Domination related parameters in the generalized lexicographic product of graphs

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Abstract

In this paper we begin an exploration of several domination-related parameters (among which are the total, restrained, total restrained, paired, outer connected and total outer connected domination numbers) in the generalized lexicographic product (GLP for short) of graphs. We prove that for each GLP of graphs there exist several equality chains containing these parameters. Some known results on standard lexicographic product of two graphs are generalized or/and extended. We also obtain results on well $\mu$-dominated GLP of graphs, where $\mu$ stands for any of the above mentioned domination parameters. In particular, we present a characterization of well $\mu$-dominated GLP of graphs in the cases when $\mu$ is the domination number or the total domination number.

Keywords: total/restrained/acyclic/paired/outer-connected domination; generalized lexicographic product; equality chains.

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1 Introduction

One of the fastest growing areas within graph theory is the study of domination. Many variants of the basic concepts of domination have appeared in the literature. We refer to [11] for a survey of the area. As many other graph invariants, domination has been studied on different graph products. Several papers have been published in the last fifteen years concerning various types domination in the lexicographic product of two graphs, including domination (Nowakowski and Rall [14], Šumajek et al. [16] and Gözüpek et al. [10]), total and restrained domination (Zhang et al. [20]), Roman domination (Šumajek et al. [16]), rainbow domination (Šumajek et al. [17]), super domination (Dettlaff et al. [7]) and double domination (Cabrera Martínez et al. [1]). However, to the best knowledge of the author, there are no studies related to domination in graphs representable as generalized lexicographic products. This fact motivates us to begin an exploration of several domination-related parameters (among which are the total, restrained, total restrained, outer connected and total outer connected domination numbers) in the generalized lexicographic product of graphs.

We give basic terminologies and notations in the rest of this section. All graphs in this paper will be finite, simple, and undirected. We use [11] as a reference for terminology and notation which are not explicitly defined here.

In a graph $G$, for a subset $S \subseteq V(G)$ the subgraph induced by $S$ is the graph $\langle S \rangle$ with vertex set $S$ and edge set $\{xy \in E(G) \mid x, y \in S\}$. The complement $\overline{G}$ of $G$ is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of $G$. We write $K_n$ for the complete graph of order $n$ and $P_n$ for the path on $n$ vertices. Let $C_m$ denote the cycle of length $m$. For any vertex $x$ of a graph $G$, $N_G(x)$ denotes the set of all neighbors of $x$ in $G$, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of $x$ is $\deg_G(x) = |N_G(x)|$. The minimum and maximum degrees of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $S \subseteq V(G)$, let $N_G[S] = \cup_{v \in S} N_G[v]$. The distance between vertices $x$ and $y$ of a graph $G$ is denoted by $\text{dist}_G(x, y)$. An isomorphism of graphs $G$ and $H$ is a bijection $f : V(G) \to V(H)$ such that any two vertices $u$ and $v$ of $G$ are adjacent in $G$ if and only if $f(u)$ and $f(v)$ are adjacent in $H$. If an isomorphism exists between two graphs, then the graphs are called isomorphic and denoted as $G \simeq H$. We use the notation $\lbrack k \rbrack$ for $\{1, 2, \ldots, k\}$.

Let $G$ be a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and let $\Phi = (F_1, F_2, \ldots, F_n)$ be an ordered $n$-tuple of paired disjoint graphs. Denote by
$G[\Phi]$ the graph with vertex set $\bigcup_{i=1}^{n} V(F_i)$ and edge set defined as follows: (a) $F_1, F_2, \ldots, F_n$ are induced subgraphs of $G[\Phi]$, and (b) if $x \in V(F_i)$, $y \in V(F_j)$, $i, j \in [n]$ and $i \neq j$, then $xy \in E(G[\Phi])$ if and only if $ij \in E(G)$. A graph $G[\Phi]$ is called the generalized lexicographic product of $G$ and $\Phi$. If $F_i \simeq F$ for every $i = 1, 2, \ldots, n$, then $G[\Phi]$ becomes the standard lexicographic product $G[F]$. Each subset $U = \{u_1, u_2, \ldots, u_n\} \subseteq V(G[\Phi])$ such that every $i \in [n]$, is called a $G$-layer. From the definition of $G[\Phi]$ it immediately follow:

(A) (folklore) $G[\Phi] \simeq G$ if and only if $G[\Phi] = G[K_1]$. $G[F] \simeq F$ if and only if $G \simeq K_1$. If $G$ has at least two vertices, then $G[\Phi]$ is connected if and only if $G$ is connected. If $G$ is edgeless, then $G[\Phi] = \bigcup_{i=1}^{n} F_i$. For any $G$-layer $U = \{u_1, u_2, \ldots, u_n\}$ the bijection $f: V(G) \rightarrow U$ defined by $f(i) = u_i \in V(F_i)$ is an isomorphism between $G$ and $\langle U \rangle$. For any $x \in V(F_i)$ and $y \in V(F_j)$, $i \neq j$, is fulfilled $dist_{G[\Phi]}(x,y) = dist_{G}(i,j)$.

The equality $dist_{G[\Phi]}(x,y) = dist_{G}(i,j)$ will be used in the sequel without specific references.

Since for any domination-related parameter $\mu$, which we consider in this work, and for any two disjoint graphs $G_1$ and $G_2$ is fulfilled $\mu(G_1 \cup G_2) = \mu(G_1) + \mu(G_2)$ (when it is known that at least one of the left or right sides of this equality exists), we restrict our attention only on connected generalized lexicographic products. Therefore, in what follows when a graph $G[\Phi]$ is under consideration we assume that $G$ is a connected graph of order $n \geq 2$. Unless otherwise stated, we also assume that always $\Phi = (F_1, F_2, \ldots, F_n)$.

Let $\mathcal{I}$ denote the set of all mutually nonisomorphic graphs. A graph property is any non-empty subset of $\mathcal{I}$. We say that a graph $G$ has property $\mathcal{P}$ whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to $G$. For example we list some graph properties:

- $\mathcal{T} = \{H \in \mathcal{I} : \delta(H) \geq 1\}$;
- $\mathcal{F} = \{H \in \mathcal{I} : H$ is a forest\};
- $\mathcal{M} = \{H \in \mathcal{I} : H$ has a perfect matching \};
- $\mathcal{S}_k = \{H \in \mathcal{I} : \Delta(G) \leq k\}$, $k \geq 0$.
- $\mathcal{C} = \{H \in \mathcal{I} : H$ is connected\}.

Any set $S \subseteq V(G)$ such that $\langle S \rangle$ possesses the property $\mathcal{A} \subseteq \mathcal{I}$ and $\langle V(G) - S \rangle$ possesses the property $\mathcal{B} \subseteq \mathcal{I}$ is called an $(\mathcal{A}, \mathcal{B})$-set. A dominating set for a graph $G$ is a set of vertices $D \subseteq V(G)$ such that every vertex
of $G$ is either in $D$ or is adjacent to an element of $D$. A dominating $(A,B)$-set $S$ of a graph $G$ is a minimal dominating $(A,B)$-set if no set $S' \subsetneq S$ is a dominating $(A,B)$-set. The set of all minimal dominating $(A,B)$-sets of a graph $G$ is denoted by $MD_{(A,B)}(G)$. The domination number with respect to the pair $(A,B)$, denoted by $\gamma_{(A,B)}(G)$, is the smallest cardinality of a dominating $(A,B)$-set of $G$. The upper domination number with respect to the pair $(A,B)$, denoted by $\Gamma_{(A,B)}(G)$, is the maximum cardinality of a minimal dominating $(A,B)$-set of $G$. A $\gamma_{(A,B)}$ (resp., $\Gamma_{(A,B)}$) -set of a graph $G$ is every set in $MD_{(A,B)}(G)$ having cardinality $\gamma_{(A,B)}(G)$ (resp., $\Gamma_{(A,B)}(G)$). Note that:

(a) $\gamma_{(x,I)}(G)$ and $\Gamma_{(x,I)}(G)$ are known as the domination and upper domination numbers $\gamma(G)$ and $\Gamma(G)$ of $G$, respectively,

(b) $\gamma_{(S_0,I)}(G)$ and $\Gamma_{(S_0,I)}(G)$ are known as the independent domination number $i(G)$ and the independence number $\beta_0(G)$,

(c) $\gamma_{(T,I)}(G)$ and $\Gamma_{(T,I)}(G)$ are known as the total domination and upper total domination numbers $\gamma_t(G)$ and $\Gamma_t(G)$ (3),

(d) $\gamma_{(x,T)}(G)$ and $\Gamma_{(x,T)}(G)$ are known as the restrained domination and upper restrained domination numbers $\gamma_r(G)$ and $\Gamma_r(G)$ (18),

(e) $\gamma_{(T,T)}(G)$ and $\Gamma_{(T,T)}(G)$ are known as the total restrained domination and upper total restrained domination numbers $\gamma_{tr}(G)$ and $\Gamma_{tr}(G)$ (2).

(f) $\gamma_{(x,C)}(G)$ and $\Gamma_{(x,C)}(G)$ are known as the outer-connected domination and upper outer-connected domination numbers $\gamma^{oc}(G)$ and $\Gamma^{oc}(G)$ (4),

(g) $\gamma_{(T,C)}(G)$ and $\Gamma_{(T,C)}(G)$ are known as the total outer-connected domination and upper total outer-connected domination numbers $\gamma^{oc}_t(G)$ and $\Gamma^{oc}_t(G)$ (5),

(h) $\gamma_{(M,I)}(G)$ and $\Gamma_{(M,I)}(G)$ are known as the paired domination and upper paired domination numbers $\gamma_p(G)$ and $\Gamma_p(G)$ (12).

The following inequalities are folklore: $\gamma(G) \leq \gamma_t(G) \leq \min\{\gamma_p(G),\gamma_{tr}(G),\gamma^{oc}_t(G)\}$, $\gamma_r(G) \leq \gamma_{tr}(G)$, $\gamma^{oc}(G) \leq \gamma^{oc}_t(G)$ and $\gamma(G) \leq \min\{\gamma_r(G),\gamma^{oc}(G)\}$.

Not much work has been done on finding relationships between (a) the value of a given domination parameter in the standard lexicographic product and that of its factors, and (b) the values of two given domination parameters.
in the standard lexicographic product. We list those results that relate to the domination parameters above defined.

**Theorem A.** Let $G_1$ and $G_2$ be graphs with at least two vertices.

(i) (Zhang et al. [20]) If $\gamma(G_2) = 1$, then $\gamma(G_1[G_2]) = \gamma(G_1)$.

(ii) (Zhang et al. [20]) $\gamma_t(G_1[G_2]) = \gamma_t(G_1)$.

(iii) (Zhang et al. [20]) If $\delta(G_1) \geq 1$, then $\gamma_r(G_1[G_2]) = \gamma_r(G_1)$.

(iv) (Šumenjak et al. [16]) If $G_1$ and $G_2$ are connected and $\gamma(G_2) \geq 2$, then $\gamma(G_1[G_2]) = \gamma_t(G_1[G_2]) = \gamma_t(G_1)$.

In Section 2 we obtain results on the parameters $\gamma$, $\gamma_t$, $\gamma_r$, $\gamma_{tr}$, $\gamma_p$, $\gamma_{oc}$ and $\gamma_{oc}^t$ in a generalized lexicographic product. Some of them generalize or/and extend those stated in the above theorem.

To continue we need the following definition.

**Definition 1.** Let $A, B \subseteq I$. A graph $G$ is said to be well $\gamma(A,B)$-dominated if $\gamma(A,B)(G) = \Gamma(A,B)(G)$.

In a 1970 paper, Plummer [15] introduced the notion of considering graphs in which all maximal independent sets have the same size; he called a graph having this property a well-covered graph. Equivalently, a well-covered graph is one in which the greedy algorithm for constructing independent sets yields always maximum independent sets. Clearly well-covered graphs form the class of all well $i$-dominated graphs. Topp and Volkmann [19] gave a characterization of well covered generalized lexicographic product of graphs.

Well $\gamma$-dominated graphs were introduced by Finbow et al. [9]. Obviously each well $\gamma$-dominated graph is well covered. Recall the characterization of the well-dominated nontrivial lexicographic product of two graphs, which was recently obtained by Gözüpek, Hujdurović and Milanič in [10].

**Theorem B.** [10] A nontrivial lexicographic product, $G[F]$, of a connected graph $G$ and a graph $H$ is well-dominated if and only if one of the following conditions holds:

(i) $G$ is well-dominated and $F$ is complete, or

(ii) $G$ is complete and $F$ is well-dominated with $\gamma(F) = 2$. 

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In Section 3 we present results on well $\gamma_{(A,B)}$-dominated graphs; in particular we characterize well $\gamma$-dominated and well $\gamma_t$-dominated generalized lexicographic product of graphs.

We conclude in Section 4 with some open problems.

2 Seven domination parameters

Recall that the equality $\gamma_t(G) = \gamma_t(G[F])$ was proven by X. Zhang et al. [20]. The next theorem shows that the equality remains valid if we remove $F$ by $\Phi$.

**Theorem 2.** If $I$ is a $\gamma_t$-set of some $G$-layer of $G[\Phi]$, then $I$ is a $\gamma_t$-set of $G[\Phi]$. In particular, $\gamma_t(G) = \gamma_t(G[\Phi])$.

**Proof.** Let $U$ be a $G$-layer of $G[\Phi]$ and $I$ a $\gamma_t$-set of $\langle U \rangle$. By the definition of a graph $G[\Phi]$ we immediately obtain that $I$ is a total dominating set of $G[\Phi]$. Since $\langle U \rangle \simeq G$, $\gamma_t(G) \geq \gamma_t(G[\Phi])$. Now if the equality holds, then clearly $I$ is a $\gamma_t$-set of $G[\Phi]$.

For each $\gamma_t$-set $T$ of $G[\Phi]$ denote by $s_T$ the number of all $i$ for which $T$ and $V(F_i)$ have at least two elements in common. Choose now a $\gamma_t$-set $R$ of $G[\Phi]$ so that $s_R$ is minimum. Suppose $s_R \neq 0$ and $x, y \in V(F_m) \cap R$ for some $m \in [n]$. Then $|R \cap V(F_m)| = 0$ for all $F_i$'s such that $\text{lm} \in E(G)$. Consider now the set $R_1 = (R - \{y\}) \cup \{z\}$, where $z \in V(F_i)$. Obviously $R_1$ is a $\gamma_t$-set $G[\Phi]$ with $s_R > s_{R_1}$, which contradicts the choice of $R$. Thus $s_R = 0$ and then there is a $G$-layer of $G[\Phi]$, say $H$, which contains $R$. Since clearly $R$ is a total dominating set of $\langle H \rangle$ and $\langle H \rangle \simeq G$, we obtain $\gamma_t(G) \leq \gamma_t(G[\Phi])$. □

**Theorem 3.** $\gamma(G) \leq \gamma(G[\Phi])$. The equality holds if and only if there is a $\gamma$-set $I$ of $G$ such that if $i_j \in I$ is an isolated vertex of $\langle I \rangle$, then $\gamma(F_j) = 1$. If $\gamma(G) = \gamma(G[\Phi])$, then $|D \cap V(F_i)| \leq 1$, $i = 1, 2, \ldots, n$, for each $\gamma$-set $D$ of $G[\Phi]$.

**Proof.** Let $D$ be a $\gamma$-set of $G[\Phi]$ and let $F_{i_1}, F_{i_2}, \ldots, F_{i_k}$ be all $F_j$'s each of which has a common vertex with $D$. Choose a $G$-layer $U = \{u_1, u_2, \ldots, u_n\}$ so that $u_i \in V(F_{i_s}) \cap D$ for $s = 1, 2, \ldots, k$. Clearly $D_1 = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ is a dominating set of $U$. Since $\langle U \rangle \simeq G$, we have $\gamma(G) = \gamma(U) \leq |D_1| \leq |D| = \gamma(G[\Phi])$.

Assume now that $\gamma(G) = \gamma(G[\Phi]) = k$. Then $D_1$ is a $\gamma$-set of $U$ and $D_1 = D$. This immediately implies $|D \cap V(F_i)| \leq 1$ for all $i \in [n]$. Since
If Corollary 6.

Let there be a \( \gamma \)-set \( I = \{i_1, i_2, ..., i_k\} \) of \( G \) such that for each isolated vertex \( i_s \) in \( I \), \( \gamma(F_{i_s}) = 1 \). But then the set \( R = \{x_{i_1}, x_{i_2}, ..., x_{i_k}\} \), where \( x_{i_s} \in V(F_{i_s}) \) has maximum degree in \( F_{i_s} \), \( s = 1, 2, ..., k \), is a dominating set of \( G[\Phi] \). Hence \( \gamma(G) \geq \gamma(G[\Phi]) \) and the required follows. \( \square \)

**Corollary 4.** If \( \gamma(F_1) = \gamma(F_2) = ... = \gamma(F_n) = 1 \), then \( \gamma(G) = \gamma(G[\Phi]) \).

By Theorem 2, Theorem 3 and the well known inequalities \( \gamma(G) \leq \gamma_l(G) \leq 2\gamma(G) \), we obtain the next inequality chain.

**Corollary 5.** \( \gamma(G) \leq \gamma(G[\Phi]) \leq \gamma_l(G[\Phi]) = \gamma_l(G) \leq 2\gamma(G) \leq 2\gamma(G[\Phi]) \).

An immediate consequence of this corollary is the following.

**Corollary 6.** If \( \gamma(G) = \gamma_l(G) \), then \( \gamma(G[\Phi]) = \gamma_l(G[\Phi]) \). If \( \gamma_l(G[\Phi]) = 2\gamma(G[\Phi]) \), then \( \gamma_l(G) = 2\gamma(G) \).

Now we concentrate on the case when all \( F_i \)'s have at least two vertices. We need the following key lemma for our purpose in this work.

**Lemma 7.** Let \( \mu \in \{\gamma, \gamma_l\} \), \( D \) be a \( \mu \)-set of \( G[\Phi] \) and \( |V(F_i)| \geq 2 \) for all \( i \in [n] \). Then the following assertions hold.

(i) \( |D \cap V(F_i)| \leq 2 \) for all \( i = 1, 2, ..., n \). If \( |D \cap V(F_j)| = 2 \) for some \( s \in [n] \), then no vertex of \( F_s \) is adjacent to a vertex of \( D - V(F_s) \) and \( D \cap V(F_s) \) is a \( \mu \)-set of \( F_s \). If \( |D \cap V(F_i)| = |D \cap V(F_j)| = 2 \), \( i \neq j \), then the distance between any vertex of \( F_i \) to any vertex of \( F_j \) is at least three.

(ii) Let \( R = \{i \mid 2 = |D \cap V(F_i)| \neq \emptyset \} \). For all \( i \in R \) let \( D \cap V(F_i) = \{z_{i1}, z_{i2}\} \) and \( x_i \in N(z_{i2}) - V(F_i) \). Then the set \( D^* = (D - \bigcup_{i \in R}\{z_{i2}\}) \cup \bigcup_{i \in R}\{x_i\} \) is a \( \mu \)-set of \( G[\Phi] \) and \( |D^* \cap V(F_i)| \leq 1 \) for all \( r \in [n] \).

(iii) \( G[\Phi] \) has a \( \mu \)-set \( U \) such that (a) \( |U \cap V(F_i)| \leq 1 \) for all \( i \in [n] \), (b) if \( \mu = \gamma \), then \( U \) is both a \( \gamma \)-set and a \( \gamma^c \)-set of \( G[\Phi] \), and (c) if \( \mu = \gamma_l \), then \( U \) is both a \( \gamma_l \)-set and a \( \gamma^c_l \)-set of \( G[\Phi] \).

(iv) Let \( \mu = \gamma \) and \( \gamma(F_i) \geq 2 \) for all \( i \in [n] \). Then a \( \gamma \)-set \( U \) (see (iii)) is a \( \nu \)-set for all \( \nu = \gamma, \gamma_l, \gamma_r, \gamma^c, \gamma^c_l \).
Proof. (i) Let for some \( j \in [n] \) is fulfilled \( D \cap V(F_j) = \{u_1, u_2, \ldots, u_r\} \), where \( r \geq 2 \). Then \( N[u_1, w] \supseteq N[u_2, \ldots, u_r] \) for any neighbor \( w \) of \( u_1 \) which is outside \( V(F_j) \). As \( D \) is a \( \mu \)-set of \( G[\Phi] \), we have \( w \not\in D \) and \( r = 2 \). Since \( w \) was chosen arbitrarily, \( N[u_1, u_2] \cap D = \{u_1, u_2\} \). But then \( \{u_1, u_2\} \) is a \( \mu \)-set of \( F_j \). Let \( |D \cap V(F_i)| = |D \cap V(F_s)| = 2 \) for some \( i, s \in [n] \), \( i \neq s \). Suppose \( z_1, z_2, z_3 \) is a shortest path in \( G[\Phi] \), where \( z_1 \in V(F_i) \) and \( z_3 \in V(F_s) \cap D \). Clearly \( z_2 \not\in V(F_i) \cup V(F_j) \cup D \). Then \( D' = (D - \{z_3\}) \cup \{z_2\} \) is a \( \mu \)-set and \( z_2 \) is a common neighbor of the vertices in \( V(F_i) \cap D \), a contradiction.

(ii) By (i) all \( x_i \)’s are paired distinct and outside \( D \), and no two of them belong to the same \( F_j \), \( j \in [n] \). Hence \( |D^*| = |D| \), \( |D^* \cap V(F_i)| \leq 1 \) for all \( r \in [n] \) and since \( N[z_{11}, z_{12}] \subset N[z_{11}, u_i] \) and \( z_{11} \) and \( u_i \) are adjacent, \( D^* \) is a \( \mu \)-set of \( G[\Phi] \).

(iii) Define a \( \mu \)-set \( U \) such that \( U = D \) when \( |D \cap V(F_i)| \leq 1 \) for all \( i \in [n] \), and \( U = D^* \) otherwise (see (ii)). Hence \( |U \cap V(F_i)| \leq 1 \) and since \( G \) is connected of order \( n \geq 2 \) and \( |V(F_i)| \geq 2 \) for all \( i \in [n] \), a graph \( V(G[\Phi]) - D \) is connected of order at least two.

(iv) Since \( \gamma(F_i) \geq 2 \) for all \( i \in [n] \), \( U \) is a total dominating set of \( G[\Phi] \) and since \( \gamma(G[\Phi]) \leq \gamma_t(G[\Phi]) \), \( U \) is a \( \gamma_t \)-set of \( G[\Phi] \). The required now immediately follows by (iii).

\[ \square \]

**Theorem 8.** Let \( |V(F_i)| \geq 2 \) for all \( i \in [n] \). Then

\[
(i) \quad \gamma(G[\Phi]) = \gamma_r(G[\Phi]) = \gamma_r^*(G[\Phi]), \\
(ii) \quad \gamma_t(G) = \gamma_t(G[\Phi]) = \gamma_t^*(G[\Phi]) = \gamma_t^*(G[\Phi]).
\]

**Proof.** Immediately by Lemma 7(iii) and Theorem 2. \[ \square \]

Let \( \mu, \nu \in \{\gamma, \gamma_r, \gamma_r^*, \gamma_t, \gamma_t^*, \gamma_t^*\} \). If the sets of all \( \mu \)-sets and all \( \nu \)-sets of a graph \( H \) coincide, then we say \( \mu(H) \) strongly equal to \( \nu(H) \), written \( \mu(H) \equiv \nu(H) \).

**Theorem 9.** Let \( |V(F_i)| \geq 3 \) for all \( i \in [n] \). Then

\[
(i) \quad \gamma(G[\Phi]) \equiv \gamma_r(G[\Phi]) \equiv \gamma_r^*(G[\Phi]), \\
(ii) \quad \gamma_t(G[\Phi]) \equiv \gamma_t^*(G[\Phi]) \equiv \gamma_t^*(G[\Phi]).
\]

**Proof.** By Lemma 7(i) we know that for any \( \mu \)-set \( D, \mu \in \{\gamma, \gamma_t\} \), of \( G[\Phi] \) is fulfilled \( |D \cap V(F_i)| \leq 2 \) for all \( i = 1, 2, \ldots, n \). Since \( n \geq 2 \) and \( |V(F_i)| \geq 3 \) for all \( i \in [n] \), \( D \) is both restrained and outer-connected. The rest immediately follows by the previous theorem. \[ \square \]
Theorem 10. If \( \gamma(F_i) \geq 2 \) for all \( i \in [n] \), then
\[
\gamma(G[\Phi]) = \gamma_r(G[\Phi]) = \gamma_i(G[\Phi]) = \gamma_tr(G[\Phi]) = \gamma^oc(G[\Phi]) = \gamma^t_r(G[\Phi]).
\]
If \( \gamma(F_i) \geq 3 \) for all \( i \in [n] \), then
\[
\gamma(G[\Phi]) \equiv \gamma_r(G[\Phi]) \equiv \gamma_i(G[\Phi]) \equiv \gamma_tr(G[\Phi]) \equiv \gamma^oc(G[\Phi]) \equiv \gamma^t_r(G[\Phi]).
\]
Proof. The first equality chain immediately follows by Lemma \( \text{iv} \). Assume now \( \gamma(F_i) \geq 3 \) for all \( i \in [n] \) and let \( D \) be any \( \gamma \)-set of \( G[\Phi] \). Lemma \( \text{vi} \) implies that \( D \) must be total dominating. The required now follows by Theorem \( \text{v} \) \( \square \)

A dominating set \( D \) of a graph \( G \) is efficient dominating if every vertex in \( V(G) \) is dominated by exactly one vertex of \( D \). Note that each efficient dominating set is a \( \gamma \)-set.

Theorem 11. Let \( D \) be a \( \gamma \)-set of \( G[\Phi] \), \( |V(F_i)| \geq 2 \) and \( |D \cap V(F_i)| \neq 1 \) for all \( i \in [n] \). Let all \( F_i \)'s for which \( |D \cap V(F_i)| \neq 0 \) be \( F_{i_1}, F_{i_2}, \ldots, F_{i_s} \). Then \( |D \cap V(F_{i_r})| = 2 \) for all \( r \in [s] \), \( \{i_1, i_2, \ldots, i_s\} \) is an efficient dominating set of \( G \) and
\[
2\gamma(G) = \gamma_i(G) = \gamma(G[\Phi]) = \gamma_r(G[\Phi]) = \gamma_tr(G[\Phi]) = \gamma^oc(G[\Phi]) = \gamma^t_r(G[\Phi]) = \gamma_p(G[\Phi]).
\]
Proof. By Lemma \( \text{vii} \) the following hold: (a) \( |D \cap V(F_i)| \in \{0, 2\} \) for all \( i \in [n] \), (b) \( |D \cap V(F_i)| = 2 \) if and only if \( i \in \{i_1, i_2, \ldots, i_s\} \), (c) \( \{i_1, i_2, \ldots, i_s\} \) is an efficient dominating set of \( G \) and (d) \( D \cap V(F_{i_r}) \) is a \( \gamma \)-set of \( F_{i_r} \) for all \( r \in [s] \). Therefore \( 2\gamma(G) = \gamma(G[\Phi]) \) and by \( \gamma_i(G[\Phi]) = \gamma_r(G) \leq \gamma_i(G) \leq \gamma_i(G[\Phi]) \), we have \( \gamma_i(G[\Phi]) = \gamma(G[\Phi]) \). In view of Theorem \( \text{vi} \) it remains to prove that \( \gamma(G[\Phi]) = \gamma_p(G[\Phi]) \).

Let \( D \cap V(F_{i_r}) = \{z_{r_1}, z_{r_2}\} \) and \( x_r \in N(z_{r_2}) - V(F_{i_r}) \), \( r \in [s] \). Since \( \{i_1, i_2, \ldots, i_s\} \) is an efficient dominating set of \( G \), the set \( D^* = (D - \cup_{r \in [s]} \{z_{r_2}\}) \cup \cup_{r \in [s]} \{x_r\} \) is a dominating \( (\mathcal{M}, \mathcal{C}) \)-set of \( G^\Phi \) of cardinality \( |D^*| = |D| = \gamma(G[\Phi]) \). Since each dominating \( (\mathcal{M}, \mathcal{C}) \)-set is a dominating \( (\mathcal{M}, \mathcal{E}) \)-set, \( \gamma_p(G[\Phi]) = \gamma(G[\Phi]) \). \( \square \)

Remark 12. In the end of the proof of the above theorem we obtain that a set \( D^* \) is a dominating \( (\mathcal{M}, \mathcal{C}) \)-set of \( G[\Phi] \). Hence under the assumptions of Theorem \( \text{vii} \) \( 2\gamma(G) = \gamma_{(\mathcal{M}, \mathcal{C})}(G[\Phi]) \). It is quite natural to call the numbers \( \gamma_{(\mathcal{M}, \mathcal{C})}(G) \) and \( \Gamma_{(\mathcal{M}, \mathcal{C})}(G) \) the paired outer connected and upper paired outer connected domination numbers of a graph \( G \).
3 Well $\gamma_{(A,B)}$-dominated graphs

We begin with an obvious but very useful observation.

**Observation 13.** Given a graph $G[\Phi]$ and properties $A, B \subseteq \mathcal{I}$. Assume that $F_1$ has a dominating $(A, B)$-set for all $l \leq |n|$, and denote by $\mathcal{D}_{G[\Phi]}(A, B)$ the family of all subsets $U$ of $V(G[\Phi])$ such that $U = \cup_{i=1}^{s} D_{i}$, where $\{l_1, l_2, ..., l_s\}$ is a maximal independent set of $G$ and $D_{i}$ is a minimal dominating $(A, B)$-set of $F_{i}$, $i = 1, 2, ..., s$. Let all elements of $\mathcal{D}_{G[\Phi]}$ be minimal dominating $(A, B)$-sets of $G[\Phi]$. Then

$$\gamma_{(A,B)}(G[\Phi]) \leq \min\{|U| \mid U \in \mathcal{D}_{G[\Phi]}(A, B)\}$$

$$= \min\{\Sigma_{r=1}^{s}\gamma_{(A,B)}(F_{r}) \mid \{i_1, i_2, ..., i_s\} \in In(G)\}$$

$$\leq \min\{\Sigma_{r=1}^{s}\gamma_{(A,B)}(F_{r}) \mid \{j_1, j_2, ..., j_k\} \text{ is an } i \text{-set of } G\} \quad (1)$$

$$\leq i(G) \max\{\gamma_{(A,B)}(F_{j}) \mid j \in [n]\}$$

and

$$\Gamma_{(A,B)}(G[\Phi]) \geq \max\{|U| \mid U \in \mathcal{D}_{G[\Phi]}(A, B)\}$$

$$= \max\{\Sigma_{r=1}^{s}\Gamma_{(A,B)}(F_{r}) \mid \{i_1, i_2, ..., i_s\} \in In(G)\}$$

$$\geq \max\{\Sigma_{r=1}^{s}\Gamma_{(A,B)}(F_{r}) \mid \{j_1, j_2, ..., j_k\} \text{ is a } \beta_{0} \text{-set of } G\} \quad (2)$$

$$\geq \beta_{0}(G) \min\{\Gamma_{(A,B)}(F_{j}) \mid j \in [n]\},$$

where $In(G)$ is the set of all maximal independent sets of $G$.

As a first consequence of this observation a necessary condition for a generalized lexicographic product of graphs to be well $\gamma_{(A,B)}$-dominated follows.

**Corollary 14.** Under the conditions and notation of Observation 13 if $G[\Phi]$ is well $\gamma_{(A,B)}$-dominated, then all $F_{i}$’s are well $\gamma_{(A,B)}$-dominated and $\gamma_{(A,B)}(G[\Phi]) = \Sigma_{r=1}^{s}\gamma_{(A,B)}(F_{r})$ for each $\{i_1, i_2, ..., i_s\} \in In(G)$.

**Proof.** By (1) and (2) we have $\gamma_{(A,B)}(G[\Phi]) \leq \min\{\Sigma_{r=1}^{s}\gamma_{(A,B)}(F_{r}) \mid \{i_1, i_2, ..., i_s\} \in In(G)\}$ and $\max\{\Sigma_{r=1}^{s}\Gamma_{(A,B)}(F_{r}) \mid \{i_1, i_2, ..., i_s\} \in In(G)\} \leq \Gamma_{(A,B)}(G[\Phi])$, respectively. Since $G[\Phi]$ is well $\gamma_{(A,B)}$-dominated, $\gamma_{(A,B)}(G[\Phi]) = \min\{\Sigma_{r=1}^{s}\gamma_{(A,B)}(F_{r}) \mid \{i_1, i_2, ..., i_s\} \in In(G)\} = \max\{\Sigma_{r=1}^{s}\Gamma_{(A,B)}(F_{r}) \mid \{i_1, i_2, ..., i_s\} \in In(G)\} = \Gamma_{(A,B)}(G[\Phi])$. This equality chain immediately implies the required. \qed
Corollary 15. Under the conditions and notation of Observation 13, assume that $\gamma_{(A,B)}(F_1) = \gamma_{(A,B)}(F_2) = \ldots = \gamma_{(A,B)}(F_n)$ and $\Gamma_{(A,B)}(F_1) = \Gamma_{(A,B)}(F_2) = \ldots = \Gamma_{(A,B)}(F_n)$. Then

$$\gamma_{(A,B)}(G[\Phi]) \leq i(G)\gamma_{(A,B)}(F_1) \leq \beta_0(G)\Gamma_{(A,B)}(F_1) \leq \Gamma_{(A,B)}(G[\Phi]). \quad (3)$$

If $G[\Phi]$ is well $\gamma_{(A,B)}$-dominated, then $G$ is well covered.

Proof. The middle inequality is a consequence of $i(G) \leq \beta_0(G)$ and $\gamma_{(A,B)}(G) \leq \Gamma_{(A,B)}(G)$. The left and right inequalities follow immediately by (1) and (2), respectively. If $G[\Phi]$ is well $\gamma_{(A,B)}$-dominated, then the inequality chain becomes equality chain implying $i(G) = \beta_0(G)$. \qed

To formulate our next result, we need the following domination parameters.

(h) $\gamma_{(F,\mathcal{I})}(G)$ and $\Gamma_{(F,\mathcal{I})}(G)$ are the acyclic domination and upper acyclic domination numbers $\gamma_a(G)$ and $\Gamma_a(G)$ (13),

(i) $\gamma_{(S_k,\mathcal{I})}(G)$ and $\Gamma_{(S_k,\mathcal{I})}(G)$ are the $k$-dependent domination and upper $k$-dependent domination numbers $\gamma^k(G)$ and $\Gamma^k(G)$ (15).

Remark 16. Let the pair $(A, B)$ be one of $(\mathcal{I}, \mathcal{I})$, $(F, \mathcal{I})$, $(S_k, \mathcal{I})$, $(\mathcal{I}, \mathcal{T})$, $(\mathcal{T}, \mathcal{I})$, $(\mathcal{T}, \mathcal{T})$ and $(\mathcal{M}, \mathcal{I})$. In addition if $(A, B)$ is one of the last four pairs, let $\delta(F_i) \geq 1$ for all $i \in [n]$. Then the assumptions of Observation 13 are fulfilled. Therefore the inequality chains (1) and (2) as well as Corollary 14 and Corollary 15 are valid for such a pair $(A, B)$.

The next result shows that the left and right inequalities in the chain (3) become equalities in the case when $(A, B) = (S_0, \mathcal{I})$ or equivalently, when $\gamma_{(A,B)} = i$ and $\Gamma_{(A,B)} = \beta_0$.

Theorem 17. (15) when $G[\Phi] = G[\mathcal{F}]$) If $i(F_1) = i(F_2) = \ldots = i(F_n)$, then $i(G[\Phi]) = i(G)i(F_1)$. If $\beta_0(F_1) = \beta_0(F_2) = \ldots = \beta_0(F_n)$, then $\beta_0(G[\Phi]) = \beta_0(G)\beta_0(F_1)$. \[19\]

Proof. By Remark 16 and Corollary 15 we know that $i(G[\Phi]) \leq i(G)i(F_1)$ and $\beta_0(G[\Phi]) \geq \beta_0(G)\beta_0(F_1)$. Let now $I$ be any maximal independent set of $G[\Phi]$ and let $F_i, F_{i+1}, \ldots, F_s$ be all $F_i$'s each of which has a vertex in $I$. Choose $u_v \in V(F_i)$ arbitrarily, $r = 1, 2, \ldots, s$, and consider any $G$-layer $H$ containing all vertices of $U = \{u_{i_1}, u_{i_2}, \ldots, u_{i_s}\}$. Clearly $U$ is a maximal...
independent set of $H \simeq G$; hence $\beta_0(G) \geq |U| \geq i(G)$. It remains to note that obviously $I \cap V(F_{i_r})$ is maximal independent set of $F_{i_r}$, $r \in [s]$. Therefore $i(G[\Phi]) \geq i(G) i(F_1)$ and $\beta_0(G[\Phi]) \leq \beta_0(G) \beta_0(F_1)$.

The following characterization of well covered generalized lexicographic product of graphs is due to Topp and Volkmann [19].

**Theorem C.** [19] The generalized lexicographic product $G[\Phi]$ is a well covered graph if and only if all $F_i$'s are well covered and $\sum_{i=1}^s \beta_0(F_{i_r}) = \sum_{i=1}^s \beta_0(F_{i_r})$ for every two maximal independent sets $\{i_1, i_2, \ldots, i_s\}$ and $\{j_1, j_2, \ldots, j_l\}$ of $G$.

Next we present a characterization of well $\gamma$-dominated generalized lexicographic product of graphs. For a graph $G[\Phi]$ and any minimal dominating set $R$ of $G$ let $I_R = \{i \mid \deg_R(i) = 0 \text{ and } \gamma(F_i) \geq 2\}$.

**Theorem 18.** Let $G[\Phi]$ be such that $|V(F_i)| \geq 2$ for all $i \in [n]$. Then $G[\Phi]$ is a well $\gamma$-dominated graph if and only if the following assertions hold.

(i) $F_i$ is well $\gamma$-dominated with $\gamma(F_i) \leq 2$ for all $i \in [n]$.

(ii) there is a number $k$ such that for each minimal dominating set $R$ of $G$, $|R| + |I_R| = k$.

**Proof.** $\Rightarrow$ Let $D_j$ be a $\Gamma$-set of $F_j$ and $D'_j$ a $\gamma$-set of $G - N[V(F_j)]$ for some $j \in [n]$. Then $D_j \cup D'_j$ is a minimal dominating set of $G[\Phi]$. Since $G[\Phi]$ is well dominated, $D_j \cup D'_j$ is a $\gamma$-set of $G[\Phi]$. Now using Lemma 7(i), we obtain that $|D_j| \leq 2$ and $D_j$ is a $\gamma$-set of $F_j$. Thus, (i) is satisfied.

Let $R = \{i_1, i_2, \ldots, i_s\}$ be an arbitrary minimal dominating set of $G$ and let $U = \{u_{i_1}, u_{i_2}, \ldots, u_{i_s}\}$ be a $G$-layer of $G[\Phi]$ such that $u_i$ belongs to some minimal dominating set of $F_i$, $i = 1, 2, \ldots, n$. Since $\langle U \rangle \simeq G$, $R_1 = \{u_{i_1}, u_{i_2}, \ldots, u_{i_s}\}$ is a minimal dominating set of $U$. If $R_1$ is a dominating set of $G[\Phi]$, then clearly $R_1$ is a minimal dominating set of $G[\Phi]$; hence $I_R = \emptyset$ and $|R| + |I_R| = |R_1|$. Since $G[\Phi]$ is well $\gamma$-dominated, $|R| + |I_R| = \gamma(G[\Phi]) = \Gamma(G[\Phi])$. Now let $I_R = \{j_1, j_2, \ldots, j_l\}$. Since $\gamma(F_{j_r}) = \Gamma(F_{j_r}) = 2$, for all $r \in [l]$ (by (i)), there is $v_{j_r} \in V(F_{j_r})$ such that $u_{j_r}$ belongs to a $\Gamma$-set of $F_{j_r}$. But then $R_1 \cup \{v_{j_1}, v_{j_2}, \ldots, v_{j_l}\}$ is a $\Gamma$-set of $G[\Phi]$ and as $G[\Phi]$ is $\gamma$-well dominated, $\gamma(G[\Phi]) = \Gamma(G[\Phi]) = |R_1| + l = |R| + |I_R|$. $\Leftarrow$ Let $D$ be an arbitrary minimal dominating set of $G[\Phi]$ and $F_{i_1}, F_{i_2}, \ldots, F_{i_s}$ be all $F_i$'s each of which has an element in common with $D$. Clearly $R = \{i_1, i_2, \ldots, i_s\}$ is a minimal dominating set of $G$. Assume $D$ and $F_{i_r}$ have more
than one element in common. By (i), there are exactly two vertices belonging to both $D$ and $F_i$. But then for each $j \in N(i_r)$, $D \cap F_j$ is empty. Therefore $i_r \in I_R$, which implies $|D| = |R| + |I_R|$. Now by (ii), $|D| = k$ and since $D$ was chosen arbitrarily, $G[\Phi]$ is well $\gamma$-dominated.

By the proof of the above theorem we obtain the next result.

**Corollary 19.** If $G[\Phi]$ is well dominated and $|V(F_i)| \geq 2$ for all $i \in [n]$, then for each minimal dominating set $R$ of $G$, $|R| + |I_R| = \gamma(G[\Phi])$.

**Theorem 20.** ([$\dagger\dagger$] when $G[\Phi] = G[F]$) Let $G[\Phi]$ be such that $|V(F_i)| \geq 2$ for all $i \in [n]$, $\gamma(F_1) = \gamma(F_2) = .. = \gamma(F_n)$ and $\Gamma(F_1) = \Gamma(F_2) = .. = \Gamma(F_n)$. Then $G[\Phi]$ is well $\gamma$-dominated if and only if one of the following conditions holds:

(i) $G$ is well-dominated and all $F_i$’s are complete, or

(ii) $G$ is complete and $F_i$ is well $\gamma$-dominated with $\gamma(F_i) = 2$ for all $i \in [n]$.

**Proof.** $\Rightarrow$ Assume first that $G[\Phi]$ is well $\gamma$-dominated. Using Remark 16 by Corollary 14 and Corollary 15 we have $i(G) = \beta_0(G)$ and $\Gamma(F_1) = \gamma(F_1)$. Let $I = \{1, 1_2, ..., 1_n\}$ be an arbitrary $i$-set of $G$ and $D_j$ an arbitrary $\gamma$-set of $F_j$, $j = 1, 2, .., n$. Then clearly $D = \cup_{r=1}^s D_i$ is a $\gamma$-set of $G[\Phi]$. Now by Lemma 7(i) it follows that $\gamma(F_1) \leq 2$. If $\gamma(F_1) = 2$, then by Lemma 7(i) it follows that each $i$-set of $G$ is efficient dominating. This fact allow us to conclude that a graph $G$ is complete.

So, let $\gamma(F_1) = 1$. We already know that $\Gamma(F_1) = \gamma(F_1)$. Hence all $\langle F_i \rangle$’s are complete. Let $U$ be a $G$-layer of $G[\Phi]$, $R_1$ a $\gamma$-set of $U$ and $R_2$ a $\Gamma$-set of $U$. Since all $\langle F_i \rangle$’s are complete, both $R_1$ and $R_2$ are minimal dominating sets of $G[\Phi]$ and since $G[\Phi]$ is well dominated, $R_1$ and $R_2$ have the same cardinality. Thus, $U$ is well $\gamma$-dominated. It remains to note that $G \simeq U$.

$\Leftarrow$ If (ii) is valid, then obviously $\gamma(G[\Phi]) = \Gamma(G[\Phi]) = 2$. So, suppose (i) is true and let $T_1$ and $T_2$ be different minimal dominating sets of $G[\Phi]$. Since all $\langle F_i \rangle$’s are complete, there are two $G$-layers, say $U_1$ and $U_2$, which contain $T_1$ and $T_2$, respectively. Clearly $T_i$ is a minimal dominating set of $\langle U_i \rangle \simeq G$, $i = 1, 2$. Since $G$ is well covered, $|T_1| = |T_2|$ and we are done. 

A characterization of well $\gamma_l$-dominated generalized lexicographic product of graphs follows.
Theorem 21. Given a graph $G[\Phi]$ with $\delta(F_i) \geq 1$ for all $i \in [n]$. Then $G[\Phi]$ is a well $\gamma_t$-dominated graph if and only if $G$ is complete and for all $i \in [n]$, $F_i$ is well $\gamma_t$-dominated with $\gamma_t(F_i) = 2$. Moreover, if $G[\Phi]$ is a well $\gamma_t$-dominated, then $\gamma_t(G[\Phi]) = 2$.

Proof. $\Rightarrow$ Let $I = \{l_1, l_2, \ldots, l_s\}$ be an arbitrary maximal independent set of $G$ and $D_j$ an arbitrary $\Gamma_t$-set of $F_j$, $j = 1, 2, \ldots, n$. Then clearly $D = \bigcup_{r=1}^{s} D_r$ is a $\gamma_t$-set of $G[\Phi]$. Lemma 7(i) now implies that $I$ is an efficient dominating set of $G$ and $D_r$ is a $\gamma_t$-set of $F_r$ with $\gamma_t(F_r) = 2$ for all $r \in [s]$. Since $I$ was chosen arbitrarily and each vertex of $G$ belongs to some maximal independent set of $G$, we can conclude that (a) all $F_i$'s are well $\gamma_t$-dominated graphs with $\gamma_t(F_i) = 2$, and (b) all maximal independent sets of $G$ are efficient dominating. The latter means that a graph $G$ is complete. Finally, by Theorem 2, $\gamma_t(G[\Phi]) = \gamma_t(G) = 2$.

$\Leftarrow$ Obviously each minimal total dominating set of $G[\Phi]$ has cardinality 2. $\square$

Now we need the following obvious but useful observation.

Observation 22. Given a graph $G[\Phi]$ with $\delta(F_i) = 0$ and $|V(F_i)| \geq 2$ for all $i \in [n]$. Then a set $T$ is a minimal total dominating set of $G[\Phi]$ if and only if $T$ is a minimal total dominating set of some $G$-layer of $G[\Phi]$. In particular, $\Gamma_t(G) = \Gamma_t(G[\Phi])$.

Theorem 23. Given a graph $G[\Phi]$ with $\delta(F_i) = 0$ and $|V(F_i)| \geq 2$ for all $i \in [n]$. Then $G[\Phi]$ is a well $\gamma_t$-dominated graph if and only if $G$ is well $\gamma_t$-total dominated.

Proof. By Theorem 2 we have $\gamma_t(G[\Phi]) = \gamma_t(G)$, and by Observation 22 - $\Gamma_t(G) = \Gamma_t(G[\Phi])$. Therefore $\gamma_t(G[\Phi]) = \Gamma_t(G[\Phi])$ if and only if $\gamma_t(G) = \Gamma_t(G)$. $\square$

In [10] Gözüpek, Hujdurović and Milanić posed the following problem.

Problem D. [10] Characterize the nontrivial lexicographic product graphs that are well $\gamma_t$-dominated.

The previous two theorems together give us the following characterization result.
**Theorem 24.** Let $G[F]$ be such that $|V(G)|, |V(F)| \geq 2$ and $G$ connected. Then $G[F]$ is well-$\gamma_t$-dominated if and only if one of the following conditions holds:

(i) $G$ is complete and $F$ is well $\gamma_t$-dominated with $\gamma_t(F) = 2$.

(ii) $G$ is well $\gamma_t$-dominated and $\delta(F) = 0$.

4 Open problems

We conclude the paper by listing some interesting problems and directions for further research.

- Find results on well $\mu$-dominated graphs, where $\mu$ is at least one of $\gamma_r, \gamma^{oc}, \gamma_{tr}, \gamma^a, \gamma_p, \gamma_k, k \geq 1$. In particular, characterize the generalized lexicographic product graphs that are well $\mu$-dominated.

- Characterize/describe those graphs $G$ having an efficient dominating set of cardinality $\gamma_t(G)/2$ (see Theorem 11). Such graphs are all circulants $C(4k + 2; \{1, 2, \ldots, k\} \cup \{n - 1, n - 2, \ldots, n - k\}, k \geq 1$.

- Find results on dominating $(M, C)$-sets (see Remark 12).

- Characterize/describe those generalized lexicographic product of graphs $G[\Phi]$ for which at least one of the following holds: $\gamma_{(A,B)}(G[\Phi]) = i(G)\gamma_{(A,B)}(F_1)$ and $\beta_0(G)\Gamma_{(A,B)}(F_1) = \Gamma_{(A,B)}(G[\Phi])$ (see Corollary 15).

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