TOTAL CURVATURES OF HOLONOMIC LINKS

TOBIAS EKHOLM AND OLA WEISTRAND

Abstract. A differential geometric characterization of the braid-index of a link is found. After multiplication by $2\pi$, it equals the infimum of the sum of total curvature and total absolute torsion over holonomic representatives of the link.

Upper and lower bounds for the infimum of total curvature over holonomic representatives of a link are given in terms of its braid- and bridge-index. Examples showing that these bounds are sharp are constructed.

1. Introduction

Let $C_k$ denote the disjoint union of $k$ circles. An isotopy class of embeddings $C_k \rightarrow \mathbb{R}^3$ will be called a $(k$-component) link. A 1-component link will be called a knot.

A collection of loops $c: C_k \rightarrow \mathbb{R}^3$ is holonomic if it arises as the 2-jet extension of a function $f: C_k \rightarrow \mathbb{R}$. That is, in coordinates $(x_1, x_2, x_3)$ on $\mathbb{R}^3$, $c(t) = (f(t), f'(t), f''(t))$, $t \in C_k$. Vassiliev [7] introduced holonomic loops in knot theory and proved that any tame link has a holonomic representative. Later, Birman and Wrinkle [2] proved that if two holonomic embeddings $C_k \rightarrow \mathbb{R}^3$ are isotopic then they are isotopic through holonomic embeddings.

The curvature $\kappa(L)$ and curvature-torsion $(\kappa + \tau)(L)$ of a link $L$ are the infima of total curvature and of the sum of total curvature and total absolute torsion, respectively, over all embeddings representing $L$. These invariants were defined by Milnor [4], [5]. He proved that if $L$ is any link then $\kappa(L) = 2\pi$ bridge($L$), where bridge($L$), the bridge-index of $L$, equals the minimal number of over-arcs in a link-diagram representing $L$, and $(\kappa + \tau)(L)$ is an integral multiple of $2\pi$.

Further results on the curvature-torsion invariant were found by Honma and Saeki [3]. They proved if $L$ is any link then $(\kappa + \tau)(L) \leq 2\kappa(L) - 2\pi$ and showed that the difference between $2\pi$ braid($L$), where braid($L$), the braid-index of $L$, equals the minimal number of strands in a closed braid representing $L$, and $(\kappa + \tau)(L)$ may be strictly positive (and actually arbitrarily large).

In this paper, we shall study the holonomic curvature $\kappa_{\text{hol}}(L)$ and holonomic curvature-torsion $(\kappa + \tau)_{\text{hol}}(L)$. For a link $L$, $\kappa_{\text{hol}}(L)$ and $(\kappa + \tau)_{\text{hol}}(L)$ are the infima of total curvature and of the sum of total curvature and total absolute torsion, respectively, over all holonomic embeddings representing $L$ (see Section 2).

Theorem 1. For any link $L$,

$$(\kappa + \tau)_{\text{hol}}(L) = 2\pi \text{braid}(L).$$

Theorem 1 is proved in Section 3. It gives a differential geometric characterization of the braid-index.
Theorem 2. For any link $L$,

$$A(\text{braid}(L) - \text{bridge}(L)) + 2\pi \text{bridge}(L) \leq \kappa_{\text{hol}}(L) \leq 2\pi \text{braid}(L),$$

(1)

where $2A$ is the area on the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ of the region defined by the inequality $y^2 - 4xz > 0$.

Theorem 2 is proved in Section 4. The constant $A = (1.29\ldots)\pi$ can be expressed in terms of elliptic integrals.

For links $L$ with $\text{bridge}(L) = \text{braid}(L)$ the inequalities (1) are equalities. Hence, the upper bound in (1) is best possible. So is the lower bound:

Proposition 1. For any $m \geq 2$, $j \geq 1$ there exists a knot $K(m, j)$ such that $\text{bridge}(K(m, j)) = m$, $\text{braid}(K(m, j)) = m + j$, and $\kappa_{\text{hol}}(K) = Aj + 2\pi m$.

Proposition 1 is proved in Section 6. Knots with properties as stated in Proposition 1 are defined in Section 5.

Theorem 2 gives rise to the following question:

Question 1. Is it true that

$$A(\text{braid}(L) - \text{bridge}(L)) + 2\pi \text{bridge}(L) = \kappa_{\text{hol}}(L),$$

for all links $L$?

An affirmative answer would give a differential geometric characterization of the difference between braid- and bridge-index. A negative answer would give a new link invariant: $\kappa_{\text{hol}} - 2\pi \text{bridge}$.

The space $E_{\text{hol}}$ of holonomic embeddings $C_k \to \mathbb{R}^3$ is a subspace of the space $E$ of all embeddings. The result of Birman and Wrinkel mentioned above implies that the inclusion $E_{\text{hol}} \subset E$ induces an isomorphism on $\pi_0$. Theorems 1 and 2 shows that there are (in a sense large) open subsets of $E$ which do not intersect $E_{\text{hol}}$.

2. Holonomic curvature and curvature-torsion

2.1. Holonomic curvature. If $c$ is a continuous closed space curve we denote its total curvature, as defined in [4], p.251, by $\int_c \kappa \, ds$.

Lemma 1. Let $c$ be a continuous closed space curve. Then

$$\int_c \kappa \, ds = \frac{1}{2} \int_{S^2} \mu(c, v) \, dA(v),$$

(2)

where $dA$ is the area form on the unit sphere $S^2 \subset \mathbb{R}^3$ and $\mu(c, v)$ is the number of local maxima of the function $t \mapsto \langle c(t), v \rangle$. (The symbol $\langle , \rangle$ denotes the standard inner-product on $\mathbb{R}^3$.)

Proof. This is Theorem 3.1 in [4].

Remark 1. Equation (2) could be taken as definition of total curvature.

Remark 2. If $c$ is a space curve with continuous unit tangent vector $e_1$ then the total curvature of $c$ equals the length on the unit sphere $S^2 \subset \mathbb{R}^3$ of the curve traced out by $e_1$.

If $c$ is a twice continuously differentiable curve with everywhere defined curvature function $\kappa$ then the total curvature of $c$ equals the integral over $c$ of $\kappa \, ds$, where $ds$ is the arclength-element.
Definition 1. The holonomic curvature of a link $L$ is the infimum of $\int_{c} \kappa \, ds$ over all holonomic representatives $c$ of $L$.

2.2. Holonomic curvature-torsion. A three times continuously differentiable curve $c: [a, b] \rightarrow \mathbb{R}^3$ is called non-degenerate if $c'(t)$ and $c''(t)$ are linearly independent for every $t \in [a, b]$. If $c$ is non-degenerate then its curvature and torsion functions $\kappa$ and $\tau$ are defined and $\kappa > 0$.

Definition 2. The sum of total curvature and total absolute torsion of a non-degenerate curve $c$ is

$$\int_{c} (\kappa + |\tau|) \, ds,$$

where $ds$ is the arclength-element.

We shall be concerned with holonomic curves and so would like to describe the set of functions $f: C_2 \rightarrow \mathbb{R}$ with 2-jet extensions which are non-degenerate curves.

Definition 3. A function $f: C_2 \rightarrow \mathbb{R}$ is torsion-generic if its associated holonomic curve is non-degenerate.

Proposition 2. The set of non-torsion-generic functions $C_2 \rightarrow \mathbb{R}^3$ has codimension 2 in the space of all smooth functions $C_2 \rightarrow \mathbb{R}^3$. (That is, non-torsion-generic functions can be avoided in generic 1-parameter families of functions and in generic 2-parameter families they appear at isolated points.)

Proof. Let $(t, x_0, x_1, x_2, x_3, x_4) \in S^1 \times \mathbb{R}^4$ be coordinates on the jet-space $J^4(S^1, \mathbb{R})$ and let $N$ be the locus of the equations

$$x_1x_3 - x_2^2 = 0, \quad x_1x_4 - x_2x_3 = 0, \quad x_2x_4 - x_3^2 = 0.$$

Then $N$ is an algebraic subvariety of $J^4(S^1 \times \mathbb{R}^4)$. The non-singular part of $N$ is a submanifold of codimension 2 and the singular part of $N$ equals $V_1 \cup V_2$ where $V_1$ is the locus of $x_2 = x_3 = x_4 = 0$ and $V_2$ the locus of $x_1 = x_2 = x_3 = 0$. A function $f: S^1 \rightarrow \mathbb{R}$ is non-torsion-generic if and only if its 4-jet extension $j^4f$ satisfies $j^4f(S^1) \cap N \neq \emptyset$. Applying the jet-transversality theorem, the proposition follows.

Definition 4. The holonomic curvature-torsion of a link $L$ is the infimum of $\int_{c} (\kappa + |\tau|) \, ds$ over all torsion-generic holonomic representatives $c$ of $L$.

3. Proof of Theorem 1

Let $(x_1, x_2, x_3)$ be coordinates on $\mathbb{R}^3$ and let $j \geq 1$ be an integer. Let $f: C_j \rightarrow \mathbb{R}$ be a generic (as defined in 2) Proposition 1, (i), (a)) and torsion-generic function such that the embedding $c: C_j \rightarrow \mathbb{R}^3, c(t) = (f(t), f'(t), f''(t)), t \in C_j$ represents the link $L$.

Let $n$ be the number of local maxima of $f$. By performing a sequence of holonomic second Reidemeister moves (more precisely, moves as shown in 2, Fig. 1e) we can assure that any local maximum of $f$ is larger than the largest local minimum. The projection of the holonomic curve of $f$ onto the $x_1x_2$-plane, endowed with over/under-information, is then a closed braid on $n$ strands. Thus, $n \geq \text{braid}(L)$. Since the linking number of $c$ and the $x_1$-axis is $-n$, it follows from 2 Theorem 3 that
\[ \int_c (\kappa + |\tau|) \, ds \geq 2\pi n. \]

Noting that the set of generic functions is open and dense in the space of all functions \(C_k \rightarrow \mathbb{R}\), we conclude that \((\kappa + \tau)_{\text{hol}}(L) \geq 2\pi \text{braid}(L)\).

Theorem 1 in [3] implies that \(L\) has a holonomic representative such that its diagram \(D\) in the \(x_1, x_2\)-plane is the diagram of a closed braid on \(\text{braid}(L)\) strands. (The diagram \(D\) has all its negative crossings in the upper half-plane \((x_2 > 0)\) and all positive ones in the lower.) It is clear that we can deform \(D\) into a new diagram \(D'\) which represents \(L\), which is such that, when looked upon as a collection of plane curves, it has nowhere vanishing curvature, and which satisfies conditions (a)-(d) of Proposition 1, (i) in [7]. Then (ii) of Proposition 1 in [7] implies that \((\kappa)\) of \(L\) is torsion-generic.

It will be convenient to look upon \(g\) as a function defined on a collection of intervals rather than on \(C_k\). That is, \(g: \bigsqcup_{i=1}^n [0, M_i] \rightarrow \mathbb{R}, \ g(0) = g(M_i) \) for \(1 \leq i \leq n\).

Let \(k > 0\) be a real number. Then the collection of holonomic curves \((c(k))(t) = (g(kt), k g'(kt), k^2 g''(kt)), t \in \bigsqcup_{i=1}^n [0, \frac{M_i}{k}]\) also represents \(L\). As \(k \rightarrow 0\), \(c(k)\) approaches a collection of curves in the \(x_1, x_2\)-plane with nowhere vanishing curvature and total tangential degree \(2\pi \text{braid}(L)\). It follows that

\[
\lim_{k \rightarrow 0} \int_{c(k)} (\kappa + |\tau|) \, ds = 2\pi \text{braid}(L).
\]

Thus, \((\kappa + \tau)_{\text{hol}}(L) \leq 2\pi \text{braid}(L)\). \(\square\)

4. PROOF OF THEOREM 3

The last part of the proof of Theorem 3 implies that \(\kappa_{\text{hol}}(L) \leq 2\pi \text{braid}(L)\). To prove the other inequality

\[
\text{A}(\text{braid}(L) - \text{bridge}(L)) + 2\pi \text{bridge}(L) \leq \kappa_{\text{hol}}(L)
\]

we use the following lemma:

Lemma 2. Let \(f: S^1 \rightarrow \mathbb{R}\) have at least \(2n\) local extrema and only non-degenerate critical points. Then for almost every \(v = (x, y, z) \in S^2 \subset \mathbb{R}^3\) such that \(y^2 - 4xz > 0\), the function

\[
h_v(t) = x f(t) + y f'(t) + z f''(t)
\]

has at least \(n\) local maxima.

Proof. Let \(a \in \mathbb{R}\) be a coordinate on \(\mathbb{R}P^1, a \rightarrow [1, a]\), where \([\xi, \eta]\) are homogeneous coordinates on \(\mathbb{R}P^1\). Consider the function \(g_a(t) = f(t) + a f'(t)\). The function \(f\) has at least \(n\) non-degenerate local maxima and \(n\) non-degenerate local minima. Between any two local maxima (minima) there is a local minimum (maximum). Thus, if \(a \neq 0\) then \(g_a\) changes sign between the critical points of \(f\) and hence \(g_a\) has at least \(n\) local maxima (minima). (The latter is of course also true for the case \(a = 0\) and \(a = \infty\), \(g_\infty(t) = f'(t).\))

Consider the plane curve \(t \mapsto c(t) = (f(t), f'(t)), t \in S^1\) (which is regular since the critical points of \(f\) are non-degenerate). If \(t \in S^1\) is a critical point of \(g_a\), then the tangent line of \(c\) at \(c(t)\) is \([-a, 1]\). Moreover, \(t\) is a degenerate critical point of \(g_a\) if and only if the curvature \(\kappa(t)\) of \(c\) at \(c(t)\) is zero.
Let \( T(t) \in \mathbb{R}P^1 \) denote the tangent line of \( c(t) \). If \( \kappa^{-1}(0) = U \subset S^1 \) then \( T(U) \) is exactly the set of critical values of the map \( T: S^1 \to \mathbb{R}P^1 \). By Sard’s theorem, \( T(U) \subset \mathbb{R}P^1 \) is a set of measure zero. Thus, for almost every \( a \in \mathbb{R} \), \( g_a \) has at least \( n \) non-degenerate local maxima (minima).

Consider the function
\[
g_{(a,b)} = (f + af') + b(f'' + a f''') = f + (a + b) f' + ab f, \quad (a, b) \in \mathbb{R}^2.
\]

It follows from the above that for almost every \( (a, b) \in \mathbb{R}^2 \), \( g_{(a,b)} \) has at least \( n \) local maxima. The map \( \mathbb{R}^2 \to \mathbb{R}P^2 \), \( (a, b) \mapsto [1, a + b, ab] \) is a diffeomorphism onto its image when restricted to \( \{(a, b) : a > b\} \). The image consists of points \( [1, \eta, \zeta] \in \mathbb{R}P^2 \) such that \( \eta^2 - 4\zeta > 0 \). This means that for almost every line \( [1, \eta, \zeta] \in \mathbb{R}P^2 \) such that \( \eta^2 - 4\zeta > 0 \), the function \( f + \eta f' + \zeta f''' \) has at least \( n \) local maxima (minima).

Taking preimages of the double cover \( \mathbb{S}^2 \to \mathbb{R}P^2 \), \( (x, y, z) \mapsto [x, y, z] \) we obtain the statement in the lemma.

We now proceed with the proof of (3): Let \( t \mapsto c(t) = (f(t), f'(t), f'''(t)) \) for some generic \( f: C \to \mathbb{R} \) be a holonomic representative of \( L \). Let \( C^1, \ldots, C^j \) be the components of \( C \). Let \( f_i \) and \( c_i \) be the restrictions of \( f \) and \( c \), respectively, to \( C^i \). Let \( n_i \) be the number of local maxima of \( f_i \) and let \( n = \sum_{i=1}^j n_i \). Then, as in the proof of Theorem 4, \( n \geq \text{braid}(L) \).

Since all critical points of a generic function are non-degenerate, Lemma 2 implies that for almost every direction \( v = (x, y, z) \) such that \( y^2 - 4xz > 0 \), \( \mu(c_i, v) \geq n_i \). For \( v \) in the complementary region \( (y^2 - 4xz \leq 0) \) such that the projection onto the orthogonal complement of \( v \) gives a generic link diagram (a set of full measure), \( \sum_{i=1}^j \mu(c_i, v) \geq \text{bridge}(L) \), by definition of \( \text{bridge}(L) \). By Lemma 4,

\[
\int_c \kappa \, ds = \frac{1}{2} \sum_{i=1}^j \int_{\mathbb{S}^2} \mu(c_i, v) \, dA(v) \geq A \text{braid}(K) + (2\pi - A) \text{bridge}(K).
\]

5. The knots \( K(m, j) \)

**Definition 5.** For \( j \geq 1 \), define the knot \( K(2, j) \) to be the class of the (holonomic) embedding in Figure \( 4 \).

Note that \( K_{2,1} \) is the figure eight knot.

**Definition 6.** For \( j \geq 1 \) and \( m \geq 2 \), define the knot \( K(m, j) \) to be the connected sum of \( K(2, j) \) and \( m - 2 \) right-handed trefoil knots (see Figure \( 3 \)).

6. Proof of Proposition 1

The proof is a combination of lemmas which are stated here and proved later.

**Lemma 3.** For \( m \geq 2 \) and \( j \geq 1 \), the knot \( K(m, j) \) satisfies \( \text{bridge}(K(m, j)) = m \) and \( \text{braid}(K(m, j)) = m + j \).

Lemma 3 is proved in Section 5.3.

To finish the proof, holonomic representatives of \( K(m, j) \) with total curvature arbitrarily close to \( Aj + 2\pi m \) must be constructed. When this has been done for
Figure 1. Holonomic representatives of the knots $K(2,j)$

$m = 2$ the other cases are easy. We therefore restrict attention to $K(2,j)$. Before going into details, the argument will be outlined:

In Lemma 4, we construct a family of non-closed holonomic curves with the following two properties: First, any curve in the family can be closed in such way that the result is a closed holonomic curve arbitrarily close to representatives of $K(2,j)$. Second, the infimum of the total curvatures of the curves in the family is $Aj + 2\pi$.

The curves in this family are however degenerate at the points where they have to be perturbed to give representatives of $K(2,j)$. Hence, there is no guarantee that the required perturbation (even if it is very small) will give a small change in total curvature.

In Lemma 5, we overcome this problem: By deforming the above curves in a specific way, a new family of curves is produced. The new family have the properties of the old family and, in addition to that, its members are curves which are non-degenerate where they must be perturbed to give representatives of $K(2,j)$. This non-degeneracy warrants that the total curvature is continuous under small perturbations.

Finally, we pick a curve in the new family with total curvature close enough to $Aj + 2\pi$ and make a small enough perturbation. In Lemma 6, we prove that it is possible to close the perturbed curve in such a way that the result is a holonomic representative of $K(2,j)$ with total curvature as close as required to $Aj + 4\pi$.

We now go into details: Let $M \geq 0$ and let $a_1, \ldots, a_j$ be points in $[0, M]$. For $i = 1, \ldots, j$, let $J_i = [a_i - 2, a_i + 2]$ and $I_i = [a_i - 1, a_i + 1]$. Assume that the $J_i \cap J_l = \emptyset$ for $i \neq l$ and that $J_i \subset (0, M)$ for every $i$.

For odd $n \geq 3$, let $f_{j,n}: [0, M] \to \mathbb{R}$ be a function which satisfies

$$f_{j,n}(t) = -(t - a_i)^n \quad \text{for} \ t \in I_i, \ 1 \leq i \leq j,$$

which has constant derivative in neighborhoods of the endpoints of $[0, M]$, and which has associated holonomic curve is as in Figure 2. (Note that the associated holonomic curve is non-closed).

For $k > 0$, let $c(j, n, k)$ be the holonomic curve associated to the function $t \mapsto f_{j,n}(kt)$, $t \in \left[0, \frac{M}{k}\right]$. Note that for $n > 3$, the curvature function of $c(j, k, n)$ is not defined at the points $\frac{4k}{n}$, where $c(j, n, k)$ has contact of order $n - 2$ with a line in the $x_3$-direction.
Lemma 4.
\[ \lim_{k \to 0} \int_{c(j,n,k)} \kappa \, ds = A(n)j + 2\pi, \]
where \(2A(n)\) is the area on the unit sphere of the region \(\{(x,y,z) \in S^2 : y^2 - 4\left(\frac{n-2}{n-1}\right)xz > 0\}\).

Lemma 4 is proved in Section 6.2.

Let \(p: [0,M] \to \mathbb{R}\) be a function which satisfies
\[ p(t) = \begin{cases} -(t - a_i)^3 & \text{for } t \in I_i, \ 1 \leq i \leq j, \\ 0 & \text{for } t \in [0,M] - \bigcup_{i=1}^j I_i. \end{cases} \]

For \(\delta > 0\), let \(c(j,n,k,\delta)\) denote the holonomic curve associated to the function \(t \mapsto f_{j,n}(kt) + \delta p(kt), t \in [0,\frac{M}{k}]\). Clearly, the curve \(c(j,n,k,\delta)\) is non-degenerate on \(\left[\frac{a_i-1}{k}, \frac{a_i+1}{k}\right], 1 \leq i \leq j\), if \(\delta > 0\) is small enough.

Lemma 5. Let \(0 < a < 1\). If \(n \geq 2 \left(1 - \frac{1}{2a^2}\right)^{-1} + 2\) and \(0 \leq k \leq 1\) then
\[ \lim_{\delta \to 0} \int_{c(j,n,k,\delta)} \kappa \, ds \leq \int_{c(j,n,k)} \kappa \, ds + 6B(a)j, \]
where \(2B(a)\) is the area on the unit sphere of the region \(\{(x,y,z) \in S^2 : |x| \leq a \text{ or } |y| \leq a \text{ or } |z| \leq a\}\).

Lemma 5 is proved in Section 6.3.

Let \(q: [0,M] \to \mathbb{R}\) be a function which satisfies
\[ q(t) = \begin{cases} (t - a_i) & \text{for } t \in I_i, \ 1 \leq i \leq j, \\ 0 & \text{for } t \in [0,M] - \bigcup_{i=1}^j I_i. \end{cases} \]

For \(\beta > 0\), let \(c(j,n,k,\delta,\beta)\) denote the holonomic curve associated to the function \(t \mapsto f(kt) + \delta p(kt) + \beta q(kt), t \in [0,\frac{M}{k}]\). Fix a small \(\delta > 0\) such that \(c(j,n,k,\delta)\) is non-degenerate along the support of \(t \mapsto p(kt)\). This non-degeneracy implies that the total curvature is continuous under small perturbation and therefore:
\[ \lim_{\beta \to 0} \int_{c(j,n,k,\delta,\beta)} \kappa \, ds = \int_{c(j,n,k,\delta)} \kappa \, ds. \]
As mentioned in the outline above, we need to close the holonomic curve \( c(j, n, k, \delta, \beta) \) so that the result is a holonomic representative of the knot \( K(2, j) \). It is easy to see that this is possible (see Figures 3 and 4). The next lemma allows us to control the total curvature when closing the curve.

**Lemma 6.** Let \( c, u, d > 0 \) be given constants and let \( s, \alpha > 0 \). Then there exist \( S > 0 \) and functions \( f, g : [-s, S + s] \rightarrow \mathbb{R} \) such that:

(i) For \( t \in [-s, 0] \), \( f(t) = ut + c \) and \( g(t) = -dt - c \).

(ii) For \( t \in [S, S + s] \), \( f(t) = -d(t - S) + c \) and \( g(t) = u(t - S) - c \).

(iii) For all \( t \), \( |f'(t)| \leq \max(u, d) \) and \( |g'(t)| \leq \max(u, d) \).

(iv) The total curvatures of the holonomic curves associated to \( f \) and \( g \) respectively are smaller than \( \pi + \alpha \).

Lemma 6 is proved in Section 6.4.

We are now in position to prove Proposition 1: The statements about bridge- and braid-index is just Lemma 3.

Let \( \epsilon > 0 \) be given. We must find a holonomic representative of \( K(m, j) \) with total curvature less than \( Aj + 2\pi m + \epsilon \). We start with the case \( K(2, j) \):

Choose \( a > 0 \) such that \( B(a) < \frac{\epsilon}{48j} \). Choose \( n \geq 2 \left( 1 - \sqrt{1 - 2a^2} \right)^{-1} + 2 \) such that \( A(n) - A \leq \frac{\epsilon}{8j} \). By Lemma 5 it is possible to find \( k > 0 \) such that

\[
\int_{c(j,n,k)} \kappa \, ds \leq Aj + 2\pi + \frac{\epsilon}{4}.
\]

Lemma 6 then implies that for \( \delta > 0 \) small enough

\[
\int_{c(k,n,\delta)} \kappa \, ds \leq Aj + 2\pi + \frac{\epsilon}{2},
\]

and \( c(j, n, k, \delta) \) is non-degenerate along the support of \( t \mapsto q(kt), t \in [0, \frac{M}{k}] \). By equation (4), for \( \beta > 0 \) small enough

\[
\int_{c(j,n,k,\delta,\beta)} \kappa \, ds \leq Aj + 2\pi + \frac{3\epsilon}{4}.
\]

Finally, we close \( c(k, n, \delta, \beta) \) as in Figure 4 (we add the dashed part). If \( u \) in Figure 4 is small enough then the resulting holonomic curve is a representative of \( K(2, j) \). Using Lemma 3, we can assure that it has total curvature less than \( Aj + 4\pi + \epsilon \). This proves the Proposition for \( K(2, j) \).

The general case follows by noting that it is possible to add \( m - 2 \) holonomic trefoils to a holonomic representative of \( K(2, j) \), with total curvature smaller than \( Aj + 4\pi + \frac{\epsilon}{2} \), in such a way that the increase in total curvature is smaller than \( 2\pi(m - 2) + \frac{\epsilon}{2} \) (see Figure 5).
6.1. **Proof of Lemma 3.** Consider the representative of $K(2, j)$ presented in Figure 4. If $j$ third Reidemeister moves, moving all crossings except the three rightmost into the upper half plane are followed by $j$ first Reidemeister moves removing the $j$ uppermost crossings then the resulting (non-holonomic) embedding has exactly two maxima in the $x_1$-direction. It follows that $\text{bridge}(K(2, j)) \leq 2$ but $K(2, j)$ is knotted. Thus, $\text{bridge}(K(2, j)) = 2$.

The bridge-index of the trefoil knot is 2 and hence $\text{bridge}(K(m, j)) = m$, by the additivity properties of bridge-indices.

Looking at the projection of the representative of $K(2, j)$ in Figure 4 onto the $x_2x_3$-plane, we see that $\text{braid}(K(m, j)) \leq m + j$.

A straightforward induction shows that the HOMFLY-polynomial $P_{K(2,j)}(z, v)$ of $K(2, j)$, written as a Laurent polynomial in $v$ with coefficients in $\mathbb{Z}[z, z^{-1}]$, is of the form

$$P_{K(2,j)}(z, v) = v^{-2j} + \cdots + v^2,$$

where $\cdots$ indicate a Laurent polynomial in $v$ with terms of degrees strictly between the degrees of the terms written out. The HOMFLY-polynomial of the right-handed trefoil is $(2 + z^2)v^2 - v^4$. Since the HOMFLY-polynomial is multiplicative under
connected sum,
\[ P_{K(m,j)} = (2 + z^2)^{m-2} v^{2(m-j-2)} + \cdots + (-1)^{m-2} v^{2(2m-3)} \]
Now, for any knot $K$, \( \text{braid}(K) \geq \frac{1}{2} \text{v-span}(P_K(z,v)) + 1 \) (see Morton \[3\]). Hence, \( \text{braid}(K(m,j)) \geq m + j \).

6.2. **Proof of Lemma 4.** Fix $n$. Let $U(k)$ denote the union of intervals
\[ U(k) = \left[ 0, \frac{M}{k} \right] \setminus \bigcup_{i=1}^j \left[ \frac{a_i - 1}{k}, \frac{a_i + 1}{k} \right]. \]
As $k \to 0$ the restriction of the curve $c(j,n,k)$ to $U(k)$, approaches parts of a plane curve with all its curvature concentrated at intersections with the $x_1$-axis. In the limit, this part of the curve contributes $2\pi$ to the total curvature (see Figure 2).

On $[\frac{a_i - 1}{k}, \frac{a_i + 1}{k}]$, $c(j,n,k)$ is a translate of the holonomic curve associated to $t \mapsto -k^nt^n$ on $[-\frac{1}{n}, \frac{1}{n}]$. It is straightforward to see that the factor $-k^n$ does not affect the total curvature of this curve. Hence, in the limit as $k \to 0$, the contribution to the total curvature from each of these parts of $c(j,n,k)$ is
\[ \int_{b(n)} \kappa \, ds, \]
where $b(n)$ is the curve $(t^n, nt^{n-1}, n(n-1)t^{n-2})$, $t \in \mathbb{R}$.

We use Lemma 4 to evaluate the integral (5). For a unit vector $v = (x, y, z)$, let
\[ h_v(t) = xt^n + nyt^{n-1} + n(n-1)zt^{n-2}, \]
It is straightforward to check that $h_v$ has one local maximum for almost every $v$ such that $y^2 - 4 \left(\frac{n-2}{n-1}\right) xz > 0$ and no local maximum for $v$ in the complementary region on $S^2$.

6.3. **Proof of Lemma 3.** Assume $n \geq 5$. Let
\[ J = \bigcup_{i=1}^j \left[ \frac{a_i - 2}{k}, \frac{a_i + 2}{k} \right], \quad I = \bigcup_{i=1}^j \left[ \frac{a_i - 1}{k}, \frac{a_i + 1}{k} \right], \quad \text{and } U = \left[ 0, \frac{M}{k} \right] \setminus I \]
The function $t \mapsto p(kt)$, $t \in \left[ 0, \frac{M}{k} \right]$ has support in $J$ and the curve $c(j,n,k)$ can be taken non-degenerate in $J - I$. Therefore,
\[ \lim_{\delta \to 0} \int_{c(j,n,k,\delta)} \kappa \, ds = \int_{c(j,n,k)} \kappa \, ds, \]
where $|U|$ denotes restriction to $U$.

We are left with the image of $I$. On the intervals in $I$, $t \mapsto f_n(kt) + \delta p(kt)$ is a translate of the function $t \mapsto -k^nt^n - \delta t^3$, $t \in \left[ -\frac{1}{k}, \frac{1}{k} \right]$. Using Lemma 3 as in the proof of Lemma 3 we need to calculate the number of maxima of the function
\[ h_v(t) = k^n t^{n-2} \left( xt^2 + (n-1)yt + (n-1)(n-2)zt \right) + \delta t \left( xt^2 + 2yt + 2z \right), \]
for $v = (x, y, z) \in S^2$. Taking derivatives and assuming $x \neq 0$ we must calculate the number of zeros of the polynomial
\[ h'_v(t) = k^n t^{n-3} \left( t^2 + (n-1)\frac{y}{x} t + (n-1)(n-2)\frac{z}{x} \right) + 3\delta \left( t^2 + 2\frac{y}{x} t + 2\frac{z}{x} \right) \]
\[ = k^n t^{n-3} g(t) + 3\delta r(t), \]
where the last equality is used to define the quadratic polynomials $g$ and $r$. 
Below we make several observations which together give an estimate on the total curvature of the image of \( I \):

Observation 1: For any \( v \), \( h'_v \) cannot have more than 6 real zeros. This follows from Descartes' Lemma (see for example Benedetti and Risler [1], Proposition 1.1.10).

Observation 2: If \( y^2 - 2xz < 0 \) then neither \( g \) nor \( r \) have any real zeros. Hence, \( h'_v \) does not have real zeros in this case.

Observation 3: If \( y^2 - 4 \left( \frac{n-2}{n-1} \right) xz > 0 \) and \( xz < 0 \) then both \( g \) and \( r \) have two real zeros on opposite sides of \( t = 0 \). Moreover, the zeros of \( r \) are closer to \( t = 0 \) than those of \( g \) and \( t \mapsto t^{n-3}g(t) \) has a maximum at \( t = 0 \). It follows that \( h'_v \) has exactly two real zeros in this case.

Observation 4: If \( y^2 - 4 \left( \frac{n-2}{n-1} \right) xz < 0 \) and \( y^2 - 2xz < 0 \) then \( xz > 0 \), \( g \) has no real zeros, and \( r \) has two real zeros on the same side of \( t = 0 \). Denote the zeros of \( r \) by \( \theta_1 \leq \theta_2 \).

Observation 5: If \( y^2 - 4 \left( \frac{n-2}{n-1} \right) xz < 0 \) and \( xz > 0 \) then \( t \mapsto t^{n-3}g(t) \) has one local maximum at \( t = \phi \), \( r \) has two real zeros \( \theta_1 \leq \theta_2 \), and \( \phi, \theta_1 \) and \( \theta_2 \) all lie on the same side of \( t = 0 \).

If \( v \) satisfies \(|x| > a, |y| > a, \) and \( |z| > a \), and if \( n \) is large enough then \(|\phi| > \max(|\theta_1|, |\theta_2|)| \): The derivative of \( t \mapsto t^{n-3}g(t) \) is

\[
\frac{d}{dt} (t^{n-3}g(t)) = (n-1)t^{n-4} \left( t^2 + (n-2)\frac{y}{x}t + (n-2)(n-3)\frac{z}{x} \right),
\]

which has zeros at

\[
t = 0 \quad \text{and} \quad t = \frac{n-2}{2} \left( -\frac{y}{x} \pm \frac{1}{|x|} \sqrt{y^2 - 4 \left( \frac{n-3}{n-2} \right) xz} \right).
\]

It is straightforward to check that the non-zero zeros have distance at least \( \sqrt{\frac{2}{|x|}} (1 - \sqrt{1 - 2a^2}) \) from \( t = 0 \). The zeros of \( r \) are \( t = -\frac{n-2}{2} \pm \frac{1}{|x|} \sqrt{y^2 - 2xz} \) which have distance at most \( \frac{2}{|x|} \) from \( t = 0 \). Thus, if

\[
n > 2 \left( 1 - \sqrt{1 - 2a^2} \right)^{-1} + 2
\]

then \(|\phi| > \max(|\theta_1|, |\theta_2|)| \).

Observation 6: If \( \delta > 0 \) is small enough and \( v \) satisfies \(|x| > a, |y| > a, \) and \( |z| > a \) then, with \( v \) as in Observation 4 or Observation 5, \( h'_v \) is monotonic on \([\theta_1, \theta_2]\) (which implies that \( h'_v \) has no zeros if \( v \) is as in Observation 4 and no more than two zeros if \( v \) is as in Observation 5):

The derivative \( h''_v \) of \( h'_v \) is

\[
h''_v(t) = k^n n(n-1)t^{n-4} \left( t^2 + (n-2)\frac{y}{x}t + (n-2)(n-3)\frac{z}{x} \right) + 6\delta \left( t + \frac{y}{x} \right)
\]

where the last equality serves as a definition of \( l \). We must check that this expression does not change sign on \([\theta_1, \theta_2]\). On this interval \(|r'(t)| < \frac{2\delta}{\alpha} \sqrt{1 - 2a^2} \) and \(|l(t)| > |\frac{1}{2} \left( 1 - \sqrt{1 - 2a^2} \right) | > 0 \). It follows that \( h'_v \) is monotonic on \([\theta_1, \theta_2]\) for \( \delta \) small enough.

We collect the observations above to a proof: Assume \( v \) is such that \(|x| > a, |y| > a, \) and \(|z| > a \). Then Observations 2, 4, and 6 imply that if \( y^2 - 4 \left( \frac{n-1}{n-2} \right) xz < 0 \) then \( h_v \) does not have any maxima, and Observations 3, 5, and 6 imply that if
Then Observation 1 implies that $h_v$ has at most 6 maxima. Thus, by Lemma 3,

$$\lim_{\delta \to 0} \int_{c(\delta,n,k)} \kappa \, ds \leq \int_{c(n,k)} \kappa \, ds + 6B(a)j.$$  

(7)

The lemma follows from (6) and (7).

6.4. **Proof of Lemma 6.** We construct the function $f$ (the function $g$ can be constructed in a similar way):

Let $(x_1,x_2,x_3)$ be coordinates on $\mathbb{R}^3$. To fulfill condition (iv), we must find $f$ such that the length of the curve $c_1(t)$, $0 \leq t \leq S$ traced out by the unit tangent vector of $t \mapsto c(t) = (f(t), f'(t), f''(t))$ on $\mathbb{S}^2$ is arbitrary close to $\pi$.

Note that $c'(t) = (f'(t), f''(t), f'''(t))$. Condition (i) implies that $e_1(0) = (1,0,0)$ and condition (ii) that $e_1(S) = (0,0,-1)$.

We first make a non-smooth model: Let $w = \min(u, d)$ and let $K \geq 1$ (ultimately we shall take $K$ very large). Let $S = \sqrt{\frac{2K}{uw}} (u + d)$ and let

$$h(t) = \begin{cases} -\frac{u}{\pi} & \text{for } 0 \leq t \leq \sqrt{\frac{2K}{uw}} u \\ \frac{uw}{\sqrt{2K}} & \text{for } \sqrt{\frac{2K}{uw}} u < t \leq \sqrt{\frac{2K}{uw}} (u + d). \end{cases}$$

Take $f'''(t) = h(t)$ on $[0,S]$ and $f'''(t) = 0$ for other $t$. With $f''(0) = 0$ and $f'(0) = u$ this defines $f'$ on $[-s,S+s]$. It is easy to check that

$$\frac{|f'''|}{\sqrt{(f')^2 + (f'')^2}} \leq \frac{8w}{d\sqrt{K}}.$$  

(8)

for all $t \in [-s,S+s]$. Inequality (8) implies that the curve $e_1$ lies entirely inside a band around the equator of $\mathbb{S}^2$ in the $x_1\,x_2$-plane of height (in the $x_3$-direction) less than $\frac{8w}{d\sqrt{K}}$. It follows that the length of the curve $e_1$ approaches $\pi$ as $K \to \infty$.

It is clearly possible to smooth the above model keeping the total curvature close to $\pi$.

This construction gives the derivative of the desired function. Denote this derivative $k(t)$. If $\int_0^S k(t) \, dt = r > 0$ then we take $s' = S + \frac{r}{K}$ and let $f(t) = c + \int_0^t k(t) \, dt$ on $[0,S]$ and $f(t) = -d(t-S) + r$ on $[S,s']$ and we get a function satisfying (a)-(d) on $[-s,s'+s]$. If $\int_0^S k(t) \, dt = r < 0$ we can proceed in a similar way, changing the function close to $t = 0$.

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**Dept. of Math. Uppsala University, S-751 06 Uppsala, Sweden**

**E-mail address:** tobias@math.uu.se