CLASSIFICATION OF EINSTEIN EQUATIONS WITH COSMOLOGICAL CONSTANT IN WARPED PRODUCT SPACE-TIME

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Received (Day Month Year)
Revised (Day Month Year)

We classify all warped product space-times in three categories as i) generalized twisted product structures, ii) base conformal warped product structures and iii) generalized static space-times and then we obtain the Einstein equations with the corresponding cosmological constant by which we can determine uniquely the warp functions in these warped product space-times.

Keywords: Einstein equation; Space-time; Cosmological constant.

Mathematics Subject Classification 2010: 83F05, 35Q76, 53C25.

1. Introduction

Twisted product space-times are generalizations of warped product space-times in that the warping function may depend on the points of both warp factors [12]. Base conformal warped product space-times have metrics in the form of a mixture of a conformal metric on the base and a warped metric. These metrics with considerations about their curvatures are very frequent in different aspects of physics, such as relativity theory, extra-dimensional theories (Kaluza-Klein, Randall-Sundrum), string and super-gravity theories, quantum gravity and also the study of spectrum

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of Laplace-Beltrami operators on \( p \)-forms. On the other hand, a standard static space-time is a Lorentzian warped product space-time where the warp function is defined on a Riemannian manifold and acts on the negative definite metric on an open interval of the real numbers. Different aspects of these space-times have been previously studied on many issues, such as geodesic equation, geodesic completeness, geodesic connectedness, causal structure, curvature conditions, etc (see for example \[3\]–[6]).

On the other hand, the Einstein equations are considered as the basic equations in General Relativity. In fact, finding the Einstein equations is of prime importance in any general relativistic model of gravitation and cosmology. In recent years, a large amount of attention has been paid on the modification of Einstein equations \[7\], specially in models with space-time dimensions other than four \[24\]. In this line of activity, we have previously obtained the Einstein’s equation in \((m + n)D\) and \((1 + n)D\) multidimensional space-time with multiply-warped product metric \((\bar{M}, \bar{g})\) \[27\], \[28\]. Specially, we have discussed on the origin of \(4D\) cosmological constant as an emergent effect of higher dimensional warped spaces.

In this work, we intend to obtain the Einstein’s equations with cosmological constant in the warped product space-times categorized as i) generalized twisted product structures, ii) base conformal warped product structures and iii) generalized static space-times. In the beginning, we recall that a pseudo-Riemannian manifold \((M_1, g)\) is conformal to the pseudo-Riemannian manifold \((M_2, h)\), if and only if there exists \(\eta \in C^\infty(M_1)\) such that \(g = e^{\eta} h\). Also, we will call a doubly twisted product as a base conformal warped product, when the functions \(\psi\) and \(\phi\) depend only on the points of \(M_1\).

2. Preliminaries

**Definition 1.** Let \((M_1, g)\) and \((M_2, h)\) are two pseudo-Riemannian manifolds, where \(M_1\) is a \(m\)-dimensional manifold and \(M_2\) is a \(n\)-dimensional manifold and \(f\) be a positive smooth function on \(M_1\). The local coordinates on \(\bar{M} = M_1 \times fM_2\) are \((x^i, x^\alpha)\), where \((x^i)\) and \((x^\alpha)\) are the local coordinates on \(M_1\) and \(M_2\), respectively. Also suppose that, \(\bar{g}\) is a pseudo-Riemannian metric on \(\bar{M}\) which is defined by

\[
\bar{g} = g_{ij}dx^i dx^j + f^2(x^k) h_{\alpha\beta} dx^\alpha dx^\beta,
\]

which is given by its local components as:

\[
\bar{g}_{ij} = g_{ij}, \quad \bar{g}_{\alpha\beta} = f^2(x^k) h_{\alpha\beta}, \quad \bar{g}_{i\alpha} = 0,
\]

where \(g_{ij}(x^k)\) are the local components of \(g\) and \(h_{\alpha\beta}(x^\mu)\) are the local components of \(h\). Here, and in the sequel, we use the following ranges for indices: \(i, j, k, ... \in \{1, ..., m\}\); \(\alpha, \beta, ... \in \{m + 1, ..., m + n\}\); \(a, b, c, ... \in \{1, ..., n + m\}\).

We denote by \(\bar{G}\) the Einstein gravitational tensor field of \((\bar{M}, \bar{g})\), that is, we have,

\[
\bar{G} = \bar{Ric} - \frac{1}{2} \bar{S} \bar{g},
\]
where $\bar{\text{Ric}}$ and $\bar{S}$ are Ricci tensor and scaler curvature of $\bar{M}$, respectively.

### 3. Generalized twisted product structures

Twisted products are generalizations of warped products, namely the warping function may depend on the points of both factors. Twisted products are useful in the study of Einstein’s equations, when we use special conditions of this products.

Let $(M_1, g)$ and $(M_2, h)$ be two pseudo-Riemannian manifolds and let $f : M_1 \times M_2 \to \mathbb{R}^+$ be a positive function on $M_1 \times M_2$. The product manifold $ar{M} = M_1 \times M_2$ endowed with the metric $\bar{g} = g \oplus f^2 h$ is called a twisted product. The same terminology of base and fiber applies in this case, whereas $f$ is the twisting function (sometimes it is also referred to as the warping function).

For simplicity in some of the expressions, we use $\xi = \log f$ instead of $f$.

**Proposition 2** ([1]). Let $\bar{M} = M_1 \times_f M_2$ be a twisted product we have

\[
\begin{align*}
\bar{\text{Ric}}(\partial_i, \partial_j) &= \text{Ric}^{M_1}(\partial_i, \partial_j) - n(\partial_i(\xi)\partial_j(\xi) + H_\xi(\partial_i, \partial_j)), \\
\bar{\text{Ric}}(\partial_i, \partial_\alpha) &= (1-n)\partial_\alpha(\partial_i(\xi)), \\
\bar{\text{Ric}}(\partial_\alpha, \partial_\beta) &= \text{Ric}^{M_2}(\partial_\alpha, \partial_\beta) + (2-n)(\partial_\alpha(\xi)\partial_\beta(\xi) + H_\xi(\partial_\alpha, \partial_\beta)) \\
&\quad + (n-2)f(\partial_\alpha, \partial_\beta)\bar{g}(\nabla \xi, \nabla \xi) - \bar{g}(\partial_\alpha, \partial_\beta)\Delta \xi,
\end{align*}
\]

where $H$ is the Hessian tensor of $\xi$.

Now let $(\bar{M}, \bar{g})$ be a space-time, where $\bar{M} = M \times_f I$ is a $(n+1)$-dimensional twisted product manifold, $M$ is a $n$-dimensional manifold, $g$ is a Lorentz metric on $M$, $I$ is a 1-dimensional manifold and $f : M \times I \to \mathbb{R}^+$ be a positive function on $M \times I$. Also, suppose that $\bar{g}$ is a semi-Riemannian metric on $\bar{M}$, given by its local components

\[
\begin{align*}
\bar{g}(\partial_i, \partial_j) &= g(\partial_i, \partial_j), \\
\bar{g}(\partial_i, \frac{\partial}{\partial f}) &= 0, \\
\bar{g}(\frac{\partial}{\partial f}, \frac{\partial}{\partial f}) &= f^2,
\end{align*}
\]

where $g(\partial_\alpha, \partial_\beta)$ are the local components of $g$. Hence, the metric defined by $\bar{g}$ has the form

\[
ds^2 = g(\partial_i, \partial_j)dx^i dx^j + f^2 dt^2.
\]

**Theorem 3** ([1]). Let $\bar{M} = M_1 \times_f M_2$ be a twisted product of $(M_1, g)$ and $(M_2, h)$ with twisting function $f$ and $\text{dim} M_2 > 1$. Then, $\text{Ric}(\partial_i, \partial_\alpha) = 0$ if and only if $M_1 \times_f M_2$ can be expressed as a warped product, $M_1 \times_\psi M_2$ of $(M_1, g)$ and $(M_2, h)$ with a warping function $\psi$, where $\hat{h}$ is a conformal metric tensor to $h$. 
Proposition 4 ([29]). Let $\bar{M} = M \times_f I$ be a twisted product. Let $\partial_i, \partial_j$ are local components of vectors tangent to the base $M$ and let $\partial_t$ be local components of vector tangent to the fiber. Then the Ricci tensor is:

$$\bar{\text{Ric}}(\partial_i, \partial_j) = \text{Ric}^M(\partial_i, \partial_j) - \frac{H_f(\partial_i, \partial_j)}{f},$$
$$\bar{\text{Ric}}(\partial_j, \partial_t) = 0,$$
$$\bar{\text{Ric}}(\partial_t, \partial_t) = -\frac{\Delta^M f}{f},$$

(7)

where $\text{Ric}^M(\partial_i, \partial_j)$ are the local components of Ricci tensor on $(M, g)$.

Now, in what followed, according a straightforward calculation, we obtain Einstein equations and cosmological constant on $\bar{M} = M \times_f I$.

Proposition 5. The Einstein gravitational tensor field of $(\bar{M}, \bar{g})$, have following equations:

$$\bar{G}(\partial_i, \partial_j) = G(\partial_i, \partial_j) - \frac{H_f(\partial_i, \partial_j)}{f} + \frac{\Delta^M f}{f} g(\partial_i, \partial_j),$$
$$\bar{G}(\frac{\partial t}{f}, \frac{\partial t}{f}) = -\frac{1}{2} f^2 S^M - \frac{\Delta^M f}{f} + f \Delta^M f,$$
$$\bar{G}(\frac{\partial t}{f}, \partial_i) = 0,$$

(8)

Proof. By the equation (3), and a straightforward computation according the argument in (7) we will have the desired result. 

Let $(\bar{M}, \bar{g})$ be the twisted product space. Suppose that the Einstein gravitational tensor field $\bar{G}$ of $(\bar{M}, \bar{g})$ satisfies the Einstein equations with cosmological constant $\bar{\Lambda}$ as

$$\bar{G} = -\bar{\Lambda}\bar{g}.$$  

(9)

Now according to the Proposition 5 and a straightforward computation to Einstein equation $\bar{G} = -\bar{\Lambda}\bar{g}$, we have the following theorem.

Theorem 6. The Einstein equations on $(\bar{M}, \bar{g})$ with cosmological $\bar{\Lambda}$ are equivalent with the following equations

$$\bar{\Lambda} = \frac{\Delta^M f}{f} \left( \frac{1}{f^2} (1 - \frac{n}{2}) - \frac{1}{2} \right),$$
$$G(\partial_i, \partial_j) = \frac{H_f(\partial_i, \partial_j)}{f} - \frac{\Delta^M f}{f} \left( \frac{1}{f^2} (1 - \frac{n}{2}) - \frac{1}{2} \right) g(\partial_i, \partial_j).$$

(10)
Now, by demanding for a constant $\bar{\Lambda}$, the warp function $f$ is obtained for a given dimension $n$, as a solution of the following differential equation

$$\frac{\Delta^M f}{f} \left( \frac{1}{f^2} \left( 1 - \frac{n}{2} \right) - \frac{1}{2} \right) = a,$$  \hspace{1cm} (11)$$

where $a$ is a constant which can take negative, zero and positive values corresponding to negative, zero and positive cosmological constants.

4. Special base conformal warped product

Doubly-twisted product is a usual product of pseudo-Riemannian manifolds $M_1 \times M_2$ with $g = f_1^2 g \oplus f_2^2 h$, then $\bar{M} = M_1 \times_{(f_1, f_2)} M_2$ is called as doubly-twisted product. When $f_1 = 1$ then $M_1 \times f M_2$ is a twisted product which we discussed in the last section. Doubly twisted product is as a base conformal warped product when the functions $f_1$ and $f_2$ only depend on the points of $M_1$. In this section we consider a subclass of base conformal warped product called as special base conformal warped products.

**Definition 7.** Let $(M_1, g)$ and $(M_2, h)$ be $m$ and $n$ dimensional pseudo Riemannian manifolds, respectively. Then $\bar{M} = M_1 \times M_2$ is an $(m+n)$-dimensional pseudo Riemannian manifold with smooth functions $f_1 : M_1 \to (0, \infty)$ and $f_2 : M_1 \to (0, \infty)$. The base conformal warped product is the product manifold $\bar{M} = M_1 \times M_2$ furnished with the metric tensor $\bar{g} = f_1^2 g \oplus f_2^2 h$ defined by

$$\bar{g} = (f_1 \circ \pi)^2 \pi^* g \oplus (f_2 \circ \sigma)^2 \sigma^* h.$$  \hspace{1cm} (12)$$

We will denote this structure by $M_1 \times_{(f_1, f_2)} M_2$. The function $f_2 : M_1 \to (0, \infty)$ is called the warping function and the function $f_1 : M_1 \to (0, \infty)$ is said to be the conformal factor.

If $f_1 = 1$ and $f_2$ is not identically 1, then we obtain a singly warped product. If both $f_1 = 1$ and $f_2 = 1$, then we have a product manifold. If neither $f_1$ nor $f_2$ is constant, then we have a nontrivial base conformal warped product. If $(M_1, g)$ and $(M_2, h)$ are both Riemannian manifolds, then $M_1 \times_{(f_1, f_2)} M_2$ is also a Riemannian manifold. We call $M_1 \times_{(f_1, f_2)} M_2$ as a Lorentzian base conformal warped product if $(M_2, h)$ is Riemannian and either $(M_1, g)$ is Lorentzian or else $(M_1, g)$ is a one-dimensional manifold with a negative definite metric $-dt^2$.

**Proposition 8** ([30]). Let $\partial_i, \partial_j$ and $\partial\alpha, \partial\beta$ are local components of vectors tangent to the base $M_1$ and fiber $M_2$, respectively. Then, the Ricci tensor of $M_1 \times_{(f_1, f_2)} M_2$ satisfies
Definition 10. A \((n+1)\) -dimensional generalized Robertson-Walker (GRW) space-time with \(n > 1\) is a Lorentzian manifold which is the base conformal warped product \(M = I \times (f_1, f_2)\) \(M\) of an open interval \(I\) of the real line \(\mathbb{R}\) and a Riemannian \(n\)-manifold \((M, g)\) endowed with the Lorentzian metric

\[
\bar{g} = -f_1(t)^2 \pi^*(dt^2) + f_2(t)^2 \sigma^*(g),
\]

we can write

\[
\bar{g} = -dt^2 + f_2(t)^2 h_{\alpha\beta} dx^\alpha dx^\beta,
\]

where \(\pi\) and \(\sigma\) denote the projections onto \(I\) and \(M\), respectively, and \(f_1, f_2\) are positive smooth function on \(I\). In a classical Robertson-Walker space-time, the fiber is three dimensional and of constant sectional curvature, and the warping function \(f_1\) and \(f_2\) are arbitrary. The following formula can be directly obtained from the previous result and noting that on a multiply generalized Robertson-Walker space-time \(\text{grad}_f f = -f', \| \text{grad}_f f \|^2 = -f'^2, g(\vec{\frac{\partial}{\partial t}}, \vec{\frac{\partial}{\partial t}}) = -1, H^f(\vec{\frac{\partial}{\partial t}}, \vec{\frac{\partial}{\partial t}}) = f''\) and \(\Delta f = -f''\), we denote the usual derivative on the real interval \(I\) by the prime notation (i.e.,') from now on.

**Proposition 9** ([30]). The scalar curvature \(S\) of \(M_1 \times_{(f_1, f_2)} M_2\) is given by

\[
f_1^2 S = S_{M_1} + S_{M_2} - \frac{f_1^2}{f_2^2} - 2(m-1) \frac{\Delta_{M_1} f_1}{f_1} - 2n \frac{\Delta_{M_2} f_2}{f_2} - (m-4)(m-1) \frac{g(\nabla_{M_1} f_1, \nabla_{M_1} f_1)}{f_1^2} - 2n \frac{\nabla_{M_1} f_1 \nabla_{M_1} f_1}{f_1^2} - n(n-1) \frac{g(\nabla_{M_2} f_2, \nabla_{M_2} f_2)}{f_2^2}.
\]

**4.1. Generalized Robertson-Walker (GRW) space-time**

**Theorem 10.**
Proposition 11 ([31]). Let $\bar{M} = I \times_{(f_1, f_2)} M_2$ be the base conformal warped product manifold.

\[
\bar{\text{Ric}}(\partial_t, \partial_t) = -nf'' f^2, \\
\bar{\text{Ric}}(\partial_t, \partial_\alpha) = 0, \\
\bar{\text{Ric}}(\partial_\alpha, \partial_\beta) = \text{Ric}^{M_2}(\partial_\alpha, \partial_\beta) - f_2^2 g(\partial_\alpha, \partial_\beta) \left( -\frac{f''}{f} - (n-1)\left(\frac{f'}{f}\right)^2 \right),
\]

where $\text{Ric}^{M_2}(\partial_\alpha, \partial_\beta)$ are the local components of Ricci tensor in $(M_2, h)$.

Proposition 12 ([31]). Let $\bar{M} = I \times_{(f_1, f_2)} M_2$ be the base conformal warped product manifold. Then scalar curvature $\bar{S}$ of $(\bar{M}, \bar{g})$ admits the following expression,

\[
\bar{S} = \frac{S_{M_2}}{f_2^2} + 2nf'' f^2 + n(n-1)\left(\frac{f'}{f}\right)^2.
\]

(18)

So we will have:

Proposition 13. Let $\bar{G}$ be the Einstein gravitational tensor field of $(\bar{M}, \bar{g})$, then we have following equations:

\[
\bar{G}(\partial_t, \partial_t) = -\frac{1}{2} \left( -\frac{S_{M_2}}{f_2^2} - n(n-1)\left(\frac{f'}{f}\right)^2 \right), \\
\bar{G}(\partial_t, \partial_\alpha) = 0, \\
\bar{G}(\partial_\alpha, \partial_\beta) = G(\partial_\alpha, \partial_\beta) - f_2^2 g(\partial_\alpha, \partial_\beta) \left( (n-1)\frac{f''}{f} - (n-1)(1 - \frac{n}{2})\left(\frac{f'}{f}\right)^2 \right),
\]

where $G(\partial_\alpha, \partial_\beta)$ are the local components of the Einstein gravitational tensor field $G$ of $(M_2, h)$ and $G(\partial_t, \partial_t)$ are the local components of the Einstein gravitational tensor field $G$ of $(I, g)$.

Theorem 14. The Einstein equations on $(\bar{M}, \bar{g})$ with cosmological $\bar{\Lambda}$ are equivalent with the following equations

\[
\bar{\Lambda} = \frac{1}{2} n(n-1)\frac{f''}{f_2}, \tag{20}
\]

\[
G(\partial_\alpha, \partial_\beta) = f_2^2 g(\partial_\alpha, \partial_\beta)(n-1)(\frac{n}{2} - 1)\left( \frac{f''}{f} - \left(\frac{f'}{f}\right)^2 \right).
\]

Now, by demanding for a constant $\bar{\Lambda}$, the warp function $f$ is obtained for a given dimension $n$, as the unique solution of the following differential equation

\[
\frac{1}{2} n(n-1)\frac{f''}{f_2} = a, \tag{21}
\]

where $a$ is a constant which can take negative, zero and positive values corresponding to negative, zero and positive cosmological constants.
5. Generalized standard static space-times

In this section we study a standard static space-time which is \(n\)-dimensional Lorentzian product. It is a generalization of a Einstein static universe. The results are important in solutions of the Einstein equations. Partial results are physical motivation for cosmological constant problems.

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold and \(f : M \to (0, \infty)\) be a smooth function. Then \((n + 1)\)-dimensional product manifold \(\bar{M} = (a, b) \times M\) furnished with the metric tensor \(\bar{g} = -f^2 dt^2 \oplus g_M\) is called a standard static space-time and is denoted by \(f(a,b) \times M\), where \(dt^2\) is the Euclidean metric tensor on \((a, b)\) and \(-\infty \leq a \leq b \leq \infty\).

We have in local components
\[
\bar{g}(\partial \alpha, \partial \beta) = g(\partial \alpha, \partial \beta),
\]
\[
\bar{g}(\partial \alpha, \frac{\partial t}{f}) = 0,
\]
\[
\bar{g}(\frac{\partial t}{f}, \frac{\partial t}{f}) = -f^2.
\]

**Proposition 15** \([6]\). Let \(f(a,b) \times M\) be a standard static space-time. We have following equation for scalar curvatures of the space-time
\[
\bar{S} = S_M - 2 \frac{\Delta^M(f)}{f},
\]
where Ricci tensor have following equations
\[
\bar{Ric}(\frac{\partial t}{f}, \frac{\partial t}{f}) = \frac{\Delta^M(f)}{f},
\]
\[
\bar{Ric}(\frac{\partial t}{f}, \partial \alpha) = 0,
\]
\[
\bar{Ric}(\partial \alpha, \partial \beta) = Ric^M(\partial \alpha, \partial \beta) - \frac{H^f(\partial \alpha, \partial \beta)}{f},
\]

So we easily deduce the following:

**Proposition 16.** The Einstein gravitational tensor field of \((\bar{M}, \bar{g})\), have following equations:
\[
\bar{G}(\frac{\partial t}{f}, \frac{\partial t}{f}) = \frac{1}{2} f^2 S^M + \frac{\Delta^M f}{f} - f \Delta^M f,
\]
\[
\bar{G}(\partial \alpha, \partial \beta) = G(\partial \alpha, \partial \beta) - \frac{H^f(\partial \alpha, \partial \beta)}{f} + \frac{\Delta^M f}{f} g(\partial \alpha, \partial \beta),
\]
\[
\bar{G}(\frac{\partial t}{f}, \partial \iota) = 0,
\]
CLASSIFICATION OF EINSTEIN EQUATIONS WITH COSMOLOGICAL CONSTANT IN WARPED PRODUCT SPACE-TIME

Theorem 17. The Einstein’s equations on \((\bar{M}, \bar{g})\) with cosmological \(\bar{\Lambda}\) are expressed with the following equations

\[
\bar{\Lambda} = \frac{\Delta^M f}{f^2} \left( f^2 (1 - \frac{n}{2} - \frac{1}{2}) \right), \\
G(\partial\alpha, \partial\beta) = \frac{H_f(\partial\alpha, \partial\beta)}{f} - \frac{\Delta^M f}{f} \left( f^2 (1 - \frac{n}{2} - \frac{1}{2}) \right) g(\partial\alpha, \partial\beta). \tag{26}
\]

Now, by demanding for a constant \(\bar{\Lambda}\), the warp function \(f\) is obtained for a given dimension \(n\), as a solution of the following differential equation

\[
\frac{\Delta^M f}{f} \left( f^2 (1 - \frac{n}{2} - \frac{1}{2}) \right) = a, \tag{27}
\]

where \(a\) is a constant which can take negative, zero and positive values corresponding to negative, zero and positive cosmological constants.

6. Conclusions

In this paper, we have studied three classes of warped product space-times as: generalized twisted product structures, base conformal warped product structures and generalized static space-times. Then, we obtained Einstein equations with cosmological constant in these warped product space-times. We may conclude that any Einstein equations with cosmological constant in warped product space-times lies within these three categories. Therefore, a variety of gravitational and cosmological models with warped product structure can be categorized according to the corresponding warp functions.

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