Research Article

Vanishing Theorems on Compact Hyper-kähler Manifolds

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We prove that if $B$ is a $k$-positive holomorphic line bundle on a compact hyper-kähler manifold $M$, then $H^p(M, \Omega^q \otimes B) = 0$ for $p > n + \lfloor k/2 \rfloor$ with $q$ a nonnegative integer. In a special case, $k = 0$ and $q = 0$, we recover a vanishing theorem of Verbitsky’s with a little stronger assumption.

1. Introduction

A hyper-kähler manifold is an oriented $4n$-dimensional Riemannian manifold with a special holonomy group $Sp(n) \subset SO(4n)$. The holonomy group of a Kähler manifold is $U(n) \subset SO(2n)$ and the unitary group $U(n)$ is exactly the subgroup of $SO(2n)$ that preserves a complex structure, which together with a compatible Riemannian metric defines a symplectic form. Hence a Kähler manifold can also be defined as a Riemannian manifold with compatible symplectic and complex structure. By the same reasoning, $Sp(n) \subset SU(2n)$ is a subgroup exactly preserving three complex structures $I, J, K$ with $IJ = -JI = K$. As the names suggests, a hyper-kähler manifold is also characterized as a Riemannian manifold with three compatible complex structures $I, J, K$ and a compatible symplectic form which is Kähler with respect to each one of $I, J, K$.

A hyper-kähler manifold $M$ is called irreducible if $H^1(M) = 0$ and $H^2(M) = C$. By Bogomolov’s decomposition theorem for a Kähler manifold with trivial canonical class ([1, 2]), any hyper-kähler manifold is biholomorphic to a product of irreducible hyper-kähler manifolds and a hyper-kähler complex torus up to finite cover. On any hyper-kähler manifold $M$ there is a symmetric bilinear form $q$, called Beauville-Bogomolov-Fujiki form, which takes positive values on the Kähler cone $\mathcal{K}$ of $(M, I)$. The closure of the dual Kähler cone $\mathcal{K}^\vee := \{ x \in H^{1,1}(M, \mathbb{R}) \mid q(x, y) > 0, \forall y \in \mathcal{K} \}$ is denoted by $\overline{\mathcal{K}}^\vee$. In [3], Verbitsky established the following vanishing theorem for a compact irreducible hyper-kähler manifold.

**Theorem 1** (Verbitsky, 2007, [3]). Let $M$ be a compact irreducible hyper-kähler manifold of real dimension $4n$, and let $L$ be a holomorphic line bundle on $M$. If $0 \neq c_1(L) \in \overline{\mathcal{K}}^\vee$, in particular, if $L$ is a positive line bundle, then

$$H^p(M, L) = 0, \quad \text{for } p > n.$$  

Verbitsky’s proof of the above theorem is a clever use of the symmetric pole of the complex structures and the holomorphic Bochner-Kodaira-Nakano-type identity, which appeared already in [5]. In the proof he used assumption of irreducibility of the hyper-kähler manifold. In this paper we use a different method to establish some vanishing theorems for more general hypercomplex Kähler manifolds. To get the flavor, we state the following result.

**Theorem 2.** Let $M$ be a compact hyper-kähler manifold of real dimension $4n$, and let $L$ be a holomorphic line bundle on $(M, I)$. If $L$ is a positive line bundle, then for any $q \geq 0$

$$H^p(M, \Omega^q \otimes L) = 0, \quad \text{for } p > n.$$  

Note that we donot assume that $M$ is irreducible. We recover Theorem 1 if $q = 0$ while a stronger assumption. During the proof of Theorem 2, the Kähler metric of $(M, I)$ is changed; the new Kähler metric is not necessarily hyper-kähler. So we develop our theory on hypercomplex Kähler
2 Geometry

manifolds and deal with hyper-kähler manifolds as their special examples. After deriving the Bochner-Kodaira-Nakano identities from Section 2 to Section 4, we get our main results finally in Section 5.

2. Preliminary

A Hermitian manifold $M$ is a complex manifold with an integrable complex structure $I$ and a Riemannian metric $g$ satisfying the compatible condition $g(IX, IY) = g(X, Y)$ for any $X, Y \in TM$. It is called a Kähler manifold if in addition the 2-form $\omega$ defined by $\omega(X, Y) = -g(X, IY)$ is sympletic form. $\omega$ is also called the Kähler form associated with $g$ and $g$ is called a Kähler metric. On the other hand, if we start with a symplectic manifold $M$ equipped with a symplectic form $\omega$, then $M$ is a Kähler manifold if and only if there is a $\omega$-compatible integrable complex structure. Recall that a complex structure $I$ is called $\omega$-compatible if $\omega$ is $I$-invariant $\omega(IX, IY) = \omega(X, Y)$ for any $X, Y \in TM$ and $I$-tamed $\omega(X, IX) > 0$ if $X \neq 0$. To show that two definitions of Kähler manifolds are equivalent, it suffices to note that $g(X, Y) = \omega(X, IY)$ is a Hermitian metric in the latter definition.

Definition 3. A Riemannian manifold $M$ with Riemannian metric $g$ is called a hyper-kähler manifold if it admits three integrable complex structures $I, J, K$ with $IJ = -JI = K$ such that $g$ is a Kähler metric with respect to each one of $I, J, K$. We called $g$ a hyper-kähler metric.

Proposition 4. Let $(M, \omega, I)$ be a Kähler manifold with Levi-Civita connection $\nabla$. Then $M$ is hyper-kähler if there are integrable complex structures $I, J, K$ with $IJ = -JI = K$ and $J, K$ satisfying the following conditions:

(i) $\omega(IX, KY) = \omega(X, IY)$;

(ii) $J, K$ are parallel: $\nabla J = \nabla K = 0$.

Proof. Let $g(X, Y) = \omega(X, IY)$; then $g$ is a Kähler metric by the assumptions. $g$ is Hermitian relative to $I$ if $g(X, Y) = g(JX, JY)$, which is $\omega(X, KY) = \omega(X, IY)$. Let $\omega(X, Y) = -g(X, JY) = -\omega(X, KY)$. Clearly $\omega_I$ is nondegenerate. It is well known that $d\omega_I = 0$ if $\nabla I = 0$. Hence $g$ is Kähler with respect to $I$ if (i) and (ii) are true. $g$ is Kähler with respect to $K$ follows in the same way if (i) and (ii) are true. ☐

Proposition 5. Let $M$ be a complex symplectic manifold with an integrable complex structure $I$ and an $I$-invariant symplectic structure $\omega$. Let $g(X, Y) = \omega(X, IY)$ for any $X, Y \in TM$. Then $g$ is a Lorentz Hermitian metric and for any point $x \in M$ there exists a local holomorphic coordinate $(w^1, \ldots, w^n)$ around $x$ such that

$$g = \sum_{jk} \left( \pm \delta_{jk} + O \left( |w|^2 \right) \right) dw^j \overline{dw}^k.$$  

Proof. If moreover $I$ is $\omega$-partible, then $M$ is a Kähler manifold and there is a proof in [6] for this special case. The general case in our proposition follows in the same way. Since $\omega$ is a symplectic form, $g$ is nondegenerate but not necessarily positive definite, hence a Lorentz Hermitian metric. We could find local holomorphic coordinates $(z^j)$ at $x$ such that $g_{jk}(x) = \pm \delta_{jk}$; in other words, we could write locally

$$g = \sum_{jk} \left( \pm \delta_{jk} + a_{jk}w_z + a_{jk}\overline{w}^z \right) \overline{dz^j} \overline{dz}^k;$$

$$\omega = i \sum_{jk} \left( \pm \delta_{jk} + a_{jk}w_z + a_{jk}\overline{w}^z \right) \overline{dz}^j \wedge dz^k.$$  

Let us make a holomorphic change of coordinates

$$z_k = w_k + \frac{1}{2} \sum b_{km} w_m w_k.$$  

Then in the new coordinate we have

$$g = \sum_{jk} \left( \pm \delta_{jk} + i \sum l (a_{kj}w_l + a_{kl}\overline{w}_l + b_{lj}w_l) \right) dw^j \overline{dw}^k.$$  

Choose $b_{kj} = -a_{kj}$. Since $g$ and $\omega$ are real, we have

$$\overline{a}_{kj} = \overline{a}_{jk}.$$  

Since $d\omega = 0$, in particular $d\omega(x) = 0$,

$$a_{jk} = a_{kj}.$$  

From (8) and (9) we have

$$\overline{b}_{kj} = -\overline{a}_{kj} = -a_{kj};$$

therefore, locally $g = \sum_{jk} (\pm \delta_{jk} + O(|w|^2)) dw^j \overline{dw}^k$. ☐

A complex manifold $M$ with integrable complex structures $I, J, K$ is called a hypercomplex manifold if $IJ = I = K$, and $(I, J, K)$ is called a hypercomplex structure (Verbitsky has studied hypercomplex manifolds and hypercomplex Kähler manifolds in a series of papers [3, 7, 8]). Obata proved that on a hypercomplex manifold $(M, I, J, K)$ there exists a unique torsion-free connection such that $I, J, K$ are parallel [9]:

$$\nabla I = \nabla J = \nabla K = 0.$$  

Such a connection is called an Obata connection. If $M$ is a hyper-kähler manifold, clearly the Levi-Civita connection is exactly the Obata connection of the underlying hypercomplex manifold.

The following proposition is cited from [8].
Proposition 6. Let \((M, I, J, K)\) be a hypercomplex manifold. At any point \(x \in M\), there exists a holomorphic (with respect to \( I \)) local coordinate \((z^i)\) around \(x\) with \(z^i(x) = 0\) such that
\[
I(z) = I_0 + O(\left|z^i\right|^2) I^i;
\]
\[
J(z) = J_0 + O(\left|z^i\right|^2) J^i;
\]
\[
K(z) = K_0 + O(\left|z^i\right|^2) K^i,
\]
where \(I_0, J_0,\) and \(K_0\) are the constant complex structures.

Proof. We could choose a normal coordinate \((z^i)\) at \(x\) with \(z^i(x) = 0\) for the Obata connection \(V\); then the Christoffel symbols of \(V\) vanish at \(x\). Since at \(x\)
\[
\nabla I = dI = \nabla J = dJ = \nabla K = dK = 0,
\]
\[
I(z) = I_0 + O(\left|z^i\right|^2) I^i \text{ with } I_0 = I(0) \text{ for example.} \]

3. Bochner-Kodaira-Nakano-Type Identities

3.1. Generalized Hodge Identities for Differential Forms. Let \((M, I, J, K)\) be a compact hypercomplex manifold of real dimension \(4n\) and \((M, I)\) a Kähler manifold with Kähler metric \(g\). There are naturally associated three nondegenerate 2-forms:
\[
\omega_I (\cdot, \cdot) = g (\cdot, I \cdot),
\]
\[
\omega_J (\cdot, \cdot) = g (\cdot, J \cdot),
\]
\[
\omega_K (\cdot, \cdot) = g (\cdot, K \cdot).
\]
Let \(\xi^1, I \xi^1, J \xi^1, K \xi^1, \ldots, \xi^n, I \xi^n, J \xi^n, K \xi^n\) be a real unit orthogonal coframe of the cotangential bundle \(T^* M\) at a fixed point \(x \in M\). Then using the Darboux theorem, we can write the Kähler form \(\omega_I\) locally as
\[
\omega_I = \sum_{j=1}^{n} \left( \xi^j \wedge I \xi^j + I \xi^j \wedge K \xi^j \right).
\]
Accordingly,
\[
\omega_J = \sum_{j=1}^{n} \left( \xi^j \wedge J \xi^j + K \xi^j \right),
\]
\[
\omega_K = \sum_{j=1}^{n} \left( \xi^j \wedge K \xi^j + J \xi^j \right).
\]
Choose holomorphic coframes relative to the complex structure \(I\),
\[
\theta^j = \xi^j - i K \xi^j, \quad \theta^{j+n} = J \xi^j - i K \xi^j, \quad j = 1, \ldots, n,
\]
with antiholomorphic coframes
\[
\bar{\theta}^j = \xi^j + i K \xi^j, \quad \bar{\theta}^{j+n} = J \xi^j + i K \xi^j, \quad j = 1, \ldots, n.
\]
Then we have the following pointwise action at \(x \in M\):
\[
I \theta^j = i \theta^j, \quad I \bar{\theta}^j = -\bar{\theta}^j, \quad j = 1, \ldots, 2n;
\]
\[
J \theta^j = \bar{\theta}^{j+n}, \quad J \bar{\theta}^j = -\theta^j, \quad j = 1, \ldots, n;
\]
\[
K \theta^j = -i \bar{\theta}^{j+n}, \quad K \bar{\theta}^j = i \theta^j, \quad j = 1, \ldots, n.
\]
Using holomorphic and antiholomorphic coframes, we could rewrite the 2-forms \(\omega_I, \omega_J, \omega_K\) locally as
\[
\omega_I = \frac{i}{2} \sum_{j=1}^{n} \left( \theta^j \wedge \bar{\theta}^j + \theta^{j+n} \wedge \bar{\theta}^{j+n} \right),
\]
\[
\omega_J = \frac{1}{2} \sum_{j=1}^{n} \left( \theta^j \wedge \theta^{j+n} + \bar{\theta}^j \wedge \bar{\theta}^{j+n} \right),
\]
\[
\omega_K = \frac{i}{2} \sum_{j=1}^{n} \left( \theta^j \wedge \bar{\theta}^{j+n} - \theta^j \wedge \theta^{j+n} \right).
\]
Thus
\[
\varphi = \omega_J + i \omega_K = \sum_{j=1}^{n} \theta^j \wedge \theta^{j+n}
\]
is a holomorphic (2,0)-form with respect to the complex structure \(I\).

For convenience, when talking about holomorphic structure of \(M\), we always mean that it is relative to the complex structure \(I\) if without special mention in the rest of this paper, though \(I, J,\) and \(K\) have symmetric and equal roles.

Theorem 7. There exists a local holomorphic coordinate \((z^i, \ldots, z^n)\) around \(x\) such that
\[
\omega_I = \sum_{jk} \left( \delta_{jk} + O(\left|z^i\right|^2) \right) dz^j \wedge dz^k;
\]
\[
\omega_J = \sum_{jk} \left( \delta_{jk} + O(\left|z^i\right|^2) \right) \left( dz^j \wedge dz^k + dz^j \wedge dz^{k+n} + dz^k \wedge dz^{j+n} \right);
\]
\[
\omega_K = \sum_{jk} \left( \delta_{jk} + O(\left|z^i\right|^2) \right) \left( dz^j \wedge dz^k + dz^j \wedge dz^{k+n} - dz^j \wedge dz^{k+n} \right).
\]

Proof. From [10], we know that there exists a local holomorphic coordinate system \((z^i)\) with respect to the complex structure \(I\) such that its action is local constant: \(Idz^j = idz^j\) with \(\theta^j = dz^j\), which coincides with the pointwise action we considered in (20). By Proposition 5, there exists a local
holomorphic coordinate system \((z^i)\) such that (25) holds. By Proposition 6 and (21), (22),
\[
\begin{align*}
J dz^i &= dz^i + O \left( |z|^2 \right) \\
K dz^i &= i dz^i + O \left( |z|^2 \right) \\
K dz^i &= -idz^i + O \left( |z|^2 \right)
\end{align*}
\] (28)

From (28) and the definitions of \(\alpha_j\) and \(\omega_K\), we conclude (26) and (27).

Let \(d : \Omega^p(M) \to \Omega^{p+1}(M)\) be the de Rham differential operator on \(M\), and let \(d^* = (-1)^p d\). Note that the complex structures \(I, J, K\) on the tangent bundle naturally induce operator actions on the vector fields and the differential forms. Take \(I\) for an example. For \(\alpha, \beta \in \Omega(M)\), the action of \(I\) on differential forms is defined by
\[
I(\alpha \wedge \cdots \beta) = I\alpha \wedge \cdots \wedge I\beta. \tag{29}
\]
The Dolbeault operators \(\partial, \bar{\partial}\) and \(d, d^*\) are related by
\[
d = \partial + \bar{\partial}, \quad d^* = -i (\partial - \bar{\partial}). \tag{30}
\]

Accordingly, the complex structures \(J, K\) also induce complex differential operators:
\[
J = I^{-1} d I = \partial J + \bar{\partial} J, \quad K = -i d K = \partial K + \bar{\partial} K, \tag{31}
\]
where \(\partial J = \partial K, \bar{\partial} J = \bar{\partial} K\) are similar notions. Like \(d, \partial, \bar{\partial}\), it is easy to check that the operators \(d_j, d_K, \partial_j, \bar{\partial}_K\) also satisfy the graded Leibniz rule: for any \(\xi, \eta \in \Omega(M)\),
\[
d_j (\xi \wedge \eta) = d_j \xi \wedge \eta + (-1)^{|\xi|} \xi \wedge d_j \eta, \tag{32}
\]
where \(|\xi|\) is the degree of \(\xi\).

For each ordered set of indices \(A = \{\alpha_1, \ldots, \alpha_p\}\), denote the index length by \(|A| = p\); we write
\[
\theta^A = \theta^\alpha_1 \wedge \cdots \wedge \theta^\alpha_p, \quad \bar{\theta}^A = \bar{\theta}^\alpha_1 \wedge \cdots \wedge \bar{\theta}^\alpha_p, \tag{33}
\]
and denote by \(\tilde{A} = \{\alpha_{p+1}, \ldots, \alpha_n\}\) the complementary of \(A\) so that
\[
\theta^A \wedge \theta^{\tilde{A}} = (-1)^{|A|} \tau(A) \theta^\alpha \wedge \cdots \wedge \theta^{2n}, \tag{34}
\]
where \(\tau(A)\) takes value 1 if \(A\) is an even permutation and \(-1\) otherwise. The Hodge star operator \(*\) is given by
\[
* (f \theta^A \wedge \bar{\theta}^B) = 2^{\frac{|A|+|B|-2n}{2}} (-1)^{|A|} e_{AB} f \theta^{\tilde{A}} \wedge \bar{\theta}^{\tilde{B}}, \tag{35}
\]
where \(f\) is a function and the signature factor
\[
e_{AB} = (-1)^{\nu(2n-1)+2(2n-p)|A|+\tau(A)+\tau(B)}. \tag{36}
\]

Given two \((p, q)\)-forms
\[
\xi = \frac{1}{p!q!} \sum_{A,B} \epsilon_{AB} \theta^A \wedge \bar{\theta}^B, \quad \eta = \frac{1}{p!q!} \sum_{A,B} \eta_{AB} \bar{\theta}^A \wedge \theta^{\tilde{B}}, \tag{37}
\]
their pointwise inner product is defined by
\[
\langle \xi, \eta \rangle = \xi \wedge \ast \eta = \left( \frac{1}{p!q!} \sum_{A,B} \epsilon_{AB} \eta_{AB} \right) \frac{1}{(2n)!} \omega_n^{2n}. \tag{38}
\]

Since \(M\) is compact, we can consider the Hermitian inner product on each \(\Omega^{p,q}(M)\) defined by
\[
\langle \xi, \eta \rangle = \int_M \langle \xi, \eta \rangle, \quad \xi, \eta \in \Omega^{p,q}(M), \tag{39}
\]

For each \(k, l = 1, \ldots, 2n\), let \(e_k : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)\) be the wedge operator defined by
\[
e_k (\eta) = \theta_k \wedge \eta; \tag{40}
\]
and \(\overline{e}_k : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)\) are similarly given by
\[
\overline{e}_k (\eta) = \bar{\theta}_k \wedge \eta. \tag{41}
\]

Let \(i_k\) and \(\bar{i}_k\) be the adjoints of \(e_k\) and \(\overline{e}_k\) with respect to the inner product (40), respectively. They are called contraction operators. Then for any \(k, l = 1, \ldots, 2n\),
\[
e_k i_l + i_k e_k = 0; \tag{42}
\]
\[
e_k i_l + i_k e_k = 2. \tag{43}
\]

If \(k \neq l\),
\[
e_k i_l + i_k e_k = 0. \tag{44}
\]

The three equations above reflect how to commute the actions of wedges and contractions. The following Proposition 8 gives the commutation relations between contract actions and complex structure actions on differential forms. Based on them, it is easy to get the commutation relations between the actions of wedges and complex structures.

**Proposition 8.** Acting on the differential forms as linear operators, the contractions \(i_k\) and the complex structures \(I, J, K\) satisfy the commuting relations
\[
i_k I = -i I i_k, \quad i_k \bar{J} = i \bar{J} i_k; \tag{45}
\]
\[
i_k J = J i_k, \quad i_k i_k n = J i_k; \tag{46}
\]
\[
i_k K = i K i_k, \quad i_k n K = -i K i_k. \tag{47}
\]
By definition in (32) and the commutation relations in Proposition 8, we have the following expressions of differential operators via contraction and wedge operators:

\[
\partial = \sum_{k} (e_{k} \partial_{k} + e_{k+n} \partial_{k+n}),
\]

(47)

\[
\partial^{*} = -\sum_{k} (\partial_{k} e_{k} + \partial_{k+n} e_{k+n});
\]

(48)

\[
\partial_{j} = \sum_{k} (\bar{e}_{k+n} \partial_{k} - \bar{e}_{k} \partial_{k+n}),
\]

(49)

\[
\partial_{K}^{*} = \sum_{k} (\bar{e}_{k+n} \partial_{k} + \bar{e}_{k} \partial_{k+n}),
\]

(50)

Let \( L = L_{J}, L_{I}, L_{K} \) be the operators from \( \Omega^{*}(E) \) to \( \Omega^{*}(E) \) defined by the wedge with the 2-forms \( \omega = \omega_{I}, \omega_{J}, \omega_{K} \), respectively, and \( \Lambda = \Lambda_{J}, \Lambda_{I}, \Lambda_{K} \) their adjoint operators:

\[
L_{J} = \frac{i}{2} \sum_{k} (e_{k} \partial_{k} + \bar{e}_{k} \partial_{k+n}),
\]

(51)

\[
L_{I} = \frac{1}{2} \sum_{k} (e_{k} \partial_{k+n} + \bar{e}_{k} \partial_{k+n}),
\]

(52)

\[
L_{K} = \frac{1}{2} \sum_{k} (\bar{e}_{k} \partial_{k} - e_{k} \partial_{k+n}),
\]

\[
L_{K} = \frac{i}{2} \sum_{k} (\bar{e}_{k} \partial_{k} - e_{k} \partial_{k+n}).
\]

(53)

The following identities in Lemma 9 are called Hodge identities [6]; they play fundamental roles in Kähler geometry. Their proof is reduced from an arbitrary Kähler manifold to the Euclidean Kähler plane via using Proposition 5. The main observation is that any intrinsically defined identity that involves the Kähler metric together with its first derivatives and which is valid for the Euclidean metric is also valid on a Kähler manifold, since by Proposition 5, a Kähler metric is osculate order 2 to the Euclidean metric everywhere.

**Lemma 9** (Hodge identities).

\[
[\Lambda, \partial] = i\partial^{*}, \quad [\Lambda, \partial^{*}] = -i\partial^{*}.
\]

(54)

For a proof of this lemma, please refer to [6, pages 111–114].

**Proposition 10.** Let \((M, I, J, K)\) be a compact hypercomplex manifold such that \((M, I)\) is a Kähler manifold; then

\[
[\Lambda_{J}, \partial_{I}] = \partial_{I}^{*}, \quad [\Lambda_{I}, \partial_{I}] = \partial_{I}^{*};
\]

(55)

\[
[\Lambda_{K}, \partial_{I}] = -\partial_{I}^{*}, \quad [\Lambda_{K}, \partial_{I}] = -\partial_{I}^{*};
\]

(56)

\[
[\Lambda_{J}, \partial_{J}] = i\partial_{J}^{*}, \quad [\Lambda_{J}, \partial_{J}] = -i\partial_{J}^{*};
\]

(57)

\[
[\Lambda_{K}, \partial_{J}] = i\partial_{J}^{*}, \quad [\Lambda_{K}, \partial_{J}] = -i\partial_{J}^{*};
\]

(58)

\[
[\Lambda_{J}, \partial_{K}] = i\partial_{K}^{*}, \quad [\Lambda_{J}, \partial_{K}] = -i\partial_{K}^{*};
\]

(59)

\[
[\Lambda_{K}, \partial_{K}] = \partial_{K}^{*}, \quad [\Lambda_{K}, \partial_{K}] = \partial_{K}^{*};
\]

(60)

\[
[\Lambda_{J}, \partial_{K}] = -i\partial^{*}, \quad [\Lambda_{J}, \partial_{K}] = i\partial^{*}.
\]

(61)

**Proof.** The idea of the proof is the same as that of Lemma 9; since by Theorem 7, the 2-forms \( \omega_{I} \) and \( \omega_{K} \) are osculate order 2 to the constant 2-forms everywhere. The proof reduces to the Euclidean Kähler plane. We follow the same lines of the proof of Lemma 9 as in [6, pages 111–114].

Note every equation in the right column follows by taking conjugate of the equation in the same row of the left column, so it suffices to establish the equations in one column. Here we only give a proof of the left equation of (54). The rest of equations are proved in the same way.

By (50), we have

\[
2[\Lambda_{J}, \partial] = \sum_{k,l} [i_{k} i_{k+n} + i_{k} i_{k+n}, \partial_{l} e_{l}]
\]

(62)

\[
= \sum_{k} [i_{k} i_{k+n} + i_{k} i_{k+n}, \partial_{k} e_{k} + \partial_{k+n} e_{k+n}]
\]

\[
+ \sum_{l \neq k, n} [i_{k} i_{k+n} + i_{k} i_{k+n}, \partial_{l} e_{l}].
\]

Note that

\[
[i_{k} i_{k+n}, \partial_{k} e_{k}] = [i_{k} i_{k+n}, \partial_{k+n} e_{k+n}] = 0,
\]

and if \( l \neq k, k + n \), we have

\[
[i_{k} i_{k+n}, \partial_{l} e_{l}] = [i_{k} i_{k+n}, \partial_{l} e_{l}] = 0.
\]

Therefore

\[
2[\Lambda_{J}, \partial] = \sum_{k} ([i_{k} i_{k+n}, \partial_{k} e_{k}] + [i_{k} i_{k+n}, \partial_{k+n} e_{k+n}]).
\]

(63)

It is not difficult to check that

\[
[i_{k} i_{k+n}, e_{k}] = -2i_{k+n},
\]

(64)

\[
[i_{k} i_{k+n}, e_{k+n}] = 2i_{k};
\]

(65)
hence
\begin{align}
[i_{k+n}, \partial_k e_k] &= \partial_k [i_{k+n}, e_k] = -2 \partial_k i_{k+n}, \\
[i_{k+n} \partial_k e_k] &= \partial_{k+n} [i_{k+n}, e_{k+n}] = 2 \partial_{k+n} i_{k+n}.
\end{align}
(67)

From (65), (67) we get
\[ [\Lambda_j, \partial] = \sum_k (- \partial_k i_{k+n} + \partial_k i_{k+n}) . \]
(68)

Note that the right hand side of (68) is the conjugation of the second equation of (48); we arrive at the first equation of (54).

3.2. Twisted Bochner-Kodaira-Nakano-Type Identities. In this section we will consider the differential operators acting on bundle-valued differential forms. Suppose that \((M, I, J, K)\) is a compact hypercomplex manifold with Kähler manifold with Kähler metric \(g\). Given a holomorphic vector bundle \(E\) over \(M\) with Hermitian metric \(h\), there exists a unique connection \(D\), called Chern connection, which is compatible with the metric \(h\) and satisfying \(D'' = \tilde{\delta}\). Here \(D = D' + D''\) and
\[ D': \Omega^p(E) \rightarrow \Omega^{p+1}(E) , \]
\[ D'': \Omega^p(E) \rightarrow \Omega^{p+1}(E) \]
are its components.

Let \(\{s_a\}\) be a holomorphic frame of \(E\). For any \(E\)-valued differential forms \(\xi = \sum_a \xi^a \otimes s_a\) and \(\eta = \sum_b \eta^b \otimes s_a\) of \(\Omega^p(E)\), we define their local inner product
\[ \langle \xi, \eta \rangle = \sum_{a,b} h_{ab} \xi^a \wedge \eta^b \]
(70)

and global inner product
\[ \langle \xi, \eta \rangle_E = \int_M \langle \xi, \eta \rangle . \]
(71)

Denote the adjoint operators of \(D', D''\) with respect to the inner product by \(\delta', \delta''\). Let \(D'_j : \Omega^p(E) \rightarrow \Omega^{p+1}(E)\) be the composition of
\[ J^{-1} \otimes id_E : E \otimes \Omega^p(M) \rightarrow E \otimes \Omega^{p+1}(M) , \]
(72)

\(D'\) and \(J \otimes id_E\), and denote the adjoint operator of \(D'_j\) with respect to \((,)_E\) by \(\delta'_j\). The operators \(D''_j, \delta''_j, D'_K, \delta'_K, D''_K, \delta''_K\) are defined in the same way. The operators \(*, d, \Lambda, L, \Lambda_j, L_j, \Lambda_K, L_K\) extend naturally to \(\Omega^p(E)\). The proof of the following commuting relations among these operators acting on \(\Omega^p(E)\) follows from Proposition 10.

**Proposition 11.**
\[ [\Lambda, D'] = i\delta', \]
\[ [\Lambda, D''] = -i\delta''; \]
\[ [\Lambda_j, D'] = \delta''_j; \]
\[ [\Lambda, D''] = \delta''_j; \]
\[ [\Lambda_j, D''] = -\delta''_j; \]
\[ [\Lambda_j, D''_K] = i\delta''_j, \]
\[ [\Lambda_j, D''_K] = -i\delta''_j; \]
\[ [\Lambda_j, D''_K] = -i\delta''_j, \]
\[ [\Lambda_j, D''_K] = i\delta''_j. \]

The Chern curvature tensor \(\Theta\) of \(E\) is an \(\text{End}(E)\)-valued differential form defined by
\[ (D''D' + D'D'') \xi = \Theta \wedge \xi = e(\Theta) \xi \in \Omega^{p+1}(E), \]
(74)
\[ \xi \in \Omega^p(E). \]

The holomorphic Laplacian \(\Delta' = D'\delta' + \delta'D'\) and antiholomorphic Laplacian \(\Delta'' = D''\delta'' + \delta''D''\) are related by the classical Bochner-Kodaira-Nakano identity [11]:
\[ \Delta'' - \Delta' = [e(i\Theta), \Lambda], \]
(75)

it plays a fundamental role in establishing many important vanishing theorems. Our naive motivation is to get more vanishing theorems by using the chances given by the other two complex structures \(I\) and \(K\) for a hypercomplex manifold. To this aim we define the following self-adjoint operators:
\[ \Delta'_j = D'_j \delta'_j + \delta'_j D'_j, \]
\[ \Delta''_j = D''_j \delta''_j + \delta''_j D''_j . \]
(76)

Let \(\Theta_I, \Theta_K\) be the curvature components corresponding to the operators \(D''D'_j + D'_j D''\) and \(D''D'_K + D'_K D''\), respectively. We have the following twisted Bochner-Kodaira-Nakano-type identities.

**Proposition 12.**
\[ \Delta'' - \Delta'_j = [e(\Theta_I), \Lambda_j], \]
\[ \Delta'' - \Delta''_K = -[e(\Theta_K), \Lambda_K] . \]
(77)

**Proof.** By Proposition 11,
\[ \Delta'' - \Delta'_j = D'' \delta''_j + \delta''_j D'' - (D'_j \delta'_j + \delta'_j D'_j) \]
\[ = - D'' [\Lambda_j, D'_j] - [\Lambda_j, D'_j] D'' \]

\[ -D_j' [\Lambda_j, D'''] - [\Lambda_j, D'''] D_j' \]
\[ = -D''\Lambda_j D_j' + D'''D_j \Lambda_j - \Lambda_j D_j' D'' + \Lambda_j D_j' D''' + D_j' \Lambda_j D'' - D_j' \Lambda_j D''' + D_j' D'' \Lambda_j - \Lambda_j D_j' D'' + D''D_j \Lambda_j \]
\[ = (D''D_j' + D_j' D''') \Lambda_j - \Lambda_j (D''D_j' + D_j' D''') \]
\[ = [\psi (\Theta_j), \Lambda_j] . \]

(78)

The second equation follows in the same way.

Recall that \( \varphi = \omega_1 + i \omega_K \); correspondingly, we define \( L_{\varphi} = L_j + i L_K \), \( D_{\varphi}' = \frac{1}{2} (D_j' - i D_K' \) and \( D_{\varphi}'' = \frac{1}{2} (D_j'' - i D_K''). \)

Then \( \Lambda_{\varphi} = \Lambda_j - i \Lambda_K \), \( D_{\varphi}' = \frac{1}{2} (\delta_j' + i \delta_K' \), and \( D_{\varphi}'' = \frac{1}{2} (\delta_j'' + i \delta_K'' \) are their adjoint operators, respectively.

Proposition 13.

\[ [\Lambda_{\varphi}, D'] = \delta_{\psi}', \quad \Lambda_{\varphi}, D'''] = -\delta''; \]
\[ [\Lambda_{\varphi}, D'''] = \delta_{\psi}', \quad \Lambda_{\varphi}, D''] = -\delta'. \]

Proof. By Proposition 11,

\[ [\Lambda_{\varphi}, D'] = \frac{1}{2} (\Lambda_j, D' - i [\Lambda_K, D']) \]
\[ = \frac{1}{2} (\delta_j'' + i \delta_K'') = \delta''', \quad [\Lambda_{\varphi}, D'''] = \frac{1}{2} (\Lambda_j, D'' - i [\Lambda_K, D'']) \]
\[ = \frac{1}{2} (\Lambda_j, D'') + i (\Lambda_K, D''') - [\Lambda_K, D''] \]
\[ = -\delta'''. \]

(80)

The rest of equations are proved in the same way.

Let \( \Theta_{\varphi} \) be the curvature component corresponding to the operator \( D''D_{\varphi}' + D_{\varphi}' D''' \). Let \( \Lambda_{\varphi}'' = \Lambda_j + i \Lambda_K \) be another twisted Laplacian operator. By using Proposition 13, it is easy to prove the following Bochner-Kodaira-Nakano-type identity.

Proposition 14.

\[ \Lambda'' - \Lambda_{\varphi}'' = \psi (\Theta_{\varphi}), \Lambda_{\varphi} . \]

(82)

4. Local Expressions of Bochner-Kodaira-Nakano Identities

Let \((M, I, J, K)\) be a compact hypercomplex manifold and suppose that \((M, I)\) is a Kähler manifold with Kähler metric \( g \). Given a holomorphic vector bundle \( E \) of rank \( r \) over \( M \) with Hermitian metric \( h \), let \( D = D_j^j + D_{\varphi}'' \) be the Chern connection of \( E \) with \( D'' = \partial \). Write \( \partial' = \partial + \theta \),

\[ D' = \partial + \theta, \]

(83)

where \( \theta \in \Omega^{1,0}(End(E)) \) is the connection matrix. By the compatible condition of the connection \( D \) and the metric \( h \), we have

\[ dh = h\theta + \tilde{\theta} h; \]

(84)

by comparing the type, we get

\[ \partial h = h\theta, \quad \tilde{\theta} h = \tilde{\theta} h; \]

(85)

it follows that

\[ \theta = h^{-1} \partial h. \]

(86)

From \( \partial^2 = \tilde{\partial}^2 = 0 \), we know that

\[ \partial \theta = -\theta \wedge \theta, \]

(87)

and the Chern curvature \( \Theta \) is given by

\[ \Theta = \theta \wedge \theta. \]

(88)

Let \( \{z^i\} \) be the local holomorphic coordinate of \( M \) such that the holomorphic coframes in (18) are represented by \( \theta^i = dz^i \).

Let \( \{s^i\} \) be a holomorphic frame and \( \{s^j\} \) be another dual frame of \( E \). Let \((g_{\psi})\) and \((h_{\varphi})\) be the Hermitian metrics on \( M \) and \( E \), respectively, and their inverses denoted, respectively, by \((g^{\psi})\) and \((h^{\varphi})\). Then the connection and curvature could be expressed by

\[ \theta = \sum_{a,\beta} g^{a\beta} s_a \otimes s^\beta, \quad \Theta = \sum_{a,\beta} \Theta_{\psi}^{a\beta} s_a \otimes s^\beta \]

(89)

with

\[ \Theta^a_{\beta} = \sum_{\gamma, j} R^a_{\gamma j} \partial h^{\gamma j} ; \]

(90)

where

\[ R^a_{\gamma j} = -\sum_{i} g^{a i} \partial h^{\gamma j} g_{i\psi} + \sum_{\gamma, \lambda, \mu} h^{\gamma j} \partial h^{\lambda \mu} \eta_{\gamma \psi} h_{ij} . \]

(91)
Proposition 15. Consider
\[ \Theta_J = \bar{\Theta}_J; \Theta_K = i\Theta_J; \Theta_\Psi = \Theta_J. \] (92)

Proof. For any \( \xi = \sum \xi^\alpha \otimes s_\alpha \in \Omega^{p, q}(E) \),
\[ D'_J \xi = \sum \left( \partial_\alpha \xi^\alpha \right) s_\alpha + \sum \left( J^{-1} \partial^{\alpha} \xi \right) s_\beta \]
\[ = \left( \partial_\alpha + J^{-1} \partial_\beta \right) \xi \]
\[ = \left( \partial_\alpha + \partial_\beta \right) \xi. \]
Therefore,
\[ D'_J D'' \xi = \sum \left( \partial_\alpha \xi^\alpha \right) s_\alpha + \sum \left( J^{-1} \partial^{\alpha} \xi \right) s_\beta \]
\[ - \sum \left( J^{-1} \partial^{\alpha} \xi \right) s_\beta. \] (94)

Since \( DJ = D'J = D''J = 0 \), we have \( D_J = \bar{D}_J \); hence
\[ \bar{\partial}_\alpha \xi^\alpha = \bar{D}_J \left( J^{-1} \partial_\beta \xi \right) = J^{-1} \bar{\partial}_\beta \xi, \]
\[ \partial_\beta \xi = J^{-1} \bar{\partial}_\beta \xi. \] (95)
Thus
\[ \bar{\partial}_\alpha \xi^\alpha + \partial_\beta \xi^\beta = 0. \] (96)

From (94) and (96), we conclude the first equation of (92).

From (21) and (22), clearly \( \Theta_K = \bar{\Theta}_K = i\Theta_J \). Since by definition
\[ \Theta_\Psi = \frac{1}{2} \left( \bar{\Theta}_J - i\bar{\Theta}_K \right), \]
we have \( \Theta_\Psi = \Theta_J \). \( \square \)

From (21), (22), and (90), we have the following local expressions of connection and curvature:
\[ (\Theta_J)^{\alpha}_{\beta} = \sum \frac{n \partial_\alpha}{\partial z^j} d\bar{z}^{j+n} - \sum \frac{n \partial_\alpha}{\partial z^{j+n}} d\bar{z}^j, \]
\[ (\Theta_J)^{\alpha}_{\beta} = \sum \frac{n \partial_\alpha}{\partial z^{j+n}} d\bar{z}^{j+n} - \sum \frac{n \partial_\alpha}{\partial z^j} d\bar{z}^j. \] (98)

Using (91), we could write the curvature components of \( \Theta_J \) simply as
\[ (\Theta_J)^{\alpha}_{\beta} = - \sum \sum \frac{n R^{\alpha}_{\beta j k}}{\partial_\beta \partial_\alpha} d\bar{z}^j \wedge d\bar{z}^{j+n} \]
\[ + \sum \sum \frac{n R^{\alpha}_{\beta j+k}}{\partial_\beta \partial_\alpha} d\bar{z}^j \wedge d\bar{z}^i. \] (99)

Therefore,
\[ \Theta_J = \sum \sum \left( -R^{\alpha}_{\beta j k} d\bar{z}^j \wedge d\bar{z}^{j+n} \otimes s_\alpha \otimes s_\beta \right) \]
\[ + R^{\alpha}_{\beta j+k} d\bar{z}^j \wedge d\bar{z}^i \otimes s_\alpha \otimes s_\beta. \] (100)

The proof of the following lemma is simple via using the commuting relations (43), (45), and (44); we omit it here for brevity.

Lemma 16. Let the operators \( e_k, \bar{e}_k \) be the wedge operators defined as in (41),(42) and \( i_k, \bar{i}_k \) their adjoint operators; then for any integer \( 1 \leq p, q, k \leq 2n \),
\[ [\bar{e}_p \bar{e}_q, i_k] = 0; \] (101)
\[ [e_p \bar{e}_q, i_k] = \begin{cases} 0, & \text{if } p, q \neq k; \\
-2\epsilon_{p k}^q, & \text{if } p = k, q \neq k; \\
-2\epsilon_{p k}^q, & \text{if } q = k, p \neq k; \\
4 - 2\epsilon_{p k}, & \text{if } q = k, p = k; \end{cases} \] (102)
\[ [\bar{e}_p \bar{e}_q, i_k] = \begin{cases} 0, & \text{if } p, q \neq k, k + n; \\
-2\epsilon_{p k}^q, & \text{if } p = k, q \neq k, k + n; \\
2\epsilon_{p k}^q, & \text{if } q = k, p \neq k, k + n; \\
-2\epsilon_{p k}^q, & \text{if } q = k, p = k, k + n; \\
4 - 2\epsilon_{p k}, & \text{if } q = k, p = k, n; \\
-4 + 2\epsilon_{p k}, & \text{if } q = k, p = k, n. \end{cases} \] (103)

For any \( E \)-valued differential form \( \xi \in \Omega^{p,q}(E) \), write
\[ \xi = \sum \xi^\alpha \otimes s_\alpha = \sum \xi_{p,Q,a} \theta^p \wedge \bar{\theta}^Q \otimes s_\alpha, \] (104)
where the lengths \(|P| = p\) and \(|Q| = q\), and
\[ \xi^\alpha = \sum \xi_{p,Q,a} \theta^p \wedge \bar{\theta}^Q \in \Omega^{p,q}(M). \] (105)

Proposition 17.
\[ (e_k \xi, \xi) = \sum \xi_{p,Q,a} \xi_{p,Q,a}, \]
\[ (e_k \xi, \xi) = \sum \xi_{p,Q,a} \xi_{p,Q,a}. \] (106)

Proof. Note that \( i_k (\theta^p \wedge \bar{\theta}^Q) \) is of the same type as \( \theta^p \wedge \bar{\theta}^Q \) for \( k \neq P \). Since \( (e_k i_k u, u) = (i_k e_k u, u) = (\theta^p \wedge \theta^Q, \theta^p \wedge \theta^Q, \theta^k \wedge \theta^k \wedge \theta^k \wedge \bar{\theta}^Q) = 2(\theta^p \wedge \bar{\theta}^Q, \theta^p \wedge \bar{\theta}^Q) \), Proposition 17 follows. \( \square \)

The formula in the following proposition appeared in Section 4 of [11] without proof. (Note that the expression is a little different since it uses unit orthogonal frame in [11] while here \( (\theta^p, \theta^p) = 2 \)). We will give a very simple proof here.
**Proposition 18.** Consider

\[
\frac{1}{2} \langle [e(i\Theta), \Lambda] \xi, \xi \rangle = \sum_{\alpha, \beta, P, Q} \sum_{(S_T) = (T_T) = (Q)} R^\alpha_{\beta|P, Q} \xi_{P, S_T, \alpha} \xi_{P, T_T, \beta}
\]

(107)

By (89) and (90),

\[
e(i\Theta) = \sum_{\alpha, \beta, p, q} \sum_{k \in \mathbb{N}} \bar{e}_p e_q \xi_{\alpha} \otimes s_\beta
\]

(108)

Recall that \(\Lambda = (i/2) \sum_k i_k^k\), hence

\[
[e(i\Theta), \Lambda] \xi = -\frac{1}{2} \sum_{p, q, \alpha, \beta, k, P, Q} R^\alpha_{p, q, k, P, Q} \left( [\bar{e}_p, e_q, i_k^k] \xi^\beta \right) \xi_\alpha
\]

(109)

By (102) of Lemma 16,

\[
[e(i\Theta), \Lambda] \xi = \sum_{\alpha, \beta, k, P, Q, k} R^\alpha_{\beta|P, Q} \left( \bar{e}_k e_k \xi^\beta \right) \xi_\alpha
\]

(110)

\[
\sum_{\alpha, \beta, p, q, k} R^\alpha_{\beta|P, Q} \left( e_p e_q i_k^k \xi^\beta \right) \xi_\alpha
\]

\[
-2 \sum_{\alpha, \beta, k, P, Q} R^\alpha_{\beta|P, Q} \left( \bar{e}_k e_k \xi^\beta \right) \xi_\alpha
\]

Taking inner products of both sides of the above equation with \(\xi\), and using (106), we get immediately (107). \(\square\)

**Proposition 19.** Consider

\[
\frac{1}{2} \langle [e(\Theta), \Lambda_j] \xi, \xi \rangle
\]

(111)

Using the expression (100) of the curvature \(\Theta_j\), we have

\[
e(\Theta_j)
\]

(112)

Recall that, by (51),

\[
\Lambda_j = \frac{1}{2} \sum_{k=1}^n \left( i_k^{k+n} + i_k^{k+n} \right)
\]

(113)

By (101) of Lemma 16, for any integers \(1 \leq p, q, k \leq 2n\), we have

\[
[\bar{e}_p e_q, i_k^{k+n}] = 0.
\]

(114)
Therefore
\[c(\Theta_j), \Lambda_j] \xi = \frac{1}{2} \sum_{\alpha, \beta=1}^{r} \sum_{p=1}^{n} \left(-R_{\beta(p,q)}^\alpha \xi_{\gamma(p,q)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)} \right) \xi_{\beta(p,q)}^\alpha \]
\[= R_{\beta(p,q)}^\alpha \xi_{\gamma(p,q)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)}^\alpha \] \(s_{\alpha} \).
(115)

By (103) of Lemma 16, if neither \(p\) nor \(q\) takes values \(k, k+n\), we have
\[\left[\bar{\xi}_p, \bar{\xi}_q, \bar{\xi}_k \right] = 0; \]
(116)

hence
\[c(\Theta_j), \Lambda_j] \xi = \sum_{p=1}^{r} \sum_{k=1}^{2n} \sum_{\alpha, \beta=1}^{r} \left(-R_{\beta(p,q)}^\alpha \xi_{\gamma(p,q)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)} \right) \xi_{\beta(p,q)}^\alpha \xi_{\gamma(q,p)}^\alpha \] \(s_{\alpha} \).
(117)

Take \(\sum_{p=k,q=k}^{r} \) for an example; here it means in the summation of (115) we add only a restricted condition \(p = k, q \neq k\). In others words, it is a summation whose terms are the same as in (115), where indices \(\alpha, \beta, P, Q, p, q, k\) vary with the same range as in (115) except that \(p = k, q \neq k\).

By (103) of Lemma 16, for \(j \neq k, k+n\), we have
\[\left[\bar{\xi}_j, \bar{\xi}_k, \bar{\xi}_k \right] = 2 \bar{\xi}_j \bar{\xi}_k; \]
(118)

hence
\[\sum_{p=k,q=k}^{r} \left(R_{\beta(p,q)}^\alpha \xi_{\gamma(p,q)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)} \right) \xi_{\beta(p,q)}^\alpha \xi_{\gamma(q,p)}^\alpha \] \(s_{\alpha} \).
(119)

By the same reasons,
\[\sum_{p=k,q=k}^{r} \left(R_{\beta(p,q)}^\alpha \xi_{\gamma(p,q)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)} \right) \xi_{\beta(p,q)}^\alpha \xi_{\gamma(q,p)}^\alpha \] \(s_{\alpha} \).
(120)

Since by (103) of Lemma 16, \(\left[\bar{\xi}_p, \bar{\xi}_p, \bar{\xi}_k \right] = 4 - 2\bar{\xi}_k \)
\[\sum_{p=k,q=k}^{r} \left(R_{\beta(p,q)}^\alpha \xi_{\gamma(p,q)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)} \right) \xi_{\beta(p,q)}^\alpha \xi_{\gamma(q,p)}^\alpha \] \(s_{\alpha} \).
(121)

Note that (119) could be rewritten as
\[\sum_{p=k,q\neq k}^{r} \left(R_{\beta(p,q)}^\alpha \xi_{\gamma(p,q)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)} \right) \xi_{\beta(p,q)}^\alpha \xi_{\gamma(q,p)}^\alpha \] \(s_{\alpha} \).
(122)

we remark where in the first summation of right hand side of (122) we have changed the index \(k\) to \(q\) (note that \(p, q, k\) have equal positions since all of them vary form 1 to \(n\)). Similarly, (120) could be rewritten as
\[\sum_{p\neq k,q=k}^{r} \left(R_{\beta(p,q)}^\alpha \xi_{\gamma(p,q)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)} \right) \xi_{\beta(p,q)}^\alpha \xi_{\gamma(q,p)}^\alpha \] \(s_{\alpha} \).
(123)

Combining (123), (122), and (121), we get from (117) that
\[c(\Theta_j), \Lambda_j] \xi = \sum_{\alpha, \beta=1}^{r} \sum_{\gamma=1}^{n} \sum_{p=1}^{2n} \left(R_{\beta(p,q)}^\alpha \xi_{\gamma(p,q)} + R_{\beta(q,p)}^\alpha \xi_{\gamma(q,p)} \right) \xi_{\beta(p,q)}^\alpha \xi_{\gamma(q,p)}^\alpha \] \(s_{\alpha} \).
(124)
If we rearrange the summation range of indices $p, q$, we get

$$
[e \left( \Theta_j \right), \Lambda_j] \xi
= \frac{r}{a, b = 1, p, q = 1; P, Q} \sum_{\alpha, \beta = 1}^{2n} \sum_{\alpha, \beta = 1}^{2n} R^a_{\beta \rho \sigma} E^{\rho \sigma}_{q, r} \xi^p \eta^q - 2 \frac{r}{a, b = 1, p, q = 1; P, Q} \sum_{\alpha, \beta = 1}^{2n} \sum_{\alpha, \beta = 1}^{2n} R^a_{\rho \kappa \nu} \xi^p \eta^q \delta_{a, \beta}
+ \frac{r}{a, b = 1, p, q = 1; P, Q} \sum_{\alpha, \beta = 1}^{2n} \sum_{\alpha, \beta = 1}^{2n} \left( R^a_{\rho \sigma \tau} - R^a_{\rho \sigma \tau} \right) \xi^p \eta^q \delta_{a, \beta}
+ \frac{r}{a, b = 1, p, q = 1; P, Q} \sum_{\alpha, \beta = 1}^{2n} \sum_{\alpha, \beta = 1}^{2n} \left( R^a_{\rho \sigma \tau} - R^a_{\rho \sigma \tau} \right) \xi^p \eta^q \delta_{a, \beta}
\times \left[ e \left( \Theta_j \right), \Lambda_j \right],
\tag{125}
$$

Taking inner products of both sides of (125) with $\xi$, and using (106), we get immediately (111).

By Propositions 12 and 14, (127) and (128), we get the proof of Proposition 19, in particular, from (114), we have

$$
\left[ e \left( \Theta_j \right), \Lambda_j \right] = \frac{1}{2} \sum_{k=1}^{n} \sum_{i, k+1} \left[ e \left( \Theta_j \right), \Lambda_j \right] = -\left[ e \left( \Theta_j \right), \Lambda_j \right],
\tag{127}
$$

and similarly

$$
\left[ e \left( \Theta_j \right), \Lambda_j \right] = \frac{1}{2} \sum_{k=1}^{n} \sum_{i, k+1} \left[ e \left( \Theta_j \right), \Lambda_j \right] = 2 \left[ e \left( \Theta_j \right), \Lambda_j \right].
\tag{128}
$$

By Propositions 12 and 14, (127) and (128), we get the following:

**Proposition 20.** Consider

$$
\Delta^\prime \Lambda_j = \left[ e \left( \Theta_j \right), \Lambda_j \right],
\tag{129}
$$

and $\Lambda_j$ as an example. If $\xi$ is an arbitrary $E$-valued $(p, q)$-form, then an integration by part of the formula

$$
\left( \left( \Delta^\prime \Lambda_j \right) \xi, \xi \right)_E = \left( \left[ e \left( \Theta_j \right), \Lambda_j \right] \xi, \xi \right)_E
\tag{132}
$$

and noting that $(\Delta^\prime \Lambda_j \xi) = \| D^\prime \xi \|_E^2 + \| \delta^\prime \xi \|_E^2 \geq 0$ yield

$$
\| D^\prime \xi \|_E^2 + \| D^\prime \xi \|_E^2 \geq \sum_{\alpha = 1}^{2n} \left( \left| \left[ e \left( \Theta_j \right), \Lambda_j \right] \xi, \xi \right|_E \right).
\tag{134}
$$

If $\xi \in \mathcal{H}^p(M, E)$, then $\xi$ is harmonic and hence $\Delta^\prime \Lambda_j \xi = 0$ by the Hodge theory. Furthermore if $\left[ \Theta_j, \Lambda_j \right]$ is positive definite everywhere on $\Omega^p(M, E)$, then $\xi = 0$. Hence $\mathcal{H}^P(M, E) \equiv \mathcal{H}^P(M, \Omega^P(M)) \equiv \mathcal{H}^P(M, E) = 0$. Thus we get a vanishing cohomology group. Therefore, using (132) to prove a vanishing theorem for $E$-valued Dolbeault's cohomology groups, the key point is to find conditions under which the operator $\left[ \Theta_j, \Lambda_j \right]$ is positive definite.

We can see from above reasoning, the second and third formulae of Proposition 20 produce new no vanishing theorems since their right hand sides are the same up to a positive constant. For a hypercomplex Kähler manifold $(M, I, J, K)$, the three complex structures $I, J, K$ have symmetric roles; however, they are not independent of each other and related by $IJ = -JI = K$. This may account that only two Bochner-Kodaira-Nakano identities, the formula (75) and one formula of Proposition 20, produce different vanishing theorems. Note however, that the computations of the proof (though we donot give its proof) of the last formula of Proposition 20 are simpler than the other two equations.

By (89), the Chern curvature form of $E$ is given by

$$
\Theta = \Theta(E) = \sum_{\alpha, \beta} R^a_{\rho \beta \alpha} \delta_{\alpha, \beta} \otimes s^\beta
= \sum_{\alpha, \beta, j, k} R^a_{\beta j k} dz^j \wedge d\bar{z}^k \otimes s_{\alpha} \otimes s^\beta.
\tag{135}
$$

The first Chern class $c_1(E) \in \mathcal{H}^2(M, \mathbb{R})$ is a cohomology class which has a representation via using the Chern curvature form

$$
c_1(E) = \frac{1}{2\pi} \text{Tr}_E \left(i\Theta(E)\right) = \frac{i}{2\pi} \Theta \left( \det(E) \right).
\tag{136}
$$

Conversely, any 2-form representing the first Chern class $c_1(E)$ is in fact the Chern curvature form of some Hermitian metric on $\det E$ (up to a constant). In local coordinate we have

$$
i\Theta \left( \det(E) \right) = \sum_{j, k} R^a_{j k} dz^j \wedge d\bar{z}^k = -\partial \bar{\partial} \log \det \left(h_{a\bar{b}} \right)
\tag{137}
$$

with $R^a_{j k} = \sum_{\alpha} R^a_{\alpha j k}$. In particular, if $E$ is a line bundle, then its curvature form represents its first Chern class up to a constant $1/2\pi$.

In [13], we introduce the following notion for semipositive holomorphic vector bundles. Based on it and the formula (75) for Kähler manifolds, we get some new vanishing theorems.
Definition 21. A holomorphic vector bundle $E$ of rank $r$ with Hermitian metric $h$ on a compact complex manifold $M$ of complex dimension $n$ is called $(k, s)$-positive for $1 \leq s \leq r$ if the following holds for any $x \in M$: for any $s$-tuple vectors $v' \in V, 1 \leq j \leq s$, where $V = E_x$ (resp., $T_xM$), the Hermitian form

$$Q_x(*, *) = i\Theta(E) \left( \sum_{j=1}^{s} \langle \Theta_J, \Lambda_J \rangle \otimes \psi_j, \sum_{j=1}^{s} \langle \Theta_J, \Lambda_J \rangle \otimes \psi_j \right),$$

(138)

defined on $W_{\theta}^s$ is semipositive and the dimension of its kernel is at most $k$, where $W = T_xM$ (resp., $E_x$).

Clearly the $(0, s)$-positivity is equivalent to the Demainly $s$-positivity [11] and the Nakano positivity [14] is equivalent to the $(0, s)$-positivity if $s \geq \min(n, r)$. The $(0, 1)$-positivity is equivalent to the Griffiths positivity. For general integer $k$, the $(k, 1)$-positivity is a semipositive version of the Griffiths positivity [6]. A holomorphic vector bundle $E$ of arbitrary rank is called Griffiths $k$-positive if it is $(k, 1)$-positive.

Theorem 22. Let $M$ be a compact hypercomplex Kähler manifold of dimension $4n$ and let $E$ be a Hermitian holomorphic vector bundle of rank $r$ on $M$ such that $E$ is $(k, s)$-positive. Then

(i) $H^p(M, E) = 0$, for $p > k$ and $s \geq \min(2n - p + 1, r)$;

(ii) if in addition $k \leq 2n - 1$, then for $s \geq \min(2n - p + 1, r)$ and any nonnegative integer $p$, $H^{2n}(M, \Omega^p \otimes E) = 0$.

(139)

Proof. (i) follows from Theorem 3.9 of [13] together with the fact that the anticanonical bundle of $M$ is trivial. It suffices to prove (ii). For any $E$-valued $(p, 2n)$-form $\xi = \sum_{P, Q} \xi_{PQ} \theta^P \wedge \bar{\theta}^Q \wedge \cdots \wedge \bar{\theta}^Q \wedge \otimes_{\alpha} \in \Omega^{p, 2n}(E)$, where $Q = 12 \cdots 2n$ is a fixed index. By Definition 5.1, the Hermitian form $i\Theta(E)$ is semipositive on $\Omega^{p, 2n}(E)$ if $s \geq \min(2n - p + 1, r)$; we could diagonalize it in some local orthogonal frames such that $R_{a \bar{B}P} = \lambda_{aP}^{\bar{B}} \delta_{aP} \delta_{P\bar{B}}$. Here $(\lambda_{aP})_{1 \leq a \leq 2n, 1 \leq P \leq s}$ are nonnegative and for a fixed $\alpha$, without loss of generality, we assume that $\lambda_{a1} \leq \lambda_{a2} \leq \cdots \leq \lambda_{a2n}$ with $\lambda_{a2n} \geq 0$; in particular $\lambda_{a2n} > 0$ since $k \leq 2n - 1$. Put $\Lambda = \min(\lambda_{a2n} \leq 1 \leq \alpha \leq r)$. Then $\Lambda$ is a positive number. Note that in the present situation the two first terms cancel each other in the first summation and the last summation vanishes in the three big summations of the Bochner-Kodaira-Nakano formula (III). Hence

$$\frac{1}{2} \langle \Theta(J), \Lambda_J, \xi, \xi \rangle = \sum_{\alpha, P} \sum_{1 \leq j \leq n} \left( \Lambda_{aP}^{\alpha} \right)^2 + \sum_{j=n} \Lambda_{j+n}^{\alpha} \left( \xi_{PQ} \right)^2 \geq \sum_{P} \left( \Lambda_{aP}^{\alpha} + \Lambda_{a2n}^{\alpha} + \cdots + \Lambda_{a2n}^{\alpha} \right) \left( \xi_{PQ} \right)^2 \geq \lambda \left( \sum_{P} \sum_{\alpha} \left| \xi_{PQ} \right|^2 \right) \geq \lambda |\xi|^2.$$  

(140)

Thus $[e(\Theta), \Lambda_J]$ is positive definite on $E$-valued $(p, 2n)$-forms. So we have $H^{2n}(M, \Omega^p \otimes E) = 0$ for any $p$ and $s \geq \min(2n - p + 1, r)$. 

A holomorphic line bundle $B$ on $M$ is called $k$-positive if there is a Hermitian metric on $B$ such that its first Chern class $c_1(B)$ is semipositive and has at least $n - k$ positive eigenvalues [14, 15]. $E$ is a holomorphic line bundle (denoted it by $B$ for the difference), then in Definition 21 only $(k, 1)$-positivity is applicable for $B$, and clearly $B$ is $(k, 1)$-positive (or Griffiths $k$-positive) if and only if it is $k$-positive, since the first Chern class has a representation by its Chern curvature form up to a positive constant.

Theorem 23. Let $B$ be a $k$-positive holomorphic line bundle on a compact hypercomplex Kähler manifold $M$. Then

(i) $H^p(M, \Omega^q \otimes B) = 0$, for $p + q > 2n + k$;

(ii) $H^p(M, \Omega^q \otimes B) = 0$, for $p > n + [k/2]$ and any nonnegative integer $q$.

Proof. We get (i) by using the Gigante-Girbau vanishing theorem on Kähler manifolds [15]; a simple proof is given in Theorem 2.4 of [13]. (i) is proved via using (75) and changing the Kähler metric on $M$. (ii) is proved in the same way as (i) by using the Bochner-Kodaira-Nakano identities in Proposition 20. Here we give a proof of (ii) in the following paragraph.

Choose a holomorphic local coordinate system at each point $x \in M$, which diagonalizes simultaneously the Hermitian form $i\Theta(x)$ and $\Theta(x)$ since both of them are semipositive, such that

$$\omega(x) = i \sum_{j=1}^n \mu_j(x) dz^j \wedge d\bar{z}^j,$$

(141)

$$i\Theta(x) = \sum_{j=1}^n v_j(x) dz^j \wedge d\bar{z}^j.$$

Without loss of generality, assume that $v_1(x) / \mu_1(x) \leq \cdots \leq v_{2n}(x) / \mu_{2n}(x)$. Then for any $(p, q)$-form $\xi = \sum_{P, Q} \xi_{PQ} \theta^P \wedge \bar{\theta}^Q \wedge \omega \wedge \cdots \wedge \omega \wedge \otimes_{\alpha} \in \omega^{p, q}(E)$, the last big summation vanishes in the formula (III). $\langle \Theta(J), \Lambda_J, \xi, \xi \rangle$ is expressed by the first two big summations in (III) in the following way:

$$\langle \Theta(J), \Lambda_J, \xi, \xi \rangle(x) = \sum_{|P|=p,|Q|=q} \sum_{p_1 \leq n} \gamma_{p_1}(x) + \sum_{p_1 \neq q, p_1 \leq n} \gamma_{p_1}(x)$$

(140)
Geometry

\[ + \sum_{p \in \Omega, 1 \leq p \leq n} \frac{v_{p+n}(x)}{\mu_{p+n}(x)} (x) \sum_{p=1}^{2n} \frac{v_p(x)}{\mu_p(x)} (x) \frac{\mu_{p} - \mu_{p+n}}{p - n} (x) \frac{\xi}{\kappa \theta} [\xi, \xi] \geq \frac{1}{2} \kappa^2. \]

(142)

Observe that if the ratio \( v_j(x)/\mu_j(x) \) varies small when \( j \) varies, for example, in the extreme case when all \( v_j(x)/\mu_j(x) \) are equal; then \( [i \Theta_\theta(E), \Lambda](x) \) is positive when \( p > n \). This observation tells us that if we choose the Kähler metric \( \omega \) properly such that the eigenvalues of \( i \Theta_\theta(E) \) vary mildly relative to \( \omega \), then we can deduce the positivity of \([i \Theta_\theta(E), \Lambda] \).

Since \( B \) is \( k \)-positive if and only if its curvature \( i \Theta \) is semi-positive with rank \( i \Theta \geq n - k \), we may have a special choice of the Kähler metric of \( M \) with \( \tilde{\omega} := \tilde{\omega}_j := i \Theta + k \omega \) for some positive number \( k \). Then \( M \) is a compact hypercomplex manifold and \( (M, I) \) is still Kähler manifold with the new Kähler form \( \tilde{\omega} \). From now on we consider \( M \) as a new hypercomplex Kähler manifold with Kähler metric \( \tilde{\omega} \). Correspondingly, we have three new nondegenerate 2-forms, \( \tilde{\omega}_j, \tilde{\omega}_j, \tilde{\omega}_k \). Let \( \tilde{\lambda}_j, \tilde{\lambda}_j, \tilde{\lambda}_k \) be the associated adjoint operators of multiplication by \( \tilde{\omega}_j, \tilde{\omega}_j, \tilde{\omega}_k \). We could get the Bochner-Kodaira-Nakano identities with respect to \( \tilde{\omega}_j, \tilde{\omega}_j, \tilde{\omega}_k \) as in Section 4. Then the eigenvalues of \( \Theta \) relative to \( \tilde{\omega} \) are \( r_j(x) \) with

\[ r_j(x) = \frac{v_j(x)}{\kappa \mu_j(x) + v_j(x)} = \frac{v_j(x)}{\kappa + v_j(x)} \frac{\mu_j(x)}{\mu_j(x)} \]

\[ = \frac{1}{1 + \left( \frac{v_j(x)}{\mu_j(x)} \right) / \kappa} \]

\[ = 1 - \frac{1}{1 + \left( \frac{v_j(x)}{\mu_j(x)} \right) / \kappa}. \]

(143)

Fix a point \( x \in M \) and assume that rank(\( \Theta_\theta \)) = \( 2n - s \geq 2n - k \). Then \( 0 = r_s(x) < r_{s+1}(x) < \cdots < r_{2n}(x) \). Thus \( r_j(x) = 0 \) for \( j \leq s \). If we choose \( \kappa \to 0^+ \), then \( r_j(x) \to 1 \) for all \( s + 1 \leq j \leq 2n \). If \( p > n + [k/2] \), then \( p > n + (k/2) \geq k \geq s \) and

\[ \lim_{\kappa \to 0^+} \left[ 2 \left( r_1(x) + \cdots + r_p(x) \right) - (r_1(x) + \cdots + r_{2n}(x)) \right] \]

\[ = 2 \left( p - s \right) - (2n - s) \]

\[ \geq 2 \left( p - \left( n + \frac{k}{2} \right) \right) \]

\[ \geq 2 \left( 1 + \frac{k}{2} - \frac{k}{2} \right) \geq 2. \]

(144)

Since \( M \) is compact, we can use a finite cover by open sets, such that, on each open set, if \( \kappa \) is a sufficiently small positive number, we may have on each open neighborhood and hence everywhere on \( M \) the following:

\[ \left( \left[ \Theta_j, \tilde{\lambda}_j \right] \xi, \xi \right) \geq \frac{1}{2} \kappa^2. \]

Therefore if \( p > n + [k/2] \), then \[ [\Theta_j, \tilde{\lambda}_j] \) is positive on \( \Omega^q(B) \) and (ii) of Theorem 23 follows.

\[ \Box \]

Corollary 24. Let \( B \) be a holomorphic \( k \)-positive line bundle on a compact hyper-Kähler manifold \( M \). Then

(i) \( H^q(M, \Omega^q \otimes B) = 0 \) for \( p + q > 2n + k \);

(ii) \( H^q(M, \Omega^q \otimes B) = 0 \), for \( p > n + [k/2] \) and any nonnegative integer \( q \).

In particular, if \( B \) is a positive holomorphic line bundle, we have

(i) \( H^q(M, \Omega^q \otimes B) = 0 \), for \( p + q > 2n \);

(ii) \( H^q(M, \Omega^q \otimes B) = 0 \), for \( p > n \) and any nonnegative integer \( q \).

In [13], we proved that, on a compact Kähler manifold, any \( k \)-ample line bundle is \( k \)-positive. So Corollary 24 is also applicable to \( k \)-ample line bundle. In particular, if \( k = 0 \), we get Theorem 2 for algebraic hyper-Kähler manifolds.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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