Absolute convergence of the free energy of the BEG model in the disordered region for all temperatures

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Abstract. We analyze the $d$-dimensional Blume–Emery–Griffiths model in the disordered region of parameters and we show that its free energy can be explicitly written in terms of a series which is absolutely convergent at any temperature in an unbounded portion of this region. As a byproduct we also obtain an upper bound for the number of $d$-dimensional fixed polycubes of size $n$.

Keywords: classical phase transitions, general equilibrium models, series expansions

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1. The model. Notations and results

The Blume–Emery–Griffiths (BEG) model is a spin-one system, introduced in the 1970s in order to explain some of the physical properties of $^3$He–$^4$He mixtures [1] and since then it has attracted a lot of attention and has been used in several applications such as ternary fluids [2, 3], phase transitions in UO$_2$ [4] and DyVO$_4$ [5] and phase changes in microemulsion [6].

The BEG model is defined in the $d$-dimensional cubic lattice $\mathbb{Z}^d$ by supposing that in each site $x \in \mathbb{Z}^d$ there is a random variable $\sigma_x$ (the spin at $x$) taking values in the set $\{0, \pm 1\}$. For $U \subset \mathbb{Z}^d$, a spin configuration $\sigma_U$ in $U$ is a function $\sigma_U : U \to \{0, \pm 1\} : x \mapsto \sigma_x$ and $\Sigma_U$ will denote the set of all spin configurations in $U$. Given a finite set $\Lambda \subset \mathbb{Z}^d$ (typically a cubic box centered at the origin of $\mathbb{Z}^d$), the Hamiltonian of the system in $\Lambda$ (with zero boundary conditions and zero magnetic field) has the following expression:

$$H_\Lambda(\sigma_\Lambda) = -\sum_{\{x,y\} \subset \Lambda} (\sigma_x \sigma_y + Y \sigma_x^2 \sigma_y^2) \delta_{|x-y|=1} - 2dX \sum_{x \in \Lambda} \sigma_x^2, \quad (1)$$

where $|\cdot|$ is the usual $L^1$ norm in $\mathbb{Z}^d$, $\delta_{|x-y|=1}$ is the Kronecker symbol (i.e. $\delta_{|x-y|=1} = 1$ if $|x-y|=1$ and zero otherwise) and $X, Y \in \mathbb{R}$.

The parameter space $XY$ of the model is generally partitioned in three distinct regions (ferromagnetic, disordered, anti-quadrupolar) according to the ground state configurations of the Hamiltonian (see for instance [7]). In this note, we will focus our attention in the disordered region, namely

$$D = \{(X, Y) \in \mathbb{R}^2 : X < 0, 1 + 2X + Y < 0\},$$

where the unique ground state is the constant configuration $\sigma_x = 0$, for all $x \in \Lambda$.

The probability $P_\Lambda(\sigma_\Lambda)$ (i.e. the finite volume Gibbs measure) of a configuration $\sigma_\Lambda \in \Sigma_\Lambda$ is defined as

$$P_\Lambda(\sigma) = \frac{e^{-\beta H_\Lambda(\sigma_\Lambda)}}{Z_\Lambda(X, Y, \beta)}$$

where $\beta$ the inverse of the temperature in units of the Boltzmann constant and the normalization constant $Z_\Lambda(X, Y, \beta)$ is the partition function of the model given by

$$Z_\Lambda(X, Y, \beta) = \sum_{\sigma_\Lambda \in \Sigma_\Lambda} e^{-\beta H_\Lambda(\sigma_\Lambda)} \quad (2)$$

Finally, the free energy (in fact, the pressure) in the thermodynamic limit is defined as
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\[ f(X, Y, \beta) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_{\Lambda}(X, Y, \beta), \]  

where $|\Lambda|$ is the cardinality of $\Lambda$. The limit in the rhs of (3), when taken for instance in the van Hove sense, is well defined and it is independent of the sequence $\Lambda \uparrow \mathbb{Z}^d$ and the boundary conditions, see e.g. [8].

In [9] it is shown that for a certain class of spin systems interacting via a pair potential (which includes the BEG model) the free energy can be written in terms of a series which is absolutely convergent at any temperature if suitable conditions on the pair potential are satisfied. Concerning specifically the BEG model, in [10] it is proved that there exists a region of parameters $D_{\text{Dob}} \subset D$ where the Dobrushin uniqueness criterium is satisfied for all temperature. In the present note, using the scheme described in [9] and a recent tree graph inequality proved in [11], we establish an optimal region $D_{\text{analytic}} \subset D_{\text{Dob}}$ where the free energy of the BEG model can be explicitly written as an absolutely convergent series of analytic functions for all temperature, ruling out the presence of phase transitions of any order in this region. Our main result can be summarized by the following theorem.

**Theorem 1.** Let $d \geq 2$ and let $D_{\text{analytic}} \subset D$ be the region whose boundary is the polygonal curve

\[
X = \begin{cases} 
-k(Y + 1), & \text{if } Y \geq 0 \\
(k-1)Y - k, & \text{if } -1 < Y < 0 \\
\bar{k}(Y - 1), & \text{if } Y \leq -1
\end{cases}
\]

where $k = \frac{50.56d-1}{2d}$ and $\bar{k} = \frac{30.52d-1}{2d}$ (see figure 1). If $(X, Y) \in D_{\text{analytic}}$, then the free energy of the BEG model defined in (3) can be explicitly written as an absolutely convergent series of analytic functions at any temperature.

The rest of this note is organized as follows. In section 2 we prove theorem 1 and in section 3 we use the technique developed in section 2 (in particular lemma 2) to obtain an upper bound on the number $A_n$ of $d$-dimensional polycubes of size $n$ (for the definition of a $d$-dimensional polycube we refer the reader to section 3).

**2. Proof of theorem 1**

We start by rewriting the partition function of the model defined in (2) performing a so-called high temperature polymer expansion. From (1) and (2), we have

\[ Z_{\Lambda}(X, Y, \beta) = \sum_{\sigma_{\Lambda} \in \Sigma_{\Lambda}} e^{2d\beta X \sum_{x \in \Lambda} \sigma_x^2} \beta \sum_{\{x, y\} \subset \Lambda} (\sigma_x \sigma_y + Y\sigma_x^2 \sigma_y^2) \delta_{|x-y|=1}.
\]

By Mayer expansion the second exponential in the rhs of (5) can be written as
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\[ \beta \sum_{\{x,y\} \in \Lambda} (\sigma_x \sigma_y + Y \sigma_x^2 \sigma_y^2) \delta_{|x-y|} = \prod_{\{x,y\} \in \Lambda} \left[ e^{\beta (\sigma_x \sigma_y + Y \sigma_x^2 \sigma_y^2) \delta_{|x-y|}} - 1 + 1 \right] \]

\[ = \sum_{g \in G_\Lambda} \prod_{\{x,y\} \in E_g} (e^{\beta (\sigma_x \sigma_y + Y \sigma_x^2 \sigma_y^2) \delta_{|x-y|}} - 1) \]

where \( G_\Lambda \) is the set of all possible graphs (connect or not) with vertex set \( \Lambda \) and given \( g \in G_\Lambda \), \( E_g \) denotes its edge set. Let \( \Pi_n(\Lambda) \) be the set of all partitions of \( \Lambda \) having \( n \) elements \((n = 1, 2, \ldots, |\Lambda|)\), then, denoting shortly

\[ F_{xy} = e^{\beta (\sigma_x \sigma_y + Y \sigma_x^2 \sigma_y^2) \delta_{|x-y|}} - 1, \]

it is a standard observation in the framework of the cluster expansion techniques (see for instance [8, 12]), that

\[ \sum_{g \in G_\Lambda} \prod_{\{x,y\} \in E_g} F_{xy} = \sum_{n=1}^{|\Lambda|} \sum_{\{R_1, \ldots, R_n\} \in \Pi_n(\Lambda)} \prod_{l=1}^n \left( \sum_{g \in G_{R_l}} \prod_{\{x,y\} \in E_g} F_{xy} \right). \]

where in the rhs \( G_{R_l} \) denotes the set of all connected graphs with vertex set \( R_l \). So, we have

\[ e^{\beta \sum_{\{x,y\} \in \Lambda} (\sigma_x \sigma_y + Y \sigma_x^2 \sigma_y^2) \delta_{|x-y|}} = \sum_{n=1}^{|\Lambda|} \sum_{\{R_1, \ldots, R_n\} \in \Pi_n(\Lambda)} \prod_{l=1}^n \rho(R_l, \sigma_{R_l}) \]

where, for \( R \subset \Lambda \)

\[ \rho(R, \sigma_R) = \begin{cases} 1 & \text{if } |R| = 1 \\ \sum_{g \in G_R} \prod_{\{x,y\} \in E_g} (e^{\beta (\sigma_x \sigma_y + Y \sigma_x^2 \sigma_y^2) \delta_{|x-y|}} - 1) & \text{if } |R| \geq 2 \end{cases} \]

We have therefore the following representation for the partition function (5)

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\[
Z_\Lambda(X, Y, \beta) = \sum_{\sigma_\Lambda \in \Sigma_\Lambda} e^{2d \beta X \sum_{x \in \Lambda} \sigma_x^2} \sum_{n=1}^{\Lambda} \sum_{\{R_1, \ldots, R_n\} \in \Pi_\Lambda} \prod_{l=1}^{n} \rho(R_l, \sigma_{R_l})
\]

\[
= \sum_{\sigma_\Lambda \in \Sigma_\Lambda} \sum_{n=1}^{\Lambda} \sum_{\{R_1, \ldots, R_n\} \in \Pi_\Lambda} \prod_{l=1}^{n} e^{2d \beta X \sum_{x \in R_l} \sigma_x^2} \rho(R_l, \sigma_{R_l})
\]

\[
= \sum_{n=1}^{\Lambda} \sum_{\{R_1, \ldots, R_n\} \in \Pi_\Lambda} \prod_{l=1}^{n} \left( \sum_{\sigma_{R_l} \in \Sigma_{R_l}} \rho(R_l, \sigma_{R_l}) e^{2d \beta X \sum_{x \in R_l} \sigma_x^2} \right),
\]

(6)

Defining

\[
\bar{\rho}(R) = \sum_{\sigma_R \in \Sigma_R} \rho(R, \sigma_R) e^{2d \beta X \sum_{x \in R} \sigma_x^2},
\]

the partition function (6) can be rewritten as

\[
Z_\Lambda(X, Y, \beta) = \sum_{n=1}^{\Lambda} \sum_{\{R_1, \ldots, R_n\} \in \Pi_\Lambda} \bar{\rho}(R_1) \cdots \bar{\rho}(R_n).
\]

Notice that, if \(|R| = 1\) and hence \(R = \{x\}\) with \(x \in \Lambda\), we have

\[
\bar{\rho}(R) = \bar{\rho}(\{x\}) = \sum_{\sigma_x \in \{0, \pm1\}} e^{2d \beta X \sigma_x^2} = 1 + 2e^{2d \beta X}.
\]

Moreover observe that for any \(R \subset \mathbb{Z}^d\) such that \(|R| \geq 2\) and for any \(g \in G_R\) the factor

\[
\prod_{\{x, y\} \in E_g} (e^{\beta (\sigma_x \sigma_y + Y \sigma_x^2 \sigma_y^2) \delta_{|x-y|}} - 1)
\]

is equal to zero whenever \(\sigma_x = 0\) for some \(x \in R\). Thus, defining

\[
\Sigma_R = \{\sigma_R \in \Sigma_R : \sigma_x = \pm1, \ \forall x \in R\},
\]

we can write, for \(|R| \geq 2\)

\[
\bar{\rho}(R) = \sum_{\sigma_R \in \Sigma_R} \rho(R, \sigma_R) e^{2d \beta X \sum_{x \in R} \sigma_x^2} \rho(R, \sigma_R) e^{2d \beta X |R|}.
\]

(7)

Hence setting

\[
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\]
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\[
\xi(R) = \begin{cases} 
1 & \text{if } |R| = 1 \\
\frac{e^{2\beta X}}{1 + 2e^{2\beta X}} \sum_{\sigma_R \in \Sigma_R} \sum_{g \in GR} \prod_{\{x,y\} \in E_g} (e^{\beta (\sigma_x \sigma_y + Y)} \delta_{|x-y|=1} - 1) & \text{if } |R| \geq 2
\end{cases}
\]

(8)

we can rewrite the partition function as

\[
Z_{\Lambda}(X,Y,\beta) = (1 + 2e^{2\beta X})^{|\Lambda|} \sum_{n=1}^{\infty} \sum_{\{R_1,\ldots,R_n\} \in \Pi_n(\Lambda)} \xi(R_1) \cdots \xi(R_n)
\]

\[
= (1 + 2e^{2\beta X})^{|\Lambda|} \left( 1 + \sum_{n \geq 1} \sum_{\{R_1,\ldots,R_n\} \in \Pi_n(\Lambda)} \xi(R_1) \cdots \xi(R_n) \right)
\]

\[
= (1 + 2e^{2\beta X})^{|\Lambda|} \left( 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\{R_1,\ldots,R_n\} \in \Lambda^n} \xi(R_1) \cdots \xi(R_n) \right)
\]

\[
\equiv (1 + 2e^{2\beta X})^{|\Lambda|} \Xi_{\Lambda}(X,Y,\beta),
\]

where \(\Xi_{\Lambda}(X,Y,\beta)\) is the grand canonical partition function of an abstract polymer model, in which the polymers \(R\) are finite subsets of \(\mathbb{Z}^d\) with cardinality greater than 1, with activity \(\xi(R)\) given by (8) and with incompatibility relation being the non-empty intersection (see e.g. [13–15]).

Therefore, the free energy of the system (in the finite volume \(\Lambda\)) is given by

\[
f_{\Lambda}(X,Y,\beta) = \frac{1}{|\Lambda|} \log Z_{\Lambda}(X,Y,\beta) = \log(1 + 2e^{2\beta X}) + P_{\Lambda}(X,Y,\beta),
\]

where

\[
P_{\Lambda}(X,Y,\beta) = \frac{1}{|\Lambda|} \log \Xi_{\Lambda}(X,Y,\beta).
\]

Since \(\log(1 + 2e^{2\beta X})\) is analytic for all \(\beta \in \mathbb{R}\), it is enough to study the absolute convergence of the pressure \(P_{\Lambda}(X,Y,\beta)\) of the polymer gas described above as a function of \(\beta\) in order to get information about analyticity of the free energy in the thermodynamic limit.

The conditions for the absolute convergence and boundness (uniformly in \(\Lambda\)) of the pressure of an abstract polymer gas such as the one described above have been studied since a long time (see for instance [13] and references therein). We apply here the Fernandez–Procacci (FP) criterion [14], according to which the pressure \(P_{\Lambda}(X,Y,\beta)\)
Absolute convergence of the free energy of the BEG model in the disordered region for all temperatures can be written as series which converges absolute uniformly bounded in the volume $\Lambda$, as long as the condition below is satisfied

$$
\sum_{n \geq 2} e^{an} \sup_{z \in \mathbb{Z}^d} \sum_{R \subset \mathbb{Z}^d, |R|=n} |\xi(R)| \leq e^a - 1.
$$

where $\xi(R)$ is defined in (8) and $a > 0$ is an arbitrary parameter to be optimized. Choosing $a = \log 2$ (which is not far from the optimal value) and using translational invariance, this condition becomes

$$
\sum_{n \geq 2} 2^n \sum_{R \subset \mathbb{Z}^d, 0 \in R, |R|=n} |\xi(R)| \leq 1.
$$

(9)

where $0 \in R$ means that $R$ contains the origin of $\mathbb{Z}^d$. Setting

$$
\alpha(X, \beta) = \frac{e^{2d/\beta}}{1 + 2e^{2d/\beta}},
$$

we have

$$
\sum_{R \subset \mathbb{Z}^d, 0 \in R, |R|=n} |\xi(R)| \leq |\alpha(X, \beta)|^n \sum_{R \subset \mathbb{Z}^d, 0 \in R, |R|=n} \sum_{\sigma_R \in \Sigma_R} \prod_{g \in G_R} \prod_{(x,y) \in E_g} (e^{\beta(s_{x}\sigma+Y)}\delta_{|x-y|=1} - 1).
$$

(11)

We now need an upper bound for the factor

$$
\left| \sum_{g \in G_R} \prod_{(x,y) \in E_g} (e^{-\beta V_{xy}} - 1) \right|
$$

(12)

where we have denoted shortly

$$
V_{xy} = -(\sigma_x \sigma_y + Y)\delta_{|x-y|}
$$

(13)

Let us first remark that the pair potential $V_{xy}$ defined in (13) enjoys the so-called stability property (see e.g. [16]) accordingly to the following lemma.

**Lemma 1.** For any $R \subset \mathbb{Z}^d$ and for any $\sigma_R \in \Sigma_R$ it holds that

$$
\sum_{\{x,y\} \subset R} V_{xy} \geq -h(Y)|R|
$$

(14)

where

$$
h(Y) = \begin{cases} 
  d(1+Y) & \text{if } Y > -1 \\
  0 & \text{if } Y \leq -1 
\end{cases}
$$

(15)
The case $Y < -1$ is trivial. Indeed, when $Y < -1$ we have $V_{xy} \geq 0$ for all $\sigma_{\{x,y\}} \in \Sigma_{\{x,y\}}$ and therefore, for any $R$ and any $\sigma_R \in \Sigma_R$

$$\sum_{\{x,y\} \subset R} V_{xy} \geq 0$$

This concludes the proof of the lemma. □

Let us now go back to the problem of finding an upper bound for (12). As long as the pair potential $V_{xy}$ is stable (i.e. satisfies (14)), it is long known that efficient estimates on factors of the form (12) involving sum over connected graphs can be obtained via tree graph identities and tree graph inequalities. Here we will use a recent tree graph inequality due to Procacci and Yuhjtman (proposition 1 in [11]). By such an inequality we can bound in our present case, for any $R \subset \mathbb{Z}^d$ such that $|R| = n$ and any $\sigma_R \in \Sigma_R$,

$$\left| \sum_{g \in G_R} \prod_{\{i,j\} \in E_g} (e^{-\beta V_{xy}} - 1) \right| \leq e^{\beta n h(Y)} \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_{\tau}} (1 - e^{-\beta |V_{xy}|})$$

$$\leq e^{\beta n h(Y)} \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_{\tau}} (1 - e^{-\beta (1+|Y|) \delta_{|x-y|1}})$$

$$= e^{\beta n h(Y)} \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_{\tau}} (1 - e^{-\beta (1+|Y|) \delta_{|x-y|1}}). \quad (16)$$

where $T_R$ denotes the set of all tree graphs with vertex set $R$ (i.e. connected graphs without loops) and $T^n_R$ is the subset of $T_R$ formed by all trees with maximum degree less than or equal to $2d$.

Turning back now to (11), first observe that the right hand side of the inequality (16) has no dependence on the spins and thus we can freely sum in the r.h.s. of (16) over all configurations in $\Sigma_R$ getting a factor $2^{|R|} = 2^n$. Secondly, notice that

$$1 - e^{-\beta (1+|Y|) \delta_{|x-y|1}} = (1 - e^{-\beta (1+|Y|)}) \delta_{|x-y|1}$$

implies that, for any tree $\tau \in T^n_R$

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\[
\prod_{\{x,y\} \in E_r} (1 - e^{-\beta(1+|Y|)} \delta_{|x-y|1}) = (1 - e^{-\beta(1+|Y|)})^{|R|-1} \prod_{\{x,y\} \in E_r} \delta_{|x-y|1}
\]

These two facts, together with (16) imply that (11) can be written as

\[
\sum_{R \subset \mathbb{Z}^d, |R|=n} \prod_{\tau \in T^*_n \{x,y\} \in E_r} \delta_{|x-y|1} \sum_{\{x_1,\ldots,n\} \subset \mathbb{Z}^d} x_1 = 0, x_i \neq x_j \prod_{\{i,j\} \in E_r} \delta_{|x_i-x_j|1} = 1
\]

(17)

Set now

\[
C_n = \sum_{R \subset \mathbb{Z}^d, |R|=n} \sum_{\tau \in T^*_n \{x,y\} \in E_r} \prod_{\{i,j\} \in E_r} \delta_{|x_i-x_j|1}
\]

(18)

and notice that

\[
\sum_{R \subset \mathbb{Z}^d, |R|=n} \prod_{\tau \in T^*_n \{x,y\} \in E_r} \delta_{|x-y|1} \sum_{\{x_1,\ldots,n\} \subset \mathbb{Z}^d} x_1 = 0, x_i \neq x_j \prod_{\{i,j\} \in E_r} \delta_{|x_i-x_j|1} = \frac{1}{(n-1)!} \sum_{\{x_1,\ldots,n\} \subset \mathbb{Z}^d} x_1 = 0, x_i \neq x_j \prod_{\{i,j\} \in E_r} \delta_{|x_i-x_j|1} \frac{1}{(n-1)!} \sum_{\tau \in T^*_n} w_{\tau}
\]

(19)

So, letting \( T^*_n \) denote the set of all trees with vertex set \([n]\) and degrees \(d_i\) such that \(1 \leq d_i \leq 2d\), we have

\[
C_n = \frac{1}{(n-1)!} \sum_{\tau \in T^*_n} \sum_{\{x_1,\ldots,n\} \subset \mathbb{Z}^d} x_1 = 0, x_i \neq x_j \prod_{\{i,j\} \in E_r} \delta_{|x_i-x_j|1} = \frac{1}{(n-1)!} \sum_{\tau \in T^*_n} w_{\tau}
\]

where we have set

\[
w_{\tau} = \sum_{\{x_1,\ldots,n\} \subset \mathbb{Z}^d} x_1 = 0, x_i \neq x_j \prod_{\{i,j\} \in E_r} \delta_{|x_i-x_j|1}
\]

The following lemma provides an upper bound for \( w_{\tau} \).

**Lemma 2.** Given a tree \( \tau \in T^*_n \) with degree \(d_1, \ldots, d_n\) at vertices \(1, \ldots, n\), we have

\[
w_{\tau} \leq \frac{2d!}{(2d-d_1)!} \prod_{i=2}^{n} \frac{(2d-1)!}{(2d-d_i)!}
\]

(20)

**Proof.** We will consider \( \tau \) as rooted in 1. We recall that a vertex \( j \neq 1 \) of \( \tau \) such that \( d_j = 1 \) is called a leaf of \( \tau \). Moreover, given a vertex \( j \) of \( \tau \) with degree \( d_j \), we denote by \( j \) the (unique) vertex of \( \tau \) which is the father of \( j \) in \( \tau \) and we denote by \( j_1, \ldots, j_{d_j-1} \)
the vertices of $\tau$ which are the children of $j$. Let $I = \{i \in [n] : i > 1$ and $d_i > 1\}$. Let $J$ be the subset of $I$ formed by the vertices of $\tau$ whose children are all leaves. For each $j \in J$, we can perform the sum over $x_{j_1}, \ldots, x_{j_{d_j} - 1}$ and we get

$$\sum_{\substack{(x_{j_1}, \ldots, x_{j_{d_j} - 1}) \in (\mathbb{Z}_d)^{d_j - 1} \delta_{x_{j_1} \neq x_{j}, \ldots, x_{j_{d_j} - 1} \neq x_{j_k}}} \prod_{s=1}^{d_j - 1} \delta_{[x_{j_s} - x_{j_k}] = (2d - 1) (2d - 2) \cdots (2d - (d_j - 1))} = \frac{(2d - 1)!}{(2d - d_j)!}$$

Then the sum over all coordinates $\{x_{j_1}, \ldots, x_{j_{d_j} - 1}\}_{j \in J}$ produce (at most) a factor

$$\prod_{j \in J} \frac{(2d - 1)!}{(2d - d_j)!}$$

and we are left with a ‘defoliated’ tree $\tau' \subset \tau$ such that now all $j \in J$ are leaves. Iterating this procedure, observing that when $j = 1$ the sum over the coordinate its children (when they are all leaves) produces the factor

$$2d(2d - 1) \cdots (2d - d_1 + 1) = \frac{(2d)!}{(2d - d_1)!}$$

we get

$$w_t \leq \frac{(2d)!}{(2d - d_1)!} \prod_{j \in I} \frac{(2d - 1)!}{(2d - d_j)!}$$

where the inequality is due to the fact that we are not taking into account that we could produce cycles when we sum over the coordinates $\{x_j\}_{j \in J}$. Now observing that when $j \notin I$ (i.e. $d_j = 1$ and $j \neq 1$) we have that

$$\frac{(2d - 1)!}{(2d - d_j)!} = 1$$

inequality (20) follows. \(\square\)

Inequality (20) will be used in the next section to obtain an upper bound on the number of $d$-dimensional fixed polycubes of size $n$. For the purpose of this section it is sufficient to use a simplified (and slightly worse) estimate for $w_t$ easily derived from (20). Indeed, from (20) we have that

$$w_t \leq \frac{2d!}{(2d - d_1)!} \prod_{i=2}^{n} \frac{(2d - 1)!}{(2d - d_i)!} \leq 2d \prod_{i=1}^{n} (2d - 1)^{d_i - 1} = 2d(2d - 1)^{n-2}$$

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where in the last inequality we have used that for any tree \( \tau \in T_n \) it holds that \( \sum_{i=1}^{n} d_i = 2n - 2 \). Therefore we can bound

\[
C_n \leq \frac{2d}{(n-1)!} (2d-1)^{n-2} \sum_{\tau \in T_n^*} 1
\]

\[
\leq \frac{2d}{(n-1)!} (2d-1)^{n-2} \sum_{\tau \in T_n^*} 1
\]

\[
= 2d \frac{n^{n-2}}{(n-1)!} (2d-1)^{n-2}
\]

\[
\leq \frac{n^{n-2}}{(n-1)!} (2d)^{n-1}
\]

(22)

where in the third line we have used Cayley formula (which says that \( \sum_{\tau \in T_n^*} 1 = n^{n-2} \)).

In conclusion lhs of (17) is bounded as

\[
\sum_{R: |R| = n} |\xi(R)| \leq [2\alpha(X, \beta)e^{\beta h(Y)}]^{n} [2d(1 - e^{-\beta(1+|Y|)})]^{n-1} \cdot \frac{n^{n-2}}{(n-1)!}
\]

By Stirling formula, we get the inequality \( \frac{n^{n-2}}{(n-1)!} \leq \frac{e^{n-1}}{n} \), which used in the above expression, implies that condition (9) is satisfied provided that

\[
\sum_{n=2}^{\infty} [4\alpha(X, \beta)e^{\beta h(Y)}]^{n} \frac{e^{n-1}}{n} [2d(1 - e^{-\beta(1+|Y|)})]^{n-1} \leq 1,
\]

or

\[
4\alpha(X, \beta)e^{\beta h(Y)} \sum_{n=2}^{\infty} \frac{1}{n} [8d\alpha(X, \beta)e^{\beta h(Y)}(1 - e^{-\beta(1+|Y|)})]^{n-1} \leq 1.
\]

(23)

When \( \beta = 0 \), the condition (23) is trivially satisfied. Let us thus suppose \( \beta > 0 \) and let \( \delta = \alpha(X, \beta)e^{\beta h(Y)} \) and \( \epsilon = (1 - e^{-\beta(1+|Y|)}) \). Using the fact that \( \sum_{n=2}^{\infty} \frac{r^{n-1}}{n} = -\frac{\log(1-r)}{r} \) for \(|r| < 1\), the inequality above will be satisfied, as long as

\[
4\delta \left[ -8d\delta \epsilon - \log (1 - 8d\delta \epsilon) \right] \leq 1,
\]

or, equivalently,

\[
4\delta \left[ -1 - \log (1 - 8d\delta \epsilon) \right] \leq 1.
\]

(24)

Since \( \delta \) is a function of the stability constant \( h(Y) \) defined in (15) which assumes two values, depending on \( Y \), we will analyze the two cases separately.
We start by analyzing (24) when $Y > -1$. In this case, $\hat{h}(Y) = d(1 + Y)$ and so $e^{2\beta X}e^{\beta h(Y)} = e^{\beta d(2X + (1 + Y))}$. Since in the disordered region $1 + 2X + Y < 0$, we have $\delta \leq 1$. So, (24) holds if

$$- \log (1 - 8de\delta \epsilon) \leq \frac{5}{4}. \tag{25}$$

Notice that the function $f(x) = -\ln[(1 - x)^{1/x}]$ is increasing for $0 < x < 1$ with $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 1^-} f(x) = +\infty$. The solution of $f(x) = 5/4$ is (slightly) greater than 0.37137. And so, the condition (25) holds, provided that

$$\delta \epsilon \leq \frac{0.37137}{8de} \leq \frac{1}{58.57d}.$$

Calling $-d[2X + (1 + Y)] = k_1$, $1 + |Y| = k_2$, and $58.57d = C_+$, the condition above becomes

$$e^{-k_1 \beta} (1 - e^{-k_2 \beta}) \leq \frac{1}{C_+}.$$

Note that $k_2 \geq 0$ by definition and $k_1 \geq 0$ since we are in the disordered phase where $2X + (1 + Y) < 0$. Observe moreover that the maximum of the function $g(\beta) = e^{-k_1 \beta}(1 - e^{-k_2 \beta})$ as $\beta$ varies in the interval $(0, +\infty)$ is attained at $\beta = \beta_c$ where $\beta_c$ is the solution of $e^{-k_2 \beta} = \frac{k_1}{k_1 + k_2}$. So, for any $\beta > 0$ we have

$$g(\beta) \leq \left(e^{-k_2 \beta}ight)^{\frac{k_1}{k_1 + k_2}} (1 - e^{-k_2 \beta}) = \left(\frac{k_1}{k_1 + k_2}\right)^{\frac{k_1}{k_2}} \frac{k_2}{k_1 + k_2} \leq \frac{k_2}{k_1 + k_2}.$$

Therefore, if $\frac{k_2}{k_1 + k_2} \leq \frac{1}{C_+}$, or equivalently, $k_1 \geq (C_+ - 1)k_2$, the condition for convergence will be satisfied for every $\beta \geq 0$. Taking into account the expressions for $k_1$ and $k_2$, we have

$$-d[2X + (1 + Y)] \geq (C_+ - 1)(1 + |Y|),$$

namely,

$$X \leq -\frac{(C_+ - 1)}{2d}(1 + |Y|) - \frac{1}{2}(1 + Y).$$

Since we have dependency on $|Y|$, we have to consider the cases $Y \geq 0$ and $-1 < Y < 0$ separately (remember that we are currently considering the case $Y > -1$). For $Y \geq 0$, we have

$$X \leq -\left[\frac{(C_+ - 1)}{2d} + \frac{1}{2}\right] (1 + Y)$$

and for $-1 < Y < 0$, we have

$$X \leq -\left[\frac{(C_+ - 1)}{2d} + \frac{1}{2}\right] + \left[\frac{(C_+ - 1)}{2d} - \frac{1}{2}\right] Y.$$

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Let us now turn to the case when \( Y \leq -1 \). In this case the stability constant is \( h(Y) = 0 \) and thus, recalling that \( \delta = \alpha(X, \beta)e^{\beta h(Y)} \) and that \( \alpha(X, \beta) = e^{\frac{e^{2d}X}{1+2e^{2d}X}} \), we have that \( \delta \leq \frac{1}{3} \). Therefore, the condition (24) becomes

\[
-\log (1 - 8de\delta^e) \leq \frac{7}{4},
\]

namely,

\[
\delta e \leq \frac{0.7127}{8de} \leq \frac{1}{30.52d}.
\]

Letting \( k_1 = 2d|X|, \ k_2 = 1 - Y \) and \( C_- = 30.52d \) and proceeding as in the first case, we have the following condition

\[
X \leq -\frac{(C_- - 1)}{2d}(1 - Y).
\]

Summarizing, we conclude that the free energy is analytic for all \( \beta \) for \( (X, Y) \) in the portion of \( D \) whose boundary is the polygonal curve

\[
X = \begin{cases} 
-k(Y + 1), & \text{if } Y \geq 0 \\
(k - 1)Y - k, & \text{if } -1 < Y < 0 \\
\bar{k}(Y - 1), & \text{if } Y \leq -1 
\end{cases}
\]

where \( k = \frac{59.57d - 1}{2d} \) and \( \bar{k} = \frac{30.52d - 1}{2d} \), and this concludes the proof of theorem 1.

3. A remark on the numbers of \( d \)-dimensional polycubes of size \( n \)

We conclude this note by showing that lemma 2 proved in the previous section can be used to obtain an upper bound for the number \( A_n \) of fixed \( d \)-dimensional polycubes of size \( n \). We recall that a \( d \)-dimensional polycube of size \( n \) is a connected set of \( n \) unit cubical cells on the lattice \( \mathbb{Z}^d \), where connectivity is through \((d - 1)\)-faces. Let us denote by \( \mathcal{P}_n \) the set of all \( d \)-dimensional polycubes of size \( n \) in \( \mathbb{Z}^d \). Two polycubes are considered equivalent if one can be transformed into the other by a translation. A class of equivalence of polycubes is called a ‘fixed polycube’. Let us denote by \( A_n \) the number of fixed \( d \)-dimensional polycubes of size \( n \) and let us explain how obtain an upper bound for \( A_n \).

First of all let us give the following definition. A finite set of vertices \( Q \) in \( \mathbb{Z}^d \) is called an animal if either \( |Q| = 1 \) or if \( |Q| \geq 2 \) and for any partition \( Q = A \cup B \), there exist \( x \in A \) and \( y \in B \) such that \( |x - y| = 1 \). Let us denote by \( \mathcal{A}_n \) the set of all animals in \( \mathbb{Z}^d \) with \( n \) vertices. Clearly there is a one-to-one correspondence between \( \mathcal{P}_n \) and

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\( \hat{A}_n \). Indeed to each \( Q = \{x_1, \ldots, x_n\} \in \hat{A}_n \) we can associate in an one-to-one manner the polycube \( p = \{c_1, \ldots, c_n\} \) in the dual lattice \((\mathbb{Z}^d)^*\), such that each cube \( c_i \in p \) is centered in \( x_i \).

Let us define

\[
A^*_n = \sum_{Q \in \hat{A}_n; 0 \in Q} 1.
\]

Then, since there are \( n \) possible choices of the position of the origin,

\[
A_n = \frac{A^*_n}{n}
\]  \hspace{1cm} (26)

is the number of fixed polyominoes (polycubes in three or more dimensions) of size \( n \).

Let us consider the infinite graph \( G^d \) with set of vertices \( \mathbb{Z}^d \) and set of edges formed by the nearest neighbor pairs of \( \mathbb{Z}^d \). Let us denote by \( T^d \) the set of all finite subgraphs of \( G^d \) which are trees. Then observe that any animal \( Q \) in \( \mathbb{Z}^d \) contains at least one spanning tree in \( T^d \). Therefore,

\[
A^*_n = \sum_{Q \in \hat{A}_n; 0 \in Q} 1 \leq \sum_{\tau \in T^d; |V_\tau| = n} 1
\]

where \( \tau \) denotes a spanning tree, with \( E_\tau \) and \( V_\tau \) its sets of edges and vertices, respectively.

Now recall that we have denoted by \( T^*_n \) the set of all trees with vertex set \([n]\) and degrees \( d_i \) such that \( 1 \leq d_i \leq 2d \). Recall also that, given two vertices \( x_i, x_j \) in \( \mathbb{Z}^d \), \( \delta_{|x_i-x_j|} = 1 \) if \( x_i, x_j \) are neighbors in \( \mathbb{Z}^d \) and \( \delta_{|x_i-x_j|} = 0 \), otherwise. Then we can write

\[
\sum_{\tau \in T^d; |V_\tau| = n} 1 \leq \sum_{\tau \in T^d; |V_\tau| = n} \prod_{\{x_i, x_j\} \subseteq \mathbb{Z}^d; \tau \in T^*_n} \delta_{|x_i-x_j|}
\]

\[
= \frac{1}{(n-1)!} \sum_{(x_1, \ldots, x_n) \subseteq \mathbb{Z}^d; \tau \in T^*_n} \prod_{\{i, j\} \subseteq \tau} \delta_{|x_i-x_j|}
\]

\[
= \frac{1}{(n-1)!} \sum_{\tau \in T^*_n} \prod_{\{i, j\} \subseteq \tau} \delta_{|x_i-x_j|}
\]

\[
= \frac{1}{(n-1)!} \sum_{\tau \in T^*_n} w_\tau
\]

where \( w_\tau \) is defined in (19). Now given a tree \( \tau \in T^*_n \) with degree \( d_1^\tau, \ldots, d_n^\tau \) at vertices \( 1, \ldots, n \), we can use the bound (21) and hence we get

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\[ \sum_{\tau \in \mathcal{T}_d} \sum_{\begin{subarray}{c} \tau \in \mathcal{T}_d \vspace{1pt} \\ 0 \leq V_{\tau} \leq 1 \end{subarray}} \left| V_{\tau} \right| = n \leq 2 \vspace{1pt} \\
\sum_{\tau \in \mathcal{T}_d^*} \sum_{d_1 + \cdots + d_n = n} \frac{(2d - 1)!}{(2d - d_i)!} \sum_{d_i \text{ fixed}} 1 \vspace{1pt} \\
= \frac{2d}{(n-1)!} \sum_{d_1 + \cdots + d_n = n} \frac{(2d - 1)!}{(2d - d_i)!} \prod_{i=1}^{n} (n - 2)! \\
= \frac{2d}{(n-1)!} \sum_{d_1 + \cdots + d_n = n} \left( \frac{(2d - 1)!}{(2d - d_i)!} \prod_{i=1}^{n} (d_i - 1)! \right) \\
= \frac{2d}{(n-1)!} \sum_{s_1 + \cdots + s_n = n} \prod_{i=1}^{n} \left( \frac{2d - 1}{s_i} \right). \]

Now observe that

\[ \sum_{s_1 + \cdots + s_n = n} \prod_{i=1}^{n} \left( \frac{2d - 1}{s_i} \right) = \binom{(2d - 1)n}{n-2}. \]

Indeed, suppose that we have a set \( A = \bigcup_{i=1}^{n} A_i \) and each \( A_i \) has cardinality \( 2d - 1 \). Hence, for fixed numbers \( s_1, \ldots, s_n \) such that \( 0 \leq s_i \leq 2d - 1 \), we have that \( \prod_{i=1}^{n} \left( \frac{2d - 1}{s_i} \right) \) is the number of ways to choose \( s_1 \) objects from \( A_1 \), \( s_2 \) objects from \( A_2 \), \ldots, \( s_n \) objects from \( A_n \). In this way we have chosen \( s_1 + \cdots + s_n = n - 2 \) objects in \( A \) which is \( \binom{(2d - 1)n}{n-2} \). Therefore we have obtained the bound

\[ A_n^* \leq \frac{2d}{(n-1)!} \binom{(2d - 1)n}{n-2} \]

and consequently, recalling (26), an upper bound for the number \( A_n \) of fixed \( d \)-dimensional polycubes of size \( n \), namely,

\[ A_n \leq A_n^{LLP} = \frac{2d}{n(n-1)} \binom{(2d - 1)n}{n-2}. \hspace{1pt} \hspace{1pt} \hspace{1pt} \hspace{1pt} \hspace{1pt} \hspace{1pt} \hspace{1pt} (27) \]

The bound (27) is slightly better than previous bounds available in the literature, at least for \( d \geq 3 \) (see e.g. theorem 9 in [17, 18]). In particular, in section 2 of the very recent paper [18] Barequet and Shalah claim that \( A_n \) is bounded from above by the number of binary sequences with \( n - 1 \) ones and \( (2d - 2)n \) zeros and hence

\[ A_n \leq A_n^{BS} = \binom{(2d - 1)n}{n-1} \]

so that the ratio

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\[ \frac{A_{n}^{LLP}}{A_{n}^{BS}} = \frac{2d(2d-1)}{(2d-2)n+1} \frac{(2d-2)n+2}{(2d-2)n+1} \]

goes to zero as \( O(1/n^2) \) when \( n \to \infty \).

4. Conclusions

In this paper we analyze the \( d \)-dimensional, with \( d \geq 2 \), Blume–Emery–Griffiths model (with Hamiltonian given by (1)) in the disordered region of parameters \( D = \{(X, Y) \in \mathbb{R}^2 : X < 0, 1 + 2X + Y < 0\} \) and we obtain a region \( D_{\text{analytic}} \subset D \) where the finite volume free energy of the model can be written, at any nonnegative inverse temperature \( \beta \), in terms of an absolutely convergent series, uniformly bounded in the volume. This implies that its thermodynamic limit is an analytic function of \( \beta \), for all nonnegative \( \beta \), ruling out the possibility of a phase transition in the region \( D_{\text{analytic}} \). This result has been obtained via a high temperature polymer expansion of the partition function of the model combined with some recent results related to tree graph inequalities. We believe that our estimates are nearly optimal compatibly with the cluster expansion techniques used here. In confirmation of this, we also obtain, as a byproduct, a slight improvement on the upper bound for the number of \( d \)-dimensional fixed polycubes of size \( n \). On the other hand, we think that to enlarge sensibly the region of \( D_{\text{analytic}} \) in which no phase transition occurs for all nonnegative \( \beta \), in particular in such a way to include negative values of the parameter \( X \) near the line \( X = 0 \) and \( Y < -1 \), new ideas are necessary.

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