Analysis on Laakso graphs with application to the structure of transportation cost spaces

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Abstract
This article is a continuation of our article in Dilworth et al. (Can J Math 72:774–804, 2020). We construct orthogonal bases of the cycle and cut spaces of the Laakso graph $L_n$. They are used to analyze projections from the edge space onto the cycle space and to obtain reasonably sharp estimates of the projection constant of $\text{Lip}_0(L_n)$, the space of Lipschitz functions on $L_n$. We deduce that the Banach–Mazur distance from $\text{TC}(L_n)$, the transportation cost space of $L_n$, to $\ell^1_N$ of the same dimension is at least $(3n - 5)/8$, which is the analogue of a result from [op. cit.] for the diamond graph $D_n$. We calculate the exact projection constants of $\text{Lip}_0(D_{n,k})$, where $D_{n,k}$ is the diamond graph of branching $k$. We also provide simple examples of finite metric spaces, transportation cost spaces on which contain $\ell^3_\infty$ and $\ell^4_\infty$ isometrically.

Keywords Analysis on Laakso graphs · Arens–Eells space · Diamond graphs · Earth mover distance · Kantorovich–Rubinstein distance · Laakso graphs · Lipschitz-free space · Transportation cost · Wasserstein distance

Mathematics Subject Classification Primary 46B03; Secondary 30L05 · 42C10 · 46B07 · 46B85
1 Introduction

1.1 Definitions and background

Let \((X, d)\) be a metric space. Consider a real-valued finitely supported function \(f\) on \(X\) with a zero sum, that is, \(\sum_{v \in \text{supp} f} f(v) = 0\). A natural and important interpretation of such a function, is considering it as a transportation problem: one needs to transport certain product from locations where \(f(v) > 0\) to locations where \(f(v) < 0\).

One can easily see that \(f\) can be represented as

\[
f = a_1(1_{x_1} - 1_{y_1}) + a_2(1_{x_2} - 1_{y_2}) + \cdots + a_n(1_{x_n} - 1_{y_n}),
\]

where \(a_i \geq 0, x_i, y_i \in X\), and \(1_u(x)\) for \(u \in X\) is the indicator function of \(u\), defined by

\[
1_u(x) = \begin{cases} 
1 & \text{if } x = u, \\
0 & \text{if } x \neq u.
\end{cases}
\]

We call each such representation a transportation plan for \(f\), and it can be interpreted as a plan of moving \(a_i\) units of the product from \(x_i\) to \(y_i\). The cost of the transportation plan (1) is defined as \(\sum_{i=1}^n a_i d(x_i, y_i)\).

**Remark 1** It is worth mentioning that in our discussion transportation plans are allowed to be fake plans, in the sense that there is no product in \(x_i\) in order to make the delivery to \(y_i\). To see what we mean consider a metric space containing three distinct points \(x, y, z\). Then \((1_x - 1_y) + (1_y - 1_z) + (1_z - 1_x)\) is a transportation plan for function 0 (null transportation problem, nothing is needed or available), although there is no product in \(x\) to be delivered to \(y\). However, it is easy to show that the defined below optimal transportation plans can be implemented.

We denote the real vector space of all transportation problems by \(\text{TP}(X)\). We introduce the transportation cost norm (or just transportation cost) \(\|f\|_{\text{TC}}\) of a transportation problem \(f\) as the infimum of costs of transportation plans satisfying (1). Using the triangle inequality and compactness it is easy to show that the infimum of costs of transportation plans for \(f\) is attained. A transportation plan for \(f\) whose cost is equal to \(\|f\|_{\text{TC}}\) is called an optimal transportation plan. The completion of the normed space \((\text{TP}(X), \|\cdot\|_{\text{TC}})\) is called a transportation cost space and is denoted by \(\text{TC}(X)\).

We use the standard terminology of Banach space theory [4], graph theory [7], and the theory of metric embeddings [25].

Transportation cost spaces are of interest in many areas and are studied under many different names (we list some of them in the alphabetical order: Arens–Eells space, earth mover distance, Kantorovich–Rubinstein distance, Lipschitz-free space, Wasserstein distance). We prefer to use the term transportation cost space since it makes the subject of this work instantly clear to a wide circle of readers and it also reflects the historical approach leading to these notions (see [15,16]). Interested readers
can find a review of the main definitions, notions, facts, terminology and historical notes pertinent to the subject in [22, Section 1.6].

By a pointed metric space we mean a metric space \((X, d_X)\) with a base point, denoted by \(O\). For a pointed metric space \(X\) with a base point at \(O\) by \(\text{Lip}_0(X)\) we denote the space of all Lipschitz functions \(f : X \to \mathbb{R}\) satisfying \(f(O) = 0\). It is not difficult to check that \(\text{Lip}_0(X)\) is a Banach space with respect to the norm \(\|f\| = \text{Lip}(f)\) (\(\text{Lip}(f)\) is the Lipschitz constant of \(f\)). As is well known \(\text{TC}(X)^* = \text{Lip}_0(X)\) (see e.g. [25, Section 10.2]).

One of the main goals of this paper is to study the geometry of the spaces \(\text{TC}(X)\). We are interested mostly in the case where \(X\) is finite. We would like to mention that for finite \(X\), the space \(\text{TC}(X)\) is an \(\ell_1\)-like space in the sense that it has three qualities which make it close to \(\ell_{|X|^{-1}}\).

1. It has a 1-complemented subspace isometric to \(\ell_{\lceil |X|/2 \rceil}\), see [17] (a weaker version was proved earlier in [8]).
2. It admits a linear embedding into \(L_1[0, 1]\) with distortion \(\leq C \ln |X|\), see [5,9,13]. Although this result is known since 2003, it seems that the only source where one can find its published proof is [3, Theorem 15].
3. It is a quotient of \(\ell^d_1\) with \(d \leq |X|^2\), see [23]. Another proof and a more precise statement can be found in Sect. 7.

However, \(\text{TC}(X)\) is isometric to \(\ell_{|X|^{-1}}\) if and only if \(X\) is a weighted tree. This result can be derived from the general result of [6]. Apparently the finite case of this result can be considered as folklore, for convenience of the readers we give a direct proof of the “only if” part (for finite case) in Sect. 7, the “if” part can be found in [8, Proposition 2.1].

One of the important problems about transportation cost spaces is the following [8, Problem 2.6]:

**Problem 2** It would be very interesting to find a condition on a finite metric space \(M\) which is equivalent to the condition that the space \(\text{TC}(M)\) is Banach–Mazur close to \(\ell^n_1\) of the corresponding dimension. It is not clear whether it is feasible to find such a condition.

In [8] we investigated this problem for large recursive families of graphs which include well-known families of diamond and Laakso graphs.

The main goal of this paper is further development of analysis in the space of functions on diamond and Laakso graphs in order to sharpen results of [8]. Let us remind the definitions of these families of graphs.

**Definition 3** (Diamond graphs) Diamond graphs \(\{D_n\}_{n=0}^\infty\) are defined recursively: The diamond graph of level 0 has two vertices joined by an edge of length 1 and is denoted by \(D_0\). The diamond graph \(D_n\) is obtained from \(D_{n-1}\) in the following way. Given an edge \(uv \in E(D_{n-1})\), it is replaced by a quadrilateral \(u, a, v, b\), with edges \(ua, av, vb, bu\). (See Fig. 1.)

Apparently Definition 3 was first introduced in [12].

Let us count some parameters associated with the graphs \(D_n\). Denote by \(V(D_n)\) and \(E(D_n)\) the vertex set and edge set of \(D_n\), respectively. Note that:
Definition 4 (Multibranche diamonds) For any integer $k \geq 2$, we define $D_{0,k}$ to be the graph consisting of two vertices joined by one edge. For any $n \in \mathbb{N}$, if the graph $D_{n-1,k}$ is already defined, the graph $D_{n,k}$ is defined as the graph obtained from $D_{n-1,k}$ by replacing each edge $uv$ in $D_{n-1,k}$ by a set of $k$ independent paths of length 2 joining $u$ and $v$. We endow $D_{n,k}$ with the shortest path distance. We call $\{D_{n,k}\}_{n=0}^{\infty}$ diamond graphs of branching $k$, or diamonds of branching $k$.

Definition 4 was introduced in [20]. Note that:

(a) $|E(D_{n,k})| = (2k)^n$.

(b) $|V(D_{n+1,k})| = |V(D_n)| + 2|E(D_n)|$.

Hence $|V(D_n)| = 2(1 + \sum_{i=0}^{n-1} 4^i)$.

Definition 5 (Laakso graphs) The Laakso graph $L_0$ has two vertices joined by an edge of length 1 and is denoted $L_0$. The Laakso graph $L_n$ is obtained from $L_{n-1}$ according to the following procedure. Each edge $uv \in E(L_{n-1})$ is replaced by the graph $L_1$ exhibited in Fig. 2, the vertices $u$ and $v$ are identified with the vertices of degree 1 of $L_1$.

Definition 5 was introduced in [19], where an idea of Laakso [18] was used. Note that:
Fig. 2 Laakso graph $\mathcal{L}_1$

(a) $|E(\mathcal{L}_n)| = 6^n$.
(b) $|V(\mathcal{L}_{n+1})| = |V(\mathcal{L}_n)| + 4|E(\mathcal{L}_n)|$.

Hence $|V(\mathcal{L}_n)| = 2 + 4 \sum_{i=0}^{n-1} 6^i$.

Diamond and Laakso graphs play important roles in Metric Geometry as examples/counterexamples to many natural questions. One of the reasons for interest in the families of graphs introduced in Definitions 3-5 is that their bilipschitz embeddability characterizes non-superreflexive Banach spaces [14,24,26]. In [21] it was shown that Laakso graphs are incomparable with diamond graphs in the following sense: elements of none of these families admit bilipschitz embeddings into the other family with uniformly bounded distortions.

We need the following description of $\text{TC}(X)$ in the case where $X$ is a vertex set of an unweighted graph with its graph distance. Let $G = (V(G), E(G)) = (V, E)$ be a finite graph. Let $\ell_1(E)$, $\ell_2(E)$, and $\ell_\infty(E)$ be the spaces of real-valued functions on $E$ with the norms $\|f\|_1 = \sum_{e \in E} |f(e)|$, $\|f\|_2 = \left(\sum_{e \in E} |f(e)|^2\right)^{\frac{1}{2}}$, and $\|f\|_\infty = \max_{e \in E} |f(e)|$, respectively. We also consider the inner product $\langle f, g \rangle$ associated with $\|f\|_2$.

We consider an arbitrary chosen orientation on $E$, so each edge of $E$ is a directed edge. We denote by $e^+$ and $e^-$ the head and tail of an oriented edge $e$, respectively. The choice of orientation affects some of the objects which we introduce, but does not affect the final results. Such orientation is usually called reference orientation.

For a directed cycle $C$ in $E$ (we mean that the cycle can be “walked around” following the direction, which is not related with the orientation of $E$) we introduce the signed indicator function of $C$ by
\[ \chi_C(e) = \begin{cases} 1 & \text{if } e \in C \text{ and its orientations in } C \text{ and } G \text{ are the same} \\ -1 & \text{if } e \in C \text{ but its orientations in } C \text{ and } G \text{ are different} \\ 0 & \text{if } e \notin C. \end{cases} \quad (2) \]

The cycle space \( Z(G) \) of \( G \) is the subspace of \( \ell_1(E) \) spanned by the signed indicator functions of all cycles in \( G \). The orthogonal complement of \( Z(G) \) in \( \ell_2(E) \) is called the cut space.

We will use the fact [25, Proposition 10.10] that \( \text{TC}(G) \) for unweighted graphs \( G \) is isometrically isomorphic to the quotient of \( \ell_1(E) \) over \( Z(G) \):

\[ \text{TC}(G) = \ell_1(E)/Z(G) \quad (3) \]

The paper [23] contains a generalization of (3) for weighted graphs, and thus for arbitrary finite metric spaces.

For convenience of the readers we give a simple proof of (3).

**Proof** Observe that if \( G = (V, E) \) is endowed with a reference orientation, each function \( f \in \ell_1(E) \) can be regarded as transportation plan given by

\[ \sum_{e \in E} f(e)(1_{e^-} - 1_{e^+}), \]

and the cost of this plan is \( \|f\|_1 \) (note that \( f(e) \) can be negative, so this transportation plan is not necessarily in the form (1)).

In turn, each such transportation plan gives (after summation) the transportation problem which it solves. Thus (for any fixed reference orientation) there is a natural linear map \( T : \ell_1(E) \to \text{TP}(G) = \text{TC}(G) \) (we consider finite graphs). The statement in the previous paragraph implies that \( \|Tf\|_{\text{TC}} \leq \|f\|_1 \).

It remains to show that for each transportation problem \( x \in \text{TC}(G) \) there is \( f \in \ell_1(E) \), such that \( T f = x \) and \( \|f\|_1 = \|x\|_{\text{TC}} \).

Let \( \sum_{i=1}^n a_i (1_{x_i} - 1_{y_i}) \) be an optimal transportation plan for \( x \). Since pairs \( x_i, y_i \) do not necessarily form edges, this optimal transportation plan does not immediately and naturally correspond to a vector in \( \ell_1(E) \). Nevertheless, by the definition of a graph distance, for each such pair \( x_i, y_i \), we can find a shortest path \( u_{0,i}, u_{1,i}, \ldots, u_{m(i),i} \) in \( G \) with \( u_{0,i} = x_i, u_{m(i),i} = y_i \), each pair \( u_{j-1,i}u_{j,i} \) \( (j = 1, \ldots, m(i)) \) being an edge in \( G \), and \( m(i) = d(x_i, y_i) \).

Then, as is easy to see,

\[ \sum_{i=1}^n \sum_{j=1}^{m(i)} a_i (1_{u_{j-1,i}} - 1_{u_{j,i}}), \]

is also an optimal transportation plan for \( x \) and this plan corresponds to a vector \( f \) in \( \ell_1(E) \) with \( \|f\|_1 = \|x\|_{\text{TC}} \).

The correspondence is the following: \( f(e) = 0 \) is \( e \) is not of the form \( u_{j-1,i}u_{j,i} \) for some \( i \) and \( j \), and \( f(e) = \theta(e, i, j)a_i \) if \( e \) is of the form \( u_{j-1,i}u_{j,i} \), where \( \theta(e, i, j) = 1 \) if \( u_{j-1,i} \) is the tail of \( e \) and \( \theta(e, i, j) = -1 \) if \( u_{j-1,i} \) is the head of \( e \). It remains to observe that the kernel of \( T \) is \( Z(G) \). \( \square \)
1.2 Results from [8] on iteratively defined graphs

Let us recall two results from [8] which are relevant to the present work.

A directed graph \( B \) having two distinguished vertices which we call top and bottom, generates a recursive family \( \{ B_n \}_{n=0}^\infty \) as follows:

- The graph \( B_0 \) consists of one directed edge.
- For \( n \geq 1 \), \( B_n \) is obtained from \( B_{n-1} \) by replacing each edge by a copy of \( B \), identifying bottom of \( B \) with the tail of the edge and top of \( B \) with the head of the edge. Edges of \( B_n \) inherit their directions from the corresponding copies of \( B \).

In [8] we considered the recursive families corresponding to directed graphs \( B \) satisfying certain natural conditions listed in [8, Section 4.1], which include the multibranching diamond and Laakso graphs defined above.

**Theorem A** [8, Theorem 4.2] If the directed graph \( B \) satisfies the conditions of [8, Section 4.1] and \( \{ B_n \}_{n=0}^\infty \) is the corresponding recursively defined family then the Banach–Mazur distance to \( \ell^d_1 \) satisfies

\[
d_{BM}(TC(B_n), \ell^d_1) \geq \frac{cn}{\ln n}
\]

for \( n \geq 2 \) and some absolute constant \( c > 0 \), where \( d(n) \) is the dimension of \( TC(B_n) \).

The \( \ln n \) factor in Theorem A was removed for the case of multibranching diamond graphs and an upper bound was also proved.

**Theorem B** [8, Theorem 6.10] The Banach–Mazur distance \( d_{n,k} \) from the transportation cost space \( TC(D_{n,k}) \) to the \( \ell^N_1 \) space of the same dimension satisfies

\[
4n + 4 \geq d_{n,k} \geq \frac{k - 1}{2k} n.
\]

1.3 Statement of results

Our main goal is to investigate the analogue of Theorem B for the Laakso graph \( \mathcal{L}_n \). In Sect. 5 we prove the lower bound of \( (3n - 5)/8 \) for the Banach–Mazur distance from \( TC(\mathcal{L}_n) \) to \( \ell^N_1 \) (Corollary 17). This removes the \( \ln n \) factor of Theorem A and is the analogue of the lower bound in Theorem B. However, we have not succeeded in proving a comparable (e.g. \( O(n^a) \)) upper bound. The obstacle to proving an analogue of the upper bound in Theorem B is explained in Sect. 7.

Our analysis of \( TC(\mathcal{L}_n) \) is based on the fact (see (3)) that \( TC(G) \) is isometrically isomorphic to \( E(G)/Z(G) \). In Sect. 3 we construct orthogonal basis vectors for the cycle and cut spaces and in Sect. 2.3 we compute their norms. They are used in Sect. 3 to construct a projection \( P_n \) from the edge space onto the cycle space of relatively small norm (Theorem 11). In Sect. 4 we show that \( P_n \) is close to being of minimal norm (Theorem 15). To prove this, we use the method of invariant projections as in
Grünbaum [11], Rudin [27] and Andrew [2], and analyze projections that are invariant with respect to a certain group of isometries of the edge space.

Let $X$ be a finite-dimensional normed space and let $X_1$ be any subspace of $\ell_\infty$ that is isometrically isomorphic to $X$. Recall that the projection constant of $X$, denoted $\lambda(X)$, is defined by

$$\lambda(X) = \inf \{ \|P\| : P : \ell_\infty \to \ell_\infty \text{ is a projection with range } X_1 \}.$$  

(Note that $\lambda(X)$ is independent of the choice of $X_1$.)

In Sect. 5 we deduce from Theorems 11 and 15 reasonably sharp estimates of the projection constant of the space of Lipschitz functions on $L_n$ (Theorem 16). We also present the results described above on the transportation cost space of $L_n$. In Sect. 6 we sharpen the proof of Theorem B from [8] to obtain the exact projection constant of the space of Lipschitz functions on $D_{n,k}$.

In Sect. 7, for the convenience of the reader we give a direct proof in the finite case that if $TC(X)$ is isometric to $\ell_1^{\lfloor X \rfloor - 1}$, then $X$ is a weighted tree and make a comment on the number of extreme points in the unit ball of $TC(M)$.

Section 8 is devoted to simple examples of finite metric spaces, transportation cost spaces on which contain $\ell^3_\infty$ and $\ell^4_\infty$ isometrically. Earlier, more complicated finite spaces with this property were provided in [17]. It is an open question whether there exist a finite metric space $M$ such that $TC(M)$ contains $\ell^5_\infty$ isometrically.

2 Preliminaries

2.1 Definitions and notation needed for the proofs

Let us fix some notation for the Laakso graph $L_n$. We denote the edge, cycle, and cut spaces of $L_n$ by $E_n$, $Z_n$ and $C_n$ respectively. The usual $\ell_1, \ell_2$, and $\ell_\infty$ norms on $E_n$ are denoted $\|\cdot\|_1, \|\cdot\|_2$, and $\|\cdot\|_\infty$. The usual inner product is denoted $\langle \cdot, \cdot \rangle$.

The edges of $L_1$ are labelled as in Fig. 3. We shall fix the reference orientation indicated by the arrows.

For the induction arguments which are used it will be convenient to label the 6 sub-$L_{n-1}$’s of $L_n$ as $A, \ldots, F$ as shown in Fig. 4. For $n \geq 2$, the edges of $L_n$ inherit a reference orientation from $L_1$ as indicated by the arrows in Fig. 4. The edges of $L_n$ are oriented from ‘bottom’ to ‘top’ in Fig. 4.

For each $1 \leq j \leq n$, we shall use the term ‘sub-$L_j$’ to refer to any of the copies of $L_j$ contained in $L_n$.

2.2 The cycle and cut spaces of $L_n$

For each $1 \leq j \leq n$ and for each given sub-$L_j$, $Z_n$ contains the signed indicator function of the outer cycle (see Fig. 3) contained in the given sub-$L_j$. The collection of all such signed indicator functions is easily seen to be an algebraic basis of $Z_n$. Counting the total number of sub-$L_j$’s, it follows that $\dim Z_n = (6^n - 1)/5$, and
hence \( \dim C_n = (4 \cdot 6^n + 1)/5 \) since \( C_n \) is the orthogonal complement of \( Z_n \). However, this basis of \( Z_n \) is difficult to work with because it is not orthogonal.

We shall now construct orthogonal bases for \( Z_n \) and \( C_n \) which will be used later to analyze projections onto \( Z_n \).

\( n = 1 \): A vector in the edge space will be denoted by a vector

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6
\end{bmatrix},
\]
where $x_i$ denotes the coefficient on the edge labelled $i$ (see Fig. 3).

Note that $\dim Z_1 = 1$ and $\dim C_1 = 5$. It is easily seen that $Z_1$ is spanned by

$$h_1 = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 \end{bmatrix}.$$  \hfill (4)

$C_1$, which is the orthogonal complement of $Z_1$, is easily seen to be spanned by the row vectors (which are orthogonal) of the following matrix:

$$\begin{bmatrix}
-1 & 1 & 1 & 1 & 1 & -1 \\
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
1 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \\
\end{bmatrix}.$$  \hfill (5)

Note that these 6 vectors form an orthogonal basis of $E_1$.

$n = 2$: $\mathcal{L}_2$ is formed from $\mathcal{L}_1$ by replacing each edge of $\mathcal{L}_1$ by a copy of $\mathcal{L}_1$. Similarly, the edge vectors of $\mathcal{L}_2$ are obtained by replacing each coefficient $x_i$ of an edge vector of $\mathcal{L}_1$ by the entries of a 6-dimensional vector.

In this way a vector in $E_1$ generates a vector in $E_2$ according to the following replacement rule: for each $x \in \mathbb{R}$,

$$x \mapsto \begin{bmatrix} x/2 & x/2 & x/2 & x/2 & x \end{bmatrix}.$$  

We will describe this process of replacement as ‘propagation’.

Define $f_1 \in C_1$ as follows:

$$f_1 = \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}.$$  

Note that

$$f_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

which expresses $f_1$ as the average of 2 indicator functions of paths connecting the bottom vertex of $\mathcal{L}_1$ to the top vertex. Hence $h_1$ propagates to an average of two signed indicator functions of cycles in $\mathcal{L}_2$. In particular, $h_1$ propagates to a vector $h_2$ in $Z_2$.

In addition to this vector, each of the 6 copies of $\mathcal{L}_1$ supports a ‘new’ cycle vector given by

$$\begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 \end{bmatrix}.$$  

(Its coefficients on the other five copies of $\mathcal{L}_1$ are all zero.) Note that this vector is orthogonal to the propagated vector since it is orthogonal to $f_1$.

The 5 basis vectors of $C_1$ propagate to form basis vectors of $C_2$. In addition, supported on each of the six copies of $\mathcal{L}_1$ we obtain 4 ‘new’ orthogonal cut vectors given by the row vectors of the following matrix:
Note that the row vectors are orthogonal to $f_1$. Hence the new cut vectors are orthogonal to the propagated cut vectors. The 5 propagated cut vectors and the 24 new cut vectors together form an orthogonal basis of the cut space $C_2$.

$n \geq 3$: This is similar to the case $n = 2$. The orthogonal bases of $Z_{n-1}$ and $C_{n-1}$ propagate to collections of orthogonal vectors in $Z_n$ and $C_n$. In addition, each of the $6^{n-1}$ copies of $L_1$ supports one new cycle vector and 4 new cut vectors as above.

Let us check these claims. The claimed bases of $Z_n$ and $C_n$ are orthogonal and have the correct cardinality. So it suffices to check they they are contained in $Z_n$ and $C_n$ respectively. For $n \geq 2$, let $h_n$ be the propagation of $h_{n-1}$ and let $f_n$ be the propagation of $f_{n-1}$ (see Fig. 5). It suffices to check that $h_n \in Z_n$. A straightforward induction shows that $f_n$ is the average of $2^n$ indicator functions of paths joining the bottom and top vertices of $L_n$. Hence (see Fig. 6) $h_n$ is the average of $2^{n-1}$ signed indicator functions of large cycles in $L_n$. In particular, $h_n \in Z_n$ as desired.

Recalling that $C_n$ is the orthogonal complement of $Z_n$, the orthogonality of the basis guarantees that the claimed basis of $C_n$ is indeed contained in $C_n$.

### 2.3 Norms of cycle and cut vectors

Note that

$$
\|f_1\|_1 = 4, \quad \|f_1\|_2^2 = 3.
$$

For $n \geq 2$, define $f_n \in E_n$ inductively as shown in Fig. 5. Note that

$$
\|f_n\|_1 = 4\|f_{n-1}\|_1 = 4^n, \quad \|f_n\|_2^2 = 3\|f_{n-1}\|_2^2 = 3^n.
$$

Recall from (4) that $h_1 \in Z_1$ was defined by

$$
h_1 = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 \end{bmatrix}.
$$

Now define $g_1 \in C_1$ by

$$
g_1 = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix},
$$

and, for $n \geq 2$, define $g_n \in C_n$ and $h_n \in Z_n$ inductively as shown in Fig. 6. Note that $h_n$ is the cycle vector obtained from $h_1$ by repeated propagation, $g_n$ is the cut vector obtained from $g_1$ by repeated propagation,

$$
\|g_n\|_1 = 6\|f_{n-1}\|_1 = \frac{3}{2}4^n, \quad \|g_n\|_2^2 = 6\|f_{n-1}\|_2^2 = 2 \cdot 3^n,
$$
and

\[ \|h_n\|_1 = 4\|f_{n-1}\|_1 = 4^n, \quad \|h_n\|_2^2 = 4\|f_{n-1}\|_2^2 = \frac{4}{3} \cdot 3^n. \]

Hence, in particular,

\[ \frac{\|g_n\|_1}{\|g_n\|_2^2} = \frac{\|h_n\|_1}{\|h_n\|_2^2} = \left( \frac{4}{3} \right)^{n-1}. \tag{6} \]

Note that each sub-\(L_j\) supports a unique \(Z_n\) basis vector \(H_j\) of the form \(h_j\) and a unique \(C_n\) basis vector \(G_j\) of the form \(g_j\). To justify this claim, let \(L_j\) be a sub-\(L_j\) of \(L_n\). For \(j = 1\), \(G_1\) and \(H_1\) are the ‘new’ \(g_1\) and \(h_1\) basis vectors supported on \(L_1\) arising in the passage from \(Z_{n-1}\) to \(Z_n\) and \(C_{n-1}\) to \(C_n\) described above. For \(j > 1\), note that \(L_j\) evolves from a unique sub-\(L_1\) of \(L_{n-1-j}\), \(L'_1\) say. Let \(G'_1\) and \(H'_1\) be the \(g_1\) and \(h_1\) basis vectors supported on \(L'_1\). Propagating \(G'_1\) and \(H'_1\) repeatedly \((j - 1)\) times produces basis vectors \(G_j\) and \(H_j\) of the form \(g_j\) and \(h_j\) that are supported on \(L_j\) as claimed.

The next two lemmas will be used in Sect. 3.

Lemma 6 Let \(1 \leq j \leq n\) and let \(H_j\) and \(G_j\) be supported in some sub-\(L_j\), \(L_j\), say. Then, for every edge vector \(e\) belonging to \(L_j\), we have

1. \(\langle e, H_j \rangle = 0 \iff \langle e, G_j \rangle < 0\).
2. If \(\langle e, H_j \rangle \neq 0\) then \(\langle e, G_j \rangle > 0\) and \(|\langle e, H_j \rangle| = \langle e, G_j \rangle\).

Proof (1) From Fig. 6, note that \(\langle e, H_j \rangle = 0\) if and only if \(e\) belongs to the \(A\) or \(F\) sub-\(L_{j-1}\) of \(L_j\) if and only if \(\langle e, G_j \rangle < 0\).

(2) If \(\langle e, H_j \rangle \neq 0\) then \(e\) belongs to the \(B, C, D\) or \(E\) sub-\(L_{j-1}\). From Fig. 6, note that \(\langle e, G_j \rangle > 0\) and \(|\langle e, H_j \rangle| = \langle e, G_j \rangle\). \qed
To state the next lemma, let us first fix some notation. For $1 \leq j \leq n$, let $L_j$ be a sub-
$\mathcal{L}_j$ of $\mathcal{L}_n$ such that $(L_j)_{j=1}^n$ is an increasing chain, i.e., $L_1 \subset L_2 \subset \cdots \subset L_n = \mathcal{L}_n$.
Let $S_j$ be the set of edge vectors contained in $L_j$, so that $(S_j)_{j=1}^n$ is also increasing.
Finally, for $1 \leq j \leq n$, let $G_j$ and $H_j$ be the cut and cycle basis vectors corresponding
to $L_j$ (of the form $g_j$ and $h_j$).

**Lemma 7** Let $1 \leq j \leq n$. Then for every $e \in S_1$, we have

$$
\langle e, G_j \rangle = \left(\frac{1}{2}\right)^{\alpha_j} \text{sgn}(\langle e, G_j \rangle) \quad \text{and} \quad \langle e, H_j \rangle = \left(\frac{1}{2}\right)^{\alpha_j} \text{sgn}(\langle e, H_j \rangle),
$$

where $\alpha_1 = 0$ and, for $j \geq 2$, $\alpha_j$ is the cardinality of the set $\{1 \leq r < j : S_{r-1} \subset \text{supp}(H_r)\}$ (here $\text{sgn}(0) = 0$).

**Proof** The result clearly holds for $j = 1$. So suppose that the result holds for $j = j_0$,
where $1 \leq j_0 < n$. For $1 \leq j \leq n$, let $F_j$ be the vector of the form $f_j$ corresponding
to $L_j$. From Fig. 6, we have

$$
|\langle e, G_{j_0+1} \rangle| = \langle e, F_{j_0} \rangle.
$$

If $S_{j_0-1} \subset \text{supp}(H_{j_0})$, then $\alpha_{j_0+1} = \alpha_{j_0} + 1$ and, from Fig. 5,

$$
\langle e, F_{j_0} \rangle = \frac{1}{2} \langle e, F_{j_0-1} \rangle = \frac{1}{2} |\langle e, G_{j_0} \rangle|,
$$

(where $\langle e, F_0 \rangle = 1$ by convention in the case $j_0 = 1$). So by the inductive hypothesis,

$$
\langle e, G_{j_0+1} \rangle = \frac{1}{2} |\langle e, G_{j_0} \rangle| \text{sgn}(\langle e, G_{j_0+1} \rangle) = \left(\frac{1}{2}\right)^{\alpha_{j_0+1}} \text{sgn}(\langle e, G_{j_0+1} \rangle)
$$
as desired. On the other hand, if $S_{j_0-1}$ is disjoint from $\text{supp}(H_{j_0})$, then $\alpha_{j_0+1} = \alpha_{j_0}$ and from Fig. 5,

$$\langle e, F_{j_0} \rangle = \langle e, F_{j_0-1} \rangle = |\langle e, G_{j_0} \rangle|.$$ 

So by the inductive hypothesis,

$$\langle e, G_{j_0+1} \rangle = |\langle e, G_{j_0} \rangle| \text{sgn}(\langle e, G_{j_0+1} \rangle) = \left(\frac{1}{2}\right)^{\alpha_{j_0+1}} \text{sgn}(\langle e, G_{j_0+1} \rangle)$$

as desired. The stated result for $\langle e, H_j \rangle$ follows from the result for $\langle e, G_j \rangle$ and Lemma 6. 

3 A projection onto the cycle space

In this section we define a projection $P_n$ from $E_n$ onto its cycle space $Z_n$ which has relatively small (linear in $n$, i.e., logarithmic in $\dim(E_n)$) norm on $(E_n, \| \cdot \|_1)$.

Let us first observe that the orthogonal projection $P_n$ of $E_n$ onto $Z_n$ has large (exponential in $n$) norm on $(E_n, \| \cdot \|_1)$.

**Proposition 8**

$$\|P_n\|_1 \geq \left(\frac{4}{3}\right)^{n-1}.$$ 

**Proof** Let $e$ be the edge vector in $Z_n$ corresponding to the ‘lowest’ edge (with respect to the ‘bottom’ to ‘top’ orientation) in the sub-$\mathcal{L}_{n-1}$ labelled as $B$. Then $\langle e, h_n \rangle = 1$ and $\langle e, h \rangle = 0$ if $h \neq h_n$ is any other basis vector of $Z_n$. Hence, using (6),

$$\|P_n\|_1 \geq \|P_n(e)\|_1 = \langle e, h_n \rangle \frac{\|h_n\|_1}{\|h_n\|_2} = \left(\frac{4}{3}\right)^{n-1}.$$ 

The definition of $P_n$ is inductive. $P_1$ is the orthogonal projection.

Suppose $n \geq 2$. We start the definition of $P_n$ by setting $P_n(g_n) = 0$ and $P_n(g) = 0$ for every cut vector $g$ in the orthogonal basis of $C_n$ which is not of the form $g_j$ for some sub-$\mathcal{L}_j$ ($1 \leq j \leq n-1$). This is to be expected as we shall show in the next section that this holds for any projection which is invariant with respect to a natural group of isometries of $E_n$. Thus, to complete the definition, it suffices to define $P_n(g_j)$ for each sub-$\mathcal{L}_j$.

We shall label the six sub-$\mathcal{L}_{n-1}$’s as $A, \ldots, F$ as shown in Fig. 4. On $A$ and $F$ we define $P_n$ to be a copy of $P_{n-1}$. So it suffices to define $P_n(g_j)$ for all $g_j$ supported on a sub-$\mathcal{L}_j$ contained in $B$, $C$, $D$ or $E$. The definition of $P_n(g_j)$ will proceed backwards from $j = n-1$ to $j = 1$. 
Let $S_{n-1}$ be the set of edge vectors of any one of $B, C, D$ or $E$. Now let $S_{n-2}$ be the set of edge vectors of any one of the 6 sub-$L_{n-2}$’s supported in $S_{n-1}$. Continue in this way to obtain a chain $S_{n-1} \supset S_{n-2} \supset \cdots \supset S_1$. Finally, let $e$ be one of the 6 edge vectors contained in $S_1$. Note that $S_1$ uniquely determines the chain $(S_j)_{j=1}^{n-1}$ and that every edge vector $e$ in the support of $B, C, D, E$ determines a unique choice of $S_1$.

For each $1 \leq j \leq n-1$, let $G_j$ denote the $g_j$ cut vector and let $H_j$ denote the $h_j$ cycle vector corresponding to the sub-$L_j$ supported on $S_j$. We shall define $P_n(G_j)$ inductively along the chain $(S_j)_{j=1}^{n-1}$ starting with $j = n-1$. By varying the chain we define $P_n(G_j)$ for every cut vector in the orthogonal basis of $C_n$ which is of the form $G_j$ for some sub-$L_j$ $(1 \leq j \leq n-1)$. Since each sub-$L_j$ occurs in several different chains, we must also check that $P_n(G_j)$ is well-defined.

The motivating idea behind this definition is a ‘balancing’ of certain norms which is described in (iv) below. However, since the proof is lengthy and not particularly intuitive, we will describe the strategy before going into the details. The definition of $P_n(G_j)$ will involve a sequence of vectors $(X_j)_{j=n}^{1}$ and sequences of scalars $(x_j)_{j=n}^{1}$ and $(a_j)_{j=n}^{1}$, which are defined inductively. The strategy behind the definition of $P_n$ and the proof of Theorem 11 below is as follows:

(i) $X_j$ is completely determined by $S_{j-1}$ and is defined inductively as a linear combination of $H_j, H_{j+1}, \ldots, h_n$.

(ii) The definition of $X_j$ given by (10) has two cases, depending on whether or not $S_{j-1}$ is contained in the support of $H_j$ (equivalently, whether or not $e \in \text{supp}(H_j)$).

(iii) The choice of $a_j$ as defined by (9) ensures that $X_j$ has roughly the same $\| \cdot \|_1$ norm in both cases.

(iv) Hence $P_n(G_j)$, as defined by (8), has roughly the same norm in both cases of the definition of $X_{j+1}$. It is this balancing which ultimately leads to a projection of relatively small norm. (Note also that $P_n(G_j)$ is a certain linear combination of $H_{j+1}, H_{j+2}, \ldots, h_n$.)

(v) The choice of $a_j$ ensures that $\|X_j\|_1 \leq x_j:=(1-a_j)x_{j+1}$.

(vi) It is shown in Lemma 9 that $X_1 = P_n(e)$, and hence $\|P_n(e)\|_1 \leq x_1$. This is the key estimate in the proof of Theorem 11.

(vii) $(x_j)_{j=n}^{1}$ satisfies a recurrence relation which is solved in Lemma 10. This leads to the estimate $\|P_n\|_1 \leq (n+1)/2$, which is proved in Theorem 11.

Let us now go through the details of the definition of $P_n(G_j)$ starting with $j = n-1$. Set

$$X_n = \text{sgn}(\langle e, h_n \rangle) \frac{h_n}{\|h_n\|^2_2} \quad \text{and} \quad x_n = \|X_n\|_1 = \left(\frac{4}{3}\right)^{n-1}, \quad (7)$$

where $\text{sgn}(a)$ is the sign of $a$. Define

$$P_n \left( \frac{G_{n-1}}{\|G_{n-1}\|^2_2} \right) = a_{n-1}X_n.$$
where \( a_{n-1} \) is defined by the equation

\[
(1 - a_{n-1}) \left( \frac{4}{3} \right) = \left( \frac{1}{2} + a_{n-1} \right) \left( \frac{4}{3} \right)^{n-1} + \left( \frac{4}{3} \right)^{n-2}.
\]

(Note that, in fact, \( a_{n-1} = -1/8 \).) Now set

\[
X_{n-1} = \begin{cases} 
\left( \frac{1}{2} + a_{n-1} \right) X_n + \text{sgn}(\langle e, H_{n-1} \rangle) \frac{H_{n-1}}{\|H_{n-1}\|_2^2}, & e \in \text{supp}(H_{n-1}), \\
(1 - a_{n-1}) X_n, & e \not\in \text{supp}(H_{n-1}).
\end{cases}
\]

Since \( 1 - a_{n-1} = 9/8 > 0 \) and \( \frac{1}{2} + a_{n-1} = 3/8 > 0 \), the triangle inequality and (6) give

\[
\|X_{n-1}\|_1 \leq \left[ \left( \frac{1}{2} + a_{n-1} \right) \|X_n\|_1 + \frac{\|H_{n-1}\|_1}{\|H_{n-1}\|_2^2} \right] \vee (1 - a_{n-1}) \|X_n\|_1
\]

\[
= \left[ \left( \frac{1}{2} + a_{n-1} \right) \left( \frac{4}{3} \right)^{n-1} + \left( \frac{4}{3} \right)^{n-2} \right] \vee (1 - a_{n-1}) \left( \frac{4}{3} \right)^{n-1}
\]

\[
= (1 - a_{n-1}) \left( \frac{4}{3} \right)^{n-1}
\]

\[
= (1 - a_{n-1}) \|X_n\|_1.
\]

Set \( x_{n-1} = \|X_{n-1}\|_1 \). Then \( \|X_{n-1}\|_1 \leq x_{n-1} \).

Let us now turn to the inductive step, which is similar to the case \( j = n-1 \). Suppose that \( 1 \leq j < n-1 \) and that \( X_{j+1}, x_{j+1}, \) and \( P_n(G_{j+1}) \) have been defined with \( \|X_{j+1}\|_1 \leq x_{j+1} \). Now define

\[
P_n \left( \frac{G_j}{\|G_j\|_2^2} \right) = a_j X_{j+1},
\]

where \( a_j \) is defined by the equation

\[
(1 - a_j)x_{j+1} = \left( \frac{1}{2} + a_j \right) x_{j+1} + \left( \frac{4}{3} \right)^{j-1}.
\]

Set

\[
X_j = \begin{cases} 
\left( \frac{1}{2} + a_j \right) X_{j+1} + \text{sgn}(\langle e, H_j \rangle) \frac{H_j}{\|H_j\|_2^2}, & e \in \text{supp}(H_j), \\
(1 - a_j) X_{j+1}, & e \not\in \text{supp}(H_j).
\end{cases}
\]

It is worth observing that, for \( j \geq 2 \), \( X_j \) does not depend on the particular choice of \( e \) from \( S_1 \). Hence, for \( j \geq 1 \), \( P_n(G_j) \) defined by (8) is also independent of the choice of \( e \) from \( S_1 \). Hence, for \( j \geq 1 \), \( P_n(G_j) \) defined by (8) is also independent of the choice of \( e \) from \( S_1 \).
as required. But we prove below (Lemma 9) that \( X_1 = P_n(e) \), which does depend on the choice of \( e \).

We prove in Lemma 10 below that \( \frac{1}{2} + a_j > 0 \) and \( 1 - a_j > 0 \). Hence, by the triangle inequality and (6),

\[
\|X_j\|_1 \leq \left[ \left( \frac{1}{2} + a_j \right) \|X_{j+1}\|_1 + \frac{\|H_j\|}{\|H_j\|_2^2} \right] \vee (1 - a_j)\|X_{j+1}\|_1
\]

\[
\leq \left[ \left( \frac{1}{2} + a_j \right) x_{j+1} + \left( \frac{4}{3} \right)^{j-1} \right] \vee (1 - a_j)x_{j+1}
\]

\[
= (1 - a_j)x_{j+1}.
\]

Finally, set \( x_j = (1 - a_j)x_{j+1} \) to complete the inductive step.

To check that \( P_n(G_j) \) as given by (8) is well-defined, we need to check that it depends only on \( \text{supp}(G_j) = S_j \). To see this, note that \( S_j \) determines its ‘ancestors’ \( S_{j+1}, \ldots, S_{n-1} \) uniquely. Moreover, the definition of \( X_{j+1} \) (see (10) and replace \( j \) by \( j + 1 \)) actually depends only on \( S_j \) since \( \text{sgn}(\langle e, H_{j+1} \rangle) \) is simply the (constant) sign of \( H_{j+1} \) on \( S_j \). Hence \( P_n(G_j) \) is indeed well-defined.

By considering every chain \( S_{n-1} \supset S_{n-2} \supset \cdots \supset S_1 \), we define \( S(g) \) for every cut vector \( g \) of the form \( g_j \) for some sub-\( \mathcal{L}_j \).

The definition of \( P_n \) is now complete. (Recall that we started the definition by setting \( P_n(g_n) = 0 \) and \( P_n(g) = 0 \) for all other cut vectors \( g \) in the orthogonal basis of \( C_n \) described above.)

**Lemma 9** \( P_n(e) = X_1 \).

**Proof** Using Lemma 6 and the fact (see (8)) that

\[
P_n \left( \frac{G_j}{\|G_j\|_2^2} \right) = a_jX_{j+1},
\]

we can combine the two cases in the definition (10) of \( X_j \) as follows:

\[
X_j = \left( \frac{1}{2} \right)^{\varepsilon_j} X_{j+1} + \text{sgn}(\langle e, G_j \rangle) \frac{P_n(G_j)}{\|G_j\|_2^2} + \text{sgn}(\langle e, H_j \rangle) \frac{H_j}{\|H_j\|_2^2},
\]

where

\[
\varepsilon_j = \begin{cases} 
1, & S_{j-1} \subset \text{supp}(H_j), \\
0, & S_{j-1} \cap \text{supp}(H_j) = \emptyset
\end{cases}
\]

and setting \( \text{sgn}(0) = 0 \). After repeated application of this formula, starting at \( j = 1 \) and ending at \( j = n - 1 \), and then substituting (see (7))

\[
X_n = \text{sgn}(\langle e, h_n \rangle) \frac{h_n}{\|h_n\|_2^2},
\]
we obtain

\[
X_1 = \sum_{j=1}^{n-1} \frac{1}{2} \left( \frac{1}{2} \right)^{\alpha_j} \left[ \text{sgn}(\langle e, G_j \rangle) \frac{P_n(G_j)}{\|G_j\|_2^2} + \text{sgn}(\langle e, H_j \rangle) \frac{H_j}{\|H_j\|_2^2} \right] + \left( \frac{1}{2} \right)^{\alpha_n} \text{sgn}(\langle e, h_n \rangle) \frac{h_n}{\|h_n\|_2^2},
\]

where \( \alpha_1 = 0 \) and, for \( j \geq 2 \), \( \alpha_j \) is the cardinality of the set \( \{1 \leq r < j : S_{r-1} \subset \text{supp}(H_r)\} \). By Lemma 7, for \( 1 \leq j \leq n-1 \),

\[
\langle e, G_j \rangle = \frac{1}{2} \alpha_j \text{sgn}(\langle e, G_j \rangle)
\]

and

\[
\langle e, H_j \rangle = \frac{1}{2} \alpha_j \text{sgn}(\langle e, H_j \rangle)
\]

and

\[
\langle e, h_n \rangle = \frac{1}{2} \alpha_n \text{sgn}(\langle e, h_n \rangle).
\]

Hence

\[
X_1 = \sum_{j=1}^{n-1} \left[ \langle e, G_j \rangle \frac{P_n(G_j)}{\|G_j\|_2^2} + \langle e, H_j \rangle \frac{H_j}{\|H_j\|_2^2} \right] + \langle e, h_n \rangle \frac{h_n}{\|h_n\|_2^2}
\]

\[
= P_n \left( \sum_{j=1}^{n-1} \left( \frac{e}{G_j} \left( \frac{G_j}{\|G_j\|_2} \right) + \frac{e}{H_j} \left( \frac{H_j}{\|H_j\|_2} \right) \right) \right) + \frac{e}{h_n} \left( \frac{h_n}{\|h_n\|_2} \right)
\]

\[
= P_n(e).
\]

To see the last line of the above, note that if \( \langle e, g \rangle \neq 0 \), for \( g \) belonging to the othogonal basis of \( C_n \), then either \( g = G_j \) for some \( 1 \leq j \leq n-1 \) or \( P_n(g) = 0 \). This is because we began the definition of \( P_n \) by setting \( P_n(g_n) = 0 \) and \( P_n(g) = 0 \) for every cut vector \( g \) in the orthogonal basis of \( C_n \) which is not of the form \( g_j \) for some sub-L_j.

On the other hand, if \( g \) is of the form \( g_j \) and \( \langle e, g \rangle \neq 0 \) then \( \text{supp}(g) = S_j \), i.e., \( g = G_j \). Similarly, if \( \langle e, h \rangle \neq 0 \), for \( h \) belonging to the orthogonal basis of \( Z_n \), then either \( h = H_j \) or \( h = h_n \). So the above expression for \( X_1 \) is simply \( P_n \) applied to the expansion of \( e \) with respect to the othogonal basis of \( E_n \).

\[\square\]

**Lemma 10** \( x_1 = \frac{n+1}{2} \) and \( \min(1 - a_j, \frac{1}{2} + a_j) > 0 \) for \( 1 \leq j < n - 1 \).
Proof Recall that \(x_n = (4/3)^{n-1}\) (see (7)) and that, for \(1 \leq j \leq n-1\), \(x_j\) satisfies the recurrence
\[
x_j = (1 - a_j)x_{j+1} = \left(\frac{1}{2} + a_j\right)x_{j+1} + \left(\frac{4}{3}\right)^{j-1}
\]
which serves to define \(a_j\) for \(1 \leq j \leq n-1\). Hence
\[
x_j = \frac{1}{2}[(1 - a_j)x_{j+1} + \left(\frac{1}{2} + a_j\right)x_{j+1} + \left(\frac{4}{3}\right)^{j-1}]
\]
\[
= \frac{3}{4}x_{j+1} + \frac{1}{2} \left(\frac{4}{3}\right)^{j-1}.
\]
The solution to this recurrence is
\[
x_j = \frac{n + 2 - j}{2} \left(\frac{4}{3}\right)^{j-1}.
\]
Note that
\[
a_jx_{j+1} = x_{j+1} - x_j = \left(\frac{4}{3}\right)^{j} \left[-\frac{1}{4} + \frac{n - j}{8}\right].
\]
Hence \(a_{n-1} = -1/8\), \(a_{n-2} = 0\), and \(0 < a_j < 1\) for \(1 \leq j \leq n - 3\). In all cases
\[
\min(1 - a_j, \frac{1}{2} + a_j) > 0.
\]

Theorem 11 \(\|P_n\|_1 \leq \frac{n + 1}{2}\).

Proof Recall that \(P_1\) is the orthogonal projection onto \(Z_1\):
\[
P_1(e_i) = \begin{cases} \pm\frac{1}{4}(e_2 + e_3 - e_4 - e_5), & i = 2, 3, 4, 5 \\ 0, & i = 1, 6. \end{cases}
\]

Clearly, \(\|P_1\|_1 = 1\). Now suppose \(n \geq 2\). If \(e\) is an edge vector belonging to the \(A\) or \(F\) sub-\(\mathcal{L}_{n-1}\), then, by the inductive hypothesis,
\[
\|P_n(e)\|_1 \leq \|P_{n-1}\|_1 \leq \frac{n}{2}.
\]
On the other hand, if \(e\) belongs to the \(B, C, D\) or \(E\) sub-\(\mathcal{L}_{n-1}\), then \(P_n(e) = X_1\) for the chain \((S_j)_{j=1}^{n-1}\) with \(e \in S_1\), so by Lemma 10,
\[
\|P_n(e)\|_1 = \|X_1\|_1 \leq x_1 = \frac{n + 1}{2}.
\]
Hence
\[ \| P_n \|_1 = \max_e \| P_n(e) \|_1 \leq \frac{n + 1}{2}. \]

\[ \square \]

4 Invariant Projections

In this section we prove that the projection \( P_n \) constructed in the previous section is close to being optimal. First we show that we may restrict attention to projections that are ‘invariant’ with respect to a certain group of isometries of \( E_n \). Then we show that \( P_n \) is close to being optimal in the sense that its operator norm is of the same order.

First, let us define a group of isometries of \((E_n, \| \cdot \|_2)\). To that end, let us say that a cut vector \( g \) belonging to the orthogonal basis of the cut space \( C_n \) is \textbf{special} if \( g \) is of the form \( g_j \) for some sub-\( L_j \) for \( 1 \leq j \leq n \). We shall say that \( g \) is \textbf{non-special} if \( g \) is not of the form \( g_j \) and \( g \) is not the unique cut vector propagated by \( \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \).

If \( g \) is a non-special cut vector then there will be a smallest sub-\( L_j \) (\( 1 \leq j \leq n \)) which contains its support. Let us call this the \textbf{support sub-}\( L_j \) of \( g \). Let \( \psi \) be the natural isometry of \( E_n \) induced by interchanging \( \{ g > 0 \} \) and \( \{ g < 0 \} \). Since there are three types of non-special vector, namely those cut vectors propagated by the second, third, and fourth rows of (5), \( \psi \) is effectuated by either (a) interchanging the \( B \) and \( C \) sub-\( L_j - 1 \) of its support (using the inductively defined isomorphism between \( B \) and \( C \) and \( L_j - 1 \)), or (b) interchanging the \( D \) and \( E \) sub-\( L_j - 1 \), or (c) interchanging the \( A \) and \( F \) sub-\( L_j - 1 \). Note that \( Z_n \) and \( C_n \) are \( \psi \)-invariant subspaces.

Similarly, each \( Z_n \) basis vector \( h \) has a support sub-\( L_j \). Let \( \phi \) be the natural isometry induced by interchanging \( \{ h > 0 \} \) and \( \{ h < 0 \} \). Then \( \phi \) is effectuated by interchanging the \( B \) and \( E \) sub-\( L_j - 1 \) and the \( C \) and \( D \) sub-\( L_j - 1 \) of the support sub-\( L_j \) of \( h \). Note that \( Z_n \) and \( C_n \) are \( \phi \)-invariant subspaces.

Note that \( \phi^* = \phi^{-1} \) and \( \psi^* = \psi^{-1} \) when considered as isometries of the Euclidean space \( (E_n, \| \cdot \|_2) \).

Let \( G \) be the (finite) group generated by the collection of all \( \psi \) and \( \phi \) isometries. Let \( Q \) be any projection form \( E_n \) onto \( Z_n \). Then
\[ P = \frac{1}{|G|} \sum_{\theta \in G} \theta^{-1} Q \theta \]
satisfies \( \| P \|_1 \leq \| Q \|_1 \), and \( P \theta = \theta P \) for all \( \theta \in G \). Moreover, \( P \) is also a projection onto \( Z_n \) since \( Z_n \) and \( C_n \) are \( \theta \)-invariant for each \( \theta \in G \).

\textbf{Lemma 12} If \( g \) is non-special or if \( g \) is the (unique) cut vector propagated by \( \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \) then \( P(g) = 0 \).
Proof Since $P(g) \in Z_n$ it suffices to show that $\langle P(g), h \rangle = 0$ for every $h$ belonging to the basis of $Z_n$. If $\text{supp}(g) \subseteq \text{supp}(h)$ then $\psi_g(h) = h$. So

$$\langle P(g), h \rangle = \langle P(g), \psi_g(h) \rangle = \langle \psi_g(P(g)), h \rangle = \langle P(\psi_g(g)), h \rangle = -\langle P(g), h \rangle.$$ 

On the other hand, if $\text{supp}(h) \subseteq \text{supp}(g)$ or $\text{supp}(h) \cap \text{supp}(g) = \emptyset$ then $\phi_h(g) = h$ and $\phi_h(h) = g$. So

$$\langle P(g), h \rangle = \langle P(\phi_h(g)), h \rangle = \langle \phi_h(P(g)), h \rangle = \langle P(g), \phi_h(h) \rangle = -\langle P(g), h \rangle.$$ 

Hence, in both cases, $\langle P(g), h \rangle = 0$. $\square$

Lemma 13 If $g$ is a special cut vector then $P(g) \in \text{span}\{h : \text{supp}(g) \subseteq \text{supp}(h)\}$. In particular, $P(g_n) = 0$.

Proof If $\text{supp}(h) \subseteq \text{supp}(g)$ or $\text{supp}(h) \cap \text{supp}(g) = \emptyset$, then, as above, $\langle P(g), h \rangle = 0$, which gives the result. $\square$

The following lemma will be needed in the proof of Theorem 15 below.

Lemma 14 Let $(H_j)_{j=1}^n$ be a chain of cycle vectors such that $H_j$ is of type $h_j$ and $\text{supp}(H_j) \subset \text{supp}(H_{j+1})$ for each $1 \leq j < n$. Then

$$\| \sum_{j=1}^n a_j H_j \|_1 \geq \frac{3}{4} \sum_{j=1}^n |a_j| \| H_j \|_1$$

for all scalars $(a_j)_{j=1}^n$.

Proof Note that, for each $2 \leq j \leq n$,

$$\| H_j \|_{\text{supp}(H_{j-1})} \|_1 = \frac{1}{8} \| H_j \|_1.$$ 

Hence

$$\| \sum_{j=1}^n a_j H_j \|_1 = |a_n| \| H_n \|_{\text{supp}(H_n) \setminus \text{supp}(H_{n-1})} \|_1 + \| \sum_{j=1}^{n-1} a_j H_j + a_n H_n \|_{\text{supp}(H_{n-1})} \|_1$$

$$\geq |a_n| \| H_n \|_1 + \| \sum_{j=1}^{n-1} a_j H_j \|_1 - 2 |a_n| \| H_n \|_{\text{supp}(H_{n-1})} \|_1$$

$$= \frac{3}{4} |a_n| \| H_n \|_1 + \| \sum_{j=1}^{n-1} a_j H_j \|_1.$$ 

Iterating this calculation yields the result. $\square$
Theorem 15 Let $Q$ be any projection from $E_n$ onto $Z_n$. Then $\|Q\|_1 \geq \frac{3}{8}(n + 1)$.

Proof Let $P$ be the invariant projection associated to $Q$. We shall prove that $\|P\|_1 \geq \frac{3}{8}(n + 1)$, which implies the result since $\|P\|_1 \leq \|Q\|_1$.

The analysis of $P$ is very similar to the analysis of $P_n$ in the previous section. In particular, we will define an auxiliary sequence of vectors $(X_j)_{j=1}^n$ and an auxiliary sequence of scalars $(x_j)_{j=1}^n$. The goal is to construct a chain $(S_j)_{j=1}^{n-1}$, with $S_0 = \{e\}$, such that $\|P(e)\|_1$ is large, i.e., comparable to $\|P_n\|_1$. This is a chain which (roughly speaking) maximizes $\|X_j\|_1$ at each bifurcation.

To that end, we shall inductively define a chain of cycle vectors $(H_j)_{j=1}^n$ such that $H_j$ is of type $h_j$ and $\text{supp}(H_j) \supset \text{supp}(H_{j+1})$ for each $1 \leq j < n$. To start the induction, set $H_n = h_n$. To simplify the calculation of the norm we define an equivalent norm $||| \cdot |||$ on $\text{span}(H_j)_{j=1}^n$ which is easier to work with:

$$||| \sum_{j=1}^n a_j H_j ||| = \sum_{j=1}^n |a_j| \|H_j\|_1.$$  

By Lemma 14

$$\| \sum_{j=1}^n a_j H_j \|_1 \leq \| \sum_{j=1}^n a_j H_j \|_1 \leq \frac{4}{3} \sum_{j=1}^n a_j H_j \|_1.$$  

Inductively, we define vectors $(X_j)_{j=1}^n$ and a decreasing chain $S_{n-1} \supset S_{n-2} \supset \cdots \supset S_1$ such that $S_j$ is the support of a sub-$L_j$. To start the inductive definition, set

$$X_n = \frac{H_n}{\|H_n\|_2^2} \quad \text{and} \quad x_n = \|X_n\| = \|X_n\|_1 = \left(\frac{4}{3}\right)^{n-1}$$

and let $S_{n-1} \subset \{h_n > 0\}$. Set

$$x_{n-1} = \|X_n - P \left( \frac{G_{n-1}}{\|G_{n-1}\|_2^2} \right) \| \wedge \frac{X_n}{2} + P \left( \frac{G_{n-1}}{\|G_{n-1}\|_2^2} \right) + \frac{H_n}{\|H_{n-1}\|_2^2} \|.$$  

Averaging the two vectors above and using convexity of $\| \cdot \|,$

$$x_{n-1} \geq \frac{3}{4} x_n + \frac{1}{2} \frac{H_n}{\|H_{n-1}\|_2^2} \|$$

$$= \frac{3}{4} \|X_n\| + \frac{1}{2} \frac{H_n}{\|H_{n-1}\|_2^2} \| \quad \frac{1}{4}$$

$$= \frac{3}{4} x_n + \frac{1}{2} \left( \frac{4}{3} \right)^{n-2}$$
by (6). If
\[ x_{n-1} = \frac{X_n}{2} + P \left( \frac{G_{n-1}}{\|G_{n-1}\|_2} \right) + \frac{H_{n-1}}{\|H_{n-1}\|_2}, \]
set
\[ X_{n-1} = \frac{X_n}{2} + P \left( \frac{G_{n-1}}{\|G_{n-1}\|_2} \right) + \frac{H_{n-1}}{\|H_{n-1}\|_2^2} \]
and choose \( S_{n-2} \subset \{ H_{n-1} > 0 \} \). Otherwise, set
\[ X_{n-1} = X_n - P \left( \frac{G_{n-1}}{\|G_{n-1}\|_2} \right) \]
and choose \( S_{n-2} \subset S_{n-1} \) disjoint from \( \text{supp}(H_{n-1}) \).

We now describe the inductive step which is similar to the case \( j = n - 1 \). Suppose
that \( 1 \leq j < n - 1 \) and that \( S_i, X_i \) and \( x_i \) have been defined for \( i = j + 1, \ldots, n \) with
\( S_j \subset S_{j+1} \subset \cdots \subset S_n \) and with \( X_i \in \text{span}\{ H_k : i \leq k \leq n \} \). Set \( x_i = \|X_i\| \)

Let \( G_j \) and \( H_j \) be the cut and cycle vectors whose support sub-\( \mathcal{L}_j \) is \( S_j \). Note that,
by Lemma 13,
\[ P(G_j) \in \text{span}\{ H_i : j + 1 \leq i \leq n \} \]
and hence
\[ x_j = \|X_{j+1} - P \left( \frac{G_j}{\|G_j\|_2} \right) \| = \frac{3}{4} X_{j+1} + P \left( \frac{G_j}{\|G_j\|_2} \right) + \frac{H_j}{\|H_j\|_2^2} \]
is well-defined. Moreover, by convexity,
\[ x_j \geq \frac{3}{4} x_{j+1} + \frac{1}{2} \frac{H_j}{\|H_j\|_2^2} \]
\[ = \frac{3}{4} x_{j+1} + \frac{1}{2} \frac{\|H_j\|_1}{\|H_j\|_2^2} \]
(since \( X_{j+1} \in \text{span}\{ H_k : j + 1 \leq k \leq n \} \))
\[ = \frac{3}{4} x_{j+1} + \frac{1}{2} \left( \frac{4}{3} \right)^{j-1}. \]

If
\[ x_j = \frac{X_{j+1}}{2} + P \left( \frac{G_j}{\|G_j\|_2} \right) + \frac{H_j}{\|H_j\|_2^2}, \]
set

\[ X_j = \frac{X_{j+1}}{2} + P \left( \frac{G_j}{\|G_j\|_2^2} \right) + \frac{H_j}{\|H_j\|_2^2} \]

and choose \( S_{j-1} \subset \{H_j > 0\} \). Otherwise, set

\[ X_j = X_{j+1} - P \left( \frac{G_j}{\|G_j\|_2^2} \right) \]

and choose \( S_{j-1} \subset S_j \) disjoint from \( \text{supp}(H_j) \). Note that in both cases we have \( X_j \in \text{span}\{H_k : j \leq k \leq n\} \) as required. This completes the inductive definition.

Note that \( S_0 = \{e\} \) for some edge vector \( e \). Moreover, using Lemma 6 we can combine both cases to obtain, for \( 1 \leq j \leq n-1 \),

\[ X_j = \left( \frac{1}{2} \right)^{\varepsilon_j} X_{j+1} + \text{sgn}(\langle e, G_j \rangle) \frac{P_n(G_j)}{\|G_j\|_2^2} + \text{sgn}(\langle e, H_j \rangle) \frac{H_j}{\|H_j\|_2^2}, \]

where

\[ \varepsilon_j = \begin{cases} 1, & S_{j-1} \subset \text{supp}(H_j), \\ 0, & S_{j-1} \cap \text{supp}(H_j) = \emptyset. \end{cases} \]

Arguing as in the proof of Lemma 9 it follows that

\[ X_1 = \sum_{j=1}^{n-1} \left[ \langle e, G_j \rangle \frac{P(G_j)}{\|G_j\|_2^2} + \langle e, H_j \rangle \frac{H_j}{\|H_j\|_2^2} + \langle e, H_n \rangle \frac{H_n}{\|H_n\|_2^2} \right] \]

\[ = P \left( \sum_{j=1}^{n-1} \left[ \langle e, G_j \rangle \frac{G_j}{\|G_j\|_2^2 \|G_j\|_2} + \langle e, H_j \rangle \frac{H_j}{\|H_j\|_2^2 \|H_j\|_2} + \langle e, H_n \rangle \frac{H_n}{\|H_n\|_2^2 \|H_n\|_2} \right] \right) \]

\[ = P(e) \]

To see this, note that \( P(g_n) = 0 \) by Lemma 13 and \( P(g) = 0 \) by Lemma 12 unless \( g \) is a special cut vector of the form \( g_j \) for some sub-L. Note also that if \( h \) is of the form \( h_j \) and \( g \) is of the form \( g_j \) for some sub-L, then \( \langle e, h \rangle \neq 0 \) only if \( h = H_j \) \((1 \leq j \leq n)\) and \( \langle e, g \rangle \neq 0 \) only if \( g = G_j \) \((1 \leq j \leq n)\). So the above expression for \( X_1 \) is simply \( P \) applied to the expansion of \( e \) with respect to the orthogonal basis of \( E_n \).

Finally,

\[ \|P\|_1 \geq \|P(e)\|_1 = \|X_1\|_1 \geq \frac{3}{4} \|X_1\| = \frac{3}{4} x_1 \geq \frac{3}{4} \left( \frac{n+1}{2} \right). \]
The last inequality follows from the solution of the recurrence in Lemma 10 since
\[ x_j \geq \frac{3}{4} x_{j+1} + \frac{1}{2} \left( \frac{4}{3} \right)^{j-1}, \quad x_n = \left( \frac{4}{3} \right)^{n-1}. \]
\[ \square \]

5 Applications to the transportation cost space of $\mathcal{L}_n$

**Theorem 16** The projection constant of $\text{Lip}_0(\mathcal{L}_n)$ satisfies
\[ \frac{3n - 5}{8} \leq \lambda(\text{Lip}_0(\mathcal{L}_n)) \leq \frac{n + 3}{2}. \]

**Proof** Note that $\text{Lip}_0(\mathcal{L}_n) = (\text{TC}(\mathcal{L}_n))^*$ is isometrically isomorphic to $(C_n, \|\cdot\|_\infty) \subset (E_n, \|\cdot\|_\infty)$ by (3), since $C_n = Z_n^\perp$. Let $P_n$ be the projection from $(E_n, \|\cdot\|_1)$ onto $Z_n$ constructed in Sect. 3. Then $I - P_n^*$ is a projection from $(E_n, \|\cdot\|_\infty)$ onto $Z_n^\perp = C_n$. Thus,
\[ \lambda(\text{Lip}_0(\mathcal{L}_n)) \leq \|I - P_n^*\| \leq 1 + \|P_n\| \leq 1 + \frac{n + 1}{2} = \frac{n + 3}{2}. \]

Now suppose $Q$ is any projection from $(E_n, \|\cdot\|_\infty)$ onto $C_n$. Then $I - Q^*$ is a projection from $(E_n, \|\cdot\|_1)$ onto $Z_n$. So, by Theorem 15,
\[ \|Q\| \geq \|I - Q^*\| - 1 \geq \frac{3}{8} (n + 1) - 1 = \frac{3n - 5}{8}. \]

So $\lambda(\text{Lip}_0(\mathcal{L}_n)) \geq (3n - 5)/8$. \[ \square \]

**Corollary 17** The Banach–Mazur distance from $\text{TC}(\mathcal{L}_n)$ to $\ell_1^N$, where $N(n) = (4 \cdot 6^n + 1)/5$ is the dimension of $\text{TC}(\mathcal{L}_n)$, satisfies
\[ d_{BM}(\text{TC}(\mathcal{L}_n), \ell_1^N) \geq (3n - 5)/8. \]

**Proof** By duality,
\[ d_{BM}(\text{TC}(\mathcal{L}_n), \ell_1^N) = d_{BM}(\text{Lip}_0(\mathcal{L}_n), \ell_\infty^N) \geq \lambda(\text{Lip}_0(\mathcal{L}_n)) \geq \frac{3n - 5}{8}. \]
\[ \square \]

**Remark 18** The interpretation of this corollary in terms of transportation costs is as follows. For each $1 \leq j \leq N$, let $x_j$ be any transportation plan on $\mathcal{L}_n$ of unit cost. Then there exists an absolutely convex combination $\sum_{j=1}^N a_j x_j (\sum_{j=1}^N |a_j| = 1)$ such
that
\[ \| \sum_{j=1}^{N} a_j x_j \|_{TC} \leq \frac{8}{3n-5} \quad (n \geq 2). \]

In contrast to the diamond graphs \( D_n \) [8, Theorem 6.5], we have not been able to prove a good upper bound for the Banach–Mazur distance from \( TC(\mathcal{L}_n) \) to \( \ell_1^N \). However, we have the following matching upper bound for a linear embedding of \( TC(\mathcal{L}_n) \) into \( \ell_1 \).

**Corollary 19** There exists \( X_n \subset (E_n, \|\cdot\|_1) \) such that \( d_{BM}(TC(\mathcal{L}_n), X_n) \leq \frac{n+3}{2} \).

**Proof** Let \( P_n \) be the projection constructed in Sect. 3. Then, setting \( X_n = \ker P_n \), Theorem 11 yields
\[ d_{BM}(TC(\mathcal{L}_n), X_n) = d_{BM}((E_n/Z_n, \|\cdot\|_1), X_n) \leq \| I - P_n \|_1 \leq \frac{n+3}{2}. \]

\[ \square \]

**Remark 20** Actually, as we remarked in the Introduction, for every finite metric space \( X, TC(X) \) admits a linear embedding into \( L_1[0, 1] \) with distortion \( \leq C \ln |X| \), see [5,9,13]. Corollary 19 is just a slightly more precise statement of this fact for \( TC(\mathcal{L}_n) \).

For the diamond graph \( D_n \), the transportation cost space \( TC(D_n) \) has a natural monotone Schauder basis which leads to a matching upper bound for the Banach–Mazur distance. The difficulty in obtaining the same result for \( TC(\mathcal{L}_n) \) stems from the fact that the orthogonal basis of \( C_n \) constructed above is not a Schauder basis in the \( TC(\mathcal{L}_n) \) norm. In fact, the collection of special cut vectors \( g_j \) in \( (C_n, \|\cdot\|_{TC}) \) does not admit a bounded biorthogonal system (uniformly in \( n \)).

To make this precise, for each \( 1 \leq j \leq n-1 \), let \( g_i^j \) \( (1 \leq i \leq 6^{n-j}) \) be an enumeration of the \( 6^{n-j} \) basis vectors supported on a sub-\( \mathcal{L}_j \). Note that \( TC(\mathcal{L}_n) \) is isometrically isomorphic to \( (C_n, \|\cdot\|_{TC}) \), where \( \|\cdot\|_{TC} \) denotes the quotient norm of \( (E_n, \|\cdot\|_1)/Z_n \).

**Proposition 21** Suppose \( g_n^* \in (C_n, \|\cdot\|_\infty) \) satisfies
\[ g_n^*(g_n) = \| g_n \|_{TC} \quad \text{and} \quad g_n^*(g_i^j) = 0 \quad (1 \leq j \leq n-1, 1 \leq i \leq 6^{n-j}). \]

Then \( \| g_n^* \|_{\infty} \geq (4/3)^{n-1} \).

**Proof** Note that \( \| g_n \|_{TC} = \| g_n \|_1 \). This follows easily from convexity since each \( h \in Z_n \) has a symmetric distribution relative to \( g_n \) (see Fig. 6) and so \( \| g_n + h \|_1 \geq \| g_n \|_1 \). (In fact, one can show that \( \| g_i^j \|_{TC} = \| g_i^j \|_1 \) for all \( i, j \) but this is not needed for the proof.) Note also that (see Figs. 5 and 6)
\[ \| f_n - \frac{1}{2} g_n \|_1 = \frac{3}{4} \| f_n \|_1. \]
Applying this to each sub-$\mathcal{L}_{n-1}$ of $\mathcal{L}_n$ (see Fig. 6) gives

$$\|g_n - \frac{1}{2} \sum_i \varepsilon_{n-1}^i g_{n-1}^i \|_1 = \frac{3}{4} \|g_n\|_1$$

for some choice of signs $\varepsilon_{n-1}^i = \pm 1$. Repeating this argument, we get

$$\left\| g_n - \frac{1}{2} \left[ \sum_{j=1}^{n-1} \left( \frac{3}{2} \right)^{n-1-j} \left( \sum_i \varepsilon_j^i g_j^i \right) \right] \right\|_1 = \left( \frac{3}{4} \right)^{n-k} \|g_n\|_1$$

for some choice of $\varepsilon_j^i \in \{-1, 0, 1\}$. In general, we get for each $1 \leq k \leq n-1$,

$$\left\| g_n - \frac{1}{2} \left[ \sum_{j=k}^{n-1} \left( \frac{3}{2} \right)^{n-1-j} \left( \sum_i \varepsilon_j^i g_j^i \right) \right] \right\|_1 = \left( \frac{3}{4} \right)^{n-k} \|g_n\|_1$$

for some choice of $\varepsilon_j^i \in \{-1, 0, 1\}$. Hence

$$\|g_n - \frac{1}{2} \left[ \sum_{j=1}^{n-1} \left( \frac{3}{2} \right)^{n-1-j} \left( \sum_i \varepsilon_j^i g_j^i \right) \right] \|_{TC} \leq \left( \frac{3}{4} \right)^{n-1} \|g_n\|_1 = \left( \frac{3}{4} \right)^{n-1} \|g_n\|_{TC}.$$  \hspace{1cm} (11)

The desired result follows. \hfill \Box

### Remark 22

The proof shows that the collection of special cut vectors $g_j$ does not admit a bounded biorthogonal system (uniformly in $n$) for its span in $(C_n, \| \cdot \|_1)$. In particular, the orthogonal basis of $C_n$ constructed above is not a Schauder basis (uniformly in $n$) in $(C_n, \| \cdot \|_1)$.

Moreover, (11) show that the equivalence constant of the basis of $\| \cdot \|_1$-normalized (or $\| \cdot \|_{TC}$-normalized) special cut vectors with the unit vector basis of $\ell_1$ is at least $(4/3)^{n-1}$.

On the other hand, the orthogonal basis of $Z_n$ constructed above is a monotone Schauder basis for $(Z_n, \| \cdot \|_1)$. This allows an estimate from above for $d_{BM}(Z_n, \ell_1^N)$.

### Proposition 23

$$d((Z_n, \| \cdot \|_1), \ell_1^N) \leq 2n,$$

where $N = \dim(Z_n) = (6^n - 1)/5$.

**Proof** For $1 \leq j \leq n$, let $H_j = (h_j^i)_{i=1}^{6^n-j}$ be an enumeration of the $Z_n$ basis vectors of the form $h_j$ for some sub-$\mathcal{L}_j$. Since each $h_j^i$ is symmetric on its support sub-$\mathcal{L}_j$ it follows by convexity that $(\delta_{j=0}^{n-1} H_{n-j})$ is a monotone basis of $(Z_n \| \cdot \|)$. Moreover, $\{h_j^i/\|h_j^i\|_1 : 1 \leq i \leq 6^n-j\}$ is 1-equivalent to the unit vector basis of $\ell_1^6$ since these vectors have disjoint supports. Let $x \in Z_n$ and write $x = \sum_{k=0}^{n-1} x_k$, where
In this section we sharpen some of the results of [8, Section 6].

6 Multi-branching diamond graphs

In this section we sharpen some of the results of [8, Section 6].

**Theorem 24**  For each $k \geq 2$ and $n \geq 1$,

$$\lambda(\text{Lip}_0(D_{n,k})) = \frac{2k - 2}{2k - 1} n^2 + \frac{4k^2 - 6k + 3}{(2k - 1)^2} + \frac{2k - 2}{(2k - 1)^2} (2k)^n.$$

In particular, for $k = 2$ and $n \geq 1$,

$$\lambda(\text{Lip}_0(D_n)) = \frac{2n}{3} + \frac{7}{9} + \frac{2}{9} 4^{-n}.$$

**Proof**  Let us recall the representation of $D_{n,k}$ used in [8]. We identify the edge space of $D_{n,k}$ with a subspace of $L_1[0,1]$ as follows. For $n = 1$ and $1 \leq j \leq k$ we identify the pair of edge vectors of the $j^{th}$ path of length 2 from the ‘top’ to the ‘bottom’ vertex with the $L_1$-normalized indicator functions $2k1_{(j-1)/k,(j-1)/(2k)}$ and $2k1_{(2j-1)/(2k),j/k}$. For $n \geq 2$, the edge space of $D_{n,k}$ is obtained from that of $D_{n,k-1}$ by subdividing the intervals corresponding to edge vectors of $D_{n,k-1}$ into $2k$ subintervals each of length $(2k)^{-n}$. Each of the $k$ consecutive disjoint pairs of $L_1$-normalized indicator functions of the subintervals corresponds to each pair of edge vectors of the $k$ paths of length 2 from the top and bottom vertices of the copy of $D_{1,k}$ which replaces the edge vector of $D_{n-1,k}$ corresponding to the interval of length $(2k)^{n-1}$ which is subdivided. We have now identified the edge vectors of $D_{n,k}$ with the $L_1$-normalized indicator functions

$$e_{n,j} = (2k)^{n}1_{((j-1)/(2k)^n,j/(2k)^n)} \quad (1 \leq j \leq (2k)^n).$$

A basis for the cycle space corresponds to the $L_\infty$-normalized system $\bigcup_{i=1}^{n} \{g_{i,j} : 1 \leq j \leq (2k)^{i-1}(k-1)\}$, where, setting $j = a(k-1) + b$ with $0 \leq a \leq (2k)^{i-1} - 1$ and $1 \leq b \leq k - 1$,

$$g_{i,j} = (2k)^{-i}(e_{i,a2^k+2b-1} + e_{i,a2^k+2b} - e_{i,a2^k+2b+1} - e_{i,a2^k+2b+2}).$$

For $k \geq 3$, note that $g_{i,j}$ overlaps with $g_{i,j+1}$ when $b \leq k - 2$, and hence this is not an orthogonal basis.

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An orthogonal basis for the cut space corresponds to the $L_\infty$-normalized system
\[\{h_0\} \cup \bigcup_{i=1}^n \{h_{i,j} : 1 \leq j \leq (2k)^i\},\]
where $h_0 = 1_{[0,1]}$, and
\[h_{i,j} = (2k)^{-i} (e_{i,2j-1} - e_{i,2j}).\]

Let $G$ be the group of automorphisms of the edge space generated by those automorphisms which interchange (by translations) the intervals $\{g_{i,j} > 0\}$ and $\{g_{i,j} < 0\}$ or the sets $\{h_{i,j} > 0\}$ and $\{h_{i,j} < 0\}$. Then (as observed in [8]) arguing as in Lemma 12, the orthogonal projection $P_{n,k}$ onto the cut space is the unique $G$-invariant projection onto the cut space. First, let us compute the $\| \cdot \|_1$-norm of $P_{n,k}$. Note that
\[P_{n,k}(e_{n,1}) = h_0 + \frac{1}{2} \sum_{i=1}^n (2k)^i h_{i,1}.\]

An elementary calculation which we omit yields
\[\|P_{n,k}(e_{n,1})\|_1 = \frac{2k - 2}{2k - 1} n + \frac{4k^2 - 6k + 3}{(2k - 1)^2} + \frac{2k - 2}{(2k - 1)^2} \frac{1}{(2k)^n}.\]

Now suppose $1 \leq j \leq (2n)^k$. For $1 \leq i \leq n$, let $\text{supp}(e_{n,j}) \subset \text{supp}(h_{i,r(i)})$. Then
\[P_{n,k}(e_{n,j}) = h_0 + \frac{1}{2} \sum_{i=1}^n \text{sgn}(\langle e_{n,j}, h_{i,r(i)} \rangle)(2k)^i h_{i,r(i)}.\]

So $P_{n,k}(e_{n,j})$ has the same distribution as $P_{n,k}(e_{n,1})$. In particular, $\|P_{n,k}(e_{n,j})\|_1 = \|P_{n,k}(e_{n,1})\|_1$. Hence
\[\|P_{n,k}\|_1 = \max_{1 \leq j \leq (2n)^k} \|P_{n,k}(e_{n,j})\|_1 = \|P_{n,k}(e_{n,1})\|_1.\]

Finally, since $P_{n,k}$ is the unique $G$-invariant projection onto the cut space and is self-adjoint,
\[\lambda(\text{Lip}(D_{n,k})) = \|P_{n,k}\|_\infty = \|P_{n,k}\|_1 = \frac{2k - 2}{2k - 1} n + \frac{4k^2 - 6k + 3}{(2k - 1)^2} + \frac{2k - 2}{(2k - 1)^2} \frac{1}{(2k)^n}.\]

As a corollary, we get an improvement on [8, Theorem 6.10].

**Corollary 25** For each $n \geq 1$ and $k \geq 2$, the Banach–Mazur distance $d_{n,k}$ from the transportation cost space $\text{TC}(D_{n,k})$ to the $\ell_1^N$ space of the same dimension satisfies
\[d_{n,k} \geq \frac{2k - 2}{2k - 1} n + \frac{4k^2 - 6k + 3}{(2k - 1)^2} + \frac{2k - 2}{(2k - 1)^2} \frac{1}{(2k)^n}.\]
7 Characterization of finite trees in terms of their transportation cost spaces

The following result is well known.

**Proposition 26** Let $M$ be a finite metric space with $n$ elements. The space $\text{TC}(M)$ is isometric to $\ell_1^{n-1}$ if and only if $M$ is a weighted tree (the weight of an edge is the distance between its ends) with its shortest path distance.

Apparently for finite metric spaces it is folklore. The earliest proof of the “if” part we are aware of is [10, Corollary 3.6]. Its more general version for infinite metric spaces was proved in [6]. Our goal is to give a direct proof of the “only if” part. A simple direct proof of the “if” part can be found in [8, Proposition 2.1].

**Proof** We suppose that $\text{TC}(M)$ is isometric to $\ell_1^{n-1}$ and prove that this implies that $T$ is isometric to a weighted tree.

We may and shall identify the metric space $M$ with a complete weighted graph, whose vertex set is $M$ and for which the weight of an edge is the distance between its ends. In such a case the metric of $M$ coincides with the weighted graph distance of this graph.

An edge $uv$ in this weighted graph is called *essential* if and only if $d(u, v) < d(u, w) + d(w, v)$ for every $w \in M \setminus \{u, v\}$, or, equivalently, if the weighted graph distance of this graph will change if the edge $uv$ is deleted.

It is well known (and easy to check) that for a finite metric space a vector $f$ is an extreme point of the unit ball of $\text{TC}(M)$ if and only if $f = (1_u - 1_v)/d(u, v)$ for some essential edge $uv$ in the described weighted graph (this result is known in a more general form [1], in which it is far from being easy).

Since $\ell_1^{n-1}$ has $(n-1)$ symmetric pairs of extreme points, we conclude that the weighted graph corresponding to $M$ has $(n-1)$ essential edges. Since it is clear that the set of essential edges has to connect the graph, we get that the set of essential edges in $M$ forms a spanning tree. Recalling the definition of essential edges, we derive that the metric of $M$ is the distance of the weighted tree formed by essential edges. \(\square\)

**Corollary 27** The space $\text{TC}(M)$ with $|M| = n$ has between $(n-1)$ and $\frac{n(n-1)}{2}$ symmetric pairs of extreme points and thus is a quotient of $\ell_1^d$ for $(n-1) \leq d \leq \frac{n(n-1)}{2}$.

**Proof** In fact, the number of essential edges in a weighted connected simple graph with $n$ vertices can be any number between $(n-1)$ and $\frac{n(n-1)}{2}$. This follows from the following easy observations: (a) All edges in an unweighted (equivalently, a weighted graph with all weights equal to 1) connected simple graph are essential, and the number of such edges can be any number between $(n-1)$ and $\frac{n(n-1)}{2}$. (b) Essential edges induce a connected spanning graph, and thus there should be at least $(n-1)$ of them. \(\square\)
8 Isometric copies of $\ell_3^\infty$ and $\ell_4^\infty$ in TC($M$) on finite metric spaces

One of the results of [17] is a construction of finite metric spaces for which TC($M$) contains isometric copies of $\ell_3^\infty$ and $\ell_4^\infty$. The goal of this last section is to provide a simpler constructions of such spaces. We show that

1. There exists a 6-point set $T$ such that TC($T$) contains $\ell_3^\infty$ isometrically.
2. There exists an 8-point set $F$ such that TC($F$) contains $\ell_4^\infty$ isometrically.

Below we describe the metric spaces and the transportation problems spanning $\ell_3^\infty$ and $\ell_4^\infty$, respectively. We leave it as an exercise the straightforward verification of the equality

$$\left\| \sum_{i=1}^{k} \theta_i f_i \right\| = 1$$

for $k = 3$ or $k = 4$, and $\theta_i = \pm 1$.

| Table 1 Distances | a | b | c | d | e | f |
|-------------------|---|---|---|---|---|---|
| a | 0 | 1 | 1 | 1 | 1/2 | 1/2 |
| b | 1 | 0 | 1 | 1 | 1/2 | 1/2 |
| c | 1 | 1 | 0 | 1 | 1/2 | 1/2 |
| d | 1 | 1 | 1 | 0 | 1/2 | 1/2 |
| e | 1/2 | 1/2 | 1/2 | 1/2 | 0 | 1 |
| f | 1/2 | 1/2 | 1/2 | 1/2 | 1 | 0 |

| Table 2 Values of transportation problems | a | b | c | d | e | f |
|------------------------------------------|---|---|---|---|---|---|
| $f_1$ | 1/2 | $-1/2$ | 1/2 | $-1/2$ | 0 | 0 |
| $f_2$ | 1/2 | 1/2 | $-1/2$ | $-1/2$ | 0 | 0 |
| $f_3$ | 0 | 0 | 0 | 0 | 1 | $-1$ |

| Table 3 Distances | a | b | c | d | e | f | g | h |
|-------------------|---|---|---|---|---|---|---|---|
| a | 0 | 1 | 1 | 1 | 1/2 | 1/2 | 1/2 | 1/2 |
| b | 1 | 0 | 1 | 1 | 1/2 | 1/2 | 1/2 | 1/2 |
| c | 1 | 1 | 0 | 1 | 1/2 | 1/2 | 1/2 | 1/2 |
| d | 1 | 1 | 1 | 0 | 1/2 | 1/2 | 1/2 | 1/2 |
| e | 1/2 | 1/2 | 1/2 | 1/2 | 0 | 1 | 1 | 1 |
| f | 1/2 | 1/2 | 1/2 | 1/2 | 1 | 0 | 1 | 1 |
| g | 1/2 | 1/2 | 1/2 | 1/2 | 1 | 1 | 0 | 0 |
| h | 1/2 | 1/2 | 1/2 | 1/2 | 1 | 1 | 1 | 0 |
The description of the metric space $T$ (Table 1):
The description of three transportation problems on $T$ spanning $\ell_3^\infty$ (Table 2):
The description of the metric space $F$ (Table 3):
The description of four transportation problems on $F$ spanning $\ell_4^\infty$ (Table 4):

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