Application of spherical convex bodies to Wulff shape

Marek Lassak

Abstract. We present some relationships between the diameter, width and thickness of a reduced convex body on the $d$-dimensional sphere. We apply the obtained properties to recognize if a Wulff shape in the Euclidean $d$-space is self-dual.

Keywords: spherical geometry, lune, convex body, diameter, width, thickness, constant width, constant diameter, reduced body, Wulff shape

MSC: 52A55, 82D25

1 Introduction

Our subject is spherical geometry (see the monographs [10] and [14]) and its application for some methods of recognizing if a Wulff shape in Euclidean $d$-space $E^d$ is self-dual.

In the space $E^{d+1}$, where $d \geq 2$, take the unit sphere $S^d$ centered at the origin. The intersection of $S^d$ with any $(k+1)$-dimensional Euclidean space, where $0 \leq k \leq d-1$, is called a $k$-dimensional subsphere of $S^d$. For $k = 1$ we call it a great circle, and for $k = 0$ a pair of antipodes. If different points $a, b \in S^d$ are not antipodes, by the arc $ab$ connecting them we mean the “shorter” part of the great circle containing $a$ and $b$, i.e., this part which does not contain any pair of antipodes. By the spherical distance $|ab|$, or shortly distance, of these points we understand the length of the arc connecting them.

The intersection of $S^d$ with any half-space of $E^{d+1}$ is called a hemisphere of $S^d$. In other words, a hemisphere of $S^d$ is the set of points of $S^d$ in distances at most $\frac{\pi}{2}$ from a point $c$ called the center of this hemisphere. It is denoted by $H(c)$. Two hemispheres whose centers are antipodes are called opposite hemispheres.

Let a set $C \subset S^d$ do not contain any pair of antipodes. We say that $C$ is convex provided together with every two points it contains the whole arc connecting them. The smallest convex set containing a given set $A \subset S^d$ is called the convex hull of $A$. If the interior of a closed convex set $C \subset S^d$ is non-empty, we call $C$ a convex body.

If a hemisphere $H$ contains a convex body $C$ and if $p \in \text{bd}(H) \cap C$, we say that $H$ supports
$C$ at $p$ or that $H$ is a supporting hemisphere of $C$ at $p$. If at every boundary point of a convex body $C \subset S^d$ exactly one hemisphere supports $C$, then $C$ is called smooth. We call $C$ strictly convex if its boundary $\text{bd}(C)$ does not contain any arc.

If hemispheres $G$ and $H$ of $S^d$ are different and not opposite, then $L = G \cap H$ is called a lune of $S^d$. This notion is considered in many books and papers. The parts of the boundaries $\text{bd}(G)$ and $\text{bd}(H)$ contained in $G \cap H$ are denoted by $G/H$ and $H/G$, respectively. Clearly, $(G/H) \cup (H/G)$ is the boundary of the lune $G \cap H$. Points of $(G/H) \cap (H/G)$ are called corners of the lune $G \cap H$. By the thickness $\Delta(L)$ of the lune $L = G \cap H$ we mean the spherical distance of the centers of $G/H$ and $H/G$. We have the following obvious claim.

**Claim 1.** Any $(d-1)$-dimensional subsphere of $S^d$ passing through a point $c \in S^d$ dissects the hemisphere $H(c)$ into two lunes of thickness $\pi/2$.

For any convex body $C \subset S^d$ and any hemisphere $K$ supporting $C$ we define the width of $C$ determined by $K$ as the minimum thickness of a lune $K \cap K'$ over all hemispheres $K' \neq K$ supporting $C$ and we denote it by $\text{width}_K(C)$. By the thickness (often also called minimum width) $\Delta(C)$ of $C$ we mean the minimum of $\text{width}_K(C)$ over all hemispheres $K$ supporting $C$. Clearly, $\Delta(C)$ is nothing else but the thickness of a “narrowest” lune containing $C$. We say that $C$ is of constant width if all its widths are equal. These notions and a few properties of lunes and convex bodies in $S^d$ are given in [4] and [7]. Here is an additional property.

**Claim 2.** Let $C \subset S^d$ be a convex body. We have $\Delta(C) \leq \text{diam}(C)$. Moreover, if $\text{diam}(C) = \Delta = w$ and if $w \leq \pi/2$, then $C$ is of constant width $w$.

**Proof.** We get the first assertion by the definition of $\Delta(C)$ and the inequality

$$\max\{\text{width}_K(C) : K \text{ is a supporting hemisphere of } K\} \leq \text{diam}(C)$$

resulting from Theorem 3 and Proposition 1 of [4].

In order to show the second assertion, we apply Theorem 3 of [4] which says that if $\text{diam}(C)$ is at most $\pi/2$, then it is nothing else but the maximum of the widths of $C$. Moreover, having in mind that $\Delta(C)$ is the minimum width of $C$, from the assumption that $\text{diam}(C) = \Delta(C) = w$ we conclude that all widths of $C$ are equal $w$, which means that $C$ is of constant width $w$. 

In Section 2 we recall the notion of a spherical reduced spherical body and we present some relationships between the thickness and the diameter of reduced bodies. Section 3 is devoted to applications of these results for recognizing if a Wulff shape is self-dual.
2 Thickness and diameter of reduced spherical bodies

We say that a convex body $R \subset S^d$ is reduced if $\Delta(Z) < \Delta(R)$ for each convex body $Z$ being a proper subset of $R$. Some properties of spherical reduced bodies are given in [4], [6], [7], [8] and [9]. This notion is analogous to the notion of a reduced convex body in $E^d$; for instance see the survey article [5]. Clearly, every spherical body of constant width is reduced.

**Proposition 1.** For every reduced body $R \subset S^d$ such that $\Delta(R) \leq \frac{\pi}{2}$ we have $\text{diam}(R) \leq \frac{\pi}{2}$. Moreover, if $\Delta(R) < \frac{\pi}{2}$, then $\text{diam}(R) < \frac{\pi}{2}$.

**Proof.** Assume that $\Delta(R) \leq \frac{\pi}{2}$. Since $R$ is compact, there are points $p, q \in \text{bd}(R)$ such that $|pq| = \text{diam}(R)$. Applying Proposition 3.5 of [6] which holds also for $R \subset S^d$ (the proof is the same as for $d = 2$), we conclude that $R \subset H(p)$. Hence $|pq| \leq \frac{\pi}{2}$. Consequently, $\text{diam}(R) \leq \frac{\pi}{2}$.

If $\Delta(R) < \frac{\pi}{2}$, then in the proof of Proposition 3.5 of [6] $R$ does not contain any corner of $L$, and $L$ of thickness below $\frac{\pi}{2}$. Thus the considered now $R$ is a subset of the interior of $H(p)$. By the compactness of $R$, there exists a $\lambda < \frac{\pi}{2}$ such that $R \subset \{x : |xp| < \lambda \}$. Consequently, $|pq| \leq \lambda < \frac{\pi}{2}$, which implies $\text{diam}(R) \leq \lambda < \frac{\pi}{2}$. \(\square\)

The special case of the first assertion of Proposition 1 for $d = 2$ is in the observation just before Proposition 1 of [8]. By the way, the first assertion of Proposition 1 of [8] is a special case for $d = 2$ of the first statement of our Claim 2 and the second assertion of Proposition 1. Let us add that the proof of Proposition 1 of [8] is different from our approach, it applies Theorem 1 of [8] specific for $d = 2$.

The following proposition generalizes Theorem 4.3 of [6] from $S^2$ up to $S^d$. Our proof is analogous (this time we must apply Proposition 1 of [7]). Here a more detailed consideration supplemented by a figure is presented.

**Proposition 2.** If a reduced convex body $R \subset S^d$ fulfills $\Delta(R) \geq \frac{\pi}{2}$, then $R$ is a body of constant width $\Delta(R)$.

**Proof.** We consider two cases.

Case 1, when $\Delta(R) > \frac{\pi}{2}$. We apply Proposition 1 of [7] that every spherical reduced convex body of thickness over $\frac{\pi}{2}$ is smooth, and next Theorem 5 of [4], that every smooth reduced body is of constant width.

Case 2, when $\Delta(R) = \frac{\pi}{2}$. Our assertion that $R$ is of constant width $\Delta(R) = \frac{\pi}{2}$ means that $\text{width}_G(R) = \frac{\pi}{2}$ for every hemisphere $G$ supporting $R$. 


Assume the contrary assertion that there exists a hemisphere $K$ supporting $R$ such that $\text{width}_K(R) \neq \frac{\pi}{2}$. By the definition of the thickness, from $\Delta(R) = \frac{\pi}{2}$ we see that $\text{width}_K(R) < \frac{\pi}{2}$ is impossible. Thus our contrary assertion is nothing else but $\text{width}_K(R) > \frac{\pi}{2}$ (under this form of the contrary assertion we present Fig. 1 for $d = 2$ showing the hemisphere $K$ from the front by the orthogonal outside look on $S^2$).

Take any $K^*$ described in Part III of Theorem 1 of [4] and the center $t$ of $K^*/K$ (see also Corollary 2 there). Since $K$ supports $R$, a point $e$ of $R$ belongs to $\text{bd}(K)$. Thus $e \in K/K^*$. By Proposition 3.5 of [6] the hemisphere $M$ with center $e$ contains $R$. Hence $|te| \leq \frac{\pi}{2}$. Consider the lune $L = K \cap K^*$. From $\text{width}_K(R) > \frac{\pi}{2}$ we conclude that $\Delta(L) > \frac{\pi}{2}$. Hence the only points of $K/K^*$ at a distance at most $\frac{\pi}{2}$ from $t$ are the corners of $L$. So $e$ is a corner of $L$. Therefore $e \in \text{bd}(K)$.

By Claim 1 we see that $\text{bd}(K)$ dissects $M$ into two lunes of thickness $\frac{\pi}{2}$. Since one of them is $K\cap M$, we have $\Delta(K\cap M) = \frac{\pi}{2}$. This and $\Delta(K\cap K^*) \leq \Delta(K\cap M)$ imply $\Delta(K\cap K^*) \leq \frac{\pi}{2}$, which contradicts $\text{width}_K(R) > \frac{\pi}{2}$. Consequently, our contrary assertion assumed at the beginning of Case 2 is false. Hence $\text{width}_G(R) = \frac{\pi}{2}$ for every hemisphere $G$ supporting $R$, which means that $R$ is of constant width also in Case 2.

Thanks to Theorem 4 of [7] which says that every spherical body of constant width $w$ has diameter $w$, from Proposition 2 we obtain the following corollary.

**Corollary 1.** For every reduced body $R \subset S^d$ fulfilling $\Delta(R) \geq \frac{\pi}{2}$ we have $\text{diam}(R) = \Delta(R)$. 
Observe that this corollary is not true without the assumption that the body is reduced.

**Corollary 2.** If \( \Delta(R) < \text{diam}(R) \) for a reduced body \( R \subset S^d \), then both these numbers are below \( \frac{\pi}{2} \). Moreover, \( R \) is not a body of constant width.

**Proof.** The inequality \( \Delta(R) \geq \frac{\pi}{2} \) is impossible, since then by Corollary 1 we have \( \text{diam}(R) = \Delta(R) \), which contradicts the assumption of our statement. Hence \( \Delta(R) < \frac{\pi}{2} \). Then by the second part of Proposition 1 we have \( \text{diam}(R) < \frac{\pi}{2} \). So both these numbers are below \( \frac{\pi}{2} \).

In order to show the second assertion, assume the opposite that \( R \) is of constant width. Then the maximum and minimum widths of \( R \) are equal. Therefore \( \text{diam}(R) = \Delta(R) \). This contradicts the assumption of our corollary. Consequently, \( R \) is not of constant width. \( \square \)

**Proposition 3.** Let \( R \subset S^d \) be a reduced body. Then

\begin{enumerate}
  \item \( \Delta(R) = \frac{\pi}{2} \) if and only if \( \text{diam}(R) = \frac{\pi}{2} \),
  \item \( \Delta(R) \geq \frac{\pi}{2} \) if and only if \( \text{diam}(R) \geq \frac{\pi}{2} \),
  \item \( \Delta(R) > \frac{\pi}{2} \) if and only if \( \text{diam}(R) > \frac{\pi}{2} \),
  \item \( \Delta(R) \leq \frac{\pi}{2} \) if and only if \( \text{diam}(R) \leq \frac{\pi}{2} \),
  \item \( \Delta(R) < \frac{\pi}{2} \) if and only if \( \text{diam}(R) < \frac{\pi}{2} \).
\end{enumerate}

**Proof.** Let us show (a). If \( \Delta(R) = \frac{\pi}{2} \), then by Corollary 1 we have \( \text{diam}(R) = \frac{\pi}{2} \). Now assume that \( \text{diam}(R) = \frac{\pi}{2} \). Then by the first assertion of Claim 2 we get \( \Delta(R) \leq \frac{\pi}{2} \). Moreover, by the contrapositive of the second assertion of Proposition 1 we get \( \Delta(R) \geq \frac{\pi}{2} \). Consequently, \( \Delta(R) = \frac{\pi}{2} \).

Let us show (b). By Corollary 1 the inequality \( \Delta(R) \geq \frac{\pi}{2} \) implies \( \text{diam}(R) = \Delta(R) \) and thus \( \text{diam}(R) \geq \frac{\pi}{2} \). The opposite implication is the contrapositive of the second assertion of Proposition 1.

From (a) and (b) we get (c). It implies (d). We obtain (e) as the contrapositive of (b). \( \square \)

After Part 4 of [7], we say that a convex body \( D \subset S^d \) of diameter \( \delta \) is of constant diameter \( \delta \) provided for every \( p \in \text{bd}(D) \) there exist \( p' \in \text{bd}(D) \) such that \( |pp'| = \delta \).

Next proposition is convenient to apply in the proof of the forthcoming Theorem.

**Proposition 4.** The following conditions are equivalent:

\begin{enumerate}
  \item \( C \subset S^d \) is a reduced body with \( \Delta(C) = \frac{\pi}{2} \),
  \item \( C \subset S^d \) is a reduced body with \( \text{diam}(C) = \frac{\pi}{2} \),
  \item \( C \subset S^d \) is a body of constant width \( \frac{\pi}{2} \),
  \item \( C \subset S^d \) is of constant diameter \( \frac{\pi}{2} \).
\end{enumerate}
Proof. The equivalence of (1) and (2) results from (a) of Proposition 3. By Proposition 2 we conclude that (1) implies (3). The opposite implication is obvious since every body of constant width is a reduced body. The equivalence of (3) and (4) follows by the fact (being a particular case of Theorem 5 of [7]) that a convex body $D \subset S^2$ is of constant diameter $\pi$, if and only if $D$ is of constant width $\frac{\pi}{2}$.

By part (b) of Proposition 3 we conclude the following variant of Proposition 2. Here we also add a consequence of the equivalence of (2) and (3) from Proposition 4.

**Corollary 3.** If a reduced convex body $R \subset S^d$ fulfills $\text{diam}(R) \geq \frac{\pi}{2}$, then $R$ is a body of constant width $\text{diam}(R)$. It is also a body of constant diameter $\text{diam}(R)$.

## 3 An application for recognizing if a Wulff shape is self-dual

Wulff [13] defined a geometric model of a crystal equilibrium, later named *Wulff shape*. The literature concerning this subject is very wide. For instance see the monograph [11] and the articles [3], [12].

For a given Wulff shape $W_\gamma$ in $E^d$, Han and Nishimura [11] (we follow their notation) consider the *dual Wulff shape* $W_\gamma$, where $\bar{\gamma}(\theta) = 1/w(-\theta)$. It is denoted by $DW_\gamma$. If $W_\gamma = DW_\gamma$, it is called a *self-dual Wulff shape*. Next, these authors apply the classical notion of the central projection $\alpha_N$ from the open hemisphere $H(N)$ centered at a point $N \in S^d$ (see also [3], especially section 2.3) into the hyperplane $P(N) \subset E^{d+1}$ supporting $S^d$ at $N$. This hyperplane may be treated as the $E^d$ with the origin $N$. The image of a Wulff shape $W_\gamma$ on $P(N)$ (which is treated as an $E^d$) under the inverse projection $\alpha_N^{-1}$ onto $H(N)$ is called the *spherical convex body induced by $W_\gamma$*. Han and Nishimura [11] prove that a Wulff shape $W_\gamma \subset E^d$ is self-dual if and only if the induced spherical convex body is a spherical body of constant width $\frac{\pi}{2}$. By this result of Han and Nishimura, from Proposition 4 we obtain our theorem.

**Theorem.** The following conditions are equivalent to the statement that the Wulff shape $W_\gamma$ is self-dual:

- the spherical convex body induced by $W_\gamma$ is of constant width $\frac{\pi}{2}$,
- the spherical convex body induced by $W_\gamma$ is a reduced body of thickness $\frac{\pi}{2}$,
- the spherical convex body induced by $W_\gamma$ is a reduced body of diameter $\frac{\pi}{2}$,
- the spherical convex body induced by $W_\gamma$ is a body of constant diameter $\frac{\pi}{2}$.

Let us add that the equivalence to the second conditions gives the positive answer to the question put by Han and Nishimura at the end of [2].
On $S^2$ we get some spherical bodies of the form given in Theorem as a particular case of the example given on p. 95 of [6] by taking there any non-negative $\kappa < \frac{\pi}{2}$ and $\sigma = \frac{\pi}{4} - \frac{\kappa}{2}$. The following example presents a wider class of spherical bodies of constant diameter (an thus being also of other kinds considered in Theorem).

**Example.** Take a triangle $v_1v_2v_3 \subset S^2$ of diameter at most $\frac{\pi}{2}$ (see Fig. 2). Put $\kappa_{12} = |v_1v_2|, \kappa_{23} = |v_2v_3|, \kappa_{31} = |v_3v_1|, \sigma_1 = \frac{\pi}{4} + \frac{\kappa_{12}}{2} - \frac{\kappa_{31}}{2}, \sigma_2 = \frac{\pi}{4} - \frac{\kappa_{12}}{2} + \frac{\kappa_{23}}{2}, \sigma_3 = \frac{\pi}{4} - \frac{\kappa_{23}}{2} - \frac{\kappa_{31}}{2}$. Prolong the following sides: $v_1v_2$ up to $w_{12}w_{21}$ with $v_1 \in w_{12}v_2$, $v_2v_3$ up to $w_{23}w_{32}$ with $v_2 \in w_{23}v_3$, $v_3v_1$ up to $w_{31}w_{13}$ with $v_3 \in w_{31}v_1$ such that $|v_1w_{12}| = |v_1w_{13}| = \sigma_1$, $|v_2w_{21}| = |v_2w_{23}| = \sigma_2$, $|v_3w_{31}| = |v_3w_{32}| = \sigma_3$. Draw six pieces of circles: with center $v_1$ of radius $\sigma_1$ from $w_{12}$ to $w_{13}$ and of radius $\frac{\pi}{2} - \sigma_1$ from $w_{21}$ to $w_{31}$, with center $v_2$ of radius $\sigma_2$ from $w_{23}$ to $w_{21}$ and of radius $\frac{\pi}{2} - \sigma_2$ from $w_{32}$ to $w_{31}$, with center $v_3$ of radius $\sigma_3$ from $w_{31}$ to $w_{32}$ and of radius $\frac{\pi}{2} - \sigma_3$ from $w_{13}$ to $w_{23}$. Clearly, the convex hull of these pieces of circles is a body of constant diameter $\frac{\pi}{2}$.

![FIGURE 2. A spherical body of constant diameter](image)

Generalizing, take a convex odd-gon $v_1 \ldots v_n \subset S^2$ of diameter at most $\frac{\pi}{2}$. Put $\kappa_i = |v_iv_{i+(n-1)/2}|$ (modulo $n$) for $i = 1, \ldots, n$. Let $\sigma_i = \frac{\pi}{4} + \Sigma_{i=1}^{n} s_i \kappa_i$, where $s_i = 1$ if $\frac{n+1-i}{2} \leq \frac{n}{4}$ for $n$ of the form $3+4k$ and $|\frac{n}{4} - i| \leq \frac{n}{4}$ for $n$ of the form $5+4k$ (where $k = 0, 1, 2, \ldots$) and $s_i = -1$ in the opposite case. Prolong each diagonal $v_iv_{i+(n-1)/2}$ up to the arc $w_{i \cdot i+(n-1)/2w_{i+(n-1)/2}}$ such that $v_i \in w_{i \cdot i+(n-1)/2w_{i+(n-1)/2}}$ and $|v_iw_i \cdot i+(n-1)/2| = \sigma_i$. For $i = 1, \ldots, n$ draw two pieces of circles with center $v_i$ of radius $\sigma_i$ from $w_{i+(n-1)/2 \cdot i}$ to $w_{i+(n+1)/2 \cdot i}$ and of radius $\frac{\pi}{2} - \sigma_i$ from $w_{i \cdot i+(n-1)/2}$ to $w_{i+(n+1)/2}$. The convex hull of our $2n$ pieces is a body of constant diameter $\frac{\pi}{2}$ on $S^d$.

Similarly to Example from [7] we may construct bodies of constant diameter $\frac{\pi}{2}$ on $S^d$. 
References

[1] H. Han and T. Nishimura, Self-dual Wulff shapes and spherical convex bodies of constant width $\pi/2$, *J. Math. Soc. Japan* **69** (2017), 1475–1484.

[2] H. Han and T. Nishimura, Wulff shapes and their duals, www.kurims.kyoto-u.ac.jp/kyodo/kokyuroku/contents/pdf/2049-04.pdf.

[3] H. Han and T. Nishimura, Strictly convex Wulff shapes and $C^1$ convex integrands, *Proc. Amer. Math. Soc.* **145** (2017), no. 9, 3997-4008.

[4] M. Lassak, Width of spherical convex bodies, *Aequationes Math.* **89** (2015), 555–567.

[5] M. Lassak and H. Martini, Reduced convex bodies in Euclidean space – a survey, *Expositiones Math.* **29** (2011), 204–219.

[6] M. Lassak and M. Musielak, Reduced spherical convex bodies, *Bull. Pol. Ac. Math.* **66** (2018), 87–97.

[7] M. Lassak and M. Musielak, Spherical bodies of constant width, *Aequationes Math.* **92** (2018), 627–640.

[8] M. Lassak and M. Musielak, Diameter of reduced spherical bodies, *Fasciculi Math.*, **61** (2018), 103–108.

[9] M. Musielak, Covering a reduced spherical body by a disk, *Ukr. Math. J.*, to appear (see also arXiv:1806.04246).

[10] A. Papadopoulos, On the works of Euler and his followers on spherical geometry, *Ganita Bharati* **36** (2014), no.1, 53–108.

[11] A. Pimpinelli and J. Vilain, Physics of Crystal Growth, Monographs and Texts in Statistical Physics, Cambridge University Press, Cambridge, New York, 1998.

[12] J. E. Taylor, J. E. Cahn and C. A. Handwerker, Geometric models of crystal growth, *Acta Metallurgica et Materialia* **40** (1992), 1443–1474.

[13] G. Wulff, Zur Frage der geschwindindigkeit des wachstrums und derauflössung der krystallflaschen, *Z. Kristallographie and Mineralogie* **34** (1901), 449–530.

[14] G. Van Brummelen, Heavenly mathematics. The forgotten art of spherical trigonometry. Princeton University Press (Princeton, 2013).

Marek Lassak
University of Science and Technology
al. Kaliskiego 7, Bydgoszcz 85-796
e-mail: lassak@utp.edu.pl