Anomalous self-similarity in two-dimensional turbulence

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(March 3, 2008)

Abstract

Our velocity measurements on a quasi-two-dimensional turbulent flow in a rapidly rotating annulus yield an inverse cascade with $E(k) \sim k^{-2}$ rather than the expected $E(k) \sim k^{-5/3}$. The probability distribution functions for longitudinal velocity differences, $\delta v(r) = v(x + r) - v(x)$, are self-similar (scale independent) but strongly non-Gaussian, which suggests that the coherent vortices play a significant role. The structure functions, $\langle [\delta v(r)]^p \rangle \sim r^{\zeta_p}$, exhibit anomalous scaling: $\zeta_p = \frac{p}{2}$ rather than $\zeta_p = \frac{p}{3}$ as in the 1941 Kolmogorov theory.
Kolmogorov’s 1941 theory of turbulence (K41) assumes self-similarity of a homogeneous, isotropic turbulent flow, that is, the statistics of the difference between velocities measured at two points, \( \delta v(r) = v(x+r) - v(x) \) (where \( v \) is along the line connecting the two points) are independent of the separation \( r \). Self-similarity can also be determined from the scaling of structure functions, \( S_p(r) \equiv \langle [\delta v(r)]^p \rangle \sim r^{\zeta_p} \); for any self-similar flow, \( \zeta_p \) will vary linearly with \( p \). In particular, Kolmogorov theory predicts the existence of an inertial range with power law scaling with \( \zeta_p = \frac{p}{3} \).

Experiments and numerical simulations have shown that three-dimensional (3D) turbulence is not self-similar. The observed deviations from a linear dependence of \( \zeta_p \) on \( p \) are attributed to the stretching and folding of vortices. As a vortex stretches, it shrinks in lateral extent until it collapses onto a singular line. Consequently, the dissipation rate varies with length scale.

Since vortex stretching is not allowed in two dimensions, self-similarity of 2D turbulence is a separate question from self-similarity of 3D turbulence. The tendency of 2D flows to form large coherent vortices could produce intermittent rather than self-similar statistics. However, a recent numerical simulation of 2D turbulence suggests self-similar behavior. Three recent experiments have given differing answers to the question of self-similarity in 2D turbulent flows. Measurements on a magnetically driven shallow fluid layer yielded a clear inverse energy cascade, and self-similar behavior was indicated by both the Probability Distribution Functions (PDFs) for \( \delta v \) and by the scaling of the structure function. The PDFs were Gaussian throughout the inertial range, and structure functions yielded \( \zeta_p = p/3 \), in accord with the K41 prediction. In contrast, two experiments on turbulence in soap films yielded intermittent rather than self-similar flow. In one experiment the PDFs changed in shape from stretched exponentials at small separations to Gaussian at large separations, and it was concluded that “the simultaneous presence of both inverse energy and forward enstrophy cascades appears necessary to observe intermittency in 2D turbulence.”

Here we report on experiments that use rotation to two-dimensionalize a continuously driven turbulent flow. We observe both an inverse energy cascade and a forward enstrophy cascade; the latter will be discussed elsewhere. The inverse cascade dynamics are characterized by a family of scale-invariant PDFs for \( \delta v \) that have a strongly non-Gaussian shape. Log-log plots of the \( p \)th-order structure functions \( S_p(r) \) as a function of \( S_3(r) \) (a method called Extended Self-Similarity (ESS)) shows the Kolmogorov behavior, \( \zeta_p = p/3 \); this scaling must follow for self-similar flows if \( \zeta_{p=0} = 0 \). However, our graphs of \( S_p \) as a function of \( r \) suggest \( \zeta_p = p/2 \), confirming self-similar behavior with no intermittency, but contrasting with the Kolmogorov prediction. We now describe the experiments and then present the results.

Our apparatus consists of an annular tank filled with water and covered with a solid lid; the inner diameter of the tank is 21.6 cm and the outer diameter is 86.4 cm. Flow in the annulus is produced by continuously pumping water in a closed circuit through two concentric rings of 120 holes each in the bottom of the tank; the source ring is at 18.9 cm and the sink ring is at 35.1 cm. The radially outward flux couples with the Coriolis force to generate a counter-rotating jet, which is wide and turbulent over a wide range of parameters. The rapid rotation leads by the Taylor-Proudman theorem to a flow that is essentially 2D except in the thin Ekman boundary layers at the top and bottom surfaces.
The bottom of the tank is conical, increasing in depth from 17.1 cm at the inner radius to 20.3 cm at the outer radius; this sloping “beta plane”, a first order approximation of the variation of the Coriolis force with latitude, breaks the symmetry between co-rotating and counter-rotating jets.

In the present experiments the tank rotates at 11.0 rad/s, sufficiently fast to produce essentially 2D flow, and the flux is 150 cm$^3$/s, sufficiently large to produce an inverse energy cascade. The azimuthal velocity is measured using hot film probes which are inserted through the top lid and extend 1 cm into the water on opposite sides of the tank, midway between the inner and outer walls. Each probe was sampled at 150 Hz for periods of two hours, giving $10^6$ data points per probe for each run, and the measurements were repeated four times, yielding a total of $8 \times 10^6$ data points. Using the maximum velocity ($U_{max} = 22$ cm/s) as the velocity scale and the distance between the forcing rings ($L = 16.2$ cm) as the integral length scale, we calculate the Reynolds number to be $35 \times 10^3$ and the Rossby number ($U_{max}/2\Omega L$, where $\Omega$ is the rotation rate) to be 0.05.

Instantaneous 2D velocity fields were obtained using a Particle Image Velocimetry (PIV) system with a horizontal light sheet at mid-fluid depth and a rotating camera above the tank. For each flow condition, 50 instantaneous velocity fields were obtained, equivalent to approximately $2 \times 10^5$ velocity values at the radius of the hot film probes. Though this sample size is inadequate for higher order statistics, the spatial information provided by the PIV measurements complements the long velocity time series obtained with the hot film probes. We also consider PIV measurements at the same rotation rate at a higher pumping rate of 550 cm$^3$/s ($Re = 100 \times 10^3$).

Two vorticity profiles obtained with the PIV system are shown in Fig. 1. When the pump is first turned on, the flow consists of rings of cyclonic and anti-cyclonic vortices that form above the outlet and inlet rings, respectively. Vortices of like sign immediately begin merging and growing in size, and an anticyclonic jet forms between the two rings. At long times the inverse energy cascade leads to large vortices; the vortices are larger at higher pumping rates, as Fig. 1 illustrates.

We compute energy power spectra from the velocity time series data assuming Taylor’s frozen turbulence hypothesis, which is applicable because the turbulent intensity (ratio of the rms velocity fluctuation to the mean velocity) is less than 10%. The spectra contain a region with $E(k) \sim k^{-2}$ (Fig. 2), in contrast with Kraichnan’s prediction of $E(k) \sim k^{-5/3}$ for the inverse cascade [10]. From the PIV data for the higher pumping rate, energy spectra for the azimuthal velocity at the probe radius show a scaling consistent with those obtained from the time series data, as shown in the inset of Fig. 2. At high wavenumbers, the power spectra are consistent with the previously observed $E(k) \sim k^{-n}$, where $3 \leq n \leq 4$ for the forward (enstrophy) cascade [1], but our spectral range is too small to deduce an exponent value.

The histogram of the velocity differences $\delta v(r)$ in Fig. 3 demonstrates self-similar behavior: data for different separations $r$ collapse onto a single curve. The velocity differences are normalized by their standard deviation $(\delta v)_{rms}$ and the probability by its maximum value, $P_{max}$. The velocity differences for both the forward and inverse cascades fall on a single curve, which is far from Gaussian, in contrast to the observations for other quasi-two-dimensional flows [5–7] but in accord with a recent numerical simulation of 2D turbulence [2]. The enhanced probability in the tails of our PDF is likely due to the strong velocity differences...
that arise as coherent vortices pass the probes, as noted by She et al. [11].

We have shown that the energy spectrum scales anomalously, that is, different from K41 theory (Fig. 2). Another indication of anomalous behavior is the scaling of the standard deviation of the velocity differences, $\delta v_{\text{rms}}$. Our data suggest $\delta v(r) \sim r^{0.5}$ for both pumping rates (cf. insets of Fig. 3) rather than the $\delta v(r) \sim r$ of K41 theory.

Now we examine the higher order structure functions, plotting $S_p$ as a function of $r$, as shown in Fig. 4(a). There is a scaling region (labeled A) from about 2 cm to 8 cm, and some indication of another scaling region (labeled B) for $r < 2$ cm; the transition between the two regions can be seen more clearly in the plot of $S_{10}/r^{5.5}$ vs. $r$ in the inset.

The location of the transition in the structure functions (2 cm) does not agree with the transition (at 5 cm) in the energy spectra of Fig. 2. However, the region where the two disagree is the range where the measurements from our two hot film probes begin to disagree with each other. The presence of spurious peaks in this range as well as the disagreement with the PIV scaling, point to a bias in the hot film measurements that is amplified by the correlation function, but to which the velocity differences are insensitive. This bias might come from a systematic error due to the calibration of the hot film probes, which would cancel out when differences are taken, but which would be amplified by the correlation. It is also possible that the scaling regions do not exactly coincide, as discussed in [1] (p. 62).

One test for the existence of an inverse energy cascade in 2D turbulence is the sign of $S_3$: Kolmogorov’s four-fifths law can be written for anisotropic flows as $\varepsilon = -\frac{1}{2} \nabla_r (|\delta v(r)|^2 \delta v(r))$ ($v$ is the full 3D velocity vector, and $\varepsilon$ is the mean rate of energy transfer) (Ref. [1], p. 88). For the three-dimensional forward cascade $\varepsilon$ is positive; a negative value of $\varepsilon$ corresponds to an inverse cascade. If the anisotropy is not too strong the longitudinal $S_3$ dominates the transverse $S_3$, and one can obtain the sign of $\varepsilon$ from measurements of the longitudinal structure function alone. In our flows, $S_3 > 0$ up to $r \approx 10$ cm, which suggests that the inverse energy cascade stops at that point. This length corresponds visually to the scale of the largest vortices seen in the flow field. Thus region A corresponds to the inverse cascade. Region B corresponds to lengths smaller than the distance (2 cm) between the forcing holes in the outer ring and may correspond to the forward enstrophy cascade.

The now standard way of extracting structure function exponents from data with limited inertial range is the technique introduced by Benzi et al. [8] called Extended Self Similarity (ESS), where the $\zeta_p$ values are given by the slope of log-log plots of $S_p$ vs. $S_3$. The ESS plots of our data, Fig. 4(b), exhibit power law scaling through both ranges A and B, for lengths $0.5 < r < 15$ cm. Since $S_3$ should contain the same trends as the other structure functions, it is not surprising that the switching from region A to B is lost by this method.

The exponent values $\zeta_p$ deduced from the plot of $S_p$ [region A in Fig. 4(a)] and from the ESS plot [Fig. 4(b)] are compared in Fig. 5 with the K41 prediction, $\zeta_p = p/3$. Figure 4(a) yields (in region A), $\zeta_{p,A} = p/2$, in contrast with theory, although $\zeta_{p,ESS} = p/3$, as it must if the flow is self-similar. Previous experiments on quasi-2D flows have not found a well defined scaling region for $S_p$ with $r$. In applying ESS the magnetically driven fluid experiment yielded $\zeta_p = p/3$, as we found, but the soap film experiments [6,7] obtained a strong departure from linearity for $\zeta_p$.

In summary, a quasi-two-dimensional turbulent flow generated by strong rotation is found to be self-similar but to exhibit anomalous scaling of the energy, $E(k) \sim k^{-2}$, and the
structure function, $S_p \sim r^{p/2}$. A possible explanation of the $p/2$ scaling may be a merging driven by the radial velocity shear, independent of the azimuthal velocity fluctuations. This merging would yield an inverse energy flux $\varepsilon \sim \delta v_s (\delta v_\parallel)^2 / r$ where $v_s$ might be the shear velocity, $v_\parallel$ the azimuthal velocity and $r$ the azimuthal separation. For a scale-invariant $\varepsilon$, the resulting $p^{th}$ order velocity structure function would have the scaling exponent $p/2$. Although the anomalous scaling is also seen in the slope of the energy spectrum (Fig. 2), it is missed if one only conducts an Extended Self Similarity analysis.

The work at the University of Texas was supported by a grant from the Office of Naval Research. Z-S. S. acknowledges partial support from the Minister of Education in China and the Natural Science Foundation of China.
FIG. 1. Vorticity field at Reynolds number 35 000 \((Q = 150 \text{ cm}^3/\text{s})\) and 100 000 \((Q = 550 \text{ cm}^3/\text{s})\) for rotation rate 11.0 rad/s. Vortices with light (dark) center are cyclonic (anticyclonic). The vortices are advected by the clockwise jet.
FIG. 2. Energy spectra with a dotted line showing the Kraichnan $k^{-5/3}$ inverse cascade and a solid line showing $k^{-2}$ behavior. The sharp spectral peaks correspond to harmonics of the tank’s rotation rate, not to dynamics of the flow.
FIG. 3. Normalized probability distribution function for the velocity differences, demonstrating self-similar behavior: data for different separations ($r = 0.6, 4.6, 9.2, 17.3$ cm) collapse onto a single curve. The standard deviation of the velocity differences (see insets) scales as $r^{0.5}$ for $2 \leq r \leq 10$ cm for both pumping rates.
FIG. 4. Even order structure functions (a) as a function of $r$ and (b) as a function of $S_3$ (an Extended Self Similarity plot). The graph of $S_{10}/r^{5.5}$ in the inset in (a) emphasizes the sharpness of the bend in $S_{10}$ at $r \simeq 2$ cm. In (b) the dark symbols indicate the region from which the values of $\zeta_p$ were extracted.

FIG. 5. Scaling exponents $\zeta_p$ as a function of $p$ for region A of Fig. 4(a), and from the Extended Self Similarity analysis, Fig. 4(b). Region A, the inverse energy cascade region yields $\zeta_p = (0.50 \pm 0.03)p$, while the ESS scaling shows the expected $\zeta_p = (0.33 \pm 0.01)p$ scaling over the forward and inverse cascade ranges. The error bars are obtained by taking the standard deviation of the spread from the eight separate data sets of $10^6$ points.
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