Abstract

In this paper, using geometric properties of the field rotation parameters, we present a solution of Smale’s Thirteenth Problem on the maximum number of limit cycles for Liénard’s polynomial system. We also generalize the obtained result and present a solution of Hilbert’s Sixteenth Problem on the maximum number of limit cycles surrounding a singular point for an arbitrary polynomial system. Besides, we consider a generalized Liénard’s cubic system with three finite singularities, for which the developed geometric approach can complete its global qualitative analysis: in particular, it easily solves the problem on the maximum number of limit cycles in their different distribution. We give also an alternative proof of the main theorem for the generalized Liénard’s system applying the Wintner–Perko termination principle for multiple limit cycles and discuss some other results concerning this system.

Keywords: planar polynomial dynamical system; Liénard’s polynomial system; generalized Liénard’s cubic system; Hilbert’s sixteenth problem; Smale’s thirteenth problem; field rotation parameter; bifurcation; limit cycle

1 Introduction

We consider planar dynamical systems

\[ \dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \]  

(1.1)

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where $P_n(x, y)$ and $Q_n(x, y)$ are polynomials with real coefficients in the real variables $x$, $y$, and, first of all, a special case of (1.1): classical Liénard’s polynomial system of the form

$$
\dot{x} = y, \quad \dot{y} = -x + \mu_1 y + \mu_2 y^2 + \mu_3 y^3 + \ldots + \mu_{2k} y^{2k} + \mu_{2k+1} y^{2k+1}. \quad (1.2)
$$

The main problem of qualitative theory of such systems is Hilbert’s Sixteenth Problem on the maximum number and relative position of their limit cycles, i.e., closed isolated trajectories of (1.1). This problem was formulated as one of the fundamental problems for mathematicians of the XX century, however it has not been solved even in the simplest (quadratic, cubic, etc.) cases of the polynomial systems. In this paper, we suggest a new geometric approach to solving the problem in the case of Liénard’s system (1.2). In this special case, it is considered as Smale’s Thirteenth Problem becoming one of the main problems for mathematicians of the XXI century [17], [25].

In Section 2 of this paper, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we present a solution of Smale’s Thirteenth Problem for Liénard’s polynomial system (1.2). In Section 3, by means of the same geometric approach, we generalize the obtained result and present a solution of Hilbert’s sixteenth problem on the maximum number of limit cycles surrounding a singular point for an arbitrary polynomial system. In Section 4, we consider a generalized Liénard’s cubic system with three finite singularities, for which the developed geometric approach can complete its global qualitative analysis: in particular, it easily solves the problem on the maximum number of limit cycles in their different distribution. In this section, we give also an alternative proof of the main theorem for the generalized Liénard’s system applying the Wintner–Perko termination principle for multiple limit cycles and discuss some other results concerning this system.

2 Liénard’s polynomial system

System (1.2) and more general Liénard’s systems have been studied in numerous works (see, for example, [1], [2], [4], [7], [13], [15], [18]–[22], [24]). It is easy to see that (1.2) has the only finite singularity: an anti-saddle at the origin. At infinity, system (1.2) for $k \geq 1$ has two singular points: a node at the “ends” of the $y$-axis and a saddle at the “ends” of the $x$-axis. For studying the infinite singularities, the methods applied in [1] for Rayleigh’s and van der Pol’s equations and also Erugin’s two-isocline method developed in [11] can be used. Following [11], we will study limit cycle bifurcations of (1.2) by means of a canonical system containing only the field rotation parameters of (1.2). It is valid the following theorem.
Theorem 2.1. Liénard’s polynomial system (1.2) with limit cycles can be reduced to the canonical form
\[ \dot{x} = y \equiv P, \quad \dot{y} = -x + \mu_1 y + y^2 + \mu_3 y^3 + \ldots + y^{2k} + \mu_{2k+1} y^{2k+1} \equiv Q, \]  
where \( \mu_1, \mu_3, \ldots, \mu_{2k+1} \) are field rotation parameters of (2.1).

Proof. Vanish all odd parameters of (1.2),
\[ \dot{x} = y, \quad \dot{y} = -x + \mu_2 y^2 + \mu_4 y^4 + \ldots + \mu_{2k} y^{2k}, \]  
and consider the corresponding equation
\[ \frac{dy}{dx} = -x + \mu_2 y^2 + \mu_4 y^4 + \ldots + \mu_{2k} y^{2k} \equiv F(x, y). \]  
Since \( F(x, -y) = -F(x, y) \), the direction field of (2.3) (and the vector field of (2.2) as well) is symmetric with respect to the \( x \)-axis. It follows that for arbitrary values of the parameters \( \mu_2, \mu_4, \ldots, \mu_{2k} \) system (2.2) has a center at the origin and cannot have a limit cycle surrounding this point. Therefore, without loss of generality, all even parameters of system (1.2) can be supposed to be equal, for example, to one: \( \mu_2 = \mu_4 = \ldots = \mu_{2k} = 1 \) (they could be also supposed to be equal to zero).

To prove that the rest (odd) parameters rotate the vector field of (2.1), let us calculate the following determinants:
\[ \Delta_{\mu_1} = PQ'_{\mu_1} - QP'_{\mu_1} = y^2 \geq 0, \]
\[ \Delta_{\mu_3} = PQ'_{\mu_3} - QP'_{\mu_3} = y^2 \geq 0, \]
\[ \ldots \]
\[ \Delta_{\mu_{2k+1}} = PQ'_{\mu_{2k+1}} - QP'_{\mu_{2k+1}} = y^2 \geq 0. \]

By definition of a field rotation parameter [5], for increasing each of the parameters \( \mu_1, \mu_3, \ldots, \mu_{2k+1} \), under the fixed others, the vector field of system (2.1) is rotated in positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2.1) is rotated in negative direction (clockwise).

Thus, for studying limit cycle bifurcations of (1.2), it is sufficient to consider canonical system (2.1) containing only its odd parameters, \( \mu_1, \mu_3, \ldots, \mu_{2k+1} \), which rotate the vector field of (2.1). The theorem is proved.

By means of canonical system (2.1), let us study global limit cycle bifurcations of (1.2) and prove the following theorem.

Theorem 2.2. Liénard’s polynomial system (1.2) has at most \( k \) limit cycles.
Proof. According to Theorem 2.1, for the study of limit cycle bifurcations of system (1.2), it is sufficient to consider canonical system (2.1) containing only the field rotation parameters of (1.2): $\mu_1, \mu_3, \ldots, \mu_{2k+1}$.

Vanish all these parameters:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + y^{2k}. \quad (2.4)$$

System (2.4) is symmetric with respect to the $x$-axis and has a center at the origin. Let us input successively the field rotation parameters into this system beginning with the parameters at the highest degrees of $y$ and alternating with their signs. So, begin with the parameter $\mu_{2k+1}$ and let, for definiteness, $\mu_{2k+1} > 0$:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + y^{2k} + \mu_{2k+1} y^{2k+1}. \quad (2.5)$$

In this case, the vector field of (2.5) is rotated in positive direction (counterclockwise) turning the origin into a nonrough unstable focus.

Fix $\mu_{2k+1}$ and input the parameter $\mu_{2k-1} < 0$ into (2.5):

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + \mu_{2k-1} y^{2k-1} + y^{2k} + \mu_{2k+1} y^{2k+1}. \quad (2.6)$$

Then the vector field of (2.6) is rotated in opposite direction (clockwise) and the focus immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of $y$ changes) generating a stable limit cycle. Under further decreasing $\mu_{2k-1}$, this limit cycle will expand infinitely, not disappearing at infinity (because of the parameter $\mu_{2k+1}$ at the higher degree of $y$).

Denote the limit cycle by $\Gamma_1$, the domain outside the cycle by $D_1$, the domain inside the cycle by $D_2$ and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a “trajectory concentration” surrounding the origin. It is clear that, under decreasing the parameter $\mu_{2k-1}$, a semi-stable limit cycle cannot appear in the domain $D_2$, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation.

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain $D_1$. Suppose it appears in this domain for some values of the parameters $\mu^{*}_{2k+1} > 0$ and $\mu^{*}_{2k-1} < 0$. Return to initial system (2.4) and change the inputting order for the field rotation parameters. Input first the parameter $\mu_{2k-1} < 0$:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + \mu_{2k-1} y^{2k-1} + y^{2k}. \quad (2.7)$$

Fix it under $\mu_{2k-1} = \mu^{*}_{2k-1}$. The vector field of (2.7) is rotated clockwise and the origin turns into a nonrough stable focus. Inputting the parameter
\( \mu_{2k+1} > 0 \) into (2.7), we get again system (2.6), the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle \( \Gamma_1 \) will immediately appear from infinity, more precisely, from a separatrix cycle of the Poincaré circle form containing infinite singularities of the saddle and node types [I]. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing \( \mu_{2k+1} \) to the value \( \mu_{2k+1}^* \). It follows that there are no values of \( \mu_{2k-1}^* < 0 \) and \( \mu_{2k+1}^* > 0 \), for which a semi-stable limit cycle could appear in the domain \( D_1 \).

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (2.6) for any values of the parameters \( \mu_{2k-1} \) and \( \mu_{2k+1} \) of different signs. Obviously, if these parameters have the same sign, system (2.6) has no limit cycles surrounding the origin at all.

Let system (2.6) have the unique limit cycle \( \Gamma_1 \). Fix the parameters \( \mu_{2k+1} > 0 \), \( \mu_{2k-1} < 0 \) and input the third parameter, \( \mu_{2k-3} > 0 \), into this system:

\[
\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-3} y^{2k-3} + y^{2k-2} + \ldots + \mu_{2k+1} y^{2k+1}. \tag{2.8}
\]

The vector field of (2.8) is rotated counterclockwise, the focus at the origin changes the character of its stability and the second (unstable) limit cycle, \( \Gamma_2 \), immediately appears from this point. Under further increasing \( \mu_{2k-3} \), the limit cycle \( \Gamma_2 \) will join with \( \Gamma_1 \) forming a semi-stable limit cycle, \( \Gamma_{12} \), which will disappear in a “trajectory concentration” surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to \( \Gamma_{12} \)? It is clear that such a limit cycle cannot appear neither in the domain \( D_1 \) bounded on the inside by the cycle \( \Gamma_1 \) nor in the domain \( D_3 \) bounded by the origin and \( \Gamma_2 \) because of increasing the distance between the spiral coils filling these domains under increasing the parameter \( \mu_{2k-3} \).

To prove impossibility of the appearance of a semi-stable limit cycle in the domain \( D_2 \) bounded by the cycles \( \Gamma_1 \) and \( \Gamma_2 \) (before their joining), suppose the contrary, i.e., for some set of values of the parameters, \( \mu_{2k+1}^* > 0 \), \( \mu_{2k-1}^* < 0 \), and \( \mu_{2k-3}^* > 0 \), such a semi-stable cycle exists. Return to system (2.4) again and input first the parameters \( \mu_{2k-3} > 0 \) and \( \mu_{2k+1} > 0 \):

\[
\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-3} y^{2k-3} + y^{2k-2} + \ldots + \mu_{2k+1} y^{2k+1}. \tag{2.9}
\]

Both parameters act in a similar way: they rotate the vector field of (2.9) counterclockwise turning the origin into a nonrough unstable focus.

Fix these parameters under \( \mu_{2k-3} = \mu_{2k-3}^* \), \( \mu_{2k+1} = \mu_{2k+1}^* \) and input the parameter \( \mu_{2k-1} < 0 \) into (2.9) getting again system (2.8). Since, on our assumption, this system has two limit cycles for \( \mu_{2k-1} > \mu_{2k-1}^* \), there exists some value of the parameter, \( \mu_{2k-1}^{12} (\mu_{2k-1}^* < \mu_{2k-1}^{12} < 0) \), for which a semi-stable limit cycle, \( \Gamma_{12} \), appears in system (2.8) and then splits into a stable cycle,
\(\Gamma_1\), and an unstable cycle, \(\Gamma_2\), under further decreasing \(\mu_{2k-1}\). The formed domain \(D_2\) bounded by the limit cycles \(\Gamma_1\), \(\Gamma_2\) and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle \(\Gamma_2\) will contract and the exterior stable limit cycle \(\Gamma_1\) will expand under decreasing \(\mu_{2k-1}\). The distance between the spirals of the domain \(D_2\) will naturally increase, what will prohibit from the appearance of a semi-stable limit cycle in this domain for \(\mu_{2k-1} < \mu_{2k-1}^{12}\).

Thus, there are no such values of the parameters, \(\mu_{2k+1}^* > 0\), \(\mu_{2k-1}^* < 0\), and \(\mu_{2k-3}^* > 0\), for which system (2.8) would have an additional semi-stable limit cycle. Obviously, there are no other values of the parameters \(\mu_{2k+1}\), \(\mu_{2k-1}\), and \(\mu_{2k-3}\) for which system (2.8) would have more than two limit cycles surrounding the origin. Therefore, two is the maximum number of limit cycles for system (2.8). This result agrees with [24], where it was proved for the first time that the maximum number of limit cycles for Liénard’s system of the form

\[
\dot{x} = y, \quad \dot{y} = -x + \mu_1 y + \mu_3 y^3 + \mu_5 y^5 \tag{2.10}
\]

was equal to two.

Suppose that system (2.8) has two limit cycles, \(\Gamma_1\) and \(\Gamma_2\) (this is always possible if \(\mu_{2k+1} \gg -\mu_{2k-1} \gg \mu_{2k-3} > 0\)), fix the parameters \(\mu_{2k+1}, \mu_{2k-1}, \mu_{2k-3}\) and consider a more general system than (2.8) (and (2.10)) inputting the fourth parameter, \(\mu_{2k-5} < 0\), into (2.8):

\[
\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-5} y^{2k-5} + y^{2k-4} + \ldots + \mu_{2k+1} y^{2k+1}. \tag{2.11}
\]

Under decreasing \(\mu_{2k-5}\), the vector field of (2.11) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating the third (stable) limit cycle, \(\Gamma_3\). Under further decreasing \(\mu_{2k-5}\), \(\Gamma_3\) will join with \(\Gamma_2\) forming a semi-stable limit cycle, \(\Gamma_{23}\), which will disappear in a “trajectory concentration” surrounding the origin; the cycle \(\Gamma_1\) will expand infinitely tending to the Poincaré circle at infinity.

Let system (2.11) have three limit cycles: \(\Gamma_1\), \(\Gamma_2\), \(\Gamma_3\). Could an additional semi-stable limit cycle appear under decreasing \(\mu_{2k-5}\), after splitting of which system (2.11) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear neither in the domain \(D_2\) bounded by the cycles \(\Gamma_1\) and \(\Gamma_2\) nor in the domain \(D_3\) bounded by the origin and \(\Gamma_3\) because of increasing the distance between the spiral coils filling these domains under decreasing \(\mu_{2k-5}\). Consider two other domains: \(D_1\) bounded on the inside by the cycle \(\Gamma_1\) and \(D_3\) bounded by the cycles \(\Gamma_2\) and \(\Gamma_3\). As before, we will prove impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters \(\mu_{2k+1}^* > 0\), \(\mu_{2k-1}^* < 0\), \(\mu_{2k-3}^* > 0\), and \(\mu_{2k-5}^* < 0\), such a semi-stable cycle exists. Return to system
(2.4) again, input first the parameters $\mu_{2k}-5 < 0$, $\mu_{2k-1} < 0$ and then the parameter $\mu_{2k+1} > 0$:

$$
\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-5}y^{2k-5} + \ldots + \mu_{2k-1}y^{2k-1} + y^2 + \mu_{2k+1}y^{2k+1}. \quad (2.12)
$$

Fix the parameters $\mu_{2k-5}$, $\mu_{2k-1}$ under the values $\mu^*_{2k-5}$, $\mu^*_{2k-1}$, respectively. Under increasing $\mu_{2k+1}$, the node at infinity will change the character of its stability, the separatrix behaviour of the infinite saddle will be also changed and a stable limit cycle, $\Gamma_1$, will immediately appear from the Poincaré circle at infinity $\mathbb{P}$. Fix $\mu_{2k+1}$ under the value $\mu^*_{2k+1}$ and input the parameter $\mu_{2k-3} > 0$ into (2.12) getting system (2.11).

Since, on our assumption, (2.11) has three limit cycles for $\mu_{2k-3} < \mu^*_{2k-3}$, there exists some value of the parameter $\mu^*_{2k-3}$ ($0 < \mu^*_{2k-3} < \mu^*_{2k-3}$) for which a semi-stable limit cycle, $\Gamma_{23}$, appears in this system and then splits into an unstable cycle, $\Gamma_2$, and a stable cycle, $\Gamma_3$, under further increasing $\mu_{2k-3}$. The formed domain $D_3$ bounded by the limit cycles $\Gamma_2$, $\Gamma_3$ and also the domain $D_1$ bounded on the inside by the limit cycle $\Gamma_1$ will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there.

All other combinations of the parameters $\mu_{2k+1}$, $\mu_{2k-1}$, $\mu_{2k-3}$, and $\mu_{2k-5}$ are considered in a similar way. It follows that system (2.11) has at most three limit cycles. If we continue the procedure of successive inputting the odd parameters, $\mu_{2k-7}, \ldots, \mu_3$, $\mu_1$, into system (2.4), it is possible first to obtain $k$ limit cycles ($\mu_{2k+1} \gg -\mu_{2k-1} \gg \mu_{2k-3} \gg -\mu_{2k-5} \gg \mu_{2k-7} \gg \ldots$) and then to conclude that canonical system (2.1) (i.e., Liénard’s polynomial system (1.2) as well) has at most $k$ limit cycles. The theorem is proved.

3 An arbitrary polynomial system

Let us consider an arbitrary polynomial system

$$
\dot{x} = P_n(x, y, \mu_1, \ldots, \mu_k), \quad \dot{y} = Q_n(x, y, \mu_1, \ldots, \mu_k) \quad (3.1)
$$

containing $k$ field rotation parameters, $\mu_1, \ldots, \mu_k$, and having an anti-saddle at the origin. Generalizing the main result of the previous section on the maximum number of limit cycles surrounding a singular point in Liénard’s polynomial system (1.2), we prove the following theorem.

**Theorem 3.1.** Polynomial system (3.1) containing $k$ field rotation parameters and having a singular point of the center type at the origin for the zero values of these parameters can have at most $k-1$ limit cycles surrounding the origin.
Proof. Vanish all parameters of (3.1) and suppose that the obtained system
\[ \dot{x} = P_n(x, y, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, 0, \ldots, 0) \] (3.2)
has a singular point of the center type at the origin. Let us input successively
the field rotation parameters, \( \mu_1, \ldots, \mu_k \), into this system.

Suppose, for example, that \( \mu_1 > 0 \) and that the vector field of the system
\[ \dot{x} = P_n(x, y, \mu_1, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, 0, \ldots, 0) \] (3.3)
is rotated counterclockwise turning the origin into a stable focus under in-
creasing \( \mu_1 \).

Fix \( \mu_1 \) and input the parameter \( \mu_2 \) into (3.3) changing it so that the field of
the system
\[ \dot{x} = P_n(x, y, \mu_1, \mu_2, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, 0, \ldots, 0) \] (3.4)
would be rotated in opposite direction (clockwise). Let be so for \( \mu_2 < 0 \).
Then, for some value of this parameter, a limit cycle will appear in system
(3.4). There are three logical possibilities for such a bifurcation: 1) the limit
cycle appears from the focus at the origin; 2) it can also appear from some
separatrix cycle surrounding the origin; 3) the limit cycle appears from a
so-called “trajectory concentration”. In the last case, the limit cycle is semi-
stable and, under further decreasing \( \mu_2 \), it splits into two limit cycles (stable
and unstable), one of which then disappears at (or tends to) the origin and the
other disappears on (or tends to) some separatrix cycle surrounding this point.
But since the stability character of both a singular point and a separatrix cycle
is quite easily controlled \([11]\), this logical possibility can be excluded. Let us
choose one of the two other possibilities: for example, the first one, the so-
called Andronov–Hopf bifurcation. Suppose that, for some value of \( \mu_2 \), the
focus at the origin becomes non-rough, changes the character of its stability
and generates a stable limit cycle, \( \Gamma_1 \).

Under further decreasing \( \mu_2 \), three new logical possibilities can arise: 1) the
limit cycle \( \Gamma_1 \) disappears on some separatrix cycle surrounding the origin;
2) a separatrix cycle can be formed earlier than \( \Gamma_1 \) disappears on it, then it
generates one more (unstable) limit cycle, \( \Gamma_2 \), which joins with \( \Gamma_1 \) forming
a semi-stable limit cycle, \( \Gamma_{12} \), disappearing in a “trajectory concentration”
under further decreasing \( \mu_2 \); 3) in the domain \( D_1 \) outside the cycle \( \Gamma_1 \) or in
the domain \( D_2 \) inside \( \Gamma_1 \), a semi-stable limit cycle appears from a “trajectory
concentration” and then splits into two limit cycles (logically, the appearance
of such semi-stable limit cycles can be repeated).

Let us consider the third case. It is clear that, under decreasing \( \mu_2 \), a semi-
stable limit cycle cannot appear in the domain \( D_2 \), since the focus spirals filling
this domain will untwist and the distance between their coils will increase because of the vector field rotation. By contradiction, we can prove that a semi-stable limit cycle cannot appear in the domain \(D_1\). Suppose it appears in this domain for some values of the parameters \(\mu_1^* > 0\) and \(\mu_2^* < 0\). Return to initial system (3.2) and change the inputting order for the field rotation parameters. Input first the parameter \(\mu_2 < 0\):

\[
\begin{align*}
\dot{x} &= P_n(x, y, \mu_2, 0, \ldots, 0), \\
\dot{y} &= Q_n(x, y, \mu_2, 0, \ldots, 0).
\end{align*}
\]

Fix it under \(\mu_2 = \mu_2^*\). The vector field of (3.5) is rotated clockwise and the origin turns into an unstable focus. Inputting the parameter \(\mu_1 > 0\) into (3.5), we get again system (3.4), the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle, \(\Gamma_1\), will appear from some separatrix cycle. The limit cycle \(\Gamma_1\) will contract, the outside spirals winding onto this cycle will untwist and the distance between their coils will increase under increasing \(\mu_1\) to the value \(\mu_1^*\). It follows that there are no values of \(\mu_2^* < 0\) and \(\mu_1^* > 0\), for which a semi-stable limit cycle could appear in the domain \(D_1\).

The second logical possibility can be excluded by controlling the stability character of the separatrix cycle \([11]\). Thus, only the first possibility is valid, i.e., system (3.4) has at most one limit cycle.

Let system (3.4) have the unique limit cycle \(\Gamma_1\). Fix the parameters \(\mu_1 > 0\), \(\mu_2 < 0\) and input the third parameter, \(\mu_3 > 0\), into this system supposing that \(\mu_3\) rotates its vector field counterclockwise:

\[
\begin{align*}
\dot{x} &= P_n(x, y, \mu_1, \mu_2, \mu_3, 0, \ldots, 0), \\
\dot{y} &= Q_n(x, y, \mu_1, \mu_2, \mu_3, 0, \ldots, 0).
\end{align*}
\]

Here we can have two basic possibilities: 1) the limit cycle \(\Gamma_1\) disappears at the origin; 2) the second (unstable) limit cycle, \(\Gamma_2\), appears from the origin and, under further increasing the parameter \(\mu_3\), the cycle \(\Gamma_2\) joins with \(\Gamma_1\) forming a semi-stable limit cycle, \(\Gamma_{12}\), which disappears in a “trajectory concentration” surrounding the origin. Besides, we can also suggest that: 3) in the domain \(D_2\) bounded by the origin and \(\Gamma_1\), a semi-stable limit cycle, \(\Gamma_{23}\), appears from a “trajectory concentration”, splits into an unstable cycle, \(\Gamma_2\), and a stable cycle, \(\Gamma_3\), and then the cycles \(\Gamma_1\), \(\Gamma_2\) disappear through a semi-stable limit cycle, \(\Gamma_{12}\), and the cycle \(\Gamma_3\) disappears through the Andronov–Hopf bifurcation; 4) a semi-stable limit cycle, \(\Gamma_{34}\), appears in the domain \(D_2\) bounded by the cycles \(\Gamma_1\), \(\Gamma_2\) and, for some set of values of the parameters, \(\mu_1^*, \mu_2^*, \mu_3^*\), system (3.6) has at least four limit cycles.

Let us consider the last, fourth, case. It is clear that a semi-stable limit cycle cannot appear neither in the domain \(D_1\) bounded on the inside by the cycle \(\Gamma_1\) nor in the domain \(D_3\) bounded by the origin and \(\Gamma_2\) because of increasing the distance between the spiral coils filling these domains under increasing the parameter \(\mu_3\). To prove impossibility of the appearance of a semi-stable limit
cycle in the domain $D_2$, suppose the contrary, i.e., for some set of values of the parameters, $\mu_1^* > 0$, $\mu_2^* < 0$, and $\mu_3^* > 0$, such a semi-stable cycle exists. Return to system (3.2) again and input first the parameters $\mu_3 > 0$, $\mu_1 > 0$:

$$\dot{x} = P_n(x, y, \mu_1, \mu_3, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_3, 0, \ldots, 0).$$  \hspace{1cm} (3.7)$$

Fix these parameters under $\mu_3 = \mu_3^*$, $\mu_1 = \mu_1^*$ and input the parameter $\mu_2 < 0$ into (3.7) getting again system (3.6). Since, on our assumption, this system has two limit cycles for $\mu_2 > \mu_2^*$, there exists some value of the parameter, $\mu_2^{12}$ ($\mu_2^* < \mu_2^{12} < 0$), for which a semi-stable limit cycle, $\Gamma_{12}$, appears in system (3.6) and then splits into a stable cycle, $\Gamma_1$, and an unstable cycle, $\Gamma_2$, under further decreasing $\mu_2$. The formed domain $D_2$ bounded by the limit cycles $\Gamma_1$, $\Gamma_2$ and filled by the spirals will enlarge, since, on the properties of a field rotation parameter, the interior unstable limit cycle $\Gamma_2$ will contract and the exterior stable limit cycle $\Gamma_1$ will expand under decreasing $\mu_2$. The distance between the spirals of the domain $D_2$ will naturally increase, what will prohibit from the appearance of a semi-stable limit cycle in this domain for $\mu_2 < \mu_2^{12}$.

Thus, there are no such values of the parameters, $\mu_1^* > 0$, $\mu_2^* < 0$, $\mu_3^* > 0$, for which system (3.6) would have an additional semi-stable limit cycle. Therefore, the fourth case cannot be realized. The third case is considered absolutely similarly. It follows from the first two cases that system (3.6) can have at most two limit cycles.

Suppose that system (3.6) has two limit cycles, $\Gamma_1$ and $\Gamma_2$, fix the parameters $\mu_1 > 0$, $\mu_2 < 0$, $\mu_3 > 0$ and input the fourth parameter, $\mu_4 < 0$, into this system supposing that $\mu_4$ rotates its vector field clockwise:

$$\dot{x} = P_n(x, y, \mu_1, \ldots, \mu_4, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \ldots, \mu_4, 0, \ldots, 0).$$  \hspace{1cm} (3.8)$$

The most interesting logical possibility here is that when the third (stable) limit cycle, $\Gamma_3$, appears from the origin and then, under preservation of the cycles $\Gamma_1$ and $\Gamma_2$, in the domain $D_3$ bounded on the inside by the cycle $\Gamma_3$ and on the outside by the cycle $\Gamma_2$, a semi-stable limit cycle, $\Gamma_45$, appears and then splits into a stable cycle, $\Gamma_4$, and an unstable cycle, $\Gamma_5$, i.e., when system (3.8) for some set of values of the parameters, $\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*$, has at least five limit cycles. Logically, such a semi-stable limit cycle could also appear in the domain $D_1$ bounded on the inside by the cycle $\Gamma_1$, since, under decreasing $\mu_4$, the spirals of the trajectories of (3.8) will twist and the distance between their coils will decrease. On the other hand, in the domain $D_2$ bounded on the inside by the cycle $\Gamma_2$ and on the outside by the cycle $\Gamma_1$ and also in the domain $D_4$ bounded by the origin and $\Gamma_3$, a semi-stable limit cycle cannot appear, since, under decreasing $\mu_4$, the spirals will untwist and the distance between their coils will increase. To prove impossibility of the appearance of a semi-stable limit cycle in the domains $D_3$ and $D_1$, suppose the contrary, i.e., for some set of values of the parameters, $\mu_1^* > 0$, $\mu_2^* < 0$, $\mu_3^* > 0$, and $\mu_4^* < 0,
such a semi-stable cycle exists. Return to system (3.2) again, input first the parameters $\mu_4 < 0$, $\mu_2 < 0$ and then the parameter $\mu_1 > 0$:

$$
\dot{x} = P_n(x, y, \mu_1, \mu_2, \mu_4, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, \mu_4, 0, \ldots, 0). \tag{3.9}
$$

Fix the parameters $\mu_4$, $\mu_2$ under the values $\mu_4^*, \mu_2^*$, respectively. Under increasing $\mu_1$, a separatrix cycle is formed around the origin generating a stable limit cycle, $\Gamma_1$. Fix $\mu_1$ under the value $\mu_1^*$ and input the parameter $\mu_3 > 0$ into (3.9) getting system (3.8).

Since, on our assumption, system (3.8) has three limit cycles for $\mu_3 < \mu_3^*$, there exists some value of the parameter $\mu_3^{23} (0 < \mu_3^{23} < \mu_3^*)$ for which a semi-stable limit cycle, $\Gamma_23$, appears in this system and then splits into an unstable cycle, $\Gamma_2$, and a stable cycle, $\Gamma_3$, under further increasing $\mu_3$. The formed domain $D_3$ bounded by the limit cycles $\Gamma_2$, $\Gamma_3$ and also the domain $D_1$ bounded on the inside by the limit cycle $\Gamma_1$ will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there.

All other combinations of the parameters $\mu_1$, $\mu_2$, $\mu_3$, and $\mu_4$ are considered in a similar way. It follows that system (3.8) has at most three limit cycles. If we continue the procedure of successive inputting the field rotation parameters, $\mu_5$, $\mu_6$, . . . , $\mu_k$, into system (3.2), it is possible to conclude that system (3.1) can have at most $k - 1$ limit cycles surrounding the origin. The theorem is proved.

4 A generalized Liénard’s system

In [14], we considered a generalized Liénard’s cubic system of the form:

$$
\dot{x} = y, \quad \dot{y} = -x + (\lambda - \mu) y + (3/2) x^2 + \mu x y - (1/2) x^3 + \alpha x^2 y. \tag{4.1}
$$

This system has three finite singularities: a saddle $(1, 0)$ and two antisaddles $(0, 0)$ and $(2, 0)$. At infinity system (4.1) can have either the only nilpotent singular point of fourth order with two closed elliptic and four hyperbolic domains or two singular points: one of them is a hyperbolic saddle and the other is a triple nilpotent singular point with two elliptic and two hyperbolic domains. We studied global bifurcations of limit and separatrix cycles of (4.1), found possible distributions of its limit cycles and carried out a classification of its separatrix cycles. We proved also the following theorems.

**Theorem 4.1.** The foci of system (4.1) can be at most of second order.

**Theorem 4.2.** System (4.1) has at least three limit cycles.
Using the results obtained in [14] and applying the approach developed in this paper, we can easily prove a much stronger theorem.

**Theorem 4.3.** System (4.1) has at most three limit cycles with the following their distributions: ((1, 1), 1), ((1, 2), 0), ((2, 1), 0), ((1, 0), 2), ((0, 1), 2), where the first two numbers denote the numbers of limit cycles surrounding each of two anti-saddles and the third one denotes the number of limit cycles surrounding simultaneously all three finite singularities.

Theorem 4.3 agrees, for example, with the earlier results by Iliev and Perko [16], but it does not agree with a quite recent result by Dumortier and Li [6] published in the same journal. The authors of both papers use very similar methods: small perturbations of a Hamiltonian system. In [16], the zeros of the Melnikov functions are studied and, in particular, it is proved that at most two limit cycles can bifurcate from either the interior or exterior period annulus of the Hamiltonian under small parameter perturbations giving a generalized Liénard system. In [6], zeros of the Abelian integrals are studied and it is “proved” that at most four limit cycles can bifurcate from the exterior period annulus. Thus, Dumortier and Li “obtain” a configuration of four big limit cycles surrounding three finite singularities together with the fifth small limit cycle which surrounds one of the anti-saddles.

The result by Dumortier and Li [6] also does not agree with the Wintner–Perko termination principle for multiple limit cycles [11], [23]. Applying the method as developed in [3], [8]–[13], we can show that system (4.1) cannot have neither a multiplicity-three limit cycle nor more than three limit cycles in any configuration. That will be another proof of Theorem 4.3 (the same approach can be applied to proving Theorems 2.2 and 3.1 as well). But first let us formulate the Wintner–Perko termination principle [23] for the polynomial system

\[ \dot{x} = f(x, \mu), \]  

(4.2μ)

where \( x \in \mathbb{R}^2; \ \mu \in \mathbb{R}^n; \ f \in \mathbb{R}^2 \) (\( f \) is a polynomial vector function).

**Theorem 4.4 (Wintner–Perko termination principle).** Any one-parameter family of multiplicity-m limit cycles of relatively prime polynomial system (4.2μ) can be extended in a unique way to a maximal one-parameter family of multiplicity-m limit cycles of (4.2μ) which is either open or cyclic.

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of (4.2μ), which is typically a fine focus of multiplicity m, or on a (compound) separatrix cycle of (4.2μ), which is also typically of multiplicity m.
The proof of this principle for general polynomial system (4.2\(\mu\)) with a vector parameter \(\mu \in \mathbb{R}^n\) parallels the proof of the planar termination principle for the system
\[
\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda)
\]
with a single parameter \(\lambda \in \mathbb{R}\) (see [11], [23]), since there is no loss of generality in assuming that system (4.2\(\mu\)) is parameterized by a single parameter \(\lambda\); i.e., we can assume that there exists an analytic mapping \(\mu(\lambda)\) of \(\mathbb{R}\) into \(\mathbb{R}^n\) such that (4.2\(\mu\)) can be written as (4.2\(\mu(\lambda)\)) or even (4.2\(\lambda\)) and then we can repeat everything, what had been done for system (4.2\(\lambda\)) in [23]. In particular, if \(\lambda\) is a field rotation parameter of (4.2\(\lambda\)), it is valid the following Perko’s theorem on monotonic families of limit cycles.

**Theorem 4.5.** If \(L_0\) is a nonsingular multiple limit cycle of (4.2\(\lambda_0\)), then \(L_0\) belongs to a one-parameter family of limit cycles of (4.2\(\lambda\)); furthermore:

1) if the multiplicity of \(L_0\) is odd, then the family either expands or contracts monotonically as \(\lambda\) increases through \(\lambda_0\);

2) if the multiplicity of \(L_0\) is even, then \(L_0\) befurcates into a stable and an unstable limit cycle as \(\lambda\) varies from \(\lambda_0\) in one sense and \(L_0\) disappears as \(\lambda\) varies from \(\lambda_0\) in the opposite sense; i.e., there is a fold bifurcation at \(\lambda_0\).

**Proof of Theorem 4.3.** The proof is carried out by contradiction. Suppose that system (4.1) with three field rotation parameters, \(\lambda\), \(\mu\), and \(\alpha\), has three limit cycles around, for example, the origin (the case when limit cycles surround another focus is considered in a similar way). Then we get into some domain in the space of these parameters which is bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles.

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by a field-rotation parameter, according to Theorem 4.5, we will obtain a monotonic curve which, by the Wintner–Perko termination principle (Theorem 4.4), terminates either at the origin or on some separatrix cycle surrounding the origin. Since we know absolutely precisely at least the cyclicity of the singular point (Theorem 4.1) which is equal to two, we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate.
If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, on the same principle (Theorem 4.4), this again contradicts to Theorem 4.1 not admitting the multiplicity of limit cycles higher than two. Moreover, it also follows from the termination principle that neither the ordinary separatrix loop nor the eight-loop cannot have the multiplicity (cyclicity) higher than two (in that way, it can be proved that the cyclicity of three other separatrix cycles [14] is at most two). Therefore, according to the same principle, there are no more than two limit cycles in the exterior domain surrounding all three finite singularities of (4.1). Thus, system (4.1) cannot have neither a multiplicity-three limit cycle nor more than three limit cycles in any configuration. The theorem is proved.

So, we have found two approaches to solving Smale’s Thirteenth and Hilbert’s Sixteenth Problems. Both these approaches are based on the application of field rotation parameters which determine limit cycle bifurcations of polynomial systems.

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