Balanced Folding Over a Polygon and Euler Numbers

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Abstract

In this paper we introduced a new folding over a polygon we called it balanced folding, then we proved that for a balanced folding of a simply connected surface $M$ there is a subgroup of the group of all homeomorphisms of $M$ that acts transitively on the 2-cells of $M$. Also we explored the relationship between balanced folding and covering spaces. Finally we obtained a general relation of the Euler number of surfaces which may balance folded over a polygon and we also listed all the possibilities if $M$ is a sphere balanced folded over a triangle and we gave the subgroup mentioned above in each case.

Keywords: Surface; Cellular folding; Singularities; Cayley color graph; 1-Transitive; Universal covering; Euler number

Introduction

Let $K$ and $L$ be cellular complexes and $f : |K| \rightarrow |L|$ a continuous map. Then $f : K \rightarrow L$ is a cellular map if

(i) for each cell $e \in K$, $f(e)$ is a cell in $L$,

(ii) $\dim (f(e)) \leq \dim(e), \ [1].$

A cellular map $f : K \rightarrow L$ is a cellular folding iff

(i) for each i-cell $\sigma \in K$, $f(\sigma)$ is an i-cell in $L$, i.e., $f$ maps i-cells to i-cells,

(ii) if $\sigma$ contains $n$ vertices, then $f(\sigma)$ must contains $n$ distinct vertices, \ [2].

A cellular folding $f : K \rightarrow L$ is neat if $L^0 : L^n-1$ consists of a single n-cell, interior $L$. The set of all cellular folding of $K$ into $L$ is denoted by $C(K, L)$ and the set of all neat foldings of $K$ into $L$ by $N(K, L)$.

If $f \in C(K, L)$, then $x \mapsto f(x)$ is said to be a singularities of $f$ iff $f$ is not a local homeomorphism at $x$. The set of all singularities of $f$ corresponds to the "folds" of the map. This set associates a cell decomposition $C_{\sigma}$ of $M$. If $M$ is a surface, then the edges and vertices of $C_{\sigma}$ form a graph $\Gamma_{\sigma}$ embedded in $M$ [3].

Now there is a graph $K_{\sigma}$ associated to $C_{\sigma}$ in a natural way. In fact the vertices of $K_{\sigma}$ are just the n-cells of $C_{\sigma}$, and its edges are the (n-1)-cells. If $e \in C_{\sigma}$ is (n-1)-cell, then $e$ lies in the frontiers of exactly two n-cells $\sigma, \sigma'$. We then say that $\sigma$ is an edge in $K_{\sigma}$, with end points $\sigma, \sigma'$. The graph $K_{\sigma}$ can be realized as a graph $\tilde{K}_{\sigma}$ embedded in $M$ as follows. For each n-cell $\sigma$, choose any point $\bar{\sigma} \in \sigma$. If the n-cells $\sigma, \sigma'$ are end points of $e$, then we can join $\bar{\sigma}$ to $\bar{\sigma}'$ by an arc $\bar{\epsilon}$ in $M$ that runs from $\bar{\sigma}$ through $\sigma$ and $\sigma'$ to $\bar{\sigma}'$, crossing $e$ transversely at a single point. Trivially, the correspondence $\sigma \rightarrow \bar{\sigma}$, $e \rightarrow \bar{\epsilon}$ is a graph isomorphism. Figure 1 illustrates this relationship in case $n=2$.

It should be noted that the graph $K_{\sigma}$ may have more than one edge joining a given pair of vertices. For instance, consider the cellular folding $f$ of the torus into itself with the cellular subdivision shown in Figure 2. The graph $K_{\sigma}$ has just two vertices but two edges, see Figure 2.

Balanced Folding

Definition 1: Let $M$ be a compact connected surface, and $P_n$ a cell complex having $n$ 0-cells, $n$ 1-cells and just one 2-cell. Again a continuous map $f : M \rightarrow P_n$ is called neat folding if there is a cell decomposition $C_{\sigma}$ of $M$ such that:

(i) $f$ is a cellular map of $C_{\sigma}$ onto $C(P_n)$.

(ii) for each closed cell $\sigma$ of $C_{\sigma}$, $f(\sigma)$ is a homeomorphism of $\sigma$ onto a closed cell of $C(P_n)$.

To avoid trivial cases, we require that each 0-cell of $M$ is an end point of more than two 1-cells. Thus the 0-cells and 1-cells of this decomposition form a finite graph $\Gamma_{\sigma}$ without loops (but possibly with multiple edges) and $f$ folds $M$ along the edges or 1-cells of $\Gamma_{\sigma}$ [4].

Let $f : M \rightarrow P_n$ be a neat folding. Then for any n-cells $A$ and $B$ there is a homeomorphism $f_{AB} : A \rightarrow B$ given by $f_{AB}(a)=b$ iff $a=b(\sigma)$, where $a \in A$ and $b \in B$. We can always extend $f_{AB}$ to a homeomorphism, $f \circ f_{AB} : A \rightarrow B$, but there need not exist an extension to any open neighbourhoods of $A$ and $B$. The following two examples explore this fact.

Example 1: Let $M$ be a sphere, then there is a graph $f$ on $M$ by $f(\sigma)=\sigma$, $\sigma=1,2, \ldots, 9$.

Example 2: Let $M$ be a compact connected surface, and $P_n$ a cell complex having $n$ 0-cells, $n$ 1-cells and just one 2-cell. Again a continuous map $f : M \rightarrow P_n$ is called neat folding if there is a cell decomposition $C_{\sigma}$ of $M$ such that:

(i) $f$ is a cellular map of $C_{\sigma}$ onto $C(P_n)$.

(ii) for each closed cell $\sigma$ of $C_{\sigma}$, $f(\sigma)$ is a homeomorphism of $\sigma$ onto a closed cell of $C(P_n)$.

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Now, it can be checked that a homeomorphism \( f_{AB}: A \to B \), (where \( A \) and \( B \) are the 2-cells shaded in Figure 4) cannot be extended to a homeomorphism of any neighborhoods \( \overline{A}, \overline{B} \) of \( A, B \) respectively. This is because the valencies of the vertices of the 2-cell \( A \) are 12, 4, 4 while those of \( B \) are 12, 8, 4.

**Definition 2:** We will call a neat folding \( f: M \to P_n \) a balanced folding if for all 2-cells \( A, B \) and each homeomorphism \( f_{AB}: A \to B \) given by \( f_{AB}(a) = b \) iff \( f(a) = f(b) \), we can extend \( f_{AB} \) to a homeomorphism for any neighbourhoods \( \overline{A}, \overline{B} \) of \( A, B \) respectively.

We denote the set of all balanced foldings of \( M \) into \( P_n \) by \( \mathcal{B}(M, P_n) \).

**Example 3:** Let \( M \) be a sphere partitioned by the cells shown in Figure 5. The valencies of the vertices of each 2-cells are 4, 6 and 8.

A neat folding \( f \) may be defined from \( M \) to a polygon \( P_n \). The vertices are labeled in such a way that vertices with the same image under \( f \) are labeled alike.

If we considered any 2-cells \( A \) and \( B \) of \( M \) (e.g. the shaded 2-cells in Figure 5) then, it can be checked that a homeomorphism \( f_{AB}: A \to B \), (where \( A \) and \( B \) are the 2-cells shaded in Figure 5) can be extended to a homeomorphism of any neighbourhoods \( \overline{A}, \overline{B} \) of \( A, B \) respectively. This is because the vertices of the 2-cells \( A \) and \( B \) have the same valencies. It follows that \( f \) is balanced.

The Properties of the Graph \( K_f \) of Neat Folding

Let \( f \in \mathcal{N}(M, P_n) \), then \( K_f \) has the following special features.

(a) Edge coloring: Let \( e_1, e_2, \ldots, e_n \) be the 1-cells of \( P_n \), we can regard the indices \( i, i = 1, 2, \ldots, n \) “colors”. Each edge of \( K_f \) is mapped by \( f \) to one
of these. We may then give $K$, an edge-coloring by assigning to each edge $e$ of $K$, the color $i$ of its image $f(e)=e$.

(b) Sources and sinks: If $A$ and $B$ are 2-cells of $M$ with a common 1-cell in their frontiers, then $A$ and $B$ are given opposite orientations by $f$. It follows that each edge of the graph $K$, may be oriented in such a way that every vertex is either a source or a sink (where a vertex $u$ is a source if all the oriented edges with $u$ as a vertex begin at $u$, and is a sink if all the edges end at $u$), see Figure 6. For such a graph, every circuit has an even number of edges (and hence of vertices).

(c) Regularity: If $f\in \mathcal{N}(M,P)$, then so every 2-cell of $M$ is mapped homeomorphically by $f$ onto interior $P$, then the graph $K$, is regular. This follows immediately from the fact that the 1-cells in the frontier of each 2-cell is 1-1 correspondence under $f$ with those of $P$. It is also worth observing that every color $i$ occurs once in the set of colored edges at each vertex of $K$. Consequently, the valency of each vertex of $K$, is the cardinality of the set of 1-cells of $P$, that is to say, of the set of colors.

The properties of the graph $K$, we have already discussed suggest that in certain cases the graph $K$, may be a Cayley color graph. In the following we can show that this is indeed the case for a large class of balanced foldings.

Note first that, for any map $f: M \to N$, we can associate a group $G(f)$ namely the group of all homeomorphisms $h: M \to M$ such that $f\circ h=f$. In case $f\in \mathcal{N}(M,P)$, we may ask whether the induced action of $G(f)$ on the 2-cells of $M$ is transitive. In particular, we ask whether there is a subgroup $H(f)$ of $G(f)$ that acts 1-transitively on the set of 2-cells.

The Action of the Group of Homeomorphisms on the 2-Cells

**Theorem 1:** Let $M$ be a simply connected surface, $f: M \to P$ be a balanced folding then there is a subgroup $H(f)$ of $G(f)$ that acts 1-transitively on the set of 2-cells of $M$. Moreover $K$, is a Cayley color graph of the group $H(f)$.

**Proof:** Let $f\in \mathcal{B}(M,P)$. Let $A,B$ be 2-cells of the cell decomposition of $M$ associated by $f$. Then $f_{AB} : A \to B$ extends to a homeomorphism $f_{\tilde{A}B} : \tilde{A} \to \tilde{B}$, where $f_{\tilde{A}B}$ extends to a homeomorphism $f_{\tilde{A}B} : \tilde{A} \to \tilde{B}$, such that $f_{\tilde{A}B}$ agrees on $A \cap \tilde{B}$. Iterate this procedure to extend $f_{\tilde{A}B}$ to a map $f_{\tilde{A}B} : M \to M$.

The existence and uniqueness of this extension are guaranteed by the fact that $M$ is 1-connected.

Now, to prove that $f_{\tilde{A}B}$ is onto, let $y \in M$ a non-singular point.

Then $y$ belongs to a 2-cell $\sigma$. Let $B_{1}, B_{2},..., B_{k}=\sigma$, be a sequence of 2-cells such that $B_{i}, B_{i+1}$ are contiguous, $i=1,2,...,k$. The sequence $B_{1}, B_{2},..., B_{k}$ of 2-cells is the image under $f_{\tilde{A}B}$ of a unique sequence $A_{1}, A_{2},..., A_{k}=\sigma$ of 2-cells such that $A_{i}, A_{i+1}$ are contiguous, $i=1,2,...,k$.

Then there are open neighborhoods $\tilde{C}$ and $\tilde{D}$ of $C$ and $D$ such that $f_{\tilde{A}B}$ extends to a homeomorphism $f_{\tilde{A}B} : \tilde{C} \to \tilde{D}$, where $f_{\tilde{A}B}$ and $f_{\tilde{A}B}$ agree on $\tilde{A} \cap \tilde{B}$. Iterate this procedure to extend $f_{\tilde{A}B}$ to a map $f_{\tilde{A}B} : M \to M$.

**Example 4:** Let $M = S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$, be the unit sphere in the Euclidean 3-space. Let $f: M \to M$, be given by $f(x,y,z) = (x,z,y)$, i.e. a rotational by $\pi/2$ about the $z$-axis. $f$ is a local homeomorphism of the Euclidean 3-space. Let $\tilde{A} = (0,0,1)$, $\tilde{B} = (0,1,0)$, $\tilde{C} = (1,0,0)$, $\tilde{D} = (0,0,-1)$. Then $f_\tilde{A}B$ is a local homeomorphism of the Euclidean 3-space.

We now explore the relationship between balanced foldings and the covering maps.

**Theorem 2:** Let $f: M \to N$, be such that $f: M \to N$ is balanced. Then the valencies of the vertices are invariant under any of the extended homeomorphisms $f_{\tilde{A}B}$. If $f_{\tilde{A}B}$ is balanced, then the valencies of the vertices are invariant under any of the extended homeomorphisms $f_{\tilde{A}B}$. In particular, if $f\in \mathcal{N}(M,P)$, then $f\in \mathcal{B}(M,P)$.

Moreover, if $M$ is simply connected, then $H(f)$ will act 1-transitively on the set of 2-cells of $M$ and $K$, is a Cayley color graph of the group $H(f)$.
element of $H(\tilde{f})$ such that $\tilde{h}(\lambda) = \tilde{b}$ and let $h = \theta(\tilde{h})$. Then $h(\Lambda) = B$, and there is only one such element of $H(f)$.

It should be noted that if $p : M \to M$ is a covering map, and $\tilde{f} = fop$, where $f \in \text{N}(M, P)$, then $\tilde{f} \in \beta(M, P)$ implies that $f \in \beta(M, P)$.

**Example 5:** Let $M = P_n(R)$ be the real projective, n-space, and let $P_n$ be the n-polygon $|t \in R^{n+1} : \sum_{i=1}^{n+1} t_i = 1, 0 \leq t_i \leq 1|$. Define $f : M \to P_n$ by $f(t(x)) = [x_1, \ldots, x_{n+1}] / |x|$. Then $\tilde{M}$ may be identified with $S^n$, and $p : M \to M$ is given by $p(x) = [x]$. In this case $G(p) = Z_n$ is generated by the reflexions $g : R^{n+1} \to R^{n+1}, g(x_1, \ldots, x_{n+1}) = (x_{n+2}, \ldots, x_1)$ and $\tilde{f}(x) = (fop)(x) = ([x_1, \ldots, x_{n+1}] / |x|)$ as above.

**Theorem 3:** Let $\tilde{f}$ and $f$ be as in Theorem 1 such that $G(p) \triangleleft H(\tilde{f})$. Let $\gamma : L \to M$ be a regular covering. Then $H(\gamma)$, where $g = \gamma o f$, acts 1-transitively on the set of 2-cells of $L$ by $g$.

Proof: Since $M$ is simply connected, for any other covering map $\psi : L \to M$ there exists a universal covering map $h : M \to L$ such that $h o f = \psi$ (Figure 8).

Now $G(p) \cong \Pi_1(M)$ and $G(h) \cong \Pi_1(L)$. Since $\gamma : L \to M$ is regular $\gamma \Pi_1(L, x) < \Pi_1(M, x)$, where $\gamma(y) = x$. There is isomorphism $\Phi : G(p) \to \Pi_1(M)$ and $\Psi : G(h) \to \Pi_1(L)$ such that following diagram is commutative (Figure 9).

It follows from elementary group theory that, since $\Pi_1(L)$ is embedded in $\Pi_1(M)$ as a normal subgroup, then $G(h)$ is embedded by $\alpha$ in $G(p)$ as a normal subgroup. But $G(p) \triangleleft H(\tilde{f})$ by assumption. Hence $G(h) \triangleleft H(\tilde{f})$ and Theorem 2 can be applied for $g$, yielding that $G(g) = H(\tilde{f}) / G(h)$ acts 1-transitively on the set of 2-cells of $L$ by $g$.

**Euler Numbers of Balanced Folding onto a Polygon**

**General considerations**

Let $f \in \text{N}(M, N)$, where $M$ and $N$ are surfaces. To avoid too many complications, let us suppose that $M$ is compact, connected and without boundary, and let $N$ be connected.

Since $M$ is compact the graph $\Gamma_f$ is a finite graph. Let $\Gamma_f$ divides $M$ into $k$ 2-cells, or faces, $A_1, A_2, \ldots, A_k$. In this case $\sum |A_i| = 1, \ldots, k$ is a homeomorphism onto the interior of $N$.

We can triangulate $N$ by a simplicial complex $T_N$ such that every vertex of the cell decomposition $C$ of $\partial N$ is a vertex of $T_N$. Let $T_M$ be the triangulation of $M$ induced by $f$.

Consider the faces $A_1, \ldots, A_k$ and their closures $B_1, \ldots, B_k$. Thus $e(B_i) = e(N)$, $i = 1, \ldots, k$, where $e(X)$ is the Euler number of $X$. If we now calculate the Euler number $e(M)$ of $M$, using the triangulation $T_M$, then we can compare $e(M)$ with $\sum_{i=1}^{k} e(B_i) = k e(N)$. We note that for each vertex of $\Gamma_f$ with valency $\nu$ exactly $\nu$ vertices have been counted in the calculation of the Euler number $\nu e(N)$ of the disjoint union of $B_1, \ldots, B_k$. Likewise, every edge of $\Gamma_f$ appears twice in these calculations. Figure 10 which shows the neighborhood of a vertex with valency 4, may help to clarify these remarks.

Thus to obtain $e(M)$ from $\sum_{i=1}^{k} e(B_i)$ we must subtract $(\nu - 1)$ for each vertex of $\Gamma_f$ (of valency $\nu$) and add the number of edges of $\Gamma_f$. The first of these is $V - nk$, where $V$ is the number of vertices of $\Gamma_f$ and $n$ is the number of vertices of $\partial N$. The second is equal to $\frac{nk}{2}$. We conclude that:

$$e(M) = ke(N) + V - \frac{nk}{2}$$

(1)

The case in which $N$ is the disc $D^2$, $e(N) = 1$ and each 2-cell $A$ of $M$ is homeomorphic to $D^2$. Thus equation (1) now reduces to

$$2e(M) = k(2 - n) + 2V$$

(2)
Balanced folding over a polygon

Equations (1) and (2) can be refined slightly when $f$ is balanced. In this case, if we label the vertices of the polygon $P_n$ as $V_1, \ldots, V_n$, then each vertex in the set $f^{-1}(V_j)$ has the same valency $2q_j$, $j=1, \ldots, n$.

It follows that $f^{-1}(V_j)$ contains $\frac{k}{2q_j}$ elements. Thus the number of vertices of $\Gamma_f$ is:

$$V = \frac{k}{2} \sum_{j=1}^{n} \frac{1}{q_j}$$  \hspace{1cm} (3)

Hence for a balanced folding over a disc, equation (3) may be reduced to

$$2e(M) = k \left(2 - n\right) + k \sum_{j=1}^{n} \frac{1}{q_j} = k \left(2 - n\right) + \frac{k}{q_1} + \frac{k}{q_2} + \ldots$$

$$= \frac{k}{q_1} + \frac{k}{q_2} + \ldots + 1 \geq 1$$  \hspace{1cm} (4)

Certain cases of equation (4) are of special interest. For instance, let $n=3$, so that $M$ is triangulated by $\Gamma_f$, and equation (4) becomes

$$2e(M) = k \left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} - 1\right)$$

Thus if $M$ is a sphere, then $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} > 1$ and $k \geq 4$. The only possibilities are listed in the following Table 1:

| $q_1$ | $q_2$ | $q_3$ | $k$ | $H(f)$ |
|-------|-------|-------|-----|--------|
| 2     | 2     | $p, p \geq 1$ | 4$p$ | $D_{2p}$ |
| 2     | 3     | 3     | 24  | $O$    |
| 2     | 3     | 4     | 48  | $O$    |
| 2     | 3     | 5     | 120 | $I$    |

Table 1: Possibilities list.

Figure 11: Triangulation.

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