Generalized Bernstein Type Operators on Unbounded Interval and Some Approximation Properties

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Abstract. In the present paper, we construct a new family of Bernstein type operators on infinite interval by using exponential function $a^x$. We study some approximation results for these new operators on the interval $[0, \infty)$.

1. Introduction

In approximation theory, the most important basic result was given by Karl Weierstrass. In 1912, S.N. Bernstein \cite{1} introduced the sequence of operators to give a constructive proof of the Weierstrass approximation theorem. In 1950 a new generalization of Bernstein’s polynomials to the infinite interval was given by O. Szász \cite{8}. The uniform convergence of a sequence of linear positive operators to continuous functions was introduced by Bohman \cite{3} and Korovkin \cite{7}. The Bernstein operators are defined as follows:

\[ B_n(f; x) = \sum_{r=0}^{n} \binom{n}{r} x^r (1-x)^{n-r} f \left( \frac{r}{n} \right), \quad x \in [0, 1] \quad \text{and} \quad n \in \mathbb{N}. \tag{1} \]

For detailed study we can see \cite{6}.

O. Szász \cite{8} introduced the following operators on the infinite interval as follows:

\[ S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{k}{n} f \left( \frac{k}{n} \right), \quad f \in C[0, \infty), \quad n \geq 1. \tag{2} \]

In this paper we construct a new family of Bernstein type operators on an infinite interval and obtain some important approximation results for these operators. In next section, a new family of operators is constructed by using the exponential function $a^x$ for the interval $[0, \infty)$. We use the symbol log $a$ for log, $a$ throughout the paper.

Stirling’s Formula:

\[ n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n. \tag{3} \]
Definition 1.1. By \( f(n) = o(g(n)) \), we mean \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \).

2. Construction of operators and some auxiliary results

We define the following operators:

\[
S^*_u(f; x) = a^{-ux} \sum_{v=0}^{\infty} \frac{(ux \log a)^v}{v!} f\left( \frac{v}{u \log a} \right), \quad f \in C[0, \infty), \quad u > 0 \text{ and } a > 1. \tag{4}
\]

For \( a = e \) or \( u = n / \log a \), then it reduces to (2).

Definition 2.1. A set of continuous functions \( S^*_u(f; x) \) is said to be convergent uniformly to the value \( M \) at \( x = \alpha \) as \( u \to \infty \) if \( S^*_u(f; x_n) \to M \), whenever \( x_n \to \alpha \) and \( u_n \to \infty \) as \( n \to \infty \).

Lemma 2.2. For \( u > 0 \) and \( \lambda > 0 \), we have

\[
\sum_{|v-u \log a| \geq \lambda} \frac{(u \log a)^v}{v!} < \lambda^2 u \log a(a^u), \quad \text{for } a > 1. \tag{5}
\]

Proof. We have the following identity which is easily verified:

\[
\sum_{v=0}^{\infty} (v-u \log a)^2 \frac{(u \log a)^v}{v!} = u \log a(a^u), \tag{6}
\]

and then it follows that

\[
\lambda^2 \sum_{|v-u \log a| \geq \lambda} \frac{(u \log a)^v}{v!} \geq \sum_{v=0}^{\infty} (v-u \log a)^2 \frac{(u \log a)^v}{v!} = u \log a(a^u).
\]

The proof is completed. \( \Box \)

Lemma 2.3. For \( u \geq 0 \), the following inequality holds:

\[
\sum_{v=0}^{\infty} |v-u \log a| \frac{(u \log a)^v}{v!} \leq \sqrt{u \log a(a^u)} \tag{7}
\]

Proof. By using Schwarz’s inequality and (6), we have

\[
\left( \sum_{v=0}^{\infty} |v-u \log a| \frac{(u \log a)^v}{v!} \right)^2 \leq \left( \sum_{v=0}^{\infty} (v-u \log a)^2 \frac{(u \log a)^v}{v!} \right) \left( \sum_{v=0}^{\infty} \frac{(u \log a)^v}{v!} \right) = u \log a(a^u)(a^u) = u \log a(a^{2u}).
\]

The proof is completed. \( \Box \)

Note that

\[
\sum_{v=0}^{\infty} (v-u \log a) \frac{(u \log a)^v}{v!} = 0. \tag{8}
\]
Therefore, for a positive integer $u$, we have
\[
\sum_{v=0}^{\infty} |v - u \log a| \frac{(u \log a)^v}{v!} = \sum_{v \leq u \log a} (u \log a - v) \frac{(u \log a)^v}{v!} + \sum_{v \geq u \log a} (v - u \log a) \frac{(u \log a)^v}{v!}
\]
\[
= 2 \sum_{v \leq u \log a} (u \log a - v) \frac{(u \log a)^v}{v!}
\]
\[
= 2u \log a \sum_{v \leq u \log a} \frac{(u \log a)^v}{v!} - 2 \sum_{v \leq u \log a} \frac{(u \log a)^v}{v!}
\]
\[
= 2u \log a \sum_{v \leq u \log a} \frac{(u \log a)^v}{v!} - 2 \sum_{v \leq u \log a} \frac{(u \log a)^v}{(v - 1)!}
\]
\[
= 2u \log a \sum_{v \leq u \log a} \frac{(u \log a)^v}{v!} - 2u \log a \sum_{v \leq u \log a - 1} \frac{(u \log a)^v}{v!}
\]
\[
= 2u \log a \frac{(u \log a)^{u \log a}}{u \log a!}. \quad (9)
\]

Now by using Stirling formula (3), we have
\[
\sum_{v=0}^{\infty} |v - u \log a| \frac{(u \log a)^v}{v!} = 2 \sqrt{u \log a} \frac{e^{u \log a}}{\sqrt{2\pi}}
\]
\[
= 2 \sqrt{u \log a} a^{u \log a}
\]

Thus, except for a constant factor, the estimate (7) is the sharpest possible.

3. Main results

In this section, we will study some convergence results, pointwise as well as uniformly convergence and also obtain a Voronovskaja type theorem.

Theorem 3.1. Suppose that $f(x)$ is bounded in every finite interval, if $f(x) = o(x^k)$ for some $k > 0$ and if $f(x)$ is continuous at a point $\alpha > 0$, then $S_n^*(f;x)$ converges uniformly to $f(x)$ at $x = \alpha$.

Proof.
\[
a^{a^k} \left| S_n^*(f;x) - f(x) \right| = \sum_{v=0}^{\infty} \left| f(v/u \log a) - f(x) \right| \frac{(ux \log a)^v}{v!}
\]
\[
= \sum_{|v/u \log a - x| \leq \delta} + \sum_{|v/u \log a - x| \geq \delta}
\]
\[
= L_1 + L_2, \quad \text{(say)}.
\]

Let
\[
\max |f(x) - f(\beta)| = m^*(\delta, \beta) = m^*(\delta), \quad \text{for} \quad |x - \beta| \leq \delta,
\]
then \( m'(\delta) \to 0 \) as \( \delta \to 0 \). Now

\[
f(v/u \log a) - f(x) = f(v/u \log a) - f(\alpha) + f(\alpha) - f(x),
\]

and

\[
\frac{v}{u \log a} - \beta = \frac{v}{u \log a} - x + x - \beta.
\]

In the sum \( L_1 \), \( |v/u \log a - x| \leq \delta \), hence from (10)

\[
\left| \frac{v}{u \log a} - \beta \right| \leq 2\delta
\]

and

\[
|f(v/u \log a) - f(x)| \leq m'(2\delta) + m'(\delta) \leq 2m'(2\delta).
\]

Hence

\[
|L_1| < 2m'(2\delta) \sum_{v=0}^{\infty} \frac{(ux \log a)^v}{v!} = 2m'(2\delta)u^x a^x.
\]

Next write

\[
L_2 = \sum_{v < u \log a(x-\delta)} + \sum_{v > u \log a(x+\delta)} = L_3 + L_4, \quad (\text{say}).
\]

Then

\[
L_3 < \sum_{v < u \log a(x-\delta)} \frac{(ux \log a)^v}{v!} |f(v/u \log a) - f(x)|.
\]

Let

\[
\sup |f(x)| = V(\delta), \quad \text{for} \quad x \leq \delta.
\]

Then

\[
L_3 < 2V(\alpha + \delta) \sum_{ux \log a - v \alpha \log e} \frac{(ux \log a)^v}{v!}.
\]

By using Lemma 2.2 with \( \lambda = u\delta \log a \), we have

\[
L_3 < 2V(\alpha + \delta) \frac{ux \log a (a^{ux})}{(u \log a)^2 \delta^2} = 2V(\alpha + \delta) \frac{x a^{ux}}{(u \log a) \delta^2}.
\]

Now, assuming \( u \log a(x + \delta) > k \),
Finally, we have

Theorem 3.2. If \( f \) satisfies the Lipschitz-type condition

\[
|f(x_1) - f(x_2)| < M \frac{|x_1 - x_2|^\rho}{(x_1 + x_2)^\gamma}, \quad 0 < x_1 < x_2 < \infty,
\]

where \( M, \rho \) are constants, \( 0 < \rho \leq 1 \), then

\[
|S_a^u(f; x) - f(x)| \leq M (u \log a)^{-\frac{\rho}{\gamma}},
\]

converges uniformly for \( 0 < x < \infty \), as \( u \to \infty \).

Proof. First of all, we consider the case \( \rho = 1 \).

\[
L_4 = o \left( \sum_{v > u \log a \delta} \frac{(ux \log a)^v}{v!} \left( \frac{v}{u \log a} \right)^k \right)
= o \left( \sum_{v > u \log a \delta} \frac{x^k (ux \log a)^{v-k}}{(v-k)!} \right)
= o \left( \sum_{v > u \log a \delta - k} \frac{x^k (ux \log a)^{v}}{\mu!} \right).
\]

Again by using Lemma 2.2, with \( \lambda = u \delta \log a - k > 0 \), we get

\[
L_4 = o \left( \frac{x^k (ux \log a)^{\delta \log a - k}}{(u \delta \log a - k)^2} \right) = o \left( \frac{u \log a (a^\mu)}{(u \delta \log a - k)^2} \right), \quad \text{as } u \to \infty.
\]

Finally, we have

\[
S_a^u(f; x) - f(x) = o \left\{ m(2\delta) + \frac{1}{u \delta^2 \log a} + \frac{u \log a}{(u \delta \log a - k)^2} \right\}.
\]

Now for a fixed \( \delta \), letting \( u \to \infty \), we have

\[
\limsup |S_a^u(f; x) - f(x)| \leq o(m(2\delta)), \quad u \to \infty, \quad \text{for } |x - a| \leq \delta,
\]

which gives the desired result. \( \square \)

Theorem 3.2. If \( f(x) \) satisfies the Lipschitz-type condition

\[
|f(x_1) - f(x_2)| < M \frac{|x_1 - x_2|^\rho}{(x_1 + x_2)^\gamma}, \quad 0 < x_1 < x_2 < \infty,
\]

where \( M, \rho \) are constants, \( 0 < \rho \leq 1 \), then

\[
|S_a^u(f; x) - f(x)| \leq M (u \log a)^{-\frac{\rho}{\gamma}},
\]

converges uniformly for \( 0 < x < \infty \), as \( u \to \infty \).
by Lemma 2.3. Thus, the proof is completed for $\rho = 1$.

Now from Hölder’s inequality, for $0 < \rho < 1$, we have

$$a^{\mu x}[S_\mu (f; x) - f(x)] = \sum_{\nu=0}^{\infty} \frac{(ux \log a)^{\nu(1-\rho)}}{(\nu!)^{1-\rho}} (ux \log a)^{\rho \nu} \left( |f(v/u \log a) - f(x)| \right)^{\rho} \leq \left( \sum_{\nu=0}^{\infty} \frac{(ux \log a)^{\nu}}{(\nu!)^{1-\rho}} \right)^{1-\rho} \left( \sum_{\nu=0}^{\infty} \frac{(ux \log a)^{\nu}}{(\nu!)^{1-\rho}} \right)^{\rho}.$$  

Now by (11), we have

$$a^{\mu x}[S_\mu (f; x) - f(x)] < a^{(1-\rho)\mu x} M \left( \sum_{\nu=0}^{\infty} \frac{(ux \log a)^{\nu}}{(\nu!)^{1-\rho}} |v/u \log a - x| \right)^{\rho} \leq Ma^{(1-\rho)\mu x} \left( \frac{1}{u \log a} \sqrt[n]{u \log a (u^{\mu x})} \right)^{\rho} = M(u \log a)^{-\frac{\rho}{n}} a^{\mu x}.$$  

Thus, the proof is completed. \(\square\)

Example 3.3. Let

$$f(x) = \begin{cases} \frac{c-x}{x} & \text{for } 0 \leq x \leq c \\ 0 & \text{for } x \geq c, \end{cases}$$

where $c$ is a positive constant and the condition (11) is fulfilled. Furthermore

$$S_\mu (f; x) - f(c) = S_\mu^*(f; x) = a^{-nc} \sum_{v \leq uc \log a} \frac{(uc \log a)^{\nu}}{v!} (c - v/u \log a)$$

$$= \frac{1}{u \log a} a^{-nc} \sum_{v \leq uc \log a} \frac{(uc \log a)^{\nu}}{v!} (uc \log a - v) \frac{(uc \log a)^{\nu}}{v!}.$$

Let $[uc] = r$ and multiply by $(u \log a)^{\frac{1}{r}}$ on both sides. Then

$$(u \log a)^{\frac{1}{r}} S_\mu^*(f; c) > (u \log a)^{\frac{1}{r}} a^{-r} \sum_{v=0}^{r \log a} (r \log a - v) \frac{(r \log a)^{\nu}}{v!}.$$  

By using (9), we have

$$\sum_{v=0}^{r \log a} (r \log a - v) \frac{(r \log a)^{\nu}}{v!} = \frac{(r \log a)^{r \log a+1}}{(r \log a)!} \sim \left( \frac{r \log a}{2\pi} \right)^{\frac{1}{r}} a^r,$$

since $a^r = e^{r \log a}$. Thus,

$$\liminf_{u \to \infty} (u \log a)^{\frac{1}{r}} S_\mu^*(f; c) > 0,$$

which proves that the order of estimate in Theorem 3.2 is sharpest possible for $\rho = 1$.

Theorem 3.4. If $f(x)$ is continuous in $(0, \infty)$, then $S_\mu^*(f; x) \to f(x)$ uniformly in the interval $(0, \infty)$. 

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Proof. Suppose $f(x)$ is continuous in $(0, \infty)$. Let

$$x = \frac{1}{\log a} \log \left( \frac{1}{t} \right), \quad 0 \leq t \leq 1,$$

$$f(x) = f \left( \frac{1}{\log a} \log \frac{1}{t} \right) = \psi(t)$$

is continuous in $0 \leq t \leq 1$.

Now for a given $\epsilon > 0$, we can find a polynomial

$$\sum_{r=0}^{n} b_r t^r = P_n(t),$$

so that

$$|\psi(t) - P_n(t)| < \epsilon.$$ 

It follows that

$$|f(x) - P_n(a^{-x})| < \epsilon, \quad x \in (0, \infty).$$

Now by (4) for $P_n(a^{-x})$, we have

$$S_n^*(P_n; x) = a^{-ux} \sum_{n=0}^{\infty} \frac{(ux \log a)^v}{v!} \sum_{r=0}^{n} b_r a^{-rv} \log a \log (a^{-r/u \log a})^v \sum_{v=0}^{n} b_r a^{-ux(1-u^{-v} \log a)}.$$

Hence, $S_n^*(P_n; x) \to P_n(a^{-x})$ uniformly in the interval $(0, \infty)$, as $u \to \infty$.

Furthermore

$$f(x) = P_n(a^{-x}) + \epsilon_n(x), \quad |\epsilon_n(x)| < \epsilon,$$

and

$$S_n^*(f; x) = S_n^*(f - P_n; x) + S_n^*(P_n; x).$$

Here,

$$|S_n^*(f - P_n; x)| < \epsilon,$$

so that

$$|S_n^*(f; x) - f(x)| < \epsilon + |S_n^*(P_n; x) - f(x)|$$

$$< \epsilon + |S_n^*(P_n(a^{-x}); x) - P_n(a^{-x})| + |P_n(a^{-x}) - f(x)|.$$

Thus, the proof is completed.
Theorem 3.5. If \( f(x) \) is \( r \)-times differentiable, \( f^{(r)}(x) = o(x^k) \) as \( x \to \infty \), for some \( k > 0 \), and if \( f^{(r)}(x) \) is continuous at a point \( \alpha \), then \( S_u^{(r)}(f;x) \) converges uniformly to \( f^{(r)}(x) \) at \( x = \alpha \).

Let \( 1/u = h \). We use the following notations

\[
\begin{align*}
\Delta f(vh) &= f(vh + 1h) - f(vh), \\
\Delta^2 f(vh) &= f(vh + 2h) - 2f(vh + 1h) + f(vh).
\end{align*}
\]

\( \vdots \) \quad (12)

\[
\begin{align*}
\Delta^r f(vh) &= \Delta \Delta^{r-1} f(vh) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(v + k h), \quad r \geq 0.
\end{align*}
\]

\( \vdots \) \quad (13)

Let \( a^{-x/h} \frac{d}{dx} Q_h(f;x) = \sum_{v=0}^{\infty} \Delta^r f(vh/\log a) \frac{1}{v!} \left( \frac{x \log a}{h} \right)^v \left( \frac{\log a}{h} \right)^r \).

Differentiation gives

\[
\begin{align*}
\frac{d}{dx} Q_h(f;x) &= a^{-x/h} \sum_{v=0}^{\infty} \frac{1}{(v-1)!} \left( \frac{x \log a}{h} \right)^{v-1} \frac{\log a}{h} f(vh/\log a) \\
&- \frac{\log a}{h} a^{-x/h} \sum_{v=0}^{\infty} \frac{1}{v!} \left( \frac{x \log a}{h} \right)^v f(vh/\log a) \\
&= \frac{\log a}{h} a^{-x/h} \sum_{v=0}^{\infty} \frac{1}{v!} \left( \frac{x \log a}{h} \right)^v \Delta f(vh/\log a),
\end{align*}
\]

and the result follows by the induction.

It is known that

\[
\left( \frac{\log a}{h} \right)^r \Delta^r(f(vh)) = f^{(r)}(\mu),
\]

where

\[
vh < \mu < (v + r)h.
\]

\[
D_s Q_h(f;x) - f^{(r)}(x) = a^{-x/h} \sum_{v=0}^{\infty} \left( \frac{x \log a}{h} \right)^v \Delta^r(f(vh/\log a) - f^{(r)}(x)) \left( \frac{x \log a}{h} \right)^v \\
= a^{-x/h} \left\{ \sum_{|vh/\log a - x| \leq \delta} + \sum_{|vh/\log a - x| > \delta} \right\},
\]

where \( |x - a| < \delta \). Now using the same technique as in the proof of Theorem 3.1, we obtain the above theorem.
Theorem 3.7. Let $f(x)$ be bounded in every finite interval and differentiable at a point $\alpha > 0$. If $f(x) = o(x^k)$ for some $k > 0$, then
\[
\lim_{u \to \infty} (u \log a)^{1/2} \{ S'_u(f(\alpha); x) - f(\alpha) \} = 0.
\]

Proof. Let
\[
\max \left| \frac{f(\alpha + h) - f(\alpha)}{h} - f'(\alpha) \right| = \eta(\delta, \alpha) = \eta(\delta).
\]
Then $\eta(\delta) \to 0$ as $\delta \to 0$. We may write
\[
f(\alpha + h) - f(\alpha) = hf'(\alpha) + he(\alpha, h),
\]
where $|e(\alpha, h)| \leq \eta(\delta)$ for $|h| \leq \delta$.

Thus
\[
S'_u(f(\alpha); x) - f(\alpha) = a^{-u\alpha} \sum_{v=0}^{\infty} \frac{(u\alpha \log a)^v}{v!} \left\{ \left( \frac{v}{u \log a} - \alpha \right) f'(\alpha) + \left( \frac{v}{u \log a} - \alpha \right) e_v(u) \right\},
\]
where
\[
|e_v(u)| \leq \eta(\delta) \text{ for } \left| \frac{v}{u \log a} - \alpha \right| \leq \delta.
\]
Now by using formula (8), we have
\[
S'_u(f(\alpha); x) - f(\alpha) = \frac{1}{u \log a} a^{-u\alpha} \sum_{v=0}^{\infty} \frac{(u\alpha \log a)^v}{v!} \left( \sum_{|v-\alpha u \log a| \leq \delta u \log a} + \sum_{|v-\alpha u \log a| > \delta u \log a} \right).
\]
On the same technique as in the proof of Theorem 3.1, and using Lemma 2.3, we can get the desired result.

The following result is generalize to higher derivatives. We restrict here that $f''(\alpha)$ exists and the same result proved for Bernstein polynomials (see [2], [9]).

Theorem 3.8. Let $f(x)$ be bounded in every finite interval and twice differentiable at a point $\alpha > 0$. If for some $k > 0$,
\[
f(x) = o(x^k),
\]
then
\[
\lim_{u \to \infty} u \log a \left[ S'_u(f(\alpha); x) - f(\alpha) \right] = \frac{1}{2} \alpha f''(\alpha).
\]

Proof. We restrict here to the case that $f''(\alpha)$ exists. Thus
\[
f(\alpha + h) - f(\alpha) = hf'(\alpha) + \frac{1}{2} h^2 \{ f''(\alpha) + e(\alpha, h) \},
\]
where
\[
|e(\alpha, h)| \leq \mu(\delta) \text{ for } |h| \leq \delta, \text{ and } \mu(\delta) \to 0, \delta \to 0.
\]
Now
\[ S_n(f(a); x) - f(a) = a^{-u} \sum_{v=0}^{\infty} \frac{(ua \log a)^v}{v!} \left\{ \left( \frac{v}{u \log a} - a \right) f'(a) + \frac{1}{2} \left( \frac{v}{u \log a} - a \right)^2 f''(a) \right\} \]
\[ + a^{-u} \sum_{v=0}^{\infty} \frac{(ua \log a)^v}{v!} \frac{1}{2} \left( \frac{v}{u \log a} - a \right)^2 \varepsilon_v(u), \]
where
\[ |\varepsilon_v(u)| \leq \mu(\delta) \quad \text{for} \quad \left| \frac{v}{u \log a} - a \right| \leq \delta. \]  

(14)

From (6) and (8), we get
\[ S_n(f(a); x) - f(a) = \frac{a}{2u \log a} f''(a) + \frac{a^{-u}}{2(u \log a)^2} \sum_{v=0}^{\infty} \frac{(ua \log a)^v}{v!} (v - au \log a)^2 \varepsilon_v(u), \]
or
\[ u \log a [S_n(f(a); x) - f(a)] = \frac{a}{2} f''(a) + \frac{a^{-u}}{2u \log a} \sum_{v=0}^{\infty} \frac{(ua \log a)^v}{v!} (v - au \log a)^2 \varepsilon_v(u). \]

Write
\[ \sum_{v=0}^{\infty} \frac{(ua \log a)^v}{v!} (v - au \log a)^2 \varepsilon_v(u) = \sum_{|v - au \log a| \leq \delta u \log a} + \sum_{|v - au \log a| > \delta u \log a} = T_1 + T_2, \quad \text{(say)}. \]

Then from (14) and (6), we have
\[ |T_1| < \mu(\delta) au \log a (a^u). \]  

(15)

Hence
\[ \frac{a^{-u}}{2u \log a} |T_1| < \frac{1}{2} \delta u(a). \]

Next write
\[ T_2 = \sum_{v < au \log a(a-\delta)} + \sum_{v > au \log a(a+\delta)} = T_3 + T_4, \quad \text{(say)} \]
and also note that
\[ \frac{1}{2} \left( \frac{v}{u \log a} - a \right)^2 \varepsilon_v(u) = f \left( \frac{v}{u \log a} \right) - f(a) - \left( \frac{v}{u \log a} - a \right) f'(a) \]
\[ - \frac{1}{2} \left( \frac{v}{u \log a} - a \right)^2 f''(a). \]

Let
\[ \sup |f(x)| = M(a), \quad x \leq a. \]

Then
\[ |T_3| < \left\{ 2M(a) + a|f'(a)| + \frac{1}{2} a^2 |f''(a)| \right\} \sum_{v < au \log a(a-\delta)} (u \log a)^2 \alpha^2 \frac{(ua \log a)^v}{v!}. \]
Now by using the formula (see e.g. [4], p.200)

\[ \sum_{v=\alpha}^{\infty} e^{-u} \frac{(u \alpha)_{v}}{v!} = o \left( \exp \left( \frac{-1}{3} \delta^2 u \right) \right), \quad u \to \infty, \]

it follows that

\[ \sum_{u \alpha \log a - \infty}^{\infty} e^{-u} \frac{(u \alpha \log a)_{v}}{v!} = o \left( \exp \left( \frac{-1}{3} \delta^2 u \log a \right) \right). \]

Hence

\[ \frac{a^{-u \alpha}}{u \log a} T_3 = o \left( (u \log a) \exp \left( \frac{-1}{3} \delta^2 u \log a \right) \right) \]

\[ = o \left( u \log a \left( a^{-\frac{1}{3} \delta^2 u} \right) \right). \]

In view of \( f(x) = o(x^k) \), for \( v > u \alpha \log a \) and \( k \geq 2 \), we have

\[ (v - u \alpha \log a)^2 e_v(u) = o \left( (u \log a)^2 \frac{v^k}{(u \log a)^k} \right). \]

Therefore,

\[ T_4 = o \left( \sum_{v=\alpha}^{\infty} \frac{(u \alpha \log a)_{v}}{v!} \frac{v^k}{(u \log a)^k} \right) \]

\[ = o \left( \sum_{v=\alpha}^{\infty} (u \log a)^2 \frac{(u \alpha \log a)_{v-k}}{(v-k)!} \right) \]

\[ = o \left( (u \log a)^2 \exp \left( u \alpha \log a - \frac{\delta^2}{3 \alpha} u \log a \right) \right). \]

Thus

\[ \frac{a^{-u \alpha}}{u \log a} T_4 = o \left( (u \log a) \exp \left( \frac{-1}{3 \alpha} \delta^2 u \log a \right) \right) \]

\[ = o \left( u \log a \left( a^{-\frac{1}{3} \delta^2 u} \right) \right). \]

From (15) and (16), we finally get

\[ \limsup \left| u \log a \left[ S_u^* f(x; \alpha) - f(x) \right] - \frac{\alpha}{2} f''(\alpha) \right| \leq \delta. \]

But \( \delta \) is arbitrarily small, hence the proof is completed. \( \square \)

**Remark 3.9.** From a well-known property of the Beta function

\[ \left( \begin{array}{c} n \\ v \end{array} \right) \int_0^1 t^v (1-t)^{n-v} dt = \frac{1}{n+1}, \quad v = 0, 1, 2, \ldots n; \]
we have

\[ \int_0^1 B_n(t) dt = \frac{1}{n+1} \sum_{v=0}^{n} f(v/n). \]

So that, for any Riemann integrable function

\[ \int_0^1 B_n(t) dt \to \int_0^1 f(t) dt. \]

Similarly,

\[ \int_0^\infty S_u^*(f; x) dx = \sum_{v=0}^{\infty} \frac{(u \log a)^v}{v!} f(v/u \log a) \int_0^\infty e^{-ux \log a} x^v dx \]

\[ = \frac{1}{u \log a} \sum_{v=0}^{\infty} f(v/u \log a), \]

the interchange of integration and summation is legitimate if the series \( \sum f(v/u \log a) \) is convergent.

Thus, the formula

\[ \int_0^\infty S_u^*(f; x) dx = \frac{1}{u \log a} \sum_{v=0}^{\infty} f(v/u \log a) \]

is valid if both sides exist.

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