Bernoulli disjointness and maximally almost periodic groups

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Abstract

We show that any discrete group containing an infinite, normal, maximally almost periodic subgroup has the Bernoulli Disjointness Property, or BDJ. Also, any group containing enough infinite normal subgroups to separate points has the BDJ. Lastly, any group admitting a minimal free proximal flow has the BDJ.

1 Introduction

All groups and spaces considered in this paper are Hausdorff. Fix a topological group \( G \).

Recall that a \( G \)-flow is a compact Hausdorff space equipped with a continuous (left) \( G \)-action \( a : G \times X \to X \). When the action is understood, we suppress the notation and simply write \( g \cdot x \) or \( gx \) for \( a(g, x) \). If \( X \) and \( Y \) are \( G \)-flows, a \( G \)-map from \( X \) to \( Y \) is a map \( \varphi : X \to Y \) which respects the \( G \)-actions. A subflow of \( X \) is any \( Z \subseteq X \) which is non-empty, closed, and invariant under the action. The flow \( X \) is minimal if the only subflow of \( X \) is \( X \) itself; equivalently, \( X \) is minimal iff every orbit is dense.

If \( \{X_i : i \in I\} \) is a family of \( G \)-flows, then \( \prod_{i \in I} X_i \) is a \( G \)-flow, where \( g(x_i)_{i \in I} = (gx_i)_{i \in I} \). The projection map \( \pi_i \) onto the \( i^{th} \) coordinate is a \( G \)-map. The following notion was first isolated by Furstenberg in [5].

**Definition 1.1.** Let \( X_0 \) and \( X_1 \) be \( G \)-flows. We say that \( X_0 \) and \( X_1 \) are disjoint if for any subflow \( Y \subseteq X_0 \times X_1 \) with \( \pi_i(Y) = X_i \) for each \( i \in \{0, 1\} \), then \( Y = X_0 \times X_1 \).

One useful observation is that if \( X_0 \) and \( X_1 \) are disjoint, then at least one of them must be minimal; if \( Y_0 \) and \( Y_1 \) are proper subflows of \( X_0 \) and \( X_1 \), respectively, then the subflow \( Y_0 \times X_1 \cup X_0 \times Y_1 \) shows that \( X_0 \) and \( X_1 \) are not disjoint.

In this paper, we will mostly be interested in discrete groups. If \( G \) is a discrete group, the Bernoulli shift is the flow \( 2^G \), where given \( f \in 2^G \) and \( g, h \in G \), we have \( g \cdot f(h) = f(hg) \).
The following notion is due to Glasner and Weiss [6], though the definition we give here is slightly different.

**Definition 1.2.** A discrete group $G$ has the *Bernoulli Disjointness Property*, or BDJ, if every minimal $G$-flow is disjoint from the Bernoulli shift.

Suppose $\pi : X \to Y$ is a surjective $G$-map. Then if $Z$ is some flow with $X$ and $Z$ disjoint, then also $Y$ and $Z$ are disjoint. It is a fact (see [3]) that there is a universal minimal flow $M(G)$, a minimal flow which admits a $G$-map onto any other minimal flow, and this property defines $M(G)$ up to isomorphism. These observations give us the following fact.

**Fact 1.3.** For $G$ a discrete group, the following are equivalent.

1. $G$ has the BDJ.
2. $M(G)$ and $2^G$ are disjoint.

**Remark.** In [6], Glasner and Weiss consider countable discrete groups and say that a countable group $G$ has the BDJ iff every metrizable minimal $G$-flow is disjoint from the Bernoulli shift. It can be shown that for countable groups, this definition and ours coincide.

Furstenberg shows in [5] that $\mathbb{Z}$ has the BDJ; the recent paper of Glasner and Weiss [6] shows that among countable discrete groups, the amenable groups and the residually finite groups have the BDJ. They do this by introducing various notions of “disjointness” that a flow might enjoy and proving various theorems comparing them. One such notion that we will use here considers covering a minimal flow by certain translates of an open set, where some constraint is placed on the translates allowed.

**Definition 1.4.**

1. Let $D \subseteq G$ be finite. A set $S \subseteq G$ is *$D$-spaced* if for any $g \neq h \in S$, we have $Dg \cap Dh = \emptyset$.
2. Let $G$ be any group. A minimal flow $X$ has the *Dynamical Disjointness Property*, or DDJ, if for any finite $D \subseteq G$ and any non-empty open $A \subseteq X$, there is some $D$-spaced set $S \subseteq G$ with $S^{-1}A = X$.

Glasner and Weiss show that for a given metrizable minimal flow $X$, if $X$ has DDJ, then $X$ is disjoint from the Bernoulli shift. The first main theorem of this paper strengthens this implication significantly. Recall that a flow $X$ is said to be free if for any $x \in X$ and $g \in G \setminus \{1_G\}$, we have $gx \neq x$. If $F \subseteq G$ is finite, then we say that $X$ is $F$-free if for any $x \in X$ and $g \in F \setminus \{1_G\}$, we have $gx \neq x$. It turns out that for $G$ to be BDJ, one only needs “adequately many” minimal DDJ flows.
Definition 1.5. Let $G$ be a discrete group. We say that $G$ has adequate dynamical disjointness, or ADDJ, if for every finite $F \subseteq G$, there is a minimal $F$-free flow $X$ with the DDJ.

In particular, any group admitting a minimal free DDJ flow has the ADDJ.

Theorem 1.6. Let $G$ be a discrete group with the ADDJ. Then $G$ has the BDJ.

The main application of this theorem concerns maximally almost periodic groups, or maxap for short; these are groups which admit a continuous injective homomorphism into a compact group. Among the discrete groups, all abelian groups and all residually finite groups are maximally almost periodic.

If we demand that the homomorphism have dense image, then the Bohr compactification of $G$ is the largest such homomorphism, which we denote by $\varphi_G : G \to bG$. Here “largest” refers to the following universal property: if $K$ is a compact group and $\pi : G \to K$ is a homomorphism, then there is a unique continuous homomorphism $\psi : bG \to K$ so that $\pi = \psi \circ \varphi_G$. For $g \in G$ and $p \in bG$, we can set $g \cdot p = \varphi_G(g)p$ to endow $bG$ with the structure of a minimal $G$-flow. When $G$ is maxap, this action is free.

Very recently, Glasner and Weiss have independently shown that maxap groups have the BDJ. While our main theorem provides another proof of this result, we only require $G$ to have some infinite, normal, maxap subgroup.

Theorem 1.7. Let $G$ be a discrete group containing an infinite, normal, maxap subgroup. Then $G$ has the ADDJ, hence also the BDJ.

We also show that another class of discrete groups generalizing the residually finite groups has the BDJ. The following theorem is much easier to prove, but makes crucial use of the fact that to show ADDJ, one only needs finite fragments of freeness.

Theorem 1.8. Let $G$ be a discrete group, and suppose for every finite $F \subseteq G$, there is an infinite normal subgroup $H \subseteq G$ with $F^{-1}F \cap H = 1_G$. Then $G$ has the ADDJ, hence also the BDJ.

Glasner and Weiss have recently used Theorem 1.6 to show that $C^*$-simple groups have the BDJ. A group is said to be $C^*$-simple if the action of the group on its Furstenberg boundary, the universal strongly proximal flow, is free. Recall that the Furstenberg boundary is in some sense an indicator of by how much a group is or is not amenable; a group is amenable iff the Furstenberg boundary is trivial. Glasner and Weiss show that $C^*$-simple groups have the BDJ by arguing that the Furstenberg boundary has the DDJ. This led the current author to consider the case of proximal flows. Recall that a $G$-flow $X$ is proximal iff for any $x, y \in X$, there is a net $g_i$ from $G$ with $(g_i x, g_i y) \to \Delta_X$, the diagonal. In particular, strongly proximal flows, hence the Furstenberg boundary, are proximal.
Theorem 1.9. Every proximal minimal flow has the DDJ. Hence if \( G \) admits a free proximal flow, then \( G \) has the BDJ.

One application of the BDJ worth pointing out concerns a problem in topological dynamics often attributed to Ellis [4], but likely due to Furstenberg (see [6]). Given a topological group \( G \), two somewhat “canonical” \( G \)-flows one can form are the Samuel compactification \( S(G) \), the Gelfand space of the bounded right uniformly continuous functions from \( G \) to \( \mathbb{C} \), and the enveloping semigroup of the universal minimal flow, denoted \( E(M(G)) \). For more on these flows, see [1].

Problem 1.10. Suppose \( G \) is a topological group. Can \( S(G) \) and \( E(M(G)) \) be isomorphic?

If the right completion of \( G \) is compact, then all of \( S(G) \), \( M(G) \) and \( E(M(G)) \) are isomorphic to this completion. In [7], Pestov conjectured that this trivial case is the only possibility.

Conjecture 1.11 ([7], p. 4163). Suppose \( G \) is not precompact. Then \( S(G) \not\cong E(M(G)) \).

For discrete groups, one can show (first see section III, part 5 of [5], then [1] or [6]) that if \( G \) has the BDJ, then \( S(G) \not\cong E(M(G)) \).

Corollary 1.12. Let \( G \) be a discrete group as in either Theorem 1.7 or Corollary 1.8. Then \( S(G) \not\cong E(M(G)) \).

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2 Proof of Theorem 1.6

In this section, fix a discrete group \( G \) with the ADDJ. Our goal is to show that any subflow of \( M(G) \times 2^G \) with full projection onto each coordinate is the whole product. Notice that if \( \alpha \in 2^G \) has dense orbit, then for any \( p \in M(G) \), the flow \( G \cdot (p, \alpha) \) has full projection onto each coordinate. So it suffices to show that \( G \cdot (p, \alpha) \subseteq M(G) \times 2^G \) is dense.

So fix \( p \in M(G) \) and \( \alpha \in 2^G \) as above. We also fix \( F \subseteq G \) finite, \( \beta \in 2^F \), and \( N_\beta := \{ \gamma \in 2^G : \gamma|_F = \beta \} \). Then \( N_\beta \) is a typical basic open neighborhood of \( 2^G \). Setting \( S_\beta := \{ g \in G : g \cdot \alpha \in N_\beta \} \), it is enough to show that \( S_\beta \cdot p \subseteq M(G) \) is dense. We first prove a general lemma valid for any group, where given a minimal \( G \)-flow \( X \), \( x \in X \), and \( S \subseteq G \) we provide a sufficient condition for \( S \cdot x \subseteq X \) to be dense.

Definition 2.1. Let \( X \) be a \( G \)-flow, and let \( A \subseteq X \) be nonempty open. For \( x \in X \), we set \( \text{Ret}(x, A) := \{ g \in G : gx \in A \} \).
We will frequently make use of the fact that when $X$ is a minimal flow and $A \subseteq X$ is nonempty open, the set $\text{Ret}(x, A)$ is syndetic; where $S \subseteq G$ is syndetic if for some finite $D \subseteq G$, we have $Dg \cap S \neq \emptyset$ for every $g \in G$. Given $S \subseteq G$ and finite $D \subseteq G$, we say $S$ is $D$-syndetic if $Dg \cap S \neq \emptyset$ for every $g \in G$.

**Lemma 2.2.** Let $X$ be a minimal $G$-flow, and let $\{A_0, ..., A_{k-1}\}$ be an open cover of $X$. Let $S \subseteq G$, and suppose that for every finite $D \subseteq G$ and every $i < k$, there is $g \in G$ so that $(D \cap \text{Ret}(x, A_i))g \subseteq \text{Ret}(x, A_i) \cap S$. Then $S \cdot x \subseteq X$ is dense.

**Proof.** For some $i < k$, suppose $B \subseteq A_i$ is nonempty open. We will find $g \in S$ with $gx \in B$. Find finite $D \subseteq G$ so that $\text{Ret}(x, B)$ is $D$-syndetic.

Call a subset $D_0 \subseteq D$ $A_i$-maximal if both of the following hold:

1. For some $g \in G$, $D_0g \subseteq \text{Ret}(x, A_i)$.
2. For every $g \in G$, if $D_0g \subseteq \text{Ret}(x, A_i)$, then $D_0g = Dg \cap \text{Ret}(x, A_i)$.

Fix some $A_i$-maximal $D_0 \subseteq D$ with $D \cap \text{Ret}(x, A_i) \subseteq D_0$. Notice that we can find $g \in G$ with $D_0g \subseteq \text{Ret}(x, A_i) \cap S$; to do this, find $h \in G$ with $D_0h = Dh \cap \text{Ret}(x, A_i)$, then use our standing assumptions to find $k \in G$ with $(D_0h \cap \text{Ret}(x, A_i))k = D_0hk \subseteq \text{Ret}(x, A_i) \cap S$. Then $g = hk$ is as desired.

Since $\text{Ret}(x, B)$ is $D$-syndetic, there is $h \in D$ with $hg \in \text{Ret}(x, B)$. So $hg \in \text{Ret}(x, A_i)$, and since $D_0$ is $A_i$-maximal, we must have $h \in D_0$. In particular, $hg \in S \cap \text{Ret}(x, B)$ as desired. 

**Remark.** Suppose $X$, $x$, and $\{A_0, ..., A_{k-1}\}$ are as in Lemma 2.2. Let $Y$ is another minimal flow with $\pi : Y \to X$ a $G$-map, and let $y \in Y$ with $\pi(y) = x$. Then if $S \subseteq G$ satisfies the hypotheses of the lemma, then also $S \cdot y \subseteq Y$ is dense. This is because all the hypotheses about $S$ are also valid regarding $Y$, $y$, and $\{\pi^{-1}(A_0), ..., \pi^{-1}(A_{k-1})\}$. In particular, we have $\text{Ret}(y, \pi^{-1}(A_i)) = \text{Ret}(x, A_i)$.

We now return to the situation at hand, where $G$ is a discrete group with ADDJ and $p$, $\alpha$, $F$, $\beta$, $N_\beta$, and $S_\beta$ are as defined before Definition 2.1. Using the ADDJ, let $X$ be a minimal $F^{-1}F$-free flow with the DDJ. We can find an open cover $\{A_0, ..., A_{k-1}\}$ with the property that for each $g \in F^{-1}F \setminus \{1_G\}$ and each $i < k$, we have $gA_i \cap A_i = \emptyset$. This implies that for any $x \in X$, the sets $\text{Ret}(x, A_i)$ are $F$-spaced.

We now fix a $G$-map $\pi : M(G) \to X$.

**Proposition 2.3.** With notation as above, then for any finite $D \subseteq G$ and $i < k$, there is a finite $F$-spaced set $W \subseteq G$ so that for any $g \in G$, there is $h \in G$ with $(D \cap \text{Ret}(\pi(p), A_i))h \subseteq Wg \cap \text{Ret}(\pi(p), A_i)$.
Proof. Fix \( D \) and \( i < k \) as above. Find an open set \( B \subseteq X \), \( \pi(p) \in B \), with the property that \((D \cap \text{Ret}(\pi(p), A_i))B \subseteq A_i\). Because \( X \) is a DDJ flow, there is an FD-spaced set \( \{h_j : j < \ell\} \) with \( X = \bigcup_{j<\ell} h_j^{-1}B \). Set \( W = \bigcup_{j<\ell}(D \cap \text{Ret}(\pi(p), A_i))h_j \). Notice that since \( \text{Ret}(\pi(p), A_i) \) is \( F \)-spaced and since \( \{h_j : j < \ell\} \) is \( F \)-spaced, we have that \( W \) is \( F \)-spaced.

Now let \( g \in G \). Then \( h_jg\pi(p) \in B \) for some \( j < \ell \). It follows that \((D \cap \text{Ret}(\pi(p), A_i))h_jg \subseteq Wg \cap \text{Ret}(\pi(p), A_i)\) as desired.

We can now complete the proof of Theorem 1.6 by showing that \( S_\beta \cdot p \subseteq M(G) \) is dense. To do this, let the flow \( X \) and the open cover \( \{A_i : i < k\} \) be as in Proposition 2.3. Fix some finite \( D \subseteq G \) and \( i < k \), and produce \( W \) as in Proposition 2.3. Now since \( W \) is \( F \)-spaced, we can form \( \delta \in 2^{FW} \), where if \( f \in F \) and \( w \in W \), we set \( \delta(fw) = \beta(f) \). As \( \alpha \) has dense orbit, there is \( g \in G \) with \( g\alpha|_{FW} = \delta \); in particular, this implies that \( Wg \subseteq S_\beta \). The conclusion now follows from Lemma 2.2.

3 Constructing DDJ flows

We first isolate a property of flows which implies the DDJ.

Definition 3.1. Let \( G \) be an infinite discrete group, and let \( X \) be a \( G \)-flow. We say that \( X \) has open recurrence if for every nonempty open \( A \subseteq X \), there is non-empty open \( B \subseteq A \) so that \( \{g \in G : gB \subseteq A\} \) is infinite.

Proposition 3.2. Let \( X \) be a minimal flow with open recurrence. Then \( X \) has the DDJ.

Proof. Fix \( D \subseteq G \) finite and \( A \subseteq X \) non-empty open. Let \( B \subseteq A \) be a non-empty open subset witnessing open recurrence. As \( X \) is minimal, find \( g_0, \ldots, g_{k-1} \in G \) with \( X = \bigcup_{i<k} g_iB \). We will find \( h_0, \ldots, h_{k-1} \in G \) with \( X = \bigcup_{i<k} h_iA \) so that \( \{h_i^{-1} : i < k\} \) is \( D \)-spaced. Set \( h_0 = g_0 \). If \( h_0, \ldots, h_{i-1} \) have been chosen, we choose some \( h_i \) so that \( h_i \notin h_jD^{-1}D \) for any \( j < i \) and so that \( g_iB \subseteq h_iA \).

Proof of Theorem 1.7. Let \( H \subseteq G \) be an infinite, normal, maxap subgroup. Let \( Y \) be a minimal free \( G/H \)-flow; then \( Y \) is also a minimal \( G \)-flow.

We will now construct a minimal \( G \)-flow \( X \) which will be a subflow of \( (Y \times bH)^G \); the shift action here is similar to the action on the Bernoulli shift. Given \( \gamma \in (Y \times bH)^G \), we will often write \( \gamma = \gamma_0 \times \gamma_1 \), where \( \gamma_0 \in Y^G \) and \( \gamma_1 \in bH^G \). A warning about the notation: given \( g, h \in G \) and \( \gamma \in (Y \times bH)^G \), note the difference between \( g \cdot (\gamma_0(h)) \) and \( g \cdot \gamma_0(h) := \gamma_0(hg) \). A similar warning for \( \gamma_1 \) also applies.

So now consider the collection

\[
Z = \{\gamma \in (Y \times bH)^G : \forall a, b \in G \forall h \in H \; \gamma_0(ab) = a \cdot (\gamma_0(b)) \text{ and } \gamma_1(hb) = h \cdot (\gamma_1(b))\}.
\]
Then $Z$ is non-empty, closed, and $G$-invariant, so let $X$ be any minimal subflow of $Z$. Notice that $X$ is free. To see this, fix $\gamma \in Z$ and $a \in G \setminus \{1_G\}$. If $a \not\in H$, we have $a \cdot \gamma_0(a) = \gamma_0(a^2) = a \cdot (\gamma_0(a)) \neq \gamma_0(a)$. If $a \in H$, then similarly $a \cdot \gamma_1(a) = a \cdot (\gamma_1(a)) \neq \gamma_1(a)$.

Consider a basic open neighborhood of $X$ of the form

$$A := \{ \gamma \in X : \gamma_0(g_i) \in A_i \text{ and } \gamma_1(g_i) \in B_i \},$$

where $A_i \subseteq Y$ and $B_i \subseteq bH$. For each $i < k$, let $\varphi_i : H \to H$ be the automorphism $\varphi_i(h) = g_i hg_i^{-1}$. We continuously extend $\varphi_i$ to an automorphism of $bH$. Now in the topological group $bH$, find open $C_i \subseteq B_i$ and an open symmetric neighborhood of the identity $U$ with $UC_i \subseteq B_i$. Then find an open symmetric neighborhood of the identity $V$ so that $\varphi_i(U) \subseteq U$ for each $i < k$.

Now we define

$$B := \{ \gamma \in X : \gamma_0(g_i) \in A_i \text{ and } \gamma_1(g_i) \in C_i \}.$$

Notice that if $\gamma \in B$ and $h \in H$ with $\varphi_h(h) \in V$, then

$$h \cdot \gamma_1(g_i) = \gamma_1(g_i h) = \gamma_1(\varphi_i(h) g_i) = \varphi_i(h) \cdot (\gamma_1(g_i)) \in B_i.$$

It follows that for such $h \in H$, we have $h \cdot B \subseteq A$. Since $\varphi_h^{-1}(V)$ is infinite, this shows that $X$ has open recurrence. \hfill $\square$

**Proof of Theorem 1.8** Let $F \subseteq G$ be finite; to show the ADDJ, it is enough by Theorem 1.6 to construct some minimal $F$-free DDJ flow. By our assumption on $G$, fix $H$ an infinite normal subgroup with $F^{-1}F \cap H = \{1_G\}$. In particular, this implies that the quotient map $\pi : G \to G/H$ is injective on $F$.

Let $X$ be a minimal free $G/H$-flow. Then $X$ is also a minimal $F$-free $G$-flow. Let $A \subseteq X$ be open. Then for any $h \in H$, we have $h \cdot A = A$. It immediately follows that $X$ has open recurrence. \hfill $\square$

# Proximal flows

This section is dedicated to the proof of Theorem 1.9. The first folklore fact we need is Proposition 4.2, which says that for any given pair of points in a proximal flow, an abundance of group elements bring them close together.

**Definition 4.1.** Fix $T \subseteq G$.

1. $T$ is **thick** if for every finite $F \subseteq G$, there is $g \in G$ with $Fg \subseteq T$.

2. $T$ is **syndetic** if $G \setminus T$ is not thick; equivalently, if there is a finite $F \subseteq G$ with $FT = G$. 

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3. $T$ is piecewise syndetic if $T = T_1 \cap T_2$ with $T_1$ thick and $T_2$ syndetic; equivalently, there is a net $g_i$ in $G$ with $g_i \cdot \chi_T \to \chi_S$ for some syndetic $S \subseteq G$.

4. $T$ is thickly syndetic if $G \setminus T$ is not piecewise syndetic; equivalently, if for every finite $F \subseteq G$, there is a syndetic $S \subseteq G$ so that $FS \subseteq T$.

**Proposition 4.2.** Let $X$ be a minimal proximal flow, and let $\rho$ be a continuous pseudometric on $X$. For each $\epsilon > 0$, set $T_\epsilon := \{ g \in G : \rho(gx, gy) < \epsilon \}$. Then for any $x, y \in X$ and $\epsilon > 0$, $T_\epsilon$ is thickly syndetic.

**Proof.** First we show that for every $\delta > 0$, the set $T_\delta$ is syndetic. If this were not the case, we could find for each finite $D \subseteq G$ some $g_D \in G$ with $\rho(hg_Dx, hg_Dy) \geq \delta$ for each $h \in D$. Passing to a convergent subnet, we obtain $x', y' \in X$ so that $\rho(gx', gy') \geq \delta$ for every $g \in G$, contradicting proximality.

Now fix $F \subseteq G$ finite and $\epsilon > 0$. Then there is $\delta > 0$ so that whenever $\rho(x, y) < \delta$ and $f \in F$, we have $\rho(fx, fy) < \epsilon$. It follows that $FT_\delta \subseteq T_\epsilon$. \hfill $\Box$

**Proposition 4.3.** Let $X$ be a minimal flow of weight $|G| = \kappa$, and fix $x \in X$ and $D \subseteq G$ finite. Then there is a $D$-spaced set $S \subseteq X$ so that for any thickly syndetic set $T \subseteq G$, we have $(S \cap T) \cdot x \subseteq X$ dense.

**Proof.** We can find $\kappa$-many disjoint thick sets $\{ T_\alpha : \alpha < \kappa \}$ using Theorem 2.4 of [2] (the assumption there that $\kappa$ is regular is not needed here). By replacing $T_\alpha$ with $\bigcap_{g \in F} g^{-1}T_\alpha$ if necessary, we can assume that $FT_\alpha \cap FT_\beta = \emptyset$ for $\alpha < \beta < \kappa$.

Fix an enumeration $\{ U_\alpha : \alpha < \kappa \}$ of a basis for $X$. Let $S_\alpha$ be a maximal $D$-spaced subset of $\text{Ret}(x, U_\alpha)$. Notice that $S_\alpha$ is syndetic. Set $S = \bigcup_{\alpha < \kappa} S_\alpha \cap T_\alpha$.

To see that $S$ is as desired, let $T \subseteq G$ be thickly syndetic. Then for any $\alpha < \kappa$, the set $S_\alpha \cap T_\alpha$ is piecewise syndetic, so $T \cap S_\alpha \cap T_\alpha \neq \emptyset$. For any $g \in T \cap S_\alpha \cap T_\alpha$, we have $gx \in U_\alpha$. \hfill $\Box$

**Proof of Theorem 1.9.** Letting $\kappa = |G|$, every minimal $G$-flow is an inverse limit of flows of weight at most $\kappa$. As the DDJ is preserved under inverse limits, we may assume that $X$ has weight at most $\kappa$.

Fix finite $D \subseteq G$. Pick $x \in X$, and let $S \subseteq G$ be a $D$-spaced set as guaranteed by Proposition 4.3. We will show that for any $y \in X$, $S \cdot y \subseteq X$ is dense.

If this failed for some $y \in X$, we can find a pseudometric $\rho$ and an open ball $B_\rho(z, \epsilon) := \{ w \in X : \rho(z, w) < \epsilon \}$ so that $B_\rho(z, 2\epsilon) \cap S \cdot y = \emptyset$. However, letting $T_\epsilon$ be as in Proposition 4.2 we can find $g \in S \cap T_\epsilon$ with $gx \in B_\rho(z, \epsilon)$, implying that $gy \in B_\rho(z, 2\epsilon)$, a contradiction. \hfill $\Box$
References

[1] D. Bartošová and A. Zucker, Fraïssé structures and a conjecture of Furstenberg, Comment. Math. Univ. Carolin., to appear.

[2] T.J. Carlson, N. Hindman, J. McLeod, and D. Strauss, Almost disjoint large subsets of semigroups. Topol. Appl., 155(5) (2008), 433–444.

[3] R. Ellis, Universal minimal sets, Proc. Amer. Math. Soc., 11 (1960), 540–543.

[4] R. Ellis, Lectures in Topological Dynamics, W.A. Benjamin, 1969.

[5] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. Math. Systems Theory, 1 (1967), 1–49.

[6] E. Glasner and B. Weiss, On the disjointness property of groups and a conjecture of Furstenberg, arXiv:1807.08493

[7] V. Pestov, On free actions, minimal flows, and a problem by Ellis, Trans. Amer. Math. Soc., 350 (10), (1998), 4149–4165.

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