Randomness and Irreversibility
in
Quantum Field Theory.

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Quantum fluctuations, through quantum corrections, have the potential to lead to irreversibility in quantum field theory. We consider the virtual “charge” distribution generated by quantum corrections in the leading log, short range approximation, and adopt for it a statistical interpretation. This virtual charge density has fractal structure, and it is seen that, independently of whether the theory is or is not asymptotically free, it describes a system where the equilibrium state is at its classical limit ($\hbar \to 0$). We also present a simple analysis of how diffusion of the charge density proceeds as a function of the distance at which the system is probed.

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Quantum fluctuations are perhaps the most fundamental feature of quantum field theory, affecting fields, their sources and the vacuum in which they evolve. Induced by the fact that quantum fluctuations are to a certain extent random \[1\], we can expect them to have an impact on questions of irreversibility. Namely, since true randomness impairs one’s ability to carry out an exact reconstruction of the past history of the system, one can expect some irreversibility at the microscopic (quantum field) level because of the presence of quantum fluctuations in this domain. On the other hand, as is well known, quantum fluctuations through their breaking of canonical scale–invariance, induce non–trivial scaling in physical parameters and turn them into scale dependent, effective parameters, whose scale dependence is described by the appropriate renormalization group equations (RGE) \[2\].

There is then a relationship between irreversibility and non–trivial renormalization group scaling.

In fact, in quantum field theory one associates with the quantum–corrected interaction energy a charge density which results from taking into account (through quantum corrections) not only simple two body processes with a precise and fixed impact parameter, but also the effects of many body interactions with impact parameter similar or smaller than the distance scale at which we probe the system. The measured physical potential then is subject to fluctuations, and it is impossible to specify the exact dependence of the potential on the individual components of the virtual charge density. The effect of quantum fluctuations on the scaling properties of the potential and the associated effective charge density, are then a manifestation of statistical indeterminacy.

To handle questions relating to the effective charge density we need to introduce some statistical notions into the quantum field theory. The fluctuations in the potential are related to the ones in the charge density through the Poisson equation satisfied by the potential and the charge density. Under very general circumstances the induced density can be interpreted as a probability density \[3\], and the needed statistical framework just unfolds in front of our eyes.

We will compute the effective interaction energy and derive from it the charge density.
The effective interaction energy is obtained by solving its RGE. Since energy has canonical
dimension of inverse time and no anomalous dimension, its RGE \[4\] has the solution,

\[ V(\lambda r_0, g_0, a) = \lambda^{-1} V(r_0, \bar{g}_0(\lambda), a). \]

Here \( V \) is the interaction energy, \( r_0 \) is the distance between the interacting sources, and \( a \) is
a reference distance. The quantity \( \lambda \) is the scale parameter, \( \bar{g}_0(\lambda) \) is the effective coupling
and satisfies the RGE \( \lambda \partial \bar{g}_0 / \partial \lambda = -\beta(\bar{g}_0) \). For a massless mediating field (such as photons or
gravitons), the effective interaction energy for two point particles separated by a distance \( a \)
is given by \( V(a, g_0, a) = C g_0^2 / 4 \pi a^4 \). Computing to 1–loop order, where \( \beta = -\beta_0 g_0^3 \), and choosing
\( \lambda = r/a \), we get (for \( r \) close to \( a \) and after exponentiating)

\[ V(r, g_0, a) = C g_0^2 / 4 \pi a^{-\sigma} r^{-1+\sigma}. \]

\( \sigma \) is related to the \( \beta \)–function for the coupling \( g_0 \), and to one–loop is given by \( \sigma = -2\beta_0 g_0^2 \)
and we have taken it to be a constant.\[1\] \[2\] For the case of QED one recognizes here the first
term in the short distance expansion of the famous Uehling potential; in QCD one recognizes
the expression for the interquark potential \[3\].

Potential theory guarantees that eq. (2) satisfies a Poisson equation whose right hand
side is proportional to the “charge” density dictated by quantum corrections. This charge
density can be understood \[3\] as a “probability density”, and used to derive information
on the physics of the virtual cloud as a many body (statistical) system. For our isotropic
potential, the density is

\[ \rho(r) = A' r^{-3+\sigma} \]

\[ \] 1Strictly speaking, \( \beta \) is a function of the distance at which the system is probed. This can be
made explicit by computing it in a mass dependent subtraction procedure \[5\]. However, it will be
sufficient for our present purposes to take it to be a constant.

\[ \] 2Note that one can also arrive at the result shown in eq. (2) by direct solution of the RGE for
the propagator of the particle mediating the force.
where $A'$ is a constant. For $\rho(r)$ to be a probability density, we need that it be a positive and integrable function on its support. How this is implemented depends on the sign and size of $\sigma$ (see below).

From eq. (3) we see that $\rho(r)$ is the solution to the functional equation

$$\rho(\lambda r) = \lambda^\beta \rho(r)$$

with $\beta = -3 + \sigma$. Thus $\rho(r)$ describes a fractal distribution of charge embedded in a 3–dimensional configuration space; the Hausdorff (or fractal) dimension is $d_f = +\sigma$. This is not surprising, since $\sigma$ has its origin in the deviation from canonical scaling of the effective coupling due to quantum fluctuations. Furthermore, since $\sigma$ changes as we change the size of the domain on which we probe the virtual cloud, it turns out that the Hausdorff dimension also changes: in truth we are then dealing with a multifractal. For QCD and Quantum Gravity in their asymptotically free regime, $\sigma$ is positive. In QED and other non-asymptotically free theories, $\sigma$ is negative.

Normalizability of the probability density, leads to

$$\rho = Ar^{-3+\sigma}$$

with $A = \frac{\sigma}{4\pi} R_0^{-\sigma}$ for asymptotically free theories, and $A = -\frac{\sigma}{4\pi} r_0^{-\sigma}$ when $\sigma < 0$. Here $R_0$ denotes an IR–cutoff, necessary in the case of positive $\sigma$, in order to ensure the finiteness of the probability distribution. For non–asymptotically free theories $r_0$ is an UV–cutoff necessary because of the same reason. The IR–cutoff can be identified in QCD with, e.g., a typical hadronic size; the UV–cutoff $r_0$ can be identified in QED with the Compton wavelength for the electron, or for the heaviest degree of freedom considered. In either class of theories the resulting probability density is of the Pareto type; this is a natural consequence of the renormalization group origin for $\rho(r)$, which is ultimately responsible

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3Because it is related to the number of degrees of freedom that contribute at the scale on which we study the system.
for the functional equation in (H) and the associated scaling: scaling leads to Levy–type
distributions, which turn Pareto in some limit \[8\]. Notice also that since \(\sigma\) vanishes in the
classical \((\hbar \to 0)\) limit, then both probability densities go to zero in this limit.

One may now introduce a quantity

\[
S = -k \int d^3\tilde{r} \rho(r) \log [C_N \rho(r)]
\]

(6)

with the properties of a fine–grained entropy, and study the “equilibrium configurations”
that these distributions support. We will identify the configurations (as in information
theory and thermodynamics) corresponding to the maximum entropy with the equilibrium
configurations of the system. In other words, we will assume that this fine–grained entropy
reveals a preferred “direction” for stability in the evolution of the quantum field system. In
eq. (H), the integral extends over the full support of the variable \(r\). The constant \(C_N\) needs
to be introduced because \(\rho(r)\) is a dimensionful quantity, and we do not have the equivalent
of a Nernst theorem to set a reference valid for all physical systems.

When \(\sigma\) is positive

\[
S^{(\sigma>0)} = 1 - \frac{3}{\sigma} - \log \frac{\sigma}{4\pi} C_N + 3 \log R_0 ,
\]

(7)

and for negative \(\sigma\) one gets,

\[
S^{(\sigma<0)} = 1 - \frac{3}{\sigma} - \log \frac{-\sigma}{4\pi} C_N + 3 \log r_0 .
\]

(8)

The “entropy constant”, \(C_N\), may be chosen so as to eliminate the dependence of
the entropy on the cutoff. The “Nernst–volume” corresponds to the extreme volume that
can be physically reachable. These entropies are shown in figs. 1 and 2.

For non–asymptotically free theories, the maximum of the entropy happens when \(\sigma\) goes
to 0 from the left. This means either the classical limit, in the sense that \(\hbar \to 0\), or that
one probes the system at distances \(so large\) that no quantum fluctuation contributes to \(\sigma\).
It corresponds to a first order phase transition.\[4\]

\[4\] Naturally, this also happens when the interaction is turned off and \(g_0^2 \sim 0\).
For asymptotically free theories the situation is different. The entropy has a maximum at $\sigma = 3$ and a discontinuity at $\sigma = 0$. The maximum value of the entropy occurs when $\sigma$ is close to the non–perturbative regime, where we can not trust the approximations made in this paper; in perturbative regimes, and since matter fluctuations contribute negatively to $\sigma$, the most stable state corresponds to a configuration (or system size) where matter states cannot be excited and do not contribute. Because of the decoupling theorem [10], this occurs at the maximum–possible size of the system: that is, at its IR–limit!

Thus, for both, asymptotically–free and non–asymptotically–free theories one sees that the maximum of the entropy is attained at the classical limit.

The randomness in the virtual cloud will affect the way into which it self–organizes and spreads into space–time. The fractal nature of the probability density leads one to expect some form of fractal brownian motion to be present. The parameter controlling this is $\sigma$, as may be seen by examining some basic properties of the coefficient of diffusion derived from $\rho(r)$.

Taking the collision probability among the components of the virtual cloud to be $\rho(r)$, the virtual charges will execute random walks with individual steps controlled by $\rho(r)$; the net effect of these processes is the diffusion of the virtual cloud outside of a ball of radius $R$, and within which the original interaction is operative.

As is well known [11], after $N$–steps, the probability that a “particle” be at a position between $\vec{r}$ and $\vec{r} + d\vec{r}$, is given by

$$W(r)d^3\vec{r} = \left[4\pi D t\right]^{-3/2} \exp \left[-\frac{r^2}{4D t}\right] d^3\vec{r}$$

(9)

where $D$ is the coefficient of diffusion,

$$D = \frac{n}{6} \left\langle \vec{r}^2 \right\rangle ,$$

$n$ is the number of steps per unit time and $N = nt$ is the total number of steps. $\left\langle \vec{r}^2 \right\rangle$ is the second moment of the probability density $\rho(r)$. When $D = 0$, there is no diffusion.

From a radial distance $s_0$ to a radial distance $s_1 > s_0$, the second moment of $\rho(r)$ is
\[ \langle r^2 \rangle = \frac{4\pi}{2 + \sigma} A \left( s_1^{2+\sigma} - s_0^{2+\sigma} \right), \]  

(10)

and therefore, in some cases, the coefficient of diffusion diverges (naively) as \( s \) goes to infinity. This was to be expected, since our probability distribution is the limit of a Levy–type distribution, and hence its moments diverge.

When \( \sigma > 0 \), we can take \( s_1 = R_0 \) and set \( s_0 = s \), an unspecified but fixed distance. The coefficient of diffusion is given by

\[ D^{(\sigma > 0)} = \frac{n}{6} \frac{\sigma}{2 + \sigma} s^2 \left[ \left( \frac{R_0}{s} \right)^2 - \left( \frac{s}{R_0} \right)^\sigma \right]. \]  

(11)

This has the property that it vanishes both when \( \sigma \to 0 \) (e.g., as in the classical limit) and when \( s \to R_0 \). In the limit when \( s/R_0 \ll 1 \), \( D^{(\sigma > 0)} \to \frac{n}{6} \frac{\sigma}{2+\sigma} R_0^2 \), and we see the divergence associated with Levy–Pareto distributions. A plot of \( D^{(\sigma > 0)} \) is shown in fig. 3. From the above, it follows that in the classical and infrared limits there is no diffusion.

In non–asymptotically free theories, we take \( s_0 = r_0 \) (the UV–cutoff) and now leave \( s \) unspecified. Then,

\[ D^{(\sigma < 0)} = \frac{n}{6} \frac{-\sigma}{2 + \sigma} s^2 \left[ \left( \frac{s}{r_0} \right)^\sigma - \left( \frac{r_0}{s} \right)^2 \right]. \]  

(12)

As before, \( \lim_{\sigma \to 0} D^{(\sigma < 0)} = 0 \); also, when \( s \) approaches the UV–cutoff, \( D^{(\sigma < 0)} \) goes to zero. Hence no diffusion of the cloud takes place at either the classical limit, or at the UV–cutoff. Figure 4 shows a plot of \( D^{(\sigma < 0)} \). When the limit of \( s \) well into the IR is taken, we see that there are two possible regimes, depending on whether \( |\sigma| < 2 \) or \( |\sigma| > 2 \). When \( |\sigma| < 2 \), \( D^{(\sigma < 0)} \) diverges as \( \left( \frac{s}{r_0} \right)^{2-|\sigma|} \), whereas when \( |\sigma| > 2 \), \( D^{(\sigma < 0)} \) tends to the limit \( \frac{n}{6} r_0^2 \); the first case is the signature of a Levy–Pareto distribution, while the second indicates the presence of a classical brownian motion process. Notice also that for \( |\sigma| \leq 1.5 \), at a given value of \( \sigma \), there always is a maximum value of the diffusion coefficient that happens when \( s/r_0 \) falls on the apex of the surface plotted in Figure 4; at these values the virtual cloud diffuses at the fastest rate.

In summary, we have exploited the randomness associated with quantum fluctuations, and applied a statistical interpretation to the virtual charge they generate. This has in-
...teresting and apparently deep consequences which stem from the scaling properties of the interaction energy. In particular, fractal behavior in a form related to the Levy–Pareto form of the charge density, leads to systems where the maximum entropy is associated with the classical limit of the quantum system; in other words, the classical regime is the most stable, and the one preferred by the system. As is the case for other many body systems, diffusion is a familiar mechanism for the approach to equilibrium (although perhaps not the only one). In concordance to what follows from studying the entropy, the virtual cloud diffuses with a coefficient of diffusion proportional to $\bar{\hbar}$, and thus diffusion of the cloud stops as the classical limit is reached.

We see that, under very general conditions, irreversibility is contained in quantum field systems, and that this irreversibility is encountered as the system size grows. Furthermore, the quantum field system tends to “relax” into a classical system.

The ideas presented here may also find application in the study of intermittency, turbulence, phase transitions, multiparticle physics and other complex phenomena in quantum field theory and the early universe.

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FIGURES

FIG. 1. Entropy for a generic asymptotically free theory. See text for notation.

FIG. 2. Entropy for a generic non-asymptotically free theory. See text for notation.

FIG. 3. The diffusion coefficient as a function of $s/R_0$ and $\sigma$ for an asymptotically free theory. Here $s$ is a fixed distance representing the maximum size on which the system is studied, subject to the constraint that it be smaller than the IR–cutoff $R_0$.

FIG. 4. The diffusion coefficient as a function of $s/r_0$ and $\sigma$ for a non-asymptotically free theory. Here $s$ is a fixed distance representing the maximum size on which the system is studied. $r_0$ represents the UV–cutoff.