no-kick cases we can have a succinct way of characterising the effects of the kicks. We use Markov Chain Monte Carlo (MCMC) to fit the parametric model to both final particle distributions. The form of our parametric model for the final spatial distribution of MSPs was chosen based on previous studies and also such that there were not excessive residuals between the model fits and the simulations. Our model of the final distributions consists of four components: a spherically symmetric bulge, a potentially non-spherical bulge also known as a bar, a long bar and a disk. The spherically symmetric bulge component uses the Hernquist model [38]:

$$\rho_{\text{Hernquist}}(r) \propto \frac{1}{(r/a_b) (1 + r/a_b)^3}$$  \hspace{1cm} (2.7)

where $r^2 = x^2 + y^2 + z^2$ and $a_b$ is a free parameter. Our initial conditions models, based on those of Fujii et al. [33], also include a component distributed according the Hernquist model which would naturally have different parameter values to the one we fitted after 10 Gyr of evolution. Our bar model is distributed as:

$$\rho_{\text{bar}}(R_s) \propto K_0(R_s) \times \begin{cases} 1 & R \leq R_{\text{end}} \\ \exp(-(R - R_{\text{end}})^2/h_{\text{end}}^2) & R > R_{\text{end}} \end{cases}$$  \hspace{1cm} (2.8)

where $K_0$ is the modified Bessel function of the second kind and where:

$$R_s = \left( R_{\perp}^{C_\parallel} + \left( \frac{|z|}{z_b} \right) C_\parallel \right)^{1/C_\parallel}$$  \hspace{1cm} (2.9)

$$R_{\perp} = \left( \left( \frac{|x|}{x_b} \right)^{C_{\perp}} + \left( \frac{|y|}{y_b} \right)^{C_{\perp}} \right)^{1/C_{\perp}}.$$  \hspace{1cm} (2.10)

The free parameters are $C_{\parallel}, C_{\perp}, x_b, y_b, z_b$ and $R_{\text{end}}$ with $h_{\text{end}}$ fixed at $\sqrt{\frac{1}{2}}$ kpc. The effective radius is $R_s$; the scale lengths are $x_b, y_b,$ and $z_b$; and $C_{\perp}$ and $C_{\parallel}$ are the face-on and edge-on shape parameters. The bar shape is elliptical in the corresponding direction when $C_{\perp}, C_{\parallel} = 2$, diamond-shaped when $C_{\perp}, C_{\parallel} < 2$, and boxy when $C_{\perp}, C_{\parallel} > 2$. The Gaussian function with scale length $h_{\text{end}}$ in eq. (2.8) truncates the bar at radius $R_{\text{end}}$. The modified Bessel function was also used in Cao et al. [39] to model the distribution of red clump giants, but with no cutoff and with $C_{\parallel} = 4$ and $C_{\perp} = 2$. For the long bar we use [40]:

$$\rho_{\text{long bar}}(x, y, z) \propto \exp\left(-\left( \left( \frac{|x|}{x_{\text{in}}} \right)^{C_{\perp,\text{in}}} + \left( \frac{|y|}{y_{\text{in}}} \right)^{C_{\parallel,\text{in}}} \right)^{1/C_{\perp,\text{in}}} \right) \exp\left(-\frac{|z|}{z_{\text{in}}} \right) \text{Cut} \left( \frac{R - R_{\text{out}}}{\sigma_{\text{out}}} \right) \text{Cut} \left( \frac{R_{\text{in}} - R}{\sigma_{\text{in}}} \right)$$  \hspace{1cm} (2.11)
where $C_{\perp lb}, x_{lb}, y_{lb}, z_{lb}, R_{out}$ and $R_{in}$ are free parameters, $\sigma_{out} = \sigma_{in} = \sqrt{\frac{\pi}{2}}$ kpc and:

$$\text{Cut} (x) = \begin{cases} \exp(-x^2) & x > 0 \\ 1 & x \leq 0 \end{cases}$$

Note that Fujii et al. [33] did not need to include any bar components in their initial conditions as the bar components evolve naturally over the 10 Gy from their Hernquist bulge, disk and halo model initial conditions.

Finally, we have a disk with a central hole:

$$\rho_{\text{disk}}(x, y, z) \propto \exp\left(-\frac{R^2}{2\sigma_r^2}\right) \exp(-|z|/z_0) H(x, y)$$

where $\sigma_r$ and $z_0$ are free parameters and for the hole we use the form adopted by Freudenreich [41]:

$$H(x, y) = 1 - \exp\left(-\left(R_H/O_R\right)^{\alpha}\right)$$

with:

$$R_H^2 = (x)^2 + (\epsilon y)^2$$

where $\epsilon$, $O_R$ and $O_N$ are also free parameters. Note that Fujii et al. [33] did not have a hole in their initial disk model. This hole evolves naturally through the process of the formation of the bar over their 10 Gy simulation.

The total number of particles in our simulations are fixed. So we do not have to include the number of particles as part of our likelihood. Therefore the probability of having an N-body particle at position $x, y, z$ will be proportional to the density of our model ($\rho$) at $x, y, z$. We have for each component of the model a parameter giving the probability a particle is from that component. We treat the probability of an N-body particle being from a component of the density distribution as parameters. These parameters, $P(\text{Disk})$, $P(\text{Bar})$, $P(\text{Hernquist})$ and $P(\text{Long Bar})$, have a Dirichlet prior [42]. This prior constrains

$$P(\text{Disk}) + P(\text{Bar}) + P(\text{Hernquist}) + P(\text{Long Bar}) = 1$$

and is uniformly distributed over any values of these parameter satisfying this condition. The log-likelihood is then:

$$\log(L) = \sum_i \log(\rho(x_i, y_i, z_i))$$

where $x_i$, $y_i$ and $z_i$ are the coordinates of a particle, $N$ is the number of particles, and $\rho$ is the density of the model:

$$\rho(x, y, z) = P(\text{Disk})\rho_{\text{disk}}(x, y, z) + P(\text{Bar})\rho_{\text{bar}}(x, y, z) + P(\text{Hernquist})\rho_{\text{Hernquist}}(x, y, z) + P(\text{Long Bar})\rho_{\text{long bar}}(x, y, z)$$

All scale parameters are given a prior so they are uniform in $\log(\theta)$ where $\theta \in \{a_0, x_b, y_b, z_b, x_{lb}, y_{lb}, z_{lb}, \sigma_r, z_0, O_R\}$ and this implies $\rho(\theta) \propto 1/\theta$. In calculating the likelihood, we do not include particles for which $R > 12$ kpc or $|z| > 3$. It can be seen in Fujii et al. [33] that the scale height of the disk may start to decline between $10 \lesssim R \lesssim 15$ kpc. We also don’t want the fit to be affected by particles that may have been kicked well out of the galaxy. The likelihood ($L$) is insensitive to being multiplied by a constant but that
constant has to be the same for all parameters of our combined model. To accommodate this we normalize each density component such that
\[
\int_{R \leq 12 \text{ kpc}, |z| \leq 3 \text{ kpc}} \rho_i(x, y, z) \, dx \, dy \, dz = 1
\]
where \( i \in \{\text{disk, bar, Hernquist, long bar}\} \). This integral is estimated with importance sampling. We use a set of random numbers which are transformed into the points at which we evaluate the density models in order to estimate the normalization constant. In order to stabilize the estimation of the likelihood function these numbers are always the same every time we perform the importance sampling within a particular chain.

After running the N-body simulations, we shift the coordinates of the particles so that the center of mass is at the origin, then rotate so the bar is along the x-axis. The bar angle is estimated using the method described in Fujii et al. [33]. However, we add four parameters that we expect to be near zero to allow a further shift in the center and clockwise rotation of the model. These are \( \alpha \), \( x_{\text{center}} \), \( y_{\text{center}} \) and \( z_{\text{center}} \), with the latter three parameters in parsecs, so:
\[
\begin{align*}
  x_{\text{data}} &= \cos(\alpha)x + \sin(\alpha)y + x_{\text{center}}/1000 \\
  y_{\text{data}} &= -\sin(\alpha)x + \cos(\alpha)y + y_{\text{center}}/1000 \\
  z_{\text{data}} &= z + z_{\text{center}}/1000
\end{align*}
\]
where \( x_{\text{data}}, y_{\text{data}} \) and \( z_{\text{data}} \) are coordinates in the coordinate system of the N-body simulation. In estimating the peculiar velocity distribution scale parameter, \( \hat{\sigma}_v \), above, we assumed \( x \approx x_{\text{data}}, y \approx y_{\text{data}} \) and \( z \approx z_{\text{data}} \).

To compare our model fits to the final N-body simulations we sample from the model directly, then we bin the samples. We used a simple MCMC to sample from the bar and long bar. Since the disk (excluding the hole) can be turned into an invertible cumulative distributions in \( R \) and \( z \), and the Hernquist bulge in \( r \), we used inverse transform sampling for those. The disk hole was handled by accepting the disk samples with probability \( H(x, y) \), i.e. we used rejection sampling for the disk.

Our MCMC algorithm used to fit the models is similar to that of Foreman-Mackay et al. [43] with a mixture of the Differential Evolution [44] and snooker updates [45]. We use a simple annealing method in which we divide the log-likelihood by a temperature \( T \) which is gradually reduced to 1. This occurs during the first half of each Markov chain, which we discard. See appendix B for more details.

### 3 Results

In table 1 we present the kick velocities \( \sigma_k \) that produce peculiar velocity distributions close to \( \sigma_v = 77 \pm 6 \text{ km s}^{-1} \) as estimated by P20. We display the rotation curves at \( t = 10 \text{ Gyr} \) for the three initial condition models (MWa, MWb and MWc0.8) in figure 1. The central values in table 1 were used to run N-body simulations with a larger number of particles to which we fitted a parametric model. The fitted parameters are shown in tables 2, 3 and 4. There are three potential sources of uncertainty in the model parameters: the posterior density function, variation in the likelihood between chains as a result of the importance sampling method used to estimate the normalization constant for each model component, and the possibility that Markov chains may get stuck in different local likelihood maxima. We found that for most parameters the posterior distributions overlapped significantly or,
### Table 1

| Kick At Beginning | Uniform Kick Rate |
|------------------|-------------------|
| MWa              | 93 ± 10           |
| MWb              | 97 ± 10           |
| MWc0.8           | 95 ± 10           |

Kick velocity Maxwell distribution \((\sigma_k \text{ in km/s})\) parameters that produce a peculiar velocity distribution where \(\sigma_v = 77 \pm 6\text{ km/s}\).
| Parameter          | No Kick | Kick At Beginning | Uniform Kick Rate |
|--------------------|---------|-------------------|-------------------|
| $P$(Disk)          | 0.499±0.006−0.005 | 0.418±0.006−0.006 | 0.365±0.006−0.004 |
| $P$(Bar)           | 0.354±0.004−0.005 | 0.3220±0.019−0.002 | 0.364±0.005−0.007 |
| $P$(Hernquist)     | 0.0145±0.0013−0.0018 | 0.059±0.003−0.002 | 0.0647±0.0022−0.0023 |
| $P$(Long Bar)      | 0.1323±0.0025−0.0025 | 0.201±0.007−0.007 | 0.206±0.007−0.010 |
| $\sigma_r$ (kpc)  | 4.92±0.04−0.03 | 5.66±0.05−0.04 | 6.07±0.05−0.08 |
| $z_0$ (kpc)        | 0.2238±0.0008−0.0027 | 1.203±0.023−0.013 | 1.222±0.018−0.004 |
| $O_R$ (kpc)        | 2.82±0.08−0.11 | — | — |
| $O_N$              | 4.3±0.3−0.2 | — | — |
| $\epsilon$        | 0.780±0.020−0.013 | — | — |
| $a_b$ (kpc)        | 0.205±0.009−0.010 | 0.509±0.019−0.010 | 0.555±0.022−0.021 |
| $C_{\perp}$        | 1.84±0.02−0.05 | 1.831±0.013−0.015 | 1.821±0.017−0.015 |
| $C_{\parallel}$    | 3.08±0.11−0.11 | 2.49±0.03−0.03 | 2.542±0.019−0.019 |
| $x_b$ (kpc)        | 0.557±0.008−0.008 | 0.685±0.005−0.005 | 0.673±0.007−0.011 |
| $y_b$ (kpc)        | 0.367±0.006−0.003 | 0.455±0.003−0.003 | 0.445±0.006−0.011 |
| $z_b$ (kpc)        | 0.2557±0.0014−0.0010 | 0.3328±0.0014−0.0015 | 0.3155±0.0026−0.0029 |
| $R_{end}$ (kpc)    | 2.00±0.05−0.06 | 4.6±0.4 | 4.9±0.6 |
| $\Delta \alpha$ (deg) | — | −0.34±0.15−0.14 | −0.07±0.13−0.13 |
| $\Delta x_{center}$ (pc) | — | 0.8±1.0 | −0.7±1.0−1.0 |
| $\Delta y_{center}$ (pc) | — | −1.8±0.9−0.9 | −0.3±0.7−0.7 |
| $\Delta z_{center}$ (pc) | — | −0.9±0.6−0.5 | 2.7±0.4−0.5 |
| $x_{lb}$ (kpc)     | 5.5±5 | 2.78±0.03−0.03 | 2.46±0.04−0.03 |
| $y_{lb}$ (kpc)     | 1.43±0.16−0.07 | 2.78±0.04−0.04 | 2.49±0.05−0.04 |
| $z_{lb}$ (kpc)     | 0.315±0.029−0.010 | 0.541±0.006−0.011 | 0.495±0.010−0.014 |
| $C_{L,lb}$         | 0.88±0.04−0.11 | 1.764±0.026−0.025 | 1.983±0.028−0.026 |
| $R_{out}$ (kpc)    | 2.86±0.04−0.05 | 7.46±0.04−0.04 | 8.37±0.06−0.06 |
| $R_{in}$ (kpc)     | 1.676±0.018−0.016 | 2.09±0.03−0.04 | 2.03±0.07−0.12 |

Table 2. Fitted parameters for model MWa. We have used the median of the MCMC chains for the central value and also included 68% confidence intervals.

likelihood between chains, we also show the standard deviation in $-2 \log(L)$. Our choice of $\rho_{\text{bar}}(R_s) \propto K_0(R_s)$ is clearly preferred over the others. The worst form, where $\rho_{\text{bar}}(R_s) \propto \exp(R_s)$, was entirely removed with $P$(Bar) = 0 and the long bar component taking over the fit in the central region.
where $\theta_{\text{kicked},i}$ are the bulge parameters, $x_b$, $y_b$, and $z_b$ for the kicked distribution and $\theta_i$ are

\begin{equation}
\theta_{\text{kicked},i} = \alpha \theta_i + \beta
\end{equation}

Table 3. Fitted parameters for model MWb.
| Parameter        | No Kick       | Kick At Beginning | Uniform Kick Rate |
|------------------|---------------|-------------------|-------------------|
| $P$(Disk)        | $0.555^{+0.003}_{-0.009}$ | $0.467^{+0.012}_{-0.010}$ | $0.357^{+0.013}_{-0.003}$ |
| $P$(Bar)         | $0.318^{+0.004}_{-0.005}$ | $0.262^{+0.003}_{-0.005}$ | $0.305^{+0.003}_{-0.002}$ |
| $P$(Hernquist)   | $0.0099^{+0.0006}_{-0.0005}$ | $0.0063^{+0.0021}_{-0.0025}$ | $0.0619^{+0.0019}_{-0.0020}$ |
| $P$(Long Bar)    | $0.116^{+0.014}_{-0.004}$ | $0.210^{+0.014}_{-0.010}$ | $0.277^{+0.005}_{-0.017}$ |
| $\sigma_r$ (kpc) | $5.49^{+0.03}_{-0.04}$ | $6.11^{+0.07}_{-0.09}$ | $6.99^{+0.06}_{-0.19}$ |
| $z_0$ (kpc)      | $0.2246^{+0.0007}_{-0.0022}$ | $1.266^{+0.036}_{-0.028}$ | $1.262^{+0.026}_{-0.022}$ |
| $O_R$ (kpc)      | $2.5^{+0.4}_{-0.16}$ | — | — |
| $O_N$            | $4.9^{+2.6}_{-1.1}$ | — | — |
| $\epsilon$      | $0.82^{+0.12}_{-0.12}$ | — | — |
| $a_b$ (kpc)      | $0.220^{+0.006}_{-0.006}$ | $0.72^{+0.01}_{-0.04}$ | $0.74^{+0.03}_{-0.03}$ |
| $C_{\perp}$      | $1.88^{+0.03}_{-0.05}$ | $1.861^{+0.020}_{-0.020}$ | $1.909^{+0.024}_{-0.022}$ |
| $C_{\|}$         | $3.37^{+0.05}_{-0.05}$ | $2.69^{+0.03}_{-0.04}$ | $2.69^{+0.03}_{-0.03}$ |
| $x_b$ (kpc)      | $0.551^{+0.012}_{-0.014}$ | $0.657^{+0.006}_{-0.007}$ | $0.620^{+0.006}_{-0.007}$ |
| $y_b$ (kpc)      | $0.342^{+0.007}_{-0.008}$ | $0.434^{+0.005}_{-0.007}$ | $0.378^{+0.005}_{-0.003}$ |
| $z_b$ (kpc)      | $0.2395^{+0.0015}_{-0.0018}$ | $0.3217^{+0.0023}_{-0.0023}$ | $0.2845^{+0.0025}_{-0.0018}$ |
| $R_{end}$ (kpc)  | $2.03^{+0.07}_{-0.10}$ | $4.81^{+0.29}_{-0.25}$ | $5.5^{+1.1}_{-0.9}$ |
| $\Delta \alpha$ (deg) | — | $-0.79^{+0.18}_{-0.18}$ | $0.32^{+0.13}_{-0.13}$ |
| $\Delta x_{center}$ (pc) | — | $1.4^{+1.4}_{-1.3}$ | $1.0^{+1.2}_{-1.1}$ |
| $\Delta y_{center}$ (pc) | — | $-0.2^{+0.9}_{-0.9}$ | $0.5^{+0.8}_{-0.8}$ |
| $\Delta z_{center}$ (pc) | — | $2.4^{+0.7}_{-0.6}$ | $-0.5^{+0.5}_{-0.5}$ |
| $x_{lb}$ (kpc)   | $3^{+7}_{-1}$ | $2.533^{+0.023}_{-0.022}$ | $2.533^{+0.020}_{-0.021}$ |
| $y_{lb}$ (kpc)   | $1.5^{+0.4}_{-0.2}$ | $2.584^{+0.027}_{-0.025}$ | $2.41^{+0.021}_{-0.022}$ |
| $z_{lb}$ (kpc)   | $0.335^{+0.006}_{-0.005}$ | $0.564^{+0.019}_{-0.016}$ | $0.530^{+0.010}_{-0.011}$ |
| $C_{\perp,lb}$  | $0.95^{+0.07}_{-0.11}$ | $1.964^{+0.029}_{-0.029}$ | $1.954^{+0.022}_{-0.022}$ |
| $R_{out}$ (kpc)  | $3.0^{+0.1}_{-0.4}$ | $7.88^{+0.07}_{-0.09}$ | $9.13^{+0.13}_{-0.08}$ |
| $R_{in}$ (kpc)   | $1.61^{+0.03}_{-0.06}$ | $1.91^{+0.05}_{-0.07}$ | $1.60^{+0.05}_{-0.03}$ |

Table 4. Fitted parameters for model MWc0.8.

the corresponding parameters in the non-kicked case. The $\alpha$ and $\beta$ were found by performing a least squared fit for the values given in tables 2, 3, and 4. We also did a similar fit for $C_{\perp}$ and $C_{\|}$. The results are shown in table 6 and figure 5. A prediction for the Milky Way bar parameters found in ref. [39] are shown in table 7. The predicted line of sight contours for the kicked and unkicked Milky Way bar are shown in figure 6.
Figure 1. The rotation curves at $t = 10$ Gyr for the three initial condition models MWa, MWb and MWc0.8. They are consistent with the local circular velocity of the Sun which is measured to be $238 \pm 15$ km s$^{-1}$ [47]. The distance between the Sun and the Galactic Center is approximately 8 kpc.

| $\rho_{\text{bar}}(R_s)$ | Mean $-2(\Delta \log(L))$ | Standard Deviation |
|------------------------|--------------------------|-------------------|
| $K_0(R_s)$             | 0                        | 739               |
| $\exp(-R_s)$           | 3581                     | 1606              |
| $\sech^2(R_s)$         | 12795                    | 1985              |
| $\exp(-0.5R_s^2)$      | 19369                    | 391               |
| $(1 + R_s^n)^{-1}$      | 20403                    | 953               |
| $\exp(R_s^n)$          | 57114                    | 3152              |

Table 5. Change in mean $-2\log(L)$ using different bar models $\rho_{\text{bar}}(R_s)$. The mean was taken over all the samples in the MCMC chains.

| Parameter   | $\alpha_i$ | $\beta_i$ |
|-------------|-------------|-----------|
| $x_b, y_b, z_b$ | $1.13 \pm 0.05$ | $0.03 \pm 0.02$ |
| $C_{\perp}, C_{\parallel}$ | $0.56 \pm 0.02$ | $0.81 \pm 0.05$ |

Table 6. Least square fit values with 68% confidence intervals for eq. (3.1) fitted to the points shown in figure 5.
Figure 2. Final density map of particles with no kick, a kick at the beginning and a uniform kick rate generated from the MWa initial conditions. Every second row shows the fitted model.

| Parameter | $x_b$ (kpc) | $y_b$ (kpc) | $z_b$ (kpc) | $C_\perp$ | $C_\parallel$ |
|-----------|-------------|-------------|-------------|-----------|-------------|
| Not kicked | 0.67        | 0.29        | 0.27        | 2         | 4           |
| Kicked    | 0.79±0.02   | 0.36±0.01   | 0.35±0.01   | 1.93±0.02 | 3.05±0.03   |

Table 7. Predictions with 68% confidence intervals for the kicked spatial distribution for the Milky Way bar model found in ref. [39] using eq. (3.1) and the parameter values given in table 6.
Figure 3. Profile along $x$, $y$ and $z$ axes with kicks occurring at the beginning for the case generated from the MWa initial conditions. We show both the final $N$-body simulation data and data simulated using the final fitted model. For the final fitted model we show the mean number of particles in each bin and the standard deviation.
Figure 4. Final flux distribution in Galactic coordinates generated from the MWa initial conditions. The contours for each distribution are at $1, 2, 4, 8$ and $16$ times the mean in this region. The Sun is placed at a distance of $7.9\,\text{kpc}$, at an angle relative to the bar of $20^\circ$ and at a height of $15\,\text{pc}$.

Figure 5. Simulation parameters with 68% confidence interval bands for straight line model fits. The closed symbol values are obtained from values given in tables 2, 3, and 4. The predictions for the ref. [39] model of the Milky Way Galaxy are given as open symbols.
Figure 6. Line of sight contours for model of the Milky Way bar given by Cao et al. [39]. Both the model and its kicked version, obtained from the parameters in table 7, are shown. As in ref. [39] the Galactic Centre is taken to be 8.13 kpc away and the angle of the bulge to be 29°. Contours are from 1 to 32 times the mean of the corresponding case given in steps of factors of 2.

4 Discussion and conclusions

Our goal was to investigate the effect of neutron star birth kicks on the distribution of MSPs in the Galactic center. We began by running $N$-body simulations with small populations of particles kicked with a range of scales in order to estimate the required Maxwellian kick to produce a peculiar velocity distribution similar to that of resolved gamma-ray MSPs. We then reran the simulations with a larger number of particles at the required kick velocity scale and used MCMC to fit the data with a model.

We used three initial condition models intended to approximate the Milky Way, these were the MWa, MWb and MWc0.8 initial condition models of Fujii et al. [33]. Our results were consistent with theirs as can be seen, for example, by comparing our figure 1 to the top left hand panels of their figures 1, 2, and 3 which are also consistent with the observed circular rotation velocity at the Sun’s location. In Cao et al. [39] the bar scale lengths for a modified Bessel function of the second kind model fitted to red clump giant data are 0.67, 0.29 and 0.27 for the $x$, $y$ and $z$ axes respectively, with the parameters $C_{\parallel}$ and $C_{\perp}$ fixed at 4 and 2. Our fits to $N$-body models without kicks find $(x_b, y_b, z_b)$ of (0.56, 0.37, 0.26) for MWa, (0.55, 0.34, 0.24) for MWb and (0.55, 0.34, 0.24) for MWc0.8. We also have $C_{\parallel}$ between 3 and 4, producing a boxy structure in $x$-$z$ and $y$-$z$, this is visible in figure 2 for the MWa case and figures 12 and 13 for the MWb and MWc0.8 cases. The other shape parameter $C_{\perp}$ was relatively close to 2 in all cases, resulting in a more elliptical shape in $x$-$y$. We found it was necessary to extend the bar structure using the long bar component given in eq. (2.11).
Without it the bar scale parameters would be larger, but the bar would not be long enough to explain the structure seen for 2 kpc $\lesssim |x| \lesssim 4$ kpc in figure 3 in the MWa, kicked at the beginning case, and figures 14, 15, 16, 17 and 18 in the uniform kick rate, MWb and MWc0.8 cases. We find the disk scale height to be 0.22 kpc for the MWa, MWb, and MWc0.8 unknicked cases. This is at the lower end of the range of 220 to 450 pc given in Bland-Hawthorn and Gerhard [47]. There is also a small spherically symmetric Hernquist component $\sim 1\%$ of the particles in the region of interest for the unknicked cases.

Eckner et al. [31] argued using the virial theorem that kicks $\langle v^2 \rangle \lesssim (70$ km s$^{-1})^2$ would lead to a “smoothing” of the distribution of 700–900 pc. We show the effect of a 400 pc and an 800 pc Gaussian smoothing on the fitted bulge (bar plus Hernquist bulge) distribution in figures 7 and 8 for the kicked at the beginning case. The uniform kick rate case is shown in figure 21. Those figures also show the bulge component of the models with and without kicks for comparison. It is clear that a Gaussian smoothing kernel will remove the peak that survives in the N-body simulations of kicked distributions and, particularly for the 800 pc case, will produce an apparently spherically symmetric bulge. From the peculiar velocity data we inferred kicks that are larger than assumed by Eckner et al. [31] with $\sigma_0$ at around 80–100 km s$^{-1}$ ($\langle v^2 \rangle = 3 \sigma_0^2$ for a Maxwell distribution where the angular brackets signify the mean value) so the smoothing effect of the Gaussian would be even more severe. We show an even smaller Gaussian smoothing of 200 pc in figures 9 and 10 for the kicked at beginning case and in figure 22 for the uniform kick rate case. We also show in figure 11 the profile for particles with smaller kick scales between 0 km s$^{-1}$ and 80 km s$^{-1}$ for the kicked at the beginning case. The corresponding uniform kick rate case is shown in figure 23. In these two figures, each kick scale has only $4 \times 10^5$ particles; therefore, to reduce noise, the bins in each of the other two dimensions are twice as big as in previous single dimensional plots, and particles within 0.5 kpc (previously 0.25 kpc) of the axis are included. The profiles along the $z$ axis in particular show that there is a reduction in the slope as the kick velocities increase, along the other two axes the general increase in scaleheight is seen as a reduction in density. These results demonstrates that Gaussian smoothing is not a good way of modelling a kicked version of a boxy bulge/bar template.

In every case, the bar fitted to the kicked data is both broader, with larger scale parameters $x_b$, $y_b$ and $z_b$, and less boxy, with smaller $C_{\parallel}$. For example, for the MWa initial conditions $(x_b, y_b, z_b)$ increases from $(0.56, 0.37, 0.26)$ to $(0.69, 0.46, 0.33)$ and $(0.67, 0.45, 0.32)$ for the kick at beginning case and the uniform kick rate case respectively, while $C_{\parallel}$ declines from 3.08 to 2.49 and 2.54. The spherically symmetric Hernquist bulge increases from $\sim 1\%$ of the particles to 6\% for MWa. Like the bar, it becomes broader with $a_0$ increasing from around 0.2 kpc to 0.51 kpc and 0.56 kpc. For the other two models similar changes occur, $P$(Long Bar) and $a_0$ both increase significantly. In MSP model A1 of P20, the disk parameters were $\sigma_r = 4.5^{+0.5}_{-0.4}$ kpc and $z_0 = 0.71^{+0.11}_{-0.09}$. In the current article, after being kicked, the disk scale heights $z_0$ of all models increase from 0.22 kpc to $\gtrsim 1$ kpc, while $\sigma_r$ is in the range 5–7 kpc. However, we find that in all kicked cases, except for MWb with kicks occurring at the beginning, the long bar behaves like a relatively thin disk component. We have $x_{lb} \approx y_{lb}$, $C_{\perp, lb} \sim 2$ and $R_{out} \gtrsim 7$, resulting in a density $\sim \exp(-R/R_0)$ in $R$ for scalelength $R_0$. These scalelengths would then be between approximately 2.4 kpc and 2.8 kpc. For comparison, in Bland-Hawthorn and Gerhard [47] the Milky Way disk scalelength is reported as 2.6 ± 0.5 kpc. The exponential scaleheights of these “long bars” range between about 0.5–0.6 kpc. In figure 3, for the MWa kicked at the beginning case and in figures 14, 16, 17 and 18 for the uniform kick rate and MWb, and MWc0.8 case, there may be, to varying degrees, an excess of kicked particles over
the model in the region of $2 \text{kpc} \lesssim |x| \lesssim 4 \text{kpc}$. This can be seen as a correlated run of roughly two standard deviation difference between the prediction and the data in this region.

Although, a more complicated parametric model could remedy these fit defects, our main aim in this article was to estimate the effect of the MSP kicks on their distribution in the Milky Way Galactic bar. As can be seen for the MWa case from figure 4, the line of sight integral of the models are a good match to the simulations. In particular the noisy simulation contours scatter in an unbiased way around the smooth model contours. Similar results can be seen for the MWb and MWc0.8 cases in figures 19 and 20 respectively.

We used the same bar parametric model as Cao et al. [39] who fit the best fit parameters to the red clump luminosity density distribution of the Galactic bar measured by the Optical Gravitational Lensing Experiment (OGLE) III survey. Comparing to their fit, our final simulation fits of the unkicked particles had somewhat different bulge parameters. But, there appears to be a linear relationship between the unkicked scale parameters $x_b$, $y_b$, and $z_b$ and their kicked counterparts. Similarly, there appears to be a linear relationship between $C_\perp$, $C_\parallel$, and their kicked counterparts. Therefore, we were able to estimate the Milky Way Galactic bar kicked parameters as shown in figure 5 and table 7. The residuals in this least squares fit are partially due to the systematic error of our model misfit and also the Cao et al. [39] model misfit to the red clump data. As can be seen, there is more scatter in the $x_b$ parameter. This is not unexpected due to the already mentioned degeneracy with the long bar. Also, as can be seen, estimating the Milky Way bar kicked $x_b$ parameter did involve a reasonable amount of extrapolation and so future simulations which have a larger $x_b$ will be needed to check it. A made-to-measure [48, 49] approach may be needed. This would also be advantageous as it could take into account the X/peanut shaped morphology of the bar [40, 50, 51] as done in ref. [52]. There is some preliminary evidence that the X-shape may improve the fit to the Fermi-LAT gamma-ray data [10].

In conclusion, we used $N$-body simulations to explore the effect of a Maxwell distributed kick on the distribution of MSPs in the Galactic center. We find that while a 700–900 pc Gaussian smoothing of the stellar mass would be too aggressive, the bulge distribution of the kicked particles is slightly broader and less boxy. From these results, we expect the GCE to deviate by a small amount from the stellar mass spatial distribution in the Galactic center. Also, as can seen from table 7, we would not expect the GCE to appear spherically symmetric due to the MSP kicks as that would require $x_b = y_b = z_b$ and $C_\perp = C_\parallel = 2$ which are far from our inferred points relative to their error bars.

The amount of spatial smoothing of the bulge MSPs will depend on the proportion of MSPs in the bulge that are made from the recycling channel and the proportion that are made from the accretion induced collapse channel. Motivated by similarities between the bulge and disk population seen by P20 we have assumed this mixture has the same proportions as the disk MSPs. If the GCE is due to bulge MSPs, its morphology could be used to check our smoothing prediction by comparing if there are any deviations between the GCE morphology and the stellar spatial distribution. A complication to this approach would be the possibility of some smearing of the GCE due to cosmic ray electron diffusion [53, 54]. An additional complication is that if the MSP is spun up by a captured star then the MSP spatial distribution would be proportional to the stellar density squared [8, 31]. We have been assuming that the MSPs formed in a binary system and so have a spatial distribution proportional to the stellar density. Eventually, once the bulge MSPs are resolved [30, 55], comparing their spatial distribution to the stellar distribution should provide independent information to more robustly estimate the natal kick distribution.
Figure 7. Final profile along $x$, $y$ and $z$ axes with kicks occurring at the beginning generated from the MWa initial conditions. Here we show the fitted bulge components, which consist of the bar plus Hernquist bulge, as well as the no kick bulge smoothed with 400 pc and 800 pc Gaussians. We show both $N$-body simulation data and data simulated using the fitted model. For the fitted model we show the mean number of particles in each bin and the standard deviation.

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Figure 8. Final bulge, consisting of bar plus Hernquist bulge, flux distribution in Galactic coordinates generated from the MWa initial conditions. We also show the no kick bulge smoothed with 400 pc and 800 pc Gaussians. The contours for each distribution are at 1, 2, 4, 8 and 16 times the mean in this region. The Sun is placed at a distance of 7.9 kpc, at an angle relative to the bar of 20° and at a height of 15 pc.
Figure 9. Final profile along $x$, $y$ and $z$ axes with kicks occurring at the beginning generated from the MWa initial conditions. Here we show the fitted bulge components as well as the no kick bulge smoothed with a 200 pc Gaussian. We show both $N$-body simulation data and data simulated using the fitted model. For the fitted model we show the mean number of particles in each bin and the standard deviation.
Figure 10. Final bulge flux distribution in Galactic coordinates generated from the MWa initial conditions. We also show the no kick bulge smoothed with a 200 pc Gaussian. The contours for each distribution are at 1, 2, 4, 8 and 16 times the mean in this region. The Sun is placed at a distance of 7.9 kpc, at an angle relative to the bar of 20º and at a height of 15 pc.
Figure 11. Final profile along $x$, $y$ and $z$ axes with kicks occurring at the beginning for kicks between 0 km s$^{-1}$ and 80 km s$^{-1}$ in 20 km s$^{-1}$ increments. Generated from the MWa initial conditions.
A Uniform kick rate, MWb, and MWc0.8 figures

As the uniform kick rate, MWb, and MWc0.8 gave similar results to the MWa kicked at beginning case we have moved many of their figures to this appendix.

Figure 12. Final density map of particles with no kick, a kick at the beginning and a uniform kick rate generated from the MWb initial conditions. Every second row shows the fitted model.
Figure 13. Final density map of particles with no kick, a kick at the beginning and a uniform kick rate generated from the MWc0.8 initial conditions. Every second row shows the fitted model.
Figure 14. Final profile along $x$, $y$ and $z$ axes with a uniform kick rate generated from the MWa initial conditions. We show both $N$-body simulation data and data simulated using the fitted model. For the fitted model we show the mean number of particles in each bin and the standard deviation.
Figure 15. Final profile along $x$, $y$ and $z$ axes with kicks occurring at the beginning generated from the MWb initial conditions. We show both $N$-body simulation data and data simulated using the fitted model. For the fitted model we show the mean number of particles in each bin and the standard deviation.
Figure 16. Final profile along $x$, $y$ and $z$ axes with a uniform kick rate generated from the MWb initial conditions. We show both $N$-body simulation data and data simulated using the fitted model. For the fitted model we show the mean number of particles in each bin and the standard deviation.
Figure 17. Final profile along $x$, $y$ and $z$ axes with kicks occurring at the beginning generated from the MWe0.8 initial conditions. We show both $N$-body simulation data and data simulated using the fitted model. For the fitted model we show the mean number of particles in each bin and the standard deviation.
Figure 18. Final profile along $x$, $y$ and $z$ axes with a uniform kick rate generated from the MWc0.8 initial conditions. We show both $N$-body simulation data and data simulated using the fitted model. For the fitted model we show the mean number of particles in each bin and the standard deviation.
Figure 19. Final flux distribution in Galactic coordinates generated from the MWb initial conditions. The contours for each distribution are at 1, 2, 4, 8 and 16 times the mean in this region. The Sun is placed at a distance of 7.9 kpc, at an angle relative to the bar of 20° and at a height of 15 pc.

Figure 20. Final flux distribution in Galactic coordinates generated from the MWc0.8 initial conditions. The contours for each distribution are at 1, 2, 4, 8 and 16 times the mean in this region. The Sun is placed at a distance of 7.9 kpc, at an angle relative to the bar of 20° and at a height of 15 pc.
Figure 21. Final profile along $x$, $y$ and $z$ axes with a uniform kick rate generated from the MWa initial conditions. Here we show the fitted bulge components, consisting of the bar plus Hernquist bulge, as well as the no kick bulge smoothed with 400 pc and 800 pc Gaussians. We show both $N$-body simulation data and data simulated using the fitted model. For the fitted model we show the mean number of particles in each bin and the standard deviation.
Figure 22. Final profile along $x$, $y$ and $z$ axes with a uniform kick rate generated for the MWa initial conditions. Here we show the fitted bulge components as well as the no kick bulge smoothed with a 200 pc Gaussian. We show both $N$-body simulation data and data simulated using the fitted model. For the fitted model we show the mean number of particles in each bin and the standard deviation.
Figure 23. Final profile along $x$, $y$ and $z$ axes with a uniform kick rate for kicks between 0 km s$^{-1}$ and 80 km s$^{-1}$ in 20 km s$^{-1}$ increments. Generated from the MWa initial conditions.
B Markov chain Monte Carlo method

In previous work, such as ref. [30], we used for our MCMC the adaptive Metropolis algorithm of Haario et al. [56]. In this article, however, we found it was necessary to replace adaptive Metropolis algorithm with an alternative algorithm to ensure rapid convergence to the peak likelihood region of the parameter space. The MCMC algorithm used in this article is similar to that of Foreman-Mackay et al. [43] with a mixture of the Differential Evolution [44] and snooker updates [45]. Instead of performing a single random walk through the parameter space where proposed moves are accepted with a probability based on the likelihood of the current point of the chain and the proposed next point, we use an ensemble of $K$ “walkers” where a proposed update for a walker $j$ depends on the distribution of the other walkers.

A single step of the stretch move update suggested in Foreman-Mackay et al. [43] involves updating the $K$ walkers sequentially. Let $x_j$ be the state of walker $j$, an update for walker $x_j$ is performed as follows:

1. Draw $k$ from $1, 2, \ldots, K$ where $k \neq j$.
2. Draw $z$ from the probability density function with parameter $a$ (Foreman-Mackay et al. [43] suggest $a = 2$):
   \[
g(z) \propto \begin{cases} 
1 & z \in \left[\frac{1}{a}, a\right] \\
0 & \text{otherwise}
\end{cases}
\]
   (B.1)
3. Calculate proposal $y = x_k + z(x_j - x_k)$
4. Calculate acceptance probability $r$:
   \[
r = z^{d-1} \frac{p(y)}{p(x_j)}
\]
   (B.2)
   where $d$ is the number of dimensions.
5. Set $x_j = y$ with probability $\min(1, r)$

We found better results using a mixture of two alternative updates: 80% the Differential Evolution update of ter Braak [44] and 20% the snooker update of ter Braak and Vrugt [45].

To update walker $x_j$ using the Differential Evolution update:

1. Draw $k$ and $l$ from $1, 2, \ldots, K$ where $k \neq j$, $l \neq j$ and $l \neq k$.
2. Propose $y = x_j + \gamma(x_k - x_l) + e$ where $\gamma$ is a parameter and where $e$ is drawn from a small $d$ dimensional symmetric probability distribution.
3. Calculate acceptance probability $r$:
   \[
r = \frac{p(y)}{p(x_j)}
\]
   (B.3)
4. Set $x_j = y$ with probability $\min(1, r)$.

We drew $e$ from a $d$ dimensional Gaussian with standard deviation $10^{-5}$ in each dimension. In case the likelihood distribution had multiple modes, we used $\gamma = 1$ with probability 0.1 as suggested by ter Braak [44], otherwise we used the default value of $\gamma = 2.38/\sqrt{2d}$. 

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Using the snooker update, we update $x_j$ as follows:

1. Draw $k$, $l$ and $m$ from $1, 2, \ldots, K$ with no index repeated or equal to $j$

2. Calculate the orthogonal projections $\text{proj}_{x_j-x_k}(x_l)$ and $\text{proj}_{x_j-x_k}(x_m)$, where:

\begin{equation}
\text{proj}_u(v) = \frac{v \cdot u}{u \cdot u} 
\end{equation}

3. Propose $y = x_j + \gamma_s \left( \text{proj}_{x_j-x_k}(x_l) - \text{proj}_{x_j-x_k}(x_m) \right)$ where $\gamma_s$ is a parameter

4. Calculate acceptance probability $r$:

\begin{equation}
r = \frac{p(y)}{p(x_j)} \frac{|y-x_k|^{d-1}}{|x_j-x_k|^{d-1}} 
\end{equation}

5. Set $x_j = y$ with probability $\min(1, r)$

We use $\gamma_s = 2.38/\sqrt{2}$ as suggested by ter Braak and Vrugt [45].

We also used a simple annealing method in which we divide the log-likelihood by a temperature $T$ which is gradually reduced to 1. The posterior probability density for a parameter set $\theta$ at MCMC iteration $t$ is:

\begin{equation}
p(\theta|\text{N-body data}) \propto p(\theta)L^{1/T_t} 
\end{equation}

where $p(\theta)$ is the prior, $L$ is the likelihood, and $T_t$ is the temperature. We used a linearly decreasing log($T$) from log(1000) to log(1) during the first half of each Markov chain, which we discard. This method allows the algorithm to explore a broad region in the parameter space, while slowly converging to the desired posterior distribution where $T = 1$. This appeared to help the Markov chains avoid getting stuck in local likelihood maxima.

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