On a certain class of operator algebras and their derivations

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Abstract

Given a von Neumann algebra $M$ with a faithful normal finite trace, we introduce the so called finite tracial algebra $M_f$ as the intersection of $L_p$-spaces $L_p(M, \mu)$ over all $p \geq 1$ and over all faithful normal finite traces $\mu$ on $M$. Basic algebraic and topological properties of finite tracial algebras are studied. We prove that all derivations on these algebras are inner.

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1 Introduction

In the present paper we introduce a new class of algebras, the so-called finite tracial algebras, which are defined as the intersection of non commutative $L_p$-spaces $L_p(M, \mu)$ \[1\] over all $p \in [1, \infty)$ and over all faithful normal finite (f.n.f.) traces $\mu$ on a von Neumann algebra $M$. Equivalently, a finite tracial algebra $M_f$ is the intersection of all non commutative Arens algebras $L^\omega(M, \mu) = \bigcap_{p \geq 1} L_p(M, \mu)$, over all f.n.f. traces $\mu$. It is known that Arens algebras are metrizable locally convex *-algebras with respect to the topology generated by the system of $L_p$-norms for a fixed trace. Algebraic and topological properties of Arens algebras have been investigated in the papers \[1\]-\[3\], \[7\], \[10\].

In the present paper we study basic properties of finite tracial algebras with the topology generated by all $L_p$-norms $\{\| \cdot \|_p^\mu\}$, where $p \in [1, \infty)$ and $\mu$ runs over all f.n.f. traces on the given von Neumann algebra $M$. We prove that a finite tracial algebra $M_f$ is metrizable or reflexive if and only if the center of the von Neumann algebra $M$ is finite dimensional; in this case $M_f$ coincides with an appropriate Arens algebra. We also give a necessary and sufficient condition for $M_f$ to coincide (as a set) with $M$. But even in this case one has a new topology on the von Neumann algebra $M$. We obtain also a description of the dual space for the algebra $M_f$.

Finally we prove that every derivation on a solid subalgebra of the Arens algebra $L^\omega(M, \tau)$ is inner. In particular we obtain that the algebra $M_f$ admits only inner derivations.

Throughout the paper we consider a von Neumann algebra $M$ with a f.n.f. trace. Therefore $M$ is a finite von Neumann algebra and thus all closed densely defined operators affiliated with $M$ are measurable with respect to $M$, i.e. the set of all such operators coincides with the algebra $S(M)$ of all measurable operators and hence also with the algebra $LS(M)$ of all locally measurable operators affiliated with $M$; moreover the center of $S(M) = LS(M)$ coincides with the set of operators affiliated with the center of $M$.

2 Preliminaries

Let $M$ be a von Neumann algebra with the positive cone $M^+$ and let $1$ denote the identity operator in $M$. 


A positive linear functional $\mu$ is called a finite trace if $\mu(u^* xu) = \mu(x)$ for all $x \in M$ and each unitary operator $u \in M$.

A finite trace $\mu$ is said to be faithful if for $x \in M^+$, $\mu(x) = 0$ implies that $x = 0$.

A finite trace $\mu$ is normal if given any monotone net $\{x_\alpha\}$ increasing to $x \in M$, one has $\mu(x) = \sup \mu(x_\alpha)$.

Let $\tau$ be a fixed faithful normal finite (f.n.f.) trace on a von Neumann algebra $M$. The Radon — Nikodym theorem [13, Theorem 14] implies that given any f.n.f. trace $\mu$ on $M$, there exists a positive operator $h \in L^1(M, \tau)$ affiliated with the center of $M$ such that $\mu(x) = \tau(hx)$ for all $x \in M$. This operator $h$ is called the Radon — Nikodym derivative of the trace $\mu$ with respect to the trace $\tau$ and it is denoted as $\frac{d\mu}{d\tau}$.

We recall [13, 15] that given a f.n.f. trace $\tau$ on a von Neumann algebra $M$, the space $L^p(M, \tau)$, $p \in [1, \infty)$, is defined as

$$L^p(M, \tau) = \{x \in S(M) : |x|^p \in L^1(M, \tau)\}.$$ 

The space $L^p(M, \tau)$ equipped with the norm $\|x\|_p = (\tau(|x|^p))^{1/p}$ is a Banach space and its dual space coincides with $L^q(M, \tau)$ where $\frac{1}{p} + \frac{1}{q} = 1$, and the duality is given by

$$\langle x, a \rangle = f_a(x) = \tau(ax),$$

for all $f_a \in L^p(M, \tau)^*$, $a \in L^q(M, \tau)$ (see [15, Theorem 4.4]).

Following [10] consider the intersection

$$L^\omega(M, \tau) = \bigcap_{p \in [1, \infty)} L^p(M, \tau).$$

It is known (see also [1, 3, 7]), that $L^\omega(M, \tau)$ is a complete locally convex $*$-algebra with respect to the topology $t^\tau$ generated by the system of norms $\{\| \cdot \|_p\}_{p \in [1, \infty)}$.

Each operator $a \in \bigcup_{q \in (1, \infty)} L^q(M, \tau)$ defines a continuous linear functional $f_a$ on $(L^\omega(M, \tau), t^\tau)$ by the formula $f_a(x) = \tau(ax)$, and conversely given an arbitrary continuous linear functional $f$ on the algebra $(L^\omega(M, \tau), t^\tau)$ there exists an element $a \in \bigcup_{q \in (1, \infty)} L^q(M, \tau)$ such that $f(x) = \tau(ax)$. 

3
3 Finite Tracial Algebras

Let $M$ be a finite von Neumann algebra. Denote by $\mathcal{F}$ the set of all f.n.f. traces on $M$ and from now on suppose that $\mathcal{F} \neq \emptyset$.

Consider the space

$$M_f = \bigcap_{\mu \in \mathcal{F}} \bigcap_{p \in [1, \infty)} L_p(M, \mu) = \bigcap_{\mu \in \mathcal{F}} L^\omega(M, \mu).$$

On the space $M_f$ one can consider the topology $t$, generated by the system of norms $\{\| \cdot \|_p^\mu : \mu \in \mathcal{F}, p \in [1, \infty)\}$.

Since each Arens algebra $L^\omega(M, \mu)$, $\mu \in \mathcal{F}$, is a complete locally convex topological $*$-algebra in $S(M)$ from the above definition one easily obtains the following

**Theorem 3.1.** $(M_f, t)$ is a complete locally convex topological $*$-algebra.

**Definition.** The topological $*$-algebra $M_f$ is called the finite tracial algebra with respect to the von Neumann algebra $M$.

**Remark.** Finite tracial algebras present examples of so called GW$^*$-algebras in the sense of [12].

Recall (see [12]) that a topological $*$-algebra $(A, t_A)$ is called GW$^*$-algebra, if $A$ has a $W^*$-subalgebra $B$ with $(1 + x^*x)^{-1} \in B$ for all $x \in A$ and the unit ball of $B$ if $t_A$-bounded.

The finite tracial algebra $M_f$ is a GW$^*$-algebra. Since $M \subset M_f$ it is sufficient to show that the unit ball in $M$ is $t$-bounded in $M_f$.

Let $x \in M$, $\|x\|_\infty \leq 1$. For $\mu \in \mathcal{F}$, and $1 \leq p < \infty$ we have

$$\|x\|_p^\mu = \|x1\|_p^\mu \leq \|x\|_\infty \|1\|_p^\mu \leq \mu(1)^{\frac{1}{p}},$$

i. e. $\|x\|_p^\mu \leq \mu(1)^{\frac{1}{p}}$ for all $x \in M$, $\|x\|_\infty \leq 1$. This means that the unit ball of $M$ is $t$-bounded in $M_f$. Therefore $M_f$ is a GW$^*$-algebra.

The algebra $M_f$ contains $M$ but it is a rather small algebra, since it is contained in all $L^p(M, \mu)$ for all $p \geq 1$ and f.n.f. traces $\mu$ on $M$. The following result gives necessary and sufficient conditions for $M_f$ to coincide with $M$.

**Theorem 3.2.** For a finite von Neumann algebra $M$ the following conditions are equivalent

i) $M_f = M$;

ii) $M$ is a finite sum of homogeneous type $I_n$, $n \in \mathbb{N}$ von Neumann algebras.
The proof of this theorem consists of several auxiliary propositions which are interesting in their own right. Let us start with the commutative case.

**Proposition 3.1.** Let $M$ be a von Neumann algebra with a faithful normal trace and let $Z$ be its center. Then the center of the algebra $M_f$ coincides with $Z$, i.e. $Z(M_f) = Z$. In particular if $M$ is commutative then $M_f = M$.

**Proof.** Let $M$ be a von Neumann algebra with a faithful normal finite trace $\tau$, and $\tau(1) = 1$.

Consider $x \in Z(M_f)$, $x \geq 0$, and let $x = \int_0^\infty \lambda d\varepsilon$ be the spectral resolution of $x$. Since $x \in Z(M_f)$ and $M \subset M_f$, we have that $e_\lambda \in Z$ for all $\lambda \in \mathbb{R}$. Passing if necessary to the element $\varepsilon 1 + x$ we may suppose without loss of generality that $e_1 = 0$.

For $n \in \mathbb{N}$ set

$$p_n = e_{(n+1)^2} - e_n$$

and

$$y = \sum_{n \in \mathbb{N}} n^2 p_n.$$ 

Since $xp_n \geq n^2 p_n$ for all $n \in \mathbb{N}$, we have that $0 \leq y \leq x$ and hence $y \in M_f$.

Let

$$F = \{ n \in \mathbb{N} : t_n = \tau(p_n) \neq 0 \}$$

and

$$h = \sum_{n \in F} \frac{1}{n^2 t_n} p_n \in Z(S(M)).$$

Since

$$\bigvee_{n=1}^m p_n = \bigvee_{n=1}^m (e_{(n+1)^2} - e_n^2) = \sum_{n=1}^m (e_{(n+1)^2} - e_n^2) = e_{(m+1)^2} - e_1 = e_{(m+1)^2} \uparrow 1,$$

one has that

$$\bigvee_{n=1}^\infty p_n = 1.$$

Therefore there exists $h^{-1} \in S(M)$. Further we have

$$\tau(h) = \sum_{n \in F} \frac{1}{n^2 t_n} \tau(p_n) = \sum_{n \in F} \frac{1}{n^2 t_n} t_n = \sum_{n \in F} \frac{1}{n^2} \leq \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty,$$
i.e. \( h \in L_1(M, \tau) \).

Put \( \mu(\cdot) = \tau(h\cdot) \). Since \( y \in M_f \), it follows that \( y \in L_1(M, \mu) \). Therefore \( \mu(y) < \infty \).

On the other hand

\[
hy = \sum_{n \in F} \frac{1}{n^2} p_n \sum_{n \in \mathbb{N}} n^2 p_n = \sum_{n \in F} \frac{1}{t_n} p_n,
\]

and thus

\[
\mu(y) = \tau(hy) = \sum_{n \in F} \frac{1}{t_n} \tau(p_n) = \sum_{n \in F} \frac{1}{t_n} = \sum_{n \in F} 1 = |F|,
\]

where \( |F| \) is the cardinality of the set \( F \). Since \( \mu(y) < \infty \) this implies that \( F \) is a finite set. Let \( k = \max\{n : n \in F\} \). Then \( \tau(p_n) = 0 \) for all \( n > k \), and since \( \tau \) is faithful we have that \( p_n = 0 \) for all \( n > k \), i.e. \( e_{(n+1)^2} = e_{n^2} \). But \( e_{n^2} \uparrow 1 \) and thus \( e_{n^2} = 1 \) for all \( n > k \). This means that \( 0 \leq x \leq (k+1)^2 1 \), i.e. \( x \in Z \).

The proof is complete. \( \blacksquare \)

**Proposition 3.2.** Let \( M \) be a type \( I_n \), \( n \in \mathbb{N} \) von Neumann algebra. Then \( M_f = M \).

**Proof.** By [14, Ch. V, Theorem 1.27] the von Neumann algebra \( M \) of type \( I_n \) (\( n \in \mathbb{N} \)) can be represented as \( M = Z \otimes B(H_n) \), where \( Z \) is the center \( M \) and \( H_n \) is the \( n \)-dimensional Hilbert space. Put \( \mathcal{F}_Z = \{ \tau|_Z : \tau \in \mathcal{F} \} \). Therefore from Proposition 3.1 we obtain

\[
M_f = \bigcap_{p \in [1, \infty)} \bigcap_{\tau \in \mathcal{F}} L_p(M, \tau) = \bigcap_{p \in [1, \infty)} \bigcap_{\mu \in \mathcal{F}_Z} L_p(Z, \mu) \otimes B(H_n) =

= \left( \bigcap_{p \in [1, \infty)} \bigcap_{\mu \in \mathcal{F}_Z} L_p(Z, \mu) \right) \otimes B(H_n) = Z_f \otimes B(H_n) =

= Z \otimes B(H_n) = M,
\]

i.e. \( M_f = M \).

The proof is complete. \( \blacksquare \)

**Proposition 3.3.** Let \( M \) be a finite von Neumann algebra which is isomorphic to the direct sum of an infinite number of homogeneous type \( I_n \) \( (n \in \mathbb{N}) \) von Neumann algebras. Then \( M_f \neq M \).
Proof. Suppose that \( M = \sum_{k \in K} \oplus M_k \), where \( K \) is an infinite subset of \( \mathbb{N} \), and \( M_k \) is a homogeneous type \( I_k \) von Neumann algebra.

Since the set \( K \) is infinite, there exists a sequence \( \{k_n\} \subset K \) such that \( k_n \geq 2^n \) for all \( n \in \mathbb{N} \). We have that

\[
M_{k_n} = Z_{k_n} \otimes B(H_{k_n}),
\]

where \( Z_{k_n} \) is the center of \( M_{k_n} \). Therefore the algebra \( M \) contains a subalgebra *-isomorphic to the algebra \( N = \sum_{n \in \mathbb{N}} \oplus N_n \).

Hence, without loss of generality we may assume that \( M = \sum_{n \in \mathbb{N}} \oplus N_n \), where \( N_n = B(H_{2^n}) \) is the algebra of all \( 2^n \times 2^n \) matrices over \( \mathbb{C} \). On each \( N_n \) we consider the unique tracial state (i. e. normalized f.n.f. trace) \( \mu_n \) and define on \( M \) the following f.n.f. trace

\[
\tau(x) = \sum_{n \in \mathbb{N}} 2^{-n} \mu_n(x_n),
\]

where \( x = \sum_{n \in \mathbb{N}} \oplus x_n \in M \). Then every f.n.f. trace \( \mu \) on \( M \) has the form

\[
\mu(x) = \tau(hx) = \sum_{n \in \mathbb{N}} 2^{-n} \mu_n(h_n x_n) = \sum_{n \in \mathbb{N}} 2^{-n} \alpha_n \mu_n(x_n),
\]

where

\[
h = \sum_{n \in \mathbb{N}} \oplus h_n = \sum_{n \in \mathbb{N}} \oplus \alpha_n 1_n \in L_1(M, \tau),
\]

i. e. \( \alpha_n > 0, \ n \in \mathbb{N}, \sum_{n \in \mathbb{N}} 2^{-n} \alpha_n < \infty \).

Take a minimal projection \( p_n \) in each \( N_n = B(H_{2^n}) \). Then \( \mu_n(p_n) = \frac{1}{2^n} \).

Consider the unbounded element \( x = \sum_{n \in \mathbb{N}} \oplus np_n \) in \( S(M) \setminus M \) and let us prove that \( x \in M_f \). For every f.n.f. trace \( \mu \) on \( M \) one has that

\[
\mu(x^p) = \sum_{n \in \mathbb{N}} 2^{-n} \alpha_n \mu_n(n^p p_n) = \sum_{n \in \mathbb{N}} 2^{-n} \alpha_n n^p 2^{-n} < \infty,
\]
because \( n^p2^{-n} < 1 \) for sufficiently large \( n \in \mathbb{N} \). Therefore \( x \in L_p(M, \mu) \) for all \( p \geq 1 \) and every f.n.f. trace \( \mu \in \mathcal{F} \), i.e. \( x \in M_f \).

The proof is complete. \( \square \)

**Proposition 3.4.** Let \( M \) be a type II\(_1\) von Neumann algebra with a f.n.f. trace \( \tau \). Then \( M_f \neq M \).

**Proof.** Suppose that the trace \( \tau \) is normalized, i.e. \( \tau(1) = 1 \), and denote by \( \Phi \) the canonical center-valued trace on \( M \). Since \( M \) is of type II\(_1\) there exists a projection \( p_1 \) such that

\[
p_1 \sim 1 - p_1.
\]

Therefore from \( \Phi(p_1) + \Phi(p_1^+) = \Phi(1) = 1 \) and \( \Phi(p_1) = \Phi(p_2) \) we obtain that

\[
\Phi(p_1) = \Phi(p_1^+) = \frac{1}{2} 1.
\]

Suppose that we have constructed mutually orthogonal projections \( p_1, p_2, \ldots, p_n \) in \( M \) such that

\[
\Phi(p_k) = \frac{1}{2^k} 1, \quad k = 1, n.
\]

Set \( e_n = \sum_{k=1}^{n} p_k \). Then \( \Phi(e_n^+) = \frac{1}{2^n} 1 \). Now take a projection \( p_{n+1} \leq e_n^+ \) such that

\[
p_{n+1} \sim e_n^+ - p_{n+1},
\]

i.e.

\[
\Phi(p_{n+1}) = \frac{1}{2^{n+1}} 1.
\]

In this manner we obtain a sequence \( \{p_n\}_{n \in \mathbb{N}} \) of mutually orthogonal projections such that

\[
\Phi(p_n) = \frac{1}{2^n} 1, \quad n \in \mathbb{N}.
\]

It is clear that \( \tau(p_n) = \tau(\Phi(p_n)) = \frac{1}{2^n}, \quad n \in \mathbb{N} \).

From

\[
\sum_{n=1}^{\infty} ||np_n||_1^\tau = \sum_{n=1}^{\infty} \tau(np_n) = \sum_{n=1}^{\infty} \frac{n}{2^n} < \infty,
\]

it follows that the element \( x = \sum_{n=1}^{\infty} np_n \) belongs to \( L_1(M, \tau) \), and it is unbounded, i.e. \( x \notin M \).
On the other hand for an arbitrary central element \( h \in L_1(M, \tau), h > 0, \) and \( n \in \mathbb{N} \) we have
\[
\tau(hp_n) = \tau(\Phi(hp_n)) = \tau(h\Phi(p_n)) = \tau(h \frac{1}{2^n} 1) = \frac{1}{2^n} \tau(h).
\]
Therefore for an arbitrary f.n.f. trace \( \mu \) on \( M \) with \( \frac{d\mu}{d\tau} = h \) we have
\[
\mu(|x|^p) = \mu(x^p) = \tau(h^p) = \tau(h \sum_{n=1}^{\infty} n^p p_n) = \sum_{n=1}^{\infty} n^p \tau(hp_n) = \tau(h) \sum_{n=1}^{\infty} n^p < \infty,
\]
i.e. \( x \in L_p(M, \mu) \) for all \( p \geq 1 \) and every f.n.f. trace \( \mu \). Therefore \( x \in M_f \backslash M \).

The proof is complete. ■

Proof of Theorem 3.2. The implication \((i) \Rightarrow (ii)\) follows from Propositions 3.3 and 3.4, while \((ii) \Rightarrow (i)\) follows from Propositions 3.2.

The proof is complete. ■

Now let us describe continuous linear functionals on the space \((M_f, t)\).

**Theorem 3.3.** Given any \( \mu \in \mathcal{F} \), \( 1 < q < \infty \), and \( a \in L_q(M, \mu) \) the functional \( \varphi(x) = \mu(xa) \), \( x \in M_f \), is a continuous linear functional on \((M_f, t)\). Conversely for any continuous linear functional \( \varphi \) on \((M_f, t)\) there exist \( \mu \in \mathcal{F} \), \( 1 < q < \infty \), \( c > 0 \) such that
\[
\varphi(x) = \mu(xa), \ x \in M_f.
\]

**Proof.** Let \( \mu \in \mathcal{F} \), \( 1 < q < \infty \), \( a \in L_q(M, \mu) \). Put
\[
\varphi_a(x) = \mu(xa), \ x \in M_f.
\]
Take \( p \in \mathbb{R} \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Since
\[
|\varphi_a(x)| = |\mu(xa)| \leq ||a||_q ||x||_p^\mu
\]
for all \( x \in M_f \), one has that \( \varphi_a \) is a continuous linear functional on \((M_f, t)\).

Conversely, let \( \varphi \) be a continuous linear functional on \((M_f, t)\). By [16, Corollary 1 on p.43] there exist \( \mu \in \mathcal{F} \), \( 1 \leq p < \infty \), \( c > 0 \), such that
\[
|\varphi(x)| \leq c ||x||_p^\mu
\]
for all $x \in M_f$. Since $M \subset M_f$ and $M$ is $\| \cdot \|_p^\mu$-dense in $L_p(M, \mu)$, the functional $\varphi$ can be uniquely extended onto $L_p(M, \mu)$. By [15, Theorem 4.4] there exists $a \in L_q(M, \mu), \frac{1}{p} + \frac{1}{q} = 1$, such that

$$\varphi(x) = \mu(xa)$$

for all $x \in L_p(M, \mu)$. In particular

$$\varphi(x) = \mu(xa)$$

for all $x \in M_f$, i.e. $\varphi = \varphi_a$.

The proof is complete. ■

If the von Neumann algebra $M$ is a factor then it has a unique (up to a scalar multiple) f.n.f. trace $\mu$. In this case the finite tracial algebra $M_f$ coincides with the Arens algebra $L^\omega(M, \mu)$ and the topology $t$ merges to the topology $t^\mu$ generated by the system of norms $\{\| \cdot \|_p^\mu\}_{p \geq 1}$. The following theorem describes the general case where this phenomenon occurs.

Recall some notions from the theory of linear topological spaces. Let $E$ be a locally convex linear topological space. An absolutely convex absorbing set in $E$ is called a barrel. If each barrel in $E$ is a neighborhood of zero, then $E$ is said to be a barreled space.

It is known ([16], Theorem 2, p.200) that every reflexive locally convex space is barreled.

**Theorem 3.4.** Let $M$ be a finite von Neumann algebra and suppose that $\mathcal{F} \neq \emptyset$ is the family of all f.n.f. traces on $M$. The following conditions are equivalent:

(i) $M_f = L^\omega(M, \mu)$ for some (and hence for all) $\mu \in \mathcal{F}$;

(ii) $(M_f, t)$ is metrizable;

(iii) $(M_f; t)$ is reflexive;

(iv) the center $Z$ of $M$ is finite dimensional, i.e. $M = \sum_{i=1}^m M_i$, where all $M_i$ are $I_n$-factors or $II_1$-factors.

**Proof.** Suppose that $Z$ is finite dimensional. Then $M$ is a finite direct sum of factors $M_i$, $i = 1, k$. Then for each factor $M_i$ the algebras $(M_i)_f$ and $L^\omega(M_i, \mu_i)$ coincide and the topology $t_i$ is the same as $t_i^\mu_i$. Therefore

$$M_f = (\sum_{i=1}^n M_i)_f = \sum_{i=1}^n (M_i)_f = \sum_{i=1}^n L^\omega(M_i, \mu_i) = L^\omega(M, \mu),$$
where \( \mu = \sum_{i=1}^{n} \mu_i \in \mathcal{F} \), i.e. \( M_f = L^\omega(M, \mu) \).

Now since the topology \( t^\mu \) on the Arens algebra \( L^\omega(M, \mu) \) is metrizable \[1\] it follows that \( t = t^\mu \) is also metrizable.

It is known \[2\] that for finite traces \( \mu \) the Arens algebra \( (L^\omega(M, \mu), t^\mu) \) is reflexive and hence \( (M_f, t) \) is also reflexive.

Therefore \((iv)\) implies \((i), (ii)\) and \((iii)\).

\((i) \Rightarrow (iv)\). Suppose that \( M_f = L^\omega(M, \mu) \) for an appropriate \( \mu \in \mathcal{F} \). Then there exists a sequence of mutually orthogonal projections \( \{p_n\} \) in \( Z \) such that \( p_n \neq 0 \) for all \( n \in \mathbb{N} \). Since the trace \( \mu \) is finite one has that \( \sum_{k=1}^{\infty} \mu(p_k) < \infty \) and hence there is a subsequence \( \{n_k : k \in \mathbb{N}\} \) such that \( \mu(p_{n_k}) \leq \frac{1}{2^k} \) for all \( k \).

Set
\[
x = \sum_{k=1}^{\infty} kp_k
\]

For \( p \geq 1 \) we have
\[
\mu(|x|^p) = \sum_{k=1}^{\infty} k^p \mu(p_k) \leq \sum_{k=1}^{\infty} k^p \frac{1}{2^k} < \infty,
\]
and hence \( x \in L^\omega(M, \mu) = M_f \).

On the other hand \( x \) is a central element in \( M_f \) and Proposition 3.1 implies that \( x \in Z(M_f) = Z \subset M \). But it is clear that the element \( x \) is unbounded, i.e. \( x \notin M \). The contradiction shows that \( Z \) is finite dimensional.

\((ii) \Rightarrow (iv)\). Suppose that \( (M_f, t) \) is metrizable. By Theorem 3.1 it is complete and hence it is a Fre'chet space. In particular the center of \( M_f \) which coincides with \( Z_f \) is also a Fre'chet space. By Proposition 3.1 \( Z_f = Z \) and hence \( Z \) is a Fre'chet space with respect to the induced topology \( t_z = t|_Z \).

Consider the identity mapping
\[
I : (Z, \| \cdot \|_\infty) \rightarrow (Z, t_z)
\]
where \( \| \cdot \|_\infty \) is the operator norm on \( Z \). From the inequalities
\[
\|x\|_\mu \leq C_\mu^p \|x\|_\infty
\]
(where \( C_\mu^p \) is an appropriate constant for each \( p \geq 1, \mu \in \mathcal{F} \)) it follows that the mapping \( I \) is continuous. Since \( (Z, t_z) \) is a Fre'chet space, from Banach...
Theorem on the inverse operator ([10], Chapter II, Section 5) we obtain that the inverse mapping

\[ I^{-1} : (Z, t_z) \rightarrow (Z, \| \cdot \|_{\infty}) \]

is also continuous. This means that for some \( p \in [1, \infty) \) and an appropriate \( \mu \in \mathcal{F} \) there exists a constant \( K^\mu_p \) such that

\[ \|x\|_{\infty} \leq K^\mu_p \|x\|_p^\mu \]

(1)

for all \( x \in Z \) ([10], Theorem 1, p. 42).

Now suppose that \( \dim Z = \infty \). There exists a sequence \( \{p_n\} \) of projections in \( Z \) such that \( p_n \uparrow 1 \), \( p_n \neq p_{n+1} \). Thus \( p_n^\perp \neq 0 \), \( \tau(p_n^\perp) \rightarrow 0 \), i.e. \( \|p_n^\perp\|_p^\mu \rightarrow 0 \). From the inequality (1) we obtain that \( \|p_n^\perp\|_{\infty} \rightarrow 0 \).

On the other hand \( \|p_n^\perp\|_{\infty} = 1 \). This contradiction implies that \( Z \) is finite dimensional.

\((iii) \Rightarrow (iv)\). Suppose that \( M_f \) is reflexive. Then the center \( Z(M_f) = Z \) is also reflexive as a closed subspace of a reflexive space.

The set

\[ B = \{x \in Z : \|x\|_{\infty} \leq 1\} \]

is a barrel in \( (Z, t) \) and since \( Z \) is reflexive, we have that \( B \) is a neighborhood of zero in \( Z \). Therefore there exist \( p \geq 1 \), \( \mu \in \mathcal{F} \) and \( \varepsilon > 0 \) such that

\[ \{x \in Z : \|x\|_p^\mu \leq \varepsilon\} \subseteq B \]

i.e.

\[ \|x\|_{\infty} \leq \varepsilon^{-1} \|x\|_p^\mu \]

for all \( x \in Z \). From this as above it follows that \( Z \) is finite dimensional.

The proof is complete. \( \blacksquare \)

**Remark.** In the von Neumann algebra \( M \) the operator topology is stronger than the topology \( t \), \( t \) is stronger than \( \tau \), and \( \tau \) is stronger than each \( L_p \)-norm topology for any \( p \geq 1 \).

### 4 Derivations on Finite Tracial Algebras

Derivations on unbounded operator algebras, in particular on various algebras of measurable operators affiliated with von Neumann algebras, appear to be a very attractive special case of general unbounded derivations on operator algebras.
Let $A$ be an algebra over the complex numbers. A linear operator $D : A \to A$ is called a derivation if it satisfies the identity $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (Leibniz rule). Each element $a \in A$ defines a derivation $D_a$ on $A$ given as $D_a(x) = ax - xa$, $x \in A$. Such derivations $D_a$ are said to be inner derivations.

In [4] we have investigated and completely described derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra $M$ and on its various subalgebras. Recently the above conjecture was also confirmed for the type I case in the paper [8] by a representation of measurable operators as operator valued functions. Another approach to similar problems in $AW^*$-algebras of type I was suggested in the recent paper [9].

In the paper [3] we have proved the spatiality of derivations on the noncommutative Arens algebra $L^ω(M, τ)$ associated with an arbitrary von Neumann algebra $M$ and a faithful normal semi-finite trace $τ$. Moreover if the trace $τ$ is finite then every derivation on $L^ω(M, τ)$ is inner.

In this section we prove that each derivation on a finite tracial algebra is inner.

The following result is an immediate corollary of [6, Proposition 3.6].

**Lemma 4.1.** Let $M$ be a von Neumann algebra with a faithful normal trace $τ$. Given any derivation $D : M \to L^ω(M, τ)$ there exists an element $a \in L^ω(M, τ)$ such that $D(x) = ax - xa$, $x \in M$.

Further we need also the following assertion from [8, Proposition 6.17].

**Lemma 4.2.** Let $A$ be a $*$-subalgebra of $LS(M)$ such that $M \subseteq A$ and $A$ is solid (that is, if $x \in LS(M)$ and $y \in A$ satisfy $|x| \leq |y|$ then $x \in A$). If $ω \in LS(M)$ is such that $[ω, x] \in A$ for all $x \in A$, then there exists $ω_1 \in A$ such that $[ω, x] = [ω_1, x]$ for all $x \in A$.

The main result of this section is the following theorem.

**Theorem 4.1.** Let $M$ be a von Neumann algebra with a faithful normal finite trace $τ$. If $A \subseteq L^ω(M, τ)$ is a solid $*$-subalgebra such that $M \subseteq A$, then every derivation on $A$ is inner.

**Proof.** Since $A \subseteq L^ω(M, τ)$, by Lemma 4.1 there exists an element $a \in L^ω(M, τ)$ such that $D(x) = ax - xa$, $x \in M$. (2)
Let us show that in fact

\[ D(x) = ax - xa, \text{ for all } x \in A. \]  

Consider \( x \in A, \ x \geq 0 \). Then \((1 + x)^{-1} \in M\). From the Leibniz rule it follows that for each invertible \( b \in A \) one has

\[ D(b) = -bD(b^{-1})b. \]

Therefore

\[ D(x) = D(1 + x) = -(1 + x)D((1 + x)^{-1})(1 + x). \]

On the other hand since \((1 + x)^{-1} \in M\) the equality (2) implies that

\[ D((1 + x)^{-1}) = a(1 + x)^{-1} - (1 + x)^{-1}a. \]

Therefore

\[ -(1 + x)D((1 + x)^{-1})(1 + x) = -(1 + x)[a(1 + x)^{-1} - (1 + x)^{-1}a](1 + x) = \]

\[ = -(1 + x)a + a(1 + x) = ax - xa, \]

i.e.

\[ D(x) = ax - xa, \ x \in A, \ x \geq 0. \]

Since each element from \( A \) is a finite linear combination of positive elements, we obtain the equality (3) for arbitrary \( x \in A \).

Now since \( A \) is a solid *-subalgebra in \( L^\omega(M, \tau) \) containing \( A \), Lemma 4.2 implies that the element \( a \) implementing the derivation \( D \) may be chosed from the algebra \( A \), i.e.

\[ D(x) = ax - xa, \ x \in A \]

for an appropriate \( a \in A \).

The proof is complete. ■

Since the algebra \( M_f \) is a solid *-subalgebra of \( L^\omega(M, \tau) \) and contains \( M \), we obtain the following result.

**Corollary 4.1.** If \( M \) is a von Neumann algebra with a faithful normal trace, then every derivation on \( M_f \) is inner.
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