Features of the Motion of Ultracold Atoms in Quasiperiodic Potentials

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Abstract—We consider here quasiperiodic potentials on the plane, which can serve as a “transitional link” between ordered (periodic) and chaotic (random) potentials. As can be shown, in almost any family of quasiperiodic potentials depending on a certain set of parameters, it is possible to distinguish a set (in the parameter space) where, according to a certain criterion, potentials with features of ordered potentials arise, and a set where we have potentials with features of random potentials. These sets complement each other in the complete parameter space, and each of them has its own specific structure. The difference between “ordered” and “chaotic” potentials will manifest itself, in particular, in the transport properties at different energies, which we consider here in relation to systems of ultracold atoms. It should be noted here also that the transport properties of particles in the considered potentials can be accompanied by the phenomena of “partial integrability” inherent in two-dimensional Hamiltonian systems.

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1. INTRODUCTION

In this paper, we consider applications of relatively recent results in the theory of quasiperiodic functions on the plane to the dynamics of ultracold atoms in quasiperiodic potentials. The general theory of quasiperiodic functions, the origins of which go back to the studies of H. Bohr and A.S. Besicovich (see, for example [1, 2]), is currently a classical area of mathematics and mathematical physics. In different papers, actually, one can find slightly different definitions of a quasiperiodic function. Here we will call a quasiperiodic function on the plane any function \( f(x, y) \), obtained by restricting a smooth \( d \)-periodic function \( F(z^1, ..., z^d) \) to a generic affine embedding \( \mathbb{R}^d \subset \mathbb{R}^2 \). In this case, the dimension \( d \) will be called the number of quasiperiods of the corresponding quasiperiodic function on the plane.

The questions we consider here are mainly related to the geometry of the regions defined by the relation

\[ f(x, y) \leq \epsilon_0 \] (1.1)

for different energy values \( \epsilon_0 \). The function \( f(x, y) \) will play the role of a potential in which the two-dimensional dynamics of a particle is considered. It is easy to see that if the dynamics is purely classical, then such regions play the role of “accessibility regions” in which a particle with energy \( \epsilon_0 \) may appear. As we will see below, the complexity of the motion of a particle, inherent in the general case of any Hamiltonian dynamics, can add up to the complexity of the geometry of the corresponding “regions of accessibility”, which may have certain “chaotic properties.”

In addition to the geometry of the “accessibility regions” for a particle with a given energy, we will also be interested in describing the changes in this geometry when changing the parameters that define the function \( f(x, y) \) in real systems. In particular, it may be of interest to control the dynamics of a gas of particles in quasiperiodic potentials by changing such parameters.

It is easy to see that the geometry of the domains (1.1) is in fact closely related to the geometry of the level lines

\[ f(x, y) = \epsilon_0 \] (1.2)

of the function \( f(x, y) \), bounding these domains for the corresponding values of \( \epsilon_0 \). We can also say that each of the regions (1.1) is the union of the level lines (1.2) for all \( \epsilon_0 \leq \epsilon_0 \).

The general problem of describing the geometry of the level lines of quasiperiodic functions on the plane was set by S.P. Novikov in the early 1980s (see [3]) and then was actively studied in his topological school [4–11]. In its original setting, it was focused on the study of the geometry of the level lines of functions with three quasiperiods, which is actually equivalent to
The geometry of the intersection lines of an arbitrary 3-periodic two-dimensional surface in \( \mathbb{R}^3 \) by planes of a given direction. In this setting, Novikov's problem is most directly related to the problem of describing the dynamics of electrons in metals with complex Fermi surfaces, which, in turn, plays an important role in describing galvanomagnetic phenomena in metals (see, for example [12–18]). We note here that the results obtained in the study of Novikov's problem in this case turn out to be very important for the description of galvanomagnetic phenomena in metals in the most general situation. In particular, it is possible to define nontrivial topological numbers observable in the conductivity of normal metals with complex Fermi surfaces in strong enough magnetic fields [19]. In addition, a complete classification of possible types of dynamics (trajectories) of electrons on the Fermi surface allowed to give also a complete description of various asymptotic behaviors of conductivity in metals with arbitrary Fermi surfaces in the limit of strong magnetic fields (see, for example [20–22]).

In general, Novikov's problem for an arbitrary number of quasiperiods is closely related to the theory of foliations and the theory of dynamical systems on manifolds. In particular, this connection plays an important role in the study of Novikov's problem in the case of four quasiperiods (see [23, 24]), which allows to obtain a number of fundamental results for this case.

In this paper, we try to use, in the most complete way, the results obtained so far in the study of Novikov's problem to describe the main features of the particle dynamics in the potentials associated with this problem. In addition, the analysis of the results obtained for Novikov's problem allows, in fact, to propose a criterion dividing all quasiperiodic potentials (with three or more quasiperiods) into two types, namely, potentials with topologically regular level lines and potentials with chaotic level lines. If we consider quasiperiodic potentials as a model of the transition between periodic and random potentials, then the potentials of the first type can be classified as potentials in which the dynamics retains some topological integrability, and potentials of the second type as potentials in which the dynamics is close to the dynamics in random potentials. In this sense, only potentials of the second type can represent models of random potentials.

In fact, quasiperiodic potentials arise quite often as members of a family (and this will be so in this work), i.e. we usually have a whole family of functions \( V(x, y, U) \), depending smoothly on a number of additional parameters \( U = (U^1, ..., U^n) \). In this situation, it is important that the subsets of the parameter space corresponding to potentials of different types have, as a rule, very specific structure. Namely, potentials of the first type, being stable with respect to small variations of parameters, form an open subset in the complete space of parameters. Moreover, this subset is a union of a (finite or countable) number of pairwise disjoint regions (stability zones) each of which is determined by its individual value of some topological invariant (a tuple of integers). On the contrary, potentials of the second type form a subset of fractal type complementary to the set described above.

It can be seen, therefore, that in order to create quasiperiodic potentials that are most alike to truly random potentials, it is necessary to find a rather complicated set in the parameter space defining the family of quasiperiodic potentials under consideration. It can also be noted here that the larger the number of quasiperiods the richer becomes the structure of this set.

We believe that the most convenient systems for the experimental study of dynamics in quasiperiodic potentials are systems of ultracold atoms in optical traps, where such potentials can be easily constructed by superposition of several standing waves. It must be said that, despite the special technique for creating such potentials, they, in fact, have all the features of Novikov's problem in the general setting, so such systems allow us to study all essential aspects of the problem under consideration.

As we have already said, here we study the dynamics of particles of different energies in the potentials described above. Thus, the presented results will be directly related to the description of transport phenomena in the limit of almost noninteracting atoms in traps with quasiperiodic potentials. This limit, in fact, arises quite often in the case of low concentration of (neutral) atoms trapped in a trap, as well as small radius of their interaction. In our case, we should naturally require that the mean free path of the atoms is greater than the typical length at which the global geometric features of their trajectories manifest themselves. What is also essential, as is known, two-dimensional Hamiltonian systems have, as a rule, very special dynamics (see [25, 26]), being integrable at sufficiently low energy levels and passing to chaotic regimes with increasing energy. In our case, we will be able to observe how this circumstance agrees with the geometry of the level lines of our potentials.

2. GENERAL ANALYTICAL RESULTS. REGULAR AND RANDOM POTENTIALS

As is well known, the most common method for creating external potentials for atoms in optical traps is a superposition of standing waves from additional laser sources (see, for example, [27–31]). In the leading approximation, such potentials usually have a finite number of harmonics, i.e., they can be represented as the sum of a finite number of sinusoidal potentials. Despite this circumstance, as we will see below, in the situations we are considering, such potentials already
have sufficient complexity, and to describe the atomic dynamics in them, it is necessary to describe the complete picture that arises in the study of the general Novikov’s problem.

It is easy to see that the problem under consideration is rapidly becoming more complicated with an increase in the number of quasiperiods. As we have already said, rigorous analytical results exist at the moment only for the case of three and four quasiperiods. For comparison, it is convenient here to briefly consider also the situations of “one” and “two” quasiperiods corresponding to periodic potentials depending on a single coordinate, and doubly periodic potentials on the plane, respectively.

In our situation, the case of “one” quasiperiod will correspond in reality to the presence of a very simple potential, which can often be written approximately in the form

\[ V(x, y) = V_0 \sin kx, \]

where \( T = 2\pi/k \) is the ordinary period of the potential along the \( x \) axis. The creation of potentials of this type is the simplest one from the technical point of view, and, certainly, such potentials are widely used in systems of cold atoms.

The level lines of potential (2.1), obviously, represent vertical straight lines, and the motion of atoms in such a potential occurs in straight vertical stripes at \( E < V_0 \). In fact, it is easy to see in this situation, that the motion of an atom is confined to a vertical strip provided

\[ E - p_y^2/2m < V_0, \]

and in this case, the atom performs periodic oscillations along the \( x \) axis and moves uniformly along the \( y \) axis. If the above condition is violated, the atom obviously moves along a periodic trajectory having non-zero mean inclination with respect to the \( y \) axis. It is easy to see that similar conditions can also be written for a uniformly moving coordinate system, which also allows to give a similar description of the motion of atoms in potentials of the form

\[ V(x, y, t) = V_0 \sin (k(x - ut)). \]

It can be seen here that at sufficiently low atomic energies and the value of the velocity \( u \), the moving potential carries out a complete “transportation” of the atomic gas along the \( x \) axis. In the general case, with a significant spread of energies, the moving potential allows only a partial “transportation” of the atomic gas.

The geometry of the level lines of a doubly periodic potential on the plane (with two independent periods \( e_1 \) and \( e_2 \)) also has a relatively simple description. As in the case of “one” quasiperiod, the values of the potential \( V(x, y) \) lie here in some closed interval \([V_{\text{min}}, V_{\text{max}}]\).

It is easy to see that, for generic potentials, the level lines of the potential are closed for values of \( E \) sufficiently close to \( V_{\text{min}} \) or \( V_{\text{max}} \). In the first case, however, closed level lines bound areas of lower potential values, while in the second case they bound areas of larger values. In the generic case, we have here two different values \( V_1, V_2 \):

\[ V_{\text{min}} < V_1 < V_2 < V_{\text{max}}, \]

such that for all fixed values of \( V(x, y) \) lying in the interval \((V_1, V_2)\) the corresponding levels contain open (non-closed) components. All open level components (lines) of the potential \( V(x, y) \) are, in this case, periodic curves having the same mean direction in the plane (for all levels in the interval \((V_1, V_2)\)). The average direction of the open level lines can be any integer direction, i.e. a direction given by a vector of the form

\[ l = n_1e_1 + n_2e_2 \]

with some integers \( n_1, n_2 \). Note that it is natural to demand the numbers \( n_1, n_2 \) to be relatively prime, given up to a common sign.

All open level lines for a given value of the potential can be divided into a finite number of families, such that all lines in one family pass into each other when shifted by some period of \( V(x, y) \). The number of such different families is always even (although it can vary within the interval \((V_1, V_2)\) for sufficiently complex potentials).

Thus, it can be seen that the motion of atoms in generic periodic potentials occurs in bounded regions at \( V_{\text{min}} < e_0 < V_1 \), in periodic stripes (as well as, possibly, isolated bounded regions) at \( V_1 < e_0 < V_2 \), in the plane with excluded bounded domains (and possibly isolated bounded areas) at \( V_2 < e_0 < V_{\text{max}} \) and in the whole plane at \( e_0 > V_{\text{max}} \). In the case of adiabatic shifts of the potential in the plane, the atoms in the bounded regions shift with them, while the atoms in the periodic stripes move along with the stripes only when the shift is in the direction perpendicular to the stripes. Here, in contrast to the case of “one” quasiperiod, however, the shift of the potential along the direction of the stripes also affects the motion of atoms, since the stripes now have a nontrivial shape.

For non-generic periodic potentials (for example, having elements of rotational symmetry), the values \( V_1 \) and \( V_2 \) may coincide. Such potentials do not have open level lines, but at the level \( V_1 = V_2 \) they have a singular net (Fig. 1), separating the areas of smaller values of \( V(x, y) \) from the areas of its larger values. Atoms move in bounded regions at \( V_{\text{min}} < e_0 < V_1 = V_2 \) and in the entire plane with excluded bounded regions (and, possibly, isolated bounded domains) at \( V_1 = V_2 < e_0 < V_{\text{max}} \).

Methods for creating periodic potentials in a plane naturally imply the presence of a certain finite number of parameters describing such potentials. For example, when making a potential by using a superposition of two sinusoidal standing waves (with or without generation of higher harmonics), such parameters can be
the orientations of both sinusoids, their amplitudes, periods, positions of the maxima, possibly the angle between their polarizations and the relative phase shift. It is easy to see that in this case the shift of the maxima of any of the sinusoids actually leads to a shift of the resultant potential as a whole and does not qualitatively change the picture we are considering.

As for more general (continuous) variations of the parameters described above, one important common feature can be noted here. Namely, generic potentials form an open set, i.e. the relation $V_2 > V_1$ is stable with respect to an arbitrary small change in parameters. On the contrary, the condition $V_1 = V_2$ is unstable and can break down under an arbitrarily small variation of the parameters of the general form. In addition, the numbers $(n_1, n_2)$, relating the average directions of the open level lines to the periods of the potential, are also locally stable (although $e_1$ and $e_2$ can change with a change in parameters), and can change only when the potential passes through a non-generic situation ($V_1 = V_2$). Thus, it can be seen that the complete space of parameters can generally be divided into regions corresponding to different integer pairs $(n_1, n_2)$ and separated by boundaries on which the relation $V_1 = V_2$ holds.

Here we would also like to note that despite the relatively simple description of the “accessibility regions” in periodic potentials at any energy values, the conservative dynamics of cold atoms in such potentials can have very nontrivial properties (see, for example, [32–34]).

We now turn to considering potentials with larger numbers of quasiperiods, which are the main subject of our work.

The creation of quasiperiodic potentials in systems of ultracold atoms using superposition of standing waves has also attracted interest from both the theoretical and the experimental points of view. In particular, such potentials were considered both in the case of three-dimensional (see, for example, [35]) and in the case of two-dimensional (see [36–39]) optical lattices for atoms trapped in magneto-optical traps (see also a review on the creation, confinement and monitoring of the behavior of gases of ultracold atoms, including that in potentials of various shapes, in [40]). It can also be noted that quasiperiodic (quasicrystalline) structures in two-dimensional systems of interacting ultracold atoms can arise even in the absence of special external modulation (see, for example, [41]).

As we have said above, here we are interested in quasiperiodic potentials for two-dimensional systems of ultracold atoms. As we have also already noted, we will especially focus here on the cases of 3 and 4 quasi-periods, for which profound analytical results are known to date. In this section, we will simply formulate the rigorous analytical results obtained so far for such potentials. In the next section, we will consider in more detail the features of the geometry arising here, with specific examples.

As we have already said, Novikov’s problem for the case of three quasiperiods, which we consider here in the most detail, is currently most thoroughly studied. In the considered method of creating potentials in the plane, this situation corresponds to the potentials obtained by superimposing three sinusoidal standing waves oriented at different angles (with or without the presence of higher harmonics) (Fig. 2).

It should be noted here, of course, that a superposition of three (or more) standing waves in lattices of cold atoms can be used to create not only quasiperiodic, but also interesting periodic potentials in two-dimensional systems (in particular, such a scheme was proposed in the work [42] to create hexagonal (honeycomb) lattices and in [43] to create tri-hexagonal (Kagome) lattices). In this case, the wave numbers of the corresponding waves must, generally speaking, satisfy a number of special additional conditions. In our situation, we will assume that the wave numbers of the potentials we are considering will not have exact periods in the plane, which can arise only for special values of the parameters.

Fig. 1. Periodic net of singular level lines separating areas of the lower values from areas of the larger values of a non-generic periodic potential.

Fig. 2. Superposition of three standing waves in a plane with the formation of a potential with three quasi-periods (schematically). (The vectors $\eta_i$ indicate the directions of the wave fronts, and the vectors $a_j$ are the shifts between the maxima of their amplitudes.)
As we have already said, the results presented below will be based only on quasiperiodic properties of the potentials arising in our case, therefore, many additional details of their origin will not, in fact, play any essential role.

It is easy to see that a potential generated by three sinusoidal waves (and higher harmonics)

\[ V(\mathbf{r}) = \sum_{i=1}^{3} V_{i} \cos(k_{i0} \mathbf{r} + \delta_{i}) + \ldots, \]

represents a restriction of the periodic in \( \mathbb{R}^3 \) function

\[ V(X^1, X^2, X^3) = \sum_{i=1}^{3} V_{i} \cos X^i + \ldots \]

to an affine embedding \( \mathbb{R}^2 \to \mathbb{R}^3 \), given by the formulas

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\to
\begin{pmatrix}
  k_{(1)}^1 x + k_{(2)}^2 y + \delta_1 \\
  k_{(1)}^1 x + k_{(2)}^2 y + \delta_2 \\
  k_{(1)}^1 x + k_{(2)}^2 y + \delta_3
\end{pmatrix} =
\begin{pmatrix}
  k_{(1)}^1 \mathbf{r} + \delta_1 \\
  k_{(1)}^1 \mathbf{r} + \delta_2 \\
  k_{(1)}^1 \mathbf{r} + \delta_3
\end{pmatrix}. \tag{2.2}
\]

It can be noted here that the potentials differing only in shifts of the maxima of the standing waves (\( \delta_i \)) correspond to the same function \( V(X^1, X^2, X^3) \) in \( \mathbb{R}^3 \), and their difference is due only to a change in the affine embedding \( \mathbb{R}^2 \to \mathbb{R}^3 \), under which the plane \( \mathbb{R}^2 \) is shifted in \( \mathbb{R}^3 \) keeping its direction fixed.

As we have already said, even quasiperiodic potentials having a small number of harmonics are already sufficiently complex to observe all aspects of the general Novikov’s problem for three quasi-periods; for this reason, we will consider the problem described above on the grounds of the general results for functions with three quasi-periods. Below we will formulate, in the most convenient form for us, a number of fundamental results for the general Novikov’s problem following from [4, 5]. Similar statements for artificially created potentials in the plane, in fact, were given in [44], where electron transport phenomena in such potentials in the presence of a strong magnetic field were studied. Note, however, that in [44], the main role was played by the geometry of the level lines of quasiperiodic potentials, rather than by that of the regions of lower values, which we consider here.

It is convenient for us to start with the remark that, although this does not happen in the general case, the potentials obtained by the method considered here can also be periodic. This situation arises whenever the corresponding plane \( \mathbb{R}^2 \subset \mathbb{R}^3 \) is integral (rational), that is contains two independent integer vectors in \( \mathbb{R}^3 \).

It is easy to see that the corresponding potentials are everywhere dense among all potentials of our interest, while their periodic properties are determined by the values of the parameters \( (k_1, k_2, k_3) \). To describe the level lines of such potentials, all previously made statements about periodic potentials can be used. However, it should be noted here right away that the overwhelming majority of these potentials will have very large (in absolute value) periods \( e_1 \) and \( e_2 \). As a consequence, the above-described properties of the level lines of such potentials will be observed only on very large scales, while on smaller scales their level lines may have completely different nontrivial properties that are more important for describing the observed experimental data. As a result, for potentials with large periods, as a rule, more informative will be more general statements about the level lines of quasiperiodic potentials, which we cite below.

At the same time, it turns out that the role of the everywhere dense set of periodic potentials in the parameter space is, in fact, extremely important in the study of Novikov’s problem for the quasiperiodic potentials we are considering. Below we will formulate an extremely important statement concerning small deformations of periodic potentials and following from the results [4, 5]. Note here that in [4, 5] it is actually assumed that certain regularity conditions (of general position) are fulfilled, which we do not formulate here in detail, assuming that they are always satisfied for arising in reality physical potentials. In this case, in the simplest form, the corollaries we need from [4, 5] can be formulated as follows:

Let some complete set of parameters

\[ U_0 = (k_{(1)}^0, k_{(2)}^0, k_{(3)}^0, V_0^1, V_0^2, V_0^3, \ldots) \]

correspond to some periodic potential in the plane. Then there is an open neighborhood \( \Omega \) of the point \( U_0 \)

1 One of the main conditions imposed in [4, 5] on the function \( V(X^1, X^2, X^3) \) and the rational direction of embedding \( \mathbb{R}^2 \to \mathbb{R}^3 \), is the absence in all planes of this direction of singular level lines connecting two different singular points of the potential, at least at one of the energy levels in the interval of the existence of open level lines. For Morse functions \( V(X^1, X^2, X^3) \) and rational directions of embedding \( \mathbb{R}^2 \to \mathbb{R}^3 \) in general position, this condition is satisfied. In some physical examples, this condition may actually be violated due to the imposition of certain additional special symmetries. As we have already said, we do not assume here the special presence of such additional symmetries on the considered families of potentials. In this case, the violation of this condition an our only for some special rational directions of embedding, which, in reality, does not change the described picture in the corresponding family of quasi periodic potentials.
corresponding to some values 
up to the sign) the triple ( 
Note here that, for embeddings (2.2) of maximal irra-
In particular, such potentials 
associated with the described transformations are, in a 
that number of positions \( N \), by which the line is shifted, is 
the same for all open lines on this level, and is deter-
only by the selected transformation.

The triplet of numbers \( (N^1, N^2, N^3) \), determined at 
three successive shifts of the first, second and third 
waves, respectively, can be represented as 
\[ (N^1, N^2, N^3) = M(m^1, m^2, m^3), \]
where \( M \in \mathbb{Z} \) and \( (m^1, m^2, m^3) \) is an irreducible integer 
triple. The number \( M \) is always even and also has a 
topological origin (the number of “equivalence 
classes” of open level lines in the plane, and the triple 
\( (m^1, m^2, m^3) \) coincides with the one introduced earlier 
up to a sign.

Thus, we can see that in the space of our param-
ters we can distinguish a set of “stability zones” \( \Omega_{m,M} \), 
parameterized by integer triples \( m = (m^1, m^2, m^3) \) (and 
also by numbers \( M \)). The numbers \( (m^1, m^2, m^3) \), 
generally speaking, do not run through the entire set of inte-
ger (relatively prime) triples, but their number is 
generally infinite, and they can take arbitrarily large 
values. Each “stability zone” represents some open 
region in the parameter space with a piecewise smooth 
boundary (the boundaries of different zones can be 
adjacent to each other). The set of “stability zones” is 
a rather rich structure in the parameter space, in par-
cular, the “stability zones” contain all the values of 
parameters corresponding to the emergence of peri-
odic potentials \( V(x, y, U) \) (see footnote 1).

It can be seen that the above description of open 
level lines of potentials arising in the zones \( \Omega_{m,M} \), is 
rather simple and very informative (especially in the 
case of small values of \( (m^1, m^2, m^3) \)). In particular, it 
gives much more information about the behavior of 
level lines of arising in \( \Omega_{m,M} \) periodic potentials with 
large periods than can be obtained from the fact of 
their periodicity. As the values of \( (m^1, m^2, m^3) \) grow, 
the sizes of the stability zones \( \Omega_{m,M} \) decrease, and this 
also makes the strips containing open level lines (and 
the corresponding areas of lower values) wider. Along 
with this, the above description also refers to larger 
and larger scales in the plane, giving less detailed 
information about the geometry of the level lines at 
smaller scales (Fig. 5). As we will see below, in this sit-
uation, the behavior of the level lines (on small scales) 
can no longer be described in such a simple way and 

\[ (m^1k_1 + m^2k_2 + m^3k_3, l(U)) = 0. \] (2.3)

Note here that, for embeddings (2.2) of maximal irra-
tionality degree, relation (2.3) uniquely determines 
(up to the sign) the triple \( (m^1, m^2, m^3) \).

The triples \( (m^1, m^2, m^3) \) have, actually, a topologi-
cal origin and can also be introduced in another way. 
Namely, we recall that among the parameters of our 
potentials, there are the positions of the maxima of the 
standing waves used to create the potential. In the pic-
ture we are now considering, the shift of an individ-
ual standing wave is no longer equivalent to a simple shift 
of the resulting potential and represents a somewhat 
more complex transformation. It is also easy to see that 
the shift of the front of a standing wave (perpendicular 
to itself) by the period of this wave is equivalent to the 
identical transformation. In general, the full set of all 
such transformations forms a three-parameter group 
\( (T^3) \) containing simple shifts as an algebraic subgroup. 
At the same time, simple shifts form an everywhere 
dense set in the considered group of transformations 
for generic potentials; therefore, all such potentials 
associated with the described transformations are, in a 
sense, mutually related. In particular, such potentials 
have similar level lines for any energy value \( \epsilon_0 \).

If we now consider a generic potential \( V(x, y, U) \) 
corresponding to some values \( U \in \Omega \), and fix a level \( \epsilon_0 \),

Fig. 4. The shift of regular open level lines of a quasiperiodic 
potential under a shift of the maxima of one of the standing 
waves by a full period in the direction of the phase growth.
has much more complex (chaotic) properties. As we will also see, in limiting cases such behavior can lead to completely chaotic behavior of the level lines of quasiperiodic potentials, which has complex chaotic properties at all scales.

Based on the properties of the level lines of quasiperiodic potentials in stability zones, it can be seen here that the corresponding regions $V(x, y, U) < e_0$ should also possess the same properties if the level lines $V(x, y, U) = e_0$ are open (Fig. 6).

Namely, if the level $e_0$ contains open level lines, then any open connected domain $V(x, y, U) < e_0$ also lies in a straight strip of finite width and passes through it. The mean direction of such a strip, obviously, coincides with the mean direction of the open level lines and is given by Eq. (2.3). We note here that now both the level lines and the regions of lower values of the potential are no longer periodic for the general values of the parameters. As can be seen, however, properties (A1), (A2) give here a certain analogy with the case of periodic potentials.

It is easy to see that transport phenomena in a potential with parameters lying in one of the stability zones can have an explicit anisotropy. This property should, as a rule, be observed if the ensemble of particles placed in such a potential contains particles with energies corresponding to the emergence of open level lines of the potential. It can be seen, in fact, that such anisotropy can also be observed under more general assumptions, in particular, in systems of strongly interacting particles or in the hydrodynamic approximation.

Here, however, we must immediately make an important remark about transport phenomena in the case we are considering. Namely, in stability zones with large values of the numbers $(m^1, m^2, m^3)$ the width of the strips containing the domains $V(x, y, U) < e_0$ (in the presence of open level lines at $V(x, y, U) = e_0$) becomes rather large, and the shape of such regions becomes more and more complex, which is (very schematically) shown at Fig. 6. As a consequence of this, the motion of particles in such regions becomes more and more complicated, gradually acquiring the features of wandering in a random potential.

We also note here that the above-described movement of the open level lines (and the “regions of accessibility” restricted by them) under the shifts of the maxima of each of the standing waves depends in the most significant way on the values $(m^1, m^2, m^3)$. In particular, the speed of movement of the “accessibility regions” at large values of $(m^1, m^2, m^3)$ can significantly exceed the speed of the maxima of standing waves. The latter circumstance can, in fact, play an essential role for many questions of the transport of atoms in optical lattices (see, for example, [45] and the references therein). Moreover, such a movement, generally speaking, produces an incomplete transportation of atoms in the corresponding direction due to separation of some closed areas from the “accessibility areas” and joining of others to them in the process of their movement.

We also point out here one more important circumstance. Namely, as we have already said, the presence of a stability zone in the space of parameters means, naturally, the preservation of the picture determined by it under sufficiently small variations of the parameters. In fact, as follows from the topological considerations, this picture is also stable under much more general variations of the potential, in particular, those arising on finite scales provided that they are small enough. Thus, it can be seen that the above description of the geometry of the regions $V(x, y, U) < e_0$, as well as the features of transport phenomena, is stable under disturbances or “defects” of a sufficiently small magnitude. This circumstance is also important in the presence of additional (non-quasiperiodic) slowly varying poten-

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**Fig. 5.** Complication of the geometry of “topologically regular” open level lines of a quasiperiodic potential with increasing values of $(m^1, m^2, m^3)$ (schematically).

**Fig. 6.** The region of smaller values of a quasiperiodic potential, lying in a straight strip of finite width and passing through it (schematically).
tials, which are also often present in experimental techniques. The above description (with the same numbers \((m^1, m^2, m^3)\)) is preserved in general in regions where the maximum change of such potentials does not exceed the energy interval of the existence of open level lines of the potential \(V(x, y, U)\). It should be said here that the estimate of the admissible variations of the potential \(V(x, y, U)\) also rapidly decreases with increase in numbers \((m^1, m^2, m^3)\).

Despite the fact that the set of all stability zones \(\Omega_{m, M}\) forms an open cover of an everywhere dense set in the parameter space, it, generally speaking, does not cover it entirely, and quasiperiodic potentials with three quasi-periods can have, as we said, level lines that are more complex than those described above \([6, 10]\). It can be said, nevertheless, that the “regular” situation described above is, in a sense, basic for the case of three quasi-periods, while the more complex behavior of the level lines requires a special construction of the corresponding potential. For a more complete description of the picture arising in the general case, we present here, based on the results of \([7, 11]\), a number of important statements about the structure of the level lines of potentials with three quasiperiods of the most general form. In fact, together with the above statements, the results presented below represent, in a certain sense, a complete theory of the level lines of potentials with three quasiperiods in the plane.

We note here at once that all the potentials obtained from 3-periodic functions by means of an embedding \(\mathbb{R}^2 \subset \mathbb{R}^3\) can in fact be divided into 3 types. Namely, first of all, as we have already seen, for certain values of \((k^1_1, k^2_1, k^3_1)\) the corresponding potential \(V(x, y, U)\) may actually turn out to be doubly periodic. Here we call such potentials type I potentials. The second possibility is that \(V(x, y, U)\), while not being doubly periodic, still has one (up to a factor) period in the plane \(\mathbb{R}^2\). Such potentials will be called type II potentials. Like potentials of type I, they arise on an everywhere dense set of zero measure in the space of parameters \((k^1_1, k^2_2, k^3_3)\). Finally, into the family of potentials of type III we include potentials that do not have exact periods in the plane \(\mathbb{R}^2\). In our case, only potentials of type III are in general position and correspond to a set of the full measure in the parameter space. In the statements about the level lines of quasiperiodic potentials formulated below, we will assume that the corresponding potentials are of type II or III, since potentials of type I have already been actually considered above. Under the assumptions made, the following statements can be formulated.

Let \(V(x, y, U)\) be a potential with three quasi-periods taking values in the interval \([V_{\min}(U), V_{\max}(U)]\). Then:

(B1) Open level lines of \(V(x, y, U)\) exist in a connected interval \([V_1(U), V_2(U)]\),

\[ V_{\min}(U) < V_1(U) \leq V_2(U) < V_{\max}(U), \]

which can degenerate to a single point \(V_0(U) = V_1(U) = V_2(U)\).

(B2) Whenever open level lines appear in a finite interval \([V_1(U), V_2(U)]\), they have Properties (A1), (A2), presented above.

(B3) In the case when the interval \([V_1(U), V_2(U)]\) shrinks to a single point \(V_0(U) = V_1(U) = V_2(U)\), the open level lines arising at the corresponding level can either satisfy conditions (A1), (A2) (this occurs at the boundaries of the stability zones \(\Omega_{m, M}\)), or have more complex chaotic behavior (this occurs at the accumulation points of an infinite number of zones \(\Omega_{m, M}\) with unboundedly increasing values of \((m^1, m^2, m^3)\)).

It can be seen, therefore, that the appearance of open level lines in “stability zones” does not at all resemble the similar phenomenon for truly random potentials, where, as a rule, open level lines arise at a single energy level (if we consider random potentials from the point of view of the percolation theory, see e.g. \([46, 47]\)). Also, the potentials arising at the boundaries of the “stability zones,” although they have open level lines only for one value of \(\epsilon_0\), are not very suitable for the role of random potentials due to the “too regular behavior” of their open level lines. Therefore, it can be seen that it is natural to consider only potentials with chaotic level lines as models of a random potential. As we said above, in the case of three quasi-periods, such potentials always arise at accumulation points of “stability zones” with an increasingly complex geometry of open level lines, so that there is always a passage from the “regular” behavior to the “chaotic” one in this case.

It should also be noted here that chaotic level lines arising in the case of type II potentials are very different from those of type III potentials. Namely, chaotic level lines of potentials of type II always have the form of curves with an asymptotic direction in the plane \(\mathbb{R}^2\) (see \([10]\)). Thus, such level lines resemble to some extent “regular” level lines described above, and, in general, pass through the plane along some fixed direction. The difference between the two cases lies in the fact that the deviations of a chaotic level line of a type II potential in the direction perpendicular to the asymptotic one are not necessarily bounded, so the
line may not be enclosed in any straight strip of finite width. In this case, the same can be said also about the corresponding regions of lower values of the corresponding potential

$$V(x, y, U) \leq V_0,$$

namely, they have the form of "strips" passing through plane in some fixed direction. The width of these strips, however, can vary unlimitedly in their different parts, and they cannot be enclosed in straight strips of a fixed width. When the boundary value $\epsilon_0$ is shifted downward by an arbitrarily small amount, the region of lower values of the corresponding potential consists of strongly elongated bounded regions, and when $\epsilon_0$ is shifted upward, this region becomes the entire plane with strongly elongated bounded regions removed (and, possibly, with bounded regions not associated with the main component and lying inside the excluded regions added).

One may wonder, therefore, to what extent type II potentials with chaotic level lines can be considered as a model of a random potential. In a sense, they can be considered as an intermediate case between "regular" and "chaotic" potentials.

Chaotic level lines arising in the case of type III potentials are much more complex and "sweep" the entire plane $\mathbb{R}^2$ in a chaotic manner (Fig. 7). A similar behavior is exhibited in this case also by the regions of the smaller values $V(x, y, U) \leq V_0$. When the boundary value $\epsilon_0$ is shifted downward by an arbitrarily small amount, the region of lower values of the corresponding potential splits into rather complicated bounded regions, and when $\epsilon_0$ is shifted upward, this region becomes the entire plane from which bounded areas of complex shape are excluded (and, possibly, additional bounded areas that are not associated with the main component and lying inside the excluded areas added). An important circumstance here is that the linear dimensions of such regions grow according to a power law when approaching the value of $\epsilon_0$ with a fractional exponent ($|\epsilon - \epsilon_0|^{-\alpha}$), which makes such potentials alike random potentials (see, for example, [48, 49]). It should be noted, however, that here, in the general case, a certain anisotropy can be observed, namely, the presence of two different growth exponents $\alpha$ and $\beta$, in a certain direction in the plane and in the one perpendicular to it ($0 < \alpha, \beta < 1$). In general, the stochastic properties of such level lines are quite complex and are currently the subject of intensive research (see, for example [10, 11, 50–65]).

The randomness of the level lines of a potential formed by three standing waves is invariant under shifts of the phases $\delta_i$ (i.e., the positions of the standing wave maxima) with the remaining parameters fixed. In this case, the value of $V_0$, as well as the geometric features of the chaotic level lines (in particular, the degrees $\alpha$ and $\beta$), are preserved in a natural way under such shifts. The change in the "accessibility areas" $V(x, y) \leq V_0$ during the shifts of the maxima of the standing waves is accompanied here by their rather complex movement, as well as numerous rearrangements at their boundaries. Generally speaking, the transportation of atomic gas with an adiabatic change in the positions of the maxima of standing waves in this situation must be computed separately for each such potential.

In this work, we will try to represent the described level lines and the corresponding areas of lower potential values in the most visual way. In addition, as we said above, we will be interested in the dynamics of ultracold atoms in the regions we have described. Especially interesting in this case, in our opinion, is the superposition of the properties of chaotic dynamics itself and the chaotic properties of the "accessibility regions" for this dynamics.
Here we would like to note one more important property of the “chaotic” level lines of potentials \( V(x, y, U) \) with three quasi-periods, namely, the presence of their segments where the level line (as well as the corresponding region of smaller values \( V(x, y, U) \leq V_0 \)) passes “very close” to itself (Fig. 8). More precisely, by considering larger and larger areas in the plane, we can find segments of such a level line that are arbitrarily close to each other. As a consequence, when considering the semiclassical dynamics of atoms with energies close to the corresponding level \( V_0 \), it is always necessary to consider also the effects of tunneling from one part of the “accessibility region” to another near such places.

Another consequence of the above circumstance is that, in contrast to the situation in stability zones, here the global geometry of the domains \( V(x, y, U) \leq V_0 \) is unstable with respect to arbitrarily small local variations of the potential \( V(x, y) \) and can vary greatly (on large scales) in the presence of arbitrarily small perturbations or defects. As a consequence, the transport properties of particles in such potentials can also strongly depend on the presence of such defects in the plane of the potential. In the presence of additional slowly varying potentials, for example, of the type

\[
V(x, y) = ax^2, \quad a \to 0,
\]

the global geometry of open level lines and the corresponding regions of lower values of the resulting potential will most often resemble the corresponding geometry for a smooth potential on large scales and behave in a typical “chaotic” manner on small scales.

According to the general conjecture of Novikov, in the case of three quasi-periods, the emergence of potentials with chaotic level lines can occur only at a set of measure zero and, moreover, of the fractal codimension strictly greater than one, in the full space of parameters. It can be seen, therefore, that experimental construction of a potential with three quasi-periods which is close in properties to a random one in the sense described above requires a very special choice of parameters specifying such a potential. Figure 9 gives an example of the location of stability zones in the space of essential parameters under the restriction of the potential

\[
\cos X^1 + \cos X^2 + \cos X^3
\]

to two-dimensional planes for all possible affine embeddings \( \mathbb{R}^2 \to \mathbb{R}^3 \). In this case, only the direction of the embedding is essential, which can be specified by a unit vector in \( \mathbb{R}^3 \) orthogonal to the corresponding plane \( \mathbb{R}^2 \). The endpoints of such vectors lie on the unit sphere \( S^2 \), and thus all stability zones can be viewed as regions on the unit sphere. It can be seen that the union of (an infinite number of) such regions defines a rather complex set on the sphere, and the complement to it has the properties of a fractal. We also note here that Novikov’s conjecture has not yet been rigorously proven, although it has been confirmed in a number of serious numerical experiments.

Let us now formulate the analytical results known to date for potentials with four quasi-periods. It must be said at once that Novikov’s problem for the case of four quasi-periods is more complicated in comparison with the case of three quasi-periods. At the same time, the case of four quasi-periods may turn out to be very important in the formulation we are considering in connection with the problem of modulation of two-dimensional quasicrystals in systems of cold atoms.

In the setting we are considering, potentials with four quasi-periods are obtained as a result of a superposition of four independent sinusoidal standing waves (possibly with the generation of higher harmonics, Fig. 10) and can be written in the following general form

\[
V(r) = \sum_{i=1}^{4} V_i \cos(k_i r + \delta_i) + \ldots.
\]
The quasiperiodic properties of the potential are determined by the parameters \((k_{(1)}, k_{(2)}, k_{(3)}, k_{(4)})\) specifying the affine embedding \(\mathbb{R}^2 \to \mathbb{R}^4\), according to the formulas

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k_{(1)}^1 x + k_{(1)}^2 y + \delta_1 \\ k_{(2)}^1 x + k_{(2)}^2 y + \delta_2 \\ k_{(3)}^1 x + k_{(3)}^2 y + \delta_3 \\ k_{(4)}^1 x + k_{(4)}^2 y + \delta_4 \end{pmatrix} - \begin{pmatrix} k_{(1)}^0 r + \delta_1 \\ k_{(2)}^0 r + \delta_2 \\ k_{(3)}^0 r + \delta_3 \\ k_{(4)}^0 r + \delta_4 \end{pmatrix}.
\]

We present here some simplified consequences from the results of [23, 24], which are analogous to the first of the statements formulated above for the case of three quasiperiods. As before, we will not consider here in detail all the regularity conditions (of general position) imposed in [23, 24], and we will assume that they are always satisfied for real potentials. Then, from the results of [23, 24], the following statement follows:

(C1) open level lines of \(V(x, y, U)\) exist in a finite connected energy interval \([V_1(U), V_2(U)]\):

\[V_{\min}(U) < V_1(U) \leq V_2(U) < V_{\max}(U);\]

(C2) all open level lines of \(V(x, y, U)\) lie in straight strips of finite width and pass through them (Fig. 3);

(C3) the mean direction \(l(U)\) of strips containing open level lines of the potential \(V(x, y, U)\) is defined in the entire domain \(\Omega\) by some (irreducible) integer quadruple \((m^1, m^2, m^3, m^4)\) by the relation

\[m^1 k_{(1)} + m^2 k_{(2)} + m^3 k_{(3)} + m^4 k_{(4)}, l(U)) = 0.\]

As in the case of three quasiperiods, the quadruples \((m^1, m^2, m^3, m^4)\) are actually of topological origin and can be defined similarly to the earlier definition of the triples \((m^1, m^2, m^3)\) given in terms of the shift transformations of the parameters \(\delta_r\).

Also, as in the case of three quasi-periods, the union of the stability zones (in the general case) does not cover here the entire parameter space. The complement to this union forms a complex set parametrizing potentials with chaotic level lines. It must be said that both the features of the chaotic behavior of the level lines and the structure of the corresponding set in the space of parameters almost have not been studied to date. In particular, it can be expected that the set of parameters corresponding to potentials with chaotic level lines has a nonzero measure here.

Thus, in families of potentials with four quasiperiods, the construction of potentials with the properties of truly random potentials might be simpler from the experimental point of view. Note that, as in the case of three quasi-periods, each “chaotic” potential here is the limit of getting more and more complicated “regular” potentials due to the accumulation of an infinite number of “stability zones” near the point \(U_0\) defining this potential.

When passing to potentials with larger numbers of quasi-periods given by the embeddings

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k_{(1)}^1 x + k_{(1)}^2 y + \delta_1 \\ \vdots \\ k_{(d)}^1 x + k_{(d)}^2 y + \delta_d \end{pmatrix} - \begin{pmatrix} k_{(1)}^0 r + \delta_1 \\ \vdots \\ k_{(d)}^0 r + \delta_d \end{pmatrix},
\]

it should be noted right away that there are currently no rigorous analytic results for the case \(d > 4\). It can be seen, however, that in this case too, quasiperiodic potentials can have a “regular” behavior of open level lines. In general, we will say that a potential with \(d\) quasiperiods has regular open level lines if the following conditions are satisfied:

(D1) Open level lines of \(V(x, y)\) exist in a finite connected energy interval

\[V_{\min} < V_1 \leq V(x, y) \leq V_2 < V_{\max};\]

(D2) All open level lines of \(V(x, y, U)\) lie in straight strips of finite width and pass through them (Fig. 3).

(D3) The mean direction \(l\) of strips containing open level lines of the potential \(V(x, y)\) is determined, for some (irreducible) integer vector \((m^1, ..., m^d)\), by the relation

\[(m^1 k_{(1)} + ... + m^d k_{(d)}, l) = 0.\]
Potentials with $d$ quasiperiods with regular behavior of open level lines arise, in particular, whenever they are formed by means of sufficiently small (quasiperiodic) additions to potentials with a smaller number of quasiperiods (and regular behavior of open level lines). But in reality, such situations are not limited to this, and it is possible to construct a huge number of examples of very complex potentials with a large number of quasi-periods and regular behavior of open level lines in the sense described above.

It can be shown that in the most general case there is a huge set of situations when for a generic potential from some family of quasiperiodic potentials $V(x, y, U)$ properties (D1)–(D3) take place and, moreover, such a situation is locally stable. In this case we have a stability zone in the space of parameters, such that conditions (D1)–(D3) are satisfied at all its points for some values $V_1(U)$ and $V_2(U)$ and the same $(m^1, \ldots, m^d)$. The boundaries of the zone $\Omega_{(m^1, \ldots, m^d)}$ are actually determined then by the condition $V_1(U) = V_2(U)$.

Returning to the considered method of creating quasiperiodic potentials in systems of cold atoms (a superposition of standing waves), we can show, as before, that the numbers $(m^1, \ldots, m^d)$ can be determined in a purely topological way (by successive shifts of the maxima of the standing waves by a period and observation of the corresponding shifts of open level lines).

In the general case, it can be stated that when creating quasiperiodic potentials it is natural to divide them, according to the behavior of their level lines, into potentials that retain certain properties of ordered potentials, and potentials approaching random potentials. The potentials of the first type are stable and arise in some open regions in the space of their parameters, while each of these regions is determined by the respective value of the topological invariant an integer vector $(m^1, \ldots, m^d)$. Potentials of the second type are unstable and arise in rather complex sets of fractal type (the complements to the union of domains $\Omega_{(m^1, \ldots, m^d)}$). To construct potentials with the properties of random potentials, it is necessary to fix a set of domains $\Omega_{(m^1, \ldots, m^d)}$ in the parameter space. Considering the potentials in the complement, it can be expected that the corresponding potentials with a large number of quasi-periods can, in fact, serve as one of the models of random potentials, since the complexity of the behavior of their open level lines increases very rapidly with an increase in the number of quasi-periods. In conclusion of this section, we note that the emergence of random potentials has also been considered in systems of optical lattices for ultracold atoms (see, for example [66, 67]).

3. PARTICLE DYNAMICS IN POTENTIALS OF DIFFERENT TYPES

The main subject of this section is the dynamics of ultracold atoms in a two-dimensional plane in the presence of a quasiperiodic potential $V(x, y)$. In the leading approximation, such dynamics can be considered in a classical way provided it is assumed that the atoms have fairly well-defined trajectories in $\mathbb{R}^2$. An exception in this case may be the motion of atoms at special segments of trajectories, where quantum tunneling from one segment of the trajectory to another can play an important role. As we have already said, we will be primarily interested in the dynamics of atoms with energies corresponding to the presence of open level lines of the considered potentials. To apply the semiclassical description, we must therefore assume the relation $\hbar/\sqrt{2MV} \ll a$, where $a$ is the typical value of the periods of standing waves used in experiment. In our situation we assume also that a sufficiently large number of atoms in the ensemble have energies corresponding to the presence of open level lines, which also implies the relation $T = V$. Thus, in our case, we must also put $\hbar/\sqrt{2M} \ll a$. In the general case, the forces acting on atoms in optical traps can contain both conservative and dissipative parts (see, for example, [68–71]). We will consider here the classical dynamics of noninteracting atoms in the nondissipative limit, which gives a good approximation to the real dynamics of heavy atoms in many important situations.

As can be seen in this case, the features of the geometry of quasiperiodic potentials should manifest themselves most of all in the dynamics of particles at energies lying in the intervals of the presence of open level lines of the potential. Here we will focus specifically on the study of such dynamics. It is easy to see that the features of this dynamics should naturally manifest themselves in the transport properties of an ultracold gas in the presence of particles of the corresponding energies in the ensemble.

As we saw in the previous section, quasiperiodic potentials can in fact be divided into two types according to the behavior of their open level lines. At the same time, this division is most directly related to the dynamics of particles in such potentials, since it determines the geometry of the “regions of accessibility” for particles with certain energies. As a consequence of this, one can expect a difference in the dynamics of particles in the potentials of these two types, which is observable in the study of systems of ultracold atoms.

As we mentioned above, dynamics in two-dimensional Hamiltonian systems has one more feature, namely, it is integrable at low energy levels and becomes chaotic at high energies (see [25, 26]). At intermediate energy levels, the phase space of a system is divided into regions where the integrable case takes place, and domains where chaotic dynamics occurs.
In particular, one can observe the effects of gradual chaotization of dynamics, when a particle can “stick” to invariant two-dimensional tori for a long time, performing rare “jumps” (Levy flights) between different tori, as well as other similar effects. This feature of the dynamics has been studied in systems of cold atoms both for free atoms moving in periodic potentials (see [33, 34]), and in the presence of additional interaction of the atom motion with internal degrees of freedom (see [72, 73]).

In this work, we do not consider the internal degrees of freedom of atoms and focus only on the motion of the atom as a whole. As we have already said, we will be interested here in the dynamics of atoms with energies corresponding to the appearance of open level lines at the corresponding potentials. This is the area in which it is especially interesting for us to observe the combination of different modes (integrability, its complication, and transition to chaotic dynamics) in different parts of the phase space. It can be immediately noted that, since open level lines of a quasiperiodic potential also appear at some “intermediate” (between the minimum and maximum) values, the “gradual chaotization” of dynamics described above will often arise precisely near such level lines. It can also be said that such chaotization must, of course, correlate with the nontrivial geometry of the “accessible regions” of the motion of atoms in the potentials we are considering. In particular, for developed chaotization, the motion of atoms should be close to diffusion, so that the transport properties of an atomic gas are determined by the diffusion of atoms in regions of a given geometry. Under conditions of “intermediate” chaotization, one can expect that the geometry of the accessibility regions significantly affects the Levy flights between two-dimensional tori, which, of course, is also determinant when considering the transport properties of an atomic gas. In addition, in the situation under consideration, two-dimensional tori corresponding to integrable dynamics can cut (three-dimensional) manifolds of constant energy in a nontrivial way, so that we can also observe nonintegrable dynamics localized in different parts of the “accessibility regions.” The geometry of such areas and their location is, of course, also related to the general geometry of the “regions of accessibility” for a given particle energy. On the whole, as it is easy to see, in the model we are considering, it is the diffusion or Levy flight regimes that determine the transport properties of an atomic gas that can make it possible to observe the differences between the “regular” and “chaotic” quasiperiodic potentials introduced above.

Below we give the results of a numerical study of the classical dynamics of particles in the potentials considered here. As a model, we consider here potentials with three quasiperiods obtained by the restrictions of the potential

$$V(x, y) = \cos(a_1x + b_1y + c_1) + \cos(a_2x + b_2y + c_2) + \cos(a_3x + b_3y + c_3)$$

(3.1)

to the planes defined by different (isometric) embeddings (2.2). It must be said that, from the point of view of Novikov’s problem, this family of potentials contains absolutely all situations that can occur for potentials with three quasiperiods; namely, we will find here both stable “regular” potentials with very different features of the geometry of open level lines, and a rich set of “chaotic” potentials. It is worth to mention just one feature of the potentials from this set, which is as follows. For any “regular” potential arising in the described family, the interval of existence of open level lines is symmetric with respect to zero, i.e. we always have the relation $V_1(U) = -V_2(U)$ for the values $V_1(U)$ and $V_2(U)$ introduced above. Similarly, for all “chaotic” potentials of this family, open level lines arise exactly at the zero energy level ($V_0 = 0$). This feature is specific for this family of potentials and, generally speaking, does not take place in the most general case. In all other respects, the geometric properties of the level lines of potentials from the above family reflect the most general situation. Since, as we have already said, we are interested in the dynamics of particles in the areas of the appearance of open potential level lines, we will often study such a dynamics here at the value $\varepsilon = 0$. Thus, all potentials will have the form

$$V(x, y) = \cos(a_1x + b_1y + c_1) + \cos(a_2x + b_2y + c_2) + \cos(a_3x + b_3y + c_3)$$

(3.1)

with some coefficients $a_i$, $b_i$, and $c_i$. In fact, since the choice of the direction of the coordinate axes in the plane $\mathbb{R}^2$ will not matter much to us, we will assume that the $x$-axis coincides with the intersection line of $\mathbb{R}^2$ with the plane ($X^1$, $X^2$) in $\mathbb{R}^3$. So we can always put here $a_3 = 0$. 

Fig. 11. The potential (3.2) with “regular” open level lines from the largest stability zone in Fig. 9. (The filled areas correspond to the values $V(x, y) \leq 0$.)
As we have already mentioned above, the type of the potential is determined in our case only by the direction of the embedding $\mathbb{R}^2 \to \mathbb{R}^3$ and can be obtained from the diagram shown in Fig. 9. It can be seen that by choosing the embedding parameters, we can easily implement any of the situations interesting for us.

To compare the dynamics in potentials of different types, we present here the results for three potentials, the first two of which have “regular” open level lines and belong to rather large stability zones in Fig. 9, and the third has “chaotic” open level lines, which appear only at the zero energy value.

Fig. 12. Examples of invariant tori corresponding to a relatively simple dynamics of atoms in the “regular” potential (3.2) at zero total energy.

Fig. 13. Examples of invariant tori defining more complex dynamics of atoms in the “regular” potential (3.2) at zero total energy.
Our first series of computations refers to the potential with coefficients

\[ a_1 = -0.12251993420338196, \]
\[ b_1 = -0.2250221718850486, \]
\[ c_1 = 1.5505542426422338, \]

which lies inside the largest zone in Fig. 9 with topological numbers \((m_1, m_2, m_3) = (1, 0, 0)\). It must be said that we are intentionally considering separately a potential from this zone here, since the latter actually differs somewhat from the other zones. Namely, in addition to the above-mentioned “partial” integrability inherent in two-dimensional Hamiltonian systems, here there is an additional closeness to the integrable situation, due to the fact that the center of this zone corresponds to a potential integrable at all energies, which is

\[ V(x, y) = \cos x + \cos y. \]

As a consequence of this, integrable dynamics can arise here for richer sets of initial conditions than it can for potentials from other zones. As we will actually see, this assumption is confirmed; in particular, we failed to find here a well-pronounced diffusion dynamics at

\[ a_2 = 0.9924660526802913, \]
\[ b_2 = -0.02777898711920925, \]
\[ c_2 = 0.12374024573075965, \]
\[ a_3 = 0, \]
\[ b_3 = 0.9739575709622912, \]
\[ c_3 = 3.1548761694687415 \]
the level $\epsilon = 0$. This dynamics arises only with a noticeable increase in the particle energy. At the same time, the emergence of potentials from this particular zone is the most expected in the experiment in comparison with others due to the significant size of this zone.

As we have already said, we restrict ourselves here to considering three-dimensional manifolds in the phase space by fixing the total energy of a particle. The potential we are considering has open level lines in rather wide energy interval

$$-0.7493 \leq V \leq 0.7493$$

(approximately). We first consider the dynamics of particles with energy $\epsilon = 0$. As we said above, we can expect here the appearance of large regions where the dynamics is in fact integrable and occurs on two-dimensional tori embedded in the phase space. This is exactly what happens at the level $\epsilon = 0$, moreover, by carefully choosing the initial data, one can find the regions of their values, where the dynamics on the tori corresponds to a rather simple motion in the coordinate space (Fig. 12).

By varying the initial data at the level $\epsilon = 0$ (for the same potential), one can, however, discover (smaller) regions in which the geometry of invariant tori becomes more and more complicated, which also leads to complication of the motion of particles in the coordinate space (Fig. 13).

Choosing the initial data in an even more special way (at the level $\epsilon = 0$), one can also see even more complicated surgery of invariant tori, where particles “stick” for a rather long time to simpler invariant tori and “jump” from one of these tori to another (Levy flights) at certain times (Fig. 14).

The regimes shown in Figs. 12–14, correspond to the dynamics of particles with zero total energy. As we have already said, for this potential we failed to find diffusion regimes at $\epsilon = 0$; however, they appear with increasing particle energy (Fig. 15). In fact, a clearly pronounced diffusion behavior arises here at energies for which the open level lines of the potential have already disappeared, and the region of accessibility extends infinitely in two dimensions. It is interesting that the diffusion dynamics here retains, nevertheless, a pronounced anisotropy, preserving the memory of the mean direction of the open level lines of the potential. It can also be noted here that even at these levels there remain quite a lot of invariant tori, and the diffusion dynamics has, at the same time, the form of Levy flights with “sticking” to invariant tori. As we have already said, this behavior is apparently characteristic only for potentials from the zone with $(m^1, m^2, m^3) = (1, 0, 0)$ (and the zones identical with it) due to the circumstances mentioned above. In particular, we will present below a description of the dynamics in a potential from another large stability zone, which, apparently, is characteristic for most potentials with “regular” level lines.

With a further increase in energy, the motion of particles in the potential gradually changes from diffusion motion to ballistic motion. We should note here, however, that ballistic motion in quasiperiodic potentials also has, apparently, noticeable features. In particular, for potentials from the family under consideration, even at rather high energies, a fairly large part of the phase volume is occupied by ballistic trajectories of certain
directions. For the potential we are considering, three main directions can be distinguished at once, namely the directions of the level lines of the three cosines in the formula (3.1), along which ballistic motion occurs already at sufficiently low energies (Fig. 16). With increasing energy, the number of such directions increases and ballistic trajectories with directions other than the three indicated ones can also be observed.

In addition to the “purely” ballistic trajectories described above, one can also observe trajectories consisting of long ballistic sections of the indicated directions, connected by short sections “switching” between two directions (Fig. 17). Like the “purely” ballistic trajectories, such trajectories should also introduce specific features into transport phenomena at the corresponding particle energies. It can be seen, however, that with an increase in the number of the corresponding ballistic directions, as well as the complication of the geometry of the “quasi-ballistic” trajectories, it will be more and more difficult to detect such properties.

Thus, it can be seen that transport phenomena caused by the ballistic motion of atoms also reveal the geometric structure of quasiperiodic potentials. In comparison with the geometry of open level lines, however, this structure is simpler and is directly related to the harmonics generating the potential. As we will see below, this property of ballistic motion manifests itself in fact for potentials of all types, and, in this sense, the corresponding transport phenomena almost do not distinguish between “regular” quasiperiodic potentials from “chaotic” ones. Also, in contrast to the case of diffusion motion, which gives a well-observed contribution to the transport processes associated with the geometry of open level lines, the experimental observation of transport contributions from ballistic trajectories of stable directions can be more complicated due to the addition of a large number of such contributions at high energies. On the other hand, ballistic motion in quasiperiodic potentials is also, apparently, a fundamental property of such potentials, in particular, ballistic directions also play an important role in quantum dynamics in potentials of this type (see [38]).

As we have already noted above, diffusion dynamics, as well as dynamics including distant jumps
between different types of localized dynamics, which give us the most information about the type and topological parameters of a potential, in this example arise mainly at energies lying above the interval of existence of open level lines. In this case, however, such dynamics keeps the “memory” of the geometry of open level lines of the potential and gives a strongly anisotropic contribution of the corresponding direction to transport phenomena at these energies. This feature, as we have already said above, is apparently associated with “additional reasons” for the appearance of integrable dynamics in this stability zone, leading to an increase in the phase volume filled with such dynamics.

The second series of our computations refers to the potential with coefficients

\[
\begin{align*}
    a_1 &= -0.6194151736623348, \\
    b_1 &= -0.44502823229775823, \\
    c_1 &= 1.4421279589366298, \\
    a_2 &= 0.7850635914605004,
\end{align*}
\]

**Fig. 19.** Examples of invariant tori corresponding to the dynamics of atoms in the case of the “regular” potential (3.3) at zero total energy.

**Fig. 20.** Non-integrable dynamics in a region separated from the rest of the phase space in the case of the “regular” potential (3.3) at zero total energy.

**Fig. 21.** Rare Levy flights in the case of the “regular” potential (3.3) at zero total particle energy.
lying inside the zone in Fig. 9 with topological numbers $(m_1, m_2, m_3) = (1, 1, 1)$ (Fig. 18). Like the previous one, potential (3.3) has a fairly large energy interval

$$-0.7548 \leq V \leq 0.7548,$$

containing open level lines of the potential.

Here, in fact, to observe most of the described regimes, it is sufficient to study the dynamics of particles at the energy $\epsilon = 0$. In particular, we can also observe here the presence of invariant tori of varying complexity (Fig. 19), as well as regions in the phase space, separated by such tori (Fig. 20).

$$b_2 = -0.3511272752829312,$$
$$c_3 = 0.8986352554278761,$$
$$a_3 = 0,$$
$$b_3 = 0.823807932117983,$$
$$c_3 = 2.3379002628621635,$$

lying inside the zone in Fig. 9 with topological numbers $(m_1, m_2, m_3) = (1, 1, 1)$ (Fig. 18). Like the previous one, potential (3.3) has a fairly large energy interval

$$-0.7548 \leq V \leq 0.7548,$$

containing open level lines of the potential.

Here one can also observe such a phenomenon as incomplete separation of the energy level by invariant tori, when there are “gaps” between the tori, allowing the trajectory to leave an “almost isolated” region at certain times. This situation is presented in the coordinate space by long wandering paths of particles in cer-
tain areas with very rare transitions (Levy flights) between them (Fig. 21).

Finally, in certain areas of the initial data at zero energy level, one can also observe here much more complex Levy flights (Fig. 22), turning into diffusion modes (Fig. 23). As we have already said, we expect that the presence of clearly pronounced diffusion dynamics among other regimes in the interval of the existence of open level lines is in fact a general phenomenon for “regular” quasiperiodic potentials, if there are no special reasons suppressing such dynamics (as in the previous case). As it is easy to see, the geometry of the accessible regions for particles of fixed energy has here the most direct influence on the geometry of Levy flights and diffusion dynamics. As in the previous case, a further increase in energy leads to the appearance of ballistic trajectories in the considered potential. The main stable directions of such trajectories (Fig. 24) are also determined here simply by the directions of the level lines of cosines present in (3.1), and are not related, in fact, with the type of arising potential. Ballistic trajectories of stable directions, as we have already noted above, occupy a finite phase volume at a fixed energy level.

With a further increase in the particle energy, the number of stable directions of ballistic trajectories increases. In addition, as in the previous case, a lot of “quasi-ballistic” trajectories also arise here, with rather long ballistic segments joined by short transitions between them (Fig. 25). As in the previous case, it can be noted that the geometric features of the contribution of ballistic trajectories to transport phenomena become more and more “blurred” with an increase in the number of stable directions of such trajectories, as well as with a complication of the geometry of “quasi-ballistic” trajectories. As for determining the type of potential, as well as its stable topolog-

Fig. 24. Ballistic trajectories of the “main” directions in the case of the “regular” potential (3.3) ($\epsilon = 4$).

Fig. 25. “Almost” ballistic trajectories in the case of the “regular” potential (3.3) ($\epsilon = 4$).
atical parameters (numbers \((m^1, m^2, m^3)\)), they, as in the previous case, are best determined by the contribution of diffusion trajectories, as well as trajectories containing long “hops” (Levy flights) between segments of almost integrable or localized dynamics. It can be noted that in this example (in contrast to the previous one), the corresponding dynamics arises mostly in the interval of the existence of open level lines of the potential. We expect that this property should actually manifest itself for most types of “regular” potentials (in the absence of additional reasons for increasing the phase volume occupied by the integrable dynamics) created by the method under consideration.

Our last series of computations refers to the potential with coefficients

\[
\begin{align*}
a_1 &= -0.6190763027420052, \\
b_2 &= -0.2572674789786692, \\
c_2 &= 0.7853308419916342, \\
b_2 &= -0.2028039536788493, \\
c_3 &= 0.8662242771884692, \\
a_3 &= 0, \\
b_3 &= 0.9448195598272575, \\
c_3 &= 2.950743051151684,
\end{align*}
\]

having “chaotic” level lines (Fig. 26). In this case, open level lines, as we have already said, exist only at the value \(V_0 = 0\).

As in the previous two cases, in this case, at the energy level \(\epsilon = 0\) we can also see regions corresponding to the motion along tori of relatively simple (Fig. 27) as well as more complex geometry (Fig. 28).

In addition, in certain regions of initial data, one can observe non-integrable dynamics, which is bounded, however, by some invariant tori in the manifold \(\epsilon = 0\). Such dynamics is also easily distinguishable from other types when it is projected onto the coordinate space (Fig. 29).
As in the previous cases, for certain initial conditions at the zero energy level, one can also observe the “sticking” of the particle trajectory to invariant tori for a rather long time, interspersed with Levy flights at certain moments (Fig. 30).

Also, as in the previous case, by changing the initial data, we can achieve the complication of the geometry of the tori and the transition to the diffusion dynamics of particles (Fig. 31). Diffusion dynamics here is also limited to the “region of accessibility,” which now has a completely different geometry and itself has, in a sense, “diffusion properties.” It must be said that, apparently, in the classical limit, the transport properties of particles at the zero energy level here are close to the transport properties of localized (although in large regions) particles, since the probability of distant diffusion in the region under consideration is very small. It can be noted, however, that in the case of three quasi-
periods, such regions always contain segments of the boundary that are very close to each other, where quantum tunneling should be possible. In this case, it is quantum tunneling that, apparently, should play an important role for transport phenomena at $\epsilon = 0$.

In general, with a gradual increase in the energy of particles in the ensemble, the transport properties of an atomic gas in the described potential should (in the classical limit) change significantly when particles with positive energies appear in the ensemble. Indeed, with increasing energy, the accessibility regions expand and become non-simply connected, in contrast to the case $\epsilon = 0$. We can say that, in a certain sense, such regions have the property of “percolation.” At the same time, they retain for some time also certain “diffusion” form, which should manifest itself in the transport properties of an atomic gas. As we can see in Fig. 32, the diffusion properties of the particle dynamics in such potentials rapidly increase with an increase in the value of $\epsilon$. As the energy of the particles decreases, the “accessible regions” become bounded regions in the plane. Thus, one can see here a noticeable difference from the potentials with “regular” level lines, where the picture does not change significantly when the particle energy is varied near zero.

With a significant increase in energy, ballistic trajectories with stable directions appear for the potential (3.4) as well (Fig. 33). As in the previous two cases, the main stable directions are the directions of the level lines of the cosines constituting the potential (3.4) according to (3.1). The corresponding trajectories appear at the lowest energy levels; with a further increase in energy, the number of such directions increases. As in the previous two cases, the final phase volume is also occupied by “quasi-ballistic” trajectories, consisting of long segments of ballistic trajectories joined by short intermediate segments (Fig. 34). As we have already said above, to determine the geometric features of a “chaotic” potential, as in the “regular” case, it seems most appropriate to study the contribution of the “diffusion” and “jumping” trajectories appearing at “intermediate” energy values.
4. CONCLUSIONS

The paper considers questions related to the geometry of quasiperiodic potentials on a plane, the methods of their creation, their dependence on control parameters, and the dynamics of semiclassical particles in such potentials. The main considerations are related to the situation of the emergence of such potentials in systems of ultracold atoms in magneto-optical traps, although our results are actually valid for the most general types of quasiperiodic potentials. It is shown that, in the general case, quasiperiodic potentials on the plane can be naturally divided into two main classes, namely, potentials with a “regular” behavior of open level lines and potentials with a “chaotic” behavior of open level lines. In each family of quasiperiodic potentials, depending smoothly on some set of parameters, potentials from these classes are parametrized by sets having different structures, complementing each other in the full space of parameters. Namely, the first set has the form of a union of (countably many) regions with piecewise smooth boundaries, while the second has fractal properties. According to the behavior of open level lines, the former potentials can be attributed the “regular” type (approaching periodic potentials), while the latter can be considered as a model of random potentials. The non-dissipative dynamics of ultracold atoms in the considered potentials is integrable at lower energy levels, gradually becoming chaotic with increasing the energy of atoms. As a rule, in the interval of the existence of open level lines of the potential, both types (integrable and chaotic) of dynamics are present, and the properties of the chaotic dynamics substantially depend on the geometry of the potential level lines. The study of the transport properties of an atomic gas in quasiperiodic potentials in the presence of particles with the
corresponding energies in the ensemble can thus allow observing the differences between potentials of both types and also provide more detailed information on the geometry of their open level lines.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.
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