SU(2)-IRREDUCIBLY COVARIANT AND EPOSIC CHANNELS

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Abstract. In this paper we introduce EPOSIC channels, a class of SU(2)-covariant quantum channels. We give their definition, a Kraus representation of them, and compute their Choi matrices. We show that these channels form the set of extreme points of all SU(2)-irreducibly covariant channels. We also compute their complementary channels, and their dual maps. As application of these channel, we get a new example of positive map that is not completely positive.

INTRODUCTION

Given two representations of a compact group $G$ on Hilbert spaces $K, E$, one can construct $G$-covariant quantum channels. Indeed, the $G$-space $K \otimes E$ decomposes into a direct sum of $G$-irreducible invariant subspaces $\{H_i\}_{i \in I}$. For each $i \in I$, the $G$-equivariant inclusion $\alpha_i : H_i \to K \otimes E$ provides (by a standard construction) two $G$-covariant channels $\Phi_1^i : \text{End}(H_i) \to \text{End}(K)$ and $\Phi_2^i : \text{End}(H_i) \to \text{End}(E)$. Unless we make further assumptions on the $G$-spaces $K, E$ these channels remain difficult to study. To simplify the study of these channels, we choose $K$ and $E$ to be $G$-irreducible spaces. In this case, the channels $\Phi_1^i, \Phi_2^i$ are called $G$-irreducibly covariant channels [6]. If $G = SU(n)$ then Clebsch-Gordan formula [14, ch.5] describes the decomposition of the $SU(n)$-space $K \otimes E$ into $SU(n)$-irreducible subspaces and gives their multiplicity. Studying $SU(n)$-equivariant maps between spaces of the form $K \otimes E$, is in general difficult, due to the fact that the multiplicity of $SU(n)$-irreducible subspaces of $K \otimes E$ might be greater than one. However, in case of $SU(2)$,
multiplicity of any $SU(2)$-irreducible subspace in $K \otimes E$ is always one, simplifying the study of such maps.

In this paper we introduce EPOSIC channels, a class of $SU(2)$-covariant quantum channels. We show that if $H, K$ are $SU(2)$-irreducible spaces then EPOSIC channels from $End(H)$ into $End(K)$ forms the extreme points of the convex set of all $SU(2)$-irreducibly covariant channels form $End(H)$ into $End(K)$. The first section contains definitions, lemmas and propositions from both representation theory and quantum information theory. In Section 2, we introduce our examples of quantum channels and exhibit some of their properties. In section 3, we get Kraus operators and the Choi matrix of EPOSIC channel. Section 4 shows that the set of all $SU(2)$-irreducibly covariant channels forms a simplex and that its extreme points are nothing but EPOSIC channels. In section 5 we compute a complementary channel and the dual map for EPOSIC channel. The paper ends with an application of the EPOSIC channels where we get an example of positive map that is not completely positive.

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1. BACKGROUND DEFINITION AND RESULTS

In this section, we recall standard notions and fix the notations we will then use. We assume all vector spaces are finite dimensional. A Hilbert space $H$ is a complex vector space endowed with an inner product $\langle \cdot \rangle_H$. If $H, K$ are Hilbert spaces, then $End(H, K)$ denotes the vector space of linear maps from $H$ to $K$. It is a Hilbert space endowed with Hilbert-Schmidt inner product given by $\langle A|B \rangle_{End(H, K)} = tr(A^*B)$ for $A, B \in End(H, K)$. For $y \in H$, let $y^*$ denote the linear form given by $y^*(z) = \langle y|z \rangle_H$. If $x \in K$ then $xy^* \in End(H, K)$ denotes the map $xy^*(z) = \langle y|z \rangle_H x$, for any $z \in H$. Recall that $\{xy^* : x \in K, y \in H\}$ forms a set of generators of $End(H, K)$. As usual,
we write $\text{End}(H)$ for $\text{End}(H, H)$ and $I_H$ for the identity map on $H$. To a Hilbert space $H$, we associate the conjugate space $\overline{H}$, the space $\overline{H}$ is a vector space with the same underlying abelian group as $H$ and with scalar multiplication $(\lambda, v) \mapsto \overline{\lambda v}$. The inner product on $\overline{H}$ is then defined by $\langle h_1 | h_2 \rangle_{\overline{H}} = \langle h_2 | h_1 \rangle_H$. One can easily verify that $\text{End}(\overline{H}) = \overline{\text{End}(H)}$ and the inner products of the two spaces coincide.

1.1. The $G$-equivariant maps.

**Definition 1.1.** Let $G$ be a topological group. A representation of $G$ on a Hilbert space $H$ is a continuous homomorphism $\pi_H : G \to GL(H)$ that makes $H$ a $G$-space. If $\pi_{H(g)}$ is a unitary operator for each $g \in G$ then $\pi_H$ is called unitary representation. We restrict the use of the symbol $\rho_H$ to an irreducible representation on $H$.

If $W$ is a subspace of $H$ such that $\pi_{H(g)}(w) \in W$ for any $g \in G$. Then $\pi_H$ can be made into a representation of $G$ on $W$, called subrepresentation. In such case, $W$ is called $G$-invariant subspace of $H$.

There are standard methods for constructing a new representation from given ones. For example if $\pi_H, \pi_K$ are two representations of $G$ on $H, K$, then the maps given by $g \mapsto \pi_H(g) \otimes \pi_K(g)$ and $g \mapsto l(g)$ where $l(g)(A) = \pi_K(g)A\pi_H(g^{-1})$, for $A \in \text{End}(H, K)$, are representations of $G$ on the Hilbert spaces $H \otimes K$ and $\text{End}(H, K)$ respectively.

If $\pi : G \to GL(H)$ is a representation of $G$ then $\tilde{\pi} : G \to GL(\overline{H})$ defined by $\tilde{\pi}(g) = \pi^t(g^{-1})$ for each $g \in G$ ($t$ denotes the transpose map), and $\overline{\pi} : G \to GL(\overline{H})$ defined by $\overline{\pi}(g) = \overline{\pi(g)}$ for each $g \in G$, are representations of $G$ on the Hilbert space $\overline{H}$. These representations are known as the contragredient and the conjugate representation of $G$ respectively. These two representations on $\overline{H}$ coincide if $\pi$ is unitary representation.

**Definition 1.2.** Let $G$ be a group and $\pi_H : G \to GL(H)$ is a representation of $G$ on the Hilbert space $H$, the commutant of $\pi_H(G)$, denoted by $\pi_H(G)'$, is the set $\{ T \in \text{End}(H) : T\pi_H(g) = \pi_H(g)T \forall g \in G \}$.
Remark 1.3. If $\pi_H$ is a representation of $G$ on a Hilbert space $H$, and $W$ is a subspace of $H$, then $W$ is a $G$-invariant subspace of $H$ if and only if $q_W$, the orthogonal projection on $W$, belongs to $\pi_H(G)'$.

Definition 1.4. Let $G$ be a group, and $\pi_H, \pi_K$ be representations of $G$ on the Hilbert spaces $H, K$. A linear map $\alpha : H \rightarrow K$ is said to be $G$-equivariant if $\pi_K(g)\alpha = \alpha \pi_H(g)$ for all $g \in G$. We denote the set of $G$-equivariant maps from $H \rightarrow K$ by $\text{End}(H, K)^G$.

Example 1.5. Let $G$ be a compact group, and $\pi_H, \pi_K$ be representations of $G$ on the Hilbert spaces $H, K$, then the following are standard constructions of $G$-equivariant maps.

1. If $\alpha : H \rightarrow K$ is a $G$-equivariant then so is $\alpha^* : K \rightarrow H$.

2. The map $\text{flip}_K^H : H \otimes K \rightarrow K \otimes H$ given by linearly extending $h \otimes k \rightarrow k \otimes h$ is $G$-equivariant map. It is a unitary with adjoint $(\text{flip}_K^H)^* = \text{flip}_K^H$ and satisfy $\text{Tr}_K(\text{flip}_K^H A \text{flip}_K^H) = \text{Tr}_K(A)$ for any $A \in \text{End}(H \otimes K)$. The map $\text{Tr}_K$ denotes the partial trace [18, p.19], and it is defined on the generators set of $\text{End}(H) \otimes \text{End}(K)$ by $A_1 \otimes A_2 \mapsto tr(A_2)A_1$.

3. The natural isomorphism

$$T : \text{End}(K) \otimes \text{End}(\overline{H}) \rightarrow \text{End}(K \otimes \overline{H})$$

defined by the map taking $A \otimes B$ to $T(A \otimes B)(k \otimes h) = A(k) \otimes B(h)$ and extending linearly, is a $G$-equivariant map.

4. The map $\text{Vec} : \text{End}(H, K) \rightarrow K \otimes \overline{H}$ [18, p.23] defined to be the unique linear extension of $\text{Vec}(xy^*) = x \otimes \overline{y}$, is an example of $G$-equivariant map. It represents any element in $\text{End}(H, K)$ as a vector in the tensor product space $K \otimes \overline{H}$. The map $\text{Vec}$ is a unitary in the sense that

$$\langle A \mid B \rangle_{\text{End}(H, K)} = \langle \text{Vec}(A) \mid \text{Vec}(B) \rangle_{K \otimes \overline{H}}$$
for any \( A, B \in \text{End}(H, K) \).

(5) Let \( B^* : \text{End}(H) \rightarrow \mathbb{C} \) denote the map \( B^*X = \langle B|X \rangle_{\text{End}(H)} \). Then the Choi-Jamiolkowski map

\[
C : \text{End}(\text{End}(H), \text{End}(K)) \rightarrow \text{End}(K \otimes \overline{H})
\]

defined by taking \( AB^* \) to \( A \otimes \overline{B} \), for \( A \in \text{End}(K) \), \( B \in \text{End}(H) \) and extending linearly, is a unitary \( G \)-equivariant map. The map \( C \) assigns for each \( \Phi \) a unique matrix \( C(\Phi) \), known as Choi matrix of \( \Phi \), that can be computed using the formula \( C(\Phi) = \sum_{ij} \Phi(E_{ij}) \otimes E_{ij} \) where \( E_{ij} \) are the standard basis for \( \text{End}(H) \) \([4]\).

Remark 1.6. The Choi-Jamiolkowski map \( C \) is the composition of the map

\[
\text{Vec} : \text{End}(\text{End}(H), \text{End}(K)) \rightarrow \text{End}(K) \otimes \overline{\text{End}(H)} = \text{End}(K) \otimes \text{End}(\overline{H})
\]

and the natural isomorphism \( T : \text{End}(K) \otimes \overline{\text{End}(H)} \rightarrow \text{End}(K \otimes \overline{H}) \).

Proposition 1.7. Let \( G \) be a group, and \( \pi_H, \pi_K \) be representations of \( G \) on the Hilbert spaces \( H, K \). The set of \( G \)-equivariant maps \( \text{End}(H, K)^G \) forms a subspace of \( \text{End}(H, K) \). In particular, it is a convex set.

1.2. \( G \)-Covariant quantum channels.

Let \( H, K \) be Hilbert spaces. Recall that quantum channel \( \Phi : \text{End}(H) \rightarrow \text{End}(K) \) is a completely positive trace preserving map \([18\text{, p}54]\), and that such a channel has several equivalent representations:

Proposition 1.8. \([18\text{, p}54]\) Let \( H, K \) be Hilbert spaces and \( \Phi : \text{End}(H) \rightarrow \text{End}(K) \) be a quantum channel, then

(1) A Stinespring representation (dilation) of \( \Phi \) is a pair \((E, \alpha)\) consisting of a Hilbert space \( E \) (an environment space), and an isometry \( \alpha : H \rightarrow K \otimes E \).
such that \( \Phi(A) = \text{Tr}_E(\alpha A \alpha^*) \) for any \( A \in \text{End}(H) \). The map \( \text{Tr}_E \) denotes the partial trace over \( E \).

(2) A Kraus representation of \( \Phi \) is a set of operators \( T_j \in \text{End}(H, K) \), called Kraus operators, that satisfy \( \Phi(A) = \sum_{j=1}^{k} T_j A T_j^* \) and \( \sum_{j=1}^{k} T_j^* T_j = I_H \).

(3) Choi representation of \( \Phi \), it states that \( \Phi \) is channel if and only if its Choi matrix \( C(\Phi) \) is a positive matrix satisfies \( \text{Tr}_K(C(\Phi)) = I_{\Pi} \).

For any quantum channel a Stinespring and a Kraus representation always exist [IS p.54], but neither is unique. However, the Choi representation gives a unique characterization of a quantum channel. The following proposition [IS p.51-p.54] gives some relations between these different representations of a quantum channel.

**Proposition 1.9.** Let \( H, K \) be Hilbert spaces, and \( \Phi : \text{End}(H) \rightarrow \text{End}(K) \) be a quantum channel. Then

1. If \((E, \alpha)\) is a Stinespring representation of \( \Phi \), and \( \{ e_j : 1 \leq j \leq d_E \} \) is an orthonormal basis of \( E \) then the set of maps \( T_j : H \rightarrow K, 1 \leq j \leq d_E \) given by \( T_j = (I_K \otimes e_j^*) \alpha \) forms a Kraus operators of \( \Phi \).
2. If \( \{ T_j : 1 \leq j \leq k \} \) is a Kraus operators of \( \Phi \), then the Choi matrix of \( \Phi \) is given by \( C(\Phi) = \sum_{j=1}^{k} \text{Vec}(T_j) \text{Vec}(T_j)^* \).

**Remark 1.10.** As a corollary to the last proposition and Theorem 5.3 in [IS p.51], the rank of the Choi matrix of \( \Phi \) gives an achievable lower bound for both the number of any Kraus operators, and of the dimension of any environment space.

**Remark 1.11.** The three representations of a quantum channel are exist in general for any completely positive map, see [PI ch.4] and [3]. In the case of quantum channels, more conditions come up as a consequence of being trace preserving map.

**Definition 1.12.** Let \( G \) be a group, and \( \pi_H, \pi_K \) be representations of \( G \) on the Hilbert spaces \( H, K \) respectively. A quantum channel \( \Phi : \text{End}(H) \rightarrow \text{End}(K) \) is
said to be $G$-covariant if it is a $G$-equivariant map from $\mathrm{End}(H)$ into $\mathrm{End}(K)$. i.e

\begin{equation}
\Phi(\pi_H(g)A\pi_H^*(g)) = \pi_K(g)\Phi(A)\pi_K^*(g)
\end{equation}

for each $A \in \mathrm{End}(H)$ and $g \in G$. If both $\pi_H, \pi_K$ are irreducible representations, then $\Phi$ is called $G$-irreducibly covariant channel.

The set of all $G$-covariant channels from $\mathrm{End}(H)$ to $\mathrm{End}(K)$ is denoted by $QC_G(\pi_H, \pi_K)$.

**Proposition 1.13.** Let $G$ be a group, and $\pi_H, \pi_K$ be representations of $G$ on the Hilbert spaces $H, K$. Then the set $QC_G(\pi_H, \pi_K)$ is a convex set.

**Proposition 1.14.** Let $G$ be a group, and $\pi_H, \pi_K$ be representations of $G$ on the Hilbert spaces $H, K$. Let $\Phi : \mathrm{End}(H) \rightarrow \mathrm{End}(K)$ be a quantum channel given by a Stinespring representation $(E, \alpha)$. If $\alpha : H \rightarrow K \otimes E$ is a $G$-equivariant map then $\Phi$ is a $G$-covariant channel.

Recall that for a Hilbert space $H$, the set of all density operators on $H$, denoted by $D(H)$, is $\{\varrho \in \mathrm{End}(H) : \varrho \geq 0, \mathrm{tr}(\varrho) = 1\}$.

**Proposition 1.15.** [4] If $G$ is a group, and $\pi_H, \pi_K$ are representations of $G$ on the Hilbert spaces $H, K$. Let $\Phi : \mathrm{End}(H) \rightarrow \mathrm{End}(K)$ be a linear map. Then

1. The map $\Phi$ is $G$-equivariant map if and only if $C(\Phi) \in (\pi_K \otimes \pi_H(G))'$.

2. If $\pi_H$ is an irreducible representation, then $\Phi$ is $G$-covariant channel if and only if $\frac{1}{d_H}C(\Phi) \in (\pi_K \otimes \pi_H(G))' \bigcap D(K \otimes \overline{\Pi})$.

For further information on these results, we refer the reader to [1], [18], [11], [6], [4] and [13].

2. EPOSIC CHANNELS

The goal of this section is to introduce our example of a class of $SU(2)$-irreducibly covariant channels. The first two subsections contain all the results that are needed to construct the channels.
2.1. SU(2)-irreducible representations and SU(2)-equivariant maps.

2.1.1. The irreducible representations of SU(2). For \( m \in \mathbb{N} \), let \( P_m \) denote the space of homogeneous polynomials of degree \( m \) in the two variables \( x_1, x_2 \). It is a complex vector space of dimension \( m + 1 \) with a basis consists of \( \{ x_1^i x_2^{m-i} : 0 \leq i \leq m \} \). The space \( P_{-1} \) will denote the zero vector space. For any \( m \in \mathbb{N} \), the compact group \( SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\} \) has a representation \( \rho_m \) on \( P_m \) given by

\[
(\rho_m(g)f)(x_1,x_2) = f((x_1,x_2)g) = f(ax_1 - \bar{b}x_2, bx_1 + \bar{a}x_2)
\]

for \( f \in P_m \) and \( g \in SU(2) \).

Proposition 2.1. [14, p.181], [2, p.85-p.86], and [16, p.276-p.279].

1. For \( m \in \mathbb{N} \), \( \rho_m \) is a unitary representation with respect to the inner product on \( P_m \) given by

\[
\langle x_1^l x_2^{m-l}, x_1^k x_2^{m-k} \rangle_{P_m} = l! (m - l)! \delta_{lk}
\]

2. The set \( \{ \rho_m : m \in \mathbb{N} \} \) constitutes the full list of the irreducible representations of \( SU(2) \).

To facilitate the computations, we choose the orthonormal basis of \( P_m \) given by the functions \( \{ f^m_l = a^l_{m,l} x_1^l x_2^{m-l} : 0 \leq l \leq m \} \), where \( a^l_{m,l} = \frac{1}{\sqrt{l!(m-l)!}} \). This basis is called the canonical basis of the \( SU(2) \)-irreducible space \( P_m \). The corresponding standard basis of \( End(P_m) \) will be \( \{ E_{lk} = f^m_{l-1} f^m_{k-1} : 1 \leq l, k \leq m + 1 \} \).

It follows directly from the definition that the action of \( g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in SU(2) \) on the canonical basis for \( P_m \) will be given by

\[
\rho_m(g)(f^m_l) = a^l_{m,l} (ax_1 - \bar{b}x_2)^l (bx_1 + \bar{a}x_2)^{m-l}
\]
In particular, for \( g_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), we have \( \rho_m(g_0)(f^m_l) = (-1)^lf^m_{m-l} \) for any \( 0 \leq l \leq m \).

The element \( g_0 \) will play a special role in constructing an \( SU(2) \)-equivariant unitary map of \( P_m \) onto \( \overline{P}_m \). We fix our notation for \( g_0 \) to be \( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

**Definition 2.2.** For \( m \in \mathbb{N} \), define the endomorphisms

1. \( \Theta_m : P_m \longrightarrow \overline{P}_m \) by \( \Theta_m \left( \sum_{l=0}^{m} \lambda_l f^m_l \right) = \sum_{l=0}^{m} \lambda_l \cdot f^m_l \), where \( \cdot \) is the multiplication in \( P_m \).

2. \( J_m : P_m \longrightarrow \overline{P}_m \) by \( J_m = \Theta_m \rho_m(g_0) \).

**Proposition 2.3.** For \( m \in \mathbb{N} \),

1. \( \Theta_m \) is a unitary isomorphism that satisfies \( \overline{\rho_m(g)} \Theta_m = \Theta_m \rho_m(g) \) for any \( g \in SU(2) \).

2. \( J_m \) is an \( SU(2) \)-equivariant unitary isomorphism from \( P_m \) onto \( \overline{P}_m \).

**Proof.**

The first assertion is straightforward. For the second one, note that \( J_m \) is a composition of unitary maps, so is unitary. As \( g_0 g = \bar{g}g_0 \) for any \( g \in SU(2) \), we have

\[
J_m \rho_m(g) = \Theta_m \rho_m(g_0) \rho_m(g) = \Theta_m \rho_m(g_0) \rho_m(g) = \overline{\rho_m(g)} \Theta_m \rho_m(g_0) = \overline{\rho_m(g)} J_m.
\]

Note that on a basis element \( f^m_l \) of \( P_m \), \( 0 \leq l \leq m \), we have

\[
(2.4) \quad J_m(f^m_l) = (-1)^l f^m_{m-l}, \quad J_m^*(f^m_l) = (-1)^{m-l} f^m_{m-l}
\]

Recall the definition of the *flip* map in Example 1.5. By direct computations on the basic elements \( f^m_l \otimes f^n_j \) of \( P_m \otimes P_n \), we obtain

**Proposition 2.4.** For \( m, n \in \mathbb{N} \), the map \( \text{flip}^m_{P_m}(J_m \otimes I_{P_n}) : P_m \otimes P_n \longrightarrow P_n \otimes \overline{P}_m \)

is an \( SU(2) \)-equivariant isomorphism that satisfies

\[
\text{flip}^m_{P_m}(J_m \otimes I_{P_n}) = (I_{P_n} \otimes J_m) \text{flip}^n_{P_n}
\]
2.1.2. Clebsch-Gordan decomposition and SU(2)-equivariant maps on \( P_m \otimes P_n \).

For \( m, n \in \mathbb{N} \), let \( \rho_m, \rho_n \) be the corresponding irreducible representations of \( SU(2) \) on \( P_m \) and \( P_n \) respectively. The new constructed representation by taking the tensor product of \( \rho_m \) and \( \rho_n \) is not necessarily irreducible. Clebsch Gordan Decomposition formula \([2, \text{p.87}]\), gives the decomposition of \( \rho_m \otimes \rho_n \) into irreducible representations, namely \( \rho_m \otimes \rho_n = \bigoplus_{h=0}^{\min\{m, n\}} \rho_{m+n-2h} \). Note that the corresponding decomposition for \( P_m \otimes P_n \) will be given by the formula

\[
P_m \otimes P_n \cong \bigoplus_{h=0}^{\min\{m, n\}} P_{m+n-2h} \tag{2.5}
\]

Since the representations \( \rho_{m+n-2h} \) are irreducible, by Schur Lemma \([13, \text{p.13}]\), the direct sum in the last formula will be an orthogonal direct sum.

To obtain a concrete representation of \( P_m \otimes P_n \), let \( x := (x_1, x_2) \), \( y := (y_1, y_2) \), \( P_m := P_m(x) \), and \( P_n := P_n(y) \). We embed the tensor product \( P_m(x) \otimes P_n(y) \) into \( \mathbb{C}[x, y] \) as follows. Define the map : \( P_m(x) \times P_n(y) \to \mathbb{C}[x, y] \) by \( (f(x), g(y)) \mapsto f(x)g(y) \), it is a bilinear map hence extends to a linear \( T : P_m(x) \otimes P_n(y) \to \mathbb{C}[x, y] \) taking \( f(x) \otimes g(y) \) to \( f(x)g(y) \). Let \( P_{m,n} \) denote the vector space of polynomials in \( x \) and \( y \) of bi-degree \((m, n)\) (homogeneous polynomials of degree \( m \) in \( x = (x_1, x_2) \)) and degree \( n \) in \( y = (y_1, y_2) \)). The space \( P_{m,n} \) has a basis consist of \( \{x^s y^t : 0 \leq s \leq m, 0 \leq t \leq n\} \). Since the map \( T \) takes the basis of \( P_m \otimes P_n \) to a basis in \( P_{m,n} \), it is an isomorphism. Henceforth, we will use \( P_{m,n} \) as a concrete representation of \( P_m \otimes P_n \). Using this identification, we define the following polynomial maps

\[
\Delta_{xy} : P_m \otimes P_n \to P_{m+1} \otimes P_{n-1} \quad \Delta_{yz} : P_m \otimes P_n \to P_{m-1} \otimes P_{n+1} \\
\Delta_{xy}(f(x, y)) = \left( x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} \right) f(x, y) \quad \Delta_{yz}(f(x, y)) = \left( y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} \right) f(x, y)
\]

\[
\Gamma_{xy} : P_m \otimes P_n \to P_{m+1} \otimes P_{n+1} \quad \Omega_{xy} : P_m \otimes P_n \to P_{m-1} \otimes P_{n-1} \\
\Gamma_{xy}(f(x, y)) = (x_1 y_2 - y_1 x_2) f(x, y) \quad \Omega_{xy}(f(x, y)) = \left( \frac{\partial}{\partial x_1} \frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2} \frac{\partial}{\partial y_1} \right) f(x, y)
\]
for \( f(x, y) \in P_m \otimes P_n \).

**Remark 2.5.** Let \( f(x, y) \in P_m \otimes P_n \), then by direct computations, we have:

1. For \( g(x, y) \in P_{m+1} \otimes P_{n-1} \),
   \[
   \langle \Delta_{xy} (f(x, y)) | g(x, y) \rangle_{P_{m+1} \otimes P_{n-1}} = \langle f(x, y) | \Delta_{yx} (g(x, y)) \rangle_{P_m \otimes P_n}.
   \]
2. For \( g(x, y) \in P_{m+1} \otimes P_{n+1} \),
   \[
   \langle \Gamma_{xy} (f(x, y)) | g(x, y) \rangle_{P_{m+1} \otimes P_{n+1}} = \langle f(x, y) | \Omega_{xy} (g(x, y)) \rangle_{P_m \otimes P_n}.
   \]

By the remark above and [12, p.47], we have:

**Proposition 2.6.** The operators \( \Delta_{xy} \), \( \Delta_{yx} \), \( \Gamma_{xy} \) and \( \Omega_{xy} \) are \( SU(2) \)-equivariant maps that satisfy \( \Delta_{xy}^* = \Delta_{yx} \) and \( \Gamma_{xy}^* = \Omega_{xy} \).

**Theorem 2.7.** [12, chp.3] For a polynomial \( f(x, y) \) of bi-degree \((m, n)\) we have

\[
f(x, y) = \sum_{h=0}^{\min\{m,n\}} c_{m,n,h} \Gamma_{xy}^h \Delta_{yx}^{n-h} \Delta_{xy}^{n-h} \Omega_{xy}^h (f(x, y))
\]

where the coefficients \( c_{m,n,h} \) are determined by induction as follows: \( c_{m,0,0} = 1 \) and for \( n \geq 1 \)

\[
c_{m,n,h} = \begin{cases} 
\frac{1}{(m+1)n} c_{m+1,n-1,h} & h=0 \\
\frac{1}{(m+1)n} [c_{m-1,n-1,h-1} + c_{m+1,n-1,h}] & 0 < h < n \\
\frac{1}{(m+1)n} c_{m-1,n-1,h-1} & h=n
\end{cases}
\]

2.2. **Forming the \( SU(2) \)-equivariant isometry \( \alpha_{m,n,h} \).**

We now define an isometry \( \alpha_{m,n,h} \) that will be used in constructing our examples of \( SU(2) \)-covariant channels. We also find a computational formula for this isometry.

2.2.1. **The isometry \( \alpha_{m,n,h} \).**

**Definition 2.8.** For \( m, n, h \in \mathbb{N} \), with \( 0 \leq h \leq \min \{n, m\} \), let \( \alpha_{m,n,h} : P_{m+n-2h} \rightarrow P_m \otimes P_n \) be the map defined by

\[
\alpha_{m,n,h}(f(x_1, x_2)) = \sqrt{c_{m,n,h}} \Gamma_{xy}^h \Delta_{yx}^{n-h}(f(x_1, x_2))
\]
where $f(x_1, x_2)$ is a homogeneous polynomial in $x_1, x_2$ of degree $m + n - 2h$.

As a direct result of Proposition 2.6 and Theorem 2.7, we have the following lemma:

**Lemma 2.9.** For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{n, m\}$,

1. The adjoint map of $\alpha_{m,n,h}$ is given by $\alpha_{m,n,h}^*: P_m \otimes P_n \rightarrow P_{m+n-2h}$ where

$$\alpha_{m,n,h}^*(f(x, y)) = \sqrt{c_{m,n,h}} \Delta_{xy}^{n-h} \Omega_{xy}^h (f(x, y))$$

for any $f(x, y) \in P_m \otimes P_n$.

2. $\sum_{h=0}^{\min\{m,n\}} \alpha_{m,n,h} \alpha_{m,n,h}^* = I_{P_m \otimes P_n}$.

**Proposition 2.10.** For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, the map $\alpha_{m,n,h}$ is an $SU(2)$-equivariant isometry.

**Proof.**

By Proposition 2.6 the maps $\Gamma_{xy}^h, \Delta_{yx}^{n-h}$ are $SU(2)$-equivariant, so $\alpha_{m,n,h}$ is. To show that $\alpha_{m,n,h}$ is an isometry, let $0 \leq h, s \leq \min\{m, n\}$. The map $\alpha_{m,n,h}^* \alpha_{m,n,s} : P_{m+n-2s} \rightarrow P_{m+n-2h}$ is an $SU(2)$-equivariant map, thus by Schur Lemma [13, p.13], we have

$$\alpha_{m,n,h}^* \alpha_{m,n,s} = \begin{cases} 0 & \text{if } h \neq s \\ \lambda I_{P_{m+n-2h}} & \text{if } h = s \end{cases}$$

(2.6)

It remains to show that $\lambda = 1$. By Lemma 2.9 we have:

$$\sum_{s=0}^{\min\{m,n\}} \alpha_{m,n,s} \alpha_{m,n,s}^* = I_{P_m \otimes P_n}.$$

Thus, for any $0 \leq h \leq \min\{m, n\}$

$$\alpha_{m,n,h}^* = \alpha_{m,n,h}^* I_{P_m \otimes P_n} = \alpha_{m,n,h}^* \sum_{s=0}^{\min\{m,n\}} \alpha_{m,n,s} \alpha_{m,n,s}^* = \sum_{s=0}^{\min\{m,n\}} \alpha_{m,n,h}^* \alpha_{m,n,s} \alpha_{m,n,s}^*$$

By Equation (2.6) above, we get

$$\alpha_{m,n,h}^* = \alpha_{m,n,h}^* \alpha_{m,n,h} \alpha_{m,n,h}^* = \lambda \alpha_{m,n,h}^*$$
Since $\alpha_{m,n,h} \neq 0$, we have $\alpha^*_{m,n,h} \neq 0$ and $\lambda = 1$. □

The following straightforward corollary describes the $SU(2)$-equivariant projections of $P_m \otimes P_n$ onto the $SU(2)$ subspaces $W_{m+n-2h} = \alpha_{m,n,h} (P_{m+n-2h})$. Note that by the equivariance of $\alpha_{m,n,h}$ and by Schur Lemma [13, p.13], the space $W_{m+n-2h}$ is an irreducible.

**Corollary 2.11.** Let $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, and $\alpha_{m,n,h}$ be as in Definition 2.8. Then

1. $\alpha_{m,n,h} \alpha^*_{m,n,h}$ is the orthogonal projection of $P_m \otimes P_n$ onto the $SU(2)$-irreducible subspace $W_{m+n-2h} = \alpha_{m,n,h} (P_{m+n-2h}) \cong P_{m+n-2h}$.
2. $P_m \otimes P_n = \bigoplus_{h=0}^{\min\{m,n\}} W_{m+n-2h}$, where the subspaces $W_{m+n-2h}$ are mutually orthogonal.

A similar result can be proved for the $SU(2)$-space $P_m \otimes \overline{P}_n$. We start by constructing isometric embedding in the following lemma. Recall that $\overline{P}_n$ is an $SU(2)$-irreducible space under the contragredient representation $\overline{\rho}_n$.

**Lemma 2.12.** For $m, n \in \mathbb{N}$,

1. The map $I_{P_m} \otimes J_n : P_m \otimes P_n \rightarrow P_m \otimes \overline{P}_n$ is an $SU(2)$-equivariant unitary isomorphism whose inverse is $I_{P_m} \otimes J_n^*$.
2. The map $\eta_{m,n,h} = (I_{P_m} \otimes J_n) \alpha_{m,n,h}$ is an $SU(2)$-equivariant isometry from $P_{m+n-2h}$ into $P_m \otimes \overline{P}_n$.

Using the embedding above, we obtain the decomposition of $P_m \otimes \overline{P}_n$ as follows:

**Corollary 2.13.** Let $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$. Let $\eta_{m,n,h}$ be defined as in Lemma 2.12. Then $\eta_{m,n,h} \eta^*_{m,n,h}$ is the orthogonal projection of $P_m \otimes \overline{P}_n$ onto the $SU(2)$-subspace $V_{m+n-2h} = \eta_{m,n,h} (P_{m+n-2h}) \cong P_{m+n-2h}$. Moreover, $P_m \otimes \overline{P}_n = \bigoplus_{h=0}^{\min\{m,n\}} V_{m+n-2h}$, where $V_{m+n-2h}$, $0 \leq h \leq \min\{m,n\}$ are mutually orthogonal $SU(2)$-irreducible subspaces.
2.2.2. A computational formula for $\alpha_{m,n,h}$.

In this subsection, we derive a computational formula for the isometry $\alpha_{m,n,h}$, and prove some related corollaries. For the rest of this paper, we will systematically use the following notations without further mention.

For $m, n, h, i, j \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, $0 \leq i \leq m + n - 2h$, and $0 \leq j \leq n$.

Let

- $k_1(i) := \max\{0, -m + i + h\}$.
- $k_2(i) := \min\{i, n - h\}$.
- $B(i) := \{j : k_1(i) \leq j \leq k_2(i) + h\}$.
- $l_{ij} := i - j + h$.
- $c_{m,n,h}$ is as defined in Theorem 2.7.

**Lemma 2.14.** For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, let $r = m + n - 2h$, and \{f_i^r : 0 \leq i \leq r\} be the canonical basis for $P_r$. Then for $0 \leq i \leq r$,

$$
\alpha_{m,n,h}(f_i^r) = \sum_{s=0}^{h} \sum_{j=k_1(i)+s}^{k_2(i)+s} \beta_{i,s,j}^{m,n,h} f_i^m \otimes f_j^n
$$

where $\beta_{i,s,j}^{m,n,h} = (-1)^s \sqrt{\frac{c_{m,n,h} \prod \frac{m!}{i!} \frac{n!}{j!} \frac{(m-h)!}{(i-j+h)!}}{r!}}$.

**Proof.**

Recall that the canonical basis element of $P_r$ is $f_i^r = a_i^{r} x_1^i x_2^{r-i}$ where $a_i^{r} = \frac{1}{\sqrt{i!(r-i)!}}$.

As

$$
\Delta_{y_1}^{n-h} = \left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right)^{n-h} = \sum_{t=0}^{n-h} \binom{n-h}{t} \frac{\partial^t}{\partial x_1^t} \frac{\partial^{n-h-t}}{\partial x_2^{n-h-t}},
$$

and

$$
\Gamma_{xy}^{h} = (x_1 y_2 - y_1 x_2)^h = \sum_{s=0}^{h} (-1)^s \binom{h}{s} (x_1 y_2)^{h-s} (x_2 y_1)^s.
$$
We have by the definition of \( \alpha_{m,n,h} \) (see Definition 2.8), that for \( 0 \leq i \leq r \)
\[
\alpha_{m,n,h}(f^r_i) = \sqrt{c_{m,n,h}} a^r_i \sum_{s=0}^{h} \sum_{t=0}^{n-h} (-1)^s \binom{h}{s} \binom{n-h}{t} x_1^{h-s} x_2^{s} y_1^{(t+s)} y_2^{n-(t+s)} \frac{\partial^t}{\partial x_1^t} \frac{\partial^{n-h-t}}{\partial x_2^{n-h-t}} x_2^{r-i}
\]

But
\[
\frac{\partial^t}{\partial x_1^t} \frac{\partial^{n-h-t}}{\partial x_2^{n-h-t}} x_2^{r-i} = \begin{cases} 
\frac{r!(r-i)!}{(i-t)!(m-h-i+t)!} x_1^{i-t} x_2^{m+t-h-i} & \text{if } -m + i + h \leq t \leq i \\
0 & \text{otherwise}
\end{cases}
\]

Thus, we can rewrite the above sum as
\[
\alpha_{m,n,h}(f^r_i) = \sqrt{c_{m,n,h}} a^r_i \sum_{s=0}^{h} \sum_{t=\max(0,-m+i+h)}^{\min(i,n-h)} (-1)^s \binom{h}{s} \binom{n-h}{t} \frac{r!(r-i)!}{(i-t)!(m-h-i+t)!} x_1^{h-s} x_2^{s} y_1^{(t+s)} y_2^{n-(t+s)}
\]

Changing the summation variable in the inner sum to \( j = s + t \), we obtain
\[
\alpha_{m,n,h}(f^r_i) = \sqrt{c_{m,n,h}} a^r_i \sum_{s=0}^{h} \sum_{j=k_1(j)+s}^{k_2(j)+s} (-1)^s \binom{h}{s} \binom{n-h}{j} \binom{m-h}{j-s} \binom{i-j+s}{(i-t)!} x_1^{l_{ij}} x_2^{m-l_{ij}} y_1^{j} y_2^{n-j}
\]

Finally, as \( \frac{a^r_i}{a_{m,n} a^r_i} = \sqrt{\frac{j! n! (m-L_{ij})!}{L_{ij}! (r-i)!}} \), we have
\[
\alpha_{m,n,h}(f^r_i) = \sqrt{c_{m,n,h}} \sum_{s=0}^{h} \sum_{j=k_1(j)+s}^{k_2(j)+s} (-1)^s \binom{h}{s} \binom{n-h}{j} \binom{m-h}{j-s} \binom{i-j+s}{(i-t)!} f_{l_{ij}}^{m} \otimes f_j^n
\]
\[
= \sum_{s=0}^{h} \sum_{j=k_1(j)+s}^{k_2(j)+s} \beta^{m,n,h}_{i,s,j} f_{l_{ij}}^{m} \otimes f_j^n
\]

The following corollary gives a more compact form of the above formula for \( \alpha_{m,n,h}(f^r_i) \).
Corollary 2.15. For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, let $r = m + n - 2h$. Then

$$\alpha_{m,n,h}(f'_i) = \sum_{j \in B(i)} \varepsilon^j_{(m,n,h)} f^m_{i_j} \otimes f^n_{j}$$

where $\varepsilon^j_i = \min\{h, j, j + m - i - h\} \sum_{s = \max\{0, j - i, j + h - n\}}^h \beta^m_{i, s, j} \cdot$

Proof.

Let $A = \{(s, j) : 0 \leq s \leq h, k_1(i) + s \leq j \leq k_2(i) + s\}$, and $A_j = \{s : (s, j) \in A\}$. Then $A_j = \{s : 0 \leq s \leq h, j - k_2(i) \leq s \leq j - k_1(i)\}$

$$= \{s : \max\{0, j - k_2(i)\} \leq s \leq \min\{h, j - k_1(i)\}\}$$

$$= \{s : \max\{0, j - i, j + h - n\} \leq s \leq \min\{h, j, j + m - i - h\}\}.$$  

Observe that $A_j \subseteq (s, j) : s \in A_j$}, where the index $j$ range from $k_1(i)$ at $s = 0$ to $k_2(i) + h$ at $s = h$. Thus, by Lemma 2.14, we have

$$\alpha_{m,n,h}(f'_i) = \sum_{(s, j) \in A} \beta^m_{i, s, j} f^m_{i_j} \otimes f^n_{j} = \sum_{j = k_1(i)}^{k_2(i) + h} \sum_{s = \max\{0, j - i, j + h - n\}}^h \beta^m_{i, s, j} f^m_{i_j} \otimes f^n_{j}$$

$$= \sum_{j = k_1(i)}^{k_2(i) + h} \beta^m_{i, s, j} f^m_{i_j} \otimes f^n_{j} = \sum_{j \in B(i)} \varepsilon^j_{(m,n,h)} f^m_{i_j} \otimes f^n_{j} \square$$

We now state and prove identities for the coefficients $\varepsilon^j_{(m,n,h)}$. When $m, n, h$ are clear from the context, we write $\varepsilon^j_{(m,n,h)}$ as $\varepsilon^j_i$.

Lemma 2.16. For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, let $r = m + n - 2h$. Then, the coefficients $\varepsilon^j_i := \varepsilon^j_{(m,n,h)}$ satisfy

1. $\varepsilon^j_i = (-1)^h \varepsilon^{n-j}_{r-i}$, for $0 \leq i \leq r$ and $j \in B(i)$.
2. $\varepsilon^{i+h}_{i} = \beta_{i, h, h+i}^m$, for any $i \leq n - h$.
3. For $n - h \leq i \leq r$, we have $\varepsilon^i_n = \beta_{h, h+i}^m = (-1)^h \beta_{i, h, h+i}^m \neq 0$.
4. For $j \in B(0)$, we have $\varepsilon^j_0 = \beta_{0, j, j}^m = (-1)^j \beta_{0, j, j}^m \neq 0$. 
Proof.

(1) Let $g_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. By the $SU(2)$-equivariance of $\alpha_{m,n,h}$, we have

$$\left(\rho_m(g_0) \otimes \rho_n(g_0)\right)\alpha_{m,n,h}(f_i^r) = \alpha_{m,n,h}(f_i^r)$$

for $0 \leq i \leq r$. As $\rho^*_m(g_0)(f_i^r) = (-1)^{r-i}f_{r-i}^r$, $\rho_m(g_0)(f_i^m) = (-1)^if_{m-r-i}$, and $\rho_n(g_0)(f_i^n) = (-1)^if_{n-i}^n$, by Corollary 2.15, the above equation can be written as

$$(-1)^h \sum_{t=k_1(i)}^{k_2(r-i)+h} \varepsilon_{r-i} t f_i^m \otimes f_t^n = \sum_{j=k_1(i)}^{k_2(i)+h} \varepsilon_{i} j f_i^m \otimes f_j^n$$

Let $t = n - s$. Since $h + k_2(r - i) = n - k_1(i)$ and $h + k_1(r - i) = n - k_2(i)$, we have

$$(-1)^h \sum_{t=k_1(i)}^{k_2(i)+h} \varepsilon_{r-i} t f_i^m \otimes f_t^n = \sum_{j=k_1(i)}^{k_2(i)+h} \varepsilon_{i} j f_i^m \otimes f_j^n$$

Since the vectors $f_i^m \otimes f_j^n$ are linearly independent, the last equality implies that

$$(-1)^h \varepsilon_{r-i} t = \varepsilon_{i} j$$

for each $k_1(i) \leq j \leq k_2(i) + h$.

The claims (2), (3), and (4) follow by direct computations. \qed

Using the fact that $\alpha_{m,n,h}$ is an isometry, one can prove that the coefficients $c_{m,n,h}$ in Theorem 2.14 are given by

$$c_{m,n,h} = \frac{(m-h)!^2}{(m+n-2h)! m! n! \left( \sum_{h=0}^{m} \binom{h}{k} \binom{m}{h-k} \binom{k}{h} \right)^2}$$

Consequently, we have the following corollary

**Corollary 2.17.** Let $m, n, h \in \mathbb{N}$ be such that $0 \leq h \leq \min\{m, n\}$. Then

1. The coefficients $c_{m,n,h}$ in Theorem 2.14 satisfy $c_{m,n,h} = \frac{(m-h)!^2}{(n-h)!^2} c_{n,m,h}$.

2. For $0 \leq i \leq m + n - 2h$ and $j \in B(i)$, we have

   a. $\beta_{i,h-s,j}^{n,m,h} = (-1)^h \beta_{i,s,j}^{m,n,h}$, for any $0 \leq s \leq h$.
   b. $\varepsilon_{i}^j(n,m,h) = (-1)^h \varepsilon_{i}^j(m,n,h)$. 

2.3. EPOSIC channels.

The $SU(2)$-equivariant isometry $\alpha_{m,n,h}$ in Proposition 2.10 will induce an $SU(2)$-covariant quantum channel $\Phi_{m,n,h} : \text{End}(P_{m+n-2h}) \to \text{End}(P_m)$ that has a Stinespring representation $(P_n, \alpha_{m,n,h})$, see Proposition 1.8. According to Definition 1.12, $\Phi_{m,n,h}$ is $SU(2)$-irreducibly covariant channel. The following proposition records this result.

**Proposition 2.18.** For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, let $r = m + n - 2h$ and $\alpha_{m,n,h}$ be as in Definition 2.8. The map $\Phi_{m,n,h} : \text{End}(P_r) \to \text{End}(P_m)$ defined by

$$\Phi_{m,n,h}(A) = \text{Tr}_{P_n}(\alpha_{m,n,h}A\alpha_{m,n,h}^*) \quad A \in \text{End}(P_r)$$

is an $SU(2)$-irreducibly covariant channel.

**Definition 2.19.** Let $r, m \in \mathbb{N}$, and $\mathcal{E}(r, m) = \{(n, h) \in \mathbb{N}^2 : r = m + n - 2h, 0 \leq h \leq \min\{m, n\}\}$. For $(n, h) \in \mathcal{E}(r, m)$, We call the quantum channel $\Phi_{m,n,h} : \text{End}(P_r) \to \text{End}(P_m)$, defined in Proposition 2.18 an EPOSIC channel.

Recall that by Proposition 1.13, the set $QC_{SU(2)}(\rho_r, \rho_m)$ of all $SU(2)$-irreducibly covariant channels from $\text{End}(P_r)$ into $\text{End}(P_m)$, is a convex set. As we will show in Section 4 the set of all EPOSIC channels form $\text{End}(P_r)$ into $\text{End}(P_m)$ forms the extreme points of $QC_{SU(2)}(\rho_r, \rho_m)$, justifying the nomenclature EPOSIC.

We denote by $EC(r, m)$ the set of all EPOSIC channels form $\text{End}(P_r)$ into $\text{End}(P_m)$, and abbreviate $EC(m, m)$ to $EC(m)$.

**Lemma 2.20.** Let $r, m \in \mathbb{N}$, then

$$\mathcal{E}(r, m) = \{(r + m - 2l, m - l) \in \mathbb{N}^2 : 0 \leq l \leq \min\{r, m\}\}$$

Proof.

Let $\mathcal{B} = \{(r + m - 2l, m - l) \in \mathbb{N}^2 : 0 \leq l \leq \min\{r, m\}\}$. If $(r + m - 2l, m - l) \in \mathcal{B}$ for some $0 \leq l \leq \min\{r, m\}$, set $n_0 = r + m - 2l$, and $h_0 = m - l$. Then $(n_0, h_0)$
satisfies \( r = n_0 + m - 2h_0 \) and \( 0 \leq h_0 \leq \min\{m, n_0\} \). (note that \( h_0 \leq n_0 \) since \( h_0 \leq h_0 + (r - l) = m - l + r - l = m + r - 2l = n_0 \)). Thus, \( (r + m - 2l, m - l) = (n_0, h_0) \in \mathcal{E}(r, m) \). Conversely, if \((n, h) \in \mathcal{E}(r, m)\), set \( l_0 = m - h \). Then, since \( r = n + m - 2h \), we have \( n = r - m + 2h = r + m - 2l_0 \). As \( 0 \leq h \leq \min\{m, n\} \), we have \( 0 \leq l_0 \leq m \) and \( l_0 \leq l_0 + (n - h) = (m - h) + (n - h) = m + n - 2h = r \). Hence, \( 0 \leq l_0 \leq \min\{r, m\} \), and \((n, h) = (r + m - 2l_0, m - l_0) \in \mathcal{B}\). □

As a consequence of the above lemma, we have:

**Proposition 2.21.** Let \( r, m \in \mathbb{N} \), then

\[
EC(r, m) = \{\Phi_{m,r+2l,m-l}, 0 \leq l \leq \min\{r, m\}\}
\]

**Definition 2.22.** [5, p.125] Let \( H, K \) be two finite dimensional spaces. Then a quantum channel \( \Phi : \text{End}(H) \rightarrow \text{End}(K) \) is said to be unital if it satisfies that \( \Phi(I_H) = \frac{d_H}{d_K} I_K \).

**Remark 2.23.**

(1) According to a result that is due to A. Holevo [6], any \( G \)-irreducibly covariant channel is unital. Thus, EPOSIC channels are unital.

(2) Since the coefficient \( c_{m,0,0} \) in Theorem 2.7 is equal to 1, then using the natural isomorphism to identify the spaces \( P_m \otimes P_0 \) and \( P_m \) in Definition 2.8, we have that \( \alpha_{m,0,0} \) is the identity map on \( P_m \). Thus, \( \Phi_{m,0,0} \) is the identity map on \( \text{End}(P_m) \).

3. A Kraus Representation and the Choi Matrix of EPOSIC Channels

In this section, we use the same notation introduced in Section 2.2.2. Recall that for \( k \in \mathbb{N} \) the set \( \{f_s^k: 0 \leq s \leq k\} \) denotes the canonical basis for \( P_k \).
3.1. A Kraus representation of EPOSIC channel.

**Definition 3.1.** Let \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \). For \( 0 \leq j \leq n \), we define \( T_j : P_{m+n-2h} \to P_m \) by

\[
T_j = (I_{P_m} \otimes f_j^*) \alpha_{m,n,h}
\]

By Proposition 1.9, the set \( \{T_j : 0 \leq j \leq n\} \) is a Kraus representation of \( \Phi_{m,n,h} \). We call the Kraus operators defined above, EPOSIC Kraus operators. By direct computations using Corollary 2.15 and Definition 3.1, we have:

**Proposition 3.2.** Let \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \), let \( \{T_j : 0 \leq j \leq n\} \) be the EPOSIC Kraus operators of \( \Phi_{m,n,h} \). Then for \( 0 \leq j \leq n \)

\[
T_j(f_i^r) = \begin{cases} 
\varepsilon_j^i f_{l_{i,j}}^m & \text{if } j \in B(i) \\
0 & \text{otherwise}
\end{cases}
\]

**Corollary 3.3.** Let \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \), let \( r = m + n - 2h \). For \( 0 \leq i_1, i_2 \leq r \), we have

\[
\Phi_{m,n,h}(f_{i_1}^r f_{i_2}^*) = \sum_{j \in B(i_1) \cap B(i_2)} \varepsilon_{i_1}^j \varepsilon_{i_2}^j f_{l_{i_1,j}}^m f_{l_{i_2,j}}^m
\]

**Corollary 3.4.** Let \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \), and \( r = m + n - 2h \). Let \( D_k \) denote the set of all diagonal operators of \( \text{End}(P_k) \). Then \( \Phi_{m,n,h}(D_r) \subseteq D_m \).

The \( SU(2) \) equivariance property of \( \alpha_{m,n,h} \) implies the symmetry relations for the \( T_j \)'s. This relation is given in the next proposition whose proof follows easily by the following lemma.

**Lemma 3.5.** Let \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \), \( r = m + n - 2h \), and \( \{T_j : 0 \leq j \leq n\} \) be the EPOSIC Kraus operators of \( \Phi_{m,n,h} \). Then for any \( 0 \leq i \leq r \), \( 0 \leq l \leq m \), and \( 0 \leq j \leq n \), we have

\[
\langle f_i^m | T_j(f_i^r) \rangle_{P_m} = (-1)^h \langle f_{m-l}^m | T_{n-j}(f_{r-i}^r) \rangle_{P_m}
\]
where \( \{f^k_s : 0 \leq s \leq k\} \) is the canonical basis for \( P_k, k \in \{r, m\} \).

**Proof.**

Fix \( j \in \mathbb{N} \) such that \( 0 \leq j \leq n \), then we have one of the following cases:

- If \( j \in B(i) \) then by Lemma 2.16, we have \( \varepsilon^j_i = (-1)^h \varepsilon^{n-j}_{r-i} \), and since \( l = l_{ij} \) if and only if \( m - l = l_{(r-i)(n-j)} \), it follows that
  \[
  \langle f^m_l | T_j(f^r_i) \rangle_{P_m} = \varepsilon^j_i \delta_{l_{ij}} = \langle f^m_{m-l} | T_{n-j}(f^r_{r-i}) \rangle_{P_m}
  \]

- If \( j \notin B(i) \), then \( n - j \notin n - B(i) = B(r - i) \) and both of \( T_j(f^r_i) \), \( T_{n-j}(f^r_{r-i}) \) are zero. This implies that the identity also holds in this case.

□

**Proposition 3.6.** Let \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min \{m, n\} \), and \( \{T_j : 0 \leq j \leq n\} \) be the EPOSIC Kraus operators of \( \Phi_{m,n,h} \). Then for each \( 0 \leq j \leq n \), we have

\[
\rho_m(g_0) T_j \rho_r(g_0) = (-1)^j T_{n-j}
\]

where \( g_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

The relation in Proposition 3.6 can be translated into a symmetry relation for the vectors representing the operators \( T_j \) in \( P_m \otimes P_r \), see Example 1.5, to get

(3.1) \[ \text{Vec}(T_{n-j}) = (-1)^j (\rho_m(g_0) \otimes \tilde{\rho}_r(g_0)) \text{Vec}(T_j) \]

It is worth noticing that even though \( \Phi \) is a \( G \)-covariant channel, its Kraus operators are not necessary \( G \)-equivariant.

**Notations** : Let \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min \{m, n\} \), and \( r = m = n - 2h \). For any \( 0 \leq j \leq n \), let \( l_c(j) = \max \{0, h - j\} \), \( l_f(j) = \min \{r - j + h, m\} \), and \( R(j) = \{l : l_c(j) \leq l \leq l_f(j)\} \).

**Remark 3.7.** Using the same notation above, we have:
(1) For fixed $j$ such $0 \leq j \leq n$, and for $i$, $0 \leq i \leq r$, we have $j \in B(i) \iff l_{ij} \in R(j)$.

(2) By Proposition 3.2 and (1) above, we have that the Kraus operator $T_j$ can be written as

$$T_j = \sum_{l \in R(j)} \epsilon^j_{l+j-h} f^m_l J^r_{l+j-h}.$$ 

(3) Using (2) above, we have $\text{Vec}(T_j) = \sum_{l \in R(j)} \epsilon^j_{l+j-h} f^m_l \otimes f^r_{l+j-h}$.

The first part of the following proposition can be proved by taking the adjoint of both sides of the equation in Remark 3.7 (2), while the second part is direct computations.

**Proposition 3.8.** Let $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$ and $r = m + n - 2h$. Let $\{T_j : 0 \leq j \leq n\}$ be the EPOSIC Kraus operators for $\Phi_{m,n,h}$ and $T_j^*$ denotes the adjoint map of $T_j$. Then for $0 \leq j \leq n$ we have

(1) $T_j^* = \sum_{l \in R(j)} \epsilon^j_{l+j-h} f^r_{l+j-h} f^m_l$ i.e.

$$T_j^* (f^m_l) = \begin{cases} \epsilon^j_{l+j-h} f^r_{l+j-h} & \text{if } l \in R(j) \\ 0 & \text{else} \end{cases}$$

(2) $\text{flip}_{P_r} (J_m \otimes J_r^*) (\text{Vec}(T_{n-j})) = (-1)^{m+j} \text{Vec}(T_j^*)$.

where $\{f^s_k : 0 \leq s \leq k\}$, is the canonical basis for $P_k$, and $J_k$ as in Definition 2.2, $k \in \{r, m\}$.

3.2. **The Choi matrix of $\Phi_{m,n,h}$.**

Recall that by Corollary 2.13 the space $P_m \otimes P_r$ decomposes into an orthogonal direct sum of $SU(2)$-irreducible subspaces $V_{m+r-2l}$, $0 \leq l \leq \min\{m, r\}$. The maps

$$\eta_{m,r,l} : V_{m+r-2l} \rightarrow P_m \otimes P_r,$$

$0 \leq l \leq \min\{m, r\}$ are isometries such that the final supports $q_{m,r,l} = \eta_{m,r,l} \eta_{m,r,l}^*$ are the mutually orthogonal projections onto $V_{m+r-2l}$, $0 \leq l \leq \min\{m, r\}$. These projections satisfy

$$\sum_{l=0}^{\min\{m,r\}} q_{m,r,l} = I_{P_m \otimes P_r}.$$ By Schur Lemma [13, p.13], we get:
Proposition 3.9. [17] Let $\rho_m, \rho_r$ be the irreducible representation of $SU(2)$ on $P_m, P_r$ respectively. Then $(\rho_m \otimes \tilde{\rho}_r (SU(2)))'$ is an abelian algebra that is generated by the projections on the $SU(2)$-subspaces $V_{m+r-2l}$ of $P_m \otimes P_r$, where $0 \leq l \leq \min\{m, r\}$.

By the last Proposition and Proposition [1.15] we have:

Corollary 3.10. For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, and $r = m + n - 2h$, the Choi matrix of $\Phi_{m,n,h}$ is generated by $\{q_{m,r,l} = \eta_{m,r,l} \eta_{m,r,l}^* : 0 \leq l \leq \min\{m, r\}\}$.

Remark 3.11. For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, if $r = m + n - 2h$ then since $0 \leq m - h \leq \min\{m, r\}$, the set $\{q_{m,r,l} : 0 \leq l \leq \min\{m, r\}\}$ always contains the projection $q_{m,r,m-h} = \eta_{m,r,m-h} \eta_{m,r,m-h}^*$.

The following lemma will be used to prove Proposition 3.13.

Lemma 3.12. Let $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, and $r = m + n - 2h$. Then the operator $C(\Phi_{m,n,h}) \eta_{m,r,m-h}$ is non zero.

Proof.

Let $\{f_s^k : 0 \leq s \leq k\}$ denote the canonical basis of $P_k$ where $k \in \{r, n, m\}$. To show that $C(\Phi_{m,n,h}) \eta_{m,r,m-h}$ is non zero, it is enough to show that $C(\Phi_{m,n,h}) (\eta_{m,r,m-h}(f_s^m)) \neq 0$. By Corollary 3.3 we have

$$C(\Phi_{m,n,h}) = \sum_{i_1, i_2=0}^{r} \Phi_{m,n,h}(f_{i_1}^r, f_{i_2}^r)^* \otimes f_{i_1}^r f_{i_2}^r = \sum_{i_1, i_2=0}^{r} \sum_{j \in B(i_1) \cap B(i_2)} \varepsilon_{ij}^r f_{i_1 j}^m f_{i_2 j}^m \otimes f_{i_1}^r f_{i_2}^r.$$ 

By Lemma 2.12 and Corollary 2.16 we have

$$\eta_{m,r,m-h}(f_s^m) = (I_{P_m} \otimes J_r) \alpha_{m,r,m-h}(f_s^m) = (I_{P_m} \otimes J_r) \left( \sum_{t \in B(0)} \varepsilon_{0}^t (m,r,m-h) f_{lt}^m \otimes f_t^r \right)$$

$$= \sum_{t \in B(0)} (-1)^t \varepsilon_{0}^t (m,r,m-h) f_{lt}^m \otimes f_{t-l}^r$$

$$= \sum_{t \in B(0)} \lambda_t f_{m-l+t}^m \otimes f_{r-t}^r$$

where $\lambda_t > 0$. 

Hence, we get:

\[
C(\Phi_{m,n,h}) \left( \eta_{m,r,m-h}(f^n_0) \right) = \sum_{i_1,i_2=0}^r \sum_{j \in B(i_1) \cap B(i_2)} \varepsilon_{i_1}^j \varepsilon_{i_2}^j f_{i_1}^m f_{i_2}^m f_{r-t}^r \left( \sum_{t \in B(0)} \lambda_t f_{m-(t+h)}^m \otimes f_{r-t}^r \right)
\]

\[
= \sum_{i_1,i_2=0}^r \sum_{j \in B(i_1) \cap B(i_2)} \lambda_t \varepsilon_{i_1}^j \varepsilon_{i_2}^j \left( f_{i_1}^m f_{i_2}^m f_{m-(t+h)}^m \right) \otimes \left( f_{r-t}^r f_{r-t}^r \right) f_{r-t}^r
\]

But, for \(0 \leq i_1, i_2 \leq r, j \in B(i_1) \cap B(i_2)\) and \(t \in B(0)\), we have:

\[
f_{i_1}^m f_{i_2}^m f_{m-(t+h)}^m \otimes f_{r-t}^r = \begin{cases} 
  f_{i_1}^n f_{i_1}^r & \text{if } i_2 = r-t, j = n \\
  0 & \text{otherwise}
\end{cases}
\]

As for \(t \in B(0), n \in B(i_1) \cap B(r-t)\) if and only if \(n-h \leq i_1 \leq r\), we have

\[
C(\Phi_{m,n,h}) \left( \eta_{m,r,m-h}(f^n_0) \right) = \sum_{i_1=n-h}^r \sum_{t \in B(0)} \lambda_t \varepsilon_{i_1}^n \varepsilon_{r-t}^n f_{i_1}^m f_{r-t}^r
\]

Assume that

\[
0 = C(\Phi_{m,n,h}) \left( \eta_{m,r,m-h}(f^n_0) \right) = \sum_{i_1=n-h}^r \sum_{t \in B(0)} \lambda_t \varepsilon_{i_1}^n \varepsilon_{r-t}^n f_{i_1}^m f_{r-t}^r
\]

then by the linearity independence of \(\{f_l^m \otimes f_i^r : 0 \leq l \leq m, 0 \leq i \leq r\}\), we would have that \(\sum_{t \in B(0)} \lambda_t \varepsilon_{i_1}^n \varepsilon_{r-t}^n = 0\) for \(n-h \leq i_1 \leq r\).

As by Corollary 2.16 \(\varepsilon_{i_1}^n \neq 0\) for \(n-h \leq i_1 \leq r\), we would have \(\sum_{t \in B(0)} \lambda_t \varepsilon_{i_1}^n \varepsilon_{r-t}^n = 0\). But by Corollary 2.16 we have \(\varepsilon_{r-t}^n = (-1)^h \theta_t\) with \(\theta_t > 0\) holds for \(t \in B(0) = \{t : 0 \leq t \leq m-h\}\). Therefore, \(\sum_{t \in B(0)} \lambda_t \varepsilon_{r-t}^n = (-1)^h \sum_{t=0}^{m-h} \lambda_t \theta_t \neq 0\) which is a contradiction. \(\square\)

**Proposition 3.13.** Let \(m, n, h \in \mathbb{N}\) with \(0 \leq h \leq \min\{m,n\}\), and \(r = m + n - 2h\).

Then the Choi matrix of the EPOSIC channel \(\Phi_{m,n,h}\) is equal to \(\frac{r+1}{n+1} q_{m,r,m-h}\) where \(q_{m,r,m-h}\) is the projection of \(P_m \otimes P_r\) onto the \(SU(2)\)-invariant subspace of dimension \(n + 1\).
Proof.

By Corollary 3.10, we have $C(\Phi_{m,n,h}) = \sum_{l=0}^{\min\{m,r\}} \lambda_l q_{m,r,l}$ where $q_{m,r,l} = \eta_{m,r,l} \eta_{m,r,l}^*$, $0 \leq l \leq \min\{m, r\}$ are the mutually orthogonal projections onto $V_{m+r-2l}$. Consequently, $\text{rank}(C(\Phi_{m,n,h})) = \sum_{l=0, \lambda_l \neq 0}^{\min\{m, r\}} \text{rank}(q_{m,r,l})$.

By Lemma 3.12, we have $\lambda_{m-h} \neq 0$, hence $\text{rank}(C(\Phi_{m,n,h})) \geq \text{rank}(q_{m,r,m-h}) = \dim(V_n) = n+1$. As by Remark 1.10 $\text{rank}(C(\Phi_{m,n,h})) \leq n+1$, we obtain that $C(\Phi_{m,n,h}) = \lambda_{m-h} q_{m,r,m-h}$. Taking the trace of both side of the equation, we get

$$r + 1 = tr(C(\Phi_{m,n,h})) = \lambda_{m-h} tr(q_{m,r,m-h}) = \lambda_{m-h} n + 1$$

and $\lambda_{m-h} = \frac{r+1}{n+1}$.

□

Remark 3.14. Note that the above proposition establishes a one to one corresponding between $EC(r, m)$ and the projections on the $SU(2)$-subspaces of $P_m \otimes \overline{P}_r$ $(\Phi_{m,m+r-2l,m-l} \longleftrightarrow \frac{r+1}{m+r-2l+1} q_{m,r,l})$.

Corollary 3.15. For $r, m \in \mathbb{N}$, there are exactly $\min\{r, m\} + 1$ elements in $EC(r, m)$.

Recall that for $r, m \in \mathbb{N}$, the set $\text{End}(\text{End}(P_r), \text{End}(P_m))^{SU(2)}$ denotes the vector space of $SU(2)$-equivariant maps $\Phi$ from $\text{End}(P_r)$ to $\text{End}(P_m)$. The following corollaries are direct consequence of Proposition 3.13.

Corollary 3.16. Let $r, m \in \mathbb{N}$, let $\rho_r$ and $\rho_m$ be the irreducible representations of $SU(2)$ on $P_r$ and $P_m$. Then $EC(r, m)$ is a spanning set of $\text{End}(\text{End}(P_r), \text{End}(P_m))^{SU(2)}$.

Proof.

By Proposition 3.9 and Proposition 3.13, the commutant

$$(\rho_m \otimes \tilde{\rho}_r (SU(2)))' = \left\{ \sum_{l=0}^{\min\{r,m\}} \lambda_l q_{m,r,l} : \lambda_l \in \mathbb{C} \right\}$$
\[
= \left\{ \min\{r,m\} \sum_{l=0}^{\min\{r,m\}} \mu_l C(\Phi_{m,m+r-2l,m-l}) : \mu_l \in \mathbb{C} \right\} \\
= \text{Span} \{ C(EC(r,m)) \}
\]

The result now follows from Proposition 1.15.

**Corollary 3.17.** Let \( r, m \in \mathbb{N} \). Then \( QC(\rho_r, \rho_m) \) is the convex hull of \( EC(r,m) \).

**Proof.**

Let \( \Phi \in QC(\rho_r, \rho_m) \). Since \( \Phi \) is \( SU(2) \)-equivariant map then by Corollary 3.16, we have

\[
\Phi = \sum_{l=0}^{\min\{r,m\}} \lambda_l \Phi_{m,m+r-2l,m-l} \quad \lambda_l \in \mathbb{C}
\]

(3.2)

It remains to show that \( 0 \leq \lambda_l \) and \( \sum_{l=0}^{\min\{r,m\}} \lambda_l = 1 \). By Remark 3.14, we have

\[
C(\Phi) = \sum_{l=0}^{\min\{r,m\}} \frac{r+1}{m+r-2l+1} \lambda_l q_{m,r,l} \quad \text{where } q_{m,r,l} , 0 \leq l \leq \min\{r,m\} \text{ are mutually orthogonal projections of } P_m \otimes \overline{P}_r.
\]

By the orthogonality of \( q_{m,r,l,s} \) and the positivity of \( C(\Phi) \) (see Proposition 1.8), we have \( \lambda_l \geq 0 \) for \( 0 \leq l \leq \min\{m,r\} \). Since both \( \Phi \) and \( \Phi_{m,m+r-2l,m-l} \) are trace preserving, choosing any state \( \varrho \in D(P_r) \), we have

\[
1 = tr(\varrho) = tr(\Phi(\varrho)) = \sum_{l=0}^{\min\{r,m\}} \lambda_l tr(\Phi_{m,m+r-2l,m-l}(\varrho)) = \sum_{l=0}^{\min\{r,m\}} \lambda_l.
\]

**4. The Extreme Points of \( SU(2) \)-Irreducibly Covariant Channels**

In this section, we show that \( EC(r,m) \), the set of all EPOSIC channels from \( End(P_r) \) to \( End(P_m) \), forms the set of the extreme points of \( QC(\rho_r, \rho_m) \). We also show that any completely positive \( SU(2) \)-equivariant map \( \Phi : End(P_r) \rightarrow End(P_m) \) is a multiple of \( SU(2) \)-covariant channel. Recall that \( EC(r,m) = \{ \Phi_{m,m+r-2l,m-l}, 0 \leq l \leq \min\{r,m\} \} \).

**Proposition 4.1.** For \( r, m \in \mathbb{N} \), the set of extreme points in \( QC(\rho_r, \rho_m) \) is \( EC(r,m) \).
Proof.

Since by Corollary [3.17] we have $QC(\rho_r, \rho_m) = Conv(E(r, m))$, it is enough to show that any element in $QC(\rho_r, \rho_m)$ is uniquely written as linear combination of elements of $E(r, m)$. Let $\Psi \in QC(\rho_r, \rho_m)$ such that $\sum_{l=0}^{\min\{r,m\}} \lambda_l \Phi_l = \Psi = \sum_{l=0}^{\min\{r,m\}} \mu_l \Phi_l$ where $\Phi_l = \Phi_{m,m+r-2l,m-l}$, then by Proposition [3.13] and the orthogonality of $q_{m,r,l}$, $0 \leq l \leq \min\{m, r\}$, we have

$$\frac{r+1}{m+r-2l+1} \lambda_l q_{m,r,l} = q_{m,r,l} C(\Psi) = \frac{r+1}{m+r-2l+1} \mu_l q_{m,r,l}$$

Thus, $\lambda_l = \mu_l$, and $E(r, m)$ are extreme points of $QC(\rho_r, \rho_m)$. To complete the proof, note that since any extreme point of $QC(\rho_r, \rho_m)$ can not be written as a linear combination of elements of $QC(\rho_r, \rho_m)$ other than itself, then any extreme point of $QC(\rho_r, \rho_m)$ must be in $E(r, m)$.

As $E(r, m)$ is a spanning set for both the $SU(2)$-irreducibly equivariant maps and the $SU(2)$-irreducibly covariant channels, we have the following corollary:

**Corollary 4.2.** For $r, m \in \mathbb{N}$, any completely positive $SU(2)$-equivariant map $\Phi : End(P_r) \to End(P_m)$ is a multiple of an $SU(2)$-irreducibly covariant channel.

Proof.

By Corollary [3.16] we have $\Phi = \sum_{l=0}^{\min\{r,m\}} \lambda_l \Phi_{m,r+m-2l,m-l}$ for some $\lambda_l \in \mathbb{C}$. Since $\Phi$ is completely positive the coefficients $\lambda_l$ are non-negative (otherwise, $C(\Phi) = \sum_{l=0}^{\min\{r,m\}} \frac{r+1}{m+r-2l+1} \lambda_l q_{m,r,l}$ will have a negative eigenvalue). Let $\lambda = \sum_{l=0}^{\min\{r,m\}} \lambda_l$. If $\lambda = 0$ then $\lambda_l = 0$ for all $0 \leq l \leq \min\{r,m\}$, and $\Phi = 0$ is a multiple of any $SU(2)$-irreducibly covariant channel. If $\lambda \neq 0$, then $\Psi = \sum_{l=0}^{\min\{m,r\}} \lambda_l \Phi_{m,m+r-2l,m-l}$ is a convex combination of EPOSIC channels. Thus, by corollary [3.17] $\Psi$ is an $SU(2)$-irreducibly covariant channel, and $\Phi = \lambda \Psi$. □
5. SOME MAPS RELATED TO EPOSIC CHANNEL

In this section, given an EPOSIC channel $\Phi_{m,n,h}$, we construct a complementary channel $\tilde{\Phi}_{m,n,h}$. We also give the condition for the dual map $\Phi^*_{m,n,h}$ to be a quantum channel.

5.1. A complementary channel of $\Phi_{m,n,h}$.

Let us first recall the notion of complementary channels [8]. Given three Hilbert spaces $H, K, E$ and a linear isometry $\alpha : H \rightarrow K \otimes E$ one associates two quantum channels

$$\Phi : \text{End}(H) \rightarrow \text{End}(K) \quad \text{and} \quad \Psi : \text{End}(H) \rightarrow \text{End}(E)$$

defined for $A \in \text{End}(H)$ by

$$\Phi(A) = Tr_E(\alpha A \alpha^*) \quad \text{and} \quad \Psi(A) = Tr_K(\alpha A \alpha^*)$$

The maps $\Phi$ and $\Psi$ are called mutually complementary. For any quantum channel, a complementary channel always exists, see Proposition 1.8. However, due to the fact that Stinespring representation (dilation) is not unique, there can be many candidates for “the” complementary channel. In [8], Holevo clarifies in what sense the complementary map is unique. He showed that if $(E, \alpha)$ and $(E', \alpha')$ are two Stinespring representations (dilations) of $\Phi : \text{End}(H) \rightarrow \text{End}(K)$ then there exist a partial isometry $J : E \rightarrow E'$ such that $\alpha' = (I_K \otimes J)\alpha$, and $\alpha = (I_K \otimes J^*)\alpha'$. It follows that if $\Phi_E$, $\Phi_{E'}$ are complementary channels of $\Phi$, then they are equivalent in the sense that there exist a partial isometry $J : E \rightarrow E'$ such that $\Phi_E(\varrho) = J^*\Phi_{E'}(\varrho)J$, and $\Phi_{E'}(\varrho) = J\Phi_{E}(\varrho)J'$ for any $\varrho \in D(H)$. Stinespring representations with minimal dimensionality of the space $E$ are called minimal dilation, and any two minimal dilations are isometric. By Remark 1.10, the Stinespring representation with an environment space that satisfies $\text{dim}(E) = \text{rank}(C(\Phi))$ is a minimal dilation.
Remark 5.1. [8, p.96] If $G$ is a group, and $\pi_H, \pi_K$ are representations of $G$ on the Hilbert spaces $H, K$. Then $\Phi : \text{End}(H) \to \text{End}(K)$ is $G$-covariant channel if and only if any complementary channel of $\Phi$ is $G$-covariant.

The following proposition will be used below to construct a complementary channel $\tilde{\Phi}_{m,n,h}$ of $\Phi_{m,n,h}$. Recall the notations in Section 2.2.2.

Proposition 5.2. For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$

$$\text{flip}_{P_n}^{P_m} \alpha_{m,n,h} = (-1)^h \alpha_{n,m,h}$$

where $\alpha_{m,n,h}$ is the isometry in Definition 2.8.

Proof.

Let $\{f_s^k : 0 \leq s \leq k\}$ be the canonical bases for $P_k$, $k \in \{r, m, n\}$. Let $B_{m,n,h}(i) = \{j : \max\{0, -m + i + h\} \leq j \leq \min\{i + h, n\}\}$ (the set $B(i)$ associated to $\alpha_{m,n,h}$), and $B_{n,m,h}(i) = \{j : \max\{0, -n + i + h\} \leq j \leq \min\{i + h, m\}\}$ (the set $B(i)$ associated to $\alpha_{n,m,h}$), see Corollary 2.15. By Corollary 2.17 and since $j \in B_{m,n,h}(i)$ if and only if $l_{ij} \in B_{n,m,h}(i)$, we have for $0 \leq i \leq r$,

$$\text{flip}_{P_n}^{P_m} \alpha_{m,n,h}(f^r_i) = \sum_{j \in B_{m,n,h}(i)} \varepsilon_{ij}^{l_{ij}(n,m,h)} f^n_j \otimes f^m_{l_{ij}} = (-1)^h \sum_{l_{ij} \in B_{n,m,h}(i)} \varepsilon_{ij}^{l_{ij}(n,m,h)} f^n_j \otimes f^m_{l_{ij}}$$

By taking the sum over $l = l_{ij}$, we get

$$\text{flip}_{P_n}^{P_m} \alpha_{m,n,h}(f^r_i) = (-1)^h \sum_{l \in B(i)} \varepsilon_{il}^{l(n,m,h)} f^r_{i-l} \otimes f^m_l = (-1)^h \sum_{l \in B(i)} \varepsilon_{il}^{l(n,m,h)} f^n_{il} \otimes f^m_l$$

$$= (-1)^h \alpha_{n,m,h}(f^r_i)$$

$\square$
Using the proposition above and the equation $\text{Tr}_K(\text{flip}_K^mA\text{flip}_K^b) = \text{Tr}_K(A)$ for any $A \in \text{End}(H \otimes K)$, the following corollary becomes straightforward.

**Corollary 5.3.** The channel $\Phi_{n,m,h}$ is a complementary channel for $\Phi_{m,n,h}$.

The following corollary to Proposition 5.2 will be used in the proof of Corollary 5.6. Recall that by Proposition 2.3, $J_m : P_m \rightarrow P_m$ is a unitary map.

**Corollary 5.4.** For $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$,

$$\text{flip}_{P_n}(J_m \otimes J_n^*)\eta_{m,n,h} = (-1)^h \eta_{n,m,h}$$

**Proof.**

By Lemma 2.12, we have

$$\text{flip}_{P_n}(J_m \otimes J_n^*)\eta_{m,n,h} = \text{flip}_{P_n}(J_m \otimes J_n^*)(I_{P_m} \otimes J_n)\alpha_{m,n,h}$$

$$= \text{flip}_{P_n}(J_m \otimes J_n^*)\alpha_{m,n,h}$$

Thus, by Propositions 2.4 and 5.2 we have

$$\text{flip}_{P_n}(J_m \otimes J_n^*)\eta_{m,n,h} = (I_{P_n} \otimes J_m)\text{flip}_{P_n}\alpha_{m,n,h} = (-1)^h(I_{P_n} \otimes J_m)\alpha_{n,m,h}$$

$$= (-1)^h \eta_{n,m,h}.$$

\[\Box\]

5.2. **The dual map of $\Phi_{m,n,h}$.**

For Hilbert spaces $H, K$ and a linear map $\Phi : \text{End}(H) \rightarrow \text{End}(K)$, the dual map of $\Phi$ is defined to be the unique map $\Phi^* : \text{End}(K) \rightarrow \text{End}(H)$ such that $\langle B | \Phi(A) \rangle_{\text{End}(K)} = \langle \Phi^*(B) | A \rangle_{\text{End}(H)}$ for all $A \in \text{End}(H), B \in \text{End}(K)$. One can easily check that if $\Phi : \text{End}(H) \rightarrow \text{End}(K)$ is a quantum channel, then

- $\Phi^*$ is a quantum channel if and only if $\Phi(I_H) = I_K$. 


• If \( \{T_j : 1 \leq j \leq d\} \) are Kraus operators for \( \Phi \), then \( \{T_j^* : 1 \leq j \leq d\} \) are Kraus operators for \( \Phi^* \).

• If \( G \) is a group and \( \pi_H, \pi_K \) are representations of \( G \) on the Hilbert spaces \( H, K \), then \( \Phi \) is \( G \)-equivariant map if and only if \( \Phi^* \) is \( G \)-equivariant.

To obtain a relation between \( \Phi_{m,n,h} \) and \( \Phi^*_{m,n,h} \), we examine their Choi matrices. Recall that by Proposition 3.8, we have

\[
\text{flip}_{P_m} (J_m \otimes J_r^*) (\text{Vec}(T_{n-j})) = (-1)^{m-j} \text{Vec}(T_j^*)
\]

**Proposition 5.5.** For \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \), let \( T_{m,r} = \text{flip}_{P_m} (J_m \otimes J_r^*) \).

Then \( C(\Phi^*_{m,n,h}) = T_{m,r} C(\Phi_{m,n,h}) T_{m,r}^* \).

**Proof.**

Let \( \{T_j : 0 \leq j \leq n\} \) be the EPOSIC Kraus operators for \( \Phi_{m,n,h} \). As \( \{T_j^* : 0 \leq j \leq n\} \) are Kraus operators for \( \Phi^*_{m,n,h} \), then by Propositions 1.9 and 3.8 we have

\[
C(\Phi_{m,n,h}) = \sum_{j=0}^{n} \text{Vec}(T_j^*) (\text{Vec}(T_j^*))^* = \sum_{j=0}^{n} (T_{m,r} \text{Vec}(T_{n-j}) (T_{m,r} \text{Vec}(T_{n-j})))^*
\]

\[
= \sum_{j=0}^{n} T_{m,r} (\text{Vec}(T_{n-j}) (\text{Vec}(T_{n-j})))^* T_{m,r}^*
\]

\[
= T_{m,r} \left( \sum_{j=0}^{n} \text{Vec}(T_{n-j}) (\text{Vec}(T_{n-j}))^* \right) T_{m,r}^*
\]

\[
= T_{m,r} C(\Phi_{m,n,h}) T_{m,r}^*.
\]

**Corollary 5.6.** For \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \), let \( r = m + n - 2h \). Then

\[
\Phi^*_{m,n,h} = \frac{r+1}{m+1} \Phi_{r,n,n-h}
\]

**Proof.**

It suffices to show that \( C(\Phi^*_{m,n,h}) = \frac{r+1}{m+1} C(\Phi_{r,n,n-h}) \). By Proposition 5.5 this is equivalent to \( T_{m,r} C(\Phi_{m,n,h}) T_{m,r}^* = \frac{r+1}{m+1} C(\Phi_{r,n,n-h}) \), and by Proposition 3.13 it is
equivalent to $T_{m,r} q_{m,r,m-h} T_{m,r}^* = q_{r,m,m-h}$ where $q_{m,r,m-h}$, $q_{r,m,m-h}$ are the projections on the $SU(2)$-irreducible subspaces of $P_m \otimes P_r$ and $P_r \otimes P_m$ respectively.

By Corollary 5.4, one has

$$T_{m,r} q_{m,r,m-h} T_{m,r}^* = T_{m,r} \eta_{m,r,m-h} \eta_{m,r,m-h}^*$$

$$= \eta_{r,m,m-h} \eta_{r,m,m-h}^* = q_{r,m,m-h}. \square$$

**Remark 5.7.** The dual map for $\Phi_{m,n,h}$ is a channel if and only if $n = 2h$, in which case $\Phi_{m,2h,h}$ is $\Phi_{m,2h,h}$.

6. Application in Operator Algebra: An Example of Positive Non-Completely Positive Map

In this section, using EPOSIC channels, we derive a new example of positive map that is not completely positive. We begin with reviewing notions we need.

**Definition 6.1.** Let $H$ and $K$ be Hilbert spaces, a linear map $\Phi : \text{End}(H) \rightarrow \text{End}(K)$ is said to be

1. positive map, if $\Phi(A) \geq 0$ for any positive matrix $A \in \text{End}(H)$.
2. $n$-positive map, if $\Phi \otimes I_n$ is positive where $\Phi \otimes I_n : \text{End}(H) \otimes M_n \rightarrow \text{End}(K) \otimes M_n$ given by $A \otimes B \mapsto \Phi(A) \otimes B$ and extended by linearity.
3. completely positive map, if it is $n$-positive for each $n \geq 1$.

Clearly, any completely positive map is automatically positive, but the converse is not true. In fact there are some examples of positive, non completely positive maps. Here, we use the EPOSIC channels $EC(1,m)$, $m \in \mathbb{N} \setminus \{0\}$ to derive a new example of these maps. Recall that for $m \in \mathbb{N} \setminus \{0\}$, the $EC(1,m)$ consists of two EPOSIC channels from $End(P_1)$ to $End(P_m)$, namely $EC(1,m) = \{ \Phi_{m,m+1,m}, \Phi_{m,m-1,m-1} \}$.

Recall also that $P_1$ has the canonical basis $\{ f_0^1, f_1^1 \}$ where $f_0^1(x_1, x_2) = x_2$, and $f_1^1(x_1, x_2) = x_1$. 

Lemma 6.2. Let \( h \in P_1 \) with \( \|h\| = 1 \). Then

1. There exist \( g_h \in SU(2) \) such that \( \rho_1(g_h)(f^1_0) = h \).
2. If \( \Phi : \text{End}(P_1) \rightarrow \text{End}(P_m) \) is an \( SU(2) \)-equivariant map then the matrices \( \Phi(hh^*) \) and \( \Phi(E_{11}) \) have the same eigenvalues.

Proof.

(1) Since \( h \) is a unit element in \( P_1 \) then \( h \) can be written as \( u_0f^1_0 + u_1f^1_1 \) for some \( u_0, u_1 \in \mathbb{C} \) that satisfy \( u_0^2 + u_1^2 = 1 \). Let \( g_h = \begin{bmatrix} \bar{u}_0 & u_1 \\ -\bar{u}_1 & u_0 \end{bmatrix} \in SU(2) \).

Then by Equation 2.1 we have \( (\rho_1(g_h)f^1_0)(x_1, x_2) = f^1_0(u_0x_1 - \bar{u}_1x_2, u_1x_1 + u_0x_2) = u_1x_1 + u_0x_2 = u_0f^1_0(x_1, x_2) + u_1f^1_1(x_1, x_2) = h(x_1, x_2) \).

(2) By item (1), \( hh^* = \rho_1(g_h)f^1_0f^1_0^* \rho^*_1(g_h) = \rho_1(g_h)E_{11}\rho^*_1(g_h) \), and by equivariance property of \( \Phi \) we have that \( \Phi(hh^*) = \Phi(\rho_1(g_h)E_{11}\rho^*_1(g_h)) = \rho_1(g_h)\Phi(E_{11})\rho^*_1(g_h) \) which gives the result.

\( \square \)

By direct computations using the formula of \( \varepsilon^1_i \) (Corollary 2.15), and the equation in Corollary 3.3 one can show:

Lemma 6.3. For \( m \in \mathbb{N} \setminus \{0\} \)

1. \( \Phi_{m,m+1,m}(E_{11}) = \sum_{j=0}^{m} \frac{2(m-j+1)}{(m+1)(m+2)} E_{m-j+1,m-j+1} \).
2. \( \Phi_{m,m-1,m}(E_{11}) = \sum_{j=0}^{m-1} \frac{2(j+1)}{m(m+1)} E_{m-j,m-j} \).

Proposition 6.4. For \( m \in \mathbb{N} \setminus \{0\} \) and \( \alpha \in \mathbb{R} \), the map \( \Phi_{m,m+1,m} - \alpha \Phi_{m,m-1,m-1} \) is a positive map if and only if \( \alpha \leq \frac{1}{m+2} \).

Proof.

Let \( A \) be a positive matrix in \( \text{End}(P_1) \). By the spectral theorem there exist an orthonormal basis \( \{x_1, x_2\} \) of \( P_1 \), and non-negative numbers \( \lambda_1, \lambda_2 \) such that
\[ A = \sum_{i=1}^{2} \lambda_i x_i x_i^*. \] To show that \( \Phi := \Phi_{m,m+1,m} - \alpha \Phi_{m,m-1,m-1} \) is positive, it suffices to show the positivity of \( \Phi(x_i x_i^*) \), and by Lemma 6.2 this is equivalent to check the positivity of \( \Phi(E_{11}) \). Note that by Lemma 6.3 we have

\[
\Phi(E_{11}) = \Phi_{m,m+1}(E_{11}) - \alpha \Phi_{m,m-1}(E_{11}) \geq 0
\]

if and only if

\[
\sum_{j=0}^{m} \frac{2(m-j+1)}{(m+1)(m+2)} E_{m-j+1,m-j+1} - \alpha \sum_{j=0}^{m-1} \frac{2(j+1)}{m(m+1)} E_{m-j,m-j} \geq 0
\]

if and only if

\[
\frac{2}{m+2} E_{m+1,m+1} + \sum_{j=0}^{m-1} \left[ \frac{2(m-j)}{(m+1)(m+2)} - \frac{2(j+1)}{m(m+1)} \right] E_{m-j,m-j} \geq 0
\]

Thus, \( \Phi(E_{11}) \geq 0 \) if and only if \( \alpha \leq \min\{ \frac{m(m-j)}{(m+2)(j+1)} : 0 \leq j \leq m-1 \} \). Since the map \( f(t) = \frac{(m-t)}{t+1} \) is decreasing map for \( 0 \leq t \leq m-1 \), then \( \min\{ \frac{m(m-j)}{(m+2)(j+1)} : 0 \leq j \leq m-1 \} = \frac{1}{m+2} \). Consequently, \( \Phi(E_{11}) \geq 0 \) if and only if \( \alpha \leq \frac{1}{m+2} \).

Using the formula for \( \varepsilon_i^r \) in Corollary 2.15 we get:

**Lemma 6.5.** For \( m \in \mathbb{N} \setminus \{0\} \), we have

\[
(1) \quad \varepsilon_0^0(m,1,0) = \sqrt{\frac{m}{m+1}}, \quad \varepsilon_1^1(m,1,0) = \sqrt{\frac{1}{m+1}}, \quad \varepsilon_0^0(m,1,1) = \sqrt{\frac{1}{m+1}}, \quad \text{and} \quad \varepsilon_1^1(m,1,1) = -\sqrt{\frac{m}{m+1}}.
\]

**Proposition 6.6.** For \( m \in \mathbb{N} \setminus \{0\} \), and \( \alpha > 0 \), the map \( \Phi_{m,m+1,m} - \alpha \Phi_{m,m-1,m-1} \) is not completely positive.

**Proof.**

Let \( \Phi = \Phi_{m,m+1,m} - \alpha \Phi_{m,m-1,m-1} \). We show that \( -\frac{2\alpha}{m} \) is an eigenvalue of \( C(\Phi) \) with a corresponding eigenvectors \( v = \sqrt{\frac{m}{m+1}} f_0^m \otimes f_1^1 + f_1^m \otimes f_1^1 \). By Proposition 3.13

\[
C(\Phi_{m,m+1,m}) = \frac{2}{m+2} \eta_{m,1,0} \eta_{m,1,0}^*, \quad \text{and} \quad C(\Phi_{m,m-1,m-1}) = \frac{2}{m} \eta_{m,1,1} \eta_{m,1,1}^*. \]

Thus,

\[
C(\Phi_{m,m+1,m})(v) = \frac{2}{m+2} \eta_{m,1,0} \eta_{m,1,0}^* (\sqrt{\frac{m}{m+1}} f_0^m \otimes f_1^1) + f_1^m \otimes f_1^1)
\]
SU(2)-irreducibly covariant and Eposic Channels

\[ \frac{2}{m+2} \eta_{m,1,0} \left[ (-\sqrt{m} \varepsilon_{1}^{1}(m,1,0) + \varepsilon_{0}^{0}(m,1,0)) f_{1}^{m+1} \right] \]

\[ = \frac{2}{m+2} \eta_{m,1,0} \left[ (-\sqrt{\frac{m}{m+1}} + \sqrt{\frac{m}{m+1}}) f_{1}^{m+1} \right] \text{ (Lemma 6.5)} \]

\[ = \frac{2}{m+2} \eta_{m,1,0} \left[ 0 \times f_{1}^{m+1} \right] = 0. \]

Similarly,

\[ C(\Phi)(v) = \frac{2}{m} \eta_{m,1,1} \eta_{m,1,1} \left( \sqrt{m} (f_{0}^{m} \otimes f_{1}^{1}) + f_{1}^{m} \otimes f_{1}^{1} \right) \]

\[ = \frac{2}{m+2} \eta_{m,1,1} \left[ (-\sqrt{m} \varepsilon_{0}^{1}(m,1,1) + \varepsilon_{0}^{0}(m,1,1)) f_{0}^{m-1} \right] \]

\[ = \frac{2}{m} \left( \frac{m+1}{m} \right) \eta_{m,1,1} \left( f_{0}^{m-1} \right) \text{ (Lemma 6.5)} \]

\[ = \frac{2}{m} \left( \frac{m+1}{m} \right) \left[ \frac{1}{m} \sum_{j=0}^{l} (-1)^{j} \varepsilon_{j}^{0} f_{1-j}^{m} \otimes f_{1-j}^{1} \right] \]

\[ = \frac{2}{m} \left( \frac{m+1}{m} \right) \left( \frac{1}{\sqrt{m+1}} f_{1}^{m} \otimes f_{1}^{1} + \sqrt{\frac{m}{m+1}} f_{0}^{m} \otimes f_{0}^{1} \right) \]

\[ = \frac{2}{m} \left( \frac{m+1}{m} \right) f_{1}^{m} \otimes f_{1}^{1} + \sqrt{\frac{m}{m+1}} f_{0}^{m} \otimes f_{0}^{1} = \frac{2}{m} v. \]

Thus, we have \( C(\Phi)(v) = C(\Phi_{m,m+1,m})(v) - \alpha C(\Phi_{m-1,m,m-1})(v) = -\frac{2\alpha}{m} v. \)

Hence, \(-\frac{2\alpha}{m}\) is a negative eigenvalue of \( C(\Phi) \) for any \( \alpha > 0. \)

It is straightforward to show that if \( \Phi \) is \( n \)-positive then it is \( s \)-positive for \( 1 \leq s \leq n \). Thus, combining the result of the proposition above and Choi result [11, p.35] about the \( n \)-positivity, we get the following:

**Corollary 6.7.** For \( 0 < \alpha \) and \( m \in \mathbb{N} \setminus \{0\} \) the map \( \Phi_{m,m+1,m} - \alpha \Phi_{m-1,m,m-1} \) is not \( n \)-positive for any \( n > 1. \)

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For any $m, n \in \mathbb{N}$ and $0 \leq h \leq \min\{m, n\}$, let $r = m + n - 2h$ then

- $c_{m,n,h} = \frac{(m-h)!^2}{(m+n-2h)! \cdot m! \cdot n! \cdot \left( \sum_{k=0}^{h} \frac{(h)^2}{(k)! \cdot \binom{m}{k} \cdot \binom{n}{k}} \right)}$.

- $\beta_{i,n,h}^{m,n,h} = (-1)^{s} \sqrt{c_{m,n,h} \cdot m! \cdot n! \cdot \left( \frac{h}{s} \right) \left( \frac{n-h}{s} \right) \left( \frac{m-h}{s} \right)}$.

- $k_1(i) = \max\{0, -m + i + h\}$, $k_2(i) = \min\{i, n - h\}$, and $l_{ij} = h + i - j$.

- $\varepsilon_{i}^{j}(m,n,h) = \sum_{s=\max\{0, i-j, j-h-n\}}^{\min\{h, i+j+m-1-h\}} \beta_{i,s,j}^{m,n,h}$.

- $B(i) = \{j : k_1(i) \leq j \leq k_2(i) + h\}$.

- $\{ f_{i}^{m} = a_{m}^{l} x_{1}^{l} x_{2}^{m-l} : 0 \leq l \leq m \}$ where $a_{m}^{l} = \frac{1}{\sqrt{l! \cdot (m-l)!}}$.

- $J_{m}(f_{i}^{m}) = (-1)^{l} f_{m-l}^{m}$ and $J_{m}^{*}(f_{i}^{m}) = (-1)^{m-l} f_{m-l}^{m}$.

- $P_{m} \otimes P_{n} \cong \bigoplus_{h=0}^{\min\{m,n\}} P_{m+n-2h}$.

- $\alpha_{m,n,h}(f_{i}^{r}) = \sum_{j \in B(i)} \varepsilon_{i}^{j}(m,n,h) f_{i}^{m} \otimes f_{j}^{n}$.

- $\eta_{m,n,h} = (I_{m} \otimes J_{n}) \alpha_{m,n,h} : P_{m+n-2h} \rightarrow P_{m} \otimes P_{n}$.

- $C(\Phi_{m,n,h}) = \frac{r+1}{n+1} q_{m,r,m-h}$. 