In this paper, we study a matching market model on a bipartite network where agents on each side arrive and depart stochastically by a Poisson process. For such a dynamic model, we design a mechanism that decides not only which agents to match, but also when to match them, to minimize the expected number of unmatched agents. The main contribution of this paper is to achieve theoretical bounds on the performance of local mechanisms with different timing properties. We show that an algorithm that waits to thicken the market, called the Patient algorithm, is exponentially better than the Greedy algorithm, i.e., an algorithm that matches agents greedily. This means that waiting has substantial benefits on maximizing a matching over a bipartite network. We remark that the Patient algorithm requires the planner to identify agents who are about to leave the market, and, under the requirement, the Patient algorithm is shown to be an optimal algorithm. We also show that, without the requirement, the Greedy algorithm is almost optimal. In addition, we consider the 1-sided algorithms where only an agent on one side can attempt to match. This models a practical matching market such as a freight exchange market and a labor market where only agents on one side can make a decision. For this setting, we prove that the Greedy and Patient algorithms admit the same performance, that is, waiting to thicken the market is not valuable. This conclusion is in contrast to the case where agents on both sides can make a decision and the non-bipartite case by [Akbarpour et al., Journal of Political Economy, 2020].
1 Introduction

Matching markets arise in many applications such as marriage and dating market \([22]\), paired kidney exchange \([4]\), and ride-hailing system \([12, 35]\). In a matching market, which can be modeled as a network with agents (vertices) and edges, a social planner designs a mechanism that finds an acceptable matching on the network. In a dynamic matching market, agents are allowed to arrive and depart over time. A market is then changed dynamically over time, in which a social planner designs a mechanism that chooses how to match agents.

A dynamic matching market has been studied extensively in theory \([21, 25, 4]\) and practice \([14, 17]\). Recently, Akbarpour et al. \([4]\) introduced a seminal matching market model with arrivals and departures. In their model, agents arrive at and depart from the market according to the Poisson process. The planner observes the network and chooses a matching, aiming to minimize the number of unmatched agents. One of the key feature in their model is that the planner must decide not only which agents to match, but also \textit{when} to match them. Akbarpour et al. \([4]\) showed that the choice of when to match agents has large effects on performance. Specifically, they introduced two simple mechanisms with different timing properties, \textit{Greedy} and \textit{Patient}. They provided theoretical guarantees for these mechanisms, that suggests waiting has substantial benefits on maximizing a matching over the network.

This paper focuses on a \textit{bipartite matching market} where the network is a bipartite graph. Agents in the market are divided into two separated groups, and a matching is formed between the two groups. A bipartite matching market is one of the most popular matching markets in practice; a labor market matches a worker to a task, and a ride-hailing market matches a taxi to a passenger \([13, 32]\).

We propose a bipartite matching market model with arrivals and departures as a variant of Akbarpour et al.’s (non-bipartite) matching model. We aim at designing \textit{local} algorithms in the sense that they look only at the neighbors of an agent which attempts to match, rather than at the global network structure. Local algorithms can be viewed as a mechanism that each agent individually decides to find a partner. In a bipartite matching market, agents in two separated groups have different roles, and agents on one side often have no right to make a decision. For example, in a freight exchange market between shippers and carriers, some platforms such as Wtransnet\(^4\) only allow carriers to choose shipments. For another, in a competitive labor market, only workers submit job applications to companies, and companies make final decisions. Thus, it is natural to consider the situation when agents on only one side have a right to make a decision. Such a setting is called a \textit{1-sided market} \([2]\). We also consider the situation when agents on both sides can make a decision, called a \textit{2-sided market}, that also appears in practice such as a marriage and dating market and a freight exchange market like Cargopedia\(^5\).

1.1 Our Contributions

In this work, we evaluate the performance of simple local mechanisms on a bipartite matching market to measure the impact of waiting time in the 1-sided/2-sided markets. Our main contributions are summarized as follows:

- We introduce a formal framework of 1-sided/2-sided bipartite market model with arrivals and departures. We propose algorithms with different timing properties, \textit{Greedy} and \textit{Patient} algorithms, for the 1-sided and 2-sided markets, respectively. We present almost optimal bounds on the performance of these algorithms. Our results show that waiting to thicken the market is highly valuable for the 2-sided market, while it is not true for the 1-sided market.

- We provide lower bounds on the performance of any matching algorithms. We show that, if the planner does not know the information when an agent departs, any algorithm suffers a loss exponentially larger than that of an omniscient algorithm where the information is available.

Let us describe our results in more detail.

\textbf{Model} \quad In our model, agents in two classes arrive at Poisson rates \(\lambda_a\) and \(\lambda_b\), respectively, and a pair of two agents in different classes are compatible with probability \(p\). Each agent departs at a Poisson rate, normalized to 1. The planner chooses a matching on the current network, and matched agents leave the market. The planner aims to minimize the proportion of the expected unmatched agents (called \textit{the loss}). This setting is a variant of a matching market model by Akbarpour et al. \([4]\), where each agent arrives at Poisson rate \(\lambda\) and edges are formed between any pair of agents with probability \(p\).

In this paper, we consider two simple mechanisms, \textit{Greedy} and \textit{Patient}, for a bipartite matching market. The \textit{Greedy} algorithm attempts to match an agent upon her arrival, while the \textit{Patient} algorithm attempts to match only an urgent

\(^4\)https://www.wtransnet.com/ [Online; accessed 24-January-2021]
\(^5\)https://www.cargopedia.net/ [Online; accessed 23-December-2019]
Table 1: Summary of the loss when $T, \lambda, \lambda_a, \lambda_b \to \infty$. We denote $d = \lambda p$ and $d_i = \lambda_i p$ for $i \in \{a, b\}$, which are all constants.

(a) Lower bounds of the loss, and upper bounds of the loss when $\lambda_a = \lambda_b$

| Setting          | Lower bound | Upper bound ($d_a = d_b$) |
|------------------|-------------|---------------------------|
| non-bipartite    |             |                           |
| Greedy           | $\frac{1}{2^d + 1}$ | $\log(2)$ $d$            |
| Patient          | $\frac{e^{-a}}{d_a + 1}$ | $e^{-\frac{d_a}{d}}$     |
| 2-sided          |             |                           |
| Greedy           | $\max \left\{ \Delta, \frac{1}{2d_a + d_b + 1} \right\}$ (cf. [18]) | $2 \log(d_a + 3)$ $d_a$ |
| Patient          | $\frac{1}{2} \left( \frac{e^{-a}}{d_a + 1} + \frac{e^{-a}}{d_b + 1} \right)$ | $e^{-O(d_a)}$            |
| 1-sided          |             |                           |
| Greedy           | $\max \left\{ \Delta, \frac{1}{1 + 2d_a + d_b} \right\}$ | $2 \log(d_a + 3)$ $d_a$ |
| Patient          | $\max \left\{ \Delta, \frac{\log d_b}{d_a + d_b} \right\}$ | $\frac{2}{d_a}$         |

(b) Upper bounds when $\lambda_a \neq \lambda_b$. In the 2-sided market, we assume $\lambda_a \geq \lambda_b$, and in the 1-sided market, we assume that agents with rate $\lambda_a$ are inactive.

| Setting          | Total | Loss $\lambda_a$-side: $L_a$ | Loss $\lambda_b$-side: $L_b$ |
|------------------|-------|-------------------------------|-------------------------------|
| 2-sided ($d_a \geq d_b$) | Greedy | $\Delta + 2 \log(d_a + 3)$ $d_a + d_b$ | $\frac{d_a - d_b}{d_a} + \frac{\log(d_a + 3)}{d_a}$ | $e^{-\max\{d_a - d_b, \frac{d_a + d_b}{2}\}}$ |
| Patient          | $\Delta + \log(d_a + 3)$ $d_a + d_b$ | $\log(d_a + 3)$ $d_a$ | $\frac{d_a - d_b}{d_a}$ $d_a$ |
| 1-sided ($d_a \geq d_b$) | Greedy | $\Delta + 2 \log(d_a + 3)$ $d_a + d_b$ | $\frac{\log(d_a + 3)}{d_a}$ | $\frac{|d_a - d_b|}{d_a} + \frac{\log(d_a + 3)}{d_a}$ |
| Patient          | $\frac{\log(d_a + 3)}{d_a}$ | $\frac{|d_a - d_b|}{d_a} + \frac{\log(d_a + 3)}{d_a}$ |
| 1-sided ($d_a < d_b$) | Greedy | $\frac{\log(d_a + 3)}{d_a}$ | $\frac{|d_a - d_b|}{d_a} + \frac{\log(d_a + 3)}{d_a}$ |
| Patient          | $\frac{\log(d_a + 3)}{d_a}$ | $\frac{|d_a - d_b|}{d_a} + \frac{\log(d_a + 3)}{d_a}$ |

Theoretical Guarantee Our main contributions are to derive theoretical bounds on the Greedy and Patient algorithms in the 2-sided and 1-sided markets, respectively. The obtained guarantees are summarized as in Tables 1(a) and 1(b). We here denote $d_i = \lambda_ip$ for $i \in \{a, b\}$ and $\Delta = \frac{|d_a - d_b|}{d_a + d_b}$. We remark that lower bounds for the 2-sided market model were also derived by Jiang [18].

Let us first consider the balanced case, that is, when $\lambda_a = \lambda_b$, implying that $d_a = d_b$ and $\Delta = 0$ in Table 1(a). Table 1(a) shows that the loss of the 2-sided Greedy algorithm is $O \left( \frac{1}{d_a} \right)$, ignoring a logarithmic factor in $d_a$, while the 2-sided Patient algorithm has the loss $e^{-O(d_a)}$. Thus waiting to match agents allows us to achieve exponentially small loss, which is a similar consequence to the non-bipartite matching market [3]. In contrast, the 1-sided market leads to a different conclusion. In fact, both of the Greedy and Patient algorithms have the same loss, which is $O \left( \frac{1}{d_a} \right)$, ignoring a logarithmic factor in $d_a$. This means that waiting to match agents is not valuable in the 1-sided market, and other information such as the graph structure is necessary to achieve smaller loss.

The situation changes when $d_a \neq d_b$. For better understanding, we evaluate the proportion of unmatched agents on both sides separately, which are the losses $L_a$ and $L_b$ of $\lambda_a$-side and $\lambda_b$-side, respectively, in Table 1(b). Note that the total loss is equal to $\frac{d_a}{d_a + d_b}L_a + \frac{d_b}{d_a + d_b}L_b$. We see from Table 1(b) that the larger side, i.e., the side with $\max\{d_a, d_b\}$, has a constant loss of $O \left( \frac{1}{\max\{d_a, d_b\}} \right)$ in every market. This factor is unavoidable since a bipartite graph is unbalanced. Our results say that, except for the unavoidable loss, we suffer only the loss of $O \left( \frac{1}{\max\{d_a, d_b\}} \right)$ on the large side in every market.
In the 2-sided market when \( d_a \neq d_b \), the smaller side of the Patient algorithm has exponentially smaller loss than that of the Greedy algorithm. This again indicates that waiting to thicken the market in the 2-sided market is beneficial. In contrast, both of 1-sided Greedy and Patient algorithms have the same loss as the 2-sided Greedy algorithms.

We remark that, in the 1-sided Greedy algorithm, agents on one side do not attempt to match. Hence, it has less opportunity to make a partner compared to the 2-sided Greedy algorithm, which implies that the 1-sided Greedy algorithm seems to have larger loss. However, our results show that their losses have the same order. On the other hand, in the 1-sided Patient algorithm, since an active agent delays her decision, she is allowed to have more neighbors. Hence the 1-sided Patient algorithm intuitively has smaller loss than the 1-sided Greedy algorithm. However, our results show that their losses have the same order. In fact, Table 1 shows that the loss of the 1-sided Patient algorithm is strictly worse than the 2-sided one when \( d_a = d_b \).

Another contribution of this paper is to evaluate the loss of optimal algorithms. We show that any algorithm suffers a loss of at least \( 1/(2d_a + d_b + 1) \) if it does not know the departure information. In other words, no matter how long each agent waits, the loss must be at least \( 1/(2d_a + d_b + 1) \). Thus the Greedy algorithm is almost optimal, up to a logarithmic factor in \( d_a \). In contrast, if we know the departure information, we prove that the loss of any algorithm is at least \( \frac{1}{2} \left( e^{-d_a} + e^{-d_b} \right) \). Thus, since the loss of the Patient algorithm is \( e^{-O(d_a)} \) when \( d_a = d_b \), waiting to match agents suffices to achieve optimal loss.

**Technical Highlights**  
The key observation for bounding the loss is that the number of agents in the market determines the loss of matching algorithms. This is observed in a non-bipartite market as well. In our bipartite markets, in particular, the loss on one side is determined by the number of agents on the other side; an agent is likely to be matched if there are many agents on the other side, and the number of agents on the same side does not matter.

In the 2-sided market, since the Greedy algorithm attempts to match agents as soon as possible, the number of available agents on both sides is reduced rapidly when \( d_a \) and \( d_b \) grow (the market is thin). Since the market has no edges under the Greedy algorithm, all urgent agents perish, which are counted as the loss. On the other hand, the Patient algorithm attempts to match only urgent agents, which implies that the number of agents on both sides will remain large even when \( d_a \) and \( d_b \) increase (the market is thick). This allows the planner to find a pair to an urgent agent, which reduces the loss. We remark that, in the case when \( d_a > d_b \), since the number of agents on the \( \lambda_b \)-side is small compared to the one on the \( \lambda_a \)-side, agents on the \( \lambda_a \)-side is hard to find a partner even if the market is thick, which worsens the loss of the larger side.

A similar observation can be applied to the 1-sided market. As observed in the 2-sided market, the market size of active agents (i.e., agents who can make a decision) will be thin under the Greedy algorithm, while it will be thick under the Patient algorithm. However, as we will see, the number of inactive agents decreases rapidly under both algorithms when \( d_a \) and \( d_b \) grow. This causes large loss for both the algorithms.

The above observation can be formalized with Markov chain. That is, the dynamics of our proposed algorithms can be modeled as continuous-time Markov chains determined by a pair of market sizes on both sides. We first show that the loss of the proposed algorithms can be expressed as the pool sizes in the stationary distribution of the Markov chain. Moreover, we prove that, for each of the proposed algorithms, the pool sizes in the stationary distribution highly concentrate around some values, which allows us to upper-bound the loss of the algorithms.

The most challenging part is to find the concentration of the pool sizes in the steady state. The primary technical tool is the balance equations of Markov chains. The balance equation describes the probability flux in and out of a given set of states. For a non-bipartite matching model, a Markov chain is of a simple form on the set of non-negative integers, and hence we can naturally apply the balance equations. On the other hand, our Markov chain is defined on 2-dimensional space, i.e., each state is a pair of market sizes. This requires us to choose a set of states for the balance equations more carefully. In fact, we need to adopt different strategies for each of the proposed algorithms. See Section 4 for the details.

### 1.2 Related Work

There have been many studies on dynamic matching with different applications in the literature of economics, computer science, and operations research. In particular, the problem has received much attention recently in online advertising. We refer readers to Mehta [25] for a detailed survey on Ad display. Another example is a paired kidney exchange market, which requires simultaneous transplantation between different patients [34, 30]. Other applications include crowd sourcing platform [20], task allocation [11, 5, 33], house allocation [1, 31], school choice [15], and real-time ride sharing [28].
Stochastic Matching Market  In the classical setting of the dynamic bipartite matching problem (e.g., [21][25][2][16]), the vertex set on one side is given in advance. Emek et al. [14] studied the setting where agents on both sides arrive randomly. We note that their model assumes that each agent stays in the market until she gets matched. They evaluated algorithms to find a maximum-weight matching incorporating waiting cost. Ashlagi et al. [7] generalized the results by Emek et al. [14] to apply ride-sharing platforms. Mertikopoulos et al. [26] further extended to investigate the trade-off issue on waiting time and matching costs. They showed that waiting to thicken the market is profitable, but no algorithm can find an optimal matching with both objectives. Loertscher et al. [29] proposed a batching algorithm (batch auctions) that matching occurs periodically in a bipartite network. They indicate that a lower frequency of matching can increase the market thickness. Manshadi and Ro [24] introduced a model where agent arrival distributions are endogenous.

As a generalization of a paired kidney exchange market, the dynamic matching problem was extended to finding disjoint 3-way circles and chains [19][3][6][8][34]. Anderson et al. [5] and Ashlagi et al. [8] analyzed the expected waiting time in the market.

Matching Market with Departures  A matching market where each agent is allowed to leave has been studied in various settings. Johari et al. [19] and Ashlagi et al. [7] studied a matching model where agents would depart after a constant time after arrival.

Akbarpour et al. [4] introduced a dynamic model where each vertex arrives and departs stochastically on a general (i.e., non-bipartite) network. Our model is a bipartite variant of their model. Jiang [18] gave lower bounds on the loss of algorithms in the 2-sided market. Beccara et al. [10] studied a model where heterogeneous agents arrive randomly.

We remark that, in the previous work, waiting to thicken the market is shown to be more beneficial than myopic algorithms such as the Greedy algorithm. However, interestingly, in the 1-sided market, we obtain a different conclusion that waiting to thicken the market is not beneficial.

1.3 Organization

The paper is organized as follows. We introduce our models and algorithms in Section 2. Section 3 summarizes our main theorems. In Section 4, we present the proof outline to obtain upper bounds of the loss of the proposed algorithms. The details follow in Sections 5–7. The conclusion is given in Section 8.

2 Model Definition

This section defines a continuous-time stochastic model for a bipartite matching market in a formal way. The model runs in the time interval [0, T].

Bipartite Matching Market  In a market, there are two types of agents, say type a and b. An agent of both types arrives at the market according to the Poisson process with rates \( \lambda_a \) and \( \lambda_b \), respectively. Thus, in any interval \([t, t + 1]\), \( \lambda_a \) (resp., \( \lambda_b \)) new agents of type a (resp., type b) enter the market in expectation. We denote by \( U_t \) and \( V_t \) the sets of agents of the two types at time \( t \), respectively, called the pools of the market. We also refer to \( A_t = |U_t| \) and \( B_t = |V_t| \) as the pool sizes at time \( t \). We assume that the market is empty at the beginning, i.e., \( U_0 = 0 \) and \( V_0 = 0 \).

When an agent \( v \) of some type arrives at the market, she forms edges to agents of the other type, that represents her compatible partners. An edge is formed with a probability \( p \). We denote by \( E_t \) the set of edges between \( U_t \) and \( V_t \) at time \( t \). Thus the market forms a bipartite graph, denoted by \( G_t = (U_t, V_t, E_t) \), at time \( t \). We assume that edges persist over time until one of the end vertices leaves. For an agent \( v \in U_t \cup V_t \), we use \( N_t(v) \) to denote the set of neighbors of \( v \) in \( G_t \). We also denote \( U = \bigcup_{t=0}^{T} U_t \), \( V = \bigcup_{t=0}^{T} V_t \), and \( E = \bigcup_{t=0}^{T} E_t \).

Each agent can stay in the market for a while. Her staying time is modeled as an independent Poisson process. Without loss of generality, we can normalize time so that the Poisson process has rate one. That is, if an agent \( v \) enters the market at time \( t_0 \), she can stay until time \( t_0 + Y \) where \( Y \) is an exponential-distribution random variable with mean 1. We say that \( v \) becomes critical at time \( t_0 + X \). An agent \( v \in U_t \cup V_t \) leaves the market at time \( t \) if \( v \) becomes critical at time \( t \) and/or \( v \) is matched to another agent at time \( t \). Thus \( v \) leaves the market at some time \( t_1 \) where \( t_0 \leq t_1 \leq t_0 + X \). We call \( t_1 - t_0 \) the sojourn of \( v \), denoted \( s(v) \).

Define \( d_a = \lambda_a p \) and \( d_b = \lambda_b p \) to be the density of the market. Then a bipartite matching market is defined by the tuple \((\lambda_a, \lambda_b, p)\) or equivalently, \((d_a, d_b, p)\).
Matching Algorithms  A set of edges $M_t \subseteq E_t$ is a matching if there are no edges that share the same vertices. A matching algorithm, at any time $t$, selects (possibly empty) matching $M_t$ in the current graph $G_t$, and the end vertices of the edges of $M_t$ leave the market immediately. We assume that any matching algorithm, at any time $t_0$, only knows the current graph $G_t$ for $t \leq t_0$ and does not know the future information on $G_t$ for $t > t_0$. We measure the performance of a matching algorithm by the loss defined as follows. Let $ALG(T)$ be the set of agents matched by an algorithm $ALG$ by time $T$.

**Definition 1** (Loss of Matching Algorithms). The loss $L_a$ of a matching algorithm $ALG$ is defined by the ratio of the expected number of perished agents in $U$ over the expected number of agents in $U$, that is,

$$L_a(ALG) = \frac{E[(U - ALG(T) \cap U - U_T)]}{E[|U|]} = \frac{E[(U - ALG(T) \cap U - U_T)]}{\lambda_a T}.$$  

The loss $L_b(ALG)$ can be defined similarly. The total loss $L(ALG)$ is then defined as

$$L(ALG) = \frac{E[(U \cup V - ALG(T) - U_T \cup V_T)]}{(\lambda_a + \lambda_b)T} = \lambda_a L_a(ALG) + \lambda_b L_b(ALG).$$

By definition, the loss is in $[0, 1]$. Minimizing the loss is equivalent to maximizing the size of matchings by the algorithm. We note that we focus only on the cost of being unmatched and ignore the waiting cost, as we here assume that the latter is negligible compared to the former.

2-sided/1-sided Greedy and Patient Algorithms  In this paper, we focus on local algorithms, in the sense that they look only at the neighbors of agents which attempt to match, rather than at the global network structure. Local algorithms can be designed by defining how each agent behaves to make a decision during her staying time.

We consider two simple agent behaviors with different timing properties, greedy and patient, defined as follows. We also introduce an agent who does nothing in the market.

**Greedy agent:** An agent $v$ is greedy if, immediately after $v$ enters the market at time $t$, the agent $v$ is matched to an arbitrary agent in $N_t(v)$ whenever $N_t(v) \neq \emptyset$. If $N_t(v) = \emptyset$, then $v$ does not make a decision.

**Patient agent:** An agent $v$ is patient if, at her critical time $t$, the agent $v$ is matched to an arbitrary agent in $N_t(v)$ whenever $N_t(v) \neq \emptyset$.

**Inactive agent:** An agent $v$ is inactive if $v$ does not make a decision in the market. That is, $v$ leaves the market if $v$ becomes critical and/or $v$ is asked by some neighbor to match during her staying time.

In the 2-sided matching algorithms, we assume that all agents are homogeneous, greedy or patient. In the 1-sided matching algorithms, we assume that all agents on one side are inactive and that either all agents on the other side are greedy or patient. Our proposed mechanisms are summarized as follows.

**Greedy$_2$:** Every agent is greedy.

**Patient$_2$:** Every agent is patient.

**Greedy$_1$:** Every agent in $V$ is greedy, while every agent in $U$ is inactive.

**Patient$_1$:** Every agent in $V$ is patient, while every agent in $U$ is inactive.

Note that Patient$_i$ for $i = 1, 2$ can only be applied when the planner knows the departure information.

Optimal and Omniscient Algorithms  We compare the performance of a matching algorithm to that of an optimal algorithm and an omniscient algorithm. Let $OPT$ be the algorithm that minimizes the loss over a time period $T$, assuming that the algorithm does not know the future information (or even the departure information). The omniscient algorithm $OMN$ is the algorithm that finds the maximum matching, provided full information about the future, i.e., it knows the full realization of the graph $G_t$ for all $0 \leq t \leq T$. By definition, we easily observe that $L(OPT) \geq L(OMN)$. Moreover, we have, for any $i = 1, 2$, $L(Greedy$_i$) \geq L(OPT)$ and $L(Patient$_i$) \geq L(OMN)$.

We remark that it does not make sense to define optimal and omniscient algorithms for the 1-sided markets. In fact, if a 1-sided algorithm allows active agents to make decision at any time during her stay, then any 2-sided algorithm can be simulated as 1-sided algorithms. This is because the decision of her partner can be regarded as her decision. Thus optimal and omniscient algorithms for the 1-sided markets are identical with those in the 2-sided setting, respectively.
3 Main Results

Our purpose is to evaluate the performance of the proposed algorithms to measure the impact of waiting time on the performance. We estimate their losses for the case of large markets with sparse graphs in the steady state, that is, in the case when $\lambda_a, \lambda_b \to \infty$, $d_a$ and $d_b$ are kept constants, and $T \to \infty$. In this section, we present our main theorems. Proof sketches for upper-bounding the loss will be described in the next section.

3.1 2-sided Matching Algorithms

For the 2-sided matching algorithms, we have the following upper bounds.

**Theorem 3.1.** For a bipartite matching market $(d_a, d_b, p)$ with $d_a \geq d_b$, we have

\[
L_a(\text{Greedy}_2) \leq \frac{d_a - d_b}{d_a} + \frac{\log (d_b + 3)}{d_a} \quad \text{and} \quad L_b(\text{Greedy}_2) \leq \frac{\log (d_b + 3)}{d_b}
\]

when $\lambda_a, \lambda_b, T \to \infty$. Therefore, when $\lambda_a, \lambda_b, T \to \infty$, it holds that

\[
L(\text{Greedy}_2) \leq \frac{d_a - d_b}{d_a + d_b} + \frac{2\log (d_b + 3)}{d_a + d_b}.
\]

**Theorem 3.2.** For a bipartite matching market $(d_a, d_b, p)$ with $d_a \geq d_b$, we have

\[
L_a(\text{Patient}_2) \leq \frac{d_a - d_b}{d_a} + \frac{\log (d_b + 3)}{d_a} \quad \text{and} \quad L_b(\text{Patient}_2) \leq e^{-\max\{d_a-d_b,\frac{d_a}{1+\lambda_b}\}}
\]

when $\lambda_a, \lambda_b, T \to \infty$. Therefore, when $\lambda_a, \lambda_b, T \to \infty$, it holds that

\[
L(\text{Patient}_2) \leq \frac{d_a - d_b}{d_a + d_b} + \frac{\log (d_b + 3)}{d_a + d_b} + e^{-\max\{d_a-d_b,\frac{d_a}{1+\lambda_b}\}}.
\]

If $d_a = d_b$, then we have a better bound:

\[
L(\text{Patient}_2) \leq 2e^{-Cd_a} \quad \text{for some constant } C.
\]

We observe from Theorems 3.1 and 3.2 that, when $d_a = d_b$, $L(\text{Greedy}_2) = O\left(\frac{\log d_a}{d_a}\right)$ and $L(\text{Patient}_2) = e^{-O(d_a)}$. It indicates that the upper bound of $L(\text{Patient}_2)$ is exponentially smaller than that of $L(\text{Greedy}_2)$. On the other hand, when $d_a > cd_b$ for some constant $c > 1$, we see that $L_a(\text{Greedy}_2) \approx L_a(\text{Patient}_2)$, while $L_b(\text{Patient}_2)$ is exponentially better than $L_b(\text{Greedy}_2)$.

Furthermore, we provide lower bounds for optimal algorithms.

**Theorem 3.3.** For a bipartite matching market $(d_a, d_b, p)$ such that $d_a \geq d_b \geq 1$ and $p < 1/10$, it holds that

\[
L(\text{OPT}) \geq \max\left\{\frac{1}{1 + 2d_a + d_b + 2d_a^2/\lambda_d + d_b^2/\lambda_b}, \frac{d_a - d_b}{d_a + d_b}\right\},
\]

\[
L(\text{OMN}) \geq \max\left\{\frac{1}{2} \left(\frac{e^{-(d_a+d_b)p}}{1 + d_a + d_b^2/\lambda_a + 1 + d_b + d_b^2/\lambda_b}, \frac{d_a - d_b}{d_a + d_b}\right)\right\}.
\]

It is not difficult to see that the loss $\frac{d_a - d_b}{d_a + d_b}$ is unavoidable in every market. Since agents in two classes arrive at Poisson rates $\lambda_a$ and $\lambda_b$, the expected numbers of agents on both sides in the time interval $[0, T]$ are $\lambda_a T$ and $\lambda_b T$. Then it cannot have a matching of size greater than $\lambda_b T$ when $\lambda_a \geq \lambda_b$, which implies that $(\lambda_a - \lambda_b)T$ agents in $U$ can never be matched during the process. Thus the loss of the omniscient algorithm is at least $\frac{d_a - d_b}{d_a + d_b}$. See Lemma 7.1 in Section 7.

Since $L(\text{Greedy}_2) \geq L(\text{OPT})$ and $L(\text{Patient}_2) \geq L(\text{OMN})$, they give lower bounds for $L(\text{Greedy}_2)$ and $L(\text{Patient}_2)$, respectively. When $\lambda_a$ and $\lambda_b$ are sufficiently large, (4) in Theorem 3.3 implies that $L(\text{OPT}) \geq \frac{1}{2} \left(\frac{d_a - d_b}{d_a + d_b} + \frac{1}{1 + 2d_a + d_b}\right)$. Thus, the upper bound of Theorem 3.1 is nearly optimal, up to logarithmic factors. Moreover, by (5), when $d_a = d_b$, we obtain $L(\text{OMN}) \geq \frac{e^{-d_a}}{1 + d_a}$. This implies that the upper bound when $d_a = d_b$ in Theorem 3.2 is nearly optimal.
Remark. Jiang [13] gave lower bounds on \( L(\text{OPT}) \) and \( L(\text{OMN}) \). Compared to his bound, our bound of \( L(\text{OPT}) \) is of a simpler form with the same order, while that of \( L(\text{OMN}) \) is better. In fact, Jiang [13] showed that
\[
L(\text{OPT}) \geq \frac{1 - \log(1 - p) \left( \lambda_a - \lambda_b \right) \frac{\lambda_a - \lambda_b}{\lambda_a + \lambda_b}}{1 - \log(1 - p) \left( \lambda_a + \lambda_b \right)}.
\]
Using the fact that \(-p - 2p^2 \leq \log(1 - p) \leq -p\) for \( p \leq 1/2\), the RHS is at least \( \frac{1}{1 + (d_a + d_b) + 2(d_a + d_b)p} \), which is of the same order as \([4]\). On the other hand, the lower bound of \( L(\text{OMN}) \) by Jiang [13] is
\[
L(\text{OMN}) \geq \frac{1 - \log(1 - p) \left( \lambda_a - \lambda_b \right) \frac{\lambda_a - \lambda_b}{\lambda_a + \lambda_b} - (\log(1 - p))^2 \lambda_a \lambda_b}{1 - \log(1 - p) \left( \lambda_a + \lambda_b \right)}.
\]
This is worse than our bound \([5]\), as, for example, when \( \lambda_a = \lambda_b \), the numerator is equal to \(1 - \left( \frac{\log(1 - p)}{p} \right)^2 \), which is negative when \( d_a \geq 1 \).

### 3.2 1-sided Matching Algorithms

For the 1-sided matching algorithms Greedy\(_1\) and Patient\(_1\), we obtain the same upper bounds of their losses as below. Recall that we assume that agents with density \( d_a \) are inactive.

**Theorem 3.4.** Let ALG\(_1\) be either Greedy\(_1\) or Patient\(_1\). For a bipartite matching market \((d_a, d_b, p)\), it holds that, if \( d_a \geq d_b \), then
\[
L_a(\text{ALG}_1) \leq \frac{d_a - d_b}{d_a} + \frac{\log(d_b + 3)}{d_a} \quad \text{and} \quad L_b(\text{ALG}_1) \leq \frac{\log(d_b + 3)}{d_b},
\]
and, if \( d_a < d_b \), then
\[
L_a(\text{ALG}_1) \leq \frac{\log(d_b + 3)}{d_a} \quad \text{and} \quad L_b(\text{ALG}_1) \leq \frac{d_b - d_a}{d_b} + \frac{\log(d_b + 3)}{d_b},
\]
when \( \lambda_a, \lambda_b, T \to \infty \). Therefore, when \( \lambda_a, \lambda_b, T \to \infty \), we have
\[
L(\text{ALG}_1) \leq \frac{|d_b - d_a|}{d_a + d_b} + \frac{2 \log(d_b + 3)}{d_a + d_b}.
\]
We can see that the bound of \( L(\text{Greedy}_1) \) has the same order as that of \( L(\text{Greedy}_2) \), while the bound of \( L(\text{Patient}_1) \) is worse than that of \( L(\text{Patient}_2) \). In fact, \( L(\text{Patient}_1) \) is lower-bounded by the following theorem, which shows that \( L(\text{Patient}_1) \) is strictly worse than \( L(\text{Patient}_2) \) when \( d_a = d_b \). In addition, both of Greedy\(_1\) and Patient\(_1\) have the same performances.

**Theorem 3.5.** For a bipartite matching market \((d_a, d_b, p)\) with \( d_a, d_b \geq 1 \) and \( p < 1/10 \), it holds that
\[
L(\text{Greedy}_1) \geq \max \left\{ \frac{1}{2(1 + d_b + d_b^2/\lambda_b)} \cdot \frac{|d_a - d_b|}{d_a + d_b} \right\},
\]
\[
L(\text{Patient}_1) \geq \max \left\{ \frac{\log(d_b + d_b^2/\lambda_b)}{d_a + d_b + d_b^2/\lambda_a + d_b^2/\lambda_b} \cdot \frac{|d_a - d_b|}{d_a + d_b} \right\}.
\]

### 4 Performance Analysis: Proof Overview

In this section, we describe the proof outline of Theorems 3.1, 3.2, and 3.4. See Section 6 for more detailed proofs, and the proofs of Theorems 3.3 and 3.5 may be found in Section 7. The proof outline is basically similar to that for a non-bipartite matching model by Akbarpour et al. [4], but it requires new techniques with more rigorous analysis.

As mentioned in Introduction, we first observe that dynamics of our proposed algorithms can be formulated as continuous-time Markov chains determined by the pool sizes \((A_t, B_t)\), that is, a pair of non-negative integers. The chains are shown to have unique stationary distributions \(\pi\). It means that, in the long run, the distribution of the pool sizes is converged to \(\pi\). Moreover, we can see that the loss of the proposed algorithms can be expressed as the pool sizes, and therefore, it suffices to estimate the pool sizes in the steady state to bound the loss.

Our main contribution is to show that, for each of the proposed algorithms, the pool sizes in the steady state highly concentrate around some values. The primary technical tool is the balance equations of Markov chains. In the rest of this section, we will explain proof outlines for each algorithm in a bit more detail.
4.1 2-sided Greedy Algorithm

Let us first discuss the 2-sided Greedy algorithm Greedy₂. Recall that we may assume by symmetry that \( \lambda_a \geq \lambda_b \). When we run Greedy₂, the bipartite graph \( G_t = (U_t, V_t, E_t) \) almost always has no edges. Since each agent’s staying time follows the Poisson process with mean 1, the rate that some agent in \( U_t \) (resp., \( V_t \)) becomes critical is \( \lambda_t \) (resp., \( B_t \)). Critical agents perish with probability one, which implies that the expected number of perished agents on each side at time \( t \) is \( \mathbb{E}[A_t] \) and \( \mathbb{E}[B_t] \). Since the distribution of \( (A_t, B_t) \) converges to the stationary distribution \( \pi \) of the corresponding Markov chain after the long run, \( \mathbb{E}[A_t] \) (resp., \( \mathbb{E}[B_t] \)) is approximated by \( \mathbb{E}_{(A,B) \sim \pi}[A] \) (resp., \( \mathbb{E}_{(A,B) \sim \pi}[B] \)) when \( T \rightarrow \infty \). Thus the losses \( L_a(\text{Greedy}_2) \) and \( L_b(\text{Greedy}_2) \) are roughly equal to \( \frac{\mathbb{E}_{(A,B) \sim \pi}[A]}{\lambda_a} \) and \( \frac{\mathbb{E}_{(A,B) \sim \pi}[B]}{\lambda_b} \), respectively, and the total loss \( L(\text{Greedy}_2) \) is \( \frac{1}{\lambda_a + \lambda_b} \mathbb{E}_{(A,B) \sim \pi}[A + B] \).

In what follows, we obtain upper bounds of \( \mathbb{E}_{(A,B) \sim \pi}[A] \) and \( \mathbb{E}_{(A,B) \sim \pi}[B] \). We show that, under Greedy₂, the probability that the pool size \( A_t \) (resp., \( B_t \)) in the steady state is larger than some value \( k_2 \) (resp., \( \ell_2 \)) is small. More specifically, we prove the following proposition.

**Proposition 4.1.** There exist \( k_2 \) and \( \ell_2 \), where \( k_2 \leq \lambda_a - \lambda_b + \frac{\lambda_a \log(d_a + 3)}{d_a} \) and \( \ell_2 \leq \frac{\lambda_b \log(d_b + 3)}{d_b} \), such that, for any \( \sigma \geq 1 \), we have

\[
\Pr_{(A,B) \sim \pi}[A \geq k_2 + \sigma + 1] \leq O(\lambda_a) e^{-\frac{\sigma^2}{\sigma + \lambda_a}} \quad \text{and} \quad \Pr_{(A,B) \sim \pi}[B \geq \ell_2 + \sigma + 1] \leq O(\lambda_b) e^{-\frac{\sigma^2}{\sigma + \lambda_b}}.
\]

The first inequality of the above proposition says that the probability that \( A \) is larger than \( k_2 \) drops exponentially. For example, if we set \( \sigma = \Theta(\sqrt{\lambda_a \log \lambda_a}) \), we see that \( \Pr_{(A,B) \sim \pi}[A \geq k_2 + \sigma + 1] \) is constant. Therefore, the pool sizes are concentrated with high probability in the region depicted as in Figure 1.

Proposition 4.1 implies that \( \mathbb{E}_{(A,B) \sim \pi}[A] \leq k_2 + o(\lambda_a) \) by setting \( \sigma = \Theta(\sqrt{\lambda_a \log \lambda_a}) \). Since \( L_a(\text{Greedy}_2) \approx \frac{1}{\lambda_a} \mathbb{E}_{(A,B) \sim \pi}[A] \), this implies that

\[
L_a(\text{Greedy}_2) \leq \frac{1}{\lambda_a} \left( \lambda_a - \lambda_b + \frac{\lambda_a \log(d_a + 3)}{d_a} + o(\lambda_a) \right) = \frac{d_a - d_b}{d_a} + \frac{\log(d_a + 3)}{d_a} + o(1).
\]

Similarly, by setting \( \sigma = \Theta(\sqrt{\lambda_b \log \lambda_b}) \), we have \( \mathbb{E}_{(A,B) \sim \pi}[B] \leq \ell_2 + o(\lambda_b) \), and hence we obtain \( L_b(\text{Greedy}_2) \approx \frac{1}{\lambda_b} \mathbb{E}_{(A,B) \sim \pi}[B] \leq \frac{\log(d_a + 3)}{d_a} + o(1) \). This shows Theorem 3.1. See Section 6.2 for the details.

Here is an intuition behind the values \( k_2 \) and \( \ell_2 \). In the unit-time interval, \( \lambda_a \) (resp., \( \lambda_b \)) new agents in \( U \) (resp, \( V \)) enter the market in expectation, and Greedy₂ attempts to make a matching between them. We observe that the size of a maximum matching in the time interval is at most \( \lambda_b \), and hence at least \( \lambda_a - \lambda_b \) agents in \( U \) cannot get matched. Thus, it is unavoidable that at least \( \lambda_a - \lambda_b \) agents in \( U \) enter the market in expectation. This is an intuitive reason why \( \mathbb{E}_{(A,B) \sim \pi}[A] \) is larger than \( \mathbb{E}_{(A,B) \sim \pi}[B] \) by \( \lambda_a - \lambda_b \). The number of the other agents in the market is \( O \left( \frac{\lambda_a \log d_a}{d_a} \right) \), which is small, compared to the case when all agents are inactive, in which the expected pool size is \( \lambda_a \). This is because the Greedy algorithm matches agents as soon as possible, which reduces the number of agents in the pool.

4.2 2-sided Patient Algorithm

Under the 2-sided Patient algorithm Patient₂, conditioned on \( A_t \) and \( B_t \), the graph \( G_t \) is a random bipartite graph with vertex sets \( A_t \) and \( B_t \) where an edge is formed with probability \( p \). The rate that some agent in \( U_t \) (resp., \( V_t \)
becomes critical is $A_t$ (resp., $B_t$). Since $G_t$ is a random bipartite graph, a critical agent in $U_t$ (resp., $V_t$) perishes with probability $(1 - p)^{B_t}$ (resp., $(1 - p)^{A_t}$). Therefore, the expected numbers of perished agents on both sides at time $t$ are $E[A_t(1 - p)^{B_t}]$ and $E[B_t(1 - p)^{A_t}]$, respectively. Since the distribution of $(A_t, B_t)$ converges to the stationary distribution $\pi$ in the long run, they are approximated by $E_{(A,B)\sim \pi}[A(1 - p)^B]$ and $E_{(A,B)\sim \pi}[B(1 - p)^A]$ when $T \to \infty$. Thus $L_a(\text{Patient}_2)$ and $L_b(\text{Patient}_2)$ are roughly equal to $\frac{1}{\lambda_a} E_{(A,B)\sim \pi}[A(1 - p)^B]$ and $\frac{1}{\lambda_b} E_{(A,B)\sim \pi}[B(1 - p)^A]$, respectively.

We show that, under Patient$_2$, the pool sizes $(A, B)$ in the steady state are roughly between $k_2 \leq A \leq \lambda_a$ and $\ell_2 \leq B \leq \lambda_b$ with high probability, where we recall that $k_2$ and $\ell_2$ are the values found in Proposition 4.1. More formally, we show the following, saying that the probability that the pool sizes are out of the region $S$ drops exponentially. See Figure 2(a) for your help.

**Proposition 4.2.** For any $\sigma_a, \sigma_b \geq 1$, there exist $k_2$ and $\ell_2$, where $k_2 - \sigma_b \leq k_2 \leq k_2$ and $\ell_2 - \sigma_a \leq \ell_2 \leq \ell_2$, such that

$$
Pr_{(A,B)\sim \pi}[(A, B) \not\in S] \leq O(\lambda_a \lambda_b^2) e^{-\frac{\sigma^2}{2(\sigma_a + \lambda_a)}} + O(\lambda_a^2 \lambda_b) e^{-\frac{\sigma^2}{2(\sigma_b + \lambda_b)}},
$$

where $S = \{(i, j) \mid k_2 - \sigma_a \leq i \leq \lambda_a + \sigma_a, \ell_2 - \sigma_b \leq j \leq \lambda_b + \sigma_b\}$.

By setting $\sigma_a = \Theta(\sqrt{\lambda_a \log \lambda_a})$ and $\sigma_b = \Theta(\sqrt{\lambda_b \log \lambda_b})$ in Proposition 4.2, we can upper-bound $E_{(A,B)\sim \pi}[A(1 - p)^B]$ and $E_{(A,B)\sim \pi}[B(1 - p)^A]$, which shows the first part of Theorem 3.2.

We remark that the expected pool size in the steady state is larger than about $(k_2, \ell_2)$ with high probability, which means that the pool sizes are larger than those of the 2-sided Greedy algorithm. In other words, waiting to match makes the market thicker after the long run.

When $d_a = d_b$, the bounds can be improved by further narrowing the concentration region. We first show that the probability that the sum of the pool sizes $A_t + B_t$ in the steady state is at most about $(\lambda_a + \lambda_b)/2$ is small. Under Patient$_2$, $\lambda_a + \lambda_b$ agents enter the pool in unit time interval, and at most $A_t + B_t$ agents leave the pool. Thus, if $A_t + B_t$ is much smaller than $\lambda_a + \lambda_b$, then the number of entering agents is larger than that of leaving agents, which increases the pool size. Hence, in the steady state, the sum of the pool sizes cannot be small.

**Proposition 4.3.** For any $\sigma \geq 1$, it holds that

$$
Pr_{(A,B)\sim \pi}[A + B \leq \left(\frac{\lambda_a + \lambda_b}{2} - 2\right) - \sigma - 1] \leq O(\lambda_a + \lambda_b) e^{-\frac{\sigma^2}{2(\sigma_a + \lambda_a)}}.
$$

Moreover, we prove that the probability that the difference of the pool sizes $A_t - B_t$ in the steady state is at least $\lambda_a/2$ is small. Formally, we prove the following.

**Proposition 4.4.** Suppose that $d_a = d_b \geq 3$ and $p < 1/10$. Suppose that $\sigma_a$ satisfies that $1 \leq \sigma_a \leq \lambda_a$. For any $\sigma_d \geq 1$, it holds that

$$
Pr_{(A,B)\sim \pi}[A - B \geq \frac{\lambda_a + \sigma_a}{2} + \sigma_d] \leq e^{-\frac{\sigma_d}{2}} + O(\lambda_a) e^{-\frac{\sigma^2}{2(\sigma_a + \lambda_a)}}.
$$

for some constant $0 < \sigma_d < 1$.

Therefore, we can see that the probability that the pool sizes are out of some region $S'$ is small (see Figure 2(b)), which allows us to upper-bound $E_{(A,B)\sim \pi}[A(1 - p)^B]$ and $E_{(A,B)\sim \pi}[B(1 - p)^A]$, which proves the second part of Theorem 3.2.

### 4.3 1-sided Matching Algorithms

The proof outline is similar to the 2-sided case, but 1-sided algorithms require a more involved analysis to show the concentration. We remark that we need to solve recursive equations with additive terms.

In a similar way to the 2-sided case, we can observe that $L(\text{Greedy}_1)$ is approximated by $\frac{1}{\lambda_a + \lambda_b} E_{(A,B)\sim \pi}[A + B]$, and $L(\text{Patient}_1)$ is by $\frac{1}{\lambda_a + \lambda_b} E_{(A,B)\sim \pi}[A + B(1 - p)^A]$, where $\pi$ is the stationary distribution of the corresponding Markov chains.

For Greedy$_1$, we show that the pool size $A_t$ of inactive agents in the steady state is highly concentrated around some value $k_1$, where $k_1 = \max \{\lambda_a - \lambda_b, 0\} + \frac{1}{d_s} \log(\lambda_a - \lambda_b + 1)$. On the other hand, letting $\ell_1 = (1 - p)^{-\sigma_a}(\lambda_b - \lambda_a + k_1)$ for $\sigma_a \geq 1$, the probability that the pool size $B_t$ of greedy agents in the steady state is larger than $\ell_1$ is small (See Figure 3(a)). The formal statement may be found as below, which proves Theorem 3.4 for Greedy$_1$. 

10
Proposition 4.5. There exists a number $k_1$, where $k_1 \leq \max\{\lambda_a - \lambda_b, 0\} + \frac{\lambda_b \log(d_b + 3)}{d_b}$, such that, for any $\sigma_a \geq 1$,

$$\Pr_{(A,B) \sim \pi} [A \geq k_1 + \sigma_a + 1] \leq O(\lambda_a) e^{-\sigma_a^2}, \quad \Pr_{(A,B) \sim \pi} [A \leq k_1 - \sigma_a - 1] \leq O(\lambda_a) e^{-\frac{\sigma_a^2}{\lambda_a}},$$

and, for any $\sigma_a, \sigma_b \geq 1$, we have

$$\Pr_{(A,B) \sim \pi} [B \geq \ell_1 + \sigma_b + 1] \leq O(\lambda_b) e^{-\frac{\sigma_b^2}{2(\sigma_a + \lambda_a) \lambda_b}} + O(\lambda_b^2 \lambda_a^2) e^{-\frac{\sigma_b^2}{\lambda_b}},$$

where $\ell_1 = (1 - p)^{-\sigma_a} (\lambda_b - \lambda_a + k_1)$.

For Patient1, we prove the following, asserting that the pool size $(A, B)$ in the steady state is highly concentrated around $(k_1, \lambda_b)$, where $k_1$ is defined as above for Greedy1. See Figure 3(b). This fact proves Theorem 5.4 for Patient1.

Proposition 4.6. For any $\sigma_a, \sigma_b \geq 1$, there exist $k_1$ and $k_1$ such that $k_1 - \sigma_b < k_1 < k_1 < \min\{k_1 + \sigma_b, \lambda_a\}$ and it holds that

$$\Pr_{(A,B) \sim \pi} [(A, B) \notin S] \leq O(\lambda_a) e^{-\frac{\sigma_a^2}{2(\sigma_a + \lambda_a) \lambda_b}} + O(\lambda_a^2 \lambda_b^2) e^{-\frac{\sigma_a^2}{\lambda_a}},$$

where

$$S = \{(k,j) \mid k_1 - \sigma_a \leq k \leq k_1 + \sigma_a, \lambda_b - \sigma_b \leq j \leq \lambda_b + \sigma_b\}.$$

Interestingly, for both of Greedy1 and Patient1, the size of inactive agents has the same concentrated value $k_1$.

5 Stationary Distribution of Markov Chains

In this section, we formulate our stochastic models as continuous-time Markov chains determined by the pool sizes, and show that the chains have unique stationary distributions.
5.1 Formulation by Markov Chains

We first review continuous-time Markov chains. We refer to Norris et al. [27] for the details. Let $Z_t$ be a continuous-time Markov chain on the ground set $\Omega$. For any two states $i, j \in \Omega$, we denote by $r_{i\to j}$ the transition rate from $i$ to $j$. Then $r_{i\to j} \geq 0$. The rate matrix $Q \in \mathbb{R}^{11 \times 11}$ is defined as

$$Q_{ij} = \begin{cases} r_{i\to j} & \text{if } i \neq j \\ \sum_{k \neq i} -r_{i\to k} & \text{otherwise.} \end{cases}$$

Then the transition probability in $t$ units of time is $e^{tQ} = \sum_{i=0}^{\infty} \frac{t^iQ^i}{i!}$. Let $P_t = e^{tQ}$ be the transition probability matrix of the Markov chain in $t$ time units. We say that a distribution $\pi : \Omega \to \mathbb{R}_+$ is a stationary distribution of the Markov chain if $P_t \pi = \pi$ for any $t \geq 0$.

For each of our algorithms, we define a Markov chain where each state corresponds to the pool sizes on both sides. That is, each state is a pair of non-negative integers. Specific transitions for each algorithm can be found in Sections 6.2–6.5.

We here observe that, for each of our algorithms, the pool sizes are Markovian. Moreover, we show that the corresponding Markov chains have unique stationary distributions.

**Theorem 5.1.** For Greedy$_i$ and Patient$_i$ for $i = 1, 2$ and any $0 \leq t_0 \leq t_1$, it holds that

$$\Pr[(A_{t_0}, B_{t_0}) \mid (A_i, B_i) \text{ for } 0 \leq t < t_1] = \Pr[(A_{t_0}, B_{t_0}) \mid (A_i, B_i) \text{ for } t_0 \leq t < t_1]. \quad (7)$$

Moreover, the corresponding Markov chains have unique stationary distributions.

The proof is given in Section 5.2.

This section concludes with definitions on a Markov chain, which will be used in the subsequent sections. For a Markov chain on the ground set $\Omega$ with stationary distribution $\pi$, the balance equations say that, for any $X \subseteq \Omega$,

$$\sum_{i \in X, j \notin X} \pi(i)r_{i\to j} = \sum_{i \in X, j \notin X} \pi(j)r_{j\to i}. \quad (8)$$

We denote $z_t(i) = \Pr[Z_t = i] \text{ for any } i \in \Omega \text{ and any time } t \geq 0$. The total variation distance between $z_t$ and $\pi$ is defined by $\|z_t - \pi\|_{\text{TV}} = \sum_{i \in \Omega} |z_t(i) - \pi(i)|$. For any $\epsilon > 0$, we define the mixing time as

$$\tau_{\text{mix}}(\epsilon) = \inf \{t \mid \|z_t - \pi\|_{\text{TV}} \leq \epsilon\}.$$

5.2 Proof of Theorem 5.1

In this section, we prove Theorem 5.1. To prove the theorem, we use the ergodic theorem (see, e.g., [27]). For a Markov chain $Z_t$ on the ground set $\Omega$, it is irreducible if for any pair of states $i, j \in \Omega$, $j$ is reachable from $i$ with positive probability. For any $i \in \Omega$ with $Z_{t_0} = i$, we say that $i$ is positive recurrent if

$$\mathbb{E}[\inf \{t \geq T_1 \mid Z_t = i\} \mid Z_{t_0} = i] < \infty$$

where $T_1$ is the first time it jumps out of $i$. The ergodic theorem asserts that an irreducible Markov chain has a unique stationary distribution if and only if it has a positive recurrent state.
Proof of Theorem 5.7. It is not difficult to see that (7) holds for each of the algorithms as explained in Section 5.1.

We here prove that the corresponding Markov chains have unique stationary distributions. By the ergodic theorem, it suffices to show that the chains are irreducible, and has a positive recurrent state.

First, we show that the chain \((A_t, B_t)\) is irreducible under any algorithm. Indeed, every state \((i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+\) is reachable from the initial state \((0, 0)\) with positive probability, because it happens when \(i\) and \(j\) agents arrives at the market, respectively, with no acceptable transactions. The state \((0, 0)\) is reachable from any state \((i, j)\) with positive probability, because it happens when all the agents leave the market without having new agents. Thus the chain \((A_t, B_t)\) is irreducible under any algorithm.

In the rest of the proof, we show that the state \((0, 0)\) is positive recurrent. We consider an algorithm that agents on both sides are inactive. A pair of pool sizes under this algorithm also forms a Markov chain, denoted by \((\tilde{A}_t, \tilde{B}_t)\). Then \(\tilde{A}_t \geq A_t\) and \(\tilde{B}_t \geq B_t\) hold at any time \(t\). In particular, \(A_t = 0\) if \(A_t = 0\) and \(B_t = 0\) if \(B_t = 0\). Therefore, if the state \((0, 0)\) in the chain \((\tilde{A}_t, \tilde{B}_t)\) is positive recurrent, the state \((0, 0)\) is also positive recurrent for the chain \((A_t, B_t)\) of each algorithm.

Since \(\tilde{A}_t\) and \(\tilde{B}_t\) are independent, we can decompose the chain \((\tilde{A}_t, \tilde{B}_t)\) into two independent Markov chains \(\{\tilde{A}_t\}\) and \(\{\tilde{B}_t\}\). Akbarpour et al. [4] showed that the state 0 is positive recurrent for \(\{\tilde{A}_t\}\), i.e.,

\[
\mathbb{E} \left[ \inf \{t \geq t_0 : \tilde{A}_t = 0 \mid \tilde{A}_{t_0} = 0 \} \right] < \infty.
\]

Thus the proof is complete.

6 Upper Bounds of Loss

In the subsequent sections, we prove Theorems 3.1, 3.2, and 3.4 on the upper bounds of the loss, following the proof outline in Section 4.

6.1 Preliminaries for Upper-Bounding Loss

In the proofs of Propositions 4.1–4.6 in Section 4, we develop recursive equations with the balance equations on the stationary distribution \(\pi\). In this section, we summarize technical lemmas for solving recursive equations, where the proofs of these lemmas are deferred to Appendix A.

Lemma 6.1. Let \(k^*\) be an integer. Suppose that, for any \(k \geq k^*\),

\[
f(k+1) \leq \exp \left( -\frac{k-k^*}{k+\eta} \right) f(k)
\]

for some constant \(\eta\), and \(f(k^*) \leq 1\).

(i) For any \(\sigma \geq 1\), it holds that

\[
\sum_{k=k^*+\sigma+1}^{\infty} f(k) = O \left( \frac{k^*+\eta+\sigma}{\sigma} \right) \exp \left( -\frac{\sigma^2}{\sigma+k^*+\eta} \right).
\]

(ii) For any \(\sigma \geq 1\) with \(\sigma = O(k^*+\eta)\), it holds that

\[
\sum_{k=k^*+\sigma+1}^{\infty} kf(k) = O \left( (k^*+\eta)^3 \right) \exp \left( -\frac{\sigma^2}{\sigma+k^*+\eta} \right).
\]

Lemma 6.2. Let \(k^*\) be an integer. Suppose that, for any \(k \leq k^*\),

\[
f(k-1) \leq \exp \left( -\frac{k^*-k}{\eta} \right) f(k)
\]

for some constant \(\eta\), and \(f(k^*) \leq 1\). Then, for any \(\sigma \geq 1\),

\[
\sum_{k=0}^{k^*-\sigma} f(k) = O(\eta) e^{-\frac{\sigma^2}{\eta}}.
\]
When we run the 2-sided greedy algorithm, the bipartite graph $G$

We also solve recursive equations with small additive terms.

**Lemma 6.3.** Let $k^*$ be an integer. Suppose that, for any $k \geq k^*$,

$$g(k + 1) \leq \alpha_k g(k) + \beta_k,$$

where $\alpha_k \leq \exp(-\frac{k - k^*}{k + \eta})$,

for some constant $\eta$, and $g(k^*) \leq 1$. Then, for any $\sigma \geq 1$, we have

$$\sum_{k=k^*+\sigma+1}^{\infty} g(k) = O(k^* + \eta)e^{-\frac{\sigma^2}{2(\sigma + \eta)}} + O(k^* + \eta)\sum_{k=k^*}^{\infty} \beta_k.$$

### 6.2 Greedy$_2$

When we run the 2-sided greedy algorithm, the bipartite graph $G_t = (U_t, V_t, E_t)$ almost always has no edges. Since each agent’s staying time follows the Poisson process with mean 1, the rate that some agent in $U_t$ (resp., $V_t$) becomes critical is $A_t$ (resp., $B_t$). Critical agents perish with probability one, which implies that the expected number of perished agents in $U_t$ at time $t$ is $A_t$ (resp., $B_t$). Hence we have

$$L_a(Greedy_2) = \frac{1}{\lambda_a T}E \left[ \int_0^T A_t dt \right] \quad \text{and} \quad L_b(Greedy_2) = \frac{1}{\lambda_b T}E \left[ \int_0^T B_t dt \right].$$

They are roughly equal to $\frac{E(A,B)\sim\pi[A]}{\lambda_a}$ and $\frac{E(A,B)\sim\pi[B]}{\lambda_b}$, respectively, in the stationary distribution $\pi$, which is formally stated as follows.

**Lemma 6.4.** For a bipartite matching market $(d_a, d_b, p)$ with $d_a \geq d_b$, let $\pi$ be the stationary distribution of the Markov chain of Greedy$_2$. For any $\epsilon > 0$ and $T > 0$, it holds that

$$L_a(Greedy_2) \leq \frac{E(A,B)\sim\pi[A]}{\lambda_a} + \frac{\tau_{\max}(\epsilon)}{T} + 6\epsilon + \frac{1}{\lambda_a} 2^{-6\lambda_a},$$

$$L_b(Greedy_2) \leq \frac{E(A,B)\sim\pi[B]}{\lambda_b} + \frac{\tau_{\max}(\epsilon)}{T} + 6\epsilon + \frac{1}{\lambda_b} 2^{-6\lambda_b},$$

where $\tau_{\max}(\epsilon)$ is the mixing time of the Markov chain.

The proof of Lemma 6.4 is given in Section 6.6.1 Note that as $\lambda_a$, $\lambda_b$, and $T$ grow, the last three terms become negligible.

**Markov Chain** We formally define a Markov chain for the 2-sided Greedy algorithms. Let $(A_t, B_t)$ be a continuous-time Markov Chain on $\mathbb{Z}_+ \times \mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of non-negative integers. For any $(k, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and $(k', j') \in \mathbb{Z}_+ \times \mathbb{Z}_+$, we denote by $r_{(k, j) \rightarrow (k', j')}$ the transition rate from $(k, j)$ to $(k', j')$.

For any pair of pool sizes $(k, j)$, the Markov chain transits only to the states $(k + 1, j), (k, j + 1), (k - 1, j), \text{and} (k, j - 1)$. See Figure 4. It transits to $(k + 1, j)$ or $(k, j + 1)$ when a new agent arrives and she does not get matched, and to $(k - 1, j)$ or $(k, j - 1)$ when either a new agent arrives and she gets matched with some agent in the pool, or
some agent in the pool perishes. Therefore, the transition rates are defined by

\[
\begin{align*}
    r_{k,j \to (k+1,j)} &= \lambda_a (1-p)^j, \\
    r_{k,j \to (k,j+1)} &= \lambda_b (1-p)^k, \\
    r_{k,j \to (k-1,j)} &= k + \lambda_b (1-(1-p)^k), \\
    r_{k,j \to (k,j-1)} &= j + \lambda_a (1-(1-p)^j).
\end{align*}
\]

### Concentration of Pool Sizes

By Lemma 6.4, we can upper-bound the loss of Greedy₂ by bounding the expected pool sizes \( \mathbb{E}_{(A,B) \sim \pi} [A] \) and \( \mathbb{E}_{(A,B) \sim \pi} [B] \) in the stationary distribution \( \pi \). We show that, under Greedy₂, the probability that the pool size \( A_t \) (resp., \( B_t \)) in the steady state is larger than some value \( k_2 \) (resp., \( \ell_2 \)) is exponentially small, where \( k_2 \leq \lambda_a - \lambda_b + \frac{\lambda_a \log (d_a + 3)}{d_a} \) and \( \ell_2 \leq \frac{\lambda_b \log (d_b + 3)}{d_b} \).

#### Proposition 4.1

There exist \( k_2 \) and \( \ell_2 \), where \( k_2 \leq \lambda_a - \lambda_b + \frac{\lambda_a \log (d_a + 3)}{d_a} \) and \( \ell_2 \leq \frac{\lambda_b \log (d_b + 3)}{d_b} \), such that, for any \( \sigma \geq 1 \), we have

\[
\Pr_{(A,B) \sim \pi} [A \geq k_2 + \sigma + 1] \leq O(\lambda_a) e^{-\frac{\sigma^2}{2 \lambda_a}} \quad \text{and} \quad \Pr_{(A,B) \sim \pi} [B \geq \ell_2 + \sigma + 1] \leq O(\lambda_b) e^{-\frac{\sigma^2}{2 \lambda_b}}.
\]

The proof of Proposition 4.1 is based on the balance equations. For any integer \( k \geq 0 \), let \( X = \{ (i,j) \mid 0 \leq i \leq k, j \geq 0 \} \). We apply the balance equation (8) with the set \( X \). Then we obtain

\[
\sum_{j=0}^{\infty} \lambda_a (1-p)^j \pi(k, j) = \left( k + 1 + \lambda_b (1-(1-p)^{k+1}) \right) \sum_{j=0}^{\infty} \pi(k+1, j).
\]

Since the LHS is at most \( \lambda_a \sum_{j=0}^{\infty} \pi(k, j) \), it holds that

\[
\sum_{j=0}^{\infty} \pi(k+1, j) \leq \frac{\lambda_a}{k + 1 + \lambda_b (1-(1-p)^{k+1})} \sum_{j=0}^{\infty} \pi(k, j).
\]

Thus we have a recursive relationship for \( \sum_{j=0}^{\infty} \pi(k, j) \) for \( k = 0, 1, 2, \ldots \). By solving this recursive equation, we have the first part of Proposition 4.1. The second one is analogous. See Section 6.2.1 for the complete proof.

Proposition 4.1 implies that \( \mathbb{E}_{(A,B) \sim \pi} [A] \leq k_2 + o(\lambda_a) \) by setting \( \sigma = \Theta(\sqrt{\lambda_a \log \lambda_a}) \). Since \( L_a(\text{Greedy}_2) \approx \frac{1}{\lambda_a} \mathbb{E}_{(A,B) \sim \pi} [A] \) by Lemma 6.4, this implies that

\[
L_a(\text{Greedy}_2) \leq \frac{1}{\lambda_a} \left( \lambda_a - \lambda_b + \frac{\lambda_a \log (d_a + 3)}{d_a} + o(\lambda_a) \right) = \frac{d_a - d_b}{d_a} + \frac{\log (d_a + 3)}{d_a} + o(1).
\]

Similarly, by setting \( \sigma = \Theta(\sqrt{\lambda_b \log \lambda_b}) \), we have \( \mathbb{E}_{(A,B) \sim \pi} [B] \leq \ell_2 + o(\lambda_b) \), and hence

\[
L_b(\text{Greedy}_2) \leq \frac{1}{\lambda_b} \left( \frac{\lambda_b \log (d_b + 3)}{d_b} + o(\lambda_b) \right) = \frac{\log (d_b + 3)}{d_b} + o(1).
\]

This shows Theorem 3.1. See Section 6.2.2 for the details.

#### 6.2.1 Concentration of Pool Sizes: Proof of Proposition 4.1

We define \( k_2 \) as the value that satisfies \( \lambda_a = k_2 + \lambda_b (1-(1-p)^{k_2}) \), and \( \ell_2 \) as the value that satisfies \( \lambda_b = \ell_2 + \lambda_a (1-(1-p)^{\ell_2}) \). Since we assume that \( \lambda_a \geq \lambda_b \), it holds that \( k_2 \geq \ell_2 \). Moreover, these values can be approximated with \( \lambda_a \) and \( \lambda_b \).

#### Lemma 6.5

Suppose that \( \lambda_a \geq \lambda_b \) and \( d_a \geq d_b > 0 \). It holds that

\[
\max \left\{ \lambda_a - \lambda_b, \frac{\lambda_a}{1+d_a} \right\} \leq k_2 \leq \lambda_a - \lambda_b + \frac{1}{p} \log (d_b + 3),
\]

\[
\frac{\lambda_b}{1+d_a} \leq \ell_2 \leq \frac{\lambda_b}{d_b} \log (d_b + 3).
\]
Proof. We first estimate $k_2$. We denote $k_{\text{max}} = \lambda_a - \lambda_b + \frac{1}{p} \log(d_b + 3)$ and $k_{\text{min}} = \max \left\{ \lambda_a - \lambda_b, \frac{\lambda_a}{1 + d_b} \right\}$. We define a function $f$ by $f(k) = k + \lambda_b(1 - e^{-p}) - \lambda_a$. Then the function $f$ is a non-decreasing function, and $f(k_2) = 0$. Since $f(k) \geq k + \lambda_b(1 - e^{-p}) - \lambda_a$, it holds that
\[
f(k_{\text{max}}) \geq \lambda_a - \lambda_b + \frac{\lambda_b}{d_b} \log(d_b + 3) + \lambda_b \left( 1 - e^{-\log(d_b + 3)} \right) - \lambda_a = \frac{\lambda_b}{d_b} \log(d_b + 3) - \frac{\lambda_b}{d_b + 3} \geq 0
\]
since $\log(d_b + 3) \geq 1$. Moreover, since $f(k) \leq k + \lambda_b(1 - (1 - pk)) - \lambda_a = k + \lambda_b pk - \lambda_a$, it holds that
\[
f \left( \frac{\lambda_a}{1 + d_b} \right) \leq \frac{\lambda_a}{1 + d_b} + \lambda_b p \left( \frac{\lambda_a}{1 + d_b} - \lambda_a \right) = 0.
\]
We also have
\[
f(\lambda_a - \lambda_b) = (\lambda_a - \lambda_b) + \lambda_b(1 - e^{-p(\lambda_a - \lambda_b)}) - \lambda_a = -\lambda_b e^{-p(\lambda_a - \lambda_b)} < 0.
\]
Thus the inequality for $k_2$ holds.

The inequality for $\ell_2$ can be proved similarly. In fact, by letting $f'(k) = k + \lambda_a(1 - (1 - p)k) - \lambda_b$, we obtain
\[
f' \left( \frac{\lambda_b}{d_b} \log(d_b + 3) \right) \geq (\lambda_a - \lambda_b) + \frac{\lambda_b}{d_b} \log(d_b + 3) - \frac{\lambda_a}{d_b + 3} \geq (\lambda_a - \lambda_b) + \frac{\lambda_b}{d_b + 3} - \frac{\lambda_a}{d_b + 3} > 0,
\]
since $\log(d_b + 3) \geq 1$. Thus the inequality holds.

We now prove Proposition 4.4 using $k_2$ and $\ell_2$ defined as above.

Proof of Proposition 4.4 Let $k_2, \ell_2$ be the values defined as above, that is, they satisfy $\lambda_a = k_2 + \lambda_b(1 - (1 - p)k_2)$ and $\lambda_b = \ell_2 + \lambda_a(1 - (1 - p)^{\ell_2})$. By the balance equation (8) with $X = \{ (i, j) \mid 0 \leq i \leq k, j \geq 0 \}$ for any $k \geq 0$, it holds that
\[
\sum_{j=0}^{\infty} \lambda_a(1 - p)^j \pi(k, j) = (k + 1 + \lambda_b(1 - (1 - p)^{k+1})) \sum_{j=0}^{\infty} \pi(k + 1, j).
\]
Since the LHS of (11) is at most $\lambda_a \sum_{j=0}^{\infty} \pi(k, j)$, we obtain
\[
\frac{\sum_{j=0}^{\infty} \pi(k + 1, j)}{\sum_{j=0}^{\infty} \pi(k, j)} \leq \frac{\lambda_a}{k + 1 + \lambda_b(1 - (1 - p)^{k+1})}.
\]
By the definition of $k_2$, we have, for $k \geq k_2$,
\[
\frac{\lambda_a}{k + 1 + \lambda_b(1 - (1 - p)^{k+1})} = \frac{\lambda_a}{k + 1 + k_2 + \lambda_b(1 - (1 - p)^{k+1}) - \lambda_b(1 - (1 - p)^{k_2}) + \lambda_a} \leq \frac{\lambda_a}{k - k_2 + \lambda_a}.
\]
Hence, for any $k \geq k_2$,
\[
\frac{\sum_{j=0}^{\infty} \pi(k + 1, j)}{\sum_{j=0}^{\infty} \pi(k, j)} \leq \frac{\lambda_a}{k - k_2 + \lambda_a} = 1 - \frac{k - k_2}{k - k_2 + \lambda_a} \leq \exp \left( -\frac{k - k_2}{k - k_2 + \lambda_a} \right).
\]
We apply Lemma 6.1(iii) with $k^* = k_2$ and $\eta = \lambda_a - k_2$, which implies that, for any $\sigma \geq 1$,
\[
\sum_{k=k_2+\sigma+1}^{\infty} \sum_{j=0}^{\infty} \pi(k, j) \leq O \left( \frac{\sigma + \lambda_a}{\sigma} \right) \exp \left( -\frac{\sigma^2}{\sigma + \lambda_a} \right) = O \left( \lambda_a \right) \exp \left( -\frac{\sigma^2}{\sigma + \lambda_a} \right)
\]
where we note $O \left( \frac{\sigma + \lambda_a}{\sigma} \right) = O \left( \lambda_a \right)$ since $\sigma \geq 1$. This proves the first inequality in the proposition. The second inequality is analogous.

We remark that, in the proof of Proposition 4.4 using Lemma 6.1(ii) instead of (i), we obtain
\[
\sum_{k \geq k_2+\sigma+1} \sum_{j \geq 0} \pi(k, j) = O \left( \lambda_a^3 \right) e^{-\frac{\sigma^2}{\sigma + \lambda_a}}
\]
when $\sigma = O(\lambda_a)$.
6.2.2 Bounding the Loss of Greedy₂

We are ready to bound $E_{(A,B) \sim \pi} [A]$ and $E_{(A,B) \sim \pi} [B]$, respectively. This shows Theorem 3.1 combined with Lemma 6.4.

**Theorem 6.6.** For a bipartite matching market $(d_a, d_b, p)$ with $d_a \geq d_b$, let $\pi$ be the stationary distribution of the Markov chain of Greedy₂. It holds that

$$
\frac{E_{(A,B) \sim \pi} [A]}{\lambda_a} \leq \frac{d_a - d_b}{d_a} + \frac{\log(d_b + 3)}{d_a} + o(1),
$$

$$
\frac{E_{(A,B) \sim \pi} [B]}{\lambda_b} \leq \frac{\log(d_b + 3)}{d_b} + o(1).
$$

**Proof.** Define $\sigma_a = \Theta(\sqrt{A \log \lambda_a})$. We observe that

$$
E_{(A,B) \sim \pi} [A] = \sum_{k \geq 0} k \sum_{j \geq 0} \pi(k, j) \leq k_2 + \sigma_a + \sum_{k \geq k_2 + \sigma_a + 1} k \sum_{j \geq 0} \pi(k, j).
$$

Since $\sigma_a = O(\lambda_a)$, the last term is bounded by a constant from (12). Hence we obtain

$$
E_{(A,B) \sim \pi} [A] \leq k_2 + O\left(\sqrt{\lambda_a \log \lambda_a}\right).
$$

Therefore, by Lemma 6.5

$$
\frac{E_{(A,B) \sim \pi} [A]}{\lambda_a} \leq \frac{k_2 + o(\lambda_a)}{\lambda_a} \leq \frac{1}{\lambda_a} \left(\lambda_a - \lambda_b + \frac{1}{p \log(d_b + 3)}\right) + o(1) = \frac{d_a - d_b}{d_a} + \frac{\log(d_b + 3)}{d_a} + o(1).
$$

Similarly, we see that

$$
\sum_{j \geq 0} \sum_{k \geq 0} \pi(k, j) \leq \ell_2 + O\left(\sqrt{\lambda_b \log \lambda_b}\right).
$$

and, by Lemma 6.5 it holds that

$$
\frac{E_{(A,B) \sim \pi} [B]}{\lambda_b} \leq \frac{\ell_2 + o(\lambda_b)}{\lambda_b} \leq \frac{1}{\lambda_b} \left(\frac{1}{p} \log(d_b + 3)\right) + o(1) = \frac{\log(d_b + 3)}{d_b} + o(1).
$$

This completes the proof. \hfill \Box

6.3 Patient₂

Similarly to the case of Greedy₂, we first show that the loss can be formulated using the expected pool sizes in the steady state.

Under Patient₂, conditional on $A_t$ and $B_t$, the graph $G_t$ is a random bipartite graph with vertex sets $A_t$ and $B_t$ where an edge is formed with probability $p$. The rate that some agent in $U_t$ (resp., $V_t$) becomes critical is $A_t$ (resp., $B_t$). Since $G_t$ is a random bipartite graph, a critical agent in $U_t$ (resp., $V_t$) perishes with probability $(1 - p)^{B_t}$ (resp., $(1 - p)^{A_t}$). Therefore, we have

$$
L_a(\text{Patient}_2) = \frac{1}{\lambda_a} T E \left[ \int_{t=0}^{T} A_t (1 - p)^{B_t} dt \right] \quad \text{and} \quad L_b(\text{Patient}_2) = \frac{1}{\lambda_b} T E \left[ \int_{t=0}^{T} B_t (1 - p)^{A_t} dt \right].
$$

They are roughly equal to $\frac{1}{\lambda_a} E_{(A,B) \sim \pi} [A(1 - p)^B]$ and $\frac{1}{\lambda_b} E_{(A,B) \sim \pi} [B(1 - p)^A]$, respectively, where $\pi$ is the stationary distribution of the corresponding Markov chain, as stated below. The proof is deferred to Section 6.6.2.

**Lemma 6.7.** For a bipartite matching market $(d_a, d_b, p)$ with $d_a \geq d_b$, let $\pi$ be the stationary distribution of the Markov chain of Patient₂. For any $\epsilon > 0$ and $T > 0$,

$$
L_a(\text{Patient}_2) \leq \frac{E_{(A,B) \sim \pi} [A(1 - p)^B]}{\lambda_a} + \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{6 \epsilon}{p} + \frac{2^{-6\lambda_a}}{\lambda_a},
$$

$$
L_b(\text{Patient}_2) \leq \frac{E_{(A,B) \sim \pi} [B(1 - p)^A]}{\lambda_b} + \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{6 \epsilon}{p} + \frac{2^{-6\lambda_b}}{\lambda_b},
$$

where $\tau_{\text{mix}}(\epsilon)$ is the mixing time of the Markov chain.

Note that, by setting $\epsilon$ small enough so that $\frac{\epsilon}{p} = \frac{6 \epsilon}{p} + \frac{2^{-6\lambda_a}}{\lambda_a}$, the last three terms are negligible.
We first observe that the probability that the pool size when some agent in the pool leaves the market without getting matched to another agent, and to 

To prove Proposition 4.2, we show the following two lemmas. 

Theorem 6.11), which implies the first part of Theorem 3.2 by Lemma 6.7. See Section 6.3.1 for the details with the 

For any \( \sigma_a, \sigma_b \geq 1 \), there exist \( k_2, \ell_2 \) where \( k_2 - \sigma_b \leq k_2 \leq k_2 \) and \( \ell_2 - \sigma_a \leq \ell_2 \leq \ell_2 \), such that 

\[
\Pr_{(A,B) \sim \pi} [(A,B) \not\in S] \leq O(\lambda_a \lambda_b^2) e^{-\frac{\sigma_b^2}{2(\sigma_a + \lambda_b)}} + O(\lambda_a^2 \lambda_b) e^{-\frac{\sigma_a^2}{2(\sigma_b + \lambda_a)}},
\]

where \( S = \{(i,j) \mid k_2 - \sigma_a \leq i \leq \lambda_a + \sigma_a, \ell_2 - \sigma_b \leq j \leq \lambda_b + \sigma_b\} \). 

The above proposition shows that the probability \( \Pr_{(A,B) \sim \pi} [(A,B) = (k,\ell)] \) decreases exponentially outside of the region \( S \), which means that the expected pool sizes \( E_{(A,B) \sim \pi} [A] \) and \( E_{(A,B) \sim \pi} [B] \) are roughly between \( k_2 \) and \( \lambda_a \) and between \( \ell_2 \) and \( \lambda_b \), respectively. Therefore, we can upper-bound \( E_{(A,B) \sim \pi} [A] \) and \( E_{(A,B) \sim \pi} [B] \) (see Theorem 6.11), which implies the first part of Theorem 3.2 by Lemma 6.7. See Section 6.3.1 for the details with the proof of Proposition 4.2. 

When \( a = b \), the bounds can be improved by further narrowing the concentration region. We first show Proposition 4.3 in Section 6.3.2 that says that the probability that the sum of the pool sizes \( A_t + B_t \) in the steady state is at most about \( \lambda_a + \lambda_b \) is small. Note that this holds even when \( \lambda_a \neq \lambda_b \). Moreover, we prove Proposition 4.4 when \( a = b \), the probability that the difference of the pool sizes \( A_t - B_t \) in the steady state is large is small. By the two propositions, together with Lemma 6.7, we can upper-bound the loss (Theorem 6.12), which implies the second part of Theorem 3.2. See Section 6.3.3 for the details with the proof of Proposition 4.2. 

6.3.1 Loss of the Unbalanced Case: Proof of Proposition 4.2 

To prove Proposition 4.2, we show the following two lemmas. 

We first observe that the probability that the pool size \( A_t \) (resp., \( B_t \)) in the steady state is larger than \( \lambda_a \) (resp., \( \lambda_b \)) is small. This can be shown by applying the balance equation with the same set of states as in the proof of Proposition 4.1. 

Lemma 6.8. For any \( \sigma \geq 1 \), it holds that 

\[
\Pr_{(A,B) \sim \pi} [A \geq \lambda_a + \sigma + 1] \leq O(\lambda_a) e^{-\frac{\sigma^2}{2(\sigma_a + \lambda_b)}} \quad \text{and} \quad \Pr_{(A,B) \sim \pi} [B \geq \lambda_b + \sigma + 1] \leq O(\lambda_b) e^{-\frac{\sigma^2}{2(\sigma_a + \lambda_b)}}.
\]
Proof. By the balance equation (8) with \( X = \{(i, j) \mid 0 \leq i \leq k, j \geq 0\} \) for any \( k \geq 0 \), it holds that
\[
\lambda_a \sum_{j=0}^k \pi(k, j) = \sum_{j=0}^k (k + 1 + j(1 - (1 - p)^{k+1})) \pi(k + 1, j) \geq (k + 1) \sum_{j=0}^k \pi(k + 1, j). \tag{13}
\]
Therefore, for \( k \geq \lambda_a - 1 \), we have
\[
\frac{\sum_{j=0}^k \pi(k, j)}{\sum_{j=0}^k \pi(k, j)} \leq \frac{\lambda_a}{k + 1} = 1 - \frac{k - \lambda_a + 1}{k + 1} \leq \exp \left( -\frac{k - \lambda_a + 1}{k + 1} \right).
\]
Hence, by applying Lemma 6.1 (9) with \( k^* = \lambda_a - 1 \) and \( \eta = 1 \), we have, for any \( \sigma \geq 1 \),
\[
\sum_{k=\lambda_a+\sigma+1}^\infty \sum_{j=0}^k \pi(k, j) = O \left( \frac{\lambda_a + \sigma}{\sigma} \right) \exp \left( -\frac{\sigma^2}{\sigma + \lambda_a} \right).
\]
This proves the first inequality of the lemma as \( \sigma \geq 1 \). The second one is analogous.

We remark that, using Lemma 6.1 in the proof of Lemma 6.8 we obtain
\[
\sum_{k=\lambda_a+\sigma_a+1}^\infty \sum_{j=0}^k \pi(k, j) = O \left( \frac{\lambda_a}{\sigma_a} \right) e^{-\frac{\sigma_a^2}{\sigma_a + \lambda_a}}, \tag{14}
\]
\[
\sum_{j=\lambda_b+\sigma_b+1}^\infty \sum_{k=0}^j \pi(k, j) = O \left( \frac{\lambda_b}{\sigma_b} \right) e^{-\frac{\sigma_b^2}{\sigma_b + \lambda_b}}, \tag{15}
\]
when \( \sigma_a = O(\lambda_a) \) and \( \sigma_b = O(\lambda_b) \).

We next show that the pool size \( A_k \) (resp., \( B_k \)) is not so small in the steady state. For any \( \sigma_a, \sigma_b \geq 1 \), define \( k_2 \) to be the value that satisfies \( \lambda_a = k + (\lambda_b + \sigma_b)(1 - (1 - p)^k) \), and \( \ell_2 \) to be the value that satisfies \( \lambda_b = \ell + (\lambda_a + \sigma_a)(1 - (1 - p)^\ell) \).

We first observe the following relationship between \( k_2, \ell_2 \) and \( k_2, \ell_2 \).

**Lemma 6.9.** For any \( \sigma_a, \sigma_b \geq 1 \), define \( k_2 \) and \( \ell_2 \) as above. Then \( k_2 - \sigma_b \leq k_2 \leq k_2 + \sigma_b < k_2 \) and \( \ell_2 - \sigma_a \leq \ell_2 < \ell_2 \).

Proof. Define a function \( f(k) = k + (\lambda_b + \sigma_b)(1 - (1 - p)^k) - \lambda_a \), which is a non-decreasing function. Note that \( f(k_2) = 0 \). Since \( f(k_2 - \sigma_b) = -\sigma_b(1 - p)^{k_2 - \sigma_b} \) from the definition of \( k_2 \), we obtain \( k_2 - \sigma_b < k_2 \). Since \( f(k_2) > 0 \), we also have \( k_2 < k_2 \). The inequality for \( \ell_2 \) is analogous.

Using \( k_2 \) and \( \ell_2 \) defined as above, we can lower-bound the expected pool sizes. We remark that we need to solve recursive equations with additive terms.

**Lemma 6.10.** For any \( \sigma_a, \sigma_b \geq 1 \) such that \( \sigma_a = O(\lambda_a) \) and \( \sigma_b = O(\lambda_b) \), define \( k_2 \) and \( \ell_2 \) as above. Then it holds that, for any \( \sigma \geq 1 \),
\[
\Pr_{(A,B) \sim \pi} \left[ A \leq k_2 - \sigma \right] \leq O(\lambda_a) e^{-\frac{\sigma^2}{\sigma_a + \lambda_a}} + O(\lambda_a^2) e^{-\frac{\sigma^2}{\sigma_b + \lambda_b}} + O(\lambda_b^2) e^{-\frac{\sigma^2}{\sigma_a + \lambda_a}}.
\]

Proof. By the balance equation (13) with \( X = \{(i, j) \mid 0 \leq i \leq k, j \geq 0\} \) for any \( k \geq 0 \), it holds that
\[
\lambda_a \sum_{j=0}^k \pi(k, j) = \sum_{j=0}^k (k + 1 + j(1 - (1 - p)^{k+1})) \pi(k + 1, j) \leq (k + 1 + (\lambda_b + \sigma_b)(1 - (1 - p)^k + 1) \sum_{j=0}^k \pi(k + 1, j) + \sum_{j=\lambda_a+\sigma_a+1}^\infty (k + 1 + j) \pi(k + 1, j). \tag{16}
\]
To simplify the notation, for any \( k \geq 0 \), define
\[
g(k) = \sum_{j=0}^k \pi(k, j), \quad \alpha_k = \frac{\lambda_a}{\lambda_a} \sum_{j=\lambda_a+\sigma_a+1}^\infty (k + 1 + j) \pi(k + 1, j), \quad \text{and} \quad \beta_k = \frac{1}{\lambda_a} \sum_{j=\lambda_b+\sigma_b+1}^\infty (k + 1 + j) \pi(k + 1, j).
\]
Then, since the LHS of (16) is at least $\lambda_a g(k)$, (16) implies the following recursive relationship for any $k \geq 0$:

$$g(k) \leq \alpha_{k+1} g(k+1) + \beta_{k+1}.$$ 

We observe that, for $0 \leq k \leq k_2$,

$$\alpha_k = \frac{k + (\lambda_a + \sigma_b)(1 - (1 - p)^k) - k_2 - (\lambda_a + \sigma_b)(1 - (1 - p)^{k_2}) + \lambda_a}{\lambda_a} \leq 1 - \frac{k_2 - k}{\lambda_a} \leq \exp\left(-\frac{k_2 - k}{\lambda_a}\right) \leq \exp\left(-\frac{k_2 - k}{\lambda_a + k}\right).$$

Moreover, by Lemma 6.8 with (15), when $\sigma_b = O(\lambda_b)$,

$$\sum_{k=0}^{k_2} \beta_k = \frac{1}{\lambda_a} \sum_{k=0}^{k_2} k \sum_{j=\lambda_b+\sigma_b+1}^{\infty} \pi(k, j) + \frac{1}{\lambda_a} \sum_{k=0}^{k_2} j \sum_{j=\lambda_b+\sigma_b+1}^{\infty} \pi(k, j) \leq O(\lambda_b) \exp\left(-\frac{\sigma_b^2}{\sigma_b + \lambda_b}\right) + \frac{1}{\lambda_a} O(\lambda_b^2) \exp\left(-\frac{\sigma_b^2}{\sigma_b + \lambda_b}\right) \leq O(\lambda_b^2) \exp\left(-\frac{\sigma_b^2}{\sigma_b + \lambda_b}\right)$$

where the second inequality holds since $k_2 \leq \lambda_a$ and the last inequality follows from that $\lambda_a \geq \lambda_b$. Therefore, by Lemma 6.3 with $k^* = k_2$ and $\eta = \lambda_a$, we have that

$$\sum_{k=0}^{k_2 - \sigma - 1} g(k) = O(\lambda_a) e^{-\frac{\sigma_b^2}{\sigma_b + \lambda_a}} + O(\lambda_a^2) e^{-\frac{\sigma_b^2}{\sigma_b + \lambda_b}}.$$ 

This proves the first inequality of the lemma. The second one is analogous.

Proposition 4.2 easily follows from Lemmas 6.8 and 6.10 by defining $\sigma$ to be $\sigma_a$ or $\sigma_b$.

We are ready to bound the loss in the unbalanced case. The following theorem, together with Lemma 6.7, provides upper bounds on $L_a(\text{Patient}_2)$ and $L_b(\text{Patient}_2)$, which shows the first part of Theorem 3.2

**Theorem 6.11.** For a bipartite matching market $(d_a, d_b, p)$ with $d_a \geq d_b$, let $\pi$ be the stationary distribution of the Markov chain of Patient$_2$. Then it holds that

$$E_{(A, B)} \sim \pi[A(1 - p)^B] \leq (1 + o(1)) \left(\frac{d_a - d_b}{d_a} + \frac{\log(d_b + 3)}{d_a}\right) + o(1),$$

$$E_{(A, B)} \sim \pi[B(1 - p)^A] \leq (1 + o(1)) e^{-\max\left\{d_a - d_b, \frac{d_a}{\lambda_b}\right\}} + o(1).$$

**Proof.** Define $\sigma_a = \Theta(\sqrt{\lambda_a \log(\lambda_a \lambda_b)})$ and $\sigma_b = \Theta(\sqrt{\lambda_b \log(\lambda_a \lambda_b)})$. Let $S = \{(i, j) \mid k_2 - \sigma_a \leq i \leq \lambda_a + \sigma_a, k_2 - \sigma_b \leq j \leq \lambda_b + \sigma_b\}$. It follows that

$$E_{(A, B)} \sim \pi[A(1 - p)^B] \leq \sum_{(i, j) \in S} i(1 - p)^j \pi(i, j) + \sum_{(i, j) \notin S} i \pi(i, j).$$

(17)

The second term is bounded by

$$\sum_{(i, j) \notin S} i \pi(i, j) \leq (\lambda_a + \sigma_a) \sum_{(i, j) \notin S} \pi(i, j) + \sum_{i \geq \lambda_a + \sigma_a + 1} \sum_{j \geq 0} i \sum_{(i, j) \notin S} \pi(i, j).$$

By the definitions of $\sigma_a$ and $\sigma_b$, this is upper-bounded by a constant by Proposition 4.2 with (14). Moreover, the first term of (17) is bounded by the definition of $S$ as follows:

$$\sum_{(i, j) \in S} i(1 - p)^j \pi(i, j) \leq \max_{(i, j) \in S} i(1 - p)^j \leq (\lambda_a + \sigma_a)(1 - p)^{k_2 - \sigma_a}.$$
Since $\ell_2 > \ell_2 - \sigma_a$ by Lemma [6.9] we have
\[
(1 - p)^{\ell_2 - \sigma_b} \leq (1 - p)^{\ell_2 - \sigma_a - \sigma_b} \leq \frac{(1 - p)^{\ell_2}}{1 - \sigma_a \rho - \sigma_b \rho} \leq (1 + o(1))(1 - p)^{\ell_2}.
\]
Since $\lambda_a (1 - p)^{\ell_2} = \lambda_a - \lambda_b + \ell_2$ by the definition of $\ell_2$, we have
\[
\mathbb{E}_{(A, B) \sim \pi}[A(1 - p)^B] \leq (1 + o(1))(\lambda_a + \sigma_a)(1 - p)^{\ell_2} + O(1)
\]
\[
\leq (1 + o(1))(\lambda_a - \lambda_b + \ell_2 + o(\lambda_a)) + O(1).
\]
We note that, since $\lambda_a = \frac{d_a}{\sigma_a} \lambda_b$ and $d_a, d_b$ are constants, we may assume that $\lambda_a \leq C \lambda_b$ for some constant $C$, and hence $\sigma_a = o(\lambda_a)$. Therefore, since $\ell_2 \leq \log(d_b + 3)/p$ by Lemma [6.5] it holds that
\[
\mathbb{E}_{(A, B) \sim \pi}[A(1 - p)^B] \leq (1 + o(1)) \left(\frac{d_a - d_b}{d_a} + \frac{\log(d_b + 3)}{d_a}\right) + o(1).
\]
Similarly, it follows that
\[
\mathbb{E}_{(A, B) \sim \pi}[B(1 - p)^A] \leq \sum_{(i, j) \in S} j(1 - p)^j \pi(i, j) + \sum_{(i, j) \notin S} j \pi(i, j) \leq \max_{(i, j) \in S} (j(1 - p)^i) + O(1)
\]
\[
\leq (\lambda_b + \sigma_b)(1 - p)^{\ell_2 - \sigma_b} + O(1),
\]
where the second inequality follows from Proposition [4.2] with (15). It follows from Lemma [6.9] that $(1 - p)^{\ell_2 - \sigma_a} = (1 + o(1))(1 - p)^{\ell_2}$. Moreover, we have $(1 - p)^{\ell_2} \leq e^{-\ell_2} \leq e^{-\max\left\{d_a - d_b, \frac{d_b}{\sigma_a}\right\}}$ by Lemma [6.5]. Hence, we obtain
\[
\mathbb{E}_{(A, B) \sim \pi}[B(1 - p)^A] \leq (1 + o(1))(\lambda_b + \sigma_b)(1 - p)^{\ell_2} + O(1)
\]
\[
\leq (1 + o(1))(\lambda_b + o(\lambda_b)) e^{-\max\left\{d_a - d_b, \frac{d_b}{\sigma_a}\right\}} + O(1).
\]
Therefore, it holds that
\[
\mathbb{E}_{(A, B) \sim \pi}[B(1 - p)^A] \leq (1 + o(1)) e^{-\max\left\{d_a - d_b, \frac{d_b}{\sigma_a}\right\}} + o(1).
\]
This completes the proof.

\section*{6.3.2 Loss of the Balanced Case: Proofs of Propositions [4.3] and [4.4]}

We here bound the loss of Patient$_2$ when $\lambda_a = \lambda_b$. To this end, we first present further concentration bounds on the sum of the pool sizes, which follows even when $\lambda_a > \lambda_b$. The proof can be done by taking another balance equation with $X = \{(i, j) \mid i + j \geq 0, j \geq 0\}$ for any $h \geq 0$.

\textbf{Proposition 4.3.} For any $\sigma \geq 1$, it holds that
\[
\Pr_{(A, B) \sim \pi} \left[A + B \leq \left(\frac{\lambda_a + \lambda_b}{2} - 2\right) - \sigma - 1\right] \leq O(\lambda_a + \lambda_b) e^{-\frac{\sigma^2}{\lambda_a + \lambda_b}}.
\]

\textbf{Proof.} For any $h \geq 0$, the balance equation [8] with $X = \{(i, j) \mid i + j \leq h, i \geq 0, j \geq 0\}$ implies that
\[
(\lambda_a + \lambda_b) \sum_{i, j : i + j = h} \pi(i, j) = (h + 1) \sum_{i, j : i + j = h + 1} \pi(i, j) + \sum_{i, j : i + j = h + 2} (h + 2 - i(1 - p)^j) \pi(i, j).
\]
Hence we have
\[
(\lambda_a + \lambda_b) \sum_{i, j : i + j = h} \pi(i, j) \leq (2h + 3) \max \left\{\sum_{i, j : i + j = h + 1} \pi(i, j), \sum_{i, j : i + j = h + 2} \pi(i, j)\right\}.
\]
We denote $g(h) = \sum_{i, j : i + j = h} \pi(i, j)$ for $h \geq 0$. Then the above inequality can be written as
\[
(\lambda_a + \lambda_b) g(h) \leq (2h + 3) \max \{g(h + 1), g(h + 2)\}.
\]
We first observe that the function 
\[
g(h) = \frac{g(n_0)g(n_1)\ldots g(n_{\ell})}{g(n_{\ell})} \leq \exp \left( -\sum_{j=0}^{\ell-1} \frac{m'-n_j}{\lambda_a + \lambda_b} \right)
\]
\[
\leq \exp \left( -\sum_{j=0}^{m'-h} \frac{2j}{\lambda_a + \lambda_b} \right) = \exp \left( -\frac{1}{\lambda_a + \lambda_b} \frac{(m'-h)^2}{4} \right).
\]

Therefore, for any \( \sigma \geq 1 \), it holds that
\[
\sum_{h=0}^{m'-\sigma} g(h) \leq \sum_{h=0}^{m'-\sigma} \exp \left( -\frac{1}{\lambda_a + \lambda_b} \frac{(m'-h)^2}{4} \right) \leq \sum_{h=\sigma}^{\infty} \exp \left( -\frac{1}{\lambda_a + \lambda_b} \frac{h^2}{4} \right)
\]
\[
\leq \exp \left( -\frac{\sigma^2}{4(\lambda_a + \lambda_b)} \right) = O(\lambda_a + \lambda_b) \exp \left( -\frac{\sigma^2}{4(\lambda_a + \lambda_b)} \right),
\]

where the third inequality follows from Lemma A.1 in Section A. 

In what follows, we assume that \( \lambda_a = \lambda_b \) and \( \sigma_a = \sigma_b = O(\sqrt{\lambda_a \log \lambda_a}) \). Then \( (\lambda_a + \lambda_b)/2 = \lambda_a \).

**Proposition 4.4.** Suppose that \( d_a = d_b \geq 3 \) and \( p < 1/10 \). Suppose that \( \sigma_a \) satisfies that \( 1 \leq \sigma_a \leq \lambda_a \). For any \( \sigma_d \geq 1 \), it holds that
\[
\Pr_{(A,B) \sim \pi} \left[ A - B \geq \frac{\lambda_a + \sigma_d}{2} + \sigma_d \right] \leq \frac{e^{-c_a \sigma_d} + O(\lambda_a) e^{-\frac{\sigma_d^2}{4(\lambda_a + \lambda_b)}}}{1 - e^{-c_a \sigma_d}}.
\]

for some constant \( 0 < c_a < 1 \).

**Proof.** We denote \( \overline{\lambda_a} = \lambda_a + \sigma_a \) and \( \overline{\lambda_b} = \lambda_b + \sigma_b \). Note that \( \overline{\lambda_d} = \overline{\lambda_b} \) by the assumption. For any \( z \geq \overline{\lambda_a}/2 \), consider the balance equation with \( X = \{(i, j) \mid i = j + z, i \geq 0, j \geq 0\} \), implying that
\[
\sum_{i,j:i=j+z} (\lambda_a + j(1-p)^z) \pi(i, j) = \sum_{i,j:i=j+z} (\lambda_b + i(1-p)^z) \pi(i, j).
\]

For \( z \geq \overline{\lambda_a}/2 \), define
\[
M_z = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid i = j + z, 0 \leq j \leq \overline{\lambda_b}, z \leq i \leq \overline{\lambda_a}\},
\]
\[
M_{z}^c = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid i = j + z, (i, j) \notin M_z\}.
\]

We also define
\[
g(z) = \sum_{(i,j) \in M_z} \pi(i, j) \quad \text{and} \quad \beta_z = (\lambda_a + \lambda_b) \sum_{(i,j) \in M_z} \pi(i, j).
\]

We first observe that the function \( f(j) = j(1-p)^z \) is maximized when \( j = \frac{1}{1-p} \log \frac{1}{1-p} \), whose value is at most \( 1/(pe) \).

This implies that, for \( (i, j) \in M_z \), \( \lambda_a + j(1-p)^z \leq \lambda_a + \frac{1}{pe} (1-p)^z \leq \lambda_a + \lambda_b \). Hence the LHS of (18) is at most
\[
\left( \lambda_a + \frac{1}{pe} (1-p)^z \right) g(z) + \beta_z \leq \left( \lambda_a + \frac{1}{pe} (1-p)^{\overline{\lambda_a}/2} \right) g(z) + \beta_z
\]
then \( z \geq \overline{\lambda_a}/2 \). Moreover, since the function \( i(1-p)^{i-z} \) is minimized when \( i = \overline{\lambda_a} \) for \( z \leq i \leq \overline{\lambda_a} \), the RHS of (18) is at least
\[
\left( \lambda_b + \overline{\lambda_a}(1-p)^{\overline{\lambda_a}-z} \right) g(z+1) \geq \left( \lambda_b + \overline{\lambda_a}(1-p)^{\overline{\lambda_a}/2} \right) g(z+1)
\]
We now estimate $\Pr$.

We now observe that

we obtain

since $\lambda_b + \lambda_a(1-p)\bar{\lambda}_a^{-z} \geq \lambda_b$.

**Claim 1.** There exists some constant $c_d > 0$ such that $\alpha_z < 1 - c_d$ for any $z \geq \bar{\lambda}_a/2$.

**Proof of Claim** It holds that

$$\alpha_z = 1 - \frac{\lambda_a - \frac{1}{pe}}{\lambda_b + \lambda_a(1-p)\bar{\lambda}_a^{-z}/2}(1-p)\bar{\lambda}_a/2.$$  

We now observe that

$$\lambda_a - \frac{1}{pe} \geq \lambda_a \left(1 - \frac{1}{d_a e}\right),$$

$$\lambda_b + \lambda_a(1-p)\bar{\lambda}_a^{-z} \leq \lambda_b + \lambda_a e^{-p\lambda_a/2} \leq \lambda_b + \frac{\lambda_a}{2} \leq \lambda_a + \lambda_b,$$

$$(1-p)\bar{\lambda}_a/2 \leq e^{-\lambda_a(p+p^2)} - e^{-(d_a + d_a)p} \geq e^{-2d_a},$$

where the second one follows since $e^{-d_a/2} \leq 1/2$ if $d_a \geq 3$ and $\bar{\lambda}_a \leq 2\lambda_a$ and the last one follows since $p < 1/10$. 

Hence, we obtain

$$\alpha_z \leq 1 - \frac{1}{2}d_a e^{-2d_a}.$$  

Therefore, there exists some constant $c_d > 0$ such that $\alpha_z < 1 - c_d$ for any $z \geq \bar{\lambda}_a/2$. \hfill $\Box$

Applying the inequality repeatedly, we have

$$g(z + 1) \leq \left(\prod_{i=\bar{\lambda}_a/2}^{\bar{\lambda}_a} \alpha_i\right) g(k^*) + \frac{1}{\lambda_b} \sum_{i=\bar{\lambda}_a/2}^{\bar{\lambda}_a} \left(\prod_{j=i+1}^{\bar{\lambda}_a} \alpha_j\right) \beta_i \leq (1 - c_d)^{z-\bar{\lambda}_a/2} + \frac{1}{\lambda_b} \sum_{i=\bar{\lambda}_a/2}^{\bar{\lambda}_a} \beta_i.$$  

Since

$$\sum_{i=\bar{\lambda}_a/2}^{\bar{\lambda}_a} \beta_i \leq (\lambda_a + \lambda_b) \sum_{i=\bar{\lambda}_a/2}^{\bar{\lambda}_a} \sum_{i,j \in M} \pi(i, j) \leq (\lambda_a + \lambda_b) \sum_{i \geq \bar{\lambda}_a} \sum_{j=0}^{\infty} \pi(i, j),$$

and $\lambda_a = \lambda_b$, we have

$$g(z + 1) \leq (1 - c_d)^{z-\bar{\lambda}_a/2} + 2M,$$

where $M = \sum_{i=\bar{\lambda}_a}^{\infty} \sum_{j=0}^{\infty} \pi(i, j)$.

We now estimate $\Pr_{(A,B) \sim \pi} \left[A - B \geq \bar{\lambda}_a^2 + \sigma_d\right]$ for $\sigma_d \geq 1$, which is denoted by $P$ for simplicity. It holds that

$$P \leq \sum_{z=\bar{\lambda}_a/2+\sigma_d}^{\bar{\lambda}_a} g(z) + \sum_{i=\bar{\lambda}_a}^{\infty} \sum_{j=0}^{\infty} \pi(i, j) = \sum_{z=\bar{\lambda}_a/2+\sigma_d}^{\bar{\lambda}_a} g(z) + M.$$  

From the above discussion, we have

$$\sum_{z=\bar{\lambda}_a/2+\sigma_d}^{\bar{\lambda}_a} g(z) \leq \sum_{z=\bar{\lambda}_a/2+\sigma_d}^{\bar{\lambda}_a} (1 - c_d)^{z-\bar{\lambda}_a/2} + \bar{\lambda}_a M.$$
We observe that
\[
\sum_{z=\bar{\lambda}_a/2+\sigma_d}^{\bar{\lambda}_a} (1-c_d)z^{-\bar{\lambda}_a/2} \leq \sum_{z=\bar{\lambda}_a/2+\sigma_d}^{\bar{\lambda}_a} e^{-c_d(z-\bar{\lambda}_a/2)} \leq \sum_{z=\sigma_d}^{\infty} e^{-c_dz} \leq \frac{e^{-c_d\sigma_d}}{1-e^{-c_d}}.
\]
Therefore, we have
\[
P \leq \frac{e^{-c_d\sigma_d}}{1-e^{-c_d}} + (\bar{\lambda}_a + 1)M.
\]
By Lemma 6.8 it holds that \(M = O(\lambda_a) e^{-\sigma_d^2/2\lambda_a}.\) Thus the lemma holds.

The above lemma implies that, if we set \(\sigma_d = \frac{1}{c_d} \log \lambda_a\), then \(\frac{e^{-c_d\sigma_d}}{1-e^{-c_d}} = O(1/\lambda_a).\)

Recall Proposition 4.2 that, for \(\sigma_a, \sigma_b \geq 1\), define \(S = \{(i, j) \mid \bar{\lambda}_2 - \sigma_a \leq i \leq \lambda_a + \sigma_a, \bar{\lambda}_2 - \sigma_b \leq j \leq \lambda_b + \sigma_b\}.

**Theorem 6.12.** For a bipartite matching market \((d_a, d_b, p)\) with \(d_a = d_b\), let \(\pi\) be the stationary distribution of the Markov chain of Patient. Then it holds that
\[
\frac{\mathbb{E}_{(A,B) \sim \pi}[A(1-p)^B]}{\lambda_a} = \frac{\mathbb{E}_{(A,B) \sim \pi}[B(1-p)^A]}{\lambda_b} \leq (1+o(1))e^{-Cd_a} + o(1)
\]
for some constant \(C\).

**Proof.** Define \(\sigma_a = \sigma_b = \Theta(\sqrt{\lambda_a \log \lambda_a}), \sigma_h = \Theta(\sqrt{\lambda_a \log \lambda_a}),\) and \(\sigma_d = \frac{1}{c_d} \log \lambda_a\) where \(c_d\) is a constant in Proposition 4.4.

Let
\[
S' = \{(i, j) \in S \mid \lambda_a - 2 - \sigma_h \leq i + j, i - \bar{\lambda}_a/2 - \sigma_d \leq j \leq i + \bar{\lambda}_a/2 + \sigma_d\}
\]

It follows that
\[
\frac{\mathbb{E}_{(A,B) \sim \pi}[A(1-p)^B]}{\lambda_a} \leq \sum_{(k,j) \in S'} (k(1-p)^j)\pi(k,j) + \sum_{(k,j) \notin S'} k\pi(k,j). \tag{19}
\]

The second term is bounded by
\[
\sum_{(k,j) \notin S} k\pi(k,j) \leq (\lambda_a + \sigma_a) \sum_{(k,j) \notin S'} \pi(k,j) + \sum_{k \geq \lambda_a + \sigma_a + 1} k \sum_{j \geq 0} \pi(k,j).
\]

By the definitions of \(\sigma_a, \sigma_b\) and \(\sigma_h\), this is upper-bounded by a constant by Propositions 4.2, 4.3, and 4.4 with \(\lambda_a\) and \(\lambda_b\).

By the definition of \(S'\), for any \((i, j) \in S\), we have
\[
\frac{1}{2} \left( \lambda_a - 2 - \sigma_h - \sigma_d - \bar{\lambda}_a/2 \right) \leq i \leq \bar{\lambda}_a.
\]

Let \(z = \frac{1}{2}(\lambda_a - 2 - \sigma_h - \sigma_d - \bar{\lambda}_a/2)\). Since \(\sigma_h = \Theta(\sqrt{\lambda_a \log \lambda_a})\) and \(\sigma_d = O(\log \lambda_a)\), it holds that \(z \geq C\lambda_a\) for some constant \(C\). Hence the first term of \(19\) is bounded as follows.
\[
\sum_{(k,j) \in S} k(1-p)^j\pi(k,j) \leq \max_{(k,j) \in S} (k(1-p)^j) \leq \bar{\lambda}_a(1-p)^z \leq \lambda_a e^{-Cd} + o(\lambda_a).
\]

Therefore,
\[
\frac{\mathbb{E}_{(A,B) \sim \pi}[A(1-p)^B]}{\lambda_a} \leq e^{-Cd} + o(1).
\]
This completes the proof. 

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6.4 Greedy₁

The proof outline is similar to the 2-sided Greedy algorithm. However, 1-sided algorithms require a more involved analysis to show the concentration.

Recall that, in Greedy₁, we denote by \( A_t \) the pool size of inactive agents at time \( t \), and by \( B_t \) the pool size of greedy agents at time \( t \). Similarly to the 2-sided Greedy algorithms, it suffices to bound the expected pool sizes \( E_{(A,B) \sim \pi}[A] \) and \( E_{(A,B) \sim \pi}[B] \) as below. The proof of Lemma 6.13 is presented in Section 6.6.1.

**Lemma 6.13.** For a bipartite matching market \((d_a, d_b, p)\), let \( \pi \) be the stationary distribution of the Markov chain of Greedy₁. For any \( \varepsilon > 0 \) and \( T > 0 \), it holds that

\[
L_a(\text{Greedy}_1) \leq \frac{E_{(A,B) \sim \pi}[A]}{\lambda_a} + \frac{\tau_{\text{mix}}(\varepsilon)}{T} + 6\varepsilon + \frac{1}{\lambda_a} 2^{-6m},
\]

\[
L_b(\text{Greedy}_1) \leq \frac{E_{(A,B) \sim \pi}[B]}{\lambda_b} + \frac{\tau_{\text{mix}}(\varepsilon)}{T} + 6\varepsilon + \frac{1}{\lambda_b} 2^{-6m},
\]

where \( \tau_{\text{mix}}(\varepsilon) \) is the mixing time of the Markov chain.

**Markov Chain** We here define a Markov chain \((A_t, B_t)\) on \( \mathbb{Z}_+ \times \mathbb{Z}_+ \) for the 1-sided Greedy algorithms. In contrast to Greedy₂, agents in \( U_t \) are inactive, and hence new agents in \( U_t \) enter the market without making a pair. Specifically, for any pair of pool sizes \((k, j)\), the Markov chain transits only to the states \((k + 1, j), (k, j + 1), (k - 1, j), \) and \((k, j - 1)\). See Figure 4. It transits to \((k + 1, j)\) when a new agent arrives in \( U_t \), to \((k, j + 1)\) when a new agent arrives in \( V_t \) and she does not get matched. Moreover, it transits to \((k - 1, j)\) when either a new agent arrives in \( V_t \) and she gets matched with some agent in \( U_t \), or some agent in \( U_t \) perishes, and to \((k, j - 1)\) when some agent in \( V_t \) perishes. Therefore, we have

\[
r_{(k,j)\to(k+1,j)} = \lambda_a,
\]

\[
r_{(k,j)\to(k,j+1)} = \lambda_b(1 - p)^k,
\]

\[
r_{(k,j)\to(k-1,j)} = k + \lambda_b(1 - (1 - p)^k),
\]

\[
r_{(k,j)\to(k,j-1)} = j.
\]

**Concentration of Pool Sizes** We show that the pool size \( A_t \) of inactive agents in the steady state is highly concentrated around some value \( k_1 \). On the other hand, letting \( \ell_1 = (1 - p)^{-\sigma_a} (\lambda_b - \lambda_a + k_1) \) for \( \sigma_a \geq 1 \), the probability that the pool size \( B_t \) of greedy agents in the steady state is larger than \( \ell_1 \) is small. More formally, we prove the following proposition.

**Proposition 4.5.** There exists a number \( k_1 \), where \( k_1 \leq \max\{\lambda_a - \lambda_b, 0\} + \frac{\lambda_b \log(d_b+3)}{d_b} \), such that, for any \( \sigma_a \geq 1 \),

\[
\Pr_{(A,B) \sim \pi} [A \geq k_1 + \sigma_a + 1] \leq O(\lambda_a) e^{-\frac{\sigma^2_a}{2\lambda_a}},
\]

\[
\Pr_{(A,B) \sim \pi} [A \leq k_1 - \sigma_a - 1] \leq O(\lambda_a) e^{-\frac{\sigma^2_a}{2\lambda_a}},
\]

and, for any \( \sigma_a, \sigma_b \geq 1 \), we have

\[
\Pr_{(A,B) \sim \pi} [B \geq \ell_1 + \sigma_b + 1] \leq O(\lambda_b) e^{-\frac{\sigma^2_a}{2\lambda_b} + \frac{\sigma^2_b}{2\lambda_b}} + O(\lambda_a \lambda_b^2 e^{-\frac{\sigma^2_a}{2\lambda_a}}),
\]

where \( \ell_1 = (1 - p)^{-\sigma_a} (\lambda_b - \lambda_a + k_1) \).

The first inequality of Proposition 4.5 can be shown in a similar way to Proposition 4.1. Under Greedy₁, we further show that \( \Pr_{(A,B) \sim \pi} [A \leq k_1 - \sigma_a - 1] \) also decreases exponentially. Thus the pool size \( A_t \) of inactive agents in the steady state is highly concentrated around \( k_1 \). Since \( A_t \) is not so small in the steady state, a greedy agent in \( V_t \) is likely to match an agent in \( U_t \), and hence the pool size \( B_t \) of greedy agents becomes small in the steady state. This intuition can be confirmed by taking the balance equations carefully. We remark that we need to solve recursive equations with additive terms. See Section 6.4.1 for the details.

If we set \( \sigma_a = \Theta(\sqrt{\lambda_a \log(\lambda_a \lambda_b)}) \) and \( \sigma_b = \Theta(\sqrt{\lambda_b \log \lambda_b}) \), Proposition 4.5 together with Lemma 6.13 implies that

\[
L_a(\text{Greedy}_1) \approx \frac{1}{\lambda_a} E_{(A,B) \sim \pi}[A] \leq \frac{1}{\lambda_a} (k_1 + \sigma_a) + o(1),
\]

\[
L_b(\text{Greedy}_1) \approx \frac{1}{\lambda_b} E_{(A,B) \sim \pi}[B] \leq \frac{1}{\lambda_b} (\ell_1 + \sigma_b) + o(1).
\]

This shows Theorem 3.4 for Greedy₁.
6.4.1 Concentration of Pool Sizes: Proof of Proposition 4.5

The goal of this subsection is to show Proposition 4.5. We first show the first part of Proposition 4.5 as below.

Lemma 6.14. For any \( \sigma_a \geq 1 \), it holds that

\[
\Pr_{(A,B) \sim \pi} [A \geq k_1 + \sigma_a + 1] \leq O(\lambda_a) e^{-\frac{\sigma_a^2}{\sigma_a + \lambda_a}} \quad \text{and} \quad \Pr_{(A,B) \sim \pi} [A \leq k_1 - \sigma_a - 1] \leq O(\lambda_a) e^{-\frac{\sigma_a^2}{\sigma_a}}.
\]

Proof. By the balance equation (8) with \( X = \{(i,j) \mid 0 \leq i \leq k, j \geq 0\} \) for any \( k \geq 0 \), it holds that

\[
\lambda_a \sum_{j=0}^{\infty} \pi(k,j) = (k + 1 + \lambda_b(1 - (1 - p)^{k+1})) \sum_{j=0}^{\infty} \pi(k+1,j).
\]

Hence we obtain

\[
\frac{\sum_{j=0}^{\infty} \pi(k+1,j)}{\sum_{j=0}^{\infty} \pi(k,j)} \leq \frac{\lambda_a}{k + 1 + \lambda_b(1 - (1 - p)^{k+1})}.
\]

Recall that \( k_1 \) satisfies \( \lambda_a = k_1 + \lambda_b(1 - (1 - p)^{k_1}) \). We can follow the proof of Proposition 4.1 using Lemma 6.1(9), which implies that, for any \( k \geq k_1 \) and \( \sigma_a \geq 1 \), we have

\[
\sum_{k=k_1+\sigma_a+1}^{\infty} \sum_{j=0}^{\infty} \pi(k,j) = O\left(\frac{\sigma_a + \lambda_a}{\sigma_a}\right) \exp\left(-\frac{\sigma_a^2}{\sigma_a + \lambda_a}\right).
\]

This proves the first part of the lemma as \( \sigma_a \geq 1 \).

Similarly, for \( k \leq k_1 \), we have by (20),

\[
\frac{\sum_{j=0}^{\infty} \pi(k-1,j)}{\sum_{j=0}^{\infty} \pi(k,j)} = \frac{k + \lambda_b(1 - (1 - p)^k)}{\lambda_a} \leq \frac{k - k_1 + \lambda_b(1 - (1 - p)^{k_1}) - \lambda_b(1 - (1 - p)^k) + \lambda_a}{\lambda_a} \leq 1 - \frac{k_1 - k}{\lambda_a} e^{-\frac{k_1 - k}{\lambda_a}}.
\]

Therefore, it follows from Lemma 6.2 with \( k^* = k_1 \) and \( \eta = \lambda_a \) that, for any \( \sigma_a \geq 1 \), we have

\[
\sum_{k=k_1+\sigma_a+1}^{\infty} \sum_{j=0}^{\infty} \pi(k,j) = O(\lambda_a) e^{-\frac{\sigma_a^2}{\sigma_a}}.
\]

This proves the second part of the lemma.

We remark that, by using Lemma 6.1(ii) with \( k^* = k_1 \) and \( \eta = \lambda_a - k_1 \) in the above proof, it follows that, if \( \sigma_a = O(\lambda_a) \),

\[
\sum_{k=k_1+\sigma_a+1}^{\infty} k \sum_{j=0}^{\infty} \pi(k,j) = O(\lambda_a^3) e^{-\frac{\sigma_a^2}{\sigma_a + \lambda_a}}.
\]

We next prove that, if the pool size \( A_t \) in the steady state is larger than about \( k_1 \), the pool size \( B_t \) cannot be larger than about \( \ell_1 \).

Lemma 6.15. For \( \sigma_a \geq 1 \), we define \( \ell_1 = \lambda_b(1 - p)^{k_1 - \sigma_a} \). Then, for any \( \sigma_b \geq 1 \), we have

\[
\Pr_{(A,B) \sim \pi} [B \geq \ell_1 + \sigma_b + 1, A \geq k_1 - \sigma_a] \leq O(\lambda_b) e^{-\frac{\sigma_a^2}{2(\sigma_a + \lambda_a)} + O(\lambda_a^2) e^{-\frac{\sigma_a^2}{\sigma_a}}}.
\]
We observe that, for $j \geq k$, Therefore, by Lemma 6.3 with $X = \{i, \ell \mid i \geq 0, 0 \leq \ell \leq j\}$ implies that
\[
\sum_{k=0}^{\infty} \lambda_b (1-p)^k \pi(k,j) = (j+1) \sum_{k=0}^{\infty} \pi(k,j+1).
\]
(22)
Define $K = k_1 - \sigma_b$. The LHS can be transformed as follows.
\[
\sum_{k=0}^{\infty} \lambda_b (1-p)^k \pi(k,j) = \sum_{k=0}^{\infty} \lambda_b (1-p)^k \pi(k,j) + \sum_{k=0}^{K-1} \lambda_b (1-p)^k \pi(k,j)
\]
\[
\leq \lambda_b (1-p)^K \sum_{k=0}^{\infty} \pi(k,j) + \lambda_b \sum_{k=0}^{K-1} \pi(k,j).
\]
To simplify the notation, define
\[
g(j) = \sum_{k=K}^{\infty} \pi(k,j), \quad \alpha_j = \frac{\lambda_b (1-p)^K}{j+1} \text{ and } \beta_j = \frac{\lambda_b \sum_{k=0}^{K-1} \pi(k,j)}{j+1}.
\]
Note that $\alpha_j = \ell_1/(j+1)$. Then (22) implies the following recursive relationship for any $j \geq 0$:
\[
g(j+1) \leq \alpha_j g(j) + \beta_j.
\]
We observe that, for $j \geq \ell_1 - 1$,
\[
\alpha_j = \frac{\ell_1}{j+1} = 1 - \frac{j - \ell_1 + 1}{j+1} \leq \exp \left( -\frac{j - \ell_1 + 1}{j+1} \right).
\]
Moreover, by Theorem 6.14
\[
\sum_{j=\ell_1}^{\infty} \beta_j \leq \frac{\lambda_b}{\ell_1 + 1} \sum_{k=0}^{\infty} \pi(k,j) \leq O(\lambda_a \lambda_b) e^{-\frac{\sigma_b^2}{\lambda_b}}.
\]
Therefore, by Lemma 6.3 with $k^* = \ell_1 - 1$ and $\eta = 1$, we have that
\[
\sum_{j=\ell_1 + \sigma_b + 1}^{\infty} g(j) = O(\ell_1) e^{-\frac{\sigma_b^2}{\pi_b \pi_a \theta}} + O(\ell_1 \lambda_a \lambda_b) e^{-\frac{\sigma_b^2}{\pi_a}} = O(\lambda_b) e^{-\frac{\sigma_b^2}{\pi_b \lambda_b}} + O(\lambda_b \lambda_b^2) e^{-\frac{\sigma_b^2}{\pi_a}},
\]
since $\ell_1 \leq \lambda_b$. This proves the lemma.

The above lemma, together with Lemma 6.14 implies the second part of Proposition 4.5.

The balance equation (22) in the proof of Lemma 6.15 implies the following, which will be used in bounding the loss.

Lemma 6.16. For any $\sigma_b \geq 1$, it holds that
\[
\sum_{j=\lambda_b + \sigma_b + 1}^{\infty} j \sum_{k=0}^{\infty} \pi(k,j) = O \left( \lambda_b^3 \right) e^{-\frac{\sigma_b^2}{\pi_b + \lambda_b}}.
\]

Proof. The balance equation (22) in the proof of Lemma 6.15 implies that
\[
\lambda_b \sum_{k=0}^{\infty} \pi(k,j) \geq (j+1) \sum_{k=0}^{\infty} \pi(k,j+1).
\]
Then, for $j \geq \lambda_b$, it holds that
\[
\sum_{k=0}^{\infty} \pi(k,j+1) \leq \sum_{k=0}^{\infty} \pi(k,j) \leq 1 - \frac{j - \lambda_b + 1}{j+1} \leq \exp \left( -\frac{j - \lambda_b + 1}{j+1} \right).
\]
By Lemma 6.1 with $k^* = \lambda_b - 1$ and $\eta = 1$, we have
\[
\sum_{j=\lambda_b + \sigma_b + 1}^{\infty} j \sum_{k=0}^{\infty} \pi(k,j) = O \left( \lambda_b^3 \right) \exp \left( -\frac{\sigma_b^2}{\sigma_b + \lambda_b} \right).
\]
6.4.2 Bounding the Loss of Greedy₁

It follows from Proposition 4.5 that the loss of Greedy₁ can be bounded. This proves Theorem 3.4 for Greedy₁. We remark that, since $\lambda_a = \frac{d_a}{d}$ and $d_a, d_b$ are constants, we may assume that $\lambda_a \leq C \lambda_b$ for some constant $C$.

**Theorem 6.17.** For a bipartite matching market $(d_a, d_b, p)$, let $\pi$ be the stationary distribution of the Markov chain of Greedy₁. Then, if $d_a \geq d_b$, it holds that

$$\frac{E(\pi B)}{\lambda_a} \leq \frac{d_a - d_b}{d_a} + \frac{\log(d_b + 3)}{d_a} + o(1),$$

$$\frac{E(\pi B)}{\lambda_b} \leq (1 + o(1)) \frac{\log(d_b + 3)}{d_b} + o(1).$$

If $d_a < d_b$, then

$$\frac{E(\pi A)}{\lambda_a} \leq \frac{\log(d_b + 3)}{d_a} + o(1),$$

$$\frac{E(\pi B)}{\lambda_b} \leq (1 + o(1)) \left( \frac{d_a - d_b}{d_b} + \frac{\log(d_b + 3)}{d_b} \right) + o(1).$$

**Proof.** Define $\sigma_a = \Theta \left( \sqrt{\lambda_a \log(\lambda_a \lambda_b)} \right)$ and $\sigma_b = \Theta \left( \sqrt{\lambda_b \log \lambda_b} \right)$. It follows that

$$E(\pi A) = \sum_{k \geq 0} k \sum_{j \geq 0} \pi(k, j) \leq k_1 + \sigma_a + \sum_{k \geq k_1 + \sigma_a + 1} k \sum_{j \geq 0} \pi(k, j).$$

Since $\sigma_a = O(\lambda_b)$, the last term is $O(1)$ by (21). Hence $E(\pi A)$ is upper-bounded by $k_1 + O(\sqrt{\lambda_a \log(\lambda_a \lambda_b)})$. Therefore, we obtain

$$\frac{E(\pi A)}{\lambda_a} \leq \frac{k_1}{\lambda_a} + o(1). \quad (23)$$

Similarly, it holds that

$$E(\pi B) = \sum_{j \geq 0} \sum_{k \geq 0} \pi(k, j) \leq \ell_1 + \sigma_b + (\lambda_b + \sigma_b) \sum_{j = \ell_1 + \sigma_b + 1}^{\infty} \sum_{k \geq j + \lambda_b + \sigma_a + 1} \pi(k, j) + o(1) \leq \ell_1 + O(\sqrt{\lambda_b \log \lambda_b}),$$

where the second inequality follows from Lemma 6.16 and the third inequality follows since the second term is a constant by Proposition 4.5. Since $\ell_1 = \lambda_b (1 - p)^{-1}$, we have

$$E(\pi B) \leq \lambda_b (1 - p)^{k_1 - \sigma_a} + o(\lambda_b).$$

We note that $\lambda_b (1 - p)^{k_1} = \lambda_b - \lambda_b + k_1$ by definition of $k_1$, and $(1 - p)^{-\sigma_a} \leq \frac{1}{1 - \sigma_a} = 1 + o(1)$. Hence we obtain

$$\frac{E(\pi B)}{\lambda_b} \leq (1 + o(1)) \frac{\lambda_b - \lambda_a + k_1}{\lambda_b} + o(1). \quad (24)$$

We now express (23) and (24) using $d_a$ and $d_b$ with Lemma 6.5. If $\lambda_a \geq \lambda_b$, then $k_1 = k_2$, and hence, by Lemma 6.5, we see that $k_1 = \lambda_b \lambda_a - \lambda_a + \frac{1}{p} \log(d_a + 3)$. Therefore, we have

$$\frac{E(\pi A)}{\lambda_a} \leq \frac{d_a - d_b}{d_a} + \frac{\log(d_b + 3)}{d_a} + o(1)$$

and

$$\frac{E(\pi B)}{\lambda_b} \leq (1 + o(1)) \frac{\log(d_b + 3)}{d_b} + o(1).$$

Otherwise, i.e., if $\lambda_a < \lambda_b$, then we see that $k_1 = \ell_2 \leq \frac{1}{p} \log(d_b + 3)$ by Lemma 6.5, implying that

$$\frac{E(\pi A)}{\lambda_a} \leq \frac{\log(d_b + 3)}{d_a} + o(1)$$

and

$$\frac{E(\pi B)}{\lambda_b} \leq (1 + o(1)) \left( \frac{d_a - d_b}{d_b} + \frac{\log(d_b + 3)}{d_b} \right) + o(1).$$

This completes the proof. \[\square\]
6.5 Patient₁

Recall that, in Patient₁, we denote by \( A_t \) the pool size of inactive agents at time \( t \), and by \( B_t \) the pool size of greedy agents at time \( t \). Similarly to the other algorithms, it suffices to bound the expected pool sizes in the steady state.

**Lemma 6.18.** For a bipartite matching market \((d_a, d_b, p)\), let \( \pi \) be the stationary distribution of the Markov chain of Patient₁. For any \( \epsilon > 0 \) and \( T > 0 \),

\[
\begin{align*}
L_a(\text{Patient}_1) &\leq \frac{\mathbb{E}_{(A,B) \sim \pi}[A]}{\lambda_a} + \frac{\tau_{\text{mix}}(\epsilon)}{T} + 6\epsilon + \frac{2-6\lambda_a}{\lambda_a}, \\
L_b(\text{Patient}_1) &\leq \frac{\mathbb{E}_{(A,B) \sim \pi}[B(1-p)^A]}{\lambda_b} + \frac{\tau_{\text{mix}}(\epsilon)}{T} + 6\epsilon + \frac{2-6\lambda_b}{\lambda_b},
\end{align*}
\]

where \( \tau_{\text{mix}}(\epsilon) \) is the mixing time of the Markov chain.

The proof of Lemma 6.18 is given in Section 6.6.2.

**Markov Chain** We here define a Markov chain \((A_t, B_t)\) on \( \mathbb{Z}_+ \times \mathbb{Z}_+ \) for the 1-sided Patient algorithms. In contrast to Patient₂, agents in \( U_t \) are inactive, and hence new agents in \( U_t \) stay in the market without making a pair. Specifically, for any pair of pool sizes \((k, j)\), the Markov chain transits only to the states \((k+1, j)\), \((k, j+1)\), \((k-1, j)\), \((k, j-1)\), and \((k-1, j-1)\). See Figure 5. It transits to \((k+1, j)\) or \((k, j+1)\) when a new agent arrives, to \((k-1, j)\) or \((k, j-1)\) when some agent in the pool leaves the market without getting matched to another agent, and to \((k-1, j-1)\) when some agent in \( V_t \) leaves the market with getting matched to another agent. Thus we have

\[
\begin{align*}
\mathbb{P}(r(k,j) \rightarrow (k+1,j)) &= \lambda_a, \\
\mathbb{P}(r(k,j) \rightarrow (k,j+1)) &= \lambda_b, \\
\mathbb{P}(r(k,j) \rightarrow (k-1,j)) &= k, \\
\mathbb{P}(r(k,j) \rightarrow (k-1,j-1)) &= \left(j(1-(1-p)^k)\right), \\
\mathbb{P}(r(k,j) \rightarrow (k,j-1)) &= \left(j(1-p)^k\right).
\end{align*}
\]

**Concentration of Pool Sizes** For Patient₁, we prove that the pool size \((A_t, B_t)\) in the steady state is highly concentrated around \((k_1, \lambda_b)\), where we recall that \( k_1 \) is defined to be the value that satisfies \( \lambda_a = k_1 + \lambda_b(1-(1-p)^{k_1}) \).

See Section 6.4

**Proposition 4.6.** For any \( \sigma_a, \sigma_b \geq 1 \), there exist \( \bar{k}_1 \) and \( \bar{k}_1 \) such that \( k_1 - \sigma_a < \bar{k}_1 < k_1 < \bar{k}_1 < \min\{k_1 + \sigma_b, \lambda_a\} \) and it holds that

\[
\Pr_{(A,B) \sim \pi} \left[(A,B) \not\in S\right] \leq O(\lambda_a)e^{-\frac{\sigma^2}{\pi_a \sigma_a}} + O(\lambda_a^2 \lambda_b + \lambda_b^3) e^{-\frac{\sigma^2}{\pi_b \lambda_b}} ,
\]

where \( S = \{(k,j) \mid \bar{k}_1 - \sigma_a \leq k \leq \bar{k}_1 + \sigma_a, \lambda_b - \sigma_b \leq j \leq \lambda_b + \sigma_b\} \).

To prove Proposition 4.6, we first show that the pool size \( B_t \) of patient agents is highly concentrated around \( \lambda_b \) in the steady state. This means the pool size of patient agents remains large in the steady state. Therefore, since many patient agents attempt to match inactive agents, the number of inactive agents becomes small, which concentrates around \( k_1 \). The details are given in Section 6.5.1.

If we set \( \sigma_a = \Theta(\sqrt{\lambda_a \log \lambda_a}) \) and \( \sigma_b = \Theta(\sqrt{\lambda_b \log(\lambda_b \lambda_a)}) \), then Proposition 4.6 and Lemma 6.18 imply that

\[
\begin{align*}
L_a(\text{Patient}_1) &\leq \frac{1}{\lambda_a} \mathbb{E}_{(A,B) \sim \pi}[A] \leq \frac{1}{\lambda_a} (k_1 + o(\lambda_a)) , \\
L_b(\text{Patient}_1) &\leq \frac{1}{\lambda_b} \mathbb{E}_{(A,B) \sim \pi}[B(1-p)^A] \leq \frac{1}{\lambda_b} \left(\lambda_b(1-p)^{k_1} - o(\lambda_b)\right) ,
\end{align*}
\]

which shows Theorem 3.4 for Patient₁. See Section 6.5.2 for the complete proof.

### 6.5.1 Concentration of Pool Sizes: Proof of Proposition 4.6

In this subsection, we show Proposition 4.6

For \( \sigma_b \geq 1 \), define \( \bar{k}_1 \) by the value that satisfies \( \lambda_a = \bar{k}_1 + (\lambda_b - \sigma_b)(1-(1-p)^{\bar{k}_1}) \), and \( \bar{k}_1 \) by the value that satisfies \( \lambda_a = \bar{k}_1 + (\lambda_b + \sigma_b)(1-(1-p)^{\bar{k}_1}) \). Note that \( \bar{k}_1 \) and \( \bar{k}_1 \) depend on \( \sigma_b \).

We first observe the relationship between \( \bar{k}_1, \bar{k}_1 \), and \( k_1 \). By definition, \( \bar{k}_1 > k_1 > \bar{k}_1 \) holds.
We remark that, using Lemma 6.1 (ii) to (25) in the proof of Lemma 6.20, we obtain which proves the second part of the lemma.

Therefore, by Lemma 6.1 (i) with The inequality for \( \sigma \) proves the first part of the lemma as

Define a function \( f(k) = k + (\lambda_b - \sigma_b)(1 - (1 - p)^k) - \lambda_a \), which is a non-decreasing function. Note that \( f(\overline{k}_1) = 0 \). Since \( f(k_1 + \sigma_b) = \sigma_b(1 - p)^{k_1} > 0 \) from the definition of \( k_1 \), we obtain \( \overline{k}_1 \leq k_1 + \sigma_b \). Since \( f(\lambda_a) > 0 \) and \( f(k_1) < 0 \), we also have \( k_1 \leq \overline{k}_1 \leq \lambda_a \).

The inequality for \( \overline{k}_1 \) holds in a similar way. If we define a function \( f'(k) \) by \( k + (\lambda_b + \sigma_b)(1 - (1 - p)^k) - \lambda_1 \), then \( f' \) is a non-decreasing function with \( f'(k_1) = 0 \), and we have \( f'(k_1 - \sigma_b) = -\sigma_b(1 - p)^{k_1} < 0 \). Also, \( \overline{k}_1 \leq k_1 \) holds since \( f'(k_1) = \sigma_b(1 - (1 - p)^k) > 0 \).

In the rest of this section, we prove \([6]\) in Proposition 4.6. We first show that the pool size \( B_t \) of patient agents in the steady state is highly concentrated around \( \lambda_b \).

**Lemma 6.20.** For any \( \sigma_b \geq 1 \), we have

\[
\Pr_{(A, B) \sim \pi} [B \geq \lambda_b + \sigma_b + 1] \leq O(\lambda_b) e^{-\frac{\sigma_b^2}{2\lambda_b}} \quad \text{and} \quad \Pr_{(A, B) \sim \pi} [B \leq \lambda_b - \sigma_b - 1] \leq O(\lambda_b) e^{-\frac{\sigma_b^2}{2\lambda_b}}.
\]

**Proof.** By the balance equation (8) with \( X = \{(i, \ell) \mid i \geq 0, 0 \leq \ell \leq j\} \) for \( j \geq 0 \), it holds that

\[
\lambda_b \sum_{k=0}^{\infty} \pi(k, j) = (j + 1) \sum_{k=0}^{\infty} \pi(k, j + 1). \tag{25}
\]

If \( j \geq \lambda_b - 1 \), then we obtain

\[
\frac{\sum_{k=0}^{\infty} \pi(k, j+1)}{\sum_{k=0}^{\infty} \pi(k, j)} = \frac{\lambda_b}{j+1} = 1 - \frac{j - \lambda_b + 1}{j + 1} \leq \exp \left( -\frac{j - \lambda_b + 1}{j + 1} \right).
\]

Therefore, by Lemma 6.1(i) with \( k^* = \lambda_b - 1 \) and \( \eta = 1 \), we have, for any \( \sigma_b \geq 1 \),

\[
\sum_{j=\lambda_b+\sigma_b+1}^{\infty} \sum_{k=0}^{\infty} \pi(k, j) = O \left( \frac{\lambda_b + \sigma_b}{\sigma_b} \right) \exp \left( -\frac{\sigma_b^2}{2\lambda_b} \right).
\]

This proves the first part of the lemma as \( \sigma_b \geq 1 \).

Moreover, if \( j \leq \lambda_b \), it holds by (25) that

\[
\frac{\sum_{k=0}^{\infty} \pi(k, j-1)}{\sum_{k=0}^{\infty} \pi(k, j)} = \frac{j}{\lambda_b} = 1 - \frac{\lambda_b - j}{\lambda_b} \leq \exp \left( -\frac{\lambda_b - j}{\lambda_b} \right).
\]

By Lemma 6.2 with \( k^* = \eta = \lambda_b \), for any \( \sigma_b \geq 1 \),

\[
\sum_{j=0}^{\lambda_b-\sigma_b-1} \sum_{k=0}^{\infty} \pi(k, j) = O(\lambda_b) e^{-\frac{\sigma_b^2}{2\lambda_b}},
\]

which proves the second part of the lemma.

We remark that, using Lemma 6.1(ii) to (25) in the proof of Lemma 6.20 we obtain

\[
\sum_{j=\lambda_b+\sigma_b+1}^{\infty} \sum_{k=0}^{j} \pi(k, j) = O \left( \frac{\lambda_b^2}{\sigma_b} \right) \exp \left( -\frac{\sigma_b^2}{2(\lambda_b + \lambda_a)} \right).
\]

We next prove that, if \( B_t \) is larger than about \( \lambda_b \), the pool size \( A_t \) cannot be larger than about \( \overline{k}_1 \) in the steady state.

**Lemma 6.21.** For any \( \sigma_a \geq 1 \) and \( \sigma_b \geq 1 \), we have

\[
\Pr_{(A, B) \sim \pi} [A \geq \overline{k}_1 + \sigma_a + 1, B \geq \lambda_b - \sigma_b] \leq O(\lambda_a) e^{-\frac{\sigma_b^2}{2(\lambda_a + \lambda_b)}} + O(\lambda_a^2 \lambda_b) e^{-\frac{\sigma_a^2}{2\lambda_a}}.
\]
We observe that

\[ \sum_{j=0}^\infty k^j \leq \frac{1}{1-k} \]

Therefore, by applying Lemma 6.3 with \( k \), we have

\[ (k + 1 + (\lambda_b - \sigma_b)(1 - (1 - p)^{k+1})) \sum_{j=\lambda_b-\sigma_b}^\infty \pi(k+1,j). \]

The LHS is at most

\[
\lambda_a \sum_{j=0}^\infty \pi(k,j) \leq \lambda_a \sum_{j=\lambda_b-\sigma_b}^\infty \pi(k,j) + \beta_k, \quad \text{where } \beta_k = \lambda_a \sum_{j=0}^{\lambda_b-\sigma_b-1} \pi(k,j).
\]

Let

\[ g(k) = \sum_{j=\lambda_b-\sigma_b}^\infty \pi(k,j) \quad \text{and} \quad \alpha_k = \frac{\lambda_a}{k + 1 + (\lambda_b - \sigma_b)(1 - (1 - p)^{k+1})}. \]

Then, if \( k \geq \bar{k}_1 \), we have

\[ \alpha_k g(k) + \beta_k \geq g(k+1). \]

We observe that

\[ \alpha_k \leq \frac{\lambda_a}{k - \bar{k}_1 + \lambda_a} \leq \exp \left( -\frac{k - \bar{k}_1}{k - \bar{k}_1 + \lambda_a} \right). \]

Moreover, by Lemma 6.20,

\[ \sum_{k=\bar{k}_1}^\infty \beta_k \leq \lambda_a \sum_{j=0}^{\lambda_b-\sigma_b-1} \sum_{k=\lambda_b}^\infty \pi(k,j) = O(\lambda_a \lambda_b e^{-\frac{\sigma^2}{\lambda_b}}). \]

Therefore, by applying Lemma 6.3 with \( k^* = \bar{k}_1 \) and \( \eta = \lambda_a - \bar{k}_1 \), we have

\[
\sum_{k=\bar{k}_1+1}^\infty g(k) = O(\lambda_a e^{-\frac{\sigma^2}{\lambda_b + \eta}}) + O(\lambda_a) \sum_{k=\bar{k}_1}^\infty \beta_k = O(\lambda_a e^{-\frac{\sigma^2}{\lambda_b + \eta}}) + O(\lambda_a \lambda_b) e^{-\frac{\sigma^2}{\lambda_b}}.
\]

Moreover, if \( B_j \) is smaller than about \( \lambda_b \), then \( A_t \) cannot be smaller than about \( \bar{k}_1 \) in the steady state.

**Lemma 6.22.** For any \( \sigma_a \geq 1 \) and \( \sigma_b \geq 1 \), we have

\[
\Pr_{(A,B)\sim \pi} [A \leq \bar{k}_1 - \sigma_a - 1, B \leq \lambda_b + \sigma_b] = O(\lambda_a e^{-\frac{\sigma^2}{\lambda_b + \eta}}) + O(\lambda_a \lambda_b + \lambda_b^3) e^{-\frac{\sigma^2}{\lambda_b + \eta}}.
\]

**Proof.** The balance equation (27) can be transformed as follows.

\[
\lambda_a \sum_{j=0}^\infty \pi(k-1,j) = \sum_{j=0}^\infty (k + j(1 - (1 - p)^k)) \pi(k,j) \leq (k + (\lambda_b + \sigma_b)(1 - (1 - p)^k)) \sum_{j=\lambda_b+\sigma_b}^\infty \pi(k,j) + \sum_{j=\lambda_b+\sigma_b+1}^\infty (k + j) \pi(k,j).
\]

The LHS is at least

\[ \lambda_a \sum_{j=0}^{\lambda_b+\sigma_b} \pi(k-1,j). \]

Define

\[ \alpha_k = \frac{k + (\lambda_b + \sigma_b)(1 - (1 - p)^k)}{\lambda_a} \quad \text{and} \quad \beta_k = \frac{1}{\lambda_a} \sum_{j=\lambda_b+\sigma_b+1}^\infty (k + j) \pi(k,j). \]
Then, for \( k \leq k_1 \), we have
\[
\sum_{j=0}^{\lambda_b + \sigma_b} \pi(k-1, j) \leq \alpha_k \sum_{j=0}^{\lambda_b + \sigma_b} \pi(k, j) + \beta_k.
\]

We will apply Lemma 6.3. We observe that
\[
\alpha_k \leq \frac{k - k_1 + \lambda_a}{\lambda_a} \leq \exp \left( -\frac{k_1 - k}{\lambda_a} \right).
\]

It follows from Lemma 6.20 and (26) that
\[
\sum_{k=0}^{k_1} \beta_k \leq \frac{k_1}{\lambda_a} \sum_{j=\lambda_b + \sigma_b + 1}^{\infty} \sum_{k=0}^{\infty} \pi(k, j) + \frac{1}{\lambda_a} \sum_{j=\lambda_b + \sigma_b + 1}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \pi(k, j)
\]
\[
= O(\lambda_b) \exp \left( -\frac{\sigma_b^2}{\sigma_a + \lambda_b} \right) + O \left( \frac{\lambda_b^3}{\lambda_a} \right) \exp \left( -\frac{\sigma_b^2}{\sigma_b + \lambda_b} \right)
\]
\[
= O \left( \lambda_b + \frac{\lambda_b^2}{\lambda_a} \right) \exp \left( -\frac{\sigma_b^2}{\sigma_b + \lambda_b} \right).
\]

Therefore, by Lemma 6.3
\[
\sum_{k=0}^{k_1 - \sigma_a - 1} \sum_{j=0}^{\infty} \pi(k, j) = O(\lambda_a) e^{-\frac{\sigma_b^2}{2(\sigma_a + \lambda_b)\sigma_a}} + O(\lambda_a) \sum_{k=0}^{k_1} \beta_k = O(\lambda_a) e^{-\frac{\sigma_b^2}{2(\sigma_a + \lambda_b)\sigma_a}} + O \left( \lambda_a \lambda_b + \lambda_b^3 \right) e^{-\frac{\sigma_b^2}{\sigma_b + \lambda_b}}.
\]

The above two lemmas, together with Lemma 6.20 imply Proposition 4.6.

We also present the following lemma, which will be used in bounding the loss in the next subsection.

**Lemma 6.23.** For any \( \sigma_a \geq 1 \), we have
\[
\sum_{k=0}^{\infty} \sum_{j=0}^{\lambda_a + \sigma_a + 1} \pi(k, j) = O \left( \lambda_a^3 \right) e^{-\frac{\sigma_b^2}{\sigma_a + \lambda_a}}.
\]

**Proof.** The balance equation (27) implies that
\[
\lambda_a \sum_{j=0}^{\infty} \pi(k, j) \geq (k + 1) \sum_{j=0}^{\infty} \pi(k + 1, j).
\]

If \( k \geq \lambda_a - 1 \), then it holds that
\[
\frac{\sum_{j=0}^{\infty} \pi(k + 1, j)}{\sum_{j=0}^{\infty} \pi(k, j)} \leq \frac{\lambda_a}{k + 1} = 1 - \frac{k - \lambda_a + 1}{k + 1} \leq \exp \left( -\frac{k - \lambda_a + 1}{k + 1} \right).
\]

Hence, by applying Lemma 6.1, we obtain
\[
\sum_{k=0}^{\infty} \sum_{j=0}^{\lambda_a + \sigma_a + 1} \pi(k, j) = O \left( \lambda_a^3 \right) \exp \left( -\frac{\sigma_b^2}{\sigma_a + \lambda_a} \right).
\]

### 6.5.2 Bounding the Loss of Patient1

It follows from Proposition 4.6 that the loss of Patient1 can be upper-bounded as below. This with Lemma 6.18 proves Theorem 3.4 for Patient1.
Theorem 6.24. For a bipartite matching market \((d_a, d_b, p)\), let \(\pi\) be the stationary distribution of the Markov chain of Patient. Then, if \(d_a \geq d_b\), it holds that

\[
\frac{\mathbb{E}(A,B) \sim \pi [A]}{\lambda_a} \leq \frac{d_a - d_b}{d_a} + \frac{\log(d_b + 3)}{d_a} + o(1),
\]

\[
\frac{\mathbb{E}(A,B) \sim \pi [B(1 - p)^A]}{\lambda_b} \leq (1 + o(1)) \frac{\log(d_b + 3)}{d_b} + o(1),
\]

If \(d_a < d_b\), then

\[
\frac{\mathbb{E}(A,B) \sim \pi [A]}{\lambda_a} \leq \frac{\log(d_b + 3)}{d_a} + o(1),
\]

\[
\frac{\mathbb{E}(A,B) \sim \pi [B(1 - p)^A]}{\lambda_b} \leq (1 + o(1)) \left( \frac{d_b - d_a}{d_b} + \frac{\log(d_b + 3)}{d_b} \right) + o(1).
\]

Proof. Define \(\sigma_a = \Theta(\sqrt{\lambda_a \log \lambda_a})\) and \(\sigma_b = \Theta(\sqrt{\lambda_b \log(\lambda_a \lambda_b)})\). Let \(S = \{(k, j) \mid \bar{k}_1 - \sigma_a \leq k \leq \bar{k}_1 + \sigma_a, \lambda_b - \sigma_b \leq j \leq \lambda_b + \sigma_b\}\). It holds that

\[
\mathbb{E}(A,B) \sim \pi [A] \leq \sum_{(k,j) \in S} k \pi(k,j) + \sum_{(k,j) \notin S} \pi(k,j),
\]

(28)

\[
\mathbb{E}(A,B) \sim \pi [B(1 - p)^A] \leq \sum_{(k,j) \notin S} j(1 - p)^k \pi(k,j) + \sum_{(k,j) \notin S} j \pi(k,j).
\]

(29)

The second term of (28) is divided as follows.

\[
\sum_{(k,j) \notin S} k \pi(k,j) \leq (\lambda_a + \sigma_a) \sum_{(k,j) \notin S} \pi(k,j) + \sum_{k \geq \lambda_a + \sigma_a + 1} k \sum_{j \geq 0} \pi(k,j).
\]

The first term is bounded by a constant from Proposition 4.6 while the second term is bounded by a constant by Lemma 6.23. In addition, the second term of (29) can be transformed to

\[
\sum_{(k,j) \notin S} j \pi(k,j) \leq (\lambda_b + \sigma_b) \sum_{(k,j) \notin S} \pi(k,j) + \sum_{j \geq \lambda_b + \sigma_b + 1} j \sum_{k \geq 0} \pi(k,j).
\]

Both the terms are bounded by constants from Proposition 4.6 and (26). Therefore, we obtain

\[
\mathbb{E}(A,B) \sim \pi [A] \leq \sum_{(k,j) \in S} k \pi(k,j) + O(1) \quad \text{and} \quad \mathbb{E}(A,B) \sim \pi [B(1 - p)^A] \leq \sum_{(k,j) \in S} j(1 - p)^k \pi(k,j) + O(1).
\]

Since \(\lambda_b - \sigma_b \leq j \leq \lambda_b + \sigma_b\) and \(\bar{k}_1 - \sigma_a \leq k \leq \bar{k}_1 + \sigma_a\) in \(S\), it holds that

\[
\mathbb{E}(A,B) \sim \pi [A] \leq \bar{k}_1 + \sigma_a + O(1) \leq k_1 + \sigma_a + \sigma_b + O(1)
\]

since \(\bar{k}_1 \leq k_1 + \sigma_b\) by Lemma 6.19. Moreover, it holds that

\[
\mathbb{E}(A,B) \sim \pi [B(1 - p)^A] \leq (\lambda_b + \sigma_b)(1 - p)^{k_1 - \sigma_a} + O(1)
\]

\[
\leq (\lambda_b + o(\lambda_b))(1 - p)^{k_1 - \sigma_a - \sigma_b} + O(1)
\]

\[
\leq (1 + o(1))(1 - p)^{k_1} + O(1)
\]

where the second inequality follows from that \(\bar{k}_1 \geq k_1 - \sigma_b\) by Lemma 6.19 and the third inequality follows since \((1 - p)^{\sigma_a - \sigma_b} \leq 1 + \sigma_b\) by Lemma 6.19 and the third inequality follows since \(1 - p)^{\sigma_a - \sigma_b} \leq 1 + o(1)\). In summary, since \(\sigma_a = o(\lambda_a)\) and \(\sigma_b = o(\lambda_b)\), we obtain

\[
\frac{\mathbb{E}(A,B) \sim \pi [A]}{\lambda_a} \leq \frac{k_1}{\lambda_a} + o(1) \quad \text{and} \quad \frac{\mathbb{E}(A,B) \sim \pi [B(1 - p)^A]}{\lambda_b} \leq (1 + o(1))(1 - p)^{k_1} + o(1),
\]

where we note that, since \(\lambda_a = \frac{d_a}{d_b} \lambda_b\) and \(d_a\) and \(d_b\) are constants, we may assume that \(\lambda_a \leq C \lambda_b\) for some constant \(C\). They are identical with (23) and (24) in the proof of Theorem 6.17 respectively. Following the left part in the proof of Theorem 6.17 we complete the proof. \(\square\)
6.6 Reducing to Estimating Pool Sizes in the Steady State

In this section, we prove Lemmas 6.4, 6.7, 6.13, and 6.18 in Sections 6.2 and 6.5.

We denote by \( \tilde{A}_t \) and \( \tilde{B}_t \) the pool sizes when an algorithm does nothing and no agents ever get matched (see also the proof of Theorem 5.1). It is easy to see that \( A_t \leq \tilde{A}_t \) and \( B_t \leq \tilde{B}_t \) for any \( t \).

Akbarpour et al. [4] calculated the probability that the pool size is a given integer \( \ell \) for a (non-bipartite) matching market where each agent is inactive. This can be directly applied to the bipartite matching markets, as \( \tilde{A}_t \) and \( \tilde{B}_t \) are independent.

**Proposition 6.1.** (Akbarpour et al. [4]) For any \( t \geq 0 \), it holds that

\[
\Pr[\tilde{A}_t = \ell] \leq \frac{\lambda_a^\ell e^{\ell}}{\ell!} \quad \text{and} \quad \Pr[\tilde{B}_t = \ell] \leq \frac{\lambda_b^\ell e^{\ell}}{\ell!}.
\]

Therefore, we have

\[
E[\tilde{A}_t] = (1 - e^{-t})\lambda_a \leq \lambda_a \quad \text{and} \quad E[\tilde{B}_t] = (1 - e^{-t})\lambda_b \leq \lambda_b.
\]

We can bound the expected sizes of \( A_t \) and \( B_t \) for a Markov chain \((A_t, B_t)\) such that \( A_t \leq \tilde{A}_t \) and \( B_t \leq \tilde{B}_t \) for any \( t \).

**Proposition 6.2.** (Akbarpour et al. [4]) Let \((A_t, B_t)\) be the Markov chain such that \( A_t \leq \tilde{A}_t \) and \( B_t \leq \tilde{B}_t \) for any \( t \). Let \( \pi \) be the stationary distribution, and \( \tau_{\max}(\epsilon) \) be the mixing time. For any \( t \geq \tau_{\max}(\epsilon) \), it holds that

\[
E[A_t] \leq E_{(A, B) \sim \pi}[A] + 6\epsilon\lambda_a + 2^{-6\lambda_a} \quad \text{and} \quad E[B_t] \leq E_{(A, B) \sim \pi}[B] + 6\epsilon\lambda_b + 2^{-6\lambda_b}.
\]

**6.6.1 Proofs of Lemmas 6.4 and 6.13**

We prove Lemma 6.4 as Lemma 6.13 can be proved similarly.

By Proposition 6.1 we have \( E[A_t] \leq E[\tilde{A}_t] \leq \lambda_a \). This implies that

\[
L_a(\text{Greedy}_2) = \frac{1}{\lambda_a T} \mathbb{E} \left[ \int_{t=0}^{T} A_t dt \right] \leq \frac{1}{\lambda_a T} \lambda_a \tau_{\max}(\epsilon) + \frac{1}{\lambda_a T} \int_{t=\tau_{\max}(\epsilon)}^{T} \mathbb{E}[A_t] dt.
\]

Therefore, it follows from Proposition 6.2 that

\[
L_a(\text{Greedy}_2) \leq \frac{\tau_{\max}(\epsilon)}{T} + 6\epsilon + \frac{1}{\lambda_a} 2^{-6\lambda_a} + \frac{E_{(A, B) \sim \pi}[A]}{\lambda_a}.
\]

The argument for \( L_b(\text{Greedy}_2) \) is symmetrical, and hence Lemma 6.4 holds.

**6.6.2 Proofs of Lemmas 6.7 and 6.18**

We first prove Lemma 6.7.

**Proof of Lemma 6.7** It follows from Proposition 6.1 that \( E[A_t(1 - p)^{B_t}] \leq E[A_t] \leq E[\tilde{A}_t] \leq \lambda_a \). Hence it follows that

\[
L_a(\text{Patient}_2) = \frac{1}{\lambda_a} \left( \mathbb{E} \left[ \int_{t=0}^{T} A_t(1 - p)^{B_t} dt \right] \right) = \frac{\tau_{\max}(\epsilon)}{T} + \frac{1}{\lambda_a T} \int_{t=\tau_{\max}(\epsilon)}^{T} \mathbb{E}[A_t(1 - p)^{B_t}] dt.
\]

Let \( t \geq \tau_{\max}(\epsilon) \). We denote \( p_{ij} = \Pr[(A_t, B_t) = (i, j)] \) for simplicity. It holds that

\[
E[A_t(1 - p)^{B_t}] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij}(1 - p)^j \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij}(1 - p)^j + \sum_{i=6\lambda_a + 1}^{\infty} i \sum_{j=0}^{\infty} p_{ij}.
\]

\[
\leq \frac{6\lambda_a}{\lambda_a} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (\pi(i, j) + \epsilon)(1 - p)^j + \sum_{i=6\lambda_a + 1}^{\infty} \Pr[A_t \geq i]
\]

\[
\leq E_{(A, B) \sim \pi}[A(1 - p)^{B}] + 6\epsilon\lambda_a \sum_{j=0}^{\infty} (1 - p)^j + \sum_{i=6\lambda_a + 1}^{\infty} \Pr[\tilde{A}_t \geq i]
\]

\[
\leq E_{(A, B) \sim \pi}[A(1 - p)^{B}] + 6\epsilon\lambda_a \sum_{j=0}^{\infty} (1 - p)^j + 2^{-6\lambda_a}
\]

\[
\leq E_{(A, B) \sim \pi}[A(1 - p)^{B}] + \frac{6\epsilon\lambda_a}{p} + 2^{-6\lambda_a},
\]

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where the second inequality follows from the definition of the mixing time, and the third inequality follows since $\Pr[A_t \geq i] \leq \Pr[A_t \geq i]$ for any $i$ by the definition of $A_t$, and the fourth inequality comes from the fact that $\sum_{i=6\lambda_a+1}^{\infty} \Pr[A_t \geq i] \leq 2^{-6\lambda_b}$ (see [11]). Therefore, we have

$$L_a(\text{Patient}_2) \leq \frac{E_{A,B} \sim \pi [A(1-p)^B]}{\lambda_a} + \frac{\tau_{\min}(\epsilon)}{T} + 6\epsilon + \frac{2^{-6\lambda_b}}{\lambda_a}.$$ 

Similarly, $E[B_t(1-p)^A] \leq E_{A,B} \sim \pi [B(1-p)^A] + 6\frac{\epsilon\lambda_a}{p} + 2^{-6\lambda_b}$, which proves Lemma 6.7.

We can use a similar argument for proving Lemma 6.18. It holds that $E[B_t(1-p)^A] \leq E_{A,B} \sim \pi [B(1-p)^A] + 6\frac{\epsilon\lambda_a}{p} + 2^{-6\lambda_b}$. Moreover, by Proposition 6.2, $E[A_t] \leq E_{A,B} \sim \pi [A] + 6\epsilon\lambda_a + 2^{-6\lambda_b}$. This gives Lemma 6.18.

## 7 Lower Bounds

In this section, we prove Theorems 3.3 and 3.5. Recall that OPT is the optimal algorithm under the assumption that the algorithm does not know the future information (or even the departure information), and OMN is the algorithm that can use full information about the future. Although the optimal solution is computationally hard to obtain, we can lower-bound them by estimating the loss which any matching algorithm suffers.

Let $(\zeta_a, \zeta_b)$ be the expected pool size of an arbitrary algorithm. In the market, the expected rate that some agent gets critical is $\zeta_a + \zeta_b$. When the planner does not observe critical agents, all critical agents perish with probability one. Hence, the loss is equal to $\frac{\zeta_a + \zeta_b}{\lambda_a + \lambda_b}$ in this case.

We compute the fraction of agents who form no edges upon arrival and during their sojourn. If an agent has no edges, then no matching algorithm can match her. Hence the fraction of such agents gives a lower bound on the loss of any matching algorithm. As will be seen below, the fraction can be expressed as a function of $\zeta_a$ and $\zeta_b$. Taking the worst case as $\zeta_a$ and $\zeta_b$ vary, we obtain a lower bound of $L(OPT)$, which shows the first part of Theorems 3.3. The proof to lower-bound $L(OMN)$ is similar, except for that we cannot use the fact that $\frac{\zeta_a + \zeta_b}{\lambda_a + \lambda_b}$. Instead, we utilize the basic observation that $\zeta_a \leq \lambda_a$ and $\zeta_b \leq \lambda_b$.

For the 1-sided matching algorithms, we adopt a similar idea. Under Greedy, each agent in $U$ is inactive, and hence she does not get matched if she has no edges to agents who will arrive after her arrival. Thus, for an agent in $U$, we compute the fraction of agents who do not form edges to agents arriving after her (she may form edges to agents in the pool upon her arrival). Similarly, a greedy agent does not get matched if she forms no edges to agents upon her arrival, and a patient agent does not if she has no edges at her departure. Computing the fraction of such agents gives lower bounds of $L(\text{Greedy}_1)$ and $L(\text{Patient}_1)$. This shows Theorem 3.5.

Before going to each case, we prove the following, which was also shown in Jiang [18]. Since $L(\text{OPT}) \geq L(\text{OMN})$, this implies that $\frac{d_a - d_b}{d_a + d_b}$ is a lower bound of each algorithm.

**Lemma 7.1.** For $d_a \geq d_b$, it holds that

$$L(\text{OMN}) \geq \frac{d_a - d_b}{d_a + d_b}.$$

**Proof.** We first note that the bipartite graph $G_t = (U_t, V_t, E_t)$ at time $t$ can have a matching of size at most $\min\{A_t, B_t\}$. This means that the number of matched agents at time $t$ is at most $2 \min\{A_t, B_t\}$. Hence the total number of matched agents is at most

$$2 \int_0^T \min\{A_t, B_t\} dt \leq 2 \min\left\{ \int_0^T A_t dt, \int_0^T B_t dt \right\} = 2 \min\{\zeta_a, \zeta_b\} T$$

where we define $\zeta_a := E_{t \sim \text{unif}[0,T]}[A_t]$ and $\zeta_b := E_{t \sim \text{unif}[0,T]}[B_t]$. Since $\zeta_a \leq \lambda_a$ and $\zeta_b \leq \lambda_b$, this implies that the total number of matched agents is at most $2\lambda_b T$ as $\lambda_a \geq \lambda_b$ Therefore, the loss is

$$L(\text{OMN}) \geq 1 - \frac{2\lambda_b T}{(\lambda_a + \lambda_b) T} = \frac{d_a - d_b}{d_a + d_b}.$$

This completes the proof. □
7.1 Lower Bound for $L(\text{OPT})$

We here prove the first part of Theorem 3.3. Let $\zeta_a := E_{t \sim \text{unif}[0,T]}[A_t]$ and $\zeta_b := E_{t \sim \text{unif}[0,T]}[B_t]$. Since OPT does not know the departure information, each critical agent perishes with probability 1. Therefore, it holds that

$$L(\text{OPT}) = \frac{1}{\lambda_a T + \lambda_b T} \mathbb{E} \left[ \int_0^T A_t + B_t dt \right] = \frac{\zeta_a + \zeta_b}{\lambda_a + \lambda_b}. \quad (30)$$

Below we bound $L(\text{OPT})$ in a different way. Suppose that an agent $v$ arrives at time $t_0 \sim \text{unif}[0,T]$. If an agent $v$ does not form any edge during her sojourn, she must perish, which are counted in the loss. Hence the loss is lower bounded by $\Pr[N(v) = \emptyset]$, where $N(v)$ is the set of neighbors of $v$. Recall that $s(v)$ denotes the staying time of the agent $v$. We also denote $U_{t_0,t_0+t}$ (resp., $V_{t_0,t_0+t}$) the set of agents in $U$ (resp., $V$) that arrive in the time interval $[t_0,t_0+t]$ for $t \geq 0$. By definition, we obtain

$$\Pr[N(v) = \emptyset] \geq \frac{\lambda_a}{\lambda_a + \lambda_b} \int_0^\infty \Pr[s(v) = t] \mathbb{E}[(1-p)^{|B_t|}] \mathbb{E}[(1-p)^{|V_{t_0,t_0+t}|}] dt \quad + \quad \frac{\lambda_b}{\lambda_a + \lambda_b} \int_0^\infty \Pr[s(v) = t] \mathbb{E}[(1-p)^{|A_t|}] \mathbb{E}[(1-p)^{|U_{t_0,t_0+t}|}] dt \quad \geq \quad \frac{\lambda_a}{\lambda_a + \lambda_b} \int_0^\infty e^{-t(1-p)^{\zeta_a}(1-p)^{\lambda_a}} dt \quad + \quad \frac{\lambda_b}{\lambda_a + \lambda_b} \int_0^\infty e^{-t(1-p)^{\zeta_b}(1-p)^{\lambda_b}} dt,$$

where the second inequality follows from the Jensen’s inequality. Since $1-p \geq e^{-p^2}$ for $p < 1/10$, we have

$$\int_0^\infty e^{-t(1-p)^{\zeta_a}(1-p)^{\lambda_a}} dt \geq e^{-\zeta_a(p+p^2)} \int_0^\infty e^{-t(1+\lambda_a(p+p^2))} dt \geq \frac{e^{-\zeta_a(p+p^2)}}{1 + d_a + d_a^2/\lambda_b}. \quad (31)$$

Similarly, it holds that

$$\int_0^\infty e^{-t(1-p)^{\zeta_b}(1-p)^{\lambda_b}} dt \geq \frac{e^{-\zeta_b(p+p^2)}}{1 + d_b + d_b^2/\lambda_a}. \quad (32)$$

Thus $L(\text{OPT})$ is lower bounded by the maximum of the RHSes of (30) and (32). First suppose that the former term of (32) is smaller than the latter. For a fixed $\zeta_a$, the worst case is attained when

$$\frac{\zeta_a}{\lambda_a + \lambda_b} + \frac{\zeta_b}{\lambda_a + \lambda_b} = 1 - \zeta_b(p+p^2) \quad \Rightarrow \quad \zeta_b = \frac{\lambda_a + \lambda_b}{1 + d_a + 2d_b/\lambda_a + 2d_b^2/\lambda_b} \left( 1 - \frac{\zeta_a(1 + d_b + d_b^2/\lambda_b)}{\lambda_a + \lambda_b} \right).$$

Note that $\zeta_b$ may be negative, but this gives a lower bound. This implies that

$$L(\text{OPT}) \geq \frac{1}{1 + d_a + 2d_b + d_a^2/\lambda_a + 2d_b^2/\lambda_b} - \frac{\zeta_a}{\lambda_a + \lambda_b} \frac{1 + d_b + d_b^2/\lambda_b}{\lambda_a + \lambda_b} + \frac{\zeta_a}{\lambda_a + \lambda_b} \geq \frac{1}{1 + d_a + 2d_b + d_a^2/\lambda_a + 2d_b^2/\lambda_b}$$

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We prove the second part of Theorem 3.3. Since
\[ L \leq 1 + 2d_a + d_b + 2d_a^2/\lambda_a + d_b^2/\lambda_b, \]
where the inequality follows from the Jensen’s inequality.

Thus we obtain
\[
L(\text{OPT}) \geq \min \left\{ \frac{1}{1 + 2d_a + d_b + 2d_a^2/\lambda_a + d_b^2/\lambda_b}, \frac{1}{1 + d_a + 2d_b + 2d_a^2/\lambda_a + 2d_b^2/\lambda_b} \right\}
\]
which follows since
\[ L \geq \min \{1, d \} \quad \text{for an agent} \quad v. \]

This proves the first part of Theorem 3.3.

\[ \square \]

7.2 Lower Bound for \( L(\text{OMN}) \)

We prove the second part of Theorem 3.3. Since \( L(\text{OMN}) \geq \Pr[N(v) = 0] \) for an agent \( v \), we can use the lower bound (31) as for \( L(\text{OPT}) \). Hence we have

\[
L(\text{OMN}) \geq \Pr[N(v) = 0] \geq \frac{\lambda_a}{\lambda_a + \lambda_b} \cdot \frac{e^{-\zeta_a(p+p^2)}}{1 + d_a + d_b^2/\lambda_b} + \frac{\lambda_b}{\lambda_a + \lambda_b} \cdot \frac{e^{-\zeta_b(p+p^2)}}{1 + d_a + d_b^2/\lambda_a}
\]

where the second to the last inequality follows since \( \lambda_a \geq \zeta_a \) and \( \lambda_b \geq \zeta_b \) by Proposition 6.1, and the last inequality follows since \( \gamma \alpha + (1-\gamma)\beta \geq (\alpha + \beta)/2 \) for any \( \alpha \geq \beta \geq 0 \) and \( 1/2 \leq \gamma \leq 1 \). Thus the proof of the second part of Theorem 3.3 is complete.

\[ \square \]

7.3 Lower Bound for \( L(\text{Greedy}_1) \)

Let \( \zeta_a := \mathbb{E}_{t \sim \text{unif}[0,T]}[A_t] \) and \( \zeta_b := \mathbb{E}_{t \sim \text{unif}[0,T]}[B_t] \). Since Greedy_1 does not know the departure information, it holds that

\[ L(\text{Greedy}_1) = \frac{\zeta_a + \zeta_b}{\lambda_a + \lambda_b}. \]  

Suppose that an agent \( v \) arrives at time \( t_0 \sim \text{unif}[0,T] \). Let \( X_v \) be an event that \( v \) is matched to no one. If \( v \) is an inactive agent in \( U \), \( X_v \) happens if \( v \) does not form an edge to agents after her arrival. Hence we can write

\[
\Pr[X_v] \geq \int_{t=0}^{\infty} \Pr[s(v) = t][\mathbb{E}[v(1-p)^{T_{t\sim\text{unif}}}] ] dt
\]

\[
\geq \int_{t=0}^{\infty} e^{-t}(1-p)^{\lambda a t} dt \quad \text{(by the Jensen’s inequality)}
\]

\[
\geq \int_{t=0}^{\infty} e^{-t-\lambda a t(p+p^2)} dt \quad \text{(since} \quad 1 - p \geq e^{-p^2/p^2} \text{ for } p < 1/10) \]

\[
= \frac{1}{1 + \lambda_b(p + p^2)}. \]

On the other hand, if \( v \) is a greedy agent in \( V \), then \( X_v \) happens if \( v \) does not form an edge to agents upon her arrival. In this case, we can write

\[
\Pr[X_v] = \int_{t=0}^{\infty} \Pr[s(v) = t][\mathbb{E}[v(1-p)^{T_{t\sim\text{unif}}}] ] dt \geq \int_{t=0}^{\infty} e^{-t}(1-p)\zeta_a dt = e^{-\zeta_a(p+p^2)},
\]

where the inequality follows from the Jensen’s inequality.
Therefore, the total loss is lower bounded by the following.

\[ L(\text{Greedy}_1) \geq \frac{1}{\lambda_a + \lambda_b} \max \left\{ \zeta_a, \frac{\lambda_a}{1 + \lambda_b(p + p^2)} \right\} + \frac{1}{\lambda_a + \lambda_b} \max \left\{ \zeta_b, \lambda_b e^{-\zeta_a(p + p^2)} \right\}. \]

The worst case is attained when

\[ \zeta_a = \frac{\lambda_a}{1 + \lambda_b(p + p^2)} \quad \text{and} \quad \zeta_b = \lambda_b e^{-\zeta_a(p + p^2)}. \]

First suppose that \( \lambda_a \geq \lambda_b \). Then, since \( \zeta_b \geq 0 \), we have

\[ L(\text{Greedy}_1) \geq \frac{1}{\lambda_a + \lambda_b} \frac{1}{1 + \lambda_b(p + p^2)} + \frac{1}{2 \lambda_a + \lambda_b} \frac{1}{1 + \lambda_b(p + p^2)}. \tag{35} \]

Next suppose that \( \lambda_a < \lambda_b \). In this case,

\[ \zeta_b = \lambda_b e^{-\zeta_a(p + p^2)} = \lambda_b \exp \left( - \frac{(p + p^2)\lambda_a}{1 + \lambda_b(p + p^2)} \right) \geq \lambda_b \left( 1 - \frac{(p + p^2)\lambda_a}{1 + \lambda_b(p + p^2)} \right) = \lambda_b \left( 1 + \frac{(p + p^2)(\lambda_b - \lambda_a)}{1 + \lambda_b(p + p^2)} \right). \]

Therefore, the total loss is

\[ L(\text{Greedy}_1) \geq \frac{\lambda_a}{\lambda_a + \lambda_b} + \frac{\lambda_b}{1 + \lambda_b(p + p^2)} + \frac{\lambda_a}{\lambda_a + \lambda_b} \frac{1}{1 + \lambda_b(p + p^2)} \geq \frac{1}{1 + \lambda_b(p + p^2)} + \frac{1}{2 \lambda_a + \lambda_b} \frac{1}{1 + \lambda_b(p + p^2)} + \frac{db}{\lambda_a + \lambda_b} \]

since \( \frac{d_a}{\lambda_a + \lambda_b} \geq \frac{1}{2} \) for \( db \geq 1 \). This, together with (35), proves the first part of Theorem 3.5.

\[ \square \]

7.4 Lower Bound of \( L(\text{Patient}_1) \)

Let \( \zeta_a := \mathbb{P}(t \sim \text{unif}[0,T])[A_t] \) and \( \zeta_b := \mathbb{E}[t \sim \text{unif}[0,T]][B_t] \). Suppose that an agent \( v \) arrives at time \( t_0 \sim \text{unif}[0,T] \). Let \( X_v \) be an event that \( v \) is matched to no one. For an inactive agent \( v \) in \( U, \hat{X}_v \), happens if \( v \) does not form an edge to any agent. In this case, we can write

\[ \mathbb{P}[X_v] \geq \int_{t=0}^{\infty} \mathbb{P}[s(v) = t] \mathbb{E}[(1 - p)^{|B_t|}] \mathbb{E}[(1 - p)^{|V_{t_0 + t}|}] dt \geq \int_{t=0}^{\infty} e^{-t(1 - p)\zeta_v + \lambda_b t} dt \geq \frac{e^{-\zeta_b(p + p^2)}}{1 + \lambda_b(p + p^2)}, \]

where the second inequality follows from the Jensen’s inequality, and the last one holds since \( 1 - p \geq e^{-p - p^2} \) for \( p < 1/10 \). Hence the expected number of unmatched agents in \( U \) is at least \( \lambda_a e^{-\zeta_a(p + p^2)} \). Also, \( \mathbb{E}[A_t] \geq \zeta_a \) holds, since agents in \( U \) are inactive. For a patient agent \( v \) in \( V, X_v \), happens if \( v \) does not form an edge to any agent at her departure. Hence, it holds that

\[ \mathbb{P}[X_v] \geq \int_{t=0}^{\infty} \mathbb{P}[s(v) = t] \mathbb{E}[(1 - p)^{|A_{t_0 + t}|}] dt \geq \int_{t=0}^{\infty} e^{-t(1 - p)\zeta_v} dt \geq e^{-\zeta_b(p + p^2)}, \]

where we again use the Jensen’s inequality and the fact that \( 1 - p \geq e^{-p - p^2} \) for \( p < 1/10 \). Hence the expected number of unmatched agents in \( V \) is at least \( \lambda_b e^{-\zeta_a(p + p^2)} \).

Therefore, we obtain a lower bound of the loss:

\[ L(\text{Patient}_1) \geq \frac{1}{\lambda_a + \lambda_b} \max \left\{ \zeta_a, \frac{\lambda_a}{1 + \lambda_b(p + p^2)} \right\} + \frac{\lambda_b}{\lambda_a + \lambda_b} \frac{e^{-\zeta_a(p + p^2)}}{1 + \lambda_b(p + p^2)}. \]

The worst case is attained when

\[ \zeta_a = \frac{\lambda_a}{1 + \lambda_b(p + p^2)}. \]

In this case,

\[ L(\text{Patient}_1) \geq \frac{1}{\lambda_a + \lambda_b} \left( \zeta_a + \lambda_b e^{-\zeta_a(p + p^2)} \right). \]

The RHS is minimized when \( \zeta_a = \frac{1}{p + p^2} \log(d_b + d_b p) \). Hence, we have

\[ L(\text{Patient}_1) \geq \frac{\log(d_b + d_b^2/\lambda_b)}{d_a + d_b + d_a p + d_b p} + \frac{1}{d_a + d_b}. \]
8 Conclusion

In this work, we studied a bipartite matching market model with arrivals and departures. We proposed 1-sided/2-sided local algorithms with different timing properties, the Greedy and Patient algorithms. We achieved both upper and lower bounds on the performance of these algorithms, which are shown to be almost tight. In addition, we provided lower bounds on the performance of any matching algorithms.

Our results indicate that waiting to thicken the market is highly valuable for the balanced 2-sided market, which is a similar conclusion to Akbarpour et al. [4] and Beccara et al. [10]. Even when the market is not balanced, the loss of the smaller side can be made much smaller by waiting. On the other hand, waiting is not valuable for the 1-sided market. It means that, to improve the loss in the 1-sided market, the departure information is not so beneficial, and other information, such as the time that her neighbors arrive/depart, is necessary to obtain smaller loss.

Our models are simple and developed from theoretical interest. Although our models carry practical implications on waiting, they ignore some aspects of practical settings in reality. For example, the probability that two agents are compatible is set to be constant over time \([0, T]\) and among agents. However, the probability may be affected by past information [29]. Also, our model ignores waiting costs. It would be interesting to incorporate such practical settings into our model and analyze the performance of matching algorithms.

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A Lemmas for Solving Recursions

In this section, we prove lemmas used in Section 6. We first provide the following lemmas by Akbarpour et al. [4] with basic observation. Then we prove Lemmas 6.1, 6.2, and 6.3 in the subsequent sections.

Lemma A.1. (Akbarpour et al. [4]) For any $a, b \geq 0$, we have

$$\sum_{i=a}^{\infty} e^{-bi^2} = \frac{e^{-ba^2}}{1 - e^{-2ab}} \leq \frac{e^{-ba^2}}{\min\{ab, 1/2\}}.$$  

Lemma A.2. (Akbarpour et al. [4]) For any $a, b \geq 0$, we have

$$\sum_{i=a}^{\infty} i e^{-bi^2} \leq \frac{e^{-ba^2}(2ab + 4)}{b^2}.$$  

Lemma A.3. It holds that, for any $x \geq 0$,

$$\frac{1}{1 - e^{-1/x}} \leq \frac{1}{\frac{x}{x-\frac{1}{2}}} = O(x).$$

Proof. Since $e^{-x} \leq 1 - x + \frac{x^2}{2}$, we have

$$\frac{1}{1 - e^{-1/x}} \leq \frac{1}{\frac{x}{x-\frac{1}{2}}} = O(x).$$

A.1 Proof of Lemma 6.1

By definition, it holds that

$$f(k + 1) \leq \exp\left(-\frac{k - k^*}{k + \eta}\right) f(k) \leq \cdots \leq \exp\left(-\sum_{i=k^*}^{k} \frac{i - k^*}{i + \eta}\right) f(k) \leq \exp\left(-\sum_{i=0}^{k-k^*} \frac{i}{i + k^* + \eta}\right).$$

This implies that, for any $\sigma \geq 1$, we have

$$\sum_{k=k^*+\sigma+1}^{\infty} f(k) \leq \sum_{k=k^*+\sigma}^{\infty} \exp\left(-\frac{1}{k + \eta} (k - k^*)^2\right) \leq \sum_{k=k^*+\sigma}^{\infty} \exp\left(-\frac{1}{k + k^* + \eta} k^2\right) \leq \sum_{k=k^*+\sigma}^{\infty} \exp\left(-\frac{1}{1 + \frac{k^* + \eta}{\sigma}} k\right)$$

$$= \frac{1}{1 - \exp\left(-\frac{\sigma}{\sigma + k^* + \eta}\right)} \exp\left(-\frac{\sigma^2}{\sigma + k^* + \eta}\right).$$

Since Lemma A.3 implies that

$$\frac{1}{1 - \exp\left(-\frac{\sigma}{\sigma + k^* + \eta}\right)} = O\left(\frac{\sigma + k^* + \eta}{\sigma}\right),$$

this proves the first part of the lemma.

Similarly, it holds that

$$\sum_{k=k^*+\sigma+1}^{\infty} kf(k) \leq \sum_{k=k^*+\sigma}^{\infty} (k + 1) \exp\left(-\frac{1}{k + \eta} (k - k^*)^2\right)$$

$$\leq \sum_{k=k^*+\sigma}^{\infty} k \exp\left(-\frac{1}{k + k^* + \eta} k^2\right) + (k^* + 1) \sum_{k=k^*+\sigma}^{\infty} \exp\left(-\frac{1}{k + k^* + \eta} k^2\right)$$

$$= \sum_{k=k^*+\sigma}^{\infty} k \exp\left(-\frac{1}{k + k^* + \eta} k^2\right) + O(k^* (k^* + \eta)) \exp\left(-\frac{\sigma^2}{\sigma + k^* + \eta}\right),$$

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where the last equality follows from (9). The first term is bounded by Lemma A.2 as follows.

\[
\sum_{k=\sigma}^{\infty} k \exp \left( -\frac{1}{k + k^* + \eta} k^2 \right) \leq \sum_{k=\sigma}^{\infty} k \exp \left( -\frac{1}{1 + \frac{k^* + \eta}{\sigma}} k^2 \right) \leq \left( \frac{\sigma + k^* + \eta}{\sigma} \right)^2 \left( \frac{\sigma^2}{\sigma + k^* + \eta} + 4 \right) e^{-\frac{\sigma^2}{\sigma + k^* + \eta}}
\]

\[
= \left( \frac{\sigma + k^* + \eta}{\sigma} \right) \left( 2\sigma^2 + 4\sigma + 4k^* + 4\eta \right) e^{-\frac{\sigma^2}{\sigma + k^* + \eta}}.
\]

Therefore, when \( \sigma = O(k^* + \eta) \) and \( \sigma \geq 1 \), it is \( O((k^* + \eta)^3) e^{-\frac{\sigma^2}{\sigma + k^* + \eta}} \). This proves the second part of the lemma.

**A.2 Proof of Lemma 6.2**

The proof is similar to Lemma 6.1. By definition, it holds that

\[
f(k - 1) \leq \exp \left( -\frac{k^* - k}{\eta} \right) f(k) \leq \cdots \leq \exp \left( -\frac{k^* - k}{\eta} \right) f(k^*)
\]

\[
\leq \exp \left( -\frac{1}{\eta} \sum_{i=0}^{k^* - k} i \right) \leq \exp \left( -\frac{1}{\eta} (k^* - k)^2 \right).
\]

This implies by Lemma A.1 that, for any \( \sigma \geq 1 \), we have

\[
\sum_{k=0}^{k^* - \sigma - 1} f(k) \leq \sum_{k=1}^{k^* - \sigma} \exp \left( -\frac{1}{\eta} (k^* - k)^2 \right) \leq \sum_{k=\sigma}^{\infty} \exp \left( -\frac{1}{\eta} k^2 \right) \leq e^{-\frac{\sigma}{\eta}} \frac{1}{\min\{\sigma/\eta, 1/2\}} = O(\eta) e^{-\frac{\sigma}{\eta}}.
\]

**A.3 Proof of Lemma 6.3**

Applying the given inequality repeatedly, we have

\[
g(k + 1) \leq \alpha_k g(k) + \beta_k \leq \alpha_k \alpha_{k-1} g(k - 1) + \alpha_k \beta_{k-1} + \beta_k
\]

\[
\leq \alpha_k \alpha_{k-1} \alpha_{k-2} g(k - 2) + \alpha_k \alpha_{k-1} \beta_{k-2} + \alpha_k \beta_{k-1} + \beta_k
\]

\[
\leq \cdots \leq \left( \prod_{i=k^*}^{k} \alpha_i \right) g(k^*) + \sum_{i=k^*}^{k} \left( \prod_{j=i+1}^{k} \alpha_j \right) \beta_i.
\]

For \( k_1, k_2 \in \mathbb{N} \), we denote

\[
\gamma(k_1, k_2) = \prod_{i=k_1}^{k_2} \alpha_i.
\]

When \( k_1 > k_2 \), define \( \gamma(k_1, k_2) = 0 \). Since \( g(k^*) \leq 1 \), we have for any \( k \geq k^* \),

\[
g(k + 1) \leq g(k^*, k) + \sum_{i=k^*}^{k} \gamma(i + 1, k) \beta_i.
\]

(36)

Summing up \( g(k) \)'s of \( (36) \), we have

\[
\sum_{k=k^*+\sigma}^{\infty} g(k) \leq \sum_{k=k^*+\sigma}^{\infty} g(k^*, k) + \sum_{k=k^*+\sigma}^{\infty} \sum_{i=k^*}^{k} \gamma(i + 1, k) \beta_i.
\]

Since the second term is equal to

\[
\sum_{k=k^*+\sigma}^{\infty} \sum_{i=k^*}^{k} \gamma(i + 1, k) \beta_i = \sum_{i=k^*}^{\infty} \sum_{k=k^*+\sigma}^{\infty} \gamma(i + 1, k) \beta_i,
\]

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we obtain
\[ \sum_{k=k^*+\sigma+1}^{\infty} g(k) \leq \sum_{k=k^*+\sigma}^{\infty} \gamma(k^*, k) + \sum_{i=k^*+\sigma}^{\infty} \sum_{k=k^*+\sigma}^{\infty} \gamma(i+1, k) \beta_i. \]  \hspace{1cm} (37)

To show Lemma 6.3, we provide a sequence of claims to bound each term of (37).

**Claim 2.** For any \( j \geq 0 \), it holds that
\[ \gamma(k^* + j, k) \leq \exp \left( -\frac{1}{2(k + \eta)} (k - k^* + j)(k - k^* - j) \right). \]

**Proof of Claim 2.** We have
\[
\prod_{i=k^* + j}^{k} \alpha_k \leq \prod_{i=k^* + j}^{k} \exp \left( -\frac{i - k^*}{i + \eta} \right) = \exp \left( -\sum_{i=k^* + j}^{k} \frac{i - k^*}{i + \eta} \right) \leq \exp \left( -\frac{1}{2(k + \eta)} (k - k^* + j)(k - k^* - j) \right).
\]

The first term of (37) is then bounded as follows.

**Claim 3.** It holds that
\[ \sum_{k=k^*+\sigma}^{\infty} \gamma(k^*, k) = O \left( k^* + \eta \right) e^{-\frac{\sigma^2}{2(\sigma + k^* + \eta)}}. \]  \hspace{1cm} (38)

**Proof of Claim 3.** Using Claim 2 with \( j = 0 \), we have
\[
\sum_{k=k^*+\sigma}^{\infty} \gamma(k^*, k) \leq \sum_{k=k^*+\sigma}^{\infty} \exp \left( -\frac{1}{2(k + \eta)} (k - k^*)^2 \right) = \sum_{k=\sigma}^{\infty} \exp \left( -\frac{k}{2(k + k^* + \eta)} \right) \leq \sum_{k=\sigma}^{\infty} \exp \left( -\frac{1}{2(1 + \frac{k^* + \eta}{\sigma})} \right) = O \left( \frac{1}{b} \right) e^{-b \sigma},
\]
where \( b = \frac{1}{2(1 + \frac{\sigma + k^* + \eta}{\sigma})} = \frac{\sigma}{2(\sigma + k^* + \eta)} \). Since \( \sigma \geq 1 \),
\[ O \left( \frac{1}{b} \right) = O \left( \frac{\sigma + k^* + \eta}{\sigma} \right) = O(k^* + \eta), \]
which proves the claim.

We next bound the second term of (37). To do it, we provide the following claims.

**Claim 4.** For any \( j \geq 0 \), we have
\[ \sum_{k=k^*+\sigma+j}^{\infty} \gamma(k^* + \sigma + j, k) = O \left( k^* + \eta \right), \]  \hspace{1cm} (39)
where \( b_j = \frac{\sigma + j}{2(\sigma + j + k^* + \eta)} \).
Proof of Claim \[4\] The proof is similar to Claim \[3\]. It follows from Claim \[2\] that
\[
\sum_{k=k^*+\sigma+j}^{\infty} \gamma(k^* + \sigma + j, k) \leq \sum_{k=k^*+\sigma+j}^{\infty} \exp \left( -\frac{1}{2(k + \eta)} ((k - k^*)^2 - (\sigma + j)^2) \right)
\]
\[
= \sum_{k=\sigma+j}^{\infty} \exp \left( -\frac{1}{2(k + \sigma + \eta)} (k^2 - (\sigma + j)^2) \right)
\]
\[
= \sum_{k=\sigma+j}^{\infty} \exp \left( \frac{(\sigma + j)^2}{2(k + \sigma + \eta)} \right) \exp \left( -\frac{1}{2(k + \sigma + \eta)} k^2 \right)
\]
\[
\leq \exp \left( \frac{(\sigma + j)^2}{2(\sigma + j + k^*)} \right) \sum_{k=\sigma+j}^{\infty} \exp \left( -\frac{1}{2 \left( \frac{k^* + \eta}{\sigma + j} \right)} k \right)
\]
\[
= e^{b_j \gamma + j} e^{-b_j (\sigma + j)} \frac{1}{1 - e^{-b_j}} = \frac{1}{1 - e^{-b_j}},
\]
where \( b_j = \frac{1}{2(1 + \frac{\sigma+j}{\sigma+\eta})} = \frac{\sigma+j}{2(\sigma+j + k^* + \eta)} \). By Lemma \[A.3\] we see that
\[
\frac{1}{1 - e^{-b_j}} = O \left( \frac{\sigma + j + k^* + \eta}{\sigma + j} \right) = O(k^* + \eta),
\]
since \( \sigma + j \geq 1 \). This proves the claim. \( \square \)

We are ready to bound the second term of \( \{57\} \).

Claim 5. It holds that
\[
\sum_{i=k^*}^{\infty} \sum_{k=k^*+\sigma}^{\infty} \gamma(i + 1, k) \beta_i \leq O(k^* + \eta) \sum_{k=k^*}^{\infty} \beta_i.
\]

Proof of Claim \[5\] Since \( \gamma(i + 1, k) \leq \gamma(k^* + \sigma, k) \) if \( i \leq k^* + \sigma - 1 \), it follows that
\[
\sum_{i=k^*}^{\infty} \sum_{k=k^*+\sigma}^{\infty} \gamma(i + 1, k) \beta_i = \sum_{i=k^*}^{k^*+\sigma-1} \sum_{k=k^*+\sigma}^{\infty} \gamma(i + 1, k) \beta_i + \sum_{i=k^*+\sigma}^{\infty} \sum_{k=k^*+\sigma}^{\infty} \gamma(i + 1, k) \beta_i
\]
\[
\leq \sum_{k=k^*+\sigma}^{\infty} \gamma(k^* + \sigma, k) \sum_{i=k^*}^{k^*+\sigma-1} \beta_i + \sum_{j=\sigma}^{\infty} \sum_{k=k^*+j+1}^{\infty} \gamma(k^* + j + 1, k) \beta_{k^*+j}.
\]

By Claim \[4\]
\[
\sum_{k=k^*+\sigma}^{\infty} \gamma(k^* + \sigma, k) = O(k^* + \eta) \text{ and } \sum_{k=k^*+j+1}^{\infty} \gamma(k^* + j + 1, k) = O(k^* + \eta).
\]

Hence, it holds that
\[
\sum_{i=k^*}^{\infty} \sum_{k=k^*+\sigma}^{\infty} \gamma(i + 1, k) \beta_i \leq O(k^* + \eta) \sum_{i=k^*}^{k^*+\sigma-1} \beta_i + O(k^* + \eta) \sum_{j=\sigma}^{\infty} \sum_{i=k^*}^{\infty} \beta_{k^*+j} = O(k^* + \eta) \sum_{i=k^*}^{\infty} \beta_i.
\]

\( \square \)

Lemma \[6.3\] follows from Claims \[4\] and \[5\].