Singular Holomorphic Foliations by Curves II: Negative Lyapunov Exponent

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Abstract

Let \( \mathcal{F} \) be a holomorphic foliation by Riemann surfaces defined on a compact complex projective surface \( X \) satisfying the following two conditions: the singular points of \( \mathcal{F} \) are all hyperbolic; \( \mathcal{F} \) is Brody hyperbolic. Then we establish cohomological formulas for the Lyapunov exponent and the Poincaré mass of an extremal positive \( dd^c \)-closed current tangent to \( \mathcal{F} \). If, moreover, there is no nonzero positive closed current tangent to \( \mathcal{F} \), then we show that the Lyapunov exponent \( \chi(\mathcal{F}) \) of \( \mathcal{F} \), which is, by definition, the Lyapunov exponent of the unique normalized positive \( dd^c \)-closed current tangent to \( \mathcal{F} \), is a strictly negative real number. As an application, we compute the Lyapunov exponent of a generic foliation with a given degree in \( \mathbb{P}^2 \).

Keywords Holomorphic foliation · Hyperbolic singularity · Poincaré metric · \( dd^c \)-Closed current · Holonomy cocycle · Lyapunov exponent

Mathematics Subject Classification Primary 37F75 · 37A30; Secondary 57R30 · 58J35 · 58J65 · 60J65

1 Introduction

Let \( X \) be a compact Kähler surface, and denote by \( \text{Tan}(X) \) its tangent bundle. Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a holomorphic foliation by curves on \( X \) with the set of singularities \( E \). Recall that the foliation \( \mathcal{F} \) is given by an open covering \( \{U_j\} \) of \( X \) and holomorphic
vector fields \( v_j \in H^0(\cup_j, \text{Tan}(X)) \) with isolated singularities (i.e. isolated zeroes) such that

\[
v_j = g_{jk} v_k \quad \text{on} \quad \cup_j \cap \cup_k
\]

for some non-vanishing holomorphic functions \( g_{jk} \in H^0(\cup_j \cap \cup_k, \mathcal{O}_X^*) \). Its leaves are locally integral curves of these vector fields. The set of singularities \( E \) of \( \mathcal{F} \) is precisely the union of the zero sets of these local vector fields. The set \( E \) is finite.

We say that a singular point \( a \in E \) is \textit{linearizable} if there are local holomorphic coordinates \((z, w) \in \mathbb{C}^2 \) centered at \( a \) such that the leaves of \( \mathcal{F} \) are integral curves of a vector field

\[
Z(z, w) = z \frac{\partial}{\partial z} + \lambda w \frac{\partial}{\partial w}
\]

with some complex number \( \lambda \neq 0 \).

The analytic curves \( \{z = 0\} \) and \( \{w = 0\} \) are called \textit{separatrices} at \( a \). If moreover, \( \lambda \notin \mathbb{R} \), then we say that \( a \) is \textit{hyperbolic}.

The functions \( g_{jk} \) form a multiplicative cocycle and hence give a cohomology class in \( H^1(X, \mathcal{O}_X^*) \), that is, a holomorphic line bundle on \( X \). It is called the \textit{cotangent bundle} of \( \mathcal{F} \), and is denoted by \( \text{Cotan}(\mathcal{F}) \). Its dual \( \text{Tan}(\mathcal{F}) \), represented by the inverse cocycle \( \{g_{jk}^{-1}\} \), is called the \textit{tangent bundle} of \( \mathcal{F} \).

Recall also that a positive \( dd^c \)-closed current \( T \) of bidegree \((1, 1) \) on \( X \) is \textit{directed} by the foliation \( \mathcal{F} \) (or equivalently, \textit{tangent to} \( \mathcal{F} \)) if \( T \wedge \Phi = 0 \) for every local holomorphic 1-form \( \Phi \) defining \( \mathcal{F} \). The directed \( dd^c \)-closed currents are generalizations of the \textit{foliations cycles} introduced by Sullivan [44].

Two fundamental concepts associated to \( \mathcal{F} \) are its holonomy cocycle \( \mathcal{H} \) and the convex cone of \textit{positive} \( dd^c \)-\textit{closed currents} directed by \( \mathcal{F} \). They are geometric objects. An important characteristic number which relates these two concepts is the Lyapunov exponent \( \chi(T) \) of an extremal positive \( dd^c \)-closed current \( T \) directed by \( \mathcal{F} \). In the first article of this series, we have established an effective sufficient condition for the existence of the Lyapunov exponents of Brody hyperbolic foliations. This class of foliations was first introduced in our joint-work with Dinh and Sibony [23]. Now we continue, in the second article of this series, the study of the Lyapunov exponents. More specifically, we investigate the interplay between the dynamical and geometric interpretations of these characteristic numbers, and we determine whether these numbers are positive/zero/negative. The presence of singular points makes our analysis delicate.

Let \( \Omega = \Omega(\mathcal{F}) \) be the sample-path space consisting of all continuous paths \( \omega : \mathbb{R} \to X \setminus E \) with image fully contained in a single leaf. Let \( g_P \) be the leafwise Poincaré metric on \( \mathcal{F} \). Consider the corresponding harmonic positive measure

\[
\mu := T \wedge g_P \quad \text{on} \quad X \setminus E, \quad \text{and} \quad \mu(E) = 0.
\]

The \textit{Poincaré mass} of \( T \) is, by definition, the mass \( \|\mu\| := \int_X d\mu \) of the measure \( \mu \). When all points \( a \in E \) are linearizable, by [21, Proposition 4.2] \( \|\mu\| \) is finite. For \( x \in X \setminus E \), the restriction of \( g_P \) on the leaf \( L_x \) passing through \( x \), generates the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Diagram of the foliation \( \mathcal{F} \).}
\end{figure}
corresponding leafwise Brownian motion. This Markov process defines a canonical probability measure, the Wiener measure $W_x$ on $\Omega$ which gives full mass to the subspace $\Omega_x$ consisting of all paths $\omega \in \Omega$ with $\omega(0) = x$ (see Sect. 2.4 below). Now we recall the following existence theorem for the Lyapunov exponents.

**Theorem 1.1** ([38, Theorem 1.1]) Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a holomorphic foliation by Riemann surfaces defined on a Hermitian compact complex projective surface $X$ satisfying the following two conditions:

- its singularities $E$ are all hyperbolic;
- $\mathcal{F}$ is Brody hyperbolic.

Let $T$ be a positive $dd^c$-closed current directed by $\mathcal{F}$ which does not give mass to any invariant analytic curve. Assume, in addition, that $T$ is an extremal element in the convex cone of all positive $dd^c$-closed currents directed by $\mathcal{F}$.

Then

1. $T$ admits the (unique) **Lyapunov exponent** $\chi(T)$ given by the formula

$$\chi(T) := \int_X \left( \int_{\Omega} \log \| H(\omega, 1) \| \, dW_x(\omega) \right) \, d\mu(x). \quad (1.3)$$

2. For $\mu$-almost every $x \in X \setminus E$, we have

$$\lim_{t \to \infty} \frac{1}{t} \log \| H(\omega, t) \| = \chi(T)$$

for almost every path $\omega \in \Omega$ with respect to $W_x$.

In fact, assertion (1) is a consequence of the so-called **integrability of the holonomy cocycle**. Assertion (2) says that the characteristic number $\chi(T)$ measures heuristically the exponential rate of convergence of leaves toward each other along leafwise Brownian trajectories (see Candel [9], Deroin [15] for the nonsingular case). Therefore, Theorem 1.1 gives a dynamical characterization of $\chi(T)$.

Now we discuss the geometric aspect of $\chi(T)$. Let $H^{1,1}(X)$ denote the Dolbeault cohomology group of real smooth $(1, 1)$-forms on $X$. For a real smooth closed $(1, 1)$-form $\alpha$ on $X$, let $[\alpha]$ be its class in $H^{1,1}(X)$. The cup-product $\sim$ on $H^{1,1}(X) \times H^{1,1}(X)$ is defined by

$$([\alpha], [\beta]) \mapsto [\alpha] \sim [\beta] := \int_X \alpha \wedge \beta,$$

where $\alpha$ and $\beta$ are real smooth closed forms. The last integral depends only on the classes of $\alpha$ and $\beta$. The bilinear form $\sim$ is non-degenerate and induces a canonical isomorphism between $H^{1,1}(X)$ and its dual $H^{1,1}(X)^*$ (Poincaré duality). In the definition of $\sim$ one can take $\beta$ smooth and $\alpha$ a current in the sense of de Rham. So, $H^{1,1}(X)$ can be defined as the quotient of the space of real closed $(1, 1)$-currents by the subspace of $d$-exact currents. Recall that an $(1, 1)$-current $\alpha$ is real (resp. $dd^c$-closed) if $\alpha = \tilde{\alpha}$ (resp. $dd^c\alpha = 0$). Assume that $\alpha$ is a real $dd^c$-closed $(1, 1)$-current. Then by the $dd^c$-lemma, the integral $\int_X \alpha \wedge \beta$ is also independent of the choice of
\( \beta \) smooth and closed in a fixed cohomology class. So, using the above isomorphism, one can associate to such \( \alpha \) a class \( \{ \alpha \} \) in \( H^{1,1}(X) \). For a complex line bundle \( \mathbb{E} \) over \( X \), let \( c_1(\mathbb{E}) \) denote the cohomology Chern class of \( \mathbb{E} \). This is an element in \( H^{1,1}(X) \).

Our first main result gives cohomological formulas for \( \chi(T) \) and \( \| \mu \| \) in terms of the geometric quantity \( T \) and some characteristic classes of \( \mathcal{F} \).

**Theorem 1.2** (Theorem A) Under the assumption of Theorem 1.1, the following identities hold

\[
\chi(T) = -c_1(\text{Nor}(\mathcal{F})) \sim \{ T \}, \\
\| \mu \| = c_1(\text{Cotan}(\mathcal{F})) \sim \{ T \}.
\]

Here \( \text{Nor}(\mathcal{F}) := \text{Tan}(X) / \text{Tan}(\mathcal{F}) \) stands for the normal bundle of \( \mathcal{F} \), where \( \text{Tan}(\mathcal{F}) \) and \( \text{Cotan}(\mathcal{F}) \) are as usual the tangent bundle and the cotangent bundle of \( \mathcal{F} \).

On the other hand, in collaboration with Dinh and Sibony, we have recently proved that in most interesting cases, the directed positive \( dd^c \)-closed current exists uniquely (up to a multiplicative constant).

**Theorem 1.3** ([24, Theorem 1.1]) Let \( \mathcal{F} \) be a holomorphic foliation by Riemann surfaces with only hyperbolic singularities in a compact Kähler surface \( X \). Assume that \( \mathcal{F} \) admits no directed positive closed current. Then there exists a unique positive \( dd^c \)-closed current \( T \) of Poincaré mass 1 directed by \( \mathcal{F} \).

Theorem 1.3 implies strong ergodic properties for the foliation \( \mathcal{F} \). It is worthy noting that when \( X = \mathbb{P}^2 \) the theorem was obtained by Fornæss–Sibony [27], see also Dinh–Sibony [19] and Pérez-Garrandés [42].

Suppose now that \( \mathcal{F} \) is a holomorphic foliation by Riemann surfaces with only hyperbolic singularities in a compact projective surface \( X \) such that \( \mathcal{F} \) admits no directed positive closed current. So the assumptions of both Theorem 1.1 and 1.3 are fulfilled. By Theorem 1.3, let \( T \) be the unique directed positive \( dd^c \)-closed current \( T \) whose the Poincaré mass is equal to 1.

**Definition 1.4** The Lyapunov exponent of the foliation \( \mathcal{F} \), denoted by \( \chi(\mathcal{F}) \), is by definition, the real number \( \chi(T) \) given by Theorem 1.1.

When we explore the dynamical system associated to a foliation \( \mathcal{F} \), the sign of its Lyapunov exponent is a crucial information. Indeed, the positivity/negativity of \( \chi(\mathcal{F}) \) corresponds to the repelling/attracting character of a typical leaf along a typical Brownian trajectory. Here is our second main result.

**Theorem 1.5** (Theorem B) Under the assumption of Theorem 1.3, assume in addition that \( X \) is projective. Then \( \chi(\mathcal{F}) \) is a negative nonzero real number.

Roughly speaking, Theorem B says that in the sense of ergodic theory, generic leaves have the tendency to wrap together towards the support of the unique normalized directed positive \( dd^c \)-closed current.
Now we apply the above results to the family of singular holomorphic foliations on \( \mathbb{P}^2 \) with a given degree \( d > 1 \). Recall that the degree is the number of tangencies of the foliation with a generic line. This family can be identified with a Zariski dense open set \( \mathcal{U}_d \) of some projective space. By Brunella [6], if \( \mathcal{F} \in \mathcal{U}_d \) with the properties that all the singularities of \( \mathcal{F} \) are hyperbolic and that \( \mathcal{F} \) does not possess any invariant algebraic curve, then \( \mathcal{F} \) admits no nontrivial directed positive closed current, and hence \( \mathcal{F} \) satisfies the assumptions of both Theorems A and B. By Jouanolou [31] and Lins Neto-Soares [34], these properties are satisfied for \( \mathcal{F} \) in a set of full Lebesgue measure of \( \mathcal{U}_d \). Consequently, Theorems A and B apply and give us the following result. It can be applied to every generic foliation in \( \mathbb{P}^2 \) with a given degree \( d > 1 \).

**Corollary 1.6** Let \( \mathcal{F} = (\mathbb{P}^2, \mathcal{L}, E) \) be a singular foliation by curves on the complex projective plane \( \mathbb{P}^2 \). Assume that all the singularities are hyperbolic and that \( \mathcal{F} \) has no invariant algebraic curve. Then

\[
\chi(\mathcal{F}) = -\frac{d + 2}{d - 1},
\]

where \( d \) is the degree of \( \mathcal{F} \).

To the best of our knowledge, this is the first family of singular holomorphic foliations for which one knows to compute the Lyapunov exponents.

Now we discuss the relations between our results and previous works. Candel [8] introduces the Euler class of a directed positive harmonic current in the context of compact Riemann surface laminations endowed with a conformal metric. So his notion generalizes the intersection of the Chern class of a holomorphic line bundle with a directed positive harmonic current considered in Theorem A, but only when the foliation is nonsingular. Moreover, in the context of general compact laminations, Candel introduces in [9] the notion of one-dimensional cocycle. This notion has the advantage of not using any structure of complex line bundles. He obtains an integral formula for the Lyapunov exponent of such a cocycle with respect to a directed positive harmonic current. This result has been generalized for higher dimensional cocycles in [35]. On the other hand, a version of Theorem A for exceptional minimal sets has been established by Deroin [15, Appendix A]. Concerning Theorem B, the negativity of \( \chi(\mathcal{F}) \) has been proved by Deroin and Kleptsyn [16, Theorem B] in the context of transversally conformal foliations (without singularities), see also Baxendale [2] for a related result. Formula (1.5) has already been found out by Deroin and Kleptsyn [16, Proposition 3.12], but under the strong hypothetical additional assumption that there is an exceptional minimal set.

One of the main ingredients in our proof of Theorem A is some precise and delicate estimates on the variations of the holonomy cocycle and the clustering mass of a directed positive \( dd^c \)-closed current near the singularities. In particular, Proposition 4.6 below plays a decisive role in our approach, it gives a stronger estimate than those obtained previously in [24, 27, 38] etc. The proof of Proposition 4.6 is partly based on our last work [38]. However, it also requires new techniques which rely on estimates on heat diffusions associated to the leafwise Poincaré metric \( g_P \).
If we move away all the dynamical aspects of Theorem A, then its proof boils down to some continuity problems of the wedge-product of a positive $dd^c$-closed $(1, 1)$-current with another (not necessarily positive) $(1, 1)$-current whose potential is unbounded. This type of problems is quite new in the intersection theory of currents since up to now only the case with bounded potential is understood (see e.g. [18]). In this vein, other ingredients in our proof of Theorem A are some tools from complex geometry such as the regularization of curvature currents of singular Hermitian holomorphic line bundle (see [4]), the positivity of the cotangent bundle with respect to the leafwise Poincaré metric (see [5]).

To prove the negativity of the Lyapunov exponent (i.e. Theorem B), we use Hahn–Banach separation theorem following an idea of Sullivan [44] and Ghys [29]. However, in order to carry out this plan in the context of singular foliations, we introduce an adapted normed space whose norm is taken with respect to a natural weight function $W$. This function reflects the singularities of the considered foliation. We make a full use of the key fact obtained in Proposition 4.6 that $W$ is $\mu$-integrable. A systematic study on the variations of the holonomy cocycle near the singularities of the foliation is also needed.

Understanding the dynamics of singular holomorphic foliations is a challenging big program. We hope that some of the techniques developed in this paper may be extended to singular holomorphic foliations in higher dimensions. The reader is invited to consult Păun-Sibony [41] for a fruitful discussion on the link between value distribution theory and positive currents directed by singular holomorphic foliations.

The paper is organized as follows. In Sect. 2, we recall some basic elements of singular holomorphic foliations. Section 3 is devoted to a geometric study of the transversal metric and the holonomy variations near singularities, which leads to a natural weight function $W$. Based on this study, Sect. 4 develops a dynamical study of the holonomy variations as well as some estimates on the clustering mass of a directed positive $dd^c$-closed current near singularities. Two proofs of the first half of Theorem A (i.e. identity (1.4a)) are given at the end of the section. The last half of Theorem A (i.e. identity (1.4b)) is proved in Sect. 5. Section 6 is devoted to the proof of Theorem B. Finally, we conclude the article with some open questions and remarks.

After I had finished the first version of this article in November 2018, Deroin informed me that independently with Kleptsyn, they had obtained a similar result but possibly under a stronger hypothesis on $\mathcal{F}$. A crucial ingredient of their approach is my Theorem 1.1.

Notation. Recall that $d$, $d^c$ denote the real differential operators on $X$ defined by $d:=\partial + \overline{\partial}$, $d^c := \frac{1}{2\pi i} (\partial - \overline{\partial})$ so that $dd^c = \frac{i}{\pi} \partial \overline{\partial}$. Throughout the article, we denote by $D$ the unit disc in $\mathbb{C}$. For $r > 0$ we denote by $rD$ the disc in $\mathbb{C}$ with center 0 and with radius $r$. The letters $c$, $c'$, $c_0$, $c_1$, $c_2$ etc. denote positive constants, not necessarily the same at each occurrence. The notation $\gtrsim$ and $\lesssim$ means inequalities up to a multiplicative constant, whereas we write $\approx$ when both inequalities are satisfied. Let $O$ and $o$ denote the usual Landau asymptotic notations.

Let $\log^* (\cdot) := 1 + |\log (\cdot)|$ be a log-type function.
2 Background

Let $X$ be a compact complex surface endowed with a smooth Hermitian metric $g_X$. Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a holomorphic foliation by Riemann surfaces, where the set of singularities $E$ is finite. For a recent account on singular holomorphic foliations, the reader is invited to consult the survey articles [20, 26, 37, 40].

2.1 Poincaré Metric and Brody Hyperbolicity

Let $g_P$ be the Poincaré metric on the unit disc $\mathbb{D}$, defined by

$$g_P(\zeta) := \frac{2}{(1 - |\zeta|^2)^2}i d\zeta \wedge d\overline{\zeta}, \quad \zeta \in \mathbb{D}, \quad \text{where } i := \sqrt{-1}.$$  

A leaf $L$ of the foliation is said to be hyperbolic if it is a hyperbolic Riemann surface, i.e., it is uniformized by $\mathbb{D}$. For any point $x \in X \setminus E$, let $L_x$ be the leaf passing through $x$. The foliation is said to be hyperbolic if all its leaves are hyperbolic. For a hyperbolic leaf $L_x$, consider a universal covering map

$$\phi_x : \mathbb{D} \rightarrow L_x \quad \text{such that } \phi_x(0) = x. \quad (2.1)$$

This map is uniquely defined by $x$ up to a rotation on $\mathbb{D}$. Then, by pushing forward the Poincaré metric $g_P$ on $\mathbb{D}$ via $\phi_x$, we obtain the so-called Poincaré metric on $L_x$ which depends only on the leaf. The latter metric is given by a positive $(1, 1)$-form on $L_x$ that we also denote by $g_P$ for the sake of simplicity.

For simplicity we still denote by $g_X$ the Hermitian metric on leaves of the foliation $(X \setminus E, \mathcal{L})$ induced by the ambient Hermitian metric $g_X$. Consider the function $\eta : X \setminus E \rightarrow [0, \infty]$ defined by

$$\eta(x) := \sup \{ \|d\phi(0)\| : \phi : \mathbb{D} \rightarrow L_x \text{ holomorphic such that } \phi(0) = x \}.$$  

Here, for the norm of the differential $d\phi$ we use the Poincaré metric on $\mathbb{D}$ and the Hermitian metric $g_X$ on $L_x$. Clearly, we have

$$\eta(x) = \|d\phi_x(0)\| \quad \text{for } x \in X \setminus E. \quad (2.2)$$

We recall the following relation between $g_X$ and the Poincaré metric $g_P$ on leaves

$$g_X(y)|_{L_x} = \eta^2(y)g_P(y) \quad \text{for } y \in L_x, x \in X \setminus E. \quad (2.3)$$

In [23] the following class of foliations is introduced.

**Definition 2.1** A foliation $\mathcal{F} = (X, \mathcal{L}, E)$ is said to be Brody hyperbolic if there is a constant $c > 0$ such that $\eta(x) \leq c$ for all $x \in X \setminus E$.  

\[ \square \] Springer
Remark 2.2 Note that if the foliation \( F \) is Brody hyperbolic then it is hyperbolic. But the converse statement does not hold in general. The Brody hyperbolicity is equivalent to the non-existence of holomorphic non-constant maps \( \mathbb{C} \rightarrow X \) such that out of \( E \) the image of \( \mathbb{C} \) is locally contained in a leaf, see [26, Theorem 15]. If \( F \) admits no directed positive closed current, then it is Brody hyperbolic.

As a consequence of (2.3), the Poincaré norm \( |df|_p \) of the differential \( df \) for a differentiable function \( f : L_x \rightarrow \mathbb{C} \) is given by

\[
|df|_p(y) := \eta(y)|df(y)| \quad \text{for} \quad y \in L_x,
\]

where \( |df(y)| \) denotes the Euclidean norm of \( df(y) \).

2.2 Poincaré Laplacian \( \Delta_F \) and the Heat Diffusions, Directed Positive Harmonic Currents Versus Harmonic Measures

A (directed) \( p \)-form (resp. a (directed) \( (p,q) \)-form) on \( F \) can be seen on the flow box \( \mathbb{U} \simeq \mathbb{B} \times \mathbb{T} \) as a \( p \)-form (resp. \( (p,q) \)-form) on \( \mathbb{B} \) depending on the parameter \( t \in \mathbb{T} \). For \( 0 \leq p \leq 2 \) (resp. for \( 0 \leq p, q \leq 1 \)), denote by \( \mathcal{D}^p_F(\mathcal{F}) \) (resp. \( \mathcal{D}^{p,q}_F(\mathcal{F}) \)) the space of \( p \)-forms (resp. \( (p,q) \)-form) \( f \) with compact support in \( X \setminus E \) satisfying the following property: \( f \) restricted to each flow box \( \mathbb{U} \simeq \mathbb{B} \times \mathbb{T} \) is a \( p \)-form (resp. \( (p,q) \)-form) of class \( \mathcal{C}^l \) on the plaques whose coefficients and all their derivatives up to order \( l \) depend continuously on the plaque. The norm \( \| \cdot \|_{\mathcal{C}^l} \) on this space is defined as in the case of real manifold using a locally finite atlas of \( \mathcal{F} \). We also define \( \mathcal{D}^p_F(\mathcal{F}) \) (resp. \( \mathcal{D}^{p,q}_F(\mathcal{F}) \)) as the intersection of \( \mathcal{D}^k_F(\mathcal{F}) \) (resp. \( \mathcal{D}^{k,q}_F(\mathcal{F}) \)) for \( l \geq 0 \).

In particular, a sequence \( f_j \) converges to \( f \) in \( \mathcal{D}^p_F(\mathcal{F}) \) (resp. in \( \mathcal{D}^{p,q}_F(\mathcal{F}) \)) if these forms are supported in a fixed compact set of \( X \setminus E \) and if \( \| f_j - f \|_{\mathcal{C}^l} \rightarrow 0 \) for every \( l \). A (directed) current of degree \( p \) (or equivalently, of dimension \( 2 - p \)) on \( F \) is a continuous linear form on the space \( \mathcal{D}^{2-p}_F(\mathcal{F}) \) with values in \( \mathbb{C} \). We often write for short \( \mathcal{D}(\mathcal{F}) \) instead of \( \mathcal{D}^0(\mathcal{F}) \). A directed \( (p + q) \)-current is said to be of bidegree \( (p, q) \) (or equivalently, of bidimension \( (1 - p, 1 - q) \)) if it vanishes on forms of bidegree \( (1 - p', 1 - q') \neq (p, q) \).

A form \( f \in \mathcal{D}^{1,1}_F(\mathcal{F}) \) is said to be positive if its restriction to every plaque is a positive measure in the usual sense.

Definition 2.3 (Garnett [28], see also Sullivan [44]). Let \( T \) be a directed current of bidegree \( (0, 0) \) on \( F \).

- \( T \) is said to be positive if \( T(f) \geq 0 \) for all positive forms \( f \in \mathcal{D}^{1,1}_F(\mathcal{F}) \).
- \( T \) is said to be harmonic if \( dd^c T = 0 \) in the weak sense (namely, \( T(dd^c f) = 0 \) for all functions \( f \in \mathcal{D}(\mathcal{F}) \)).

Let \( \mathbb{U} \) be any flow box of \( F \) outside the singularities and denote by \( V_\alpha \) the plaques of \( F \) in \( \mathbb{U} \) parametrized by \( \alpha \) in some transversal \( \Sigma \) of \( \mathbb{U} \). On the flow box \( \mathbb{U} \), a positive harmonic current \( T \) directed by \( F \) (or equivalently, tangent to \( F \)) has the form

\[
T|_\mathbb{U} = \int_{\alpha \in \Sigma} h_\alpha[V_\alpha] d\nu(\alpha),
\]
where \( h_\alpha \) is a positive harmonic function on \( V_\alpha \), and \([V_\alpha]\) denotes the current of integration on the plaque \( V_\alpha \), and \( \nu \) is a Radon measure on \( \Sigma \) (see e.g. [21, Proposition 2.3]).

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a hyperbolic foliation. The leafwise Poincaré metric \( g_\mathcal{P} \) induces the corresponding Laplacian \( \Delta_\mathcal{P} \) on leaves such that

\[
\frac{dd^c f}{|L_x|} = \frac{1}{\pi} \Delta_\mathcal{P} f \cdot g_\mathcal{P}, \quad \text{on } L_x \text{ for all } f \in \mathcal{D}(\mathcal{F}). \tag{2.5}
\]

It is worthy noting that the left-hand side \( \frac{dd^c f}{|L_x|} \) belongs to \( \mathcal{D}^{1,1}(\mathcal{F}) \). A positive finite Borel measure \( \mu \) on \( X \) which does not give mass to \( E \) is said to be harmonic if

\[
\int_X \Delta_\mathcal{P} f \, d\mu = 0
\]

for all functions \( f \in \mathcal{D}(\mathcal{F}) \).

For every point \( x \in X \setminus E \), consider the heat equation on \( L_x \)

\[
\frac{\partial p(x, y, t)}{\partial t} = \Delta_{p,y} p(x, y, t), \quad \lim_{t \to 0} p(x, y, t) = \delta_x(y), \quad y \in L_x, \ t \in \mathbb{R}_+.
\]

Here \( \delta_x \) denotes the Dirac mass at \( x \), \( \Delta_{p,y} \) denotes the Laplacian \( \Delta_\mathcal{P} \) with respect to the variable \( y \), and the limit is taken in the sense of distribution, that is,

\[
\lim_{t \to 0^+} \int_{L_x} p(x, y, t) f(y) g_\mathcal{P}(y) = f(x)
\]

for every compactly supported smooth function \( f \) in \( L_x \).

The smallest positive solution of the above equation, denoted by \( p(x, y, t) \), is called the heat kernel. Such a solution exists because \((L_x, g_\mathcal{P})\) is complete and of bounded geometry (see, for example, [11, 12]). The heat kernel gives rise to a one parameter family \( \{D_t : t \geq 0\} \) of diffusion operators defined on bounded Borel measurable functions on \( X \setminus E \) :

\[
D_t f(x) := \int_{L_x} p(x, y, t) f(y) g_\mathcal{P}(y), \quad x \in X \setminus E. \tag{2.6}
\]

We record here the semi-group property of this family:

\[
D_0 = \text{id} \quad \text{and} \quad D_t 1 = 1 \quad \text{and} \quad D_{t+s} = D_t \circ D_s \quad \text{for } t, s \geq 0, \tag{2.7}
\]

where \( 1 \) denotes the function which is identically equal to 1.

We also denote by \( \Delta_\mathcal{P} \) the Laplacian on the Poincaré disc \((\mathbb{D}, g_\mathcal{P})\), that is, for every function \( f \in \mathcal{C}^2(\mathbb{D}) \),

\[
\frac{1}{\pi} (\Delta_\mathcal{P} f) g_\mathcal{P} = dd^c f \quad \text{on } \mathbb{D}. \tag{2.8}
\]
For every function $f \in C^1(D)$, we also denote by $|df|_P$ the length of the differential $df$ with respect to $g_P$, that is,

$$|df|_P = |df| \cdot g_P^{-1/2} \quad \text{on} \quad D,$$

(2.9)

where $|df|$ denotes the Euclidean norm of $df$. Let $\text{dist}_P$ denote the Poincaré distance on $(D, g_P)$.

Using the map $\phi_x : D \to L_x$ given in (2.1), the following identity relates the diffusion operators in $L_x$ and those in the Poincaré disc $(D, g_P)$: For $x \in X$ and for every bounded measurable function $f$ defined on $L_x$,

$$D_t(f \circ \phi_x) = (D_t f) \circ \phi_x, \quad \text{on} \quad D \quad \text{for all} \quad t \in \mathbb{R}^+.$$  (2.10)

Here $\{D_t : t \geq 0\}$ on the left-hand side is the family of the heat diffusion associated to the Poincaré disc, see [35, Proposition 2.7] for a proof.

Recall that a positive finite measure $\mu$ on the $\sigma$-algebra of Borel sets in $X$ with $\mu(E) = 0$ is said to be ergodic if for every leafwise saturated Borel measurable set $Z \subset X$, $\mu(Z)$ is equal to either $\mu(X)$ or 0. A directed positive harmonic current $T$ is said to be extremal if $T = T_1 + T_2$ for directed positive harmonic current $T_1, T_2$ implies that $T_1 = \lambda T$ for some $\lambda \in [0, 1]$.

The following result gives the link between directed positive harmonic currents, directed positive $dd^c$-closed currents and harmonic measures (see e.g. Theorem 2.9 and Proposition 2.17 in [40], see also [21]).

**Proposition 2.4** Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a hyperbolic foliation with linearizable singularities $E$ in a compact complex surface $X$.

1. Every directed positive harmonic current extends, by trivial extension across $E$, to a $dd^c$-closed current on $X$. In other words, directed positive harmonic currents are equivalent to directed positive $dd^c$-closed currents.
2. The relation $T \mapsto \mu$ given in (1.2) is a one-to-one correspondence between the convex cone of positive harmonic currents $T$ directed by $\mathcal{F}$ and the convex cone of harmonic measures $\mu$.
3. $T$ is extremal if and only if $\mu$ given in (1.2) is ergodic.
4. Each harmonic measure $\mu$ is $D_t$-invariant, i.e,

$$\int_X D_t f \, d\mu = \int_X f \, d\mu, \quad f \in L^1(X, \mu), \quad t \in \mathbb{R}^+.$$

**2.3 Local Model for Hyperbolic Singularities, Regular and Singular Flow Boxes**

To study $\mathcal{F}$ near a hyperbolic singularity $a$, we use the following local model introduced in [22]. In this model, a neighborhood of $a$ is identified with the bidisc $\mathbb{D}^2$, and the restriction of $\mathcal{F}$ to $\mathbb{D}^2$, i.e., the leaves of $(\mathbb{D}^2, \mathcal{L}, \{0\})$ coincide with the restriction to $\mathbb{D}^2$ of the integral curves of a vector field

$$Z(z, w) = z \frac{\partial}{\partial z} + \lambda w \frac{\partial}{\partial w} \quad \text{with some} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$  (2.11)
Fig. 1 On the left-hand side, the sector $\Pi_x$ with vertex $I_x$, which gives rise to the parametrization $\psi_x : \Pi_x \to L_x \cap \mathbb{D}^2$, is depicted. On the right-hand side, the canonical sector $\mathcal{S}$, which gives rise to the parametrization $\tilde{\psi}_\alpha : \mathcal{S} \to L_\alpha$ for every $\alpha \in \Lambda$, is given. In fact, $\mathcal{S}$ is the image of $\Pi_x$ by the translation mapping $I_x$ to the vertex 0 of $\mathcal{S}$.

For $x = (z, w) \in \mathbb{D}^2 \setminus \{0\}$, define the holomorphic map $\psi_x : \mathbb{C} \to \mathbb{C}^2 \setminus \{0\}$

$$\psi_x(\zeta) := \left(ze^{i\zeta}, we^{i\lambda\zeta}\right) \quad \text{for} \quad \zeta \in \mathbb{C}.$$ (2.12)

It is easy to see that $\psi_x(\mathbb{C})$ is the integral curve of $Z$ which contains $\psi_x(0) = x$. Write $\zeta = u + iv$ with $u, v \in \mathbb{R}$. The domain $\Pi_x := \psi_x^{-1}(\mathbb{D}^2)$ in $\mathbb{C}$ is defined by the inequalities

$$(\text{Im}\lambda)u + (\text{Re}\lambda)v > \log |w| \quad \text{and} \quad v > \log |z|.$$ 

So, $\Pi_x$ defines a sector in $\mathbb{C}$. It contains 0 since $\psi_x(0) = x$, see the left-hand side part of Fig. 1.

The leaf of $\mathcal{F}$ through $x$ contains the Riemann surface

$$\hat{L}_x := \psi_x(\Pi_x) \subset L_x.$$ (2.13)

In particular, the leaves in a singular flow box are parametrized using holomorphic maps $\psi_x : \Pi_x \to L_x$.

Now let $\mathcal{F} = (X, \mathcal{L}, E)$ be a foliation on a Hermitian compact complex surface $(X, g_X)$. Assume as usual that $E$ is finite and all points of $E$ are linearizable. Let dist be the distance on $X$ induced by the ambient metric $g_X$. We only consider flow boxes which are biholomorphic to $\mathbb{D}^2$. A regular flow box $\mathcal{U}$ is a flow box with foliated chart $\Phi : \mathcal{U} \to \mathbb{B} \times \Sigma$ outside the singularities, where $\mathbb{B}$ and $\Sigma$ are open sets in $\mathbb{C}$. For each $\alpha \in \Sigma$, the Riemann surface $V_\alpha := \Phi^{-1}(\mathbb{B} \times \{\alpha\})$ is called a plaque of $\mathcal{U}$. Singular flow boxes are identified to their models $(\mathbb{D}^2, \mathcal{L}, \{0\})$ as described above. For $\mathcal{U} := \mathbb{D}^2$ and $s > 0$, let $s\mathcal{U} := (s\mathbb{D})^2$. For each singular point $a \in E$, we fix a singular flow box.
Fig. 2 A singular flow box (in gray) around the origin 0 in coordinates $(z, w) \in \mathbb{D}^2$, and a regular flow box on the upper right-hand side.

There exists a constant $c > 1$ with the following properties.

(1) $\eta \leq c$ on $X$, $\eta \geq c^{-1}$ outside the singular flow boxes $\bigcup_{a \in E} \frac{1}{4} U_a$ and

$$c^{-1} \cdot s \log^* s \leq \eta(x) \leq c \cdot s \log^* s$$

for $x \in X \setminus E$ and $s := \text{dist}(x, E)$.

Lemma 2.5 We keep the above hypotheses and notation. Suppose in addition that $\mathcal{F}$ is Brody hyperbolic. Then there exists a constant $c > 1$ with the following properties.
(2) For every \( x \) in a singular box which is identified with \((\mathbb{D}^2, \mathcal{L}, \{0\})\) as above, for every \( \zeta \in \Pi_x \),

\[
c^{-1} \cdot \frac{id\zeta \wedge d\bar{\zeta}}{(\log^* (\psi_x(\zeta)))^2} \leq (\psi_x^* g_p)(\zeta) \leq c \cdot \frac{id\zeta \wedge d\bar{\zeta}}{(\log^* (\psi_x(\zeta)))^2}.
\]

We revisit the local model by describing its special flow box structure. Notice that if we flip \( z \) and \( w \), we replace \( \lambda \) by \( \lambda^{-1} \). Since \((\text{Im}\lambda)(\text{Im}\lambda^{-1}) < 0\), we may assume below that the axes are chosen so that \( \text{Im}\lambda > 0 \). Consider the ring \( \mathbb{A} \) defined by

\[
\mathbb{A} := \left\{ \alpha \in \mathbb{C}, \ e^{-2\pi \text{Im}\lambda} < \left| \alpha \right| \leq 1 \right\}.
\]

Define also the sector \( \mathbb{S} \) by

\[
\mathbb{S} := \{ \zeta = u + iv \in \mathbb{C}, \ v > 0 \text{ and } (\text{Im}\lambda)u + (\text{Re}\lambda)v > 0 \}.
\]

Note that the sector \( \mathbb{S} \) is contained in the upper half-plane \( \mathbb{H} := \{ u + iv, \ v > 0 \} \), see the right-hand side part of Fig. 1.

For \( \alpha \in \mathbb{C}^* \), consider the following holomorphic map \( \tilde{\psi}_\alpha : \mathbb{C} \rightarrow (\mathbb{C}^*)^2 \):

\[
\tilde{\psi}_\alpha : = \left( e^{i(\zeta + \log|\alpha|/\text{Im}\lambda)}, \alpha e^{i\lambda(\zeta + \log|\alpha|/\text{Im}\lambda)} \right) \text{ for } \zeta = u + iv \in \mathbb{C}.
\]

Note that the map \( \tilde{\psi}_\alpha \) is injective because \( \lambda \notin \mathbb{R} \). Let \( \mathcal{L}'_\alpha \) be the image of \( \tilde{\psi}_\alpha \). This is a Riemann surface immersed in \( \mathbb{C}^2 \). Note also that

\[
|z| = e^{-v} \quad \text{and} \quad |w| = e^{-(\text{Im}\lambda)u - (\text{Re}\lambda)v}.
\]

It is easy to check the following properties:

1. \( \mathcal{L}'_\alpha \) is tangent to the vector field \( Z \) given in (2.11) and is a submanifold of \( \mathbb{C}^* \).
2. \( \mathcal{L}'_{\alpha_1} \) is equal to \( \mathcal{L}'_{\alpha_2} \) if \( \alpha_1/\alpha_2 = e^{2k\lambda \pi} \) for some \( k \in \mathbb{Z} \) and they are disjoint otherwise. In particular, \( \mathcal{L}'_{\alpha_1} \) and \( \mathcal{L}'_{\alpha_2} \) are disjoint if \( \alpha_1, \alpha_2 \in \mathbb{A} \) and \( \alpha_1 \neq \alpha_2 \).
3. The union of \( \mathcal{L}'_\alpha \) is equal to \( \mathbb{C}^* \) for \( \alpha \in \mathbb{C}^* \), and then also for \( \alpha \in \mathbb{A} \).
4. The intersection \( \mathcal{L}_\zeta := \mathcal{L}'_\alpha \cap \mathbb{D}^2 \) of \( \mathcal{L}'_\alpha \) with the unit bidisc \( \mathbb{D}^2 \) is given by the same equations as in the definition of \( \mathcal{L}'_\alpha \) but with \( \zeta \in \mathbb{S} \). Moreover, \( \mathcal{L}_\zeta \) is a connected submanifold of \( \mathbb{D}^2 \). In particular, it is a leaf of \( (\mathbb{D}^2, \mathcal{L}, \{0\}) \).

### 2.4 Measure Theory on Sample-Path Spaces

In this subsection we follow the presentation given in Sects. 2.2, 2.4 and 2.5 in [35], which are, in turn, inspired by Garnett’s theory of leafwise Brownian motion in [28] (see also [9, 11]). This exposition is equivalent to that given in [38, Sect. 2.4].

We first recall the construction of the Wiener measure \( W_0 \) on the Poincaré disc \((\mathbb{D}, g_P)\). Let \( \Omega_0 \) be the space consisting of all continuous paths \( \omega : [0, \infty) \rightarrow \mathbb{D} \) with
\( \omega(0) = 0 \). A cylinder set (in \( \Omega_0 \)) is a set of the form

\[
C = C([t_i, B_i] : 1 \leq i \leq m) := \{ \omega \in \Omega_0 : \omega(t_i) \in B_i, \quad 1 \leq i \leq m \}.
\]

where \( m \) is a positive integer and the \( B_i \)'s are Borel subsets of \( \mathbb{D} \), and \( 0 < t_1 < t_2 < \cdots < t_m \) is a set of increasing times. In other words, \( C \) consists of all paths \( \omega \in \Omega_0 \) which can be found within \( B_i \) at time \( t_i \). Let \( \mathcal{F}_0 \) be the \( \sigma \)-algebra on \( \Omega_0 \) generated by all cylinder sets. For each cylinder set \( C := C([t_i, B_i] : 1 \leq i \leq m) \) as above, define

\[
W_x(C) := \left( D_{t_1}(1_{B_1} D_{t_2-t_1}(1_{B_2} \cdots 1_{B_{m-1}} D_{t_m-t_{m-1}}(1_{B_m}) \cdots )) \right)(x),
\] (2.16)

where, \( 1_{B_i} \) is the characteristic function of \( B_i \) and \( D_t \) is the diffusion operator given by (2.6) where \( p(x, y, t) \) is the heat kernel of the Poincaré disc \( (\mathbb{D}, g_P) \). It is well-known that \( W_0 \) can be extended to a unique probability measure on \( (\Omega_0, \mathcal{F}_0) \). This is the canonical Wiener measure at 0 on the Poincaré disc.

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a hyperbolic Riemann surface foliation with the set of singularities \( E \). Consider the leafwise Poincaré metric \( g_P \). Recall from Introduction that \( \Omega := \Omega(\mathcal{F}) \) is the space consisting of all continuous paths \( \omega : [0, \infty) \to X \setminus E \) with image contained in a single leaf. This space is called the sample-path space associated to \( \mathcal{F} \). Observe that \( \Omega \) can be thought of as the set of all possible paths that a Brownian particle, located at \( \omega(0) \) at time \( t = 0 \), might follow as time progresses. For each \( x \in X \setminus E \), let \( \Omega_x = \Omega_x(\mathcal{F}) \) be the space of all continuous leafwise paths starting at \( x \in X \setminus E \) in \( \mathcal{F} \), that is,

\[
\Omega_x := \{ \omega \in \Omega : \omega(0) = x \}.
\]

For each \( x \in X \setminus E \), the following mapping

\[
\Omega_0 \ni \omega \mapsto \phi_x \circ \omega \quad \text{maps} \quad \Omega_0 \quad \text{bijectively onto} \quad \Omega_x,
\] (2.17)

where \( \phi_x : \mathbb{D} \to L_x \) is given in (2.1). Using this bijection we obtain a natural \( \sigma \)-algebra \( \mathcal{A}_x \) on the space \( \Omega_x \), and a natural probability (Wiener) measure \( W_x \) on \( \mathcal{A}_x \) as follows:

\[
\mathcal{A}_x := \{ \phi_x \circ A : A \in \mathcal{A}_0 \} \quad \text{and} \quad W_x(\phi_x \circ A) := W_0(A), \quad A \in \mathcal{A}_0,
\] (2.18)

where \( \phi_x \circ A := \{ \phi_x \circ \omega : \omega \in A \} \subset \Omega_x \).

For any function \( F \in L^1(\Omega_x, \mathcal{A}_x, W_x) \), the expectation of \( F \) at \( x \) is the number

\[
\mathbb{E}_x[F] := \int_{\Omega_x} F(\omega) dW_x(\omega).
\] (2.19)

It is well-known (see [11, Proposition C.3.8]) that for any bounded Borel measurable function \( f \) on \( L_x \),

\[
\mathbb{E}_x[f(\bullet(t))] = (D_t f)(x), \quad t \in \mathbb{R}^+,
\] (2.20)
where \( f(\bullet(t)) \) is the function given by \( \Omega \ni \omega \mapsto f(\omega(t)) \).

### 2.5 Holonomy Cocycle

Now we recall from [38, Sects. 2.5, 3] the holonomy cocycle as well as its properties of a hyperbolic foliation \( \mathcal{F} = (X, \mathcal{L}, E) \) on a Hermitian complex surface \( X \). For each point \( x \in X \setminus E \), let \( \operatorname{Tan}_x(X) \) (resp. \( \operatorname{Tan}_x(L_x) \subset \operatorname{Tan}_x(X) \)) be the tangent space of \( X \) (resp. \( L_x \)) at \( x \). For every transversal \( S \) at a point \( x \) (that is, \( x \in S \)), let \( \operatorname{Tan}_x(S) \) denote the tangent space of \( S \) at \( x \).

Now fix a point \( x \in X \setminus E \) and a path \( \omega \in \Omega_x \) and a time \( t \in \mathbb{R}^+ \), and let \( y := \omega(t) \). Fix a transversal \( S_x \) at \( x \) (resp. \( S_y \) at \( y \)) such that the complex line \( \operatorname{Tan}_x(S_x) \) is the orthogonal complement of the complex line \( \operatorname{Tan}_x(L_x) \) in the Hermitian space \( (\operatorname{Tan}_x(X), (g_X)_x) \) (resp. \( \operatorname{Tan}_y(S_y) \) is the orthogonal complement of \( \operatorname{Tan}_y(L_y) \) in \( (\operatorname{Tan}_y(X), (g_X)_y) \)). Let \( \operatorname{hol}_{\omega,t} \) be the holonomy map along the path \( \omega|_{[0,t]} \) from an open neighborhood of \( x \) in \( S_x \) onto an open neighborhood of \( y \) in \( S_y \). The derivative \( D\operatorname{hol}_{\omega,t} : \operatorname{Tan}_x(S_x) \to \operatorname{Tan}_y(S_y) \) induces the so-called holonomy cocycle \( \mathcal{H} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+ \) given by

\[
\mathcal{H}(\omega, t) := \|D\operatorname{hol}_{\omega,t}(x)\|.
\]

The last map depends only on the path \( \omega|_{[0,t]} \), in fact, it even depends only on the homotopy class of this path. In particular, it is independent of the choice of transversals \( S_x \) and \( S_y \). We see easily that

\[
\mathcal{H}(\omega, t) = \lim_{z \to x, \, z \in S_x} \frac{\operatorname{dist}(\operatorname{hol}_{\omega,t}(z), y)}{\operatorname{dist}(z, x)}.
\]

The following result gives an explicit expression for \( \mathcal{H} \) near a singular point \( a \) using the local model \((D^2, \mathcal{L}, \{0\})\) introduced in Sect. 2.3.

**Lemma 2.6** ([38, Proposition 3.1]) Let \( D^2 \) be endowed with the Euclidean metric. For each \( x = (z, w) \in D^2 \), consider the function \( \Phi_x : \Pi_x \to \mathbb{R}^+ \) as follows. For \( \xi \in \Pi_x \), consider a path \( \omega \in \Omega \) such that

\[
\omega(t) = \psi_x(t\xi) = (ze^{i\xi t}, we^{i\lambda\xi t}) \subset D^2
\]

for all \( t \in [0, 1] \) (see (2.12) above). So \( \omega(0, 1] \subset \Pi_x \). In fact, it always exists since \( \Pi_x \) is convex and \( 0 \in \Pi_x \) as \( x \in D^2 \). Define \( \Phi_x(\xi) := \mathcal{H}(\omega, 1) \). Then

\[
\Phi_x(\xi) = |e^{i\xi}| |e^{i\lambda\xi}| \frac{\sqrt{|z|^2 + |\lambda w|^2}}{\sqrt{|ze^{i\xi}|^2 + |\lambda we^{i\lambda\xi}|^2}}.
\]

### 2.6 Chern Curvature, Chern Class

Let \((L, h)\) be a singular Hermitian holomorphic line bundle on \( X \). If \( e_L \) is a holomorphic frame of \( L \) on some open set \( U \subset X \), then the function \( \varphi \) defined by \( |e_L|^2_h = \exp(-2\varphi) \) is called the local weight of the metric \( h \) with respect to \( e_L \).
If the local weights \( \varphi \) are in \( L^1_{\text{loc}}(U) \), then (Chern) curvature current of \( (L, h) \) denoted by \( c_1(L, h) \) is given by \( c_1(L, h)|_U = dd^c \varphi \). This is a \((1, 1)\)-closed current. Its class in \( H^{1,1}(X) \) is called the Chern class of \( L \). If we fix a smooth Hermitian metric \( h_0 \) on \( L \), then every singular metric \( h \) on \( L \) can be written \( h = e^{-2\varphi} h_0 \) for some function \( \varphi \). We say that \( \varphi \) is the global weight of \( h \) with respect to \( h_0 \). Clearly, \( c_1(L, h) = c_1(L, h_0) + dd^c \varphi \).

### 2.7 Transversal Metric, Transversal Form, Change of Variables

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a holomorphic foliation on a Hermitian complex surface \( X \). Consider the normal bundle \( \text{Nor}(\mathcal{F}) = \text{Tan}(X)/\text{Tan}(\mathcal{F}) \) of \( \mathcal{F} \). For \( x \in X \setminus E \) and a vector \( u_x \in \text{Tan}_x(X) \), let \([u_x]\) denotes its class in \( \text{Nor}_x(\mathcal{F}) \). We also identify \([u_x]\) with the set \( u_x + \text{Tan}_x(\mathcal{F}) \subset \text{Tan}_x(X) \) Note that a (local) smooth section \( w \) of \( \text{Nor}(\mathcal{F}) \) can be locally written as \( w_x = [u_x] \) for some smooth vector field \( u \).

Consider the following metric \( g^\perp_X \) on the normal bundle \( \text{Nor}(\mathcal{F}) \) :

\[
\|w_x\|_{g^\perp_X} := \min_{u_x \in [w_x]} \|u_x\|_{g_X}, \quad \text{for} \quad w_x \in \text{Nor}(\mathcal{F})_x, \quad x \in X \setminus E. \tag{2.22}
\]

Note that \( u_x \) achieving the minimum in (2.22) is uniquely determined by \([w_x]\). We call \( g^\perp_X \) the transversal metric associated with \( \mathcal{F} \) and the ambient metric \( g_X \).

Fix a smooth Hermitian metric \( g_0 \) on \( \text{Nor}(\mathcal{F}) \). There is a global weight function \( \varphi \) on \( X \) such that \( g^\perp_X = e^{-2\varphi} g_0 \).

Let \( w \) be a (smooth) section of \( \text{Nor}(\mathcal{F}) \) over a small open subset \( U \subset X \) such that \( \|w_x\|_{g^\perp_X} = 1 \) for \( x \in U \). Let \( \beta \) be the dual form of \( w \) with respect to \( g_x \), defined by

\[
\langle \beta_x, v_x \rangle = g_X(v_x, u_x) \quad \text{for all} \quad x \in U \quad \text{and} \quad v_x \in \text{Tan}_x(X).
\]

Here by shrinking \( U \) if necessary, we may assume that \( u \) is a vector field on \( U \) such that \( w_x = [u_x] \) for \( x \in U \). The transversal form associated with \( \mathcal{F} \) and the ambient metric \( g_X \) is the smooth positive \((1, 1)\)-form denoted by \( \omega^\perp_X \) defined on \( U \) by

\[
\omega^\perp_X := i \beta \wedge \bar{\beta}. \tag{2.23}
\]

Patching this form over all such open sets \( U \), we obtain a \((1, 1)\)-smooth form \( \omega^\perp_X \) well-defined on \( X \setminus E \).

In each regular flow box \( U \) with foliated chart \( \Phi : U \rightarrow \mathbb{B} \times \Sigma \) with \( 0 \in \mathbb{B} \), consider the following volume form \( \Upsilon \) on \( \Sigma \)

\[
\Upsilon(\alpha) := (\Phi_\ast \omega^\perp_X)(0, \alpha) \quad \text{for} \quad \alpha \in \Sigma. \tag{2.24}
\]

Here \( \mathbb{B} \times \Sigma \) is an open subset of \( \mathbb{C} \times \mathbb{C} \). Then there exists a unique function \( \varphi_U : U \rightarrow \mathbb{R} \) such that

\[
(id \xi \wedge d \bar{\xi}) \wedge \Phi_\ast(\omega^\perp_X)(\xi, \alpha) = \exp \left( -2\varphi_U(\Phi^{-1}(\xi, \alpha)) \right) \cdot (id \xi \wedge d \bar{\xi}) \wedge \Upsilon(\alpha)
\]

for \( (\xi, \alpha) \in \mathbb{B} \times \Sigma \). \tag{2.25}
In fact, if we assume that $B$ is a simply connected domain in $\mathbb{C}$, for example, $B$ is a disc in $\mathbb{C}$ centered at 0, then for $x \in U$ we write $(\zeta, \alpha) = \Phi(x)$, and obtain that

$$\varphi_U(x) = - \ln \mathcal{H}(\omega, 1),$$

(2.26)

where $\mathcal{H}$ is the holonomy cocycle (see (2.21)), $\omega : [0, 1] \to L_x$ is a path in a plaque of $U$ going from $y := \Phi^{-1}(0, \alpha)$ to $x$. Note that $\varphi_U(\Phi^{-1}(0, \alpha)) = 0$ for $\alpha \in \Sigma$.

We infer from (2.23), (2.24) and (2.25) the following change of variables for integrals with directed forms:

$$\langle h, \omega^{1/\kappa}_X \rangle = \int_{\Omega} \left( \int_B \exp \left( -2 \varphi_U(\Phi^{-1}(\zeta, \alpha)) \right) (\Phi_* h)(\zeta, \alpha) \right) \Upsilon(\alpha)$$

for $h \in \mathcal{D}^{1,1}(\mathcal{F})$.

(2.27)

In each singular flow box $U_a$ with $a \in E$, we use the local model $U_a \simeq \mathbb{D}^2$ and consider $\Phi : U_a \to \mathbb{B} \times \Sigma = \mathbb{S} \times A$ which is the inverse map of $\mathbb{S} \times A \ni (\zeta, \alpha) \mapsto \tilde{\psi}_{\alpha}(\zeta) \in (\mathbb{D} \setminus \{0\})^2$, where $\tilde{\psi}_{\alpha}$ is given by (2.14). Consequently, using (2.24) and (2.26), we can find a volume form $\Upsilon$ on $A$ and a function $\varphi_a : (\mathbb{D} \setminus \{0\})^2 \to \mathbb{R}$ such that

$$\left( id \zeta \wedge d \tilde{\zeta} \right) \wedge \Phi_*(\omega^{1/\kappa}_X)(\zeta, \alpha) = \exp \left( -2 \varphi_a(\zeta, \alpha) \right) \cdot \left( id \zeta \wedge d \tilde{\zeta} \right) \wedge \Upsilon(\alpha)$$

for $(\zeta, \alpha) \in \mathbb{S} \times A$.

(2.28)

We also obtain a similar formula as (2.27) for the singular flow box $U_a$.

### 2.8 Specializations $\kappa_x$ and the Curvature Density $\kappa$

We recall some notions and results from [35, Sect. 9.1]. Fix a point $x \in X \setminus E$ and let $\phi_x : \mathbb{D} \to L = L_x$ be the universal covering map given in (2.1). Consider the function $\kappa_x : \mathbb{D} \to \mathbb{R}$ defined by

$$\kappa_x(\zeta) := \log \|\mathcal{H}(\phi_x \circ \omega, 1)\|, \quad \zeta \in \mathbb{D},$$

(2.29)

where $\omega \in \Omega_0$ is any path such that $\omega(1) = \zeta$. This function is well-defined because $\mathcal{H}(\phi_x \circ \omega, t)$ depends only on the homotopy class of the path $\omega|_{[0, t]}$ and $\mathbb{D}$ is simply connected. Following [35], $\kappa_x$ is said to be the specialization of the holonomy cocycle $\mathcal{H}$ at $x$.

Next, we recall from [35] two conversion rules for changing specializations in the same leaf. For this purpose let $y \in L_x$ and pick $\xi \in \phi_x^{-1}(y)$. Since the holonomy cocycle is multiplicative (see [38, eq. (2.11)]), the first conversion rule (see [35, identity (9.6)]) states that

$$\kappa_y \left( \frac{\zeta - \xi}{1 - \xi \bar{\zeta}} \right) = \kappa_x(\zeta) - \kappa_x(\xi), \quad \zeta \in \mathbb{D}.$$

(2.30)
Consequently, since $\Delta p$ is invariant with respect to the automorphisms of $\mathbb{D}$, it follows that

$$\Delta p \kappa_\lambda(0) = \Delta p \kappa_\lambda(\xi). \quad (2.31)$$

By [35, identities (9.5) and (9.8)], we have that

$$\kappa_\lambda(0) = 0 \quad \text{and} \quad \mathbb{E}_x[\log \|H(\bullet, t)\|] = (D_t \kappa_\lambda)(0), \quad t \in \mathbb{R}^+, \quad (2.32)$$

where $(D_t)_{t \in \mathbb{R}^+}$ is the family of diffusion operators associated with $(\mathbb{D}, g_P)$.

Consider the function $\kappa : X \setminus E \rightarrow \mathbb{R}$ defined by

$$\kappa(x) := (\Delta p \kappa_\lambda)(0) \quad \text{for} \quad x \in X \setminus E. \quad (2.33)$$

**Remark 2.7** Let $\mathbb{U}, \mathbb{U}'$ be two flow boxes, each of them may be either regular or singular one, not necessarily taken from the cover $\mathcal{U}$, such that $\mathbb{U} \cap \mathbb{U}'$ is a regular flow box. We infer from (2.26), (2.29) and (2.30) that $\varphi_{\mathbb{U}} - \varphi_{\mathbb{U}'} = \text{const on } \mathbb{U} \cap \mathbb{U}'$. Therefore, $|d\varphi_{\mathbb{U}}|_p, \Delta p \varphi_{\mathbb{U}}$ are globally well-defined on $X \setminus E$, that is, they do not depend on the choice of $\mathbb{U}$. Moreover, we deduce from (2.29), (2.31) and (2.33) that

$$\kappa(x) = -(\Delta p \varphi_{\mathbb{U}})(x) \quad \text{for} \quad x \in \mathbb{U}. \quad (2.34)$$

Now let $L = \text{Nor}(\mathcal{F})$. Suppose that $L$ is trivial over a flow box $\mathbb{U} \simeq \mathbb{B} \times \Sigma$, i.e. $L|_U \simeq \mathbb{U} \times \mathbb{C} \simeq \mathbb{B} \times \Sigma \times \mathbb{C}$. Consider the holomorphic section $e_L$ on $U$ defined by $e_L(x) := (x, 1)$. Let $\varphi$ be the local weight of $(L, g^+_X)$ with respect to $e_L$. Equality (2.21) is rewritten as follows

$$\frac{\|e_L(y)\|_{g^+_X}}{\|e_L(x)\|_{g^+_X}} = \frac{e^{-\varphi(y)}}{e^{-\varphi(x)}},$$

where $x$ and $y$ are on the same plaque in the flow box $\mathbb{U} \simeq \mathbb{B} \times \Sigma$. Consequently,

$$dd_c^C \log \|H(\omega, t)\|_{L_x} = \left(dd_c^C \log \left(\frac{\|e_L(y)\|_{g^+_X}}{\|e_L(x)\|_{g^+_X}}\right)\right)|_{L_x} = -dd_c^C \varphi(y)|_{L_x}. \quad (2.35)$$

Combining (2.29), (2.30), (2.31), (2.33) and (2.35), we obtain that $\varphi - \varphi_{\mathbb{U}} = \text{const on } \mathbb{U}$, and hence

$$\kappa(x) = -(\Delta p \varphi)(x) \quad \text{for} \quad x \in X \setminus E. \quad (2.36)$$

That is why we call the function $\kappa$ the curvature density of $\text{Nor}(\mathcal{F})$. 
3 Preliminary Results

3.1 Transversal Metric and Holonomy Variations Near Singularities

Fix an ambient Hermitian metric $g_X$ on $X$. Fix a smooth Hermitian metric $g_0$ on the normal bundle $\text{Nor}(\mathcal{F})$ of $\mathcal{F}$. Let $g^\perp_X$ be the metric on $\text{Nor}(\mathcal{F})$ defined by (2.22). So there is a global weight function $f : X \to [-\infty, \infty)$ such that $g^\perp_X = g_0 \exp(-2f)$. We know that the weight function $f$ is smooth outside $E$.

**Lemma 3.1**

(1) If $g'_X$ is another smooth Hermitian metric on $X$ and $g'^\perp_X$ is the transversal metric associated with $F$ and $g'_X$ (see (2.22)), then there is a constant $c > 1$ such that $c^{-1}g^\perp_X \leq g'^\perp_X \leq cg^\perp_X$ on $X$.

(2) There is a constant $c > 0$ such that $\omega^\perp_X \leq cg_X$ for $x \in X \setminus E$, where $\omega^\perp_X$ is the transversal form given in (2.23).

(3) Suppose that $g_X$ coincides with the Euclidean metric in a local model near a singular point $a$ of $\mathcal{F}$. Then for $x = (z, w) \in U_a \simeq \mathbb{D}^2$,

$$f(x) = \frac{1}{2} \log (|z|^2 + |\lambda w|^2) + \text{a smooth function in } x. \quad (3.1)$$

(4) There is a constant $c > 0$ such that

$$|df(x)| \leq c(\text{dist}(x, E))^{-1}g_X(x) \quad \text{and} \quad |dd^c f(x)| \leq c(\text{dist}(x, E))^{-2}g_X(x)$$

for $x \in X \setminus E$.

Here, in assertions (2) and (4), $g_X$ also denotes the fundamental form associated to the Hermitian metric $g_X$.

**Proof**

Proof of assertion (1). Since there is a constant $c' > 1$ such that $c'^{-1}g_X \leq g'_X \leq c'g_X$ on $X$, assertion (i) follows immediately from (2.22).

Proof of assertion (2). Consider first the case where $g_X$ coincides with the Euclidean metric in a local model near a singular point $a$ of $\mathcal{F}$. Since the vector $(z, \lambda w)$ is tangent to the leaf $L_x$ at $x = (z, w)$, the unit vector field

$$N_x := \frac{1}{\sqrt{|z|^2 + |\lambda w|^2}} \left( -\frac{\bar{\lambda} \bar{w}}{\partial z} + \frac{\bar{z}}{\partial w} \right)$$

is normal to $L_x$ at $x$. Using this and (2.23), we get assertion (2) in this case.

The general case is similar using assertion (1).

Proof of assertion (3). Consider the local holomorphic section $e_L$ given by $(z, w) \mapsto \frac{\partial}{\partial z}$ of $\text{Tan}(X)$ over $U_a \simeq \mathbb{D}^2$. This section induces a holomorphic section $\tilde{e}_L(x) = e_L(x)/\text{Tan}(\mathcal{F})_x$ of $\text{Nor}(\mathcal{F})$ over $U_a$. We have, for $x = (z, w) \in \mathbb{D} \times (\mathbb{D} \setminus \{0\})$,

$$\exp(-\varphi(x)) = |\tilde{e}_L(x)|_{g^\perp_X} = \frac{1}{\sqrt{|z|^2 + |\lambda w|^2}} |\lambda w|.$$
Hence, for \( x = (z, w) \in \mathbb{D}^2 \setminus \{(0, 0)\} \),
\[
c_1(\text{Nor}(\mathcal{F}), g_\mathcal{X}^\perp)(x) = dd^c \varphi(x) = dd^c_{z,w} \log \sqrt{|z|^2 + |\lambda w|^2}.
\] (3.2)

Moreover, in the local model with coordinates \((z, w)\) associated to the singular point \(E \ni a \simeq (0, 0) \in \mathbb{D}^2\), it follows from (3.2) that
\[
c_1(\text{Nor}(\mathcal{F}), g_0) + dd^c f = c_1(\text{Nor}(\mathcal{F}), g_\mathcal{X}^\perp)(x) = \frac{1}{2} dd^c_{z,w} \log \sqrt{|z|^2 + |\lambda w|^2}.
\]

Consequently, assertion (3) follows.

**Proof of assertion (4).** We only need to prove this assertion in a local model for a singular point \(a \in E\), that is, to prove the lemma for \(x \in U_a \simeq D^2\).

If \(g_X|_{U_a}\) coincides with the standard Euclidean metric on \(D^2\), assertion (4) is an immediate consequence of assertion (3). For the general case we argue as in the proof of Lemma 3.2 (2) below. \(\Box\)

The following result gives precise variations up to order 2 of \(H\) near a singular point \(a\) using the local model \((D^2, \mathcal{L}, \{0\})\) introduced in Sect. 2.3.

**Lemma 3.2 (1)** Let \(D^2\) be endowed with the Euclidean metric. Then, there is a constant \(c > 1\) such that for every \(x = (z, w) \in (\frac{1}{2} D^2)^2\), we have that
\[
c^{-1} \log^\star \|z, w\| \leq |d\kappa_x(0)| \leq c \log^\star \|z, w\|,
\]
\[
-\frac{|z|^2|w|^2}{(|z|^2 + |w|^2)^2} \left( \log^\star \|z, w\| \right)^2 \leq \Delta_P \kappa_x(0)
\]
\[
\leq -c^{-1} \frac{|z|^2|w|^2}{(|z|^2 + |w|^2)^2} \left( \log^\star \|z, w\| \right)^2,
\]
where the function \(\kappa_x\) is defined in (2.29).

(2) Let \(D^2\) be endowed with a smooth Hermitian metric. Then there is a constant \(c > 1\) such that for every \(x = (z, w) \in (\frac{1}{2} D^2)^2\), we have that
\[
c^{-1} \log^\star \|z, w\| \leq |d\kappa_x(0)| \leq c \log^\star \|z, w\|,
\]
\[
|\Delta_P \kappa_x(0)| \leq c \frac{|z|^2|w|^2}{(|z|^2 + |w|^2)^2} \left( \log^\star \|z, w\| \right)^2
\]
\[
+ c \sqrt{|z|^2 + |w|^2} \left( \log^\star \|z, w\| \right)^2.
\]

**Proof** Proof of assertion (1). Let \(\Phi_x\) be the function defined in Lemma 2.6. For \(\xi \in \mathbb{C}\) with \(|\xi| \ll 1\) let \(\xi(\xi) \in \Pi_x\) be such that
\[
\psi_x(\xi(\xi)) = \phi_x(\xi).
\]
Differentiating both sides and using (2.2) yield that
\[ |\xi'(0)| = \frac{|d\phi_x(0)|}{|d\psi_x(0)|} = \frac{\eta(x)}{|d\psi_x(0)|}. \]

This, Lemma 2.6 and (2.29) together imply that
\[ |d\kappa_x(0)| = \left| \frac{d \log \|H(\phi_x \circ \omega, 1)\|_\zeta=0}{d\zeta} \right| = \left| \frac{d \log \Phi_x(\xi(\zeta))}{d\zeta} \right|_{\zeta=0} \]
\[ = \frac{\eta(x)|d \log \Phi_x(0)|}{|(d\psi_x)(0)|}. \tag{3.3} \]

Using (2.3), a similar argument shows that
\[ (\Delta_p \kappa_x)(0) = \frac{\eta^2(x)|dd^c \log \Phi_x(0)|}{|(d\psi_x)(0)|^2}. \tag{3.4} \]

We infer from (2.12) that
\[ |(d\psi_x)(0)| \approx \|(z, w)\| = \text{dist}(x, E). \tag{3.5} \]

Next, observe that \( \partial \log |e^{i\xi}| + \partial \log |e^{i\lambda \xi}| = O(1) \). A straightforward computation shows that
\[ \partial \log (|ze^{i\xi}|^2 + |\lambda we^{i\lambda \xi}|^2) = \frac{\partial|ze^{i\xi}|^2 + \partial|\lambda we^{i\lambda \xi}|^2}{(|ze^{i\xi}|^2 + |\lambda we^{i\lambda \xi}|^2)} \]
\[ = \frac{-i(|ze^{i\xi}|^2 + \bar{\lambda} |\lambda we^{i\lambda \xi}|^2)}{(|ze^{i\xi}|^2 + |\lambda we^{i\lambda \xi}|^2)}. \tag{3.6} \]

Therefore, it follows that
\[ |d \log \Phi_x(0)| = |d \log |e^{i\xi}| + d \log |e^{i\lambda \xi}| - \frac{1}{2} d \log (|ze^{i\xi}|^2 + |\lambda we^{i\lambda \xi}|^2) | = O(1). \]

Inserting this and (3.5) into the first equality in (3.3) and applying Lemma 2.5, the first inequality of assertion (1) follows.

To prove the second inequality of assertion (1), note that \( i \partial \bar{\partial} \log |e^{i\xi}| = i \partial \bar{\partial} \log |e^{i\lambda \xi}| = 0 \). Therefore, we infer from Lemma 2.6 that
\[ i \partial \bar{\partial} \log \Phi_x(\zeta) = -\frac{1}{2} i \partial \bar{\partial} \log (|ze^{i\xi}|^2 + |\lambda we^{i\lambda \xi}|^2). \]

To compute the right-hand side of the last line, we use (3.6):
\[ \partial \bar{\partial} \log (|ze^{i\xi}|^2 + |\lambda we^{i\lambda \xi}|^2) = \frac{-i \partial \left(|ze^{i\xi}|^2 + \bar{\lambda} |\lambda we^{i\lambda \xi}|^2 \right) \wedge d\bar{\zeta}}{(|ze^{i\xi}|^2 + |\lambda we^{i\lambda \xi}|^2)}. \]
\[-i (|ze^{i\zeta}|^2 + \lambda |we^{i\lambda \zeta}|^2) d\bar{\zeta} \wedge \partial (|ze^{i\zeta}|^2 + |\lambda we^{i\lambda \zeta}|^2)\]
\[-\frac{(|ze^{i\zeta}|^2 + |\lambda |we^{i\lambda \zeta}|^2)(|ze^{i\zeta}|^2 + |\lambda we^{i\lambda \zeta}|^2) - |ze^{i\zeta}|^2 + \lambda |\lambda we^{i\lambda \zeta}|^2}{(|ze^{i\zeta}|^2 + |\lambda we^{i\lambda \zeta}|^2)^2} \, d\zeta \wedge d\bar{\zeta}.
\]

We have shown that
\[
i d_\beta \log \Phi_x(\zeta) = \frac{-|\lambda - 1|^2}{2} \frac{|ze^{i\zeta}|^2 |\lambda we^{i\lambda \zeta}|^2}{(|ze^{i\zeta}|^2 + |\lambda we^{i\lambda \zeta}|^2)^2} i d\zeta \wedge d\bar{\zeta}.
\]

Inserting this and (3.5) into the second equality in (3.3) and applying Lemma 2.5, the second inequality of assertion (1) follows.

**Proof of assertion (2).** The first inequality can be proved similarly as in assertion (1). We only prove the second inequality.

First consider the case where \(\mathbb{D}^2\) is endowed with a constant Hermitian metric \(g_X\), that is, \(g_X(x) = g_X(0)\) for \(x \in \mathbb{D}^2\). This case is analogous to the case of Euclidean metric in assertion (1), although the computation is slightly more involved. We only give a sketchy proof here. There are \(a_1, a_2, b_1, b_2 \in \mathbb{C}\) such that \(a_1 b_2 \neq b_1 a_2\) and that
\[g_X(0)(Z, W) = |a_1 Z + b_1 W|^2 + |a_2 Z + b_2 W|^2 \quad \text{for} \quad (Z, W) \in \mathbb{C}^2.
\]

For \(x = (z, w) \in \mathbb{D}^2 \setminus \{(0, 0)\}\), \(\kappa(x)\) is determined by
\[-dd^c \log (|a_1 Z + b_1 W|^2 + |a_2 Z + b_2 W|^2)(0)|_{\mathbb{C}^2} = \kappa(x)g_P(x),
\]
where \((Z(\zeta), W(\zeta)) = (ze^{i\zeta}, we^{i\lambda \zeta}), \zeta \in \mathbb{C}\) (see (2.12)). Write
\[\pi dd^c \log (|a_1 Z + b_1 W|^2 + |a_2 Z + b_2 W|^2) = \frac{i\gamma \wedge \bar{\gamma}}{(|a_1 Z + b_1 W|^2 + |a_2 Z + b_2 W|^2)^2},
\]
where \(\gamma := (a_1 Z + b_1 W)(a_2 dZ + b_2 dW) - (a_2 Z + b_2 W)(a_1 dZ + b_1 dW)\). Using this and applying Lemma 2.5, a straightforward computation shows that \(\kappa(x)\) satisfies the second inequality of assertion (2).

Consider the general case. Let \(g'_X\) be a Hermitian metric on \(X\) such that in a local model \(U_x \simeq \mathbb{D}^2\) near every singular point \(a \in E\) it coincides with a constant Hermitian metric on \(\mathbb{D}^2\). We see that there is a smooth function \(\varphi\) on \(\mathbb{D}^2 \setminus \{(0, 0)\}\) such that \(g'_X = e^{-2\varphi}g_X\) on \(\mathbb{D}^2 \setminus \{(0, 0)\}\). Let \(\kappa'_X\) be the function given in (2.29) using \(g'_X\) instead of \(g_X\). By (2.25), (2.26) and (2.29), one has
\[\kappa_X(\zeta) = \kappa'_X(\zeta) + (\varphi \circ \phi_X)(\zeta) \quad \text{for} \quad \zeta \in \mathbb{D}.
\]
So we get
\[ \Delta_p \kappa_x(0) = \Delta_p \kappa'_x(0) + (\Delta_p \varphi)(x) \quad \text{for} \quad x \in X \setminus E. \quad (3.7) \]

Observe that \( g_X(x) = g_X(0) + O(\|x\|) \) for \( x \in (\frac{1}{2} \mathbb{D})^2 \). Consequently, \( \varphi(x) = O(\|x\|^{-1}) \). Therefore, by Lemma 2.5, \( (\Delta_p \varphi)(x) = O((\|x\| (\log \|x\|)^2)) \). Note that \( \Delta_p \kappa'_x(0) \) satisfies the second estimate of assertion (1). Inserting this in (3.7) and invoking the above estimate for \( \Delta_p \kappa'_x(0) \), the second inequality of assertion (2) follows. \( \square \)

### 3.2 Lyapunov Exponent and Weight Function \( W \)

We keep the hypotheses and notation of Theorem 1.1. For \( t \in \mathbb{R}^+ \), consider the function \( F_t : X \setminus E \to \mathbb{R} \) defined by
\[ F_t(x) := \int_\Omega \log \| H(\omega, t) \| dW_x(\omega) \quad \text{for} \quad x \in X \setminus E. \quad (3.8) \]

By (1.3) the Lyapunov exponent \( \chi(T) \) can be rewritten as
\[ \chi(T) := \int_X F_1(x) d\mu(x) = \frac{1}{T} \int_X F_t(x) d\mu(x), \quad (3.9) \]
where the measure \( \mu \) is given in (1.2). The following result is needed.

**Theorem 3.3** There is a constant \( c > 0 \) such that
\[ |F_1(x)| \leq c \log^* \text{dist}(x, E) \quad \text{for} \quad x \in X \setminus E. \quad (3.10) \]

Moreover, the following inequality holds:
\[ \int_X \log^* \text{dist}(x, E) d\mu(x) < \infty. \quad (3.11) \]

**Proof** (3.10) follows from Proposition 3.3 and Lemma 4.1 in [38].

Theorem 1.4 in [38] gives (3.11). \( \square \)

Define a weight function \( W : X \setminus E \to \mathbb{R}^+ \) as follows. Let \( x \in X \setminus E \). If \( x \) belongs to a regular flow box then \( W(x) := 1 \). Otherwise, if \( x = (z, w) \) belongs to a singular flow box \( \mathbb{U}_a \simeq (\mathbb{D}^2, \mathcal{L}, \{0\}), a \in E \), which is identified with the local model with coordinates \( (z, w) \), then
\[ W(x) := \log^* \| (z, w) \| + \frac{|z|^2|w|^2}{(|z|^2 + |w|^2)^2} (\log^* \| (z, w) \|)^2. \quad (3.12) \]

Note that
\[ 1 \leq \log^* \text{dist}(x, E) \leq W(x) \leq 2(\log^* \text{dist}(x, E))^2. \]
4 Cohomological Formula for the Lyapunov Exponent

In this section we first prove Proposition 4.6 which plays a crucial role in this article. Next, we prove the first identity of Theorem A. Inspired by Definition 8.3 in Candel [9], the following notion is needed.

Definition 4.1 A real-valued function $h$ defined on $\mathbb{D}$ is called weakly moderate if there is a constant $c > 0$ such that

$$\log |h(\xi) - h(0)| \leq c \text{ dist}_P(\xi, 0) + c,$$

$\xi \in \mathbb{D}$.

Remark 4.2 The notion of weak moderateness is weaker than the notion of moderateness given in [9, 39].

The usefulness of weakly moderate functions is illustrated by the following Dynkin type formula.

Lemma 4.3 Let $f \in C^2(\mathbb{D})$ be such that $f$, $|df|_P$ (see (2.9)) and $\Delta_P f$ (see (2.8)) are weakly moderate functions. Then

$$(D_t f)(0) - f(0) = \int_0^t (D_s \Delta_P f)(0)ds, \quad t \in \mathbb{R}^+.$$ 

Proof Lemma 5.1 in [39] shows that if $f$, $|df|_P$ and $\Delta_P f$ are moderate functions, then

$$(D_t f)(0) - f(0) = \int_0^t (D_s \Delta_P f)(0)ds, \quad t \in \mathbb{R}^+, \quad \xi \in \mathbb{D}.$$ 

Now under the weaker hypothesis that $f$, $|df|_P$ and $\Delta_P f$ are weakly moderate, the same proof shows that the above equality holds for $\xi = 0$. Hence, the result follows.

The next lemma shows us how deep a leaf can go into a singular flow box before the hyperbolic time $R$.

Lemma 4.4 [38, Lemma 3.2] There is a constant $c > 0$ with the following property. Let $x = (z, w) \in (1/2\mathbb{D})^2$ and $\xi \in \mathbb{D}$ be such that $\phi_x(t\xi) \in (1/2\mathbb{D})^2$ for all $t \in [0, 1]$. Write $y := \phi_x(\xi)$ and $R := \text{dist}_P(\xi, 0)$. Then there exists $\zeta \in \Pi_x$ (see (2.12) above) such that $y = (ze^{\xi}, we^{\lambda\xi})$ and that

$$|\zeta| \leq e^{cR} |\log \|x\| |.$$ 

The following result gives an estimate on the expansion rate up to order 2 of $\mathcal{H}(\omega, \cdot)$ in terms of $\text{dist}_P(\cdot, 0)$ and the distance $\text{dist}(x, E)$.
Proposition 4.5 There is a constant $c > 0$ such that for every $x \in X \setminus E$ and every $\xi \in \mathbb{D}$,

$$\left| \kappa_x(\xi) - \kappa_x(0) \right| \leq c \log^* \text{dist}(x, E) \cdot \exp\left(c \text{ dist}_P(\xi, 0)\right),$$

$$\left| d\kappa_x(\xi) \right|_P - \left| d\kappa_x(0) \right|_P \leq c \log^* \text{dist}(x, E) \cdot \exp\left(c \text{ dist}_P(\xi, 0)\right),$$

$$\left| \Delta_P \kappa_x(\xi) - \Delta_P \kappa_x(0) \right| \leq c(\log^* \text{dist}(x, E))^2 \cdot \exp\left(c \text{ dist}_P(\xi, 0)\right).$$

**Proof** Fix $\xi \in \mathbb{D}$ and set $y := \phi_x(\xi)$. Let $\mathcal{U}$ be the finite cover of $X$ by regular and singular flow boxes given in Sect. 2.3. We consider three steps.

**Step 1** If there is a singular flow box $U$ which contains the whole segment $\{\phi_x(t\xi) : t \in [0, 1]\}$, then the proposition is true for $c = c_1$, where $c_1 > 0$ is a constant large enough.

Write $x = (z, w)$ and $R := \text{dist}_P(0, \xi)$. By Lemma 4.4, we may write $y = (ze^\xi, we^{\lambda^*})$ for some $\xi \in \mathbb{C}$ such that

$$|\xi| \leq e^{c_2 R}. \tag{4.1}$$

Inserting this into the expression for the holonomy map given in Lemma 2.6, a straightforward computation shows that

$$\left| \kappa_x(\xi) - \kappa_x(0) \right| = \left| \ln \Phi_x(\xi) \right| \leq c_3 |\log \|x\||e^{c_3 R}$$

for a constant $c_3 > 0$ independent of $x$ and $y$.

Next, we deduce from the first inequalities of Lemma 3.2 (2) that

$$\left| d\kappa_x(\xi) \right|_P - \left| d\kappa_x(0) \right|_P \leq \left| d\kappa_x(\xi) \right|_P + \left| d\kappa_x(0) \right|_P \lesssim \log^* \|x\|$$

$$+ \log^* \|y\| \leq c_3 |\log \|x\||e^{c_3 R},$$

for a constant $c_3 > 0$ independent of $x$ and $y$, where the last inequality holds by (4.1).

By (2.31), we have $\Delta_P \kappa_x(0) = \Delta_P \kappa_x(\xi)$. Consequently, we deduce from the second inequalities of Lemma 3.2 (2) and (4.1) that

$$\left| \Delta_P \kappa_x(\xi) - \Delta_P \kappa_x(0) \right| \leq \left| \Delta_P \kappa_x(0) \right| + \left| \Delta_P \kappa_x(\xi) \right| \lesssim (\log^* \|x\|)^2$$

$$+ \left(\log^* \|y\|\right)^2 \leq c_3 (\log \|x\|)^2 e^{c_3 R},$$

for a constant $c_3 > 0$ independent of $x$ and $y$.

Choosing $c_1 > c_3$ large enough, Step 1 follows from the above estimates.

**Step 2** If the whole segment $\{\phi_x(t\xi) : t \in [0, 1]\}$ is contained in a single regular flow box $U \in \mathcal{U}$, then

$$|\kappa_x(t\xi)| \leq c_4$$

$$\text{and } |d\kappa_x(t\xi)|_P \leq c_4$$

$$\text{and } \left| \Delta_P \kappa_x(t\xi) \right| \leq c_4 \text{ for all } t \in [0, 1].$$

Here $c_4 > 0$ is a constant independent of $x$ and $y$. In particular, the proposition is true in this case for $c = c_1$, where $c_1 > 0$ is a constant large enough.
Observe that the geodesic segment \( \{ \phi_x(t\xi) : t \in [0, 1] \} \) is contained in the unique plaque of \( \mathcal{U} \) which passes through \( x \). Moreover, by Lemma 2.5, \( \eta \approx 1 \) on \( \mathcal{U} \). This, combined with the description of the holonomy map on the regular flow box \( \mathcal{U} \), implies the above three estimates. Therefore, choosing \( c_1 > c_4 \) large enough, we have that

\[
\log^* \text{dist}(x, E) \geq c_4. 
\]

This proves the proposition in Step 2.

**Step 3 Proof of the proposition in the general case.**

We only prove the last inequality of the proposition. The other two inequalities can be proved similarly. Consider the family of all finite subdivisions of \([0,1]\) into intervals \([t_{j-1}, t_j]\) with \(1 \leq j \leq n\) such that \(t_0 = 0\), \(t_n = 1\) and that each segment \(\{ \phi_x(t\xi) : t \in [t_{j-1}, t_j]\}\) is contained in a single (regular or singular) flow box \(\mathcal{U}_j\) for each \(j\). Fix a member of this family such that the number \(n\) is smallest possible. We may assume without loss of generality that \(n > 1\) since the case \(n = 1\) follows either from Step 1 (if \(\mathcal{U}_1\) is singular) or from Step 2 (if \(\mathcal{U}_1\) is regular). The minimality of \(n\) implies that all \(\phi_x(t_1\xi), \ldots, \phi_x(t_{n-1}\xi)\) belong to the union of all regular flow boxes of \(\mathcal{Y}\). Therefore, there is a constant \(r_0 > 0\) independent of \(x\) and \(y\) such that

\[
\text{dist}_P(t_j\xi, t_{j+1}\xi) \geq r_0, \quad 1 \leq j \leq n - 1.
\]

Thus

\[
n \leq 1 + r_0^{-1} \text{dist}_P(\xi, 0) = 1 + r_0^{-1} R. \tag{4.2}
\]

Moreover, there is a constant \(c_5 > 1\) independent of \(\omega\) such that

\[
1 \leq (\log^* \text{dist}(\phi_x(t_j\xi), E))^2 \leq c_5, \quad 1 \leq j \leq n - 1.
\]

Using this and applying Step 1 to each singular box in the family \((\mathcal{U}_j)^n\) and applying Step 2 to each regular flow box in the above family, we obtain that

\[
\begin{align*}
|\Delta_P K_x(t_1\xi) - \Delta_P K_x(0)| &\leq c_1 \left( \log^* \text{dist}(x, E) \right)^2 \cdot \exp \left( c_1 \text{dist}_P(0, t_1\xi) \right), \\
|\Delta_P K_x(t_j\xi) - \Delta_P K_x(t_{j-1}\xi)| &\leq c_1 c_5 \exp \left( c_1 \text{dist}_P(t_{j-1}\xi, t_j\xi) \right), \quad 2 \leq j \leq n.
\end{align*}
\]

Summing up the above estimates, we get that

\[
\sum_{j=1}^n |\Delta_P K_x(t_j\xi) - \Delta_P K_x(t_{j-1}\xi)| \leq c_1 \left( \log^* \text{dist}(x, E) \right)^2 \cdot \exp \left( c_1 \text{dist}_P(0, t_1\xi) \right)
\]

\[
+ \sum_{j=2}^n c_1 c_5 \exp \left( c_1 \text{dist}_P(t_{j-1}\xi, t_j\xi) \right).
\]

On the other hand, we have that

\[
|\Delta_P K_x(\xi) - \Delta_P K_x(0)| \leq \sum_{j=1}^n |\Delta_P K_x(t_j\xi) - \Delta_P K_x(t_{j-1}\xi)|.
\]
This, coupled with the previous estimate, gives that

\[
\left| \Delta_P \kappa_x(\xi) - \Delta_P \kappa_x(0) \right| \leq c_1 \left( \log^* \text{dist}(\omega(x, E)) \right)^2 \cdot \exp \left( c_0 \text{dist}_P(0, t_1\xi) \right) \\
+ \sum_{j=2}^{n} c_1 c_5 \exp \left( c_1 \text{dist}_P(t_{j-1}\xi, t_j\xi) \right).
\]

(4.3)

Since \( \log^* \text{dist}(x, E) \geq 1 \) for all \( x \in X \setminus E \), the right-hand side of the last line is dominated by a constant times \( \log^* \text{dist}(x, E) \) times

\[
\sum_{j=1}^{n} \exp \left( c_1 \text{dist}_P(t_{j-1}\xi, t_j\xi) \right) \leq n \cdot \exp \left( c_1 \text{dist}_P(0, \xi) \right),
\]

where the last inequality holds because of the identity

\[
\text{dist}_P(0, \xi) = \sum_{j=1}^{n} \text{dist}_P(t_{j-1}\xi, t_j\xi).
\]

Inserting (4.2) into the right-hand side of the last inequality and choosing \( c > c_1 \) large enough, we find that its left hand side is bounded by \( c \exp (cR) \). So the right-hand side of (4.3) is also bounded by a constant times \( \left( \log^* \text{dist}(x, E) \right)^2 \cdot \exp \left( c \text{dist}_P(0, \xi) \right) \), and the proof is thereby completed. \( \square \)

The following result relates the Lyapunov exponent \( \chi(T) \) to the function \( \kappa \) defined in (2.33). It plays the key role in this article.

**Proposition 4.6** Under the hypotheses and notations of Theorem 1.1, the integrals \( \int_X |\kappa(x)| d\mu(x) \) and \( \int_X W(x) d\mu(x) \) are bounded, and the following identity holds

\[
\chi(T) = \int_X \kappa(x) d\mu(x).
\]

The novelty of this proposition is that the weight \( W(x) \) behaves like \( (\log^* \text{dist}(x, E))^2 \) when \( (z, w) \) satisfies \( |z| \approx |w| \), whereas our previous work [38] (see Theorem 3.3 above) only provides the \( \mu \)-integrability of the less singular weight \( \log^* \text{dist}(x, E) \).

**Proof** We divide the proof into 2 steps.

**Step 1** Assume in addition that the ambient metric \( g_X \) is equal to the Euclidean metric in a local model near every singular point of \( \mathcal{F} \). By Proposition 4.5, \( \kappa_x, |d\kappa_x|_P \) and \( \Delta_P \kappa_x \) are weakly moderate functions on \( \mathbb{D} \). Consequently, applying Lemma 4.3 yields that

\[
(D_1\kappa_x)(0) - \kappa_x(0) = \int_0^1 (D_s(\Delta_P \kappa_x))(0) ds.
\]

(4.4)
By (2.32) and (3.8), the left-hand side of (4.4) is equal to
\[ E_x[\log \mathcal{H}(\omega,1)] = F_1(x), \]
which is finite because of (3.10). On the other hand, by (2.30) and (2.33), the right-hand side of (4.4) can be rewritten as
\[ \int_0^1 (D_s \kappa)(x) ds. \]
Consequently, integrating both sides of (4.4) with respect to \( \mu \), we get that
\[ \int_X F_1(x) d\mu(x) = \int_X \left( \int_0^1 (D_s \kappa)(x) ds \right) d\mu(x). \quad (4.5) \]
Since we know by (3.9), (3.10) and (3.11) that the left integral is bounded and is equal to \( \chi(T) \), it follows that right-side double integral is also bounded.

On the one hand, by the second inequalities in Lemma 3.2 (1) and (2.33), \( \kappa(x) \leq 0 \) for every \( x \) in a singular flow box of a singular point \( a \in E \). On the other hand, using regular flow boxes we see easily that \( \kappa(x) \leq c_0 \) for every \( x \) outside the union of all singular flow boxes, where \( c_0 > 0 \) is a constant. Therefore,
\[ \kappa(x) \leq c_0 \quad \text{for} \quad x \in X \setminus E. \quad (4.6) \]
Moreover, \( D_s \) is a positive contraction for \( s \in \mathbb{R}^+ \) (see the second identity in (2.7) and recall that \( p(x,y,t) > 0 \)). Consequently, by Fubini’s theorem, we infer that for almost every \( s \in [0,1] \) with respect to the Lebesgue measure,
\[ \int_X (D_s \kappa)(x) d\mu(x) \]
is bounded. Consequently, by (4.6) and (2.7) and the obvious inequality \( |\kappa| \leq 2c_0 - \kappa \),
\[ \int_X (D_s |\kappa|)(x) d\mu(x) \leq 2c_0 \int_X (D_s 1)(x) d\mu(x) - \int_X (D_s \kappa)(x) d\mu(x) < \infty. \]
Hence, by Proposition 2.4 (4), we infer that
\[ \int_X |\kappa(x)| d\mu(x) = \int_X (D_s |\kappa|)(x) d\mu(x) < \infty. \]
Thus, we obtain that
\[ \int_X \kappa(x) d\mu(x) = \int_X (D_s \kappa)(x) d\mu(x). \]
Inserting this in (4.5) and using Fubini’s theorem, we get that
\[ \chi(T) = \int_0^1 \left( \int_X \kappa(x) d\mu(x) \right) ds = \int_X \kappa(x) d\mu(x). \]

Finally, using (3.12) and the second inequalities of Lemma 3.2 (1) and inequality (3.11), we infer that
\[ \int_X W(x) d\mu(x) \lesssim \int_X |\kappa(x)| d\mu(x) + \int_X \log^* \text{dist}(x, E) d\mu(x) < \infty. \quad (4.7) \]

The proof of Step 1 is thereby completed.

**Step 2** The general case. By (4.5), we have that
\[ \int_X F_1(x) d\mu(x) = \int_X \left( \int_0^1 (D_s \kappa)(x) ds \right) d\mu(x). \]

By the second inequalities of Lemma 3.2 (2), there is a constant \( c > 0 \) such that
\[ |\kappa(x)| \leq c W(x) \quad \text{for} \quad x \in X \setminus E. \]

Hence,
\[ \int_X |\kappa(x)| d\mu(x) \leq c \int_X W(x) d\mu(x). \]

By (4.7), the right hand-side is bounded. We infer that the left hand-side is also bounded. Hence, we conclude the proof as in Step 1. \( \square \)

**Corollary 4.7** Under the hypotheses and notations of Theorem 1.1, the following identities hold:
\[ \kappa g_P = -c_1(\text{Nor}(\mathcal{F}), g_X^\perp) \quad \text{on} \quad X \setminus E, \quad \text{and} \]
\[ \int_X \kappa(x) d\mu(x) = -\int_X c_1(\text{Nor}(\mathcal{F}), g_X^\perp) \wedge T. \]

**Proof** By (2.36) we obtain that
\[ \kappa g_P = -(\Delta_P \varphi) g_P = -dd^c \varphi = -c_1(\text{Nor}(\mathcal{F}), g_X^\perp) \quad \text{on} \quad \mathbb{U}. \]

The first identity follows. Since \( \int_X |\kappa(x)| d\mu(x) < \infty \) by Proposition 4.6, Integrating both sides of the first identity over \( X \setminus E \) gives the second identity. \( \square \)

**Proof of the first identity of Theorem A** As in the proof of Proposition 4.6 we divide the proof into 2 steps.

**Step 1** Assume in addition that the ambient metric \( g_X \) is equal to the Euclidean metric in a local model near every singular point of \( \mathcal{F} \).

Fix a smooth Hermitian metric \( g_0 \) on the normal bundle \( \text{Nor}(\mathcal{F}) \) of \( \mathcal{F} \). So there is a global weight function \( f : X \to (-\infty, \infty) \) such that \( g_X^\perp = g_0 \exp (-2f) \). We know that the weight function \( f \) is smooth outside \( E \). Using a finite partition of the
unity on $X$ and applying Lemma 3.1 (see (3.1)), we can construct a family of smooth functions $(f_\epsilon)_{0<\epsilon\ll1}$ on $X$ such that $f_\epsilon$ converges uniformly to $f$ in $C^2$-norm on each regular flow box as $\epsilon \to 0$ and that in a local model with coordinates $(z, w)$ associated to each singular point $a \in E$,

$$f_\epsilon - \frac{1}{2} \log (|z|^2 + |\lambda w|^2 + \epsilon^2) = f - \frac{1}{2} \log (|z|^2 + |\lambda w|^2) \quad \text{on} \quad \mathbb{D}^2.$$ (4.8)

For every $0 < \epsilon \ll 1$ we endow $\text{Nor}(\mathcal{F})$ with the metric $g_\epsilon := g_0 \exp (-2f_\epsilon)$. Since $g_\epsilon$ is smooth and the current $T$ is $dd^c$-closed, it follows that

$$c_1(\text{Nor}(\mathcal{F})) \sim \{T\} = \int_X c_1(\text{Nor}(\mathcal{F}), g_\epsilon) \wedge T.$$ (4.9)

Let $\kappa_\epsilon : X \setminus E \to \mathbb{R}$ be the function defined by

$$-c_1(\text{Nor}(\mathcal{F}), g_\epsilon)(x)|_{L_\epsilon} = \kappa_\epsilon(x)g_P(x).$$ (4.10)

This, combined with (1.2), implies that

$$-c_1(\text{Nor}(\mathcal{F}), g_\epsilon) \wedge T = \kappa_\epsilon d\mu.$$ (4.11)

Since $f_\epsilon$ converges uniformly to $f$ in $C^2$-norm on compact subsets of $X \setminus E$ as $\epsilon \to 0$, it follows that $\kappa_\epsilon$ converge pointwise to $\kappa$ $\mu$-almost everywhere. Hence, we get that

$$-\int_{X \setminus (\bigcup_{a \in E} U_a)} c_1(\text{Nor}(\mathcal{F}), g_\epsilon) \wedge T = \int_{X \setminus (\bigcup_{a \in E} U_a)} \kappa_\epsilon(x) d\mu(x)$$

$$\rightarrow \int_{X \setminus (\bigcup_{a \in E} U_a)} \kappa(x) d\mu(x) \quad \text{as} \quad \epsilon \to 0.$$ (4.12)

We will show that on each singular flow box $U_a \simeq \mathbb{D}^2$,

$$-\int_{U_a} c_1(\text{Nor}(\mathcal{F}), g_\epsilon) \wedge T \to \int_{U_a} \kappa(x) d\mu(x) \quad \text{as} \quad \epsilon \to 0.$$ (4.12)

Taking (4.12) for granted, we combine it with the previous limit and get that

$$-\int_X c_1(\text{Nor}(\mathcal{F}), g_\epsilon) \wedge T \to \int_X \kappa(x) d\mu(x)$$

$$= -\int_X c_1(\text{Nor}(\mathcal{F}), g_\epsilon^\perp) \wedge T \quad \text{as} \quad \epsilon \to 0,$$

where the last equality follows from Corollary 4.7. We deduce from this and (4.9) that

$$c_1(\text{Nor}(\mathcal{F})) \sim \{T\} = \int_X \kappa(x) d\mu(x).$$
By Proposition 4.6, the right-hand side is $\chi(T)$. Hence, the last equality implies the desired identity of the theorem.

Now it remains to prove (4.12). We need the following result which gives a precise behaviour of $\kappa_\epsilon$ near a singular point $a$ using the local model $(\mathbb{D}^2, \mathcal{L}, (0))$ introduced in Sect. 2.3.

**Lemma 4.8** There is a constant $c > 1$ such that for every $0 < \epsilon \ll 1$ and for every $x = (z, w) \in (\frac{1}{2}\mathbb{D})^2$, we have that

$$-c\left(\frac{|z|^2|w|^2}{(|z|^2 + |w|^2 + \epsilon^2)^2} + (|z|^2 + |w|^2) \log^* \| (z, w) \| \right)^2 \leq \kappa_\epsilon(x)$$

$$\leq \left( -c^{-1}\frac{|z|^2|w|^2}{(|z|^2 + |w|^2 + \epsilon^2)^2} + c(|z|^2 + |w|^2) \log^* \| (z, w) \| \right)^2.$$

**Proof** Since $g_\epsilon = g_0 \exp(-2f_\epsilon)$ we get that

$$c_1(Nor(\mathcal{F}), g_\epsilon) = c_1(Nor(\mathcal{F}), g_0) + dd^c f_\epsilon = dd^c f_\epsilon + \text{a smooth } (1, 1)\text{-form independent of } \epsilon.$$

This, together with (3.1), (4.8) and (4.10), imply that

$$\kappa_\epsilon(x)g_P(x) = -\frac{1}{2}dd^c \log \left( |z|^2 + |\lambda w|^2 + \epsilon^2 \right)(x)|_{L_\epsilon}$$

$$+ \text{a smooth } (1, 1)\text{-form independent of } \epsilon.$$

Using the parametrization (2.12) the pull-back of the first term of the right-hand side by $\psi_x$ is

$$-\frac{1}{2}dd^c \log \left( |ze^{i\xi}|^2 + |\lambda we^{i\lambda \xi}|^2 + \epsilon^2 \right)(0),$$

whereas the pull-back of the second term of the right-hand side by $\psi_x$ is $O(|z|^2 + |w|^2)d\xi \wedge d\bar{\xi}$. A straightforward computation as in the end of the proof of Lemma 3.2 shows that the former expression is equal to

$$-\frac{|\lambda - 1|^2}{4\pi} \frac{|z|^2|w|^2}{(|z|^2 + |\lambda w|^2 + \epsilon^2)^2} \frac{i}{2} d\xi \wedge d\bar{\xi}.$$

Using this and (3.5), and applying Lemma 2.5, the result follows. \qed

We resume the proof of (4.12). By Lemmas 3.2 and 4.8, there is a constant $c > 1$ such that for $(z, w) \in U_a \simeq \mathbb{D}^2$ that

$$|\kappa_\epsilon(z, w)| \leq c\left(\frac{|z|^2|w|^2}{(|z|^2 + |w|^2 + \epsilon^2)^2} + (|z|^2 + |w|^2) \log^* \| (z, w) \| \right)^2 \leq c^2|\kappa(z, w)| + c^2.$$
Recall from the proof of Proposition 4.6 that \( \int_{U_\alpha} |\kappa(x)|d\mu(x) < \infty \). On the other hand, \( \kappa_\epsilon \) converge pointwise to \( \kappa \) \( \mu \)-almost everywhere as \( \epsilon \to 0 \). Consequently, by Lebesgue dominated convergence,

\[
\lim_{\epsilon \to 0} \int_{U_\alpha} \kappa_\epsilon d\mu = \int_{U_\alpha} \kappa d\mu.
\]

This and (4.10) imply (4.12). The proof of Step 1 is thereby completed.

**Step 2 The general case** (two proofs).

Consider a general ambient Hermitian metric \( \hat{g}_X \) on \( X \). We keep the notations introduced in Step 1, for example, let \( g_X \) be the metric considered in Step 1. Let \( \chi(T) \) (resp. \( \hat{\chi}(T) \)) be the Lyapunov exponent of \( T \) when we use the ambient metric \( g_X \) (resp. \( \hat{g}_X \)). We only need to show that \( \chi(T) = \hat{\chi}(T) \). There is a constant \( c > 1 \) such that

\[
c^{-1}g_X \leq \hat{g}_X \leq cg_X.
\]

Consequently, we infer from (2.21) that

\[
c^{-2}\|\mathcal{H}(\omega, t)\|_{g_X} \leq \|\mathcal{H}(\omega, t)\|_{\hat{g}_X} \leq c^2\|\mathcal{H}(\omega, t)\|_{g_X}
\]

for every \( x \in X \setminus E \) and every path \( \omega \in \Omega_x \) and every time \( t \in \mathbb{R}^+ \). Hence,

\[
\lim_{t \to \infty} \left( \frac{1}{t} \log \|\mathcal{H}(\omega, t)\|_{g_X} - \frac{1}{t} \log \|\mathcal{H}(\omega, t)\|_{\hat{g}_X} \right) = 0.
\]

By Theorem 1.1 (2) applied to \( \mu \)-almost every \( x \in X \setminus E \) and to almost every path \( \omega \in \Omega \) with respect to \( W_x \), we get from the above equality that \( \chi(T) = \hat{\chi}(T) \).

Here is an alternative proof of Step 2 which is of independent interest. By Proposition 4.6 and Corollary 4.7 and, the desired equality \( \chi(T) = \hat{\chi}(T) \) amounts to the equality

\[
\int_X c_1(\text{Nor}(\mathcal{F}), g_X^\perp) \land T = \int_X c_1(\text{Nor}(\mathcal{F}), \hat{g}_X^\perp) \land T.
\]

Consider the global weight function \( \hat{f} : X \to [-\infty, \infty) \) satisfying \( \hat{g}_X^\perp = g_0 \exp (-2\hat{f}) \). So

\[
c_1(\text{Nor}(\mathcal{F}), g_X^\perp) = c_1(\text{Nor}(\mathcal{F}), g_0) + dd^c f \quad \text{and} \quad c_1(\text{Nor}(\mathcal{F}), \hat{g}_X^\perp) = c_1(\text{Nor}(\mathcal{F}), g_0) + dd^c \hat{f}.
\]

The above equality is reduced to

\[
\int_X dd^c f \land T = \int_X dd^c \hat{f} \land T.
\]
Note that both $f$ and $\hat{f}$ are smooth outside $E$. Consider the function $\tilde{f} : X \to [-\infty, \infty]$ given by $\tilde{f} = f - \hat{f}$. Fix $\epsilon_0 > 0$ small enough. There is a constant $c > 1$ such that for every $0 < \epsilon < \epsilon_0$, there is a smooth function $\theta_{\epsilon} : X \to [0, 1]$ such that

$$\theta_{\epsilon}(x) = 0 \text{ for dist}(x, E) < \epsilon/2, \quad \theta_{\epsilon}(x) = 1 \text{ for dist}(x, E) > \epsilon$$

and

$$|d\theta_{\epsilon}| \leq c\epsilon^{-1}, \quad |dd^c\theta_{\epsilon}| \leq c\epsilon^{-2}.$$  \hspace{1cm} (4.15)

Fix $0 < \epsilon < \epsilon_0$. Since $T$ is $dd^c$-closed and both $f$ and $\hat{f}$ are smooth on a neighborhood of the support of $\theta_{\epsilon}$, we have that

$$\int_X dd^c(\theta_{\epsilon}\tilde{f}) \wedge T = 0.$$

By Proposition 4.6 and Corollary 4.7 and (4.13), we get that

$$\lim_{\epsilon \to 0} \int_X \theta_{\epsilon} dd^c \tilde{f} \wedge T = \hat{\chi}(T) - \chi(T).$$

Therefore, equality (4.14) is reduced to showing that each of the following terms

$$\int_X d\tilde{f} \wedge d^c\theta_{\epsilon} \wedge T, \quad \int_X d^c\tilde{f} \wedge d\theta_{\epsilon} \wedge T, \quad \int_X \tilde{f} dd^c\theta_{\epsilon} \wedge T.$$ \hspace{1cm} (4.16)

tends to 0 as $\epsilon$ tends to 0.

By Lemma 3.1 (1) and (4) applied to $g_X$ and $\hat{g}_X$, we get a constant $c > 0$ such that

$$|\tilde{f}(x)| \leq c \quad \text{and} \quad |d\tilde{f}(x)| \leq c(\text{dist}(x, E))^{-1}g_X(x) \text{ for } x \in X \setminus E.$$

Using this and the properties (4.15) of the functions $\theta_{\epsilon}$, (4.16) is reduced to showing that

$$\lim_{\epsilon \to 0} \epsilon^{-2} \int_{x \in X : \epsilon/2 < \text{dist}(x, a) < \epsilon} T \wedge g_X = 0 \quad \text{for all} \quad a \in E.$$

But this inequality is equivalent to the vanishing of the Lelong number of $T$ at $a$ which has been established in [36]. Hence, the general case is completed. $\square$

**Remark 4.9** There is an alternative proof of Step 1 of Proposition 4.6 which is based on the regularization as in the proof of the first identity of Theorem A and a cohomological argument. This new method uses the monotone convergence theorem instead of the Lebesgue dominated convergence. However, it still relies on Theorem 3.3.

## 5 Cohomological Formula for the Poincaré Mass

In this section we prove the last identity (1.4b) of Theorem A. As an application of this theorem, we compute the Lyapunov exponent of a generic foliation with degree $d > 1$ in $\mathbb{P}^2$ (Corollary C).
First, we keep the hypotheses and notations of Theorem 1.1. Let \( \mathcal{F} \) be given by an open covering \( \{U_j\} \) of \( X \) and holomorphic vector fields \( v_j \in H^0(U_j, \mathrm{T}an(X)) \) with isolated singularities satisfying (1.1) for some non-vanishing holomorphic functions \( g_{jk} \in H^0(U_j \cap U_k, \mathcal{O}_X^*) \). Recall that \( \mathrm{T}an(\mathcal{F}) \) is the holomorphic line bundle on \( X \) associated to the multiplicative cocycle \( g_{jk} \). Consider the singular Hermitian metric \( h \) on \( \mathrm{T}an(\mathcal{F}) \) defined by

\[
h(v_j(x), v_j(x)) := \|v_j(x)\|_p^2,
\]

where on the right-hand side \( \|v_j(x)\|_p \) denotes the length of the tangent vector \( v_j(x) \) measured with respect to the leafwise Poincaré metric \( g_\rho \), that is, using (2.1) and (2.2)

\[
\|v_j(x)\|_p = \|\phi_x^*(v_j(x))\| = \frac{\|v_j(x)\|_{g_X}}{\|d\phi_x(0)\|} = \frac{\|v_j(x)\|_{g_X}}{\eta(x)}.
\]

Let \( h^* \) be the dual metric of \( h \) on \( \mathrm{T}an(\mathcal{F}) \). By Brunella [5, Theorem 1.1], the curvature \( c_1(\mathrm{T}an(\mathcal{F}), h^*) \) is a positive closed current on \( X \). Fix a smooth Hermitian metric \( h_0 \) on \( \mathrm{T}an(\mathcal{F}) \). Let \( h_0^* \) be the dual metric of \( h_0 \) on \( \mathrm{T}an(\mathcal{F}) \). Hence, there is a upper-semi continuous function \( \psi : X \to [-\infty, \infty) \) such that \( h^* = e^{-2\psi} h_0^* \). So the above result of Brunella says that

\[
c_1(\mathrm{T}an(\mathcal{F}), h^*) = dd^c \psi + c_1(\mathrm{T}an(\mathcal{F}), h_0^*) \geq 0.
\]

The following result is needed.

**Lemma 5.1** (1) We have \( c_1(\mathrm{T}an(\mathcal{F}), h^*) \rangle_{L^1} = g_\rho(x) \) for \( x \in X \setminus E \).

(2) \( \psi \) is smooth function outside \( E \), and

\[
\psi(x) = \log \log \|\text{dist}(x, E)\| + O(1) \quad \text{for} \quad x \in X \setminus E.
\]

**Proof of assertion (1).** Let \( U = \cup_j \) be a flow box containing \( x \). Let \( 0 < r < 1 \) be so small such that \( u \) is well-defined on \( \phi_x(\mathbb{D}_r) \subset U \). Let \( v := v_j \) be the vector field associated with \( U \). Let \( \zeta \in \mathbb{D}_r \) and write \( y := \phi_x(\zeta) \). Since \( \phi_y \) is the compose of the disc-automorphism \( \mathbb{D} \ni z \mapsto \frac{e^{v \zeta}}{1 + z \zeta} \) and \( \phi_x \), we infer from (5.2) that

\[
\|v(y)\|_p = \|\phi_x^*(v(y))\| = \frac{\|v(y)\|_{g_X}}{\|d\phi_x(\zeta)\| (1 - |\zeta|^2)}.
\]

Observe that as \( v \) is a holomorphic vector field, \( \|v(\phi_x(\zeta))\|_{g_X} \) is the modulus of a non vanishing holomorphic function. Therefore,

\[
 dd^c \log \| v(\phi_x(\zeta)) \|_p = - dd^c \log (1 - |\zeta|^2) = g_\rho(\zeta) \quad \text{for} \quad \zeta \in r \mathbb{D}.
\]

Since the left-hand side is equal to

\[
\phi^*_x( - c_1(\mathrm{T}an(\mathcal{F}), h))(\zeta) = \phi^*_x(c_1(\mathrm{T}an(\mathcal{F}), h^*))(\zeta),
\]

assertion (1) follows.

\( \square \) Springer
Proof of assertion (2). Let $g_X$ be the metric on $\text{Tan}(\mathcal{F})$ induced by $g_X$. So there is a global weight function $f: X \to [-\infty, \infty)$ such that $g_X = h_0 \exp(-2f)$. We know that the weight function $f$ is smooth outside $E$.

Suppose without loss of generality that $g_X$ coincides with the Euclidean metric in a local model near every singular point $a$ of $\mathcal{F}$. Consider the local holomorphic section $e_L$ given by $(z, w) \mapsto z \partial_z + \lambda w \partial_w$ of $\text{Tan}(\mathcal{F})$ over $U_a \simeq \mathbb{D}^2$. We have, for $x = (z, w) \in \mathbb{D}^2 \setminus \{(0, 0)\}$,

$$\exp(-\psi(x)) = |e_L(x)|_{g_X} = \sqrt{|z|^2 + |\lambda w|^2}.$$

Hence, for $x = (z, w) \in \mathbb{D}^2 \setminus \{(0, 0)\},$

$$c_1(\text{Tan}(\mathcal{F}), g_X)(x) = dd^c\phi(x) = -dd^c_{z,w} \log \sqrt{|z|^2 + |\lambda w|^2}.$$

Moreover, in the local model with coordinates $(z, w)$ associated to the singular point $E \ni a \simeq (0, 0) \in \mathbb{D}^2$, we get from the last equalities that

$$c_1(\text{Nor}(\mathcal{F}), h_0) + dd^c f = c_1(\text{Tan}(\mathcal{F}), g_X)(x) = -\frac{1}{2} dd^c_{z,w} \log (|z|^2 + |\lambda w|^2).$$

Consequently, it follows that for $x = (z, w) \in U_a \simeq \mathbb{D}^2$,

$$f(x) = -\frac{1}{2} \log (|z|^2 + |\lambda w|^2) + \text{a smooth function in } x.$$

So, there is a constant $c > 1$ such that for every local holomorphic section $v$ of $\text{Tan}(\mathcal{F})$ over an open set $U$, we have

$$c^{-1} \|v(x)\|_{g_X} \leq \|v(x)\|_{h_0 \text{dist}(x, E)} \leq \|v(x)\|_{g_X} \quad \text{for } x \in U.$$

On the other hand, we deduce from (5.1) and (2.3) that

$$\|v_j(x)\|_{g_X} = \|v_j(x)\|_{h_0 e^{-\psi(x)} \eta(x)} = \|v_j(x)\|_{h_0 e^{-\psi(x)} \eta(x)} \quad \text{for } x \in U_j.$$

This, coupled with the last inequalities, yields that

$$c^{-1} \text{dist}(x, E) \leq e^{-\psi(x)} \eta(x) \leq c \text{dist}(x, E) \quad \text{for } x \in X.$$

By Lemma 2.5, $\eta(x) \approx \text{dist}(x, E) \log^* \text{dist}(x, E)$. Hence, we get the conclusion of assertion (2). □

We infer from Lemma 5.1 (1) and (1.2) that

$$c_1(\text{Cotan}(\mathcal{F}), h^*) \wedge T = \mu \quad \text{on } X \setminus E. \quad (5.4)$$
Multiplying \( h_0^* \) by a positive constant, we may assume without loss of generality that \( \psi \leq -1 \).

**Lemma 5.2** There exist a decreasing sequence of positive real numbers \((\epsilon_n)\) and a decreasing sequence of smooth negative quasi-psh functions \( \psi_n \) with the following properties: \( \epsilon_n \downarrow 0 \), and \( \psi_n \downarrow \psi \) on \( X \), and \( \psi_n \) converge to \( \psi \) locally uniformly on \( X \setminus E \) as \( n \to \infty \), and

\[
dd^c \psi_n + c_1(\operatorname{Cotan}(\mathcal{F}), h_0^*) + \epsilon_n g_X \geq 0 \quad \text{in the current sense on } X, \quad \text{for every } n.
\]

**Proof** Lemma 5.1 says in particular that the Lelong number of \( \psi \) vanishes everywhere.

Using this we may perform Błocki–Kołodziej’s regularization [4, Theorem 2] to obtain the conclusion of the lemma. However, since the unbounded locus of \( \psi \) is exactly \( E \), which is a finite set, we prefer to give here a simpler direct argument for the reader’s convenience.

Fix a finite cover \( \mathcal{V} := (V_p)_{p \in I} \) of (singular and regular) flow boxes of \( X \) such that \( V_p \subseteq U_p, \ p \in I \), where \( \mathcal{U} := (U_p)_{p \in I} \) is the cover introduced in Sect. 2.3. Fix an \( \epsilon > 0 \). We choose a smooth function \( f_p \) in \( U_p \) such that

\[
0 \leq dd^c f_p - c_1(\operatorname{Cotan}(\mathcal{F}), h_0^*) \leq \epsilon g_X \quad \text{in } U_p.
\]

Then the function \( \varphi_p := \psi + f_p \) is plurisubharmonic in \( U_p \). Every flow box \( U_p, \ p \in I \), can be regarded as a bidisc \( \mathbb{D}^2 \). we fix a regularization by convolution for functions defined on \( U_p \) as follows. Fix a smooth radial function \( \rho: \mathbb{C}^2 \to [0, \infty) \) with compact support in \( \mathbb{D}^2 \) such that \( \int_{\mathbb{C}^2} \rho(x) \operatorname{Leb}(x) = 1 \), where \( \operatorname{Leb} \) denotes the Lebesgue measure of \( \mathbb{C}^2 \). For a function \( u: U_p \simeq \mathbb{D}^2 \to [-\infty, \infty) \) and a number \( \delta > 0 \), we consider the regularized function

\[
u_\delta(x) := (u \ast \rho_\delta)(x) = \int_{\mathbb{C}^2} u(x - \delta y) \rho(y) \operatorname{Leb}(y) \quad \text{for } x \in U_p \simeq \mathbb{D}^2.
\]

Here, \( \rho_\delta(x) := \delta^{-4} \rho(x/\delta) \), and on the right-hand side the function \( u \) is extended to \( \mathbb{C}^2 \) by setting it equal to 0 outside \( \mathbb{D}^2 \). So if \( u \) is plurisubharmonic, then so is \( u_\delta \) and \( u_\delta \) decrease to \( u \) as \( \delta \downarrow 0 \). Moreover, if \( u \) is continuous then \( u_\delta \) converge to \( u \) locally uniformly.

Since for two different (regular or singular) flow boxes \( U_p, \ U_q \in \mathcal{U} \), we always have \((U_p \cap U_q) \cap E = \emptyset\), it follows that

\[
\varphi_p,\delta - \varphi_q,\delta \quad \text{converge locally uniformly in } U_p \cap U_q \text{ to } f_p - f_q.
\]

Let \( \chi_p \) be a smooth function in \( U_p \) such that \( \chi_p = 0 \) on \( V_p \) and \( \chi_p = -1 \) away from a compact subset of \( U_p \). So there is a constant \( c > 0 \) such that \( dd^c \chi_p \geq -cg_X \). For \( \delta > 0 \) consider

\[1\) The unbounded locus of \( \psi \) is by definition the set of all points \( a \in X \) such that \( \psi \) is unbounded on every neighborhood of \( a \).
\( \varphi_\delta := \max_{p \in I} (\varphi_{p,\delta} - f_p + \epsilon c^{-1} \chi_p). \)

Therefore, we deduce from (5.6) that for \( \delta > 0 \) small enough, the values on the set \( \{ \chi_p = -1 \} \) do not contribute to the maximum. Hence, \( \varphi_\delta \) is continuous. If we consider regularized maximum (see e.g. [14]) in place of maximum, we obtain smooth function \( \varphi_\delta \). This, combined with (5.5), implies that

\[ dd^c \varphi_\delta + c_1 (\text{Cotan}(\mathcal{F}), h_0^X) + 2 \epsilon g_X \geq 0. \]

Moreover, \( \varphi_\delta \searrow \psi \) as \( \delta \searrow 0 \). Now for each \( n \), we set \( \epsilon := \epsilon_n / 2 \) and \( \psi_n := \varphi_\delta \) with \( \delta > 0 \) small enough. Since \( \psi \leq 1 \), \( \varphi_n \) can be chosen to be negative. This completes the proof. \( \square \)

By Lemma 5.2 \( \psi_n \leq 0 \) and \( \psi_n \searrow \psi \) as \( n \) tends to infinity. This, coupled with Lemma 5.1 (2), gives a constant \( c \) independent of \( n \) such that

\[ |\psi_n| \leq |\psi| \leq \log^* \log^* \text{dist}(x, E) + c. \]  

For \( n \geq 1 \) define the following measure on \( X \):

\[ \nu_n := (dd^c \psi_n + c_1 (\text{Cotan}(\mathcal{F}), h_0^X) + \epsilon_n g_X) \wedge T. \]  

By Lemma 5.2 \( \nu_n \) are all positive finite.

**Lemma 5.3** There is a smooth function \( W_\bullet : X \setminus E \to \mathbb{R}^+ \) and a constant \( c > 1 \) such that

\[ c^{-1} \log^* \log^* \text{dist}(x, E)) \leq W_\bullet(x) \leq c \log^* \log^* \text{dist}(x, E), \]

\[ |dW_\bullet(x)|_{L^1} \leq c (\text{dist}(x, E))^{-1} (\log^* \text{dist}(x, E))^{-1}, \]

\[ |dd^c W_\bullet(x)|_{L^1} \leq c (\text{dist}(x, E))^{-2} (\log^* \text{dist}(x, E))^{-2} \]

for all \( x \in X \setminus E \).

**Proof** Let \( \rho : [0, 1] \to [0, 1] \) be a smooth increasing function such that \( \rho(t) = 0 \) for \( t \leq 0 \) and \( \rho(t) = 1 \) for \( t \geq 1 \). Consider the “regularized min” function

\[ m(t, s) := t - \rho(t - s). \]

We check easily that \( m \) is smooth and its first and second derivatives are uniformly bounded and

\[ \min\{t, s\} - 1 \leq m(t, s) \leq \min\{t, s\} + 1. \]

Consider a function \( W_\bullet : X \setminus E \to \mathbb{R}^+ \) which is smooth outside the singular flow boxes \( \mathbb{U}_a \) for \( a \in E \) and which is defined in a local model near a singular point \( a \) by

\[ W_\bullet(x) := \log m(- \log |z|, - \log |w|) \quad \text{for} \quad x = (z, w) \in \mathbb{D}^2 \simeq \mathbb{U}_a. \]
Using (2.14) and (2.15) for \( x = \tilde{\psi}_\alpha(\zeta) \) with \( \zeta = u + iv \in \mathbb{C} \) and \( \alpha \in \mathbb{A} \), we get that
\[
W_\bullet(x) = \log m(v, (\text{Im}\lambda)u + (\text{Re}\lambda)v).
\]

Using the above properties of \( m \), a straightforward computation shows that the first derivative and the second one with respect to \( u, v \) satisfy
\[
\begin{align*}
D \log m(v, (\text{Im}\lambda)u + (\text{Re}\lambda)v) &= O((\min\{v, (\text{Im}\lambda)u + (\text{Re}\lambda)v\})^{-1}), \\
D^2 \log m(v, (\text{Im}\lambda)u + (\text{Re}\lambda)v) &= O((\min\{v, (\text{Im}\lambda)u + (\text{Re}\lambda)v\})^{-2}).
\end{align*}
\]

Note that
\[
\log^* \text{dist}(x, E) \approx \log \|(z, w)\| \approx \min\{ -\log |z|, -\log |w| \} \approx m(-\log |z|, -\log |w|)
\]
and \( \tilde{\psi}_\alpha'(\zeta) \approx \|(z, w)\| \). Combining these and (2.14), the result follows. \( \square \)

**Lemma 5.4** For every \( 0 < \epsilon \ll 1 \), there is a smooth function \( \theta_\epsilon : X \to [0, 1] \) such that \( \theta_\epsilon(x) = 0 \) if \( \text{dist}(x, E) \leq \epsilon^2 \), and \( \theta_\epsilon(x) = 1 \) if \( \text{dist}(x, E) \geq \epsilon \), and
\[
\begin{align*}
|d\theta_\epsilon(x)|_{L^1} &\leq c|\log \epsilon|^{-1}(\text{dist}(x, E))^{-1}, \\
|dd^c\theta_\epsilon(x)|_{L^1} &\leq c|\log \epsilon|^{-2}(\text{dist}(x, E))^{-2}
\end{align*}
\]
for all \( x \in X \setminus E \). Here \( c > 1 \) is a constant independent of \( \epsilon \).

**Proof** First we construct \( \theta_\epsilon \) on a singular flow box \( \mathbb{U}_a, a \in E \). Let \( \rho : \mathbb{R} \to [0, 1] \) be a smooth function such that \( \rho(t) = 1 \) for \( t \leq 1 \) and \( \rho(t) = 0 \) for \( t \geq 2 \). For \( x = (z, w) \in \mathbb{D}^2 \simeq \mathbb{U}_a \), we can find \( \zeta = u + iv \in \mathbb{C} \) and \( \alpha \in \mathbb{A} \) such that \( x = \tilde{\psi}_\alpha(\zeta) \) using (2.14) and (2.15). Now we set
\[
\theta_\epsilon(x) := \rho\left(\frac{m(v, (\text{Im}\lambda)u + (\text{Re}\lambda)v)}{|\log \epsilon|}\right),
\]
where \( m \) is the the “regularized min” function constructed in Lemma 5.3. For \( x \in X \setminus \bigcup_{a \in E} \mathbb{U}_a \), we simply set \( \theta_\epsilon(x) = 1 \).

As in Lemma 5.3 we check that all desired properties are satisfied. \( \square \)

**Lemma 5.5** There is a constant \( c > 0 \) such that
\[
\int_X W_\bullet(x) d\nu_n(x) < c \quad \text{for} \quad n \geq 1.
\]

**Proof** By (5.8), \( \nu_n \) does not give mass to \( E \) as \( T \) is a positive \( dd^c \)-current of bidimension \( (1, 1) \). On the other hand, by Lemma 5.4, \( \theta_\epsilon \leq 1 \setminus E \) and \( \theta_\epsilon \to 1 \setminus E \) on \( X \) as \( \epsilon \downarrow 0 \), and the convergence is local uniform on \( X \setminus E \). Therefore, we get that
\[
\int_X W_\bullet(x) d\nu_n(x) = \lim_{\epsilon \to 0} \int_X \theta_\epsilon W_\bullet(x) d\nu_n(x).
\]

\( \square \) Springer
To prove the lemma we need to show that the integral in the right-hand side is bounded independently of $n$ and $\epsilon$. By (5.8), this integral is equal to

$$\langle \theta_\epsilon W_\bullet, dd^c \psi_n \wedge T \rangle + \langle \theta_\epsilon W_\bullet, c_1(\cotan(\mathcal{F}), h_0^*) \wedge T \rangle + \epsilon_n \langle \theta_\epsilon W_\bullet, g_X \wedge T \rangle.$$  

Using the inequality $0 \leq \theta_\epsilon \leq 1$ of Lemma 5.4, and using the second inequality of Lemma 5.3, the second term is bounded and the last term tends to 0 as $n$ tends to infinity. Hence, it suffices to show that the first term is bounded. Since $\theta_\epsilon W_\bullet$ is a smooth function on $X$, by Stokes’ theorem the first term is equal to

$$\langle dd^c (\theta_\epsilon W_\bullet) \wedge T, \psi_n \rangle + \langle \partial (\theta_\epsilon W_\bullet) \wedge \partial T, \psi_n \rangle + \langle \partial (\theta_\epsilon W_\bullet) \wedge -\partial T, \psi_n \rangle =: I_1 + I_2 + I_3.$$  

Now we show that $I_1$, $I_2$ and $I_3$ are all uniformly bounded independently of $n$ and $\epsilon$.

We expand $I_1$ and obtain that

$$I_1 = \langle \theta_\epsilon dd^c W_\bullet \wedge T, \psi_n \rangle + \langle \partial \theta_\epsilon \wedge \partial W_\bullet \wedge T, \psi_n \rangle + \langle \partial \theta_\epsilon \wedge W_\bullet \wedge T, \psi_n \rangle$$

$$=: I_{11} + I_{12} + I_{13} + I_{14}.$$  

Using the inequality $0 \leq \theta_\epsilon \leq 1$ of Lemma 5.4 and using the fourth inequality of Lemma 5.3 and (5.7), we see that

$$|I_{11}| \lesssim \int (\text{dist}(x, E))^{-2}(\log^* \text{dist}(x, E))^{-2}(\log^* \text{dist}(x, E))g_X(x)|_{L_x} \wedge T(x).$$  

Using this and (1.2) and (2.3), and applying Lemma 2.5, we infer that $|I_{11}|$ is bounded by $\int_X \log^* \log^* \text{dist}(x, E)d\mu(x)$, which is finite by (3.11). Hence $I_{11}$ is uniformly bounded independently of $n$ and $\epsilon$.

Using the estimate on $dd^c \theta_\epsilon$ of Lemma 5.4 and the fact that this form has a support in $\{x \in X : \epsilon^2 \leq \text{dist}(x, E) \leq \epsilon\}$ and the estimate on $W_\bullet$ of Lemma 5.3 and (5.7), we deduce that

$$|I_{14}| \lesssim \int (\text{dist}(x, E))^{-2}(\log^* \text{dist}(x, E))^{-2}(\log^* \text{dist}(x, E))^2g_X(x)|_{L_x} \wedge T(x).$$  

Using this and (1.2) and (2.3), and applying Lemma 2.5, we infer that $|I_{14}|$ is bounded by $\int_X (\log^* \log^* \text{dist}(x, E))^2d\mu(x)$, which is finite by (3.11). Hence $I_{14}$ is uniformly bounded independently of $n$ and $\epsilon$.

Similarly, using the estimate on $d\theta_\epsilon$ of Lemma 5.4 and the estimate on $dW_\bullet$ of Lemma 5.3 and (5.7), we infer that $I_{12}$ and $I_{13}$ are bounded. We have shown that $I$ is uniformly bounded independently of $n$ and $\epsilon$.

Let $\tau$ be a $(1, 0)$-form defined almost everywhere with respect to $\mu = T \wedge g_P$ on $X \setminus E$ such that $\partial T = \tau \wedge T$. In a regular flow box $U$ as in (2.4), we see that $\tau = h^{-1}_\alpha \partial h_\alpha$ on the plaque passing through $\alpha \in \Sigma$ for $\nu$-almost every $\alpha$. Using (5.7) and the estimate on $\theta_\epsilon, d\theta_\epsilon$ of Lemma 5.4 and the estimate on $W_\bullet, dW_\bullet$ of Lemma 5.3,
we see that

$$|I_2| \leq \int_X O(\text{dist}(x, E)^{-1})(\log^* \text{dist}(x, E))^{-1} (\log^* \log^* \text{dist}(x, E))^2) \wedge \tau \wedge T.$$  

Applying the Cauchy-Schwarz inequality, and then using (1.2), (2.3) and Lemma 2.5, we infer that

$$|I_2|^2 \leq \left( \int_X (\text{dist}(x, E))^{-2}(\log^* \text{dist}(x, E))^{-2}(\log^* \log^* \text{dist}(x, E))^2 \cdot T \wedge g_X \right)$$

$$ \cdot \left( \int_X (\log^* \log^* \text{dist}(x, E))^2 i \tau \wedge \bar{\tau} \wedge T \right)$$

$$\lesssim \left( \int_X (\log^* \log^* \text{dist}(x, E))^2 d\mu(x) \right) \left( \int_X (\log^* \log^* \text{dist}(x, E))^2 i \tau \wedge \bar{\tau} \wedge T \right).$$

By [27, Proposition 3] (see also the proof of [40, Proposition 5.28]), we get the following inequality

$$i \tau \wedge \bar{\tau} \wedge T \leq T \wedge g_P = \mu \quad \text{on} \quad X \setminus E.$$

Putting these estimates together, we get that

$$|I_2| \leq \int_X (\log^* \log^* \text{dist}(x, E))^2 d\mu(x).$$

So by (3.11), $I_2$ is uniformly bounded independently of $n$ and $\epsilon$. The same argument also shows that $I_3$ is uniformly bounded independently of $n$ and $\epsilon$. The proof is thereby completed. \qed

Now we arrive at the

**Proof of the second identity of Theorem A** Since the continuous functions $\psi_n$ converge to $\psi$ locally uniformly on $X \setminus E$ as $n \to \infty$, we deduce from (5.4) and (5.8) that $\nu_n \to \mu$ on $X \setminus E$.

On the other hand, consider the space $C^\bullet(X)$ of all continuous function $f : X \setminus E \to \mathbb{R}$ such that

$$\|f\|_\bullet := \sup_{x \in X \setminus E} \frac{|f(x)|}{W_\bullet(x)} < \infty.$$  

Note that $C^\bullet(X)$ endowed with the norm $\| \cdot \|_\bullet$ is a Banach space. By Lemma 5.5, $\nu_n$ are continuous linear forms on $C^\bullet(X)$ with norms $\leq c$. Hence, every cluster limit of this sequence is also a continuous linear form on $C^\bullet(X)$ with norm $\leq c$. Let $\nu$ be such a cluster limit. We will show that $\nu = \mu$. Indeed, the previous paragraph shows that $\nu = \mu$ on $X \setminus E$. Moreover, by (1.2), $\mu$ does not give mass to $E$. As the function $W_\bullet$ is integrable with respect to $\nu$, $\nu$ does not give mass to $E$. Hence, $\nu = \mu$ on $X$. \qed
Consequently, \( v_n \to v \) on \( \mathcal{C}^\bullet(X) \). Applying this convergence to the function \( 1 \) and using (5.4) and (5.8), we get that

\[
\langle dd^c \psi_n, T \rangle + \langle c_1(\text{Cotan}(\mathcal{F}), h_0^\ast), T \rangle + \epsilon_n \langle g_X, T \rangle \to \langle 1, \mu \rangle \quad \text{as} \quad n \to \infty.
\]

The right-hand side is \( \| \mu \| \). On the other hand, on the left-hand side,

\[
\langle c_1(\text{Cotan}(\mathcal{F}), h_0^\ast), T \rangle = c_1(\text{Cotan}(\mathcal{F})) \cdot \{ T \}
\]

since \( h_0^\ast \) is a smooth metric. Moreover, \( \langle dd^c \psi_n, T \rangle = 0 \) because \( T \) is \( dd^c \)-closed, and \( \epsilon_n \langle g_X, T \rangle = O(\epsilon_n) \to 0 \).

Therefore, the above limit implies identity (1.4b).

\[\square\]

**End of the proof of Corollary C** (see also [16, Proposition 3.12]) By Theorem 1.3 let \( T \) be the unique directed positive \( dd^c \)-closed current such that \( \mu \) given by (1.2) is a probability measure. Therefore, it follows from identity (1.4b) that \( c_1(\text{Cotan}(\mathcal{F})) \sim \{ T \} = 1 \). On the other hand, it is well-known (see e.g. [7]) that \( \text{Nor}(\mathcal{F}) \) is equal to \( \mathcal{O}(d + 2) \) and \( \text{Cotan}(\mathcal{F}) \) is equal to \( \mathcal{O}(d - 1) \). Therefore, by identity (1.4a),

\[
\chi(\mathcal{F}) = \chi(T) = -\{ c_1(\text{Nor}(\mathcal{F})) \} \sim \{ T \} = -(d + 2)\{ c_1(\mathcal{O}(1)) \} \sim \{ T \}
\]

\[
= -\frac{d + 2}{d - 1} \cdot c_1(\text{Cotan}(\mathcal{F})) \sim \{ T \}
\]

\[
= -\frac{d + 2}{d - 1}.
\]

Hence, the result follows. \[\square\]

### 6 Negative Lyapunov Exponent

This section is devoted to the proof of Theorem B. By Theorem 1.2 we may assume without loss of generality that \( g_X \) coincides with the Euclidean metric in a local model for every singular flow box \( U_\alpha \) with \( \alpha \in E \). Consider the space \( \mathcal{C}^\bullet(X) \) of all continuous functions \( f : X \setminus E \to \mathbb{R} \) such that

\[
\sup_{x \in X \setminus E} \frac{|f(x)|}{W(x)} < \infty,
\]

where \( W(x) \) is as usual given in (3.12). Consider the norm

\[
\| f \|:= \sup_{x \in X \setminus E} \frac{|f(x)|}{W_\ast(x)} < \infty,
\]

where

\[
W_\ast(x) := \log^* \text{dist}(x, E) \cdot W(x) \quad \text{for} \quad x \in X \setminus E.
\]

By Theorem 1.3, let \( T \) be the unique directed positive \( dd^c \)-closed current and let \( \mu \) be the measure associated to \( T \) by (1.2). So \( \mu \) is a probability measure. Consider the function \( \kappa \) defined in (2.33).
Proposition 6.1  (1) The function $\kappa$ belongs to $\mathscr{C}^*(X)$.
(2) $\mathcal{D}(X \setminus E)$ is a dense subspace of $\mathscr{C}^*(X)$.

\textbf{Proof} Proof of assertion (1). Since the Poincaré metric $g_\rho$ is leafwise smooth and transversally continuous on $X \setminus E$, we deduce that $\kappa \in \mathscr{C}(X \setminus E)$. Next, we infer from Lemma 2.5 that in a regular flow box $\kappa(x) : = \Delta_{P\kappa}(0)$ is a bounded function. Since $W(x) \geq 1$, it follows that $\kappa(x) \lesssim W(x)$ in this case.

On the other hand, by Lemma 3.2 (1) we see that for a point $x = (z, w)$ in the local model of a singular flow box $\mathbb{U}_a \simeq \mathbb{D}^2$, $a \in E$,

$$\kappa(x) \lesssim \frac{|z|^2 |w|^2}{(|z|^2 + |w|^2)^2} (\log^* \|(z, w)\|)^2 \lesssim W(x).$$

So in all cases $\kappa(x) \lesssim W(x)$. Hence, $\kappa$ belongs to $\mathscr{C}^*(X)$.

\textbf{Proof of assertion (2).} Let $f$ be a function in $\mathscr{C}^*(X)$. For $0 < \epsilon < \epsilon_0$ consider the function $f_\epsilon : = \theta_\epsilon f$, where the family $(\theta_\epsilon)_{0 < \epsilon < \epsilon_0}$ is given in (4.15). Using the first two properties of (4.15) and the assumption $f \in \mathscr{C}(X \setminus E)$ with $\sup_{x \in X \setminus E} \frac{|f(x)|}{W(x)} < \infty$, we see that $f_\epsilon \in \mathscr{C}_0(X \setminus E)$ and

$$\|f - f_\epsilon\|_* \leq (\log \epsilon)^{-1} \left( \sup_{x \in X \setminus E : \text{dist}(x, E) \leq \epsilon} \frac{|f(x)|}{W(x)} \right) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Hence, $\mathscr{C}_0(X \setminus E)$ is a dense subspace of $\mathscr{C}^*(X)$. Since $\mathcal{D}_0(X \setminus E)$ is a dense subspace of $\mathscr{C}_0(X \setminus E)$, assertion (2) follows.

In the next key lemma we use Hahn–Banach separation theorem following an idea of Sullivan [44] and Ghys [29]. In the case of transversally conformal foliations without singularities, the idea had been used in [16]. Note that by Proposition 4.6, $\int_X \kappa(x) d\mu(x)$ is well-defined.

Lemma 6.2 Suppose that $\int_X \kappa(x) d\mu(x) \geq 0$. Then there exist a sequence of smooth real-valued functions $\psi_n$ compactly supported in $X \setminus E$ and a sequence of $(\epsilon_n) \subset \mathbb{R}^+$ such that $\epsilon_n \searrow 0$ and that

$$\frac{\kappa(x) - \Delta_P \psi_n(x)}{W_*(x)} \geq -\epsilon_n \quad \text{for all} \quad x \in X \setminus E.$$

\textbf{Proof} Consider the space $\mathcal{I}(X)$ of all function $u : X \setminus E \to \mathbb{R}$ such that there is a smooth real-valued function $f$ compactly supported in $X \setminus E$ (i.e. $f \in \mathcal{D}(X \setminus E)$) such that $u = \Delta_P f$. We see that $\mathcal{I}(X)$ is a subspace of $\mathscr{C}^*(X)$. Let $\overline{\mathcal{I}}(X)$ be the closure of $\mathcal{I}(X)$ in $\mathscr{C}^*(X)$. Consider the cone $\mathscr{C}^+(X)$ of all functions $f \in \mathscr{C}^*(X)$ such that $f(x) \geq 0$ everywhere. Let $\mathcal{Q}(X)$ be the quotient of $\mathscr{C}^*(X)$ by $\mathcal{I}(X)$ and $\pi : \mathscr{C}^*(X) \to \mathcal{Q}(X)$ the canonical projection. Let $\overline{\mathcal{I}}^+(X)$ be the closure of $\pi(\mathscr{C}^+(X))$ in $\mathcal{Q}(X)$. By Proposition 6.1 (1), the conclusion of the lemma is equivalent to the fact that $\pi(\kappa) \in \mathcal{Q}(X)$ belongs to $\overline{\mathcal{I}}^+(X)$. Suppose the contrary in order to get a contradiction. By Hahn–Banach separation theorem applied to $\mathcal{Q}(X)$, there...
exists a continuous linear functional \( \hat{v} : \mathcal{D}(X) \to \mathbb{R} \) such that \( v \geq 0 \) on \( \mathcal{C}^+(X) \) and \( \hat{v}(\pi(\kappa)) < 0 \). So the continuous linear functionnal \( v := v \circ \pi : \mathcal{C}^+(X) \to \mathbb{R} \) satisfies that \( v \geq 0 \) on \( \mathcal{C}^+(X) \) and \( v(\kappa) < 0 \) and \( v = 0 \) on \( \mathcal{I}(X) \). Since we know by Proposition 6.1 (2) that \( \mathcal{D}(X \setminus E) \) is dense in \( \mathcal{C}^+(X) \), it is also dense in \( \mathcal{D}(\mathcal{F}) \), and hence we infer that \( v \) is a nonzero finite positive measure such that the restriction of \( v \) to \( X \setminus E \) defines a nonzero positive harmonic measure. Moreover, \( v \) does not give mass to the set \( E \), since the function \( x \mapsto \log^* \text{dist}(x, E) \) belongs to \( \mathcal{C}^+(X) \), and hence the Dirac mass \( \delta_a \) at any point \( a \in E \) cannot be evaluated at this function. Consequently, by Proposition 2.4, \( v = T' \land g_P \) for a positive directed \( dd^c \)-closed current \( T' \). Therefore, it follows from Theorem 1.3 and formula (1.2) that \( v \) is equal to \( \mu \) up to a positive multiplicative constant. Hence, \( \mu(\kappa) \) which is of the same sign as \( v(\kappa) \) should be negative, and we reach a contradiction. \( \square \)

**Remark 6.3** In Lemma 6.2 we cannot use the following natural norm for \( \mathcal{C}^+(X) \):

\[
\| f \| := \sup_{x \in X \setminus E} \frac{|f(x)|}{W(x)} < \infty.
\]

Indeed, with this norm \( \mathcal{D}(X \setminus E) \) is not dense in \( \mathcal{C}^+(X) \) : the function \( x \mapsto \log^* \text{dist}(x, E) \) does not belong to the closure of \( \mathcal{D}(X \setminus E) \).

By adding a constant \( c_n \) to each \( \psi_n \), we obtain a new sequence of functions \( (\psi_n) \) still satisfying the estimate of Lemma 6.2. Note that \( \psi_n \) is not necessarily compactly supported in \( X \setminus E \). In fact, \( \psi_n \) is constant near \( E \).

Consider the volume form \( \text{Vol} \) on \( X \setminus E \) given by

\[
\int_{X \setminus E} f d\text{Vol} = \langle \omega^1_X, f g_P \rangle \quad \text{for} \quad f \in \mathcal{D}(\mathcal{F}),
\]

where the transversal form \( \omega^1_X \) is given in (2.23).

**Lemma 6.4** (1) We have that \( \langle W_*, \text{Vol} \rangle < \infty \).

(2) By adding a constant \( c_n \) to each function \( \psi_n \) given by Lemma 6.2 if necessary (see Remark 6.3), we may assume that \( \langle W_*, \mu_n \rangle = 1 \), where \( \mu_n \) is the volume form on \( X \setminus E \) defined by

\[
\mu_n := \exp (-2\psi_n) \text{Vol}.
\]

**Proof** Proof of assertion (1). We only need to work on a local model near a singularity \( a \in E \). For a point \( x \) close to \( a \), write \( x = (z, w) \). Assume without loss of generality that the fundamental form associated to \( g_X \) is equal to \( idz \land d\bar{z} + i dw \land d\bar{w} \). By Lemma 2.5, we have on \( L_x \) that \( g_P(x) \approx \frac{g_X(x)}{\|x\|^2(\log \|x\|)^2} \). Moreover, by Lemma 3.1 (1) there is a constant \( c > 0 \) such that \( g^1_X(x) \leq cg_X(x) \). We also infer from (3.12) that \( W(x) \leq 2(\log \|x\|)^2 \). Hence, by (6.1) \( W_*(x) \leq 2(\log \|x\|)^3 \). Therefore, the condition \( \langle W_*, \text{Vol} \rangle < \infty \) will follow if one can show that

\[
\int_{x \in \mathbb{D}^2} (\log \|x\|)^3 \frac{(idz \land d\bar{z}) \land (idw \land d\bar{w})}{\|x\|^2(\log \|x\|)^2} < \infty.
\]
Using the spherical coordinates, we see easily that the above integral is convergent. The proof of assertion (1) is thereby completed. **Proof of assertion (2).** It follows from formula (6.3), assertion (1) and the fact that the functions $\psi_n$ are bounded. □

Consider the finite open cover $\mathcal{U} = (U_p)_{p \in I}$ of $X$ given in Sect. 2.3. So $E \subset I$ and for every $a \in E$, $U_a$ is a singular flow box associated to $a$. For each (regular or singular) flow box $U_p \in \mathcal{U}_p$ with foliated chart $\Phi_p : U_p \to \mathbb{B}_p \times \Sigma_p$, let $\gamma_p$ be the volume form on $\Sigma_p$ and $\varphi_{U_p}$ be the real-valued function on $U_p$ given in (2.24), (2.25) and (2.28). For every $n \geq 1$ consider the function $\varphi_{n,p} : U_p \to \mathbb{R}$ given by

$$\varphi_{n,p} = \psi_n + \varphi_{U_p} \quad \text{on } U_p. \quad (6.4)$$

We make the following important observation.

**Remark 6.5** By Remark 2.7, $\Delta \varphi_{U_p}$, $|d\varphi_{U_p}|_p$, $\Delta_p \varphi_{n,p}$ and $|d\varphi_{n,p}|_p$ are independent of $p \in I$, in other words, they are well-defined functions on the whole $X \setminus E$. So we will write $\Delta_p \varphi_n$ (resp. $|d\varphi_{n,p}|_p$) instead of $\Delta_p \varphi_{n,p}$ (resp. $|d\varphi_{n,p}|_p$).

By (6.3), (6.2) and (2.27) we get

$$(\Phi_p)_* \mu_n = \exp(-2 \varphi_{n,p} \circ \Phi_p^{-1})(\Phi_p)_* (g_p) \wedge \gamma_p \quad \text{on } \mathbb{B}_p \times \Sigma_p. \quad (6.5)$$

**Lemma 6.6** For every $n \geq 1$ and every $a \in E$, in the local model for the singular flow box,

$$\int_{U_a} |\Delta_p \varphi_{n,a}|_p d\mu_n < \infty \quad \text{and} \quad \int_{U_a} |d\varphi_{n,a}|_p^2 d\mu_n < \infty.$$

**Proof** We fix $n \geq 1$. To prove the first inequality of the lemma, observe that

$$\int_{U_a} |\Delta_p \varphi_{n,a}|_p d\mu_n \leq \int_X |\Delta_p \psi_n| d\mu_n + \int_{U_a} |\Delta_p \varphi| d\mu_n.$$ 

Since $\psi_n$ is a smooth function, we have that $|\Delta_p \psi_n| \lesssim \|\psi_n\|_{C^2} < \infty$. Hence,

$$\int_X |\Delta_p \psi_n| d\mu_n \lesssim \langle 1, \mu_n \rangle \lesssim \langle W_*, \mu_n \rangle = 1.$$

By Lemma 3.2, we have $|(\Delta_p \varphi)(x)| \lesssim W(x)$ for $x \in U_a$. Using this and applying Lemma 6.4 (2) yield that

$$\int_{U_a} |\Delta_p \varphi| d\mu_n \lesssim \langle W_*, \mu_n \rangle = 1.$$

The last three estimates imply the first inequality of the lemma.
We turn to the proof of the second inequality of the lemma. For \( r > 0 \) let \( \mathbb{D}_P(r) \) denote the disc with center 0 and radius \( r \) with respect to the Poincaré metric \( g_P \) on \( \mathbb{D} \). Let \( \theta : \mathbb{D}_P(3) \to [0, 1] \) be a smooth function such that \( \theta = 1 \) on \( \mathbb{D}_P(1) \) and \( \theta = 0 \) outside \( \mathbb{D}_P(2) \) and \( |\Delta_P \theta| < 100 \). Let \( \chi : \mathbb{D}_P(2) \to \mathbb{R} \) be smooth function. Applying Stokes’ theorem on \( \mathbb{D}_P(2) \) yields that

\[
\int_{\mathbb{D}_P(2)} d\xi \exp(-2\chi) = \int_{\mathbb{D}_P(2)} \theta d\xi \exp(-2\chi).
\]

Using (2.8) and (2.9) we deduce that

\[
\int_{\mathbb{D}_P(2)} \theta |d\chi|^2_P \exp(-2\chi) g_P = \int_{\mathbb{D}_P(2)} (\Delta_P \theta) \exp(-2\chi) g_P + \int_{\mathbb{D}_P(2)} \theta (\Delta_P \chi) \exp(-2\chi) g_P.
\]

Hence,

\[
\int_{\mathbb{D}_P(2)} |d\chi|^2_P \exp(-2\chi) g_P \leq 100 \int_{\mathbb{D}_P(2)} \exp(-2\chi) g_P + \int_{\mathbb{D}_P(2)} \Delta_P \chi \exp(-2\chi) g_P.
\]

Fix \( \alpha \in \mathbb{A} \). The above inequality implies that for every \( x \in \mathcal{L}_\alpha \),

\[
\int_{\pi_x(\mathbb{D}_P(1))} |d\varphi_{n,a}|^2_P \exp(-2\varphi_{n,a}) g_P \leq 100 \int_{\phi_x(\mathbb{D}_P(2))} \exp(-2\varphi_{n,a}) g_P + \int_{\phi_x(\mathbb{D}_P(2))} |\Delta_P \varphi_{n,a}| \exp(-2\varphi_{n,a}) g_P,
\]

where \( \phi_x \) is given in (2.1). Integrating both sides of the above inequality with respect to \( g_P(x) \), \( x \in \mathcal{L}_\alpha \), we see that there is a universal constant \( c > 0 \) independent of \( \alpha \in \mathbb{A} \) and \( n \in \mathbb{N} \) such that

\[
\int_{\mathcal{L}_\alpha} |d\varphi_{n,a}|^2_P \exp(-2\varphi_{n,a}) g_P \leq c \int_{\mathcal{L}_\alpha} \exp(-2\varphi_{n,a}) g_P + c \int_{\mathcal{L}_\alpha} |\Delta_P \varphi_{n,a}| \exp(-2\varphi_{n,a}) g_P.
\]

Integrating both sides of the last inequality with respect to the volume form \( \Upsilon_a \) on \( \mathbb{A} \) and using (2.27) and (6.2), (6.3) and (6.4), we obtain that

\[
\int_{\mathbb{U}_a} |d\varphi_{n,a}|^2_P d\mu_n \leq c \int_{\mathbb{U}_a} d\mu_n + c \int_{\mathbb{U}_a} |\Delta_P \varphi_{n,a}| d\mu_n.
\]
On the other hand, by Lemma 6.4 (2), \( \langle 1, \mu_n \rangle \leq \langle W_*, \mu_n \rangle = 1 \). Hence, the second inequality of the lemma follows from the first one. \( \Box \)

**Lemma 6.7** For every \( n \geq 1 \) we have that

\[
\int_X (\Delta_P \varphi_n - |d\varphi_n|^2_P) d\mu_n = 0.
\]

By Lemma 6.6, the above integral makes sense.

**Proof** Fix \( n \geq 1 \) and a partition of unity \((f_p)_{p \in I}\) associated to \( U : \sum_{p \in I} f_p = 1 \), where the support of each function \( f_p \) is contained in \( U_p \). Using (2.8) and (2.27) and then applying Stokes’ theorem to each plaque of each regular flow box \( U_p \), we get that

\[
\frac{1}{\pi} \int_X \Delta_P f_p d\mu_n = \int_{\mathcal{B}_p \times \Sigma_p} \ddc^c (f_p \circ \Phi_p^{-1}) \exp(-2\varphi_{n,p} \circ \Phi_p^{-1}) \gamma_p \\
= \int_{\Sigma_p} \left( \int_{\mathcal{B}_p} \ddc^c (f_p \circ \Phi_p^{-1}) \exp(-2\varphi_{n,p} \circ \Phi_p^{-1}) \right) \gamma_p \\
= \int_{\Sigma_p} \left( \int_{\mathcal{B}_p} (f_p \circ \Phi_p^{-1}) \ddc^c \exp(-2\varphi_{n,p} \circ \Phi_p^{-1}) \right) \gamma_p.
\]

A straightforward computation shows that the expression in the last line is equal to

\[
\int_{\Sigma_p} \left( \int_{\mathcal{B}_p} (f_p \circ \Phi_p^{-1})(-2ddc(\varphi_{n,p} \circ \Phi_p^{-1}) + 4i \partial(\varphi_{n,p} \circ \Phi_p^{-1}) \wedge \overline{\partial(\varphi_{n,p} \circ \Phi_p^{-1})} \right) \\
\cdot \exp(-2\varphi_{n,p} \circ \Phi_p^{-1}) \gamma_p \\
= \int_{\Sigma_p} \left( \int_{\mathcal{B}_p} (f_p \circ \Phi_p^{-1})(-\Delta_P(\varphi_{n,p} \circ \Phi_p^{-1}) \\
+ |d\varphi_{n,p} \circ \Phi_p^{-1}|_P^2) \exp(-2\varphi_{n,p} \circ \Phi_p^{-1}) g_P \right) \wedge \gamma_p \\
= \int_X f_p (-\Delta_P \varphi_{n,p} + |d\varphi_{n,p}|_P^2) d\mu_n,
\]

where the first equality holds by (2.8)–(2.9), and the second one by (2.27). We have shown that for \( p \in I \setminus E \),

\[
\int_X \Delta_P f_p d\mu_n = \int_X f_p (-\Delta_P \varphi_{n,p} + |d\varphi_{n,p}|_P^2) d\mu_n. \tag{6.6}
\]

We will show that for each \( a \in E \),

\[
\int_X \Delta_P f_a d\mu_n = \int_X f_a (-\Delta_P \varphi_{n,a} + |d\varphi_{n,a}|_P^2) d\mu_n. \tag{6.7}
\]
Taking (6.7) for granted, we will complete the proof of the lemma. Indeed, by summing up (6.6) and (6.7), and using that \( \Delta_P 1 = 0 \), we infer that

\[
0 = \int_X \Delta_P 1 d \mu_n = \sum_{p \in I} \int_X \Delta_P f_p d \mu_n \\
= \sum_{p \in I \setminus E} \int_X f_p(-\Delta_P \varphi_{n,p} + |d \varphi_{n,p}|_p^2) d \mu_n + \sum_{a \in E} \int_X f_a(-\Delta_P \varphi_{n,a} + |d \varphi_{n,a}|_p^2) d \mu_n \\
= \sum_{p \in I} \int_X f_p(-\Delta_P \varphi_n + |d \varphi_n|_p^2) d \mu_n \\
= \int_X (\sum_{p \in I} f_p)(-\Delta_P \varphi_n + |d \varphi_n|_p^2) d \mu_n.
\]

Hence, the desired identity of the lemma follows.

To complete the proof it remains to establish (6.7). We work in the local model \( \mathcal{P}|_U \simeq (D^2, \mathcal{L}, \{0\}) \) associated to a point \( a \in E \). Using (2.28) and (6.4), we obtain a form \( \Upsilon_a \) which is well-defined on the distinguished transversal \( A_a \) of this model as well as function \( \varphi_{n,a} \) which is well-defined on \( D^2 \).

Let \( \rho : C \cup \{\infty\} \to \mathbb{R}^+ \) be a smooth function which satisfies

\[
\rho(|t|) = 0 \text{ for } |t| \geq 2, \quad \rho(t) = 1 \text{ for } |t| \leq 1/2, \quad 0 < \rho(t) < 1 \text{ for } 1/2 < |t| < 2.
\]

For every \( 0 < \epsilon \ll 1 \), consider the smooth function \( \rho_\epsilon : D^2 \to [0, 1] \) given by

\[
\rho_\epsilon(x) := (1 - \rho(2z/\epsilon))(1 - \rho(2w/\epsilon)) \quad \text{for} \quad x = (z, w) \in D^2. \tag{6.8}
\]

Note that \( \rho_\epsilon(x) = 1 \) if \( |z| > \epsilon \) and \( |w| > \epsilon \), \( \rho_\epsilon(x) = 0 \) if either \( |z| < \epsilon/4 \) or \( |w| < \epsilon/4 \). Moreover, we have

\[
\partial_z \rho_\epsilon = O(\epsilon^{-1})dz, \quad \partial_{\bar{z}} \rho_\epsilon = O(\epsilon^{-1})d\bar{z}, \quad \partial_w \rho_\epsilon = O(\epsilon^{-2})idz \land d\bar{z}, \quad \partial_{\bar{w}} \rho_\epsilon = O(\epsilon^{-2})idz \land d\bar{w}. \tag{6.9}
\]

Similar estimates also hold when \( z \) is replaced by \( w \).

Now fix \( \alpha \in A \) and consider the function \( (\Delta_P f_a) \rho_\epsilon \exp(-2\varphi_{n,a}) \) restricted to the Riemann surface \( \mathcal{L}_a \). Recall that \( \mathcal{L}_a \) is the image of the sector \( S \) by the map \( \tilde{\psi}_a \) given in (2.14), and that \( (z, w) = \tilde{\psi}_a(\xi) \) is related to \( \xi = u + iv \) by (2.15). Since \( f_a = 1 \) in a neighborhood of \( a = 0 \) and \( f_a \) is compactly supported in \( D^2 \), it follows from the above properties of \( \rho_\epsilon \) that the support of the function \( (\Delta_P f_a) \rho_\epsilon \exp(-2\varphi_{n,a}) \) is contained in the image by \( \tilde{\psi}_a \) of a set \( K_\alpha \) such that

\[
K_\alpha \subseteq \{(u, v) \in S : 0 < v < -c \log(\epsilon/4) \quad \text{and} \quad 0 < (\text{Im}\lambda)u + (\text{Re}\lambda)v < -c \log(\epsilon/4)\} \subset S.
\]
Here $c > 0$ is a constant that depends only on $a$. Therefore, we are able to applying Stokes’ theorem to this function on $L_\alpha$. Consequently, for each $\alpha \in A$ we have that

$$
\int_{L_\alpha} (\Delta_P f_a) \rho_\epsilon \exp(-2\varphi_{n,a}) g_P = \int_{L_\alpha} f_a \Delta_P (\rho_\epsilon \exp(-2\varphi_{n,a})) g_P.
$$

(6.10)

Since $\rho_\epsilon \to 1$ on $\mathbb{D}^2$ as $\epsilon \to 0$, the left-hand side of (6.10) tends to $\int_{L_\alpha} (\Delta_P f_a) \exp(-2\varphi_{n,a}) g_P$. On the other hand, as the difference $\Delta_P (\rho_\epsilon \exp(-2\varphi_{n,a})) - \rho_\epsilon \Delta_P (\exp(-2\varphi_{n,a}))$ is equal to 0 on every open set where $\rho_\epsilon$ is identically either to 0 or 1, we deduce from the above properties of $\rho_\epsilon$ that the above difference is nonzero only if $x \in D_\epsilon$, where

$$
D_\epsilon := \left\{ x = (z, w) \in \mathbb{D}^2 : \epsilon/4 \leq |z|, |w| \leq \epsilon \right\}.
$$

Using Lemma 3.1 (4), we infer from (6.4) and the smoothness of $\psi_n$ that

$$
|\partial \varphi_{n,p}|_p = O(1) + O(|\partial \varphi_{U_p}|_p) = O(1) + O(\log \epsilon) = O(\log \epsilon)
$$
on $U_p$,

where on the right-hand side, $O(1)$ is a bounded function on $U_p$ which depends also on $n$. Therefore, we infer from (6.9) that for $x = (z, w) \in D_\epsilon \cap L_\alpha$,

$$
\Delta_P (\rho_\epsilon \exp(-2\varphi_{n,a})) - \rho_\epsilon \Delta_P (\exp(-2\varphi_{n,a})) = O((\log \epsilon)^2) \mathbf{1}_{D_\epsilon} \exp(-2\varphi_{n,a}),
$$

where $\mathbf{1}_{D_\epsilon}$ is the characteristic function of $D_\epsilon$. Integrating both sides with respect to the form $\Upsilon_a$ on $A_a$ and using (6.3), (6.5), we get that

$$
\left| \int_{\alpha \in \mathbf{A}_a} \left( \int_{L_\alpha} f_a (\Delta_P (\rho_\epsilon \exp(-2\varphi_{n,a})) - \rho_\epsilon \Delta_P (\exp(-2\varphi_{n,a})) g_P) \Upsilon_a(\alpha) \right) \right| \\
\leq c(\log \epsilon)^2 \mathbf{1}_{D_\epsilon} \mu_n.
$$

Arguing as in the proof of Lemma 6.4, we see easily that the right-hand side is bounded by

$$
\int_{x = (z, w) \in D_\epsilon} (\log \epsilon)^2 (idz \wedge d\bar{z}) \wedge (idw \wedge d\bar{w}) \|x\|^2 (\log \|x\|)^2 = O(\epsilon^2).
$$

Therefore, we get that as $\epsilon$ tends to 0, the difference

$$
\int_{\alpha \in \mathbf{A}_a} \left( \int_{L_\alpha} f_a \Delta_P (\rho_\epsilon \exp(-2\varphi_{n,a})) g_P \Upsilon_a(\alpha) \right) \\
- \int_{\alpha \in \mathbf{A}_a} \left( \int_{L_\alpha} f_a \rho_\epsilon \Delta_P (\exp(-2\varphi_{n,a})) g_P \right) \Upsilon_a(\alpha)
$$
tends also to 0. Letting $\epsilon$ tend to 0, we infer from this and (6.10) that

$$
\int_{a \in h_a} \left( \int_{L_a} (\Delta_P f_a) \exp (-2\varphi_{n,a}) g_P \right) \gamma_a(\alpha) \\
= \int_{a \in h_a} \left( \int_{L_a} f_a \Delta_P (\exp (-2\varphi_{n,a})) g_P \right) \gamma_a(\alpha).
$$

Hence, we get (6.7) easily using (6.5). 

\[\mathbb{E}\]

End of the proof of Theorem B

Suppose in order to get a contradiction that $\chi(T) \geq 0$. So by Proposition 4.6, we have that

$$
\int_{X} \kappa(x) d\mu(x) \geq 0.
$$

By Lemma 6.4 (2), there is a subsequence of $(\mu_n)$ converging weakly to a measure $\mu'$. Consider the family of currents $T_n$ on $\mathcal{D}^{1,1}(\mathcal{F})$ defined by

$$
T_n(h) := \left\langle \exp (-2\varphi_n) \text{Vol}, h \right\rangle \quad \text{for} \quad h \in \mathcal{D}^{1,1}(\mathcal{F}),
$$

where $\text{Vol}$ and $\varphi_n$ are defined in (6.2) and (6.3). Let $h = f_n g_P$ be defined by $h = f_n g_P$. Then we get that $T_n(h) = \langle f_n, \mu_n \rangle$. On the other hand, by Lemma 6.4 (2), $\langle W_*, \mu' \rangle = 1$. So $\mu'$ does not charge any singular point. Consequently, restricting $\mu'$ to each flow box we can prove that $T_n$ converge weakly to a nonzero directed positive current $T'$ and $\mu' = T' \wedge g_P$ on $X \setminus E$.

Next, we will show that $T'$ is closed on $X \setminus E$. Taking this for granted, we see that $T'$ is a nonzero directed positive closed current. Moreover, since

$$
\langle T', g_X \rangle \leq \langle T' \wedge g_P, 1_X \rangle = \langle \mu', 1_X \rangle \lesssim \langle \mu', W_* \rangle = 1,
$$

by a classical theorem of Skoda [43], $T'$ extends trivially through $E$ to a positive closed current on $X$. This contradicts the hypothesis of the theorem, and the proof is thereby completed.

It remains to prove that $dT' = 0$ on $X \setminus E$. To do this consider $\xi \in \mathcal{D}^{1}(\mathcal{F})$. Using a partition of unity, we can write $\xi$ as a finite sum of 1-form whose support is contained a regular flow box. Therefore, we may assume that the support of $\xi$ is contained in a regular flow box $U$ with foliated chart $\Phi : U \to \mathcal{B} \times \Sigma$. Let $\gamma$ be the volume form on $\Sigma$ and $\varphi_U$ be the real-valued function on $U$ given in (2.24) and (2.25). For every $n \geq 1$, following (6.4) we consider the function $\varphi_{n,U} : U \to \mathbb{R}$ given by $\varphi_{n,U} := \psi_n + \varphi_U$ on $U$. The measures $\mu_n$ given in (6.3) can be rewritten as

$$
\mu_n = g_{X,n}^\perp \wedge g_P,
$$

where using (6.5), $g_{X,n}^\perp$ is a directed $(1, 1)$-current on $\mathcal{F}$ which can be described in the flow box $\mathcal{B} \times \Sigma \simeq U$ by

$$
(\Phi_* g_{X,n}^\perp)(\zeta, \alpha) = \exp (-2(\varphi_{n,U} \circ \Phi^{-1})(\zeta, \alpha)) \gamma(\alpha) \quad \text{for} \quad (\zeta, \alpha) \in \mathcal{B} \times \Sigma.
$$
Applying Stokes’ theorem on $\mathbb{B} \times \Sigma$ and on each plaque of $\mathbb{B} \times \{\alpha\}$ with $\alpha \in \Sigma$ and using (2.27), we get that

$$\langle d\xi, g_{X,n}^\perp \rangle = \int_{\mathbb{B} \times \Sigma} \Phi_*(d\xi) \wedge \exp (-2\varphi_n \circ \Phi^{-1}) \gamma$$

$$= \int_{\mathbb{B} \times \Sigma} \Phi_* \xi \wedge d\left( \exp (-2\varphi_n \circ \Phi^{-1}) \gamma \right)$$

$$= \int_{\Sigma} \left( \int_{\mathbb{B}} \Phi_* \xi \wedge d\left( \exp (-2\varphi_n \circ \Phi^{-1}) \gamma \right) \right)$$

$$= -2 \int_{\Sigma} \left( \int_{\mathbb{B}} \Phi_* \xi \wedge (d\varphi_n \circ \Phi^{-1}) \cdot \left( \exp (-2\varphi_n \circ \Phi^{-1}) \gamma \right) \right)$$

$$= -2 \int_X (\xi \wedge d\varphi_n) \wedge g_{X,n}^\perp.$$ 

Hence, by Cauchy–Schwarz’s inequality

$$\left| \langle d\xi, g_{X,n}^\perp \rangle \right| = 2 \int_X (\xi \wedge d\varphi_n) \wedge g_{X,n}^\perp \leq c \|\xi\|_{C^0} \int_X |d\varphi_n|_p \, d\mu_n \leq c \|\xi\|_{C^0} \left( \int_X |d\varphi_n|^2 \, d\mu_n \right)^{1/2},$$

(6.11)

where $c > 0$ is a constant depending only on $\mathbb{U}$, and $\|\xi\|_{C^0}$ is the sup-norm of $\xi$ (see Sect. 2.2). Here we recall from Lemma 2.5 (1) that on $\mathbb{U}$ the leafwise Poincaré metric $g_P$ is equivalent to the restriction of the ambient metric $g_X$ on plaques of $\mathbb{U}$. Moreover, by Remark 6.5, $|d\varphi_n|_P, \Delta_p \varphi_n, \Delta \varphi$ are independent of $\mathbb{U}$, and are well-defined on the whole $X \setminus E$. So we will omit the index $\mathbb{U}$ and rewrite them simply as $|d\varphi_n|_P, \Delta_p \varphi_n, \Delta \varphi$ respectively.

By Lemma 6.7,

$$\int_X |d\varphi_n|^2 \, d\mu_n = \int_X \Delta_p \varphi_n \, d\mu_n.$$ (6.12)

On the other hand, recall from (2.34) and Remark 2.7 that $\kappa(x) = -\Delta_p \varphi(x) = -\Delta_p \varphi(x)$ for $x \in X \setminus E$. Therefore, by Lemma 6.2, there is a sequence of $(\epsilon_n) \subset \mathbb{R}^+$ such that $\epsilon_n \downarrow 0$ and

$$\frac{-\Delta_p \varphi(x) - \Delta_p \psi_n(x)}{W_*(x)} \geq -\epsilon_n \quad \text{for} \quad x \in X \setminus E.$$

So

$$\int_X \Delta_p (-\varphi_n) \, d\mu_n = \int_X W_*(x) \frac{\Delta_p (-\varphi - \psi_n)}{W_*(x)} \, d\mu_n \geq -\int_X \epsilon_n W_*(x) \, d\mu_n.$$
Applying Lemma 6.4 (2) to the last line yields that

$$\limsup_{n \to \infty} \int_X \Delta_P \varphi_n d\mu_n \leq 0.$$ 

This, combined with (6.12), implies that

$$\limsup_{n \to \infty} \int_X |d\varphi_n^2|_P d\mu_n \leq 0.$$ 

Hence, $$\lim_{n \to \infty} \int_X |d\varphi_n^2|_P d\mu_n = 0.$$ Inserting this into the right-hand side of (6.11), we get that $$\lim_{n \to \infty} \langle d\xi, g_{\perp X} \rangle = 0.$$ Hence, $$\langle dT', \xi \rangle = 0.$$ Thus, $$T'$$ is closed on $$X \setminus E$$ as claimed. 

**Remark 6.8** It is of interest to know whether Theorems A and B still hold if $$X$$ is merely a compact Kähler surface. This is the case if we can relax the projectivity assumption in [38, Theorem 1.1]. The reader may find in [20, 26, 37, 40] some other open questions in the ergodic theory of singular holomorphic foliations.

We can also investigate Theorems A and B when the singularities of $$\mathcal{F}$$ are not necessarily hyperbolic. The reader may consult the recent articles by Chen [13] and Bacher [1] for recent results for non-hyperbolic singularities.

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