Mixing of 3-term progressions in Quasirandom Groups

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Abstract

In this note, we show the mixing of three-term progressions \((x, xg, xg^2)\) in every finite quasirandom groups, fully answering a question of Gowers. More precisely, we show that for any \(D\)-quasirandom group \(G\) and any three sets \(A_1, A_2, A_3 \subseteq G\), we have

\[
\left| \Pr_{x,y \sim G} [x \in A_1, xy \in A_2, xy^2 \in A_3] - \prod_{i=1}^3 \Pr_{x \sim G} [x \in A_i] \right| \leq \left( \frac{2}{\sqrt{D}} \right)^{1/4}.
\]

Prior to this, Tao answered this question when the underlying quasirandom group is \(SL_d(F_q)\). Subsequently, Peluse extended the result to all nonabelian finite simple groups. In this work, we show that a slight modification of Peluse’s argument is sufficient to fully resolve Gower’s quasirandom conjecture for 3-term progressions. Surprisingly, unlike the proofs of Tao and Peluse, our proof is elementary and only uses basic facts from nonabelian Fourier analysis.

1 Introduction

In this note, we revisit a conjecture by Gowers [Gow08] about mixing of three term arithmetic progressions in quasirandom finite groups. Gowers initiated the study of quasirandom groups while refuting a conjecture of Babai and Sós [BS85] regarding the size of the largest product-free set in a given finite group. A finite group is said to be \(D\)-quasirandom for a positive integer \(D > 1\) if all its non-trivial irreducible representations are at least \(D\)-dimensional. The quasirandomness property of groups can be used to show that certain "objects" related to the group "mix" well. For instance, the quasirandomness of the group \(PSL_2(F_q)\) can be used to give an alternate (and weaker) proof [DSV03] that the Ramanujan graphs of Lubotzky, Philips and Sarnak [LPS88] are expanders. Bourgain and Gambard [BG08] used quasi randomness to prove that certain other Cayley graphs were expanders.

Gowers proved that for any \(D\)-quasirandom group \(G\) and any three subsets \(A, B, C \subseteq G\) satisfying \(|A| \cdot |B| \cdot |C| \geq |G|^3/D\), there exist \(a \in A, b \in B, c \in C\) such that \(ab = c\). More generally, he proved that the number of such triples \((a, b, c) \in A \times B \times C\) such that \(ab = c\) is at least \((1 - \eta)|A| \cdot |B| \cdot |C|/|G|\) provided \(|A| \cdot |B| \cdot |C| \geq |G|^3/\eta^2 D\). In other words the set of triples of the form \((a, b, ab)\) mix well in a quasirandom group. Gowers’ proof of this result was the inspiration and the first step towards the recent optimal inapproximability result for satisfiable \(k\)LIN over nonabelian groups [BK21]. After proving

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the well-mixing of triples of the form \((a, b, ab)\) in quasirandom groups, Gowers conjectured a similar statement for triples of the form \((x, xg, xg^2)\). More precisely, he conjectured the following statement: Let \(G\) be a \(D\)-quasirandom group and \(f_1, f_2, f_3 : G \to \mathbb{C}\) such that \(\|f_i\|_\infty \leq 1\), then

\[
\left| \mathbb{E}_{x,y \sim G} [f_1(x)f_2(xy)f_3(xy^2)] - \prod_{i=1,2,3} \mathbb{E}_{x \sim G} [f_i(x)] \right| = o_D(1),
\]

where the expression \(o_D(1)\) goes to zero as \(D\) increases. The conjecture can be naturally extended to \(k\)-term arithmetic progressions and product of \(k\) functions for \(k > 3\). However, in this note we will focus on the three term case.

For the specific case of 3-term progressions, Tao [Tao13] proved the conjecture for the group \(SL_d(\mathbb{F}_q)\) for bounded \(d\) using algebraic geometric machinery. In particular, he proved that the right-hand side expression in Eq. (1) can be bounded by \(O(1/q^{3/8})\) when \(d = 2\) and \(O_d(1/q^{3/4})\) for larger \(d\). Tao’s approach relied on algebraic geometry and was not amenable to other quasirandom groups. Later, Peluse [Pel18] proved the conjecture for all nonabelian finite simple groups. She used basic facts from nonabelian Fourier analysis to prove that the right-hand side expression in Eq. (1) can be bounded by

\[
\sum_{1 \neq \rho \in \hat{G}} 1/d_\rho
\]

where \(\hat{G}\) represents the set of irreducible unitary representation of \(G\) and \(d_\rho\) the dimension of the irreducible representation \(\rho\). This latter quantity is the Witten zeta function \(\zeta_G\) of the group \(G\) minus one and can be bounded for simple finite quasirandom groups using a result due to Liebeck and Shalev [LS04].

In this paper, we show that a slight variation of Peluse’s argument can be used to prove the conjecture for all quasirandom groups with better error parameters. More surprisingly, the proof stays completely elementary and short. Specifically, we prove the following statement:

**Theorem 1.** Let \(G\) be a \(D\)-quasirandom finite group, i.e, its all non-trivial irreducible representations are at least \(D\)-dimensional. Let \(f_1, f_2, f_3 : G \to \mathbb{C}\) such that \(\|f_i\|_\infty \leq 1\) then

\[
\left| \mathbb{E}_{x,y \sim G} [f_1(x)f_2(xy)f_3(xy^2)] - \prod_{i=1,2,3} \mathbb{E}_{x \sim G} [f_i(x)] \right| \leq \left( \frac{2}{\sqrt{D}} \right)^{1/4}.
\]

## 2 Preliminaries

We begin by recalling some basic representation theory and nonabelian Fourier analysis. See the monograph by Diaconis [Dia98, Chapter 2] for a more detailed treatment (with proofs).

We will be working with a finite group \(G\) and complex-valued functions \(f : G \to \mathbb{C}\) on \(G\). All expectations will be with respect to the uniform distribution on \(G\). The *convolution* between two function \(f, h : G \to \mathbb{C}\), denoted by \(f \ast h\), is defined as follows:

\[(f \ast h)(x) := \mathbb{E}_y [f(xy^{-1})h(y)].\]

For any \(p \geq 1\), the \(p\)-norm of any function \(f : G \to \mathbb{C}\) is defined as

\[
\|f\|_p^p := \mathbb{E}_x [|f(x)|^p].
\]
For any element $g \in G$, the conjugacy class of $g$, denoted by $C(g)$, refers to the set $\{x^{-1}gx \mid x \in G\}$. Observe that the conjugacy classes form a partition of the group $G$. A function $f: G \rightarrow \mathbb{C}$ is said to be a class function if it is constant on conjugacy classes.

For any $b \in G$ we use $\Delta_b f(x) := f(x) \cdot f(xb)$. For any set $S \subset G$, $\mu_S: G \rightarrow \mathbb{R}$ denotes the scaled density function $\frac{|G|}{|S|} \mathbbm{1}_S$. The scaling ensures that $\mathbb{E}_x[\mu_S(x)] = 1$.

Given a complex vector space $V$, we denote the vector space of linear operators on $V$ by $\text{End}(V)$. This space is endowed with the following inner product and norm (usually referred to as the Hilbert-Schmidt norm):

For $A, B \in \text{End}(V)$, $\langle A, B \rangle_{HS} := \text{Trace}(A^*B)$ and $\|A\|_{HS}^2 := \langle A, A \rangle_{HS} = \text{Trace}(A^*A)$. This norm is known to be submultiplicative (i.e, $\|AB\|_{HS} \leq \|A\|_{HS} \cdot \|B\|_{HS}$).

**Representations and Characters:** A representation $\rho: G \rightarrow \text{End}(V)$ is a homomorphism from $G$ to the set of linear operators on $V$ for some finite-dimensional vector space $V$ over $\mathbb{C}$, i.e., for all $x, y \in G$, we have $\rho(xy) = \rho(x)\rho(y)$. The dimension of the representation $\rho$, denoted by $d_\rho$, is the dimension of the underlying $\mathbb{C}$-vector space $V$. The character of a representation $\rho$, denoted by $\chi_\rho: G \rightarrow \mathbb{C}$, is defined as $\chi_\rho(x) := \text{Trace}(\rho(x))$.

The representation $1: G \rightarrow \mathbb{C}$ satisfying $1(x) = 1$ for all $x \in G$ is the trivial representation. A representation $\rho: G \rightarrow \text{End}(V)$ is said to reducible if there exists a non-trivial subspace $W \subset V$ such that for all $x \in G$, we have $\rho(x)W \subset W$. A representation is said to be irreducible otherwise. The set of all irreducible representations of $G$ (up to equivalences) is denoted by $\hat{G}$.

For every representation $\rho: G \rightarrow \text{End}(V)$, there exists an inner product $\langle \cdot, \cdot \rangle_V$ over $V$ such that every $\rho(x)$ is unitary (i.e, $\langle \rho(x)u, \rho(x)v \rangle_V = \langle u, v \rangle_V$ for all $u, v \in V$ and $x \in G$). Hence, we might wlog. assume that all the representations we are considering are unitary.

The following are some well-known facts about representations and characters.

**Proposition 2.**
1. The group $G$ is abelian iff $d_\rho = 1$ for every irreducible representation $\rho$ in $\hat{G}$.
2. For any finite group $G$, $\sum_{\rho \in \hat{G}} d_\rho^2 = |G|$.
3. [Orthogonality of characters] For any $\rho, \rho' \in \hat{G}$ we have: $\mathbb{E}_x \left[\chi_\rho(x)\overline{\chi_{\rho'}(x)}\right] = 1[\rho = \rho']$.

**Definition 3** (quasirandom groups). A nonabelian group $G$ is said to be $D$-quasirandom for some positive integer $D > 1$ if all its non-trivial irreducible representations $\rho$ satisfy $d_\rho \geq D$.

**Nonabelian Fourier analysis:** Given a function $f: G \rightarrow \mathbb{C}$ and an irreducible representation $\rho \in \hat{G}$, the Fourier transform is defined as follows:

$$\hat{f}(\rho) := \mathbb{E}_x[f(x)\rho(x)].$$

The following proposition summarizes the basic properties of Fourier transform that we will need.

**Proposition 4.** For any $f, h: G \rightarrow \mathbb{C}$, we have the following
1. [Fourier transform of trivial representation]
   \[ \hat{f}(1) = \mathbb{E}_x[f(x)]. \]

2. [Convolution]
   \[ \hat{f} \ast \hat{h}(\rho) = \hat{f}(\rho) \cdot \hat{h}(\rho). \]

3. [Fourier inversion formula]
   \[ f(x) = \sum_{\rho \in \hat{G}} d_\rho \cdot \langle \hat{f}(\rho), \rho(x) \rangle_{\text{HS}}. \]

4. [Parseval’s identity]
   \[ \|f\|_2^2 = \sum_{\rho \in \hat{G}} d_\rho \cdot \|\hat{f}(\rho)\|_{\text{HS}}^2. \]

5. [Fourier transform of class functions] For any class function \( f : G \to \mathbb{C} \), the Fourier transform satisfies
   \[ \hat{f}(\rho) = c \cdot I_{d_\rho} \]
   for some constant \( c = c(f, \rho) \in \mathbb{C} \). In other words, the Fourier transform is a scaling of the Identity operator \( I_{d_\rho} \).

The following claim (also used by Peluse [Pel18]) observes that the scaled density function \( \mu_{gC(g)} \) has a very simple Fourier transform since it is a translate of the class function \( \mu_{C(g)} \)

**Claim 5.** For any \( g \in G \) and \( \rho \in \hat{G} \) we have:

\[ \hat{\mu}_{gC(g)}(\rho) = \frac{\chi_\rho(g)}{d_\rho} \cdot \rho(g) \]

where \( C(g) \) refers to the conjugacy class of \( g \). Moreover, \( \|\hat{\mu}_{gC(g)}\|_{\text{HS}}^2 = \frac{\|\chi_\rho(g)\|^2}{d_\rho} \)

**Proof.** We begin by observing that

\[ \hat{\mu}_{gC(g)}(\rho) = \mathbb{E}_x [\mu_{gC(g)}(x) \cdot \rho(x)] = \mathbb{E}_x [\mu_{gC(g)}(gx) \cdot \rho(gx)] = \mathbb{E}_x [\mu_{gC(g)}(gx) \cdot \rho(g) \cdot \rho(x)] = \rho(g) \cdot \chi_\rho(g) \cdot \rho(x) \]

where

\[ \|\hat{\mu}_{gC(g)}\|_{\text{HS}}^2 = \frac{\|\chi_\rho(g)\|^2}{d_\rho} \]

On the other hand, as \( \mu_{C(g)} \) is a class function, we have \( \hat{\mu}_{C(g)}(\rho) = c \cdot I_{d_\rho} \) for some constant \( c \in \mathbb{C} \). The constant \( c \) can be determined by taking trace on either side of \( cI_{d_\rho} = \hat{\mu}_{C(g)} = \mathbb{E}_x [\mu_{C(g)}(x) \cdot \rho(x)] \)

and noting that \( \text{Trace}(\rho(x)) = \chi_\rho(g) \) as follows:

\[ c \cdot d_\rho = \mathbb{E}_x [\mu_{C(g)}(x) \cdot \chi_\rho(x)] = \mathbb{E}_x [\mu_{C(g)}(x)] \cdot \chi_\rho(g) = \chi_\rho(g). \]
Hence, \( c = \frac{x_{\rho}(g)}{d_{\rho}} \) and \( \hat{\mu}_{gC(g)} = \frac{x_{\rho}(g)}{d_{\rho}} \cdot \rho(g) \). Lastly we have,

\[
\|\hat{\mu}_{gC(g)}\|_{HS}^2 = \left\| \frac{x_{\rho}(g)}{d_{\rho}} \cdot \rho(g) \right\|_{HS}^2
\]

\[
= \frac{|x_{\rho}(g)|^2}{d_{\rho}^2} \cdot \text{Tr} (\rho(g) \cdot \rho(g))
\]

\[
= \frac{|x_{\rho}(g)|^2}{d_{\rho}^2} \cdot d_{\rho}
\]

(By unitariness of \( \rho(g) \))

The key property of \( D \)-quasirandom groups that we will be using is the following inequality due to Babai, Nikolov and Pyber, the proof of which we provide for the sake of completeness.

**Lemma 6 ([BNP08]).** If \( G \) is a \( D \)-quasirandom group and \( f_1, f_2 : G \rightarrow \mathbb{C} \) such that either \( f_1 \) or \( f_2 \) is mean zero then

\[
\| f_1 \ast f_2 \|_2 \leq \frac{1}{\sqrt{D}} \cdot \| f_1 \|_2 \cdot \| f_2 \|_2.
\]

**Proof.**

\[
\| f_1 \ast f_2 \|_2^2 = \sum_{\rho \in \hat{G}} d_{\rho} \| \hat{f}_1 \ast \hat{f}_2(\rho) \|_{HS}^2
\]

\[
= \sum_{\rho \in \hat{G}} d_{\rho} \| \hat{f}_1(\rho) \cdot \hat{f}_2(\rho) \|_{HS}^2
\]

\[
\leq \sum_{\rho \in \hat{G}} d_{\rho} \| \hat{f}_1(\rho) \|_{HS}^2 \cdot \| \hat{f}_2(\rho) \|_{HS}^2
\]

(By submultiplicativity of norm)

\[
= \sum_{1 \neq \rho \in \hat{G}} d_{\rho} \| \hat{f}_1(\rho) \|_{HS}^2 \cdot \| \hat{f}_2(\rho) \|_{HS}^2
\]

(By mean zeroness)

\[
\leq \frac{1}{D} \cdot \sum_{1 \neq \rho \in \hat{G}} d_{\rho}^2 \| \hat{f}_1(\rho) \|_{HS}^2 \cdot \| \hat{f}_2(\rho) \|_{HS}^2
\]

(By \( D \)-quasirandomness)

\[
\leq \frac{1}{D} \left( \sum_{1 \neq \rho \in \hat{G}} d_{\rho} \| \hat{f}_1(\rho) \|_{HS}^2 \right) \cdot \left( \sum_{1 \neq \rho \in \hat{G}} d_{\rho} \| \hat{f}_2(\rho) \|_{HS}^2 \right)
\]

\[
\leq \frac{1}{D} \cdot \| f_1 \|_2^2 \cdot \| f_2 \|_2^2.
\]

\[\blacksquare\]

The following is a simple corollary of Lemma 6.

**Corollary 7.** If \( G \) is \( D \)-quasirandom; \( f : G \rightarrow \mathbb{C} \) has zero mean and \( \| f \|_{\infty} \leq 1 \) then

\[
\mathbb{E}_b \left[ \| \Delta_b f(x) \| \right] \leq \frac{1}{\sqrt{D}}.
\]
Proof. Let \( f'(x) := f(x^{-1}) \). We have,

\[
\mathbb{E}_b \left[ \mathbb{E}_x [\Delta_b f(x)] \right] = \mathbb{E}_b \left[ \mathbb{E}_x [f(x)f(b)] \right]
\]

\[
= \mathbb{E}_b \left[ \mathbb{E}_x [f'(x^{-1})f(x)] \right]
\]

\[
= \mathbb{E}_b \left[ |f' \ast f(b)| \right]
\]

\[
\leq \mathbb{E}_b \left[ |f' \ast f(b)|^2 \right]^{1/2}
\]

(Because of Cauchy-Schwarz inequality)

\[
= \| f' \ast f \|_2
\]

\[
\leq \frac{1}{\sqrt{D}} \cdot \| f' \|_2 \cdot \| f \|_2
\]

(By Lemma 6)

\[
\leq \frac{1}{\sqrt{D}}.
\]

(Since \( \| f \|_2 \leq \| f \|_\infty \leq 1 \)).

\[\square\]

### 3 Proof of Theorem 1

The following proposition is where we deviate from Peluse’s proof [Pel18]. We give an elementary proof for every quasirandom group while Peluse proved the same result for simple finite groups using the result of Liebeck and Shalev [LS04] to bound the Witten zeta function \( \zeta_G \) for simple finite groups.

**Proposition 8.** Let \( G \) be a \( D \)-quasirandom group. Let \( f : G \rightarrow \mathbb{C} \) such that \( \| f \|_\infty \leq 1 \), \( \mathbb{E}[f] = 0 \) and \( f_b \) is the mean zero component of the function \( \Delta_b f \) (i.e., \( f_b(x) = \Delta_b f(x) - \mathbb{E}_x[\Delta_b f(x)] \)). Then

\[
\mathbb{E}_{g,b} \left[ \mathbb{E}_x [\Delta_b f(x) \cdot (f_{g^{-1}bg} \ast \mu_{g^{-1}C(g^{-1})})(x)] \right] \leq \frac{1}{\sqrt{D}}.
\]

**Proof.** Let us denote the expression on the L.H.S. as \( \Gamma \). We use simple manipulations and previously stated facts to simplify the expression.

\[
\Gamma^2 \leq \mathbb{E}_{g,b} \left[ \| \Delta_b f \|_2 \cdot \| (f_{g^{-1}bg} \ast \mu_{g^{-1}C(g^{-1})}) \|_2 \right]^2
\]

(By Cauchy-Schwarz inequality)

\[
\leq \mathbb{E}_{g,b} \left[ \| f_{g^{-1}bg} \ast \mu_{g^{-1}C(g^{-1})} \|_2 \right]^2
\]

(Since \( \| \Delta_b f \|_2 \leq 1 \))

\[
\leq \mathbb{E}_{g,b} \left[ \| f_{g^{-1}bg} \ast \mu_{g^{-1}C(g^{-1})} \|_2 \right]^2
\]

(By Cauchy Schwarz inequality)

\[
= \mathbb{E}_{g,b} \left[ \sum_{1 \neq \rho \in G} d_\rho \cdot \| \hat{f}_{g^{-1}bg}(\rho) \cdot \hat{\mu}_{g^{-1}C(g^{-1})}(\rho) \|_{\text{HS}}^2 \right]
\]

(By Parseval’s identity & \( \hat{f}_{g^{-1}bg}(1) = 0 \))

\[
\leq \mathbb{E}_{g,b} \left[ \sum_{1 \neq \rho \in G} d_\rho \cdot \| \hat{f}_{g^{-1}bg}(\rho) \|_{\text{HS}}^2 \cdot \| \hat{\mu}_{g^{-1}C(g^{-1})}(\rho) \|_{\text{HS}}^2 \right]
\]

(By submultiplicativity of norm)
Theorem 1. This part of the proof is similar to the corresponding

\textit{Proof of} \cite{Pel18} \textit{we use to give a sharper bound.}

being that we have a complete characterization of the Fourier

expression that appears in the paper of Peluse \cite{E}

we assume

-quasirandom group.

\[ g \in G \]

\[ \sum_{1 \neq \rho \in \hat{G}} |\hat{f}_b(\rho)|^2 \cdot |\chi_{\rho}(g)|^2 \]

Now using the fact that \( gb \in G \) for a fixed \( g \) and a uniformly random \( b \)

in \( G \), we can simply the above expression as follows.

\[
\Gamma^2 \leq \sum_{1 \neq \rho \in \hat{G}} E_g \left[ |\chi_{\rho}(g)|^2 \cdot E_b \left[ ||\hat{f}_b(\rho)||_{HS}^2 \right] \right]
\]

\[
= \sum_{1 \neq \rho \in \hat{G}} E_b \left[ ||\hat{f}_b(\rho)||_{HS}^2 \right] \cdot E_g \left[ |\chi_{\rho}(g)|^2 \right]
\]

\[
= \sum_{1 \neq \rho \in \hat{G}} E_b \left[ ||\hat{f}_b(\rho)||_{HS}^2 \right]
\]

\[
= E_b \left[ \sum_{1 \neq \rho \in \hat{G}} ||\hat{f}_b(\rho)||_{HS}^2 \right].
\]

Finally, we use the fact that all the terms in the summation are non-negative and the group \( G \) is a

\( D \)-quasirandom group.

\[
\Gamma^2 \leq \frac{1}{D} \cdot E_b \left[ \sum_{1 \neq \rho \in \hat{G}} d_\rho \cdot ||\hat{f}_b(\rho)||_{HS}^2 \right]
\]

\[
= \frac{1}{D} \cdot E_b \left[ ||f_b||_2^2 \right]
\]

(Because \( ||f_b||_2^2 \leq 1 \)).

The proof of this lemma is similar to the proof of the BNP inequality (Lemma 6). The key difference

being that we have a complete characterization of the Fourier transform of \( \mu_{gC(g)} \) from Claim 5 which

we use to give a sharper bound.

We are now ready to prove the main Theorem 1. This part of the proof is similar to the corresponding

expression that appears in the paper of Peluse \cite{Pel18}, which is in turn inspired by Tao’s adaptation of

Gowers’ repeated Cauchy-Schwarz trick to the nonebeian setting. We, however, present the entire

proof for the sake of completeness.

\textit{Proof of Theorem 1.} Let us denote the L.H.S. of the expression by \( \Theta_{f_1,f_2,f_3}^4 \). Without loss of generality

we assume \( E[f_3] = 0 \). Now we have,

\[
\Theta_{f_1,f_2,f_3}^4 = \left| E_{x,y} \left[ f_1(x)f_2(xy)f_3(xy^2) \right] \right|^4
\]

\[
= \left| E_{x,z} \left[ f_1(xz^{-1})f_2(x)f_3(xz) \right] \right|^4
\]

(Change of variables: \( x \leftarrow xy, z \leftarrow y \)).
\[
\begin{align*}
\leq & \left| \mathbb{E}_{x,z_1,z_2} \left[ f_1(xz_1^{-1}f_1(xz_2^{-1})f_3(xz_1)f_3(xz_2) \right] \right|^2 \\
& \quad \text{(Cauchy-Schwarz over } x; \|f_2\|_\infty = 1 \text{ and expansion)} \\
= & \left| \mathbb{E}_{y,z,a} \left[ f_1(y)f_1(ya)f_3(yz^2)f_3(yza^{-1}z) \right] \right|^2 \\
& \quad \text{(Change of variables: } y \leftarrow xz_1^{-1}, z \leftarrow z_1, a \leftarrow z_1z_2^{-1}) \\
\leq & \left| \mathbb{E}_{y,a,z_1,z_2} \left[ \Delta_{z_1^{-1}a^{-1}z_1} f_3(yz_1^2) \cdot \Delta_{z_2^{-1}a^{-1}z_2} f_3(yz_2^2) \right] \right| \\
& \quad \text{(Cauchy-Schwarz over } y,a; \|f_1\|_\infty \leq 1). \\
\end{align*}
\]

Now, using the following change of variables, \(z \leftarrow z_1, x \leftarrow yz_1^2, b \leftarrow z_1^{-1}a^{-1}z_1, g \leftarrow z_1^{-1}z_2\), we get

\[
\begin{align*}
\Theta^4_{f_1,f_2,f_3} & \leq \left| \mathbb{E}_{x,b,g} \left[ \Delta_b f_3(x) \cdot \Delta_{g^{-1}bg} f_3(xz^{-1}gzh) \right] \right| \\
& = \left| \mathbb{E}_{x,b,g} \left[ \Delta_b f_3(x) \cdot \Delta_{g^{-1}bg} f_3(xz^{-1}gzh) \right] \right| \\
& = \left| \mathbb{E}_{x,b,g} \left[ \Delta_b f_3(x) \cdot \mathbb{E}_{a} [\Delta_{g^{-1}bg} f_3(xa^{-1}) \cdot \frac{|G|}{|C(g^{-1})|} \mu_{g^{-1}C(g^{-1})}(a)] \right] \right| \\
& = \left| \mathbb{E}_{x,b,g} \left[ \Delta_b f_3(x) \cdot \mathbb{E}_{a} [\Delta_{g^{-1}bg} f_3(xa^{-1}) \cdot \mu_{g^{-1}C(g^{-1})}(a)] \right] \right| \\
& = \left| \mathbb{E}_{x,b,g} \left[ \Delta_b f_3(x) \cdot \Delta_{g^{-1}bg} f_3 \ast \mu_{g^{-1}C(g^{-1})}(x) \right] \right|.
\end{align*}
\]

We now separate the function \(\Delta_{g^{-1}bg} f_3\) from its the mean zero part as follows: Let \(\Delta_{g^{-1}bg} f_3 = f'_{g^{-1}bg} + f_{g^{-1}bg}\) where \(f'_{g^{-1}bg} = \mathbb{E}_x [\Delta_{g^{-1}bg} f_3(x)]\) and \(f_{g^{-1}bg}(x) = \Delta_{g^{-1}bg} f_3(x) - f'_{g^{-1}bg}\).

\[
\begin{align*}
\Theta^4_{f_1,f_2,f_3} & \leq \left| \mathbb{E}_{x,b,g} \left[ \Delta_b f_3(x) \cdot (f_{g^{-1}bg} + f'_{g^{-1}bg}) \cdot \mu_{g^{-1}C(g^{-1})}(x) \right] \right| \\
& \leq \mathbb{E}_{b,g} \left[ \left| \mathbb{E}_x \left[ \Delta_b f_3(x) \cdot f_{g^{-1}bg} \cdot \mu_{g^{-1}C(g^{-1})}(x) \right] \right| \\
& \quad \quad + \mathbb{E}_{b,g} \left[ \left| \mathbb{E}_x \left[ \Delta_b f_3(x) \cdot f'_{g^{-1}bg} \cdot \mu_{g^{-1}C(g^{-1})}(x) \right] \right| \right] \right| \\
& \leq \frac{1}{\sqrt{D}} + \mathbb{E}_{b,g} \left[ \left| \mathbb{E}_x \left[ \Delta_b f_3(x) \right] \right| \cdot \|f'_{g^{-1}bg}\|_\infty \right] \\
& \quad \quad \text{(Using Proposition 8 to bound the first expectation)} \\
& = \frac{1}{\sqrt{D}} + \mathbb{E}_{b,g} \left[ \left| \mathbb{E}_x \left[ \Delta_b f_3(x) \right] \right| \cdot |f'_{g^{-1}bg}| \right] \\
& \leq \frac{1}{\sqrt{D}} + \mathbb{E}_{b} \left[ \left| \mathbb{E}_x \left[ \Delta_b f_3(x) \right] \right| \right] \\
& \quad \quad \text{(Using } |f'_{g^{-1}bg}| \leq 1) \\
& \leq \frac{2}{\sqrt{D}}, \\
& \quad \quad \text{(By Corollary 7 and } \|f_3\|_\infty \leq 1). \\
\end{align*}
\]
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