Abstract

In this paper we consider the category $\mathcal{C}(\tilde{\mathfrak{k}}, \tilde{\mathfrak{H}})$ of the $(\tilde{\mathfrak{k}}, \tilde{\mathfrak{H}})$-modules, including all the Verma modules, where $\tilde{\mathfrak{k}}$ is some compact Lie algebra and $\tilde{\mathfrak{H}}$ some Cartan subgroup, $\tilde{\mathfrak{k}}$ and $\tilde{\mathfrak{H}}$ are the corresponding affine Lie algebra and the affine Cartan group, respectively. To this category we apply the Zuckermann functor and its derivatives. By using the determinant bundle structure, we prove the natural duality of the Zuckermann derived functors, deduce a Borel-Weil-Bott type theorem on decomposition of the nilpotent part cohomology.
1 Introduction

Let us consider loop groups associated with compact Lie groups. Considering the (infinite dimensional) affine analogy of the classical theories for finite dimensional Lie groups [1]-[3], we have in our situation the following diagram of categories and functors

\[ \mathcal{C}(\hat{\mathfrak{t}}, \hat{H}) \xrightarrow{R^S} \mathcal{C}(\hat{\mathfrak{t}}, \hat{H}) \bowtie \]

\[ \bigcup \]

\[ \mathcal{C}(\hat{\mathfrak{h}}, \hat{H}) \rightarrow \mathcal{C}(\hat{\mathfrak{b}}, \hat{H}) \xrightarrow{H} \mathcal{O} \cong \bigoplus_\chi \mathcal{O}_\chi \]

\[ \bigcup \]

\[ \mathcal{A}(\hat{\mathfrak{t}}, \hat{B}) = \mathcal{K} \bowtie \]

\[ \{ M_\lambda = U(\hat{\mathfrak{k}}) \otimes_{U(\hat{\mathfrak{b}})} L_{\lambda - \bar{\rho}} \} \]

(Verma modules)

where

\[ I(W) := \text{Hom}_{U(\hat{\mathfrak{b}})}(U(\hat{\mathfrak{t}}), W)[\hat{\mathfrak{h}}], \]

\[ H(W) := \text{Hom}_{U(\hat{\mathfrak{b}})}(U(\hat{\mathfrak{t}}), W)[\hat{\mathfrak{h}}], \]

\[ S(V) := V[\hat{\mathfrak{t}}], \text{ the Zuckerman functor,} \]

\[ V^\sim \text{ is the maximal } U(\hat{\mathfrak{h}})\text{-locally finite and } \hat{H}\text{-semisimple module in the algebraic dual } V^\ast, \]

\[ V^\aleph \text{ the maximal } U(\hat{\mathfrak{t}})\text{-locally finite and } \hat{\mathfrak{t}}\text{-semisimple module in } V^\ast, \]

\[ \mathcal{C}(\hat{\mathfrak{h}}, \hat{H}) \text{ the category of finite dimensional semisimple } (\hat{\mathfrak{h}}, \hat{H})\text{-modules}, \]

\[ \mathcal{C}(\hat{\mathfrak{b}}, \hat{H}) \text{ the category of the } U(\hat{\mathfrak{h}})\text{-locally finite, } \hat{H}\text{-semisimple and } U(\tilde{n}_\pm)\text{-locally finite } (\hat{\mathfrak{b}}, \hat{H})\text{-modules}, \]

\[ \mathcal{C}(\hat{\mathfrak{t}}, \hat{H}) \text{ the category of the } U(\hat{\mathfrak{h}})\text{-locally finite, } \hat{H}\text{-semisimple } (\hat{\mathfrak{t}}, \hat{H})\text{-modules}, \]

\[ \mathcal{O} \text{ the category of } U(\hat{\mathfrak{h}})\text{-locally finite, } \hat{H}\text{-semisimple } U(\tilde{n}_\pm)\text{-locally finite } (\hat{\mathfrak{t}}, \hat{H})\text{-modules}, \]

\[ \mathcal{A}(\hat{\mathfrak{t}}, \hat{B}) \text{ the category of } U(\hat{\mathfrak{h}})\text{-locally finite, } \hat{H}\text{-semisimple finite dimensional } \hat{H}\text{-isotropic and } U(\tilde{n}_\pm)\text{-locally finite } (\hat{\mathfrak{t}}, \hat{B})\text{-modules,} \]

\[ \mathcal{C}(\hat{\mathfrak{t}}, \hat{K}) \text{ the category of } (\hat{\mathfrak{t}}, \hat{K})\text{-modules.} \]
The functor $I$ shows that the category $\mathcal{C}(\tilde{k}, \tilde{H})$ has enough injective objects; hence, one can define the Zuckerman derived functors

$$R^i S(V) := \bigoplus_{\lambda \in P} H^i(\tilde{k}, \tilde{H}; V_\lambda^* \otimes V) \otimes V_\lambda.$$ 

For the objects from the subcategory $\mathcal{O}$ one can restrict on the injective in $\mathcal{O}$ resolutions, i.e.

$$R^i (S|_{\mathcal{O}}) \cong (R^i S)|_{\mathcal{O}}.$$ 

By using the determinant bundle see for example [3] structure

$$\text{det}(\tilde{k}/\tilde{h}) = \wedge^{\max}(\tilde{k}/\tilde{h}),$$

we can easily prove the natural duality for relative Lie cohomology

$$H^i(\tilde{k}, \tilde{H}; V) \times H^{\max-i}(\tilde{k}, \tilde{H}; W) \to \mathbb{C},$$

for a dual pair of modules $V$ and $W$, and the natural duality of Zuckerman derived functors

$$R^i S(V) \times R^{\max-i} S(W \otimes \varepsilon_{\tilde{k}/\tilde{h}}) \to \mathbb{C},$$

or in a weaker form, an isomorphism

$$R^i S(V) \cong R^{\max-i} S(V^\sim)^\sim.$$ 

It is easy then to deduce a Borel-Weil-Bott type theorem

$$R^i S(H(L_{-w,\lambda+\tilde{\rho}})) \cong \delta_{i,s(w)} V_{-\lambda+\tilde{\rho}},$$

where $s(w) = \sum \dim g_\alpha + l - 1 + \sum a_i$ and $\tilde{\rho}$ is the sum of the fundamental weights. Finally, it is easy to deduce an affine analogue of the Kostant theorem on decomposition for the cohomology of the nilpotent part

$$H^i(\tilde{n}_+; V_\lambda) = \bigoplus_{w \in W, s(w)=i} L_{w(\lambda+\tilde{\rho})-\tilde{\rho}}.$$
2 Notations

First of all we recall some notations from the theory of finite dimensional semisimple Lie algebras:

- $\mathfrak{g}$ a complex finite dimensional Lie algebra of rank $l$,
- $\mathfrak{h}$ some Cartan subalgebra, $\text{dim} \mathfrak{h} = l$,
- $\Delta$ the root system corresponding to the pair $(\mathfrak{g}, \mathfrak{h})$,
- $\Delta_+$ the system of positive roots,
- $\mathfrak{n}_\pm = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ the nilpotent part,
- $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ the simple roots,
- $\theta$ the maximal positive root,
- $(.,.)$ a nondegenerate invariant bilinear form, inducing an isomorphism $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$ and a form on $\mathfrak{h}^*$ normed by the following condition $(\theta|\theta) = 2$. Hence $(\alpha|\alpha) = 2$ for all long roots from $\Delta$,
- $\check{\alpha} = \frac{2}{(\alpha|\alpha)}$ the dual root corresponding to $\alpha$,
- $\check{\theta} = \sum a_i \check{\alpha}_i$, where $a_1, \ldots, a_l \in \mathbb{N}$,
- $\check{\Lambda}_i : (\check{\Lambda}_i, \alpha_j) = \delta_{ij}$ the fundamental weights,
- $Q = \sum_{\alpha \in \Delta} \mathbb{Z} \alpha$ the root lattice in $\mathfrak{h}^*$,
- $L = \sum_{\alpha \in \Delta} \mathbb{Z} \alpha$ the sublattice of the long roots,
- $\nu^{-1}(L) = \check{Q} = \sum_{\alpha \in \Delta} \mathbb{Z} \check{\alpha}$ the co-root lattice,
- $P = \sum_{i=1}^l \mathbb{Z} \check{\Lambda}_i$ the weight lattice,
- $P_+ = \sum_{i=1}^l \mathbb{Z}_+ \check{\Lambda}_i$ the positive chamber of fundamental weights,
- $\rho = \sum_{i=1}^l \check{\Lambda}_i$ the sum of fundamental weights,
- $g = 1 + \rho(\check{\theta}) = 1 + \sum_{i=1}^l \check{a}_i$ the Coxeter number,
- $r_\alpha, \alpha \in \Delta$ the reflections defined by $r_\alpha(\lambda) = \lambda - 2(\lambda, \check{\alpha}) \alpha$,
- $W = \langle r_\alpha, \alpha \in \Delta \rangle \subset \text{GL}(\mathfrak{h}^*)$ the Weyl group generated by reflections, acting on $\Delta, Q, P, etc., . . .$

Recall now the corresponding notations for the theory of affine Lie algebras:

- $\mathcal{L} = \mathbb{C}[z, z^{-1}]$ the algebra of Laurent polynomials in variable $z \in \mathbb{C} \setminus \{0\}$,
- $\tilde{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) := \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$ the Laurent loop algebra,
- $\langle , \rangle_z := I \otimes (.,.)$ the invariant Hermitian structure,
\[ c(X, Y) := \text{Res}_{z=0} \left( \frac{d}{dz} X | Y \right) \text{ the antisymmetric 2-cocycle}, \]
\[ \hat{\mathfrak{g}} \text{ the central extension defined by the cocycle } c(\cdot, \cdot) \text{ and the central element } c, \]
\[ 0 \to \mathbb{C}.c \to \hat{\mathfrak{g}} \to \bar{\mathfrak{g}} \to 0, \]
\[ d = d \otimes 1 \in \text{Der } \mathbb{C}[z, z^{-1}] \to \text{Der } \hat{\mathfrak{g}}, \text{ acting on } \hat{\mathfrak{g}} \text{ as } z \frac{d}{dz} \text{ and commuting with the central element } c, \]
\[ \hat{\mathfrak{g}} = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d \text{ the affine Lie algebra, } \]
\[ [X(\cdot) + \alpha c + \beta d, Y(\cdot) + \alpha_1 c + \beta_1 d] := ([X, Y](\cdot) + \beta z \frac{d}{dz} Y - \beta_1 z \frac{d}{dz} X) + \text{Res}_{z=0} \left( \frac{d}{dz} X | Y \right)_z c, \]
\[ \langle X(\cdot) + \alpha c + \beta d | Y(\cdot) + \alpha_1 c + \beta_1 d \rangle = \text{Res}_{z=0} (z^{-1} \langle X | Y \rangle_z) + \alpha \beta_1 + \alpha_1 \beta, \]
\[ \tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d \text{ some Cartan affine subalgebra, where } \mathfrak{h} \text{ is some finite dimensional) Cartan subalgebra of } \mathfrak{g}, \]
\[ \tilde{\mathfrak{g}}_\tilde{\alpha} = \{ X \in \tilde{\mathfrak{g}}; [H, X] = \tilde{\alpha}(H) X, H \in \tilde{\mathfrak{h}} \} \text{ a root space,} \]
\[ \Delta = \{ \tilde{\alpha}; \tilde{\mathfrak{g}}_\tilde{\alpha} \neq 0 \} = \tilde{\Delta}_W \cup \tilde{\Delta}_I \text{ the root system,} \]
\[ \tilde{\Delta}_W = \{ \tilde{\alpha} = \alpha + n \delta; \alpha \in \Delta, n \in \mathbb{Z} \} \text{ the real roots,} \]
\[ \tilde{\delta} \text{ such a root that } \delta(d) = c^* (d) = 1, \delta|_{\mathfrak{h} + \mathbb{C}c} = 0, \]
\[ \Delta_I = \{ n \delta; n \in \mathbb{Z} \setminus \{0\} \} \text{ the imaginary roots,} \]
\[ \dim_{\mathbb{C}} \tilde{\mathfrak{g}}_\tilde{\alpha} = \begin{cases} 1 & \text{if } \alpha \in \tilde{\Delta}_W \\ l & \text{if } \alpha \in \tilde{\Delta}_I \end{cases}, \]
\[ \mathfrak{n}_\pm = n_+ \oplus \sum_{n>0} z^n \otimes \mathfrak{g} \text{ the nilpotent parts,} \]
\[ \tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- = \tilde{\mathfrak{h}} \oplus \sum_{\tilde{\alpha} \in \tilde{\Delta}} \tilde{\mathfrak{g}}_{\tilde{\alpha}} \text{ the Cartan decomposition,} \]
\[ \delta = d^* \in \tilde{\mathfrak{h}}^*, \delta(d) = c^*(d) = 1, \delta|_{\mathfrak{h} + \mathbb{C}c} = 0, \]
\[ \tilde{\Pi} = \{ \alpha_0 := \delta - \theta, \alpha_1, \ldots, \alpha_l \} \text{ the simple roots,} \]
\[ \tilde{\Pi} = \{ \tilde{\alpha}_0 := c - \theta, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_l \} \text{ the simple co-roots,} \]
\[ \{ \tilde{\Lambda}_0, \ldots, \tilde{\Lambda}_l; \tilde{\Lambda}_i = \Lambda_i + \tilde{a}_i \Lambda_0 \} \text{ the fundamental weights, normed as } \langle \tilde{\Lambda}_i | \tilde{\alpha}_j \rangle = \delta_{ij}, \]
\[ \tilde{\Delta}_+ = \Delta_+ \cup \{ k \delta + \alpha; \alpha \in \Delta, k \in \mathbb{Z}_+ \} \cup \{ k \delta; k \in \mathbb{Z}_+ \}, \]
\[ \tilde{P} = \sum_{i=0}^l \mathbb{Z} \tilde{\Lambda}_i \text{ the weight lattice,} \]
\[ \tilde{P}_+ = \sum_{i=0}^l \mathbb{Z}_+ \tilde{\Lambda}_i \text{ the chamber of the dominant weights,} \]
\[ \tilde{\rho} = \sum_{i=0}^l \tilde{\Lambda}_i = \rho + g \tilde{\Lambda}_0 \text{ the sum of the fundamental weights,} \]
\[ \tilde{Q} = \mathbb{Z} \delta \oplus \tilde{Q} \text{ the affine root lattice,} \]
\[ \tilde{A} = (\alpha_j(\tilde{\alpha}_i))_{i,j=0,1,\ldots,l} \text{ the generalized Cartan matrix,} \]
\[ \tilde{W} = W \ltimes \tilde{Q} \text{ the affine Weyl group, generated by the reflections } r_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{\Delta}, \]
3 **Category** $\mathcal{C}(\tilde{\mathfrak{h}}, \tilde{H})$

Assume $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}}$ to be an affine Lie algebra associated to a compact complex Lie algebra $\mathfrak{k} = k_R \otimes \mathbb{C}$, where $k_R$ is the real Lie algebra of some compact connected Lie group $K$. Suppose that 2-cocycle $c(.,.)$ is integral. In this case the associated Lie algebra extension can be lifted to an extension of the corresponding Lie groups

$$1 \rightarrow \mathbb{T} \rightarrow \tilde{K} \rightarrow \mathcal{L}K \cong C^\infty(S^1, K) \rightarrow 1.$$  

For each subgroup $H \subset K$, consider the corresponding central extension

$$1 \rightarrow \mathbb{T} \rightarrow \tilde{H} \rightarrow H \rightarrow 1.$$  

In particular, if $H$ is some Cartan subgroup of $K$ we have an affine extension $\tilde{H}$ of Cartan subgroup $H$. In this case, the affine Cartan subgroup $\tilde{H}$ is also finite dimensional.

The pair $(\tilde{\mathfrak{k}}, \tilde{H})$ is called to be **compatible**, if there is such a homomorphism of affine Lie groups $\tilde{H} \rightarrow \text{Aut} \tilde{\mathfrak{k}}$ that the corresponding homomorphism of affine Lie algebras $\tilde{\mathfrak{h}} \rightarrow \text{End} \tilde{\mathfrak{k}}$ is coincided with the adjoint representation $\text{ad}_{\tilde{\mathfrak{h}}}(\tilde{\mathfrak{k}})$ and that it induces the adjoint representation $\text{Ad}_{\tilde{\mathfrak{h}}}(\tilde{H})$ in $\tilde{\mathfrak{k}}$.

For such a fixed compatible pair $(\tilde{\mathfrak{k}}, \tilde{H})$ we define $\mathcal{C}(\tilde{\mathfrak{k}}, \tilde{H})$ as the category the objects of which have simultaneously $(\tilde{\mathfrak{k}}, \tilde{\mathfrak{h}})$-module (see [3]) and $\tilde{H}$-module structures, which are compatible in the sense

$$h(Xv) = ((\text{Ad} h)X).hv, \forall h \in \tilde{H}, \forall v \in V, X \in \tilde{\mathfrak{k}}.$$  

Remark that all the categories $\mathcal{C}(\tilde{\mathfrak{h}}, \tilde{H}), \mathcal{C}(\tilde{\mathfrak{k}}, \tilde{H}), \mathcal{C}(\tilde{\mathfrak{f}}, \tilde{K}), \ldots$, raised in the introduction are the particular cases of this definition. Following M. Duflo and M.
3 CATEGORY \( \mathcal{C}(\tilde{\mathfrak{k}}, \tilde{H}) \)

Vergne [2] we define now the derived Zuckermann functors and then prove their duality in the category \( \mathcal{C}(\tilde{\mathfrak{k}}, \tilde{H}) \).

Denote by \( F(\tilde{K}) = \bigoplus_{\lambda \in \tilde{P}} (V_\lambda^* \otimes V_\lambda) \) the associated algebra of matrix elements of the standard modules, where the sum runs over the set of weight lattice \( \tilde{P} \). It is also the \((\tilde{\mathfrak{k}}, \tilde{H})\)-module of regular functions on \( \tilde{K} \). Denote by \( r \) and \( l \) the right and left regular representations of \( \tilde{K} \) (or \( \tilde{k} \)) on \( F(\tilde{K}) \), respectively. Let \( V \) be a \((\tilde{\mathfrak{k}}, \tilde{H})\)-module with the action \( \tilde{\theta} \) of \( \tilde{k} \) (or \( \tilde{\mathfrak{k}} \)) in \( V \). Then we define a \((\tilde{\mathfrak{k}}, \tilde{H})\)-module structure on \( F(\tilde{K}; V) := F(\tilde{K}) \otimes V \) by

\[
(\psi(X)f)(k) = \tilde{\theta}(X)f(k) + (l(X)f)(k), \quad \forall x \in \tilde{\mathfrak{k}}, \forall k \in \tilde{K},
\]

\[
(\psi(h)f)(k) = \tilde{\theta}(h)f(h^{-1}k), \quad \forall k, h \in \tilde{H}.
\]

This action commutes with the \((\tilde{\mathfrak{k}}, \tilde{H})\)-module structure \( \psi \). The right regular representation \( r \) (of algebra \( \tilde{\mathfrak{k}} \), or of group \( \tilde{K} \)) also commutes with the \((\tilde{\mathfrak{k}}, \tilde{H})\)-module structure \( \psi \) and finally

\[
r(k)e(X)r(k^{-1}) = (kX).
\]

The representation \( \Gamma \) of the Algebra \( \tilde{\mathfrak{k}} \) and the representation \( r \) of the group \( \tilde{K} \) can be continued on

\[
\text{Hom}(\Lambda^*\tilde{\mathfrak{k}}, F(\tilde{K}; V)) \cong F(\tilde{K}; \text{Hom}(\Lambda^*\tilde{\mathfrak{k}}, V)).
\]

It is easy to prove (see [2], p. 468) that for every \( Y \in \tilde{\mathfrak{k}} \), one can define \( I(Y) \) by

\[
(I(Y)\mu)(k) = i(k.Y)\mu(k), \quad \forall k \in \tilde{K}, \forall \mu \in F(\tilde{K}, \text{Hom}(\Lambda^*\tilde{\mathfrak{k}}, V))
\]

and that \( \Gamma(Y) - r(Y) = d_{\Omega}I(Y) + I(Y)\circ d \). Thus \( \Gamma \) and \( r \) induces a \((\tilde{\mathfrak{k}}, \tilde{K})\)-module structure on the Zuckermann derived functors

\[
\Gamma^i(V) := \bigoplus_{\lambda \in \tilde{P}} H^i(\tilde{\mathfrak{k}}, \tilde{H}; V_\lambda^* \otimes V) \otimes V_\lambda.
\]

In the next section, by using of the general ideas of homological algebra [3], we show that for connected \( \tilde{K} \), \( \Gamma^0(V) \) coincides with the Zuckermann functor \( S(V) := V[\tilde{\mathfrak{k}}] \) from the category \( \mathcal{C}(\tilde{\mathfrak{k}}, \tilde{H}) \) into the category \( \mathcal{C}(\tilde{\mathfrak{k}}, \tilde{K}) \).

Remark that in the second part of this contribution we shall prove a version of the Peter-Weyl theorem, which asserts that

\[
S(U(\tilde{\mathfrak{k}})^*) \cong \bigoplus_{\lambda \in \tilde{P}} V_\lambda^* \otimes V_\lambda = F(\tilde{K}).
\]
Thus $F(\tilde{K})$ can be interpreted as the algebra of regular functions on $\tilde{K}$, as in the classical case, [5], [7].

Recall that $\tilde{k}/\tilde{h}$ is infinite dimensional space $L[z, z^{-1}] \otimes (\tilde{k}/h)$. But it is well-defined the determinant fiber bundle structure on $\det(\tilde{k}/\tilde{h}) := \bigwedge^{\max}(\tilde{k}/\tilde{h})$ (see for example [5]), in case where $h$ is a Cartan subalgebra. Because the corresponding group $H$ is compact, $\det(\tilde{k}/\tilde{h})$ admits also a $(\tilde{k}, \tilde{H})$-module structure $\varepsilon_{\tilde{k}/\tilde{h}}$ with the trivial differential.

Suppose that $\langle ., . \rangle$ is a $(\tilde{k}, \tilde{H})$-invariant pairing of $(\tilde{k}, \tilde{H})$-modules $V, W \in C(\tilde{k}, \tilde{H})$.

**Theorem 3.1** For each $(\tilde{k}, \tilde{H})$-invariant pairing of $V, W \in C(\tilde{k}, \tilde{H})$, there exists a $(\tilde{k}, \tilde{K})$-invariant pairing of type

$$\langle \Gamma^i(V), \Gamma^{\max-i}(W \otimes \varepsilon_{\tilde{k}/\tilde{h}}) \rangle \to \mathbb{C}.$$

**Proof.** Let us denote by $dW_K$ the Wiener measure on $\tilde{K}$. Then each $\tilde{H}$-invariant $i^{th}$ relative cocycle

$$\omega_1 \otimes f_1 \in \bigwedge^i(\tilde{k}/\tilde{h})^* \otimes F(\tilde{K}; V)$$

can be $(\tilde{k}, \tilde{H})$-invariantly paired with any relative cocycle $\omega_2 \otimes f_2 \otimes \zeta$ of degree 1 from

$$\bigwedge^{\max-i}(\tilde{k}/\tilde{h})^* \otimes F(\tilde{K}; W) \otimes \bigwedge^{\max}(\tilde{k}, \tilde{h})$$

by the formula

$$\langle \omega_1 \otimes f_1, \omega \otimes f_2 \otimes \zeta \rangle = \langle \omega_1 \wedge \omega_2, \zeta \rangle \int_{\tilde{K}} \langle f_1(k), f_2(k) \rangle dW_K(k).$$

This pairing is compatible with the differential $d$ of the complex of relative $(\tilde{k}, \tilde{H})$-valued cocycles, i.e. for $\tilde{H}$-invariant elements $\mu_1 \in \bigwedge^i(\tilde{k}/\tilde{h})^* \otimes F(\tilde{K}; V)$ and $\mu_2 \in \bigwedge^{\max-i}(\tilde{k}, \tilde{h})^* \otimes F(\tilde{K}) \otimes \bigwedge^{\max}(\tilde{k}, \tilde{h})$, we have

$$\langle d\mu_1, \mu_2 \rangle + (-1)^i \langle \mu_1, d\mu_2 \rangle = 0.$$

The last gives thus a $(\tilde{k}, \tilde{K})$-invariant pairing of the Zuckermann derived functors of type

$$\langle \Gamma^i(V), \Gamma^{\max-i}(W \otimes \varepsilon_{\tilde{k}/\tilde{h}}) \rangle \to \mathbb{C}.$$

$\square$
4 Duality of the Zuckermann derived functors

We develop in this section the formal homological algebra on the categories $\mathcal{C}(\tilde{h}, \tilde{H})$, $\mathcal{C}(\tilde{k}, \tilde{H})$, $\mathcal{C}(\tilde{k}, \tilde{K})$ etc.

First of all, it is easy to see that the category $\mathcal{C}(\tilde{k}, \tilde{H})$ has enough injective objects. Really, for every object $W$ of $\mathcal{C}(\tilde{k}, \tilde{H})$, the object $I(W) = \text{Hom}_{\mathcal{C}(\tilde{k})}(U(\tilde{k}), W)$ is injective in the category $\mathcal{C}(\tilde{k}, \tilde{H})$. Furthermore every injective object in $\mathcal{C}(\tilde{k}, \tilde{H})$ can be presented as a direct summand of an object of this type $I(W)$. More exactly, we have $V \hookrightarrow I(V)$. It is easy to see that if $F \in \mathcal{C}(\tilde{k}, \tilde{K})$, then $F \in \mathcal{C}(\tilde{k}, \tilde{H})$ and the exact covariant additive functor $F \otimes (\cdot)$ maps the injective objects into the same ones, commuting with the Zuckermann functor $S(V) = V[\tilde{k}]$. Hence, we have the isomorphism

$$R^i S(F \otimes V) \cong F \otimes R^i S(V).$$

In particular, to each resolution $0 \to V \to I^*$ of type

$$0 \to V \to I^0 \to I^1 \to \ldots$$

we can apply the functor $U(\tilde{k}) \otimes (\cdot)$ and obtain the commutativity of $R^i S$ with $U(\tilde{k}) \otimes (\cdot)$ and also with the multiplication functor $m$ (i.e. the $U(\tilde{k})$-action), following the commutative diagram

$$\begin{array}{ccc}
U(\tilde{k}) \otimes R^i S(V) & \cong & R^i S(U(\tilde{k}) \otimes V) \\
 \downarrow m & & \downarrow R^i S(m) \\
R^i S(V) & & \end{array}$$

More particularly, if the submodule $V \subset V_1 \oplus V_2 \in \mathcal{C}(\tilde{k}, \tilde{H})$ is stable under $Z(\tilde{k}) = U(\tilde{k})^\sharp$ and if for some element $u \in Z(\tilde{k})$, the operator $u - \chi(u)$ acts nilpotently on $V$, then so it acts also on the derived functors $R^i S(V)$.

Consider the finite dimensional injective $\tilde{H}$-invariant Koszul resolution

$$0 \to \mathbb{C} \to \text{Hom}_{\mathcal{C}(\tilde{h})}(U(\tilde{h}), \wedge^0(\tilde{h})^*) \xrightarrow{d} \ldots$$
and apply the tensor multiplication with $V$ on the right. We have then an injective resolution of $V$

$$0 \to V \to \text{Hom}_{U(\tilde{h})}(U(\tilde{\ell}), \wedge^0(\tilde{\ell}/\tilde{h})^* \otimes V)^{d_\ell} \to \ldots$$

$$\ldots \to \text{Hom}_{U(\tilde{h})}(U(\tilde{\ell}), \wedge^{\max}(\tilde{\ell}/\tilde{h})^* \otimes V) \to 0.$$ Applied the Zuckermann functor $S(V) = V[\tilde{\ell}]$ and taken the $\tilde{H}$-invariant parts, we have the derived functors as the cohomology groups of complex

$$R^i S(V) = \text{Coh}^i(I(\wedge^* (\tilde{\ell}/\tilde{h})^* \otimes V)\tilde{H}, d_V).$$

Remark that

$$\text{Hom}_{U(\tilde{h})}(U(\tilde{\ell}), \wedge^i(\tilde{\ell}/\tilde{h}) \otimes V) \cong \text{Hom}_{U(\tilde{h})}(\wedge^i(\tilde{\ell}/\tilde{h}), U(\tilde{\ell})^* \otimes V).$$

Hence, applying the functor $S$, we have

$$I(\wedge^i(\tilde{\ell}/\tilde{h})^* \otimes V) \cong \text{Hom}_{U(\tilde{h})}(\wedge^i(\tilde{\ell}/\tilde{h}), R(\tilde{\ell}) \otimes V),$$

where $R(\tilde{\ell}) = \bigoplus_{\lambda \in \tilde{P}} V_\lambda \otimes V_\lambda^*$ is the maximal $(\tilde{\ell}, \tilde{H})$-bimodule in $U(\tilde{\ell})^*$, i.e.

$$I(\wedge^i(\tilde{\ell}/\tilde{h})^* \otimes V) \cong \bigoplus_{\lambda \in \tilde{P}} \text{Hom}_{U(\tilde{h})}(\wedge^i(\tilde{\ell}/\tilde{h}), V \otimes V_\lambda^*) \otimes V_\lambda.$$

Therefore, we have the equivalence of functors

$$R^i S(V) \cong H(V) := \bigoplus_{\lambda \in \tilde{P}} H^i(\tilde{\ell}, \tilde{H}; V_\lambda \otimes V) \otimes V_\lambda^*,$$

$$R^{\max-i} S(V) \cong G(V) := \bigoplus_{\lambda \in \tilde{P}} H^{\max-i}(\tilde{\ell}, \tilde{H}; V_\lambda^* \otimes V_\lambda^*) \otimes V_\lambda^*.$$

Applying Theorem 1, we have now an equivalence of functors

$$T_V : R^i S(V) \cong R^{\max-i} S(V^\sim).$$

Hence we obtain the result

**Theorem 4.1** $V \to T_V$ is an equivalence of functors.
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5 Category \( \mathcal{O} \)

By using the trivial action of the nilpotent part \( \tilde{n}_+ \) of affine Lie algebras \( \tilde{b} := \tilde{h} \oplus \tilde{n}_+ \), one can, in place of the category \( \mathcal{C}(\tilde{\mathfrak{k}}, \tilde{\mathcal{H}}) \) and the functor \( I \) consider the category \( \mathcal{C}(\tilde{b}, \tilde{\mathcal{H}}) \) and the functor \( H \),

\[
H(W) := \text{Hom}_{U(\tilde{h})}(U(\tilde{\mathfrak{k}}), W).
\]

Thus we can use the subcategory \( \mathcal{O} \) in \( \mathcal{C}(\tilde{\mathfrak{k}}, \tilde{\mathcal{H}}) \), which has also enough injective objects. This is a more economic way of computing the derived functors. Remark that the most important Verma modules belong too to this subcategory. We have deal with this situation in proving the Borel-Weil-Bott type theorem.
we see that in the category $\mathcal{O}$ all the objects are acyclic, i.e. if $V$ is an injective object in $\mathcal{O}$, then $R^iS(V)$ vanishes unless $i = 0$. In the category $\mathcal{O}$, all the objects have finite injective cohomological dimension. Then by induction on this dimension one proves that $(R^iS)|_{\mathcal{O}} \cong R^i(S)|_{\mathcal{O}}$. This means that computing the Zuckermann derived functors for objects in $\mathcal{O}$ is independent from the bigger category $\mathcal{C}(\tilde{\mathfrak{k}}, \tilde{H})$.

6 Borel-Weil-Bott and Kostant type theorems

Recall that by $\tilde{P}$ we denote the weight lattice of our compact complex affine Lie algebra, $\tilde{P}_+$, the Weyl chamber of the dominant weights. The Weyl group $\tilde{W}$ acts on $\tilde{P}$ and every regular weight $\mu$ can be uniquely presented in form $-w\lambda + \tilde{\rho}$, where $\lambda \in \tilde{P}_+$ and $w \in \tilde{W}$. Recall that the length function is defined by $l(w) := \#\{\lambda \in \tilde{\Pi}; w\lambda = -\lambda\} \leq l + 1$. If $\varpi \in W$ and $l(\varpi) = l + 1$, then $\varpi\tilde{\Pi} = -\tilde{\Pi}$.

By definition, we have $\tilde{\rho} = \sum_{i=0}^{l} \lambda_i$ and

$$-\varpi\tilde{\rho} + \tilde{\rho} = 2\sum_{i=0}^{l} \lambda_i.$$

Remark that although $\dim_{\mathbb{C}} g_{h\delta} = l, \forall k \in \mathbb{Z} \setminus (0)$, where $\delta \in \mathfrak{h}^*, \delta(d) = 1$ and $\delta|_{\mathfrak{h} + Cc} = 0$, we have always $\dim L_{-\varpi\tilde{\rho} + \tilde{\rho}} = 1$.

Let us introduce a function

$$s(w) = \begin{cases} l(w) & \text{if } w\alpha_0 \neq -\alpha_0 \\ l(w) + l - 1 + \sum_i a_i & \text{if } w\alpha_0 = -\alpha_0. \end{cases}$$

Theorem 6.1 For every regular integral dominant weight $\lambda \in \tilde{P}_+$ and for every element $w \in \tilde{W}$,

$$R^iS(H(L_{-w\lambda + \tilde{\rho}})) = \begin{cases} 0 & \text{if } i \neq s(w) \\ V_{-w\lambda + \tilde{\rho}} & \text{if } i = s(w). \end{cases}$$

Proof. As in the classical case, we prove the theorem by induction on the length of elements of the affine Weyl group $\tilde{W}$.

(a) First of all we verify the assertion in the case of maximal length elements, $l(w) = l + 1, w\tilde{\Pi} = -\tilde{\Pi}$. Hence $w = \varpi$. 


From the discussed properties of the functor $H$ one deduces easily that
\[ H(L_{-\varpi \lambda}) \cong H(L_{-\varpi \tilde{\varrho} + \tilde{\varrho}} \otimes V_{-\lambda + \tilde{\varrho}})_{\chi_\lambda}, \]
where the index $\chi_\lambda$ means the submodule, where the operators $z - \chi_\lambda(z)$ act nilpotently, for all $z \in \mathcal{Z}(\tilde{\mathfrak{g}}) = \text{cent} \ U(\tilde{\mathfrak{g}})$, and $\chi_\lambda$ the infinitesimal character corresponding to $\lambda$.

In Section 3 we have seen that
\[ R^i S(W_{\chi_\lambda}) \cong (R^i S(W))_{\chi_\lambda}, W \in \mathcal{C}(\tilde{\mathfrak{g}}, \tilde{H}), \]
\[ R^i S(W \otimes V_\lambda) \cong R^i S(W) \otimes V_\lambda. \]

Hence, we have
\[ R^i S(H(L_{-\varpi \lambda + \tilde{\varrho}})) \cong R^i S((H(L_{-\varpi \tilde{\varrho} + \tilde{\varrho}}) \otimes V_{-\lambda + \tilde{\varrho}}))_{\chi_\lambda} \]
\[ \cong (R^i S(H(L_{-\varpi \tilde{\varrho} + \tilde{\varrho}}) \otimes V_{-\lambda + \tilde{\varrho}}))_{\chi_\lambda} \cong (R^i S(H(L_{-\varpi \tilde{\varrho} + \tilde{\varrho}}))) \otimes V_{-\lambda + \tilde{\varrho}})_{\chi_\lambda}. \]

Therefore it is enough to compute $R^i S(H(L_{-\varpi \tilde{\varrho} + \tilde{\varrho}}))_{\chi_\lambda}$. We have
\[ R^i S(H(L_{-\varpi \tilde{\varrho} + \tilde{\varrho}})) = \left( \bigoplus_{\lambda \in \mathcal{P}} V_\lambda \otimes H^i(\tilde{n}; L_{-\varpi \tilde{\varrho} + \tilde{\varrho}} \otimes V^*) \right)_{\chi_0} \]
\[ = V_0 \otimes H^i(\tilde{n}; L_{-\varpi \tilde{\varrho} + \tilde{\varrho}} \otimes V_0^*)_{\mathcal{H}} \cong V_0 \otimes H^i(\tilde{n}; L_{-\varpi \tilde{\varrho} + \tilde{\varrho}}), \]
where $\tilde{n} := \tilde{n}_+ \oplus \tilde{n}_-$, $\wedge^n \tilde{n}_+$ has no vectors of weight $-\varpi \tilde{\varrho} + \tilde{\varrho}$, if $i \neq s(w)$. If $i = s(w)$, we have
\[ \wedge^{s(w)} \tilde{n} = (\wedge^{s(w)} \tilde{n})_{-\varpi \tilde{\varrho} + \tilde{\varrho}} \oplus \ldots \]
and
\[ H^{s(w)}(\tilde{n}; L_{-\varpi \tilde{\varrho} + \tilde{\varrho}}) = \mathbb{C}. \]

The theorem is verified in this case. Remark that here we have the non-triviality of the cohomology group in dimension $s(w)$, while in the classical case the non-triviality in the dimension $l(w)$.

(b) Suppose that $l(w) < l + 1$ and that the theorem is verified for all $w' \in \tilde{W}$, such that
\[ l(w') \geq l(w) + 1. \]
Consider $\tilde{Q} = w\tilde{P}$. Then $w\lambda$ is $\tilde{Q}$-dominant, integral and regular. Because $\tilde{Q} \neq w\tilde{\Delta}_+$, there exists a $\tilde{Q}$-simple root $\alpha$ from $\tilde{\Delta}_+$, such that $\langle \lambda, \alpha \rangle > 0$. Because
\[ s_\alpha \tilde{Q} \supseteq \tilde{Q} - \{\alpha\}, \]
then $w\lambda$ and $s_\alpha w\lambda$ are $P_1$-dominant, where $\tilde{P}_1 = \tilde{Q} - \{\alpha\}$.
Lemma 6.1 Suppose that $\lambda \in \tilde{P}$ is regular integral weight. Then there exists a unique root system $Q$ such that $\lambda$ is dominant weight. Suppose that there exists a $Q$-simple root $\alpha \in \Delta_+ \cup Q$ such that $\langle \lambda, \alpha \rangle > 0$ and that the both elements $\lambda$ and $\lambda' = s_\alpha \lambda$ are $(Q - \{\alpha\})$-dominant. Then in the category $O$ there exists such an object $E$ that:

(i) the sequence

$$0 \to M_\lambda \to E \to M_{\lambda'} \to 0$$

is exact and

(ii) $R^i S(E) = 0, \forall i = 0, 1, 2, \ldots$, where $M_\lambda = U(\tilde{\mathfrak{e}}) \otimes_{U(\tilde{\mathfrak{b}})} L_{\lambda - \tilde{\rho}}$ is the Verma modules associated to $\lambda$.

Proof. This lemma is an affine analogue of Lemma 6.2 from [3] and its proof does not require an essential change. □

End of the proof of Theorem [6.1].

From this lemma, by changing $\lambda \mapsto w\lambda$, there exists a module $E \in O$ satisfying (i) and (ii). Then $E$ has the finite dimensional weight spaces with respect to $\mathfrak{h}$. Hence we have an exact sequence

$$0 \to M_{\sim w\lambda} \to E^\sim \to M_{\sim w\lambda} \to 0.$$

In virtue of the duality theorem, we have

$$R^i S(E^\sim) = R^i S(E)^{\sim} = 0, \forall i \in \mathbb{N}.$$

From the long exact sequence, one has

$$R^i S(M_{\sim w\lambda}^-) \cong R^{i+1} S(M_{s_\alpha w\lambda}^-).$$

Remark that

$$M_{w\lambda}^- = H(L_{-w\lambda - \tilde{\rho}}),$$

$$M_{\sim w\lambda}^- = H(L_{-s_\alpha w\lambda - \tilde{\rho}}) .$$

Hence

$$R^i S(H(L_{-w\lambda - \tilde{\rho}})) \cong R^{i+1} S(H(L_{-s_\alpha w\lambda - \tilde{\rho}})) = \begin{cases} 0 & \text{if } i + 1 \neq s(s_\alpha w) \\ V_{-\lambda + \tilde{\rho}} & \text{if } i + 1 = s(s_\alpha w) , \end{cases}$$
i.e.
\[ R^i S(H(L_{-w\lambda + \tilde{\rho}})) \cong \begin{cases} 0 & \text{if } i \neq s(w) \\ V_{-\lambda + \tilde{\rho}} & \text{if } i = s(w). \end{cases} \]

The theorem is completely proved. □

As usually, changing \( \lambda \) by \( \lambda + \tilde{\rho} \), we have
\[
\delta_{i,s(w)} V_{-\lambda} = R^i S(H(L_{-w(\lambda + \tilde{\rho})} + \tilde{\rho})) \\
\cong V_{-\lambda} \otimes H^i(\tilde{n}; L_{-w(\lambda + \tilde{\rho})} + \tilde{\rho} \otimes V_{\lambda}') \check{H} \\
\cong V_{-\lambda} \otimes \text{Hom}_{U(\tilde{h})}(L^*_{-w(\lambda + \tilde{\rho}) - \tilde{\rho}}, H^i(\tilde{n}; V_{-\lambda}^*)') \check{H}.
\]

Hence we have just an affine analogue of the Kostant theorem on cohomology of the nilpotent part.

**Theorem 6.2**

\[ H^i(\tilde{n}_+; V) \cong \bigoplus_{w \in W, s(w) = i} L_{w(\lambda + \tilde{\rho}) - \tilde{\rho}} \]

**Remark.** We have deal with \( (\tilde{t}, \tilde{H}) \)-modules and the central characters of type \( e^\lambda \) and the infinitesimal characters of type \( \chi_\lambda \). A similar situation with \( (\tilde{g}, \tilde{K}) \)-modules with the central characters of Harish-Chandra type \( \theta_\lambda \) and the infinitesimal characters \( \chi_\lambda \) gives us an algebraic realization of the discrete series representations for loop groups. This idea will be devoted the next part of our contribution.

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