A new representation of $\alpha$-openness, $\alpha$-continuity, $\alpha$-irresoluteness, and $\alpha$-compactness in $L$-fuzzy pretopological spaces

Abstract: This paper presents a new representation of $\alpha$-openness, $\alpha$-continuity, $\alpha$-irresoluteness, and $\alpha$-compactness based on $L$-fuzzy $\alpha$-open operators introduced by Nannan and Ruiying [1] and implication operation. The proposed representation extends the properties of $\alpha$-openness, $\alpha$-continuity, $\alpha$-irresoluteness, and $\alpha$-compactness to the setting of $L$-fuzzy pretopological spaces based on graded concepts. Moreover, we introduce and establish the relationships among the new concepts.

Keywords: $L$-fuzzy pretopology, $L$-fuzzy $\alpha$-open operator, $L$-fuzzy $\alpha$-openness degree, $L$-fuzzy $\alpha$-continuity degree, $L$-fuzzy $\alpha$-irresoluteness degree, $L$-fuzzy $\alpha$-compactness degree

PACS: 03E72, 54A40, 54C20

1 Introduction

Continuity is an important concept in topology, which has developed extensively with the emergence of fuzzy mathematics. In [2, 3], Šostak considered the degrees to which a mapping is continuous, open, and closed between two $(L, M)$-fuzzy topological spaces (including the fuzzifying case) for the first time. Subsequently, the degrees of continuity, openness, and closeness of mappings between $L$-fuzzifying topological spaces were discussed in detail by Pang [4]. Later on, Liang and Shi [5] clarified the relationship among these degrees and the degree of compactness and connectedness in the case of $(L, M)$-fuzzy setting.

Recently, Shi [6] measured preopenness and semiopenness of $L$-subset by introducing the concepts of $L$-fuzzy preopen operators and $L$-fuzzy semiopen operators, respectively. In [7], Shi and Li used $L$-fuzzy semiopen operators to introduce and characterize the semicommutativity. Later on in [8] the degree of preconnectedness was introduced with the help of $L$-fuzzy preopen operators. In addition, he used Shi’s operators to define new operators such as $L$-fuzzy semipreopen operators [9] and $L$-fuzzy F-open operators [10]. These
operators have proved to be of great importance in studying the characteristics of many concepts of $L$-fuzzy topology (see [11–14]).

In [1], Nannan and Ruiying introduced $L$-fuzzy $a$-open operators in $L$-fuzzy topological spaces and used it to study $L$-fuzzy $a$-compactness. Moreover, the concept of open cover and $a$-fuzzy $a$-compact are given and its related properties are discussed. Also, the relationship between $L$-fuzzy $a$-compactness and fuzzy $a$-compactness are discussed.

This paper first discusses some important properties of $L$-fuzzy $a$-open operators. It then introduces $a$-openness, $a$-continuity, $a$-irresoluteness, and $a$-compactness degree based on the implication operation and $L$-fuzzy $a$-open operators. Further, some important properties of $a$-openness, $a$-continuity, $a$-irresoluteness, and $a$-compactness degree were extended to the setting of $L$-fuzzy pretopology based on graded concepts. Moreover, it presents a systematic discussion on the relationship among the new concepts.

2 Preliminaries

In the sequel, $X \neq \emptyset$, and $L$ refers to a completely distributive De Morgan algebra (briefly, CDAA). Let $1_L$ and $0_L$ denote the greatest and smallest elements of $L$, respectively. For each $u, v \in L$, the element $u$ is wedge below $v$ [15], written $u \triangleleft v$, if for each $D \subseteq L$, $\bigvee D \geq v$ yields to $w \geq u$ for some $w \in D$. We say the complete lattice $L$ is completely distributive (briefly, CD) if and only if $v = \bigvee \{u \in L \mid u \triangleleft v\}$ for any $v \in L$. A member $u \in L$ is said to be co-prime if $u \leq v \lor w$ yields to $u \leq v$ or $u \leq w$. $P(L)$ and $J(L)$ refer to the family of non-unit prime members and non-zero co-prime members of $L$ respectively. The greatest minimal family and the greatest maximal family of $v \in L$ are denoted by $a(v)$ and $\beta(v)$ respectively. Moreover, $a^*(v) = a(v) \cap J(L)$ and $\beta^*(v) = \beta(v) \cap P(L)$. By $L^X$ we refer to the set of all $L$-subsets on $X$. $2^L$ denotes the collection of all finite sub-collections of $U \subseteq L^X$. Evidently, $L^X$ is a CDAA when it inherits the structure of the lattice $L$ in a natural way, by defining $\bigvee, \bigwedge, \leq$ and $\prime$ pointwisely. Further, $\{x_u \mid u \in J(L)\}$ denotes the collection of non-zero co-primes of $L^X$.

For each CDAA $L$, there exists an implication operation $\mapsto: L \times L \longrightarrow L$ as the right adjoint for the meet operation $\wedge$ is defined by

$$u \mapsto v = \bigvee \{w \in L \mid u \wedge w \leq v\}.$$ 

Further, the operation $\leftrightarrow$ is given by

$$u \leftrightarrow v = (u \mapsto v) \wedge (v \mapsto u).$$

The following lemma lists some important properties of implication operation.

Lemma 2.1. [16] Let $(L, \bigvee, \bigwedge)$ be a CD lattice and $\mapsto$ be the implication operation corresponding to $\wedge$. Then for all $u, v, w \in L$, \{ui\}$_{i \in I}$, and \{vi\}$_{i \in I} \subseteq L$, we have the following statements:

\[(11)\] $(u \mapsto v) \wedge w \iff u \wedge w \leq v$.

\[(12)\] $u \leq v \iff u \mapsto v = 1_L$.

\[(13)\] $(u \mapsto (v \mapsto w)) = (u \wedge v) \mapsto w$.

\[(14)\] $(w \mapsto u) \wedge (u \mapsto v) \leq w \mapsto v$.

\[(15)\] $w \mapsto u \leq (u \mapsto v) \mapsto (w \mapsto v)$.

\[(16)\] $u \mapsto \bigvee_{i \in I} u_i = \bigvee_{i \in I} (u_i \mapsto u_i)$, hence $u \mapsto v \leq u \mapsto w$ whenever $v \leq w$.

\[(17)\] $\forall_{i \in I} u_i \mapsto v = \bigvee_{i \in I} (u_i \mapsto v)$, hence $u \mapsto w \geq v \mapsto w$ whenever $u \leq v$.

An $L$-fuzzy inclusion [17, 18] on $X$ is defined by the function $\zeta: L^X \times L^X \longrightarrow L$, where $\zeta(A_1, A_2) = \bigwedge_{x \in X} (A'_1(x) \vee A'_2(x))$. We shall denote an $L$-fuzzy inclusion by $[A_1 \zeta A_2]$. For each function $f : X \longrightarrow Y$ and $\zeta \subseteq L^L$, the next
equality is defined in [19]:
\[
\bigwedge_{y \in Y} \left\{ f^{-1}(A)(y) \lor \bigvee_{B \in \mathcal{C}} B(y) \right\} = \bigwedge_{x \in X} \left\{ B'(x) \lor \bigvee_{B \in \mathcal{C}} f^{-1}(B)(x) \right\}.
\]

An \( L \)-topological space (briefly, \( L \)-ts) is a pair \((X, \tau)\), where the subfamily \( \tau \subseteq L^X \) contains \( 0_L, 1_L \), and closed for any suprema and finite infs. Elements of \( \tau \) are called open \( L \)-subsets and their complements are called closed \( L \)-subsets. For an \( L \)-subset \( A \) of an \( L \)-topological space \((X, \tau)\) we denote by \( \bar{A} \) and \( A^c \) the closure and the interior of \( A \), respectively.

**Definition 2.2.** [16, 20–22] An \( L \)-fuzzy pretopology is given by the function \( \sigma : L^X \rightarrow L \) satisfies the following statements:

\begin{enumerate}
\item[(\( \mathcal{O}1 \))] \( \sigma(1_L) = \sigma(0_L) = 1_L \).
\item[(\( \mathcal{O}2 \))] \( \sigma \left( \bigvee_{i \in I} A_i \right) \geq \bigwedge_{i \in I} \sigma(A_i), \forall \{A_i\}_{i \in I} \subseteq L^X \).
\end{enumerate}

For any \( L \)-subset \( A \subseteq L^X \), \( \sigma(A) \) refers to the degree of openness of \( A \). \( \sigma^*(A) = \sigma(A') \) is the closeness degree of \( A \). The pair \((X, \sigma)\) is said to be an \( L \)-fuzzy pretopological space (briefly, \( L \)-pfts). A function \( f : (X, \sigma_1) \rightarrow (Y, \sigma_2) \) is said to be \( L \)-fuzzy continuous with respect to \( L \)-pfts’s \((X, \sigma_1)\) and \((Y, \sigma_2)\) if and only if \( \sigma_1(f^{-1}(B)) \geq \sigma_2(B) \) for each \( B \subseteq L^Y \), where \( f^{-1}(B)(x) = B(f(x)) \).

**Definition 2.3.** [1] Let \( \sigma \) be an \( L \)-fpt on \( X \) and let the mapping \( \mathcal{A} : L^X \rightarrow L \) defined as follows:

\[
\mathcal{A}(A) = \bigvee_{B \subseteq A} \left\{ \sigma(B) \land \bigvee_{x_u \in \bar{A}} \left\{ \sigma(C) \land \bigwedge_{y_r \in \bar{C}, y_r \leq B} \sigma(D')(y_r) \right\} \right\}.
\]

In this case, \( \mathcal{A} \) is the induced \( L \)-fuzzy \( \alpha \)-open operator by \( \sigma \). \( \mathcal{A}(A) \) is called the degree of \( \alpha \)-openness of \( A \) and \( \mathcal{A}^*(A) = \mathcal{A}(A') \) can be regarded as the \( \alpha \)-closeness degree of \( A \).

**Corollary 2.4.** If \( \sigma \) is an \( L \)-fpt on \( X \) and \( A \subseteq L^X \), then:

\[
\mathcal{A}(A) = \bigvee_{B \subseteq A} \left\{ \sigma(B) \land \bigvee_{x_u \in \bar{A}} \left\{ \sigma(C) \land \bigwedge_{y_r \in \bar{C}} \text{Cl}^U(B)(y_r) \right\} \right\},
\]

where \( \text{Cl}^U \) refers to the \( L \)-fuzzy closure operator induced by \( \sigma \) (see [23]).

**Theorem 2.5.** [1] Let \( \sigma \) be an \( L \)-fpt on \( X, A \subseteq L^X \), and \( u \in \mathfrak{J}(L) \), then \( A \in \mathcal{A}_u \), if and only if \( A \) is an \( \alpha \)-open set in \( \mathcal{A}_u \), where \( \mathcal{A}_u = \{ A \in L^X \mid \mathcal{A}(A) \geq u \} \).

**Theorem 2.6.** Let \( \sigma : L^X \rightarrow \{0_L, 1_L\} \) be an \( L \)-pfts and let \( \mathcal{A} : L^X \rightarrow \{0_L, 1_L\} \) be the corresponding \( L \)-\( \alpha \)-open operator. Then \( \mathcal{A}(A) = 1_L \), if and only if \( A \) is \( \alpha \)-open \( L \)-subset.

**Proof.** We can prove the theorem by using the following fact:

\[
\mathcal{A}(A) = 1_L \iff \bigvee_{B \subseteq A} \left\{ \sigma(B) \land \bigvee_{x_u \in \bar{A}} \left\{ \sigma(C) \land \bigwedge_{y_r \in \bar{C}} \text{Cl}^U(B)(y_r) \right\} \right\} = 1_L
\]

\[
\iff \exists B \subseteq A \text{ such that } \sigma(B) = 1_L \text{ and } \bigwedge_{x_u \in \bar{A}} \left\{ \sigma(C) \land \bigwedge_{y_r \in \bar{C}} \text{Cl}^U(B)(y_r) \right\} = 1_L
\]

\[
\iff \exists B \subseteq A \text{ such that } \sigma(B) = 1_L \text{ and } \forall x_u \in \bar{A}, \exists C \text{ with } x_u \in \bar{C} \text{ such that } \sigma(C) = 1_L
\]

and \( \bigwedge_{y_r \in \bar{C}} \text{Cl}^U(B)(y_r) \)

\[
\iff \exists B \subseteq A \text{ such that } \sigma(B) = 1_L \text{ and } \forall x_u \in \bar{A}, \exists C \text{ with } x_u \in \bar{C} \text{ such that } \sigma(C) = 1_L
\]
and \( \forall y_v \in C, Cl^p(B(y_v)) = 1_L \)
\( \Leftrightarrow \exists B \preceq A \text{ such that } \sigma(B) = 1_L \) and \( \forall x_u \preceq A, \exists C \text{ with } x_u \preceq C \text{ such that } \sigma(C) = 1_L \)
and \( C \preceq Cl^p(B) \)
\( \Leftrightarrow \exists B \in \sigma, B \preceq A \preceq (\bar{B})^\circ \)
\( \Leftrightarrow A \text{ is } \alpha\text{-open} \text{-} L\text{-subset.} \)

Where \( ^\circ \) refer to the closure and the interior operator, respectively.

**Theorem 2.7.** Let \( \sigma \) be an L-fpt on \( X \) and let \( \mathcal{A} \) be its corresponding L-fuzzy \( \alpha\)-open operator. Then \( \sigma(A) \leq \mathcal{A}(A) \) for all \( A \in L^X \).

**Proof.** The proof can be obtained from the following inequality:

\[
\mathcal{A}(A) = \bigvee_{B \preceq A} \left\{ \sigma(B) \land \bigwedge_{x_u \preceq A} \left( \bigvee_{x_v \preceq C} \left\{ \sigma(C) \land \bigwedge_{y_v \preceq C} Cl^p(B(y_v)) \right\} \right) \right\}
\]
\[
\geq \sigma(A) \land \bigwedge_{x_u \preceq A} \left( \bigvee_{x_v \preceq C} \left( \bigwedge_{y_v \preceq C} Cl^p(A(y_v)) \right) \right)
\]
\[
\geq \sigma(A) \land \bigwedge_{x_u \preceq A} \left( \bigvee_{x_v \preceq C} \left( \bigwedge_{y_v \preceq C} Cl^p(A(y_v)) \right) \right)
\]
\[
= \sigma(A) \land \sigma(A) \land 1_L
\]
\[
= \sigma(A).
\]

**Corollary 2.8.** Let \( \sigma \) be an L-fpt on \( X \) and let \( \mathcal{A} \) be its corresponding L-fuzzy \( \alpha\)-open operator. Then \( \sigma^*(A) \leq \mathcal{A}^*(A) \) for all \( A \in L^X \).

**Theorem 2.9.** Let \( \mathcal{A} : L^X \rightarrow L \) be an L-fuzzy \( \alpha\)-open operator induced by L-fpt \( \sigma \) on \( X \). Then \( \mathcal{A} \) satisfies the following conditions:

1. \( \mathcal{A}(\bigcup_{i \in I} A_i) \leq \bigwedge_{i \in I} \mathcal{A}(A_i) \) for any \( \{A_i\}_{i \in I} \subseteq L^X \).

**Proof.** The proof of (1) is clear. To prove (2), suppose that \( w \in L \) and \( w \preceq \bigwedge_{i \in I} \mathcal{A}(A_i) \). Then for any \( i \in I \), there is \( B_i \preceq A_i \) such that

\[
w \preceq \sigma(B_i) \quad \text{and} \quad w \preceq \bigwedge_{x_u \preceq A_i} \left( \bigvee_{x_v \preceq C_i} \left\{ \sigma(C_i) \land \bigwedge_{y_v \preceq C_i} (\sigma(D'))' \right\} \right),
\]
i.e., \( w \preceq \sigma(B_i) \) and for any \( i \in I \) and \( x_u \preceq A_i \), there is \( C_i \in L^X \) such that \( x_u \preceq C_i \), \( w \preceq \sigma(C_i) \) and \( w \preceq \bigwedge_{x_u \preceq C_i} \bigvee_{y \in D} B_i \left( \sigma(D') \right)' \). Hence

\[
w \preceq \bigwedge_{i \in I} \sigma(B_i) \leq \sigma \left( \bigvee_{i \in I} B_i \right), \quad w \preceq \bigwedge_{i \in I} \sigma(C_i) \leq \sigma \left( \bigvee_{i \in I} C_i \right),
\]
and

\[
w \preceq \bigwedge_{i \in I} \bigwedge_{y \in C_i} \bigvee_{y \in D} B_i \left( \sigma(D') \right)'.
\]
This shows $\mathcal{A}(A_i) \geq \bigwedge_{i \in I} \mathcal{A}(A_i)$. 

In the following definition, we use $L$-fuzzy $\alpha$-open operators to introduce generalized definitions for $L$-fuzzy $\alpha$-open, $L$-fuzzy $\alpha$-continuous and $L$-fuzzy $\alpha$-irresolute functions.

**Definition 2.10.** If $(X, \sigma_1)$ and $(Y, \sigma_2)$ are $L$-fpt's and $f : (X, \sigma_1) \rightarrow (Y, \sigma_2)$ is a function, then:

1. $f$ is an $L$-fuzzy $\alpha$-open function iff $\sigma_1(A) \leq \mathcal{A}(f^{-1}(A))$ for any $A \in L^X$.
2. $f$ is an $L$-fuzzy $\alpha$-continuous function iff $\sigma_2(B) \leq \mathcal{A}_1(f^{-1}(B))$ holds for any $B \in L^Y$.
3. $f$ is an $L$-fuzzy $\alpha$-irresolute function iff $\sigma_2(B) \leq \mathcal{A}_1(f^{-1}(B))$ holds for any $B \in L^Y$.

**Corollary 11.** If $(X, \sigma_1)$ and $(Y, \sigma_2)$ are $L$-fpt's and $f : (X, \sigma_1) \rightarrow (Y, \sigma_2)$ is a function, then:

1. $f$ is an $L$-fuzzy $\alpha$-continuous function iff $\sigma_2(B) \leq \mathcal{A}_1(f^{-1}(B))$ for any $B \in L^Y$.
2. $f$ is an $L$-fuzzy $\alpha$-irresolute function iff $\sigma_2(B) \leq \mathcal{A}_1(f^{-1}(B))$ for any $B \in L^Y$.

**Definition 2.12.** [24] For an $L$-fpt $\sigma$ on $X$ and an $L$-subset $A \in L^X$, the degree of fuzzy compactness $\text{com}(A)$ of $A$ is given by:

$$\text{com}(A) = \bigwedge_{\mathcal{U} \subseteq L^X} \left\{ \left\{ \bigwedge_{B \in \mathcal{U}} \sigma(B) \wedge \bigvee_{x \in X} \left( A' \vee \bigvee_{B \in \mathcal{V}} B \right)(x) \right\} \implies \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \bigvee_{x \in X} \left( A' \vee \bigvee_{B \in \mathcal{V}} B \right)(x) \right\}.$$

In this case, an $L$-subset $A$ is said to be fuzzy compact if and only if $\text{com}(A) = 1_L$.

**Definition 2.13.** [1] Let $\sigma$ be an $L$-fpt on $X$. An $L$-subset $A \in L^X$ is called $\alpha$-compact if

$$\bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \wedge \bigwedge_{x \in X} \left( A' \vee \bigvee_{B \in \mathcal{V}} B \right)(x) \leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( A' \vee \bigvee_{B \in \mathcal{V}} B \right)(x)$$

for every $\mathcal{U} \subseteq L^X$. 
**Definition 2.14.** [25, 26] For an $L$-pt $x$ on $X$, $u \in L \setminus \{1_L\}$ and $A \subseteq L^u$, a family $\mathcal{U} \subseteq L^u$ is said to be an $a_u$-cover of $A$ if for each $x \in X$, we have $u \in a(A'(x) \vee \bigvee_{B \in \mathcal{U}} B(x))$. The family $\mathcal{U}$ is said to be a strong $a_u$-cover of $A$ if $u \in a(\bigcap_{x \in X} (A'(x) \vee \bigvee_{B \in \mathcal{U}} B(x)))$.

**Definition 2.15.** [25, 26] For an $L$-pt $x$ on $X$, $u \in L \setminus \{1_L\}$ and $A \subseteq L^u$, a family $\mathcal{U} \subseteq L^u$ is said to be a $Q_u$-cover of $A$ if for each $x \in X$, we have $A'(x) \vee \bigvee_{B \in \mathcal{U}} B(x) \geq u$.

**Definition 2.16.** [25, 26] For an $L$-pt $x$ on $X$, $u \in L \setminus \{1_L\}$ and $A \subseteq L^u$, a family $\mathcal{U} \subseteq L^u$ is called:

1. a $u$-shading of $A$ if for each $x \in X$, $(A'(x) \vee \bigvee_{B \in \mathcal{U}} B(x)) \leq u$.
2. a strong $u$-shading of $A$ if $\bigcap_{x \in X} (A'(x) \vee \bigvee_{B \in \mathcal{U}} B(x)) \leq u$.
3. a $u$-remote family of $A$ if for each $x \in X$, $(A(x) \wedge \bigwedge_{B \in \mathcal{U}} B(x)) \geq u$.
4. a strong $u$-remote family of $A$ if $\bigvee_{x \in X} (A(x) \wedge \bigwedge_{B \in \mathcal{U}} B(x)) \geq u$.

3 Degree of $\alpha$-openness, $\alpha$-continuity and $\alpha$-irresoluteness for functions between $L$-fpts’s

In this section, we will introduce the notions of $\alpha$-openness, $\alpha$-continuity, and $\alpha$-irresoluteness degree for functions between $L$-fpts’s. Further, we will discuss their properties.

**Definition 3.1.** If $(X, \sigma_1)$ and $(Y, \sigma_2)$ are $L$-fpts’s and $f : (X, \sigma_1) \rightarrow (Y, \sigma_2)$ is a function, then:

1. the $\alpha$-openness degree of $f$ with respect to $\sigma_1$ and $\sigma_2$ is defined by
   
   \[ ao(f) = \bigwedge_{A \in L^X} \left\{ \mathcal{A}_1(A) \rightarrow \mathcal{A}_2(f^{-1}(A)) \right\}. \]

2. the continuity degree of $f$ with respect to $\sigma_1$ and $\sigma_2$ is defined by
   
   \[ ac(f) = \bigwedge_{B \in L^Y} \left\{ \sigma_2(B) \rightarrow \mathcal{A}_1(f^{-1}(B)) \right\}. \]

3. the irresoluteness degree of $f$ with respect to $\sigma_1$ and $\sigma_2$ is defined by
   
   \[ ai(f) = \bigwedge_{B \in L^Y} \left\{ \mathcal{A}_2(B) \rightarrow \mathcal{A}_1(f^{-1}(B)) \right\}. \]

**Definition 3.2.** For any two $L$-fpts’s $(X, \sigma_1)$ and $(Y, \sigma_2)$ and any bijective function $f : (X, \sigma_1) \rightarrow (Y, \sigma_2)$, the $\alpha$-homomorphism degree of $f$ with respect to $\sigma_1$ and $\sigma_2$ is given by

\[ \alpha \text{-Hom}(f) = ai(f) \wedge ao(f). \]

**Remark 3.3.**

1. Based on (2) of Lemma 2.1, $ac(f) = 1_L$ implies to $\mathcal{A}_1(f^{-1}(B)) \geq \sigma_2(B)$ for all $B \in L^Y$. This is exactly the definition of $\alpha$-continuous function. The cases $ao(f) = 1_L$ and $ai(f) = 1_L$ can be shown similarly. Thus (2) and (3) in Definition 3.1 are precisely the $\alpha$-open and $\alpha$-irresolute function’s definition as in the sense of Definition 2.10.

2. For the identity function $i : (X, \sigma_1) \rightarrow (X, \sigma_1)$, we have $ai(i) = ao(i) = \alpha \text{-Hom}(i) = 1_L$.

By using Definition 3.1 and Corollary 2.11, we can state the following corollary.
Corollary 3.4. If \((X, \sigma_1)\) and \((Y, \sigma_2)\) are L-fpts's and \(f : (X, \sigma_1) \rightarrow (Y, \sigma_2)\) is a function, then:

1. the \(\alpha\)-continuity degree of \(f\) is characterized by
   \[
   ac(f) = \bigwedge_{B \in L^Y} \left\{ \alpha_2'(B) \rightarrow \alpha_1'(f^-(B)) \right\}.
   \]

2. the \(\alpha\)-irresoluteness degree of \(f\) is characterized by
   \[
   ai(f) = \bigwedge_{B \in L^Y} \left\{ \alpha_2'(B) \rightarrow \alpha_1'(f^-(B)) \right\}.
   \]

Definition 3.5. For any function \(f : (X, \sigma_1) \rightarrow (Y, \sigma_2)\) where \((X, \sigma_1)\) and \((Y, \sigma_2)\) are two L-fpts's, the \(\alpha\)-closeness degree of \(f\) is given by
\[
ac(f) = \bigwedge_{A \in L^X} \left\{ \alpha_2'(A) \rightarrow \alpha_1'(f^+(A)) \right\}.
\]

Theorem 3.6. If \(f : (X, \sigma_1) \rightarrow (Y, \sigma_2)\) and \(g : (Y, \sigma_2) \rightarrow (Z, \sigma_3)\) are two functions where \((X, \sigma_1)\), \((Y, \sigma_2)\) and \((Z, \sigma_3)\) are three L-fpts's, then:

1. \(ai(f) \land ai(g) \leq ai(g \circ f)\).
2. \(ao(f) \land ao(g) \leq ao(g \circ f)\).
3. \(acl(f) \land acl(g) \leq acl(g \circ f)\).

Proof. Since the proof of (2) and (3) is clear, we only prove (1). By using Definition 3.1 and Lemma 2.1 (4), we obtain
\[
ai(f) \land ai(g) = \bigwedge_{B \in L^Y} \left\{ \alpha_2'(B) \rightarrow \alpha_1'(f^-(B)) \right\} \land \bigwedge_{C \in L^X} \left\{ \alpha_2'(C) \rightarrow \alpha_2'(g^+(C)) \right\}
\]
\[
\leq \bigwedge_{C \in L^X} \left\{ \alpha_2'(g^+(C)) \rightarrow \alpha_1'(f^+(g^+(C))) \right\} \land \bigwedge_{C \in L^X} \left\{ \alpha_2'(C) \rightarrow \alpha_2'(g^+(C)) \right\}
\]
\[
= \bigwedge_{C \in L^X} \left\{ \left( \alpha_2'(g^+(C)) \rightarrow \alpha_2'(g^+(C)) \right) \rightarrow \alpha_1'((g \circ f)^+(C)) \right\} \land \left( \alpha_2'(C) \rightarrow \alpha_2'(g^+(C)) \right)
\]
\[
= ai(g \circ f).
\]

By using Definition 3.2 and Theorem 3.6, we have the following corollary.

Corollary 3.7. Let \((X, \sigma_1)\), \((Y, \sigma_2)\) and \((Z, \sigma_3)\) be L-fpts's, \(f : X \rightarrow Y\) and \(g : Y \rightarrow Z\) be two bijective functions. Then \(a-Hom(f) \land a-Hom(g) \leq a-Hom(g \circ f)\).

Theorem 3.8. Let \((X, \sigma_1)\), \((Y, \sigma_2)\) and \((Z, \sigma_3)\) be L-fpts's and \(g : Y \rightarrow Z\) be a surjective function. Then:

1. \(ao(g \circ f) \land ai(f) \leq ao(g)\).
2. \(acl(g \circ f) \land ai(f) \leq acl(g)\).

Proof. (1) Since \(f\) is a surjective function, we have \((g \circ f)^-(f^+(B)) = g^-(B)\) for each \(B \in L^Y\). By using (4) of Lemma 2.1, we get
\[
ao(g \circ f) \land ai(f) = \bigwedge_{A \in L^X} \left\{ \alpha_2'(A) \rightarrow \alpha_2'(g \circ f^-(A)) \right\} \land \bigwedge_{B \in L^Y} \left\{ \alpha_2'(B) \rightarrow \alpha_2'(f^+(B)) \right\}
\]
Proof. From the bijectivity of Theorem 3.10. If $g$ and $\alpha$ are bijective functions, we get $f \circ g \circ f^{-1} = f \circ \alpha = \alpha \circ f$. Similarly, the following theorem is true.

(1) $\alpha(g \circ f) \leq \alpha(g) \circ \alpha(f)$.
(2) $\alpha(g \circ f) \leq \alpha(f \circ g)$.

Theorem 3.9. Given three L-fpTs $(X, \sigma_1)$, $(Y, \sigma_2)$ and $(Z, \sigma_3)$. If $f : (X, \sigma_1) \rightarrow (Y, \sigma_2)$ is an injective function and $g : Y \rightarrow Z$ is any function, then

(1) $\alpha(g \circ f) \leq \alpha(g) \circ \alpha(f)$.
(2) $\alpha(g \circ f) \leq \alpha(f \circ g)$.

Theorem 3.10. If $f : (X, \sigma_1) \rightarrow (Y, \sigma_2)$ is a bijective function where $(X, \sigma_1)$ and $(Y, \sigma_2)$ are two L-fpTs, then

(1) $\alpha(f) = \bigwedge_{A \in L^X} \left\{ \omega_2(f^{-1}(A)) \Rightarrow \omega_1(A) \right\}$.
(2) $\alpha(f) = \bigwedge_{B \in L^Y} \left\{ \omega_2(f^-(B)) \Rightarrow \omega_2(B) \right\}$.
(3) $\alpha(f^{-1}) = \alpha(f) = \alpha(f) = \alpha(f) = \alpha(f)$.

Proof. The proof of (2) is similar to (1), we only prove (1) and (3).

(1) From the bijectivity of $f$, we get $f^{-1}(f(A)) = A$ for any $A \in L^X$, and $f^{-1}(f^-(B)) = B$ for any $B \in L^Y$. It follows that

$$\bigwedge_{A \in L^X} \left\{ \omega_2(f^{-1}(A)) \Rightarrow \omega_1(A) \right\} = \bigwedge_{A \in L^X} \left\{ \omega_2(f^{-1}(A)) \Rightarrow \omega_1(f^{-1}(f(A))) \right\} \geq \bigwedge_{B \in L^Y} \left\{ \omega_2(B) \Rightarrow \omega_1(f^{-1}(B)) \right\} = \bigwedge_{B \in L^Y} \left\{ \omega_2(f^{-1}(B)) \Rightarrow \omega_1(f^{-1}(B)) \right\} \geq \bigwedge_{A \in L^X} \left\{ \omega_2(f^{-1}(A)) \Rightarrow \omega_1(A) \right\}.$$

Hence

$$\alpha(f) = \bigwedge_{B \in L^Y} \left\{ \omega_2(B) \Rightarrow \omega_1(f^{-1}(B)) \right\} = \bigwedge_{A \in L^X} \left\{ \omega_2(f^{-1}(A)) \Rightarrow \omega_1(A) \right\}.$$

(3) Since $f$ is a bijective function, we get $(f^{-1})^{-1}(A) = f^{-1}(A)$ and $f^{-1}(A') = f^{-1}(A)'$ for any $A \in L^X$. Therefore

$$\alpha(f^{-1}) = \bigwedge_{A \in L^X} \left\{ \omega_1(A) \Rightarrow \omega_2((f^{-1})^{-1}(A)) \right\} = \bigwedge_{A \in L^X} \left\{ \omega_1(A) \Rightarrow \omega_2(f^{-1}(A)) \right\}.$$
Definition 4.1. Let $\alpha$ be an $L$-fuzzy open operator. In the following definition, we present the degree of $\alpha$-compactness based on implication operation as a new generalization of $\alpha$-compactness.

**Definition 4.1.** Let $(X, \sigma)$ be an $L$-fpts. For any $A \in L^X$, let

$$\text{Com}_\sigma(A) = \bigwedge_{U \subseteq L^X} \left\{ \mathcal{A}(U) \mapsto \left( \bigcap_{x \in X} \left( A^c \cup U \right) \cup \bigcup_{V \in 2(U)} \left( A \cup V \right) \right) \right\}$$

Then $\text{Com}_\sigma(A)$ is said to be the degree of $\alpha$-compactness of $A$ with respect to $\sigma$. By using Theorem 2.9, we have $\text{Com}_\sigma(A) = \text{Com}(A)$ for any $A \in L^X$.

**Theorem 4.2.** Let $\tau$ be an $L$-pt on $X$ and $A \in L^X$. An $L$-subset $A$ is fuzzy $\alpha$-compact if and only if $\text{Com}_\tau(A) = 1_L$, where the mapping $\chi_\tau : L^X \rightarrow L$ is given by

$$\chi_\tau(A) = \begin{cases} 1_L, & \text{if } A \in \tau; \\ 0_L, & \text{if } A \notin \tau. \end{cases}$$

**Proof.** Let $\tau$ be an $L$-pt on $X$. It is clear that $\chi_\tau$ is $L$-fpt. An $L$-subset $A \in L^X$ is fuzzy $\alpha$-open set with respect to $\tau$ if and only if $\mathcal{A}_\tau(A) = 1_L$. Based on the definition of fuzzy $\alpha$-compactness, we have an $L$-subset $A \in L^X$ is fuzzy $\alpha$-compact such that for any collection $U \subseteq L^X$, we have that

$$\mathcal{A}_\tau(U) \leq \left( A \cap U \right) \cup \bigcup_{V \in 2(U)} \left( A \cup V \right).$$
By using Lemma 2.1, A is fuzzy $\alpha$-compact if and only if for any collection $\cup \subseteq L^X$, we have

$$\varphi_\alpha(\cup) \mapsto \left( A \subseteq \bigvee \cup \mapsto \bigvee_{\forall \in 2^{\{\|\}} \left( A \subseteq \bigvee \forall \right) \right) = 1_L.$$ 

This result together with the definition of $aCoM(\alpha)$ yields to $aCoM(\alpha) = 1_L$. \hfill $\square$

**Theorem 4.3.** Let $\sigma$ be an $L$-fpt on X and $A \in L^X$. An L-subset A is L-fuzzy $\alpha$-compact if and only if $aCoM(A) = 1_L$.

**Proof.** Based on Definition 4.1 and Lemma 2.1, the conclusion is straightforward. \hfill $\square$

**Theorem 4.4.** For any $L$-fpt $\sigma$ on X and $A \in L^X$, we have $aCoM(A) \succeq CoM(A)$.

**Proof.** Straightforward. \hfill $\square$

**Lemma 4.5.** For any $L$-fpt $\sigma$ on X and $A \in L^X$, we have $aCoM(A) \succeq u$ if and only if

$$\varphi(\cup) \land \left( A \subseteq \bigvee \cup \land u \leq \bigvee_{\forall \in 2^{\{\|\}}} \left( A \subseteq \bigvee \forall \right) \right),$$

for any $\cup \subseteq L^X$.

**Proof.** For every $u \in L, A \in L^X$ and $\cup \subseteq L^X$, we have

$$aCoM(A) \succeq u \iff \bigwedge_{\cup \subseteq L^X} \left( \varphi(\cup) \mapsto \left( A \subseteq \bigvee \cup \mapsto \bigvee_{\forall \in 2^{\{\|\}}} \left( A \subseteq \bigvee \forall \right) \right) \succeq u \right)$$

$$\iff \varphi(\cup) \mapsto \left( A \subseteq \bigvee \cup \mapsto \bigvee_{\forall \in 2^{\{\|\}}} \left( A \subseteq \bigvee \forall \right) \right) \succeq u$$

$$\iff \left( \varphi(\cup) \land \left( A \subseteq \bigvee \cup \right) \right) \mapsto \bigvee_{\forall \in 2^{\{\|\}}} \left( A \subseteq \bigvee \forall \right) \succeq u$$

$$\iff \varphi(\cup) \land \left( A \subseteq \bigvee \cup \land u \leq \bigvee_{\forall \in 2^{\{\|\}}} \left( A \subseteq \bigvee \forall \right) \right).$$

\hfill $\square$

**Theorem 4.6.** For any $L$-fpt $\sigma$ on X and $A \in L^X$, we have $aCoM(A) \succeq u$ if and only if

$$\bigvee_{B \in \mathcal{M}} \varphi'(B) \vee \left\{ \bigvee_{x \in X} \left( A(x) \land \bigwedge_{B \in \mathcal{M}} B(x) \right) \right\} \vee u' \geq \bigvee_{N \in 2^{\{\|\}} x \in X} \left( A(x) \land \bigwedge_{B \in N} B(x) \right),$$

for each $\mathcal{M} \subseteq L^X$.

**Proof.** Based on the definition of $\varphi'$ and Lemma 2.1, the proof is clear. \hfill $\square$

**Theorem 4.7.** For any $L$-fpt $\sigma$ on X and $A \in L^X$, we have

$$aCoM(A) = \bigvee \left\{ u \in L \mid \bigwedge_{B \in \cup} \varphi(B) \land \left( A \subseteq \bigvee \cup \right) \land u \leq \bigvee_{\forall \in 2^{\{\|\}}} \left( A \subseteq \bigvee \forall \right), \forall \cup \subseteq L^X \right\}.$$

**Proof.** By using Lemma 2.1, we have $aCoM(A)$ as the upper bound of

$$\left\{ u \in L \mid \bigwedge_{B \in \cup} \varphi(B) \land \left( A \subseteq \bigvee \cup \right) \land u \leq \bigvee_{\forall \in 2^{\{\|\}}} \left( A \subseteq \bigvee \forall \right), \forall \cup \subseteq L^X \right\}.$$
By using the Definition 4.1, we have

\[
\text{aCom}(A) \leq \bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \mapsto \left( [A \subseteq \bigvee \mathcal{U}] \mapsto \bigvee_{\mathcal{V} \in \mathcal{U}} [A \subseteq \bigvee \mathcal{V}] \right) \\
= \left( \bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \land [A \subseteq \bigvee \mathcal{U}] \mapsto \bigvee_{\mathcal{V} \in \mathcal{U}} [A \subseteq \bigvee \mathcal{V}] \right),
\]

for each \( \mathcal{U} \subseteq L^X \). By applying the properties of the operation “\( \mapsto \)”, we have

\[
\bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \land [A \subseteq \bigvee \mathcal{U}] \land \text{aCom}(A) \leq \bigvee_{\mathcal{V} \in \mathcal{U}} [A \subseteq \bigvee \mathcal{V}],
\]

and hence

\[
\text{aCom}(A) \in \bigvee \left\{ u \in L \mid \bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \land [A \subseteq \bigvee \mathcal{U}] \land u \leq \bigvee_{\mathcal{V} \in \mathcal{U}} [A \subseteq \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \right\}.
\]

Therefore, we completed the proof. \( \square \)

**Theorem 4.8.** For any \( \text{L-fpt} \ \sigma \) on \( X \) and \( A_1, A_2 \in L^X \), we have

\[
\text{aCom}(A_1 \lor A_2) \geq \text{aCom}(A_1) \land \text{aCom}(A_2).
\]

**Proof.** We can prove the theorem by using the next inequality:

\[
\text{aCom}(A_1 \lor A_2) = \bigvee \left\{ u \in L \mid \bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \land [A_1 \lor A_2 \subseteq \bigvee \mathcal{U}] \land u \leq \bigvee_{\mathcal{V} \in \mathcal{U}} [A_1 \lor A_2 \subseteq \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \right\}
\]

\[
\leq \bigvee_{\mathcal{V} \in \mathcal{U}} [A_1 \lor A_2 \subseteq \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X
\]

\[
= \bigvee \left\{ u \in L \mid \bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \land [A_1 \subseteq \bigvee \mathcal{U}] \land [A_2 \subseteq \bigvee \mathcal{U}] \land u \leq \bigvee_{\mathcal{V} \in \mathcal{U}} [A_1 \subseteq \bigvee \mathcal{V}] \land [A_2 \subseteq \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X \right\}
\]

\[
\geq \bigvee_{\mathcal{V} \in \mathcal{U}} [A_1 \subseteq \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X
\]

\[
= \text{aCom}(A_1) \land \text{aCom}(A_2).
\]

\( \square \)

**Theorem 4.9.** For any \( \text{L-fpt} \ \sigma \) on \( X \) and \( A_1, A_2 \in L^X \), we have

\[
\text{aCom}(A_1 \land A_2) \geq \text{aCom}(A_1) \land \mathcal{A}(A_2).
\]
Proof. We can prove the theorem by using the next inequality
\[
\alpha\text{Com}(A_1 \land A_2) = \bigvee \left\{ u \in L \mid \bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \land \left[ A_1 \land A_2 \subseteq \bigvee u \right] \right\} \\
\land u \leq \bigvee \left\{ u \in L \mid \bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \land \left[ A_1 \land A_2 \subseteq \bigvee \mathcal{U} \right] \right\} \\
= \bigvee \left\{ u \in L \mid \bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \land \left[ A_1 \subseteq A_2 \lor \bigvee u \right] \right\} \\
\land u \leq \bigvee \left\{ u \in L \mid \bigwedge_{B \in \mathcal{U}} \mathcal{A}(B) \land \left[ A_1 \subseteq A_2 \lor \bigvee \mathcal{U} \right] \right\} \\
\geq \alpha\text{Com}(A_1) \land \mathcal{A}(A_2).
\]

\[\square\]

Corollary 4.10. For any L-fpt \( \sigma \) on \( X \) and \( A \in L^X \), we have
\[
\alpha\text{Com}(A) \geq \alpha\text{Com}(1_L) \land \mathcal{A}(A').
\]

Theorem 4.11. For any L-fts's \( (X, \sigma_1) \) and \( (X, \sigma_2) \) such that \( \sigma_1 \leq \sigma_2 \) and for any \( A \in L^X \), we have \( \alpha\text{Com}_{\sigma_1}(A) \leq \alpha\text{Com}_{\sigma_1}(A) \).

Corollary 4.12. For any L-fpts \((X, \sigma)\) with the base or the subbase \( \mathcal{B} \), we have \( \alpha\text{Com}(A) \leq \alpha\text{Com}_{\mathcal{B}}(A) \), for any \( A \in L^X \).

Theorem 4.13. For any L-fpts's \((X, \sigma_1)\) and \( (Y, \sigma_2) \), if \( f : (X, \sigma_1) \rightarrow (Y, \sigma_2) \) is an L-fuzzy \( \alpha \)-irresolute function, then
\[
\alpha\text{Com}_{\sigma_1}(f^{-1}_L(C)) \geq \alpha\text{Com}_{\sigma_1}(C),
\]
for every \( C \in L^Y \).

Proof. For each \( C \in L^X \), we have
\[
\alpha\text{Com}_{\sigma_1}(f^{-1}_L(C)) = \bigvee \left\{ u \in L | \mathcal{A}(u) \land \left[ f^{-1}_L(C) \subseteq \bigvee u \right] \right\} \\
\land u \leq \bigvee \left\{ u \in L | \mathcal{A}(u) \land \left[ f^{-1}_L(C) \subseteq \bigvee \mathcal{U} \right] \right\} \\
\geq \bigvee \left\{ u \in L | \mathcal{A}(u) \land \left[ \mathcal{C} \subseteq \bigvee f^{-1}_L(u) \right] \right\} \\
\land u \leq \bigvee \left\{ u \in L | \mathcal{A}(u) \land \left[ \mathcal{C} \subseteq \bigvee \mathcal{U} \right] \right\} \\
\geq \bigvee \left\{ u \in L | \mathcal{A}(u) \land \left[ \mathcal{C} \subseteq \bigvee \mathcal{U} \right] \right\} \\
\land u \leq \bigvee \left\{ u \in L | \mathcal{A}(u) \land \left[ \mathcal{C} \subseteq \bigvee \mathcal{U} \right] \right\} \\
\geq \alpha\text{Com}_{\sigma_1}(C).
\]

\[\square\]

Theorem 4.14. For any L-fpts's \((X, \sigma_1)\) and \((Y, \sigma_2)\), if \( f : (X, \sigma_1) \rightarrow (Y, \sigma_2) \) is an L-fuzzy \( \alpha \)-continuous function, then
\[
\text{Com}_{\sigma_1}(f^{-1}_L(C)) \geq \alpha\text{Com}_{\sigma_1}(C),
\]
for every $C \subseteq L^X$.

**Proof.** For each $C \subseteq L^X$, we have

$$\text{Com}_{\sigma}(f^*_L(C)) = \bigvee \left\{ u \in L|\sigma(\emptyset) \land \left[ f^*_L(C) \subseteq \bigvee Y \right] \land u \subseteq L^Y \right\}$$

$$\land u \subseteq \bigvee_{\emptyset \subseteq Y \subseteq L^X} \left[ f^*_L(C) \subseteq \bigvee Y \right], \forall Y \subseteq L^X$$

$$\geq \bigvee \left\{ u \in L|\sigma_1(f^*_L(\emptyset)) \land \left[ C \subseteq \bigvee \emptyset \right] \right\}$$

$$\land u \subseteq \bigvee_{\emptyset \subseteq Y \subseteq L^X} \left[ C \subseteq \bigvee Y \right], \forall Y \subseteq L^X$$

$$= \text{aCom}_{\sigma}(C).$$

\[\square\]

**Theorem 4.15.** For any $L$-fpt $\sigma$ on $X$, $A \subseteq L^X$ and $u \subseteq L \setminus \{\emptyset\}$, the next statements are equivalent:

1. $\text{aCom}(A) \supseteq u$.
2. For each $v \subseteq P(L)$, $v \supseteq u$, every strong $v$-shading $\emptyset$ of $A$ with $\sigma(\emptyset) \subseteq v$ has a finite sub-collection $V$ which is a strong $v$-shading of $A$.
3. For each $v \subseteq P(L)$, $v \supseteq u$, every strong $v$-shading $\emptyset$ of $A$ with $\sigma(\emptyset) \subseteq v$, there exists a finite sub-collection $V$ of $\emptyset$ and $w \subseteq \emptyset^v(\emptyset)$ such that $V$ is a $w$-shading of $A$.
4. For each $v \subseteq P(L)$, $v \supseteq u$, every strong $v$-shading $\emptyset$ of $A$ with $\sigma(\emptyset) \subseteq v$, there exists a finite sub-collection $V$ of $\emptyset$ and $w \subseteq \emptyset^v(\emptyset)$ such that $V$ is a strong $w$-shading of $A$.
5. For each $v \subseteq J(L)$, $v \supseteq u'$, every strong $v$-remote collection $W$ of $A$ with $\sigma^v(W) \subseteq v'$ has a finite sub-collection $R$ which is a strong $v$-remote collection of $A$.
6. For each $v \subseteq J(L)$, $v \supseteq u'$, every strong $v$-remote collection $W$ of $A$ with $\sigma^v(W) \subseteq v'$, there exists a finite sub-collection $R$ of $W$ and $w \subseteq \emptyset^v(W)$ such that $R$ is a $w$-remote collection of $A$.
7. For each $v \subseteq J(L)$, $v \supseteq u'$, every strong $v$-remote collection $W$ of $A$ with $\sigma^v(W) \subseteq v'$, there exists a finite sub-collection $R$ of $W$ and $w \subseteq \emptyset^v(W)$ such that $R$ is a strong $w$-remote collection of $A$.
8. For each $v \subseteq u$, $u \subseteq a(v)$, $v, w \neq \emptyset$, every $Q_r$-cover $\emptyset \subseteq (\emptyset)_r$ of $A$ has a finite sub-collection $V$ which is a $Q_w$-cover of $A$.
9. For each $v \subseteq u$, $w \subseteq a(v)$, $v, w \neq \emptyset$, every $Q_r$-cover $\emptyset \subseteq (\emptyset)_r$ of $A$ has a finite sub-collection $V$ which is a strong $a_w$-cover of $A$.
10. For each $v \subseteq u$, $w \subseteq a(v)$, $v, w \neq \emptyset$, every $Q_r$-cover $\emptyset \subseteq (\emptyset)_r$ of $A$ has a finite sub-collection $V$ which is a strong $a_w$-cover of $A$.
11. For each $v \subseteq u$, $w \subseteq a(v)$, $u \neq \emptyset$, every strong $a_v$-cover $\emptyset \subseteq (\emptyset)_v$ of $A$ has a finite sub-collection $V$ which is a $Q_w$-cover of $A$.
12. For each $v \subseteq u$, $w \subseteq a(v)$, $u \neq \emptyset$, every strong $a_v$-cover $\emptyset \subseteq (\emptyset)_v$ of $A$ has a finite sub-collection $V$ which is a strong $a_w$-cover of $A$.
13. For each $v \subseteq u$, $w \subseteq a(v)$, $u \neq \emptyset$, every strong $a_v$-cover $\emptyset \subseteq (\emptyset)_v$ of $A$ has a finite sub-collection $V$ which is a strong $a_w$-cover of $A$.

**Theorem 4.16.** For any $L$-fpt $\sigma$ on $X$, $A \subseteq L^X$, and $u \subseteq L \setminus \{\emptyset\}$, if $a(w \land s) = a(w) \land a(s)$ for each $w, s \subseteq L$, then the next statements will be equivalent:
The following theorem and its corollary verify the relationship between \(a\)-irresoluteness degree and \(a\)-compactness degree.

**Theorem 4.17.** If \(f : (X, \sigma_1) \to (Y, \sigma_2)\) is a function between two \(L\)-fpts's \((X, \sigma_1)\) and \((Y, \sigma_2)\), then

\[
a\text{Com}_{\sigma_1}(A) \land ai(f) \leq a\text{Com}_{\sigma_2}(f^{-1}(A))
\]

for any \(A \in L^Y\).

**Proof.** Suppose that \(u_1 \in L^X\) with \(u_1 \leq a\text{Com}_{\sigma_1}(A) \land ai(f)\). Then

\[
u_1 \leq a\text{Com}_{\sigma_2}(A)
\]

and

\[
u_1 \leq \bigwedge_{u \in L^X} \left\{ \left( \bigwedge_{A_1 \in \mathcal{L}_u} \mathcal{A}_1(A_1) \land \bigwedge_{x \in X} \left( A'_1 \lor \bigvee_{A_1 \in \mathcal{L}_u} A_1 \right)(x) \right) \implies \bigvee_{\mathcal{L}_u \in L^X} \left( A'_1 \lor \bigvee_{A_1 \in \mathcal{L}_u} A_1 \right)(x) \right\}
\]

Then for any \(B \in L^Y\) and \(\mathcal{L} \subseteq L^X\), we have

\[
u_1 \leq a\mathcal{L}_u(B) \implies a\mathcal{L}_u(f^{-1}(B))
\]

By Lemma 2.1 (1), we have

\[
u_1 \leq \bigwedge_{W \in \mathcal{L}_u} \left( \bigwedge_{x \in X} \left( A'_1 \lor \bigvee_{W \in \mathcal{L}_u} W \right)(x) \right) \leq \bigvee_{\mathcal{L}_u \in L^X} \left( A'_1 \lor \bigvee_{W \in \mathcal{L}_u} W \right)(x).
\]

To prove

\[
u_1 \leq a\text{Com}_{\sigma_2}(f^{-1}(A))
\]

\[
= \bigwedge_{\mathcal{L}_u \in L^Y} \left\{ \left( \bigwedge_{B_1 \in \mathcal{L}_u} a\mathcal{L}_u(B_1) \land \bigvee_{y \in Y} \left( f^{-1}(A)' \lor \bigvee_{B_1 \in \mathcal{L}_u} B_1 \right)(y) \right) \right\}
\]

for all \(\mathcal{L} \subseteq L^Y\), let \(f^{-1}(W) = \{ f^{-1}(B_1) \mid B_1 \in \mathcal{L} \} \subseteq L^X\). Then, we have

\[
u_1 \leq a\mathcal{L}_u(f^{-1}(B_1)) \land \bigwedge_{y \in Y} \left( f^{-1}(A)' \lor \bigvee_{B_1 \in \mathcal{L}_u} B_1 \right)(y)
\]

\[
\leq a\mathcal{L}_u(f^{-1}(B_1)) \land \bigwedge_{y \in Y} \left( f^{-1}(A)' \lor \bigvee_{B_1 \in \mathcal{L}_u} B_1 \right)(y)
\]
\[
\begin{align*}
-u_1 & \land \bigwedge_{B_1 \in W} \mathcal{A}_f(f^{-\ast}(B_1)) \land \bigwedge_{x \in X} \left( A' \lor \bigvee_{B_1 \in W} f^{-\ast}(B_1) \right)(x)
= u_1 & \land \bigwedge_{A_1 \in f^{-\ast}(W)} \mathcal{A}_f(A_1) \land \bigwedge_{x \in X} \left( A' \lor \bigvee_{A_1 \in f^{-\ast}(W)} (A_1) \right)(x)
\leq & \bigvee_{\mathcal{W} \in 2^{\mathcal{W}(W)}} \bigwedge_{x \in X} \left( A' \lor \bigvee_{A_1 \in W} (A_1) \right)(x)
= & \bigvee_{\mathcal{D} \in 2(W)} \bigwedge_{x \in X} \left( f^{-\ast}(A)' \lor \bigvee_{B_1 \in D} B_1 \right)(x)
= & \bigvee_{\mathcal{D} \in 2(W)} \bigwedge_{y \in Y} \left( f^{-\ast}(A)' \lor \bigvee_{B_1 \in D} B_1 \right)(y).
\end{align*}
\]

By using Lemma 2.1 (1), we know
\[
\begin{align*}
u_1 & \leq \left( \bigwedge_{B_1 \in W} \mathcal{A}_f(B_1) \land \bigwedge_{y \in Y} \left( f^{-\ast}(A)' \lor \bigvee_{B_1 \in W} B_1 \right)(y) \right)
\implies & \bigvee_{\mathcal{D} \in 2(W)} \bigwedge_{y \in Y} \left( f^{-\ast}(A)' \lor \bigvee_{B_1 \in D} B_1 \right)(y).
\end{align*}
\]
Thus
\[
\begin{align*}
u_1 & \leq \bigwedge_{\mathcal{W} \subseteq \mathcal{W}(W)} \left( \bigwedge_{B_1 \in W} \mathcal{A}_f(B_1) \land \bigwedge_{y \in Y} \left( f^{-\ast}(A)' \lor \bigvee_{B_1 \in W} B_1 \right)(y) \right)
\implies & \bigvee_{\mathcal{D} \in 2(W)} \bigwedge_{y \in Y} \left( f^{-\ast}(A)' \lor \bigvee_{B_1 \in D} B_1 \right)(y) = a\text{Com}_{\mathcal{A}_f}(f^{-\ast}(A)).
\end{align*}
\]
Since \( u_1 \) is arbitrary, we have \( a\text{Com}_{\mathcal{A}_f}(A) \land \alpha(f) \leq a\text{Com}_{\mathcal{A}_f}(f^{-\ast}(A)) \). The proof is completed. \( \square \)

**Corollary 4.18.** For any surjective function \( f : (X, \sigma_1) \rightarrow (Y, \sigma_2) \) where \( (X, \sigma_1) \) and \( (Y, \sigma_2) \) are L-fpts's, we have
\[
a\text{Com}_{\mathcal{A}_f}(1_{L_{\sigma_1}}) \land \alpha(f) \leq a\text{Com}_{\mathcal{A}_f}(1_{L_{\sigma_2}}).
\]

**Acknowledgement:** The authors would like to thank Deanship of Scientific Research at Majmaah University for supporting this work under Project Number No. 1440-91. Moreover, we are extremely grateful to the editors and the anonymous referees for their valuable comments and suggestions in improving this paper.

**References**

[1] Nannan L. and Ruizing W., Regular open set in L-fuzzy topology, *Pure and Applied Mathematics*, 2016, 32, no. 4, 409-415 (in Chinese).

[2] Šostak A. P., Fuzzy categories related to algebra and topology, *Tatra Mount Math Publ.*, 1999, 16, 159-185

[3] Šostak A. P., *Categorical structures and their applications*, Chapter: L-valued categories: generalities and examples related to algebra and topology, 2004, 291-311, World scientific publ.

[4] Pang B., Degrees of continuous mappings, open mappings, and closed mappings in L-fuzzifying topological spaces, *Journal of Intelligent and Fuzzy Systems*, 2014, 27, no. 2, 805-816

[5] Liang C.-Y. and Shi F.-G., Degree of continuity for mappings of \((L, M)\)-fuzzy topological spaces, *Journal of Intelligent and Fuzzy Systems*, 2014, 27, no. 5, 2665-2677

[6] Shi F.-G., L-fuzzy semiopenness and L-fuzzy preopeness, *J. Nonlinear Sci. Appl.*, Accepted.

[7] Shi F.-G. and Li R.X., Semicompactness in L-fuzzy topological spaces, *Annals of Fuzzy Mathematics and Informatics*, 2011, 2, 163-169

[8] Ghaeeeb A., Preconnectedness degree of L-fuzzy topological spaces, *International Journal of Fuzzy Logic and Intelligent Systems*, 2011, 11, no. 1, 54-58
[9] Ghareeb A., *L*-fuzzy semi-preopen operator in *L*-fuzzy topological spaces, *Neural Computing and Applications*, 2012, 21, no. 1, 87-92.
[10] Ghareeb A., A new form of *F*-compactness in *L*-fuzzy topological spaces, *Mathematical and Computer Modelling*, 2011, 54, no. 9, 2544-2550.
[11] Ghareeb A. and Shi F.-G., SP-compactness and SP-connectedness degree in *L*-fuzzy pretopological spaces, *Journal of Intelligent and Fuzzy Systems*, 2016, 31, no. 3, 1435-1445.
[12] Al-Omeri W. F., Khalil O. H., and Ghareeb A., Degree of (*L, M*)-fuzzy semi-precontinuous and (*L, M*)-fuzzy semi-preirresolute functions, *Demonstratio Mathematica*, 2018, 51, no. 1, 182-197.
[13] Ghareeb A. and Al-Omeri W. F., New degrees for functions in (*L, M*)-fuzzy topological spaces based on (*L, M*)-fuzzy semiopen and (*L, M*)-fuzzy preopen operators, *Journal of Intelligent and Fuzzy Systems*, 2019, 36, no. 1, 787-803.
[14] Ghareeb A., Degree of *F*-irresolute function in (*L, M*)-fuzzy topological spaces, *Iranian Journal of Fuzzy Systems*, 2019, DOI: 10.22111/IJFS.2019.4431, Article in press.
[15] Raney G. N., A subdirect-union representation for completely distributive complete lattices, *Proceedings of the American Mathematical Society*, 1953, 4, no. 4, 518-522.
[16] Hähling U. and Šostak A. P., *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory*, Chapter: Axiomatic Foundations Of Fixed-Basis Fuzzy Topology, 1999, 123-272, Springer US, Boston, MA.
[17] Šostak A. P., *General Topology and its Relations to Modern Analysis and Algebra*, Chapter: On compactness and connectedness degrees of fuzzy sets in fuzzy topological spaces, 1988, 123-272, Heldermann Verlag, Berlin.
[18] Šostak A. P., Two decades of fuzzy topology: basic ideas, notions, and results, *Russian Mathematical Surveys*, 1989, 44, no. 6, 1-25.
[19] Shi F.-G., A new definition of fuzzy compactness, *Fuzzy Sets and Systems*, 2007, 158, no. 13, 1486-1495.
[20] Kubiak T., *On fuzzy topologies*, 1985, Ph.D. Thesis, Adam Mickiewicz, Poznan, Poland.
[21] Rodabaugh S. E., *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory*, Chapter: Categorical Foundations of Variable-Basis Fuzzy Topology, 1999, 273-388, Springer US, Boston, MA.
[22] Šostak A. P., On a fuzzy topological structure, *Rendiconti Circolo Matematico Palermo*, 1985, 11(Suppl. Ser. II), 89-103.
[23] Shi F.-G., *L*-fuzzy interiors and *L*-fuzzy closures, *Fuzzy Sets and Systems*, 2009, 160, no. 9, 1218-1232.
[24] Shi F.-G. and Liang C., Measures of compactness in *L*-fuzzy pretopological spaces, *Journal of Intelligent and Fuzzy Systems*, 2014, 26, no. 3, 1557-1561.
[25] Shi F.-G., Countable compactness and the Lindelöf property of *L*-fuzzy sets, *Iranian Journal of Fuzzy Systems*, 2004, 1, no. 1, 79-88.
[26] Shi F.-G., A new notion of fuzzy compactness in *L*-topological spaces, *Information Science*, 2005, 173, no. 1-3, 35-48.