Almost monotonicity formula for H-minimal Legendrian surfaces in the Heisenberg group

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Abstract
We prove an almost monotonicity formula for H-minimal Legendrian Surfaces (also called contact stationary Legendrian immersions or Hamiltonian stationary immersions) in the Heisenberg Group $\mathbb{H}^2$. From this formula we deduce a Bernstein-Liouville type theorem for H-minimal Legendrian Surfaces. We also present some possible range of applications of this formula.

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1 INTRODUCTION

1.1 Main results

Conserved quantities and monotonicity formula, which are integrated versions of conservation laws, are fundamental notions in the calculus of variations. These identities are reflecting the existence of groups of symmetries of the underlying Lagrangian most of the time in locally isotropic spaces. The most illustrative example is maybe the monotonicity formula for minimal immersion of a surfaces in a Euclidean space $\mathbb{R}^n$. This identity says the following: let $\Phi$ be a minimal immersion of a surface $\Sigma$ without boundary into $\mathbb{R}^n$, then the following conservation law holds

$$\forall r > 0 \frac{1}{r^2} \int_{\rho<r} dvol_{\Sigma} = \int_{\rho<r} \frac{|(\nabla \rho)^\perp|^2}{\rho^2} dvol_{\Sigma} + \pi \text{Card}(\Phi^{-1}(0)), \quad (1.1)$$

where $\rho$ is the distance function to the origin of the space, $dvol_{\Sigma}$ denotes the volume form on $\Sigma$ induced by the immersion $\Phi$, $(\nabla^\Sigma \rho)^\perp$ is the projection to the normal co-dimension two plane

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to the immersion of the gradient of \( \rho \) and \( \text{Card}(\Phi^{-1}(0)) \) is the number of pre-images of the origin 0 by the immersion \( \Phi \) (see for instance [19]). This formula which holds as well for much weaker mathematical objects critical point of the area than minimal immersions (such as \textit{stationary varifolds} for instance) is the starting point for the analysis of the variation of the area in Euclidean or even Riemannian spaces.

The purpose of the present work is to look for the natural replacement of (1.1) in a “fully anisotropic environment” that is, in an ambient space where all directions are not equal (even infinitesimally). The most symmetric (or isotropic) model for anisotropy is maybe given by the Heisenberg groups \( \mathbb{H}^n \) of dimension \( 2n + 1 \) in which one direction is “forbidden” while perfect isotropy holds in the \( 2n \) remaining ones. The coordinates in \( \mathbb{H}^n \) will be denoted \((z_1, \ldots, z_{2n}, \varphi)\) where the last coordinate \( \varphi \) is called the \textit{Legendrian coordinate}. The so called \textit{Horizontal Hyperplanes} \( H \) are generated at every points by the following \( 2n \) vectors

\[
X_i := \frac{\partial}{\partial z_{2i-1}} - z_{2i} \frac{\partial}{\partial \varphi}, \quad Y_i := \frac{\partial}{\partial z_{2i}} + z_{2i-1} \frac{\partial}{\partial \varphi} \quad \text{for } i = 1 \cdots n
\]

We take these vectors to realize an orthonormal frame in such a way that the tangent map \( \pi_* \) to the canonical projection \( \pi \) from \( T\mathbb{H}^n \) into \( T\mathbb{R}^{2n} \) given by

\[
\pi_* X_i = \frac{\partial}{\partial z_{2i-1}}, \quad \pi_* Y_i = \frac{\partial}{\partial z_{2i}} \quad \text{and} \quad \pi_* \frac{\partial}{\partial \varphi} = 0
\]

realizes at every point an isometry from \( H \) into \( T\mathbb{R}^{2n} \). An immersion \( \Lambda \) of a surface \( \Sigma \) into \( \mathbb{H}^n \) is called \textit{Legendrian} if it is tangent to \( H \) at every point. This is also equivalent to the following \textit{contact condition}

\[
\Lambda^* \alpha = 0 \quad \text{where} \quad \alpha := -d\varphi + \sum_{i=1}^n z_{2i-1} dz_{2i} - z_{2i} dz_{2i-1}.
\]

The projection by \( \pi \) of a \textit{Legendrian immersion} \( G := \pi \circ \Lambda \) generates obviously\(^1\) a \textit{Lagrangian immersion} of \( \mathbb{R}^{2n} \) that is, an immersion into \( \mathbb{R}^{2n} \) satisfying

\[
G^* \omega = 0 \quad \text{where} \quad \omega := 2 \sum_{i=1}^n dz_{2i-1} \wedge dz_{2i}.
\]

The reciprocal holds as well locally but we prefer to work with Legendrian immersion rather than Lagrangian immersion because the global existence of the Legendrian coordinates is necessary for monotonicity formula to hold for the area functional as observed first by R. Schoen and J. Wolfson in [17] (see a counterexample pp. 192–193). Lagrangian surfaces with a global Legendrian lift are also called \textit{exact Lagrangian surfaces}.

Critical points to the area functional for arbitrary compactly supported perturbations \textbf{among Legendrian immersions} are called \textit{Hamiltonian stationary Legendrian immersions} or simply \textit{H-minimal Legendrian Surfaces}. Since \( \pi_* \) is an isometry from \( H \) into \( T\mathbb{R}^{2n} \) and since locally a one to one correspondence holds between Legendrian and Lagrangian immersions, such an immersion is projected by \( \pi \) onto a critical point of the area among Lagrangian surfaces. These surfaces have been considered first by Oh under the name of \textit{Hamiltonian Stationary Surfaces}\(^1\)

\(^1\)The fact that \( \pi_* \) is an isometry from \( H \) into \( T\mathbb{R}^{2n} \) and that \( \Lambda \) is an immersion everywhere tangent to \( H \) preserves the fact that \( G := \pi \circ \Lambda \) is an immersion, moreover \( d\alpha = \pi^* \omega \) and \( \Lambda^* \alpha = 0 \) gives \( \Lambda^* \pi^* \omega = G^* \omega = 0 \).
For such surfaces the mean-curvature vector is given by the image by the canonical complex structure \( i \) of \( \mathbb{R}^{2n} \cong \mathbb{C}^n \) of the gradient along the surface of a harmonic \( 2\pi \mathbb{Z} \) multivalued function \( \beta \), the Lagrangian angle, whose differential is generating the Maslov class of the Lagrangian surface (see also [17]). We shall now concentrate mostly on the case \( n = 2 \). In this particular dimension Lagrangian surfaces are characterized by the fact that the action of \( i \) realizes an isometry between the tangent 2-space and the orthogonal 2-space to the surface. In conformal parametrization a Lagrangian immersion is stationary if and only if the multivalued function \( \beta \) (well defined up to a multiple of \( 2\pi \)) satisfies:

\[
\begin{cases}
i \Delta G = 2 \nabla \beta \cdot \nabla G \\
\Delta \beta = 0
\end{cases}
\]  

(1.2)

An important quantity in the Heisenberg group is given by \( r := (\rho^4 + 4\varphi^2)^{1/4} \) where \( \rho \) denotes the Euclidean distance in \( \mathbb{R}^4 \) to the origin. The function \( r \) is called the Folland-Korányi gauge. It defines the left invariant homogeneous Heisenberg distance equivalent to the Carnot Carathéodory distance related to the minimal length among horizontal geodesics between two points.

Our main result in the present work is the following result.

**Theorem 1.1 (Almost Monotonicity).** There exists a universal constant \( C > 0 \) such that for any smooth H-minimal Legendrian proper immersion \( \Lambda \) of an oriented surface \( \Sigma \) without boundary into \( \mathbb{H}^2 \), we have

\[
\forall r < 1 \quad C^{-1} \left[ \vartheta_0 + \int_{r<r/2} \frac{|(\nabla^\Sigma r)^\perp|^2}{r^2} \, d\text{vol}_\Sigma \right] \leq \frac{1}{r^2} \int_{r<r} d\text{vol}_\Sigma \leq C \int_{1/2<r<2} d\text{vol}_\Sigma,
\]  

(1.3)

where

\[
\vartheta_0 := 2\pi \text{ Card} \Lambda^{-1}(0),
\]

where \( r := (\rho^4 + 4\varphi^2)^{1/4} \) is the Folland-Korányi gauge and where \( (\nabla^\Sigma r)^\perp \) denotes the projection of the gradient of the Folland-Korányi gauge onto the orthogonal 2-plane to the tangent space of the immersion \( \Lambda \) within the horizontal plane \( H \).

The Inequality (1.3) is the counterpart for H-minimal Legendrian immersion of the monotonicity identity (1.1) for minimal surfaces in \( \mathbb{R}^n \). In fact inequality (1.3) is the consequence of the conservation law (3.19) for H-minimal Legendrian immersion which corresponds to the conservation law (4.19) for classical minimal surfaces. The comparison between the two conservation laws has however some limits. The variation of the area without Legendrian constraint (in the

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2 The Euler Lagrange equation (1.2) can also be formulated as follows: there exists an \( S^1 \)-valued harmonic map \( u \) on \( \Sigma \) equipped with the induced metric by the immersion \( G \) such that

\[
dw^\Sigma(u \nabla^\Sigma G) = 0.
\]

3 The author would like to point out that in the sub-Riemannian framework the denomination H-minimal could lead to confusion and interpreted as “Horizontal minimal”. This last denomination is referring to a very different notion of surfaces with horizontal mean curvature equal to zero. This is again a second order but sub-elliptic PDE this time. This is very different from the Hamiltonian Minimal equation (1.2).
Euclidean space) generates a 2nd order PDE while obviously the Legendrian stationary equation (1.2) is of order 3.

We expect theorem (1.1) to hold in the Heisenberg group \( \mathbb{H}^n \) of arbitrary dimension as well. A control from above and below of the area density for exact stationary Lagrangian maps of surfaces has first been given by Schoen and Wolfson in [17], (proposition 3.2).4

A consequence of the conservation law (3.19) is the following Bernstein-Liouville type theorem for H-minimal Legendrian immersion which says roughly that if such an immersion is asymptotically at infinity a Lagrangian plane then it must be a Lagrangian plane.

**Theorem 1.2** (Bernstein-Liouville type theorem). Let \( \Lambda \) be a smooth H-minimal proper Legendrian immersion of an oriented surface \( \Sigma \) without boundary into \( \mathbb{H}^2 \). Assume \( \Lambda \) is asymptotically a plane at infinity that is,

\[
\lim_{r \to +\infty} \frac{1}{r} \int_{r < t < 2r} \frac{1}{r} \, d\text{vol}_\Sigma = 2\pi \quad \text{and} \quad \rho \to 1 \quad \text{as} \quad r \to +\infty, \tag{1.4}
\]

then \( \Lambda(\Sigma) \) is a Lagrangian plane in \( \mathbb{C}^2 \approx \mathbb{H}^2 \cap \{\sigma = 0\} \).

### 1.2 Motivations

As mentioned above, the notion of surfaces, critical point for the area under Legendrian constraints, has been first introduced by Oh. The first main analytic works involving variations under pointwise Lagrangian/Legendrian constraints were undertaken by Schoen and Wolfson ([17, 18]) for the area and by Minicozzi ([6, 7]) for the area and for the Willmore Functional. The main motivation in [18] was to generate Minimal Lagrangian Surfaces realizing some Lagrangian homotopy/homology class or within a given Hamiltonian isotopy class by the mean of variational methods. This remarkable project has been “hindered” by the discovery by the two authors of the possible existence of isolated singularities (the nowadays called Schoen-Wolfson cones) “preventing” even in the \( S^2 \) case the multivalued Lagrangian harmonic angle function \( \beta \) above to be constant and hence the surface to be minimal Lagrangian (the existence of branched points, the other possible singularities, being totally inoffensive on that respect). The possible existence of these singularities for Lagrangian minimizing 2-spheres in some spherical Lagrangian integer homology class of a K3 surface has been confirmed in [20].

It has to be noted that beside the study of its “flow counterpart” - the Lagrangian Mean curvature flow - a rather limited set of works has been devoted to the variational aspect of the area

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4 Observe that in the proof of Proposition 3.2 given in [18] the computation of \( \text{div}_0(X_{\eta(t,\theta)}) \) at the bottom of page 14 holds away from the origin and is not correct if extended throughout the origin as done in [18]. For a smooth H-minimal Legendrian immersion for instance the asymptotic expansion of the vector field \( X_{\partial} \) generated by the Hamiltonian \( \partial \) and whose flow is preserving the contact distribution is of the form

\[
X_{\partial} = -\frac{1}{\rho} \frac{\partial}{\partial \rho} + O(1)
\]

where the quantity \( \rho^2 := x^2 + y^2 \) corresponds to \( \sqrt{s} \) in [18]. Hence the divergence of \( X_{\partial} \) generates a Dirac mass at the each of the pre-images of the origin. The weights in front of these Dirac masses all have the same sign (their value is in fact \( -2\pi \) in the smooth setting but might vary if the H-minimal Legendrian is not exactly an immersion and has Schoen-Wolfson \( p - q \) type conical singularities then the weights should be \( -2\pi \sqrt{pq} \) and are “responsible” for the lower bound in the area density control. Hence the derivation of the almost monotonicity formula in [18] is problematic to the author.
under Lagrangian/Legendrian constraint since the publication of the above mentioned break-
through by Schoen and Wolfson. The author of the present work see two main reasons that could
shed light on this observation: the Hamiltonian stationary condition assuming a-priori very lit-
tle regularity is analytically particularly challenging in comparison to the stationarity condition
without constraint (i.e., minimal surface theory). The second reason is that the cornerstone to
any variational studies (in situations where the maximum principle is not operative) is the exis-
tence of conservation laws and monotonicity formula. The almost monotonicity formula that was
available until now was the one in [18]. Its proof goes through very intricate arguments such as
the resolution of some hyperbolic equation and is henceforth difficult to grasp and to reproduce
in slightly different context (the issue raised in the footnote 3 is also bringing some fundamen-
tal additional difficulties). The proof of the almost monotonicity formula we are giving below is
in the contrary using direct computations which are specific to this 3rd order problem. The
main strategy followed by the author of the present work was to develop computations taking
into account the group structure of the Heisenberg group and to “eliminate” the Lagrangian
multi-valued angle function which is nothing but a Lagrange multiplier coming from the Leg-
endrian pointwise constraint. To conclude the comments on theorem 1.1, one could see the
almost monotonicity formula proved below as one of the very first example of monotony for
anisotropic problems. The project of studying the variations of anisotropic Lagrangians launched
by Almgren and by Allard (see [1]) is confronted to the major difficulty of the absence of known
almost monotonicity formula in almost every cases. In absence of such almost monotonic quan-
tity, performing minmax or even simply minimization procedures becomes very challenging
(see [3]).

One of the main motivation of the author for the study of area variations under pointwise Leg-
dendrian constraint is coming from his project to solve the Willmore conjecture using a more PDE
approach exposed in the still unsupplied working notes [15].

To every immersions $\bar{\Phi}$ of an oriented connected two-dimensional surfaces $\Sigma$ into $S^3$ one asso-
ciates the unit Gauss vector $\vec{n}$ which is the unit vector both tangent to the 3—sphere and orthogonal
to the surface in the positive direction. In other words $\bar{G}_{\bar{\Phi}} := \bar{\Phi} \wedge \vec{n}$ is a map taking values into the
Grassmann Manifolds $Gr(2,4)$ of simple unit two vectors of $\mathbb{R}^4$ and giving the oriented normal two
space to the immersion. This manifold is in fact isometric and inherits a natural complex struc-
tures which makes it isomorphic to $CP^1 \times CP^1$. The immersion $\bar{G}$ has of course a lift $\bar{\Lambda} = (\bar{\Phi}, \vec{n})$
in the Stiefel Manifold $V_2(\mathbb{R}^4)$ of orthonormal 2-frames in $\mathbb{R}^4$.

The first fundamental observation which is behind the whole project is the following corre-
spondence: For any immersion $\bar{\Phi}$ the Gauss map $\bar{G}_{\bar{\Phi}}$ is Lagrangian in $CP^1 \times CP^1$ moreover every
local Lagrangian deformation is the Gauss map of an immersion into $S^3$ (see [15]). Parallelly the
map $\bar{\Lambda}$ is Legendrian for the contact structure in $V_2(\mathbb{R}^4)$ given by the Liouville 1-form

$$\alpha = \bar{\Phi} \cdot d\vec{n} - \vec{n} \cdot d\bar{\Phi}.$$ 

The above correspondence between immersions into $S^3$ and Lagrangian (resp. Legendrian)
Gauss immersions into the Kähler-Einstein manifold $Gr(2,4)$ (resp. the Sasaki-Einstein manifold
$V_2(\mathbb{R}^4)$ ) goes even further while considering the variation of the area. Precisely, if the immersion
$\bar{\Phi}$ is regular isotopic to a minimal surface, then $\bar{G}_{\bar{\Phi}}$ is minimal Lagrangian if and only if there
exists $t \in [0, 2\pi]$ such that

$$\bar{\Phi}_t := \cos t \bar{\Phi} + \sin t \vec{n}$$
is a minimal immersion into $S^3$ (see [4, 10] and [15]) and the Morse index of $\bar{\Phi}$, is the Hamiltonian Morse index\(^5\) of $\bar{\Phi}$, plus 1 (see [2]).

This one to one dictionary between the two paradigms is definitively fascinating but a-priori does not give any convincing reasons so far why it could be more handful to study the space of Lagrangian immersions into $Gr(2, 4)$ (that looks definitively more intricate) instead of immersions into $S^3$. There are nevertheless good reasons for doing so and they are explained in [15]. One of them is a bit technical but might be appealing for specialists in the field: the fifth direction\(^6\) of the canonical family in the minmax operation introduced by Marques and Neves in order to identify the non zero genus minimal surface of $S^3$ of lowest area disappears while working at the level of the Gauss map. The author then is proposing in [15] a four-dimensional minmax problem on the space of Lagrangian Immersions which is proved to be non trivial and whose width is conjectured in [15] to be achieved by the Gauss map of the Clifford torus which is a double covering of the Clifford torus in $Gr(2, 4)$. This minmax problem is based on the homology of the isometric group of $Gr(2, 4)$. The question hence remains how to prove that the width is indeed achieved by a Lagrangian minimal surface of Hamiltonian index 4.

In a series of works, part of them written in collaboration with Alessandro Pigati, the author has introduced a method for performing arbitrary minmax operations for the area of closed oriented surfaces $\Sigma$ in arbitrary ambient closed Riemannian manifolds $(N^n, h)$ (see [11–14]). The strategy consists roughly at working at a rather high level of regularity by considering immersions of surfaces in rather small Sobolev spaces of immersions such as $W^{2,4}_{imm}(\Sigma, N^n)$ or $W^{3,2}_{imm}(\Sigma, N^n)$ for instance and by adding to the area a “smoothing term” preceded by a small parameter called “viscosity parameter”. The main reason for considering such a smoothed or “enhanced Lagrangian” is to have the possibility to rely on the framework of Palais deformation theory in infinite dimensional spaces in order to prove the realization of the considered minmax operation by a saddle point.

In this approach it is also important for this smoothing to be of “tensor type” so that the group of diffeomorphism of the surfaces still remains an invariance group (i.e., gauge group) for the “enhanced Lagrangian” as for the area itself. Typically one uses the $L^p$ norm of the second fundamental form of the immersion as smoother:

$$A^\varepsilon(\bar{\Phi}) = \int_\Sigma dvol_{\bar{g}_\bar{\Phi}} + \varepsilon^2 \int_\Sigma |\bar{\mathcal{I}}_{\bar{\Phi}}|^p dvol_{\bar{g}_\bar{\Phi}}$$

where $g_{\bar{\Phi}}$ and $\bar{\mathcal{I}}_\bar{\Phi}$ are respectively the first and the second fundamental forms of the immersion $\bar{\Phi}$ of $\Sigma$ in $N^n$ and $dvol_{\bar{g}_\bar{\Phi}}$ is the volume form of $g_{\bar{\Phi}}$. One proves (see [14]) that for $p > 2$ the previous Lagrangian satisfies the Palais-Smale property (modulo the action of the diffeomorphism group of $\Sigma$).

Once a saddle point is achieved for the enhanced Lagrangian $A^\varepsilon$, the task is to make the small parameter go to zero and eventually prove that the sequence of associated saddle points for $A^\varepsilon$

\(^5\)The Hamiltonian Morse index of a minimal Lagrangian Surface is the number of independent Hamiltonian negative directions for the area.

\(^6\)This fifth direction corresponds to the most negative eigenvalue of the Jacobi operator for any minimal surface in $S^3$ it is of multiplicity one. In the canonical family this is the direction corresponding to the flow of the gradient of the signed distance to the surface and obviously generates after some time “focal type singularities” even starting from an analytic surface. This last fact is the main reason why the framework adopted by Marques and Neves is the one of Varifolds and $\mathbb{Z}_2$-cycles and also why the minmax strategy proposed in [5] is very much of codimension one nature.
converge to a saddle point for the area which is realizing the optimum for the corresponding minmax operation. This is achieved in the above mentioned works and the method moreover provides with a regularity result on the limiting surface, a control of both the topology and the Morse index.\(^8\) It is important to insist at this stage on the fact that the *viscosity method* is “insensitive” to the co-dimension that is to say: increasing codimension does not result in an increase in the difficulty of its implementation.

The author in [16] is developing the *viscosity method under Legendrian constraint* in order to implement the minmax operation proposed in [15] and give a PDE proof of the Willmore conjecture in three dimensions. The almost monotonicity formula theorem 1.1 is the cornerstone of the whole proof.\(^7\),\(^8\)

## 2 Preliminaries

Recall that the Lagrangian projection \(G := \pi \circ \Lambda\) from \(\mathbb{H}^2\) into \(\mathbb{C}^2\) of a conformal Hamiltonian stationary Legendrian immersion \(\Lambda\) satisfies locally in conformal charts (see [18])

\[
i \Delta G = 2 \nabla \beta \cdot \nabla G.
\]

Taking the scalar product respectively with \(G\) and \(iG\) gives

\[
\begin{aligned}
\div (G \cdot i \nabla G) &= \nabla \beta \cdot |G|^2 \\
\div (G \cdot \nabla G) - |\nabla G|^2 &= -2 \nabla \beta \cdot (G \cdot i \nabla G)
\end{aligned}
\]

(2.1)

Since \(G\) is Lagrangian we have that the multiplication by \(i\) realizes an isometry between the tangent and the normal planes to the immersion \(G\). Hence

\[
\div (G \cdot i \nabla \perp G) = 0.
\]

Hence we introduce the pull-back by \(\Lambda\) of the Legendrian coordinate equal to zero at the origin of \(\mathbb{H}^2\) and given by

\[
d\varphi := G \cdot idG
\]

We denote \(\rho^2 := |G|^2\). With this notation (2.1) becomes (independently of coordinates)

\[
\begin{aligned}
\nabla^\Sigma \beta \cdot \nabla^\Sigma \rho^2 &= \Delta^\Sigma \varphi \\
\nabla^\Sigma \beta \cdot \nabla^\Sigma \varphi &= 1 - 4^{-1} \Delta^\Sigma \rho^2.
\end{aligned}
\]

(2.2)

where \(\nabla^\Sigma\) and \(\Delta^\Sigma\) denote respectively the *gradient operator* and the *negative Laplace Beltrami operator* for the induced metric by \(G\) (or \(\Lambda\)) on \(\Sigma\).

\(^7\)The limiting minimal surface is a smooth immersion with possibly isolated branched points.

\(^8\)The genus of the limiting minimal surface is bounded by the maximal genus needed to realize the minmax operation and the Morse index of the minimal surface is bounded by the homological (resp. cohomological or homotopical) dimension of the minmax operation.
The Lagrangian angle multivalued function $\beta$ plays the role of a Lagrangian multiplier and satisfies the Euler Lagrange equation

$$\Delta^\Sigma \beta = 0.$$  

(2.3)

In the conformal chart we are considering, the metric induced by the immersion writes $G^* g_{\mathbb{C}^2} = e^{2\lambda} [dx_1^2 + dx_2^2]$ where $\lambda$ is a smooth real valued function. Since $(e^{-\lambda} \partial_x G, e^{-\lambda} \partial_y G, i e^{-\lambda} \partial_x G, i e^{-\lambda} \partial_y G)$ realizes an orthonormal frame of $G^* T \mathbb{C}^2$ we have

$$\rho^2 = |G|^2 = e^{-2\lambda} |G \cdot \nabla G|^2 + e^{-2\lambda} |G \cdot i \nabla G|^2 = |2^{-1} \nabla^\Sigma \rho^2|^2 + |
abla^\Sigma \varphi|^2$$

from which we deduce

$$1 = |\nabla^\Sigma \rho|^2 + \rho^{-2} |\nabla^\Sigma \varphi|^2.$$  

(2.4)

### 3  | DENSITY BOUND AND DIRICHLET ENERGY BOUND OF THE ARCTANGENT OF THE PHASE

We call the phase the scaling invariant quantity in the Heisenberg group given by

$$\sigma := \frac{2\varphi}{\rho^2}.$$  

We shall be frequently using the following identity which is a direct consequence of the definition

$$1 + \sigma^2 = \frac{\rho^4}{\rho^4}.$$  

The purpose of the present section is to prove that, for any H-minimal Legendrian immersion, the sum of the density at the origin with the Dirichlet energy of $\arctan \sigma$ in a unit ball for the Folland-Korányi distance is controlled by a constant times the area contained in the surrounding dyadic annulus. Precisely we have

**Lemma 3.1.** There exists a universal constant $C_0 > 0$ such that for any $\Lambda$ smooth $H$–minimal Legendrian immersion of an oriented surface $\Sigma$ into $\mathbb{H}^2$, denoting $\sigma := 2\varphi/\rho^2$ we have

$$2\pi \text{Card}_\Sigma \Lambda^{-1}(0) + \int_{r<1} \left| \frac{\nabla^\Sigma \sigma}{1 + \sigma^2} \right|^2 d\text{vol}_\Sigma \leq C_0 \int_{1<\rho<2} d\text{vol}_\Sigma.$$  

(3.1)

**Proof of lemma 3.1.** Using (2.2) away from $\rho = 0$

$$\nabla^\Sigma \sigma \cdot \nabla^\Sigma \varphi = \left[ 2\rho^{-2} \nabla^\Sigma \varphi - 2\varphi \rho^{-4} \nabla^\Sigma \rho^2 \right] \cdot \nabla^\Sigma \varphi = \frac{1}{2\rho^2} \left[ 4 - \Delta^\Sigma \rho^2 \right] - \frac{2\varphi}{\rho^4} \Delta^\Sigma \varphi$$

$$= 2\rho^{-2} - 2\rho^{-2} \Delta^\Sigma \rho^2 - \rho^{-4} \Delta^\Sigma \varphi^2 + 2\rho^{-4} |\nabla^\Sigma \varphi|^2$$

$$= 2\rho^{-2} - 4\rho^{-4} 2\rho^2 \Delta^\Sigma \rho^2 - \rho^{-4} \Delta^\Sigma \varphi^2 + 2\rho^{-4} |\nabla^\Sigma \varphi|^2$$
\[= 2 \rho^{-2} - 4^{-1} \rho^{-4} \left[ \Delta^\Sigma \rho^4 - 2 |\nabla^\Sigma \rho^2|^2 + 4 \Delta^\Sigma \varphi^2 \right] + 2 \rho^{-4} |\nabla^\Sigma \varphi|^2 \]
\[= 2 \rho^{-2} + 2 \rho^{-2} |\nabla^\Sigma \rho|^2 + 2 \rho^{-4} |\nabla^\Sigma \varphi|^2 - 4^{-1} \rho^{-4} \Delta^\Sigma r^4 \]
\[= 4 \rho^{-2} - 4^{-1} \rho^{-4} \Delta^\Sigma r^4 = 4 \left( \frac{\sqrt{1 + \sigma^2}}{r^2} - 4^{-1} \frac{1 + \sigma^2}{r^4} \right) \Delta^\Sigma r^4, \quad (3.2) \]

where we used (2.4). This implies assuming \( \rho \neq 0 \) and dividing by \( 1 + \sigma^2 \)

\[\frac{\nabla^\Sigma \sigma}{1 + \sigma^2} \cdot \nabla^\Sigma \beta = \frac{4}{r^2} \frac{1}{\sqrt{1 + \sigma^2}} \Delta^\Sigma r^4 = \frac{4}{r^2} \frac{1}{\sqrt{1 + \sigma^2}} \left( \frac{1}{4} \text{div}^\Sigma \left[ \frac{\nabla^\Sigma r^4}{r^4} \right] - 4 |\nabla \log r|^2 \right) \]
\[= \frac{4}{r^2} \frac{1}{\sqrt{1 + \sigma^2}} - 4 |\nabla^\Sigma \log r|^2 - \Delta^\Sigma \log r \quad (3.3)\]

Since

\[\begin{cases}
d\varphi = z_2 dz_1 - z_1 dz_2 + z_4 dz_3 - z_3 dz_4 \\
\rho d\rho = z_1 dz_1 + z_2 dz_2 + z_3 dz_3 + z_4 dz_4,
\end{cases}\]

this implies

\[|\nabla^H \varphi| = \rho, \quad |\nabla^H \rho| = 1 \quad \text{and} \quad \nabla^H \varphi \cdot \nabla^H \rho = 0. \quad (3.4)\]

We deduce the length of the horizontal gradient of the Folland - Korányi gauge

\[|\nabla^H r|^2 = \frac{\rho^6}{r^6} |\nabla^H \rho|^2 + \frac{|\nabla^H \varphi|^2}{r^6} = \frac{\rho^2}{r^2} \rho^4 + 4 \varphi^2 \frac{\rho^2}{r^4} = \frac{\rho^2}{r^2}. \quad (3.5)\]

hence we have

\[|\nabla^H r|^2 = \frac{\rho^2}{r^2} = \frac{1}{\sqrt{1 + \sigma^2}}. \quad (3.6)\]

Combining (3.3) and (3.6) gives away from \( \rho = 0 \), since \( \Delta^\Sigma \beta = 0 \),

\[\text{div}^\Sigma (\text{arctan} \sigma \nabla^\Sigma \beta + \nabla^\Sigma \log r) = 4 \left( \frac{|(\nabla^\Sigma r)^\perp|^2}{r^2} \right). \quad (3.7)\]

Assume now \( \rho = 0 \) only happens at the points \( p \) where \( r(p) = 0 \), then, since we have a smooth Legendrian immersion we deduce that

\[\text{div}^\Sigma (\text{arctan} \sigma \nabla^\Sigma \beta + \nabla^\Sigma \log r) = 4 \left( \frac{|(\nabla^\Sigma r)^\perp|^2}{r^2} \right) + \sum_{r(p) = 0} \theta_0(p) \delta_p. \quad (3.8)\]

where, for \( \varepsilon > 0 \) small enough and fixed

\[\theta_0(p) := \lim_{t \to 0} \int_{\{t = t\} \cap B_\varepsilon(p)} \frac{\partial \rho}{r} \, dl_\Sigma.\]
where $\nu$ is the unit normal tangent to $\Sigma$ pointing out of the set $r \leq t$. In conformal coordinates $x = (x_1, x_2)$ for the associated Lagrangian immersion $G$ assuming that $r \circ \Lambda(0) = 0$ we have the existence of an orthonormal basis of $C^2$ such that $J(\tilde{e}_1) = \tilde{e}_2$ and $J(\tilde{e}_3) = \tilde{e}_4$ and

$$G(x) = e^{i(0)} (x_1 \tilde{e}_1 + x_2 \tilde{e}_3) + O(|x|^2)$$

This gives

$$G^*d\varphi = G^*(z_2 dz_1 - z_1 dz_2 + z_4 dz_3 - z_3 dz_4) = O(|x|^2).$$

Hence

$$\varphi(x) = O(|x|^3), \quad \frac{\rho}{r} = 1 + o(1) \quad \text{and} \quad \nabla^\Sigma r = \nabla^\Sigma (1 + o(1)). \quad (3.9)$$

This gives

$$\vartheta_0(p) := \lim_{t \to 0} \int_{\{r = t\} \cap B_\varepsilon(p)} \frac{\partial_r r}{\nabla^\Sigma} \, dl_\Sigma \quad \lim_{t \to 0} \int_{\rho = t} \frac{\partial_\rho \rho}{\rho} \, dl_\Sigma = 2\pi \quad (3.10)$$

We claim that the following holds on the whole surface $\Sigma$, without assuming anymore that $\rho = 0$ only happens at the points $p$ where $r(p) = 0$,

$$\text{div}^\Sigma (\arctan \sigma \nabla^\Sigma \beta + \nabla^\Sigma \log r) = 4 \frac{|(\nabla^\Sigma r)^\perp|^2}{r^2} + 2\pi \sum_{p \in r^{-1}(0)} \delta_p. \quad (3.11)$$

Let $\varepsilon > 0$ and consider $\phi \in C_0^\infty(\Sigma \setminus r^{-1}([0, \varepsilon]))$ and let $\delta > 0$. Observe that $\arctan \sigma \nabla^\Sigma \beta + \nabla^\Sigma \log r$ extends smoothly on $\Sigma \setminus r^{-1}([0, \varepsilon])$. On $\Sigma$ we define

$$\chi_{\delta, \rho} := \chi(\rho/\delta)$$

where $\chi$ is the cut-off function on $\mathbb{R}_+$ defined by

$$\chi(t) = \begin{cases} 1 & \text{for } t < 1 \\ 0 & \text{for } t > 2. \end{cases}$$

Observe that $\arctan \sigma \nabla^\Sigma \beta + \nabla^\Sigma \log r$ extends continuously on $\Sigma \setminus r^{-1}([0, \varepsilon])$ and for any $\delta > 0$ we write, using (3.8),

$$- \int \nabla^\Sigma \phi \left( \arctan \sigma \nabla^\Sigma \beta + \nabla^\Sigma \log r \right) + 4 \phi \frac{|(\nabla^\Sigma r)^\perp|^2}{r^2} \, dvol_\Sigma \quad = - \int \nabla^\Sigma (\chi_{\delta, \rho} \phi) \left( \arctan \sigma \nabla^\Sigma \beta + \nabla^\Sigma \log r \right) + 4 \chi_{\delta, \rho} \phi \frac{|(\nabla^\Sigma r)^\perp|^2}{r^2} \, dvol_\Sigma. \quad (3.12)$$

We have respectively, since $G$ is an immersion in $C^2$

$$\int_{\{r < 2\delta : \rho > \varepsilon \} \cap \text{Supp}(\phi)} dvol_\Sigma = O(\delta^2), \quad \|\nabla^\Sigma (\chi_{\delta, \rho} \phi)\|_\infty \leq C_\phi \delta^{-1}, \quad (3.13)$$
\[
\left\| \arctan \nabla \Sigma \beta + \nabla \Sigma \log r \right\|_{L^\infty(\text{supp}(\phi))} < C_\phi, \quad \left\| \chi_{\delta, \rho} \phi \frac{|(\nabla \Sigma r)^\perp|^2}{r^2} \right\|_{L^\infty(\text{supp}(\phi))} < C_\phi,
\]  
(3.14)

where \( C_\phi \) is independent of \( \delta \). Combining (3.12), (3.13), and (3.14) and making \( \delta \) converge to zero gives

\[
\int_\Sigma \nabla \Sigma \phi \left( \arctan \nabla \Sigma \beta + \nabla \Sigma \log r \right) + 4 \phi \frac{|(\nabla \Sigma r)^\perp|^2}{r^2} \, d\text{vol}_\Sigma = 0,
\]

and this proves that (3.11) holds on the whole surface \( \Sigma \).

Using again (2.2) we obtain away from \( \rho = 0 \)

\[
\begin{align*}
\rho^3 \nabla \Sigma \beta \cdot \nabla \Sigma r &= \rho^3 \nabla \Sigma \beta \cdot \nabla \Sigma \rho + \nabla \Sigma \rho^2 \cdot \nabla \Sigma \beta = 2^{-1} \rho^2 \Delta \Sigma \phi + 2 \phi \left( 1 - 4^{-1} \Delta \Sigma \rho^2 \right) \\
&= 2 \phi + 2^{-1} \text{div} \left( \rho^2 \nabla \Sigma \phi - \phi \nabla \Sigma \rho^2 \right) = 2 \phi + 4^{-1} \text{div} \left( \rho^4 \nabla \Sigma \sigma \right) \\
&= \sigma \rho^2 + 4^{-1} \text{div} \left( r^4 \frac{\nabla \Sigma \sigma}{1 + \sigma^2} \right) = \sigma \rho^2 + \frac{\sigma}{\sqrt{1 + \sigma^2}} + 4^{-1} \text{div} \left( r^4 \frac{\nabla \Sigma \sigma}{1 + \sigma^2} \right). 
\end{align*}
\]

(3.15)

Similarly as above while proving (3.8) on the whole \( \Sigma \), we prove that (3.15) holds on \( \Sigma \) away from \( r = 0 \). Let \( \chi \) be a smooth cut-off function on \( \mathbb{R}_+ \) defined above. Multiplying (3.8) by \( \chi(r) \) and integrating over \( \Sigma \) gives

\[
- \int_\Sigma \arctan \sigma \chi'(r) \nabla \Sigma \beta \cdot \nabla \Sigma r \, d\text{vol}_\Sigma - \int_\Sigma \chi'(r) \frac{|\nabla \Sigma r|^2}{r} = 4 \int_\Sigma \chi(r) \frac{|(\nabla \Sigma r)^\perp|^2}{r^2} \, d\text{vol}_\Sigma + \theta_0,
\]

where \( \theta_0 = 2\pi \text{Card}(\Lambda^{-1}(\{0\})) \). Multiplying now (3.15) by \( - \arctan \sigma \chi'(r) \rho^{-3} \) and integrating over \( \Sigma \) gives

\[
- \int_\Sigma \arctan \sigma \chi'(r) \nabla \Sigma \beta \cdot \nabla \Sigma r \, d\text{vol}_\Sigma = - \int_\Sigma \arctan \sigma \frac{\chi'(r)}{r} \frac{\sigma}{\sqrt{1 + \sigma^2}} \, d\text{vol}_\Sigma \\
- \int_\Sigma \arctan \sigma \frac{\chi'(r)}{4} \frac{\sigma}{r^3} \frac{\nabla \Sigma \sigma}{1 + \sigma^2} \, d\text{vol}_\Sigma.
\]

(3.17)

Integrating by parts gives for the second term of the r.-h.-s. of (3.17)

\[
- \int_\Sigma \arctan \sigma \frac{\chi'(r)}{4} \frac{\sigma}{r^3} \frac{\nabla \Sigma \sigma}{1 + \sigma^2} \, d\text{vol}_\Sigma \\
= \int_\Sigma \frac{|\nabla \Sigma \sigma|^2}{(1 + \sigma^2)^2} \frac{\chi'(r)}{r} \, d\text{vol}_\Sigma + \frac{1}{4} \int_\Sigma \arctan \sigma \chi''(r) \rho \nabla \Sigma r \cdot \nabla \Sigma \sigma \frac{|\nabla \Sigma \sigma}{1 + \sigma^2} \, d\text{vol}_\Sigma \\
- \frac{3}{4} \int_\Sigma \arctan \sigma \chi'(r) \nabla \Sigma r \cdot \nabla \Sigma \sigma \frac{|\nabla \Sigma \sigma}{1 + \sigma^2} \, d\text{vol}_\Sigma.
\]

(3.18)
Combining (3.16), (3.17) and (3.18) gives then
\[
\int_\Sigma \frac{|\nabla^\Sigma \sigma|^2}{(1+\sigma^2)^2} \frac{\chi'(r)}{4} \, d\text{vol}_\Sigma + \frac{1}{4} \int_\Sigma \text{arctan} \sigma \chi''(r) \frac{\nabla^\Sigma \sigma \cdot \nabla^\Sigma \sigma}{1+\sigma^2} \, d\text{vol}_\Sigma
\]
\[-\frac{3}{4} \int_\Sigma \text{arctan} \sigma \frac{\chi'(r)}{r} \frac{\nabla^\Sigma \sigma \cdot \nabla^\Sigma \sigma}{1+\sigma^2} \, d\text{vol}_\Sigma
\]
\[-\int_\Sigma \text{arctan} \frac{\chi'(r)}{r} \frac{\sigma}{\sqrt{1+\sigma^2}} \, d\text{vol}_\Sigma - \int_\Sigma \chi'(r) \frac{|\nabla^\Sigma r|^2}{r^2} \, d\text{vol}_\Sigma \]
\[= 4 \int_\Sigma \frac{|(\nabla^\Sigma r)\perp|^2}{r^2} \, d\text{vol}_\Sigma + \theta_0. \tag{3.19}\]

Observe that we have
\[
\frac{\nabla^\Sigma \sigma}{1+\sigma^2} = \frac{2}{\rho^2} \frac{\nabla^\Sigma \varphi}{1+\sigma^2} - 4 \frac{\rho^2}{\rho^3} \frac{\nabla^\Sigma \rho}{1+\sigma^2} = 2 \frac{\rho^2}{r^4} \nabla^\Sigma \varphi - 2 \frac{\sigma}{\sqrt{1+\sigma^2}} \frac{\rho}{r^2} \nabla^\Sigma \rho, \tag{3.20}\]
and recall from (2.4) that
\[|\nabla^\Sigma \rho|^2 + \rho^{-2} |\nabla^\Sigma \varphi|^2 = 1.\]

Hence we deduce that
\[
\left| \frac{\nabla^\Sigma \sigma}{1+\sigma^2} \right|^2 \leq 8 \left[ \frac{\rho^6}{r^8} + \frac{\rho^2}{r^4} \right] \leq \frac{16}{r^2}. \tag{3.21}\]

Combining (3.19) and (3.21) gives the existence of a universal constant $C_0$ such that
\[
2\pi \text{Card} \Lambda^{-1}(0) + \int_{r<1} \frac{|(\nabla^\Sigma r)\perp|^2}{r^2} \, d\text{vol}_\Sigma \leq C_0 \int_{1<r<2} \, d\text{vol}_\Sigma. \tag{3.22}\]

Recall
\[
r^3 J(\nabla^H r) = \rho^3 J(\nabla^H \rho) + 2 \varphi J(\nabla^H \varphi) = -\rho^2 \nabla^H \varphi + 2 \rho \varphi \nabla^H \rho = -\rho^4 \nabla^H \left( \frac{\varphi}{\rho^2} \right). \tag{3.23}\]

Since the immersion of $\Sigma$ by $G$ is Lagrangian, the complex form $J$ is sending the tangential part of $\nabla^H r$ (that we have denoted $\nabla^\Sigma r$) to the normal directions to the surface and vice versa that is, the normal part to the surface of $\nabla^H r$ (that we have denoted $(\nabla^\Sigma r)\perp$) to the tangential directions. Hence we have in particular
\[
r^3 J(\nabla^\Sigma r)\perp = -\rho^4 \nabla^\Sigma \left( \frac{\varphi}{\rho^2} \right). \tag{3.24}\]

Hence
\[
(\nabla^\Sigma r)\perp = \frac{1}{2} \frac{\rho^4}{r^4} J(\nabla^\Sigma \sigma) = \frac{1}{2} J \left( \frac{\nabla^\Sigma \sigma}{1+\sigma^2} \right). \tag{3.25}\]

Combining (3.22) and (3.25) gives the lemma 3.1.
PROOF OF THE MAIN THEOREM 1.1

Let $0 < r < 1$. Observe that the lower bound in (1.3) is a direct consequence of (3.1) after rescaling at $r$. We now prove the upper bound in (1.3).

We consider a smooth cut-off function $\chi$ on $\mathbb{R}_+$ such that

$$
\chi(t) = \begin{cases} 
1 & \text{for } t < 1, \\
0 & \text{for } t > 2,
\end{cases} \quad \chi' \leq 0 \quad \text{on } \mathbb{R}_+ \quad \text{and} \quad \chi' \equiv -3/2 \quad \text{on } [5/4, 7/4].
$$

Replacing in (3.19) $\chi$ by $\chi(\gamma/r)$ and shifting the first term of the l.h.s to the r.h.s gives

$$
\begin{align*}
\frac{1}{4} \int_\Sigma & \arctan \sigma \, r^{-2} \, \chi''(\gamma/r) \, r \, \nabla^\Sigma \gamma \cdot \frac{\nabla^\Sigma \sigma}{1 + \sigma^2} \, dvol_\Sigma - \frac{3}{4} \int_\Sigma \arctan \sigma \, r^{-1} \, \chi'(\gamma/r) \, \nabla^\Sigma \gamma \cdot \frac{\nabla^\Sigma \sigma}{1 + \sigma^2} \, dvol_\Sigma \\
& - \int_\Sigma \frac{\chi'(\gamma/r)}{r} \, \left[ \frac{\sigma \, \arctan \sigma}{\sqrt{1 + \sigma^2}} + |\nabla^\Sigma \gamma|^2 \right] \, dvol_\Sigma \\
& = 4 \int_\Sigma \chi(\gamma/r) \left[ \frac{|\nabla^\Sigma \gamma|^2}{r^2} \right] \, dvol_\Sigma - \int_\Sigma \frac{|\nabla^\Sigma \sigma|^2}{(1 + \sigma^2)^2} \, \frac{r \, \chi'(\gamma/r)}{4} \, dvol_\Sigma + \theta_0 \\
& = 4 \int_\Sigma \chi(\gamma/r) \left[ \frac{|\nabla^\Sigma \gamma|^2}{r^2} \right] \, dvol_\Sigma.
\end{align*}
$$

Recall from (3.6) that

$$
\frac{1}{\sqrt{1 + \sigma^2}} = |\nabla^H \gamma|^2 = |\nabla^\Sigma \gamma|^2 + |(\nabla^\Sigma \gamma)^\perp|^2,
$$

Hence (4.1) becomes

$$
\begin{align*}
\frac{1}{4} \int_\Sigma & \arctan \sigma \, r^{-2} \, \chi''(\gamma/r) \, r \, \nabla^\Sigma \gamma \cdot \frac{\nabla^\Sigma \sigma}{1 + \sigma^2} \, dvol_\Sigma - \frac{3}{4} \int_\Sigma \arctan \sigma \, r^{-1} \, \chi'(\gamma/r) \, \nabla^\Sigma \gamma \cdot \frac{\nabla^\Sigma \sigma}{1 + \sigma^2} \, dvol_\Sigma \\
& - \int_\Sigma \frac{\chi'(\gamma/r)}{r} \, \left[ \frac{\sigma \, \arctan \sigma}{\sqrt{1 + \sigma^2}} + |\nabla^\Sigma \gamma|^2 \right] \, dvol_\Sigma = 4 \int_\Sigma \chi(\gamma/r) \left[ \frac{|\nabla^\Sigma \gamma|^2}{r^2} \right] \, dvol_\Sigma - \int_\Sigma \frac{|\nabla^\Sigma \gamma|^2}{(1 + \sigma^2)^2} \, \frac{r \, \chi'(\gamma/r)}{4} \, dvol_\Sigma + \theta_0.
\end{align*}
$$

Observe that for any $\sigma \in \mathbb{R}$

$$
\left( \frac{\sigma \, \arctan \sigma + 1}{\sqrt{1 + \sigma^2}} \right)' = \frac{\arctan \sigma}{(1 + \sigma^2)^{3/2}}.
$$

We deduce that

$$
\forall \sigma \in \mathbb{R} \qquad 1 \leq \frac{\sigma \, \arctan \sigma + 1}{\sqrt{1 + \sigma^2}} \leq \frac{\pi}{2}.
$$

This gives in particular the existence of a universal constant $C_1$ for $r < 1/2$, using the fact that $\chi' \leq 0$ and $\chi' = -3/2$ on $[5/4, 7/4]$

$$
\begin{align*}
r^{-2} \int_{5r/4 < \gamma < 7r/4} \, dvol_\Sigma & \leq C_1 \int_{r < 2r} \left| \frac{\nabla^\Sigma \sigma}{1 + \sigma^2} \right|^2 \, dvol_\Sigma + C_1 \theta_0 + C_1 \, r^{-1} \int_{r < \gamma < 2r} \left| \frac{\nabla^\Sigma \sigma}{1 + \sigma^2} \right| \, dvol_\Sigma.
\end{align*}
$$
Using Cauchy Schwartz together with (3.1) we obtain
\[
\int_{5r/4 < \tau < 7r/4} dvol_\Sigma \leq C_1 \int_{1 < \tau < 2} dvol_\Sigma + C_2 \left[ \int_{1 < \tau < 2} dvol_\Sigma \right]^{1/2} \left[ \int_{r < \tau < 2r} dvol_\Sigma \right]^{1/2},
\]
where the C_i's denote universal constants. Let
\[
A : = \sup_{r < 1/2} \int_{5r/4 < \tau < 7r/4} dvol_\Sigma.
\]
Observe that for \(5 \times 49 < 2 \times 25\) we have
\[
\int_{r < \tau < 2r} dvol_\Sigma \leq \int_{r < \tau < 7r/5} dvol_\Sigma + \int_{7r/5 < \tau < 49r/25} dvol_\Sigma + \int_{49r/25 < \tau < 343r/125} dvol_\Sigma \leq CA (4.7)
\]
where \(C > 0\) is universal. Apply now (4.6) for \(r = 4/7\) we have
\[
\int_{5/7 < \tau < 1} dvol_\Sigma \leq C_1 \int_{1 < \tau < 2} dvol_\Sigma + \frac{7}{4} C_2 \left[ \int_{1 < \tau < 2} dvol_\Sigma \right]^{1/2} \left[ \int_{4/7 < \tau < 8/7} dvol_\Sigma \right]^{1/2}. (4.8)
\]
This implies in particular
\[
\int_{5/7 < \tau < 1} dvol_\Sigma \leq C'_1 \int_{1 < \tau < 2} dvol_\Sigma + C'_2 \left[ \int_{1 < \tau < 2} dvol_\Sigma \right]^{1/2} \left[ \int_{4/7 < \tau < 5/7} dvol_\Sigma \right]^{1/2}. (4.9)
\]
Observe that \(5/7 < 7/8\) and that \((5/7)^2 < 4/7\), hence
\[
\int_{4/7 < \tau < 5/7} dvol_\Sigma \leq \frac{49^2}{20^2} A (4.10)
\]
Inserting this fact in (4.9) gives finally
\[
\int_{5/7 < \tau < 2} dvol_\Sigma \leq C''_1 \int_{1 < \tau < 2} dvol_\Sigma + C''_2 \left[ \int_{1 < \tau < 2} dvol_\Sigma \right]^{1/2} A^{1/2} (4.11)
\]
We apply (4.6) to \(r = 20/49\) and we obtain for the same reasons as above
\[
\int_{20/49 < \tau < 2} dvol_\Sigma \leq C^{(3)}_1 \int_{1 < \tau < 2} dvol_\Sigma + C^{(3)}_2 \left[ \int_{1 < \tau < 2} dvol_\Sigma \right]^{1/2} A^{1/2}. (4.12)
\]
We iterate this procedure until reaching $5 \times 49 r < 2 \times 25$ for getting

$$
\int_{10/49 < r < 2} dvol_{\Sigma} \leq C_* \int_{1 < r < 2} dvol_{\Sigma} + C_* \left[ \int_{1 < r < 2} dvol_{\Sigma} \right]^{1/2} A^{1/2}, \quad (4.13)
$$

where $C_* > 0$ is universal. Combining (4.7) and (4.13) gives finally

$$
\sup_{r < 1} r^{-2} \int_{r < r < 2} dvol_{\Sigma} \leq C'_* \int_{1 < r < 2} dvol_{\Sigma} + C'_* \left[ \int_{1 < r < 2} dvol_{\Sigma} \right]^{1/2} A^{1/2} + C A. \quad (4.14)
$$

where $C'_* > 0$ is universal. We deduce from (4.6) and (4.14)

$$
A \leq C \int_{1/2 < r < 2} dvol_{\Sigma} + C \left[ \int_{1/2 < r < 2} dvol_{\Sigma} \right]^{1/2} A^{1/2}. \quad (4.15)
$$

where $C > 0$ is universal. This last inequality implies the upper bound in (1.3) after observing that

$$
\frac{1}{r^2} \int_{r < r} dvol_{\Sigma} = \frac{1}{r^2} \left[ \sum_{j=0}^{\infty} \int_{4^{-j-1} r < r < 4^{-j} r} dvol_{\Sigma} \right] \leq \sum_{j=0}^{\infty} 2 4^{-2j} A \leq CA. \quad (4.16)
$$

This concludes the proof of theorem 1.1.

Observe that a similar approach holds for the proof of the classical monotonicity formula (1.1). Indeed, the minimal surface equation in $\mathbb{R}^n$ is $\Delta_{\Sigma} \Phi = 0$ hence we deduce

$$
\Delta_{\Sigma} \rho^2 = 2 |\nabla_{\Sigma} \Phi|^2 = 4. \quad (4.17)
$$

away from $\rho = 0$ this last equation implies

$$
\Delta_{\Sigma} \log \rho^2 = \frac{4}{\rho^2} - 4 \frac{|\nabla_{\Sigma} \rho|^2}{\rho^2}. \quad (4.18)
$$

Since with the above notations $1 = |\nabla \rho|^2 = |\nabla_{\Sigma} \rho|^2 + |(\nabla_{\Sigma} \rho)^\perp|^2$, the previous identity implies as when passing from (3.3) to (3.11)

$$
\frac{1}{4} \Delta_{\Sigma} \log \rho^2 = \frac{|(\nabla_{\Sigma} \rho)^\perp|^2}{\rho^2} + \pi \sum_{p \in \rho^{-1}(0)} \delta_p. \quad (4.19)
$$

Integrating over $\rho < r$ and using again (4.17) gives

$$
\frac{1}{r^2} \int_{\rho < r} dvol_{\Sigma} = \frac{1}{4 r^2} \int_{\rho < r} \Delta_{\Sigma} \rho^2 dvol_{\Sigma} = \frac{1}{4} \int_{\rho = r} \frac{\delta_{\rho} \rho^2}{\rho^2} dl_{\Sigma} = \int_{\rho < r} \frac{|(\nabla \rho)^\perp|^2}{\rho^2} dvol_{\Sigma} + \pi \text{Card}(\Phi^{-1}(0)). \quad (4.20)
$$

This implies the monotonicity formula (1.1).
5  |  PROOF OF THE BERNSTEIN-LIOUVILLE THEOREM 1.2

Modulo a translation (by the Heisenberg group action) we can assume that the surface is passing though the origin and that, using the above notations, we have

\[ \theta_0 \geq 2\pi. \]  \hspace{1cm} (5.1)

Since by assumption

\[ \limsup_{r \to +\infty} \frac{1}{r^2} \int_{r<r<2r} dvol_\Sigma < +\infty, \]  \hspace{1cm} (5.2)

we deduce from lemma 3.1

\[ \left| \int_\Sigma \left| \frac{\nabla \Sigma \sigma}{1+\sigma^2} \right|^2 dvol_\Sigma \right| < +\infty. \]

This implies in particular that

\[ \lim_{r \to +\infty} \int_{r<r<2r} \left| \frac{\nabla \Sigma \sigma}{1+\sigma^2} \right|^2 dvol_\Sigma = 0. \]  \hspace{1cm} (5.3)

We fix \( \varepsilon > 0 \) and we consider a smooth cut-off function \( \chi \) on \( \mathbb{R}_+ \) such that

\[ \chi(t) = \begin{cases} 
1 & \text{for } t < 1 \\
0 & \text{for } t > 2 
\end{cases}, \quad \chi' \leq 0 \quad \text{on } \mathbb{R}_+ \quad \text{and} \quad \chi' \equiv -1 \quad \text{on } [1+\varepsilon, 2-\varepsilon]. \]

and we can also assume that \( \|\chi'\|_\infty \) is uniformly bounded independently of \( \varepsilon \). We shall denote \( C_\varepsilon := \|\chi''\|_\infty \) (which clearly is not bounded independently of \( \varepsilon \)). Replacing in (3.19) \( \chi \) by \( \chi_\varepsilon(t/r) \), using (3.25) and shifting the first term of the l.h.s to the r.h.s gives

\[ \frac{1}{4} \int_\Sigma \arctan \sigma r^{-2} \chi_\varepsilon'(t/r) \frac{\nabla \Sigma \sigma}{1+\sigma^2} dvol_\Sigma - \frac{3}{4} \int_\Sigma \arctan \sigma r^{-2} \chi_\varepsilon'(t/r) \frac{\nabla \Sigma \sigma}{1+\sigma^2} dvol_\Sigma \]

\[ - \int_\Sigma \frac{\chi_\varepsilon(t/r)}{r} \sigma \arctan \sigma + 1 \sqrt{1+\sigma^2} dvol_\Sigma = \int_\Sigma \frac{\chi'(t/r)}{r} \frac{|\nabla \Sigma \sigma|^2}{(1+\sigma^2)^2} dvol_\Sigma - \int_\Sigma \frac{\chi'(t/r)}{4r} \frac{|\nabla \Sigma \sigma|^2}{(1+\sigma^2)^2} dvol_\Sigma \]

\[ - \int_\Sigma \frac{|\nabla \Sigma \sigma|^2}{r} \frac{\chi'(t/r)}{4} dvol_\Sigma + \theta_0. \]

We have respectively for \( \varepsilon > 0 \) fixed

\[ \frac{1}{4} \left| \int_\Sigma \arctan \sigma r^{-2} \chi_\varepsilon''(t/r) \frac{\nabla \Sigma \sigma}{1+\sigma^2} dvol_\Sigma \right| \]

\[ \leq C_\varepsilon \left[ \int_{r<r<2r} \frac{r^2}{r^4} dvol_\Sigma \right]^{1/2} \left[ \int_{r<r<2r} \left| \frac{\nabla \Sigma \sigma}{1+\sigma^2} \right|^2 dvol_\Sigma \right]^{1/2} \to 0 \quad \text{as } r \to +\infty, \]  \hspace{1cm} (5.5)
and
\[
\left| \frac{3}{4} \int_{\Sigma} \arctan \frac{\sigma}{r^{-1}} \frac{1}{r^2} \frac{\nabla^2 \sigma}{1 + \sigma^2} \, d\text{vol}_\Sigma \right| \leq \| \chi' \|_\infty \left[ \int_{r < r < 2r} \frac{1}{r^2} \, d\text{vol}_\Sigma \right]^{1/2} \left[ \int_{r < r < 2r} \left( \frac{\nabla^2 \sigma}{1 + \sigma^2} \right)^2 \, d\text{vol}_\Sigma \right]^{1/2} \rightarrow 0 \quad \text{as } r \rightarrow +\infty. \tag{5.6}
\]

From the assumption of the theorem we have that \( \sigma \) converges uniformly towards 0 as \( r \rightarrow +\infty \). Hence
\[
\lim_{r \rightarrow +\infty} - \int_{\Sigma} \frac{\chi'(r/r)}{r/r} \frac{\sigma \arctan \frac{\sigma}{r} + 1}{\sqrt{1 + \sigma^2}} \, d\text{vol}_\Sigma + \int_{r < r < 2r} \frac{\chi'(r/r)}{r/r} \, d\text{vol}_\Sigma = 0. \tag{5.7}
\]

Because of (5.2) we have the existence of \( r_k \rightarrow +\infty \) such that
\[
\frac{1}{r_k^2} \int_{r_k < r < (1+\varepsilon)r_k} \, d\text{vol}_\Sigma + \frac{1}{r_k^2} \int_{(2-\varepsilon)r_k < r < 2r_k} \, d\text{vol}_\Sigma \leq C \varepsilon. \tag{5.8}
\]

Hence, since \( \| \chi' \|_\infty \) is bounded independently of \( \varepsilon \), we deduce from (5.7), (5.8)
\[
\lim_{\varepsilon \rightarrow 0} \limsup_{k \rightarrow +\infty} \left| - \int_{\Sigma} \frac{\chi'(r/r_k)}{r_k/r} \frac{\sigma \arctan \frac{\sigma}{r} + 1}{\sqrt{1 + \sigma^2}} \, d\text{vol}_\Sigma - \int_{r_k < r < 2r_k} \frac{1}{r_k/r} \, d\text{vol}_\Sigma \right| = 0, \tag{5.9}
\]
which implies by assumption
\[
\lim_{\varepsilon \rightarrow 0} \limsup_{k \rightarrow +\infty} \left| - \int_{\Sigma} \frac{\chi'(r/r_k)}{r_k/r} \frac{\sigma \arctan \frac{\sigma}{r} + 1}{\sqrt{1 + \sigma^2}} \, d\text{vol}_\Sigma - 2\pi \right| = 0. \tag{5.10}
\]
Combining now the fact that \( \theta_0 \geq 2\pi \) together with (5.4), (5.5), (5.6), (5.10) and the fact that \( \chi' \leq 0 \), we obtain
\[
0 = \int_{\Sigma} \left| \frac{\nabla^2 \sigma}{1 + \sigma^2} \right|^2 \, d\text{vol}_\Sigma = \int_{\Sigma} \frac{|(\nabla^2 r)|^2}{r^2} \, d\text{vol}_\Sigma. \tag{5.11}
\]
Hence \( \nabla^2 r = \nabla^2 H \) and \( \Lambda \) is a H-minimal Legendrian smooth conical immersion. The only smooth conical H-minimal Legendrian immersions are Lagrangian planes (see [18]). This concludes the proof of theorem 1.2.

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