Computing the ideal class monoid of an order

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Abstract

There are well known algorithms to compute the class group of the maximal order \( \mathcal{O}_K \) of a number field \( K \) and the group of invertible ideal classes of a non-maximal order \( R \). In this paper we explain how to compute also the isomorphism classes of non-invertible ideals of an order \( R \) in a finite product of number fields \( K \). In particular we also extend the above-mentioned algorithms to this more general setting. Moreover, we generalize a theorem of Latimer and MacDuffee providing a bijection between the conjugacy classes of integral matrices with given minimal and characteristic polynomials and the isomorphism classes of lattices in certain \( \mathbb{Q} \)-algebras, which under certain assumptions can be explicitly described in terms of ideal classes.

1 Introduction

Let \( K \) be a number field and \( R \) an order in \( K \). There are well known algorithms to compute the ideal class group \( \text{Pic}(R) \) when \( R \) is the ring of integers \( \mathcal{O}_K \) of \( K \), also known as the maximal order, see for example [Coh93]. This information can be used to efficiently compute the group \( \text{Pic}(R) \) of invertible ideal classes of a non-maximal order \( R \), as is explained in [KP05].

On the other hand not much is known about non-invertible ideals and, in particular, it is not known how to compute the monoid of all ideal classes of \( R \), which we will denote \( \text{ICM}(R) \). In the literature one can find results about the local isomorphism classes of ideals. More precisely, one studies the genus of an ideal, which is its isomorphism class after localizing at a rational prime \( p \), or its weak equivalence class which is its isomorphism class after localizing at a prime ideal \( p \) of \( R \). For the notion of genus we refer to [Rei70] and [Rei03], while for results about the weak equivalence classes we cite [DTZ62]. It is important to mention that these two apparently different notions are actually equivalent, as pointed out in [LW85, Section 5].

In the present paper we exhibit:

• an algorithm to compute the monoid of isomorphism classes of fractional ideals of an order \( R \) in a finite product of number fields \( K \), see Theorem 4.6, Proposition 5.1 and the algorithms in Section 6, and

• a bijection between the set of conjugacy classes of integral matrices with given square-free minimal polynomial \( m \) and characteristic polynomial \( c \) and the \( R \)-isomorphism classes of \( \mathbb{Z} \)-lattices in a certain \( \mathbb{Q} \)-algebra, where \( R \) is an order in a certain product of number fields, see Theorem 8.1. Under certain assumptions on the polynomials \( c \) and \( m \), we can reduce such a description to an ideal class monoid computation and hence produce representatives of each conjugacy class, see Corollary 8.2.

Theorem 8.1 is a generalization of the main result of [LM33] where it is analyzed the case when \( c \) is square-free. Their theorem was then reproved with a different method under the extra assumption that \( c \) is irreducible in [Tau49]. The author recently discovered that Theorem 8.1 has independently been proved in [Hus17] in more generality.
The present paper is structured as follows. In Section 2 we recall the definitions of an order $R$ and a fractional ideal in a product of number fields $K$ and we prove some basic results, which are well-known in the case that $K$ is a number field. In Section 3 we introduce isomorphisms of fractional ideals, and the monoid that the corresponding classes form, called the ideal class monoid $ICM(R)$. Since it is hard to compute the $ICM(R)$ directly, in Section 4 we relax the notion of isomorphism to a local one, called weak equivalence. We explain how to effectively check whether two fractional ideals are weakly equivalent and how to algorithmically reconstruct $ICM(R)$ once we have computed $Pic(S)$, for every over-order $S$ of $R$, and the monoid of weak equivalence classes $W(R)$. In Section 5 we give the pseudo-code of the algorithms described in the previous sections and in Section 6 we present our results about computing conjugacy classes of integral matrices. The algorithms have been implemented in [BCP97] and the code is available on the webpage of the author.

Another application, namely computing isomorphism classes of abelian varieties defined over a finite field belonging to an isogeny class determined by a square-free Weil polynomial, is discussed in [Mar18b].

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2 Orders

In what follows, the word ring will mean commutative ring with unit. An order is a reduced ring $R$, which is free and finitely generated as a $\mathbb{Z}$-module. Let $K$ be the total quotient ring of an order $R$, that is, the localization of $R$ at the multiplicative set of non-zero divisors. Then $K$ is an étale algebra over $\mathbb{Q}$ with $R \otimes_{\mathbb{Z}} \mathbb{Q} = K$, and in particular $K$ is a finite product of number fields, say $K = K_1 \times \ldots \times K_r$. The set of orders in $K$ contains a maximal element with respect to the inclusion relation. This order, denoted $\mathcal{O}_K$, is the integral closure of $\mathbb{Z}$ in $K$ and it is usually referred to as the maximal order or the ring of integers of $K$. Note that $\mathcal{O}_K = \mathcal{O}_{K_1} \times \ldots \mathcal{O}_{K_r}$, where $\mathcal{O}_{K_i}$ is the maximal order of $K_i$. Indeed, $\mathcal{O}_K$ contains $\prod_i \mathcal{O}_{K_i}$, so $\mathcal{O}_K$ is a product of orders $S_i$ in $K_i$ and by maximality it follows that $S_i = \mathcal{O}_{K_i}$. There are well known algorithms to compute each $\mathcal{O}_{K_i}$, see for example [Coh93, Chapter 6], and in what follows we will assume that we can compute $\mathcal{O}_K$.

From now on $R$ will be an order in $K$. A finitely generated sub-$R$-module $I$ of $K$ is called a fractional $R$-ideal if $I \otimes_{\mathbb{Z}} \mathbb{Q} = K$. Such an $I$ is a finitely generated free $\mathbb{Z}$-module of the same rank as $R$, and so we can find $x_1, \ldots, x_n \in K$, where $n = \sum_{i=1}^r [K_i : \mathbb{Q}]$, such that

$$I = x_1 \mathbb{Z} \oplus \ldots \oplus x_n \mathbb{Z}. $$

In particular, if $I \subseteq R$ then the quotient $R/I$ is finite. We denote by $\mathcal{I}(R)$ the set of all fractional ideals of $R$. Observe that for every fractional $R$-ideal $I$, there exists a non-zero divisor $x \in K$ such that $xI$ is an ideal of $R$. Moreover, every ideal of $R$ containing a non-zero divisor is a fractional $R$-ideal. The fractional $R$-ideals that are rings are called over-orders of $R$. Since $\mathcal{O}_K$
and \( R \) have the same rank as free abelian groups, the quotient \( \mathcal{O}_K / R \) is finite and thus there are only finitely many over-orders of \( R \).

Given two fractional \( R \)-ideals \( I \) and \( J \), the product \( IJ \), the sum \( I + J \), the intersection \( I \cap J \), and the ideal quotient

\[(I : J) = \{ x \in K : xJ \subseteq I \}\]

are fractional \( R \)-ideals. In particular, ideal multiplication induces on \( \mathcal{S}(R) \) the structure of a commutative monoid with unit element \( R \). A useful property of the ideal quotient is the following lemma.

**Lemma 2.1.** Let \( I, J, L \) be fractional \( R \)-ideals, then

\[(I : J) : L = (I : JL) \cdot (J : L).
\]

**Proof.** Let \( x \in ((I : J) : L) \), then \( xL \subseteq (I : J) \) and so \( xJL \subseteq I \), which means that \( x \in (I : JL) \). Conversely, if \( x \in (I : JL) \), then \( xJL \subseteq I \) and hence \( xL \subseteq (I : J) \) and so \( x \in ((I : J) : L) \).

If \( I \) is a fractional \( R \)-ideal then \( (I : I) \) is a sub-ring of \( K \) containing \( R \). Hence it is an over-order of \( R \) and, in particular, it is the biggest over-order of \( R \) for which \( I \) is a fractional ideal. It is called the *multiplicator ring* of \( I \).

**Lemma 2.2.** The over-orders of \( R \) are precisely the idempotents of \( \mathcal{S}(R) \), that is, the fractional \( R \)-ideals \( S \) such that \( SS = S \).

**Proof.** Let \( S \) be an over-order of \( R \). Then \( S \) is multiplicatively closed and contains 1, so \( SS = S \). Conversely, let \( S \) be an idempotent fractional ideal of \( R \). Let \( T = (S : S) \) be the multiplicator ring of \( S \). As \( SS = S \) we have \( S \subseteq T \) and hence \( S \) is a finitely generated idempotent \( T \)-ideal. By the determinant trick it must be generated by an idempotent element \( e \) of \( T \). As \( S \) has full rank over \( \mathbb{Z} \) we must have \( e = 1 \), that is \( S = T \). In particular, \( S \) is an over-order of \( R \).

We will denote by \( \text{Tr}_{K/Q} \), or simply \( \text{Tr} \) when no confusion can arise, the *trace form* on \( K \), which associates to every \( x \in K \) the trace of the matrix of the multiplication by \( x \). For every fractional \( R \)-ideal \( I \), we define the *trace dual ideal* as \( I^t = \{ x \in K : \text{Tr}(xI) \subseteq \mathbb{Z} \} \). Given a \( \mathbb{Z} \)-basis \( \{ x_i \} \) of \( I \), we have \( I^t = x_1^t \mathbb{Z} + \ldots + x_n^t \mathbb{Z} \), where \( \{ x_j^t \} \) is the *trace dual basis*, which is characterized by \( \text{Tr}(x_i x_j^t) = 1 \) or 0 according to if \( i = j \) or \( i \neq j \). Observe that \( I^t \) is a fractional \( R \)-ideal and that the map \( x \mapsto \varphi_x \), where \( \varphi_x(y) = \text{Tr}(xy) \) is an isomorphism from \( I^t \) to \( \text{Hom}_\mathbb{Z}(I, \mathbb{Z}) \). In the next lemma we will summarize some well known properties of the trace dual ideal.

**Lemma 2.3.** Let \( R \) be an order in \( K \), let \( I \) and \( J \) be two fractional \( R \)-ideals and let \( x \) be in \( K^* \).

\[
\begin{align*}
(1) & \quad I^t I = I, \\
(2) & \quad I \subseteq J \iff J^t \subseteq I^t, \\
(3) & \quad (I \cap J)^t = I^t + J^t, \\
(4) & \quad (xI)^t = \frac{1}{x} J^t, \\
(5) & \quad (I : J) = (I^t)^t, \\
(6) & \quad (I : J) = (J^t : I^t), \\
(7) & \quad S = (I : I) \iff II^t = S^t.
\end{align*}
\]
Let \( p \) be a prime ideal of \( R \) which is also a fractional \( R \)-ideal. Since the integral domain \( R/p \) is finite, we see that \( p \) is a maximal ideal. Conversely, if \( m \) is a maximal ideal of \( R \) then it contains the prime \( p \) which is the characteristic of the field \( R/m \), and hence \( m \) is a fractional \( R \)-ideal. We will refer to the maximal ideals of \( R \) as the primes of \( R \). Since for any fractional \( R \)-ideal \( I \) contained in \( R \) the quotient \( R/I \) is finite, we deduce that there exists only a finite number of primes of \( R \) containing \( I \).

A fractional \( R \)-ideal \( I \) is said to be invertible in \( R \) if \( IJ = R \), for some fractional \( R \)-ideal \( J \).

**Remark 2.4.** Note that we could equivalently say that \( I \) is invertible in \( R \) if and only if there exists an \( R \)-ideal \( J \) and a non-zero divisor \( d \) such that \( IJ = dR \). This characterization allows us to talk about invertible ideals in any ring.

**Lemma 2.5.** Let \( I \) be a fractional \( R \)-ideal which is invertible in \( R \). Then \( R \) is the multiplicator ring of \( I \).

**Proof.** Put \( S = (I : I) \). Since \( I \) is an \( R \)-module we have \( R \subseteq S \) and using \( I = SI \) we deduce that

\[
R = I(R : I) = SI(R : I) = SR = S.
\]

The following lemmas are useful for understanding how invertible ideals behave with respect to localizations at primes.

**Lemma 2.6.** [Kap49, Theorem 12.3] Let \( T \) be a Noetherian ring. Then \( T \) is a principal ideal ring if and only if every maximal ideal is principal.

**Lemma 2.7.** [Gil92, Proposition 7.4] Let \( T \) be a ring with finitely many maximal ideals and let \( I \) be a \( T \)-ideal. Then \( I \) is invertible in \( T \) if and only if \( I \) is principal and generated by a non-zero divisor.

**Lemma 2.8.** Let \( I \) be a fractional \( R \)-ideal. Then \( I \) is invertible in \( R \) if and only if \( Ip \) is principal for every prime \( p \) of \( R \).

**Proof.** Since \( (R : I)p = (R_p : I_p) \), if \( I \) is invertible then \( I_p \) is invertible and hence principal by Lemma 2.7. Conversely, we want to show that \( I(R : I) = R \). Consider the inclusion map \( \iota : I(R : I) \subseteq R \) and let \( p \) be a prime of \( R \). By hypothesis there exists \( x \) such that \( I_p = xR_p \). Note that \( x \) is a non-zero divisor since \( I \) is a fractional ideal. Then

\[
I_p(R_p : I_p) = xR_p(R_p : xR_p) = R_p,
\]

which implies that \( \iota \) is an isomorphism locally at every prime \( p \). Hence it is so also globally and the ideal \( I \) is invertible in \( R \).

**Corollary 2.9.** Let \( p \) be a prime of \( R \). Then \( p \) is invertible if and only if \( Rp \) is a principal ideal ring.

**Proof.** By Lemma 2.6 and Lemma 2.7 we deduce that if \( p \) is an invertible prime of \( R \), then \( pR_p \) is principal, which implies that \( R_p \) is a principal ideal ring. Conversely, since \( pR_q = R_q \) for every prime \( q \neq p \), it follows that if \( pR_p \) is principal then by Lemma 2.8 \( p \) is invertible.

Observe that \( \mathcal{O}_K \) is the only order in \( K \) whose ideals are all invertible. We introduce now some classes of orders which are particularly well behaved in terms of invertibility of ideals.
Proposition 2.10. \cite[Proposition 2.7]{BL94} Let $R$ be an order with trace dual $R^t$. The following are equivalent:

(a) for every fractional $R$-ideal $I$, we have $(R : (R : I)) = I$;

(b) for every fractional $R$-ideal $I$, we have $(I : I) = R$ if and only if $I$ is invertible;

(c) $R^t$ is invertible in $R$.

An order satisfying one of the equivalent conditions of (2.10) is called Gorenstein. This definition is equivalent to the usual one, see \cite[Theorem 6.3]{Bas63}. Observe that $\mathcal{O}_K$ is Gorenstein, but there are Gorenstein orders which are not maximal. One class of examples of Gorenstein orders are the monogenic orders, which are of the form $\mathbb{Z}[x]/(f)$, where $f$ is a monic polynomial with integer coefficients and non-zero discriminant.

Corollary 2.11. Monogenic orders are Gorenstein.

Proof. Let $f$ be a monic polynomial with integer coefficients and distinct roots, that is, with non-vanishing discriminant and let $R$ be the order $\mathbb{Z}[x]/(f)$. Put $\alpha = x \mod f$. Then $R = \mathbb{Z}[\alpha]$ and $R^t = (1/f'(\alpha))R$. See \cite[Proposition 3-7-12]{Wei98} and notice that we don't need $f$ irreducible, but just with distinct roots. See also \cite[Example 2.8]{BL94}. Hence $R^t$ is invertible and the statement follows from Proposition 2.10.

An order $R$ is called a Bass order if every over-order of $R$ is Gorenstein, or equivalently, if the $R$-module $\mathcal{O}_K/R$ is cyclic, that is, if $\mathcal{O}_K = R + xR$ for some $x \in \mathcal{O}_K$. For a proof and other equivalent characterizations, see for example \cite[Theorem 2.1]{LW85} and Proposition 3.7. Observe that every order in a quadratic number field is a Bass order.

3 Ideal classes

Recall that for an order $R$ in $K$ we denote by $\mathcal{I}(R)$ the commutative monoid of fractional $R$-ideals.

Definition 3.1. Let $R$ be an order in $K$. The ideal class monoid of $R$ is

$$\text{ICM}(R) = \mathcal{I}(R)/\sim,$$

where $I \sim J$ if and only if $I$ and $J$ are isomorphic as $R$-modules. We will denote the ideal class of $I$ with $[I]$.

The name is justified by the fact that $\text{ICM}(R)$ inherits the commutative monoid structure of $\mathcal{I}(R)$, as will become evident with Corollary 3.4.

Lemma 3.2. Let $R$ be an order in $K$. Consider an $R$-module morphism $\varphi : I \to J$, where $I$ and $J$ are two fractional $R$-ideals, then $\varphi$ is a multiplication by some $\alpha \in K$.

Proof. Since $K = R \otimes \mathbb{Q}$, the map $\varphi$ induces a $K$-linear morphism $\hat{\varphi} : I \otimes \mathbb{Q} \to J \otimes \mathbb{Q}$. Since $I \otimes \mathbb{Q} = J \otimes \mathbb{Q} = K$ and $K$ is the total quotient ring of $R$, the morphism $\hat{\varphi}$ is completely determined by the image of 1, say $\hat{\varphi}(1) = \alpha$. Since $\hat{\varphi}|_I = \varphi$, also $\varphi$ is the multiplication by $\alpha$.

Corollary 3.3. Let $I$ and $J$ be two fractional $R$-ideals. Then we have a natural identification

$$\text{Hom}_R(I,J) = (J : I).$$

In particular, if $p$ is a prime of $R$ then every $R_p$-linear morphism $\varphi : I_p \to J_p$ is a multiplication by some $\alpha$ in the total quotient ring of $R_p$. 

5
Proof. The first statement follows immediately from Lemma 3.2. For the second, we just need to observe that \( (J : I)_p = (I_p : I_p) \).

Corollary 3.4. Two fractional \( R \)-ideals \( I \) and \( J \) are isomorphic if and only if there exists an \( \alpha \in K^\times \) such that \( I = \alpha J \).

The group \( \mathcal{P}(R) \) of principal fractional \( R \)-ideals acts by multiplication on \( \mathcal{J}(R) \) and we have that
\[
\text{ICM}(R) = \mathcal{J}(R) / \mathcal{P}(R).
\]
Observe that every fractional ideal in \( \mathcal{P}(R) \) is invertible in \( R \), so we can consider the quotient of invertible fractional \( R \)-ideals by \( \mathcal{P}(R) \), which will inherit a group structure.

Definition 3.5. Let \( R \) be an order in \( K \). The Picard group of \( R \) is
\[
\text{Pic}(R) = \{ I \in \mathcal{J}(R) \text{ invertible in } R \} / \mathcal{P}(R).
\]
Since being invertible is a property of the ideal class, we can conclude that \( \text{Pic}(R) \subseteq \text{ICM}(R) \). Observe that equality holds if and only if \( R = \mathcal{O}_K \).

Since \( \mathcal{O}_K \) is a finite product of Dedekind domains, we have that every ideal can be written in a unique way as a product of prime ideals, see for example [Rei03, Theorem 22.24]. For every invertible fractional ideals of non-maximal order \( R \), we can find an isomorphic one, say \( I \), which is coprime with the conductor \( \mathcal{f} = (R : \mathcal{O}_K) \). This implies that \( I \mathcal{O}_K \cap R = I \) and hence it follows that \( I \) admits a factorization into a product of primes of \( R \). But this is not true if we look at non-invertible ideals.

The following lemma proves that the multiplicator ring is an invariant of the ideal class.

Lemma 3.6. Let \( R \) be an order in \( K \). If two fractional \( R \)-ideals \( I \) and \( J \) are isomorphic then they have the same multiplicator ring.

Proof. By Lemma 3.2 there exists \( x \in K^\times \) such that \( I = xJ \). Hence \( (I : I) = (xJ : xJ) = (J : J) \), where the last equality is an immediate consequence of the definition of a quotient ideal.

It follows that
\[
\text{ICM}(R) \supseteq \bigsqcup \text{Pic}(S)
\]
where the disjoint union is taken over the set of over-orders \( S \) of \( R \).

Recall that a commutative monoid is called Clifford if it is a disjoint union of groups. For other equivalent definitions of a commutative Clifford monoid see [ZZ94, Section 1] or [Hel40, Chapter IV].

Proposition 3.7. The following are equivalent:

\begin{enumerate}
\item \( R \) is Bass,
\item the inclusion in \((1)\) is an equality,
\item \( \text{ICM}(R) \) is Clifford.
\end{enumerate}

Proof. \([a] \Rightarrow [b] \) If \( R \) is Bass then every over-order is Gorenstein and in particular every fractional \( R \)-ideal \( I \) is invertible in its own multiplicator ring \( S \). This means that \( \{ I \} \) is in \( \text{Pic}(S) \) and \([b]\) holds.

\([b] \Rightarrow [c] \) This is obvious.
Write $\text{ICM}(R) = \bigsqcup e G_e$, where $e$ runs over the set of idempotent elements of $\text{ICM}(R)$, and $G_e$ denotes the group with unit $e$. Let $J$ be a fractional $R$-ideal representing $e$. Then there exists $x \in K^\times$ such that $xJ^2 = J$. Put $S = xJ$. Then

$$S^2 = x^2J^2 = x(xJ^2) = xJ = S.$$ 

Note that $S$ is another representative of the class $e$ and by Lemma 2.2 it is an over-order of $R$. Now let $T$ be any over-order of $R$. We want to show that $T^t$ is invertible in $T$. Say that the class representing $T^t$ lies in $G_e$ where $e = \{S\}$. Then $T^t$ is invertible in $S$ and, since the multiplicator ring of $T^t$ is $T$, by Lemma 2.5 we have that $S = T$. □

**Remark 3.8.** If $K = K_1 \times \ldots \times K_r$, with $K_i$ number fields, then $\Theta_K = \prod_i \Theta_{K_i}$ and

$$\text{Pic}(\Theta_K) = \text{Pic}(\Theta_{K_1}) \times \ldots \times \text{Pic}(\Theta_{K_r}).$$

There are well-known algorithms to compute each $\text{Pic}(\Theta_{K_i})$, see [Ste08]. Note that if $S$ is an over-order of $R$, then the extension map $I \to 1S$ induces a surjective group homomorphism $\text{Pic}(R) \to \text{Pic}(S)$, see for example [DTZ62, Corollary 2.1.11]. In particular, if $S = \Theta_K$ we have an exact sequence

$$0 \to R^\times \to \Theta_K^\times \to \frac{(\Theta_K/\jmath)^\times}{(R/\jmath)^\times} \to \text{Pic}(R) \to \text{Pic}(\Theta_K) \to 0, \tag{2}$$

where $\jmath$ is the conductor of $R$, defined as the quotient ideal $(\Theta_K : R)$. The exactness of (2) is classical for the case when $r = 1$, that is, when $K$ is a number field. The case $r > 1$ has been proved only recently in [JP16]. The results contained in [KP05] describe how to compute the mid-term of (2) in the case $r = 1$ and they can be extended word-by-word to the general case. Since $\Theta_K^\times = \prod_i \Theta_{K_i}^\times$ and there are well known algorithms to compute each $\Theta_{K_i}^\times$, we deduce that we can effectively compute $\text{Pic}(R)$ and $R^\times$.

### 4 Weak equivalence classes

The following result was proved in [DTZ62] in the particular case of an integral domain.

**Proposition 4.1.** Let $I$ and $J$ be two fractional $R$-ideals. The following are equivalent:

(a) $I_p$ and $J_p$ are isomorphic for every prime $p$ of $R$;

(b) $1 \in (I : J)(J : I)$;

(c) $I$ and $J$ have the same multiplicator ring, say $S$, and there exists an ideal $L$ invertible in $S$ such that $I = JL$.

**Proof.** (a)⇒(b) Let $p$ be a prime of $R$. By Corollary 3.3 there exists a non-zero divisor $x$ in the total quotient ring of $R_p$ such that $I_p = xJ_p$, which in turn implies that

$$(I_p : J_p) = (xJ_p : J_p) = x(J_p : J_p) = \frac{1}{x}(J_p : J_p).$$

Theorem therefore

$$((I : J)(J : I))_p = (I_p : J_p)(J_p : I_p) = x(J_p : J_p)\frac{1}{x}(J_p : J_p) = (J_p : J_p),$$

which clearly contains $1$. Hence, the natural inclusion $(J : I)(I : J) \subseteq (J : J)$ is locally surjective at $p$. Since the choice of $p$ was arbitrary we conclude that $(J : I)(I : J) = (J : J)$ and in particular that $1 \in (J : I)(I : J)$.
By definition of quotient ideal we have that \((I : J)(J : I) \subseteq (I : I)\) and that \((I : J)(J : I) \subseteq (I : J)\). Since \((I : J)(J : I)\) has a structure of both \((I : I)\) and \((J : J)\)-module and contains 1, it follows that
\[
(I : I) = (I : J)(J : I) = (J : J),
\]
that is, \(I\) and \(J\) have the same multiplicator ring and \((I : J)\) and \((J : I)\) are inverse to each other. The following inclusions
\[
I = I(I : I) = I(I : J)(J : I) \subseteq J(I : J) \subseteq I
\]
are therefore equalities and in particular \(I = LJ\) for \(L = (I : J)\).

Let \(L'\) be any invertible ideal in \(R\) such that \(L'S = L\). Note that such an \(L'\) exists since the extension map \(\text{Pic}(R) \to \text{Pic}(S)\) is surjective, as we explain in Remark 3.8. The localization \(L'_p\) at any prime \(p\) of \(R\) is principal by Lemma 2.7, say \(L'_p = xR_p\). Then \(I_p = xJ_p\) and hence \(I_p \cong J_p\).

**Definition 4.2.** If two fractional \(R\)-ideals \(I\) and \(J\) satisfy the equivalent conditions of Proposition 4.1 we say that they are weakly equivalent. Denote by \(\mathcal{W}(R)\) the set of weak equivalence classes and by \([I]\) the weak equivalence class of a fractional \(R\)-ideal \(I\). Given any over-order \(S\) of \(R\) let \(\mathcal{W}(S)\) be the subset of \(\mathcal{W}(R)\) consisting of the weak equivalence classes \([I]\) such that \((I : I) = S\).

Note that \(\mathcal{W}(R)\) inherits the structure of a commutative monoid from \(\mathcal{S}(R)\). Consider the partition
\[
\mathcal{W}(R) = \bigsqcup \mathcal{W}(S),
\]
where the disjoint union is taken over all the over-orders \(S\) of \(R\). By Proposition 4.1(b) an ideal is invertible if and only if it is weakly equivalent to its multiplicator ring and hence we have that \(\mathcal{W}(S) = \{[S]\}\) if and only if \(S\) is Gorenstein.

**Remark 4.3.** Let \(p\) be a rational prime number and put \(R(p) = R \otimes \mathbb{Z}(p)\). Similarly, for fractional \(R\)-ideals \(I\) and \(J\), put \(I(p) := I \otimes \mathbb{Z}(p)\) and \(J(p) := J \otimes \mathbb{Z}(p)\). The ideals \(I\) and \(J\) are said to belong to the same genus if and only if \(I(p)\) and \(J(p)\) are isomorphic as \(R(p)\)-modules for every rational prime \(p\). Note that \(R(p)\) is a semi-local ring and hence by Lemma 2.7 fractional ideals are invertible if and only if they are principal and generated by a non-zero divisor. An easy modification of the proof of Proposition 4.1 shows that \(I\) and \(J\) are weakly equivalent if and only if they belong to the same genus. This equivalence was already noticed in [LW85, Section 5]. The notion of genus is classical and it has been widely studied in the literature, see for example [Rei03, Section 7] and [Rei70, Section 6]. For example, it is known that \(I\) and \(J\) are in the same genus if and only if they are isomorphic after tensoring with the \(p\)-adic completion \(\mathbb{Z}_p\), which in turn holds if and only if the quotients \(I/p^kI\) and \(J/p^kJ\) are isomorphic for an integer \(k\), that only depends on \(R(p)\). We prefer to work with the notion of weak equivalence introduced above, since part (b) of Proposition 4.1 implies that checking whether two ideals are weakly equivalent can be performed in polynomial time.

**Corollary 4.4.** Let \(I\) and \(J\) be fractional ideals. Then
\[
[I] = [J] \iff [I'] = [J']
\]
and
\[
[I] = [J] \iff [I'] = [J'].
\]
Proof. Note that $I = xJ$ if only if $I^t = (1/x)J^t$, which gives the first equivalence. The second equivalence follows from the equality $(I : J)(J : I) = (J^t : I^t)(I^t : J^t)$ and part (b) of Proposition 4.1.

Note that two fractional ideals $I$ and $J$ which are invertible in $R$ are isomorphic if and only if $IJ^{-1}$ is principal. Since being weakly equivalent is a necessary condition for being isomorphic, we can also reduce the isomorphism problem between non-invertible ideals to a principal ideal problem.

**Corollary 4.5.** Let $I$ and $J$ be two weakly equivalent fractional $R$-ideals, and let $S$ be their multiplicator ring. Then

$$I = (I : J)J,$$

and $(I : J)$ is a fractional ideal invertible in $S$. In particular, $I \simeq J$ if and only if $(I : J)$ is a principal fractional $S$-ideal.

Proof. Let $S$ be the multiplicator ring of $I$ and $J$. If $I$ and $J$ are weakly equivalent, we show in the proof of Proposition 4.1 that $(I : J)$ is invertible in $S$ and $I = (I : J)J$. In particular, if $(I : J)$ is principal, then $I$ and $J$ are isomorphic.

Conversely, if $I = xJ$ for some $x \in K^*$ then

$$(I : J) = (xJ : J) = x(J : J) = xS,$$

which concludes the proof.

Finally, knowing the weak equivalence classes allows us to reconstruct the isomorphism classes. Let $S$ be an over-order of $R$ and define

$$\text{ICM}(S) = \{(I) \in \text{ICM}(R) \text{ s.t. } (I : I) = S\},$$

so that we get

$$\text{ICM}(R) = \bigsqcup \text{ICM}(S),$$

where the disjoint union is taken over all the over-orders $S$ of $R$.

**Theorem 4.6.** Let $R$ be an order in $K$. For every over-order $S$ of $R$, the action of $\text{Pic}(S)$ on $\text{ICM}(S)$ induced by ideal multiplication is free and

$$\overline{\text{ICM}}(S) = \text{ICM}(S)/\text{Pic}(S).$$

More concretely, if

$$\overline{\text{ICM}}(S) = \{[I_1], \ldots, [I_r]\} \quad \text{and} \quad \text{Pic}(S) = \{[J_1], \ldots, [J_s]\},$$

with the $I_i$’s pairwise not weakly equivalent and the $J_j$’s pairwise not isomorphic then

$$\text{ICM}(S) = \{[I_iJ_j] : 1 \leq i \leq r, 1 \leq j \leq s\}$$

and the fractional ideals $I_iJ_j$ are pairwise not isomorphic.

Proof. Let $I$ be a fractional $R$-ideal with multiplicator ring $S$. Then $[I] = [I_i]$ for some $i$, that is, there exists a fractional ideal $I$ invertible in $S$ such that $I = I_iJ$. Let $j$ be the index such that $[J] = [J_i]$. It follows that $[I] = [I_iJ_i]$. It remains to prove that if $[I_iJ_j] = [I_kJ_h]$, that is $I_iJ_j = xI_kJ_h$ for some $x \in K^*$, then $i = k$ and $j = h$. Multiplying by $(S : J)$ on both sides we
get by Proposition 4.1(c) that \( I_i \) is weakly equivalent to \( I_k \), that is, \( i = k \). To conclude, it is enough to prove that if \( I = I/J \) with \( J \) and \( I \) both having multiplicator ring \( S \) and \( I \) invertible in \( S \), then \( J = S \). We will prove that this is true locally at every prime of \( S \). Since \( J \) is invertible, we have by Lemma 2.7 that \( f_p = yS_p \) for some non-zero divisor \( y \). Hence it follows that
\[
I_p = yS_pI_p = yI_p
\]
which implies that both \( y \) and \( 1/y \) are in \( (I_p : I_p) = S_p \). Therefore we again have \( I_p = S_p \). \( \square \)

**Remark 4.7.** Fixing the multiplicator ring is a key point in using the previous proposition. Let \( R = \mathbb{Z}[\alpha] \), where \( \alpha \) is a root of \( f(x) = x^2 - 8x - 8 \). Note that \( \mathcal{O}_K = \mathbb{Z}[\alpha/2] \) and \( |\mathcal{O}_K : R| = 2 \). Consider the invertible \( R \)-ideal \( p = (5, \alpha) \) and the conductor \( \mathfrak{f} = (2, \alpha) \) of \( R \). It is easy to verify that \( \text{Pic}(\mathcal{O}_K) \) is trivial, while \( \text{Pic}(R) = \mathbb{Z}/2\mathbb{Z} \) with generator the ideal class of \( p \). It follows that the product of \( \mathfrak{f}p = \mathfrak{f} \) and, in particular, that the action of \( \text{Pic}(R) \) on the whole \( \text{ICM}(R) \) is not free.

Using Theorem 4.6 we can compute the ideal class monoid of an order \( R \) if we know all its over-orders, their Picard groups and the weak equivalence class monoid. For the first issue, by Lemma 2.2 it is enough to look at the idempotent \( R \)-modules of the finite quotient \( \mathcal{O}_K/R \). In the end of Section 3 we discussed how to compute the Picard group of a possibly non-maximal order. Finally, in the next section we will describe how to compute \( \mathcal{W}(R) \). See Section 6 for the corresponding algorithms.

## 5 Computing the weak equivalence class monoid

The following results are inspired by DTZ62, where the authors produce similar results in the particular case of an integral domain. Let \( R \) be an order in \( K \). Recall that we can partition \( \mathcal{W}(R) \) as the disjoint union of \( \mathcal{W}(S) \) where \( S \) runs through the set of over-orders of \( R \). We will now describe a method to compute \( \mathcal{W}(S) \). Observe that when \( S \) is not Gorenstein there are always at least two distinct classes in \( \mathcal{W}(S) \), namely \(|S| \) and \(|S'| \).

**Proposition 5.1.** Let \( T \) be any over-order of \( S \) such that \( S'T \) is invertible in \( T \). Let \( \mathfrak{f} \) be an ideal contained in \( S \) such that \( \mathfrak{f} \subseteq (I : \mathfrak{f}) \). Then every class in \( \mathcal{W}(S) \) has a representative \( I \) satisfying \( \mathfrak{f} \subseteq I \subseteq T \).

**Proof.** Let \( I' \) be any fractional ideal with \((I' : I') = S \). By Lemma 2.3, we have that \( I'(I')^T = S'I \) and hence it follows that \( I'T \) is invertible in \( T \). Let \( J \) be a representative of the pre-image under the surjective map \( \text{Pic}(S) \to \text{Pic}(T) \) of the class of \( (I : I'T) \) and put \( I = I'J \). Note that \( |I'| = |I| \) in \( \mathcal{W}(S) \) and that \( IT = T \), which implies that \( I \subseteq T \).

On the other hand, as \( \mathfrak{f}T = \mathfrak{f} \) we get that
\[
\mathfrak{f}I = \mathfrak{f}TI = \mathfrak{f}T = \mathfrak{f},
\]
and, since \( \mathfrak{f} \subseteq (I : I) \), we obtain that \( \mathfrak{f} = \mathfrak{f}I \subseteq I \), and we can conclude that \( \mathfrak{f} \subseteq I \subseteq T \). \( \square \)

The previous proposition tells us that in order to compute the representatives of \( \mathcal{W}(S) \) we can look at the sub-\( S \)-modules of the finite quotient \( T/\mathfrak{f} \). One possible choice is to take \( T = \mathcal{O}_K \) and \( \mathfrak{f} = (S : \mathcal{O}_K) \), but to gain in efficiency we want to keep the quotient as small as possible. The natural choice is to take \( T \) the smallest over-order of \( S \) with \( S'T \) invertible in \( T \) and as \( \mathfrak{f} \), the colon ideal \((S : T)\), which is the biggest fractional \( T \)-ideal in \( S \).
Remark 5.2. Given orders $S \subseteq T$, let $\mathfrak{f} = (S : T)$. If $S$ is Gorenstein then by Lemma 2.1 and Proposition 2.10 it follows that

$$(\mathfrak{f} : \mathfrak{f}) = (S : T(S : T)) = (S : (S : T)) = T.$$  

If $S$ is not Gorenstein then the multiplicator ring of $\mathfrak{f}$ might still be equal to $T$. This for example must be the case when $T = \mathfrak{o}_K$. The multiplicator ring of $\mathfrak{f}$ can also be strictly bigger than $T$, as Example 5.3 shows. If we assume that $S^i T$ is invertible in $T$, as required in Proposition 5.1, then $(\mathfrak{f} : \mathfrak{f}) = T$, because $\mathfrak{f} = (S^i T)^i$ and each ideal has the same multiplicator ring as its trace dual.

Example 5.3. Put $f = x^6 - 4x^5 + 11x^4 - 24x^3 + 55x^2 - 100x + 125$ and $K = \mathbb{Q}(x)/(f)$. Denote by $a$ the image of $x$ in $K$. Consider the orders

$$S = \mathbb{Z} \oplus a\mathbb{Z} \oplus 2a^2\mathbb{Z} \oplus \left(\frac{1}{2}a^2 + \frac{1}{2}a^3\right)\mathbb{Z} \oplus (a^2 + a^4)\mathbb{Z} \oplus (a^2 + a^5)\mathbb{Z}$$

and

$$T = \mathbb{Z} \oplus a\mathbb{Z} \oplus a^2\mathbb{Z} \oplus \left(\frac{1}{2}a^2 + \frac{1}{2}a^3\right)\mathbb{Z} \oplus a^4\mathbb{Z} \oplus a^5\mathbb{Z}.$$

Then $S \subseteq T$ with index 2 and the multiplicator ring of $\mathfrak{f} = (S : T)$ is the maximal order

$$\mathfrak{o}_K = \mathbb{Z} \oplus a\mathbb{Z} \oplus a^2\mathbb{Z} \oplus \left(\frac{1}{2}a^2 + \frac{1}{2}a^3\right)\mathbb{Z} \oplus a^4\mathbb{Z} \oplus \left(\frac{1}{2}a^4 + \frac{1}{2}a^3\right)\mathbb{Z}$$

and it is easy to check that $|\mathfrak{o}_K : T| = 2$.

Remark 5.4. Let $I'$ be a fractional $R$-ideal. As in the proof of Proposition 5.1, let $J$ be a representative of the pre-image under the extension map $\text{Pic}(R) \to \text{Pic}(\mathfrak{o}_K)$ of $|(\mathfrak{o}_K : I'\mathfrak{o}_K)|$ and put $I = I'J$. Then $|J| = |I'|$ and $I\mathfrak{o}_K = \mathfrak{o}_K$ which implies

$$\mathfrak{f} \subseteq I \subseteq \mathfrak{o}_K,$$

where $\mathfrak{f}$ is the conductor of $R$, that is $\mathfrak{f} = (R : \mathfrak{o}_K)$. So if the quotient $\mathfrak{o}_K/\mathfrak{f}$ is not too big we can look directly at its sub-$R$-modules in order to get all representatives of the classes of $W(R)$. One can also obtain all the over-orders of $R$ by computing the multiplicator rings of the representatives of $W(R)$.

Let $T$ be an over-order of $S$ such that $S^i T$ is an invertible fractional $T$-ideal. Choose primes $p_1, \ldots, p_r$ of $S$ and positive integers $e_1, \ldots, e_r$ such that $T \subseteq (\mathfrak{f} : \mathfrak{f})$, where

$$\mathfrak{f} = p_1^{e_1} \cdots p_r^{e_r}.$$  

Note that such $\mathfrak{f}$ satisfies the hypothesis of Proposition 5.1 and, moreover, $\mathfrak{f} \subseteq (S : T)$, since $(S : T)$ contains all fractional $T$-ideals contained in $S$. It follows that the primes $p_i$ must be non-invertible.

By the Chinese Remainder Theorem there is a ring isomorphism

$$\frac{S}{\mathfrak{f}} \cong \frac{S}{p_1^{e_1}} \times \cdots \times \frac{S}{p_r^{e_r}},$$  

which, after taking the tensor product with $T$, becomes

$$\frac{T}{\mathfrak{f}} \cong \frac{T}{p_1^{e_1} T} \times \cdots \times \frac{T}{p_r^{e_r} T}.$$  

(3)
Observe that the isomorphism (3) is compatible with ideal multiplication and hence it respects weak equivalences. In particular, we can compute \( \mathcal{W}(S_p) \) by looking at the sub-\( S \)-modules of the "local" quotient \( T/p_i^e_i T \) up to weak equivalence. Then we can "patch" them together via the isomorphism (3) and hence reconstruct all the representatives of \( \mathcal{W}(S_p) \). If \( r > 1 \) this tells us that we can split the computation of \( \mathcal{W}(S_p) \) and hence potentially obtain a more efficient algorithm. The next two remarks will tell us that we can further improve the algorithm by ignoring or reducing some factors in (3) if the corresponding primes \( p_i \) satisfy certain conditions.

**Remark 5.5.** Let \( p_i \) be one the primes appearing in (3). If the \( S/p_i \)-vector space \( S^I/p_i^e_i S^I \) is one-dimensional, then by Nakayama’s Lemma we have that \( S^I \) is locally principal at \( p_i \). It follows that each fractional ideal \( I \) with multiplicator ring \( S \), that is \( II' = S^I \), will be locally invertible at \( p_i \), or, in other words, \( \mathcal{W}(S_p) \) is trivial.

**Remark 5.6.** Let \( p \) be one of the primes appearing in the decomposition (3). Observe that \( T_p \) has only finitely many primes \( P_1, \ldots, P_m \), which are exactly the ones lying above \( p \). Assume that \( m < q \), where \( q = \#(S/p) \). Then by [DCD00, Lemma 4] for each ideal \( I \) of \( S_p \) such that \( IT_p \) is invertible there exists \( x \in I \) such that \( IT_p = xT_p \). This implies that

\[
S_p \subseteq \frac{1}{x} I \subseteq \frac{1}{x} IT_p = T_p.
\]

This means that, if we also assume that \( S^I T \) is invertible in \( T \), we can find all the classes of \( \mathcal{W}(S_p) \) in the quotient \( T_p/S_p \) and this quotient might be smaller than \( T/p_i^e_i T \).

### 6 Algorithms

In this section we present the pseudo-code for the algorithms presented in the previous sections. The implementation in Magma [BCP97] is available on the author’s webpage. We will use without mentioning a lot of algorithms for abelian groups, which can all be found in [Coh93, Section 2.4].

**Algorithm 1:** Computing over-orders of a given order

**Input:** An order \( R \) in a \( \mathbb{Q} \)-étale algebra \( K \);
**Output:** A list \( \mathcal{L}^o \) containing the over-orders of \( R \);

- Compute the maximal order \( \mathcal{O}_K \) of \( K \);
- Compute the quotient as abelian groups \( q : \mathcal{O}_K \to Q := \mathcal{O}_K/R \);
- Initialize an empty list \( \mathcal{L}^o \);

for each \( H' \subseteq Q \) do

- Put \( S := \left\langle q^{-1}(H') \right\rangle \);
- if \( SS = S \) and \( S \not\in \mathcal{L}^o \) then
  - Append \( S \) to \( \mathcal{L}^o \);
end

return \( \mathcal{L}^o \);

**Theorem 6.1.** Algorithm 1 is correct.

**Proof.** This follows from the fact that the over-orders of \( R \) are precisely the idempotent fractional \( R \)-ideals contained in \( \mathcal{O}_K \) and containing \( R \), as shown in Lemma 2.2. \qed
Algorithm 2: Computing representatives of the classes in $\mathcal{W}(S)$ for an order $S$

**Input:** An order $S$ in a $\mathbb{Q}$-étale algebra $K$;

**Output:** A list $\mathcal{L}^w$ of the representatives of the weak equivalence classes of ideals with endomorphism ring $S$, that is $\mathcal{W}(S)$;

Compute the trace dual ideal $S^t$;

Initialize an empty list $\mathcal{L}^w$;

**if** $1 \in S^t(S : S^t)$ **then**

- Append $S$ to $\mathcal{L}^w$;

**else**

- Using Algorithm 1 find an over-order $T$ of $S$ such that $1 \in S^t(T : S^t T)$;

- Put $\mathfrak{f} := (S : T)$;

- Consider the quotient $q := T \rightarrow T/\mathfrak{f}$;

- **for each** $H' \subseteq Q$ **do**

  - Put $I := \langle q^{-1}(H') \rangle_S$;

  - **if** there is no $J \in \mathcal{L}^w$ such that $1 \in (I : J)(J : I)$ **then**

    - Append $I$ to $\mathcal{L}^w$;

- end

end

**return** $\mathcal{L}^w$;

---

Theorem 6.2. Algorithm 2 is correct.

**Proof.** The correctness of the algorithm follows from Propositions 4.1 and 5.1.

---

Algorithm 3: Computing representatives of the classes in $\text{ICM}(R)$ for an order $R$

**Input:** An order $R$ in a $\mathbb{Q}$-étale algebra $K$;

**Output:** A list $\mathcal{L}^{iso}$ of the representatives of the isomorphism classes of ideals, that is $\text{ICM}(R)$;

Compute the over-orders $\mathcal{L}^o$ of $R$ using Algorithm 1;

Initialize the empty list $\mathcal{L}^{iso}$;

**for each** $S$ in $\mathcal{L}^o$ **do**

- Compute a list $\mathcal{L}^w$ of representatives of $\mathcal{W}(S)$ using Algorithm 2;

- Compute a list $\mathcal{L}^i$ of representatives of $\text{Pic}(S)$;

  - **for each** $I$ in $\mathcal{L}^w$ and each $J$ in $\mathcal{L}^i$ **do**

    - Append $IJ$ to $\mathcal{L}^{iso}$;

  end

end

**return** $\mathcal{L}^{iso}$;

---

Theorem 6.3. Algorithm 3 is correct.

**Proof.** This follows from Theorem 4.6.
7 Examples

The example contained in this section were computed with Magma [BCP97]. The code can be found on the webpage of the author.

Example 7.1. Let \( f = x^3 + 31x^2 + 43x + 77 \) and let \( \alpha \) be a root of \( f \). Consider the monogenic order defined by \( f \), say \( E = S_1 = \mathbb{Z}[\alpha] \). There are 15 over-orders of \( E \). The maximal order is \( \mathcal{O} = S_{15} = \mathbb{Z} \oplus \frac{\alpha + 5}{8} \mathbb{Z} \oplus \frac{\alpha^2 + 2\alpha + 49}{64} \mathbb{Z} \). Observe that \( [\mathcal{O} : E] = 512 \), so the only singular prime is 2. In Figure 1 and Table 1 we describe the over-orders with the weak equivalence classes and Picard groups. It can be verified that the orders \( S_2, S_3, S_5, S_7, S_8, S_9, S_{10}, S_{14} \) are precisely the non-Gorenstein over-orders of \( S_1 \) and that there are no other non-invertible weak equivalence classes apart from \( S_{11} \). This implies that \( \#W(E) = 23 \) and, using the information about the Picard groups, we can deduce that \( \#ICM(E) = 59 \).

![Figure 1: The lattice of inclusions of the over-orders with indexes and number of weak equivalence classes from Example 7.1](image)

| i | \( \mathbb{Z} \)-basis of \( S_i \) | \( [\mathcal{O} : S_i] \) | Pic(\( S_i \)) |
|---|---|---|---|
| 1 | 1, \( \alpha \), \( \alpha^2 \) | 512 | \( \mathbb{Z}/4\mathbb{Z} \) |
| 2 | 1, \( \alpha \), \( \alpha + 1 \) | 256 | \( \mathbb{Z}/4\mathbb{Z} \) |
| 3 | 1, \( \alpha \), \( \alpha^2 + 2\alpha + 1 \) | 128 | \( \mathbb{Z}/4\mathbb{Z} \) |
| 4 | 1, \( \alpha \), \( \alpha^2 + 6\alpha + 5 \) | 64 | \( \mathbb{Z}/2\mathbb{Z} \) |
| 5 | 1, \( \alpha \), \( \alpha^2 + 2\alpha + 1 \) | 64 | \( \mathbb{Z}/4\mathbb{Z} \) |
| 6 | 1, \( \frac{\alpha + 1}{2} \), \( \frac{\alpha^2 + \alpha + 1}{4} \) | 64 | \( \mathbb{Z}/2\mathbb{Z} \) |
| 7 | 1, \( \frac{\alpha + 1}{2} \), \( \frac{\alpha^2 + 2\alpha + 1}{4} \) | 32 | \( \mathbb{Z}/2\mathbb{Z} \) |
| 8 | 1, \( \alpha \), \( \frac{\alpha^2 + 10\alpha + 9}{16} \) | 32 | \( \mathbb{Z}/2\mathbb{Z} \) |
| 9 | 1, \( \alpha \), \( \frac{\alpha^2 + 2\alpha + 1}{16} \) | 32 | \( \mathbb{Z}/4\mathbb{Z} \) |
| 10 | 1, \( \frac{\alpha + 1}{2} \), \( \frac{\alpha^2 + 2\alpha + 1}{16} \) | 16 | \( \mathbb{Z}/2\mathbb{Z} \) |
| 11 | 1, \( \frac{\alpha + 1}{2} \), \( \frac{\alpha^2 + 2\alpha + 17}{32} \) | 8 | 1 |
| 12 | 1, \( \frac{\alpha + 1}{2} \), \( \frac{\alpha^2 + 10\alpha + 25}{32} \) | 8 | 1 |
| 13 | 1, \( \frac{\alpha + 1}{2} \), \( \frac{\alpha^2 + 2\alpha + 17}{16} \) | 8 | \( \mathbb{Z}/2\mathbb{Z} \) |
| 14 | 1, \( \frac{\alpha + 1}{2} \), \( \frac{\alpha^2 + 2\alpha + 49}{64} \) | 4 | 1 |
| 15 | 1, \( \alpha + 5 \), \( \frac{\alpha^2 + 2\alpha + 49}{64} \) | 1 | 1 |
Example 7.2. Let \( f = x^3 - 1000x^2 - 1000x - 1000 \) and let \( \alpha \) be a root of \( f \). Consider the monogenic order defined by \( f \), say \( E = S_1 = \mathbb{Z}[\alpha] \). There are 16 over-orders of \( E \). The maximal order is \( \Theta = S_{16} = \mathbb{Z} + \alpha \mathbb{Z} + \alpha^2 \mathbb{Z} \). Observe that \( |\Theta : E| = 1000 \), so the singular primes are 2 and 5. In Figure 2 and Table 2 we describe the over-orders with the weak equivalence classes and Picard groups.

![Figure 2: The lattice of inclusions of the over-orders with indexes and number of weak equivalence classes from Example 7.2.](image)

| i  | \( \mathbb{Z} \)-basis of \( S_i \) | \( |\Theta : S_i| \) | Pic(\( S_i \)) |
|----|----------------------------------|----------------|----------------|
| 1  | \( 1, \alpha, \alpha^2 \)       | 1000           | \( \mathbb{Z}/2 \mathbb{Z} \times \mathbb{Z}/1000 \mathbb{Z} \) |
| 2  | \( 1, \alpha, \frac{\alpha^4}{7} \) | 500            | \( \mathbb{Z}/1000 \mathbb{Z} \) |
| 3  | \( 1, \alpha, \frac{\alpha^2 + 2\alpha}{4} \) | 250            | \( \mathbb{Z}/8880 \mathbb{Z} \) |
| 4  | \( 1, \alpha, \frac{\alpha^3}{5} \) | 200            | \( \mathbb{Z}/8880 \mathbb{Z} \) |
| 5  | \( 1, \frac{\alpha}{7}, \frac{\alpha^2}{4} \) | 125            | \( \mathbb{Z}/1776 \mathbb{Z} \) |
| 6  | \( 1, \alpha, \frac{\alpha^4}{10} \) | 100            | \( \mathbb{Z}/1776 \mathbb{Z} \) |
| 7  | \( 1, \alpha, \frac{\alpha^2 + 10\alpha}{20} \) | 70             | \( \mathbb{Z}/1776 \mathbb{Z} \) |
| 8  | \( 1, \alpha, \frac{\alpha^3 + 10\alpha}{25} \) | 4              | \( \mathbb{Z}/1776 \mathbb{Z} \) |
| 9  | \( 1, \frac{\alpha}{7}, \frac{\alpha^2}{10} \) | 25             | \( \mathbb{Z}/1776 \mathbb{Z} \) |
| 10 | \( 1, \alpha, \frac{\alpha^2 + 10\alpha}{100} \) | 20             | \( \mathbb{Z}/444 \mathbb{Z} \) |
| 11 | \( 1, \alpha, \frac{\alpha^3 + 10\alpha}{100} \) | 10             | \( \mathbb{Z}/444 \mathbb{Z} \) |
| 12 | \( 1, \frac{\alpha}{5}, \frac{\alpha^2}{25} \) | 8              | \( \mathbb{Z}/1776 \mathbb{Z} \) |
| 13 | \( 1, \frac{\alpha}{7}, \frac{\alpha^2 + 10\alpha}{100} \) | 5              | \( \mathbb{Z}/1776 \mathbb{Z} \) |
| 14 | \( 1, \frac{\alpha}{5}, \frac{\alpha^2}{50} \) | 4              | \( \mathbb{Z}/1776 \mathbb{Z} \) |
| 15 | \( 1, \frac{\alpha}{3}, \frac{\alpha^2 + 10\alpha}{100} \) | 2              | \( \mathbb{Z}/1776 \mathbb{Z} \) |
| 16 | \( 1, \frac{\alpha}{10}, \frac{\alpha^2}{100} \) | 1              | \( \mathbb{Z}/1776 \mathbb{Z} \) |

It can be verified that \( S_2, S_4, S_5, S_7, S_9, S_{10}, S_{14} \) are precisely the non-Gorenstein over-orders of \( S_1 \), so we also have the weak equivalence classes corresponding to \( S_i \), for \( i = 2, 4, 6, 7, 9, 10, 14 \). But unlike the previous example there are two other weak equivalence classes, represented by the ideals \( I = 50\mathbb{Z} + 10\alpha \mathbb{Z} + 5\alpha^2 \mathbb{Z} \) and \( J = 20\mathbb{Z} + 10\alpha \mathbb{Z} + 2\alpha^2 \mathbb{Z} \), both with multiplicative ring \( S_6 \). This means that \( \# \mathcal{W}(E) = 25 \) and using the information about the Picard groups of the over-orders we can deduce that \( \# \text{ICM}(E) = 69116 \).

Example 7.3. Consider the following irreducible polynomials \( f_1 = x^3 + 4x + 7 \) and \( f_2 = x^3 - 9x^2 - 3x - 1 \) and defined \( f = f_1 f_2 \). Put \( K_1 = \mathbb{Q}[x] / (f_1) \), \( K_2 = \mathbb{Q}[x] / (f_2) \) and \( K = \mathbb{Q}[x] / (f) \) \( \cong K_1 \times K_2 \). For \( i = 1, 2 \) denote by \( R_i \) the monogenic order \( \mathbb{Z}[x] / (f_i) \) and by \( \Theta_{K_i} \) the maximal order in \( K_i \). It is easy to verify that \( |\Theta_{K_i} : R_i| = 2 \) for both \( i = 1 \) and \( i = 2 \). Therefore for both \( i = 1 \) and \( i = 2 \) the only over-order of \( R_i \) is \( \Theta_{K_i} \) and hence \( R_i \) is a Bass order. In particular, it follows that

\[
\text{ICM}(R_i) = \text{Pic}(R_i) \cup \text{Pic}(\Theta_{K_i}).
\]
We can check that \( \text{Pic}(R_1) \) is trivial and hence \( \mathcal{O}_K \) is a principal ideal domain, since the extension map \( \mathcal{O}_K \rightarrow \mathcal{O}_K \) induces a surjective group homomorphism from \( \text{Pic}(R) \) to \( \text{Pic}(\mathcal{O}_K) \), see Remark 3.6.

\[
\text{ICM}(R_1) = \{ \{R_1\}, \{\mathcal{O}_K\} \}.
\]

On the other hand, \( \text{Pic}(R_2) \) and \( \text{Pic}(\mathcal{O}_{K_2}) \) are both isomorphic to \( \mathbb{Z}/3\mathbb{Z} \) and generated respectively by \( I = (69R_2 + (28 + a_2 + a_2^3)R_2) \) and \( J = 1\mathcal{O}_{K_2} \), where \( a_2 \) is a primitive root of \( K_2 \). It follows that

\[
\text{ICM}(R_2) = \{ \{R_2\}, \{I\}, \{I^2\}, \{\mathcal{O}_{K_2}\}, \{J\}, \{J^2\} \}.
\]

The situation for \( R \) is much more complicated as it easily seen from Figure 3.

**Figure 3:** The lattice of inclusions of the over-orders with indexes and number of weak equivalence classes describe in Example 7.3.

First of all \( R \) is not a product of orders of \( K_1 \) and \( K_2 \) and it can be computed that the index of \( R \) in the maximal order \( \mathcal{O}_{K_1} \times \mathcal{O}_{K_2} \) of \( K \) is 21312 = \( 2^3 \cdot 3^2 \cdot 37 \). We computed that \( R \) has 84 over-orders of which only 48 are Gorenstein and hence \( R \) is not Bass. We run our algorithm for the ideal class monoid and we found that \#ICM(\( R \)) = 852.

8 Conjugacy classes of integral matrices

Recall that two \( N \times N \) matrices \( A \) and \( B \) with entries in \( \mathbb{Z} \) are conjugate if there exists \( O \in GL_N(\mathbb{Z}) \) such that \( OAO^{-1} = B \). If this is the case, then \( A \) and \( B \) have the same minimal polynomial \( m \) and the same characteristic polynomial \( c \). The converse is not true in general. We will write \( \{A\}_- \) for the conjugacy class of the matrix \( A \). In what follows we will describe how to compute the representatives of the conjugacy classes when the minimal polynomial is square-free. Our result is a generalization of [LM33], where the authors treat the case \( m = c \), which was then re-proved with a different method in [Tat49], with the extra assumption that...
m = c is irreducible. Note that Theorem 8.1 has independently been proved in [Hus17] in more generality.

Let \( f_1, \ldots, f_r \) be a collection of distinct irreducible monic polynomials with integer coefficients and let \( e_1, \ldots, e_r \) be positive integers such that \( m = \prod f_i, \; c = \prod f_i^{e_i} \). Put \( N = \deg(c) \) and denote by \( \mathcal{M}_{m,c}(\mathbb{Z}) \) the set of integral \( N \times N \) matrices with minimal and characteristic polynomials \( m \) and \( c \), respectively.

For every \( i = 1, \ldots, r \) put \( K_i = \mathbb{Q}[x]/(f_i) \) and let \( \Delta_i \) be the diagonal embedding of \( K_i \) into \( K_i^{e_i} \). Define \( \Delta \) as the product map \( \prod_i \Delta_i \) with codomain \( K = \prod_i K_i^{e_i} \). Observe that the order \( R_0 = \mathbb{Z}[x]/(m) \) has total quotient ring the \( \mathbb{Q} \)-algebra \( \prod_i K_i \). Denote with \( R \) the image of \( R_0 \) in \( K \) via \( \Delta \) and put \( \alpha = \Delta(x \mod(m)) \). Let \( \mathcal{L}(R, K) \) be the set of full lattices in \( K \) which are \( R \)-modules and pick \( I \) in \( \mathcal{L}(R, K) \). Fix a \( \mathbb{Z} \)-basis \( \overline{w} = \{ w_1, \ldots, w_n \} \) of \( I \). The \( R \)-linear endomorphism of \( I \) given by multiplication by \( \alpha \) can be represented with respect to \( \overline{w} \) by an integral matrix \( A = A(I, \overline{w}) \) which lies in \( \mathcal{M}_{m,c}(\mathbb{Z}) \). Clearly this representation depends on the choice of the \( \mathbb{Z} \)-basis of \( I \). If we change the \( \mathbb{Z} \)-basis of \( I \) by a matrix \( O \in GL_N(\mathbb{Z}) \) then the multiplication by \( \alpha \) will be represented by \( O^{-1} AO \). Hence we have a well defined map \( I \to [A]_{-\Delta} \).

**Theorem 8.1.** The association \( \Phi : I \to [A(I, \overline{w})]_{-\Delta} \) induces a bijection

\[
\Phi : \mathcal{L}(R, K) \to \mathcal{M}_{m,c}(\mathbb{Z}) / \sim \mathbb{Z} \\
\{I\} \to [A(I, \overline{w})]_{-\Delta}
\]

**Proof.** First we prove that the map \( \Phi \) is well defined, that is that if \( \varphi : I \to J \) is an \( R \)-linear isomorphism then \( \Phi(I) = \Phi(J) \). Let \( \overline{w} \) be a \( \mathbb{Z} \)-basis of \( I \) and \( \varphi(\overline{w}) \) the induced \( \mathbb{Z} \)-basis of \( J \). Since \( \varphi \) is \( R \)-linear, we have that \( A(I, \overline{w}) = A(J, \varphi(\overline{w})) \), which implies that \( \Phi(I) = \Phi(J) \).

We now prove that \( \Phi \) is injective. Let \( I \) and \( J \) be in \( \mathcal{L}(R, K) \) and fix the \( \mathbb{Z} \)-basis, say

\[
I = w_1 \mathbb{Z} \oplus \ldots \oplus w_N \mathbb{Z}
\]

and

\[
J = v_1 \mathbb{Z} \oplus \ldots \oplus v_N \mathbb{Z}.
\]

Assume that \( \Phi(I) = \Phi(J) \), that is \( A(I, \overline{w}) = O^{-1} A(J, \overline{v}) O \) for some \( O \in GL_N(\mathbb{Z}) \). By acting with \( O^{-1} \) on \( \overline{w} \) we find a new \( \mathbb{Z} \)-basis \( \overline{v} \) for \( J \) such that \( A(I, \overline{w}) = A(J, \overline{v}) \). Now the \( \mathbb{Z} \)-linear bijection \( I \to J \) defined by \( w_i \to v'_i \) commutes with multiplication by \( \alpha \), since the matrices representing the operation with respect to \( \overline{w} \) and \( \overline{v} \) are the same, and hence it is an \( R \)-linear isomorphism. Therefore \( (I) = (J) \) and \( \Phi \) is injective.

To conclude we need to prove that \( \Phi \) is also surjective. We will do this by explicitly producing a map

\[
\Psi : \mathcal{M}_{m,c}(\mathbb{Z}) \to \mathcal{L}(R, K) / \sim_R
\]

which descends to a retraction \( \overline{\Psi} \) of \( \Phi \). Let \( A \) be a matrix in \( \mathcal{M}_{m,c}(\mathbb{Z}) \). Note that since \( m \) is square-free then \( A \) is semisimple. The complex eigenvalues of \( A \) are the roots of \( f_1, \ldots, f_r \) and, for each \( i \), all roots of \( f_i \) can be identified with the element \( x \mod f_i \) of \( K_i \), which we denote \( \alpha_i \). Note that

\[
\alpha = (\alpha_{1,1}, \ldots, \alpha_{1,e_1}, \ldots, \alpha_{r,1}, \ldots, \alpha_{r,e_r}).
\]

Let

\[
v_{i,1}, \ldots, v_{i,e_i} \in K_i^{N_i},
\]

be a basis of the eigenspace corresponding to the \( i \)-th eigenvalue, that is, linearly independent vectors such that

\[
A v_{i,j_i} = \alpha_i v_{i,j_i},
\]

17
for each \( i = 1, \ldots, r \) and \( j_i = 1, \ldots, e_i \). Let \( R = e_1 + \ldots + e_r \) and consider the \( R \times N \) matrix whose rows are the vectors \( v_{i,j_i} \), and denote by \( w_k \) the \( k \)-th column, for \( k = 1, \ldots, N \). Observe that each \( w_k \) is an element of \( K \) and define

\[
I = \langle w_1, \ldots, w_N \rangle \subset K.
\]

If \( A = (a_{h,k}) \) and \( v_{i,j_i} = (v^{(1)}_{i,j_i}, \ldots, v^{(N)}_{i,j_i}) \) then

\[
w_k = (v^{(k)}_{1,1}, \ldots, v^{(k)}_{1,e_1}, \ldots, v^{(k)}_{r,1}, \ldots, v^{(k)}_{r,e_r})
\]

and it follows that

\[
\alpha w_k = (\alpha_1 v^{(k)}_{1,1}, \ldots, \alpha_1 v^{(k)}_{1,e_1}, \ldots, \alpha_r v^{(k)}_{r,1}, \ldots, \alpha_r v^{(k)}_{r,e_r})
\]

\[
= (\sum_{h=1}^N a_{k,h} v^{(h)}_{1,1}, \ldots, \sum_{h=1}^N a_{k,h} v^{(h)}_{1,e_1}, \ldots, \sum_{h=1}^N a_{k,h} v^{(h)}_{r,1}, \ldots, \sum_{h=1}^N a_{k,h} v^{(h)}_{r,e_r})
\]

\[
= \sum_{h=1}^N a_{k,h} v_h \in I,
\]

which implies that \( I \) is closed under multiplication by \( \alpha \), and hence it is an \( R \)-module. Moreover, \( [5] \) means that the multiplication by \( \alpha \) is represented by the matrix \( A \) with respect to the generators \( w_1, \ldots, w_N \). We prove now that \( I \) is a full lattice, or equivalently that the \( \mathbb{Q} \)-vector space \( V = I \otimes_{\mathbb{Z}} \mathbb{Q} \) equals \( K \). Note that \( A \) represents the \( \mathbb{Q} \)-linear map induced by multiplication by \( \alpha \) on \( V \). Since \( A \) is semisimple there is a decomposition

\[
V = W_1 \oplus \ldots \oplus W_r
\]

into \( \mathbb{Q} \)-vector spaces which are stable under the action of \( \alpha \), and possibly after renumbering we can assume that \( A|_{W_i} \) has minimal polynomial \( f_i \) and hence that \( W_i \) is a \( K_i \)-vector space. For each \( i \), since the vectors \( v_{i,1}, \ldots, v_{i,e_i} \) are linearly independent over \( K_i \), we see that \( W_i \) must have dimension \( e_i \). This concludes the proof that \( I \in \mathcal{L}(R, K) \).

Observe that the construction of \( I \) depends on the choice of eigenvectors in \( [4] \). A different choice can be attained by the action of a block-diagonal matrix \( C \) in

\[
\begin{pmatrix}
\text{GL}_{e_1}(K_1) & 0 & \ldots & 0 \\
0 & \text{GL}_{e_2}(K_2) & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \text{GL}_{e_r}(K_r)
\end{pmatrix}
\]

Observe that \( C \) induces an \( R \)-linear automorphism of \( K \) and hence its action on \( I \) will also be an \( R \)-isomorphic object of \( \mathcal{L}(R, K) \). Hence we have a well defined map \( \Psi \) which associates \( A \rightarrow \{I\} \).

If instead of \( A \) we take a conjugate matrix \( B \), it will reflect as taking an invertible \( \mathbb{Z} \)-linear combination of the eigenvectors in \( [4] \). Clearly this will not change the \( \mathbb{Z} \)-span that they generate, that is, the lattice \( I \), and hence \( \Psi \) descents to a well-defined map

\[
\Psi : \mathcal{M}_{m,c}(\mathbb{Z}) \rightarrow \mathcal{L}(R, K) / \cong_R
\]

which by construction is a retraction of \( \Phi \). This implies that \( \Phi \) is surjective and concludes the proof.
In general, the set of \( R \)-isomorphism classes in \( \mathcal{L}(R, K) \) is hard to handle, but under certain assumptions we can reduce it to an ideal class monoid computation.

**Corollary 8.2.** Let \( f \) be a square-free monic integral polynomial and put \( R = \mathbb{Z}[x]/(f) \).

(a) There is a bijection \( \mathcal{M}_{f,R}(\mathbb{Z})/\sim_{\mathbb{Z}} \rightarrow \text{ICM}(R) \).

(b) Assume that \( R \) is a Bass order. Let \( N \) be a positive integer. We have a bijection \( \mathcal{M}_{f,R}(\mathbb{Z})/\sim_{\mathbb{Z}} \rightarrow \mathcal{C} \)

where the objects of \( \mathcal{C} \) are \( R \)-modules of the form \( I_1 \oplus \cdots \oplus I_N \) where the \( I_i \) are fractional \( R \)-ideals satisfying \( (I_1 : I_i) \subseteq (I_{i+1} : I_i) \) and two such modules \( I_1 \oplus \cdots \oplus I_N \) and \( I'_1 \oplus \cdots \oplus I'_N \) are isomorphic if and only if \( (I_1 : I_i) = (I'_1 : I'_i) \) for every \( i \) and \( I_1 \cdot \cdots \cdot I_N = I'_1 \cdot \cdots \cdot I'_N \) in \( \text{ICM}(R) \).

**Proof.** Part [(a)] follows from the equality \( \text{ICM}(R) = \mathcal{L}(R, R \otimes \mathbb{Q})/ \sim_R \) proved in Theorem 7.1 and part [(b)] is a direct consequence of the classification given in [LW85, Theorem 7.1].

**Example 8.3.** Let \( f = f_1 f_2, K = \alpha \) and \( R = \mathbb{Z} \) be defined as in Example 7.3. Put \( \alpha_1 = x \mod f_1 \) and \( \alpha_2 = x \mod f_2 \) so that \( \alpha = (\alpha_1, \alpha_2) \). Furthermore denote by \( \gamma_1 \) and \( \gamma_2 \) the images of the unit elements of \( K_1 \) and \( K_2 \) respectively under the canonical isomorphism \( K = K_1 \times K_2 \). Define

\[
\beta_1 = (1, 0), \quad \beta_2 = (\alpha_1, 0), \quad \beta_3 = (0, 1), \quad \beta_4 = (0, \alpha_2), \quad \beta_5 = (0, \alpha_2^2),
\]

Observe that

\[
\mathcal{A} = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}
\]

and

\[
\mathcal{B} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}
\]

are two bases of \( K \) over \( \mathbb{Q} \). Consider the ideal \( I \) given by

\[
I = \frac{1}{444} \left( -13 - 46\alpha - 138\alpha^2 - 50\alpha^3 + 7\alpha^4 \right) \mathbb{Z} \oplus \frac{1}{222} \left( -9 - 29\alpha - 87\alpha^2 - 9\alpha^3 + 2\alpha^4 \right) \mathbb{Z} \oplus \frac{1}{888} \left( 883 - 12\alpha - 36\alpha^2 + 32\alpha^3 - 3\alpha^4 \right) \mathbb{Z} \oplus \frac{1}{888} \left( 57 + 1084\alpha + 588\alpha^2 + 168\alpha^3 - 25\alpha^4 \right) \mathbb{Z} \oplus \frac{1}{444} \left( 190 - 99\alpha - 75\alpha^2 + 5\alpha^3 + 3\alpha^4 \right) \mathbb{Z},
\]

or equivalently

\[
I = -2\beta_1 \mathbb{Z} \oplus (\beta_1 + \beta_2) \mathbb{Z} \oplus \frac{1}{2} (5\beta_1 + \beta_2 + \beta_3) \mathbb{Z} \oplus \frac{1}{2} (5\beta_1 + \beta_2 + \beta_4) \mathbb{Z} \oplus \frac{1}{2} (3\beta_1 + \beta_2 + \beta_3 + \beta_5) \mathbb{Z}.
\]

With respect to this \( \mathbb{Z} \)-basis of \( I \), the multiplication by \( \alpha \) is represented by the following integral matrix

\[
A = \begin{pmatrix}
-1 & 2 & 3 & 2 & 4 \\
-2 & -3 & 0 & 0 & -4 \\
0 & 0 & 0 & -1 & -4 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 2 & 9
\end{pmatrix}.
\]

Performing an LLL reduction of the \( \mathbb{Z} \)-basis of \( I \), we get

\[
I = \frac{1}{2} (\beta_1 - \beta_2 + \beta_3 + \beta_5) \mathbb{Z} \oplus \frac{1}{2} (-\beta_1 - \beta_2 + 2\beta_3) \mathbb{Z} \oplus \frac{1}{2} (-\beta_1 + \beta_2 + \beta_3 + \beta_5) \mathbb{Z} \oplus \frac{1}{2} (\beta_1 + \beta_2 + \beta_3) \mathbb{Z} \oplus \frac{1}{2} (-\beta_1 - \beta_2 + 2\beta_4) \mathbb{Z}
\]
and with respect to this \(\mathbb{Z}\)-basis of \(I\) the multiplication by \(\alpha\) is represented by

\[
A' = \begin{pmatrix}
5 & 1 & 4 & -1 & 2 \\
-6 & -3 & 0 & 2 & -3 \\
4 & -1 & 5 & 1 & 0 \\
2 & 3 & -4 & -2 & 2 \\
2 & 1 & 2 & 1 & 0
\end{pmatrix}.
\]

We find that for

\[
U = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
-1 & -1 & 0 & 0 & -1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0
\end{pmatrix},
\]

we have

\[
A' = U^{-1}AU.
\]

We now follow the proof of Theorem 8.1 and we construct the ideal associated to the matrix \(A\). The eigenvectors corresponding to the eigenvalues \(\alpha_1\) and \(\alpha_2\) are respectively

\[
\left(1, \frac{1}{2}(\alpha_1 - 1), \frac{1}{4}(\alpha_1 - 5 \cdot 1_1), \frac{1}{4}(\alpha_1 - 5 \cdot 1_1), \frac{1}{4}(\alpha_1 - 3 \cdot 1_1)\right)
\]

and

\[
\left(0, 0, \alpha_2, \frac{1}{2}(\alpha_2^2 + 1_2)\right).
\]

Hence we obtain

\[
\begin{align*}
w_1 &= \beta_1, & w_2 &= -\frac{1}{2}(\beta_1 + \beta_2), & w_3 &= \frac{1}{4}(-5\beta_1 - \beta_2) + \beta_3, \\
w_4 &= \frac{1}{4}(-5\beta_1 - \beta_2) + \beta_4, & w_5 &= \frac{1}{4}(-3\beta_1 - \beta_2) + \frac{1}{2}(\beta_3 + \beta_5)
\end{align*}
\]

and we put

\[
J = \langle w_1, w_2, w_3, w_4, w_5 \rangle_{\mathbb{Z}}.
\]

We find that \(I\) and \(J\) are isomorphic. More precisely, we have

\[
(-2\beta_1 + 28\beta_4 - 3\beta_5)J = I.
\]

**Remark 8.4.** Theorem 8.1 gives an answer to the conjugacy problem over the integers, that is to determine whether two integral matrices \(A\) and \(B\) with square-free characteristic polynomial are conjugate. This was already considered in [Gru80] where the author performs a series of reductions in order to translate the problem into an isomorphism test between fractional ideals of an integral domain. In this process the author has made a mistake, namely that the morphism (3) on page 107 is not a bijection. The reason is that the map from the product of the monogenic orders to \(\mathbb{R}\) is not surjective in general, which could lead to a very different output as Example 7.3 shows.
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21
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