Towards a modeling class for port-Hamiltonian systems with time-delay

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Abstract—The framework of port-Hamiltonian (pH) systems is a powerful and broadly applicable modeling paradigm. In this paper, we extend the scope of pH systems to time-delay systems. Our definition of a delay pH system is motivated by investigating the Kalman-Yakubovich-Popov inequality on the corresponding infinite-dimensional operator equation. Moreover, we show that delay pH systems are passive and closed under interconnection. We describe an explicit way to construct a Lyapunov-Yakubovich-Popov functional and discuss implications for delayed feedback.

Index Terms—Lyapunov-Krasovskii functional, port-Hamiltonian system, time-delay, Kalman-Yakubovich-Popov inequality

I. INTRODUCTION

The port-Hamiltonian (pH) framework [9], [19] constitutes an innovative energy-based model paradigm that offers a systematic approach to the interaction of (physical) systems with each other and the environment via interconnection structures. The inherent structure of pH systems encodes, among other advantages, control theoretical concepts such as passivity and stability, and facilitates structure-preserving approximation schemes [3], [7]. It applies to a large range of different applications and was recently extended to descriptor systems; see [13] and the references therein. Nevertheless, a consistent generalization to infinite-dimensional systems is not yet available. In particular, extending the pH concept to delay differential-algebraic equations (DAEs) poses significant challenges with only limited contributions in the literature that primarily focus on the stability analysis of pH systems with delayed feedback, cf. [1], [10], [12], [16], [17], [21]. A notable exception is provided in [11], which, however, deals mainly with the solvability analysis. For an overview of DDEs and their applications, we refer to [8], [18].

In this paper, we study how the pH framework can be extended to linear time-invariant delay differential equations (DDEs) of the form

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau) + Bu(t), & t > 0, \\
y(t) &= C x(t), & t > 0, \\
x(t) &= \phi(t), & t \in [-\tau, 0],
\end{align*}
\]

where \( A_0, A_1 \in \mathbb{R}^{n \times n}, B, C^\top \in \mathbb{R}^{n \times m} \). In the non-delayed case, which is obtained if either \( \tau = 0 \) or \( A_1 = 0 \) in (1), and some further technical details, the system (1) has a pH representation if and only if (1) is passive [2], [4]. In this case, a solution of the associated Kalman-Yakubovich-Popov (KYP) inequality can then be used to construct the explicit pH representation; see section II-A for further details. We follow this idea by recasting the DDE (1) as an infinite-dimensional linear system without delay [5], for which we study special solutions of the corresponding operator KYP inequality. The pH structure on the operator level is then recast as a DDE, which forms the basis for our definition of a time-delay pH system. Our main results are the following:

1) We present a novel definition of a time-delay pH system in Definition III.1 and prove that time-delay pH systems are passive and closed under interconnection, see Lemmas III.3 and III.4.

2) The condition for a DDE to be pH generalizes a well-known matrix inequality for delay-independent passivity (cf. Proposition III.8) and details the construction of the necessary Lyapunov-Krasovskii functional in Proposition III.5 and Theorem III.6. Implications for time-delayed feedback are discussed in section III-E.

We review the necessary definitions and preliminary results in section II and present our main results in section III.

Notation

The sets of nonsingular matrices, symmetric positive-definite, and symmetric positive semi-definite matrices of dimension \( n \) are denoted by \( \text{GL}_n, S^n_+, \) and \( S^n_+ \), respectively. Moreover, for a matrix \( F \in \mathbb{R}^{n \times n} \), we use the notation \( \text{sym}(F) := \frac{1}{2}(F + F^\top) \) and \( \text{skew}(F) := \frac{1}{2}(F - F^\top) \). The set of all linear and bounded operators from a Banach space \( \mathcal{X} \) to a Banach space \( \mathcal{Y} \) is denoted by \( \mathcal{L}(\mathcal{X}, \mathcal{Y}) \). If \( \mathcal{X} = \mathcal{Y} \), we simply write \( \mathcal{L}(\mathcal{X}) \).

II. PRELIMINARIES

In this section, we recall important results on pH systems and time-delay systems that we will later leverage to motivate the definition of a time-delay pH system.

A. Passive and port-Hamiltonian systems

We consider a linear time-invariant system of the form

\[
\Sigma: \quad \begin{cases} 
\dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t),
\end{cases}
\]
with \( u : \mathbb{R} \to \mathbb{R}^m, x : \mathbb{R} \to \mathbb{R}^n, y : \mathbb{R} \to \mathbb{R}^m \) are the input, state, and output of the system and matrices \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( C \in \mathbb{R}^{m \times n} \). Throughout this subsection, we assume that (2) is minimal, i.e., for all \( s \in \mathbb{C} \) the conditions
\[
\text{rank } [sI_n - A, \ B] = n = \text{rank } [sI_n - A^T, \ C^T]
\]
are satisfied. For convenience, we introduce for \( H \in \mathbb{R}^{n \times n} \) the notation
\[
\Sigma_H := \begin{bmatrix} HA & HB \\ -C & 0 \end{bmatrix} \in \mathbb{R}^{(n+m) \times (n+m)}. \tag{3}
\]
Following [13], [20], we give the following definitions.

**Definition II.1.** We consider system (2).

1. System (2) is called passive, if there exists a state-dependent storage function \( \mathcal{H} : \mathbb{R}^n \to \mathbb{R}_+ \) satisfying for any \( t_1 \geq t_0 \) the dissipation inequality
\[
\mathcal{H}(x(t_1)) - \mathcal{H}(x(t_0)) \leq \int_{t_0}^{t_1} y(t)^\top u(t) \, dt \tag{4}
\]
for trajectories \((u, x, y)\) satisfying (2).

2. System (2) is called port-Hamiltonian (pH), if there exists \( H = H^\top \in \mathcal{S}_\geq^n \) such that the dissipativity condition
\[
\text{sym}(\Sigma_H) \leq 0. \tag{5}
\]
is satisfied. In this case, we define the matrices
\[
\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} := -\text{sym}(\Sigma_H), \quad \begin{bmatrix} J & G^\top \\ -G & 0 \end{bmatrix} := \text{skew}(\Sigma_H),
\]
and call
\[
\begin{cases} Hx(t) = (J - R)x(t) + Gu(t), \\ y(t) = G^\top x(t) \end{cases} \quad \text{a (generalized state-space) pH representation of (2) with associated Hamiltonian}
\]
\[
\mathcal{H}(x) := \frac{1}{2} x^\top H x. \tag{7}
\]

Straightforward calculations show that a pH system is passive with the Hamiltonian acting as storage function. Conversely, assume that (2) is passive, minimal, and stable. Define \( \mathcal{W} : \mathbb{R}^{n \times n} \to \mathbb{R}^{(n+m) \times (n+m)} \) via
\[
\mathcal{W}(H) := \begin{bmatrix} -A^\top H - HA & C^\top - HB \\ C - B^\top H \end{bmatrix}. \tag{8}
\]
It is well-known, see for instance [20], that the KYP inequality
\[
\mathcal{W}(H) \succeq 0 \tag{9}
\]
has a solution \( H \in \mathcal{S}_\geq^n \). We immediately notice that (9) resembles the dissipativity condition (5). Thus, we have proven the following equivalence, which is already well-known in the literature, see for instance [2].

**Theorem II.2.** Assume that (2) is minimal and stable. Then the following are equivalent.

1. The system (2) is passive.
2. The system (2) is port-Hamiltonian.

**Remark II.3.** In the literature, see for instance [6], [13], [19], pH systems are often modelled in a slightly different form. Such a different formulation can be achieved via a factorization of the Hessian of the Hamiltonian of the form
\[
H = E^\top Q \in \mathcal{S}_\geq^n
\]
with matrices \( E, Q \in \mathbb{R}^{n \times n} \). Then, a system of the form
\[
\begin{cases} E \dot{x}(t) = (J - R)Qx(t) + Gu(t), \\ y(t) = G^\top Qx(t) \end{cases} \tag{10}
\]
is called a pH system, if \( J = -J^\top \) and \( Q^\top RQ \in \mathcal{S}_\geq^n \). Depending on the modelling, the factorization is often chosen as \((E, Q) = (I_n, H)\) or \((E, Q) = (H, I_n)\) (as we do in Definition II.1).

**B. Time-delay systems as abstract infinite-dimensional systems**

To recast the delay equation (1) as abstract operator differential equation, we follow [5, Sec. 3.3] and define the Hilbert space \( \mathcal{Z}_{n,\tau} := \mathbb{R}^n \times L^2([-\tau, 0]; \mathbb{R}^n) \) with standard inner product
\[
\langle [x_1, \phi_1] \rangle := \langle x_1, x_2 \rangle_{\mathbb{R}^n} + \langle \phi_1, \phi_2 \rangle_{L^2}. \tag{11}
\]
We define the linear operator \( \mathcal{A} : \text{dom}(\mathcal{A}) \subseteq \mathcal{Z}_{n,\tau} \to \mathcal{Z}_{n,\tau}, \mathcal{B} : \mathbb{R}^m \to \mathcal{Z}_{n,\tau}, \) and \( \mathcal{C} : \mathcal{Z}_{n,\tau} \to \mathbb{R}^m \) via
\[
\begin{align*}
\mathcal{A} \phi := A_0 \phi + A_1 \phi(-\tau), \\
\mathcal{B} z := Bu, \\
\mathcal{C} x := Cx.
\end{align*}
\]
with domain
\[
\text{dom}(\mathcal{A}) := \left\{ [x, \phi] \in \mathcal{Z}_{n,\tau} \mid \phi \text{ is absolutely continuous}, \right. \]
\[
\left. \frac{d}{dt} \phi \in L^2([-\tau, 0]; \mathbb{R}^n), \text{ and } \phi(0) = x \}. \tag{12}
\]
With these preparations, we can study the operator equation
\[
\begin{cases} \dot{z}(t) = \mathcal{A} z(t) + \mathcal{B} u(t), \\ y(t) = C z(t), \\ z(0) = z_0. \end{cases} \tag{13}
\]
and it is well-known from the literature, that we can study (12) instead of (1).

**C. Passive time-delay systems**

Delay-independent passivity is typically studied via a Lyapunov-Krasovskii type functional of the form
\[
\mathcal{H}(x)_{[-\tau, 0]} = x(t)^\top Q x(t) + \int_{t-\tau}^{t} x(s)^\top \Theta x(s) \, ds. \tag{13}
\]
denotes the classical function segment used in the delay literature; cf. section II-B. The following result, taken from [14, Lem. 1], provides a sufficient condition for passivity of (1). Note that we present the result with a non-strict inequality, which does not influence the line of reasoning within the proof of [14, Lem. 1].

**Lemma II.4.** Consider the delay equation (1). If there exist positive definite matrices \( Q, \Theta \in \mathcal{S}_\geq^n \) such that
\[
\begin{align*}
A_0^\top Q + QA_0 + QA_1 \Theta^{-1} A_1^\top Q + \Theta &\leq 0, \tag{14a} \\
C &= B^\top Q. \tag{14b}
\end{align*}
\]
then the Lyapunov-Krasovskii function (13) satisfies the dissipation inequality
\[
\frac{1}{2} \dot{\mathcal{H}}(x_{|t_0-\tau,t_0}) - \frac{1}{2} \mathcal{H}(x_{|t_0-\tau,t_0}) \leq \int_{t_0}^{t_1} y(t)^T u(t) \, dt \quad (15)
\]
and hence, the delay equation (1) is passive.

III. TIME-DELAY PORT-HAMILTONIAN SYSTEMS

Leveraging the previous section’s discussion, we take the following route to obtain a meaningful definition of a time-delay pH system. First, we use the operator formulation (12) of the time-delay system and assume that the associated operator KYP inequality has a (special) solution. Testing the operator KYP inequality with suitable test functions, we obtain a finite-dimensional passivity condition that we can use to formulate a time-delay pH system. The details are presented in section III-A.

A. Motivation and definition

We briefly recall the concepts of passivity of infinite-dimensional linear systems for which we closely follow the exposition from [5, Sec. 7.5]. Given a self-adjoint, nonnegative operator \( Q \in \mathcal{L}(Z_{n,T}) \), we call the system (12) impedance passive with respect to the supply rate \( s(u, y) := \langle y, u \rangle + \langle u, y \rangle \) and the storage function \( \mathcal{H}(z) = \langle z, Qz \rangle \) if for all \( t > 0 \), \( z_0 \in Z_{n,T} \) and \( u \in L^2([0, T]; \mathbb{R}^m) \) it holds that
\[
\mathcal{H}(z(t)) \leq \mathcal{H}(z_0) + \int_0^t s(u(\theta), y(\theta)) \, d\theta.
\]

From [5, Thm. 7.5.3] it follows that (12) is impedance passive if and only if for all \( u \in \mathbb{R}^m \) and \( z \in \text{dom}(A) \) the following linear operator inequality is satisfied:
\[
\langle A_2, Qz \rangle + \langle Qz, A_2 \rangle + \langle \begin{pmatrix} z \\ u \end{pmatrix}, \begin{bmatrix} 0 & QB - C^* \\ B^* Q - C & 0 \end{bmatrix} \begin{pmatrix} z \\ u \end{pmatrix} \rangle \leq 0 \quad (16)
\]
Note that if \( Q \) maps from \( \text{dom}(A) \) into \( \text{dom}(A^*) \) instead of (12) we may introduce
\[
Q_{\text{ext}} := \begin{bmatrix} A^* Q + QA & QB - C^* \\ B^* Q - C & 0 \end{bmatrix},
\]
and consider a KYP-type inequality (cf. [5, Lem 7.5.4])
\[
\begin{pmatrix} z \\ u \end{pmatrix}, Q_{\text{ext}} \begin{pmatrix} z \\ u \end{pmatrix} \leq 0 \quad (18)
\]
for any \( z \in \text{dom}(A) \) and \( u \in \mathbb{R}^m \). While this is possible if (16) holds with equality, in the following we focus on the inequality (16) and do not assume \( Q \) to map from \( \text{dom}(A) \) into \( \text{dom}(A^*) \).

Since \( Q \in \mathcal{L}(Z_{n,T}) \), let us consider the following partitioning
\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \quad Q_{11} \in \mathcal{S}_n^+, \quad Q_{12} \in \mathcal{L}(L^2([-\tau, 0]; \mathbb{R}^n), \mathbb{R}^n), \quad Q_{22} \in \mathcal{L}(L^2([-\tau, 0]; \mathbb{R}^n)).
\]

For \( u = 0 \) and \( z = (x, \phi) \in \text{dom}(A) \) with \( \phi(0) = x \) we obtain from (16)
\[
0 \geq \begin{pmatrix} A_0 x + A_1 \phi(-T) \\ \begin{bmatrix} A_0 x + A_1 \phi(-\tau) \\ Q_{11} x + Q_{12} \phi \end{bmatrix} \end{pmatrix} = \begin{pmatrix} A_0 x + A_1 \phi(-\tau) \\ Q_{11} x + Q_{12} \phi + \frac{d}{dt} \frac{2}{\tau} \int_{-\tau}^0 Q_{12} \phi \, ds \end{pmatrix} \]
which would imply
\[
0 \geq \begin{pmatrix} A_0 x + A_1 \phi(-\tau) \\ Q_{11} x + Q_{12} \phi + \frac{d}{dt} \frac{2}{\tau} \int_{-\tau}^0 Q_{12} \phi \, ds \end{pmatrix} \quad \forall x \in \mathbb{R}^n.
\]

To mimic the Lyapunov-Krasovskii functional (13) let us assume \( Q_{12} = 0 \) to obtain
\[
0 \geq (A_0 x + A_1 \phi(-\tau))^T Q_{11} x + x^T \int_{-\tau}^{0} Q_{11} x + A_1 \phi(-\tau) + \frac{d}{dt} \frac{2}{\tau} \int_{-\tau}^{0} Q_{12} \phi \, ds \quad (19)
\]
and
\[
0 \geq \int_{-\tau}^{0} (A_0^T Q_{11} + Q_{12} A_1 - Q_{22}) x + \int_{-\tau}^{0} Q_{11} A_1 \phi(-\tau) - \frac{d}{dt} \frac{2}{\tau} \int_{-\tau}^{0} Q_{12} \phi \, ds.
\]

For arbitrary \( x, \xi \in \mathbb{R}^n \) let us define \( \phi : [-\tau, 0] \to \mathbb{R}^n \) by \( \phi(s) = \frac{\xi}{\tau}(x - \xi) + x \). Then \( (x, \phi) \in \text{dom}(A) \) and the last equation implies
\[
\begin{pmatrix} x \\ \xi \end{pmatrix}^T \begin{bmatrix} -A_0^T Q_{11} - Q_{12} A_1 - Q_{22} & -Q_{11} A_1 \\ -A_1^T Q_{11} & -Q_{22} \end{bmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \geq 0.
\]

Note further that from (16) we also obtain \( C = B^* Q \) which implies \( C = B^T Q_{11} \). Moreover, observe that the above simplifications correspond to the storage function
\[
\mathcal{H}(x, \phi) = x^T \int_{-\tau}^{0} \phi(s)^T Q_{22} \phi(s) \, ds.
\]

Similar to the non-delay case, instead of an explicit representation of \( Q_{11} \) and \( Q_{22} \) in (20), we may consider a generalized state-space representation leading to the following notion of a pH delay system.

Definition III.1. A time-delay system of the form
\[
\begin{align*}
Hx(t) &= (J - R)x(t) - Zx(t - \tau) + Gu(t), \\
y(t) &= G^T x(t) \quad (21a)
\end{align*}
\]
with Hamiltonian
\[
\mathcal{H}(x_{|t-\tau,t}) = \frac{1}{2} x(t)^T H x(t) + \int_{t-\tau}^{t} x(s)^T \Theta x(s) \, ds \quad (21b)
\]
is called a port-Hamiltonian (pH) delay system, if \( H \in S^n_\omega \), \( \Theta \in S^n_\omega \), \( J = -J^T \) and
\[
\begin{bmatrix}
R - \Theta & \frac{1}{2} Z^T \\
\frac{1}{2} Z^T & \frac{1}{2} \Theta
\end{bmatrix} \in S^{2n}_\omega. \tag{22}
\]

To demonstrate Definition II.1, we consider the following example taken from [14].

**Example III.2.** For parameters \( \alpha_0, \alpha_1 \in \mathbb{R} \) and time-delay \( \tau > 0 \) we consider the scalar delay differential equation
\[
\begin{align*}
\dot{x}(t) &= -\alpha_0 x(t) - \alpha_1 x(t - \tau) + u(t) \\
y(t) &= x(t)
\end{align*}
\tag{23}
\]
with appropriate initial condition defined on \([-\tau, 0]\). We set \( H := 1, \ J := 0, \ R := \alpha_0, \ Z := \alpha_1, \ G := 1 \) and \( \Theta := \theta \) with \( \theta \geq 0 \). The matrix in (22) has the eigenvalues
\[
\lambda_1, \lambda_2 = \frac{\alpha_0}{2} \pm \sqrt{\frac{\alpha_0^2}{4} + \theta^2 - \alpha_0 \theta + \frac{\alpha_1^2}{4}},
\]
which are non-negative if and only if \( \theta \alpha_0 \geq \theta^2 + \frac{1}{2} \alpha_1^2 \). This immediately implies \( \alpha_0 \geq |\alpha_1| \geq 0 \). In particular, we conclude that for any
\[
\theta \in \left[ \frac{\alpha_0}{2} - \sqrt{\frac{\alpha_0^2}{4} - \frac{\alpha_1^2}{4}}, \frac{\alpha_0}{2} + \sqrt{\frac{\alpha_0^2}{4} - \frac{\alpha_1^2}{4}} \right]
\]
condition (22) is satisfied and thus (23) is a pH delay system.

### B. Properties

**Lemma III.3 (Passivity).** Consider the pH delay system (21). Then
\[
\frac{\partial}{\partial t} H(x|_{t-\tau, t}) \leq y(t)^T u(t), \tag{24}
\]
along any solution of (21). In particular, the pH delay system (21) is passive.

**Proof.** We obtain
\[
\frac{\partial}{\partial t} H(x|_{t-\tau, t}) = x(t)^T H \dot{x}(t) + x(t)^T \Theta x(t) - x(t - \tau)^T \Theta x(t - \tau)
\]
\[
= - \left[ x(t)^T \left[ R - \Theta \right] \left[ \frac{1}{2} Z^T \right] + x(t - \tau)^T G u(t) \right] + y(t)^T u(t).
\]
Integration of (24) yields the dissipation inequality and thus passivity of the delay pH system.

We now discuss in which sense Definition III.1 is a generalization of the pH system definition in Definition II.1. We can consider two special cases of the pH delay system (21) to retain classical pH systems. First, let \( Z = 0 \). To recover the classical Hamiltonian, we set \( \Theta = 0 \) and then observe that (22) reduces to \( R \in S^n_\omega \), i.e., to the classical condition for pH systems. On the other hand, if \( \tau = 0 \), then we can again set \( \Theta = 0 \) and notice that (22) is only satisfied for \( Z = 0 \). On the other hand, if we require
\[
\begin{bmatrix}
x^T \\
x
\end{bmatrix}^T \begin{bmatrix}
R & \frac{1}{2} Z^T \\
\frac{1}{2} Z^T & \frac{1}{2} \Theta
\end{bmatrix} \begin{bmatrix}
x \\
x
\end{bmatrix} \geq 0
\]
for any \( x \in \mathbb{R}^n \), which corresponds to the specific choice used in the proof of Lemma III.3, then we recover the condition \( R + \text{sym}(Z) \in S^n_\omega \), corresponding to the system
\[
\begin{align*}
\dot{z}(t) &= (J - \text{skew}(Z) - (R + \text{sym}(Z)) x(t) + G u(t) \\
y(t) &= G^T x(t).
\end{align*}
\]

In terms of modelling, an important feature of pH systems is that they are closed under power-conserving or dissipative interconnection. Consider two pH delay systems of the form
\[
H_i \dot{x}_i(t) = (J_i - R_i) x_i(t) - Z_i x_i(t - \tau) + G_i u_i(t), \\
y_i(t) = G_i^T x_i(t)
\]
with \( H_i \in S^{n_i}_\omega \), \( R_i, \Theta_i \in S^n_\omega \), \( -J_i^T = J_i \in \mathbb{R}^{n_i \times n_i} \) and Hamiltonians
\[
\begin{align*}
\mathcal{H}_i(x_i|_{t-\tau, t}) &= \frac{1}{2} x_i(t)^T H_i x_i(t) + \int_{t-\tau}^t x_i(s)^T \Theta_i x_i(s) \, ds
\end{align*}
\]
for \( i = 1, 2 \). Define the aggregated input and output vectors as \( \tilde{u} := [u_1^T \ u_2^T]^T \) and \( \tilde{y} := [y_1^T \ y_2^T]^T \) and consider an output-feedback of the form
\[
\tilde{u} = F \tilde{y} + w
\]
with \( F \in \mathbb{R}^{(m_1 + m_2) \times (m_1 + m_2)} \). By defining \( \tilde{x} := [x_1^T \ x_2^T]^T \) and
\[
\check{H} := \text{diag}(H_1, H_2), \quad \check{J} := \text{diag}(J_1, J_2), \quad \check{R} := \text{diag}(R_1, R_2), \quad \check{Z} := \text{diag}(Z_1, Z_2), \quad \check{G} := \text{diag}(G_1, G_2), \quad \check{\Theta} := \text{diag}(\Theta_1, \Theta_2),
\]
we obtain the interconnected (closed-loop) system
\[
\check{H} \tilde{x}(t) = (J - \check{R} + \check{G} F \check{G}^T) \tilde{x}(t) + \check{Z} \tilde{x}(t - \tau) + \check{G} w(t) \\
\tilde{y}(t) = \check{G}^T \tilde{x}(t)
\]
(25)
with combined Hamiltonian \( \check{H} := H_1 + H_2 \) given by
\[
\check{H} \left( \tilde{x}_i|_{t-\tau, t} \right) = \frac{1}{2} \tilde{x}(t)^T \tilde{H} \tilde{x}(t) + \int_{t-\tau}^t \tilde{x}(s)^T \tilde{\Theta} \tilde{x}(s) \, ds.
\]
We have thus proven the following result, which details that delay pH systems are closed under interconnection.

**Lemma III.4 (Interconnection).** Let us consider the interconnected system (25) with Hamiltonian \( \check{H} := H_1 + H_2 \). Then, the interconnected system (25) is a pH delay system if
\[
\begin{bmatrix}
\check{R} - \check{G} \text{sym}(F) \check{G}^T - \check{\Theta} & \frac{1}{2} Z^T \\
\frac{1}{2} Z^T & \frac{1}{2} \check{\Theta}
\end{bmatrix} \in S^{2(n_1+n_2)}_\omega. \tag{26}
\]
In particular, the interconnected system is a pH delay system for every power-conserving (\( \text{sym}(F) = 0 \)) and every dissipative (\( \text{sym}(F) \in S^{n_1+n_2}_\omega \)) interconnection.

### C. Construction of the Lyapunov-Krasovskii functional

While the energy (and thus the Hamiltonian) for non-delayed systems is typically obtained during the modeling process, this may not be the case for the Hamiltonian, i.e., the Lyapunov-Krasovskii functional (13), of the delay system (1). In particular, if the delay system results from delayed
feedback of a non-delayed pH system (see the forthcoming section III-E), then the matrix $\Theta \in S^r_+$ in (21b) is not available. We thus discuss in the following conditions when such a matrix $\Theta$ satisfying the delay pH condition (22) exists. We first start with a geometric analysis.

**Proposition III.5.** A necessary condition for a system of the form (21) with $\tau > 0$ to be a pH delay system is

$$\ker(R) \not\subseteq \ker(\Theta) \subseteq \ker(Z), \quad \ker(R) \cap \im(Z) = \{0\}, \quad \text{and} \quad \ker(R) \cap \im(\Theta) = \{0\}. \quad (27a)$$

$$V^\top RV =: \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad V^\top \Theta V =: \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2 & \Theta_3 \end{bmatrix}, \quad V^\top \tilde{Z} V =: \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$$

with $R_1 \in S^r_+$, resulting in

$$0 \leq \begin{bmatrix} R_1 - \Theta_1 & Z_1 & Z_2 \\ Z_1^\top & -\Theta_2 & Z_3 \\ Z_2^\top & Z_3^\top & -\Theta_4 \end{bmatrix}. \quad (27b)$$

We conclude $\Theta_4 = 0$ and $\Theta_2 = 0$, implying $\ker(R) \not\subseteq \ker(\Theta)$ and $\ker(R) \cap \im(S) = \{0\}$, and consequently $Z_3 = 0$, $Z_5 = 0$, and $Z_1 = 0$, which implies $\ker(R) \subseteq \ker(Z)$ and (27b). The inequality (22) is thus equivalent to

$$\begin{bmatrix} R_1 - \Theta_1 & Z_1 & Z_2 \\ Z_1^\top & -\Theta_2 & Z_3 \\ Z_2^\top & Z_3^\top & -\Theta_4 \end{bmatrix} \succeq 0. \quad (28)$$

We immediately obtain $R_1 - \Theta_1(\alpha) = (1-\alpha)V_1^\top V_1$ and $\Theta_1(\alpha) \in S^r_+$ for $\alpha \in (0, 1)$. Assuming $\alpha \in (0, 1)$, the Schur complement $R_1 - \Theta_1(\alpha) - Z_1^\top \Theta_1^{-1}(\alpha) Z_1$ is symmetric positive semi-definite if and only if

$$(\alpha - \alpha^2) I_r - Z_1^\top Z_1 \in S^r_+. \quad (30)$$

Let $\sigma_{\max}$ denote the largest singular value of $\tilde{Z}$. Then, it is easily seen that the Schur complement condition (30) is satisfied if and only if $\sigma_{\max} \leq \frac{1}{\tau}$. In this case, the choice $\Theta_1(\frac{1}{2})$ guarantees that the condition (22) is satisfied. We summarize our discussion in the following theorem.

**Theorem III.6.** Consider a delay equation of the form (21) and assume $\ker(R) \subseteq \ker(\Theta)$ and $\ker(R) \cap \im(Z) = \{0\}$. Let $r := \ker(R) \cap V_1 \in \mathbb{R}^{n \times r}$ such that $V_1^\top RV_1 = I_r$. If $\|V_1^\top ZV_1\|_2 \leq 1$, then for $\Theta := \frac{1}{2} R \in S^r_+$ the condition (22) is satisfied.

**Proof.** Let $V_2 \in \mathbb{R}^{n \times (n - r)}$ be a basis of $\ker(R)$ and define $V := [V_1, V_2] \in \text{GL}_n$. Set $\Theta := \frac{1}{2} R \in S^r_+$ and define $\tilde{Z} := V_1^\top Z V_1$. We then obtain (similarly as in the proof of Proposition III.5)

$$V^\top \begin{bmatrix} 0 & \Theta_1 \frac{1}{2} Z \\ -\frac{1}{2} Z^\top & \Theta_1 \frac{1}{2} Z \end{bmatrix} V = \begin{bmatrix} I_r & \tilde{Z} \\ \tilde{Z} & I_r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (31)$$

Consider an orthogonal $U \in \text{GL}_r$ that diagonalizes $\tilde{Z}$, i.e., $U^\top \tilde{Z} U = \text{diag}(\eta_1, \ldots, \eta_r)$ with $\eta_i \geq \eta_{i+1} \geq 0$ for $i = 1, \ldots, r - 1$. By assumption, we have $\eta_1 \leq 1$. The result now follows from

$$I_r - \tilde{Z} = U (I_r - \text{diag}(\eta_1, \ldots, \eta_r)) U^\top \in S^r_+,$$

which is two times the Schur complement of the non-zero submatrix on the right-hand side in (31).

Unfortunately, the condition $\|V_1^\top ZV_1\|_2 \leq 1$ is only sufficient, but not necessary, as the following example suggests.

**Example III.7.** Let $n = 2$ and consider the matrices

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$ 

With $V_1 = I_2$ we obtain $\|V_1^\top ZV_1\|_2 = \|Z\|_2 = \frac{2}{\sqrt{2}} > 1$. Nevertheless, it is easy to see that with the choice $\Theta = \text{diag}(\frac{1}{2}, \frac{1}{4})$ the condition (22) is satisfied.

**D. Comparison with the literature**

Using the results from the previous subsection, we are now in a position to relate our definition of a delay pH system with the passivity matrix inequality from [14, Lem. 1], presented in Lemma II.4. We obtain the following result.

**Proposition III.8.** Consider a pH delay system (21) and assume $S \in S^r_+$. Then, (21) fulfills the inequality (14a).
Proof. A pH delay system of the form (21) corresponds to the delay equation (1) with
\[ A_0 = H^{-1}(J - R), \quad A_1 = -H^{-1}Z, \quad G = H^{-1}B. \]
With the choice \( Q = \frac{1}{2}H \in S^n \) we obtain
\[ A_0^\top Q + QA_0 + QA_1\Theta^{-1}A_1^\top Q + S = -R + \Theta + \frac{1}{2}Z\Theta^{-1}Z^\top, \]
which is the negative of the Schur complement of (22), and hence negative semi-definite. \( \square \)

Two remarks are in order:
- On the one hand, the pH condition (22) is slightly more general, since it does not require \( \Theta \) to be nonsingular. Moreover, the explicit choice of \( H \) (and thus \( Q \)), makes the matrix inequality (15) easier to verify; see the discussion in section III-C.
- On the other hand, the explicit choice of \( H \) has an impact on \( \ker(R) \) and thus on the possible choices of \( \Theta \); cf. Proposition III.5. In particular, the dimension of \( \ker(R) \) is not independent of \( H \), but \( \dim(\ker(Z)) \) and \( \dim(\text{im}(Z)) \) are independent of \( H \), which renders (22) less flexible then (15).

E. Delayed feedback control

As important special case, we consider the standard pH system (6) and assume a delayed feedback of the form
\[ u(t) := -Fy(t - \tau) + v(t) \tag{32} \]
with \( F \in \mathbb{R}^{n \times n} \), then results in the time-delay system
\[ \begin{align*}
   H\dot{x}(t) &= (J - R)x(t) - GFG^\top x(t - \tau) + v(t), \\
   y(t) &= G^\top x(t),
\end{align*} \tag{33} \]
which is (formally) of the form (21) with \( Z = GFG^\top \). We now discuss necessary and sufficient conditions on \( G \) and \( F \) such that (33) is indeed a pH delay system. We notice that the necessary condition from Proposition III.5 is satisfied if
\[ \begin{align*}
   \ker(R) &\subseteq \ker(G^\top) = \{0\}, \\
   \ker(R) \cap \text{im}(G) &= \{0\}. \tag{34}
\end{align*} \]
Let us mention, that the second condition also shows up in the characterization of optimal controls for pH systems, see [15, Thm. 8]. Assuming that (34) holds and using Theorem III.6 we thus obtain an upper bound on the norm of the feedback gain \( F \) such that we can guarantee that the closed-loop system is still a delay pH system. \[ \square \]

IV. Conclusions

We have presented a novel definition of a time-delay pH system, which is based on a detailed investigation of the corresponding infinite-dimensional KYP inequality. We then showed that our class of delay pH systems satisfies important properties of pH systems, such as delay-independent passivity and invariance under interconnection. Moreover, the structured pH form allows for a simplified construction of Lyapunov-Krasovskii functional compared to standard results in the literature. Based on this first step, we envision multiple generalizations, including neutral and nonlinear delay systems and an extension to delay differential-algebraic equations.

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