ON THE STRATIFICATION BY X-RANKS OF A LINEARLY NORMAL ELLIPTIC CURVE $X \subset \mathbb{P}^n$

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Abstract. Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. For any $P \in \mathbb{P}^n$ the $X$-rank of $P$ is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. In this paper we give an almost complete description of the stratification of $\mathbb{P}^n$ obtained by the $X$-rank and the open $X$-rank.

Fix an integral and non-degenerate variety $X \subset \mathbb{P}^n$. For any $P \in \mathbb{P}^n$ the $X$-rank $r_X(P)$ of $P$ is the minimal cardinality of a subset $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \cdot \rangle$ denote the linear span. The $X$-rank is an extensively studied topic ([12], [7], [6], [11], and references therein). In the applications one needs only the cases in which $X$ is either a Veronese embedding of a projective space or a Segre embedding of a multiprojective space. We feel that the general case gives a treasure of new projective geometry. Up to now only for rational normal curves there is a complete description of the stratification of $\mathbb{P}^n$ by $X$-rank ([9], [12], Theorem 5.1, [6]). Here we look at the case of elliptic linearly normal curves. For any integer $t \geq 1$ let $\sigma_t(X)$ denote the closure in $\mathbb{P}^n$ of all $(t-1)$-dimensional linear spaces spanned by $t$ points of $X$. Set $\sigma_0(X) = \emptyset$. For any $P \in \mathbb{P}^n$ the border $X$-rank $b_X(P)$ is the minimal integer $t \geq 1$ such that $P \in \sigma_t(X)$, i.e. the only positive integer $t$ such that $P \in \sigma_t(X) \setminus \sigma_{t-1}(X)$. If (as always in this paper) $X$ is a curve, then $\dim(\sigma_t(X)) = \min\{n, 2t-1\}$ for all $t \geq 1$ ([1], Remark 1.6). Notice that $r_X(P) \geq b_X(P)$ and that equality holds at least on a non-empty open subset of $\sigma_t(X) \setminus \sigma_{t-1}(X)$, $t := b_X(P)$. Obviously $b_X(P) = 1 \iff P \in X \iff r_X(P) = 1$. Hence to compute all $X$-ranks it is sufficient to compute the $X$-ranks of all points of $\mathbb{P}^n \setminus X$. In this paper we look at the case of the linearly normal elliptic curves. We prove the following result.

Theorem 1. Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a linearly normal elliptic curve. Fix $P \in \mathbb{P}^n \setminus X$ and set $w := b_X(P)$. We have $2 \leq w \leq \lfloor (n+2)/2 \rfloor$. Assume $n \geq 2w+2$. Then either $r_X(P) = w$ or $r_X(P) = n+1-w$ and both cases occurs for some $P \in \sigma_w(X) \setminus \sigma_{w-1}(X)$.

The inequalities $2 \leq w \leq \lfloor (n+2)/2 \rfloor$ in the statement of Theorem 1 are obvious ([1], Remark 1.6). The case $w = 2$ and arbitrary $n$ was settled in [6], Theorem 3.13. Theorem 1 leaves partially open the cases $n = 2w$, $n = 2w-1$ and $n = 2w-2$ (in which either $r_X(P) = w$ or $r_X(P) \geq n+1-w$). If $n = 2w-1$, then either $r_X(P) = w = n+1-w$ or $r_X(P) \geq w+1$ and the latter case occurs for a non-empty codimension two subset of points of $\mathbb{P}^n$ ([11], Proposition 3). In this case we also have a non-trivial result on the set of all zero-dimensional schemes $Z \subset X$ evincing the

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border rank of the points $P$ with maximal border rank (Proposition 2). The case $n = 3$ is contained in [13] (here we have $r_X(P) \leq 3$ and in characteristic zero to get this inequality it is sufficient to quote [12], Proposition 4.1).

Following works by A. Białynicki-Birula and A. Schinzel ([4], [5]), J. Jelisiejew introduced the definition of open rank for symmetric tensors, i.e. for the Veronese embeddings of projective spaces ([10]). In the general case of $X$-rank we may translate the definition of open rank in the following way.

**Definition 1.** Fix an integral and non-degenerate variety $X \subset \mathbb{P}^n$. For each $P \in \mathbb{P}^n$ the open $X$-rank $w_X(P)$ of $P$ is the minimal integer $t$ such that for every proper closed subset $T \subset X$ there is $S \subset X$ with $\sharp(S) \leq t$ and $P \in \langle S \rangle$.

Obviously $w_X(P) \geq r_X(P)$, but often the strict inequality holds (e.g., $w_X(P) > 1$ for all $P$). For linearly normal elliptic curves we prove the following result.

**Theorem 2.** Fix integers $w > 0$ and $n \geq 2w + 2$. Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. Fix a zero-dimensional scheme $W \subset X$ and $P \subset \mathbb{P}^n$ such that $\deg(W) = w, P \in \langle W \rangle$ and $P \notin \langle W' \rangle$ for any $W' \subset W$. Fix any finite set $U \subset X$. Then there is $E \subset X \setminus U$ such that $\sharp(E) = n + 1 - w$ and $P \in \langle E \rangle$. There is no set $F \subset X$ such that $\sharp(F) \leq n - w, F \cap W = \emptyset$ and $P \in \langle F \rangle$.

As an immediate corollary of Theorem 2 we get the following result.

**Corollary 1.** Fix integers $w > 0$ and $n \geq 2w + 2$. Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. Take any $P \in \sigma_w(X) \setminus \sigma_{w-1}(X)$, i.e. with $b_X(P) = w$. Then $w_X(P) = n + 1 - w$.

We work over an algebraically closed field $\mathbb{K}$ such that $\text{char}(\mathbb{K}) = 0$. This assumption is essential in our proofs, mainly to quote [8], Proposition 5.8, which is a very strong non-linear version of Bertini’s theorem.

## 1. Preliminary lemmas

In this paper an elliptic curve is a smooth and connected projective curve with genus 1.

Fix any non-degenerate variety $X \subset \mathbb{P}^n$. For any $P \in \mathbb{P}^n$ let $S(X,P)$ denote the set of all $S \subset X$ evincing $r_X(P)$, i.e. the set of all $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Notice that every $S \in S(X,P)$ is linearly independent and $P \notin \langle S' \rangle$ for any $S' \subset S$. Now assume that $X$ is a linearly normal elliptic curve. Let $Z(X,P)$ denote the set of all zero-dimensional subschemes $Z \subset X$ such that $\deg(Z) = b_X(P)$ and $P \in \langle Z \rangle$. Lemma 1 below gives $Z(X,P) \neq \emptyset$. Fix any $Z \in Z(X,P)$. Notice that $Z$ is linearly independent (i.e. $\dim(Z) = \deg(Z) - 1$) and $P \notin \langle Z' \rangle$ for any subscheme $Z' \subsetneq Z$.

**Notation 1.** Let $C \subset \mathbb{P}^n$ be a smooth, connected and non-degenerate curve. Let $\beta(C)$ be the maximal integer such that every zero-dimensional subscheme of $C$ with degree at most $\beta(C)$ is linearly independent.

The following lemma is just a reformulation of [2], Lemma 1.

**Lemma 1.** Let $Y \subset \mathbb{P}^r$ be an integral variety. Fix any $P \in \mathbb{P}^r$ and two zero-dimensional subschemes $A, B$ of $Y$ such that $A \neq B, P \in \langle A \rangle, P \in \langle B \rangle, P \notin \langle A' \rangle$ for any $A' \subsetneq A$ and $P \notin \langle B' \rangle$ for any $B' \subsetneq B$. Then $h^1(\mathbb{P}^r, \mathcal{I}_{A\cup B}(1)) > 0$. 
Proposition 1. Fix an integer $k \leq \lfloor \beta(C)/2 \rfloor$ and any $P \in \sigma_k(C) \setminus \sigma_{k-1}(C)$. Then there exists a unique zero-dimensional scheme $Z \subset C$ such that $\deg(Z) \leq k$ and $P \in \langle Z \rangle$. Moreover $\deg(Z) = k$ and $P \not\in \langle Z' \rangle$ for all $Z' \subsetneq Z$.

Proof. The existence part is stated in [3], Lemma 1, which in turn is just an adaptation of some parts of the beautiful paper [7] ([7], Lemma 2.1.6) or of [6], Proposition 11. The uniqueness part is true by Lemma [4] and the definition of the integer $\beta(C)$. □

Lemma 2. Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve.

(i) We have $\beta(X) = n$. A scheme $Z \subset X$ with $\deg(Z) = n + 1$ is linearly independent if and only if $Z \not\in |O_X(1)|$.

(ii) Fix zero-dimensional schemes $A, B \subset X$ such that $\deg(A) + \deg(B) \leq n + 1$. If $\deg(A) + \deg(B) = n + 1$ and $A + B \in |O_X(1)|$, assume $A \cap B \neq \emptyset$, i.e. assume $A \cup B \neq A + B$. Then $\langle A \rangle \cap \langle B \rangle = \langle A \cap B \rangle$.

(iii) Fix zero-dimensional schemes $A, B \subset X$ such that $O_X(A + B) \cong O_X(1)$. Then $\dim(\langle A \rangle \cap \langle B \rangle) = \deg(A \cap B)$.

Proof. Let $F \subset X$ be a zero-dimensional subscheme. Since $X$ is projectively normal, we have $h^1(I_F(1)) = 0$ if and only if either $\deg(F) < \deg(O_X(1)) = n + 1$ or $\deg(F) = n + 1$ and $F \not\in |O_X(1)|$ (use the cohomology of line bundles on an elliptic curve). Hence we get part (i). By the Grassmann formula we also get parts (ii) and (iii). □

Lemma 3. Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a linearly normal elliptic curve. Fix $P \in \mathbb{P}^n$. Then either $b_X(P) = r_X(P)$ or $r_X(P) < b_X(P)$ is linearly independent if and only if $Z \not\in |O_X(1)|$.

Proof. Assume $b_X(P) < r_X(P)$. Fix $W$ evincing $b_X(P)$ and $S$ evincing $r_X(P)$. Assume $\langle S \rangle + \deg(W) \leq n$. Hence $S \cup W$ is linearly independent (Lemma 2), i.e. $\langle S \rangle \cap \langle W \rangle = \langle W \cap S \rangle$. Since $S$ is reduced, while $W$ is not reduced, $W \cap S \subsetneq W$. Hence $b_X(P) \leq \deg(W \cap S) < b_X(P)$, a contradiction. □

Lemma 4. Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a linearly normal elliptic curve. Fix a positive integer $w$ such that $2w \leq n + 1$. Fix $P \in \mathbb{P}^n$ and assume the existence of a zero-dimensional scheme $Z \subset X$ such that $\deg(Z) = w$, $P \in \langle Z \rangle$, while $P \not\in \langle Z' \rangle$ for all $Z' \subsetneq Z$. Then $b_X(P) = w$.

Proof. Assume $b_X(P) < w$ and take a scheme $B \in Z(X, P)$ (Proposition 11). Hence $P \in \langle B \rangle$ and $\deg(B) \leq w - 1$. Since $\deg(Z) + \deg(B) \leq n$, $Z \cup B$ is linearly independent. Hence $\langle Z \rangle \cap \langle B \rangle = \langle Z \cap B \rangle$. We have $P \in \langle Z \rangle \cap \langle B \rangle$. Since $\deg(B) < w$, we have $Z \cap B \subsetneq Z$. Hence $P \not\in \langle Z \cap B \rangle$, a contradiction. The converse part follows from Proposition 11 part (i) of Remark 11 and the inequality $2w \leq n + 1$. The last assertion follows from the first part using induction on the integer $b_X(Q)$. □

2. Proof of Theorem 11 and related results

Proposition 2. Fix an integer $k \geq 1$, a linearly normal elliptic curve $C \subset \mathbb{P}^{2k+1}$ and $P \in \mathbb{P}^{2k+1} \setminus \sigma_k(C)$.

(a) Either $\sharp(\mathcal{Z}(C, P)) \leq 2$ or $\mathcal{Z}(C, P)$ is infinite. We have $Z_1 \cap Z_2 = \emptyset$ and $\mathcal{O}_C(Z_1 + Z_2) \cong \mathcal{O}_C(1)$ for any $Z_1, Z_2 \in \mathcal{Z}(C, P)$ such that $Z_1 \neq Z_2$.

(b) If $\sharp(\mathcal{Z}(C, P)) \neq 2$, then $\mathcal{O}_C(2Z) \cong \mathcal{O}_C(1)$ for all $Z \in \mathcal{Z}(C, P)$.
(c) If \( Z(C, P) \) is infinite, then its positive-dimensional part \( \Gamma \) is irreducible and one-dimensional. Fix a general \( Z \in \Gamma \). Either \( Z \) is reduced or there is an integer \( m \geq 2 \) such that \( Z = mS_1 \) for a reduced \( S_1 \subset C \) such that \( \sharp(S_1) = (k+1)/m \).

(d) If \( P \) is general, then \( \sharp(Z(C, P)) = 2 \).

**Proof.** Since no non-degenerate curve is defective ([1], Remark 1.6), we have \( \sigma_k+1(C) = \mathbb{P}^{2k+1} \) and \( \dim(\sigma_k(C)) = 2k - 1 \). Hence \( \beta_C(P) = k + 1 \). Proposition 1 and part (i) of Lemma 2 give \( Z_1, Z_2 \in Z(C, P) \) such that \( Z_1 \neq Z_2 \). Parts (ii) and (iii) of Lemma 2 give \( \mathcal{O}_C(Z_1 + Z_2) \cong \mathcal{O}_C(1) \) and \( Z_1 \cap Z_2 = \emptyset \), proving part (a).

(i) Let \( J(C, \ldots, C) \subset C^{k+1} \times \mathbb{P}^{2k+1} \) be the abstract join of \( k+1 \) copies of \( C \), i.e. the closure in \( C^{k+1} \times \mathbb{P}^{2k+1} \) of the set of all \( (P_1, \ldots, P_{k+1}, P) \) such that \( P_i \neq P_j \) for all \( i \neq j \), the set \( \{P_1, \ldots, P_{k+1}\} \) is linearly independent and \( P \in \langle\{P_1, \ldots, P_{k+1}\}\rangle \). Since \( \sigma_{k+1}(C) = \mathbb{P}^{2k+1} \), for a general \( P \) the set \( Z(C, P) \) is finite and its cardinality is the degree of the generically finite surjection \( J(C, \ldots, C) \rightarrow \mathbb{P}^{2k+1} \) induced by the projection \( C^{k+1} \times \mathbb{P}^{2k+1} \rightarrow \mathbb{P}^{2k+1} \). Assume the existence of schemes \( Z_1, Z_2, Z_3 \in Z(C, P) \) such that \( Z_i \neq Z_j \) for all \( i \neq j \). Part (a) gives \( Z_1 \cap Z_2 = \emptyset \) and \( \mathcal{O}_C(Z_1 + Z_2) \cong \mathcal{O}_C(1) \) for all \( i \neq j \). Taking \( i = 1 \) and \( j \in \{2, 3\} \) we get \( \mathcal{O}_C(Z_2) \cong \mathcal{O}_C(Z_3) \). By symmetry we get \( \mathcal{O}_C(Z_1) \cong \mathcal{O}_C(Z_3) \) for all \( Z \in Z(C, P) \). Since \( \mathcal{O}_C(Z_1 + Z_2) \cong \mathcal{O}_C(1) \), we also get \( \mathcal{O}_C(Z_2) \cong \mathcal{O}_C(1) \) for all \( Z \in Z(C, P) \).

(ii) Now assume \( \sharp(Z(C, P)) = 1 \), say \( Z(P, X) = \{Z\} \). Fix any \( E \in |\mathcal{O}_C(1)(-Z)| \). Since \( E+Z \) is contained in a hyperplane, we have \( \langle Z \rangle \cap \langle E \rangle \neq \emptyset \). Part (iii) of Lemma 2 gives \( \dim(\langle Z \rangle \cap \langle E \rangle) = \deg(Z \cap E) \). Set \( J := \{Q \cap \langle\langle Z_1 \rangle \rangle \} \). We just saw that \( J \) is a complete projective set. For dimensional reasons the projection of \( Z \times |\mathcal{O}_C(1)(-Z)| \) into its first factor induces a dominant morphism \( u : J \rightarrow \langle Z \rangle \). Since \( J \) is complete, there is \( E \in |\mathcal{O}_C(1)(-Z)| \) such that \( u(E) = Z \). The uniqueness of \( Z \) gives \( E = Z \). Hence \( 2Z \in |\mathcal{O}_C(1)| \). Since the set of all \( Z \subset X \) such that \( 2Z \in |\mathcal{O}_C(1)| \) has dimension \( k+1 \), we get \( \sharp(Z(C, P)) = 2 \) for a general \( P \), proving part (d). Since this integer, two, is the degree of a generically finite surjection \( \gamma : J(C, \ldots, C) \rightarrow \mathbb{P}^{2k+1} \) and \( \mathbb{P}^{2k+1} \) is a normal variety, each fiber of \( \gamma \) is either infinite or with cardinality \( \leq 2 \). Therefore either \( \sharp(Z(C, P)) \leq 2 \) or \( Z(C, P) \) is infinite.

(iii) Now assume that \( Z(C, P) \) is infinite. Since any two different elements of \( Z(C, P) \) are disjoint (see step (i)), for a general \( A \in C \) there is at most one element of \( \Gamma \) containing \( A \). Hence \( \dim(\Gamma) = 1 \) and \( \Gamma \) is irreducible. Since a general point of \( C \) is contained in a unique element of \( \Gamma \), the algebraic family \( \Gamma \) of effective divisors of \( C \) is a so-called **involution** ([5], §5). Since any two elements of \( \Gamma \) are disjoint, this involution has no base points. Let \( Z \) be a general element of \( \Gamma \). Either \( Z \) is reduced or there is an integer \( m \geq 2 \) such that \( Z = mS \) with \( S \) reduced ([5], Proposition 5.8), concluding the proof of part (c).

**Proof of Theorem 4.** For any integer \( k > 0 \) such that \( \sigma_{k-1}(X) \neq \mathbb{P}^n \), we have \( r_X(Q) = k \) for a general \( Q \in \sigma_k(X) \). Hence for arbitrary \( w \leq \lfloor (n+2)/2 \rfloor \) there are points \( P \) such that \( r_X(P) = b_X(P) = w \). Fix \( w \leq -1 + n/2 \), \( P \) and \( W \) such that \( b_X(P) = w \), and \( r_X(P) > w \). Lemma 3 gives \( r_X(P) \geq n + 1 - w \). Hence to prove Theorem 1 it is sufficient to prove that \( r_X(P) = n + 1 - w \). Fix \( W \in Z(X, P) \).

Set \( B := \{Z + W \mid Z \in |\mathcal{O}_X(1)(-2W)| \} \). Hence \( B := \{B \in |\mathcal{O}_X(1)(-W)| : W \subset B \} \). Set \( S := \{Z \in |\mathcal{O}_X(1)(-W)| : P \in (Z) \} \). Since \( \deg(\mathcal{O}_X(1)(-W)) = n + 1 - w \leq n \), every element of \( |\mathcal{O}_X(1)(-W)| \) is linearly independent. However, in the definition of the set \( S \) we did not prescribed that \( P \notin \langle Z \rangle \) for all \( Z \subset Z \). Hence \( B \subset S \). Part
(i) of Lemma \(^2\) and the inequality \(r_X(P) = n + 1 - w\) if and only if there is a reduced \(S \in \mathcal{S}\).

(a) In this step we prove that \(\mathcal{B} \neq \mathcal{S}\). Fix a general subset \(E \subset X\) such that \(\mathfrak{z}(E) = n - 2w - 1\). Since \(n > 2w + 1\), we have \(E \neq \emptyset\). Hence for a general \(E\) the degree \(w\) line bundles \(\mathcal{O}_X(W)\) and \(\mathcal{O}_X(1)(-W-E)\) are not isomorphic. Hence to get \(\mathcal{B} \neq \mathcal{S}\) it is sufficient to prove the existence of a degree \(w\) zero-dimensional subscheme \(A_E\) of \(X\) such that \(E + A_E \in \mathcal{S}\). Let \(\ell_{(E)} : \mathbb{P}^n \setminus \{E\} \to \mathbb{P}^{2w+1}\) denote the linear projection from \(E\). Call \(X_E \subset \mathbb{P}^{2w+1}\) the closure of \(\ell_{(E)}(X \setminus \langle E \rangle) \cap X\) in \(\mathbb{P}^{2w+1}\). Since \(X\) is non-degenerate, \(X_E\) spans \(\mathbb{P}^{2w+1}\). Since \(X\) is a smooth curve, the rational map \(\ell_{(E)}(X \setminus \langle E \rangle) \cap X\) extends to a surjective morphism \(\psi : X \to X_E\). Since every degree \(n - 2w + 1\) zero-dimensional subscheme of \(X\) is linearly independent, \(E\) is the scheme-theoretic intersection of \(X\) with \(\langle E \rangle\). Hence \(\deg(X_E) = \deg(X) - \deg(E) = n + 1 - n + 2w + 1 = 2w + 2\). Hence \(\deg(X_E) = 2w + 2\) and \(\deg(E) = 1\). Since \(\deg(\psi) = 1\), \(X_E\) and \(X\) are birational. Hence \(X_E\) is a linearly normal elliptic curve.

Since \(X_E\) and \(X\) are smooth curves, \(\psi\) is an isomorphism. Since \(\langle E \rangle \cap X = E\) (as schemes), we have \(\psi^+(\mathcal{O}_X(1)) \cong \mathcal{O}_X(1)(-E)\). Set \(W' := \psi(W)\). For a general \(E\) we may assume \(E \cap W = \emptyset\). Hence \(W'\) is a degree \(w\) subscheme of \(X_E\) isomorphic as an abstract scheme to \(W\). Hence \(W'\) is not reduced. Fix \(W_1 \not\subseteq W'\) and call \(W_2\) the only subscheme of \(W\) such that \(\psi(W_2) = W_1\). Since \(W'\) is linearly independent, \(\ell_{(E)}(\langle W \rangle) \to \langle W' \rangle\) is an isomorphism. Since \(\ell_{(E)}|W = \psi|W\) is an isomorphism onto \(W'\) and \(P \notin \langle W_2 \rangle\), we get \(\ell_{(E)}(P) \notin \langle W_1 \rangle\). Since this is true for all \(W_1 \not\subseteq W\), Lemma \(^3\) gives that \(W'\) evinces the border \(X_E\)-rank of the point \(\ell_{(E)}(P)\). Our choice of \(E\) implies \(\mathcal{O}_{X_E}(2W') \neq \mathcal{O}_{X_E}(1)\). Hence part (b) of Proposition \(^2\) gives the existence of a unique scheme \(A \subset X_E\) such that \(A \neq W'\) and \(\ell_{(E)}(P) \in \langle A \rangle\).

Set \(A_E := \psi^{-1}(A)\). Since \(E \cap W = \emptyset\) and \(\deg(A_E) = \deg(W)\), to prove \(E + A_E \notin \mathcal{B}\) it is sufficient to prove \(A_E \neq W\), i.e. (since \(\psi\) is an isomorphism) \(W' \neq A\). We chose \(A \neq W\). Call \(X[n - 2w - 1]\) the set of all \(E\) for which \(E + A_E\) is defined.

(b) Let \(\Gamma \subseteq \mathcal{S}\) be any irreducible component of \(\mathcal{S}\) containing the irreducible algebraic family \(\{E + A_E\}_{E \in X[n - 2w - 1]}\) constructed in step (a). Let \(F\) be a general element of \(\Gamma\). Remember that to prove \(r_X(P) = n + 1 - w\) it is sufficient to find a reduced \(S \in \Gamma\). \(\Gamma\) is an irreducible algebraic family of divisors of \(X\). We have \(\dim(\Gamma) = n - 2w - 1\). By construction for a general \(E \subset X\) such that \(\mathfrak{z}(E) = n - 2w - 1\) there is \(B_E \in \Gamma\) such that \(E \subset B_E\). For general \(E\) we have \(\langle E \rangle \cap \langle W \rangle = \emptyset\).

Since \(P \notin \langle E \rangle\), the scheme \(\ell_{(E)}(W)\) is isomorphic to \(W\), \(P \in \ell_{(E)}(\langle W \rangle)\) and \(P \notin \langle W' \rangle\) for any \(W' \not\subseteq \ell_{(E)}(\langle W \rangle)\). Lemma \(^2\) gives \(\ell_{(E)}(P) \notin \sigma_k(X_E)\) for general \(E\). For general \(E\) the degree \(2k + 2\) line bundles \(\mathcal{O}_X(2W)\) and \(\mathcal{O}_X(1)(-E)\) are not isomorphic. Hence part (b) of Proposition \(^2\) applied to the curve \(X_E\), the point \(\ell_{(E)}(P)\) and the scheme \(Z := \ell_{(E)}(W)\) gives that such a divisor \(B_E\) is unique. Hence \(\Gamma\) is an involution in the classical terminology (\(\S\), \(\S\)). Assume for the moment that \(\Gamma\) has no fixed component. We get that either \(F\) is reduced (and hence parts (i) and (ii) of Theorem \(^1\) are proved for \(P\)) or there is an integer \(m \geq 2\) such that each connected component of \(F\) appears with multiplicity \(m\) (\(\S\), Proposition 5.8). Since \(F = E + A_E\) with \(E\) reduced and \(\mathfrak{z}(E) > \deg(A_E)\) this is obviously false. Hence we may assume that \(\Gamma\) has a base locus. Call \(D\) the base locus of \(\Gamma\). Hence the irreducible algebraic family \(\Gamma(-D)\) of effective divisors of \(X\) has the same dimension and it is base point free. We have \(F = D + F'\) with \(F'\) general in \(\Gamma(-D)\). Since \(\Gamma(-D)\) is an involution without base points and whose general member has at least one reduced connected component (a connected component
of \( E \), its general member \( F' \) is reduced ([8], Proposition 5.8). Since \( D \) has finite support and \( F' \) is general, we also have \( F' \cap D = \emptyset \). Fix \( O \in D_{red} \). We have \( O \notin \langle W \rangle \), because \( \deg(W \cup \{O\}) = w+1 \) and every degree \( w+1 \) subscheme of \( X \) is linearly independent. Let \( E_1 \) be the union of \( O \) and \( n-2w-2 \) general points of \( X \) \((n = 2w+2, \text{ then } E_1 = \{O\})\). Since \( O \notin \langle W \rangle \) and \( X \) is non-degenerate, we have \( \langle W \rangle \cap \langle E_1 \rangle = \emptyset \). Hence the point \( \ell_{\langle E_1 \rangle}(P) \) is contained in the linear span of the degree \( w \) subscheme \( \ell_{\langle E_1 \rangle}(W) \) of the linearly normal elliptic curve \( X_{E_1} \subset \mathbb{P}^{2w+2} \), but not in the linear span of any proper subscheme of it. Since any degree \( 2w+1 \) subscheme of \( X_{E_1} \) is linearly independent, we get \( b_{X_{E_1}}(\ell_{\langle E_1 \rangle}(P)) = w + 1 \). Since \( O \) is a base point of \( \Gamma \), we also get a one-dimensional family \( \Gamma' \) of distinct degree \( w+1 \) subschemes of \( X_{E_1} \) such that \( \ell_{\langle E_1 \rangle}(P) \) is in the linear span of each of it. Part (a) of Proposition 2 gives that these schemes are pairwise disjoint. Hence \( \deg(D) = 1 \) and \( D = \{O\} \) (as schemes). Since \( E + A_E \) has at least \( \deg(E_1) \) points with multiplicity one, at least one connected component of the general element \( F' \) of \( \Gamma' \) is reduced. Since \( F' \) is a general element of the base point free involution \( \Gamma(-D) \), \( F' \) is reduced ([8], Proposition 5.8). Since any degree \( n \) divisor of \( X \) is linearly independent, we have \( \langle E_1 \rangle \cap X = E_1 \) (scheme-theoretic intersection). Since \( \Gamma' \) has no base points, we may also assume that \( F' \cap (X_{E_1} \setminus \ell_{\langle E_1 \rangle}(X \setminus E_1)) = \emptyset \). Hence the counterimage \( F'' \) of \( F' \) in \( X \) is disjoint from \( E_1 \). Hence \( F'' \cup E_1 \) is reduced. Since \( P \in \langle F'' \cup E_1 \rangle \), we get \( r_{X}(P) \leq n + 1 - w \).

Proposition 3. Fix an integer \( k \geq 1 \) and a linearly normal elliptic curve \( X \subset \mathbb{P}^{2k+1} \). Then there are \( Q, P \in \mathbb{P}^{2k+1} \) such that \( b_{X}(Q) = b_{X}(P) = r_{X}(Q) = k + 1 \) and \( r_{X}(P) \geq k + 2 \). The set of all such points \( Q \) contains a non-empty open subset of \( \mathbb{P}^{2k+1} \), while the set of all such points \( P \) contains a non-empty algebraic subset of codimension 2 of \( \mathbb{P}^{2k+1} \).

Proof. Since \( \sigma_{k+1}(X) = \mathbb{P}^{2k+1} \), while \( \dim(\sigma_{k}(X)) = 2k - 1 \) ([1], Remark 1.6), we may take as \( Q \) a general point of \( \mathbb{P}^{2k+1} \). Now we prove the existence of points \( P \in \mathbb{P}^{n} \) such that \( r_{X}(P) > b_{X}(P) = k + 1 \) and that the set of all \( P \) such that \( b_{X}(P) = k + 1 < r_{X}(P) \) contains a codimension 2 subset of \( \mathbb{P}^{2k+1} \). Let \( U \) be the set of all degree \( k+1 \) schemes \( Z_1 \subset X \) such that \( Z_1 \) is non-reduced and \( 2Z_1 \notin [O_X(1)] \). The set \( U \) is a quasi-projective integral variety of dimension \( k+1 \). Fix any \( Z_1 \in U \). Let \( \mathcal{V}(Z_1) \) denote the set of all non-reduced \( Z_2 \in [O_X(1)(-Z_1)] \) such that \( Z_2 \cap Z_1 = \emptyset \). The set \( \mathcal{V}(Z_1) \) is a quasi-projective and integral variety of dimension \( k \). Since \( Z_1 \cap Z_2 = \emptyset \), Remark 2 shows that \( \langle Z_1 \rangle \cap \langle Z_2 \rangle \) is a single point, \( Q \). If \( b_{X}(Q) = k + 1 \), then \( \mathcal{Z}(X, Q) = \{Z_1, Z_2\} \), because \( O_X(2Z_1) \notin O_X(1) \) (Part (b) of Proposition 2).

Since neither \( Z_1 \) nor \( Z_2 \) is reduced, we get \( r_{X}(Q) > k + 1 \). Varying \( Z_2 \) for a fixed \( Z_1 \) the set of all points \( Q \) obtained in this way covers a non-empty open subset of an irreducible hypersurface of \( \langle Z_1 \rangle \). Assume \( b_{X}(Q) \leq k \) and fix \( W \in \mathcal{Z}(X, Q) \). Notice that \( P \notin \langle W' \rangle \) for any \( W' \not\subset W \). Since \( \deg(W) + \deg(Z_1) \leq n \), Lemma 1 and Lemma 2 give the existence of \( Z' \not\subset Z \) such that \( Q \in \langle Z' \rangle \). Iterating the trick taking \( Z' \) and \( W \) instead of \( Z_1 \) and \( W \) we get \( W \not\subset Z' \) and hence \( W \subset Z_1 \). Making this construction using \( Z_2 \) and \( W \) we get \( W \not\subset Z_2 \). Since \( Z_1 \cap Z_2 = \emptyset \), we obtained a contradiction.

3. Proof of Theorem 2

Proof of Theorem 2. Since \( X \) is linearly normal, for any zero-dimensional scheme \( Z \subset X \) we have \( h^1(\mathbb{P}^n, \mathcal{I}_Z(1)) = h^1(X, O_X(1)(-Z)) \). Hence \( h^1(\mathbb{P}^n, \mathcal{I}_Z(1)) > 0 \).
0 if and only if either \( \deg(Z) \geq n + 2 \) or \( \deg(Z) = n + 1 \) and \( Z \in |O_X(1)| \). Assume the existence of the set \( F \). Since \( F \cap W = \emptyset \) and \( P \in \langle W \rangle \cap \langle E \rangle \), we have \( h^1(Z_{W \cap E}(1)) > 0 \) (Lemma 1). Since \( \deg(F \cup W) \leq n \), we get a contradiction. Define \( B \) and \( S \) as in the proof of Theorem 1. Step (a) of the quoted proof works with no modification. At the end of that step we defined \( X[n-2w-1] \) and now we continue the proof of Theorem 2 in the following way.

(b) Let \( \Gamma \subseteq S \) be any irreducible component of \( S \) containing \( Z \). To prove Theorem 2 for the point \( P \) it is sufficient to find a reduced \( S \in \Gamma \) such that \( S \cap U = \emptyset \). \( \Gamma \) is an irreducible algebraic family of divisors of \( X \). We have \( \dim(\Gamma) = n - 2w - 1 \). By construction for a general \( E \subseteq X \) such that \( \sharp(\Sigma(E)) = n - 2w - 1 \) there is \( B_E \in \Gamma \) such that \( E \subseteq B_E \). For a general \( E \) we have \( \langle E \rangle \cap \langle W \rangle = \emptyset \). Since \( P \notin \langle E \rangle \), the scheme \( \ell_{\langle E \rangle}(W) \) is isomorphic to \( W, P \in \langle \ell_{\langle E \rangle}(W) \rangle \) and \( P \notin \langle W' \rangle \) for any \( W' \subseteq \ell_{\langle E \rangle}(W) \). Lemma 2 gives \( \ell_{\langle E \rangle}(P) \notin \sigma_{w-1}(X_E) \) for a general \( E \). For general \( E \) the degree \( 2w \) line bundles \( \mathcal{O}_X(2W) \) and \( \mathcal{O}_X(1)(-E) \) are not isomorphic. Hence Proposition 4 and part (i) Lemma 2 applied to the curve \( X_E \), the point \( \ell_{\langle E \rangle}(P) \) and the scheme \( Z := \ell_{\langle E \rangle}(W) \) gives that such a divisor \( B_E \) is unique. Thus \( \Gamma \) is an involution in the classical terminology \( [8], \S 5 \).

(b1) In this step we assume that \( \Gamma \) has no base points. Since \( \Gamma \) has no base points and \( U \) is a fixed finite set, a general \( S \in \Gamma \) is contained in \( X \setminus U \). Hence to conclude the proof of Theorem 2 it is sufficient to prove that a general \( S \in \Gamma \) is reduced. Either \( S \) is reduced or there is an integer \( m \geq 2 \) such that each connected component of \( S \) appears with multiplicity one \( [8], \text{Proposition 5.8} \). Since \( F = A_E \) with \( E \) reduced and \( \sharp(E) > \deg(A_E) \) this is obviously false.

(b2) Assume that \( \Gamma \) has a base locus. Call \( D \) the base locus of \( \Gamma \). Thus the irreducible algebraic family \( \Gamma(-D) \) of effective divisors of \( X \) has the same dimension and it is base point free. We have \( F = D + F' \) with \( F' \) general in \( \Gamma(-D) \). Since \( \Gamma(-D) \) is an involution without base points and whose general member has at least one reduced connected component (a connected component of \( E \)), its general member \( F' \) is reduced \( [8], \text{Proposition 5.8} \). Since \( D \) has finite support and \( F' \) is general, we also have \( F' \cap D = \emptyset \). Fix \( O \in D_{\text{red}} \). We have \( O \notin \langle W \rangle \), because \( \deg(W \cup \langle O \rangle) = w+1 \) and every degree \( w+1 \) subscheme of \( X \) is linearly independent. Let \( E_1 \) be the union of \( O \) and \( n-2w-2 \) general point of \( X \) (if \( n = w+2 \), then \( E_1 = \{ O \} \)). Since \( O \notin \langle W \rangle \) and \( X \) is non-degenerate, we have \( \langle W \rangle \cap \langle E_1 \rangle = \emptyset \). Thus the point \( \ell_{\langle E_1 \rangle}(P) \) is contained in the linear span of the degree \( w \) subscheme \( \ell_{\langle E_1 \rangle}(W) \) of the linearly normal elliptic curve \( X_{E_1} \subset \mathbb{P}^{2w+2} \), but not in the linear span of any proper subscheme of it. Since any degree \( 2w+1 \) subscheme of \( X_{E_1} \) is linearly independent, we get \( b_{X_{E_1}}(\ell_{\langle E_1 \rangle}(P)) = w+1 \). Since \( O \) is a base point of \( \Gamma \), we also get a one-dimensional family \( \Gamma' \) of distinct degree \( w+1 \) subschemes of \( X_{E_1} \) such that \( \ell_{\langle E_1 \rangle}(P) \) is in the linear span of each of it. Part (a) of Proposition 2 gives that these schemes are pairwise disjoint. Hence \( \deg(D) = 1 \) and \( D = \{ O \} \) (as schemes). Since \( E + A_E \) has at least \( \deg(E_1) \) points with multiplicity one, at least one connected component of the general element \( F' \) of \( \Gamma' \) is reduced. Since \( F' \) is a general element of the base point free involution \( \Gamma(-D), F' \) is reduced \( [8], \text{Proposition 5.8} \). Since any degree \( n \) divisor of \( X \) is linearly independent, we have \( \langle E_1 \rangle \cap X = E_1 \) (scheme-theoretic intersection). Since \( \Gamma' \) has no base points, we may also assume that \( F' \cap (X_{E_1} \setminus \ell_{\langle E_1 \rangle}(X \setminus E_1)) = \emptyset \). Hence the counterimage \( F'' \)
of $F'$ in $X$ is disjoint from $E_1$. Thus $F'' \cup E_1$ is reduced. Since $P \in \langle F'' \cup E_1 \rangle$, we get $r_X(P) \leq n + 1 - w$.

Steps (b1) and (b2) conclude the proof of Theorem 2.

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