HYPERBOLICITY OF MINIMIZERS AND REGULARITY OF VISCOSITY SOLUTIONS FOR RANDOM HAMILTON-JACOBI EQUATIONS

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Abstract. We show that for a family of randomly kicked Hamilton-Jacobi equations, the unique global minimizer is hyperbolic, almost surely. Furthermore, we prove the unique forward and backward viscosity solutions, though in general only Lipschitz, are smooth in a neighbourhood of the global minimizer. Our result generalizes the result of E, Khanin, Mazel and Sinai (8) to arbitrary dimensions, and extends the result of Iturriaga and Khanin in [12].

1. Introduction

We start by considering the inviscid Burger’s equation
\[ \partial_t u + (u \cdot \nabla) u = f^\omega(y, t), \quad y \in \mathbb{T}^d, t \in \mathbb{R}, \tag{1} \]
where \( f^\omega(y, t) = -\nabla F^\omega(y, t) \) is a random force given by the potential \( F^\omega \). The inviscid equation can be viewed as the limit of the viscous Burger’s equation as the viscosity approaches zero. For this viewpoint, we refer to [10] and the references therein. In this paper, we will focus only on the inviscid case. The theory of (1) is developed by E, Khanin, Mazel and Sinai in [8], for the one dimensional configuration space \((d = 1)\) and the “white noise” potential
\[ F^\omega(y, t) = \sum_{i=1}^{M} F_i(y, t) = \sum_{i=1}^{M} F_i(y) \dot{W}_i(t), \tag{2} \]
where \( F_i : \mathbb{T}^d \to \mathbb{R} \) are smooth functions, and \( \dot{W}_i \) are independent white noises.

In [12], Iturriaga and Khanin considered the “kicked” potential
\[ F^\omega(y, t) = \sum_{j \in \mathbb{Z}} F_j^\omega(y) \delta(t - j), \tag{3} \]
where \( F_j^\omega \) are chosen independently from the same distribution, and \( \delta(\cdot) \) is the delta function. The potential (3) can be considered as a discrete version of (2). The theory of (3) runs parallel to the theory of (2), and has the advantage of being technically simpler.
In these works, only solutions of the type \( u = \nabla \phi \) is considered, which converts (1) to the Hamilton-Jacobi equation
\[
\partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + F^\omega(y, t) = 0.
\] (4)

Hence, for the solutions of interest, it is equivalent to study the viscosity solutions of (4). Moreover, for each \( b \in \mathbb{R}^d \), we may consider solutions of (4) with \( \int \nabla \phi(y, t) dy = b \), as the vector \( b \) is invariant under the evolution.

The study of these solutions is closely related with the concept of minimizing orbits in Lagrangian systems. More precisely, for each \( b \in \mathbb{R} \), \( s < t \), \( x, x' \in \mathbb{T}^d \), we define the minimal action function by
\[
A^{\omega, b}_{s,t}(x, x') = \inf \int_s^t \frac{1}{2} (\dot{\zeta}(\tau))^2 - \dot{\zeta} \cdot b - F^\omega(\zeta(\tau), \tau) d\tau,
\] (5)

where the infimum is taken over all absolutely continuous curves \( \zeta : [s, t] \to \mathbb{T}^d \) with \( \zeta(s) = x \) and \( \zeta(t) = x' \). A curve \( \gamma : I \to \mathbb{T}^d \) is called a minimizer if the infimum in \( A^{\omega, b}_{s,t}(\gamma(s), \gamma(t)) \) is achieved at \( \gamma|[s, t] \) for all \( [s, t] \subset I \subset \mathbb{R} \). For the cases \( I = [t_0, \infty) \), \( I = (-\infty, t_0] \) or \( I = \mathbb{R} \), the curve \( \gamma \) is called a forward minimizer, a backward minimizer or a global minimizer, respectively.

Deferring the precise definitions to the next section, we roughly reinterpret the main results of [8] (using the point of view of [12]) as follows. For \( d = 1 \), \( b \in \mathbb{R} \), under some nondegeneracy conditions, the following hold almost surely.

C1. There exists unique viscosity solutions on the set \( \mathbb{T}^d \times [t_0, \infty) \) and \( \mathbb{T}^d \times (-\infty, t_0] \), up to a constant translation;
C2. For Lebesgue a.e. \( x \in \mathbb{T}^d \), there exist unique forward and backward minimizers;
C3. (“One force, one solution principle”) There exists a unique global minimizer supporting a unique invariant measure, properly interpreted.
C4. The unique invariant measure has no zero Lyapunov exponents (and hence is hyperbolic), if properly interpreted.
C5. The unique forward and backward viscosity solutions are smooth in a neighbourhood of the global minimizer. Furthermore, the graph of the gradient of the viscosity solutions is equal to the local stable and unstable manifolds of the global minimizer.

The conclusion C5 in [8] is actually stronger, but we will not discuss it in this paper.

In [12], Iturriaga and Khanin generalized conclusions C1-C3, for both the “kicked” and “white noise” potentials, to arbitrary dimensions. Their approach is variational, and is related to Aubry-Mather theory and weak KAM theory (see for example, [13], [14], [9]). The variational approach of [12] is the starting point of this paper.

We describe our main result roughly as follows (see Theorem 2.3, Theorem 2.4 for accurate statements):
Main result. For the “kicked” potential, conclusion C4 and C5 hold in arbitrary dimensions. In other words, the unique invariant measure is hyperbolic, and the viscosity solutions are smooth near the global minimizer.

Remark. As mentioned before, it is expected that methods applied to the “kicked” case in this paper should apply to the “white noise” case. We choose to convey the ideas of our method through the technically simpler “kicked” case, and treat the “white noise” case in a later work.

The proofs of C4 and C5 in the one-dimensional case depends strongly on one-dimensionality, and seem difficult to generalize. To prove C4 for arbitrary dimensions, we devise a completely different strategy. The main new ingredient is the use of the Green bundles (see [11]). These bundles have been useful in proving uniform hyperbolicity for Hamiltonian flows, and have only recently been used to study Lyapunov exponents, due to the work of Arnaud ([2], [3]). Furthermore, we relate the Green bundles to the nondegeneracy of the minimum for a variational problem, and employ techniques from variational analysis. These also seem to be new features of this paper.

Since the viscosity solutions are at most Lipshitz in general, the regularity result we obtain in C5 is highly nontrivial. Although we do not state it explicitly, these solutions has the same Hölder regularity as the potentials (if the potentials are $C^{k+\alpha}$, then the solutions are $C^{k+\alpha}$). This fact is due to the smoothness of the invariant manifolds for hyperbolic systems.

When the global minimizer is nonuniformly hyperbolic, there is no clear-cut relation between the viscosity solutions and the stable/unstable manifolds. In general, the viscosity solutions, as a product of the variational method, may bare no direct relations to the stable/unstable manifolds. Our proof of C5 from C4 uses precise information of the variational problem (more than what’s needed to prove C4), and requires a careful adaptation of the nonuniform hyperbolic theory. We would like to mention that for nonrandom systems, under the easier assumption that the global minimizer is uniformly hyperbolic, an analogous result is known (see [5]).

The outline of the paper is as follows. In section 2 we introduce the notations, recall the results of Iturriaga and Khanin in [12], and formulate the main theorems. In section 3 we describe the variational set up, and formulate some statements in variational analysis. The proofs of these statements are deferred to the end of the paper. In section 4 we define the Green bundles, and establish the connection between the variational problem and the Green bundles. In section 5 we show that the transversality of the Green bundles imply nonzero exponents, proving Theorem 2.3. Sections 4 and 5 make use of the results of Arnaud in [2] and [3]. In section 6 and 7 we prove Theorem 2.4 using variational arguments and Pesin’s theory. In the last three sections, the statements formulated in section 3 are proved.
2. Formulation of the main results

We restrict ourselves to the case of “kicked” potentials

\[ F(y, t) = \sum_{j \in \mathbb{Z}} F_j(y) \delta(t - j). \]

Here we assume that the potentials \( F_j \) are chosen independently from a distribution \( \chi \in P(C^{2+\alpha}(\mathbb{T}^d)), 0 < \alpha \leq 1 \) (some of the results hold under weaker regularity assumptions).

Since we will be considering the solutions \( \phi \) of the type \( \int \nabla \phi dy = b \), we write \( \phi(y, t) = b \cdot y + \psi(y, t) \), where \( \int \nabla \psi dy = 0 \). It’s easy to see that \( \psi \) is a viscosity solution of the Hamilton-Jacobi equation

\[ \partial_t \psi(x, t) + H^b(\nabla \psi(x, t), t) = 0, \]

with the Hamiltonian \( H^b(x, p, t) = \frac{1}{2}(p + b)^2 + F^\omega(x, t) \). The corresponding Lagrangian is given by \( L(x, v, t) = \frac{1}{2}v^2 - b \cdot v \). Note that this is exactly the Lagrangian we defined \( A_{s,t}^{\omega,b} \) with. According to the Lax-Oleinik variational principle, given \( x, x' \in \mathbb{T}^d \) and \( s < t \), we have

\[ \psi(x', t) = \inf_{x \in \mathbb{T}^d} \left\{ \psi(x, s) + A_{s,t}^{\omega,b}(x, x') \right\}. \]

Note that \( F^\omega(y, t) = 0 \) for all \( t \notin \mathbb{Z} \). As a consequence, any curve \( \zeta \) realizing the minimum in the definition of \( A_{s,t}^{\omega,b} \) must be linear between integer values of \( \zeta \). In this sense, the viscosity solutions are completely determined by their values at \( t \in \mathbb{Z} \).

For \( b \in \mathbb{R}^d, m, n \in \mathbb{Z}, m < n \), we define the discrete version of the action function by

\[ A_{m,n}^{\omega,b}(x, x') = \inf \left\{ \sum_{i=m}^{n-1} \frac{1}{2}(\bar{x}_{i+1} - \bar{x}_i)^2 - b \cdot (\bar{x}_n - \bar{x}_m) - \sum_{i=m}^{n-1} F^\omega_i(\bar{x}_i) \right\}, \]

where the infimum is taken over all \( (\bar{x}_i)_{i=m}^n \), \( \bar{x}_j \in \mathbb{R}^d \) such that \( \bar{x}_n = x \) and \( \bar{x}_m = x' \) (mod \( \mathbb{Z}^d \)). The sequence \( (\bar{x}_j)_{j=m}^n \) corresponds to the lift of the curve \( \zeta \) in (5) at integer values, and we call it a configuration following the language of twist diffeomorphisms. The discrete version of the variational principle is given by

\[ \psi(x', n) = \inf_{x \in \mathbb{T}^d} \left\{ \psi(x, m) + A_{m,n}^{\omega,b}(x, x') \right\}, \]

where \( x, x' \in \mathbb{T}^d \) and \( m < n \in \mathbb{Z} \). Throughout the paper, we may drop the subscript \( \omega \) and \( b \) when there is no risk of confusion.

The solution \( \psi(x, n) \) is closely related to the family of maps \( \Phi_j^{\omega} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d \)

\[ \Phi_j^{\omega} : \begin{bmatrix} x_j \\ v_j \end{bmatrix} \mapsto \begin{bmatrix} x_{j+1} \\ v_{j+1} \end{bmatrix} = \begin{bmatrix} x_j + v_j - \nabla F^\omega_j(x_j) \mod \mathbb{Z}^d \\ v_j - \nabla F^\omega_j(x_j) \end{bmatrix}, \]

(8)
The maps belong to the so-called standard family, and are examples of symplectic, exact and monotonically twist diffeomorphisms. For $m, n \in \mathbb{Z}$, $m < n$, denote

$$
\Phi^\omega_{m,n}(x, v) = \Phi^\omega_{m-1} \circ \cdots \circ \Phi^\omega_{m}(x, v).
$$

For any $n \in \mathbb{Z}$ and $(x_n, v_n) \in \mathbb{T}^d \times \mathbb{R}^d$, we define $(x_j, v_j) = \Phi_{n,j}(x_n, v_n)$ if $j > n$ and $(x_j, v_j) = (\Phi_{j,n})^{-1}(x_n, v_n)$ if $j < n$. We call $(x_j, v_j)_{j \geq n}$ the forward orbit of $(x_n, v_n)$, $(x_j, v_j)_{j \leq n}$ the backward orbit of $(x_n, v_n)$, and $(x_j, v_j)_{j \in \mathbb{Z}}$ the full orbit of $(x_n, v_n)$.

Assume that $(\bar{x}_j)_{j=m}^n$ is a configuration that attains the infimum in (4), then $x_j = \bar{x}_j \mod \mathbb{Z}^d$, $j = m, \cdots, n$, and $v_j = \bar{x}_{j+1} - \bar{x}_j - \nabla F^\omega_n(\bar{x}_j)$, $j = m+1, \cdots, n$ satisfy the relation $\Phi^\omega_j(x_j, v_j) = (x_{j+1}, v_{j+1})$. We also define $v_m$ by the relation $\Phi^\omega_m(x_m, v_m) = (x_{m+1}, v_{m+1})$. An orbit $(x_j, v_j)$ defines a configuration $\bar{x}_j$ which is unique up to a lift of the first point $x_m$. We say that the orbit $(x_j, v_j)_{j=m}^n$ is a minimizer for the action $A^\omega_{m,n}(x_m, x_n)$ if the orbit generates a configuration that attains the infimum in $A^\omega_{m,n}(x_m, x_n)$. We define the forward, backward and global minimizer in the same way as in the continuous case.

The following assumptions on the probability space $P(C^2(\mathbb{T}^d))$ were introduced in [12].

**Assumption 1.** For any $y \in \mathbb{T}^d$, there exists $G_y \in \text{supp } P$ s.t. $G_y$ has a maximum at $y$ and that there exists $b > 0$ such that

$$
G_y(y) - G(x) \geq b|y - x|^2.
$$

**Assumption 2.** $0 \in \text{supp } P$.

**Assumption 3.** There exists $G \in \text{supp } P$ such that $G$ has unique maximum.

We define the backward Lax-Oleinik operator $K^\omega_{m,n} : C(\mathbb{T}^d) \to C(\mathbb{T}^d)$, for $m < n$, $m, n \in \mathbb{Z}$ by the following expression:

$$
K^\omega_{m,n} \varphi(x) = \inf_{x_m \in \mathbb{T}^d} \{ \varphi(x_m) + A^\omega_{m,n}(x_m, x) \}. \tag{9}
$$

The following is proved in [12] under the weaker assumption that $F^\omega_j \in C^1(\mathbb{T}^d)$:

**Theorem 2.1.** [12]

1. Assume assumption 1 or 2 holds. For each $n_0 \in \mathbb{Z}$, for a.e. $\omega \in \Omega$, we have the following statements.
   - There exists a Lipshitz function $\psi^-(x, n)$, $n \leq n_0$, such that for any $m < n \leq n_0$,
     $$
     K^\omega_{m,n} \psi^-(x, m) = \psi^-(x, n).
     $$
   - For any $\varphi \in C(\mathbb{T})$ and $n \leq n_0$, we have that
     $$
     \lim_{m \to -\infty} \inf_{C \in \mathbb{R}} ||K^\omega_{m,n} \varphi(x) - \psi^-(x, n) - c|| = 0.
     $$
• For \( n \leq n_0 \) and Lebesgue a.e. \( x \in \mathbb{T}^d \), the gradient \( \nabla \psi^- (x, n) \) exists. Denote \( x_n^- = x \) and \( v_n^- = \nabla \psi^- (x_n^- , n) + b \); we have that \( (x_j^-, v_j^-) = (\Phi_{j,n})^{-1}(x_n^-, v_n^-) \), \( j \leq n \) is a backward minimizer.

2) Assume that assumption 3 holds. Then the conclusions for the first case hold for \( b = 0 \).

Similar theorems hold for the forward minimizers. For \( \varphi \in C(\mathbb{T}^d) \), \( m, n \in \mathbb{Z} \), \( m < n \), we define the forward Lax-Oleinik operator as follows:

\[
\tilde{K}_{m,n}^{\omega,b} \varphi(x) = \sup_{x_n \in \mathbb{T}^d} \{ \varphi(x_n) - A_{m,n}^{\omega,b}(x, x_n) \}.
\]

Under the same assumptions as in Theorem 2.1 we have that there exists a Lipschitz function \( \psi^+(x, n) \), \( n \geq n_0 \), such that \( \tilde{K}_{m,n}^{\omega,b} \psi^+(x, n) = \psi^+(x, m) \).

For \( n \geq n_0 \) and Lebesgue a.e. \( x \in \mathbb{T}^d \), \( v_n^+ = -\nabla \psi^+(x, n) + b \) exists. Write \( x_n^+ = x \); we have that \( (x_j^+, v_j^+) = \Phi_{j,n}^{\omega}(x_n^+, v_n^+) \) is a forward minimizer.

We further reduce the choice of our potential to the one generated by a finite family of smooth potentials, multiplied by i.i.d. random vectors. These potentials emulate the behaviour of the “white noise” case.

Assumption 4. Assume that

\[
F_j^{\omega}(x) = \sum_{i=1}^{M} \xi_j^i(\omega) F_i(x),
\]

where \( F_i : \mathbb{T}^d \to \mathbb{R} \) are smooth functions, and the vectors \( \xi_j^i(\omega) = (\xi_j^i(\omega))_{i=1}^M \) are identically distributed vectors in \( \mathbb{R}^M \) with an absolutely continuous distribution.

We have the following theorem from [12].

**Theorem 2.2. [12]**

1) Assume that assumption 4 holds and one of assumptions 1 and 2 holds. If

\[
(F_1, \cdots, F_M) : \mathbb{T}^d \to \mathbb{R}^M
\]

is one-to-one, then for all \( b \in \mathbb{R}^d \) and a.e. \( \omega \) there exists a unique \( (x_0^\omega, v_0^\omega) \in \mathbb{T}^d \times \mathbb{R}^d \), such that the full orbit of \( (x_0^\omega, v_0^\omega) \) is a global minimizer.

2) The same conclusion is valid if assumption 3 holds and \( b = 0 \).

As the random potential is generated by a stationary random process, the time shift \( \theta_m \) is a metric isomorphism of the probability space \( \Omega \) satisfying

\[
F^\omega(y, n + m) = F^{\theta_m \omega}(y, n), \quad m \in \mathbb{Z}.
\]

The family of maps \( \Phi_j^\omega \) then defines a random transformation \( \hat{\Phi} \) on the space \( \mathbb{T}^d \times \mathbb{R}^d \times \Omega \) given by

\[
\hat{\Phi}(x, v, \omega) = (\Phi_0^\omega (x, v), \theta \omega).
\]
Let \((x_j^\omega, v_j^\omega)\) be the global minimizer in Theorem 2.2. We have that the probability measure

\[
\nu(d(x, v), d\omega) = \delta_{x_0^\omega, v_0^\omega}(d(x, v)) P(d\omega)
\]
is invariant and ergodic under the transformation (13). The map \(D\Phi_0^\omega : \mathbb{T}^d \times \mathbb{R}^d \to Sp(2d)\) defines a cocycle over the transformation (13), where \(Sp(2d)\) is the group of all \(2d \times 2d\) symplectic matrices. Under the first part of Assumption 5 below, the Lyapunov exponents for this cocycle are well defined. Denote them by \(\lambda_1(\nu), \cdots, \lambda_{2d}(\nu)\). Due to the symplectic nature of the cocycle we have

\[
\lambda_1(\nu) \leq \cdots \leq \lambda_d(\nu) \leq 0 \leq \lambda_{d+1}(\nu) \leq \cdots \leq \lambda_{2d}(\nu).
\]

To show the Lyapunov exponents are nonzero, we require an additional assumption.

**Assumption 5.** Under assumption 4, assume first of all that \(\xi_j^1, \cdots, \xi_j^M\) are independent random variables with bounded densities, or if \(\xi_j^i\) is given by a Gaussian distribution with a nondegenerate covariance matrix. We only need to avoid the case when the density is degenerate in certain direction.

We shall replace the one-to-one condition from Theorem 2.2 with a stronger condition, requiring the map (12) is an embedding.

**Theorem 2.3.**

1. Assume that assumption 4 and 5 hold, and one of assumptions 1 or 2 holds. Assume in addition that the map (12) is an embedding. Then for all \(b \in \mathbb{R}^d\), for a.e. \(\omega\), the Lyapunov exponents of \(\nu\) satisfy

\[
\lambda_d(\nu) < 0 < \lambda_{d+1}(\nu).
\]

2. For the case \(b = 0\), assumption 1 or 2 can be replaced with the weaker assumption 3. The same conclusions hold.

A corollary of Theorem 2.3 is that for a.e. \(\omega\), the orbit \((x_k^\omega, v_k^\omega)_{k \in \mathbb{Z}}\) is uniformly hyperbolic (see [11]). With a slightly stronger regularity assumption on the map (12), there exists local unstable manifold \(W^u(x_k, v_k)\) and stable manifold \(W^s(x_k, v_k)\).

Our next theorem states that, near \((x_k, v_k)\), \(W^u\) and \(W^s\) coincide with the sets \(W_k^- = \{(x, \nabla_x \psi^-(x, k))\}\) and \(W_k^+ = \{(x, \nabla_x \psi^+(x, k)) + b\}\). The functions \(\psi^\pm\) are only Lipshitz and \(W_k^\pm\) are only defined above a full (Lebesgue) measure of \(x\). Since \(W^u\) and \(W^s\) are smooth manifolds, our theorem states that \(\psi^\pm\) are in fact smooth in a neighbourhood of \(x_k\).
Lemma 3.2. For any ties:

\[ W_0^- \cap U \subset W^u(x_0, v_0), \quad W_0^+ \cap U \subset W^s(x_0, v_0). \]

Furthermore, there exists a neighbourhood \( V(\omega) \) of \( x_0 \) such that

\( \psi^\pm(x, 0) \) are smooth on \( V(\omega) \).

(2) For the case \( b = 0 \), assumption 1 or 2 can be replaced with the weaker assumption 3. The same conclusions hold.

3. Viscosity solutions and the minimizers

In this section we will first deduce some useful properties of the action functional, and introduce a variational problem closely related to the global minimizer. The derivation of the variational problem mostly follow [12].

We say that a function \( f : T^d \to \mathbb{R} \) is \( C \)-semi-concave on \( T^d \) if for any \( x \in T^d \), there exists a linear form \( l_x : \mathbb{R}^d \to \mathbb{R} \) such that for any \( y \in T^d \),

\[ f(y) - f(x) \leq l_x(y - x) + \frac{C}{2}d(x, y)^2. \]

Here \( d(x, y) \) is understood as the distance on the torus, and the vector \( y - x \) is interpreted as any vector from \( x \) to \( y \) on the torus. The linear form \( l_x \) is called a subderivative of \( f \) at \( x \).

Lemma 3.1 ([2], Proposition 4.7.3). If \( f \) is continuous and \( C \)-semi-concave on \( T^d \), then there exists a unique \( C' > 0 \) depending only on \( C \) such that \( f \) is \( C' \)-Lipschitz.

Let \( C_j^2 = \|F_j\|_{C^2} \). The action function \( A_{m,n}^{\omega,b} \) has the following properties.

Lemma 3.2. For any \( b \in \mathbb{R}^d \), the action function \( A_{m,n}^{\omega,b} \) satisfy the following properties:

- For any \( m < k < n \), \( A_{m,n}^{\omega,b}(x, x') = \inf_{x_k \in T^d} \{ A_{m,n}^{\omega,b}(x, x_k) + A_{m,n}^{\omega,b}(x_k, x') \} \).
- If \( (x_j, v_j)_{j=m}^n \) is a minimizer, then \( A_{m,n}^{\omega,b}(x_m, x_n) = \sum_{j=m}^{n-1} A_{m,n}^{\omega,b}(x_j, x_{j+1}) \).
- The function \( A_{m,n}^{\omega,b}(x, x') \) is \( 1 \)-semi-concave in the second component, and is \( C_\omega \)-semi-concave in the first component.
- If \( (x_j, v_j)_{j=m}^n \) is a minimizer, then for any \( k \) such that \( m < k < n \), the derivatives \( \partial_2 A_{m,k}^{\omega,b}(x_m, x_k) \) and \( \partial_1 A_{k,n}^{\omega,b}(x_k, x_n) \) exist. Furthermore,

\[ \partial_2 A_{m,k}^{\omega,b}(x_m, x_k) = -\partial_1 A_{k,n}^{\omega,b} = v_k - b. \]

- If \( (x_j, v_j)_{j=m}^n \) is a minimizer, then \( -v_m + b \) is a subderivative of \( A_{m,n}^{\omega,b}(\cdot, x_n) \) and \( v_n + b \) is a subderivative of \( A_{m,n}^{\omega,b}(x_m, \cdot) \).
Proof. The first two conclusions follow directly from the definition. For the third statement, note that a function of \( x \),

\[
A^{\omega,b}_{m,m+1}(x, x') = \inf_{\tilde{x}' = x' \mod 2^d} \left\{ \frac{1}{2}(\tilde{x}' - x)^2 - b \cdot (\tilde{x}' - x) - F^\omega_m(x) \right\}.
\]

For each different lift \( \tilde{x}' \), \( \frac{1}{2}(\tilde{x}' - x)^2 - b \cdot (\tilde{x}' - x) - F^\omega_m(x) \) is \( C^2 \) with a \( C^2 \) bound \( 1 + C^\omega_m \), and hence are \( 1 + C^\omega \) semi-concave. It follows directly from the definition that minimum of \( C \)-semi-concave functions are still \( C \)-semi-concave. Similarly, we conclude that \( A^{\omega,b}_{m,m+1} \) is \( 1 \)-semi-concave in the second component.

To prove the semi-concavity in general, let \( (x_j, v_j)_{j=m}^n \) be a minimizer for \( A_{m,n}(x_m, x_n) \). Note that

\[
A_{m,n}(x'_m, x_n) - A_{m,n}(x_m, x_n) 
\leq A_{m,m+1}(x'_m, x_{m+1}) + A_{m+1,n}(x_{m+1}, x_n) - A_{m,n}(x_m, x_n) 
\leq A_{m,m+1}(x'_m, x_{m+1}) - A_{m,m+1}(x_m, x_{m+1}),
\]

and hence \( (1 + C^\omega_m) \)-semi-concavity of \( A_{m,m+1} \) implies \( (1 + C^\omega_m) \)-semi-concavity of \( A_{m,n} \) (in the first component). Similarly, the semi-concavity of \( A_{m,n} \) in the second component depends only on \( A_{n-1,n} \).

Note that if \( (x_m, v_m), (x_{m+1}, v_{m+1}) \) is a minimizer of \( A^{\omega,b}_{m,m+1}(x_m, x_{m+1}) \), we have

\[
\partial_2 A^{\omega,b}_{m,m+1} = v_{m+1} - b \quad \text{and} \quad -\partial_1 A^{\omega,b}_{m,m+1} = v_m - b.
\]

This implies the fourth conclusion.

We say that the interval \([m_0, n_0] \) has an \( \epsilon \)-narrow place for the action if there exists \([m, n] \subset [m_0, n_0] \)

1. There exists \( M_1, M_2 \subset T^d \) such that any minimizer \( (x_i)_{i=m_0}^{n_0} \) satisfy \( x_m \in M_1 \) and \( x_n \in M_2 \).
2. For any \( x_1, x_2 \in M_1 \) and \( x'_1, x'_2 \in M_2 \), we have that

\[
|A^{\omega,b}_{m,n}(x_1, x'_1) - A^{\omega,b}_{m,n}(x_2, x'_2)| \leq \epsilon.
\]

Corollary 3.3. For any \( \varphi \in C(T^d) \), the function \( K_{m,n}^{\omega,b}(x) \) is \( 1 \)-semi-concave; the function \( -\tilde{K}_{m,n}^{\omega,b}(x) \) is \( C^\omega_m \)-semi-concave.

We say that the interval \([m_0, n_0] \) has an \( \epsilon \)-narrow place for the action if there exists \([m, n] \subset [m_0, n_0] \)

1. There exists \( M_1, M_2 \subset T^d \) such that any minimizer \( (x_i)_{i=m_0}^{n_0} \) satisfy \( x_m \in M_1 \) and \( x_n \in M_2 \).
2. For any \( x_1, x_2 \in M_1 \) and \( x'_1, x'_2 \in M_2 \), we have that

\[
|A^{\omega,b}_{m,n}(x_1, x'_1) - A^{\omega,b}_{m,n}(x_2, x'_2)| \leq \epsilon.
\]

Proposition 3.4. Fix \( b \in \mathbb{R}^d \), assume that for any \( \epsilon > 0 \) there exists an \( \epsilon \)-narrow place contained in the interval \((-\infty, n_0] \) and \([n_0, \infty) \), then there exists a unique Lipschitz function \( \psi^-(x, j) \) such that for any \( n \leq n_0 \),

\[
\lim_{m \to -\infty} \sup_{\varphi \in C(T) / \mathbb{R}} \| K_{m,n}^{\omega,b}(x) - \psi^-(x, n) \| = 0.
\]
Similarly, there exists a unique Lipschitz function $\psi^+(x, j)$ such that for any $m \geq n_0$,

$$\lim_{n \to \infty} \sup_{\omega \in C(T)/\mathbb{R}} \| \tilde{K}_{m,n}^{\omega,b}(x) - \psi^+(x, m) \| = 0.$$ 

**Corollary 3.5.** For $n \leq n_0$, $\psi^-(\cdot, n)$ is 1-semi-concave; for $m \geq n_0$, $-\psi^+(\cdot, m)$ is $C_m^\infty$-semi-concave.

We have that for a.e. $\omega$, $\epsilon$–narrow places exists for one sided intervals.

**Proposition 3.6.** Assume that assumption 1 or 2 holds, then for each $b \in \mathbb{R}^d$ and $n_0 \in \mathbb{Z}$, for a.e. $\omega$, for any $\epsilon > 0$, there exists an $\epsilon$–narrow place contained in the interval $(-\infty, n_0]$ and $[n_0, \infty)$. The same holds with assumption 3 and $b = 0$.

Since the functions $\psi^\pm$ are semi-concave, by Lemma 3.1 they are Lipschitz. It follows from the Radmacher theorem that they are Lebesgue almost everywhere differentiable. The points of differentiability are tied to the minimizers.

**Proposition 3.7.**
1. If $x_0^-$ is such that $\nabla \psi^-(x_0^-, 0)$ exists, then for $v_0^- = \nabla \psi^-(x_0^-, 0) + b$, the backward orbit of $(x_0^-, v_0^-)$ is a backward minimizer.
2. If $x_0^+$ is such that $\nabla \psi^+(x_0^+, 0)$ exists, then for $v_0^+ = \nabla \psi^+(x_0^+, 0) + b$, the forward orbit of $(x_0^+, v_0^+)$ is a forward minimizer.
3. Assume that, for any $\epsilon > 0$, there exists an $\epsilon$–narrow place contained in the interval $(-\infty, 0]$ and $[0, \infty)$. Then the full orbit of $(x_0, v_0)$, where $v_0 = \nabla \psi^-(x_0, 0) + b$, is a global minimizer if and only if $x_0$ is a point of minimum for the function $\psi^-(x, 0) - \psi^+(x, 0)$.

Theorem 2.1 follows from Proposition 3.4, Proposition 3.6 and the first two statement of Proposition 3.7. Furthermore, to prove the uniqueness of the global minimizer, we only need to show that the function $\psi^-(x, 0) - \psi^+(x, 0)$ has a unique minimum on $\mathbb{T}^d$. We need to consider potentials of the type (11). Given a family of potentials $F^c_j = \sum \xi_j^c(\omega)F_i(x)$, let $c = \{\xi_0^c, \cdots, \xi_M^c\}$. We treat $c$ as a parameter of the system. In the following lemma, we will show that the function $\psi^-(x, 0) - \psi^+(x, 0)$ decompose into a semi-concave part independent of $c$ and a smooth function depending on $c$ and $x$.

**Lemma 3.8.** There exists a semi-concave function $\psi$ such that

$$\psi^-(x, 0) - \psi^+(x, 0) = \psi(x) - \sum_{i=1}^M c_i F_i(x).$$

**Proof.** The functions $A_{m,0}$ for $m < 0$ and the functions $A_{k,n}$ for $1 \leq k < n$ are independent of $c$. As a consequence we have that the functions $\psi^-(x, m)$ for $m \geq 0$ and the functions $\psi^+(x, k)$ for $k \geq 1$ are all independent of $c$. 
On the other hand, we have
\[
\psi^+(x, 0) = K_{0,1}^+ \psi^+(x, 1) = \sup_{x_1 \in \mathbb{T}^d} \{ \psi^+(x_1, 1) - A_{0,1}^b(x, x_1) \}
\]
\[
= \sup_{x_1 \in \mathbb{T}^d} \{ \psi^+(x_1, 1) - \rho_b(x, x_1) - \sum_{i=1}^M c_i F_i(x) \},
\]
where \( \rho_b(x, x') = \inf_{m \in \mathbb{Z}^d} \| x' - x + b + m \| \). It follows that
\[
\psi^-(x, 0) - \psi^+(x, 0) = \psi(x) - \sum_{i=1}^M c_i F_i(x),
\]
where \( \psi(x) = \inf_{x_1 \in \mathbb{T}^d} \{ \psi^+(x_1, 1) - \psi^+(x_1, 1) + \rho_b(x, x_1) \} \). It’s clear that \( \psi \) is a semi-concave function. \( \square \)

Proposition 3.7 and Lemma 3.8 reduces the uniqueness of the minimum for \( \psi^-(x, 0) - \psi^+(x, 0) \) to the uniqueness of the minimum for \( \psi(x) + \sum_{i=1}^M c_i F_i(x) \). The following general statement about the minimum of variational problem implies Theorem 2.2.

**Lemma 3.9.** [12] Let \( V(x, c) \) be a \( C^2 \) function and \( \psi(x) \) a continuous function. Assume in addition that, for each \( c \in \mathbb{R}^M \), \( \frac{\partial V(c)}{\partial c} : \mathbb{T}^d \to \mathbb{R}^M \) is one-to-one. Then for Lebesgue a.e. \( c \in \mathbb{R}^M \), the function
\[
H(x, c) = \psi(x) + V(x, c)
\]
has a unique minimum as a function of \( x \).

We will prove a series of progressively stronger statements about the variational problem
\[
\inf_{x \in \mathbb{T}^d} H(x, c) = \inf_{x \in \mathbb{T}^d} \{ \psi(x) + V(x, c) \},
\]
in the form of Propositions 3.10, 3.11 and 3.12. These finer properties of the variational problem lead to finer properties of the global minimizer. In particular, the proof of Theorem 2.3 uses Propositions 3.10 and 3.11 and the proof of Theorem 2.4 uses from Proposition 3.12.

The first statement says that the unique minimum of the variational problem is also a *nondegenerate* minimum. To define nondegeneracy properly, we invoke some definitions from non-smooth analysis. Assume that \( f : \mathbb{T}^d \to \mathbb{R} \) is a semi-concave function and that \( f'(x_0) \) exists. We define the *second subderivative* of \( f \) (See [15] for more background) at \( x_0 \) to be a function \( d^2 f(x_0) : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\} \cup \{\infty\} \) given by
\[
d^2 f(x_0)(w) = \lim_{\tau \to 0^+} \frac{f(x_0 + \tau w) - f(x_0) - f'(x_0)(w)}{\frac{1}{2} \tau^2}.
\]

For semi-concave functions, the second subderivative can never be \( \infty \), but \(-\infty\) is possible.
Proposition 3.10. Let $V(x, c)$ be a $C^2$ function and $\psi(x)$ a semi-concave function. Assume that $\frac{\partial V(x,c)}{\partial c} : \mathbb{T}^d \to \mathbb{R}^M$ is an embedding. Then for Lebesgue a.e. $c \in \mathbb{R}^M$,

$$H(x, c) = \psi(x) + V(x, c)$$

has a unique minimum. Furthermore, the minimum of $H(x, c)$ is nondegenerate in the sense that if $x(c) \in \mathbb{T}^d$ is the unique minimal point for $c$, then there exists a positive Borel measurable function $a : \mathbb{R}^M \to \mathbb{R}$ such that

$$d^2_x H(x(c), c)(v) \geq a(c) |v|^2, \quad v \in \mathbb{R}^d. \quad (15)$$

We also need some quantitative estimates of the function $a(c)$ in Proposition 3.10.

Proposition 3.11. Assume that $V(x, c) = -\sum_{i=1}^{M} c_i F_i(x)$ and that the map (12) is an embedding. Then Proposition 3.10 applies. Furthermore, if $\rho$ is a density satisfying assumption 5, then there exists a constant $A(F)$ depending only on $F_1, \cdots, F_M$ such that the function $a(c)$ in (15) satisfies

$$\int a(c)^{-1} \rho(c) dc < A(F).$$

The next proposition strengthens the previous two in proving a stronger sense of nondegeneracy. While (15) implies that on a small neighbourhood of $x(c)$, the function $H(x, c)$ is bounded from below by a quadratic function, Proposition 3.12 states that the size of this neighbourhood is uniform in $c$. The only cost is a small loss to the power in the integrability condition.

Proposition 3.12. Assume that $V(x, c) = -\sum_{i=1}^{M} c_i F_i(x)$ and that the map (12) is an embedding. There exists a constant $B(F)$ depending only on $(F_1, \cdots, F_M)$ and a positive Borel measurable function $b : \mathbb{R}^M \to \mathbb{R}$ with

$$\int b(c)^{-\frac{1}{2}} \rho(c) dc < B(F),$$

and a constant $r(F) > 0$ depending only on $(F_1, \cdots, F_M)$, such that

$$H(x', c) - H(x(c), c) \geq b(c) |x' - x(c)|^2, \quad |x - x(c)| \leq r(F).$$

Note that it is in fact possible to prove all our theorems using Proposition 3.12 alone. We still states Propositions 3.10 and 3.11 to stress the fact that the stronger form of nondegeneracy is only needed for the proof of Theorem 2.4.

Propositions 3.10, 3.11 and 3.12 are proved using variational analytic methods, and will be deferred to the end of the paper (Sections 8, 9 and 10).

In Sections 4 and 5 we prove Theorem 2.3 assuming Propositions 3.10 and 3.11.

In sections 7 and 8 we prove Theorem 2.4 assuming Proposition 3.12.
4. THE GREEN BUNDLE AND THE NONDEGENERACY OF THE MINIMIZER

The nondegeneracy of the minimum from Proposition 3.10 is connected with the Lyapunov exponents, via the so-called Green bundles (see [11]). Briefly speaking, Proposition 3.10 implies the transversality of the Green bundles, and the transversality of the Green bundles implies nonzero Lyapunov exponents. We will prove the first implication in this section, and the second implication in Section 5.

Let \((\delta x, \delta v)\) be the coordinates of the tangent space adapted to the coordinates \((x,v)\). At each \((x,v)\), we define the vertical space \(V(x,v) = \{(0, \delta v)\}\) and the horizontal space \(H(x,v) = \{(\delta x,0)\}\). An orbit \((x_j, v_j)_{j \in I}\) is called disconjugate if for any \(m,n \in \mathbb{Z}\), \([m,n] \subset I\), we have that \(D\Phi^\omega_{m,n}(x_m, v_m)V(x_m, v_m) \cap V(x_n, v_n) = \{0\}\). It is well known that minimizing orbits have no conjugate points.

**Lemma 4.1.** (see [7, 3]) If \((x_j, v_j)_{j \in I}\) is a minimizer, then \((x_j, v_j)_{j \in I}\) is disconjugate.

For the rest of this section, we fix a global minimizer \((x_j, v_j)_{j \in \mathbb{Z}}\). For \(n \in \mathbb{Z}\) and \(k \in \mathbb{N}\), we define a subspace \(LU_k(x_n, v_n)\) of \(T_{(x_n, v_n)}(\mathbb{T}^d \times \mathbb{R}^d)\) by

\[
LU^\omega_k(x_n, v_n) = D\Phi^\omega_{n-k,n}V(x_{n-k}, v_{n-k}).
\]

Similarly, we may define a subspace \(S_k(x_n, v_n)\) of \(T_{(x_n, v_n)}(\mathbb{T}^d \times \mathbb{R}^d)\) by

\[
LS^\omega_k(x_n, v_n) = D(\Phi^\omega_{n,n+k})^{-1}V(x_{n+k}, v_{n+k}).
\]

When we don’t need to stress the dependence on the random realization \(\omega\), we will drop the subscript \(\omega\) for \(LS_k\) and \(LU_k\).

The following statement for the case of a sequence of twist maps is due to Bialy and MacKay (6).

**Lemma 4.2.** [6] Assume that \((x_j, v_j)_{j \in \mathbb{Z}}\) is disconjugate. We have the following conclusions.

1. There exists \(d \times d\) symmetric matrices \(S_k(x_n, v_n)\) and \(U_k(x_n, v_n)\) such that \(LS_k(x_n, v_n) = \{\delta v = S_k(x_n, v_n)\delta x\}\) and \(LU_k(x_n, v_n) = \{\delta v = U_k(x_n, v_n)\delta x\}\).

2. Given two symmetric matrices \(A\) and \(B\), we say that \(A \geq B\) if \(A - B\) is positive semi-definite. We say that \(A > B\) if \(A - B\) is positive definite. We have

\[
U_1(x_n, v_n) > \cdots U_k(x_n, v_n) > \cdots > S_k(x_n, v_n) > S_1(x_n, v_n).
\]

3. There exists symmetric matrices \(S(x_n, v_n)\) and \(U(x_n, v_n)\) such that

\[
\lim_{k \to \infty} S_k(x_n, v_n) = S(x_n, v_n), \quad \lim_{k \to \infty} U_k(x_n, v_n) = U(x_n, v_n).
\]

Let \(LS(x_n, v_n) = \{\delta v = S(x_n, v_n)\delta x\}\) and \(LU(x_n, v_n) = \{\delta v = U(x_n, v_n)\delta x\}\). These subspaces are traditionally called the negative and positive Green bundles. In this paper, we will use the name stable and unstable Green bundles to avoid possible confusions with the positive and negative viscosity solutions. It follows from Lemma 4.2.
that the bundles $LS(x_n, v_n)$ and $LU(s_n, v_n)$ are invariant in the sense that

$$D\Phi_{m,n}^W LS(x_m, v_m) = LS(x_n, v_n), \quad D\Phi_{m,n}^W LU(x_m, v_m) = LU(x_n, v_n).$$

Let $m(A) = \|A^{-1}\|^{-1}$ be the co-norm of a matrix $A$. ($m(A) = 0$ if $A$ is singular). It follows from Lemma 4.2 that $U(x_n, v_n) \geq S(x_n, v_n)$. The subspaces $LS$ and $LU$ are transversal if and only if

$$m(U(x_n, v_n) - S(x_n, v_n)) > 0.$$  

The transversality of the Green bundles at the global minimizer is related to the viscosity solutions $\psi^\pm$. We will assume that the parameters $\omega$ and $b$ are chosen such that the viscosity solutions $\psi^\pm$ exist and are unique. We now state the main conclusion of this section.

**Proposition 4.3.** Assume that the function $\psi^-(\cdot, 0) - \psi^+(\cdot, 0)$ has a unique minimum at $x_0$, and that there exists $a > 0$ such that

$$d^2(\psi^- - \psi^+)(x_0, 0)(v) \geq a\|v\|^2, \quad v \in \mathbb{R}^d. \quad (16)$$

Let $(x_j, v_j)_{j=1}^n$ be the global minimizer corresponding to the minimum $x_0$. We have

$$m(U(x_0, v_0) - S(x_0, v_0)) \geq a.$$

The proof of Proposition 4.3 is split into several lemmas. The proofs of Lemma 4.4 and formula (17) below can be extracted from the context of [2]. We provide complete proofs for the convenience of the reader.

**Lemma 4.4.** Given $m < n$ in $\mathbb{Z}$, assume that $A(x_m, x_n)$ is differentiable at both $x_m$ and $x_n$, and that $(x_j, v_j)_{j=m}^n$ is the unique minimizer. Assume in addition that the orbit $(x_j, v_j)_{j=m}^n$ is disconjugate. Then the function $A_{m,n}(x, x')$ is $C^2$ on a neighbourhood of $(x_m, x_n)$. Furthermore,

$$U_{n-m}(x_m, v_m) = \partial_{22}^2 A_{m,n}(x_m, x_n), \quad S_{n-m}(x_m, v_m) = \partial_{11}^2 A_{m,n}(x_m, x_n).$$

The subderivative of a semi-concave function is upper semi-continuous as a set function. In particular, if the subderivative is unique at $x_0$, then any subderivative at $x_n \to x_0$ must converge to the derivative at $x_0$.

**Lemma 4.5** ([13], Proposition 8.7). Let $f(x)$ be a semi-concave function and $x_0$ be such that $f'(x_0)$ exists. Then for any $x_n \to x$ and $l_{x_n}$ any subderivative of $f(x)$ at $x_n$, we have

$$l_{x_n} \to f'(x_0).$$

**Proof of Lemma 4.4**. We have that $\Phi_{m,n}$ is a $C^1$ map in a neighbourhood of $(x_m, v_m)$ and that $\Phi_{m,n}(x_m, v_m) = (x_n, v_n)$. Since $(x_j, v_j)$ is a disconjugate orbit, $D\Phi_{m,n} V(x_m, v_m) \cap V(x_n, v_n) = \{0\}$. Denote

$$\Phi_{m,n}(x, v) = (x', v'), \quad D\Phi_{m,n}(x, v) = \begin{bmatrix} A_{m,n} & B_{m,n} \\ C_{m,n} & D_{m,n} \end{bmatrix} (x, v).$$
We have \( \det \frac{\partial x'}{\partial v}(x_m, v_m) = \det B_{m,n}(x_m, v_m) \neq 0 \). By the implicit function theorem, there exists unique \( C^1 \) functions \( v = v(x, x') \) and \( v' = (x, x') \) such that 
\[ \Phi_{m,n}(x, v(x, x')) = (x', v'(x, x')) \]
in a neighbourhood of \( \{(x_m, v_m)\} \times \{(x_n, v_n)\} \).

Let \((x, x')\) be sufficiently close to \((x_m, x_n)\). Let \((y_i, w_i)_{j=m}^n\) be any minimizing orbit for \( A_{m,n}(x, x') \) and denote \( v = w_m, v' = w_n \). By Lemma 3.2 and the assumption that \( A_{m,n} \) is differentiable, \( v' \rightarrow v_n \) and \( v \rightarrow v_m \) as \((x, x') \rightarrow (x_m, x_n)\). Assume that \((x, v, x'v')\) is so close to \((x_m, v_m, x_n, v_n)\) that the implicit function theorem applies. Then we have that \( v = v(x, x') \) and \( v' = v'(x, x') \) are well defined \( C^1 \) functions in \( x \) and \( x' \). Since \( v = -\partial_1 A_{m,n}(x, x') + b \) and \( v' = \partial_2 A_{m,n}(x, x') + b \), we conclude that \( A_{m,n}(x, x') \) is a \( C^2 \) function.

Viewing \( x', v' \) as functions of \((x, v)\), we have that 

\[ \partial_2 A_{m,n}(x, x'(x, v)) = v'(x, v) - b. \]

Differentiating both sides with respect to \( v \), we have

\[ \partial_{22}^2 A_{m,n}(x, x_n, v_n) = \left( \frac{\partial x'}{\partial v} \right)^{-1} \left( \frac{\partial v'}{\partial v} \right) (x_m, v_m) = (B_{m,n})^{-1}(D_{m,n})(x_m, v_m). \]

Using the definition of \( U_{m-n} \), we have \((B_{m,n})^{-1}(D_{m,n})(x_m, v_m) = U_{m-n}(x_m, v_n)\). The conclusion about \( S_{m-n}(x_n, v_n) \) can be proved similarly using the map \( \Phi_{m,n}^{-1} \).

**Lemma 4.6.** Assume that \((x_j, v_j)_{j \in \mathbb{Z}}\) is a global minimizer. Then for \( k \in \mathbb{N} \) and \( w \in \mathbb{R}^d \), we have

\[ \langle \partial_{22}^2 A_{-k,0}(x_k, x_0)w, w \rangle \geq 2d^2 \psi^-(x_0, 0)(w), \]

and

\[ \langle \partial_{11}^2 A_{0,k}(0, v_0)w, w \rangle \geq 2d^2 \psi^+(x_0, 0)(w). \]

**Proof.** Since \( K_{-k,0} \psi^-(x, -k) = \psi^-(x, 0) \), there exists \( x_{-k} \in \mathbb{T}^d \) such that

\[ \psi^-(x_0, 0) = \inf_{x' \in \mathbb{T}^d} \{ \psi^-(x', -k) + A_{-k,0}(x', x_0) \} = \psi^-(x_{-k}, -k) + A_{-k,0}(x_{-k}, x_0). \]

For any other \( x \in \mathbb{T}^d \), we have

\[ \psi^-(x, 0) = \inf_{x' \in \mathbb{T}^d} \{ \psi^-(x', -k) + A_{-k,0}(x', x) \} \leq \psi^-(x_{-k}, -k) + A_{-k,0}(x_{-k}, x). \]

Furthermore,

\[ \partial \psi^-(x_0, 0) = \partial_2 A_{-k,0}(x_{-k}, x_0). \]

Combining the last three formulas, we have

\[ \psi^-(x, 0) - \psi^-(x_0, 0) - \langle \nabla \psi^-(x_0, 0), x - x_0 \rangle \leq A_{-k,0}(x_k, x) - A_{-k}(x_{-k}, x_0) - \langle \partial_2 A_{-k,0}(x_{-k}, x_0), x - x_0 \rangle. \quad (17) \]

Take \( x - x_0 = \tau w \), divide by \( \tau^2 \) and take lower limit as \( \tau \to 0^+ \), we conclude that

\[ d^2 \psi^-(x_0, 0)(w) \leq \langle \partial_{22}^2 A(x_k, x_0)w, w \rangle. \]
The first inequality of the lemma follows.

On the other hand, we have the following statements about \( \psi^+ \): There exists \( x_k \in \mathbb{T}^d \) such that
\[
\psi^+(x_0, 0) = \psi^+(x_k, k) - A_{0,k}(x_0, x_k).
\]
\[
\nabla \psi^+(x_0, 0) = -\partial_1 A_{0,k}(x_0, x_k).
\]
For \( x \in \mathbb{T}^d \),
\[
\psi^+(x, 0) \geq \psi^+(x_k, k) - A_{0,k}(x, x_k).
\]
By a similar calculation as in the first case, we arrive at the second inequality of the lemma.

Proposition 4.3 follows from Lemma 4.4, 4.6 and (16).

5. Nonzero Lyapunov exponents

Recall that the family of maps \( \Phi^j_\omega \) may be viewed as a single transformation \( \hat{\Phi} \) acting on \( \mathbb{T}^d \times \mathbb{R}^d \times \Omega \) given by (13). Define \( S, U : \Omega \to M(d) \) by \( S(\omega) = S(x^\omega_0, v^\omega_0, \omega) \) and \( U(\omega) = U(x^\omega_0, v^\omega_0, \omega) \), where \( (x^j_\omega, v^j_\omega) \in \mathbb{Z} \) is the unique global minimizer guaranteed by Theorem 2.2. We have
\[
S(x^\omega_n, v^\omega_n) = S(\theta^n \omega), \quad U(x^\omega_n, v^\omega_n) = U(\theta^n \omega).
\]

With the assumptions of Theorem 2.3, the assumptions of Propositions 3.10 and 3.11 are satisfied for \( V(x, c) = \sum_{i=1}^M c_i F_i \). By combining Proposition 3.10 and Proposition 4.3, we obtain that there exists a positive measurable function \( a(\omega) \) such that
\[
m(U(\omega) - S(\omega)) \geq a(\omega).
\]
Moreover, using Proposition 3.11, we have
\[
\int a(\omega)^{-1} dP(\omega) \leq A(F) < \infty.
\]
It suffices to show these estimates imply Theorem 2.3.

Before discussing the Lyapunov exponents of the cocycle \( D^\omega_0 \), we need to show that it is well defined. We first describe some properties of the symplectic map \( D\Phi^j_\omega \). To abbreviate notations, we omit the superscript \( \omega \). We have
\[
D\Phi^j_\omega = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix},
\]
where
\[
A_j = I + \partial^2 F_j, \quad B_j = D_j = I, \quad C_j = -\partial^2 F.
\]
Although we have an explicit formula for the matrix, the discussions that follow only use \( \det B_j \neq 0 \) and some norm estimates.

Any symplectic matrix given in the block form \([A, B; C, D]\) has the following properties.

- \( A^T C = C^T A, B^T D = D^T B, A^T D = C^T B = I \).
\[ AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I. \]

From the explicit formula (20) for \( D\Phi^\omega \) and its inverse, we have that \( \| D\Phi^\omega \|, \| D(\Phi^\omega)^{-1} \| \leq 1 + C_\omega^\omega = \| \partial^2 F^\omega \|. \)

Since \( \mathbb{E}(1 + C_\omega^\omega) < \infty \), we have \( \mathbb{E}(\log(1 + C_0^\omega)) < \infty \). It follows that
\[
\log^+ \| D\Phi^\omega \|, \log^+ \| D(\Phi^\omega)^{-1} \| \in L^1(\nu),
\]
where \( \log^+ (x) = \max\{\log x, 0\} \). By Oseledets’ theorem for cocycles (see [4], Theorem 3.4.3), the Lyapunov exponents \( \lambda_i(\nu) \) are well defined. We say that a positive function \( g: \Omega \to \mathbb{R} \) is tempered if
\[
\lim_{n \to \pm \infty} \frac{1}{n} \log g(\theta^n \omega) = 0.
\]

We have the following lemma:

**Lemma 5.1** ([4], Lemma 2.1.5). If \( \log^+ g(\omega), \log^+ (g(\omega)^{-1}) \in L^1(d\omega) \), then \( g \) is tempered.

The inequality in the following Proposition 5.2 was proved in [3] for hyperbolic minimal measures of a twist map. In order to use this inequality to prove hyperbolicity, the new ingredient ingredient (19) is needed. The basic idea of our proof is similar to that of [3], Theorem 4.

**Proposition 5.2.** Assume that the Green bundles defined along the global minimizer \((x_j, v_j)\) satisfies (18) and (19). Then we have
\[
\lambda_{d+1}(\nu) \geq \frac{1}{2} \int \log \left( 1 + \frac{1}{C^\omega_0 + 1} m(U(\omega) - S(\omega)) \right) dP(\omega) > 0.
\]

**Remark.** Theorem 2.3 follows from this proposition.

**Proof.** The statement that the bundles \( S(x_n, v_n) \) and \( U(x_n, v_n) \) are invariant corresponds to the following statement: Given a vector \( h \in H(x_j, v_j) \), if \( D\Phi_j(h, S(x_j, v_j)) = (h', w') \), then \( w' = S(x_{j+1}, v_{j+1})h' \). To further simplify notations, let us write \( z_j = (x_j, v_j) \), \( S_j = S(z_j) \) and \( U_j = U_j(z_j) \). Expressing this relation in the matrix form, we have:
\[
S_{j+1}(A_j + B_j S_j) = C_j + D_j S_j.
\]

By the same reasoning, we have:
\[
U_{j+1}(A_j + B_j U_j) = C_j + D_j U_j.
\]

We would like to understand the matrix product \( D\Phi_{m,n}(x_m) = \Pi_{j=m}^{n-1} D\Phi_j(x_j) \) by introducing a change of coordinates. Let
\[
Q = \begin{bmatrix} I & I \\ U & S \end{bmatrix} \begin{bmatrix} (U - S)^{-\frac{1}{2}} & 0 \\ 0 & (U - S)^{-\frac{1}{2}} \end{bmatrix}.
\]
By a direct calculation, we have that $Q$ is a symplectic matrix, and
\[
Q^{-1} = \begin{bmatrix} (U - S)^{-\frac{1}{2}} & 0 \\ 0 & (U - S)^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} -S & I \\ U & -I \end{bmatrix}.
\]

Note that
\[
\begin{bmatrix} -S_{j+1} & I \\ U_{j+1} & -I \end{bmatrix} \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \begin{bmatrix} I & I \\ U_j & S_j \end{bmatrix} =
\begin{bmatrix} -S_{j+1} (A_j + B_j U_j) + (C_j + D_j U_j) & 0 \\ 0 & U_{j+1} (A_j + B_j S_j) + (C_j + D_j S_j) \end{bmatrix}
\]
\[
= \begin{bmatrix} (U_{j+1} - S_{j+1}) (A_j + B_j U_j) & 0 \\ 0 & (U_{j+1} - S_{j+1}) (A_j + B_j S_j) \end{bmatrix}
\]

The last line of the above calculation is due to (21) and (22). We obtain
\[
Q(z_{j+1})^{-1} D\Phi_j (z_j) Q(z_j) := \begin{bmatrix} M_j & 0 \\ 0 & N_j \end{bmatrix},
\]
where
\[
M_j = (U_{j+1} - S_{j+1}) (A_j + B_j U_j) (U_j - S_j)^{-\frac{1}{2}},
\]
\[
N_j = (U_{j+1} - S_{j+1}) (A_j + B_j S_j) (U_j - S_j)^{-\frac{1}{2}}.
\]

Since the matrix is symplectic, we have $(M_j)^T N_j = I$. We have the following computation:
\[
M_j^T M_j = N_j^{-1} M_j
\]
\[
= (U_j - S_j)^{\frac{1}{2}} (A_j + B_j S_j)^{-1} (A_j + B_j U_j) (U_j - S_j)^{-\frac{1}{2}}
\]
\[
= I + (U_j - S_j)^{\frac{1}{2}} (B_j^{-1} A_j + S_j)^{-1} (U_j - S_j) (U_j - S_j)^{-\frac{1}{2}}
\]
\[
= I + (U_j - S_j)^{\frac{1}{2}} (B_j^{-1} A_j + S_j)^{-1} (U_j - S_j)^{\frac{1}{2}}
\]

We claim that the matrix $B_j^{-1} A_j + S_j$ is positive definite and that the conorm
\[
m((B_j^{-1} A_j + S_j)^{-1}) \geq \frac{1}{1 + C_j^2}.	ag{23}
\]

To see this, recall that the subspace $S_1(z_j)$ is defined by $D(\Phi_j)^{-1} V(z_j)$. Take a vertical vector $(0, \delta v) \in V(z_j)$, we have that
\[
D(\Phi_j)^{-1} \begin{bmatrix} 0 \\ \delta v \end{bmatrix} = \begin{bmatrix} D_j^T & -B_j^T \\ -C_j^T & A_j^T \end{bmatrix} \begin{bmatrix} 0 \\ \delta v \end{bmatrix} = \begin{bmatrix} -B_j^T \delta v \\ A_j^T \delta v \end{bmatrix}.	ag{24}
\]

It follows that $S_1(z_j) = - (B_j^{-1} A_j)^T = - B_j^{-1} A_j$. Since $S_j > S_1(z_j)$, we have that $B_j^{-1} A_j + S_j = S_j - S_1(z_j) > 0$. Since $U_1(z_j) > U_j > S_j$, we have
\[
m((B_j^{-1} A_j + S_j)^{-1}) = \| B_j^{-1} A_j + S_j \|^{-1} \geq \| U_1(z_j) - S_1(z_j) \|^{-1}.
\]
By a calculation similar to the one in (24), we have $U_1(z_j) = \mathcal{D}_{j-1} B_{j-1}^{-1}$. From the explicit formula (20) of $D\Phi_j$, it is easy to see that $\|U_1(z_j) - S_1(z_j)\| \leq 1 + C_0^\omega$. The estimate (23) follows.

Using the estimates obtained, we have that
\begin{equation}
m(M_j^T M_j) \geq 1 + \frac{1}{1 + C_j^\omega} m(U_j - S_j)).
\end{equation}

We are now ready to estimate the Lyapunov exponent $\lambda_{d+1}(\nu)$. Since $D\Phi_{0,n} = Q(z_{n-1}) \prod_{j=n-1}^0 [M_j 0 0 N_j] Q(z_0)$, we have
\begin{align*}
(Q(z_{n-1})^{-1} D\Phi_{0,n})^T (Q(z_{n-1})^{-1} D\Phi_{0,n}) &= Q(z_0)^T \begin{bmatrix} M_{n,0}^T M_{n,0} & 0 \\ 0 & N_{n,0}^T N_{n,0} \end{bmatrix} Q(z_0),
\end{align*}
where $M_{n,0} = \prod_{j=n-1}^0 M_j$ and $N_{n,0} = \prod_{j=n-1}^0 N_j$. We have the following estimates:
\begin{equation}
m(M_{n,0}^T M_{n,0}) = m((N_{n,0}^T N_{n,0})^{-1}) \geq \prod_{j=0}^{n-1} \left( 1 + \frac{1}{1 + C_j^\omega} m(U_j - S_j) \right).
\end{equation}

Consider a vector $w \in \mathbb{R}^{2d}$ and let $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) = Q(z_0)w$. If $\tilde{w}_1 \neq 0$, then
\begin{align*}
\liminf_{n \to \infty} \frac{1}{n} \log \|Q(z_{n-1})^{-1} D\Phi_{0,n} w\|^2 &\geq \liminf_{n \to \infty} \frac{1}{n} \log \prod_{j=0}^{n-1} \left( 1 + \frac{1}{1 + C_j^\omega} m(U_j - S_j) \right) \|\tilde{w}_1\|^2 \\
&= \liminf_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left( 1 + \frac{1}{1 + C_j^\omega} m(U_j - S_j) \right).
\end{align*}

If $\tilde{w}_1 = 0$, then
\begin{align*}
\limsup_{n \to \infty} \frac{1}{n} \log \|Q(z_{n-1})^{-1} D\Phi_{0,n} w\|^2 \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left( 1 + \frac{1}{1 + C_j^\omega} m(U_j - S_j) \right).
\end{align*}
The norm $\|Q(z_{n-1})\| \leq (1 + C_{n-1}^\omega) m(U_{n-1} - S_{n-1})^{-\frac{1}{2}}$.

Since $1 \leq 1 + C_0^\omega \leq \sup_{i=1}^M \|F_i\|_{c} |\xi_0(\omega)|$ and $\mathbb{E} \xi_0 < \infty$, we have that $\log(1 + C_0^\omega) \in L^1(dP(\omega))$. On the other hand, since
\begin{equation*}
a(\omega) \leq U(\omega) - S(\omega) \leq U_1(z_0^\omega) - S_1(z_0^\omega) \leq 1 + C_0^\omega,
\end{equation*}
we know that $\log^+(U(\omega) - S(\omega))$, $\log^+(U(\omega) - S(\omega))^{-1} \in L^1(dP(\omega))$. Using Lemma 5.1, we have
\begin{align*}
\lim_{n \to \infty} \frac{1}{n} \log(1 + C_{n-1}^\omega) = 0, \quad \lim_{n \to \infty} \frac{1}{2n} \left| \log m(U(z_{n-1}^\omega) - S(z_{n-1}^\omega)) \right| = 0.
\end{align*}
We now finish the proof of the proposition. If $\tilde{w}_1 \neq 0$, using (26), (27) and the Birkhoff ergodic theorem, we have

$$
\liminf_{n \to \infty} \frac{1}{n} \log \|D\Phi_{0,n} w\| \geq \int \frac{1}{2} \log \left(1 + \frac{1}{1 + C_0^\omega} m((U - S)(z_0^\omega))\right) dP(\omega)
$$

$$
\quad - \limsup_{n \to \infty} \frac{1}{n} \left(\log(1 + C_{n-1}^\omega) - \frac{1}{2} \log m(U_{n-1}^\omega - S_{n-1}^\omega)\right)
$$

$$
\quad = \int \frac{1}{2} \log \left(1 + \frac{1}{1 + C_0^\omega} m((U - S)(z_0^\omega))\right) dP(\omega) > 0.
$$

Note that for $\tilde{w}_1 = 0$ we have

$$
\limsup_{n \to \infty} \frac{1}{n} \log \|D\Phi_{0,n} w\| \leq - \int \frac{1}{2} \log \left(1 + \frac{1}{1 + C_0^\omega} m((U - S)(z_0^\omega))\right) dP(\omega) < 0.
$$

It follows that the Lyapunov exponents are nonzero.

6. Dynamics near the global minimizer

We prove Theorem 2.4 in the following two sections. By assumption, the potentials $F_j$ are $C^{2+\alpha}$ for some $0 < \alpha \leq 1$. We shall abuse notation and denote

$$
C_j^\omega = \|F_j\|_{2+\alpha}.
$$

By Assumption 5, $\mathbb{E}(C_j^\omega) < \infty$.

As the global minimizer $(x_j, v_j)$ is hyperbolic, we will apply the theory of nonuniform hyperbolic systems (Pesin’s theory) to obtain the local stable manifolds $W^s(x_j, v_j)$ and unstable manifolds $W^u(x_j, v_j)$. We restate the unstable part of Theorem 2.4 as follows.

**Theorem 6.1.** There exists a neighbourhood $U(\omega)$ of $(x_0, v_0)$ such that

$$
W^-_0 := \{ (x, \nabla_x \psi^-(x, 0)) \} \cap U \subset W^u(x_0, v_0).
$$

Furthermore, there exists a neighbourhood $V(\omega)$ of $x_0$ such that $\psi^-(x, 0)$ are smooth on $V(\omega)$.

**Remark.** We can prove

$$
W^+_0 := \{ (x, -\nabla_x \psi^+(x, 0)) \} \cap U \subset W^s(x_0, v_0).
$$

and $\psi^+(x, 0)$ is smooth on a neighbourhood $V(\omega)$ by reversing time. Theorem 2.4 follows.

In this section, we will focus on studying the dynamics of orbits close to the global minimizer. We will use the information obtained in this section to prove Theorem 6.1 in the next section. In the theory of nonuniformly hyperbolic systems, discussions are often made simpler with the use of a special Lyapunov norm. For our purpose,
however, it is more natural to use the standard norm. We prove a version of the classical graph transform theorem in Proposition 6.4 and show that it applies to our map family $\Phi_j$. The main goal is to obtain precise estimates of the expansion/contraction of the map family in the standard norm. These estimates will be useful in the next section.

The methods used here are well known to the experts of the field; we will provide proofs for some and refer to [4] for others.

In the proof of Theorem 5.2, we obtained the following reduction of the cocycle $D\Phi_{m,n}(x_j,v_j)$. Let $z_j$ denote $(x_j,v_j)$ and

$$Q(z_j) = \begin{bmatrix} I & I \\ U(z_j) & S(z_j) \end{bmatrix} \begin{bmatrix} (U - S)(z_j)^{-\frac{1}{2}} & 0 \\ 0 & (U - S)(z_j)^{-\frac{1}{2}} \end{bmatrix},$$

we have $L(z_j) := Q(z_{j+1})^{-1}D\Phi_jQ(z_j) = \text{diag}\{M_j, N_j\}$. Here

$$M_j = (U_{j+1} - S_{j+1})^{\frac{1}{2}}(A_j + B_jU_j)(U_j - S_j)^{-\frac{1}{2}},$$

and $N_j = (M_j^T)^{-1}$.

We will also use a standard construction in nonuniform hyperbolic theory known as the tempering kernel.

**Lemma 6.2** (See [4], Lemma 3.5.7.) Assume that $g : \Omega \to \mathbb{R}$ is a tempered function, that is $\lim_{n \to \pm\infty} \frac{1}{n} \log g(\theta^n \omega) = 0$, a.e $\omega$, then for any $\epsilon > 0$ there exists positive functions $K^1, K^2 : \Omega \to \mathbb{R}$ such that

$$K^1(\omega) \leq g(\omega) \leq K^2(\omega), \quad e^{-\epsilon} \leq \frac{K^i(\theta^i \omega)}{K_i(\omega)} \leq e^\epsilon, \quad i = 1, 2, \quad \text{a.e.} \omega.$$

$$\lambda_j(\omega) = \max \left\{ \log \left( 1 + \frac{m(U_j(\omega) - S_j(\omega))}{1 + C_j^\omega} \right), 1 \right\}.$$  

According to [25], we have

$$\|M_j^{-1}\| \geq e^{\lambda_j}, \quad \|N_j\| \leq e^{-\lambda_j}.$$ 

We have $\lambda_j(\omega) = \lambda_0(\theta^j \omega)$, and $1 \geq \lambda_0 \geq m(U_0(\omega) - S_0(\omega)) \geq a(\omega)$, hence $\lambda_0(\omega)$ is integrable. Denote $\lambda = \mathbb{E}\lambda_0(\omega)$.

**Lemma 6.3.** For any $\epsilon > 0$, there exists $\kappa(\omega), K_0(\omega) > 0, a_0(\omega) > 0$ satisfying

$$e^{-\epsilon} \leq \frac{K_0(\theta^i \omega)}{K_0(\omega)}, \quad \frac{a_0(\theta^i \omega)}{a_0(\omega)} \leq \frac{\kappa_0(\theta^i \omega)}{\kappa_0(\omega)} \leq e^\epsilon$$

such that $1 + C_0(\omega) \leq K_0(\omega)$, $a_0(\omega) \leq m(U_0(\omega) - S_0(\omega))$, and

$$\prod_{k=0}^{n-1} \lambda_{\pm k}(\omega) \geq \kappa_0(\omega)e^{n(\lambda - \epsilon)}.$$
Proof. The existence of $K_0$ and $a_0$ follows from (27) and Lemma 6.2. For the existence of $\kappa_0$, we define

$$\Gamma_\pm(\omega) = \sup \left\{ 1, \frac{(1 + \lambda)^n e^{-\epsilon n}}{\prod_{k=0}^{n-1} (1 + \lambda_0(\theta^k \omega))}, n \geq 1 \right\}.$$ 

Note that

$$\min \left\{ 1, \frac{1 + \lambda}{1 + \lambda_0(\omega)} e^{-\epsilon} \right\} \leq \frac{\Gamma_\pm(\theta \omega)}{\Gamma_\pm(\omega)} \leq \max \left\{ 1, \frac{1 + \lambda_0(\omega)}{1 + \lambda} e^\epsilon \right\},$$

we have that $\log \frac{\Gamma_\pm(\theta \omega)}{\Gamma_\pm(\omega)} \in L^1(dP(\omega))$. Apply the Birkhoff ergodic theorem to $\log \frac{\Gamma_\pm(\theta \omega)}{\Gamma_\pm(\omega)}$, we obtain that $\Gamma_\pm(\omega)$ is tempered.

Apply Lemma 6.2 to $\Gamma_\pm(\omega)$, we obtain positive functions $\kappa_0^\pm(\omega) \leq \frac{1}{\Gamma_\pm(\omega)}$ with

$$e^{-\epsilon} \leq (\kappa_0^\pm(\theta \omega))/(\kappa_0^\pm(\omega)) \leq e^\epsilon.$$

Furthermore,

$$\prod_{k=0}^{n-1} e^{\lambda_{\pm k}} \geq \frac{1}{\Gamma_\pm(\omega)} e^{\lambda n} e^{-\epsilon n} \geq \kappa_0^\pm(\omega) e^{n(\lambda-\epsilon)}.$$ 

The lemma follows by taking $\kappa_0 = \min\{\kappa_0^+, \kappa_0^\pm\}$. \hfill \square

Going forward, we will choose the functions $a_0$, $K_0$, $\kappa_0$ using Lemma 6.3. However, the parameter $\epsilon$ with which we apply the lemma will be decided later. We will also denote $a_j = a_0 \circ \theta^j$, $K_j = K_0 \circ \theta^j$, $\kappa_j = \kappa_0 \circ \theta^j$.

Let $P_j$ be a map from $\mathbb{R}^d \times \mathbb{R}^d$ to a neighbourhood of $(x_j, v_j)$ defined by

$$P_j(u, s) = Q(x_j, v_j)(u, s) + (x_j, v_j).$$

For $r > 0$, we define the local diffeomorphisms $\tilde{\Phi}_j : B(0, r) \cap \mathbb{R}^d \to \mathbb{R}^d$ by

$$\tilde{\Phi}_j = P_j^{-1} \circ \Phi_j \circ P_j | B(0, r).$$

The local diffeomorphisms satisfy the following properties:

- $\tilde{\Phi}_j(0, 0) = (0, 0)$.
- $D\tilde{\Phi}_j(0, 0) = \text{diag}\{M_j, N_j\}$.
- Since $\|\Phi_j\| \leq K_j$ and $\|Q_j\|, \|Q_j^{-1}\| \leq K_j a_j^{-\frac{1}{2}}$, we have $\|\tilde{\Phi}_j\|_{1+\alpha} \leq K_j^3 a_j^{-1}$.

This setting is the most general setting on which the Hadamard-Perron theorem can be established.

Given $\rho_j > 0$ to be chosen later, we define $\sigma_j = \|D\tilde{\Phi}_j(u, s) - D\tilde{\Phi}_j(0, 0)\|_{B_{\rho_j}(0, 0)}$.

For $\gamma > 0$, we define the unstable and stable cone field by

$$C^u_\gamma = \{(u, s) \in \mathbb{R}^{2d}; \|s\| \leq \gamma \|u\|\}, \quad C^s_\gamma = \{(u, s) \in \mathbb{R}^{2d}; \|u\| \leq \gamma \|s\|\}.$$ 

We say a set $W \subset \mathbb{R}^{2d}$ is $(u, \gamma, \rho_j, j)$–admissible if there exists a $\gamma$–Lipschitz map $\varphi : \mathbb{R}^d \cap B(0, \rho_j) \to \mathbb{R}^d$ such that $\varphi(0) = 0$ and $W = \{(u, \varphi(u))\}$. A set $W \subset \mathbb{R}^{2d}$ is
(s, γ, ρ_j, j)–admissible if there exists a γ–Lipschitz map ϕ : R^d ∩ B(0, ρ_j) → R^d such that ϕ(0) = 0 and W = {((ϕ(s), s)}. Denote D_ρ = {(u, s) ∈ R^{2d}; ||u|| ≤ ρ, ||s|| ≤ ρ}.

**Proposition 6.4.** Assume that for j ≤ 0, the parameters ρ_j and σ_j satisfy the following conditions:

- e^{-λ_j} + 2σ_j < e^{-λ_j/2},
- e^{λ_j} - 2σ_j > e^{λ_j/2},
- e^{λ_j/2}ρ_j ≥ ρ_{j+1}.

Then the following hold:

1. For any (u, s) ∈ D_{γ_j, ρ_j}, we have DΦ_j(u, s)C_{u_1}^u ⊂ C_{γ_j}^{u_1}, where γ_j = e^{-λ_j/2}.

2. If W is a (u, 1, ρ_j, j)–admissible set, then G(W) := Φ_j(W) ∩ D_{ρ_{j+1}} is a (γ_j, ρ_{j+1}, j+1)–admissible set. Furthermore, for (u, s_j) ∈ W, let (u_{j+1}, s_{j+1}) = Φ_j(u, s_j), we have

   \[ ||u_{j+1}|| ≥ e^{λ_j/2}||u_j||.\]

3. If W is a (s, 1, ρ_j, j)–admissible set, then G^{-1}(W) := Φ^{-1}_j(W) ∩ D_{ρ_{j-1}} is a (γ_{j-1}, ρ_{j-1}, j-1)–admissible set. Furthermore, for (u, s_j) ∈ W, let (u_{j-1}, s_{j-1}) = Φ^{-1}_j(u, s_j), we have

   \[ ||s_{j+1}|| ≥ e^{λ_j/2}||s_j||.\]

4. Let (W_j)_{j=0}^{−∞} be a sequence of (u, 1, ρ_j, j)–admissible sets, then the map Φ_j induces a map G(W)_{j+1} := G(W_{j−1}) on the set of such sequences. We have that this sequence is a contraction in Lipshitz norm, and there exists a unique limit sequence. Furthermore, this limit sequence is C^1.

**Proof.** Let (δu, δs) ∈ R^{2d} satisfy ||δs|| ≤ ||δu||, and let (δu', δs') = DΦ_j(u, s)(δu, δs). We have

\[ ||δs'|| ≤ e^{-λ_j}||δs|| + σ_j||δu, δs||, \quad ||δu'|| ≥ e^{λ_j}||δu|| - σ_j||δu, δs||. \tag{28} \]

Using ||δs|| ≤ ||δu||, we obtain ||δs'|| ≤ γ_j||δu'||, where γ_j = e^{-λ_j+2σ_j}. This proves the first statement of our proposition.

To show the image of an unstable admissible manifold is still admissible, the calculation is similar and we refer to Proposition 7.3.5 of [4]. The expansion in u component is a consequence of (28). This proves the second statement. The third statement follows from a symmetric argument.

For the fourth statement, by Proposition 7.3.6 of [4], the map G(W_j) is a contraction with factor e^{-λ_j−1/2} at the j−th component. By Lemma 6.3, we know that \[ \Pi_{k=−n}^{−1} e^{-λ_j+k/2} ≤ \kappa_j^− e^{n(λ+ε)}. \] For ε < λ, there exists n > 0 such that all \[ \Pi_{k=−n}^{−1} e^{-λ_j+k/2} < 1. \] This implies that G^n is a uniform contraction. \[ \square \]

We can choose parameters such that the conditions of Proposition 6.4 is satisfied. First, we have the following lemma:
Lemma 6.5. Given a positive function \( \rho_0 : \Omega \rightarrow \mathbb{R} \), we define \( \rho_j(\omega) = \rho_0 \prod_{k=j}^{-1} e^{-\lambda_j/4} \) for all \( j \leq -1 \). Then for any positive function \( R : \Omega \rightarrow \mathbb{R} \) such that \( e^{-\epsilon} \leq R \circ \theta \leq e^\epsilon \), there exists \( \rho_0 \) such that

\[
\rho_j \leq R \circ \theta^j, \quad j \leq -1 \text{ and } e^{-2\epsilon} \leq \frac{\rho_0 \circ \theta}{\rho_0} \leq e^{2\epsilon}.
\]

Proof. By Lemma 6.3

\[
\rho_j = \rho_0 \prod_{k=j}^{-1} e^{-\lambda_j/4} \leq \rho_0 \kappa_0^{-1} e^{-\lambda_j/4} |j|.
\]

Take \( \rho_0 = R \kappa_0^{-1/4} \), we have that for \( \lambda > 2\epsilon \),

\[
\rho_j \leq R e^{-\lambda_j |j|} \leq Re^{\epsilon |j|} \leq R \circ \theta^j.
\]

\( \square \)

Proposition 6.6. There exists \( \rho_0 = \rho_0(\omega) > 0 \) satisfying

\[
e^{-\epsilon} \leq \frac{\rho_0(\theta \omega)}{\rho_0(\theta)} \leq e^\epsilon,
\]

such that for \( \rho_j(\omega) = \rho_0 \prod_{k=j}^{-1} e^{-\lambda_j/4} \), the conditions of Proposition 6.4 are satisfied for all \( j \leq 0 \).

Proof. It’s easy to see that the first 2 conditions of Proposition 6.4 are satisfied if \( \sigma_j < \lambda_j/8 \). Since \( \sigma_j \leq \|\tilde{\Phi}_j\|_{1+a_1} \rho_0^\alpha = K_j^3 a_j^{-1} \rho_0^\alpha \) and \( a_j/2K_j \leq \lambda_j \), it suffice to have

\[
\rho_j^\alpha \leq \frac{a_j^2}{20 K_j^4}, \quad j \leq 0.
\]

Let \( R = \frac{a_0^{2/\alpha}}{20 K_0^{4/\alpha}} \), by re-choosing the functions \( a_0, K_0 \) using Lemma 6.3 and parameter \( \epsilon/(12\alpha) \), we can guarantee \( e^{\epsilon/2} \leq R \circ \theta/R \leq e^{\epsilon/2} \). By Lemma 6.5, there exists \( \rho_0 \) such that \( \rho_j \leq R \circ \theta^j \) and \( e^{-\epsilon} \leq \rho_0 \circ \theta/\rho_0 \leq e^{\epsilon} \).

For \( \gamma < 1 \), any \( \gamma \)-admissible stable or unstable manifolds is bi-Lipshitz mapped to the horizontal direction. This relation is crucial in linking variationally defined objects and those coming from hyperbolicity. In particular, we will need the following lemma.

Lemma 6.7. Assume that for \( 0 < \gamma < 1 \), \((y, w) = P_j(u, s) \) with \((u, s) \in D_{\rho_j} \cap C_{\gamma}^s \). Then

\[
(1 - \gamma) K_j^{-\frac{1}{2}} \|s\| \leq \|y - x_j\| \leq 2a_j^{-\frac{1}{2}} \|s\|.
\]

Proof. We have

\[
\begin{bmatrix}
y - x_j \\
w - v_j
\end{bmatrix} =
\begin{bmatrix}
I & I \\
U_j & S_j
\end{bmatrix}
\begin{bmatrix}
(U_j - S_j)^{-\frac{1}{2}}s \\
(U_j - S_j)^{-\frac{1}{2}}u
\end{bmatrix},
\]
hence $y - x_j = (U_j - S_j)^{-\frac{1}{2}}(s + u)$. It follows that
\[
\|y - x_j\| \leq \|(U_j - S_j)^{-\frac{1}{2}}\|s + u\| \leq a_j^{-\frac{1}{2}}(1 + \gamma)\|s\| \leq 2a_j^{-\frac{1}{2}}\|s\|,
\]
\[
\|y - x_j\| \geq m((U_j - S_j)^{-\frac{1}{2}})\|s + u\| \geq K_j^{-\frac{1}{2}}(1 - \gamma)\|s\|.
\]

\[\square\]

7. Local smoothness of the viscosity solutions

Roughly speaking, our proof of Theorem 6.1 is based on the following observation: near a hyperbolic orbit, any orbit that does not expand exponentially in the backward time must be contained in the unstable manifold. For $j \leq 0$, we denote $W^-_j = \{(x, \nabla_x \psi^-(x, j)\}$ and $\tilde{W}_j = P^{-1}_j W^-_j$. We will first show that the orbits contained in $\tilde{W}_j$ does not “expand exponentially” in backward time, with a technical assumption that, at the last iterate, the orbit’s unstable component is dominated by its stable component.

**Proposition 7.1.** The manifolds $\tilde{W}_j$ has the following properties:

1. $\tilde{W}_j$ is a family of invariant sets for the local diffeomorphisms $\tilde{\Phi}_j$, in that $\tilde{W}_{j+1} \subset \tilde{\Phi}_j \tilde{W}_j$.
2. There exists $\tilde{C}_j > 0$ and $R_j > 0$ satisfying
   \[
e^{-\epsilon} \leq \frac{\tilde{C}_{j+1}}{\tilde{C}_j}, \quad \frac{R_{j+1}}{R_j} \leq e^\epsilon,
   \]
such that the following hold. Given $0 < \gamma < 1$ and $k \geq 0$, let $(u_j, s_j) \in \tilde{W}_j$ be a backward orbit satisfying
   \[(a)\] $\|u_{j-i}\| + \|s_{j-i}\| \leq R_{j-i}$ for all $0 \leq i \leq k$,
   \[(b)\] $\|u_{j-k}\| \leq \gamma \|s_{j-k}\|$
   then we have
   \[
   \|s_{j-k}\| \leq (1 - \gamma)^{-1}\tilde{C}_je^{\epsilon k}\|s_j\|
   \]
   for all $k \geq 0$ such that $\|u_j\|' + \|s_j\|' \leq R_j$.

The proof of Proposition 7.1 relies on the following properties of the viscosity solutions.

**Lemma 7.2.** For $(y_0, w_0) \in W^-_0(x_0, v_0)$, denote $(y_j, w_j) = \Phi^{-1}_{j, 0}(y_0, w_0)$ for all $j < 0$. Then
\[
\Delta \psi(y_j, j) - \Delta \psi(x_j, j) \leq \Delta \psi(y_0, 0) - \Delta \psi(x_0, 0).
\]
Proof. Since the orbits \((x_k, v_k)_{k \leq 0}\) and \((y_k, w_k)_{k \leq 0}\) are backward minimizers, we have
\[
\psi^-(x_0, 0) = \psi^-(x_j, j) + A_{j,0}(x_j, x_0), \quad \psi^-(y_0, 0) = \psi^-(y_j, j) + A_{j,0}(y_j, y_0).
\]
Furthermore, since \((x_k, v_k)_{k \geq j}\) is also a forward minimizer, we have
\[
\psi^+(x_j, 0) = \psi^+(x_0, 0) - A_{j,0}(x_j, x_0).
\]
It follows from the definition of \(\psi^+\) that
\[
\psi^+(y_j, 0) \geq \psi^+(y_0, 0) - A_{j,0}(y_j, y_0).
\]
Hence
\[
\Delta \psi(y_0, 0) - \Delta \psi(x_0, 0) = \psi^-(y_0, 0) - \psi^+(y_0, 0) - \psi^-(x_0, 0) + \psi^+(x_0, 0)
\]
\[
= \psi^-(y_j, 0) - (\psi^+(y_0, 0) - A_{j,0}(y_j, y_0)) - \psi^-(x_j, 0) + (\psi^+(x_0, 0) - A_{j,0}(x_j, x_0))
\]
\[
\geq \psi^-(y_j, 0) - \psi^+(y_j, 0) - \psi^-(x_j, 0) + \psi^+(x_j, 0).
\]
\[\square\]

Proposition 7.3. There exists \(b(\omega) > 0\) with
\[
\int b(\omega)^{-\frac{1}{2}} dP(\omega) < \infty
\]
and a constant \(r(F) > 0\) depending only on \((F_1, \ldots, F_M)\), such that
\[
b(\omega)\|y - x_0\|^2 \leq \Delta \psi(y, 0) - \Delta \psi(x_0, 0) \leq (1 + C_0^\omega)\|y - x_0\|^2, \quad \text{for } \|y - x_0\| \leq r(F).
\]
Proof. The lower bound follows from Proposition 3.12 and the upper bound is a consequence of the semi-concavity (Lemma 3.2).

Apply Lemma 6.2 to \(b(\omega)\), we obtain function \(0 < b_0(\omega) \leq b(\omega)\) with \(e^{-\epsilon} \leq b_0(\theta \omega)/b_0(\omega) \leq e^\epsilon\). Write \(b_j(\omega) = b_0(\theta^j \omega)\). We now prove Proposition 7.1 using what we have obtained.

Proof of Proposition 7.1. Let \(R_j = K_j^{-1}a_j^{-\frac{1}{2}} r(F) \leq \|P_j\|^{-1} r(F)\). Let \((u_j, s_j) \in \tilde{W}_j\) be a backward orbit satisfying the conditions (a) and (b) in Proposition 7.1 and let \((y_j, w_j) = P_j(u_j, s_j)\), we have \(\|u_j\| + \|s_j\| \leq R_j\) implies \(\|y_j - x_j\| \leq r(F)\). By Lemma 7.2 and Proposition 7.3 we have that for \(j \leq 0\) and \(k \geq 0\),
\[
\frac{1}{2} b_{j-k} \|y_{j-k} - x_{j-k}\|^2 \leq \Delta \psi(y_{j-k}, j - k) - \Delta \psi(x_{j-k}, j - k)
\]
\[
\leq \Delta \psi(y_j, j) - \Delta \psi(x_j, j) \leq K_j \|y_j - x_j\|^2.
\]
By Lemma 6.7, we have
\[
\|y_j - x_j\| \leq 2a^{-\frac{1}{2}} \|s_j\|, \quad \|s_{j-k}\| \leq (1 - \gamma)^{-1} K_j^{\frac{1}{2}} \|y_j - x_j\|.
\]
Combine all three inequalities, we have
\[ \|s_{j-k}\| \leq \frac{1}{2}(1 - \gamma)^{-1}K_j^{-\frac{3}{2}}b_j^{-\frac{1}{2}}\|y_j - x_j\| \leq \frac{1}{2}(1 - \gamma)^{-1}K_j^{-\frac{3}{2}}b_j^{-\frac{1}{2}}\|y_j - x_j - s_{j-k}\|. \]
\[ \text{Let } \tilde{C}_j = K_j^{-\frac{3}{2}}b_j^{-\frac{1}{2}}a_j^{-\frac{1}{2}}, \text{ we obtain } \|s_{j-k}\| \leq (1 - \gamma)^{-1}\tilde{C}_je^{\epsilon k}\|s_j\|. \]
By re-choosing the functions \( K_0, b_0, a_0 \) again if necessary, we also have \( e^{-\epsilon} \leq \tilde{C}_{j+1}/\tilde{C}_j \leq e^\epsilon \). \( \square \)

We now prove Theorem 6.1 using Proposition 7.1.

Proof of Theorem 6.1. The proof proceeds in two steps. In the first step, we show that any backward orbit sufficiently close to the global minimizer must be contained in the unstable cone, i.e. \( \{(u, s) \in \mathbb{R}^{2d}, \|s\| \leq \|u\|\} \). In the second step, we show that any backward orbit that are contained in the unstable cone for every iterate must be contained in the unstable manifold.

Step one. By Lemma 6.2, we can choose \( \rho_0 \) satisfying conditions of Proposition 6.3 and \( \rho_j \leq R_j/2 \). Define
\[ \bar{r}_0 = \tilde{C}_0^{-3}K_0^{-a_0\rho_0^2}, \]
we claim that any \((u_0, s_0) \in \tilde{W}_0 \cap D_{\bar{r}_0}\) must satisfy
\[ \|s_0\| \leq \|u_0\|. \]
It suffices to show that \( \|u_0\| \leq \|s_0\| \) and \( \|s_0\| \leq \rho_0 \) implies \( \|s_0\| \geq \bar{r}_0 \). In this case, \((u_0, s_0)\) is contained in a \((s, 1, \rho_0, 0)-\text{admissible set. Let}\)
\[ m = \min\{i \geq 0 : (u_{-i}, s_{-i}) \notin D_{\rho_{-i}}\} - 1. \]
By Proposition 6.4, for all \( 0 \leq i \leq m \), \((u_{-i}, s_{-i})\) is contained in a \((s, \gamma_{-i}, \rho_{-i}, -i)-\text{admissible set, hence}\)
\[ \|u_{-i}\| \leq \gamma_{-i}\|s_{-i}\|. \]
By the definition of \( m \) and \( \|\Phi_j\| \leq K_j^{-3}a_j^{-1} \), we have
\[ K_{-m}^{-3}a_{-m}\rho_{-m} \leq \|s_{-m}\| \leq \rho_{-m}. \]
Apply (29), we have
\[ K_{-m}^{-3}a_{-m}\rho_{-m} \leq \|s_{-m}\| \leq (1 - \gamma_{-m})^{-1}\tilde{C}_0e^{\epsilon k}\|s_0\|. \]
We have \( (1 - \gamma_{-m})^{-1} = (1 - e^{-\lambda_{-m}/2})^{-1} \leq \lambda_{-m}^{-1} \leq a_{-m}^{-1}. \) It follows that
\[ K_{-m}^{-3}a_0^{2} \leq \tilde{C}_0e^{5\epsilon}\|s_0\|\rho_{-m}^{\frac{3}{5}}. \] (30)
Furthermore, by Proposition 6.4 part (3), and \( \|s_{-m}\| \leq \rho_{-m} \), we have
\[ \prod_{i=-m}^{-1} e^{\lambda_{i}/2}\|s_0\| \leq \|s_{-m}\| \leq \rho_{-m} = \rho_0 \prod_{i=-m}^{-1} e^{-\lambda_{i}/4}. \]
It follows that \( \|s_0\|^2 \leq \rho_{k}^{3} \), hence
\[ \|s_0\|^2 \leq \rho_{0}^{-\frac{2}{3}}\rho_{-m}. \] (31)
On the other hand, by Lemma 6.3
\[ \kappa_0^{\frac{1}{2}} e^{(\lambda-\epsilon)m/2} \|s_0\| \leq \rho_{-m} \leq 1. \]

We may choose \( \epsilon \) sufficiently small such that \( \frac{10e}{\lambda-\epsilon} < \frac{1}{3} \). We have
\[ e^{5\epsilon m} \leq \kappa_0^{-\frac{5\epsilon m}{\lambda-\epsilon}} \|s_0\|^{-\frac{10e}{\lambda-\epsilon}} \leq \kappa_0^{-\frac{5}{3}} \|s_0\|^{-\frac{1}{3}}. \]

By (30) and (31), we have
\[ K_0^{-3}a_0^2 \leq \bar{C}_0 e^{5\epsilon m} \|s_0\|^{-1} = \bar{C}_0 \kappa_0^{-\frac{2}{3}} \|s_0\|^{-\frac{1}{3}} \leq K_0^{-3}a_0^{-6} \|s_0\|^{-\frac{2}{3}}, \]

hence
\[ \|s_0\| \geq \bar{C}_0^{-3} K_0^{-9} a_0^{-6} \rho_0^{-2}. \]

This concludes the first step.

*Step two.* For \( j \leq 0 \), define \( \tilde{r}_j(\omega) = \tilde{r}_0(\theta^j \omega) \). We may apply step one to \( \theta^j \omega \) instead of \( \omega \), and obtain that for any \((u_j, s_j) \in \tilde{W}_j \cap D_{\tilde{r}_j} \), \( \|s_j\| \leq \|u_j\| \). Define
\[ r_0 = \tilde{r}_{-1} K_{-1}^{-3} a_{-1}^{-1} \]

and \( r_j = r_0 \prod_{k=j}^{1-1} e^{-\lambda_k/2} \) for all \( j \leq -1 \). Similar to Lemma 6.5 by choosing a small \( \epsilon \), we have
\[ r_j \leq \tilde{r}_{j+1} K_{j+1}^{-3} a_{j+1}^{-1} \]

for all \( j \leq 0 \).

For any \((u_0, s_0) \in D_{r_0} \cap \tilde{W}_0 \), using \( \|D\tilde{\Phi}_{-1}\| \leq K_{-1}^{3} a_{-1}^{-1} \), we have \((u_{-1}, s_{-1}) \in D_{\tilde{r}_{-1}} \cap \tilde{W}_{-1} \), and by step one \( \|s_{-1}\| \leq \|u_{-1}\| \). It follows that \((u_{-1}, s_{-1}) \) is contained in a \((u, 1, \tilde{r}_{-1}, -1)\)-admissible set, and by Proposition 6.4
\[ \|u_{-1}\| \leq e^{-\lambda_{-1}/2} \|u_0\| \leq e^{-\lambda_{-1}/2} r_0 \leq \tilde{r}_{-2} K_{-1}^{-3} a_{-2}. \]

This procedure can be continued indefinitely. It follows that, for all \( j \leq 0 \), \((u_j, s_j) \) is contained in a \((u, 1, \tilde{r}_j, j)\)-admissible set. Take a family \( W_j \) of admissible manifolds that contain \((u_j, s_j) \). The fact that \((u_j, s_j) \) is a backward orbit implies \((u_j, s_j) \in (G^n W)_j \) for all \( n \geq 0 \). Let \( n \to \infty \), we conclude that \((u_j, s_j) \in \lim_{n \to \infty} G(W)_j = W^n \), the unstable manifolds.

The first part of Theorem 6.1 follows from taking \( U(\omega) = P_0 D_{r_0} \). To prove the second statement, note that the subderivative \( \partial_x \psi^-(x, 0) \) is upper semi-continuous as a set function. Since \( \nabla \psi^-(x_0, 0) \) exists, there exists a neighbourhood \( V(\omega) \) of \( X_0 \) such that for all \( y \in V(\omega) \), \( \partial_y \psi^-(y, 0) \in U(\omega) \). Using the first part of the theorem, we have \((y, \nabla \psi^- (y, 0)) \in W^n_0 \) for almost every \( y \in V(\omega) \). Since \( W^n \) is \( C^1 \), we conclude that \( \nabla \psi^- \) is also \( C^1 \) and \( \psi^- \) is \( C^2 \).

We have proved all our conclusions modulo Propositions 3.10, 3.11 and 3.12.
8. Generic nondegeneracy of the minimum

In this section we prove Proposition 3.10. Recall that we have
\[ H(x, c) = \psi(x) + V(x, c), \]
where \( \psi \) is a semi-concave function and \( V(x, c) \) is \( C^2 \). We assume that for every \( c \), the map \( \partial_c V(\cdot, c) \) is an embedding from \( \mathbb{T}^d \) to \( \mathbb{R}^M \). We also assume that there is \( K > 0 \) such that \( \|\partial^2_{cc} V(x, c)\| \leq K \) for all \( x \) and \( c \).

First we have the following lemma:

**Lemma 8.1.** Assume that \( \partial_c V(\cdot, c) : \mathbb{T}^d \to \mathbb{R}^M \) is an embedding. Then there exists \( U_1, \ldots, U_k \subset \mathbb{T}^n \) such that \( \mathbb{T}^d = \bigcup_{j=1}^k U_j \), and for each \( U_j \), there exists a projection \( \Pi_j : \mathbb{R}^M \to \mathbb{R}^d \) given by \( \Pi_j(c_1, \ldots, c_M) = (c_{i_1}, \ldots, c_{i_d}) \) for some indices \( \{i_1, \ldots, i_d\} \subset \{1, \ldots, M\} \), and a continuous positive function \( \mu_j : \mathbb{R}^M \to \mathbb{R} \) such that
\[ m(\partial_x (\Pi_j \circ \partial_c V)(x, c)) \geq \mu_j(c), \quad \forall x \in U_j. \]

Write \( G(c) = \inf_{x \in \mathbb{T}^d} H(x, c) \). We have the following results.

**Lemma 8.2.** \( G \) is a semi-concave function. For any \( c \in \mathbb{R}^M \), if \( x(c) = \arg\min_x \{\psi(x) + V(x, c)\} \), then \( \partial_c V(x(c), c) \) is a subderivative of \( G \) at \( c \). Conversely, if \( l_c \) is a subderivative of \( G \) at \( c \), then \( l_c \in \text{conv}_{x \in X(c)} \{\partial_c V(x, c)\} \). Here \( \text{conv} \) denote the convex hull of a set.

**Proof.** For any \( c', c \in \mathbb{R}^M \), let \( x(c') \in X(c') \) and \( x(c) \in X(c) \), we have that
\[ G(c') - G(c) = H(x(c'), c') - H(x(c), c') \leq H(x(c), c') - H(x(c), c) \]
\[ = V(x(c), c') - V(x(c), c) \leq \langle \partial_c V(x(c), c), c' - c \rangle + \|\partial^2_{cc} V\| |c_2 - c_1|^2. \]

It follows that \( G(c) \) is semi-concave and \( \partial_c V(x(c)) \) is a subderivative of \( G(c) \) at \( c \).

For the converse, we first show that \( X(c) \) as a set function is upper semi-continuous in the sense that if \( c_n \to c \), \( x_n \in X(c_n) \) and \( x_n \to x \), then \( x \in X(c) \). Indeed,
\[ G(c) = \lim_{n \to \infty} G(c_n) = \lim_{n \to \infty} \psi(x_n) + V(x_n, c_n) = \psi(x) + V(x, c), \]
hence \( x \in X(c) \).

We now argue by contradiction. Assume that \( l_c \notin \text{conv}_{x \in X(c)} \{\partial_c V(x, c)\} \), then there exists a vector \( v \in \mathbb{R}^M \) such that \( \langle l_c, v \rangle > \langle \partial_c V(x, c), v \rangle \) for all \( x \in X(c) \). Then we have
\[ G(c - tv) \leq G(c) - \langle l_c, tv \rangle + \frac{1}{2} t^2 K |v|^2, \]
\[ G(c) \leq G(c - tv) + \langle \partial_c V(x_t, c - tv), tv \rangle + \frac{1}{2} t^2 K |v|^2, \]
where \( x_t \in X(c - tv) \). Since the domain for \( x \) is compact, there exists \( t_n \to 0^+ \) such that \( x_n := x_{t_n} \to x \in X(c) \). Combine the two formulas, we have
\[ G(c - t_nv) \leq G(c - t_nv) - \langle l_c, t_nv \rangle + \langle \partial_c V(x_n, c - t_nv), t_nv \rangle + t_n^2 K |v|^2. \]
Divide both sides by \( t_n \) and take \( t_n \to 0^+ \), we obtain
\[
\langle l_c, v \rangle \leq \langle \partial_c V(x, c), v \rangle,
\]
a contradiction. \( \square \)

**Corollary 8.3.** For almost every \( c \in \mathbb{R}^M \), \( G(c) \) is differentiable, there is a unique \( x(c) \in \arg\min_x H(x, c) \), and \( \partial_c V(x(c), c) = dG(c) \).

**Proof.** Since a semi-concave function is almost everywhere differentiable, for almost every \( c \), the subderivative of \( G(c) \) is unique. Since \( \partial_c V(\cdot, c) \) is one-to-one, there is at most one \( x(c) \) such that \( \partial_c V(x(c), c) = dG(c) \). \( \square \)

**Lemma 8.4.** Let \( \varphi : \mathbb{T}^d \to \mathbb{R} \) be a semi-concave function and \( x_0 \in \arg\min \varphi(x) \), then \( \varphi \) is differentiable at \( x_0 \) and \( d\varphi(x_0) = 0 \).

**Proof.** Let \( l_0 \in \partial\varphi(x_0) \), we have that for any \( x \in \mathbb{T}^n \),
\[
0 \leq \varphi(x) - \varphi(x_0) \leq l_0(x - x_0) + C|x - x_0|^2.
\]
Take \( x - x_0 = tv \) for \( t > 0, v \in \mathbb{R}^n \), we have \( 0 \leq tl_0(v) + Ct^2|v|^2 \). Divide by \( t \) and let \( t \to 0^+ \), we have that \( l_0(v) \geq 0 \) for any \( v \in \mathbb{R}^n \). Similarly, if we take \( t < 0 \), and let \( t \to 0^+ \), we have that \( l_0(v) \leq 0 \). It follows that \( \varphi \) has only 0 as a subderivative at \( x_0 \). As a consequence, \( \varphi \) is differentiable at \( x_0 \) and the derivative is 0. \( \square \)

The following corollary is an immediate consequence of the last lemma.

**Corollary 8.5.** Any \( x(c) \in \arg\min_x H(x, c) \) satisfies \( \partial_x H(x(c), c) = 0 \).

Given any function \( g : \mathbb{T}^n \to \mathbb{R} \) differentiable at \( x \), we define the second order difference to be
\[
\nabla^2 g(x)(\Delta x) = g(x + \Delta x) - g(x) - \langle dg(x), \Delta x \rangle.
\]
We have
\[
d^2 g(x)(v) = \liminf_{\tau \to 0^+} \frac{2\nabla^2 g(x)(\tau v)}{\tau^2}.
\]
There is a duality between the second subderivative of \( V \) in \( x \) and the second subderivative of \( G \).

**Lemma 8.6 (Duality).** Given any \( x \in \mathbb{T}^d \) and \( c \in \mathbb{R}^M \), we call \((x, c)\) a conjugate pair if \( \partial_c V(x, c) = dG(c), \partial_x H(x, c) = 0 \) and \( G(c) = H(x, c) \) (in particular, all derivatives must exist). We have that for any \( v \in \mathbb{R}^d \) and \( w \in \mathbb{R}^M \),
\[
\frac{1}{2}d^2 H(x, c)(v) \geq \frac{1}{2}d^2 G(c)(w) - \langle \partial^2_{xx} V(x, c)v, w \rangle - K|v|^2.
\]
Proof. Since \((x, c)\) is a conjugate pair, we have that \(G(c) = H(x, c), \partial_x H(x, c) = 0\) and \(\partial_c V(x, c) = dG(c)\).

\[
\nabla^2 H(x, c)(\Delta x) = H(x + \Delta x, c) - H(x, c) - \langle \partial_x H(x, c), \Delta x \rangle = H(x + \Delta x, c) - H(x, c)
\]
\[
\geq G(c + \Delta c) - G(c) - \langle \partial_c V(x + \Delta x, c), \Delta c \rangle - K(\Delta c)^2
\]
\[
= G(c + \Delta c) - G(c) - \langle \partial_c V(x, c), \Delta c \rangle - \langle \partial_c V(x + \Delta x, c) - \partial_c V(x, c), \Delta c \rangle - K(\Delta c)^2
\]
\[
= \nabla^2 G(c)(\Delta c) - \langle \partial_c V(x + \Delta x, c) - \partial_c V(x, c), \Delta c \rangle - K(\Delta c)^2.
\]

Here \(K = \|\partial^2_{cc} V(x, c)\|c_0\) as before.

Take \(\Delta x = \tau v, \Delta c = \tau w\), divide by \(\tau^2\), and take \(\tau \to 0^+\), we have:

\[
\frac{1}{2} \nabla^2 H(x, c)(v) \geq \frac{1}{2} \nabla^2 G(c)(w) - \langle \partial_{xx} V(x, c)v, w \rangle - K|v|^2.
\]

\[\square\]

To finally prove Proposition \ref{prop:existence-minimizers}, we need the following result from convex analysis.

**Theorem 8.7** (Alexandrov Theorem). \[\square\] A convex function on \(\mathbb{R}^n\) is twice differentiable almost everywhere.

In a neighbourhood of \(c_0\), a \(K\)-semi-concave function \(G(c)\) can always be made concave by subtracting the quadratic function \(K|c - c_0|^2\). It follows that a semi-concave function is also twice differentiable almost everywhere.

**Proof of Proposition 8.10**. According to Corollaries \ref{cor:existence-minimizers} and \ref{cor:existence-minimizers-2}, for almost every \(c \in \mathbb{R}^n\), \(G\) is differentiable at \(c\), there exists unique \(x(c) \in \arg \min_x H(x, c), \partial_c V(x, c) = dG(c)\) and \(\partial_c H(x(c), c) = 0\). In other words, \((x(c), c)\) is a conjugate pair. Furthermore, by Theorem 8.7, \(d^2 G(c)\) exists and is a symmetric bilinear form. Assume that \(d^2 G(c)(w) = \frac{1}{2} \langle A(c)w, w \rangle\), we have that

\[
\sup_{\|w\|=1} 2|d^2 G(c)(w)| = \|A(c)\|.
\]

There exists \(1 \leq j \leq k\) such that \(x(c) \in U_j\). By Lemma \ref{lem:convex-conjugate} we have that for any \(v \in \mathbb{R}^d, w \in \mathbb{R}^M\),

\[
d^2 V(x(c), c)(v) \geq d^2 G(c)(w) - 2\langle \partial^2_{xx} V(x(c), c)v, w \rangle - 2K|v|^2.
\]

By Lemma \ref{lem:coordinate-projection}, there is a coordinate projection \(\Pi_j : \mathbb{R}^M \to \mathbb{R}^d\) given by \((x_1, \cdots, x_M) \mapsto (x_{i_1}, \cdots, x_{i_d})\) for some indices \(\{i_1, \cdots, i_d\} \subset \{1, \cdots, M\}\). Let \(\Pi'_j : \mathbb{R}^M \to \mathbb{R}^{M-d}\) be the map to the complementary indices. We have that for any two vectors \(w_1, w_2 \in \mathbb{R}^M\),

\[
\langle w_1, w_2 \rangle = \langle \Pi_j w_1, \Pi_j w_2 \rangle + \langle \Pi'_j w_1, \Pi'_j w_2 \rangle.
\]
Choose \( w \) such that \( \Pi_j w = -\lambda \Pi_j \partial_{x^2}^2 V(x(c), c)^v \) and \( \Pi_j' w = 0 \), where \( \lambda > 0 \) is a parameter. We have
\[
- (\partial_{x^2}^2 V(x(c), c)^v, w) = \lambda (\Pi_j \partial_{x^2}^2 V(x(c), c)^v, \Pi_j \partial_{x^2}^2 V(x(c), c)^v) = \lambda |\Pi_j \partial_{x^2}^2 V(x(c), c)^v|^2.
\]
Let \( M(c) = \sup_{x \in \mathbb{T}^d} \|\partial_{x^2}^2 V(x, c)\| \), then \( \mu_j(c)|v| \leq |\Pi_j \partial_{c^2} V(x)^v| \leq M(c)|v| \). It follows that
\[
d^2_x V(x(c), c)^v \geq -\lambda^2 (\|A(c)\| + 2K)M(c)^2|v|^2 + 2\lambda \mu_j(x)^2|v|^2.
\]
Choose
\[
\lambda(c) = \frac{\mu_j(c)^2}{(\|A(c)\| + 2K)M(c)^2},
\]
we have that
\[
d^2_x V(x(c), c)^v \geq \mu_j^2(c)\lambda(c)|v|^2.
\]
We now choose
\[
a(c) = \frac{(\inf_{1 \leq j \leq k} \mu_j(c))^4}{M(c)^2}(\|A(c)\| + 2K)^{-1}
\]
(33)
and the proposition follows.

\[ \square \]

9. A Quantitative Alexandrov Theorem

In this section we prove Proposition 3.11.

For \( V(x, c) = -\sum_{i=1}^{M} c_i F_i(x) \), \( \partial_c V = (F_1, \cdots, F_M) \) is independent of \( c \). It follows that we can choose \( \mu_j(c) = \mu_j \) and \( M_j(c) = M_j \) independent of \( c \). Furthermore, the constant \( K = 0 \) in (33). We have that there exists \( \alpha > 0 \) such that \( a(c) = \alpha \|A(c)\|^{-1} \).

By Lemma 8.2 any subderivative of \( G(c) \) is contained in the set \( \text{conv}_{x \in X(c)} \{\partial_c V(x, c)\} \), a subset of \( B := \text{conv}_{x \in \mathbb{T}^d} \{ (F_1, \cdots, F_M)(x) \} \). Since \( \bigcup_{x \in \mathbb{T}^d} (F_1, \cdots, F_M)(x) \) is a compact set, so is \( B \).

To prove Proposition 3.11 it suffices to show that, for a density \( \rho \) satisfying assumption 5,
\[
\int \|A(c)\| \rho(c) dc < A(F).
\]
The preceding formula follows from the following “quantitative Alexandrov theorem”.

**Theorem 9.1.** Assume \( f : \mathbb{R}^M \rightarrow \mathbb{R} \) is a convex function such that there exists a bounded set \( B \) satisfying the following condition: for any \( c \in \mathbb{R}^n \), we have \( \partial f(c) \subset B \). Here \( \partial f(c) \) denote the set of all subderivative of \( f \) at \( c \).

Let \( \rho : \mathbb{R}^M \rightarrow \mathbb{R}^+ \) be a probability density satisfying assumption 5, and let \( A(c) \) denote the hessian matrix of \( f \) at \( c \), which exists almost everywhere by Theorem 8.7.

We have
\[
\int \|A(c)\| \rho(c) dc < A(F).
\]

We first prove a lemma for one-dimensional functions.
Lemma 9.2. Assume that \( g : \mathbb{R} \to \mathbb{R} \) is a convex function, \( \partial g(t) \subset (a, b) \), then

\[
\int |g''(t)| dt \leq b - a.
\]

**Proof.** For a one dimensional convex function, \( g' \) exists almost everywhere and is monotone. Without loss of generality, we may assume that \( g' \) is increasing. Again we have \( g' \) is almost everywhere differentiable and

\[
\int g''(t) dx \leq \lim_{N \to \infty} g'(N) - g'(-N) \leq b - a.
\]

\( \square \)

**Proof of Theorem 9.1.** Given a symmetric matrix \( A \), let \( v \) be such that \( |v| = 1 \) and \( \|A^\frac{1}{2} v\| = \|A^\frac{1}{2}\| \). Write \( v = a_1 e_1 + \cdots + a_M e_M \), we say that \( A \) is \( i \)-positive if \( |a_i| = \max_{1 \leq k \leq n} |a_k| \). There may exist multiple \( i \)'s such that \( A \) is \( k \)-positive.

It follows from the maximality of \( |a_i| \) that \( |a_i| \geq \frac{1}{\sqrt{M}} \). Denote \( w = a_i e_i \), we have \( |v - w| = 1 - a_i^2 \), and

\[
|A^\frac{1}{2} w| \geq |A^\frac{1}{2} v| - |A^\frac{1}{2}(v - w)| \geq \|A^\frac{1}{2}\| \left( 1 - \sqrt{1 - a_i^2} \right).
\]

It follows that

\[
|A^\frac{1}{2} e_i| \geq \frac{1 - \sqrt{1 - a_i^2}}{a_i^2} \|A^\frac{1}{2}\| \geq \beta_M \|A^\frac{1}{2}\|
\]

where \( \beta_M > 0 \) is a constant depending only on \( M \).

For each \( 1 \leq i \leq n \), define a function \( \varphi_i : \mathbb{R}^n \to \mathbb{R} \) by

\[
\varphi_i(c) = \begin{cases} \|A(c)\|, & d^2 f(c) \text{ exists and is } i \text{-positive;} \\ 0, & \text{otherwise.} \end{cases}
\]

We have \( \|A(c)\| \leq \sum_{i=1}^M \varphi_i(c) \).

Since \( f(c_1, \ldots, c_M) \) considered as a function of \( c_i \) (with \( \hat{c}_i \) as parameters) is convex, and \( \partial^2_{c_i} f(x) = \langle A(c) e_i, e_i \rangle \), it follows that

\[
\int |A(c)^\frac{1}{2} e_i|^2 dc = \int \langle A(c) e_i, e_i \rangle dc \leq \text{diam}(B).
\]

We have

\[
\int \varphi_i(c) dc_i \leq \int \|A(c)\| dc_i \leq \frac{1}{\beta_M^2} \int \langle A(c) e_i, e_i \rangle dc \leq \text{diam}(B)/\beta_M^2.
\]

Apply Fubini theorem, we have

\[
\int \varphi_i(c) \rho(c) dc \leq \int \hat{\rho}(\hat{c}_i) d\hat{c}_i \int \varphi_i(c) \rho_i(c_i) dc_i \leq \|\rho_i\|_{L^\infty} \text{diam}(B)/\beta_M^2 \int \hat{\rho}_i(\hat{c}_i) d\hat{c}_i.
\]

Define \( A(F) = \sum_{1 \leq i \leq M} \|\rho_i\|_{L^\infty} \|\hat{\rho}_i\|_{L^1} \text{diam}(B)/\beta_n^2 \) and the theorem follows. \( \square \)
10. Uniform Nondegeneracy of the Minimum

We prove Proposition 3.12 in this section. Instead of looking at the limiting behaviour, we attempt to get quantitative estimates of the second order difference $\nabla^2 H(x, c)$ instead of the subderivative.

In the proof of Lemma 8.6, we obtained the following estimate (32):

$$\nabla^2 H(x, c)(\Delta x) \geq \nabla^2 G(c)(\Delta c) - \langle \partial_c V(x + \Delta x, c) - \partial_c V(x, c), \Delta c \rangle - K(\Delta c)^2.$$

We now use the fact that $V(x, c) = \sum_{i=1}^M c_i F_i(x)$, $K = 0$ and $\partial_c V = DF$, where $F = (F_1, \cdots, F_M)$. We have

$$\nabla^2 H(x, c)(\Delta x) \geq \nabla^2 G(c)(\Delta c) - \langle DF(x)\Delta x, \Delta c \rangle + C(F)\|\Delta x\|^2\|\Delta c\|,$$

where $C(F) = \|F\|_{C^2}$.

We have the following improvement of Theorem 9.1:

**Lemma 10.1.** Assume that $f : \mathbb{R}^M \to \mathbb{R}$ is a convex function such that all subderivatives are contained in a bounded set $B$. Let $\rho$ be a density satisfying Assumption 5. Then for $i = 1, \cdots, M$, there exist positive measurable functions $g_i : \mathbb{R}^M \to \mathbb{R}^+$ and a constant $C$ depending only on $\rho$ and $B$ satisfying

$$\int \sqrt{g_i(c)} d\rho(c) < C,$$

such that

$$\nabla^2 f(c)(te_i) \leq g_i(c)t^2.$$

Before proving Lemma 10.1, let us prove a simple statement about non-decreasing functions of one variable:

**Lemma 10.2.** Let $h : \mathbb{R} \to \mathbb{R}$ be a right continuous non-decreasing function satisfying $h(b) - h(a) \leq B$ for all $a < b$. Let

$$g(t) := \sup_{r > 0} \left\{ \frac{1}{2r} (h(t + r) - h(t - r)) \right\} = \sup_{r > 0} \left\{ \frac{1}{2r} \int_{t-r}^{t+r} dh \right\},$$

where $dh$ is the Stieltjes integral associated to $h$. Let $| \cdot |$ denote the one-dimensional Lebesgue measure, there exists a constant $C'$ such that

$$|\{ s : g(s) \geq t \}| \leq C' B/t.$$

**Proof.** The proof is similar to the standard proof of the weak $(1,1)$–inequality for the Hardy-Littlewood maximal principle. We have that for any $s \in \{ g(s) \geq t \}$, there exists $I_s = [s - r_s, s + r_s]$ such that $t|I_s| \leq \int_{I_s} dh$. Since $\bigcup_s I_s \supset \{ g(s) \geq t \}$, using the Besicovitch covering lemma, there exists a subcover $\{ I_i \}$ with multiplicity at most $C'$. We have

$$|\{ s : g(s) \geq t \}| \leq \frac{1}{t} \sum_i \int_{I_i} dh \leq \frac{C'}{t} \int_{\mathbb{R}} dh \leq \frac{C' B}{t}.$$
Proof of Lemma 10.1. Note that \( f(c + te_i) \) is a single variable convex function, and hence its first derivative \( h_{c,i}(t) := \langle df(c + te_i), e_i \rangle \) is defined almost everywhere and is increasing. We extend the function to be a right continuous function defined everywhere. By Rademacher’s theorem, \( f(c + be_i) - f(c + ae_i) = \int_a^b h_{c,i}(t) dt \). Moreover, by our assumption,
\[
h_{c,i}(b) - h_{c,i}(a) \leq \text{diam}(B), \quad \text{for any } a < b,
\]
where \( B \) is the compact set that contains all subderivatives of \( f \).

Define
\[
\bar{g}_i(c) = \sup_{r > 0} \left\{ \frac{1}{2r} (h_{c,i}(r) - h_{c,i}(-r)) \right\}.
\]
By Lemma 10.2, there exists \( C' > 0 \) such that for all \( t > 0 \),
\[
|\{s : \bar{g}_i(c + se_i) > t\}| \leq C' \text{diam}(B)/t,
\]
where \( |\cdot| \) stands for the one-dimensional Lebesgue measure. Define \( g_i(c) = \max\{\bar{g}_i, 1\} \).

Since \( g_i \geq 1 \), we have that
\[
\int_{-\infty}^{\infty} \sqrt{g_i(c + se_i)} \, ds = \int_{1}^{\infty} |\{t : \sqrt{\bar{g}_i(c + se_i)} > t\}| \, dt
\]
\[
\leq C' \text{diam}(B) \int_{1}^{\infty} t^{-2} \, dt \leq C' \text{diam}(B).
\]
We now have
\[
\int \sqrt{g_i(c)} \, d\rho(c) \leq \int_{\mathbb{R}^M-1} \int_{\mathbb{R}} \sqrt{g_i(c)} \, d\rho_i(c_i) \, d\hat{\rho}_i(\hat{c}_i) \leq C' \text{diam } B \|\rho_i\|_{L^\infty} \|\hat{\rho}_i\|_{L^1} := C.
\]
We have
\[
\nabla^2 f(c)(te_i) = f(c + te_i) - f(c) - \langle df(c), te_i \rangle = \int_0^t \langle df(c + se_i) - df(c), e_i \rangle \, ds
\]
\[
\leq \int_0^t 2sg_i(c) \, ds \leq g_i(c)t^2.
\]

Proof of Proposition 7.3. This proof is similar to that of Proposition 3.10. Using Lemma 8.1, there exists a projection \( \Pi_j \) such that \( m(\Pi_j DF(x)) \geq \mu_j > 0 \). Let 
\( h = -\lambda \Pi_j (DF(x)\Delta x) \), where \( \lambda \) is a parameter to be chosen later. We have \( \|h\| \leq \lambda M_j \|\Delta x\| \) and
\[
\langle DF(x)\Delta x, -h \rangle \geq \lambda \|\Pi_j DF(x)\Delta x\|^2.
\]

We make a simple observation about inner products in \( \mathbb{R}^M \). Let \( a, b \in \mathbb{R}^n \) satisfy \( \langle a, b \rangle > 0 \). Assume that \( b = \sum t_k e_k \), then there exists \( i \in \{1, \ldots, M\} \) such that
\( t_i \langle a, e_i \rangle \geq \frac{1}{M} \langle a, b \rangle \). Indeed, choose \( i \) such that
\( t_i \langle a, e_i \rangle = \max_k \{t_k \langle a, e_k \rangle\} \) and the inequality follows.
Apply the above observation to $DF(x)\Delta x$ and $-h$, we obtain that there exists $1 \leq i \leq M$ and a vector $te_i$ with $\|te_i\| \leq \|h\|$ and
\[
\langle DF\Delta x, te_i \rangle \geq \frac{\lambda}{M} \|\Pi_j DF(x)\Delta x\|^2 \geq \frac{\lambda \mu_i^2}{M} \|\Delta x\|^2.
\]
Denote $\Delta c = -te_i$. Note that the function $G(c)$ is concave. Apply Lemma 10.1 to $G$, we have that there exists functions $g_i(c)$ such that $\nabla G(c)(\Delta c) \geq -g_i(c)\|\Delta c\|^2$. By (34), we have
\[
\nabla_x^2 H(x,c)(\Delta x) \geq \nabla^2 G(c)(\Delta c) - \langle DF(x)\Delta x, \Delta c \rangle - C\|\Delta x\|^2\|\Delta c\|
\geq -g_i(c)\|\Delta c\|^2 + \frac{\lambda \mu_i^2}{M} \|\Delta x\|^2 - C(F)\|\Delta x\|^2\|\Delta c\|
\geq \left( -g_i(c)\lambda^2 M_j^2 + \frac{\lambda \mu_i^2}{M} - C(F)\lambda M_j \|\Delta x\| \right) \|\Delta x\|^2.
\]
Let $\lambda = \frac{\mu_i^2}{4Mg_i(c)M_j^2}$ and assume $\|\Delta x\| \leq \frac{\mu_j^2}{4MC(F)M_j} := r(F)$, we obtain that
\[
\nabla_x^2 H(x,c)(\Delta x) \geq \frac{\lambda \mu_i^2}{4M} = \frac{\mu_j^2}{16M^2 g_i(c)M_j^2}, \quad \|\Delta x\| \leq r(F).
\]
Define
\[
b(c) = \frac{\min_j \mu_j^4}{8M^2 \max_i \{g_i(c)\} \max_j M_j^2},
\]
we have $\nabla_x H(x,c)(\Delta x) \geq \frac{1}{2}b(c)\|\Delta x\|^2$. This implies the lower bound in Proposition 7.3. The upper bound follows directly from semi-concavity.

It remains to prove the integrability property of $b(\omega)$. Since $b(c) = C''/\max_i \{g_i(c)\}$, where $C'' = (\min_i \mu_j^4)/(8M^2 \max_j M_j)$, we have
\[
\int b(c)^{-\frac{1}{2}}d\rho(c) \leq \sqrt{C''} \int \sum_{i=1}^M g_i(c)^{\frac{1}{2}}d\rho(c) \leq MC\sqrt{C''}.
\]
Denote $\hat{\xi} = \{\xi_j(\omega), j \neq 0\}$, we have
\[
\int b(\omega)^{-\frac{1}{2}}d\rho(\omega) \leq \int \int b(\xi_0)^{-\frac{1}{2}}dP(\xi_0)\hat{\xi}(\omega)dP(\omega) \leq MC\sqrt{C''}.
\]

\[\square\]

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