NO AREA LAW IN QCD

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Wilson’s area law in QCD is critically examined. It is shown that the expectation value of the Wilson loop integral \( \exp(\int iA_\mu dx^\mu) \) in the strong coupling limit vanishes when we employ the conjugate Wilson action which has a proper QED action in the continuum limit. The finite value of Wilson loop with the Wilson action is due to the result of the artifact. The fact that his area law is obtained even for QED simply indicates that the area law is unphysical.

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I. INTRODUCTION

Since Wilson \(^{11}\) proposed Wilson loop in evaluating the gauge field theories on the lattice, it has been commonly accepted that Wilson’s area law is the basic conceptual ingredients to understand a confining mechanism in QCD. Wilson’s formulation of the lattice version of QCD is interesting, and in addition it can be used for evaluating the lattice QCD in terms of numerical calculations. Indeed, there are many calculations of the lattice QCD simulations which confirm the confining mechanism of Wilson’s criteria \(^{2}\).

In his paper, Wilson presented an area law which can suggest a confining potential of the following type \(^{1}\)

\[
V(R) \simeq \frac{\ln g^2}{a^2} R
\]  

(1)

where \(a\) denotes a lattice constant. Therefore, it means that the QCD confining potential is a linear rising one. Further, this shape of the potential is well reproduced by the lattice QCD simulations \(^{3}\).

However, one may have somewhat an uneasy feeling on eq.(1). That is, eq.(1) does not have a proper dimension since the lattice constant (units) cannot play a role of a physical quantity. In fact, if we take the continuum limit of eq.(1), then the right hand side goes to infinity, and therefore it becomes unphysical. Here, we should explain the continuum limit itself in order to avoid any confusions. This is closely related to the way one solves field theory. When we wish to solve field theory models, the space and time are of course from \(-\infty\) to \(\infty\). However, we cannot solve the field theory models in this space and time, and therefore normally we put the theory into a box with its length \(L\). In this case, we can solve field theory models within the box and after calculations we should make the length \(L\) much larger than any other scales of the models, which is called thermodynamic limit. If this continuum field theory model can be solved in some way or the other, then we can obtain physical observables after the thermodynamic limit is taken.

Sometimes people want to solve the field theory by cutting the space and time into a lattice. Wilson’s way of solving QCD is just this lattice field theory. In this case, we should cut the space and time by the site number \(N\), and therefore the lattice constant \(a\) becomes

\[
a = \frac{L}{N}
\]  

(2)

where it is to be noted that the site number \(N\) is not a particle number of field theory model. In this case, it is clear that the lattice constant is not a physical quantity since it depends as to how one wishes to cut the space and time. In fact, one can see that the potential [eq.(1)] indeed diverges when one wants to cut the space as small as possible to simulate the continuum space, keeping the box length \(L\) finite. Therefore, it is clear that, physically, eq.(1) does not make sense.

Here, it should be important to note that the lattice constant \(a\) in solid state physics has a completely different meaning from the continuum field theory since the constant \(a\) of the lattice is fixed and finite. In this case, therefore, there is no ambiguity of cutting the space, and thus the thermodynamic limit is the only concern in the calculation of physical observables.

In this paper, we show that the finite value of the Wilson loop is an artifact due to the Wilson action since it should inevitably pick up unphysical contributions to the loop integral. A similar work has been done by Grady \(^{4}\) who shows that the action different from the Wilson action gives no area law in SU(2) lattice gauge theory. There are some works which critically examine lattice calculations \(^{3,4}\), but we do not go into details here.

II. AMBIGUITIES IN AVERAGING PROCEDURE

Now, we should discuss the ambiguities in evaluation of the expectation values. The discussion here will be used in the later sections to understand the way of evaluating the average of a Wilson loop. As examples, we employ the distribution function \(p_{\pm}(A)\) as

\[
p_{\pm}(A) = \exp(h e^{\pm iA}) = \sum_{k=0}^{\infty} \frac{h^k}{k!} e^{\pm ikA}
\]  

(3)
where $h$ is a constant. This type of the distribution function is often used in mathematics in order to pick up some integer from certain complicated functions.

Now, we make an expectation value of $A$ and obtain

$$\langle A \rangle_\pm = \sum_{k=0}^{\infty} \frac{h^k}{k!} \int_{-X}^{X} e^{i A + i k A} dA / \int_{-X}^{X} p_\pm(A) dA = 0 \quad (4)$$

where $X$ should be eventually set to infinity. Eq.(4) shows that there is no finite expectation value of $A$ with the distribution function of $p_\pm(A)$ in eq.(3).

However, if we make expectation values of $e^{i n A}$ where $n$ is a positive integer, we obtain

$$\langle e^{i n A} \rangle_+ = \sum_{k=0}^{\infty} \frac{h^k}{k!} \int_{-X}^{X} e^{i n A + i k A} dA / \int_{-X}^{X} p_+(A) dA = 0 \quad (5)$$

$$\langle e^{i n A} \rangle_- = \sum_{k=0}^{\infty} \frac{h^k}{k!} \int_{-X}^{X} e^{i n A - i k A} dA / \int_{-X}^{X} p_-(A) dA = \frac{h^n}{n!}. \quad (6)$$

The expectation value of $\langle e^{i n A} \rangle_-$ survives even though the expectation value of $\langle e^{i n A} \rangle_+$ vanishes. The last result [eq.(6)] shows just the same trick as Wilson used in obtaining a finite expectation value of the loop integral, and one notices that the procedure is just to pick up the integer value $n$ in the function of $e^{i n A}$.

### III. WiloN’s Area Law

Now, we come to the area law in Wilson’s paper. Most of the lattice formulations of the field theory are well written in his paper, and QCD on a lattice itself is quite interesting. In particular, he introduced Wilson’s loop and evaluated it in terms of Euclidean path integral formulation. Here, we repeat the discussion in his paper using the same notations. The expectation value of the Wilson loop is written as

$$\langle \exp \left( i \sum_p B_{\mu\nu} \right) \rangle_+ = Z_+^{-1} \prod_{m, \mu} \int_{-\pi}^{\pi} dB_{\mu\nu}$$

$$\times \exp \left( i \sum_p B_{\mu\nu} + \frac{1}{2g^2} \sum_{\mu\nu} \epsilon^{I \mu\nu} \right) \quad (7)$$

where the partition function $Z_+$ is defined as

$$Z_+ = \prod_{m, \mu} \int_{-\pi}^{\pi} dB_{\mu\nu} \exp \left( \frac{1}{2g^2} \sum_{\mu\nu} \epsilon^{I \mu\nu} \right). \quad (8)$$

$P$ denotes a path on the lattice. Here, the dimensionless quantity $B_{\mu\nu}$ is related to the discretized vector potential $A_{\mu\nu}$ as

$$B_{\mu\nu} = a g A_{\mu\nu}. \quad (9)$$

A dimensionless form of the field strength $F_{\mu\nu}$ is defined as

$$f_{\mu\nu} = a^2 g F_{\mu\nu} = B_{\mu\nu} + B_{\mu+\nu, \nu} - B_{\mu+\nu, \nu} - B_{\nu}. \quad (10)$$

Wilson imposed the periodic boundary conditions on the vector field $A_{\mu\nu}$,

$$A_{n, \mu} = A_{n+2, \mu}$$

where $N$ is related to the box length $L$ by $L = Na$.

In the strong coupling limit, eq.(7) can be expressed as

$$\langle \exp \left( i \sum_p B_{\mu\nu} \right) \rangle_+ = Z_+^{-1} \sum_k \frac{1}{k!} \left( \frac{1}{2g^2} \right)^k$$

$$\times \prod_{m, \mu} \int_{-\pi}^{\pi} dB_{\mu\nu} \sum \sum \sum \sum$$

$$\times \exp \left( i \sum_p B_{\mu\nu} + if_{\pi_1, \pi_1} + \cdots + if_{\pi_k, \pi_k} \right). \quad (11)$$

In the same way as eq.(7), eq.(11) has a finite contribution only when the exponent in the integrand is zero. Therefore, the nonzero terms in the sum are those for which satisfy

$$\sum_p B_{\mu\nu} + f_{\pi_1, \pi_1} + \cdots + f_{\pi_k, \pi_k} = 0. \quad (12)$$

Now, if we specify the path $P$ which contains $K$ plaquette, then we should find the number of $f_{\pi_k, \pi_k}$ with $k = K$. In this case, we see that eq.(12) is indeed satisfied. Therefore, the $k$ is the number of the plaquette that are surrounded by the path $P$. If this area is denoted by $A$, then we find that $k = K = \frac{A}{\pi^2}$.

Therefore, it is easy to find

$$\langle \exp \left( i \sum_p B_{\mu\nu} \right) \rangle_+ \sim (g^2)^{-A/\pi^2} \quad (13)$$

where $A$ should be described by some physical quantity like $A = RT$ with $T$ time distance in Euclidean space.

From eq.(13), one obtains eq.(1) using the method of transfer matrix which relates the Wilson loop to the potential $\square$. However, eq.(13) has a physically improper expression. That is, the area $A$ is described in terms of the physical quantity while $a^2$ is not a physical quantity since it should be eventually put to zero. Therefore, it is by now clear that the right hand side cannot survive when one takes the continuum limit.

### IV. A Trick in Wilson’s Calculation

Why did he obtain such an unphysical result? The answer is simple. The problem is related to the ambiguity
in the definition of the partition function (8). Wilson’s area law is obtained only when the Wilson action is employed. All other actions including Grady’s action [4] cannot reproduce the area law. We will show it below by a simple action with a negative sign in the exponential.

In eq.(8) the gauge field action is expressed as

\[ S_+ = \frac{1}{2g^2} \sum_{\mu \nu} e^{i f_{\mu \nu}} \]

instead of the conventional form \( -\frac{1}{4} a^4 \sum_{\mu \nu} F_{\mu \nu}^2 \). For small \( a \), this term can be expanded as

\[ S_+ = \frac{1}{2g^2} \sum_{\mu \nu} e^{i f_{\mu \nu}} \approx \frac{1}{2g^2} \sum_{\mu \nu} (1 + i f_{\mu \nu} - \frac{1}{2} f_{\mu \nu}^2 \cdots). \]

Eqs.(13) and (18) show that for a given path \( P \) \( \langle \exp(i \sum_{\mu} B_{\mu}) \rangle_+ \) and \( \langle \exp(i \sum_{\mu} B_{\mu}) \rangle_- \) behaves differently. This is because it is impossible to satisfy both eqs.(12) and (17) simultaneously for a given path \( P \).

Here, we should make a comment on the choice of the action Wilson employed. There is some argument that the action should be Hermitian, and therefore one should take the action of \((S_+ + S_-)/2\) which is indeed Hermitian. However, this modification of the action is a matter of no significance since the important part of the action is of \(-\frac{1}{4} a^4 \sum_{\mu \nu} F_{\mu \nu}^2 \) for small \( a \). Therefore, if the Hermiticity of the action is an important factor in the evaluation of physical observables, then it indicates that the action of eq.(14) as well as the Hermitian action is physically not a proper one.

Further, if one calculates the average using the conventional action of the gauge field, \(-\frac{1}{4} a^4 \sum_{\mu \nu} F_{\mu \nu}^2 \), then one obtains a different result which has no area law. In addition, if one employs an action, for example, \( S' = \frac{1}{8g^2} \sum_{\mu \nu} e^{2i f_{\mu \nu}} \), which has a proper continuum limit, then one obtains no contribution to the Wilson loop integral.

From the above discussion, it is concluded that the Wilson loop integral [eq.(13)] is accidentally finite as an artifact of the averaging procedure. This also indicates that the potential \( V(R) \approx \frac{\ln R}{R^2} \) obtained from eq.(13) has nothing to do with physics, which confirms our point in the first section.

V. WHAT IS THEN CONFINEMENT MECHANISM

We have proved that Wilson’s area law has nothing to do with the confinement in QCD. The basic mistake of Wilson’s calculation is due to his action which inevitably picks up unphysical contributions even though Wilson’s action can be reduced to a proper gauge field action in the continuum limit. However, higher order terms of his action contribute nonperturbatively to the Wilson loop integral and generate a fictitious area law as a result. Therefore, the “confining potential” which has been believed to be confirmed by the numerical calculations of the lattice QCD should be reexamined.

From the dynamical point of view, it is obvious that gluon dynamics without fermions cannot give the confinement mechanism since gluons produce interactions between quarks which should be always Coulomb like potential. Thus, one has to consider quark and anti-quark pairs in the intermediate states since they have masses and thus can induce a new scale. Therefore, physics of the confinement should be obtained only after one solved the QCD dynamics with quarks and gluons together. In this respect, when one carries out the lattice simulations of QCD, the quenched approximation is clearly meaningless. One should take into account the fermion degrees of freedom properly.

Now, what should be a new mechanism of confining
quarks? The confinement itself must be closely related to the gauge invariance of physical observables. The color non-singlet objects should be gauge dependent, and therefore they cannot be physically observed. The quark confinement should be due to the gauge non-invariance of the color charge and, because of it, the dynamical confinement should occur accordingly. In this respect, the fact that Wilson’s area law is proved for the U(1) gauge theory as well clearly shows that his calculated result should not have any relevance to the confinement mechanism in QCD.

In this sense, it is most probable that the essence of the confinement may well be realized by the MIT bag model.

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