REGULARIZED FUNCTIONS ON THE PLANE AND NEMYTSKII OPERATORS

FUNCIONES REGULARES EN EL PLANO Y OPERADORES DE NEMYTSKII

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Received: 29/Jul/2013; Revised: 1/Nov/2013; Accepted: 8/Nov/2013

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Abstract

In this paper we show that the space of the so-called regularized functions defined on some rectangle in the plane is a Banach space. Moreover, under suitable hypotheses we give a necessary and sufficient condition for the Nemytskii operator to map the space of regularized functions into itself.

Keywords: regularized functions of two variables, Banach spaces, Nemytskii operator.

1 Introduction

The regularized functions (also called regulated functions) were introduced by Georg Aumann in 1954 [2], that is, functions of a real variable which at each point of their domain of definition admit both finite one-sided limits. In [13] the space of regularized functions on \([a, b]\) is denoted by \(G(a, b)\).

Regularized functions play an important role, for instance, in applications to differential equations with singular right-hand sides or with distributional coefficients, see [7], or to the Skorokhod problem, see [4]. In the study of the controllability of systems governed by evolution equations, as well as in existence and expansion of solutions of differential or functional equations the Nemytskii operator appears in a natural way. To the best of our knowledge, this operator has not been studied up to now in spaces of regularized functions. Given a rectangle \(I := [a, b] \times [c, d]\) in the plane \(\mathbb{R}^2\), in this paper we show that the space \(G^-(I)\) of so called left-left regularized functions \(h : I \rightarrow \mathbb{R}\) is a Banach space, and we characterize the Nemytskii operator acting in this space.
2 Regularized functions

Let \( I = [a, b] \times [c, d] \) as above, and let \( h : I \to \mathbb{R} \) be some function. Following [5] we call the function \( h_+ \) defined by

\[
h_+(t, s) = \begin{cases} 
\lim_{(x, y) \to (t^+, s^-)} h(x, y), & (t, s) \in (a, b] \times (c, d], \\
\lim_{(x, y) \to (t^+, c^+)} h(x, y), & (t, s) \in (a, b] \times \{c\}, \\
\lim_{(x, y) \to (a^+, s^-)} h(x, y), & (t, s) \in \{a\} \times (c, d], \\
\lim_{(x, y) \to (a^+, c^+)} h(x, y), & (t, s) = (a, c) 
\end{cases}
\]

the left-left regularization of \( h \). In the sequel the class of functions \( h \) for which the left-left regularization exists will be denoted by \( G^- (I; \mathbb{R}) \). Finding functions in this class is trivial: for example, any continuous function \( h : I \to \mathbb{R} \) satisfies \( h_-(t, s) \equiv h(t, s) \), and so belongs to \( G^- (I; \mathbb{R}) \).

The right-right regularization of a function \( h : I \to \mathbb{R} \) is defined in a similar way by

\[
h_+ (t, s) = \begin{cases} 
\lim_{(x, y) \to (t^+, s^+)} h(x, y), & (t, s) \in [a, b) \times (c, d), \\
\lim_{(x, y) \to (t^+, d^-)} h(x, y), & (t, s) \in [a, b) \times \{d\}, \\
\lim_{(x, y) \to (b^-, s^+)} h(x, y), & (t, s) \in \{b\} \times [c, d], \\
\lim_{(x, y) \to (b^-, d^-)} h(x, y), & (t, s) = (b, d) .
\end{cases}
\]

Similarly to the previous case we denote the class of functions which admit a right-right regularization by \( G^+ (I; \mathbb{R}) \). Finally, the class \( G^{-+} (I; \mathbb{R}) \) of left-right regularized and the class \( G^{+-} (I; \mathbb{R}) \) of right-left regularized functions are defined analogously.

It is easy to see that the classes \( G^- (I, \mathbb{R}) \) and \( G^+ (I, \mathbb{R}) \) are different. For example, the function \( h \) defined on \( I = [-1, 1] \times [-1, 1] \) by

\[
h(x, y) = \begin{cases} 
\frac{1}{x + y}, & x > 0 \text{ and } y > 0, \\
1 & \text{otherwise}
\end{cases}
\]

satisfies

\[
h_-(0, 0) = h(0, 0) = 1, \quad h_+(0, 0) = \infty
\]

and therefore belongs to \( G^+ (I; \mathbb{R}) \), but not to \( G^- (I; \mathbb{R}) \).
3 Properties of regularized functions

In this section we show that the class $G^{-}(I; \mathbb{R})$ is a Banach space.

**Proposition 3.1** The class $(G^{-}(I; \mathbb{R}), +, \cdot)$ is a linear space.

**Proof.** Given $f, g \in G^{-}(I; \mathbb{R})$ and $\alpha \in \mathbb{R}$, we have to show that $f + g \in G^{-}(I; \mathbb{R})$ and $\alpha f \in G^{-}(I; \mathbb{R})$. Denoting by $f_-$ the left-left regularization of $f$ and by $g_-$ the left-left regularization of $g$ we get for $(t, s) \in (a, b] \times (c, d]$

$$
(f + g)_-(t, s) = \lim_{(x, y) \to (t^-, s^-)} f(x, y) + \lim_{(x, y) \to (t^-, s^-)} g(x, y)
= \lim_{(x, y) \to (t^-, s^-)} [f(x, y) + g(x, y)] = f_-(t, s) + g_-(t, s).
$$

The other three cases for $(t, s) \in I$ are treated similarly, and so we have shown that $f + g \in G^{-}(I; \mathbb{R})$. For $\alpha \in \mathbb{R}$ and $(t, s) \in (a, b] \times (c, d]$ we obtain

$$
(\alpha f)_-(t, s) = \lim_{(x, y) \to (t^-, s^-)} (\alpha f)(x, y) = \alpha \lim_{(x, y) \to (t^-, s^-)} f(x, y) = \alpha f_-(t, s),
$$

and analogously for the other choices of $(t, s) \in I$. We conclude that $\alpha f \in G^{-}(I; \mathbb{R})$ which proves the assertion. □

In the sequel we consider the linear space $G^{-}(I; \mathbb{R})$ equipped with the supremum norm

$$
\|f\|_\infty := \sup \left\{ |f(x, y)| : (x, y) \in I \right\}.
$$

We have then the following result.

**Theorem 3.1** $(G^{-}(I; \mathbb{R}), \| \cdot \|_\infty)$ is a Banach space.

**Proof.** Let $\{f_n\}_{n \geq 1} \in G^{-}(I; \mathbb{R})$ be a Cauchy sequence, which means that for each $\varepsilon > 0$ there exists $N = N_\varepsilon > 0$ such that $n, m \geq N$ implies $\|f_n - f_m\|_\infty < \varepsilon$. Since

$$
\left| f_n(t, s) - f_m(t, s) \right| \leq \|f_n - f_m\|_\infty < \varepsilon
$$

for all $(t, s) \in I$, we conclude that $\{f_m(t, s)\}_{m \geq 1}$ is a Cauchy sequence in $\mathbb{R}$, and so we know that the pointwise limit

$$
f(t, s) := \lim_{m \to \infty} f_m(t, s) \quad ((t, s) \in I)
$$

exists. We claim that $f \in G^{-}(I; \mathbb{R})$ and $\lim_{m \to \infty} \|f_n - f\|_\infty = 0$. To prove this assertion fix $\varepsilon > 0$, and choose $N$ such that $n, m \geq N$ implies that $\|f_n - f_m\|_\infty < \varepsilon$. We distinguish now four possibilities for $(t, s)$:
(i) Let \((t, s) \in (a, b] \times (c, d]\). In this case we have
\[
|f_n(t, s) - f(t, s)| = |f_n(t, s) - \lim_{m \to \infty} f_m(t, s)|
\]
\[
= \lim_{(x, y) \to (t-, s-)} f_n(x, y) - \lim_{m \to \infty} \lim_{(x, y) \to (t-, s-)} f_m(x, y)
\]
\[
= \lim_{m \to \infty} \lim_{(x, y) \to (t-, s-)} |f_n(x, y) - f_m(x, y)|
\]
\[
\leq \|f_n - f_m\|_\infty < \varepsilon.
\]

(ii) In the case \((t, s) \in (a, b] \times \{c\}\) we obtain
\[
|f_n(t, c) - f(t, c)| = |f_n(t, c) - \lim_{m \to \infty} f_m(t, c)|
\]
\[
= \lim_{(x, y) \to (a+, c)} f_n(x, y) - \lim_{m \to \infty} \lim_{(x, y) \to (a+, c)} f_m(x, y)
\]
\[
= \lim_{m \to \infty} \lim_{(x, y) \to (a+, c)} |f_n(x, y) - f_m(x, y)|
\]
\[
\leq \|f_n - f_m\|_\infty < \varepsilon.
\]

(iii) The case when \((t, s) \in \{a\} \times (c, d]\) is similar to those considered above.

(iv) If \((t, s) = (a, c)\) we get
\[
|f_n(a, c) - f(a, c)| = |f_n(a, c) - \lim_{m \to \infty} f_m(a, c)|
\]
\[
= \lim_{(x, y) \to (a+, c+)} f_n(x, y) - \lim_{m \to \infty} \lim_{(x, y) \to (a+, c+)} f_m(x, y)
\]
\[
= \lim_{m \to \infty} \lim_{(x, y) \to (a+, c+)} |f_n(x, y) - f_m(x, y)|
\]
\[
\leq \|f_n - f_m\|_\infty < \varepsilon.
\]

In all cases we have shown that \(\|f_n - f\|_\infty < \varepsilon\). Moreover, since \(G^- (I; \mathbb{R})\) is a linear space we have \(f_n, f_n - f \in G^- (I; \mathbb{R})\). Therefore \(f = (f - f_n) + f_n \in G^- (I; \mathbb{R})\), and the proof is complete. \(\blacksquare\)
Clearly, in a similar way we can prove that $G^{-+}(I; \mathbb{R})$, $G^{++}(I; \mathbb{R})$ and $G^{+}(I; \mathbb{R})$ are Banach spaces with the supremum norm. We remark that we have not only $C(I; \mathbb{R}) \subseteq G^{-}(I; \mathbb{R})$, but also $BV(I; \mathbb{R}) \subseteq G^{-}(I; \mathbb{R})$, see [3].

A function $f : I \rightarrow \mathbb{R}$ is said to be left-left continuous if

$$\lim_{(x,y)\rightarrow(t,s)} f(x,y) = f(t,s) \text{ for all } x \in (a,b] \text{ and } y \in (c,d].$$

We denote by $BV^*(I; \mathbb{R})$ the subspace of $BV(I; \mathbb{R})$ of those functions which are left–left continuous on $(a,b] \times (c,d]$.

**Lemma 3.2** (cf. [3]) If $h \in BV(I; \mathbb{R})$, then $h_- \in BV^*(I; \mathbb{R})$.

### 4 The Nemytskii operator

For $I = [a, b] \times [c, d]$ as before, consider the linear space $\mathcal{F}$ of all functions $f : I \rightarrow \mathbb{R}$ and the nonlinear operator $H : \mathcal{F} \rightarrow \mathcal{F}$ defined by the formula

$$(Hf)(t, s) = h(t, s, f(t, s)),$$

where $h : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a mapping. In this case we say that $H$ is the Nemytskii operator generated by $h$. When $h$ is independent on $(t, s) \in I$, the Nemytskii operator generated by $h$ is called autonomous.

In [6] Josephy proved that the autonomous Nemytskii operator generated by $h : \mathbb{R} \rightarrow \mathbb{R}$ is a self mapping of $BV([a, b]; \mathbb{R})$ if and only if $h$ is locally Lipschitz on $\mathbb{R}$. Subsequently, several authors proved this result for many other functions spaces (see [1, 8, 9, 10, 11, 12], for example).

Here we will give conditions to assure that the Nemytskii operator defined on the space $G^{-}(I; \mathbb{R})$ maps this space into itself. To this end we consider the space $G^{-} \cdot Lip(I \times \mathbb{R}; \mathbb{R})$ of all left-left regularized functions in the first two variables and Lipschitz in the third variable, i.e. all functions $h$ which satisfy the following two conditions:

(i) The map $(t, s) \mapsto h(t, s, u)$ is a left-left regularized function for all $u \in \mathbb{R}$.

(ii) There exists $M > 0$ such that

$$|h(t, s, u) - h(t, s, v)| \leq M|u - v| \quad ((t, s) \in I). \quad (*)$$

Observe that the class $G^{-} \cdot Lip(I \times \mathbb{R}; \mathbb{R})$ is a Banach space equipped with the norm

$$\|h\|_{Lip} = \max \left\{ \|h_0\|_{\infty}, [h] \right\}, \quad (4.1)$$

*Rev.Mate.Teor.Aplic.* (ISSN 1409-2433) Vol. 21(1): 11–20, January 2014
where \( h_0 : I \to \mathbb{R} \) is defined by \( h_0(t, s) = h(t, s, 0) \) (\( H_0 \) is the Nemytskii operator generated by \( h_0 \))

\[
\left[ h \right] = \inf \left\{ M : M \text{ satisfies (\( \ast \))} \right\}.
\]  

(4.2)

Now we are going to prove our main result which is motivated by the technique used in [14].

**Main Theorem 4.1** Suppose that \( h(t, s, \cdot) : \mathbb{R} \to \mathbb{R} \) is Lipschitzian for all \( (t, s) \in I \). Then the Nemytskii operator \( H \) generated by \( h \) maps the space \( G^{-}(I; \mathbb{R}) \) into itself if and only if \( h \in G^{-}(I; \mathbb{R}) \cdot \text{Lip}(I \times \mathbb{R}; \mathbb{R}) \). Moreover, in this case the operator \( H \) is bounded.

**Proof.** Observe that if \( H \) maps \( G^{-}(I; \mathbb{R}) \) into itself then for any function \( f : I \to \mathbb{R} \) the operator \( H_f : I \to \mathbb{R} \), where \( H_f(t, s) = h(t, s, f(t, s)) \).

Hence we deduce that \( h(t, s, f(t, s)) \) is a left-left regularized function for all \( f \in \mathbb{R}^I \). Moreover, keeping in mind that \( h \in \text{Lip}(I \times \mathbb{R}; \mathbb{R}) \), we get that

\[ h \in G^{-}(I; \mathbb{R}) \cdot \text{Lip}(I \times \mathbb{R}; \mathbb{R}). \]

Conversely, for all \( (t, s) \in I \), we have: (i) If \( (t, s) \in (a, b] \times (c, d] \), then

\[
H_f(t^-, s^-) - H_f(t, s) = h(t^-, s^-, f(t^-, s^-)) - h(t, s, f(t, s))
\]

\[
= h_-(t^-, s^-, f(t^-, s^-)) - h_-(t, s, f(t, s))
\]

\[
= \lim_{(x,y) \to (t^-, c^+)} h(x, y, f(x, y)) - h(t, s, f(t, s)) = 0.
\]

(ii) If \( (t, s) \in (a, b] \times \{c\} \), then

\[
H_f(t^-, s^-) - H_f(t, s) = h(t^-, s^-, f(t^-, s^-)) - h(t, s, f(t, s))
\]

\[
= h_-(t^-, s^-, f(t^-, s^-)) - h_-(t, s, f(t, s))
\]

\[
= \lim_{(x,y) \to (t^-, c^+)} h(x, y, f(x, y)) - h(t, s, f(t, s)) = 0.
\]

(iii) For \( (t, s) \in \{a\} \times (c, d] \) we proceed in a similar way as in (ii).

(iv) If \( (t, s) = (a, c) \) the result is trivial and we omit it. Now we will show that \( H_f \) is left-left continuous. In fact, we get

Rev.Mate.Teor.Aplic. (ISSN 1409-2433) Vol. 21(1): 11–20, January 2014
\[
\lim_{(\tau, \sigma) \to (t^-, s^-)} \left| H_f(\tau, \sigma) - H_f(t, s) \right|
\]

\[
= \lim_{(\tau, \sigma) \to (t^-, s^-)} \left| h(\tau, \sigma, f(\tau, \sigma)) - h(t, s, f(t, s)) \right|
\]

\[
= \lim_{(\tau, \sigma) \to (t^-, s^-)} \left| h_-(\tau, \sigma, f(\tau, \sigma)) - h_-(t, s, f(t, s)) \right|
\]

\[
\leq M \cdot \lim_{(\tau, \sigma) \to (t^-, s^-)} \left| f(\tau, \sigma) - f(t, s) \right|
\]

\[
+ \lim_{(\tau, \sigma) \to (t^-, s^-)} \left| h(\tau, \sigma, f(t, s)) - h(t, s, f(t, s)) \right|
\]

\[
= M \cdot 0 + 0 = 0,
\]

where \( M \) is the Lipschitz constant from \((*)\). It follows that \( H \) maps \( G^-(I, \mathbb{R}) \) into itself.

Next, we prove that the operator \( H \) is bounded.

Let \( B_r = \{ f \in G^-(I, \mathbb{R}) : \| f \|_\infty \leq r \} \). Then we obtain

\[
\| H_f \|_\infty - \| H_0 \|_\infty \leq \| H_f - H_0 \|_\infty = \sup_{(t, s) \in I} \left| (H_f)(t, s) - (H_0)(t, s) \right|
\]

\[
= \sup_{(t, s) \in I} \left| h(t, s, f(t, s)) - h(t, s, 0) \right|
\]

\[
\leq M \cdot \sup_{(t, s) \in I} \left| f(t, s) \right| \leq M \cdot r.
\]

Next, we derive

\[
\| H_f \|_\infty \leq Mr + \| H_0 \|_\infty
\]

\[
= Mr + \| h(t, s, 0) \|_\infty
\]

\[
= Mr + \| H_0 \|_\infty.
\]
This completes the proof.

We point out that our results may be extended in different directions. For instance, instead of functions on a rectangle in the plane one may consider functions of several variables on a cube in finite-dimensional Euclidean space. Moreover, all constructions carry over without any change from real valued functions to functions taking their values in a Banach space.

Acknowledgement

The author would like to thank the anonymous referees and the editors for their valuable comments and suggestions. Also, I want to mention this research was partly supported by CDCHTA of Universidad de Los Andes under the project NURR-C-547-12-05-B.

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