$L^1$-determined ideals in group algebras of exponential Lie groups

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Abstract. A locally compact group $G$ is said to be $*$-regular if the natural map $\Psi : \text{Prim} C^*(G) \to \text{Prim}_* L^1(G)$ is a homeomorphism with respect to the Jacobson topologies on the primitive ideal spaces $\text{Prim} C^*(G)$ and $\text{Prim}_* L^1(G)$. In 1980 J. Boidol characterized the $*$-regular ones among all exponential Lie groups by a purely algebraic condition. In this article we introduce the notion of $L^1$-determined ideals in order to discuss the weaker property of primitive $*$-regularity. We give two sufficient criteria for closed ideals $I$ of $C^*(G)$ to be $L^1$-determined. Herefrom we deduce a strategy to prove that a given exponential Lie group is primitive $*$-regular. The author proved in his thesis that all exponential Lie groups of dimension $\leq 7$ have this property. So far no counter-example is known. Here we discuss the example $G = B_5$, the only critical one in dimension $\leq 5$.

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1 Introduction

Let $\mathcal{A}$ be Banach $*$-algebra and $C^*(\mathcal{A})$ its enveloping $C^*$-algebra in the sense of Dixmier, see Chapter 2.7 of [S]. The $C^*$-norm on $C^*(\mathcal{A})$ is given by

$$|a|_* = \sup_{\pi \in \hat{\mathcal{A}}} |\pi(a)|$$

for all $a \in \mathcal{A}$ where $\hat{\mathcal{A}}$ is the set of equivalence classes of topologically irreducible $*$-representations of $\mathcal{A}$ in Hilbert spaces. Let $\text{Prim} C^*(\mathcal{A})$ be the set of primitive ideals in $C^*(\mathcal{A})$, and $\text{Prim}_* \mathcal{A}$ the set of kernels of representations in $\hat{\mathcal{A}}$. For ideals $I$ of $C^*(\mathcal{A})$ we define their hull $h(I) = \{ P \in \text{Prim} C^*(\mathcal{A}) : P \supset I \}$ in $\text{Prim} C^*(\mathcal{A})$, and for subsets $X$ of $\text{Prim} C^*(\mathcal{A})$ their kernel $k(X) = \cap \{ P : P \in X \}$ in $C^*(\mathcal{A})$. In the sequel all ideals are assumed to be two-sided and closed in the respective norm. Closed ideals $I$ of $C^*$-algebras are automatically involutive and satisfy $I = k(h(I))$, see Proposition 1.8.2 and Theorem 2.6.1 of [S].

Recall that $\text{Prim} C^*(\mathcal{A})$ is a topological space w. r. t. the Jacobson topology, i.e., $X \subset \text{Prim} C^*(\mathcal{A})$ is closed if and only if there exists an ideal $I$ of $C^*(\mathcal{A})$ such that
Likewise we can state the according definitions of hulls and kernels for $A$ and we provide $\text{Prim}_A$ with the Jacobson topology as well. Let $I'$ denote the preimage of the ideal $I$ under the natural map $A \longrightarrow C^*(A)$. For simplicity we write $I' = I \cap A$. The map

$$\Psi : \text{Prim} C^*(A) \longrightarrow \text{Prim} A \quad \text{given by} \quad \Psi(P) = P' = P \cap A$$

is continuous and surjective and evidently satisfies $k(\Psi(X)) = k(X) \cap A$ and $h(I) \subset \Psi^{-1}(h(I'))$. The next definition is basic for the subsequent investigation.

**Definition 1.1.** A closed ideal $I$ of $C^*(A)$ is called $A$-determined if and only if the following (obviously) equivalent conditions hold:

1. $I' \subset J'$ implies $I \subset J$ for all ideals $J$ of $C^*(A)$,
2. $I' \subset P'$ implies $I \subset P$ for all $P \in \text{Prim} C^*(A)$, i.e., $h(I) = \Psi^{-1}(h(I'))$,
3. $I'$ is dense in $I$ w. r. t. the $C^*$-norm,
4. $C^*(A/I') \cong C^*(A)/I$.

In the introduction of [2] Boidol defined $*$-regularity of Banach $*$-algebras. We restate his definition and add the notion of primitive $*$-regularity.

**Definition 1.2.** A Banach $*$-algebra $A$ is called (primitive) $*$-regular if and only if every closed (primitive) ideal of $C^*(A)$ is $A$-determined.

The group algebra $L^1(G)$ of a locally compact group $G$ is a $*$-semisimple Banach $*$-algebra with bounded approximate identities. We say that $G$ is (primitive) $*$-regular if $L^1(G)$ has this property. Similarly $*$-regularity of (real) Lie algebras $g$ is defined by means of the (unique) connected, simply connected Lie group $G$ with $\text{Lie}(G) = g$.

Part (ii) of the next lemma shows that Definition 1.2 is equivalent to Boidol’s original definition, a characterization which has already been proved in [3].

**Lemma 1.3.**

(i) If $A$ is primitive $*$-regular, then $\Psi : \text{Prim} C^*(A) \longrightarrow \text{Prim} A$ is injective.

(ii) A Banach $*$-algebra $A$ is $*$-regular if and only if $\Psi$ is a homeomorphism with respect to the Jacobson topologies on $\text{Prim} C^*(A)$ and $\text{Prim} A$.

**Proof.** If $A$ is primitive $*$-regular, then $P = \overline{\Psi(P)}$ is uniquely determined by $\Psi(P)$ for all $P \in \text{Prim} C^*(A)$. This proves (i). In order to prove (ii), let us suppose that $A$ is $*$-regular. Since $\Psi$ is a continuous bijection, it suffices to prove that $\Psi$ maps closed sets onto closed sets. But if $X$ is a closed subset of $\text{Prim} C^*(A)$, then there exists a closed ideal $I$ of $C^*(A)$ such that $X = h(I)$ and we see that $\Psi(X) = h(I')$ is closed in $\text{Prim} A$ because $I$ is $A$-determined. Now we prove the opposite implication. Assume
that $\Psi$ is a homeomorphism, $I$ a closed ideal of $C^*(A)$, and $P \in \text{Prim } C^*(A)$ such that $I' \subset P'$. Define $X = h(I)$. Since $I' = k(\Psi(X))$, it follows

$$h(I') = h(k(\Psi(X))) = \overline{\Psi(X)} = \Psi(X)$$

because $\Psi$ maps closed sets onto closed sets. Now $P' \in \Psi(X)$ implies $P \in X$ so that $P \supset I$ because $\Psi$ is injective. This proves the asserted equivalence. \hfill \square

Because of its technical importance we state the following fact as a lemma, but we omit the easy proof.

**Lemma 1.4.** Let $I \subset J$ be closed ideals of $C^*(A)$ such that $I$ is $A$-determined. Then $J$ is $A$-determined if and only if the ideal $J/I$ of $C^*(A)/I = C^*(A/I')$ is $A/I'$-determined.

This lemma can be applied in the following situation: If $A$ is a closed normal subgroup of $G$ and $\hat{G} = G/A$, then $T_A f(\hat{x}) = \int_A f(xa) \, da$ defines a quotient map of Banach $*$-algebras from $L^1(G)$ onto $L^1(\hat{G})$ which extends to a quotient map from $C^*(G)$ onto $C^*(\hat{G})$, compare p. 68 of [29]. It is easy to see that $I = \ker C^*(G) T_A$ is $L^1(G)$-determined.

**Lemma 1.5.** A finite intersection of $A$-determined ideals is $A$-determined.

**Proof.** Let $I_1$ and $I_2$ be $A$-determined ideals of $C^*(A)$. Let $P \in \text{Prim } C^*(A)$ such that $I_1' \cdot I_2' \subset I_1' \cap I_2' \subset P'$. Since $P'$ is a prime ideal of $A$, it follows $I_1' \subset P'$ or $I_2' \subset P'$. Since $I_1$ and $I_2$ are $A$-determined, we obtain $I_1 \subset P$ or $I_2 \subset P$ and thus $I_1 \cap I_2 \subset P$. Consequently $I_1 \cap I_2$ is $A$-determined and the assertion of this lemma follows by induction. \hfill \square

**Remark 1.6.** Here are a few examples of $*$-regular Banach $*$-algebras: If $G$ is a connected locally compact group such that its Haar measure has polynomial growth, then $G$ is $*$-regular. Boidol proved this fact in Theorem 2 of [4] based on ideas of Dixmier in [7]. Jenkins has shown in Theorem 1.4 of [15] that connected nilpotent Lie groups have polynomial growth. If $G$ is a metabelian connected locally compact group, then $G$ is $*$-regular, see Theorem 3.5 of [2]. Moreover the following is true: If $G$ is a compactly generated, locally compact group with polynomial growth and if $w$ is a symmetric weight function on $G$ which satisfies the non-abelian-Beurling-Domar condition (BDna) of [10], then $L^1(G, w)$ is $*$-regular. Compare Proposition 5.2 and Theorem 5.8 of [10].

In the next paragraphs we formulate sufficient criteria for ideals of the group algebra $C^*(G)$ of exponential Lie groups to be $L^1(G)$-determined, see Proposition 2.12 and Proposition 4.14.

## 2 Inducing primitive ideals from a stabilizer

We shall use the concept of the adjoint algebra (double centralizer algebra) of a Banach $*$-algebra, compare Paragraph 3 of [16] and Chapter 2.3 of [28]. Let $\mathcal{C}_0(G)$ denote the
continuous functions of compact support on $G$. If $H$ is a closed subgroup of $G$, then $C_0(H)$ acts as an algebra of double centralizers on $C_0(G)$ by convolution
\[(a * f)(x) = \int_H a(h) f(h^{-1}x) \, dh\]
from the left, and by
\[(f * a)(x) = \int_H f(xh) \Delta_{G,H}(h^{-1}) a(h^{-1}) \, dh\]
from the right where $\Delta_{G,H}(h) = \Delta_H(h)\Delta_G(h)^{-1}$. These actions extend to actions of $C^*(H)$ on $C^*(G)$ such that $(a * f)^* = f^* * a^*$ and $f * (a * g) = (f * a) * g$ for all $f, g \in C^*(G)$ and $a \in C^*(H)$.

**Definition 2.1.** Let $H$ be a closed subgroup of the locally compact group $G$. If $J$ is an ideal of $C^*(H)$, then
\[\text{ind}_{G}^{H}(J) = (C^*(G) * J * C^*(G))^\perp\]
denotes the induced ideal of $C^*(G)$. If $I$ is an ideal of $C^*(G)$, then the ideal
\[\text{res}_{G}^{H}(I) = \{a \in C^*(H) : a * C^*(G) \subset I\}\]
is its restriction to $H$. An ideal $I$ of $C^*(G)$ is said to be induced from $H$ if there exists an ideal $J$ of $C^*(H)$ such that $I = \text{ind}_{G}^{H}(J)$.

If $I = \ker_{C^*(G)} \pi$ for some unitary representation $\pi$ of $G$, then $\text{res}_{G}^{H}(I) = \ker_{C^*(H)} \pi$. If $I$ is induced from $H$, then $I = \text{ind}_{G}^{H}(\text{res}_{G}^{H}(I))$. Note that $I = \text{ind}_{G}^{H}(J)$ is minimal among all ideals of $C^*(G)$ whose restriction contains $J$.

It is interesting to compare our definition of induced ideals to that of Green and Rieffel in Section 3 of [13] involving $C^*$-imprimitivity bimodules. To this end we assume that there exists a $G$-invariant measure on the homogeneous space $G/H$ so that the character $\Delta_{G,H}$ of $H$ is trivial. This is the case e.g. if $H$ is a normal subgroup of $G$. We follow the considerations of Section 4 of Rieffel’s article [30]. Note that the right action of $C_0(H)$ on $X_0 = C_0(G)$ defined in [30] coincides with that of convolution from the right because the function $\gamma = \Delta_{G,H}^{-1/2}$ used there is also trivial. The $C_0(H)$-valued inner product
\[\langle f | g \rangle_{C_0(H)}(h) = (f^* g)(h) = \int_G \overline{f(y)} g(yh) \, dy\]
defines a norm $\|f\|_{C^*(H)} = (\langle f | f \rangle_{C_0(H)})^{1/2}$ on $X_0$ where the norm on the right is the $C^*$-norm of $C^*(H)$. Further $C_0(G)$ acts on $X_0$ by convolution from the left so that
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$X_0$ becomes a $C_0(G)$-$C_0(H)$-bimodule, and $\langle f \mid g \rangle_{C_0(G)} = f \ast g^*$ defines a $C_0(G)$-valued inner product $C^*_c(G)\langle \cdot \mid \cdot \rangle$ on $X_0$. Completion of $X_0$ with respect to the norm $| \cdot |_{C^*_c(H)}$ gives a right-$C^*_c(H)$-rigged space $X$ on which $C^*_c(G)$ acts from the left. From

$$\text{ind}_H^G(J) = \overline{\text{span}} \{ C^*_c(G)\langle f \ast a \mid g \rangle : f, g \in X \text{ and } a \in J \} = X - \text{ind}_H^G(J)$$

we learn that, at least in the case of $\Delta_{G,H}$ being trivial, our definition coincides with that of Rieffel and Green.

It is well-known that for $C^*$-imprimitivity bimodules $X$ the Rieffel correspondence $X - \text{ind}_H^G$ is compatible with inducing representations in the sense that

$$(2.2) \quad X - \text{ind}_H^G(\ker \sigma) = \ker(X - \text{ind}_H^G \sigma),$$

compare Chapter 3.3 of [28]. But in general the bimodule $X$ from above is not a $C^*_c(G)$-$C^*_c(H)$-imprimitivity bimodule because the crucial equality $C^*_c(G)\langle f \mid g \rangle_{C^*_c(H)} \ast h = f \ast (g \mid h)_{C^*_c(H)}$ is not necessarily satisfied. The norms $| \cdot |_{C^*_c(H)}$ and $| \cdot |_{C^*_c(G)}$ might be different. In fact, the imprimitivity algebra of the $C^*_c(H)$-rigged space $X$ is known to be isomorphic to the covariance algebra $C^*_c(G, C_\infty(G/H))$. As we will see, Equation (2.2) holds true for the $C^*$-bimodule $X$ defined above if $G/H$ is amenable.

In analogy to results of Leptin [17] and Hauenschild, Ludwig [14] for the $L^1$-case, we will characterize those ideals $I$ of $C^*_c(G)$ which are induced from a given closed normal subgroup $H$ of $G$. This turns out to be possible if $H$ is normal and $G/H$ amenable. In order to prepare the proof of Theorem 2.6 we recall the well-known restriction-induction-lemma of Fell, see Theorem 3.1 and Lemma 4.2 of [11]. A proof can also be found on p. 32 of [19]. We presume the definition of induced representations.

**Lemma 2.3.** Let $H$ be a closed subgroup of a locally compact group $G$. Let $\pi$ be a unitary representation of $G$ and $\pi \mid H$ its restriction to $H$.

(i) If $\tau$ is a unitary representation of $H$, then the Kronecker product $\text{ind}_H^G((\pi \mid H) \otimes \tau)$ is unitarily equivalent to $\pi \otimes \text{ind}_H^G \tau$.

(ii) In particular $\text{ind}_H^G(\pi \mid H)$ is unitarily equivalent to $\pi \otimes \lambda$ where $\lambda$ denotes the left regular representation of $G$ in $L^2(G/H)$.

Note that conjugation $f^z(x) = \Delta_G(z^{-1}) f(zxz^{-1})$ for $f \in L^1(G)$ and $z \in G$ extends to a strongly continuous action of $G$ on $C^*_c(G)$ by isometric automorphisms. Using an approximate identity of $C^*_c(G)$, one can prove that every closed ideal $I$ of $C^*_c(G)$ is two-sided translation-invariant, and hence invariant under conjugation, i.e. $I^z = I$.

If $H$ is a closed normal subgroup of $G$, then $G$ acts on $H$ by conjugation $n^z = z^{-1}nz$. Further $a^z(n) = \delta(z^{-1}) a(n^{-1})$ for $a \in L^1(H)$ and $z \in G$ yields a strongly continuous, isometric action of $G$ on $C^*_c(H)$. If $I$ is a closed ideal of $C^*_c(G)$, then $J = \text{res}_H^G(I)$ is a $G$-invariant ideal of $C^*_c(H)$, i.e. $J^z = J$, because $(a \ast f)^z = a^z \ast f^z$ and $I^z = I$.
Lemma 2.4. Let $H$ be a closed normal subgroup of a locally compact group $G$. Let $\sigma$ be a unitary representation of $H$ and $\pi = \text{ind}_H^G \sigma$. Then $\pi \mid H$ is weakly equivalent to the orbit $G \cdot \sigma$ which means 

$$\ker_{C^*(H)} \pi = k(G \cdot \sigma) = \bigcap_{x \in G} \ker_{C^*(H)} x \cdot \sigma.$$ 

Proof. Let $\mathcal{S}$ be the representation space of $\sigma$. As usual $C_0^*(G, \mathcal{S})$ denotes the vector space of all continuous functions on $G$ which satisfy $\varphi(xh) = \sigma(h)^* \varphi(x)$ for $h \in H$, $x \in G$ and have compact support modulo $H$. Then $\pi = \text{ind}_H^G \sigma$ is defined in $L^2(G, \mathcal{S})$, the completion of $C_0^*(G, \mathcal{S})$ with respect to the $L^2$-norm given by integration with respect to the Haar measure of the group $G/H$. We get 

$$\pi(h)\varphi(x) = \varphi(h^{-1}x) = \sigma(h^x) \cdot \varphi(x)$$

for $h \in H$. It follows that $\pi \mid H$ is given by $\pi(a)\varphi(x) = \sigma(a^x) \cdot \varphi(x)$ for $a \in C^*(H)$. Hence $\pi$ is essentially a direct integral of the representations $\{x \cdot \sigma : x \in G\}$ so that the assertion of this lemma becomes clear.

The importance of the left regular representation $\lambda$ of $G$ in $L^2(G/H)$ has already been indicated by Lemma 2.3.

Definition 2.5. Let $H$ be a closed normal subgroup of a locally compact group $G$. An ideal $I$ of $C^*(G)$ is said to be $(G/H)^\sim$-invariant if $\pi$ is weakly equivalent to $\pi \otimes \lambda$ (in symbols $\pi \approx \pi \otimes \lambda$) for all unitary representations $\pi$ of $G$ such that $I = \ker_{C^*(G)} \pi$.

Theorem 1 of [12] shows that $\pi \approx \pi \otimes \lambda$ for at least one such $\pi$ is sufficient for $I$ to be $(G/H)^\sim$-invariant. Now we can state the announced characterization of induced ideals.

Theorem 2.6. Let $H$ be a closed normal subgroup of a locally compact group $G$ such that $G/H$ is amenable. Then there are equivalent:

(i) $I = \text{ind}_H^G (\text{res}_H^G(I))$ is induced from $H$.

(ii) $I = \ker_{C^*(G)} \pi$ is the kernel of some induced representation $\pi = \text{ind}_H^G \sigma$.

(iii) $I$ is $(G/H)^\sim$-invariant.

Proof. First we verify (i) $\Rightarrow$ (ii). Suppose that $I$ is induced from $H$. Since $J = \text{res}_H^G(I)$ is a $G$-invariant ideal of $C^*(H)$, its hull $\Omega = h(J) \subset \hat{H}$ is $G$-invariant, too. Define $\sigma = \sum_{\tau \in \Omega} \tau$ and $\pi = \text{ind}_H^G \sigma$. Lemma 2.4 implies $\ker_{C^*(H)} \pi = k(G \cdot \sigma) = k(\Omega) = J$. Hence $I = \text{ind}_H^G(J) \subset \ker_{C^*(G)} \pi$. We must prove the opposite inclusion: Let $\rho \in \hat{G}$ be arbitrary such that $I \subset \ker_{C^*(G)} \rho$. Then $k(G \cdot \sigma) = J \subset \ker_{C^*(H)} \rho$ which means that $\rho \mid H$ is weakly contained in $G \cdot \sigma$ (in symbols $\rho \mid H \ll G \cdot \sigma$). Since $G/H$ is amenable, we have $1_G \ll \lambda = \text{ind}_H^G 1_H$ and hence $\rho \otimes 1_G \ll \rho \otimes \lambda$ by Theorem 1 of [12]. For inducing
representations is continuous w. r. t. the Fell topologies of \( \hat{H} \) and \( \hat{G} \), it follows from part (ii) of Lemma 2.3 that

\[
\rho \cong \rho \otimes 1_G \ll \rho \otimes \lambda \cong \text{ind}_H^G(\rho|H) \ll \text{ind}_H^G(G \cdot \sigma) \approx \text{ind}_H^G \sigma = \pi
\]

because the representations \( \text{ind}_H^G(z \cdot \sigma) \), \( z \in G \), are all unitarily equivalent. Thus \( \ker_{C^*_{\sigma}(G)} \pi \subset \ker_{C^*_{\sigma}(G)} \rho \). Since \( \rho \) is the intersection of all primitive ideals of \( C^*_{\sigma}(G) \) containing \( \pi \) by Theorem 2.9.7 of [8], we obtain \( \rho = \ker_{C^*_{\sigma}(G)} \pi \).

Next we show (ii) \( \Rightarrow \) (iii). Suppose that \( \rho = \ker_{C^*_{\sigma}(G)} \pi \) for some \( \pi = \text{ind}_H^G \sigma \). By Lemma 2.4 we know \( \pi|H \approx G \cdot \sigma \). Thus \( \pi \otimes \lambda \cong \text{ind}_H^G(\pi|H) \approx \text{ind}_H^G \sigma = \pi \) which proves \( \rho \) to be \( (G/H)^{\sim} \)-invariant.

Finally we prove (iii) \( \Rightarrow \) (i). Suppose that \( \rho \) is \( (G/H)^{\sim} \)-invariant. Let \( J = \text{res}_H^G(\rho) \). Clearly \( \text{ind}_H^G(J) \subset \rho \). It remains to verify the opposite inclusion: Choose a unitary representation \( \pi \) of \( G \) such that \( \rho = \ker_{C^*_{\sigma}(G)} \pi \). Let \( \rho \) be arbitrary such that \( \text{ind}_H^G(J) \subset \ker_{C^*_{\sigma}(G)} \rho \). Then \( \ker_{C^*_{\sigma}(H)} \pi = J \subset \ker_{C^*_{\sigma}(H)} \rho \) and hence \( \rho|H \ll \pi|H \).

Since \( G/H \) is amenable, we get

\[
\rho = \rho \otimes 1 \ll \rho \otimes \lambda = \text{ind}_H^G(\rho|H) \ll \text{ind}_H^G(\pi|H) = \pi \otimes \lambda \approx \pi
\]

because \( \rho \) is \( (G/H)^{\sim} \)-invariant. Thus \( \rho = \ker_{C^*_{\sigma}(G)} \pi \subset \ker_{C^*_{\sigma}(G)} \rho \). Now Theorem 2.9.7 of [8] implies \( \rho = \text{ind}_H^G(J) \). The proof is complete.

The proof of (i) \( \Rightarrow \) (ii) of Theorem 2.4 shows that \( J = \text{res}_H^G(\text{ind}_H^G(J)) \) for every \( G \)-invariant ideal \( J \) of \( C^*_{\sigma}(H) \). The preceding results can be summarized as follows:

**Theorem 2.7.** Let \( H \) be a closed normal subgroup of a locally compact group \( G \) such that \( G/H \) is amenable. Induction and restriction give bijections between the set of all \( (G/H)^{\sim} \)-invariant ideals \( I \) of \( C^*_{\sigma}(G) \) and the set of all \( G \)-invariant ideals \( J \) of \( C^*_{\sigma}(H) \) which are inverses of one another.

An immediate consequence is

**Corollary 2.8.** Let \( H \) be a closed normal subgroup of a locally compact group \( G \) such that \( G/H \) is amenable. If the ideals \( \{I_k : k \in \Lambda\} \) of \( C^*_{\sigma}(G) \) are induced from \( H \), then their intersection \( I = \bigcap \{I_k : k \in \Lambda\} \) is also induced from \( H \).

**Proof.** Let \( \pi_k \) be a unitary representation of \( G \) such that \( I_k = \ker_{C^*_{\sigma}(G)} \pi_k \). We know \( \pi_k \otimes \lambda \approx \pi_k \) by Theorem 2.4. If we define \( \pi = \sum_{k \in \Lambda} \pi_k \), then \( I = \ker_{C^*_{\sigma}(G)} \pi \) and \( \pi \otimes \lambda \approx \{\pi_k \otimes \lambda : k \in \Lambda\} \approx \{\pi_k : k \in \Lambda\} \approx \pi \). Thus \( I \) induced from \( H \) by Theorem 2.4.

Suppose that \( H \) is a coabelian normal subgroup of \( G \) so that \( G/H \) is amenable as an abelian group. In this case \( (\chi \cdot f)(x) = \chi(x) f(x) \) for \( f \in L^1(G) \) extends to an isometric, strongly continuous action of the Pontryagin dual \( (G/H)^{\sim} \) on \( C^*_{\sigma}(G) \). Note that \( \pi(\chi \cdot f) = (\pi \otimes \chi)(f) \) for any unitary representation \( \pi \) of \( G \).
Corollary 2.9. Let $H$ be a coabelian normal subgroup of $G$. An ideal $I$ of $C^*(G)$ is induced from $H$ if and only if it is $(G/H)^{\sim}$-invariant in the sense that $\chi \cdot I = I$ for all $\chi \in (G/H)^{\sim}$.

Proof. Let $\pi$ be a unitary representation of $G$ such that $I = \ker_{C^*(G)} \pi$. Theorem 2.6 shows that $I$ is induced from $H$ if and only if $\pi \simeq \pi \otimes \lambda$. Since $G/H$ is abelian, it follows $\lambda \simeq \{ \chi : \chi \in (G/H)^{\sim} \}$ and hence

$$\ker_{C^*(G)} \pi \otimes \lambda = \bigcap_{\chi \in (G/H)^{\sim}} \ker_{C^*(G)} \pi \otimes \chi \subseteq \ker_{C^*(G)} \pi.$$ 

Thus we see that $\pi \otimes \lambda \simeq \pi$ if and only if $\ker_{C^*(G)} \pi \otimes \chi = \ker_{C^*(G)} \pi$ for all $\chi$. This is the case if and only if $\chi \cdot I = I$ for all $\chi \in (G/H)^{\sim}$. \hfill \qed

The preceding corollary displays a close connection to the $L^1$-results of Leptin and Hauenschild, Ludwig. In 1968 Leptin characterized the induced ideals of generalized $L^1$-algebras / twisted covariance algebras $L^1(G, A, \tau)$, see Satz 8 and Satz 9 of [14]. His results imply Theorem 2.10 and Lemma 2.11 below. In 1981 Hauenschild and Ludwig gave a different proof of Theorem 2.10 using $L^1$-$L^\infty$-duality, see Theorem 2.3 of [14]. These $L^1$-results hold true without the additional assumption of amenability.

An ideal $I$ of $L^1(G)$ is said to be induced from $H$ if there exists an ideal $J$ of $L^1(H)$ such that $I = \text{ind}_{\hat{H}}^G(J) = (L^1(G) * J * L^1(G))^-$. 

Theorem 2.10. Let $H$ be a closed normal subgroup of a locally compact group $G$. An ideal $I$ of $L^1(G)$ is induced from $H$ if and only if it is $C_\infty(G/H)$-invariant. Induction and restriction gives a bijection between the set of all $C_\infty(G/H)$-invariant ideals $I$ of $L^1(G)$ and the set of all $G$-invariant ideals $J$ of $L^1(H)$.

Here $C_\infty(G/H)$ denotes the continuous functions on $G/H$ vanishing at infinity.

Lemma 2.11. Let $H$ be a closed normal subgroup of a locally compact group $G$. If $J$ is a closed, $G$-invariant ideal of $L^1(H)$, then $J * L^1(G)$ is contained in the closure of $L^1(G) * J$. Similarly $L^1(G) * J$ is contained in the closure of $J * L^1(G)$.

This implies $I = \text{ind}_{\hat{H}}^G(J) = (J * L^1(G))^- = (L^1(G) * J)^-$. For $G$-invariant $C^*$-ideals we even know $J * C^*_r(G) = C^*_r(G) * J$ by Corollary 2.3 of the main lemma in [31].

Now we can state our first criterion for ideals of $C^*_r(G)$ to be $L^1(G)$-determined.

Proposition 2.12. Let $G$ be a locally compact group and $H$ a $*$-regular closed subgroup. If the ideal $I$ of $C^*_r(G)$ is induced from $H$, then $I$ is $L^1(G)$-determined.

Proof. Let $J = \text{res}^G_H(I)$ so that $I = \text{ind}_H^G(J)$. If $\rho \in \hat{G}$ with $I' = I \cap L^1(G) \subset \ker_{L^1(G)} \rho$, then $J' \subset \ker_{L^1(H)} \rho$. Since $H$ is $*$-regular, it follows $J \subset \ker_{C^*_r(H)} \rho$. This implies $I = \text{ind}_H^G(J) \subset \text{ind}_H^G(\ker_{C^*_r(H)} \rho) \subset \ker_{C^*_r(G)} \rho$. \hfill \qed
In the rest of this article we will focus on exponential Lie groups (i.e. connected, simply connected, solvable Lie groups $G$ such that the exponential map $\exp : g \to G$ is a global diffeomorphism). We will use the construction of irreducible representations $\pi = K(f) = \text{ind}^G_f \chi_f$ via Pukanszky / Vergne polarizations $p$ at $f$ and the bijectivity of the Kirillov map $K : g^* / Ad^*(G) \to \hat{G}$, see Chapters 4 and 6 of [1], and Chapter 1 of [1]. Mostly we regard $K$ as a map from $g^*$ onto $\hat{G}$ which is constant on coadjoint orbits.

**Lemma 2.13.** Let $G$ be an exponential Lie group with Lie algebra $g$. Let $f \in g^*$ and $q \in [g, g]^\perp \subset g^\perp$. If we define $\pi = K(f)$ and the character $\alpha(exp X) = e^{q(X)}$ of $G$, then $K(f + q)$ and $\pi \otimes \alpha$ are unitarily equivalent.

**Proof.** Let $p \subset g$ be a Pukanszky polarization at $f$, and hence also at $f + q$. Let $\chi_f$ and $\chi_{f+q}$ denote characters of $P$ with differential $f$ and $f + q$. By definition of the Kirillov map we have $\pi = \text{ind}^G_f \chi_f$ and $\rho = K(f + q) = \text{ind}^G_p \chi_{f+q}$. Now one verifies easily that $(U \varphi)(x) = \alpha(x) \varphi(x)$ defines a unitary isomorphism from $\hat{\mathcal{H}}_\pi = L^2_{\chi_f}(G)$ onto $\hat{\mathcal{H}}_\rho = L^2_{\chi_{f+q}}(G)$ such that $\rho = U(\pi \otimes \alpha)U^{-1}$. This proves our claim. \hfill \qed

The next proposition enlightens the significance of the 'stabilizer' $M$.

**Proposition 2.14.** Let $G$ be an exponential Lie group, $n$ a coabelian (nilpotent) ideal of its Lie algebra $g$, and $f \in g^\perp$. Let $M$ denote the connected subgroup of $G$ with Lie algebra $m = g_f + n$. If $\pi = K(f)$, then the primitive ideal $\ker_{C^*(G)} \pi$ is induced from the stabilizer $M$.

**Proof.** First we observe that the orbit $\text{Ad}^*(G)f$ is saturated over $m$: Let $G_l$ denote the (connected) stabilizer of $l = f \mid n$ in $G$. Since $\text{Ad}^*(G_l)f = f + m^\perp$, it follows $\text{Ad}^*(G)f = \text{Ad}^*(G)f + m^\perp$, compare p. 23 of [1]. Now the preceding lemma implies $\pi \otimes \alpha = K(f + q) = K(f) = \pi$ for all $q \in m^\perp$ and characters $\alpha(exp X) = e^{q(X)}$ of $G/M$ proving $\ker_{C^*(G)} \pi$ to be $(G/M)^\sim$-invariant. Hence $\ker_{C^*(G)} \pi$ is induced from $M$ by Corollary 2.9. \hfill \qed

# 3 The ideal theory of $*$-regular exponential Lie groups

The results of this subsection are not new. They can be found in Boidol’s paper [2], and in a more general context in [3]. For the convenience of the reader we give a short proof for the if-part of Theorem 5.4 of [2] using the results of the previous section. The following definition has been adapted from the introduction of [3].

**Definition 3.1.** Let $G$ be a locally compact group. If $A$ is a closed normal subgroup of $G$ and $G = G/A$, then $T_A$ denotes the quotient map from $C^*(G)$ onto $C^*(G/A)$. We say that a closed ideal $I$ of $C^*(G)$ is essentially induced from a $*$-regular subgroup if there exist closed subgroups $A \subset H$ of $G$ with $A$ normal in $G$ such that the following conditions are satisfied:

(i) $\ker_{C^*(G)} T_A \subset I$, 


Further there exists a decomposition $\Lambda = \ast$.

If we pass to the quotient $\dot{\Lambda}$, then it follows from Proposition 2.12 that all ideals $I$ of $\mathcal{C}^*(G)$ which are essentially induced from a $\ast$-regular subgroup are $L^1(G)$-determined.

**Definition 3.2.** Let $g$ be an exponential Lie algebra and $n = [g, g]$ its commutator ideal. We say that $g$ satisfies condition $(R)$ if the following is true: If $f \in g^*$ is arbitrary and $m = g_f + n$ is its stabilizer, then $f = 0$ on $m^\infty = \cap_{k=1}^\infty C^k\!m$. Here the $C^k\!m$ are the ideals of the descending central series. Recall that $m^\infty$ is the smallest ideal of $m$ such that $m/m^\infty$ is nilpotent.

Note that the stabilizer $m = g_f + n$ depends only on the orbit $\mathrm{Ad}^\ast(G)f$. The following observation is extremely useful: Let $f \in g^*$ and $m = g_f + n$ be its stabilizer such that $m/m^\infty$ is nilpotent. If $\gamma_1, \ldots, \gamma_r$ are the roots of $g$, then we define the ideal $\tilde{m} = \cap_{i \in S} \ker_\gamma_i$ of $g$ where $S = \{i : \ker_\gamma_i \supset n\}$. It is easy to see that $m \subseteq \tilde{m}$ and that $\tilde{m}/m^\infty$ is nilpotent, too. Further there are only finitely many ideals $\tilde{m}$ of this kind.

**Theorem 3.3.** Let $G$ be an exponential Lie group such that its Lie algebra $g$ satisfies condition $(R)$. Then any ideal $I$ of $\mathcal{C}^*(G)$ is a finite intersection of ideals which are essentially induced from a nilpotent normal subgroup. In particular $G$ is $\ast$-regular.

**Proof.** Let $I \triangleleft \mathcal{C}^*(G)$ be arbitrary. Since $I = k(h(I))$ by Theorem 2.9.7 of [8], there is a closed, $\mathrm{Ad}^\ast(G)$-invariant subset $\Lambda$ of $g^*$ such that $I = \cap \{\ker_{\mathcal{C}^*(G)}(f) : f \in \Lambda\}$. Further there exists a decomposition $\Lambda = \cup_{k=1}^r \Lambda_k$ and ideals $\{\tilde{m}_k : 1 \leq k \leq r\}$ of $g$ as in the preceding remark such that $g_f + n \subseteq \tilde{m}_k$ for all $f \in \Lambda_k$ where $n = [g, g]$. By induction in stages it follows from Proposition 2.14 that $\ker_{\mathcal{C}^*(G)}(f)$ is induced from $\tilde{M}_k$ for all $f \in \Lambda_k$. Let us define $I_k = \cap \{\ker_{\mathcal{C}^*(G)}(f) : f \in \Lambda_k\}$. Since $f = 0$ on $\tilde{m}_k^\infty$ by condition (R) and $M_k/M_k^\infty$ is nilpotent, we conclude from Corollary 2.18 that $I_k$ is essentially induced from a nilpotent (and hence $\ast$-regular) normal subgroup. Finally Lemma 4.1 implies that the ideal $I = \cap_{k=1}^r I_k$ is $L^1(G)$-determined. \qed

### 4 Closed orbits in the unitary dual of the nilradical

First we recall how to compute the $\mathcal{C}^*$-kernel of $\pi \mid N$ in the Kirillov picture, compare Theorem 9 in Section 5 of Chapter 1 in [19]. Note that the linear projection $r : g^* \longrightarrow n^*$ given by restriction is $\mathrm{Ad}^\ast(G)$-equivariant so that $r(\mathrm{Ad}^\ast(G)f) = \mathrm{Ad}^\ast(G)l$.

**Lemma 4.1.** Let $G$ be an exponential Lie group and $n$ a coabelian ideal of its Lie algebra $g$. Let $f \in g^*$, $\pi = \mathcal{K}(f) \in \hat{G}$, $l = f \mid n$, and $\sigma = \mathcal{K}(l) \in \hat{N}$. Then

\begin{equation}
\ker_{\mathcal{C}^*(N)}\pi = k(G, \sigma) = \cap_{h \in \mathrm{Ad}^\ast(G)l} \ker_{\mathcal{C}^*(N)}\mathcal{K}(h).
\end{equation}
Our main result is Proposition 4.14 which states that the primitive ideal \( \ker L ) \) closely related to the classification of simple \( L ) \)-modules, \( G \) an exponential Lie group, established by Poguntke in [26].

Let \( g \) be a Pukanszky polarization at \( f \in g^* \). By induction in stages we obtain \( \pi = \text{ind}_{g}^{c} \chi f = \text{ind}_{N}^{c} \sigma \) so that \( \ker_{C^\ast(G)} \pi = k(G \cdot \sigma) \) by Lemma 4.11. Next we assume \( g_f \subset n \). Using the concept of Vergne polarizations passing through \( n \) we see that there exists a Pukanszky polarization \( p \subset g \) at \( f \in g^* \) such that \( q = p \cap n \) is a Pukanszky polarization at \( l \in n^* \). We point out that the restriction of functions from \( G \) to \( N \) gives a linear isomorphism \( C_0^l(G) \to C_0^l(N) \) which extends to a unitary isomorphism \( U \) from \( s = L_{c}^l(G) \) onto \( s = L_{c}^l(N) \). Clearly \( U \) intertwines \( \pi \mid H \) and \( \sigma \). On the other hand \( g = g_f + n \) implies \( \text{Ad}^*(G) = \text{Ad}^*(N) \) and thus \( G \cdot \sigma = \{ \sigma \} \). This proves \( \ker_{C^\ast(N)} \pi = \ker_{C^\ast(N)} \sigma = k(G \cdot \sigma) \).}

In the sequel we suppose that \( n \) is nilpotent and coabelian. Note that the orbit \( G \cdot \sigma \subset \hat{N} \) is uniquely determined by Equality (1.2) because it is locally closed (open in its closure): Pukanszky showed in Corollary 1 of [27] that \( Ad^*(G) \) is locally closed in \( N^* \) and Brown proved in [5] that the Kirillov map of the connected, simply connected, nilpotent Lie group \( N \) is a homeomorphism.

Our main result is Proposition 4.14 which states that the primitive ideal \( \ker_{C^\ast(G)} \pi \) is \( L^1(G) \)-determined if \( G \cdot \sigma \) is closed in \( \hat{N} \). This result is a consequence of arguments closely related to the classification of simple \( L^1(G) \)-modules, \( G \) an exponential Lie group, established by Poguntke in [26].

Let \( \pi, f, \sigma, l \) be as in Lemma 4.11. It is easy to see that \( g = g_f + n \) is sufficient for \( G \cdot \sigma \) to be closed in \( \hat{N} \): Theorem 3.1.4 of [1] implies that \( \text{Ad}^*(G) \) is closed in \( n^* \) because \( N \) acts unipotently on \( n^* \). Since the Kirillov map of \( N \) is a homeomorphism, it follows that \( G \cdot \sigma = \{ \sigma \} \) is closed in \( \hat{N} \). Alternatively one can resort to the results of Moore and Rosenberg: It follows from Theorem 1 of [22] that \( \hat{N} \) is a \( T_1 \)-space so that its one-point subsets are closed. Let us give a third proof of this fact: Since \( L^1(N) \) is symmetric for nilpotent connected Lie groups \( N \) by Satz 2 of [23], it follows \( \text{Prim}_s L^1(N) = \text{Max} L^1(N) \) by (6) of [18] so that points \( \{ \sigma \} \) are closed in \( \hat{N} \) because \( N \) is \( s \)-regular.

Poguntke proved in Theorem 7 of [24] that if \( E \) is a simple \( L^1(G) \)-module and \( N \) is a connected, coabelian, nilpotent subgroup of \( G \), then there exists a unique orbit \( G \cdot \sigma \subset \hat{N} \) such that \( \text{Ann}_L^1(E) = k(G \cdot \sigma) \). More generally, Ludwig and Molitor-Braun showed in [21] that if \( T \) is a topologically irreducible, bounded representation of \( L^1(G) \), then \( \ker_{L^1(T)} = k(G \cdot \sigma) \) for some \( \sigma \in \hat{N} \).

We need the following well-known facts about simple modules and minimal hermitian idempotents. In the following irreducible means topologically irreducible.

**Lemma 4.3.** Let \( B \) be a Banach *-algebra and \( \pi \) an irreducible *-representation of \( B \) in
a Hilbert space $\mathcal{H}$.

(i) Let $\xi \in \mathcal{H}$ be non-zero. Then the subspace $\pi(B)\xi$ is non-zero and dense in $\mathcal{H}$. If $I$ is an ideal of $B$ such that $I \not\subset \ker \pi$, then $\pi(I)\xi$ is also non-zero and dense.

(ii) Suppose that the ideal $I$ of all $f \in B$ such that $\pi(f)$ has finite rank is non-zero. Then the $\pi(B)$-invariant subspace $E = \pi(I)\mathcal{H}$ generated by $\{\pi(f)\eta : f \in I, \eta \in \mathcal{H}\}$ is a simple $B$-module such that $\text{Ann}_B(E) = \ker B \pi$.

Proof. Part (i) is obvious. The proof of (ii) follows Dixmier’s proof of Théorème 2 in [7]. Let $\xi \in E$ be non-zero. For every $f \in I$ the subspace $\pi(f)\pi(B)\xi$ is dense in $\pi(f)\mathcal{H}$ so that $\pi(f)\pi(B)\xi = \pi(f)\mathcal{H}$ because $\pi(f)\mathcal{H}$ is finite-dimensional. This proves $\pi(f)\mathcal{H} \subset \pi(I)\xi$ for every $f \in I$. Thus $E = \pi(I)\mathcal{H} = \pi(I)\xi$. The rest is obvious.

A hermitian idempotent $q \in B$ satisfies $q^2 = q = q^*$. We say that $q$ is minimal in $B$ if it is non-zero and if $qBq = \mathbb{C}q$.

Lemma 4.4. Let $B$ be Banach *-algebra.

(i) Let $\pi$ be a faithful irreducible *-representation of $B$ in a Hilbert space $\mathcal{H}$. Then $q \in B$ is a minimal hermitian idempotent if and only if $\pi(q)$ is a one-dimensional irreducible *-representation.

(ii) Assume that there exist minimal hermitian idempotents in $B$. If $\pi, \rho$ are faithful irreducible *-representations of $B$, then $\pi$ and $\rho$ are unitarily equivalent.

Proof. Clearly $q$ is a hermitian idempotent if and only if $\pi(q)$ is an orthogonal projection because $\pi$ is faithful. If $\pi(q)\mathcal{H}$ is a one-dimensional, then $\pi(\mathbb{C}q) = \mathbb{C}\pi(q) = \pi(q)\pi(B)\pi(q) = \pi(qBq)$ and thus $qBq = \mathbb{C}q$ because $\pi$ is faithful. For the converse assume $qBq = \mathbb{C}q$. Since $\pi(qBq)$ and hence $\pi(q)$ acts irreducibly on $\pi(q)\mathcal{H}$, it follows that this subspace is one-dimensional.

Now we prove (ii). Let $q \in B$ be a minimal hermitian idempotent and $\pi, \rho$ faithful irreducible *-representations in Hilbert spaces $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$. Since $\pi(q)$ and $\rho(q)$ are one-dimensional orthogonal projections by (i), there exist unit vectors $\xi \in \mathcal{H}_\pi$ and $\eta \in \mathcal{H}_\rho$ such that $\pi(q) = \langle -, \xi \rangle \xi$ and $\rho(q) = \langle -, \eta \rangle \eta$. Let us consider the positive linear functionals $f_\pi, f_\rho$ on $B$ given by $f_\pi(a) = \langle \pi(a)\xi, \xi \rangle$ and $f_\rho(a) = \langle \rho(a)\eta, \eta \rangle$. Since $qBq$ is one-dimensional and $f_\pi(q) = 1 = f_\rho(q)$, it follows $f_\pi(a) = f_\pi(qaq) = f_\rho(qaq) = f_\rho(a)$ for all $a \in B$, i.e., the positive linear forms of the cyclic representations $\pi$ and $\rho$ coincide. Now Proposition 2.4.1 of [8] shows that $\pi$ and $\rho$ are unitarily equivalent.

Poguntke proved in [25] that for exponential $G$ and $\rho \in \hat{G}$ there exists some $q \in L^1(G)$ such that $\pi(q)$ is a one-dimensional orthogonal projection. Note that the canonical image of $q$ in $L^1(G)/\ker L^1(G)\pi$ is a minimal hermitian idempotent. Part (ii) of Lemma 4.4 shows us that $\ker L^1(G)\pi = \ker L^1(G)\rho$ for $\pi, \rho \in \hat{G}$ implies that $\pi$ and $\rho$ are unitarily equivalent. In particular $G$ is a type I group. Furthermore the natural map $\Psi : \text{Prim} \xi^*(G) \longrightarrow \text{Prim}_\rho L^1(G)$ is injective, which is necessary for $G$ to be primitive
$^*$-regular by Lemma 1.3.

If $E$ is a simple $\mathcal{B}$-module, then there exists a complete norm on $E$ such that $|a \cdot \xi| \leq |a| |\xi|$ for $a \in \mathcal{B}$ and $\xi \in E$: Recall that $E$ is algebraically isomorphic to $\mathcal{B}/L$ for some maximal modular left ideal $L$ which is closed in the Banach algebra $\mathcal{B}$. The quotient norm of $E \cong \mathcal{B}/L$ has the desired property. In particular we see that primitive ideals $P = \text{Ann}_\mathcal{B}(E)$ are closed. Furthermore primitive ideals are prime. Hence the set $\text{Prim} \mathcal{B}$ of all primitive ideals of $\mathcal{B}$ can be endowed with the Jacobson (hull-kernel) topology.

In the sequel we work with hermitian idempotents in the adjoint algebra $\mathcal{B}^b$ of $\mathcal{B}$, compare [16], which is also known as the multiplier or double centralizer algebra of $\mathcal{B}$.

Proposition 4.5. Let $\mathcal{B}$ be a Banach $^*$-algebra and $q \in \mathcal{B}^b$ a hermitian idempotent.

1. $q\mathcal{B}q$ is a closed $^*$-subalgebra of $\mathcal{B}$.

2. If $E$ is a simple $\mathcal{B}$-module, then there exists a unique (simple) $\mathcal{B}^b$-module structure on $E$ such that $M \cdot (a \cdot \xi) = (Ma) \cdot \xi$ for all $M \in \mathcal{B}^b$, $a \in \mathcal{B}$, and $\xi \in E$.

3. If $E$ is a simple $\mathcal{B}$-module such that $q \cdot E \neq 0$, then $q \cdot E$ is a simple $q\mathcal{B}q$-module with annihilator $\text{Ann}_{\mathcal{B}^b}(q \cdot E) = q\mathcal{B}q \cap \text{Ann}_\mathcal{B}(E) = q\text{Ann}_\mathcal{B}(E)q$.

4. The assignment $[E] \mapsto [q \cdot E]$ gives a bijection from the set of isomorphism classes of simple $\mathcal{B}$-modules $E$ such that $q \cdot E \neq 0$ onto the set of isomorphism classes of simple $q\mathcal{B}q$-modules.

5. Further $P \mapsto q\mathcal{B}q \cap P$ is a homeomorphism from the open subset $\text{Prim} \mathcal{B} \setminus h(\mathcal{B}q\mathcal{B})$ onto $\text{Prim}(q\mathcal{B}q)$ w. r. t. the Jacobson topology.

Proof. A proof of parts 1. to 4. of this proposition can also be found in [20].

1. Clearly $q\mathcal{B}q$ is a $^*$-subalgebra of $\mathcal{B}$ because $q$ is hermitian. The map $a \mapsto qaq$ is a continuous, linear projection. Its image $q\mathcal{B}q$ is closed.

2. Recall that $\mathcal{B}$ is an ideal of $\mathcal{B}^b$. Let $E$ be a simple $\mathcal{B}$-module. If $a \cdot \xi = 0$, then $\mathcal{B} \cdot (Ma) \cdot \xi = (BM) \cdot (a \cdot \xi) = 0$ which implies $(Ma) \cdot \xi = 0$. Thus $M \cdot (a \cdot \xi) = (Ma) \cdot \xi$ defines a $\mathcal{B}^b$-module structure on $E$. The rest is obvious.

3. Let $E$ be a simple $\mathcal{B}$-module. Clearly $q \cdot E$ is a $q\mathcal{B}q$-module. If $0 \neq \xi \in q \cdot E$, then $(q\mathcal{B}q) \cdot \xi = q\mathcal{B} \cdot \xi = q \cdot E$. Thus $q \cdot E$ is simple. The equality for its annihilator is clear.

4. Since any simple $\mathcal{B}$-module is isomorphic to one of the form $\mathcal{B}/L$, $L$ a maximal left ideal of $\mathcal{B}$, the isomorphism classes of simple $\mathcal{B}$-modules form a set. Note that any $\mathcal{B}$-linear map is also $\mathcal{B}^b$-linear.

The map $\alpha([E]) = [q \cdot E]$ is well-defined because any $\mathcal{B}$-linear isomorphism $\varphi$
from $E_1$ onto $E_2$ restricts to a $qBq$-linear isomorphism $\varphi'$ from $q \cdot E_1$ onto $q \cdot E_2$. Further $\alpha$ is injective because any $qBq$-linear isomorphism $\varphi' : q \cdot E_1 \to q \cdot E_2$ extends to a $B$-linear isomorphism $\varphi : E_1 \to E_2$: To see this, choose a non-zero $\xi \in q \cdot E$ and define $\varphi(a \cdot \xi) = a \cdot \varphi'(\xi)$. Finally, it remains to verify that $\alpha$ is surjective: Let $E'$ be a simple $qBq$-module. Since $qBq \subset (qBq)^b$, we can define $E_0 = B \otimes_{qBq} E' = Bq \otimes_{qBq} E'$. Observe that $q \cdot E_0 = q \otimes_{qBq} E' \cong E'$. By Zorn’s Lemma there exists a maximal $B$-invariant subspace $U$ of $E_0$ such that $U \cap q \cdot E_0 = \{0\}$. Put $E = E_0/U$. Clearly $q \cdot E \cong E'$. We claim that $E$ is simple: If $\eta \notin U$, then the $B$-invariant subspace $\bar{U} = B \cdot \eta + U$ satisfies $\bar{U} \cap q \cdot E_0 \neq \{0\}$ and hence $q \cdot \bar{U} \neq 0$. This implies $qBq \cdot \bar{U} = q \otimes_{qBq} E'$ and $BqBq \cdot \bar{U} = B \otimes_{qBq} E' = E_0$.

5. Part 3 implies $\beta(P) = qBq \cap P \in \text{Prim}(qBq)$ for all $P \in \text{Prim}(B) \setminus h(BqB)$. It follows from 4. that any simple $qBq$-module is isomorphic to one of the form $q \cdot E$, $E$ a simple $B$-module. Hence $\beta$ is surjective. We will resort to the following preliminary remark: If $P \in \text{Prim}(BqB)$ and $I$ is an ideal of $B$, then $I \subset P$ if and only if $qIQ \subset qPq$. The only-if part is obvious. Suppose $I \subsetneq P$. Choose a simple $B$-module $E$ such that $q \cdot E \neq 0$ and $P = \text{Ann}(E)$. If $a \in I$ and $a \notin P$, then $qBaq \subset qIQ$ and $qBaq \cdot E = q \cdot E \neq 0$ which proves $qIQ \subset qPq$.

In particular the preceding remark shows that $\beta$ is injective. Furthermore it is easy to see that $\beta$ is continuous: If $A' \subset \text{Prim}(qBq)$ is closed and $P \in \beta^{-1}(A')^{-}$, then $P \supsetneq \cap\{Q : qQq \in A'\}$ and hence $qPq \supseteq Q'$ for all $Q' \in A$. This shows $qPq \in A' = A'$ and thus $P \in \beta^{-1}(A')$. Finally we prove that $\beta$ is a closed map: Suppose that $A$ is a closed subset and $P$ an element of $\text{Prim} B \setminus h(BqB)$ such that $qPq \in \beta(A)^-$ which means $qPq \supset \cap\{qQq : Q \in A\} \supset qk(A)q$. Now the preliminary remark implies $P \supset k(A)$ and thus $P \in A = A$ and $\beta(P) \in \beta(A)$. This finishes our proof.

Subalgebras of the form $qBq$ for hermitian idempotents $q \in B^b$ are called corners.

The representation theory of exponential Lie groups is dominated by the fact that certain subquotients $q \ast (L^1(G)/I) \ast q$ of the group algebra turn out to be isomorphic to (twisted) weighted convolution algebras on abelian groups.

In this context the smooth terminology of twisted covariance algebras ($L^1$-version) is profitable, compare [13], [30]. By definition a twisted covariance system $(G, A, \tau)$ consists of (1) a locally compact group $G$ acting strongly continuously on a Banach $*$-algebra $A$ by isometric $*$-isomorphisms and (2) a twist $\tau$ defined on a closed normal subgroup $H$ of $G$ (i.e. a strongly continuous group homomorphism of $H$ into the group of units of the adjoint algebra $A^0$ of $A$) such that $\tau(h^x) = \tau(h)^x$ and $a^h = \tau(h)^a \tau(h)$ for all $x \in G, h \in H$, and $a \in A$. Let $C_0(G, A, \tau)$ denote the space of all continuous functions $f : G \to A$ such that $f(xh) = \tau(h)^x f(x)$ for all $x \in G, h \in H$ and such that $f$ has compact support modulo $H$. The closure $L^1(G, A, \tau)$ of $C_0(G, A, \tau)$ with respect to the norm $|f|_1 = \int_{G/H} |f(x)| \, dx$ is a Banach $*$-algebra with convolution and
involution given by
\[(f * g)(x) = \int_{G/H} f(xy)^{y^{-1}} g(y^{-1}) \, dy, \quad f^*(x) = \Delta_{G/H}(x^{-1}) (f(x^{-1})^*)^x.\]

A covariance pair \((\pi, \gamma)\) is a unitary representation \(\pi\) of \(G\) and a \(*\)-representation \(\gamma\) of \(A\) in the same Hilbert space \(\mathcal{H}\) such that \(\gamma(a^x) = \pi(x)^* \gamma(a) \pi(x)\) and \(\gamma(\tau(h)) = \pi(h)\). It is well-known that covariance pairs \((\pi, \gamma)\) correspond to \(*\)-representations of the twisted covariance algebra \(L^1(G, A, \tau)\).

**Definition 4.6.** Let \((G, A, \tau)\) be a twisted covariance system. A family \(\{A_x : x \in G\}\) of closed subspaces of \(A\) is said to be compatible with \((G, A, \tau)\) if \(\tau(h)^* A_x = A_{xh}, (A_{xy})^{y^{-1}} A_y^{-1} \subset A_x\), and \(((A_{x^{-1}})^*)^x = A_x\) for all \(x, y \in G\) and \(h \in H\).

If \(\{A_x : x \in G\}\) is compatible, then \(C_0(G, A_x, \tau) = \{f \in C_0(G, A, \tau) : f(x) \in A_x\}\) defines a subalgebra of \(C_0(G, A, \tau)\). This bears a meaning only if the \(A_x\) are chosen continuously so that \(C_0(G, A_x, \tau)\) and hence its closure \(L^1(G, A_x, \tau)\) are non-zero. One might think of \(\{A_x : x \in G\}\) as a ‘bundle’ over \(G\) and ask for trivializations.

**Definition 4.7.** Let \((G, A, \tau)\) be a twisted covariance system and \(\{A_x : x \in G\}\) a compatible family of one-dimensional subspaces of \(A\). We say that a continuous function \(v : G \to A\) is a trivialization for \(\{A_x : x \in G\}\) if \(v(x) \in A_x, |v(x)| \geq 1, v(xh) = \tau(h)^* v(x), v(xy)^{y^{-1}} v(y^{-1}) = v(x)\), and \((v(x^{-1})^*)^x = v(x)\) for \(x, y \in G, h \in H\).

**Proposition 4.8.** Let \((G, A, \tau)\) be a twisted covariance system. If \(v\) is a trivialization of the compatible family \(\{A_x : x \in G\}\) of one-dimensional subspaces of \(A\), then the subalgebra \(L^1(G, A_x, \tau)\) is isomorphic to the Beurling algebra \(L^1(G/H, w)\) given by the symmetric weight function \(w(x) = |v(x)|\).

**Proof.** One checks easily that \(\Phi(b)(x) = b(\hat{x}) v(x)\) defines an isometric isomorphism from \(L^1(G/H, w)\) onto \(L^1(G, A_x, \tau)\).

Let \(q \in A \subset L^1(G, A, \tau)\) be a hermitian idempotent. Since \((q * f * q)(x) = q^x f(x) q\) for all \(f \in L^1(G, A, \tau)\), it follows \(q * L^1(G, A, \tau) * q = L^1(G, q^2 A q, \tau)\). In Theorem 4.9 we treat the case where \(q\) is minimal and the \(q^2 A q\) are one-dimensional.

The following theorem is due to Poguntke, see part (4) and (5) of the proof of the main theorem in [25]. The idea goes back to Theorem 5 of Leptin and Poguntke in [20].

**Theorem 4.9.** Let \((\pi, \gamma)\) be a covariance pair of the twisted covariance system \((G, A, \tau)\) such that \(\gamma\) is irreducible and faithful. Suppose that there exists a minimal hermitian idempotent \(q \in A\). Then the corner \(q * L^1(G, A, \tau) * q = L^1(G, q^2 A q, \tau)\) is isometrically isomorphic to a weighted Beurling algebra \(L^1(G/H, w)\) where \(w\) is a symmetric weight function on \(G/H\).

**Proof.** By Lemma 4.4(i) there exists a unit vector \(\lambda \in \mathcal{H}\) such that \(\gamma(q) \xi = \langle \xi, \lambda \rangle \lambda\). Now \(\gamma(q^2 a q) = \pi(x)^* \gamma(q) \pi(x) \gamma(a) \gamma(q)\) implies
\[\gamma(q^2 a q) \xi = \langle \pi(x) \gamma(a) \lambda, \lambda \rangle \langle \xi, \lambda \rangle \pi(x)^{-1} \lambda.\]
For every $x \in G$ there exists some $a \in A$ such that $\langle \pi(x)\gamma(a)\lambda, \lambda \rangle$ is non-zero because $\gamma$ is irreducible. This shows $\gamma(q^*Aq) = \mathbb{C}\pi(x)^{-1}\gamma(q)$ so that $q^*Aq$ is one-dimensional. There is a unique element $v(x) \in A$ such that $\gamma(v(x))\xi = \langle \xi, \lambda \rangle\pi(x)^{-1}\lambda$. Clearly $v : G \to A$ is continuous and $|v(x)| \geq |\gamma(v(x))| = 1$. Further one computes

$$
\gamma(v(xh)) = \pi(h)^*\gamma(v(x)) = \tau(h)^*\gamma(v(x))
$$

$$
\gamma(v(x)) = \gamma \left( (v(xy)^{y^{-1}}v(y^{-1}) \right)
$$

$$
\gamma(v(x)) = \gamma((v(x^{-1})^*x)
$$

which proves that $v$ is a trivialization for $\{q^*Aq : x \in G\}$ because $\gamma$ is faithful. Now Proposition 4.8 gives the desired result. \hfill \Box

Our aim is to apply Theorem 4.9 to certain quotients of group algebras: Let $H$ be a closed normal subgroup of $G$. It is known that $L^1(G)$ is isomorphic to the twisted covariance algebra $L^1(G, L^1(H), \tau)$ with $G$-action $a^* (h) = \delta_H(x^{-1})a(hx^{-1})$ and twist $\tau(k)a(h) = a(k^{-1}h)$, compare the corollary to Proposition 1 in [13]. Suppose that $\gamma$ is an irreducible representation of $H$. The crucial assumption in Theorem 4.9 is that $\gamma$ can be completed to a covariance pair $(\pi, \gamma)$ of $(G, L^1(H), \tau)$, or equivalently, that it can be extended to a representation $\pi$ of $G$, which is only possible if $x^*\gamma$ is unitarily equivalent to $\gamma$ for all $x \in G$. If such a $\pi$ exists, then the ideal $I' = \ker L^1(H)\gamma$ is $G$- and $\tau$-invariant so that the covariance algebra $L^1(G, L^1(H)/I', \hat{\tau})$ with induced $G$-action and twist is well-defined. This algebra is isomorphic to the quotient $L^1(G)/I$ where $I = \text{ind}^G_H(I')$. To apply Theorem 4.9 it remains to find minimal hermitian idempotents in $L^1(H)/I'$.

The existence of an extension $\pi$ of $\gamma$ is guaranteed under the assumptions of

Proposition 4.10. Let $G$ be an exponential Lie group, $f \in \mathfrak{g}^*$, and $\pi = K(f) \in \hat{G}$. Suppose that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_f + \mathfrak{h}$. Let $f_0 = f|\mathfrak{h}$ and $\gamma = \mathcal{K}(f_0)$. Then $\pi|_H$ is unitarily equivalent to $\gamma$. This means that $\pi$ yields an extension of $\gamma$.

Proof. Recall that there exists a $\mathfrak{g}_f$-invariant Pukanszky polarization $\mathfrak{p}_0 \subset \mathfrak{h}$ at $f_0 \in \mathfrak{h}^*$ because $\mathfrak{g}$ is exponential, see §4, Chapter I of [19] and Chapter 5 of [1]. We shall verify that $\mathfrak{p} = \mathfrak{g}_f + \mathfrak{p}_0 \subset \mathfrak{g}$ defines a Pukanszky polarization at $f \in \mathfrak{g}^*$: Clearly $[\mathfrak{p}, \mathfrak{p}] \subset [\mathfrak{g}_f, \mathfrak{g}_f] + [\mathfrak{g}_f, \mathfrak{p}_0] + [\mathfrak{p}_0, \mathfrak{p}_0] \subset \mathfrak{p} \cap \ker f$. Note that $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{h}$ and $h_{f_0} = \mathfrak{g}_f \cap \mathfrak{h}$. Using the canonical isomorphisms $\mathfrak{h}/h_{f_0} \cong \mathfrak{g}/\mathfrak{g}_f$ and $\mathfrak{p}_0/h_{f_0} \cong \mathfrak{p}/\mathfrak{g}_f$ we conclude that $\dim \mathfrak{g}/\mathfrak{g}_f = \frac{1}{2} \dim \mathfrak{p}/\mathfrak{g}_f$. It remains to prove that $\text{Ad}^* (P)f = f + \mathfrak{p}^\perp$. If $h \in \mathfrak{p}^\perp$, then $h_0 = h|_{\mathfrak{h}} \in \mathfrak{p}_0^\perp \subset \mathfrak{h}^*$. Since $\mathfrak{p}_0$ is a Pukanszky polarization at $f_0$, there exists some $x \in P_0$ such that $\text{Ad}^* (x)f_0 = f_0 + h_0$. This implies $\text{Ad}^* (x)f = f + h$ because $\mathfrak{g} = \mathfrak{g}_f + \mathfrak{h}$. Thus $\text{Ad}^* (P)f = f + \mathfrak{p}^\perp$.

Fix a relatively $G$-invariant measure on $G/P$. There exists a unique relatively $H$-invariant measure on $H/P_0$ such that the canonical $H$-equivariant diffeomorphism $H/P_0 \to G/P$ is measure-preserving. The modular functions of these measures satisfy
$\Delta_{G,P}|H = \Delta_{H,R}$. Now it follows that $\varphi \mapsto \varphi|H$ defines a unitary isomorphism from $L^2(G, \chi_f)$ onto $L^2(H, \chi_{f_0})$ which intertwines $\pi|H$ and $\gamma$. This completes the proof. \qed

The next theorem results from the achievements of Poguntke in [26] concerning the parametrization of simple $L^1(G)$-modules. In [21] Ludwig and Molitor-Braun gave a simplified proof of Theorem 4.11 which in particular avoids projective representations. The decisive idea of Ludwig and Molitor-Braun may be recapitulated as follows: If $\mathcal{H}/\mathcal{N}$ is chosen to be a vector space complement to $\mathcal{M}/\mathcal{N}$ instead of $\mathcal{K}/\mathcal{N}$ as in [26], then one ends up directly with a commutative subquotient.

Recall that any simple $L^1(G)$-module can be regarded as an $L^1(G)^b$-module. In particular, if $\mathcal{N}$ is a closed subgroup of $G$, then $\mathcal{E}$ becomes an $L^1(\mathcal{N})$-module so that $\text{Ann}_{L^1(\mathcal{N})}(\mathcal{E})$ is defined.

Let $(G, \mathcal{A})$ be a covariance system. A $G$-invariant ideal $J$ of $\mathcal{A}$ is called $G$-prime if $J_1 J_2 \subset J$ for $G$-invariant ideals $J_1$, $J_2$ of $\mathcal{A}$ implies $J_1 \subset J$ or $J_2 \subset J$. If $\mathcal{N}$ is a closed normal subgroup of $G$, then one has the covariance system $(G, L^1(\mathcal{N}))$ with $G$-action $a^*(n) = \delta_N(x^{-1})a(n^{-1})x.$

**Theorem 4.11.** Let $\mathcal{N}$ be a closed, connected, coabelian, nilpotent subgroup of the exponential Lie group $G$.

1. If $\mathcal{E}$ is a simple $L^1(G)$-module, then $\mathcal{J} = \text{Ann}_{L^1(\mathcal{N})}(\mathcal{E})$ is $G$-prime.

2. Conversely let $J$ be a $G$-prime ideal of $L^1(\mathcal{N})$. Define $I = \text{ind}_{\mathcal{N}}^G(J)$. The simple $L^1(G)$-modules $\mathcal{E}$ such that $J \subset \text{Ann}_{L^1(\mathcal{N})}(\mathcal{E})$ are in a canonical bijection with the simple modules of $\mathcal{B} = L^1(G)/I$. Moreover, there exist hermitian idempotents $q \in \mathcal{B}^b$ such that the corner $q \ast \mathcal{B} \ast q$ is commutative and such that $q \ast \mathcal{E} \neq 0$ exactly for those simple $\mathcal{B}$-modules $\mathcal{E}$ with $J = \text{Ann}_{L^1(\mathcal{N})}(\mathcal{E})$.

**Proof.**

1. Recall that $\lambda(x)f(y) = f(x^{-1}y)$ defines a group homomorphism from $G$ into the unitary group of $L^1(G)^b$. Since $a^*\xi = \lambda(x^{-1})(a(\lambda(x)\xi))$ for $a \in L^1(\mathcal{N}), x \in G$, and $\xi \in \mathcal{E}$, it follows that $J$ is $G$-invariant. Now let $J_1$, $J_2$ be $G$-invariant ideals of $L^1(\mathcal{N})$ such that $J_1 \ast J_2 \subset J = \text{Ann}_{L^1(\mathcal{N})}(\mathcal{E})$. Then

$$\text{ind}_{\mathcal{N}}^G(J_1) \ast \text{ind}_{\mathcal{N}}^G(J_2) \subset \left( L^1(G) \ast J_1 \ast L^1(G) \ast J_2 \ast L^1(G) \right)$$

$$\subset \left( L^1(G) \ast J_1 \ast J_2 \ast L^1(G) \right) \subset \text{Ann}_{L^1(G)}(\mathcal{E}).$$

The first inclusion is obvious and the second one results from Lemma 2.11. For the third one we use the fact that $\text{Ann}_{L^1(G)}(\mathcal{E})$ is closed. Since this ideal is prime, it follows $\text{ind}_{\mathcal{N}}^G(J_k) \subset \text{Ann}_{L^1(G)}(\mathcal{E})$ for $k = 1$ or 2. Finally we obtain $J_k \subset \text{Ann}_{L^1(\mathcal{N})}(\mathcal{E})$ because $\mathcal{E}$ is a simple $L^1(G)$-module.
2. Let $J$ be a $G$-prime ideal of $L^1(N)$ and $I = \text{ind}^G_N(J)$. In order to prove the first assertion of 2, it suffices to verify that $J \subset \text{Ann}_{L^1(N)}(E)$ if and only if $I \subset \text{Ann}_{L^1(G)}(E)$. The only-if part is obvious. Suppose that $I \subset \text{Ann}_{L^1(G)}(E)$. Then $L^1(G)\cdot(J\cdot E) \subset \text{ind}^G_N(J)\cdot E = I\cdot E = 0$ implies $J\cdot E = 0$ because $E$ is simple. This means $J \subset \text{Ann}_{L^1(N)}(E)$.

Next we prove the existence of appropriate hermitian idempotents in the adjoint algebra of $B = L^1(G)/I$: Generalizing a theorem of Poguntke in [26], Ludwig and Molitor-Braun proved in Theorem 1.1.6 of [21] that there exists a unique orbit $G\cdot\sigma$ in $\tilde{N}$ such that $J = k(G\cdot\sigma)$. Since the Kirillov map of $N$ is bijective, we can choose $l \in n^*$ such that $K(l) = \sigma$ and $f \in g^*$ such that $f|n = l$. We stress that the definition of the stabilizer $m = g_f + n$ depends only on the orbit $\text{Ad}^*(G)l$, i.e., on the $G$-prime ideal $J$. As usual $M$ denotes the closed, connected subgroup of $G$ with Lie algebra $m$. In addition we fix a closed, connected subgroup $H$ of $G$ containing $N$ such that $H/N$ is complementary to $M/N$ in the vector space $G/N$. In particular $G = G_fH$.

The ideal $I' = \text{ind}^H_f(J)$ is invariant under the $G$-action $b^x(h) = \delta_H(x^{-1})b(hx^{-1})$ and the twist $\tau(k)b(h) = b(k^{-1}h)$ in $L^1(H)$. Since $I = \text{ind}^H_f(I')$, the quotient $B = L^1(G)/I$ can be identified with $L^1(G,L^1(H)/I',\hat{\tau})$.

Let $f_0 = f|\mathfrak{h} \in \mathfrak{h}^*$ and $\gamma = K(f_0) \in \widehat{H}$. Since $\mathfrak{g} = \mathfrak{g}_f + \mathfrak{h}$, Proposition 4.10 implies that $\pi = K(f)$ furnishes an extension of $\gamma$. Note that $\mathfrak{h}f_0 = \mathfrak{g}_f \cap \mathfrak{h} \subset \mathfrak{n}$. This shows that $\text{Ad}^*(H)f_0$ is saturated over $\mathfrak{n}$, i.e., $\text{Ad}^*(H)f_0 = f_0 + \mathfrak{n}^\perp$. In particular $\ker_{L^1(H)}\gamma$ is invariant under multiplication by characters of $H/N$, and hence $C_\infty(H/N)$-invariant. Now Theorem 4.10 implies $\ker_{L^1(H)}\gamma = \text{ind}^H_f(J) = I'$ because $\ker_{L^1(N)}\gamma = k(G\cdot\sigma) = J$ by Lemma 4.1. We have shown that $\gamma$ yields a faithful irreducible representation of $A = L^1(H)/I'$ which admits an extension $\pi$.

Let us fix an arbitrary minimal hermitian idempotent $q \in A = L^1(H)/I'$. Since $H$ is an exponential Lie group, the existence such idempotents is guaranteed by Poguntke’s results in [25]. Finally Theorem 4.9 shows that the corner $q\ast B \ast q$ is commutative as it is isomorphic to a weighted Beurling algebra $L^1(G/H,w)$ on the commutative group $G/H$.

Let $E$ be a simple $L^1(G)$-module such that $J \subset \text{Ann}_{L^1(N)}(E)$. It remains to be shown that $q\cdot E \neq 0$ if and only if $J = \text{Ann}_{L^1(N)}(E)$. The subsequent proof of the if-part is from [26]. We begin with a preliminary remark: Let $Q_0'$ denote the ideal of all $b \in L^1(H)$ such that $\gamma(b)$ has finite rank. Clearly $I' \subset Q_0'$ and $q \notin Q_0' \setminus I'$. Let $\mathfrak{H}$ denote the representation space of $\gamma$, and $F$ the simple module associated to $\gamma$ in the sense of Lemma 4.3. It is known that $\gamma(Q_0')$ is equal to the algebra of all finite rank operators $A$ of $\mathfrak{H}$ such that $A(\mathfrak{H}) \subset F$ and $A^*(\mathfrak{H}) \subset F$, compare Théorème 2. of Dixmier in [7]. From this we deduce $(Q_0'/I') \ast b \ast (Q_0'/I') = Q_0'/I'$.
for all $b \in Q_0' \setminus I'$. In particular we see that either $Q_0' \subset \text{Ann}_{L^1(H)}(E)$ or $b \cdot E \neq 0$ for all $b \in Q_0' \setminus I'$.

Suppose that $J = \text{Ann}_{L^1(N)}(E)$. Since $\text{Ad}^*(H)f_0$ is saturated over $n$, it follows that $\gamma \otimes \alpha$ is unitarily equivalent to $\gamma$ for all $\alpha \in (H/N)^\sim$. Thus $Q_0'$ and hence its closure $Q'$ are $(H/N)^\sim$-invariant. By Theorem 2.10 there exists an $H$-invariant ideal $R$ of $L^1(N)$ such that $Q' = \text{ind}_{N}^{H}(R)$. Note that $R$ is not contained in $J$. Thus $Q = \text{ind}_{N}^{H}(R) = \text{ind}_{H}^{N}(Q')$ is not contained in $\text{Ann}_{L^1(G)}(E)$ because $E$ is simple. Consequently $Q_0'$ is not contained in $\text{Ann}_{L^1(H)}(E)$ so that $q \cdot E \neq 0$ by the preliminary remark.

In order to prove the only-if-part we suppose $q \cdot E \neq 0$. The preceding remark implies $Q_0' \cap \text{Ann}_{L^1(H)}(E) \subset I'$. Now we conclude $Q' \cdot \text{Ann}_{L^1(H)}(E) \subset I'$. Since $I'$ is $G$-prime and $Q'$, $\text{Ann}_{L^1(H)}(E)$ are $G$-invariant ideals, we get $\text{Ann}_{L^1(H)}(E) \subset I'$ because $Q' \not\subset I'$. Let $a \in \text{Ann}_{L^1(N)}(E)$. Since $L^1(H) \ast a \ast L^1(H)$ is contained in $I' = \ker_{L^1(H)} \gamma$, it follows $a \in \ker_{L^1(N)} \gamma = k(G \cdot \sigma) = J$ because $\gamma$ is irreducible.

Let $J$ a given $G$-prime ideal of $L^1(N)$. In combination with part 4. of Proposition 4.5 the preceding theorem shows that the equivalence classes of all simple $L^1(G)$-modules $E$ with annihilator $\text{Ann}_{L^1(N)}(E) = J$ are in a one-to-one correspondence with the characters of the commutative Beurling algebra $q \ast (L^1(G)/I) \ast q \cong L^1(G/H, w)$.

Here we are content with this rough description and deliberately renounce more delicate questions such as obtaining estimates for the weight $w$, which can be found in [26].

**Corollary 4.12.** If $E$, $F$ are simple $L^1(G)$-modules with $\text{Ann}_{L^1(G)}(E) \subset \text{Ann}_{L^1(G)}(F)$ and $J = \text{Ann}_{L^1(N)}(E) = \text{Ann}_{L^1(N)}(F)$, then $E$ and $F$ are isomorphic.

**Proof.** Note that $E$ and $F$ can be regarded as $\mathcal{B}$-modules where $\mathcal{B} = L^1(G) / \text{ind}_{N}^{H}(J)$. Let $q \in \mathcal{B}$ be a hermitian idempotent as in part 2. of Theorem 1.11. By definition $q \cdot E$ and $q \cdot F$ are non-zero. Hence they are simple modules over the commutative $\ast$-algebra $q \ast \mathcal{B} \ast q$ with annihilators $\text{Ann}_{q \ast \mathcal{B} \ast q}(q \cdot E) \subset \text{Ann}_{q \ast \mathcal{B} \ast q}(q \cdot F)$, compare Proposition 1.5. It results from Schur’s Lemma that $q \cdot E$ and $q \cdot F$ are one-dimensional, have the same annihilator, and are thus isomorphic. Finally Proposition 1.5.4. shows that $E$ and $F$ are isomorphic as $\mathcal{B}$-modules, and also as $L^1(G)$-modules.

**Corollary 4.13.** If $\pi, \rho \in \hat{G}$ are irreducible such that $\ker_{L^1(G)} \pi \subset \ker_{L^1(G)} \rho$ and $\ker_{L^1(N)} \pi = \ker_{L^1(N)} \rho$, then $\pi$ and $\rho$ are unitarily equivalent.

**Proof.** Let $E$, $F$ denote the simple $L^1(G)$-modules associated to $\pi$, $\rho$ respectively in the sense of Lemma 4.3 (ii). By definition $\text{Ann}_{L^1(G)}(E) \subset \text{Ann}_{L^1(G)}(F)$ and $J = \text{Ann}_{L^1(N)}(E) = \text{Ann}_{L^1(N)}(F)$. Thus $E$ and $F$ are isomorphic by Corollary 4.12. This means $\ker_{L^1(G)} \pi = \text{Ann}_{L^1(G)}(E) = \text{Ann}_{L^1(G)}(F) = \ker_{L^1(G)} \rho$. Finally $\pi$ and $\rho$ are unitarily equivalent by Lemma 4.4 (ii).
These preparations make it easy to prove the main result of this section.

**Proposition 4.14.** Let $G$ be an exponential Lie group and $N$ a closed, connected, coabelian, nilpotent subgroup. Let $\pi \in \hat{G}$ and $G\sigma$ be the unique $G$-orbit in $\hat{N}$ such that $k(G\sigma) = \ker C^*(\pi)$. If $G\sigma$ is closed in $\hat{N}$, then $\ker C^*(\pi)$ is $L^1(G)$-determined.

**Proof.** Let $\rho$ be in $\hat{G}$ such that $\ker L^1(G) \pi \subset \ker L^1(G) \rho$. Restricting to $N$ we obtain $\ker L^1(N) \pi \subset \ker L^1(N) \rho$. Since $N$ is $*$-regular as a connected nilpotent Lie group, it follows $k(G\sigma) = \ker C^*(N) \pi \subset \ker C^*(N) \rho$. This yields $\ker C^*(N) \pi = \ker C^*(N) \rho$ because the orbit $G\cdot \sigma$ is closed. Finally Corollary 4.13 implies that $\pi$ and $\rho$ are unitarily equivalent so that in particular $\ker C^*(G) \pi = \ker C^*(G) \rho$. \hfill $\square$

However, the preceding results are limited to the case when $G\sigma$ is closed in $\hat{N}$.

**Remark 4.15.** Let $N$ be a coabelian, nilpotent subgroup of $G$ and $\pi \in \hat{G}$ such that $G\sigma$ is not closed in $\hat{N}$. To prove that $\ker C^*(G) \pi$ is $L^1(G)$-determined, one must show that $\ker C^*(G) \pi \not\subset \ker C^*(G) \rho$ implies $\ker L^1(G) \pi \not\subset \ker L^1(G) \rho$ for all $\rho \in \hat{G}$. Note that $J = \ker L^1(N) \pi$ is $G$-prime and define $I = \mathrm{ind}^G_N(I)$. To avoid trivialities we can assume $\ker L^1(N) \pi \subset \ker L^1(N) \rho$ so that $\pi$ and $\rho$ factor to representations of $B = L^1(G)/I$. In addition we suppose that $\ker L^1(N) \pi \not\subset \ker L^1(N) \rho$. Such representations $\rho$ are likely to exist if $G\sigma$ is not closed. If $q \in B^k$ is a hermitian idempotent as in Theorem 4.11 then $\rho(q) = 0$. This means that restriction to the subquotient $qBq$ is not appropriate for proving $\ker L^1(G) \pi \not\subset \ker L^1(G) \rho$ in this case.

## 5 A strategy to prove primitive $*$-regularity

Let $G$ be an exponential Lie group and $\mathfrak{a}$ a coabelian nilpotent ideal of its Lie algebra $\mathfrak{g}$. In order to prove that $G$ is primitive $*$-regular, one must show that $\ker C^*(G) \pi$ is $L^1(G)$-determined for all $\pi \in \hat{G}$, i.e., according to Definition 4.1 one must prove that

$$\ker C^*(G) \pi \not\subset \ker C^*(G) \rho \quad \text{implies} \quad \ker L^1(G) \pi \not\subset \ker L^1(G) \rho$$

for all $\rho \in \hat{G}$. Let $f, g \in \mathfrak{g}^*$ such that $\pi = \mathcal{K}(f)$ and $\rho = \mathcal{K}(g)$. Since the Kirillov map of $G$ is a homeomorphism with respect to the Jacobson topology on the primitive ideal space $\mathrm{Prim} C^*(G)$ and the quotient topology on the coadjoint orbit space $\mathfrak{g}^*/\mathrm{Ad}^*(G)$, the relation for the $C^*$-kernels is equivalent to $\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^{-}$. From the preceding subsections we extract the following observations:

1. Let $\mathfrak{a}$ be a non-trivial ideal of $\mathfrak{g}$ such that $f = 0$ on $\mathfrak{a}$. Let $A$ be the connected subgroup of $G$ with Lie algebra $\mathfrak{a}$. Since $\pi = 1$ on $A$, we can pass over to a representation $\hat{\pi}$ of the quotient $G = G/A$. It follows from Lemma 4.4 that $\ker C^*(G) \pi$ is $L^1(G)$-determined if and only if $\ker C^*(G) \hat{\pi}$ is $L^1(\hat{G})$-determined. Often $\hat{G}$ is known to be primitive $*$-regular by induction. If this is the case for all proper quotients $\hat{G}$ of $G$, then we can assume that $f$ is in general position, i.e., $f \neq 0$ on all non-trivial ideals $\mathfrak{a}$ of $\mathfrak{g}$. 
2. If the stabilizer $m = g_f + n$ is nilpotent, then $\ker_{C^* (G)} \pi$ is $L^1 (G)$-determined by Propositions 2.12 and 2.14 because $M$ is $*\text{-regular}$.

3. If $g = g_f + n$, then $\ker_{C^* (G)} \pi$ is $L^1 (G)$-determined by Proposition 4.14. Here and in the sequel $f$ denotes the restriction of $f$ to $n$.

4. If $\text{Ad}^* (G) g'$ is not contained in the closure of $\text{Ad}^* (G) f'$, then it follows $\ker_{C^* (N)} \pi \not\subset \ker_{C^* (N)} \rho$ because the Kirillov map is a homeomorphism. Since $N$ is $*\text{-regular}$, we obtain $\ker_{L^1 (N)} \pi \not\subset \ker_{L^1 (N)} \rho$ and hence $\ker_{L^1 (G)} \pi \not\subset \ker_{L^1 (G)} \rho$.

Lemma 5.1. If there exists a one-codimensional nilpotent ideal $n$ of $g$, then $G$ is primitive $*\text{-regular}$.

Proof. Let $f \in g^*$ be arbitrary and $\pi = K (f)$. The assumption $\dim g / n = 1$ implies that either $g = g_f + n$, or that $m = g_f + n = n$ is nilpotent. Clearly the preceding remarks show that $\ker_{C^* (G)} \pi$ is $L^1 (G)$-determined.

Definition 5.2. Let $f \in g^*$ be in general position. As before $r : g^* \to n^*$ is given by $r (g) = g' = g | n$. We define $\Omega$ as the $r$-preimage of the closure of $\text{Ad}^* (G) f'$ in $n^*$. Note that $\Omega$ is a closed subset of $g^*$ containing $\text{Ad}^* (G) f$ and that $g \in \Omega$ if and only if $g'$ is in the closure of $\text{Ad}^* (G) f'$. We say that $g$ is critical for the orbit $\text{Ad}^* (G) f$ if $g \in \Omega \setminus (\text{Ad}^* (G) f)^\ominus$. By Proposition 4.14 we can even assume $\text{Ad}^* (G) g' \neq \text{Ad}^* (G) f'$.

In order to prove the primitive $*\text{-regularity of } G$ it thus suffices to verify the following two assertions:

1. Every proper quotient $\hat{G}$ of $G$ is primitive $*\text{-regular}$.

2. If $f \in g^*$ is in general position such that the stabilizer $m = g_f + n$ is a proper, non-nilpotent ideal of $g$ and if $g \in g^*$ is critical for the orbit $\text{Ad}^* (G) f$, then it follows $\ker_{L^1 (G)} \pi \not\subset \ker_{L^1 (G)} \rho$.

Let $d_1, \ldots, d_m$ be a coexponential basis for $m$ in $g$. We obtain a diffeomorphism from $\mathbb{R}^m$ onto $G / M$ by composing the smooth map $E (s) = \exp (s_1 d_1) \cdots \exp (s_m d_m)$ with the quotient map $G \to G / M$. Define $\tilde{f} = f | m$, $\tilde{f}_s = \text{Ad}^* (E (s)) \tilde{f}$ in $m^*$, and $\tilde{\pi}_s = K (\tilde{f}_s)$ in $\tilde{M}$.

Two properties of $\pi$ and their counterpart in the Kirillov picture are worth mentioning. First $\pi | M$ is reducible. By Lemma 4.1 we know that $\pi | M$ is weakly equivalent to the subset $\{ \tilde{\pi}_s : s \in \mathbb{R}^m \}$ of $\tilde{M}$. In the orbit picture $\text{Ad}^* (G) \tilde{f}$ decomposes into the disjoint union of the orbits $\{ \text{Ad}^* (M) \tilde{f}_s : s \in \mathbb{R}^m \}$.

Secondly, $\ker_{C^* (G)} \pi$ is induced from $M$ by Proposition 2.14. Hence $\ker_{C^* (G)} \pi \subset \ker_{C^* (G)} \rho$ is equivalent to the corresponding inclusion in $C^* (M)$. The same holds true in $L^1 (M)$. In the Kirillov picture we have $\text{Ad}^* (G) \tilde{f} = \text{Ad}^* (G) f + m^\perp$ so that $g$ is in $(\text{Ad}^* (G) f)^\ominus$ if and only if $\tilde{g}$ is in the closure of $\text{Ad}^* (G) \tilde{f}$. 


In analogy to Definition 5.2 we define $\tilde{\Omega} \subset m^*$ and critical $\tilde{g}$ for the orbit $\text{Ad}^*(G)\tilde{f}$ in $m^*$. We say that $\tilde{f}$ is in general position if $f(a) \neq 0$ on any non-trivial ideal $a$ of $g$ such that $a \subset m$. Now it is easy to see that we can replace the second assertion by the following equivalent one:

3. Let $m$ be a proper, non-nilpotent ideal of $g$ such that $m \supset n$. If $\tilde{f} \in m^*$ is in general position such that $m = m_f + n$ and if $\tilde{g} \in m^*$ is critical for the orbit $\text{Ad}^*(G)\tilde{f}$, then the relation

$$\bigcap_{s \in \mathbb{R}^m} \ker L_1(M) \tilde{\pi}_s \not\subset \ker L_1(M) \tilde{\rho}$$

holds for the representations $\tilde{\pi}_s = K(\tilde{f}_s)$ and $\tilde{\rho} = K(\tilde{g})$.

In this situation producing functions $c \in L^1(M)$ such that $\pi_s(c) = 0$ for all $s$ and $\rho(c) \neq 0$ turns out to be a great challenge.

**Remark 5.4.** The stabilizer $m = g_f + n$ has a remarkable algebraic property. Note that the ideal $[m, 3n] = [m_f, 3n]$ is contained in $\ker f$. If in addition $f$ is in general position, then it follows $[m, 3n] = 0$ so that $3n \subset 3m$.

**Lemma 5.5.** If $g$ is an exponential Lie algebra such that $[g, g]$ is commutative, then $G$ is $*$-regular.

**Proof.** Let $f \in g^*$ be arbitrary. If $a$ denotes the largest ideal of $g$ contained in $\ker f$, then $\tilde{f}$ on $\hat{g} = g/a$ is in general position. By Remark 5.4 we obtain $\hat{n} = [\hat{g}, \hat{g}] = 3n \subset 3m$. Thus the quotient $\hat{m}$ of $m = g_f + n$ is 2-step nilpotent so that $g$ satisfies condition (R). Now Theorem 3.3 yields the assertion of this lemma. \qed

### 6 A non-$*$-regular example

Let $g$ be an exponential Lie algebra of dimension $\leq 5$. In view of Lemma 5.1 and 5.3 we assume that the nilradical $n$ (the maximal nilpotent ideal) of $g$ is not commutative and of dimension $\leq 3$, i.e., $n = \langle e_1, e_2, e_3 \rangle$ is a 3-dimensional Heisenberg algebra. Further we suppose that $f \in g^*$ is general position (so that $f(e_3) \neq 0$) and that the stabilizer $m = g_f + n$ is a proper, non-nilpotent ideal. These assumptions imply that $g$ has a basis $d, e_0, \ldots, e_3$ satisfying the commutator relations $[e_1, e_2] = e_3, [e_0, e_1] = -e_1, [e_0, e_2] = e_2, [d, e_2] = e_2, [d, e_3] = e_3$. The algebra $g$ and the stabilizer $m = \langle e_0, \ldots, e_3 \rangle$ are specified as $g = b_5$ and $m = g_{4,0}(0)$ in the (complete) list of all non-symmetric Lie algebras up to dimension 6 in [24], whereas symmetry is equivalent to $*$-regularity by Theorem 10 of [26].

We work in coordinates of the second kind w. r. t. the above Malcev basis, given by the diffeomorphism $\Phi(t, x) = \exp(te_0) \exp(x_1e_1) \exp(x_2e_2 + x_3e_3)$ from $\mathbb{R}^4$ onto $M$. 

Denote $f|\mathfrak{m}$ again by $f$ and let $f_s = \text{Ad}^*(\exp(sd))f$. By choosing an appropriate representative on the orbit $\text{Ad}^*(G)f$ we can achieve $f(e_3) = 1$ and $f(e_1) = f(e_2) = 0$. Now we compute

$$
\text{Ad}^* (\exp(sd) \Phi(t, x)) f(e_0) = f(e_0) - x_1x_2,
$$

$$(e_1) = e^t x_2,$$

$$(e_2) = -e^{-s} e^{-t} x_1,$$

$$(e_3) = e^{-s}.$$ 

These formulas for the coadjoint representation suggest to define the $\text{Ad}^*(M)$-invariant polynomial function

$$p = e_0 e_3 - e_1 e_2 - f(e_0)e_3$$

on $\mathfrak{m}^*$ such that $p(h) = 0$ for all $h \in X = \text{Ad}^*(G)f = \cup \{\text{Ad}^*(M)f_s : s \in \mathbb{R}\}$. Here $e_{\nu}$ is interpreted as the linear function $e_{\nu}(h) = h(e_{\nu})$ on $\mathfrak{m}^*$ and $f(e_0)$ is a constant. Note that $p$ is even $\text{Ad}^*(G)$-semi-invariant. The closure of the orbit $X = \text{Ad}^*(G)f$ in $\mathfrak{m}^*$ can be characterized by means of the $\text{Ad}^*(M)$-invariant polynomial $p$.

**Lemma 6.1.** Let $g \in \Omega \subset \mathfrak{m}^*$. Then $g \in \overline{X}$ if and only if $p(g) = 0$.

**Proof.** The only-if-part is obvious because $p(h) = 0$ for all $h \in X$. Let us prove the opposite direction. Let $g \in \Omega$ such that $p(g) = 0$. We must distinguish four cases. First we assume $g(e_3) \neq 0$. Since $g \in \Omega$, it follows $g(e_3) > 0$. Without loss of generality we can assume $g(e_3) = 1 = f(e_3)$ and $g(e_1) = g(e_2) = 0$. Now $p(g) = 0$ implies $g(e_0) = f(e_0)$ so that $g \in X$. Next we consider the case $g(e_3) = g(e_2) = 0$ and $g(e_1) \neq 0$. If we define $s_n = n$, $x_{n1} = (g(e_0) - f(e_0))/g(e_1)$, and $x_{n2} = g(e_1)$, then it follows $f_n \rightarrow g$ for

$$f_n = \text{Ad}^* (\exp(s_n d) \Phi(0, x_n)) f$$

in $X$ so that $g \in \overline{X}$. The third case is $g(e_3) = g(e_1) = 0$ and $g(e_2) \neq 0$. If we set $s_n = n$, $x_{n1} = -e^{2n}g(e_2)$, and $x_{n2} = -e^{-n}(f(e_0) - g(e_0))/g(e_2)$, then it follows $f_n \rightarrow g$. Finally we assume $g(e_{\nu}) = 0$ for $1 \leq \nu \leq 3$. In this case $s_n = n$, $x_{n1} = e^{n/2}$, and $x_{n2} = e^{-n/2}(f(e_0) - g(e_0))$ yields $f_n \rightarrow g$ so that $g \in \overline{X}$. 

The preceding lemma implies that the set of critical linear functionals is given by $\Omega \setminus \overline{X} = \{ g \in \mathfrak{m}^* : g(e_3) = 0 \text{ and } g(e_1)g(e_2) \neq 0 \}$. Let us compute the relevant unitary representations: Using $p = \langle e_0, e_2, e_3 \rangle$ as a Pukanszky polarization at $f_s \in \mathfrak{m}^*$ for all $s \in \mathbb{R}$, one computes that $\pi_s = K(f_s) = \text{ind}_0^M \chi_{f_s}$ in $L^2(\mathbb{R})$ is infinitesimally given by

$$d\pi_s(e_0) = f_0 + \xi D_\xi - i/2,$$

$$d\pi_s(e_1) = -D_\xi,$$

$$d\pi_s(e_2) = e^{-s} \xi,$$

$$d\pi_s(e_3) = e^{-s}.$$ 

Here $\check{e}_\nu = -ie_{\nu}$ is in the complexification $\mathfrak{m}_\mathbb{C}$ of $\mathfrak{m}$, $\xi$ is the multiplication operator and $D_\xi = -i\partial_\xi$ is the differential operator in $L^2(\mathbb{R})$. We observe that these equations
bear a striking resemblance to the formulas for $\text{Ad}^*(\Phi(t,x))f(e_\rho)$: simply substitute $e^{-t}x_1$ by $\xi$ and $e^t x_2$ by $D_\xi$. On the other hand, if $g \in \Omega \setminus \overline{X}$, then $n$ is a Pukanszky polarization at $g \in \mathfrak{m}^*$ and $\rho = \mathcal{K}(g) = \text{ind}_M^\Sigma \chi_g$ in $L^2(\mathbb{R})$ is given by

$$
\begin{align*}
  d\rho(\dot{e}_0) &= -D_\xi, \\
  d\rho(\dot{e}_1) &= e^\xi g_1, \\
  d\rho(\dot{e}_2) &= e^{-\xi} g_2, \\
  d\rho(\dot{e}_3) &= 0.
\end{align*}
$$

Symmetrization gives a linear isomorphism from the symmetric algebra $\mathcal{S}(\mathfrak{m}_\Sigma) = \mathcal{P}(\mathfrak{m}^*)$ onto the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_\Sigma)$ of $\mathfrak{m}_\Sigma$, which maps the subspace of $\text{Ad}(M)$-invariant polynomials onto the center $Z(\mathfrak{m}_\Sigma)$ of $\mathcal{U}(\mathfrak{m}_\Sigma)$, compare Chapter 3.3 of [9]. Note that $p \in \mathcal{P}(\mathfrak{m}^*)^{\text{Ad}(M)}$ corresponds to

$$
W = \dot{e}_3 \dot{e}_0 - \frac{1}{2}(\dot{e}_2 \dot{e}_1 + \dot{e}_1 \dot{e}_2) - f_0 \dot{e}_3 = \dot{e}_3 \dot{e}_0 - \dot{e}_2 \dot{e}_1 - (f_0 - \frac{i}{2})\dot{e}_3
$$

in $Z(\mathfrak{m}_\Sigma)$. One verifies easily that $d\tau(W) = p(h)$ holds for all $h \in \mathfrak{m}^*$ and $\tau = \mathcal{K}(h)$. For the Lie algebra $\mathfrak{m}$ under consideration the symmetrization map coincides with the so-called Duflo isomorphism so that $d\tau(W) = p(h)$ can also be seen as a consequence of Théorème 2 of [9].

Furthermore we recall that if $\lambda(m)a(y) = a(m^{-1}y)$ denotes the left regular representation of $M$ in $L^2(M)$, then

$$
d\lambda(X)a(y) = \left. \frac{d}{dt} \right|_{t=0} a(\exp(-tX)y)
$$

defines a representation of $\mathfrak{m}$ in $\mathcal{C}^\infty_0(M)$, which extends to $\mathcal{U}(\mathfrak{m}_\Sigma)$. Note that $\mathcal{U}(\mathfrak{m}_\Sigma)$ acts as an associative algebra of right invariant vector fields. Let us write $V \ast a = d\lambda(V)a$ for $V \in \mathcal{U}(\mathfrak{m}_\Sigma)$ and $a \in \mathcal{C}^\infty_0(M)$. It is known that $\tau(V \ast a) = d\tau(V)\tau(a)$ holds for all $V, a$ and all unitary representations $\tau$ of $M$.

**Lemma 6.2.** If $g \in \Omega \setminus \overline{X}$ and $\rho = \mathcal{K}(g)$, then $\bigcap_{x \in \mathbb{R}} \ker \text{L}^1(M) \pi_s \not\subset \ker \text{L}^1(M) \rho$. In particular $G$ is primitive *-regular.

**Proof.** Since $\mathcal{C}^\infty_0(M)$ is dense in $L^1(M)$, there exists a function $a \in \mathcal{C}^\infty_0(M)$ such that $\rho(a) \neq 0$. Now $b = W \ast a$ satisfies $\pi_s(b) = d\pi_s(W)\pi_s(a) = p(f)_s\pi_s(a) = 0$ for all $s \in \mathbb{R}$ and $\rho(b) = d\rho(W)\rho(a) = p(g)\rho(a) \neq 0$ because $g \not\in \overline{X}$. \hfill \Box

A priori this result does not seem unlikely because the nature of $X$ is essentially different from that of typical non-*-regular subsets of $\mathfrak{m}^*/\text{Ad}^*(M)$. In the preceding lemma $X/\text{Ad}^*(M)$ is a graph over $3\mathfrak{m}^*$ in the sense that the orbit $\text{Ad}^*(M)h$ is uniquely determined by $h \mid 3\mathfrak{m}$ for all $h \in X$. Whereas basic examples of non-*-regular subsets $X$ consist of linear functionals $h \in \mathfrak{m}^*$ over a common character $\zeta = h \mid 3\mathfrak{m}$ of the center such that the set of limit points of $X/\text{Ad}^*(M)$ in $\mathfrak{m}^*/\text{Ad}^*(M)$ is not empty.
Since $\mathfrak{g} = \mathfrak{b}_5$ is the only exponential Lie algebra in dimension $\leq 5$ such that there exist $f \in \mathfrak{g}^*$ in general position with non-nilpotent, proper stabilizer and critical functionals $g \in \mathfrak{g}^*$ w. r. t. the orbit $\text{Ad}^*(G)f$, it follows from Lemma 6.2 that all exponential Lie groups up to dimension 5 are primitive $*$-regular.

Note that in the particular case $\mathfrak{g} = \mathfrak{b}_5$ the relation $\cap_s \ker L_1(M) \pi_s \not\subset \ker L_1(M) \rho$ implies $\cap_s \ker U(m_G) \pi_s \not\subset \ker U(m_G) \rho$, but in general, as one might expect, the features of the universal enveloping algebra do not suffice for this purpose. However, we anticipate that $\text{Ad}(M)$-invariant polynomials $p$ corresponding to elements $W \in Z(m_G)$ will play an important role in further investigations of primitive $*$-regularity.

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