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INTEGRALS WITH A KERNEL IN THE SOLUTION OF
NONLINEAR EQUATIONS IN N DIMENSIONS

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We consider iterations for solving the nonlinear equation $F(x) = 0$ in the $N$ dimensional Banach space, $1 \leq N \leq +\infty$, which use "integral information with a kernel". This information consists of the "standard information" $F^{(j)}(x_d), j = 0, 1, \ldots, s$ and the integral $\int_0^1 g(t) F(x_d + ty_d) dt$ where $s \geq 1$, $x_d$ is an approximation to the solution and $y_d$ depends on the standard information. We show there exists an iteration with order $2s + 1 + \delta N, 1$ and prove its optimality.
1. INTRODUCTION

We want to approximate the simple solution $\alpha$ of the nonlinear equation

$$F(x) = 0$$

where

$F: D \to B_2$, $D$ is an open convex subset of $B_1$, $B_1$ and $B_2$ are $N$-dimensional Banach spaces, $1 \leq N \leq + \infty$ and $[F'(\alpha)]^{-1}$ is a bounded operator. This problem is often solved by construction of the sequence of successive approximations to $\alpha$ using the standard information on $F$

$$\mathcal{M}_{s} = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d)\},$$

where $x_d$ is a close approximation to $\alpha$.

In previous papers we investigated another kind of information, namely the integral information

$$\mathcal{M}_{s-1,s} = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d), \int_0^1 F(x_d + ty_d)dt\},$$

where $s \geq 1$ and $y_d$ depends only on the standard information (see Kacewicz [75a] and [75b]),

We showed there exists an iteration of maximal order $s + 3 - \delta$ (for optimally chosen $y_d$), where

$$\delta = \begin{cases} 
0 & \text{if } N = 1 \text{ or } s \geq 2 \\
1 & \text{otherwise}
\end{cases}$$
Since the maximal order of iterations using the standard information is equal to \( s + 1 \), the use of the integral increases the maximal order by \( 2 - \delta \).

In this paper we consider more general kind of integral information, namely integral information with a kernel.

(1.2) \[ \mathcal{M}_{-1,s} = \mathcal{M}_{-1,s}(x_{d}; F) = \{F(x_{d}), F'(x_{d}), \ldots, F^{(s)}(x_{d}), \int_0^1 g(t) F(x_{d} + ty_{d}) dt\}, \]

where

\[ s \geq 1, y_{d} = y_{d}(x_{d}, F(x_{d}), F'(x_{d}), \ldots, F^{(s)}(x_{d})), g = g(t) \text{ is a complex function of a complex variable such that } \int_0^1 |g(t)| dt < +\infty. \]

Note that if \( g(t) = 1 \) then \( \mathcal{M}_{-1,s} = \mathcal{M}_{-1,s} \). The question is how the maximal order of iteration depends on \( g \).

In Section 2 we define the iteration \( \mathcal{I}_{-1,s}^G \) which uses \( \mathcal{M}_{-1,s}^G \) for optimally chosen \( y_{d} \) (see Section 4) and is of order \( \min(s+1+m, 2s+1+\delta_{N,1}) \) (see Section 3 and Corollary 1 in Section 4), where \( m \) is an integer depending on \( g \) (defined in Section 2) and \( \delta_{ij} \) is the Kronecker delta. In Section 4 we prove the iteration \( \mathcal{I}_{-1,s}^G \) is maximal. Furthermore we show there exists a polynomial \( g = g(t) \) independent on \( F \) such that \( m = s+\delta_{N,1} \). Since for such \( g \) the order is equal to \( 2s+1+\delta_{N,1} \), the value of the integral with a kernel, which is represented by the vector of size \( N \), increases the maximal order by \( s+\delta_{N,1} \).

In Section 5 we show that for \( N \) sufficiently large the iteration \( \mathcal{I}_{-1,1}^G \) has smaller complexity index than any interpolatory iteration \( \mathcal{I}_{0,k} \), which uses the information \( \mathcal{M}_{k} \), \( k \geq 1 \), under some assumptions on the cost of computing the value of function, its derivatives, and the integral. In
Section 6 we give examples of the function $g$ and show some connections between information $G_{-1,s}$ and certain two-point information without memory.

2. DEFINITION OF THE ITERATION $I_{-1,s}^G$

We shall use the notation

$$I_j = \int_0^1 g(t) t^{s+j} \, dt, \quad \forall j \geq 1.$$  \hfill (2.1)

Let us define

$$(2.2) \quad B_0 = \{g = g(t): I_1 = 0\}$$

$$(2.3) \quad B_1 = \{g = g(t): I_1 \neq 0, I_2 = 0\},$$

$$(2.4) \quad B_m = \left\{g = g(t): I_1 \neq 0, I_2 \neq 0, \frac{I_k}{I_1} = \left(\frac{i_2}{I_1}\right)^{k-1} \quad k = 2,3,\ldots,m, \right. \right.$$

$$\left. \left. \frac{I_{m+1}}{I_1} \neq \left(\frac{i_2}{I_1}\right)^m \right\} \text{ for } m \geq 2.$$  \hfill (2.5)

Note that $B_m \neq 0, \forall m$. Indeed, the function

$$g(t) = t - \frac{s+2+m}{s+3+m}$$

belongs to $B_m$ for $m = 0,1$. For $m \geq 2$ we can find a function $g$ for which

$$(2.5) \quad I_j = 1, \; j = 1,2,\ldots,m, \; I_{m+1} = 2.$$  \hfill (2.5)

Suppose $g$ is of the form
(2.6) \[ g(t) = \sum_{i=0}^{m} g_i t^i \]

Then the equalities (2.5) give us the system of linear equations on \( g_i \), \( i = 0, 1, \ldots, m \)

(2.7) \[ \sum_{i=0}^{m} \frac{1}{s+j+i+1} g_i = 1 + \delta_{j, m+1} \quad j = 1, 2, \ldots, m+1. \]

Since the matrix \[ \begin{bmatrix} \frac{1}{s+j+i+1} \end{bmatrix} \]
\( i = 0, 1, \ldots, m \)
\( j = 1, \ldots, m+1 \)
is symmetric and positive definite, the coefficients \( g_i \) exist and hence \( B_m \neq 0, \forall m \).

In the remaining part of this paper we often use the notation \( h = \varphi(x; F) \)
which means that \( h \) is the approximation of \( \alpha \) obtained by one step of the iteration \( \varphi \) based on \( x \) and a certain information on \( F \). Recall that if \( z = I_{0,s}(x; F) \), where \( I_{0,s} \) means the maximal interpolatory iteration which uses the standard information \( \mathcal{I}_s \) for \( s \geq 1 \), then

\[ \lim_{x \to \varphi} \frac{z - \alpha}{(\alpha - x)^{s+1}} = \frac{F(s+1)(\alpha)}{(s+1)! F'(\alpha)} \quad \text{for } N = 1 \]

and

\[ \lim_{x \to \varphi} \frac{\|z - \alpha\|}{\|\alpha - x\|^{s+1}} \leq \|F'(\alpha)^{-1} F(s+1)(\alpha)\| \quad \text{for } N \geq 2. \]

We define now the iteration \( I_{-1,s}^g \) which uses the information \( \mathcal{I}_{-1,s}^g \) for \( y_d \) given by

\[ y_d = \begin{cases} \text{arbitrary} & \text{if } m = 0 \\ z_d - x_d & \text{if } m = 1 \\ \frac{I_1}{I_2}(z_d - x_d) & \text{if } m \geq 2, \end{cases} \]

where \( x_d \) is an approximation to the solution \( \alpha, z_d = I_{0,s}(x_d; F) \) and \( g \in B_m \).
The next approximation \( h_d = l_{1,s}^{d-1}(x_d; \cdot) \) in \( l_{1,s}^{d-1} \) is defined as a zero of the polynomial \( w = w(x) = w(x; x_d, \cdot) \),

\[
(2.9) \quad w(h_d; x_d, \cdot) = 0
\]

(with a criterion of its selection, e.g., the nearest zero to \( x_d \)), where \( w \) is given as follows.

**Case I.** \( N = 1. \)

\[
(2.10) \quad w(x; x_d, \cdot) = F(x_d) + F'(x_d)(x-x_d) + \ldots + \frac{1}{s!} F^{(s)}(x_d)(x-x_d)^s +
\]

\[
+ A(x_d, \cdot)(x-x_d)^{s+1},
\]

where

\[
(2.11) \quad A(x_d, \cdot) = \begin{cases} 
0 & \text{if } m = 0 \\
\frac{1}{y_d^{s+1}} \left( \int_0^1 g(t)F(x_d+ty_d)dt - \sum_{i=0}^s \frac{1}{i!} F^{(i)}(x_d)y_d^i \int_0^1 g(t)t^i dt \right) & \text{otherwise}
\end{cases}
\]

**Case II.** \( 2 \leq N \leq +\infty. \)

\[
(2.12) \quad w(x; x_d, \cdot) = F(x_d) + F'(x_d)(x-x_d) + \ldots + \frac{1}{s!} F^{(s)}(x_d)(x-x_d)^s +
\]

\[
+ c \left[ \int_0^1 g(t) F(x_d+ty_d)dt - \sum_{i=0}^s \frac{1}{i!} F^{(i)}(x_d)y_d^i \int_0^1 g(t)t^i dt \right],
\]

where

\[
c = \begin{cases} 
0 & \text{if } m = 0 \\
\frac{1}{I_1} & \text{if } m = 1 \\
\frac{1}{I_1}^{s+1} & \text{if } m \geq 2
\end{cases}
\]
Note that to find a good approximation of $h_d$ in numerical practice it is possible to perform a few Newton steps on the equation (2.9).

We see that for $m = 0$ $I_{-1,s}^g$ is equal to the well known interpolatory iteration $I_{0,s}$ which uses the standard information $R_s$ and is of order $s+1$. Hence we assume that $m \geq 1$.

One can verify that the polynomial $w$ satisfies the following interpolatory conditions. For $N = 1$,

$$w^{(j)}(x_d) = F^{(j)}(x_d) \quad j = 0,1,\ldots,s$$

$$\int_0^1 g(t) w(x_d + ty_d) dt = \int_0^1 g(t) F(x_d + ty_d) dt.$$

For $2 \leq N \leq +\infty$,

$$w(x_d) = F(x_d) + O(||\nu-x_d||^{s+1})$$

$$w^{(j)}(x_d) = F^{(j)}(x_d) \quad j = 1,2,\ldots,s$$

$$\int_0^1 g(t) w(x_d + ty_d) dt = \int_0^1 g(t) F(x_d + ty_d) dt + O(||\nu-x_d||^{s+1}).$$

3. CONVERGENCE OF THE ITERATION $I_{-1,s}^g$

If the function $F$ is sufficiently smooth in the neighborhood of the zero $\alpha$, then from (2.10), (2.11), (2.12) and due to the special form of $y_d$ given by (2.8) we have

$$(3.1) \quad F(x) - w(x; x_d, F) = R(x),$$

where
for $N = 1$

$$R(x) = \begin{cases} 
\frac{1}{(s+2)} \cdot \frac{F(s+2)}{(s+2)} (x-x_d)^{s+2} + O((x-x_d)^{s+3}) + \\
+ O((z_d-x_d)^2(x-x_d)^{s+1}) \\
& \text{if } m = 1 \\
+ \frac{1}{(s+k)} \cdot F(s+k) (x-x_d)^{s+1} [(x-x_d)^k - (z_d-x_d)^k] + \\
+ \frac{1}{(s+1+m)} \cdot [F(s+1+m) (x-x_d)^{s+1} - F(s+1+m) (x-x_d)^{s+1+m}] + \\
+ 0((x-x_d)^{s+2} + 0((z_d-x_d)^{m+1}(x-x_d)^{s+1}) & \text{if } m \geq 2, 
\end{cases}$$

for $2 \leq N \leq + \infty$

$$R(x) = \begin{cases} 
\frac{1}{(s+1)} \cdot [F(s+1)(x-x_d)^{s+1} - F(s+1)(x-x_d)^{s+1}] + \\
+ \frac{1}{(s+2)} \cdot F(s+2) (x-x_d)^{s+2} + O(||x-x_d||^{s+3}) + \\
+ 0(||z_d-x_d||^{s+3}) & \text{if } m = 1 \\
+ \sum_{k=0}^{m-1} \frac{1}{(s+1+k)} \cdot [F(s+1+k)(x-x_d)^{s+1+k} - F(s+1+k)(x-x_d)^{s+1+k}] + \\
+ \frac{1}{(s+1+m)} \cdot [F(s+1+m)(x-x_d)^{s+1+m} - F(s+1+m)(x-x_d)^{s+1+m}] + \\
I_m = \left( \frac{I_1}{I_2} \right)^m (x-x_d)^{s+1} & \text{if } m \geq 2 
\end{cases}$$

From the Brouwer fix point theorem for $N < + \infty$ or the Schauder fix point theorem for $N = + \infty$ (see Ortega and Rheinboldt [70], p. 164), from the definition (2.9) of $I_{-1,s}^G$ and (3.1), (3.2) and (3.3) we get the following theorem about convergence of $I_{-1,s}^G$. In Section 4 we shall use the result below to establish the order of $I_{-1,s}^G$. 

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Theorem 1

Let the iteration $I_{-1,s}^g$ be defined by (2.9) and $g \in B_m$. If the function $F$ is sufficiently smooth in the neighborhood of its simple zero $\alpha$, then the approximation $h_d = I_{-1,s}^g(x_d; F)$ is well defined for $x_d$ sufficiently close to $\alpha$ and

(i) For $N = 1$

$$
\lim_{x_d \to \alpha} \frac{h_d - \alpha}{(\alpha - x_d) \min(s+1+m, 2s+2)} = \begin{cases} 
    \frac{D_{s+1+m}}{\min(s+1+m, 2s+2)} & \text{if } m = 0, 1 \\
    - \delta_{m,s+1} D_{s+1} \cdot D_{s+2} + \left(1 - \left(\frac{I_1}{I_2}\right)^m \frac{I_{m+1}}{I_1}\right) \cdot D_{s+1+m} & \text{if } 2 \leq m \leq s+1 \\
    - D_{s+1} \cdot D_{s+2} & \text{if } m > s+1
\end{cases}
$$

(ii) For $2 \leq N < +\infty$

$$
\lim_{x_d \to \alpha} \frac{\|h_d - \alpha\|}{\|\alpha - x_d\| \min(s+1+m, 2s+2)} = \begin{cases} 
    \frac{\|D_{s+1}\|}{\min(s+1+m, 2s+2)} & \text{if } m = 0 \\
    \delta_{s,1} \cdot 2 \|D_2\|^2 + \|D_{s+2}\| & \text{if } m = 1 \\
    \frac{5_{m,s} \cdot (s+1) \|D_{s+1}\|^2}{\min(s+1+m, 2s+2)} & \text{if } 2 \leq m \leq s \\
    (s+1) \cdot \|D_{s+1}\|^2 & \text{if } m > s
\end{cases}
$$

where $D_k = \frac{1}{k} [F'(\alpha)]^{-1} F(k)(\alpha)$. ■

Since $x_d$ is an arbitrary point, the theorem above describes the behavior of the function $h = I_{-1,s}^g(x; F)$ in the neighborhood of the zero $\alpha$ of $F$. 


4. ORDER OF INFORMATION $\Psi_{-1,s}^g$ AND MAXIMALITY OF THE ITERATION $I_{-1,s}^g$

In this section we show that the iteration $I_{-1,s}^g$ has order equal to $\min(s+1+m, 2s+1+n)$ whenever $g \in B_m$. We prove that this order is maximal and $y_d$ given by (2.8) is optimal.

For this purpose we define the order of iteration and the order of information as in Wozniakowski [75b].

Let $\mathcal{F}$ be a class of functions $F, F: D_F \rightarrow B_2, D_F \subset B_1$, $\dim(B_1) = \dim(B_2) = N$ which have a simple zero $\alpha = \alpha(F)$ and are analytic in its neighborhood. Let $\{x_d\}$ be a sequence converging to $\alpha$, $\lim x_d = \alpha$. We shall say that $\{F_d\} \subset \mathcal{F}$ is equal to $F \in \mathcal{F}$ with respect to $\Psi_{-1,s}^g$ iff

\begin{align*}
(4.1) & \quad F_d(\alpha_d) = 0, \lim_{d} \alpha_d = \alpha, \\
(4.2) & \quad \lim_{d} F_d^{(k)}(\alpha) = G_d^{(k)}(\alpha), \quad k = 0, 1, \ldots, \\
& \quad \text{where } G \in \mathcal{F}, G(\alpha) = 0, \\
(4.3) & \quad \Psi_{-1,s}^g(x_d; F) = \Psi_{-1,s}^g(x_d; F_d) \quad \forall d, i.e., \\
& \quad F_d^{(k)}(x_d) = F_d^{(k)}(x_d), \quad k = 0, 1, \ldots, s, \\
& \quad \int_0^1 g(t)F(x_d + ty_d)dt = \int_0^1 g(t)F_d(x_d + ty_d)dt.
\end{align*}

The order of information $p = p(\Psi_{-1,s}^g)$ is a real number such that

\[
p(\Psi_{-1,s}^g) = \begin{cases} 
\sup A & \text{if } A \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]
where

\[ A = \{ p \geq 1 : \forall \{ x_d \}, \lim x_d = \alpha, \forall F \in \mathcal{F}, F(\alpha) = 0, \forall \{ F_d \} \text{ equal to } F \text{ it is true that} \]

\[ \lim_{d \to \infty} \frac{||h_d - \alpha||}{||x_d - \alpha||^{p-\delta}} = 0, \forall \varepsilon > 0 \} . \]

Let \( \psi_{-1,s} \) be an iteration which uses the information \( \gamma_{-1,s} \). The order of iteration \( \psi_{-1,s} \), \( p = p(\psi_{-1,s}) \) is a real number such that

\[ p(\psi_{-1,s}) = \begin{cases} \sup B & \text{if } B \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \]

where

\[ B = \{ p \geq 1 : \forall \{ x_d \}, \lim x_d = \alpha, \forall F \in \mathcal{F}, F(\alpha) = 0, \forall \{ F_d \} \text{ equal to } F \text{ it is true that} \]

\[ \lim_{d \to \infty} \frac{||h_d - \alpha||}{||x_d - \alpha||^{p-\delta}} = 0, \forall \varepsilon > 0 \text{ where } h_d = \psi_{-1,s}(x_d;F) \} \]

(see Woźniakowski [75b]).

Woźniakowski [75a] proved that the order of information is equal to the maximal order of convergence. We shall use this property to show \( \gamma_{-1,s} \) is maximal.

We now prove the theorem about order of information \( \gamma_{-1,s} \).

**Theorem 2**

Let \( \gamma_{-1,s} \) be the integral information with a kernel

\[ \gamma_{-1,s}(x_d;F) = \{ F(x_d), F'(x_d), \ldots, F(s)(x_d), \int_{0}^{1} [g(t) F(x_d + ty_d)] dt \}
\]

where

\[ s \geq 1, \ y_d = y_d(x_d, F(x_d), \ldots, F(s)(x_d)), g = g(t) \]
is a complex function of a complex variable such that \( \int_0^1 |g(t)| \, dt < +\infty \) and \( g \in B_m \). Then

\[
p(\mathfrak{M}_{-1,s}, s) \leq \min(s+1+m, 2s+1+\delta_{N,1})
\]

Furthermore, if

\[
y_d = \begin{cases} 
  \text{arbitrary} & \text{if } m = 0, \\
  z_d - x_d & \text{if } m = 1, \\
  \frac{I_1}{I_2} (z_d - x_d) & \text{if } m \geq 2,
\end{cases}
\]

where \( z_d = I_{0,s}(x_d; F) \)

then

\[
p(\mathfrak{M}_{-1,s}, s) = \min(s+1+m, 2s+1+\delta_{N,1}).
\]

**Proof**

We shall prove the first part of Theorem 2, i.e., we shall show that there exist \( F \in \mathfrak{H}, F(\alpha) = 0, \{x_d\}, \lim x_d = \alpha \) and \( \{F_d\} \) equal to \( F, F_d(\alpha_d) = 0 \) such that

\[
(4.4) \quad \lim_{d \to \infty} \frac{||x_d - \alpha||}{\min(s+1+m, 2s+1+\delta_{N,1})} > 0.
\]

We consider two cases.

**Case I.** \( N = 1 \)

Let \( F \in \mathfrak{H}, F(\alpha) = 0 \) and \( e_d = \alpha - x_d \), where \( \lim e_d = 0 \). We set \( d \)

\[
F_d(x) = F(x) + (x-x_d)^{s+1} [(x-x_d)^m - y_d b_d], \quad \forall d,
\]

where

\[
y = b_d = 0 \text{ for } m = 0 \text{ and }
\]

\[
\gamma = \begin{cases} 
  0 & \text{if } \lim_{d \to \infty} \left| 1 - \frac{y_d}{e_d} \cdot \frac{I_2}{I_1} \right| = 0 \\
  m-1 & \text{otherwise},
\end{cases}
\]
\[ b_d = y_d^{m-\gamma} \frac{I_{m+1-\gamma}}{I_1} \quad \text{for } m \geq 1. \]

One can verify that \( \{F_d\} \) is equal to \( F \). Moreover,

\[
|\alpha - \alpha_d| = \begin{cases} e_d^{s+1} & \text{if } m = 0 \\ c_d |F_d(\alpha)| = \begin{cases} e_d^{s+1} & \text{if } m = 0 \\ e_d^{m-\gamma} y_d^{m-\gamma} \frac{I_{m+1-\gamma}}{I_1} & \text{otherwise,} \end{cases} \end{cases}
\]

where

\[
F_d(\alpha_d) = 0 \quad \text{and} \quad \lim_{d} c_d = c > 0.
\]

From above we have

\[
(4.6) \quad \lim_{d} \frac{|\alpha - \alpha_d|}{|e_d|^{s+1+m}} > 0, \quad \forall m.
\]

This proves \( (4.4) \) for \( m = 0 \) or \( 1 \). Hence assume \( m \geq 2 \). Let us now consider the functions \( \{F_d\} \) given by \( (4.5) \) with \( \gamma = m-1 \). This means that

\[
F_d(x) = F(x) + (x - x_d)^{s+1}(x - x_d - y_d \frac{I_2}{I_1}).
\]

Let \( z_d \) be defined by

\[
z_d = x_d + \frac{I_2}{I_1} y_d, \quad \forall d,
\]

and let the function \( F \) and the sequence \( \{x_d\} \) be such that

\[
\lim_{d} \frac{|z_d - \alpha|}{|e_d|^{s+1}} > 0.
\]

Then \( \{F_d\} \) is equal to \( F \) and

\[
(4.7) \quad \lim_{d} \frac{|\alpha - \alpha_d|}{|e_d|^{2s+2}} > 0.
\]

Hence, \( (4.6) \) and \( (4.7) \) prove \( (4.4) \) for \( N = 1 \).
Case II. $2 \leq N \leq +\infty$

Since the equality $p(-1,s) \leq \min(s+1+m,2s+2)$ holds for $N = 1$ it also holds for any $2 \leq N \leq +\infty$. Hence, we want now show that for $2 \leq N \leq +\infty$

$p(-1,s) \leq 2s+1$, i.e., that (4.4) holds for $m > s$. It suffices to consider the case $N < +\infty$. Let $z_d = x_d + \frac{I_2}{I_1} y_d$, $\forall d$, $z_d = z_d(x_d,F(x_d),\ldots,F^{(s)}(x_d))$.

If there exist $F \in \mathcal{F}$, $F(\alpha) = 0$ and $\{x_d\}$, $\lim _d x_d = \alpha$ such that

$$
\lim _d \frac{||x_d - \alpha||}{||x_d - \alpha||} > 0,
$$

then the family of functions

$$
F_d(x) = F(x) + [(x_1-x_{1d})^{s+1}(x_2-z_{2d})^T, 0,\ldots,0]^T, \forall d
$$

is equal to $F$ with respect to $\mathfrak{m}_{-1,s}$ and (4.4) holds for zeros $\alpha_d$ of $F_d$.

In the formula above, $x_{1d}$, $z_{2d}$ denote the components of vectors $x_d$, $z_d$ respectively such that

$$
\lim _d \frac{||x_d - \alpha||}{||x_d - \alpha||} > 0 \text{ and } \lim _d \frac{||\alpha - z_d||}{||\alpha - z_d||} > 0,
$$

and

$$
x = [x_1,\ldots,x_N]^T.
$$

Hence assume

$$
(4.8) \lim _d \frac{||x_d - \alpha||}{||x_d - \alpha||} = 0 \text{ for any } F \text{ and } \{x_d\}.
$$

Let the sequence $\{x_d\}$ satisfy the conditions

(i) $\lim _d x_d = \alpha$, $x_{1d} \neq \alpha_1$, $x_{2d} \neq \alpha_2$, $\lim _d \frac{\alpha_1-x_{1d}}{\alpha_2-x_{2d}} = 1$, $x_{id} = \alpha_i$ for $i = 3,4,\ldots,N$,

where $F(\alpha) = 0$. 
From the assumptions above, it follows that \( y_{2d} \) can be equal zero only for a finite number of \( d \), hence without loss of generality we can assume that \( y_{2d} \neq 0 \) \( \forall d \).

Let us define

\[
F_d(x) = F(x) + \left[ (x_1 - x_{1d})^{s+1} - \frac{y_{1d}}{s+1} (x_2 - x_{2d})^{s+1}, 0, \ldots, 0 \right]^T,
\]

One can verify that \( \{F_d\} \) is equal to \( F \). From (4.9) it follows that

\[
\|\alpha - \alpha_d\| = h \|F_d(\alpha)\| = \hat{h}_d |a_d (z_{2d} - \alpha_2) - (z_{1d} - \alpha_1)|
\]

\[
= |a_d (z_{2d} - x_{2d})^s + a_d (z_{2d} - x_{2d})^{s-1} (z_{1d} - x_{1d}) + \ldots
\]

\[
\ldots + a_d (z_{2d} - x_{2d})^s (z_{1d} - x_{1d})^{s-1} + (z_{1d} - x_{1d})^s|,
\]

where

\[
\lim_{d \to \infty} \hat{h}_d = h > 0, F_d(\alpha_d) = 0, a_d = \frac{\alpha_1 - x_{1d}}{\alpha_2 - x_{2d}} (\lim_{d \to \infty} a_d = 1).
\]

It can be verified that there exists a function \( F \) and \( \{x_d\} \) satisfying the (i) condition such that

\[
\lim_{d \to \infty} \frac{|a_d (z_{2d} - \alpha_2) - (z_{1d} - \alpha_1)|}{\|\alpha - x_d\|^{s+1}} > 0.
\]

Indeed, otherwise (due to the similar argument which was used by Kacewicz [75b]) the iteration \( \varphi \) for the solution of the nonlinear scalar equation \( f(y) = 0 \) defined as follows

\[
\varphi(x, f) = x_{2d}(x_d, F(x_d), \ldots, F^{(s)}(x_d)) - z_{1d}(x_d, F(x_d), \ldots, F^{(s)}(x_d))
\]
where $\beta_d$ is close to the solution (but not equal),

$$F(x) = [x_1, f(x_2), x_3, \ldots, x_N]^T$$

and

$$x_d = [\beta_d - I_0, s(\beta_d; f), \beta_d, 0, \ldots, 0]^T$$

has the order of convergence greater than $s+1$, i.e., greater than the order of used information, which is a contradiction.

Finally, from (4.11) and (4.10) follows the inequality (4.4) for $m > s$, which means that $p(N^{-1}, s) \leq 2s+1$. This proves Case II and also the first part of Theorem 2.

We shall prove the second part of Theorem 2. We want to show that for arbitrary $F \in \mathcal{F}$, $F(\alpha) = 0$, $\{x_d\}$, $\lim_{d} x_d = \alpha$, $\{F_d\}$ equal to $F$, $F_d(\alpha_d) = 0$ we have

$$\lim_{d} \frac{||\alpha - \alpha_d||}{||x_d - \alpha|| \min(s+1+m, 2s+1+\delta N, 1)} < +\infty.$$  \hspace{1cm} (4.12)

Since $||\alpha - \alpha_d||$ is at least of order $s+1$, (4.12) holds for $m = 0$. Assume $m \geq 1$.

Since $\{F_d\}$ is equal to $F$ we have

$$\lim_{d} \frac{||F_d(\alpha)||}{||x_d - \alpha|| \min(s+1+m, 2s+1+\delta N, 1)} < +\infty.$$  \hspace{1cm} (4.13)

where the polynomial $w = w(x; x_d, F)$ is given by (2.10) for $N = 1$ and (2.12) for $2 \leq N \leq +\infty$.

From (3.2) for $N = 1$ and (3.3) for $2 \leq N \leq +\infty$ we get

$$\lim_{d} \frac{||x_d - \alpha||}{||x_d - \alpha|| \min(s+1+m, 2s+1+\delta N, 1)} = 0.$$  \hspace{1cm} (4.14)

Hence (4.12) holds which completes the proof of the Theorem 2.
Since
\[ \|I_{-1, s}^{-1} (x_d; F) - \alpha_d\| \leq \|I_{-1, s}^{-1} (x_d; F) - \alpha\| + \|\alpha - \alpha_d\| \]
we get from Theorem 1 and (4.14)
\[ \lim_{d} \frac{\|I_{-1, s}^{-1} (x_d; F) - \alpha_d\|}{\|x_d - \alpha\| \min(s+1+m, 2s+1+N, 1)} < + \infty \]
for any \( F \in \Omega, F(\alpha) = 0, \{x_d\}, \lim_{d} x_d = \alpha, \) and \( \{F_d\} \) equal to \( F, F_d(\alpha_d) = 0. \) Hence, from the definition of the order of iteration and Theorem 2 we have

**Corollary 1**

Let \( g \in B_m \). Then
\[ p(I_{-1, s}^{-1}) = \min(s+1+m, 2s+1+N, 1). \]

From Corollary 1 and Theorem 2 there follows immediately

**Corollary 2**

Let \( I_{-1, s}^{-1} \) be the class of iterations which use information \( I_{-1, s}^{-1} \). Then
\[ p(I_{-1, s}^{-1}) = \sup_{\varphi_{-1, s} \in I_{-1, s}^{-1}} p(\varphi_{-1, s}) \]
i.e., the iteration \( I_{-1, s}^{-1} \) is maximal.

Note that the order of information and at the same time order of iteration \( I_{-1, s}^{-1} \) is maximized and equal to \( 2s+1+N, 1 \) iff \( m > s+N, 1 \). Thus, for the function \( g \) chosen such that \( m = s+N, 1 \) (see (2.6) and (2.7)) one additional value of the integral which is represented by \( N \) new data increases the order by \( s+N, 1 \).
5. COMPLEXITY INDEX

We want to compare the complexity indices of the iterations $I_{-1,s}^g$ and $I_{0,k}$. The complexity index $z$ is defined by

$$z = z(\varphi;F) = \frac{c(\mathcal{M};F) + c(\varphi)}{\log p}$$

where $\varphi$ is an iteration of order $p$ which use the information $\mathcal{I}$, $c(\mathcal{M};F)$ is the information cost and $c(\varphi)$ is the combinatory cost (see Traub and Woźniakowski [75]). For the integral information with a kernel the cost $c(\mathcal{M};F)$ consists of the costs of the standard information $c(\mathcal{M};F)$ and the computed integral $c(I)$. Let us assume that $m = s+\delta_{N,1}$. Then $p(I_{-1,s}^g) = 2s+1+\delta_{N,1}$ and one can verify that $z(I_{-1,s}^g;F) < z(I_{0,k}^F)$ iff

$$c(I) < \frac{\log(2s+1+\delta_{N,1})}{\log(k+1)} c(\mathcal{M}_k;F) - c(\mathcal{M}_s;F) + \frac{\log(2s+1+\delta_{N,1})}{\log(k+1)} c(I_{0,k}^F) - c(I_{-1,s}^g).$$

Let $2 \leq N < +\infty$ and $c(F(i))$ denote the cost of computing $F(i)(x)$. $c(F(i))$ depends on the total number of arithmetical operations as well as on the cost of data access (which is usually greater than the cost of single arithmetical operation). Let $c(F) = N$. Then we assume that $c(I) = O(N)$ and since $F(i)(x)$ can be represented in general by $O(N^{i+1})$ scalar function evaluations, assume that $c(F(i)) = O(N^{i+1})$. Since the information costs $c(\mathcal{M}_k;F)$ and $c(\mathcal{M}_{-1,s};F)$ are of order $N^{k+1}$ and $N^{s+1}$ respectively and the combinatory costs $c(I_{0,k})$ and $c(I_{-1,s})$ are increasing functions of $k$ and $s$ respectively, we have for large $N$

$$\min_{k \geq 1} z(I_{0,k}^F) = z(I_{0,1}^F)$$
(5.3) \( \min_{s \geq 1} z(I_{-1,s}^g;F) = z(I_{-1,1}^g;F) \).

However, it should be stressed that if \( c(F^{(i)}) \) is essentially less than \( N^{i+1} \) then (5.2) and (5.3) are not necessarily true. Under our assumptions

\[
(\log 3-1)c(M_1^g;F) + \log 3 c(I_{0,1}^g) - c(I_{-1,1}^g) = O(N^2)
\]

which means that (5.1) holds for large \( N \). From here, (5.2) and (5.3), it follows that \( I_{-1,1}^g \) has smaller complexity index than any iteration \( I_{0,k}, k \geq 1 \) and any \( I_{-1,s}, s \geq 2 \).

6. EXAMPLES

1. Let \( g(t) \equiv 1 \). Then \( m = 2 \) and order \( p(I_{-1,s}^g) = \min(s+3,2s+1+\delta_{N,1}) = s+3-\delta \)

where

\[
\delta = \begin{cases} 
0 & \text{if } N = 1 \text{ or } s \geq 2 \\
1 & \text{otherwise}, 
\end{cases}
\]

which agrees with Kacewicz's [75b] result.

2. Let \( N = 1 \) and \( g(t) = \delta(t-1) \), where \( \delta \) is a generalized function such that \( \forall t \in \mathbb{R} \)

\[
\int_{-\infty}^{+\infty} \delta(t-1)F(t)dt = F(1)
\]

for any function \( F \) with bounded support (see Gel'fand and Shilov [64]). Then the information is of the form

\[
\eta_{s}^{g} = \{F(x_d), \ldots, F^{(s)}(x_d), F(x_d + y_d)\}.
\]

Note that \( I_j = 1, \forall j \) and hence \( I_k = \left( \sqrt{I_1} \right)^{k-1} \), \( \forall k \). Then formally we can set
m = +∞ and the order of information \( p_{-1,s}^G \) is equal to \( \min(s+1+\infty,2s+2) = 2s+2 \), which agrees with the optimal order of this special Hermitian information (see Woźniakowski [75b]).

3. Let \( N = 1 \) and \( g(t) = \delta_k(t-1) \), where \( \int_{-\infty}^{+\infty} \delta_k(t-1)f(t)dt = F^{(k)}(1) \) for any sufficiently smooth \( F \) with bounded support.

Then the information is of the form

\[
\mathcal{M}_{-1,s}^G = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d)，F^{(k)}(x_d+y_d)\}
\]

and it was considered by Brent [74]. It is easy to see that if \( k > s+1 \) then \( I_j = 0 \), hence \( m = 0 \) and the order is equal to \( s+1 \). If \( k \leq s+1 \) then \( I_j = \frac{(s+j)!}{(s+j-k)!} \), hence \( m = 2 \) and order is equal to \( s+3 \) which agrees with Brent's result.

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