New analytic bending, buckling, and free vibration solutions of rectangular nanoplates by the symplectic superposition method

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New analytic bending, buckling, and free vibration solutions of rectangular nanoplates with combinations of clamped and simply supported edges are obtained by an up-to-date symplectic superposition method. The problems are reformulated in the Hamiltonian system and symplectic space, where the mathematical solution framework involves the construction of symplectic eigenvalue problems and symplectic eigen expansion. The analytic symplectic solutions are derived for several elaborated fundamental subproblems, the superposition of which yields the final analytic solutions. Besides Lévy-type solutions, non-Lévy-type solutions are also obtained, which cannot be achieved by conventional analytic methods. Comprehensive numerical results can provide benchmarks for other solution methods.

Nanoplates play an important role in the micro- and nano-scale technology, with applications to nano sensors, resonators, storage components, micro switches, etc. Mechanical behaviors, including bending, buckling, and free vibration, are frequently encountered in response to external excitations during the use of nanoplates. Accordingly, investigations on such behaviors are crucial for understanding the mechanical properties as well as providing guidelines for structural safety designs of relevant devices. In order to avoid enormous computational efforts when carrying out discrete atomistic or molecular dynamics simulation, some continuum theories considering scale effects, which were not incorporated in classical plate theories, have been proposed, including the couple stress elasticity theory, strain gradient theory, micro-morphic theory, surface energy incorporated continuum theory, etc. One of the well-accepted models is the non-local continuum theory by Eringen, which assumes that the stress at a point is a function of the strains at all the other points in the domain. Lu et al. established the non-local elastic plate theories based on Eringen’s theory, where the basic equations for the non-local Kirchhoff and Mindlin plate theories were derived, and the bending and free vibration problems of a rectangular nanoplate with simply supported edges were solved. It was shown that, for very small-sized plates, the influences of non-local effects on the mechanical properties are considerable.

Many studies have been conducted on modeling two-dimensional plate-like structures using the nonlocal plate theories. Some notable progresses by numerical methods are briefly reviewed in the following. Pradhan and Murmu explored the small-scale effect on the buckling analysis of biaxially compressed simply supported single-layered graphene sheets (SLGS) by computing the buckling loads using the differential quadrature method (DQM), which was also employed for vibration analysis of SLGS embedded in elastic medium. Malekzadeh and Shojae extended a two-variable refined plate theory for the free vibration of nanoplates, where different types of boundary conditions (BCs) were studied by the DQM. Farajpour et al. used the DQM to analyze the buckling of higher-order and lower-order nonlocal strain gradient theory based orthotropic micro/nanoscale plates. Mohammadalehi et al. investigated the vibration features of rectangular viscoelastic nanoplates with variable thickness by the DQM. Ghadiri et al. applied the generalized DQM to investigate the thermo-mechanical vibration of orthotropic cantilever and propped cantilever nanoplates. The method was also adopted by Ebrahimi et al. to analyze the thermo-mechanical vibration of rotating nonlocal nanoplates. Phadikar and Pradhan
reported finite element formulations for nonlocal elastic Euler–Bernoulli beam and Kirchhoff plate, and analyzed bending, vibration, and buckling of simply supported nonlocal plates. Bahu and Patel developed an improved quadrilateral finite element for nonlinear second-order strain gradient elastic Kirchhoff plates based on the nonlocal theory. Necir developed the hierarchical finite element method for size-dependent free vibration analysis of Mindlin nano-plates with curvilinear plan-forms. Akgoz and Civalek employed the discrete singular convolution method for the free vibration and bending analysis of nano-scaled graphene sheets having sector shape. Babaei and Shahidi investigated the buckling behavior of quadrilateral SLGS under bi-axial compression by the Galerkin method, where the buckling loads of nanoplates with different geometrical parameters were obtained. Rahimi et al. studied the thermoelastic damping of in-plane vibration of the functionally graded nanoplates using the Galerkin method. Based on three-dimensional nonlocal elasticity theory, Shahrbabaki developed novel trigonometric series to be used as approximating functions in the Galerkin based approach in dealing with free vibration problems of rectangular nanoplates. Aghababaei and Reddy addressed the bending and vibration characteristics of isotropic and orthotropic nanoplates using the spline finite strip method (FSM). Sarrami-Foroushani and Azhari examined the vibration and buckling characteristics of single and multi-layered graphene sheets by the FSM. Wang presented highly accurate solutions for free vibration and eigen buckling of rectangular nanoplates with the iterative separation-of-variable (iSOV) method. Thanh et al. conducted the bending, buckling, and vibration analyses of microplates via the isogeometric method with couple stress theory, and further extended the method to the thermal buckling and post-buckling analyses of functionally graded micro-plates with porosities.

Although various effective numerical methods have been developed to study the mechanical behaviors of nanoplates, it is still important to explore new analytic methods because they cannot only provide benchmark theoretical solutions of permanent interests, but can also explicitly capture the relationships among different mechanical quantities, thus can serve as useful tools for validation of numerical methods, rapid parameter analyses, and efficient structural designs. However, due to the recognized mathematical difficulties in solving the complex boundary-value problems of governing higher-order partial differential equations (PDEs), the applicability of conventional analytic methods is generally restricted to some specific cases such as Navier-type and Lévy-type rectangular nanoplates, i.e., those fully simply supported or with at least two parallel edges simply supported. Some representative studies in this regard are briefly reviewed here. Aghababaei and Reddy reformulated the third-order shear deformation plate theory using the nonlocal theory, and presented analytical solutions of bending and free vibration of a simply supported rectangular nanoplate. Aksener and Aydogdu used Navier-type solution and Lévy-type solution for vibration and buckling of simply supported nanoplates and those with two opposite edges simply supported. Sumelka presented fractional calculus as a new formulation to study the nonlocal Kirchhoff–Love plates, taking the case of simply supported plate as an illustrative example. Based on Reddy’s nonlocal third-order shear deformation plate theory, Hosseini-Hashemi et al. obtained Lévy-type solutions for buckling and vibration problems of rectangular nanoplates. Ilkhani et al. applied the wave propagation approach to determine the natural frequencies of rectangular nanoplates with two opposite edges simply supported. Jamalpoor et al. adopted the Navier approach to solve free vibration and biaxial buckling of double-magneto-electro-elastic nanoplate-systems subjected to initial external electric and magnetic potentials. Moradi-Dasjerdi et al. applied the Navier approach at the free vibration analysis of nanocomposite sandwich plates reinforced with CNT aggregates. Arefi et al. adopted the Navier-form solutions to analyze the free vibration of a sandwich nano-plate including FG core and piezoelectric face-sheets. Cornacchia et al. obtained the Navier solutions for vibration and buckling of Kirchhoff nanoplates using second-order strain gradient theory. Besides, Yang et al. utilized the Bessel functions to settle the bending problems of circular nanoplates under concentrated and uniform loads.

In recent years, we have proposed an analytic symplectic superposition method for mechanics problems of plates and shells based on classical theories, which proved to be widely applicable to bending, buckling, and vibration problems. The solution procedure involves three main steps, i.e., converting an original problem into several elaborated subproblems, solving the subproblems within the Hamiltonian system by the symplectic approach, and superposition of the subproblems for the final solution. Specifically, the symplectic eigenvalue problems of a Hamiltonian matrix are introduced, followed by symplectic eigen expansion, to yield the analytic solutions of the subproblems, which are exclusive mathematical techniques in the symplectic space rather than in the traditional Euclidean space. However, since the governing equations of the nanoplate problems are much more complex than those of the classical plate problems, there has been almost no research on developing the symplectic superposition method for analytic modeling of similar issues. In the following, for the first time, the symplectic superposition method is extended to obtain the analytic bending, buckling, and free vibration solutions of rectangular nanoplates with all combinations of clamped and simply supported edges, including both Lévy-type and non-Lévy-type solutions. Comprehensive benchmark results are presented to show fast convergence and high accuracy of the present solutions by excellent agreement with those obtained by the finite element method (FEM) and other numerical methods in the open literature. The effects of the nonlocal parameter and plate dimensions on the mechanical behaviors of the nanoplates are well examined with the present analytic solutions. Some useful conclusions are drawn to reflect the small-scale effects that are not captured in classical theories.
Governing equation for bending, buckling, and free vibration of nanoplates in the Hamiltonian system

Based on the nonlocal theory by Eringen\textsuperscript{10}, the transformed differential constitutive equation\textsuperscript{27} is

\[
\sigma^n_{ij} - \mu \nabla^2 \sigma^n_{ij} = \sigma_{ij} = S_{ijkl} \varepsilon_{kl}
\]

(1)

where \(\sigma_{ij}, \sigma^n_{ij}, S_{ijkl},\) and \(\varepsilon_{kl}\) denote the components of local stress tensor, nonlocal stress tensor, fourth order stiffness tensor and strain tensor, respectively, \(\mu = (e_0 l)^2\) is the nonlocal parameter depending on the internal characteristic length, \(l\), and an experimentally defined material constant, \(e_0\). For isotropic thin nanoplates, we have

\[
\left\{ \begin{array}{l}
\sigma^n_{xx} \\
\sigma^n_{yy} \\
\sigma^n_{xy}
\end{array} \right\} - \mu \nabla^2 \left\{ \begin{array}{l}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{array} \right\} = \left\{ \begin{array}{l}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{array} \right\} = \left\{ \begin{array}{ccc}
\frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\
\frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\
0 & 0 & \frac{E}{2(1+\nu)}
\end{array} \right\} \left\{ \begin{array}{l}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{xy}
\end{array} \right\}
\]

(2)

where \(E\) and \(\nu\) are Young's modulus and Poisson's ratio, respectively. The stress resultants are defined as

\[
M^n = \left\{ \begin{array}{l}
M^n_x \\
M^n_y \\
M^n_{xy}
\end{array} \right\} = \int_{-h/2}^{h/2} \left\{ \begin{array}{l}
\sigma^n_{xx} \\
\sigma^n_{yy} \\
\sigma^n_{xy}
\end{array} \right\} \, dz
\]

(3)

where \(h\) is the thickness of the nanoplate. The moment component resultants are thus

\[
\left\{ \begin{array}{l}
M^n_x \\
M^n_y \\
M^n_{xy}
\end{array} \right\} - \mu \nabla^2 \left\{ \begin{array}{l}
M_x \\
M_y \\
M_{xy}
\end{array} \right\} = \left\{ \begin{array}{ccc}
D & 0 & 0 \\
0 & D & 0 \\
0 & 0 & \frac{D}{2(1+\nu)}
\end{array} \right\} \left\{ \begin{array}{l}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{xy}
\end{array} \right\}
\]

(4)

where \(D = Eh^3/12(1-\nu^2)\) is the bending stiffness.

The nonlocal theory-based quadratic functional for a nanoplate within the domain \(\Omega\) resting on a two-parameter elastic foundation as shown in Fig. 1\textsuperscript{a,b} is written\textsuperscript{27,28,31} as

\[
\Pi_{total} = \iint_{\Omega} \left[ U(x,y) - T(x,y) + V(x,y) - Q(x,y) \right] \, dx\, dy
\]

(5)

in which
\[ U(x, y) = \frac{1}{2} \left[ \frac{\partial^2 w}{\partial x^2} \right]^2 + 2\nu \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2(1 - \nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + k_w \left\{ w^2 + \mu \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] \right\} \right) \]

\[ T(x, y) = \frac{1}{2} m_0 \omega^2 \left\{ w^2 + \mu \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] \right\} + m_2 \left\{ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right\} \right) \]

\[ V(x, y) = \frac{1}{2} \left\{ N_y \left[ \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] + N_x \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} + N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 \]

\[ Q(x, y) = q(x, y) \left[ w - \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \right] \]

Here, \( x \) and \( y \) are the coordinate variables; \( w \) is the transverse deflection of the nanoplate; \( k_w \) and \( k_p \) are Winkler and Pasternak foundation coefficients, respectively; \( m_0 = \int_{-h/2}^{h/2} \rho \, dz \), and \( m_2 = \int_{-h/2}^{h/2} \rho z^2 \, dz \), with \( \rho \) being the mass density of the nanoplate; \( N_x \) and \( N_y \) are the membrane forces along the \( x \) and \( y \) directions, respectively; \( q(x, y) \) is the transverse external load. Putting \( \mu = 0 \), the quadratic functional for the classical thin plate model is obtained.

The variation of the governing equation (5) about \( w \) yields

\[ -R_x \frac{\partial^4 w}{\partial x^4} - R_y \frac{\partial^4 w}{\partial y^4} - R_{xy} \frac{\partial^4 w}{\partial x \partial y^3} + \bar{R}_x \frac{\partial^2 w}{\partial x^2} + \bar{R}_y \frac{\partial^2 w}{\partial y^2} + (\bar{m}_0 - k_w)w + (1 - \mu \nabla^2)q = 0 \]

where \( R_x = D + \mu \left( k_p + N_x - \bar{m}_x \right), \quad R_y = D + \mu \left( k_p + N_y - \bar{m}_y \right), \quad R_{xy} = R_x + R_y, \quad \bar{R}_x = k_p + N_x - \bar{m}_x, \quad \bar{R}_y = k_p + N_y - \bar{m}_y, \quad \bar{m}_0 = \omega^2 m_0, \) and \( \bar{m}_2 = \omega^2 m_2. \)

Define the generalized displacement vector

\[ \mathbf{q} = (w, \theta_x)^T \]

where

\[ \theta_x = -\frac{\partial w}{\partial x} \]

The corresponding generalized force vector is

\[ \mathbf{p} = \frac{\partial \Pi_{total}}{\partial \mathbf{q}} = \left\{ \begin{array}{c} V_x^n \\ M_x^n \end{array} \right\} \]

where \( (\cdot) = \partial(\cdot) / \partial x, \) and

\[ V_x^n = -R_x \frac{\partial w^3}{\partial x^3} - (R_{xy} - D_y) \frac{\partial w^3}{\partial x \partial y^2} + \bar{R}_x \frac{\partial w}{\partial x} - \mu q \]

\[ M_x^n = -R_x \frac{\partial w^2}{\partial x^2} - D_v \frac{\partial w^2}{\partial y} - \mu q \]

are respectively the nonlocal equivalent shear force and nonlocal bending moment in the cross sections perpendicular to the \( x \) axis. By coordinate exchange, we have

\[ V_y^n = -R_y \frac{\partial w^3}{\partial y^3} - (R_{xy} - D_x) \frac{\partial w^3}{\partial x^2 \partial y} + \bar{R}_y \frac{\partial w}{\partial y} - \mu q \]

\[ M_y^n = -R_y \frac{\partial w^2}{\partial x^2} - D_v \frac{\partial w^2}{\partial x} - \mu q \]

A new set of quantities excluding the external load are introduced as
\[
V_x = -R_y \frac{\partial w}{\partial x^3} - (R_y - Dv) \frac{\partial w}{\partial x \partial y^2} + \hat{R}_x \frac{\partial w}{\partial x} \\
M_x = -R_y \frac{\partial w^2}{\partial x^2} - Dv \frac{\partial w^2}{\partial y^2} \\
V_y = -R_x \frac{\partial v}{\partial y^3} - (R_x - Dv) \frac{\partial v}{\partial x^2 \partial y} + \hat{R}_y \frac{\partial v}{\partial y} \\
M_y = -R_x \frac{\partial v^2}{\partial y^2} - Dv \frac{\partial v^2}{\partial x^2}
\]

From Eq. (12),
\[
\frac{\partial w}{\partial x} = -\theta_x
\]

From Eq. (19) and the second equation of Eq. (18),
\[
\frac{\partial \theta_x}{\partial x} = \frac{M_x}{R_x} + \frac{Dv \partial^2 w}{\partial y^2}
\]

From Eq. (19) and the first two equations of Eq. (18),
\[
\frac{\partial M_x}{\partial x} = V_x - (R_y - 2Dv) \frac{\partial^2 \theta_x}{\partial y^2} + \hat{R}_x \theta_x
\]

From Eq. (10) and the first two equations of Eq. (18),
\[
\frac{\partial V_x}{\partial x} = \left[ R_y - \frac{(Dv)^2}{R_x} \right] \frac{\partial^2 w}{\partial y^2} - \hat{R}_y \frac{\partial^2 w}{\partial y^2} + (k_w - \bar{m}_0)w - \frac{Dv}{R_x} \frac{\partial^2 M_x}{\partial y^2} - (1 - \mu \nabla^2)q
\]

Equations (19–22) are written in matrix form as
\[
\frac{\partial Z}{\partial x} = HZ + f
\]

where \( Z = [w, \theta_x, V_x, M_x]^T \) is referred to as the state vector; \( H = \begin{bmatrix} F & G \end{bmatrix} \), with
\[
Q = \begin{bmatrix} R_y - \frac{(Dv)^2}{R_x} & \frac{\partial^2 w}{\partial y^2} - \hat{R}_y \frac{\partial^2 w}{\partial y^2} + (k_w - \bar{m}_0)w - \frac{Dv}{R_x} \frac{\partial^2 M_x}{\partial y^2} - (1 - \mu \nabla^2)q \\ 0 & -R_x \end{bmatrix}
\]
\[
f = [0, 0, (1 - \mu \nabla^2)q, 0]^T
\]

is the transverse external force vector that only exists in a bending problem. \( H \) is a Hamiltonian operator matrix satisfying \( H^T = JHF \), where \( J = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \) is the unit symplectic matrix with 2×2 unit matrix \( I_2 \); accordingly, Eq. (23) gives the governing equation for bending, buckling, and free vibration of nanoplates in the Hamiltonian system.

**Fundamental analytic solutions in the symplectic space**

In applying the symplectic superposition method, an original problem is converted into superposition of several elaborated subproblems that are solved in the symplectic space, whose solutions are referred to as the fundamental analytic solutions in this study.

Taking bending of a fully clamped nanoplate as an example, Fig. 1c–f schematically shows the symplectic superposition of the problem, where the nanoplate (Fig. 1c) has length \( a \) and width \( b \), with the axes \( ox \) and \( oy \) along the plate edges. Corresponding to the bending moments excluding the external load, as expressed in Eq. (18), we denote the BCs \( w = 0, M_x = 0 \) at \( x = 0, a \) and \( w = 0, M_y = 0 \) at \( y = 0, b \) by “S”. In comparison, the actual simply supported conditions of a nanoplate, denoted by “S”, imply \( w = 0, M_y = 0 \) at \( x = 0, a \) and \( w = 0, M_y = 0 \) at \( y = 0, b \). The first subproblem (Fig. 1d) is for a transversely loaded nanoplate with all edges S-supported. For the second subproblem (Fig. 1e), the same S-supported nanoplate is driven by a pair of nonzero \( M_x |_{x=0} \) and \( M_y |_{x=a} \) that are expanded as \( \sum_{n=1,2,3,...} E_n \sin (\alpha_n x) \) and \( \sum_{n=1,2,3,...} F_n \sin (\beta_n y) \), respectively. For the third subproblem (Fig. 1f), the same S-supported nanoplate is driven by a pair of nonzero \( M_y |_{y=0} \) and \( M_y |_{y=b} \) that are expanded as \( \sum_{n=1,2,3,...} G_n \sin (\alpha_n x) \) and \( \sum_{n=1,2,3,...} H_n \sin (\beta_n y) \), respectively. Here, \( \alpha_n = n\pi / a \), \( \beta_n = n\pi / b \); \( E_n, F_n, G_n, \) and \( H_n \) are the series expansion coefficients, which will be determined later. The BCs are thus
\[
w_{x=0,a} = 0, M_{x|x=0,a} = 0 \\
w_{y=0,b} = 0, M_{y|y=0,b} = 0
\]

for the first subproblem,
\[
M_x|_{x=0} = \sum_{n=1,2,3,\ldots}^{\infty} E_n \sin (\beta_n y), \quad M_x|_{x=a} = \sum_{n=1,2,3,\ldots}^{\infty} E_n \sin (\beta_n y)
\]

\[
w|_{x=0,a} = 0, \quad w|_{y=0,b} = 0, \quad M_f|_{y=0,b} = 0
\]

for the second subproblem, and

\[
w|_{x=0,a} = 0, \quad M_x|_{x=0,a} = 0, \quad w|_{y=0,b} = 0
\]

\[
M_f|_{y=0} = \sum_{n=1,2,3,\ldots}^{\infty} G_n \sin (\alpha_n x), \quad M_f|_{y=b} = \sum_{n=1,2,3,\ldots}^{\infty} H_n \sin (\alpha_n x)
\]

for the third subproblem.

All the three subproblems come down to a general problem for a nanoplate with a pair of opposite edges \( \overline{S} \)-supported. Taking the nanoplate \( \overline{S} \)-supported at \( y=0 \) and \( y=b \) as an example, the homogeneous equation of Eq. (23) is

\[
\frac{\partial Z}{\partial x} = HZ
\]

The validity of variable separation in the symplectic space\(^{32} \) gives

\[
Z = X(x)Y(y)
\]

where \( X(x) \) is a function of \( x \), and

\[
Y(y) = [w(y), \theta_x(y), V_x(y), M_z(y)]^T
\]

is a vector with argument \( y \). Substituting Eq. (28) into Eq. (27) yields

\[
\frac{dX(x)}{dx} = \xi X(x)
\]

and

\[
HY(y) = \xi Y(y)
\]

with the eigenvalue \( \xi \) and the eigenvalue \( Y(y) \). With the BCs at \( y=0 \) and \( y=b \), we obtain the eigenvalues and eigenvectors:

\[
\xi_{n1} = -\xi_{n2} = -\xi_{n3} = -\xi_{n4} = \sqrt{\frac{\hat{R}_x + \hat{R}_y \beta_n^2 - \sqrt{(\hat{R}_y + \hat{R}_x \beta_n^2 + \hat{R}_z^2)^2 - 4\hat{R}_z [\hat{R}_y^2 (\hat{R}_y + \beta_n^2 \hat{R}_z) + k_w - \bar{m}_0]}}{2\hat{R}_z}}
\]

and

\[
Y_{ni}(y) = \sin (\beta_n y) \left[ 1, -\xi_n, \frac{k_w - \bar{m}_0 + \beta_n^2 (\beta_n^2 \hat{R}_y + \hat{R}_y^2 \beta_n^2 + \hat{R}_z^2) + \frac{\partial \xi_n^2}{\xi_n} \beta_n^2 \hat{R}_y + \hat{R}_y^2 \xi_n^2}{\beta_n^2 \hat{R}_y + \hat{R}_y^2 \xi_n^2} \right]^T
\]

for \( n = 1, 2, 3, \ldots \) and \( i = 1, 2, 3, \) or 4. Accordingly, we have

\[
HY(y) = Y(y) M
\]

where \( M = \text{diag} \{ \xi_{n1}, \xi_{n2}, \xi_{n3}, \xi_{n4}, \ldots \} \) and \( Y(y) = \{ Y_{n1}(y), Y_{n2}(y), Y_{n3}(y), Y_{n4}(y), \ldots \}^T \). Substituting Eqs. (28) and (34) into the Eq. (23), we have

\[
\frac{dX(x)}{dx} = MX(x) + G
\]

where \( X(x) = \{ X_{n1}(x), X_{n2}(x), X_{n3}(x), X_{n4}(x), \ldots \}^T \), and \( G = \{ g_{n1}, g_{n2}, g_{n3}, g_{n4}, \ldots \}^T \) is the column matrix of the expansion coefficients of \( f \), satisfying \( f = Y(y) G \). Utilizing the eigenvectors' conjugacy and orthogonality, \( G \) is determined by \( \int_0^b Y(y)^T \overline{Y}(y) Gdy = \int_0^b Y(y)^T fdy \).

For the uniform load with intensity \( q_{\overline{o}} \)

\[
g_{ni} = -\frac{q_{\overline{o}} \xi_n [1 - \cos (n\pi \xi_n)]}{n\pi \beta_n \hat{R}_y + \hat{R}_y^2 \xi_n^2}
\]

for \( n = 1, 2, 3, \ldots (i = 1, 2, 3, \text{ or 4}) \), where the script “\( \overline{o} \)” corresponds to uniform load. For the concentrated load with intensity \( q_c \) at \( (x_0, y_0) \),
where the script "c" corresponds to concentrated load. From Eq. (35), we then obtain

\[ X_n^{c}(x) = c_n e^{i n x} - \frac{q x_e}{b} \frac{[1 - \cos (n \pi)]}{[k_w - m_0 - \xi_{ni}^2 + \beta_n^2 (\beta_n^2 p_y + \tilde{R}_y)]} \delta(x - x_0) \] (37)

and

\[ X_n^{c}(x) = c_n e^{i n x} - \frac{q x_e H(x - x_0) \sin (\gamma_0 \beta_n) e^{i \gamma_0 (x - x_0)}}{b} \frac{[1 - \cos (n \pi)]}{[k_w - m_0 - \xi_{ni}^2 + \beta_n^2 (\beta_n^2 p_y + \tilde{R}_y)]} \delta(x - x_0) \] (39)

for the cases with uniform and concentrated loads, respectively. Here, \( H(x - x_0) \) is the Heaviside step function, \( c_n \) are the constants to be determined by imposing the remaining BCs at \( x = 0 \) and \( x = a \). The solution of the Eq. (23) is thus expressed by

\[ Z = \sum_{n=1,2,3,...}^{\infty} \sum_{i=1}^{4} X_n Y_{ni} \] (40)

The deflection solution of the nanoplate \( S \)-supported at \( y = 0 \) and \( y = b \), denoted by \( w_S(x, y) \), is thus obtained as

\[ w_S(x, y) = \sum_{n=1,2,3,...}^{\infty} \sum_{i=1}^{4} \sin(\beta_n y) X_n(x) \] (41)

Substituting Eq. (41) into Eq. (24) to determine the constants, we obtain the deflection solution, \( w_1(x, y) \), of the first subproblem. For the uniform loading, we have

\[ \frac{w_1(x, y)}{b} = \sum_{n=1}^{\infty} 2 \ell_n \frac{[\cos (n \pi) - 1]}{n \pi} \psi_{n3} \left\{ \left( \cosh \left( (2 \pi - 1) \gamma_{n3}/2 \right) \right) \text{sech} \left( \gamma_{n3}/2 \right) - 1 \right\} \psi_{n3} \] (42)

where \( \phi = b/a, \xi = b^2 \ell_n/D, x = x/a, y = y/b, \gamma_{n3} = a \xi e_n, Y_{n3} = a \xi e_n, \psi_{n1} = b^4 [k_w - m_0 + \beta_n^2 (\beta_n^2 p_y + \tilde{R}_y) - \tilde{R}_y \xi_{n1}^2]/D, \) and \( Y_{n3} = b^4 [k_w - m_0 + \beta_n^2 (\beta_n^2 p_y + \tilde{R}_y) - \tilde{R}_y \xi_{n3}^2]/D. \) For the concentrated loading, we have

\[ \frac{w_2(x, y)}{a} = \sum_{n=1}^{\infty} 2 \phi \ell_n \frac{[\cos (n \pi) - 1]}{n \pi} \psi_{n3} \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) + \gamma_{n3} \psi_{n3} \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \] (43)

where \( \phi = b/a, \tilde{R}_y = y_0/b, x_0 = x_0/a, \) and \( \ell_n = b q_x/D. \)

Equating \( q_x \) or \( q_y \) with zero, and imposing the BCs in Eq. (25), we obtain the deflection solution of the second subproblem, denoted by \( w_2(x, y) \), as

\[ \frac{w_2(x, y)}{a} = \sum_{n=1,2,3,...}^{\infty} \frac{\sin(n \pi y)}{(\gamma_{n3} y_0) \left( \gamma_{n3} - y_0 \right)} \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{E_n} + \] (44)

\[ \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{\tilde{F}_n} \]

where \( E_n = a E_n/\tilde{R}_y, \) and \( F_n = a F_n/\tilde{R}_y. \)

For the third subproblem, incorporating Eq. (26), we obtain the deflection solution, denoted by \( w_3(x, y) \), following the second subproblem, i.e.,

\[ \frac{w_3(x, y)}{b} = \sum_{n=1}^{\infty} \frac{\sin(n \pi y)}{(\gamma_{n3} y_0) \left( \gamma_{n3} - y_0 \right)} \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{E_n} + \] (45)

\[ \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{\tilde{F}_n} \]

where \( E_n = a E_n/\tilde{R}_y, \) and \( F_n = a F_n/\tilde{R}_y. \)

The deflection solution for the uniform loading is

\[ w_1(x, y) = \sum_{n=1}^{\infty} \frac{\sin(n \pi y)}{(\gamma_{n3} y_0) \left( \gamma_{n3} - y_0 \right)} \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{E_n} + \] (44)

\[ \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{\tilde{F}_n} \]

where \( E_n = a E_n/\tilde{R}_y, \) and \( F_n = a F_n/\tilde{R}_y. \)

For the third subproblem, incorporating Eq. (26), we obtain the deflection solution, denoted by \( w_3(x, y) \), following the second subproblem, i.e.,

\[ \frac{w_3(x, y)}{b} = \sum_{n=1}^{\infty} \frac{\sin(n \pi y)}{(\gamma_{n3} y_0) \left( \gamma_{n3} - y_0 \right)} \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{E_n} + \] (45)

\[ \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{\tilde{F}_n} \]

where \( E_n = a E_n/\tilde{R}_y, \) and \( F_n = a F_n/\tilde{R}_y. \)

For the third subproblem, incorporating Eq. (26), we obtain the deflection solution, denoted by \( w_3(x, y) \), following the second subproblem, i.e.,

\[ \frac{w_3(x, y)}{b} = \sum_{n=1}^{\infty} \frac{\sin(n \pi y)}{(\gamma_{n3} y_0) \left( \gamma_{n3} - y_0 \right)} \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{E_n} + \] (45)

\[ \left\{ \left( \cosh(\gamma_{n3}) \right) \sinh \left( \gamma_{n3}(1 - x_0) \right) - \cosh(\gamma_{n3}) \sinh \left( \gamma_{n3}(1 - x_0) \right) \right\} \right|_{\tilde{F}_n} \]
Table 1. Convergence study for bending, buckling, and free vibration solutions of the square nanoplates with \( b = 10 \text{ nm} \), \( \mu = 1 \text{ nm}^2 \), and \( k_p = 20 \) under different BCs.

Analytic solutions for rectangular nanoplates with combinations of clamped and simply supported edges

Superposing the fundamental analytic solutions given in “Fundamental analytic solutions in the symplectic space” section, analytic bending, buckling, and free vibration solutions of rectangular nanoplates with combinations of clamped and simply supported edges can be obtained, provided that the BCs are satisfied.

Denoting the clamped edge by “C”, a fully clamped (CCCC) nanoplate resting on an elastic foundation is first solved, where zero rotation conditions should be satisfied at each edge. Therefore, the following equations hold:

\[
\sum_{i=1}^{3} \frac{\partial w_i}{\partial x} \bigg|_{x=0,a} = 0, \quad \sum_{i=1}^{3} \frac{\partial w_i}{\partial y} \bigg|_{y=0,b} = 0
\]  \hspace{1cm} (46)

For a uniformly loaded CCCC nanoplate, substituting Eqs. (42), (44), (45) into Eq. (46) and expanding the existed polynomials as sine series, using the orthogonality in the trigonometric series, we have

\[
\frac{\gamma_{m1} \coth(\gamma_{m1}) - \gamma_{m3} \coth(\gamma_{m3})}{\gamma_{m1}^2 - \gamma_{m3}^2} G_m + \frac{\gamma_{m1} \csch(\gamma_{m1}) - \gamma_{m3} \csch(\gamma_{m3})}{\gamma_{m1}^2 - \gamma_{m3}^2} H_m
\]

\[
+ \sum_{n=1,2,3,\ldots} \frac{2\phi \pi \tau^2}{(m^2 \pi^2 + \gamma_{m1}^2)(m^2 \pi^2 + \gamma_{m3}^2)} \left[ \frac{G_n}{n \pi \psi_{m1} \psi_{m3}} - \cos(\tau \pi) \Pi_n \right]
\]

\[
= \frac{2\phi \pi \tau^2}{n \pi \psi_{m1} \psi_{m3}} \left[ \gamma_{m1} \psi_{m3} \tanh\left(\frac{\gamma_{m1}}{2}\right) + \gamma_{m3} \psi_{m1} \tanh\left(\frac{\gamma_{m3}}{2}\right) \right]
\]  \hspace{1cm} (47)

for \( x = 0 \) (\( m = 1, 2, 3, \ldots \)),

\[
\frac{\gamma_{m1} \csch(\gamma_{m1}) - \gamma_{m3} \csch(\gamma_{m3})}{\gamma_{m1}^2 - \gamma_{m3}^2} G_m + \frac{\gamma_{m1} \coth(\gamma_{m1}) - \gamma_{m3} \coth(\gamma_{m3})}{\gamma_{m1}^2 - \gamma_{m3}^2} H_m
\]

\[
+ \sum_{n=1,2,3,\ldots} \frac{2\phi \pi \tau^2 \cos(\tau \pi)}{(m^2 \pi^2 + \gamma_{m1}^2)(m^2 \pi^2 + \gamma_{m3}^2)} \left[ \frac{G_n}{n \pi \psi_{m1} \psi_{m3}} - \cos(\tau \pi) \Pi_n \right]
\]

\[
= \frac{2\phi \pi \tau^2}{n \pi \psi_{m1} \psi_{m3}} \left[ \gamma_{m1} \psi_{m3} \tanh\left(\frac{\gamma_{m1}}{2}\right) + \gamma_{m3} \psi_{m1} \tanh\left(\frac{\gamma_{m3}}{2}\right) \right]
\]  \hspace{1cm} (48)

for \( x = a \) (\( m = 1, 2, 3, \ldots \)),

| Mechanical quantity | Number of series terms | BC | CCC | CCS | SCSC | SCSS | SSSS |
|---------------------|------------------------|----|-----|-----|------|------|------|
| \( 10^9 Dw(a/2, b/2) / (qb^2) \) | 10 | 70.575 | 85.081 | 106.25 | 100.52 | 129.81 | 163.76 |
|                      | 20 | 70.532 | 85.028 | 106.22 | 100.51 | 129.81 | 163.76 |
|                      | 40 | 70.527 | 85.022 | 106.22 | 100.51 | 129.81 | 163.76 |
|                      | 80 | 70.526 | 85.021 | 106.22 | 100.51 | 129.81 | 163.76 |
|                      | 120 | 70.526 | 85.021 | 106.216 | 100.510 | 129.810 | 163.760 |
| \( 10^9 Dw(a/2, b/2) / (qb^2) \) | 40 | 330.37 | 351.37 | 380.94 | 373.81 | 413.82 | 460.12 |
|                      | 80 | 330.52 | 351.53 | 381.10 | 373.97 | 413.98 | 460.28 |
|                      | 160 | 330.55 | 351.56 | 381.14 | 374.01 | 414.02 | 460.32 |
|                      | 240 | 330.56 | 351.57 | 381.14 | 374.02 | 414.02 | 460.33 |
|                      | 320 | 330.57 | 351.57 | 381.15 | 374.02 | 414.03 | 460.33 |
|                      | 400 | 330.57 | 351.57 | 381.15 | 374.02 | 414.03 | 460.33 |
| \( -N_{cr}^{k_{pe}^{2}} / D \) | 3 | 34.352 | 29.844 | 24.271 | - | - | - |
|                      | 5 | 34.359 | 29.848 | 24.272 | - | - | - |
|                      | 10 | 34.359 | 29.848 | 24.273 | - | - | - |
|                      | 20 | 34.359 | 29.848 | 24.273 | - | - | - |
| \( a b^2 \sqrt{\rho h / D} \) | 3 | 32.240 | 28.620 | 24.418 | - | - | - |
|                      | 5 | 32.251 | 28.626 | 24.421 | - | - | - |
|                      | 10 | 32.252 | 28.626 | 24.421 | - | - | - |
|                      | 20 | 32.252 | 28.626 | 24.421 | - | - | - |
| Buckling | BC | Critical buckling load (N/m) | \(a=b\) (nm) |
|----------|----|-------------------------------|-----------------|
| SSSS     |    | 4.99 8.08 10.77 14.65 18.51 22.35 |
| Present  | \(\gamma = 1.84\) nm² | 1.08381 0.653020 0.435623 0.264407 0.175096 0.123826 |
| Molecular dynamics² | | 1.0837 0.6536 0.4331 0.2609 0.1714 0.1191 |

| Vibration | BC | Fundamental frequency (THz) | \(a=b\) (nm) |
|-----------|----|-----------------------------|-----------------|
| SSSS      |    | 10 15 20 25 30 35          | 1st 2nd 3rd 4th |
| Present   | \(\gamma = 0.10\) nm² | 0.166204 0.0525765 0.0297640 0.0191059 0.0132896 0.00977343 |
| Molecular dynamics² | | 0.1162438 0.0534719 0.0307422 0.0180318 0.0133060 0.0104205 |

Table 2. Comprehensive comparison of the critical buckling load and fundamental frequency solutions.

\[
\frac{\gamma m_1 \coth (\gamma m_1) - \gamma m_3 \coth (\gamma m_3)}{\gamma m_1 - \gamma m_3} + \frac{2 \Phi m }{\gamma m_1 - \gamma m_3} \sum_{n=1,2,3,...} \left[ \frac{2 \Phi m n \pi^2}{(m^2 \pi^2 + \gamma n_1^2)} \right] = \sum_{n=1,2,3,...} \left[ \frac{16 \Phi_0}{m \pi \gamma n_3} \sin \left( \frac{m \pi}{2} \right) \sin \left( \frac{n \pi}{2} \right) \left[ \psi_{m1} \gamma n_3^2 (m^2 \pi^2 + \gamma n_1^2) + \psi_{m3} \gamma n_3^2 (m^2 \pi^2 + \gamma n_3^2) \right] \right]
\]

\[
\frac{\gamma m_1 \csch (\gamma m_1) - \gamma m_3 \csch (\gamma m_3)}{\gamma m_1 - \gamma m_3} + \frac{\gamma m_3 \coth (\gamma m_3) - \gamma m_1 \coth (\gamma m_1)}{\gamma m_1 - \gamma m_3} + \frac{2 \Phi m n \pi^2 \cos (m \pi) }{\gamma m_1 - \gamma m_3} \sum_{n=1,2,3,...} \left[ \frac{2 \Phi m n \pi^2 \cos (m \pi) }{(m^2 \pi^2 + \gamma n_1^2)} \right] = \sum_{n=1,2,3,...} \left[ \frac{16 \Phi_0}{m \pi \gamma n_3} \cos (m \pi) \sin \left( \frac{n \pi}{2} \right) \sin \left( \frac{m \pi}{2} \right) \left[ \psi_{m1} \gamma n_3^2 (m^2 \pi^2 + \gamma n_1^2) + \psi_{m3} \gamma n_3^2 (m^2 \pi^2 + \gamma n_3^2) \right] \right]
\]
for \( y = b (m = 1, 2, 3, \ldots) \).

For a CCCC nanoplate under concentrated load, the only differences are on the right-hand sides of Eqs. (47–50), which become

\[
\frac{2\phi_{2c}^2}{\psi_{n1}\psi_{m3}} \sin \left( m\pi\bar{y}_0 \right) \left\{ \psi_{m1}y_{m3}^2 \text{csch}(y_{m3}) \sinh \left[ y_{m3}(x_0 - 1) \right] + \psi_{m3}y_{m1}^2 \text{csch}(y_{m1}) \sinh \left[ y_{m1}(x_0 - 1) \right] \right\}
\]

(51)

\[
\frac{2\phi_{2c}^2}{\psi_{m1}\psi_{n3}} \sin \left( m\pi\bar{y}_0 \right) \left[ \psi_{m1}y_{m3}^2 \text{csch}(y_{m3}) \sin (y_{m3}x_0) + \psi_{m3}y_{m1}^2 \text{csch}(y_{m1}) \sin (y_{m1}x_0) \right]
\]

(52)

\[
- \sum_{n=1,2,3,\ldots}^{\infty} 4\pi n \phi_{2c}^2 \sin \left( n\pi\bar{y}_0 \right) \left[ \psi_{n1}y_{n3}^2 (m^2\pi^2 + \gamma_{n1}^2) + \psi_{n3}y_{n1}^2 (m^2\pi^2 + \gamma_{n3}^2) \right]
\]

(53)

\[
- \sum_{n=1,2,3,\ldots}^{\infty} 4\pi n \phi_{2c}^2 \cos \left( n\pi\bar{y}_0 \right) \left[ \psi_{n1}y_{n3}^2 (m^2\pi^2 + \gamma_{n1}^2) + \psi_{n3}y_{n1}^2 (m^2\pi^2 + \gamma_{n3}^2) \right]
\]

(54)

for Eqs. (47–50), respectively.

For the bending problem, equating \( N_x, N_y, \) and \( \omega \) with zero, the constants \( \bar{T}_n, \bar{F}_m, \bar{S}_n, \) and \( \bar{P}_n \) \((n = 1, 2, 3, \ldots)\) are obtained by solving the nonhomogeneous equations (47–50) for the case of uniform load or incorporating Eqs. (51–54) for the case of concentrated load. Substituting the constants into Eqs. (44) and (45), followed by summation of Eqs. (42)/(43), (44), and (45), the final bending solution is obtained. For the buckling (free vibration) problem, equating \( q_x, q_y, N_x, N_y, k_w, \) and \( k_p \) \((q_x, q_y, k_w, \) and \( k_p)\) with zero, the buckling loads (natural frequencies) are determined by equating with zero the determinant of the coefficient matrix of the homogeneous simultaneous equations of Eqs. (47–50). Substituting the nonzero constant solutions into the Eqs. (44) and (45), and conducting summation, the buckling (vibration) mode shapes are obtained.
For the nanoplates with any other combinations of clamped and simply supported edges, the solutions can be obtained by relaxation of BCs from the above derivations. By equating $H_n = 0$ or $F_n = -2\mu q_n \cos (n\pi x) - 1/(-n\pi)$, and eliminating Eq. (50), we have three sets of simultaneous linear equations for the bending solutions of a CCCS nanoplate under concentrated or uniform loads. By imposing $F_n = H_n = 0$ or $F_n = H_n = -2\mu q_n \cos (n\pi x) - 1/(-n\pi)(n = 1, 2, 3, \ldots)$, and eliminating Eqs. (48) and (50), we obtain the bending solutions of a CCCS nanoplate under concentrated or uniform loads. Similar treatments yield the bending solutions of SCSC, SCSS, and SSSS nanoplates. Here, an anti-clockwise four-letter notation, starting from the edge $x = 0$, has been used to label a nanoplate under different BCs. The buckling (free vibration) solutions are obtained in a similar way after equating $q_{0c}, q_{c}, \omega, k_w$ and $K_p\left(q_{0c}, q_{c}, N_s, N_y, k_w\right)$ with zero.

### Comprehensive numerical results and discussion

Comprehensive numerical results of the nanoplates with combinations of clamped and simply supported edges are presented to confirm the validity of the developed method, and, more importantly, to provide benchmark solutions for future comparison.

The convergence study is carried out and the results are shown in Table 1 for the square nanoplates with $b = 10$ nm, $\mu = 1$ nm$^2$, and $K_p = k_p b^2 / D = 20$ under different BCs, including the central bending deflections, $10^5 D w (a/2, b/2) / (q_c b^2)$, and $10^5 D w (a/2, b/2) / (q_c b^2)$, critical buckling load factors, $-N_x^{\text{critical}} b^2 / D$, and fundamental frequency parameters, $\omega b^2 \sqrt{\rho h / D}$, where the convergent results with the accuracy of five significant figures are marked in bold. It is found that only 10 terms, at most, yield the convergence to the last digit of five significant figures for the buckling and free vibration solutions in this study, but 80 and 320 terms, at most, are needed to achieve the same accuracy for the bending solutions with uniform and concentrated loads, respectively. In Table 2, the present buckling and free vibration solutions are compared with their counterparts available in the literature by molecular dynamics simulation$^{14}$, Rayleigh–Ritz method$^{28}$, and iSOV method$^{31}$, respectively, confirming the validity of the adopted nonlocal theory and the present method. The parameters adopted are as follows$^{2,3,5,55,56}$: $\rho = 2250$ kg/m$^3$, $E = 1$ TPa, $\nu = 0.16$, and $h = 0.34$ nm. It should be noted that Young's moduli may be significantly different in the directions along short and long sides of a rectangular nanoplate$^{35,56}$, which

| $\mu$ | $k_p$ | BC | CCCC | CCSS | SCSS | SCSS | SSSS |
|-------|-------|-----|------|------|------|------|------|
| 0     | Present | 497.93 | 541.68 | 609.34 | 589.07 | 687.68 | 810.08 |
| FEM   | 497.92 | 541.68 | 609.34 | 589.06 | 687.69 | 810.10 |
| 5     | 462.99 | 499.37 | 553.69 | 538.53 | 615.51 | 708.21 |
| 10    | 432.95 | 463.69 | 508.30 | 496.63 | 558.39 | 631.11 |
| 15    | 432.95 | 433.18 | 470.48 | 461.29 | 511.94 | 570.59 |
| 20    | 406.83 | 406.75 | 434.84 | 431.05 | 473.37 | 521.70 |
| 0     | 483.29 | 523.84 | 585.63 | 567.65 | 656.63 | 765.62 |
| 5     | 432.96 | 466.24 | 515.57 | 502.04 | 571.59 | 655.00 |
| 10    | 392.31 | 420.31 | 460.98 | 450.37 | 506.74 | 573.33 |
| 15    | 358.75 | 382.81 | 417.15 | 408.56 | 455.54 | 510.40 |
| 20    | 330.57 | 351.57 | 381.15 | 374.02 | 414.03 | 460.33 |
| 0     | 469.55 | 507.27 | 563.90 | 547.87 | 603.05 | 691.14 |
| 5     | 406.61 | 472.35 | 508.42 | 478.21 | 526.64 | 609.31 |
| 10    | 358.64 | 384.36 | 421.72 | 411.99 | 463.85 | 525.25 |
| 15    | 320.85 | 342.96 | 374.72 | 366.70 | 410.41 | 461.82 |
| 20    | 290.29 | 309.66 | 337.23 | 330.43 | 386.11 | 412.20 |
| 0     | 456.62 | 491.76 | 543.90 | 529.54 | 603.05 | 691.14 |
| 5     | 383.30 | 411.67 | 453.28 | 442.20 | 506.35 | 569.64 |
| 10    | 330.30 | 354.07 | 388.63 | 379.65 | 427.65 | 484.61 |
| 15    | 290.20 | 310.64 | 340.16 | 323.62 | 373.45 | 421.74 |
| 20    | 258.79 | 276.71 | 302.46 | 295.99 | 331.47 | 373.35 |
| 0     | 444.44 | 477.27 | 525.42 | 512.50 | 579.76 | 659.60 |
| 5     | 362.52 | 388.94 | 427.49 | 417.35 | 471.05 | 534.86 |
| 10    | 306.11 | 328.24 | 360.35 | 352.01 | 396.69 | 449.81 |
| 15    | 264.90 | 283.90 | 311.45 | 304.37 | 342.62 | 388.11 |
| 20    | 233.47 | 250.13 | 274.24 | 268.09 | 301.53 | 341.30 |

Table 4. Bending deflections, $10^5 D w (a/2, b/2) / (q_c b^2)$, of centrally concentrate-loaded square nanoplates with $b = 10$ nm under different BCs.
is not considered for a 5 nm × 2.5 nm nanoplate in reference and thus in the present study for comparison purpose.

To provide more comprehensive benchmark solutions, we have tabulated the bending, buckling, and free vibration solutions of CCCC, CCCS, CCSS, SCSC, SCSS, and SSSS nanoplates in Tables 3, 4, 5 and 6, with a total of 3000 numerical results presented. The central bending deflections,  \( D_{w} \) \((a/2, b/2)/q_u b^4\) and  \( D_{w} \) \((a/2, b/2)/q_c b^2\), of the six types of square nanoplates with \( b = 10 \text{ nm} \) are tabulated in Tables 3 and 4 for the cases of uniform and concentrated loads, respectively, with \( k_w = k_w b^4/D = 200, k_p = k_p b^2/D = 0, 5, 10, 15, \) and 20. The critical buckling load factors are tabulated in Table 5 for \( N_y/N_x = 1, 2, 3, 4, \) and 5. The frequency parameters are tabulated in Table 6 for the first five modes. In each of Tables 3, 4, 5 and 6, five different nonlocal parameters are examined, i.e., \( \mu = 0, 1, 2, 3, \) and 4 nm². The results by ABAQUS software based on the FEM are also presented in each table, which correspond to the classical thin plate theory and are valid for the cases with \( \mu = 0 \). In ABAQUS, the thickness-to-width ratio of the nanoplates is 10⁻³, and the S4R thin shell element with uniform size of \( b/200 \) is taken. Satisfactory agreement between the present solutions and their counterparts by the FEM is observed, further validating the present method.

Figuring out the effects of the nonlocal parameter and others on the mechanical behavior of a nanoplate is helpful for researchers to determine the nonlocal parameter and to design the structure. Accordingly, quantitative parameter analyses are implemented with the analytic solutions obtained in this study. Defining the critical buckling load ratio as the ratio of the nonlocal theory- to classical theory-based critical buckling loads, Fig. 2a plots the nonlocal parameter dependent ratios of the six types of square nanoplates with \( b = 10 \text{ nm} \) and \( N_y/N_x = 1 \). The decrease of all six lines reveals that the nonlocal effect reduces the critical buckling loads of the nanoplates, and, compared with classical plates, a nanoplate with stronger constraints shows a greater reduction of its critical buckling load. The length effects on the critical buckling load ratios of square SSSS and CCCC nanoplates with different nonlocal parameters are illustrated in Fig. 2b. With the increase of length, it is found that the critical buckling load ratios unanimously increase, and gradually approach 1 that corresponds to the

| \( \mu \) | \( N_y/N_x \) | BC | CCCC | CCCS | CCSS | SCSC | SCSS | SSSS |
|-------|---------|-----|------|------|------|------|------|------|
| 0     | 1       | Present | 52.345 | 42.547 | 32.052 | 37.800 | 26.280 | 19.739 |
|       |         | FEM    | 52.352 | 42.552 | 32.055 | 37.802 | 26.280 | 19.740 |
|       | 2       | Present | 34.696 | 27.973 | 21.274 | 26.231 | 17.022 | 13.159 |
|       |         | FEM    | 34.700 | 27.975 | 21.275 | 26.232 | 17.023 | 13.158 |
|       | 3       | Present | 25.835 | 20.762 | 15.865 | 20.062 | 12.573 | 9.8696 |
|       |         | FEM    | 25.837 | 20.764 | 15.866 | 20.064 | 12.573 | 9.8697 |
|       | 4       | Present | 20.551 | 16.490 | 12.634 | 16.236 | 9.9633 | 7.8957 |
|       |         | FEM    | 20.553 | 16.492 | 12.635 | 16.238 | 9.9637 | 7.8960 |
|       | 5       | Present | 17.051 | 13.671 | 10.491 | 13.634 | 8.2493 | 6.5798 |
|       |         | FEM    | 17.053 | 13.672 | 10.491 | 13.635 | 8.2497 | 6.5800 |

Table 5. Critical buckling load factors, \(-N_x^{\text{critical}}b^2/D\), of square nanoplates with \( b = 10 \text{ nm} \) under different BCs.
cases of classical plates, which suggests that the nonlocal effect matters for nanoscale plates, but may be negligible for larger-scale plates such that the classical theory can well capture their behaviors.

Defining the frequency ratio as that of the nonlocal theory- to classical theory-based frequencies, Fig. 2c plots the nonlocal parameter dependent fundamental frequency ratios of the six types of square nanoplates with $b = 10 \text{ nm}$. The decrease of all six lines reveals that the nonlocal effect reduces the fundamental frequencies of the nanoplates, and, compared with classical plates, a nanoplate with stronger constraints generally shows a greater reduction of its fundamental frequency. The length effects on the fundamental frequency ratios of square SSSS and CCCC nanoplates with different nonlocal parameters are illustrated in Fig. 2d, where the ratios show a dramatic increase, and approach 1 with length, which again suggests that the nonlocal effect plays a significant role for nanoscale plates, but is negligible for macroscale plates.

**Concluding remarks**

With an up-to-date symplectic superposition method, the analytic bending, buckling, and free vibration solutions of rectangular nanoplates with combinations of clamped and simply supported edges are obtained based on Kirchhoff plate theory and Eringen's nonlocal theory. Compared with conventional analytic methods such as the semi-inverse methods, the present method describes the problems in the Hamiltonian system, and yields the analytic solutions by the mathematical techniques in the symplectic space in a rigorous step-by-step way, without predetermining solution forms, which enables one to seek new analytic solutions. After validation of the present method by the other methods, comprehensive benchmark results are presented for both Lévy-type and non-Lévy-type nanoplates, including the bending deflections, critical buckling loads, and natural frequencies. Quantitative parameter analyses are implemented with the analytic solutions to gain insight into the behaviors and to provide reference for structural designs of nanoplates. It should be noted that the present work studies linear free vibration since the adopted superposition technique is applicable in linear regime with small deformation, but the studies on nonlinear vibration are definitely worthy of further exploration, which may constitute our follow-up work.

| $\mu$ | Mode | BC | CCCC | CCCS | CCSS | SCSC | SCSS | SSSS |
|-------|------|----|------|------|------|------|------|------|
| 0     | 1    | Present | 35.985 | 31.826 | 27.054 | 28.951 | 23.646 | 19.739 |
|       | FEM  | 35.987 | 31.826 | 27.053 | 28.950 | 23.644 | 19.736 |
| 2     | 1    | Present | 73.394 | 63.331 | 60.539 | 54.743 | 51.674 | 49.348 |
|       | FEM  | 73.404 | 63.335 | 60.542 | 54.744 | 51.673 | 49.346 |
| 3     | 1    | Present | 73.394 | 71.076 | 60.786 | 69.327 | 58.646 | 78.957 |
|       | FEM  | 73.404 | 71.085 | 60.789 | 69.335 | 58.649 | 78.947 |
| 4     | 1    | Present | 108.22 | 100.79 | 92.836 | 94.585 | 86.135 | 98.696 |
|       | FEM  | 108.23 | 100.80 | 92.836 | 94.587 | 86.130 | 98.711 |
| 5     | 1    | Present | 131.58 | 116.36 | 114.56 | 110.22 | 100.27 | 128.31 |
|       | FEM  | 131.62 | 116.38 | 114.58 | 110.23 | 100.29 | 128.30 |

Table 6. Frequency parameters, $\omega b^2 \sqrt{\rho h/D}$, of square nanoplates with $b = 10 \text{ nm}$ under different BCs.
Figure 2. (a) Critical buckling load ratios versus the nonlocal parameter of square nanoplates under different BCs. (b) Critical buckling load ratios versus the length of square SSSS and CCCC nanoplates with different nonlocal parameters. (c) Fundamental frequency ratios versus the nonlocal parameter of square nanoplates under different BCs. (d) Fundamental frequency ratios versus the length of square SSSS and CCCC nanoplates with different nonlocal parameters.
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### Author contributions
X.Z. conducted derivations and wrote the original draft. M.H. analyzed data and validated derivations. D.A. analyzed and visualized data. C.Z. analyzed data. R.L. conceived the idea of the study, proposed the methodology, supervised the project, and revised the original draft. All authors reviewed and approved the final manuscript.

### Competing interests
The authors declare no competing interests.

### Additional information

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