Meta Distribution of Downlink SIR in a Poisson Cluster Process-based HetNet Model

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Abstract

The performance analysis of heterogeneous cellular networks (HetNets), that relied mostly on the homogeneous Poisson point process (PPP) for the spatial distribution of the users and base stations (BSs), has seen a major transition with the emergence of the Poisson cluster process (PCP)-based models. With the combination of PPP and PCP, it is possible to construct a general HetNet model which can capture formation of hotspots and spatial coupling between the users and the BSs. While the downlink coverage analysis of this model in terms of the distribution of the received downlink signal-to-interference ratio (SIR) is well understood by now, more fine grained analysis in terms of the meta distribution of SIR is an open problem. In this letter, we solve this problem by deriving the meta distribution of the downlink SIR assuming that the typical user connects to the BS providing the maximum received power.

Index Terms

Poisson cluster process, Poisson point process, meta distribution, stochastic geometry, cellular networks.

I. INTRODUCTION

The last few years have seen two major enhancements in the baseline approach to the modeling and analysis of cellular networks using stochastic geometry. (i) Enhancements in the model: While the baseline network models relied on homogeneous PPPs to model the spatial distribution of the BSs and users, the recent efforts have focused on using more sophisticated point processes to capture the spatial couplings between the locations of the BSs and users. A key set of works in this direction is based on the PCPs which along with PPPs result in a more general HetNet model with the PPP-based baseline network model being its special case. (ii) Enhancements in the metrics: The conventional analyses of HetNets using stochastic geometry have focused on the coverage probability which is the complementary cumulative distribution function (CCDF) of the signal-to-interference-and-noise-ratio (SINR). While coverage is a useful first-order metric, it does not provide any information on the variation of SINR over the network. To obtain a more fine-gained information on the SINR performance of the network, it is important to characterize the meta distribution of SINR from which the SINR-coverage can be obtained as a special case. While the meta distribution of SINR has been extensively studied for the baseline PPP-based HetNet models, this characterization for the PCP-driven general HetNet model, proposed in [2], is not known, which is the main objective of this letter.

Prior Art. The coverage analysis of the PPP-based cellular models is fairly mature by now (see [1] and the references therein). The meta distribution of SIR was first studied in [3] for the Poisson bipolar and cellular networks. It was subsequently extended to the K-tier PPP-based HetNet model in [4], [5]. On the modeling side, a more general HetNet model based on the combination of PPPs and PCPs was recently proposed in [2], [6], [7]. While this model yields several spatial configurations of cellular network that are of practical interest (including the baseline PPP-based model as its special case), its analytical treatment thus far has been limited to the coverage probability. In this letter, we derive the meta distribution of the downlink SIR for this model.

Contributions. In this letter, we consider a general K-tier HetNet model where the BSs and the user locations are modeled as either a PCP or a homogeneous PPP. For this model, we characterize the meta distribution of downlink SIR of the typical user assuming that the network is operating in an interference-limited regime and the typical user connects to the BS providing maximum received power averaged over fading. To enable the analysis, we construct an equivalent single tier cellular network by projecting the BS point processes in R^2 on the positive half line R^+ that will have the same distribution of SIR as the original 2-D K-tier network. Although the equivalence of the analyses using this single tier network in R^+ and the K-tier HetNet in R^2 is quite well-known for the PPP-based model (see [8]), this letter makes the first attempt to develop this approach for the new PCP-based HetNet models. While this alternate analytical framework for the general HetNet model is novel in its own right, we use this framework to derive the exact analytical expressions of the b-th moment of the conditional success probability for the typical user under Rayleigh fading. The exact expression of the meta distribution being computationally infeasible, we use the b-th moments to provide an accurate beta approximation of the CCDF of the meta distribution.
We model a HetNet as a $K$-tier cellular network in which BSs in the $i$-tier are distributed as a point process $\{x\} \equiv \Phi_i \subset \mathbb{R}^2$ and transmit with power $P_i$, which is assumed to be fixed for all the BSs in $\Phi_i$. The point process $\Phi_i$ is either a homogenous PPP with intensity $\lambda_i$ or a PCP. We denote the index sets of the BS tiers being modeled as PPP and PCP by $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively with $|\mathcal{K}_1| + |\mathcal{K}_2| = K$. While PPP, used as a baseline spatial model for cellular networks [1] needs no introduction, we define PCP for completeness as follows.

**Definition 1.** A PCP $\Phi_i(\lambda_p, \bar{m}_i, f_i)$ for $i \in \mathcal{K}_2$ is defined as:

$$\Phi_i(\lambda_p, \bar{m}_i, f_i) = \bigcup_{x \in \Phi_{\bar{m}_i}} z + \mathcal{B}^\sharp_i,$$

where $\Phi_{\bar{m}_i}$ is the parent PPP with intensity $\lambda_p$, and $\mathcal{B}^\sharp_i$ is the offspring point process. The offspring point process is a sequence of independently and identically distributed (i.i.d.) random variables with probability density function (PDF) $f_i(s)$. The number of points in $\mathcal{B}^\sharp_i$ is Poisson distributed with mean $\bar{m}_i$.

We further assume that the offspring points are isotropically distributed around the cluster center. Thus the joint PDF of the radial coordinates of the offspring points with respect to the cluster center is denoted as: $f_i(s, \theta_s) = f_i(1)(s) \frac{1}{\theta_s}$. That said, the PDF of the distance of a point $\Phi_i$ from the origin given its cluster center at $z \in \Phi_{\bar{m}_i}$ is given by: $f_{\bar{d}_i}(|r|z) = f_{\bar{d}_i}(r|z)$, where $|z| = z$. For the numerical results, we choose a special case of PCP, known as Thomas cluster process (TCP) where the offspring points in $\mathcal{B}^\sharp_i$ are distributed normally around the origin, i.e., $f_i(s) = \frac{1}{\sigma_i^2} \exp \left(-\frac{\|s\|^2}{2\sigma_i^2}\right)$. Here $\sigma_i^2$ is the cluster variance. When $\Phi_i$ is a TCP, the conditional distance distribution given $|z| = z$ is Ricean with PDF:

$$f_{\bar{d}_i}(x|z) = \frac{x}{\sigma_i^2} e^{-\frac{x^2}{2\sigma_i^2}} I_0 \left(\frac{2xz}{\sigma_i^2}\right), x, z \geq 0, i \in \mathcal{K}_2,$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind with order zero. We now focus on the user point process which is denoted as $\Phi_u$. We consider two types of users in the network:

- **Type 1** (independent user and BS point processes): $\Phi_u$ follows a stationary distribution independent of the BS point processes.
- **Type 2** (coupled user and BS point processes): $\Phi_u$ is a PCP with the same parent point process as that of $\Phi_q$ for some $q \in \mathcal{K}_2$ with cluster variance $\sigma_q^2$.

We now focus on the typical user in this network. Since the network is stationary, we can assume that the typical user is located at the origin. It should be noted that the selection of the typical user in Type 2 implies the selection of a cluster of $\Phi_q$ as well. We denote the center of this BS cluster by $z_0$. As a consequence, $\Phi_q$ is always conditioned on having a cluster $z_0 + \mathcal{B}^\sharp_q$. Thus the typical user perceives the palm version of $\Phi_q$, which, by Slivnyak’s theorem [9] is equivalent to $\Phi_q \cup z_0 + \mathcal{B}^\sharp_q$ where $\Phi_q$ and $z_0 + \mathcal{B}^\sharp_q$ are independent. For Type 1 users, this construction does not arise since the selection of the typical user does not impose any restriction on the BS distributions. In order to unify the analyses of Type 1 and Type 2 users, we define $\Phi_0$ as a set of BSs whose locations are coupled with that of the typical user as follows:

$$\Phi_0 = \begin{cases} \emptyset; & \text{Type 1,} \\ z_0 + \mathcal{B}^\sharp_q; & \text{Type 2.} \end{cases}$$

The BS point process perceived by the typical user can be defined as the superposition of $K+1$ BS point processes: $\Phi = \bigcup_{i \in \mathcal{K}} \Phi_i$, where $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \{0\}$. The downlink SIR of the typical user is denoted as:

$$\text{SIR} = \frac{\sum_{i \in \mathcal{K}} \sum_{x \in \Phi_i \setminus \{x^*\}} P_k h_k \|x\|^{-\alpha}}{\sum_{i \in \mathcal{K}} \sum_{x \in \Phi_i \setminus \{x^*\}} P_i h_i \|x\|^{-\alpha}},$$

where $\{h_k\}$ is an i.i.d. sequence of random variables where $h_k$ is the fading coefficient associated with the link between the typical user and the BS at $x$. We assume Rayleigh fading i.e. $h_k \sim \exp(1)$ and $\alpha > 2$ is the path loss exponent. Here $\|x^*\|$ is the location of the serving BS which is the BS that provides the maximum received power averaged over fading. Thus $x^* = \arg \max_{x \in \Phi_0} P_k \|x\|^{-\alpha},$ (4)

where $x_k^* = \arg \max_{x \in \Phi_k} P_k \|x\|^{-\alpha}$ is the location of the candidate serving BS in $\Phi_k$. In this letter, we are interested in a fine-grained analysis of SIR in terms of its meta distribution which is defined as follows.

**Definition 2.** The meta distribution of SIR is the CCDF of the conditional success probability $P_\text{s}(\beta) \triangleq P(\text{SIR} > \beta|\Phi)$, i.e.,

$$\bar{F}(\beta, \theta) = P(P_\text{s}(\beta) > \theta), \beta, \theta \in \mathbb{R}^+, \theta \in (0, 1].$$

Due to the ergodicity of $\Phi$, $\bar{F}(\beta, \theta)$ can be interpreted as the fraction of links in each realization of $\Phi$ that have an SIR greater than $\beta$ with probability at least $\theta$. According to Def. 2 the standard coverage probability [2] is the mean of $P_\text{s}(\theta)$ obtained by integrating [4] over $\theta \in [0,1].$
III. META DISTRIBUTION OF SIR

In this section, we will construct the equivalent single tier representation of the $K+1$ tier network defined in Section III by projecting $\Phi \subset \mathbb{R}^2$ on $\mathbb{R}^+$. For a PPP-based model, the equivalent network in $\mathbb{R}^+$ remains analytically tractable [8] because of the application of the mapping theorem [9], which is stated as follows.

**Theorem 1.** If $\Phi$ is a PPP in $\mathbb{R}^d$ with intensity $\lambda(x)$ and $f : \mathbb{R}^d \to \mathbb{R}^s$ is a measurable map with $\Lambda(f^{-1}\{y\}) = 0$, $\forall y \in \mathbb{R}^s$, then $f(\Phi) = \bigcup_{x \in \Phi} \{f(x)\}$ is a PPP with intensity measure $\Lambda(B') = \int_{f^{-1}(B')} \lambda(x)dx$, for all compact $B' \subset \mathbb{R}^s$.

Following Thm. [11] since $\{z\} = \Phi_i, \subset \mathbb{R}^2 (\forall i \in K_2)$ is a homogeneous PPP with intensity $\lambda_p$, then $\Phi_i \equiv \{\|x\|\} \subset \mathbb{R}^+ \text{ conditioned on } \Phi_i$, is an inhomogeneous PPP in $\mathbb{R}^+$ with density and intensity:

$$\lambda_{p_i}(z) = 2\pi \lambda_p z, \quad \bar{\lambda}_i(z) = \pi \lambda_p z^2, \quad z > 0,$$

(6)

respectively. Since Theorem [11] does not hold when $\Phi_i (\forall i \in K_2)$ is PCP, the projection of $\Phi$ on $\mathbb{R}^+$ cannot be handled on similar lines of [8]. The key enabler of our analysis is the following property of PCP which allows the application of Theorem 1 to $\Phi_i$ for $i \in K_2$.

**Lemma 1.** If $\Phi_i \equiv \{x\} \subset \mathbb{R}^2$ is a PCP, then the sequence $\Phi_i \equiv \{\|x\|\} \subset \mathbb{R}^+$ conditioned on $\Phi_i$, is an inhomogeneous PPP with density and intensity:

$$\bar{\lambda}_i(x|\Phi_i) = \bar{m}_i \sum_{x \in \Phi_i} F_{d_i}(x|z), \quad \bar{\lambda}(x|\Phi_i) = \bar{m}_i \sum_{z \in \Phi_i} f_{d_i}(x|z).$$

(7)

**Proof:** See [2, Prop. 1].

Following the same argument, for TYPE 2 users, $\Phi_0 = \{\|z_0\| \equiv z_0\}$ is also a PPP conditioned on $z_0$ with intensity $m_0 f_{d_0}(x|z_0) = m_0 f_{d_0}(x|z_0)$. Hence, we begin our analysis by first conditioning on the locations of the points in every parent PPP, i.e., $\Phi_i, \forall i \in K_2' \equiv K_2 \cup \{0\}$. Following Lemma 1 we have a sequence of BS PPPs $\{\Phi_i, i \in K_2'\}$ in $\mathbb{R}^+$ conditioned on $\bigcup_{i \in K_2'} \Phi_i$.

Let us define $\Phi_i = \{P_i^{-1}\|x\|^\alpha, \forall x \in \Phi_i\}$ as the projection of $\Phi_i (\forall i \in K_1 \cup K_2')$ on $\mathbb{R}^+$. For $i \in K_1$, using Theorem 1 the density of this 1-D inhomogeneous PPP $\Phi_i$ is

$$\bar{\lambda}_i(x) = \int_0^{2\pi} \int_0^{(P_i x_1)^{1/\alpha}} \lambda(x', x) dx' d\theta = \pi \lambda_i (x P_i)^{\frac{\alpha}{\alpha - 1}}.$$  

For $\Phi_i (i \in K_2')$ conditioned on $\Phi_i$, the density of $\Phi_i = \{P_i^{-1} x_0, \forall x \in \Phi_i\}$ is:

$$\bar{\lambda}_i(x|\Phi_i) = \int_0^{(P_i x_1)^{1/\alpha}} \sum_{x \in \Phi_i} f_{d_i}(x'|z) dx' = \bar{m}_i \sum_{z \in \Phi_i} F_{d_i}((x P_i)^{\frac{\alpha}{\alpha - 1}}|z).$$

Using the superposition theorem for PPP [9], the density function of the 1-D PPP $\Phi = \bigcup_{i \in K_2'} \Phi_i$, which is the projection of the $K+1$ tier HetNet $\Phi$, can be obtained as follows:

$$\bar{\lambda}(x|\bigcup_{i \in K_2'} \Phi_i) = \sum_{i \in K_1} \Lambda_i(x) + \sum_{i \in K_2'} \Lambda_i(x|\Phi_i)$$

$$= \sum_{i \in K_1} \pi \lambda_i (x P_i)^{\frac{\alpha}{\alpha - 1}} + \sum_{i \in K_2'} \bar{m}_i \sum_{z \in \Phi_i} F_{d_i}((x P_i)^{\frac{\alpha}{\alpha - 1}}|z),$$

(8)

and intensity:

$$\bar{\lambda}(x|\bigcup_{i \in K_2'} \Phi_i) = \frac{d}{dx} \bar{\lambda}(x|\bigcup_{i \in K_2'} \Phi_i)$$

$$= \sum_{i \in K_1} \pi \lambda_i x^{\frac{\alpha}{\alpha - 1}} + \sum_{i \in K_2'} \bar{m}_i \sum_{z \in \Phi_i} P_i^{\frac{1}{\alpha} x^{\frac{\alpha}{\alpha - 1}}} f_{d_i}((x P_i)^{\frac{\alpha}{\alpha - 1}}|z).$$

(9)

We are now in a position to define SIR of the typical user as:

$$\text{SIR} = \frac{h_2 \cdot (\dot{x}^*)^{-1}}{\sum_{x \in \Phi, x > x^*} h_2 x^{-1}},$$

where $x^* = \arg \min_{x \in \Phi} x$ is the point in $\Phi$ closest to the origin. The equivalence of the SIR-s expressed in terms of $\Phi$ and $\bar{\Phi}$ is formally stated in the following proposition.

**Proposition 1.** The SIR of a typical user in the $K+1$ tier HetNet $\Phi \subset \mathbb{R}^2$ with the BSs of the $i$-th tier transmitting at power $P_i$ and max-power based user association (defined in [8]) has the same distribution as that of a single tier 1-D network $\tilde{\Phi} \subset \mathbb{R}^+$ with nearest BS association, where $\tilde{\Phi}$ is an inhomogeneous PPP with intensity $\tilde{\lambda}(x|\bigcup_{i \in K_2' \cup \{0\}} \Phi_i)$, or equivalently, density
The distribution of \( \hat{\Phi} \) being unknown, the main contribution of the paper is to use the fact that conditional version of \( \hat{\Phi} \) given \( \cup_{i\in K} \hat{\Phi}_i \) is a PPP. We will then leverage the tractability of the PPP under the conditioning of the parent PPs and decondition with respect to \( \cup_{i\in K} \hat{\Phi}_i \) at the last step of the analysis. Since \( \hat{\Phi}_i | \cup_{i\in K} \hat{\Phi}_i \) is a PPP, the PDF of \( \hat{\alpha}^* \) is given by [2]:

\[
 f_{\hat{\alpha}}(x) = \lambda(x | \cup_{i\in K} \hat{\Phi}_i) \exp\left(-\Lambda(x | \cup_{i\in K} \hat{\Phi}_i)\right), \quad x > 0.
\]

The direct calculation of the meta distributions being infeasible even for the baseline PPP-based models [4], we first derive the expressions of the \( b \)-th order moments of \( P_s(\beta) \): \( M_b(\beta) \triangleq E[P_s(\beta)^b] \). Note that the coverage probability of the typical user in this setting, which was studied in our previous work [2], is a special case of this result and can be obtained directly by setting \( b = 1 \).

**Theorem 2.** The \( b \)-th moment of \( P_s(\beta) \), \( b \in \mathbb{C} \) can be expressed as:

\[
 M_b(\beta) = \sum_{i\in K_1} \pi \lambda_i \frac{2^b}{\alpha} P_i^\frac{1}{\alpha} \int_0^\infty Q(r) \prod_{j\in K_2} PG_{\Phi_j}(r) r^{\frac{1}{\alpha} - 1} dr + \sum_{i\in K_2} \int_0^\infty Q(r) \prod_{j\in K_1 \setminus \{i\}} PG_{\Phi_j}(r) SP_{\Phi_i}(r) dr,
\]

where

\[
 Q(r) = \exp\left(-\pi r^{\frac{1}{\alpha}} \sum_{j\in K_1} \lambda_j P_j^{\frac{1}{\alpha} - 1} f_j(r) (xP_j^{\frac{1}{\alpha}} | z) dx - \tilde{m}_j f_{\tilde{m}_j} \left((rP_j)^{\frac{1}{\alpha}} | z\right)\right),
\]

where \( f_j(r) \) is the hypergeometric function and

\[
 PG_{\Phi_j}(r) := \mathbb{E}\left[\prod_{z \in \Phi_j} g_j(r, z)\right], \quad (14a)
\]

and

\[
 SP_{\Phi_i}(r) := \mathbb{E}\left[\sum_{z \in \Phi_i} \rho_i(r, z) \prod_{z' \in \Phi_j} g_i(r, z')\right], \quad (14b)
\]

are the probability generating functional (PGFL) and sum-product functional (SPFL) of \( \Phi_j \) and \( \Phi_i \), respectively \((i, j) \in K_2^2\) with

\[
 g_j(r, z) = \exp\left(-\tilde{m}_j P_j^{\frac{1}{\alpha}} r^{\frac{1}{\alpha} - 1} \int_{\mathbb{R}} u(r, x) f_{\tilde{m}_j}(\langle xP_j^{\frac{1}{\alpha}} | z\rangle) dx - \tilde{m}_j f_{\tilde{m}_j} \left((rP_j)^{\frac{1}{\alpha}} | z\right)\right),
\]

\[
 \rho_i(r, z) = \tilde{m}_i P_i^{\frac{1}{\alpha}} r^{\frac{1}{\alpha} - 1} f_{\tilde{m}_i} \left((rP_i)^{\frac{1}{\alpha}} | z\right),
\]

where \( u(r, x) = (1 - (1 + \beta r/x)^{-b}) \).

**Proof:** See Appendix A.

We note that \( P_s(\theta) \) in (13) is expressed in terms of the PGFL and SPFL of \( \hat{\Phi}_p \). Hence we are left with deriving the expressions of PGFL and SPFL of \( \hat{\Phi}_p \), for \( i \in K_2 \). When \( i \in K_2 \), the PGFL and SPFL of \( \hat{\Phi}_p \) are known since it is a PPP [2] Lemmas 5.6). For Type 2 users, the PGFL and SPFL of \( \hat{\Phi}_p \) can be obtained by deconditioning over \( z_0 \), i.e. \( PG_{\hat{\Phi}_p}(r) = E_{z_0}[g_0(r, z_0)] \) and \( SP_{\hat{\Phi}_p}(r) = E_{z_0}[\rho_0(r, z_0)g_0(r, z_0)] \). We summarize the expressions of PGFL and SPFL in the following lemmas.

**Lemma 2.** The PGFL of \( \hat{\Phi}_p \), is given as:

\[
 PG_{\hat{\Phi}_p}(r) = \begin{cases}
 \exp\left(-\int_0^\infty 2\pi \lambda_i z (1 - g_i(r, z)) dz\right), & i \in K_2, \\
 \int_0^\infty g_i(r, z) f_{\tilde{m}_i}(z|0) dz, & i = 0, \text{Type 2}.
\end{cases}
\]

**Lemma 3.** The SPFL of \( \hat{\Phi}_p \), is given as:

\[
 SP_{\hat{\Phi}_p}(r) = \begin{cases}
 \int_0^\infty 2\pi \lambda_i z g_i(r, z) dz \exp\left(-\int_0^\infty 2\pi \lambda_i z' (1 - g_i(r, z')) dz'\right), & i \in K_2, \\
 \int_0^\infty \rho_i(r, z) g_i(r, z) f_{\tilde{m}_i}(z|0) dz, & i = 0, \text{Type 2}.
\end{cases}
\]

The final expression of \( M_b(\beta) \) is obtained by substituting \( PG_{\hat{\Phi}_p}(r) \) and \( SP_{\hat{\Phi}_p}(r) \) given by (16) and (17) to (12). The accuracy of these expressions for a two tier network is verified with the Monte Carlo simulations in Figs. 1(a) and 1(b). Note that we can also derive \( M_b(\beta) \) on similar lines of [2] by conditioning on the association to the BSs of the \( i \)-th tier \((i \in K)\). However, the single tier projection presented in this letter offers an alternate and more compact derivation of \( M_b(\beta) \). We note that \( M_1(\beta) \) and \( M_2(\beta) \) for Type 1 and 2 users converge to \( M_1(\beta) \) and \( M_2(\beta) \) for the baseline PPP model (i.e. where \( \Phi_1 \) and \( \Phi_2 \) are PPPs).
Gil-Pelaez theorem as:

\[ \bar{F}(\beta, \theta) = \frac{1}{\pi} \int_0^\infty \text{Im}(e^{-jt \theta} M_{jt}(\beta)) \frac{\text{d}t}{t}, \]

where \( \text{Im}(z) \) denotes the imaginary part of \( z \in \mathbb{C} \). As it can be readily observed, the expression of the exact meta distribution is not computationally efficient since it requires integration over the imaginary moments. Hence, following the approach of [3], [10], we approximate \( \bar{F}(\beta, \theta) \) with a beta-kernel \( \bar{F}(\beta, \theta) \approx 1 - \frac{1}{B(\theta_1, \theta_2)} \int_0^\theta \theta_1^{-1}(1 - t)^{\theta_2-1} \text{d}t \), where \( B(\cdot, \cdot) \) is the beta function and \((\theta_1, \theta_2)\) is given by solving the following system of equations:

\[ M_1 = \frac{\theta_1}{\theta_1 + \theta_2} \quad \text{and} \quad M_2 = \frac{\theta_2^2}{(\theta_1 + \theta_2)^2} \left( \frac{\theta_2}{\theta_1 + \theta_2 + 1} + 1 \right). \]

In Fig. 1(c) we plot \( \bar{F}(\beta, \theta) \) for a specific network configuration for Type 1 and Type 2 users. Clearly, the beta approximation of \( \bar{F}(\beta, \theta) \) is reasonably tight for a wide range of \( \beta \). Further, we observe that \( \bar{F}(\beta, \theta) \) of Type 2 users is greater than \( \bar{F}(\beta, \theta) \) in the baseline PPP-based model and \( \bar{F}(\beta, \theta) \) of Type 1 users is less than \( \bar{F}(\beta, \theta) \) in the baseline PPP-based model for all \( \beta, \theta \). This ordering is the same as the ordering observed for coverage probability (see Fig. 1(a) and [2 Sec. IV]). However, it is a stronger result than the ordering of coverage. This implies that for a given \( \beta \) and the same user density, there exists more number of Type 2 users satisfying SIR > \( \beta \) than Type 1 users in the network.

Fig. 1. Meta distribution of SIR for Type 1 and Type 2 users in a two-tier network. Details of the network configuration: \( K = 2, K_1 = \{1\}, K_2 = \{2\}, q = 2 \) for Type 2, \( \alpha = 4, P_2 = 10^2 P_1, \lambda_{p2} = 2.5 \text{ km}^{-2}, \lambda_{p1} = 1 \text{ km}^{-2}, m_2 = 4, \text{ and } \sigma_2 = \sigma_0 \). Markers indicate the values obtained from Monte Carlo simulations. The solid and dotted arrows in Fig. 1(a) indicate the shift of the quantities with the increase in cluster size \((\sigma_2 = \{20, 40, 60\} \text{ m})\). For Fig. 1(c) \( \sigma_2 = 40 \text{ m} \).

(For all \( i \in K \) and \( \Phi_u \) are homogeneous PPPs) as \( \sigma_2 \) increases. This is because of the fact that the PCP weakly converges to a homogeneous PPP as the cluster size tends to infinity [6 Sec. IV-B]. Further, for the two-tier network considered in Fig 1 following [7], it is possible to show that \( M_b(\beta), \forall b \in \mathbb{C} \) remains the same if \((\lambda_1, \lambda_{p2}, \sigma_2)\) is replaced by \((\lambda_1/\bar{\kappa}, \lambda_{p2}/\bar{\kappa}, \sigma_2 \bar{\kappa})\) for \( \bar{\kappa} > 0 \).

A. Approximation of Meta Distributions

From the \( b \)-th moment of the conditional success probability, the meta distribution of SIR can be obtained by using the Gil-Pelaez theorem as:

\[ M_b(\beta, \theta) \approx \frac{1}{\pi} \int_0^\infty \text{Im}(e^{-jt \theta} M_{jt}(\beta)) \frac{\text{d}t}{t}, \]

where \( \text{Im}(z) \) denotes the imaginary part of \( z \in \mathbb{C} \). This implies that for a given \( \beta \) and the same user density, there exists more number of Type 2 users satisfying SIR > \( \beta \) than Type 1 users in the network.
IV. Conclusion

In this letter, we characterized the meta distribution of the downlink SIR for the typical user in the general K-tier HetNet model where the BSs are distributed as a PPP or a PCP. The main technical contribution is the accurate derivation of the b-th order moments of the conditional success probability. The key enabling step of the analysis is to condition on the parent point process of the BS PCPs which allows us to treat the PCPs as inhomogeneous PPPs. Under this conditioning, we obtain a sequence of BS PPPs in \( \mathbb{R}^2 \) which are projected on \( \mathbb{R}^+ \) to construct a single tier equivalence of the multi-tier HetNet. Using this single tier network, we present a compact derivation of the b-th order moments of the conditional success probability by applying the PGFL and SPFL of the parent PPPs. We finally use the moments of the conditional success probability to compute a beta approximation of the meta distribution of SIR.

**Appendix**

A. Proof of Theorem 2

From (10),

\[
P_s(\theta) = \mathbb{P}\left( h_{x^*} > \beta \bar{x}^* \sum_{x < \bar{x}^*} h_x x^{-1} \right)
\]

\[
\overset{(a)}{=} \mathbb{E}\left[ \exp\left( - \beta \bar{x}^* \sum_{x < \bar{x}^*} h_x x^{-1} \right) \right]
\]

\[
= \mathbb{E}\left[ \prod_{x < \bar{x}^*} (1 + \beta \bar{x}^* x^{-1})^{-1} \right].
\]

Here (a) follows from the CCDF of exponential distribution and the last step follows from the fact that \( \{h_x\} \) is a sequence of i.i.d. exponential random variables. Now,

\[
P_s(\beta)|_{\bigcup_i \bar{K}_2(0)} \bar{\Phi}_{p_i} = \mathbb{E}\left[ \prod_{x > \bar{x}^*} (1 + \beta \bar{x}^* x^{-1})^{-b} \bigcup_i \bar{\Phi}_{p_i} \right]
\]

\[
\overset{(a)}{=} \exp\left( - \int_{\bar{x}^*}^{\infty} \left( 1 - \frac{\beta \bar{x}^*}{x} \right)^{-b} \lambda(x) \bigcup_i \bar{\Phi}_{p_i} \right) dx
\]

\[
\overset{(b)}{=} \int_0^{\infty} \exp\left( - \int r \left( \lambda(x) \bigcup_i \bar{\Phi}_{p_i} \right) dx \right) \exp\left( - \bar{\Lambda}(r) \bigcup_i \bar{\Phi}_{p_i} \right) dr
\]

\[
= \int_0^{\infty} \exp\left( - \int r \left( \lambda(x) \bigcup_i \bar{\Phi}_{p_i} \right) dx - \bar{\Lambda}(r) \bigcup_i \bar{\Phi}_{p_i} \right) \left( \sum_{i \in \bar{K}_1} \frac{\pi \lambda_i^2}{\alpha} P_{p_i}^{\bar{\phi}_i} r^{\frac{-1}{\alpha}} \right) dr
\]

\[
+ \sum_{i \in \bar{K}_2} \sum_{z \in \bar{\Phi}_{p_i}} P_{p_i}^{\bar{\phi}_i} r^{\frac{-1}{\alpha}} f_{d_i}\left((r P_{p_i})^{\bar{\phi}_i}|z\right) dr.
\]

**\( T_1(\bigcup_{i \in \bar{K}_2} \bar{\Phi}_{p_i}) + T_2(\bigcup_{i \in \bar{K}_2} \bar{\Phi}_{p_i}), \)**

where

\[
T_1(\bigcup_{i \in \bar{K}_2} \bar{\Phi}_{p_i}) = \sum_{i \in \bar{K}_1} \pi \lambda_i^2 \frac{2}{\alpha} P_{p_i}^{\bar{\phi}_i} \int_0^{\infty} \exp\left( - \int r \left( \lambda(x) \bigcup_i \bar{\Phi}_{p_i} \right) dx - \bar{\Lambda}(r) \bigcup_i \bar{\Phi}_{p_i} \right) r^{\frac{-1}{\alpha}} dr,
\]

\[
T_2(\bigcup_{i \in \bar{K}_2} \bar{\Phi}_{p_i}) = \sum_{i \in \bar{K}_2} \sum_{z \in \bar{\Phi}_{p_i}} \int_0^{\infty} \sum_{d_i} \frac{1}{\alpha} f_{d_i}\left((r P_{p_i})^{\bar{\phi}_i}|z\right) \times \exp\left( - \int r \left( \lambda(x) \bigcup_i \bar{\Phi}_{p_i} \right) dx - \bar{\Lambda}(r) \bigcup_i \bar{\Phi}_{p_i} \right) r^{\frac{-1}{\alpha}} dr.
\]

Here (a) follows from the PGFL of the PPP (see Lemma 2). (b) is obtained by deconditioning over \( \bar{x}^* \) whose PDF is given by (11). We are left with deconditioning \( T_1 \) and \( T_2 \) w.r.t. the distributions of the parent point processes for \( i \in \bar{K}_2 \) and \( z_0 \) for TYPE 2 users. We now derive the expressions of \( \mathbb{E}[T_1(\bigcup_{i \in \bar{K}_2} \bar{\Phi}_{p_i})] \) and \( \mathbb{E}[T_2(\bigcup_{i \in \bar{K}_2} \bar{\Phi}_{p_i})] \) as follows:

\[
\mathbb{E}[T_1(\bigcup_{i \in \bar{K}_2} \bar{\Phi}_{p_i})] = \sum_{i \in \bar{K}_1} \pi \lambda_i^2 \frac{2}{\alpha} P_{p_i}^{\bar{\phi}_i} \mathbb{E}\left[ \exp\left( - \int r \left( \lambda(x) \bigcup_i \bar{\Phi}_{p_i} \right) dx - \bar{\Lambda}(r) \bigcup_i \bar{\Phi}_{p_i} \right) \right] r^{\frac{-1}{\alpha}} dr
\]

\[
\mathbb{E}[T_2(\bigcup_{i \in \bar{K}_2} \bar{\Phi}_{p_i})] = \sum_{i \in \bar{K}_2} \sum_{z \in \bar{\Phi}_{p_i}} \mathbb{E}\left[ \exp\left( - \int r \left( \lambda(x) \bigcup_i \bar{\Phi}_{p_i} \right) dx - \bar{\Lambda}(r) \bigcup_i \bar{\Phi}_{p_i} \right) \right] f_{d_i}\left((r P_{p_i})^{\bar{\phi}_i}|z\right) r^{\frac{-1}{\alpha}} dr.
\]
\[ \begin{align*}
&= \sum_{i \in K_1} \frac{\pi \lambda_i}{\alpha} P_i^\frac{2}{\alpha} r_i^\frac{\alpha}{2} \int_0^\infty \prod_{j \in K_1} \exp \left( -\int_r^\infty u(r, x) \pi \lambda_j \frac{2}{\alpha} P_j^\frac{2}{\alpha} x^\frac{\alpha}{2} -1 \, dx - \pi \lambda_j (r P_j)^\frac{\alpha}{2} \right) \\
&\times \prod_{j \in K_2} \mathbb{E} \left[ \prod_{z \in \Phi_{p_j}} \exp \left( -\int_r^\infty \tilde{m}_j P_j^\frac{1}{\alpha} r^\frac{1}{\alpha} \frac{1}{\alpha} u(r, x) f_{d_j}((r P_j)^\frac{1}{\alpha} | z) \, dx \right) - \tilde{m}_j F_{d_j}((r P_j)^\frac{1}{\alpha} | z) \right] 
\end{align*} \]

where the last step is obtained by substituting \( \tilde{\Lambda} \) and \( \tilde{\lambda} \) with (8) and (9), respectively. Applying the same substitution,

\[ \begin{align*}
&\mathbb{E} \left[ T_2 \left( \bigcup_{i \in K_2} \hat{\Phi}_{p_i} \right) \right] = \sum_{i \in K_2} \int_0^\infty \tilde{m}_i \mathbb{E} \left[ \sum_{z \in \Phi_{p_i}} P_i^\frac{1}{\alpha} r_i^\frac{1}{\alpha} f_{d_i}((r P_i)^\frac{1}{\alpha} | z) \right. \\
&\left. \times \exp \left( -\int_r^\infty u(r, x) \tilde{\lambda}(x | \bigcup_{i \in K_2} \hat{\Phi}_{p_i}) \, dx - \tilde{\Lambda}(r | \bigcup_{i \in K_2} \hat{\Phi}_{p_i}) \right) \right] \, dr \\
&= \sum_{i \in K_2} \int_0^\infty \sum_{z \in \Phi_{p_i}} \tilde{m}_i P_i^\frac{1}{\alpha} r_i^\frac{1}{\alpha} f_{d_i}((r P_i)^\frac{1}{\alpha} | z) \exp \left( -\int_r^\infty u(r, x) \pi \lambda_j \frac{2}{\alpha} P_j^\frac{2}{\alpha} x^\frac{\alpha}{2} -1 \, dx - \pi \lambda_j (r P_j)^\frac{\alpha}{2} \right) \\
&\quad + \sum_{j \in K_2} \sum_{z \in \Phi_{p_j}} \tilde{m}_j P_j^\frac{1}{\alpha} r_j^\frac{1}{\alpha} f_{d_j}((x P_j)^\frac{1}{\alpha} | z) \, dx - \sum_{j \in K_2} \pi \lambda_j (r P_j)^\frac{\alpha}{2} \\
&\quad \left. + \sum_{j \in K_2} \sum_{z \in \Phi_{p_j}} \tilde{m}_j F_{d_j}((r P_j)^\frac{1}{\alpha} | z) \right] \, dr \\
&= \sum_{i \in K_2} \int_0^\infty \prod_{j \in K_1} \exp \left( -\int_r^\infty u(r, x) \pi \lambda_j \frac{2}{\alpha} P_j^\frac{2}{\alpha} x^\frac{\alpha}{2} -1 \, dx - \pi \lambda_j (r P_j)^\frac{\alpha}{2} \right) \mathbb{E} \left[ \sum_{z \in \Phi_{p_i}} \tilde{m}_i P_i^\frac{1}{\alpha} r_i^\frac{1}{\alpha} -1 \right. \\
&\left. \times f_{d_i}((r P_i)^\frac{1}{\alpha} | z) \prod_{j \in K_2 \setminus \{i\}} \prod_{z \in \Phi_{p_j}} \exp \left( -\int_r^\infty u(r, x) \tilde{m}_j P_j^\frac{1}{\alpha} r_j^\frac{1}{\alpha} f_{d_j}((x P_j)^\frac{1}{\alpha} | z) \, dx \right. \\
&\left. - \tilde{m}_j F_{d_j}((r P_j)^\frac{1}{\alpha} | z) \right] \, dr \\
&= \sum_{i \in K_2} \int_0^\infty Q(r) \prod_{j \in K_2 \setminus \{i\}} P \mathbb{E} \left[ \sum_{z \in \Phi_{p_i}} P_i^\frac{1}{\alpha} r_i^\frac{1}{\alpha} -1 \tilde{m}_i f_{d_i}((r P_i)^\frac{1}{\alpha} | z) \right. \\
&\left. \prod_{z \in \Phi_{p_i}} \exp \left( -\int_r^\infty u(r, x) \tilde{m}_i P_i^\frac{1}{\alpha} r_i^\frac{1}{\alpha} -1 f_{d_i}((x P_i)^\frac{1}{\alpha} | z) \, dx - \tilde{m}_i F_{d_i}((r P_i)^\frac{1}{\alpha} | z) \right) \right] \, dr.
\end{align*} \]

The expression spanning over the last two lines can be identified as \( \mathcal{S}_{\Phi_{p_i}}(r) \) (see (14b)). In the above expressions, \( Q(r) \) can be further simplified to (13).

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