Abstract.
We describe applications of (perturbed) conformal field theories to two-dimensional disordered systems. We present various methods of study:

(i) A direct method in which we compute the explicit disorder dependence of the correlation functions for any sample of the disorder. This method seems to be specific to two dimensions. The examples we use are disordered versions of the Abelian and non-Abelian WZW models. We show that the disordered WZW model over the Lie group \( G \) at level \( k \) is equivalent at large impurity density to the product of the WZW model over the coset space \( G^C/G \) at level \((-2h^\text{v})\) times an arbitrary number of copies of the original WZW model.

(ii) The supersymmetric method is introduced using the random bond Ising model and the random Dirac theory as examples. In particular, we show that the relevant algebra is the affine \( OSp(2N|2N) \) Lie superalgebra, an algebra with zero superdimension.

(iii) The replica method is introduced using the random phase sine-Gordon model as example. We describe particularities of its renormalization group flow.

(iv) A variationnal approach is also presented using the random phase sine-Gordon model as example.

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1 Introduction.

These lectures will probably look too elementary to experts in disordered systems. It will perhaps be more appropriate as an introduction to this field for conformal field theorists.

The plan of these notes is the following:

1-Introduction. We introduced the types of problems we will be concerned with. We mention a few open problems; the most challenging being the description of the quantum Hall transition.

2-A few basic techniques. In this section we have gathered the basic tools which are usually used. This includes the supersymmetric method and the replica method.

3-The direct method. There, we show that due to specific properties of two-dimensional field theories, it is (sometimes) possible to compute directly the disorder dependence of the correlation functions and then to average over the disorder. Examples are provided by WZW models coupled to random vector potentials, a particular case being the massless Dirac model coupled to a random vector potential. We discuss the Abelian and non-Abelian cases and show in both cases that the disorder can be factorized. At large impurity density, the disordered WZW model is conformally invariant being equivalent to the product of the WZW model over the non-compact space $G^C/G$ times an arbitrary number of copies of the pure version of the original WZW model.

4-The supersymmetric method. We take the random Ising model as the simplest example for illustrating the supersymmetric method. We show that it can be formulated as a Gross-Neveu type model but over the Lie superalgebra $OSp(2N|2N)$, and we use this formulation to recover its properties usually obtained by the replica method. We discuss its connection with the random Dirac theory: a model which has been introduced for describing the quantum Hall transition. These examples illustrate the fact that non-unitary conformal field theories based on affine Lie superalgebras with zero superdimension provide candidates for critical theories for gaussian disordered systems.

5-The replica method. We take the random phase sine-Gordon model as example. Its large distance behavior in the low temperature phase is still controversial. We present the approach based on the renormalization group and the symmetric replica trick. We also discuss a large $N$ version of this model.

6-The variational method. We again use the random phase sine-Gordon model to illustrate this method. This approach consists in finding a good variational ansatz at fixed disorder and then averaging. It provides exact lower bound to the free energy.

Besides stochastic differential equations, cf. Polyakov’s lecture in this volume, there are at least two classes of disordered problems which we can study.

i) The first ones concern statistical systems with random coupling constants, cf eg. [1]. Imagine having a physical system with two kinds of degree of freedom with two very different relaxation times. The degrees of freedom, called spin variables, which have the shortest relaxation time will thermalize well before the other degrees of freedom, which we call impurities. The impurities are also called quenched variables because they are not in thermal equilibrium with the spin variables. We consider a large number of such physical systems with different realizations of impurity configurations. Since the spin variables are in thermal equilibrium but not the impurities we are interested in the average over the impurity configurations of the free energy of the thermalized spin systems.

A classical example is provided by the Ising model with random bond coupling constants. The thermalized spin variables $\sigma_i$, defined on the vertices of a lattice, take two values $\sigma_i = \pm$. 

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The effect of the impurities is represented by the randomness of the interaction $J_{ij}$ which couple neighbourhood spins. At fixed disorder, its partition function is:

$$Z[J] = \sum_{\{\sigma_i\}} \exp \left( - \sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j \right)$$

A probability distribution $P[J]$ is given to the random coupling constants. The average thermal properties will then be given by the average of the logarithm of the partition function: $\log Z[J]$.

In the continuum limit and close to the critical point, i.e. in its scaling limit, it is known from prehistory that the Ising model is described by a massive real Majorana fermion with mass $m \sim \tau = \frac{(T_C - T)}{T_C}$ with $T_C$ the critical temperature. In presence of randomness in the bond interaction, the mass becomes a random function of the space position. Therefore, at fixed disorder, the action describing the random bond Ising model in its continuous limit is:

$$S[m(x)] = \int \frac{d^2x}{4\pi} \left( \overline{\Psi} i \partial \Psi + m(x) \overline{\Psi} \Psi \right)$$

with $\Psi$ a Majorana fermion. The random mass $m(x)$ is coupled to the energy operator as it should be in order to represent the randomness of the bond interaction. This action is of the form:

$$S[g(x)] = S_* + \int d^2x \ g(x) \Phi(x) \ (1)$$

where $S_*$ is the action at the renormalization group fixed point describing the pure system, and $\Phi(x)$ a scalar field. As formulated in eq.(1), the problem can also be interpreted as a random field problem.

The Ising model with random bond interactions provides one of the simplest examples of disordered system in 2d [2, 3, 4]. We will study it in details in the following (using the supersymmetric method). Besides the disordered WZW models, we will also study another example (using the replica method): the random phase sine-Gordon model. It has been introduced for describing a large variety of two dimensional disordered physical systems [5, 6, 7, 8]. Contrary to the random bond Ising model which is now well understood, there is up to now no consensus on the large distance behavior of the random phase sine-Gordon model. We will present the various suggestions which have been made concerning this behavior.

ii) The second class of problems deals with particles moving in random potential, cf eq. [9]. The random potential then represents the impurities that the particles, e.g. the electrons, encountered while travelling in a metal. The property of the medium will be characterized by the behavior of the electrons in this disordered surrounding. For example, its conductivity properties will depend on whether the averaged wave functions are localized or not. These problems are generically called localization problems.

We are thus interested in the averaged properties of Hamiltonians of the form:

$$H = H_0 + V$$

where $H_0$ is a fixed Hamiltonian, and $V$ a random perturbation. A standard example is: $H_0$ a pure kinetic term, $H_0 = -\partial_x^2$ and $V = V(x)$ a potential representing the interaction with the underlying medium.
Quantities of interest would be e.g. the averaged density of states which is given by the average of the trace of the Green function:

$$\rho(E) = \frac{1}{\pi} \text{Im} \, tr \left( \frac{1}{H - E - i\epsilon} \right)$$

(2)

More generally, we are interested in computing averages of products of Green functions. Indeed, computing the conductivity tensor $\sigma_{\mu\nu}$ using the Kubo formula requires computing the average of the product of an advanced and a retarded Green function.

Let me introduce a recent localization/delocalization problem (which actually was the motivation for these lectures). It concerns the transition between the plateaux of the integer quantum Hall effect, cf [9, 10] and references therein. Without going into any details, we recall that the quantum Hall effect is characterized by the quantization of the transverse Hall resistance $R_{xy}$ to inverse integer values in unit of $(h/e^2)$. This integer quantization remains valid when the magnetic field varies on some domain, which are called plateaux. On these plateaux the longitudinal resistance $R_{xx}$ vanishes, but it becomes non-zero in the regime separating two plateaux. The values of the resistance $R_{xy}$ and $R_{xx}$ as a function of the magnetic field is represented in Fig.1.

![Fig.1 : The transverse and longitudinal Hall resistances $R_{xy}$ and $R_{xx}$ as a function of the magnetic field $B$.](image)

The integer quantum Hall effect (IQHE) is correctly described by considering a 2d gas of non-interacting electrons subject to a transverse magnetic field. As is well known, the energy spectrum is then given by the Landau levels, which are highly degenerate. In a pure system without any impurities, the Hall resistance $R_{xy}$ will be a step-function as a function of the Fermi level. The plateaux occur when the Fermi level of the electrons gas is between two Landau levels.

However, since the degeneracy of a landau level is field dependent, (this degeneracy is $\phi/\phi_0$ where $\phi$ is the magnetic flux through the system and $\phi_0 = h/e$ the flux quantum), the
Hall resistance in a pure system is a linear function of the magnetic field $B$. More precisely:

$$R_{xy}^{\text{pure}} = \left(\frac{h}{e^2}\right) \phi \frac{N_e}{\phi_0} = \left(\frac{h}{e^2}\right) \frac{1}{\nu}$$

where $N_e$ is the number of electrons in the system and $\nu = \frac{N_e \phi_0}{\phi}$ is called the filling factor. The explanation of the plateaux for $R_{xy}$ as a function of the magnetic field $B$ and not of the Fermi level requires considering the effect of the impurities [11, 12]. The impurity potential splits the degeneracy of the Landau levels, which now form energy bands. The states whose energy is in the border of these bands are localized while those which are in the center of the bands are delocalized. The localized states then serve as reservoir for the electrons which stabilized the Fermi level between two Landau bands while the magnetic field varies. This stabilization of the Fermi level give rise to the plateaux. It is expected that the delocalized states are present only at the center of the band; that is there is not a band of delocalized states. The presence of delocalized states is necessary for having a non-zero conductivity.

![Fig. 2: The density of states in two consecutive Landau bands.](image)

Between two plateaux the Fermi level is (almost) in the middle of a Landau band. Since the delocalized states are in the middle of this band, the transition between two plateaux of the integer quantum Hall effect is thus a transition between localized and delocalized states. (De)localized states are charactized by a (in)finite localization length. Near this transition, this localization length $\xi(E)$ behaves as follows:

$$\xi(E) \sim |E - E_c|^{-\nu}$$

with $E_c$ the band center energy. The density of states is non-singular at the transition point [13]. Various mesurements, as well as numerical simulations, gave access to the exposant $\nu \sim 2.3 \pm 0.1 \sim 7/3$ [14, 13, 13]. The behavior (3) has to be compared to the standard Anderson localization in two dimensions which predicts that any tiny amount of disorder will localize all the states.

What is the field theory describing the quantum Hall transition is still an open question, cf ref. [10] for a recent discussion. The problem consists in determining the appropriate infrared fixed point, and therefore seems to be a “simple” exercise in conformal field theory. Unfortunately, this infrared fixed point is a strong coupling problem in all the models introduced so far; it is therefore difficult to attack. A sigma model approach was introduced in...
ref. [16, 17]. A very elegant network model was developed in ref. [18]. A supersymmetric spin chain, which arises as an anisotropic limit of the network model, was introduced in ref. [10]. A model closely related to the random bond Ising model, namely the random Dirac model, has also been proposed for describing the IQHE localization/delocalization transition [19]. We will present it in section 4.

2 A few basic techniques.

These problems can be analyzed with similar techniques. The two standard ways of studying them use either the replica trick, cf e.g. [1] or the supersymmetric method [20]. In this section we present a very brief introduction to these methods. (The similarity of the techniques used to study disordered statistical model and localization problems is apparent at a first naive level, but more refined studies of both subjects reveal their differences.)

As it will become clear below, the supersymmetric method applies to systems which are gaussian at fixed disorder, while the replica trick has a larger domain of applicability since it is not restricted to free theory. Perturbatively the replica and supersymmetric methods are equivalent. However, a few non-perturbative results have been obtained using the supersymmetric method. Clearly, once these disordered problems have been reformulated in a field theoretical language, all the standard field theoretical methods (e.g. renormalization group, mean field approximation, variational approximation, large $N$ developments, etc...) are available.

In the following sections we will illustrate these methods on two examples: the random bond Ising model and the random phase sine-Gordon model.

• The supersymmetric method.

The supersymmetric method [20] applies to models which are gaussian at fixed disorder. We will present it in the random Hamiltonian problem. As above let us consider a random hermitian Hamiltonian

$$H = H_0 + V,$$

where $H_0$ is a fixed Hamiltonian and $V$ a random potential. We are interested in computing averages of products of advanced or retarded Green functions:

$$\frac{1}{(H_E - i\epsilon)} \frac{1}{(H_{E'} + i\epsilon)} \ldots$$

with $\epsilon \to 0^+$. We have set $H_E = H - E$. The supersymmetric method is based on a gaussian integral representation of the Green functions. E.g.:

$$\frac{1}{H_E - i\epsilon} = \frac{1}{\text{Det}[H_E - i\epsilon]} \int d\psi \exp (\pm i\psi^*(H_E - i\epsilon)\psi)$$

with $\psi$ complex Grassmannian variables. In eq. (4), we have the freedom to choose the $\pm$ sign. There are no convergence problems in this fermionic integral. To be able to average over the disorder, we need to re-exponentiate the determinant $\text{Det}[H_E - i\epsilon]$. The inverse of this determinant can be written as a gaussian integral over complex bosonic variables:

$$\frac{1}{\text{Det}[H_E - i\epsilon]} = \int d\phi \exp (-i\phi^*(H_E - i\epsilon)\phi)$$

(5)
The prefactor \((-i)\) in the exponential is chosen by requiring the convergence of the gaussian integral. Gathering these gaussian integrals, we obtain the supersymetric representation of the Green functions:

\[
\left( \frac{1}{H - i\epsilon} \right) \left( \frac{1}{H' - i\epsilon} \right) = \int d\psi d\phi \, \psi \psi^* \exp \left( -i\psi^* (H_E - i\epsilon) \psi - i\phi^* (H_E - i\epsilon) \phi \right) \tag{6}
\]

\[
\int d\psi d\phi \, \psi R \psi^* \exp \left( -i\psi^* \left( H_E - i\epsilon \right) \psi R - i\phi^* \left( H_E - i\epsilon \right) \phi \right) \tag{7}
\]

In the equation representing product of two Green functions, the indices \(A, R\) refers to the advanced or retarded Green functions. Note that due to the convergence requirement, the sign in front of the \((i\epsilon)\) factor differs in the advanced and retarded sector. More generally, products of \(N\) Green functions are represented by integrals over \(N\) couples of superpartners \((\psi, \phi)\) with appropriate choice of the sign factor according to the convergence condition.

In the form (6), the average over the disorder can be easily done. Assume for simplicity that the random potential has a gaussian distribution:

\[P[V] = \exp \left( -\frac{1}{2\sigma} tr(V^2) \right) dV\]

Consider for example the average of one Green function. The term coupled to the random potential is \(\frac{1}{2\sigma} tr(V^2) + i tr(V(\phi^* - \psi \psi^*))\). After integration over \(V\), the averaged Green functions becomes:

\[
\left( \frac{1}{H - i\epsilon} \right) \left( \frac{1}{H' + i\epsilon} \right) = \int d\psi d\phi \, \psi \psi^* \exp \left( -S_{eff} \right) \tag{7}
\]

with

\[S_{eff} = i\psi^* (H_0 - E - i\epsilon) \psi + i\phi^* (H_0 - E - i\epsilon) \phi - \frac{\sigma}{2} tr (\phi \phi^* - \psi \psi^*)^2\]

The bosonic and fermionic sectors are coupled by the disorder. This effective action is easily generalized when considering products of higher number of Green functions.

Notice that if \(H\) is a quantum hamiltonian for a physical system in \(d\) spacial dimensions, ie. a \((d + 1)\) system, than the supersymimic action describes a \(d\) dimensional quantum field theory. In particular, since the quantum Hall effect is a \((2 + 1)\) system its localization/delocalization transition would be described by a \(2d\) field theory.

\[\bullet\] The replica method.

Let us now recall briefly the starting point of the replica method, cf [1]. We are interested in computing the average of the free energy, or of correlation functions. Since the free energy is proportional to the logarithm of the partition function \(Z[J]\), we have to compute the average of \(\log Z[J]\). The replica method is based on the identity:

\[\log Z[J] = \lim_{n \to 0} \frac{Z^n[J] - 1}{n} = \frac{d}{dn} \left. Z^n[J] \right|_{n=0}\]

In other words, we compute the quenched averages of one disordered system by introducing \(n\) copies of this system, in all of which the random variables take the same values, and then computing the averaged partition function of this new system at \(n = 0\).
This method leads immediately to a criteria selecting the systems in which the disorder is relevant. This criteria is called the Harris criteria.

Let us describe a disordered system in the continuum limit by the action:

\[ S[g(x)] = S_* + \sum_a \int d^2x \, g^a(x)\Phi_a(x) \]  

(9)

where \( S_* \) is the action at the renormalization group fixed point describing the pure system at its critical point, and \( \Phi_a(x) \) are scalar fields. When the coupling constant \( g^a(x) \) are independent of \( x \) and not random, the action \( S[g(x)] \) describes the behavior of the statistical model near the critical point in its scaling limit. In this case the field \( \Phi_a(x) \) is relevant, and therefore will influence the large distance behavior of the pure system, if its scaling dimension is less than two: \( \text{dim}(\Phi_a) < 2 \). This is changed when disorder is present.

Assume for simplicity that the random variables \( g^a(x) \) are gaussian variables with one and two-point functions:

\[ \overline{g^a(x)} = 0 \quad \text{and} \quad \overline{g^a(x)g^b(y)} = \delta^{ab}\sigma_a \delta(x-y) \]  

(10)

Before averaging over the disorder, the partition function is

\[ Z[g(x)] = \int DX \, \exp(-S[g(x)]) \]

and the \( n \) replicated one becomes:

\[ Z^n[g(x)] = \int \prod_r DX_r \, \exp \left( -\sum_r S_r^* - \sum_a \int d^2x \, g^a(x) \sum_r \Phi_a^r(x) \right) \]

where the index \( r \) runs from 1 to \( n \), \( S_r^* \) is the action for the \( r \)-th replic and \( \Phi_a^r(x) \) is the field \( \Phi_a(x) \) in the \( r \)-th copy. To have introduced the replicated copies allows us to easily average over the disorder. Assuming the gaussian measure \( \overline{[g(x)]} \) for \( g^a(x) \), we obtain:

\[ \overline{Z^n[g(x)]} = \int \prod_r DX_r \, \exp(-S_{\text{eff}}) \]  

(11)

with

\[ S_{\text{eff}} = \sum_r S_r^* + \frac{1}{2} \sum_a \sigma_a \int d^2x \, \sum_r \Phi_a^r(x)\Phi_a^s(x) \]  

(12)

The replica are now coupled by the disorder. The field perturbing the uncorrelated action \( \sum_r S_r^* \) is quadratic in terms of the \( \Phi_a(x) \)'s. It will be relevant if \( 2 \text{dim}(\Phi_a(x)) < 2 \). (A more careful study is needed to analyse the short distance singularity in \( \Phi_a^r(x)\Phi_a^s(x) \) when \( r = s \). Hence:

\[ \text{dim } \Phi_a(x) < 1 \Rightarrow \text{relevant disorder.} \]

This is the Harris criteria in two dimensions. Note that for a field to be relevant in a quenched system its dimension has to be half what is required in the pure system. If these dimensions depend on parameters the Harris criteria provides a first glance to the phase diagram.

The quenched correlation functions are defined by:

\[ \left\langle \mathcal{O}(x) \cdots \right\rangle \left\langle \cdots \right\rangle = \left( \frac{1}{Z[g(x)]} \int DX \, e^{-S[g(x)]} \, \mathcal{O}(x) \cdots \right) \cdots \]
The replica method can also be used to compute the averaged correlation functions. It provides a way to re-exponentiate the inverse partition function. The rules are the following:

\[
\langle \mathcal{O}(x) \rangle = \langle \mathcal{O}(x) \rangle_{n=0}
\]
\[
\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \langle \mathcal{O}(x) \mathcal{O}(y) \rangle_{n=0}
\]
\[
\langle \mathcal{O}(x) \rangle \langle \mathcal{O}(y) \rangle = \langle \mathcal{O}(x) \mathcal{O}(y) \rangle_{n=0}, \ r \neq s
\]

(... etc ....)

In particular, the average of the connected Green functions are represented by the correlation function of \( \frac{1}{n} \sum_{r} \mathcal{O}(x) \). The simplest way to prove eq.(13) consists in introducing sources in the quenched partition function.

3 The direct method : the random vector potential model.

As a warm up we introduce a very simple gaussian model, which evidently can be completely solved. (We will later use a few of its properties in our study of the more interesting random phase sine-Gordon model.)

In two dimensions it is sometimes possible to compute exactly field theory partition functions and correlation functions with sources. In this section, we also described how this property can be applied to very particular disordered 2d systems. The example we choose consists in a non-Abelian generalization of the gaussian model based on the WZW models. It provides one of the rare disordered models which can be solved directly without relying on the supersymmetric or replica method. This solution relies on the fact that the disorder dependence can be factorized.

• The gaussian model.

The bosonic form of its action is defined to be:

\[
S_0[\Phi, A] = \int \frac{d^2x}{4\pi} \left( \frac{1}{2} (\partial_\mu \Phi)^2 + i A_\mu \epsilon_{\mu\nu} \partial_\nu \Phi \right),
\]
\[
= \int \frac{d^2x}{\pi} \left( \frac{1}{2} \partial_z \Phi \partial_\zeta \Phi + i (A_\zeta \partial_\Phi + A_\Phi \partial_\zeta) \right).
\]

(14)

where \( A_\mu \) is a random quenched variable with measure:

\[
P[A] = \exp \left[ -\frac{1}{2\sigma} \int \frac{d^2x}{4\pi} A_\mu A_\mu \right] = \exp \left[ -\frac{1}{\sigma} \int \frac{d^2x}{\pi} A_\zeta A_\zeta \right]
\]

(15)

In eq.(14) we introduced the complex coordinate \( z = x + iy, \zeta = x - iy \) on the plane.

Since it is a gaussian model, it can be easily solved without using the replica trick or the supersymmetric formulation, but by directly computing the quenched averages. Let us introduce the Hodge decomposition of \( A_\mu : A_\mu = \partial_\mu \xi + \epsilon_{\mu\nu} \partial_\nu \eta \). The fields \( \xi \) and \( \eta \) decouple in the measure (15) :

\[
P[A] = P[\xi; \eta] = \exp \left[ -\frac{1}{2\sigma} \int \frac{d^2x}{4\pi} \left( (\partial_\mu \xi)^2 + (\partial_\mu \eta)^2 \right) \right]
\]

(16)
The action $S_0[\Phi, A]$ is independent of $\xi$ and therefore the field $\xi$ is irrelevant. This fact was expected since the field $\xi$ represents a pure gauge whereas the physically relevant quantity is the field strength $F = \epsilon_{\mu\nu} \partial_\mu A_\nu = (\partial_\mu \partial_\mu) \eta$. Moreover, the field $\eta$ can be absorbed into a translation of $\Phi$:

$$S_0[\Phi, A_\mu = \epsilon_{\mu\nu} \partial_\nu \eta] = S_0[\Phi + i \eta, A = 0] + \int \frac{d^2 x}{4\pi} \frac{1}{2} (\partial_\mu \eta)^2.$$  \hfill (17)

The fact that the field $\eta$ can be absorbed into a shift of $\Phi$ does not mean that the quenched correlation functions are identical to those in the pure system. Using eq.(17), we have

$$\left\langle \prod_n e^{i \alpha_n \Phi(x_n)} \right\rangle_{A_\mu = \epsilon_{\mu\nu} \partial_\nu \eta} = e^{i \sum_n \alpha_n \eta(x_n)} \left\langle \prod_n e^{i \alpha_n \Phi(x_n)} \right\rangle_{A = 0}.$$

It factorizes into the product of the correlation functions in the pure system times a simple function of the impurities. However, the average of this function is not irrelevant since the variables $\eta$ have long-range correlations: $\eta(x) \eta(y) = -\pi \sigma (\partial_x \partial_y)^{-1}(x, y) = -\sigma \log |x - y|^2$. In particular, it changes the values of the critical exponents.

Averages of products of correlation functions can then be computed. One has:

$$\left\langle \prod_n e^{i \alpha_n \Phi(x_n)} \right\rangle_A \cdots \left\langle \prod_m e^{i \beta_m \Phi(y_m)} \right\rangle_A = \left\langle e^{i \sum_n \alpha_n \eta(x_n) + \cdots + \sum_m \beta_m \eta(y_m)} \right\rangle \times \left\langle \prod_n e^{i \alpha_n \Phi(x_n)} \right\rangle_{A = 0} \cdots \left\langle \prod_m e^{i \beta_m \Phi(y_m)} \right\rangle_{A = 0}$$

where $\langle \cdots \rangle$ refers to the $\eta$-correlation functions with the free field measure (16). All the correlators are therefore given by gaussian integral. We get:

$$\left\langle \prod_n e^{i \alpha_n \Phi(x_n)} \right\rangle_A \cdots \left\langle \prod_m e^{i \beta_m \Phi(y_m)} \right\rangle_A = \prod_{n,m} |x_n - y_m|^{-2\sigma \alpha_n \beta_m} \times \prod_{n < n'} |x_n - x_{n'}|^{2\alpha_n \alpha_n'} \cdots \prod_{m < m'} |y_m - y_{m'}|^{2\beta_m \beta_{m'}}$$  \hfill (18)

Clearly, conformal invariance is unbroken in the random abelian case. The dimensions $\Delta$ of the vertex operators $\exp(i \alpha \Phi(x))$ in the quenched theory are:

$$\Delta_{\text{quenched}}^\alpha = \Delta_{\text{pure}}^\alpha - \sigma \alpha^2.$$  

It is interesting to look at the connected Green functions of the current $J_z = i \partial \Phi$. It turns out that all these quenched connected correlations are chiral; in particular, they are holomorphic and only depend on $z = x + iy$. For example, we have:

$$\left\langle i \partial \Phi(z) \right\rangle = 0,$$

$$\left\langle i \partial \Phi(z) i \partial \Phi(w) \right\rangle_{\text{conn.}} = 2\pi \left( \partial^{\frac{1}{2}} \right)_{z,w} = \frac{1}{(z - w)^2}$$

More generally, one verifies that:

$$\left\langle i \partial \Phi(z_1) \cdots i \partial \Phi(z_P) \right\rangle_{\text{conn.}} = \left\langle i \partial \Phi(z_1) \cdots i \partial \Phi(z_P) \right\rangle_{A = 0}. \hfill (19)$$

In other words, the averages of these connected correlation functions are unaffected by the disorder. This property is not true for the average of disconnected correlation functions of $J_z = i \partial \Phi$. 

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A more complete derivation of these Ward identities which follows from a $U(1)$ symmetry is presented in the section devoted to the study of the random phase sine-Gordon model. It is an interesting exercise to decipher the operator product algebra of these models.

**The non-Abelian case: factorization.**

We now turn to the non-Abelian case which provides an exceptional quenched model which can be solved directly, without using any replica or supersymmetric method. This is done by factorizing the disorder dependence. We derive it directly from the definition of the quenched correlation functions. This result was also found in ref. [21] using the supersymmetric method, (it appears while we were preparing these notes).

The non-Abelian model appears naturally when generalizing the previous model, after fermionization, to a $N$-component model. Indeed, using standard bosonization/fermionization rules, $\bar{\psi} \gamma^\mu \psi = \epsilon^\mu_\nu \partial_\nu \phi$, and $\bar{\psi} i \bar{\phi} \psi = \frac{1}{2} (\partial_\mu \phi)^2$, we can rewrite the action (14) in an equivalent fermionic form:

$$S_0 = \int \frac{d^2 x}{\pi} \left( \bar{\psi} (i \partial + A) \psi \right).$$

It describes a Dirac fermion coupled to a random magnetic field.

The non-Abelian generalization describes $N$ massless Dirac fermions minimally coupled to a random non-abelian gauge field $\bar{\psi}_{-j}, \psi_{+j}, j = 1, \cdots , N$. The action is defined as:

$$S[A] = \int \frac{d^2 x}{\pi} \left( \bar{\psi}_{-j} \left( \partial_\mu \delta^j_k + A^j_{+k} \right) \psi^k_+ + \bar{\psi}_{-j} \left( \partial_\mu \delta^j_k + A^j_-k \right) \psi^k_- \right)$$

where $A^j_{+k} = i \sum_a A^a_z (t^a)^j_k$, with $(A^a_z)^* = A^a_z$, is the gauge field. Here the hermitian matrices $t^a$ form the $N$-dimensional representation of $SU(N)$. We denote by $f^{abc}$ the $SU(N)$ structure constants: $[t^a, t^b] = i f^{abc} t^c$. The Dirac fermions take values in this $N$-dimensional representation. The gauge field is assumed to be a quenched variable with the gaussian measure:

$$P[A] = \exp \left[ - \frac{1}{\sigma} \int \frac{d^2 x}{\pi} \sum_a A^a_z A^a_z \right].$$

At fixed disorder the partition function is:

$$Z[A] = e^{W[A]} = \int D\psi e^{-S[A]} = Det[i \bar{\psi} + A]$$

It can be expressed in terms of the WZW action [24], see below eq.(43).

The way to factorize the disorder consists in implementing a chiral gauge transformation. So we parametrized the vector potential by an element $G$ of the complexified group $SU(N)^C$ as follows:

$$A_\tau = G^{-1} (\partial_\tau G) = -A^*_z$$

This is always possible on the sphere provided the connexion $A$ is regular enough. This parametrization is unique up to left multiplication by a constant group element. Notice that the number of degree of freedom is preserved by this transformation: in the $A$ or $G$ parametrization there are $2dimSU(N)$ degree of freedom. We then have $\psi_- (\partial_{\tau} + A^-_z) \psi_+ = \psi_0 \gamma^\mu \partial_\mu \psi_1$.
\[(\psi G^{-1}) \partial_z (G \psi) + \overline{\psi} (\partial_z + A_z) \psi = (\overline{\psi} G^*) \partial_z (G^{-1} \psi) \]. We can therefore absorbed the dependence on \( A \) by transforming the fermions as:

\[
\begin{align*}
\hat{\psi}_- &= \psi G^{-1} ; \quad \hat{\overline{\psi}}_- = \overline{\psi} G^* \\
\hat{\psi}_+ &= G \psi_+ ; \quad \hat{\overline{\psi}}_+ = G^{-1} \overline{\psi}_+
\end{align*}
\] (24)

This is the chiral gauge transformation. It maps the action (20) into the free Dirac action for the gauge transformed fermions. We denote it \( S_{\text{free}}[\hat{\psi}] \). The Jacobian for the transformation \( \psi \to \hat{\psi} \) is non-trivial but given by the chiral anomaly:

\[
\left| D\psi \overline{D\hat{\psi}} \right| = \frac{\text{Det}[i\partial + /A]}{\text{Det}[i\partial]} = Z[A]
\]

The extra factor \( \text{Det}[i\partial] \) is irrelevant when computing the quenched correlation function but relevant in the evaluation of the conformal anomaly. The crucial point is that this Jacobian is equal to the partition function at fixed disorder. Therefore, the partition and the Jacobian simplifies when computing the correlation functions at fixed disorder, and we get:

\[
\langle \psi_- \cdots \rangle_A = \frac{1}{Z[A]} \int D\psi \ e^{-S[A]} \psi_- \cdots = \int D\hat{\psi} \ e^{-S_{\text{free}}[\hat{\psi}]} (\hat{\psi}_- G) \cdots
\] (25)

This is the announced factorization property: the correlation functions at fixed disorder factorize into the product of a correlation function in the free Dirac theory times an explicitly known functional of the vector potential.

We now have to average over the disorder. For this it is convenient, if not necessary, to change variable from \( A \) to \( G \). The Jacobian is known, see eq. [25]:

\[
DA = DG \ \exp \left( 2h^v S_{\text{wzw}}(GG^*) \right)
\] (26)

where \( DG \) is the Haar measure on \( SU(N)^C \), \( S_{\text{wzw}}(GG^*) \) is the WZW action and \( h^v \) is the dual Coxeter number, equals to \( N \) in the \( SU(N) \) case. Since only the product \( GG^* \) enters into the Jacobian, we decompose \( SU(N)^C \) as \( (SU(N)^C/SU(N)) \times SU(N) \). For \( G \) this means:

\[
G = H U \quad \text{with} \quad H \in SU(N)^C/SU(N), \quad U \in SU(N)
\]

The Haar measure \( DG \) factorizes on the Haar measure on \( (SU(N)^C/SU(N)) \) and \( SU(N) \). Thus, we finally obtain:

\[
DA = DUDH \ \exp \left( 2h^v S_{\text{wzw}}(HH^*) \right)
\] (27)

In eq. [27] the part related to the compact space \( SU(N) \) only involves the Haar measure, while the part related to the non-compact symmetric space \( (SU(N)^C/SU(N)) \) involves the WZW action. Although the WZW action seems to appears with a ‘wrong’ sign in eq. [27], the non-compact sigma model is well defined since the metric on \( (SU(N)^C/SU(N)) \) is negative definite.

For an arbitrary value of the disorder strength \( \sigma \) we also have to include the factor (21) into the measure for \( A \). This factor couples \( U \) and \( H \). But things simplify at \( \sigma = \infty \). For this value of the coupling constant, we have:

\[
P_{\sigma=\infty}[A] \ DA = DUDH \ \exp \left( 2h^v S_{\text{wzw}}(HH^*) \right)
\] (28)

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The compact and non-compact sectors are now independent. Moreover, the field $U$ of the compact sector completely decouples from the quenched correlation functions of the local spinless neutral operators as e.g. $(\psi_- \overline{\psi}_+)$. Indeed, we have:

$$\langle (\psi_- \overline{\psi}_+) \cdots \rangle_{\sigma=\infty} = \int DH \, D\hat{\psi} \, e^{2h^vS_{wzw}(HH^*)-S_{free}[\hat{\psi}]} \, (\overline{\psi}_- \, HH^* \, \overline{\psi}_+) \cdots$$

(29)

Notice that a necessary condition for the cancelation of the $U$-dependence in eq.(29) is to consider $SU(N)$ scalar operators. I.e. we have to sum over the color indices of the fermions $\psi_-$ and $\overline{\psi}_+$ in order to multiply $U$ and $U^*$ to get $UU^* = 1$.

The free Dirac action plus the non-compact sigma model are conformally invariant. This is clear for the free massless Dirac theory. The sigma-model on the non-compact space $(SU(N)^C/SU(N))$ defined by the WZW action was studied in ref.[25] in connexion with path integral construction of the coset models. There it is was shown that it is a non-unitary conformal field theory which carries a representation of the affine Kac-Moody algebra of negative level $\tilde{k} = -2h^v$. Therefore, the Virasoro central charge of the non-compact sector is:

$$C = \frac{(-2h^v)\dim su(N)}{(-2h^v) + h^v} = 2\dim SU(N) = 2(N^2 - 1)$$

(30)

The scaling dimension of the primary fields $\phi_R$ which belong to a representation $R$ are given as usual by:

$$\dim[\phi_R] = 2 - \frac{\text{Cas}(R)}{2((-2h^v) + h^v)} = -2 \frac{\text{Cas}(R)}{2h^v}$$

(31)

with $\text{Cas}(R)$ the Casimir operator in $R$. The negative value of this dimension is an echo of the non-unitary character of the theory.

As an application of eq.(29) we can derive the scaling dimension of the operator $(\psi_- \overline{\psi}_+)$ in the quenched correlation functions. It is the sum of the scaling dimensions of the operator $(\overline{\psi}_- \overline{\psi}_+)$ in the free Dirac theory plus the one of the operator $(HH^*)$ in the vector representation $(\underline{a})$ in the non-compact WZW model. Explicitly:

$$\dim_{\sigma=\infty}[ (\psi_- \overline{\psi}_+) ] = 1 - 2 \frac{\text{Cas}(\underline{a})}{(-2N)} = 1 - \frac{N^2 - 1}{N^2}$$

This agrees with ref.[22, 21].

In summary, we have shown that at $\sigma = \infty$ we have the following equivalence, valid in the local spinless neutral sector:

$$\overline{\text{Dirac}}_{\sigma=\infty} = (\text{Free Dirac})^N \times (SU(N)^C/SU(N))_{\tilde{k}=-2h^v}$$

with $N$ the number of correlations we are averaging. This equivalence is a direct consequence of the factorization property eq.(28,29) of the random vector potential model.

- Quenched current correlation functions.

The previous factorization property is really useful only at infinite coupling $\sigma$. Here we describe how averages of current correlation functions can be computed for any value of $\sigma$. We also show that, as in the Abelian gaussian model, the averages of the connected current correlation functions are unaffected by the disorder [23].
We are thus interested in computing the quenched average of the correlation functions of the currents \( J^a_\mu = (\Psi \gamma_\mu t^a \Psi) \). Explicitly, their components are:

\[
J^a_z = \psi_{-j}(t^a)_k^j \psi^k_+ \\
J^a_\tau = \overline{\psi}_{-j}(t^a)_k^j \overline{\psi}^k_+
\]

(32)

While these currents are conserved in the pure system, they are not conserved once the disorder has been turned on. However, as explained in ref.[23], the quenched theory still possesses a \( su(N) \) symmetry. The currents generating this symmetry are represented by the insertion of the following operators \( J^a_\mu \) in the quenched averages:

\[
J^a_\mu = \pi \frac{\delta}{\delta A^a_\mu}
\]

Although the fields \( J^a_\mu \) are not conserved, their quenched correlation functions can be computed. This relies on the Polyakov-Wiegman (PW) formula [24] for the effective action \( W[A] \) defined in eq.(22). It can be exactly computed by integrating its anomalous transformation under a chiral gauge transformation. Indeed, let \( G_\mu = \sum_n t^a G^a_n[A] \) with \( G^a_\mu[A] = \langle J^a_\mu \rangle_A = \pi \frac{\delta W[A]}{\delta A^a_\mu} \) be the generating functions of the connected current Green function in the pure system. They satisfy the anomalous Ward identities [24],

\[
\partial_\tau G_\tau + \partial_z G_z + [A_z, G_\tau] + [A_\tau, G_z] = 0, \\
\partial_\tau G_\tau - \partial_z G_z + [A_z, G_\tau] - [A_\tau, G_z] = 2 F_{\tau z}[A],
\]

(33)

with \( F_{\tau z}[A] = \partial_z A_\tau - \partial_\tau A_z + [A_z, A_\tau] \). Eqs.(33) completely specify \( G_\mu[A] \). The solution of eqs.(33) can written as:

\[
G_z[A] = A_z - \frac{1}{\partial_z + ad.A_z} \partial_z A_\tau
\]

(34)

\[
G_\tau[A] = A_\tau - \frac{1}{\partial_\tau + ad.A_\tau} \partial_\tau A_z
\]

(35)

Here \( ad.A_z \) denoted the adjoint action of \( A_z \). Notice that \( G_z[A] \) is local in \( A_z \) but non-local in \( A_\tau \).

Knowing explicitly the correlation functions for any impurity sample, it is a priori possible to take the quenched average. Let us first concentrate on the average of the chiral correlation functions involving only currents of the same chirality; e.g. involving only \( J^a_\tau \). Consider first the average of products of one-point functions. Since the quenched average is defined by \( A^a(x)A^b(y) = \sigma \pi \delta^{ab}\delta(x - y) \) and since \( \langle J^a_\mu \rangle_A \) are linear in \( A_z \), these quenched correlations can be computed using Wick’s theorem applied on \( A \). For example, the two-point and three-point functions are:

\[
\langle J^a_\tau(z_1)J^b_\tau(z_2) \rangle_A = 2\sigma \pi \delta^{ab} \left( \frac{1}{\partial_\tau} \right)_{(z_1,z_2)} = \frac{2\sigma \delta^{ab}}{(z_1 - z_2)^2}
\]

\[
\langle J^a_\tau(z_1)J^b_\tau(z_2)J^c_\tau(z_3) \rangle_A = i(\pi \sigma)^2 f^{abc} \left( \frac{1}{\partial_\tau} \right)_{(z_1,z_2)} \left[ \left( \frac{1}{\partial_\tau} \right)_{(z_1,z_3)} - \left( \frac{1}{\partial_\tau} \right)_{(z_2,z_3)} \right] + (cyclic \ permutation)
\]

\[
= 3\sigma^2 \left( \frac{1}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)} \right)
\]

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More generally, the average of products of one-point functions is the sum of connected correlations which can be expressed in terms of the correlation functions of the pure system:

$$\langle J_{z_1}^{α_1}(z_1) \cdots J_{z_M}^{α_M}(z_M) \rangle_{A}^{\text{connected}} = Mσ^{M-1} \langle J_{z_1}^{α_1}(z_1) \cdots J_{z_M}^{α_M}(z_M) \rangle_0$$

(36)

Here, $\langle \cdots \rangle_0$ denote the pure correlation functions. They are known exactly [26]. The relation between the quenched correlations and their connected parts is the usual one.

More interesting is the average of products of correlations with insertion of the conserved currents $J^a$ since they encode the underlying symmetry algebra. We find:

$$\langle J_{z_1}^{α_1}(z_1) \cdots J_{z_M}^{α_M}(z_M) \rangle_{A} = \langle J_{z_1}^{α_1}(z_1) \cdots J_{z_M}^{α_M}(z_M) \rangle_{A}^{\text{connected}} + \sum_{j=1}^{M} \langle J_{z_1}^{α_1}(z_1) \cdots J_{z_j}^{α_j}(z_j) \cdots J_{z_M}^{α_M}(z_M) \rangle_{A}^{\text{connected}}$$

$$+ \cdots + \langle J_{z_1}^{α_1}(z_1) \cdots J_{z_M}^{α_M}(z_M) \rangle_{A}^{\text{connected}}$$

(37)

The hatted fields have to be omitted. Here, we assumed that there is no insertion of one-point functions. The formula (37) is actually simpler in words: it is obtained by distributing the currents $J_{z_1}^{α_1}(z_1), \cdots, J_{z_M}^{α_M}(z_M)$ among the pure correlators in all possible way, each counted only once. Notice that all the chiral quenched correlation functions are purely algebraic, without any logarithmic correction.

Eq. (37) with no insertion of $J^a_z$ shows that the connected correlation function of the fields $J^a_z$ are unaffected by the disorder.

From eq. (37) we read the operator product expansion of the fields. The currents $J^a_z$ satisfy:

$$J_{z_1}^{α_1}(z_1)J_{z_2}^{α_2}(z_2) = \frac{δ^{α_1 α_2}}{(z_1 - z_2)^2} + \frac{iJ^{α_1 α_2}}{z_1 - z_2} J_{z_2}^{α_2}(z_2) + \text{reg.}$$

(38)

Therefore, the quenched conserved currents satisfy the commutation relations of a Kac-Moody algebra, exactly as the currents in the pure system do.

The operator product expansion between the conserved currents $J^a_z$ and the correlators $\langle J_{z_1}^{α_1}(w_1)J_{z_2}^{α_2}(w_2) \cdots \rangle_A$ are:

$$\langle J_{z_1}^{α_1}(z)J_{z_2}^{α_2}(w_1)J_{z_3}^{α_3}(w_2) \cdots \rangle_A \sim \sum_j \frac{δ^{α_j}}{(z - w_j)^2} \langle J_{z_1}^{α_1}(w_1)J_{z_2}^{α_2}(w_2) \cdots \rangle_A$$

$$+ \sum_j \frac{iJ^{α_j}}{z - w_j} \langle J_{z_1}^{α_1}(w_1)J_{z_2}^{α_2}(w_2) \cdots \rangle_A + \text{reg.}$$

(39)

These operator product expansions are unusual in conformal field theory with Kac-Moody symmetry. In particular, they imply that the fields $\langle J_{z_1}^{α_1}(w_1)J_{z_2}^{α_2}(w_2) \cdots \rangle_A$ are not associated to highest weight vector representations. We don’t know to which category of representations they correspond to.

Contrary to the chiral quenched correlation functions which are easy to compute, the averages of correlation functions involving fields of opposite chiralities are difficult, if not
impossible, to evaluate directly. This is due to the fact that the generating functions \( G_z[A] \)
and \( G_{\bar{z}}[A] \) are non-local in \( A_z \) and \( A_{\bar{z}} \) respectively. A naive perturbative expansion is spoilt
by untractable divergences. However, since the model is asymptotically free in the infrared,
one can used renormalization group techniques to evaluate few two-point functions [23].

**Generalization to any WZW model.**
The previous discussion generalizes easily to any WZW models. So let us consider a
WZW model over the Lie group \( G \) at level \( k \). Its action is [37, 26]:
\[
S_{wzw}(g) = k \int \frac{d^2x}{8\pi} tr(\partial_z g^{-1}\partial_{\bar{z}} g) + k\Gamma(g)
\]
where \( \Gamma(g) \) is the WZW topological term. We define the action of the disordered WZW
model to be the WZW action in presence of sources. Namely :
\[
S_{wzw}(g; A) = S_{wzw}(g) - k \int \frac{d^2x}{2\pi} \left( tr(A_z J_z + A_{\bar{z}} J_{\bar{z}}) + tr(g A_z g^{-1} A_{\bar{z}}) - tr(A_z A_{\bar{z}}) \right)
\]
(40)

where \( J_z, J_{\bar{z}} \) are the WZW currents and \( A_z, A_{\bar{z}} \) the random variables. We assume that the
probability distribution \( P[A] \) is given by eq.(21).

As a functional of \( g \) and \( A \) the action (40) satisfies the “Polyakov-Wiegman” relation :
\[
S_{wzw}(g; A) = S_{wzw}(g^h; A^h) - S_{wzw}(h^*h; A^h), \quad \text{with} \quad A^h_z = h A_z h^{-1} + h \partial_z h^{-1}
\]
\[
g^h = hgh^*
\]
(41)

for any element \( h \) of the complexified group \( G^C \).

If we parametrize the impurity connexion as in previous section by \( A_{\bar{z}} = G^{-1}\partial_{\bar{z}} G \) with
\( G \in G^C \); this action action has been cooked up such that :
\[
S_{wzw}(g; A) = S_{wzw}(g^G) - S_{wzw}(GG^*), \quad \text{with} \quad A_{\bar{z}} = G^{-1}\partial_{\bar{z}} G = -A_z^*
\]
(42)

This follows from eq.(41) and the fact that \( A^G = 0 \).

This has two consequences. First we can explicitly evaluate the partition function for
any sample of the disorder. Namely, we have :
\[
Z[A] = \int Dg e^{-S_{wzw}(g; A)} = \exp(S_{wzw}(GG^*))
\]
(43)

In the last equation we used the fact that for the Haar measure on \( G \) we have :
\( \int dg f(g_1 g_2 g_3) = \int dg f(g) \) for any \( g_1, g_2 \in G^C \) and any function regular enough.

The second consequence concerns the correlation functions of the \( G \)-scalar local operators
at fixed disorder. For example consider the operator
\[
\Phi_R(z, \bar{z}) = \sum_i \bar{\phi'}_{R,i}(\bar{z}) \phi^i_R(z)
\]
where \( \phi^i_R(z) \) the primary fields in the representation \( R \). These operators are invariant under
the global \( G \) symmetry. In the path integral formulation, they are represented by the insertion
of \( tr(\rho_R(g)) \), ie :
\[
\Phi_R(z, \bar{z}) \equiv tr(\rho_R(g))
\]
(44)
where $\rho_R$ denotes the representation $R$ of $G$. We then have:

$$\langle \text{tr}(\rho_R(g) \cdots) \rangle_A = \int \frac{Dg}{Z[A]} e^{-S_{wzw}(g;A)} \text{tr}(\rho_R(g)) \cdots$$

$$= \int \frac{Dg}{Z[A]} e^{-S_{wzw}(gG) + S_{wzw}(GG^*)} \text{tr}(\rho_R(g)) \cdots$$

where we have used eq.(42). Now we see from eq.(43) that the partition function $Z[A]$ cancels against $\exp(S_{wzw}(GG^*))$. Therefore, using the invariance of the Haar measure, we obtain:

$$\langle \text{tr}(\rho_R(g) \cdots) \rangle_A = \int Dg e^{-S_{wzw}(g)} \text{tr}(\rho_R(g)) \rho_R(GG^*)^{-1} \cdots \quad (45)$$

We have used the cyclicity of the trace in order to reconstitute the product $GG^*$. In other words, we have used the fact that the operator $\text{tr}(\rho_R(g))$ is $G$ invariant. This is important because it proves that the $G$-dependence of these correlation functions projects from $G \rightarrow G/C$. Indeed, let us factorized $G \in G^C$ as:

$$G = H U \quad \text{with} \quad H \in G^C/G \quad \text{and} \quad U \in G \quad (46)$$

Then since $UU^* = 1$, we have:

$$\langle \text{tr}(\rho_R(g) \cdots) \rangle_A = \int Dg e^{-S_{wzw}(g)} \text{tr}(\rho_R(g)) \rho_R(HH^*)^{-1} \cdots \quad (47)$$

We thus have proved that the correlation functions of the neutral local operators factorize into the WZW correlation functions times an explicitly known function of the $G^C/G$ component of the impurity connexion. This is analogue to the factorization of the previous section.

Using the factorization (47) we can now average over $A$. As before, it is convenient to change variable from $A$ to $G = HU$. The Jacobian is given in eq.(27). In particular at $\sigma = \infty$, we obtain:

$$\langle \text{tr}(\rho_R(g) \cdots) \rangle_A \cdots = \int DH Dg e^{2hS_{wzw}(HH^*) - S_{wzw}(g)} \text{tr}(\rho_R(g)) \rho_R(HH^*)^{-1} \cdots \quad (48)$$

Therefore, we have proved the following equivalence, valid in the $G$-neutral sector:

$$\langle WZW^{(k)}_{G^C/G} \rangle_W = (WZW)^N \times \left( WZW^{(k=-2h^*)}_{G^C/G} \right) \quad (49)$$

where $N$ is the number of correlation functions we are averaging. These are conformal field theories. The conformal dimensions of the operators in the disordered theory can be evaluate using the formula (31).

### 4 The supersymmetric method: the random bond Ising model.

In this section, we apply the supersymmetric method to study the random bond Ising model and the closely related Dirac theory coupled to a random potential and a random mass. These models have been analysed using the replica method in ref.[3, 4, 5].
As we will soon explain, the randomness of the bond interaction has a marginal effect on the large distance behavior of the Ising model. This allows us to use renormalization group techniques to study the infrared behavior of this random model. At criticality, the disorder only induces logarithmic corrections to the pure system. For example, the averages of the spin correlation functions at criticality are [3, 4]:

\[ \langle \sigma(R)\sigma(0) \rangle^N \sim \left( \frac{a}{R} \right)^{N/4} (\log(R/a))^{N(N-1)/8} \]

These have to be compared to the pure correlation function which are:

\[ \langle \sigma(R)\sigma(0) \rangle^N \sim \left( \frac{a}{\tau} \right)^{N/4}. \]

Close to criticality, the specific heat of the Ising model possesses a logarithmic divergence:

\[ C_v = \log(\tau) \text{ with } \tau \sim \frac{(T_c - T)}{T_c}. \]

In the disordered model, the behavior of the specific heat near criticality reads [2]:

\[ C_v \sim \frac{1}{g_a} \log \left( 1 + g_a \log \left( \frac{1}{\tau} \right) \right) \]

where \( g_a \) represents the strength of the disorder. This \( \log(\log \tau) \) behavior has to be compared to the \( (\log \tau) \) behavior in the pure system.

The perturbative study of the Dirac theory with a random potential and a random mass is very similar to the perturbative study of the random mass Ising model. However, the crucial difference is that contrary to the randomness of the mass, the randomness of the potential is marginally relevant.

Although the supersymmetric method is only applicable to models which are gaussian at fixed disorder, it is certainly a good starting point for disentangling properties of a large class of disordered conformal field theories. The study of random bond Ising model clearly reveals that an appropriate algebraic framework for studying gaussian disordered systems at criticality should be based on affine Lie superalgebra with zero superdimension. In the case of the random Ising model, this algebra is \( OSp(2N|2N) \). The fact that the algebra has zero superdimension, i.e. has an equal number of bosonic and fermionic generators, ensures that the Virasoro central charge of the Sugawara stress-tensor is zero. This is needed by construction for a disordered system. The vanishing of the central charge does not imply the triviality of the theory since it is non-unitary. The general framework should probably be based on the Wess-Zumino-Witten models, or their cosets, on Lie superalgebras which, like \( OSp(2N|2N) \), have equal numbers of bosonic and fermionic generators. The relevance of affine Lie superalgebras was independently realized in ref.[21].

- **The model.**

In its scaling limit (near criticality) the Ising model is described by a massive real Majorana fermion with mass \( \bar{m} \sim \tau = \frac{(T_c - T)}{T_c} \) with \( T_c \) the critical temperature. In presence of disorder, the mass becomes a function of the space position. The random Ising model is defined by the action \( (z = x + iy) \):

\[ S[m(x)] = \int \frac{d^2x}{4\pi} \left( \psi \partial_z \psi + \overline{\psi} \partial_{\overline{z}} \overline{\psi} + im(x)\overline{\psi}\psi \right) \quad (50) \]

where \( \psi \) are grassmanian variables. The mass \( m(x) \) is coupled to the energy operator : \( \epsilon(x) = i \overline{\psi}(x)\psi(x) \). It is chosen to be a random quenched variable with a gaussian measure :

\[ P[m] = \exp \left( -\frac{1}{4g} \int \frac{d^2x}{2\pi} (m(x) - \bar{m})^2 \right) \quad (51) \]
The energy operator $\epsilon(x)$ has dimension one. The Harris criteria thus tell us that randomness in the bond interaction is marginal in the 2d Ising model. As we will see, it is not exactly marginal but only marginally irrelevant.

Since, at fixed disorder the action (54) is a free quadratic action, we can use the supersymmetric formulation to study the quenched model. In order to compute averages of product of correlation functions, we need to introduce an arbitrary number of copies of the fermions and of their supersymmetric partners. For simplicity, let us restrict to the case of two copies. The generalization to an arbitrary number of copies will be given later. Let us denote by $\psi_1$ and $\psi_2$ the two real Majorana fermions, and introduce the complex fermions $\psi_{\pm}$ by:

$$\psi_{\pm} = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2) \quad (52)$$

At fixed disorder, the fermionic action is:

$$S_F = S_1[m(x)] + S_2[m(x)]$$

$$\begin{align*}
S_F &= \int \frac{d^2x}{4\pi} \left( \psi_1 \partial_z \psi_1 + \psi_2 \partial_z \psi_2 + \overline{\psi}_1 \partial_z \overline{\psi}_1 + \overline{\psi}_2 \partial_z \overline{\psi}_2 + im(x) (\overline{\psi}_1 \psi_1 + \overline{\psi}_2 \psi_2) \right) \\
&= \int \frac{d^2x}{2\pi} \left( \psi_- \partial_z \psi_+ + \overline{\psi}_- \partial_z \overline{\psi}_+ + i \frac{m(x)}{2} (\overline{\psi}_- \psi_+ - \psi_- \overline{\psi}_+) \right) \quad (53)
\end{align*}$$

By definition this action can be used to compute products of two Ising correlation functions. Namely:

$$\langle A(z) \cdots \rangle_{m(x)} \langle B(w) \cdots \rangle_{m(x)} = \frac{1}{Z[m(x)]} \int D\psi_1 D\psi_2 A_1(z) \cdots B_2(w) \cdots e^{-S_F}$$

where $\langle \cdots \rangle_{m(x)}$ refers to the Ising correlation function at fixed disorder. $A_1(z)$ and $B_2(w)$ are the expressions of the Ising operators $A(z), B(w)$ in the first and second copies. The partition function is a determinant:

$$Z[m(x)] = \int D\psi_{\pm} e^{-S_F} = Det \left( -\frac{i}{2} \partial_z \quad \frac{i}{2} \overline{\partial}_z \right) = Det H_{\text{Dirac}} \quad (54)$$

Its inverse can be represented as a path integral over bosonic complex fields $\eta$ and $\gamma$:

$$\frac{1}{Z[m(x)]} = \int D\eta D\gamma e^{-S_B}$$

with

$$S_B = \int \frac{d^2x}{2\pi} \left( \eta \partial_z \gamma + \eta \overline{\partial}_z \overline{\gamma} + i \frac{m(x)}{2} (\overline{\eta} \gamma - \eta \overline{\gamma}) \right) \quad (55)$$

Since the Dirac Hamiltonian $H_D$, as defined in (54), is purely imaginary, $H_D^\dagger = -H_D$, the integral over the bosonic variables is an integral of an imaginary gaussian if we defined the complex conjugation by $\eta^* = \overline{\gamma}, \gamma^* = \overline{\eta}$. To insure absolute convergence, we could add the term $\epsilon \int \frac{d^2x}{2\pi} (\overline{\eta} \gamma + \eta \overline{\gamma})$ in the action $S_B$.

The total action is $S_{tot} = S_F + S_B$. By construction, products of Ising correlation functions at fixed disorder can be rewritten as:

$$\langle A(z) \cdots \rangle_{m(x)} \langle B(w) \cdots \rangle_{m(x)} = \int D\psi_{\pm} D\eta D\gamma A_1(z) \cdots B_2(w) \cdots e^{-S_{tot}}$$
The action $S_{\text{tot}}$ is supersymmetric. The supersymmetric transformation can be defined as:

\[
\delta \psi_- = \eta, \quad \delta \psi_+ = 0
\]

\[
\delta \eta = 0, \quad \delta \gamma = -\psi_+
\]

Similarly for the $\overline{\psi}_\pm, \cdots$. It is easy to check that the kinetic term in $S_{\text{tot}}$ is susy invariant:

\[
\delta (\psi_- \partial z \psi_+ + \eta \partial z \gamma) = 0.
\]

Moreover the field coupled to the mass terms in $S_{\text{tot}}$ is susy exact:

\[
(\overline{\psi}_- \psi_+ - \psi_- \overline{\psi}_+ + \eta \gamma - \eta \gamma) = \delta (\overline{\psi}_- \gamma - \psi_- \gamma)
\]

This insures that the partition function defined by $S_{\text{tot}}$ is independent of the random mass $m(x)$, exactly as BRS symmetry ensures gauge invariance in gauge theories. We actually have a stronger result: $S_{\text{tot}}$ is susy-exact. This is similar to topological theory; however, the difference with topological theory is that we are interested in correlation functions of non susy-closed operators. As we will explain below, this choice of the supersymmetric transformation is not unique.

**The effective action.**

We now set $\overline{m} = 0$. I.e. we consider the effect of the disorder at criticality. Averaging over the quenched variables with the gaussian measure (51) leads to the effective action:

\[
S_{\text{eff}} = \int \frac{d^2x}{2\pi} (\psi_- \partial z \psi_+ + \overline{\psi}_- \partial z \overline{\psi}_+ + \eta \partial z \gamma + \overline{\eta} \partial z \gamma) + \frac{g}{8} \int \frac{d^2x}{\pi} \Phi_{\text{pert}}
\]

with

\[
\Phi_{\text{pert}} = (\overline{\psi}_- \psi_+ - \psi_- \overline{\psi}_+ + \eta \gamma - \eta \gamma)^2
\]

\[
= 2(\overline{\psi}_1 \psi_1)(\overline{\psi}_2 \psi_2) + \text{(susy partners)}.
\]

Clearly this action is supersymmetric since the total action $S_{\text{tot}}$ was supersymmetric for any values of the disorder before averaging.

This action can be viewed as a perturbation of the (non-unitary) conformal field theory specified by the action $S_*$. This fixes the normalization of the fields to be:

\[
\psi_-(z) \psi_+(w) \sim \frac{1}{z-w}, \quad \gamma(z) \eta(w) \sim \frac{1}{z-w}
\]

The Virasoro algebra is the standard one. The Virasoro central charge is zero. Note that since the fermions $\psi_\pm$ have dimension half, the perturbing field $\Phi_{\text{pert}}$ has dimension two. It is therefore marginal.

The conformal field theory specified by $S_*$ is invariant under a supersymmetric algebra whose conserved currents are:

\[
G_\pm(z) = \eta(z) \psi_\pm(z), \quad \tilde{G}_\pm(z) = \gamma(z) \psi_\pm(z)
\]

\[
K(z) = \eta^2(z), \quad \tilde{K}(z) = \gamma^2(z)
\]

\[
J(z) =: \psi_-(z) \psi_+(z), \quad H(z) =: \gamma(z) \eta(z)
\]
There are four fermionic currents, which are generators of supersymmetric transformations, and four bosonic ones. They form a representation of the affine $OSp(2|2)$ current algebra at level one. The root system of $OSp(2|2)$ is given below:

\[ \otimes G_+ \quad \otimes \hat{G}_+ \]

\[ \bullet K \quad \bullet J \bullet H \quad \bullet \hat{K} \]

\[ \otimes G_- \quad \otimes \hat{G}_- \]

The black dots refer to the bosonic generators, while the symbols $\otimes$ represent the fermionic roots. The non-trivial operator product expansions of the currents are the following:

\[ J(z)J(w) \sim \frac{1}{(z-w)^2} \quad ; \quad H(z)H(w) \sim \frac{-1}{(z-w)^2} \]
\[ J(z)G_\pm(w) \sim \frac{\pm 1}{(z-w)}G_\pm(w) \quad ; \quad J(z)\hat{G}_\pm(w) \sim \frac{\pm 1}{(z-w)}\hat{G}_\pm(w) \]
\[ H(z)G_\pm(w) \sim \frac{1}{(z-w)}G_\pm(w) \quad ; \quad H(z)\hat{G}_\pm(w) \sim \frac{-1}{(z-w)}\hat{G}_\pm(w) \quad (60) \]
\[ H(z)K(w) \sim \frac{2}{(z-w)}K(w) \quad ; \quad H(z)\hat{K}(w) \sim \frac{-2}{(z-w)}\hat{K}(w) \]

\[ \hat{G}_\pm(z)G_\mp(w) \sim \frac{1}{(z-w)^2} + \frac{1}{(z-w)}(H(w) \pm J(w)) \]
\[ \hat{K}(z)K(w) \sim \frac{2}{(z-w)^2} + \frac{4}{(z-w)}H(w) \]
\[ G_-(z)G_+(w) \sim \frac{1}{(z-w)}K(w) \quad ; \quad \hat{G}_-(z)\hat{G}_+(w) \sim \frac{1}{(z-w)}\hat{K}(w) \]
\[ K(z)\hat{G}_\pm(w) \sim \frac{-2}{(z-w)}G_\pm(w) \quad ; \quad \hat{K}(z)G_\pm(w) \sim \frac{2}{(z-w)}\hat{G}_\pm(w) \]

The perturbing field can be written in terms of these fields:

\[ \Phi_{pert} = 2 \left[ J J - \overline{H}H + \frac{1}{2}(\overline{K}\hat{K} + \overline{\hat{K}}K) + \overline{G_-}\hat{G}_+ - \overline{G_-}G_+ + \overline{G_+}\hat{G}_- - \overline{G_+}G_- \right] \]

In other words, the perturbation \( \Phi_{pert} \) is a current-current perturbation. It preserves the $OSp(2|2)$ symmetry.

The generalization to $2N$ copies is obvious. We introduce $2N$ copies of Majorana fermions $\psi_\alpha, \alpha = 1, \cdots, 2N$, or $N$ copies of Dirac fermions $\psi_\alpha^\dagger, \alpha = 1, \cdots, N$, and $2N$ copies of their supersymmetric partners. The perturbing field is similarly obtained by a gaussian integral from the free random theory:

\[ \Phi_{pert} = 2 \sum_{\alpha<\beta} (\overline{\psi}_\alpha \psi_\alpha)(\overline{\psi}_\beta \psi_\beta) + (susy partners) \quad (61) \]

The symmetry algebra is now extended to the affine $OSp(2N|2N)$ current algebra. The perturbing field is still bilinear in the currents. Hence, the random Ising model is described by a Gross-Neveu model on the Lie superalgebra $OSp(2N|2N)$. In particular it is an integrable model, with factorizable scattering, Yangian symmetry, etc ...
• Renormalization group computation.

The action $S_{\text{eff}}$ describes an interacting theory. Although the perturbing field has dimension two, and therefore is naively marginal, it breaks conformal invariance. To see it requires computing the beta function. As explained in the Appendix, the beta function at one loop is encoded in the operator product expansion (OPE) of the perturbing field. Since $S_\ast$ is a free gaussian theory, this OPE is easily computed using Wick’s theorem and eqs. (58). We obtain:

$$\Phi_{\text{pert}}(z)\Phi_{\text{pert}}(w) = \frac{8}{|z-w|^2} \Phi_{\text{pert}}(w) + \text{irrelevant terms}.$$ 

Using the formula (116) of the Appendix, we get the beta function:

$$\dot{g} = \beta(g) = -g^2 + \cdots$$

(62)

The dots refer to higher loop corrections. We recognize the beta function for asymptotically free theory. It is easily integrated, giving the coupling constant flow:

$$g_R = \frac{g_a}{1 + g_a \log(R/a)}$$

where $g_a$ is the value of the coupling constant at the lattice cut-off $a$. Since $g_a$ is positive by definition (it is the strength of the disorder), $g_R$ decreases at large distances $R$. In other words, the theory is asymptotically free in the infrared regime.

The beta function is independent of the number of copies, i.e., of $N$.

• Quenched correlation functions.

The fact that the random Ising model is asymptotically free in the infrared regime is important since it allows us to compute the large distance behavior of two point quenched correlation functions using the renormalization group, (using formula (117) of the Appendix for the anomalous dimensions).

We recall the OPE between the energy operator and the spin field. The fusion rules are $\epsilon \times \sigma = \sigma$. More precisely, we have:

$$\epsilon(z)\sigma(w) = i(\bar{\psi}\psi)(z)\sigma(w) = \frac{C_{\epsilon\sigma\sigma}}{|z-w|}\sigma(w) + \cdots$$

with $C_{\epsilon\sigma\sigma}^2 = 1/4$.

Let us consider the quenched average of products of two point functions of the spin field, i.e. $\langle \sigma(R)\sigma(0) \rangle^N$. Consider first the case $N = 1$. In the supersymmetric effective theory, the quenched correlation function $\langle \sigma(R)\sigma(0) \rangle$ is represented by the two point function of the spin field $\sigma_1$ in the first copy. Its OPE with the perturbing field is:

$$\Phi_{\text{pert}}(z)\sigma_1(w) = \frac{1}{|z-w|}O(w) + \cdots$$

where $O(w)$, an operator of dimension $(1/8 + 1)$, is irrelevant compared to $\sigma_1$ which has dimension $1/8$. Therefore, the anomalous dimension of $\sigma_1$ at one loop is $\gamma^{(1)} = 1/8 + \cdots$, up to irrelevant terms. Hence:

$$\langle \sigma(R)\sigma(0) \rangle \sim \left(\frac{1}{R}\right)^{1/4}$$

(63)
It has the same infrared behavior as the spin-spin correlation function in the pure system at the critical temperature.

Consider now the general case \( N \) arbitrary. In the supersymmetric effective action, the quenched correlation function \( \langle \sigma(R)\sigma(0) \rangle^N \) is represented by the two point function of the operator \( \mathcal{O}^{(N)}(z) \), product of the spin fields in the \( N \) first copies:

\[
\mathcal{O}^{(N)}(z) = \sigma_1(z) \cdots \sigma_N(z)
\]

When computing the OPE between the perturbing field \( \Phi_{\text{pert}} \) and \( \mathcal{O}^{(N)}(z) \) only the first term in eq. (61),

\[
2 \sum_{\alpha<\beta} (\overline{\psi}_\alpha \psi_\alpha)(\overline{\psi}_\beta \psi_\beta),
\]

gives a relevant contribution. When computing the OPE between this term and \( \mathcal{O}^{(N)}(z) \) we can contract any pair of products of \( (\overline{\psi}_\alpha \psi_\alpha)(\overline{\psi}_\beta \psi_\beta) \) with the pair of product of spin fields \( \sigma_\alpha \sigma_\beta \). E.g.

\[
(\overline{\psi}_1 \psi_1)(\overline{\psi}_2 \psi_2) \sigma_1(z) \sigma_2(z) \cdots \sigma_N(z)
\]

Thus we have:

\[
\Phi_{\text{pert}}(z)\mathcal{O}^{(N)}(w) = - \frac{N(N-1)C_{\sigma\sigma}^2}{|z-w|^2} \mathcal{O}^{(N)}(w) + \text{irrelevant terms}.
\]

Therefore, the anomalous dimension of \( \mathcal{O}^{(N)}(w) \) is

\[
\gamma^{(N)} = \frac{N}{8} - \frac{N(N-1)C_{\sigma\sigma}^2}{4} g + \cdots
\]

Recall that \( C_{\sigma\sigma}^2 = 1/4 \). It implies:

\[
\langle \sigma(R)\sigma(0) \rangle^N_{ga} = \exp\left(-2 \int_{ga}^{R} \gamma^{(N)} / \beta d\sigma\right) \langle \sigma(a)\sigma(0) \rangle^N_{gR}
\]

\[
\sim \left( \frac{a}{R} \right)^{N/4} (\log(R/a))^{N(N-1)/8}
\]

There are logarithmic corrections to the pure system.

**• The specific heat near criticality.**

In order to compute the behavior of the mean specific heat near the critical point we consider a non zero value of \( \overline{m} \):

\[
\overline{m} = \frac{T_C - T}{T_C} = \tau
\]

The specific heat \( C_v \) per unity of volume \( C_v = \frac{1}{V_{\text{vol}}}. \left( \frac{\partial E}{\partial T} \right) \) can be expressed in terms of the connected correlation function of the energy operator \( \epsilon(x) \):

\[
C_v = \int d^2x \langle \epsilon(x)\epsilon(0) \rangle_{\text{conn}}.
\]

Near criticality, its behavior is given by the infrared singular behavior of \( \langle \epsilon(x)\epsilon(0) \rangle_{\text{conn}} \) at zero mass. The infrared cut-off is specified by the mass scale \( 1/\overline{m} \). Thus:

\[
C_v \sim \int^{1/\overline{m}} d^2x \left( \frac{\epsilon(x)\epsilon(0)}{\text{conn}} \right)_{\overline{m}=0}
\]

(65)
In the supersymmetric effective theory, the connected two-point correlation function for the energy operator is represented by the two-point function of the operator \( \epsilon'(x) = \epsilon_1(x) - \epsilon_2(x) \) where \( \epsilon_{1,2}(x) \) refer to the energy operators in the first and second copies:

\[
2\langle \epsilon(x)\epsilon(0) \rangle_{\text{conn.}} = \langle \epsilon'(x)\epsilon'(0) \rangle
\]

To evaluate the behavior of \( C_v \) as \( \overline{m} \to 0 \) requires evaluating the infrared behavior of \( \langle \epsilon'(x)\epsilon'(0) \rangle \). This can be done using the renormalization group by computing the anomalous dimension of \( \epsilon'(x) \). The OPE between \( \epsilon'(x) \) and the perturbing field \( \Phi_{\text{pert}} \) follows from the fusion rule, \( \epsilon \times \epsilon = 1 + \epsilon \), in the Ising model:

\[
\Phi_{\text{pert}}(z)\epsilon'(w) = 2\frac{2}{|z-w|^2}\epsilon'(w) + \cdots
\]

Formula (117) of the Appendix gives:

\[
\gamma_{\epsilon'}(g) = 1 + \frac{1}{2}g + \cdots
\]

Therefore, at large distances we have:

\[
\langle \epsilon'(R)\epsilon'(0) \rangle \sim \left( \frac{a}{R} \right)^2 \frac{1}{(1 + g_a \log(R/a))}
\]

This gives the critical behavior of the specific heat:

\[
C_v \sim \int_1^{1/\tau} d^2x \langle \epsilon(x)\epsilon(0) \rangle_{\text{conn.}}
\sim \frac{1}{g_a \log \left( 1 + g_a \log \left( \frac{1}{\tau} \right) \right)}
\]

(66)

This \( \log(\log \tau) \) behavior has to be compared to the \( (\log \tau) \) behavior in the pure system.

Miscellaneous remarks on the random Dirac theory

A model very closely connected to the random bond Ising model has been introduced in connection with the quantum Hall transition [19]. It is a model of Dirac fermions coupled to a random potential, or more generally, to a random vector potential, a random mass and a random scalar potential. Its action is:

\[
S = \int \frac{d^2x}{2\pi} \left( \psi_- (\partial_x + A_x) \psi_+ + \bar{\psi}_-(\partial_x + A_x) \bar{\psi}_+ 
+ \frac{i}{2} m(x) (\psi_- \psi_+ - \psi_+ \psi_-) + \frac{i}{2} V(x) (\bar{\psi}_- \psi_+ + \psi_- \bar{\psi}_+) \right)
\]

The random variables \( A, m \) and \( V \) have a gaussian distribution with width \( g_A, g_M \) and \( g_V \). Let us denote by \( \Phi_A, \Phi_M \) and \( \Phi_V \) the perturbing fields coupled to the constants \( g_A, g_M \) and \( g_V \) after averaging over the disorder. In the two copy sector, we can write them in terms of the \( OSp(2|2) \) currents as follows:

\[
\Phi_A = (\bar{H} - \bar{J})(H - J)
\]

\[
\frac{1}{2}(\Phi_V + \Phi_M) = 2\bar{G}_+ G_- - 2\bar{G}_- G_+ + \bar{K} \bar{K} + \bar{K} K
\]

\[
\frac{1}{2}(\Phi_V - \Phi_M) = 2HH - 2JJ + 2\bar{G}_- G_+ - 2\bar{G}_+ G_-
\]

These remarks arisen from discussions we had with Martin Zirnbauer.
It is an interesting exercise to compute the beta function at one-loop for these coupling constants. We find:

\[
\begin{align*}
\beta_A &= \dot{g}_A = 32 g_M g_V + \cdots \\
\beta_V - \beta_M &= \dot{g}_V - \dot{g}_M = 8 (g_V + g_M)^2 + \cdots \\
\beta_V + \beta_M &= \dot{g}_V + \dot{g}_M = 8(g_V + g_M)(g_V - g_M) + 8g_A(g_V + g_M) + \cdots
\end{align*}
\]

For generic value of the initial coupling constants this describes complicated flows. In the particular case with \( g_A = g_V = 0 \), we recover the beta function of the random bond Ising model.

For \( g_A = g_M = 0 \), we have \( \dot{g}_V = 8g_V^2 \). This means that a random potential is marginally relevant in the Dirac theory [19]. The coupling constant \( g_V \) grows at large distances. The infrared fixed is expected to describe the quantum Hall effect transition. However, being a strong coupling problem no concrete description of it has been proposed.

Note that in this perturbative computation we put all the epsilons which distinguished between advanced and retarded sectors equal to zero. In other words, we did not distinguished whether we are computing averages of product of only advanced Green functions, or averages of product of advanced and retarded Green functions. It is likely that the infrared behavior will not be the same in these two cases. It is tempting to conjecture that it will be trivial when we consider only advanced Green functions but non-trivial when mixed products of advanced and retarded Green functions are considered.

Being formulated as current-current perturbation of a first order free theory, we can implement a Hubbard-Stratonovitch transformation on the random Dirac theory in order to produce a dual theory whose coupling constant is \( 1/g \). The transformation goes as follows: one first represents the current-current interaction by a gaussian integral over an auxiliary gauge connexion \( A \), and then integrate over the free fermions and bosons. This produces a dual theory whose fundamental fields are the connexion \( A \), or equivalently the group element \( G \) such that \( A_z = G^{-1} \partial_z G \), and whose fundamental coupling constant is \( 1/g \). However, in the mixed sector with both advanced and retarded Green functions, this naive transformation seems to be spoiled by divergences in the gaussian integrals.

5 The replica method : the random phase sine-Gordon model.

In this section, we apply the replica method to the random phase sine-Gordon model. It is a random version of the 2d XY model. It has also been used to describe other random physical systems; e.g. the 2d XY model in a random field [3, 6], interfacial roughening transition [7], randomly pinned flux lines in supraconductors [8], etc...

Let us first introduce the model from the XY point of view. Recall that the partition function of the XY model is defined as:

\[
Z = \sum_{\{h_i\}} \exp \left( -\frac{K}{2} \sum_{i,j} (h_i - h_j)^2 \right)
\]

It describes the thermodynamics of a fluctuating surface. The variables \( h_i \) are interpreted as the height of the surface above a base plane. In the pure system the \( h_i \) take integer values. The disordered model is defined by the same partition function but the height variables \( h_i \)
take the values \( h_i = d_i + n_i \) with \( n_i \) integers and \( d_i \) some random variables. Similarly as for the standard XY model, we can formulate this problem in an alternative form by using the Poisson summation formula: \( \sum_n f(n) = \sum_m \int d\Phi f(\Phi)e^{i2\pi m\Phi} \). Introducing the variables \( \Phi_i \), one gets:

\[
Z = \int D\Phi \sum_{\{m_i\}} \exp \left( -\frac{K}{2} \sum_{(i,j)} (\Phi_i - \Phi_j)^2 + i2\pi \sum_j m_j (\Phi_j - d_j) \right)
\]

In the continuum limit, the variables \( \Phi_i \) are replaced by a field \( \Phi(x) \) and the discrete Laplacian becomes a continuous one. Near the critical point, only the first harmonic of the local potential is relevant. Keeping only this first harmonics gives the action:

\[
S = \int \frac{d^2x}{4\pi} \left( \frac{K}{2}(\partial_\nu \Phi)^2 - \Delta \cos(\Phi(x) - d(x)) \right)
\]

with \( d(x) \) a random quenched fields. This is the model we will use to illustrate the replica method.

As we explain below, this model as two different phase: a high temperature phase in which the disorder is irrelevant and a low temperature phase in which it is relevant. In the high temperature phase nothing interesting happens. In the low temperature phase, the large distance behavior of the system could a priori be different than in the pure system. However, a precise description of this behavior is still missing. For example, the behavior of the quenched correlation functions of \( \Phi(x) \) is still controversial. One finds that at large distance we have:

\[
\langle [\Phi(x) - \Phi(0)]^2 \rangle = A(\log |x|) + B(T - T_c)^2(\log |x|)^2
\]

However, renormalization group computations predict a non-vanishing \( B \neq 0 \), while variational approaches give \( B = 0 \). In the sequel we will present both approaches.

- **The model.**

Let us first rewrite the action with more appropriate notations. We also need to slightly generalize it by introducing a random potential in addition to the random phase. The action reads:

\[
S(\Phi|A_\mu; \xi) = \int \frac{d^2x}{4\pi} \left( \frac{K}{2}(\partial_\nu \Phi)^2 - A_\nu \partial_\nu \Phi - \xi(x)e^{i\Phi} - \xi^*(x)e^{-i\Phi} \right) \tag{68}
\]

The constant \( K \) is proportional to the inverse temperature, \( K \propto 1/T \). In the formula, \( A_\nu(x) \) and \( \xi(x) \) are random quenched variables with gaussian measure:

\[
P[A] = \exp \left[ -\frac{1}{2g} \int \frac{d^2x}{4\pi} A_\mu A_\mu \right]
\]

\[
P[\xi] = \exp \left[ -\frac{1}{2\sigma} \int \frac{d^2x}{4\pi} \xi \xi^* \right] \tag{69}
\]

In absence of disorder, the dimension of the vertex operator \( \exp(i\Phi) \) is: \( \dim(e^{i\Phi}) = \frac{1}{K} \). The Harris criteria then tell us that there are two different phases:

- **low temperature phase** \( (K > K_c = 1) \) \( \Rightarrow \) disorder is relevant
- **high temperature phase** \( (K < K_c = 1) \) \( \Rightarrow \) disorder is irrelevant

26
Recall that $K \propto 1/T$. At high temperature, the large distance behavior is identical to that of the pure system. At the critical temperature $K = K_c$, the disorder has a marginal effect. At low temperature, the large distance behavior is a priori different.

### 6. Symmetries.

Similarly as for the random gaussian model, the random phase sine-Gordon model possesses a $U(1)$ symmetry whose Noether current corresponds to insertion of $\partial_\mu \Phi$ in the quenched connected correlation functions. This symmetry in particular implies that all the quenched connected correlation functions of $\partial_\mu \Phi$ are unaffected by the disorder.

Just as for the Gaussian model, we can decompose $A_\mu$ as $A_\mu = \partial_\mu \Lambda + \epsilon_{\mu\nu} \partial_\nu \zeta$. Then the field $\zeta$ decouples from the action and from the measure. So we can set it to zero, and we have:

$$P[A] = \exp\left[-\frac{1}{2g} \int \frac{d^2 x}{4\pi} (\partial_\mu \Lambda)^2\right], \quad \text{with} \quad A_\mu = \partial_\mu \Lambda$$  \hspace{1cm} (70)

We will denote by $S(\Phi|\Lambda; \xi)$ the action (68) with $A_\mu = \partial_\mu \Lambda$.

The simplest way to visualize this symmetry consists in introducing sources for $\partial_\mu \Phi$ in the action (68):

$$S(\Phi|\Lambda; \xi) \rightarrow S(\Phi|J_\mu; \Lambda; \xi) = S(\Phi|\Lambda; \xi) - \int \frac{d^2 x}{4\pi} J_\mu \partial_\mu \Phi$$

Let us decompose $J_\mu$ as $J_\mu = \partial_\mu \rho + \epsilon_{\mu\nu} \partial_\nu \eta$. The field $\eta$ decouples and only the field $\rho$ is relevant. The $U(1)$ symmetry follows from the fact that the $\rho$ and $\Lambda$ dependence can be absorbed into a shift of $\Phi$. Namely:

$$S(\Phi|J_\mu; \Lambda, \xi) = S(\Phi - \frac{\rho + \Lambda}{K}|J_\mu = \Lambda = 0, \tilde{\xi}) - \frac{1}{2K} \int \frac{d^2 x}{4\pi} (\partial_\mu (\rho + \Lambda))^2$$  \hspace{1cm} (71)

with $\tilde{\xi} = \xi e^{i(\rho + \Lambda)/K}$ and $J_\mu = \partial_\mu \rho$. As a consequence, the generating function of the connected correlation functions of $\partial_\mu \Phi$, which is log $Z[J_\mu, \Lambda, \xi]$, can be written as:

$$\log Z[J_\mu, \Lambda, \xi] = -\frac{1}{2K} \int \frac{d^2 x}{4\pi} (\partial_\mu (\rho + \Lambda))^2 + \log Z[J_\mu = \Lambda = 0, \tilde{\xi}]$$  \hspace{1cm} (72)

Now, notice that $\tilde{\xi}$ and $\xi$ have identical distribution. Therefore, averaging (72) over the disorder implies:

$$\overline{\log Z[J_\mu, \Lambda, \xi]} = -\frac{1}{2K} \int \frac{d^2 x}{4\pi} (\partial_\mu \rho)^2 + \text{const.}$$

where the constant is independent of $J_\mu$. This shows that the quenched average of the connected correlation functions of the $U(1)$ current are identical to the correlation functions of this current in absence of disorder. In other words, the connected correlation functions are protected from the disorder by the $U(1)$ symmetry.

It is worth noticing that the presence of the disorder restore the $U(1)$ symmetry which is absent in the pure sine-Gordon model.

In connection with this $U(1)$ symmetry, the model possesses the remarkable property that the $g$-dependence of the correlation functions of the vertex operators can be factorized.
Consider eq. (71) in absence of source: $J_\mu = 0$. The fact that the $\Lambda$-dependence can be absorbed in a translation of $\Phi$ implies for the correlation functions that:

$$\langle e^{i\alpha_1\Phi(x_1)} \cdots \rangle_{\Lambda, \xi} = \left( \prod_p e^{i\alpha_p \Lambda(x_p)} \right) \left( \langle e^{i\alpha_1\Phi(x_1)} \cdots \rangle_{\Lambda=0, \xi} \right)$$

Let $G_{\alpha_1,\cdots}(x_1,\cdots | g, \sigma)$ be the quenched correlations of the vertex operator $\exp(i\alpha\Phi)$:

$$G_{\alpha_1,\cdots}(x_1,\cdots | g, \sigma) = \langle e^{i\alpha_1\Phi(x_1)} \cdots \rangle_{\Lambda, \xi, \Lambda, \xi} = \langle \cdots \rangle_{\Lambda, \xi}$$

Integrating over $\Lambda$ using the free field gaussian measure (70), and using the fact that $\hat{\xi}$ and $\xi$ have the same measure, we deduce:

$$G_{\alpha_1,\cdots}(x_1,\cdots | g, \sigma) = \prod_{p<q} |x_p - x_q|^2 g \alpha_p \alpha_q / K^2 \ G_{\alpha_1,\cdots}(x_1,\cdots | g = 0, \sigma)$$

Equivalently,

$$\partial_g G_{\alpha_1,\cdots}(x_1,\cdots | g, \sigma) = \left( \sum_{p<q} \frac{\alpha_p \alpha_q}{K^2} \log(|x_p - x_q|^2) \right) G_{\alpha_1,\cdots}(x_1,\cdots | g, \sigma)$$

This identity will be useful for analyzing the renormalization group equations.

- **The effective action.**
Since the action (68) is not free we cannot rely on the supersymmetric method but only on the replica trick. Therefore, as explained in the introduction, we consider $n$ copies of the system with the same disorder and then average over the disorder. This gives the following effective action:

$$S_{eff} = \frac{d^2 x}{4\pi} \left( \frac{K}{2} \sum_r (\partial_\mu \Phi^r)^2 - \frac{g}{2} \sum_{r,s} (\partial_\mu \Phi^r)(\partial_\mu \Phi^s) - 2\sigma \sum_{r \neq s} \exp(i(\Phi^r - \Phi^s)) \right)$$

Let us first look at the kinetic term. It is of the form $\int \frac{d^2 x}{4\pi} \left( \frac{1}{2} (\partial_\mu \Phi^r)(\partial_\mu \Phi^s) \right)$ with

$$G_{rs} = K \delta_{rs} - g = (K - g) \delta_{rs} - g(1 - \delta_{rs})$$

Since the interaction only involves the difference of the replicated fields, it is convenient to decompose the kinetic terms as:

$$\frac{1}{2} (\partial_\mu \Phi^r)(\partial_\mu \Phi^s) = \frac{(K - ng)}{2n} (\partial_\mu (\sum_r \Phi^r))^2 + \frac{K}{4n} \sum_{r \neq s} (\partial_\mu (\Phi^r - \Phi^s))^2$$

Note that the field $\sum_r \Phi^r$ decouples. Its correlation functions are therefore unaffected by the disorder. Since these correlation functions represent the averages of the connected correlation functions of the $U(1)$ current, we recover the previous result obtained from the $U(1)$ Ward identities. In particular, $\partial_\mu (\sum_r \Phi^r)$ is a $U(1)$ conserved current. We could also use this decomposition to derive eq. (73).

The kinetic term (77) fixes the normalization of the vertex operators. In particular, we find the dimension of the perturbing field: $\text{dim} \left( e^{i(\Phi^r - \Phi^s)} \right) = \frac{2}{K}$. It is independent of $\sigma$. It is relevant for $K > 1$; we thus recover the Harris criteria.
This decomposition has a Lie algebraic interpretation. It corresponds to the decomposition of $U(n) = SU(n) \times U(1)$. The second term in (77) can be rewritten as $\frac{K}{2}(\partial \bar{\varphi})^2$ where $\bar{\varphi}$ takes values in the Cartan subalgebra of $SU(n)$. In terms of $\bar{\varphi}$, the perturbing field in (77) reads: $\sum \sigma \exp(i\bar{\varphi})$ where the sum extends over all the $SU(n)$ roots. At $K = K_c$, the fields $\exp(i\bar{\varphi})$ have dimension two, and the perturbing field is a current-current interaction. The action (76) at $K = K_c$ is therefore equivalent to the $SU(n)$ Gross-Neveu model. We will later see that at this point the model is asymptotically free in the infrared. Hence at the critical temperature $K = K_c$, the disorder only induces logarithmic corrections.

- Renormalization group.

The renormalization group allows to perturbatively analyse the behavior of the system in the low temperature phase. We will do a one loop computation, which is valid close to the critical temperature, i.e. $\frac{K-K_c}{K_c} \ll 1$.

As we are already familiar with, the one-loop beta functions are encoded in the OPE of the fields. Let us introduce the following notation:

$$O_1 = \sum_{r\neq s} \exp(i(\Phi^r - \Phi^s))$$

$$O_2 = \frac{1}{2n} \sum_{r,s} i\partial_z(\Phi^r - \Phi^s)i\partial_z(\Phi^r - \Phi^s)$$

The field $O_1$ is the perturbing field associated to the coupling constant $\sigma$. The field $O_2$ is one of the kinetic field. Notice that $O_2 = (i\partial_z \bar{\varphi})(i\partial_z \bar{\varphi})$. It is necessary to introduce it since it is generated from $O_1$ by OPE. Indeed we have:

$$O_1(z)O_1(0) = \frac{2(n-2)}{|z|^2/2}O_1(0) + \frac{2n}{|z|^2/2}O_2(0) + \text{irrelevent terms},$$

$$O_2(z)O_1(0) = \frac{2}{|z|^2}O_1(0) + \text{irrelevent terms},$$

$$O_2(z)O_2(0) = \text{irrelevent terms}.$$ 

For $n = 1$ these are the familiar OPE of the Kosterlitz-Thouless transition. Note the change of behavior between $n < 2$ or $n > 2$. The first OPE indicates how $\sigma$ is renormalized, but the second equation shows that the kinetic term needs also to be renormalized.

Using the formula (116) of the Appendix, we get the beta functions:

$$\beta_\sigma = 2\left(\frac{K-K_c}{K}\right)\sigma + (n-2)\sigma^2 + \cdots$$

$$\beta_K = \frac{n}{2}\sigma^2 + \cdots$$

This shows that the coupling to $A_\mu$ would have been generated at one loop even if we did not start with it. Since the field ($\sum_r \Phi^r$) decouples, the coupling ($K-ng$) is unrenormalized (at any order in perturbation theory). Thus: $\beta_K = n\beta_g$. Setting $n = 0$ as required by the replica trick, we get:

$$\beta_\sigma = 2\left(\frac{K-K_c}{K}\right)\sigma - 2\sigma^2 + \cdots$$

$$\beta_g = \frac{1}{2}\sigma^2 + \cdots$$

$$\beta_K = 0$$
So, \( K \) is unrenormalized at \( n = 0 \). It may appears surprising that the coupling \( g \) is renormalized although we know exactly the \( g \)-dependence of any correlation functions. We will later see that this is not in contradiction with the renormalization group equations.

From equation (81), we immediatly see that in the low temperature phase (\( K > K_c \)), the beta function \( \beta_\sigma \) possesses a non trivial infrared zero at \( \sigma_\ast \):

\[
\sigma_\ast = \left( \frac{K - K_c}{K_c} \right) + \cdots \quad \text{for} \quad \left( \frac{K - K_c}{K_c} \right) \ll 1. \tag{82}
\]

Notice the fact the other beta function \( \beta_g \) does not vanish at \( \sigma_\ast \):

\[
\beta_g(\sigma_\ast) = \frac{1}{2} \left( \frac{K - K_c}{K_c} \right)^2 + \cdots \tag{83}
\]

Hence, even at \( \sigma_\ast \) the coupling \( g \) will continue to flow. We may characterize such pseudo-fixed point as a “run away fixed point”. This is a particularity of the model which has direct consequences on the correlation functions.

• Quenched correlation functions.

Let us now analyse the renormalization group equations. As we already said, the field \( A_\mu \) would have been generated at one loop if not present initially. But on other hand we know that the dependence on \( g \) can be completely disentangled. This apparent conflict should have an effect on the renormalization group equations.

Consider the correlation functions \( G_{\alpha_1,\ldots}(x_1,\cdots|g,\sigma) \) defined in eq.(73). They satisfy the renormalization group equations :

\[
\left[ \sum_p x_\nu_p \frac{\partial}{\partial x_\nu_p} + \sum_p \gamma_p(g,\sigma) - \beta_\sigma(\sigma) \frac{\partial}{\partial \sigma} - \beta_g(\sigma) \frac{\partial}{\partial g} \right] G_{\alpha_1,\ldots}(x_1,\cdots|g,\sigma) = 0
\]

where \( \gamma_p(g,\sigma) \) are the anomalous dimensions. But the \( g \)-dependence is explicitly known, therefore using eq.(75) we get :

\[
\left[ \sum_p x_\nu_p \frac{\partial}{\partial x_\nu_p} + \sum_p \gamma_p(g,\sigma) - \beta_\sigma(\sigma) \frac{\partial}{\partial \sigma} - \beta_g(\sigma) \sum_{p<q} \frac{\alpha_p \alpha_q}{K^2} \log(|x_p - x_q|^2) \right] G_{\alpha_1,\ldots}(x_1,\cdots|g,\sigma) = 0
\]

Note that it is now possible to set \( g \) equal to zero. In particular at the infrared fixed point \( \sigma_\ast \), in which \( \beta_\sigma(\sigma_\ast) = 0 \), we get :

\[
\left[ \sum_p x_\nu_p \frac{\partial}{\partial x_\nu_p} + \sum_p \gamma_p(\sigma_\ast) - \beta_g(\sigma_\ast) \sum_{p<q} \frac{\alpha_p \alpha_q}{K^2} \log(|x_p - x_q|^2) \right] G_{\ast\alpha_1,\ldots}(x_1,\cdots|g,\sigma) = 0 \tag{84}
\]

with \( \beta_g(\sigma_\ast) \) given in eq.(83). The effect of the \( g \)-flow is to add the extra logarithmic term in the renormalization group equations (84).

This can be used to compute two-point functions at the infrared fixed point. Consider,

\[
G_1(x) = \frac{\langle \exp \left( i\alpha(\Phi(x) - \Phi(0)) \right) \rangle_{\sigma_\ast}}{\langle \exp \left( i\alpha(\Phi(0)) \right) \rangle_{\sigma_\ast}}
\]

\[
G_2(x) = \frac{\langle \exp \left( i\alpha\Phi(x) \right) \rangle \langle \exp \left( -i\alpha\Phi(0) \right) \rangle_{\sigma_\ast}}{\langle \exp \left( i\alpha(\Phi(0)) \right) \rangle_{\sigma_\ast}}
\]

30
Let $\gamma_{1,2}(\sigma_*)$ be their anomalous dimensions at the infrared fixed point. The renormalization group equation gives:

$$G_{1,2}(x) = |x|^{-2\gamma_{1,2}(\sigma_*)} \exp\left(-\frac{\alpha^2 \beta_g(\sigma_*)}{2K^2}(\log |x|)^2\right)$$  \hfill (85)

Notice the $(\log |x|)^2$ correction which arises from the renormalization of $g$. This term is independent of the anomalous dimension.

The anomalous dimensions are even in $\alpha$ and vanishes at $\alpha = 0$, therefore:

$$\gamma_{1,2}(\sigma_*) = \frac{\alpha^2}{K} \rho_{1,2}(\sigma_*) + \mathcal{O}(\alpha^4), \quad \text{with} \quad \rho_1(\sigma_*) = 1 + \mathcal{O}\left(\frac{K - K_c}{K_c}\right)$$

Expanding in power of $\alpha^2$, gives the two point functions of $\Phi$:

$$\left\langle \frac{(|\Phi(x) - \Phi(0)|^2)}{\left\langle |\Phi(x) - \Phi(0)|^2\right\rangle}\right\rangle = \frac{2\rho_1(\sigma_*)}{K} \log |x| + \frac{\beta_g(\sigma_*)}{2K^2}(\log |x|)^2$$  \hfill (86)

$$\left\langle \frac{\left\langle |\Phi(x) - \Phi(0)|^2\right\rangle}{\left\langle |\Phi(x) - \Phi(0)|^2\right\rangle}\right\rangle = \frac{2\rho_2(\sigma_*)}{K} \log |x| + \frac{\beta_g(\sigma_*)}{2K^2}(\log |x|)^2$$  \hfill (87)

Note that the $(\log |x|)^2$ cancels in the connected correlation function $\overline{\left\langle |\Phi(x) - \Phi(0)|^2\right\rangle}_{\text{conn}}$ as it should be, since this connected correlation function is unaffected by the disorder.

There is a crossover from a $(\log |x|)$ to a $(\log |x|)^2$ behavior. For $|x| < R_{\text{cross}}$ we have a usual $(\log |x|)$ behavior, while for $|x| > R_{\text{cross}}$ we have a $(\log |x|)^2$ behavior. The crossover length is $R_{\text{cross}}$ with

$$\log R_{\text{cross}} \sim \frac{2K\rho_1(\sigma_*)}{\beta_g(\sigma_*)} \sim \frac{K^2}{(K - K_c)^2}, \quad \text{for} \quad (K - K_c) \ll 1.$$  

$R_{\text{cross}}$ is exponentially large close to the phase transition. However eqs.(86,87) are true to all order in perturbation theory if we can rely on the renormalization group in the replica symmetric approach.

We can use eqs.(86,87) to find an estimate of the width of the surface for a system of finite length $L$, i.e. to find an estimate of $\overline{\left\langle |\Phi(0)|^2\right\rangle}_L$. We define it as the integral of the Fourier transform $\Gamma(q)$ of $|\Phi(0)|^2$ using $1/L$ as ultraviolet cutoff. At short momenta $q \ll 1/R_{\text{cross}}$, the propagator $\Gamma(q)$ is dominated by the Fourier transform of the $(\log |x|)^2$ term; i.e. $\Gamma(q) \sim \beta_g(\sigma_*) \left(\frac{\log^2 q^2}{q^2}\right)$. Thus:

$$Q(L) = \overline{\left\langle |\Phi(0)|^2\right\rangle}_L = \int_{1/L} d^2q \Gamma(q) \sim \beta_g(\sigma_*) (\log L)^2$$  \hfill (88)

for $L \gg R_{\text{cross}}$. This has to be compare with the pure case which gives a $(\frac{1}{K} \log L^2)$ behavior.

- **A large $N$ model.**

The formula (86, 87) are still controversial, theoretically as well as numerically. Variational approaches, (part of which we will describe below), predict a $(\log |x|)$ behavior [27, 28, 29]. The RG flows was also found to be unstable again asymmetric replica perturbations [30, 31]. The numerical verifications of (84) are also not settled: a $(\log |x|)$ behavior was found in ref.[32, 33], while more recent simulations [34] seem to indicate a $(\log |x|)^2$ behavior. In view of this conflict, and since the variational approaches are argued to be
exact for systems with a large number of components \[^{35}\] in ref. \[^{36}\] we studied a large \(N\) version of the model \[^{68}\]. Using RG computations based on the (a priori symmetric) replica trick, we find that our large \(N\) model possesses a non-trivial infrared fixed point in which the correlation functions have the form \[^{69} \,^{70}\] but with the \((\log |x|)^2\) term suppressed by a factor \((1/N^2)\) compared to the \((\log |x|)\) term.

To introduce the large \(N\) version of \[^{68}\], it is convenient to fermionize it. The fermionic form of the random phase sine-Gordon model is a massless Thirring model coupled to a quenched potential \(A_\mu\) and a random phase \(\xi\). To define its large \(N\) version, we need to introduce \(N\) Dirac fermions with components \(\psi^k_\pm\) and \(\bar{\psi}^k_\pm\) with \(k = 1, \cdots, N\). Let \(z = x + iy\) and \(\bar{z} = x - iy\) be the complex coordinates on the plane. The action is:

\[
S^{(N)} = \int \frac{d^2x}{\pi} \left( \sum_{k=1}^{N} (\psi^-_{-\kappa} \partial_\kappa \psi^k_+ + \bar{\psi}^-_{-\kappa} \partial_\kappa \bar{\psi}^k_+) - \frac{a}{N} (\sum_{k=1}^{N} \psi^-_{-\kappa} \psi^k_+) (\sum_{k=1}^{N} \bar{\psi}^-_{-\kappa} \bar{\psi}^k_+) \right) - \int \frac{d^2x}{\pi} \left( A_\pi (\sum_{k=1}^{N} \psi^-_{-\kappa} \psi^k_+) + A_\tau (\sum_{k=1}^{N} \bar{\psi}^-_{-\kappa} \bar{\psi}^k_+) + \xi (\sum_{k=1}^{N} \psi^-_{-\kappa} \psi^k_+) + \xi^* (\sum_{k=1}^{N} \bar{\psi}^-_{-\kappa} \bar{\psi}^k_+) \right)
\]

In absence of disorder, it is conformally invariant. The random potential \(A_\mu = (A_\pi, A_\tau)\) is coupled to the \(U(1)\) currents \(J_\pi = \sum_{k=1}^{N} \psi^-_{-\kappa} \psi^k_+\) and \(J_\tau = \sum_{k=1}^{N} \bar{\psi}^-_{-\kappa} \bar{\psi}^k_+\) of the unperturbed theory. At \(\xi = 0\), the random potential does not break conformal invariance.

There are a priori many ways to generalize the action \[^{68}\] to a large \(N\) version. The action \[^{89}\] has been designed in such way as (i) to keep the number of disordered variables fixed, (ii) to preserve the exact conformal invariance in absence of disorder, and (iii) to parallel as much as possible standard properties of large \(N\) models.

The fermionic action \[^{88}\] can be bosonized back using non-Abelian bosonization \[^{37}\]. As usual, since the pure system describes \(N\) Dirac fermions, the pure bosonized theory will be described by a \(su(N)\) Wess-Zumino-Witten (WZW) model at level one plus a massless free field. The \(su(N)\) WZW model at level one possesses primary fields taking values in the \((N - 1)\) fundamental representations of \(su(N)\). Let \(\phi_\kappa^k\) and \(\phi_\kappa^{k*}\) be the chiral WZW primary fields which take values in the defining representation of \(su(N)\) and in its complex conjugate. Their conformal weights are both equal to \(\left(\frac{N-1}{2N}\right)\). Let us denote by \(\varphi\) the gaussian free field. The original fermions \(\psi^k_\pm\) can be written as the product of these WZW primary fields by a vertex operator of the gaussian model. Namely, \(\psi^k_+ = \phi_\kappa^k e^{i\lambda \varphi}\) and \(\psi^-_{-\kappa} = \phi_\kappa^{k*} e^{-i\lambda \varphi}\), and similarly for the other chiral components \(\bar{\psi}^k_+\) and \(\bar{\psi}^-_{-\kappa}\). The bosonic form of the action \[^{89}\] is:

\[
S^{(N)} = S_{wzw} + \frac{K}{2} \int \frac{d^2x}{4\pi} (\partial_\nu \varphi)^2 - \int \frac{d^2x}{4\pi} \left( A_\nu(x) \partial_\nu \varphi + \xi(x) \Phi e^{i\lambda \varphi} + \xi^*(x) \Phi^* e^{-i\lambda \varphi} \right)
\]
\[ \beta_g(\sigma_*) \text{ given by :} \]
\[ \beta_g(\sigma_*) = \frac{1}{8N^3} \left( \frac{K - K_c}{K_c} \right)^2 + \cdots, \quad \text{for } N \gg 1, \quad \frac{K - K_c}{K_c} \ll 1 \]

The occurrence of this factor \( \frac{1}{N^3} \) could explain why the \( (\log |x|)^2 \) term does not manifest itself in the variational approaches.

6 The variational method : the random phase sine-Gordon model.

Various variational approaches to disorder systems have been proposed. They can be applied after or before disorder averaging. In the first case, one first averages over the disorder using the replica trick, and then implements a variational method on the replicated model, cf. eg \[53\]. In the second case, one applies a variational method on any sample of fixed disorder, and then averages over the disorder. We will present the second method using the random phase sine-Gordon model as example, following ref.\[29\]. But it can clearly be applied to other models. The variational method at fixed disorder leads to an exact bound to the averaged free energy.

Consider the partition function
\[ Z[\xi(x)] = \int D\Phi \exp(-S(\Phi|\xi)), \]
where \( S(\Phi|\xi) \) is the action \[58\] at \( A_\mu = 0 \), which we recall:
\[ S(\Phi|\xi) = \int \frac{d^2x}{4\pi} \left( \frac{K}{2} (\partial_\nu \Phi)^2 - \xi(x)e^{i\Phi} - \xi^*(x)e^{-i\Phi} \right) \]

At fixed disorder, we approximate the partition function by a gaussian action:
\[ S_0(\Phi|G) = \frac{1}{2} \int d^2x \int d^2y \; \Phi(x)G^{-1}(x-y)\Phi(y) = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \; \hat{\Phi}(k)\hat{G}^{-1}(-k)\hat{\Phi}(k) \]

Here the hatted quantities refer to the Fourier transforms. Notice that the ansatz \[91\] is choosen to be translation invariant, and that the gaussian is centered around the origin. We choose a gaussian ansatz otherwise the computations are undoable. The kernel \( G(x-y) \) is the variational parameter.

The approximated partition function is \( Z_0 = \int D\Phi \exp(-S_0) \). Let \( F \) and \( F_0 \) be the respective free energies, ie. \( Z = e^{-F} \) and \( Z_0 = e^{-F_0} \). For any realization of the disorder and for any choice of \( S_0 \), the following inequality holds:
\[ F \leq F_0 + \langle (S - S_0) \rangle_0 \] (92)
where \( \langle \cdots \rangle_0 \) refers to the expectation values with the measure \( S_0 \). This is proved using the following inequality:
\[ \frac{Z}{Z_0} = \langle e^{-(S-S_0)} \rangle_0 \geq e^{-\langle (S-S_0) \rangle_0} \]

Therefore, the best approximated action \( S_0 \) will be find by minimazing \( (F_0 + \langle (S - S_0) \rangle_0) \). That is:
\[ \frac{\delta}{\delta G(k)} \left( F_0 + \langle (S - S_0) \rangle_0 \right) = 0. \] (93)
The inequality (92) gives an upper bound to the free energy.

Let us apply this general setup to our example. As it is formulated the method is more appropriate to study the free energy than the correlation functions. Thus we consider the system in a box of finite volume of size $Vol. = L^2$ and look for the $L$-dependence of the free energy. That is, we use the variational free energy to analyse the finite size effects.

Since $S_0$ is Gaussian, $F_0$ and $\langle (S - S_0) \rangle_0$ are easily computed. The free energy $F_0$ per unit of volume is given by the logarithm of a determinant:

$$\frac{F_0}{Vol.} = -\frac{1}{2} \int d^2k \log \hat{G}(k)$$

while for the expectation value of $(S - S_0)$ per unit of volume we have:

$$\frac{\langle (S - S_0) \rangle_0}{Vol.} = \frac{\pi K}{2} \int \frac{d^2k}{(2\pi)^2} k^2 \hat{G}(k) - \frac{e^{-Q/2}}{Vol.} \int \frac{d^2x}{4\pi} (\xi(x) + \xi^*(x))$$

(94)

with

$$Q = \langle \Phi^2(0) \rangle = G(0) = \int \frac{d^2k}{(2\pi)^2} \hat{G}(k)$$

(95)

This parameter is naturally interpreted as the width of the system.

Notice that the expectation value (94) only depends on a particular moment of $\xi(x)$. It does not depend on all the details of the disorder configuration. It is this particular fact which makes the variational approach doable.

The variational equations (93) are simple to compute. They determine the kernel $\hat{G}(k)$:

$$\hat{G}(k) = \frac{4\pi}{K} \left( \frac{1}{k^2 + M^2} \right)$$

(96)

where the effective mass is:

$$M^2 = \frac{1}{\pi K} \frac{e^{-Q/2}}{(Vol.)^{1/2}} D_0 \quad \text{with} \quad D_0 = \frac{1}{(Vol.)^{1/2}} \int \frac{d^2x}{4\pi} (\xi(x) + \xi^*(x))$$

(97)

We have introduced the prefactor $(Vol.)^{1/2}$ in the definition of $D_0$ to make it scale invariant. Since the effective mass $M^2$ depends on $Q$, which is a functional of the kernel $\hat{G}(k)$, eqs. (95, 96) form a set of non linear coupled equations for $\hat{G}(k)$ which we rewrite below:

$$Q = \frac{4\pi}{K} \int \frac{d^2k}{(2\pi)^2} \left( \frac{1}{k^2 + M^2} \right) \quad \text{with} \quad M^2 = \frac{1}{\pi K} \frac{e^{-Q/2}}{(Vol.)^{1/2}} D_0$$

(98)

The effective mass depends on the disorder through its moment $D_0$. This moment can be either positive or negative, with a symmetric probability distribution around the origin. Thus we have to study separately the two cases: $D_0 > 0$ or $D_0 < 0$. This analysis was done in ref. [29].

For $D_0 > 0$, the effective mass is real and the Green function $\hat{G}(k)$ has no pole. Therefore, for $Q$ we get:

$$Q = \int_{1/a}^{L} \frac{d^2k}{(2\pi)^2} \hat{G}(k) = \frac{1}{K} \log \left( \frac{a^{-2} + M^2}{L^{-2} + M^2} \right)$$

(99)
where \( k_{UV} = 1/a \) and \( k_{IR} = 1/L \) are the ultraviolet and infrared cutoff. Recall that the volume is \( Vol. = L^2 \). Inserting this expression in the definition of the effective mass gives:

\[
M^2 = \frac{D_0}{\pi K L} \left( \frac{a^{-2} + M^2}{L^{-2} + M^2} \right)^{-1/2K}
\]  

(100)

The effective mass vanishes as \( L \to \infty \), but with different power of \( L \) for \( K < K_c = 1 \) or \( K > K_c = 1 \). These differences arise from the different behavior of the last term in eq.(100). For \( K < K_c \), we have \( M^2 \ll (1/L)^2 \), and therefore the last term in eq.(100) behaves like \((1/L)^{1/K}\). For \( K > K_c \), the mass term dominates, \( M^2 \gg (1/L)^2 \), and therefore, the last term in eq.(100) behaves like \( M^{1/K} \). Hence, for \( D_0 > 0 \) we obtain:

\[
M^2(L) \sim \left( \frac{1}{L} \right)^{1+\frac{1}{K}} ; \quad Q(L) \sim \frac{1}{K} \log L^2 \quad \text{for} \quad K < K_c
\]  

(101)

\[
(M^2(L))^{1-\frac{1}{K}} \sim \left( \frac{1}{L} \right)^{1-\frac{1}{K}} ; \quad Q(L) \sim \frac{1}{2K-1} \log L^2 \quad \text{for} \quad K > K_c
\]  

(102)

For \( D_0 < 0 \), since the situation is quite different since the effective mass square is negative. The Green function \( \hat{G}(k) \) now possesses a pole at \( k^2 = -M^2 \). In order to analyse the effect of this pole we have to remember that for a system in a box of length \( L \), the momenta are quantized to discrete values : \( (k_x, k_y) = \left( \frac{2\pi n_x}{L}, \frac{2\pi n_y}{L} \right) \), with \( (n_x, n_y) \) integers. So in the key equation (98) the integral is actually a discret sum : \( \int \frac{d^2k}{(2\pi)^2} \to \frac{1}{L^2} \sum_{n_x, n_y} \). In eq. (98), we separate the first terms which correspond to \( (n_x = \pm, n_y = \pm) \) from the others which we approximate by an integral. We obtain:

\[
Q(L) = \frac{4\pi}{KL^2} \frac{4}{(2\pi L)^2 + M^2} + \frac{4\pi}{K} \int_{1/a} \frac{d^2k}{(2\pi)^2} \left( \frac{1}{k^2 + M^2} \right)
\]

\[
= \frac{4\pi}{KL^2} \frac{4}{(2\pi L)^2 + M^2} + \frac{1}{K} \log \left( L^2/a^2 \right)
\]  

(103)

where we have neglected the \( M^2 \) dependence in the last integral. Once again, the effective mass behaves differently as \( L \to \infty \) for \( K < K_c \) and \( K > K_c \). These different behaviors are distinguished by the relative importance between the two terms in eq.(103). Indeed, suppose that \( M^2 \ll 1/L^2 \). Then the first term in (103) is irrelevant and therefore \( Q(L) \sim \frac{1}{K} \log L^2 \). However, inserting this value of \( Q \) in the definition of \( M^2 \) as a function of \( Q \), cf eq.(98), we deduce then that \( M^2(L) \sim (1/L)^{1+\frac{1}{K}} \). Thus consistency of the hypothesis \( M^2 \ll 1/L^2 \) requires \( K < 1 \). When \( M^2 \) becomes of order \( 1/L^2 \), the first term in (103) dominates and \( M^2 \) remains frozen to this values. The expression of \( M^2 \) as a function of \( Q \) then tell us that \( Q(L) \sim \log L^2 \). Summarizing, for \( D_0 < 0 \) we get:

\[
M^2(L) \sim \left( \frac{1}{L} \right)^{1+\frac{1}{K}} ; \quad Q(L) \sim \frac{1}{K} \log L^2 \quad \text{for} \quad K < K_c
\]  

(104)

\[
M^2(L) \sim \left( \frac{1}{L} \right)^2 ; \quad Q(L) \sim \log L^2 \quad \text{for} \quad K > K_c
\]  

(105)

Notice that in the high temperature phase \( K < K_c \), the behavior of \( Q(L) \) is the same for \( D_0 \) positive or negative, while this behavior is different in the low temperature phase.
These behaviors have been obtained at fixed disorder. We can now average over the disorder. Since the effective mass was only a function of $D_0$ which is symmetrically distributed around the origin, the average value of $Q(L)$ is half of the sum of its values for $D_0$ positive and negative. Hence, at large $L$, we have:

$$Q(L) \sim \frac{1}{K} \log L^2 \quad \text{for } K < K_c$$

$$Q(L) \sim \frac{K}{2K - 1} \log L^2 \quad \text{for } K > K_c$$

The averaged behavior at high temperature is the same as in absence of disorder. That is, the disorder is irrelevant in the high temperature phase as we found in the previous section using the replica approach. But the disorder is relevant in the low temperature phase. However the result obtained for $Q(L)$ in the low temperature phase disagree with the result obtained sing the symmetric replica trick, cf eq.(88).

This variational method gives poor results for the correlation functions. Indeed, since we choose the Gaussian ansatz (91) to be centered around the origin, the variational one-point functions vanishes: $\langle \Phi(x) \rangle_0 = 0$. But the connected correlation are unaffected by the disorder since it is protected by the $U(1)$ symmetry. Therefore, the variational two-point function is also unaffected by the disorder, which is probably not realistic. A more appropriate ansatz could be to choose a gaussian action not centered around the origin.

Also, this variational approach does not take one-loop effect into account. This could be the origin of the disagrement between the renormalization group and the variational approaches.

### 7 Replica symmetry breaking or not?

In this section we very shortly describe how replica symmetry breaking is incorporated in the renormalization group perturbative approach based on the replica method. The aim is not to present all the details and subtilities of the replica symmetry breaking, (there already exist extensive reviews on the subject), but only to introduce the main steps.

As example we choose the minimal conformal models perturbed by random bond interaction. The replica symmetric was studied in [38], while the study of the theory with replica symmetry breaking was done in [39]. The basic examples are the random bond Ising or Potts models.

The simplest minimal model is the Ising model, whose disordered version was studied above in the supersymmetric approach. In the scaling limit, random bond interaction is represented by a perturbation by the energy operator $\epsilon(x)$ with a random coupling constant. In the Kac classification this operator is the $\Phi_{12}$ operator. Thus we consider the random models:

$$S[g(x)] = S_\ast + \int d^2 x g(x) \Phi_{12}(x)$$

where $S_\ast$ represents the action of the corresponding minimal conformal model, e.g the Ising or Potts models. As we have seen, in the Ising model the disorder is marginal and only induces logarithmic corrections. The dimension of $\Phi_{12}$ in the Potts model is $\dim(\Phi_{12}) = \frac{4}{5} < 1$. It is therefore a strictly relevant disordered perturbation.
This models were studied in ref.[38] using perturbative renormalization group computer in the a priori symmetric replicated theory.

After replica, the effective action is:

\[ S_{\text{eff}} = \sum_z S_z^r + \sigma \int d^2x \sum_{r \neq s} \Phi_{12}^r(x)\Phi_{12}^s(x) \]  \hspace{1cm} (108)

In eq.(108), one chooses to restrict the sum to \( r \neq s \) since in the OPE of two \( \Phi_{12}(x) \) operators, which is given by the fusion rule \( \Phi_{12} \times \Phi_{12} = 1 + \Phi_{13} \), only the identity arises with a singular term. The compatibility of this hypothesis with the renormalization group has to be checked, and this is not a priori clear.

Replica symmetry breaking is incorporated in two steps [40]. One first promotes the coupling constant \( \sigma \) to a matrix \( \sigma_{rs} \), and then consider \( \sigma_{rs} \) at \( n = 0 \) has a hierarchical Parisi matrix. This amounts to parametrize it by a diagonal element and a function \( \sigma(x) \) with \( x \in [n,1], \ n \rightarrow 0 : \)

\[ \sigma \rightarrow \sigma_{rs} \rightarrow (\tilde{\sigma}, \sigma(x)) \]  \hspace{1cm} (109)

The multiplication law of two Parisi matrices parametrized by \( (\tilde{\sigma}_1, \sigma_1(x)) \) and \( (\tilde{\sigma}_2, \sigma_2(x)) \) is then defined by :

\[ (\tilde{\sigma}_1, \sigma_1(x)) \cdot (\tilde{\sigma}_2, \sigma_2(x)) = (\tilde{h}, h(x)) \]  \hspace{1cm} (110)

with

\[ h = \tilde{\sigma}_1 \tilde{\sigma}_2 - \sigma_1 \sigma_2 \]  \hspace{1cm} (111)

\[ h(x) = -n\sigma_1(x)\sigma_2(x) + (\tilde{\sigma}_1 - \sigma_1)\sigma_2(x) + (\tilde{\sigma}_2 - \sigma_2)\sigma_2(x) \]

\[ -\int_n^x dy(\sigma_1(x) - \sigma_1(y))(\sigma_2(x) - \sigma_2(y)) \]  \hspace{1cm} (112)

where \( f_n = \int f(x) \).

One can now study the consequences of this ansatz in the renormalization group. This is done by first computing the renormalization group equations with the matrix \( \sigma_{rs} \) and then implementing Parisi’s ansatz. As usual the one loop beta functions \( \beta_{rs} = \dot{\sigma}_{rs} \) are encoded in the operator product expansions. However, non replica symmetric fixed points in the random minimal conformal models only appear at two loops.

The two-loop beta function was computed in ref.[39], using an epsilon expansion. The central charge is parametrized as :

\[ c = 13 - 6(\alpha_+^2 + \alpha_-^2) \quad \text{with} \quad \alpha_+^2 = \frac{4}{3} - \epsilon \]

The case \( \epsilon = 0 \) corresponds to the Ising model \( c = \frac{1}{2} \). The Potts model corresponds to \( \epsilon = \frac{2}{15} \). The dimension of the \( \Phi_{12} \) primary field is then :

\[ \text{dim}(\Phi_{12}) = 1 - \frac{3}{2} \epsilon \]

For \( \epsilon = 0 \) the disorder is marginal, while for \( \epsilon \ll 1 \) the disorder is slightly relevant. This allows to implement an \( \epsilon \)-expansion, and in particular to determine the non-trivial fixed point in an \( \epsilon \)-expansion. Since we assumed that the diagonal matrix elements of \( \sigma_{rs} \) vanish,
in Parisi’s ansatz the matrix $\sigma_{rs}$ is represented only be a function $\sigma(x)$. The vanishing of the beta functions then determines the possible fixed point function $\sigma_*(x)$. According to ref. \[39\], the fixed point equation reads:

$$3\epsilon \sigma_*(x) - 2\bar{\sigma}_* \sigma_*(x) - \int_0^x dy (\sigma_*(x) - \sigma_*(y))^2 + \sigma_*(x) + \bar{\sigma}_*^2 \sigma_*(x) = 0$$  \hfill (113)

with $\bar{\sigma}_* = \int_0^1 dx \sigma_*(x)$.

The replica symmetric solution corresponds to $\sigma_*(x) = \text{const}$:

$$\sigma_*(x) = \text{const.} = \frac{3}{2}\epsilon + \frac{9}{4}\epsilon^2 + \cdots$$

The non replica symmetric solution is given by a solution for which $\sigma_*(x)$ grows linearly for $0 < x < x_1$ and then remains constant for $x_1 < x < 1$. This is called a one-step replica symmetry breaking. It is found by deriving eq.(113) with respect to $x$ as many times as necessary. Explicitly, one has:

$$\sigma_*(x) = \begin{cases} \frac{1}{2}x, & \text{if } 0 < x < x_1, \\ \frac{1}{2}x_1, & \text{if } x_1 < x < 1. \end{cases} \text{ with } x_1 = \frac{9}{2}\epsilon + \frac{27}{2}\epsilon^2 + \cdots.$$  

This solution is not present at one-loop. It has been checked that it is a stable solution of eq.(113).

The symmetric and non-symmetric solution can be distinguished by analysing the anomalous dimensions at the corresponding infrared fixed points. For example, the dimension of the energy operator at the infrared fixed point differs in the two solutions by two percent up to $O(\epsilon^3)$. There is up to now no numerical evidence in favor of the non-symmetric solution \[41\].

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8 Appendix: Renormalization group and OPE.

In this appendix we gather standard informations about the renormalization group in two dimension, cf eg. \[42\] \[43\].

Consider partition functions and correlation functions computed with the measure $\int D\phi \exp(-S)$ with a perturbed action:

$$S = S_* + \sum_i g^i \int d^2 x \Phi_i(x)$$  \hfill (114)

where $\Phi_i(x)$ are relevent primary operators of dimension $h_i$. Suppose that these fields satisfy the following operator product expansion:

$$\Phi_i(x)\Phi_j(y) = \frac{C_{ij}^k}{|x-y|^{h_i+h_j-h_k}} \Phi_k(y) + \cdots$$  \hfill (115)

Then, the beta function at one loop is:

$$\dot{g}^i = \beta^i(g) = (2 - h_i)g^i - \pi \sum_{jk} C_{ij}^k g^j g^k + \cdots$$  \hfill (116)
No summation in the first term but summation over \( j, k \) in the second. The summation is over all the relevant fields generated by the product operator expansion.

In the same way, if \( \mathcal{O}_\alpha \) is a set of operators with OPE with the perturbing field \( \Phi_i \) given by the structure constant \( C_{i\alpha\alpha'} \). Then the matrix of anomalous dimensions \( (\gamma = -a\partial_a \log Z) \) is:

\[
\gamma_{\alpha\alpha'}^\alpha = h_\alpha \delta_{\alpha\alpha'} + 2\pi \sum_j C_{j\alpha\alpha'}^\alpha g^j + \cdots \tag{117}
\]

Note that we have: \( \gamma_j^i = 2\delta_j^i - \partial_i \beta^j \).

The renormalization group equations are:

\[
\left[ \sum_a x_a^\nu \frac{\partial}{\partial x_a^\nu} + \sum_a \alpha_a(g) - \sum_j \beta_j(g) \frac{\partial}{\partial g^j} \right] \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_P(x_P) \rangle = 0 \tag{118}
\]

For the two point functions, the integrated version of the RG equation reads:

\[
\langle \mathcal{O}(R)\mathcal{O}(0) \rangle_{g(a)} = \langle \mathcal{O}(a)\mathcal{O}(0) \rangle_{g(R)} \exp \left( -2 \int_{g(a)}^{g(R)} \frac{\gamma_\Phi(g)}{\beta(g)} \right) \tag{119}
\]

where \( \gamma_\Phi \) is the \( \gamma \)-function for \( \Phi \).

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