Gravity on the Bloch Brane

Adalto R. Gomes

Departamento de Ciências Exatas, Centro Federal de Educação Tecnológica do Maranhão, 65025-001 São Luís, Maranhão, Brazil
E-mail: argomes@cefet-ma.br

ABSTRACT: Bloch branes were introduced previously and are constructed in a system described by two real scalar fields coupled with gravity in (4, 1) dimensions in warped spacetime involving one extra dimension. This work investigates gravity on such thick branes with internal structure and focuses on the effects of massive graviton modes localization on the brane and to what extent they might reproduce the 4d gravity at a scale before escaping into the extra dimension. In this way gravitational measurements on the brane could reveal the existence of the extra dimension on some scales, with possible applications on brane cosmology.

KEYWORDS: Classical Theories of Gravity, Brane models.
1. Introduction

Domain walls as extended defects in field theory have been used in high energy physics for representing brane scenarios with extra dimensions. Considered as domain walls with an internal structure, the Bloch brane is constructed in a $(4,1)$ model of two scalar fields coupled with gravity. The interaction among the fields depends on a real parameter, which generalizes the standard $\phi^4$ model, changing the way the scalar field self-interacts. In this way the brane is generated dynamically and the asymptotic bulk metric is a slice of an $AdS_5$ space. This and other thick brane models are alternatives to the infinitely thin brane models (see, for instance, Ref. and extensions), there constructed with non-dynamical source terms in the action.

In a previous work the internal structure of the Bloch brane was analyzed in terms of the matter energy density as a function of the extra dimension. It was found that in a certain region of the coupling parameter, an splitting of the defect in two interfaces occurs. In that work the effect of gravity on the internal structure of the domain wall was made evident as an induced attraction between the internal interfaces that form the defect.

This intriguing possibility has led us to refine the former investigations and search for resonance states for our $r$ small. We found that this occurs in the region of parameter where the brane splitting is more accentuated as we report on section 3. In the same section we considered the Newtonian potential for the whole range of the $r$ parameter with the aim to study gravity localization. This is the main issue of the present work. Before studying this problem, however, let us first review some of the results of the Bloch brane model. It is described by the action

$$S = \int d^4x \, dy \sqrt{|g|} \left[ -\frac{1}{4} R + \frac{1}{2} \partial_a \phi \partial^a \phi + \frac{1}{2} \partial_a \chi \partial^a \chi - V(\phi, \chi) \right]$$

(1.1)

where $g = \det(g_{ab})$ and the metric

$$ds^2 = g_{ab}dx^a dx^b$$

$$= e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$$

(1.2)
describes a background with 4-dimensional Poincare symmetry with \( y \) as the extra dimension. Here \( a, b = 0, 1, 2, 3, 4 \), and \( e^{2A} \) is the warp factor. We suppose that the scalar fields and the warp factor only depend on the extra coordinate \( y \).

The action given by Eq. (1.1) leads to the following coupled differential equations for the scalar fields \( \phi(y), \chi(y) \) and the function \( A(y) \) from the warp factor:

\[
\phi'' + 4A'\phi' = \frac{\partial V(\phi,\chi)}{\partial \phi}, \tag{1.3}
\]

\[
\chi'' + 4A'\chi' = \frac{\partial V(\phi,\chi)}{\partial \chi}, \tag{1.4}
\]

\[
A'' = -\frac{2}{3}(\phi'^2 + \chi'^2), \tag{1.5}
\]

\[
A'^2 = \frac{1}{6}(\phi'^2 + \chi'^2) - \frac{1}{3}V(\phi,\chi), \tag{1.6}
\]

where prime stands for derivative with respect to \( y \).

The potential used is

\[
V(\phi,\chi) = \frac{1}{8}\left[\left(\frac{\partial W}{\partial \phi}\right)^2 + \left(\frac{\partial W}{\partial \chi}\right)^2\right] - \frac{1}{3}W^2, \tag{1.7}
\]

where

\[
W(\phi,\chi) = 2\phi - \frac{2}{3}\phi^3 - 2r\phi\chi^2, \tag{1.8}
\]

and \( r \) is a real parameter. The particular relation from Eq. (1.7) between \( V \) and \( W \) leads to a description of the scalar fields in terms of a coupled set of first-order differential equations, with analytical expressions for \( \phi(y) \) and \( \chi(y) \) identical to the obtained in Refs. [8, 9] for flat spacetime.

The first-order differential equations which also solve the equations of motion are [10, 11]

\[
\phi' = \frac{1}{2}\frac{\partial W}{\partial \phi}, \tag{1.9}
\]

\[
\chi' = \frac{1}{2}\frac{\partial W}{\partial \chi}, \tag{1.10}
\]

and

\[
A' = -\frac{1}{3}W. \tag{1.11}
\]

Similar investigations where also done in Ref. [12].

In the present case the former equations can be solved analytically leading to the following results

\[
\phi(y) = \tanh(2ry), \tag{1.12}
\]
\[ \chi(y) = \sqrt{\left(\frac{1}{r} - 2\right)} \sech(2ry) \quad (1.13) \]

and

\[ A(y) = \frac{1}{9r} \left[ (1 - 3r) \tanh^2(2ry) - 2 \ln \cosh(2ry) \right] \quad (1.14) \]

Graphical analysis from these solutions for several values of the parameter \(0 < r < 0.5\) show that the thickness of the defect increases with decreasing \(r\), with \(A(0) = 0\) implying canonical normalization for the 4-dimensional metric in the transverse space on \(y = 0\) [4]. Note also from Eq. (1.13) that for the limit \(r = 0.5\) the one-field scenario is recovered.

2. Fluctuations and massive modes

We consider now the effective 4-dimensional gravitational fluctuations in the conformally flat background discussed previously, as well as that of scalar fields around solutions (Eqs. 1.12-1.14):

\[ ds^2 = e^{2A(y)} (\eta_{\mu\nu} + \epsilon h_{\mu\nu}) dx^\mu dx^\nu - dy^2 \quad (2.1) \]

and we set \( \phi \to \phi + \epsilon \tilde{\phi} \) and \( \chi \to \chi + \epsilon \tilde{\chi} \), where \( h_{\mu\nu} = h_{\mu\nu}(x, y) \), \( \tilde{\phi} = \tilde{\phi}(x, y) \) and \( \tilde{\chi} = \tilde{\chi}(x, y) \) represent small perturbations. On the transverse traceless gauge the metric perturbation separates from the scalars [10], leading to

\[ \tilde{h}''_{\mu\nu} + 4 A' \tilde{h}'_{\mu\nu} = e^{-2A} \Box \tilde{h}_{\mu\nu}, \quad (2.2) \]

with \( \Box \) the (3,1)-dimensional d’Alembertian.

This equation can be decoupled by separating the 4-dimensional plane wave perturbations in a form of plane waves from the extra dimension contribution. We introduce a new variable \( z \) that turns the metric into a conformal one. This changes the equation for the extra dimension contribution of the metric perturbations in a Schroedinger-like form, where no single derivative terms are present. The new conformal coordinate \( z \) is defined by \( dz = e^{-A(y)} dy \). The separation of variables taken as

\[ \tilde{h}_{\mu\nu}(x, z) = e^{ip\cdot x} e^{-\frac{3}{2}A(z)} \psi_{\mu\nu}(z), \quad (2.3) \]

turns Eq. (2.2) into a Klein-Gordon equation for the 4-dimensional components of the transverse-traceless \( \tilde{h}_{\mu\nu} \), with the remaining Schroedinger-like equation

\[ -\frac{d^2 \psi_m(z)}{dz^2} + V_{sch}(z) \psi_m(z) = m^2 \psi_m(z) \quad (2.4) \]

for its fifth-dimensional component, where we dropped the \( \mu\nu \) indices for the wavefunctions, now labeled by their corresponding energy \( m^2 \). Here the potential is given by

\[ V_{sch}(z) = \frac{3}{2} A''(z) + \frac{9}{4} A'(z) \quad (2.5) \]
Particularly interesting is that in this new conformal variable the appropriate inner-product for the wavefunctions $\psi(z)$ is the conventional quantum mechanical one \[13\]; in this way we can interpret a normalized $|\psi_m(z)|^2$ as the probability for finding a massive mode at the position labeled by $z$.

The solutions of Eq.(2.4) for general $m$ form a tower of states associated with the massive modes responsible for the gravitational interaction in the semiclassical theory we are considering. Note that the l.h.s. of Eq.(2.4) can be written as a positive definite Hermitean operator, meaning absence of tachyonic excitations \[14\] with a zero mode (normalized $(m = 0)$ state) given by

$$\psi_0(z) = Ne^{3A(z)/4},$$

(2.6)

where $N$ is a normalization factor.

Graphical analysis from Eq.(2.3) shows that for some region of parameters $V_{sch}(z)$ is a “volcano-type potential” with a true minimum at $y = 0$ for $r^* < r < 0.5$, whereas for $r < r^*$ there appears a splitting of the potential with two separate minima. In Fig. 1 we reproduced the figure from Ref. 4 in order to ease the comparison with the results from next section. As the potential goes to zero as $|z| \to \pm\infty$, the possibility for the model to reproduce 4-dimensional gravity can be investigated. We are particularly interested on massive gravity on the Bloch brane, mainly on the effect of the massive spectrum of excitations for the localization of gravity on the brane. The form of the potential shows that besides the zero mode there are no other normalized states, with the massive states escaping from the brane as plane waves.

**Figure 1:** Plots of the Schroedinger-like potential $V_{sch}(z)$ for $r = 0.05$ (thinner trace) and $r = 0.10$. Case $r = 0.30$ (thicker trace) corresponds to $1/2 V_{sch}(z)$ to ease comparison.

The zero mode is responsible for gravity localization on the brane, whereas the massive modes introduce modifications on the Newtonian potential where in some scale gravity on the brane turns out to be essentially 5-dimensional.

We numerically studied the Newtonian potential for two unit test masses separated by a distance $R$. In order to obtain the Newtonian potential one sums the tower of Kaluza-
Klein excitations to the usual contribution coming from the zero-mode as \[13\]

\[ U(R) = G \frac{1}{R} + \frac{1}{M^4} \int_0^\infty dm \frac{e^{-mR}}{R} |\psi_m(0)|^2, \tag{2.7} \]

where the 4-dimensional coupling \( G = M_4^{-2} \), \( M_4 \) is the fundamental 5-dimensional Planck scale and the integration is considered at the brane position \( z = 0 \) on the thin brane case. Separating from the Einstein-Hilbert action the 4-dimensional part from the extra dimensional one leads to an expression relating the two scales as \[13\]

\[ M_4^2 = M_5^3 \int_{-\infty}^{+\infty} dz e^{-3A(z)/2}. \tag{2.8} \]

In this way one can see that those scales are related to the integral of the squared wavefunction for the zero-mode - as we have imposed \( \psi_0(0) = 1 \) we cannot associate a true probability for \( \psi_0(z) \). That is the reason one can say integrability of the zero-mode is equivalent to a non-null 4-dimensional gravitational constant. Note that this does not guarantee localization of gravity in all scales, as the massive modes can modify sensibly the gravitational potential on the brane. However, see from Eq.\( (2.7) \) that the higher massive modes are exponentially suppressed due to the Yukawa coupling. In this way gravity localization and the reproducibility of 4-dimensional gravity on the brane is strongly related to the decoupling of the soft KK modes on the brane. In order to make easier the numerical approach to obtain the function \( U(R) \), see that the integral on the continuum modes is already saturated at \( m \sim 10/R \).

Eq.\( (2.8) \) shows that \( |\psi_m(0)|^2 \) is of fundamental importance in order to achieve a true description of the Newtonian potential. As the massive modes are on the form of plane waves, normalization can be considered in a box of fixed length. The conformal transformation from \( y \) to \( z \) is done in order to achieve a distribution of points on the \( z \) variable with fixed step with a controlled precision. With the function \( y(z) \) one can obtain \( A(z) \) and the Schrödinger potential \( V_{sch}(z) \). The massive modes are obtained using the Numerov method \[15\], whose efficiency was evaluated after comparison of the \( m = 0 \) case with the expression given by Eq. \( (2.7) \) for the zero-mode. For each value of the parameter \( r \) we have chosen a box extending from \( z = -z_{max} \) to \( z = z_{max} \), with \( z_{max} \) chosen for each \( r \) such that the potential \( V_{sch} \) was near to the asymptotic value zero. This and a sufficiently large number of points turns out to be necessary in order to obtain better efficiency for comparison with the zero mode. We also expect that the normalization procedure for higher massive modes turns out to have higher confidence as the box collects larger number of wavelengths. We conclude from Fig. 1 that for lower values of \( r \) one needs boxes with larger \( z_{max} \). The Numerov method consists of a discretization of the Schrödinger equation in order to get a recurrence formula after choosing a fixed value for the non-normalized wavefunction at \( z = 0 \).
Table I: Ratio between 4-dimensional coupling and fundamental 5-dimensional Planck scale for some values of $r$. Note the greater importance for $M_4$ for smaller values of $r$. Third column contains the normalized zero-mode on the brane obtained using Eq. 2.6. This can be compared with the corresponding values for $m = 0$ obtained with Numerov method and displayed on Fig. 2.

| $r$  | $M_4^2/M_5^2$ | $|\psi_0(0)|^2$ (Eq. 2.6) |
|------|---------------|--------------------------|
| 0.05 | 14.67         | 0.0681                   |
| 0.10 | 8.85          | 0.113                    |
| 0.15 | 6.62          | 0.151                    |
| 0.20 | 5.43          | 0.184                    |
| 0.30 | 4.13          | 0.241                    |
| 0.50 | 3.04          | 0.329                    |

Fig. 2 shows the normalized probability for massive modes on the brane as a function of the masses of the modes for $r = 0.3$ and $r = 0.05$. First note that in the limit $m \to 0$, for the chosen parameters displayed on the same figure, Numerov method gives $|\psi_0(0)|^2 = 0.247$ for $r = 0.3$ and $|\psi_0(0)|^2 = 0.0690$ for $r = 0.05$. Comparing those values with Table I, this results in a relative error for determining the zero-mode of 2.5% and 1.3%, respectively. This shows that our method has higher precision for lower values of $r$. In particular, for higher values of $r$ this error is difficult to reduce considerably as the Schrödinger potential goes slowly to zero in comparison to the volcano-like potential for lower values of $r$. Note also that the near-zero mode scattering states occur on the brane at higher probability, with the higher massive modes tending to a fixed probability. The asymptotic behaviour for higher massive modes shows that exponential suppression on Eq. 2.7 is effective. The appearance of resonances could be identified as isolated peaks on energies below the maximum peak of the volcano potential and be responsible for a modification of the Newtonian potential obtained further in this section. From the figure one can see few evidence of resonances for $r = 0.3$. However, for $r = 0.05$ there is an oscillation on the curve for small masses, indicating the presence of resonances for values of $r$ smaller than a critical one. In this way we see that the splitting of the Schrödinger potential for lower values of $r$, besides leading to an internal structure on the brane, also leads to a richer distribution of the massive modes. Extensions of Fig. 2 for higher masses up to $m = 1000$ show that the probabilities tend to constant values on both cases, indicating similar contributions to the 4-dimensional gravity on short distances, as we will see in the following.

To better quantify the results, we are interested on the $L$ parameter, defined as the exponent of the R-dependence of the Newtonian potential,

$$U(R) = \frac{A}{R^L},$$

where $A$ is independent of $R$, and is related to the 5-dimensional gravitational constant $G_5$. From the former equation we get $L = -\frac{\partial (\log U)}{\partial (\log R)}$. The dependence of the exponent $L$ as a function of $\log R$ has an interesting interpretation. It describes the dimensionality
Figure 2: Normalized probability for finding massive modes on the brane as a function of the mass of the modes, for $r = 0.3$ (left) and $r = 0.05$ (right). $Z_{max}$ is for box thickness, $N$ the number of points and $\delta z$ the step on $z$ variable.

of the gravitational interaction as a function of the scale of separation between two test masses.

Note, however, from Eq.(2.7) that in order to obtain the Newtonian potential, an important information is the relation between the two scales $M_2^2$ and $M_3^2$. Table I displays some results for their ratio depending on the $r$ parameter. Note from the Table that the 4-dimensional coupling constant $G = 1/M_4^2$ is more pronounced for larger values of $r$, indicating that the gravity localization is favored for those parameters. In particular, we see that when the field $\chi$ is tuned to zero for the limit $r = 0.5$ we see that $G$ achieves the maximum value compared to the fundamental scale. In this way we can associate the introduction of the additional field $\chi$ and the consequently richer structure of the brane to a larger modification of 4-dimensional gravity due to the extra dimension.

Some results for $L$ as a function of $\log R$ can be seen in Fig. 3 for coupling parameter $r = 0.05$ and $r = 0.3$. From the figure one can see that the exponent $L$ interpolates between the usual 4-dimensional gravity $L = 1$ to a pure 5-dimensional gravity $L = 2$. The extra-dimensional characteristic is revealed on the brane for short separation between the test
masses, whereas the usual Newtonian potential is recovered for the whole scale of distances above a certain transition region. In Fig. 3 one can see that the explicit contribution for $L$ coming from the continuum modes would induce a transition region with higher and lower values around the $L = 2$ region. In this way one can see that in the present model the normalized zero mode is essential for a proper recovering of the 4-dimensional gravity on the brane. Note also that on the transition region, one gets a higher dimensionality for lower values of $r$. If we compare this with Fig. 1 we see that the splitting of the Schrödinger potential favors the extra dimension to be revealed on the brane.

Note that the Bloch brane scenario leads to different limit than RS model for the Newtonian potential on short distance scales. As one knows, the $AdS$ case for RS model gives $L = 3$ for those scales. However, the continuum modes decouple and the zero-mode is normalized in both scenarios, leading to corrections to Newton’s law that are suppressed on large distance scales.

Figure 3: Gravity localization on the brane for $r = 0.3$ and $r = 0.05$: for large scales the Newtonian potential is essentially 4-dimensional.

3. Conclusions

In this work we have studied the Bloch brane, with particular interest on signals from the extra dimension on the Newtonian potential. The spectrum analysis for the massive modes indicates a richer structure for low values for the parameter $r$, where some resonances do appear. This richer structure can be related to the splitting of the Schrödinger potential observed on Ref. 4. Also the resonance structure was revealed due to an improvement of the numerical normalization of the massive modes in a box and increased the understanding
of the Bloch brane. Our results for the ratio $M_2^2/M_3^2$ indicate that modifications coming from the extra dimension turns to be more important for lower values of the $r$ parameter. This observation was confirmed from the Newtonian potential analysis where we have considered the change of the potential decay law $U \sim 1/R^L$ with the increase of the separation $R$ between two unit masses. We analyzed numerically the continuous change of the exponent $L$ with log($R$) indicating the change of a pure 4-dimensional character for the gravitational interaction on the brane to a 5-dimensional one as one moves to shorter separation between the masses. In this way our result addresses the effort for experimental determination of gravitational interaction on short distances. In the Bloch brane scenario experimental modifications for the usual Newtonian law would indicate the presence of the extra dimension, for the parameter $r$ very small.

4. Acknowledgements

The author thanks D. Bazeia and F.A. Brito for suggestions and stimulating discussions and CNPq for financial support.

References

[1] V.A. Rubakov and M.E. Shaposhnikov, Phys. Lett. B 125, 136 (1983).
[2] M. Visser, Phys.Lett. B 159, 22(1985).
[3] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999); [arXiv:hep-th/9906064].
[4] D. Bazeia and A.R. Gomes, JHEP 0405 (2004) 012; [hep-th/0403141].
[5] M. Gremm, Phys. Lett. B 478, 434 (2000); hep-th/9912060.
[6] A. Melfo, N. Pantoja, and A. Skirzewski, Phys. Rev. D 67, 105003 (2003); gr-qc/0211081.
[7] C. Csaki, J. Erlich, and T. Hollowood, Phys.Rev.Lett. 84 (2000) 5932-5935; [arXiv:hep-th/0002161].
[8] D. Bazeia, M.J. dos Santos and R.F. Ribeiro, Phys. Lett. A208, 84 (1995).
[9] D. Bazeia and F.A. Brito, Phys. Rev. D 61, 105019 (2000).
[10] O. DeWolfe, D.Z. Freedman, S.S. Gubser, and A. Karch, Phys. Rev. D 62, 046008 (2000).
[11] K. Skenderis and P. Townsend, Phys. Lett. B 468, 46 (1999); [arXiv:hep-th/9909070].
[12] M. Cvetic and H.H. Soleng, Phys. Rept. 282 (1997) 159; [arXiv:hep-th/9604090].
[13] C. Csaki, J. Erlich, T. Hollowood and Y. Shirman, Nucl.Phys. B581 (2000) 309-338; [arXiv:hep-th/0001033].
[14] D. Bazeia, C. Furtado and A.R. Gomes, J. Cosmol. Astropart. Phys. 0402 (2004) 002; [Arxiv: hep-th/0308034].
[15] M. Mueller and H. Huber, *Solution of the 1-D Schroedinger Equation*, http://www.mapleapps.com