Rainbow Turán problems for paths and forests of stars

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Abstract

For a fixed graph $F$, we would like to determine the maximum number of edges in a properly edge-colored graph on $n$ vertices which does not contain a rainbow copy of $F$, that is, a copy of $F$ all of whose edges receive a different color. This maximum, denoted by $\text{ex}^*(n, F)$, is the rainbow Turán number of $F$, and its systematic study was initiated by Keevash, Mubayi, Sudakov and Verstraëte in 2007 [9]. We determine $\text{ex}^*(n, F)$ exactly when $F$ is a forest of stars, and give bounds on $\text{ex}^*(n, F)$ when $F$ is a path with $k$ edges, disproving a conjecture in [9].

1 Introduction

For a fixed graph $F$, we would like to determine the maximum number of edges in a properly edge-colored graph on $n$ vertices which does not contain a rainbow copy of $F$, that is, a copy of $F$ all of whose edges receive a different color. This maximum, denoted by $\text{ex}^*(n, F)$, is the rainbow Turán number of $F$, and its systematic study was initiated by Keevash, Mubayi, Sudakov and Verstraëte in 2007 [9]. Among other things they proved that when $F$ has chromatic number at least 3, then

$$\text{ex}^*(n, F) = (1 + o(1))\text{ex}(n, F)$$

where $\text{ex}(n, F)$ is the (usual) Turán number of $F$. They also showed that

$$\text{ex}^*(n, K_{s,t}) = O(n^{2-1/s})$$

where $K_{s,t}$ is the complete bipartite graph with classes of size $s$ and $t$. This research was continued by Das, Lee and Sudakov [5], who partially answered a question from [9] on even cycles (this case has an interesting connection to additive number theory). In this paper, we determine $\text{ex}^*(n, F)$ exactly when $F$ is a forest of stars, and give bounds on $\text{ex}^*(n, F)$ when $F$ is a path with $l$ edges, disproving a conjecture in [9].

Our methods also yield short proofs of the classic results on Erdős and Gallai on the (usual) Turán numbers of matchings [6], and of some recent results of Lidický, Liu and Palmer [10] on the Turán numbers of forests of stars.

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2 Matchings

Write $M_k$ for a matching with $k$ edges. The usual Turán number for matchings was determined by Erdős and Gallai [6], who proved the following. Define $G_{n,k} = (V,E)$ to be the graph containing a clique $G_k$ on vertex set $V_k \subset V$, where $|V| = n$, $|V_k| = k$, and in which each $v \in V_k$ is joined to every vertex of $W = V \setminus V_k$. Then

$$\text{ex}(n, M_k) = \max\{e(G_{n,k-1}), e(K_{2k-1})\} = \max\left\{ \binom{k-1}{2} + (k-1)(n-k+1), \binom{2k-1}{2} \right\} = n(k-1) + O(k^2),$$

and, for sufficiently large $n$, $G_{n,k-1}$ is the unique extremal graph. The second term of the maximum is necessary since a clique on $2k-1$ vertices also contains no $M_k$, and for small $n$ it has more edges than $G_{n,k-1}$.

In other words, for sufficiently large $n$, $\text{ex}(n, M_k) = \binom{k-1}{2} + (k-1)(n-k+1)$. Rather surprisingly, the same is true for $\text{ex}^*(n, M_k)$. First we establish a weak version of this result.

Although both the next two theorems are special cases of the results in the next section, their proofs will serve as templates for what follows.

**Theorem 1.**

$$\text{ex}^*(n, M_k) = n(k-1) + O(k^2).$$

**Proof.** Suppose $G = (V,E)$ has the maximum number of edges such that there exists a proper edge-coloring $\chi$ of $G$ with no rainbow $M_k$. Then $G$ must contain a rainbow $M_{k-1}$, on vertex set $A$, say. Write $B = V \setminus A$, $C \subset A$ for those vertices of $A$ which send at least $t = 2k$ edges to $B$, and set $c = |C|$.

We must have $c \leq k-1$, or else we could greedily build a rainbow matching from $A$ to $B$ of size $k$ as follows. First choose an edge $c_1b_1 \in E$, where $c_1 \in C$ and $b_1 \in B$, where without loss of generality $\chi(c_1b_1) = 1$. Then choose an edge $c_2b_2 \in E$ of a different color, say $\chi(c_2b_2) = 2$, where $c_2 \in C$ and $b_2 \in B$ with $b_2 \neq b_1$. This is possible since $d(c_2) \geq 3$. Continuing, we finally choose $c_kb_k \in E$ with $\chi(c_kb_k) = k$, which is possible since $d(c_k) \geq 2k-1$ (we have $k-1$ vertices $b_1, \ldots, b_{k-1}$ and $k-1$ edge colors to avoid). Alternatively, the inequality $c \leq k-1$ follows on observing that if any edge $c_i c_j$ of our $M_{k-1}$ has two vertices from $C$, then $c_i c_j$ can be replaced by two edges $c_i b_i$ and $c_j b_j$ of new colors.

At least (and in fact, exactly) $k-1-c$ of the edges of our $M_{k-1}$ contain no vertex of $C$; write $M'$ for this set of edges. We claim that $G' = G[B]$ is $(k-1-c)$-colorable. Indeed, it is $(k-1-c)$-colored by $\chi$. For if $e \in E(G')$ has a color not appearing among the colors of $M'$, we can form a rainbow copy of $M_k$ by starting with $M'$ and $e$, and then greedily extending from the vertices of $C$ as above (at the last stage we have $k-1$ colors and at most $(c-1)+2 \leq (k-2)+2 = k$ vertices to avoid). Consequently, the maximum degree in $G[B]$
is at most $k - 1 - c$, and so $e(G[B]) \leq \frac{k-1-c}{2}(n - 2(k - 1))$. Therefore,

$$
e(G) \leq \left(\frac{2(k - 1)}{2}\right) + (2(k - 1) - c)(2k - 1) + c(n - 2(k - 1)) + \frac{k - 1 - c}{2}(n - 2(k - 1))$$

$$= (k - 1)(6k - 5) - c(2k - 1) + \frac{k - 1 + c}{2}(n - 2(k - 1))$$

$$\leq (k - 1)(6k - 5) + (k - 1)(n - 2(k - 1))$$

$$= n(k - 1) + (k - 1)(4k - 3).$$

Next we refine this argument to get an exact result, at least for sufficiently large $n$.

**Theorem 2.** For $n \geq 9k^2$,

$$
ex^*(n, M_k) = \left(\frac{k - 1}{2}\right) + (k - 1)(n - k + 1).$$

**Proof.** We already know that $\text{ex}^*(n, M_k) \geq \text{ex}(n, M_k) = \left(\frac{k - 1}{2}\right) + (k - 1)(n - k + 1)$, so we only need to show that $\text{ex}^*(n, M_k) \leq \left(\frac{k - 1}{2}\right) + (k - 1)(n - k + 1)$. To this end, suppose again that $G = (V, E)$ has the maximum number of edges such that there exists a proper edge-coloring $\chi$ of $G$ with no rainbow $M_k$. Following the proof of Theorem 1, we see that we must have $c = k - 1$, since otherwise

$$e(G) \leq \frac{2k - 3}{2}(n - 2(k - 1)) + (k - 1)(6k - 5) \leq \left(\frac{k - 1}{2}\right) + (k - 1)(n - k + 1),$$

as long as $n \geq 9k^2$. Armed with this information, we deduce that $G[(A \cup B) \setminus C]$ contains no edges. Otherwise, if $e \in E(G[(A \cup B) \setminus C])$, we could greedily extend $e$ to a rainbow matching $M_k$ using the vertices of $C$. Consequently,

$$e(G) \leq \left(\frac{|C|}{2}\right) + |C|(|A| - |C| + |B|) = \left(\frac{k - 1}{2}\right) + (k - 1)(n - k + 1).$$

We remark that this method can be used to prove Erdős and Gallai’s result that $\text{ex}(n, M_k) = \left(\frac{k - 1}{2}\right) + (k - 1)(n - k + 1)$, at least for sufficiently large $n$. Rather than elaborate here, we note that the theorem is a special case of the result of Lidický, Liu and Palmer on star forests, which we will reprove in the next section. Note also that our argument avoids Hall’s theorem.
3 Forests of stars

In this section we address the rainbow Turán number of a forest \( F \) where each component is a star. In this case, the Turán number was determined by 

We give a new proof of this result at the end of this section.

Let \( F \) be a forest of \( k \) stars \( S_1, S_2, \ldots, S_k \) such that \( e(S_j) \leq e(S_{j+1}) \) for each \( j \). We will construct a family of \( n \)-vertex graphs that each have a proper edge-coloring with no rainbow copy of \( F \). For \( 0 \leq c \leq k-1 \), define \( f(c) \) to be

\[
f(c) = \left( \sum_{i=1}^{k-c} e(S_i) \right) - 1.
\]

The graph \( H_F(n, c) \) is defined as follows. For \( c = k-1 \), we connect a set \( C \) of \( c = k-1 \) universal vertices to an edge-maximal graph \( H \) of maximum degree \( f(c) = f(k-1) = e(S_1) - 1 \) on the remaining \( n - k + 1 \) vertices. (A universal vertex is one that is joined to every other vertex, so that in particular \( G[C] \) is a clique.) When \( c \leq k-2 \), we connect a set \( C \) of \( c \) universal vertices to an edge-maximal \( f(c) \)-edge-colorable graph \( H \) on \( n - c \) vertices.

Note the slight distinction in the definition of the subgraph \( H \) in the two cases \( c = k-1 \) and \( c \leq k-2 \). In both cases, it is easy to see that \( H \) can only contain \( k-c-1 \) of the stars in \( F \). The remaining \( c+1 \) stars must each use at least one vertex from \( C \), which is impossible. Therefore, in both cases, \( H_F(n, c) \) does not contain a rainbow copy of \( F \).

When \( c = k-1 \), the subgraph \( H \) is \( (e(S_1) - 1) \)-regular when either \( n-c \) or \( e(S_1) - 1 \) is even. Otherwise, \( H \) has one vertex of degree \( e(S_1) - 2 \) and \( n-k \) vertices of degree \( e(S_1) - 1 \). Therefore, the total number of edges in \( H_F(n, k-1) \) is

\[
e(H_F(n, k-1)) = \left( \frac{k-1}{2} \right) + (k-1)(n-k+1) + \left\lfloor \frac{(e(S_1) - 1)(n-k+1)}{2} \right\rfloor.
\]

When \( c \leq k-2 \), there are exactly \( \left\lfloor \frac{n-c}{2} \right\rfloor \) edges of each color in \( H \), so that \( H \) has \( f(c) \left\lfloor \frac{n-c}{2} \right\rfloor \) edges. Therefore, the total number of edges in \( H_F(n, c) \) is

\[
e(H_F(n, c)) = \left( \frac{c}{2} \right) + c(n-c) + f(c) \left\lfloor \frac{n-c}{2} \right\rfloor
\]

\[
= \left( \frac{c}{2} \right) + c(n-c) + \left( \sum_{i=1}^{k-c} e(S_i) \right) - 1 \left\lfloor \frac{n-c}{2} \right\rfloor.
\]

Consequently, for all \( c \leq k-1 \), the number of edges in the graph \( H_F(n, c) \) is

\[
e(H_F(n, c)) = cn + \frac{1}{2} \left( \sum_{i=1}^{k-c} e(S_i) \right) n + O(1).
\]  \hspace{1cm} (1)

Furthermore, the subgraph \( H \) of \( H_F(n, c) \) has average degree \( f(c) - \epsilon \), where \( \epsilon < 1 \).

Of particular interest is the construction \( H_F(n, 0) \), which is simply an edge-maximal \( (e(F) - 1) \)-edge-colored graph, since \( f(0) = e(F) - 1 \).
The key to our analysis is the following technical lemma, which allows us to restrict our attention to the family $H_F(n, c)$.

**Lemma 3.** Let $F$ be a forest of $k$ stars. Suppose that $G$ is an edge-maximal properly edge-colored graph on $n$ vertices containing no rainbow copy of $F$. Then, for sufficiently large $n$, $G$ is isomorphic to one of the graphs $H_F(n, c)$.

Before turning to the proof of this lemma, we explain its use in the proof of our main result, Theorem 4. Specifically, suppose we have proved Lemma 3 and consider a fixed forest of stars $F$. In order to find the extremal graphs for a rainbow copy of $F$, we just need to determine the value of $c = c(F)$ that maximizes the number of edges $e(H_F(n, c))$ of $H_F(n, c)$.

For example, when $F$ is a forest of stars each of size 1 (i.e., a matching), then, for large $n$, the sum in (1) is maximized when $c = k - 1$. Therefore, for large $n$, an edge-maximal properly edge-colored graph $G$ containing no rainbow copy of $F$ must be isomorphic to $H_F(n, k - 1)$.

In this case, $f(k - 1) = e(S_1) - 1 = 0$ (this holds whenever $F$ contains a star of size 1), so that $G$ consists of a universal set of size $k - 1$ joined to an independent set of size $n - k + 1$. This reproves Theorem 2.

It turns out that, for every $F$, the maximum of $e(H_F(n, c))$ is attained at either $c = 0$ or $c = k - 1$.

**Theorem 4.** Let $F$ be a forest of $k$ stars. Suppose that $G$ is an edge-maximal properly edge-colored graph on $n$ vertices containing no rainbow copy of $F$. Then, for sufficiently large $n$,

1) if $F$ contains no star of size 1, then $G$ is isomorphic to $H_F(n, 0)$;

2) otherwise, $G$ is isomorphic to the larger of $H_F(n, 0)$ and $H_F(n, k - 1)$.

**Proof.** First consider the case when $F$ contains no star of size 1. In this case, if $F$ contains at least one star of size at least 3, then, for sufficiently large $n$, the right hand side of (1) is maximized when $c = 0$. Therefore, by Lemma 3 $G$ must be isomorphic to $H_F(n, 0)$ (for large $n$).

If every star in $F$ has size 2, then the sum of the two main terms in (1) is constant over all $c \leq k - 1$, so we need to examine the error term. In both the cases $c = k - 1$ and $c \leq k - 2$, we have

$$e(H_F(n, c)) = \binom{c}{2} + c(n - c) + (2(k - c) - 1) \left\lfloor \frac{n - c}{2} \right\rfloor.$$  

Simple computations show that this is maximized at $c = 0$. Therefore, $G$ must be isomorphic to $H_F(n, 0)$.

To summarize, if $F$ contains no star of size 1, $G$ must be isomorphic to $H_F(n, 0)$, if $n$ is sufficiently large. As already mentioned, this extremal graph is just an edge-maximal graph that is properly edge-colored with $f(0) = e(F) - 1$ colors.

Now suppose that $F$ contains a star of size 1. Write $s \geq 1$ for the number of stars of size 1, $t$ for the number of stars of size 2, and $p = k - s - t$ for the number of stars of size at least 3 in $F$. If $p = 0$, then we should clearly take $c = k - 1$ to maximize the sum of the two main terms in (1). Consequently, we may assume $p > 0$. We now have three estimates for
the number of edges in $H_F(n, c)$, depending on the value of $c$. If $c < p$ (and $p > 0$), then

$$e(H_F(n, c)) = cn + \frac{1}{2} \left(s + 2t + \left(\sum_{i=s+t+1}^{k-c} e(S_i)\right) - 1\right) n + O(1),$$

which is maximized (for large $n$) when $c = 0$ (as each $e(S_i)$ in the above sum is at least 3). Thus, when $c < p$ (and $p > 0$), we should take $c = 0$, and then

$$e(H_F(n, c)) = \frac{1}{2} \left(s + 2t + \left(\sum_{i=s+t+1}^{k} e(S_i)\right) - 1\right) n + O(1). \quad (2)$$

If next $p \leq c < p + t$, then

$$e(H_F(n, c)) = cn + \frac{1}{2} (s + 2(t - (c - p)) - 1) n + O(1) = \frac{1}{2} (s + 2t + 2p - 1)n + O(1), \quad (3)$$

which (for large $n$) is clearly smaller than (2) if $p > 0$. If lastly $p + t \leq c \leq p + t + s - 1 = k - 1$, then

$$e(H_F(n, c)) = cn + \frac{1}{2} (s - (c - (p + t)) - 1)n + O(1) = \frac{1}{2} (s + t + p + c - 1) n + O(1),$$

which is maximized (for large $n$) when $c = k - 1$. (We remind the reader that in the case we are considering, $f(k - 1) = e(S_1) - 1 = 0$, so that both constructions of $H_F(n, c)$ coincide when $c = k - 1$.) Thus, when $p + t \leq c \leq p + t + s - 1 = k - 1$, we should take $c = k - 1 = s + t + p - 1$, and then

$$e(H_F(n, c)) = (s + t + p - 1)n + O(1) = (k - 1)n + O(1),$$

which is larger than (3) when $n$ is large. Therefore, for sufficiently large $n$, the number of edges in $H_F(n, c)$ is maximized when $c$ is either 0 or $k - 1$. \hfill \square

The choice of $c$ to maximize the sum of the two main terms in (1) can be illustrated as follows (see Table 1). Write down a row of $k$ 2s, and underneath this row, write down the star sizes $e(S_k), e(S_{k-1}), \ldots, e(S_1)$ in decreasing order. Next, take the sum of the first $c$ entries in the top row and the last $k - c$ entries in the bottom row, where $c \leq k - 1$. This sum represents twice the coefficient of $n$ in (1).

We now turn our attention to the proof of Lemma 3. We begin with a simple lemma.
Lemma 5. Fix positive integers $d$ and $\Delta$ and a constant $0 \leq \epsilon < 1$. If $G$ is a graph with average degree at least $d - \epsilon$ and maximum degree at most $\Delta$, then the number of vertices in $G$ of degree less than $d$ is at most

$$\frac{\Delta - d + \epsilon}{\Delta - d + 1}n.$$  

In particular, the number of vertices in $G$ of degree at least $d$ is $\Omega(n)$ (i.e. at least $Cn$ where $C = C(d, \Delta, \epsilon) > 0$).

Proof. The sum of the degrees in $G$ is at least $(d - \epsilon)n$. On the other hand, if $x$ is the number of vertices of degree less than $d$ in $G$, then the sum of the degrees in $G$ is at most $(d - 1)x + \Delta(n - x)$.

Combining these two estimates and solving for $x$ gives the result. $\square$

We are now ready to prove Lemma 3.

Proof of Lemma 3. Let $G$ be as in the statement of the theorem, and let $C$ be the set of vertices in $G$ of degree at least $3e(F)$. Write $c = |C|$. Observe that $c \leq k - 1$, since otherwise we could greedily embed the components of $F$ into $G$, using the vertices of $C$ as their centers.

The subgraph $G' = G[V \setminus C]$ has maximum degree at most $3e(F)$. Since $G$ has at least as many edges as the graph $H_F(n, c)$, it follows that $G'$ must have average degree at least $f(c) - \epsilon$, for some $\epsilon < 1$. Therefore, by Lemma 5, the subgraph $G'$ has at least $\Omega(n)$ vertices of degree

$$f(c) = \left( \sum_{i=1}^{k-c} e(S_i) \right) - 1.$$  

Now suppose (for a contradiction) that $G'$ has a vertex $v$ of degree greater than $f(c)$. Then we can form a rainbow copy of $F$ in $G$ as follows. Choose $k-c-1$ vertices of $G'$ of degree $f(c)$ that are at distance at least 3 from each other and from $v$ (this is possible since the maximum degree is constant). We can build a rainbow forest of the stars $S_1, S_2, \ldots, S_{k-c-1}$ on these vertices, since these stars use $f(c) + 1 - e(S_{k-c})$ edge colors. The vertex $v$ has degree at least $f(c) + 1$, so it is incident to at least $f(c) + 1 - (f(c) + 1 - e(S_{k-c})) = e(S_{k-c})$ unused colors. Therefore, we can extend the rainbow forest to include $S_{k-c}$. Finally, the remaining $c$ stars of $F$ can be greedily embedded using the vertices in $C$ as their centers, so that $G$ contains a rainbow copy of $F$. This is a contradiction. Therefore, $G'$ has maximum degree at most $f(c)$. When $c = k - 1$ we are done, since we have shown that $G$ has at most as many edges as $H_F(n, k - 1)$.

Let us now consider the case $c \leq k - 2$. The lower bound $e(G) \geq e(H_F(n, c))$ shows that the number of edges in $G'$ is at least

$$f(c) \left\lfloor \frac{n - c}{2} \right\rfloor \geq f(c) \left( \frac{n - c}{2} \right) - \left\lfloor \frac{f(c)}{2} \right\rfloor.$$
In particular, $G'$ has $n - O(1)$ vertices of degree $f(c)$, since $G'$ has maximum degree $f(c)$. We claim that $G'$ must be colored with $f(c)$ edge colors. Suppose, for a contradiction, that $G'$ is colored with at least $f(c) + 1$ colors. Then there is a color class, say red, with at most

$$\frac{1}{f(c) + 1} \left\lfloor \frac{n - c}{2} \right\rfloor$$

edges. Therefore, there are $\Omega(n)$ vertices in $G'$ of degree $f(c)$ that are not incident to a red edge.

Since $c \leq k - 2$, the sum in $f(c)$ has at least two terms, so that

$$2e(S_1) \leq e(S_1) + e(S_2) \leq \sum_{i=1}^{k-c} e(S_i) = f(c) + 1.$$ 

As $e(S_1)$ is an integer, this implies that $e(S_1) \leq \lceil f(c)/2 \rceil$.

We now embed $S_1$ in $G'$ using a red edge. If $n - c$ is even, then every vertex in $G'$ has degree $f(c) \geq \lceil f(c)/2 \rceil$, so we can choose a vertex $v$ incident to a red edge and embed $S_1$ using that red edge.

When $n - c$ is odd, $G'$ may contain vertices of degree less than $f(c)$. Consider a red edge $uv$ and observe that at least one of the vertices $u$ and $v$ (say $v$) has degree at least $\lceil f(c)/2 \rceil$; otherwise the number of edges in $G'$ is less than $f(c) \left\lfloor \frac{n - c}{2} \right\rfloor$. Therefore, we can embed $S_1$ using the red edge $uv$ with $v$ as the center.

Now, among the vertices not incident to red edges, pick $k - c - 1$ vertices of degree $f(c)$ that are at distance at least 3 from each other and from the center $v$ of $S_1$. Using these vertices as centers, we can greedily build a rainbow forest of stars $S_2, S_3, \ldots, S_{k-c}$, since we have only used at most $e(S_1) - 1$ of the $f(c)$ colors incident to these vertices. Finally, the remaining $c$ stars of $F$ can be greedily embedded using the vertices in $C$ as their centers, so that $G$ contains a rainbow copy of $F$. This is a contradiction. Therefore, $G'$ is properly $f(c)$-edge-colored.

We now give a new proof of the result of Lidický, Liu and Palmer on the Turán number of forests of stars.

We begin by describing the extremal graph for the forest of stars $S_1, S_2, \ldots, S_k$, where $e(S_j) \leq e(S_{j+1})$ for each $j$. Let $H'_F(n, i)$ be the graph obtained by connecting a set of $i$ universal vertices to an edge-maximal graph of maximal degree $e(S_{k-i}) - 1$ on $n - i$ vertices. Observe that if one of $e(S_{k-i}) - 1$ or $n - i$ is even, and $n$ is large enough, then $H'$ is $(e(S_{k-i}) - 1)$-regular. If both are odd, then $H$ has exactly one vertex of degree $e(S_{k-i}) - 2$, and $n - i - 1$ vertices of degree $e(S_{k-i}) - 1$. Each of the graphs $H'_F(n, i)$ is $F$-free, since otherwise each of the $i + 1$ stars $S_k, S_{k-1}, \ldots, S_{k-i}$ must use at least one vertex from the universal set of size $i$, which is impossible.

**Theorem 6** (Lidický, Liu, Palmer [10]). Let $F$ be a forest of $k$ stars $S_1, S_2, \ldots, S_k$, such that $e(S_j) \leq e(S_{j+1})$ for each $j$. Then

$$\text{ex}(n, F) = \max_{0 \leq i \leq k-1} \left\{ i(n - i) + \binom{i}{2} + \left\lceil \frac{e(S_{k-i}) - 1(n - i)}{2} \right\rceil \right\}.$$ 

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Proof. Note that $G$ has at least as many edges as $H'_p(n,i)$ for all $i \leq k - 1$. Suppose that $G$ has a set $C$ of $c$ vertices of degree at least $e(F)$. We must have $c \leq k - 1$, since otherwise we could greedily embed $F$ from the vertices of $C$. Let $G' = G[V \setminus C]$ be the graph on the remaining $n - c$ vertices. The maximum degree of $G'$ is less than $e(F)$. First let us suppose that $c = k - 1$. In this case, we claim that the maximum degree of $G'$ is at most $e(S_1) - 1$. Indeed, if there is a vertex $v$ of higher degree, then we can embed $S_1$ into $G'$ using $v$, and complete the forest $F$ by greedily embedding the stars $S_2, S_3, \ldots, S_k$ using the vertices of $C$ as their centers.

Next suppose that $c < k - 1$. Suppose (for a contradiction) that $e(S_{k-c-1}) = e(S_{k-c})$. Comparing $G$ to $H'_p(n,c+1)$, we see that $G'$ must have average degree at least $e(S_{k-c-1}) - \epsilon = e(S_{k-c}) - \epsilon$. Therefore, by Lemma 5 the graph $G'$ contains $\Omega(n)$ vertices of degree at least $e(S_{k-c})$. Now we can embed $F$ as follows. Choose $k - c$ vertices of $G'$ of degree $e(S_{k-c})$ that are at distance at least 3 from each other. We can embed the stars $S_1, S_2, \ldots, S_{k-c}$ on these vertices. Next we can greedily embed the remaining stars $S_{k-c+1}, \ldots, S_k$ into $G$ using the vertices of $C$ as their centers; a contradiction.

Therefore, we may assume that $e(S_{k-c-1}) < e(S_{k-c})$. By comparing $G$ to $H'_p(n,c)$, we see that $G'$ must have average degree at least $e(S_{k-c}) - 1$. Therefore, by Lemma 5 the graph $G'$ contains $\Omega(n)$ vertices of degree at least $e(S_{k-c}) - 1$. Now suppose that $G'$ has a vertex $v$ of degree greater than $e(S_{k-c}) - 1$. Then we can embed $F$ as follows. Choose $k - c - 1$ vertices of $G'$ of degree $e(S_{k-c}) - 1$ that are at distance at least 3 from each other and from $v$. We can embed the stars $S_1, S_2, \ldots, S_{k-c-1}$ on these vertices, since $e(S_{k-c}) - 1 \geq e(S_{k-c-1})$. Next we embed the star $S_{k-c}$ at $v$, and then greedily embed the remaining stars $S_{k-c+1}, \ldots, S_k$ into $G$ using the vertices of $C$ as their centers; a contradiction. Therefore, the maximum degree of $G'$ is $e(S_{k-c}) - 1$.  

\[ \square \]

4 Paths

In this paper, $P_l$ will denote a path with $l$ edges, which we will call a path of length $l$. The usual Turán number for paths was determined asymptotically by Erdős and Gallai [6], and exactly by Faudree and Schelp [7]. Erdős and Gallai proved that, given a path length $l$, if $l$ divides $n$ then

\[ \text{ex}(n,P_l) = \frac{n}{l} \binom{l}{2} = \frac{l-1}{2} n, \]

and the unique extremal graph is the disjoint union of $\frac{n}{l}$ copies of $K_l$. We briefly recall the proof. First we show that any graph $G$ with minimum degree at least $\delta$ contains a path of length $2\delta$ (provided of course that $2\delta < n$). Next, consider a graph $G$ of order $n$ with more than $\frac{l-1}{2} n$ edges (i.e., of average degree greater than $l - 1$). By repeatedly removing a vertex of minimum degree, we can show that $G$ must contain a subgraph $H$ whose minimum degree is at least $\frac{l}{2}$, and so $H$ contains a path of length $l$.

Following this approach for the rainbow Turán problem therefore requires us to find a rainbow path of length $cd$ in a graph of minimum degree $\delta$. To this end, we have the following theorem, which generalizes a result of Gyárfás and Mhalla [8], and is itself a special case of a
theorem of Babu, Chandran and Rajendraprasad [2]. For completeness, we provide a short proof of the result we need, which is less technical than the proof in [2].

**Theorem 7.** Let $G$ be a graph with minimum degree $\delta = \delta(G)$. Then any proper edge-coloring of $G$ contains a rainbow path of length at least $\frac{2}{3}\delta$.

**Proof.** Suppose that $c$ is a proper edge-coloring of $G$. Take a longest rainbow path $P = v_0v_1 \cdots v_l$ in $G$, of length $l$. Without loss of generality, $c(v_{i-1}v_i) = i$ for each $i$ (i.e., the $i^{th}$ edge of $P$ receives color $i$). Write $s_o$ for the number of edges colored with colors $1, \ldots, l$ that $v_0$ sends to vertices outside $P$, and note that $v_0$ can send no other edges outside $P$, or else $P$ could be extended. Also write $s_i$ for the number of edges of colors $1, \ldots, l$ that $v_0$ sends to other vertices of $P$ (including $v_1$), and write $s^x$ for the number of edges of other colors that $v_0$ sends to vertices of $P$. Finally, define $t_o, t_i$ and $t^x$ to be the analogous quantities for $v_l$.

Observe now that

$$s_o + s_i \leq l,$$

since $c$ is a proper coloring, that

$$s_i + s^x \leq l,$$

since there are exactly $l$ vertices on $P$ other than $v_0$, and that

$$s_o + t^x \leq l,$$

since if $v_iv_l \in E(G)$ with $c(v_iv_l) > l$ then there is no $w \not\in V(P)$ with $c(wv_0) = c(v_iv_{i+1}) = i+1$, or else $wv_0v_1 \cdots v_iv_{i-1}v_{i+1} \cdots v_{i+2}$ would be a rainbow path in $G$ of length $l+1$. Analogous inequalities hold for $t_o, t_i$ and $t^x$.

Consequently, combining (1), (2) and (3) with the minimum degree condition, we have

$$2\delta \leq (s_o + s_i + s^x) + (t_o + t_i + t^x) = (s_i + s^x) + (s_o + t^x) + (t_o + t_i) \leq l + l + l = 3l,$$

so that $l \geq \frac{2}{3}\delta$, as desired. \qed

We remark that the constant $\frac{2}{3}$ cannot be improved in general. To see this, let $G$ be the disjoint union of $r$ copies of $K_4$, and properly 3-color the edges of each $K_4$ (there is a unique way to do this, up to isomorphism). Then $\delta(G) = 3$, and the longest rainbow path in $G$ has length 2. However, Chen and Li [4], and independently Mousset [12], proved that a proper edge-coloring of $K_n$ contains a rainbow path of length $\frac{3}{4}n - o(n)$. It is widely believed (see [1]) that a proper edge-coloring of $K_n$ in fact contains both a rainbow path and a rainbow cycle of length $n - o(n)$, and perhaps even a rainbow path of length $n - 2$. However, Maamoun and Meyniel [11] showed that we are not always guaranteed a rainbow path of length $n - 1$. In their construction, $n = 2^k$, and we identify the vertices of $K_{2^k}$ with the points of the Boolean cube $\{0, 1\}^k$. If we now color each edge $uv$ with color $u - v \neq 0$, a monochromatic path $v_0v_1 \cdots v_{n-1}$ of length $n - 1$ in $K_n$ would involve all possible colors (except for 0), so that

$$v_0 - v_{n-1} = \sum_{i=0}^{n-2} (v_{i+1} - v_i) = \sum_{0 \neq x \in \{0,1\}^k} x = \sum_{x \in \{0,1\}^k} x = 0,$$
which implies that $v_0 = v_{n-1}$, a contradiction.

A slight modification of the proof of Theorem 7 yields a short proof of the full result of Babu, Chandran and Rajendraprasad [2] mentioned above. Their result deals with general (not necessarily proper) edge-colorings, in which, given an edge-colored graph $G$, $\theta(G)$ is the minimum number of distinct colors seen at each vertex. Clearly $\theta(G) = \delta(G)$ if the coloring is proper.

**Theorem 8.** Let $G$ be an edge-colored graph in which every vertex is incident to at least $\theta = \theta(G)$ edge-colors. Then $G$ contains a rainbow path of length at least $\frac{2}{3} \theta$.

**Proof.** We follow the proof of Theorem 7, with a slight change in the definitions of $s_o$, $s_i$ and $s^\times$. This time, $s_o$ is the number of colors of edges that $v_0$ sends to vertices outside $P$ (as before, each of these colors already occurs on $P$), and $s^\times$ is the number of colors not seen on $P$ which occur as the colors of edges $v_0$ sends to $P$. Now $s_i$ is the number of colors from 1 to $l$ that occur as colors of edges $v_0$ sends to $P$ and which are not counted in $s_o$. The rest of the proof goes through as before, with $\delta$ replaced by $\theta$. \qed

Returning to the problem at hand, we can use Theorem 7 to obtain a bound on the rainbow Turán number of paths.

**Theorem 9.** For each fixed $l \geq 1$, we have

$$\frac{l - 1}{2} n \sim \text{ex}(n, P_l) \leq \text{ex}^*(n, P_l) \leq \left\lceil \frac{3l - 2}{2} \right\rceil n.$$

**Proof.** We will make use of the standard fact that a graph $G$ of average degree more than $2d$ contains a subgraph $H$ of minimum degree at least $d + 1$. This is proved by repeatedly removing a vertex of minimum degree from $G$.

First, suppose that $l$ is even, and write $l = 2k$. Let $G$ be a graph of order $n$ with more than $\frac{3l - 2}{2} n = (3k - 1)n$ edges (and so of average degree more than $2(3k - 1)$). Then $G$ contains a subgraph $H$ of minimum degree at least $3k$, which by Theorem 7 contains a rainbow path of length $2k = l$.

Second, suppose that $l$ is odd, and write $l = 2k + 1$. Let $G$ be a graph of order $n$ with more than $\frac{3l - 1}{2} n = (3k + 1)n$ edges (and so of average degree more than $2(3k + 1)$). Then $G$ contains a subgraph $H$ of minimum degree at least $3k + 2$, which by Theorem 7 contains a rainbow path of length $2k + 1 = l$. \qed

For small values of $l$, one can do considerably better. It is trivial that $\text{ex}^*(n, P_1) = \text{ex}(n, P_1) = 0$ and that $\text{ex}^*(n, P_2) = \text{ex}(n, P_2) = \left\lfloor \frac{n}{2} \right\rfloor$. When $l = 3$, we have the following simple result.

**Theorem 10.** Suppose that $n$ is divisible by 4. Then $\text{ex}^*(n, P_3) = \frac{3n}{2} = \frac{3}{2} \text{ex}(n, P_3) + O(1)$.

**Proof.** The example already shown, namely $\frac{n}{4}$ disjoint copies of properly 3-colored $K_4$S, shows that $\text{ex}^*(n, P_3) \geq \frac{3n}{2}$. For the other direction, suppose that $G = (V, E)$ is a graph with more than $\frac{3n}{2}$ edges and no rainbow $P_3$, and select $v \in V$ with $d(v) \geq 3$ (there must be at least
one such $v$). Then the neighbors $v_1, \ldots, v_r$ of $v$ can only be adjacent to each other, since if $v_i w \in E$ with $vw \not\in E$ then $wv_i v_j$ is a rainbow $P_3$ for some $j$ (chosen so that the colors of $v_i w$ and $vv_j$ are different). Moreover, if $d(v) \geq 4$, then $G[v \cup \Gamma(v)]$ is a star, since if $v_i v_j \in E$ then $v_j v_i v_k$ is a rainbow $P_3$, where this time $k$ has been chosen so that $v_i v_j$ and $v_i v_k$ receive different colors. Consequently, if $d(v) \geq 3$, then $G_v = G[v \cup \Gamma(v)]$ is a component of $G$ whose average degree is at most 3, so we may remove it and apply induction.

For $P_4$, we have the following theorem.

**Theorem 11.** If $n$ is divisible by 8, then $\text{ex}^*(n, P_4) = 2n$. In general, $\text{ex}^*(n, P_4) = 2n + O(1)$.

**Proof.** The lower bound comes from the proper edge-coloring of $K_{4,4}$ illustrated in Figure 1, which contains no rainbow $P_4$. (To see this, note that in the given coloring, any 4-cycle containing two identically-colored edges must in fact be 2-colored, so that every 4-cycle contains either 2 or 4 colors. Now suppose (to the contrary) that $xyzst$ is a rainbow $P_4$. Then the cycle $xyzsx$ must contain all 4 colors, so that edges $st$ and $sx$ must receive the same color, which is impossible since they are adjacent.) Next, if $n = 8k$, then the disjoint union of $k$ such edge-colored $K_{4,4}$s has $2n$ edges and no rainbow $P_4$. Consequently, $\text{ex}^*(n, P_4) \geq 2n$ if $8 \mid n$, and $\text{ex}^*(n, P_4) \geq 2n + O(1)$ in general.

![Figure 1: A proper edge-coloring of $K_{4,4}$ with no rainbow $P_4$](image)

For the upper bound, we show that every proper edge-coloring of an $n$-vertex graph $G$ with $m > 2n$ edges contains a rainbow $P_4$. As noted before, $G$ contains a subgraph $G'$ of minimum degree at least 3, since otherwise we can repeatedly remove vertices of degrees 1 and 2 so that the average degree increases. Furthermore, $G'$ has average degree greater than 4. Therefore, $G'$ has a vertex $v$ of degree at least 5. We will show that $G'$ contains a rainbow $P_4$. The proof now splits into two cases.
Case 1: $G'$ contains a rainbow $P_3$ ending at $v$. This case is illustrated in Figure 2; let the rainbow $P_3$ be $P = vxyz$, where edges $vx, xy$ and $yz$ are colored $1, 2$ and $3$ respectively. Since $v$ has degree at least $5$, it must be adjacent to at least $2$ vertices not on $P$; suppose these vertices are $s$ and $t$. If either of the edges $vs$ and $vt$ receives a color other than 2 or 3, then we have a rainbow $P_4$. Now suppose that $c(vs) = 2$ and $c(vt) = 3$, where $c$ denotes the color of the edge. If $v$ is adjacent to any other vertex $u$ not on $P$, then since $c(uv)$ would have to be different from 1, 2 and 3, the edge $uv$ with $P$ forms a rainbow $P_4$. Otherwise, the vertex $v$ has degree 5 and is adjacent to both $y$ and $z$. Without loss of generality, suppose $c(vy) = 4$ and $c(vz) = 5$.

Suppose that the vertex $z$ is adjacent to $x$. Note that $c(xz)$ cannot be 1, 2 or 3, and so $svxyz$ is a rainbow $P_4$. If $z$ is not adjacent to $x$, then $z$ is adjacent to a vertex $w$ not on $P$ (possibly $w = s$ or $w = t$) as the minimum degree of $G'$ is at least 3. We know that $c(wz)$ cannot be 3 or 5; if $c(wz) = 1$ then $wzvyx$ is a rainbow $P_4$, while if $c(wz) = 2$ then $wzyvx$ is a rainbow $P_4$. However, if $c(wz)$ is not 1, 2 or 3, then $vxyzw$ is a rainbow $P_4$. Accordingly, this completes the proof in Case 1.

Case 2: $G'$ contains no rainbow $P_3$ ending at $v$. Since $\delta(G') \geq 3$, $G'$ contains a rainbow $P_2$ ending at $v$; let this path be $vxy$, where $c(vx) = 1$ and $c(xy) = 2$. The vertex $y$ has degree at least 3; if $y$ were adjacent to two vertices $s$ and $t$ other than $v$ and $x$, then one of edges $ys$ and $yt$ would receive color 3, creating a rainbow $P_3$ ending at $v$. Consequently, the degree of $y$ is 3 and $y$ is adjacent to $v$ and a new vertex $z$. Furthermore, $c(yz) = 1$, and, without loss of generality, $c(yv) = 3$. Let $P$ be the path $vxyz$.

The vertex $z$ is adjacent to at most one vertex $w$ not on $P$ and the edge $zw$ must receive color 3 to avoid the rainbow $P_3 vyzw$ ending at $v$. Consequently, $z$ is adjacent to at least one of $v$ or $x$. The proof now splits into three sub-cases.

Case 2A: $z$ is adjacent to $x$ and a new vertex $w$. This case is illustrated on the left of Figure 3. Edge $xz$ cannot receive any of colors 1, 2 or 3, and so $vxyzw$ is a rainbow $P_3$ ending at $v$.

Case 2B: $z$ is adjacent to $v$ and a new vertex $w$. This case is illustrated in the center of Figure 3. Edge $vz$ must receive color 2 to avoid the rainbow $P_3 vzyx$ ending at $v$. Now, if $w$ were adjacent to two vertices $s$ and $t$ other than $v, x, y$ and $z$, then one of edges $ws$ and
$wt$ would receive color other than 2 and 3, creating a rainbow $P_3$ ending at $v$. Therefore, there is at least one edge from $w$ to $v$, $x$, or $y$. Such an edge cannot receive colors 1, 2, or 3. If $wv$ is an edge, then $vwyz$ is a rainbow $P_3$; if $wx$ is an edge, then $vxwz$ is a rainbow $P_3$; if $wy$ is an edge, then $vxyw$ is a rainbow $P_3$. In all cases we have found a rainbow $P_3$ ending at $v$.

**Case 2C:** $z$ is adjacent to both $v$ and $x$. This case is illustrated on the right of Figure 3. In this case, the vertices $v, x, y, z$ induce a properly 3-edge-colored $K_4$ as otherwise we can easily find a rainbow $P_3$ ending at $v$. We will exploit the resulting symmetry in the three colors 1, 2 and 3. The vertex $v$ must be adjacent to a new vertex $u$, and, without loss of generality, $c(uv) = 4$. If the vertex $u$ is adjacent to a new vertex $w$, then we may assume that $c(uw) = 1$, and then $wuvzx$ would be a rainbow $P_4$. Otherwise, $u$ is adjacent to at least two of $x, y$ and $z$; suppose it is adjacent to $x$. Then $c(ux)$ cannot be 1, 2, 3 or 4, and then $xuvzy$ is a rainbow $P_4$.

Thus, in all three sub-cases we obtain either a rainbow $P_3$ ending at $v$ (leading us to Case 1), or a rainbow $P_4$ in $G'$.

![Figure 3: No rainbow $P_3$ ends at a vertex $v$ of degree at least 5](image)

Keevash, Mubayi, Sudakov and Verstraëte conjectured that the extremal example for rainbow $P_5$ is a disjoint union of cliques of size $c(l)$, where $c(l)$ is chosen as large as possible so that $K_{c(l)}$ can be properly edge-colored with no rainbow $P_l$. It is not hard to show that a properly edge-colored $K_5$ must contain a rainbow $P_4$, so that $c(4) = 4$. Consequently, the conjecture implies that $\text{ex}^*(n, P_4) = \frac{3n}{2} + O(1)$, which is false, as our theorem shows.

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