A NEW WAY TO EVALUATE MOY GRAPHS

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Abstract. This paper is concerned with evaluation of $\mathfrak{sl}_N$-webs from a graph-theoretical point of view: We give an interpretation of the evaluation of $\mathfrak{sl}_N$ webs in terms of colorings. This is very close from the approach of Cautis, Kamnitzer and Morrison, but we provide a non-local and algebra-free definition of the degree associated with a coloring. In particular we do not use skew Howe duality. As a counter-part we are only concerned with closed webs. We prove that this new evaluation coincides with the classical evaluation of MOY graphs by checking some skein relations. As a consequence, we prove a formula which relates the $\mathfrak{sl}_N$ and $\mathfrak{sl}_{N-1}$-evaluations of MOY graphs.

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1. Introduction

MOY graphs and MOY graph evaluation have been introduced by Murakami Ohtsuki and Yamada [MOY98] to provide a combinatorial and computational approach to the $\mathfrak{sl}_N$-invariant of links [RT90]. The edges of these graphs are meant to represent some wedge powers of the fundamental representation of the Hopf algebra $U_q(\mathfrak{sl}_N)$ and the vertices correspond to some intertwiners. A MOY graph can therefore be interpreted as an endomorphism of $\mathbb{C}(q)$ the trivial $U_q(\mathfrak{sl}_N)$-module. The evaluation of a MOY graph is the image of 1 under this endomorphism.

Murakami, Ohtsuki and Yamada gave a combinatorial way to evaluate MOY graphs using colorings and state sums and gave some skein relations satisfied by the evaluation. Kim [Kim03] and Morrison [Mor07] conjectured that (a Karoubi-completion of) the category of MOY graphs is equivalent to that of finite dimensional $U_q(\mathfrak{sl}_N)$-modules. This has been proved by Cautis, Kamnitzer and Morrison [CKM14].

The combinatorial evaluation of MOY graphs has been used to study the categorification of the $\mathfrak{sl}_N$-invariant (see for example [KR08a], [KR08b], [Wu14], [MSV09], [LZ14], Qiu14, [CKM14]).

In this paper, we give an alternative definition of the evaluation of MOY graphs. This new definition is very close from that of [CKM14]; just like Cautis, Kamnitzer and Morrison, we count all possible colorings of a MOY graphs taking into account a certain degree. As we only work with closed MOY graphs, many simplifications

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occur, giving us a global definition of the degree (this implies in particular that we do not need their tags). In other word, this paper can be seen as a combinatorial rewriting of part of [CKM14] in the case of closed MOY graphs.

From this new evaluation, one can deduce a skein relation which relates $\mathfrak{s}l_N$ and $\mathfrak{s}l_{N-1}$-evaluations of MOY graphs (this formula can be as well derived from the diagrammatic description of the Gel’fand-Tsetlin functor in the PhD thesis of Morrison [Mor07, Chapter 4]).

We hope that our fully-combinatorial approach to the evaluation of closed MOY graphs can be applied to foam-theoretic categorifications of the $\mathfrak{s}l_N$-invariant. In particular, we think that it could helpful to get rid of the “ladder” formalism used, see for example [QR14].

Organization of the paper. In section [2] we define our evaluation $\langle \cdot \rangle_{\text{col}}$ of $\mathfrak{s}l_N$-webs and we state in Theorem 2.6 that our evaluation agrees with the one defined in [MOY98]. We explain that in order to prove the theorem, it is enough to show that $\langle \cdot \rangle_{\text{col}}$ satisfies some skein relations.

In section [3] we introduce degrees of partitions of ordered sets and show a relatively technical lemma about this notion which is the key point of the proof of Theorem 2.6.

In section [4] we prove that $\langle \cdot \rangle_{\text{col}}$ satisfies the skein relations. Finally in section [5] we state and prove Proposition 5.3 which related the $\mathfrak{s}l_N$ evaluation and the $\mathfrak{s}l_{N-1}$ evaluation of MOY graphs.

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2. Evaluation of $\mathfrak{s}l_N$-webs

Definition 2.1. Let $N$ be a positive integer. An $\mathfrak{s}l_N$-web or simply web is an oriented, trivalent, plane graph $\Gamma = (V, E)$ with possibly some vertex-less loops, whose edges are labeled with elements of $\mathbb{Z}$ such that for every vertex $v$ of $\Gamma$, we have:

$$\sum_{v \rightarrow e} \lambda(e) = \sum_{v \leftarrow e} \lambda(e) \mod N,$$

where $\lambda : E \rightarrow [1, N]$ is the labelling of the edges. Two $\mathfrak{s}l_N$-webs $\Gamma_1$ and $\Gamma_2$ are considered to be equivalent if one can obtain one from the other by reversing the orientations of some edges $(e)_{e \in E'}$ and replacing their labels by $N - \lambda(e)$.

Definition 2.2. A MOY graph is a trivalent oriented plane graph $\Gamma$ with possibly some vertex-less loops, whose edges are labeled with elements of $\mathbb{N}$, such that for every vertex the labels and the orientations look locally like:

$$\begin{align*}
\begin{array}{c}
a + b \\
\downarrow & \downarrow \\
\quad & \quad \\
\left\{ \begin{array}{c}
a \\
b \\
\end{array} \right. & \left\{ \begin{array}{c}
a \\
b \\
\end{array} \right. \\
\end{array}
\end{align*}
$$

or

If $\Gamma$ is a MOY graph, the writhe of $\Gamma$, denoted by $w(\Gamma)$, is the algebraic number of circles obtained, when one replaces every edge with label $i$ of $\Gamma$ by $i$ parallel copies (see Figure [1]).

---

1Actually, if the label of an edge is not in $[0, N] := \{0, 1, \ldots, N\}$, the web will be pretty uninteresting. However, for technical reasons, it is convenient to allow labelling by all relative integers.
Figure 1. Computation of the writhe of a MOY graph: here we have $w(\Gamma) = 2 - 2 = 0$

Definition 2.3. A coloring of a web $\Gamma = (V, E)$ is a function $c$ from $E$ to $\mathcal{P}(\llbracket 1, N \rrbracket)$ such that:

(E) for every edge $e$, $\# c(e) = \lambda(e)$,

(V) for every vertex $v$, the multiset:

$$\bigcup_{v \to e} c(e) \cup \bigcup_{v \leftarrow e} c(e)$$

is a multiple of $\llbracket 1, N \rrbracket$, where $\overline{c(e)}$ is the complement of $c(e)$ in $\llbracket 1, N \rrbracket$.

If two webs $\Gamma_1$ and $\Gamma_2$ are equivalent and $c_1$ is a coloring of $\Gamma_1$, there is a canonical coloring of $\Gamma_2$ obtained by replacing $c_1(e)$ by its complement in $\llbracket 1, N \rrbracket$ for every edge $e$ whose orientation has been reversed.

Remark 2.4. (1) If some labels of an $\mathfrak{sl}_N$-web $\Gamma$ are not in $\llbracket 0, N \rrbracket$, then the web $\Gamma$ admit no coloring.

(2) If $\Gamma$ is a MOY graph (and hence can be thought of as an $\mathfrak{sl}_N$-web for all $N$), the condition (V) of a coloring is equivalent to saying that around each vertex, the colors of the two edges with the smallest labels form a partition of the color of the edge with the greatest label. Therefore, for each element $i$ in $\llbracket 1, N \rrbracket$, the flow of $i$ is preserved around each vertex and one can see the coloring of $\Gamma$ as a collection of connected cycles colored by element of $\llbracket 1, N \rrbracket$ such that:

- any two cycles with the same colors are disjoint,
- an edge $e$ belongs to exactly $\lambda(e)$ cycles.

Definition 2.5. A bicolor $b$ is a subset of $\llbracket 1, N \rrbracket$ with exactly two elements. The greatest color (for the natural order on $\llbracket 1, N \rrbracket$) is denoted by $b^+$, the other one by $b^-$. If $(\Gamma, c)$ is a colored web and $b$ is a bicolor, the state $(\Gamma, c)_b$ is the collection of oriented circles obtained from $\Gamma$ by erasing every edge $e$ such that the cardinal of the intersection of $c(e)$ and $b$ is different from 1 and reversing the orientation of every edge $e$ such that $\lambda(e) \cap b = \{b^-, b^-\}$. The degree $d(\Gamma, c)_b$ of a state $(\Gamma, c)_b$ is equal to the algebraic number of circles in $(\Gamma, c)_b$. The degree $d(\Gamma, c)$ of a colored web $(\Gamma, c)$ is the sum of the degree of all the possible states.

Theorem 2.6. Let $\Gamma$ be a web, the evaluation $\langle \Gamma \rangle$ of $\Gamma$ given in [MOY98] is equal to:

$$\langle \Gamma \rangle_{\text{col}} := \sum_{c \text{ coloring of } \Gamma} q^{d(\Gamma, c)}.$$

Remark 2.7. (1) It is worthwhile to note that, if $\Gamma_1$ and $\Gamma_2$ are two equivalent webs, $c_1$ a coloring of $\Gamma_1$ and $c_2$ the corresponding coloring of $\Gamma_2$, then for
every bicolor $b$, the states $(\Gamma_1, c_1)_b$ and $d(\Gamma_2, c_2)_b$ are equal. It follows that $d(\Gamma_1, c_1) = d(\Gamma_2, c_2)$ and $(\Gamma_1)_{\text{col}} = (\Gamma_2)_{\text{col}}$.

(2) Let $\Gamma$ be a web. The graph $\Gamma'$ obtained by removing all edges labelled by 0 or $N$ is a web and we have $(\Gamma)_{\text{col}} = (\Gamma')_{\text{col}}$.

(3) The theorem can be seen as a generalization of [Rob13, Theorem 1.11], one can therefore wonder if the other result of [Rob13] remains true in our context, namely: are all colorings of a web $\Gamma$ Kempe equivalent?

The evaluation of MOY graphs $\langle \cdot \rangle$ is multiplicative with respect to the disjoint union and it satisfies the following skein relations:

$$\langle \bigcirc \rangle_k = \binom{N}{k}_q.$$  

$$\langle \begin{array}{c} i \\ j \\ k \\ i+j+k \end{array} \rangle = \langle \begin{array}{c} i \\ j \\ k \\ i+j+k \end{array} \rangle.$$  

$$\langle \begin{array}{c} m+n \\ m \\ m+n \\ m+n \end{array} \rangle = \binom{m+n}{m}_q \langle \begin{array}{c} m+n \\ m \end{array} \rangle.$$  

$$\langle \begin{array}{c} m+n \\ n \end{array} \rangle = \binom{N-m}{n}_q \langle \begin{array}{c} m \end{array} \rangle.$$  

$$\langle \begin{array}{c} m+1 \\ m+1 \\ m+1 \\ m \\ m+1 \\ m \\ 1 \\ 1 \end{array} \rangle = \langle \begin{array}{c} 1 \\ 1 \end{array} \rangle + [N-m-1]_q \langle \begin{array}{c} m-1 \\ m-1 \end{array} \rangle.$$  

$$\langle \begin{array}{c} 1 \\ 1 \\ m \\ m \\ m \\ m-1 \\ 1 \end{array} \rangle = \binom{m-1}{n}_q \langle \begin{array}{c} l \\ m-1 \\ l-1 \\ m+1-1 \\ m \end{array} \rangle + \binom{m-1}{n-1}_q \langle \begin{array}{c} m-1 \\ m+1-1 \\ l \\ m \end{array} \rangle.$$  

$$\langle \begin{array}{c} n+k \\ n+l \\ n+k-m \\ m+l \\ m+1-k \\ m \end{array} \rangle = \sum_{j=0}^{\infty} \binom{l}{k-j}_q \langle \begin{array}{c} m-j \\ m+j \\ n \\ n+m-j \\ n+m+j \end{array} \rangle.$$  

In the previous formulas, $q$ is a formal variable, $[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}$ and 

$$\binom{k}{l}_q := \begin{cases} 0 & \text{if } k < 0, \text{ or } l < 0 \text{ or } l > k, \\ \frac{[k]_q!}{[l]_q! \cdot [k-l]_q!} & \text{else,} \end{cases}$$  

where $[i]_q! = \prod_{j=1}^{i} [j]_q$.  

\footnote{Formally, one should as well remove the vertices such that all adjacent edges are labelled by 0 or $N$.}
Lemma 2.10. Let \((\Gamma, c)\) to (7) and for every coloring \((\Gamma, c)\) where:

The degree if we fix a bicolor, the state \((\Gamma, c)\) is defined by:

\[
d(\Gamma, c)_b := C_+ - C_- + \frac{1}{2}(TR - TL - BR + BL)
\]

where:

- \(C_+\) and \(C_-\) are the numbers of positively and negatively oriented circles.
- \(TR\) is the number of arcs with both ends on the top (i.e., in \(\mathbb{R} \times \{1\}\)) rightwards oriented.
- \(TL\) is the number of arcs with both ends on the top leftwards oriented.
- \(BR\) is the number of arcs with both ends on the bottom (i.e., in \(\mathbb{R} \times \{0\}\)) rightwards oriented.
- \(TL\) is the number of arcs with both ends on the bottom leftwards oriented.

The degree of a colored web \((\Gamma, c)\) is equal to \(d(\Gamma, c) := \sum_b \text{bicol} d(\Gamma, c)_b\).

If \(\Gamma\) is an open web and \(c_\partial\) a coloring of its boundary, we define:

\[
\langle \Gamma \rangle_{c_\partial} := \sum_{c \text{ coloring of } \Gamma} q^{d(\Gamma, c)}.
\]

Lemma 2.10. Let \((\Gamma, c)\) be a colored open web obtained by stacking \((\Gamma', c')\) and \((\Gamma'', c'')\) one onto the other. For every bicolor \(b\), we have \(d((\Gamma, c)_b) = d((\Gamma', c')_b) + d((\Gamma'', c'')_b)\). It follows that we have: \(d((\Gamma, c)) = d((\Gamma', c') + d((\Gamma'', c'')\).

Sketch of the proof. If we fix a bicolor, the degree \(d((\Gamma, c)_b)\) can be seen as a the integral of a (normalized) curvature along the arcs. With this point of view, the lemma simply follows from the Chasles relation. \(\blacksquare\)

Remark 2.11. From Lemma 2.10, we deduce that in order to check that \(\langle \cdot \rangle_{c_\partial}\) satisfies the relations (1) to (7), it is enough to check that \(\langle \cdot \rangle_{c_\partial}^\text{col}\) satisfies the relations (1) to (7) and for every coloring \(c_\partial\) of the common boundary. 

\[\text{Or compose, if one think of open webs as morphism in a suitable category.}\]

\[\text{All the webs involved in a local relation have the same boundary.}\]
3. Partitions and $q$-identities

The aim of this section is to introduce degrees of partitions of ordered set and to prove Lemma 3.5 from which Theorem 2.6 will follow.

**Definition 3.1.** Let $(X, <)$ be a finite totally ordered set and $Y$ and $Z$ two disjoint subsets of $X$. The degree $d(Y \sqcup Z)$ of $Y \sqcup Z$ is the integer defined by the formula:

$$d(Y \sqcup Z) = \# \{(y, z) \in (Y \times Z) | y < z\} - \# \{(y, z) \in (Y \times Z) | y > z\}.$$

**Lemma 3.2.** Let $n$ and $m$ be two integers such that $m + n \geq 1$. The following relation holds:

$$\binom{m + n}{m, n}_q = q^{+m} \binom{m + n - 1}{m, n - 1}_q + q^{-n} \binom{m + n - 1}{m - 1, n}_q$$

**Proof.** If $m$ or $n$ is negative, the relation reads $0 = 0$. We suppose that $m$ and $n$ are non-negative. This can be thought of in terms of degree of partition. The left-hand side counts partitions such that 1 is in $Y$, the second counts partitions such that 1 is in $Z$. We can as well prove this equality directly:

$$\binom{m + n}{m, n}_q = \frac{(q^{+m}[n] + q^{-n}[m])[m + n - 1]}{[m]![n]!} = \frac{(q^{+m}[n])[m + n - 1]}{[m]![n]!} + \frac{(q^{-n}[m])[m + n - 1]}{[m]![n]!} = q^{+m} \binom{m + n - 1}{m, n - 1}_q + q^{-n} \binom{m + n - 1}{m - 1, n}_q$$

Note that $\binom{m + n}{m, n}_q$ is entirely determined by the formula of Lemma 3.2 and the fact that for all $k \geq 0$, $\binom{k}{0}_{q} = 1$.

The following lemma is not strictly necessary, however we do think that it enlightens the relation between degree of disjoint union and quantum binomials.

**Lemma 3.3.** We consider $(X, <)$ a finite totally ordered set with $m + n$ element. Let $\mathcal{P}_{m,n}(X)$ the set of partition $Y \sqcup Z$ of $X$ such that $\#Y = m$ and $\#Z = n$. The following relation holds:

$$\sum_{Y \sqcup Z \in \mathcal{P}_{m,n}(X)} q^{d(Y \sqcup Z)} = \binom{m + n}{m, n}_q$$

**Proof.** The statement actually does not depends on $X$. Hence we may suppose that $X = [1, m + n]$ with the natural order. Let us write:

$$p_{m,n} = \sum_{Y \sqcup Z \in \mathcal{P}_{m,n}([1,m+n])} q^{d(Y \sqcup Z)} = \binom{m + n}{m, n}_q$$

For every positive integer $k$, we have $p_{k,0} = p_{0,k} = 1$. We have:

$$p_{m+1,n+1} = \sum_{Y \sqcup Z \in \mathcal{P}_{m+1,n+1}([1,m+n+2])} q^{d(Y \sqcup Z)}$$
Lemma 3.4. Let $m$ be an integer. It satisfies the same recursion formula so the quantum binomial (and have the same initial values). This proves that for all $m$ and $n$ we have $p_{m,n} = \binom{m+n}{m}_q$. \hfill \Box

The following observation will be very useful for proving Lemma 3.5.

Lemma 3.5. Let $X$ and $Y$ be two disjoint subsets of $[1, M]$ and $k$ be an integer of $[1, M - 1]$. Let us write $X_1 = X \cap [1, k]$, $Y_1 = Y \cap [1, k]$, $X = X \cap [k + 1, N]$ and $Y = Y \cap [k + 1, M]$. The following relation holds:

$$d(X \cup Y) = d(X_1 \cup Y_1) + d(X_2 \cup Y_2) + \#X_1 \#Y_2 - \#Y_1 \#X_2.$$ 

Proof. It follows from the definition. \hfill \Box

The following lemma is the key ingredient to prove theorem 2.6. It should be compared to [CKM13] Proof of relation 2.10.

Lemma 3.5. Let us fix $X$ and $Y$ two disjoint subsets of in $[1, M]$, such that $\#X = \#Y + l$ with $l \geq 0$. For every integer $k_1$, the following relation holds:

$$\sum_{X = X_1 \cup X_2 \atop \#X_1 = k_1} q^{d(X_1 \cup X_2) + d(Y \cup X_1)} = \sum_{j_2 = 0}^{\infty} \left( k_1 - j_2 \atop l - k_1 + j_2 \right)_q \sum_{Y = Y_1 \cup Y_2 \atop \#Y_2 = j_2} q^{d(Y_1 \cup Y_2) + d(Y_2 \cup X)}
$$

Proof. The proof is done by induction on the cardinal of $\#X + \#Y$. If $\#X + \#Y = 0$, the relation reads 1 = 1. For the induction, we need to be careful and stay in the case where $\#X \geq \#Y$. Suppose that $\#X - \#Y > 1$, then removing the smallest element of $X \cup Y$ gives us two sets $X'$ and $Y'$ such that $\#X' - \#Y' > 0$. Let us now consider the extreme case, where $\#X = \#Y \geq 1$. We can distinguish two situations:

- The lowest (or the highest) element of $X \cup Y$ is in $Y$, then removing this element gives us two sets $X'$ and $Y'$ such that $\#X - \#Y = 1$.
- The lowest element and the highest element of $X \cup Y$ are in $Y$, in this case we can find an element $k$ in $[1, M]$ such that, if we define $X' := X \cap [1, k]$, $Y' := Y \cap [1, k]$, $X'' := X \cap [k + 1, M]$, $Y'' := Y \cap [k + 1, M]$, we have $\#X' = \#Y'$, $\#X'' = \#Y''$ and none of these sets is empty.

We first suppose that $\#X - \#Y = l \geq 1$ and the lowest element $t$ of $X \cup Y$ is in $X$. Let us write $X' := X \setminus \{t\}$. We have:

$$\sum_{X = X_1 \cup X_2 \atop \#X_1 = k_1} q^{d(X_1 \cup X_2) + d(Y \cup X_1)} = \sum_{X = X_1 \cup X_2 \atop \#X_1 = k_1} q^{d(X_1 \cup X_2) + d(Y \cup X_1)} + \sum_{X = X_1 \cup X_2 \atop \#X_1 = k_1} q^{d(X_1 \cup X_2) + d(Y \cup X_1)}
$$

$$= \sum_{X' = X'_1 \cup X'_2 \atop \#X'_2 = k_1 - 1} q^{d(X'_1 \cup X'_2) + d(Y \cup X'_1)} + \sum_{X = X_1 \cup X_2 \atop \#X_1 = k_1} q^{d(X_1 \cup X_2) + d(Y \cup X_1)} - \sum_{X = X_1 \cup X_2 \atop \#X_1 = k_1} q^{d(X_1 \cup X_2) + d(Y \cup X_1)}$$

$$= q^{\#X'} + q^{\#X}.$$
\[
\sum_{j_2=0}^\infty q^{l-k_1} \left( k_1 - 1 - j_2 \ v_{l-1} \ v_{k_1-1} + j_2 \right) \sum_{Y = Y_1 \cup Y_2 \over \#Y_2 = j_2} q^d(Y_1 \cup Y_2) + d(Y_2 \cup X') \\
+ \sum_{j_2=0}^\infty q^{-k_1} \left( k_1 - j_2 \ v_{l-1} \ v_{l-1-k_1} + j_2 \right) \sum_{Y = Y_1 \cup Y_2 \over \#Y_2 = j_2} q^d(Y_1 \cup Y_2) + d(Y_2 \cup X') \\
= \sum_{j_2=0}^\infty \left( q^{l-k_1} \left( k_1 - 1 - j_2 \ v_{l-1} \ v_{l-1-k_1} + j_2 \right) + q^{-k_1} \left( k_1 - j_2 \ v_{l-1} \ v_{l-1-k_1} + j_2 \right) \right) \\
\cdot \sum_{Y = Y_1 \cup Y_2 \over \#Y_2 = j_2} q^d(Y_1 \cup Y_2) + d(Y_2 \cup X') \\
= \sum_{j_2=0}^\infty q^{-j_2} \left( k_1 - j_2 \ v_{l} \ v_{l-k_1} + j_2 \right) \sum_{Y = Y_1 \cup Y_2 \over \#Y_2 = j_2} q^d(Y_1 \cup Y_2) + d(Y_2 \cup X') \\
= \sum_{j_2=0}^\infty \left( k_1 - j_2 \ v_{l} \ v_{l-k_1} + j_2 \right) \sum_{Y = Y_1 \cup Y_2 \over \#Y_2 = j_2} q^d(Y_1 \cup Y_2) + d(Y_2 \cup X) 
\]

The case where the greatest element of \( X \cup Y \) is in \( X \) is analogue. We suppose now that \( \#X - \#Y = l \geq 0 \) and the lowest element \( t \) of \( X \cup Y \) is in \( Y \). Let us write \( Y' := Y \setminus \{t\} \). We have:

\[
\sum_{j_2=0}^\infty \left( k_1 - j_2 \ v_{l} \ v_{l-k_1} + j_2 \right) q^d(Y_1 \cup Y_2) + d(Y_2 \cup X) \\
= \sum_{j_2=0}^\infty \left( k_1 - j_2 \ v_{l} \ v_{l-k_1} + j_2 \right) \sum_{Y = Y_1 \cup Y_2 \over \#Y_2 = j_2} q^d(Y_1 \cup Y_2) + d(Y_2 \cup X) \\
+ \sum_{j_2=0}^\infty \left( k_1 - j_2 \ v_{l} \ v_{l-k_1} + j_2 \right) \sum_{Y = Y_1 \cup Y_2 \over \#Y_2 = j_2} q^d(Y_1 \cup Y_2) + d(Y_2 \cup X) \\
= \sum_{j_2=0}^\infty \left( k_1 - j_2 \ v_{l} \ v_{l-k_1} + j_2 \right) \sum_{Y' = Y_1 \cup Y'_2 \over \#Y'_2 = j_2} q^d(Y_1 \cup Y'_2) + d(Y'_2 \cup X) + j_2 \\
+ \sum_{j_2=0}^\infty \left( k_1 - j_2 \ v_{l} \ v_{l-k_1} + j_2 \right) \sum_{Y' = Y'_1 \cup Y'_2 \over \#Y'_2 = j_2} q^d(Y'_1 \cup Y'_2) + d(Y'_2 \cup X) - (\#X - l - j_2 + #X) \\
= \sum_{j_2=0}^\infty \left( k_1 - j_2 \ v_{l} \ v_{l-k_1} + j_2 \right) \sum_{Y' = Y_1 \cup Y'_2 \over \#Y'_2 = j_2} q^d(Y_1 \cup Y'_2) + d(Y'_2 \cup X) + j_2 \\
+ \sum_{j_2=0}^\infty \left( k_1 - (j_2 + 1) \ v_{l} \ v_{l-k_1} + (j_2 + 1) \right) \sum_{Y' = Y_1 \cup Y'_2 \over \#Y'_2 = j_2} q^d(Y'_1 \cup Y'_2) + d(Y'_2 \cup X) + l + j_2 + 1 \\
= \sum_{j_2=0}^\infty \left(q^j \right) \left( k_1 - j_2 \ v_{l} \ v_{l-k_1} + j_2 \right) + q^{l+j_2+1} \left( k_1 - (j_2 + 1) \ v_{l} \ v_{l-k_1} + (j_2 + 1) \right) 
\]
We suppose now that the greatest and the lowest element of \(k\) are in \(Y\) and that \(l = 0\). We use the notations explained before and we write \(k' = \#X' = \#Y'\) and \(k'' = \#X'' = \#Y''\). For readability it is convenient to set \(l' = l'' = 0\). We have:

\[
\sum_{j_2=0}^{\infty} \left( k_1 - j_2 \right) \sum_{q = Y \subseteq Y_2}^{l + 1} q d(Y_1 \cup Y_2) + d(Y_2 \cup X) = \sum_{q = Y \subseteq Y_2}^{l + 1} q d(Y_1 \cup Y_2) + d(Y_2 \cup X)
\]

\[
= \sum_{k_1 + k'' = k_1} \sum_{Y' = Y_1 \cup Y_2}^{\#Y_2 = k''} q d(Y_1 \cup Y_2') + d(Y_2' \cup X') + d(Y_2' \cup X''') + (k'' - k_1)k'' - k'' = k'' k'' - k'' k''
\]

\[
= \sum_{k_1 + k'' = k_1} \sum_{Y' = Y_1 \cup Y_2}^{\#Y_2 = k''} q d(Y_1 \cup Y_2') + d(Y_2' \cup X') + d(Y_2' \cup X'')
\]

The case were the greatest element of \(X \cup Y\) is in \(Y\) is analogue.
Lemma 4.1. Let $\Gamma$ be a web, and $\Gamma'$ the web obtained by deleting all the edges labelled by 0 or $N$. We have:

$$
\langle \Gamma \rangle_{\text{col}} = \langle \Gamma' \rangle_{\text{col}}.
$$

Proof. We clearly have a one-one correspondence between the colorings of $\Gamma$ and of $\Gamma'$ since an edge labeled 0 can only be colored by $\varnothing$ and an edge labeled $N$ can only be colored by $[1,N]$. Furthermore, the edges of $\Gamma$ labeled by 0 and $N$ never appear in any state for any coloring, this means that the state $(\Gamma, e)_b$ and $(\Gamma', e')_b$ are equal (where $c$ and $c'$ are two colorings corresponding one to another). This proves that $\langle \Gamma \rangle_{\text{col}} = \langle \Gamma' \rangle_{\text{col}}$. \hfill $\Box$

Lemma 4.2. It is enough to check relations (3) and (7).

Proof. We prove that (3) and (4) follow from (3). We suppose that relation (3) holds. We have

$$
\langle \begin{array}{c} k \\ \text{col} \end{array} \rangle = \langle \begin{array}{c} N-k \\ \text{col} \end{array} \rangle = \langle \begin{array}{c} N \\ \text{col} \end{array} \rangle q^{k}. \quad (3)
$$

This proves relation (3) holds.

$$
\langle \begin{array}{c} m+n \\ \text{col} \end{array} \rangle = \langle \begin{array}{c} N-m \\ \text{col} \end{array} \rangle = \langle \begin{array}{c} N \\ \text{col} \end{array} \rangle q^{m+n} \quad (4)
$$

This proves relation (4) holds.

We prove that relations (5), (6) and (7) follow from (7). We suppose that (7) holds.

Relation (5) is a special case of relation (7): by setting $n = 1$, $m = l'$, $k = l' + n' - 1$ and $m' = 1$ in (7), we obtain (5) with all the labels replaced by labels with $'$. 

Relation (6) is a special case of relation (5): by setting $l = 1$, $m = N - m'$, and $n = N - m' - 1$ in (5), we obtain (6) with all the labels replaced by labels with $'$. 

4. Checking the skein relations

$$
q^{d(X_1 \cup X_2) + d(Y' \cup X_1') + d(X''_1 \cup X''_2) + d(Y'' \cup X''_1)}
$$

$$
= \sum_{k_1 + k_1'' = k_1} \sum_{X=X_1 \cup X_2} \sum_{X''=X''_1 \cup X''_2} q^{d(X_1 \cup X_2) + d(Y' \cup X_1') + d(X''_1 \cup X''_2) + d(Y'' \cup X''_1) + k_1(k'' - k''') - (k' - k')k'' + k'k'' - k'k'''}
$$

$$
= \sum_{k_1 + k_1'' = k_1} \sum_{X=X_1 \cup X_2} \sum_{X''=X''_1 \cup X''_2} q^{d(X_1 \cup X_2) + d(Y' \cup X_1') + d(X''_1 \cup X''_2) + d(Y'' \cup X''_1)}
$$

$$
= \sum_{X=X_1 \cup X_2} q^{d(X_1 \cup X_2) + d(Y \cup X_1)}
$$
Relation (3) is a special case of relation (7); by setting \( m = n = 0, l = m' + n' \) and \( k = m' \) in (7), we obtain (3) with all the labels replaced by labels with ‘.

**Lemma 4.3.** The following relation holds:

\[
\begin{pmatrix}
  i & j & k \\
  i + j + k
\end{pmatrix}_{\text{col}} + \begin{pmatrix}
  i & j & k \\
  i + j + k
\end{pmatrix}_{\text{col}} = \begin{pmatrix}
  i & j & k \\
  i + j + k
\end{pmatrix}_{\text{col}}
\]

where \( c_0 \) is any coloring of the boundary (see footnote on page 3).

**Proof.** Let \( \Gamma \) be the open web on the left and \( \Gamma' \) the open web on the right. The boundary consists of four points, \( \tau_1, \tau_2, \tau_3 \) and \( \tau_3 \) on the top and \( \beta \) on the bottom. Let us consider a coloring \( c_\beta \) of the boundary (that is any application \( \{ \tau_1, \tau_2, \tau_3, \beta \} \to [1, N] \) such that \( \#c_\beta(\tau_1) = i, \#c_\beta(\tau_2) = j, \#c_\beta(\tau_3) = k \) and \( \#c_\beta(\beta) = i + j + k \).

Suppose \( X_1 := c_\beta(\tau_1), X_2 := c_\beta(\tau_2) \) and \( X_3 := c_\beta(\tau_3) \) form a partition of \( c_\beta(\beta) \). Then there exists a unique coloring \( c_\Gamma \) of \( \Gamma \) and a unique coloring \( c'_\Gamma \) of \( \Gamma' \) compatible with \( c_\beta \). We need to compare the degree \( d(\Gamma, c) \) and \( d(\Gamma', c') \). The values of \( d(\Gamma, c)_b \) and \( d(\Gamma', c')_b \) for every bicolor \( b \) are given in Table 1. We have \( d(\Gamma, c)_b = d(\Gamma', c')_b \) for all \( b \). Hence \( d(\Gamma, c) = d(\Gamma', c') \) and \( (\Gamma, c)_\text{col} = (\Gamma', c')_\text{col} \).

Suppose \( c_\beta(\tau_1), c_\beta(\tau_2) \) and \( c_\beta(\tau_3) \) do not form a partition of \( c_\beta(\beta) \). Then there is no coloring of \( \Gamma \) inducing \( c_\beta \) and no coloring of \( \Gamma' \) inducing \( c_\beta \). Hence we have \( (\Gamma, c)_\text{col} = (\Gamma', c')_\text{col} = 0 \).

Finally we have

\[
\begin{pmatrix}
  i & j & k \\
  i + j + k
\end{pmatrix}_{\text{col}} + \begin{pmatrix}
  i & j & k \\
  i + j + k
\end{pmatrix}_{\text{col}} = \begin{pmatrix}
  i & j & k \\
  i + j + k
\end{pmatrix}_{\text{col}}
\]

for all colorings \( c_\beta \) of the boundary.

**Lemma 4.4.** The following relation holds:

\[
\begin{pmatrix}
  m & m + n \\
  n + k - m & m + l - k
\end{pmatrix}_{\text{col}} = \sum_{j=0}^{\infty} (k - j + l - k + j)_{\text{col}}
\]

where \( c_0 \) is any coloring of the boundary.

**Proof.** Let us denote by \( \Gamma \) the open web on the left-hand side of the relation and by \( (\Gamma_j)_{j \in \mathbb{N}} \) the open webs on the right-hand side of the relation.

Let \( c \) be a coloring of \( \Gamma \) and \( i \) an element of \([1, N]\). We consider the set \( E_i \) of edges \( e \) of \( \Gamma \) such that \( i \) is in \( c(e) \). Due to the flow condition on MOY graphs (see Remark 2.3), the following configurations \( E_i \) are the only possible ones (the solid edges are in \( E_i \) the others not):

\[
\begin{array}{ccccccc}
\text{C} & \text{X}_1 & \text{X}_2 & \text{Y} & \text{Z} & \text{T} & \text{R} \\
\end{array}
\]

This gives us a partition of \([1, N]\) into 7 sets: \( C, X_1, X_2, Y, Z, T, \) and \( R \). The coloring of the boundary induced by \( c \) is:
the intersection of the coloring of the two points on the right minus the set 

From this we easily deduce that we have:

\[
5 \text{ can recover } \Gamma.
\]

If we are given a coloring of the boundary (which extends to a coloring of } \Gamma), we can recover } C, R, Y, Z, T \text{ and } X := X_1 \sqcup X_2.

On the other hand, if we are given a partition of } [1, N] \text{ into 6 sets } C, X, Y, Z, T \text{ and } R \text{ such that there cardinals } c, x, y, z, t \text{ and } r \text{ satisfy:

- } c + y + z = m,
- } c + y + t = n,
- } c + x + x_2 + t = n + l,
- } c + x_1 + x_2 + z = m + l,
- } x_1 + t = k,
- } c + r + x_1 + x_2 + y + z + t + r = N.

From this we easily deduce that we have:

\[
x_1 + x_2 - y = l, \quad x_1 + x_2 = n + l - c - t \quad \text{and} \quad x_1 = k - t.
\]

If we are given a coloring of the boundary (which extends to a coloring of } \Gamma), we can recover } C, R, Y, Z, T \text{ and } X := X_1 \sqcup X_2.

On the other hand, if we are given a partition of } [1, N] \text{ into 6 sets } C, X, Y, Z, T \text{ and } R \text{ such that there cardinals } c, x, y, z, t \text{ and } r \text{ satisfy:

- } c + y + z = m,
- } c + y + t = n,
- } c + x + t = n + l,
- } c + x + z = m + l
- } \text{ and } t \leq k,

\[5\text{For example the set } C \text{ is the intersection of the colorings of the four points and the set } X \text{ is} \]

the intersection of the coloring of the two points on the right minus the set } C.
We have given in Figure 2, we can compute $\Gamma$ is stronger than the condition to be extendable to $\Gamma$ and $R$ extend to $\Gamma$, the equality simply says $0 = 0$.

Every partition of $\mathbf{C}$ into six sets $C, X, Y_1, Y_2, Z, T$ and $R$ (we name them $c, x, y_1, y_2, z, t$ and $r$):

- $c + y_1 + y_2 + z = m$,
- $c + y_1 + y_2 + t = n$,
- $c + x + t = n + l$,
- $c + x + z = m + l$,
- $y_2 + t = j$,
- $c + r + x_1 + x_2 + y + z + t + r = N$.

From this we easily deduce that we have:

$$x - (y_1 + y_2) = l, \quad x = n + l - c - t \quad \text{and} \quad y_1 = j - t.$$

If we are given a coloring of the boundary (which extend to a coloring of $\Gamma_j$), we can recover $C, X, Z, T, R$ and $Y := Y_1 \sqcup Y_2$ with the same strategy as for $\Gamma$.

If we are given a partition of $[1, N]$ into six sets $C, X, Y, Z, T$ and $R$ such that their cardinals $c, x, y, z, t$ and $r$ satisfy

- $c + y_1 + y_2 + z = m$,
- $c + y_1 + y_2 + t = n$,
- $c + x + t = n + l$,
- $c + x + z = m + l$,
- $t \leq j$,

every partition of $Y$ into two sets $Y_1$ and $X_2$, such that $\#Y_2 = j - t$ provides a coloring of $\Gamma_j$. This means, that there exist exactly $\binom{n + l - c - t}{k - t}$ such colorings.

Note that if $j > k$, the coefficient multiplying $\Gamma_j$ is equal to zero. Hence we may suppose that $j \leq k$. In this case the condition for a coloring to be extendable to $\Gamma$ is stronger than the condition to be extendable to $\Gamma_j$. If a coloring $c_0$ does not extend to $\Gamma$, the equality simply says $0 = 0$.

Let us suppose that $c_0$ extends to $\Gamma$. We denote $C, R, X, Z, T, Y$ the partition of $[1, N]$ such that:

- The top right point is colored by $C \sqcup Y \sqcup Z$,
- The top left point is colored by $C \sqcup X \sqcup T$,
- The bottom right point is colored by $C \sqcup Y \sqcup T$,
- The bottom left point is colored by $C \sqcup X \sqcup Z$.

We have $\#X = \#Y + l$.

The colorings of $\Gamma$ which induce $c_0$ on the boundary are given by partitions $X_1 \sqcup X_2$ of $X$ such that $X_1$ has $k - t$ elements. If we fix $c$ such a coloring (notations are given in Figure 2), we can compute $d((\Gamma, c)_b)$ for every bicolour $b$. The computations
are done in Table 2. From the table we deduce that $d((\Gamma, c)_{b}) = \delta(c_{\partial}) + d(X_1 \sqcup X_2) + d(Y \sqcup X_1)$ where $\delta(c_{\partial})$ is a constant depending only on $c_{\partial}$. We obtain:

$$\langle \Gamma \rangle_{\text{col}} = q^{\delta(c_{\partial})} \sum_{X=X_1\sqcup X_2\#X_1=k-t} q^{d(X_1\sqcup X_2)+d(Y\sqcup X_1)}.$$
Table 3. Computations of $d(\Gamma_j, c_j)_b$. The red cells emphasize the contributions which do not only depend on $c_0$.

5. A new of skein relation

Now that we know that $\langle \cdot \rangle_{\text{col}}$ and $\langle \cdot \rangle$ coincide on MOY graph, we might denote both by $\langle \cdot \rangle_N$. We would like now to relate $\mathfrak{sl}_N$-evaluations of MOY graphs for different $N$’s.

**Definition 5.1.** Let $\Gamma$ be a closed MOY graph and $A = \{\alpha_1, \ldots, \alpha_k\}$ a collection of disjoint oriented cycles in $\Gamma$. We denote by $\Gamma^A_N$ the MOY graph obtained by reversing the orientations of every edge included in $A$ and replacing the label $i$ of such an edge by $N - i$.

**Remark 5.2.** The MOY graphs $\Gamma$ and $\Gamma^A_N$ are equivalent as $\mathfrak{sl}_N$-webs but not as $\mathfrak{sl}_{N-1}$-webs (see Remark 2.7).

**Proposition 5.3.** Let $\Gamma$ be a MOY graph. The following equality holds:

$$\langle \Gamma \rangle_N = \sum_{A \text{ collection of disjoint cycles}} q^{-w(\Gamma^A_N)} \langle \Gamma^A_N \rangle_{N-1}$$

$$= \sum_{A \text{ collection of cycles}} q^{w(\Gamma^A)} \langle \Gamma^A_N \rangle_{N-1}.$$

**Proof.** We only prove

$$\langle \Gamma \rangle_N = \sum_{A \text{ collection of cycles}} q^{-w(\Gamma^A_N)} \langle \Gamma^A \rangle_{N-1}$$

since the other equality follows from the symmetry of $\langle \cdot \rangle$ in $q$ and $q^{-1}$. This equality is a consequence from the following observation: If $c$ is a coloring of $\Gamma$, we can...
consider the set $A(c)$ of disjoint cycles which consists of the edges whose colorings contain $N$. The web $\Gamma^{A(c)}$ inherits a coloring $c'$ such that none of the colors of the edges contain $N$. The degree $d_N$ of the coloring $c'$ as a coloring of an $\mathfrak{sl}_N$-web is equal to $d_{N-1} - w(\Gamma A^N)$ where $d_{N-1}$ is the degree of the coloring $c'$ as a coloring of an $\mathfrak{sl}_{N-1}$-web:

$$d_N - d_{N-1} = \sum_{j=1}^{N-1} d(\Gamma A^N, c')(j,N) = -w(\Gamma A^N).$$

Conversely, if $A$ is a collection of disjoint cycles of $\Gamma$, and $c'$ a coloring of $\Gamma A^N$ as a $\mathfrak{sl}_N$-web, the $\mathfrak{sl}_N$-web $\Gamma$ inherits a coloring $c$ and the previous equality holds. □

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