Noise Prevents Singularities in Linear Transport Equations

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Abstract

A stochastic linear transport equation with multiplicative noise is considered and the question of no-blow-up is investigated. The drift is assumed only integrable to a certain power. Opposite to the deterministic case where smooth initial conditions may develop discontinuities, we prove that a certain Sobolev degree of regularity is maintained, which implies Hölder continuity of solutions. The proof is based on a careful analysis of the associated stochastic flow of characteristics.

\textit{Keywords:} stochastic linear transport equation, no-blow-up, integrable drift, Sobolev initial conditions

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1. Introduction

Consider the stochastic linear transport equation in Stratonovich form

\[
\frac{\partial u}{\partial t} + b \cdot \nabla u + \sigma \nabla u \circ dW \frac{dt}{dt} = 0, \quad u|_{t=0} = u_0.
\]

Here \(W = (W_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion defined on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\), the drift \(b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\) is a given deterministic vector field, \(\sigma \in \mathbb{R}\) and \(u_0 : \mathbb{R}^d \to \mathbb{R}\) are given and the solution

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$u = u(x,t)$ will be a scalar random field on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ defined for $(t,x) \in [0,T] \times \mathbb{R}^d$.

We deal with the problem of singularities of $u$ starting from a regular initial condition $u_0$. When $\sigma = 0$ and $b$ is not Lipschitz, singularities may appear, in the form of discontinuities (or blow-up of derivatives), as in the simple example $d = 1$, $b(x) = -\text{sign}(x) \sqrt{|x|}$: any non-symmetric smooth initial condition $u_0$ develops a discontinuity at $x = 0$ for any $t > 0$, because there are different, symmetric, initial conditions $x_0$ for the associated equation of characteristics

$$x'(t) = b(t, x(t)), \quad x(0) = x_0$$

which coalesce at $x = 0$ at any arbitrary positive time. Opposite to the question of uniqueness of weak $L^\infty$ solutions, where positive results have been given under relatively weak assumptions on $b$, see for instance [11] and [1], it seems that good results of no blow-up are not available in the deterministic case when $b$ is not Lipschitz.

The purpose of this paper is to show that, for $\sigma \neq 0$ and $b$ of class

$$b \in L_p^q := L^q(0,T; L^p(\mathbb{R}^d, \mathbb{R}^d)),$$

$$p,q \geq 2, \quad \frac{d}{p} + \frac{2}{q} < 1 \quad (1)$$

some regularity of the initial condition is maintained, in particular discontinuities do not appear. We prove the following result.

**Theorem 1.** If $\sigma \neq 0$, (1) holds and $u_0 \in \cap_{r \geq 1} W^{1,r}(\mathbb{R}^d)$ then there exists a unique solution $u$ such that

$$P \left( u(t, \cdot) \in \cap_{r \geq 1} W^{1,r}_{\text{loc}}(\mathbb{R}^d) \right) = 1$$

for every $t \in [0,T]$.

The unique solution in this class is given by a representation formula, in terms of $u_0$, involving a weakly differentiable stochastic flow. By Sobolev embedding theorem, $u(t, \cdot)$ is $\alpha$-Hölder continuous for every $\alpha \in (0,1)$, with probability one. Hence, from smooth initial conditions, discontinuities cannot arise.

The precise formulation of the concept of solution and other details are given in the sequel.
The intuitive idea is that, opposite to the deterministic case, when $\sigma \neq 0$ the characteristics cannot meet. They satisfy the stochastic equation

$$dX_t = b(X_t, t) \, dt + \sigma dW_t$$  \hspace{1cm} (2)$$

which generates, under assumptions (1), a stochastic flow of Hölder continuous homeomorphisms, with some weak form of differentiability. The existence of an Hölder continuous stochastic flow has been proved in [12], [13], [21]. A differentiability property in terms of finite increments has been given in [13]. Here we establish Sobolev type differentiability. A similar Sobolev regularity of the flow is investigated in [20] by different tools (Malliavin calculus). See also [19].

The assumption (1) has been introduced in the framework of stochastic differential equations by [17] who have proved strong uniqueness. In the fluid dynamic literature, with $\leq$ in place of $<$, it is known as the Ladyzhenskaya-Prodi-Serrin condition. One of its main consequences is that it gives uniform bounds on gradients of solutions to an auxiliary parabolic problem (see Theorem 2 below) essential for our approach, along with good properties of the second derivatives.

The possibility that noise may prevent the emergence of singularities is an intriguing phenomenon that is under investigation for several systems. For linear transport equations with $b \in L^\infty ([0, T]; C^\alpha_b (\mathbb{R}^d, \mathbb{R}^d))$, it may be deduced from [15] (the result presented here is more general). For nonlinear systems there are negative results, like the fact that noise does not prevent shocks in Burgers equation, see [14], and positive results for special kind of singularities (collapse of measure valued solutions) for the vorticity field of 2D Euler equations, see [16], and for 1D Vlasov-Poisson equation, see [10]. Moreover, for Schrödinger equations, there are several theoretical and numerical results of great interest, see [3] - [9]. We do not list here the results concerning the restored uniqueness due to noise and address to the lecture note [14] on this subject.

After the result of Theorem 1 it remains open the question whether the solution is Lipschitz continuous (or more) when $u_0 \in W^{1,\infty} (\mathbb{R}^d)$ (or more). In dimension $d$ we think that this is a difficult question under assumption (1). The answer is positive when $b \in L^\infty ([0, T]; C^\alpha_b (\mathbb{R}^d, \mathbb{R}^d))$ because the stochastic flow is made of diffeomorphisms, see [15] and it is also positive in dimension $d = 1$ for certain discontinuous drift $b$, including for instance $b(x) = \text{sign} (x)$, see [2].
It must be emphasized that, although this “regularization by noise” may look related to the regularization produced by the addition of a Laplacian to the equation, in fact it preserves the hyperbolic structure of the equation. The equations remain reversible and the solution at time \( t \) is, in the problem treated in this paper, just given by the initial condition composed with a flow. If the initial condition has a discontinuity, the solution also has a discontinuity; no smoothing effect is introduced. However, the emergence of singularities (shocks in our case) is prevented.

The work is organized as follows. In Section 2 we present some results on regularity and approximation properties of the flow associated to the SDE (2). They are obtained via the study of an associated SDE and the regularity of its solutions. The main results are contained in Lemmas 3 and 5 while more technical results are collected in the Appendix (Section 5). In Section 3 we define weakly differentiable solutions of the SPDE and prove their existence in Theorem 10. A technical result on convergence of random fields in Sobolev spaces is left to the last Appendix. Finally, uniqueness of weakly differentiable solutions of the SPDE is proved in Section 4.

2. Convergence Results

In this section we present some technical results on an associated SDE that we will study as an intermediate step to obtain some regularity and approximation properties of the flow associated to the SDE (2). The main results are contained in Lemmas 3 and 5.

Let us start by setting the notation used and recalling some results. We will use the following auxiliary SDE, introduced in [13]:

\[
\begin{align*}
\mathrm{d}Y_t &= \lambda U(t, \gamma_0^{-1}(x)) \mathrm{d}t + (\nabla U(t, \gamma_0^{-1}(x)) + \text{Id}) \mathrm{d}W_t, \quad Y_0 = x .
\end{align*}
\]

The link between this SDE and the one presented in the introduction is given by the \( C^1 \)-diffeomorphism \( \gamma_t: Y_t = \gamma_t \circ X_t \circ \gamma_0^{-1} \), where \( \gamma_t(x) = x + U(t, x) \).

Here, \( U: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d \) is the solution of the PDE

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t U + \frac{1}{2} \Delta U + b \cdot \nabla U - \lambda U + b = 0 \\
U(T, x) = 0
\end{array} \right.
\end{align*}
\]

This PDE is well posed in the space

\[
H_{2,p}^q(T) := L^q(0, T; W^{2,p}(\mathbb{R}^d)) \cap W^{1,q}(0, T; L^p(\mathbb{R}^d)) ;
\]

we report here the precise result, given by [13, Theorem 3.3].
Theorem 2. Take \( p, q \) such that (1) holds, \( \lambda > 0 \) and two vector fields \( b, f (t, x) : \mathbb{R}^{d+1} \to \mathbb{R}^d \) belonging to \( L^q_{\infty} (T) \). Then in \( H^q_{2,p} (T) \) there exists a unique solution of the backward parabolic system
\[
\begin{aligned}
\partial_t u + \frac{1}{2} \Delta u + b \cdot \nabla u - \lambda u + f &= 0, \\
u(T, x) &= 0.
\end{aligned}
\] (4)
For this solution there exists a finite constant \( N \) depending only on \( d, p, q, T, \lambda \) and \( \|b\|_{L^q_p (T)} \) such that \( \|u\|_{H^q_{2,p} (T)} \leq N \|f\|_{L^q_p (T)} \). (5)

We will use the result of this theorem with \( f = b \).

Let \( b^n \) be a sequence of smooth vector fields converging to \( b \) in \( L^q_p \). Let \( U \) be the unique solution to the PDE (1) provided by the above Theorem and \( U^n \) the solutions obtained using the approximating vector fields \( b^n \). Lemma 12 shows that the vector fields \( U^n \) converge in \( H^q_{2,p} \) to \( U \).

In [13] is also proved the existence of Hölder flows of homeomorphisms for the two SDEs above, which we denote by \( \phi_t (\cdot) \) for the SDE (2), and \( \psi_t (\cdot) \) for (3). We will use \( \phi^t_n (\cdot) \) to denote the flows obtained for the approximating vector fields \( b^n \), and \( \psi^t_n (\cdot) \) for the flows corresponding to the auxiliaries SDEs obtained via the diffeomorphisms \( \gamma^t_n = Id + U^n(t, \cdot) \). We will use \( \phi^t_{0,n} (\cdot) \), and \( \psi^t_{0,n} (\cdot) \) for the inverse flows.

We can now state and prove the two main regularity results on the flows \( \phi^t_{0,n} \).

Lemma 3. For every \( R > 0 \), \( p \geq 1 \) and \( x, y \in B_R \),
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in B_R} E \left[ \left| \phi^t_{0,n} (x) - \phi^t_0 (y) \right|^p \right] \leq C_{p,T} \left| x - y \right|^p.
\]

In particular,
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in B_R} E \left[ \left| \phi^t_{0,n} (x) - \phi^t_0 (x) \right|^p \right] = 0. \quad (6)
\]

Proof. Step 1 (preliminary estimates). By lemma 12 we have that for every \( r > 0 \) there exist a function \( f \) s.t. \( \lim_{n \to \infty} f(n) = 0 \) and
\[
\sup_{x \in \mathbb{R}^d} \sup_{t \in [0,T]} \left| \nabla U(t, x) \right| \leq 1/2, \\
\sup_{x \in B_r} \sup_{t \in [0,T]} \left| U^n(t, x) - U(t, x) \right| \leq f(n), \\
\sup_{x \in B_r} \sup_{t \in [0,T]} \left| \nabla U^n(t, x) - \nabla U(t, x) \right| \leq f(n).
\]
Since $\phi_t(x)$ is jointly continuous in space and time, there exist an $r < \infty$ s.t. the image of $B_R \times [0,T]$ will be contained in $B_r$ for all $t \leq T$. In the following we will always take $x, y \in B_R$. It follows that

$$\left| U^n(t, \phi^n_t(x)) - U(t, \phi_t(y)) \right| \leq f(n) + \frac{1}{2} |\phi^n_t(x) - \phi_t(y)|,$$

$$\left| \nabla U^n(t, \phi^n_t(x)) - \nabla U(t, \phi_t(y)) \right| \leq f(n) + \left| \nabla U^n(t, \phi^n_t(x)) - \nabla U^n(t, \phi_t(y)) \right|.$$

To shorten notation, we will write $\phi^n$ and $\phi$ to denote $\phi^n_t(x)$ and $\phi_t(y)$, $U^n(\phi^n)$ and $U^n(x)$ to denote $U^n(t, \phi^n)$ and $U^n(0, x)$, etc. The same holds for the flows of the SDE (3). From the definition $\psi^n_t = \gamma_t \circ \phi^n_t \circ (\gamma^n_t)^{-1}$ and the properties of the diffeomorphisms $\gamma^n_t$ obtained from Lemma 12 and Remark 13 we immediately have

$$|\psi^n - \psi| \geq |\phi^n - \phi + U^n(\phi^n) - U^n(\phi)| - f(n)$$

$$\geq \frac{1}{2} |\phi^n - \phi| - f(n)$$

(7)

$$2 \left( |\psi^n - \psi| + f(n) \right) \geq |\phi^n - \phi|$$

and

$$|\psi^n - \psi| \leq \frac{3}{2} |\phi^n - \phi| + f(n).$$

(8)

**Step 2** (computations). We start by proving the convergence of the flows of the auxiliary SDE 3. By Itô formula, for any $a \geq 2$

$$\frac{1}{a} d \left| \psi^n - \psi \right|^a = \left\{ \lambda \langle (\psi^n - \psi), U^n(\phi^n) - U(\phi) \rangle_{\mathbb{R}^d} dt + \langle (\psi^n - \psi), (\nabla U^n(\phi^n) - \nabla U(\phi)) \cdot dW_t \rangle_{\mathbb{R}^d} 
+ \frac{a - 1}{2} Tr \left( [\nabla U^n(\phi^n) - \nabla U(\phi)] [\nabla U^n(\phi^n) - \nabla U(\phi)]^T \right) dt \right\}$$

$$= \left| \psi^n - \psi \right|^{a-2} \left\{ A_1 + A_2 + A_3 \right\}.$$

Let us analyze the three terms $A_1, A_2, A_3$. Using (7) we have

$$A_1 \leq \lambda |\psi^n - \psi| \left( f(n) + \frac{1}{2} |\phi^n - \phi| \right) dt$$

$$\leq \lambda |\psi^n - \psi|^2 dt + 2\lambda f(n) |\psi^n - \psi| dt.$$
Since $\nabla U^n$ is bounded (uniformly in $n$, see Lemma [12]) and by (26) $|\psi^n|^a$ belongs to $L^2(\Omega \times [0,T])$ for any $a \geq 1$, we can write $A_2 = dM^n_t$, where for every $n$, $dM^n_t$ is the differential of a zero mean martingale. As for the third term, using twice the inequality $(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)$ and the estimates of the first step, we get

$$
\frac{2}{d^2(a-1)} A_3 \leq \left\{ 2\left| \nabla U^n(\phi^n) - \nabla U^n(\phi) \right|^2 + 2f^2(n) \right\} dt
$$

$$
\leq |\phi^n - \phi|^2 dA^n_t + 2f^2(n) dt
$$

$$
\leq 8|\psi^n - \psi|^2 dA^n_t + 8f^2(n) dA^n_t + 2f^2(n) dt,
$$

where for every $n$

$$
A^n_t := 2 \int_0^t \frac{\left| \nabla U^n(\phi^n_s) - \nabla U^n(\phi)_s \right|^2}{|\phi^n_s - \phi_s|^2} 1_{\{\phi^n \neq \phi_s\}} ds
$$

(9)

is a nondecreasing adapted stochastic process, with $A^n_0 = 0$, and uniformly in $n \mathbb{E}[A^n_T] \leq C < \infty$, see Lemma [14]. Set $B^n_t := [4d^2a(a-1)]A^n_t$. From the above estimates and after renaming $M_t$ (which remains a zero mean martingale), we get

$$
d\left(e^{-B^n_t} |\psi^n - \psi|_t^a\right) \leq e^{-B^n_t} \left[a\lambda|\psi^n - \psi|^a + 2a\lambda f(n)|\psi^n - \psi|^{a-1}\right] dt
$$

$$
+ dM_t + f^2(n)e^{-B^n_t} |\psi^n - \psi|^{a-2} dB^n_t
$$

$$
+ d^2a(a-1)e^{-B^n_t} f^2(n)|\psi^n - \psi|^{a-2} dt.
$$

Integrating in time, taking the expected value, and finally the supremum over $t \in [0,T]$, we get

$$
\sup_{t \in [0,T]} \mathbb{E}\left[e^{-B^n_T} |\psi^n_t - \psi_t|^a\right] \leq \left|\psi^n_0 - \psi_0\right|^a + a\lambda \mathbb{E}\left[\int_0^T e^{-B^n_s} |\psi^n_s - \psi_s|^a ds\right]
$$

$$
+ C_{a,d,\lambda} f(n) \mathbb{E}\left[\int_0^T e^{-B^n_s} \left|\psi^n_s - \psi_s\right|^{a-1} + f(n)|\psi^n_s - \psi_s|^{a-2} ds\right]
$$

$$
+ f^2(n) \mathbb{E}\left[\int_0^T e^{-B^n_s} |\psi^n_s - \psi_s|^{a-2} dB^n_s\right].
$$

(10)

The expected value in the second line above is bounded uniformly in $n$. This fact is easily seen using for each term H"older inequality together with the
integrability properties of the flows $\psi^n$ and of the exponential of the processes $B^n_s$, provided by (26) and Lemma 14 respectively. We claim that also the expected value of the last line is bounded.

**Claim 4.** There exists a constant $C$ s.t. for every $n$ and $p \geq 0$

$$
\mathbb{E} \left[ \int_0^T e^{-B^n_s} \left| \psi^n_s - \psi_s \right|^p \, dB^n_s \right] \leq C .
$$

**Proof of the Claim.** Using the definition of $B^n_t$ we can rewrite the term on the left hand side as

$$
\mathbb{E} \left[ \int_0^T e^{-B^n_s} \left| \psi^n_s - \psi_s \right|^p \frac{\nabla U(s, \phi^n_s) - \nabla U(s, \phi_s)}{\left| \phi^n_s - \phi_s \right|^2} \, ds \right] .
$$

Using Hölder inequality, for some $\varepsilon > 1$ small (to be fixed later) and $k$ the conjugate exponent, we obtain the term

$$
\mathbb{E} \left[ \int_0^T e^{-kB^n_s} \left| \psi^n_s - \psi_s \right|^{kp} \, ds \right] ,
$$

for which we have already obtained a uniform bound, and the term

$$
\mathbb{E} \left[ \int_0^T \left| \nabla U^n(s, \phi^n_s) - \nabla U^n(s, \phi_s) \right|^{2\varepsilon} \frac{\left| \phi^n_s - \phi_s \right|^{2\varepsilon}}{\left| \phi^n_s - \phi_s \right|^{2\varepsilon}} \, ds \right] . \quad (11)
$$

For this term, we proceed as in the proof of Lemma 14. The key point is the estimate of the term

$$
\int_0^1 \mathbb{E} \left[ \int_0^T \left| \nabla^2 U^n(s, \phi^{n,r}_s) \right|^{2\varepsilon} \, ds \right] \, dr ,
$$

where

$$
\phi^{n,r}_t = rx + (1 - r)y + \int_0^t rb^n(s, \phi^n_s) + (1 - r)b(s, \phi_s) \, ds + W_t .
$$

We can conclude as in the proof of Lemma 14 if we use the result of Lemma 15. In particular, (11) is controlled by $\|b\|_{L^p_2}$. ■
We return to the proof of Lemma 3. Thanks to the uniform bounds obtained for the expectations in the second and third lines of (10), we can pass to the limit in $n$ to obtain
\[
\limsup_n \sup_{t \in [0,T]} \mathbb{E}\left[ e^{-B^a_t} |\psi^n_t - \psi_t|^a \right] 
\leq \limsup_n C_a \left( |\phi_0(x) - \phi_0(y)|^a + f(n)^a \right)
+ C_{a,\lambda} \limsup_n \mathbb{E}\left[ \int_0^T e^{-B^a_s} |\psi^n_s - \psi_s|^a \, ds \right]
\leq C_a |x - y|^a + C_{a,\lambda} \int_0^T \limsup_n \sup_{t \in [0,s]} \mathbb{E}\left[ e^{-B^a_t} |\psi^n_t - \psi_t|^a \right] \, ds.
\]

Using Gronwall lemma we get
\[
\limsup_n \sup_{t \in [0,T]} \mathbb{E}\left[ e^{-B^a_t} |\psi^n_t - \psi_t|^a \right] \leq C_{a,\lambda,T} |x - y|^a.
\] (12)

We can now get rid of the exponential factor using again Hölder inequality
\[
\limsup_n \sup_{t \in [0,T]} \mathbb{E}\left[ |\psi^n_t - \psi_t|^a \right] \leq \limsup_n \left\{ \mathbb{E}\left[ e^{2B^a_T} \right]^{1/2} \sup_{t \in [0,T]} \mathbb{E}\left[ e^{-2B^a_t} |\psi^n_t - \psi_t|^{2a} \right]^{1/2} \right\}
\leq C_{p,\lambda,T} |x - y|^a.
\]

With $a = 2p$, redefining $B^a_t$ as $1/2$ of the process defined above and using the relation (7), we can finally transport this bound to the flows $\phi^n$:
\[
\limsup_n \sup_{t \in [0,T]} \mathbb{E}\left[ |\phi^n_t - \phi_t|^p \right] \leq C_p \limsup_n \left( \sup_{t \in [0,T]} \mathbb{E}\left[ |\psi^n_t - \psi_t|^p \right] + f^p(n) \right)
\leq C_{p,\lambda,T} |x - y|^p.
\]

Remark that all the estimates found are uniform in $x, y \in B(0, R)$, so that we have obtained the desired result for the forward flows. But since the backward flows $\phi^{L,n}_0(\cdot)$ and $\phi^b_0(\cdot)$ are solution of the same SDE driven by the drifts $-\bar{b}^n$ and $-\bar{b}$, the same result holds for them too. $\blacksquare$

**Lemma 5.** For every $p \geq 1$, there exists $C_{d,p,T} > 0$ such that
\[
\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}\left[ |\nabla \phi^{L,n}_0(x)|^p \right] \leq C_{d,p,T}
\] (13)
uniformly in $n$. 9
Proof. Again, since the backward flow satisfies the same SDE of the forward flow with a drift of opposite sign, it is enough to show that the uniform bound \([13] \) holds for the forward flows. Let \( \theta^n \) and \( \xi^n \) be the derivatives of \( \phi^n \) and \( \psi^n \), respectively. Since \( \phi^n_t = (\gamma^n_t)^{-1} \circ \psi_t \circ \gamma^n_0 \), from (25) we have \( |\theta^n_t|^p \leq C_{d,p} |\xi^n_t|^p \). Therefore, we only need to show that the estimate (13) holds for the flow \( \psi^n \), which solves

\[
\, d\xi^n_t(x) = \lambda \nabla U^n(t, \phi^n_t(x)) \xi^n_t(x) \, dt + \nabla^2 U^n(t, \phi^n_t(x)) \xi^n_t(x) \, dW_t
\]

with initial condition \( \xi^n_0(x) = Id \). For the rest of the proof we take any fixed \( x \in \mathbb{R}^d \). \( \nabla U^n \) is bounded uniformly in \( n \) and the function \( \nabla^2 U \) is at least in \( L^p \), so that the last term is the differential of a martingale \( (dM^n_t) \) due to Lemma [13]. By Itô formula we have therefore

\[
d|\xi^n|^p \leq C|\xi^n|^p dt + dM^n_t + |\xi^n|^{p-2}Tr\left( [\nabla^2 U^n (t, \phi^n_t(x)) \xi^n_t] [\nabla^2 U^n (t, \phi^n_t(x)) \xi^n_t]^T\right) dt .
\]

The constant \( C \) can be chosen independently of \( n \), and the trace of the matrix in the last term above can be controlled by a constant \( C_{p,d} \), depending on \( p \) and the dimension \( d \) of the space, times \( |\xi^n_t|^2 |\nabla^2 U^n(t, \phi^n_t)|^2 \). Introduce the process

\[
A^n_t := C_{p,d} \int_0^t |\nabla^2 U^n(s, \phi^n_s)|^2 ds .
\]  

(14)

This is a continuous, adapted, non decreasing process, with \( A^n_0 = 0 \) and, due to Lemma [13], \( \mathbb{E}[A^n_t] \leq C \) uniformly in \( n \). Lemma [13] even provides the bound \( \mathbb{E}[e^{kA^n_t}] \leq C_{||U^n||} \) for any real constant \( k \). We can therefore find a bound uniform in \( n \) reasoning as in Lemma [12] We find that

\[
d e^{-A^n_t} |\xi^n|^p \leq C e^{-A^n_t} |\xi^n|^p dt + e^{-A^n_t} dM^n_t
\]

and after integrating and taking the expected value one obtains

\[
\mathbb{E}[e^{-A^n_t} |\xi^n|^p] \leq |\xi_0|^p + C \int_0^t \mathbb{E}[e^{-A^n_s} |\xi^n|^p] ds .
\]

Take the supremum over all \( t \in [0,T] \) and apply Gronwall inequality to get

\[
\sup_{t \in [0,T]} \mathbb{E}[e^{-A^n_t} |\xi^n|^p] \leq C_T |\xi_0|^p = C_{d,p,T} ,
\]

uniformly in \( n \) and \( x \in \mathbb{R}^d \). Using Hölder inequality as in the proof of the previous lemma, we finally obtain estimate (13) for the derivative of the flow \( \psi^n \), and this concludes the proof. \( \square \)
3. Main Result of Existence of Weakly Differentiable Solutions

Consider the SPDE in Stratonovich form
\[
\frac{\partial u}{\partial t} + b \cdot \nabla u + \sigma \nabla u \circ dW = 0, \quad u|_{t=0} = u_0.
\]

The Itô formulation (as explained in detail also in [15]) is
\[
du + b \cdot \nabla u \, dt + \sigma \nabla u \, dW = \frac{\sigma^2}{2} \Delta u \, dt, \quad u|_{t=0} = u_0.
\]

In this section we assume \( b \in L^p_q \), with \( p, q \) satisfying condition (1).

**Definition 6.** Assume that \( b \in L^p_q \), with \( p, q \) as in (1). We say that \( u \) is a weakly differentiable solution of the SPDE if

1. \( u : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is measurable, \( \int u(t, x) \varphi(x) \, dx \) (well defined by property 2 below) is progressively measurable for each \( \varphi \in C_0^\infty(\mathbb{R}^d) \);
2. \( P \left( u(t, \cdot) \in \cap_{r \geq 1} W^1_{r, \text{loc}}(\mathbb{R}^d) \right) = 1 \) for every \( t \in [0, T] \) and both \( u \) and \( \nabla u \) are in \( C_0^0([0, T]; \cap_{r \geq 1} L^r(\Omega \times \mathbb{R}^d)) \);
3. for every \( \varphi \in C_0^\infty(\mathbb{R}^d) \) and \( t \in [0, T] \), with probability one one has
\[
\int u(t, x) \varphi(x) \, dx + \int_0^t \int b(s, x) \cdot \nabla u(s, x) \varphi(x) \, dx \, ds
\]
\[
= \int u_0(x) \varphi(x) \, dx + \sigma \sum_{i=1}^d \int_0^t \left( \int u(s, x) \partial_{x_i} \varphi(x) \, dx \right) \, dW^i_s + \frac{\sigma^2}{2} \int_0^t \int u(s, x) \Delta \varphi(x) \, dx \, ds.
\]

**Remark 7.** The process \( s \mapsto Y^i_s := \int u(s, x) \partial_{x_i} \varphi(x) \, dx \) is progressively measurable by property 1 and satisfies \( \int_0^T |Y^i_s|^2 \, ds < \infty \) by property 2, hence the Itô integral is well defined.

**Remark 8.** The term \( \int_0^t \int b(s, x) \cdot \nabla u(s, x) \varphi(x) \, dx \, ds \) is well defined with probability one because of the integrability properties in \( (t, x) \) of \( b \) (assumptions) and \( \nabla u \) (property 2).

**Remark 9.** From 3 it follows that \( \int u(t, x) \varphi(x) \, dx \) has a continuous adapted modification, for every \( \varphi \in C_0^\infty(\mathbb{R}^d) \).
Let \( \phi_t(\omega) : \mathbb{R}^d \to \mathbb{R}^d \) be the \( \alpha \)-Hölder continuous stochastic flow of homeomorphisms, for every \( \alpha \in (0, 1) \), associated to the SDE
\[
dX^x_t = b(t, X^x_t) \, dt + dW_t, \quad X^x_0 = x
\]
constructed in [13]. The inverse of \( \phi_t \) will be denoted by \( \phi_t^{-1} \).

**Theorem 10.** Assume \( b \in L^p_q \) with \( p, q \) as in (1). If \( u_0 \in \bigcap_{r \geq 1} W^{1,r}(\mathbb{R}^d) \) then \( u(t, x) := u_0(\phi_0^{-1}(x)) \) is a weakly differentiable solution of the SPDE.

**Proof.** Step 1 (preparation). The random field \( (\omega, t, x) \mapsto u_0(\phi_0^{-1}(x)) \) is jointly measurable and \( (\omega, t) \mapsto \int u_0(\phi_0^{-1}(x)) \varphi(x) \, dx \) is progressively measurable for each \( \varphi \in C_0^\infty(\mathbb{R}^d) \). Hence part 1 of Definition 6 is true. We could prove part 2 by chain rule and Sobolev properties of \( \phi_0^{-1}(x) \). However, a direct verification of part 3 from the formula \( u(t, x) := u_0(\phi_0^{-1}(x)) \) is difficult because of lack of calculus. Hence we choose to approximate \( u(t, x) \) by a smooth field \( u_n(t, x) \); doing this, we prove both 2 and 3 by means of this approximation.

Let \( u_0^n \) be a sequence of smooth functions which converges to \( u_0 \) in \( W^{1,r}(\mathbb{R}^d) \) and uniformly on \( \mathbb{R}^d \). It is easy to check that these properties are satisfied for instance by \( u_0^n(x) = \int \theta_n(x-y) u_0(y) \, dy \) when \( \theta_n \) are usual mollifiers; for instance, the uniform convergence property comes from
\[
|u_0^n(x) - u_0(x)| \leq \int \theta_n(x-y) |u_0(y) - u_0(x)| \, dy \\
\leq C \int \theta_n(x-y) |y-x|^\alpha \, dy = C \int \theta_n(y) |y|^\alpha \, dy
\]
because \( u_0 \in C^{0,\alpha} \).

Let \( \phi^n_t(\omega) : \mathbb{R}^d \to \mathbb{R}^d \) be the stochastic flow of smooth diffeomorphisms associated to the equation
\[
dX^{x,n}_t = b_n(t, X^{x,n}_t) \, dt + dW_t, \quad X^{x,n}_0 = x,
\]
where \( b_n \) are smooth approximations of \( b \) as considered in the previous section, and let \( \phi_0^{t,n} \) be the inverse of \( \phi^n_t \). Then \( u_n(t, x) := u_0^n(\phi_0^{t,n}(x)) \) is a smooth solution of
\[
du_n + b_n \cdot \nabla u_n \, dt + \sigma \nabla u_n \, dW = \frac{\sigma^2}{2} \Delta u_n \, dt, \quad u_n|_{t=0} = u_0^n,
\]
see [18, Theorem 6.1.5], and thus it satisfies

\[
\int u_n(t, x) \varphi(x) \, dx + \int_0^t \int b_n(s, x) \cdot \nabla u_n(s, x) \varphi(x) \, dx \, ds \\
= \int u_0^n(x) \varphi(x) \, dx + \sigma \sum_{i=1}^d \int_0^t \left( \int u_n(s, x) \partial_{x_i} \varphi(x) \, dx \right) \, dW^i_s \\
+ \frac{\sigma^2}{2} \int_0^t \int u_n(s, x) \Delta \varphi(x) \, dx \, ds
\]

for every \( \varphi \in C_0^\infty(\mathbb{R}^d) \) and \( t \in [0, T] \), with probability one. We need to establish suitable bounds on \( u_n(t, x) \) and suitable convergence properties of \( u_n(t, x) \) to \( u(t, x) \) in order to apply Lemma 16 - which is the first step to obtain the regularity properties of \( u \) of point 2 of Definition 6 - and pass to the limit in the equation. More precisely, for every \( \varphi \in C_0^\infty(\mathbb{R}^d) \), \( t \in [0, T] \) and bounded r.v. \( Z \) we have

\[
\mathbb{E} \left[ Z \int u_n(t, x) \varphi(x) \, dx \right] + \mathbb{E} \left[ Z \int_0^t \int b_n(s, x) \cdot \nabla u_n(s, x) \varphi(x) \, dx \, ds \right] \\
= \int u_0^n(x) \varphi(x) \, dx + \sigma \sum_{i=1}^d \mathbb{E} \left[ Z \int_0^t \left( \int u_n(s, x) \partial_{x_i} \varphi(x) \, dx \right) \, dW^i_s \right] \\
+ \frac{\sigma^2}{2} \mathbb{E} \left[ Z \int_0^t \int u_n(s, x) \Delta \varphi(x) \, dx \, ds \right].
\]

We shall pass to the limit in each one of these terms. We are forced to use this very weak convergence due to the term

\[
\mathbb{E} \left[ Z \int_0^t \int b_n(s, x) \cdot \nabla u_n(s, x) \varphi(x) \, dx \, ds \right]
\]

where we may only use weak convergence of \( \nabla u_n \).

**Step 2** (convergence of \( u_n \) to \( u \)). We claim that, uniformly in \( n \) and for every \( r \geq 1 \),

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^d} \mathbb{E} \left[ |u_n(t, x)|^r \right] \, dx \leq C_r ,
\]

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^d} \mathbb{E} \left[ |\nabla u_n(t, x)|^r \right] \, dx \leq C_r .
\]
Let us show how to prove the second bound; the first one can be obtained in the same way. We use the representation formula for $u_n$ and Hölder inequality to obtain
\[
\left( \int_{\mathbb{R}^d} \mathbb{E} \left[ |\nabla u_n(t,x)|^r \right] \, dx \right)^2 \leq \int_{\mathbb{R}^d} \mathbb{E} \left[ |\nabla u_0^n(\phi_{0}^{t,n}(x))|^{2r} \right] \, dx \int_{\mathbb{R}^d} \mathbb{E} \left[ |\nabla \phi_{0}^{t,n}(x)|^{2r} \right] \, dx.
\]
The last integral on the right hand side is uniformly bounded by (13). Also the other integral term can be bounded uniformly: changing variables (recall that all functions involved are regular) we get
\[
\int_{\mathbb{R}^d} \mathbb{E} \left[ |\nabla u_0^n(\phi_{0}^{t,n}(x))|^{2r} \right] \, dx \leq \int_{\mathbb{R}^d} |\nabla u_0^n(y)|^{2r} \mathbb{E} \left[ |J_{\phi_{0}^{t}}(y)| \right] dy,
\]
where $J_{\phi_{0}^{t}}(y)$ is the Jacobian of $\phi_{0}^{t}(y)$; this last term can be controlled using again Hölder inequality, (13) and the convergence of $u_0^n$ in $W^{1,r}$ (for every $r \geq 1$). Remark that all the bounds obtained are uniform in $n$ and $t$.

We consider now the problem of the convergence of $u_n$ to $u$. Let us first prove that, given $t \in [0,T]$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$,
\[
P - \lim_{n \to \infty} \int_{\mathbb{R}^d} u_n(t,x) \varphi(x) \, dx = \int_{\mathbb{R}^d} u(t,x) \varphi(x) \, dx \quad (18)
\]
(convergence in probability). This is the first assumption of Lemma 16 and allows also to pass to the limit in the first term of equation (15) using the uniform bound (16) and Vitali convergence theorem (we are on the compact support of the test function $\varphi$). Since
\[
\int_{\mathbb{R}^d} u_n(t,x) \varphi(x) \, dx = \int_{\mathbb{R}^d} u_0^n(\phi_{0}^{t,n}(x)) \varphi(x) \, dx \\
= \int_{\mathbb{R}^d} (u_0^n - u_0)(\phi_{0}^{t,n}(x)) \varphi(x) \, dx + \int_{\mathbb{R}^d} u_0(\phi_{0}^{t,n}(x)) \varphi(x) \, dx,
\]
using Sobolev embedding $W^{1,2d} \hookrightarrow C^{0,1/2}$ we have
\[
\left| \int_{\mathbb{R}^d} (u_n(t,x) - u(t,x)) \varphi(x) \, dx \right| \leq |u_0^n - u_0|_{L^\infty} ||\varphi||_{L^1} \\
+ C ||\varphi||_{L^\infty} \int_{B_R} |\phi_{0}^{t,n}(x) - \phi_{0}^{t}(x)|^{1/2} \, dx.
\]
The first term converges to zero by the uniform convergence of $u^n_0$ to $u_0$. To treat the second one, recall we have proved property (6). Hence

$$\lim_{n \to \infty} \mathbb{E} \left[ \int_{B_R} |\phi^{l,n}_0(x) - \phi^l_0(x)| \, dx \right] = 0$$

and thus

$$P - \lim_{n \to \infty} \int_{B_R} |\phi^{l,n}_0(x) - \phi^l_0(x)| \, dx = 0.$$ 

Property (18) is proved.

Similarly, we can show that, given $\varphi \in C^0_\infty(\mathbb{R}^d)$,

$$P - \lim_{n \to \infty} \int_0^T \int_{\mathbb{R}^d} \left( u_n(t,x) - u(t,x) \right) \varphi(x) \, dx \, dt = 0. \quad (19)$$

This implies that we can pass to the limit in the last two terms of equation (15). Indeed, property (19) implies that

$$\mathbb{E} \left[ \left( \int_0^T \int_{B_R} \left( u_n(s,x) - u(s,x) \right) \varphi(x) \, dx \right) \, dW^i_s \right] = 0$$

for each $i = 1, \ldots, d$. Moreover,

$$\mathbb{E} \left[ \left( \int_0^T \int_{\mathbb{R}^d} \left( u_n(s,x) \partial_x \varphi(x) \, dx \right) \, ds \right)^2 \right] = \mathbb{E} \left[ \left( \int_0^T \int_{B_R} \left( u_n(s,x) \partial_x \varphi(x) \, dx \right) \, ds \right)^2 \right],$$

which is uniformly bounded in $n$ due to (16). By Vitali convergence theorem we obtain that

$$\lim_{n \to \infty} \mathbb{E} \left[ Z \int_0^t \left( \int u_n(s,x) \partial_x \varphi(x) \, dx \right) \, dW^i_s \right] = \mathbb{E} \left[ Z \int_0^t \left( \int u(s,x) \partial_x \varphi(x) \, dx \right) \, dW^i_s \right].$$

The proof of convergence for the last term of equation (15) is similar.

**Step 3** (regularity of $u$). Let us prove property 2 of Definition 6. The key estimate is property (13).

Given $r \geq 1$ and $t \in [0,T]$, let us prove that $P \left( u(t, \cdot) \in W^{1,r}_{loc}(\mathbb{R}^d) \right) = 1$. We want to use Lemma 16 with $F = u$, $F_n = u_n$. Condition 1 of Lemma 16
is provided by (18). It is clear that \( u_n(t, \cdot) \in W^{1,r}_{\text{loc}}(\mathbb{R}^d) \) for \( P \)-a.e. \( \omega \), so that condition 2 follows from the uniform bound on \( \nabla u_n \) obtained in (17). We can apply Lemma 16 and get \( u(t, \cdot) \in W^{1,r}_{\text{loc}}(\mathbb{R}^d) \) for \( P \)-a.e. \( \omega \).

Let us prove the second part of property 2 of Definition 6. We have, from Lemma 16 and (17),

\[
\mathbb{E} \left[ \int_{B_R} |\nabla u(t,x)|^r \, dx \right] \leq \limsup_{n \to \infty} \mathbb{E} \left[ \int_{B_R} |\nabla u_n(t,x)|^r \, dx \right] \leq C_r
\]

for every \( R > 0 \) and \( t \in [0,T] \). Hence, by monotone convergence we have

\[
\sup_{t \in [0,T]} \mathbb{E} \left[ \int_{\mathbb{R}^d} |\nabla u(t,x)|^r \, dx \right] \leq C_r .
\]

A similar bound can be proved for \( u \) itself: using (16), the convergence in probability proved in the previous step and Vitali convergence theorem we get that for any \( r' < r \), \( R > 0 \) and uniformly in time,

\[
\int_{B_R} \mathbb{E} \left[ |u(t,x)|^{r'} \right] \, dx = \lim_{n \to \infty} \int_{B_R} \mathbb{E} \left[ |u_n(t,x)|^{r'} \right] \, dx \leq C_r ;
\]

by monotone convergence it follows that

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^d} \mathbb{E} \left[ |u(t,x)|^{r'} \right] \, dx \leq C_r .
\]

**Step 4** (passage to the limit). Finally, we have to prove that we can pass to the limit in equation (15) and deduce that \( u \) satisfies property 3 of Definition 6. We have already proved that all terms converge to the corresponding ones except for the term \( \mathbb{E} \left[ Z \int_0^t \int b_n(s,x) \cdot \nabla u_n(s,x) \varphi(x) \, dx \, ds \right] \).

We do not want to integrate by parts, for otherwise we would have to assume something on \( \text{div} \, b \). Since \( b_n \to b \) in \( L^q_p = L^q([0,T]; L^p(\mathbb{R}^d)) \), it is sufficient to use a suitable weak convergence of \( \nabla u_n \) to \( \nabla u \). Precisely,

\[
\mathbb{E} \left[ Z \int_0^t \int b_n(s,x) \cdot \nabla u_n(s,x) \varphi(x) \, dx \, ds \right] - \mathbb{E} \left[ Z \int_0^t \int b(s,x) \cdot \nabla u(s,x) \varphi(x) \, dx \, ds \right] = I^{(1)}_n(t) + I^{(2)}_n(t)
\]
\[ I_n^{(1)}(t) = \mathbb{E} \left[ Z \int_0^t \int (b_n(s,x) - b(s,x)) \cdot \nabla u_n(s,x) \varphi(x) \, dx \, ds \right] \]

\[ I_n^{(2)}(t) = \mathbb{E} \left[ Z \int_0^t \int \varphi(x) b(s,x) \cdot (\nabla u_n(s,x) - \nabla u(s,x)) \, dx \, ds \right]. \]

We have to prove that both \( I_n^{(1)}(t) \) and \( I_n^{(2)}(t) \) converge to zero as \( n \to \infty \).

By Hölder inequality,

\[ I_n^{(1)}(t) \leq C \|b_n - b\|_{L^p([0,T];L^p(\mathbb{R}^d))} \mathbb{E} \left[ \|\nabla u_n\|_{L^{p'}([0,T];L^{p'}(\mathbb{R}^d))} \right] \]

where \( 1/p + 1/p' = 1 \) and \( 1/q + 1/q' = 1 \). Thus, from (17), \( I_n^{(1)}(t) \) converges to zero.

Let us treat \( I_n^{(2)}(t) \). Using the integrability properties shown above we have

\[ \mathbb{E} \left[ Z \int_0^t \int \varphi(x) b(s,x) \cdot (\nabla u_n(s,x) - \nabla u(s,x)) \, dx \, ds \right] = \int_0^t \mathbb{E} \left[ \int Z \varphi(x) b(s,x) \cdot (\nabla u_n(s,x) - \nabla u(s,x)) \, dx \right] \, ds. \]

The function

\[ h_n(s) := \mathbb{E} \left[ \int Z \varphi(x) b(s,x) \cdot (\nabla u_n(s,x) - \nabla u(s,x)) \, dx \right] \]

converges to zero as \( n \to \infty \) for almost every \( s \) and satisfies the assumptions of Vitali convergence theorem (we shall prove these two claims in Step 5 below). Hence \( I_n^{(2)}(t) \) converges to zero.

Now we may pass to the limit in equation (15) and get

\[ \mathbb{E} \left[ Z \int u(t,x) \varphi(x) \, dx \right] + \mathbb{E} \left[ Z \int_0^t \int b(s,x) \cdot \nabla u(s,x) \varphi(x) \, dx \, ds \right] \]

\[ = \int u_0(x) \varphi(x) \, dx + \sigma \sum_{i=1}^d \mathbb{E} \left[ Z \int_0^t \left( \int u(s,x) \partial_x^i \varphi(x) \, dx \right) \, dW^i_s \right] \]

\[ + \frac{\sigma^2}{2} \mathbb{E} \left[ Z \int_0^t \int u(s,x) \Delta \varphi(x) \, dx \, ds \right]. \]

The arbitrariness of \( Z \) implies property 3 of Definition 6.
Step 5 (auxiliary facts). We have to prove the two properties of $h_n(s)$ claimed in Step 4. Recall we may use Lemma 16 at each value of time. It gives us

$$\mathbb{E} \left[ \int_{\mathbb{R}^d} \partial_x u(s, x) \varphi(x) Z \, dx \right] = \lim_{n \to \infty} \mathbb{E} \left[ \int_{\mathbb{R}^d} \partial_x u_n(s, x) \varphi(x) Z \, dx \right]$$

for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ and bounded r.v. $Z$, at each $s \in [0, T]$. We have $b \in L^q([0, T]; L^p(\mathbb{R}^d))$, hence $b(s, \cdot) \in L^p(\mathbb{R}^d)$ for a.e. $s \in [0, T]$. The space $C_0^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. We may extend the convergence property (21) to all $\varphi \in L^p(\mathbb{R}^d)$ by means of the bounds (17) and (20). Hence $h_n(s) \to 0$ as $n \to \infty$, for a.e. $s \in [0, T]$.

Moreover, for every $\varepsilon > 0$ there is a constant $C_{Z, \varphi, \varepsilon}$ such that

$$\int_0^T h_n^{1+\varepsilon}(s) \, ds \leq C_{Z, \varphi, \varepsilon} \int_0^T \mathbb{E} \left[ \int_{B_R} |b(s, x)|^{1+\varepsilon} \left( |\nabla u_n(s, x)|^{1+\varepsilon} + |\nabla u(s, x)|^{1+\varepsilon} \right) \, dx \right] \, ds$$

$$\leq C_{Z, \varphi, \varepsilon} \|b\|_{L_q^p}^{1+\varepsilon} \left( \mathbb{E} \int_0^T \int_{B_R} |\nabla u_n(s, x)| \, dxds \right)^{\frac{1+\varepsilon}{1+\varepsilon}}$$

$$+ C_{Z, \varphi, \varepsilon} \|b\|_{L_q^p}^{1+\varepsilon} \left( \mathbb{E} \int_0^T \int_{B_R} |\nabla u(s, x)| \, dxds \right)^{\frac{1+\varepsilon}{1+\varepsilon}}$$

for a suitable $r$ depending on $\varepsilon$ (we have used Hölder inequality). The bounds (17) and (20) imply that $\int_0^T h_n^{1+\varepsilon}(s) \, ds$ is uniformly bounded. Hence Vitali theorem can be applied to prove that $I_n^{(2)}(t) = \int_0^t h_n(s) \, ds \to 0$ as $n \to \infty$. The proof is complete. 

4. Uniqueness of Weakly Differentiable Solutions

Theorem 11. Weak solutions of Definition 4 are unique.

Proof. Let $u^1$ be two weakly differentiable solutions of equation

$$\frac{\partial u}{\partial t} + b \cdot \nabla u + \sigma \nabla u \circ dW = 0, \quad u|_{t=0} = u_0. $$

Then $u := u^1 - u^2$ is a weakly differentiable solution of

$$\frac{\partial u}{\partial t} + b \cdot \nabla u + \sigma \nabla u \circ dW = 0, \quad u|_{t=0} = 0. $$

(22)
We want to prove that \( u \) is identically zero. We divide the proof in three steps.

**Step 1** (Equation for \( u^2 \)) The first step consists in proving that \( u^2 \) is also a weakly differentiable solution of

\[
\frac{\partial u^2}{\partial t} + b \cdot \nabla u^2 + \sigma \nabla u^2 \circ \frac{dW}{dt} = 0, \quad u|_{t=0} = 0 \quad (23)
\]

namely that

\[
\int u^2(t, x) \varphi(x) \, dx + \int_0^t \int b(s, x) \cdot \nabla u^2(s, x) \varphi(x) \, dx \, ds
\]

\[
= \sigma \sum_{i=1}^d \int_0^t \left( \int u^2(s, x) \partial_{x_i} \varphi(x) \, dx \right) \, dW_s^i
\]

\[
+ \frac{\sigma^2}{2} \int_0^t \int u^2(s, x) \Delta \varphi(x) \, dx \, ds
\]

for any \( \varphi \in C_0^\infty(\mathbb{R}^d) \). Let \( \theta^\varepsilon \) be a sequence of standard mollifiers. From the definition of weak solution, using \( \varphi^\varepsilon_y(x) = \theta^\varepsilon(y - x) \), we have

\[
u^\varepsilon(t, y) + \int_0^t b(s, y) \cdot \nabla u^\varepsilon(s, y) \, ds
\]

\[
+ \sigma \sum_{i=1}^d \int_0^t \partial_{y_i} u(s, y) \circ dW_s^i = \int_0^t R^\varepsilon(s, y) \, ds,
\]

\[
R^\varepsilon(s, y) = \left[ \int \left( b(s, y) - b(s, x) \right) \nabla u(s, x) \theta^\varepsilon(x - y) \, dx \right].
\]

The function \( u^\varepsilon \) is smooth in space. For any fixed \( y \), by Itô formula we have

\[
du^2(t, y) = 2u^\varepsilon(t, y) \, du^\varepsilon(t, y)
\]

\[
= -2u^\varepsilon(t, y) b(t, y) \nabla u^\varepsilon(t, y) \, dt - 2\sigma u^\varepsilon(t, y) \sum_{i=1}^d \partial_{y_i} u^\varepsilon(s, y) \circ dW_s^i
\]

\[
+ 2u^\varepsilon(t, y) R^\varepsilon(t, y) \, dt
\]

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which, rewritten in the weak formulation using a generic test function $\varphi$, reads

\[
\int u^2_{\varepsilon}(t, y)\varphi(y)\,dy + \int_0^t \int b(s, y)\nabla u^2_{\varepsilon}(s, y)\varphi(y)\,dy\,ds
+ \sigma \sum_{i=1}^d \int_0^t \left( \int \partial_i u^2_{\varepsilon}(s, y)\varphi(y)\,dy \right) \circ dW^i_s = \int_0^t 2u_{\varepsilon}(s, y)R_{\varepsilon}(s, y)\varphi(y)\,dy\,ds.
\]

We want now to pass to the limit for $\varepsilon \to 0$ in the different terms. Since for every $t$, $u_{\varepsilon} \to u$ uniformly on compact sets, by dominated convergence the first term tends to

\[
\int u^2(t, y)\varphi(y)\,dy.
\]

For the following terms, we consider $s$ fixed. Using Hölder inequality and the convergence of $\|\nabla u_{\varepsilon}\|_{L^p} \to \|\nabla u\|_{L^p}$ on compact sets (recall that $\varphi$ is of compact support) for any $p \geq 1$, we have

\[
\int b(s, y)\varphi(y) \left( \nabla u^2_{\varepsilon}(s, y) - \nabla u^2(s, y) \right)\,dy
\leq \|b(s, y)\varphi(y)\|_{L^{r'}} \|\nabla u^2_{\varepsilon}(s, y) - \nabla u^2(s, y)\|_{L^r}
\leq C_\varepsilon\|b\|_{L^{r'}} \|\nabla u^2_{\varepsilon}(s, y) - \nabla u^2(s, y)\|_{L^r} \to 0,
\]

which is enough to obtain the convergence of the second term. In the same way one obtains also the convergence of the third term. As for the term containing the commutator $R_{\varepsilon}$, we can use again Hölder inequality, the uniform convergence of $u_{\varepsilon}$, the equi–boundedness of $\nabla u_{\varepsilon}$ in $L^p$ for every $p \geq 1$, and the continuity in mean (for a.e. $y$) of the function $b \in L^p(\mathbb{R}^d)$. This proves (23).

**Step 2** (equation for $v^2$) We have that $u$ is a.s. continuous in space and time (and therefore locally bounded) and by definition of weak solution $\nabla u \in L^r([0, T] \times \mathbb{R}^d)$ for every $r \geq 1$ a.s.. It follows that $f(s) = \int \nabla u^2(s, y)\varphi(y)\,dy$ is still a.s. a function in $L^r(0, T)$. This means that, writing (23) in Itô form, the stochastic integral is a martingale and

\[
\int \mathbb{E} [u^2(t, x)] \varphi(x)\,dx + \int_0^t \int b(s, x) \cdot \nabla \mathbb{E} [u^2(s, x)] \varphi(x)\,dx\,ds
= \frac{\sigma^2}{2} \int_0^t \int \mathbb{E} [u^2(s, x)] \Delta \varphi(x)\,dx\,ds.
\]
Hence \( v(t, x) = \mathbb{E} [u^2 (t, x)] \) satisfies
\[
\int v(t, x) \varphi(x) \, dx + \int_0^t \int b(s, x) \cdot \nabla v(t, x) \varphi(x) \, dx \, ds = \frac{\sigma^2}{2} \int_0^t \int v(t, x) \Delta \varphi(x) \, dx \, ds
\]
and is fairly regular: \( v \in C^0 (\mathbb{R}_+; W^{1, r}(\mathbb{R}^d)) \) for \( r \geq 1 \). This follows by Hölder inequality because
\[
\int |\nabla v(t, x)|^r \, dx = \int |\mathbb{E}[u \nabla u]|^r \, dx \leq \left( \int |\mathbb{E}[u|^2| \mathbb{E}[|\nabla u|^2]| \right)^{r/2} \, dx \\
\leq \left( \int \mathbb{E}[|u|^{2r}]^2 \, dx \right)^{1/2} \left( \int \mathbb{E}[|\nabla u|^{2r}]^2 \, dx \right)^{1/2} \leq C,
\]
uniformly in \( t \) (similar computations provide the same result for the function \( v \)).

Thanks to its global integrability properties, using approximating functions as in the first step, one can prove that \( v \) solves
\[
\int v^2 (t, x) \, dx + \sigma^2 \int_0^t \int |\nabla v (t, x)|^2 \, dx \, ds = -2 \int_0^t \int b(s, x) \cdot \nabla v(t, x) v(t, x) \, dx \, ds . \tag{24}
\]

**Step 3** (final estimates) We want to find suitable bounds on the last term of (24) allowing to apply Gronwall inequality. This will complete the proof. For every \( t \in [0, T] \), we have
\[
\left| \int v b \cdot \nabla v \, dx \right| \leq \left( \int |\nabla v|^2 \, dx \right)^{1/2} \left( \int v^2 |b|^2 \, dx \right)^{1/2} ;
\]
\[
\int v^2 |b|^2 \, dx \leq \left( \int v^{2r} \, dx \right)^{1/r} \left( \int |b|^p \, dx \right)^{2/p} ,
\]
where \( 1/r + 2/p = 1 \) namely \( 1/r = 1 - 2/p = (p - 2)/p \):
\[
r = \frac{p}{p - 2} .
\]
One has the interpolation inequality
\[
\left( \int v^\alpha \, dx \right)^{1/\alpha} \leq \left( \int v^2 \, dx \right)^{1-s} \left( \int |\nabla v|^2 \, dx \right)^s, \quad s = \frac{\alpha - 2}{2\alpha} d.
\]

The idea of the result comes from: \( W^{s,2} \subset L^\alpha \) for \( 1/\alpha = 1/2 - s/d \), namely \( \frac{s}{d} = \frac{1}{2} - \frac{1}{\alpha} = \frac{\alpha - 2}{2\alpha} \), \( s = \frac{\alpha - 2}{2\alpha} d \); and then
\[
\left( \int v^\alpha \, dx \right)^{1/\alpha} \leq \|v\|_{W^{s,2}} \leq \|v\|_{L^2}^{1-s} \|v\|^s_{W^{1,2}}.
\]

Let us put everything together:
\[
\left( \int v^{2r} \, dx \right)^{1/r} = \left( \int v^\alpha \, dx \right)^{2/\alpha} \leq \left( \int v^2 \, dx \right)^{1-s} \left( \int |\nabla v|^2 \, dx \right)^s
\]
\[r = \frac{p}{p - 2}, \quad \alpha = 2r, \quad s = \frac{\alpha - 2}{2\alpha} d\]

namely
\[s = \frac{2r - 2}{4r} d = \frac{2\frac{p}{p - 2} - 2}{\frac{4p}{p - 2}} d = \frac{d}{p}.
\]

Thus we have proved:
\[
\int v^2 |b|^2 \, dx \leq \left( \int v^2 \, dx \right)^{1-\frac{d}{2p}} \left( \int |\nabla v|^2 \, dx \right)^{\frac{d}{2p}} \left( \int |b|^p \, dx \right)^{2/p}
\]
and we can bound the last term in (24)
\[
\left| \int v b \cdot \nabla v \, dx \right| \leq \left( \int |\nabla v|^2 \, dx \right)^{\frac{p + d}{2p}} \left( \int v^2 \, dx \right)^{\frac{p - d}{2p}} \left( \int |b|^p \, dx \right)^{1/p}.
\]

Recall that \( ab \leq \frac{a^2}{\epsilon} + \frac{b^r}{\eta}, \frac{1}{\epsilon} + \frac{1}{\eta} = 1 \). Then, with \( s = \frac{2p}{p + d}, r = \frac{2p}{p - d} \) we have
\[
\left| \int v b \cdot \nabla v \, dx \right| \leq \frac{\sigma^2}{2} \left( \int |\nabla v|^2 \, dx \right) + C \left( \int v^2 \, dx \right) \left( \int |b|^p \, dx \right)^{\frac{2}{p - d}}.
\]

Therefore
\[
\int v^2 (t, x) \, dx \leq C \int_0^t \left( \int v^2 \, dx \right) \left( \int |b|^p \, dx \right)^{\frac{2}{p - d}} \, ds
\]
hence we may apply Gronwall lemma and deduce \( \int v^2(t, x) \, dx = 0 \) if
\[
\int_0^T \left( \int |b|^p \, dx \right)^{\frac{2}{p-d}} \, ds < \infty.
\]
We know that
\[
\int_0^T \left( \int |b|^p \, dx \right)^{\frac{2}{p}} \, ds < \infty
\]
for certain \( p, q \geq 2 \) such that \( \frac{4}{p} + \frac{2}{q} < 1 \). Then
\[
\frac{2}{p-d} < \frac{q}{p}
\]
because \( 2p < qp -qd, \frac{2}{q} < 1 - \frac{4}{p} \). The proof is complete. \( \blacksquare \)

5. Appendix: Technical Lemmas

For completeness, we collect here some modifications of known results used in Section 2. We will use the notation introduced there.

**Lemma 12.** Let \( U_n \) be the solution of the PDE (4) for \( f = b = b^n \), as defined in Section 2. Then

i) \( U^n(t, x) \) and \( \nabla U^n(t, x) \) converge pointwise in \((t, x)\) to \( U(t, x) \) and \( \nabla U(t, x) \) respectively, and the convergence is uniform on compact sets;

ii) there exists a \( \lambda \) for which \( \sup_{t,x} |\nabla U^n(t, x)| \leq 1/2; \)

iii) \( \|\nabla^2 U^n(t, x)\|_{L^q_p(T)} \leq C \).

**Proof.** The result of the second point is proved in [13, Lemma 3.4] for a fixed \( n \), but inspecting the proof we see that all the bounds obtained depend on \( \|b\|_{L^q_p} \), but never on \( b \) itself. Since \( \|b^n\|_{L^q_p} \to \|b\|_{L^q_p} \), the uniformity in \( n \)
follows.

To prove the other two points, set \( V^n := U^n - U \); then
\[
\partial_t V^n + \frac{1}{2} \Delta V^n + b \cdot \nabla V^n - \lambda V^n = -(b^n - b) \cdot (Id + \nabla U^n), \quad V^n(T, x) = 0.
\]
From the bound (5) on the solution provided by Theorem 2 we obtain
\[
\|V^n\|_{H^q_{2,p}} \leq N\|b^n - b\|_{L^q_p} \to 0.
\]
It follows that $U^n \to U$ in $H^2_{2,p}$, which proves the last point. Since by Lemma 10.2 $U, U^n, \nabla U$ and $\nabla U^n$ are all Hölder continuous functions, there exists a subsequence (that we still call $U^n$) s.t. $U^n \to U$ and $\nabla U^n \to \nabla U$ for every $(t, x)$ and uniformly on compacts.

**Remark 13.** The following results hold uniformly in $n$ because, as remarked in the previous proof, all the bounds obtained depend on the norm of $b$.

1. From [13, Lemma 3.5] we have
   \[ \sup_n \sup_{t \in [0,T]} \left| \nabla (\gamma^n_t)^{-1}(\cdot) \right|_{C(\mathbb{R}^d)} \leq 2 ; \]  \hspace{1cm} (25)

2. from the uniform boundedness of the coefficients ($U^n$ and $\nabla U^n$) of the SDE (3), we get
   \[ \sup_{t \in [0,T]} \mathbb{E} \left[ \left| \psi^n_t(x) \right|^a \right] \leq C \left( 1 + |x|^a \right) . \]  \hspace{1cm} (26)

**Lemma 14.** For every $n$, both the process $A^n$ defined by (9) and the one defined by (14) are continuous, adapted, nondecreasing, with $A^n_0 = 0$, $\mathbb{E}[A^n_T] \leq C$ and for every $k \in \mathbb{R}$, $\mathbb{E}[e^{kA^n_T}] \leq C$. The constant $C$ can be chosen independently of $n$.

**Proof.** For the process defined by (9) the proof follows the same steps of the proof of [12, Lemma 7]. We only remark that the function $U$ is the solution of a different PDE, but it has the same properties in terms of regularity. Moreover, the flows $\phi$ and $\phi^n$ solve two SDEs with different drifts $b$ and $b^n$, which means that in the proof one has to use twice the result of [12, Corollary 13], once for every drift.

For the process defined by (14), the result is already contained in [12, Corollary 13].

**Lemma 15.** Let $f^n$ be a sequence of vector fields belonging to $L^q_p$, convergent to $f \in L^q_p$. Then, there exists $\varepsilon > 1$ s.t.
   \[ \mathbb{E} \left[ \int_0^T |f^n(s, \phi^n_s)|^{2\varepsilon} \, ds \right] \leq C < \infty . \]  \hspace{1cm} (27)

**Proof.** To prove the result for a fixed $n$ one can use [12, Corollary 13] and follow the proof of [12, Lemma 8 and Corollary 9], which still works due to the strict inequality in the conditions imposed on $p, q$. Then, since all the bounds only depend on the norm of $f$ but never on the function itself, one obtains that (27) is uniform in $n$. 

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6. Appendix: Sobolev Regularity of Random Fields

Let \( r \geq 1 \) be given. We recall that \( f \in W^{1,r}_{\text{loc}}(\mathbb{R}^d) \) if \( f \in L^r_{\text{loc}}(\mathbb{R}^d) \) and there exist \( g_i \in L^r_{\text{loc}}(\mathbb{R}^d), \ i = 1, ..., d, \) such that

\[
\int_{\mathbb{R}^d} f(x) \partial_{x_i} \varphi(x) \, dx = - \int_{\mathbb{R}^d} g_i(x) \varphi(x) \, dx
\]

for all \( \varphi \in C_0^\infty(\mathbb{R}^d) \). When this happens, we set \( \partial_{x_i} f(x) = g_i(x) \). From the definition and easy arguments one has the following criterion: if \( f \in L^r_{\text{loc}}(\mathbb{R}^d) \) and there exist a sequence \( \{f_n\} \subset W^{1,r}_{\text{loc}}(\mathbb{R}^d) \) such that \( f_n \rightarrow f \) in \( L^1_{\text{loc}}(\mathbb{R}^d) \) (or even in distributions) and for all \( R > 0 \) one has a constant \( C_R > 0 \) such that \( \int_{B_R} |\nabla f_n(x)|^r \, dx \leq C_R \) uniformly in \( n \), then \( f \in W^{1,r}_{\text{loc}}(\mathbb{R}^d) \). This criterion will not be used below; it is only stated for comparison with the next result.

Let now \( F : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \) be a random field. When we use below this name we always assume it is jointly measurable.

**Lemma 16.** Assume that \( F(\omega, \cdot) \in L^r_{\text{loc}}(\mathbb{R}^d) \) for \( P \)-a.e. \( \omega \) and there exist a sequence \( \{F_n\}_{n \in \mathbb{N}} \) of random fields such that

1. \( F_n(\omega, \cdot) \rightarrow F(\omega, \cdot) \) in distributions in probability, namely

\[
P - \lim_{n \to \infty} \int_{\mathbb{R}^d} F_n(\omega, x) \varphi(x) \, dx = \int_{\mathbb{R}^d} F(\omega, x) \varphi(x) \, dx
\]

for every \( \varphi \in C_0^\infty(\mathbb{R}^d) \);

2. \( F_n(\omega, \cdot) \in W^{1,r}_{\text{loc}}(\mathbb{R}^d) \) for \( P \)-a.e. \( \omega \) and for every \( R > 0 \) there exists a constant \( C_R > 0 \) such that

\[
\mathbb{E} \left[ \int_{B_R} |\nabla F_n(x)|^r \, dx \right] \leq C_R
\]

uniformly in \( n \).

Then \( F(\omega, \cdot) \in W^{1,r}_{\text{loc}}(\mathbb{R}^d) \) for \( P \)-a.e. \( \omega \),

\[
\mathbb{E} \left[ \int_{\mathbb{R}^d} \partial_{x_i} F(\cdot, x) \varphi(x) \, Z \, dx \right] = \lim_{n \to \infty} \mathbb{E} \left[ \int_{\mathbb{R}^d} \partial_{x_i} F_n(\cdot, x) \varphi(x) \, Z \, dx \right] \quad (28)
\]
for all $\varphi \in C_0^\infty (\mathbb{R}^d)$ and bounded r.v. $Z$,

$$P - \lim_{n \to \infty} \int_{\mathbb{R}^d} \partial_{x_i} F_n(\omega, x) \varphi(x) \, dx = - \int_{\mathbb{R}^d} F(\omega, x) \partial_{x_i} \varphi(x) \, dx \quad (29)$$

for all $\varphi \in C_0^\infty (\mathbb{R}^d)$, and for every $R > 0$

$$\mathbb{E} \left[ \int_{B_R} |\nabla F(x)|^r \, dx \right] \leq \limsup_{n \to \infty} \mathbb{E} \left[ \int_{B_R} |\nabla F_n(x)|^r \, dx \right]. \quad (30)$$

**Proof.** Given $R > 0$, there is a subsequence $\{n_k\}$ and a vector valued random field $G$ such that $\nabla F_{n_k}$ converges weakly to $G$ in $L^r(\Omega \times \mathbb{R}^d)$, as $k \to \infty$. Taking $R \in \mathbb{N}$, we may apply a diagonal procedure and find a single subsequence $\{n_k\}$ and vector valued random fields $G_R$, $R \in \mathbb{N}$, such that $\nabla F_{n_k}$ converges weakly to $G_R$ in $L^r(\Omega \times B_R)$, as $k \to \infty$, for each $R \in \mathbb{N}$. Using suitable test functions, one can see that $G_R = G_R'$ on $\Omega \times B_R$ if $R' > R$. Hence we have found a single vector valued random field $G$, such that $\nabla F_n$ converges weakly to $G$ in $L^r(\Omega \times \mathbb{R}^d)$, as $n \to \infty$. For this reason, to simplify notations, we omit the notation of the subsequence.

For each $\varphi \in C_0^\infty (\mathbb{R}^d)$, by assumptions 1 and 2

$$\int_{\mathbb{R}^d} F(\omega, x) \partial_{x_i} \varphi(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} F_n(\omega, x) \partial_{x_i} \varphi(x) \, dx$$

$$= - \lim_{n \to \infty} \int_{\mathbb{R}^d} \partial_{x_i} F_n(\omega, x) \varphi(x) \, dx,$$

the limits being understood in probability. For each bounded r.v. $Z$, this implies that

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \partial_{x_i} F_n(\omega, x) \varphi(x) \, Z(\omega) \, dx = - \int_{\mathbb{R}^d} F(\omega, x) \partial_{x_i} \varphi(x) \, Z(\omega) \, dx$$

in probability. This limit holds also in $L^1(\Omega)$ by Vitali convergence criterion because, by Hölder inequality,

$$\mathbb{E} \left[ \left| \int_{\mathbb{R}^d} \partial_{x_i} F_n(\omega, x) \varphi(x) \, Z(\omega) \, dx \right|^p \right]$$

$$\leq C_{R,p} \|\varphi\|_{L^\infty} \|Z\|_{L^\infty} \mathbb{E} \left[ \int_{B_R} |\partial_{x_i} F_n(\omega, x)|^p \, dx \right]$$

$$\leq C_{R,p} \|\varphi\|_{L^\infty} \|Z\|_{L^\infty} C_R$$
uniformly in $n$, for some $p > 1$, and with $R$ such that $B_R$ contains the support of $\varphi$.

From the weak convergence above, we also get that

$$\lim_{n \to \infty} \mathbb{E} \left[ \int_{\mathbb{R}^d} \partial_{x_i} F_n (\cdot, x) \varphi (x) \, Z \, dx \right] = \mathbb{E} \left[ \int_{\mathbb{R}^d} G_i (\cdot, x) \varphi (x) \, Z \, dx \right].$$

Hence

$$\mathbb{E} \left[ \int_{\mathbb{R}^d} G_i (\cdot, x) \varphi (x) \, Z \, dx \right] = - \mathbb{E} \left[ \int_{\mathbb{R}^d} F (\cdot, x) \partial_{x_i} \varphi (x) \, Z \, dx \right].$$

By the arbitrariness of $Z$ this gives us

$$\int_{\mathbb{R}^d} F (\omega, x) \partial_{x_i} \varphi (x) \, dx = - \int_{\mathbb{R}^d} G_i (\omega, x) \varphi (x) \, dx \quad (31)$$

for $P$-a.e. $\omega$. This is the identification of $G$ mentioned above, which implies the weak convergence of the full sequence $\nabla F_n$.

Identity (31) holds $P$-a.s. for every $\varphi \in C_0^\infty (\mathbb{R}^d)$ a priori given. Thus it holds $P$-a.s., uniformly on a dense countable set $D$ of test functions $\varphi$, dense for instance in $W^{1,r'}_{\text{loc}} (\mathbb{R}^d)$, $\frac{1}{r} + \frac{1}{r'} = 1$. Using the integrability properties of $F (\omega, \cdot)$ and $G_i (\omega, \cdot)$ we may extend identity (31) to all $\varphi \in W^{1,r'}_{\text{loc}} (\mathbb{R}^d)$ and thus all $\varphi \in C_0^\infty (\mathbb{R}^d)$, uniformly with respect to the good set of $\omega$ for which it holds on $D$.

Thus, from identity (31) in the stronger form just explained, we deduce - by definition - that $F (\omega, \cdot) \in W^{1,r'}_{\text{loc}} (\mathbb{R}^d)$ for $P$-a.e. $\omega$. And $\nabla F (\omega, x) = G (\omega, x)$. We immediately have (28) and (29).

We have shown that, for every function $\xi (\omega, x)$ of the form

$$\xi(\omega, x) = \sum_{k=1}^m \varphi_k(x)Z_k(\omega),$$

with $\varphi$ and $Z$ as above, we have

$$\left| \mathbb{E} \left[ \int_{B_R} \nabla F (\cdot, x) \xi (\cdot, x) \, dx \right] \right| \leq \lim_{n \to \infty} \mathbb{E} \left[ \int_{B_R} \nabla F_n (\cdot, x) \xi (\cdot, x) \, dx \right]$$

$$\leq \mathbb{E} \left[ \int_{B_R} |\xi (\cdot, x)|^{r'} \, dx \right]^{1/r'} \limsup_{n \to \infty} \mathbb{E} \left[ \int_{B_R} |\nabla F_n (\cdot, x)|^r \, dx \right]^{1/r}.$$
(in the last passage we have used Hölder inequality). The set of functions $\xi$ introduced is dense in $L^{r'}(\Omega \times B_R)$, so that

$$
\mathbb{E} \left[ \int_{B_R} |\nabla F(\cdot, x)|^r \, dx \right]^{1/r} = \|\nabla F\|_{L^{r'}(\Omega \times B_R)}
$$

\[
\leq \sup_{\|\xi\|_{L^{r'}} \leq 1} \mathbb{E} \left[ \int_{B_R} \nabla F(\cdot, x) \xi(\cdot, x) \, dx \right]
\]

\[
\leq \limsup_{n \to \infty} \mathbb{E} \left[ \int_{B_R} |\nabla F_n(\cdot, x)|^r \, dx \right]^{1/r}.
\]

This completes the proof. $\blacksquare$
References

[1] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, Invent. Math. 158 (2004) 227-260.

[2] S. Attanasio, Stochastic flows of diffeomorphisms for one-dimensional SDE with discontinuous drift, Electron. Commun. Probab. 15 (2010) 213–226.

[3] A. de Bouard, A. Debussche, On the effect of the noise on the solutions of supercritical Schrödinger equation, Probab. Theory Related Fields 123 (2002) 76–96.

[4] A. de Bouard, A. Debussche, Finite time blow-up in the additive supercritical stochastic nonlinear Schrödinger equation: the real noise case, Contemporary Math. 301 (2002) 183–194.

[5] A. de Bouard, A. Debussche, Blow-up for the stochastic nonlinear Schrödinger equation with multiplicative noise, Annals of Probab. 33 (3) (2005) 1078–1110.

[6] A. de Bouard, A. Debussche, The nonlinear Schrödinger equation with white noise dispersion, J. Funct. Anal. 259 (5) (2010) 1300–1321.

[7] A. Debussche, L. Di Menza, Numerical simulation of focusing stochastic nonlinear Schrödinger equations, Physica D 162 (2002) 131–154.

[8] A. Debussche, L. Di Menza, Numerical resolution of stochastic focusing NLS equations, Appl. Math. Letters 15 (6) (2002) 661–669.

[9] A. Debussche, Y. Tsutsumi, 1D quintic nonlinear Schrödinger equation with white noise dispersion, J. Math. Pures Appl. 96 (4) (2011) 363–376.

[10] F. Delarue, F. Flandoli, D. Vincenzi, Noise prevents collaps of Vlasov-Poisson point charges, preprint.

[11] R. J. DiPerna, P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989) 511-547.

[12] E. Fedrizzi, F. Flandoli, Pathwise uniqueness and continuous dependence for SDEs with nonregular drift, Stochastics 83 (3) (2011) 241–257.
[13] E. Fedrizzi, F. Flandoli, Hölder Flow and Differentiability for SDEs with Nonregular Drift, to appear in Stochastic Analysis and Applications (2012).

[14] F. Flandoli, Random Perturbation of PDEs and Fluid Dynamic Models, Saint Flour summer school lectures 2010, Lecture Notes in Mathematics n. 2015, Springer, Berlin (2011).

[15] F. Flandoli, M. Gubinelli and E. Priola, Well-posedness of the transport equation by stochastic perturbation, Invent. Math. 180 (1) (2010), 1–53.

[16] F. Flandoli, M. Gubinelli, E. Priola, Full well-posedness of point vortex dynamics corresponding to stochastic 2D Euler equations, Stoch. Proc. Appl., 121 (7) (2011) 1445–1463.

[17] N.V. Krylov and M. Röckner, Strong solutions to stochastic equations with singular time dependent drift, Probab. Theory Relat. Fields 131 (2005) 154–196.

[18] H. Kunita, Stochastic flows and stochastic differential equations, Cambridge studies in advanced mathematics, Cambridge university press (1990).

[19] T. Meyer-Brandis and F. Proske, Construction of strong solutions of SDE’s via Malliavin calculus, J. Funct. Anal. 258 (11) (2010) 3922–3953.

[20] S.E.A. Mohammed, T.K. Nilssen, F.N. Proske, Sobolev Differentiable Stochastic Flows of SDE’s with Measurable Drift and Applications, preprint, arXiv:1204.3867.

[21] X. Zhang, Stochastic Homeomorphism Flows of SDEs with Singular Drifts and Sobolev Diffusion Coefficients, Electronic Journal of Probability 16 (38) (2011) 1096–1116.