A PRIORI ERROR ESTIMATES FOR COMPATIBLE SPECTRAL DISCRETIZATION OF THE STOKES PROBLEM FOR ALL ADMISSIBLE BOUNDARY CONDITIONS

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Abstract. This paper describes the recently developed mixed mimetic spectral element method for the Stokes problem in the vorticity-velocity-pressure formulation. This compatible discretization method relies on the construction of a conforming discrete Hodge decomposition, that is based on a bounded projection operator that commutes with the exterior derivative. The projection operator is the composition of a reduction and a reconstruction step. The reconstruction in terms of mimetic spectral element basis-functions are tensor-based constructions and therefore hold for curvilinear quadrilateral and hexahedral meshes.

For compatible discretization methods that contain a conforming discrete Hodge decomposition, we derive optimal a priori error estimates which are valid for all admissible boundary conditions on both Cartesian and curvilinear meshes. These theoretical results are confirmed by numerical experiments. These clearly show that the mimetic spectral elements outperform the commonly used $H(\text{div})$-compatible Raviart-Thomas elements.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded contractible domain with boundary $\Gamma = \partial \Omega$. On this domain we consider the Stokes problem, consisting of the equations for conservation of momentum and for conservation of mass,

\begin{align}
\nabla \cdot \sigma &= \mathbf{f} \quad \text{on } \Omega, \\
\text{div } \mathbf{u} &= \mathbf{g} \quad \text{on } \Omega,
\end{align}

where the stress tensor $\sigma$ is given by

\begin{equation}
\sigma = -\nu \nabla \mathbf{u} + p \mathbf{I},
\end{equation}

with $\mathbf{u}$ the velocity vector, $p$ the pressure, $\mathbf{f}$ the forcing term, $\mathbf{g}$ the mass source and $\nu$ the kinematic viscosity. For analysis purposes we choose $\nu = 1$.

This paper considers the recently developed mixed mimetic spectral element method (MMSEM) [40, 41]. This compatible finite/spectral element method is based on the compatible discretization of the exterior derivative $d$ from differential geometry, which represents the vector operators, grad, curl and div. The Stokes problem expressed in terms of these vector operations is known as the vorticity-velocity-pressure (VVP) formulation, [9, 23]. For the VVP formulation, the Laplace operator is split using the vector identity, $-\Delta \mathbf{u} = \text{curl} \text{curl}^* \mathbf{u} - \text{grad}^* \text{div} \mathbf{u}$, and by introducing vorticity as auxiliary variable, $\mathbf{\omega} = \text{curl}^* \mathbf{u}$. The VVP formulation of the Stokes problem becomes

\begin{align}
\mathbf{\omega} - \text{curl}^* \mathbf{u} &= 0, \quad \text{on } \Omega \\
\text{curl } \mathbf{\omega} - \text{grad}^* \text{div} \mathbf{u} + \text{grad}^* p &= \mathbf{f}, \quad \text{on } \Omega \\
\text{div } \mathbf{u} &= \mathbf{g}, \quad \text{on } \Omega.
\end{align}

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Following [9, 40] we make a distinction between the operators grad, curl and div, that correspond to
the classical Newton-Leibnitz, Stokes circulation and Gauss divergence theorems, and the operators
-grad*, curl* and -div* that are their formal Hilbert adjoints,

\[(\vec{a}, -\text{grad}^* \vec{b}) := (\text{div} \vec{a}, \vec{b}), \quad (\vec{a}, \text{curl}^* \vec{b}) := (\text{curl} \vec{a}, \vec{b}), \quad (\vec{a}, -\text{div}^* \vec{b}) := (\text{grad} \vec{a}, \vec{b}).\]

The distinction between the two types of differential operators is made explicitly, because the
construction of our conforming finite element spaces relies on the three mentioned integration
theorems, while the mixed formulation relies on the formal Hilbert adjoint relations. While in
vector calculus this distinction is not common, in differential geometry these structures naturally
appear since they make a clear distinction between metric-free (topological) and metric-dependent
operations.

The MMSEM is a compatible discretization method that relies on the construction of a con-
forming discrete Hodge-decomposition, which implies a discrete Poincaré inequality. It requires
the development of a bounded projection operator that commutes with the exterior derivative.
The bounded projection is a composition of a reduction by means of integration and mimetic
spectral element basis functions as reconstruction.

The reduction onto \(k\)-dimensional submanifolds result in the discrete unknowns representing
integral quantities. This is one of the major differences with related methods as the Marker and
Cell scheme [32] and the lowest-order Raviart-Thomas and Nédélec compatible finite elements
[45, 48], where use is made of averaged quantities.

The basis functions, used for the reconstruction, are constructed using tensor products of one
dimensional nodal and edge interpolation basis functions [28], and therefore hold for quadrilateral
and hexahedral meshes. They belong to the class of compatible finite elements, and were con-
structed based on the mimetic framework first described in [38] and later extended in [11]. The
mimetic framework, including the mimetic spectral elements, were extensively described in [41].
This mimetic framework relies on the languages of differential geometry instead of vector calculus,
and algebraic topology as its discrete counterpart.

The use of differential geometry and algebraic topology enjoys increasing popularity for the
development of compatible schemes, [5, 6, 11, 12, 13, 21, 34, 35]. Compatible discretizations are
often combined with mixed formulations. Mixed formulations are described extensively in among
others [15, 30] and in terms of differential forms in [5, 6] for the Hodge-Laplacian and in [40] for
the VVP formulation of the Stokes problem.

The MMSEM contains compatible finite elements that are compatible with all admissible types
of boundary conditions for the Stokes problem in VVP formulation. We will show that the
method obtains optimal rates of convergence for all variables on curvilinear meshes and for all
admissible boundary conditions, i.e. standard and nonstandard. It is therefore extending the error
estimates found in literature, which are often specifically constructed for certain types of boundary
conditions, [1, 4, 10, 14, 24, 29]. To show optimal convergence a priori error estimates are derived.

This is an improvement with respect to the well-known Raviart-Thomas compatible finite ele-
ments. These are not compatible in case of Dirichlet boundary conditions and therefore lead to
suboptimal convergence behavior, as was shown in [4, 24]. This non-compatibility results in a
decrease in rate of convergence of maximal 3/2 order.

From a physical/fluid dynamics point-of-view the new method is relevant because it combines
optimal convergence with a pointwise divergence-free discretization (in absence of any mass source)
of arbitrary order on curvilinear meshes, valid for all allowable types of boundary conditions, among
which the no-slip condition.

The derived rates of convergence are confirmed using simple manufactured solution problems,
discretized on both Cartesian and curvilinear meshes. The fact that the analysis holds for all
admissible boundary conditions is also reflected in the numerical results.

This paper is organized as follows: First an introduction into differential geometry is given
and the Stokes problem is reformulated in terms of differential forms. In Section 3 the mixed
formulation is given and well-posedness is proven. In Section 4 the key properties of the mimetic
discretization are explained that lead to compatible function spaces. This includes a discussion on
the relevant properties of algebraic topology, the definitions of mimetic operators, the introduction
of mimetic spectral element basisfunctions and finally the proof of discrete well-posedness. Having formulated the conforming/compatible finite element spaces, the error estimates are developed in Section 5 and the numerical results are shown in Section 6.

2. Notation and preliminaries

2.1. Differential forms. Differential forms offer significant benefits in the construction of structure-preserving spatial discretizations. For example, the coordinate-free action of the exterior derivative and generalized Stokes theorem give rise to commuting properties with respect to mappings between different manifolds. Acknowledging and respecting these kind of commuting properties are essential for the structure preserving behavior of the mimetic method.

Only those concepts from differential geometry which play a role in the remainder of this paper will be explained. More can be found in [2, 26, 27, 41].

Let \( \Lambda^k(\Omega) \) denote a space of differential \( k \)-forms or \( k \)-forms, on a sufficiently smooth bounded \( n \)-dimensional oriented manifold \( \Omega \subset \mathbb{R}^n \) with boundary \( \Gamma = \partial \Omega \). Every element \( a \in \Lambda^k(\Omega) \) has a unique representation of the form

\[
(2.1) \quad a = \sum_I f_I(x) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k},
\]

where \( I = i_1, \ldots, i_k \) with \( 1 \leq i_1 < \cdots < i_k \leq n \) and where \( f_I(x) \) is a continuously differentiable scalar function. \( f_I(x) \in C^\infty(\Omega) \). Differential \( k \)-forms are naturally integrated over \( k \)-dimensional manifolds, i.e. for \( a \in \Lambda^k(\Omega) \) and \( \Omega_k \subset \mathbb{R}^n \), with \( k = \text{dim}(\Omega_k) \),

\[
(2.2) \quad \int_{\Omega_k} a \in \mathbb{R} \Leftrightarrow \langle a, \Omega_k \rangle \in \mathbb{R},
\]

where \( \langle \cdot, \cdot \rangle \) indicates a duality pairing between the differential form and the geometry. Note that the \( n \)-dimensional computational domain is indicated as \( \hat{\Omega} \), so without subscript. The differential forms live on manifolds and transform under the action of mappings. Let \( \Phi : \hat{\Omega} \rightarrow \Omega \) be a mapping between two manifolds. Then we can define the pullback operator, \( \Phi^* : \Lambda^k(\Omega) \rightarrow \Lambda^k(\hat{\Omega}) \), expressing the \( k \)-form on the \( n \)-dimensional reference manifold, \( \hat{\Omega} \). The mapping, \( \Phi \), and the pullback, \( \Phi^* \), are each others formal adjoints with respect to a duality pairing (2.2),

\[
(2.3) \quad \int_{\Phi(\hat{\Omega})} a = \int_{\hat{\Omega}} \Phi^*a \Leftrightarrow \langle a, \Phi(\hat{\Omega}) \rangle = \langle \Phi^*a, \hat{\Omega} \rangle,
\]

where \( \hat{\Omega}_l \) is an \( l \)-dimensional submanifold of \( \hat{\Omega} \) and \( \Omega_k = \Phi(\hat{\Omega}_l) \) a \( k \)-dimensional submanifold of \( \Omega \). A special case of the pullback operator is the trace operator. The trace of \( k \)-forms to the boundary, \( \text{tr} : \Lambda^k(\Omega) \rightarrow \Lambda^k(\partial \Omega) \), is the pullback of the inclusion of the boundary of a manifold, \( \partial \Omega \hookrightarrow \Omega \), see [41].

The wedge product, \( \wedge \), of two differential forms \( a \in \Lambda^k(\Omega) \) and \( b \in \Lambda^l(\Omega) \) is a mapping: \( \wedge : \Lambda^k(\Omega) \times \Lambda^l(\Omega) \rightarrow \Lambda^{k+l}(\Omega) \), \( k + l \leq n \). The wedge product is a skew-symmetric operator, i.e. \( a \wedge b = (-1)^{kl} b \wedge a \).

An important operator in differential geometry is the exterior derivative, \( d : \Lambda^k(\Omega) \rightarrow \Lambda^{k+1}(\Omega) \). It is induced by the generalized Stokes’ theorem, combining the classical Newton-Leibnitz, Stokes circulation and Gauss divergence theorems. Let \( \Omega_{k+1} \) be a \((k+1)\)-dimensional manifold and \( a \in \Lambda^k(\Omega) \), then

\[
(2.4) \quad \int_{\partial \Omega_{k+1}} a = \int_{\Omega_{k+1}} da \Leftrightarrow \langle a, \partial \Omega_{k+1} \rangle = \langle da, \Omega_{k+1} \rangle,
\]

where \( \partial \Omega_{k+1} \) is a \( k \)-dimensional manifold being the boundary of \( \Omega_{k+1} \). The duality pairing in (2.4) shows that the exterior derivative is the formal adjoint of the boundary operator \( \partial : \Omega_{k+1} \rightarrow \Omega_k \). The exterior derivative is independent of any metric and coordinate system. Applying the exterior derivative twice always leads to the null \((k+2)\)-form, \( d(da) = 0^{(k+2)} \) for all \( a \in \Lambda^k(\Omega) \). As a consequence, on contractible domains the exterior derivative gives rise to an exact sequence, called De Rham complex [27], and indicated by \((\Lambda, d)\),

\[
(2.5) \quad \mathbb{R} \hookrightarrow \Lambda^0(\Omega) \overset{d}{\longrightarrow} \Lambda^1(\Omega) \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \Lambda^n(\Omega) \overset{d}{\longrightarrow} 0.
\]
In vector calculus a similar sequence exists, where, from left to right for \( \mathbb{R}^3 \), the \( d \)'s denote
the vector operators grad, curl and div. The exterior derivative and wedge product are related
according to Leibnitz's rule as: for all \( a \in \Lambda^k(\Omega) \) and \( b \in \Lambda^l(\Omega) \),
\[
\text{d}(a \wedge b) = \text{da} \wedge b + (-1)^k a \wedge \text{db}, \quad \text{for } k + l < n.
\]
The pullback operator and exterior derivative possess the following commuting property,
\[
\Phi^* \text{d} = \text{d} \Phi^*, \quad \forall a \in \Lambda^k(\Omega).
\]
In this paper we will consider Hilbert spaces \( L^2\Lambda^k(\Omega) \supset \Lambda^k(\Omega) \), where in (2.1) the
functions \( f_i(x) \in L^2(\Omega) \). The pointwise inner-product of \( k \)-forms, \((\cdot, \cdot)\), is constructed
using inner products of one-forms, that is based on the inner product on vector spaces, see [26, 27].
The wedge product and inner product produce the Hodge-\( \star \) operator, \( \star : L^2\Lambda^k(\Omega) \to L^2\Lambda^{n-k}(\Omega) \), a metric operator
that includes orientation. Let \( a, b \in L^2\Lambda^k(\Omega) \), then
\[
a \wedge \star b := (a, b)\sigma,
\]
where \( \sigma \in \Lambda^n(\Omega) \) is a unit volume form, \( \sigma = \star 1 \). In geometric physics the Hodge-\( \star \) switches
between an inner-oriented description of physical variables and an outer-oriented description.
See [40, 41, 43, 52] for a thorough discussion on the concepts of inner and outer orientation.
The space of square integrable \( k \)-forms on \( \Omega \) can be equipped with a \( L^2 \) inner product,
\((\cdot, \cdot)_\Omega : L^2\Lambda^k(\Omega) \times L^2\Lambda^k(\Omega) \to \mathbb{R} \), given by,
\[
(a, b)_\Omega := \int_{\Omega} (a, b)\sigma = \int_{\Omega} a \wedge \star b.
\]
The norm corresponding to the space \( L^2\Lambda^k(\Omega) \) is \( |a|_{L^2\Lambda^k} = \sqrt{(a, a)}_{\Omega} \). Higher degree Sobolev
spaces, \( H^m\Lambda^k \), consists of all \( k \)-forms as in (2.1) where \( f_i(x) \in H^m(\Omega) \), with corresponding norms
\( |a|_{H^m\Lambda^k} \) and \( |a|_{H^{m+1}\Lambda^k} \). The Hilbert space associated to the exterior derivative \( H\Lambda^k(\Omega) \) is defined as
\[
H\Lambda^k(\Omega) = \{ a \in L^2\Lambda^k(\Omega) | \text{da} \in L^2\Lambda^{k+1}(\Omega) \}.
\]
and the norm corresponding to \( H\Lambda^k(\Omega) \) is defined as \( \|a\|_{H\Lambda^k} := \|\text{da}\|_{L^2\Lambda^{k+1}} \). The \( H\Lambda^k \)
semi-norm is the \( L^2 \)-norm of the exterior derivative, \( |a|_{H\Lambda^k} = \|\text{da}\|_{L^2\Lambda^{k+1}} \). Note that \( H^1\Lambda^k(\Omega) \subseteq H\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \), where the left equality holds for \( k = 0 \) and the right for \( k = n \). The \( L^2 \)-de Rham complex, also called Hilbert complex [16], \( (H\Lambda, d) \), is the exact sequence of maps and spaces given by
\[
\mathbb{R} \hookrightarrow H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \xrightarrow{d} 0.
\]
In terms of vector operations the Hilbert complex becomes for \( \Omega \subset \mathbb{R}^3 \),
\[
H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega),
\]
and for \( \Omega \subset \mathbb{R}^2 \), either
\[
H^1(\Omega) \xrightarrow{\text{grad}} H(\text{rot}, \Omega) \xrightarrow{\text{rot}} L^2(\Omega), \quad \text{or} \quad H^1(\Omega) \xrightarrow{\text{curl}} H(\text{curl}, \Omega) \xrightarrow{\text{div}} L^2(\Omega).
\]
The two are related by the Hodge-\( \star \) operator (2.8), see [46],
\[
\begin{align*}
&H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} L^2\Lambda^2(\Omega) \quad \text{or} \quad H^1(\Omega) \xrightarrow{\text{curl}} H(\text{curl}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \\
&L^2\Lambda^2(\Omega) \xleftarrow{\star} H\Lambda^1(\Omega) \xleftarrow{\star} H\Lambda^0(\Omega) \quad \text{or} \quad L^2(\Omega) \xleftarrow{\text{rot}} H(\text{rot}, \Omega) \xleftarrow{\text{grad}} H^1(\Omega).
\end{align*}
\]
Remark 1. The upper complex is associated with outer-oriented \( k \)-forms, i.e. \( k \)-forms that are
associated with outer-oriented manifolds, and the lower complex is associated with inner-oriented
\( k \)-forms. In this paper we mainly consider the upper complex and circumvent the lower complex
by means of integration by parts. Only the pressure and tangential velocity boundary conditions
are given on the lower complex, as we will see in the following sections.
A similar double Hilbert complex can be constructed in $\mathbb{R}^3$. Since the exterior derivative is nilpotent, it ensures that the range, $B^k := dH^k(\Omega)$, of the exterior derivative on $(k-1)$-forms is contained in the nullspace, $Z^k := \{ a \in H^k(\Omega) \mid da = 0 \}$, of the exterior derivative on $k$-forms, $B^k \subseteq Z^k$.

Every space of $k$-forms in the complex $(\Lambda, d)$ can be decomposed into the nullspace of $d$, and its orthogonal complement, $H^k(\Omega) = Z^k \oplus Z^k_\perp$. This is the Hodge decomposition, where on contractible domains $Z^k = B^k$. By the Hodge decomposition it follows that the exterior derivative is an isomorphism $d : Z^k_\perp \to B^{k+1}$.

The inner product gives rise to the Hilbert adjoint of the exterior derivative, the codifferential operator, $d^* : H^k(\Omega) \to L^2\Lambda^{k-1}(\Omega)$. Let $H^k(\Omega) = \{ a \in L^2\Lambda^k(\Omega) \mid d^* a = 0 \}$, then

$$\langle da, b \rangle_\Omega = \langle a, d^* b \rangle_\Omega, \quad \forall a \in H^k(\Omega), \ b \in H^k(\Omega).$$

In case of non-zero trace, and by combining (2.9), (2.4) and (2.6), we obtain integration by parts,

$$(a, d^* b)_\Omega = \langle da, b \rangle_\Omega - \int_{\partial\Omega} tr a \wedge tr * b.$$

Also the codifferential operator is nilpotent, $d^* (d^*(a^{[k]})) = 0$, i.e., its range is contained in its nullspace, $B^{*,k} \subseteq Z^{*,k}$, where $B^{*,k} := d^* H^k(\Omega)$ and $Z^{*,k} := \{ a \in H^k(\Omega) \mid d^* a = 0 \}$. In fact the codifferential is an isomorphism $d^* : Z^{*,k-1} \to B^{*,k+1}$, where $Z^{*,k-1}$ follows from the following Hodge decomposition, $\Lambda^k(\Omega) = Z^{*,k} \oplus Z^{*,k}_\perp$. On contractible manifolds this gives rise to the following exact sequence,

$$0 \xrightarrow{d^*} H^k(\Omega) \xrightarrow{d^*} H^k(\Omega) \xrightarrow{d^*} \cdots \xrightarrow{d^*} H^k(n)(\Omega) \xrightarrow{d^*} \mathbb{R}.$$

In vector notation from right to left the $d^*$'s denote the -grad*, curl* and -div* operators in $\mathbb{R}^3$, as were also mentioned in the introduction. However, whereas the exterior derivative is a metric-free operator, the codifferential operator is metric-dependent. The Hodge-Laplace operator, $\Delta : H^2\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega)$, is constructed as a composition of the exterior derivative and the codifferential operator,

$$\Delta a := (d^* d + dd^*) a.$$

An important inequality in stability analysis, relating the $L^2\Lambda^k$-norm and the $H^k$-norm, is Poincaré inequality.

**Lemma 1 (Poincaré inequality).** [6] Consider the Hilbert complex $(\Lambda, d)$, then the exterior derivative is a bounded bijection from $Z^{*,k-1}$ to $B^{k+1}$, and hence, by Banach’s bounded inverse theorem, there exists a constant $C_p$ such that

$$\|a\|_{H^k} \leq C_p \|a\|_{L^2\Lambda^{k+1}}, \quad \forall a \in Z^{*,k-1}.$$

Finally, for Hilbert spaces with essential boundary conditions we write, $H^k_0(\Omega) := \{ a \in H^k(\Omega) \mid \text{tr } a = 0 \}$, and for natural boundary conditions we consider the following trace map, $\text{tr } * : H^k(\Omega) \to H^{\frac{1}{2}}\Lambda^{n-k}(\partial\Omega)$.

### 2.2. Stokes problem in differential form notation.

Consider again a bounded contractible domain $\Omega \subset \mathbb{R}^n$. Because we require exact conservation of mass and because we can perform exact discretization of the exterior derivative, see Section 4.2, we use the following formulation for the Stokes problem: let $(\omega, u, p) \in \{ \Lambda^{n-2}(\Omega) \times \Lambda^{n-1}(\Omega) \times \Lambda^n(\Omega) \}$, then the VVP formulation is given by

$$\begin{align}
\omega - d^* u &= 0, \quad \text{in } \Lambda^{n-2}(\Omega), \\
\text{d}^* du + d\omega + d^* p &= f, \quad \text{in } \Lambda^{n-1}(\Omega), \\
\text{d}u &= g, \quad \text{in } \Lambda^n(\Omega).
\end{align}$$

In the VVP formulation the pressure in (2.18b) acts as a Lagrange multiplier for the constraint on velocity, (2.18c), whereas velocity in (2.18a) acts as a Lagrange multiplier for the constraint on vorticity in (2.18b).
Let $\Gamma = \partial \Omega$ be the boundary of $\Omega$, where

$$\Gamma = \Gamma_\omega \cup \Gamma_t, \quad \Gamma_\omega \cap \Gamma_t = \emptyset, \quad \text{and} \quad \Gamma = \Gamma_n \cup \Gamma_\pi, \quad \Gamma_n \cap \Gamma_\pi = \emptyset.$$ 

We will impose the tangential vorticity and normal velocity as essential boundary conditions, and the tangential velocity and the pressure plus divergence of velocity as the natural boundary conditions:

\begin{align}
(2.19a) & \quad \text{tr } \omega = 0 \quad \text{on } \Gamma_\omega, \\
(2.19b) & \quad \text{tr } u = 0 \quad \text{on } \Gamma_n, \\
(2.19c) & \quad \text{tr } *u = u_{b,t} \quad \text{on } \Gamma_t, \quad \text{with } u_{b,t} \in H^\frac{1}{2} \Lambda^1(\Gamma_t), \\
(2.19d) & \quad \text{tr } (du + p) = \Pi_b \quad \text{on } \Gamma_\pi, \quad \text{with } \Pi_b \in H^\frac{1}{2} \Lambda^0(\Gamma_\pi).
\end{align}

Then the boundary $\Gamma$ can be partitioned into four sections, $\Gamma = \bigcup_{i=1}^4 \Gamma_i$, with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, where

$$\Gamma_1 := \Gamma_t = \Gamma_n, \quad \Gamma_2 := \Gamma_t = \Gamma_\pi, \quad \Gamma_3 := \Gamma_\omega = \Gamma_n, \quad \Gamma_4 := \Gamma_\omega = \Gamma_\pi.$$

This decomposition, introduced before in [23, 36, 40], shows all admissible boundary conditions. It will also follow directly from the mixed formulation, see (3.7), Section 3.

In case, $\Gamma_1 \cap \Gamma_3, \Gamma_2 \cup \Gamma_4 = \emptyset$, no pressure boundary conditions are prescribed, and so the pressure is only determined up to an element $\tilde{p} \in Z^{*,-n}$, i.e. up to a constant. As a post processing step either the pressure in a point in $\Omega$ can be set, or a zero average pressure can be imposed; i.e. $\int_\Omega \tilde{p} = 0$. In case $\Gamma = \Gamma_4$, no velocity boundary conditions are prescribed, and so the solution of velocity is determined modulo a curl$^*$-free element, i.e. modulo $\hat{u} \in Z^{*,-n-1}$.

3. Mixed formulation

3.1. Mixed formulation of Stokes problem. The use of a mixed formulation is based on the following reasoning: We know how to discretize exactly the metric-free exterior derivative $d$, but it is less obvious how to treat the codifferential operator $d^*$.

3.1.1. Generalized Poisson problem. Take for example the generalized Poisson problem using the Hodge-Laplacian acting on $k$-forms, $(dd^* + d^*d)u = f$, on $\Omega$ with boundary $\Gamma = \partial \Omega$. A standard Galerkin approach, using integration by parts (2.14), would give; find $u \in H^{\Lambda_k}(\Omega) \cap H^*\Lambda^k(\Omega)$ with $du \in H^*\Lambda^{k+1}(\Omega)$ and $d^*u \in H^{\Lambda^{k-1}}(\Omega)$, given $f \in L^2\Lambda^k(\Omega)$, such that

$$\int_\Omega (d^*v, d^*u) + (dv, du) = (v, f), \quad \forall u \in H^{\Lambda_k}(\Omega) \cap H^*\Lambda^k(\Omega).$$

It has a corresponding minimization problem for an energy functional over the space $H^{\Lambda_k}(\Omega) \cap H^*\Lambda^k(\Omega)$. The standard Galerkin formulation is coercive, which immediately implies stability. Corresponding to this standard Galerkin formulation one usually chooses a $H^1\Lambda^k(\Omega)$-conforming approximation space. This could be a standard continuous piecewise polynomial vector space based on nodal interpolation.

However, in case of a nonconvex polyhedral or curvilinear or noncontractible domain $\Omega$, for allmost all $f$, $H^{\Lambda_k}(\Omega) \cap H^*\Lambda^k(\Omega) \not\subset H^1\Lambda^k(\Omega)$. Consequently, the solution will be stable but inconsistent in general, [20]. In other words, the solution converges to the wrong solution. Unfortunately, it seems not possible to construct $H^{\Lambda_k}(\Omega) \cap H^*\Lambda^k(\Omega)$ conforming finite element spaces. Alternatively, one proposed to use mixed formulations, [15]. In contrast to standard Galerkin, the mixed formulation uses integration by parts (2.14) to express each codifferential in terms of an exterior derivative and suitable boundary conditions.

Consequently, mixed formulations require only $H^{\Lambda_k}(\Omega)$-conforming finite element spaces, which are much easier to construct. Therefore, in all cases mixed formulations do converge to the true solution. Mixed formulations correspond to saddle point problems instead of minimization problems.

The derivation of the mixed formulation of the Poisson problem consists of three steps:

1. Introduce an auxiliary variable $\omega = d^*u$ in $H^{\Lambda^{k-1}}$, 

Proposition 1. [7] Problems (2.18)-(2.19) and (3.7) are equivalent, in the sense that any triple \((\omega, u, p) \in \{W \times V \times Q\}\) is a solution of problem (2.18)-(2.19) if and only if it is a solution of problem (3.7).
3.2. Well-posedness of mixed formulation. Before we continue first define the following nullspaces of $W$,

\[(3.8a)\quad Z_W := \{ \tau \in W \mid d\tau = 0 \},\]
\[(3.8b)\quad Z_W^\perp := \{ \tau \in W \mid d^*\tau = 0 \},\]

and consider the following decompositions, $W = Z_W \oplus Z_W^\perp$ and $W = Z_W^\perp \oplus Z_W^{\perp \perp}$. Since vorticity is defined as $\omega = d^*u$, we have $\omega \in Z_W^\perp$, and because we consider contractible domains only, it follows that $\omega \in Z_W^{\perp \perp}$. Note that for $n = 2$, $Z_W^\perp \equiv W$. A similar decomposition can be made for $V$.

Define
\[(3.8c)\quad Z_V := \{ v \in V \mid dv = 0 \},\]
then $V = Z_V \oplus Z_V^\perp$. The velocity is decomposed as $u = u_Z + u_\perp$, where $u_Z \in Z_V$ and $u_\perp \in Z_V^\perp$.

We can write the mixed formulation of (3.7) in a more general representation, using four continuous bilinear forms,

\[a(\cdot, \cdot)_\Omega := (\cdot, \cdot)_\Omega : W \times W \rightarrow \mathbb{R}, \quad b(\cdot, \cdot) := (d\cdot, \cdot)_\Omega : V \times Q \rightarrow \mathbb{R},\]
\[c(\cdot, \cdot)_\Omega := (d\cdot, \cdot)_\Omega : W \times V \rightarrow \mathbb{R}, \quad e(\cdot, \cdot)_\Omega := (d\cdot, d\cdot)_\Omega : V \times V \rightarrow \mathbb{R},\]

and three continuous linear forms

\[f(\cdot) := (\cdot, \cdot)_\Omega + \int_{\Gamma_2 \cup \Gamma_4} \text{tr} \cdot \wedge \Pi_b : V \rightarrow \mathbb{R},\]
\[g(\cdot) := (\cdot, \cdot)_\Omega : Q \rightarrow \mathbb{R}, \quad h(\cdot) := -\int_{\Gamma_1 \cup \Gamma_2} \text{tr} \cdot \wedge u_b,t : W \rightarrow \mathbb{R}.\]

The mixed formulation becomes

\[(3.9a)\quad a(\tau, \omega) - c(\tau, u) = h(\tau), \quad \forall \tau \in W,\]
\[(3.9b)\quad e(v, u) + c(\omega, v) + b(v, p) = f(v), \quad \forall v \in V,\]
\[(3.9c)\quad b(u, q) = g(q), \quad \forall q \in Q.\]

There exists continuity constants $0 < c_a, c_b, c_c, c_e < \infty$ such that

\[(3.10)\quad a(\tau, \kappa) \leq c_a ||\tau||_W ||\kappa||_W, \quad b(v, q) \leq c_b ||v||_V ||q||_Q, \quad c(\tau, v) \leq c_c ||\tau||_W ||v||_V, \quad e(v, w) \leq c_e ||v||_V ||w||_V.\]

By Cauchy-Schwarz we know that $c_a = 1$, however we write $c_a$ for generality purpose. The continuous linear forms are bounded such that

\[(3.11)\quad f(v) \leq ||f||_V ||v||_V, \quad g(v) \leq ||g||_V ||v||_V, \quad h(\tau) \leq ||h||_W ||\tau||_W.\]

At first restrict to all $v = v_Z \in Z_V$. This gives the vorticity-velocity subproblem, which is a saddle point problem:

\[(3.12a)\quad a(\tau, \omega) - c(\tau, u_Z) = h(\tau), \quad \forall \tau \in W,\]
\[(3.12b)\quad c(v_Z, \omega) = f(v_Z), \quad \forall v_Z \in Z_V.\]

**Proposition 2.** [23] System (3.12) has a unique solution $\omega, u_Z \in \{ W \times Z_V \}$ if there exists positive constants $\alpha, \gamma$, such that we have coercivity in the kernel of $W$,

\[(3.13)\quad \inf_{\tau_Z \in Z_W} \sup_{\kappa_Z \in Z_W} \frac{a(\tau_Z, \kappa_Z)}{||\tau_Z||_W ||\kappa_Z||_W} \geq \alpha, \quad \inf_{\kappa_Z \in Z_W} \sup_{\tau_Z \in Z_W} \frac{a(\tau_Z, \kappa_Z)}{||\tau_Z||_W ||\kappa_Z||_W} \geq \alpha,\]

and satisfies the following inf-sup condition for $c(\tau, v_Z)$,

\[(3.14)\quad \inf_{v_Z \in Z_V} \sup_{\tau \in W} \frac{c(\tau, v_Z)}{||\tau||_W ||v_Z||_V} \geq \gamma,\]
Proof. The proof of (3.13) is straightforward, see e.g. [15]. For (3.14), we have $c(\tau, v_\varepsilon) = (d\tau, v_\varepsilon)_\Omega$, where $d : Z^k_V \to Z^k_I$. Thus, given $v_\varepsilon \in Z^k_I$ there exists a unique $\tau_v \in Z^k_V$ such that $d\tau_v = v_\varepsilon$ and $\|\tau_v\|_W \leq c_p \|v_\varepsilon\|_V$ by Lemma 1. Therefore

$$\sup_{\tau \in W} \frac{c(\tau, v_\varepsilon)}{\|\tau\|_W} = \frac{\|v_\varepsilon\|_V^2}{\|\tau_v\|_W} \geq \frac{1}{c_p} \|v_\varepsilon\|_V.$$

□

**Proposition 3.** The full problem (3.9) has a unique solution $(\omega, u, p) \in \{W, V, Q\}$ if conditions (3.13) and (3.14) from Proposition 2 are satisfied and additionally is there exists positive constants $\varepsilon, \beta$, such that we have coercivity in the range of $V$,

$$\inf_{v_\varepsilon \in Z^k_W} \sup_{w_\varepsilon \in Z^k_W} \frac{e(v_\varepsilon, w_\varepsilon)}{\|v_\varepsilon\|_V \|w_\varepsilon\|_V} \geq \varepsilon, \quad \inf_{v_\varepsilon \in Z^k_W} \sup_{w_\varepsilon \in Z^k_W} \frac{e(v_\varepsilon, w_\varepsilon)}{\|v_\varepsilon\|_V \|w_\varepsilon\|_V} \geq \varepsilon,$$

and satisfies the following inf-sup condition for $b(v, q)$,

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta > 0.$$

**Proof.** The proof is similar to that of Proposition 2. See also [8], Section 7.1. □

So well-posedness of the Stokes problem (3.7) relies only on the Hodge decomposition and the Poincaré inequality.

**Corollary 1.** [7, 23] Problem (3.7) is well-posed according to Propositions 2 and 3. That is, for any given data $f \in L^2\Lambda^{n-1}(\Omega)$ and $g \in L^2\Lambda^n(\Omega)$ and natural boundary conditions $u_{b,t} \in H^1\Lambda^{n-1}(\Gamma_t)$ and $\Pi_p \in H^1\Lambda^n(\Gamma_p)$, there exists a unique solution $(\omega, u, p) \in W \times V \times Q$ satisfying (3.7). Moreover, this solution satisfies:

$$\|\omega\|_W + \|u\|_V + \|p\|_Q \leq C \left( \|f\|_{L^2\Lambda^{n-1}} + \|g\|_{L^2\Lambda^n} + \|u_{b,t}\|_{H^1\Lambda^{n-1}} + \|\Pi_p\|_{H^1\Lambda^n} \right),$$

where $C$ is a constant depending only on the Poincaré constant $c_p$ and the continuity constants.

4. Compatible spectral discretization

Well-posedness of the Stokes problem in VVP formulation relies solely on the Hodge decomposition and the Poincaré inequality. For a compatible discretization, these properties need to be respected as well in the finite dimensional spaces. Key ingredient to obtain a discrete Hodge decomposition and discrete Poincaré inequality is the construction of a bounded projection operator that commutes with the exterior derivative.

The compatible spectral discretization consists of three parts. First, the discrete structure is described in terms of chains and cochains from algebraic topology, the discrete counterpart of differential geometry. This discrete structure mimics many of the properties from differential geometry. Secondly, mimetic operators are introduced that relate the continuous formulation in terms of differential forms to the discrete representation based on cochains and finite dimensional differential forms. Thirdly, mimetic spectral element basis functions are described following the definitions of the mimetic operators. In this paper we address these topics only briefly. More details of the mimetic spectral element method can be found in [40, 41]. Finally, well-posedness of the discrete numerical formulation is proven and interpolation error estimates are given.

4.1. Algebraic Topology. Let $D$ be an oriented cell-complex covering the manifold $\Omega$, describing the topology of the mesh, and consisting of $k$-cells $\tau_k$, $k = 0, \ldots, n$. The two most popular classes of $k$-cells in literature to describe the topology of a manifold are either in terms of simplices, see for instance [44, 51, 53], or in terms of cubes, see [42, 52]. From a topological point of view both descriptions are equivalent, see [22]. Despite this equivalence of simplicial complexes and cubical complexes, the reconstruction maps in terms of basis functions, to be discussed in Section 4.2, differ significantly. For mimetic methods based on simplices see [5, 6, 21, 47], whereas for mimetic methods based on singular cubes see [3, 37, 39, 50]. We restrict ourselves to $k$-cubes, although we will keep calling them $k$-cells.
The ordered collection of all \( k \)-cells in \( D \) generate a basis for the space of \( k \)-chains, \( C_k(D) \). Then a \( k \)-chain, \( c_{(k)} \in C_k(D) \), is a formal linear combination of \( k \)-cells, \( \tau_{(k),i} \in D \),
\[
(4.1) \quad c_{(k)} = \sum_i c_i \tau_{(k),i}.
\]

The boundary operator on \( k \)-chains, \( \partial : C_k(D) \to C_{k-1}(D) \), is an homomorphism defined by [33, 44],
\[
(4.2) \quad \partial c_{(k)} = \partial \sum_i c_i \tau_{(k),i} := \sum_i c_i \partial \left( \tau_{(k),i} \right).
\]

The boundary of a \( k \)-cell \( \tau_{(k)} \) will then be a \((k-1)\)-chain formed by the oriented faces of \( \tau_{(k)} \). Like the exterior derivative, applying the boundary operator twice on a \( k \)-chain gives the null \((k-2)\)-chain, \( \partial \partial c_{(k)} = 0 \) for all \( c_{(k)} \in C_k(D) \). The set of \( k \)-chains and boundary operators gives rise to an exact sequence, the chain complex \((C_k(D), \partial)\),
\[
(4.3) \quad \cdots \xleftarrow{\partial} C_{k-1}(D) \xleftarrow{\partial} C_k(D) \xleftarrow{\partial} C_{k+1}(D) \xleftarrow{\partial} \cdots.
\]

Let \( B_k \) be the range and \( Z_k \) be the nullspace of \( \partial \) in \( C_k \). Then the topological Hodge decomposition of the space of \( k \)-chains is given by \( C_k = Z_k \oplus Z_k^\perp \), where \( Z_k = B_k \) on contractible domains\(^1\). The boundary operator on chains in (4.3) is a bijection that maps \( \partial : Z_k \to B_k \).

Dual to the space of \( k \)-chains, \( C_k(D) \), is the space of \( k \)-cochains, \( C^k(D) \), defined as the set of all \( k \)-linear functionals, \( c^{(k)} : C_k(D) \to \mathbb{R} \). The duality is expressed using the duality pairing \( \langle c^{(k)}, c_{(k)} \rangle := c^{(k)}(c_{(k)}) \). Note the resemblance between this duality pairing and the integration of \((k-2)\)-forms.

Let \( \{ \tau_{(k),i} \} \) form a basis of \( C_k(D) \), then there is a dual basis \( \{ \tau^{(k),i} \} \) of \( C^k(D) \), such that \( \tau^{(k),i}(\tau_{(k),j}) = \delta_{ij} \) and all \( k \)-cochains can be represented as linear combinations of the basis elements,
\[
(4.4) \quad c^{(k)} = \sum_i c^{(k)}(\tau^{(k),i}).
\]

With the duality relation between chains and cochains, we can define the formal adjoint of the boundary operator which constitutes an exact sequence on the spaces of \( k \)-cochains in the cell complex. This formal adjoint is called the coboundary operator, \( \delta : C^k(D) \to C^{k+1}(D) \), and is defined analogous to (2.4) as
\[
(4.5) \quad \langle \delta c^{(k)}, c_{(k+1)} \rangle := \langle c^{(k)}, \partial c_{(k+1)} \rangle, \quad \forall c^{(k)} \in C^k(D) \text{ and } \forall c_{(k+1)} \in C_{k+1}(D).
\]

Note that expression (4.5) is nothing but a discrete Stokes’ theorem and that the coboundary operator is nothing but a discrete exterior derivative. Also the coboundary operator satisfies \( \delta \delta c^{(k)} = 0 \), for all \( c^{(k)} \in C^k(D) \), and gives rise to an exact sequence, called the cochain complex \((C^k(D), \delta)\),
\[
(4.6) \quad \cdots \xrightarrow{\delta} C_{k-1}(D) \xrightarrow{\delta} C_k(D) \xrightarrow{\delta} C_{k+1}(D) \xrightarrow{\delta} \cdots.
\]

Let \( B^k \) be the range and \( Z^k \) be the nullspace of \( \delta \) in \( C^k \), then a Hodge decomposition of the space of \( k \)-cochains is given by \( C^k = Z^k \oplus Z^k_{\perp} \), where \( Z^k = B^k \) on contractible domains. The coboundary operator in (4.6) is a bijection that maps \( \delta : Z^k_{\perp} \to B^{k+1} \). Note the similarity between this map, that of the boundary operator on \( k \)-chains and that of the exterior derivative on \( k \)-forms.

### 4.2. Mimetic Operators

The discretization of the flow variables involves a bounded projection operator, \( \pi_h \), from the complete space \( H\Lambda^k(\Omega) \) to a conforming subspace \( \Lambda^k_h(\Omega; C_k) \subset H\Lambda^k(\Omega) \). The projection operator consists of two steps, a reduction operator, \( \mathcal{R} : H\Lambda^k(\Omega) \to C^k(D) \), that integrates the \( k \)-forms on \( k \)-chains to get \( k \)-cochains, and a reconstruction operator, \( \mathcal{I} : C^k(D) \to \Lambda^k_h(\Omega; C_k) \), to reconstruct \( k \)-forms from \( k \)-cochains using appropriate basis-functions.

\(^1\)Although ‘perpendicular’ in a topological space is not well defined, we refer to \( Z_k^\perp \) as the complement space of \( Z_k \) in \( C_k \).
These mimetic operators were already introduced before in [11, 38]. A composition of the two gives the projection operator $\pi_h = I \circ R$ as is illustrated below.

$$
\begin{array}{c}
H \Lambda^k(\Omega) \xrightarrow{\pi_h} \Lambda_h^k(\Omega; C_k) \\
R \downarrow \quad \downarrow \delta
\end{array}
\quad
C^k(D)
$$

These three operators together constitute the mimetic framework. An extensive discussion on mimetic operators can be found in [40, 41].

The reduction $R$ and reconstruction $I$ operators are defined below. The fundamental property of $R$ and $I$ is the commutation with differentiation in terms of exterior derivative and coboundary operator.

The reduction operator $R : H \Lambda^k(\Omega) \rightarrow C^k(D)$ is a homomorphism that maps differential forms to cochains. This map is defined by integration as

\begin{equation}
\langle Ra, \tau(k) \rangle := \int_{\tau(k)} a, \quad \forall a \in H \Lambda^k(\Omega), \quad \tau(k) \in C_k(D).
\end{equation}

Then for all $c(k) \in C_k(D)$, the reduction of the $k$-form, $a \in H \Lambda^k(\Omega)$, to the $k$-cochain, $a(k) \in C^k(D)$, is given by

\begin{equation}
a(k)(c(k)) := \langle Ra, c(k) \rangle = \sum_i c^i \langle Ra, \tau(k), i \rangle = \sum_i c^i \int_{\tau(k), i} a = \int_{c(k)} a.
\end{equation}

The reduction maps have a commuting property with respect to differentiation in terms of exterior derivative and coboundary operator,

\begin{equation}
Rd = \delta R, \quad \text{on } H \Lambda^k(\Omega).
\end{equation}

Since $R$ is defined by integration, (4.9) follows directly from Stokes theorem (2.4) and the duality property (4.5).

Next by definition also the reconstruction map $I : C^k(D) \rightarrow \Lambda_h^k(\Omega; C_k)$ needs to have a commuting property with respect to differentiation in terms of exterior derivative and coboundary operator,

\begin{equation}
dI = I \delta, \quad \text{on } C^k(D).
\end{equation}

The reconstruction $I$ must be the right inverse of $R$, so $RI = Id$ on $C^k(D)$, and we want it to be an approximate left inverse of $R$, so $IR = Id + O(h^p)$ on $H \Lambda^k(\Omega)$. This composition is defined as the projection operator.

**Definition 1 (Bounded projection operator).** The composition $I \circ R$ will denote the projection operator, $\pi_h := I \circ R : H \Lambda^k(\Omega) \rightarrow \Lambda_h^k(\Omega; C_k)$, allowing for a finite dimensional representation of a $k$-form,

\begin{equation}
\pi_h a := I Ra, \quad \pi_h a \in \Lambda_h^k(\Omega; C_k) \subset H \Lambda^k(\Omega).
\end{equation}

where $IRa$ is expressed as a combination of $k$-cochains and interpolating $k$-forms. The projection operator $\pi_h$ is a bounded operator if for $C < \infty$ and for all $a \in \Lambda^k(\Omega)$ we have $\|\pi_h a\|_{H \Lambda^k} \leq C\|a\|_{H \Lambda^k}$.

A proof that $\pi_h$ is indeed a projection operator is given in [41]. In Section 4.5 also boundedness is proven.

**Lemma 2 (Commutation property).** There exists a commuting property for the projection and the exterior derivative, such that

\begin{equation}
d\pi_h = \pi_h d \quad \text{on } H \Lambda^k(\Omega).
\end{equation}
Proof. Express the projection in terms of the reduction and reconstruction operator, then
\[ d\pi_h a \overset{(4.11)}{=} dITRa \overset{(4.10)}{=} I\delta Ra \overset{(4.9)}{=} I\mathcal{R} da \overset{(4.11)}{=} \pi_h da, \quad \forall a \in H\Lambda^k(\Omega). \]

Note that it is the intermediate step \( I\delta Ra \) that is used in practice for the discretization, see [40], and (4.20) on page 14.

**Corollary 2 (Discrete Hodge decomposition).** From Lemma 2 it follows that \( B_h^k := \pi_h B^k \subset B^k, Z_h^k := \pi_h Z^k \subset Z^k \) and that on contractible domains, \( Z_h^{k,\perp} := \pi_h Z^{k,\perp} \subset Z^{k,\perp} \). Then the discrete Hodge decomposition is given by \( \Lambda^k = Z_h^k \oplus Z_h^{k,\perp} \). As a consequence of Lemma 2 and the discrete Hodge decomposition we have \( Z_{W_h} \subset Z_W, Z_{V_h} \subset Z_V, dW_h \subset V_h \) and \( dV_h = Q_h \), which shows that the discretization method is compatible.

Finally, we do not restrict ourselves to affine mappings only, as is required in many other compatible finite elements, like Nédélec and Raviart-Thomas elements and their generalizations [5, 45, 48], but also allow non-affine maps such as curvilinear transfinite or isoparametric mappings of quadrilaterals or hexahedrals, [31], where \( \Phi \) and its inverse are piecewise sufficiently smooth, i.e.

\begin{align*}
(1) & \quad \Phi \text{ is a } C^{p+1}\text{-diffeomorphism}, \\
(2) & \quad |\Phi|_{W^l} \leq Ch^l, \quad l \leq p + 1, \\
(3) & \quad |\Phi^{-1}|_{W^l_\infty} \leq Ch^{-l}, \quad l \leq p + 1.
\end{align*}

This allows for better approximations in complex domains with curved boundaries, without the need for excessive refinement, while maintaining design convergence rates, [19]. This is possible since the projection operator \( \pi_h \) commutes with the pullback \( \Phi^* \),

\[ \Phi^* \pi_h = \pi_h \Phi^* \text{ on } H\Lambda^k(\Omega). \]

An extensive proof is given in [41].

### 4.3. Numerical stability

Essential ingredients in proving numerical stability are the discrete Hodge decomposition and the discrete Poincaré inequality. Because the complexes \( (H\Lambda, d) \) and \( (\Lambda_h, d) \) are each others supercomplex and subcomplex, respectively, the discrete Poincaré inequality is directly related to the Poincaré inequality in Lemma 1 and the bounded projection in Definition 1.

**Lemma 3 (Discrete Poincaré inequality).** Let \( (H\Lambda, d) \) be a bounded closed Hilbert complex, \( (\Lambda_h, d) \) a subcomplex, and \( \pi_h \) a bounded projection. Then

\[ \begin{cases} 
\|a_h\|_{H\Lambda^k} \leq c_{ph} \|da_h\|_{L^2 \Lambda^k}, \\
1 \leq c_{ph} \leq c_P.
\end{cases} \]

**Proof.** Given \( a_h \in Z_h^{k,\perp} \). From the Hodge decomposition and the bounded projection it follows that \( B_h^k \subset B^k \) and \( Z_h^k \subset Z^k \) and from the commutation relation (4.12) it follows that \( Z_h^{k,\perp} \subset Z^{k,\perp} \). Since we consider a proper subspace, Lemma 1 is still valid, with \( c_{ph} \leq c_P \).

**Theorem 1 (Discrete well-posedness).** Let \( (\Lambda_h, d) \) be a subcomplex of the closed Hilbert complex \( (H\Lambda, d) \). Then there exists constants \( \alpha_h, \beta_h, \gamma_h \), depending only on \( c_{ph} \), such that for any \( (\tau_h, v_h, q_h) \in W_h \times V_h \times Q_h \), there exists a stable finite dimensional solution \( (\omega_h, u_h, p_h) \in W_h \times V_h \times Q_h \) of the Stokes problem (3.7), with

\[ \alpha_h > \alpha > 0, \quad \beta_h > \beta > 0, \quad \gamma_h > \gamma > 0. \]

**Proof.** This is just Propositions 2 and 3 applied to the complex \( (\Lambda_h, d) \), combined with the fact that the constant in the Poincaré inequality for \( \Lambda_h^k \) is \( c_{ph} \leq c_P \) by Lemma 3. □
4.4. Mimetic spectral element basis-functions. The finite dimensional differential forms used in this paper are polynomials, based on the idea of spectral element methods, [18]. The mimetic spectral elements used here were derived independently in [28, 49], and are more extensively discussed in [41]. Only the most important properties of the mimetic spectral element method are presented here.

In spectral element methods the domain Ω is decomposed into M non-overlapping, in this case curvilinear quadrilateral or hexahedral, closed sub-domains \( Q_m \),

\[
\Omega = \bigcup_{m=1}^{M} Q_m, \quad Q_m \cap Q_l = \partial Q_m \cap \partial Q_l, \ m \neq l,
\]

where in each sub-domain a Gauss-Lobatto grid is constructed. The complete mesh is indicated by \( \mathcal{Q} := \sum_{m=1}^{M} Q_m \).

The collection of Gauss-Lobatto meshes in all elements \( Q_m \in \mathcal{Q} \) constitutes the cell complex \( D \). For each element \( Q_m \) there exists a sub cell complex, \( D_m \). Note that \( D_m \cap D_l, \ m \neq l \), is not an empty set in case they are neighboring elements, but contains all \( k \)-cells, \( k < n \), of the common boundary.

Each sub-domain \( Q_m \in \mathcal{Q} \) is mapped from the reference element, \( \hat{Q} = [-1,1]^n \), using the mapping \( \Phi_m : \hat{Q} \rightarrow Q_m \). Then all flow variables defined on \( Q_m \) are pulled back onto this reference element using the following pullback operation, \( \Phi_m^* : \Lambda^k_h(Q_m;C_k) \rightarrow \Lambda^k_h(\hat{Q};C_k) \).

The basis-functions that interpolate the cochains on the quadrilateral or hexahedral elements constructed using tensor products. It is therefore sufficient to derive interpolation functions in one dimension and use tensor products afterwards to construct \( n \)-dimensional basis functions. A similar approach was taken in [17]. Because projection operator and pullback operator commute (4.13), the interpolation functions are discussed for the reference element only. Since the mappings \( \Phi_m \) and their inverse are assumed to be sufficiently smooth, the rates of convergence for interpolation estimates on the physical elements are equal to that of the reference element. Only the constants \( C \) that will appear below will depend on the mappings \( \Phi_m \), but will be independent of the meshsize and polynomial order.

Consider a 0-form \( a \in H^0(\Omega) \) on \( \hat{Q} := \xi \in [-1,1] \), on which a cell complex \( D \) is defined that consists of \( N+1 \) nodes, \( \xi_i \), where \( -1 \leq \xi_0 \ldots < \xi_N \leq 1 \), and \( N \) edges, \( \tau_{1,i} = [\xi_{i-1},\xi_i] \), of which the nodes are their boundaries. Corresponding to this set of nodes (0-chains) there exists a projection using \( N^{th} \) order Lagrange polynomials, \( l_i(\xi) \), to approximate a 0-form, as

\[
\pi_h a = \sum_{i=0}^{N} a_i l_i(\xi).
\]

The property of Lagrange polynomials is that they interpolate nodal values. They are therefore suitable to reconstruct a 0-form form the 0-cochain \( a^{(0)} = Ra \), \( a \in \Lambda^0(\Omega) \), containing the set \( a_i = a(\xi_i) \) for \( i = 0, \ldots, N \). Lagrange polynomials are in fact 0-forms, \( l_i(\xi) \in \Lambda^0_h(\hat{Q};C_0) \). Lagrange polynomials are constructed such that their value is one in the corresponding point and zero in all other grid points,

\[
\mathcal{R} l_i(\xi) = l_i(\xi_p) = \begin{cases} 
1 & \text{if } i = p \\
0 & \text{if } i \neq p
\end{cases}.
\]

In [28, 49] a similar basis for projection of 1-forms was derived, consisting of 1-cochains and 1-form polynomials, that is called the edge polynomial, \( e_i(\xi) \in \Lambda^1_h(\hat{Q}) \). Let \( b \in L^2(\hat{Q}) \), then the projected 1-form is given by

\[
\pi_h b(\xi) = \sum_{i=1}^{N} b_i e_i(\xi),
\]
where the edge polynomial is defined as
\[ e_i(\xi) = -\sum_{k=0}^{i-1} d l_k(\xi) = \sum_{k=i}^{N} d l_k(\xi) = \frac{1}{2} \sum_{k=i}^{N} d l_k(\xi) - \frac{1}{2} \sum_{k=0}^{i-1} d l_k(\xi). \]

Let \( a_h(\xi) \in \Lambda^0_h(\hat{Q}; C_0) \) be expressed as in (4.16), then \( b_h = d a_h \in \Lambda^1_h(\hat{Q}; C_1) \) is expressed as
\[ \sum_{i=0}^{N} a_i l_i(\xi) = \sum_{i=1}^{N} (a_i - a_{i-1}) e_i(\xi) = \sum_{i=1}^{N} (\delta a^{(0)}_i) e_i(\xi) = \sum_{i=1}^{N} b_i e_i(\xi), \]
where \( \delta \) is the coboundary operator (4.5), applied to the 0-cochain \( a^{(0)}_i \). It therefore satisfies (4.10). For derivations and proofs see [28, 41, 49]. Similar to (4.17), the edge basis-functions are constructed such that when integrating \( e_i(\xi) \) over a line segment it gives one for the corresponding element and zero for any other line segment, so
\[ R e_i(\xi) = \int_{\xi_{p-1}}^{\xi_p} e_i(\xi) = \begin{cases} 1 & \text{if } i = p \\ 0 & \text{if } i \neq p. \end{cases} \]

Equations (4.17) and (4.21) show that indeed we have \( R I = I d \). The fourth-order Lagrange and third-order edge polynomials, corresponding to a Gauss-Lobatto grid with \( N = 4 \), are shown in Figures 1 and 2.

**Figure 1.** Lagrange polynomials on Gauss-Lobatto-Legendre grid.

**Figure 2.** Edge polynomials on Gauss-Lobatto-Legendre grid.

4.5. **Bounded projections and interpolation estimates.** The mimetic framework uses Lagrange, \( l_i(\xi) \in H\Lambda^0(\hat{Q}) \), and edge functions, \( e_i(\xi) \in L^2\Lambda^1(\hat{Q}) \), for the reconstruction, \( I \). Because we consider tensor products to construct higher-dimensional interpolation, it is sufficient to show that the projection operator is bounded in one dimension. A similar approach was used in [17]. Due to the way the edge functions are constructed, there exists a commuting diagram property between projection and exterior derivative,
\[
\begin{array}{ccc}
R & \longrightarrow & H\Lambda^0 \\
\downarrow \pi_h & & \downarrow \pi_h \\
\Lambda^0_h & \longrightarrow & \Lambda^1_h \\
\end{array}
\]
which gives, for \( a \in H\Lambda^0(\hat{Q}) \), the one form \( d\pi_h a = \pi_h da \in L^2\Lambda^1(\hat{Q}) \). Lagrange interpolation by itself does not guarantee a convergent approximation [25], but it requires a suitably chosen set of points, \(-1 \leq \xi_0 < \xi_1 < \ldots < \xi_N \leq 1 \). Here, the Gauss-Lobatto distribution is proposed,
because of its superior convergence behaviour. For \( a \in H^m \Lambda^0(\Omega) \), the a priori error estimate in the \( H^l \Lambda^0 \)-norm is given by [18],
\[
\|a - \pi_h a\|_{H^l \Lambda^0} \leq C h^l |a|_{H^{l+1} \Lambda^0}, \quad l = \min(N, m - 1).
\]
Equation (4.22) also implies that the projection of zero-forms is stable in the \( H^0(\hat{Q}) \), as is shown in the following proposition.

**Proposition 4.** [41] For \( a \in H^L \Lambda^0(\hat{Q}) \) and the projection \( \pi_h : H^L \Lambda^0 \to \Lambda^1_h \), there exists the following two stability estimates in \( H^L \Lambda^0 \)-norm and \( H^L \Lambda^0 \)-semi-norm:
\[
\begin{align*}
\|\pi_h a\|_{H^L \Lambda^0} &\leq C |a|_{H^L \Lambda^0}, \\
|\pi_h a|_{H^L \Lambda^0} &\leq C |a|_{H^L \Lambda^0}.
\end{align*}
\]

Now that we have a bounded linear projection of zero forms in one dimension, we can also proof boundedness of the projection of one-forms.

**Proposition 5.** Let \( a \in H^L \Lambda^0 \) and \( b = da \in L^2 \Lambda^1 \), then there exists a bounded linear projection \( \pi_h : L^2 \Lambda^1 \to \Lambda^1_h \), such that
\[
\|\pi_h b\|_{L^2 \Lambda^1} \leq C \|b\|_{L^2 \Lambda^1}.
\]

**Proof.** The proof is based on the result of the previous proposition and the commutation between the bounded projection operator and the exterior derivative, Lemma 2,
\[
\|\pi_h b\|_{L^2 \Lambda^1} = |\pi_h da|_{L^2 \Lambda^1} = |d\pi_h a|_{L^2 \Lambda^1} = |\pi_h a|_{H^L \Lambda^0} \leq C |a|_{H^L \Lambda^0} = C \|da\|_{L^2 \Lambda^1} = C \|b\|_{L^2 \Lambda^1}.
\]

Propositions 4 and 5 show that the projection \( \pi_h \) is a bounded projection operator, based on Lagrange functions and edge functions. As for zero forms using Lagrange interpolation, we can also give an estimate for the interpolation error of one forms, interpolated using edge functions.

**Proposition 6.** [41] Let \( a \in H^L \Lambda^0 \) and \( b = da \in L^2 \Lambda^1 \), the interpolation error \( b - \pi_h b \in L^2 \Lambda^1 \) is given by
\[
\|b - \pi_h b\|_{L^2 \Lambda^1} \leq C h^l |b|_{H^{l+1} \Lambda^1}, \quad l = \min(N, m - 1).
\]

The one dimensional results can be extended to the multidimensional framework by means of tensor products. This allows for the interpolation of integral quantities defined on \( k \)-dimensional cubes. Consider a reference element in \( \mathbb{R}^2 \), \( \hat{Q} = [-1,1]^2 \). Then the interpolation functions for points, lines, surfaces (2D volumes) are given by,
\[
\begin{align*}
\text{point: } & \quad P_{i,j}^{(0)}(\xi, \eta) = l_i(\xi) \otimes l_j(\eta), \\
\text{line: } & \quad L_{i,j}^{(1)}(\xi, \eta) = \{ e_i(\xi) \otimes l_j(\eta), \ l_i(\xi) \otimes e_j(\eta) \}, \\
\text{surface: } & \quad S_{i,j}^{(2)}(\xi, \eta) = e_i(\xi) \otimes e_j(\eta).
\end{align*}
\]
The approximation spaces are spanned by combinations of Lagrange and edge basis functions,
\[
\begin{align*}
H^1 \Lambda^0(\Omega) &\supset \Lambda^0_h(\mathbb{Q}; C_0) := \text{span} \left\{ P_{i,j}^{(0)} \right\}_{i=0,j=0}^{N,N}, \\
H^L \Lambda^1(\Omega) &\supset \Lambda^1_h(\mathbb{Q}; C_1) := \text{span} \left\{ (L_{i,j}^{(1)}) \right\}_{i=1,j=0}^{N,N} \times \text{span} \left\{ (L_{i,j}^{(1)})_2 \right\}_{i=0,j=1}^{N,N}, \\
L^2 \Lambda^2(\Omega) &\supset \Lambda^2_h(\mathbb{Q}; C_2) := \text{span} \left\{ S_{i,j}^{(2)} \right\}_{i=1,j=1}^{N,N}.
\end{align*}
\]
For the variables vorticity, velocity and pressure in the VVP formulation of the Stokes problem, the \( h \)-convergence rates of the interpolation errors in \( L^2 \Lambda^k \)-norm become,
\[
\begin{align*}
\|\omega - \pi_h \omega\|_{L^2 \Lambda^{n-2}} &\sim O(h^{N+s}), \quad \|u - \pi_h u\|_{L^2 \Lambda^{n-1}} = O(h^N), \quad \|p - \pi_h p\|_{L^2 \Lambda^n} = O(h^N),
\end{align*}
\]
in case the functions \((\omega, u, p)\) are sufficiently smooth, where \( s = 1 \) for \( n = 2 \) and \( s = 0 \) for \( n > 2 \).
The interpolation errors in \( H^L \Lambda^k \)-norm become,
\[
\begin{align*}
\|\omega - \pi_h \omega\|_{H^L \Lambda^{n-2}} &\sim O(h^N), \quad \|u - \pi_h u\|_{H^L \Lambda^{n-1}} = O(h^N),
\end{align*}
\]
with $N$ defined as in Section 4.4.

5. Error estimates

Next consider the finite dimensional problem: find $(\omega_h, u_h, p_h) \in \{W_h \times V_h \times Q_h\}$, given $f \in L^2\Lambda^{n-1}(\Omega)$ and $g \in L^2\Lambda^n$ and boundary conditions in (2.19), for all $(\tau_h, v_h, q_h) \in \{W_h \times V_h \times Q_h\}$, such that

\begin{align*}
(5.1a) \quad &a(\tau_h, \omega_h) - c(\tau_h, u_h) = h(\tau_h), \quad \forall \tau_h \in W_h, \\
(5.1b) \quad &e(v_h, u_h) + c(\omega_h, v_h) + b(v_h, p_h) = f(v_h), \quad \forall v_h \in V_h, \\
(5.1c) \quad &b(u_h, q_h) = g(q_h). \quad \forall q_h \in Q_h.
\end{align*}

The following theorem gives the a priori error estimates of this problem when using the compatible spectral discretization method described in the previous section. Corollary 2 showed that we have $Z_{W_h} \subset Z_W$ and $Z_{V_h} \subset Z_V$. From this it follows that we have compatible finite dimensional subspaces: $W_h \subset W$, $V_h = dV_h \oplus h^*Q_h \subset V$ and $Q_h = dV_h \subset Q$. The derivations of the error estimates are based on the methodology of [15]. The proofs are given in the subsequent propositions.

**Theorem 2 (Error estimates).** Let $(\omega, u, p)$ be the solution of the continuous problem given in (3.7) or (3.9) and $(\omega_h, u_h, p_h)$ the solution of the finite dimensional problem in (5.1). The continuous problem is well-posed by Propositions 2 and 3 and the finite dimensional problem is well-posed by Theorem 1 and Propositions 4 and 5. Furthermore, from Corollary 2 we have that for the compatible spectral discretization method, $Z_{W_h} \subset Z_W$ and $Z_{V_h} \subset Z_V$. Then the following a priori error estimates for the VVP formulation of the Stokes problem hold:

\begin{align*}
(5.2) \quad &||\omega - \omega_h||_W \leq \left(1 + \frac{c_a}{\alpha_h}\right) \left(1 + \frac{c_e}{\gamma_h}\right) \inf_{\tau_h \in W_h} ||\omega - \tau_h||_W, \\
(5.3) \quad &||u - u_h||_V \leq \left(1 + \frac{c_e}{\gamma_h}\right) \left(1 + \frac{c_a}{\beta_h}\right) \inf_{v_h \in V_h} ||u - v_h||_V + \frac{c_a}{\gamma_h} \left(1 + \frac{c_a}{\alpha_h}\right) \left(1 + \frac{c_e}{\gamma_h}\right) \inf_{\tau_h \in W_h} ||\omega - \tau_h||_W, \\
(5.4) \quad &||p - p_h||_Q \leq \left(1 + \frac{c_b}{\beta_h}\right) \inf_{q_h \in Q_h} ||p - q_h||_Q + \frac{c_a}{\gamma_h} \left(1 + \frac{c_e}{\gamma_h}\right) \left(1 + \frac{c_b}{\beta_h}\right) \inf_{v_h \in V_h} ||u - v_h||_V \\
&\quad \quad + \left(\frac{c_a}{\gamma_h} + \frac{c_e}{\beta_h}\right) \left(1 + \frac{c_a}{\alpha_h}\right) \left(1 + \frac{c_e}{\gamma_h}\right) \inf_{\tau_h \in W_h} ||\omega - \tau_h||_W.
\end{align*}

**Proof.** The proof of this Theorem will be given in a series of Propositions 7 to 10. \hfill \Box

**Proposition 7 (Vorticity error bound).** Let $\sigma_h \in Z_{W_h}^\dagger$, the error for vorticity is bounded by

\begin{align*}
(5.5) \quad &||\omega - \omega_h||_W \leq \left(1 + \frac{c_a}{\alpha_h}\right) \inf_{\omega_h \in Z_{W_h}^\dagger} ||\omega - \sigma_h||_W.
\end{align*}

**Proof.** Subtract the velocity-vorticity relation in the finite dimensional problem (5.1a) from that of the continuous problem (3.9a), we get

\[
a(\tau_h, \omega - \omega_h) - c(\tau_h, u - u_h) = 0, \quad \forall \tau_h \in Z_{W_h} \subset Z_W.
\]

Bound $\sigma_h - \omega_h \in Z_{W_h}$ using inf-sup condition (3.13), we get

\[
\begin{align*}
\alpha_h ||\sigma_h - \omega_h||_W \leq & \sup_{\tau_h \in Z_{W_h}} \frac{a(\tau_h, \sigma_h - \omega_h)}{||\tau_h||_W} \\
= & \sup_{\tau_h \in Z_{W_h}} \frac{a(\tau_h, \sigma_h - \omega) + a(\tau_h, \omega - \omega_h)}{||\tau_h||_W} \\
= & \sup_{\tau_h \in Z_{W_h}} \frac{a(\tau_h, \sigma_h - \omega) + c(\tau_h, u - u_h)}{||\tau_h||_W}.
\end{align*}
\]
The last term vanishes since $\tau_h \in Z_{Wh}$ and $Z_{Wh} \subset Z_{W}$, hence $\alpha_h \|\omega_h - \omega_i\|_W \leq c_a \|\omega - \sigma_h\|_W$. By the triangle inequality and the infimum over all $\sigma_h \in Z^\perp_{Wh}$, we obtain (5.5). \hfill \Box

**Proposition 8 (Velocity error bound).** Let $s_h \in Z^\perp_{Vh}$, the error for velocity is bounded by

\begin{equation}
    \|u - u_h\|_V \leq \left(1 + \frac{c_b}{\gamma_h}\right) \inf_{s_h \in Z^\perp_{Vh}} \|u - s_h\|_V + \frac{c_a}{\gamma_h} \left(1 + \frac{c_c}{\gamma_h}\right) \inf_{\sigma_h \in Z^\perp_{Wh}} \|\omega - \sigma_h\|_W.
\end{equation}

**Proof.** Use the inf-sup condition (3.14) to bound $s_h - u_h \in Z_V$,

$$
\gamma_h \|s_h - u_h\|_V \leq \sup_{\tau_h \in Z_{Wh}} \frac{c(\tau_h,s_h - u_h)}{\|\tau_h\|_W} = \sup_{\tau_h \in Z_{Wh}} \frac{c(\tau_h,s_h - u) + c(\tau_h,u - u_h)}{\|\tau_h\|_W}
$$

\begin{equation*}
= \sup_{\tau_h \in Z_{Wh}} \frac{c(\tau_h,s_h - u) + a(\tau_h,\omega - \omega_h)}{\|\tau_h\|_W}
\end{equation*}

\begin{equation*}
\leq c_c \|u - s_h\|_V + c_a \|\omega - \omega_h\|_W.
\end{equation*}

By triangle inequality, estimate (5.5) and the infimum over all $s_h \in Z^\perp_{Vh}$, we obtain (5.6). \hfill \Box

**Proposition 9 (Pressure error bound).** The error for pressure is bounded by

\begin{equation}
    \|p - p_h\|_Q \leq \left(1 + \frac{c_b}{\beta_h}\right) \inf_{q_h \in Q_h} \|p - q_h\|_Q + \frac{c_a}{\beta_h} \left(1 + \frac{c_c}{\gamma_h}\right) \inf_{s_h \in Z^\perp_{Vh}} \|u - s_h\|_V
\end{equation}

\begin{equation*}
+ \left(\frac{c_a}{\gamma_h} + \frac{c_c}{\beta_h}\right) \left(1 + \frac{c_a}{\alpha_h}\right) \inf_{\sigma_h \in Z^\perp_{Wh}} \|\omega - \sigma_h\|_W.
\end{equation*}

**Proof.** Subtract (5.1b) from (3.9b), we get

$$
\mathcal{L}(\omega - \omega_h,v_h) + \mathcal{E}(v_h,u - u_h) + \mathcal{B}(v_h,p - p_h) = 0, \quad \forall v_h \in V_h.
$$

So for $q_h \in Q_h$ we have

$$
b(v_h, q_h - p_h) = -\mathcal{L}(\omega - \omega_h,v_h) - \mathcal{E}(v_h,u - u_h) - \mathcal{B}(v_h,p - q_h).
$$

Use this and the inf-sup condition (3.16) to bound $q_h - p_h \in Q_h$,

\begin{equation*}
\beta_h \|q_h - p_h\|_Q \leq \sup_{v_h \in V_h} \frac{b(v_h,q_h - p_h)}{\|v_h\|_V} = \sup_{v_h \in V_h} \frac{-\mathcal{L}(\omega - \omega_h,v_h) - \mathcal{E}(v_h,u - u_h) - \mathcal{B}(v_h,p - q_h)}{\|v_h\|_V}
\end{equation*}

\begin{equation*}
\leq c_c \|\omega - \omega_h\|_W + c_a \|u - u_h\|_V + c_b \|p - q_h\|_Q.
\end{equation*}

By triangle inequality, estimates (5.5) and (5.6), and the infimum over all $q_h \in Q_h$, we obtain (5.7). \hfill \Box

Next we replace the infimums over $\sigma_h \in Z^\perp_{Wh}$ and $s_h \in Z^\perp_{Vh}$ by best approximation errors.

**Proposition 10.** The terms $\inf_{\tau_h \in W_h} \|\omega - \tau_h\|_W$ and $\inf_{s_h \in Z^\perp_{Vh}} \|u - s_h\|_V$ are bounded by the best approximation estimates $\inf_{\tau_h \in W_h} \|\omega - \tau_h\|_W$ and $\inf_{v_h \in V_h} \|u - v_h\|_V$, using the inf-sup conditions (3.16) and (3.14), as

\begin{equation}
    \inf_{\sigma_h \in Z^\perp_{Wh}} \|\omega - \sigma_h\|_W \leq \left(1 + \frac{c_c}{\gamma_h}\right) \inf_{\tau_h \in W_h} \|\omega - \tau_h\|_W,
\end{equation}

\begin{equation}
    \inf_{s_h \in Z^\perp_{Vh}} \|u - s_h\|_V \leq \left(1 + \frac{c_b}{\beta_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_V.
\end{equation}
Prove. Take \( \tau_h \in W_h \), then there exists a \( \kappa_h \in W_h \) such that

\[
c(\kappa_h, v_{Z_h}) = c(\omega - \tau_h, v_{Z_h}), \quad \forall v_{Z_h} \in Z_{V_h}.
\]

This is equivalent to

\[
c(\kappa_h + \tau_h, v_{Z_h}) = c(\omega, v_{Z_h}) = f(v_{Z_h}), \quad \forall v_{Z_h} \in Z_{V_h},
\]

which shows that \( \sigma_h = \kappa_h + \tau_h \in \mathbb{Z}_h \). We can bound \( \|\kappa_h\|_W \) using the discrete inf-sup condition as follows

\[
\gamma_h \|\kappa_h\|_W \leq \sup_{v \in Z_{V_h}} \frac{c(\kappa_h, v)}{\|v\|_V} = \sup_{v \in Z_{V_h}} \frac{c(\omega - \tau_h, v)}{\|v\|_V} \leq c_c \|\omega - \tau_h\|_V.
\]

By triangle inequality and since \( \tau_h \in W_h \) was arbitrary, we find (5.8). A similar proof holds for (5.9) (see also [15], Proposition 2.5).

Additionally, following section 7.7.6 in [10], we have the following \( L^2A^k(\Omega) \) error estimates for the curl of vorticity and divergence of velocity.

**Proposition 11.** The errors of the curl of vorticity and divergence of velocity are bounded by their best approximation estimates,

\[
(5.10) \quad \|d(\omega - \omega_h)\|_{L^2A^{n-1}} \leq \inf_{\tau_h \in W_h} \|d(\omega - \tau_h)\|_{L^2A^{n-1}},
\]

\[
(5.11) \quad \|d(u - u_h)\|_{L^2A^n} \leq \inf_{v_h \in V_h} \|d(u - v_h)\|_{L^2A^n}.
\]

**Proof.** Choose \( v = v_{Z_h} \in Z_{V_h} \) in (3.9b) and (5.1b) and subtract these. Set \( v_{Z_h} = d\tau_h \), this gives the orthogonality relation

\[
(d(\omega - \omega_h), d\tau_h)_{\Omega} = 0, \quad \forall \tau_h \in W_h.
\]

Substitute this into the following Cauchy-Schwarz inequality,

\[
\|d(\omega - \omega_h)\|_{L^2A^{n-1}}^2 = (d(\omega - \omega_h), d(\omega - \tau_h))_{\Omega} \\
\leq \|d(\omega - \omega_h)\|_{L^2A^{n-1}} \|d(\omega - \tau_h)\|_{L^2A^{n-1}}, \quad \forall \tau_h \in W_h,
\]

and (5.10) follows. Next choose \( q_h = dv_h \in Q_h \) in (3.9c) and (5.1c) and subtract these. Then (5.11) follows again from the Cauchy-Schwarz inequality.

Because the projections of respectively \( \omega, u, \) and \( p \), belong to the finite dimensional subspaces \( W_h \subset W, V_h \subset V \) and \( Q_h \subset Q \), the best approximation errors can be bounded using the interpolation errors,

\[
\inf_{\tau_h \in W_h} \|\omega - \tau_h\|_W \leq \|\omega - \pi_h \omega\|_W, \quad \inf_{v_h \in V_h} \|u - v_h\|_V \leq \|u - \pi_h u\|_V, \quad \inf_{q_h \in Q_h} \|p - q_h\|_Q \leq \|p - \pi_h p\|_Q,
\]

and therefore we obtain the following optimal a priori error estimates,

\[
(5.12) \quad \|\omega - \omega_h\|_W = \mathcal{O}(h^N), \quad \|u - u_h\|_V = \mathcal{O}(h^N), \quad \|p - p_h\|_Q = \mathcal{O}(h^N).
\]

So the convergence rates for the approximations are equal to those of the interpolations, (4.27), (4.28), thus we obtained optimal convergence. The error estimates were obtained independent of the chosen types of boundary conditions.

**Remark 2.** In contrast to [4, 24] and [10], Table 7.5, where Raviart-Thomas elements were used, the proposed compatible method has provably optimal convergence also with standard velocity boundary conditions and with non-affine mappings.
6. Numerical Results

Now that the compatible spectral discretization method and its a priori error estimates are derived, we perform a series of test problems to show optimal convergence behavior. Purpose of the testcases is to show convergence behavior in case of various boundary conditions and in case of curvilinear meshes. In all cases we show optimal convergence.

The first three testcases originate from a recent paper by Arnold et al [4], where suboptimal convergence is shown for normal velocity - tangential boundary conditions in vector Poisson and Stokes problems, when using Raviart-Thomas elements [48]. Since Raviart-Thomas elements are the most popular \( H(\text{div}, \Omega) \) conforming elements, we compare our method to these results.

6.1. Vector Poisson problems. Figure 3 shows the result of the vector Poisson problem (3.5) on \( \Omega = [0,1]^2 \) with coordinates \( x := (x,y) \), for a 1-form \( u \in H^1(\Omega) \), where \( \Gamma = \Gamma_2 \), i.e. with tangential velocity - divergence-free boundary conditions \((\text{tr} \star u = 0, \text{tr} \star du = 0)\). The corresponding solution is given by

\[
 u^{(1)} = -v(x) \, dx + u(x) \, dy
 = -(2 \sin \pi x \cos \pi y) \, dx + (\cos \pi x \sin \pi y) \, dy .
\]

Both Raviart-Thomas and mimetic spectral element methods show optimal convergence rates. All results of this and the following two problems where obtained on the same quadrilateral mesh of \( 2^n \times 2^n \) subsquares, \( n = 1, 2, 3, 4, \ldots \) just like the reference solutions from [4].

Figure 3. Comparison of the \( h \)-convergence between Raviart-Thomas and Mimetic spectral elements for the 2D 1-form Poisson problem with tangential velocity - divergence-free boundary conditions.

Figure 4 shows again results for the vector Poisson problem for a 1-form, but now in combination with normal velocity - tangential velocity boundary conditions \((\text{tr} u = 0, \text{tr} \star u = 0)\), so \( \Gamma = \Gamma_1 \). The corresponding manufactured solution is

\[
 u^{(1)} = -v(x) \, dx + u(x) \, dy
 = -(\sin \pi x \sin \pi y) \, dx + (\sin \pi x \sin \pi y) \, dy .
\]
The compatible spectral discretization method again shows optimal convergence, as was expected from the above analysis. The Raviart-Thomas elements only show suboptimal convergence in case of velocity boundary conditions. This suboptimality was proven in [4]. Especially for $\omega_h$ and $d\omega_h$ the current method outperforms the Raviart-Thomas elements, with a difference in rate of convergence of $\frac{3}{2}$.

6.2. Stokes problems. The same difference in convergence behavior is found for the Stokes problem, where $\Gamma = \Gamma_1$, i.e. with normal velocity - tangential velocity boundary conditions, see Figure 5. Again $\Omega$ is the unit square, and the velocity and pressure fields are given by

$$u^{(1)} = -v(x) \, dx + u(x) \, dy$$

$$= - (2y^2(y - 1)^2x(2x - 1)(x - 1)) \, dx + (-2x^2(x - 1)^2y(2y - 1)(y - 1)) \, dy,$$

$$p^{(2)} = p(x) \, dx \wedge dy = (x - \frac{1}{2})^5 + (y - \frac{1}{2})^5) \, dx \wedge dy.$$

While for velocity both methods show optimal convergence, for pressure a difference of $\frac{1}{2}$ is noticed in the rate of convergence and for vorticity and the curl of vorticity again a difference in rate of convergence of $\frac{3}{2}$ is revealed. The error in divergence of velocity is not shown here for the Stokes problem, because the method is pointwise divergence-free up to machine precision. Special attention to this property is given in [40].

We would like to remark is that the results shown in Figure 5 are independent of the kind of boundary conditions used. Table 1 shows the results of vorticity for all types of admissible boundary conditions.

The next testcase reveals the optimal convergence in case of higher-order approximation on curvilinear quadrilateral meshes for all admissible types of boundary conditions. The manufactured solution Stokes problem is given on a curvilinear domain, defined by the mapping $(x, y) = \Phi(\xi, \eta)$.
A PRIORI ERROR ESTIMATES FOR COMPATIBLE SPECTRAL DISCRETIZATION OF THE STOKES PROBLEM

Figure 5. Comparison of the $h$-convergence between Raviart-Thomas and Mimetic spectral element projections for the 2D Stokes problem with normal velocity - tangential velocity boundary conditions.

Table 1. This table shows the vorticity error $\| \omega - \omega_h \|_{L^2,\Lambda^0}$ obtained using the four types of boundary conditions given in (2.19). The results are obtained on an uniform Cartesian mesh with $N = 2$ and $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$. All four cases show third order convergence.

| normal velocity | tangential velocity | vorticity | vorticity | convergence rate |
|-----------------|---------------------|-----------|-----------|------------------|
| tangential velocity | pressure | normal velocity | pressure |
| 1.0280e-04 | 1.0109e-04 | 1.0630e-04 | 1.0035e-04 | 3.14 |
| 1.2445e-05 | 1.2410e-05 | 1.2364e-05 | 1.2375e-05 | 3.05 |
| 1.5424e-06 | 1.5426e-06 | 1.5399e-06 | 1.5416e-06 | 3.01 |
| 1.9238e-07 | 1.9247e-07 | 1.9230e-07 | 1.9255e-07 | 3.00 |

1.0280e-04 1.0109e-04 1.0630e-04 1.0035e-04 3.14
1.2445e-05 1.2410e-05 1.2364e-05 1.2375e-05 3.05
1.5424e-06 1.5426e-06 1.5399e-06 1.5416e-06 3.01
1.9238e-07 1.9247e-07 1.9230e-07 1.9255e-07 3.00

(6.5a) $x(\xi, \eta) = \frac{1}{2} + \frac{1}{2} (\xi + \frac{1}{10} \cos(2\pi \xi) \sin(2\pi \eta))$, 
(6.5b) $y(\xi, \eta) = \frac{1}{2} + \frac{1}{2} (\eta + \frac{1}{10} \sin(2\pi \xi) \cos(2\pi \eta))$.

A 6 × 6 element $N = 6$ mesh is show in Figure 6. Each side of the domain has a different type of boundary condition, so $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, as shown in the same figure and listed in (2.19). The solutions of vorticity $\omega \in \Lambda^0(\Omega)$, velocity $u \in \Lambda^1(\Omega)$ and pressure $p \in \Lambda^2(\Omega)$ are given by

(6.6a) $\omega(0) = \frac{3}{2} \pi \sin(\frac{3}{2} \pi x) \sin(\frac{3}{2} \pi y)$,
(6.6b) $u^{(1)} = - (\cos(\frac{3}{2} \pi x) \sin(\frac{3}{2} \pi y)) \, dx + \left(2 \sin(\frac{3}{2} \pi x) \cos(\frac{3}{2} \pi y) \right) \, dy$,
(6.6c) $p^{(2)} = (\sin(\pi x) \sin(\pi y)) \, dx \wedge dy$.

They lead to nonzero body force $f \in \Lambda^1(\Omega)$ and mass source $g \in \Lambda^2(\Omega)$. Figure 6 shows the convergence of the vorticity $\omega_h \in \Lambda^0_h(\Omega; C_0)$, velocity $u_h \in \Lambda^1_h(\Omega; C_1)$ and pressure $p_h \in \Lambda^2_h(\Omega; C_2)$.
The errors for the vorticity and velocity are measured in the $H^k$-norm, i.e. $\|\omega - \omega_h\|_{H^0}$, and $\|u - u_h\|_{H^1}$, respectively, and the error of the pressure is given in the $L^2$-norm. In Figure 6 convergence rates are added which show the optimal $h$-convergence behavior of the Stokes problem on a curvilinear domain with curvilinear grid and all four types of boundary conditions.

![Diagram of computational domain with a 6x6 element mesh](image)

**Figure 6.** Upper left figure show the computational domain with a 6x6 element mesh of $N = 6$. Furthermore the velocity, vorticity and pressure $h$-convergence results are shown of Stokes problem (6.6). All variables are tested on grids with $N = 2, 4, 6$ and 8.

### 7. Concluding remark

Optimal approximation of the Stokes problem for all admissible boundary conditions essentially hinges on the construction of a conforming discrete Hodge decomposition, $\Lambda^k_h = Z^k_h \oplus Z^{k,\perp}_h$ and a discrete Poicaré inequality, that are based on the bijection of the exterior derivative on the conforming subspace, $d : Z^k_h \rightarrow B^{k+1}_h$. Ensuring these properties result in a compatible discretization method, and relied on the construction of a bounded projection operator, $\pi_h : \Lambda^k(\Omega) \rightarrow \Lambda^k_h(\Omega; C_k)$, that commutes with the exterior derivative, $\pi_h d = d \pi_h$. So the compatibility is based on the bijection of the codomain operator, $\delta : Z^{k,\perp}_h \rightarrow B^{k+1}_h$, and the construction of interpolatory basis functions. From this it follows that, $B^{k+1}_h \subset B^{k+1}_h$, $Z^k_h \subset Z^k$ and $Z^{k,\perp}_h \subset Z^{k,\perp}_h$. From these properties the rest follows.

For piecewise sufficiently smooth mappings, the optimal conference rates hold on curvilinear grids as well, since the pullback operator of the map from a curvilinear domain to the Cartesian frame commutes with the projection operator. Any projection (discretization) with these properties will yield similar results as described in this paper.

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REFERENCES

[1] H. Abboud, F. E. Chami, and T. Sayah. A priori and a posteriori estimates for three-dimensional Stokes equations with nonstandard boundary conditions. *Numerical methods for partial differential equations*, 2011.

[2] R. Abraham, J. Marsden, and T. Ratiu. *Manifolds, tensor analysis, and applications*, volume 75 of *Applied Mathematical Sciences*. Springer, second edition.

[3] D. Arnold, D. Boffi, and R. Falk. Quadrilateral H(div) finite elements. *SIAM journal of numerical analysis*, 42(6):2429–2451, 2005.

[4] D. Arnold, R. Falk, and J. Gopalakrishnan. Mixed finite element approximation of the vector Laplacian with Dirichlet boundary conditions. *Mathematical Models & Methods in Applied Sciences*, 22(9), 2012.

[5] D. Arnold, R. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numerica*, pages 1–155, 2006.

[6] D. Arnold, R. Falk, and R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bulletin of the American mathematical society*, 47:281–354, 2010.

[7] C. Bernardi and N. Chorfi. Spectral discretization of the vorticity, velocity, and pressure formulation of the Stokes problem. *SIAM journal of numerical analysis*, 44(2):826–850, 2006.

[8] P. Bochev. A discourse on variational and geometric aspects of stability of discretizations. Technical report, Sandia National Laboratories, 2003.

[9] P. Bochev and M. Gunzburger. A locally conservative mimetic least-squares finite element method for the Stokes equations. In I. Lirkov, S. Margenov, and J. Wasniewski, editors, *Proceedings of LSSC 2009*, volume 5910 of *Springer Lecture Notes in Computer Science*, page 637644, 2009.

[10] P. Bochev and M. Gunzburger. *Least-Squares Finite Element Methods*, volume 166 of *Applied Mathematical Sciences*. Springer, 2009.

[11] P. Bochev and J. Hyman. Principles of mimetic discretizations of differential operators. In D. Arnold, P. Bochev, R. Lehoucq, R. Nicolaides, and M. Shashkov, editors, *Compatible Discretizations*, volume 142 of *IMA Volumes in Mathematics and its Applications*, pages 89–119. Springer, 2006.

[12] A. Bossavit. On the geometry of electromagnetism. *Journal of Japanese Society of applied electromagnetics and mechanics*, 6:17–28, 114–123, 233–240, 318–326, 1998.

[13] A. Bossavit. Computational electromagnetism and geometry. *Journal of Japanese Society of applied electromagnetics and mechanics*, 7, 1999, 2000.

[14] J. Bramble and P. Lee. On variational formulations for the Stokes equations with nonstandard boundary conditions. *RAIRO modelisation mathematique et analyse numerique*, 28(7):903–919, 1994.

[15] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[16] J. Brüning and M. Lesch. Hilbert complexes. *Journal of Functional Analysis*, 108:88–132, 1992.

[17] A. Buffa, J. Rivas, G. Sangalli, and R. Vazquez. Isogeometric discrete differential forms in three dimensions. *SIAM journal of numerical analysis*, 49:818–844, 2011.

[18] C. Canuto, M. Hussaini, A. Quarteroni, and T. Zang. *Spectral Methods, fundamentals in single domains*. Springer, 2006.

[19] P. Ciarlet and P.-A. Raviart. Interpolation theory over curved elements, with applications to finite element methods. *Comput. Methods Appl Mech. Engrg.*, 1:217–249, 1972.

[20] M. Costabel. A coercive bilinear form for Maxwell’s equations. *Journal of Mathematical Analysis and Applications*, 157:527–541, 1991.

[21] M. Desbrun, A. Hirani, M. Leok, and J. Marsden. Discrete Exterior Calculus. *arXiv:math/0508341*, 2005.

[22] J. Dieudonné. *A History of Algebraic and Differential Topology, 1900 – 1960*, Birkhäuser, 1989.

[23] F. Dubois. Vorticity-velocity-pressure formulation for the Stokes problem. *Mathematical methods in the applied sciences*, 25:1091–1119, 2002.

[24] F. Dubois, M. Salaüın, and S. Salmon. First vorticity-velocity-pressure numerical scheme for the Stokes problem. *Comput. Methods Appl Mech. Engrg.*, 192:4877–4907, 2003.

[25] P. Erdős and P. Vértesi. On the almost everywhere divergence of Lagrange interpolatory polynomials for arbitrary system of nodes. *Acta mathematica Academiae Scientiarum Hungaricae* Tomus, 36(1-2):71–89, 1980.

[26] H. Flanders. *Differential forms with applications to the physical sciences*. Dover publications, 1989.

[27] T. Frankel. *The geometry of physics, an introduction*. Cambridge University Press, 2006.

[28] M. Gerritsma. Edge functions for spectral element methods. In J. Hesthaven and E. Ronquist, editors, *Spectral and High Order Methods for Partial Differential Equations*, pages 199–208. Springer, 2011.

[29] V. Girault. Incompressible finite element methods for Navier-Stokes equations with nonstandard boundary conditions in $\mathbb{R}^3$. *Mathematics of computation*, 51(183):55–74, July 1988.

[30] V. Girault and P. Raviart. *Finite Element Methods for Navier-Stokes equations*. Springer-Verlag, 1986.

[31] W. Gordon and C. Hall. Transfinite element methods: blending-function interpolation over arbitrary curved element domains. *Numerische Mathematik*, 21(2):109–129, 1973.

[32] F. Harlow and J. Welch. Numerical calculations of time dependent viscous incompressible flow of fluid with a free surface. *Physics of fluids*, 8(12):2182–2189, 1965.

[33] A. Hatcher. *Algebraic topology*. Cambridge University Press, 2002.

[34] R. Hiptmair. Discrete Hodge operator. *Numerische Mathematik*, 90:265–289, 2001.

[35] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, pages 237–339, 2002.
[36] T. Hughes and L. Franca. A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity/pressure spaces. *Comp. Meth. Appl. Mech. Eng.*, 65:85–96, 1987.

[37] J. Hyman, J. Morel, M. Shashkov, and S. Steinberg. Mimetic finite difference methods for diffusion equations. *Computational Geosciences*, 6(3-4):333–352, 2002.

[38] J. Hyman and J. Scovel. Deriving mimetic difference approximations to differential operators using algebraic topology. Technical report, Los Alamos National Laboratory, 1988.

[39] J. M. Hyman and S. Steinberg. The convergence of mimetic methods for rough grids. *Computers and Mathematics with applications*, 47(10-11):1565–1610, 2004.

[40] J. Kreeft and M. Gerritsma. Mixed mimetic spectral element method for Stokes flow: a pointwise divergence-free solution. *Submitted to Journal Computational Physics, arXiv:1201.4409*, 2012.

[41] J. Kreeft, A. Palha, and M. Gerritsma. Mimetic framework on curvilinear quadrilaterals of arbitrary order. *submitted to J. Found. Comp. Math., arXiv:1111.4304*, 2011.

[42] W. Massey. *A Basic Course in Algebraic Topology*. Springer, 1991.

[43] C. Mattiussi. A reference discretization strategy for the numerical solution of physical field problems. *Advances in Imaging and Electron Physics*, 121:144–276, 2000.

[44] J. Munkres. *Elements of algebraic topology*. Westview Press, 1984.

[45] J. Nédélec. Mixed Finite Elements in $\mathbb{R}^3$. *Numerische Mathematik*, 35:315–341, 1980.

[46] A. Palha and M. Gerritsma. Mimetic least-squares spectral/hp finite element method for the Poisson equation. In I. Lirkov, S. Margenov, and J. Waśniewski, editors, *Large-Scale Scientific Computing*, volume 5910/2010 of *Lecture notes in Computer science*, pages 662–670, 2010.

[47] F. Rapetti and A. Bossavit. Whitney forms of higher degree. *SIAM journal of numerical analysis*, 47(3):2369–2386, 2009.

[48] P. Raviart and J. Thomas. A mixed finite element method for 2nd order elliptic problems. *Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics*, 606:292–315, 1977.

[49] N. Robidoux. Polynomial Histopolation, Superconvergent Degrees Of Freedom, And Pseudospectral Discrete Hodge Operators. *Unpublished: http://www.cs.laurentian.ca/nrobidoux/prints/super/_histogram.pdf*, 2008.

[50] N. Robidoux and S. Steinberg. A discrete vector calculus in tensor grids. *Computational Methods in Applied Mathematics*, 1:1–44, 2011.

[51] I. Singer and J. Thorpe. *Lecture notes on elementary topology and geometry*. Springer, 1976.

[52] E. Tonti. On the formal structure of physical theories. *Monograph of the Italian National Research Council*, 1975.

[53] H. Whitney. *Geometric Integration Theory*. Dover Publications, 1957.