Quantum correlations from Brownian diffusion of chaotic level-spacings

S.N. Evangelou∗ and D.E. Katsanos
Department of Physics, University of Ioannina, Ioannina 45110, Greece

Quantum chaos is linked to Brownian diffusion of the underlying quantum energy-level-spacing sequences. The level-spacings viewed as functions of their order execute random walks which imply uncorrelated random increments of the level-spacings while the integrability to chaos transition becomes a change from Poisson to Gauss statistics for the level-spacing increments. This universal nature of quantum chaotic spectral correlations is numerically demonstrated for eigenvalues from random tight binding lattices and for zeros of the Riemann zeta function.

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Over the last couple of decades a general consensus has emerged concerning remnants of classical chaos in quantum physics. Since the smooth and reversible quantum evolution allows no chaos what is nowadays called quantum chaos had to be traced elsewhere, usually in the stationary eigenvalues of a quantum system [1]. A key quantity which was introduced to distinguish between quantum chaos and quantum integrability is the distribution function of level-spacings between nearest eigenvalues [1, 2]. The quantum chaotic behavior is then defined as the resemblance of the level-spacing statistics to those obtained from Gaussian random matrices having highly correlated levels, which are described by random matrix theory (RMT) [2, 3, 4, 5]. The major characteristic of quantum chaos is the so-called “level-repulsion” between energy levels which is due to the dramatic lowering of symmetries in chaotic systems, while the corresponding level-spacing distribution obeys the Wigner surmise. On the contrary, quantum systems with integrable classical analogues are described by Poisson level-spacing distribution function which implies uncorrelated random level-spacings with possible “degeneracies” related to the presence of symmetries. Quantum chaos has been experimentally observed for the irregular energy spectra in atom-optics billiards [6, 7].

The present era of quantum computation, where the exploitation of quantum information resources came to the forefront, requires a better understanding of spectral noise arising from quantum chaotic systems. This noise could be related to quantum correlations, such as entanglement which is measured via von Neumann entropy and usually violates Bell inequalities [8, 9]. For such a purpose the energy level-spacing statistical distribution function is insufficient to describe fully a quantum system. In order to reveal more about the nature of quantum spectral noise one must examine level-to-level correlations. This was done in [10] where they conjecture that quantum chaotic energy spectra are characterized by $1/f$ noise, which is also verified by numerical data [10]. In this paper we do not look only correlations between the energy eigenvalues but also view the nearest level-spacings as functions of their order, so that they can imitate discrete time series with their order corresponding to time. We provide evidence that in quantum chaos the classically chaotic temporal evolution is replaced by diffusion of the level-spacing dynamics. In other words, we can interpret the evolution of level-spacings (not the energies themselves) as that of fictitious particles executing random walks in one-dimensional space. Moreover, we arrive at a more precise definition of quantum chaos which includes quantum correlations in addition to statistical behavior.

The energy level series of a quantum system make up an irregular continuous but non-differentiable self-affine fractal curve described by a fractal dimension $D$. For quantum chaotic levels the noisy curve is space filling with $D = 2$ while for energy levels from quantum integrable systems the absence of correlations implies less irregular curves with fractal dimension $D = 1.5$. The corresponding power-spectra $P(f) \propto f^{-\alpha}$ are characterized by a power-law exponent $\alpha$ which obeys the dimensional relation $D = (5 - \alpha)/2$ [11]. Therefore, quantum chaos implies $1/f$ spectral noise with $\alpha = 1$ while quantum integrability gives $1/f^2$ noise with $\alpha = 2$, respectively [11]. In order to provide a better understanding for the nature of correlations in quantum chaotic energy spectra we ask the question: “What is the nature of correlations when dealing with the level-spacings, instead?” Our main finding is that the correlations in the level-spacing sequences of quantum chaotic spectra are Brownian. Nevertheless, these are quantum correlations which occur in addition to the unpredictability associated to the classical chaotic behavior. Therefore, a link can be established between quantum chaos and quantum correlations such as entanglement [8]. These facts are illustrated for two basic examples of quantum chaotic behavior, the weakly disordered lattice and the Riemann zeros. Furthermore, quantum chaos is related to a Gaussian distribution for the level-spacing increments while quantum integrability is tight up to a Poisson distribu-

∗e-mail: sevagel@cc.uoi.gr
the corresponding Fourier analysis of these energy-level series can be seen with the space filling chaotic curve approaching a line. In Fig. 2, the integrable curve approaching $D = 1.5$ and $D$ close to 1 for the chosen quasi-integrable model. The dimensions shown in the figure differ from the expected values since they were obtained by box-counting algorithms which are usually not very accurate for such self-affine curves. Quantum correlations exist in (a) for the maximally noisy quantum chaotic curves. The quantum correlations die out in (b),(c).

In Fig. 1 energy level signals from electrons in lattices as a function of the level-index $n$ are displayed. The $n$th level is expressed as a sum of level-spacings, beginning from the left end of the spectrum up to the $n$th level, via

$$\varepsilon_n = \sum_{i=1,n} (S_i - \langle S \rangle).$$

From each level-spacing $S_i$ we have subtracted a local mean level-spacing $\langle S \rangle$ by following the usual unfolding procedure for the density of states to become constant on average everywhere in the spectrum. The chaotic, quasi-integrable and integrable energy-level series can be seen to display very different curve roughness. Very rough curves with $D$ approaching 2 are obtained for chaotic levels, the roughness is smaller with $D$ close to 1.5 for integrable ones while it becomes much lower, closer to $D = 1$, for the studied quasi-integrable system where the corresponding curve approaches a line. In Fig. 2 the corresponding Fourier analysis of these energy-level signals is shown via the power-spectrum

$$P(f) = \frac{1}{N} \left| \sum_{n=1}^{N} \varepsilon_n \exp \left( -i \frac{2\pi f n}{N} \right) \right|^2,$$

averaged over many configurations. We show the energy levels from weakly disordered solids and Riemann zeroes which give $1/f$ noise. Our findings for the presence of $1/f$ power-spectra for quantum chaotic levels agree with the conjecture of Ref. [10]. We emphasize that quantum correlations exist only for the maximally noisy spectrum of quantum chaotic levels of Fig. 1(a). The integrable system of Fig. 1(c) does not show any correlations described by a less rough curve with Brownian $1/f^2$ noise. The quasi-integrable system of Fig. 1(b) does not display any quantum correlations either. Below we shall demonstrate that Brownian noise describes quantum correlations for chaotic systems not for the energy levels but for the level-spacing sequences, instead.

The $1/f$ noise found to describe the quantum chaotic levels is a common law for many classical complex phenomena where very rare intense effects require the cooperation of various favorable cases. It can be also viewed as a scaling law since a change of the intensity combined with contraction in time leaves its form unchanged. The $1/f$ law is very old and appears in many cases ranging from fluid flows, stock market price indices, earthquakes, etc. [12]. However, despite of its generality its origin remains a mystery without any widely accepted general mechanism for its interpretation. The most common explanation of $1/f$ noise relies on the extension of Lorentzian correlation function $\gamma/\left(\omega^2 + \gamma^2\right)$ from usual random telegraph noise of a single fluctuator with exponential relaxation $\gamma$, to several fluctuators with a broad distribution of relaxation rates. Although hard to justify physically if $\log(\gamma)$ is uniformly distributed the average over Lorentzians with the chosen broad distribution of $\gamma$’s gives $1/f$ noise [12]. Another mechanism of $1/f$ noise seems more appropriate for quantum chaotic lev-
It was proposed (see for example Ref. [14]) that a possible origin of $1/f$ noise can be Brownian correlated interevents between narrow pulses. Therefore, in order to obtain $1/f$ noise for the chaotic levels $\varepsilon_n$ the corresponding level-spacing $S_n$ which appear in Eq. (1) should follow a Brownian random walk and the level-spacing increments (spacings of spacings) $S_n - S_{n-1}$ to be random uncorrelated having white type of noise. This can be understood rather easily since Brownian $1/f^2$ noise of the levels arises from uncorrelated $f^0$ white noise of the level-spacings, similarly $1/f$ noise observed for chaotic levels can arise from uncorrelated white noise level-spacing increments.

Our main results for the level-spacing series $S_n = \varepsilon_n - \varepsilon_{n-1}$ are displayed in Fig. 3 together with the corresponding level-spacing distributions $P(S)$ on the left of the figure. It is well-known that from the chaotic series the obtained $P(S)$ obeys the Wigner surmise, the less irregular integrable series is described by simple Gauss distributions for both cases. In the integrable case the statistics is Poisson while a similar result of different Poisson type is obtained for the quasi-integrable statistics in the inset.

We summarize our numerical computations of eigenvalues responsible for the level-spacings. First, we consider the weakly disordered Anderson model studied on $L^3$ sites of a 3D lattice. The diagonal matrix elements of the corresponding are random taken from a uniform probability distribution of zero mean and variance $W^2/12$ and the off-diagonal matrix elements connecting nearest neighbors are unity. The short ranged and sparse real symmetric matrices are diagonalized to obtain long averaged power spectra which are shown in Fig. 2 for an intermediate semi-Poisson distribution $P(S)$. The difference between the amount of noise displayed in the various cases is immediately obvious. In order to show that quantum correlations are included in the chaotic level-spacings which follow Brownian motion we have looked at the statistical distribution of the level-spacing differences $S_n - S_{n-1}$. For chaotic levels it should be Gaussian and for integrable levels of Poisson type which correspond to correlated and uncorrelated levels, respectively. The two behaviors are clearly distinguished in Fig. 4. We have verified this picture by computing all the levels, the level-spacings and the increments of level-spacings in lattice models and the Riemann zeros. In Fig. 4 we show the statistics of the level-spacing increments $S_n - S_{n-1}$.

For weakly disordered lattices where $\beta = 1$ and 30000 Riemann zeroes where $\beta = 2$ the data can be described by simple Gauss distributions for both cases. In the integrable case the statistics is Poisson while a similar result of different Poisson type is obtained for the quasi-integrable statistics in the inset.
energy window $[0,2]$ which include approximately 2000 eigenvalues. We have estimated the power spectrum laws 

$$\alpha = 2.0 \pm 0.07$$

for the chaotic case, $\alpha = 1.52 \pm 0.04$ for the integrable case while a law close to $\alpha = 1.62 \pm 0.08$ was found for the critical case (not shown).

We have also diagonalized a simpler critical model replacing the critical case of 3D disordered systems to consider the behavior at the borderline from integrability to chaos. This is the one-dimensional quasi-integrable Harper model \cite{13} where the site potential is $V_n = 2 \cos(2\pi \sigma n)$ with irrational $\sigma = (\sqrt{5} - 1)/2$. This model displays critical behavior with multifractal spectra and wave functions. We find that the averaged power spectra gave power-law estimates close to $\alpha = 3.3$ to 3.8 as the system size increased. The level-spacings of adjacent Riemann complex zeroes \cite{10} also exhibit fluctuations similar to the ones of the energy spectra of electrons in disordered lattices. The Riemann hypothesis states that all non-trivial zeros of the Riemann $\zeta(s) = \sum_n n^{-s}$ function have real part equal to 1/2 with $\zeta(\frac{1}{2}) = 1/2 + \epsilon_n$. Although numerical computations agree with Riemann hypothesis a full theoretical proof is not known. However, the RMT is believed to be valid in this case so that the spectrum of Riemann zeroes represents a hypothetical quantum chaotic system in the unitary universality class described by GUE, which corresponds to broken time-reversal invariance via a magnetic field. The $P(S)$ statistics for small $S$ in this case is $\propto S^\beta$ with $\beta = 2$, compared to $\beta = 1$ for the orthogonal case of disordered systems. In RMT language we have a crossover from GOE to GUE statistics. In Fig. 2 we demonstrate $1/f$ noise also for sequences of the Riemann zeroes. We summarize that our findings for fractal fluctuations and $1/f$ noise for the chaotic spectra are linked to quantum fluctuations.

It is reasonable to expect that fingerprints of chaos present in the classical time evolution can be also found in the stationary energy levels of the corresponding quantum system. In addition to classical chaos the chaotic spectra should also contain some form of quantum correlations. Therefore, it is worth searching for the two ingredients of quantum chaos: “correlations” for the quantum part and “randomness” for the chaotic part. We find that both appear in the self-affine fractal noise of the spectral curves which is exemplified via the random walk behavior found for the level-spacings.

Although quantum evolution is very different from that of classical point particles in phase space Brownian diffusion also occurs in the quantum case. In Hilbert space level-spacing evolution is very different from that of classical point particles. In Hilbert space level-spacing evolution is very different from that of classical point particles. In Hilbert space level-spacing evolution is very different from that of classical point particles. In Hilbert space level-spacing evolution is very different from that of classical point particles. In Hilbert space level-spacing evolution is very different from that of classical point particles. In Hilbert space level-spacing evolution is very different from that of classical point particles. In Hilbert space level-spacing evolution is very different from that of classical point particles. In Hilbert space level-spacing evolution is very different from that of classical point particles. In Hilbert space level-spacing evolution is very different from that of classical point particles.

In summary, starting from the identification of proposed $1/f$ spectral fluctuations in quantum chaos numerical evidence was given in favor of random walk behavior for the level-spacings. We also argued that the definition of quantum chaos requires both “quantum correlations” and “randomness”. Our numerical study suggests a connection between chaotic motion of classical particles in phase space and motion of the level-spacings between eigenvalues in Hilbert space. In the quantum case Brownian correlations are shown to contribute to quantum effects. Although the level-spacing distribution is widely accepted as the best diagnostic tool of quantum chaos there is no direct theory attached to it, as it exists for example for other level correlation functions via RMT (see however the recent study of \cite{15}). This might explain why the simple random walk behavior of level-spacings in quantum chaotic systems, where quantum correlations and unpredictable behavior are combined was not paid much attention before.

We have resolved numerically important issues concerning the nature of spectral correlations and noise in quantum chaos. Common signatures of quantum chaos such as “level-repulsion” and “spectral rigidity” arise from quantum “correlations” which appear in chaotic level-spacings. If combined with “randomness”, which might have a classical chaotic origin, make up what we call quantum chaos. On the other hand, integrability usually concerns uncorrelated random levels where the level-spacings obey Poisson distribution. We should add that integrability occurs either for pure ballistic systems where quantum correlations exist without randomness or strongly disordered systems where due to localization in random positions randomness exists without correlations since localized wave functions do not communicate. Quantum chaos requires both ingredients, “quantum correlations” and “randomness”, simultaneously, for example both of them are present in RMT spectra.

In summary, starting from the identification of proposed $1/f$ spectral fluctuations in quantum chaos numerical evidence was given in favor of random walk behavior for the level-spacings. We also argued that the definition of quantum chaos requires both “quantum correlations” and “randomness”. Our numerical study suggests a connection between chaotic motion of classical particles in phase space and motion of the level-spacings between eigenvalues in Hilbert space. In the quantum case Brownian correlations are shown to contribute to quantum effects. Although the level-spacing distribution is widely accepted as the best diagnostic tool of quantum chaos there is no direct theory attached to it, as it exists for example for other level correlation functions via RMT (see however the recent study of \cite{15}). This might explain why the simple random walk behavior of level-spacings in quantum chaotic systems, where quantum correlations and unpredictable behavior are combined was not paid much attention before.

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