The thermodynamical properties of dark energy with the equation of state $\omega = \omega_0 + \omega_1 z$

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The thermodynamical properties of dark energy are usually investigated with the equation of state $\omega = \omega_0 + \omega_1 z$. Recent observations show that our universe is accelerating, and the apparent horizon and the event horizon vary with redshift $z$. When definitions of the temperature and entropy of a black hole are used to the two horizons of the universe, we examine the thermodynamical properties of the universe which is enveloped by the apparent horizon and the event horizon respectively. We show that the first and the second laws of thermodynamics inside the apparent horizon in any redshift are satisfied, while they are broken down inside the event horizon in some redshift. Therefore, the apparent horizon for the universe may be the boundary of thermodynamical equilibrium for the universe like the event horizon for a black hole.

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I. INTRODUCTION

Many cosmological observations [1, 2, 3, 4], such as the type Ia Supernova(SN Ia), Wilkinson Microwave Anisotropy Probe(WMAP), the Sloan Digital Sky Survey(SDSS) etc., support that our universe is undergoing an accelerated expansion. This accelerated expansion is always attributed to the dark energy with negative pressure which induces repulsive gravity [5, 6]. Except for the property of negative pressure, we do not know too much about the components and other properties of the dark energy. What is more, we have no idea about how the dark energy evolves. Various dark energy models have been proposed to describe the evolution, all of which must be constrained by astronomical observations. In these models, the equation of state $\omega = p/\rho$ plays a key role and can reveal the nature of dark energy which accelerates the universe, where $p$ and $\rho$ are the pressure and energy density of the dark energy respectively. Different equations of state lead to different dynamical changes and may influence the evolution of our universe. $\omega$ and its time derivative with respect to Hubble time $\omega = d\omega/d(ln a)$ (where $a$ is scale factor) are currently constrained by the distance measurements of SN Ia, and the current observational data constrain the range of equation of state as $-1.38 < \omega < -0.82$ [7]. The cosmological constant model with equation of state $\omega = -1$, which is considered as the most acceptable candidate for dark energy has been investigated in [8]. The Quintessence scalar field with $\omega$ in the range from $-1/3$ to $-1$ has been investigated in [9, 10, 11]. The Phantom scalar field with $\omega < -1$, which can introduce the negative kinetic energy and violate the well known energy condition, has been found in [12].

In order to get more information on dark energy, it has been widely discussed from the thermodynamical viewpoints, such as thermodynamics of dark energy with constant $\omega$ in the range $-1 < \omega < -1/3$ [13], $\omega = -1$ in the de Sitter space-time and anti-de Sitter space-time [14], $\omega < -1$ in the Phantom field [15, 16] and the generalized chaplygin gas [17] and so on. More discussions on the thermodynamics of dark energy can be found in [18, 19, 20, 21].

In this paper, we investigate the thermodynamical properties of an accelerated expanding universe driven by dark energy with the equation of state $\omega = \omega_0 + \omega_1 z$ and the Hubble parameter $H^2(z) = H_0^2\Omega_0 m(1 + z)^3 +$
\((1 - \Omega_{0m})(1 + z)^3(1 + \omega_0 - \omega_1)e^{3\omega_1 z}\) (where \(H_0\) is the Hubble constant and \(\Omega_{0m}\) is the parameter of matter density). This model is in agreement with the observations in low redshift, where the parameter \(\omega_0 = -1.25 \pm 0.09\) and \(\omega_1 = 1.97 \pm 0.08\) are suggested by [23].

Though combining quantum mechanics with general relativity, it was discovered that a black hole can emit particles. The temperature of a black hole is proportional to its surface gravity and the entropy is also proportional to its surface area [23, 24]. The Hawking temperature is given as \(T_H = \frac{\kappa}{2\pi}\), where \(\kappa\) is the surface gravity of a black hole, and the entropy of a black hole is \(S = A/4\), where \(A\) is its surface area. Hawking temperature, black hole entropy, and the black hole mass satisfy the first law of thermodynamics \(dM = TdS\) [23]. If a charged black hole is rotating, the thermodynamical first law is expressed as \(dM = TdS + \Omega dJ + V_0 dQ\), where \(\Omega\), \(J\), \(V_0\), \(Q\) are the dragged velocity, angular momentum, electric potential, and charge of a black hole respectively. The event horizon of a stationary black hole is considered as the system boundary [20], inside which the black hole should maintain thermodynamical equilibrium. If our universe can be considered as a thermodynamical system [27, 28, 29, 30], the thermodynamical properties of the black hole can be generalized to space-time enveloped by the apparent horizon or the event horizon. The thermodynamical properties of the universe may be similar to those of the black hole, the thermodynamical laws should be satisfied.

This paper is organized as follows: In Sec II, we examine the thermodynamical first and the second laws on the apparent horizon with \(\omega = \omega_0 + \omega_1 z\). In Sec III, we similarly examine the thermodynamical laws on the event horizon. In Sec IV, we draw some conclusions and discussions.

II. THE THERMODYNAMICAL LAWS ON THE APPARENT HORIZON

The Friedmann-Robertson-Walker metric is [31]

\[
g_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & a^2/(1 - kr^2) \end{pmatrix}. \tag{1}\]

If setting

\[
\tilde{r} = ar, \tag{2}
\]
the dynamical apparent horizon can be determined by the relation

\[
f = g^{ab}\tilde{r}_a\tilde{r}_b = 1 - (H^2 + k/a^2)\tilde{r}^2 = 0, \tag{3}\]

so we can get the apparent horizon radius

\[
\tilde{r}_A = ar_A = 1/\sqrt{H^2 + k/a^2}. \tag{4}\]

The WMAP data suggest that our universe is spatially flat with \(k = 0\), so the apparent horizon is \(\tilde{r}_A = 1/H\) in this case. The surface gravity on the apparent horizon is

\[
\kappa = -\frac{f'}{2}|_{r=r_A} = 1/\tilde{r}_A. \tag{5}\]

and the temperature is

\[
T_A = \kappa/2\pi = 1/2\pi\tilde{r}_A. \tag{6}\]

In this paper, we concentrate on a more general model with

\[
H^2(z) = H^2_0(1+z)^3 + (1-\Omega_{0m})\Omega(z) = (1+z)^3(1+\omega_0 - \omega_1) e^{3\omega_1 z}. \tag{7}\]

As we just want to know more properties about dark energy, we take \(\Omega_{0m} = 0\). And the Hubble parameter becomes

\[
H^2(z) = H^2_0(1+z)^3(1+\omega_0 - \omega_1) e^{3\omega_1 z}. \tag{8}\]

In this section we use temperature, entropy and energy to examine the first and the second laws of thermodynamics on the apparent horizon. The temperature and entropy of the universe on the apparent horizon are \(T_A = 1/2\pi\tilde{r}_A\) and \(S = A/4\) respectively.

When calculating the flux of energy through the area \(A = 4\pi\tilde{r}_A^2 = 4\pi/\Omega\) of the apparent horizon, we treat the horizon as a static surface, but allowing it to vary slowly with time. The amount of energy flux [15] crossing the apparent horizon within the time interval \(dt\) is

\[
dE_A = 4\pi\tilde{r}_A^2T_nk^n k^bdt = 4\pi\tilde{r}_A^2 (\rho + P) dt = -\frac{3}{2}H_0^{-1} \frac{1 + \omega_0 + \omega_1 z}{1 + z} \times (1 + z)^{-\frac{3}{2}(1+\omega_0 - \omega_1)} e^{\frac{3}{2}\omega_1 z} dz. \tag{9}\]

The entropy of the apparent horizon is

\[
S_A = A/4 = 4\pi\tilde{r}_A^2/4 = \pi\tilde{r}_A^2, \tag{10}\]

and its differential form is

\[
dS_A = 2\pi\tilde{r}_A d\tilde{r}_A. \tag{11}\]

Thus we can get

\[
T_A dS_A = d\tilde{r}_A = -\frac{3}{2}H_0^{-1} \frac{1 + \omega_0 + \omega_1 z}{1 + z} \times (1 + z)^{-\frac{3}{2}(1+\omega_0 - \omega_1)} e^{\frac{3}{2}\omega_1 z} dz. \tag{12}\]

From Eq.(9) and Eq.(12), we obtain the result \(-dE_A = T_A dS_A\), so the first law of thermodynamics on the apparent horizon of this redshift-dependent model is confirmed.

The entropy of the universe inside the apparent horizon is related to its energy and pressure through

\[
TdS_I = dE_I + PdV. \tag{13}\]

The energy inside the apparent horizon is \(E_I = 4\pi\tilde{r}_A^3 \rho/3\) and the volume is \(V = 4\pi\tilde{r}_A^3/3\). So we get

\[
dS_I = \pi\tilde{r}_A (1 + 3\omega) d\tilde{r}_A, \tag{14}\]
\( z = \omega_0 + \omega_1 z \) and \( \dot{r}_A = 1/H = H_0^{-1}(1 + z)^{-1/2}(1 + \omega_0 - \omega_1)e^{-3/2\omega_1 z} \).

Thus we can get the derivative of \( S_I \) with respect to \( z \)

\[
\frac{dS_I}{dz} = \frac{-3\pi H_0^{-2}}{2}(1 + \omega_0 + \omega_1 z)^2 \times (1 + \omega_0 + \omega_1 z)^{-3(1 + \omega_0 - \omega_1) - 1} e^{-3\omega_1 z}.
\tag{15}
\]

For the entropy of the apparent horizon, we get

\[
S_A = \pi r_A^2 = \pi H_0^{-2}(1 + z)^{-3(1 + \omega_0 - \omega_1)} e^{-3\omega_1 z},
\tag{16}
\]

and the derivation of \( S_A \) with respect to \( z \) is

\[
\frac{dS_A}{dz} = -3\pi H_0^{-2}(1 + \omega_0 + \omega_1 z)(1 + z)^{-3(1 + \omega_0 - \omega_1) - 1} e^{-3\omega_1 z}.
\tag{17}
\]

The total entropy varies with redshift \( z \), so its derivation is

\[
\frac{d(S_I + S_A)}{dz} = \frac{-3\pi H_0^{-2}}{2}(1 + \omega_0 + \omega_1 z)^2 \times (1 + \omega_0 + \omega_1 z)^{-3(1 + \omega_0 + \omega_1) - 1} e^{-3\omega_1 z}.
\tag{18}
\]

Considering the relation \( a = 1/(1 + z) \), we get

\[
\frac{d(S_I + S_A)}{da} = \frac{3\pi H_0^{-2}}{2}(1 + \omega_0 + \omega_1 z)^2 \times (1 + \omega_0 + \omega_1 z)^{-3(1 + \omega_0 - \omega_1) + 1} e^{-3\omega_1 z} \geq 0.
\tag{19}
\]

We find that the total entropy of the apparent horizon does not decrease with time and the second thermodynamical law is satisfied on the apparent horizon.

III. THE THERMODYNAMICAL LAWS ON THE EVENT HORIZON

The event horizon of the universe is the position of the greatest distance that an particle can reach at a particular cosmic epoch, so the definition of the event horizon is

\[
r_E = a \int_0^\infty \frac{dt}{a} = \frac{1}{1 + z} \int_0^{-1} (-\frac{1}{H})dz
\]

\[
= -\frac{1}{1 + z} \int_0^{-1} H_0^{-1}(1 + z)^{-3(1 + \omega_0 - \omega_1)} \times e^{-\frac{3}{2}\omega_1 z} dz.
\tag{20}
\]

The energy flux through the event horizon can be similarly expressed as

\[
- dE_E = 4\pi r_E^2 \rho(1 + \omega)dt = \frac{-3}{2} r_E^2 H_0(1 + \omega_0 + \omega_1 z) \times (1 + z)^{3(1 + \omega_0 - \omega_1)} e^{-\frac{3}{2}\omega_1 z} dz.
\tag{21}
\]

Meanwhile, we get

\[
dE_E = \frac{3}{2} r_E^2 H_0(1 + \omega_0 + \omega_1 z) \times (1 + z)^{3(1 + \omega_0 - \omega_1)} e^{-\frac{3}{2}\omega_1 z} dz.
\tag{22}
\]

The entropy of the event horizon is

\[
dS_E = 2\pi r_E dE_E = \frac{2\pi r_E}{dz} dE_E dz.
\tag{23}
\]

Using Hawking temperature \( dE_E \) as a function of \( z \) with \( \omega_0 = -1 \) and \( \omega_1 = 2 \), we obtain

\[
T_E dS_E = dE_E.
\tag{24}
\]
The Hawking temperature, entropy together with the black hole mass follow the thermodynamical first law $dT = T dS$, and the Einstein equation can be derived from the thermodynamical law [30]. As the thermodynamical laws hold on the event horizon of a stationary black hole, the black hole keeps thermodynamical equilibrium inside it.

The apparent horizon of a black hole is a two-dimensional surface for which the outgoing orthogonal null geodesics have zero divergence [32]. A black hole in asymptotically flat space-time is defined as a region so that no casual signal (i.e., a signal propagating at a veloc-
The apparent horizon is on the apparent horizon. We find that the thermodynamical entropy variation of the geometrical entropy due to the heat through apparent horizon area, we may not always hold on the event horizon.

When a spherically symmetric black hole falling into it, the thermodynamical laws are not fit for non-stationary black hole. When a black hole is non-stationary, the apparent horizon is always not consistent with the event horizon and the thermodynamical laws may not always hold on the event horizon.

The universe is accelerating and should be not stationary. Due to the heat through apparent horizon area, we have calculated a variation of the geometrical entropy on the apparent horizon. We find that the thermodynamical relation is \( -dE = TdS \). In addition, we also examine the changes of the total entropy with the time \( d(S + S_A)/da \geq 0 \), therefore, the second law of thermodynamics is evidently confirmed. While the thermodynamical laws on the event horizon do not hold on, the results are in agreement with the work of Wang [13].

When the apparent horizon can be regarded as a thermodynamical system which has associated entropy \( S = A/4 \) with the temperature \( T = 1/2\pi r \), the differential form of the first thermodynamical law on the apparent horizon can be written to the differential form of the Friedmann equation [33]. In a word, the apparent horizon is a good boundary of keeping thermodynamical properties, and the universe has a thermal equilibrium inside it.

These properties disappear on the event horizon. Maybe the usual definition of temperature and entropy are not well-defined, or it is non-equilibrium of thermodynamics inside the universe.

In FIG. 4, in the low redshift, the changing rate of the apparent horizon and the event horizon is not significant. The thermodynamical description of horizons [13] will be approximately valid. The apparent horizon always exists and usually is inside the event horizon. In FIG. 5, the event horizon is outside apparent horizon at higher redshift phase and the universe is undergoing accelerated expansion. But in this model the redshift becomes smaller than zero and the equation of state is \( \omega < -1 \), the universe may have a superaccelerated expanding process and the apparent horizon is outside the event horizon. This model is similar with the Phantom field which will lead to undesirable future singularity and violate the weak energy condition.

In summary, it is interesting to investigate the universe from the thermodynamical viewpoints, although thermodynamical properties need to be explored more deeply in the future.

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