DISCRETE N-BARRIER MAXIMUM PRINCIPLE FOR A
LATTICE DYNAMICAL SYSTEM ARISING
IN COMPETITION MODELS

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(Communicated by Hirokazu Ninomiya)

ABSTRACT. In the present paper, we show that an analogous N-barrier maximum principle (see [3, 7, 5]) remains true for lattice systems. This extends the results in [3, 7, 5] from continuous equations to discrete equations. In order to overcome the difficulty induced by a discretized version of the classical diffusion in the lattice systems, we propose a more delicate construction of the N-barrier which is appropriate for the proof of the N-barrier maximum principle for lattice systems. As an application of the discrete N-barrier maximum principle, we study a coexistence problem of three species arising from biology, and show that the three species cannot coexist under certain conditions.

1. Introduction and main results. This paper introduces a generalization of the N-barrier maximum principle (NBMP) from second order differential operators, such as [3, 6], to the second order difference operators in the following boundary value problem for the two-component lattice dynamical system ([12, 13])

\[
\begin{aligned}
\frac{1}{h^2} (u(x + h) - 2u(x) + u(x - h)) + \theta u'(x) + u (1 - u - a_1 v) &= 0, \\
\frac{d}{h^2} (v(x + h) - 2v(x) + v(x - h)) + \theta v'(x) + k v (1 - a_2 u - v) &= 0,
\end{aligned}
\]

\[
(u, v)(-\infty) = (1, 0), \quad (u, v)(+\infty) = (0, 1),
\]

where \( x \in \mathbb{R} \) and the parameters \( h, d, k, a_1, a_2 \) are positive constants. For the limiting case \( h \to 0^+ \), the NBMP of \((BVP)\) has been established in [3, 6]. \((BVP)\) is a discrete version of itself with \( h \to 0 \), which arises in finding a traveling wave solution of the form

\[
(u(y, t), v(y, t)) = (u(x), v(x)), \quad x = y - \theta t
\]
to the Lotka-Volterra system of two competing species

$$\begin{align*}
&\text{(LV)} \\
&\begin{cases}
  u_t = u_{yy} + u(1 - u - a_1 v), & y \in \mathbb{R}, \ t > 0, \\
  v_t = d v_{yy} + k v(1 - a_2 u - v), & y \in \mathbb{R}, \ t > 0,
\end{cases}
\end{align*}$$

(2)

where $$u(y,t)$$ and $$v(y,t)$$ represent the density of the two species $$u$$ and $$v$$, respectively, and $$\mathbb{R}$$ is the habitat of the two species. For the problem arising in ecology as to which species will survive in a competitive system, traveling wave solutions serve an important role in understanding the competition mechanism of species. In (1), $$\theta$$ is the propagation speed of the traveling wave, which is an important index to understand the competition mechanism. When $$\theta > 0$$ in (BVP), $$u$$ survives and $$v$$ dies out eventually; when $$\theta < 0$$ in (BVP), $$v$$ survives and $$u$$ dies out eventually.

Clearly, (2) has four constant equilibria: $$e_1 = (0,0)$$, $$e_2 = (1,0)$$, $$e_3 = (0,1)$$ and $$e_4 = (u^*, v^*)$$, where $$(u^*, v^*) = \left( \frac{1 - a_1}{1 - a_1 a_2}, \frac{1 - a_2}{1 - a_1 a_2} \right)$$ is the intersection of the two lines $$1 - u - a_1 v = 0$$ and $$1 - a_2 u - v = 0$$ whenever it exists. The asymptotic behavior of solutions $$(u(x,t), v(x,t))$$ for (2) with initial conditions $$u(x,0), v(x,0) > 0$$ can be classified into the following four cases.

**Theorem 1.1 ([10]).** Suppose that $$(u(y,t), v(y,t))$$ is the solution of (2) with the entire space $$\mathbb{R}$$ replaced by a bounded domain in $$\mathbb{R}$$ under the zero Neumann boundary conditions. Then for initial conditions $$u(x,0), v(x,0) > 0$$, we have

(i) $$a_1 < 1 < a_2 \Rightarrow \lim_{t \to \infty} (u(y,t), v(y,t)) = (1,0);$$

(ii) $$a_2 < 1 < a_1 \Rightarrow \lim_{t \to \infty} (u(y,t), v(y,t)) = (0,1);$$

(iii) $$a_1 > 1, a_2 > 1 \Rightarrow (1,0)$$ and $$(0,1)$$ are locally stable equilibria;

(iv) $$a_1 < 1, a_2 < 1 \Rightarrow \lim_{t \to \infty} (u(y,t), v(y,t)) = (u^*, v^*).$$

In this paper, we restrict ourselves to the case where one of (i), (ii), (iii) and (iv) in Theorem 1.1 occurs. To establish the discrete NBMP for (BVP), we first observe that, without loss of generality we may assume $$\theta \geq 0$$ by letting $$\tilde{x} = -x$$ and interchanging the boundary conditions at $$\pm \infty$$. Throughout this paper we shall assume unless otherwise stated, that $$\theta \geq 0$$. The sign of $$\theta$$ determines which species is stronger and can survive in the ecological system (BVP).

The main contribution of the discrete NBMP for (BVP) is to provide a priori lower bounds for the linear combination of the components of $$(u(x), v(x))$$. More precisely, our discrete NBMP gives an affirmative answer to the following question.

**Q:** For any $$h > 0$$, can we establish the discrete NBMP for (BVP), i.e., can we find nontrivial lower bounds depending on the parameters in (BVP) of $$\alpha u(x) + \beta v(x)$$, where $$(u(x), v(x))$$ solves (BVP) and $$\alpha, \beta$$ are arbitrary positive constants?

When $$h \to 0^+$$ and $$d = 1$$, upper and lower bounds of $$u(x) + v(x)$$ can be given by the classical elliptic maximum principle ([4]). When $$h \to 0^+$$ and $$d \neq 1$$, an affirmative answer to an even more general problem of estimating $$\alpha u(x) + \beta v(x)$$ is given in [3].

From an economic point of view, one motivation for addressing the above question is as follows. Suppose that the two species $$U$$ and $$V$$ in (BVP) are commercial farming animals or cash crops which are grown for profit. Let $$u$$ and $$v$$ represent the units of $$U$$ and $$V$$, respectively. The price of each $$U$$ unit is $$\alpha$$ and each $$V$$ unit is
\[ \beta. \] Then the total value of \( \mathcal{U} \) and \( \mathcal{V} \) is \( \alpha u + \beta v \) and it is at least the amount given by the lower bound in the discrete NBMP.

One of the main ingredients of the proof is to express the second order difference operator in (BVP) in terms of a convolution operator (see Lemma 2.2 (ii) in Section 2) with Gaussian-like kernel \( \psi(x) \) defined by Definition 2.1 (iv) in Section 2. Using Lemma 2.2 (ii), (BVP) can be written as

\[
(BVP') \quad \begin{cases} 
(u \ast \psi)''(x) + \theta u'(x) + u (1 - u - a_1 v) = 0, & x \in \mathbb{R}, \\
\frac{dv}{dx} \psi''(x) + \theta v'(x) + k v (1 - a_2 u - v) = 0, & x \in \mathbb{R}, \\
(u, v)(-\infty) = (1, 0), & (u, v)(+\infty) = (0, 1).
\end{cases}
\]

By using the monotonicity of solutions established in [12], we prove the discrete NBMP for (BVP'). Let \( p(x) = \alpha u(x) + \beta v(x) \) and \( q(x) = \alpha u(x) + d\beta v(x) \). Our main result reads as follows.

**Theorem 1.2 (Discrete NBMP).** Assume that \( \theta \geq 0 \) and that \( \alpha \) and \( \beta \) are arbitrary constants. Let \( u(x), v(x) \in C^2(\mathbb{R}) \) with \( u'(x) < 0 \) and \( v'(x) > 0 \). Suppose that \( (u(x), v(x)) \) satisfies (BVP'). We have for \( x \in \mathbb{R}, \)

\( q(x) \geq \lambda_1 - \alpha \sqrt{Mh} - \beta d \sqrt{M} \),

\( p(x) \geq \bar{\lambda}_1 - \alpha \sqrt{\bar{M}h} - \beta \sqrt{\bar{M}h} = \Lambda(\alpha, \beta), \)

where \( \lambda_1 \) is given by (100), \( M \) and \( \bar{M} \) are defined by (79) and (80), and \( \bar{\lambda}_1 \) is given by

\[
\bar{\lambda}_1 = \left( \min \left\{ \alpha \hat{u}, \beta \hat{v} \right\} \right) \max \{1, d\} - \max \left\{ 2 \left( \alpha \sqrt{\bar{M}h} + \frac{\beta}{d} \sqrt{\bar{M}h} \right), \alpha \sqrt{\bar{M}h} + \beta \sqrt{\bar{M}h} \right\}
\]

In (3), \( \hat{u} \) and \( \hat{v} \) are given by (84) and (85), respectively.

**Remark 1 (Theorem 1.2).** From the proof of Theorem 1.2 (see Section 2, Section 3, and Section 4), we see that the boundary conditions at \( \pm \infty \) in (BVP') can be interchanged and Theorem 1.2 remains true. Furthermore, we assume \( u'(x) < 0 \) and \( v'(x) > 0 \). In fact, for Theorem 1.2 to be true, we only need the assumption that the signs of \( u'(x) \) and \( v'(x) \) are opposite. In addition, we without loss of generality assume in Theorem 1.2 that the wave speed \( \theta \geq 0 \) due to the fact that we see that the proof of Theorem 1.2 remains valid if \( \theta \) is replaced by \( |\theta| \).

It turns out from the construction of the N-barrier (see Section 3) and the proof of Theorem 1.2 (see Section 4) that an analogous discrete NBMP remains true for more general nonlinearity or reaction terms in (BVP'). In addition, it is easy to see that a trivial upper bound of \( p(x) \) is

\[ p(x) = \alpha u(x) + \beta v(x) \leq \alpha + \beta, \quad x \in \mathbb{R}, \]

where \((u(x), v(x))\) solves (BVP) without assuming the monotonicity of \( u(x) \) and \( v(x) \). Employing our N-barrier method, a sharper upper bound of \( p(x) \) can be found by using the monotonicity of \( u(x) \) and \( v(x) \) established in [12].

We remark that the NBMP in [3] is recovered by letting \( h \to 0^+ \) in Theorem 1.2.

Under certain restrictions on the parameters, we obtain nonexistence of traveling wave solutions for the Lotka-Volterra system of three competing species ([11]), i.e.
nonexistence of solutions of the following problem in $\mathbb{R}$ (see Theorem 1.3)

$$\begin{align*}
\text{(N)} \quad & \frac{1}{h^2}(u(x+h) - 2u(x) + u(x-h)) + \theta u_x + u(\sigma_1 - c_{11} u - c_{12} v - c_{13} w) = 0, \\
& \frac{d_2}{h^2}(v(x+h) - 2v(x) + v(x-h)) + \theta v_x + v(\sigma_2 - c_{21} u - c_{22} v - c_{23} w) = 0, \\
& \frac{d_3}{h^2}(w(x+h) - 2w(x) + w(x-h)) + \theta w_x + w(\sigma_3 - c_{31} u - c_{32} v - c_{33} w) = 0,
\end{align*}$$

as an application of the discrete NBMP. This application makes our discrete NBMP more biologically appealing. Here $u(x)$, $v(x)$ and $w(x)$ represent the density of the three species $u$, $v$ and $w$ respectively; $d_i$ ($i = 2,3$), $\sigma_i$, $c_{ij}$ ($i = 1, 2, 3$), and $c_{ij}$ ($i,j = 1, 2, 3$, $i \neq j$) are the diffusion rates, the intrinsic growth rates, the intra-specific competition rates, and the inter-specific competition rates respectively. Except the propagation speed of the traveling wave $\theta$, these parameters are all assumed to be positive.

From the viewpoint of the study of competitive exclusion ([1, 14, 15, 17, 20, 22]) or competitor-mediated coexistence ([2, 19, 21]), (N) originates from the investigation of the problem when one exotic species (say, $w$) invades the ecological system of two native species (say, $u$ and $v$) that are competing in the absence of $w$.

For the continuous case when $h \to 0$ and $w$ is absent, (N) becomes a two-species system with the asymptotic behavior at $\pm \infty$

$$(u,v)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0\right), \quad (u,v)(\infty) = \left(0, \frac{\sigma_2}{c_{22}}\right).$$

Under the condition of strong competition (or the bistable condition)

$$\frac{\sigma_1}{c_{11}} > \frac{\sigma_2}{c_{21}}, \quad \frac{\sigma_2}{c_{22}} > \frac{\sigma_1}{c_{12}},$$

such a system admits a unique monotone solution $(u(x), v(x))$ with $u(x)$ being monotonically decreasing and $v(x)$ being monotonically increasing in $x$ ([16, 18]). However, the situation changes dramatically for the discrete case $h > 0$. Under the same condition of strong competition (6) and boundary conditions (5), there exists no traveling solution of (N) with $w$ being absent and $\theta \neq 0$ if $d_1$ and $d_2$ are sufficiently small ([12]).

On the other hand, under the monostable condition (i.e. $u$ is stronger than $v$ and competitive exclusion occurs)

$$\frac{\sigma_1}{c_{11}} > \frac{\sigma_2}{c_{21}}, \quad \frac{\sigma_2}{c_{22}} < \frac{\sigma_1}{c_{12}},$$

(N) with $w$ being absent admits a solution $(u(x), v(x))$ satisfying $u'(x) < 0$ and $v''(x) > 0$ if and only if $\theta \geq \theta_{\text{min}}$ for some constant threshold $\theta_{\text{min}} > 0$. A similar conclusion can be drawn for the monostable condition

$$\frac{\sigma_1}{c_{11}} < \frac{\sigma_2}{c_{21}}, \quad \frac{\sigma_2}{c_{22}} > \frac{\sigma_1}{c_{12}},$$

under which $v$ is stronger than $u$ and competitive exclusion occurs ([13]). Under either (7) or (8), we see that when $w$ is absent in (N), $u(x)$ and $v(x)$ dominate the neighborhoods around $x = -\infty$ and $x = \infty$ respectively. This fact leads us to consider the situation that when $w$ as an exotic species invades (N), the wave
profile of \( w(x) \) remains pulse-like, i.e. \( w(\pm \infty) = 0 \) and \( w(x) > 0 \) for \( x \in \mathbb{R} \), if the three species coexist since \( w \) will prevail over \( u \) and \( v \) only on the region where \( u \) and \( v \) are not too dominant. Under certain conditions on the parameters, this conjecture turns out to be true for the continuous case \( h \to 0 \). As indicated in [8, 9, 4], existence of (N) has been proved by means of the tanh method as well as numerical experiments. When \( h \to 0 \), nonexistence of (N) under certain conditions on the parameters has been established by using the NBMP for (N) with \( h \to 0 \) ([4, 3]). In this paper, we also apply the discrete NBMP (Theorem 1.2) to show the following nonexistence of solutions to (N) for some \( h > 0 \).

**Theorem 1.3 (Nonexistence of 3-species waves).** Let

\[
M = \max \{M_1|_{\theta=1}, M_2|_{\theta=1}\}, \quad \bar{M} = \max \{\bar{M}_1|_{\theta=1}, \bar{M}_2|_{\theta=1}\},
\]

\[
M^* = \max \{M_1^*|_{\theta=1}, M_2^*|_{\theta=1}\}, \quad \bar{M}^* = \max \{\bar{M}_1^*|_{\theta=1}, \bar{M}_2^*|_{\theta=1}\},
\]

where

\[
M_1 = M(a_1, 1, 1, \theta), \quad \bar{M}_1 = M(a_2, d, k, \theta),
\]

\[
M_2 = 8K(a_1, 1, 1, \theta, 1/2), \quad \bar{M}_2 = 8K(a_2, d, k, \theta, 1/2),
\]

\[
M_1^* = M^*(1, \sigma_2, \sigma_1, \sigma_3, c_{12}, c_{11}, c_{13}, \theta), \quad \bar{M}_1^* = M^*(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta),
\]

\[
M_2^* = 8K(1, \sigma_2, \sigma_1, c_{12}, c_{11}, c_{13}, \theta, 1/2), \quad \bar{M}_2^* = 8K(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta, 1/2).
\]

Here

\[
K(a, d, k, \theta, z_1 - z_2) = 8 + \frac{1}{4d} \left(10\theta + k(1 + 4a)(4 + z_1 - z_2)\right),
\]

\[
M(a, d, k, \theta) = \frac{3k^2}{16\theta^2}(1 + 4a)^2 + 29\frac{k}{\theta}(1 + 4a) + 160\frac{d}{\theta} + 54,
\]

and

\[
M^*(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta)
\]

\[
= \frac{2d_2}{\theta} \left(2\tilde{K}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta, 1) + 2\tilde{J}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta) + 4\tilde{K}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta, 2)\right) + \frac{3}{\theta^2} \left(\frac{\sigma_2^2}{4c_{22}} + c_{21} \frac{\sigma_1}{c_{11}} + c_{23} \frac{\sigma_3}{c_{22} c_{33}}\right)^2,
\]

where

\[
\tilde{J}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta)
\]

\[
= 4\frac{\sigma_2}{c_{22}} \tilde{J}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta) + \frac{2\theta}{d_2} \frac{\sigma_2^2}{c_{22}^2}
\]

\[
+ \frac{4}{d_2} \left(\frac{\sigma_2^2}{4c_{22}} + c_{21} \frac{\sigma_1}{c_{11}} + c_{23} \frac{\sigma_3}{c_{22} c_{33}}\right)
\]

\[
\tilde{K}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta, 2)
\]

\[
= \frac{2\sigma_2}{c_{22}} \tilde{J}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta)
\]

\[
+ \frac{1}{d_2} \left(\frac{\theta}{2} \frac{\sigma_2^2}{c_{22}^2} + 2 \left(\frac{\sigma_2^2}{4c_{22}} + c_{21} \frac{\sigma_1}{c_{11}} + c_{23} \frac{\sigma_3}{c_{22} c_{33}}\right)\right),
\]
According to the assumptions in Theorem 1 are given in the Appendix (Section 1) the nonexistence result of three species in Theorem 1. It follows from the assumptions in Theorem 1Λ to establish Theorem 2. Theorem 2 is devoted to the construction of the 5 some Remark 2 (Theorem 2): It is readily seen that when the intrinsic growth rate σ cannot coexist in the ecological system if σ is sufficiently small when we fix other parameters. Ecologically, this means that when the intrinsic growth rate σ is small enough, the three species u, v and w cannot coexist in the ecological system (N) under certain parameter regimes.

The remainder of this paper is organized as follows. We collect in Section 2 some preliminary results including the L2 estimates of u’ and v’ which turn out to be crucial in proving Theorem 1. Section 3 is devoted to the construction of the N-barrier for (BVP). In Section 4, we make use of the N-barrier constructed in Section 3 to establish Theorem 1. As an application of Theorem 1, we show in Section 5 the nonexistence result of three species in Theorem 1. Finally, the elementary proofs of certain results in Section 2 are given in the Appendix (Section 5).
2. Preliminaries. Throughout this paper, we use the notations collected in the following definition.

Definition 2.1. Given $h > 0$ and $u(x), \rho(x) \in C^0(\mathbb{R})$, we define

(i) (Forward difference operator)
$$A^h_\rho(x) = \frac{\rho(x + h) - \rho(x)}{h};$$
(ii) (Steklov average)
$$S^h_\rho(x) = \frac{1}{h} \int_x^{x+h} \rho(z) dz;$$
(iii) (Gaussian-like kernel with compact support)
$$\phi(x) = \begin{cases} 
  h + x, & \text{if } -h \leq x \leq 0, \\
  h - x, & \text{if } 0 \leq x \leq h, \\
  0, & \text{otherwise};
\end{cases}$$
(iv) (Rescaled $\phi$)
$$\psi'(x) = \begin{cases} 
  \frac{1}{h^2}, & \text{if } -h < x < 0, \\
  -\frac{1}{h^2}, & \text{if } 0 < x < h, \\
  0, & \text{otherwise},
\end{cases}$$

![Figure 1](image.png)

Figure 1. $\phi(x)$ in (11) and $\psi(x) = \frac{\phi(x)}{h^2}$ in Definition 2.1 (iv).

(v) (Convolution with $\phi$)
$$(\rho * \phi)(x) = \int_{-\infty}^{\infty} \rho(z) \phi(x - z) dz.$$
From Definition 2.1 (i) and (ii), it is readily seen that \( \frac{d}{dx} S_\rho(x) = A_\rho(x) \). For other fundamental properties concerning the functions given in Definition 2.1 that will be used in the subsequent sections, see Lemma 2.2 below and its proof in the Appendix (Section 5). Without causing confusion as to what we refer to, we use \( S_\rho(x) \) instead of \( S^h_\rho(x) \) and use \( A_\rho(x) \) or \( A(x) \) instead of \( A^h_\rho(x) \).

Lemma 2.2. Assume \( \rho \in C^2(\mathbb{R}) \). Then

(i) \( (\rho \ast \psi)'(x) = \frac{1}{h} \int_{x-h}^{x} A_\rho(z) \, dz \);  
(ii) \( (\rho \ast \phi)''(x) = \rho(x + h) - 2\rho(x) + \rho(x - h) \);  
(iii) \( \int_{z_2}^{z_1} (\rho \ast \psi)''(x) \, dx = \frac{1}{h} \left( (S_\rho(z_1)) - (S_\rho(z_1)) - (S_\rho(z_2) - S_\rho(z_2 - h)) \right) \).

For convenience of notation, we let

\[
\begin{align*}
  f(u, v) & = u(1 - u - a_1 v);  \\
  g(u, v) & = k v (1 - a_2 u - v).  
\end{align*}
\]

Suppose that \( (u(x), v(x)) \) satisfies the boundary conditions

\[
(u, v)(-\infty) = (1, 0), \quad (u, v)(+\infty) = (0, 1)
\]

in \((BVP^*)\) with \( u'(x) < 0 \) and \( v'(x) > 0 \). Then

\[
|g(u, v)| = |k v (1 - a_2 u - v)| = k \left| v - \frac{1}{2} \right|^2 + \frac{1}{4} - a_2 u v \leq k \left( \left| v - \frac{1}{2} \right|^2 + \frac{1}{4} + |a_2 u v| \right) \leq k \left( \frac{1}{4} + a_2 \right).  
\]

On the other hand, we obtain

\[
|f(u, v)| \leq \frac{1}{4} + a_1  
\]

by letting \( k = 1 \) and replacing \( a_2 \) with \( a_1 \) in (17). To establish Proposition 1 below, we make use of (17) and (18). Proposition 1 (i) and (ii) are used to prove the \( L^2 \) estimates of \( u' \) and \( v' \) given by Proposition 1 (iii).

Proposition 1. Let \( u(x), v(x) \in C^2(\mathbb{R}) \) with \( u'(x) < 0 \) and \( v'(x) > 0 \). Suppose that \( (u(x), v(x)) \) satisfies \((BVP^*)\). Then

(i) \( |(u \ast \psi)'(x)| \leq J(a_1, 1, 1, \theta) \) and \( |(v \ast \psi)'(x)| \leq J(a_2, d, k, \theta) \) for \( x \in \mathbb{R} \);  
(ii) for any \( z_1, z_2 \in \mathbb{R} \) with \( z_1 > z_2 \),

\[
\left| \int_{z_2}^{z_1} (u \ast \psi)'(x) \, u'(x) \, dx \right| \leq K(a_1, 1, 1, \theta, z_1 - z_2)  
\]

and

\[
\left| \int_{z_2}^{z_1} (v \ast \psi)'(z) \, v'(z) \, dz \right| \leq K(a_2, d, k, \theta, z_1 - z_2);  
\]

(iii) further assume that \( 0 < z_1 - z_2 \leq 1 \), we have

\[
\int_{z_2}^{z_1} |u'(z)|^2 \, dz \leq M(a_1, 1, 1, \theta) := M_1  
\]

and

\[
\int_{z_2}^{z_1} |v'(z)|^2 \, dz \leq M(a_2, d, k, \theta) := \tilde{M}_1,  
\]
where
\[ J(a, d, k, \theta) = 4 + \frac{1}{2d} (2 \theta + k (1 + 4a)), \]
\[ K(a, d, k, \theta, z_1 - z_2) = 8 + \frac{1}{4d} (10 \theta + k (1 + 4a) (4 + z_1 - z_2)), \]
\[ M(a, d, k, \theta) = \frac{3}{16} k^2 (1 + 4a)^2 + 29 \frac{k}{\theta} (1 + 4a) + 160 \frac{d}{\theta} + 54. \]

**Proof.** It will turn out from the following proof that the estimates of \( u \) follows immediately from those of \( v \) by letting \( a_2 = a_1 \) and \( d = k = 1 \) in (23)~(25) since the estimates in Proposition 1 (i), (ii), and (iii) are shown by considering each equation in (BVP*) separately. Therefore, it suffices to show the estimates for \( v \) in (i), (ii), and (iii).

To prove Proposition 1 (i), if \( 0 < h < 1 \) we can find \( N \in \mathbb{N} \) such that \( Nh \leq 1 \) and \((N + 1)h > 1 \). Then we have \( \frac{1}{Nh} < 2 \). Indeed, since \( 0 < h < 1 \), there exists \( p \in \mathbb{N} \) such that \( \frac{1}{p+1} \leq h < 1 \) or \( \frac{p}{p+1} \leq ph < 1 \), which leads to \( \frac{p}{p+1} \leq ph \leq Nh \leq 1 \). Thus \( 1 \leq \frac{1}{Nh} \leq \frac{p+1}{p} < 2 \). For \( m = 0, 1, \cdots, N \), we have
\[
\int_{z_2 - mh}^{z_1} (A(x) - A(x - h)) \, dx = \int_{z_2 - mh}^{z_2 - mh} A(x) \, dx - \int_{z_2 - mh}^{z_1 - h} A(x) \, dx \nonumber
\]
\[
= \int_{z_1 - h}^{z_1 - mh} A(x) \, dx - \int_{z_2 - mh}^{z_1 - mh} A(x) \, dx. \tag{26}
\]
From Definition 2.1 (i), it is easy to see that when \( A(x) = \frac{v(x + h) - v(x)}{h} \), we have
\[
\frac{1}{h^2} (v(x + h) - 2 v(x) + v(x - h)) = \frac{1}{h} (A(x) - A(x - h)). \tag{27}
\]
Let \( z_1, z_2 \in \mathbb{R} \) with \( 0 < z_1 - z_2 \leq 1 \). Integrating the second equation in (BVP) from \( z_2 - mh \) to \( z_1 \) and using (27) lead to
\[
0 = \frac{d}{h} \int_{z_2 - mh}^{z_1} \left( A(x) - A(x - h) \right) \, dx + \int_{z_2 - mh}^{z_1} \theta v' (x) \, dx + \int_{z_2 - mh}^{z_2} \theta v' (x) \, dx \nonumber
\]
\[
= \frac{d}{h} \left( \int_{z_1}^{z_1 - mh} A(x) \, dx - \int_{z_2 - (m+1)h}^{z_1 - mh} A(x) \, dx \right) + \int_{z_1}^{z_2} \theta v' (x) \, dx \nonumber
\]
\[
+ \int_{z_2 - mh}^{z_1} g(u(x), v(x)) \, dx, \tag{28}
\]
where we have used (26). Let \( z_3 = z_2 - Nh \). Since \( z_1 - z_2 \leq 1 \) and \( Nh \leq 1 \), we have \( z_1 - z_3 = z_1 - z_2 + Nh \leq 2 \). Due to
\[
\left| \int_{z_2 - mh}^{z_1} \theta v' (x) \, dx \right| + \left| \int_{z_2 - mh}^{z_1} g(u(x), v(x)) \, dx \right| \nonumber
\]
\[
\leq \theta + (z_1 - (z_2 - mh)) k \left( \frac{1}{4} + a_2 \right) \leq \theta + k (z_1 - z_3) \left( \frac{1}{4} + a_2 \right), \tag{29}
\]
it follows from (28) that
\[
\frac{1}{h} \left| \int_{z_2 - mh}^{z_1} \left( A(x) - A(x - h) \right) \, dx \right| \leq \frac{1}{d} \left( \theta + k (z_1 - z_3) \left( \frac{1}{4} + a_2 \right) \right). \tag{30}
\]
On the other hand, summation of (26) from \( m = 0 \) to \( m = N - 1 \) gives
\[
\sum_{m=0}^{N-1} \frac{1}{h} \int_{z_2-mh}^{z_1} (A(x) - A(x-h)) \, dx
= \sum_{m=0}^{N-1} \frac{1}{h} \left( \int_{z_1-h}^{z_1} A(x) \, dx - \int_{z_2-(m+1)h}^{z_2-mh} A(x) \, dx \right)
= \frac{N}{h} \int_{z_1-h}^{z_1} A(x) \, dx
- \frac{1}{h} \left( \int_{z_2-h}^{z_2} A(x) \, dx + \int_{z_2-2h}^{z_2-h} A(x) \, dx + \cdots + \int_{z_2-Nh}^{z_2-(N-1)h} A(x) \, dx \right)
= \frac{N}{h} \int_{z_1-h}^{z_1} A(x) \, dx - \frac{1}{h} \int_{z_2-Nh}^{z_2} A(x) \, dx. \tag{31}
\]
Dividing the last equation by \( N \), we use (30) and the fact that \( \frac{1}{Nh} < 2 \) to obtain
\[
\frac{1}{h} \left| \int_{z_1-h}^{z_1} A(x) \, dx \right|
\leq \frac{1}{Nh} \left| \int_{z_2-Nh}^{z_2} A(x) \, dx \right| + \frac{1}{N} \sum_{m=0}^{N-1} \frac{1}{h} \left| \int_{z_2-mh}^{z_1} (A(x) - A(x-h)) \, dx \right|
\leq \frac{1}{Nh^2} \int_{z_2-Nh}^{z_2} (v(x+h) - v(x)) \, dx + \frac{1}{d} \left( \theta + k (z_1 - z_3) \left( \frac{1}{4} + a_2 \right) \right)
\leq \frac{1}{Nh^2} \int_{z_2-Nh}^{z_2} v(x) \, dx - \int_{z_2-Nh}^{z_2} v(x) \, dx + \frac{1}{d} \left( \theta + k (z_1 - z_3) \left( \frac{1}{4} + a_2 \right) \right)
= \frac{1}{Nh^2} \int_{z_2}^{z_2+h} v(x) \, dx - \int_{z_2}^{z_2} v(x) \, dx + \frac{1}{d} \left( \theta + k (z_1 - z_3) \left( \frac{1}{4} + a_2 \right) \right)
= \frac{1}{Nh^2} \int_{z_2}^{z_2+h} v(x) \, dx - \int_{z_2}^{z_2+h} v(x) \, dx + \frac{1}{d} \left( \theta + k (z_1 - z_3) \left( \frac{1}{4} + a_2 \right) \right)
\leq \frac{1}{Nh^2} \left( \int_{z_2}^{z_2+h} v(x) \, dx \right) + \frac{1}{Nh^2} \left( \int_{z_2}^{z_2+h} v(x) \, dx \right) + \frac{1}{d} \left( \theta + k (z_1 - z_3) \left( \frac{1}{4} + a_2 \right) \right)
\leq \frac{1}{Nh} \left( \frac{h}{h} + \frac{1}{d} \left( \theta + k (z_1 - z_3) \left( \frac{1}{4} + a_2 \right) \right) \right)
\leq 4 + \frac{1}{d} \left( \theta + 2k \left( \frac{1}{4} + a_2 \right) \right) = 4 + \frac{1}{2d} (2\theta + k (1 + 4a_2)) := J(a_2, d, k, \theta). \tag{32}
\]
Then the desired estimate
\[
|\langle v * \psi \rangle'(z_1)| = \frac{1}{h} \left| \int_{z_1-h}^{z_1} A(x) \, dx \right| \leq J(a_2, d, k, \theta) \tag{33}
\]
follows from Lemma 2.2 (i). This completes the proof of the second part of Proposition 1 (i).
Proposition 1 (ii) follows from multiplying the second equation in (BVP*) by \( v \) and integrating the resulting equation from \( z_2 \) to \( z_1 \):

\[
0 = d \int_{z_2}^{z_1} (v * \psi''(x) v(x)) dx + \int_{z_2}^{z_1} \theta v'(x)v(x) dx + \int_{z_2}^{z_1} g(u(x), v(x)) v(x) dx
\]

\[
= d (v * \psi)'(x) v(x) \bigg|_{x=z_2}^{x=z_1} - d \int_{z_2}^{z_1} (v * \psi)'(x)v'(x) dx
\]

\[
+ \frac{\theta}{2} v^2(x) \bigg|_{x=z_2}^{x=z_1} + \int_{z_2}^{z_1} g(u(x), v(x)) v(x) dx,
\]

from which we have

\[
\left| \int_{z_2}^{z_1} (v * \psi)'(x)v'(x) dx \right|
\]

\[
\leq |(v * \psi)'(z_1)v(z_1) - (v * \psi)'(z_2)v(z_2)|
\]

\[
+ \frac{1}{d} \left( \frac{\theta}{2} |v^2(z_1) - v^2(z_2)| + k (z_1 - z_2) \left( \frac{1}{4} + a_2 \right) \right)
\]

\[
\leq \left| (v * \psi)'(z_1)v(z_1) \right| + \left| (v * \psi)'(z_2)v(z_2) \right| + \frac{1}{d} \left( \frac{\theta}{2} + k (z_1 - z_2) \left( \frac{1}{4} + a_2 \right) \right)
\]

\[
\leq 2 \mathcal{J}(a_2, d, k, \theta) + \frac{1}{d} \left( \frac{\theta}{2} + k (z_1 - z_2) \left( \frac{1}{4} + a_2 \right) \right)
\]

\[
= 8 + \frac{1}{d} \left( 2\theta + k (1 + 4a_2) + \frac{\theta}{2} + k (z_1 - z_2) \left( \frac{1}{4} + a_2 \right) \right)
\]

\[
= 8 + \frac{1}{d} \left( 5\theta + k (1 + 4a_2) \left( 1 + \frac{1}{4} (z_1 - z_2) \right) \right)
\]

\[
= 8 + \frac{1}{4d} \left( 10\theta + k (1 + 4a_2) (4 + z_1 - z_2) \right)
\]

where Proposition 1 (i) has been used. We notice that it is not necessary to assume \( z_1 - z_2 \leq 1 \) for the estimates in (ii) to hold.

With the aid of Proposition 1 (i) and (ii), we prove Proposition 1 (iii). Multiplying the first equation in (BVP*) by \( v'(x) \) and integrating the resulting equation from \( z_2 \) to \( z_1 \) yield

\[
d \int_{z_2}^{z_1} (v * \psi)''(x) v'(x) dx + \int_{z_2}^{z_1} \theta |v'(x)|^2 dx = - \int_{z_2}^{z_1} g(u(x), v(x)) v'(x) dx
\]

\[
\leq \frac{1}{2\epsilon} \int_{z_2}^{z_1} g^2(u(x), v(x)) dx + \frac{\epsilon}{2} \int_{z_2}^{z_1} |v'(x)|^2 dx,
\]

where Young’s inequality \( ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2} \) with \( \epsilon > 0 \) has been used. Therefore,

\[
\left( \theta - \frac{\epsilon}{2} \right) \int_{z_2}^{z_1} |v'(x)|^2 dx \leq -d \int_{z_2}^{z_1} (v * \psi)''(x) v'(x) dx + \frac{1}{2\epsilon} \int_{z_2}^{z_1} g^2(u(x), v(x)) dx.
\]

Letting \( \epsilon = \theta \), we have

\[
\int_{z_2}^{z_1} |v'(x)|^2 dx \leq \frac{2d}{\theta} \left| \int_{z_2}^{z_1} (v * \psi)''(x) v'(x) dx \right| + \frac{1}{\theta^2} \int_{z_2}^{z_1} g^2(u(x), v(x)) dx.
\]
Now it suffices to give an estimate for the first term on the right hand side of (38). Integrating by parts yields
\[
\int_{z_2}^{z_1} (v \ast \psi)'(x) v'(x) \, dx = (v \ast \psi)'(x) v'(x) \bigg|_{z_2}^{z_1} - \int_{z_2}^{z_1} (v \ast \psi)'(x) v''(x) \, dx. \tag{39}
\]
Let \( \bar{s}_1 < z_2 < \bar{s}_2 < s_1 < z_1 < s_2 \) with \( \bar{s}_2 - \bar{s}_1 = s_2 - s_1 = 1 \) and \( s_1 - \bar{s}_2 < 1 \). Double integration of (39) with respect to \( z_1 \) from \( s_1 \) to \( s_2 \) and with respect to \( z_2 \) from \( \bar{s}_1 \) to \( \bar{s}_2 \) gives rise to
\[
\int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_1} (v \ast \psi)'(x) v'(x) \, dx \, dz_1 \, dz_2 = \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_1} (v \ast \psi)'(x) v'(x) \, dx \, dz_1 \, dz_2 - \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_1} (v \ast \psi)'(z_2) v'(z_2) \, dz_1 \, dz_2 \bigg|_{z_2}^{z_1} - \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_1} (v \ast \psi)'(x) v''(x) \, dx \, dz_1 \, dz_2, \tag{40}
\]
where integration by parts has been used. Rearranging (40) and using the fact that \( \bar{s}_2 - \bar{s}_1 = s_2 - s_1 = 1 \), it follows from Proposition 1 (ii) that
\[
\int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_1} (v \ast \psi)'(x) v'(x) \, dx \, dz_1 \, dz_2 + \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_1} (v \ast \psi)'(x) v''(x) \, dx \, dz_1 \, dz_2 = \int_{s_1}^{s_2} (v \ast \psi)'(z_1) v'(z_1) \, dz_1 - \int_{s_1}^{s_2} (v \ast \psi)'(z_2) v'(z_2) \, dz_2 \leq 2 \mathcal{K}(a_2, d, k, \theta, 1). \tag{41}
\]
From Definition 2.1 (iii) we have
\[
\phi(x - y) = \begin{cases} h + x - y, & \text{if } -h \leq x - y \leq 0, \\ h - x + y, & \text{if } 0 \leq x - y \leq h, \\ 0, & \text{otherwise}. \end{cases} \tag{42}
\]
Clearly \( \psi(x - y) = \frac{\phi(x - y)}{h^2} \) is an even function with compact support in \( y \in [x - h, x + h] \) or \( x \in [y - h, y + h] \). In order to estimate
\[
\int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_1} (v \ast \psi)'(x) v'(x) \, dx \, dz_1 \, dz_2, \tag{43}
\]
we further consider
\[
\int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_1} (v \ast \psi)'(x) v'(x) \, dx \, dz_2 \quad \text{and} \quad \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_1} (v \ast \psi)'(x) v''(x) \, dx \, dz_1 \, dz_2
\]
and
\[
\int_{z_2}^{z_1} (v \ast \psi)'(x) v'(x) \, dx - \int_{z_2}^{z_1} (v \ast \psi)'(x) v''(x) \, dx \\
= \int_{z_2}^{z_1} \int_{-\infty}^{\infty} v''(y) \psi(x - y) v'(x) \, dy \, dx - \int_{z_2}^{z_1} \int_{-\infty}^{\infty} v'(y) \psi(x - y) v''(x) \, dy \, dx \\
= \int_{z_2}^{z_1} \int_{-\infty}^{\infty} v''(y) \psi(x - y) v'(x) \, dy \, dx - \int_{z_2}^{z_1} \int_{-\infty}^{\infty} v''(y) \psi(x - y) v'(x) \, dx \, dy
\]
\[= \int_{z_2}^{z_1} \int_{y=x-h}^{y=x+h} v''(y) \psi(x - y) v'(x) \, dy \, dx - \int_{z_2}^{z_1} \int_{y=y-h}^{y=y+h} v''(y) \psi(x - y) v'(x) \, dy \, dx\]
\[= \int_{z_2}^{z_2+h} \int_{x-h}^{x+h} v''(y) \psi(x - y) v'(x) \, dy \, dx - \int_{z_2-h}^{z_2+h} \int_{x}^{x-h} v''(y) \psi(x - y) v'(x) \, dy \, dx\]
\[+ \int_{z_1-h}^{z_1} \int_{z_1}^{z_1+h} v''(y) \psi(x - y) v'(x) \, dy \, dx - \int_{z_1}^{z_1+h} \int_{x-h}^{x} v''(y) \psi(x - y) v'(x) \, dy \, dx. \quad (44)\]

(See Figure 2)

Figure 2. Integral domain of \(\int_{z_2}^{z_1} (v \ast \psi)''(x) v'(x) - (v \ast \psi)'(x) v''(x) \, dx\) in (44).

For convenience of notation, let
\[I = \int_{z_2}^{z_2+h} \int_{x-h}^{x} v''(y) \psi(x - y) v'(x) \, dy \, dx,\]
\[\mathcal{I} = -\int_{z_2-h}^{z_2+h} \int_{x}^{x-h} v''(y) \psi(x - y) v'(x) \, dy \, dx,\]
\[\mathcal{III} = \int_{z_1-h}^{z_1} \int_{z_1}^{z_1+h} v''(y) \psi(x - y) v'(x) \, dy \, dx,\]
\[\mathcal{IV} = -\int_{z_1}^{z_1+h} \int_{x-h}^{x} v''(y) \psi(x - y) v'(x) \, dy \, dx.\]

Then
\[\int_{z_2}^{z_1} (v \ast \psi)''(x) v'(x) \, dx - \int_{z_2}^{z_1} (v \ast \psi)'(x) v''(x) \, dx = I + \mathcal{I} + \mathcal{III} + \mathcal{IV}. \quad (45)\]
Performing integration by parts on $I$ gives

$$
I = \int_{z_2}^{z_2+h} v'(y) \psi(x - y) \psi'(x) \left| y = z_2 \right. \, dx + \int_{x-h}^{z_2} \int_{x-h}^{z_2+h} v'(y) \psi(x - y) \psi'(x) \, dy \, dx
$$

$$= \int_{z_2}^{z_2+h} v'(z_2) \psi(x - z_2) \psi'(x) \, dx + \int_{x-h}^{z_2} \int_{x-h}^{z_2+h} v'(y) \psi(x - y) \psi'(x) \, dy \, dx, \quad (46)
$$

where we have used the fact that $\psi(h) = 0$. Using the assumption $v'(x) < 0$, the fact that $\psi$ is even, and Proposition 1 (ii), we are led to

$$
\left| \int_{s_1}^{s_2} \int_{s_1}^{s_2} I_1(z_2) \, dz_2 \, dz_2 \right| = \int_{s_1}^{s_2} \int_{s_1}^{s_2} I_1(z_2) \, dz_2 = \int_{s_1}^{s_2} \int_{z_2}^{z_2+h} v'(z_2) \psi(x - z_2) \psi'(x) \, dx \, dz_2
$$

$$\leq \int_{s_1}^{s_2} \int_{s_1}^{\infty} v'(z_2) \psi(x - z_2) \psi'(x) \, dx \, dz_2 = \int_{s_1}^{s_2} v'(z_2) \left( \int_{-\infty}^{\infty} \psi(x - z_2) \psi'(x) \, dx \right) \, dz_2
$$

$$= \int_{s_1}^{s_2} v'(z_2) \left( \int_{-\infty}^{\infty} \psi(x - z_2 - x) \, dx \right) \, dz_2 = \int_{s_1}^{s_2} v'(z_2)(v'*\psi)(z_2) \, dz_2
$$

$$= \int_{s_1}^{s_2} (v'*\psi)'(z_2) \psi'(z_2) \, dz_2 \leq K(a_2, d, k, \theta, s_2 - s_1) = K(a_2, d, k, \theta, 1). \quad (47)
$$

On the other hand,

$$\left| \int_{s_1}^{s_2} \int_{s_1}^{s_2} I_2(z_2) \, dz_2 \, dz_2 \right| = \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_2+h} v'(y) \psi(x - y) \psi'(x) \, dy \, dz_2 \, dz_2
$$

$$\leq \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_2+h} v'(y) \psi'(x) \, \frac{1}{h^2} \, dy \, dz_2 \, dz_2
$$

$$\leq \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_2+h} v'(y) \psi'(x) \, \frac{1}{h^2} \, dy \, dz_2 \, dz_1
$$

$$+ \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_2+h} v'(y) \psi'(x) \, \frac{1}{h^2} \, dy \, dz_2 \, dz_1
$$

$$+ \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_2+h} v'(y) \psi'(x) \, \frac{1}{h^2} \, dy \, dz_2 \, dz_1
$$

$$= \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{z_2}^{z_2+h} v'(y) \psi'(x) \, \frac{1}{h^2} \, \Psi(x) \, dy \, dz_2 \, dz_1, \quad (48)
$$

(See Figure 3)

Where we have used the fact that $\psi'(x - y) = -\frac{1}{h^2}$ when $x - h < y < x$ from Definition 2.1 (iv) and $\Psi(x)$ is given by (See Figure 4)

$$
\Psi(x) = \begin{cases}
  x - s_1, & \text{if } s_1 \leq x \leq s_1 + h, \\
  h, & \text{if } s_1 + h \leq x \leq s_2, \\
  -x + s_2 + h, & \text{if } s_2 \leq x \leq s_2 + h, \\
  0, & \text{otherwise}.
\end{cases} \quad (49)
$$
Since $0 \leq \Psi \leq h$, we have
\[
\left| \int_{s_1}^{s_2} \int_{s_1}^{s_2} I_2(z_2) \, dz_2 \, dz_1 \right| \leq \int_{s_1}^{s_2} \int_{s_1}^{s_2+h} \int_{x-h}^{x} v'(y) v'(x) \frac{1}{h^2} \Psi(x) \, dy \, dx \, dz_1
\]
\[
\leq \int_{s_1}^{s_2} \int_{s_1}^{s_2+h} \int_{x-h}^{x} v'(y) v'(x) \frac{1}{h} \, dy \, dx \, dz_1 = \int_{s_1}^{s_2+h} \int_{x-h}^{x} v'(y) v'(x) \frac{1}{h} \, dy \, dx
\]
\[
= \int_{s_1}^{s_2+h} \int_{x-h}^{x-h/2} v'(y) v'(x) \frac{1}{h} \, dy \, dx + \int_{s_1}^{s_2+h} \int_{x-h/2}^{x} v'(y) v'(x) \frac{1}{h} \, dy \, dx
\]
\[
= \int_{s_1}^{s_2+h} \int_{x-h/2}^{x} v'(y - h/2) v'(x) \frac{1}{h} \, dy \, dx + \int_{s_1}^{s_2+h} \int_{x-h/2}^{x} v'(y) v'(x) \frac{1}{h} \, dy \, dx
\]
by using (48). We first estimate \( I_{22} \). It follows from (42) that when \( x-h/2 \leq y \leq x \),
\[
\psi(x-y) = \frac{\phi(x-y)}{h^2} = h - x + y \geq h - x + h/2 = \frac{1}{2h}.
\]
(51)
Therefore, by means of Proposition 1 (ii) and \( v'(x) > 0 \), we have
\[
I_{22} \leq \int_{\bar{s}_1}^{\bar{s}_2+1} \int_x^{x-\frac{h}{2}} v'(y) v'(x) 2 \psi(x-y) \, dy \, dx \\
\leq \int_{\bar{s}_1}^{\bar{s}_2+1} \int_{-\infty}^{\infty} 2 v'(y) v'(x) \psi(x-y) \, dy \, dx \\
= 2 \int_{\bar{s}_1}^{\bar{s}_2+1} (v \ast \psi)'(x) v'(x) \, dx \\
\leq 2 \mathcal{K}(a_2, d, k, \theta, \bar{s}_2 - \bar{s}_1 + 1) = 2 \mathcal{K}(a_2, d, k, \theta, 2).
\]
(52)
To estimate \( I_{21} \), we rewrite the first equation in (BVP*)
\[
d(v \ast \psi)'(x-h/2) + \theta v'(x-h/2) + g(u(x-h/2), v(x-h/2)) = 0, \quad x \in \mathbb{R}
\]
(53)
since the domain considered in (BVP*) is the entire \( \mathbb{R} \). Multiplying (53) by \( v(x) \) and integrating the resulting equation from \( \bar{s}_1 \) to \( \bar{s}_2 + 1 \), we have
\[
0 = d \int_{\bar{s}_1}^{\bar{s}_2+1} (v \ast \psi)'(x-h/2) v(x) \, dx + \int_{\bar{s}_1}^{\bar{s}_2+1} \theta v'(x-h/2) v(x) \, dx \\
+ \int_{\bar{s}_1}^{\bar{s}_2+1} g(u(x-h/2), v(x-h/2)) v(x) \, dx.
\]
(54)
When \( x-h \leq y \leq x-h/2 \leq x \), we have \( \phi(x-h/2-y) = h -(x-h/2-y) = \frac{3}{2} h - x + y \) from Definition 2.1 (iii) or (42). Thus,
\[
\psi(x-h/2-y) = \frac{1}{h^2} \phi(x-h/2-y) = \frac{1}{h^2} \left( \frac{3}{2} h - x + y \right) \\
\geq \frac{1}{h^2} \left( \frac{3}{2} h - x + x - h \right) \geq \frac{1}{h^2} \frac{h}{2} = \frac{1}{2h}.
\]
(55)
Now we use \( \bar{s}_2 - \bar{s}_1 = 1 \), Proposition 1 (i), (54) and (55) to conclude that \( I_{21} \) given by (50) can be estimated as follows:
\[
I_{21} = \int_{\bar{s}_1}^{\bar{s}_2+1} \int_{x-\frac{h}{2}}^{x} v'(y-h/2) v'(x) \frac{1}{h} \, dy \, dx = \int_{\bar{s}_1}^{\bar{s}_2+1} \int_{x-h}^{x} v'(y) v'(x) \frac{1}{h} \, dy \, dx \\
\leq 2 \int_{\bar{s}_1}^{\bar{s}_2+1} \int_{x-h}^{x-\frac{h}{2}} v'(y) v'(x) \psi(x-h/2-y) \, dy \, dx \\
\leq 2 \int_{\bar{s}_1}^{\bar{s}_2+1} v'(x) \left( \int_{-\infty}^{\infty} v'(y) \psi(x-h/2-y) \, dy \right) \, dx \\
= 2 \left[ v(x) \right]_{-\infty}^{\infty} v'(y) \psi(x-h/2-y) \left|_{x=\bar{s}_2+1} \right. \\
- \int_{\bar{s}_1}^{\bar{s}_2+1} v(x) \left( \int_{-\infty}^{\infty} v'(y) \psi(x-h/2-y) \, dy \right) \, dx \\
= 2 (v(\bar{s}_2 + 1) (v' \ast \psi(\bar{s}_2 + 1 - h/2)) - v(\bar{s}_1) (v' \ast \psi(\bar{s}_1 - h/2)))
\]
As we have done for \( \mathcal{I} \), we can perform integration by parts on \( \mathcal{II}, \mathcal{III}, \) and \( \mathcal{IV} \) to obtain

\[
\mathcal{I} = \int_{z_2}^{z_2+h} v'(z_2) \psi(x - z_2) v'(x) \, dx + \int_{z_2}^{z_2+h} v(y) \psi'(x-y) v'(x) \, dy \, dx,
\]

\[
\mathcal{II} = \int_{z_2-h}^{z_2} v'(z_2) \psi(x - z_2) v'(x) \, dx - \int_{z_2-h}^{z_2} v(y) \psi'(x-y) v'(x) \, dy \, dx,
\]

\[
\mathcal{III} = -\int_{z_1-h}^{z_1} v'(z_1) \psi(x - z_1) v'(x) \, dx + \int_{z_1-h}^{z_1+h} v(y) \psi'(x-y) v'(x) \, dy \, dx,
\]

\[
\mathcal{IV} = -\int_{z_1}^{z_1+h} v'(z_1) \psi(x - z_1) v'(x) \, dx - \int_{z_1}^{z_1+h} v(y) \psi'(x-y) v'(x) \, dy \, dx.
\]

It follows from (50), (52) and (56) that

\[
\left| \int_{z_1}^{z_2} \int_{z_2}^{z_2+h} I_2(z_2) \, dz_2 \, dz_1 \right| \leq I_{21} + I_{22} \leq 6 \mathcal{J}(a_2, d, k, \theta) + 2K(a_2, d, k, \theta, 2) - 8. \quad (57)
\]
where \( \mathcal{I} = I_1(z_2) + I_2(z_2) \) is also given here for completeness. In fact, we can include \( H_1(z_2) \) in the estimate performing in (47) as follows

\[
\int_{s_1}^{s_2} \int_{s_1}^{s_2} (I_1(z_2) + H_1(z_2)) \, dz_1 \, dz_2 = \int_{s_1}^{s_2} (I_1(z_2) + H_1(z_2)) \, dz_2
\]

\[
= \int_{s_1}^{s_2} \int_{s_2}^{s_2+h} \psi'(z_2) \psi(x - z_2) \psi'(x) \, dx \, dz_2 + \int_{s_1}^{s_2} \int_{s_2-h}^{s_2} \psi'(z_2) \psi(x - z_2) \psi'(x) \, dx \, dz_2
\]

\[
= \int_{s_1}^{s_2} \int_{s_2-h}^{s_2+h} \psi'(z_2) \psi(x - z_2) \psi'(x) \, dx \, dz_2
\]

\[
\leq \int_{s_1}^{s_2} \psi'(z_2) \left( \int_{-\infty}^{\infty} \psi'(x) \psi(z_2 - x) \, dx \right) \, dz_2
\]

\[
= \int_{s_1}^{s_2} (\psi' * \psi)(z_2) \, dz_2 = \int_{s_1}^{s_2} (\psi * \psi')(z_2) \, dz_2 \leq K(a_2, d, k, \theta, s_2 - s_1) = K(a_2, d, k, \theta, 1)
\]

(62)

Since the points \( z_1 \) and \( z_2 \) make no difference in estimating \( \mathcal{I}, II, III, \) and \( IV \), it is easy to see that \( I_1(z_2) \) and \( IV_1(z_1) \) (also \( H_1(z_2) \) and \( III_1(z_1) \)) contribute to the same bound. Therefore,

\[
\int_{s_1}^{s_2} \int_{s_1}^{s_2} (I_1(z_2) + H_1(z_2) - III_1(z_1) - IV_1(z_1)) \, dz_1 \, dz_2 \leq 2K(a_2, d, k, \theta, 1)
\]

(63)

Similarly, we see that \( I_2(z_2) \) and \( IV_2(z_1) \) (also \( H_2(z_2) \) and \( III_2(z_1) \)) lead to the same bound. Moreover, \( I_2(z_2) = -I_2(z_2) \) and \( III_2(z_1) = -IV_2(z_1) \). Indeed, it suffices to show for any \( z \in \mathbb{R} \), we have

\[
H_2(z) = III_2(z)
\]

\[
= \int_{z}^{z h} \int_{z}^{z + h} \psi'(y) \psi'(x - y) \psi'(x) \, dy \, dx
\]

\[
= \int_{x = z}^{x = z + h} \int_{y = z}^{y = z + h} \psi'(y) \psi'(x - y) \psi'(x) \, dy \, dx
\]

\[
= \int_{y = z}^{y = z + h} \int_{x = y}^{x = z + h} \psi'(y) \psi'(x - y) \psi'(x) \, dx \, dy
\]

\[
= - \int_{y = z}^{y = z + h} \int_{x = z}^{x = y} \psi'(y) \psi'(y - x) \psi'(x) \, dx \, dy
\]

\[
= - \int_{x = z}^{x = z + h} \int_{y = z}^{y = z + h} \psi'(x) \psi'(x - y) \psi'(y) \, dy \, dx
\]

\[
= - \int_{x = z}^{x = z + h} \int_{x - h}^{x} \psi'(x) \psi'(x - y) \psi'(y) \, dy \, dx
\]

(64)

where we have used the fact that \( \psi' \) is odd. Therefore, it follows from (57) that

\[
\left| \int_{s_1}^{s_2} \int_{s_1}^{s_2} (I_2(z_2) - II_2(z_2) + H_2(z_1) - IV_2(z_1)) \, dz_1 \, dz_2 \right|
\]

\[
= 2 \int_{s_1}^{s_2} \int_{s_1}^{s_2} (I_2(z_2) + H_2(z_1)) \, dz_1 \, dz_2
\]

\[
\leq 4 (6J(a_2, d, k, \theta) + 2K(a_2, d, k, \theta, 2) - 8)
\]

(65)
since $II_2(z_2)$ and $III_2(z_1)$ lead to the same bound, and $I_2(z_2) = -II_2(z_2)$. Recalling (58), (59), (60) and (61), we combine (45) together with (63) and (65) to obtain

$$\int_{z_1}^{z_2} \int_{z_1}^{z_2} (v \ast \psi)'(x) v'(x) \, dx \, dz_1 \, dz_2 - \int_{z_1}^{z_2} \int_{z_1}^{z_2} (v \ast \psi)''(x) \, dx \, dz_1 \, dz_2$$

$$= \int_{z_1}^{z_2} \int_{z_1}^{z_2} (I + II + III + IV) \, dz_1 \, dz_2$$

$$= \int_{z_1}^{z_2} \int_{z_1}^{z_2} ((I(z_2) + I(z_2)) + (II(z_2) - II(z_2)) +$$

$$(-III(z_1) + III(z_1)) + (-IV(z_1) - IV(z_1))) \, dz_1 \, dz_2$$

$$= \int_{z_1}^{z_2} \int_{z_1}^{z_2} ((I(z_2) + I(z_2) - III(z_1) - IV(z_1)) +$$

$$(I_2(z_2) - II_2(z_2) + III_2(z_1) - IV_2(z_1)) \, dz_1 \, dz_2$$

$$\leq 2K(a_2, d, k, \theta, 1) + 4 (6J(a_2, d, k, \theta) + 2K(a_2, d, k, \theta, 2) - 8). \quad (66)$$

Then the addition of (41) and (66) gives

$$\int_{z_1}^{z_2} \int_{z_1}^{z_2} (v \ast \psi)''(x) \, dx \, dz_1 \, dz_2$$

$$\leq 2K(a_2, d, k, \theta, 1) + 2 \left( 6J(a_2, d, k, \theta) + 2K(a_2, d, k, \theta, 2) - 8 \right). \quad (67)$$

From (38) and the last equation it follows that

$$\int_{z_1}^{z_2} |v'(x)|^2 \, dx = \int_{z_1}^{z_2} \int_{z_1}^{z_2} |v'(x)|^2 \, dx \, dz_1 \, dz_2$$

$$\leq \int_{z_1}^{z_2} \int_{z_1}^{z_2} \int_{z_2}^{z_1} |v'(x)|^2 \, dx \, dz_1 \, dz_2$$

$$\leq \frac{2d}{\theta} \int_{z_1}^{z_2} \int_{z_1}^{z_2} \int_{z_2}^{z_1} (v \ast \psi)''(x) v'(x) \, dx \, dz_1 \, dz_2$$

$$+ \frac{1}{\theta^2} \int_{z_1}^{z_2} \int_{z_1}^{z_2} \int_{z_1}^{z_2} \int_{z_1}^{z_2} g^2(u(x), v(x)) \, dx \, dz_1 \, dz_2$$

$$\leq \frac{2d}{\theta} \left( 2K(a_2, d, k, \theta, 1) + 2 \left( 6J(a_2, d, k, \theta) + 2K(a_2, d, k, \theta, 2) - 8 \right) \right)$$

$$+ 3 \left( \frac{k}{\theta} \right)^2 \left( \frac{1}{4} + a_2 \right)^2$$

$$= \frac{3k^2}{16a^2} \left( 1 + 8a_2 \right)^2 + 29 \frac{k}{\theta} (1 + 4a_2) + 160 \frac{d}{\theta} + 54 := \mathcal{M}(a_2, d, k, \theta), \quad (68)$$

where we have used the fact that $\delta_2 - \delta_1 = s_2 - s_1 = 1$ and $\delta_1 < s_2 < \delta_2 < s_1 < z_1 < s_2$ give $z_1 - z_2 \leq s_2 - \delta_1 = (s_2 - s_1) + (\delta_2 - \delta_1) + (s_1 - \delta_2) = 1 + 1 + (z_1 - z_2) \leq 3$. See Remark 3 for the final step of deriving (68). When $s_1 - \delta_2 < 1$, (68) gives an upper bound of the $L^2$-norm of $v'$ in $(\delta_2, s_1)$. This shows Proposition 1 (iii) and hence the proof of Proposition 1 is completed. \hfill \Box

Remark 3. Some remarks related to Proposition 1.

- Note that the assumption $z_1 - z_2 \leq 1$ is not used in deriving Proposition 1 (ii), while it is assumed in deriving Proposition 1 (iii). In addition, Proposition 1 (i) is a point-wise estimate.
When we estimate $I_{21}$ for the equation of $u$-component in (BVP*) as we have done for the equation of $v$-component in (56), the term

$$\int_{s_1}^{s_{21}+1} \theta u'(x-h/2) u(x) \, dx < 0$$

(69)

since $u' < 0$. Due to this fact, it turns out that for the $u$-component $I_{21} \leq 4 J(a_1, 1, 1, \theta) + 4 \left( \frac{1}{4} + a_1 \right) = 4 J(a_1, 1, 1, \theta) + 1 + 4 a_1$.

It follows from (35) that

$$K(a_2, d, k, \theta, z_1 - z_2) = 2 J(a_2, d, k, \theta) + \frac{1}{d} \left( \theta + k (z_1 - z_2) \left( \frac{1}{4} + a_2 \right) \right)$$

(70)

Let $\tau = \frac{1}{d} \left( 2 \theta + k (1 + 4 a_2) \right)$, we have

- as $z_1 - z_2 = 1$,

$$K(a_2, d, k, \theta, 1) = 8 + \frac{5}{4d} \left( 2 \theta + k (1 + 4 a_2) \right) = 8 + \frac{5}{4} \tau$$

$$= 2 J(a_2, d, k, \theta) + \frac{1}{4} \tau; \quad (71)$$

- as $z_1 - z_2 = 2$,

$$K(a_2, d, k, \theta, 2) = 8 + \frac{1}{4d} \left( 10 \theta + 6 k (1 + 4 a_2) \right)$$

$$= 8 + \frac{5}{4d} \left( 2 \theta + k (1 + 4 a_2) \right) + \frac{k}{4d} \left( 1 + 4 a_2 \right)$$

$$= K(a_2, d, k, \theta, 1) + \frac{k}{4d} \left( 1 + 4 a_2 \right)$$

$$= 2 J(a_2, d, k, \theta) + \frac{1}{4} \tau + \frac{k}{4d} (1 + 4 a_2)$$

$$= 2 J(a_2, d, k, \theta) + \frac{\theta}{2d} + \frac{k}{2d} (1 + 4 a_2). \quad (72)$$

From (23), (68), (71), and (72), it follows that

$$M(a_2, d, k, \theta) = \frac{2d}{\theta} \left( 2 K(a_2, d, k, \theta, 1) + 2 \left( 6 J(a_2, d, k, \theta) + 2 K(a_2, d, k, \theta, 2) - 8 \right) \right)$$

$$+ 3 \left( \frac{k}{\theta} \right)^2 \left( \frac{1}{4} + a_2 \right)^2$$

$$= \frac{2d}{\theta} \left( 4 J(a_2, d, k, \theta) + \frac{1}{2} \tau + 12 J(a_2, d, k, \theta) + 8 J(a_2, d, k, \theta) + \tau \right.$$

$$+ \frac{k}{d} \left( 1 + 4 a_2 \right) - 16 \right) + \frac{3 k^2}{16 \theta^2} \left( 1 + 4 a_2 \right)^2$$

$$= \frac{2d}{\theta} \left( 24 J(a_2, d, k, \theta) + \frac{3}{2} \tau + \frac{k}{d} \left( 1 + 4 a_2 \right) - 16 \right) + \frac{3 k^2}{16 \theta^2} \left( 1 + 4 a_2 \right)^2$$

$$= \frac{2d}{\theta} \left( 96 + 12 \tau + \frac{3}{2} \tau + \tau - \frac{2\theta}{d} - 16 \right) + \frac{3 k^2}{16 \theta^2} (1 + 4 a_2)^2$$

$$= \frac{2d}{\theta} \left( 80 + \frac{29}{2} \tau - \frac{2\theta}{d} \right) + \frac{3 k^2}{16 \theta^2} (1 + 4 a_2)^2$$
\[
= 160 \frac{d}{\theta} + 29 \frac{d}{\theta} \left( \frac{2 \theta}{d} + \frac{k}{4d} (1 + 4a_2) \right) - 4 + \frac{3 k^2}{16 \theta^2} (1 + 4a_2)^2
\]
\[
= 160 \frac{d}{\theta} + 58 + 29 \frac{k}{\theta} (1 + 4a_2) - 4 + \frac{3 k^2}{16 \theta^2} (1 + 4a_2)^2
\]
\[
= 160 \frac{d}{\theta} + 29 \frac{k}{\theta} (1 + 4a_2) + \frac{3 k^2}{16 \theta^2} (1 + 4a_2)^2 + 54.
\]

This shows the last step in deriving (68).

Let \( M_1 \) and \( \tilde{M}_1 \) be defined as in (21) and (22) in Proposition 1 respectively. To prove the following Lemma 2.3, we first find the upper bound of \(|v(x) - v(y)|\) when \(|x - y| \leq \frac{h}{2}\). Indeed, using (20) in Proposition 1 (ii) leads to

\[
\mathcal{K}(a_2, d, \theta, z_1 - z_2)
\]
\[
\geq \int_{z_2}^{x_2} (v \ast \psi)'(x) v'(x) dx = \int_{z_2}^{x_2} \int_{-\infty}^{\infty} v'(x) v'(y) \psi(x - y) dy dx
\]
\[
= \int_{z_2}^{x_2} \int_{x - h}^{x + h} v'(x) v'(y) \psi(x - y) dy dx \geq \int_{z_2}^{x_2} \int_{x - \frac{h}{2}}^{x + \frac{h}{2}} v'(x) v'(y) \psi(x - y) dy dx
\]
\[
\geq \int_{z_2}^{x_2} \int_{x - \frac{h}{2}}^{x + \frac{h}{2}} v'(x) v'(y) \frac{1}{2h} dy dx \geq \int_{z_2}^{x_2} \int_{z_2 + \frac{h}{2}}^{z_2 + \frac{h}{2}} v'(x) v'(y) \frac{1}{2h} dy dx
\]
\[
\geq \int_{z_2}^{x_2} \int_{z_2}^{z_2 + \frac{h}{2}} v'(x) v'(y) \frac{1}{2h} dy dx = \frac{1}{2h} (v(z_2 + h/2) - v(z_2))^2,
\]

where \( 0 < z_1 - z_2 \leq h/2 \), and we have used \( v'(x) > 0 \) for \( x \in \mathbb{R}, \psi(x - y) \geq \frac{1}{2h} \) in (51) and the fact that \( \psi(x - y) = \frac{\phi(x - y)}{h^2} \) has a compact support in \( y \in [x - h, x + h] \).

Consequently, when \(|x - y| \leq \frac{h}{2}\) we have

\[
|v(x) - v(y)| \leq \sqrt{2h \mathcal{K}(a_2, d, \theta, |x - y|)}.
\]

Now for \(|x - y| \leq h\), we use (74) to obtain

\[
|v(x) - v(y)| \leq \left| v(x) - v\left(\frac{x + y}{2}\right) \right| + \left| v\left(\frac{x + y}{2}\right) - v(y) \right|
\]
\[
\leq 2\sqrt{2h \mathcal{K}(a_2, d, \theta, |x - y|/2)} \leq 2\sqrt{2h \mathcal{K}(a_2, d, \theta, 1/2)}
\]
\[
= \sqrt{\tilde{M}_2 h}.
\]

where

\[
\tilde{M}_2 = 8 \mathcal{K}(a_2, d, \theta, 1/2)
\]

and we assume \(|x - y| \leq h \leq 1\) on the last inequality in (75). Similarly, it follows from (19) in Proposition 1 (ii) that when \(|x - y| \leq h \leq 1\),

\[
|u(x) - u(y)| \leq 2\sqrt{2h \mathcal{K}(a_1, 1, 1, \theta, 1/2)} = \sqrt{M_2 h}
\]

where

\[
M_2 = 8 \mathcal{K}(a_1, 1, 1, \theta, 1/2).
\]

In view of the \( L^2 \)-norm estimate in Proposition 1 (iii), (74), and (77), we establish the Sobolev type inequalities in the following lemma, which are crucial in proving our main theorem.
Lemma 2.3. Assume the same hypothesis of Proposition 1. If $|x - y| \leq h \leq 1$, then we have

(i) $|u(x) - u(y)| \leq \sqrt{M} h$ and $|v(x) - v(y)| \leq \sqrt{M} h$;

(ii) $|p(x) - p(y)| \leq \alpha \sqrt{M} h + \beta \sqrt{M} h$ and $|q(x) - q(y)| \leq \alpha \sqrt{M} h + \beta \sqrt{M} h$;

(iii) $|S_u(x) - u(x)| \leq \sqrt{M} h$ and $|S_v(x) - v(x)| \leq \sqrt{M} h$;

(iv) $|S_p(x) - p(x)| \leq \alpha \sqrt{M} h + \beta \sqrt{M} h$ and $|S_q(x) - q(x)| \leq \alpha \sqrt{M} h + \beta \sqrt{M} h$;

(v) $|S_u(x) - S_u(y)| \leq \sqrt{M} h$ and $|S_v(x) - S_v(y)| \leq \sqrt{M} h$;

(vi) $|S_p(x) - S_p(y)| \leq \alpha \sqrt{M} h + \beta \sqrt{M} h$ and $|S_q(x) - S_q(y)| \leq \alpha \sqrt{M} h + \beta \sqrt{M} h$,

where

$M = \max\{M_1|_{\theta=1}, M_2|_{\theta=1}\}$ \hspace{1cm} (79)

and

$\tilde{M} = \max\{\tilde{M}_1|_{\theta=1}, \tilde{M}_2|_{\theta=1}\}$. \hspace{1cm} (80)

The proof of Lemma 2.3 is elementary and we postpone its proof in Section 5. In other words, Lemma 2.3 asserts that when $|x - y| \leq h \leq 1$, we have

$$|p(x) - p(y)|, |S_p(x) - p(x)|, |S_p(x) - S_p(y)| \leq \alpha \sqrt{M} h + \beta \sqrt{M} h$$ \hspace{1cm} (81)

and thus

$$|u(x) - u(y)|, |S_u(x) - u(x)|, |S_u(x) - S_u(y)| \leq \sqrt{M} h$$ \hspace{1cm} (82)

by letting $v \equiv 0$ and $\alpha = 1$ in $p(x) = \alpha u(x) + \beta v(x)$.

3. Construction of the N-barrier for forward difference operator. In this section, we show how to construct an appropriate N-barrier which is used to establish Theorem 1.2. Let

$$u = \min\left(1, \frac{1}{a_2}\right), \quad v = \min\left(1, \frac{1}{a_1}\right)$$ \hspace{1cm} (83)

and $0 < \tau = \tan^{-1} \frac{v}{u} < \frac{\pi}{2}$. We set

$$\hat{u} := u - \sqrt{M} h - \sqrt{M} h \cot \tau = u - \sqrt{M} h - \sqrt{M} h \frac{u}{v} \hspace{1cm} (84)$$

$$\hat{v} := v - \sqrt{M} h - \sqrt{M} h \tan \tau = v - \sqrt{M} h - \sqrt{M} h \frac{v}{u} \hspace{1cm} (85)$$

where $M$ and $\tilde{M}$ are as defined respectively in (79) and (80). It turns out that the line $\frac{S_u}{u} + \frac{S_v}{v} = 1$ is below and parallel to the line $\frac{S_u}{u} + \frac{S_v}{v} = 1$ in the $S_uS_v$-plane. Indeed, from (84) and (85) we see $\hat{u} < u$, $\hat{v} < v$, and

$$\frac{\hat{u}}{\hat{v}} = \frac{u - \sqrt{M} h - \sqrt{M} h \frac{u}{v}}{v - \sqrt{M} h - \sqrt{M} h \frac{v}{u}} = \frac{u \left(1 - \frac{\sqrt{M} h}{u} - \frac{\sqrt{M} h}{v}\right)}{v \left(1 - \frac{\sqrt{M} h}{v} - \frac{\sqrt{M} h}{u}\right)} = \frac{u}{v}$$ \hspace{1cm} (86)
Moreover, it is easy to see from Figure 5 that the distance between the two lines is given by

\[ L = \sqrt{Mh} \sin \tau + \sqrt{Mh} \cos \tau. \]  

\[ (87) \]

**Remark 4.** Let us explain the meaning of the distance \( L \) in the proof of Theorem 1.2 (see Section 4) as follows. Let \((u(x), v(x))\) solve \((\text{BVP}^*)\) and recall Definition 2.1 (ii), i.e.

\[ S_h^\rho(x) = \frac{1}{h} \int_x^{x+h} \rho(z) \, dz. \]

Suppose that for some \( x_1, x_2 \in \mathbb{R} \),

\[ \{(S_u(x), S_v(x)) \mid x_1 \leq x \leq x_2\} \subset \left\{(S_u, S_v) \left| \frac{S_u}{u} + \frac{S_v}{v} \leq 1, S_u, S_v \geq 0 \right\} \right. \]

\[ (88) \]

When we plot \((S_u(x), S_v(x))\) or

\[ (S_u(x), S_v(x)) = \left( \frac{1}{h} \int_x^{x+h} u(z) \, dz, \frac{1}{h} \int_x^{x+h} v(z) \, dz \right) \]

\[ (89) \]

for \( x \in [x_1, x_2] \) in the first quadrant of the \( S_uS_v\)-plane, we see from Figure 5 that

\[ \{(u(x), v(x)) \mid x_1 \leq x \leq x_2\} \subset \left\{(u, v) \left| \frac{u}{\bar{u}} + \frac{v}{\bar{v}} \leq 1, u, v \geq 0 \right\} \right. \]

\[ (90) \]
whenever Lemma 2.3 (iii) is true, i.e. $$|S_u(x) - u(x)| \leq \sqrt{M}h$$ and $$|S_v(x) - v(x)| \leq \sqrt{M}h$$ hold. We see that when $$h = 0$$, (84) and (85) give $$\dot{u} = u$$ and $$\dot{v} = v$$, and (87) leads to $$L = 0$$. Then the two lines $$\frac{S_u}{\dot{u}} + \frac{S_v}{\dot{v}} = 1$$ and $$\frac{S_u}{u} + \frac{S_v}{v} = 1$$ coincide, and therefore this case reduces to that in [3, 6].

**Lemma 3.1.** Assume that $$\alpha$$ and $$\beta$$ are arbitrary positive constants. Let

$$\mathcal{F}^* = \left\{ (u, v) \mid (1 - u - a_1 v)(1 - a_2 u - v) \leq 0, u, v \geq 0 \right\}$$

be the region bounded by the lines $$1 - u - a_1 v = 0$$, $$1 - a_2 u - v = 0$$, the $$u$$-axis and the $$v$$-axis in the first quadrant of the uv-plane. Then

$$\mathcal{F}_0 := \left\{ (u, v) \mid F(u, v) = 0, u, v \geq 0 \right\} \subset \mathcal{F}^*,$$

where $$F(u, v) = \alpha u (1 - u - a_1 v) + \beta v (1 - a_2 u - v)$$. 

**Proof.** Suppose that we can find some $$(u_0, v_0) \neq (u^*, v^*), (1, 0), (0, 1)$$ in the first quadrant of the $$uv$$-plane such that $$(u_0, v_0) \in \mathcal{F}_0$$ but $$(u_0, v_0) \notin \mathcal{F}^*$$. It follows that either

$$1 - u_0 - a_1 v_0 > 0, \quad 1 - a_2 u_0 - v_0 > 0$$

(93) or

$$1 - u_0 - a_1 v_0 < 0, \quad 1 - a_2 u_0 - v_0 < 0.$$ 

(94)

However, both cases lead to a contradiction since $$F(u_0, v_0) = 0$$. Therefore, $$\mathcal{F}_0 \subset \mathcal{F}^*$$. \hfill \Box

In other words, Lemma 3.1 asserts that the graph of $$F(u, v) = 0$$ in the first quadrant of the $$uv$$-plane lies between the two lines $$1 - u - a_1 v = 0$$ and $$1 - a_2 u - v = 0$$. This fact will be used in proving Theorem 1.2.

Determining an appropriate $$N$$-barrier is crucial in establishing Theorem 1.2. To achieve this, let

$$\hat{\mathcal{R}} = \left\{ (S_u, S_v) \mid \frac{S_u}{\dot{u}} + \frac{S_v}{\dot{v}} \leq 1, S_u, S_v \geq 0 \right\},$$

(95)

where $$\dot{u}$$ and $$\dot{v}$$ are given by (84) and (85), respectively. The construction of the $$N$$-barrier consists of determining $$\lambda_2, \eta_2, \eta_1,$$ and $$\lambda_1$$ such that the four lines $$\alpha S_u + \beta d S_v = \lambda_2, \alpha S_u + \beta S_v = \eta_2, \alpha S_u + \beta S_v = \eta_1,$$ and $$\alpha S_u + \beta d S_v = \lambda_1$$ in the $$S_u S_v$$-plane satisfy the relationship

$$\mathcal{Q}_{\lambda_1} \subset \mathcal{P}_{\eta_1} \subset \mathcal{P}_{\eta_2} \subset \mathcal{Q}_{\lambda_2} \subset \hat{\mathcal{R}},$$

(96)

where

$$\mathcal{P}_\eta = \left\{ (u, v) \mid \alpha S_u + \beta S_v \leq \eta, S_u, S_v \geq 0 \right\};$$

$$\mathcal{Q}_\lambda = \left\{ (u, v) \mid \alpha S_u + \beta d S_v \leq \lambda, S_u, S_v \geq 0 \right\}.$$ 

(97) (98)

We follow the four steps below to construct the $$N$$-barrier:

1. Taking $$\lambda_2 = \min \left\{ \alpha \dot{u}, \beta \dot{v} \right\},$$ the line $$\alpha S_u + \beta d S_v = \lambda_2$$ has the intercepts $$\left( \frac{\lambda_2}{\alpha}, 0 \right)$$ and $$\left( 0, \frac{\lambda_2}{\beta d} \right)$$. It is readily seen that $$\frac{\lambda_2}{\alpha} \leq \dot{u}$$ and $$\frac{\lambda_2}{\beta d} \leq \dot{v}$$. This gives $$\mathcal{Q}_{\lambda_2} \subset \hat{\mathcal{R}}$$. 


Taking \(\eta_2 = \lambda_2 \min \left\{1, \frac{1}{d}\right\}\), the line \(\alpha S_u + \beta S_v = \eta_2\) has the intercepts
\[
\left(\frac{\eta_2}{\alpha}, 0\right) \text{ and } \left(0, \frac{\eta_2}{\beta}\right).
\]
It is readily seen that \(\frac{\eta_2}{\alpha} \leq \frac{\lambda_2}{\alpha}\) and \(\frac{\eta_2}{\beta} \leq \frac{\lambda_2}{\beta d}\). This gives \(P_{\eta_2} \subset \mathcal{Q}_{\lambda_2}\).

3. We let \(\eta_1 > 0\) satisfying
\[
\eta_2 - \eta_1 = \max \left[2 \left(\alpha \sqrt{M h} + \beta \sqrt{M h}\right), \alpha \sqrt{M h} + \beta d \sqrt{M h}\right].
\]
(99)
The line \(\alpha S_u + \beta S_v = \eta_1\) has the intercepts \(\left(\frac{\eta_1}{\alpha}, 0\right)\) and \(\left(0, \frac{\eta_1}{\beta}\right)\). We see that \(\eta_1 < \eta_2\). This gives \(P_{\eta_1} \subset P_{\eta_2}\).

4. Taking \(\lambda_1 = \eta_1 \min \{1, d\}\), the line \(\alpha S_u + \beta d S_v = \lambda_1\) has the intercepts
\[
\left(\frac{\lambda_1}{\alpha}, 0\right) \text{ and } \left(0, \frac{\lambda_1}{\beta d}\right).
\]
It is readily seen that \(\frac{\lambda_1}{\alpha} \leq \frac{\eta_1}{\alpha}\) and \(\frac{\lambda_1}{\beta d} \leq \frac{\eta_1}{\beta}\). This gives \(Q_{\lambda_1} \subset P_{\eta_1}\).

The choice of \(\eta_1\) given by (99) will be made clear in the proof of Theorem 1.2. When \(h = 0\) in (99), we have \(\eta_1 = \eta_2\) and in this case the N-barrier reduces to that in [3, 6]. The four lines \(\alpha S_u + \beta S_v = \eta_2\), \(\alpha S_u + \beta S_v = \eta_2\), \(\alpha S_u + \beta S_v = \eta_1\), and \(\alpha S_u + \beta d S_v = \lambda_1\) constructed in the four steps form the N-barrier as shown in Figure 6. From the four steps above, it follows that \(\lambda_1\) is given by
\[
\lambda_1 = \eta_1 \min \{1, d\}
\]
\[
= \left(\eta_2 - \max \left[2 \left(\alpha \sqrt{M h} + \beta \sqrt{M h}\right), \alpha \sqrt{M h} + \beta d \sqrt{M h}\right]\right) \min \{1, d\}
\]
\[
= \left(\lambda_2 \min \left\{1, \frac{1}{d}\right\} - \max \left[2 \left(\alpha \sqrt{M h} + \beta \sqrt{M h}\right), \alpha \sqrt{M h} + \beta d \sqrt{M h}\right]\right) \min \{1, d\}
\]
\[
= \left(\frac{\min \{\alpha \hat{u}, \beta \hat{d} \hat{v}\}}{\max \{1, d\}} - \max \left[2 \left(\alpha \sqrt{M h} + \beta \sqrt{M h}\right), \alpha \sqrt{M h} + \beta d \sqrt{M h}\right]\right) \min \{1, d\}.
\]
(100)

We note that when \(h = 0\), we have \(\hat{u} = u\) and \(\hat{v} = v\) by (84) and (85). Moreover, (99) leads to \(\eta_1 = \eta_2\) and hence the two lines \(\alpha S_u + \beta S_v = \eta_1\) and \(\alpha S_u + \beta S_v = \eta_2\) of the N-barrier coincide for the case \(h = 0\). In view of these facts, the case of \(h = 0\) considered in [3] can be regarded as a degenerate case.

4. **Proof of the discrete NBMP.** With the aid of the Sobolev type inequalities in Lemma 2.3 and the construction of the N-barrier in Section 3, we prove Theorem 1.2.

**Proof of Theorem 1.2.** Multiplying respectively the first and the second equations in (BVP\(^*\)) by \(\alpha\) and \(\beta\) and adding the resulting two equations, we obtain an equation involving \(p(x)\) and \(q(x)\)
\[
(q * \psi)''(x) + \theta p'(x) + F(u(x), v(x)) = 0, \quad x \in \mathbb{R},
\]
(101)
where \(F(u, v) := \alpha u (1 - u - a_1 v) + \beta k v (1 - a_2 u - v)\). First of all, when \(d \neq 1\), we claim that \(S_q(x) \geq \lambda_1\) for \(x \in \mathbb{R}\). Suppose that, contrary to our claim, there exists \(z \in \mathbb{R}\) such that \(S_q(z) < \lambda_1\). Since \(u, v \in C^2(\mathbb{R})\), by \((u, v)(-\infty) = (1, 0)\) and \((u, v)(+\infty) = (0, 1)\), we may assume \(\min_{x \in \mathbb{R}} S_q(x) = S_q(z)\). We denote respectively
by $z_1$ and $z_2$ the first points at which the solution $(u(x), v(x))$ intersects the line $S_q := \alpha S_u + d \beta S_v = \lambda_2$ in the $S_u S_v$-plane when $x$ moves from $z$ towards $\infty$ and $-\infty$ (as shown in Figure 6). We integrate (101) with respect to $x$ from $z$ to $z_2$ to obtain

$$0 = \int_z^{z_2} (q * \psi)''(x) \, dx + \theta \int_z^{z_2} p'(x) \, dx + \int_z^{z_2} F(u(x), v(x)) \, dx$$

$$= \frac{1}{h} ((S_q(z_1) - S_q(z_1 - h)) - (S_q(z) - S_q(z - h))) + \theta (p(z_1) - p(z))$$

$$+ \int_z^{z_2} F(u(x), v(x)) \, dx$$

(102)

by using $(iii)$ of Lemma 2.2. On the other hand we have:

- For any $h > 0$

$$S_q(z) - S_q(z - h) \leq 0$$

(103)

since $\min_{x \in \mathbb{R}} S_q(x) = S_q(z)$.

- According to the construction of the N-barrier in Section 3, $z_1$ lies above the line $S_p = \eta_2$ while $z$ lies below the line $S_p = \eta_1$. This leads to $S_p(z_1) - S_p(z) > \eta_2 - \eta_1$. Applying $(iv)$ and $(vi)$ of Lemma 2.3 gives

$$p(z_1) - p(z) = (p(z_1) - S_p(z_1)) + (S_p(z_1) - S_p(z)) + (S_p(z) - p(z))$$

$$> - \left( \alpha \sqrt{M \bar{h}} + \beta \sqrt{M \bar{h}} \right) + (\eta_2 - \eta_1) - \left( \alpha \sqrt{M \bar{h}} + \beta \sqrt{M \bar{h}} \right)$$

$$= \eta_2 - \eta_1 - 2 \left( \alpha \sqrt{M \bar{h}} + \beta \sqrt{M \bar{h}} \right) \geq 0.$$  

(104)

The last inequality holds due to (99).

- It follows from the construction of the N-barrier in Section 3 that $\lambda_2 > \eta_2 > \eta_1$, which gives $\eta_2 - \eta_1 < \lambda_2 - \lambda_1$. Since $z_1$ is on $S_q = \lambda_2$ and $z$ lies below $S_q = \lambda_1$, we have $S_q(z_1) = \lambda_2$ and $S_q(z) < \lambda_1$. Accordingly, by means of (99) again, we have

$$\alpha \sqrt{M \bar{h}} + \beta d \sqrt{M \bar{h}} \leq \eta_2 - \eta_1 < \lambda_2 - \lambda_1 < S_q(z + (z_1 - z)) - S_q(z).$$  

(105)

However, if $z_1 - z \leq h \leq 1$, $(vi)$ of Lemma 2.3 implies $S_q(z_1) - S_q(z) \leq \alpha \sqrt{M \bar{h}} + \beta d \sqrt{M \bar{h}}$, which contradicts (105). As a result, we conclude that $z_1 - z > h$, which leads to $z < z_1 - h < z_1$ or $z_1 - h \in (z, z_1)$. Since $z_1 - h \in (z, z_1)$, $z$ satisfies $\min_{x \in \mathbb{R}} S_q(x) = S_q(z)$, and $z_1$ is the first point at which the solution $(u(x), v(x))$ intersects the line $\alpha S_u + d \beta S_v = \lambda_2$ when $x$ moves from $z$ towards $\infty$, the line $S_q = \alpha S_u + d \beta S_v$ passing through the point $z_1 - h$ is below and parallel to the line $\alpha S_u + d \beta S_v = \lambda_2$ passing through $z_1$ in the $S_u S_v$-plane. Hence we have

$$S_q(z_1) - S_q(z_1 - h) > 0.$$  

(106)

- $(iii)$ of Lemma 2.3 gives

$$|S_u(x) - u(x)| \leq \sqrt{M \bar{h}}, \quad |S_v(x) - v(x)| \leq \sqrt{M \bar{h}}.$$  

(107)
which assert that we can control the distance between \( S_u(x) \) and \( u(x) \) (\( S_u(x) \) and \( v(x) \)). This, together with the fact that the region \( \mathcal{R} \) is given by (96) and \( \{ (S_u(x), S_u(x)) \mid z \leq x \leq z_1 \} \subset \mathcal{R} = \{(S_u, S_v) \mid \frac{S_u}{u} + \frac{S_v}{v} \leq 1, \ S_u, S_v \geq 0 \} \),

we are led to (see also Figure 5 and Remark 4)

\[
\{ (u(x), v(x)) \mid z \leq x \leq z_1 \} \subset \left\{ (u, v) \mid \frac{u}{u} + \frac{v}{v} \leq 1, \ u, v \geq 0 \right\} := \mathcal{R}. \tag{109}
\]

It is readily seen that the quadratic curve \( F(u, v) = 0 \) passes through the points \((0, 0), (1, 0), (0, 1), \) and \((u^*, v^*)\). We note that \((u^*, v^*)\) may not be in the first quadrant in the \( uv\)-plane. Let \( \mathcal{F}^*_0 = \left\{ (u, v) \mid F(u, v) \geq 0, \ u, v \geq 0 \right\} \).

By Lemma 3.1, we have \( \mathcal{F}^*_0 = \left\{ (u, v) \mid F(u, v) = 0, \ u, v \geq 0 \right\} \subset \mathcal{F}^* \), where \( \mathcal{F}^* \) is given in Lemma 3.1. Also, it is easy to see that \( F(u, v) < 0 \) for large \( u \) and \( v \), and \( \mathcal{F}^*_0 \) is the region bounded by \( \mathcal{F}^*_0 \), the \( u \)-axis and the \( v \)-axis. Moreover,

\[
\{ (u(x), v(x)) \mid z \leq x \leq z_1 \} \subset \mathcal{R} \subset \mathcal{F}^*_0. \tag{110}
\]

These lead us to

\[
\int_z^{z_1} F(u(x), v(x)) \, dx > 0. \tag{111}
\]

Combining (103), (104), (106) and (111), we obtain

\[
\frac{1}{h} \left( (S_q(z_1) - S_q(z_1 - h)) - (S_q(z) - S_q(z - h)) \right) + \theta (p(z_1) - p(z))
+ \int_z^{z_1} F(u(x), v(x)) \, dx > 0,
\]

which contradicts (102). Consequently, \( S_q(x) \geq \lambda_1 \) for \( x \in \mathbb{R} \) is proved for \( d \neq 1 \).

For the case \( d = 1 \), the proof is elementary. In this case we clearly have \( q = p \).

Multiplying respectively the first and the second equations in (BVP) by \( \alpha \) and \( \beta \) and adding the resulting equations, we obtain

\[
\frac{1}{h^2} \left( p(x + h) - 2p(x) + p(x - h) \right) + \theta p'(x) + F(u(x), v(x)) = 0, \quad x \in \mathbb{R}. \tag{113}
\]

Analogously to the case of \( d \neq 1 \), we assume that there exists \( \tilde{z} \in \mathbb{R} \) such that \( p(\tilde{z}) < \lambda_1 \) and \( \min_{x \in \mathbb{R}} p(x) = p(\tilde{z}) \). Due to \( \min_{x \in \mathbb{R}} p(x) = p(\tilde{z}) \), we have \( p'(\tilde{z}) = 0 \), \( p(\tilde{z} + h) \geq p(\tilde{z}) \) and \( p(\tilde{z} - h) \geq p(\tilde{z}) \). Thus \( p(\tilde{z} + h) - 2p(\tilde{z}) + p(\tilde{z} - h) \geq 0 \). Since \( (u(\tilde{z}), v(\tilde{z})) \) is in the interior of \( \mathcal{R} \) (defined in (109)) and \( \mathcal{R} \subset \mathcal{F}^*_0 \) as shown in (110), we have \( F(u(\tilde{z}), v(\tilde{z})) > 0 \). These together give

\[
\frac{1}{h^2} \left( p(\tilde{z} + h) - 2p(\tilde{z}) + p(\tilde{z} - h) \right) + \theta p'(\tilde{z}) + F(u(\tilde{z}), v(\tilde{z})) > 0,
\]

which contradicts (113). Thus, \( p(x) \geq \lambda_1 \) for \( x \in \mathbb{R} \) when \( d = 1 \).

As a result, we arrive at \( S_q(x) \geq \lambda_1 \) for \( x \in \mathbb{R} \).

(i) follows immediately from the fact that \( S_q(x) \geq \lambda_1 \) for \( x \in \mathbb{R} \) and (iv) of Lemma 2.3, i.e.

\[
q(x) \geq S_q(x) - \alpha \sqrt{M/h} - \beta d \sqrt{M/h} \geq \lambda_1 - \alpha \sqrt{M/h} - \beta d \sqrt{M/h}. \tag{115}
\]
\( \lambda_2 = \alpha \hat{u} \), \( \eta_2 = \lambda_2 / d \).

\( \lambda_2 = \alpha \hat{u} \), \( \eta_2 = \lambda_2 \).

\( \lambda_2 = \beta d \hat{v} \), \( \eta_2 = \lambda_2 / d \).

\( \lambda_2 = \beta d \hat{v} \), \( \eta_2 = \lambda_2 \).

**Figure 6.** N-barrier for the case \( a_1, a_2 > 1 \) in the \( S_uS_v \)-plane. Dashed black curves: the solution \((u(x), v(x))\) of \((BVP^*)\); black lines: \( 1 - u - a_1 v = 0 \) and \( 1 - a_2 u - v = 0 \); green curve: \( F(u, v) = 0 \); magenta line (above): \( \frac{u}{\hat{u}} + \frac{v}{\hat{v}} = 1 \), where \( \hat{u} \) and \( \hat{v} \) are given by (83); magenta line (below): \( \frac{u}{\hat{u}} + \frac{v}{\hat{v}} = 1 \), where \( \hat{u} \) and \( \hat{v} \) are given by (84) and (85); blue line (above): \( \alpha S_u + \beta d S_v = \lambda_2 \), where \( \lambda_2 = \min \{ \alpha \hat{u}, \beta d \hat{v} \} \); red lines: \( \alpha S_u + \beta S_v = \eta_2 \) (above), where \( \eta_2 = \lambda_2 \min \{ 1, 1/d \} \), and \( \alpha S_u + \beta S_v = \eta_1 \) (below), where \( \eta_1 \) satisfies (99); blue line (below): \( \alpha S_u + \beta d S_v = \lambda_1 \), where \( \lambda_1 = \eta_1 \min \{ 1, d \} \).

It is easy to see that \((ii)\) can be established by letting \( \beta = \frac{\hat{\beta}}{d} \) in \((i)\) and (100). The proof of Theorem 1.2 is completed. \( \square \)
5. **Nonexistence of three species.** In this section, we prove Theorem 1.3 by contradiction. When $w(x)$ is a given function with a known upper bound in (N), we follow the proof of Proposition 1 for the first and the second equations of (N) and obtain $L^2$-norm estimates as in (21) and (22). Using these $L^2$-norm estimates, we recover Lemma 2.3 with new $M$ and $\tilde{M}$. Finally, by applying new Lemma 2.3 and using [H1] and [H2], we mimic the proof of Theorem 1.2 to obtain a lower bound of $c_{31} u(x) + c_{32} v(x)$, which contradicts [H3].

**Proof of Theorem 1.3.** Suppose to the contrary that there exists a positive solution to (N), say $(u(x), v(x), w(x))$. Since $w(x) > 0$ for $x \in \mathbb{R}$ and $w(\pm \infty) = 0$, we can find $x_0 \in \mathbb{R}$ such that $\max x \in \mathbb{R}$, which leads to $w(x_0 + h) - 2w(x_0) + w(x_0 - h) \leq 0$ and $w'(x_0) = 0$. Then it follows from the third equation of (N) that

$$\sigma_3 - c_{31} u(x_0) - c_{32} v(x_0) - c_{33} w(x_0) \geq 0,$$

which gives

$$w(x) \leq w(x_0) \leq \frac{1}{c_{33}}(\sigma_3 - c_{31} u(x_0) - c_{32} v(x_0)) < \frac{\sigma_3}{c_{33}}, \quad x \in \mathbb{R}. \quad (117)$$

Following the proof of Proposition 1 for the first and the second equations of (N) with $w(x)$ being a given function bounded by $\frac{\sigma_3}{c_{33}}$ as shown in (117), we obtain the $L^2$-norm estimates as in (21) and (22) with $M_1$ and $\tilde{M}_1$ replaced by $M_1^*$ and $\tilde{M}_1^*$ respectively, where

$$M_1^* = M^*(d_2, \sigma_1, \sigma_2, \sigma_3, c_{12}, c_{13}, \theta)$$

and

$$\tilde{M}_1^* = M^*(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta). \quad (119)$$

In (118) and (119), $M^*(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta)$ is given by

$$M^*(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta) = \frac{2}{d_2}(2K(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta, 1) + 2\tilde{J}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta)$$

$$+4\tilde{K}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta, 2)) + \frac{3}{\theta^2}(\sigma_1^2 + c_{21} \sigma_2 + c_{23} \sigma_3)(\frac{\sigma_3^2}{4c_{22}} + \frac{\sigma_1}{c_{11} c_{22}} + \frac{\sigma_2}{c_{22} c_{33}})^2,$$

where

$$\tilde{J}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta) = \frac{4\sigma_1^2}{c_{22}}\tilde{J}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta) + \frac{2\theta}{d_2} \sigma_2^2$$

$$+ \frac{4}{d_2} \left(\sigma_2^2 \frac{1}{4c_{22}} + \frac{c_{21}}{c_{11} c_{22}} \sigma_2 + c_{23} \sigma_3 \frac{c_3}{c_{22}^2} \right),$$

$$\tilde{K}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta, 2) = \frac{2\sigma_2}{c_{22}} \tilde{J}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta)$$

$$+ \frac{1}{d_2} \left(\frac{\theta}{2 \sigma_2^2} \sigma_2^2 + 2 \left(\frac{\sigma_2^2}{4c_{22}} + \frac{c_{21}}{c_{11} c_{22}} \sigma_2 + c_{23} \sigma_3 \frac{c_3}{c_{22}^2} \right) \right),$$
and
\[ \tilde{f}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta) = \frac{4\sigma_2}{c_{22}} + \frac{1}{d_2^2} \left( \frac{\theta \sigma_2}{c_{22}} + 2 \left( \frac{\sigma_2^2}{4c_{22}} + \frac{\sigma_1}{c_{11} c_{22}} + \frac{c_{23}}{c_{22} c_{33}} \right) \right). \]

Also, we obtain the $L^2$-norm estimates as in (78) and (76) with $M_2$ and $\tilde{M}_2$ replaced by $M^*_2$ and $\tilde{M}^*_2$ respectively, where
\[ M^*_2 = 8 \tilde{K}(1, \sigma_1, \sigma_2, \sigma_3, c_{12}, c_{11}, c_{13}, \theta, 1/2) \] (120)
and
\[ \tilde{M}^*_2 = 8 \tilde{K}(d_2, \sigma_1, \sigma_2, \sigma_3, c_{21}, c_{22}, c_{23}, \theta, 1/2). \] (121)

Now we define
\[ M^* = \max\{M_1^*_{\theta=1}, M_2^*_{\theta=1}\} \] (122)
and
\[ \tilde{M}^* = \max\{\tilde{M}_1^*_{\theta=1}, \tilde{M}_2^*_{\theta=1}\} \] (123)
as in (79) and (80), and recover Lemma 2.3 with $M$ and $\tilde{M}$ replaced by $M^*$ and $\tilde{M}^*$, respectively.

Substituting (117) into the first and the second equations of (N) leads to
\[
\begin{aligned}
\frac{1}{h^2} (u(x+h) - 2u(x) + u(x-h)) + \theta u_x + u (\sigma_1 - c_{13} \sigma_3 c_{33} - c_{11} u - c_{12} v) &\leq 0, \\
\frac{d_2}{h^2} (v(x+h) - 2v(x) + v(x-h)) + \theta v_x + v (\sigma_2 - c_{23} \sigma_3 c_{33} - c_{21} u - c_{22} v) &\leq 0,
\end{aligned}
\]
where $x \in \mathbb{R}$. By means of [H1] and [H2], from (124) it follows that
\[
\begin{aligned}
\frac{1}{h^2} (u(x+h) - 2u(x) + u(x-h)) + \theta u_x + u (1 - u - a_1 v) &\leq 0, & x \in \mathbb{R}, \\
\frac{d_2}{h^2} (v(x+h) - 2v(x) + v(x-h)) + \theta v_x + kv (1 - a_2 u - v) &\leq 0, & x \in \mathbb{R}.
\end{aligned}
\]
(125)
Addition of the two inequalities in (125) results in a analogous inequality as in (102) (but with equality replaced by inequality), i.e.
\[
0 \geq \int_z^{z_1} (q * \psi')'(x) \, dx + \theta \int_z^{z_1} p'(x) \, dx + \int_z^{z_1} F(u(x), v(x)) \, dx \\
= \frac{1}{h} \left((S_q(z_1) - S_q(z_1 - h)) - (S_q(z) - S_q(z - h))\right) + \theta (p(z_1) - p(z)) \\
+ \int_z^{z_1} F(u(x), v(x)) \, dx.
\] (126)

We proceed by mimicking the proof of Theorem 1.2 with minor modification that $M$ and $\tilde{M}$ in Lemma 2.3 are replaced by $M^*$ and $\tilde{M}^*$ respectively. Finally, we have
\[ p(x) = \alpha u(x) + \beta v(x) \geq \check{\lambda}_1 = \frac{\min \{ \alpha \check{u}, \beta \check{v} \}}{\max \{ 1, d \}} - \max \left[ 2 \left( \alpha \sqrt{M^* h} + \beta \sqrt{\tilde{M}^* h} \right), \alpha \sqrt{M^* h} + \beta \sqrt{\tilde{M}^* h} \right]\] (127)
where
\[ \check{\lambda}_1 = \left( \frac{\min \{ \alpha \check{u}, \beta \check{v} \}}{\max \{ 1, d \}} - \max \left[ 2 \left( \alpha \sqrt{M^* h} + \beta \sqrt{\tilde{M}^* h} \right), \alpha \sqrt{M^* h} + \beta \sqrt{\tilde{M}^* h} \right] \right) \min \{ 1, d \} \] (128)
Letting $\alpha = c_{31}$ and $\beta = c_{32}$ in (127), we obtain a lower bound of $c_{31} u(x) + c_{32} v(x)$, i.e.,

$$c_{31} u(x) + c_{32} v(x) \geq \Lambda^*(c_{31}, c_{32}, \sigma_3), \quad x \in \mathbb{R}.$$  \hfill (129)

However, [H3] then yields

$$c_{31} u(x) + c_{32} v(x) \geq \sigma_3, \quad x \in \mathbb{R},$$  \hfill (130)

which contradicts (116). This completes the proof. $\square$

**Appendix.** We prove Lemma 2.2 and Lemma 2.3.

**Proof of Lemma 2.2.** We first show that (ii) follows from (i) by differentiation and then show that (iii) follows from (ii) by integration.

Using Lemma 2.2 (i), Definition 2.1 (i), (iv) and (v), we have

$$(\rho * \phi)''(x) = h^2 (\rho * \psi)''(x) = h \frac{d}{dx} \int_{x-h}^{x} A_\rho(z) \, dz = \frac{d}{dx} \int_{x-h}^{x} (\rho(z + h) - \rho(z)) \, dz$$

$$= \rho(x + h) - \rho(x) - (\rho(x) - \rho(x - h)) = \rho(x + h) - 2 \rho(x) + \rho(x - h).$$  \hfill (131)

This shows that Lemma 2.2 (ii) holds. Integrating the above equation from $z_2$ to $z_1$ and using Definition 2.1 (ii) we obtain

$$\int_{z_2}^{z_1} (\rho * \phi)''(x) \, dx = \int_{z_2}^{z_1} (\rho(x + h) - 2 \rho(x) + \rho(x - h)) \, dx$$

$$= \int_{z_2}^{z_1} (\rho(x + h) - \rho(x)) \, dx - \int_{z_2}^{z_1} (\rho(x) - \rho(x - h)) \, dx$$

$$= \left( \int_{z_2 + h}^{z_1 + h} \rho(x) \, dx - \int_{z_2}^{z_1} \rho(x) \, dx \right) - \left( \int_{z_2}^{z_2 - h} \rho(x) \, dx - \int_{z_2 - h}^{z_1 - h} \rho(x) \, dx \right)$$

$$= \left( \int_{z_2}^{z_1 + h} \rho(x) \, dx - \int_{z_2}^{z_2 + h} \rho(x) \, dx \right) - \left( \int_{z_2}^{z_1} \rho(x) \, dx - \int_{z_2 - h}^{z_2 - h} \rho(x) \, dx \right)$$

$$= \left( \int_{z_2}^{z_1 + h} \rho(x) \, dx - \int_{z_2}^{z_1} \rho(x) \, dx \right) - \left( \int_{z_2}^{z_2 + h} \rho(x) \, dx - \int_{z_2 - h}^{z_2 - h} \rho(x) \, dx \right)$$

$$= h \left( (S_\rho(z_1) - S_\rho(z_1 - h)) - (S_\rho(z_2) - S_\rho(z_2 - h)) \right),$$  \hfill (132)

from which Lemma 2.2 (iii) follows by using Definition 2.1 (iii) and (iv). Now it remains to show Lemma 2.2 (i). Indeed, by Definition 2.1 (i), we have

$$\frac{1}{h} \int_{x-h}^{x} A_\rho(z) \, dz = \frac{1}{h^2} \int_{x-h}^{x} ((\rho(z + h) - \rho(z)) \, dz$$

$$= \frac{1}{h^2} \left( \int_{x}^{x+h} \rho(z) \, dz - \int_{x-h}^{x} \rho(z) \, dz \right).$$  \hfill (133)

On the other hand, Definition 2.1 (iii) $\sim$ (v) lead to

$$(\rho * \psi)'(x) = (\psi' * \rho)(x) = \int_{-\infty}^{\infty} \psi'(z) \rho(x-z) \, dz$$

$$= \frac{1}{h^2} \int_{-h}^{0} \rho(x-z) \, dz - \frac{1}{h^2} \int_{0}^{h} \rho(x-z) \, dz$$
The proof is elementary. We use the Hölder inequality and is completed.

Thus we have shown

\[ \text{Lemma 184 CHIUN-CHUAN CHEN, TING-YANG HSIAO AND LI-CHANG HUNG} \]

Proposition is given by \((M)\) where \(\psi\) can be taken as

\[ \text{Lemma 184 CHIUN-CHUAN CHEN, TING-YANG HSIAO AND LI-CHANG HUNG} \]

In deriving (134), we have used the fact that \((\rho * \psi)'(x) = (\psi' * \rho)(x)\). Since \(\psi(x)\) can be taken as

\[ \psi(x) = \begin{cases} 
  \frac{h + x}{h^2}, & \text{if } -h \leq x \leq 0, \\
  \frac{h - x}{h^2}, & \text{if } 0 \leq x \leq h, \\
  0, & \text{otherwise}, 
\end{cases} \]

such that \(\psi'(x)\) satisfies Definition 2.1 (iv), we have

\[ (\rho * \psi)'(x) = \frac{d}{dx} \int_{-\infty}^{x} \rho(z)\psi(x - z) \, dz = \frac{d}{dx} \int_{-\infty}^{x} \rho(x - z)\psi(z) \, dz \]

\[ = \int_{-\infty}^{x} \rho'(x-z)\psi(z) \, dz + \int_{x}^{x+h} \rho'(z)\psi(x-z) \, dz \]

\[ = \left( \rho(z)\psi(x-z) \right)_{z=x-h}^{z=x} + \int_{x-h}^{x} \rho(z)\psi'(x-z) \, dz \]

\[ = (\rho(x)\psi(0) - \rho(x-h)\psi(h)) + \rho(x+h)\psi(-h) \]

\[ = (\rho * \psi')(x) = (\psi' * \rho)(x). \] (135)

Now (133) and (134) lead to \((\rho * \psi)'(x) = \frac{1}{h} \int_{x-h}^{x} \rho(z) \, dz\). Thus we have shown Lemma 2.2 (i). The proof of Lemma 2.2 is completed. \(\square\)

**Proof of Lemma 2.3.** The proof is elementary. We use the Hölder inequality and Proposition 1 (iii) to find

\[ |u(x) - u(y)| = \left| \int_{y}^{x} u'(z) \, dz \right| \leq \left( \int_{y}^{x} |u'(z)|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{y}^{x} 1^2 \, dz \right)^{\frac{1}{2}} \leq \sqrt{M_1 h}, \]

where \(M_1 = M(a_1, 1, 1, \theta)\) is determined in (21) and

\[ M(a, d, k, \theta) = \frac{3}{16 \theta} k^2 (1 + 4a)^2 + 29 \frac{k}{\theta} (1 + 4a) + 160 \frac{d}{\theta} + 54 \]

is given by (25). On the other hand, using (77) we find

\[ |u(x) - u(y)| \leq \sqrt{M_2 h}. \] (137)
Recall that $M_2$ is defined in (78) as

$$M_2 = 8 \mathcal{K}(a_1, 1, 1, \theta, 1/2),$$

where

$$\mathcal{K}(a, d, k, \theta, z_1 - z_2) = 8 \frac{1}{4d} \left(10 \theta + k (1 + 4a) (4 + z_1 - z_2) \right)$$

is given by (78). We observe that $M_1$ has three fractions with $\theta$ or $\theta^2$ in the denominator, while $M_2$ is a linear function of $\theta$. This fact allows us to obtain a sharper upper bound of $|u(x) - u(y)|$. In other words, when $\theta \geq 1$, we have

$$|u(x) - u(y)| \leq \sqrt{M_1 h} \leq \sqrt{M_1 h}_{|\theta=1}$$

by using (136). When $0 \leq \theta \leq 1$, it follows from (137) that

$$|u(x) - u(y)| \leq \sqrt{M_2 h} \leq \sqrt{M_2 h}_{|\theta=1}. \tag{139}$$

These amount to

$$|u(x) - u(y)| \leq \max\{\sqrt{M_1 h}_{|\theta=1}, \sqrt{M_2 h}_{|\theta=1}\} = \sqrt{M h}. \tag{140}$$

The inequality $|v(x) - v(y)| \leq \sqrt{M h}$ can be shown in the same manner. This proves (i). Since $|p(x) - p(y)| = |\alpha (u(x) - u(y)) + \beta (v(x) - v(y))|$ and $|q(x) - q(y)| = |\alpha (u(x) - u(y)) + \beta d (v(x) - v(y))|$, (ii) immediately follows from (i). To see (iii), employing (i) gives

$$|S_u(x) - u(x)| = \left| \frac{1}{h} \int_x^{x+h} (u(y) - u(x)) \, dy \right| \leq \frac{1}{h} \int_x^{x+h} \sqrt{M h} \, dy = \sqrt{M h}. \tag{141}$$

The inequality $|S_v(x) - v(x)| \leq \sqrt{M h}$ can be shown in a similar manner. By means of (iii), (iv) can be proved:

$$|S_p(x) - p(x)| = \left| \frac{\alpha}{h} \int_x^{x+h} (u(y) - u(x)) \, dy + \frac{\beta}{h} \int_x^{x+h} (u(y) - u(x)) \, dy \right|$$

$$\leq \alpha \sqrt{M h} + \beta \sqrt{M h}, \tag{142}$$

from which we see the inequality $|S_q(x) - q(x)| \leq \alpha \sqrt{M h} + \beta d \sqrt{M h}$ also holds.

To prove (v), without loss of generality we let $0 < x - y = \hat{\delta} \leq h$. Then

$$|S_u(x) - S_u(y)| = \left| \frac{1}{h} \int_x^{x+h} u(z) \, dz - \frac{1}{h} \int_y^{y+h} u(z) \, dz \right|$$

$$= \frac{1}{h} \left| \int_{y+\hat{\delta}}^{y+\hat{\delta}+h} u(z) \, dz - \int_y^{y+h} u(z) \, dz \right|$$

$$= \frac{1}{h} \left| \int_y^{y+h} u(z + \hat{\delta}) \, dz - \int_y^{y+h} u(z) \, dz \right|$$

$$\leq \frac{1}{h} \int_y^{y+h} \left| u(z + \hat{\delta}) - u(z) \right| \, dz \leq \sqrt{M h},$$
where we have used (i). It is easy to see that $|S_u(x) - S_v(y)| \leq \sqrt{M} h$ can be proved in the same manner. Finally, we use (v) to prove (vi):

$$
|S_u(x) - S_v(y)| = |\alpha S_u(x) + \beta d S_v(x) - \alpha S_v(y) - \beta d S_v(y)|
\leq \alpha |S_u(x) - S_v(y)| + \beta d |S_v(x) - S_v(y)|
\leq \alpha \sqrt{M} h + \beta d \sqrt{M} h.
$$

(143)

By letting $d = 1$ in the above inequality, we find $|S_u(x) - S_v(y)| \leq \alpha \sqrt{M} h + \beta \sqrt{M} h$. The proof of the lemma is completed.

\section*{Acknowledgments}
The authors are grateful to the anonymous referees for many helpful comments and valuable suggestions on this paper.

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Received November 2018; 1st revision February 2019; final revision June 2019.

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