Classification of non-Kähler surfaces and locally conformally Kähler geometry

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Abstract
Enriques-Kodaira classification treats non-Kähler surfaces as a special case within the Kodaira’s framework. We prove the classification results for non-Kähler complex surfaces not relying on the machinery of Enriques-Kodaira classification. We deduce the classification theorem of non-Kähler surfaces from the Buchdahl-Lamari theorem. We also prove that all non-Kähler surfaces which are not of class VII are locally conformally Kähler.

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¹Liviu Ornea and Victor Vuletescu are partially supported by a grant of Ministry of Research and Innovation, CNCS - UEFISCDI, project number PN-III-P4-ID-PCE-2016-0065, within PNCDI III.
²Misha Verbitsky is partially supported by the Russian Academic Excellence Project '5-100' and CNPq - Process 313608/2017-2.

Keywords: locally conformally Kähler, surfaces, Kato surface, elliptic fibration.
2010 Mathematics Subject Classification: 53C55, 32E05, 32E10.
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Classification of non-Kähler surfaces

1 Introduction

This paper appeared as a distillation of a lecture course on complex surfaces given in 2008 and 2012 in Moscow Independent University. The main reference on complex surfaces is the great book by Barth, Hulek, Peters and Van de Ven [BHPV]. This book offers a powerful narrative, but (as it often happens with great books) some plots are nested within more plots, and it sometimes becomes hard to separate a particular strain from the polyphonic discourse.

For a forthcoming book on locally conformally Kähler (LCK) manifolds the first named two authors needed a classification of LCK structures on (a posteriori, non-Kähler) surfaces. It was easier (and more enlightening) to prove the non-Kähler part of the Kodaira-Enriques classification directly along with the classification of LCK structures.¹ There are (almost) no new results of this paper, but most of the proofs are different from those given in the literature, such as [BHPV].

We tried to keep this exposé self-contained. With the exception of two results from Lamari’s paper [L], we invoke only general notions of complex algebraic geometry, found, for example, in [D2].

The main focus of this paper is elliptic non-Kähler surfaces. We give a new proof that they are principal elliptic bundles in the orbifold category (Theorem 4.2) - a result originally proven by Brînzănescu in [Bri1] (see also [Bri2]) - , and locally conformally Kähler (Theorem 4.15) - a result due to Belgun, [Bel]. We prove that all these surfaces are Vaisman, giving a new proof of Belgun’s classification of Vaisman surfaces ([Bel]).

¹For a definition and an introduction to LCK structures, see Subsection 1.2.
We prove that all non-Kähler non-elliptic surfaces are of class VII (Theorem 5.1). There is enough good literature ([DOT], [Dl1], [Na1], [Na2], [T4]) on class VII surfaces for us to give less attention to this case.

We state the known classification results up to the GSS conjecture and give a new proof of Brunella’s theorem on existence of LCK metrics on Kato surfaces. Together with the GSS conjecture (still not fully proven) this would imply that all non-Kähler complex surfaces are LCK, with the exception of some of the Inoue surfaces ([Bel]).

1.1 Buchdahl-Lamari theorem

In [Bu, L], N. Buchdahl and A. Lamari have proven a result previously known only from the Kodaira-Enriques classification of complex surfaces.

**Theorem 1.1:** Let $M$ be a compact complex surface. Then the first Betti number $b_1(M)$ is odd if and only if $M$ is non-Kähler. ■

Its direct proof, however, simplifies this classification significantly. In this paper we attempt to recover most of the Kodaira-Enriques classification for non-Kähler surfaces using the Buchdahl-Lamari theorem and the following intermediate result (Theorem 1.3), which was used by Lamari to prove Theorem 1.1. Our proof is different from the classical one, found in [BHPV], in a few aspects: we do not rely on birational arguments and classification of the elliptic surfaces due to Kodaira. Also, we aim to classify the locally conformally Kähler structures on complex surfaces.

For an introduction to currents and their applications in differential geometry, see [D2]. Recall that “currents” on $M$ are functionals on the space of differential forms on $M$ with compact support which are continuous in $C^\infty$-topology. A differential form $\alpha$ defines a current $\tau \mapsto \int_M \tau \wedge \alpha$. This allows us to consider the differential forms as a subspace of currents.

**Remark 1.2:** The usual operators and constructions of Kähler geometry (for example, $d, d^\ast, \partial, \bar{\partial}$, Laplacian, the Hodge decomposition) extend from differential forms to currents in a natural way. The corresponding cohomology (de Rham, Dolbeault, Bott-Chern) for currents is equal to those of differential forms ([D2]).
A (1,1)-form on a complex manifold $M$, $\dim \mathbb{C} M = n$ is positive if it is defined by a pseudo-Hermitian form with non-negative eigenvalues, and strictly positive when it is Hermitian. An $(n-1, n-1)$-form is positive if it is a product of $n-1$ positive forms. A (1,1)-current is called positive if it is non-negative on any positive $(n-1, n-1)$-form. The notion of positivity for forms is compatible with that of currents ([D2]).

**Theorem 1.3:** Let $M$ be a compact complex non-Kähler surface. Then there exists a non-zero positive, exact (1,1)-current $\Theta$ on $M$.

**Proof:** [L, Theorem 6.1].

**Remark 1.4:** The existence of an exact, non-zero positive (1,1)-current $\Theta$ immediately implies that the surface $M$ is non-Kähler. Indeed, suppose that $M$ admits a Kähler form $\omega$. Then $\int_M \omega \wedge \Theta > 0$, because $\omega \wedge \Theta$ is a non-zero measure on $M$, called the mass measure ([D2, Chapter III, Remark 1.15]). However, the strict inequality is impossible since $\Theta$ is exact.

### 1.2 Locally conformally Kähler surfaces

Let us recall that a complex manifold $M$, $\dim \mathbb{C} M > 1$, is called locally conformally Kähler (LCK) if it admits a Hermitian form $\omega$ such that $d\omega = \theta \wedge \omega$, where $\theta$ is a closed 1-form, see [DO]. Then any cover $\tilde{M} \rightarrow M$ such that $\tau^* \omega = df$ is exact is Kähler, with the Kähler form given by $\tilde{\omega} := e^{-f} \tau^* \omega$. When $\theta$ is exact, $M$ is called globally conformally Kähler. However, when $\theta$ is not exact, and $M$ is compact, $M$ is non-Kähler ([Va1]). If, in addition, there exists a holomorphic conformal flow $\rho : \mathbb{C} \rightarrow \text{Aut}(\tilde{M})$ acting on the Kähler cover $(\tilde{M}, \tilde{\omega})$ non-isometrically, $M$ is called Vaisman.

**Remark 1.5:** Since $(\tilde{M}, \tilde{\omega})$ is Kähler, any LCK manifold can be obtained as a quotient of a Kähler manifold by a discrete, proper action of a group of holomorphic automorphisms, acting on $\omega$ by holomorphic homotheties.\(^2\)

\(^2\)For historical reasons, “positivity” for differential forms is understood in French sense: 0 is “positive.” We idly suggest the term “French-positive”, to avoid confusion.

\(^3\)Notice that any conformal holomorphic map of a connected Kähler manifold $\varphi : (M, \tilde{\omega}) \rightarrow (M, \tilde{\omega})$, with $\dim \mathbb{C} M > 1$, takes $\tilde{\omega}$ to $f \tilde{\omega}$, with $df = 0$ because $d\varphi^* \tilde{\omega} = df \wedge \omega = 0$. Therefore, any conformal automorphism of a Kähler manifold is a homothety.
Thus, an LCK manifold can be defined as a quotient of a Kähler manifold \((\tilde{M}, \tilde{\omega})\) by a discrete, proper group acting by holomorphic homotheties.

**Definition 1.6:** Let \(M\) be a compact complex surface with \(b_1(M) = 1\). It is called a **class VII surface** if its Kodaira dimension is \(\kappa(M) = -\infty\).

From the Kodaira-Enriques classification ([BHPV]) it follows that all minimal non-Kähler surfaces not of class VII are elliptic. We prove this result (independently from the rest of Kodaira-Enriques classification) in Theorem 4.2. We prove that all elliptic surfaces are Vaisman when they are minimal.

**Theorem 1.7:** Let \(M\) be a compact, non-Kähler surface, which is not of class VII. Then \(M\) admits an LCK structure, and a Vaisman one if \(M\) is minimal.

**Proof:** By Theorem 5.1, \(M\) is elliptic when it is minimal, and by Theorem 4.15, it is Vaisman. By [Tr] and [Vu1], the blow-up at points preserves the LCK class, in particular a blow-up of an LCK surface remains LCK. Therefore, a surface is LCK if its minimal model is LCK, for example elliptic.

For class VII surfaces, a complete classification is not known, but it would follow from the so-called “Global Spherical Shell conjecture” (GSS conjecture), which claims that any minimal class VII surface \(M\) with \(b_2 > 0\) contains an open complex subvariety \(U \subset M\) biholomorphic to a neighbourhood of the standard sphere \(S^3 \subset \mathbb{C}^2\) such that \(M \setminus U\) is connected. Surfaces which satisfy this conjecture are called **Kato surfaces**.

Brunella has shown that all Kato surfaces are LCK ([Bru2] and Section 6). Bogomolov’s theorem on class VII surfaces with \(b_2(M) = 0\) (see [Bo1, Bo2, LYZ, T1]) implies that they are either Inoue surfaces or Hopf surfaces. The Hopf surfaces are LCK ([GO, Bel, OV1]), and among the three classes of Inoue surfaces, two are LCK, and the third contains a subclass which does not admit an LCK structure ([Tr, Bel]).

The modern proof of Bogomolov’s classification theorem (due to A. Teleman and Li-Yau-Zhang) is based on gauge theory. Using gauge-theoretic methods, A. Teleman was able to prove the GSS conjecture for minimal class VII surfaces with \(b_2 = 1\) ([T3]). Extending this approach to \(b_2 > 1\), Teleman was also able to prove that any class VII manifold with \(b_2(M) = 2\) contains a cycle of rational curves, hence can be smoothly deformed to a blown-up
Hopf surface ([T4]).

Once the GSS conjecture is proven, this finishes the classification of LCK surfaces. If it is true, all non-Kähler surfaces are LCK, except a particular class of Inoue surfaces, which is non-LCK by results of F. Belgun ([Bel]).

2 Cohomology of non-Kähler surfaces

2.1 Bott-Chern cohomology of a surface

In this section, we repeat some of the arguments about the Bott-Chern cohomology previously given in [T2] and [ATV]. Recall that any compact complex manifold admits a Gauduchon metric in any conformal class of Hermitian metrics ([Ga]). By definition, a Gauduchon metric on an n-dimensional complex manifold is a Hermitian metric with Hermitian form $\omega$ satisfying $dd^c(\omega^{n-1}) = 0$, where $d^c = IdI^{-1}$ is the twisted differential (here, as elsewhere, $I$ denotes the complex structure operator extended to differential forms multiplicatively).

**Definition 2.1:** The Bott-Chern cohomology group of a complex manifold is $H^{p,q}_{BC}(M) := \frac{\ker d}{\text{im} dd^c}$. $\Omega^{p,q-1}(M) \xrightarrow{dd^c} \Omega^{p,q}(M) \xrightarrow{d} \Omega^{p+q+1}(M)$.

**Remark 2.2:** By the $dd^c$-lemma, on a compact Kähler manifold the Bott-Chern cohomology groups are equal to the de Rham and (hence) the Dolbeault cohomology groups (Remark 1.2). As usual, we denote the space of global $(p,q)$-forms on $M$ by $\Lambda^{p,q}(M)$.

It is well known (and not hard to see) that the complex

$\Lambda^{p-1,q-1}(M) \xrightarrow{dd^c} \Lambda^{p,q}(M) \xrightarrow{d} \Lambda^{p+q+1}(M)$

is elliptic; see [KS, Proposition 5]. This implies that the Bott-Chern cohomology is finite-dimensional on any compact complex manifold.

**Theorem 2.3:** Let $M$ be a compact non-Kähler surface. Then the kernel of the natural map $P : H^{1,1}_{BC}(M) \to H^2(M)$ is 1-dimensional.

**Proof.** **Step 1:** Let $\omega$ be a Gauduchon metric on $M$. Consider the differential operator $D : f \mapsto dd^c(f) \wedge \omega$ mapping functions to 4-forms.
Clearly, $D$ is elliptic and its index is the same as the index of the Laplacian: $\text{ind} \, D = \text{ind} \, \Delta = 0$, hence $\text{dim ker} \, D = \text{dim coker} \, D$. The Hopf maximum principle implies that ker $D$ only contains constants, hence coker $D$ is 1-dimensional. However, $\int_M D(f) = \int_M dd^c(f) \land \omega = \int_M f dd^c \omega = 0$. This implies that a 4-form $\kappa$ belongs to im $D$ if and only if $\int_M \kappa = 0$.

Step 2: Let $\alpha$ be a closed $(1,1)$-form. Define the degree $\deg_\omega \alpha := \int_M \omega \land \alpha$. Since $\int_M dd^c f \land \omega = 0$, this defines a map $\deg_\omega : H_{11}^{BC}(M, \mathbb{R}) \to \mathbb{R}$. Given a closed $(1,1)$-form $\alpha$ of degree 0, the form $\alpha' := \alpha - dd^c(D^{-1}(\alpha \land \omega))$ satisfies $\alpha' \land \omega = 0$, in other words, it is an $\omega$-primitive $(1,1)$-form. For $\omega$-primitive forms, one has $\alpha' \land \alpha' = -|\alpha'|^2 \omega \land \omega$, giving

$$\int_M \alpha' \land \alpha' = -\|\alpha'\|^2_\omega$$  

(2.1)

which is impossible when $\alpha'$ is a non-zero vector in ker $P$, because $\alpha'$ is exact. Therefore, any vector of zero degree in ker $P \subset H_{11}^{BC}(M, \mathbb{R})$ vanishes. This implies that any two vectors in ker $P$ are proportional. ■

2.2 First cohomology of non-Kähler surfaces

The Dolbeault cohomology of a complex manifold $M$ is denoted by $H^{p,q}(M)$. Clearly, $H^{p,0}(M)$ coincides with the space of holomorphic $p$-forms on $M$. On a compact Kähler manifold, holomorphic $p$-forms are closed because they are harmonic. On non-Kähler manifolds, this is generally false. However, this is true on compact complex surfaces.

Lemma 2.4: All holomorphic 1-forms on a compact complex surface are closed.

Proof: Let $\alpha \in \Lambda^{1,0}(M)$ be a holomorphic 1-form. Then $\overline{\partial} \alpha = 0$, because it is holomorphic, and by the same reason $d \alpha$ is a holomorphic, exact $(2,0)$-form. Then $d \alpha \land d \overline{\alpha}$ is a positive $(2,2)$-form, giving $0 = \int_M d \alpha \land d \overline{\alpha} = \|d \alpha\|^2$, Then $d \alpha = 0$, and $\alpha$ is closed. ■

Theorem 2.5: Let $M$ be a non-Kähler manifold and $\Theta$ a non-zero exact positive $(1,1)$-current, which exists by Theorem 1.3. Let $d \alpha = \beta$ be a real $(1,1)$-form in the same Bott-Chern cohomology class. Then:
(i) Denote the space of holomorphic 1-forms on $M$ by $H^{1,0}(M)$, and let $\overline{H^{1,0}(M)}$ denote its complex conjugate (the space of antiholomorphic forms). By Lemma 2.4, holomorphic 1-forms are closed, therefore, there is a natural map $H^{1,0}(M) \oplus \overline{H^{1,0}(M)} \longrightarrow H^1(M, \mathbb{C})$ mapping the sum of a holomorphic and an antiholomorphic form to its cohomology class. We claim that this map is injective, and, moreover, for appropriate choice of $\alpha$ the form $\theta := \alpha^{1,0} - \alpha^{0,1}$ is closed and satisfies

$$H^1(M) = H^{1,0}(M) \oplus \overline{H^{1,0}(M)} \oplus \langle [\theta] \rangle = H^1(M), \quad (2.2)$$

that is, $H^1(M)$ is generated by cohomology classes of holomorphic and antiholomorphic forms and $\theta$.

(ii) Since all holomorphic forms are closed, the antiholomorphic forms are $\overline{\partial}$-closed and have Dolbeault classes. This gives a natural embedding $H^{1,0}(M) \hookrightarrow H^{0,1}(M)$. We claim that $H^{0,1}(M)$ is generated by $\overline{H^{1,0}(M)}$ and the Dolbeault class $[\theta^{0,1}]$, giving $H^{0,1}(M) = \overline{H^{1,0}(M)} \oplus \langle [\theta^{0,1}] \rangle$.

**Proof of (i):** A non-zero linear combination of holomorphic and antiholomorphic forms is closed and never exact. Indeed, if $df$ is a linear combination of holomorphic and antiholomorphic forms, then $d'df = 0$, hence $f$ is a globally defined harmonic function on a compact manifold. Such a function has to be constant by maximum principle. Therefore, the natural map $H^{1,0}(M) \oplus \overline{H^{1,0}(M)} \xrightarrow{\kappa} H^1(M, \mathbb{C})$ is injective. To prove Theorem 2.5 (i) it remains to show that its image has codimension 1, and prove that $H^1(M, \mathbb{C})$ is generated by the class of $\theta$ and the image of $\kappa$.

Clearly, the kernel of the natural map $\partial : H^{0,1}(M) \longrightarrow H^{1,1}_{BC}(M)$ coincides with $\overline{H^{1,0}(M)}$. Its image is contained in the one-dimensional kernel of the map $P : H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ (Theorem 2.3). Therefore, the image of $\kappa$ has codimension at most 1. The class $\partial \alpha^{0,1} \in H^{1,1}_{BC}(M)$ has non-zero degree, because $\partial \alpha^{0,1} + \overline{\partial \alpha^{0,1}} = \beta$, hence $\partial \alpha^{0,1}$ generates ker $P = \langle [\beta] \rangle = \langle [\Theta] \rangle$.

Replacing $\alpha^{0,1}$ by a Dolbeault cohomologous class if necessary, we can always assume that $\beta = \partial \alpha^{0,1}$ and $\overline{\partial \alpha^{0,1}} = 0$. Since $d\alpha$ is a real (1,1)-form, $\theta := \alpha^{1,0} - \alpha^{0,1}$ is closed. Since its (0,1)-part has the same Dolbeault cohomology class as $-\alpha^{0,1}$, the form $\theta$ is not exact. Therefore, it generates the cokernel of the embedding $H^{1,0}(M) \oplus \overline{H^{1,0}(M)} \xrightarrow{\kappa} H^1(M, \mathbb{C})$. We proved (2.2).
Proof of (ii): Consider the map $\partial : H^{0,1}(M) \rightarrow H^{1,1}_{BC}(M)$. By Step 1, $\partial(\theta^{0,1})$ is non-zero in $H^{1,1}_{BC}(M)$. Since $\partial : H^{0,1}(M) \rightarrow H^{1,1}_{BC}(M)$ vanishes on antiholomorphic forms, the Dolbeault cohomology class of $\theta^{0,1}$ does not belong to $H^{1,0}(M)$, giving an injective map

$$H^{1,0}(M) \oplus \langle \theta^{0,1} \rangle \rightarrow H^{0,1}(M). \quad (2.3)$$

Now, the kernel of $\partial : H^{0,1}(M) \rightarrow H^{1,1}_{BC}(M)$ is generated by antiholomorphic forms. This gives an exact sequence

$$0 \rightarrow H^{1,0}(M) \rightarrow H^{0,1}(M) \rightarrow H^{1,1}_{BC}(M) \quad (2.4)$$

The image of $\partial : H^{0,1}(M) \rightarrow H^{1,1}_{BC}(M)$ is at most 1-dimensional by Theorem 2.3, hence it is generated by $\partial(\theta^{0,1})$. Using (2.4), we obtain that the injective map (2.3) is actually surjective. We proved Theorem 2.5 (ii).

2.3 Second cohomology of non-Kähler surfaces

Claim 2.6: Let $M$ be a compact complex surface. Then:

$$H^{0,2}(M) = H^{2,0}(M) = H^{0}(\Omega^{2}(M)).$$

Proof: By Serre’s duality, $H^{0,2}(M) = H^{0}(K_{M})^\ast$, which has the same dimension as $H^{0}(\Omega^{2}(M)) = H^{0}(K_{M}) = H^{2,0}(M)$. The natural map $R : H^{2,0}(M) \rightarrow H^{0,2}(M)$ is injective, because its kernel is formed by $\partial$-exact holomorphic forms $\alpha = \partial \beta$, but for such $\alpha$ one has $0 = \int_{M} d\beta \wedge \bar{\alpha} = \int_{M} \alpha \wedge \bar{\alpha} = \|\alpha\|^2$, which is impossible unless $\alpha = 0$. As shown above, these spaces have the same dimension, hence $R$ is an isomorphism.

Corollary 2.7: ([BHPV, Theorem IV.2.8]) The Hodge-de Rham-Frölicher spectral sequence of a compact complex surface degenerates in $E_{1}$.

Proof: This argument is standard: the Hodge-de Rham-Frölicher degenerates if the dimension does not drop.

We have proven the degeneration for $H^{1}(M)$ in Theorem 2.5. Serre’s and Poincaré duality give the degeneration of this spectral sequence in the page $E_{1}$ for the $H^{3}(M)$ term. Now it has to degenerate in the page $E_{1}$ in $H^{2}(M)$, because for each instance where $d_{k} \neq 0$, with $d_{k} : E^{p,q}_{k-1} \rightarrow E^{p-k+1,q+k}_{k-1}$ non-zero, the group $E^{p,q}_{k-1}$ is replaced by ker $d_{k}$ and $E^{p-k+1,q+k}_{k-1}$ by coker $d_{k}$. Then
the total dimension of the space $\bigoplus_{p+q=d} E_{k-1}^{p,q}$ and $\bigoplus_{p+q=d} E_{k-1}^{p-k+1,q+k}$ would decrease. However, for $d = 1$ and $d = 3$ it is already minimal, hence $d_k = 0$ for all $k \geq 2$. ■

Remark 2.8: We have shown that $H^2(M, \mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$ and $H^2(M) = H^{0,2}(M)$, same as for Kähler surfaces (Claim 2.6).

Lemma 2.9: Let $M$ be a compact complex surface. Then the natural map $H^{1,1}_{BC}(M) \to H^{1,1}(M)$ (from Bott-Chern cohomology to Dolbeault cohomology) is surjective.

Proof: Let $\alpha \in \Lambda^{1,1}(M)$ be a $\overline{\partial}$-closed form representing a Dolbeault cohomology class $[\alpha] \in H^{1,1}(M)$. Clearly, $\alpha$ can be represented by a closed $(1,1)$-form if and only if the Bott-Chern class $[\beta] = \overline{\partial} \alpha$ vanishes. Indeed, if $\beta = \partial \overline{\partial} \eta$, then $\alpha - \partial \eta$ is $\partial$-closed.

Since the Hodge - de Rham - Frölicher spectral sequence degenerates in $E_1$, and $\beta$ is $\partial$-exact and $\overline{\partial}$-closed, one has $\beta = \overline{\partial} \gamma$, for some $\gamma \in \Lambda^{2,0}(M)$. However, the cokernel of the map $\overline{\partial} : \Lambda^{0,1}(M) \to \Lambda^{2,0}(M)$ is generated by holomorphic 2-forms, and they are all closed (Claim 2.6), hence $\overline{\partial}(\Lambda^{2,0}(M)) = \overline{\partial}(\Lambda^{1,0}(M))$, and the Bott-Chern class of $\beta \in \overline{\partial}(\Lambda^{2,0}(M))$ vanishes. ■

Proposition 2.10: ([BHPV, Theorem IV.2.14]) Let $M$ be a compact non-Kähler surface. Then the intersection form on the image of $H^{1,1}_{BC}(M, \mathbb{R})$ in de Rham cohomology is negative definite.

Proof: Fix a Gauduchon metric $\omega$ on $M$. Let $[\alpha] \in H^{1,1}(M)$ be a cohomology class. By Lemma 2.9, we can represent $[\alpha]$ by a closed $(1,1)$-form $\alpha$. Consider the degree functional $\deg_{\omega} : H^{1,1}_{BC}(M, \mathbb{R}) \to \mathbb{R}$ defined in Subsection 2.1. Since $\deg_{\omega}(\Theta) > 0$ for an exact $(1,1)$-current $\Theta$, any cohomology class $[\alpha] \in H^{1,1}(M) \subset H^2(M)$ can be represented by a closed $(1,1)$-form $\alpha$ with $\deg_{\omega} \alpha = 0$. Acting as in the proof of Theorem 2.3, we find $f \in C^\infty(M)$ such that $\alpha - dd^c f$ is primitive. Replacing $\alpha$ by $\alpha - dd^c f$, we obtain $\int_M \alpha \wedge \alpha = -\|\alpha\|^2_{\omega} < 0$ as in (2.1). ■

Remark 2.11: In [Bri2, Theorem 2.37] (see also [BF1, BF2]) it was shown that the space $H^{1,1}(M) \cap H^2(M, \mathbb{Q})$ (the rational Neron-Severi group) of a
complex surface with algebraic dimension 0 has negative definite intersection form. This important result also follows from Proposition 2.10

2.4 Vanishing of products of holomorphic 1-forms

Proposition 2.10 has an interesting corollary.

**Proposition 2.12:** Let $M$ be a non-Kähler, compact, complex surface. Then for any holomorphic 1-forms $\alpha, \beta$, the product $\alpha \wedge \beta$ vanishes, and the product $\alpha \wedge \bar{\beta}$ is exact.

**Proof.**

**Step 1:** Let $\alpha, \beta$ be holomorphic 1-forms. Then $\alpha \wedge \beta$ is a holomorphic 2-form, with $\int_M \alpha \wedge \beta \wedge \overline{\alpha} \wedge \overline{\beta} > 0$ unless $\alpha \wedge \beta = 0$.

**Step 2:** The intersection form on $H^{1,1}(M)$ is negative definite by Proposition 2.10. Therefore, $\int_M \eta \wedge \overline{\eta} = 0$ implies $\eta = 0$ for any $\eta \in H^{1,1}(M)$. Take $\eta = \alpha \wedge \overline{\alpha}$ and $\rho = \beta \wedge \overline{\beta}$. Then clearly $\int_M \eta \wedge \overline{\eta} = \int_M \rho \wedge \overline{\rho} = 0$, which implies $\int_M \alpha \wedge \beta \wedge \overline{\alpha} \wedge \overline{\beta} = 0$, hence $\alpha \wedge \beta = 0$ (Step 1).

**Step 3:** To see that $\eta := \alpha \wedge \overline{\beta}$ is exact, we use Proposition 2.10 again. Unless $\mu := \alpha \wedge \overline{\beta}$ is exact, one would have $\int_M \mu \wedge \overline{\mu} < 0$, but $\int_M \mu \wedge \overline{\mu} = 0$ as we have already shown in Step 2. Therefore, $\mu$ is exact. ■

2.5 Structure of multiplication in $H^1(M)$ for non-Kähler surface without curves

Further on, we shall need the following lemma.

**Lemma 2.13:** Let $M$ be a compact non-Kähler surface, and $\Theta$ an exact real (1,1)-form. Then there exists a closed form $\theta \in \Lambda^1(M)$ such that $d^c(\theta) = \Theta$.

**Proof:** Without restricting generality, we can assume that $\Theta$ is an exact real (1,1)-form representing a non-zero class in $H^{1,1}_{BC}(M)$. By Theorem 2.5 (ii), the cohomology class of $\Theta$ belongs to the image of the natural map $\partial : H^0(M) \to H^{1,1}_{BC}(M)$, and can be expressed as $\partial(\tilde{\theta}^{0,1})$, where $\tilde{\theta}$ is a closed form. Therefore, $\Theta - d^c(\tilde{\theta}) = d^c df$ for some function $f \in C^\infty M$. Take $\theta := \tilde{\theta} + df$. Then $\theta$ is a closed 1-form such that $d^c \theta = \Theta$. ■
In this subsection we prove the following structure theorem for the multiplication in $H^1(M)$. It is a posteriori true in the general situation, but we need it only for surfaces without complex curves.

**Theorem 2.14:** Let $M$ be a compact non-Kähler surface without curves, and $\theta \in \Lambda^1(M)$ a closed 1-form such that $d^c(\theta)$ is non-zero in $H^{1,1}_{BC}(M, \mathbb{R})$ (Lemma 2.13). Denote by $W \subset \Lambda^1(M)$ the subspace generated by holomorphic and antiholomorphic forms. We identify $W$ with its image in $H^1(M) = W \oplus \langle \theta \rangle$ (Theorem 2.5). Then

(i) The multiplication $W \wedge W \to H^2(M)$ vanishes. The multiplication $W \wedge \theta \to H^2(M)$ is injective.

(ii) Let $\theta^c := I(\theta)$. Then for any non-zero $x \in W$, the form $x \wedge \theta^c$ is closed and represents a non-zero element of $H^2(M)$.

(iii) The Poincaré pairing on the spaces $W \wedge \theta$ and $W \wedge \theta^c \subset H^2(M)$ vanishes. The Poincaré pairing between these two subspaces is non-degenerate.

**Proof.**

**Step 1:** Multiplication $W \wedge W \to H^2(M)$ vanishes by Proposition 2.12. Let us prove that for any non-zero $x \in W$, the form $x \wedge \theta^c$ is closed and represents a non-zero element of $H^2(M)$.

Without restricting the generality, we may assume that $W \neq 0$. For any $x \in W$, the closed (1,1)-form $x \wedge \bar{x}$ is homologous to zero (Proposition 2.12). Indeed, one has $(x \wedge \bar{x})^2 = 0$, and the Poincare pairing on the image of $H^{1,1}_{BC}(M)$ in the de Rham cohomology is negative definite by Proposition 2.10.

For any non-zero holomorphic form $x$, the form $\Theta := x \wedge \bar{x}$ is positive, non-zero and exact. We fix such $\Theta$, and fix a closed form $\tilde{\theta}$ such that $d^c \tilde{\theta} = \Theta$ as in Theorem 2.5 (ii). Choose any $y \in W$. Then $d(y \wedge \theta^c) = y \wedge x \wedge \bar{x} = 0$ (Proposition 2.12) hence $y \wedge \theta^c$ is closed.

**Step 2:** The injectivity of $W \xrightarrow{x \wedge \theta} H^2(M)$ and $W \xrightarrow{x \wedge \theta^c} H^2(M)$ would follow if we prove that the formula

$$x, y \mapsto \int_M x \wedge y \wedge \theta \wedge \theta^c, \quad x, y \in W$$

defines a non-degenerate pairing on $W$, these two spaces is non-degenerate, hence (i) and (ii) follows from (iii).
The Poincaré pairing on the images $W \wedge \theta$ of $W \wedge \theta^c$ in $H^2(M)$ vanishes because $\theta \wedge \theta = \theta^c \wedge \theta^c = 0$. To prove that the Poincaré pairing between $W \wedge \theta$ and $W \wedge \theta^c$ is non-degenerate, take a holomorphic 1-form $x \in W$. To finish the proof of Theorem 2.14 (iii) it would suffice to show that the integral $\int_M \sqrt{-1} x \wedge \overline{x} \wedge \theta \wedge \theta^c$ is positive. The form $\sqrt{-1} x \wedge \overline{x} \wedge \theta \wedge \theta^c$ is positive, because it is a product of a $(2,0)$-form and its complex conjugate. It is non-zero if $x$ is not proportional to the $(1,0)$-part of $\theta$, denoted $\theta^{1,0}$.

**Step 3:** It remains to show that $\theta^{1,0}$ is not proportional to a holomorphic form; this is where we use that $M$ has no holomorphic curves. In this case the zero set of any 1-form $x \in W$ is a finite set.

Suppose that $x \in W$ is holomorphic and proportional to $\theta^{1,0}$. Then there exists a smooth function $\alpha$ defined outside of the zero set $S$ of $x$ such that $\theta^{1,0} = \alpha x$. Without restricting generality, we may assume that $d\theta^{1,0} = x \wedge \overline{x}$ (Step 1). Then $\overline{\partial} \alpha = \overline{x}$, which gives $dd^c \alpha = 0$ outside of $S$. Then $\alpha$ is locally the real part of a holomorphic function; using the Hartogs extension theorem, we obtain that $\alpha$ is smooth and defined globally on $M$. Then $\alpha = 0$ by maximum principle, because $dd^c$ is elliptic.

**Corollary 2.15:** Let $M$ be a compact surface without curves, and $W \subset H^1(M, \mathbb{C})$ the subspace generated by holomorphic and antiholomorphic forms. Then $W$ is a rational subspace, that is, there exists a subspace $W_\mathbb{Q} \subset H^1(M, \mathbb{Q})$ such that $W = W_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$.

**Proof:** For $x \in H^1(M)$, denote by $L_x : H^1(M) \rightarrow H^2(M)$ the map $y \mapsto x \wedge y$. Without restricting generality, we may assume $W \neq 0$; then $\dim W \geq 2$. By Theorem 2.14, $\text{rk } L_\theta = \dim W \geq 2$ and $\text{rk } L_x = 1$ for $x \in W$. Therefore $W$ is the space of all $x \in H^1(M)$ such that the $\text{rk } L_x = 1$. Since the multiplication in $H^*(M)$ is defined over $\mathbb{Q}$, the subspace $W \subset H^1(M)$ is rational.

**Remark 2.16:** Corollary 2.15 is true for all surfaces. We leave its proof as an exercise to the reader.
3 Barlet spaces

Barlet spaces are spaces of cycles, that is, closed complex analytic subvarieties of given dimension in a given complex manifold with multiplicities (positive integers) assigned to their irreducible components. They are similar but distinct from the Douady spaces, which are spaces of closed complex analytic subspaces (possibly with nilpotents in the structure sheaf).

For more details on Barlet spaces and their properties, please see the book [BM].

Let $M$ be a metric space. The (Gromov-)Hausdorff metric on the set $C$ of closed subsets of $M$ is defined as follows: $d(X,Y)$ is infimum of all $\varepsilon$ such that $X$ belongs to $\varepsilon$-neighbourhood of $Y$, and $Y$ belongs to $\varepsilon$-neighbourhood of $X$. When $M$ is compact, the corresponding topology on $C$ is independent from the choice of metric on $M$ as long as topology of $M$ remains the same. It is called the (Gromov-)Hausdorff topology.

It is important for the sequel that the Barlet space of cycles is a reduced complex analytic space, with the topology which is compatible with the (Gromov-)Hausdorff topology on the set of all closed subvarieties.

For each $k \geq 0$, the Barlet space $\mathcal{B}_k(M)$ of $k$-cycles on a manifold $M$ comes equipped with a universal family of cycles, and their supports form a closed complex-analytic subvariety $\mathcal{B}_k^m(M) \subset M \times \mathcal{B}(M)$, the “marked Barlet space” of pairs “complex analytic cycle and a point in its support”. The forgetful map $\mathcal{B}_k^m(M) \rightarrow \mathcal{B}_k(M)$ is equidimensional of relative dimension $k$, and the forgetful map $\Psi : \mathcal{B}_k^m(K) \rightarrow M$ is complex-analytic. In particular, for any compact irreducible component $Z$ in a Barlet space, and its marked counterpart $Z^m$, the image $\Psi(Z^m)$ is a complex analytic subvariety by Remmert’s proper mapping theorem. Geometrically, this means that for any compact complex-analytic family $Z$ of cycles in $M$, the union of supports of all these cycles is a complex subvariety in $M$.

Further on, we shall use the following result, which was proven for curves in [Ve2, Corollary 2.19], and for divisors in [Bar].

**Theorem 3.1:** Let $(M,I)$ be a compact complex manifold, and $\omega$ a Hermitian form. Assume that $dd^c(\omega^k) = 0$ for some integer $k > 0$. Then any connected component $Z$ of the Barlet space $\mathcal{B}_k(M)$ of $k$-cycles in $M$ is compact.

**Proof:** By Bishop’s theorem ([B]), a (Gromov-)Hausdorff limit of a
family of compact complex subvarieties is complex if its Hermitian volume stays bounded. Since the space of closed subvarieties in $M$ with (Gromov-)Hausdorff topology is compact, this implies that the set of closed compact $k$-dimensional subvarieties with volume bounded by a constant $C \in \mathbb{R}$ is compact.

Therefore, to prove compactness it would suffice to show that the volume $\text{Vol}(S)$ is constant as a function of $[S] \in Z$.

Let $X$ denote the marked family associated with $Z$, and $\pi_M : X \to M$, $\pi_X : X \to Z$ the forgetful maps. Then the volume function $\text{Vol} : Z \to \mathbb{R}^{>0}$ can be expressed as $\text{Vol} = (\pi_X)_* \pi_M^* \omega_k$, where $(\pi_X)_*$ denotes the pushforward of a differential form (generally speaking, the pushforward is not a form, but it is well defined as a current).

Let $k$ be dimension of the cycles parametrized by $Z$. Since pullback and pushforward of differential forms commute with $d$, $d^c$, this gives $d\!d^c \text{Vol} = (\pi_X)_* \pi_M^* (d\!d^c \omega_k)$ (see e.g. [KV, (8.12)], [Ve1, Theorem 2.10] or [I1, Proposition 1.9]). Therefore, Vol satisfies $d\!d^c (\text{Vol}) = 0$ whenever $d\!d^c (\omega_k) = 0$.

Functions which satisfy $d\!d^c f = 0$ are called pluriharmonic. Using local $d\!d^c$-lemma, it is easy to see that any pluriharmonic function is locally a sum of a holomorphic and antiholomorphic function.

By Bishop’s theorem, the set $\text{Vol}^{-1}([-\infty, C])$ is compact for all $C \in \mathbb{R}$, hence $-\text{Vol}$ has a maximum somewhere in $X$. However, a pluriharmonic function which admits a maximum is necessarily constant by E. Hopf’s strong maximum principle. Therefore, Vol is constant on each connected component of the Barlet space. Now, each of these components is compact by Bishop.

Applying this result to the space of curves on a complex surface and using the Gauduchon form $\omega$ (Subsection 2.1), we obtain the following useful corollary.

**Corollary 3.2:** Let $M$ be a compact complex surface, $\mathcal{B}_1(M)$ the Barlet space of 1-cycles on $M$ and $Z$ its connected component. Then $Z$ is compact.
4 Non-Kähler elliptic surfaces

4.1 The Gauss-Manin connection

For the sequel, we recall some basic facts about the Gauss-Manin connection. We may refer to [Gri] or [Voi]. Let \( \pi : M \rightarrow B \) be a smooth, proper map of smooth manifolds. By Ehresmann’s theorem, the fibers of \( \pi \) are diffeomorphic. Then \( \pi \) is a locally trivial fibration, hence for any fixed \( k \) the \( k \)-th cohomology of its fibers form a local system, called **Gauss-Manin local system**. By the Riemann-Hilbert correspondence, the category of local systems is equivalent to the category of vector bundles equipped with a flat connection.

The bundle associated with the Gauss-Manin local system is called the **Gauss-Manin bundle**, and the connection the **Gauss-Manin connection**. It can be constructed as follows.

Let \( T_{\text{vert}} M \subset TM \) be the bundle of fiberwise tangent vectors. By definition, an Ehresmann connection \( e \) on \( M \) is a decomposition \( TM = T_{\text{vert}} M \oplus T_{\text{hor}} M \), that is, a choice of such a splitting. Identifying \( T_{\text{hor}} M \) and the pull-back \( \pi^* TB \), we may consider the pullback of a vector field \( X \in TB \) as a vector field \( X_e \in T_{\text{hor}} M \).

A section of a vector bundle associated with the fiberwise cohomology of \( M \) is given by a fiberwise closed differential form \( \eta \in \Lambda^k M \). The Lie derivative \( \text{Lie}_{X_e} \eta \) is closed on fibers of \( \pi \) for any vector field on \( B \) lifted to a horizontal vector field \( X_e \) on \( M \). Indeed, the corresponding diffeomorphisms map fibers to fibers and fiberwise closed forms to fiberwise closed forms.

Since different choices \( e, e' \) of the Ehresmann connection result in the vector fields \( X_e, X_{e'} \) which satisfy \( Y := X_e - X_{e'} \in T_{\text{vert}} M \), and the form \( \text{Lie}_Y \eta \) is fiberwise exact, the cohomology class of the restriction of \( \text{Lie}_{X_e} \eta \) is independent from the choice of the Ehresmann connection.

Let now \( [\eta] \) be the collection of the cohomology classes of \( \eta \) on all fibers of \( \pi \), considered as a section of the Gauss-Manin bundle. Define

\[
\nabla_X[\eta] := [\text{Lie}_{X_e} \eta], \tag{4.1}
\]

where \( [\text{Lie}_{X_e} \eta] \) is the collection of the cohomology classes of \( \text{Lie}_{X_e} \eta \) on all fibers of \( \pi \). This formula defines the Gauss-Manin connection \( \nabla \).

We needed this observation to prove the following lemma.
Lemma 4.1: Let $\pi : M \rightarrow B$ be a smooth fibration, and $\eta$ a $p$-form which is closed on the fibers of $\pi$. Using the pullback map, we consider $\pi^*(\Lambda^*B)$ as a subspace in $\Lambda^*(M)$. Assume that $d\eta$ belongs to $\pi^*(\Lambda^2B) \wedge \Lambda^{p-1}M \subset \Lambda^{p+1}(M)$. Then the section of the Gauss-Manin bundle, corresponding to $[\eta]$, is parallel.

Proof: We use the formula (4.1): $\nabla_X[\eta] = [\text{Lie}_X \eta]$. Cartan’s formula gives $\text{Lie}_X \eta = i_X d\eta + d(i_X \eta)$, the second term on the right hand side is exact, and the first vanishes on fibers because $d\eta \in \pi^*(\Lambda^2B) \wedge \Lambda^{p-1}M$. ■

4.2 Elliptic fibrations on non-Kähler surfaces

For a version of the following theorem (with a different proof), see [Bri2, Theorem 3.17] and [Bri1].

Theorem 4.2: Let $M$ be a non-Kähler compact complex surface admitting a non-constant 1-dimensional family of divisors. Then $M$ admits a holomorphic, surjective map $\pi : M \rightarrow S$ to a curve, and the general fibers of $\pi$ are isomorphic elliptic curves, homologous to 0. If, moreover, $M$ is minimal, then all fibers of $\pi$ are elliptic curves.

Proof. Step 1: We may assume that all the divisors in our family are irreducible curves (if all components in a divisor do not move, the divisor itself does not move). Let $S$ be the base of the family. This is a reduced irreducible complex space of dimension 1, and the curves in the family with multiplicity 1 form an analytic family of 1-cycles on $M$. By the universal property of the Barlet spaces, the family is induced from the universal one via a map $S \rightarrow \mathfrak{B}_1(M)$. Replace $S$ with the irreducible component of $\mathfrak{B}_1(M)$ that contains the image of the map. Then as shown in Corollary 3.2, each connected component of $\mathfrak{B}_1(M)$ is compact. Therefore $S$ is compact, the support $S^m \subset M \times S$ of our family is irreducible, and its image $\Psi(S^m) \subset M$ under the projection $\Psi : M \times S \rightarrow M$ is an irreducible closed subvariety. Since the family is not constant, $\Psi(S^m)$ cannot be a curve, so $\Psi(S^m) = M$.

All complex curves $C \subset M$ with the fundamental class $[C] \neq 0$ have negative self-intersection by Proposition 2.10. Therefore if we let $U \subset S$ be the dense open subset parametrizing cycles that are irreducible with multiplicity one, and let $U^m \subset S^m$ be it preimage in $S^m$, then the proper map
\[ \Psi : S^m \to M \] is injective on \( U^m \). First of all, this means that \( S \) is of dimension exactly 1, and \( S^m \) is of dimension 2. Secondly, \( \Psi \) is one-to-one over the complement \( M \setminus \Phi_S \), and since \( S^m \setminus U^m \) is a finite disjoint union of “bad” fibers of the projection \( S^m \to S \), \( \Psi(S^m \setminus U^m) \subset M \) is at most one-dimensional. Lastly, note that while these “bad” fibers are different as cycles, they might have common irreducible components, so they might intersect. However, if a curve \( C \subset S^m \) is contracted by \( \Psi \), then it must lie in \( S^m \setminus U^m \), so each of its connected components lies in a bad fiber. But by definition, \( \Psi \) is injective on each of the fibers including bad ones, so \( C \) cannot exist.

Thus \( \Psi : S^m \to M \) is a proper map with finite fibers that is one-to-one over the complement to a subvariety of positive codimension. Since \( M \) is smooth, thus normal, \( \Psi \) is an isomorphism, so we obtain a holomorphic map \( \pi : M \cong S^m \to S \) to an irreducible compact curve. Taking its Stein factorization, we may assume that \( S \) is normal, thus smooth.

**Step 2:** Let now \( C \) be a general fiber of \( \pi \). Using the standard isomorphism \( \pi^*(\Omega^1 S) = N^*_C M \) between the pullback of the cotangent sheaf and the conormal bundle to the fibers of \( \pi \) we obtain an exact sequence, sometimes called “adjunction formula”:

\[
0 \to \pi^*(\Omega^1 S) \to \Omega^1 M \to \Omega^1_{\pi} M \to 0,
\]

where \( \Omega^1_{\pi} M \) is the bundle of holomorphic differentials on fibers of \( \pi \). This sequence makes sense on the smooth locus of \( \pi \), but we need it only in a neighbourhood of the generic fiber \( C \).

The exact sequence (4.2) gives \( K_{M|C} = K_C \), that is, the canonical bundle to \( M \) restricted to \( C \) gives the canonical bundle to \( C \). Since the self-intersection of \( C \) is zero, \( C \) is homologous to 0 (Proposition 2.10). This gives \( \int_C c_1(K_M) = 0 \), hence the degree of \( K_C \) is equal 0. This implies that \( C \) is an elliptic curve.

**Step 3:** Return to the map \( \pi : M \to S \) constructed above. At this step we are going to prove that all its smooth fibers are isomorphic.

Denote by \( S_0 \subset S \) its smooth locus, and let \( H \) be the Gauss-Manin bundle associated with the first cohomology of the fibers of \( \pi \). The closed form \( \theta \) gives a non-zero cohomology class in each of the elliptic curves \( \pi^{-1}(s) \) (Step 3), hence it gives a section of \( H \). Being closed, it is constant with respect to the Gauss-Manin connection. The form \( d^\theta = \Theta \) belongs to \( \pi^* \Lambda^2 B \). Then
\(I(\theta)\) is closed on the fibers of \(\pi\). By Lemma 4.1, \(I(\theta)\), and hence the Hodge components of \(\theta\), define a parallel section of the Gauss-Manin bundle.\(^1\)

We obtain a basis \(\theta^{1,0}\) and \(\theta^{0,1}\) for \(H\). Therefore, the variation of Hodge structures induced by periods of the elliptic curves on \(H\) is trivial. This implies that the corresponding elliptic fibration is isotrivial.

Note that isotriviality implies local triviality by a theorem of Grauert and Fischer ([FG]), hence the fibration \(\pi\) is locally trivial over \(S_0\).

**Step 4:** It remains to show that each fiber of \(\pi\) contains an elliptic curve, and nothing else if \(M\) is minimal.

Take a critical value \(s \in S \setminus S_0\) of the map \(\pi\), and choose a small neighborhood \(U \subset s\) that contains no other critical values, and such that \(M_U = \pi^{-1}(U)\) admits a retraction onto the special fiber \(C_s = \pi^{-1}(s)\) (this retraction is constructed in [Pe], [C], see also [Mor]). Choose also a point \(u\) in the punctured disc \(U^o = U \setminus \{s\}\), a point \(m \in C_u\) in its preimage \(C_u = \pi^{-1}(u) \subset M_U\), and denote \(M'_U = \pi^{-1}(U^o) = M_U \setminus C_s \subset M_U\).

Since \(C_s \subset M\) is of real codimension at least 2, the map \(\pi_1(M'_U, m) \to \pi_1(M_U, m)\) is surjective, and then so is the map \(H_1(M'_U, \mathbb{Z}) \to H_1(M_U, \mathbb{Z}) \cong H_1(C_s, \mathbb{Z})\) and its rational version

\[H_1(M'_U, \mathbb{Q}) \to H_1(M_U, \mathbb{Q}) \cong H_1(C_s, \mathbb{Q}). \tag{4.3}\]

By the Leray spectral sequence, we have \(H_1(M'_U, \mathbb{Q}) \cong H_1(C_u, \mathbb{Q}) \oplus \mathbb{Q}\langle \alpha \rangle\), where \(\alpha\) is any class with non-trivial image \(\pi(\alpha) \in \mathbb{Q} \cong H_1(U^o, \mathbb{Q})\). In particular, it induces a canonical specialization map \(H_1(C_u, \mathbb{Q}) \to H_1(C_s, \mathbb{Q})\) and its counterpart in homology with coefficients in \(\mathbb{Z}\). We can then take a multisection \(\sigma: \tilde{U} \to M_U\) of the projection \(\pi: M_U \to U\) — that is, a finite cover \(\tilde{U} \to U\) of our disc \(U\) ramified at \(s \in U\) that factors through \(\pi\) — and let \(\alpha = \sigma(1)\) be the image of the generator 1 \(\in \mathbb{Q}\) under the map \(\sigma: \mathbb{Q} = H_1(U^o, \mathbb{Q}) \to H_1(M'_U, \mathbb{Q})\). Then the map (4.3) annihilates \(\alpha\), so that the specialization map \(H_1(C_u, \mathbb{Q}) \to H_1(C_s, \mathbb{Q})\) is still surjective.

Next, note that since \(U\) is a disc, the Kähler form \(\omega_S\) is \(dd^c\)-exact on \(U\), so that \(\omega_S = \overline{\partial} \partial \varphi\), for some \(\varphi \in C^\infty U\). Replacing \(\theta^{1,0}\) by \(\gamma := \theta^{1,0} - \partial \varphi\), we obtain a \((1,0)\)-form \(\gamma\) with the same restriction to each fiber, but now it

\(^1\)The same conclusion is implied by Schmid’s fixed part theorem ([Sch, Theorem 7.22]): for any polarizable variation of Hodge structures over a quasiprojective base, the \((1,0)\) and \((0,1)\)-parts of a parallel section \(\theta\) are also constant with respect to the Gauss-Manin connection.
satisfies
\[ d\gamma = d\theta^{1,0} - d\partial\varphi = \omega_S - \overline{\partial}\partial\varphi = 0. \]

Therefore, \( \gamma \) is holomorphic and closed on \( M_U \), and as such, it has a de Rham cohomology class \([\gamma]\).

Denote \( P = H_1(C_u, \mathbb{Z}) \), \( P' = H_1(C_s, \mathbb{Z}) \), and consider the maps
\[ P \to P' \to \mathbb{C}, \quad (4.4) \]
where the first arrow is the specialization map, and the second one is obtained by evaluating \([\gamma]\). We know from Step 2 that \( C_u \) is an elliptic curve \( E \), so that \( P \cong \mathbb{Z}^2 \) is a rank-2-lattice, and the map \((4.4)\) is the period map that identifies \( E \cong \mathbb{C}/P \). In particular, \( P \to \mathbb{C} \) is injective. Then so is the specialization map \( P \to P' \), and since \( P \otimes \mathbb{Q} \to P' \otimes \mathbb{Q} \) is surjective, we see that \( P' \otimes \mathbb{Q} \cong P \otimes \mathbb{Q} \), and \( P \subset P' \) is a subgroup of some finite index \( n \).

Denote \( G = P'/P \), note that being a quotient of \( H_1(M_U, \mathbb{Z}) \cong P' \), it is also a quotient of \( \pi_1 \), and let \( \eta : M'_U \to M_U \) be the corresponding unramified \( n \)-fold Galois cover with the Galois group \( G \). By definition, the map \( H_1(M'_U, \mathbb{Z}) \to H_1(M_U, \mathbb{Z}) = P' \) factors through \( P \). Consider the map \( \pi \circ \eta : M'_U \to U \), and take its Stein factorization \( \pi \circ \eta = \nu \circ \eta' \) into a finite ramified covering \( \nu : U' \to U \) and a holomorphic map \( \pi' : M'_U \to U' \) with connected fibers. Then for any \( u \in U^\circ \), the map \( \pi(C_u) \to \pi(M_U) \to G \) is trivial by construction, so that \( \eta \) splits over \( C_u \), and \( C'_u = \eta^{-1}(C_u) \) is the disjoint union of \( n \) copies of \( E \cong C_u \) transitive permuted by \( G \). By definition, these components correspond to points in the set \( \nu^{-1}(u) \), so that \( \nu : U' \to U \) is an \( n \)-fold unramified cover over \( U^\circ \subset U \). Since \( M'_U \) is connected, so is \( U' \), and then being normal hence smooth, it must be a disc, and \( \nu : U' \to U \) must be the standard \( n \)-fold cover \( z \mapsto z^n \). Therefore \( G \cong \mathbb{Z}/n\mathbb{Z} \) is a cyclic group whose generator acts on \( U' \) via the root of unity.

Now let \( C'_u = \eta^{-1}(C_u) \), choose a point \( m \in C'_u \), and consider the Albanese-type map \( \text{Alb} : M'_U \to E = \mathbb{C}/P \) sending a point \( m' \in M'_U \) to the integral \( \int_l \eta^*\gamma \), where \( l : [0, 1] \to M'_U \) is any path connecting \( m \) and \( m' \) (modulo \( P \), the integral does not depend on the choice of the path). Then \( \text{Alb} \) identifies each component of the curve \( C'_u \), \( u \in U^\circ \) with \( E \), and the product map \( \text{Alb} \times \pi' : M'_U \to E \times U' \) is a proper map that is an isomorphism over \( \nu^{-1}(U^\circ) \). Thus it is a surjective rational map, \( \pi_1(M'_U, m) \cong P \) is abelian, and \( H^1(E, \mathbb{Z}) \cong H^1(E \times U', \mathbb{Z}) \to H^1(M'_U, \mathbb{Z}) \cong H^1(C'_u, \mathbb{Z}) \) is an isomorphism. Moreover, \( C'_s \to E \) is a proper surjective map, so that \( C'_s \) has a component \( C \) that dominates \( E \), thus has genus \( \geq 2 \), and possibly some other components. But if we let \( C''_s \) be
the normalization of $C_s'$, then the map $H^1(E, Z) \cong H^1(C_s', Z) \to H^1(C_s'', Z)$ is surjective. Therefore $H^1(E, Z) \cong H^1(C_s', Z)$ so that $C \cong E$, while all the other components are rational curves with no $H^1$. If $M$ is minimal, the other components do not exist and $M' \cong E \times U'$.

To finish the proof, it remains to notice that $G \in \mathbb{Z}/n\mathbb{Z}$ acts on $M'U$ without fixed points, and since it fixes the only non-rational component $C \cong E$ of $C_s'$, it must also act on $E$ without fixed points. Therefore the generator $1 \in P'/P \cong \mathbb{Z}/n\mathbb{Z}$ acts by shifts by an $n$-torsion element $e \in E = \mathbb{C}/P$, the map $P' \to \mathbb{C}$ is injective, so that $P' \cong \mathbb{C}^2$, and $C_s = C_s'/G$ consists of an elliptic curve $E' = E/G = \mathbb{C}/P'$ and some rational curves (or nothing in the minimal case).

**Corollary 4.3**: Let $M$ be a non-Kähler, compact, complex surface with $b_1(M) > 3$. Then $M$ admits an elliptic fibration.

**Proof**: Theorem 2.5 implies that $\text{dim } H^{1,0}(M) = \frac{b_1(M) - 1}{2}$, and hence $\text{dim } H^0(\Omega^1(M)) \geq 2$. By Proposition 2.12, all globally defined holomorphic 1-forms on $M$ are pointwise proportional. Consider the rank 1 sheaf $L \subset \Omega^1(M)$ generated by holomorphic 1-forms. If $\text{dim } H^{1,0}(M) = \text{dim } H^0(L) > 1$, the zero divisors of the sections of this sheaf form a continuous family of curves on $M$, and Theorem 4.2 can be applied.

**Remark 4.4**: Note that the result we have proven does not imply that the map $\pi$ is a submersion. Indeed, $\pi$ may have multiple fibers. The basic local example is given by Kodaira log-transform. Take an elliptic curve $E = \mathbb{C}/P$, and a torsion point $e \in E$ of some order $n$, and consider the quotient $M_U(E, e) = (E \times U')/(\mathbb{Z}/n\mathbb{Z})$, where $U'$ is the unit disc, and the generator $1 \in \mathbb{Z}/n\mathbb{Z}$ acts by translation by $e$ on $E$, and by multiplication with the $n$-th root of unity on $U'$. Then the action has no fixed points, so that $M_U(E, e)$ is smooth, and we have a holomorphic map $M_U(E, e) \to U = U'/(\mathbb{Z}/n\mathbb{Z})$, where $U$ is again a disc. As we saw in the proof of Step 4 of Theorem 4.2, this local example is universal: whenever $M$ is minimal, all the varieties $M_U = M'_U/(\mathbb{Z}/n\mathbb{Z})$ are of this type.

### 4.3 Isotrivial elliptic fibrations

**Proposition 4.5**: Let $M$ be a non-Kähler compact complex surface, admitting an elliptic fibration $\pi : M \to S$. Denote by $C$ a general fiber of $M$. 

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Then there is a natural holomorphic action of $C$ on $M$, transitive on fibers and free on non-multiple fibers.

**Proof:** Two pairs of points $(a, b)$ and $(a_1, b_1)$ on a smooth genus 1 curve $T$ are called **rationally equivalent** if there exists a translation automorphism of $T$ mapping $(a, b)$ to $(a_1, b_1)$. Clearly, the set $\text{Jac}(T)$ of such pairs up to rational equivalence is a curve isomorphic to $T$, with a fixed point $(x, x)$, which we can consider as the group unit. We can consider $T$ as a torsor over the group $\text{Jac}(T)$. 

Consider the relative Jacobian $\text{Jac}(M)$ of $M$ over $B ([L])$, that is, a quotient of the product $M \times_B M$ by rational equivalence: two points $x, y \in M \times_B M$ are rationally equivalent if they can be connected by rational curves. Locally, we can interpret the relative Jacobian using the relative Albanese map (see Remark 4.4 and Step 4 of Theorem 4.2).

The map $\pi : \text{Jac}(M) \to S$ is a locally trivial fibration with a globally defined section, hence it is trivial. The natural fiberwise action of $\text{Jac}(M) = C \times S$ on $M$ defines the action on $M$, which is free and transitive on general fibers. This action is transitive on all fibers. To see this, consider the set

$\{(x, y, c) \in M \times S M \times S \text{Jac}(M), \quad (x, y) \sim c\}$

This set is complex analytic in $M \times_S M \times_S \text{Jac}(M)$. By Remmert’s proper mapping theorem, its projection $V$ to $M \times_S M$ is closed. Since $V$ contains all regular points of the projection $M \times_S M \to S$, one has $V = M \times_S M$.

**Definition 4.6:** Let $M$ be a complex manifold equipped with an action of a compact complex torus $T$, with the orbits of the same dimension. Assume that the quotient map $\pi : M \to S$ is well defined. Then $\pi : M \to S$ is called an **isotrivial toric fibration**, and an **isotrivial elliptic fibration** if $\dim_C T = 1$.

**Remark 4.7:** In some literature, “isotrivial elliptic fibration” is a fibration with all smooth fibers isomorphic to the same elliptic curve and the only singularities in multiple fibers. In [Bri2] and [Bri1], the same notion is called “a quasi-bundle”.

**Remark 4.8:** Let $\pi : M \to S$ be an isotrivial elliptic fibration over a curve

\footnotetext{2}{A torsor over a group $G$ is a set with a free and transitive $G$-action.}
classification of non-Kähler surfaces

S, with fiber C. We associate with π the following orbifold structure on S. For any multiple fiber \( R = \pi^{-1}(s) \) of π, consider the group \( \Gamma_R \subset C \), obtained as the kernel of the natural action of C on R. Taking the quotient \( M/\Gamma_R \), we obtain another fibration, which is locally trivial in a neighbourhood of s. A smooth section of this fibration in a neighbourhood of s gives a \( \Gamma_R \)-invariant multisection \( \tilde{U} \rightarrow U \) of \( \pi^{-1}(U) \rightarrow U \). Then U is obtained as a finite quotient of \( \tilde{U} \), and the lifting of π to \( \tilde{U} \) is locally trivial.

**Remark 4.9:** We obtained that any isotrivial elliptic bundle \( \pi : M \rightarrow S \) over a curve defines an orbifold structure on this curve, and π is locally trivial in this orbifold structure. The same argument would work for any base S regardless of dimension.

To simplify the notation and the arguments, we shall deal with the smooth orbifold fibrations in the same way as for the smooth ones, and use the standard terminology instead of adding “orbifold smooth” everywhere. Most of the standard results and constructions in smooth category are extended to the orbifold category in the usual way; for the only exception of importance to us, see below Remark 4.12.

The topology and geometry of isotrivial toric fibrations and principal toric fibrations was explored in some depth in [Hö]. For our present purposes we shall need the following theorem. Notice that the isotrivial toric fibration can be considered as a locally trivial principal toric fibration if we work in the orbifold category. Following [Hö], we define the Chern classes associated with isotrivial toric fibrations as follows.

**Theorem 4.10:** Let \( T(S) \) be the sheaf of \( T \)-valued smooth functions on S. Let \( \tilde{T} = \mathbb{R}^n \) the universal cover of T, and denote by \( \tilde{T}(S) \) the sheaf of smooth \( \tilde{T} \)-valued functions on S. Denote by \( \mathbb{Z}^n(S) \) the constant sheaf with fiber \( \mathbb{Z}^n \). Then the following exact sequence of sheaves

\[
0 \rightarrow \mathbb{Z}^n(S) \rightarrow \tilde{T}(S) \rightarrow T(S) \rightarrow 0
\]

gives an exact sequence in cohomology

\[
0 = H^1(S, \tilde{T}(S)) \rightarrow H^1(S, T(S)) \rightarrow H^2(S, \mathbb{Z}^n(S)) \rightarrow H^1(S, \tilde{T}(S)) = 0
\]

and a bijective correspondence between the set \( H^1(T(S)) \) of principal toric fibrations and the cohomology group \( H^2(\mathbb{Z}^n(S)) = H^2(S, \mathbb{Z})^n \).
Proof: Clear (see [Hö] for details).

**Definition 4.11:** Let $\pi: M \to S$ be an isotrivial toric fibration, with $T = \mathbb{R}^n/\mathbb{Z}^n$ its fiber. The $n$ cohomology classes associated to $\pi$ as in Theorem 4.10 are called the *Chern classes of the isotrivial toric fibration*.

**Remark 4.12:** In the case when $\pi: M \to S$ is an isotrivial elliptic fibration over a compact curve, one has $H^2(S, \mathbb{Z}) = \mathbb{Z}$. Therefore a toric fibration is uniquely determined by the vector $c_1(\pi) \in \mathbb{Z}^2 = H^2(S, \mathbb{Z})^2$. Recall that a vector $v$ in a lattice $\mathbb{Z}^n$ is called *primitive* if $v$ is not divisible by an integer $n > 1$. For any primitive vector $v$ in a lattice $\Lambda = \mathbb{Z}^2$, there exists $w \in \Lambda$ such that $\Lambda = \langle w, v \rangle$. Therefore, the Chern classes of an isotrivial elliptic fibration over a curve are determined (up to an automorphism of $\Lambda$) by the biggest $n \in \mathbb{Z}$ such that $c_1(\pi)$ is divisible by $n$. Using this observation, we shall understand $c_1(\pi)$ as a non-negative integer; it is equal to zero when $v = 0$ and equal to the largest integer divisor of $v$ when $v \neq 0$. Note that if $S$ is an orbifold, then the first Chern class is still perfectly well-defined and classifies isotrivial elliptic fibrations, but it is no longer necessarily true that $H^2(S, \mathbb{Z}) = \mathbb{Z}$ — it might also have some torsion.

Now we shall prove that an isotrivial elliptic fibration $\pi: M \to S$ over a curve with non-torsion $c_1(\pi)$ is always obtained as a quotient of the total space of an ample $\mathbb{C}^*$-bundle. We need the following preliminary topological statement.

**Claim 4.13:** Let $\pi: M \to S$ be an isotrivial elliptic fibration over a compact curve $S$, and $C$ its general fiber. Assume that $c_1(\pi)$ is not torsion. Then the rank of the natural map $H_1(C) \to H_1(M)$ is 1. Moreover, there exists a normal subgroup $G \subset \pi_1(M)$ with $\pi_1(M)/G = \mathbb{Z}$ such that the corresponding $\mathbb{Z}$-cover is infinite on $C$.

**Proof:** Consider the exact sequence

$$
\begin{align*}
H_2(S, \mathbb{Q}) &\xrightarrow{\delta} H_1(C, \mathbb{Q}) \xrightarrow{\psi} H_1(M, \mathbb{Q}) \to H_1(S, \mathbb{Q}) \to 0
\end{align*}
$$

obtained from the Leray spectral sequence of the fibration. It is easy to see that $\delta$ is dual to the Chern class of $\pi$. Therefore, the map $\psi: H_1(C, \mathbb{Q}) \to H_1(M, \mathbb{Q})$ has rank 1 when $c_1$ it not torsion and 2 when it is. Therefore, there ex-
ists an element \( v \in \pi_1(C) \) such that its image in \( \pi_1(M) \) has infinite order. Take a homomorphism \( H_1(M, \mathbb{Z}) \to \mathbb{Z} \) which is non-zero on \( \psi(H_1(C)) \).

Since \( H_1(M) = \frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]} \), the map \( v \) defines a group homomorphism \( \pi_1(M) \to \mathbb{Z} \) which is non-trivial on \( \pi_1(C) = H_1(C, \mathbb{Z}) \).

**Proposition 4.14:** Let \( \pi : M \to S \) be an isotrivial elliptic fibration over a compact curve \( S \), and \( C \) its general fiber. Consider a \( \mathbb{Z} \)-cover \( \varphi : \tilde{M} \to M \) such that \( \varphi^{-1}(C) \to C \) is an infinite cover. Then \( \tilde{M} \) is the total space of a principal \( \mathbb{C}^* \)-bundle associated with a line bundle \( L \) on \( M \). Moreover, the surface \( M \) is obtained as the quotient of the total space \( \text{Tot}(L^\natural) \) of non-zero vectors in \( L \) by a holomorphic automorphism \( q : S \to S \) acting on \( L \) equivariantly and linearly on all fibers. Finally, the automorphism \( q : S \to S \) has finite order, and \( |c_1(L)| \) is equal to \( c_1(\pi) \).

**Proof:** Since \( \varphi^{-1}(C) \to C \) is a \( \mathbb{Z} \)-cover, the space \( \tilde{M} \) is a \( \mathbb{C}^* \)-bundle over \( S \). Denote the associated vector bundle by \( L \). Then \( \tilde{M} = \text{Tot}(L^\natural) \), and \( M \) is obtained from \( \text{Tot}(L^\natural) \) as a quotient by an cyclic group of automorphisms generated by an automorphism \( \tilde{q} : \tilde{M} \to \tilde{M} \) commuting with the \( \mathbb{C}^* \)-action. Then \( \tilde{q} \) defines a holomorphic automorphism of \( S \). However, all automorphisms of \( \mathbb{C}^* \) commuting with the \( \mathbb{C}^* \)-action are given by a multiplication by a number. Therefore, \( \tilde{q} \) gives a holomorphic section \( \lambda \in \text{Hom}(L, q^*(L)) \). The fibers of \( \pi : \tilde{M} \to S \) are compact, hence \( q \) has to be of finite order.

The Chern class of \( L \) is obtained from the \( U(1) \)-bundle associated with \( L \), and this bundle is homotopy equivalent to \( \tilde{M} \), hence \( c_1(L) \) is equal to \( c_1 \) of the circle bundle associated with the \( \mathbb{C}^* \)-bundle \( \tilde{M} \to S \). However, \( c_1 \) of \( \tilde{M} \to S \) is by construction equal with \( \pm c_1(\pi) \) (the \( \pm \) sign is due to the ambiguity of the definition of \( c_1 \) of the toric fibration, see Remark 4.12).

### 4.4 Blanchard theorem and LCK structure on non-Kähler elliptic surfaces

The main result of the present section is the following theorem, proven at the end of this subsection.

**Theorem 4.15:** ([Bel, Theorem 1]) Let \( M \) be a compact, non-Kähler, minimal surface admitting an elliptic fibration. Then \( M \) is Vaisman.
We use Blanchard theorem (see [Bl] or [R]), which is applied to the present case as follows.

**Theorem 4.16:** Let \( \pi : M \to S \) be an isotrivial elliptic fibration. Then \( M \) is Kähler if and only if \( c_1(\pi) \) is torsion.

**Proof:** Suppose that \( c_1(\pi) \) is torsion, and consider the exact sequence

\[
0 \to H^1(S, \mathbb{Q}) \to H^1(M, \mathbb{Q}) \to H^1(C, \mathbb{Q}) \to H^2(S, \mathbb{Q})
\]

obtained from the Leray spectral sequence. As shown in the proof of Claim 4.13, one has \( \delta^* = 0 \) if and only if \( c_1(\pi) \) is torsion. However, if \( \delta^* \neq 0 \), the pullback \( \pi^*(\omega_S) \) of the volume form is exact, which is impossible when \( M \) is Kähler by Remark 1.4.

Assume now that \( c_1(\pi) \) is torsion, or equivalently, that \( \delta^* = 0 \). Using Proposition 4.14, we obtain \( M \) as a quotient of a total bundle \( \text{Tot}(L^\circ) \) by an automorphism acting on the base \( S \) as \( \varphi : S \to S \) and on the fibers as a constant endomorphism \( \lambda : L \to \varphi^*L \), \( |\lambda| \neq 0 \). Since \( c_1(L) = \pm c_1(\pi) \) is torsion, this bundle admits a flat Hermitian connection. Denote by \( \rho \) its monodromy action. Since \( \rho \) acts on the fibers of \( L \) isometrically, we can choose a \( \rho \)-invariant Kähler metric \( \omega_F \) on the fibers of \( L^\circ/\lambda \), which are elliptic curves.

Extending \( \omega_F \) to \( M \) using the flat Ehresmann connection on \( \pi : M \to S \), we obtain a closed, positive \((1,1)\)-form \( \hat{\omega}_F \) on \( M \) which is strictly positive on the fibers of \( \pi \); the sum \( \omega := \hat{\omega}_F + \pi^*(\omega_S) \) is a positive, closed, strictly positive almost everywhere \((1,1)\)-form. Then \( \omega^2 > 0 \), in contradiction with the negativity of the intersection form on \( H_{1,1}(M) \) (Proposition 2.10).

Let us prove Theorem 4.15 (see also [Va2, Theorem 3.5 and the subsequent Remark] for the regular case, and also [Vu2]). If \( M \) is non-Kähler, it is obtained as a quotient of \( \text{Tot}(L^\circ) \) by \( \lambda : L \to \varphi^*L \), where \( c_1(L) \neq 0 \) by Theorem 4.16. Replacing \( L \) by its dual bundle if necessary, we may assume that \( L \) is ample. Choose a Hermitian structure on \( L \) such that its curvature is a positive \((1,1)\)-form on \( S \), and let \( \psi \in C^\infty(\text{Tot}(L^\circ)) \) be the function \( \psi(v) = |v|^2 \).

We want the isomorphism \( \lambda : L \to \varphi^*L \) to have constant length. Since \( \varphi \) is of finite order, we can always represent the curvature \( \Theta \) of \( L \) by a \( \varphi \)-invariant form. Using the \( dd^c \)-lemma as usual, we choose a metric on \( L \) with curvature \( \Theta \). Then the curvature of \( \text{Hom}(L, \varphi^*L) \) is \( \varphi^*(\Theta) - \Theta = 0 \). Therefore, \( \text{Hom}(L, \varphi^*L) \) is a flat unitary bundle; any section \( \lambda \in \text{Hom}(L, \varphi^*L) \)
gives a holomorphic section of a flat unitary bundle, hence it is of constant length.

Then $d\bar{d}c(\psi)$ is a Kähler form on $\text{Tot}(L^\alpha)$ ([Bes, (15.19)]). Then $\lambda$ acts on $\text{Tot}(L^\alpha)$ by holomorphic homotheties, hence the quotient $M = \text{Tot}(L^\alpha)/\langle \lambda \rangle$ is LCK. Also, this quotient is Vaisman, because the standard action of $\mathbb{C}^*$ on $\text{Tot}(L^\alpha) = \tilde{M}$ is by holomorphic homotheties, [KO].

5 Class VII surfaces

In this section we prove the following theorem.

**Theorem 5.1:** Let $M$ be a non-Kähler compact minimal complex surface. Then $M$ is isotrivial elliptic or belongs to class VII (or both).

**Proof.** **Step 1:** As $b_1(M) > 0$, and $H_1(M, \mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$, there exists a subgroup of any given finite index $r$ in $\pi_1(M)$. Denote the corresponding cover by $\sigma : M_1 \rightarrow M$. Then $M_1$ is a compact manifold. Since $\chi(\mathcal{O}_M) = \frac{c_1^2 + c_2}{12}$ is expressed through the curvature, one has $\chi(\mathcal{O}_{M_1}) = d\chi(\mathcal{O}_M) \geq d$ for any $d$-sheeted cover $\sigma : M_1 \rightarrow M$. Unless $\chi(\mathcal{O}_M) = 0$, we can find a cover $M_1$ with $\chi(\mathcal{O}_{M_1}) < -3$ or $\chi(\mathcal{O}_{M_1}) > 3$. In the first case, $M_1$ is elliptic by Corollary 4.3. In the second case, the canonical bundle $K_{M_1}$ satisfies $\dim H^0(K_{M_1}) \geq 2$, hence it has sections which give a continuous family of divisors. Therefore, for $\chi(\mathcal{O}_M) \neq 0$, the finite cover $M_1$ has a continuous family of divisors. Then $M$ also has such a continuous family, and it is elliptic by Theorem 4.2.

**Step 2:** Assume that $b_1(M) = 3$, but $M$ is not elliptic. Then $\chi(\mathcal{O}_M) = 0$, hence $h^{1,1}(M) = 2$ and $h^{0,2}(M) = 1$, and the same is true for all non-ramified covers of $M$. We are going to prove that $M$ contains no curves.

First, we prove that $c_1(M)^2 = (K_M)^2 = 0$. Otherwise $c_1(M)^2 < 0$, so $K_M$ is non-trivial, and since $h^{0,2}(M) = 1$, we have $K_M = \mathcal{O}(D)$ for some effective divisor $D = \sum_i a_i D_i$. If for some $i$ we have $(D_i)^2 < 0$ and $(K_M \cdot D_i) < 0$, then $D$ is a $(-1)$-curve, and this is not possible since $M$ is minimal. If $(D_i)^2 = 0$, then $D_i$ is homologous to 0 by Proposition 2.10. Thus in any case, $(K_M \cdot D_i) = 0$, and then $(K_M)^2 = \sum_i a_i (K_M, D_i) = 0$, a contradiction.
Note that $c_1(M)^2 = 0$ implies that

$$0 = \chi(\mathcal{O}_M) = \frac{c_1(M)^2 + c_2(M)}{12} = \frac{c_2(M)}{12} = \frac{\chi(M)}{12},$$

hence $b_2(M) = b_1(M) + b_3(M) - b_0(M) - b_4(M) = 6 - 2 = 4$. Moreover, the same holds for any unramified cover $M_1$ of $M$.

Now assume that $M$ contains an irreducible curve $C$. Then by the adjunction formula, its arithmetic genus is $p_a(C) = (C)^2 + (K_M \cdot C) + 1$, and since $(C)^2 \leq 0$ and $(K_M)^2 = 0$, we have $(K_M \cdot C) = 0$, hence $p_a(C) \leq 1$. Then $C$ is either smooth rational, or rational with a single node, or rational with a single cusp, or smooth elliptic, and in any case, $b_1(C) \leq 2$. Since $b_1(M) = 3 > b_1(C)$, we have an unramified cover $M_1 \to M$ of any degree $d$ that splits over $C$, thus contains non-intersecting curves $C_1, \ldots, C_d$ with the same self-intersection $l = (C_i)^1$, $i = 1, \ldots, d$.

Then if $l < 0$, the classes of the curves $C_i$ span a $d$-dimensional subspace in $H^2(M_1, \mathbb{Q})$, so that $d \leq b_1(M_1) = 4$, and this is a contradiction since $d$ was arbitrary.

Otherwise $l = 0$, and all the curves $C_i$ are homologous to 0. Let $D$ be their union, and consider the short exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_M_1 \to \bigoplus_i \mathcal{O}_{C_i} \to 0. \quad (5.1)$$

The corresponding long exact sequence

$$C = H^0(\mathcal{O}_{M_1}) \to C^d = \bigoplus_i H^0(\mathcal{O}_{C_i}) \to H^1(\mathcal{O}(-D)).$$

shows that that dim $H^1(M_1, L) \geq d - 1$, where $L := \mathcal{O}(-D)$. However, $\chi(L) = \chi(\mathcal{O}_{M_1}) = 0$ because $c_1(L) = [D] = 0$, and for any line bundle $L'$ on $M_1$, we have dim $H^0(M_1, L') \leq 1$ — otherwise $M_1$, hence also $M$ carries a non-trivial family of divisors and is elliptic by Theorem 4.2. Thus dim $H^0(M_1, L)$ and dim $H^2(M_1, L) = \dim H^0(M_1, K_{M_1} \otimes L^*)$ are at most 1, and $d \leq 3$. This is again a contradiction.

**Step 3:** We can prove now that all surfaces with $b_1(M) = 3$ and $\chi(\mathcal{O}_M) = 0$ are elliptic. Arguing by absurd, we may assume $M$ is non-elliptic, but in this case $M$ has no complex curves (Step 2). Consider the space $W \subset H^1(M)$ generated by holomorphic and antiholomorphic forms. Let $\alpha \in \Lambda^{1,0}(M)$ be
a holomorphic 1-form generating the space $H^{1,0}(M)$. Fix $x \in M$ and define the Albanese map by taking $y \in M$ to $\int_\gamma \alpha$, where $\gamma$ is a path connecting $x$ to $y$. This map depends on the choice of the path $\gamma$, hence it gives a map $\text{Alb} : M \to \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is the period lattice of $\alpha$, that is, the set $\{ \int_v \alpha \mid v \in H_1(M, \mathbb{Z}) \}$. By Corollary 2.15, $W$ is rational. Therefore, the integral $\int_v \alpha$ vanishes on one of the generators in $H_1(M, \mathbb{Z})$, and $\Lambda$ is a rank 2 lattice in $\mathbb{C}$. It is easy to see that it is discrete. Then the Albanese map gives a fibration $M \to \mathbb{C}/\Lambda$, and $M$ contains a continuous family of curves, hence it is elliptic.

**Step 4:** It remains to show that $M$ is of class VII or elliptic if $\chi(\mathcal{O}_M) = 0$. When $b_1(M) = 3$, the surface $M$ is elliptic (Step 3), and the same holds if $b_1(M) \geq 5$ by Corollary 4.3. Therefore, we may assume that $b_1(M) = h^{0,1}(M) = 1$ and the same is true for all covers $M_1$ of $M$, ramified or unramified. Unless $M$ is elliptic, we have $\chi(\mathcal{O}_M) = 0$, hence $h^{0,2}(M) = 0$. To prove that $M$ is class VII one has to check that $H^0(K^n_M) = 0$ for any $n > 0$.

If $K^n_M$ admits a non-zero section, then $K^n_{M_1}$ admits a non-zero section for some finite ramified cover $\sigma : M_1 \to M$. Indeed, given a section $\alpha$ of $K^n_{M_1}$, with zero divisor $C$, locally around a point $x \in M\setminus C$, one can take its $n$-th degree root $\sqrt[n]{\alpha}$ and obtain a section of $K_M$. The sheaf of such sections is finite and locally constant on $M\setminus C$, hence it becomes trivial after passing to a finite cover $\widehat{M}\setminus C \to M\setminus C$. In a neighbourhood $U$ of $x \in C$, we interpret $\sqrt[n]{\alpha}$ as a multivalued function with values in $K_M$, which can be trivialized in $U$. The graph of $\sqrt[n]{\alpha} \subset U \times C$ can be glued to $\widehat{M}\setminus C$, giving a ramified cover $\sigma : M_1 \to M$ and a section of $K_{M_1}$. Then $h^{0,2}(M_1) \neq 0$, hence $M_1$ is elliptic (Step 3). Therefore, $M = \sigma(M_1)$ contains a continuous family of holomorphic curves; it is elliptic by Theorem 4.2. Isotriviality of the elliptic fibration follows from Proposition 4.5.

6 Brunella’s theorem: all Kato surfaces are LCK

The Kato surfaces are also called “Global spherical shell surfaces”. By definition a Kato surface is a surface $M$ which possesses a **global spherical shell**, that is, an open subset $U \subset M$ which is biholomorphic to a neighbour-
hood of $S^3$ in $\mathbb{C}^2$ and such that $M \setminus U$ is connected. In this section we prove Brunella’s theorem, showing that all Kato surfaces are locally conformally Kähler.

For the original definition and early works on Kato surfaces see [Ka1, Ka2, Ka3, Dl1], and for Brunella’s original proof, [Bru1] and [Bru2].

It is not hard to see that all Kato surfaces are deformations of a blow-up of a Hopf surface. To see this, we consider the following explicit construction of Kato surfaces.

Let $M$ be a Kato surface, and $S \subset U \subset M$ the corresponding 3-sphere. Consider the map $\chi : \pi(M) \to \mathbb{Z}$ mapping a path $\gamma$ to the intersection index $\gamma \cap S$. Clearly, $\chi$ is a group homomorphism. Denote the corresponding $\mathbb{Z}$-cover by $\tilde{M}$, and the preimages of $S$ in $\tilde{M}$ by $S_i$, $i \in \mathbb{Z}$ (these preimages can be enumerated, because the deck transform group $\mathbb{Z}$ acts on the set of preimages of $S$ freely and transitively). Denote by $M_i$ the subset of $\tilde{M}$ situated between $S_i$ and $S_{i-1}$. Clearly, each $M_i$ is a fundamental domain of the deck transform action.

Each $M_i$ has two boundary components, with $S_i$ pseudo-convex and $S_{i-1}$ pseudo-concave. Gluing a ball $B$ to $S_{i-1}$, we obtain a manifold $\tilde{M}_i = B \bigsqcup_{S_{i-1}} M_i$ which is compact and has strictly pseudoconvex boundary. By Grauert’s solution of Levi problem ([Gra, Section 2, Theorem 1]), $\tilde{M}_i$ is holomorphically convex. Using Remmert’s reduction theorem [Re], we obtain a proper map with connected fibers $p : \tilde{M}_i \to X$, where $X$ is Stein. Since the
neighbourhood of the boundary of $X$ is biholomorphic to a neighbourhood of $S^3 \subset B$. Hartogs theorem implies that $X$ is in fact biholomorphic to $B$. Then $p : \hat{M}_i \to X$ is a bimeromorphic, holomorphic map to a ball, and is obtained by a sequence of a blow-ups.

Following Dloussky [Dl1], we define the **Kato data** on a closed ball $B \subset \mathbb{C}^2$ as a bimeromorphic, holomorphic map $\hat{B} \to B$, together with an open subset $B_0 \subset \hat{B}$ and a biholomorphism from $B_0$ to an open ball. Then the complement $\hat{B} \setminus B_0$ has two smooth boundary components, which are both isomorphic to $S^3 \subset B$ and can be glued together to obtain a compact complex surface.

We have just proved the following theorem, originally due to Ma. Kato. ([Ka1]).

**Theorem 6.1:** Let $M$ be a Kato surface, and $S \subset M$ its global spherical shell. Then $M$ can be obtained from the Kato data by gluing two boundary components of $\hat{B} \setminus B_0$ as above.

**Remark 6.2:** Using the same arguments, it is possible to show that the Kato surface $M$ is not minimal unless $\hat{B}$ is obtained by blowing up a point $x_0 \in B$, then blowing up points which lie on exceptional divisors ([Dl2]). Since a blow-up of an LCK manifold is LCK, we can always assume that $M$ is minimal and $\hat{B}$ is obtained from $B$ by successive blow-ups in $0 \in B$.

**Remark 6.3:** Suppose that the Kato data satisfy the assumptions in Remark 6.2. Then in the construction of the Kato surface $M$ from the Kato data, we can always replace the open ball $B$ of radius 1 by a ball $B(r) \subset B$ of radius $r$ and the ball $B_0$ by its image $\Psi(B(r))$. Clearly, the resulting Kato surface is biholomorphic to $M$. Similarly, the ball $B$ can be replaced by any holomorphically convex domain $U \subset \mathbb{C}^2$ with smooth boundary containing the origin 0.

**Definition 6.4:** Choose $\varepsilon > 0$, and let $\max_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ be a smooth, convex function, monotonous in both variables, which satisfies $\max_\varepsilon(x, y) = \max(x, y)$ whenever $|x - y| > \varepsilon$. Then $\max_\varepsilon$ is called a **regularized maximum** ([D1]). It is easy to see that the regularized maximum of two plurisubharmonic functions is again plurisubharmonic. This construction allows one to “glue” plurisubharmonic functions and the corresponding Kähler metrics.
Now we can prove Brunella’s theorem.

**Theorem 6.5:** Let $M$ be a Kato surface. Then $M$ is locally conformally Kähler.

**Proof.**

**Step 1:** Let $\pi : \hat{B} \to B$ and $B_0 \subset \hat{B}$ be the Kato data. Denote by $\Psi : B \to B_0$ the corresponding biholomorphic equivalence. Choose a Kähler metric $\hat{\omega}$ on $\hat{B}$.

Brunella’s theorem is proved by finding a metric $\hat{\omega}$ on $\hat{B}$ with the following automorphic condition. Consider the space $\tilde{M}$ obtained by gluing $\mathbb{Z}$ copies of $\hat{B} \setminus B_0 = M_i$ as above. A Kähler metric $\tilde{\omega}$ on $\tilde{M}$ is called $\mathbb{Z}$-automorphic if the deck transform group mapping $M_i$ to $M_j$ acts on $(\tilde{M}, \tilde{\omega})$ by homotheties.

To obtain such a form we need to find a Kähler form $\hat{\omega}$ on $\hat{B}$ such that $\hat{\omega}|_{B_0}$ is equal to $\Psi^*\hat{\omega}$ in a neighbourhood of the boundary of $B_0$. If this is true, $\Psi$ acts by homotheties in a neighbourhood of $S$. Then the restriction of $\hat{\omega}$ to $\hat{B} \setminus B_0 = M_i$ can be extended to a $\mathbb{Z}$-automorphic Kähler form on $\tilde{M} = \bigcup_{i \in \mathbb{Z}} M_i$.

This is the strategy we follow, except that we replace $B$ by another strictly pseudoconvex domain as in Remark 6.3.

**Step 2:** Using the local $dd^c$-lemma, we can find a smooth function $\varphi$ on $B_0$ such that $dd^c\varphi = \hat{\omega}|_{B_0}$. Adding an appropriate plurisubharmonic function if necessary, we may assume that $\varphi$ reaches its minimum in an interior point $x \in B_0$. Adding an appropriate constant, we can assume that $U_0 := \varphi^{-1}(-\infty, 0)$ has compact closure with smooth strictly pseudoconvex boundary. Let $U$ be the closure of $\Psi^{-1}(U_0)$, and $\hat{U}$ the preimage of $U$ under the bimeromorphic contraction $\hat{B} \to B$. We obtain $M$ by gluing two boundary components of $\hat{U} \setminus U_0$ as in Remark 6.3.

**Step 3:** The map $\pi : \hat{U} \to U$ is a proper holomorphic map of manifolds of the same dimension. Therefore, the pushforward of a positive $(p,p)$-form is a positive $(p,p)$-current.

Let $\pi_*\hat{\omega}$ be the pushforward of $\hat{\omega}$, considered as a current on $U$. Using the $dd^c$-lemma for currents, we obtain $\pi_*\hat{\omega} = dd^cf$, where $f$ is a plurisubharmonic function on $U$ which is smooth outside of the singularities of $\pi$.

Denote by $R$ the boundary of $U \subset \mathbb{C}^2$. Then $f$ is smooth and strictly...
plurisubharmonic in a neighbourhood of $R$. Another plurisubharmonic function in a neighbourhood of $R$ is obtained by taking $f_1 := (\Psi^{-1})^* \varphi$, where $\varphi$ is the Kähler potential on $U_0$ constructed in Step 2.

Rescaling $f$ if necessary and adding a constant, we may assume that $-\varepsilon < f \big|_R < 0$ and $|df|_R \ll \varepsilon$. Let $A$ be a sufficiently big positive number, and $0 < \delta \ll \varepsilon$. Then the regularized maximum $\max_{\delta} f$ and $Af_1$ is equal to $Af_1$ in a very small neighbourhood of $R$ (because $f$ is negative on $R$ and $f_1 = 0$ on $R$), and equal to $f$ in a neighbourhood $V$ of $R_\varepsilon := Af_1^{-1}(-2\varepsilon)$ because $|df| \ll A|df_1|$ and as $Af_1$ goes to $-2\varepsilon$, $f$ does not go below $-\varepsilon$.

Replacing $\hat{\omega}$ by $dd^c \max_{\delta}(f, f_1)$ on the annulus between $R$ and $R_{2\varepsilon}$, we obtain a Kähler form $\hat{\omega}_1$. Since $\max_{\delta}(f, f_1) = f_1$ in a neighbourhood of $R$, the map $\Psi : (U, \hat{\omega}_1) \rightarrow (U_0, \hat{\omega}_1)$ acts by homothety mapping a neighbourhood of $R$ with the metric $\hat{\omega}_1$ isometrically to a neighbourhood of $\Psi(R)$ with the metric $A\hat{\omega}_1$ (however, $A\hat{\omega}_1 = A\hat{\omega}$ outside the annulus bounded by $R$ and $R_{2\varepsilon}$). We have constructed an LCK metric on any Kato surface.

**Remark 6.6:** The argument used in the proof of Brunella’s theorem is valid for any open ball in $\mathbb{C}^n$, giving a generalization of Kato manifolds to arbitrary dimension. These manifolds, were explored in [IOP]. By Brunella’s theorem, they also admit an LCK structure.

**Remark 6.7:** Prior to publishing his general result in [Bru2], M. Brunella constructed examples of LCK metrics on Enoki surfaces ([Bru1]), which are special cases of class VII surfaces admitting a global spherical shell. Even before, LCK metrics on some Kato surfaces (hyperbolic Inoue and parabolic Inoue) also appeared as anti-self-dual bihermitian metrics, obtained by twistor methods in the work of A. Fujiki and M. Pontecorvo, see [FP1, FP2]. In [FP3] possible values of the Lee class (cohomology class of the Lee form $\theta \in \Lambda^1(M)$) of these LCK structures is discussed; see also [AD]. As M. Pontecorvo pointed out, [Po], it is implicitly shown there that these Lee classes are different from the ones constructed by Brunella.

**Acknowledgments:** We are grateful to Georges Dloussky and Matei Toma for a careful reading of an early version of this paper and to Eduardo Esteves for an interesting discussion about elliptic surfaces. Stefan Nemirovski and Iku Nakamura helped with reference and corrections to the first version of this paper. Multiple thanks to Dmitry Kaledin and the anonymous referee for many insightful comments and useful suggestions which improved many
proofs.

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