Reducibility of 1D quantum harmonic oscillator perturbed by a quasiperiodic potential with logarithmic decay

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Abstract

In this paper we prove an infinite dimensional KAM theorem, in which the assumptions on the derivatives of the perturbation in [24] are weakened from polynomial decay to logarithmic decay. As a consequence, we can apply it to 1D quantum harmonic oscillators and prove the reducibility of the linear harmonic oscillator, \( T = -\frac{d^2}{dx^2} + x^2 \), on \( L^2(\mathbb{R}) \) perturbed by the quasi-periodic in the time potential \( V(x, \omega t; \omega) \) with logarithmic decay. This proves the pure-point nature of the spectrum of the Floquet operator \( K \), where

\[
K := -i \sum_{k=1}^{n} \omega_k \frac{\partial}{\partial \theta_k} - \frac{d^2}{dx^2} + x^2 + \varepsilon V(x, \theta; \omega)
\]

is defined on \( L^2(\mathbb{R}) \otimes L^2(\mathbb{T}^d) \), and the potential \( V(x, \theta; \omega) \) has logarithmic decay as well as its gradient in \( \omega \).

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1. Introduction and main results

1.1. Statement of the results

In this paper we consider the linear equation

\[ i\partial_t u = -\partial_x^2 u + \varepsilon V(x, \omega t; \omega) u, \quad u = u(t, x), \quad x \in \mathbb{R}, \]  

(1.1)

where \( \varepsilon > 0 \) is a small parameter and the frequency vector \( \omega \) of the forced oscillations is regarded as a parameter in \( \Pi := [0, 2\pi)^n \). We assume that the potential \( V : \mathbb{R} \times \mathbb{T}^n \ni (x, \theta; \omega) \mapsto \mathbb{R} \) is \( C^3 \) smooth in all its variables, and analytic in \( \theta \) where \( \mathbb{T}^n = \mathbb{R}^n/2\pi \mathbb{Z}^n \) denotes the \( n \)-dimensional torus. For \( \rho > 0 \), the function \( V(x, \theta; \omega) \) analytically extends to the domain \( \mathbb{T}^n_{\rho} = \{(a + bi) \in \mathbb{C}^n/2\pi \mathbb{Z}^n : |b| < \rho\} \) as well as its gradient in \( \omega \), and satisfies

\[ |V(x, \theta; \omega)|, |\partial_x V(x, \theta; \omega)| \leq C(1 + \ln(1 + |x|^2))^{-23}, \]  

(1.2)

\[ |\partial_x V(x, \theta; \omega)|, |\partial_x \partial_{\theta} V(x, \theta; \omega)| \leq C, \]  

(1.3)

\[ |\partial_x^2 V(x, \theta; \omega)|, |\partial_x^2 \partial_{\theta} V(x, \theta; \omega)| \leq C, \]  

(1.4)

where \( (x, \theta; \omega) \in \mathbb{R} \times \mathbb{T}^n_{\rho} \times \Pi, \beta \geq 2(n + 2), j = 1, \cdots, n \) and \( C > 0 \).

**Theorem 1.1.** Assume that \( V \) satisfies (1.2)–(1.4) and \( \beta \geq 2(n + 2) \). Then, there exists \( \varepsilon_0 \) such that for all \( 0 \leq \varepsilon < \varepsilon_0 \) there exists \( \Pi_{\varepsilon} \subset [0, 2\pi)^n \) of a positive measure with the asymptotically full measure: \( \text{Meas}(\Pi_{\varepsilon}) \to (2\pi)^n \) as \( \varepsilon \to 0 \), such that for all \( \omega \in \Pi_{\varepsilon} \), the linear Schrödinger equation (1.1) reduces, in \( L^2(\mathbb{R}) \), to a linear equation with constant coefficients (with respect to the time variable).

**Remark 1.2.** Very recently, Bambusi has posted two preprints \([2, 3]\). Using techniques from pseudo-differential calculus, Bambusi \([2, 3]\) proved that the equation (1.1) is reducible even when the potential \( V(x, \theta; \omega) \) grows sub-quadratically under some additional assumptions about the potential.

Similar to \([24]\), the above theorem has two direct corollaries. As a preparation, we define the harmonic oscillator \( T = -\frac{d^2}{dx^2} + x^2 \) and its related Sobolev space. Let \( p \geq 2 \) and denote \( \ell^{2,p} \) as the Hilbert space of all real-valued functions \( f \) with

\[ \|f\|_{\ell^{2,p}}^2 = \sum_{j \geq 1} j^p |w_j|^2 < \infty. \]

The operator \( T \) has the eigenfunctions \( (h_j)_{j \geq 1} \)—the so-called Hermite functions—which satisfy

\[ Th_j = (2j - 1)h_j, \quad \|h_j\|_{L^2(\mathbb{R})} = 1, \quad j \geq 1, \]  

(1.5)

and form a Hilbertian basis of \( L^2(\mathbb{R}) \). Let \( u = \sum_{j \geq 1} u_j h_j \) be a typical element of \( L^2(\mathbb{R}) \). Then \( (u_j) \in \ell^{2,p} \), if and only if

\[ u \in \mathcal{H}^p := \{ u \in L^2(\mathbb{R}) : x^{\alpha_1} \partial^{\alpha_2} u \in L^2(\mathbb{R}) \text{ for } 0 \leq \alpha_1 + \alpha_2 \leq p \}. \]

For the function \( f \in \mathcal{H}^p(\mathbb{R}) \), we define

\[ \|f\|_{\ell^{2,p}}^2 = \sum_{0 \leq \alpha_1 + \alpha_2 \leq p} \|x^{\alpha_1} \partial^{\alpha_2} f\|_{L^2}^2 < \infty. \]


Then we have:

**Corollary 1.3.** Assume that $V$ and $\partial_\nu V$, $j = 1, \ldots, n$, are $C^\infty$ in $x$, and there exists a constant $C > 0$ such that for all $\nu \geq 1$, $x \in \mathbb{R}$ and $|\theta| < \rho$,

$$
|V(x, \theta; \omega)|, |\partial_\nu V(x, \theta; \omega)| \leq C(1 + \ln(1 + x^2))^{-1},
$$

$$
|\partial^2_\nu V(x, \theta; \omega)|, |\partial^2_\nu V(x, \theta; \omega)| \leq C.
$$

Let $p \geq 0$, $u_0 \in H^p$ and $\beta \geq 2(n + 2)$. Then there exists $\varepsilon_0 > 0$ so that for all $0 \leq \varepsilon < \varepsilon_0$ and $\omega \in \Pi$, there exists a unique solution $u \in C(\mathbb{R}, H^p)$ of (1.1) so that $u(0) = u_0$. Moreover, $u$ is almost-periodic in time and we have the bounds

$$(1 - \varepsilon C)||u_0||_{H^p} \leq ||u(t)||_{H^p} \leq (1 + \varepsilon C)||u_0||_{H^p},$$

with $t \in \mathbb{R}$, for some $C = C(p, \omega)$.

Consider the Floquet operator on $L^2(\mathbb{R}) \otimes L^2(\mathbb{T}^n)$

$$K := -i \sum_{k=1}^n \omega_k \partial_\theta \frac{d^2}{dx^2} + x^2 + \varepsilon V(x, \theta; \omega),$$

then we have:

**Corollary 1.4.** Assume that $V$ satisfies the same conditions as in theorem 1.1 and $\beta \geq 2(n + 2)$. There exists $\varepsilon_0 > 0$ so that for all $0 \leq \varepsilon < \varepsilon_0$ and $\omega \in \Pi$, the spectrum of the Floquet operator $K$ is a pure point.

### 1.2. Related results

As in [1] the equations (1.1) can be generalized into a time-dependent Schrödinger equation

$$i \partial_t \psi(t) = (A + \varepsilon P(\omega t))\psi(t),$$

where $A$ is a positive self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and the perturbation $P$ is an operator-valued function from $\mathbb{T}^n$ into the space of symmetric operators on $\mathcal{H}$. The Floquet operator associated with (1.7) is defined by

$$K_F := -i \omega \cdot \partial_\theta + A + \varepsilon P(\theta) \quad \text{on} \quad \mathcal{H} \otimes L^2(\mathbb{T}^n).$$

It is well known that the long-time behaviour of the solution $\psi(t)$ of the time-dependent Schrödinger equation (1.7) is closely related to the spectral properties of the Floquet operator $K_F$ (see Wang [44]). It has been proved in [4, 9–11, 22, 26, 37, 38] that the Floquet operator $K_F$ is of the pure-point spectra, or no absolutely continuous spectra where $P$ is bounded. When $P$ is unbounded, the first result was obtained by Bambusi and Graffi [1], where they considered the time-dependent Schrödinger equation

$$i \partial_t \psi(x, t) = H(t)\psi(x, t), x \in \mathbb{R}; \quad H(t) := -\frac{d^2}{dx^2} + Q(x) + \varepsilon V(x, \omega t), \quad \varepsilon \in \mathbb{R},$$

where $Q(x) \sim |x|^\alpha$ with $\alpha > 2$ as $|x| \to \infty$, and $|V(x, \theta)||x|^{-\beta}$ is bounded as $|x| \to \infty$ for some $\beta < \frac{\alpha - 2}{2}$. This proves the pure-point nature of the spectrum of the Floquet operator

$$K_F := -i \omega \cdot \partial_\theta - \frac{d^2}{dx^2} + Q(x) + \varepsilon V(x, \theta).$$
on \( L^2(\mathbb{R}) \otimes L^2(\mathbb{T}^n) \) for small \( \varepsilon \). Liu and Yuan [35] solved the limit case when \( \alpha > 2 \) and \( \beta \leq \frac{\alpha - 2}{2} \), which can be applied to the so-called quantum version of the Duffing oscillator

\[
\dot{\psi}(x, t) = \left(-\frac{d^2}{dx^2} + x^4 + \varepsilon V(\omega t)\right)\psi(x, t), \quad x, \varepsilon \in \mathbb{R}.
\]

The results in [1] and [35] did not include the case \( Q(x) = x^2 \) (see (1.8)), which is the so-called quantum harmonic oscillator (\( \alpha = 2 \)). The quantum harmonic oscillator is the quantum-mechanical analogue of the classical harmonic oscillator. Because an arbitrary potential can usually be approximated as a harmonic potential in the vicinity of a stable equilibrium point, it is one of the most important model systems in quantum mechanics.

In [17], Enss and Veselic proved that if \( \omega \) is rational, the Floquet operator \( K \) (see (1.6)) has a pure-point spectrum when the perturbing potential \( V \) is bounded and has a sufficiently fast decay at infinity. In [44] Wang proved that the spectrum of the Floquet operator \( K \) is pure-point where the perturbing potential \( V(x, \theta) = \sum_{j \in \mathbb{N}} e^{i \omega_j x} \cos \theta_j \) has exponential decay. Grebert and Thomann [24] improved the results in [44] from exponential decay to polynomial decay. In this paper, we improve the results in [24] from polynomial decay to logarithmic decay. But we know nothing about \( K \) when \( \alpha = 2 \) and \( \beta = 0 \), i.e., \( Q(x) \sim |x|^2 \) as \( |x| \to \infty \) and \( V(x, \theta) \) is only bounded (except the special case when \( V(x, \theta) \) is independent of \( x \), see [24]). This problem was first posed by Eliasson in [12] and is still open now.

When \( V \) is unbounded, (see [2, 3, 22, 25, 45].

### 1.3. The main idea for proving theorem 1.1

As in [1, 15] and [24], the proof of theorem 1.1 is closely related to an infinite dimensional KAM theorem. Since the formulation of this abstract theorem is technical and lengthy, we postpone it to section 2, see theorem 2.2 (KAM Theorem). Let us first explain the main idea and techniques for proving theorem 2.2.

We begin with a parameter-dependent family of analytic Hamiltonians of the form

\[
H = N(y, z, \xi; \xi) + P(\theta, y, z, \xi; \xi),
\]

where \( \theta, y \in \mathbb{T}^n \times \mathbb{R}^n \) and \( z = (z_j)_{j \geq 0}, \xi = (\xi_j)_{j \geq 0} \) are infinitely many variables, \( \omega(\xi) = (\omega_j(\xi)) \in \mathbb{R}^n, \Omega(\xi) = (\Omega_j(\xi)) \in \mathbb{R}^n \) and \( \xi \in \Pi \subset \mathbb{R}^n \). For our applications we suppose \( \Omega_j(\xi) = 2j - 1 \) for simplicity and \( (l, \Omega(\xi)) \neq 0, \forall \ 1 \leq |l| \leq 2 \). Our aim is to find a suitable real analytic symplectic coordinate transformation \( \Phi \) such that \( H \circ \Phi = N^* + P^* \) where \( N^* \) has a similar form to \( N \), and \( P^* \) is analytic and globally of order 3.

Actually, following the formulation of the KAM theorem given in [40] (see also [24]), it is sufficient to verify that there exist \( \xi \in \Pi \) with a big Lebesgue measure, which satisfy

\[
|\langle k, \omega_j(\xi) \rangle + \langle l, \Omega_j(\xi) \rangle| \geq \frac{(l_{\omega_j} \xi)}{1 + |k|^3}, \tag{1.9}
\]

where \( \omega_j(\xi), \Omega_j(\xi) \) are the frequencies in the \( \nu \)-th KAM step.

To obtain (1.9) in most cases, we need some assumptions for \( X_F \) such as [30] and [40] (see theorem 1 in [30] and its applications to 1D wave equations and 1D harmonic oscillators with a smooth nonlinearity of type \( P = \frac{1}{2} \int_{\mathbb{R}} \psi(|u * \xi(x)|^2; a) dx \), where \( u * \xi \) is the convolution
with a smooth real-valued function $\xi$, vanishing at infinity). In [40], Pöschel required a similar condition on $X_0$, i.e.

$$X_0 : \mathcal{P}^0 \rightarrow \mathcal{P}^\beta, \quad \beta > p. \quad (1.10)$$

See [41] for its application to a nonlinear wave equation.

The assumption (1.10) is smartly weakened in [24]. Using the Töplitz–Lipschitz techniques from Eliasson and Kuksin in [16], Grébert and Thomann assumed a weaker regularity on $P$—more clearly,

$$\left\| \frac{\partial P}{\partial w_j} \right\|_{\mathcal{D}(D_{l,r})} \leq \frac{r}{(1 + j)^\beta} \left\| (P)^*_r,1 \right\|_{\mathcal{D}(D_{l,r})},$$

for all $j, l \geq 1$ and $w_j = z_j, z_j$, where $\beta > 0$ and $\| \cdot \|_{\mathcal{D}(D_{l,r})}$ stands for either $\| \cdot \|_{\mathcal{D}(D_{l,r})}$ or $\| \cdot \|_{\tilde{\mathcal{D}}(D_{l,r})}$. To recover this assumption at each step, they noticed that $F$ satisfied an even better estimate, i.e.

$$\left\| \frac{\partial F}{\partial w_j} \right\|_{\mathcal{D}(D_{l,r})} \leq \frac{r}{(1 + j)^\beta} \left\| (F)^*_r,1 \right\|_{\mathcal{D}(D_{l,r})},$$

for all $j, l \geq 1$ and $w_j = z_j, z_j$.

In this paper, we further weaken the assumptions on $P$ which satisfy the logarithmic decay, i.e.

$$\left\| \frac{\partial P}{\partial w_j} \right\|_{\mathcal{D}(D_{l,r})} \leq \frac{r}{\ln j} \left\| (P)^*_r,1 \right\|_{\mathcal{D}(D_{l,r})},$$

for all $j, l \geq 1$ and $w_j = z_j, z_j$. The index $\beta > 0$ is apparently different from $\beta > 0$ in [24], which we will explain in the following. As in [24], we obtain a better estimate for $F$, i.e.

$$\left\| \frac{\partial F}{\partial w_j} \right\|_{\mathcal{D}(D_{l,r})} \leq \frac{r}{\ln j} \left\| (F)^*_r,1 \right\|_{\mathcal{D}(D_{l,r})},$$

for all $j, l \geq 1$ and $w_j = z_j, z_j$.

The shift in normal frequencies in the next step $\Omega_j^+ (\xi) = \Omega_j (\xi) + \tilde{\Omega}_j (\xi)$ satisfies the much weaker estimate

$$|\tilde{\Omega}_j| \leq \alpha (1 + \ln j)^{-2j}, j \geq 1, \quad (1.11)$$

comparing it with the corresponding one in [24], which is

$$|\tilde{\Omega}_j| \leq \alpha j^{-2j}, j \geq 1. \quad (1.12)$$

The weaker estimate (1.11) brought up new problems in the measure estimates. To solve it we introduce a new small divisor condition

$$|k, \omega (\xi) + (l, \Omega (\xi))| \geq \frac{(l) \alpha}{\exp (|k|/\beta)}, \quad \beta \geq 2\tau \geq 2(n + 2), \quad (1.13)$$

which will be explained in a heuristic way in the following.
For simplicity, we suppose $\omega(\xi) = \xi$ and $\Omega_j(\xi) = 2j - 1 + O(\alpha(1 + \ln j)^{-2\beta})(j \geq 1)$. Our main problem is to estimate the set

$$
\bigcup_{k \leq b} \left\{ \xi \in \Pi : |f_{k,b}(\xi)| < \frac{\alpha b}{\Delta(|k|)} \right\},
$$

where $f_{k,b}(\xi) := \langle k, \omega(\xi) \rangle + \Omega_j(\xi) - \Omega_j(\xi) = \langle k, \xi \rangle + 2b + O(\alpha(1 + \ln j)^{-2\beta}),$

$i = j + b$ and $\Delta(|\cdot|)$ will be chosen later. From a straightforward computation (see 1.11), if $|f_{k,b}(\xi)| \geq \alpha b \Delta^{-1}(|k|) \geq \alpha b \Delta^{-1}(|k|)$, where $\Delta(|k|) \geq \Delta(|k|)$ and $|\langle k, \xi \rangle + 2b| \geq 2\alpha b \Delta^{-1}(|k|)$. For the rest,

$$
\operatorname{Meas} \left\{ \xi \in \Pi : |f_{k,b}(\xi)| < \frac{\alpha b}{\Delta(|k|)} \right\} \leq \frac{\alpha|k|^2}{\Delta(|k|)} \cdot \exp \left\{ \alpha \left( \Delta(|k|) \right) \right\}.
$$

(1.14) explains (1.13) since we set $\Delta(l) = l^\tau$, and thus $\Delta(l) = \exp(l^{1/\beta})$. The index $\beta \geq 2(n + 2)$ comes from lemma 5.1($\beta \tau > 0$) and $\beta = \nu \tau \geq 2\tau \geq 2(n + 2)$ (see theorem 6.1 for $\tau \geq n + 2$). In the KAM proof, we set $\nu \geq 2$ for simplicity.

We remark that the structure (1.11) or (1.12) is not necessary for some evolution equations. Based on the work [16], Berti, Biasco and Procesi in [5, 6] found a remarkable structure for $\Omega_j^1(\xi) = \Omega_j(\xi) + \bar{\Omega}_j(\xi)$ in the 1D derivative wave equation where $\bar{\Omega}_j(\xi) = a_+(\xi) + O(1/j)(j \gg 1)$ and $a_+(\xi)$ is independent of $j$ (similar for $j < 0$). However, we do not know whether there exists a similar or weaker structure for the 1D harmonic oscillator, which would be a potential way to solve Eliasson’s problem.

To finish the proof of theorem 1.1 we will follow the scheme given by Eliasson and Kuksin in [15]. As in [15] the equation (1.1) is rewritten into an autonomous Hamiltonian system in an extended phase space $P^2 := T^n \times \mathbb{R}^n \times \ell^{2,2} \times \ell^{2,2}$, with the Hamiltonian function $H = N + eP$, where

$$
N := N(\omega) = \sum_{\omega \in \mathbb{N}} |\omega|^2 + \sum_{j \geq 1} (2j - 1)z_0z_j
$$

and

$$
P(\theta, z, \bar{z}) = \int_{\mathbb{R}} V(x, \theta, \omega)(\sum_{j \geq 1} z_j \bar{h}_j)(\sum_{j \geq 1} \bar{z}_j h_j)dx
$$

is quadratic in $(z, \bar{z})$. Here, the external parameters are the frequencies $\omega = (\omega)_{\omega \in \omega} \in \Pi := [0, 2\pi)^n$, and the normal frequencies $\Omega_j = 2j - 1$ are independent of $\omega$. To apply theorem 2.2 in the above Hamiltonian we need to check that all the assumptions in theorem 2.2 are satisfied, and thus finish the proof of theorem 1.1. In fact, we need an improved estimate on $h_0(x)$. For simplicity we define the weighted $L^2$ norm of $h_0(x), \|$Z Wang and Z Liang\| Nonlinearity 30 (2017) 1405
\[ \left\| h_n(x) \right\| = \left( \int_R \frac{h_n^2(x)}{(1 + \ln(1 + x^2))^{2\delta_i}} \, dx \right)^{\frac{1}{2}}, \]

with \( \delta_i > 0 \).

**Lemma 1.5.** Suppose \( h_n(x) \) satisfies (1.5), then for any \( n \geq 1 \),

\[ \left\| h_n(x) \right\| \leq \frac{C_{\delta_i}}{(1 + \ln n)^{\delta_i}}, \]

where \( C_{\delta_i} \) is a constant depending on \( \delta_i \) only.

At the end of this section we give a quick description of the recent development in KAM theory. For the KAM results with bounded perturbations, see [19, 27, 29, 30, 33, 34] for 1D-NLS; for high-dimensional NLS see the milestone work by Eliasson and Kuksin [16], where they found and defined a \( \text{Töplitz–Lipschitz} \) property and used it to control the shift of the normal frequencies. See [18] and [39] for the recent development in nd-NLS. For nd-beam equations see [20] and [21] for the nonlinearity \( g(u) \) and see [13] and [14] for the more general nonlinearity \( g(x, u) \). Adapting the techniques in [16] and [24], Grébert and Paturel built a KAM for the Klein–Gordon equation on \( S^d \) in [23]. We remark that an earlier result for NLW and NLS on the compact Lie groups via the Nash–Moser techniques can be found in [7] (see also [8]), in which Berti, Corsi and Procesi proved the existence of quasi-periodic solutions without a linear stability.

For the unbounded perturbations, the first KAM results were obtained by Kuksin [31, 32] for a KdV with analytic perturbations (see also Kappeler and Pöschel [28]). See [36] and [47] for recent progress on the 1D derivative nonlinear Schrödinger equation (DNLS), the Benjamin–Ono equation and the reversible DNLS equation.

**A plan of the proof of theorem 1.1.** In section 2, we give an abstract KAM theorem (theorem 2.2) which will be used to prove theorem 4.1, a Hamiltonian formulation of theorem 1.1, in section 4. In section 3, we prove the weighted \( L^2 \) estimates on the Hermite basis. The KAM proof of theorem 2.2 is deferred to section 5, while the measure estimates are presented in section 6. We put two technical inequalities in the appendix.

**Notations.** \( \langle \cdot, \cdot \rangle \) is the standard scalar product in \( \mathbb{R}^d \) or \( \mathbb{R}^\infty \). \( \| \cdot \| \) is an operator-norm or \( \ell^2 \)-norm. \( |\cdot| \) will, in general, denote a supremum norm with a notable exception. For \( l = (l_1, l_2, \ldots, l_k, \ldots) \in \mathbb{Z}^\infty \), so that only a finite number of coordinates are nonzero, we denote by \( |l| = \sum_{j=1}^{\infty} |l_j| \) its length, \( \langle l \rangle = \max \{1, \sum_{j=1}^{\infty} |l_j| \} \). We use the notations \( \mathbb{Z}_+ = \{1, 2, \ldots \} \) and \( \mathbb{N} = \{0, 1, 2, \ldots \} \).

The notation ‘\( \ll \)’ used below means \( \leq \) modulo a multiplicative constant that, unless otherwise specified, only depends on \( n \).

We set \( Z = \{(k, l) \in \mathbb{Z}^d : |l| \leq 2 \} \subset \mathbb{Z}^d \times \mathbb{Z}^\infty \). Denote by \( \Delta_{l \delta} \) the difference operator in the variable \( \xi \), \( \Delta_{l \delta} f = f(\cdot, \xi) - f(\cdot, \eta) \), where \( f \) is a real function.

We denote \( \ell^2_{\mathbb{C}} \) as the Hilbert space of all complex sequences \( w = (w_j)_{j \in \mathbb{N}} \) with \( \|w\|_2^2 = \sum_{j=1}^{\infty} |w_j|^2 < \infty \). We denote \( \ell^p_{\mathbb{C}} \) as the space of all real (complex) sequences with the finite norm \( |w|_p = \sum_{j=1}^{\infty} |w_j|^p \). The notation Meas stands for the Lebesgue measure in \( \mathbb{R}^n \).
2. KAM theorem

2.1. KAM theorem

Following the exposition in [24, 27] and [40], we consider small perturbations of a family of infinite-dimensional integrable Hamiltonians \( N(y, u, v; \xi) \) with the parameter \( \xi \) in the normal form

\[
N = \sum_{i \in \mathbb{Z}^n} \omega_i(\xi)y_i + \frac{1}{2} \sum_{j \geq 1} \Omega_j(\xi)(u_j^2 + v_j^2),
\] (2.1)
on the phase space \( M^0 := T^n \times \mathbb{R}^n \times \ell_{\delta}^2 \times \ell_{\delta}^2 \) with coordinates \((\theta, y, u, v)\). The ‘internal’ frequencies, \( \omega = (\omega_i)_{i \in \mathbb{Z}^n} \), as well as the ‘external’ ones, \( \Omega = (\Omega_j)_{j \geq 1} \), are real-valued and depend on the parameter \( \xi \in \Pi \subset \mathbb{R}^n \), and \( \Pi \) is a compact subset of \( \mathbb{R}^n \) of positive Lebesgue measure.

The symplectic structure on \( M^0 \) is the standard one given by \( \sum_{i} \theta_i \wedge dy_i + \sum_{j} du_j \wedge dv_j \).

The Hamiltonian equations for the motion of \( N \) are therefore

\[
\dot{\theta} = \omega(\xi), \quad \dot{y} = 0, \quad \dot{u} = \Omega(\xi)v, \quad \dot{v} = -\Omega(\xi)u,
\]

where for any \( j \geq 1 \), \( (\Omega(\xi)u)_j = \Omega_j u_j \). Hence, for any parameter \( \xi \in \Pi \), on the \( n \)-dimensional invariant torus,

\[ T_0 = T^n \times \{0\} \times \{0\} \times \{0\}, \]

the flow is rotational with internal frequencies \( \omega(\xi) \). In the normal space, described by the \((u,v)\) coordinates, we have an elliptic equilibrium at the origin, whose frequencies are \( \Omega(\xi) = (\Omega_j)_{j \geq 1} \). Hence, for any \( \xi \in \Pi \), \( T_0 \) is an invariant, rotational, linearly stable torus for the Hamiltonian \( N \).

Our aim is to prove the persistence of this torus under the small perturbations \( N + P \) of the integrable Hamiltonian \( N \) for a large Cantor set of parameter values \( \xi \). To this end, we make assumptions about the frequencies of the unperturbed Hamiltonian \( N \) and the perturbation \( P \).

We need some notations for simplification. In the sequel, we use the distance

\[
|\Omega - \Omega'|_{2,0,\Pi} = \sup_{\xi \in \Pi} \sup_{j \geq 1} (1 + \ln j)^{2|\delta|} |\Omega_j(\xi) - \Omega'_j(\xi)|,
\]

and the semi-norm,

\[
|\Omega|_{2,0,\Pi} = \sup_{\xi, \eta \in \Pi} \sup_{j \geq 1} \frac{(1 + \ln j)^{2|\delta|} |\Delta_{\xi,\eta}\Omega_j|}{|\xi - \eta|}.
\]

**Assumption \( A \) (frequencies):**

(A1) The map \( \xi \mapsto \omega(\xi) \) between \( \Pi \) and its image \( \omega(\Pi) \) is a homeomorphism which, together with its inverse, is Lipschitz continuous.

(A2) There exists a real sequence \((\Omega_j)_{j \geq 1}\), independent of \( \xi \in \Pi \), of the form \( \Omega_j = a_j j + a_j \) with \( a_n, a_j \in \mathbb{R} \) and \( a_n \neq 0 \), so that \( \xi \mapsto (\Omega_j - \Omega_j)_{j \geq 1} \) is a Lipschitz continuous map on \( \Pi \) with values in \( \ell_{\delta}^{2\delta} (\delta < 0) \). More clearly, for \( \xi \in \Pi, |\Omega - \Omega'|_{2,2,\Pi} \leq M_1 \) with \( M_1 > 0 \).

(A3) For all \((k, l) \in \mathbb{Z} \),
Meas\{\{\xi : k \cdot \omega(\xi) + l \cdot \Omega(\xi) = 0\}\} = 0.

and for all \(\xi \in \Pi,\)

\[ l \cdot \Omega(\xi) = 0, \quad \forall 1 \leq |l| \leq 2. \]

The second set of assumptions concerns the perturbing Hamiltonian \(P\) and its vector field, \(X_P = (\partial_x P, -\partial_y P, \partial_z P, -\partial_{\bar{z}} P).\) We use the notation \(i_X X_P\) for \(X_P\) evaluated at \(\xi.\) Finally, we denote by \(\mathcal{M}_C^p\) the complexification of the phase space \(\mathcal{M}^p, \mathcal{M}_C^p = (\mathbb{C}/2\pi \mathbb{Z})^n \times \mathbb{C}^n \times \ell_2^{2p} \times \ell_2^{\bar{2}p}.\) Note that at each point of \(\mathcal{M}_C^p,\) the tangent space is given by

\[ T\mathcal{M}_C^p := \mathbb{C}^n \times \mathbb{C}^n \times \ell_2^{2p} \times \ell_2^{\bar{2}p}. \]

To state the assumptions about the perturbation we need to introduce some domains and norms. For \(s, r > 0,\) we define the complex neighbourhood of the \(\mathbb{C}/2\pi \mathbb{Z}\) neighbourhoods

\[ D(s, r) = \{ |\theta| < s \} \times \{ |y| < r^2 \} \times \{ ||u||_p + ||v||_p < r \} \subset \mathcal{M}_C^p. \]

Here, for \(a \in \mathbb{R}^n\) or \(\mathbb{C}^n,\) \(||a||_2 = \max_j |a_j|\) and \(p \geq 2.\) Let \(r > 0, \) then for \(W = (X, Y, U, V)\) in \(\mathcal{P}_C^p\) we denote

\[ ||W||_r = ||X|| + \frac{1}{r^2}||Y|| + \frac{1}{r}(||U||_p + ||V||_p). \]

We then define the norms

\[ \|P\|_{D(s, r)} := \sup_{D(s, r) \times \Pi} |P|, \quad \|P\|_{D(s, r)}^\xi := \sup_{\xi, \eta \in \Pi} \frac{|\Delta\xi P|}{|\xi - \eta|}, \]

and we define the semi-norms

\[ ||X_P||_{D(s, r)} := \sup_{D(s, r) \times \Pi} |X_P|, \quad ||X_P||_{D(s, r)}^\xi := \sup_{\xi, \eta \in \Pi} \frac{|\Delta\xi X_P|}{|\xi - \eta|}. \]

In the sequel, we will often work in the complex coordinates \(z = \frac{1}{\sqrt{2}}(u - iv), \bar{z} = \frac{1}{\sqrt{2}}(u + iv).\) Notice that this is not a canonical change of variables, and in the variables \((\theta, y, z, \bar{z}) \in \mathcal{M}_C^p,\) the symplectic structure reads \(\sum_{j, \ell < \delta} d\theta_j \wedge dy_j + i \sum_{j \geq \delta} dz_j \wedge d\bar{z}_j,\) and the Hamiltonian in normal form is

\[ N = \sum_{1 \leq j < \delta} \omega_j(\xi) y_j + \sum_{j \geq \delta} \Omega_j(\xi) z_j \bar{z}_j. \]

**Assumption B (Perturbation):**

**(B1)** We assume that there exists \(s, r > 0\) so that

\[ X_P : D(s, r) \times \Pi \to \mathcal{P}_C^p. \]

Moreover, \(i_X X_P(\cdot, \xi)\) is analytic in \(D(s, r)\) for each \(\xi \in \Pi.\) \(i_X P\) and \(i_X X_P\) are uniformly Lipschitz on \(\Pi\) for each \(w \in D(s, r).\)

Similar to [24], we denote \(\Gamma^{\beta}_{r, D(s, r)}\) as the following: Let \(\beta > 0, \) then we say that \(P \in \Gamma^{\beta}_{r, D(s, r)}\) if \(\langle P\rangle_{r, D(s, r)} + \langle P\rangle_{r, D(s, r)}^\xi < \infty,\) where the norm \(\langle \cdot \rangle_{r, D(s, r)}\) is defined by the conditions
\[ \|P\|_{D_n(r)} \leq r^2 \langle P \rangle_{r,D_n(r)}, \]
\[ \max_{1 \leq j \leq n} \left\| \frac{\partial P}{\partial y_j} \right\|_{D_n(r)} \leq \langle P \rangle_{r,D_n(r)}, \]
\[ \left\| \frac{\partial P}{\partial w_j} \right\|_{D_n(r)} \leq \frac{r}{(1 + \ln j)^\xi} \langle P \rangle_{r,D_n(r)}, \quad \forall \ j \geq 1 \quad \text{and} \quad w_j = z_j, \]
\[ \left\| \frac{\partial^2 P}{\partial w_j \partial w_l} \right\|_{D_n(r)} \leq \frac{1}{(1 + \ln j)^\xi(1 + \ln l)^\xi} \langle P \rangle_{r,D_n(r)}, \quad \forall \ j, l \geq 1 \quad \text{and} \quad w_j = z_j, \]

and the semi-norm \( \langle \rangle \) is defined by the conditions
\[ \|P\|_{D_n(r)}^2 \leq r^2 \langle P \rangle_{r,D_n(r)}, \]
\[ \max_{1 \leq j \leq n} \left\| \frac{\partial P}{\partial y_j} \right\|_{D_n(r)}^2 \leq \langle P \rangle_{r,D_n(r)}, \]
\[ \left\| \frac{\partial P}{\partial w_j} \right\|_{D_n(r)} \leq \frac{r}{(1 + \ln j)^\xi} \langle P \rangle_{r,D_n(r)}, \quad \forall \ j \geq 1 \quad \text{and} \quad w_j = z_j, \]
\[ \left\| \frac{\partial^2 P}{\partial w_j \partial w_l} \right\|_{D_n(r)} \leq \frac{1}{(1 + \ln j)^\xi(1 + \ln l)^\xi} \langle P \rangle_{r,D_n(r)}, \quad \forall \ j, l \geq 1 \quad \text{and} \quad w_j = z_j, \]

\( (B2) \ P \in \Gamma_{r,D_n(r)}^\beta \) for some \( \beta = \epsilon \gamma \geq \epsilon(n + 2) \) where \( \epsilon \geq 2 \).

**Remark 2.1.** In the application to the 1D quantum harmonic oscillator we will choose \( \beta \geq 2(n + 2) \), which is not the best choice. However, we have no intention of obtaining the optimal one for \( \beta \).

We set \( |\omega|_1 \leq M, \ |\omega^{-1}|_1 \leq L \) and
\[ |\omega|_1^\xi + \|\Omega\|_{\xi, \Pi}^2 \leq M, \tag{3.2} \]
where \( |\omega|_1^\xi = \sup_{\xi \in \Pi} \max_{1 \leq k \leq n} \frac{|\omega_{k,e}|}{|k - e|}. \)

**Theorem 2.2 (KAM).** Suppose that \( N \) is a family of Hamiltonians of the form (2.2) defined on the phase space \( M^p \) with \( p \geq 2 \) depending on parameters \( \xi \in \Pi \) so that assumption \( A \) is satisfied. Then, there exist \( \gamma > 0 \) and \( s > 0 \), so that for every perturbation \( H = N + P \) of \( N \) which satisfies assumption \( B \) and the smallness condition
\[ \varepsilon = (\|X_P\|_{D_n(r)} + \langle P \rangle_{r,D_n(r)}) + \frac{\alpha}{M} (\|X_P\|^2_{D_n(r)} + \langle P \rangle^2_{r,D_n(r)}) \leq \gamma \alpha^5, \]
for some \( r > 0 \) and \( 0 < \alpha \leq 1 \), the following holds.

There exist
(i) a Cantor set \( \Pi_\alpha \subset \Pi \) with \( \text{Meas}(\Pi \setminus \Pi_\alpha) \to 0 \) as \( \alpha \to 0 \);
(ii) a Lipschitz family of real analytic, symplectic coordinate transformations
(iii) a Lipschitz family of new normal form

\[ N^* = \sum_{j=1}^{n} \omega_j(\xi) \eta_j + \sum_{j \geq 1} \Omega_j(\xi) \xi_j \bar{\xi}_j \]

defined on \( D(s/2, r/2) \times \Pi_\alpha \) such that

\[ H \circ \Phi = N^* + P^*, \]

where \( P^* \) is analytic on \( D(s/2, r/2) \) and globally of order 3 at \( T_0 \). So the Taylor expansion of \( P^* \) only contains monomials \( \gamma^m \xi^q \bar{\xi}^s \) with \( 2|m| + |q + s| \geq 3 \). Moreover, each symplectic coordinate transformation is close to the identity

\[ \| \Phi - Id \|_{C^0(D(s/2, r/2))} \ll \epsilon e^{1/2}, \quad (2.4) \]

and the new frequencies are close to the original ones

\[ |\omega^* - \omega|_{\Pi_\alpha} + \| \Omega^* - \Omega \|_{L^2(\Pi_\alpha)} \ll \epsilon e, \quad (2.5) \]

and the new frequencies satisfy a nonresonance condition

\[ |k \cdot \omega^*(\xi) + l \cdot \Omega^*(\xi)| \geq \frac{\alpha}{2} \exp(|k|^{1/2}), \quad k \geq 2, \quad (k, l) \in \mathbb{Z}, \quad \xi \in \Pi_\alpha. \quad (2.6) \]

**Remark 2.3.** As a consequence, for each \( \xi \in \Pi_\alpha \) the torus \( \Phi(T_0) \) is invariant under the flow of the perturbed Hamiltonian \( H = N + P \) and all these tori are linearly stable.

**Remark 2.4.** We remark that the small divisor condition (2.6) is similar to the Bruno condition for the Lagrangian tori in a finite dimension.

### 3. Estimates on the eigenfunctions in a weighted \( L^2 \) norm

In this section we will prove lemma 1.5. A well-known fact is that \( h_n(x) = (n! \, 2^n \pi^{1/4})^{-1/2} e^{-x^2} H_n(x) \) where \( H_n(x) \) is the Hermite polynomial of degree \( n \), and \( h_n(x) \) is an even or odd function of \( x \) according to whether \( n \) is odd or even (see Titchmarsh [42]). The proof of lemma 1.5 is based upon Langer’s turning-point theory, as presented in chapter 22.27 of [43] (see [46]). For simplicity we define the weighted \( L^2 \) norms of \( h_n(x) \) on \( \mathbb{R} \) and \( \mathbb{R}_+ \), which are

\[ \| h_n(x) \| = \left( \int_{\mathbb{R}} \frac{h_n^2(x)}{(1 + \ln(1 + x^2))^{1/2}} \, dx \right)^{1/2}, \]

and

\[ \| h_n(x) \|_+ = \left( \int_{\mathbb{R}_+} \frac{h_n^2(x)}{(1 + \ln(1 + x^2))^{1/2}} \, dx \right)^{1/2} \]

with \( \delta_1 > 0 \).

From the symmetry \( \| h_n(x) \|^2 = 2 \| h_n(x) \|^2_+ \), and thus we only need to estimate \( \| h_n(x) \|_+ \).

In the following, we assume \( n \) to be sufficiently large. As in [46],

\[ \Phi : D(s/2, r/2) \times \Pi_\alpha \longrightarrow D(s, r); \]
\[ h_n(x) = (\lambda_n - x^2) - \frac{1}{2} i \pi \zeta \frac{1}{\pi} H_1^1(\xi) + (\lambda_n - x^2) - \frac{1}{2} i \pi \zeta \frac{1}{\pi} H_1^1(\xi) \mathcal{O}(1) \]

where \( \zeta(x) = \int_x^1 \sqrt{\lambda_n - t^2} \, dt \) with \( X^2 = \lambda_n \). We only need to estimate \( \psi_1(x) \), since the estimate for \( \psi_2(x) \) is even better. Let

\[ Q(y) = \begin{cases} -\int_y^1 \sqrt{1 - s^2} \, ds, & \text{if } y < 1, \\ i \int_y^1 \sqrt{s^2 - 1} \, ds, & \text{if } y > 1. \end{cases} \]

We have \( \zeta(x) = \lambda_n Q(y) \). By lemma 2.2 in [46], it holds that for any \( K > 1 \)

\[ Q(y) \sim -1(1 - y)^{3}, \quad \text{for } 0 \leq y \leq 1, \]

\[ -i Q(y) \sim (y - 1)^{3}, \quad \text{for } 1 \leq y \leq K, \]

\[ -i Q(y) \sim y^2, \quad \text{for } y \geq K. \]

Recall that \( \frac{1}{\pi} H_1^1(\xi) \) satisfies the following ([42]):

1. When \( \zeta = z < 0, \frac{1}{\pi} H_1^1(\xi) = \frac{2}{3} e^{-z\pi i} (J_1(z) + J_2(z)) \) and

\[ \zeta \frac{1}{\pi} H_1^1(\xi) = \begin{cases} 2\pi e^{-z\pi i} (\cos(z - \pi/4) + \mathcal{O}(z^{-1})), & z \to \infty, \\ 2\pi e^{-z\pi i} \Gamma(2/3) e^{3z}, & z \to 0, \end{cases} \]

(3.1)

2. When \( \zeta = iw \) and \( w \geq 0, \frac{1}{\pi} H_1^1(\xi) = \frac{2}{\pi} e^{-z\pi i} K_1(w) \) and

\[ \zeta \frac{1}{\pi} H_1^1(\xi) = \begin{cases} \mathcal{O}(e^{-w}), & w \to \infty, \\ 2\pi e^{-\pi i} \Gamma(1/3) w^{-1/3} + \mathcal{O}(w^{1/3}), & w \to 0. \end{cases} \]

(3.2)

If \( n \) is large, then

\[
\left\| h_n \right\|_1^2 \lesssim 2 \int_0^{+\infty} \frac{|\psi_1(x)|^2}{(1 + \ln(1 + x^2))^{2\delta_1}} \, dx + 2 \int_0^{+\infty} \frac{|\psi_2(x)|^2}{(1 + \ln(1 + x^2))^{2\delta_1}} \, dx \\
\leq C \int_0^{+\infty} \frac{|\zeta \frac{1}{\pi} H_1^1(\xi)|^2}{(1 - y^2)^{1/2}} \, dy.
\]

Lemma 1.5 is a direct corollary from the following lemma.

**Lemma 3.1.** There exists \( C > 0 \) such that for large \( n \),

\[
\int_0^{+\infty} \frac{|\zeta \frac{1}{\pi} H_1^1(\xi)|^2}{(1 - y^2)^{1/2} (1 + \ln(1 + y^2X^2))^{2\delta_1}} \, dy \leq C \cdot 2^{2\delta_1} \frac{1}{(1 + \ln n)^{2\delta_1}}.
\]

(3.3)
Proof. We split the integral into three parts

\[
\left( \int_0^1 + \int_1^K + \int_K^{+\infty} \right) \frac{\left| \frac{1}{\pi} H_n^{(1)}(\xi) \right|^2}{|1 - y^2|^{\frac{1}{2}} (1 + \ln(1 + y^2 X^2))^{2\alpha}} \, dy = I_1 + I_2 + I_3
\]

and estimate them separately.

(1) When \(0 \leq y \leq 1\) and \(\zeta = -z < 0\), we split the integral \(I_1\) into two parts \(I_1 = I_{11} + I_{12}\).

Applying the first relation of (3.1) to \(I_{11}\) and the second to \(I_{12}\),

\[
I_{11} \leq \int_0^{1 - X^{-\frac{1}{2}}} \frac{C}{|1 - y^2|^{\frac{1}{2}} (1 + \ln(1 + y^2 X^2))^{2\alpha}} \, dy
\]

\[
\leq \left( \int_0^{X^{-\frac{1}{2}}} + \int_1^{1 - X^{-\frac{1}{2}}} \right) \frac{C}{|1 - y^2|^{\frac{1}{2}} (1 + \ln(1 + y^2 X^2))^{2\alpha}} \, dy
\]

\[
\leq C \left( X^{-\frac{1}{2}} + \frac{1}{(1 + \ln(1 + X))^{2\alpha}} \right) \leq C \cdot \frac{2^{2\alpha}}{(\ln n)^{2\alpha}},
\]

and

\[
I_{12} \leq C \int_0^{1 - X^{-\frac{1}{2}}} \frac{\zeta^2}{|1 - y^2|^{\frac{1}{2}} (1 + \ln(1 + y^2 X^2))^{2\alpha}} \, dy
\]

\[
\leq \left( \int_1^{1 - X^{-\frac{1}{2}}} \right) \frac{C X^2 (1 - y)^{\frac{1}{2}}}{|1 - y^2|^{\frac{1}{2}} (1 + \ln(1 + y^2 X^2))^{2\alpha}} \, dy \leq C \cdot \frac{2^{2\alpha}}{(\ln n)^{2\alpha}}.
\]

(2) When \(1 \leq y \leq K\) and \(w = -i\zeta > 0\), we split the integral \(I_2\) into two parts \(I_2 = I_{21} + I_{22}\).

Applying the first relation of (3.2) to \(I_{21}\) and the second to \(I_{22}\), we obtain

\[
I_{21} \leq \int_1^{1 + X^{-\frac{1}{2}}} \frac{C e^{-2X(y-1)^{\frac{1}{2}}}}{|1 - y^2|^{\frac{1}{2}} (1 + \ln(1 + y^2 X^2))^{2\alpha}} \, dy
\]

\[
\leq \frac{C}{(\ln n)^{2\alpha}} \int_1^{1 + X^{-\frac{1}{2}}} \frac{e^{-2X(y-1)^{\frac{1}{2}}}}{(y-1)^{\frac{1}{2}}} \, dy \leq \frac{C \omega_n^{-\frac{1}{2}}}{(\ln n)^{2\alpha}} \leq \frac{C}{(\ln n)^{2\alpha}},
\]

and

\[
I_{22} \leq \int_1^{1 + X^{-\frac{1}{2}}} \frac{C X^2 (y-1)^{\frac{1}{2}}}{|1 - y^2|^{\frac{1}{2}} (1 + \ln(1 + y^2 X^2))^{2\alpha}} \, dy
\]

\[
\leq \frac{C}{(1 + \ln(1 + X^2))^{2\alpha}} \leq \frac{C}{(\ln n)^{2\alpha}}.
\]

(3) When \(y \geq K\) and \(w = -i\zeta > 0\), we apply the first relation of (3.2) to \(I_3\),
\[
I_n \leq \int_1^\infty \frac{C e^{-2y}}{|1 - y^2|^2 (1 + \ln(1 + y^2 X^2)^{2n})} dy \\
\leq \int_1^\infty \frac{C e^{-2Xy^2}}{y(1 + \ln(1 + y^2 X^2)^{2n})} dy \\
\leq \frac{C}{(1 + \ln(1 + X^2)^{2n})} \leq \frac{C}{(1 + \ln n)^{2n}}.
\]

Combining the above estimates together, we obtain \((3.3)\).

\[\Box\]

## 4. Application to quantum harmonic oscillators

In this section, we will apply theorem 2.2 to our model equation \((1.1)\) and prove the results stated in section 1. For the readers’ convenience, we rewrite the equation

\[
i\theta \dot{u} = -\partial_t^2 u + x^2 u + \epsilon V(x, \omega t; \omega) u, \quad u = u(t,x), \ x \in \mathbb{R}, \quad (4.1)
\]

where the potential \(V: \mathbb{R} \times \mathbb{T}^n \times \Pi \ni (x, \theta; \omega) \mapsto \mathbb{R}\) is \(C^3\) smooth in all its variables and analytic in \(\theta\). For \(\rho > 0\), the function \(V(x, \theta; \omega)\) analytically in \(\theta\) extends to the domain \(\mathbb{T}_n^\rho\) as well as its gradient in \(\omega\), and satisfies \((1.2)-(1.4)\) with \(\beta \geq 2(n + 2)\).

In the following, we will follow the scheme developed by Eliasson and Kuksin in [15]. Expand \(u\) and \(\bar{u}\) on the Hermite basis \(\{h_j\}_{j \geq 1}, u = \sum_{j \geq 1} z j h_j, \quad \bar{u} = \sum_{j \geq 1} \xi j h_j\). Thus, \((4.1)\) can be written as a nonautonomous Hamiltonian system

\[
\begin{align*}
\dot{z}_j &= -(2j - 1)z_j - i\epsilon \frac{\partial}{\partial z_j} \bar{P}(t, z, \bar{z}), \ j \geq 1, \\
\dot{\xi}_j &= i(2j - 1)\xi_j + i\epsilon \frac{\partial}{\partial \xi_j} \bar{P}(t, z, \bar{z}), \ j \geq 1,
\end{align*}
\]

where

\[
\bar{P}(t, z, \bar{z}) = \int_\mathbb{R} V(x, \omega t; \omega)(\sum_{j \geq 1} z_j h_j)(\sum_{j \geq 1} \xi_j h_j) dx,
\]

and \((z, \bar{z}) \in \ell^{2.2} \times \ell^{2.2}\). As [15] and [24], we write \((4.2)\) as an autonomous Hamiltonian system in an extended phase space \(\mathcal{P}^2 := \mathbb{T}^n \times \mathbb{R}^n \times \ell^{2.2} \times \ell^{2.2}\),

\[
\begin{align*}
\dot{z}_j &= -(2j - 1)z_j - i\epsilon \frac{\partial}{\partial z_j} \bar{P}(\theta, z, \bar{z}), \ j \geq 1, \\
\dot{\xi}_j &= i(2j - 1)\xi_j + i\epsilon \frac{\partial}{\partial \xi_j} \bar{P}(\theta, z, \bar{z}), \ j \geq 1, \\
\dot{\theta}_j &= \omega_j, \quad j = 1, 2, \ldots, n, \\
\dot{y}_j &= -\epsilon \frac{\partial}{\partial \theta_j} \bar{P}(\theta, z, \bar{z}), \quad j = 1, 2, \ldots, n,
\end{align*}
\]

with the Hamiltonian function \(H = N + \epsilon P\), where
\[ N := N(\omega) = \sum_{1 \leq j \leq n} \omega_j y_j + \sum_{j \geq 1} (2j - 1)z_j \xi_j. \]

and
\[ P(\theta, z, \tilde{\xi}) = \int_{\mathbb{R}} V(x, \theta; \omega)(\sum_{j \geq 1} z_j h_j)(\sum_{j \geq 1} \tilde{z}_j h_j) dx \]
is quadratic in \((z, \tilde{\xi})\). Here the external parameters are the frequencies \(\omega = (\omega_j)_{1 \leq j \leq n} \in \Pi := [0, 2\pi)^n\) and the normal frequencies \(\Omega_j = 2j - 1\) are independent of \(\omega\). We remark that the first three equations of (4.3) are independent of \(y\) and equivalent to (4.2).

Similar to [24], we have:

**Theorem 4.1.** Assume that \(V\) satisfies all the conditions in theorem 1.1 and \(\beta \geq 2(n + 2)\). Then there exists \(\varepsilon_0\) such that for all \(0 \leq \varepsilon < \varepsilon_0\) there exist

(i) \(\Pi_\varepsilon \subset [0, 2\pi)^n\) of positive measure and \(\text{Meas}(\Pi_0) \rightarrow (2\pi)^n\) as \(\varepsilon \rightarrow 0\);

(ii) a Lipschitz family of real analytic, symplectic and linear coordinate transformations \(\Phi: \Pi_\varepsilon \times \mathcal{P}^0 \mapsto \mathcal{P}^0\) of the form
\[
\Phi_\varepsilon(y, \theta, \zeta) = (y + \frac{1}{2} \zeta \cdot M_\varepsilon(\theta) \zeta, \theta, L_\varepsilon(\theta) \zeta)
\]
where \(\zeta = (z, \tilde{\xi})\), \(M_\varepsilon(\theta)\) and \(L_\varepsilon(\theta)\) are linear bounded operators from \(\ell^2 \times \ell^2\) into itself for all \(p \geq 0\), and \(L_\varepsilon(\theta)\) is invertible;

(iii) a Lipschitz family of new normal forms
\[
N^*(\omega) = \sum_{1 \leq j \leq n} \omega_j y_j + \sum_{j \geq 1} \Omega_j^*(\omega) z_j \xi_j;
\]
such that on \(\Pi_\varepsilon \times \mathcal{P}^0\), \(H \circ \Phi = N^*\).

Moreover, the new external frequencies are close to the original ones, \(\|\Omega^* - \Omega\|_{\ell^2 \times \ell^2} \leq c \varepsilon\), and the new frequencies satisfy a nonresonant condition, i.e.
\[ |k \cdot \omega + l \cdot \Omega^*(\omega)| \geq \frac{\alpha}{2} \frac{\|l\|}{\exp(k \|l\|)}, \quad \varepsilon \geq 2, \ (k, l) \in \mathbb{Z}, \]
for some \(\alpha > 0\) and \(\omega \in \Pi_\varepsilon\).

**Proof.** As [24], assumption \(A\) is clear. We now check that assumption \(B\) holds. Firstly, we need to check that the condition (4.3) is satisfied.

Note that \(\frac{\partial V}{\partial \omega} \in \ell^2\), which is the \(k\)th coefficient of the decomposition of \(V(x, \theta; \omega) \tilde{u}\) in the Hermite basis. It follows that \(\frac{\partial V}{\partial \omega} \in \ell^2\), and if only if \(V(x, \theta; \omega) \tilde{u} \in \mathcal{H}^2\). From \(|V| \leq C\), \(|\partial_x V| \leq C\), \(|\partial_x^2 V| \leq C\), \(\tilde{u} \in \mathcal{H}^2\) and a straightforward computation, we have \(V(x, \theta; \omega) \tilde{u} \in \mathcal{H}^2\). This implies that \(\frac{\partial V}{\partial \omega} \in \ell^2\). Similarly to \(|\partial_x V| \leq C\), \(|\partial_x (\partial_x V)| \leq C\), \(|\partial_x^2 (\partial_x V)| \leq C\) and \(\tilde{u} \in \mathcal{H}^2\), we obtain \(\frac{\partial V}{\partial \omega} \in \ell^2\), and thus (B1) is satisfied.

In the following, we turn to (B2) in assumption \(B\). From (1.2), lemma 1.5 and a straightforward computation,
From the conditions (1.2)–(1.4) and a similar computation we obtain

\[ \left\| \frac{\partial Z}{\partial t} \right\|_{L^2(D_{\delta r},)} \leq C_{\delta} \frac{r}{(1 + \ln k)^{3/2}}. \]

Similarly,

\[ \left\| \frac{\partial^2 P}{\partial t \partial t} \right\|_{L^2(D_{\delta r},)} = \sup_{D_{\delta r},} \left| \int_{D_{\delta r},} V(x, \theta; \omega) \xi h dx \right| \leq \frac{C_{\delta}}{(1 + \ln k)^{3/2}}. \]

From the conditions (1.2)–(1.4) and a similar computation we obtain

\[ \left\| \frac{\partial Z}{\partial t} \right\|_{L^2(D_{\delta r},)} \leq \frac{C_{\delta} r}{(1 + \ln k)^{3/2}} \quad \text{and} \quad \left\| \frac{\partial^2 P}{\partial t \partial t} \right\|_{L^2(D_{\delta r},)} \leq \frac{C_{\delta}}{(1 + \ln k)^{3/2}}. \]

It follows that \( P \in \Gamma_{s,\delta r} \) with \( s = \rho \) and \( \beta \geq 2(n + 2) \).

For our application to theorem 1.1, we will choose \( M = 2\pi, \beta = \delta(n + 2) \) with \( \epsilon \gg 2 \). A straightforward computation shows that

\[ \left\| X_{\rho} \right\|_{L^2(D_{\rho,\delta r},)} + \left\langle \varepsilon P \right\rangle_{L^2(D_{\rho,\delta r},)} \leq \frac{2\epsilon(n, \rho)}{\rho} \leq \gamma \alpha \epsilon, \]

if we choose \( \alpha = \frac{1}{2\pi} \) and \( \epsilon \leq \epsilon_0 := \left( \frac{2^{n+2}}{2^{n+2}} \right)^2 \).

Therefore theorem 2.2 applies with \( p = 2 \). Following [24] we have:

1. The symplectic coordinate transformation \( \Phi \) is quadratic and thus it is defined on the whole phase space and has the form

\[ \Phi(y, \theta, \zeta) = (y + \frac{1}{2} \zeta \cdot M_{\rho}(\theta) \zeta, \theta, L_{\rho}(\theta) \zeta); \]

2. The new normal form still has the same frequency vector \( \omega \);
3. The new Hamiltonian reduces to the new normal form, i.e. \( H^t = 0 \);
4. The symplectic coordinate transformation \( \Phi^t \) which is defined by theorem 4.1 on each \( P^t \), extends to \( P^t := \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}^{2\ell} \times \mathbb{R}^{2\ell} \).

We complete the proof of theorem 4.1. Meanwhile, theorem 1.1 follows directly from theorem 4.1.

\[ \square \]

**Proof of corollary 1.3.** See [24].

**Proof of corollary 1.4.** We follow the scheme developed in [24]. Firstly, we write the solution \( u(t, x) \) of (4.1) with the initial datum \( u_0(x) = \sum_{j=1}^N \varphi_j(0) h_j(x) \) as \( u(t, x) = \sum_{j=1}^N \varphi_j(t) h_j(x) \) with
(z, \bar{z})(t) = L_\omega(\delta t) (z(0) e^{-i\Omega t}, \bar{z}(0) e^{i\Omega t})

and \((z'(0), \bar{z}'(0)) = L_\omega(1) (z(0), \bar{z}(0))\). From the structure of \(L_\omega(\delta t)\)—more clearly \((L_\omega(\delta t))^2 = (L_\omega(\delta t))_{11} = 0\)—we then have

\[ u(t, x) = \sum_{j \neq 1} \psi_j(\delta t, x) e^{-i\Omega t}, \]

where \(\psi_j(\delta t, x) = \sum_{j \neq 1} (L_\omega(\delta t))^1_{j1}(0) h_j(x)\). In particular, the solutions are all almost periodic in time with a nonresonant frequency vector \((\omega, \Omega^*)\). By a straightforward computation we can prove that \(\psi_j(\omega, x) e^{-i\Omega t}\) solves (4.1), if and only if \(k \cdot \omega + \Omega^* k j\) is an eigenvalue of \(K\) with the eigenfunction \(\psi_j(\omega, x) e^{ik \cdot \theta}\). This shows that the spectrum set of the Floquet operator \(K\) equals \(\{k \cdot \omega + \Omega^* k j | k \in \mathbb{Z}^n, j \geq 1\}\), and thus we complete the proof.

5. Proof of KAM theorem

5.1. The linearized equation

Let \(H = N + P\) be a Hamiltonian, where

\[ N = \sum_{1 \leq j \leq n} \omega_j(\delta \xi) y_j + \sum_{j \neq 1} \Omega_j(\xi) z_j \bar{z}_j, \]

and \(P\) satisfies assumption \(B\) in section 2. The aim in this section is to put \(N + P\) into a new normal form \(N + P,\) such that \(P + 1\) is much smaller than \(P\). To do this, we need to solve the homological equation

\[ \{F, N\} + \bar{N} = R, \]

where \(R\) is the second order Taylor approximation of \(P,\)

\[ R = \sum_{2 | m+q+| \leq 2} R_{mqq} e^{i|\delta| y_m z_q \xi^q}, \]

with \(R_{mqq} = P_{mqq,}\) and \(F\) has a similar form as \(R,\)

\[ F = \sum_{2 | m+q+| \leq 2} F_{mqq} e^{i|\delta| y_m z_q \xi^q}. \]

From [40], we have the following:

**Lemma 5.1.** Suppose \(|\omega|^2 + ||\Omega||^2 \leq M\) uniformly on \(\Omega\) and

\[ |(k, \omega(\xi)) + (l, \Omega(\xi))| \geq \frac{(l, \alpha)}{A_k}, \quad (k, l) \in \mathbb{Z}, \]

where \(\alpha > 0\) and \(A_k = e^{\beta |\delta|/\beta} (\beta > \tau).\) Then, the linearized equation \(\{F, N\} + \bar{N} = R\) has a solution \(F, \bar{N}\) satisfying \(|F| = 0, \bar{N} = |R| = \sum_{2 | m+q+| \leq 2} R_{mqq} y_m z_q \xi^q,\) and

\[ ||X_k|| \leq ||X_k||, \quad ||X_k||_{\delta, 1, 0} \leq \frac{c(n, \beta) e^{\beta |\delta|/\alpha}}{\alpha} ||X_k||, \]

\[ ||X_k||_2 \leq ||X_k||_2, \quad ||X_k||_{2, 1, 0} \leq \frac{c(n, \beta) e^{\beta |\delta|/\alpha}}{\alpha} (||X_k||_2^2 + M||X_k||). \]
for $0 < \sigma \leq s$, $t_1 = \frac{\sigma}{\sigma + r}$, and the short hand $\| \cdot \|_{D(r,s)}$ is used.

Introduce the space $\Gamma^3_{r,D(r,s)} \subset \Gamma^3_{r,D(r,s)}$, endowed with the norm $\langle \cdot \rangle_{r,D(r,s)}^3 + \langle \cdot \rangle_{r,D(r,s)}^1$, defined by the following conditions:

\[
\|F\|_{D(r,s)} \leq \epsilon^2 \langle F\rangle_{r,D(r,s)}^3, \quad \max_{1 \leq j \leq n} \left\| \frac{\partial F}{\partial y_j} \right\|_{D(r,s)} \leq \langle F\rangle_{r,D(r,s)}^3,
\]
\[
\left\| \frac{\partial F}{\partial y_j} \right\|_{D(r,s)} \leq \frac{r}{j(1 + \ln j)^3} \langle F\rangle_{r,D(r,s)}^3, \quad \forall j \geq 1 \text{ and } w_j = z_j, \quad \forall j \geq 1 \text{ and } w_j = \bar{z}_j.
\]
\[
\left\| \frac{\partial^2 F}{\partial y_j \partial y_l} \right\|_{D(r,s)} \leq \frac{\langle F\rangle_{r,D(r,s)}^3}{(1 + |j - l|)(1 + \ln j)^3(1 + \ln l)^3}, \quad \forall j, l \geq 1 \text{ and } w_j = z_j, \quad \forall j, l \geq 1 \text{ and } w_j = \bar{z}_j.
\]

**Lemma 5.2.** Assume that the frequencies satisfy

\[
|\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle| \geq \frac{\langle I \rangle_{r,D(r,s)}^3}{A_k}, \quad (k, l) \in \mathbb{Z}, \tag{5.5}
\]

where $\alpha > 0$ and $A_k = \mathbb{E}[\frac{1}{\beta}]$ uniformly on $\Pi$. Let $F, \bar{N}$ be given in the above lemma and $R \in \Gamma^3_{r,D(r,s)}$, then for any $0 < \sigma \leq s$, we have $F \in \Gamma^3_{r,D(\sigma-s,r)}$ such that

\[
\langle F\rangle_{r,D(\sigma-s,r)}^3 \leq \frac{c(n, \beta) e^{\frac{2(z^n)}{\alpha}}}{\alpha} \langle R\rangle_{r,D(r,s)}^3,
\]
\[
\langle F\rangle_{r,D(\sigma-s,r)}^1 \leq \frac{c(n, \beta) e^{\frac{2(z^n)}{\alpha}}}{\alpha} \left( M \langle R\rangle_{r,D(r,s)} + \langle R\rangle_{r,D(r,s)}^3 \right)
\]

with $\eta = \frac{\sigma}{\sigma + r}$, and

\[
\langle \bar{N}\rangle_{r,D(\sigma-s,r)} \leq \langle R\rangle_{r,D(r,s)}, \quad \langle \bar{N}\rangle_{r,D(\sigma-s,r)}^3 \leq \langle R\rangle_{r,D(r,s)}^3.
\]

**Proof.** Our aim is to solve the homological equation (5.2) and find a solution $F$ for it. A straightforward computation shows that the coefficients in (5.4) are given by

\[
iF_{\eta\eta q} = \begin{cases} \frac{R_{\eta\eta q}}{k \cdot \omega + (q - \bar{q}) \Omega}, & \text{if } |k| + |q - \bar{q}| = 0, \\ 0, & \text{otherwise}. \tag{5.6} \end{cases}
\]

As [24], in the following we will use the notation $q_j = (0, \cdots, 0, 1, 0, \cdots)$, where ‘1’ is in the $j$th position and $q_1 = q + q_1$. The variables $z$ and $\bar{z}$ play exactly the same role; therefore, it is enough to study the derivatives in $z$. We first show that

\[
\langle F\rangle_{r,D(\sigma-s,r)}^3 \leq \frac{c(n, \beta) e^{\frac{2(z^n)}{\alpha}}}{\alpha} \langle R\rangle_{r,D(r,s)}^3
\]

From $|R_{\eta\eta q, 0}| \leq \frac{\langle R\rangle_{r,D(r,s)} e^{\frac{2(z^n)}{\alpha}}}{(1 + \ln j)(1 + \ln l)^3}$, (5.5) and (5.6),
\[ |F_{kq0}| \leq \frac{A_k(R)_{k,D,k,r}e^{-|k|\sigma}}{\alpha |j - l|(1 + \ln j)^3(1 + \ln l)^3} \leq \frac{A_k(R)_{k,D,k,r}e^{-|k|\sigma}}{\alpha(1 + |j - l|)(1 + \ln j)^3(1 + \ln l)^3}. \]

Therefore,

\[
\left| \frac{\partial^2 F}{\partial z_2 \partial z_l} \right|_{D(s-\sigma,r)} \leq \sum_{k \in \mathbb{Z}^n} |F_{kq0}| e^{\delta |k|^{\sigma}} \leq \sum_{k \in \mathbb{Z}^n} \frac{A_k(R)_{k,D,k,r}e^{-|k|\sigma}}{\alpha(1 + |j - l|)(1 + \ln j)^3(1 + \ln l)^3} \cdot \frac{c(n, \beta)e^{2\xi\rho}}{\alpha(1 + |j - l|)(1 + \ln j)^3(1 + \ln l)^3} \langle R \rangle_{k,D,s,r}. \tag{5.7} \]

From lemma A.3 and a similar computation, we have

\[
\left| \frac{\partial F}{\partial z_2} \right|_{D(s-\sigma,r)} \leq \frac{c(n, \beta)e^{2\xi\rho}}{\alpha(1 + j)(1 + \ln j)^3} \langle R \rangle_{k,D,s,r}. \tag{5.8} \]

Similarly,

\[
\left| \frac{\partial F}{\partial z_l} \right|_{D(s-\sigma,r)} \leq \frac{c(n, \beta)e^{2\xi\rho}}{\alpha} \langle R \rangle_{k,D,s,r}. \tag{5.9} \]

and

\[
\|F\|_{D(s-\sigma,r)} \leq \frac{c(n, \beta)e^{2\xi\rho}}{\alpha} r^2 \langle R \rangle_{k,D,s,r}. \tag{5.10} \]

The above estimates (5.7)–(5.10), show us that

\[
\langle F \rangle_{k,D,s-\sigma,r}^+ \leq \frac{c(n, \beta)e^{2\xi\rho}}{\alpha} \langle R \rangle_{k,D,s,r}. \]

It remains to check the estimates with the Lipschitz semi-norms. As in [40], for \( |k| + |q_j - q_l| \neq 0 \) define \( \delta_{k,l} = k \cdot \omega + \Omega_j - \Omega_l \) and \( \Delta = \Delta_{\delta, \omega} \). Then we have

\[
i \Delta F_{kq0} = \delta_{k,l}(\eta) \Delta R_{kq0} + R_{kq0}(\xi) \Delta \delta_{k,l}, \]

and

\[
- \Delta \delta_{k,l}^{-1} = \frac{\langle k, \Delta \omega \rangle + \Delta \Omega_j - \Delta \Omega_l}{\delta_{k,l}(\xi) \delta_{k,l}(\zeta)}. \]

By the small divisor assumptions and a direct computation, we have
Now we go to estimate \( \langle F \rangle_{r, D(s - \sigma, r)}^{+} \). We only estimate \( \left\| \frac{\partial^{2} F}{\partial z_{j} \partial \tilde{z}_{l}} \right\|_{\Theta(s - \sigma, r)}^{\theta} \). Note \( \frac{\partial^{2} F}{\partial z_{j} \partial \tilde{z}_{l}} = \sum_{k \in \mathbb{Z}} F_{0,k} \bar{w}_{k}^{l,j} \) and \( \Delta \frac{\partial^{2} F}{\partial z_{j} \partial \tilde{z}_{l}} = \sum_{k \in \mathbb{Z}} \Delta F_{0,k} \bar{w}_{k}^{l,j} \), it follows that for \((\theta, y, z, \bar{z}) \in D(s - \sigma, r)\),

\[
\frac{|\Delta F_{0,k}|}{|\xi - \eta|} \leq \sum_{k} \frac{|k| A_{k}^{2}}{\alpha (j - l)^{\alpha}} \left( \frac{|R_{0,k}|}{|\xi - \eta|} + \frac{M^{2} (R_{0,k})^{2}}{\alpha} \right) e^{\theta (s - \sigma)}.
\]

Combining with \(|R_{0,k}| \leq \frac{(R_{0,k}) e^{-\theta t}}{(1 + l)^{\alpha}(1 + j)^{\alpha}}\) and \(|\bar{w}_{k}^{l,j}| \leq \frac{\bar{w}_{k}^{l,j}}{(1 + l)^{\alpha}(1 + j)^{\alpha}}\), we deduce that

\[
\left\| \frac{\partial^{2} F}{\partial z_{j} \partial \tilde{z}_{l}} \right\|_{\Theta(s - \sigma, r)}^{\theta} \leq \frac{c(n, \beta) e^{\theta (s - \sigma)}}{\alpha (1 + l)^{\alpha}(1 + j)^{\alpha}(1 + j - l)} \left( \frac{M}{\alpha} (R)_{r, D(s, r)} + (R)_{r, D(s, r)}^{\theta} \right).
\]

A similar computation for other terms provides

\[
\langle F \rangle_{r, D(s - \sigma, r)}^{+} \leq \frac{c(n, \beta) e^{\theta (s - \sigma)}}{\alpha} \left( \frac{M}{\alpha} (R)_{r, D(s, r)} + (R)_{r, D(s, r)}^{\theta} \right).
\]

The estimates for \( \tilde{N} \) are similar and we omit the details.

Now we turn to the estimates on the Poisson bracket.

**Lemma 5.3.** Let \( R \in \Gamma_{r, D(s, r)}^{\beta} \) and \( F \in \Gamma_{r, D(s - \sigma, r)}^{\beta} \) both be of degree 2, i.e. \( R, F \) are of the forms (5.3) and (5.4), respectively. Then, for any \( 0 < 2 \sigma < s \),

\[
\langle [R, F] \rangle_{r, D(s - 2\sigma, r)}^{\beta} \leq \frac{1}{\sigma} \langle R \rangle_{r, D(s, r)}^{\beta} \langle F \rangle_{r, D(s - 2\sigma, r)}^{\beta},
\]

\[
\langle [R, F] \rangle_{r, D(s - 2\sigma, r)}^{\beta} \leq \frac{1}{\sigma} \langle [R, F] \rangle_{r, D(s, r)}^{\beta} + \langle F \rangle_{r, D(s - 2\sigma, r)}^{\beta}.
\]

**Proof.** For simplicity we denote \( (R) := \langle R \rangle_{r, D(s, r)}^{\beta} \), \( (R)^{\beta} := \langle R \rangle_{r, D(s, r)}^{\beta} \) and \( (F)^{+} := \langle F \rangle_{r, D(s - \sigma, r)}^{\beta} \). Note that
\[
\{R, F\} = \sum_{k=1}^{n} \left( \frac{\partial R}{\partial \theta_k} \frac{\partial F}{\partial y_k} - \frac{\partial R}{\partial y_k} \frac{\partial F}{\partial \theta_k} \right) + i \sum_{j=1}^{\beta} \left( \frac{\partial R}{\partial z_j} \frac{\partial F}{\partial \bar{z}_j} - \frac{\partial R}{\partial \bar{z}_j} \frac{\partial F}{\partial z_j} \right),
\]

and it remains to estimate each term of this expansion and its derivatives.

We first prove (5.11). From Cauchy and the basic inequality \[\sum_{j=1}^{\beta} \frac{1}{\beta(1 + \ln j)^\alpha} \leq 1(\beta \geq 1),\] we have

\[
\|\{R, F\}\|_{D(\tau-2\tau, r)} \leq \frac{r^2 (R)^\gamma}{\sigma}. \tag{5.13}
\]

Similarly,

\[
\max_{1 \leq j \leq n} \left\| \frac{\partial}{\partial y_j} \{R, F\} \right\|_{D(\tau-2\tau, r)} \leq \frac{(R)^\gamma}{\sigma}. \tag{5.14}
\]

Write

\[
\frac{\partial \{R, F\}}{\partial z_j} = \sum_{k=1}^{n} \left( \frac{\partial^2 R}{\partial \theta_k \partial z_j} \frac{\partial F}{\partial y_k} + \frac{\partial R}{\partial \theta_k} \frac{\partial^2 F}{\partial y_k \partial z_j} - \frac{\partial^2 R}{\partial y_k \partial \theta_k} \frac{\partial F}{\partial z_j} - \frac{\partial R}{\partial y_k} \frac{\partial^2 F}{\partial \theta_k \partial z_j} \right) + i \sum_{j=1}^{\beta} \left( \frac{\partial^2 R}{\partial z_j \partial \bar{z}_j} \frac{\partial F}{\partial \bar{z}_k} + \frac{\partial R}{\partial z_j} \frac{\partial^2 F}{\partial \bar{z}_k \partial \bar{z}_j} - \frac{\partial^2 R}{\partial \bar{z}_k \partial \bar{z}_j} \frac{\partial F}{\partial z_j} - \frac{\partial R}{\partial \bar{z}_k} \frac{\partial^2 F}{\partial \bar{z}_k \partial z_j} \right) := (I) + (II). \tag{5.15}
\]

By a direct computation it holds that

\[
\| (I) \|_{D(\tau-2\tau, \frac{3}{4})} \leq \sum_{k=1}^{n} \left( \frac{r(R)(F)^\gamma}{\sigma(1 + \ln j)^\alpha} + \frac{4r(R)(F)^\gamma}{\sigma j(1 + \ln j)^\alpha} + \frac{4r(R)(F)^\gamma}{\sigma(1 + \ln j)^\alpha} + \frac{r(R)(F)^\gamma}{\sigma j(1 + \ln j)^\alpha} \right) \leq \frac{r(R)(F)^\gamma}{\sigma(1 + \ln j)^\alpha} \tag{5.16}
\]

From lemma A1, \[\| (II) \|_{D(\tau-2\tau, \frac{3}{4})} \leq \frac{r(R)(F)^\gamma}{(1 + \ln j)^\alpha}.\] Thus

\[
\left\| \frac{\partial}{\partial z_j} \{R, F\} \right\|_{D(\tau-2\tau, \frac{3}{4})} \leq \frac{r(R)(F)^\gamma}{\sigma(1 + \ln j)^\alpha}. \tag{5.16}
\]

By the same method and lemma A1 we obtain

\[
\left\| \frac{\partial^2}{\partial z_j \partial \bar{z}_l} \{R, F\} \right\|_{D(\tau-2\tau, \frac{3}{4})} \leq \frac{(R)(F)^\gamma}{\sigma(1 + \ln j)^\alpha(1 + \ln j)^\alpha}. \tag{5.17}
\]

Together with (5.13)–(5.17), (5.11) is proved.
For the Lipschitz norm estimates, we first estimate \( \| \frac{\partial}{\partial x} (R, F) \|_{D^{(2, \nu)}(\mathbb{T}^d)} \). Note that
\[
\frac{\partial^2 R}{\partial \theta \partial z} \left. \right|_{D^2} \leq \frac{1}{\sigma} \frac{\partial R}{\partial z} \left. \right|_{D^{(2, \nu)}(\mathbb{T}^d)} \leq \frac{r(R)}{\sigma(1 + \ln j)^3},
\]
where
\[
\frac{\partial^2 R}{\partial \theta \partial z} \left. \right|_{D^2} \leq \frac{1}{\sigma} \frac{\partial R}{\partial z} \left. \right|_{D^{(2, \nu)}(\mathbb{T}^d)} \leq \frac{r(R)}{\sigma(1 + \ln j)^3},
\]
and
\[
\left\| \frac{\partial F}{\partial \theta} \right\|_{D^{(2, \nu)}(\mathbb{T}^d)} \leq \langle F \rangle^+ \text{ and } \left\| \frac{\partial F}{\partial \theta} \right\|_{D^{(2, \nu)}(\mathbb{T}^d)} \leq \langle F \rangle^{+, \nu}. \]
Hence
\[
\left\| \frac{\partial F}{\partial \theta} \right\|_{D^{(2, \nu)}(\mathbb{T}^d)} \leq \frac{r((R)^{+} \langle F \rangle^+ + \langle R \rangle \langle F \rangle^{+, \nu})}{\sigma(1 + \ln j)^3}.
\]
For the other terms in (I) in (5.15) the estimates are similar. Thus,
\[
\| (I) \|_{D^{(2, \nu)}(\mathbb{T}^d)} \leq \frac{r((R)^{+} \langle F \rangle^+ + \langle R \rangle \langle F \rangle^{+, \nu})}{\sigma(1 + \ln j)^3}.
\]
For (II) we only estimate \( \left\| \sum_{k \geq 1} \frac{\partial^2 R}{\partial \theta \partial z_k} \frac{\partial F}{\partial \theta_k} \right\|_{D^{(2, \nu)}(\mathbb{T}^d)} \). From lemma A1,
\[
\left\| \sum_{k \geq 1} \frac{\partial^2 R}{\partial \theta \partial z_k} \frac{\partial F}{\partial \theta_k} \right\|_{D^{(2, \nu)}(\mathbb{T}^d)} \leq \sum_{k \geq 1} \left( \frac{\langle R \rangle^k}{(1 + \ln j)^3(k + 1 + \ln k)^3} \right) \left( \frac{\langle F \rangle^+}{(1 + \ln j)^3(k + 1 + \ln k)^3} \right)
\leq \frac{1}{(1 + \ln j)^3} \langle R \rangle \langle F \rangle^+ + \langle R \rangle \langle F \rangle^{+, \nu}.
\]
Other similar estimates result in
\[
\| (II) \|_{D^{(2, \nu)}(\mathbb{T}^d)} \leq \frac{r((R)^{+} \langle F \rangle^+ + \langle R \rangle \langle F \rangle^{+, \nu})}{\sigma(1 + \ln j)^3}.
\]
Therefore,
\[
\left\| \frac{\partial}{\partial z_j} (R, F) \right\|_{0, k = 2, \sigma, r} ^2 \leq \frac{r}{\sigma(1 + \ln j)j^d} \langle (R) \rangle^2 (F) + \langle (F) \rangle + \langle (F) \rangle^2 .
\]

To obtain (5.12) we need some other estimates, and the proofs are similar; however, we omit them for simplicity.

5.2. Phase flow

In this section, we study the Hamiltonian flow generated by \( F \in \Gamma_{r, Dk(\sigma, r)} \) which is globally of degree 2. Namely, we consider the system
\[
\begin{aligned}
(\theta(t), y(t), z(t), \xi(t)) &= X_F(\theta(t), y(t), z(t), \xi(t)), \\
(\theta(0), y(0), z(0), \xi(0)) &= (\theta^0, y^0, z^0, \xi^0).
\end{aligned}
\]

Lemma 5.4. Let \( 0 < 3 \sigma < s \) and \( F \in \Gamma_{r, Dk(\sigma, r)} \) be of degree 2. Assume that
\[
\langle (F) \rangle_{r, Dk(\sigma, r)} ^2 < C_\sigma.
\]

Then the solution of equation (5.18) with the initial condition \((\theta(t), y(t), z(t), \xi(t)) \in D(s - 3 \sigma, \frac{\sigma}{2})\) satisfies \((\theta(t), y(t), z(t), \xi(t)) \in D(s - 2 \sigma, \frac{\sigma}{2})\) for all \( 0 \leq t \leq 1 \), and we have the estimates
\[
\sup_{0 \leq t \leq 1} \left| \frac{\partial (\theta(t))}{\partial \theta^0} \right| \leq \frac{r}{\sigma(1 + \ln j)j^d} \langle (F) \rangle_{r, Dk(\sigma, r)} ^2 ,
\]
\[
\sup_{0 \leq t \leq 1} \left| \frac{\partial (\gamma(t))}{\partial \gamma^0} \right| \leq \frac{1}{(1 + \ln j)^d(1 + \ln k)^d(1 + |j - k|)} \langle (F) \rangle_{r, Dk(\sigma, r)} ^2 + \delta_{jk} ,
\]
\[
\sup_{0 \leq t \leq 1} \left| \frac{\partial (\gamma(t))}{\partial \gamma^0} \right| \leq \frac{1}{\sigma} \langle (F) \rangle_{r, Dk(\sigma, r)} ^2 + \delta_{jk} ,
\]
\[
\sup_{0 \leq t \leq 1} \left| \frac{\partial^2 (\gamma(t))}{\partial \gamma^0 \partial \gamma^0} \right| \leq \frac{1}{\sigma(1 + \ln j)^d(1 + \ln i)^d(1 + |j - k|)} \langle (F) \rangle_{r, Dk(\sigma, r)} ^2 .
\]

with \( w_k = z_k \) or \( z_k \) and \( w_k^0 = \gamma_k^0 \) or \( \gamma_k^0 \), \( k = 1, 2, \ldots \).

Before we give the proof of lemma 5.4, we introduce a space of infinite dimensional matrices with decaying coefficients. We denote by \( M \) the set of infinite matrices \( A : \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{C}) \) with values in the space of complex \( 2 \times 2 \) matrices and
\[
|A| := \sup_{i,j \geq 1} \|A_{ij}\|_{HS} < \infty ,
\]
\[
M_{\text{HS}}^2 := \sum_{i,j=1}^2 |M_{ij}|^2,
\]
where \(M \in \mathcal{M}_{2 \times 2}(\mathbb{C})\). For \(\beta > 0\) we define \(\mathcal{M}_\beta\) as the subset of \(\mathcal{M}\) such that \([A]_\beta < \infty\), where the norm \([\cdot]_\beta\) is given by the condition
\[
\sup_{\zeta \in \Pi} \|A\|_{\text{HS}} \leq \frac{[A]_\beta}{(1 + \ln i)^{\beta}(1 + \ln j)^{\beta}(1 + |i - j|)} \quad i,j \geq 1.
\]
In the following lemma, we will suppress the parameter \(\zeta\) for simplicity.

**Lemma 5.5.** Let \(A, B \in \mathcal{M}_\beta\) where \(\zeta \in \Pi\). Then \(A \cdot B \in \mathcal{M}_\beta\) and \([A \cdot B]_\beta \leq [A]_\beta [B]_\beta\).

**Proof.** For all \(j,l \geq 1\), \((A \cdot B)_{jl} = \sum_k A_{jk} B_{kl}\). Thus, for \(\zeta \in \Pi\),
\[
\|\|A\|_{\text{HS}} \leq \sum_{\zeta \in \Pi} \|A\|_{\text{HS}} \|B\|_{\text{HS}}
\]
\[
\leq \sum_{\zeta \in \Pi} \frac{[A]_\beta [B]_\beta}{(1 + \ln j)^{\beta}(1 + \ln k)^{\beta}(1 + |j - k|)} \frac{1}{(1 + \ln i)^{\beta}(1 + \ln l)^{\beta}(1 + |i - l|)}
\]
\[
\leq \frac{[A]_\beta [B]_\beta}{(1 + \ln j)^{\beta}(1 + \ln l)^{\beta}(1 + |j - l|)}.
\]

The last inequality comes from \(\beta \geq 1\), lemma A1, and a similar discussion as lemma 3.6 in [24].

**Proof of lemma 5.4.** For simplicity, we introduce the notations \(\zeta = (z_j, \zeta_j)\) and \(\zeta = (\zeta_j)_{j \in \mathbb{Z}}\). Then \(F\) reads
\[
F(\theta, y, \zeta) = a_0(\theta) + a_1(\theta) \cdot y + b(\theta) \cdot \zeta + \frac{1}{2} (B(\theta) \zeta) \cdot \zeta
\]
with \(a_0(\theta) = F(\theta, 0, 0)\), \(a_1(\theta) = \nabla_{\theta} F(\theta, 0, 0)\), \(b(\theta) = \nabla_{\zeta} F(\theta, 0, 0)\) and \(B = (B_{ij})\) is the infinite matrix where
\[
B_{ij}(\theta) = \begin{pmatrix}
\frac{\partial^2 F}{\partial z_i \partial \zeta_j}(\theta, 0, 0) & \frac{\partial^2 F}{\partial \zeta_i \partial \zeta_j}(\theta, 0, 0) \\
\frac{\partial^2 F}{\partial z_i \partial \zeta_j}(\theta, 0, 0) & \frac{\partial^2 F}{\partial \zeta_i \partial \zeta_j}(\theta, 0, 0)
\end{pmatrix}.
\]

The flow \(X^\prime\) exists for \(0 \leq t \leq 1\) and maps \(D(s - 3\sigma, r/4)\) into \(D(s - 2\sigma, r/2)\). In the sequel, we write
\[
(\theta(t), y(t), \zeta(t)) = X^\prime(t)(\theta_0, y_0, \zeta_0).
\]
From the equation \(\dot{\theta} = \nabla_{\theta} F(\theta, y, \zeta)\) and (5.19), we have the bound \(\sup_{0 \leq t \leq 1} |\dot{\theta}(t)| < s - 2\sigma\).
We now turn to the equation in \( \zeta \). To solve
\[
\dot{\zeta}(t) = J \nabla F(\theta, y, \zeta(t)) = b_\ell(t) + \mathcal{B}(t)\zeta(t), \quad \zeta(0) = \zeta_0,
\]
where \( b_\ell(t) = Jb(\theta(t)) \) and \( \mathcal{B}(t) = JB(\theta(t)) \) with
\[
J = \text{diag} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}_{j \geq 1},
\]
we iterate the integral formulation of the problem
\[
\zeta(t) = \zeta^0 + \int_0^t (b_\ell(t_1) + \mathcal{B}(t_1)\zeta(t_1))dt_1,
\]
and formally obtain
\[
\zeta(t) = b^\infty(t) + (1 + \mathcal{B}^\infty(t))\zeta^0, \quad (5.27)
\]
where
\[
b^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k-1} \mathcal{B}(t_j)b_\ell(t_k)dt_k \cdots dt_1, \quad (5.28)
\]
and
\[
\mathcal{B}^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k} \mathcal{B}(t_j)dt_j \cdots dt_1. \quad (5.29)
\]
It is clear that there exists \( C > 0 \) so that
\[
\sup_{0 \leq t \leq 1} \| \mathcal{B}(t) \|_{L^2(\ell^2)} \leq C,
\]
and thus, for all \( 0 \leq t \leq 1 \) the series (5.28) converges by
\[
\| b^\infty(t) \|_{L^2} \leq \sup_{0 \leq t \leq 1} \| b_\ell(t) \|_{L^2} \sum_{k \geq 1} \frac{(4(\mathcal{F})^+)^{k-1}}{k!} \leq e \sup_{0 \leq t \leq 1} \| b_\ell(t) \|_{L^2}.
\]
Similarly for \( 0 \leq t \leq 1 \), \( \| \mathcal{B}^\infty(t) \|_{L^2(\ell^2)} \leq 4e(\mathcal{F})^+ \). As a conclusion, the formula (5.27) makes sense. In fact, we can say more about \( \mathcal{B}^\infty(t) \). For \( |\mathcal{F}| < s - 2\sigma \),
\[
|B_{y_j}^{11}| = |B_{y_j}^{21}| = \left| \frac{\partial^2 F}{\partial \bar{z}_j \partial y_j}(\theta, 0, 0) \right| \leq \frac{\langle \mathcal{F} \rangle^+}{(1 + \ln i)^2(1 + \ln j)^2(1 + |i - j|)}.
\]
Similar estimates hold for \( B_{y_j}^{12}, B_{y_j}^{21} \) and \( B_{y_j}^{22} \). Recall that \( \mathcal{B}(t) = J\mathcal{B}(\theta(t)) \) and \( |\mathcal{F}| < s - 2\sigma \) for \( 0 \leq t \leq 1 \). It follows that \( \mathcal{B}(t) \in M_\beta \) and \( \sup_{0 \leq t \leq 1} |\mathcal{B}(t)|_\beta \leq \langle \mathcal{F} \rangle^+ \). Hence, by lemma 5.5 and (5.29),
\[
\sup_{0 \leq \epsilon \leq 1} [B^\epsilon(t)]_\beta \leq e^{(F)^+} - 1 \leq e^{(F)^+}_{\tau, D(\kappa - \sigma, r)}. \tag{5.30}
\]

In the following we study the equation in \( y, \dot{y}(t) = -\nabla_\theta F(\theta, y, \zeta(t)), \ y(0) = y_0 \). From (5.24),
\[
\dot{y}(t) = f(t) + g(t)y(t), \ y(0) = y_0,
\]
where \( f(t) = -\nabla_\theta a_0(\theta(t)) - \nabla_\theta b(\theta(t)) \zeta - \frac{1}{2} \nabla_\theta B(\theta(t)) \zeta \cdot \zeta \) and \( g(t) = -\nabla_\theta \nabla_\theta t F(\theta, 0, 0) \). As above, we formally have
\[
y(t) = f^\infty(t) + (1 + g^\infty(t))y_0, \tag{5.31}
\]
where
\[
f^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k-1} g(t_j)f(t_k)dt_k \cdots dt_1,
\]
and
\[
g^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k} g(t_j)dt_k \cdots dt_1.
\]

From Cauchy we have
\[
\sup_{0 \leq \epsilon \leq 1} \|g(t)\| \leq \frac{1}{\sigma} \max_{1 \leq j \leq n} \left| \frac{\partial F}{\partial y_j}(\theta(t), 0, 0) \right| \leq \frac{1}{\sigma} (F)^+_{\tau, D(\kappa - \sigma, r)} := \kappa,
\]
which follows that for \( 0 \leq t \leq 1 \),
\[
\|f^\infty(t)\| \leq \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k-1} \kappa^{k-1}\|f(t)\|dt_k \cdots dt_1
\]
\[
\leq \sup_{0 \leq \epsilon \leq 1} \|f(t)\| \sum_{k \geq 1} \frac{\kappa^{k-1}}{k!} \leq \sup_{0 \leq \epsilon \leq 1} \|f(t)\|.
\]

Similarly for \( 0 \leq t \leq 1 \),
\[
\|g^\infty(t)\| \leq \frac{(F)^+}{\sigma}. \tag{5.32}
\]

Therefore (5.31) makes sense.

Now we turn to show the estimates on the solutions of the equations (5.18). By (5.27),
\[
\nabla_{\xi, k}^\epsilon(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta_{ij} + B^\epsilon_{ij}(t), \tag{5.33}
\]
and (5.30) we have (5.21). From \( y(t) = f^\infty_k(t) + y_0 + \sum_{1 \leq j \leq n} g^\infty_j(t)y_0 \) and (5.32) we obtain (5.22). In the following, we give the estimates (5.20) and (5.23).
Since $g$ and $g_\infty$ do not depend on $\zeta$, we obtain that $\frac{\partial g_\infty}{\partial \zeta_j} = \frac{\partial g_\infty}{\partial \zeta_i}$. Now, by the definition of $f_\infty$, we deduce that for $0 \leq t \leq 1$, 

$$
\left\| \frac{\partial g(t)}{\partial \zeta_j} \right\| = \left\| \sum_{k=1}^{n} \int_{0}^{1} \cdots \int_{0}^{r_{k-1}} \prod_{l \in \mathcal{K} \setminus k-1} g(t_l) \frac{\partial f_j(t_l)}{\partial \zeta_j} \right\|
\leq \sum_{k=1}^{n} \int_{0}^{1} \cdots \int_{0}^{r_{k-1}} \prod_{l \in \mathcal{K} \setminus k-1} g(t_l) \left\| \frac{\partial f_j(t_l)}{\partial \zeta_j} \right\| \, dt_k \cdots dt_l
\leq \left( \sum_{k=1}^{n} \int_{0}^{1} \cdots \int_{0}^{r_{k-1}} B_k \right) \left( \sum_{k=1}^{n} \int_{0}^{1} \cdots \int_{0}^{r_{k-1}} \left\| \frac{\partial f_j(t_l)}{\partial \zeta_j} \right\| \right) \, dt_k \cdots dt_l
\leq \sup_{0 \leq t \leq 1} |\nabla_{\zeta_j} f(t)|.
$$

From a straightforward computation for all $1 \leq l \leq n$ we have

$$
\nabla_{\zeta_j} f_l(t) = -\partial_0 b_l(\theta(t)) - \sum_{i \neq l} \partial_0 B_{ij}(\theta(t)) \zeta_j(t), \quad \text{with } b_l(\theta) = \nabla_{\zeta} F(\theta, 0, 0).
$$

(5.34)

By Cauchy we obtain that

$$
\sup_{0 \leq t \leq 1} |\partial_0 b_l(\theta(t))| \leq \frac{1}{r} \sup_{|\theta| < 2\sigma \, 0} \left| \nabla_{\zeta} F(\theta, 0, 0) \right| \leq \frac{1}{\sigma} \frac{r(F)_{r, D_{k-\sigma}}}{(1 + \ln k)^{\frac{\sigma}{2}}}.
$$

For the second term in (5.34) we obtain by Cauchy

$$
\sup_{0 \leq t \leq 1} \|\partial_0 B_{ij}(\theta(t))\| \leq \frac{(F)_{r, D_{k-\sigma}}}{\sigma(1 + \ln k)^{\frac{\sigma}{2}}}.
$$

Thus,

$$
\|\nabla_{\zeta_j} f_l(t)\| \leq \frac{r(F)_{r, D_{k-\sigma}}}{\sigma(1 + \ln k)^{\frac{\sigma}{2}}} \left( \sum_{i \neq l} \frac{(F)_{r, D_{k-\sigma}}}{\sigma(1 + \ln k)^{\frac{\sigma}{2}}} \right) \|\zeta\| \leq \frac{r(F)_{r, D_{k-\sigma}}}{\sigma(1 + \ln k)^{\frac{\sigma}{2}}} \left( \sum_{i \neq l} (1 + \ln k)^{\frac{\sigma}{2}} \right)
$$

Further, from $\nabla_{\zeta_j} f_l(t) = \sum_{i \neq l} \nabla_{\zeta_j} \zeta_i \nabla_{\zeta_i} f_l(t)$, we have
\begin{equation}
\| \nabla_\zeta f(t) \| \leq \sum_{k \geq 1} \| \nabla_\zeta \zeta_k \| \| \nabla_\zeta f(t) \|
\end{equation}

\begin{equation}
\leq \sum_{k \geq 1, k \neq j} \frac{\langle F^+_r(e, \bar{D}_{k-j}, r) \rangle}{(1+|k-j|)(1+\ln k)^d(1+\ln j)^d} - \frac{r(F^+_r(e, \bar{D}_{k-j}, r))}{\sigma(1+\ln k)^d} \left( 1 + \sum_{k \geq 1, k \neq j} \frac{1}{(1+|k-j|)(1+\ln k)^d(1+\ln j)^d} \right)
\end{equation}

\begin{equation}
\leq \frac{r(F^+_r(e, \bar{D}_{k-j}, r))}{\sigma(1+\ln j)^d}.
\end{equation}

The above third inequality comes from the lemma A1 and \( \beta \geq 1 \). It then follows that

\begin{equation}
\sup_{0 \leq t \leq 1} \left\| \frac{\partial y(t)}{\partial \zeta_j^0} \right\| \leq \frac{r(F^+_r(e, \bar{D}_{k-j}, r))}{\sigma(1+\ln j)^d}.
\end{equation}

It remains to show (5.23). First, we have

\begin{equation}
\sup_{0 \leq t \leq 1} \left\| \nabla_\zeta \nabla_\zeta f(t) \right\| \leq \sup_{0 \leq t \leq 1} \| \nabla_\zeta \nabla_\zeta f(t) \|.
\end{equation}

Note \( \| \nabla_\zeta \nabla_\zeta f(t) \| = \| \nabla_\zeta B_j(\theta(t)) \| \) and use Cauchy in \( \theta \),

\begin{equation}
\frac{\partial^2 y_j(t)}{\partial \zeta_j^0 \partial \zeta_i^0} \leq \frac{\langle F^+_r(e, \bar{D}_{k-j}, r) \rangle}{\sigma(1+\ln i)^d(1+\ln j)^d(1+|i-j|)}.
\end{equation}

\begin{equation}
\square
\end{equation}

Similarly, we have:

**Lemma 5.6.** Under the assumptions of lemma 5.4 and the condition \( \langle F^+_r(e, \bar{D}_{k-j}, r) \rangle \leq C \sigma \), the solution of (5.18) satisfies

\begin{equation}
\sup_{0 \leq t \leq 1} \left| \frac{\partial y_j(t)}{\partial \zeta_j^0} \right|^2 \leq \frac{r(F^+_r(e, \bar{D}_{k-j}, r))}{\sigma(1+\ln j)^d},
\end{equation}

\begin{equation}
\sup_{0 \leq t \leq 1} \left| \frac{\partial w_j(t)}{\partial \zeta_j^0} \right|^2 \leq \frac{1}{(1+\ln j)^d(1+\ln k)^d(1+|j-k|)} \langle F^+_r(e, \bar{D}_{k-j}, r) \rangle,
\end{equation}

\begin{equation}
\sup_{0 \leq t \leq 1} \left| \frac{\partial y_j(t)}{\partial y_j^0} \right|^2 \leq \frac{1}{\sigma(F^+_r(e, \bar{D}_{k-j}, r))},
\end{equation}

\begin{equation}
\sup_{0 \leq t \leq 1} \left| \frac{\partial^2 y_j(t)}{\partial \zeta_j^0 \partial \zeta_i^0} \right|^2 \leq \frac{1}{\sigma(1+\ln j)^d(1+\ln i)^d(1+|i-j|)} \langle F^+_r(e, \bar{D}_{k-j}, r) \rangle.
\end{equation}

with \( w_k = z_k \) or \( x_k \) and \( w_k^0 = x_k^0 \) or \( x_k^0 \), \( k = 1, 2, \ldots \).
The proof of lemma 5.4 implies

**Corollary 5.7.** The time 1 map \( X^1_F \) reads

\[
\begin{pmatrix}
\theta \\
y \\
\zeta
\end{pmatrix} \mapsto \begin{pmatrix}
K(\theta) \\
L(\theta, \zeta) + M(\theta) \zeta + S(\theta)y \\
T(\theta) + U(\theta)\zeta
\end{pmatrix}
\]

where \( L(\theta, \zeta) \) is quadratic in \( \zeta \), \( M(\theta) \) and \( U(\theta) \) are bounded linear operators from \( \ell_2^p \times \ell_2^p \) into \( \mathbb{R}^n \) and \( \ell_2^p \times \ell_2^p \), respectively, and \( S(\theta) \) is a bounded linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

5.3. Composition estimates

**Proposition 5.8.** Let \( 0 < \eta < 1/8 \) and \( 0 < \sigma < s \), \( R \in \Gamma^\beta_{\varphi, D(s-2\sigma, 4\eta r)} \) and \( F \in \Gamma^{\beta, +}_{r, D(s-\sigma, r)} \) with an \( F \) of degree 2. Assume that

\[
\langle F \rangle_{r, D(s-\sigma, r)}^+ + \langle F \rangle_{r, D(s-\sigma, r)}^+ \leq C\sigma^2.
\]  

Then, \( R \circ X^1_F \in \Gamma^\beta_{\varphi, D(s-5\sigma, 4\eta r)} \) and

\[
\langle R \circ X^1_F \rangle_{\varphi, D(s-5\sigma, 4\eta r)} \leq \langle R \rangle_{\varphi, D(s-2\sigma, 4\eta r)}^+ \langle F \rangle_{r, D(s-\sigma, r)}^+.
\]

**Remark 5.9.** In proposition 5.8, we do not require an \( R \) of degree 2.

**Proof.** In the sequel, we use the notation

\[
(\theta, y, z, \zeta) = X^1_F((\theta^0, y^0, z^0, \zeta^0)).
\]

From (5.35) and \( \langle \lambda \rangle_{\varphi, D(s-4\eta r)} \leq \eta^{-2} \langle \lambda \rangle_{\varphi, D(s-\sigma, r)} \) it follows that

\[
\langle F \rangle_{\varphi, D(s-\sigma, 4\eta r)} \leq \sigma
\]

which will be used later. Now, by (5.35) it is easy to show that \( X^1_F \) maps \( D(s-5\sigma, 4\eta r) \) into \( D(s-2\sigma, 4\eta r) \) and thus,

\[
\| R \circ X^1_F \|_{D(s-5\sigma, 4\eta r)} \leq (4\eta r)^2 \langle R \rangle_{\varphi, D(s-2\sigma, 4\eta r)}^+.
\]

By the Leibniz rule, for all \( 1 \leq j \leq n \),

\[
\frac{\partial}{\partial y_j} \left( R \circ X^1_F \right) = \sum_{k=1}^n \frac{\partial R}{\partial y_k} \frac{\partial X^1_F}{\partial y_j}.
\]

From the definition \( \| \frac{\partial}{\partial y_k} \|_{D(s-2\sigma, 4\eta r)} \leq \langle R \rangle_{\varphi, D(s-2\sigma, 4\eta r)} \), and by lemma 5.4,
\[
\sup_{0 \leq t \leq 1} \left| \frac{\partial y(t)}{\partial \eta_j^0} \right| \leq \frac{1}{\sigma} (F)_{4y, D(s-\sigma, 4\eta y)}^+ + \delta_k.
\]

Thus, from (5.38) we obtain
\[
\left| \frac{\partial (R \circ X^j_1)}{\partial \eta_j^0} \right| \leq (R)_{y, D(s-2\sigma, 4\eta y)}^+ (F)_{4y, D(s-\sigma, 4\eta y)}^+.
\]

(5.40)

For \( j \geq 1 \), the derivatives in \( \eta_j^0 \) read
\[
\frac{\partial (R \circ X^j_1)}{\partial \eta_j^0} = \sum_{k=1}^{n} \frac{\partial R(X^1_k)}{\partial \eta_k} \frac{\partial y_k}{\partial \eta_j^0} + \sum_{k \geq 1} \left( \frac{\partial R(X^1_k)}{\partial \eta_k} \frac{\partial \eta_k}{\partial \eta_j^0} + \frac{\partial R(X^1_k)}{\partial \eta_k} \frac{\partial \eta_k}{\partial \eta_j^0} \right) := (I) + (II).
\]

From (5.38),
\[
|I| \leq \sum_{k=1}^{n} \frac{\eta y}{\sigma (1 + \ln j)^3} (R)_{y, D(s-2\sigma, 4\eta y)}^+ (F)_{4y, D(s-\sigma, 4\eta y)}^+ \leq \frac{\eta y}{(1 + \ln j)^3} (R)_{y, D(s-2\sigma, 4\eta y)}^+.
\]

and
\[
|II| \leq \sum_{k \geq 1} \left( \left( \frac{\partial R(X^1_k)}{\partial \eta_k} \right) \left| \frac{\partial \eta_k}{\partial \eta_j^0} \right| + \left( \frac{\partial R(X^1_k)}{\partial \eta_k} \right) \left| \frac{\partial \eta_k}{\partial \eta_j^0} \right| \right) \leq \sum_{k \geq 1} \left( \frac{\eta y}{(1 + \ln j)^3 (1 + \ln k)^3 (1 + |j-k|)} + \delta_k \right) \frac{\eta y}{(1 + \ln j)^3} (R)_{y, D(s-2\sigma, 4\eta y)}^+ \leq \frac{\eta y}{(1 + \ln j)^3} (R)_{y, D(s-2\sigma, 4\eta y)}^+. \tag{5.8}
\]

Thus
\[
\left| \frac{\partial (R \circ X^j_1)}{\partial \eta_j^0} \right| \leq \frac{\eta y}{(1 + \ln j)^3} (R)_{y, D(s-2\sigma, 4\eta y)}^+. \tag{5.41}
\]

We now estimate \( \left| \frac{\partial^2 (R \circ X^j_1)}{\partial \eta_j^0 \partial \eta_j^0} \right|_{D(s-5\sigma, \eta y)} \). The derivatives read
\[
\frac{\partial^2 (R \circ X^j_1)}{\partial \eta_j^0 \partial \eta_j^0} = (I_1) + (I_2) + (I_3) + (I_4)
\]
with

\[ (I_1) = \sum_{k,l=1}^{n} \frac{\partial^2 R(X_p^l) \partial y_l \partial y_k}{\partial y_l \partial z_k} \quad (I_2) = \sum_{k=1}^{n} \frac{\partial R(X_p^l) \partial^2 y_k}{\partial y_l \partial z_k} \]

\[ (I_3) = \sum_{k \geq 1 \atop l \geq 1} \sum_{i=1}^{n} \frac{\partial^2 R(X_p^l) \partial y_l \partial z_k}{\partial y_l \partial z_k} + \sum_{k \geq 1 \atop l \geq 1} \sum_{p \geq 1} \frac{\partial^2 R(X_p^l) \partial z_p \partial z_k}{\partial z_p \partial z_k} + \sum_{k \geq 1 \atop l \geq 1 \atop p \geq 1} \frac{\partial^2 R(X_p^l) \partial z_p \partial z_k}{\partial z_p \partial z_k} \]

\[ = (I_a) + (I_b) + (I_c) \]

and

\[ (I_4) = \sum_{k \geq 1 \atop l \geq 1} \sum_{i=1}^{n} \frac{\partial^2 R(X_p^l) \partial y_l \partial z_k}{\partial y_l \partial z_k} \]

We give a detailed estimation for \((I_3)\). From Cauchy, (5.38) and lemma 5.4,

\[ \| (I_3) \|_{D_{\sigma-5,\nu}} \lesssim \sum_{k \geq 1 \atop l \geq 1} \sum_{i=1}^{n} \left| \frac{\partial^2 R \circ X_p^l}{\partial y_l \partial z_k} \right| \left| \frac{\partial y_l}{\partial y_l} \right| \left| \frac{\partial z_k}{\partial z_k} \right| \]

\[ \lesssim \sum_{k \geq 1 \atop l \geq 1} \sum_{i=1}^{n} (n \nu)^2 \left| \frac{\partial R}{\partial z_k} \right| \left| \frac{\partial z_k}{\partial z_k} \right| \]

\[ \cdot \left( \frac{\langle F \rangle_{4g,\nu}^{+} D_{\sigma-2,4\nu}}{(1 + \ln \nu)^2(1 + \ln \nu)^3(1 + j - k)} + \delta_k \right) \]

\[ \lesssim \frac{\langle R \rangle_{4g,\nu}^{+} D_{\sigma-2,4\nu}}{(1 + \ln \nu)^2(1 + \ln \nu)^3}. \]

In the same way,

\[ \| (I_4) \|_{D_{\sigma-5,\nu}} \lesssim \sum_{k \geq 1 \atop l \geq 1} \sum_{i=1}^{n} \left| \frac{\partial^2 R \circ X_p^l}{\partial y_l \partial z_k} \right| \left| \frac{\partial z_k}{\partial z_k} \right| \]

\[ \cdot \left( \frac{\langle F \rangle_{4g,\nu}^{+} D_{\sigma-2,4\nu}}{(1 + \ln \nu)^2(1 + \ln \nu)^3(1 + j - k)} + \delta_k \right) \]

\[ \lesssim \frac{\langle R \rangle_{4g,\nu}^{+} D_{\sigma-2,4\nu}}{(1 + \ln \nu)^2(1 + \ln \nu)^3}. \]

Similarly, we have

\[ \| (I_5) \|_{D_{\sigma-5,\nu}} \lesssim \frac{\langle R \rangle_{4g,\nu}^{+} D_{\sigma-2,4\nu}}{(1 + \ln \nu)^2(1 + \ln \nu)^3}. \]
Therefore,

\[ \| (I_1) \|_{D^{(1-5\eta, \eta')}} \leq \frac{(R)_{ij, D_k(-2\eta, 4\eta')}}{(1 + \ln i)^\gamma(1 + \ln j)^\gamma}. \]

A similar computation provides us

\[ \| (I_2) \|_{D^{(1-5\eta, \eta')}} \leq \frac{(R)_{ij, D_k(-2\eta, 4\eta')}}{(1 + \ln i)^\gamma(1 + \ln j)^\gamma}, \]

\[ \| (I_3) \|_{D^{(1-5\eta, \eta')}} \leq \frac{(R)_{ij, D_k(-2\eta, 4\eta')}}{(1 + \ln i)^\gamma(1 + \ln j)^\gamma}, \]

\[ \| (I_4) \|_{D^{(1-5\eta, \eta')}} \leq \frac{(R)_{ij, D_k(-2\eta, 4\eta')}}{(1 + \ln i)^\gamma(1 + \ln j)^\gamma}. \]

This results in

\[ \left\| \frac{\partial^2 (R \circ X^p_j)}{\partial \zeta_j^0 \partial \zeta_j^0} \right\|_{D^{(1-5\eta, \eta')}} \leq \frac{(R)_{ij, D_k(-2\eta, 4\eta')}}{(1 + \ln i)^\gamma(1 + \ln j)^\gamma}. \]  \hspace{1cm} (5.42)

By (5.39)–(5.42), (5.36) holds. We omit the proof of (5.37), which is similar by using the estimates of the lemma 5.6 instead.

**Lemma 5.10.** Assume \( P \) satisfies assumption \( B \), and consider its Taylor approximation \( R \) of the form (5.3). Then, for all \( \eta > 0 \),

\[ \| X_k \|_{r, D^{(k, r)}} \leq \| X^p_k \|_{r, D^{(k, r)}}, \]

and

\[ \| X_P - X_k \|_{r, D^{(k, r)}} \leq \eta \| X^p_k \|_{r, D^{(k, r)}}. \]

We have an analogous result for the norm \( \langle \cdot \rangle_{r, D^{(k, r)}} \).

**Lemma 5.11.** Let \( P \in \Gamma^{(2)}_{r, D^{(k, r)}} \) and consider its Taylor approximation \( R \) of the form (5.3). Then, for all \( \eta > 0 \),

\[ \langle R \rangle_{ij, D^{(k, r)}} \leq \langle P \rangle_{ij, D^{(k, r)}} \]

and

\[ \langle P - R \rangle_{ij, D^{(k, r)}} \leq \eta \langle P \rangle_{ij, D^{(k, r)}}. \]

We omit the proofs for the above two lemmas.

### 5.4. The KAM step

Let \( N \) be a Hamiltonian in normal form as in (5.1), which reads in the variables \((\theta, y, z, \xi)\),

\[ N = \sum_{1 \leq j \leq n} \omega_j(\xi) + \sum_{j \geq 1} \sum_{\beta} \Omega_j(\xi) \xi^\beta \xi^\beta, \]

1436
and suppose that assumption $A$ is satisfied. Consider a perturbation $P$ which satisfies assumption $B$ for some $r, s > 0$. Then choose $0 < \eta < 1/8$, $0 < \sigma < s$ and assume that
\[
\langle P\rangle_{\ell^{-D(s,r)}} + \|X_P\|_{\ell^{-D(s,r)}} + \frac{\alpha}{M}(\langle P\rangle_{\ell^{-D(s,r)}}^2 + \|X_P\|_{\ell^{-D(s,r)}}^2) \leq \frac{\alpha^2 \eta^2 e^{-\gamma \beta}}{MC_0},
\] (5.43)
where $\theta = \frac{r}{\beta - \gamma}$, and $C_0$ is a large constant depending only on $\nu, \tau$ and $\gamma$.

5.4.1 Estimates on the new error term. We estimate the new error term $P_+$ given by the formula
\[
P_+ = (P - R) \circ X_1^t + \int_0^1 \{R(t), F\} \circ X_1^t dt,
\] (5.44)
where $R(t) = (1 - t)\bar{N} + tR$.

**Lemma 5.12.** Assume (5.43). Then there exists $c(n, \beta) > 0$ so that for all $0 \leq \lambda \leq \alpha / M$,
\[
\langle P\rangle_{\ell^{-D(s-5\nu, \nu)}}^\lambda + \|X_P\|_{\ell^{-D(s-5\nu, \nu)}}^\lambda \leq \frac{c(n, \beta) e^{\frac{\lambda^2}{4 \alpha \beta}}}{\alpha \eta^2}(\langle P\rangle_{\ell^{-D(s,r)}}^\lambda + \|X_P\|_{\ell^{-D(s,r)}}^\lambda)^2 + \eta(\langle P\rangle_{\ell^{-D(s,r)}}^\lambda + \|X_P\|_{\ell^{-D(s,r)}}^\lambda)^2).
\]

We divide this into two lemmas; from [40], we have:

**Lemma 5.13.** Assume (5.43), then
\[
\|X_P\|_{\ell^{-D(s-5\nu, \nu)}}^\lambda \leq \frac{c(n, \beta) e^{\frac{\lambda^2}{4 \alpha \beta}}}{\alpha \eta^2}(\langle X_P\rangle_{\ell^{-D(s,r)}}^\lambda)^2 + \eta(\langle X_P\rangle_{\ell^{-D(s,r)}}^\lambda)^2).
\]

**Lemma 5.14.** Assume (5.43), then
\[
\langle P\rangle_{\ell^{-D(s-5\nu, \nu)}}^\lambda \leq \frac{c(n, \beta) e^{\frac{\lambda^2}{4 \alpha \beta}}}{\alpha \eta^2}(\langle P\rangle_{\ell^{-D(s,r)}}^\lambda)^2 + \eta(\langle P\rangle_{\ell^{-D(s,r)}}^\lambda)^2.
\]

**Proof.** By (5.44), proposition 5.8 and lemma 5.11, we have
\[
\langle (P - R) \circ X_1^t \rangle_{\ell^{-D(s-5\nu, \nu)}} = \langle (P - R) \circ X_1^t \rangle_{\ell^{-D(s-5\nu, \nu)}} + \lambda \langle (P - R) \circ X_1^t \rangle_{\ell^{-D(s-5\nu, \nu)}} \leq \langle P - R \rangle_{\ell^{-D(s-2\nu, 4\nu)}}^\lambda + \lambda(\langle P - R \rangle_{\ell^{-D(s-2\nu, 4\nu)}}^\lambda)^2 \leq \eta(\langle P\rangle_{\ell^{-D(s,r)}}^\lambda)^2 + \eta(\langle P\rangle_{\ell^{-D(s,r)}}^\lambda)^2.
\]

On the other hand, by the same method,
\[
\left\{ \int_0^1 \{R(t), F\} \circ X_1^t dt \right\}_{\ell^{-D(s-5\nu, \nu)}}^\lambda \leq \langle \{R(t), F\} \rangle_{\ell^{-D(s-2\nu, 4\nu)}}^\lambda.
\] (5.45)

Note that $R(t) = (1 - t)\bar{N} + tR$ and $\bar{N} = [R]$, and from lemma 5.3 we obtain
\[
\langle \langle [R], F \rangle \rangle \lambda_{r, D, (r, r)} \leq \eta^{-2} \langle \langle [R], F \rangle \rangle_{r, D, (r, r)}^{-2} \\
\leq \frac{1}{\sigma \eta^2} (\langle [R] \rangle_{r, D, (r, r)}^{\lambda} (F)_{r, D, (r, r), \sigma, \rho} + \langle [R] \rangle_{r, D, (r, r)}^{\lambda} (F)_{r, D, (r, r), \rho, \sigma}).
\]

where we use \( \langle \rangle \rangle_{r, D, (r, r)} \leq \eta^{-2} \langle \rangle_{r, D, (r, r)} \) for \( 0 < \eta < 1 \). Similarly
\[
\langle [R], F \rangle \rangle_{r, D, (r, r)} \leq \frac{1}{\sigma \eta^2} (\langle [R] \rangle_{r, D, (r, r)}^{\lambda} (F)_{r, D, (r, r), \sigma, \rho} + \langle [R] \rangle_{r, D, (r, r)}^{\lambda} (F)_{r, D, (r, r), \rho, \sigma}).
\]

Thus, by lemma 5.2 and \( 0 \leq \lambda \leq \alpha / M \),
\[
(5.45) \leq \frac{1}{\sigma \eta^2} (\langle P \rangle_{r, D, (r, r)}^{\lambda} (F)_{r, D, (r, r), \sigma, \rho} + \langle P \rangle_{r, D, (r, r)}^{\lambda} (F)_{r, D, (r, r), \rho, \sigma})
\leq \frac{c(n, \beta)}{\alpha \sigma \eta^2} (\langle P \rangle_{r, D, (r, r)}^{\lambda} (F)_{r, D, (r, r), \sigma, \rho} + \lambda \langle P \rangle_{r, D, (r, r)}^{\lambda} (F)_{r, D, (r, r), \sigma, \rho} \sigma^2 + \lambda \langle P \rangle_{r, D, (r, r)}^{\lambda} (F)_{r, D, (r, r), \rho, \sigma})
\leq \frac{c(n, \beta)}{\alpha \sigma \eta^2} \sigma^2 \| (P)_{r, D, (r, r)} \|.
\]

\[\Box\]

5.4.2. Estimates on the frequencies.

**Lemma 5.15.** There exists \( K \) and \( 0 < \alpha < \alpha \) so that
\[
| (k, \omega(\xi)) + (l, \Omega(\xi)) | \geq \frac{(l) \alpha}{A_k}, \quad |k| \leq K, \quad |l| \leq 2,
\]

where \( A_k = e^{|k| \beta / (3 > \tau)} \).

**Proof.** Note that \( \omega = \omega + \Omega \), \( \Omega_{\tau} = \Omega + \Omega \). Since \( \Omega(\xi) = \frac{\partial N}{\partial y}(0, 0, 0, 0, \xi) \), we obtain that
\[
| \Omega | = \sup_{D, (r, r) \times \Omega} \left| \frac{\partial N}{\partial y} \right| \leq \| X \|_{r, D, (r, r)} \leq \| X \|_{r, D, (r, r)}.
\]

On the other hand, \( \Omega_{\tau}(\xi) = \frac{\partial N}{\partial y}(0, 0, 0, 0, \xi) \), thus
\[
\| \Omega_{\tau} \|_{2, 1} \leq \sup_{D, (r, r) \times \Omega} \left| \frac{\partial N}{\partial y} \right| (1 + \ln j)^{2, 3} \leq \langle N \rangle_{r, D, (r, r)} \leq \langle P \rangle_{r, D, (r, r)}.
\]

Therefore,
\[
\| \Omega_{\tau} \|_{2, 1} \leq \| X \|_{r, D, (r, r)} + \langle P \rangle_{r, D, (r, r)}.
\]

Similarly, for the Lipschitz norms we obtain
\[
\| \Omega_{\tau} \|_{2, 1} \leq \| X \|_{r, D, (r, r)} + \langle P \rangle_{r, D, (r, r)}.
\]

Discussing different cases we easily obtain
\[
| (k, \Omega) + (l, \Omega) | \leq \| X \|_{r, D, (r, r)} + \langle P \rangle_{r, D, (r, r)}.
\]
If we choose $\alpha + \frac{\omega_c(\xi)}{\Omega} \leq CK \max_{|k| \leq K} A_k(\|X_P\|_{D(\xi, \rho)} + \langle P \rangle_{D(\xi, \rho)})$, then for $|k| \leq K$,

$$
|\langle k, \omega_c(\xi) \rangle + \langle I, \Omega_c(\xi) \rangle| \geq |\langle k, \omega_c(\xi) \rangle + \langle I, \Omega_c(\xi) \rangle| - |\langle k, \Omega(\xi) \rangle + \langle I, \Omega(\xi) \rangle|
$$

$$
\geq \frac{(l)\alpha}{A_k} - C|k|\langle X_P\rangle_{D(\xi, \rho)} + \langle P \rangle_{D(\xi, \rho)}
$$

$$
\geq \frac{(l)\alpha}{A_k}
$$

with $\alpha_+ = \alpha - \alpha$.

It remains to show that $\alpha_+ > 0$. This will be done in the KAM iteration below (see (5.49)).

5.4.3. The iterative lemma. Denote $P_0 = P$ and $N_0 = N$. Then at the $\nu$th step of the Newton scheme, we have a Hamiltonian $H_\nu = N_\nu + P$ where

$$
N_\nu = \sum_{j=1}^{a} \omega_j(\xi) y_j + \sum_{j=1}^{b} \Omega_j(\xi) z_j^2
$$

We will show that there exists a symplectic coordinate transformation $\Phi : \Pi_{\nu+1} \rightarrow \Pi_{\nu}$ such that $H_{\nu+1} = H_\nu \circ \Phi_{\nu+1} = N_{\nu+1} + P_{\nu+1}$ satisfies the same assumptions with $\nu + 1$ in place of $\nu$, where the new normal form $N_{\nu+1}$ is associated with the new frequencies given by

$$
\omega_{\nu+1,j} = \omega_{\nu,j} + \Omega_{\nu,j}, \quad \Omega_{\nu+1,j} = \Omega_{\nu,j} + \bar{\omega}_{\nu,j} and P_{\nu+1} is given by
$$

$$
P_{\nu+1} = (P_\nu - R_\nu) \circ \Phi_{\nu}^{-1} + \int_0^{t_\nu + 1} \{R_\nu(t), F_\nu\} \circ \Phi_{\nu}^{-1} dt
$$

with $R_\nu(t) = (1 - t)N_\nu + tR_\nu$.

Let $c_1$ be twice the maximum of all constants obtained during the KAM step. Set $r_0 = r, s_0 = s, \alpha_0 = \alpha$ and $M_0 = M$. For $\nu \geq 0$ set

$$
\alpha_\nu = \frac{\alpha_0}{2^{\nu}}(1 + 2^{-\nu}), \quad M_\nu = M_0(2 - 2^{-\nu}), \quad \lambda_\nu = \frac{\alpha_\nu}{M_\nu},
$$

and

$$
\epsilon_{\nu+1} = \frac{c_1 \epsilon_0}{\alpha_\nu}, \quad \sigma_\nu = \frac{8 \cdot 700^{\nu-1}}{\ln \epsilon_\nu}, \quad \eta_\nu = \frac{99}{\epsilon_0 \epsilon_\nu}, \quad s_{\nu+1} = s_\nu - 5\sigma_\nu, \quad r_{\nu+1} = \eta_\nu r_\nu, \quad \beta = \epsilon \nu (\nu \geq 2),
$$

and $D_\nu = D(s_\nu, r_\nu)$.

The initial conditions are chosen in the following way: $\sigma_0 = s_0/48 \ll 1$, so that $s_0 > s_1 > \cdots \geq s_0/2$, and assume $\sigma_0 \leq \gamma_0 \alpha_0^5$ with $\gamma_0 \leq \min\{1/5\alpha_0, 1/\epsilon_0\}$, where $c_2(s_0) = \exp \left( -\frac{48 \cdot 700^{\nu-1}}{\ln s_0} \right)$. Furthermore, we define $K_\nu = K_0(\frac{36}{25})^\nu$ with $K_0 = \frac{1}{4} \ln \frac{700}{4 \epsilon_0}$.

Lemma 5.16. Iterative lemma: suppose that $H_\nu = N_\nu + P_\nu$ is given on $D_\nu \times \Pi_\nu$, where $N_\nu = \sum_{j \leq n} \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) z_j^2$ is a normal form satisfying

$$
|\omega_j|_{\Pi_\nu} + |\Omega_j|_{\Pi_\nu} \leq M_\nu,
$$

$$
|\langle k, \omega_c(\xi) \rangle + \langle I, \Omega_c(\xi) \rangle| \geq \frac{(l)\alpha}{A_k}, \quad (k, I) \in \mathbb{Z},
$$

$$
(5.47)
$$
on \( \Pi_\nu \) and

\[
(P)_{\nu+1}^{\nu} + \|X\|_{L^2}^{\nu} \lesssim \varepsilon_\nu. \tag{5.48}
\]

Then there exists a Lipschitz family of real analytic symplectic coordinate transformations

\[ \Phi_{\nu+1}: D_{\omega+1} \times \Pi_{\nu+1} \mapsto D_\nu \]

with a closed subset \( \Pi_{\nu+1} = \Pi_\nu \setminus \bigcup_{k \geq K_\nu} \mathcal{R}_{kl}^{\nu+1}(\alpha_{\nu+1}) \) of \( \Pi_\nu \), where

\[
\mathcal{R}_{kl}^{\nu+1}(\alpha_{\nu+1}) = \left\{ \xi \in \Pi_\nu : \left| \langle k, \omega_\nu(\xi) \rangle + \langle l, \Omega_\nu(\xi) \rangle \right| < \frac{\langle l \rangle}{A_k} \right\}
\]

such that for \( H_{\nu+1} = H_\nu \circ \Phi_{\nu+1} = N_{\nu+1} + P_{\nu+1} \), the same assumptions (5.47) and (5.48) are satisfied with \( \nu + 1 \) in place of \( \nu \).

**Proof.** By induction one verifies that

\[
\varepsilon_\nu \leq \frac{\alpha^2 \eta^2}{M_c \varepsilon_0}, \quad \eta = \frac{\tau}{\beta - \tau} = \frac{1}{t - 1}.
\]

So the smallness condition (5.43) at the \( \nu \)th KAM step is satisfied, and there exists a transformation \( \Phi_{\nu+1}: D_{\omega+1} \times \Pi_{\nu+1} \mapsto D_\nu \) taking \( H_\nu \) into \( H_{\nu+1} = N_{\nu+1} + P_{\nu+1} \). From lemma 5.12, the new error term satisfies the estimate

\[
\langle P_{\nu+1} \rangle_{\nu+1}^{\nu+1} + \|X\|_{L^2}^{\nu+1} \lesssim \left( \frac{\varepsilon_\nu^{(L)}}{\alpha_\nu \eta_\nu^2} + \eta \varepsilon_\nu \right) \lesssim \varepsilon_{\nu+1}.
\]

In view of (5.47) the Lipschitz semi-norm of the new frequencies is bounded by

\[
\|\omega_{\nu+1}\|_{L^2}^{\nu+1} + \|\Omega_{\nu+1}\|_{L^2}^{\nu+1} \lesssim M_\nu + \frac{\varepsilon_\nu^{(L)}}{\alpha_\nu} \lesssim M_\nu \left( 1 + \frac{1}{2^{2+2}} \right)
\]

where the second inequality is from (5.46). Finally, one verifies that \( \alpha_\nu - \alpha_{\nu+1} \geq c K_\nu K^{\nu+3}_{\nu_3} \varepsilon_\nu \), hence

\[
\alpha_\nu - \alpha_{\nu+1} \geq c K_\nu K^{\nu+1}_{\nu_3} (\langle P \rangle_{\nu+1}^{\nu} + \|X\|_{L^2}^{\nu}). \tag{5.49}
\]

Therefore, by lemma 5.15, the small divisor estimates hold for the new frequencies with parameter \( \alpha_{\nu+1} \) up to \( |k| \leq K_\nu \). Removing from \( \Pi_\nu \) the union of the resonance zones \( \mathcal{R}_{kl}^{\nu+1}(\alpha_{\nu+1}) \) for \( |k| > K_\nu \), we obtain the parameter domain \( \Pi_{\nu+1} \subset \Pi_\nu \) with the required properties.

**5.4.4. Proof of theorem 2.2.** We follow the proofs in [24] and [40]. For the readers’ convenience, we use the same notations as in [24]. Firstly, as [40], we have the estimates.
Lemma 5.17. For $\nu \geq 0$,
\[
\frac{1}{\alpha_0} \| \Phi_{\nu + 1} - id \|_{D_{\nu + 1}} + \| D\Phi_{\nu + 1} - ID \|_{D_{\nu + 1}} \leq c_1 e^{\frac{4}{\nu - 1} \alpha_0^{-1} \varepsilon_0},
\]
\[
|\omega_{\nu + 1} - \omega_0|_D^{1/2}, \quad \| \Omega_{\nu + 1} - \Omega_0 \|_{D_{\nu + 1}} \leq c_1 \varepsilon_0.
\]

Now, suppose the assumptions of theorem 2.2 are satisfied. To apply the iterative lemma (lemma 5.16) with $\nu = 0$, set $s_0 = s$, $r_0 = r$, ..., $N_0 = N$, $P_0 = P$ and $\gamma_0 = \gamma$, $\alpha_0 = \alpha$, $M_0 = M$. The smallness condition is satisfied, because
\[
\varepsilon = (P \lambda \epsilon_0)^{1/2} + \| X(P \lambda \epsilon_0)^{1/2} \|_{D_0} \leq \gamma_0 \epsilon_0^5 = \varepsilon_0.
\]

The small divisor conditions are satisfied by setting $\Pi_0 = \Pi \bigcup_{k,l} R_{kl}^{0}$. Then, the iterative lemma applies, and we obtain a decreasing sequence of domains $D_{\nu} \subset \Pi_\nu$ and transformations $\Phi^\nu = \Phi_0 \circ \cdots \circ \Phi_{\nu}$, $D_{\nu} \times \Pi_{\nu-1} \rightarrow D_{\nu-1}$ for $\nu \geq 1$, such that $H \circ \Phi^\nu = N_{\nu} + P_{\nu}$. Moreover, the estimates in lemma 5.17 hold.

From lemma 5.17 we have $\| D\Phi^\nu \|_{D_\nu} \leq 1 + c_1 e^{\frac{4}{\nu - 1} \alpha_0^{-1} \varepsilon_0}$, and thus
\[
\| D\Phi^\nu \|_{D_\nu} \leq (1 + 2^{-\nu/2}) \leq 2,
\]
for all $\nu \geq 0$. Similarly, we have $\| D\Phi^\nu \|_{D_\nu} \leq 2$. Thus, $\| \Phi^\nu - \Phi^0 \|_{D_0} \leq \| \Phi^\nu - id \|_{D_\nu}$. So $\Phi^\nu$ converges uniformly on $D_{\nu} \times \Pi_{\nu-1} = D(s/2) \times \Pi_\nu$ to a Lipschitz continuous family of real analytic torus embeddings $\Phi : T^n \rightarrow \Pi_\nu$.

From (5.51) we obtain $\| \Phi^\nu - \Phi^0 \|_{D_\nu} \leq 2 c_1 \| \alpha_0^{-1} \varepsilon_0^{1/2} \|_{D_0}$. It follows that
\[
\| \Phi^\nu - id \|_{D_\nu} \leq \sum_{\nu} 2 c_1 \| \alpha_0^{-1} \varepsilon_0^{1/2} \|_{D_0} \leq \frac{1}{\nu}.
\]
Notice (5.50), the estimate (2.4) holds on $D(s/2) \times \Pi_\nu$. A similar discussion in [24] shows us that the estimate (2.4) can be extended to the domain $D(s/2, r/2)$. The estimates (2.5) and (2.6) are simple and we omit the details.

Note that $\Phi$ is analytic on $D(s/2, r/2)$, and we deduce that $H \circ \Phi = N^* + P^*$ analytic on $D(s/2, r/2)$. We need to prove that $\partial_{s} P^* = \partial_{r} P^* = 0$. $\partial_{s}^2 P^* = \partial_{r}^2 P^* = 0$ on $D(s/2) \times \Pi_\nu$. In the following, we only give the proof for $\partial_{s}^2 P^* = 0$ and omit the proofs for the others.

Note that $\| \partial_{s}^2 P \|_{D(s/2)} \leq \varepsilon_0$ and $\| \partial_{s}^2 (P_k - P_{k+1}) \|_{D(s/2)} \leq \varepsilon_{k+1} \leq \varepsilon_0$. It then follows that
\[
\| \partial_{s}^2 P \|_{D(s/2)} \leq \sum_{k} \| \partial_{s}^2 (P_k - P_{k+1}) \|_{D(s/2)} \leq \varepsilon_0
\]
and so,
\[
\| \partial_{s}^2 P \|_{D(s/2)} \leq \| \partial_{s}^2 P \|_{D(s/2)} + \| \partial_{s}^2 (P_k - P_{k+1}) \|_{D(s/2)} \leq \varepsilon_0
\]
for all $\nu$, which means that $\partial_{s}^2 P^* = 0$ on $D(s/2) \times \Pi_\nu$. \hfill \qed

6. Measure estimates

In this section we prove the measure estimates.
Theorem 6.1. Let $\omega_{\nu}$, $\Omega_{\nu}$ for $\nu \geq 0$ be Lipschitz maps on $\Pi$ satisfying
\[ |\omega_{\nu} - \omega|, \|\Omega_{\nu} - \Omega\|_{2,3} \leq \alpha, \quad |\omega_{\nu} - \omega|^2, \|\Omega_{\nu} - \Omega\|_{2,3}^2 \leq \frac{1}{2L}, \]
and define the sets $\mathcal{R}_\nu^0(\alpha)$ as in lemma 5.16 choosing $\tau \geq n + 2$. Then,
\[ \text{Meas}(\Pi \setminus \Pi_{\nu}) \leq \text{Meas}(\bigcup \mathcal{R}_\nu^0(\alpha)) \rightarrow 0, \quad \text{as } \alpha \rightarrow 0. \]
In estimating the measure of the resonance zones, it is not necessary to distinguish between the various perturbations $\omega_{\nu}$ and $\Omega_{\nu}$ of the frequencies, since only the size of the perturbation matters. Therefore, we write $\omega'$, $\Omega'$ for all of them, and we have
\[ |\omega' - \omega|, \|\Omega' - \Omega\|_{2,3} \leq \alpha, \quad |\omega' - \omega|^2, \|\Omega' - \Omega\|_{2,3}^2 \leq \frac{1}{2L}. \]
Similarly, we write $\mathcal{R}_0^0$ rather than $\mathcal{R}_0^0$ for the various resonance zones. The proof of theorem 6.1 requires a couple of lemmas.

Lemma 6.2. For $l \in \Lambda = \{l : 1 \leq |l| \leq 2\}$,
\[ \ln(1 + \langle l \rangle) \geq \frac{1}{8} \|l\|_{2,3} \|l\|_{2,3}, \]
where $\|l\|_{2,3} = \sup_{j \geq 1} |l_j|(1 + \ln j)^{2,3}$.

Proof. We only prove the most complicated case, i.e. $l = (\ldots, 1, \cdots, -1, \cdots)$. In other words, $l_i = 1$, $l_j = -1$ with $i < j$. Set $b = \langle l \rangle = j - i$.

Case 1: $b \geq 2e$. Clearly,
\[ \|l\|_{2,3} \|l\|_{2,3} \leq \left(1 + \ln j \right)^{2,3} \leq (1 + \ln (i + b))^{2,3}. \]
If $b \geq i$, then $\|l\|_{2,3} \|l\|_{2,3} \leq 2^{2,3}(\ln b)^{2,3}$. It follows that $\ln(\langle l \rangle) \geq \frac{1}{2} \|l\|_{2,3} \|l\|_{2,3}^{2,3}$. If $b \leq i$, it follows that $i \leq j \leq 2i$. From a straightforward computation,
\[ \|l\|_{2,3} \|l\|_{2,3} \leq \left(1 + \ln 2 \right)^{2,3} \leq 2^{2,3}. \]
We obtain $\ln(\langle l \rangle) \geq 1 \geq \frac{1}{2} \|l\|_{2,3} \|l\|_{2,3}^{2,3}$.

Case 2: $1 \leq b \leq 2e$. Similarly, $\frac{1}{4} \|l\|_{2,3} \|l\|_{2,3}^{2,3} \leq 1$. It follows that
\[ \ln(1 + \langle l \rangle) \geq 2 \geq \frac{1}{8} \|l\|_{2,3} \|l\|_{2,3}^{2,3}. \]
For other cases the proofs are similar. □

Lemma 6.3. If $\mathcal{R}_0^0 = \emptyset$ and $k = 0$, $l \in \Lambda$, then $|k| \geq c_3\langle l \rangle$, where $0 < \alpha \leq \min\{1, \frac{1}{2}|a_1|\}$, and $c_3$ is a constant depending on $a_1, M, M_0$ where $a_1$ are defined in assumption $A$. 

1442
Proof. Case 1: \( i = (\cdots, 1, \cdots, -1, \cdots) \). In other words, \( i_1 = 1, i_j = -1, i < j \).
\[
\langle k, \omega' \rangle + \langle l, \Omega' \rangle = \frac{(\Omega_1 - \Omega_2)}{k} + \frac{(\Omega_1 - \Omega_2 - \Omega_3 + \Omega_2)}{k} + \frac{(\Omega_1 - \Omega' - \Omega)}{k} + \frac{\langle k, \omega' \rangle}{l}.
\]
where \( |l_1| > |a_1| || i - j = |a_1| \langle l \rangle \) and \( |l_1| \leq 2M_1 \) (note \( \delta < 0 \)). From \( || \Omega' - \Omega ||_{2,3} \leq \alpha \), it follows that
\[
|l_2| = |(l_1, \Omega' - \Omega)| \leq \alpha(1 + \ln \delta)^{-23} + \alpha(1 + \ln J)^{-23} \leq 2 \alpha.
\]
Thus, \( |l_2 + J_3 + J_4| \leq 2M_1 + 2 \alpha + |k| (\alpha + M) \). If \( \mathcal{R}'_M = \phi \), then there exists \( \xi \in \Pi \) so that
\[
|\langle k, \omega'(\xi) \rangle + \langle l, \Omega' (\xi) \rangle | < \frac{\alpha(l)}{A_k}.
\]
Thus,
\[
\frac{\alpha(l)}{A_k} > |\langle k, \omega' \rangle + \langle l, \Omega' \rangle|
\geq |l_1| - |l_2 + J_3 + J_4|
\geq |a_1| \langle l \rangle - (2M_1 + 2 \alpha + |k| (\alpha + M)).
\]
If \( 0 < \alpha \leq \min \{ \frac{1}{2}, \frac{1}{2} |a_1| \} \), then \( \langle l \rangle \leq \frac{1}{2} |a_1| (2M + M + 3) \) or \( |k| \geq \frac{|a_1|}{22M + M + 3} \langle l \rangle := \langle l \rangle c_3 \). For other cases the proofs are similar. \( \square \)

From [40], we have:

**Lemma 6.4.** If \( |k| \geq 8LM || l \||_{2,3} \) then
\[
\text{Meas}(\mathcal{R}'_M(\alpha)) \leq \frac{c_4}{A_k},
\]
where \( c_4 = C_4L'^{-1} \rho^{-1} c_3^{-1} \) with \( \rho = \text{diam}(\Pi) \).

Similar to [40], let
\[
L_\ast = \frac{LM}{c_2 C_3}, \quad K_\ast = 8LM \max \frac{||l||_{2,3} + c(\tau, \iota, \delta)}{\rho},
\]
with \( c(\tau, \iota, \delta) \) defined in (6.3) below, we have:

**Lemma 6.5.** If \( |k| \geq K_\ast \) or \( || l ||_{2,3} \geq L_\ast \), \( l \in \Lambda \), then for \( k \neq 0 \),
\[
\text{Meas}(\mathcal{R}'_M(\alpha)) \leq \frac{c_4}{A_k}.
\]

**Proof.** The proof is followed by lemma 6.2 and a straightforward computation. \( \square \)

**Remark 6.6.** The same holds for \( k = 0, l = 0 \).

Next we consider the ‘resonance classes’ \( \mathcal{R}'_M(\alpha) = \bigcup_{\iota \in \Lambda} \mathcal{R}'_M(\alpha) \), where the star indicates that we exclude the finitely many resonance zones with 0 \( \leq |k| < K_\ast \) and 0 \( < || l ||_{2,3} < L_\ast \).

Without loss of generality we suppose \(-1 \leq \delta < 0 \). If \( \delta < -1 \), then we set \( \delta = -1 \).

**Lemma 6.7.** \( \text{Meas}(\mathcal{R}'_M(\alpha)) \leq \frac{c_4}{|k|^\mu} \), where \( \mu = \frac{\delta}{\delta - 1} \).
Proof. Write $\Lambda = \Lambda' \cup \Lambda^*$, where $\Lambda^*$ contains those $l \in \Lambda$ with two nonzero components of the opposite sign, and $\Lambda'$ contains the rest. It is easy to obtain $\text{Meas}(\bigcup_{l \in \Lambda'} R_{kl}^*(\alpha)) \leq \frac{c \alpha |k|^2 \alpha}{A_k}$.

Now we turn to the minus case. For $l \in \Lambda^*$, we have $(l, \Omega') = \Omega'_l - \Omega'_j$ and $(l) = |i - j|$ and up to an irrelevant sign, $l$ is uniquely determined by two integers $i \neq j$. We may suppose that $i - j = b > 0$. Then, for $|k| \geq K, \geq (((\tau + 1)\epsilon) + 1)!$,

$$\langle k, \omega' \rangle + a_b | \leq \frac{\alpha b}{A_k} + 2\alpha (1 + \ln j)^{-2\beta} + 2M_j j^\beta.$$

Therefore,

$$R_{kl}^*(\alpha) = \left\{ \xi : |\langle k, \omega' (\xi) \rangle + (l, \Omega' (\xi)) | < \frac{\alpha b}{A_k} \right\}$$

$$\subset Q_{bkl} := \left\{ \xi : |\langle k, \omega' (\xi) \rangle + a_b | < \frac{\alpha b}{|k|^\tau} + 2\alpha (1 + \ln j)^{-2\beta} + 2M_j j^\beta \right\}.$$

Moreover, $Q_{bkl} \subset Q_{bkl'}$ for $j \geq j_0$. For a fixed $b \leq c_3^{-1}|k|$, we obtain

$$\text{Meas}(\bigcup_{j = j_0}^* R_{kl}^*(\alpha)) \leq \sum_{j = j_0}^* \text{Meas}(R_{kl}^*(\alpha)) + \text{Meas}(Q_{bkl})$$

$$\leq \frac{c_4 \alpha b}{A_k} + c_5 \left( \frac{\alpha b}{|k|^\tau + 1} + \frac{2\alpha (1 + \ln j)^{-2\beta} + 2M_j j^\beta}{|k|} \right).$$

(6.2)

Choose $j_0 = \max \{ \exp(\frac{\tau - 1}{\gamma}), \frac{\alpha^2}{\gamma |k|^\frac{1 - \tau}{\gamma}} \}$, where $\gamma$ will be fixed at the end. By computation, if we choose

$$|k| \geq K, \geq c(\tau, \iota, \delta) := \max \left\{ 2^{2\gamma}, \left( \iota (\tau + 1 + \frac{1 - \tau}{\delta}) \right) + 1 \right\},$$

then

(6.3)

$$(6.2) \leq c_6 \left( \frac{\alpha^{1 + \gamma}}{|k|^\tau} + \frac{\alpha}{|k|^\tau} + \frac{\alpha^\gamma}{|k|^\tau} \right).$$

Note $0 < -\delta \leq 1$, if we choose $\gamma = \frac{\delta}{\tau - 1}$, then $(6.2) \leq c_8 \frac{\alpha^\delta}{|k|^\tau}$. Summing over $b$,

$$\text{Meas}(\bigcup_{l \in \Lambda}^* R_{kl}^*(\alpha)) \leq \frac{c_6 |k|^2 \alpha}{A_k} + c_7 \frac{\alpha^\mu}{|k|^\tau - 1} \leq c_3 \frac{\alpha^\mu}{|k|^\tau - 1}.$$
From remark 6.6, if \( k \neq 0, \ l_0 = 0 \), \( \operatorname{Meas}(\mathcal{R}_{kl_0}^k(\alpha)) \leq \frac{c_0}{A_k} \), where we define \( \mathcal{R}_{kl_0}^k(\alpha) = \bigcup_{|k| \geq K_k} \mathcal{R}_{kl_0}^k(\alpha) \). Note the choice of \( K_k \) and \( |k| \geq K_n \), and we deduce that for \( l_0 = 0 \),

\[
\operatorname{Meas}(\mathcal{R}_{kl_0}^k(\alpha)) \leq \frac{c_0}{|k|^{-1}} \leq \frac{c_0^{\nu'}}{|k|^{-1}}.
\]

Thus, we have:

**Lemma 6.8.** For \( |k| \geq K_n \), \( \operatorname{Meas}(\mathcal{R}_{kl_0}^k(\alpha)) \leq \frac{c_0^{\nu'}}{|k|^{-1}} \).

**Lemma 6.9.** There exists a finite subset \( \mathcal{X} \subset \mathcal{Z} \) and a constant \( \tilde{c}_1 \) such that

\[
\operatorname{Meas}(\bigcup_{(k,l) \in \mathcal{X}_0} \mathcal{R}_{kl_0}^k(\alpha)) \leq \tilde{c}_1 \rho^{\mu-1} \alpha^\mu
\]

for all sufficiently small \( \alpha \). The constant \( \tilde{c}_1 \) and the index set \( \mathcal{X} \) are monotone functions of the domain \( \Pi \): they do not increase for closed subsets of \( \Pi \). In particular,

\[
\mathcal{X}_0 \subset \{(k,l) : 0 \leq |k| < \bar{K}_0 : 16LM + c(\tau, \iota, \delta), 0 < \|l\|_{\mathcal{H}} \leq L_n, l \in \Lambda \}.
\]

**Proof.** The proof is from lemma 6.7 with \( \tau \geq n + 2 \), and is similar to [40].

If we set \( l_0 = 0 \) as above, then we obtain a similar lemma.

**Lemma 6.10.** There exists a finite subset \( \mathcal{X}_1 \subset \mathcal{Z} \) and a constant \( \tilde{c}_2 \) such that

\[
\operatorname{Meas}(\bigcup_{(k,l) \in \mathcal{X}_0} \mathcal{R}_{kl_0}^k(\alpha)) \leq \tilde{c}_2 \rho^{\mu-1} \alpha^\mu
\]

for all sufficiently small \( \alpha \). The constant \( \tilde{c}_2 \) and the index set \( \mathcal{X}_1 \) are monotone functions of the domain \( \Pi \): they do not increase for closed subsets of \( \Pi \). In particular, \( \mathcal{X}_1 \subset \{(k,l) : 0 \leq |k| < \bar{K}_0 \} \).

**Proof of theorem 6.1.** If we choose

\[
\gamma_0 \leq \min \left\{ \left( \frac{1}{4c_1} \right)^{10} \cdot \left( \frac{1}{8c_0M} \right)^4 \cdot c_2(s_0), \frac{1}{4c_1} \exp( -2\bar{K}_0^2 \right) \right\},
\]

then \( K_0 \geq \bar{K}_0 \). Thus, when \( \nu \geq 1 \) and \( l \in \Lambda \), \( \operatorname{Meas}(\mathcal{R}_{kl_0}^k(\alpha)) = 0 \). Since \( \mathcal{X}_1 \) is finite, assumption \( \mathcal{A} \) implies \( \operatorname{Meas}(\bigcup_{(k,l) \in \mathcal{X}_1} \mathcal{R}_{kl_0}^k(\alpha)) \to 0 \) as \( \alpha \to 0 \). Combined with lemma 6.9, we have

\[
\operatorname{Meas}(\bigcup_{l \in \Lambda} \mathcal{R}_{kl_0}^k(\alpha)) \to 0, \quad \text{as } \alpha \to 0. \tag{6.4}
\]

The proof for \( l_0 = 0 \) is similar. We have

\[
\operatorname{Meas}(\bigcup_{k \neq 0} \mathcal{R}_{kl_0}^k(\alpha)) \to 0, \quad \text{as } \alpha \to 0. \tag{6.5}
\]

Combined with (6.4) and (6.5) we complete the proof. □
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Appendix

**Lemma A1.** For $j \geq 1$ and $\beta > 1$, there exists a constant $C(\beta)$ independent of $j$ such that,
\[
\sum_{l \geq 1} \frac{1}{(1 + |j - l|)(1 + \ln l)^{\beta}} \leq C(\beta).
\]

**Proof.** The summation is divided into three parts,
\[
\left( \sum_{1 \leq l \leq j/2} + \sum_{j/2 < l < j} + \sum_{l > j} \right) \frac{1}{(1 + |j - l|)(1 + \ln l)^{\beta}} := (I) + (II) + (III).
\]

Hence the result is followed by the following facts:

\[
(I) \leq \frac{j}{2} \frac{1}{(1 + \ln 2)^{\beta}} \leq C(\beta),
\]
\[
(II) \leq \sum_{1 \leq l \leq j/2} \frac{1}{k(1 + \ln(j - k - 1))^3} \leq \sum_{1 \leq k \leq j/2} \frac{1}{k(1 + \ln k)^3} \leq C(\beta),
\]
\[
(III) \leq \sum_{k \geq 1} \frac{1}{(1 + k)(1 + \ln(j + k))^3} \leq \sum_{k \geq 1} \frac{1}{(1 + k)(1 + \ln k)^3} \leq C(\beta). \quad \square
\]

**Remark A2.** If $\beta \geq 2$, then $\sum_{l \geq 1} \frac{1}{(1 + |j - l|)(1 + \ln l)^{\beta}} \leq C$, where $C$ is independent of $j$ and $\beta$.

**Lemma A3.** For any $j, l \geq 1$, $p \geq 2$ and $\beta \geq 1$,
\[
\sum_{l \geq 1} \frac{(1 + j)^2}{l^p(1 + |j - l|)(1 + \ln l)^{2\beta}} \leq C.
\]

**Proof.**
\[
\sum_{l \geq 1} \frac{(1 + j)^2}{l^p(1 + |j - l|)(1 + \ln l)^{2\beta}} \leq \sum_{2 \leq l} \frac{(1 + j)^2}{l^p(1 + |j - l|)(1 + \ln l)^{2\beta}} + \sum_{l \geq 2} \frac{(1 + j)^2}{l^p(1 + |j - l|)(1 + \ln l)^{2\beta}} = (I) + (II).
\]

We estimate $(I)$ and $(II)$ respectively. For $(I)$, note $|j - l| \geq j/2$ and $p \geq 2$, we have
\[
(I) \leq \sum_{2 \leq l} \frac{(1 + j)^2}{l^p(1 + j/2)^{2\beta}} \leq 4 \sum_{l \geq 1} \frac{1}{l^p} \leq C.
\]

For $(II)$, from $p \geq 2$ and $\beta \geq 1$,
(II) \[ \sum_{2 \leq j \leq l} \frac{(1+j)^2}{j^2(1+j-1)^2(1+\ln l)^{2\beta}} \leq \sum_{2 \leq j \leq l} \frac{(1+j)^2}{(j/2)^2(1+j-1)^2(1+\ln l)^{2\beta}} \leq C \sum_{i \geq 1} \frac{1}{(1+j-1)^2(1+\ln l)^{2\beta}} \]

Lemma 7.1 \[ \leq C. \]

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