Glassy behaviour in disordered systems with non-relaxational dynamics

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We show that a family of disordered systems with non-relaxational dynamics may exhibit “glassy” behavior at nonzero temperature, although such a behavior appears to be ruled out by a face-value application of mean-field theory. Nevertheless, the roots of this behavior can be understood within mean-field theory itself, properly interpreted. Finite systems belonging to this family have a dynamical regime with a self-similar pattern of alternating periods of fast motion and trapping.

The dynamics of disordered physical systems exhibits “glassy” features such as ergodicity breaking, slow dynamics and aging. These systems are usually modelled by purely relaxational stochastic processes satisfying detailed balance. In this situation, glassiness arises from the complexity of the energy landscape: the representative point strives to go downhill in energy along the maximal slope while receiving random kicks from the thermal noise, and gets trapped into deeper and deeper energy valleys.

Disordered physical systems have often been considered a paradigm for complex behaviour in other fields, especially biology. However, there are no compelling reasons, in these fields, to restrict oneself to purely relaxational dynamics. It is therefore important to know if the glassy properties exhibited by purely relaxational dynamical models are also present in the general case and, if so, whether the mechanisms responsible for glassy dynamics in non-relaxational systems are completely different from those acting in purely relaxational ones, even when the violation of detailed balance is small. If they were different, the picture that has evolved to explain the behaviour of disordered systems would be entirely irrelevant as soon as non-dissipative forces are turned on.

Hertz et al., Parisi and especially Crisanti and Sompolinsky (CS) showed some years ago that spin-glass behaviour was destroyed in several mean-field disordered models by an arbitrarily small but generic violation of detailed balance.

We show in this paper that there is a wide class of disordered systems in which glassy behavior resists non-relaxational perturbations (even at $T > 0$), provided that either the initial condition is properly chosen, or that the size of the system is finite. (We shall not deal here with effects that are exclusive of the zero temperature case.) This behavior should be relevant for infinite systems in a finite dimensional space. The reason for this robustness can be understood, at least qualitatively, by a suitable interpretation of mean-field theory as follows.

The stability of each energy (resp. free-energy) saddle can be characterized by the lowest eigenvalue $\lambda_{\text{min}}$ of the Hessian. A class of models, of which that of Sherrington and Kirkpatrick (SK) is the best-known representative, are marginal in the sense that in almost all states $\lambda_{\text{min}}$ goes to zero in the thermodynamical limit, and the single-state spin-glass susceptibility diverges. It is then not surprising that in such purely marginal models a small (but still $O(N)$) perturbation may completely change their dynamics, as found in.

However, there is another class of models (currently thought to mimic ‘fragile’ structural glasses) having many non-marginal states with non-zero $\lambda_{\text{min}}$ and for which the spin-glass susceptibility within such states is finite. Non-marginal states look locally much like ferromagnetic states or the retrieval states in the Hopfield model. Now, it is easy to convince oneself that the stability properties of a non-marginal state cannot be dramatically altered by the combined effects of arbitrarily small non-relaxational forces and thermal noise.

Consider for definiteness the typical case of the $p$-spin spherical spin-glass. The energy landscape of the system features many saddle points of the energy function at different energy-density values $\mathcal{E} < 0$. One can identify a threshold value $\mathcal{E}_{\text{th}}$ such that only the saddle points with $\mathcal{E} < \mathcal{E}_{\text{th}}$ are minima. The lowest eigenvalue $\lambda_{\text{min}}$ at each saddle point is proportional to the depth of the state beneath the threshold $\mathcal{E}_{\text{th}} - \mathcal{E}$. The only marginal saddle points are those just below the threshold, unlike the case of marginal models for which all states are marginal (or, in other words, for which there are no minima deep below the threshold). From the reasoning above, we might expect that only near-threshold minima will be destabilized by infinitesimal asymmetries, whereas the deeper a state, the more robust it will be.

In the $p$-spin spherical glass, purely relaxational dynamics starting from a random initial condition exhibits
“aging” \[14\]: when \( N = \infty \), the system keeps touring a region just above the threshold, moving slower and slower but without ever getting completely trapped. For large but finite \( N \) (or for an infinite system in any finite dimensionality) the system penetrates a time-dependent amount below the threshold, and still ages due to the increasing depth of the traps it finds. In this purely relaxational case, the dynamics is qualitatively similar to that of a marginal model.

Now, just as a non-relaxational perturbation destroys aging in a marginal model, it also seems to destroy aging \textit{around the threshold} in a non-marginal one. However, in non-marginal models, there are an infinite number of deeper states that remain stable in the presence of the perturbation. Hence, as soon as a finite system (or an infinite system in finite dimensions) is able to penetrate below the threshold, it rediscovers the glassy features which had been destroyed above and near the threshold.

In what follows we confirm this scenario for non-marginal models with non-relaxational dynamics. We first give evidence that mean-field dynamics (\( N = \infty \)), starting from a random initial condition, yields for long times a time-translational invariant solution for the correlation and response functions even at small asymmetries (no aging), and that the correlation functions decay to zero (no ergodicity breaking), confirming CS \[5\].

We then show, always within mean-field dynamics, that there are initial conditions such that the correlations do not decay to zero, in the presence of non-relaxational perturbations, even at \( T > 0 \). This confirms the existence of stable regions with ergodicity breaking. Obviously, such regions are never found if the system is infinite and starts from a random configuration.

Finally, in order to estimate the time scales involved for trapping in a large but finite system, we performed simulations.

We consider a system of \( N \) variables \( s = (s_1, \ldots, s_N) \), subject to forces \( F_i \) given by

\[
F_i(s) = \sum_{\{j_1, \ldots, j_{p-1}\}} J^{i \ldots j_{p-1}}_i s_{j_1} \cdots s_{j_{p-1}},
\]

(1)

where the couplings are random Gaussian variables. For different sets of indices \( \{i, j_1, \ldots, j_{p-1}\} \) the \( J^i \)'s are uncorrelated, while for permutations of the same set of indices they are correlated so that

\[
\overline{F_i(s')F_j(s)} = \delta_{ij} f_1(q) + s'_i s_j f_2(q)/N,
\]

(2)

where \( q = (s \cdot s')/N \). In the purely relaxational case one has \( f_2(q) = f_1(q) \). We consider here \( f_2(q) = \alpha f_1(q) \), where \( f_1(q) = pq^{p-1}/2 \). The purely relaxational case has symmetric couplings \( J^{i \ldots j_{p-1}}_i \) under \( i \leftrightarrow j_k \) (\( \alpha = 1 \)) while for uncorrelated \( J^{i \ldots j_{p-1}}_i \) and \( J^{j_1 \ldots j_{p-1}}_i \); \( \alpha = 0 \).

We consider: (i) the continuous spherical model \(|s|^2 = N\), with Langevin dynamics \( \dot{s}_i = -F_i(s) - z(t)s_i + h_i(t) + \eta_i(t) \), where \( \eta \) is a white noise of variance \( 2T = 2/\beta \), \( z(t) \) is a Lagrange multiplier enforcing the constraint and \( h_i(t) \) is an external field (usually set to zero); (ii) the Ising \( s_i = \pm 1 \) model with Metropolis dynamics, in which a randomly chosen spin flips with probability \( \min[1, \exp(-2\beta s_i F_i)] \). The “energy” \( \mathcal{E} \) is defined in all cases by the expression \( N\mathcal{E}(s) = - (\mathbf{F} \cdot s)/p \) to which only the symmetric part of the force contributes. We monitor the autocorrelation function \( C(t, t') \equiv \langle s(t) \cdot s(t') \rangle/N \) and the response function \( G(t, t') \equiv \langle \delta (s_i(t)) / \delta h_i(t') \rangle/N \).

Both models are \textit{marginal} for \( p = 2 \). The dynamics of the asymmetric \( p = 2 \) spherical and \( \pm 1 \) (SK) models were studied numerically and by mean-field theory (large \( N \)) by CS, who found no glassiness at finite \( T \). Here we concentrate on the \textit{non-marginal} case \( p > 2 \).

We have solved numerically the mean-field equations (exact for \( N \rightarrow \infty \)) for the correlation and the response function of the spherical (\( p = 3 \)) model, starting from a random initial configuration \[14\]. The result for \( \alpha = 0.96 \) and \( T = 0.2 \) is shown in Fig. 1. The corresponding curves for \( \alpha = 1 \) are shown in the inset. On can see that there is an initial regime in which the solution exhibits aging as for \( \alpha = 1 \), followed by a crossover to time-translational invariance (no aging). The closer \( \alpha \) is to one, the longer the (interrupted) aging regime lasts. We have not found any evidence for a critical asymmetry (\( \alpha_c < 1 \)) beyond which the system ages forever although we cannot rule out a transition for \( \alpha \) even closer to one. The correlations decay to zero ruling out ergodicity-breaking also as in CS.

In order to prove that there are trapping regions for \( \alpha < 1, T > 0 \), we have studied the mean-field dynamics starting from a configuration with given (low) energy,
but otherwise random. This amounts to solving a static problem for the initial condition, at a temperature $T'$ tuned to give the desired value of the energy. Figure 2 shows that for $\alpha > 0.86$ and sufficiently small initial energies the system remains trapped at non-zero temperature (forever, if $N = \infty$).

![Figure 2](image)

FIG. 2. $C(t_w + \tau, t_w)$ vs. $\tau$ for $t_w = 100$ at $T = 0.1$ starting from an initial condition with low energy-density ($T' = 0.1$). From top to bottom the asymmetry parameter equals $\alpha = 0.88, 0.86, 0.84, 0.82, 0.8$.

$C > q_{E\alpha}$ (FDT) and $X = \text{constant} < 1$ for $C < q_{E\alpha}$. It is remarkable that the plot crosses over smoothly, as $\alpha \to 1$, to the one holding for the relaxational case (note that $X[C] \leq 1$, $\forall C$, $\forall \alpha$). In the inset we plot the FD ratio starting from low energy, for $\alpha = 0.8$ (untrapped) and $\alpha = 0.9$ (trapped).

In conclusion, the mean-field analysis reveals that the present model has trapping regions that cannot be reached when $N = \infty$ because the time needed to fall into them diverges with $N$ and, by the same token, that the time needed to escape from a trap also diverges. The question how typical falling and escaping times scale with $N$ is beyond the present analytical tools. We have thus performed numerical simulations, choosing for convenience the ±1 version of the $(p = 3)$ model.

Figure 4 shows the typical behaviour at $T = 0.01$ and $\alpha = 0.5$, for $N = 50$. The system alternates between periods of trapping and periods of rapid motion at high energy. A blow-up in time of the same run shows that the overall appearance of the graph is self-similar. The longest trapping time is of the order of the total observation time, which indicates a broad distribution of release times. We have found such behaviour, for $N = 50$, in a region in the $(T, \alpha)$ plane bounded by $\alpha = 0.4$ at $T \sim 0$ and $T = T_c \sim 0.05$ at $\alpha = 1$. Traps are visited once showing that there is a large time-span between the smallest falling time and the maximal (‘equilibration’) trapping time. For larger system sizes, $N = 100, 200$ falling and escaping times increase with $N$, as expected.

This behaviour is reminiscent of the non-relaxational dynamics of a particle in a random velocity field where anomalous diffusion is due to broad, Lévy-stable distributions of trapping times. Indeed, this model likely to be close to a microscopical realisation of the related ‘trap model’ that has been fruitfully used to describe aging in.

We have therefore shown that aging and ergodicity breaking resist non-relaxational perturbations in the dynamics of disordered systems if they have non-marginal states. Finite-size fully connected systems (relevant, e.g., for modelling biological networks) may thus exhibit striking aging effects. This is also likely to be the case for finite-dimensional systems whose mean-field limit is non-marginal.

Let us also mention that we have checked that systems with random forces deriving from a potential and strongly perturbed by random non-potential forces correlated with a different range exhibit aging phenomena even at the mean-field level.

Further scenarios can be envisaged: it is likely that some no-go results obtained almost a decade ago did not exhaust all possibilities of nature and that this direction is open for further research.
FIG. 4. a. Instantaneous energy $E$ vs. time (Monte-Carlo sweeps) in the $\pm 1 \ (p = 3)$ model for $\alpha = 0.5$ and $T = 0.01$. b. A blow-up of the first 10,000 sweeps.

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[16] These equations are derived by standard functional methods via a saddle-point integration. The propagator self-energy $\Sigma(t, t')$ becomes equal to $f_2[C(t, t')]G(t, t')$, while the noise renormalization vertex $D(t, t')$ becomes equal to $f_1[C(t, t')]$ in the limit $N \to \infty$. See, e.g. J-P Bouchaud, L. Cugliandolo, J. Kurchan and M. Mézard; Physica A226, 243 (1996).
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