Quantum relative modular functions

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Abstract

Let $H \trianglelefteq G$ be a closed normal subgroup of a locally compact quantum group. We introduce a strictly positive group-like element affiliated with $L^\infty(G)$ that, roughly, measures the failure of $G$ to act measure-preservingly on $H$ by conjugation. The triviality of that element is equivalent to the condition that $G$ and $G/H$ have the same modular element, by analogy with the classical situation. This condition is automatic if $H \leq G$ is central, and in general implies the unimodularity of $H$.

We also describe a bijection between strictly positive group-like elements $\delta$ affiliated with $C_0(G)$ and quantum-group morphisms $G \to (\mathbb{R}, +)$, with the closed image of the morphism easily described in terms of the spectrum of $\delta$. This then implies that property-(T) locally compact quantum groups admit no non-obvious strictly positive group-like elements.

Key words: locally compact quantum group; modular element; modular function; unimodular

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Introduction

The initial motivation for the present paper was the well-known result that nilpotent locally compact groups are unimodular (e.g. [18, p.318, Corollary 2]). Several proofs exist in the literature (with more references cited in Section 2), but one naive strategy that comes to mind would be as follows: given that nilpotence means that the ascending central series

$$\{1\} \leq Z(G) \leq \cdots \leq G$$

is finite, perhaps one can employ induction by starting with the abelian group $Z(G)$ (which is of course unimodular) and then lifting unimodularity along cocentral quotients $G \to G/H$ (i.e. quotients by a central closed subgroup). In short, one would need

Claim A cocentral quotient $G/H$ of a locally compact group is unimodular if and only if $G$ is.

Since unimodularity simply means that the modular function [15, §2.4] is trivial, this suggests possible generalization:

Claim For any cocentral quotient $\pi : G \to G/H$, the modular function of $G$ is obtained from that of $G/H$ by restriction along $\pi$.

All of this is true and follows easily enough from standard material on modular functions (e.g. from [15, Theorem 2.51]), though I have not seen these precise statements. Couched in these terms, though, the statements generalize easily to the framework of locally compact quantum groups [24, 25, 23, 39, 31, 32, 26], since all of the ingredients are present. To summarize, postponing the notation and terminology until Section 1:
• For a locally compact quantum group $G$ there is a modular element $\delta_G$ [24, §7] affiliated with the von Neumann algebra $L^\infty(G)$, to be thought of as the inverse of the usual modular function (see Remark 2.2 for why this convention is convenient).

• There are notions of (closed [36, Definition 2.6]) normal [37, Definition 2.10] and central [20, Definition 2.3] quantum subgroups $H \leq G$.

• As well as quotient groups $G/H$ by closed normal quantum subgroups [37, Theorem 2.11].

All of this allows the formulation of one of the main results below (see Section 2 and Corollary 2.3):

**Theorem** Given a closed central quantum subgroup $H \leq G$ of a locally compact quantum group, the modular elements of $G$ and $G/H$ coincide.

In particular, a cocentral quotient $G/H$ is unimodular if and only if $G$ is. ■

More generally, there is a very satisfying way of measuring the discrepancy from the previous theorem’s conclusion. Summarizing Theorems 2.12 and 2.14 and Propositions 2.16 and 2.17:

**Theorem** Let $H \leq G$ be a closed normal quantum subgroup. The modular elements $\delta_G$ and $\delta_{G/H}$ strongly commute, so their ratio $\delta := \delta_G\delta_{G/H}^{-1}$ is again a strictly positive element affiliated with $L^\infty(G)$, group-like in the sense that $\Delta_G(\delta) = \delta \otimes \delta$.

That element is trivial precisely when two canonical operator valued weights from $L^\infty(G)$ to its von Neumann subalgebra $L^\infty(G/H)$ coincide. This condition

• is the quantum analogue of $G$ acting measure-preservingly by conjugation on $H$;

• is automatic when $H \leq G$ is central;

• and entails the unimodularity of $H$. ■

The element $\delta$ in the statement above is the relative modular function alluded to in the title of the paper (see Definition 2.13): the phrase is meant to indicate that it is relative to an embedding $H \leq G$ rather than absolute, attached to $G$ alone.

On a different note but still on the topic of strictly positive group-like elements $\delta$ affiliated with $L^\infty(G)$, we have (Proposition 3.2, Proposition 3.5 and Theorem 3.7)

**Theorem** Every strictly positive group-like element $\delta$ affiliated with $L^\infty(G)$ induces a quantum-group morphism $G \to \mathbb{R}$ whose closed image is precisely the closed subgroup

$$\{ \log t \mid 0 < t \in \text{Sp}(\delta) \} \subseteq (\mathbb{R}, +).$$ ■

An immediate consequence (Theorem 3.9 below) is the following generalization of the unimodularity of property-(T) quantum groups [9, Theorem 6.1]:

**Theorem** If the LCQG $G$ has property (T) then the only strictly positive group-like element affiliated with $L^\infty(G)$ is 1. ■

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1 Preliminaries

Inner products are linear in the second variable, and for vectors \( v, w \) in a Hilbert space \((\mathcal{H}, \langle \cdot | \cdot \rangle)\) and an operator \( A \in B(\mathcal{H}) \) we write

\[
\omega_{v,w}(A) := \langle v | Aw \rangle = \langle vA | w \rangle = \langle A^*v | w \rangle.
\]

Any number of sources cover the needed operator-algebra background: [4, 29], etc. Assorted standard notation:

- \( B(\cdot) \) and \( K(\cdot) \) denote the algebras of bounded and respectively compact operators on a Hilbert space.
- \( M(\cdot) \) is the multiplier algebra of a \( C^* \)-algebra [4, §II.7.3].
- For \( C^* \)-algebras \( A \) and \( B \) the space \( \text{Mor}(A,B) \) of morphisms from \( A \) to \( B \) consists (as in [38, Introduction], [11, §1.1], [12, §2], etc.) of those linear, bounded, multiplicative \( * \)-maps \( f : A \to M(B) \) that are non-degenerate in the sense that \( f(A)B \) is norm-dense in \( B \).

We also depict \( \pi \in \text{Mor}(A,B) \) as arrows:

\[ A \xrightarrow{\pi} B. \]

- \( M_* \) is the predual [4, §III.2.4] of a von Neumann algebra, \( M_+ \) its positive cone (set of positive elements), and \( \tilde{M}_+ \) its extended positive part [16, Definition 1.1].
- the tensor-product symbol ‘\( \otimes \)’ has contextual meaning: between \( C^* \)-algebras it denotes the minimal (or spatial) \( C^* \) tensor product [4, §II.9.1.3], between \( W^* \)-algebras it is the von-Neumann-flavored spatial tensor product of [4, §III.5.1.4], the Hilbert-space tensor product when appropriate, etc.

For the needed material on locally compact quantum groups we refer mainly to [24, 25, 23] (with more precise citations below, as needed). The first of these also has an introductory overview of the necessary weight and modular theory; [34, 33] are other good sources for this latter topic. Of particular interest are the operator-valued weights of [16, 17], covered also in [34, §IX.4].

To recall, briefly, the main concept of interest ([25, Definition 1.1]):

**Definition 1.1** A locally compact quantum group \( \mathbb{G} \) (occasionally abbreviated LCQG) is a pair \((M, \Delta)\) where

- \( M \) is a von Neumann algebra, \( M_* \), denoted also by \( L^\infty(\mathbb{G}) \),
- \( \Delta = \Delta_\mathbb{G} \) is a \( W^* \) morphism \( M \to M \otimes M \), coassociative in the sense that
  \[
  (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta : M \to M \otimes M \otimes M.
  \]
- we assume the existence of
  - (a) a *left Haar weight* on \( M \): a normal, semifinite and faithful (n.s.f. for short) weight \( \varphi = \varphi_\mathbb{G} \), left-invariant in the sense that
    \[
    \varphi((\omega \otimes \text{id})\Delta(x)) = \omega(1)\varphi(x)
    \]
    for all \( \omega \in M_* \) and
    \[
    x \in M_+^\varphi := \{ x \in M_+ | \varphi(x) < \infty \}.
    \]
(b) similarly, a right Haar weight \( \psi = \psi_G \), right-invariant:

\[
\psi((\text{id} \otimes \omega)\Delta(x)) = \omega(1)\psi(x), \forall \omega \in M_\ast, \forall x \in m^+_\psi.
\]

Also central to the discussion is the following object ([24, Terminology 7.16]).

**Definition 1.2** The modular element \( \delta = \delta_G \) of an LCQG \( G \) is the unique (possibly unbounded) operator that is

- **strictly positive** in the sense that its spectral resolution [30, Theorem 13.30] assigns the zero projection to \( \{0\} \) (equivalently: it has dense range [24, p.841]);
- **affiliated** with \( L^\infty(G) \) in the sense that its spectral projections belong to that von Neumann algebra;

and such that

\[
\psi_G(\cdot) = \varphi_{G,\delta}(\cdot) := \varphi_G\left(\delta^{1/2} \cdot \delta^{1/2}\right).
\] (1-1)

Other notation pertinent to quantum groups:

- \( L^2(G) = L^2(G, \varphi_G) \) is the Hilbert space carrying the GNS representation attached to the left Haar weight \( \varphi \), equipped with \( \Lambda = \Lambda_G : n_\varphi \rightarrow L^2(G) \).
- \( C_0(G) \subset L^\infty(G) \) is the reduced function algebra of \( G \), associated to \( L^\infty(G) \) in [25, §1.2] (with the notation \( M = L^\infty(G), M_u = C_0(G) \)) and studied extensively in [24].
- \( C_0^u(G) \) is the universal function algebra, constructed in [23, §4].
- \( \hat{G} \) is the dual LCQG: [25, §1.1] for the von Neumann version and [24, §8] for the \( C^* \) counterpart.
- \( W \in W_G \) is the multiplicative unitary of [24, Proposition 3.17]: it is defined as an operator on \( L^2(G) \otimes L^2(G) \) by

\[
W^*(\Lambda(x) \otimes \Lambda(y)) = \Lambda \otimes \Lambda(\Delta(y)(x \otimes 1)),
\]

it implements the comultiplication by

\[
\Delta(x) = W^*(1 \otimes x)W,
\]

and belongs to

\[
M(C_0(G) \otimes C_0(\hat{G})) \subset L^\infty(G) \otimes L^\infty(\hat{G}).
\]

- \( S = S_G \) and \( R = R_G \) are the antipode and unitary antipode of \( G \) respectively [24, Terminology 5.42].

**Remark 1.3** [24, Proposition 7.10] says that \( \delta_G \) is in fact also affiliated with the \( C^* \)-algebra \( C_0(G) \) in the sense of [38, Definition 1.1]. Furthermore, it lifts along the surjection \( C_0^u(G) \rightarrow C_0(G) \) to a strictly positive element \( \delta_u = \delta_{u,G} \) affiliated with the universal function algebra [23, Proposition 10.1].

**Notation 1.4** We denote the affiliation relation, in either the \( C^* \) or \( W^* \) setting, by primed containment symbols: \( \in' \) and \( \ni' \).
1.1 Morphisms

LCQG morphisms have many incarnations; for a review of the theory the reader can consult, for instance, [27] (where many of the issues were initially settled), [23, §12] or [11, §1.3]. In particular, attached to such a morphism $\pi : \mathbb{H} \to \mathbb{G}$ we have a right action

$$\pi_r : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{H})$$

as well as a left one,

$$\pi_l : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{H}).$$

Throughout the paper, closed quantum subgroups $\mathbb{H} \leq \mathbb{G}$ are as in [36, Definition 2.6], referred to as closed in the sense of Vaes in [11, Definition 3.1] (to distinguish from a formally weaker version due to Woronowicz): those for which the dual morphism $\widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ corresponds to a comultiplication-intertwining embedding

$$L^\infty(\widehat{\mathbb{H}}) \subseteq L^\infty(\widehat{\mathbb{G}}).$$

The centrality of a quantum subgroup (or more generally, of a morphism) can be cast as the following paraphrase of [20, Definition 2.3]:

**Definition 1.5** A morphism $\pi : \mathbb{H} \to \mathbb{G}$ is central if the diagram

$$\begin{array}{ccc}
L^\infty(\mathbb{G}) & \xrightarrow{\pi_r} & L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{H}) \\
\pi_l & \downarrow & \downarrow \text{id} \\
L^\infty(\mathbb{H}) & \xrightarrow{\pi_r} & L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{H}) \\
\end{array}$$

commutes. ♦

For a closed quantum subgroup $\iota : \mathbb{H} \leq \mathbb{G}$ one can define the left and right quantum homogeneous $\mathbb{G}$-spaces (e.g. [36, Definition 4.1]):

$$L^\infty(\mathbb{G}/\mathbb{H}) := \{ x \in L^\infty(\mathbb{G}) \mid \iota_r(x) = x \otimes 1 \}$$

$$L^\infty(\mathbb{H}\backslash\mathbb{G}) := \{ x \in L^\infty(\mathbb{G}) \mid \iota_l(x) = 1 \otimes x \}.\quad (1-3)$$

Morphisms of locally compact quantum groups preserve unitary antipodes; this is well known, but we set out the claim here in precisely the form needed below (see e.g. [21, equation (2.2b)]).

**Lemma 1.6** For an LCQG morphism $\pi : \mathbb{H} \to \mathbb{G}$ the diagrams

$$\begin{array}{ccc}
L^\infty(\mathbb{G}) & \xrightarrow{\pi_r} & L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{H}) \\
R_G & \downarrow & R_G \otimes R_H \\
L^\infty(\mathbb{G}) & \xrightarrow{\pi_l} & L^\infty(\mathbb{H}) \otimes L^\infty(\mathbb{G}) \\
\pi_l & \downarrow \text{flip} & \downarrow \text{flip} \\
L^\infty(\mathbb{H}) & \xrightarrow{\pi_r} & L^\infty(\mathbb{H}) \otimes L^\infty(\mathbb{H}) \\
\end{array}$$

and

$$\begin{array}{ccc}
L^\infty(\mathbb{G}) & \xrightarrow{\pi} & L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{H}) \\
R_G & \downarrow & R_H \otimes R_G \\
L^\infty(\mathbb{G}) & \xrightarrow{\pi_r} & L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{H}) \\
\pi_l & \downarrow \text{flip} & \downarrow \text{flip} \\
L^\infty(\mathbb{H}) & \xrightarrow{\pi_l} & L^\infty(\mathbb{H}) \otimes L^\infty(\mathbb{G}) \\
\end{array}$$

commute.
Proof This follows from [27, Theorems 5.3 and 5.5], which describe \( \pi_r \) and \( \pi_l \) in terms of a single object attached to the morphism \( \pi \) (a bicharacter, in the language of [27, §3]).

In the discussion below, we follow [24] in denoting by

- \( \sigma_t \) (or \( \sigma_{G,t} \) when wishing to emphasize the group) the modular automorphism group of a left Haar weight [24, §1.3];
- \( \sigma'_t = \sigma'_{G,t} \) the modular group of a right Haar weight [24, p.846];
- \( \tau_t = \tau_{G,t} \) the scaling group of \( G \) [24, Terminology 5.42] and by \( \nu = \nu_G \) its scaling constant [24, Terminology 7.16].

In addition to the antipode-intertwining properties noted in Lemma 1.6, it will also be useful to record the compatibility between \( \pi_{l,r} \) and these one-parameter groups.

**Lemma 1.7** For an LCQG morphism \( \pi : \mathbb{H} \to G \) we have

\[
\pi_l \sigma_{G,t} = (\tau_{\mathbb{H},t} \circ \sigma_{G,t}) \pi_l : L^\infty(G) \to L^\infty(\mathbb{H}) \otimes L^\infty(G). \tag{1-4}
\]

\[
\pi_r \sigma'_{G,t} = (\sigma'_{G,t} \circ \tau_{\mathbb{H},-t}) \pi_r : L^\infty(G) \to L^\infty(G) \otimes L^\infty(\mathbb{H}). \tag{1-5}
\]

\[
\pi_l \tau_{G,t} = (\tau_{\mathbb{H},t} \circ \tau_{G,t}) \pi_l : L^\infty(G) \to L^\infty(\mathbb{H}) \otimes L^\infty(G). \tag{1-6}
\]

\[
\pi_r \tau_{G,t} = (\tau_{G,t} \circ \tau_{\mathbb{H},t}) \pi_r : L^\infty(G) \to L^\infty(G) \otimes L^\infty(\mathbb{H}). \tag{1-7}
\]

Proof The style of proof is the same for all of these, so we focus on (1-4).

All three one-parameter groups lift to the universal quantum-group function algebras \( C_0^u(G) \) (and analogue for \( \mathbb{H} \)) of [23]: see [23, §8] for the modular groups \( \sigma \) and \( \sigma' \) and [23, §9] for \( \tau \).

At the universal level we have [23, Proposition 9.2]

\[
\Delta_G \sigma_{G,t}^u = (\tau_{G,t}^u \circ \sigma_{G,t}^u) \Delta_G : C_0^u(G) \to M(C_0^u(G) \otimes C_0^u(G)). \tag{1-8}
\]

Now apply the universal incarnation

\[
\pi^u : C_0^u(G) \to M(C_0(\mathbb{H})^u)
\]

of \( \pi \) ([27, §4], [23, §12]) to the left leg of (1-8) to obtain

\[
\pi_l^u \sigma_{G,t}^u = (\pi^u \tau_{G,t}^u \circ \sigma_{G,t}^u) \Delta_G : C_0^u(G) \to M(C_0(\mathbb{H})^u \otimes C_0^u(G)),
\]

where

\[
\pi_l^u := (\pi^u \circ \text{id}) \Delta_G.
\]

Next, use the scaling-group-intertwining property

\[
\pi^u \tau_{G,t}^u = \tau_{\mathbb{H},t}^u \pi^u
\]

of \( \pi^u \) (which follows, for instance, from [27, Proposition 3.10]) on the right-hand side to produce

\[
\pi_l^u \sigma_{G,t}^u = (\tau_{\mathbb{H},t}^u \pi^u \circ \sigma_{G,t}^u) \Delta_G
\]

\[
= (\tau_{\mathbb{H},t}^u \circ \sigma_{G,t}^u) \pi_l^u.
\]
Finally, to conclude, note that this reduces precisely to the desired identity (1-4), because

\[
\begin{array}{ccc}
C_0^u(\mathbb{G}) & \xrightarrow{\pi^u} & M(C_0(\mathbb{H})^u \otimes C_0^u(\mathbb{G})) \\
\downarrow & & \downarrow \\
L^\infty(\mathbb{G}) & \xrightarrow{\pi_l} & L^\infty(\mathbb{H}) \otimes L^\infty(\mathbb{G})
\end{array}
\]

commutes [23, Proposition 12.1].

An immediate consequence of Lemma 1.7 and the definitions of the quantum homogeneous spaces $\mathbb{G}/\mathbb{H}$ and $\mathbb{H}\backslash\mathbb{G}$:

**Corollary 1.8** For any closed locally compact quantum subgroup $\mathbb{H} \leq \mathbb{G}$

1. $L^\infty(\mathbb{G}/\mathbb{H}) \subseteq L^\infty(\mathbb{G})$ is invariant under $\tau_{\mathbb{G},t}$ and $\sigma'_{\mathbb{G},t}$;
2. and similarly, $L^\infty(\mathbb{H}\backslash\mathbb{G}) \subseteq L^\infty(\mathbb{G})$ is invariant under $\tau_{\mathbb{G},t}$ and $\sigma_{\mathbb{G},t}$.

\[\blacksquare\]

## 2 Relative modular elements

One of the main results of this section (to be strengthened later, when more language has been introduced) is

**Theorem 2.1** Let $\mathbb{H} \leq \mathbb{G}$ be a closed central subgroup of a locally compact quantum group. The modular element of $\mathbb{G}$ coincides with that of $\mathbb{G}/\mathbb{H}$.

**Remark 2.2** To put Theorem 2.1 into some perspective, with centrality being the last of a series of progressively more stringent conditions, note that

- If $\mathbb{H} \leq \mathbb{G}$ is a closed normal subgroup then the modular element $\delta_{\mathbb{G}}$ of $\mathbb{G}$ restricts to $\delta_{\mathbb{H}}$ in the sense of [5, Definition 3.3] (by [5, Theorem 3.4 and Corollary 3.9]).

The restriction terminology employed there is chosen so that classically it specializes back to what one would guess. The modular function of a locally compact group $\mathbb{G}$ is typically denoted by $\Delta_{\mathbb{G}}$ or plain $\Delta$ ([3, §A.3], [13, §1.4], [15, §2.4], etc.). Here, in order to avoid confusion with the comultiplication, we write

$$
\delta_{\mathbb{G}}(x) := \Delta(x)^{-1}, \ x \in \mathbb{G}.
$$

This is compatible with the previous use of the symbol $\delta$, in the general context of quantum groups: on the one hand we have the relation (1-1) between left and right Haar weights, while on the other hand, classically, we have

$$
d\mu_{\text{right}}(x) = \Delta(x)^{-1}d\mu_{\text{left}}(x)
$$

by [15, Proposition 2.31].

As the name suggests, then, $\delta_{\mathbb{G}}$ restricting to $\delta_{\mathbb{H}}$ as in [5, Definition 3.3] means precisely that $\delta_{\mathbb{H}} = \delta_{\mathbb{G}}|_{\mathbb{H}}$ for ordinary locally compact groups.
• If furthermore \( H \leq G \) is unimodular, it follows that the modular function \( f \) factors through \( G \to G/H \), in the sense that
\[
\delta_G^H \in L^\infty(G/H), \forall t \in \mathbb{R};
\]
in other words, \( \delta \) is affiliated with the von Neumann subalgebra \( L^\infty(G/H) \subseteq L^\infty(G) \). This follows from [5, Theorem 3.4, condition (2)] and classically it means that the morphism
\[
\delta_G : G \to (\mathbb{R}^\times, \cdot)
\]
factors through \( G/H \).

• Finally, it takes centrality to ensure that that factorization in fact coincides with the modular function
\[
\delta_{G/H} : G/H \to (\mathbb{R}^\times, \cdot).
\]

Before moving on to the proof of Theorem 2.1, note the following immediate consequence.

**Corollary 2.3** If \( H \leq G \) is a closed central subgroup of a locally compact quantum group then \( G \) is unimodular if and only if \( G/H \) is.

As yet another consequence, we have the unimodularity of nilpotent locally compact (classical) groups. The result is well known, but the proofs one encounters tend to be different in flavor: [18, Corollary 2, p.318] leverages some structure results on nilpotent groups, while [3, Example A.3.7] uses (via [3, Exercise A.8.10]) the fact that nilpotent groups have subexponential growth.

**Corollary 2.4** Nilpotent locally compact groups are unimodular.

**Proof** Filter the nilpotent group \( G \) with its ascending central series
\[
\{1\} \leq Z(G) \leq \cdots \leq G
\]
(finite, by the nilpotence assumption), and proceed by induction on the length of that series: the base case of abelian groups is trivial, and the induction step passes from a quotient to a central extension using Corollary 2.3.

For a closed quantum subgroup \( H \leq G \) we will work with the two operator-valued weights \( \pi_l \) and \( \pi_r \) defined by
\[
L^\infty(G) \xrightarrow{\pi_l} L^\infty(G) \otimes L^\infty(H) \xrightarrow{id \otimes \phi_H} L^\infty(G)_+, \tag{2-1}
\]
with the ‘l’ subscript indicating left invariance or mapping to the left coset space, and similarly,
\[
L^\infty(G) \xrightarrow{\pi_r} L^\infty(H) \otimes L^\infty(G) \xrightarrow{\psi_H \otimes id} L^\infty(G)_+, \tag{2-2}
\]

**Lemma 2.5** For a closed quantum subgroup \( H \leq G \) of a closed quantum subgroup we have
\[
R_G \circ \pi_l = \pi_r \circ R_G \quad \text{and} \quad R_G \circ \pi_r = \pi_l \circ R_G.
\]
Proof That $R_G$ interchanges $L^\infty(G/\mathbb{H})_+$ and $L^\infty(\mathbb{H}ackslash G)_+$ follows from Lemma 1.6 (applied to the embedding morphism $\iota: \mathbb{H} \leq G$) and the definition (1-3) of the two quantum homogeneous spaces (see also [21, Proposition 3.3]).

As for the substance of the statement, it too is an immediate consequence of Lemma 1.6: to obtain $R_G \circ T_l = T_r \circ R_G$, for instance, apply $\psi_\mathbb{H}$ to the right-hand leg of the top diagram in Lemma 1.6 and use the fact that (by definition!) $\varphi_\mathbb{H}$ is nothing but $\psi_\mathbb{H} \circ R_\mathbb{H}$. The other equation follows similarly from the second diagram.

For a closed normal quantum subgroup $\mathbb{H} \leq G$ the two homogeneous spaces coincide (and this in fact characterizes normality; [21, §4], [37, Theorem 2.11]):

$$\mathbb{H} \text{ normal} \iff L^\infty(G/\mathbb{H}) = L^\infty(\mathbb{H}\backslash G).$$

In that case $G/\mathbb{H}$ is an LCQG in its own right and $R_G$ restricts to $R_{G/\mathbb{H}}$. Lemma 2.5 thus implies

**Lemma 2.6** For a closed normal quantum subgroup $\mathbb{H} \trianglelefteq G$ of a closed quantum subgroup we have

$$R_{G/\mathbb{H}} \circ T_l = T_r \circ R_G \quad \text{and} \quad R_{G/\mathbb{H}} \circ T_r = T_l \circ R_G.$$  

Recall [8, Proposition] also that for closed normal quantum subgroups we have a Weyl-type “disintegration formula”

$$\varphi_G = \varphi_{G/\mathbb{H}} \circ T_l.$$  

(2-3)

Naturally, since left Haar weights are only determined up to positive scaling, the content of this claim is that the right-hand side of (2-3) is left-invariant. Having fixed a left Haar weight $\varphi$ though, we are making the convention that the corresponding right Haar weight $\psi$ is determined by it: $\psi = \varphi \circ R$. The following observation says that this switch from left to right Haar weights is compatible with the operator-valued weights $T$.

**Lemma 2.7** For a closed, normal quantum subgroup $\mathbb{H} \trianglelefteq G$ of a locally compact quantum group we have

$$\varphi_G = \varphi_{G/\mathbb{H}} \circ T_l \iff \psi_G = \psi_{G/\mathbb{H}} \circ T_r.$$  

(2-4)

**Proof** This follows from the various intertwining properties of the unitary antipode(s), already noted above: suppose we have scaled the left Haar weights so that the left hand equation holds. We then have

$$\psi_G = \varphi_G \circ R_G \quad \text{by convention}$$

$$= \varphi_{G/\mathbb{H}} \circ T_l \circ R_G \quad \text{by assumption}$$

$$= \varphi_{G/\mathbb{H}} \circ R_{G/\mathbb{H}} \circ T_r \quad \text{Lemma 2.6}$$

$$= \psi_{G/\mathbb{H}} \circ T_r \quad \text{again by convention}.$$  

This concludes the proof.

**Proof of Theorem 2.1** Under the centrality assumption $\mathbb{H}$ will in particular be abelian (in the sense that $L^\infty(\mathbb{H})$ is cocommutative) and hence unimodular, so its left and right Haar weights coincide: $\varphi_\mathbb{H} = \psi_\mathbb{H}$. $\mathbb{H}$ is furthermore normal so that

$$L^\infty(\mathbb{H}\backslash G) = L^\infty(G/\mathbb{H}),$$
and the two operator-valued weights $T_l$ and $T_r$ introduced in (2-1) and (2-2) coincide:

$$T := T_l = T_r.$$  

According to Lemma 2.7 we can scale the various Haar weights so that

$$\varphi_{G/H} \circ T = \varphi_G \quad \text{and} \quad \psi_{G/H} \circ T = \psi_G. \quad (2-5)$$

We have

$$\nu_G^{\frac{1}{2}it^2} \delta_G^t = (D\psi_G : D\varphi_G) t \quad [35, \text{Proposition 4.4}] \quad \text{and} \quad [24, \text{Proposition 7.12 (6)}]$$

$$= (D\psi_{G/H} \circ T : D\varphi_G \circ T) t \quad \text{by (2-5)}$$

$$= (D\psi_{G/H} : D\varphi_{G/H}) t \quad [16, \text{Theorem 4.7}]$$

$$= \nu_{G/H}^{\frac{1}{2}it^2} \delta_{G/H}^t \quad \text{analogous to the first equality.}$$

This is already sufficient to draw the desired conclusion

$$\delta_G^t = \delta_{G/H}^t,$$

since given a positive real $\lambda$ and a positive (possibly unbounded) operator $\delta$, the latter can be recovered from $u_t := \lambda^{it^2} \delta^t$: the logarithm $\log \delta$ (obtained by applying $\log$ to the positive operator $\delta$ as usual, via functional calculus [30, Theorem 13.24]) can be obtained [34, §A.3] as

$$i \log \delta \xi = \lim_{t \to 0} \frac{\delta^t - 1}{t} \xi = \lim_{t \to 0} \frac{u_t - 1}{t} \xi$$

for $\xi$ ranging over a dense subspace of the ambient Hilbert space. ■

We also record the following remark, obtained in passing in the course of the above proof.

**Corollary 2.8** If $H \leq G$ is a central, closed, normal quantum subgroup of a locally compact quantum group the scaling constants of $G$ and $G/H$ coincide.

**Proof** The proof of Theorem 2.1 actually shows that

$$\nu_G^{\frac{1}{2}it^2} \delta_G^t = \nu_{G/H}^{\frac{1}{2}it^2} \delta_{G/H}^t, \forall t \in \mathbb{R}$$

and then concludes that the $\delta$ factors coincide: $\delta_G^t = \delta_{G/H}^t$. The $\nu$ factors must thus also coincide:

$$\nu_G^{\frac{1}{2}it^2} = \nu_{G/H}^{\frac{1}{2}it^2}, \forall t \in \mathbb{R},$$

which of course implies $\nu_G = \nu_{G/H}$. ■

**Remark 2.9** By way of bolstering the intuitive plausibility of Theorem 2.1, it might be instructive to consider the classical setup whereby $G$ is a connected Lie group. In that case we know [15, Proposition 2.30] that

$$\delta_G(x) = \Delta_G(x)^{-1} = \det Ad(x),$$

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where \( Ad : G \to GL(Lie(G)) \) is the adjoint action. Choose a decomposition
\[
Lie(G) = Lie(H) \oplus V
\]
and a compatible basis that will give matrix expressions for adjoint-action operators. The centrality of \( H \) then ensures that
\[
Ad(x) = \begin{pmatrix} I & * \\ 0 & Ad(\pi) \end{pmatrix},
\]
where
\[
G \ni x \mapsto \pi \in G/H.
\]
Plainly, the determinant of (2-6) equals that of its lower right-hand block, hence Theorem 2.1 in this case.

Remark 2.9 also suggests what is needed in order to extend Theorem 2.1 to normal (non-central) closed quantum subgroups. In that case, (2-6) takes the form
\[
Ad(x) = \begin{pmatrix} Ad(x|_H) & * \\ 0 & Ad(\pi) \end{pmatrix},
\]
where \( Ad(x|_H) \) denotes the adjoint action by \( x \) on \( Lie(H) \). Taking determinants we thus have
\[
\delta_G(x) = \det Ad(x) = \det Ad(x|_H) \cdot \det Ad(\pi) = \det Ad(x|_H) \cdot \delta_{G/H}(\pi).
\]
The “correction factor” away from Theorem 2.1 is thus \( \det Ad(x|_H^{-1}) \). Its quantum counterpart, for normal \( H \leq G \), will be a measure of how far apart the two operator-valued weights \( T_l, T_r \) are from each other; it was their coincidence that captured the triviality of the upper left-hand block in (2-6). Measuring this discrepancy between \( T_r \) and \( T_l \) is precisely what the Radon-Nikodym derivative \( (DT_r : DT_l)_t \) of [17, Definition 6.2] is designed to do, so that construction features below.

As [35, Proposition 5.5] makes clear, such Radon-Nikodym derivatives ought to be intimately related to how one of the operator-valued weights \( T_{l,r} \) evolves under the modular group of the other. The following result examines this.

**Lemma 2.10** For a closed locally compact quantum group \( H \leq G \) we have
\[
T_l \sigma'_{G,t} = \nu_{H}^{t} \sigma'_{G,t} T_l
\]
and similarly,
\[
T_r \sigma_{G,t} = \nu_{H}^{t} \sigma_{G,t} T_r
\]

**Proof** Note first that the right-hand sides actually make sense: by Corollary 1.8 the modular group \( \sigma'_{G,t} \) leaves the codomain
\[
L^\infty(G/H)_+ \subseteq L^\infty(G)_+
\]
of \( T_l \) invariant, and similarly for \( T_r \). The two arguments being entirely parallel, we only run through the first. Denoting by \( \pi : H \to G \) the embedding:
\[
T_l \sigma'_{G,t} = (id \otimes \varphi_H) \pi_r \sigma'_{G,t} \quad \text{by definition}
\]
\[
= (id \otimes \varphi_H)(\sigma'_{G,t} \otimes \tau_{H,-t}) \pi_r \quad (1-5)
\]
\[
= \nu_{H}^{t} (\sigma'_{G,t} \otimes \varphi_H) \pi_r \quad [24, \text{Proposition 6.8 (3)}]
\]
\[
= \nu_{H}^{t} \sigma'_{G,t} T_l \quad \text{by the definition of } T_l \text{ again}.
\]
This concludes the proof of (2-9).
Note, in passing, that for normal quantum subgroups Weyl disintegration transports over to scaling constants.

**Proposition 2.11** For a closed, normal quantum subgroup $H \subseteq G$ of a locally compact quantum group we have

$$\nu_G = \nu_H \nu_{G/H}.$$ 

**Proof** Throughout the proof we assume we have fixed Haar weights on $G$ and $G/H$ so that both conditions in (2.4) hold (as that result says we may):

$$\varphi_{G/H} \circ \mathcal{T}_l = \varphi_G \quad \text{and} \quad \psi_{G/H} \circ \mathcal{T}_r = \psi_G.$$ 

By definition ([24, Proposition 6.8 and Terminology 7.16]), $\nu_{G/H}$ can be expressed by

$$\varphi_{G/H} \circ \sigma'_{G/H,t} = \nu_{G/H} \varphi_{G/H}.$$ 

Now precompose both sides with $\mathcal{T}_l$:

$$\nu_{G/H} \varphi_G = \nu_{G/H} \varphi_{G/H} \mathcal{T}_l \quad (2.11)$$

$$= \varphi_{G/H} \circ \sigma'_{G/H,t} \mathcal{T}_l \quad (2.12)$$

$$= \varphi_{G/H} \circ \sigma'_{G,H,t} \mathcal{T}_l \quad [16, \text{Theorem 4.7}]$$

$$= \nu^{-1}_H \varphi_{G/H} \circ \mathcal{T}_l \sigma'_{G,t} \quad (2.9)$$

$$= \nu^{-1}_H \varphi_G \sigma'_{G,t} \quad (2.11) \text{again}$$

$$= \nu^{-1}_H \nu_{G} \varphi_G \quad [24, \text{Proposition 6.8 (3)}].$$

This gives the desired result $\nu_{G/H} = \nu^{-1}_H \nu_G$. □

In light of [35, Proposition 5.5], Lemma 2.10 is strongly suggestive of Theorem 2.12 below. In the statement, we refer to the modular group $\sigma^T_t$ of an operator-valued weight $T$ on $L^\infty(G)$; recall that for an operator-valued weight $T : M \to \hat{N}$ that modular group is

$$\sigma^T_t := \sigma_t^\theta \circ T |_{N^\circ}, \theta \text{ any n.s.f. weight on } N.$$ 

this is [17, Definition 6.2 (1)], relying on the fact that by [17, Proposition 6.1 (1)] the definition does not depend on $\theta$.

**Theorem 2.12** Let $H \subseteq G$ be a closed, normal locally compact quantum subgroup. There is a strictly positive element $\delta = \delta_{G/H}$ affiliated with the relative commutant

$$L^\infty(G/H)^c = L^\infty(G) \cap L^\infty(G/H)'$$

such that

$$(D\mathcal{T}_r : D\mathcal{T}_l)_t = \nu_H \delta_{G/H}^t$$ 

(2.13)

and

$$\sigma^T_{G,s}(\delta^t) = \nu_H \delta^t, \forall s, t \in \mathbb{R}.$$ 

(2.14)
**Proof** By [17, Definition 6.2], the Radon-Nikodym derivative \((DT^t : DT)_t\) between two operator-valued weights is simply \((D\omega T^t : D\omega T)_t\) for any n.s.f. weight \(\omega\) (since that derivative does not depend on \(\omega\) [17, Proposition 6.1]). We are thus free to choose the weight conveniently:

\[
(DT_r : DT_l)_t = (D\varphi_{G/H} \circ T_r : D\varphi_{G/H} \circ T_l)_t, \quad t \in \mathbb{R}.
\]

Assuming (2-11) (as we will), the right-hand weight is nothing but \(\varphi_G\), and its modular group is \(\sigma_{G,t}\). Under that group, the other weight evolves as follows:

\[
\varphi_{G/H} \circ T_r \sigma_{G,t} = \nu^{-t}_{H} \varphi_{G/H} \circ \sigma_{G,t} T_r \quad (2-10)
\]

\[
= \nu^{-t}_{H} \varphi_{G/H} \circ \varphi_{G/H} \circ T_r \quad [16, \text{Theorem 4.7}]
\]

\[
= \nu^{-t}_{H} \varphi_{G/H} \circ T_r \quad (\varphi_{G/H} \text{ invariant under its own modular group}).
\]

Now [35, Proposition 5.5, (ii) \(\Rightarrow\) (iv)] shows that

\[
(D\varphi_{G/H} \circ T_r : D\varphi_{G/H} \circ T_l)_t = \nu^{\frac{1}{2}u^2 \delta_{it}},
\]

i.e. (2-13). That these elements are actually in the relative commutant of \(L^\infty(G/H)\) is a general feature of cocycle derivatives between operator-valued weights ([17, Proposition 6.1] again).

As for (2-14), it follows from (2-13) and the cocycle property of the Radon-Nikodym derivatives [17, Proposition 6.3 (2)]:

\[
(DT_r : DT_l)_{s+t} = (DT_r : DT_l)_s \sigma_T^T (DT_r : DT_l)_t.
\]

We now have the object, alluded to in the discussion following Remark 2.9, that captures the discrepancy between \(T_r\) and \(T_l\):

**Definition 2.13** Let \(H \trianglelefteq G\) be a closed, normal, locally compact quantum subgroup.

The **relative modular element** \(\delta_{G,H}\) is the positive element affiliated with

\[
L^\infty(G/H)^c = L^\infty(G) \cap L^\infty(G/H)' \quad (\text{provided by Theorem 2.12, determined by})
\]

\[
(DT_r : DT_l)_t = \nu^{\frac{1}{2}u^2 \delta_{it}}_{H}.
\]

We are now ready to generalize Theorem 2.1 to non-central quantum subgroups and provide the quantum counterpart to (2-8).

**Theorem 2.14** For a closed, normal, locally compact quantum subgroup \(H \trianglelefteq G\) we have

\[
\delta_G = \delta_{G,H} \delta_{G/H}. \quad (2-16)
\]

**Proof** Since

- \(\delta_{G/H}\), which is affiliated with \(L^\infty(G/H)\);
- and \(\delta_{G,H}\), affiliated with the relative commutant \(L^\infty(G/H)^c\) by Theorem 2.12,
the two strongly commute \[2, \S 11.5\] in the sense that the spectral projections of one commute with those of the other. The strong product of \[34, \text{discussion following Definition IX.2.11}\] thus makes sense and is again a positive (unbounded, typically) operator; this is the meaning of the right-hand side of (2-16).

On the one hand, we have

\[(D\psi_G : D\varphi_G)_t = \nu_G^{\frac{1}{2}it^2} \delta_G^t \]  

(2-17)

by \[35, \text{Propositoin 5.5}\] and \[24, \text{Proposition 6.8 (3)}\]. On the other,

\[(D\psi_G : D\varphi_G)_t = (D\psi_G/H \circ \mathcal{T}_r : D\varphi_G/H \circ \mathcal{T}_l)_t \quad \text{Lemma 2.7} \]

\[(= (DT_r : DT_l)_t(D\psi_G/H \circ \mathcal{T}_r : D\varphi_G/H \circ \mathcal{T}_l)_t) \quad \text{[34, Theorem VIII.3.2]} \]

\[(= (DT_r : DT_l)_t(D\psi_G/H : D\varphi_G/H)_t) \quad \text{[16, Theorem 4.7 (2)]} \]

\[= \nu_G^{\frac{1}{2}it^2} \delta_G^t \]  

(2-15)

\[= \nu_G^{\frac{1}{2}it^2} \delta_G^t G \oplus \nu_G^{\frac{1}{2}it^2} \delta_G^t H \quad \text{as in (2-17), applied to } G/H \]

\[= \nu_G^{\frac{1}{2}it^2} \delta_G^t G \oplus \nu_G^{\frac{1}{2}it^2} \delta_G^t H \quad \text{Proposition 2.11.} \]

A comparison with (2-17) delivers the conclusion.

\vspace{1cm}

**Remark 2.15** Proposition 2.11 was not, strictly speaking, necessary in the proof of Theorem 2.14, for we could have reversed the implication as in the proof of Theorem 2.1: upon obtaining the equality

\[\nu_G^{\frac{1}{2}it^2} \delta_G^t = \nu_H^{\frac{1}{2}it^2} \nu_G^{\frac{1}{2}it^2} \delta_G^t G \oplus \nu_G^{\frac{1}{2}it^2} \delta_G^t H \]

the quadratic and linear factors automatically separate to give

\[\nu_G^{\frac{1}{2}it^2} = \nu_H^{\frac{1}{2}it^2} \nu_G^{\frac{1}{2}it^2} \nu_G^{\frac{1}{2}it^2} G \oplus \nu_G^{\frac{1}{2}it^2} \delta_G^t H \]

(i.e. Proposition 2.11) and the target equation (2-16).

It will be convenient, for future reference, to collect a few assorted general remarks on relative modular elements.

**Proposition 2.16** Let \[\iota : H \triangleleft G\] be a closed, normal locally compact quantum subgroup and \[\delta = \delta_G \oplus H\] the relative modular element of Definition 2.13. The following assertions hold.

1. \[\Delta_G(\delta) = \delta \otimes \delta.\]

2. \[\tau_{G, t}(\delta) = \delta \text{ and } R_G(\delta) = \delta^{-1}.\]

3. \[L^\infty(G) \ni \delta \xrightarrow{\iota^*} \delta \otimes \delta_H \oplus \delta \in' L^\infty(G) \otimes L^\infty(H), \]

where primed belonging symbols denote affiliation, per Notation 1.4.

4. similarly,

\[L^\infty(G) \ni \delta \xrightarrow{\iota^*} \delta_H \otimes G \in' L^\infty(H) \otimes L^\infty(G). \]

(2-18)

(2-19)
\[
\begin{align*}
\sigma_{G,t}^T(\delta) &= \sigma_{G,t}^T(\delta) = \nu_G^t \delta. \tag{2-20} \\
\sigma_{G,t}^T &= \delta^i t \sigma_{G,t}^T(\cdot) \delta^{-i t}. \tag{2-21}
\end{align*}
\]

**Proof** Item (1) follows from

- the analogous statement ([24, Proposition 7.12 (1)]) for the plain modular elements \( \delta_G \) and \( \delta_{G/H} \), which in the context of Theorem 2.14 strongly commute;
- together with (2-16);
- and the fact that the embedding

\[
L^\infty(G/H) \subseteq L^\infty(G)
\]

intertwines the comultiplications \( \Delta_{G/H} \) and \( \Delta_G \).

The argument is very similar for part (2): analogous statements hold for \( \delta_G \) and \( \delta_{G/H} \) [24, Proposition 7.12 (2)], the inclusion (2-22) intertwines both scaling groups and unitary antipodes [1, Proposition A.5], and we can again apply (2-16).

To obtain (2-18), note that

- \( \iota_r(\delta_G) = \delta_G \otimes \delta_H \) [5, Theorem 3.4, Corollary 3.9];
- \( \iota_r(\delta_{G/H}) = \delta_G \otimes 1 \) because \( \delta_{G/H} \) is affiliated with (1-3);
- hence the conclusion, per (2-16).

(2-19) is a consequence of (2-18), Lemma 1.6 and the fact that unitary antipodes turn all modular elements (absolute or relative) into their inverses ([24, Proposition 7.12 (2)] and part (2) of this proposition).

The last equality in (2-20) is nothing but (2-14), whereas the first equality will follow once we have (2-21); it thus remains to prove the latter. For that purpose, note that for

\[
a \in L^\infty(G/H)^c
\]

we have

\[
\begin{align*}
\sigma_{G,t}^T(a) &= \sigma_{G,t}^T(a) \quad \text{by [17, Definition 6.2]} \\
&= \sigma_{G,t}^T(a) \quad (2-11) \tag{2-21} \\
&= \delta_G^i t \sigma_{G,t}^T(a) \delta_{G/H}^{-i t} \quad [24, Proposition 7.12 (5)] \\
&= \delta_G^i t \sigma_{G,H}^{\delta_G^T}(a) \delta_{G/H}^{-i t} \quad (2-11) \text{ again} \\
&= \delta_G^i t \sigma_{G,t}^T(a) \delta_{G/H}^{-i t} \quad \text{once more [17, Definition 6.2]} \\
&= \delta_G^i t \sigma_{G,H}^{\delta_G^T}(a) \delta_{G/H}^{-i t} \delta^{-i t} \quad \text{Theorem 2.14} \\
&= \delta^i t \sigma_{G,t}^T(a) \delta^{-i t} \quad \text{because the middle factor is in the commutant}\ L^\infty(G/H)^c.
\end{align*}
\]

This concludes the proof of (6) and the result as a whole. ■
The block decomposition (2-7) suggests that Theorem 2.1 ought to generalize past central subgroups, to the case when the upper left-hand block has trivial determinant. We isolate that situation.

**Proposition 2.17** For a closed, normal locally compact quantum subgroup \( \iota : \mathbb{H} \subseteq G \) the two operator-valued weights

\[
T_l, T_r : L^\infty(G) \to L^\infty(G/\mathbb{H})_+ = L^\infty(\mathbb{H}\backslash G)_+
\]

coincide if and only if the relative modular element \( \delta_{G \mathbb{H}} \) of Definition 2.13 is 1.

Furthermore, in that case \( \mathbb{H} \) is unimodular.

**Proof** The two operator-valued weights coincide precisely when \( (DT_r : DT_l)_t = 1 \) [17, Theorem 6.5]. That this is equivalent to

\[
\delta^t_{G \mathbb{H}} = 1, \quad \forall t \in \mathbb{R} \iff \delta = 1
\]

then follows from (2-13) and (2-14).

As for the unimodularity of \( \mathbb{H} \), it too follows from \( \delta_{G \mathbb{H}} = 1 \) by (2-18).

**Definition 2.18** Let \( \mathbb{H} \subseteq G \) be a closed, normal locally compact quantum subgroup. We say that the conjugation (or adjoint) action of \( G \) on \( \mathbb{H} \) is measure-preserving if the equivalent conditions of Proposition 2.17 hold.

Alternative phrasing: \( G \) acts measure-preservingly (by conjugation).

As hinted above, we have the following immediate consequence of Theorem 2.14 (and Definition 2.18):

**Corollary 2.19** Let \( \mathbb{H} \subseteq G \) be a closed normal quantum subgroup of an LCQG.

\( G \) acts measure-preservingly on \( \mathbb{H} \) if and only if \( G \) and \( G/\mathbb{H} \) have the same modular element.

**Remark 2.20** Once more, the terminology of Definition 2.18 is meant to convey the analogy to the classical case. To see this, let \( \mathbb{H} \subseteq G \) be a closed normal subgroup of an ordinary locally compact group and denote left Haar measures by \( \mu \) and as before, the classical modular function \( \Delta \) by \( \delta(\cdot^{-1}) \).

The disintegration formula (2-3) and the relation

\[
d\mu_G(y \cdot y^{-1}) = \delta_G(y)d\mu_G, \quad y \in G
\]

(and its analogue for \( G/\mathbb{H} \)) easily show that

\[
d\mu_{\mathbb{H}}(y \cdot y^{-1}) = \frac{\delta_G(y)}{\delta_{G/\mathbb{H}}(y)}d\mu_{\mathbb{H}} = \delta_{G \mathbb{H}}(y)d\mu_{\mathbb{H}} \quad (2-16).
\]

This delivers the classical version of Corollary 2.19, with the phrase ‘acts measure-preservingly’ being assigned its straightforward meaning.

### 3 Modular elements as morphisms

Classically, the inverse modular function \( \delta_G \) is a continuous morphism \( G \to (\mathbb{R}_{>0}, \cdot) \). This is also true in the quantum setting, for \( \delta_G, \delta_{G \mathbb{H}}, \) and more broadly. Echoes of these remark can be seen in [5, Remark 5.2] or the proof of [9, Theorem 6.1], though not quite stated as such. We outline the matter here with some elaboration for future reference, including one application appearing below.

Following the terminology of [24, §7], and by analogy with the standard phrase in use in the theory of Hopf algebras (e.g. [28, Definition 1.3.4]):
Definition 3.1 Let $G$ be an LCQG. A strictly positive element $\delta$ affiliated with $L^\infty(G)$ or $C_0(G)$ or $C^*_0(G)$ is group-like if $\Delta(\delta) = \delta \otimes \delta$.

In terms of bounded operators only, this is equivalent to
$$\Delta(\delta^it) = \delta^it \otimes \delta^it, \forall t \in \mathbb{R}.$$ ♦

The following observation merely collects together a number of ready-made results.

Proposition 3.2 Let $G$ be a locally compact quantum group. The following sets of objects are in mutual bijection

(a) morphisms $G \to (\mathbb{R}, +)$;

(b) strictly positive group-like elements affiliated with $C^*_0(G)$;

(c) strictly positive group-like elements affiliated with $C_0(G)$;

(d) strictly positive group-like elements affiliated with $L^\infty(G)$.

Proof Recall Notation 1.4: $\varepsilon'$ denotes the affiliation relation. $C^*$ morphisms extend to affiliated operators [38, Theorem 1.2]; we will use this implicitly in the sequel.

(a) $\leftrightarrow$ (b). There is an comultiplication-preserving isomorphism
$$C_0(\mathbb{R}) \ni \exp \overset{\varepsilon'}{\mapsto} \text{id}_{\mathbb{R}_{>0}} \in' C_0(\mathbb{R}_{>0})$$ (dual to the usual exponential identification of the groups $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, \cdot)$).

Because $\delta$ is strictly positive there is also a unique morphism $C_0(\mathbb{R}_{>0}) \rightsquigarrow C^*_0(\mathbb{G})$ sending $\text{id}_{\mathbb{R}_{>0}}$ to $\delta$ [22, Proposition 6.5]. Composing with (3-1) this gives, for every strictly positive group-like $\delta \in' C^*_0(\mathbb{G})$, a unique morphism
$$\pi^u : C_0(\mathbb{R}) \rightsquigarrow C^*_0(\mathbb{G})$$

sending $\exp \mapsto \delta$ and intertwining the comultiplications $\Delta_\mathbb{R}$ and $\Delta_G$ in the sense that
$$\Delta_G \pi^u = (\pi^u \otimes \pi^u) \Delta_\mathbb{R}.$$ Such a map $\pi^u$ is one of the equivalent ways of specifying a quantum-group morphism $G \to \mathbb{R}$ [27, Theorem 4.8], so we are done.

(b) $\leftrightarrow$ (c). Strictly positive group-likes affiliated with $C^*_0(\mathbb{G})$ project to such along the surjection $C^*_0(\mathbb{G}) \twoheadrightarrow C_0(\mathbb{G})$. Conversely, they lift uniquely along the same map as explained in the proof of [23, Proposition 10.1].

(c) $\leftrightarrow$ (d). One direction is clear, $C_0(\mathbb{G})$ being contained in $L^\infty(\mathbb{G})$ via a coproduct-preserving inclusion. Conversely, strictly positive group-likes affiliated with $L^\infty(\mathbb{G})$ are in fact $C^*$-affiliated with $C_0(\mathbb{G})$, as in the proof of [24, Proposition 7.10].

This concludes the proof. ■

Notation 3.3 For a strictly positive group-like $\delta$ affiliated with $L^\infty(\mathbb{G})$ or $C_0(\mathbb{G})$ or $C^*_0(\mathbb{G})$ we write $\delta$ for the corresponding morphism $G \to \mathbb{R}$ attached to it via Proposition 3.2. ♦

Corollary 3.4 Let $G$ be an LCQG. Strictly positive group-like elements $\delta \in' C_0(\mathbb{G})$ are invariant under the scaling group and satisfy $R_G(\delta) = \delta^{-1}$. ♦
We know from Proposition 3.2 that $\delta$ is the image of the canonical group-like $\exp \in' C_0(\mathbb{R})$ through a comultiplication-intertwining morphism

$$C_0(\mathbb{R}) \rightarrow C_0(G).$$

Such morphisms also intertwine the scaling groups and unitary antipodes [23, Remark 12.1], so it suffices to verify the claim for the universal strictly positive group-like

$$\delta := \exp \in' C_0(\mathbb{R});$$

that verification is immediate, hence the conclusion.

$\mathbf{C^*}$-affiliated elements have an accompanying notion of spectrum [38, equation (1.20)], which by [38, discussion following Theorem 1.6] specializes back to the usual concept for concrete unbounded, normal operators on Hilbert spaces (which is the situation we are concerned with here).

For positive group-likes the spectrum has some very pleasant properties.

**Proposition 3.5** For an LCQG $G$ the strictly positive portion

$$\text{Sp}_{>0}(\delta) := \text{Sp}(\delta) \setminus \{0\}$$

of the spectrum of a strictly positive group-like $\delta \in' C_0(G)$ is a closed subgroup of the multiplicative group $(\mathbb{R}_{>0}, \cdot)$.

**Proof** The spectrum of a positive (possibly unbounded) operator is a closed subset of $\mathbb{R}_{\geq 0}$, hence the (topological) closure claim. It remains to argue that $\text{Sp}_{>0}(\delta)$ is closed under multiplication and inversion.

It is a simple application of the spectral theorem (e.g. [30, Theorem 13.30]) to show that for a strictly positive $T \in' B(\mathcal{H})$

- the positive spectrum $\text{Sp}_{>0}(T^{-1})$ is

$$\text{Sp}_{>0}(T)^{-1} := \{t^{-1} \mid t \in \text{Sp}_{>0}(T)\}$$

- and similarly, the spectrum $\text{Sp}(T \otimes T)$ is the closure of

$$\{st \mid s, t \in \text{Sp}(T)\}.$$

Since

- we have a morphism $\Delta_G \in \text{Mor}(C_0(G), C_0(G)^{\otimes 2})$ sending $\delta$ to $\delta \otimes \delta$;

- and a morphism $R_G$ sending $\delta \mapsto \delta^{-1}$ by Corollary 3.4 ($R_G$ is anti-multiplicative, but this makes no difference here);

- for a morphism $\pi \in \text{Mor}(A, B)$ the spectrum of an image $\pi(T)$ is contained in that of $T$ for any $A$-affiliated $T$ [38, equation (1.21)],

the conclusion follows.
Let $\delta \in \ell' C_0(G)$ be a strictly positive group-like. The closed subgroup
\[ \text{Sp}(\delta)_{>0} \leq (\mathbb{R}_{>0}, \cdot) \] (3-2)
of Proposition 3.5 has an alternative interpretation as a group-theoretic invariant attached to the morphism $\hat{\delta} : G \to \mathbb{R}$ in Notation 3.3. Every quantum-group morphism has a closed image, introduced in [19, Definition 4.2]; we paraphrase that discussion as follows.

**Definition 3.6** Let $\pi : \mathbb{H} \to G$ be a morphism of LCQGs. The closed image $\text{im} \pi$ of $\pi$ is the smallest closed quantum subgroup $\iota : \text{im} \pi \leq G$ for which $\pi$ admits a factorization
\[ \text{H} \xrightarrow{\pi} \text{im} \pi \xrightarrow{\iota} G. \]

**Theorem 3.7** Let $G$ be an LCQG, $\delta \in \ell' C_0(G)$ a strictly positive group-like, and $\hat{\delta} : G \to \mathbb{R}$ the morphism associated to it as in Notation 3.3.

The closed image $\overline{\text{im} \delta} \leq (\mathbb{R}, +)$ is precisely $\log \text{Sp}_{>0}(\delta)$, i.e. the image of the closed subgroup (3-2) under the logarithm isomorphism $(\mathbb{R}_{>0}, \cdot) \cong (\mathbb{R}, +)$.

**Proof** Since $\delta$ is strictly positive, its functional calculus allows the application of the logarithm to produce a self-adjoint element $\log \delta$ [22, Definition 7.16]. The same goes for $\log \exp = \text{id}_\mathbb{R} \in \ell' C_0(\mathbb{R})$, and since $\hat{\delta}$ intertwines these logarithm operations [22, Proposition 6.17] and by definition sends $\exp \mapsto \delta$, we have
\[ \hat{\delta}(\text{id}_\mathbb{R}) = \log \delta. \]
In short, then, $\hat{\delta}$ is the unique [22, Proposition 6.5] morphism sending $\text{id}_\mathbb{R}$ to $\log \delta$.

Write
\[ \mathbb{H} := \log \text{Sp}_{>0}(\delta) \leq \mathbb{R}. \]
[22, Result 6.16] says that $\mathbb{H}$ is precisely the spectrum of $\log \delta$, so that by [22, Theorem 3.4] $\hat{\delta}$ factors as
\[ C_0(\mathbb{R}) \xrightarrow{\delta} C_0(\mathbb{H}) \xrightarrow{\hat{\delta}} M(C^*_0(G)), \]
where the top left arrow is the obvious restriction map and the top right map is one-to-one. That injectivity in particular means that there is no further factorization through any smaller quotients of $C_0(\mathbb{R})$, meaning precisely what was sought: $\mathbb{H} \leq \mathbb{R}$ the smallest close subgroup factoring $\hat{\delta}$. ■

The application alluded to at the beginning of the section has to do with property $(T)$ for LCQGs; this is a quantum version of the classical familiar concept (e.g. [3, Definition 1.1.3]). Early references in the quantum setting are [14, Definition 3.1] for discrete quantum groups and, say, [7, Definition 3.1] and [10, §6] for the general concept. We also refer to [12, 9] (which will be cited more heavily shortly) and their sources for further information. In brief ([7, Definitions 2.3 and 3.1]):
Definition 3.8 Let $G$ be an LCQG.

(1) Let $U \in M(C_0(G) \otimes K(H))$ be a unitary $G$-representation in the sense, say, of [12, Definition 2.1]. A net $\zeta_i \in H$ of unit vectors is almost invariant (also: constitutes an almost-invariant vector) if

$$\|U(\eta \otimes \zeta_i) - \eta \otimes \zeta_i\| \to 0$$

for all $\eta \in L^2(G)$.

(2) Similarly, having fixed $U$ again, a vector $\zeta \in H$ is invariant if

$$U(\eta \otimes \zeta) = \eta \otimes \zeta, \ \forall \eta \in L^2(G).$$

(3) $G$ has property (T) if every unitary representation that has almost-invariant vectors in fact has non-zero invariant vectors.

Theorem 3.9 below is a slight generalization of the fact that property-(T) quantum groups are unimodular. This latter result has appeared before a number of times: [14, Proposition 3.2] proves the claim for discrete quantum groups, [6, Theorem 6.3] handles second-countable locally compact quantum groups, and [9, Theorem 6.1] proves the general result for arbitrary property-(T) LCQGs. These all deal with the specific group-like element $\delta_G$; among them, the first and third both bear similarities to the argument below.

Theorem 3.9 For an LCQG $G$ with property (T) the only strictly positive group-like $\delta \in' C_0(G)$ is 1.

Proof Consider $\delta \in' C_0(G)$ as in the statement. Proposition 3.2 and Theorem 3.7 provide us with a morphism $G \to \mathbb{R}$ whose closed image is

$$\mathbb{H} := \{\log t | 0 \neq t \in \text{Sp}(\delta)\} \leq (\mathbb{R}, +).$$

But the closed image of the resulting morphism $G \to \mathbb{H}$ is then all of $\mathbb{H}$ essentially by definition. It follows that $G \to \mathbb{H}$ has dense image in the sense of [12, Definition 2.8] (see [12, discussion following the statement of Theorem A.1]), and hence $\mathbb{H}$ also has property (T) by [12, Theorem 5.7].

Being classical abelian and property-(T), $\mathbb{H}$ must be compact [3, Theorem 1.1.6] and hence trivial because it is a subgroup of $(\mathbb{R}, +)$. We are now done: the spectrum of the strictly-positive operator $\delta$ is $\{1\}$, so $\delta = 1$. ■

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