Optimal error estimates for homogenization of linear elasticity systems on perforated domains

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Abstract

In the present work, we established optimal error estimates for linear elasticity systems in periodically perforated domains. We obtain the error estimate $O(\varepsilon)$ for a $C^{1,1}$ domain and $O(\varepsilon \ln(1/\varepsilon))$ for a Lipschitz domain. This work may be regarded as an extension of \cite{9, 10}.

Key words. Homogenization; perforated domain; elasticity systems; optimal error estimate.

1 Introduction

In this paper, we are aimed to establish optimal convergence rates for linear elasticity systems in periodically perforated domains. Consider the operator

$$L_\varepsilon = -\text{div}(A_\varepsilon(x)\nabla) = -\frac{\partial}{\partial x_i} \left( a_{ij}^{\alpha\beta}(\frac{x}{\varepsilon}) \frac{\partial u_\varepsilon}{\partial x_j} \right), \quad x \in \varepsilon \omega, \ v > 0,$$

where $A_\varepsilon(x) = A(x/\varepsilon)$, $A(y) = \{ a_{ij}^{\alpha\beta}(y) \}_{1 \leq i,j,\alpha,\beta \leq d}$ for $y \in \omega, d \geq 2$, and $\omega \subset \mathbb{R}^d$ is an unbounded Lipschitz domain with 1-periodic structure. We denote the $\varepsilon$-homothetic set $\{ x \in \mathbb{R}^d : x/\varepsilon \in \omega \}$ by $\varepsilon \omega$. For a bounded domain $\Omega \subset \mathbb{R}^d$, consider the following mixed boundary value problem

$$\begin{cases}
L_\varepsilon u_\varepsilon = F & \text{in } \Omega_\varepsilon, \\
\sigma_\varepsilon(u_\varepsilon) = 0 & \text{on } S_\varepsilon, \\
u_\varepsilon = g & \text{on } \Gamma_\varepsilon,
\end{cases}$$

where $\sigma_\varepsilon = -\vec{n} A(x/\varepsilon) \nabla$ and $\vec{n}$ denotes the outward unit normal to $\Omega_\varepsilon$, and $\Omega_\varepsilon = \Omega \cap \varepsilon \omega$, $S_\varepsilon = \partial \Omega_\varepsilon \cap \Omega$, $\Gamma_\varepsilon = \partial \Omega_\varepsilon \cap \partial \Omega$. The coefficient matrix $A(y)$ satisfies the following structure conditions:

$$\begin{cases}
A(y) \text{ is real, measurable, 1-periodic,} \\
a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y) = a_{\alpha j}^{i\beta}(y), \\
\mu_0 |\xi|^2 \leq a_{ij}^{\alpha\beta}(y) \xi_i^{\alpha} \xi_j^{\beta} \leq \mu_1 |\xi|^2
\end{cases}$$

for $y \in \omega$ and any symmetric matrix $\xi = \{ \xi^{\alpha}_i \}_{1 \leq i,\alpha \leq d}$, where $\mu_0, \mu_1 > 0$.

The following qualitative homogenization results are well known (see for example \cite{3, 4}). Let $F \in H^{-1}(\Omega; \mathbb{R}^d)$, and $u_\varepsilon$ be the weak solution to (2). Then we conclude that $l^*_\varepsilon u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega; \mathbb{R}^d)$, and $l^*_\varepsilon A(x/\varepsilon) \nabla u_\varepsilon \rightharpoonup \tilde{A} \nabla u_0$ weakly in $L^2(\Omega; \mathbb{R}^{d^2})$, where $u_0$ is the solution to the effective (homogenized) equation.
\[ \begin{align*}
\mathcal{L}_0 u_0 & \equiv -\text{div}(\hat{A} \nabla u_0) = F \quad \text{in } \Omega, \\
u_0 & = g \quad \text{on } \partial \Omega.
\end{align*} \] (4)

The matrix \( \hat{A} = \{ \hat{a}_{ij}^{\alpha\beta} \}_{1 \leq i,j,\alpha,\beta \leq d} \) is defined by

\[ \hat{a}_{ij}^{\alpha\beta} = \int_{Y \cap \omega} a_{ik}^{\alpha\gamma}(y) \frac{\partial X_j^{\gamma\beta}}{\partial y_k} dy, \] (5)

where \( X_j^{\beta} = \{ X_j^{\gamma\beta} \}_{1 \leq \gamma \leq d} \) is the weak solution to the following cell problem

\[ \begin{align*}
\text{div}\{ A(y)(\nabla X_j^{\beta}) \} & = 0 \quad \text{in } Y \cap \omega, \\
\bar{n} \cdot A(y) \nabla X_j^{\beta} & = 0 \quad \text{on } Y \cap \partial \omega, \\
X_j^{\beta} & - y e^\beta := \chi_j^{\beta} \in H^1_{\text{per.}}(\omega; \mathbb{R}^d), \\
\int_{Y \cap \omega} \chi_j^{\beta} dy & = 0,
\end{align*} \] (6)

and the notation \( \int_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} \) represents the average of integral with \( |\Omega| \) being the volume of \( \Omega \).

We impose \( l^+ \) to denote the characteristic function of \( \omega \), and assume that \( \omega \) is connected, \((\mathbb{R}^d \setminus \omega) \cap Y \subset \subset Y\) in which \( Y \) is the unit cube; and that any two connected components of \( \mathbb{R}^d \setminus \omega \) are separated by some positive distance. Specifically, if \( \mathbb{R}^d \setminus \omega = \bigcup_{k=1}^\infty H_k \) in which \( H_k \) is connected and bounded for each \( k \), then there exists a constant \( g^\omega \) such that

\[ 0 < g^\omega \leq \inf \text{dist}(H_i, H_j). \] (7)

We now state the main result of this paper.

**Theorem 1.1** (optimal error estimates). Suppose that \( \mathcal{L}_\varepsilon \) satisfies (3). Let \( u_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d) \) and \( u_0 \in H^1(\Omega; \mathbb{R}^d) \) be weak solutions of (2) and (4), respectively.

- If \( \Omega \) is a Lipschitz domain, \( F \in L^2(\Omega; \mathbb{R}^d) \) and \( g \in H^1(\partial \Omega; \mathbb{R}^d) \), then there holds

\[ \| u_\varepsilon - u_0 \|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \ln(1/\varepsilon) \left( \| F \|_{L^2(\Omega)} + \| g \|_{H^1(\partial \Omega)} \right). \] (8)

- If \( \Omega \) is a \( C^{1,1} \) domain, \( F \in L^2(\Omega; \mathbb{R}^d) \) and \( g \in H^{3/2}(\partial \Omega; \mathbb{R}^d) \), then we have

\[ \| u_\varepsilon - u_0 \|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \left( \| F \|_{L^2(\Omega)} + \| g \|_{H^{3/2}(\partial \Omega)} \right). \] (9)

where \( C \) depends on \( \mu_0, \mu_1, d, \omega \) and the character of \( \Omega \).

Here the space \( H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d) \) denotes the closure in \( H^1(\Omega_\varepsilon; \mathbb{R}^d) \) of smooth vector-valued functions vanishing on \( \Gamma_\varepsilon \).

Homogenization in the perforated domain has been considered for decades, and most of the papers studied the qualitative results, for example [2, 3, 4, 9]. There was also some quantitative outcomes. O. Oleinik, A. Shamaev and G. Yosifian [9, Theorem 1.2] obtained the following estimate for elasticity systems:

\[ \| u_\varepsilon - u_0 - \varepsilon \chi_\varepsilon \nabla u_0 \|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{1/2} \left( \| g \|_{H^{3/2}(\partial \Omega)} + \| F \|_{H^1(\Omega)} \right). \] (10)
under the assumption that coefficient matrix $a_{ij}^{\alpha\beta}(y)$ is piece-wise smooth and $\Omega$ is a smooth domain. In 2017, B. Russell [10] improved the result (10) by obtaining
\[
\|u_{\varepsilon} - u_0 - \varepsilon \chi_{\varepsilon} \nabla u_0\|_{H^1(\Omega_{\varepsilon})} \leq C\varepsilon^{1/2}\|g\|_{H^1(\Omega)}
\]
in the case that $F = 0$, $a_{ij}^{\alpha\beta}$ satisfies (3) and $\Omega$ is a Lipschitz region. Therefore, this paper can be regarded as an extension of [9] and [10]. The innovation in the present work involves how to deal with the source term. To make it clearly, we outline the main ideas as follows.

Consider the two-scale expansion $w_{\varepsilon} = u_{\varepsilon} - v_{\varepsilon}$ with $v_{\varepsilon} = u_0 + \varepsilon \chi_{\varepsilon} \varphi$, we obtain that
\[
\int_{\Omega_{\varepsilon}} A(x/\varepsilon) \nabla w_{\varepsilon} \nabla w_{\varepsilon} dx = \int_{\Omega} (l^+_\varepsilon - \theta \psi'_\varepsilon) F \tilde{w}_{\varepsilon} dx + \text{“other terms”}
\]
(see Lemma 3.3), where $\psi'_\varepsilon$ is a cut-off function defined in (14), and $\tilde{w}_{\varepsilon}$ is an extension of $w_{\varepsilon}$ by Lemma 2.3. The term $T$ may be decomposed into
\[
\int_{\Omega} (1 - \psi'_\varepsilon) l^+_\varepsilon F \tilde{w}_{\varepsilon} dx, \quad \text{and} \quad \int_{\Omega} (l^+_\varepsilon - \theta) F \tilde{w}_{\varepsilon} \psi'_\varepsilon dx,
\]
which can be regarded as the layer part and the co-layer part, respectively. By noting that $\int_{\Omega} (l^+_\varepsilon - \theta) dx = 0$, we could construct an auxiliary equation (21) which is similar to that the flux corrector satisfies, it follows that
\[
| \int_{\Omega} (l^+_\varepsilon - \theta) F \tilde{w}_{\varepsilon} \psi'_\varepsilon dx | \leq C\varepsilon \| \nabla F \|_{H^{-1}(\Omega)} \| \tilde{w}_{\varepsilon} \psi'_\varepsilon \|_{H^0(\Omega)}.
\]
Observing the fact that $\| \nabla F \|_{H^{-1}(\Omega)} \leq C\| F \|_{L^2(\Omega)}$, one may clearly lower the regularity assumption on $F$, while the layer part is in fact a “good” term, and it will be given by $O(\varepsilon)$ through an application of Poincaré’s inequality down to $\varepsilon$ scales (see Lemma 2.5). The “other terms” can be reduced to the layer $\| \nabla u_0 \|_{L^2(\Omega_{\varepsilon})}$ and the co-layer $\| \nabla^2 u_0 \|_{L^2(\Omega_{\varepsilon})}$ type estimates, and then we obtain the convergence rate $O(\varepsilon^{1/2})$ by using energy methods as in [10] and [12]. Besides, the condition that $F \in L^2(\Omega; \mathbb{R}^d)$ instead of $F \in H^1(\Omega; \mathbb{R}^d)$ enables us to employ the duality argument meaningfully, which was initially introduced by T. Suslina [15] to derive the optimal error estimates for elliptic systems. The weighted-type estimates for smoothing operator $S_\varepsilon$ at $\varepsilon$ scale (see [17, Lemmas 3.2 and 3.3]) are proved to be an effective tool to handle the problem in the case of nonsmooth domains, which is originally developed by the second author [17]. Finally, involving the related area, we refer the reader to [2, 6, 7, 8, 11, 13, 16] and the references therein for more results.

2 Preliminaries

**Definition 2.1.** Fix a nonnegative function $\zeta \in C_0^\infty(B(0,1/2))$, and $\int_{\mathbb{R}^d} \zeta(x) dx = 1$. Define the smoothing operator for $f \in L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$ as
\[
S_\varepsilon(f)(x) = f \ast \zeta_\varepsilon(x) = \int_{\mathbb{R}^d} f(x - y) \zeta_\varepsilon(y) dy,
\]
where $\zeta_\varepsilon(y) = \varepsilon^{-d} \zeta(y/\varepsilon)$. 

Lemma 2.2. Let $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Then for any $\varpi \in L^p_{per}(\mathbb{R}^d)$,

$$
\|\varpi(\cdot / \varepsilon) S_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq C\|\varpi\|_{L^p(Y)} \|f\|_{L^p(\mathbb{R}^d)},
$$

where $C$ depends on $d$. Moreover, if $f \in W^{1,p}(\mathbb{R}^d)$ for some $1 < p < \infty$, then we have

$$
\|S_\varepsilon(f) - f\|_{L^p(\mathbb{R}^d)} \leq C\varepsilon\|\nabla f\|_{L^p(\mathbb{R}^d)};
$$

where $C$ also depends on $d$.

Proof. See [12, Lemmas 2.1 and 2.2].

Lemma 2.3 (extension property). Let $\Omega$ and $\Omega_0$ be a bounded Lipschitz domains with $\bar{\Omega} \subset \Omega_0$ and $\text{dist}(\partial\Omega_0, \Omega) > 1$. Then, for $0 < \varepsilon < 1$, there exists a linear extension operator $P_\varepsilon : H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d) \to H^1_0(\Omega_0; \mathbb{R}^d)$ such that

$$
\|P_\varepsilon w\|_{H^1_0(\Omega_0)} \leq C_1\|w\|_{H^1(\Omega_\varepsilon)},
$$

$$
\|\nabla P_\varepsilon w\|_{L^2(\Omega_0)} \leq C_2\|\nabla w\|_{L^2(\Omega_\varepsilon)},
$$

$$
\|e(P_\varepsilon w)\|_{L^2(\Omega_0)} \leq C_3\|e(w)\|_{L^2(\Omega_\varepsilon)}
$$

for some constants $C_1, C_2, C_3$ depending only on $\Omega$ and $\omega$, where $e(w)$ denotes the symmetric part of $\nabla w$, i.e., $e(w) = \frac{1}{2}[\nabla w + (\nabla w)^T]$.

Proof. See [9, p.50, Theorem 4.3].

Lemma 2.4. There exists a constant $C$ independent of $\varepsilon$ such that

$$
\|w\|_{H^1(\Omega_\varepsilon)} \leq C\|e(w)\|_{L^2(\Omega_\varepsilon)}
$$

for any $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$.

Proof. See [9, p.53, Theorem 4.5].

Actually, Lemma 2.4 is Korn’s first inequality for periodically perforated domains and it follows from Lemma 2.3 and Korn’s first inequality.

Lemma 2.5. For $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$,

$$
\|\tilde{w}\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon\|\nabla \tilde{w}\|_{L^2(\Omega)},
$$

where $O_{4\varepsilon} = \{x \in \Omega : \text{dist}(x, \partial\Omega) < 4\varepsilon\}$ and $C$ depends on $d, \Omega$ and $\omega$.

Proof. See [10, Lemma 3.4].

Lemma 2.6. Suppose $B = \{b_{ij}^{\alpha\beta}\}_{1 \leq i,j,\alpha,\beta \leq d}$ is 1-periodic and satisfies $b_{ij}^{\alpha\beta} \in L^2_{\text{loc}}(\mathbb{R}^d)$ with

(i) $\frac{\partial}{\partial y_i} b_{ij}^{\alpha\beta} = 0$,

(ii) $\int_{\mathbb{R}^d} b_{ij}^{\alpha\beta} = 0$.

There exists $E = \{E_{kij}^{\alpha\beta}\}_{1 \leq i,j,k,\alpha,\beta \leq d}$ with $E_{kij}^{\alpha\beta} \in H^1_{\text{loc}}(\mathbb{R}^d)$ that is 1-periodic and satisfies

$$
\frac{\partial}{\partial y_k} E_{kij}^{\alpha\beta} = b_{ij}^{\alpha\beta} \text{ and } E_{ij}^{\alpha\beta} = -E_{kij}^{\alpha\beta}.
$$

Proof. See [7] and also [8, Lemma 3.1].

Lemma 2.7. Suppose that $A$ satisfies (3), then the effective matrix $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ defined in (5) satisfies

$$
\begin{cases}
\hat{a}_{ij}^{\alpha\beta}(y) = \hat{a}_{ij}^{\alpha\beta}(y) = \hat{a}_{ij}^{\alpha\beta}(y), \\
\hat{\mu}_0 |\xi|^2 \leq \hat{a}_{ij}^{\alpha\beta}(y) |\xi|^2 \leq \hat{\mu}_1 |\xi|^2
\end{cases}
$$

for any symmetric matrix $\xi = \{\xi_i^{\alpha}\}_{1 \leq i,\alpha \leq d}$, where $\hat{\mu}_0, \hat{\mu}_1 > 0$ depends on $\mu_0$ and $\mu_1$.

Proof. See either [7] or [9].
3 Convergence rates of $\varepsilon^{1/2}$ order

In this section, we derive the error estimate $O(\varepsilon^{1/2})$ by estimating the integral $\int_{\Omega_\varepsilon} A(x/\varepsilon) \nabla w_\varepsilon \nabla \psi_\varepsilon dx$. At first, we introduce two cut-off functions. Let $O_\varepsilon = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \varepsilon \} = \Omega \setminus \Sigma_\varepsilon$, and $\psi'_\varepsilon, \psi_\varepsilon \in C^\infty_0(\Omega)$ satisfy

\[
\begin{aligned}
0 \leq \psi_\varepsilon, \psi'_\varepsilon & \leq 1 \quad \text{for} \quad x \in \Omega, \\
\text{supp}(\psi_\varepsilon) & \subseteq \Omega \setminus O_{2\varepsilon}, \quad \text{supp}(\psi'_\varepsilon) \subseteq \Omega \setminus O_\varepsilon, \\
\psi_\varepsilon & = 1 \quad \text{in} \quad \Omega \setminus O_{4\varepsilon}, \quad \psi'_\varepsilon = 1 \quad \text{in} \quad \Omega \setminus O_{2\varepsilon}, \\
\max\{ |\nabla \psi_\varepsilon|, |\nabla \psi'_\varepsilon| \} & \leq C\varepsilon^{-1}.
\end{aligned}
\]

By the above definition, it’s known that $\psi_\varepsilon(1 - \psi'_\varepsilon) = 0$ in $\Omega$.

Let $w_\varepsilon = u_\varepsilon - v_\varepsilon$ and $v_\varepsilon = u_0 + \varepsilon \chi_\varepsilon \varphi$, where $\varphi = S_\varepsilon(\psi_\varepsilon \nabla u_0)$. The following is the main result of this section.

**Theorem 3.1.** Suppose that $\mathcal{L}_\varepsilon$ satisfies (3), and $u_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$, $u_0 \in H^1(\Omega; \mathbb{R}^d)$ are weak solutions of (2) and (4), respectively. Let $\Omega$ be a Lipschitz domain and $F \in L^2(\Omega; \mathbb{R}^d)$, $g \in H^1(\partial \Omega; \mathbb{R}^d)$. Then we have

\[
\begin{aligned}
\|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq C\varepsilon^{1/2}\left( \|F\|_{L^2(\Omega)} + \|g\|_{H^1(\partial \Omega)} \right), \\
\|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon)} & \leq C\varepsilon^{1/2}\left( \|F\|_{L^2(\Omega)} + \|g\|_{H^1(\partial \Omega)} \right),
\end{aligned}
\]

where $C$ depends on $\mu_0, \mu_1, d, \omega$ and the character of $\Omega$.

**Lemma 3.2.** For $\phi \in H^1_0(\Omega; \mathbb{R}^d)$, we have

\[
\int_{\Omega_\varepsilon} A(x/\varepsilon) \nabla w_\varepsilon \nabla \phi dx = \int_{\Omega_\varepsilon} (l^+_\varepsilon - \theta) F \phi dx + \int_{\Omega_\varepsilon} (\theta \hat{A} - l^+_\varepsilon A^\varepsilon)(\nabla u_0 - \varphi) \nabla \phi dx \\
+ \int_{\Omega_\varepsilon} (\theta \hat{A} - l^+_\varepsilon A^\varepsilon - l^+_\varepsilon A^\varepsilon \nabla \chi_\varepsilon) \varphi \nabla \phi dx - \varepsilon \int_{\Omega_\varepsilon} l^+_\varepsilon A^\varepsilon \chi_\varepsilon \nabla \varphi \nabla \phi dx,
\]

in which $\chi_\varepsilon = \chi_\varepsilon$ in $\varepsilon \omega$ and $\chi_\varepsilon = 0$ in $\mathbb{R}^d \setminus \varepsilon \omega$.

**Proof.** In view of $u_\varepsilon$ and $u_0$ being solutions to (2) and (4), respectively, we note that for $\phi \in H^1_0(\Omega)$ there hold

\[
\int_{\Omega_\varepsilon} A(x/\varepsilon) \nabla u_\varepsilon \nabla \phi dx = \int_{\Omega_\varepsilon} F \phi dx = \int_{\Omega} l^+_\varepsilon F \phi dx, \\
\int_{\Omega} \hat{A} \nabla u_0 \nabla \phi dx = \int_{\Omega} F \phi dx,
\]

where we use the fact that $\phi \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ in (17). Combining the definition of $w_\varepsilon$ and equalities (17) and (18), we see that

\[
\begin{aligned}
\int_{\Omega_\varepsilon} A^\varepsilon \nabla w_\varepsilon \nabla \phi dx & = \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla \phi dx - \int_{\Omega_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla \phi dx \\
& = \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla \phi dx - \int_{\Omega} \hat{A} \nabla u_0 \nabla \phi dx + \int_{\Omega} \theta \hat{A} \nabla u_0 \nabla \phi dx - \int_{\Omega_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla \phi dx \\
& = \int_{\Omega} l^+_\varepsilon F \phi dx - \int_{\Omega} \theta F \phi dx + \int_{\Omega} \theta \hat{A} \nabla u_0 \nabla \phi dx - \int_{\Omega} \theta \hat{A} \varphi \nabla \phi dx + \int_{\Omega} \theta \hat{A} \varphi \nabla \phi dx \\
& - \int_{\Omega} l^+_\varepsilon A^\varepsilon \nabla \chi_\varepsilon \varphi \nabla \phi dx - \varepsilon \int_{\Omega_\varepsilon} l^+_\varepsilon A^\varepsilon \chi_\varepsilon \nabla \varphi \nabla \phi dx
\end{aligned}
\]
By a routine calculation, the left-hand side of (19) is equal to
\[
\int_\Omega (l^+ - \theta) F \phi dx + \int_\Omega (\tilde{\theta} \tilde{A} - l^+ A^\varepsilon)(\nabla u_0 - \varphi) \nabla \phi dx \\
+ \int_\Omega (\tilde{\theta} \tilde{A} - l^+ A^\varepsilon - l^+ A^\varepsilon \nabla \tilde{\chi}_\varepsilon) \varphi \nabla \phi dx - \varepsilon \int_\Omega l^+ A^\varepsilon \tilde{\chi}_\varepsilon \nabla \varphi \nabla \phi dx,
\]
and this completes the proof. \(\square\)

**Lemma 3.3.** Let \(\Omega \subset \mathbb{R}^d\) be a bounded Lipschitz domain. Suppose that \(u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)\) and \(u_0 \in H^1(\Omega; \mathbb{R}^d)\) are weak solutions of (2) and (4), respectively. Then we obtain
\[
\|\nabla w_\varepsilon\|_{L^2(\Omega)} \leq C \left\{ \varepsilon \left( \|F\|_{L^2(\Omega)} + \|E \nabla \varphi\|_{L^2(\Omega)} + \|l^+ \tilde{\chi}_\varepsilon \nabla \varphi\|_{L^2(\Omega)} \right) \\
+ \|\nabla u_0 - \varphi\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^2(O_{2\varepsilon})} \right\},
\]
where \(\tilde{\chi}_\varepsilon\) is defined as in Lemma 3.2, and the constant \(C\) depends on \(\mu_0, \mu_1, d\) and \(\omega\).

**Proof.** Firstly, we point out that it’s easy to verify that \(w_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon; \mathbb{R}^d)\). By the extension operator in Lemma 2.3, we can extend \(w_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon; \mathbb{R}^d)\) to \(\tilde{w}_\varepsilon \in H^1_0(\Omega_0; \mathbb{R}^d)\). Then we take \(w_\varepsilon\) and \(\psi'_\varepsilon \tilde{w}_\varepsilon\) as the test function in (17) and (18), respectively. By a similar calculation as we did in Lemma 3.2, we obtain
\[
\int_{\Omega_\varepsilon} A(x/\varepsilon) \nabla w_\varepsilon \nabla w_\varepsilon dx = \int_{\Omega} (l^+ + \theta \psi'_\varepsilon) \tilde{F} \tilde{w}_\varepsilon dx - \theta \int_{\Omega} \tilde{A} \nabla u_0 \nabla [(1 - \psi'_\varepsilon) \tilde{w}_\varepsilon] dx \\
+ \int_{\Omega_\varepsilon} \tilde{l}^+ A(x/\varepsilon) \nabla \varphi - \nabla u_0 \nabla \tilde{w}_\varepsilon dx \\
+ \int_{\Omega} \tilde{\theta} \tilde{A} - \tilde{l}^+ A(x/\varepsilon) - l^+ A(x/\varepsilon) \nabla \tilde{\chi}_\varepsilon \nabla \varphi \tilde{w}_\varepsilon dx \\
- \varepsilon \int_{\Omega} \tilde{l}^+ A(x/\varepsilon) \tilde{\chi}_\varepsilon \nabla \varphi \nabla \tilde{w}_\varepsilon dx
\]
=:I_1 + I_2 + I_3 + I_4 + I_5.

We will compute each \(I_i\) for \(i = 1, 2, 3, 4, 5\). In terms of \(I_1\), we have
\[
I_1 = \int_{\Omega} (l^+ + \theta) \tilde{F} \tilde{w}_\varepsilon \psi'_\varepsilon dx + \int_{\Omega} (1 - \psi'_\varepsilon) l^+ \tilde{F} \tilde{w}_\varepsilon dx :: I_{11} + I_{12}.
\]
Since \(\text{supp}(1 - \psi'_\varepsilon) = O_{2\varepsilon}\) and Lemma 2.5, we have
\[
|I_{12}| \leq \int_{O_{2\varepsilon}} \|F\|_{L^2(O_{2\varepsilon})} \|\tilde{w}_\varepsilon\|_{L^2(O_{2\varepsilon})} \leq C\varepsilon \|F\|_{L^2(\Omega)} \|\nabla \tilde{w}_\varepsilon\|_{L^2(\Omega)}.
\]
To deal with the first term \(I_{11}\), we consider the equation
\[
\begin{cases}
-\Delta \Psi(y) = l^+(y) - \theta & \text{in } Y, \\
\int_Y \Psi dy = 0, & \Psi \in H^1_{\text{per}}(Y).
\end{cases}
\]
According to $\int_Y (l^+(y) - \theta) dy = 0$, it’s known that (21) has a solution $\Psi \in H^1_{per}(Y)$, and by Schauder’s estimates we obtain $|\nabla \Psi|_{C^{0,\alpha}(Y)} \leq C\|l^+ - \theta\|_{L^\infty(Y)}$. This gives

$$|I_{11}| = | - \int_\Omega \Delta_y \Psi(F\bar{w}_\varepsilon\psi'_\varepsilon) dx| = | - \varepsilon \int_\Omega \text{div}_x(\nabla_y \Psi)(F\bar{w}_\varepsilon\psi'_\varepsilon) dx|$$

$$= \varepsilon \int_\Omega \nabla_y \Psi(\nabla(\bar{w}_\varepsilon\psi'_\varepsilon)) dx \leq C\varepsilon \|\nabla F\|_{H^{-1}(\Omega)}\|\bar{w}_\varepsilon\psi'_\varepsilon\|_{H^1(\Omega)} + \varepsilon \|F\|_{L^2(\Omega)}\|\nabla(\bar{w}_\varepsilon\psi'_\varepsilon)\|_{L^2(\Omega)}$$

$$\leq C\varepsilon \|F\|_{L^2(\Omega)}\|\nabla(\bar{w}_\varepsilon\psi'_\varepsilon)\|_{L^2(\Omega)} + C\varepsilon \|F\|_{L^2(\Omega)}\|\nabla \bar{w}_\varepsilon\|_{L^2(\Omega)},$$

where $y = x/\varepsilon$, we use the fact that $\|\nabla F\|_{H^{-1}(\Omega)} \leq C\|F\|_{L^2(\Omega)}$ and Lemma 2.5 in the last two inequalities. Hence, $|I_1| \leq C\varepsilon \|F\|_{L^2(\Omega)}\|\nabla \bar{w}_\varepsilon\|_{L^2(\Omega)}$.

By the properties of $\hat{A}(\xi)$ in Lemma 2.7, we see that

$$|I_2| = \left| \theta \int_\Omega \hat{A}\nabla u_0\nabla[(1 - \psi'_\varepsilon)\bar{w}_\varepsilon] dx \right| \leq C \int_\Omega \|\nabla u_0\|\|\nabla[(1 - \psi'_\varepsilon)\bar{w}_\varepsilon]\| dx$$

$$\leq C\|\nabla u_0\|_{L^2(O_{2\varepsilon})}\|\nabla \bar{w}_\varepsilon\|_{L^2(\Omega)},$$

where we use Hölder’s inequality, $\text{supp}(1 - \psi'_\varepsilon) = O_{2\varepsilon}$ and Lemma 2.5 in the last inequality. Due to the boundedness of $A(x/\varepsilon)$ and $\hat{A}$, it follows from Hölder’s inequality that

$$|I_3| \leq C\|\nabla u_0 - \varphi\|_{L^2(\Omega)}\|\nabla \bar{w}_\varepsilon\|_{L^2(\Omega)}.$$

If setting $b(y) = \theta \hat{A} - l^+ A(y) - l^+ A(y)\nabla \hat{X}$, then it follows from Lemma 2.6 that

$$|I_4| = \left| \int_\Omega b(x/\varepsilon)\varphi \nabla \bar{w}_\varepsilon dx \right| = \varepsilon \int_\Omega \frac{\partial}{\partial x_k} \{E_{kij}(x/\varepsilon)\} \varphi \frac{\partial \bar{w}_\varepsilon}{\partial x_i} dx$$

$$= \varepsilon \int_\Omega E_{kij} \frac{\partial}{\partial x_k} \left( \varphi \frac{\partial \bar{w}_\varepsilon}{\partial x_i} \right) dx = \varepsilon \int_\Omega E_{kij} \frac{\partial \varphi_j}{\partial x_k} \frac{\partial \bar{w}_\varepsilon}{\partial x_i} dx$$

$$\leq \varepsilon \|E_{kij}\|_{L^2(\Omega)}\|\nabla \bar{w}_\varepsilon\|_{L^2(\Omega)},$$

where we employ the anti-symmetric property of $E$ in the fourth step. Then we turn to the last term $I_5$, and acquire that

$$|I_5| = \varepsilon \int_\Omega l^+_{\varepsilon}(x/\varepsilon)\hat{X}_\varepsilon \nabla \varphi \nabla \bar{w}_\varepsilon dx \leq C\varepsilon \|l^+_{\varepsilon}\|_{L^\infty(\Omega)}\|\nabla \varphi\|_{L^2(\Omega)}\|\nabla \bar{w}_\varepsilon\|_{L^2(\Omega)}.$$

Combining the above estimates for $I_i$ with $i = 1, 2, 3, 4, 5$ and the assumption (3) and Lemmas 2.3, 2.4, we consequently obtain the desired estimate (20).

**Lemma 3.4.** Assume the same conditions as in Lemma 3.3, we have the following estimates

$$\|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\left\{ \|\nabla u_0\|_{L^2(O_{2\varepsilon})} + \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega\setminus O_{2\varepsilon})} + \varepsilon \|F\|_{L^2(\Omega)} \right\}$$

and

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon)} \leq Cr_0\left\{ \|\nabla u_0\|_{L^2(O_{2\varepsilon})} + \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega\setminus O_{2\varepsilon})} + \varepsilon \|F\|_{L^2(\Omega)} \right\},$$

where $r_0 = \text{diam}(\Omega)$, and $C$ depends only on $\mu_0, \mu_1, d$ and $\omega$. □
Proof. According to Lemma 3.3, to see the estimate (22), it suffices to show \( \|E\nabla \varphi\|_{L^2(\Omega)}, \|I^1_\varepsilon \chi \nabla \varphi\|_{L^2(\Omega)} \) and \( \|\nabla u_0 - \varphi\|_{L^2(\Omega)} \). From Lemma 2.2 and the properties of cut-off functions in (14), it follows that

\[
\|E\nabla \varphi\|_{L^2(\Omega)} + \|I^1_\varepsilon \chi \nabla \varphi\|_{L^2(\Omega)} \leq C \|\nabla^2 u_0\|_{L^2(\Omega;\mathbb{R}^d)} + C\varepsilon^{-1}\|\nabla u_0\|_{L^2(\Omega;\mathbb{R}^d)}.
\]

By the same token, we also have the following inequalities

\[
\|\nabla u_0 - \varphi\|_{L^2(\Omega)} \leq \|\psi_\varepsilon \nabla u_0 - S_\varepsilon (\psi_\varepsilon \nabla u_0)\|_{L^2(\Omega)} + \|(1 - \psi_\varepsilon) \nabla u_0\|_{L^2(\Omega)}
\]

\[
\leq C\varepsilon\|\nabla (\psi_\varepsilon \nabla u_0)\|_{L^2(\Omega)} + C\|(1 - \psi_\varepsilon) \nabla u_0\|_{L^2(\Omega)}
\]

\[
\leq C\varepsilon\|\nabla^2 u_0\|_{L^2(\Omega;\mathbb{R}^d)} + C\|\nabla u_0\|_{L^2(\Omega;\mathbb{R}^d)}.
\]

Thus, we have proved the estimate (22). Then we turn to see (23), and on account of Lemma 2.2 and Poincaré’s inequality, there holds

\[
\|\chi \varphi\|_{L^2(\Omega;\varepsilon)} \leq C\|\varphi\|_{L^2(\Omega)} \leq C\|\psi_\varepsilon \nabla u_0\|_{L^2(\Omega)}
\]

\[
\leq C\varepsilon\|\nabla u_0\|_{L^2(\Omega;\mathbb{R}^d)} + \|\nabla^2 u_0\|_{L^2(\Omega;\mathbb{R}^d)}),
\]

and we have completed the proof.

Lemma 3.5. Suppose the same conditions as in Theorem 3.1. Then we have

\[
\|\nabla u_0\|_{L^2(\Omega;\varepsilon)} \leq C\varepsilon^{-1/2}\left(\|g\|_{H^1(\partial\Omega)} + \|F\|_{L^2(\Omega)}\right),
\]

\[
\|\nabla^2 u_0\|_{L^2(\Omega;\mathbb{R}^d)} \leq C\varepsilon^{-1/2}\left(\|g\|_{H^1(\partial\Omega)} + \|F\|_{L^2(\Omega)}\right).
\]

Proof. The proof is standard and may be found in [12, Theorem 2.16]. It’s also quite similar to that given later in Lemma 4.3.

Proof of Theorem 3.1. Combining Lemmas 3.4 and 3.5, we may have

\[
\|e(w_\varepsilon)\|_{L^2(\Omega;\varepsilon)} \lesssim \varepsilon^{1/2}(\|F\|_{L^2(\Omega)} + \|g\|_{H^1(\partial\Omega)}),
\]

and

\[
\|u_\varepsilon - u_0\|_{L^2(\Omega;\varepsilon)} \lesssim \varepsilon^{1/2}(\|F\|_{L^2(\Omega)} + \|g\|_{H^1(\partial\Omega)}).
\]

Thus, Lemma 2.4 gives the desired estimate, and we have completed the proof.

4 Sharp error estimates

In this section, we will calculate the optimal error estimates for \( \mathcal{L}_\varepsilon \) by the duality argument. The dual operator of \( \mathcal{L}_\varepsilon = -\nabla \cdot A(\cdot/\varepsilon) \nabla \) is given by \( \mathcal{L}_\varepsilon^* = -\nabla \cdot A^*(\cdot/\varepsilon) \nabla \), while there holds \( A^* = A \) according to (3). In order to show the duality argument is independent of the symmetry condition, we still keep the notation \( \mathcal{L}_\varepsilon^* \) in the proof. For any \( \Phi \in L^2(\Omega;\mathbb{R}^d) \), we have the adjoint problem:

\[
\begin{cases}
\mathcal{L}_\varepsilon^*(\phi_\varepsilon) = \Phi & \text{in } \Omega_\varepsilon, \\
\vec{n} A_\varepsilon^* \nabla \phi_\varepsilon = 0 & \text{on } S_\varepsilon, \\
\phi_\varepsilon = 0 & \text{on } \Gamma_\varepsilon.
\end{cases}
\]
The corresponding homogenized equation is given by
\[
\begin{cases}
L^*_0 \phi_0 = -\text{div}(A^* \nabla \phi_0) = \Phi & \text{in } \Omega, \\
\phi_0 = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where the matrix \( \hat{A}^* \) is defined by
\[
\hat{A}^*_{ij} = \int_{Y \cap \omega} a^{*\alpha\beta}(y) \frac{\partial X^{*\gamma\delta}}{\partial y_k} dy,
\]
in which \( X^{*\beta}_j = \{X^{*\gamma\beta}_j\}_{1 \leq \gamma \leq d} \) is the weak solution to the following cell problem
\[
\begin{aligned}
&\text{div}\{A^*(y)(\nabla X^{*\beta}_j)\} = 0 & \text{in } Y \cap \omega, \\
&\hat{n} \cdot A^*(y) \nabla X^{*\beta}_j = 0 & \text{on } Y \cap \partial \omega, \\
&X^{*\beta}_j - y_j \epsilon^\beta = \chi^{*\beta}_j \in H^1_{\text{per}}(\omega; \mathbb{R}^d), & \int_{Y \cap \omega} \chi^{*\beta}_j dy = 0.
\end{aligned}
\]
Setting \( z_\epsilon = \phi_\epsilon - \phi_0 - \epsilon \chi^*(x/\epsilon) \varphi^* \) and \( \varphi^* = S^*_\epsilon(\psi_\epsilon \nabla \phi_0) \), by the results in Section 3, there holds
\[
\| \nabla z_\epsilon \|_{L^2(\Omega)} \lesssim \epsilon^{1/2} \| \Phi \|_{L^2(\Omega)}.
\]
Consider the quantity
\[
\int_{\Omega_\epsilon} w_\epsilon \Phi dx = \int_{\Omega_\epsilon} w_\epsilon \left[ -\nabla_{x_\epsilon}(A_{ji}(x/\epsilon) \nabla_{x_\epsilon} \phi_\epsilon) \right] dx = \int_{\Omega_\epsilon} \nabla_{x_\epsilon} w_\epsilon (A_{ji}(x/\epsilon) \nabla_{x_\epsilon} \phi_\epsilon) dx \\
= \int_{\Omega_\epsilon} A_{ij}(x/\epsilon) \nabla_{x_\epsilon} w_\epsilon \nabla_{x_\epsilon} \phi_\epsilon dx \\
= \int_{\Omega_\epsilon} A(x/\epsilon) \nabla w_\epsilon \nabla [z_\epsilon + \phi_0 + \epsilon \chi^*_\epsilon \varphi^*] dx,
\]
where \( w_\epsilon = u_\epsilon - v_\epsilon, \ v_\epsilon = u_0 + \epsilon \chi_\epsilon \varphi \) and \( \varphi = S^*_\epsilon(\psi_\epsilon \nabla u_0) \) as we defined in Section 3.

**Remark 4.1.** Due to the fact that \( \| w_\epsilon \|_{L^2(\Omega_\epsilon)} = \| l^+ w_\epsilon \|_{L^2(\Omega)} \), we see that
\[
\| w_\epsilon \|_{L^2(\Omega_\epsilon)} = \sup_{\Phi \in L^2(\Omega); \Phi \neq 0} \frac{\int_{\Omega_\epsilon} w_\epsilon \Phi dx}{\| \Phi \|_{L^2(\Omega)}}.
\]

**Lemma 4.2.** Assume that \( w_\epsilon = u_\epsilon - v_\epsilon \) with \( v_\epsilon = u_0 + \epsilon \chi_\epsilon \varphi \), where \( \varphi = S^*_\epsilon(\psi_\epsilon \nabla u_0), \ \varphi^* = S^*_\epsilon(\psi_\epsilon \nabla \phi_0) \). Let \( \delta(x) = \text{dist}(x, \partial \Omega) \). Then we have
\[
| \int_{\Omega_\epsilon} A^\epsilon \nabla w_\epsilon \nabla [\phi_0 + \epsilon \chi^*_\epsilon \varphi^*] dx |
\lesssim \epsilon \| F \|_{L^2(\Omega)} \left\{ \| \nabla \phi_0 \|_{L^2(\Omega)} + \epsilon \| \nabla^2 \phi_0 \|_{L^2(\Sigma_\epsilon)} \right\}
\]
\[
+ \| \nabla u_0 \|_{L^2(\Omega_\epsilon)} \left\{ \| \nabla \phi_0 \|_{L^2(\Omega_\epsilon)} + \epsilon \| \nabla^2 \phi_0 \|_{L^2(\Sigma_\epsilon)} + \epsilon^{3/2} \left( \int_{\Sigma_\epsilon} |\nabla \phi_0|^2 dx \right)^{1/2} \right\}
\]
\[
+ \left( \int_{\Sigma_\epsilon} |\nabla^2 u_0|^2 \delta(x) dx \right)^{1/2} \left\{ \epsilon \left( \int_{\Sigma_\epsilon} |\nabla \phi_0|^2 dx \right)^{1/2} + \epsilon^{3/2} \| \nabla^2 \phi_0 \|_{L^2(\Sigma_\epsilon)} \right\},
\]
where the notation \( \lesssim \) means \( \leq C \) with \( C \) independent of \( \epsilon \).
Proof. Our proof is inspired by [17]. If we choose \( \phi_0 \) and \( \varepsilon \chi_2^* \varphi^* \) as the test functions in (16), respectively, we have

\[
\int_{\Omega_\varepsilon} A^2 \nabla w_\varepsilon \nabla \phi_0 \, dx =: I_1 + I_2 + I_3 + I_4;
\]

\[
\int_{\Omega_\varepsilon} A^2 \nabla w_\varepsilon \nabla (\varepsilon \chi_2^* \varphi^*) \, dx =: J_1 + J_2 + J_3 + J_4.
\]

In the following, we will estimate each term \( I_i, J_i \) with \( i = 1, \cdots, 4 \). For \( I_1 \) and \( J_1 \), we deal with them by the same strategy that we employed to handle the first term in Lemma 3.3, and it follows that

\[
|I_1| \lesssim \varepsilon \left\{ \| \nabla F \|_{H^{-1}(\Omega)} \| \phi_0 \|_{H^1_2(\Omega)} + \| F \|_{L^2(\Omega)} \| \nabla \phi_0 \|_{L^2(\Omega)} \right\} \lesssim \varepsilon \| F \|_{L^2(\Omega)} \| \nabla \phi_0 \|_{L^2(\Omega)},
\]

\[
|J_1| \lesssim \varepsilon \| F \|_{L^2(\Omega)} \| \nabla (\varepsilon \chi_2^* \varphi^*) \|_{L^2(\Omega)}.
\]

According to Lemmas 2.2 and 2.5, we acquire that

\[
\| \nabla (\varepsilon \chi_2^* \varphi^*) \|_{L^2(\Omega)} \lesssim \| \varphi^* \|_{L^2(\Omega)} + \varepsilon \| \nabla \varphi^* \|_{L^2(\Omega)} \lesssim \| \nabla \phi_0 \|_{L^2(\Omega)} + \varepsilon \| \nabla^2 \phi_0 \|_{L^2(\Sigma_0^c)}, \tag{33}
\]

and there holds

\[
|I_1| + |J_1| \lesssim \varepsilon \| F \|_{L^2(\Omega)} \left( \| \nabla \phi_0 \|_{L^2(\Omega)} + \varepsilon \| \nabla^2 \phi_0 \|_{L^2(\Sigma_0^c)} \right).
\]

Then we turn to consider \( I_2 \) and \( J_2 \). We note that

\[
|I_2| \leq \int_{\Omega} |\hat{\theta} - l^+ \varepsilon^2| \| \nabla u_0 - \varphi \| \| \nabla \phi_0 \| dx \lesssim \int_{\Omega} |\nabla u_0 - \varphi | \| \nabla \phi_0 \| dx
\]

\[
\lesssim \int_{\Omega} \| \nabla u_0 (1 - \psi_\varepsilon) \| \| \nabla \phi_0 \| dx + \int_{\Omega} |\nabla u_0 \psi_\varepsilon - \varphi | \| \nabla \phi_0 \| dx =: I_{21} + I_{22}.
\]

By Hölder’s inequality, it follows that

\[
I_{21} \leq \int_{\Omega_{0, \varepsilon}} \| \nabla u_0 \| \| \nabla \phi_0 \| dx \leq \| \nabla u_0 \|_{L^2(\Omega_{0, \varepsilon})} \| \nabla \phi_0 \|_{L^2(\Omega_{0, \varepsilon})},
\]

\[
I_{22} \leq \left( \int_{\Sigma_0^c} |\nabla u_0 \psi_\varepsilon - \varphi |^2 \delta(x) dx \right)^{1/2} \left( \int_{\Sigma_0^c} |\nabla \phi_0 |^2 \frac{dx}{\delta(x)} \right)^{1/2},
\]

where we employ the fact that \( \text{supp} \varphi \subset \Sigma_0^c \) in the last inequality. By Lemmas 2.2 and [17, Lemma 3.3], we have

\[
\left( \int_{\Sigma_0^c} \| \nabla u_0 \psi_\varepsilon - \varphi |^2 \delta(x) dx \right)^{1/2} \leq \varepsilon \left( \int_{\Sigma_0^c} |\nabla (\nabla u_0 \psi_\varepsilon) |^2 \delta(x) dx \right)^{1/2}
\]

\[
\lesssim \varepsilon \left( \int_{\Sigma_0^c} \| \nabla^2 u_0 \|^2 \delta(x) dx \right)^{1/2} + \varepsilon \left( \int_{\Omega_{0, \varepsilon} \setminus \Omega_{\varepsilon}} |\varepsilon^{-1} \nabla u_0 |^2 \delta(x) dx \right)^{1/2}, \tag{34}
\]

\[
\lesssim \varepsilon \left( \int_{\Sigma_0^c} \| \nabla^2 u_0 \|^2 \delta(x) dx \right)^{1/2} + \varepsilon \frac{1}{2} \| \nabla u_0 \|_{L^2(\Omega_{0, \varepsilon})},
\]

where we use the fact that \( \varepsilon \leq \delta(x) \leq 4 \varepsilon \) in \( \Omega_{0, \varepsilon} \setminus \Omega_{\varepsilon} \). Combining the above inequalities, we can derive

\[
|I_2| \lesssim \| \nabla u_0 \|_{L^2(\Omega_{0, \varepsilon})} \left( \| \nabla \phi_0 \|_{L^2(\Omega_{0, \varepsilon})} + \varepsilon \frac{1}{2} \left( \int_{\Sigma_0^c} |\nabla \phi_0 |^2 \frac{dx}{\delta(x)} \right)^{1/2} \right)
\]

\[
+ \varepsilon \left( \int_{\Sigma_0^c} \| \nabla^2 u_0 \|^2 \delta(x) dx \right)^{1/2} \left( \int_{\Sigma_0^c} |\nabla \phi_0 |^2 \frac{dx}{\delta(x)} \right)^{1/2}.
\]
According to (16), one may have

\[ |J_2| \lesssim \int_\Omega |\nabla u_0 - \varphi||\nabla (\varepsilon \check{\chi}_\varepsilon^* \varphi^*)|dx \]

\[ \leq \int_\Omega |\nabla u_0 (1 - \psi_\varepsilon)||\nabla (\varepsilon \check{\chi}_\varepsilon^* \varphi^*)|dx + \int_\Omega |\nabla u_0 \psi_\varepsilon - \varphi||\nabla (\varepsilon \check{\chi}_\varepsilon^* \varphi^*)|dx =: J_{21} + J_{22}. \]

Moreover, it follows that

\[ J_{21} \leq \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}\|\nabla (\varepsilon \check{\chi}_\varepsilon^* \varphi^*)\|_{L^2(\Omega_{4\varepsilon})} \leq \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} \left[ \|\varphi^*\|_{L^2(\Omega_{4\varepsilon})} + \varepsilon \|\nabla \varphi^*\|_{L^2(\Omega_{4\varepsilon})} \right] \]

\[ \lesssim \varepsilon \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}\|\nabla \varphi^*\|_{L^2(\Omega)}, \]

where we use Lemma 2.5 in the last step, and on account of (33),

\[ J_{21} \lesssim \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} \left( \|\nabla \phi_0\|_{L^2(\Omega_{4\varepsilon})} + \varepsilon \|\nabla^2 \phi_0\|_{L^2(\Sigma_{4\varepsilon})} \right). \tag{35} \]

By Hölder’s inequality,

\[ J_{22} \leq \left( \int_{\Sigma_{2\varepsilon}} |\nabla u_0 \psi_\varepsilon - \varphi|^2 \delta(x) dx \right)^{\frac{1}{2}} \left( \int_{\Sigma_{2\varepsilon}} |\nabla (\varepsilon \check{\chi}_\varepsilon^* \varphi^*)|^2 \frac{dx}{\delta(x)} \right)^{\frac{1}{2}}, \]

and

\[ \left( \int_{\Sigma_{2\varepsilon}} |\nabla (\varepsilon \check{\chi}_\varepsilon^* \varphi^*)|^2 \frac{dx}{\delta(x)} \right)^{\frac{1}{2}} \lesssim \left( \int_{\Sigma_{2\varepsilon}} |\nabla \check{\chi}_\varepsilon^* \varphi^*|^2 \frac{dx}{\delta(x)} \right)^{\frac{1}{2}} + \left( \int_{\Sigma_{2\varepsilon}} |\varepsilon \check{\chi}_\varepsilon \nabla \varphi^*|^2 \frac{dx}{\delta(x)} \right)^{\frac{1}{2}} \]

\[ \lesssim \left( \int_{\Sigma_{\varepsilon}} |\nabla \phi_0|^2 \frac{dx}{\delta(x)} \right)^{\frac{1}{2}} + \varepsilon^{1/2} \|\nabla^2 \phi_0\|_{L^2(\Sigma_{\varepsilon})}, \tag{36} \]

where we use the fact that \( \delta(x) \geq \varepsilon \) in \( \Sigma_{\varepsilon} \), Lemma 2.2 and [17, Lemma 3.2] in the last equality. Collecting (34), (35) and (36), we arrive at

\[ |J_2| \leq \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} \left\{ \|\nabla \phi_0\|_{L^2(\Omega_{4\varepsilon})} + \varepsilon \|\nabla^2 \phi_0\|_{L^2(\Sigma_{\varepsilon})} + \varepsilon^{1/2} \left( \int_{\Sigma_{\varepsilon}} |\nabla \phi_0|^2 \frac{dx}{\delta(x)} \right)^{\frac{1}{2}} \right\} \]

\[ + \varepsilon \left( \int_{\Sigma_{\varepsilon}} |\nabla^2 u_0|^2 \delta(x) dx \right)^{\frac{1}{2}} \left\{ \left( \int_{\Sigma_{\varepsilon}} |\nabla \phi_0|^2 \frac{dx}{\delta(x)} \right)^{\frac{1}{2}} + \varepsilon^{1/2} \|\nabla^2 \phi_0\|_{L^2(\Sigma_{\varepsilon})} \right\}. \]

Next, a similar argument as we did for \( I_4 \) in Lemma 3.3 leads to

\[ |I_3| = \varepsilon \int_\Omega E_{kij} \frac{\partial \varphi_j}{\partial x_k} \frac{\partial \phi_0}{\partial x_i} dx \lesssim \varepsilon \int_\Omega |\nabla \varphi||\nabla \phi_0| dx \]

\[ \lesssim \varepsilon \left( \int_{\Sigma_{\varepsilon}} |\nabla (\psi_\varepsilon \nabla u_0)|^2 \delta(x) dx \right)^{1/2} \left( \int_{\Sigma_{\varepsilon}} |\nabla \phi_0|^2 \frac{dx}{\delta(x)} \right)^{1/2} \]

\[ \lesssim \left\{ \varepsilon \left( \int_{\Sigma_{\varepsilon}} |\nabla^2 u_0|^2 \delta(x) dx \right)^{\frac{1}{2}} + \varepsilon^{1/2} \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} \right\} \left( \int_{\Sigma_{\varepsilon}} |\nabla \phi_0|^2 \frac{dx}{\delta(x)} \right)^{1/2}. \]
in which the third step comes from [17, Lemma 3.2] and the last step follows from (34). Similarly, Combining (34) and (36) with [17, Lemma 3.2], we may obtain
\[
|J_3| = |\varepsilon \int_{\Omega} E_{ki} \frac{\partial \varphi_i}{\partial x_k} \partial (\varepsilon \chi^*_x \varphi^*) dx| \lesssim |\varepsilon \int_{\Omega} |E||\nabla \varphi||\nabla (\varepsilon \chi^*_x \varphi^*)| dx
\]
\[
\lesssim \varepsilon \left( \int_{\Sigma_{\varepsilon}} |\nabla (\psi^* \varphi)|^2 \delta(x) dx \right)^{1/2} \left( \int_{\Sigma_{\varepsilon}} |\nabla (\varepsilon \chi^*_x \varphi^*)|^2 \frac{dx}{\delta(x)} \right)^{1/2}
\]
\[
\lesssim \left[ \varepsilon \left( \int_{\Sigma_{\varepsilon}} |\nabla^2 u|^2 \delta(x) dx \right) + \varepsilon^2 \int_{\Omega} |\nabla \phi_0|^2 L^2(\Omega) \right] \left( \int_{\Sigma_{\varepsilon}} |\nabla \phi_0|^2 \frac{dx}{\delta(x)} \right)^{1/2} + \varepsilon^{1/2} |\nabla^2 \phi_0| L^2(\Sigma_{\varepsilon}).
\]
In view of the facts
\[
|I_4| = |\varepsilon \int_{\Omega} \varepsilon A^x \chi^*_x \nabla \varphi \nabla \phi_0 dx| \lesssim |\varepsilon \int_{\Omega} |\chi^*_x||\nabla \varphi||\nabla \phi_0| dx
\]
and
\[
|J_4| = |\varepsilon \int_{\Omega} \varepsilon A^x \chi^*_x \nabla \varphi (\varepsilon \chi^*_x \varphi^*) dx| \lesssim |\varepsilon \int_{\Omega} |\chi^*_x||\nabla \varphi||\nabla (\varepsilon \chi^*_x \varphi^*)| dx,
\]
it’s known that the terms $I_4$ and $J_4$ will be controlled by the same upper bound of $I_3$ and $J_3$. Combined the estimates for $I_i$ and $J_i$, we consequently acquired the desired estimate (32) and the proof is complete.

**Lemma 4.3.** Assume that $u_0$, $\phi_0$ are solutions to the homogenized equations (4) and (27), respectively. Then there hold
\[
\int_{\Sigma_{\varepsilon}} |\nabla \phi_0|^2 \frac{dx}{\delta(x)} \leq C \ln(1/\varepsilon) ||\Phi||^2 L^2(\Omega)
\]
and
\[
\int_{\Sigma_{\varepsilon}} |\nabla^2 u_0|^2 \delta(x) dx \leq C \ln(1/\varepsilon) \left( ||g||^2 H^{1,1}(\Omega) + ||F||^2 L^2(\Omega) \right),
\]
where the constant $C$ depends on $\mu_0, \mu_1, d$ and $\Omega$.

**Proof.** Here we just verify the case for $\int_{\Sigma_{\varepsilon}} |\nabla^2 u_0|^2 \delta(x) dx \lesssim \varepsilon$, and the estimate (37) may be derived in a similar way. First, we decompose $u_0$ into two parts: $v$ and $w$, which satisfy the following systems:
\[
-\nabla \cdot (\hat{A} \nabla v) = \tilde{F} \quad \text{in} \quad \mathbb{R}^d,
\]
and
\[
\begin{cases}
-\nabla \cdot (\hat{A} \nabla w) = 0 & \text{in} \Omega, \\
w = g - v & \text{on} \partial \Omega,
\end{cases}
\]
respectively, in which $\tilde{F} = F$ in $\Omega$ and $\tilde{F} = 0$ in $\mathbb{R}^d \setminus \Omega$. In terms of (39), by the well-known fractional integral estimates and singular integral estimates for $v$ (see for example [14]), we have that
\[
||\nabla v||_{L^p(\mathbb{R}^d)} + ||\nabla^2 v||_{L^p(\mathbb{R}^d)} \leq C ||\tilde{F}||_{L^p(\mathbb{R}^d)} \quad \text{for} \quad 1 < p < d, \quad \frac{1}{p} = \frac{1}{p'} - \frac{1}{d}
\]
It follows from the divergence theorem that
\[
\int_{\partial \Omega} |\nabla v|^2 dS \leq C \left[ \int_{\Omega} |\nabla v|^2 dx + \int_{\Omega} |\nabla v||\nabla^2 v| dx \right]
\]
\[
\lesssim ||\nabla v||_{L^p(\Omega)}^2 + ||\nabla^2 v||_{L^p(\Omega)}^2.
\]
where we use Hölder’s inequality. Then we turn to the equation (40). It follows from the nontangential maximal function estimates for $L^2$-regular problem in Lipschitz domains (see [5])

\[
\| (\nabla w)^* \|_{L^2(\partial \Omega)} \leq C \left( \| \nabla_{\text{tan}} g \|_{L^2(\partial \Omega)} + \| \nabla_{\text{tan}} v \|_{L^2(\partial \Omega)} \right)
\]

\[
\lesssim \| g \|_{H^1(\partial \Omega)} + \| \nabla v \|_{L^2(\partial \Omega)},
\]

where $(\nabla w)^*$ denotes the nontangential maximal function of $\nabla w$. On account of (42), we have

\[
\| (\nabla w)^* \|_{L^2(\partial \Omega)} \lesssim \| g \|_{H^1(\partial \Omega)} + \| \nabla v \|_{L^2(\partial \Omega)} + \| \nabla^2 v \|_{L^2(\partial \Omega)}
\]

\[
\lesssim \| g \|_{H^1(\partial \Omega)} + \| F \|_{L^2(\partial \Omega)},
\]

where we use (41) in the last step.

Notice that $u_0$ solves the equation $-\text{div}(\hat{A} \nabla u_0) = F$ in $\Omega$, the following interior estimates holds for $w$:

\[
|\nabla^2 w(x)| \leq \frac{C}{\delta(x)} \left( \int_{B(x,\delta(x)/8)} |\nabla w|^2 \right)^{1/2}.
\]

Then we have

\[
\int_{\Sigma_\delta} |\nabla^2 u_0|^2 \delta(x) dx \leq \int_{\Sigma_\delta} |\nabla^2 v|^2 \delta(x) dx + \int_{\Sigma_\delta} |\nabla^2 w|^2 \delta(x) dx
\]

\[
\lesssim \| F \|^2_{L^2(\Omega)} + \int_\delta^{r_0} r^{-1} dr \int_{\partial \Omega} |(\nabla w)^*|^2 dS_r + \int_{\Sigma_{\delta_0}} |\nabla w|^2 dx,
\]

\[
\lesssim \| F \|^2_{L^2(\Omega)} + \ln(1/\varepsilon) \| (\nabla w)^* \|^2_{L^2(\partial \Omega)} + \int_{\Sigma_{\delta_0}} |\nabla w|^2 dx.
\]

For the last term in the right-hand side above, one may derive that

\[
\int_{\Sigma_{\delta_0}} |\nabla w|^2 dx \lesssim \| g \|^2_{H^{1/2}(\partial \Omega)} + \| v \|^2_{H^{1/2}(\partial \Omega)} \lesssim \| g \|^2_{H^1(\partial \Omega)} + \| F \|^2_{L^2(\partial \Omega)},
\]

where we use (41) and (42). Consequently, combining the above estimates (44), (45) and (46) will lead to

\[
\int_{\Sigma_\delta} |\nabla^2 u_0|^2 \delta(x) dx \lesssim \ln(1/\varepsilon) \left( \| g \|^2_{H^1(\partial \Omega)} + \| F \|^2_{L^2(\Omega)} \right).
\]

We have completed the proof. \hfill \Box

**Lemma 4.4.** Assume that $w_\varepsilon = u_\varepsilon - v_\varepsilon$ with $v_\varepsilon - u_0 + \varepsilon \chi \varphi$, where $\varphi = S_\varepsilon(\psi_\varepsilon \nabla u_0)$ and $\varphi^* = S_\varepsilon(\psi_\varepsilon \nabla \varphi_0)$. Then there holds

\[
\left| \int_{\Omega_\varepsilon} A^\varepsilon \nabla w_\varepsilon \nabla [\varphi_0 + \varepsilon \chi \varphi^*] dx \right| \lesssim \varepsilon \ln(1/\varepsilon) \| \Phi \|^2_{L^2(\Omega)} \left( \| g \|^2_{H^1(\partial \Omega)} + \| F \|^2_{L^2(\Omega)} \right).
\]

**Proof.** We can arrive at Lemma 4.4 immediately by combining Lemmas 4.2 and 4.3. \hfill \Box

**The proof of Theorem 1.1.** In view of Theorem 3.1 for $w_\varepsilon$ and $z_\varepsilon$, we have

\[
\int_{\Omega_\varepsilon} A(x/\varepsilon) \nabla w_\varepsilon \nabla z_\varepsilon dx \lesssim \varepsilon \| \Phi \|^2_{L^2(\Omega)} \left( \| g \|^2_{H^1(\partial \Omega)} + \| F \|^2_{L^2(\Omega)} \right).
\]
On account of (31), (47), Remark 4.1 and Lemma 4.4, we obtain that
\[ \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \ln(1/\varepsilon) \left( \|g\|_{H^1(\partial \Omega)} + \|F\|_{L^2(\Omega)} \right). \]

It’s not hard to observe that
\[ \|\chi_\varepsilon \varphi\|_{L^2(\Omega_\varepsilon)} \leq \|\nabla u_0\|_{L^2(\Omega)} \leq C \left( \|F\|_{L^2(\Omega)} + \|g\|_{H^1(\partial \Omega)} \right), \]
and the stated estimate (8) follows from
\[ \|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon)} \leq \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon \|\chi_\varepsilon \varphi\|_{L^2(\Omega_\varepsilon)}. \]

To show (9), we will proceed a similar but simpler argument used for (8). In Lemma 4.2, we do not require the weighted inequalities any more. Instead, one may apply the $H^2$ theory to the desired estimate (9) directly, and the proof is complete. \qed

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