SCHWARZ LEMMA FOR CONICAL KÄHLER METRICS
WITH GENERAL CONE ANGLES

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Abstract. The Schwarz–Pick lemma is a fundamental result in complex analysis. It is well-known that Yau generalized it to the higher dimensional manifolds by applying his maximum principle for complete Riemannian manifolds. Jeffres obtained Schwarz lemma for volume forms of conical Kähler metrics, based on a barrier function and the maximum principle argument. In this note, we generalize Jeffres’ result to general cone angles including the case when the pullback of the metric would blows up along the divisors.

1. Introduction

The Schwarz–Pick lemma states that any holomorphic map between the unit disks in the complex plane decreases the Poincaré metrics. After that, Ahlfors [Ahl38] generalized it to a holomorphic map from the unit disk to a hyperbolic Riemann surface. For higher dimensions, Yau [Yau78] showed that any holomorphic map from complete Kähler manifold whose Ricci curvature is bounded from below to a Hermitian manifold whose holomorphic bisectional curvature is bounded by a negative constant decreases the metric up to a multiplicative constant. Also, he showed that, under similar conditions on curvatures, any holomorphic map decreases the volume forms up to a multiplicative constant. Both results essentially based on his maximum principle for complete Riemannian manifolds. Later on, many generalizations obtained in various geometric settings.

In this note, we focus on the conical Kähler metrics, for short, cone metrics. Let $X$ be a compact Kähler manifold of dimension $n$, $D$ be a smooth divisor on $X$, and $\beta$ be a real number satisfying $0 < \beta < 1$. The cone metric $\omega$ with cone angle $2\pi\beta$ along $D$ is a Kähler metric on $X \setminus D$ which is locally quasi-isometric to the standard cone metric

$$\omega_\beta := \frac{\beta^2}{|z|^{2(1-\beta)}} \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} + \sum_{i=2}^{n} \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i,$$

and satisfies some regularity conditions. (For a precise definition of the cone metric, see Definition 2.2.) The notion of cone metrics plays an important role in recent advances in Kähler geometries, in particular Kähler-Einstein problems, for instance see [CDS15a, CDS15b, CDS15c, Tia15].

To state the theorems, we use the following setups and notations.

Setups 1.1. Let $X$ and $Y$ be compact Kähler manifolds, $D \subset X$, $E \subset Y$ be smooth divisors, and $f : X \to Y$ be a surjective holomorphic map satisfying

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Let \( f^*(E) = kD \) with \( k \in \mathbb{Z}_{>0} \). Let \( \omega_X \) (resp. \( \omega_Y \)) be a cone metric with cone angle \( 2\pi \alpha \) (resp. \( 2\pi \beta \)) along \( D \) (resp. \( E \)) on \( X \) (resp. \( Y \)). Let \( s \in H^0(X, \mathcal{O}_X(D)) \) be a holomorphic section of the line bundle \( \mathcal{O}_X(D) \) whose zero divisor is \( D \) and \( h \) be a smooth Hermitian metric on it satisfying \( |s|^2_h \leq 1 \). Let \( C > 0 \) be an upper bound for the Chern curvature of \( h \) i.e. \( \sqrt{\text{tr}}R_h \leq C \omega_X \). For a Kähler form \( \omega \), we will denote by \( \text{Ric}(\omega) \) the Ricci curvature of \( \omega \), \( R(\omega) \) the scalar curvature of \( \omega \), and \( \text{Bisect}(\omega) \) the bisectional curvature of \( \omega \).

Schwarz lemma for the cone metrics obtained by Jeffres [Jef00a] is states as follows.

**Theorem 1.2** ([Jef00a, Theorem]). Assume that \( \dim X = \dim Y = n \), the cone angles satisfy \( \alpha \leq \beta \) and there exists non-negative constants \( A, B \geq 0 \) satisfying

\[
R(\omega_X) \geq -A, \quad \text{Ric}(\omega_Y) \leq -B \omega_Y < 0.
\]

Then, the volume forms satisfy

\[
f^*\omega_Y^n \leq \left( \frac{A}{B} \right)^n \omega_X^n \quad \text{on } X \setminus D.
\]

Since the cone metric is not complete on \( X \setminus D \), we cannot apply the maximum principle argument directly. Jeffers overcame this difficulty by using a barrier function, called “Jeffres’ trick”. However, his original proof seems to need more assumptions on the regularity of the cone metrics along \( D \) as in Definition 2.1 (see the proof of Proposition 3.3).

In this note, we will generalize this theorem to a general cone angle and prove a Schwarz lemma for cone metrics.

**Theorem 1.4** (Volume forms). Assume that \( \dim X = \dim Y = n \) and the curvature condition (1.3) holds.

(a) Suppose \( \alpha \leq k\beta \). Then we have

\[
f^*\omega_Y^n \leq \left( \frac{A}{nB} \right)^n \omega_X^n \quad \text{on } X \setminus D.
\]

(b) Suppose \( \alpha > k\beta \). Then we have

\[
f^*\omega_Y^n \leq \left( \frac{A + (\alpha - k\beta)C}{nB} \right)^n \frac{\omega_X^n}{|s|^2(\alpha - k\beta)} \quad \text{on } X \setminus D.
\]

We remark that the condition \( \alpha \leq k\beta \) on cone angles in the statement (a) is weaker than assumptions in Theorem 1.2.

**Theorem 1.5** (Metrics). Assume that there exists non-negative constants \( A, B \geq 0 \) such that the curvatures satisfy the following:

\[
Ric(\omega_X) \geq -A \omega_X, \quad \text{Bisect}(\omega_Y) \leq -B < 0.
\]

(a) Suppose \( \alpha \leq k\beta \). Then we have

\[
f^*\omega_Y \leq \frac{A}{B} \omega_X \quad \text{on } X \setminus D.
\]
(b) Suppose $\alpha > k\beta$. Then we have
\[
 f^*\omega_Y \leq \frac{A + (\alpha - k\beta)C}{B} \frac{\omega_X}{|s|^{2(\alpha-k\beta)}} \text{ on } X \setminus D.
\]

If the cone angle satisfies $\alpha > k\beta$, the pullback $f^*\omega_Y$ has singularities along $D$. In fact, even in a one-dimensional case, the pullback of the standard cone metric $\omega_\beta = \left(\beta^2/|w|^{2(1-\beta)}\right) \sqrt{-1}dw \wedge d\overline{w}/2$ by $f : z \mapsto w = z^k$ is given by
\[
f^*\omega_\beta = \beta^2 k^2 |z|^{2(\beta-1)} \frac{\sqrt{-1}}{2} \frac{dz \wedge d\overline{z}}{2},
\]
therefore we have
\[
f^*\omega_\beta = \frac{\beta^2 k^2}{\alpha^2} \frac{1}{|z|^{2(\beta-\alpha)}},
\]
which is singular if $\alpha > k\beta$.

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2. Cone metrics

In this section, we recall the definition of cone metrics following [Don12, Section 4]. Let $X$ be a compact Kähler manifold of dimension $n$, $D$ be a smooth divisor on $X$, and $\beta$ be a real number satisfying $0 < \beta < 1$. We first remark that if we take a local holomorphic chart $(U, (z^1, \ldots, z^n))$ satisfying $D \cap U = \{z^1 = 0\}$, the standard cone metric $\omega_\beta$ induces a distance function $d_\beta$ on $U$ which is expressed as
\[
d_\beta(z, w) = \left( |z^1|^\beta - |w^1|^\beta \right)^2 + |z^2-w^2|^2 + \cdots + |z^n-w^n|^2 \right)^{1/2},
\]
where $z = (z^1, \ldots, z^n), w = (w^1, \ldots, w^n)$. Here, we take a suitable branch of $z^\beta$.

**Definition 2.1** ($C^{2,\alpha,\beta}$-functions). Let $\alpha$ be a constant satisfying $0 < \alpha < \min\{1/\beta - 1, 1\}$. We define the regularities of functions along $D$ as follows.

1. A function $f$ on $X$ is said to be of class $C^{\alpha,\beta}$ if for any local holomorphic chart $(U, (z^1, \ldots, z^n))$ satisfying $D \cap U = \{z^1 = 0\}$, $f$ is an $\alpha$-Hölder continuous function on $U$ with respect to the distance function $d_\beta$.

   This definition is equivalent to the following statement which is the original definition in [Don12]. We set $\tilde{f}$ by $\tilde{f}(\xi, z^2, \ldots, z^n) := f(|\xi|^{1/\beta-1} \xi, z^2, \ldots, z^n)$. Then $\tilde{f}$ is an $\alpha$-Hölder continuous function with respect to $\xi, z^2, \ldots, z^n$ with respect to the Euclidean distance.

2. A $(1,0)$-form $\tau$ is said to be of class $C^{\alpha,\beta}$ if
\[
|z^1|^{1-\beta} \tau \left( \frac{\partial}{\partial z^1} \right) \in C^{\alpha,\beta},
\]
\[
\tau \left( \frac{\partial}{\partial z^i} \right) \in C^{\alpha,\beta} \quad \text{for } i = 2, \ldots, n
\]
A (1, 1)-form $\sigma$ is said to be of class $C^{\alpha, \beta}$ if
\[
|z^1|^{2(1-\beta)} \sigma \left( \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^1} \right) \in C^{\alpha, \beta},
\]
\[
|z^1|^{1-\beta} \sigma \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^i} \right) \in C^{\alpha, \beta} \quad \text{for } i = 2, \ldots, n,
\]
\[
|z^1|^{1-\beta} \sigma \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right) \in C^{\alpha, \beta} \quad \text{for } i, j = 2, \ldots, n.
\]

A function $f$ is said to be of class $C^{2,\alpha,\beta}$ if $f$, $\partial f$, $\overline{\partial f}$, $\sqrt{-1} \partial \overline{\partial f}$ are of class $C^{\alpha, \beta}$.

**Definition 2.2** (Cone metrics). A closed positive $(1, 1)$-current $\omega$ on $X$ is called a cone metric with cone angle $2\pi \beta$ along $D$ if it satisfies the following three conditions:

(i) $\omega$ is a Kähler metric on $X \setminus D$

(ii) For each point $x \in D$, there exists a local holomorphic chart $(U, (z^1, \ldots, z^n))$ satisfying $D \cap U = \{z^1 = 0\}$ such that $\omega$ is quasi-isometric to $\omega_{\beta}$ on $U \setminus D$, that is, there exists a constant $C = C_U > 0$ such that
\[
\frac{1}{C} \omega_{\beta} \leq \omega \leq C \omega_{\beta} \quad \text{on } U \setminus D.
\]

Here, $\omega_{\beta}$ is the standard cone metric defined by
\[
\omega_{\beta} := \frac{\beta^2}{|z|^{2(1-\beta)}} \sqrt{-1} dz^1 \wedge d\bar{z}^1 + \sum_{i=2}^{n} \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i.
\]

(iii) There exists a smooth Kähler form $\omega_0$ on $X$, and a $C^{2,\alpha,\beta}$-function $\varphi$ such that
\[
\omega = \omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi.
\]

In [Jef00a], the regularity condition (iii) does not assumed. However, we assume here.

A typical example of the cone metric is $\omega := \omega_0 + \delta \sqrt{-1} \partial \overline{\partial} |s|^2 h$, where $\omega_0$ is a smooth Kähler metric on $X$, $\delta$ is a sufficiently small constant, $s \in H^0(X, \mathcal{O}_X(D))$ is a holomorphic section of the line bundle $\mathcal{O}_X(D)$ whose zero divisor is $D$, and $h$ is a smooth Hermitian metric.

**3. Proof of the theorems**

To prove the theorem, we need the following Laplacian estimates which are obtained by [Che68, Lu68]. For the readers convenience, we prove here.

**Proposition 3.1.** Let $X, Y$ be (not necessarily compact) Kähler manifolds, and $f : X \to Y$ be a holomorphic map. Let $\omega_X$ (resp. $\omega_Y$) be a smooth Kähler metric on $X$ (resp. $Y$). We set $v := f^* \omega_Y / \omega_X^n$, and $u := \text{tr}_{\omega_X}(f^* \omega_Y)$.
(a) Suppose that there exists non-negative constants $A, B \geq 0$ satisfying $R(\omega_X) \geq -A$, $\text{Ric}(\omega_Y) \leq -B\omega_Y$, and $\dim X = \dim Y = n$. Then we have
\begin{align*}
\Delta_{\omega_X} \log v &\geq nBv^{1/n} - A, \\
\Delta_{\omega_X} v &\geq v(nBv^{1/n} - A).
\end{align*}

(b) Suppose that there exists non-negative constants $A, B \geq 0$ satisfying $\text{Ric}(\omega_X) \geq -A\omega_X$, $\text{Bisect}(\omega_Y) \leq -B\omega_Y$. Then we have
\begin{align*}
\Delta_{\omega_X} \log u &\geq Bu - A, \\
\Delta_{\omega_X} u &\geq u(Bu - A).
\end{align*}

Proof. Let $(z^1, \ldots, z^n)$ and $(w^1, \ldots, w^n)$ be normal coordinates on $X$ and $Y$ respectively. We set
\begin{align*}
\omega_X &= \sqrt{-1} \sum_{i,j} g_{i,j} dz^i \wedge d\overline{z}^j, \\
\omega_Y &= \sqrt{-1} \sum_{\alpha,\beta} h_{\alpha,\beta} dw^\alpha \wedge d\overline{w}^\beta.
\end{align*}

(a) $v$ is locally denoted as
\begin{equation}
(3.2) \quad v = \frac{f^* \omega^Y}{\omega^n_X} = \frac{\det(h_{\omega_Y} \circ f) \det J(f)^2}{\det(g_{\overline{\sigma}})}
\end{equation}

where $J(f)$ is the Jacobian of $f$. Therefore, on $\Omega := \{ x \in X \mid \det J(f)(x) \neq 0 \}$, we obtain
\begin{align*}
\sqrt{-1} \partial \overline{\partial} \log v &= f^* \sqrt{-1} \partial \overline{\partial} \log \det(h_{\alpha,\beta}) + \sqrt{-1} \partial \overline{\partial} \log \det(g_{i,j}) - \sqrt{-1} \partial \overline{\partial} \log \det J(f)^2 \\
&= f^*(-\text{Ric}(\omega_Y)) + \text{Ric}(\omega_X).
\end{align*}

By the assumption on curvatures and the inequality of arithmetic and geometric means, we have the following estimates on $\Omega$:
\begin{align*}
\Delta_{\omega_X} \log v &= \text{tr}_{\omega_X} \left( \sqrt{-1} \partial \overline{\partial} \log v \right) = \text{tr}_{\omega_X} (f^*(-\text{Ric}(\omega_Y))) + R(\omega_X) \\
&\geq B \text{tr}_{\omega_X} (f^*\omega_Y) - A \\
&\geq nBv^{1/n} - A, \\
\Delta_{\omega_X} v &= \Delta_{\omega_X} e^{\log v} = e^{\log v} \left( |\nabla \log v|_{\omega_X}^2 + \Delta_{\omega_X} \log v \right) \\
&\geq v \Delta_{\omega_X} \log v \\
&\geq v(nBv^{1/n} - A).
\end{align*}

By continuity, the last inequality holds on the whole $X$.

(b) We set
\begin{equation}
(3.1) \quad f^*\omega_Y = \sqrt{-1} \sum_{i,j} h_{i,j} dz^i \wedge d\overline{z}^j := \sqrt{-1} (h_{\omega_Y} \circ f)(\partial_i f^\alpha)(\partial_j \overline{f}^\beta) dz^i \wedge d\overline{z}^j,
\end{equation}
and denote $R_{ijkl}$ and $S_{ijk\ell}$ by the curvature tensor of $\omega_X$ and $\omega_Y$ respectively. Then we have the following inequalities, which are our assertion (b).

\begin{align*}
\Delta_{\omega_X} \log \text{tr}_{\omega_X} (f^*\omega_Y) &= \frac{\Delta_{\omega_X} \text{tr}_{\omega_X} (f^*\omega_Y)}{\text{tr}_{\omega_X} (f^*\omega_Y)} - \frac{|\nabla \text{tr}_{\omega_X} (f^*\omega_Y)|^2}{\text{tr}_{\omega_X} (f^*\omega_Y))^2} \\
&= \frac{1}{\text{tr}_{\omega_X} (f^*\omega_Y)} \left( \langle \text{Ric}(\omega_X), f^*\omega_Y \rangle_{\omega_X} - g^{k\ell} g^{i\ell} \partial_i \partial_k f^\alpha (\partial_j f^\beta) (\partial_k f^\gamma) (\partial_i f^\delta) S_{\alpha\beta\gamma\delta} \\
&+ g^{k\ell} g^{i\ell} \partial_i \partial_k f^\alpha (\partial_j f^\beta) \right) - \frac{|\nabla \text{tr}_{\omega_X} (f^*\omega_Y)|^2}{\text{tr}_{\omega_X} (f^*\omega_Y))^2} \\
&= \frac{1}{\text{tr}_{\omega_X} (f^*\omega_Y)} \left( \langle \text{Ric}(\omega_X), f^*\omega_Y \rangle_{\omega_X} - g^{k\ell} g^{i\ell} \partial_i \partial_k f^\alpha (\partial_j f^\beta) (\partial_k f^\gamma) (\partial_i f^\delta) S_{\alpha\beta\gamma\delta} \\
&+ \frac{1}{(\text{tr}_{\omega_X} (f^*\omega_Y))^2} \left( \text{tr}_{\omega_X} (f^*\omega_Y) g^{k\ell} g^{i\ell} \partial_i \partial_k f^\alpha (\partial_j f^\beta) \right) - |\nabla \text{tr}_{\omega_X} (f^*\omega_Y)|^2_{\omega_X} \\
&\geq \frac{1}{\text{tr}_{\omega_X} (f^*\omega_Y)} \left( \langle \text{Ric}(\omega_X), f^*\omega_Y \rangle_{\omega_X} - g^{k\ell} g^{i\ell} \partial_i \partial_k f^\alpha (\partial_j f^\beta) (\partial_k f^\gamma) (\partial_i f^\delta) S_{\alpha\beta\gamma\delta} \\
&\geq Bu - A.
\end{align*}

In the second line from the bottom, we used the following inequality:

\begin{align*}
|\nabla \text{tr}_{\omega_X} (f^*\omega_Y)|^2_{\omega_X} &= g^{i\ell} (\partial_i g^{k\ell} h^*_{k\ell}) (\partial_j g^{\alpha\beta} h^*_{\alpha\beta}) = g^{i\ell} g^{k\ell} \partial_i \partial_k f^\alpha (\partial_j f^\beta) \\
&= \sum_{i,k,p,\alpha,\beta} (\partial_i \partial_k f^\alpha) (\partial_j f^\beta) (\partial_k f^\gamma) (\partial_i f^\delta) S_{\alpha\beta\gamma\delta} \\
&\leq \sum_{k,p,\alpha,\beta} \left( |\partial_i f^\beta| |\partial_i f^\alpha| \left( \sum_i |\partial_i \partial_k f^\alpha|^2 \right)^{1/2} \left( \sum_j |\partial_j \partial_p f^\beta|^2 \right)^{1/2} \right) \\
&= \left( \sum_{k,\alpha} |\partial_k f^\alpha| \left( \sum_i |(\partial_i \partial_k f^\alpha)|^2 \right)^{1/2} \right)^2 \\
&\leq \left( \sum_{i,\beta} |\partial_i f^\beta| \left( \sum_{i,k,\alpha} |(\partial_i \partial_k f^\alpha)|^2 \right) \right) \\
&= \text{tr}_{\omega_X} (f^*\omega_Y) g^{k\ell} g^{i\ell} \partial_i \partial_k f^\alpha (\partial_j f^\beta).
\end{align*}

Here, we used the Cauchy-Schwarz inequalities.

The next proposition is the so-called “Jeffres’ trick”.

Since \( \varepsilon \), we have
\[
\therefore \text{there exists a constant } C > 0 \text{ on } X \text{ belongs to } X \setminus D \text{ if } 0 < 2\gamma < \alpha \beta.
\]

**Proof.** We assume that \( u_{\delta} \) takes maximum at \( x_0 \in D \). Let \((U, (z^1, \ldots, z^n)) \) be a holomorphic chart centered at \( x_0 \) satisfying \( D \cap U = \{z^1 = 0\} \). By the definition of \( x_0 \), for any \( x = (z, 0, \ldots, 0) \in U \), we have
\[
\frac{|u(x) - u(x_0)|}{\alpha} \geq \frac{\varepsilon |s|^{2\gamma}}{\beta} \geq \frac{\varepsilon |z|^{2\gamma}}{C |z|^{\alpha \beta}}.
\]

Since \( 0 < 2\gamma < \alpha \beta \), the right hand side goes to \( \infty \) as \( z \to 0 \). This contradicts with the definition of \( C^{\alpha \beta} \).

Theorem 1.4 and Theorem 1.5 can be shown in a similar manner. We only prove Theorem 1.4 here.

**Proof of Theorem 1.4 (a).** Since \( f \) can be represented as \((w^1, \ldots, w^n) = ((z^1)^k, f_2(z), \ldots, f_n(z)) \) such that \( D = \{z^1 = 0\} \) and \( E = \{w^1 = 0\} \), the direct computation gives that \( f \) is locally Hölder continuous continuous with respect to \( d_a \) and \( d_{\beta} \) if \( \alpha < k \beta \). Combining with (3.2) and the definition of the cone metrics, \( v := f^*\omega_X^n/\omega_X^n \) is a \( C^{\alpha, \beta} \) function for some \( 0 < \sigma < 1 \). By Proposition 3.3, all maximum points of \( v_{\delta} := v + \varepsilon |s|^{2\gamma} \) belong to \( X \setminus D \) where \( \gamma \) is sufficiently small. Since \( v_{\varepsilon} \) is smooth on \( X \setminus D \), we can apply the maximum principle argument to \( v_{\varepsilon} \).

The direct computation show that
\[
\sqrt{-1} \partial \overline{\partial}\varepsilon |s|^{2\gamma} = \sqrt{-1} \partial \overline{\partial}\varepsilon \log |s|^2 = |s|^{2\gamma} \gamma + 2 \gamma \sqrt{-1} \partial \log |s|^2 + \Delta_{\omega_X} |s|^{2\gamma} \geq -C.
\]

Therefore, there exists a constant \( C > 0 \) (which is independent of \( \varepsilon \)) satisfying
\[
\Delta_{\omega_X} |s|^{2\gamma} \geq -C.
\]

Let \( x_0 \in X \setminus D \) be a maximum point of \( v_{\varepsilon} \). At this point, by Proposition 3.1 (a), we have
\[
0 \geq \Delta_{\omega_X} v_{\varepsilon} = \Delta_{\omega_X} v + \varepsilon \Delta_{\omega_X} |s|^{2\gamma} \geq v(nBu^{1/n} - A) - \varepsilon C.
\]

Simple calculus show that the function \( t \mapsto t^n(nBu^{1/n} - A) - \varepsilon C \) takes non-positive values exactly on some bounded interval \([0, T_\varepsilon]\) and \( T_\varepsilon \to A/(nB) \) as \( \varepsilon \to 0 \). It follows that
\[
v_{\varepsilon}(x_0) = v(x_0) + \varepsilon |s|^{2\gamma}(x_0) \leq T_\varepsilon^n + \varepsilon \sup_x |s|^{2\gamma}.
\]

Since the right hand side does not depend on \( x_0 \) and \( x_0 \) is any maximum point of \( v_{\varepsilon} \), this inequality holds on whole \( X \). Therefore, we have the following inequality
\[
v = v_{\varepsilon} - \varepsilon |s|^{2\gamma} \leq v_{\varepsilon} \leq T_\varepsilon^n + \varepsilon \sup_x |s|^{2\gamma}
\]
on \( X \). By taking \( \varepsilon \to 0 \), we obtain \( v \leq (A/(nB))^n \). \( \square \)
Proof of Theorem 1.4 (b). By definition of the cone metric, we can easily see that for any \(\varepsilon > 0\),

\[
v_{\varepsilon} := |s|_{h}^{2(\ell + \varepsilon)} v = |s|_{h}^{2(\ell + \varepsilon)} f_{*}^{\omega_{Y}^{n}} \frac{\omega_{X}^{n}}{\omega_{X}^{n}}
\]
tends to 0 as \(x\) approaches to \(D\), where \(\ell := \alpha - k\beta > 0\). Then, combining the Laplacian estimate in Proposition 3.1 (a), we have

\[
\Delta_{\omega_{X}} \log v_{\varepsilon} = -(\ell + \varepsilon) \text{tr}_{\omega_{X}} (\sqrt{-1} R_{h}) + \Delta_{\omega_{X}} \log v \\
\geq -(\ell + \varepsilon) C - A + nBv^{1/n},
\]

\[
\Delta_{\omega_{X}} v_{\varepsilon} \geq v_{\varepsilon} (- (\ell + \varepsilon) C - A + nBv^{1/n}).
\]

If \(x_{0} \in X\) is a maximum of \(v_{\varepsilon}\), we can assume that \(x_{0} \in X \setminus D\). At this point, by applying the maximum principle, we have

\[
v(x_{0}) \leq \left( \frac{A + (\ell + \varepsilon) C}{nB} \right)^{n}.
\]

Therefore, we get

\[
v_{\varepsilon}(x_{0}) \leq |s|_{h}^{\ell + \varepsilon}(x_{0}) \left( \frac{A + (\ell + \varepsilon) C}{nB} \right)^{n} \leq \left( \frac{A + (\ell + \varepsilon) C}{nB} \right)^{n}.
\]

Since the right hand side does not depend on \(x_{0}\), this inequality holds on \(X\). Taking \(\varepsilon \to 0\), we obtain

\[
|s|_{h}^{2f_{*}^{\omega_{Y}^{n}} \omega_{X}^{n}} \leq \left( \frac{A + \ell C}{nB} \right)^{n}.
\]

\[\square\]

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