Excited Heavy Baryons and Their Symmetries I: Formalism

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This is the first of two papers to study a new emergent symmetry which connects orbitally excited heavy baryons to the ground states in the combined heavy quark and large $N_c$ limit. The existence of this symmetry is shown in a model-independent way, and different possible realizations of the symmetry are discussed. It is also proved that this emergent symmetry commutes with the large $N_c$ spin-flavor symmetry.

I. INTRODUCTION

Quantum Chromodynamics is now almost universally accepted as the theory which governs strong interaction. This theory has been repeatedly tested against experiment, with great success. Due to its non-abelian nature, however, the QCD coupling gets strong at low energy, and the dynamics become nonperturbative and intractable. As a result, much of our quantitative understanding of low energy hadron properties are based on symmetry considerations. The most notable of these schemes is chiral perturbation theory, which is based on the fact that, when the light quark masses $m_q \to 0$, the QCD Lagrangian is invariant under the chiral symmetry group $SU(n_f)_L \times SU(n_f)_R$. In the real world, the light masses are not zero; nevertheless chiral symmetry survives as an approximate symmetry of QCD, with symmetry breaking terms of order $p/\Lambda$ or $m_\pi/\Lambda$, where $m_\pi$ is the pion mass, $p$ is the scale of external probes, and $\Lambda$ the chiral symmetry breaking scale. Despite being just an approximate symmetry, chiral symmetry nevertheless provides strong constraints on low energy pion dynamics.

Important insights into some states in QCD comes from emergent symmetries which are not symmetries (not even approximate symmetries) of the QCD Lagrangian, but emerge as symmetries of the states in the Hilbert space of an effective theory obtained by taking certain limits. A famous example of such emergent symmetries is the heavy quark symmetry $[1-3]$ for heavy hadron states containing a single heavy quark with mass $m_Q \gg \Lambda_{QCD}$. Heavy quark spin symmetry ensures that states related by a heavy quark spin flip, like $(B, B^*)$ and $(\Sigma_b, \Sigma_b^*)$, are degenerate. Moreover, heavy quark flavor symmetry implies that the brown mucks (i.e., the light degrees of freedom) of heavy hadrons are insensitive to the mass or the flavor of the heavy quark. This guarantees that the $B \to D^{(*)}$ and $A_0 \to \Lambda_0$ semileptonic form factor (which are usually referred as Isgur-Wise form factors) are normalized to unity at the point of zero recoil, where the initial and final hadrons have the same velocity. Such absolute normalizations of form factors have profound implications in experimental determination of the CKM matrix element $V_{cb}$. The combined heavy quark spin-flavor symmetry is described by the symmetry group $SU(2n_f)$ where $n_f$ is the number of heavy flavors, and this symmetry is broken by corrections proportional to powers of $\Lambda_{QCD}/m_Q [4,5]$.

Note that heavy quark symmetry is not a symmetry of the QCD Lagrangian; if it were, the $B$ and $D$ mesons, related by heavy quark flavor symmetry, would be degenerate. However, if our interest is restricted to states with a single heavy quark, one can perform a spacetime-dependent phase redefinition such that the heavy quark mass and spin drop out of the Lagrangian. In other words, while heavy quark symmetry is not a symmetry of the QCD Lagrangian, it is a symmetry of the Lagrangian of heavy quark effective theory, which describes only states with a single heavy quark.

Another well-known emergent symmetry is the light quark spin-flavor symmetry for baryons in the large $N_c$ limit. The large $N_c$ limit was first studied by ’t Hooft for mesons [3] and was subsequently extended to baryons by Witten [5]. They studied how QCD amplitudes involving various numbers of mesons and baryons scale with the number of color $N_c$ when $N_c$ is large. GPUs and Sakita [6,7] realized that, for large $N_c$ baryons, the spin symmetry $SU(2)$ and flavor symmetry $SU(n_f)$ (where $n_f$ is the number of light flavors) are combined and enlarged into the spin-flavor symmetry group $SU(2n_f)$. (This spin-flavor symmetry was systematically re-analyzed by other groups; see Refs. [13].) It was shown that the low-lying baryon spectrum in the large $N_c$ limit contains a tower of states with $(I, J) = (0, 0), (1, 1), \ldots$ when $N_c$ is even, and $(4/3, 2), (3/2, 2), \ldots$ when $N_c$ is odd. In the latter case one can identify the $(4/3, 2)$ state as the nucleon, and the $(3/2, 2)$ as the $\Delta(1232)$ resonance. Moreover, it can be shown that the splittings between these tower states are of order $1/N_c$. As a result, when $N_c \to \infty$, the nucleon, $\Delta$ and all the other states in the tower collapse into degeneracy, signifying the emergence of the symmetry $SU(2n_f)$. Similarly, in the heavy baryon (baryon with a single heavy quark) sector, this spin-flavor symmetry decrees that the $\Sigma_Q^{(*)}-\Lambda_Q$ splitting vanishes in the large $N_c$
limit. Again, note that this light quark spin-flavor symmetry is an emergent symmetry in the sense that it is not a symmetry of the QCD Lagrangian, but only a symmetry of the QCD Hamiltonian of states with unit baryon number.

We have recently reported \[14\] a new emergent symmetry of QCD which emerges in the heavy baryon (baryon with a single heavy quark) sector in the combined heavy quark and large $N_c$ limit. This contracted $U(4)$ symmetry (or more generally $U(d+1)$ in a theory with $d$ spatial dimensions) connects the ground state heavy baryon to some of its orbitally excited states, which become degenerate with the ground state as $m_Q \to \infty$ and $N_c \to \infty$. As a result, static properties such as the axial current couplings and the moments of the weak form factors of these orbitally excited states can be related to their counterparts of the ground state. While many of these results have been discussed before in the literature in the context of particular models, they were first presented as model-independent symmetry predictions in Ref. \[14\].

After the publication of Ref. \[14\], we realized that this contracted $U(4)$ can be further enlarged into a contracted $O(8)$ symmetry (or more generally $O(2d+2)$ in a theory with $d$ spatial dimensions). Moreover, this “symmetric realization” is only one of two possible realizations of the emergent symmetry. In the “symmetry broken realization“, the symmetry is broken down to contracted $O(4) \times O(4)$. This paper is the first of two papers where we will report these and other new progresses on these emergent symmetries in the combined heavy quark and large $N_c$ limits, as well as discuss the results reported in Ref. \[14\] in more details. This paper will focus on the formalism and the different realizations of the emergent symmetry, while phenomenological applications and corrections to the symmetry will be discussed in the next paper \[15\].

This paper is organized as follow: In Sec. II, we will briefly review the bound state picture, a class of models which exhibits this same contracted $O(2d+2)$ symmetry and which motivates our studies. While the bound state picture is not logically related to QCD, it provides a simple physical picture of the origins of this new symmetry. The bound state picture treats the heavy baryon as a bound state of an ordinary baryon and a heavy meson and thus is a model. However, emergent symmetries are often first recognized in models. For example, heavy quark symmetry was first discovered in quark models, while the large $N_c$ spin-flavor symmetry was first realized in the Skyrme model. Hence it is useful to first consider a model which embodies the correct symmetries to get a feeling of the physical picture before launching a formal discussion of the symmetry in QCD language.

After examining the logical foundation of the bound state picture in Sec. III, we begin the main task of this paper and study the emergent symmetry in the context of QCD. In Sec. IV, we discuss the relative sizes of different contributions to the QCD Hamiltonian. Then in Sec. V we introduce the kinematic variables of the bound state picture in the context of QCD, and show that many conclusions of the bound state picture can be justified in the model independent manner. The generators of the emergent symmetry will be formally introduced in Sec. VI, and in Sec. VII and VIII we will focus on the “symmetric realization” and show that in this case the QCD Hamiltonian is that of a three-dimensional simple harmonic oscillator by considering multiple commutation relations in the combined heavy quark and large $N_c$ limit. Following this is a short discussion in Sec. IX, while Sec. X and XI will discuss the “symmetry broken realization” and the inclusion of spin and isospin effects. Then the paper concludes with a short preview of the companion paper \[16\], which is under preparation and will discuss phenomenological issues and higher order corrections to the symmetry predictions.

II. THE BOUND STATE PICTURE OF A HEAVY BARYON

The bound state picture \[16\] regards a heavy baryon as a bound state of a heavy meson and a light baryon (a baryon without any valence heavy quarks); the latter often treated as a chiral soliton, i.e., a topologically nontrivial configuration of the classical meson fields. In particular, the lightest charmed baryon $\Lambda_c$ is regarded as the bound state of the heavy mesons $D$ or $D^*$ (which are degenerate in the heavy quark limit) and a nucleon. In the following, we will focus on the model described in Refs. \[17\] \[18\], which will be referred to as the simple bound state model as it is the simplest model with correct behaviors in the heavy quark and large $N_c$ limit. However, we emphasize that one can make a similar analysis on other versions of the bound state picture, and the symmetry properties should be qualitatively the same as long as these models are consistent with heavy quark symmetry and obey the usual large $N_c$ counting rules.

In QCD, a heavy baryon is a complicated bound state, with the quarks interacting through strongly coupled gauge dynamics, and with quark-antiquark pairs popping in and out of the vacuum, etc. — a highly intractable problem. The bound state picture replaces it (in an ad hoc manner) with the much-simpler problem of a two-body bound state. Moreover, the problem further simplifies in the heavy quark limit, where the heavy meson becomes infinitely massive, and the large $N_c$ limit, where the nucleon mass $m_N$ grows like $N_c$. For concreteness, we will adopt the prescription...
(only for this section) that the heavy quark limit is taken before the large \( N_c \) limit.\footnote{In the real world, the heavy meson masses \( m_B \sim 5 \) GeV, \( m_D \sim 1.8 \) GeV while the nucleon mass \( m_N \sim 1 \) GeV. So as far as heavy baryon kinematics is concerned, the real world is closer to the heavy quark limit than the large \( N_c \) limit, justifying our ordering of the limits.}  Taking the heavy quark limit first, the reduced mass of the two-body system \( \mu \sim m_N \sim N_c \rightarrow \infty \) in this combined heavy quark–large \( N_c \) limit. As a result, the kinetic term, which is suppressed by \( 1/\mu \), vanishes, and the wave function does not spread but is instead localized at the bottom of the potential. (When \( \mu \rightarrow \infty \), the absolute square of the wave function will be a Dirac delta distribution at the bottom of the potential.) Consequently, a small attraction between the heavy meson and the nucleon is sufficient to ensure the existence of a bound state.

What is the potential \( V(x) \) between a heavy meson and a nucleon? Without resorting to models, we do not know much about the potential except the fact that, by the usual large \( N_c \) counting rules\footnote{This contracted U(4) is different from the contracted SU(4) group of the light quark spin-flavor symmetry\cite{1}.}, \( V(x) \) is of order \( N_c^0 \). However, let us assume that \( V(x) \) has a global minimum at the origin, i.e., the heavy meson sits on the top of (the center of) the nucleon. In this case, the wave function will be highly localized around the origin, and the potential can be approximated by \( V(x) = V_0 + \frac{k}{2} x^2 \), which includes only the first two terms in the Taylor expansion of \( V(x) \). For the origin to be a global minimum, we need \( V_0 < 0 \) and \( \kappa > 0 \). In this case, when the bound state is the ground state of the simple harmonic oscillator, it is a \( \Lambda_Q \). On the other hand, excited states in the simple harmonic oscillator are orbitally excited heavy baryons. With explicit wave functions, coupling constants and form factors for transitions between different states can be calculated in a straightforward manner.

However, it remains to be seen whether the assumptions that \( V_0 < 0 \) and \( \kappa > 0 \) are justified. In the simple bound state model\footnote{Note that this U(3) group contains the rotational SO(3) subgroup, generated by \( L_i = -i \epsilon_{ijk} T_{jk} \) with \([L_i, L_j] = i \epsilon_{ijk} L_k\). When \( N_c \rightarrow \infty \) and the excited states become degenerate with the ground state, the annihilation and creation operators \( a_j \) and \( a_j^\dagger \) are generators of the emergent symmetry. The additional commutation relations are}

\[ [a_j, T_{kl}] = \delta_{kj} T_{il} - \delta_{il} T_{kj}, \]

(2.1)

Note that this U(3) group contains the rotational SO(3) subgroup, generated by \( L_i = -i \epsilon_{ijk} T_{jk} \) with \([L_i, L_j] = i \epsilon_{ijk} L_k\). When \( N_c \rightarrow \infty \) and the excited states become degenerate with the ground state, the annihilation and creation operators \( a_j \) and \( a_j^\dagger \) (\( i, j = 1, 2, 3 \)) also become generators of the emergent symmetry. The additional commutation relations are

\[ [a_j, T_{kl}] = \delta_{kj} a_i, \quad [a_i^\dagger, T_{kl}] = -\delta_{il} a_k^\dagger, \quad [a_j, a_i^\dagger] = \delta_{ij} \mathbf{1}, \]

(2.2)

where \( \mathbf{1} \) is the identity operator. These sixteen generators \( \{T_{ij}, a_i, a_i^\dagger, \mathbf{1}\} \) form the minimal spectrum generating algebra of a three-dimensional harmonic oscillator, i.e., the smallest algebra which contains the symmetry group U(3) and connects all eigenstates of a three-dimensional simple harmonic oscillator. It is related to the usual U(4) algebra, generated by \( T_{ij} \) (\( i, j = 1, 2, 3, 4 \)) satisfying commutation relations (2.1) by the following limiting procedure:

\[ a_j = \lim_{R \rightarrow \infty} T_{4j}/R, \quad a_i^\dagger = \lim_{R \rightarrow \infty} T_{i4}/R, \quad \mathbf{1} = \lim_{R \rightarrow \infty} T_{44}/R^2. \]

(2.3)

Such a limiting procedure is called a group contraction, and hence the group generated by \( \{T_{ij}, a_i, a_i^\dagger, \mathbf{1}\} \) is called a contracted U(4) group.\footnote{This contracted U(4) is different from the contracted SU(4) group of the light quark spin-flavor symmetry\cite{1}.}
The contracted U(4) minimal spectrum generating algebra can be enlarged to contain the extra operators \( S_{ij} = a_ia_j \) and \( S_{ij}^\dagger = a_i^\dagger a_j^\dagger \) \((i, j = 1, 2, 3)\). The new commutation relations are

\[
[S_{ij}, S_{kl}] = [S_{ij}, a_l] = 0, \quad [S_{ij}, a_k^\dagger] = a_i\delta_{jk} + a_j\delta_{ik}, \quad [S_{ij}, T_{kl}] = S_{il}\delta_{jk} + S_{jl}\delta_{ik}, \\
[S_{ij}, S_{kl}^\dagger] = T_{ik}\delta_{jl} + T_{jk}\delta_{il} + T_{il}\delta_{jk} + T_{jl}\delta_{ik},
\]

and the commutation relations involving \( S_{ij}^\dagger \) can be obtained through Hermitian conjugation. As a result, these 28 generators \( \{S_{ij}, S_{ij}^\dagger, T_{ij}, a_i, a_i^\dagger, 1\} \) form a closed operator algebra, which is actually a contracted O(8) algebra. This contracted O(8) is generated by the creation and annihilation operators, all possible bilinears, as well as the commutation relations involving \( S \) and \( S^\dagger \); the commutator of two trilinears, for example, will be a quadrilinear, and the algebra will not close (or will contain an infinite number of generators). As a result, this contracted O(8) algebra is called maximal spectrum generating algebra of the three-dimensional simple harmonic oscillator. Again, as \( N_c \to \infty \) and \( \omega \to 0 \), the excited states become degenerate with the ground state and the contracted O(8) become the symmetry group of the bound state picture. The relationship between all the algebraic structures discussed above is summarized in the following chain:

\[
\begin{array}{c|c|c|c|c}
\text{SO(3)} & \subset & \text{U(3)} & \subset & \text{contracted U(4)} & \subset & \text{contracted O(8)} \\
\{L_j\} & || & \{T_{ij}\} & || & \{S_{ij}, S_{ij}^\dagger, T_{ij}, a_i, a_i^\dagger, 1\} & || & \{S_{ij}, S_{ij}^\dagger, T_{ij}, a_i^\dagger, a_j, 1\} \\
\text{symmetry group} & & \text{symmetry group} & & \text{minimal spectrum} & & \text{maximal spectrum} \\
of\text{any} & & \text{of 3-D simple} & & \text{generating algebra,} & & \text{generating algebra,} \\
\text{central potential} & & \text{harmonic oscillator} & & \text{symmetry subgroup} & & \text{symmetry group} \\
& & & & \text{as } \omega \to 0. & & \text{as } \omega \to 0. \\
\end{array}
\]

We have shown that the contracted O(8) is a symmetry of the bound state picture. In the more general case of a bound state picture with \( d \) spatial dimensions, it is clear that the symmetry is described by a similarly contracted O(2d + 2) group with a contracted U(d + 1) subgroup. While we have been focusing on the simple bound state picture, this symmetry is actually a generic feature of all variants of the bound state picture as long as the models embody heavy quark symmetry as \( N_c \to \infty \), and obey the large \( N_c \) scaling rules as \( N_c \to \infty \). However, it is not obvious that the physical picture is reasonably reasonable. This will be addressed in the next section.

### III. THE FOUNDATION OF THE BOUND STATE PICTURE

Questions may be raised about the foundation of the bound state picture on several different levels. On the technical level, one may question the description of a nucleon as a classical pion distribution in the simple bound state model. Because we have infinitely many species of mesons in the large \( N_c \) limit, there is no reason why all other mesons besides pions should be ignored. This question can be resolved by including more light mesons in the model. This is the motivation behind Ref. [22], where the effects of the \( \rho \) and \( \omega \) vector mesons are included, leading to results which are numerically improved at the expense of more parameters and much more mathematical complexities. Since we are interested in the generic features of the bound state picture, we will only remark that including extra meson states does not change the physics qualitatively. However, in the large \( N_c \) limit there is an infinite number of mesons, and each meson has infinitely many coupling constants.

A more serious technical issue of concern is the modeling of the interaction between the heavy meson and the classical light meson fields (which make the nucleon). In the simple bound state picture, the heavy mesons interact

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3 A note on the literature: as far as the authors can discern, among all the literature on the bound state picture for heavy baryon, only the simple bound state picture [14, 15] makes the explicit statement that the binding potential is simple harmonic in the combined heavy quark and large \( N_c \) limit. In none of these works on the bound state picture was the spectrum generating algebras discussed, nor was the observation that in the combined limit they become the symmetry group of an emergent symmetry. Both of these points were first explicitly raised in Ref. [14]. On the other hand, the appearance of an emergent symmetry in the bound state picture does not depend on the details on the model, as long as the model embodies the heavy quark and large \( N_c \) symmetry. Consequently, the emergent symmetry is an implicit feature of all viable bound state models.
with the classical background pion configuration through a truncated chiral Lagrangian, which is the most general interaction Lagrangian which respects chiral symmetry truncated to the leading order of the chiral expansion $p/\Lambda$, where $p$ is the pion momentum. While it is justifiable to use this truncated Lagrangian in low momentum pion processes (where $p \ll \Lambda$), there is little justification for such truncation here, as the chiral soliton contains pionic modes over a wide range of momentum, and in general $p/\Lambda$ is not a small expansion parameter. As a result, the truncated Lagrangian is an ad hoc description of the interaction between the heavy meson and the classical pion fields. The situation is even worse in models where the $\rho$ and $\sigma$ vector mesons are included. Since the interactions involving these vector mesons are not well-constrained by symmetry (unlike pionic interactions, which are severely constrained by chiral symmetry), their interactions with heavy mesons are only described by phenomenological Lagrangians of an entirely ad hoc nature.

However, these technological issues are not fundamental and do not alter the conceptual issues about the bound state picture. For example, it seems likely that, as far as the emergent symmetry is concerned, the description of the light baryon as a chiral soliton is not essential. The essence of the bound state picture is that the heavy baryon can be regarded as a bound state with potential $V(x) \sim N^0$ and reduced mass $\mu \sim N_c$. The details of the interaction are inessential as far as the symmetry is concerned. This naturally leads us to ask the question whether one can recast the analysis in such a form that chiral solitons are not invoked. As we will see below, the answer to this question is affirmative.

A more severe conceptual criticism of the simple bound state model is the use of point particle quantum mechanics when both the heavy meson and the nucleon are extended objects. Assuming that the bound state picture is reasonable, the mean square relative displacement of the nucleon from the heavy meson is $3/(2\mu\omega) \sim N^{-1/2}_c$, which vanishes as $N_c \to \infty$, while the size of the nucleon has a smooth non-zero large $N_c$ limit. Hence the heavy meson will be jigging well inside the nucleon near its center, and it is not obvious that point particle quantum mechanics is applicable.

Lastly, the connection of the bound state picture to QCD is obscure. To address this philosophical concern, one can only try to reproduce the emergent symmetry directly from QCD. This is the purpose of both our previous paper [14] and this paper. We will see that in a model-independent way, one can show that this contracted O(8) symmetry is not only a symmetry of the bound state picture, but in fact a symmetry of QCD.

IV. DISSECTING THE QCD HAMILTONIAN FOR HEAVY BARYONS

Due to the conservation of baryon number and heavy quark number (in the heavy quark limit), it is legitimate to restrict our attention to the heavy baryon Hilbert space, i.e., the subspace with both heavy quark number and baryon number equal to unity. In the combined heavy quark and large $N_c$ limit, this subspace is well-defined. We introduce the small power counting parameter, $\lambda$, to quantify the deviation from the combined limit. It is defined as:

$$\lambda \sim \frac{\Lambda_{\text{QCD}}}{m_Q} \frac{1}{N_c}. \quad (4.1)$$

with the ratio $N_c\Lambda_{\text{QCD}}/m_Q$ arbitrary. In other words, both $1/m_Q$ and $1/N_c$ corrections are of order $\lambda$, while order $\lambda^2$ corrections include those scale like $1/m_Q^2$, $1/m_Q N_c$ and $1/N_c^2$, etc.

In the heavy baryon Hilbert space, it is useful to decompose the QCD Hamiltonian $H$ in the following way:

$$H = H_Q + H_\ell, \quad \text{where } H_Q = m_Q + \tilde{H}_Q \text{ and } H_\ell = m_N + \tilde{H}_\ell. \quad (4.2)$$

The heavy quark part of the QCD Hamiltonian $H_Q$ contains the heavy quark mass $m_Q$, as well as the heavy quark kinetic and interaction terms denoted by $\tilde{H}_Q$. Since $H_Q$ involves only a single quark (namely the heavy quark), it at most scales like $N_c^0 \sim \lambda^0$. (In contrast, both $m_Q$ and $m_N$ are large and of order $\lambda^{-1}$. ) As we are only interested in states with a single heavy quark, by performing a Foldy-Wouthuysen transformation $\tilde{H}_Q$ can be expanded in powers of $1/m_Q$ in heavy quark effective theory:

$$\tilde{H}_Q = gA^0 + \frac{\vec{P}_Q^2}{2m_Q} - g \frac{S_Q \cdot B}{2m_Q} + \mathcal{O}(m_Q^{-2}), \quad (4.3)$$

where $\vec{P}_Q$ is the three-dimensional heavy quark momentum:

$$\vec{P}_Q = \int d^4x \vec{h}(x) \vec{D} h(x), \quad (4.4)$$
with \( h(x) \) being the heavy quark field in heavy quark effective theory, and \( \vec{D} \) is the three-dimensional covariant derivative. Note that the chromomagnetic term \( S_Q \cdot B \) is suppressed by the heavy quark mass \( m_Q \). While the definitions of \( m_Q \) and \( \vec{P}_Q \) are ambiguous since the heavy quark mass is not uniquely defined, these ambiguities are of order unity (\( \lambda^0 \)) while the heavy mass and momentum are typically large (\( m_Q, \vec{P}_Q \sim \lambda^{-1} \)). As a result, the relative ambiguities are small and one can rewrite Eq. (4.3) as

\[
\hat{H}_Q = g A^0 + \frac{\vec{P}_Q^2}{2m_Q} - g \frac{S_Q \cdot B}{2m_Q} + O(\lambda^2).
\] (4.5)

Similarly, the light part \( \hat{H}_\ell \) contains the nucleon mass \( m_N \) (which is proportional to \( N_c \)) and \( \hat{H}_\ell \), which represents the change in the energy of the brown muck (i.e., the light degrees of freedom of the heavy baryon) when one of the light quarks is replaced by a heavy quark. We cannot write down a simple expression for \( \hat{H}_\ell \) as we have done for \( \hat{H}_Q \), but it is easy to see that it scales like \( N_c \). The reasoning is as follows: the interaction energy between any two quarks is of order \( N_c^{-1} \) by the standard large \( N_c \) counting rules. Since the replacing of a light quark with a heavy quark in a baryon breaks \( N_c - 1 \) light quark–light quark interactions and replaces them with \( N_c - 1 \) light quark–heavy quark interactions, we have in the large \( N_c \) limit,

\[
\hat{H}_\ell \sim \text{(number of interactions modified)} \times \text{(change of energy in each interaction)} \sim N_c \times N_c^{-1} \sim N_c^0 \sim \lambda^0. \quad (4.6)
\]

The light Hamiltonian \( \hat{H}_\ell \) contains all the dynamics of the brown muck as well as its interaction with the heavy quark. In general, it can depend not only on its position \( \vec{x} \) and momentum \( \vec{p} \) relative to the heavy quark, but also all kinds of internal degrees of freedom which correspond to different modes of excitation. In comparison, the Hamiltonian \( \hat{H}_{\text{bs}} \) of a two-particle bound state, in general, can be decomposed in the following form:

\[
\hat{H}_{\text{bs}} = \hat{H}_{\text{kin}} + \hat{H}_{\text{pot}} + \hat{H}_{\text{exc}}, \quad (4.7)
\]

where \( \hat{H}_{\text{kin}} \) is a kinetic term which depends only on \( \vec{p} \), \( \hat{H}_{\text{pot}} \) is a potential term which only depends on \( \vec{x} \), and \( \hat{H}_{\text{exc}} \) represents possible internal excitations and commutes with both \( \vec{x} \) and \( \vec{p} \). The issue becomes whether \( \hat{H}_\ell \) can be recast in this form of Eq. (4.7). This is the question which we will be attempting to answer in the next four sections.

**V. KINEMATICS AND THE KINETIC ENERGY**

In the previous section, we have decomposed the QCD Hamiltonian, \( \hat{H} \), into a heavy part \( \hat{H}_Q \) and a light part \( \hat{H}_\ell \). Our next step will be to perform similar decompositions for the kinematic variables; namely, the momentum and position operators. We will reproduce the two-body kinematics of the bound state picture using QCD operators. Recall that our aim is to demonstrate the existence of an emergent symmetry in QCD itself. In order to achieve this goal in a model-independent manner, one cannot simply assume the kinematic variables in the bound state picture are well defined. Instead we need to construct these kinematic variables from QCD operators without reference to any model.

While the total momentum of any given heavy baryon system \( \vec{P} \) is a well-defined quantity, in general there is no unambiguous way to separate the momentum into a heavy quark contribution and a brown muck contribution. However, in the heavy quark limit, the heavy quark momentum \( \vec{P}_{Q} \) in Eq. (4.4) is a well-defined QCD operator (up to corrections of relative order \( \lambda \)), and one can define the brown muck momentum \( \vec{P}_\ell \) as \( \vec{P} - \vec{P}_{Q} \). Lastly, the QCD based position operators of the whole system \( \hat{X} \), of the heavy quark \( \hat{X}_{Q} \), and of the brown muck \( \hat{X}_{\ell} \) are defined as the conjugate operators of the respective momentum operators:

\[
[X_j, P_k] = -i\delta_{ij}, \quad [X_{Q,j}, P_{Q,k}] = -i\delta_{ij}, \quad [X_{\ell,j}, P_{\ell,k}] = -i\delta_{ij}, \quad [x_j, p_k] = -i\delta_{ij}, \quad (5.1)
\]

where \( X_j \) is the \( j \)-th component of \( \hat{X} \), etc.. In the last equality, \( \vec{x} \) is defined as the relative position operator \( \hat{X}_{\ell} - \hat{X}_{Q} \), and \( \vec{p} \) is its conjugate operator. The relationship between these eight operators (four momenta and four positions) are summarized in the following diagram:

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4 Both the heavy quark mass \( m_Q \) and the heavy quark velocity \( v \) are well defined up to order \( \lambda^0 \) ambiguities. For a discussion of these ambiguities and their theoretical implications, see Refs. 24-28.
\[
\vec{P} = \vec{P}_\ell + \vec{P}_Q = \vec{p},
\]
\[
\vec{X} = \vec{X}_\ell - \vec{X}_Q = \vec{x},
\]
where vertical arrows represent conjugations. Of course these look just like the analogous relations in the bound state picture, but recall that the point of introducing these operators is to see whether the bound state picture dynamics can be reproduced directly from operators in QCD with no model-dependent assumptions.

By construction, the heavy quark momentum and position operators commute with the brown muck counterparts.

\[
[X_{Qj}, X_{\ell k}] = [X_{Qj}, P_{\ell k}] = [P_{Qj}, X_{\ell k}] = [P_{Qj}, P_{\ell k}] = 0. \tag{5.3}
\]

The center-of-mass position \(\vec{X}\) is an unknown linear combination of \(\vec{X}_Q\) and \(\vec{X}_\ell\). Similarly, the relative momentum \(\vec{p}\) is an unknown linear combination of \(\vec{P}_Q\) and \(\vec{P}_\ell\). The operators are defined in such a way that the relative kinematic variables commute with the center-of-mass counterparts.

\[
[X_j, x_k] = [X_j, p_k] = [P_j, x_k] = [P_j, p_k] = 0, \tag{5.4}
\]

which in turn implies the following linear relations:

\[
\vec{X} = \hat{\alpha} \vec{X}_\ell + \hat{\beta} \vec{X}_Q, \quad \vec{p} = \hat{\beta} \vec{P}_\ell - \hat{\alpha} \vec{P}_Q, \tag{5.5}
\]

with \(\hat{\alpha}\) and \(\hat{\beta}\) being operators which commute with all the momentum and position operators and satisfy \(\hat{\alpha} + \hat{\beta} = 1\), the identity operator. We emphasize that all of these operators are defined from the QCD operators \(\vec{P}\) and \(\vec{P}_Q\) through linear combinations and conjugations. As a result, all of them are QCD operators.

What are the operators \(\hat{\alpha}\) and \(\hat{\beta}\)? To answer this question, one needs to look at the dynamics of the system and study the QCD Hamiltonian \(\mathcal{H}\). In particular, we will study the commutators of \(\mathcal{H}\) with the position operators.

First, consider the commutator \([X_j, \mathcal{H}]\).

\[
[X_j, \mathcal{H}] = i\vec{X}_j = i\frac{P_j}{\mathcal{H}}, \tag{5.6}
\]

where the second equality is from Poincare invariance. Note that the \(P_j/\mathcal{H}\) is well defined as \(\mathcal{H}\) commutes with \(P_j\). Moreover, since \(\mathcal{H} = M + \mathcal{H}\), where \(M = m_Q + m_N \sim \lambda^{-1}\) while \(\mathcal{H} \sim \lambda^0\), one can replace \(1/\mathcal{H}\) with \(1/M\) with relative correction of order \(\lambda\). As a result,

\[
[X_j, \mathcal{H}] = iP_j/M + \mathcal{O}(\lambda^2), \tag{5.7}
\]

and consequently we have the following double commutators:

\[
[X_k, [X_j, \mathcal{H}]] = -\delta_{jk}/M, \quad [x_k, [X_j, \mathcal{H}]] = 0, \tag{5.8}
\]

where all \(\mathcal{O}(\lambda^{-2})\) corrections are dropped. Note that the first equality is exactly what one expects if one starts with the nonrelativistic Hamiltonian, \(\vec{P}^2/2M\). As a result, we say that the kinetic mass of the heavy baryon, defined to be the reciprocal of the double commutator with the position operator, is \(M = m_Q + m_N\) in the combined heavy quark and large \(N_c\) limit, with possible corrections of order unity. These double commutators will be important in the determinations of \(\hat{\alpha}\) and \(\hat{\beta}\).

Next, let us study the commutator \([X_{Qj}, \mathcal{H}]\). Out of the four contributions to the QCD Hamiltonian \(\mathcal{H}\) in Eq. (5.2), \(m_Q\) and \(m_N\) are \(c\)-numbers, and the light operator \(\mathcal{H}_\ell\) commutes with \(X_{Qj}\). Thus,

\[
[X_{Qj}, \mathcal{H}] = [X_{Qj}, \mathcal{H}_Q] = [X_{Qj}, gA^0 + \frac{\vec{P}_Q^2}{2m_Q} - g \frac{S_Q \cdot B}{2m_Q}] = i\frac{P_j}{m_Q} + \mathcal{O}(\lambda^2), \tag{5.9}
\]

5For the reader who does not find the above equality obvious, recall that \(\mathcal{H}\) commutes with \(P_j\) (this is one of the defining commutation relations of the Poincare group), and hence the rest mass \(M\), defined by \(\mathcal{H} = \sum P_j^2 + M^2\), is Poincare invariant. Moreover, being the energy of the whole system at \(P_j = 0\), \(M\) is given by the QCD Hamiltonian \(\mathcal{H}\) in Eq. (5.2); as a result \(M = m_Q + m_N\) up to corrections of order \(\lambda^0\). As a result, \([X_j, \mathcal{H}] = i\frac{P_j}{m_Q} = i\frac{P_j}{m_Q} (\sum_k P_k^2 + M^2)^{1/2} = iP_j (\sum_k P_k^2 + M^2)^{-1/2} = i\frac{P_j}{\mathcal{H}}\).
as $A^0$ and $B$ are light operators and commute with $\tilde{X}_Q$. On the other hand, by ignoring the $O(\lambda^2)$ corrections, one has the following double commutators:

$$[X_{Qk}, [X_{Qj}, \mathcal{H}]] = -\delta_{jk}/m_Q, \quad [X_{\ell k}, [X_{Qj}, \mathcal{H}]] = [X_{Qk}, [X_{\ell j}, \mathcal{H}]] = 0, \quad (5.10)$$

where the Jacobi identity has been used to show the vanishning of the last double commutator. Note that the first equality states that, in the heavy quark limit, the kinetic mass of the heavy quark is $m_Q$.

This, together with Eq. (5.10), implies the following decomposition of $\mathcal{H}$:

$$\mathcal{H} = m_Q + m_N + \tilde{\mathcal{H}}_{Q,kin} + \tilde{\mathcal{H}}_{\ell,kin} + \tilde{\mathcal{H}}_{pot}, \quad (5.15)$$

where the second equality comes from Eq. (5.14). The potential energy term $\tilde{\mathcal{H}}_{pot} \sim \lambda^0$ does not depend on any momentum operators (but can and does depend on the position operators; see the following section). Using Eqs. (5.13), these two kinetic terms can be recast in terms of the center-of-mass and relative momenta.

$$\tilde{\mathcal{H}}_{kin} = \tilde{\mathcal{H}}_{Q,kin} + \tilde{\mathcal{H}}_{\ell,kin} = \frac{\vec{P}_Q^2}{2m_Q} + O(\lambda^2), \quad \tilde{\mathcal{H}}_{\ell,kin} = \frac{\vec{P}_\ell^2}{2m_N} + O(\lambda^2), \quad (5.16)$$

where $M = m_Q + m_N$ is the total mass and $\mu = m_Q m_N/(m_Q + m_N)$ will be referred as the reduced mass of the system. (Note that both $M$ and $\mu$ are of order $\lambda^{-1}$.) The reduced mass $\mu$ can be interpreted as the kinetic mass of the relative coordinate. Indeed, from the obtained value of $\tilde{\alpha}$ and $\tilde{\beta}$, one can verify that

$$[x_k, [x_j, \mathcal{H}]] = -\delta_{jk}/\mu + O(\lambda^2) \quad (5.17)$$

is in agreement with Eq. (5.15). In other words, in the combined heavy quark and large $N_c$ limit, the kinetic terms of the heavy baryon system are those of a nonrelativistic bound state of two point particles with $m_Q$ and $m_N$.

One may wonder what is the point of this whole exercise of reproducing elementary two-particle quantum mechanics, with apparently no new results. However, remember there is no a priori justification of treating a heavy baryon as the bound state of two point particles. In particular, the brown muck is not a point particle; it has a substantial size and complicated internal structures with the possibility of excitations. A formalism such as the bound state picture which treats the brown muck as though it were a point particle requires justification from QCD. Our goal is
to demonstrate the existence of an emergent symmetry in QCD itself (not merely in a model), we have to work with operators $\hat{P}$ and $\hat{E}_0$, which are QCD operators, and construct out of them all other kinematic operators. That our $\alpha$ and $\beta$ are identical with that in a nonrelativistic point particle treatment means that we have succeeded in providing a justification of the latter treatment in the combined heavy quark and large $N_c$ limit.

The apparently trivial double commutator in Eq. (5.14), which states that the kinetic term of the brown muck is nonrelativistic in the combined limit, is in fact not completely trivial. In the presence of a heavy quark, the kinetic mass of a composite object like the brown muck may depend on the relative position of the brown muck to the heavy quark. What we find instead is a constant kinetic mass $m_N$ in the combined limit. Physically this reflects the fact that the system in question is weakly bound; the interaction term $H$, which is of order $\lambda^0$, is much smaller than the masses of the constituents which are of order $\lambda^{-1}$. In a weakly bound state, the kinetic mass of the whole system is the sum of the kinetic masses of the constituents. Since the kinetic mass of the whole system $M$ and that of the heavy quark $m_Q$ are position independent, the kinetic mass of the brown muck is also independent of its position.

\section{VI. THE GENERATORS OF THE EMERGENT SYMMETRY}

In the previous section, we have analyzed the kinetic terms of the QCD Hamiltonian, $H$, by studying its double commutators with the position operators. One may also want to analyze the potential term $\hat{H}_\text{pot}$ by studying the double commutators of $H$ with the momentum operators. Unfortunately, this strategy actually provides very limited information.

As Poincare invariance demands that $[P_j, H] = 0$, it immediately follows that $[P_{k,j}, H] = -[P_{Q,j}, H]$. Moreover,

$$[p_j, H] = \hat{\beta}[P_{f,j}, H] - \hat{\alpha}[P_{Q,j}, H] = (\hat{\alpha} + \hat{\beta})[P_{f,j}, H] = [P_{f,j}, H],$$

(6.1)

where again we have used $\hat{\alpha} + \hat{\beta} = 1$. This reflects the simple observation that $\hat{H}_\text{pot}$ can depend on the relative position $\vec{x}$ but not the center-of-mass position of the whole system $\vec{X}$.

Recall that we have made the following decomposition: $H = m_Q + m_N + \hat{H}_\text{kin} + \hat{H}_\text{pot}$, with the last two terms both being of order unity or less. Both $m_Q$ and $m_N$ are $c$-numbers and commute with any operator, and it is easy to see that $\hat{H}_\text{kin}$ commutes with all momentum operators from its expression in Eq. (5.17), which does not depend on any of the position operators. As a result, $\hat{H}_\text{pot}$ is the only term which may have non-vanishing commutators with the momentum operators. The commutators of interest are $[p_j, \hat{H}_\text{pot}], [p_k, [p_j, \hat{H}_\text{pot}]], [p_k, [p_j, [p_i, \hat{H}_\text{pot}]]], \text{ etc.}$ We can say very little about these multiple commutators except that they are all (at most) of the same order as $\hat{H}_\text{pot}$, i.e., of order unity.

Of particular interest is the double commutator, whose significance lies in the following definition of the \textit{spring operator} $\hat{\kappa}$:

$$\hat{\kappa}\delta_{jk} = -[p_k, [p_j, H]] = -[p_k, [p_j, \hat{H}_\text{pot}]].$$

(6.2)

Let $|G\rangle$ be the ground state of the QCD Hamiltonian $H$ \textit{in the heavy baryon Hilbert space}, and $E_0$ be its mass, satisfying $(H - E_0)|G\rangle = 0$. The \textit{spring constant} $\kappa$ is then defined as $\langle G|\hat{\kappa}|G\rangle$. Since

$$\hat{\kappa} = -[p_j, [p_j, H]] = -[p_j, |H - E_0\rangle] = 2p_j(|H - E_0\rangle p_j - p_j|H - E_0\rangle) - (H - E_0)p_jp_j,$$

(6.3)

it is easy to see that $\kappa$ is positive by inserting a complete set of states $\{|n\rangle\}$.

$$\kappa = 2\langle G|p_j(H - E_0)p_j|G\rangle = 2\sum_n |\langle n|p_j|G\rangle|^2(E_n - E_0) > 0.$$  

(6.4)

Now we are ready to define the generators of the emergent symmetry; namely, the creation and annihilation operators.

$$\hat{a} = \sqrt{\frac{\mu \omega}{2}} \hat{\vec{x}} + i \sqrt{\frac{1}{2\mu \omega}} \hat{\vec{p}}, \quad \hat{a}^\dagger = \sqrt{\frac{\mu \omega}{2}} \hat{\vec{x}} - i \sqrt{\frac{1}{2\mu \omega}} \hat{\vec{p}},$$

(6.5)

\footnotesize{In the following two equations, the repeated index $j$ is \textit{not} being summed over.}
where \( \mu = m_Q m_N / (m_Q + m_N) \) is the reduced mass and \( \omega = \sqrt{\kappa / \mu} \) will be referred to as the natural frequency of the heavy baryon. With \( \kappa \) of order unity and \( \mu \sim \lambda^{-1} \) in the combined heavy quark and large \( N_c \) limit, \( \omega \) vanishes — an important result which does not depend on the order in which the two limits are taken. If the heavy quark limit is taken first, \( \mu \sim N_c \) and \( \omega \sim N_c^{-1/2} \to 0 \) as \( N_c \to \infty \). On the other hand, if the large \( N_c \) limit is taken first, \( \mu \sim m_Q \) and \( \omega \sim m_Q^{-1/2} \to 0 \) as \( m_Q \to \infty \).

The natural frequency \( \omega \) plays a central role in heavy baryon dynamics in the combined heavy quark and large \( N_c \) limit. As we will see below, \( \omega \) is the coefficient of the only term in the QCD Hamiltonian which breaks the emergent symmetry at leading order of \( \lambda \). As a result, all the physical properties (masses, coupling constants, form factors, etc.) of the low-lying baryons can be expressed in terms of \( \omega \). Alternatively, if one can determine the value of \( \omega \) by measuring some physical observable which depends on \( \omega \) (e.g., the mass of the first excited heavy baryon), then one can predict the values of many other physical observables.

### VII. CONSTRAINTS ON THE QCD HAMILTONIAN

Let us recall that our analysis is based on \( \lambda \) counting of quantities describing heavy baryon dynamics in the combined limit. The reduced mass can be expressed as the double commutator in Eq. (6.18):

\[
\delta_{jk} / \mu = -[x_k, [x_j, \mathcal{H}]] \sim \lambda,
\]

where higher order terms in \( \lambda \) are dropped. Similarly, the spring constant \( \kappa \) is the ground state expectation value of the double commutator in Eq. (6.2):

\[
\delta_{jk}\kappa = -[p_k, [p_j, \mathcal{H}]] \sim \lambda^0, \quad \kappa = \langle G | \kappa | G \rangle.
\]

As a result, the natural frequency \( \omega = \sqrt{\kappa / \mu} \sim \lambda^{1/2} \). This is a notable feature: recall that the expansion in \( \lambda \) embodies the expansions in both \( \Lambda_{QCD} / m_Q \) and \( 1/N_c \). We rarely encounter situations in which fractional powers of \( \Lambda_{QCD} / m_Q \) or \( 1/N_c \) arise for direct physical observables. Here, however, we have found that powers like \( \lambda^{1/2} \) do arise naturally. In fact, we will see that the natural expansion parameter will be \( \lambda^{1/2} \) instead of \( \lambda \). This ultimately reflects the interplay of two heavy scales (namely \( m_Q \) and \( m_N \)).

The double commutation relations in Eqs. (7.1) constrain the possible forms of the QCD Hamiltonian \( \mathcal{H} \). Note, however, that both double commutation relations are satisfied by replacing \( \mathcal{H} \) with \( \mathcal{H}_{\text{SHO}} \), the Hamiltonian of the bound state picture, which is just the simple harmonic oscillator.

\[
\mathcal{H}_{\text{SHO}} = \frac{\vec{p}^2}{2\mu} + \frac{\kappa \vec{x}^2}{2} - \frac{3\omega}{2} = \omega \vec{a}^\dagger \cdot \vec{a}.
\]

Moreover, the contracted O(8) symmetry mentioned above is precisely the maximal spectrum generating algebra of \( \mathcal{H}_{\text{SHO}} \) and becomes an emergent symmetry as \( \omega \to 0 \) as \( \lambda \to 0 \) in the combined limit. On the other hand, to demonstrate that this contracted O(8) is a symmetry of QCD in this limit, one needs to show that the generators of the contracted O(8) commute with the QCD Hamiltonian \( \mathcal{H} \), or equivalently, show that

\[
\mathcal{H} = \mathcal{H}_{\text{SHO}} + \mathcal{H}_{\text{exc}} + \ldots,
\]

where \( \mathcal{H}_{\text{exc}} \) commutes with \( \vec{a} \) and \( \vec{a}^\dagger \) in the combined limit, i.e., \( [a_j, \mathcal{H}_{\text{exc}}] = [a^\dagger_j, \mathcal{H}_{\text{exc}}] = 0 \), and represents the possibility of internal excitations of the brown muck. On the other hand, the ellipses are possible corrections to the simple harmonic Hamiltonian \( \mathcal{H}_{\text{SHO}} \) and should be negligible in comparison to \( \mathcal{H}_{\text{SHO}} \) in the combined limit. In other words, we need to show that, relative to \( \mathcal{H}_{\text{SHO}} \), these correction terms are suppressed by powers of \( \lambda \), and hence can be dropped in the combined limit.

Before we embark the power counting of \( \mathcal{H} \), we need to clarify the counting of powers for the kinematic variables. For example, consider the simple harmonic Hamiltonian \( \mathcal{H}_{\text{SHO}} = \omega \vec{a}^\dagger \cdot \vec{a} \), which apparently is of order \( \lambda^{1/2} \) as \( \omega \sim \lambda^{1/2} \). However, \( \mathcal{H}_{\text{SHO}} \) can also be written as \( \frac{\vec{p}^2}{2\omega} + \frac{\kappa \vec{x}^2}{2} - \frac{3\omega}{2} \), where the kinetic term, being suppressed by \( 1 / \mu \), is apparently of order \( \lambda \), while the potential term, with coefficient \( \kappa \sim \lambda^0 \), is apparently of order unity. The origins of this apparent discrepancy lie in the relationship between the operators \( (\vec{x}, \vec{p}) \) and \( (\vec{a}, \vec{a}^\dagger) \).

\[
\vec{x} = \sqrt{\frac{1}{2\mu \omega}} (\vec{a} + \vec{a}^\dagger), \quad \vec{p} = -i \sqrt{\frac{\mu \omega}{2}} (\vec{a} - \vec{a}^\dagger).
\]
Since $\mu \omega \sim \lambda^{-1/2}$, one cannot simultaneously set $\vec{x}$, $\vec{p}$, $\vec{a}$, and $\vec{a}^\dagger$ to the same order in $\lambda$.

In the following, we will make the prescription that $\vec{a}, \vec{a}^\dagger \sim \lambda^0$, which implies $\vec{x} \sim \lambda^{1/4}$ and $\vec{p} \sim \lambda^{-1/4}$, and show that it is self-consistent. As we will see later in this paper, this prescription will lead to the “symmetric realization” of the emergent symmetry with a contracted O(8) symmetry group. We can make the following a posteriori justification for this “symmetric prescription”. Instead of studying the power counting of the operators $\vec{x}$, $\vec{p}$, $\vec{a}$ and $\vec{a}^\dagger$, one can study the power counting of the matrix elements of $\vec{x}$, $\vec{p}$, $\vec{a}$ and $\vec{a}^\dagger$ between the low-lying states of $\mathcal{H}$. Power counting on matrix elements, which are $c$-numbers, is free of the aforementioned ambiguities. If the low-lying states of $\mathcal{H}$ are simple harmonic states as suggested by the bound state picture, then the matrix elements of $\vec{a}$ and $\vec{a}^\dagger$ are indeed of order unity, while the matrix elements of $\vec{x}$ and $\vec{p}$ are not. As the subsequent discussion verifies this picture, we will have justified our prescription a posteriori.

Now we are ready to show that in the combined limit $\mathcal{H}$ has the form presented in Eq. (7.3). We will achieve this in three steps. In the remainder of this section, we will verify the fact that all possible triple commutators of $\mathcal{H}$ with $\vec{a}$ and $\vec{a}^\dagger$ vanish in the combined limit. More specifically, we will show that these triple commutators go to zero more quickly than $\mathcal{H}_{\text{SHO}}$, which scales like $\lambda^{1/2}$. Then in the following section, we will show how the vanishings of these commutators imply that $\mathcal{H}$ can be at most bilinear in $\vec{a}$ and $\vec{a}^\dagger$:

$$\mathcal{H} = \hat{C}\vec{a}^\dagger \cdot \vec{a} + \hat{D}\vec{a} \cdot \vec{a} + \hat{D}^\dagger\vec{a}^\dagger \cdot \vec{a} + \mathcal{H}_{\text{exc}},$$

where $\mathcal{H}_{\text{exc}}$ commutes with both $\vec{a}$ and $\vec{a}^\dagger$. Lastly, $\hat{C}$ and $\hat{D}$ can be determined from the double commutation relations in Eqs. (7.1).

The relevant triple commutators of $\mathcal{H}$ with $\vec{a}$ and $\vec{a}^\dagger$ are the following operators:

$$t^{(0)} = [a_i, [a_j, [a_k, \mathcal{H}]]], \quad t^{(1)} = [a_i^\dagger, [a_j, [a_k, \mathcal{H}]]],$$

$$t^{(2)} = [a_i^\dagger, [a_j^\dagger, [a_k, \mathcal{H}]]], \quad t^{(3)} = [a_i^\dagger, [a_j^\dagger, [a_k^\dagger, \mathcal{H}]]].$$

Using the Jacobi identity and the commutators of $a_j$ and $a_j^\dagger$, it is easy to show that the values of these $t^{(a)}$ triple commutators do not depend on the ordering of the $a_j$’s and $a_j^\dagger$’s. For example,

$$[a_i^\dagger, [a_j, [a_k, \mathcal{O}]]] = -[a_j, [[a_k, \mathcal{O}], a_i^\dagger]] - [[a_k, \mathcal{O}], [a_i, a_j]] = [a_j, [a_i^\dagger, [a_k, \mathcal{O}]]] = -[a_j, [a_i, [a_k, \mathcal{O}]]].$$

Since $t^{(0)} = -(t^{(3)})^\dagger$ and $t^{(1)} = -(t^{(2)})^\dagger$, it suffices to show that $t^{(0)} = t^{(1)} = 0$ in the combined limit.

Since $\vec{a}$ and $\vec{a}^\dagger$ are linear combinations of $\vec{x}$ and $\vec{p}$, $t^{(0)}$ and $t^{(1)}$ can be expressed as linear combinations of the triple commutators of $\mathcal{H}$ with $\vec{x}$ and $\vec{p}$:

$$t^{(0)} = \frac{2^{-3/2}}{3} (T^{(3)} + 3iT^{(2)} - 3T^{(1)} - iT^{(0)}), \quad t^{(1)} = \frac{2^{-3/2}}{3} (T^{(3)} + iT^{(2)} + T^{(1)} + iT^{(0)}),$$

where

$$T^{(0)} = (\mu \omega)^{-3/2} [p_i, [p_j, [p_k, \mathcal{H}]]], \quad T^{(1)} = (\mu \omega)^{-1/2} [p_i, [p_j, [x_k, \mathcal{H}]]],$$

$$T^{(2)} = (\mu \omega)^{1/2} [p_i, [x_j, [x_k, \mathcal{H}]]], \quad T^{(3)} = (\mu \omega)^{3/2} [x_i, [x_j, [x_k, \mathcal{H}]]].$$

Again note that the values of these $T^{(a)}$ triple commutators do not depend on the ordering of the $x_j$’s and $p_j$’s. We have shown in Sec. VI that the triple $p$ commutator is at most of order unity, and hence $T^{(0)} \sim \mathcal{O}(\lambda^{3/4})$. All of the other three triple commutators are also small as $[x_k, \mathcal{H}] = ip_k/\mu + \mathcal{O}(\lambda^2)$. The first term gets killed by the following commutations and does not contribute to the triple commutators. So $T^{(1)} \sim \lambda^{3/4}$, $T^{(2)} \sim \lambda^{7/4}$ and $T^{(3)} \sim \lambda^{3/4}$ — all vanish faster than $T^{(0)}$. As a result, both $t^{(0)}$ and $t^{(1)}$ vanish at least as fast as $T^{(0)} \sim \lambda^{3/4}$, and are negligible when compared to $\mathcal{H}_{\text{SHO}} \sim \lambda^{1/2}$ in the combined limit.

VIII. THE QCD HAMILTONIAN IN THE COMBINED LIMIT

In the previous section, we have verified that all triple commutators $t^{(a)}$ vanish faster than $\mathcal{H}_{\text{SHO}} \sim \lambda^{1/2}$ in the combined limit. In other words, if we only keep terms up to those of order $\lambda^{1/2}$, all these triple commutators are zero.

$$[a_i, [a_j, [a_k, \mathcal{H}]]] = [a_i^\dagger, [a_j, [a_k, \mathcal{H}]]] = [a_i^\dagger, [a_j^\dagger, [a_k, \mathcal{H}]]] = [a_i^\dagger, [a_j^\dagger, [a_k^\dagger, \mathcal{H}]]] \leq \mathcal{O}(\lambda^{-3/4}).$$ (8.1)
Now the vanishings of these triple commutators at order $\lambda^{1/2}$ imply that to this order the double commutators commute with $\vec{a}$ and $\vec{a}^\dagger$:

$$[a_j, [a_k^\dagger, \mathcal{H}]] = [a_j^\dagger, [a_k, \mathcal{H}]] = -\dot{C}\delta_{jk}, \quad [a_j^\dagger, [a_k, \mathcal{H}]] = 2\dot{D}\delta_{jk}, \quad [a_j, [a_k, \mathcal{H}]] = 2\dot{D}^\dagger\delta_{jk},$$

(8.2)

where the operators $\dot{C}$ and $\dot{D}$ commute with both $\vec{a}$ and $\vec{a}^\dagger$. Note that $\dot{C}$ is hermitian while $\dot{D}$ is in general non-hermitian. The above relations can be recast as:

$$[a_j, \dot{B}] = 0, \quad [a_j^\dagger, \dot{B}] = 0, \quad \text{where} \quad \dot{B} = [a_k^\dagger, \mathcal{H}] + \dot{C}a_k^\dagger + 2\dot{D}a_k.$$

(8.3)

However, parity invariance of $\mathcal{H}$ implies that there does not exist any parity odd operator which commutes with both $\vec{a}$ and $\vec{a}^\dagger$. So $\dot{B}$ vanishes and

$$[a_k^\dagger, \mathcal{H}] + \dot{C}a_k^\dagger + 2\dot{D}a_k = 0.$$

(8.4a)

Similarly, one has

$$[a_k, \mathcal{H}] - \dot{C}a_k - 2\dot{D}^\dagger a_k^\dagger = 0.$$

(8.4b)

Notice that, while in Eqs. (8.2) we expressed the double commutators of $\mathcal{H}$ in terms of $\dot{C}$ and $\dot{D}$, in Eqs. (8.4) we managed to express the single commutators of $\mathcal{H}$ in term of $\dot{C}$ and $\dot{D}$. We can make one more step and express $\mathcal{H}$ itself in terms of $\dot{C}$ and $\dot{D}$ by noting that Eqs. (8.4) imply

$$[a_j, \dot{A}] = 0, \quad [a_j^\dagger, \dot{A}] = 0, \quad \text{where} \quad \dot{A} = \mathcal{H} - \dot{C}\vec{a}^\dagger \cdot \vec{a} + \dot{D}\vec{a} \cdot \vec{a} + \dot{D}^\dagger\vec{a}^\dagger \cdot \vec{a}^\dagger,$$

(8.5)

where $\dot{A}$ commutes with both $\vec{a}$ and $\vec{a}^\dagger$. Lastly, by renaming $\dot{A}$ as $\mathcal{H}_{\text{exc}}$, we recover Eq. (7.3):

$$\mathcal{H} = \dot{C}\vec{a}^\dagger \cdot \vec{a} + \dot{D}\vec{a} \cdot \vec{a} + \dot{D}^\dagger\vec{a}^\dagger \cdot \vec{a}^\dagger + \mathcal{H}_{\text{exc}}.$$

(8.6)

We have succeeded in showing that the vanishings of the triple commutators of $\mathcal{H}$ imply that $\mathcal{H}$ is at most a bilinear in $\vec{a}$ and $\vec{a}^\dagger$. While the above derivation looks rather complicated, the essence of the statement is very intuitive: if $\mathcal{H}$ contains, for instance, trilinear terms in $\vec{a}$ and $\vec{a}^\dagger$, the triple commutators will read off the coefficients of these trilinear terms and hence will not vanish. What we have achieved, through the derivation above, is to realize this intuition in a rigorous manner.

We have shown that, up to order $\lambda^{1/2}$, $\mathcal{H}$ is a bilinear in $\vec{a}$ and $\vec{a}^\dagger$. Now we will make the final step in this demonstration and determine the form of the operators $\dot{C}$ and $\dot{D}$. Clearly if $\dot{C} = \omega$ and $\dot{D} = 0$, $\mathcal{H}$ will simply be the sum of the simple harmonic Hamiltonian $\mathcal{H}_{\text{SHO}}$ and a possible excitation term $\mathcal{H}_{\text{exc}}$ which commutes with $\vec{a}$ and $\vec{a}^\dagger$. Our goal is to show that these are indeed the values of $\dot{C}$ and $\dot{D}$.

One can re-express $\mathcal{H}$ in terms of $\vec{x}$ and $\vec{p}$:

$$\mathcal{H} = \frac{(\dot{C} - \dot{D}^+)\vec{p}^2 + \mu\omega(\dot{C} + \dot{D}^+)\vec{x}^2}{2} - \frac{3\dot{C}}{2} - \frac{\dot{D}^-}{2}(\vec{x} \cdot \vec{p} + \vec{p} \cdot \vec{x}) + \mathcal{H}_{\text{exc}},$$

(8.7)

where $\dot{D}^+ = \dot{D} + \dot{D}^\dagger$ and $i\dot{D}^- = \dot{D} - \dot{D}^\dagger$. Then $\dot{C}$, $\dot{D}^+$ and $\dot{D}^-$ can be deduced by using the double commutation relations Eqs. (7.1):

$$[p_j, [x_k, \mathcal{H}]] = 0 \quad \Rightarrow \quad \dot{D}^- = 0,$$

$$[x_j, [x_k, \mathcal{H}]] = -\delta_{jk}/\mu \quad \Rightarrow \quad \dot{C} - \dot{D}^+ = \omega,$$

$$[p_j, [p_k, \mathcal{H}]] = -\delta_{jk}\kappa \quad \Rightarrow \quad \dot{C} + \dot{D}^+ = \kappa/\mu = \omega\kappa/\kappa.$$

The last two equalities lead to:

\[ \text{This is actually a special case of the following general result. Let } \mathcal{O} \text{ be an arbitrary operator with vanishing } m \text{-th multiple commutators with } \vec{a} \text{ and } \vec{a}^\dagger. \text{ Then one can prove by induction that } \mathcal{O} \text{ is at most } (m - 1)\text{-linear in } \vec{a} \text{ and } \vec{a}^\dagger. \]
\[ \hat{C} = \omega (\hat{\kappa}/\kappa + 1)/2, \quad \hat{D}_\pm = \omega (\hat{\kappa}/\kappa - 1)/2, \] (8.8)

where the spring constant \( \kappa \) is the ground state expectation value of the spring operator \( \hat{\kappa} \) introduced in Eq. (8.2). As a result, when acting on the heavy baryon ground state, \( \hat{\kappa}/\kappa = 1 \), which in turn implies \( \hat{C} = \omega \) and \( \hat{D}_\pm = 0 \). For a general heavy baryon state (not necessarily the ground state), however, \( \hat{\kappa} \) is not identical to its ground state expectation value \( \kappa \). However, it is true not merely for the ground state, but also for states in the ground state band, which is the subspace spanned by states of the form \( (a_\mu^L)^n (a_\nu^R)^n (a_\lambda^z)^n |G\rangle \). We therefore conclude that, in the ground state band, \( \hat{C} = \omega \), \( \hat{D}_\pm = 0 \), and \( \mathcal{H} \) has the simple harmonic form:

\[ \mathcal{H} = \frac{p^2}{2\mu} + \frac{\kappa x^2}{2} - \frac{3\omega^2}{2} + \mathcal{H}_{\text{exc}} + \mathcal{O}(\lambda) = \omega \hat{a}^\dagger \cdot \hat{a} + \mathcal{H}_{\text{exc}} + \mathcal{O}(\lambda). \] (8.9)

This Hamiltonian clearly reduces to a three-dimensional simple harmonic oscillator with reduced mass \( \mu \), spring constant \( \kappa \), and hence natural frequency \( \omega \). The maximal spectrum generating algebra of this Hamiltonian is a contracted \( \text{O}(8) \) algebra, and in the combined limit when \( \omega \sim \lambda^{1/2} \rightarrow 0 \), the contracted \( \text{O}(8) \) becomes an emergent symmetry. Again, we emphasize that this emergent symmetry is a symmetry of QCD in the heavy baryon sector — not only that of the bound state picture or any other models.

\[ \begin{align*}
\hat{C} &= \omega (\hat{\kappa}/\kappa + 1)/2, \\
\hat{D}_\pm &= \omega (\hat{\kappa}/\kappa - 1)/2,
\end{align*} \]

\[ \mathcal{H} = \frac{p^2}{2\mu} + \frac{\kappa x^2}{2} - \frac{3\omega^2}{2} + \mathcal{H}_{\text{exc}} + \mathcal{O}(\lambda) = \omega \hat{a}^\dagger \cdot \hat{a} + \mathcal{H}_{\text{exc}} + \mathcal{O}(\lambda). \]

IX. REVIEW AND DISCUSSION

In summary, we have demonstrated that the contracted \( \text{O}(8) \) symmetry seen in the bound state picture is in fact a symmetry of QCD. Near the combined limit there exists a band of low-lying heavy baryons, labeled by \((n_x, n_y, n_z)\), the number of excitation quanta in the \( x, y \) and \( z \) directions (or alternatively \((N, L, L_z)\), where \( N = n_x + n_y + n_z \) is the total number of excitation quanta, \( L \) the orbital angular momentum and \( L_z \) its \( z \)-component). For each state the excitation energy is \((n_x + n_y + n_z)\omega = N\omega\). As \( \lambda \rightarrow 0 \) in the combined limit, \( \omega \rightarrow 0 \) and the entire band become degenerate. Our discussion can be generalized in a straightforward manner to the case with \( d \) spatial dimensions, with an emergent contracted \( \text{O}(2d + 2) \) symmetry group.

While the demonstration was rather long, the basic idea is very simple. We started by constructing the kinematic variables \( x_j \) and \( p_j \), which are not \textit{a priori} well defined in QCD (Sec. V). Since our goal is to study the symmetry in QCD itself, we cannot merely assume that the kinematic variables are well defined (as in models), but need to show that they are legitimate QCD operators. After defining these kinematic variables in QCD, we showed, by considering triple commutators (Sec. VII), that the QCD Hamiltonian \( \mathcal{H} \) is a bilinear of these creation and annihilation operators up to a certain order in the \( \lambda \) expansion (Sec. VIII). Lastly, the “coefficients” (which are formally operators) of the bilinear terms in \( \mathcal{H} \) are fixed by considering double commutators.

It may seem strange that a symmetry of QCD is only applicable to a certain subspace (namely, the ground state band of the heavy baryon subspace) of the whole QCD Hilbert space. This, however, is a typical feature for emergent symmetries. The heavy quark symmetry is only applicable to states containing a heavy quark \( \bar{q}q \), and the light quark spin-flavor symmetry in the large \( N_c \) limit is only relevant for baryonic states \( \bar{q}_L q_R q_L \). While the symmetries of the QCD Lagrangian is applicable to all states in the QCD Hilbert space, emergent symmetries are not symmetries of the QCD Lagrangian and may be applicable only to particular subspaces.

We will end this section by briefly returning to the bound state picture and discuss some intricate issues on its relationship to our formalism. We noted in Sec. III that there were conceptual problems associated with the bound state picture, particularly in regard to treating the brown muck as though it were a point particle. The possible problem was that the characteristic size of the brown muck distribution is \( L_\mu^2 \sim \Lambda_{\text{QCD}} \sim \lambda^0 \) while the bound state wave function had a typical size of \( L_\text{wf}^2 \sim (\mu\omega)^{1/2} \sim \lambda^{1/2} \) which is characteristically narrower in position space. In spite of this concept we have shown that as far as kinematics and symmetries are concerned, the point particle description correctly reproduces the QCD result. It turns out that the comparison between the characteristic size of the brown muck and the scale of the wave function is not the appropriate comparison. In fact, there are three distance scales in this problem: If the brown muck is to be approximated by a point particle, the point particle should be located at the center-of-mass of the brown muck in order to reproduce the correct kinematics. According to Ref. \[8\], the brown
muck can be studied under the Hartree picture, which becomes exact in the large \( N_c \) limit. In the Hartree picture, one can easily see that the center-of-mass of the brown muck is much better localized than each individual quark. The size of the wave function of each individual quark \( L_q \) is comparable to the size of the whole brown muck, which is of order unity. On the other hand, the fluctuation of the position of center-of-mass \( L_{CM}^2 \) of the brown muck is smaller by a factor of \( \sqrt{N_c} \) by the central limit theorem. As a result, \( L_{CM}^2 \sim L_q^2/N_c \sim \lambda \), which is much smaller than the typical spread of the wave function. The different scales form the following hierarchy:

\[
L_q^2 \sim \lambda^0 \implies L_{w1}^2 \sim \lambda^{1/2} \implies L_{CM}^2 \sim \lambda.
\]  

(9.1)

In other words, the bound state wave function cannot resolve the fluctuation of the center-of-mass of the brown muck. This provides an intuitive explanation why, despite its huge size, the brown muck can be approximated by a point particle without drastically altering the kinematics and the symmetries of the system.

**X. The Symmetry Broken Realization**

Recall that our demonstration of the simple harmonic form of the QCD Hamiltonian \( \mathcal{H} \) depends on the symmetric prescription for the \( \lambda \) power counting introduced in Sec. VII, that the creation and annihilation operators should be counted as order unity in \( \mathcal{H} \). This prescription is in turn a posteriori justified by the fact that the matrix element of \( a_j^\dagger \) and \( a_j \) between states in the ground state band are indeed of order unity. While this confirms the self-consistency of this symmetric prescription of \( \lambda \) counting rules, it does not preclude the possible existence of other self-consistent counting schemes. In this section, we will briefly describe other possible realizations of this emergent symmetry.

To gain physical insight, it is useful to think in terms of the bound state picture. Clearly our simple harmonic oscillator obtained by assuming that \( \tilde{a}_a, \tilde{a}_a^\dagger \sim \lambda^0 \) implies that the ground state expectation value \( \langle G|x^2|G\rangle \sim \lambda^{1/2} \) vanishes in the combined limit. This reflects that, as the reduced mass \( \mu \to \infty \), the center of the brown muck gets more and more localized around the heavy quark. This is the scenario where the origin of the relative position space \( \tilde{x} = 0 \) minimizes the potential energy globally. However, it does not need to be the case. The potential may have a “mexican hat” shape with the global minimum at \( r = |\tilde{x}| = r_0 > 0 \), where by naturalness \( r_0 \sim 1/\Lambda_{QCD} \sim \lambda^0 \). In such a case, in the combined limit the relative wave function will be a shell sharply peaked around \( r = r_0 \). As a result, \( \langle G|x^2|G\rangle = r_0^2 \sim \lambda^0 \) and the symmetric prescription is clearly inapplicable. Instead this “mexican hat” scenario corresponds to a different realization of the emergent symmetry, which hereinafter will be referred to as the “symmetry broken realization”.

We will sketch how one may study the emergent symmetry in this “mexican hat” scenario, where there are two modes of low-energy excitations. Firstly, there are orbital excitations along the bottom of the potential well at \( r = r_0 \), described by the Hamiltonian \( \mathcal{H}_L \sim \hat{L}^2/2\hat{L} \). The moment of inertia \( \hat{I} = \mu r_0^2 \sim \lambda^{-1} \) in the combined limit. As a result, as \( \lambda \to 0 \), \( \mathcal{H}_L \to 0 \) and the whole tower of orbitally excited states collapses into degeneracy. The symmetry group of \( \mathcal{H}_L \) with finite moment of inertia \( \hat{I} \) is the rotational group \( O(3) \), and the spectrum generating algebra is contracted \( O(4) \) (also known as \( E_3 \), the three dimensional Euclidean group; cf. Sec. III.4 of Ref. [27]). Secondly, there are radial excitations around \( r = r_0 \), which can be studied through a formalism similar to what we constructed in previous sections to study the symmetric realization. It turns out that the radial excitations are also simple harmonic in the combined limit, but only as in a one-dimensional oscillator again with \( \omega \sim \lambda^{1/2} \). The spectrum generating algebra in this case is again a contracted \( O(4) \) (generated by \( a, a^\dagger, a^2, a^2 \dagger, a^\dagger a, \) and \( 1 \)). Near the combined limit where \( \lambda \) is small, the rotational excitation energies \( \sim 1/2\hat{I} \sim \lambda \) are much smaller than the radial excitation energies \( \sim \omega \sim \lambda^{1/2} \). Hence the spectrum consists of a tower of equally spaced simple harmonic levels with splitting \( \omega \), with each level further split into a tower of rotor states with splitting \( \sim 1/2\hat{I} \). As \( \lambda \to 0 \), all these orbital and radially excited states become degenerate with the ground state, and the spectrum generating algebra contracted \( O(4) \times O(4) \) becomes the emergent symmetry group in this symmetry broken realization.

This contracted \( O(4) \times O(4) \) group in the symmetry broken realization is a subgroup of the contracted \( O(8) \) in the symmetric realization. Actually this is very reminiscent of spontaneous symmetry breaking in field theory. Note that orbital excitations are light in the combined limit. As a result, one can construct a wave function sharply peaked at

---

8 Actually Ref. [8] used the Hartree picture to study a baryon, not the brown muck of a heavy baryon. However, the difference between a brown muck of a heavy baryon, with \( N_c - 1 \) light quarks, and a light baryon with \( N_c \) light quarks, becomes negligible in the large \( N_c \) limit. (Of course, the brown muck is different from a baryon as the former is not a color singlet, but the extra color charge is neutralized by the heavy quark.)
$x = y = 0, z = r_0$ which is degenerate with the ground state (and hence is itself also a legitimate ground state) as $\lambda \to 0$. Such a ground state breaks the contracted $O(8)$ in the symmetric realization to the contracted $O(4) \times O(4)$ in the asymmetric realization. This explains the terminologies, “symmetric” and “symmetry broken” realizations.

While this symmetry broken realization of the emergent symmetry may not be as aesthetically appealing as the symmetric counterpart, we emphasize that it is a viable logical possibility and there is no theoretical justification of a priori rejecting this possibility. However, these two realizations are phenomenologically distinguishable, as least when $\lambda \to 0$. In the symmetric realization the excitation energy of the second excited state is twice that of the first excited state, where in the symmetry broken realization the ratio is 3. In the real world, the first excited charmed baryon is around 330 MeV heavier than the ground state and about 200 MeV beneath the D-N dissociation threshold. If future experiments find the second excited charmed baryon beneath this dissociation threshold, one would be very tempted to rule out the asymmetric realization.

In this section, we have compared the two possible realizations of the emergent symmetry: the symmetric realization when the potential is globally minimized at the origin, and the symmetry broken realization when the global minimum of the potential is away from the origin. (Actually it is logically possible that there is more than one global minimum — one at the origin, while the other is not. This scenario, however, is so extremely unnatural and requires such fine-tuning that we will not consider it further in this paper.) While the preceding discussion relied on the bound state picture, one can rephrase it in QCD language in a manner analogous to our analysis of the symmetric realization represented in previous sections. The symmetry broken realization has the interesting feature of light states of excitation energies $\sim \lambda$, which originate from the orbital revolution around the bottom of the “mexican hat” potential. But let us be reminded that there are other possible rotational modes for a baryon which has nothing to do with orbital revolution. For example, one can rotate the brown muck itself (not around the heavy quark) in space or in isospace, which quantum mechanically correspond to spin and isospin excitations. The moment of inertia is of order $N_c$, which is degenerate with the ground state (and hence is itself also a legitimate ground state) as $\lambda \to 0$. Such a ground state breaks the contracted $O(8)$ in the symmetric realization to the contracted $O(4)$ limit. This is the well-known large $N_c$ spin-flavor symmetry $\mathfrak{f}^{\mathfrak{f}}$, relating $\Delta(1323)$ to the nucleon and $\Sigma^0_Q$ to $\Lambda_Q$. These rotational modes of the brown muck have little to do with its relative motion relative to the heavy quark. As a result, intuitively we expect this large $N_c$ spin-flavor symmetry to commute with the emergent symmetry (in either realization). It will be the goal of the next section to demonstrate that this intuition is indeed correct.

**XI. INCLUSION OF THE SPIN AND ISOSPIN EFFECTS**

So far we have neglected the spin and isospin of the heavy baryon, which consists of a single heavy quark and $N_c - 1$ valence light quarks. For concreteness we will only consider the cases where $N_c$ is an odd number, so there is an even number of valence light quarks in a heavy baryon. In QCD with two light flavors, each light quark is isospin-1/2, and as a result the brown muck can be of isospin $I = 0, 1, \ldots (N_c - 1)/2$. The isospin symmetry is described by an $SU(2)_I$ group, generated by

$$I^a = \int d^3 x \sum_{k=1}^{N_c-1} q_k^\dagger r^a q_k, \quad a = 1, 2, 3, \quad \text{(11.1)}$$

where the summation is over all the valence light quarks. Similarly, light quarks are spin-1/2 fermions; a brown muck with $N_c - 1$ light quarks without any orbital angular momentum between them can be of spin $S_\ell = 0, 1, \ldots (N_c - 1)/2$. (Note that in the heavy quark limit the heavy quark spin $S_Q$ decouples from the rest of the system. As a result, the brown muck spin $S_\ell$ is conserved and is a good quantum number.) The brown muck spin symmetry is also described by an $SU(2)_{S_\ell}$ group, generated by

$$S_\ell^i = \int d^3 x \sum_{k=1}^{N_c-1} q_k^\dagger \sigma^i q_k, \quad i = 1, 2, 3. \quad \text{(11.2)}$$

Both isospin and brown muck spin symmetries are symmetries of the QCD Lagrangian (the latter only in the heavy quark limit), and their generators $I^a$ and $S_\ell^i$ satisfy these commutation relations:

\footnote{To make such a conclusion, however, one has to check if the corrections higher order in $\lambda$ are small. Unfortunately, such corrections are likely to be substantial.}
\[ [I^a, I^b] = i\epsilon^{abc} I^c, \quad [S^a_I, S^b_I] = i\epsilon^{ijk} S^k_I, \quad [I^a, S^b_I] = 0. \quad (11.3) \]

It was realized in 1993 by several different collaborations \[12, 13\] that for large \( N_c \) baryons, the separate isospin and brown quark spin symmetries, described by the product group \( SU(2)_I \times SU(2)_S \), get combined and enlarged into an emergent spin-flavor symmetry. This spin-flavor symmetry is described by a contracted \( SU(4) \) group, generated by \( \{ X^{ai}, I^a, S^b_I \} \), satisfying the following commutation relations:

\[ [X^{ai}, I^b] = i\epsilon^{abc} X^{ci}, \quad [X^{ai}, S^b_I] = i\epsilon^{ijk} X^{ak}, \quad [X^{ai}, X^{bj}] = 0. \quad (11.4) \]

with \( X^{ai} \) being the axial current couplings:

\[ X^{ai} = \int d^3x \sum_{k=1}^{N_c-1} q^I_k \tau^a \sigma^i q_k, \quad a, i = 1, 2, 3. \quad (11.5) \]

Since all generators commute with \( K^2 \), where \( K \) is the vector sum of \( I \) and \( S_I \), all the states with the same \( K^2 \) are degenerate. Of particular interest are the \( K = 0 \) states, for which \( I = 0 \) and hence \( (I, S_I) = (0, 0), (1,1), \ldots, ((N_c-1)/2, (N_c-1)/2) \). Phenomenologically one can identify the \((0,0)\) state as \( \Lambda_Q \) and the \((1,1)\) state as \( \Sigma_Q^{(*)} \), and the \( \Sigma_Q^{(*)} - \Lambda_Q \) splitting is \( 1/N_c \) suppressed. These analyses have been extended to orbitally excited baryons in Refs. \[28–31\], although orbitally excited baryons containing charm or bottom quarks were only briefly discussed.

Note that \( X^{ai}, I^a \) and \( S^b_I \) act only on the light quarks. In other words, they represent the internal degrees of freedom of the brown muck, while leaving the heavy quark alone. On the other hand, the creation and annihilation operators which generate the contracted \( O(8) \) symmetry represent collective excitations; \( i.e. \), all the light quarks are excited in a correlated manner relative to the heavy quark. Intuitively, one expects these two modes of excitations to be independent of each other in the large \( N_c \) limit, and hence the light quark spin-flavor symmetry group should commute with the contracted \( O(8) \) group. \[3\]

To show that this piece of intuition is correct, we will prove the following general result: Let \( J_I \) be a local operator which acts only on the brown muck, \( i.e. \), \( [X^{ai}_Q, J_I] = [P^{ai}_Q, J_I] = 0 \). Then the forward matrix element of \( [a_j, J_I] \) and \( [a_j^I, J_I] \) all vanish up to order \( \lambda \). For forward matrix element we mean the kinematic condition that there exists an inertial frame such that both the initial and final states are at rest (carrying zero total momentum).

To see that this is true, note that

\[ [a_j, J_I] = \sqrt{\frac{1}{2\mu\omega}} [x_j, J_I] + i \sqrt{\frac{1}{2\mu\omega}} [p_j, J_I], \quad [a_j^I, J_I] = \sqrt{\frac{1}{2\mu\omega}} [x_j, J_I] - i \sqrt{\frac{1}{2\mu\omega}} [p_j, J_I] \quad (11.6) \]

and the vanishing of the \( \bar{x} \) and \( \bar{p} \) commutators on the right-hand side imply the vanishing of the \( \bar{a} \) and \( \bar{a}^I \) commutators on the left-hand side. It is straightforward to show from the definitions of the kinematic variables that

\[ \bar{x} = \frac{1}{\alpha}(\bar{X} - \bar{X}_Q), \quad \bar{p} = \hat{\beta}\bar{P} - \bar{P}_Q, \quad (11.7) \]

where \( \alpha \) and \( \hat{\beta} \) are the operator-valued coefficients introduced in Eq. \(5, 3\). Note that, since \( \alpha \) is a \( c \)-number up to order \( \lambda \), \( 1/\alpha \) is well defined up to the same order. Since \( [X^{ai}_Q, J_I] = [P^{ai}_Q, J_I] = 0 \), one has

\[ [x_j, J_I] = \frac{1}{\alpha} [x_j, J_I], \quad [p_j, J_I] = \hat{\beta} [p_j, J_I] \quad (11.8) \]

We have expressed the commutators of the relative kinematic variables \( \bar{x} \) and \( \bar{p} \) in terms of their center-of-mass counterparts \( \bar{X} \) and \( \bar{P} \). However, since \( J_I \) is a local operator without any intrinsic momentum scale, \( [X_j, J_I] = 0 \). On the other hand, \( [P_J, J_I] \) in general does not vanish as the local operator is translated in position space by \( \bar{P} \). The forward matrix element, however, trivially vanishes. As a result, both terms in Eqs. \(11.6\) are zero, and the proof is completed.

---

\(10\) Do not confuse the axial current couplings \( X^{ai} \) with \( X_j \), the center-of-mass position!

\(11\) We note in passing that models based on the bound state picture have often implicitly assumed that spin-flavor degrees of freedom commute with the orbital ones. In the simple bound state model \[14, 18\], for example, this assumption was built in by using the same profile function to describe the light baryon as a chiral soliton.
Two comments are in place here. First, by saying that the commutators \([a_j, J_\ell] \) and \([a_j^\dagger, J_\ell] \) vanish, we actually mean the more precise statement that these commutators are smaller than the typical matrix element of \( J_\ell \) by at least an order in \( \lambda \) — the order at which it is no longer justifiable to treat \( \hat{\sigma} \) and \( \hat{\beta} \) as c-numbers. Second, by reversing the role of the brown muck and the heavy quark, it is straightforward to show that, for a local operator \( J_Q \) which acts only on the heavy quark, \( \chi \), \( \{ X_{\ell j}, J_Q \} = \{ P_{\ell j}, J_Q \} = 0 \), the forward matrix element of \([a_j, J_Q]\) and \([a_j^\dagger, J_Q] \) also vanish up to order \( \lambda \). Both of these results reflect the intuitive statement that any excitation which involves only one of the constituents but does not transfer any momentum will not change the relative motion. These results will be useful when we consider the heavy baryon matrix elements of pionic current or weak interaction currents in the companion paper \[13\].

Returning to our discussion of spin and isospin effects, since the spin-flavor symmetry generators \( X^{ai} \) act only on the light degrees of freedom, they commute with the creation and annihilation operators implying that all of the states below are degenerate in the combined limit. The states are labeled by the quantum numbers \((N, L, S_\ell, J_\ell)\), where \( N \) is the number of excitation quanta in the simple harmonic oscillator, \( L \) is the orbital angular momentum, \( S_\ell \) is the spin of the brown muck (which is always equal to the isospin \( \mathbf{I}_Q \)), \( L \) is the orbital angular momentum of the brown muck. The total spin of the whole heavy baryon \( J = S_Q + J_\ell \) is the vectorial sum of \( J_\ell \) with the heavy quark spin \( S_Q \). \[12\]

\[
\begin{align*}
N = 2 & \quad L = 0, 2 \quad \Lambda_{Q2}^{(s)} \quad (J_\ell = 0, 2) \quad & X^{ai} & \quad \Sigma_{Q2}^{(s)} \quad (J_\ell = 1, 2, 3) \quad & X^{ai} & \quad \ldots \\
N = 1 & \quad L = 1 \quad \Lambda_{Q1}^{(s)} \quad (J_\ell = 1) \quad & X^{ai} & \quad \Sigma_{Q1}^{(s)} \quad (J_\ell = 0, 1, 2) \quad & X^{ai} & \quad \ldots \\
N = 0 & \quad L = 0 \quad \Lambda_{Q} \quad (J_\ell = 0) \quad & X^{ai} & \quad \Sigma_{Q}^{(s)} \quad (J_\ell = 1) \quad & X^{ai} & \quad \ldots \\
\end{align*}
\]

The \( \Lambda \) sector
\[
I = S_\ell = 0
\]

The \( \Sigma \) sector
\[
I = S_\ell = 1
\]

That \( X^{ai} \) and \( a_j^\dagger \) commute means that one can construct, say, the \( \Sigma_{Q1}^{(s)} \) state through \( a_j^\dagger (X^{ai} \Lambda_{Q}) = a_j^\dagger \Sigma_{Q1}^{(s)} \), or the opposite order \( X^{ai}(a_j^\dagger \Lambda_{Q}) = X^{ai} \Lambda_{Q1}^{(s)} \), and both constructions give the same state \( \Sigma_{Q1}^{(s)} \).

Recall that we have shown in Eq. (3.9) that the QCD Hamiltonian in the heavy baryon sector can be written as the sum of the simple harmonic part \( H_{\text{SHO}} \), and the internal excitation Hamiltonian \( H_{\text{exc}} \), with possible corrections of order \( \lambda \). \[13\] Among the different contributions to \( H_{\text{exc}} \) is \( H_I = \alpha I^2 \), the Hamiltonian describing the low-lying spin-flavor excitations. The large \( N_c \) spin-flavor symmetry implies \( \sigma \sim N_c^{-1} \sim \lambda \). As a result, \( H_I \sim \lambda \) and one can move it from \( H_{\text{exc}} \) to \( H_{\text{SHO}} \) and rewrite Eq. (3.9) as follows:

\[
H = H_{\text{SHO}}' + H_{\text{exc}}' + \mathcal{O}(\lambda), \quad H_{\text{SHO}}' = H_{\text{SHO}} + H_I,
\]

and \( H_{\text{exc}}' \) is the Hamiltonian for other internal excitation modes excluding the low-lying spin-flavor excitations.

---

12 Here we have many different angular momenta adding in different ways. We will clarify their meanings by comparing a heavy baryon to a multi-electronic atom. If the heavy quark is the analogy of the heavy nucleus and the electron cloud corresponds to the brown muck, then \( S_\ell \) corresponds to the electronic spin \( S \), \( L \) corresponds to the orbital angular momentum \( L \), and \( J_\ell = S_\ell + L \) translates into \( J = S + L \). The heavy quark spin \( S_Q \) is the counterpart of the nuclear spin \( I \), and the total spin of the heavy baryon given by \( J = S_Q + J_\ell \) is the analogy of the \( J \)-spin, \( J = I + J \).

13 We are working with the symmetric realization here. The case for asymmetric realization can be studied in a similar way.
The Hamiltonian $H_{\text{SHO}}'$ describes the low-lying heavy baryon spectroscopy up to order $\lambda$ corrections. (Note that these corrections are of the same order as $H_I$.) Under $H_{\text{SHO}}'$, each of the simple harmonic bound states of $H_{\text{SHO}}$ is split by $H_I$ into a whole tower of states with different $I$. The masses of the heavy baryon states under $H_{\text{SHO}}'$ are

$$m = \Lambda_Q + N\omega + \sigma I^2 + \mathcal{O}(\lambda),$$

(11.11)

where $N = n_x + n_y + n_z$ is the total number of excitation quanta, and we are adopting the customary abuse of notation that the symbol of a state also represent its mass, e.g., $\Lambda_k = m_{\Lambda_k} = 5624$ MeV. This mass relation implies that the orbital excitation energies are the same in the $\Lambda$ sector ($I = 0$) and the $\Sigma$ sector ($I = 1$), i.e.,

$$\omega_\Sigma - \omega_\Lambda \equiv (\Sigma_{Q1}^{(s)} - \Sigma_Q^{(s)}) - (\Lambda_{Q1}^{(s)} - \Lambda_Q^{(s)}) = \mathcal{O}(\lambda).$$

(11.12)

Formally it also implies spin-flavor excitation energies of the $N = 1$ states are the same as their $N = 0$ counterparts:

$$2\sigma_1 - 2\sigma_0 \equiv (\Sigma_{Q1}^{(s)} - \Lambda_{Q1}^{(s)}) - (\Sigma_Q^{(s)} - \Lambda_Q^{(s)}) = \mathcal{O}(\lambda).$$

(11.13)

Unfortunately this relation is actually devoid of information, as both $\sigma_1$ and $\sigma_0$ are of order $\lambda$, which is the order of the leading order corrections.

Note that the qualitative features of the low-lying heavy baryon spectrum is correctly given by $H_{\text{SHO}}'$, which specifies the low-energy spectroscopy up to order $\lambda^{1/2}$. The not-yet-specified correction terms at order $\lambda$ are small when compared to $H_{\text{SHO}}'$ and hence can only perturb the spectrum but not change it qualitatively. On the other hand, to make quantitative predictions about heavy baryon spectroscopy (or other heavy baryon dynamical properties) at order $\lambda$, one has to determine the explicit forms of the order $\lambda$ correction terms — a task which we will undertake in our next paper [15].

**XII. CONCLUDING REMARKS**

In this paper, we have shown that there is an emergent symmetry in the heavy baryon Hilbert subspace in the combined heavy quark and large $N_c$ limits. This emergent symmetry can either be realized as a contracted $O(8)$ or a contracted $O(4) \times O(4)$, and in either case it relates the orbitally excited states with the ground state. Both realizations of this emergent symmetry have interesting spectroscopic implications: in the former case the spectrum is that of a three-dimensional simple harmonic oscillator, while in the latter case the spectrum is that of a heavy rigid rotor with simple harmonic radial excitations. Moreover, we have also shown that this emergent symmetry commutes with the light quark spin-flavor symmetry for large $N_c$ baryons.

While the main purpose of this paper is to discuss the results reported in Ref. [14] in a more detailed fashion, there are several places where the formalism in this paper differs from the original treatment in Ref. [14]. We list the most important differences below for comparison:

- In Ref. [14], we have always taken the heavy quark limit before the large $N_c$ limit. In this paper, the combined limit is taken by keeping $N_c\Lambda_{\text{QCD}}/m_Q$ fixed at an arbitrary value. This is a more general treatment, as it includes the possibilities of taking the heavy quark limit both before and after the large $N_c$ limit, and a whole range of other possible limiting procedures.

- Since the heavy quark limit is taken first in Ref. [14], one does not need to distinguish the relative momentum $\vec{p}$ from the brown muck momentum $\vec{P}_t$. On the other hand, in the paper we are keeping the ratio $N_c\Lambda_{\text{QCD}}/m_Q$ arbitrary, and the distinction between $\vec{p}$ and $\vec{P}_t$ should not be overlooked. Here we have presented the analysis of the kinematics in full generality, while the result of Ref. [14] can be recovered by setting $\alpha = 0$ and $\beta = 1$.

- We have realized that the emergent symmetry group reported in Ref. [14], namely the contracted $U(4)$, is only a subgroup of the full symmetry group in the symmetric realization — contracted $O(8)$. Moreover, we have studied the symmetry broken realization for the sake of generality.

We want to emphasize once more that, even though we seem to be dealing with a quantum mechanical potential model, the formalism is actually completely field theoretical, and the kinematic variables are QCD operators. Why can one reduce a field theoretical problem into a quantum mechanical framework? The answer lies in the separation of scales: both constituents, namely the heavy quark and the brown muck, have mass of order $\lambda^{-1}$, while the interaction is only of order unity and hence is very weak compared to the mass scale represented by the reduced mass $\mu$. The separation of scales makes it possible to make an expansion in powers of $\lambda$ and have an effective field theory which includes only the lowest excitation modes, which in this case is the motion of the brown muck relative to the heavy quark (and the possible spin-flavor excitations). This falls under a category of effective field theory which is referred to as “rigorous potential models” in Ref. [32], of which the most notable example is nonrelativistic QCD (NRQCD)
NRQCD describes quarkonium states, which are heavy quark–heavy antiquark bound states. While the mass of each constituent is \( m_Q \), the three-momentum of the relative motion is only of the order \( \alpha_s m_Q \), and the kinetic energy is of an even lower order \( \alpha^2 m_Q \), where \( \alpha_s \) is the QCD coupling constant at scale \( m_Q \). Since \( \alpha_s \) is small, we have a separation of scales which allows us to expand the Hamiltonian in powers of \( \alpha_s \). In particular, since the kinetic energy scale is much smaller than the three-momentum scale, it is natural to impose a stronger cutoff on energies than on three-momenta in the effective field theory. The resultant effective theory is local in time but not in space, i.e., a potential. Potential models constructed with such a philosophy are rigorous in the sense that they are related to the original field theory through Wilsonian renormalization. Our treatment of heavy baryons is similar to NRQCD, with \( \lambda^{1/2} \) playing the role of \( \alpha_s^2 \). The similarity is even more apparent when one realizes that our formalism, just like NRQCD, can be viewed as a nonrelativistic expansion. In NRQCD \( v^2 \sim \alpha_s^2 \), while in our case \( v^2 \sim \lambda^{1/2} \).

We will end on a cautionary note with a comparison with another “rigorous potential model”, namely the nucleon-nucleon effective field theory. The deuteron is a nonrelativistic bound state, with the binding energy \( \sim 2 \) MeV order of magnitude smaller than the nucleon mass \( \sim 1 \) GeV. However, the large \( N_c \) counting rules would have suggested a very different picture. Since baryon-baryon interaction is of order \( N_c \), both the binding energy \( V_0 \) and the spring constants \( \kappa \) of a deuteron are of order \( N_c \). With reduced mass \( \mu \sim N_c \), large \( N_c \) scaling rules suggest that the excitation energy \( \omega = \sqrt{\kappa/\mu} \sim \alpha_s^0 \), which is much smaller than the binding energy \( V_0 \). This implies the existence of many bound states beneath the dissociation threshold (the number of bound states should be of order \( V_0/\omega \sim N_c \)), and the low-lying bound states should be deeply bound. This is in blatant disagreement with the deuteron in the real world, which is barely bound with a tiny binding energy in comparison to \( \Lambda_{QCD} \): \( V_0 \sim 2 \) MeV in the triplet channel, and the singlet channel is not even bound. It turns out that there are many different physical contributions to the binding energy. Each of these contributions may be of order \( N_c \), but it happens that they almost cancel completely, and the numerical value for \( V_0 \) turns out to be accidentally small. This illustrates a fundamental limitation of counting schemes: a physical quantity may carry a numerical value very different from what the formal power counting suggests due to accidental cancelation or appearances of unnaturally large or small coefficients. One should be aware of the possibilities of such accidents and carefully check if the physical picture suggested by the counting rules resembles the real world.

For our analysis of the heavy baryon, the binding energy \( V_0 \) is formally of order unity, while the excitation energy \( \omega \) is of order \( \lambda^{1/2} \). As a result, one expects the number of bound states to be of order \( \lambda^{-1/2} \sim N_c^{1/2} = \sqrt{3} \) in the real world. Experimentally \( V_0 \) and \( \omega \) have been determined to be about 625 MeV and 330 MeV, respectively. These numbers are at least compatible with the picture suggested in this paper. With an expansion parameter as large as \( \lambda^{1/2} \sim 1/\sqrt{3} \sim 0.6 \), however, the expansion series may converge slowly and one needs to include corrections due to higher order terms before one can make any quantitative predictions. As a result, it is imperative to make a careful study of the higher order corrections.

In summary, we have introduced an effective theory to study excited heavy baryons which makes the existence of the emergent symmetry manifest. What are the phenomenological implications of this emergent symmetry? What does this symmetry tell us about the strong decays of excited baryons and the weak decay form factors? And most importantly, are these symmetry predictions safe against corrections due to higher order terms in the \( \lambda \) counting? All of these issues will be discussed in an upcoming paper [3].

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