On the F-index and F-coindex of the line graphs of the
subdivision graphs

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Abstract
The aim of this work is to investigate the F-index and F-coindex of the line graphs of
the cycle graphs, star graphs, tadpole graphs, wheel graphs and ladder graphs using
the subdivision concepts. F-index of the line graph of subdivision graph of square grid
graph, 2D-lattice, nanotube and nanotorus of $TUC_4C_8[p,q]$ are also investigated here.

Keywords: Topological index, F-index, F-coindex, line graphs, subdivision graphs,
cycle graph, star graph, square grid graph, tadpole graphs, wheel graphs, ladder
graphs, 2D-lattice of $TUC_4C_8[p,q]$, nanotube of $TUC_4C_8[p,q]$, nanotorus of
$TUC_4C_8[p,q]$.

1. Introduction

Topological indices are numerical quantities associated with different graph param-
eters. These are used to correlate chemical structure of molecular graphs with various
physical properties, chemical reactivities and biological activities. By molecular graph
we mean a simple graph, representing the carbon-atom skeleton of an organic molecule. Let $G$
be a simple graph with vertex set $V(G)$, edge set $E(G)$ and $d(u)$ denotes the de-
gree of a vertex $u$ in $G$. An edge between vertices $u$ and $v$ is denoted by $uv$. The set of
vertices which are adjacent to a vertex $u$ is denoted by $N(u)$ and is known as neighbour-
hood of $u$. Clearly, $d(u) = |N(u)|$. Degree based topological indices have been subject
to study since the introduction of Randić index in 1975. Although the first degree-
based topological indices are the Zagreb indices [1], these were initially intended for
the study of total $\varphi$-electron energy [2] and were included among the topological in-
dices much later. The first and second Zagreb indices are respectively defined as

$$M_1(G) = \sum_{u\in V(G)} d^2(u) = \sum_{uv\in E(G)} [d(u) + d(v)],$$

and

$$M_2(G) = \sum_{uv\in E(G)} d(u)d(v).$$

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In the same paper [1] where Zagreb indices were introduced, Gutman and Trinajstić indicated that another term of the form $\sum_{u \in V(G)} d^3(u)$ influences the total $\varphi$-electron energy. But this remained unstudied by the researchers for a long time, except for a few occasions [3, 4, 5] until the publication of an article by Furtula and Gutman in 2015 and so they named it “forgotten topological index” or F-index in short [6]. Thus, F-index of a graph $G$ is defined as

$$F(G) = \sum_{u \in V(G)} d^3(u) = \sum_{uv \in E(G)} [d^2(u) + d^2(v)].$$

F-index for different graph operations has been studied De et al. [7]. Extremal trees with respect to F-index have been studied by Abdo et al. [8].

While considering contribution of non adjacent pair of vertices in computing the weighted Wiener polynomials of certain composite graphs, Došlic [9] introduced quantities named as Zagreb co-indices. Thus the first and second Zagreb co-indices are respectively defined as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d(u) + d(v)]$$

and

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} [d(u)d(v)].$$

In a similar manner, F-coindex is defined as

$$\overline{F}(G) = \sum_{uv \notin E(G)} [d^2(u) + d^2(v)].$$

De et al. [10] have shown that F-coindex can predict the octanol water partition coefficients of molecular structures very efficiently. They have also studied the F-coindex of graph operations. Trees with minimum F-coindex have been found by Amin and Nayeem [11].

Complement of the graph $G$ is a simple graph $\overline{G}$ with same vertex set $V$ and there is an edge between the vertex $u, v$ in $\overline{G}$ if and only if there is no edge between $u, v$ in $G$. It is evident that the F-coindex of $G$ is not same as F-index of $\overline{G}$, because during computation of F-coindex, degrees of the vertices are considered as in $G$.

The subdivision graph $S(G)$ is the graph obtained from $G$ by replacing each of its edge by a path of length 2, or equivalently, by inserting an additional vertex into each edge of $G$. The line graph of the graph $G$, denoted by $L(G)$, is the simple graph whose vertices are the edges of $G$, with $ef$ belong to $E(L(G))$ when $e$ and $f$ are incident to a common vertex in $G$. We use $P_n, C_n, S_n$ and $W_n$ to denote path, cycle, star and wheel graphs on $n$ vertices respectively. The tadpole graph $T_{n,k}$ is the graph obtained by joining a cycle graph $C_n$ to a path of length $k$. The ladder graph $L_n$ is given by $L_n = P_2 \square P_n$, the cartesian product of $P_2$ with $P_n$. The graph obtained via this definition has the advantage of looking like a ladder, having two rails and $n$ rungs between them. In a similar manner, the square grid graph $G(m, n)$ is defined by $G(m, n) = P(m) \square P(n)$. Thus $L(n) = G(2, n)$. 
Study on different topological indices for line graphs of subdivision graphs have been done by many researchers in the recent past. Ranjini et al. [12] computed Zagreb index and Zagreb co-index of the line graphs of the subdivision graphs of $T(n, k), L(n)$ and $W(n + 1)$. Su and Xu [13] generalized the idea of Ranjini et al. and presented the Schur-bound for the general sum-connectivity co-index. Nadeem et al. [14] studied the $ABC_4$ and $GA_5$ indices of the line graph of tadpole, wheel and ladder graphs using the notion of subdivision. Recently Nadeem et al. [15] obtained expressions for certain topological indices for the line graph of subdivision graphs of 2D-lattice, nanotube and nanotorus of $TUC_4C_8[p,q]$, where $p$ and $q$ denote the number of squares in a row and number of rows of squares, respectively in 2D-lattice, nanotube and nanotorus. 2D-lattice, nanotube and nanotorus of $TUC_4C_8[p,q]$ are depicted in Figure 1. The order and size of 2D-lattice of $TUC_4C_8[p,q]$ are $4pq$ and $6pq - p - q$ respectively. Again the order and size of nanotube of $TUC_4C_8[p,q]$ are $4pq$ and $6pq - p$ respectively. Hosamani and Zafar [16] have studied some more topological indices of the line graphs of the subdivision graphs of the above mentioned nano-structures. Here we have computed the F-index and F-coindex of the line graphs of subdivision graph of $C(n), S(n), T(n, k), L(n)$ and $W(n + 1)$ in Section 2. We also study the line graph of subdivision graphs of 2D-lattice, nanotube and nanotorus of $TUC_4C_8[p,q]$ and calculate the F-index of the line graph of subdivision of 2D-lattice, nanotube and nanotorus of $TUC_4C_8[p,q]$ in Section 3.

2. F-index and F-coindex of some graph using subdivision concept

**Theorem 1.** Let $G$ be the line graph of the subdivision graph of the cycle $C_n$ with $n$ vertices. Then F-index of $G$ is $F(G) = 16n$.

**Proof.** The line graph of the subdivision graph of the cycle $C_n$ is $C_{2n}$, i.e., $G = C_{2n}$ (See Figure 2). Hence

$$F(G) = \sum_{i=1}^{2n} 2^3 = 2n \cdot 2^3 = 16n.$$
\textbf{Theorem 2.} Let $G$ be the line graph of the subdivision graph of the cycle $C_n$ with $n$ vertices. Then $F$-coindex of $G$ is \( F(G) = 16n^2 - 16n \).

\textit{Proof.} As before, $G = C_{2n}$. Each vertex of $G$ is of degree two. Also each vertex of $G$ is non-adjacent to $2n - 2$ two degree vertex. Hence
\[
\sum_{u \in V(G)} \sum_{v \in N(u)} [d^2(u) + d^2(v)] = 2n(2n - 2)(2^2 + 2^2).
\]

Since one edge is shared by a pair of vertices,
\[
F(G) = \frac{1}{2}2n(2n - 2)(2^2 + 2^2) = 16n^2 - 16n.
\]

\hfill \Box

\textbf{Theorem 3.} Let $G$ be the line graph of the subdivision graph of the star $S_n$ with $n$ vertices (See Figure 3). Then $F$-index of $G$ is $F(G) = n(n - 1)(n^2 - 3n + 3)$.

\textit{Proof.} The line graph of the subdivision graph of the star contains $n - 1$ vertices of degree $n - 1$ and $n - 1$ vertices of degree one. Hence
\[
F(G) = \sum_{v \in V(G)} d^3(v)
\]
\[
= (n - 1)(n - 1)^3 + (n - 1)1^3
\]
\[
= (n - 1)(n^3 - 3n^2 + 3n - 1 + 1)
\]
\[
= n(n - 1)(n^2 - 3n + 3).
\]
Theorem 4. Let $G$ be the line graph of the subdivision graph of the star $S_n$ with $n$ vertices. Then $\bar{F}(G) = (n-1)(n-2)(n^2 - 2n + 3)$.

Proof. Each one degree vertex, i.e., pendant vertex in $L(S(S_n))$ is non-adjacent with $n-2$ one degree vertex and $n-2$ vertices of degree $n-1$. All non-pendant vertices are adjacent to each other (See Figure 3). Hence

$$\bar{F}(G) = \frac{1}{2}(1^2 + 1^2)(n-2)(n-1) + (n-1)(n-2)(1^2 + (n-1)^2)$$

$$= (n-1)(n-2)(n^2 - 2n + 3).$$

□

Theorem 5. For the line graph of the subdivision graph of a tadpole graph, $F(L(S(T_n,k))) = 16n + 16k + 50$.

Proof. The subdivision graph $S(T_n,k)$ contains $2(n+k)$ edges, so that the line graph contains $2(n+k)$ vertices, out of which three vertices are of degree 3, one vertex is of degree 1 and the remaining $2n+2k-4$ vertices are of degree 2 (See Figure 4). Hence,

$$F(L(S(T_n,k))) = 3.3^1 + 1.1^3 + (2n+2k-4).2^3$$

$$= 16n + 16k + 50.$$  

□

Theorem 6. The $F$-coindex of the line graph $L(S(T_n,k))$ is $16(n+k)^2 - 57$.

Proof. The line graph $L(S(T_n,k))$ contains a subgraph $P_{2k-1}$. $L(S(T_n,k))$ contains only 3 vertices of degree 3. Let $u, v_1, v_2$ be those vertices of degree 3, among which the vertex $u$ is attached to the path $P_{2k-1}$. The vertex $u$ is not adjacent to $2k-3$ vertices of degree 2 and the pendant vertex among the $2k-1$ vertices on $P_{2k-1}$. The neighbor of $u$ on $P_{2k-1}$ is not adjacent with $2k-4$ vertices of degree 2 and the pendant vertex among the $2k-1$ vertices on $P_{2k-1}$, and so on. The vertices $v_1$ and $v_2$ are not adjacent with $2n+2k-5$ vertices of degree 2 and also with the pendant vertex. The vertex $u$ is not adjacent with $2n-2$ vertices of degree 2 in $L[S(C_n) + e]$, where $e$ is the edge adjacent to $S(C_n)$. Also, $2n-2$ vertices of degree 2 in $L[S(C_n) + e]$ are not adjacent.
with any of the vertices on the path $P_{2k-1}$. Out of these $2n-2$ vertices in $L[S(C_n) + e]$, $2n-4$ vertices of degree 2 are non adjacent with $2n-5$ vertices of degree 2 and the remaining 2 vertices which are adjacent to $v_1$ and $v_2$ have $2n-4$ non adjacent vertices in $L[S(C_n) + e]$ of degree 2. Hence,

\[
\overline{F}(L(S(T_{n,k}))) = (2k - 2)(1^2 + 2^2) + (2n - 2)(1^2 + 2^2) + 3(2k + 2n - 5)(2^2 + 3^2) \\
+ (2n - 2)(2k - 2)(2^2 + 2^2) + 3(1^2 + 3^2) \\
+ [(2^2 + 2^2) + 2(2^2 + 2^2) + \ldots + (2k - 4)(2^2 + 2^2)] \\
+ [(2^2 + 2^2) + 2(2^2 + 2^2) + \ldots + (2n - 4)(2^2 + 2^2)] \\
= 5(2k - 2) + 5(2n - 2) + 39(2n + 2k - 5) + 8(2n - 2)(2k - 2) \\
+ 30 + 4(2k - 4)(2k - 3) + 4(2n - 4)(2n - 3) \\
= 16n^2 + 16k^2 + 32nk - 57 \\
= 16(n + k)^2 - 57. 
\]

\[\Box\]

**Theorem 7.** The $F$-index of $L[S(W_{n+1})]$ is $n(n^3 + 81)$.

**Proof.** The line graph $L[S(W_{n+1})]$ contains $4n$ vertices. Out of these $4n$ vertices, $3n$ vertices are of degree 3 and $n$ vertices are of degree $n$ (See Figure 5).

Hence

\[F(G) = 3n.3^3 + n.n^3 = n(n^3 + 81).\]

\[\Box\]

**Theorem 8.** The $F$-coindex of $L[S(W_{n+1})]$ is $2n(n^3 + 56n - 56)$.

**Proof.** As seen in Theorem 7, $L(S(W_{n+1}))$ has total $4n$ vertices, out of which $3n$ vertices are of degree 3 and $n$ vertices are of degree $n$. Out of the $3n$ three degree vertices, $2n$ vertices are not adjacent to the $3n - 4$ vertices of degree 3 and $n$ vertices of degree $n$.

Hence their contribution in the $F$-coindex is

\[
2n(3n - 4)(3^2 + 3^2) + 2n.n(3^2 + n^2) \\
= 108n^2 - 144n + 18n^2 + 2n^3. 
\]

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The remaining $n$ vertices of degree 3 are non adjacent with $n - 1$ vertices of degree $n$ and $3n - 3$ vertices of degree 3. Hence their contribution to the F-coindex is
\[
\begin{align*}
n(n - 1)(3^2 + n^2) + n(3n - 3)(3^2 + 3^2) \\
= (n^2 - n)(9 + n^2) + 18n(3n - 3) \\
= n^4 - n^3 + 63n^2 - 63n.
\end{align*}
\]

Also each of the $n$ vertices of degree $n$ are non adjacent with $3n - 1$ vertices of degree 3. So, their contribution to F-coindex is
\[
\begin{align*}
n(3n - 1)(3^2 + n^2) \\
= (3n^2 - n)(9 + n^2) \\
= 3n^4 + 27n^2 - 9n - n^3.
\end{align*}
\]

Hence,
\[
\sum_{u \in V(G)} \sum_{v \notin N(u)} [(d(u))^2 + (d(v))^2] = 2n^3 + 18n + 108n^2 - 144n + n^4 - n^3 + 63n^2 - 63n \\
+ 3n^4 + 27n^2 - n^3 - 9n \\
= 4n(n^3 + 56n - 56).
\]

Since each edge is shared by a pair of vertices,
\[
\overline{F}(L(S(W_{n+1}))) = 2n(n^3 + 56n - 56).
\]

\begin{flushright}
\qed
\end{flushright}

**Theorem 9.** The F-index of $L(S(L_n))$ is $162n - 260$.

**Proof.** $L(S(L_n))$ contains total $6n - 4$ vertices (See Figure 6). Out of these $6n - 4$ vertices, 8 vertices are of degree 2 and the remaining $6n - 12$ vertices are of degree 3. Hence,
\[
F(L(S(L_n))) = 8.2^3 + (6n - 12).3^3 = 162n - 260.
\]

\begin{flushright}
\qed
\end{flushright}
**Theorem 10.** The F-coindex of the line graph $L(S(L_n))$ is $324n^2 - 832n + 532$.

*Proof.* As stated earlier, $L(S(L_n))$ has $6n - 4$ vertices in total, among which 8 vertices are of degree two and all other vertices are of degree three. Out of the $6n - 12$ three degree vertices, the 4 vertices which are adjacent to the corner vertices are not adjacent to $6n - 14$ three degree vertices and 7 two degree vertices. Hence their contribution in the F-coindex is

$$4(6n - 14)(3^2 + 3^2) + 20(2^2 + 2^2).$$

Contribution of the rest of the three degree vertices, each of which is not adjacent to $6n - 15$ three degree vertices and 8 two degree vertices, is

$$(6n - 16)(6n - 15)(3^2 + 3^2) + 8(6n - 16)(2^2 + 3^2).$$

Each of the four corner vertices of degree two is not adjacent to $6n - 13$ three degree vertices and 6 two degree vertices. Hence their contribution in the F-coindex is

$$4(6n - 13)(2^2 + 3^2) + 24(2^2 + 2^2).$$

Contribution of the rest of the two degree vertices is

$$4(6n - 12)(2^2 + 3^2) + 20(2^2 + 2^2).$$

Hence,

$$\sum_{u \in V(G)} \sum_{v \notin N(u)} [(d(u))^2 + (d(v))^2] = 18(6n - 16)(6n - 15) + 104(6n - 16) + 72(6n - 14) + 364 + 52(6n - 12) + 160 + 52(6n - 13) + 192 = 648n^2 - 1668n + 1064.$$  

Since one edge is shared by a pair of vertices,

$$F(L(S(L_n))) = \frac{1}{2}(648n^2 - 1668n + 1064) = 324n^2 - 832n + 532.$$

\[\square\]

3. **F-index of some nanostructure**

**Theorem 11.** Let $G$ be a line graph of the subdivision graph of a square grid graph with $mn$ vertices (See Figure 7). Then $F$-index of $G$ is $F(G) = 256mn - 350m - 350n + 440$.

*Proof.* The line graph of the subdivision graph of a square grid graph with $mn$ vertices, i.e., $(m-1)(n-1)$ number of squares contains 8 two degree vertices, $6(n-2) + 6(m-2)$ three degree vertices and $4(m-2)(n-2)$ four degree vertices. Hence,
Theorem 12. Let $G$ be a line graph of the subdivision graph of a 2D-lattice of $TUC_4C_8[p, q]$ (See Figure 8). Then $F$-index of $G$ is $F(G) = 324pq - 130p - 130q$.

Proof. The line graph of the subdivision graph of a 2D-lattice of $TUC_4C_8[p, q]$ contains $4p + 4q$ two degree vertices and $12pq - 6p - 6q$ three degree vertices. Hence,

$$F(G) = \sum_{v \in V(G)} d^3(v)$$

$$= (4p + 4q)^3 + (12pq - 6p - 6q)^3$$

$$= 324pq - 162p - 162q + 32p + 32q$$

$$= 324pq - 130p - 130q.$$

\[\Box\]
Theorem 13. Let $G$ be a line graph of the subdivision graph of $TUC_4C_8[p,q]$ nanotube (See Figure 9). Then $F$-index of $G$ is $F(G) = 324pq - 130p + 2q$.

Proof. The line graph of the subdivision graph of $TUC_4C_8[p,q]$ nanotube contains $2q$ pendant vertices, $4p$ two degree vertices and $12pq - 6p$ three degree vertices. Hence,

$$F(G) = \sum_{v \in V(G)} d^3(v)$$

$$= 2q.1^3 + 4p.2^3 + (12pq - 6p).3^3$$

$$= 2q + 32p + 27(12pq - 6p)$$

$$= 324pq - 130p + 2q.$$ 

\[\square\]

Theorem 14. Let $G$ be a line graph of the subdivision graph of $TUC_4C_8[p,q]$ nanotorus (See Figure 10). Then $F$-index of $G$ is $F(G) = 324pq + 2p + 2q$.

Proof. The line graph of the subdivision graph of $TUC_4C_8[p,q]$ nanotorus contains $2p + 2q$ pendant vertices, and $12pq$ three degree vertices. Hence,

$$F(G) = \sum_{v \in V(G)} d^3(v)$$

$$= (2p + 2q).1^3 + 12pq.3^3$$

$$= 324pq + 2p + 2q.$$ 

\[\square\]

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