COUPLED PAINLEVÉ SYSTEMS IN DIMENSION FOUR WITH AFFINE WEYL GROUP SYMMETRY OF TYPES $A_4^{(2)}$ AND $A_1^{(1)}$

Abstract. We find a two-parameter family of coupled Painlevé systems in dimension four with affine Weyl group symmetry of type $A_4^{(2)}$. For a degenerate system of $A_4^{(2)}$ system, we also find a one-parameter family of coupled Painlevé systems in dimension four with affine Weyl group symmetry of type $A_1^{(1)}$. We show that for each system, we give its symmetry and holomorphy conditions. These symmetries, holomorphy conditions and invariant divisors are new.

1. Introduction

In [19], we find a four-parameter (resp. three-parameter) family of ordinary differential systems in dimension four with affine Weyl group symmetry of type $A_7^{(2)}$ (resp. $A_5^{(2)}$). These systems are equivalent to the polynomial Hamiltonian systems, and can be considered to be 2-coupled Painlevé systems in dimension four.

We will complete the study of the below problem in a series of papers, for which this paper is the third, resulting in a series of equations for the remaining affine root systems of type $A_{2d+2}^{(2)}$.

Problem For each affine root system $X_i^{(2)}$ with affine Weyl group $W(X_i^{(2)})$, find a system of differential equations for which $W(X_i^{(2)})$ acts as its Bäcklund transformations.

This paper is the stage in this project where we find a 2-parameter family of coupled Painlevé systems in dimension four with affine Weyl group symmetry of type $A_1^{(2)}$ given by

$$
\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z} \tag{1}
$$

with the polynomial Hamiltonian

$$
H = 2H_{II}(x, y, t, \alpha_1) + H_{II}^{auto}(z, w, t, \alpha_0) + xw + 2yzw \tag{2}
$$

$$
= 2xy^2 + 2x^2 + 2tx - 2\alpha_1 y + z^2 w - \frac{w^2}{2} + \alpha_0 z + xw + 2yzw.
$$

Here $x, y, z$ and $w$ denote unknown complex variables, and $\alpha_0, \alpha_1, \alpha_2$ are complex parameters satisfying the relation:

$$
\alpha_0 + 2\alpha_1 + 2\alpha_2 = 1. \tag{3}
$$

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The symbol $H_{II}$ denotes the second Painlevé Hamiltonian given by
\[ H_{II}(x, y, t, \alpha_1) = xy^2 + x^2 + tx - \alpha_1 y, \]
and the symbol $H_{II}^{\text{auto}}$ denotes the autonomous version of the second Painlevé system given by
\[ H_{II}^{\text{auto}}(z, w, t, \alpha_0) = z^2 w - \frac{w^2}{2} + \alpha_0 z. \]
Of course, the Hamiltonian itself is the first integral.

We remark that for this system we tried to seek its first integrals of polynomial type with respect to $x, y, z, w$. However, we can not find. Of course, the Hamiltonian $H$ is not its first integral.

We also remark that the system (1) can be obtained by connecting the pair of the invariant divisors $(x + y + w + t, y)$ and $(x - z, z)$ for the canonical variables $(x, y, z, w)$ (see figure 1) in the system of type $A_{4}^{(1)}$ (see section 5 in [10]).

This is the second example which gave higher-order Painlevé equations of type $A_{4}^{(2)}$.

**Problem** It is still an open question whether the system (1) is equivalent to Ramani’s equation of type $A_{4}^{(2)}$.

For a degenerate system of $A_{4}^{(2)}$ system, we also find a one-parameter family of coupled Painlevé systems in dimension four with affine Weyl group symmetry of type $A_{1}^{(1)}$.

We show that for each system, we give its symmetry and holomorphy conditions. These symmetries, holomorphy conditions and invariant divisors are new.

2. **Symmetry and holomorphy conditions**

In this section, we study the symmetry and holomorphy conditions of the system (1). These properties are new.

**Theorem 2.1.** The system (1) admits the affine Weyl group symmetry of type $A_{2}^{(2)}$ as the group of its Bäcklund transformations, whose generators $s_0, s_1, s_2$ defined as follows: with the notation $(\ast) := (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2)$:

\[
    s_0 : (\ast) \rightarrow (x, y, z + \frac{\alpha_0}{w}, w, t; -\alpha_0, \alpha_1 + \alpha_0, \alpha_2),
\]
\[
    s_1 : (\ast) \rightarrow (x, y - \frac{\alpha_1}{x + z^2}, z, w - \frac{2\alpha_1 z}{x + z^2}, t; \alpha_0 + 2\alpha_1, -\alpha_1, \alpha_2 + \alpha_1),
\]
\[
    s_2 : (\ast) \rightarrow (x + \frac{2\alpha_2 y}{f_2}, y - \frac{\alpha_2}{f_2}, z + \frac{\alpha_2}{f_2}, w, t; \alpha_0, \alpha_1 + 2\alpha_2, -\alpha_2),
\]
where $f_2 := x + y^2 + w + t$.

We note that the Bäcklund transformations of this system satisfy
\[
    s_i(g) = g + \frac{\alpha_i}{f_i} \{f_i, g\} + \frac{1}{2!} \left( \frac{\alpha_i}{f_i} \right)^2 \{f_i, \{f_i, g\}\} + \cdots \quad (g \in \mathbb{C}(t)[x, y, z, w]),
\]
where Poisson bracket $\{,\}$ satisfies the relations:
\[ \{y, x\} = \{w, z\} = 1, \quad \text{the others are 0.} \]

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

**Proposition 2.2.** This system has the following invariant divisors:

| Parameter’s relation | $f_i$                  |
|----------------------|------------------------|
| $\alpha_0 = 0$       | $f_0 := w$             |
| $\alpha_1 = 0$       | $f_1 := x + z^2$       |
| $\alpha_2 = 0$       | $f_2 := x + y^2 + w + t$ |

We note that when $\alpha_0 = 0$, we see that the system (1) admits a particular solution $w = 0$, and when $\alpha_2 = 0$, after we make the birational and symplectic transformations:

\[ x_2 = x + y^2 + w + t, \; y_2 = y, \; z_2 = z + y, \; w_2 = w \]

we see that the system (1) admits a particular solution $x_2 = 0$.

**Proposition 2.3.** Let us define the following translation operators:

\[ T_1 := s_1 s_2 s_1 s_0, \quad T_2 := s_1 T_1 s_1. \]

These translation operators act on parameters $\alpha_i$ as follows:

\[ T_1(\alpha_0, \alpha_1, \alpha_2) = (\alpha_0, \alpha_1, \alpha_2) + (-2, 1, 0), \]
\[ T_2(\alpha_0, \alpha_1, \alpha_2) = (\alpha_0, \alpha_1, \alpha_2) + (0, -1, 1). \]

**Theorem 2.4.** Let us consider a polynomial Hamiltonian system with Hamiltonian $K \in \mathbb{C}(t)[x, y, z, w]$. We assume that

(A1) $\deg(K) = 6$ with respect to $x, y, z, w$.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system $r_i$ ($i = 0, 1, 2$):

\[ r_0 : x_0 = x, \; y_0 = y, \; z_0 = \frac{1}{z}, \; w_0 = -(wz + \alpha_0)z, \]
\[ r_1 : x_1 = -((x + z^2)y - \alpha_1)y, \; y_1 = \frac{1}{y}, \; z_1 = z, \; w_1 = w - 2yz, \]
\[ r_2 : x_2 = -((x + y^2 + w + t)y - \alpha_2)y, \; y_2 = \frac{1}{y}, \; z_2 = z + y, \; w_2 = w. \]

Then such a system coincides with the system (1) with the polynomial Hamiltonian (2).

By this theorem, we can also recover the parameter’s relation (3).

We note that the condition (A2) should be read that

\[ r_j(K) \quad (j = 0, 1), \quad r_2(K + y) \]

are polynomials with respect to $x_i, y_i, z_i, w_i$.  

In this section, we find a 1-parameter family of coupled Painlevé systems in dimension four with affine Weyl group symmetry of type $A_1^{(1)}$ given by

$$ dx/dt = \partial H/\partial y, \quad dy/dt = -\partial H/\partial x, \quad dz/dt = \partial H/\partial w, \quad dw/dt = -\partial H/\partial z $$

with the polynomial Hamiltonian

$$ H = H_{II}(x, y, t, \alpha_0) + H_3(z, w, t) + yzw $$

$$ = xy^2 + x^2 + tx - \alpha_0 y + \frac{z^2}{4} - \frac{w^2}{4} + yzw. $$

Here $x, y, z$ and $w$ denote unknown complex variables, and $\alpha_0, \alpha_1$ are complex parameters satisfying the relation:

$$ \alpha_0 + \alpha_1 = 1 $$

The symbol $H_3$ is given by

$$ H_3(z, w, t) = \frac{z^2}{4} - \frac{w^2}{4}. $$

Of course, the Hamiltonian itself is the first integral.
This is the second example which gave higher-order Painlevé equations of type $A_1^{(1)}$.

We note that in this case the invariant divisors are different from the ones of the second member $P_{II}^{(2)}$ of the second Painlevé hierarchy given in the paper [20].

$$\begin{array}{|c|c|c|} 
\hline 
\text{Invariant divisors} & f_0 & f_1 \\
\hline 
\text{System (12)} & x + z^2 & x + y^2 + w^2 + t \\
\hline 
P_{II}^{(2)} & y & y + t - 2w^2 + 4x(z + xw) \\
\hline 
\end{array}$$

We remark that for this system we tried to seek its first integrals of polynomial type with respect to $x, y, z, w$. However, we can not find. Of course, the Hamiltonian $H$ is not its first integral.

This system can be obtained by connecting the invariant divisors $w$ and $x + y^2 + w^2 + t$ for the canonical variables $(x, y, z, w)$ in the system (1).

**Theorem 3.1.** The system (12) admits the affine Weyl group symmetry of type $A_1^{(1)}$ as the group of its Bäcklund transformations, whose generators $s_0, s_1$ defined as follows:

$$s_0 : (\ast) \rightarrow \left( x, y - \frac{\alpha_0}{x + z^2}, z, w - \frac{2\alpha_0 z}{x + z^2}, t; -\alpha_0, \alpha_1 + 2\alpha_0 \right),$$

$$s_1 : (\ast) \rightarrow \left( x + \frac{2\alpha_1 y}{f_1}, y - \frac{\alpha_1}{f_1}, z + \frac{2\alpha_1 w}{f_1}, w, t; \alpha_0 + 2\alpha_1, -\alpha_1 \right),$$

where $f_1 := x + y^2 + w^2 + t$.

**Proposition 3.2.** This system has the following invariant divisors:

| parameter’s relation | $f_i$       |
|----------------------|-------------|
| $\alpha_0 = 0$      | $f_0 := x + z^2$       |
| $\alpha_1 = 0$      | $f_1 := x + y^2 + w^2 + t$       |

**Proposition 3.3.** Let us define the following translation operator:

$$T := s_1s_0.$$ 

This translation operator acts on parameters $\alpha_i$ as follows:

$$T(\alpha_0, \alpha_1) = (\alpha_0, \alpha_1) + (-2, 2).$$

**Theorem 3.4.** Let us consider a polynomial Hamiltonian system with Hamiltonian $K \in \mathbb{C}(t)[x, y, z, w]$. We assume that

1. $\deg(K) = 6$ with respect to $x, y, z, w$.
2. This system becomes again a polynomial Hamiltonian system in each coordinate system $r_i$ ($i = 0, 1$):

$$r_0 : x_0 = -((x + z^2)y - \alpha_0)y, \quad y_0 = \frac{1}{y}, \quad z_0 = z, \quad w_0 = w - 2yz,$$

$$r_1 : x_1 = -((x + y^2 + w^2 + t)y - \alpha_1)y, \quad y_1 = \frac{1}{y}, \quad z_1 = z + 2yw, \quad w_1 = w.$$
Then such a system coincides with the system \((12)\) with the polynomial Hamiltonian \((13)\).

By this theorem, we can also recover the parameter’s relation \((14)\).

We note that the condition \((B2)\) should be read that
\[
r_0(K), \quad r_1(K + y)
\]
are polynomials with respect to \(x_i, y_i, z_i, w_i\).

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