Coprimemappings and lonely runners

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Abstract
For $x$ real, let $\{x\}$ be the fractional part of $x$ (that is, $\{x\} = x - \lfloor x \rfloor$). The lonely runner conjecture can be stated as follows: for any $n$ positive integers $v_1 < v_2 < \cdots < v_n$, there exists a real number $t$ such that $1/(n+1) \leq \{v_it\} \leq n/(n+1)$ for $i = 1, \ldots, n$. In this paper, we prove that if $\epsilon > 0$ and $n$ is sufficiently large (relative to $\epsilon$), then such a $t$ exists for any collection of positive integers $v_1 < v_2 < \cdots < v_n$ such that $v_n < (2 - \epsilon)n$. This is an approximate version of a natural next step for the study of the lonely runner conjecture suggested by Tao. A key ingredient in our proof is a result on coprime mappings. Let $A$ and $B$ be sets of integers. A bijection $f : A \rightarrow B$ is a coprime mapping if $a$ and $f(a)$ are coprime for every $a \in A$. We show that if $A, B \subset [n]$ are intervals of length $2m$ where $m = e^{\Omega((\log\log n)^2)}$ then there exists a coprime mapping from $A$ to $B$.

1 | INTRODUCTION

Suppose $n$ runners are running on a circular track of circumference 1. It is not a race. The runners all start at the same point on the track and at the same time, and each runs at their own distinct constant speed. We say that a runner is lonely at time $t$ if the distance (along the track) to the nearest of the other runners is at least $1/n$. The lonely runner conjecture asserts that every
runner is lonely at some point in time. This problem originally arose in the context of dio-
phantine approximations and view obstruction problems [4, 16]. (The poetic formulation given
here is due to Goddyn [1].) It is easier to work with the following restatement of the conjet-
ture, which we obtain by subtracting the speed of one runner from all speeds (then one of
the runners is “standing still”). In the original problem speeds are real valued, but it is known
that the general problem can be reduced to case where all speeds are integers (see [2]). So we
henceforth consider only integer speeds. For $x$ real, let $\{x\}$ be the fractional part of $x$ (that is,
$\{x\} = x - \lfloor x \rfloor$).

**Conjecture 1.1** (Lonely Runner Conjecture). For any $n$ positive integers $v_1 < v_2 < \cdots < v_n$,

$$\exists t \in \mathbb{R} \text{ such that } 1/(n+1) \leq \{v_it\} \leq n/(n+1) \text{ for } i = 1, \ldots, n.$$  (1)

There are examples of sets of speed which “almost” break Condition (1).

**Definition 1.2.** Positive integers $v_1 < v_2 < \cdots < v_n$ are said to be a tight instance for the lonely
runner conjecture if Condition (1) holds, but only with equality for at least one $i$. In other words,
the instance $v_1 < v_2 < \cdots < v_n$ is tight if (1) holds and there does not exist $t \in \mathbb{R}$ such that $1/(n +
1) < \{v_it\} < n/(n + 1)$ for $i = 1, \ldots, n$. An instance that is neither a counterexample nor tight is a
loose instance of the lonely runner conjecture. An instance is loose if

$$\exists t \in \mathbb{R} \text{ such that } 1/(n+1) < \{v_it\} < n/(n+1) \text{ for } i = 1, \ldots, n.$$  (2)

The canonical example of a tight instance is $(1, 2, \ldots, n)$. Tight instances were studied by Goddyn
and Wong [8]. They showed that the canonical tight instance can be modified to create another
tight instance by accelerating a speed that is slightly less than $n$ — and satisfies certain num-
ber theoretic conditions — by a suitable integer factor. For example, $(1, 2, 3, 4, 5, 7, 12)$ is a tight
instance. They also showed that small sets of speeds in the canonical instance that satisfy these
conditions can be simultaneously accelerated to produce tight instances.

Tao [15, Proposition 1.5] showed that (1) holds if $v_1, \ldots, v_n \leq 1.2n$. He also suggested that proving
the conjecture holds for $v_1, \ldots, v_n \leq 2n$ is a natural target: the condition $v_1, \ldots, v_n \leq 2n$, unlike
1.2$n$, allows multiple tight instances. So the desired statement would prove the conjecture for
instances that are in the vicinity of tight instances. We prove an approximate version of this target:

**Theorem 1.3.** There exists a constant $C$ such that for sufficiently large $n$, if $n < v_n \leq 2n - \exp(C \cdot
(\log \log n)^2)$, then positive integers $v_1 < v_2 < \cdots < v_n$ are a loose instance for the lonely runner con-
jecture.

Unfortunately, as seen in the theorem statement, the underlying objective to separate tight
instances from counterexamples is not achieved.

A key ingredient in our proof of Theorem 1.3 is inspired by coprime mappings.

**Definition 1.4.** If $A, B$ are sets of integers, then a bijection $f : A \rightarrow B$ is a coprime mapping if $a$
and $f(a)$ are coprime for every $a \in A$. 

**COPRIME MAPPINGS AND LONELY RUNNERS**
Initial interest in coprime mappings was focused on the case $A = [n]$. Newman conjectured that for all $n \in \mathbb{Z}^+$ there is a coprime mapping between $[n]$ and any set of $n$ consecutive integers. This conjecture was proved by Pomerance and Selfridge [12] (after Daykin and Baines [5] and Chvátal [3] established special cases of the conjecture.) Robertson and Small [13] determined when a coprime mapping exists between $A = [n]$ (or $A = \{1, 3, 5, \ldots, 2n-1\}$) and an $n$-term arithmetic progression (AP).

More recently there has been interest in coprime mappings where neither $A$ nor $B$ contains 1. Note that if $A = \{2, \ldots, n+1\}$, the integer $s$ is the product of all primes that are at most $n+1$, and $s \in B$ then there is no coprime map from $A$ to $B$. So we must place some restriction on the set $B$ if we consider sets $A$ that do not contain 1. Larsen et al. [10] considered sets of adjacent intervals of integers. They conjectured that if $1 \leq \ell < k$ and $k \neq 3$, then there is coprime mapping from $A = \{\ell + 1, \ldots, \ell + k\}$ to $B = \{\ell + k + 1, \ldots, \ell + 2k\}$.

The application of coprime mappings that we use in the context of the lonely runner conjecture requires the further generalization to the case that $A$ and $B$ are not adjacent. For each positive integer $n$, we define the number $f(n)$ to be the smallest integer such that for all $2m \geq f(n)$ there is a coprime mapping between every pair of intervals $A, B \subseteq [n]$ with $|A| = |B| = 2m$. The Jacobsthal function $j(\ell)$ is defined to be the largest length of an interval $I$ with the property that no integer in $I$ is coprime to $\ell$. Note that we have $f(n) > \max_{\ell \leq n} j(\ell)$. The Jacobsthal function $j(\ell)$ is known to be larger than $\log(\ell)$ for infinitely many $\ell$ [7], and this gives a lower bound on $f(n)$. On the other hand, the Conjecture of Larsen et al. requires only a linear upper bound on $f(n)$. We establish the following asymptotic bound.

**Theorem 1.5.** $f(n) = \exp(O((\log\log n)^2))$.

This result is not sharp. Indeed, while this paper was under review, Pomerance proved that $f(n) = O(\log^2(n))$ [11]. As the best-known upper bound on the Jacobsthal function is $j(\ell) = O(\log^2(\ell))$, this suggests the possibility that we have $f(n) = 1 + \max_{\ell \leq n} j(\ell)$ for $n$ sufficiently large. The theory of random graphs provides support for such a conjecture as a perfect matching appears in the random graph process as soon as the minimum degree is one with high probability. This is an example of the “minimum degree phenomenon” in random graphs. (See Chapter 4 of [9] for discussion of this phenomenon and perfect matchings in random graphs.)

We also resolve the conjecture of Larsen et al.

**Theorem 1.6.** If $0 \leq \ell < k$ and $k \geq 4$, then there is coprime mapping from $A = \{\ell + 1, \ldots, \ell + k\}$ to $B = \{\ell + k + 1, \ldots, \ell + 2k\}$.

Note that there is no coprime mapping from $A = \{2, 3, 4\}$ to $B = \{5, 6, 7\}$. Thus, some condition on $k$ is required.

The remainder of this paper is organized as follows. In the next section we prove a result — Theorem 2.1 — which can be viewed as a number theoretic version of Hall’s condition. Theorem 1.5 follows immediately from Theorem 2.1. In §3, we use Theorem 2.1 to prove Theorem 1.3. In the final section, we prove the conjecture of Larsen et al. regarding coprime mappings between adjacent intervals; that is, we prove Theorem 1.6.

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1 We force the cardinality to be even, because when the cardinality is odd, $A$ and $B$ can both have a majority of even numbers, making a coprime mapping impossible.
2  |  HALL’S CONDITION

**Theorem 2.1.** There exists a constant $C$ such that the following is true for sufficiently large $n$. If $I,J \subseteq [n]$ are both sets of $2m$ consecutive integers and $2m \geq \exp(C \cdot (\log \log n)^2)$, then for all subsets $S \subseteq I$, $T \subseteq J$ that satisfy $|S| + |T| \geq 2m$, exactly one of the following happens.

1. $S = \emptyset$.
2. $T = \emptyset$.
3. $S = I \cap 2\mathbb{Z}$ and $T = J \cap 2\mathbb{Z}$.
4. there exist $s \in S$, $t \in T$ that are coprime.

In particular, if $|S| + |T| > 2m$, then there is a coprime pair.

Remark 2.2. In order to apply Hall’s Theorem to establish the existence of a coprime mapping between $I$ and $J$, it suffices to show that every pair of sets $S \subseteq I$, $T \subseteq J$ such that $|S| + |T| \geq 2m + 1$ contains a coprime pair $s$, $t$ such that $s \in S$ and $t \in T$. Thus, Theorem 1.5 follows immediately from Theorem 2.1. Note that Theorem 2.1 is stronger than necessary for this purpose as it treats the case $|S| + |T| = 2m$. This case is needed for the application to the lonely runner problem. (That is, for the proof of Theorem 1.3.)

We now turn to the proof of Theorem 2.1. Lemma 2.4 is the core of the proof. It has a weaker condition (APs) and a weaker result (2-coprime) than Theorem 2.1. For the remainder of this section, we assume $I,J \subseteq [n]$ are APs of cardinality $m$ with common difference 1 or 2.

**Definition 2.3.** Two numbers $s, t \in \mathbb{Z}^+$ are said to be 2-coprime if no prime other than 2 divides them both.

**Lemma 2.4.** There exists a constant $C$ such that the following is true for sufficiently large $n$. If $m \geq \exp(C \cdot (\log \log n)^2)$, then for all nonempty subsets $S \subseteq I$, $T \subseteq J$ such that $|S| + |T| \geq m$ there exist $s \in S$, $t \in T$ that are 2-coprime.

The main ingredients of the proof of Lemma 2.4 are the following two lemmas and one fact, where the fact is a bound on the Jacobsthal function.

**Lemma 2.5.** Let $S \subseteq I$, $T \subseteq J$ be nonempty subsets such that $|S| + |T| \geq m$, and let $r = m/|S|$. If $r \geq 16$, $m \geq 5 \log (n)^{\log_2(2r)}$, and $n$ is sufficiently large, then there exist $s \in S$, $t \in T$ that are 2-coprime.

**Lemma 2.6.** Let $S \subseteq I$, $T \subseteq J$ be nonempty subsets such that $|S| + |T| \geq m$, and let $r = m/|S|$. If $2 \leq r \leq 16$, $m \geq \log (n)^{3}$ and $n$ is sufficiently large, then there exist $s \in S$, $t \in T$ that are 2-coprime.

**Fact 2.7** (See [6]). There are positive constants $c_1, c_2$ such that for all $n > 1$ we have $j(n) \leq c_1 \cdot \omega(n)^{c_2}$; that is, among any sequence of $c_1 \cdot \omega(n)^{c_2}$ consecutive integers, there is at least one that is coprime to $n$. (Here $\omega(n)$ is the number of distinct prime divisors of $n$.)

We now prove Lemma 2.4, assuming Lemma 2.5 — Fact 2.7. We will prove Lemmas 2.5 and 2.6 immediately after. We end the section with the proof of Theorem 2.1.
Proof of Lemma 2.4. Set $C = 2c_2 \log_2(e)$, where $c_2$ is the constant in Fact 2.7. Without loss of generality, assume that $|S| + |T| = m$. Since the roles of $S$ and $T$ are the same, one can assume that $|S| \leq m/2$. That is, $r := m/|S| \geq 2$. Note that Lemma 2.4 follows immediately from either Lemma 2.5 or Lemma 2.6 (depending on the value of $r$) unless

$$\exp(C \cdot (\log \log n)^2) \leq m < 5 \log(n)^{\log_2(2r)}.$$ \hspace{1cm} (3)

(Note that we clearly have $\exp(C \cdot (\log \log n)^2) > \log(n)^3$ for $n$ sufficiently large.) It remains to prove Lemma 2.4 when $r$ and $m$ satisfy (3). To this end, we first observe that if we replace coprime with $2$-coprime, then we can extend Fact 2.7 to APs with common difference 2.

Claim. For all $n \in \mathbb{Z}^+$, any integer AP with common difference 1 or 2 and at least $c_1 \cdot \omega(n) c_2$ terms contains at least one term that is $2$-coprime to $n$. (Using the same constants as in Fact 2.7.)

Proof of Claim. Assume without loss of generality that $n$ is odd. Say the AP is $a_1, \ldots, a_r$. If the AP contains only even numbers, consider $a_1/2, a_2/2, \ldots, a_r/2$; if only odd numbers, consider $(a_1 + n)/2, (a_2 + n)/2, \ldots, (a_r + n)/2$. In any case, we have a sequence of $r$ consecutive integers, where the $i$th number is coprime to $n$ if and only if $a_i$ is coprime to $n$. Fact 2.7 implies that the new sequence has a number coprime to $n$.

Now consider $s \in S$. By the Claim, among any $c_1 \cdot \omega(s) c_2$ consecutive terms of $J$, at least one is $2$-coprime to $s$. Thus, $J$ has at least $\lceil m/(c_1 \cdot \omega(s) c_2) \rceil$ terms that are $2$-coprime to $s$. If any of these terms are in $T$, then we have the desired $2$-coprime pair, so we may assume for the sake of contradiction that they are all in $J \setminus T$. Note that, $\omega(s) \leq \log_2(s) \leq \log_2(n)$, so, for sufficiently large $n$, we have

$$\frac{m}{c_1 \omega(s) c_2} \geq \exp(C \cdot (\log \log n)^2) \frac{c_1 \log_2(n) c_2}{c_1 \omega(s) c_2}.$$ \hspace{1cm}

As this quantity is arbitrarily large for large $n$, the floor function has negligible effect, and it follows that we have

$$r = \frac{m}{|S|} = \frac{m}{|J \setminus T|} \leq \left\lceil \frac{m}{c_1 \omega(s) c_2} \right\rceil \leq (1 + o(1))c_1 \log_2(n) c_2.$$ \hspace{1cm}

Then, again appealing to (3), we have

$$m < 5 \log(n)^{\log_2(2r)} \leq 5 \log(n)^{(c_2 + o(1)) \log_2 \log_2 n} < \exp(C \cdot (\log \log n)^2) \leq m.$$ \hspace{1cm}

This is a contradiction.

We now prove Lemmas 2.5 and 2.6. The key idea is to count the non-coprime pairs in $S \times T$ by summing up $|S \cap p\mathbb{Z}||T \cap p\mathbb{Z}|$ over primes $p$ greater than 2. Note that if $S$ and $T$ are random subsets of $I$ and $J$, respectively, then we expect to have

$$\sum_{p \geq 2} |S \cap p\mathbb{Z}||T \cap p\mathbb{Z}| \approx \sum_{p \geq 2} \frac{|S|}{p} \frac{|T|}{p} = |S \times T| \sum_{p \geq 2} \frac{1}{p^2} \approx 0.2 |S \times T|,$$ \hspace{1cm}
and there are many 2-coprime pairs. Of course, we have to complete the proof for all sets $S$ and $T$ (rather than just random ones) and we do this by “zooming in” on primes $p$ for which $|S \cap p\mathbb{Z}|$ is large. We “zoom in” by looking at $S \cap p\mathbb{Z}$ and $T \setminus p\mathbb{Z}$ instead of $S$ and $T$, and we iterate this process if necessary.

Before the proof, we state a simple approximation that we use throughout this Section.

**Lemma 2.8.** Let $A$ be an integer AP with common difference $d$, and suppose $P \in \mathbb{Z}^+$ is coprime with $d$. Then for all $R \subseteq [P]$, 

$$\frac{|A \cap (R + P\mathbb{Z})|}{|R|} - \frac{|A|}{P} \in (-1, 1).$$

(4)

As a result, if $|A|/P \geq \delta^{-1}$ for some $\delta > 0$, then

$$\frac{|A \cap (R + P\mathbb{Z})|}{|A||R|/P} \in (1 - \delta, 1 + \delta).$$

(5)

**Proof.** Among every $P$ consecutive terms of $A$ (a “chunk”), exactly $|R|$ of them are in $A \cap (R + P\mathbb{Z})$. $|A|/P$ disjoint chunks can cover a subset of $A$ and $|[A]/P|$ chunks can cover a superset of $A$, so $|A \cap (R + P\mathbb{Z})|$ is between $|[A]/P||R|$ and $|[A]/P||R|$. □

**Proof of Lemma 2.5.** Let $S_0 = S$, $T_0 = T$, $I_0 = I$, $J_0 = J$. Let

$$M = \left(\frac{m}{5}\right)^{1/\log_2(2r)}.$$  

(6)

Note that by assumption,

$$M > \left(\log(n)^{\log_2(2r)}\right)^{1/\log_2(2r)} = \log(n).$$

For each $i \in \mathbb{Z}^+$, if there exists prime $p_i \notin \{2, p_1, \ldots, p_{i-1}\}, p_i \leq M$ such that

$$\frac{|S_{i-1} \cap p_i\mathbb{Z}|}{|I_{i-1} \cap p_i\mathbb{Z}|} \geq 2 \frac{|S_{i-1}|}{|I_{i-1}|},$$

define

$$S_i = S_{i-1} \cap p_i\mathbb{Z}, T_i = T_{i-1} \setminus p_i\mathbb{Z}, I_i = I_{i-1} \cap p_i\mathbb{Z}, J_i = J_{i-1} \setminus p_i\mathbb{Z}.$$ 

Let us say $k$ is the last index where these are defined. For every $0 \leq i \leq k$, define $P_i = \text{Primes} \setminus \{2, p_1, \ldots, p_i\}$ for convenience. Now we have

$$\frac{|S_i|}{|I_i|} \geq 2 \frac{|S_{i-1}|}{|I_{i-1}|} \quad \text{for } i = 1, \ldots, k,$$

(7)

$$\frac{|S_k \cap p\mathbb{Z}|}{|I_k \cap p\mathbb{Z}|} < 2 \frac{|S_k|}{|I_k|} \quad \forall p \in P_k \text{ such that } p \leq M.$$ 

(8)
Here, (7) implies that
\[
\frac{|S_k|}{|I_k|} \geq 2^k \frac{|S_0|}{|I_0|} = \frac{2^k}{r}.
\]

Since $S_i$ is a subset of $I_i$, we have $|S_k|/|I_k| \leq 1$, and hence $k \leq \log_2(r)$. Define $\Gamma = \prod_{i=1}^k p_i$. Note that
\[
M \Gamma = M \prod p_i \leq M \cdot M^k \leq M^{\log_2(2^r)} = \frac{m}{5}. \tag{9}
\]

We establish some estimates for $|I_k|$, $|J_k|$ and $|I_k \cap p\mathbb{Z}|$, $|J_k \cap p\mathbb{Z}|$ for $p \in P_k$ such that $p < M$. Since $I_k = I \cap \Gamma \mathbb{Z}$ and $|I|/\Gamma \geq 5M$, by (5) we have
\[
|I_k| \in (1 \pm 1/M)m/\Gamma. \tag{10}
\]

$I_k$ is once again an AP. For $p \in P_k$ such that $p \leq M$, $p$ does not divide the common difference of $I_k$ and $|I_k|/p \geq |I_k|/M > (1 - 1/M)m/M \Gamma \geq 5 - 5/M$. Hence by (5)
\[
|I_k \cap p\mathbb{Z}| \in \left( \left(1 - \frac{1}{5 - 5/M}\right) \frac{|I_k|}{p}, \left(1 + \frac{1}{5 - 5/M}\right) \frac{|I_k|}{p}\right). \tag{11}
\]

Similarly, $J_k$ and $J_k \cap p\mathbb{Z}$ can be bounded from both sides. Since $J_k = J \setminus p_1\mathbb{Z} \setminus \cdots \setminus p_k\mathbb{Z}$, and $|J|/\Gamma \geq 5M$, by (5), we have
\[
|J_k| \in (1 \pm 1/M)m \cdot \prod (p_i - 1)/\Gamma.
\]

Define $\Phi = m \prod (p_i - 1)/\Gamma$. Then
\[
|J_k| \in (1 \pm 1/M)\Phi. \tag{12}
\]

For $J_k \cap p\mathbb{Z}$, we apply (5) with $A = J, P = \Gamma p, R = ([\Gamma p] \cap p\mathbb{Z}) \setminus p_1\mathbb{Z} \setminus \cdots \setminus p_k\mathbb{Z}$. Since $|J|/\Gamma p \geq 5$, we have
\[
\forall p \in P_k, p \leq M, \ |J_k \cap p\mathbb{Z}| \in (0.8m|R|/|P|, 1.2m|R|/|P|)
\]
\[
= (0.8\Phi/p, 1.2\Phi/p). \tag{13}
\]

Now consider $\Phi$. Note that
\[
\Phi = m \cdot \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) \geq m \cdot \prod_{i=1}^k \left(1 - \frac{1}{i + 2}\right) = 2m/(k + 2)
\]
\[
\geq 2m / \log_2(4r) > \frac{2r|S|}{\log_2(4r)} > \frac{16}{3} |S| \quad \text{(as } r \geq 16).\]
That is, $|S| < 3\Phi/16$. Since $J_k \setminus T_k \subseteq J \setminus T$, whose cardinality is at most $|S|$, by (12), we have

\begin{align*}
\frac{|T_k|}{|J_k|} = 1 - \frac{|J_k \setminus T_k|}{|J_k|} > 1 - \frac{3\Phi/16}{(1 - 1/M)\Phi} = \frac{13/16 - 1/M}{1 - 1/M} > \frac{3}{4}; \\
|T_k| > \frac{13/16 - 1/M}{1 - 1/M} |J_k| \geq (13/16 - 1/M)\Phi \geq \frac{3\Phi}{4}.
\end{align*}

(14) (15)

With these estimates in hand, we consider the quantity

$$\lambda(p) = \frac{|S_k \cap p\mathbb{Z}| \ |T_k \cap p\mathbb{Z}|}{|S_k| \ |T_k|}$$

for every odd prime $p$. Note that the set of pairs $(s, t) \in S_k \times T_k$ that are not 2-coprime because $p$ divides both $s$ and $t$ is $(S_k \cap p\mathbb{Z}) \times (T_k \cap p\mathbb{Z})$. Therefore, it follows from pigeonhole that if $\sum_{p \in P_0} \lambda(p) < 1$, then there is a pair $(s, t)$ that is in none of these sets. Such a pair is 2-coprime. Thus, it suffices to show $\sum_{p \in P_0} \lambda(p) < 1$. In order to estimate $\lambda(p)$, we divide into cases.

- **When $p \in \{p_1, \ldots, p_k\}$**: By definition, $T_k$ has already excluded multiples of such $p$, so $\lambda(p) = 0$.
- **When $p \in P_k, p \leq M$**: By (8) and (11),

\begin{align*}
\frac{|S_k \cap p\mathbb{Z}|}{|S_k|} < 2 \frac{|I_k \cap p\mathbb{Z}|}{|I_k|} < \left(1 + \frac{1}{5 - 5/M}\right) \frac{2}{p} < \frac{5}{2p}.
\end{align*}

(16)

By (13) and (15),

$$\frac{|T_k \cap p\mathbb{Z}|}{|T_k|} \leq \frac{|I_k \cap p\mathbb{Z}|}{|T_k|} < \frac{1.2\Phi/p}{3\Phi/4} = \frac{8}{5p}.$$  

Hence, $\lambda(p) < \frac{4}{p^2}$.

- **When $p \in P_k, p > M$**: In this case, we do not have equally good individual bounds. Here we use the following simple observation:

\begin{align*}
\sum_{p \in P_k, p > M} |S_k \cap p\mathbb{Z}| < \log_M(n) |S_k|.
\end{align*}

(17)

(This follows from the fact that every number in $S_k$ is at most $n$, and so is divisible by less than $\log_M(n)$ primes greater than $M$.) Also by (4) and (14),

$$\frac{|T_k \cap p\mathbb{Z}|}{|T_k|} \leq \frac{|I_k \cap p\mathbb{Z}|}{|T_k|} < \frac{|J_k|/p + 1}{|T_k|} < \frac{4}{3p} + \frac{1}{|T_k|}.$$  

By (15) and (9),

$$|T_k| > \frac{3\Phi}{4} = \frac{3m}{4\Gamma} \prod (p_i - 1) \geq \frac{3m}{4\Gamma} \geq \frac{15M}{4}.$$
Since \( p > M \),

\[
\frac{|T_k \cap p\mathbb{Z}|}{|T_k|} < \frac{4}{3M} + \frac{4}{15M} = \frac{8}{5M}.
\]

In view of (17),

\[
\sum_{p \in P_k, p > M} \lambda(p) < \log_M(n) \cdot \frac{8}{5M} = \frac{8 \log(n)}{5M \log(M)}.
\]

Summing up all cases,

\[
\sum_{p \in P_0} \lambda(p) < \sum_{p \in P_k, p < M} \frac{4}{p^2} + \sum_{p \in P_k, p > M} \lambda(p)
\]

\[
< 4 \left( P(2) - \frac{1}{4} \right) + \frac{8 \log(n)}{5M \log(M)}
\]

\[
< 4 \left( P(2) - \frac{1}{4} \right) + \frac{8}{5 \log \log(n)} \quad \text{(as} \ M > \log(n)\text{)}
\]

\[
< 1,
\]

for \( n \) sufficiently large, where \( P(2) = \sum_{\text{prime}} p^{-2} \) is the prime zeta function. (Note that \( \log \log(n) \geq 9 \) suffices.) As this sum is less than 1, we conclude that some \( s \in S_k \subseteq S \) and \( t \in T_k \subseteq T \) are 2-coprime. \( \square \)

**Proof of Lemma 2.6.** Let \( \alpha = 1/r = |S|/m \in [1/16, 1/2] \), \( P_0 = \text{Primes} \setminus \{2\} \) and

\[
M = m^{1/3} > \log(n).
\]

Let \( P_M = P_0 \cap [M] \). For each \( p \in P_0 \), define \( \alpha_p = |S \cap p\mathbb{Z}|/m \leq \alpha \).

We begin with some estimates. Note that we have

\[
|T| \geq m - |S| = (1 - \alpha)m.
\]

For every \( p, q \in P_M \) (\( p \neq q \)), \( m/pq \geq m/M^2 = M \). By (5), we have

\[
|I \cap pq\mathbb{Z}| = (1 \pm 1/M) \frac{m}{pq} \quad \text{for all} \ p \neq q \in P_M,
\]

\[
|J \setminus p\mathbb{Z}) \cap q\mathbb{Z}| = (1 \pm 1/M) \frac{(p-1)m}{pq} \quad \text{for all} \ p \neq q \in P_M, \text{ and}
\]

\[
|I \cap p\mathbb{Z}|, |J \cap p\mathbb{Z}| \in (1 \pm 1/M) \frac{m}{p} \quad \text{for all} \ p \in P_M.
\]
A consequence of (22) and (19) is

\[ \forall p \in P_M, \ |T \setminus p\mathbb{Z}| \geq |T| - |J \cap p\mathbb{Z}| \geq (1 - \alpha - (1 + 1/M)/p)m. \quad (23) \]

As in Lemma 2.5, we have

\[ \sum_{p \in P_0 \setminus P_M} |S \cap p\mathbb{Z}| < |S| \log_M(n); \quad (24) \]

\[ \sum_{q \in P_0 \setminus P_M} |(S \cap p\mathbb{Z}) \cap q\mathbb{Z}| < |S \cap p\mathbb{Z}| \log_M(n) \text{ for all } p \in P_0 \text{ such that } \alpha_p > 0. \quad (25) \]

For \( p \in P_M \) such that \( \alpha_p > 0 \), by (4) and (23), we have

\[ \forall q \in P_0 \setminus P_M, \ \frac{|(T \setminus p\mathbb{Z}) \cap q\mathbb{Z}|}{|T \setminus p\mathbb{Z}|} \leq \frac{|J \cap q\mathbb{Z}|}{(1 - \alpha - (1 + 1/M)/p)m} \leq \frac{m/q + 1}{(1/6 - 1/(Mp))m} = \frac{6 \ M/q + 1/M^2}{M \ 1 - 6/Mp} \]

(assuming \( M \geq 15, \)) \[ < \frac{7}{M}. \quad (26) \]

(Remark: Note that it is crucial that we excluded 2 from the primes.) With these estimates in hand, we consider two cases.

**Case (a):** There exists some \( p \in P_M \) such that

\[ \alpha_p > \frac{0.24}{p(1 - 3\alpha/2)}. \quad (27) \]

Fix such \( p \in P_M \), and for \( q \in P_0 \) define

\[ \lambda_1(q) = \frac{|(S \cap p\mathbb{Z}) \cap q\mathbb{Z}|}{|S \cap p\mathbb{Z}|} \cdot \frac{|(T \setminus p\mathbb{Z}) \cap q\mathbb{Z}|}{|T \setminus p\mathbb{Z}|}. \]

In view of (23), \( T \setminus p\mathbb{Z} \) is nonempty, and so we will have the desired 2-coprime pair if \( \sum_{q \in P_0} \lambda_1(q) < 1 \). Note that when \( q = p \), \( \lambda_1(q) = 0 \). Thus, by (26), (25), (22), and (23), we have

\[ \sum_{q \in P_0} \lambda_1(q) = \sum_{q \in P_0 \setminus p\mathbb{Z}} \lambda_1(q) + \sum_{q \in P_0 \setminus P_M} \lambda_1(q) \leq \sum_{q \in P_0 \setminus p\mathbb{Z}} \frac{|I \cap pq\mathbb{Z}|}{|S \cap p\mathbb{Z}|} \cdot \frac{|J \setminus p\mathbb{Z}| \cap q\mathbb{Z}|}{|T \setminus p\mathbb{Z}|} + \left( \sum_{q \in P_0 \setminus P_M} \frac{|S \cap p\mathbb{Z}) \cap q\mathbb{Z}|}{|S \cap p\mathbb{Z}|} \right) \frac{7}{M} \]
\[
\sum_{q \in P \setminus \{p\}} \frac{(1 + 1/M)m/pq}{\alpha_km} \frac{(1 + 1/M)(p - 1)m/pq}{(1 - 1/M)(1 - \alpha - 1/p)m} + \log_M(n) \cdot \frac{7}{M} = \frac{(1 + 1/M)^2}{1 - 1/M} \frac{(p - 1)}{\alpha_p(p - \alpha_p - 1)} \left( \sum_{q \in P \setminus \{p\}} \frac{1}{q^2} \right) + \frac{7\log_M(n)}{M}.
\]

By (27),
\[
\sum_{q \in P_0} \lambda_1(q) < \frac{(1 + 1/M)^2}{1 - 1/M} \frac{(P(2) - 1/4)}{0.24} \frac{(p - 1)(1 - 3\alpha/2)}{(p - \alpha_p - 1)} + \frac{7\log_M(n)}{M} < 1
\]
for \( n \) sufficiently large (log \log(n) \geq 45 suffices), and we have the desired 2-coprime pair.

**Case (b):** For all \( p \in P_M \),
\[
\alpha_p \leq \frac{0.24}{p(1 - 3\alpha/2)}.
\]

For every \( p \in P_0 \), consider the quantity
\[
\lambda_0(p) = \frac{|S \cap p\mathbb{Z}|}{|S|} \frac{|T \cap p\mathbb{Z}|}{|T|}.
\]

We show that the sum of these terms is less than 1. We bound this term in two cases.

- **\( p \in P_M \).** In this case, by (19) and (22),
  \[
  \lambda_0(p) \leq \frac{\alpha_p}{\alpha} \frac{|J \cap p\mathbb{Z}|}{(1 - \alpha)m} < \frac{\alpha_p}{\alpha} \frac{(1 + 1/M)m/p}{(1 - \alpha)m} = \frac{(1 + 1/M)\alpha_p}{\alpha(1 - \alpha)p}.
  \]

- **\( p \in P_0 \setminus P_M \).** By (4), (assuming \( M \geq 15 \)),
  \[
  \frac{|T \cap p\mathbb{Z}|}{|T|} < \frac{m/p + 1}{(1 - \alpha)m} \leq \frac{2}{M} \left( \frac{M}{p} + \frac{1}{M^2} \right) < \frac{2.01}{M}.
  \]

Therefore, by (32), (33), (24) and (31) (noting that \( \alpha_p \leq \alpha \) and assuming \( M \geq 16 \)), we have
\[
\sum_{p \in P_0} \lambda_0(p) < \sum_{p \in P_M} \frac{17\alpha_p}{16\alpha(1 - \alpha)p} + \left( \sum_{p \in P_0 \setminus P_M} \frac{|S \cap p\mathbb{Z}|}{|S|} \right) \frac{2.01}{M} < \sum_{p \in P_M} \frac{(1 + 1/M)\alpha_p}{\alpha(1 - \alpha)p} + \log_M(n) \cdot \frac{2.01}{M} \leq (1 + 1/M) \frac{0.24}{\alpha(1 - \alpha)(1 - 3\alpha/2)} \left( \sum_{p \in P_M} \frac{1}{p^2} \right) + \frac{2.01}{\log \log(n)}.
\]

(34)
Define function $f : \alpha \mapsto \alpha(1-\alpha)(1-\frac{3\alpha}{2})$, and observe that $f$ is concave on the interval $(0, 5/9)$. It follows that $f$ takes its minimum value in the interval $[1/16, 1/2]$ at one of the end-points. As $f(1/2) = 1/16$ and $f(1/16) = 435/2^{13} > 1/19$, we have

$$\sum_{p \in P_0} \lambda_0(p) < (1 + 1/M)0.24 \cdot 19(P(2) - 0.25) + \frac{2.01}{\log \log(n)} < 1,$$

for $n$ sufficiently large. (Here $\log \log(n) \geq 28$ suffices.) □

Remark 2.9. The explicit conditions on $n$ that are sufficient for the proofs of Lemmas 2.5 and 2.6 play a role when we apply these Lemmas in §4. Note that we can establish better bounds by writing some of the conditions in terms of both $n$ and the parameter $M$ (instead of simply applying the bound $M > \log(n)$). Indeed, the conditions

$$M = \left(\frac{m}{5}\right)^{1/\log_2(2r)} > \log n \quad \text{and} \quad \frac{\log(n)}{M \log(M)} < \frac{1}{9}$$

are sufficient for Lemma 2.5, and the conditions

$$M = m^{1/3} \geq 16 \quad \text{and} \quad \frac{\log(n)}{M \log(M)} < \frac{1}{45}$$

are sufficient for Lemma 2.6.

Proof of Theorem 2.1. The four outcomes are clearly pairwise disjoint. We will show that at least one of them happens.

Let $I_1 = I \setminus 2Z$, $I_2 = I \cap 2Z$, $J_1 = J \setminus 2Z$, $J_2 = J \cap 2Z$. They are all integer APs with cardinality $m$ and common difference 2. For $i = 1, 2$, define $S_i = S \cap I_i$, $T_i = T \cap J_i$.

Since $|S_1| + |S_2| + |T_1| + |T_2| \geq 2m$, at least one of the following happens.

- Case I: $|S_1| + |T_2| > m$ (which implies $S_1, T_2 \neq \emptyset$).
- Case II: $|S_2| + |T_1| > m$ (which implies $S_2, T_1 \neq \emptyset$).
- Case III: $|S_1| + |T_2| = |S_2| + |T_1| = m$.

If $|S_1| + |T_2| \geq m$ and $S_1, T_2 \neq \emptyset$, then by Lemma 2.4, there are $s \in S_1$, $t \in T_2$ that are 2-coprime. Because $s$ is odd, that means they are coprime, so the last outcome applies. Analogously, a coprime pair exists if $|S_2| + |T_1| \geq m$ and $S_2, T_1 \neq \emptyset$. To not fall into the last outcome, we must be in Case III, with at least one of $S_1, T_2$ empty and at least one of $S_2, T_1$ empty.

- If $S_1, S_2 = \emptyset$, we are in Outcome 1.
- If $T_2, T_1 = \emptyset$, we are in Outcome 2.
- If $S_1, T_1 = \emptyset$, we are in Outcome 3.
- If $T_2, S_2 = \emptyset$, then $|S_1| = |T_1| = m$. By Lemma 2.4 there are $s \in S_1$, $t \in T_1$ that are 2-coprime.

Because both $s$ and $t$ are odd, they are coprime, so we are in Outcome 4.

In conclusion, one of the four outcomes must happen, so we are done. □
In this section, we prove Theorem 1.3.

Let $C_0$ be the constant in Theorem 2.1. Set $C$ as a constant for which the following holds for sufficiently large $n$:

$$\exp(C \cdot (\log \log n)^2) \geq 8 \left[ \exp(C_0 \cdot (\log \log (2n))^2) \right] + 6.$$ 

For convenience, let

$$k = k(n) = \frac{\exp(C \cdot (\log \log n)^2)}{2}, \quad M = M(n) = \left\lceil \exp(C_0 \cdot (\log \log (2n))^2) \right\rceil + 6.$$ 

The assumption then becomes $k \geq 4M + 3$. We will show that for sufficiently large $n$, if positive integers $v_1 < v_2 < \cdots < v_n$ satisfy that $n < v_n \leq 2n - 2k$, then Condition (2) holds.

Consider such $v_1, \ldots, v_n$. Denote $V = \{v_1, \ldots, v_n\}$. Since $v_n > n$, in particular $V \neq [n]$, so there exists a “largest missing number in $[n]$”

$$x = \max ([n] \setminus V).$$

We first claim that if $x > n - k$, then Condition (2) holds. Note that $x \neq v_i$ for all $i$, and larger multiples of $x$ are $> 2n - 2k$, which is too large to be one of $v_1, \ldots, v_n$. Letting $t = 1/x$, the quantity $\{v_it\} = \{v_i/x\}$ cannot be zero; moreover, its denominator is at most $x \leq n$. Thus, $1/(n+1) < \{v_it\} < n/(n+1)$ for all $i$.

Hence, we may assume $x \leq n - k$. Next, we claim that it is possible to cut $[n]$ into some groups of consecutive integers in ascending order, such that all but the last group has cardinality either $2M$ or $2M + 2$, the last group’s cardinality is between $2M + 2$ and $4M + 3$, and $x, x + 1$ and $x + 2$ belong to the same group. One can start with $\{1, \ldots, 2M + 2\}, \{2M + 3, \ldots, 4M + 4\}$, and continue to append groups of size $2M + 2$ by default; but if a new group would cut $x, x + 1$ and $x + 2$ apart, shrink its size by 2 to avoid the separation. There will likely be a residue of size $< 2M + 2$ at the end of $[n]$, in which case merge it with the previous group. Given that the distance between $x$ and $n$ is at least $k \geq 4M + 3$, the resulting last group would not contain $x$, hence unaffected by the potential shrinking.

Denote the groups $I_1, I_2, \ldots, I_r, I_{r+1}$ in ascending order, and say $x, x + 1, x + 2 \in I_r$ for some $r \in [r]$. Now we will also partition $\{n + 1, \ldots, 2n\}$ into groups. Let $J_r$ be the interval starting at $n + 1$ with the same cardinality as $I_r$. Let $J_{r-1}$ be the next contiguous interval, with the same cardinality as $I_{r-1}$. (That is, $J_{r-1} = [n + |I_r| + 1, n + |I_r| + 2, \ldots, n + |I_r| + |I_{r-1}|].$) Define $J_{r-2}, \ldots, J_1$ analogously. The remaining numbers would form the interval $J_{r+1}$, whose cardinality is equal to that of $I_{r+1}$.
Note inductively that for all \(1 \leq j \leq \ell\), \(\min(I_j) + \max(J_j) = 2n + 1 - |I_{\ell+1}|\). Thus,

\[
\min(I_j) + \min(J_j) = 2n + 1 - (4M + 3) - (2M + 1) \\
= 2n - 6M - 3 \\
> 2n - 2k,
\]

\[
\max(I_j) + \max(J_j) = 2n + 1 - |I_{\ell+1}| + (|J_j| - 1) \\
\leq 2n + 1 - (2M + 2) + (2M + 1) \\
= 2n.
\]

For all \(1 \leq j \leq \ell + 1\), let \(S_j = I_j \setminus V\) and \(T_j = J_j \setminus V\). We claim that if for some \(j \in [\ell]\) there exist \(s \in S_j\) and \(t \in T_j\) coprime, then Condition (2) holds. By the above calculation, \(2n - 2k < s + t\) is at least \(2/(s + t) > 2/(2n)\) away from the nearest integer. Hence \(1/(n + 1) < \{v_i T\} < n/(n + 1)\).

Define \(\alpha_j = |S_j| + |T_j|\) and \(m_j = |I_j|/2 = |J_j|/2\) for all \(1 \leq j \leq \ell + 1\). By the grouping conditions, for all \(j \leq \ell\) we have \(m_j \in \mathbb{Z}\) and \(m_j \geq M\). Note that \(\sum_{j=1}^{\ell+1} \alpha_j = n = \sum_{j=1}^{\ell+1} 2m_j\). Recall that \(I_{\ell+1} \subseteq V\) because \(x \leq n - k \leq n - |I_{\ell+1}|\), and \(J_{\ell+1} \cap V = \emptyset\) because \(v_n \leq 2n - 2k \leq 2n - |I_{\ell+1}|\). Hence, \(\alpha_{\ell+1} = 0 + |J_{\ell+1}| = 2m_{\ell+1}\). By pigeonhole, either \(\alpha_j > 2m_j\) for some \(j \in [\ell]\), or \(\alpha_j = 2m_j\) for all \(j \in [\ell]\). In the former case, according to Theorem 2.1 applied on \((2n, m_j, I_j, J_j, S_j, T_j)\), there must be a coprime pair between \(S_j\) and \(T_j\), witnessing Condition (2). In the second case, consider \(S_r\) and \(T_r\). We have \(|S_r| + |T_r| = \alpha_r = 2m_r\), so by Theorem 2.1, one of the four outcomes happens: \(S_r = \emptyset, T_r = \emptyset, (S_r = I_r \cap 2\mathbb{Z}\) and \(T_r = J_r \cap 2\mathbb{Z}\), or there is a coprime pair between \(S_r\) and \(T_r\). But as \(x \not\in V\), \(S_r \neq \emptyset\); as \(x + 1 \in V\), \(T_r \neq \emptyset\); as \(x + 1\) and \(x + 2\) have different parities and are both in \(V\), \(S_r\) misses a number of each parity. Thus, only the last outcome is possible, which also witnesses Condition (2).

## 4 COPRIME Mappings for Adjacent Intervals

In this section, we prove Theorem 1.6; that is, we show that if \(0 \leq \ell < k\) and \(k \geq 4\), then there is coprime mapping from \(A = \{\ell + 1, \ldots, \ell + k\}\) to \(B = \{\ell + k + 1, \ldots, \ell + 2k\}\).

Theorem 1.6 follows from Lemmas 2.5 and 2.6 when \(n\) is sufficiently large. We prove Theorem 1.6 for smaller values of \(n\) using a separate argument. We note in passing that there are parallels between this argument and the earlier work of Pomerance and Selfridge [12]. Both arguments make use of a statistical study of the parameter \(\phi(x)/x\), where \(\phi\) is the Euler totient function, and both proofs make use of estimates on the prime counting functions given by Rosser and Schoenfeld [14] (but our reliance on these estimates is significantly less extensive).

To handle smaller \(n\), we need some additional definitions and elementary observations. Let \(p_i\) be the \(i\)th odd prime, and set \(q_i = \prod_{j=1}^{i} p_i\). For each integer \(x\), we let \(P(x)\) be the set of odd prime
factors of $x$ and define

$$\gamma(x) = \prod_{p \in P(x)} \frac{p - 1}{p}.$$ 

Note that $\gamma(x)$ is the proportion of numbers that are $2$-coprime with $x$. (Note further that $\gamma(x) = \phi(x)/x$ when $x$ is odd.)

**Claim 4.1.** Let $J$ be an integer AP with cardinality $m$ and common difference $d$. Let $x$ be an integer such that $d$ is coprime with all elements of $P(x)$, then more than $\gamma(x)m - 2^{|P|} + 1$ numbers in $J$ are $2$-coprime with $x$.

**Proof.** For $p \in P$, let $J_p = J \cap p\mathbb{Z}$. By inclusion–exclusion,

$$\left| \bigcup_{p \in P} J_p \right| = \sum_{\emptyset \neq Q \subseteq P} (-1)^{|Q|+1} \left| \bigcap_{q \in Q} J_q \right|.$$ 

Because $P \cup \{d\}$ is mutually coprime, the cardinality of $\bigcap_{q \in Q} J_q$ can be approximated by

$$\left| \bigcap_{q \in Q} J_q \right| - \frac{m}{\prod_{q \in Q} q} \in (-1, 1).$$

Note that $m/\prod_{q \in Q} q$ is the “heuristic” cardinality as the size of $J$ gets large. By summing these terms for all nonempty subsets of $P$, the cardinality of $\bigcup_{p \in P} J_p$ can be approximated

$$\left(1 - \prod_{p \in P} \frac{p - 1}{p}\right)m = (1 - \gamma)m,$$

with an error less than $2^{|P|} - 1$. The claim follows by taking the complement. □

**Claim 4.2.** Let $J \subset [n]$ be an AP with common difference 1 or 2 such that $|J| = m \geq 2$. Let $T \subseteq J$ such that $|T| \geq (m + 1)/2$. If $x$ is an integer such that $|P(x)| \leq 1$, then there is $y \in T$ such that $x$ and $y$ are $2$-coprime.

**Proof.** Since $m \geq 2$, the set $T$ contains two consecutive elements or two elements out of three consecutive elements of $J$. In either case, at least one of these numbers is not divisible by the odd prime that divides $x$. □

With these preliminary observations in hand, we are ready to prove Theorem 1.6. Set $A_0 = A \cap 2\mathbb{Z}, A_1 = A \setminus A_0, B_0 = B \cap 2\mathbb{Z}, B_1 = B \setminus B_0$. Note that it suffices to find a 2-coprime mapping from $A_0$ to $B_1$ and another from $A_1$ to $B_0$. (As the intervals $A$ and $B$ are consecutive, we have $|A_0| = |B_1|$ and $|A_1| = |B_0|$.) Hall’s Theorem states that there is a 2-coprime mapping between $A_1$ and $B_0$ if and only if for every pair of sets $S \subseteq A_1$ and $T \subseteq B_0$ such that $|S| + |T| = |A_0| + 1 = |B_1| + 1$ there exists a 2-coprime pair $x, y$ such that $x \in S$ and $y \in T$. When $k$ is sufficiently large, we can apply
Lemmas 2.5 and 2.6 — with $n = 2k + \ell$ and $m = |A_0| = |B_1|$ or $m = |A_1| = |B_0|$ — to conclude that the desired 2-coprime pair exists, and hence the desired 2-coprime mappings exist.

Let $I$ and $J$ be disjoint APs in $[n]$ with common difference 2 such that $|I| = |J| = m > \frac{n-2}{6}$. (We take $\{I,J\} = \{A_0, B_1\}$ or $\{I,J\} = \{A_1, B_0\}$. Note that $\frac{n-2}{6} \leq \lfloor k/2 \rfloor$.) Let $S \subseteq I$ and $T \subseteq J$ such that $|S| + |T| = m + 1$ and $|S| \leq |T|$. As above we set $r = |S|/m$. We show that there is a 2-coprime pair $x, y$ such that $x \in S$ and $y \in T$. We consider two cases depending on the value of $n$. For large $n$, we appeal to Lemmas 2.5 and 2.6, and for small $n$ we provide a direct argument.

**Case 1:** $2k + \ell = n > 3 \cdot 10^7$.

First consider $2 \leq r \leq 16$. Here we apply Lemma 2.6. We clearly have $m > \log(n)^3$. As noted in Remark 2.9, the parameter $n$ is sufficiently large if the following two conditions hold:

$$m^{1/3} = M \geq 16 \quad \text{and} \quad \frac{\log(n)}{\frac{1}{3} \log(m) \cdot m^{1/3}} = \frac{\log M(n)}{M} < \frac{1}{45}.$$ 

Both conditions hold in the range in question, and Lemma 2.6 gives the desired 2-coprime pair.

Now consider $r > 16$. We first consider $3 \cdot 10^7 < n < 10^{50}$. Assume for the sake of contradiction that we do not have the desired 2-coprime pair. Then it follows from Claim 4.1 that for any $x \in S$ we have

$$15m/16 < m - |S| < |T| < m - m\gamma(x) + 2|P(x)|.$$

Observe that $|P(x)| \leq \log_3(n)$ for all integers $x$. Furthermore, as $q_{32} > 10^{50}$, the assumed restriction on $n$ implies $\gamma(x) > \gamma(q_{31}) > 1/5$. It follows that we have

$$1/5 < \gamma(x) < 1/16 + 2|P(x)|/m < 1/16 + 2\log_3(n)/((n - 2)/6) < 1/16 + 6 \cdot n^{\log_3 2 - 1}/(n - 2)/6 < 1/16 + 12/n < 1/5,$$

which is a contradiction.

It remains to consider $r > 16$ and $n > 10^{50}$. We apply Lemma 2.5 when $r < \log(n)^2$. In order to handle the large $r$ case, we first note that if $r > \log(n)^2$, then $T$ either contains primes $p_1, p_2 > k/4$ or contains $2p_1, 2p_2$ where $p_1, p_2 > k/8$ are primes. To see this, we appeal to bounds on the prime counting function $\pi(x)$ (see [14, Theorem 1]). First suppose $J$ consists of odd numbers and $a$ is the largest element of $J$. Then the number of primes in $J \cap [a - m + 1, a]$ is

$$\pi(a) - \pi(a - m) \geq \frac{a}{\log(a)} - \frac{a - m}{\log(a - m)} \left(1 + \frac{2}{\log(a - m)}\right).$$

For ease of notation, let $a - m = \eta a$, we have

$$\pi(a) - \pi(\eta a) \geq \frac{a}{\log(a)} - \frac{\eta a}{\log(\eta a)} \left(1 + \frac{2}{\log(\eta a)}\right).$$
\[ \frac{a - \eta a}{\log(a)} + \frac{\eta \log(\eta) \cdot a}{\log(\eta a) \log(a)} - \frac{2\eta a}{\log(\eta a)^2} \geq \frac{m}{\log(n)} - \frac{2n}{\log(n)^2} \geq \frac{2m}{\log(n)^2} > |S|. \]

Recalling that \(|S| + |T| \geq m + 1\), the number of elements of \(J\) that are NOT elements of \(T\) is at most \(|S| - 1\). Thus we have the two desired primes in \(T\). If \(J\) consists of even numbers, then we apply the same estimates to \(J/2 = \{x/2 : x \in J\}\) to get the desired elements \(2p_1, 2p_2\) where \(p_1\) and \(p_2\) are prime. In either case, no element of \(S\) is divisible by both \(p_1\) and \(p_2\), and we have the desired 2-coprime pair.

When \(16 < r < \log(n)^2\) and \(n > 10^{50}\) we apply Lemma 2.5. Recall that, as noted in Remark 2.9, \(n\) is sufficiently large if the following conditions hold:

\[ M = \left( \frac{m}{5} \right)^{1/\log_2(2r)} > \log n \quad \text{and} \quad \frac{\log(n)}{M \log(M)} < \frac{1}{9}. \]

These conditions hold here (indeed, we have \(M > 3\log(n)/2\)), and Lemma 2.5 applies.

**Case 2:** \(83 \leq 2k + \ell = n < 3 \cdot 10^7\).

Here we apply Claim 4.1. Note if \(x \in S\) and

\[ |T| \geq (1 - \gamma(x))m + 2^{|P(x)|} - 1, \quad (36) \]

then Claim 4.1 implies that the desired 2-coprime pair exists. As many elements of \([3 \cdot 10^7]\) have large values of \(\gamma(x)\), this observation is usually sufficient to complete the proof for \(n\) in this interval. In order to make the proof precise we consider cases. In some cases, we will need to make a more detailed study of the collection of sets \(\{P(x) : x \in S\}\).

**Case 2a:** There exists \(s \in S\) that is not divisible by 3.

As \(|T| \geq (m + 1)/2\), it suffices to show that there exists \(x \in S\) such that

\[ (\gamma(x) - 1/2)m \geq 2^{|P(x)|} - 3/2. \quad (37) \]

Set \(w_a = q_{a+1}/3 = \prod_{i=2}^{a+1} p_i\). Among numbers \(x\) such that \(3 \nmid x\) with a fixed value of \(a = |P(x)|\), \(x\) is minimized by \(w_a\) and the parameter \(\gamma(x)\) is minimized by \(\gamma(w_a)\). Set

\[ \chi_a = \frac{2^a - 3/2}{\gamma(w_a) - 1/2} \cdot 6 + 2 \]
and observe that if for a particular value of $a$ we have $n \geq \chi_a$ and there exists $s \in S \setminus 3\mathbb{Z}$ such that $|P(s)| = a$ then\footnote{We implicitly also need $\gamma(w_a) > 1/2$, which holds within our range of consideration.} we have (37) and Claim 4.1 implies that the desired 2-coprime pair exists. We refer to the following table for values of $w_a$ and $\chi_a$.

| $a$ | $w_a$ | $\gamma(w_a)$ | $\chi_a$ |
|-----|-------|----------------|----------|
| 1   | 5     | 0.8            | 12       |
| 2   | 35    | 0.6857         | 82.8     |
| 3   | 385   | 0.6234         | 318.1    |
| 4   | 5005  | 0.5754         | 1155.5   |
| 5   | 85,085| 0.5416         | 4403.6   |
| 6   | 1,616,615| 0.5131     | 28,689.1 |

Take any $s \in S \setminus 3\mathbb{Z}$ and let $a = |P(s)|$. As $w_7 > 3 \cdot 10^7$, we have $a \leq 6$. If $a \leq 2$, we have $n \geq 83 > \chi_a$ immediately. If $a \geq 3$, then we also have $n \geq s \geq w_a > \chi_a$. Thus, we always have the desired 2-coprime pair.

**Case 2b:** Every number in $S$ is divisible by 3.

As $|S| \leq |I \cap 3\mathbb{Z}| \leq (m + 2)/3$ and hence $|T| \geq (2m + 1)/3$, it suffices to show that there exists $x \in S$ such that

$$(\gamma(x) - 1/3)m \geq 2^{|P(x)|} - 4/3.$$ (38)

Proceeding as in the previous case, if there exists a value of $a$ such that

$$n \geq \kappa_a := \frac{2^a - 4/3}{\gamma(q_a) - 1/3} \cdot 6 + 2$$

and $x \in S$ such that $|P(x)| = a$, then Claim 4.1 implies that the desired 2-coprime pair exists. It follows from the following table that we have the desired condition for $a = 1, 2, 4, 5, 6, 7$.

| $a$ | $q_a$ | $\gamma(q_a)$ | $\kappa_a$ |
|-----|-------|----------------|------------|
| 1   | 3     | 0.6667         | 14         |
| 2   | 15    | 0.5333         | 82         |
| 3   | 105   | 0.4571         | 325.1      |
| 4   | 1155  | 0.4156         | 1071.9     |
| 5   | 15,015| 0.3846         | 3661.3     |
| 6   | 255,255| 0.3611     | 13,567.5   |
| 7   | 4,849,845| 0.3420   | 87,210.9   |

As $q_8 > 3 \cdot 10^7$, we do not need to consider larger values of $a$. It remains to consider the case where $|P(x)| = 3$ for all $x \in S$. Again, if $n \geq \kappa_3 = 325.1$, we will have the desired 2-coprime pair,
so assume $n \leq 325$. Note that

$$S \subseteq \{x \in [325] : |P(x)| = 3\} = \{105, 165, 195, 210, 231, 255, 273, 285, 315\}.$$ 

For (36) to not hold, we must have

$$m + 1 - |S| \leq |T| < (1 - \gamma(q_3))m + 2^3 - 1$$

$$\Rightarrow m < \frac{6 + |S|}{\gamma(q_3)}$$

$$\Rightarrow \max(S) \leq n < 6 \cdot \frac{6 + |S|}{\gamma(q_3)} + 2 < 14|S| + 81.$$ 

One can verify that this is not possible. Hence, (36) holds and we have the desired 2-coprime pair.

**Case 3:** $8 \leq 2k + \ell = n < 83$.

Here we make a more careful analysis of the collections of sets $\{P(x) : x \in S\}$. We recall Claim 4.2: If any of these sets has cardinality at most 1, then we have the desired 2-coprime pair. It follows that, as $n < 83 < q_3$, we may assume that $|P(x)| = 2$ for all $x \in S$, and so we can view $\{P(x) : x \in S\}$ as a graph $G$ on a vertex set consisting of the odd primes. Now we consider some further cases.

**Case 3a:** $G$ contains disjoint edges $p_1 p_2$ and $p_3 p_4$.

Note that in this case we have $n \geq 35$ and $m \geq 6$. Note that a number $y$ is 2-coprime with neither of the corresponding elements of $S$ if and only if $P(y)$ contains a set in $\{p_1, p_2\} \times \{p_3, p_4\}$. As 15 is the smallest such product, at most four elements out of any 15 consecutive elements of $J$ contains one of these products. It follows that we have the desired 2-coprime pair so long as $m \geq 8$. In the case $4 \leq m \leq 7$, we have $n \leq 44$ and there are at most four elements $z$ of $I \cup J$ such that $|P(z)| \geq 2$. Two of them are already in $S$, and $T$ has at least three elements, so some element $t \in T$ satisfies $|P(t)| \leq 1$, giving the desired 2-coprime pair. We finally note that if $m \leq 3$, then $n \leq 20$ and this case is not possible.

**Case 3b:** $G$ has a vertex of degree 2.

Suppose $G$ contains the edges $p_1 p_2$ and $p_1 p_3$. Note if $y$ is an element of $J$ such that $p_1 \not\in P(y)$ and $\{p_2, p_3\} \not\in P(y)$, then $y$ is 2-coprime with one of the corresponding elements of $S$.

First consider $p_1 > 3$. In this case, $n \geq 35$, which implies $m \geq 6$. Furthermore, if $p_1 > 3$, then at most 3 out of 10 consecutive elements of $J$ are not 2-coprime with the corresponding elements of $S$. It follows from these observations that we have the desired 2-coprime pair.

So, we may assume $p_1 = 3$. As $m \leq (k + 1)/2 \leq (n + 2)/4 < 22$, $J$ contains at most one element that is a multiple of $p_2 p_3$, and the number of elements in $J$ that are not
2-coprime to any of the corresponding elements of $S$ is at most $(m + 2)/3 + 1$. As $|T| \geq (m + 1)/2$, we have the desired 2-coprime pair if $m \geq 8$.

Finally, consider $m \leq 7$, which implies $n \leq 44$. Here we claim that $J$ does not contain any element that is a multiple of $p_2 p_3$. Assume for the sake of contradiction that $J$ has such an element. Given the bound on $n$, 35 is the only product of distinct odd primes in $I \cup J$ that does not include 3. So, we have $35 \in J$. Recalling that one of $I$ and $J$ consists of only even numbers, we observe that $35 \in J$ implies that $I$ consists of even numbers and $30, 42 \in S$. This is a contradiction as $A$ and $B$ are disjoint intervals. It follows that the number of elements of $J$ that are not 2-coprime to the corresponding elements of $S$ is at most $(m + 2)/3$, and we have the desired 2-coprime pair.

**Case 3c:** $G$ consists of a single edge $p_1 p_2$.

Here we observe that $|S| \leq 2$ and that at most three elements out of any five consecutive elements of $J$ is not 2-coprime with the corresponding elements of $S$. As $|T| \geq m + 1 - |S| \geq m - 1$, we have the desired 2-coprime pair when $m \geq 5$. But $m \leq 4$ implies $n \leq 26$, which implies $|S| \leq 1$ and $T = J$. As at least two out of any three consecutive elements of $J$ are 2-coprime with the corresponding elements of $S$, we have the desired 2-coprime pair when $m = 3, 4$. Finally, if $m \leq 2$ then we have $n \leq 14$ and this case is not possible.

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**JOURNAL INFORMATION**

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