ON LINEAR CHAOS IN THE SPACE OF CONVERGENT SEQUENCES

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Abstract. We show that linear chaos in the space $c(\mathbb{N})$ of convergent sequences cannot be arrived at by merely extending the weighted backward shifts in the space $c_0(\mathbb{N})$ of vanishing sequences.

Applying a newly found sufficient condition for linear chaos, we furnish concise proofs of the chaoticity of the foregoing operators along with their powers and also itemize their spectral structure.

We further construct bounded and unbounded linear chaotic operators in $c(\mathbb{N})$ as conjugate to the chaotic backward shifts in $c_0(\mathbb{Z}_+)$ via a homeomorphic isomorphism between the two spaces.

It turns out that an eerie type of chaos can lurk just behind a facade of order - and yet, deep inside the chaos lurks an even eerier type of order.

Douglas R. Hofstadter

1. Introduction

We show that linear chaos in the space $c(\mathbb{N})$ ($\mathbb{N} := \{1, 2, 3, \ldots\}$ is the set of natural numbers) of convergent sequences cannot be arrived at by merely extending the weighted backward shifts in the space $c_0(\mathbb{N})$ of vanishing sequences, bounded

$$c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w x := w(x_{k+1})_{k \in \mathbb{N}} \in c_0 \quad (|w| > 1),$$

introduced in [16] (see also [7]), as well as unbounded

$$A_w x := (w^k x_{k+1})_{k \in \mathbb{N}} \quad (|w| > 1)$$

with maximal domain

$$D(A_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in c_0 \mid (w^k x_{k+1})_{k \in \mathbb{N}} \in c_0 \right\},$$

introduced in [13].

Applying the newly found Sufficient Condition for Linear Chaos (Theorem 2.2) [10, Theorem 3.2], we furnish concise proofs of the chaoticity of the aforementioned weighted backward shifts along with their powers and also itemize their spectral structure.

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We further construct bounded and unbounded linear chaotic operators in $c(\mathbb{N})$ as conjugate to the chaotic backward shifts in $c_0(\mathbb{Z}_+) \ (\mathbb{Z}_+ := \{0,1,2,\ldots\}$ is the set of nonnegative integers) via a homeomorphic isomorphism between the two spaces.

As follows from the inclusions

$$c_0(\mathbb{N}) \subset c(\mathbb{N}) \subset l_\infty(\mathbb{N}),$$

the space $c(\mathbb{N})$ lives between the space $c_0(\mathbb{N})$, where linear chaos is known, and the space $l_\infty(\mathbb{N})$ of bounded sequences, where linear chaos does not exist.

**Remark 1.1.** Henceforth, we use the notations $c_0(\mathbb{N})$, $c(\mathbb{N})$ for the spaces of vanishing and convergent sequences over $\mathbb{N}$, respectively, and the notations $c_0(\mathbb{Z}_+)$, $c(\mathbb{Z}_+)$ for their counterparts over $\mathbb{Z}_+$. We also use the shorter notations $c_0$ and $c$ whenever the indexing set is implied contextually.

## 2. Preliminaries

The subsequent preliminaries are essential for our discourse.

### 2.1. Spaces $c_0$ and $c$

The spaces

$$c_0 := \left\{ x := (x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \mid \lim_{k \to \infty} x_k = 0 \right\}$$

of vanishing sequences and

$$c := \left\{ x := (x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \mid \exists \lim_{k \to \infty} x_k \in \mathbb{F} \right\}$$

of convergent sequences are infinite-dimensional separable Banach spaces relative to $\infty$-norm

$$x := (x_k)_{k \in \mathbb{N}} \mapsto \|x\|_\infty := \sup_{k \in \mathbb{N}} |x_k|,$$

the former being a closed hyperplane, which is a nowhere dense subspace, of the latter (see, e.g., [12, 14]).

The limit functional

$$(2.1) \quad c \ni x := (x_n)_{n \in \mathbb{N}} \mapsto l(x) := \lim_{n \to \infty} x_n \in \mathbb{F},$$

is a bounded linear functional on $c$ with $\ker l = c_0$ (see, e.g., [12, 14]).

Relative to the standard Schauder basis $\{e_n := (\delta_{nk})_{k \in \mathbb{N}}\}_{n \in \mathbb{N}}$ for $c_0$, where $\delta_{nk}$ is the Kronecker delta, each $x := (x_k)_{k \in \mathbb{N}} \in c_0$ allows the Schauder expansion

$$x = \sum_{k=1}^{\infty} c_k(x)e_k$$

with the coordinates $c_k(x) = x_k$, $k \in \mathbb{N}$.

Relative to the standard Schauder basis $\{e_n\}_{n \in \mathbb{Z}_+}$ for $c$, where

$$e_0 := (1,1,1,\ldots) \quad \text{and} \quad e_n := (\delta_{nk})_{k \in \mathbb{N}}, \ n \in \mathbb{N},$$
each \( x := (x_k)_{k \in \mathbb{N}} \in c \) has the Schauder expansion

\[
x = \sum_{k=0}^{\infty} c_k(x)e_k
\]

with the coordinates

\[
c_0(x) = l(x) \quad \text{and} \quad c_k(x) = x_k - l(x), \ k \in \mathbb{N}.
\]

See, e.g., [12, 14, 15].

2.2. Spectrum.

The spectrum \( \sigma(A) \) of a closed linear operator \( A \) in a complex Banach space \( X \) is the union of the following pairwise disjoint sets:

\[
\sigma_p(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is not injective, i.e., } \lambda \text{ is an eigenvalue of } A \},
\]

\[
\sigma_c(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is injective, not surjective, and } \overline{R(A - \lambda I)} = X \},
\]

\[
\sigma_r(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is injective and } \overline{R(A - \lambda I)} \neq X \}
\]

\((\overline{R(\cdot)})\) is the range of an operator and \( \tau \) is the closure of a set), called the point, continuous and residual spectrum of \( A \), respectively (see, e.g., [5, 12]).

2.3. Hypercyclicity and Linear Chaos.

**Definition 2.1** (Hypercyclic and Chaotic Linear Operators).

For a (bounded or unbounded) linear operator \( A \) in a (real or complex) Banach space \( X \), a nonzero vector

\[
x \in C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n)
\]

\((D(\cdot))\) is the domain of an operator, \( A^0 := I \), \( I \) is the identity operator on \( X \) is called hypercyclic if its orbit under \( A \)

\[
\text{orb}(x, A) := \{ A^n x \}_{n \in \mathbb{Z}^+}
\]

is dense in \( X \), i.e.,

\[
\overline{\text{orb}(x, A)} = X.
\]

Linear operators possessing hypercyclic vectors are said to be hypercyclic.

If there exist an \( N \in \mathbb{N} \) and a vector

\[
x \in D(A^N) \quad \text{with} \quad A^N x = x,
\]

such a vector is called a periodic point for the operator \( A \) of period \( N \). If \( f \neq 0 \), we say that \( N \) is a period for \( A \). Hypercyclic linear operators with a dense in \( X \) set \( \text{Per}(A) \) of periodic points, i.e.,

\[
\overline{\text{Per}(A)} = X,
\]

are said to be chaotic.
Examples 2.1.

1. On the infinite-dimensional separable Banach space $X := c_0$ or $X := \ell_p$ ($1 \leq p < \infty$), the classical Rolewicz weighted backward shifts

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w x := w (x_{k+1})_{k \in \mathbb{N}} \in X,$$

where $w \in \mathbb{F}$ ($\mathbb{F} := \mathbb{R}$ or $\mathbb{F} := \mathbb{C}$) with $|w| > 1$ are chaotic $[7, 16]$.

2. On the sequence space $X := \{(x_n)_{n \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \sum_{k=1}^{\infty} \left| \frac{x_k}{k} - \frac{x_{k+1}}{k+1} \right| < \infty \text{ and } \lim_{k \to \infty} \frac{x_k}{k} = 0 \}$, which is an infinite-dimensional separable Banach space relative to the norm

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto \|x\| := \sum_{k=1}^{\infty} \left| \frac{x_k}{k} - \frac{x_{k+1}}{k+1} \right|,$$

the backward shift

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Ax := (x_{k+1})_{k \in \mathbb{N}} \in X$$

is hypercyclic but not densely periodic, and hence, not chaotic $[8$, Exercise 4.1.3$]$.

3. On an infinite-dimensional separable Banach space $(X, \| \cdot \|)$, the identity operator $I$ is densely periodic but not hypercyclic, and hence, not chaotic.

Remarks 2.1.

- In the prior definition of hypercyclicity, the underlying space is necessarily infinite-dimensional and separable (see, e.g., $[8]$).

- For a hypercyclic linear operator $A$, the set $HC(A)$ of its hypercyclic vectors is necessarily dense in $X$, and hence, the more so, is the subspace $C^\infty(A) \supseteq HC(A)$.

- Observe that

$$\text{Per}(A) = \bigcup_{N=1}^{\infty} \text{Per}_N(A),$$

where

$$\text{Per}_N(A) = \ker(A^N - I), \quad N \in \mathbb{N}$$

is the subspace of $N$-periodic points of $A$.

- As immediately follows from the inclusions

$$HC(A^n) \subseteq HC(A), \quad \text{Per}(A^n) \subseteq \text{Per}(A), \quad n \in \mathbb{N},$$

if, for a linear operator $A$ in an infinite-dimensional separable Banach space $X$ and some $n \geq 2$, the operator $A^n$ is hypercyclic or chaotic, then $A$ is also hypercyclic or chaotic, respectively.

Prior to $[2, 3]$, the notions of linear hypercyclicity and chaos had been studied exclusively for continuous linear operators on Fréchet spaces, in particular for bounded linear operators on Banach spaces (for a comprehensive survey, see $[1, 8]$).
The following extension of Kitai’s criterion for bounded linear operators (see [6, 9]) is a useful shortcut for establishing hypercyclicity for (bounded or unbounded) linear operators without explicitly furnishing a hypercyclic vector as in [16].

**Theorem 2.1** *(Sufficient Condition for Hypercyclicity [2, Theorem 2.1]).*

*Let $X$ be a (real or complex) infinite-dimensional separable Banach space and $A$ be a densely defined linear operator in $X$ such that each power $A^n$ ($n \in \mathbb{N}$) is a closed operator. If there exists a set $Y \subseteq C^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n)$ dense in $X$ and a mapping $B : Y \to Y$ such that

1. $\forall x \in Y : ABx = x$ and
2. $\forall x \in Y : A^n x, B^n x \to 0$, $n \to \infty$,

then the operator $A$ is hypercyclic.*

The subsequent newly established sufficient condition for linear chaos [10], obtained via strengthening one of the hypotheses of the prior sufficient condition for hypercyclicity, serves as a shortcut for establishing chaoticity for (bounded or unbounded) linear operators without explicitly furnishing both a hypercyclic vector and a dense set of periodic points and is fundamental for our discourse.

**Theorem 2.2** *(Sufficient Condition for Linear Chaos [10, Theorem 3.2]).*

*Let $(X, \| \cdot \|)$ be a (real or complex) infinite-dimensional separable Banach space and $A$ be a densely defined linear operator in $X$ such that each power $A^n$ ($n \in \mathbb{N}$) is a closed operator. If there exists a set $Y \subseteq C^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n)$ dense in $X$ and a mapping $B : Y \to Y$ such that

1. $\forall x \in Y : ABx = x$ and
2. $\forall x \in Y \exists \alpha = \alpha(x) \in (0, 1), \exists c = c(x, \alpha) > 0 \forall n \in \mathbb{N} : \max (\|A^n x\|, \|B^n x\|) \leq c\alpha^n$,

then the operator $A$ is chaotic.*

For applications, see [11].

We also need the following statements.

**Corollary 2.1** *(Chaoticity of Powers [10, Corollary 4.3]).*

*For a chaotic linear operator $A$ in a (real or complex) infinite-dimensional separable Banach space subject to the Sufficient Condition for Linear Chaos (Theorem 2.2), each power $A^n$ ($n \in \mathbb{N}$) is chaotic.*
Theorem 2.3 (Bourdon [8, Theorem 2.54]).
For a bounded linear hypercyclic operator $A$ on an infinite-dimensional separable Banach space $X$ and a nonzero polynomial $p(\lambda) := \sum_{k=0}^{n} c_k \lambda^k$ ($n \in \mathbb{Z}_+, c_k \in \mathbb{F}$, $k = 0, \ldots, n$), the range $R(p(A))$ of the operator $p(A) := \sum_{k=0}^{n} c_k A^k$ is dense in $X$, i.e.,

$$R(p(A)) = X.$$ 

Remark 2.1. Consistently with more general necessary conditions for hypercyclicity, found in [10], the latter implies that, for a bounded linear hypercyclic operator $A$ on an infinite-dimensional separable Banach space $X$ and an arbitrary $\lambda \in \mathbb{F}$,

the range $R(A - \lambda I)$ of the operator $A - \lambda I$ is dense in $X$, i.e.,

$$R(A - \lambda I) = X.$$ 

3. Linear Chaos in $c_0$

Theorem 3.1 (Bounded Linear Chaos on $c_0$).
For an arbitrary $w \in \mathbb{F}$ with $|w| > 1$, the bounded linear weighted backward shift operator

$$c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w x := w (x_{k+1})_{k \in \mathbb{N}} \in c_0$$
on the space $c_0$ is chaotic as well as its every power $A_w^n$ ($n \in \mathbb{N}$) and, provided the underlying space is complex (i.e., $\mathbb{F} = \mathbb{C}$),

$$\sigma(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| \}$$

with

$$\sigma_p(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| < |w| \} \quad \text{and} \quad \sigma_c(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| = |w| \}.$$ 

Proof. Here, we provide a concise proof based on the Sufficient Condition for Linear Chaos (Theorem 2.2) (cf. the original proofs for hypercyclicity and chaoticity [7, 16]).

Let $w \in \mathbb{F}$ with $|w| > 1$ be arbitrary and, for the simplicity of notation, let $A := A_w$.

Consider the subspace

$$Y := c_0 := \{ x := (x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \mid \exists N \in \mathbb{N} \forall k \geq N : x_k = 0 \}$$
dense in $c_0$ (see, e.g., [12, 14]) and the mapping $B : Y \to Y$, which is the restriction to $Y$ of the bounded linear operator on $c_0$

$$c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Bx := w^{-1} (x_{k-1})_{k \in \mathbb{N}} \in c_0 \quad (x_0 := 0),$$

(the right inverse of $A$) for which

$$\|B\| = |w|^{-1} < 1$$

(3.1)

(here and henceforth, whenever appropriate, $\| \cdot \|$ stands for the operator norm) (see, e.g., [12]) and

$$ABx = x, \quad x \in Y.$$ 

(3.2)

Let us show that $\forall x \in Y \exists \alpha = \alpha(f) \in (0, 1), \exists c = c(f, \alpha) > 0 \forall n \in \mathbb{N} :$

$$\max (\|A^n x\|, \|B^n x\|) \leq c \alpha^n,$$
Let $x := (x_k)_{k \in \mathbb{N}} \in Y$ be arbitrary. Then

$$\exists N \in \mathbb{N} \forall k \geq N : x_k = 0,$$

and hence,

$$\forall n \geq N : A^n x = 0,$$

which implies that

$$\forall \alpha \in (0, 1), \exists c = c(x, \alpha) > 0, \forall n \in \mathbb{N} : \|A^n x\|_{\infty} \leq c \alpha^n.$$  

By the submultiplicativity of the operator norm, in view of (3.1), we also have:

$$\|B^n x\| \leq \|B^n\| \|x\| \leq \|B\| \|x\| = |w|^{-n} \|x\|.$$  

By the Sufficient Condition for Linear Chaos (Theorem 2.2) and the Chaoticity of Powers Corollary (Corollary 2.1), we conclude that the operator $A$ is chaotic as well as every power $A^n \ (n \in \mathbb{N})$.

Provided the underlying space is complex, the spectral part of the statement immediately follows from the fact that $A = wL$,

where

$$c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Lx := (x_{k+1})_{k \in \mathbb{N}} \in c_0$$

is the backward shift operator on $c_0$, for which

$$\sigma(L) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$$

with

$$\sigma_p(L) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \quad \text{and} \quad \sigma_c(L) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$$

(see, e.g., [5, 12]).

Lemma 3.1.

Let $w \in \mathbb{F}$ and $|w| > 1$. Then, for the weighted backward shift operator

$$A_w x := (w^k x_{k+1})_{k \in \mathbb{N}}$$

in the space $c_0$ with maximal domain

$$D(A_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in c_0 \mid (w^k x_{k+1})_{k \in \mathbb{N}} \in c_0 \right\},$$

each power

$$A_w^n x = \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n}$$

$(n \in \mathbb{N})$ with domain

$$D(A_w^n) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in c_0 \mid \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n} \in c_0 \right\}$$

is a densely defined unbounded closed linear operator and the subspace

$$C^\infty(A) := \bigcap_{n=1}^{\infty} D(A_w^n)$$

of infinite differentiable relative to $A$ vectors is dense in $c_0$. 
Proof. Let \( w \in F \) with \( |w| > 1 \) be arbitrary and, for the simplicity of notation, let \( A := A_w \).

Since
\[
A^2 x = \left( w^k w^{k+1} x_{k+2} \right)_{k \in \mathbb{N}}
\]
with domain
\[
D(A^2) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in D(A) \mid Ax \in D(A) \right\}
\]
\[
= \left\{ x := (x_k)_{k \in \mathbb{N}} \in c_0 \mid \left( w^k w^{k+1} x_{k+2} \right)_{k \in \mathbb{N}} \in c_0 \right\}
\]
and
\[
A^3 x = \left( w^k w^{k+1} w^{k+2} x_{k+3} \right)_{k \in \mathbb{N}}
\]
with domain
\[
D(A^3) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in D(A^2) \mid A^2 x \in D(A) \right\}
\]
\[
= \left\{ x := (x_k)_{k \in \mathbb{N}} \in c_0 \mid \left( w^k w^{k+1} w^{k+2} x_{k+3} \right)_{k \in \mathbb{N}} \in c_0 \right\}
\]
we infer inductively that, for each \( n \in \mathbb{N} \)
\[
A^n x = \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n}
\]
with domain
\[
D(A^n) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in c_0 \mid \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n} \in c_0 \right\}
\]

We have:
\[
D(A^{n+1}) \subseteq D(A^n), \quad n \in \mathbb{N}.
\]

Since the subspace \( c_{00} \) is dense in \( c_0 \) and
\[
c_{00} \subseteq D(A^n), \quad n \in \mathbb{N},
\]
then each power \( A^n \) \((n \in \mathbb{N})\) is densely defined and furthermore
\[
C^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n)
\]
is also dense in \( c_0 \).

Let \( n \in \mathbb{N} \) and \( e_m := (\delta_{nm})_{k \in \mathbb{N}}, \ m \in \mathbb{N} \), with \( \|e_m\|_\infty = 1, \ m \in \mathbb{N} \). Then, in view of \( |w| > 1 \),
\[
\forall m \in \mathbb{N} : \|A^n e_{n+m}\| = \left\| \left( \prod_{j=k}^{k+n-1} w^j \right) \delta_{(n+m)(k+n)} \right\|_{k \in \mathbb{N}} = \prod_{j=1}^{m+n-1} |w|^j = |w|^\left(\frac{(m+n)(m+n-1)}{2}\right) \to \infty, \ m \to \infty,
\]
which implies that the linear operator \( A^n \) is unbounded.
Let $n \in \mathbb{N}$ and a sequence \( x^{(m)} := (x^{(m)}_k)_{k \in \mathbb{N}} \) in $D(A^n)$ be such that
\[
x^{(m)} \to x := (x_k)_{m \in \mathbb{N}} \in c_0, \ m \to \infty,
\]
and
\[
A^n x^{(m)} = \left( \prod_{j=k}^{k+n-1} w^j \right) x^{(m)}_{k+n} \to y := (y_k)_{k \in \mathbb{N}} \in c_0, \ m \to \infty.
\]
Then, for each $k \in \mathbb{N}$ (see, e.g., [12, 14, 15]),
\[
x^{(m)}_k \to x_k, \ m \to \infty,
\]
and
\[
\left( \prod_{j=k}^{k+n-1} w^j \right) x^{(m)}_{k+n} \to y_k, \ m \to \infty.
\]
Whence we infer that, for each $k \in \mathbb{N},$
\[
\left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n} = y_k,
\]
which means that
\[
\left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n} \in c_0.
\]
Therefore, $x \in D(A^n)$ and $y = A^n x,$ which, by the Sequential Characterization of Closed Linear Operators (see, e.g., [12, 14]), implies the operator $A^n$ is closed. □

Theorem 3.2 (Unbounded Linear Chaos in $c_0$).
For an arbitrary $w \in \mathbb{F}$ with $|w| > 1,$ the unbounded linear weighted backward shift operator
\[
A_w x := (w^k x_{k+1})_{k \in \mathbb{N}}
\]
in the space $c_0$ with maximal domain
\[
D(A_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in c_0 \left| (w^k x_{k+1})_{k \in \mathbb{N}} \in c_0 \right. \right\}
\]
is chaotic as well as every power $A^n_w$ ($n \in \mathbb{N}$).
Furthermore, each $\lambda \in \mathbb{F}$ is an eigenvalue for $A_w$ of geometric multiplicity 1, i.e.,
\[
\dim \ker(A_w - \lambda I) = 1.
\]
In particular, provided the underlying space is complex,
\[
\sigma_p(A_w) = \mathbb{C}.
\]

Proof. Here, we also provide a concise proof based on the Sufficient Condition for Linear Chaos (Theorem 2.2) (cf. the original proof [13]).
Let $w \in \mathbb{F}$ with $|w| > 1$ be arbitrary and, for the simplicity of notation, let $A := A_w.$
Consider the subspace
\[
Y := c_{00}
\]
denote in \( c_0 \) and the mapping \( B : Y \to Y \), which is the restriction to \( Y \) of the bounded linear operator on \( c_0 \)
\[
c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Bx := \left( w^{-(k-1)}x_{k-1} \right)_{k \in \mathbb{N}} \in c_0 \quad (x_0 := 0),
\]
(the right inverse of \( A \) for which
\[
(3.3) \quad ABx = x, \ x \in Y.
\]
With
\[
c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto B^2x := \left( w^{-(k-1)}w^{-(k-2)}x_{k-2} \right)_{k \in \mathbb{N}} \quad (x_{k-2} := 0, \ k = 1, 2)
\]
and
\[
c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto B^3x := \left( w^{-(k-1)}w^{-(k-2)}w^{-(k-3)}x_{k-3} \right)_{k \in \mathbb{N}} \quad (x_{k-3} := 0, \ k = 1, 2, 3),
\]
we infer inductively that, for any \( n \in \mathbb{N} \),
\[
c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto B^n x = \left( \prod_{j=1}^{n} w^{-(k-j)} \right)_{k \in \mathbb{N}} \quad (x_k := 0, \ k = 1, \ldots, n),
\]
or equivalently, in view of
\[
\prod_{j=1}^{n} w^{-(k-j)} = w^{-\sum_{j=1}^{n} (k-j)} = w^{-nk + \frac{n(n+1)}{2}},
\]
we have:
\[
c_0 \ni (x_k)_{k \in \mathbb{N}} \mapsto B^n x = \left( w^{-nk + \frac{n(n+1)}{2}} x_{k-n} \right)_{k \in \mathbb{N}} \quad (x_k := 0, \ k = 1, \ldots, n)
\]
with
\[
\|B^n\| = |w|^{-\frac{n(n+1)}{2}}.
\]
Indeed, for any \( x := (x_k)_{k \in \mathbb{N}} \in c_0 \), in view of \( |w| > 1 \),
\[
\|B^n x\|_\infty = \sup_{k \in \mathbb{N}} \left| w^{-nk + \frac{n(n+1)}{2}} x_{k-n} \right| = \sup_{k \geq n+1} \left| w^{-nk + \frac{n(n+1)}{2}} x_{k-n} \right|
\]
\[
\leq \sup_{k \geq n+1} |w|^{-nk + \frac{n(n+1)}{2}} \sup_{k \geq n+1} |x_{k-n}| = |w|^{-n(n+1)+\frac{n(n+1)}{2}} \|x\|_\infty
\]
and hence,
\[
\|B^n\| \leq |w|^{-\frac{n(n+1)}{2}}.
\]
Further, since, for \( e_1 := (\delta_{1k})_{k \in \mathbb{N}} \) with \( \|e_1\|_\infty = 1 \),
\[
\|B^n e_1\|_\infty = \sup_{k \geq n+1} \left| w^{-nk + \frac{n(n+1)}{2}} \delta_{1(k-n)} \right| = |w|^{-n(n+1)+\frac{n(n+1)}{2}} = |w|^{-\frac{n(n+1)}{2}},
\]
we infer that
\[
\|B^n\| = |w|^{-\frac{n(n+1)}{2}}.
\]
Thus,
\[
(3.4) \quad \lim_{n \to \infty} \|B^n\|^{1/n} = \lim_{n \to \infty} |w|^{-\frac{n+1}{2}} = 0,
\]
i.e., the operator $B$ is quasinilpotent (cf. [13]), which implies that
\[ \forall x \in Y, \forall \alpha \in (0,1) \exists c = c(x, \alpha) > 0 \forall n \in \mathbb{N}: \|B^n x\| \leq c \alpha^n. \]

Let $x := (x_k)_{k \in \mathbb{N}} \in Y$ be arbitrary. Then
\[ \exists N \in \mathbb{N} \forall k \geq N: x_k = 0, \]
and hence,
\[ \forall n \geq N: A^n x = 0, \]
which implies that
\[ \forall \alpha \in (0,1) \exists c = c(x, \alpha) > 0 \forall n \in \mathbb{N}: \|A^n x\| \leq c \alpha^n. \]

By Lemma 3.1, the Sufficient Condition for Linear Chaos (Theorem 2.2), and the Chaoticity of Powers Corollary (Corollary 2.1), we conclude that the operator $A$ is chaotic as well as every power $A^n$ ($n \in \mathbb{N}$).

Here, we reproduce the proof of the spectral part of the statement given in [13].

For arbitrary $\lambda \in \mathbb{F}$ ($\mathbb{F} := \mathbb{R}$ or $\mathbb{F} := \mathbb{C}$) and $x := (x_k)_{k \in D(A)} \in \mathbb{D}$, the equation (3.5)
\[ Ax = \lambda x \]
is equivalent to
\[ (w^k x_{k+1})_{k \in \mathbb{N}} = \lambda (x_k)_{k \in \mathbb{N}}, \]
i.e.,
\[ w^k x_{k+1} = \lambda x_k, \quad k \in \mathbb{N} \]

Whence, we recursively infer that
\[ x_k = \left[ \prod_{j=1}^{k-1} \frac{\lambda}{w^{k-j}} \right] x_1 = \frac{\lambda^{k-1}}{w^{\sum_{j=1}^{k-1} (k-j)}} x_1 = \frac{\lambda^{k-1}}{w^{\frac{k(k-1)}{2}}} x_1 = \left( \frac{\lambda}{w^2} \right)^{k-1} x_1, \quad k \in \mathbb{N}, \]
where for $\lambda = 0$, $0^0 := 1$.

Considering that $|w| > 1$, for all sufficiently large $k \in \mathbb{N}$, we have:
\[ \left| \frac{\lambda}{w^2} \right|^{k-1} \leq \left( \frac{1}{2} \right)^{k-1}, \]
which implies that
\[ y := (y_k)_{k \in \mathbb{N}} := \left( \left( \frac{\lambda}{w^2} \right)^{k-1} \right)_{k \in \mathbb{N}} \in \mathbb{C}_0. \]

Further, since
\[ w^k y_{k+1} = w^k \frac{\lambda^k}{w^{\sum_{j=1}^{k} (k+1-j)}} = \frac{\lambda^k}{w^{\frac{k(k+1)}{2}}} x_1 = \left( \frac{\lambda}{w^2} \right)^k, \quad k \in \mathbb{N}, \]
we similarly conclude that
\[ (w^k y_{k+1})_{k \in \mathbb{N}} \in \mathbb{C}_0, \]
and hence,
\[ y \in D(A) \setminus \{0\}. \]
Thus, we have shown that, for any \( \lambda \in \mathbb{F} \), all solutions of equation (3.5) are of the form
\[
x := (x_k)n = cy \in D(A),
\]
where \( c \in \mathbb{F} \) is arbitrary. They form the one-dimensional subspace of \( c_0 \) spanned by the sequence \( y \), which completes the proof. \( \square \)

**Remark 3.1.** Theorem 3.1, Lemma 3.1, and Theorem 3.2 naturally extend from \( c_0(\mathbb{N}) \) to \( c_0(\mathbb{Z}_+) \) for the bounded weighted backward shifts:
\[
c_0(\mathbb{Z}_+) \ni x := (x_k)_{k \in \mathbb{Z}_+} \mapsto A_w x := w (x_{k+1})_{k \in \mathbb{Z}_+} \in c_0(\mathbb{Z}_+) \quad (|w| > 1)
\]
and the unbounded weighted backward shifts:
\[
A_w x := (w^k x_{k+1})_{k \in \mathbb{Z}_+} \quad (|w| > 1)
\]
with maximal domain
\[
D(A_w) := \left\{ x := (x_k)_{k \in \mathbb{Z}_+} \in c_0(\mathbb{Z}_+) \mid (w^k x_{k+1})_{k \in \mathbb{Z}_+} \in c_0(\mathbb{Z}_+) \right\}
\]
and the powers
\[
A_w^n x = \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n} \quad , \quad n \in \mathbb{N},
\]
defined on
\[
D(A_w^n) := \left\{ x := (x_k)_{k \in \mathbb{Z}_+} \in c_0(\mathbb{Z}_+) \mid \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n} \in c_0(\mathbb{Z}_+) \right\}
\]
(see the proof of Lemma 3.1).

In the former case, the bounded right inverse of \( A_w \) is
\[
c_0(\mathbb{Z}_+) \ni x := (x_k)_{k \in \mathbb{Z}_+} \mapsto B_w x := w^{-1} (x_{k-1})_{k \in \mathbb{Z}_-} \in c_0(\mathbb{Z}_+) \quad (x_{-1} := 0),
\]
for which \( \|B\| = |w|^{-1} < 1 \), and, in the latter case, the bonded right inverse of \( A_w \) is
\[
c_0(\mathbb{Z}_+) \ni x := (x_k)_{k \in \mathbb{Z}_+} \mapsto B_w x := \left( w^{-(k-1)} x_{k-1} \right)_{k \in \mathbb{Z}_+} \in c_0(\mathbb{Z}_+) \quad (x_0 := 0),
\]
with
\[
c_0(\mathbb{Z}_+) \ni x := (x_k)_{k \in \mathbb{Z}_+} \mapsto B_w^n x = \left( \prod_{j=1}^{n} w^{-(k-j)} \right) x_{k-n} = \left( w^{-nk+\frac{n(n+1)}{2}} x_{k-n} \right)_{k \in \mathbb{Z}_+}, \quad n \in \mathbb{N},
\]
\((x_{k-n} := 0, k = 0, 1, \ldots, n - 1)\), for which
\[
\|B_w^n\| = |w|^{-n^2 + \frac{n(n+1)}{2}} = |w|^{-\frac{n(n-1)}{2}}
\]
and hence,
\[
\lim_{n \to \infty} \|B_w^n\|^{1/n} = \lim_{n \to \infty} |w|^{-\frac{n-1}{2}} = 0,
\]
i.e., \( B_w \) is quasinilpotent (cf. the proof of Theorem 3.2).
4. Weighted Backward Shifts in \( c \)

The answer to the natural question of whether one can obtain linear chaos in the space \( c(\mathbb{N}) \) of convergent sequences by merely extending the foregoing chaotic weighted backward shifts from the space \( c_0(\mathbb{N}) \) is given in the negative by the subsequent statements.

**Proposition 4.1** (Bounded Weighted Backward Shifts on \( c \)).

*For an arbitrary \( w \in \mathbb{F} \) with \( |w| > 1 \), the bounded linear weighted backward shift operator*

\[
   c \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w x := w(x_{k+1})_{k \in \mathbb{N}} \in c
\]

*on the space \( c \) is not hypercyclic.*

**Proof.** Let \( w \in \mathbb{F} \) with \( |w| > 1 \) be arbitrary and, for the simplicity of notation, let \( A := A_w \).

It is obvious that the operator \( A \) is well defined on \( c \) and also is linear and bounded with

\[
   \|A\| = |w|.
\]

Since, for any \( x := (x_k)_{k \in \mathbb{N}} \in c \),

\[
   (A - wI)x = w(x_{k+1})_{k \in \mathbb{N}} - w(x_k)_{k \in \mathbb{N}} = w(x_{k+1} - x_k)_{k \in \mathbb{N}},
\]

and

\[
   \lim_{k \to \infty} w(x_{k+1} - x_k) = w \left( \lim_{k \to \infty} x_{k+1} - \lim_{k \to \infty} x_k \right) = w \left( l(x) - l(x) \right) = 0,
\]

we infer that

\[
   \overline{R(A - wI)} \subseteq c_0.
\]

Since \( c_0 \) is a closed proper subspace of \( c \), it is nowhere dense in \( c \) (see, e.g., \([12, 14]\)) and, as follows from the prior inclusion, so is \( \overline{R(A - wI)} \).

Hence,

\[
   \overline{R(A_w - wI)} \neq c,
\]

which, by Bourdon’s Theorem (Theorem 2.3) with \( p(\lambda) := \lambda - w \), \( \lambda \in \mathbb{F} \) (see also \([10, Proposition 4.1]\)), the latter implies that the operator \( A \) is not hypercyclic. \( \square \)

**Proposition 4.2** (Unbounded Weighted Backward Shifts in \( c \)).

*For an arbitrary \( w \in \mathbb{F} \) with \( |w| > 1 \), the unbounded linear weighted backward shift operator*

\[
   A_w x := (w^k x_{k+1})_{k \in \mathbb{N}}
\]

*in the space \( c \) with maximal domain*

\[
   D(A_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in c \mid (w^k x_{k+1})_{k \in \mathbb{N}} \in c \right\}
\]

*is not hypercyclic.*
Proof. Let $w \in F$ with $|w| > 1$ be arbitrary and, for the simplicity of notation, let $A := A_w$.

As follows from the definition, for any $x := (x_k)_{k \in \mathbb{N}} \in D(A)$,

$$y := (y_k := w^k x_{k+1})_{k \in \mathbb{N}} \in c$$

and hence, in view of $|w| > 1$,

$$x_{k+1} = w^{-k} y_k \to 0, \ k \to \infty.$$ 

Therefore, $C^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n) \subseteq D(A) \subseteq c_0$.

Since $c_0$ is a closed proper subspace of $c$, it is nowhere dense in $c$ (see, e.g., [12,14]) and, as follows from the prior inclusion, so is $C^\infty(A)$.

Hence, $C^\infty(A) \neq c$,

which immediately implies that the operator $A$ is not hypercyclic (see Remarks 2.1).

□

5. Linear Chaos in $c$

With the hypercyclicity by extension compromised, here, we construct bounded and unbounded chaotic linear operators in $c(\mathbb{N})$ based on the chaotic backward shifts in $c_0(\mathbb{Z}^+)$ via establishing a homeomorphic isomorphism between the two spaces (i.e., an isomorphism which is also a homeomorphism).

Lemma 5.1 (Homeomorphic Isomorphism).

The mapping

$$c(\mathbb{N}) \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Jx := (y_k)_{k \in \mathbb{Z}^+} \in c_0(\mathbb{Z}^+),$$

assigning to each $x := (x_k)_{k \in \mathbb{N}} \in c(\mathbb{N})$ the sequence $(y_k)_{k \in \mathbb{Z}^+} \in c_0(\mathbb{Z}^+)$ of the coordinates of $x$ relative to the standard Schauder basis $\{e_n\}_{n \in \mathbb{Z}^+}$ for $c(\mathbb{N})$, where

$$e_0 := (1,1,1,\ldots) \quad \text{and} \quad e_n := (\delta_{nk})_{k \in \mathbb{N}}, \ n \in \mathbb{N},$$

i.e.,

$$y_0 := l(x) \quad \text{and} \quad y_k := x_k - l(x), \ k \in \mathbb{N},$$

where $l$ is the limit functional, is a homeomorphic isomorphism from $c(\mathbb{N})$ to $c_0(\mathbb{Z}^+)$. 

Proof. In view of the uniqueness of the Schauder expansion, we infer that the mapping $J$ is linear and further, since, for $x \in c(\mathbb{N})$,

$$Jx = 0 \iff y_k = 0, \ k \in \mathbb{Z}^+ \iff x = \sum_{k=0}^{\infty} y_k e_k = 0 \in c(\mathbb{N}),$$

$J$ is also injective (see, e.g., [12,14,15]).

Further, for any $(y_k)_{k \in \mathbb{Z}^+} \in c_0(\mathbb{Z}^+)$, let

$$x := (y_k + y_0)_{k \in \mathbb{N}}.$$
Since
\[ \lim_{k \to \infty} y_k = 0, \]
we infer that
\[ \lim_{k \to \infty} x_k = \lim_{k \to \infty} (y_k + y_0) = y_0, \]
Thus,
\[ x \in c(\mathbb{N}) \quad \text{and} \quad Jx = y, \]
which implies that the mapping \( J \) is also surjective, and hence \( J : c(\mathbb{N}) \to c_0(\mathbb{Z}_+) \) is an isomorphism between the spaces \( c(\mathbb{N}) \) and \( c_0(\mathbb{Z}_+) \).

Since, for an arbitrary \( x := (x_k)_{k \in \mathbb{N}} \in c(\mathbb{N}), \)
\[ |l(x)| = \left| \lim_{k \to \infty} x_k \right| = \lim_{k \to \infty} |x_k| \leq \sup_{k \in \mathbb{N}} |x_k| =: \|x\|, \]
we also have:
\[ \|Jx\|_\infty := \sup_{k \in \mathbb{Z}_+} |y_k| = \max \left[ |l(x)|, \sup_{k \in \mathbb{N}} |x_k - l(x)| \right] \leq 2\|x\|, \]
Thus, the linear mapping \( J \) is bounded, and hence continuous, which, by the Inverse Mapping Theorem (see, e.g., [12, 14]), implies that so is its inverse \( J^{-1} : c_0(\mathbb{Z}_+) \to c(\mathbb{N}) \):
\[ c_0(\mathbb{Z}_+) \ni x := (y_k)_{k \in \mathbb{Z}_+} \mapsto J^{-1}x := (y_k + y_0)_{k \in \mathbb{N}} \in c(\mathbb{N}). \]
We conclude that the mapping \( J : c(\mathbb{N}) \to c_0(\mathbb{Z}_+) \) is both isomorphic and homeomorphic. \( \square \)

**Theorem 5.1** (Bounded Linear Chaos in \( c \)).
For an arbitrary \( w \in \mathbb{F} \) with \( |w| > 1 \), the bounded linear operator
\[ c \ni x := (x_k)_{k \in \mathbb{N}} \mapsto \tilde{A}_w x := w(x_{k+1} + x_1 - 2l(x))_{k \in \mathbb{N}} \in c \]
on the space \( c \) is chaotic as well every power
\[ c \ni x := (x_k)_{k \in \mathbb{N}} \mapsto \tilde{A}_w^n x = w^n(x_{k+n} + x_n - 2l(x))_{k \in \mathbb{N}}, \ n \in \mathbb{N}, \]
and, provided the underlying space is complex
\[ \sigma \left( \tilde{A}_w \right) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| \} \]
with
\[ \sigma_p \left( \tilde{A}_w \right) = \{ \lambda \in \mathbb{C} \mid |\lambda| < |w| \} \quad \text{and} \quad \sigma_c \left( \tilde{A}_w \right) = \{ \lambda \in \mathbb{C} \mid |\lambda| = |w| \}. \]

**Proof.** Let \( w \in \mathbb{F} \) with \( |w| > 1 \) be arbitrary and, for the simplicity of notation, let \( A := \tilde{A}_w \).
On \( c(\mathbb{N}) \), consider the linear operator \( \hat{A} \) defined as follows:
\[ \hat{A} := J^{-1}AJ, \]
i.e., via the commutative diagram
\[
\begin{array}{ccc}
\cd{0} (\mathbb{Z}_+) & \stackrel{A}{\longrightarrow} & \cd{0} (\mathbb{Z}_+) \\
\uparrow \downarrow \ & \ & \uparrow \downarrow \\
\mathbb{C} (\mathbb{N}) & \stackrel{\hat{A}}{\longrightarrow} & \mathbb{C} (\mathbb{N})
\end{array}
\]

Since, by (5.2),
\[
\hat{A}^n := J^{-1} A^n J, \quad n \in \mathbb{N},
\]
where
\[
c_0 (\mathbb{Z}_+) \ni y := (y_k)_{k \in \mathbb{Z}_+} \mapsto A^n y := w^n (y_{k+n})_{k \in \mathbb{Z}_+} \in c_0 (\mathbb{Z}_+)
\]
(see Remark 3.1), for any \( x := (x_k)_{k \in \mathbb{N}} \in c (\mathbb{N}) \),
\[
A^n J x = w^n (x_{k+n} - l(x))_{k \in \mathbb{Z}_+} =: (y_k)_{k \in \mathbb{Z}_+},
\]
and hence, in view of (5.1), we have:
\[
\hat{A}^n x := J^{-1} A^n J = (y_k + y_0)_{k \in \mathbb{N}} = w^n (x_{k+n} + x_n - 2l(x))_{k \in \mathbb{N}} \in c (\mathbb{N}).
\]

Observe that
\[
\lim_{k \to \infty} w^n (x_{k+n} + x_n - 2l(x)) = w^n (l(x) + x_n - 2l(x)) = w^n (x_n - l(x)), \quad n \in \mathbb{N}.
\]

Since, by Lemma 5.1, \( J : c (\mathbb{N}) \to c_0 (\mathbb{Z}_+) \) is a homeomorphic isomorphism, the operator \( \hat{A}^n (n \in \mathbb{N}) \) inherits its linearity, boundedness, chaoticity, and spectral structure directly from its conjugate \( A^n \) via \( J \). Therefore, the statement follows immediately from Theorem 3.1. \( \square \)

**Theorem 5.2** (Unbounded Linear Chaos in \( c \)).

For an arbitrary \( w \in \mathbb{F} \) with \( |w| > 1 \), the linear operator
\[
\hat{A}_w x := (w^k (x_{k+1} - l(x)) + x_1 - l(x))_{k \in \mathbb{N}}
\]
in the space \( c \) with domain
\[
D (\hat{A}_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in c \left| \left( w^k (x_{k+1} - l(x)) \right)_{k \in \mathbb{N}} \in c_0 \right. \right\}
\]
is unbounded, closed, and chaotic as well every power
\[
\hat{A}_w^n := \left( \prod_{j=k}^{k+n-1} w^j \right) (x_{k+n} - l(x)) + \left( \prod_{j=0}^{n-1} w^j \right) (x_n - l(x))_{k \in \mathbb{N}}, \quad n \in \mathbb{N},
\]
with domain
\[
D (\hat{A}_w^n) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in c \left| \left( \prod_{j=k}^{k+n-1} w^j \right) (x_{k+n} - l(x))_{k \in \mathbb{N}} \in c_0 \right. \right\}
\]
Furthermore, each \( \lambda \in \mathbb{F} \) is an eigenvalue for \( \hat{A}_w \) of geometric multiplicity \( n \), i.e.,
\[
\dim \ker (\hat{A}_w - \lambda I) = 1.
\]
Proof. Let $w \in \mathbb{F}$ with $|w| > 1$ be arbitrary and, for the simplicity of notation, let $A := A_w$.

In $c(\mathbb{N})$, consider the linear operator $\hat{A}$ defined as follows:

\begin{equation}
\hat{A} := J^{-1}AJ,
\end{equation}

i.e., via the commutative diagram

\[
\begin{array}{ccc}
\vspace{1cm}
c_0(\mathbb{Z}_+) & \supseteq D(A) & \xrightarrow{A} c_0(\mathbb{Z}_+) \\
\uparrow J & & \uparrow J \\
c(\mathbb{N}) & \supseteq D(\hat{A}) & \xrightarrow{\hat{A}} c(\mathbb{N})
\end{array}
\]

for which

\[
J \left( D(\hat{A}) \right) := D(A).
\]

Since, by (5.3),

\[
\hat{A}^n := J^{-1}A^nJ, \quad n \in \mathbb{N},
\]

where

\[
A^n y = \left( \left\{ \prod_{j=k}^{k+n-1} w^j \right\} y_{k+n} \right)_{k \in \mathbb{Z}_+}
\]

with domain

\[
D(A^n_w) = \left\{ y := (y_k)_{k \in \mathbb{Z}_+} \in c_0(\mathbb{Z}_+) \left| \left( \left\{ \prod_{j=k}^{k+n-1} w^j \right\} y_{k+n} \right)_{k \in \mathbb{Z}_+} \in c_0(\mathbb{Z}_+) \right\}
\]

(see Remark 3.1), we have:

\[
D(\hat{A}^n) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in c(\mathbb{N}) \left| \left( \left\{ \prod_{j=k}^{k+n-1} w^j \right\} (x_{k+n} - l(x)) \right)_{k \in \mathbb{N}} \in c(\mathbb{N}) \right\}
\]

and, considering that, for any $x := (x_k)_{k \in \mathbb{N}} \in D(\hat{A}^n)$,

\[
A^n J x = \left( \left\{ \prod_{j=k}^{k+n-1} w^j \right\} (x_{k+n} - l(x)) \right)_{k \in \mathbb{Z}_+} =: (y_k)_{k \in \mathbb{Z}_+},
\]

in view of (5.1),

\[
\hat{A}^n x = J^{-1}A^n J = (y_k + y_0)_{k \in \mathbb{N}}
\]

= \left( \left\{ \prod_{j=k}^{k+n-1} w^j \right\} (x_{k+n} - l(x)) + \left\{ \prod_{j=0}^{n-1} w^j \right\} (x_n - l(x)) \right)_{k \in \mathbb{N}} \in c(\mathbb{N}).

Observe that

\[
\lim_{k \to \infty} \left( \left\{ \prod_{j=k}^{k+n-1} w^j \right\} (x_{k+n} - l(x)) + \left\{ \prod_{j=0}^{n-1} w^j \right\} (x_n - l(x)) \right)
\]
Since, by Lemma 5.1, $J : c(\mathbb{N}) \to c_0(\mathbb{Z}_+)$ is a homeomorphic isomorphism, the operator $\hat{A}^n \ (n \in \mathbb{N})$ inherits its linearity, unboundedness, closedness, chaoticity, and eigenvalues along with their geometric multiplicities directly from its conjugate $A^n$ via $J$. Therefore, the statement follows immediately from Theorem 3.2. □

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