A SUPPLEMENT TO A THEOREM OF MERKER AND PORTEN: A SHORT PROOF OF HARTOGS’ EXTENSION THEOREM FOR $(n-1)$-COMPLETE COMPLEX SPACES

MIHNEA COLŢOIU

1. INTRODUCTION

The well-known Hartogs’ extension theorem states that for every open subset $D \subset \mathbb{C}^n$, $n \geq 2$, every compact subset $K \subset D$ such that $D \setminus K$ is connected the holomorphic functions on $D \setminus K$ extend to holomorphic functions on $D$. For a simple and short $\overline{\partial}$ proof see [E]. A long and very involved proof of this result on 33 pp., using Morse theory, in the spirit of Hartogs’ original idea [Ha] of moving discs to get the extension, was recently obtained by J. Merker and E. Porten [M-P1].

The Hartogs’ theorem was generalized to $(n-1)$-complete manifolds (in the sense of A. Andreotti and H. Grauert [A-G]) by A. Andreotti and D. Hill [A-H] using cohomological results ($\overline{\partial}$ method). In their forthcoming paper [M-P2] J. Merker and E. Porten observed that in the singular case “it is at present advisable to look for methods avoiding $\overline{\partial}$ methods, because such tools are not yet available” and “the essence of the present article is to transfer such an approach to $(n-1)$-complete general complex spaces, where the $\overline{\partial}$ techniques are still lacking, with some new difficulties due to singularities”. J. Ruppenthal [R] developed a $\overline{\partial}$ machinery for proving the Hartogs’ extension theorem on Stein spaces with isolated singularities.

The main result of J. Merker and E. Porten [M-P2], which generalizes Andreotti-Hill theorem [A-H] for singular spaces, can be stated as follows:

**Theorem 1.1.** Let $X$ be a normal $(n-1)$-complete space ($n = \dim X$), $D \subset X$ a relatively compact open subset, $K \subset D$ a compact subset such that $D \setminus K$ is connected. Then every holomorphic function on $D \setminus K$ can be extended to a holomorphic function on $D$.

In fact they proved this result even for the extension of meromorphic functions (previously considered in the smooth case by V. Koziarz and F. Sarkis [K-S]) but we shall consider in this short note only the holomorphic extension. The 20 pages proof of Merker and Porten [M-P2] is also based on their previous paper [M-P1] on 33 pp., so putting together one gets about 50 pages which are very technical.

We will give in this short note a 1 page proof for Theorem 1.1., using the $\overline{\partial}$ method on the resolution of singularities, especially the Takegoshi relative vanishing theorem [T], (Submitted on November 12th, 2008.)
see also [O], which gives even a more general statement valid on cohomologically \((n-1)\)-complete spaces (and without the assumption that \(D\) is relatively compact). Namely one has:

**Theorem 1.2.** Let \(X\) be a \(n\)-dimensional normal cohomologically \((n-1)\)-complete complex space, \(D \subset X\) an open subset, \(K \subset D\) a compact subset such that \(D \setminus K\) is connected. Then every holomorphic function on \(D \setminus K\) can be extended to a holomorphic function on \(D\).

The ideas of the proof are essentially contained in the paper [C-S].

**2. Proof of the result**

For the basic definitions of \(q\)-convex functions, \(q\)-complete complex space we refer to [A-G]. We also recall that a complex space \(X\) is called cohomologically \(q\)-complete if one has the vanishing of the cohomology groups \(H^i(X, F) = 0\) for every \(i \geq q\) and every \(F \in Coh(X)\). By the main result of [A-G] a \(q\)-complete space is cohomologically \(q\)-complete (a counter-example to the converse is still unknown). For a complex manifold \(X\) we denote by \(K_X\) its canonical sheaf (associated to the canonical line bundle). Let \(X\) be a complex (reduced) space and \(\pi : \tilde{X} \to X\) a resolution of singularities (which exists by [A-H-V], [B-M]). The following result, due to K. Takegoshi [T] (see also T. Ohsawa [O]), will be fundamental for our proof:

**Theorem 2.1.** Let \(\pi : \tilde{X} \to X\) be a resolution of singularities of a complex space \(X\). Then one has the following vanishing for the higher direct images:

\[
R^i \pi_* K_{\tilde{X}} = 0 \text{ if } i \geq 1
\]

Let us also recall that by Grauert’s coherence theorem [G] \(\pi_* K_{\tilde{X}}\) is a coherent sheaf on \(X\). If moreover \(X\) is assumed to be cohomologically \((n-1)\)-complete it then follows that \(H^i(X, \pi_* K_{\tilde{X}}) = 0\) if \(i \geq n - 1\). By Theorem 2.1., the maps \(H^i(X, \pi_* K_{\tilde{X}}) \to H^i(\tilde{X}, K_{\tilde{X}})\) are isomorphisms, so that one gets the vanishing of the cohomology group \(H^i(\tilde{X}, K_{\tilde{X}}) = 0\) if \(i \geq n - 1\). By Serre duality [S] one gets the vanishing of the first cohomology group with compact supports \(H^1_c(\tilde{X}, O_{\tilde{X}}) = 0\). Moreover the arguments of L. Ehrenpreis [E] (see also [H0]) show without any modification that the following holds: If \(\tilde{X}\) is a complex connected non-compact manifold such that \(H^1_c(\tilde{X}, O_{\tilde{X}}) = 0\) then for every open subset \(\tilde{D} \subset \tilde{X}\) and for every compact subset \(\tilde{K} \subset \tilde{D}\), such that \(\tilde{D} \setminus \tilde{K}\) is connected, it follows that every holomorphic function on \(\tilde{D} \setminus \tilde{K}\) can be extended to a holomorphic function on \(\tilde{D}\). Applying this result to \(\tilde{D} = \pi^{-1}(D)\) and \(\tilde{K} = \pi^{-1}(K)\), where \(\pi : \tilde{X} \to X\) is a resolution of singularities for \(X\), one gets immediately Theorem 1.2., since \(\pi_* O_{\tilde{X}} = O_X\) (by the normality of \(X\)).

**References**

[A-G] A. Andreotti and H. Grauert: *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90 (1962), 193-259.

[A-H] A. Andreotti and C.D. Hill: *E.E. Levi convexity and the Hans Lewy problem I and II*, Ann. Sc. Norm. Sup. Pisa 26 (1972), 325-363, 747-806.

[A-H-V] J.M. Aroca, H. Hironaka, J.L. Vincente: *Desingularization theorems*, Mem. Math. Jorge Juan, No. 30, Madrid, 1977.
A SUPPLEMENT TO A THEOREM OF MERKER AND PORTEN: A SHORT PROOF OF HARTOGS’ EXTENSION THEOREM

[B-M] E. Bierstone and P. Milman: Canonical desingularisation in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. 128(1997), no. 2, 207-302.

[C-S] M. Coltoiu and A. Silva: Behnke-Stein theorem on complex spaces with singularities, Nagoya J. Math. 137 (1995), 183-194.

[E] L. Ehrenpreis: A new proof and an extension of Hartogs’ extension theorem, Bull. Amer. Math. Soc. 67 (1961), 507-509.

[G] H. Grauert: Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, Inst. Hautes Études Sci. Publ. Math. no. 5, 1960, 64 pp.

[Ha] F. Hartogs: Zur theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reichen, welche nach Potenzen einer Veränderlichen fortschreiten, Math. Ann. 62 (1906), no. 1, 1-88.

[Ho] L. Hörmander: An introduction to complex analysis in several variables, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, 1966, 208 pp.

[K-S] V. Koziarz and F. Sarkis: Problème du bord dans les variétés q-convexes et phénomène de Hartogs-Bochner, Math. Ann. 321 (2001), no.3, 569-585.

[M-P1] J. Merker and E. Porten: A Morse theoretical proof of the Hartogs extension theorem, J. Geom. Anal. 17 (2007), no.3, 513-546.

[M-P2] J. Merker and E. Porten: The Hartogs’ extension theorem on (n − 1)-complete complex spaces, preprint, arXiv:0704.3210 (to appear in J. reine angew. Math).

[O] T. Ohsawa: A vanishing theorem for proper direct images, Publ. RIMS 23(1987), no. 2, 243-250.

[R] J. Ruppenthal: A \bar{\partial} theoretical proof of Hartogs’ extension theorem on Stein spaces with isolated singularities, J. Geom. Anal. 18 (2008), no.4

[S] J. P. Serre: Un théorème de dualité, Comment. Math. Helv. 29(1955), 9-26.

[T] K. Takegoshi: Relative vanishing theorems in analytic spaces, Duke Math. J. 52 (1985), no. 1, 273-279.

M. Coltoiu: Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700, București, Romania.

E-mail address: Mihnea.Coltoiu@imar.ro