GRAPH MANIFOLDS AS ENDS OF NEGATIVELY CURVED RIEMANNIAN MANIFOLDS

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Dedicated to Professor Kenji Fukaya on the occasion of his sixtieth birthday

Abstract. Let \( M \) be a graph manifold such that each piece of its JSJ decomposition has the \( H^2 \times \mathbb{R} \) geometry. Assume that the pieces are glued by isometries. Then, there exists a complete Riemannian metric on \( \mathbb{R} \times M \) which is an “eventually warped cusp metric” with the sectional curvature \( K \) satisfying \(-1 \leq K < 0\).

A theorem by Ontaneda then implies that \( M \) appears as an end of a 4-dimensional, complete, non-compact Riemannian manifold of finite volume with sectional curvature \( K \) satisfying \(-1 \leq K < 0\).

1. Introduction and Main Theorem

1.1. Ends of manifolds of negative curvature. If a non-compact manifold \( N \) is the interior of some compact manifold with boundary, then each boundary component is called an end of \( N \). Let \( N \) be a complete, non-compact Riemannian manifold of finite volume such that the sectional curvature \( K \) satisfies \(-1 \leq K < 0\). It is known by Gromov-Schroeder \([BGS]\) that \( N \) is diffeomorphic to the interior of a compact manifold, \( \bar{N} \), with boundary, \( \partial \bar{N} \). \( \partial \bar{N} \) has finitely many components, and each component is an end of \( N \).

It is a wide open question to decide which manifolds, \( M \), can appear as ends of such Riemannian manifolds. An end is a closed manifold and one general obstruction by Gromov \([G, 0.5]\) is that the simplicial volume of \( \partial \bar{N} \), hence, of each end is zero. Also, the \( \ell^2 \)-betti number and the Euler characteristic vanish (see \([Be1]\), Corollary 15.7). It is a theorem by Avramidi-Phan \([AP]\) Corollary 5\) that if the dimension of an end we concern is at most 4, then it is aspherical. In this paper we address the question: which aspherical manifolds can appear as such ends?

For example, an \( n \)-dimensional torus appears as an end of an \((n + 1)\)-dimensional hyperbolic manifold of finite volume. Other examples of ends are circle bundles over some hyperbolic manifolds of various dimension, \([E]\) (cf. \([Be2]\), \([M]\) for the complex hyperbolic versions). In
contrast to tori, such bundles will not be ends of any complete, non-compact Riemannian manifold of finite volume such that $-1 \leq K \leq -c < 0$ for some $c > 0$, since under this curvature assumption, the fundamental group of an end has to be virtually nilpotent.

In dimension 2, if a closed, aspherical and oriented manifold has zero simplicial volume, then it is a torus, and it appears as an end. In dimension 3, any closed aspherical manifold $M$ is irreducible, has an infinite fundamental group and its universal cover is $\mathbb{R}^3$, cf. [Lu]. If the simplicial volume of such $M$ is 0 then it is a graph manifold or it has the $Sol$ geometry.

A graph manifold is an aspherical, orientable closed 3-manifold whose JSJ decomposition along embedded incompressible tori/Klein bottles contains only Seifert fibred spaces. Abresh-Schroeder [AS] proved certain graph manifolds appear as ends. Our theorem will provide a large class of graph manifolds that appear as ends, and their examples are contained in our class (but for such manifold $M$, their manifold $N$ that has $M$ as an end is different from ours). Also, a 3-dimensional sol-manifold appears as an end, [P].

Other known examples are infranilmanifolds, [O], [BeK]. See also [Be] for an axiomatic construction from known examples.

1.2. Eventually warped product cusp metric. In this paper we show that a family of (3-dimensional) graph manifolds occurs as ends of complete, non-compact, Riemannian manifolds of finite volume whose sectional curvature $K$ satisfies $-1 \leq K < 0$.

To explain our strategy, we recall the following groundbreaking theorem by Ontaneda, [O]. If a (not necessarily connected) manifold $B$ is diffeomorphic to the boundary of a connected, smooth, compact manifold $N$, then we say that $B$ bounds $N$. We sometimes say $B$ bounds without specifying $N$.

**Theorem 1.1** (Ontaneda). Let $B$ be a closed manifold such that either $\dim B \leq 4$ or the Whitehead group $\text{Wh}(B)$ vanishes. Assume $\mathbb{R} \times B$ admits a complete Riemannian metric $g$ such that

1. there exists a constant $C < 0$ with the sectional curvature of $g$ satisfying $C \leq K < 0$,
2. $(-\infty, 0] \times B$ has finite $g$-volume,
3. there is $D > 0$ such that on $[D, \infty) \times B$, the metric $g$ is of the form $g = dr^2 + e^{2r} g_B$ for some Riemannian metric $g_B$ on $B$.

Then $B \sqcup B$ bounds a manifold whose interior admits a complete Riemannian metric of finite volume and the sectional curvature in $(-1, 0)$.

A metric on $\mathbb{R} \times B$ that satisfies the condition (2) is called a cusp metric and an eventually warped (cusp) metric if it satisfies (3). This theorem is stated only implicitly in [O] (see [Be], where the result is quoted in this form), since it is an intermediate claim, but a detailed
argument is given. The actual value of $C$ is not important and we can take $C = -1$ by rescaling $g$.

We will show that for a manifold $B$ in certain families there exists a Riemannian metric on $\mathbb{R} \times B$ that is an eventually warped cusp metric with $C \leq K < 0$ for some $C < 0$. Then Theorem 1.1 implies that $B$ is an end of a manifold of negative curvature.

This argument appears in [O] for the infranil manifolds (The existence of a desired metric is known by [BK]) then also is used in [Be] and [P] to construct other examples of ends.

1.3. **Graph manifolds and flip manifolds.** To illustrate the first family we handle, let $W$ be a 3-dimensional manifold which is diffeomorphic to $\Sigma \times S^1$, where $\Sigma$ is a compact surface with non-empty boundary and $S^1$ is a circle. Each boundary component of $W$ is a torus, $S^1 \times S^1$, where the first factor is a boundary component of $\Sigma$. We put an orientation to each factor. We call such $W$ a **piece**, and $\Sigma$ the **base surface** of $W$. We construct a **closed**, connected, 3-dimensional manifold $M$, which is a graph manifold, from a finite collection of pieces by gluing a pair of boundary tori by a diffeomorphism, a **gluing map**.

There are two special maps for gluing: the trivial map mapping the first factor to the first factor and the second one to the second; the **flip map** interchanging the first and second factors. We preserve the orientation of the factor. We say $M$ is a **flip-manifold**, [KL], if each gluing map is either the trivial map or the flip map.

Some remarks are in order. There are eight ways to glue a pair of boundary tori: two ways to put an orientation on each of the two $S^1$, then a trivial map or a flip map. If $\Sigma$ is a closed surface, then $\Sigma \times S^1$ is considered as a flip manifold made from one piece of two boundary components, where the gluing map is trivial.

More generally, maybe the $S^1$-fibers are not-orientable, and/or a piece is a Seifert fibered space, $S$. [S, Section 3]. Let $s_1, \ldots, s_n$ be the singular fibers of $S$ where the twist at $s_i$ is by the $q_i/p_i$ of a full twist. $(q_i, p_i)$ is called the **orbit invariant** of $s_i$, which is a pair of co-prime integers with $0 < q_i < p_i$. One can say that a Seifert fibered space is an $S^1$-bundle over a base orbifold, where the singular fibers occur at the orbifold points, while $\Sigma \times S^1$ is a (trivial) $S^1$-bundle over the surface $\Sigma$.

A **generalized flip manifold** is a generalization of a flip manifold where we allow Seifert fibered spaces in addition to products $\Sigma \times S^1$ as pieces in the definition. Of course we only consider gluing maps that are diffeomorphisms.

We call a base surface/orbifold $\Sigma$ **hyperbolic** if we can put a complete hyperbolic (orbifold) metric of finite area to the interior of $\Sigma$. We denote $\Sigma^\circ$ the interior of $\Sigma$. An $S^1$-bundle over $\Sigma^\circ$, has a Riemannian metric that is locally a product of the hyperbolic metric and $S^1$ (see...
In other words, it has the geometry of $\mathbb{H}^2 \times \mathbb{R}$, or the metric is of type $\mathbb{H}^2 \times \mathbb{R}$. We only consider pieces of this kind in this paper.

We truncate a small neighborhood of each cusp of $\Sigma^o$ such that the each boundary component of the universal cover of the truncated $\Sigma^o$ with respect to the hyperbolic metric is a horoline in $\mathbb{H}^2$. Since $\Sigma$ is diffeomorphic to the truncated $\Sigma^o$, we obtain a Riemannian metric on the $S^1$-bundle over $\Sigma$ such that each boundary torus/Klein bottle is flat. We also say this metric is of type $\mathbb{H}^2 \times \mathbb{R}$.

In this paper we say a graph manifold $M$ has a geometrization if one can put a Riemannian metric of type $\mathbb{H}^2 \times \mathbb{R}$ on all pieces such that every gluing map along the boundary tori/Klein bottle is an isometry. Some remarks are in order. We do not assume that each gluing map is a trivial map or a flip map, see Example 2.8. A metric of type $\mathbb{H}^2 \times \mathbb{R}$ on a piece is not unique. Without loss of generality, we may assume that there is a small $c > 0$ such that the length of the fibers of the pieces and the length of the boundary components of the pieces are $c$. The resulting metric on $M$ after gluing the pieces is only $C^1$. If $M$ has a geometrization or not depends only on the topology of $M$. In [BK], they use the term isometric geometrization instead of geometrization (see Remark 2.9).

We prove:

**Theorem 1.2.** Let $M$ be a graph manifold such that each piece has the geometry of $\mathbb{H}^2 \times \mathbb{R}$. Assume $M$ has a geometrization (ie, the gluing is isometric). Then there is a complete Riemannian metric $g$ on $\mathbb{R} \times M$ that is an eventually warped cusp metric with the sectional curvature $K$ satisfying $C \leq K < 0$ for some constant $C < 0$.

**Remark 1.3.** The metric in the above theorem is taken to be $C^\infty$. This is always the case for other results in this paper too.

By rescaling the metric $g$ we may always take $C = -1$. Theorem 1.2 has a generalization to high dimensional manifolds, see Theorem 1.6.

Since $\dim M = 3$, combining Theorem 1.2 and Theorem 1.4 we immediately obtain:

**Corollary 1.4** (Graph manifolds with a geometrization). Let $M$ be a graph manifold such that each piece has the geometry of $\mathbb{H}^2 \times \mathbb{R}$. Assume $M$ has a geometrization.

Then there exists a 4-dimensional, complete, non-compact Riemannian manifold $N$ of finite volume, of the sectional curvature $K$ satisfying $-1 \leq K < 0$, and with $M$ appearing as an end. More precisely, there is a compact subset $C$ in $N$ such that $N \setminus C$ has two connected components, and that each component is diffeomorphic to $M \times (0, \infty)$.

It is known that among graph manifolds $M$ whose pieces have the geometry of $\mathbb{H}^2 \times \mathbb{R}$, $M$ has a geometrization if and only if it has a
Riemannian metric of non-positive sectional curvature, \([L]\), cf. \([BS]\), \([BSc]\). Hence we can rephrase our results as follows:

**Corollary 1.5** (Graph manifolds of non-positive curvature). Let \(M\) be a closed graph manifold such that each piece has the geometry of \(\mathbb{H}^2 \times \mathbb{R}\). Assume \(M\) has a Riemannian metric of non-positive curvature. Then the conclusion of Theorem 1.2 and Corollary 1.4 holds.

### 1.4. High dimensional graph manifolds.

There are several notions of high dimensional graph manifolds (cf. \([FLS]\)) and one can prove a high dimensional version of Theorem 1.2. The main part of the proof of the theorem is by constructing a suitable Riemannian metric, which is same for the high dimensional case.

Fix \(n \geq 2\) and \(m \geq 1\). Let \(X\) be an \(n\)-dimensional complete, non-compact, hyperbolic manifold of finite volume such that the cross-section of each cusp is an \((n-1)\)-dimensional torus. Let \(\Sigma \subset X\) be a compact manifold with boundary obtained by truncating a sufficiently small neighborhood of each cusp from \(X\) so that each boundary component is a flat torus. The interior of \(\Sigma\) is diffeomorphic to \(X\). Take a Riemannian product \(W = \Sigma \times T^m\), where \(T^m\) is an \(m\)-dimensional flat torus. Each boundary component of \(W\) is an \((n + m - 1)\)-dimensional torus. We call \(W\) a piece, and \(\Sigma\) the base.

Suppose a closed \((n + m)\)-dimensional manifold \(M\) is obtained from pieces with various bases by gluing pairs of boundary components of the pieces by diffeomorphisms, then we call \(M\) a high dimensional graph manifold. We say \(M\) has geometrization if all gluing maps are isometric with respect to the product metric on the pieces.

**Theorem 1.6** (High dimensional graph manifolds). Let \(M\) be an \((n + m)\)-dimensional high dimensional graph manifold. Assume \(M\) has a geometrization. Then \(M\) carries a metric of non-positive curvature, so that \(\text{Wh}(M)\) vanishes. Also, there is a complete Riemannian metric \(g\) on \(\mathbb{R} \times M\) that is an eventually warped cusp metric with the sectional curvature \(K\) satisfying \(C \leq K < 0\) for some constant \(C < 0\).

**Remark 1.7.** We only consider a product metric on each piece, but we can formulate the result for locally product metrics as in Theorem 1.2.

As before, it follows from Theorem 1.1:

**Corollary 1.8.** Let \(M\) be an \((n + m)\)-dimensional high dimensional graph manifold. Assume it has a geometrization. Then there exists an \((n + m + 1)\)-dimensional, complete, non-compact Riemannian manifold \(N\) of finite volume, of the sectional curvature \(K\) satisfying \(-1 \leq K < 0\), and with \(M\) appearing as an end.

### 1.5. The other construction.

We discuss the other family of examples of ends. This family contains manifolds of various dimension, and in dimension 3, it contains all flip manifolds without a piece whose base
A manifold in this family is also obtained by gluing pieces along their boundary, and each boundary component is a circle bundle over a circle bundle over a hyperbolic manifold \( N \). If \( \dim N = 0 \), the boundary is a torus and we obtain graph-manifolds.

Here is a precise description. Let \( M_i, i = 1, 2 \), be \( n \)-dimensional closed hyperbolic manifolds, and \( N_i \) totally geodesic, closed submanifolds of codimension two in \( M_i \), respectively, such that \( b : N_1 \rightarrow N_2 \) is an isometry. For a sufficiently small \( \epsilon > 0 \), let \( P_i \) be \( S^1 \)-bundles over \( V_i = M_i \backslash N_\epsilon(N_i) \), respectively, with Riemannian metrics which are locally product of the hyperbolic metric on the base manifolds and the circle.

\[ \partial P_1 = P_1|_{\partial V_1} \text{ are flat torus-bundles over } N_1, \text{ respectively. We glue } P_1, P_2 \text{ along their boundaries by a bundle map whose base map is the isometry } b : N_1 \rightarrow N_2 \text{ and on the fiber it is a diffeomorphism, for example, a trivial map or a flip map, as in the graph manifold case. This gives an } (n+1) \text{-dimensional manifold, } W. \text{ If the bundle map is an isometry, then we say it satisfies the gluing condition and } W \text{ has a geometrization.} \]

Then

**Theorem 1.9** (Theorem 3.1). Assume \( W \) has a geometrization. Then \( W \) carries a metric of non-positive curvature, so that \( Wh(W) \) vanishes. Also, \( \mathbb{R} \times W \) carries a complete Riemannian metric that is an eventually warped cusp metric with \( C \leq K < 0 \) for some constant \( C < 0 \).

Combining Theorem 1.9 and Theorem 1.1 we obtain:

**Corollary 1.10** (Piecewise \( S^1 \)-bundles). Assume \( W \) has a geometrization. Then \( W \) appears as an end of an \( (n+2) \)-dimensional Riemannian manifold \( Z \) that is complete, non-compact, of finite volume, with the sectional curvature \( K \) satisfying \( -1 \leq K < 0 \).

1.6. **Gluing condition.** We examine the gluing condition for a geometrization in the case \( n = 3 \) in some details, where \( n \) is the dimension of \( M_i, i = 1, 2 \). Submanifolds \( N_i, i = 1, 2 \), are simple closed geodesics, and we denote them by \( \gamma_i \), respectively. By our assumption they have same length. Let \( m_i \) be the meridain curve for \( \gamma_i \) in \( M_i \). \( X_i \) is an \( S^1 \)-bundle over \( M_i \backslash N_\epsilon(\gamma_i) \). We denote \( \sigma_i \) the fiber circle of \( X_i \). With respect to the Riemannian metric, we can measure the monodromy (i.e., rotation) along the curve \( \gamma_i \) for \( \sigma_i \) and \( m_i \), respectively. We denote them by \( 0 \leq \theta(\sigma_i), \theta(m_i) < 2\pi \).

We now consider a bundle map \( \phi : \partial X_1 \rightarrow \partial X_2 \) such that the base map is the isometry \( b \) and that it is a flip map on the torus fiber:

\[ \phi(m_1) = \sigma_2, \quad \phi(\sigma_1) = m_2 \]
Then we can arrange $\phi$ to be an isometry, ie, the gluing condition is satisfied, if and only if
\begin{equation}
\theta(m_1) = \theta(\sigma_2), \quad \theta(\sigma_1) = \theta(m_2)
\end{equation}

We conclude the introduction with examples $\{(M_i, N_i)\}$ that admit circle bundles $X_i$ satisfying (1.1), which give $W$ of dim $W = 4$ by Theorem 1.9.

**Example 1.11** ($S^1$-bundle with a given monodromy). Take a closed, hyperbolic 3-manifold $M$ with a simple closed geodesic $\gamma$ which is non-trivial in $H_1(M, \mathbb{R})$. Let $m$ be the meridian curve of $\gamma$ in $M$, and $\theta(m)$ its monodromy along $\gamma$. Set $V = M \setminus N_\epsilon(\gamma)$ with a small $\epsilon > 0$. We will construct an $S^1$-bundle over $V$, $X$, and glue $X$ and a copy of $X$ along their boundary and obtain $W$. Let $\sigma$ denote the fiber circle. For our construction, we need to arrange $\theta(m) = \theta(\sigma)$.

For example, let $M$ be such that all of its closed geodesics are simple (such examples exist, [CR]). Taking a finite cover if necessary, we may assume that $H_1(M, \mathbb{R})$ is non-trivial [A]. Let $p : \pi_1(M) \to H_1(M, \mathbb{R})$ be the homomorphism obtained from abelianization. Take any closed (simple) geodesic $\gamma$ with $p([\gamma]) \neq 0$. Then $H_1(\gamma, \mathbb{R})$ injects to $H_1(M, \mathbb{R})$.

Set $\theta_0 = \theta(m)$. Then there is a homomorphism $h : \pi_1(M) \to S^1$ such that $h([\gamma]) = \theta_0$. Indeed we take a homomorphism $f : \mathbb{Z} \to S^1$ such that $f(p([\gamma])) = \theta_0$ then set $h = f \circ p$.

Now take an $S^1$-bundle $X$ over $M$, which is locally a Riemannian product whose monodromy representation of $\pi_1(M)$ to $S^1$ is $h$. Then $\theta(\sigma) = \theta_0$.

Now take $(M, \gamma)$ and its copy, then this pair satisfied the condition (1.1), so that Theorem 1.9 applies.

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2. **Proof of Theorem 1.2 and Theorem 1.6**

We will prove Theorem 1.2. We first treat the case where every piece in a graph manifold is the product of a circle and a surface, then discuss the general case.

We then prove Theorem 1.6. The main part of the proof overlaps with the proof of Theorem 1.2, which is Proposition 2.4.

2.1. **Geometric idea.** We first explain our method to construct a desired metric on $\mathbb{R} \times M$, where each piece of $M$ is a trivial bundle over a surface.

As the first step, it is pretty straightforward to construct a Riemannian metric of non-positive curvature on $M$. We review the metric
construction, cf. [KL]. By assumption the interior of the base surface \( \Sigma_i \) of each piece \( P_i \) has a hyperbolic metric \( g_0 \) of finite volume.

Choose a small constant \( c > 0 \). Truncate the interior of \( \Sigma_i \) with the metric \( g_0 \) at each cusp so that each boundary circle has constant geodesic curvature and has length \( c \). We identify this truncated surface with \( g_0 \) and \( \Sigma_i \).

To express the idea clearly, we first assume \( M \) has a geometrization with respect to a product metric on each piece, \( P_i = \Sigma_i \times S^1 \), namely it is a Riemannian product \( \Sigma_i \times S^1(c) \), where \( S^1(c) \) is a circle of length \( c \).

The curvature satisfies \( -1 \leq K \leq 0 \). Each boundary component of \( P_i \) is \( S^1(c) \times S^1(c) \), so that we can glue \( P_i \)'s along their torus boundaries by the prescribed gluing maps, which are isometries by our assumption, and obtain a metric of non-positive curvature on \( M \). This metric has singularity along the tori where pieces are glued, but we can smooth it out keeping the curvature condition \( C \leq K \leq 0 \) for some \( C < 0 \).

In the next step we want to put a desired metric on \( \mathbb{R} \times M \), but if we consider a warped product

\[
\mathbb{R} \times e^r M
\]

it does not work in general by the following reason. Since \( M \) is compact, volume of \( (-\infty, 0] \times M \) is finite. Also, since \( M \) satisfies \( C < K \leq 0 \), the curvature on \( \mathbb{R} \times e^r M \) satisfies \( K < 0 \), but as \( r \to -\infty \), the diameter of \( M \) tends to 0 and the curvature \( K \) tends to \(-\infty \), while \( K \) tends to \(-1 \) as \( r \) tends to \( \infty \). So, this construction violates the curvature bound from below for \( r \to -\infty \).

In the above warped product, at each \( r \in \mathbb{R} \), the manifold \( M \) is rescaled as \( e^r M \). As a remedy, we use another metric on \( M \) for the part \( r < 0 \) with \( r \) small enough. Take \( a > 0 \) such that \( e^r \leq c \) if \( r < -a \).

For each \( r \in (-\infty, -a] \), truncate the initial complete hyperbolic metric \( g_0 \) on the interior of \( \Sigma_i \) such that the boundary circle has length \( e^r \), which we denote by \( \Sigma_i(e^r) \). Take a Riemannian product \( \Sigma_i(e^r) \times S^1(e^r) \), which is the metric structure on \( P_i \) at \( r \). Each boundary component is \( S^1(e^r) \times S^1(e^r) \). Now glue them by the given isometries and obtain the metric on \( M \) at \( r \), which we denote by \( M_r \). As before we smooth out near the gluing tori. In this way we obtain a metric on \( (-\infty, -a] \times M \), which we write as \( (-\infty, -a] \times M_r \). Notice that volume of \( M_r \) is (more or less) proportional to \( e^r \), so that one expects volume of \( (-\infty, -a] \times M_r \) is finite. Also, we arrange that the curvature satisfies \( C \leq K \leq 0 \) (cf. [F]).

So, we try to interpolate \( (-\infty, -a] \times M_r \) and \( [a, \infty) \times e^r M_0 \) between \( r \in [-a, a] \), where \( M_0 \) is \( M \) with a Riemannian metric, say, constructed in the previous paragraph (maybe we rescale it by a constant). Note that the metric on \( M_0 \) is fixed for \( r \in [a, \infty) \) while \( M_r \) keeps changing for \( r \in (-\infty, a] \). Also, notice that diameter of \( M_r \) tends to \( \infty \) as \( r \to -\infty \).
Lastly, we address the issue that a metric on a piece is maybe a locally Riemannian product. The piece is topologically a trivial $S^1$-bundle, and it has a locally product metric such that the fiber circles have same length. The difference from the Riemannian product case is encoded in the monodromy representation of the fundamental group of the base $\Sigma$ into $S^1$, viewed as a group, which acts on the fiber circles by rotations. By assumption our manifold $M$ admits a geometrization, i.e., pieces are glued by isometries along the tori. In conclusion, the method we explained above will work in this generality without any change because we only use the property that the gluing maps are isometric.

2.2. Metric construction. We denote the group of isometries of $\mathbb{R}^n$ by Isom($\mathbb{R}^n$). In the following we consider a product

$$Y = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^m$$

and, for example, an element of Isom($\mathbb{R}^l$) naturally acts on $Y$ by an isometry that is trivial except on $\mathbb{R}^l$. The Euclidean metric on $\mathbb{R}^l, \mathbb{R}^m$ are denoted by $d\rho^2, dx^2$, respectively. We denote a flat torus of dimension $n$ by $T^n$. For $n = 1$, we may also write it as $S^1$.

The goal of the following few subsections is Proposition 2.4, which shows that a certain Riemannian metric $g$ that is invariant by Isom($\mathbb{R}^l$) and Isom($\mathbb{R}^m$) exists on $Y$. The proof of Proposition 2.4 is by concretely constructing a metric $g$. If tori $T^l, T^m$ are given as quotients of $\mathbb{R}^l, \mathbb{R}^m$ by isometric actions, then the metric $g$ descends to

$$X = \mathbb{R} \times \mathbb{R} \times T^l \times T^m,$$

which will be used later to prove theorems.

To define the metric $g$, we prepare several functions. Pick a $C^\infty$ function, $R$, on $\mathbb{R}$ such that

$$R(r) = \begin{cases} r & \text{if } r \leq 1, \\ 3 & \text{if } r \geq 5, \end{cases} \quad R' > 0 \text{ on } (1, 5), \quad \frac{1}{2} \leq R'' \leq 0.$$

We take a nonnegative $C^\infty$ function, $\lambda$, supported in $[-1, 1]$ and satisfying $\int_{-1}^1 \lambda(x) \, dx = 1$, then define the convolution product of $\lambda$ and a locally Lebesgue integrable function $\varphi$ on $\mathbb{R}$ by

$$\lambda * \varphi(x) := \int_{-\infty}^{\infty} \varphi(t) \lambda(x - t) \, dt.$$ 

Note that $\lambda * \varphi$ is also defined in the case where $\varphi$ is a finite Borel measure on $\mathbb{R}$. $\lambda * \varphi$ is a $C^\infty$ function.

Since $\lambda * e^t$ satisfies $(\lambda * e^t)' = \lambda * e^t$, we have $\lambda * e^t = c e^t$, where $c := (\lambda * e^t)(0)$. We put

$$\tilde{f}(s) := \begin{cases} 1 & \text{if } s \leq 0, \\ e^s & \text{if } s > 0, \end{cases}$$
then define a $C^\infty$ function by:

$$f := \lambda \ast \bar{f}.$$ 

By the definition,

$$(2.1) \quad f(s) = \begin{cases} 
1 & \text{if } s \leq -1, \\
 ce^s & \text{if } s \geq 1.
\end{cases}$$

We observe

$$(2.2) \quad \bar{f}'(s) = \begin{cases} 
0 & \text{if } s < 0, \\
e^s & \text{if } s > 0,
\end{cases}$$

$$(2.3) \quad \bar{f}''(s) = \begin{cases} 
0 & \text{if } s < 0, \\
\delta_0 & \text{if } s = 0, \\
e^s & \text{if } s > 0,
\end{cases}$$

where $\delta_0$ denotes Dirac's delta measure at 0 and we consider the distributional derivative for $\bar{f}''$. Note that $f' = \lambda \ast \bar{f}'$ and $f'' = \lambda \ast \bar{f}''$. It holds that $f \geq 1$ and $f', f'' \geq 0$.

We pick a $C^\infty$ function, $h$, on $\mathbb{R}$ such that

$$h(r) = \begin{cases} 
1 + e^r & \text{if } r \leq -1, \\
2e^r & \text{if } r \geq 1,
\end{cases} \quad h \geq 1, \ h', h'' > 0 \text{ on } \mathbb{R}.$$ 

Let $b$ be a positive constant and $F$ the $C^\infty$ function on $\mathbb{R}^2$ defined by

$$F(r, t) := b e^{R(r)} f(t - R(r)).$$

(We may take $b := 1$ in this section. $b$ is needed in the later sections.) Note that $F_t, F_u \geq 0$, where $F_t, F_u$ are the partial derivatives of $F$. Note also that $F = bc e^t$ for all $t \geq 4$.

We consider the metric

$$(2.4) \quad g = dr^2 + h(r)^2\left( dt^2 + b^2 e^{2R(r)} d\rho^2 + F(r, t)^2 d\tau^2 \right),$$

where $d\rho^2 = \sum_{\alpha=1}^l d\rho_\alpha^2$ is the $l$-dimensional Euclidean metric and $d\tau^2 = \sum_{\beta=1}^m d\tau_\beta^2$ the $m$-dimensional Euclidean metric. Let us set

$$g = \sum_{i=1}^n g_i dx_i^2,$$

i.e., $x_1 := r$, $x_2 := t$, $x_{2+\alpha} := \rho_\alpha$, and $x_{2+l+\beta} := \tau_\beta$, $g_1 := 1$, $g_2 := h(r)^2$, $g_{2+\alpha} := H(r)^2$, $H(r) := b e^{R(r)} h(r)$, $g_{2+l+\beta} := h(r)^2 F(r, t)^2$ for $\alpha = 1, 2, \ldots, l$ and $\beta = 1, 2, \ldots, m$.

Note that, for $t \geq 4$,

$$dt^2 + F(r, t)^2 d\tau^2 = dt^2 + bc e^{2t} d\tau^2$$

is a hyperbolic metric.
We calculate the Christoffel symbols
\[ \Gamma^{2}_{12} = \frac{h'}{h}, \quad \Gamma^{2+\alpha}_{1,2+\alpha} = \frac{H'}{H}, \]
\[ \Gamma^{2+l+\beta}_{1,2+l+\beta} = \frac{F_r}{F} + \frac{h'}{h}, \quad \Gamma^{2+l+\beta}_{2,2+l+\beta} = \frac{F_l}{F}, \]
\[ \Gamma^{1}_{22} = -hh', \quad \Gamma^{2+l+\beta}_{2+\alpha,2+\alpha} = -HH', \quad \Gamma^{1}_{2+l+\beta,2+l+\beta} = -h^2 FF_r - hh' F^2, \]
\[ \Gamma^{2+\alpha,2+\alpha'} = -\frac{(h')^2}{h^2}. \]

for \( \alpha = 1, 2, \ldots, l \) and \( \beta = 1, 2, \ldots, m \), which are the unique nonzero values of \( \Gamma^k_{ij} \) under the symmetry \( \Gamma^k_{ij} = \Gamma^k_{ji} \). By using the curvature tensor
\[ R_{ijkl} = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \sum_{m=1}^n (\Gamma_{im}^l \Gamma_{jk}^m - \Gamma_{jm}^l \Gamma_{ik}^m), \quad R_{ijkl} = R_{ijkl} g_{ll} \]
the sectional curvatures are expressed as
\[ K_{ij} = \frac{R_{ijji}}{g_{ii} g_{jj}} = \frac{R_{ijji}}{g_{jj}} \]
since the metric \( g \) is diagonal. Calculating it we obtain
\[ K_{12} = -\frac{h''}{h}, \quad K_{1,2+\alpha} = -\frac{H''}{H}, \]
\[ K_{1,2+l+\beta} = \frac{F_r}{F} - 2h' F_r \frac{h''}{h}, \quad K_{2,2+\alpha} = \frac{h' H'}{hH}, \]
\[ K_{2,2+l+\beta} = \frac{F_l}{h^2 F} - h' F_r \frac{(h')^2}{h^2}, \]
\[ K_{2+\alpha,2+\alpha'} = -\frac{(H')^2}{h^2}, \quad K_{2+\alpha,2+l+\beta} = \frac{F_r H'}{HH} - \frac{h' H'}{hH}, \]
\[ K_{2+l+\beta,2+l+\beta'} = \frac{F_l^2}{h^2 F^2} - 2h' F_r \frac{F^2}{hF} - \frac{F^2}{h^2 F^2} - \frac{(h')^2}{h^2} \]

for \( \alpha, \alpha' = 1, 2, \ldots, l \) and \( \beta, \beta' = 1, 2, \ldots, m \) with \( \alpha < \alpha' \) and \( \beta < \beta' \).

To estimate \( K_{ij} \)’s we need the following.

**Lemma 2.1.**

1. \( f \geq f' \).
2. \( f - 2f' + f'' \geq 0. \)

**Proof.** We see \( \bar{f} \geq \bar{f}' \) and \( \bar{f} - 2\bar{f}' + \bar{f}'' \geq 0 \) from (2.2) and (2.3). Taking the convolution product of them with \( \lambda \) yields the lemma. \( \square \)
Lemma 2.2.  
(1) \( h \geq 1, h', h'' > 0 \).
(2) \( H, H', H'' > 0 \).
(3) \( F > 0, F_t, F_u, F_r \geq 0 \).
(4) The following functions are all uniformly bounded:
\[
\frac{f'}{f}, \frac{f''}{f}, \frac{h'}{h}, \frac{h''}{h}, \frac{H'}{H}, \frac{H''}{H}, \frac{F_r}{F}, \frac{F_{rr}}{F}, \frac{F_t}{F}, \frac{F_u}{F}.
\]

Proof. (1) is obvious.
We prove (2). The derivative of \( H = bh e^R \) is
\[
H' = bh e^R R' + bh' e^R,
\]
which is positive. Differentiating it again, we see
\[
H'' = bh e^R (h R'' + h(R')^2 + 2h'R' + h') \geq bh e^R (h R'' + h').
\]
If \( r \leq 1 \), then \( h R'' + h'' = h'' > 0 \). If \( r > 1 \), then \( h R'' + h'' > -h/2 + h'' = e'' > 0 \). Therefore \( H'' \) is positive everywhere.
We prove (3). It is clear that \( F > 0 \). It follows from \( f', f'' \geq 0 \) that \( F_t, F_u \geq 0 \). We see
\[
F_r = b e^R (-f'(t - R) R' + R' f(t - R)),
\]
which is nonnegative by Lemma 2.1 and \( R' \geq 0 \).
(4) is clear.
We prove (5). The boundedness of \( f'/f \) and \( f''/f \) follow from (2.1). The boundedness of \( h'/h, h''/h, T'/T, \) and \( T''/T \) are derived from their definitions. We see
\[
\frac{H'}{H} = \frac{h R' + h'}{h}, \quad \frac{H''}{H} = \frac{h R'' + h (R')^2 + 2h'R' + h''}{h},
\]
\[
\frac{F_r}{F} = \frac{R'(f - f')}{f}, \quad \frac{F_{rr}}{F} = \frac{(R')^2(f - 2f' + f'') + R''(f - f')}{f},
\]
\[
\frac{F_t}{F} = \frac{f'}{f}, \quad \frac{F_u}{F} = \frac{f''}{f},
\]
which are all bounded. This completes the proof of the lemma. \( \square \)

Lemma 2.3. There is a constant \( C < 0 \) such that \( C \leq K < 0 \).

Proof. It suffices to verify the negativity and boundedness of \( K_{ij} \). That is readily seen from Lemma 2.2 except the negativity of \( K_{1,2+l+\beta} \). We remark that \( F_{rr} \geq 0 \) does not hold. We have
\[
K_{1,2+l+\beta} = -\frac{\varphi}{f h},
\]
where
\[
\varphi := h(R')^2(f - 2f' + f'') + h R''(f - f') + 2 R' h'(f - f') + h'' f.
\]
If \( r \leq 1 \), then \( R = r \), which together with Lemma 2.1 implies \( \varphi > 0 \).
If \( r \geq 1 \), then \( h = 2e'' \) and so
\[
\frac{\varphi}{h} = (R')^2(f - 2f' + f'') + R''(f - f') + 2 R'(f - f') + f.
\]
By $R' \geq 0, -1/2 \leq R'' \leq 0$, and by Lemma 2.1, we obtain
\[ \varphi h \geq R''(f - f') + f \geq R''f + f \geq \frac{f}{2} > 0. \]
Therefore, $K_{1,2+t+\beta}$ is negative. \qed

2.3. Properties of $g$. Let $b, c$ be the constants that previously appeared.

Proposition 2.4. For $l, m > 0$, there exists a Riemannian metric $g$ on $Y = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^m$ that is invariant by $\text{Isom}(\mathbb{R}^l)$ and $\text{Isom}(\mathbb{R}^m)$ satisfying the following (1)–(7).

(1) There is a constant $C < 0$ such that the sectional curvature $K$ satisfies $C \leq K < 0$ on $Y$.
(2) Let $T^l, T^m$ be flat tori obtained as quotients of $\mathbb{R}^l, \mathbb{R}^m$ by isometries. Then $g$ defines a metric on $\mathbb{R} \times \mathbb{R} \times T^l \times T^m$ such that the volume of the following subset is finite:
\[ \{ (r, t, \rho, \tau) \mid r \in (-\infty, -1], t \in [r - 1, 2], \rho \in T^l, \tau \in T^m \}. \]
(3) For $r \leq 0$ and $t \leq r - 1$,
\[ g = dr^2 + h(r)^2(dt^2 + b^2 e^{2r}d\rho^2 + b^2 e^{2r}dr^2). \]
(4) For $r \geq 0$ and $t \leq -1$,
\[ g = dr^2 + h(r)^2(dt^2 + b^2 e^{2R(r)}d\rho^2 + b^2 e^{2R(r)}dt^2). \]
(5) For $r \in \mathbb{R}$ and $t \geq 4$,
\[ g = dr^2 + h(r)^2(dt^2 + b^2 e^{2R(r)}d\rho^2 + b^2 c^2 e^{2t}dt^2). \]
(6) For $r \geq 5$, $g$ is a warped metric of the form:
\[ g = dr^2 + 4e^{2r} \hat{g}, \]
where $\hat{g}$ is the metric on $\mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^m$ defined by
\[ \hat{g} := dt^2 + b^2 e^6 d\rho^2 + b^2 e^6 f(t - 3)^2dt^2. \]
(7) The metric $\hat{g}$ in (6) has non-positive curvature.

Remark 2.5. (i) $C$ does not depend on $l, m$.
(ii) By (5), for all $r$ and for $t \geq 4$ the metric is
\[ g = dr^2 + h(r)^2(d_{hyp} + b^2 e^{2R(r)}d\rho^2), \]
where $d_{hyp} := dt^2+b^2 e^{2t}dt^2$ is a hyperbolic metric with $K = -1$.
(iii) In the proof of Theorem 1.2 setting $l = m = 1$, $g$ will be used to put a Riemannian metric on a neighborhood of a boundary component of $\mathbb{R} \times P$, where $P = \Sigma \times S^1$ is a piece of the flip-manifold $M$. Outside of the neighborhood, we use a metric from a hyperbolic metric on $\Sigma$, which coincides with the metric $g$ at $t = 4$ as in (ii). For Theorem 1.6 the general form of $g$ is used.
Proof. Let $g$ be the metric given by (2.4).

(1) By Lemma 2.3.

(2) Without loss of generality we may assume that $\text{vol}(T^d) = 1$, $\text{vol}(T^m) = 1$ with respect to $d\rho^2, dr^2$, respectively, since the volume of the concerned set is proportional to the product $\text{vol}(T^d)\text{vol}(T^m)$ because of the form of $g$.

For $r \leq -1$, we have $h(r) = 1 + e^r$ and $R(r) = r$. We divide the subset into two according to $t$:

(i) The part for $t \in [r + 1, 2]$. Since $t - R(r) = t - r \geq 1$, we have $f(t - R(r)) = ce^{t - R(r)}$, hence

$$g = dr^2 + (1 + e^r)^2(dt^2 + b^2 e^{2r} d\rho^2 + b^2 e^{2r} dr^2).$$

Fix $r$. The metric $dt^2 + b^2 e^{2r} d\tau^2$ is hyperbolic, and its volume for the part $t \in [r + 1, 2]$, $\tau \in T^m$ is at most $b^m c^m \int_{-\infty}^{\infty} e^{mt} dt = b^m c^m e^{2m}$. Hence volume of the part $t \in [r + 1, 2]$, $\rho \in T^l$, $\tau \in T^m$ for the metric $dt^2 + b^2 e^{2r} d\rho^2 + b^2 e^{2r} d\tau^2$ is at most $b^l + b^m c^m e^{2m}$. Now the $g$-volume for the part $r \leq -1$, $t \in [r + 1, 2]$, $\rho \in T^l$, $\tau \in T^m$ is, since $1 + e^r \leq 2$, at most $2^{l+m+1} b^l + b^m c^m e^{2m} \int_{-\infty}^{1} e^{t} dl = 2^{l+m+1} b^l + b^m c^m e^{2m-1}/l$.

(ii) The part for $t \in [-1, r + 1]$. In this part, we have $t - R(r) = t - r \in [-1, 1]$, so $f(t - R(r)) \leq ce$. The metric is

$$g = dr^2 + (1 + e^r)^2(dt^2 + b^2 e^{2r} d\rho^2 + b^2 e^{2r} f(t - R(r))^2 dr^2).$$

Since the volume of $be^r f(t - R(r))^T \rho$ is at most $b^m c^m e^m$, the volume for $(t, \tau), t \in [-1, r + 1], \tau \in T^m$ is at most $2b^m c^m e^{m - 2r}$, so that the volume of $dt^2 + b^2 e^{2r} d\rho^2 + b^2 e^{2r} f(t - R(r))^2 d\tau^2$ is at most $2^{l+m+1} b^l + b^m c^m e^{l+m/2}$. Finally, the volume of this part is, since $1 + e^r \leq 2$, at most

$$2^{l+m+2} b^l + b^m c^m e^{m} \int_{-\infty}^{-1} e^{(t-1)/2} \frac{e^{2m}}{l} + 2 \frac{2}{l+m}. \frac{e^{2m}}{l}.$$

Combining (i) and (ii), volume of the subset is at most

$$2^{l+m+1} b^l + b^m c^m e^{l+m} \frac{e^{2m}}{l} + 2 \frac{2}{l+m}.$$
The sectional curvatures for $\hat{g}$ are

$$K_{12} = K_{23} = 0, \quad K_{13} = -\frac{f''(t-3)}{f},$$

which are non-positive. This completes the proof. □

2.4. Proof of Theorem 1.2 where the pieces are products.

Proof. By assumption the graph manifold $M$ has a geometrization, i.e., each piece has a locally product Riemannian metric of type $\mathbb{H}^2 \times \mathbb{R}$, and the gluing maps are isometries. In the following, we first give an argument assuming that $M$ has a geometrization with respect to a product metric on each piece. Then we will explain that in fact our argument applies to the locally product case as well.

Step 1. Let $P_i$ be the pieces of $M$. Suppose $P_i = \Sigma_i \times S^1$. We will put a Riemannian metric on each $\mathbb{R} \times P_i$ so that they match up for gluing along boundary, which defines a Riemannian metric on $\mathbb{R} \times M$. First, put a complete, hyperbolic metric of finite volume in the interior of each $\Sigma_i$. Let $\text{vol}_{\text{hyp}}(\Sigma_i)$ denote its volume. There is a constant $L > 0$, such that the interior of each $\Sigma_i$ contains a compact subset $K_i$ homeomorphic to $\Sigma_i$ such that each connected component of $\Sigma_i \setminus K_i$ is isometric to an annulus $(-\infty, 0) \times S^1(\text{be}L)$ with the metric $dt^2 + e^{2t}d\tau^2$, i.e., the warped product $(-\infty, 0) \times \mathbb{R}(\text{be}L)$, where $S^1(a) := \mathbb{R}/a\mathbb{Z}$ is a circle of length $a > 0$.

Step 2. For each $r \in \mathbb{R}$, we consider a Riemannian product

$$K_i \times S^1(\text{be}^{R(r)-2}L)$$

then further take a “generalized” warped product with $\mathbb{R}$ as follows:

$$J_i = \mathbb{R} \times h(r) (K_i \times S^1(\text{be}^{R(r)-2}L)),$$

where at each $r$, the metric of the fiber $K_i \times S^1(\text{be}^{R(r)-2}L)$ is rescaled by $h(r)$. We say this is a generalized warped product since the metric on the fiber at $r$ depends on $r$. Then

Lemma 2.6. (1) The subset of $J_i$ for the part $r < 0$ has finite volume, which is bounded above by $8\text{be}L \text{vol}_{\text{hyp}}(\Sigma_i)$.

(2) For the part $r > 5$, $J_i$ is a warped product:

$$\left(5, +\infty\right) \times 2e^r (K_i \times S^1(\text{be}L))$$

(3) The sectional curvature of $J_i$ is bounded:

$$C \leq K < 0,$$

where $C < 0$ is the constant from Proposition 2.4.

(4) Each boundary component of $J_i$ is isometric to $\mathbb{R} \times h(r) (S^1(\text{be}^2L) \times S^1(\text{be}^{R(r)-2}L))$. 

Proof. (1) At each $r < 0$, $R(r) = r$, hence the volume of $K_i \times S^1(be^{R(r) - 2} L)$ is $\lesssim \text{vol}_{hyp}(\Sigma_i) \cdot be^{-2} L$. Since $h(r) \leq h(1) \leq 2e$ for $r \leq 0$, the volume of $J_i$ for the part $r \leq 0$ is

\[
\leq (2e)^3 \text{vol}_{hyp}(\Sigma_i) Lb \int_{-\infty}^{0} e^{r-2} dr = 8ebL \text{vol}_{hyp}(\Sigma_i).
\]

(2) Suppose $r > 5$. Then $R(r) = 3, h(r) = 2e^r$. Substitute them to the definition of the metric on $J_i$.

(3) The metric of $J_i$ is written as

\[
g = dr^2 + h(r)^2(d_{hyp} e^{2R(r)} d\rho^2),
\]

where $\rho$ is for $S^1(be^{-2})$. Now this metric and the metric that appears in Proposition 2.4 (5) are locally isometric to each other (see (ii) of the remark there), but that metric satisfies $C \leq K < 0$ for the constant $C$ in the proposition.

(4). This is because each boundary of $K_i$ is isometric to $S^1(be^2 L)$.

\[
\square
\]

Step 3. We set $l = m = 1$ in Proposition 2.4. We prepare a manifold with boundary

\[
A = \{(r, t, \rho, \tau) | r \in \mathbb{R}, t \in [R(r) - 2, 4], \rho \in S^1(Le^{-2}), \tau \in S^1(Le^{-2})\}
\]

with the metric $g$ given in (2.4):

\[
g = dr^2 + h(r)^2(dt^2 + b^2 e^{2R(r)} d\rho^2 + b^2 e^{2R(r)} f(t - R(r))^2 d\tau^2).
\]

The manifold $A$ has two boundary components, $\partial_0 A, \partial_1 A$, where $\partial_1 A$ is the component at $t = 4$ and $\partial_0 A$ at $t = R(r) - 2$. For $t = 4$, we have $f(t - R(r)) = f(4 - R(r)) = ce^{4 - R(r)}$, so that $\partial_1 A$ is isometric to

\[
\mathbb{R} \times_{h(r)} (S^1(be^{R(r) - 2} L) \times S^1(be^2 L)).
\]

Hence $\partial_1 A$ is isometric to each boundary component of every $J_i$ by Lemma 2.6 (4), so that we are able to glue $A$ to the boundary component of $J_i$ along $\partial_1 A$. By Proposition 2.4 (5) (see also the remark (ii) after that), no singularity of the metric occurs by this gluing. In this way we obtain a Riemannian manifold diffeomorphic to $P_t$ (or a Riemannian metric on $P_t$), such that

- $P_t$ is diffeomorphic to $\mathbb{R} \times (\Sigma_i \times S^1)$, where the first parameter is $r$.
- every connected component of the boundary of $P_t$ is isometric to

\[
\partial_0 A = \mathbb{R} \times_{h(r)} (S^1(be^{R(r) - 2} L) \times S^1(be^{R(r) - 2} L)),
\]

and moreover the 1-neighborhood of $\partial_0 A$ is isometric to the direct Riemannian product $\partial_0 A \times [0, 1]$ since $f(t - R(r)) = 1$ for $t \in [R(r) - 2, R(r) - 1]$. 

• volume of the subset $P_i$ for the part $r \leq -1$ is finite (since by Proposition 2.4 for the part isometric to $A$ and for $J_i$ it is by Lemma 2.6 (1)).

• $C \leq K < 0$ on $P_i$ (for $A$ by Proposition 2.4 and for $J_i$ by Lemma 2.6 (3))

• the metric on $P_i$ is a warped product w.r.t. the function $2e^r$ for $r > 5$, (for $A$ by Proposition 2.4 (6), for $J_i$ by Lemma 2.6 (2))

Step 4. Now our metric on $\mathbb{R} \times P_i$ will give a Riemannian metric on $\mathbb{R} \times M$. Indeed, by the second bullet in the above, the two boundary circles have the same length at each $r$, so that we can glue the $\mathbb{R} \times P_i$'s by the given gluing maps at each $r$.

We finish the proof by checking this metric satisfies all the properties in Theorem 1.1. By the third bullet, volume of the part $(-\infty, -1] \times M$ is finite since there are only finitely many pieces for $M$, which implies that the volume for $(-\infty, 0] \times M$ is finite since $M$ is compact. The sectional curvature $K$ satisfies $C \leq K < 0$ on $\mathbb{R} \times M$ by the fourth bullet. The metric is a warped product for $r \geq 5$ w.r.t. the function $2e^r$ and some metric $g_M$ on $M$ by the last bullet and Proposition 2.4 (6). Now we rescale the metric $g_M$ to $(1/4)g_M$, which we still denote by $g_M$, then the warping function becomes $e^r$. Then we have $g = dr^2 + e^{2r}g_M$ for $r \geq 5$. Set $D = 5$. Finally since $\dim M = 3$, we are done.

The proof of Theorem 1.2 is complete in the case without Seifert fibered spaces, provided that $M$ has a geometrization with respect to a product metric on every piece.

Locally product case. Now, suppose some pieces are only locally Riemannian product. We handle this case by following the product case, and we only explain the changes we need to make. Let $P_i = \Sigma_i \times S^1$ (the trivial bundle) be a piece which is a locally Riemannian product with respect to which $M$ has a geometrization. Let $\theta_i : \pi_1(\Sigma_i) \to S^1$ be the monodromy representation defined by the Riemannian metric on $P_i$.

No change is necessary in Step 1. In Step 2, instead of the Riemannian product $K_i \times S^1(be^{R(r)-2}L)$, we take the locally Riemannian product with respect to $\theta_i$, which we denote by

$$K_i \times_{\theta_i} S^1(be^{R(r)-2}L).$$

Accordingly we also use $K_i \times_{\theta_i} S^1(be^{R(r)-2}L)$ in the statement of Lemma 2.6 but the proof is nearly same: for example in the proof of (3), $g = dr^2 + h(r)^2(h_{hyp} + e^{2R(r)}dp^2)$ does not hold any more, but $g$ is only locally isometric to the right hand side. But this is enough since the sectional curvature depends only locally on $g$.

In Step 3, when we define the manifold $A$, we use the same definition, but the metric on $A$ is a locally product metric with respect to the
monodromy $\theta_i$ on the fiber circle for $\rho$. We call this circle $\rho$-circle in the following. Accordingly, in the description (2.5), $\partial_1 A$ becomes only a locally Riemannian product with respect to $\theta_i$ on the $\rho$-circle (which is the first $S^1$ acted by the second $S^1$ via $\rho$). This also happens in the metric description of $\partial_0 A$ in (2.6).

Finally in Step 4, the two circles in (2.6) has same length in this case, and we kept using the same monodromy $\theta_i$ on each piece $P_i$, therefore, the given gluing maps are all isometric. This finishes the proof in this case, and the proof of Theorem 1.2 for flip manifolds without Seifert space pieces is complete. □

2.5. Proof of Theorem 1.2 for the general case. We now handle a graph manifolds such that possibly some pieces are Seifert fibered spaces or fibers are non-orientable (from now on we consider a Seifert fibered space contains the latter case). The argument is identical to the previous case with non-trivial monodromy representation $\theta_i$ of $\pi_1(\Sigma_i)$ where $\Sigma_i$ is the base surface of a piece $P_i$. The only difference is that $\Sigma_i$ is maybe an orbifold and $\pi_1(\Sigma_i)$ is the orbifold fundamental group. In the following we only explain that part. A good reference for the geometry of Seifert fibered spaces is [S].

Proof. Let $P$ be a piece in $M$. Suppose $P$ is a Seifert fibered space, otherwise we do not have to change anything. We remember that when $P$ is a trivial circle bundle over a surface, we can choose the length of the fiber circle when we put a locally product Riemannian metric.

Let $\Sigma$ be the base orbifold of $P$. Let $x_1, \ldots, x_n$ be the singular points of $\Sigma$ such that the twist parameter at $x_i$ is $q_i/p_i$. Since $P$ has non-empty boundary, $P$ admits the geometry of $\mathbb{H}^2 \times \mathbb{R}$ ([S, Theorem 5.3(ii)]). We explain this part in some details (cf. [S, Proof of Theorem 5.3(ii)], [L, Lemma 2.5]). We put $\Sigma$ a complete hyperbolic orbifold metric of finite volume, then view $P$ as an $S^1$-bundle over the orbifold $\Sigma$ with a Riemannian metric that is locally isometric to $\mathbb{H}^2 \times \mathbb{R}$. The global geometry is described by the monodromy representation of $\pi_1(\Sigma)$ into the group $S^1$ if the fibers are oriented, otherwise into $S^1 \rtimes \mathbb{Z}_2$, the isometry group of a circle. Here the fundamental group is in the orbifold sense, and $\mathbb{Z}_2$ means $\mathbb{Z}/2\mathbb{Z}$.

First, assume $\Sigma$ is orientable. Let $X_i$ denote a loop around the singular point $x_i$, and $b_1, \ldots, b_n$ the curves around the punctures (boundary components) of $\Sigma$. Let $g$ be the genus of $\Sigma$ then take loops $\alpha_1, \beta_1; \cdots; \alpha_g, \beta_g$ associated to the genus such that $X_i, b_i, \alpha_i, \beta_i$ generate the fundamental group of $\Sigma$ satisfying a well-known relation (after choosing orientations of the loops suitably):

$$\prod_i [\alpha_i, \beta_i] \prod_i X_i \prod_i b_i = 1.$$
Let $\theta(\alpha_i), \theta(\beta_i), \theta(X_i), \theta(b_i)$ denote the monodromy along those loops for the $S^1$-fiber. We set for each $i$

$$\theta(X_i) = \frac{2\pi q_i}{p_i}.$$ 

We choose $\theta(b_i), \theta(\alpha_i), \theta(\beta_i)$ for each $i$ such that in $S^1 \rtimes \mathbb{Z}_2$,

$$\theta(\prod_i [\alpha_i, \beta_i] \prod_i X_i \prod_i b_i) = 1.$$ 

Then there is a locally product Riemannian metric on $P$ whose monodromy representation is $\theta$. Note that if $\theta(\alpha_i), \theta(\beta_i) \in S^1$ then since $S^1$ is abelian we always have $\theta(\prod_i [\alpha_i, \beta_i]) = 0$ in $S^1$.

Conversely, the monodromy representation induced by a locally product Riemannian metric is obtained in the above way.

If $\Sigma$ is not oriented, the relation in the fundamental group is slightly different, but the rest is same and we omit repeating it.

Note that when we put a Riemannian metric on $P$, as before we can choose the length of the $S^1$-fiber (at a regular point) as we want. Also each boundary component of $P$ is a flat torus/Klein bottle.

We take a compact subset $K$ homeomorphic to $\Sigma$ such that all singular points are contained in $K$, and that each connected component of $\Sigma \setminus K$ satisfies the same metric property as the non-generalized case described in Step 2 in the previous section. We do not need to alter the argument since we modify the metric only outside of $K$, then that $\Sigma$ is an orbifold does not cause any difference.

Now we proceed in the same way as the previous case, and complete the proof of Theorem 1.2 in general. \qed

We give an example of a flip manifold with a geometrization made from a Seifert fibered space.

**Example 2.7** (Seifert fibered space as a piece). We give an example of a Seifert fibered space that can appear as a piece in a flip manifold with a geometrization. Let $\Gamma$ be a three-punctured sphere. There is an obvious action of $G = \mathbb{Z}/3\mathbb{Z}$ rotating the three punctures with a generator $\rho$. Put a complete hyperbolic metric on $\Gamma$ which is $\rho$-invariant. Now set $\Sigma = \Gamma/G$, which is a hyperbolic orbifold with two singular points, $p_1, p_2$, and with one puncture. Take the product $\Gamma \times S^1$ and let $G$ act on it such that $\rho$ acts on $S^1$ by the rotation of $2\pi/3$. This is a free action and the quotient $(\Gamma \times S^1)/G$ is a three dimensional manifold $P$, which is a Seifert fibered space over $\Sigma$ such that the twists at $p_1, p_2$ are $1/3, 2/3$, respectively. (One can say that the twist at $p_2$ is $-1/3$). $P$ has only one boundary component, which is a Riemannian product of the fiber circle and a loop around the puncture of $\Sigma$ since the monodromy is trivial. Now, for example, we prepare another copy of this, then glue the two along their boundary by a trivial or flip map, and obtain a flip manifold which admits a geometrization.
We also record an example of a graph manifold $M$ with a geometrization whose gluing map is not a trivial nor a flip map, cf, [BSc, Example 1.5].

**Example 2.8** (Graph manifold of non-positive curvature). Consider the parallelogram of side length 1 with the angles of the corners equal to $\pi/3, 2\pi/3, \pi/3, 2\pi/3$. Choose a vertex of angle $\pi/3$ and call it $O$, then call the adjacent vertices $A, B$. The last vertex is called $D$. We obtain a flat torus $T$ gluing the sides $OA$ and $BD$; and $OB$ and $AD$. We regard $T$ as a circle bundle over a circle where the base circle is $OB$ and the fiber circle is $OA$. The monodromy with respect to the flat metric is $\pi$.

$T$ has an interesting isometry $\phi$ that is defined by mapping:

$$OA \mapsto BA, OB \mapsto OA.$$  

Notice $\phi$ is not homotopic to the trivial map nor the flip map of $T$.

We define a graph manifold using $\phi$. Let $\Sigma$ be a compact orientable surface of genus one with two boundary components, $a^+ , a^-$. Orient those two curves using the orientation of $\Sigma$. Let $P$ be a trivial circle bundle over $\Sigma$ and we put a locally product metric of type $\mathbb{H}^2 \times \mathbb{R}$ on $P$ such that the monodromy satisfies $\theta(a^+) = \pi, \theta(a^-) = \pi$. We arrange that there is a small constant $c > 0$ such that the two boundary tori $T^+, T^-$ of $P$ at $a^+, a^-$, respectively, are isometric to $T$ with the metric rescaled by $c$. Now we glue $T^+$ to $T^-$ by $\phi$, which is an isometry. $\phi$ is not a flip nor trivial map. In this way we obtain an oriented graph manifold $M$ that has a Riemannian metric of non-positive curvature.

**Remark 2.9.** As we said the property that a graph manifold $M$ has a geometrization formulated differently in [BK]. Although we put a complete, finite volume hyperbolic metric on the base surface/orbifold of a piece, they put a hyperbolic metric with a geodesic boundary (ie, if you lift it to the universal cover, then it is a geodesic in $\mathbb{H}^2$). In both settings we can see the piece as a circle bundle over the base, and it defines a monodromy representation of the fundamental group of the base into $S^1$, which coincides for the two settings. So, if $M$ admits isometric geometrization, then its monodromy representation can be used to put a locally product Riemannian metric on each piece that gives a geometrization on $M$ in our sense.

2.6. **Proof of Theorem 1.6.**

**Proof.** The proof of Theorem 1.6 is nearly identical to the proof of the version of Theorem 1.2 where each piece is a product of a surface and a circle, which is exactly the case where $l = m = 1$ in Theorem 1.6. The main body of the argument for Theorem 1.2 is Proposition 2.4, which is already shown for general $l, m$. So we do not repeat the argument, except we make one remark. Suppose $W_1 = \Sigma_1 \times T^m_1$, $W_2 = \Sigma_2 \times T^m_2$ are pieces such that $S^1_1 \times T^m_1$ and $S^1_2 \times T^m_2$ are glued by an isometry,
where $S'_1$ is a boundary torus of $\Sigma_i$. Also, suppose $\Sigma_i \subset X_i$. By taking $\Sigma_i$ larger in $X_i$ if necessary, one may assume the metric on $S_i$ is rescaled by any constant $0 < c < 1$. Also one can rescale the fibers $T^n_i$ by the same constant $c$, which leaves the gluing isometric.

It follows from Proposition 2.3(7) that $M$ carries a metric of non-positive curvature, so that \( \text{Wh}(M) \) vanishes. This completes the proof. \( \square \)

3. The other family

We discuss the other examples of manifolds that will be ends.

3.1. Construction. Let $M_1, M_2$ be $n$-dimensional closed, orientable hyperbolic manifolds with totally geodesic, orientable submanifold $N_1, N_2$, respectively, of codimension two. Assume that $N_1$ and $N_2$ are isometric by an isometry $b : N_1 \to N_2$.

The unit normal bundle of $N_1$ in $M_1$ is an $S^1$-bundle, $(X_1, N_1, S^1)$, with oriented fibers, which we also denote by
\[ X_1 = N_1 \ltimes S^1, \text{ or } X_1 = S^1 \ltimes N_1. \]
We will use this notation for bundles in this paper, which does not mean a semi-direct product of group structures.

The metric of $M_1$ induces a Riemannian metric on this bundle which is locally a Riemannian product of the hyperbolic metric on $N_1$ and $S^1$. Similarly we have an $S^1$-bundle over $N_2$, $(X_2, N_2, S^1)$, which is locally a Riemannian product.

For a sufficiently small constant $\varepsilon > 0$, the boundary of $V_1 = M_1 \setminus N_1(N)$ is canonically identified with $(X_1, N_1, S^1)$. Also, $V_2 = M_2 \setminus N_2(N)$ is identified with $(X_2, N_2, S^1)$. Suppose $S^1$-bundles over $M_1, M_2$ with Riemannian metrics which are locally product of $M_1, M_2$, respectively, and $S^1$ are given. We denote them by $(Y_1, M_1, S^1), (Y_2, M_2, S^1)$, and the restriction of them to $N_1, N_2$ by $(Y_1|N_1, N_1, S^1), (Y_2|N_2, N_2, S^1)$.

We assume $(X_1, N_1, S^1)$ is isometric to $(Y_2|N_2, N_2, S^1)$ by a bundle map $(f_1, b)$ where $b$ is the isometry between $N_1$ and $N_2$, and also $(Y_1|N_1, N_1, S^1)$ is isometric to $(X_2, N_2, S^1)$ by a bundle map $(f_2, b)$ in the same manner. It then follows that the fiber product $(X_1 \times Y_1|N_1, N_1, S^1 \times S^1)$ is isometric to the fiber product $(X_2 \times Y_2|N_2, N_2, S^1 \times S^1)$ by the flip map
\[ \phi : (n, (s_1, s_2)) \mapsto (b(n), (f_2(s_2), f_1(s_1))), \quad n \in N_1, (s_1, s_2) \in S^1 \times S^1, \]
or the trivial map
\[ \phi : (n, (s_1, s_2)) \mapsto (b(n), (f_1(s_1), f_2(s_2))), \quad n \in N_1, (s_1, s_2) \in S^1 \times S^1. \]
Note that the metric on the fiber $S^1 \times S^1$ of the two bundles is a product metric since $X_i$ are defined over $M_i$. 

□
The fiber products \((X_1 \times Y_1 | N_1, N_1, S^1 \times S^1)\) and \((X_2 \times Y_2 | N_2, N_2, S^1 \times S^1)\) are identified with the the boundary of \(Y_1|V_1\) and \(Y_2|V_2\).

Now we define

\[ W = (Y_1|V_1, V_1, S^1) \cup_{\phi} (Y_2|V_2, V_2, S^1) \]

by identifying their boundaries \((X_1 \times Y_1 | N_1, N_1, S^1), (X_2 \times Y_2 | N_2, N_2, S^1)\) using \(\phi\).

For example, if \(n = 2\) then \(N_1, N_2\) are points and \(W\) is a flip manifold. We recall the theorem from the introduction.

**Theorem 3.1** (Theorem 1.9). Assume \(W\) has a geometrization (ie, the gluing maps are isometric). Then \(W\) carries a metric of non-positive curvature, so that \(\text{Wh}(W)\) vanishes. Also, \(\mathbb{R} \times W\) carries a complete Riemannian metric that is an eventually warped cusp metric with \(C \leq K < 0\) for some constant \(C < 0\).

**Remark 3.2.** As in the construction of 3-dimensional graph manifolds, as a generalization of the theorem, one can use a finite collection of codimension 2 submanifolds \(N_1, \ldots, N_t\), each of which appears two times in the union of \(n\)-dimensional closed hyperbolic manifolds \(M_1, \ldots, M_k\) as totally geodesic, mutually disjoint, submanifolds. For a sufficiently small \(\epsilon > 0\) we remove the \(\epsilon\)-neighborhoods of \(N_i\)'s, then glue the two boundaries of \(N_i(N_i)\) by either the trivial map or the flip map. In this way we obtain a closed manifold \(W\) for which Theorem 3.1 holds.

Note that Theorem 1.2 follows from the generalized version of Theorem 3.1 if all of the base surfaces have genus at least two.

### 3.2. Gluing condition.

We discuss the condition for a flip map to be isometric in the case \(n = 3\) in some details. The \(N_i\) are simple closed geodesics, and we denote them by \(\gamma_i\). By our assumption they have same length. Let \(m_i\) be the meridean curve for \(\gamma_i\) in \(M_i\). We denote \(\sigma_i\) the fiber circle of \(X_i\). With respect to the Riemannian metric, we can measure the monodromy (i.e., rotation) along the curve \(\gamma_i\) for \(\sigma_i\) and \(m_i\), respectively. We denote them by \(0 \leq \theta(\sigma_i), \theta(m_i) < 2\pi\).

Notice that the flip map \(\phi\) is an isometry if and only if

\[
\theta(m_1) = \theta(\sigma_2), \quad \theta(\sigma_1) = \theta(m_2)
\]

In general, i.e., if \(\dim N \geq 1\), then let \(\rho_N(m_1)\) be the monodromy representation of \(\pi_1(N)\) to \(S^1\), in terms of the meridean curve \(m_1\). Let \(\rho_N(\sigma_1)\) be the monodromy representation in terms of \(\sigma_1\). Similarly we define \(\rho_N(m_2), \rho_N(\sigma_2)\). We then assume

\[
\rho_N(m_1) = b^* \rho_N(\sigma_2), \quad \rho_N(\sigma_1) = b^* \rho_N(m_2)
\]

It is an interesting question if the bundles \(X_i\) satisfying this property exists for given \((N_i, M_i)\). One sufficient condition is that \(H_1(N_i, \mathbb{Z})\) injects into \(H_1(M_i, \mathbb{Z})\) for both \(i = 1, 2\). Indeed, if so then first define a circle bundle over \(N_2\) using \(\rho_N(m_1) = b^* \rho_N(\sigma_2)\) (here, we use that \(S^1\)
is abelian), then extend it to $M_2$ (use that $H_1$ injects), which will be $X_2$. Similarly we can define $X_1$.

We realize that it is enough if $X_i$ are defined over $V_i$ for our construction. But in this case we need an additional condition since the metric on the fiber $S^1 \times S^1$ is flat, but not a Riemannian product any more. Hence the monodromy $\theta_{m_1}(\sigma_i), \theta_{m_1}(m_i)$ are not trivial in general, and we need

\[(3.3) \quad \theta_{m_1}(\sigma_1) = \theta_{m_2}(m_2), \quad \theta_{m_1}(m_1) = \theta_{m_2}(\sigma_2)\]

It turns out that if one is satisfied then the other one follows. We will assume this condition if we consider bundles that are defined only on $V_i$.

**Example 3.3.** We discuss the case that $\dim M = 2, \dim N = 1$. If $X$ is defined over $M$, then the boundary of $V$ is a torus which is a Riemannian product. But if $X$ is defined only on $M \setminus N_i(N)$, then maybe $\theta_m(\sigma) \neq 0$, and the boundary of $V$ is a flat torus, but not a product. Then we need to arrange that $\theta_m(\sigma)$ coincides for a pair of tori which are identified.

### 3.3. Outline of proof of Theorem 3.1

The proof of Theorem 3.1 is parallel to Theorem 1.2.

We denote $Y_i|V_i$ by $P_i$ and call it a piece. $N_i$ are isometric to each other by the isometry $b$, so we may write them as $N$.

We will put a metric on $J_i = \mathbb{R} \times P_i$ so that they match up for gluing by $\text{id} \times \phi$, which gives a desired metric on $\mathbb{R} \times W$ to apply Theorem 1.1. $J_i$ has a product metric using the (non-complete) hyperbolic metric on $V_i$, but there will be singularity when we glue them. So we deform the original metric near $\partial J_i$. A small neighborhood of $\partial J_i$ is diffeomorphic to $\mathbb{R} \times [0, \infty) \times ((S^1 \times S^1) \times N)$. In view of that we will construct a complete Riemannian metric $g$ of negative curvature on

\[\mathbb{R} \times \mathbb{R} \times S^1 \times S^1 \times N,\]

which is invariant by a rotation on each $S^1$. We arrange that there is a constant $a$ such that for every $r \in \mathbb{R}$ the metric on $\{r\} \times [a, \infty) \times S^1 \times S^1 \times N$, is identical to the original product metric on $P_i$ up to scaling by a constant depending on $r$ (see Proposition 3.3 (5)). Here, the identification of the metric is canonically done between the fiber bundle $(S^1 \times S^1) \times N$ and $S^1 \times S^1 \times N$ since the metric on the product is invariant by rotations on the both $S^1$-factors.

Moreover, the metric $g$ will be defined on $\mathbb{R} \times \mathbb{R} \times S^1 \times S^1 \times \mathbb{R}^{n-2}$. The factor $\mathbb{R}^{n-2}$ is identified with $N$ and $g$ is invariant by the action by $\pi_1(N)$ which acts trivially on the other factors. In this way, $(\mathbb{R} \times S^1 \times \mathbb{R}^{n-2})/\pi_1(N)$ is identified with $N_1(N) \setminus N$. The other $S^1$ is for the fiber circle in $P_i$, and we can regard $g$ as a metric on $J_i = \mathbb{R} \times P_i$.

We show the following (cf. Proposition 2.4). Recall that $S^1(a)$ is a circle of length $a$. 

Proposition 3.4. Let $c_1, c_2 > 0$ be constants. Then there is a Riemannian metric $g$ on $\mathbb{R} \times \mathbb{R} \times S^1(c_1) \times S^1(c_2) \times N$ that is invariant by rotations on each $S^1$ satisfying the following (1)–(7).

1. There is an absolute constant $C < 0$, which does not depend on $c_1, c_2$, such that $C \leq K < 0$ on $\mathbb{R} \times \mathbb{R} \times S^1(c_1) \times S^1(c_2) \times N$.

2. Volume of the following subset is finite:
\begin{equation}
\{ (r, t, \rho, \tau, n) \mid r \in (-\infty, -1], t \in [r-1, 2], \rho \in S^1(c_1), \tau \in S^1(c_2), n \in N \}.
\end{equation}

3. For $r \leq 0$ and $t \leq r - 1$,
\begin{equation}
g = dr^2 + (r^2)\left(dt^2 + b^2 e^{2r} d\rho^2 + b^2 e^{2r} \, d\tau^2 + dw^2 + \sum_{j=1}^{n-3} e^{2w} dw_j^2 \right).
\end{equation}

4. For $r \geq 0$ and $t \leq -1$,
\begin{equation}
g = dr^2 + (r^2)\left(dt^2 + b^2 e^{2R(r)} d\rho^2 + b^2 e^{2R(r)} \, d\tau^2 + dw^2 + \sum_{j=1}^{n-3} e^{2w} dw_j^2 \right).
\end{equation}

5. For $r \in \mathbb{R}$ and $t \geq a$,
\begin{equation}
g = dr^2 + (r^2)\left(dt^2 + b^2 e^{2R(r)} d\rho^2 + \sinh^2(t - 5) \, d\tau^2 + \cosh^2(t - 5)\left(dw^2 + \sum_{j=1}^{n-3} e^{2w} dw_j^2 \right) \right).
\end{equation}

6. For $r \geq 5$, the metric $g$ is a warped metric of the form:
\begin{equation}
g = dr^2 + 4e^{2r} \hat{g},
\end{equation}
where $\hat{g}$ is the metric on $\mathbb{R} \times S^1(c_1) \times S^1(c_2) \times N$ defined by
\begin{equation}
\hat{g} := dt^2 + b^2 e^6 d\rho^2 + \hat{F}(r, t)^2 \, d\tau^2 + T(t)^2\left(dw^2 + \sum_{j=1}^{n-3} e^{2w} dw_j^2 \right).
\end{equation}

Here, $\hat{F}(r, t)$ (and hence $\hat{g}$ too) is independent of $r$ for $r \geq 5$.

7. The metric $\hat{g}$ in (6) has non-positive curvature for $r \geq 5$.

We postpone proving this proposition and prove Theorem 3.1 using it.

3.4. Proof of Theorem 3.1.

Proof. First, $W$ carries a metric of non-positive curvature by Proposition 3.3(7). This implies that $Wh(W)$ vanishes, [FJ].

We now show the claim for $\mathbb{R} \times W$. We closely follow each step of the argument for Theorem 1.2. But there is one additional issue and we make a remark on that. For Theorem 1.2 each piece $P$ is a trivial bundle $\Sigma \times S^1$ (for the non-general case). We glue pieces along boundaries by isometries, and a boundary component of $P$ is $S^1 \times S^1$, where the first $S^1$ is a boundary component of $\Sigma$. On the other hand, for
Theorem 3.1. A boundary component of a piece $P$ will be an $S^1$-bundle over an $S^1$-bundle over a hyperbolic manifold $N$: $S^1 \times (S^1 \times N)$. But notice that any metric $g$ on $S^1 \times S^1 \times N$ that is invariant by rotations on both circles gives a metric to the boundary which is locally isometric to $g$. In view of this, when we construct a metric (see Section 3.5), we consider only rotationally invariant ones on a product space then descend it to a space with circle bundle structures, so that the bundle issue is not an extra problem for us. In the following, we may write $S^1 \times (S^1 \times N)$ simply as $S^1 \times S^1 \times N$.

**Step 1.** Fix a small constant $\epsilon > 0$. Set $\Sigma_i = M_i - N_\epsilon(N_i)$. The boundary of $\Sigma_i$ is a circle bundle over $N_i$. Set $P_i = (M_i - N_\epsilon(N_i)) \times S^1$.

**Step 2.** We will put a metric on $P_i$ and glue them along the boundary. Set $K_i = M_i - N_{2\epsilon}(N_i)$. Then $\Sigma_i - K_i$ is isometric to $[\epsilon, 2\epsilon) \times (S^1(2\pi) \times N)$ with the metric $dt^2 + \sinh(t)dt^2 + \cosh(t)g_N$.

**Step 3.** For each $r \in \mathbb{R}$, we consider an $S^1$-bundle which is locally a Riemannian product:

$$(M_i - N_\epsilon(N_i)) \times S^1(be^{R(r)-2}L),$$

then further take a “generalized” warped product with $\mathbb{R}$ as follows:

$$J_i = \mathbb{R} \times h(r) \{(M_i - N_\epsilon(N_i)) \times S^1(be^{R(r)-2}L)\},$$

where at each $r$, the metric of the fiber $K_i \times S^1(be^{R(r)-2}L)$ is rescaled by $h(r)$. We say this is a generalized warped product since the metric on the fiber at $r$ depends on $r$.

Then we have the following lemma. The argument is similar to Lemma 2.6 and we skip it.

**Lemma 3.5.**

1. The subset of $J_i$ for the part $r < 0$ has finite volume, which is bounded above by $2^{n+1}e^{n-1}bL\text{vol}_{hyp}(M_i)$.
2. For the part $r > 5$, $J_i$ is a warped product:

$$(5, +\infty) \times_{2\epsilon r} (\mathbb{R} \times S^1 \times S^1 \times N),$$

3. The sectional curvature of $J_i$ is bounded:

$$C \leq K < 0,$$

where $C < 0$ is the constant from Proposition 3.4.
4. Each boundary component of $J_i$ is isometric to

$$\mathbb{R} \times h(r) (S^1(be^2L) \times S^1(be^{R(r)-2}L) \times N).$$

**Step 4.** Similar. We use Proposition 3.4. We skip details.

**Step 5.** Similar. We use Proposition 3.4. We skip details.

Theorem 3.1 is proved. □
3.5. Metric construction. We are left with proving Proposition 3.4. It is done by constructing \( g \).

For any constant \( a > 5 \), we put \( \delta := \delta(a) := (a - 5)/2 \) and \( b := b(a) := c^{-1}e^{-(a-\delta)} \sinh(a-\delta-5) \), where we recall \( c = (\lambda e^t)(0) \). There is a \( C^\infty \) function, \( \tilde{F} \), on \( \mathbb{R}^2 \) such that

(i) for all \( r \) and \( t \),

\[
\tilde{F}(r, t) = \begin{cases} 
F(r, t) = be^{R(r)}f(t - R(r)) & \text{if } t \leq 5, \\
\sinh(t - 5) & \text{if } t > a, 
\end{cases}
\]

(ii) \( \tilde{F}_r, \tilde{F}_t \geq 0 \) everywhere,

(iii) \( \tilde{F}(r, t) \) is independent of \( r \) for \( r \geq 5 \).

Let us explain why such a function \( \tilde{F} \) exists. Assume \( t \geq 4 \). Since \( R \leq 3 \), we have \( t - R \geq 1 \) and so

\[
be^{R}f(t - R) = be^t \begin{cases} 
> \sinh(t - 5) & \text{if } t < a - \delta, \\
= \sinh(t - 5) & \text{if } t = a - \delta, \\
< \sinh(t - 5) & \text{if } t > a - \delta.
\end{cases}
\]

Therefore, there is a \( C^\infty \) approximation, \( \tilde{F} \), of the continuous function

\[
\begin{cases} 
be^{R(r)}f(t - R(r)) & \text{if } t \leq a - \delta, \\
\sinh(t - 5) & \text{if } t > a - \delta,
\end{cases}
\]

satisfying the required conditions.

Take a \( C^\infty \) function, \( T \), on \( \mathbb{R} \) such that

\[
T(t) = \begin{cases} 
1 & \text{if } t \leq 4, \\
\cosh(t - 5) & \text{if } t > a, 
\end{cases} \quad T \geq 1, \quad T', T'' \geq 0.
\]

For \( n \geq 2 \), we consider the metric

\[
g = \sum_{i=1}^{n} g_i \, dx_i^2,
\]

where \( x_1 := r, \ x_2 := t, \ x_3 := \rho, \) and \( x_4 := \tau, \ x_5 := w, \ x_i := w_{i-5} \) for \( i \geq 6 \), \( g_1 := 1, \ g_2 := h(r)^2, \ g_3 := H(r)^2, \ H(r) := be^{R(r)}h(r), \ g_4 := h(r)^2 \tilde{F}(r, t)^2, \ g_5 := h(r)^2T(t)^2, \ g_i := h(r)^2T(t)^2e^{2w} \) for \( i \geq 6 \). We see that

\[
g = dr^2 + h(r)^2 \left( dt^2 + b^2 e^{2R(r)} \, d\rho^2 + \tilde{F}(r, t)^2 \, d\tau^2 
+ T(t)^2 \left( dw^2 + \sum_{j=1}^{n-3} e^{2w} \, dw_j^2 \right) \right),
\]

where the term

\[
T(t)^2 \left( dw^2 + \sum_{j=1}^{n-3} e^{2w} \, dw_j^2 \right)
\]
vanishes for $n = 2$. Note that, for $t \geq a$,

$$dt^2 + \tilde{F}(r, t)^2 d\tau^2 + T(t)^2 \left( dw^2 + \sum_{j=1}^{n-3} e^{2w} dw_j^2 \right)$$

$$= dt^2 + \sinh^2(t - 5) d\tau^2 + \cosh^2(t - 5) \left( dw^2 + \sum_{j=1}^{n-3} e^{2w} dw_j^2 \right).$$

is a hyperbolic metric.

We calculate the Christoffel symbols,

$$\Gamma^i_{12} = \frac{h'}{h}, \quad \Gamma^i_{13} = \frac{H'}{H}, \quad \Gamma^i_{14} = \frac{\tilde{F}_r}{\tilde{F}} + \frac{h'}{h},$$

$$\Gamma^i_{ii} = \frac{h'}{h} \text{ for } i \geq 5,$$

$$\Gamma^i_{22} = -hh', \quad \Gamma^i_{33} = -HH', \quad \Gamma^i_{44} = -h^2 \tilde{F} \tilde{F}_r - hh' \tilde{F}^2, \quad \Gamma^i_{44} = -\tilde{F} \tilde{F}_r,$$

$$\Gamma^i_{55} = -hh'T^2, \quad \Gamma^i_{55} = -TT', \quad \Gamma^i_{55} = 1 \text{ for } i \geq 6,$$

$$\Gamma^i_{ii} = -hh' e^{2w}T^2, \quad \Gamma^i_{ii} = -e^{2w}TT', \quad \Gamma^i_{ii} = -e^{2w} \text{ for } i \geq 6,$$
and then the sectional curvatures,

\[
K_{12} = -\frac{h''}{h},
K_{13} = -\frac{H''}{H},
K_{14} = -\frac{\tilde{F}_{rr}}{\tilde{F}} - \frac{2h'\tilde{F}_r}{h\tilde{F}} - \frac{h''}{h},
K_{1j} = -\frac{h''}{h} \text{ for } j \geq 5,
K_{23} = -\frac{h'H'}{hH},
K_{24} = -\frac{\tilde{F}_{tt}}{h^2F} - \frac{h'\tilde{F}_r}{hF} - \frac{(h')^2}{h^2},
K_{2j} = -\frac{T''}{h^2T} - \frac{(h')^2}{h^2} \text{ for } j \geq 5,
K_{34} = -\frac{\tilde{F}_rH'}{FH} - \frac{h'H'}{hH},
K_{3j} = -\frac{h'H'}{hH} \text{ for } j \geq 5,
K_{4j} = -\frac{\tilde{F}_rT'}{h^2FT} - \frac{h'\tilde{F}_r}{hF} - \frac{(h')^2}{h^2} \text{ for } j \geq 5,
K_{ij} = -\frac{(T')^2}{h^2T^2} - \frac{1}{h^2T^2} - \frac{(h')^2}{h^2} \text{ for } j > i \geq 5.
\]

**Lemma 3.6.**

1. \(\tilde{F} > 0, \tilde{F}_t, \tilde{F}_{tt}, \tilde{F}_r \geq 0\).
2. \(T' \geq 1, T'' \geq 0, T''' \geq 0\).
3. The following functions are all uniformly bounded:

\[
\frac{\tilde{F}_r}{\tilde{F}}, \frac{\tilde{F}_{rr}}{\tilde{F}}, \frac{\tilde{F}_t}{\tilde{F}}, \frac{\tilde{F}_{tt}}{\tilde{F}}, \frac{T'}{T}, \frac{T''}{T}.
\]

**Proof.** (1) follows from the definition of \(\tilde{F}\) and Lemma 2.2.

(2) is clear.

We prove (3). The boundedness of \(T'/T\) and \(T''/T\) are derived from the definition of \(T\). For \(t \leq 5\), we see that \(\tilde{F} = F\) and the boundedness of \(\tilde{F}_r/F, \tilde{F}_{rr}/F, \tilde{F}_t/F, \tilde{F}_{tt}/F\) follow from Lemma 2.2. For \(t \geq 5\), we see that \(\tilde{F}\) is independent of \(r\), so that \(\tilde{F}_r = \tilde{F}_{rr} = 0\) and that \(\tilde{F}_t/F, \tilde{F}_{tt}/F\) are bounded for \(t \in [5, a]\). For \(t \geq a\), we have \(\tilde{F} = \sinh(t - 5)\), for which \(\tilde{F}_t/F, \tilde{F}_{tt}/F\) are bounded because of \(a > 5\). We thus obtain (3). This completes the proof of the lemma.

**Lemma 3.7.** There is a constant \(C < 0\) such that \(C \leq K < 0\).
Proof. The negativity and boundedness of $K_{ij}$ is readily seen from Lemmas 2.2 and 3.6 except the negativity of $K_{14}$. We remark that $\tilde{F}_{rr} \geq 0$ does not hold.

In the case where $r \geq 5$, we see that $\tilde{F}$ is independent of $r$ and then

$$K_{14} = -\frac{h''}{h},$$

which is negative and bounded by Lemma 2.2.

In the case where $r \leq 5$, we see $\tilde{F} = F$, in which case the negativity and the boundedness of $K_{14}$ are proved in the same way as in Lemma 2.3. This completes the proof. □

We are ready to prove Proposition 3.4.

Proof of Proposition 3.4. First of all, the rotational invariance of $g$ is clear by the form of $g$.

(1) By Lemma 3.7.

Checking (2) - (6) is similar to (2) - (6) of Proposition 2.4. We omit it.

We prove (7). Assume $r \geq 5$. The curvatures for the metric $\hat{g}$ are

$$K_{12} = -\frac{\hat{F}_u}{\hat{F}},$$

$$K_{1j} = -\frac{T''}{T} \quad \text{for} \quad j \geq 2,$$

$$K_{2j} = -\frac{\hat{F}_iT_i}{\hat{F}T} \quad \text{for} \quad j \geq 3,$$

$$K_{ij} = -\frac{(T'')^2 + 1}{T^2} \quad \text{for} \quad j > i \geq 3.$$

By Lemma 3.6(1)(2), all the $K_{ij}$’s are non-positive. We have proved Proposition 3.4. □

4. Questions

4.1. More complicated examples. As we explained in section 1.5, a flip manifold can be obtained as follows: take two surfaces $V_1, V_2$, remove a small neighborhood of a point $p_i$ from each of them, then consider an $S^1$-bundle over each. The boundary of each manifold is an $(S^1 \times S^1)$-bundle over a point ($p_1$ and $p_2$), and now we glum them by a flip map.

Regarding the above example, one can view $V_1$ and $V_2$ are intersecting in one point. In view of this, a similar construction can be done in dimension $2n, n \geq 1$, with a more complicated intersection pattern. The above case is for $n = 1$, and we describe the case for $n = 2$. Let $V_1$ be a closed hyperbolic 4-manifold with two, isometric, totally geodesic embedded closed 2-submanifolds $V_{12}, V_{13}$ intersecting at one point $V_{123}$.
transversally. Prepare two other copies: $V_2$ with submanifolds $V_{23}, V_{21}$; and $V_3$ with submanifolds $V_{31}, V_{32}$.

Fix a small $\epsilon > 0$, and consider an $S^1$-bundle:

$$X_1 = (V_1 \setminus N_\epsilon(V_{12} \cup V_{13})) \times S^1,$$

whose boundary is $\partial N_\epsilon(V_{12} \cup V_{13}) \times S^1$. Note that $\partial N_\epsilon(V_{12} \cup V_{13})$ is a flip manifold embedded in $V_1$:

$$(V_{12} \setminus N_\epsilon(V_{123})) \times S^1 \cup_{V_{123} \times S^1 \times S^1} (V_{13} \setminus N_\epsilon(V_{123})) \times S^1,$$

where we flip the two $S^1$-fibers in $V_{123} \times S^1 \times S^1$ when we glue the left piece to the right one. Similarly, consider $S^1$-bundles $X_2, X_3$ for $V_2, V_3$, respectively. We put a locally product metric on each $\partial X_i$.

Now from $X_1, X_2, X_3$, we define a 5-manifold

$$M^5 = (X_1 \cup X_2 \cup X_3)/\sim,$$

where $\sim$ means gluing among the boundaries of $X_1, X_2, X_3$:

$$\partial X_1 = (V_{12} \setminus N_\epsilon(V_{123})) \times S^1 \cup_{V_{123} \times S^1 \times S^1} (V_{13} \setminus N_\epsilon(V_{123})) \times S^1 \times S^1,$$

$$\partial X_2 = (V_2 \setminus N_\epsilon(V_{123})) \times S^1 \cup_{V_{123} \times S^1 \times S^1} (V_{23} \setminus N_\epsilon(V_{123})) \times S^1 \times S^1,$$

$$\partial X_3 = (V_{32} \setminus N_\epsilon(V_{123})) \times S^1 \cup_{V_{123} \times S^1 \times S^1} (V_{31} \setminus N_\epsilon(V_{123})) \times S^1 \times S^1.$$

A gluing map is described as follows for each pair $(i, j)$: use the obvious identification $V_{ij} \setminus N_\epsilon(V_{123}) = V_{ji} \setminus N_\epsilon(V_{123})$ and flip the two $S^1$-fibers. The common manifold $V_{123} \times S^1 \times S^1 \times S^1$ is shared by all of them in $M$. We assume that the identification are done by isometries.

It would be interesting to know if $M$ appears as an end (cf. [AS], see also [B] on the topology of those ends). In view of our strategy, as the first step we want to know if $M$ has a metric of non-positive curvature, but the curvature estimate becomes more subtle when we look for an eventually warped cusp metric for $\mathbb{R} \times M$.

4.2. Graph manifolds. Among graph manifolds $W$, we proved that $W$ appears as an end if it has a Riemannian metric of non-positive curvature (Corollary [L6]). See [BS] for the question to decide which graph manifolds carry Riemannian metric of non-positive curvature. Leeb [L] gave an example of graph manifold that does not have a metric of non-positive curvature. It would be interesting to know if his examples will/ will not appear as an end.

References

[AS] U. Abresch and V. Schroeder, Graph manifolds, ends of negatively curved spaces and the hyperbolic 120-cell space, J. Differential Geom. 35 (1992) 299–336.

[A] Ian. Agol, The virtual Haken conjecture. Doc. Math. 18 (2013), 1045–1087.

[AP] Grigori Avramidi, T. Tam Nguyen Phan. Half dimensional collapse of ends of manifolds of nonpositive curvature, arXiv:1608.02183

[BGS] W. Ballmann, M. Gromov and V. Schroeder, Manifolds of Nonpositive Curvature, Progress in Mathematics, Vol. 61 (Birkhäuser, 1985).
[Be] Igor Belegradek, An assortment of negatively curved ends. J. Topol. Anal. 5 (2013), no. 4, 439–449.
[Be1] Igor Belegradek, Topology of open nonpositively curved manifolds. Geometry, topology, and dynamics in negative curvature, 32-83, London Math. Soc. Lecture Note Ser., 425, Cambridge Univ. Press, Cambridge, 2016.
[Be2] Igor Belegradek, Complex hyperbolic hyperplane complements. Math. Ann. 353 (2012), no. 2, 545-579.
[BeK] I. Belegradek and V. Kapovitch. Classification of negatively pinched manifolds with amenable fundamental groups. Acta Math. 196 (2006), no. 2, 229–260.
[B] S. V. Buyalo, An example of a four-dimensional manifold of negative curvature. (Russian. Russian summary) Algebra i Analiz 5 (1993), no. 1, 193–199; translation in St. Petersburg Math. J. 5 (1994), no. 1, 171-176
[BK] S.V.Buyalo, V.L.Kobelskii. Geometrization of graph-manifolds. II. Isometric geometrization. Algebra i Analiz 7 (1995), no. 3, 96–117; translation in St. Petersburg Math. J. 7 (1996), no. 3, 387-404
[BSc] S. Buyalo, V. Schroeder. On the asymptotic geometry of nonpositively curved graphmanifolds. Trans. Amer. Math. Soc. 353 (2001), no. 3, 853-875.
[BS] S. V. Buyalo, P. V. Svetlov, Topological and geometric properties of graph manifolds. (Russian. Russian summary) Algebra i Analiz 16 (2004), no. 2, 3–68; translation in St. Petersburg Math. J. 16 (2005), no. 2, 297–340.
[CR] Ted Chinburg, Alan W. Reid, Closed hyperbolic 3-manifolds whose closed geodesics all are simple. J. Differential Geom. 38 (1993), no. 3, 545–558.
[FJ] F. T. Farrell, L. E. Jones, Stable pseudoisotopy spaces of compact nonpositively curved manifolds. J. Differential Geom. 34 (1991), no. 3, 769–834.
[FLS] Roberto Frigerio, Jean-François Lafont, Alessandro Sisto. Rigidity of high dimensional graph manifolds. Astérisque No. 372 (2015),
[F] Koji Fujiwara. A construction of negatively curved manifolds. Proc. Japan Acad. Ser. A Math. Sci. 64 (1988), no. 9, 352–355.
[G] M. Gromov, Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math. 56 (1982) 5–99.
[KL] M. Kapovich, B. Leeb. 3-manifold Groups and Nonpositive Curvature. GAFA, November 1998, Volume 8, Issue 5, pp 841–852.
[L] B. Leeb. 3-manifolds with(out) metrics of non-positive curvature. Invent. Math. 122 (1995), no. 2, 277–289.
[Lu] Wolfgang Lück. Survey on aspherical manifolds. European Congress of Mathematics, 53–82, Eur. Math. Soc., Zürich, 2010.
[M] Barry Minemyer, Real hyperbolic hyperplane complements in the complex hyperbolic plane. Adv. Math. 338 (2018), 10381076.
[O] P. Ontaneda, Riemannian Hyperbolization. arXiv:1406.1730
[P] T. T. Nguyen Phan, Nil happens. What about Sol?, arXiv:1207.1734v1.
[S] Peter Scott, The geometries of 3-manifolds. Bull. London Math. Soc. 15 (1983), no. 5, 401–487.

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