NEVANLINNA THEORY FOR HOLOMORPHIC CURVES FROM ANNULI INTO SEMI ABELIAN VARIETIES

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Abstract. In this paper, we prove a lemma on logarithmic derivative for holomorphic curves from annuli into Kähler compact manifolds. As its application, a second main theorem for holomorphic curves from annuli into semi abelian varieties intersecting with only one divisor is given.

1. Introduction

Let $f$ be an algebraically non-degenerate holomorphic curve from $\mathbb{C}$ into a semi-Abelian variety $M$ and let $D$ be an algebraic divisor on $M$. In 2002, J. Noguchi, J. Winkelmann and K. Yamanoi [10] proved that there exist a good compactification $\overline{M}$ of $M$ and an integer $k_0 = k_0(f, D)$ satisfying

$$T_f(r; c_1(D)) \leq N^{[k_0]}(r, f^*D) + O(\log T_f(r; c_1(D))) + O(\log r)$$

for all $r \in [0; +\infty)$ outside a finite Borel measure set. Here by $T_f(r; c_1(D))$ and $N^{[k_0]}(r, f^*D)$ we denote the characteristic function of $f$ with respect to the line bundle $L(D)$ in $\overline{M}$ and the counting function of divisor $f^*D$ truncated to level $k_0$ (see Section §2 for the definitions).

Adapting the method of the above three authors and using the lemma on logarithmic derivative given by Noguchi [8], recently Quang [11] has generalized the above result to the case of holomorphic curves from punctured disc $\Delta^* = \{ z \in \mathbb{C} : |z| \geq 1 \}$ into a semi-Abelian variety $M$. Also in [11], as an application of his second main theorem, Quang gave an alternative proof of Big Picard’s theorem for algebraically non degenerate mappings $f : \Delta^* \to M \setminus D$.

In this paper, we will extend these above results to the case of holomorphic curves from annuli into semi-Abelian varieties. In order to establish the second main theorem, we firstly prove a lemma on logarithmic derivative for holomorphic curves from annuli into Kähler manifold. To state our results, we recall the following.

For $R_0 > 1$, we set the annulus

$$A(R_0) = \left\{ z \in \mathbb{C} : \frac{1}{R_0} < |z| < R_0 \right\}.$$ 

Let $N$ be a compact Kähler manifold. Let $\mathcal{M}_N^*$ be the sheaf of germs of meromorphic functions on $N$ which do not identically vanish, and define a sheaf $\mathcal{U}_N^1$ by

$$0 \to \mathbb{C}^* \to \mathcal{M}_N^* \xrightarrow{\frac{d\log}{\gamma}} \mathcal{U}_N^1 \to 0,$$

where $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers.

Our lemma on logarithmic derivative is stated as follows.

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**Lemma 1.1.** Let $N$ be a compact Kähler manifold with Kähler metric $h$ and the associated form $\Omega$. Let $f: \mathbb{A}(R_0) \to N$ be a holomorphic curve from annulus $\mathbb{A}(R_0)$ ($R_0 > 1$) into $N$ and let $\omega \in H^0(N,\Omega^1)$. Setting $f^*\omega = \xi(z)dz$, we have

$$\| m_0(r,\xi) \leq O \left( \log^+ T_0,f(r,\Omega) \right) + O \left( \log^+ \frac{1}{R_0 - r} \right).$$

Here, $m_0(r,\xi)$ denotes the proximitive function of $\xi$ (see Section §2 for the definition) and the notation $\| P$ means the assertion $P$ holds for all $r \in [1; R_0)$ outside a Borel subset $E$ with $\int_E \frac{dt}{(R_0 - r)^{1+\lambda}} \leq +\infty$ for some positive number $\lambda$. We learn the technique of the proof of Lemma 1.1 from [8].

Our second main theorem in this paper is stated as follows.

**Theorem 1.2.** Let $f: \mathbb{A}(R_0) \to M$ be an algebraically non-degenerate holomorphic curve into a semi-Abelian variety $M$ and let $D$ be a reduced divisor on $M$. Then there exist a smooth equivariant compactification $\mathcal{M}$ of $M$ independent of $f$ and a natural number $k_0$ such that

$$\| T_0,f(r; c_1(D)) = N_0^{[k_0]}(r, f^*D) + O \left( \log^+ T_0,f(r; c_1(D)) + \log^+ \frac{1}{R_0 - r} \right).$$

The basic notation of this paper is from [2, 3, 4, 8, 10] and [11].

2. **Basic Notion from Nevanlinna theory and Semi Abelian varieties**

(a) Meromorphic functions on annuli.

Let $R_0 > 1$ and let $\mathbb{A}(R_0)$ be an annulus. For a divisor $\nu$ on $\mathbb{A}(R_0)$, which we may regard as a function on $\mathbb{A}(R_0)$ with values in $\mathbb{Z}$ whose support is a discrete subset of $\mathbb{A}(R_0)$, and for a positive integer $k$ (maybe $k = +\infty$), the counting function of $\nu$ is defined by

$$n_0^{[k]}(t) = \begin{cases} \sum_{1 \leq |i| \leq t} \min \{k, \nu(z)\} & \text{if } 1 \leq t < R_0 \\ \sum_{t \leq |i| < 1} \min \{k, \nu(z)\} & \text{if } \frac{1}{R_0} < t < 1 \end{cases}$$

and $N_0^{[k]}(r,\nu) = \int_1^r n_0^{[k]}(t) dt + \int_1^r n_0^{[k]}(t) dt$ $(1 < r < R_0)$.

Let $f$ be a meromorphic function on $\mathbb{A}(R_0)$. We define the proximity function by

$$m_0(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f \left( \frac{1}{r} e^{i\theta} \right) \right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta - \frac{1}{\pi} \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta.$$

The characteristic function of $f$ is defined by

$$T_0(r, f) = m_0(r, f) + N_0(r, \nu_f^\infty).$$

We note that these definition also available for multiplicative meromorphic functions.

The function $f$ is said to be admissible if it satisfies

$$\limsup_{r \to R_0} - \frac{T_0(r, f)}{-\log(R_0 - r)} = +\infty.$$
Throughout this paper, a Borel subset $E$ of $[1; R_0]$ is said to be an $\Delta_{R_0}$-set if it satisfies

$$\int_E \frac{dr}{(R_0 - r)^{\lambda+1}} < +\infty$$

for some $\lambda \geq 0$.

(b) Holomorphic curves from annuli into Kähler compact manifolds.

Let $\xi$ be a function on $\mathbb{A}(R_0)$ satisfying that

(i) $\xi$ is differentiable outside a discrete set of points,

(ii) $\xi$ is locally written as a difference of two subharmonic functions.

Then by [8 §1], we easily have

$$\int_1^t \frac{dt}{t} \int_{\mathbb{A}(t)} dd^c\xi = \frac{1}{4\pi} \int_{|z|=r} \xi(z) d\theta + \frac{1}{4\pi} \int_{|z|=1} \xi(z) d\theta - \frac{1}{2\pi} \int_{|z|=1} \xi(re^{i\theta}) d\theta - 2(\log r) \int_{|z|=1} d\xi,$$

where $dd^c\xi$ is taken in the sense of current.

Let $N$ be a Kähler compact manifold with Kähler metric $h$ and the associated form $\Omega$. Let $f : \mathbb{A}(R_0) \to N$ be a holomorphic curve. The characteristic function of $f$ with respect to $\Omega$ is defined by

$$T_{0,f}(r; \Omega) = \int_1^r \frac{dt}{t} \int_{\mathbb{A}(t)} f^*\Omega, \quad 1 < r < R_0.$$

Let $D$ be an effective divisor on $N$. We assume that $f(\mathbb{A}(R_0)) \not\subset D$. We denote $L(D)$ the line bundle determined by $D$. We fix a Hermitian fiber metric $\| \cdot \|$ in $L(D)$ with the curvature form $\omega$ representing the first Chern class $c_1(D)$ of $L(D)$. Take $\sigma \in H^0(N, L(D))$ with $\text{div}(\sigma) = D$ and $\|\sigma\| \leq 1$ (by the compactness of $N$). We set

$$T_{0,f}(r; c_1(D)) := T_{0,f}(r; \omega) = \int_1^r \frac{dt}{t} \int_{\mathbb{A}(t)} f^*\omega, \quad 1 < r < R_0,$$

which is well-defined up to an $O(1)$-term. The proximity function of $f$ with respect to $D$ is defined by

$$m_{0,f}(r; D) = \frac{1}{2\pi} \int_{|z|=r} \log \frac{1}{\|\sigma(f(z))\|} d\theta + \frac{1}{2\pi} \int_{|z|=1} \log \frac{1}{\|\sigma(f(z))\|} d\theta - \frac{1}{\pi} \int_{|z|=1} \log \frac{1}{\|\sigma(f(z))\|} d\theta.$$

Applying (2.1) to $\xi = f^* \log \|\sigma\|$, we obtain the First Main Theorem:

$$T_{0,f}(r; c_1(D)) = N_0(r, f^*D) + m_{0,f}(r; D) + O(1).$$

(c) Semi Abelian varieties and Logarithmic Jet bundle

Let $M_0$ be an Abelian variety and let $M$ be a complex Lie group admitting the exact sequence

$$0 \to (\mathbb{C}^*)^p \to M \xrightarrow{\pi} M_0 \to 0,$$

where $\mathbb{C}^*$ is the multiplicative group of non zero complex numbers. Such an $M$ is called a semi Abelian variety.

Taking the universal covering of (2.6), one gets

$$0 \to \mathbb{C}^p \to \mathbb{C}^n \to \mathbb{C}^m \to 0$$

where $\mathbb{C}^*$ is the multiplicative group of non zero complex numbers.
and an additive discrete subgroup $\Lambda$ of $\mathbb{C}^n$ such that
\[
\pi : \mathbb{C}^n \to M = \mathbb{C}^n / \Lambda, \\
\pi_0 : \mathbb{C}^n = (\mathbb{C}^n / \mathbb{C}^p) \to M_0 = (\mathbb{C}^n / \mathbb{C}^p) / (\Lambda / \mathbb{C}^p), \\
(\mathbb{C}^n)^p = \mathbb{C}^p / (\Lambda \cap \mathbb{C}^p).
\]

Take a smooth equivariant compactification $\overline{M}$ of $M$. Then the boundary divisor $\partial M$ has only simple normal crossings. Denote by $\Omega_{\partial M}$ the projections. For a $k \geq 1$ simple normal crossings. Denote by $\omega$ the following trivialization of the logarithmic tangent bundle:
\[
\omega \in \mathbb{C}
\]

Moreover, we have the logarithmic $k$-jet bundle $J_k(\mathbb{M}, \log \partial M)$ over $\overline{M}$ and a natural morphism
\[
\psi : J_k(\mathbb{M}, \log \partial M) \to J_k(\mathbb{M}).
\]

The trivialization (2.7) gives
\[
J_k(\mathbb{M}, \log \partial M) \cong \mathbb{M} \times \mathbb{C}^{nk}.
\]

Let
\[
\pi_1 : J_k(\mathbb{M}, \log \partial M) \cong \mathbb{M} \times \mathbb{C}^{nk} \to \mathbb{M}, \\
\pi_2 : J_k(\mathbb{M}, \log \partial M) \cong \mathbb{M} \times \mathbb{C}^{nk} \to \mathbb{C}^{nk}
\]
be the projections. For a $k$-jet $y \in J_k(\mathbb{M}, \log \partial M)$ we call $\pi_2(y)$ the jet part of $y$.

Let $x \in \partial K$ and let $\sigma = 0$ be a local defining equation of $\partial K$ around $x$. For a germ $g : (\mathbb{C}, 0) \to (M, 0)$ of holomorphic mappings, we denote its $k$-jet by $j_k(g)$, and write
\[
d^j\sigma(g) = \left. \frac{d^j}{d\xi^j} \right|_{\xi=0} \sigma(g(\xi)).
\]

We set
\[
J_k(\partial)_{x} = \{ j_k(g) \in J_k(\mathbb{M})_{x} \mid d^j(g) = 0, \ 1 \leq j \leq k\}, \\
J_k(\partial) = \bigcup_{x \in \partial} J_k(\partial)_{x}, \\
J_k(\partial, \log \partial M) = \psi^{-1} J_k(\mathbb{M}).
\]

$J_k(\mathbb{M}, \log \partial M)$, which is depending in general on the embedding $\partial D \hookrightarrow \mathbb{M}$ (cf. [9]). Note that $\pi_2(J_k(\partial, \log \partial M))$ is an algebraic subset of $\mathbb{C}^{nk}$, since $\pi_2$ is proper.

(d) Divisor of semi Abelian variety in general position

Let $M$ be the semi-Abelian variety as above and let $X$ be a complex algebraic variety, on which $M$ acts:
\[
(a, x) \in M \times X \to a \cdot x \in X.
\]

Let $Y$ be a subvariety embedded into a Zariski open subset of $X$.

Definition 2.8 (see [10] Definition 3.2). We say that $Y$ is generally positioned in $X$ if the closure $\overline{Y}$ of $Y$ in $X$ contains no $M$-orbit. If the support of a divisor $E$ on a Zariski open subset of $X$ is generally positioned in $X$, then $E$ is said to be generally positioned in $X$.

Definition 2.9. Let $Z$ be a subset of $M$. We define the stabilizer of $Z$ by
\[
\text{St}(Z) = \{ x \in M \mid x + Z = Z \}^0,
\]
where $\{ \cdot \}^0$ denotes the identity component.
3. Proof of Lemma on logarithmic derivative

In this section, we will give the proof for Lemma 1.1. The following is a general property of the characteristic function (see [5, Lemma 6.1.5] for reference).

**Theorem 3.1** (see [5, Lemma 6.1.5]). Let \( f : \mathbb{A}(R_0) \to V \) be a holomorphic curve into a complex projective manifold \( V \) and let \( H \) be a big line bundle on \( V \). Then
\[
T_{0,f}(r; c_1(L)) = O(T_{0,f}(r; c_1(H))) + O \left( \log \frac{1}{R_0 - r} \right),
\]
for every line bundle \( L \) on \( V \).

**Lemma 3.2.** Let \( \varphi \) be a positive monotone increasing function in \( r \in [1; R_0) \) \((R_0 > 1)\) Then for every \( \lambda > 0 \), we have
\[
\left\| \frac{d}{dr} (\varphi) \right\| \leq \left( \frac{\varphi}{R_0 - r} \right)^{\lambda + 1}.
\]

**Proof.** Let \( E = \left\{ r \in [1; R_0) : \frac{d}{dr} (\varphi) > \left( \frac{\varphi}{R_0 - r} \right)^{\lambda + 1} \right\} \). Since \( \varphi \) is a monotone increasing function, its derivative \( \frac{d}{dr} (\varphi) \) exists almost everywhere. Hence \( E \) is a Borel measurable subset of \([1; R_0)\). Then we have
\[
\int_E \frac{dt}{(R_0 - r)^{\lambda + 1}} \leq \int_E \frac{\varphi'}{\varphi^{\lambda + 1}} dt \leq \frac{1}{\lambda} \left( \frac{1}{\varphi^{\lambda}(1)} - \lim_{r \to R_0^-} \frac{1}{\varphi^{\lambda}(r)} \right) = O(1).
\]
The lemma is proved. \( \square \)

**Lemma 3.3** (see [8, Lemma 2.12]). Let \( f \) be a nonzero multiplicative meromorphic function on \( \mathbb{A}(R_0) \). Then for each positive integer \( k \) and positive number \( \epsilon \), we have
\[
\left\| m_0 \left( r, \frac{f^k}{r^\delta} \right) \right\| \leq (4 + \epsilon) \log^+ T_{0,f}(r, f) + O \left( \log^+ \frac{1}{R_0 - r} \right).
\]

**Proof.** We denote by \( \omega \) the standard complex coordinate on \( \mathbb{C} = \mathbb{C} \cup \infty \) and consider the canonical Kähler form
\[
\Psi_0 = \frac{1}{(1 + |\omega|^2)^2} \frac{i}{2\pi} d\omega \wedge d\bar{\omega}.
\]
By Griffiths-King [11, Proposition 6.9], we may choose suitably positive constants \( a, b \) and \( \delta \) \((\delta < 1)\) such that the form
\[
\Psi = \frac{a(|\omega| + |\omega|^{-1})^{2 + 2\delta}}{(\log b(1 + |\omega|^2))^{2}(\log b(1 + |\omega|^{-2}))^2} \Psi_0
\]
satisfies
\[
\text{Ric} \Psi \geq (|\omega| + |\omega|^{-1})^{-2\delta} \Psi.
\]
Since \( f \) is multiplicative, it is easy to see that \( f^* \Psi \) is well-defined. We set
\[
g = \frac{f'}{f}, \quad \zeta = \frac{a(|f| + |f|^{-1})^{2\delta}}{(\log b(1 + |f|^2))^{2}(\log b(1 + |f|^{-2}))^2} |g|^2,
\]
\[
f^* \Psi = \frac{a(|f| + |f|^{-1})^{2\delta}}{(\log b(1 + |f|^2))^{2}(\log b(1 + |f|^{-2}))^2} |g|^2 \frac{i}{2\pi} dz \wedge d\bar{z} = \zeta \frac{i}{2\pi} dz \wedge d\bar{z}.
\]
On the other hand, we have
\[
g^* \text{Ric} \Psi = dd^c \log \zeta \geq (|f| + |f|^{-1})^{-2\delta} \frac{i}{2\pi} dz \wedge d\bar{z}.
\]
and
\[ dd^c \log \zeta = f^* \text{Ric} \Psi - \delta \left( [\text{div}^0(f)] + [\text{div}^\infty(f)] \right) + [\text{div}^0(g)] - [\text{div}^\infty(g)] \]
in the sense of currents. By the definition, we have
\[ [\text{div}^\infty(g)] = \left[ \text{Supp} \left( \text{div}^0(f) + \text{div}^\infty(f) \right) \right] \geq \left[ \text{div}^0(f) \right] + \left[ \text{div}^\infty(f) \right], \]
and hence we deduce
\[ (|f| + |f|^{-1})^{-2\delta} \frac{i}{2\pi} d\zeta \wedge d\bar{\zeta} \leq (1 + \delta) \left( [\text{div}^0(f)] + [\text{div}^\infty(f)] \right) + dd^c \log \zeta. \]

Then, by the formula (2.1), we have
\[
\int_{1}^{t} \frac{dt}{t} \int_{\mathcal{A}(t)} \frac{\zeta}{(|f| + |f|^{-1})^{2\delta}} \frac{i}{2\pi} d\zeta \wedge d\bar{\zeta} \leq (1 + \delta) \left( N_0(r, \text{div}^0(f)) + N_0(r, \text{div}^0(f)) \right) + \frac{1}{4\pi} \int_{|z|=r} \log \zeta d\theta + \frac{1}{4\pi} \int_{|z|=1/r} \log \zeta d\theta - 2\log \zeta \int_{|z|=1} d^c \log \zeta - \frac{1}{2\pi} \log \zeta d\theta.
\]

By the definition of \( \zeta \), we have
\[
\frac{1}{4\pi} \int_{|z|=r} \log \zeta d\theta + \frac{1}{4\pi} \int_{|z|=1/r} \log \zeta d\theta \leq m_0(r, g) + \delta \left( m_0(r, f) + m_0 \left( r, \frac{1}{f} \right) \right) + O(1).
\]

Therefore, we get
\[
\int_{1}^{t} \frac{dt}{t} \int_{\mathcal{A}(t)} \frac{\zeta}{(|f| + |f|^{-1})^{2\delta}} \frac{i}{2\pi} d\zeta \wedge d\bar{\zeta} \leq m_0(r, g) + 2(1 + \delta) T_0(r, f) + O(1).
\]
For simplicity, we set $\Gamma = \{|z| = r\} \cup \{|z| = 1/r\}$. We now have the following estimate
\[
\| m_0(r, g) = \frac{1}{4\pi} \int_{\Gamma(r)} \log^+ \left( (\zeta(|f| + |f|^{-1}))^{-2\delta} (\log(1 + |f|^2))^2 (\log(1 + |f|^{-2}))^2 \right) d\theta + O(1)
\]
\[
\leq \frac{1}{4\pi} \int_{\Gamma(r)} \log^+ \left( (\zeta(|f| + |f|^{-1}))^{-2\delta} d\theta + \log^+ \left( m_0(r, f) + m_0(r, 1/f) \right) + O(1)
\]
\[
\leq \frac{1}{2} \log \left( 1 + \frac{1}{2\pi} \int_{\Gamma(r)} \log^+ \left( (\zeta(|f| + |f|^{-1}))^{-2\delta} d\theta \right) + 2 \log^+ T_0(r, f) + O(1)
\]
\[
= \frac{1}{2} \log \left( 1 + \frac{1}{2\pi} \left( \log \left( (\left\int_{\Lambda(r)} \zeta \left( \frac{\zeta}{|f| + |f|^{-1}} \right) i/2 \pi d\xi \right)^{2} \right) \right) + 2 \log^+ T_0(r, f) + O(1)
\]
\[
\leq \frac{1}{2} \log \left( 1 + \frac{1}{2\pi} \left( \log \left( (\left\int_{\Lambda(r)} \zeta \left( \frac{\zeta}{|f| + |f|^{-1}} \right) i/2 \pi d\xi \right)^{2} \right) \right) + 2 \log^+ T_0(r, f) + O(\log^+ 1/R_0 - r)
\]
\[
\leq \frac{1}{2} \log \left( 1 + \frac{r}{2} \left( \log \left( (\left\int_{\Lambda(r)} \zeta \left( \frac{\zeta}{|f| + |f|^{-1}} \right) i/2 \pi d\xi \right)^{4} \right) \right) + 2 \log^+ T_0(r, f) + O(\log^+ 1/R_0 - r)
\]
\[
\leq \frac{1}{2} \log \left( 1 + \frac{r}{2} \left( m_0(r, g) + 2(1 + \delta) T_0(r, f) \right)^4 \right) + 2 \log^+ T_0(r, f) + O(\log^+ 1/R_0 - r)
\]
\[
\leq 2 \log^+ m_0(r, g) + 4 \log^+ T_0(r, f) + O(\log^+ 1/R_0 - r).
\]

We note that, for every nonnegative function $\psi(r)$ and $\epsilon > 0$, $\log^+ \psi(r) \leq \epsilon \psi(r) + O(1)$. Then we have
\[
\| m_0(r, g) \leq 2\epsilon m_0(r, g) + 4 \log^+ T_0(r, f) + O(\log r) + O(1),
\]
\[\text{i.e., } \| m_0(r, g) \leq \frac{4}{1 - 2\epsilon} T_0(r, f) + O(\log r) + O(1).\]

Choosing $\epsilon = \frac{1}{2}(1 - \frac{4}{4+\epsilon})$, we get
\[
\| m_0(r, g) \leq (4 + \epsilon) T_0(r, f) + O(\log r) + O(1).
\]

The lemma is proved. $\square$
Proof of Lemma 1.1. By Weil [12, p. 101] (see also [8]), there is a multiplicative meromorphic function \( \varphi \) on \( N \) and a holomorphic one form \( \omega_1 \) on \( N \) such that

\[
\omega = d \log \varphi + \omega_1.
\]

We set

\[
f^* \omega_1 = \xi_1 dz
\]

and

\[
f^* (d \log \varphi) = \xi_2 dz,
\]

where \( \xi_2 = \frac{(\varphi \circ f')}{\varphi)}, \) Then we have

\[
m_0(r, \xi) \leq m_0(r, \xi_1) + m_0(r, \xi_2) + O(1).
\]

(3.4)

Firstly, we are going to estimate \( m_0(r, \xi_1) \). From the compactness of \( N \), there is a positive constant \( C \) such that

\[
|\omega_1(v)|^2 \leq C.h(v, v) \quad \text{for all holomorphic tangent vectors } v \in T_N.
\]

Setting \( f^* \Omega = s(z) \frac{dz \wedge d\bar{z}}{z} \), we have

\[
|\xi_1(z)|^2 \leq C.s(z).
\]

Hence, by some simple computations we have

\[
m_0(r, \xi_1) \leq \frac{1}{2\pi} \int_{|z|=r} \log(1 + |\xi_1|^2) d\theta + \frac{1}{4\pi} \int_{|z|=r} \log(1 + s(z)) d\theta + O(1)
\]

\[
\leq \frac{1}{4\pi} \int_{|z|=r} \log(1 + s(z)) d\theta + \frac{1}{4\pi} \int_{|z|=r} \log(1 + s(x)) d\theta + O(1)
\]

\[
\leq \frac{1}{2} \log \left( 1 + \frac{1}{2\pi} \int_{|z|=r} s d\theta \right) + \frac{1}{2} \log \left( 1 + \frac{1}{2\pi} \int_{|z|=r} s d\theta \right) + O(1)
\]

\[
= \frac{1}{2} \log \left( 1 + \frac{1}{2\pi} \frac{d}{dr} \int_{1 \leq |z| < r} ts dt \wedge d\theta \right)
\]

\[
+ \frac{1}{2} \log \left( 1 + \frac{r^2}{2\pi} \frac{d}{dr} \int_{1 \geq |z| > \frac{1}{2}} ts dt \wedge d\theta \right) + O(1)
\]

\[
= \frac{1}{2} \log \left( 1 + \frac{1}{2\pi} \frac{d}{dr} \int_{1 \leq |z| < r} f^* \Omega \right) + \frac{1}{2} \log \left( 1 + \frac{r^3}{2\pi} \frac{d}{dr} \int_{1 \geq |z| > \frac{1}{2}} f^* \Omega \right) + O(1)
\]

\[
\leq \frac{1}{2} \log \left( 1 + \frac{1}{2\pi} \frac{1}{R_0 - r} \int_{1 \leq |z| < r} f^* \Omega^2 \right)
\]

\[
+ \frac{1}{2} \log \left( 1 + \frac{r^3}{2\pi} \frac{1}{R_0 - r} \int_{1 \geq |z| > \frac{1}{2}} f^* \Omega^2 \right) + O(1),
\]
for all $r \in [0; R_0)$ outside an $\Delta_{R_0}$-set. Here the last inequality comes from the fact that $\int_{1 \leq |z| < r} f^* \Omega$ and $\int_{1 \leq |z| < 1} f^* \Omega$ are both monotone increasing functions in $r \in (1; R_0)$ and Lemma 3.2 is applied. Moreover, we have

$$\int_{1 \leq |z| < r} f^* \Omega = r \frac{d}{dr} \int_{1 \leq |z| < t} f^* \Omega \quad \text{and} \quad \int_{1 \geq |z| > t} f^* \Omega = r \frac{d}{dr} \int_{1 \geq |z| > t} f^* \Omega.$$ 

Then we have

$$m_0(r, \xi_1) \leq \log^+ \left( \frac{d}{dr} \int_{1 \leq |z| < t} f^* \Omega \right) + \log^+ \left( \frac{d}{dr} \int_{1 \geq |z| > t} f^* \Omega \right) + 2 \log^+ \frac{1}{R_0 - r} + O(1)$$

$$\leq 2 \log^+ \left( \frac{d}{dr} \int_{1 \leq |z| < t} f^* \Omega \right) + 2 \log^+ \frac{1}{R_0 - r} + O(1)$$

$$\leq 2 \log^+ \left( \frac{1}{R_0 - r} \right) + 2 \log^+ \frac{1}{R_0 - r} + O(1)$$

$$= 4 \log^+ \frac{1}{R_0 - r} + O(1).$$

Here, the fourth inequality holds because of Lemma 3.2.

Now, we will estimate $m_0(r, \xi_2)$. By Lemma 3.3 we have

$$\left\| m_0(r, \xi_2) = m_0 \left( \frac{(\varphi \circ f)}{\varphi \circ f} \right) \right\| \leq O \left( \log^+ T_0(r, \varphi \circ f) + \log^+ \frac{1}{R_0 - r} \right).$$

Take $\| \cdot \|_1$ and $\| \cdot \|_2$ the Hermitian fiber metrics on $L(\text{div}^0(\varphi))$ and $L(\text{div}^\infty(\varphi))$ respectively. We take $\sigma_1 \in H^0(N, L(\text{div}^0(\varphi)))$ and $\sigma_2 \in H^0(N, L(\text{div}^\infty(\varphi)))$ so that $\text{div}(\sigma_1) = \text{div}^0(\varphi)$, $\text{div}(\sigma_2) = \text{div}^\infty(\varphi)$ and $|\varphi| = \frac{\|\sigma_1\|_1}{\|\sigma_2\|_2} \leq \frac{1}{\|\sigma_2\|_2}$ (because of compactness of $N$, we may suppose that $\|\sigma_1\|_1 \leq 1$ and $\|\sigma_2\|_2 \leq 1$). Then we have

$$m_0(r, \varphi \circ f) = \frac{1}{2\pi} \int_{|z|=r} \log(1 + |\varphi \circ f|^2) d\theta + \frac{1}{2\pi} \int_{|z|=\frac{1}{2}} \log(1 + |\varphi \circ f|^2) d\theta + O(1)$$

$$\leq \frac{1}{2\pi} \int_{|z|=r} \log \frac{1}{\|\sigma_2 \circ f\|_2} d\theta + \frac{1}{2\pi} \int_{|z|=\frac{1}{2}} \log \frac{1}{\|\sigma_2 \circ f\|_2} d\theta + O(1)$$

$$= m_{0,f}(r, \text{div}^\infty(\varphi)).$$

On the other hand, we have

$$N_{0}(r, \text{div}^\infty(\varphi \circ f)) = N_{0}(r, \text{div}(\sigma_2 \circ f))$$

$$= T_{0,f}(r, c_1(\text{div}^\infty(\varphi))) - m_{0,f}(r, \text{div}^\infty(\varphi)) + O(1).$$

(3.8)
From (3.7), (3.8) and by Lemma 3.1, we have
\[ T_0(r, \varphi \circ f) \leq T_0(f(r, c_1(\text{div}^\infty(\varphi)))) + O(1) = O(T_0(f(r, \Omega))). \]
Combining the above inequality with (3.6), we get
\[ m_0(r, \xi_2) \leq O \left( \log^+ T_0(f(r, \Omega)) + \log^+ \frac{1}{R_0 - r} \right). \]  
(3.9)

From (3.4), (3.5) and (3.9) we get
\[ m_0(r, \xi) \leq O \left( \log^+ T_0(f(r, \Omega)) + \log^+ \frac{1}{R_0 - r} \right). \]
(4.2)

The lemma is proved. \(\Box\)

4. Proof of Second main theorem for holomorphic curves

We have the following lemma from [11] (which is a special case of [10, Lemma 3.14]).

**Lemma 4.1** (see [11, Lemma 4.4]). Let \( M \) be a semi-Abelian variety and let \( D \) be an algebraic divisor on \( M \) such that \( \text{St}(D) = \{0\} \). Then there exists a smooth equivariant compactification \( \overline{M} \) of \( M \) such that the closure \( \overline{D} \) of \( D \) in \( \overline{M} \) is big, generally positioned.

**(c) Proof of Theorem 1.2**

Let \( \overline{M} \) be a smooth equivariant compactification of \( M \) which is chosen as in Lemma 4.1. We may regard \( f \) as a holomorphic curve into \( \overline{M} \).

As in the Section §2, denote by \( M^*_M \) the sheaf of germs of meromorphic functions on \( M \) which do not identically vanish, and the sheaf \( U^1_M \) is defined by
\[ 0 \rightarrow \mathbb{C}^* \rightarrow M^*_M \rightarrow \mathbb{C}^* \rightarrow U^1_M \rightarrow 0. \]

Since \( M \) is a semi-Abelian variety, by taking the standard coordinates from the universal cover \( \mathbb{C}^n \rightarrow M \) of \( M \), which gives automatically sections \( \omega_i \) in \( H^0(M, U^1_M) \), we may assume that \( \omega_i \in H^0(M, U^1_M) \) for all \( 1 \leq i \leq n \).

We define functions \( \xi^i \) by setting \( f^*\omega^i = \xi^idz \). Then by Lemma 1.1 we have
\[ m(r, \xi^i) \leq O (\log^+ T_0(f(r, \Omega)) + O \left( \log^+ \frac{1}{R_0 - r} \right), \quad \forall 1 \leq i \leq n. \]

If \( D \) is a divisor on \( M \) such that \( \overline{D} \) is generally positioned in \( \overline{M} \), by Theorem 3.1 we obtain
\[ m(r, \xi^i) \leq O (\log^+ T_0(f(r, c_1(\overline{D}))) + O \left( \log^+ \frac{1}{R_0 - r} \right), \quad \forall 1 \leq i \leq n. \]  
(4.2)

**Proof of theorem 1.2** Without loss of generality we assume that \( D \) is irreducible. If \( \text{St}(D) \neq \{0\} \), by taking the quotient \( q : M \rightarrow M/\text{St}(D) \) and deal with the holomorphic curve \( q \circ f : \mathbb{A}(R_0) \rightarrow M/\text{St}(D) \) and the divisor \( D/\text{St}(D) \), then we may reduce to the case where \( D \) is irreducible and \( \text{St}(D) = \{0\} \). Thus we may assume that \( D \) is irreducible and \( \text{St}(D) = \{0\} \).

By Lemma 4.1 there exists a smooth equivariant compactification \( \overline{M} \) of \( M \), in which \( \overline{D} \) is generally positioned and big.

Let \( J_k(f) : \mathbb{A}(R_0) \rightarrow J_k(\overline{M}, \log \partial M) \cong \overline{M} \times \mathbb{C}^{nk} \) be the \( k \)-jet lifting of \( f \). We have the following claim from [11] Claim 6.1}
Claim 4.3. There exists a number \( k_0 \) such that
\[
\pi_2(J_{k_0}(D, \log \partial M)) \cap \pi_2(J_{k_0}(\mathcal{A}(R_0))^{Zar}) \neq \pi_2(J_{k_0}(\mathcal{A}(R_0))^{Zar}),
\]
where \( J_{k_0}(\mathcal{A}(R_0))^{Zar} \) is the Zariski closure of \( J_{k_0}(\mathcal{A}(R_0)) \) in \( J_k(\overline{M}, \log \partial M) \).

Since the proof of this claim is just follow the proof of [11, Claim 6.1] with the same lines, we will omit its proof here.

Let \( \{ U_\alpha \} \) be an affine open covering of \( \overline{M} \) such that
\[
L(D)|_{U_\alpha} \cong U_\alpha \times \mathbb{C}.
\]

(4.4)

We take \( \sigma \in H^0(\overline{M}, L(D)) \) so that \( \text{div}(\sigma) = D \) and take local holomorphic functions \( \sigma_\alpha = \sigma|_{U_\alpha} \) given by the trivialization \( \{ U_\alpha \} \).

We fix a Hermitian metric \( \| \cdot \| \) in \( L(D) \) and choose positive smooth functions \( h_\alpha \) on \( U_\alpha \) such that
\[
\frac{1}{\| \sigma(x) \|} = \frac{h_\alpha(x)}{|\sigma(x)|}, \quad x \in U_\alpha.
\]

By Claim 4.3 there exists a polynomial \( R(w) \) in variable
\[
w = (w_{lk}) \in \pi_2(J_{k_0}(f)(\mathcal{A}(R_0))^{Zar}) \cong \mathbb{C}^{nk_0}
\]
such that
\[
\pi_2(J_{k_0}(D, \log \partial M)) \cap \pi_2(J_{k_0}(f)(\mathcal{A}(R_0))^{Zar}) \subset \left\{ w \in \pi_2(J_{k_0}(f)(\mathcal{A}(R_0))^{Zar}) \mid R(w) = 0 \right\}
\]
\[
\neq \pi_2(J_{k_0}(f)(\mathcal{A}(R_0))^{Zar}).
\]

Then we have an equation on every \( U_\alpha \times \pi_2(J_{k_0}(f)(\mathcal{A}(R_0))^{Zar}) \) of the form:
\[
b_{a_0}\sigma_\alpha + b_{a_1}d\sigma_\alpha + \cdots + b_{a_{k_0}}d^{a_{k_0}}\sigma_\alpha = R(w),
\]
where \( b_{a_i} \) are jet differentials on \( U_\alpha \). Therefore, in every \( U_j \times \pi_2(J_{k_0}(f)(\mathcal{A}(R_0))^{Zar}) \), we have
\[
\frac{1}{\| \sigma \|} = \frac{1}{|R|} \frac{h_\alpha}{|\sigma_\alpha|} = \frac{1}{|R|} h_\alpha b_{a_0} + h_\alpha b_{a_1} \frac{d\sigma_\alpha}{\sigma_\alpha} + \cdots + h_\alpha b_{a_{k_0}} \frac{d^{a_{k_0}}\sigma_\alpha}{\sigma_\alpha}.
\]

Choose relatively compact open subsets \( U'_\alpha \) of \( U_\alpha \) so that \( \cup_\alpha U'_\alpha = \overline{M} \). For every \( \alpha \), there exist positive constants \( C_\alpha \) such that
\[
h_\alpha |b_{a_i}| \leq \sum_{\text{finite}} h_\alpha |b_{a_{i_1}a_{i_2}...a_{i_k}(x)}| \cdot |w_{lk}|^{\beta_i} \leq C_\alpha \sum_{\text{finite}} |w_{lk}|^{\beta_i}
\]
for all \( x \in U'_\alpha \).

By making \( C_\alpha \) larger if necessary, there exists \( d_\alpha > 0 \) such that for \( f(z) \in U'_\alpha \) we have
\[
h_\alpha(f(z)) |b_{a_i}(J_k(f)(z))| \leq C_\alpha \left( 1 + \sum_{1 \leq l \leq n} |w_{lk} \circ J_k(f)(z)| \right)^{d_\alpha}.
\]

Setting \( \xi_{(k)} = (w_{1k}(J_k(f)), \cdots, w_{2k}(J_k(f))) \) and \( \xi_{(k)} = w_{lk}(J_k(f)) \), for \( f(z) \in U'_\alpha \) we have
\[
\frac{1}{\| \sigma(f(z)) \|} \leq \frac{1}{|R(\xi_{(1)}(z), \cdots, \xi_{(k_0)}(z))|} \sum_{j=1}^{N} C_\alpha \left( 1 + \sum_{1 \leq l \leq n} |\xi_{(l)}(z)| \right)^{d_\alpha}.
\]
Combining Lemma 1.1, (4.6) and (4.7) we obtain

\[ \times \left( 1 + \left| \frac{d\sigma_\alpha}{\sigma_\alpha}(J_1(f)(z)) \right| + \cdots + \left| \frac{d^{k_0}\sigma_\alpha}{\sigma_\alpha}(J_{k_0}(f)(z)) \right| \right). \]

This implies that

\[ m_{0,f}(r; D) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( \frac{1}{\|\sigma(f(re^{i\theta})\|} \right) d\theta + O(1) \]

\[ \leq m_0 \left( r, \frac{1}{R(\xi(1)(z), \ldots, \xi(k_0)(z))} \right) + O \left( \sum_{1 \leq l \leq n} \frac{1}{2\pi} \int_0^{2\pi} \left( \log^+ |\xi_l|_{l} (re^{i\theta})| + \log^+ |\xi_l|_{l} (\frac{1}{r}e^{i\theta})| \right) d\theta \right) + O(1). \]

By Lemma 1.1 and by (4.2) we have

\[ \left\| m_0 \left( r, \frac{d^k \sigma_{\alpha}}{\sigma_{\alpha}} f \right) \right\| = O(\log^+ T_0,f(r; c_1(D))) + O \left( \log^+ \frac{1}{R_0 - r} \right). \]

Combining Lemma 1.1, (4.6) and (4.7) we obtain

\[ \left\| m_{0,f}(r; D) = O(\log^+ T_0,f(r; c_1(D))) \right\| + O \left( \log^+ \frac{1}{R_0 - r} \right). \]

We next estimate the counting function \(N_0(r, f^*D)\). We see that for all \(z \in \mathbb{S}(R_0)\),

\[ \text{ord}_z f^*D > k \iff J_k(f)(z) \in J_k(D, \log \partial M). \]

Then, from (4.8) we have

\[ \text{ord}_z f^*D - \min\{\text{ord}_z f^*D, k_0\} \leq \text{ord}_z \text{div}_0(R(\xi(1), \ldots, \xi(k_0))). \]

Thus

\[ N_0(r, f^*D) - N_0^{[k_0]}(r, f^*D) \leq N_0(r, \text{div}_0 R(\xi(1), \ldots, \xi(k_0))). \]

(4.9)

By the first main theorem and by Lemma 1.1 we have

\[ \left\| N_0(r, \text{div}_0 R(\xi(1), \ldots, \xi(k_0))) \right\| \leq T_0(r, R(\xi(1), \ldots, \xi(k_0))) + O(1) \]

\[ \leq O \left( \sum_{1 \leq l \leq n} T_0(r, \xi_l(1)) \right) + O(1) \]

\[ = O \left( \sum_{1 \leq l \leq n} m_0(r, \xi_l(1)) \right) + O(1) \]

\[ = O(\log^+ T_0,f(r; c_1(D))) + O \left( \log^+ \frac{1}{R_0 - r} \right). \]

(4.10)

We have

\[ T_0,f(r; c_1(D)) \leq N_0(r, f^*D) + O(1). \]

(4.11)
Combining (4.9), (4.10) and (4.11), we obtain
\[ T_{0,f}(r; c_1(D)) \leq N^{[k_0]}(r, f^*D) + O(\log^+ T_{0,f}(r; c_1(D))) + O\left(\log^+ \frac{1}{R_0 - r}\right). \]

The theorem is proved. \qed

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