A short note on Cuntz splice from a viewpoint of continuous orbit equivalence of topological Markov shifts

Kengo Matsumoto
Department of Mathematics
Joetsu University of Education
Joetsu, 943-8512, JAPAN

Abstract

Let $A$ be an $N \times N$ irreducible matrix with entries in $\{0, 1\}$. We present an easy way to find an $(N+3) \times (N+3)$ irreducible matrix $\overline{A}$ with entries in $\{0, 1\}$ such that their Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{\overline{A}}$ are isomorphic and $\det(1 - A) = -\det(1 - \overline{A})$. As a consequence, we know that two Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic if and only if the one-sided topological Markov shift $(X_A, \sigma_A)$ is continuously orbit equivalent to either $(X_B, \sigma_B)$ or $(\overline{X}_B, \overline{\sigma}_B)$.

For an $N \times N$ irreducible matrix $A$ with entries in $\{0, 1\}$, let us denote by $G(A)$ the abelian group $\mathbb{Z}^N/(1 - A^t)\mathbb{Z}^N$ and by $u_A$ the position of the class $[(1, \ldots, 1)]$ of the vector $(1, \ldots, 1)$ in the group $G(A)$. Throughout this short note, matrices are all assumed to be irreducible and not any permutation matrices. J. Cuntz in [3] has shown that the pair $(K_0(\mathcal{O}_A), [1])$ of the $K_0$-group $K_0(\mathcal{O}_A)$ of the Cuntz–Krieger algebra $\mathcal{O}_A$ and the class $[1]$ of the unit in $K_0(\mathcal{O}_A)$ is isomorphic to $(G(A), u_A)$. In [12], M. Rørdam has shown that $(G(A), u_A)$ is a complete invariant of the isomorphism class of $\mathcal{O}_A$ (see [6] for $N \leq 3$). For an $N \times N$ irreducible matrix $A = [A(i, j)]_{i, j=1}^N$ with entries in $\{0, 1\}$, the $(N+2) \times (N+2)$ irreducible matrix $A_-$ defined by

$$A_- = \begin{pmatrix}
A(1, 1) & \ldots & A(1, N-1) & A(1, N) & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
A(N-1, 1) & \ldots & A(N-1, N-1) & A(N-1, N) & 0 & 0 \\
A(N, 1) & \ldots & A(N, N-1) & A(N, N) & 1 & 0 \\
0 & \ldots & 0 & 1 & 1 & 1 \\
0 & \ldots & 0 & 0 & 1 & 1
\end{pmatrix}$$

is called the Cuntz splice for $A$, which has been first introduced in [4] by J. Cuntz, related to classification problem for Cuntz–Krieger algebras. In [4], he had used the notation $A^-$ instead of the above $A_-$. The crucial property of the Cuntz splice is that $G(A_-)$ is isomorphic to $G(A)$ and $\det(1 - A_-) = -\det(1 - A)$. The Cuntz splice

$$\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}$$

is

1
for the matrix $[1\ 1]$ is denoted by $2_-$. In the proof of the above Rørdam’s result [12, Theorem 6.5], J. Cuntz’s theorem [12, Theorem 7.2] is used which says that $O_2 \cong O_{2-}$ implies $O_A \otimes K \cong O_A \otimes K$ for all irreducible non-permutation matrices $A$. Since Rørdam has proved $O_2 \cong O_{2-}$ ([12, Lemma 6.4]), the result $O_A \otimes K \cong O_A \otimes K$ holds for all irreducible non-permutation matrices $A$. By using this result, Rørdam has also obtained that the group $G(A)$ is a complete invariant of the stable isomorphism class of $O_A$.

Let us denote by $BF(A)$ the abelian group $G(A^t) = \mathbb{Z}^N/(1-A)\mathbb{Z}^N$, which is called the Bowen–Franks group for $N \times N$ matrix $A$ ([II]). Although $BF(A)$ is isomorphic to $G(A)$ as a group, there is no canonical isomorphism between them. Related to classification theory of symbolic dynamical systems, J. Franks has shown that the pair $(BF(A), sgn(det(1-A)))$ is a complete invariant of the flow equivalence class of the two-sided topological Markov shift $(\tilde{X}_A, \tilde{\sigma}_A)$ by using Bowen–Franks’s result [II] for the group $BF(A)$ and Parry–Sullivan’s result [III] for the determinant $det(1-A)$. Combining this with the Rørdam’s result for the stable isomorphism classes of the Cuntz–Krieger algebras, $O_A$ is stably isomorphic to $O_{2-}$ if and only if $(\tilde{X}_A, \tilde{\sigma}_A)$ is flow equivalent to either $(\tilde{X}_A, \tilde{\sigma}_A)$ or $(\tilde{X}_{2-}, \tilde{\sigma}_{2-})$.

In [II], the author has introduced a notion of continuous orbit equivalence in one-sided topological Markov shifts to classify Cuntz–Krieger algebras from a viewpoint of topological dynamical system. In [II], H. Matui and the author have shown that the triple $(G(A), u_A, sgn(det(1-A)))$ is a complete invariant of the continuous orbit equivalence class of the right one-sided topological Markov shift $(X_A, \sigma_A)$. The $C^*$-algebra $O_{A-}$ is not necessarily isomorphic to $O_A$, whereas they are stably isomorphic, because the position $u_{A-}$ in $G(A-)$ generally is different from the position $u_A$ in $G(A)$. We note that the group $G(A)$ determines the absolute value $|det(1-A)|$. If $G(A)$ is infinite, $\text{Ker}(1-A)$ is not trivial so that $det(1-A) = 0$. If $G(A)$ is finite, it forms a finite direct sum $\mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z}$ for some $m_1, \ldots, m_r \in \mathbb{N}$ so that $|det(1-A)| = m_1 \cdots m_r$ (cf. [I], [II], [12]).

By [II, Lemma 3.7], we know that there is a matrix $A'$ with entries in $\{0, 1\}$ such that the triple $(G(A), u_A, sgn(det(1-A)))$ is isomorphic to $(G(A'), u_{A'}, -sgn(det(1-A')))$, which means that there exists an isomorphism $\Phi : G(A) \rightarrow G(A')$ such that $\Phi(u_A) = u_{A'}$ and $sgn(det(1-A)) = -sgn(det(1-A'))$. Following the given proof of [II, Lemma 3.7], the construction of the matrix $A'$ seems to be slightly complicated and the matrix size of $A'$ becomes much bigger than that of $A$. It is not an easy task to present the matrix $A'$ for the given matrix $A$ in a concrete way.

In this short note, we directly present an $(N+3) \times (N+3)$ matrix $\tilde{A}$ with entries in $\{0, 1\}$ such that $(G(A), u_A, sgn(det(1-A)))$ is isomorphic to $(G(\tilde{A}), u_{\tilde{A}}, -sgn(det(1-\tilde{A})))$. The matrix $\tilde{A}$ is constructed such that if $A$ is an irreducible non-permutation matrix, so is $\tilde{A}$.
We define

$$A^\circ = \begin{bmatrix}
A(1, 1) & \ldots & A(1, N-1) & A(1, N) & 0 \\
\vdots & & \vdots & \vdots & \\
A(N-1, 1) & \ldots & A(N-1, N-1) & A(N-1, N) & 0 \\
0 & \ldots & 0 & 0 & 1 \\
A(N, 1) & \ldots & A(N, N-1) & A(N, N) & 0
\end{bmatrix}$$

and

$$\bar{A} = (A^\circ)_- = \begin{bmatrix}
A(1, 1) & \ldots & A(1, N-1) & A(1, N) & 0 & 0 & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
A(N-1, 1) & \ldots & A(N-1, N-1) & A(N-1, N) & 0 & 0 & 0 \\
0 & \ldots & 0 & 0 & 1 & 0 & 0 \\
A(N, 1) & \ldots & A(N, N-1) & A(N, N) & 0 & 1 & 0 \\
0 & \ldots & 0 & 0 & 1 & 1 & 1 \\
0 & \ldots & 0 & 0 & 1 & 1 & 1
\end{bmatrix} \tag{1}$$

The operation $A \to A^\circ$ is nothing but an expansion defined by Parry–Sullivan in [11], and preserves their determinant: $\det(1 - A) = \det(1 - A^\circ)$. The following figure is a graphical expression of the matrix $\bar{A}$ from $A$.

![Graphical expression of $\bar{A}$](image)

Figure 1:

We provide two lemmas. The first one is seen in [1]. The second one is seen in [4] and [12] in a different form.

**Lemma 1** ([1, Theorem 1.3]). *The map*

$$\eta_A : (x_1, \ldots, x_{N-1}, x_N, x_{N+1}) \in \mathbb{Z}^{N+1} \to (x_1, \ldots, x_{N-1}, x_N + x_{N+1}) \in \mathbb{Z}^N$$

*induces an isomorphism $\bar{\eta}_A$ from $G(A^\circ)$ to $G(A)$ such that $\bar{\eta}_A([(1, \ldots, 1, 0)]) = u_A$.*

**Lemma 2** (cf. [4, Proposition 2], [12, Proposition 7.1]). *The map*

$$\xi_A : (x_1, \ldots, x_N) \in \mathbb{Z}^N \to (x_1, \ldots, x_N, 0, 0) \in \mathbb{Z}^{N+2}$$

*induces an isomorphism $\bar{\xi}_A$ from $G(A)$ to $G(A_-)$ such that $\bar{\xi}_A([(1, \ldots, 1, 0)]) = u_{A_-}$.*
Proof. For \( y = (y_1, \ldots, y_N) \in \mathbb{Z}^N \), put
\[
z = \begin{bmatrix}
z_1 \\
\vdots \\
z_N
\end{bmatrix} = (1 - A^t) \begin{bmatrix}
y_1 \\
\vdots \\
y_N
\end{bmatrix}.
\]
We then have
\[
\xi_A(z) = \begin{bmatrix}
z_1 \\
\vdots \\
z_N
\end{bmatrix} = (1 - A^t) \begin{bmatrix}
y_1 \\
\vdots \\
y_N
\end{bmatrix}.
\]
Hence we have \( \xi_A( (1 - A^t) \mathbb{Z}^N ) \subset (1 - A^t) \mathbb{Z}^{N+2} \) so that \( \xi_A : \mathbb{Z}^N \to \mathbb{Z}^{N+2} \) induces a homomorphism from \( G(A) \) to \( G(A_-) \) denoted by \( \xi_A \). Suppose that \( [\xi(x_1, \ldots, x_N)] = 0 \) in \( G(A_-) \) so that
\[
\begin{bmatrix}
x_1 \\
\vdots \\
x_N
\end{bmatrix} = (1 - A^t) \begin{bmatrix}
z_1 \\
\vdots \\
z_N
\end{bmatrix}
\]
for some \( (z_1, \ldots, z_{N+2}) \in \mathbb{Z}^{N+2} \). It then follows that \( z_{N+1} = 0, z_{N+2} = -z_N \) so that
\[
\begin{bmatrix}
x_1 \\
\vdots \\
x_N
\end{bmatrix} = (1 - A^t) \begin{bmatrix}
z_1 \\
\vdots \\
z_N
\end{bmatrix}.
\]
This implies \( [x_1, \ldots, x_N] = 0 \) in \( G(A) \) and hence \( \xi_A \) is injective.

For \( (x_1, \ldots, x_N, x_{N+1}, x_{N+2}) \in \mathbb{Z}^{N+2} \), we have
\[
\begin{bmatrix}
x_1 \\
\vdots \\
x_N \\
x_{N+1} \\
x_{N+2}
\end{bmatrix} = \begin{bmatrix}
x_1 \\
\vdots \\
x_N-1 \\
x_{N-1} \\
x_{N+2} \\
x_{N+1} \\
x_{N+2}
\end{bmatrix} + \begin{bmatrix}
0 \\
\vdots \\
0 \\
x_{N-1} \\
x_{N+2} \\
x_{N+1} \\
x_{N+2}
\end{bmatrix} + (1 - A^t) \begin{bmatrix}
0 \\
\vdots \\
0 \\
x_{N-1} \\
x_{N+2} \\
x_{N+1} \\
x_{N+2}
\end{bmatrix}.
\]
This implies that \( [(x_1, \ldots, x_N, x_{N+1}, x_{N+2})] = \xi_A([(x_1, \ldots, x_{N-1}, x_{N-1} - x_{N-2})]) \) in \( G(A_-) \).
Therefore \( \xi_A : G(A) \to G(A_-) \) is surjective and hence an isomorphism. In particular, we see that \( [(1, \ldots, 1, 1, 1)] = \xi_A([(1, \ldots, 1, 0)]) \) in \( G(A_-) \). \( \square \)

We have the following theorem by the preceding two lemmas.

**Theorem 3.** For an \( N \times N \) matrix \( A \) with entries in \( \{0, 1\} \), let \( \bar{A} \) be the \( (N+3) \times (N+3) \) matrix with entries in \( \{0, 1\} \) defined in (11). Then there exists an isomorphism \( \Phi : G(A) \to G(A) \) such that \( \Phi(u_A) = u_A \) and the matrices \( A, \bar{A} \) satisfy \( \det(1 - A) = -\det(1 - \bar{A}) \). If \( A \) is an irreducible non-permutation matrix, so is \( \bar{A} \).
Proof. Define \( \Phi : G(A) \to G(\tilde{A}) \) by \( \Phi = \tilde{\xi}_A \circ \tilde{\eta}_A^{-1} \) so that \( \Phi(u_A) = \tilde{\xi}_A((1, \ldots, 1)) = u_{\tilde{A}}. \) Since \( \det(1 - \tilde{A}) = -\det(1 - A^0) = -\det(1 - A) \), we see the desired assertion.

Let \( P \) be an \( N \times N \) permutation matrix coming from a permutation of the set \( \{1, 2, \ldots, N\} \). Since there exists a natural isomorphism \( \Phi_P : G(A) \to G(PAP^{-1}) \) such that \( \Phi_P(u_A) = u_{PAP^{-1}} \) and \( \det(1-A) = \det(1-PAP^{-1}) \), the triplet \( (G(A), u_A, \det(1-A)) \) does not depend on the choice of the vertex \( v_N \) in the directed graph of the matrix \( A \).

We have some corollaries.

**Corollary 4.** Let \( A \) be an irreducible non-permutation matrix with entries in \( \{0, 1\} \). Then \( O_A \) is isomorphic to \( O_{\tilde{A}} \) and \( \det(1 - A) = -\det(1 - \tilde{A}) \).

Let \( \bar{I} \) denote the matrix
\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]
which is the matrix \( \tilde{A} \) for the \( 1 \times 1 \) matrix \( A = [1] \). By the above theorem, we have

**Corollary 5.** \( (K_0(O_1), u_1) = (\mathbb{Z}, 1) \).

Hence the simple purely infinite \( C^* \)-algebra \( O_1 \) has the same K-theory as the \( C^* \)-algebra \( O_1 = C(S^1) \) of the continuous functions on the unit circle \( S^1 \) with the positions of their units, whereas \( (K_0(O_{1_\pm}), u_{1_-}) = (\mathbb{Z}, 0) \) for the matrix \( 1_- = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \) by [6] (cf. [4] p. 150).

The following corollary has been shown in [10]. Its proof is now easy by using [12].

**Corollary 6 (10, Lemma 3.7).** Let \( F \) be a finitely generated abelian group and \( u \) an element of \( F \). Let \( s = 0 \) when \( F \) is infinite and \( s = -1 \) or \( 1 \) when \( F \) is finite. Then there exists an irreducible non-permutation matrix \( A \) such that

\[
(F, u, s) = (G(A), u_A, \text{sgn}(\det(1 - A))).
\]

Proof. By [12] Proposition 6.7 (i), we know that there exists an irreducible non-permutation matrix \( A \) such that \( (F, u) = (G(A), u_A) \). If \( s = \text{sgn}(\det(1 - A)) \), the matrix \( A \) is the desired one, otherwise \( A \) is the desired one.

Let \( A \) and \( B \) be two irreducible non-permutation matrices with entries in \( \{0, 1\} \). The one-sided topological Markov shifts \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) are said to be flip continuously orbit equivalent if \( (X_A, \sigma_A) \) is continuously orbit equivalent to either \( (X_B, \sigma_B) \) or \( (X_B, \sigma_B) \). Similarly two-sided topological Markov shifts \( (\tilde{X}_A, \tilde{\sigma}_A) \) and \( (\tilde{X}_B, \tilde{\sigma}_B) \) are said to be flip flow equivalent if \( (\tilde{X}_A, \tilde{\sigma}_A) \) is flow equivalent to either \( (\tilde{X}_B, \tilde{\sigma}_B) \), or \( (\tilde{X}_B, \tilde{\sigma}_B) \). We thus have the following corollaries.

**Corollary 7.** Let \( A, B \) be irreducible and not any permutation matrices with entries in \( \{0, 1\} \).

(i) \( O_A \) is isomorphic to \( O_B \) if and only if the one-sided topological Markov shifts \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) are flip continuously orbit equivalent.
(ii) \( \mathcal{O}_A \) is stably isomorphic to \( \mathcal{O}_B \) if and only if the two-sided topological Markov shifts \( (\bar{X}_A, \bar{\sigma}_A) \) and \( (\bar{X}_B, \bar{\sigma}_B) \) are flip flow equivalent.

Let us denote by \([\mathcal{O}_A]\) the isomorphism class of the Cuntz–Krieger algebra \( \mathcal{O}_A \) as a \( C^* \)-algebra. Since \((G(A), u_A)\) is isomorphic to \((G(\bar{A}), u_{\bar{A}})\), we have \([\mathcal{O}_A] = [\mathcal{O}_{\bar{A}}]\). We regard the sign \( \text{sgn} (\det (1 - A)) \) of \( \det (1 - A) \) as the orientation of the class \([\mathcal{O}_A]\). Then we can say that the pair \(([\mathcal{O}_A], \text{sgn} (\det (1 - A)))\) is a complete invariant of the continuous orbit equivalence class of the one-sided topological Markov shift \((X_A, \sigma_A)\).

In the rest of this short note, we present another square matrix \( \tilde{A} \) of size \( N + 3 \) from a square matrix \( A = [A(i, j)]_{i,j=1}^{N} \) of size \( N \) such that \( \mathcal{O}_A \) is isomorphic to \( \mathcal{O}_{\tilde{A}} \) and \( \det (1 - A) = -\det (1 - \tilde{A}) \). Define \( (N + 3) \times (N + 3) \) matrix \( \tilde{A} \) by setting

\[
\tilde{A} = \begin{bmatrix}
A(1,1) & \ldots & A(1, N-1) & A(1, N) & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A(N-1,1) & \ldots & A(N-1, N-1) & A(N-1, N) & 0 & 0 & 0 \\
0 & \ldots & 0 & 0 & 1 & 0 & 0 \\
A(N,1) & \ldots & A(N, N-1) & A(N, N) & 0 & 1 & 0 \\
0 & \ldots & 0 & 0 & 1 & 0 & 1 \\
0 & \ldots & 0 & 0 & 0 & 1 & 1
\end{bmatrix}.
\]  

The difference between the previous matrix \( \bar{A} \) in (1) and the above matrix \( \tilde{A} \) is the only \(((N + 2), (N + 2))\)-component. Its graphical expression of the matrix \( \tilde{A} \) from \( A \) is the following figure.

By virtue of [6], we know the following proposition.

**Proposition 8.** The Cuntz–Krieger algebras \( \mathcal{O}_{\bar{A}} \) and \( \mathcal{O}_{\tilde{A}} \) are isomorphic, and \( \det (1 - \bar{A}) = \det (1 - \tilde{A}) \).

**Proof.** Let us denote by \( \bar{A}_i \) the \( i \)th row vector of the matrix \( \bar{A} \) of size \( N + 3 \). We put \( E_i \) the row vector of size \( N + 3 \) such that \( E_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) where the \( i \)th component is one, and the other components are zero. Then we have \( \bar{A}_{N+2} = E_{N+1} + \bar{A}_{N+3} \). Since
the \((N + 2)\)th row \(\tilde{A}_{N+2}\) of \(\tilde{A}\) is \(\tilde{A}_{N+2} = E_{N+1} + E_{N+3}\), and the other rows of \(\tilde{A}\) are the same as those of \(\tilde{A}\), the matrix \(\tilde{A}\) is obtained from \(\bar{A}\) by the primitive transfer

\[
\tilde{A} \xrightarrow{E_{N+1} + \bar{A}_{N+3} \to \bar{A}_{N+2}} \bar{A}
\]

in the sense of [6, Definition 3.5]. We obtain that \(\mathcal{O}_{\tilde{A}}\) is isomorphic to \(\mathcal{O}_{\bar{A}}\) by [6, Theorem 3.7], and \(\det(1 - \tilde{A}) = \det(1 - \bar{A})\) by [6, Theorem 8.4]. □

Before ending this short note, we refer to differences among the three matrices \(A_-, \bar{A}, \tilde{A}\) from a viewpoint of dynamical system. As \((G(A_-), \det(1 - A_-)) = (G(\bar{A}), \det(1 - \bar{A})) = (G(\tilde{A}), \det(1 - \tilde{A}))\), there is a possibility that their two-sided topological Markov shifts \((\bar{X}_{A_-}, \bar{\sigma}_{A_-}), (\tilde{X}_{\tilde{A}}, \bar{\sigma}_{\tilde{A}})\) are topologically conjugate. We however know that they are not topologically conjugate to each other in general by the following example. Denote by \(p_n(\bar{\sigma}_A)\) the cardinal number of the \(n\)-periodic points \(\{x \in \bar{X}_A \mid \bar{\sigma}_A^n(x) = x\}\) of the topological Markov shift \((\bar{X}_A, \bar{\sigma}_A)\). The zeta function \(\zeta_A(z)\) for \((\bar{X}_A, \bar{\sigma}_A)\) is defined by

\[
\zeta_A(z) = \exp \left( \sum_{n=1}^{\infty} \frac{p_n(\bar{\sigma}_A)}{n} z^n \right) \quad (\text{c.f.}[8]).
\]

It is well-known that the formula \(\zeta_A(z) = \frac{1}{\det(1-z\bar{A})}\) holds ([2]). Let us denote by \(2_-, \bar{2}, \tilde{2}\) the matrices \(A_-, \bar{A}, \tilde{A}\) for \([1 \ 1] \ [1 \ 1]\) respectively. It is direct to see that

\[
\zeta_{2_-}(z) = \frac{1}{1 - 4z + 3z^2 + 2z^3 - z^4}, \quad \zeta_{\bar{2}}(z) = \frac{1}{1 - 3z + 4z^2 + 2z^3 - z^4}, \quad \zeta_{\tilde{2}}(z) = \frac{1}{1 - 3z + z^2 + 2z^3 + z^4}.
\]

The zeta function is invariant under topological conjugacy so that \((\bar{X}_{2_-}, \bar{\sigma}_{2_-}), (\tilde{X}_{\bar{2}}, \bar{\sigma}_{\bar{2}}), (\tilde{X}_{\tilde{2}}, \bar{\sigma}_{\tilde{2}})\) are not topologically conjugate to each other.

This paper is a revised version of the paper entitled “Continuous orbit equivalence of topological Markov shifts and Cuntz splice” [arXiv:1511.01193v2 [math.OA]].

Acknowledgment. This work was supported by JSPS KAKENHI Grant Number 15K04896.

References

[1] R. Bowen and J. Franks, Homology for zero-dimensional nonwandering sets, Ann. Math. 106(1977), pp. 73–92.

[2] R. Bowen and O. E. Lanford III, Zeta functions of the shift transformation, Trans. Amer. Math. Soc. 112(1964), pp. 55–66.

[3] J. Cuntz, A class of \(C^*\)-algebras and topological Markov chains II: reducible chains and the \(\text{Ext}\)-functor for \(C^*\)-algebras, Invent. Math. 63(1980), pp. 25–40.

[4] J. Cuntz, The classification problem for the \(C^*\)-algebra \(O_{\bar{A}}\), Geometric methods in operator algebras, Pitman Research Notes in Mathematics Series 123(1986), pp. 145–151.
[5] J. Cuntz and W. Krieger, *A class of C*-algebras and topological Markov chains*, Invent. Math. **56**(1980), pp. 251–268.

[6] M. Enomoto, M. Fujii and Y. Watatani, *K₀-groups and classifications of Cuntz–Krieger algebras*, Math. Japon. **26**(1981), pp. 443–460.

[7] J. Franks, *Flow equivalence of subshifts of finite type*, Ergodic Theory Dynam. Systems **4**(1984), pp. 53–66.

[8] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[9] K. Matsumoto, *Orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras*, Pacific J. Math. **246**(2010), pp. 199–225.

[10] K. Matsumoto and H. Matui, *Continuous orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras*, Kyoto J. Math. **54**(2014), pp. 863–878.

[11] W. Parry and D. Sullivan, *A topological invariant for flows on one-dimensional spaces*, Topology **14**(1975), pp. 297–299.

[12] M. Rørdam, *Classification of Cuntz–Krieger algebras*, K-theory **9**(1995), pp. 31–58.