Thermal contact: mischief and time scales

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Abstract
We discuss what kind of quantum dynamics can yield thermalization. We show that any temperature measurement protocol must be sensitive to the system’s local energy scale, not just its state. Specifically, we illustrate this by showing that any temperature measurement protocol that does not resolve the system’s local Hamiltonian (such as, e.g. full state tomography) is susceptible to being fooled into measuring any value for the temperature. Since placing a system in thermal contact with a reference system at a known temperature is a way of measuring temperature, any robust notion of thermal contact must be resistant to this ‘attack’. This provides us with a necessary conditions for thermal contact between quantum systems. We then apply this condition to a scenario where one expects to find thermalization: a system repeatedly interacting with the microconstitutents of a thermal reservoir. We show such models of thermalization cannot be correct without fine-tuning.

Keywords: thermalization, thermometry, collision models, zeroth law of thermodynamics, quantum metrology

1. Introduction
In life, ‘all contact is thermal contact’ is a good rule of thumb. Place any two everyday objects in physical contact with each other and one can confidently predict that they will eventually thermalize to a common temperature—it seems that no knowledge of how exactly they interact is required. Indeed, it appears this phenomenon is highly robust to the particular details of the scenario considered.
A significant amount of work has been done investigating this robustness, characterizing the emergence of thermalization from quantum dynamics [1–9]. Indeed, use is often made of collision models to model thermalization [10–14] in quantum thermodynamics. Such scenarios consider a quantum system repeatedly interacting with (being bombarded by) the constituents of its environment one at a time. In particular the partial swap interaction described in [15] is very common [16–20]. Often collision models are studied in the continuum limit, where the duration of each interaction, $\delta t$, is taken to be vanish as the strength of each interaction (with dimensions of frequency), $g$, diverges such that $g^2 \delta t$ is held constant [10, 21–31].

Perhaps surprisingly, we will show that in the regime of rapid bombardment ($g \delta t \ll 1$), such collisional scenarios cannot constitute thermal contact without fine-tuning. In other words the system is almost never driven to thermal equilibrium with its environment by rapid bombardment. As we will argue, the reason for this non-thermalization in rapid bombardment scenarios is that the system does not have time to sense the temperature of its environment.

To demonstrate this, we will argue that any dynamics underlying thermal contact must ‘know’ both systems’ local Hamiltonians. We will then show that without fine-tuning the system must ‘learn’ these local Hamiltonians dynamically. Applying these arguments to the rapid bombardment scenario described above we will find the system does not spend enough time with each constituents of its environment to ‘learn’ its local Hamiltonian; the process which carries that information is ‘too long/complex’ and is highly suppressed. Therefore rapid bombardment cannot mediate thermal contact without fine-tuning. Finally, we will provide some example scenarios, including a comparison with [15].

2. Thermometers and local Hamiltonians

Suppose that we have access to a quantum system in a Gibbs state of unknown inverse temperature, $\beta = 1/kT$, with an unknown local Hamiltonian, $\hat{H}$. That is, $\rho_{th} = \exp(-\beta \hat{H})/Z$ where $Z = \text{Tr}(\exp(-\beta \hat{H}))$. We can ask the following question: Can we determine its temperature without knowledge of its local Hamiltonian?

An easy answer to this question is: ‘Simple! Put the system in thermal contact with a thermometer and read off the temperature’. It appears this task does not require knowledge of the system’s local Hamiltonian, just a thermometer. Since one simply needs a thermometer to determine the system’s temperature, it would seem that doing full state tomography (using arbitrarily many copies of the system) is overkill. However, state tomography alone does not give us enough information to determine the system’s temperature.

To see why, note that any thermal density matrix $\rho_{th} = \exp(-\beta \hat{H})/Z$ is invariant under the transformation,

$$\Lambda : \beta \rightarrow \lambda \beta; \quad \hat{H} \rightarrow \hat{H}/\lambda,$$

(1)

which rescales both the temperature and energy scale of the system. While the above argument is relatively obvious in hindsight, we can gain a new perspective on it by rephrasing it as the basis of an ‘attack’ on a thermodynamics experiment (here temperature measurement) as described below.

Imagine that some mischievous agent, Loki himself perhaps, breaks into our lab and replaces our system with a new one which has the same density matrix but nonetheless a different temperature. By construction we cannot notice this swap-out by just characterizing the

5 Such a transformation can indeed be physically made: for instance, Loki may swap-out a magnetic spin for one with a higher temperature while simultaneously increasing the strength of the magnetic field. Note that while this trick may cost Loki some energy (i.e. work) no additional work is required after his action.
system’s density matrix. Note that we could however detect Loki’s trick if we determine the system’s energy scale, which Loki necessarily altered in order to maintain the state’s density matrix. In fact, as we will now show this is the only way to detect Loki’s trick.

This energy scale cannot be determined from the density matrix alone. In fact, as we will now show this is the only information about the local Hamiltonian not available by characterizing the state, \( \rho_{th} \). We can determine the eigenbasis, \( \{ |n\rangle \} \), of \( \hat{H} \) by diagonalizing \( \rho_{th} \) since they commute and therefore share an eigenbasis. Moreover, we can partially determine the eigenvalues, \( \{ E_n \} \), of \( \hat{H} \) using thermal detailed balance,

\[
\frac{\langle n | \rho_{th} | n \rangle}{\langle m | \rho_{th} | m \rangle} = \exp(-\beta(E_n - E_m)).
\]

Specifically, we can determine the relative spacing between the eigenvalues of \( \hat{H} \) as,

\[
\frac{E_n - E_m}{E_j - E_k} = \frac{\log(\langle n | \rho_{th} | n \rangle) - \log(\langle m | \rho_{th} | m \rangle)}{\log(\langle j | \rho_{th} | j \rangle) - \log(\langle k | \rho_{th} | k \rangle)}.
\]

This fixes the eigenvalues of \( \hat{H} \) up to a constant offset (which can be ignored) and, crucially, an unknown scale factor. Thus, by characterizing \( \rho_{th} \) we can learn all there is to know about \( \hat{H} \) except its overall scale.

Since knowing everything but this energy scale is insufficient to determine a system’s temperature, characterizing this energy scale is a necessary aspect of any temperature measurement protocol. The only way to notice Loki’s trick is to realize he has rescaled the system’s local energy scale. This requires dependence on the system’s local Hamiltonian.

How can we reconcile this conclusion with our earlier intuition that we can easily measure the temperature of a system without knowledge of its local Hamiltonian? Our above conclusion does not imply that the experimenter ever knows anything about the system’s local Hamiltonian, what it implies is that the thermometer ‘knows’ or ‘learns’ this information. Concretely this means that valid thermometer should be able to differentiate the cases where Loki has or has not acted on our system and report different temperatures in either case. The thermometer does not need to report to the experimenter what the system’s local energy scales were in either case but it must itself be sensitive to them.

3. Thermal contact and sabotage

While our above conclusions hold for any type of temperature measurement protocol we will now particularize to a protocol closely related to the definition of temperature: place the system of interest in thermal contact with a series of thermal reference states at known temperatures and report the temperature of the reference state with which the system does not exchange heat. Applying the conclusions of the previous section to this scenario, we expect that the dynamics underlying this the thermal contact must somehow depend on the system’s local Hamiltonian. To see this however, let us first review some foundational concepts in thermodynamics.

Two systems (\( A \) and \( B \)) are defined to be in thermal contact if they are allowed to exchange heat freely and do so until they reach a thermal equilibrium, where no more heat flows between them. The zeroth law of thermodynamics states that this notion of thermal equilibrium is transitive and so can be thought of as an equivalence relation. Temperature is then defined as any monotonic labeling of these equilibria with the ordering ‘hot > cold’ defined by the direction of heat flow.

To illustrate, the textbook example of thermal contact is two systems permitted to exchange energy freely with each other while maximizing their total entropy. By a standard calculation,
this process reaches its equilibrium when $dS_A/dE_A = dS_B/dE_B$. From this we get the usual textbook definition of temperature $1/T := dS/dE$. Note this is the temperature that appears in the exponential of a Gibbs state, $\rho_{th} = \exp(-\hat{H}/kT)/Z$.

The approach we use in this paper is to take the temperature appearing in the Gibbs state as the canonical temperature defined by a process known to be thermal contact. Using this we can then judge whether any interaction constitutes thermal contact or not by whether its equilibrium condition is compatible with these canonical notions of temperature and thermal contact.

For example, suppose two thermal systems, $A$ and $C$, interact in a way that is known to be thermal contact (as described above) and it is noted that their reduced states evolve due to this contact (implying they were not in thermal equilibrium), such that we know $T_A \neq T_C$. Suppose then that $A$ and $C$ each interact in an unknown way with some other thermal system $B$ and its noted that neither system’s reduced state evolves due to this contact (implying no heat flow). If this second set of interactions were thermal contact then we would have to conclude $T_A = T_B = T_C$ while knowing $T_A \neq T_C$, which would be a violation of the zeroth law. Thus the interaction between $A$ and $B$ (and between $B$ and $C$) cannot be thermal contact, despite the fact that $A$ and $B$ are in some sense equilibrated. There are many reasons why this interaction may have failed to be thermal contact: maybe the systems did not interact freely, maybe energy was not conserved, maybe entropy was not maximized. Nonetheless, whatever equilibrium there was between $A$ and $B$ it was not thermal equilibrium, and whatever contact there was it was not thermal contact.

Note that in this example it was sufficient to check that whether the system’s reduced states evolved or not to establish if heat had flowed between them. Specifically, any heat flow would imply evolution of the system’s reduced state. Contrapositively, no evolution of the system’s reduced state would imply no heat flow. Note however that the system’s reduced state could change without heat flowing. Nonetheless, this allows us to at least partially characterize thermal contact without having to define heat rigorously, which is notoriously difficult in quantum thermodynamics. In turn, this will allow us to go ahead with out analysis with no more characterization of heat than Maxwell’s statement that ‘All heat is of the same kind’ [32].

Given the above considerations, we can now define some necessary conditions for thermal contact. Suppose two quantum systems are initially uncorrelated and thermal (in Gibbs states) with inverse temperatures $\beta_A(0)$ and $\beta_B(0)$. Imagine that these systems interact with each other for a long time and then are separated. This interaction can constitute thermal contact only if for all initial temperatures:

1. The final reduced states of both systems are thermal, with inverse temperatures $\beta_A(\infty)$ and $\beta_B(\infty)$.
2. These temperatures are the same, $\beta_A(\infty) = \beta_B(\infty)$.
3. The (reduced) systems do not evolve only if they are initially at the same temperature.

That is, $\beta_A(0) = \beta_B(0)$ if $\rho_A(t) = \rho_A(0)$ and $\rho_B(t) = \rho_B(0)$ for all $t \geq 0$.

Note again that these conditions are necessary (but not sufficient) for an interaction to be considered thermal contact. Additional conditions such as energy conservation and entropy maximization could additionally be considered and would yield stronger more restricted results.

As such in a generic thermal contact scenario between $A$ and $B$, the systems are initially thermal (with respect to their local Hamiltonians, $\hat{H}_A$ and $\hat{H}_B$) and uncorrelated, i.e. $\rho_{AB}(0) = \rho_A(0) \otimes \rho_B(0)$, with

$$\rho_A(0) = \frac{e^{-\beta_A(0)\hat{H}_A}}{Z_A(0)}, \quad \rho_B(0) = \frac{e^{-\beta_B(0)\hat{H}_B}}{Z_B(0)}.$$  

(4)
The most general completely positive trace preserving (CPTP) evolution of this joint system is given by
\[
\rho_{AB}(t) := \Phi_{AB}(t)[\rho_A(0) \otimes \rho_B(0)],
\] (5)
for some CPTP map \(\Phi_{AB}(t)\). This dynamical map will in general depend on all the details of the interaction between the two systems. This includes both the systems’ local Hamiltonians as well as any other interaction details, which we collect into the label \(\mathcal{I}\) as,
\[
\Phi_{AB}(t) = \Phi_{AB}(t, \hat{H}_A, \hat{H}_B, \mathcal{I}).
\] (6)
For instance, \(\mathcal{I}\) could include: an interaction Hamiltonian, \(\hat{H}_{AB}\), time dependent switching functions, spatial smearing functions, probability distributions for various stochastic elements, the initial temperatures of the system, and other particular details.

Note that the final reduced states depend on the parameters of the interaction as,
\[
\rho_A(\infty) = f_A(\hat{H}_A, \hat{H}_B, \rho_A(0), \rho_B(0), \mathcal{I})
\]
(7)
\[
\rho_B(\infty) = f_B(\hat{H}_A, \hat{H}_B, \rho_A(0), \rho_B(0), \mathcal{I}).
\]
If the dynamics described by (5) constitutes thermal contact then the final reduced states are both thermal,
\[
\rho_A(\infty) = \frac{e^{-\beta_A(\infty) \hat{H}_A}}{Z_A(\infty)}, \quad \rho_B(\infty) = \frac{e^{-\beta_B(\infty) \hat{H}_B}}{Z_B(\infty)},
\] (8)
with identical temperatures, \(\beta_A(\infty) = \beta_B(\infty)\).

As motivated in the section 2, we expect that in order to not be vulnerable to Loki’s swap-out trick the dynamics underlying thermal contact must be sensitive to the systems’ local Hamiltonians (outside of density matrices). At this point it is useful to distinguish two ways for the dynamics to accomplish this.

Firstly, the dynamics could ‘already know’ the systems’ local Hamiltonians through some fine-tuned co-dependence of the parameters of the interaction set up before the interaction. Explicitly this would mean that we are not allowed to pick the parameters of the interaction independently. To exclude this possibility we now assume that the parameters of the interaction (i.e. \(\beta_A(0), \beta_B(0), \hat{H}_A, \hat{H}_B, \) and \(\mathcal{I}\) are mutually independent. In particular this means that transformations of the form (1) can be performed on either system without affecting the other parameters of the interaction. Barring this fine-tuning, the second option is that the dynamics may ‘learn’ about the systems’ local Hamiltonians through their interaction. We will now explore this second option in detail.

Suppose that \(\Phi_{AB}(t)\) does not depend on \(\hat{H}_A\) such that, \(\Phi_{AB}(t) = \Phi_{AB}(t, \hat{H}_B, \mathcal{I})\). From this it follows that the systems’ final states depend on \(\hat{H}_A\) only through \(\rho_A(0)\). That is,
\[
\rho_A(\infty) = f_A(\hat{H}_B, \rho_A(0), \rho_B(0), \mathcal{I}),
\]
\[
\rho_B(\infty) = f_b(\hat{H}_B, \rho_A(0), \rho_B(0), \mathcal{I}).
\] (9)
Note that both of these states are then invariant under the transformation,
\[
\Lambda_A : \beta_A(0) \rightarrow \lambda_A \beta_A(0) ; \quad \hat{H}_A \rightarrow \hat{H}_A/\lambda_A,
\] (10)
since \(\rho_A(0) = \exp(-\beta_A(0) \hat{H}_A)/Z(0)\) is invariant under this transformation and by assumption \(\hat{H}_B, \beta_B(0), \) and \(\mathcal{I}\) are each independent of both \(\hat{H}_A\) and \(\beta_A(0)\). Using this invariance and (8) we can determine how the final temperatures of the systems transform under \(\Lambda_A\),

\[
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\]
\[ \Lambda_A : \beta_A(\infty) \rightarrow \lambda_A \beta_A(\infty); \quad \beta_B(\infty) \rightarrow \beta_B(\infty). \]  

(11)

The final temperatures of the systems transform differently and will therefore in general not be the same except for a particular fine-tuned choice of \( \lambda_A \).

From an operational point of view, any interaction like the one described above cannot constitute thermal contact since it is vulnerable to Loki's swap-out trick. For instance, suppose that system \( B \) is at a standardized temperature and that we have just 'confirmed' that system \( A \) is at the same temperature by placing it in (what we believe is) thermal contact with \( B \), noting that it does not evolve, and by invoking property (3) of thermal contact. Suppose that Loki then swaps system \( A \) with system \( C = \Lambda_A[A] \). If we then perform the same procedure using systems \( C \) and \( B \), we would 'confirm' they are at the same temperature yielding \( \beta_C = \beta_B = \beta_A \) even though \( \beta_C \neq \lambda_A \). This contradicts the zeroth law of thermodynamics, thus the interaction between \( A \) and \( B \) cannot be thermal contact.

Hence we can conclude that for an interaction to constitute thermal contact without fine-tuning the joint dynamical map, \( \Phi_{AB}(t) \), must depend explicitly on \( \hat{H}_A \). The same argument applies reversing the roles of \( A \) and \( B \), such that the joint dynamics must depend on \( \hat{H}_B \) as well.

Of course, one may think this condition is trivially satisfied; physically meaningful maps often depend on the systems' local Hamiltonians. Indeed this is true; for instance, consider master equations of the form,

\[
\frac{d}{dt} \rho_A(t) = -i \frac{\hbar}{\hbar} [\hat{H}_A, \rho_A(t)] + D_A[\rho_A(t)],
\]

\[
\frac{d}{dt} \rho_B(t) = -i \frac{\hbar}{\hbar} [\hat{H}_B, \rho_B(t)] + D_B[\rho_B(t)],
\]

(12)

where \( D_A \) and \( D_B \) describe the dynamics induced by the interaction. Clearly this time evolution explicitly depends on the local Hamiltonians. Importantly, however, as we will now discuss, dependence on the local Hamiltonians though such time-independent local terms (i.e. \([\hat{H}_A, \cdot]\) and \([\hat{H}_B, \cdot]\)) is not enough to yield thermal contact. As we will show the induced dynamics (\( D_A \) and \( D_B \)) themselves must depend on the local Hamiltonians for interaction to produce thermal contact.

To see this consider the scenario where the initial temperatures of the two systems are very close. It is reasonable to expect that they will not evolve much as they equilibrate, staying approximately thermal. We claim that by taking their initial temperatures to be arbitrarily close the reduced states can be made arbitrarily close to thermal states throughout their evolution. Specifically we claim that the local terms in (12) can be made arbitrarily small throughout the interaction.

Therefore in order for (12) to constitute thermal contact the induced dynamics (\( D_A \) and \( D_B \)) must themselves depend on the local Hamiltonians. More generally, the above argument shows that for any dynamics to constitute thermal contact it must depend on the systems’ local Hamiltonians even after assuming the systems are in (or arbitrarily near to) thermal states throughout the interaction.

At first glance, this improved condition does not seem much more difficult to satisfy. It is true that in a generic scenario, the induced dynamics will generally depend on both systems’ local Hamiltonians. However, in certain regimes, or under certain approximations this dependence may vanish. As we will soon see, the dynamics generated by rapid bombardment (which we intuitively expect to be able to produce thermalization) in fact fails to meet this condition and therefore cannot yield thermal contact.
As a final note, we reiterate that these results hold as long as there is no fine-tuning, that is as long as the independence assumption holds. Of course scenarios can be constructed that do not satisfy this assumption. For example, consider two identical magnetic spins ($\mu_A = \mu_B = \mu$) polarized by the same magnetic field ($B_A = B_B = B$) with, $\hat{H}_A = \mu B \sigma_z A$, and $\hat{H}_B = \mu B \sigma_z B$. In this case Loki’s trick cannot address each system individually. Additionally there are situations (for instance phase transitions) where the local Hamiltonians or coupling strengths are temperature dependent. Nonetheless the scope of applicability of these results is very wide and includes all situations where the local Hamiltonians and coupling strengths are set independently by fundamental considerations.

4. Rapid bombardment is not thermal contact

Consider a quantum system, $S$, interacting with an environment, $E$, composed of infinitely many identical uncoupled quantum systems, $A_i$, (called ancillas). Suppose that the system interacts unitarily with (is bombarded by) these ancillas one at a time, each for a time $\delta t$.

This scenario constitutes a collision model and is commonly used in quantum thermodynamics to model thermalization [10–14]. It seems natural to expect that the system will (or at least can) be driven to the temperature of its environment by some bombardment process. Informally, we often think of thermalization as a process where a microscopic constituent of the environment interacts with the system again without holding any memory of previous interaction, and the process is repeated until equilibration. This intuitive notion has been actually used in the literature to model thermalization processes [10, 25, 30, 31]. However, we will prove that this intuition is actually wrong: without fine tuning, rapid bombardment cannot mediate thermal contact.

Concretely, suppose that the system and environment are initially uncorrelated and thermal (with respect to their local Hamiltonians, $H_S$ and $H_E$) with inverse temperatures $\beta_S(0)$ and $\beta_E(0)$. Since the ancillas that make up the environment are uncoupled, we have that $\hat{H}_E = \sum_i \hat{H}_{A_i}$ where $\hat{H}_{A_i} = \hat{H}_A$ are the local Hamiltonians of each ancilla. From this it follows that each ancilla is in the state $\rho_A(0) = \exp(-\beta_E(0)\hat{H}_A)/Z_A(0)$ until its interaction with the system.

Therefore each time the system interacts with an ancilla it is updated by the map,

$$\phi(\delta t) [\rho_S] = \text{Tr}_A \left( e^{-i \delta t \hat{H}_A/\hbar} \rho_S \otimes \rho_A(0) e^{i \delta t \hat{H}_A/\hbar} \right),$$

where $\hat{H} = \hat{H}_S \otimes 1_A + 1_S \otimes \hat{H}_A + \hat{H}_{SA}$. Thus at a time $t = n \delta t$ the state of the system is given by $n$ applications of $\phi(\delta t)$ to the initial state $\rho_S(n \delta t) = \phi(\delta t)^n [\rho_S(0)]$.

Using the rapid repeated interaction formalism developed in [33] and [34] we can construct an interpolation scheme between the discrete time points $t = n \delta t$. Specifically we can construct the unique interpolation scheme which: (1) exactly matches the discrete dynamics, (2) is time-local and time-independent, and (3) converges as $\delta t \rightarrow 0$. The interpolated dynamics is given by the master equation $\frac{d}{dt} \rho_S(t) = \mathcal{L}_\delta t [\rho_S(t)]$, where $\mathcal{L}_\delta t := \frac{1}{\delta t} \text{Log}(\phi(\delta t))$, called the effective Liouvillian, generates time translations for the system.

Since we are interested in the rapid bombardment regime it is useful for us to expand $\phi(\delta t)$ as a series in $\delta t$, as $\phi(\delta t) = 1 + \delta t \phi_1 + \delta t^2 \phi_2 + \delta t^3 \phi_3 + \ldots$ where,

$^6$ We note that this rapid bombardment regime is relevant for investigations of strong local passivity [35].
\[ \phi_1[\rho_S] = -\frac{i}{\hbar} \text{Tr}_E\left( [\hat{H}, \rho_S \otimes \rho_A(0)] \right), \]
\[ \phi_2[\rho_S] = \frac{1}{2!} \left( -\frac{i}{\hbar} \right)^2 \text{Tr}_E\left( [\hat{H}, [\hat{H}, \rho_S \otimes \rho_A(0)]] \right), \]
\[ \phi_3[\rho_S] = \frac{1}{3!} \left( -\frac{i}{\hbar} \right)^3 \text{Tr}_E\left( [\hat{H}, [\hat{H}, [\hat{H}, \rho_S \otimes \rho_A(0)]]] \right), \] (14)

e etc. From this expansion we can expand \( \mathcal{L}_{\delta t} \) as a series as
\( \mathcal{L}_{\delta t} = \mathcal{L}_0 + \delta t \mathcal{L}_1 + \delta t^2 \mathcal{L}_2 + \delta t^3 \mathcal{L}_3 + \ldots \) where,
\[ \mathcal{L}_0 = \phi_1, \]
\[ \mathcal{L}_1 = \phi_2 - \frac{1}{2} \phi_1^2, \]
\[ \mathcal{L}_2 = \phi_3 - \frac{1}{2} (\phi_1 \phi_2 + \phi_2 \phi_1) + \frac{1}{3} \phi_1^3, \] (15)

e etc. In [36] and [33] the first terms, \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \), were computed and analyzed in detail. Specifically, it was shown in [33] that the common technique of taking the continuum limit is equivalent to only considering \( \mathcal{L}_0 + \delta t \mathcal{L}_1 \) in the above expansion.

More generally, in the rapid bombardment regime (when \( \delta t E/\hbar \ll 1 \) where \( E \) is the energy scale of \( \hat{H} \)) it often suffices to study the lowest order terms in this series. For example, if \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) determine a unique fixed point for the dynamics (i.e. \( \mathcal{L}_0 + \delta t \mathcal{L}_1 \) is full rank) then for small enough \( \delta t \) all higher order approximations will also have a unique attractive fixed point. Moreover these higher order fixed points are perturbatively near to the lower order ones for small enough \( \delta t \). If such a fixed point is established without knowledge of the systems’ local Hamiltonians it cannot be the at the temperature of the environment and thus the dynamics cannot be thermal contact.

This raises the question: at what orders in \( \delta t \) can \( \mathcal{L}_{\delta t} \) constitute thermal contact? As we will see \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) do not depend on the local Hamiltonian of the ancillas (and so cannot constitute thermal contact) whereas \( \mathcal{L}_2 \) generically depends on both \( \hat{H}_S \) and \( \hat{H}_A \).

Using the linearity of the partial trace and the commutator, we can see that the \( n \)th term in (14) involves all the ways of picking one of \( \hat{H}_S, \hat{H}_A, \) or \( \hat{H}_{SA} \) for each of the \( n \) copies of \( \hat{H} \) appearing in the expressions given by (14). We now systematically analyze each of these possible combinations which contain \( \hat{H}_A \).

Using the cyclic property of partial trace, one finds that all terms with \( \hat{H}_A \) in the outermost commutator vanish. Likewise, using the fact that \( \rho_A(0) \) is thermal and therefore commutes with \( \hat{H}_A \) one finds that all terms with \( \hat{H}_A \) in the innermost commutator vanish. Moreover, using the nested commutator identity, \( [A, [B, C]] = [B, [A, C]] \) if \( [A, B] = 0 \), one can see that all non-vanishing occurrences of \( \hat{H}_A \) must be ‘sandwiched’ on either side by an interaction Hamiltonian \( \hat{H}_{SA} \) with which it does not commute. Otherwise we could move the \( \hat{H}_A \) to either end and the term vanishes by the above arguments.

Thus all terms depending on \( \hat{H}_A \) in \( \phi_1 \) and \( \phi_2 \) vanish. Since \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) are constructed from \( \phi_1 \) and \( \phi_2 \) they do not depend on \( \hat{H}_A \) either. Thus if a unique fixed point is established by \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \), the dynamics cannot be thermal contact without fine-tuning.

In \( \phi_3 \) (and therefore in \( \mathcal{L}_2 \)) we find the first term which depends on \( \hat{H}_A \) and does not vanish, namely \( \text{Tr}_A \left( [\hat{H}_{SA}, [\hat{H}_A, [\hat{H}_{SA}, \rho_S \otimes \rho_A(0)]]] \right) \). We note that \( \mathcal{L}_2 \) also depends on \( \hat{H}_S \) non-trivially through terms like \( \text{Tr}_A \left( [\hat{H}_{SA}, [\hat{H}_S, [\hat{H}_{SA}, \rho_S \otimes \rho_A(0)]]] \right) \).
We can find an explanation for why $\hat{H}_A$ does not show up until $L_2$ by interpreting $[\hat{H}_S, \cdot]$ as evolution with respect to $\hat{H}_A$. Doing this we can see that the simplest/shortest process carrying information about the ancilla’s local Hamiltonian (and therefore its temperature) is to:

1. Interact with it (so it is not thermal anymore)
2. Let it evolve freely (bringing in its energy scale)
3. Interact with it again (to get the information out).

In the rapid bombardment regime this process ‘takes too long’ and is therefore highly suppressed.

To make this more concrete, we can consider the following scenario. Suppose that we have many pairs of thermal systems, $A$ and $B$, and we suspect that Loki carries out his swap-out trick on our $B$ systems as

$$\Lambda_B : \beta_B(0) \to \lambda \beta_B(0); \quad \hat{H}_B \to \hat{H}_B / \lambda,$$

for some $\lambda$. If we couple each of our $N$ pairs of systems together, each for a time $\delta t$, and then measure the $A$ systems, how accurate of an (unbiased) estimate can we make about the Cramer–Rao theorem to be [37]

$$\text{Var}(\lambda) \geq \frac{1}{NF(\lambda, \delta t)}$$

where $F(\lambda, \delta t)$ is the Fisher information about $\lambda$ in each of the $A$ systems after a time $\delta t$. We will now investigate the scaling of the Fisher information about $\lambda$ for small $\delta t$.

As above let us assume that the systems interact unitarily such that after a time $\delta t$, the state of each system $A$ is

$$\phi(\delta t)[\rho_A(0)] = \text{Tr}_B \left( e^{-i \delta t \hat{H}_A / \hbar} \rho_A(0) \otimes \rho_B(0) \ e^{i \delta t \hat{H}_B / \hbar} \right),$$

where $\hat{H} = \hat{H}_A \otimes I_B + I_A \otimes \hat{H}_B + \hat{H}_{AB}$. The Fisher information in this state about $\lambda$ is given by

$$F(\lambda, \delta t) = \text{Tr}(L^2 \rho_A(\lambda, \delta t))$$

where $L$ is the symmetric logarithmic derivative of $\rho_A(\lambda, \delta t)$ with respect to $\lambda$ defined by,

$$\frac{d}{d\lambda} \rho_A(\lambda, \delta t) = \frac{\rho_A(\lambda, \delta t) L + L \rho_A(\lambda, \delta t)}{2}.$$  

As noted above, $\phi(\delta t)$ does not depend on $\lambda$ until $\phi_1$ such that the left hand side of equation (18) is $O(\delta t^3)$. Since $\rho_A(\lambda, \delta t) = \rho_A(0) + O(\delta t)$ we must therefore have $L = O(\delta t^3)$. This in turn implies that $F(\lambda, \delta t) = O(\delta t^5)$. From (17) we can thus see that doing more interactions of shorter duration (e.g. $N \to 2N$ and $\delta t \to \delta t / 2$) results in significantly less information about $\lambda$.

5. Illustrative examples

First let us briefly review the widely used partial swap model of thermalization [15], in which a two level system, $S$, with $\hat{H}_S = E_S \hat{\sigma}_S$ interacts with a series of thermal ancillas, $A$, with $\hat{H}_A = E_A \hat{\sigma}_A$ and inverse temperature $\beta_A$. Note these systems are often taken to have the same energy gap, $E = E_S = E_A$. Each interaction has the systems evolve for a
duration $\delta t$ under the Hamiltonian $\dot{H}_w = hJ(\frac{1}{2}\sigma_A + \sigma_A \cdot \hat{\sigma}_A)/2$. This implements a partial swap unitary, $U(\delta t) = \cos(J\delta t) 1 - i\sin(J\delta t) U_{sw}$ where $U_{sw}$ swaps the states of $S$ and $A$ as $U_{sw}(|S\rangle \otimes |A\rangle) = |A\rangle \otimes |S\rangle$. If $J\delta t \neq n\pi$, then repeatedly interacting with these ancillas drives the system to the state $\rho_S(\infty) = \rho_A(0)$, such that $\beta_S(\infty) = \beta_A$. Thus we find thermalization even in the rapid bombardment regime, when $J\delta t \ll 1$.

But the above setup is fine-tuned with $E = E_S = E_A$. If we allow detuning so that $E_S \neq E_A$, we still find that $\rho_S(\infty) = \rho_A(0)$, but this now means $E_S \beta_S(\infty) = E_A \beta_A$. This situation produces thermal contact only when $E_S = E_A$. Any detuning between the systems will cause them not to thermalize.

Next consider an harmonic oscillator, $S$, rapidly bombarded by an environment of other harmonic oscillators, $A$, with local Hamiltonians, $H_S = h\nu_S(\hat{n}_S + 1/2)$ and $H_A = h\nu_A(\hat{n}_A + 1/2)$ and a generic quadratic interaction Hamiltonian,

$$H_{SA} = (\hat{x}_S \hat{p}_S) \begin{pmatrix} g_{xx} & g_{xp} \\ g_{px} & g_{pp} \end{pmatrix} (\hat{x}_A \hat{p}_A) = X_S^T G X_A. \quad (21)$$

Assuming that the system and ancillas are each initially in thermal states it was shown in [38] that $L_0$ and $L_1$ produce a unique fixed point for the system dynamics. As discussed in the previous section, this dynamics therefore cannot produce thermal contact without fine-tuning.

Specifically, the system is driven to the thermal state with

$$\nu_S(\infty) = \frac{\text{Tr}(G^T G)}{2 \text{det}(G)} \nu_A; \quad \nu_X = \frac{\exp(h\omega_X/\beta_X) + 1}{\exp(h\omega_X/\beta_X) - 1}. \quad (22)$$

where $\nu_X$ is a temperature monotone. In order for this interaction to constitute thermal contact we must have $\beta_S(\infty) = \beta_A$. Even when $S$ and $A$ are fine-tuned such that $\omega_S = \omega_A$ and we therefore only need $\nu_S(\infty, \omega_A)$, we only have thermal contact for a very specific family of interactions, those with $\text{Tr}(G^T G) = 2 \text{det}(G)$. The equilibration properties of such couplings are studied in [38]. On the other hand, if $S$ and $A$ are detuned (with $\omega_S \neq \omega_A$) then the interaction Hamiltonian must depend each systems’ energy scales and the ancillas temperature as $G = G(\omega_S, \omega_A, \beta_A)$ in a very specific fine-tuned way to yield thermal contact.

Finally, as a fun sanity check, consider a molecule placed in the air at room temperature ($T = 300$ K) interacting with nitrogen molecules ($m = 28$ amu) via a Van der Waals interaction (with energy scale $E = 10^{-20}$ J) as it crosses their Van der Waals radius ($r = 2.25$ A). We can estimate the duration of each interaction as,

$$\delta t = \frac{2r}{v_{\text{rms}}} = \frac{2r}{\sqrt{3kT/m}} = 0.87 \text{ ps}, \quad (23)$$

such that $\delta t E/h = 83$. Thus in this example we do not have $\delta t E/h \ll 1$. This example is not in the rapid bombardment regime. Thus the interactions of molecules in the air are (thankfully) long enough to sense the ambient temperature. The rapid bombardment regime discussed here occurs at Zeno-like time scales wherein every individual interaction the system only varies perturbatively.

6. Conclusion

In order to determine a system’s temperature, a thermometer (or any temperature measuring protocol) must be sensitive to system’s local Hamiltonian, not just its state. Since thermal contact is a way of measuring temperatures, this produces a restriction on the types of dynamics
which can constitute thermal contact. Using this restriction, we have shown that the intuitive idea of thermalization emerging out of a rapid bombardment of the microconstituents of a thermal reservoir with a system cannot be correct without fine-tuning.

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