Symmetry-breaking instability of leapfrogging vortex rings in a Bose-Einstein condensate

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Three coaxial quantized vortex rings in a Bose-Einstein condensate exhibit aperiodic leapfrogging dynamics. It is found that such circular vortex rings are dynamically unstable against deformation breaking axial rotational symmetry. The dynamics of the system are analyzed using the Gross-Pitaevskii and vortex-filament models. The dependence of the instability on the initial arrangement of the vortex rings is investigated. The system is found to be significantly unstable for a specific configuration of the three vortex rings.

I. INTRODUCTION

A vortex ring has a toroidal vorticity distribution with a torus shape, and survives for a long time after it is created. A vortex ring has a linear momentum and can therefore travel a long distance, as for the well-known example of smoke rings in air. However, in a normal fluid with viscosity, such as air or water, a vortex ring eventually decays due to the dissipation of energy and momentum. Furthermore, the toroidal vorticity distribution has an inherent instability against azimuthal-wave excitation [1, 2].

In superfluids, the situation is much simpler. Vortices are quantized, and a vortex ring is simply a circular closed loop of a quantized vortex line. Because of the absence of viscosity, a circular quantized vortex ring has an infinite lifetime in a uniform superfluid. Quantized vortex rings were first detected in superfluid helium using ion spectroscopy [3]. In Bose-Einstein condensates (BECs) of ultracold atomic gases, quantized vortex rings have been created through the decay of solitons and directly observed by imaging the density distribution [4–7]. The dynamics of quantized vortex rings in superfluids have been theoretically studied by many researchers [8–24].

A circular vortex ring in a superfluid travels at a constant velocity roughly proportional to the inverse of its radius. Such a quantized vortex ring traveling in a uniform superfluid is stationary in the moving frame of reference, and stable against perturbations. When two circular vortex rings with the same vorticity are coaxially arranged in parallel sharing the same axis, the radius of the front (rear) ring increases (decreases) and the ring decelerates (accelerates). The rear ring thus passes through the front ring, and then the roles of the two rings are reversed, which results in leapfrogging dynamics of the two vortex rings [23, 24]. For more than two coaxial vortex rings, the leapfrogging dynamics become more complicated.

In the present paper, we focus on the axial-symmetry breaking instability of leapfrogging vortex rings in superfluids. When circular vortex rings are coaxially arranged in the initial state, the system has rotational symmetry about the axis of the rings. In the ensuing leapfrogging dynamics, the axial symmetry is spontaneously broken, i.e., infinitesimal azimuthal perturbations are exponentially increased, and the leapfrogging dynamics are eventually destroyed by significant distortions of the rings. Such instabilities of coaxial vortex rings have been studied using the vortex-filament model. In Ref. [10], the dynamics of leapfrogging vortex rings were numerically studied and axial-symmetry breaking instability was shown to occur for seven vortex rings. A linear stability analysis was performed for two vortex rings by Ref. [21]. The long-time stability of two and three vortex rings was studied by Ref. [22].

In the present paper, we investigate the axial-symmetry breaking instability of three quantized vortex rings using both the Gross-Pitaevskii (GP) equation and the vortex-filament model. Three vortex rings with the same direction of vorticity are coaxially arranged, which exhibit leapfrogging dynamics. We find that the system is significantly unstable against axial-symmetry breaking for a specific initial arrangement of the vortex rings. The instability occurs both for the GP and vortex-filament models. We investigate the dependence of the instability on the initial arrangements of the vortex rings and on the interaction coefficient. We will show that the symmetry-breaking modulation significantly grows when the vortex rings form a particular configuration during the dynamics, and most unstable eigenmode is obtained.

This paper is organized as follows. Section II presents a study of the dynamics of vortex rings by solving the GP equation numerically. Section III analyzes the symmetry-breaking instability using the vortex-filament model. Section IV gives the conclusions of this study.

II. GROSS-PITAEVSKII MODEL

We consider a BEC of a dilute atomic gas at zero temperature described by the GP equation,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \nabla^2 \psi + V(r)\psi + \frac{4\pi\hbar^2a}{M} |\psi|^2 \psi,$$  \(1\)

where $$\psi(r, t)$$ is the macroscopic wave function of the condensate, $$M$$ is the mass of an atom, $$V(r)$$ is an external potential, and $$a$$ is the s-wave scattering length. We normalize the length, time, energy, and atomic density by an arbitrary length $$L$$, arbitrary time $$T$$, $$\hbar/T$$, and arbitrary
density \( n_0 \), respectively, where \( hT = ML^2 \) is satisfied. The GP equation then becomes non-dimensional as

\[
i \frac{\partial \psi}{\partial \tau} = -\frac{\nabla^2}{2} \psi + V(\mathbf{r}) \psi + g|\psi|^2 \psi,
\]

where the non-dimensional interaction coefficient is \( g = 4\pi aL^2 n_0 \). To reduce the effect of boundary conditions on the axial symmetry of the vortex rings, we use a cylindrical tube potential given by

\[
V(\mathbf{r}) = \begin{cases} 0 & (r_\perp < R_{\text{wall}}) \\ V_0 & (r_\perp \geq R_{\text{wall}}) \end{cases},
\]

where \( r_\perp = (x^2 + y^2)^{1/2} \) and \( R_{\text{wall}} \) is the radius of the cylindrical tube. Such a potential can be produced by a phase-imprinted laser beam [27]. The height \( V_0 \) of the potential wall is taken to be much larger than the chemical potential \( g|\psi|^2 \). A periodic boundary condition is imposed in the \( z \) direction.

The initial state is prepared by the imaginary-time propagation of Eq. (2), where \( i \) on the left-hand side is replaced with \(-1\). The wave function is normalized with the volume of the cylindrical tube as

\[
\int |\psi|^2 d\mathbf{r} = \pi R_{\text{wall}}^2 L_z,
\]

where \( L_z \) is the size of the system in the \( z \) direction. The density \( |\psi|^2 \) is thus almost unity inside the tube potential, when the radius \( R_{\text{wall}} \) is much larger than the healing length. After the imaginary-time propagation converges, circular vortex rings are imprinted in such a way that their symmetry axis is on the \( z \) axis. The wave function is multiplied by \( \exp[i\phi(\mathbf{r})] \), with the phase given by

\[
\phi(\mathbf{r}) = \sum_{j=1}^{N_{\text{ring}}} \sum_{n=-n_z}^{n_z} \left( \frac{\tan^{-1} \left( z - Z_j - nL_z \right) / r_\perp + R_j}{r_\perp + R_j} - \frac{\tan^{-1} \left( z - Z_j - nL_z \right) / r_\perp - R_j}{r_\perp - R_j} \right),
\]

where \( N_{\text{ring}} \) is the number of vortex rings and \( R_j \) and \( Z_j \) are the radius and \( z \) coordinate of the \( j \)th vortex ring, respectively. The first and second terms in the bracket in Eq. (5) cancel the radial flow \( \partial \phi/\partial r_\perp \) on the \( z \) axis, and the summation over integers \( n \) in Eq. (5) ensures a periodic boundary condition in the \( z \) direction for a sufficiently large \( n_z \), which reduces the initial disturbances due to the phase imprint. To further reduce the initial disturbances, we perform a short imaginary-time evolution during \( T_{\text{imag}} \) after the phase \( \phi(\mathbf{r}) \) is imprinted on the wave function. This imaginary-time evolution slightly changes the radii and \( z \) positions of the vortex rings from \( R_j \) and \( Z_j \) in Eq. (3).

We obtain the imaginary and real time evolution of the system by numerically solving Eq. (2) using the pseudospectral method [28]. The numerical mesh is typically \( 512^3 \) with a spatial discretization of \( dx = dy = dz = 0.25 \), and the system size is \( L_x \times L_y \times L_z = 128^3 \). Small noise must be added to the initial state to trigger the symmetry breaking instability. To generate moderate noise, we first set complex random numbers with a normal distribution on each numerical mesh point and eliminate large wavenumber components with \( k > k_{\text{cutoff}} \) using a Fourier transform, where \( k_{\text{cutoff}} \) is taken to be \( 10 \times 2\pi/128 \). Such low-pass filtered noise multiplied by a small number (typically 0.1) is added to the wave function after the imaginary-time evolution for \( T_{\text{imag}} \). In the following calculations, we take \( V_0 = 10, R_{\text{wall}} = 63, n_z = 50, \) and \( T_{\text{imag}} = 4 \).

Figure 1 shows the time evolution of two and three vortex rings.
FIG. 2. (color online) Dependence of the symmetry breaking dynamics on the initial distance \( d \) between the vortex rings in the rectilinear initial arrangement with radius \( R = 20 \). The dynamics are obtained by solving the GP equation with \( g = 1 \). (a) \( d = 1 \), (b) \( d = 2 \), and (c) \( d = 4 \). The dynamics in (b) is the same as that in Fig. 1(c). The isodensity surface of the density \( |\psi|^2 = 0.2 \) is shown. The size of the box is \( 64^3 \), with the origin at the center, and is seen from the +z direction. See the Supplemental Material for movies showing the dynamics.

FIG. 3. (color online) Dependence of symmetry-breaking dynamics on interaction coefficient \( g \) for vortex rings in rectilinear initial arrangement with \( d = 2 \) and \( R = 20 \). The dynamics are obtained by solving the GP equation. (a) \( g = 2 \) and (b) \( g = 0.5 \). The isodensity surfaces of the densities \( |\psi|^2 = 0.5 \) and 0.2 are shown for (a) and (b), respectively. The size of the box is \( 64^3 \) with the origin at the center, and is seen from the +z direction. See the Supplemental Material for movies showing the dynamics.
We approximate the vortex line between \( S \) and the line integral is taken along all vortex lines. For a larger value of \( g \), the healing length becomes smaller, and therefore, the spatial discretization \( dx, dy, \) and \( dz \) must be decreased and the numerical mesh must be increased. For a smaller value of \( g \), the time for which the instability emerges increases and long-time evolution is needed.

The symmetry-breaking instability demonstrated in Figs. 1-3 is a modulational instability (or dynamical instability), in which infinitesimal symmetry-breaking noise grows exponentially with time. However, in the present case, the standard linear stability analysis, i.e., Bogoliubov analysis, cannot be used, since the vortex rings move and the system is not in a stationary state. We also cannot use the Floquet analysis, since the dynamics of the three vortex rings are not periodic for the rectilinear initial arrangement.

III. VORTEX-FILAMENT MODEL

To study the symmetry-breaking dynamics of vortex rings in more detail, we employ the vortex-filament model. In this model, we focus only on the dynamics of quantized vortex lines. We assume that the fluid is incompressible and irrotational except at the vortex lines. A vortex segment located at \( r \) moves with the velocity given by the Biot-Savart law,

\[
v(r) = \frac{\kappa}{4\pi} \int \frac{(r' - r) \times dr'}{|r - r'|^3},
\]

where \( \kappa = 2\pi \) is the normalized circulation of a quantized vortex and the line integral is taken along all vortex lines.

In the numerical calculations, the nth vortex ring is represented by a sequence of \( N_p \) positions on the ring, \( S_1^{(n)}, S_2^{(n)}, \ldots, S_{N_p}^{(n)} \), and \( S_1^{(n)} = S_1^{(n)} \). The Biot-Savart integral in Eq. (6) is then rewritten as

\[
v(r) = \frac{\kappa}{4\pi} \sum_{n=1}^{N_{\text{ring}}} \sum_{j=1}^{N_p} F_j^{(n)}(r).
\]

We approximate the vortex line between \( S_j^{(n)} \) and \( S_{j+1}^{(n)} \) to be a straight line \( S_j^{(n)} + m \), and the line integral becomes

\[
F_j^{(n)}(r) = \frac{D_j^{(n)} + D_{j+1}^{(n)}}{D_j^{(n)} D_{j+1}^{(n)}},
\]

where \( D_j^{(n)} = S_j^{(n)} - r \). Since the vortex-filament model breaks down for \( |r - r'| \) smaller than the size of the vortex core \( \xi \), we omit the line integral inside the vortex core, \( |r' - r| < \xi \), in Eqs. (7) and (8), which avoids the divergence at \( r' = r \). The length \( \xi \) corresponds to the healing length in the GP model and we take \( \xi = 1 \), corresponding to \( g = 1 \). The number of points per vortex ring is taken to be \( N_{\text{ring}} = 256 \). Because the radii of the rings that we are considering are \( \approx 20 \), the distance between the adjacent points \( |S_j^{(n)} - S_{j+1}^{(n)}| \) is typically \( 2\pi \times 20 / 256 \approx 0.5 \xi \). This is in contrast to the case of liquid helium [30], in which the size of the vortex core is \( \sim 10^{-10} \) m and usually much smaller than the distance between the adjacent points \( |S_j^{(n)} - S_{j+1}^{(n)}| \). We do not implement vortex reconnections in our calculations, because we focus on the axial-symmetry breaking.

The dynamics of \( N_{\text{ring}} \times N_p \) points are obtained by numerically solving the equation of motion \( dS_j^{(n)/dt} = v(S_j^{(n)}) \) using the fourth-order Runge-Kutta method. The initial positions of the points are set to

\[
S_j^{(n)} = R_n r_j + Z_n z,
\]

where \( R_n \) and \( Z_n \) are the radius and \( z \)-coordinate of the nth vortex ring, \( r_j = x \cos(2\pi j/N_p) + y \sin(2\pi j/N_p) \) is the unit vector in the radial direction, and \( x, y \), and \( z \) are the unit vectors in Cartesian coordinate. Small initial noise is added to each point to trigger the axial symmetry breaking. To avoid numerical instability in the vortex-filament model, we cut off the large wavenumber components in each time step as follows. We perform a Fourier transform of the radius and \( z \)-coordinate as

\[
e_n^{(n)} = \sum_{j=1}^{N_p} \sqrt{(S_j^{(n)} \cdot x)^2 + (S_j^{(n)} \cdot y)^2} e^{-imj},
\]

\[
d_m^{(n)} = \sum_{j=1}^{N_p} S_j^{(n)} \cdot ze^{-imj},
\]

where \( m \) is an integer and \( e_m^{(n)} = e_n^{(n)*} \) and \( d_m^{(n)} = d_n^{(n)*} \) are satisfied. We then eliminate Fourier components with \( |m| \) larger than \( m_{\text{cutoff}} \), and perform an inverse Fourier transform as

\[
S_j^{(n)} = N_p^{-1} \sum_{|m| \leq m_{\text{cutoff}}} \left( e_m^{(n)} r_j + d_m^{(n)} z \right) e^{imj}.
\]

We take \( m_{\text{cutoff}} = 10 \) in the following calculations. Figure 4 shows the dynamics of vortex rings. In Fig. 4(a), two vortex rings with the same radius \( R_1 = R_2 = 20 \) are coaxially arranged at a distance \( Z_2 - Z_1 = 4 \) in the initial state, and they exhibit periodic leapfrogging dynamics in the time evolution. The frequency of the leapfrogging dynamics in Fig. 4(a) is similar to that in the GP model in Fig. 4(a). The period of the leapfrogging dynamics is roughly proportional to the distance between the vortex rings. In the GP model in Fig. 4(a), the distance is \( Z_2 - Z_1 = 2 \) in the initial phase imprint, though this increases during the imaginary-time evolution for \( T_{\text{imag}} \), which is why the leapfrog frequency in Fig. 4(a) is similar to that in Fig. 4(a).
FIG. 4. (color online) Dynamics of vortex rings obtained by the vortex-filament model. (a) Two vortex rings initially located with $R_1 = R_2 = 20$, $Z_1 = -30$, and $Z_2 = -26$. (b) Three vortex rings with a triangular initial arrangement, where $R_1 = 20 - 4\sqrt{3}/3$, $R_2 = R_3 = 20 + 2/\sqrt{3}$, $Z_1 = -28$, $Z_2 = -26$, and $Z_3 = -30$. (c) Three vortex rings with a rectilinear initial arrangement, where $R_1 = R_2 = R_3 = 20$, $Z_1 = -30$, $Z_2 = -26$, and $Z_3 = -22$. Vortices are seen from the $+z$ direction in the rightmost panels. Axial symmetry is retained in (a) and (b) and broken in (c). The size of the box is $64^3$ with the origin at the center. See the Supplemental Material for movies showing the dynamics [29].

initial values of $C_m \sim 10^{-3}$ originate from the initial random noise. For the triangular initial arrangement, the values of $C_m$ are suppressed below $10^{-1}$ until $t = 200$, as shown in Fig. 5(a). They oscillate and never grow for $m \neq 6$. The growth of the $m = 6$ mode is slow and only affects the long-time dynamics [23]. The time evolution of $C_m$ for the rectilinear initial arrangement in Fig. 5(b) is qualitatively different from that in Fig. 5(a). The values of $C_m$ exponentially grow in time as they oscillate, reflecting the fact that axial symmetry breaking occurs in Fig. 5(b). The value of $C_m$ with $m = 4$ is largest for $t \approx 100$ in Fig. 5(b), which gives the modulation of the rings shown in Fig. 5(c).

We note that the significant growth of $C_4$ occurs roughly periodically, e.g., at $t \approx 40$ and $t \approx 75$, as indicated by the vertical dashed lines in Fig. 5(b). The vortex-ring configurations at $t = 40$ and $t = 75$ are shown as the insets in Fig. 5(b). We find that the three vortex rings align in a rectilinear manner at these instants, and
To study the unstable modes, we perform a linear stability analysis. The symmetry-breaking modulation of the vortex rings configuration is unstable has yet to be clarified. Therefore, we can linearize the time evolution of the modulation as

\[ \mathbf{v}_m(t) = M(t)\mathbf{v}_m(0), \]

where \( M(t) \) is a 6 × 6 matrix. The modes with different \( m \) are not coupled with each other. We obtain the matrix \( M(t) \) numerically as follows. First, we set the initial vector as \( \mathbf{v}_m(0) = (\epsilon, 0, 0, 0, 0, 0)^T \) and make the initial vortex rings using Eq. (12), where \( \epsilon \ll 1 \). After the time evolution of the vortex rings, \( \mathbf{v}_m(t) \) is obtained with Eqs. (10) and (11), which gives \( (M_{11}(t), M_{21}(t), \cdots, M_{61}(t))^T = \mathbf{v}_m(t)/\epsilon \). In a similar manner, we obtain all the matrix elements of \( M(t) \).

The eigenvector of \( M(t) \) with the eigenvalue having the largest magnitude corresponds to the most unstable mode. Figure 6(c) shows the form of the most unstable eigenvector for \( m = 4 \) with the triangular initial arrangement as in Fig. 4(c). All the elements of the eigenvector can be taken to be real. In Fig. 6(a), we plot the most unstable mode on the \( c_m - d_m \) plane. At \( t = 40 \), the square (corresponding to the middle vortex ring) is located opposite the circle and triangle (corresponding to the front and rear vortex rings) across the origin. These plots move in time as the leapfrog dynamics of the vortex rings, and at \( t = 75 \), the circle (corresponding to the middle vortex ring) is located opposite the triangle and square (corresponding to the front and rear vortex rings) across the origin. These eigenvectors of the unstable mode are visualized in Fig. 6(b), where the modulation is superimposed on the unmodulated vortex rings in an exaggerated manner. We find that the modes at \( t = 40 \) and \( t = 75 \) have similar shapes, despite the order of the vortex rings changing. At both \( t = 40 \) and \( t = 75 \), the deviations from the unperturbed vortex rings in adjacent vortex rings are opposite to each other.

Figure 7 shows the dependence of the axial symmetry breaking on the initial distance \( d \) between the vortex rings for the rectilinear initial arrangement. In Fig. 7(a), the initial distance is \( d = 6 \), which is larger than \( d = 4 \) in Fig. 7(c). The axial symmetry breaking becomes significant at \( t \simeq 300 \) and the modes \( m = 2 \) or 3 seem to be the most unstable. For \( d = 3 \), as shown in Fig. 7(b), the modulation with \( m \simeq 8 \) becomes significant at \( t \gtrsim 60 \). Thus, the wavelength of the symmetry-breaking modulation increases and the growth rate of the modulation decreases with increasing \( d \), which agrees with the tendency in the GP model shown in Fig. 2.

IV. CONCLUSIONS

We have investigated the dynamics of coaxially arranged multiple vortex rings using the GP model and vortex-filament model. In both models, three vortex rings with rectilinear initial arrangement (Fig. 1(d), right-hand panel) are found to be very unstable, and the axial rotational symmetry of the system is spontaneously
The rightmost panels show the system seen from the $+z$ direction. See the Supplemental Material for movies showing the dynamics.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7}
\caption{(color online) Dependence of symmetry breaking dynamics on initial distance $d$ between vortex rings for rectilinear initial arrangement with radius $R = 20$. The dynamics are obtained by the vortex-filament model. (a) $d = 6$ with $Z_1 = -70$, $Z_2 = -64$, and $Z_3 = -58$. (b) $d = 3$ with $Z_1 = -30$, $Z_2 = -27$, and $Z_3 = -24$. The size of the box is $64^3$ with the $z$ axis at the center. $-80 < z < -16$ is shown at $t = 0$ in (a) and $-32 < z < 32$ is shown in the other boxes. The rightmost panels show the system seen from the $+z$ direction. See the Supplemental Material for movies showing the dynamics.}
\end{figure}

broken within a few leapfrogs, as shown in Figs. 1(c) and 2(c). In contrast, three vortex rings with a triangular initial arrangement (Fig. 1(d), left-hand panel) are much more stable, as shown in Figs. 1(b) and 2(b). In the GP model, we have shown that the most unstable wavelength depends on the initial distance between the vortex rings (Fig. 3) and the interaction coefficient (Fig. 4). In the vortex-filament model, we performed a Fourier analysis of the modulation of the vortex rings, and found that the symmetry-breaking modulation significantly grows when the three vortex rings are arranged in a line and the rear vortex rings are about to pass through the front ring (Fig. 5(b)). The shape of the unstable mode was obtained (Fig. 6).

Such symmetry-breaking dynamics of multiple vortex rings are difficult to observe in experiments, since coaxial vortex rings with axial rotational symmetry must be created in a controlled manner. Phase imprinting and dynamical methods may realize such an arrangement of quantized vortex rings in an atomic BEC.

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