Nambu brackets, Chern-Simons theories, quantum curves and M2-branes

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Abstract

A non-technical review on recent progress in M2-branes is given. Especially we focus on (1) historical progress in searching for the worldvolume theory of M2-branes using Nambu brackets and supersymmetric Chern-Simons theories, (2) studies of the ABJM matrix model obtained from localization using various techniques and (3) new insights of quantum curves derived from the studies of the ABJM matrix model.
1 Introduction

There have been a lot of important progress on Nambu dynamics. Here we give a non-technical review on how the idea of Nambu brackets helps in recent studies of M2-branes and what new insights on M2-branes are obtained from these studies. In this short review, we do not intend to provide a full list of topics and references, but only focus on some selected ones.

String theory is expected to be the theory of everything unifying all interactions in the nature. From the perturbative studies, it was known that there are five vacua in ten-dimensional spacetime, IIA, IIB, I, heterotic SO(32), heterotic $E_8 \times E_8$. However, a new dimension appears in the non-perturbative regime, which leads to the eleven-dimensional M-theory [1].

“M” in M-theory contains many ideas of string theory. Since M-theory unifies the five perturbative vacua of string theory, it was expected to be the mother theory of the nature. Despite its importance, the theory remains mysterious, since the theory is only defined non-perturbatively. One of crucial understandings is that the low-energy effective theory is the eleven-dimensional supergravity. Using the supergravity we can construct two types of solutions preserving half of supersymmetries. These solutions expand spatially in two dimensions and in five dimensions respectively and are electro-magnetic dual to each other. They are known as M2-branes and M5-branes, as generalizations from membranes.

As D-branes in string theory, M2-branes and M5-branes are considered to have dynamical excitations instead of just hyperplanes in the eleven-dimensional spacetime. For this reason, the branes should be described by worldvolume theories, such as field theories. Depending on the field theories, degrees of freedom behave differently when multiple branes cumulate. If we utilize the proposed AdS/CFT correspondence, the correspondence between field theories and gravity theories, the degrees of freedom for the field theories are calculated from the corresponding gravity theories [2].

We can study entropies by applying the Bekenstein-Hawking entropy formula (with numerical factors dropped)

$$S = \frac{A}{G},$$

(1.1)

to the corresponding gravity theories, which shows that the entropy of the system $S$ is given by the horizon area $A$ in the units of the Newton constant $G$. The horizon area $A$ can be calculated by solving the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = GT_{\mu\nu},$$

(1.2)

where the matter term on the right-hand side shows that the horizon area $A$ is always a function of the product of the number of branes $N$, the Newton constant $G$ and the brane
tension $T$: $A = A(NGT)$. Although both $NGT$ in the numerator and $G$ in the denominator are dimensionful parameters

$$G_{10} = l_s^3 g_s^2, \quad G_{11} = l_p^9, \quad T_{D3} = 1/(l_s^4 g_s), \quad T_{M2} = 1/l_p^3, \quad T_{M5} = 1/l_p^6,$$

the combined entropy $S = A(NGT)/G$ is dimensionless. Hence, the degrees of freedom for these branes follow respectively the power laws [2]

$$S_{D3} = N^2, \quad S_{M2} = N^{3/2}, \quad S_{M5} = N^3. \quad (1.4)$$

Since D-branes are defined to be planes on which open strings end, the excitations should be described by matrices with two indices corresponding to the two endpoints of open strings. For this reason, it is natural that the degrees of freedom follow the square law $N^2$. However, the interpretation of the $N^{3/2}$ law for M2-branes and the $N^3$ law for M5-branes remains mysterious. Furthermore, although the worldvolume theory for D-branes is known to be the supersymmetric Yang-Mills theory, those for M2-branes and M5-branes were not known. If these worldvolumes are described by gauge theories using Lie algebras, we may end up with the same $N^2$ law for degrees of freedom as D-branes, which apparently needs clarifications.

Here we will review recent progress on the worldvolume theory of M2-branes and further insights from the analysis of the worldvolume theories.

## 2 Chern-Simons theories and Nambu brackets

As explained in the introduction, although M-theory is expected to unify five vacua of string theories and considered to the mother theory of the nature, it is a mysterious theory. It is desirable to establish the worldvolume theory to describe M2-branes. In searching for the worldvolume theory, Chern-Simons theories and Nambu brackets play important roles.

Let us start with discussions on the worldvolume theory of M2-branes from the viewpoint of symmetries and gauge theories. Since M2-branes are $2 + 1$ dimensional dynamical objects residing in the $10 + 1$ dimensional spacetime, the translational invariance is broken and correspondingly eight spin-0 Nambu-Goldstone bosons appear. Simultaneously, the supersymmetry is also broken, which gives eight spin-$1/2$ Nambu-Goldstone fermions. These Nambu-Goldstone particles are balanced and consistent with the supersymmetry. However, if we try to construct a gauge theory by adding the spin-1 gauge field, the matter contents are not consistent with the supersymmetry any more.

An interesting idea was proposed by Schwarz at an early stage [3]. Recalling that the Chern-Simons theory in three-dimensional spacetime does not add any dynamical degrees
of freedom, it was proposed to regard this theory as a seed and try to supersymmetrize it. In the end, if we can supersymmetrize it to the maximally supersymmetric $\mathcal{N} = 8$ theory containing eight bosons and eight fermions, this is nothing but the field theory that describes the worldvolume of M2-branes since it contains the same matter contents and preserves the same symmetries as M2-branes. This seems a nice idea, but unfortunately we quickly face a difficulty. Before this proposal, there had already been works on supersymmetric Chern-Simons theories [4]. It was known that we can only construct supersymmetric Chern-Simons theories up to $\mathcal{N} = 3$ for general gauge groups and general representations of matters.

Another interesting suggestion was made, inspired by the Nambu brackets in Nambu dynamics. Note that the Nambu brackets defined from three physical quantities

$$\{F, G, H\} = \epsilon^{\mu\nu\rho}(\partial_\mu F)(\partial_\nu G)(\partial_\rho H),$$

satisfying the fundamental identity

$$\{A, B, \{F, G, H\}\} = \{\{A, B, F\}, G, H\} + \{F, \{A, B, G\}, H\} + \{F, G, \{A, B, H\}\}. \quad (2.2)$$

Similarly, let us define a three-algebra

$$[T^\mu, T^\nu, T^\rho] = f^{\mu\nu\rho} T^\sigma,$$

by requiring the fundamental identity of the same form

$$[A, B, [F, G, H]] = [[A, B, F], G, H] + [F, [A, B, G], H] + [F, G, [A, B, H]]. \quad (2.4)$$

According to Bagger-Lambert and Gustavsson [5–7], it turns out that, with the three-algebra, we can construct an action

$$S = \int d^3 x \left( -\frac{1}{2} D^\mu X^I D_\mu X_I + \frac{i}{2} \bar{\Psi} \Gamma^\mu D_\mu \Psi \\
+ \frac{i}{4} \bar{\Psi} \Gamma_{IJ}[X^I, X^J, \Psi] - \frac{1}{12} [X^I, X^J, X^K][X_I, X_J, X_K] + \cdots \right), \quad (2.5)$$

that enjoys both supersymmetry and gauge symmetry. This is another concrete interesting proposal, but again we face another difficulty. It was proved [8, 9] that there are only trivial three-algebras if we require the algebras to be positive-definite and finite-dimensional in solving the fundamental identity.

Returning to supersymmetric Chern-Simons theories, it was pointed out [10, 11] that the supersymmetry can be extended if the gauge groups and the representations are specially chosen. In particular [11–13], if we prepare the gauge group $U(N_1) \times U(N_2)$ and introduce two pairs of matters in the bifundamental representation with appropriate interactions, the
supersymmetry of the Chern-Simons theory enhances to $\mathcal{N} = 6$, which is called the ABJM theory. This theory describes the worldvolume of M2-branes on a background $\mathbb{C}^4/\mathbb{Z}_k$. If $N_1$ and $N_2$ are equal, the M2-branes are full-fledged, otherwise they are “fractional” [12].

This interpretation is derived from brane configurations [11,12]. The supersymmetry $\mathcal{N} = 6$ is special and has attracted much attention from the beginning. It was known [14] that the brane configuration of D3-branes on a circle in IIB string theory, with an NS5-brane and a $(1,k)$5-brane perpendicular to them and tilted relatively by an angle depending on the level $k$, enjoys the supersymmetry $\mathcal{N} = 6$. When we move to IIA string theory by T-duality and lift to M-theory, we find that the two 5-branes change to Taub-NUT spaces and combine into a background $\mathbb{C}^4/\mathbb{Z}_k$. Here when the numbers of D3-branes differ in different intervals, they are not full-fledged D2-branes in the T-duality but fractional.

This development settles the question of describing the worldvolume of M2-branes, though we keep in mind that there may be rooms for improving the ABJM theory. This theory achieves the supersymmetry $\mathcal{N} = 6$, but there should be the supersymmetry $\mathcal{N} = 8$ on the flat space $\mathbb{C}^4$ (or $k = 1$). The remaining supersymmetry is expected to be realized only non-perturbatively. It is worthwhile to ask whether three-algebras may give us hints to improve the situation. Let me introduce a couple of trials. See [15] for complete reviews on various interesting aspects.

A new Higgs mechanism leading to D2-branes with compactification was proposed in [16], where the assumption of positive-definiteness is lifted and a three-algebra of the Minkowski signature (with one negative sign) was constructed. Since we already allow one negative sign, it may not be unnatural to consider more negative signs. In fact, the partition function of the ABJM theory equips the superalgebraic structure of $U(N_1|N_2)$, which has multiple negative signs in the Killing form. Following these ideas, in [17] a three-algebra based on the superalgebra $U(N_1|N_2)$ was constructed. Unfortunately, the algebraic structure is basically equivalent to the Minkowski case and may not be particularly interesting.

Also, a modified three-algebra [18]

$$[T^a, T^b, T^c] = f^{abc} T^d,$$

(2.6)
equivalent to the $\mathcal{N} = 6$ case was constructed by introducing conjugate indices with bars, where the Chern-Simons terms in the Lagrangian are given by

$$L_{CS} = f^{a'b'd} A_{a'b'} A_{d'a'\bar{a}} + \frac{2}{3} f^{a'd} A_{a'd'} A_{d'\bar{a}c} A_{c'\bar{c}} A_{c'\bar{c}e}. $$

(2.7)

Since we already have the ABJM theory, this alone may not be very surprising. However, compared with the standard expression of the Chern-Simons terms, it is obvious that the
structure constant of the Lie algebra is constructed from two structure constants of the three-
algebra. Indeed, it is shown in appendix that, given a three-algebra, we can always construct
a Lie algebra, which implies that three-algebras are more fundamental than Lie algebras. We
hope that these attempts provide a deeper understanding of M2-branes.

3 Matrix models

So far we have reviewed that the worldvolume of M2-branes is described by the ABJM theory.
We hope to understand M2-branes from the ABJM theory. Let us ask questions whether we
can reproduce the characteristic behavior of $N^{3\over 2}$ for the degrees of freedom (1.4) and gain
more insights on M2-branes.

For this purpose, the partition function of the ABJM theory and vacuum expectation val-
ues of supersymmetric Wilson loop operators on $S^3$ are studied. These correlation functions
are originally defined by infinite-dimensional path integrals. Due to the supersymmetry, the
contributions from bosons and fermions cancel each other, which results in finite-dimensional
multiple integrals. Generally, for the half-BPS supersymmetric Wilson loop in the represen-
tation $R$, the result is given by [19]

$$
\langle s_R | e^\mu | e^{\nu} \rangle = \frac{i^{-\frac{1}{2}}(N_1^2 - N_2^2)}{N_1! N_2!} \int \frac{d\mu_m}{2\pi} \frac{d\nu_n}{2\pi} e^{i k (\sum_{m=1}^{N_1} \mu_m^2 - \sum_{n=1}^{N_2} \nu_n^2)} \times \frac{\prod_{m<n}^{N_1} (2 \sinh \frac{\mu_m - \mu_n}{2})}{\prod_{m=1}^{N_1} (2 \cosh \frac{\mu_m}{2})^2} \times \frac{\prod_{n<n'}^{N_2} (2 \sinh \frac{\nu_n - \nu_n'}{2})}{\prod_{n=1}^{N_2} (2 \cosh \frac{\nu_n}{2})^2} s_R (e^\mu | e^{\nu}),
$$

(3.1)

(with $s_R (e^\mu | e^{\nu})$ being the super Schur polynomial), which is called the ABJM matrix model.
Though the detailed expression is not necessary for remaining parts of this review, we stress
that many nice properties of the mysterious M2-branes such as the integrability [20–26] and
the open-closed duality [27,28] come exactly from this expression.

Hereafter, we consider the partition function without insertions of Wilson loops or frac-
tional branes, $Z_k(N) = \langle 1 \rangle_k (N|N)$, which is a function of only the rank $N$ and the level
$k$. Since the free energy $F_k(N) = \log Z_k(N)$ indicates the degrees of freedom, we expect
$F_k(N) \sim N^{3\over 2}$ in the large $N$ limit from the gravity analysis (1.4). Though the derivation is
not straightforward, this is the case [29]. The free energy was studied in a standard technique
for matrix models called the ’t Hooft expansion by taking the large $N$ limit with the ’t Hooft
coupling constant $\lambda = N/k$ fixed. After substituting $\lambda = N/k$ into the result $F_k(N) = \lambda^{-{3\over 2}} N^2$, we
obtain $F_k(N) = k^{-{3\over 2}} N^{3\over 2}$. 
This analysis is, however, not completely satisfactory. In the 't Hooft expansion with the level \( k \) being large, we detect a regime compatible with the string worldsheet picture. In fact, in this limit, one dimension of the M-theory background \( \mathbb{C}^4/\mathbb{Z}_k \) is compactified effectively and reduces to \( \mathbb{CP}^3 \) in string theory. If one is interested in the M-theory regime, one should separate the rank \( N \) and the level \( k \) in taking the limit. Here we have obtained the \( N^{3/2} \) behavior for the degrees of freedom by studying in the 't Hooft expansion and substituting the 't Hooft coupling \( \lambda \) afterwards. However, it is generally invalid to connect two regimes simply by substitutions, since there can be new contributions undetected in the original analysis.

Since we cannot overcome the difficulty immediately, let us continue in the 't Hooft expansion. Especially it was found [30] that we can sum up all the perturbation corrections to find the Airy function \( Z_k(N) = \text{Ai}(k^{1/3}N) \). The Airy function enjoys an integral representation

\[
\text{Ai}(N) = \int \frac{d\mu}{2\pi i} e^{\frac{1}{3}\mu^3 - N\mu},
\]

which resembles closely to the transformation between the canonical ensemble and the grand canonical ensemble

\[
Z_k(N) = \int \frac{d\mu}{2\pi i} e^{J_k(\mu) - N\mu}.
\]

Namely, for the partition function \( Z_k(N) \) of the ABJM theory, let us regard the rank \( N \) as the particle number and move to the grand canonical partition function

\[
\Xi_k(z) = \sum_{N=0}^{\infty} Z_k(N) z^N,
\]

by introducing the fugacity \( z \). After that, we define the grand potential \( J_k(\mu) = \log \Xi_k(z) \) of the chemical potential \( \mu = \log z \) by taking the logarithm (with modifications taking care of the \( 2\pi i \) periodicity of \( \mu \)). Then, the partition function is given inversely by (3.3). The Airy function is a well-known function with rather complicated behavior in the complex plane. Comparatively, the cubic function appears in the exponent in (3.2) is much simpler. By comparing (3.2) and (3.3), we are naturally led to the idea of studying in the grand canonical ensemble.

With this in mind, Marino and Putrov [31] proceeded to study in the grand canonical ensemble and found that the grand canonical partition function is given by a determinant

\[
\Xi_k(N) = \text{Det}(1 + z\hat{H}^{-1}).
\]

The spectral operator \( \hat{H} \) is constructed from the exponentiated canonical operators \((\hat{Q}, \hat{P}) = (e^{\hat{q}}, e^{\hat{p}})\) satisfying the canonical commutation relation \([\hat{q}, \hat{p}] = i\hbar\) by

\[
\hat{H} = (\hat{Q}^{1/2} + \hat{Q}^{-1/2})(\hat{P}^{1/2} + \hat{P}^{-1/2}),
\]

\[
(3.6)
\]
where the Planck constant is identified with the level $k$ by $\hbar = 2\pi k$. The determinant is taken over the whole Hilbert space of the canonical operators $\{\hat{q}, \hat{p}\}$. Interestingly, after suitable redefinitions, the spectral operator (3.6) takes the form of the defining equation of $\mathbb{P}^1 \times \mathbb{P}^1$, which will be important later.

In other words, though we have originally studied the partition function of M2-branes $Z_k(N)$ using the 't Hooft expansion, after we move to the grand canonical ensemble, the study reduces to the spectral problem of quantum mechanics. Once it reduces to the spectral problem, it can be solved by the WKB expansion, a standard method for quantum mechanics. Moreover, in the WKB expansion, the Planck constant $\hbar$ decouples from the particle number $N$, making it possible to detect the M-theory regime with fixed $k$ directly.

Indeed, the derivative of the grand potential in the large $\mu$ limit simply reduces to the classical area of the phase space,

$$
\frac{\partial [J_k(\mu)]}{\partial \mu} = \frac{\partial [\log \Xi_k(z)]}{\partial \log z} = \partial [\log \text{Det}(1 + z\hat{H}^{-1})] = \text{Tr}(1 + \hat{H}/z)^{-1}
$$

$$
= \frac{\text{Area}(H < z)}{2\pi \hbar} = C\mu^2,
$$

with a constant $C$ depending only on $k$. After integration, we obtain the grand potential

$$
J_k(\mu) = \frac{C}{3}\mu^3,
$$

which reproduces the Airy function immediately.

Moreover, from this viewpoint, the exact expression of the ABJM matrix model (3.1) may not be very important any longer. As long as the spectral operator $\hat{H}$ is a Laurent polynomial function of the exponentiated canonical operators $\{\hat{Q}, \hat{P}\}$, the boundaries of the phase space area in the classical limit should be linear functions of classical cousins of $\hat{q}, \hat{p}$ and $\mu$. For this reason, the area of the phase space is always quadratic (3.7), which gives the grand potential (3.8) after integration. In terms of the canonical ensemble, the partition function is always the Airy function $\text{Ai}(N)$ satisfying the $3/2$ power law, $N^{3/2}$.

Although we have explained that the $N^{3/2}$ law was mysterious for a long time in the introduction, once we adopt the grand canonical Fermi gas formalism [31], it is clear that the cubic behavior of the grand potential (3.8) leading to the Airy function is ubiquitous. If we consider the $N^{3/2}$ law and the Airy function to be characteristic of M2-branes, it may be natural to propose to describe the multiple M2-brane system by spectral operators instead of the original matrix model (3.1).

Finding out that the perturbative corrections sum up to the Airy function is actually just a starting point of the full exploration. We can head for non-perturbative effects of the
ABJM matrix model. There are two types of non-perturbative effects, the worldsheet instantons $e^{-\frac{\mu}{k}} \simeq e^{-\sqrt{N/k}}$ and the membrane instantons $e^{-\mu} \simeq e^{-\sqrt{Nk}}$ [32], which are interpreted respectively as fundamental strings wrapping on $\mathbb{C}P^1(\subset \mathbb{C}P^3)$ and D2-branes wrapping on $\mathbb{R}P^3(\subset \mathbb{C}P^3)$. Interestingly, it was found [33] that, although the coefficients of both the instantons are divergent at certain values of $k$, the divergences cancel each other completely and lead to a finite expression finally. Equipped with this structure, the non-perturbative effects were clarified. Finally it was found that, if we redefine the chemical potential $\mu$ suitably [34], the non-perturbative term can be expressed by the free energy of topological strings on the background of local $\mathbb{P}^1 \times \mathbb{P}^1$ [35]. More concretely, the worldsheet instantons are given by the free energy of topological strings while the membrane instantons are given by the derivative of its refinement.

Geometrically, $\mathbb{P}^1 \times \mathbb{P}^1$ is a curve of genus one with two cycles, the A-cycle and the B-cycle. We can define periods by integrating the holomorphic differential form over them. Then, it turns out that the A-period carries the information of the redefinition of the chemical potential $\mu$, while the B-period carries the information of the free energy of topological strings.

To summarize, on one hand, the grand canonical partition function of the ABJM matrix model leads to the determinant where the spectral operator is the defining equation of $\mathbb{P}^1 \times \mathbb{P}^1$. On the other hand, the grand potential is given by the free energy of topological strings on local $\mathbb{P}^1 \times \mathbb{P}^1$ after redefining the chemical potential. By removing the role of the ABJM matrix model, it was proposed generally [36] that the determinant of a spectral operator for a certain geometry is given by the free energy of topological strings on the corresponding geometry after redefining the chemical potential, which was named ST(spectral theories)/TS(topological strings) correspondence. Although this may seem just a mathematical correspondence, after we observe the natural description of M2-branes by spectral operators, the correspondence may provide a key framework to understand M2-branes.

As we have seen in this section, the grand canonical Fermi gas formalism has played a central role in understanding the ABJM matrix model. We know that, as long as the spectral operator $\hat{H}$ is given by a Laurent polynomial function of the exponentiated canonical operators $(\hat{Q}, \hat{P})$, the partition function will definitely sum up to the Airy function $\text{Ai}(N)$ leading to the $N^{\frac{3}{2}}$ behavior for the degrees of freedom. Also, the spectral operator continues to serve an important role in the analysis of non-perturbative corrections by regarding it as the defining equation of a curve. These ideas are collected by “quantum curves” [37, 38]. For this reason, we turn to study quantum curves in pursuing M2-branes.
4 Quantum curves

In this section, we explain two of our recent attempts in understanding M2-branes from the viewpoint of quantum curves. We hope that these attempts clarify non-perturbative aspects of M2-branes and unveil their mysterious structure.

4.1 ST/TS correspondence

Previously we have explained that the ST/TS correspondence may provide an important framework to study M2-branes. Despite its importance, even in the case of genus one, the explicit expressions on both sides were missing until recently. In [39], the correspondence for the case of genus one was clarified. The algebraic curve is known as the del Pezzo geometry and enjoys symmetries of the Weyl group of the exceptional Lie algebra. This structure of the Weyl group plays a crucial role in understanding the correspondence.

On the ST side, we can provide the spectral operators for all of the del Pezzo geometries. Although the del Pezzo geometries of lower ranks are toric and the quantizations are not difficult, the del Pezzo geometries of higher ranks remain mysterious. In particular, the quantum spectral operator of $E_8$ was notoriously difficult and absent also in mathematical literatures. By clarifying the group-theoretical structure, we can construct it explicitly from classical results in [40,41].

Also, on the TS side, with the group-theoretical structure clarified, we can fix some ambiguities in previous studies [42] and give the coefficients of the perturbative and non-perturbative effects explicitly [42–45]. Moreover, with the spectral operators on the ST side given explicitly, we can continue to study the A-periods extensively, which lead to the redefinitions of the chemical potential on the TS side.

4.2 Brane transitions beyond Hanany-Witten transitions

In addition to providing the ST/TS correspondence explicitly, we can return to brane configurations in string theory and study brane transitions from the viewpoint of quantum curves. It was known as the Hanany-Witten transition [46] that when a NS5-brane and a D5-brane move across each other, a D3-brane is generated. This brane transition was further extended to brane configurations with general 5-branes. The brane configuration leading to M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ in the ABJM theory is of the same type and enjoys the symmetries of the Hanany-Witten brane transition. Also, as we have explained previously, the del Pezzo geometry enjoys
the symmetries of the Weyl group of the exceptional Lie algebra. It is interesting to clarify
the relation of these two symmetries and find out implications on both sides.

This question was studied in [47, 48]. It was found that to translate quantum curves
into brane configurations using the Hanany-Witten transition [49, 50], we need to first fix the
interval which 5-branes do not move across. This is similar to the idea of fixing a reference
frame in mechanics or fixing a local patch in manifolds. Then, we can embed the Hanany-
Witten transition into the Weyl group and read off new brane transitions unknown previously
from the Weyl group. Surprisingly, the new brane transitions obtained from the Weyl group
seem to share a common feature and we can summarize it as a “local” rule for the brane
transitions [48], which, as the Hanany-Witten transition, refers to only the numbers of adjacent
D3-branes instead of the whole set of brane configurations. It is of course an interesting future
direction to clarify the scope of application and understand the rule from the field-theoretical
viewpoint.

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A Lie algebra from three algebra

In this appendix, we show that, given a modified (\(\mathcal{N} = 6\)) three-algebra [18], we can always
construct a Lie algebra.

The modified three-algebra is defined by the structure constants satisfying the fundamental
identity [18]

\[
\begin{align*}
&f^{ef\bar{g}}_b f^{\bar{c}\bar{d}}_d + f^{fe\bar{a}}_b f^{\bar{c}\bar{d}}_d + f^{e\bar{g}\bar{a}}_b f^{f\bar{c}\bar{d}}_d + f^{*\bar{g}\bar{a}}_b f^{f\bar{c}\bar{d}}_d + f^{*\bar{g}\bar{e}}_b f^{f\bar{c}\bar{d}}_d + f^{e\bar{g}\bar{e}}_b f^{f\bar{c}\bar{d}}_d = 0.
\end{align*}
\]  

(A.1)

Assuming the existence of the metric \(h^{\bar{a}\bar{b}}\), we can raise or lower the indices at will, where the
symmetry of the structure constant is given by

\[
\begin{align*}
&f^{a\bar{b}\bar{c}\bar{d}} = - f^{ba\bar{c}\bar{d}}, \quad f^{a\bar{b}\bar{c}\bar{d}} = f^{*\bar{c}\bar{d}ab}.
\end{align*}
\]  

(A.2)
Motivated by the Chern-Simons terms in (2.7), let us define the Killing form and the structure constant of the Lie algebra as

\[ G^{ac, bd} = f^{abcd}, \quad F^{ab, cd, ef} = -f^{ac, d}g f g e f d + f a e f g f g d c b. \]  

(A.3)

We regard the combination of two indices with and without bars separated by commas (such as \( a\bar{b} \)) as one adjoint index for the Lie algebra. It is easy to check that the structure constant satisfies an anti-symmetric property \( F^{ab, ef, cd} = -F^{ab, cd, ef} \). For other symmetries and identities, we need to study more carefully using the fundamental identity.

In fact, using (A.2) we can rewrite the fundamental identity (A.1) as

\[ f^{ef, gb} f^{eb, ad} + f^{ec, ab} f^{eg, bd} + f^{fb, ga} f^{ce, bd} + f^{eb, ag} f^{cf, bd} = 0. \]

By renaming the indices, we find \( f^{ca, bg} f^{eg, df} + f^{ac, dg} f^{eg, bf} + f^{ag, bd} f^{ec, gf} + f^{cg, db} f^{ea, gf} = 0 \), or

\[ f^{ca, bg} f^{ge, fd} + f^{ac, dg} f^{ge, fb} - f^{ce, fg} f^{ga, bd} - f^{ac, fg} f^{gc, db} = 0. \]  

(A.4)

This implies another anti-symmetric property of the structure constant \( F^{cd, ab, ef} = -F^{ab, cd, ef} \). Combining this with the previous symmetry, we find the total anti-symmetric property of the structure constant of the Lie algebra.

The Jacobi identity of the Lie algebra is proven similarly. For the notational simplicity, we express the two indices for the Lie algebra with the same characters such as \( a\bar{a} \). Since we assume the Killing form is given by (A.3),

\[ f^{af, \bar{a}d} = -f^{af, \bar{d}f} = -G^{ag, \bar{d}f}, \quad f^{df, \bar{g}a} = -f^{dg, \bar{g}a} = -G^{dg, \bar{f}f}, \]  

(A.5)

we obtain

\[ f^{af, \bar{a}d} G_{ff, e\bar{e}} = -\delta_g^a \delta_{\bar{d}}^e, \quad f^{df, \bar{g}a} G_{ff, e\bar{e}} = -\delta_{\bar{d}}^a \delta_g^e, \]  

(A.6)

after contracting with the inverse \( G_{ff, e\bar{e}} \). Then, we find

\[
\begin{align*}
F^{dd, a\bar{a}}_{\bar{e}e} & F^{e\bar{e}, bb, cc} = F^{dd, a\bar{a}, ff} G_{ff, e\bar{e}} F^{e\bar{e}, bb, cc} \\
& = (f^{daa}_{g g d} f^{g f, \bar{d}f} - f^{dg, \bar{f}g} f^{gg, \bar{a}d}) G_{ff, e\bar{e}} (f^{ebb}_{h f h c\bar{e}} - f^{e\bar{e}}_{h f h b\bar{e}}) \\
& = -f^{daa}_{g g d} (f^{h c\bar{d}}_{h f h c\bar{e}} - f^{e\bar{e}}_{h f h b\bar{e}}) + f^{dd}_{g g d} (f^{h b\bar{c}}_{h f h c\bar{e}} - f^{e\bar{e}}_{h f h b\bar{e}}) \\
& = -f^{daa}_{g g d} (f^{h c\bar{c}}_{h f h c\bar{e}} - f^{e\bar{e}}_{h f h b\bar{e}}) + f^{dd}_{g g d} (f^{h c\bar{d}}_{h f h c\bar{e}} - f^{e\bar{e}}_{h f h b\bar{e}}).
\end{align*}
\]  

(A.7)

This vanishes after the cyclic summation

\[ F^{dd, a\bar{a}}_{\bar{e}e} F^{e\bar{e}, bb, cc} + F^{dd, bb}_{\bar{e}e} F^{e\bar{e}, c\bar{c}, a\bar{a}} + F^{dd, c\bar{e}}_{\bar{e}e} F^{e\bar{e}, a\bar{a}, bb} = 0, \]  

(A.8)

which is nothing but the Jacobi identity.
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