Cascade Textures and SUSY $SO(10)$ GUT

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Abstract

We give texture analyses of cascade hierarchical mass matrices in supersymmetric $SO(10)$ grand unified theory. We embed cascade mass textures of the standard model fermion with right-handed neutrinos into the theory, which gives relations among the mass matrices of the fermions. The related phenomenologies, such as the lepton flavor violating processes and leptogenesis, are also investigated in addition to the PMNS mixing angles.

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1 Introduction

The neutrino oscillation experiments have suggested that there are two large mixing angles among three generations in the lepton sector while all mixing angles in the quark sector are small. It is known that the current experimental data of leptonic mixing angles \[^{[1]}\] is well approximated by the tri-bimaximal mixing \[^{[2]}\], which is given by

\[
V_{TB} = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\end{pmatrix}.
\]

Such suggestive form of the generation mixing gives us a strong motivation to study a flavor structure of the lepton sector. Actually, there are a number of proposals based on a flavor symmetry to unravel it and related phenomenologies have been elaborated \[^{[3]}\].

It has been pointed out that the neutrino Dirac mass matrix of the cascade form can lead to the tri-bimaximal mixing at the leading order in the framework of type I seesaw mechanism \[^{[4]}\]. The mass matrix of the cascade form is parametrized by

\[
M_{\text{cas}} \simeq \begin{pmatrix}
\delta & \delta & \delta \\
\delta & \lambda & \lambda \\
\delta & \lambda & 1 \\
\end{pmatrix} v, \quad \text{with} \quad |\delta| \ll |\lambda| \ll 1,
\]

and \(v\) denotes an overall mass scale. We call this kind of hierarchy and the matrix with such hierarchy, “cascade hierarchy” and “cascade matrix”, respectively. On the other hand, the down quark mass matrix of a different hierarchical form, which is

\[
M_{\text{hyb}} \simeq \begin{pmatrix}
\epsilon' & \delta' & \delta' \\
\delta' & \lambda' & \lambda' \\
\delta' & \lambda' & 1 \\
\end{pmatrix} v', \quad \text{with} \quad |\epsilon'| \ll |\delta'| \ll |\lambda'| \ll 1,
\]

can give realistic values of CKM matrix elements. The (1,1) element, \(\epsilon'\), of this matrix is smaller than all other elements but this hierarchical structure is close to the cascade form except for \(\epsilon'\). We call this type of hierarchy “hybrid cascade (H.C.) hierarchy”, and the matrix with such a hierarchy “hybrid cascade (H.C.) matrix”. The neutrino Dirac mass matrix of a cascade form gives nearly tri-bimaximal generation mixing and the down quark mass matrix of a H.C. form realizes the CKM structure. The fact gives us a strong motivation to comprehensively investigate the quark and lepton. Actually, a proposal to embed such cascade textures into a supersymmetric (SUSY) \(SU(5)\) grand unified theory and texture analyses have been presented \[^{[5]}\]. For comparison with a \(SU(5)\) case and its results, we investigate embedding cascade hierarchies into a SUSY \(SO(10)\) GUT in this paper, which is also one of fascinating grand unified models.

The paper is organized as follows: In section 2, we give a brief review of cascade hierarchies for the fermion masses and mixing angles. In section 3, we embed the cascade
hierarchies into the SUSY $SO(10)$ GUT. The texture analyses for the quark/lepton sectors are also given in the section. In section 4, we show some numerical analyses of our model. In section 5, we give a comment on the proton decay. Section 6 is devoted to the summary. Appendix A gives a discussion about constraints on the structure of right-handed neutrino mass matrix.

2 Cascade hierarchies for fermion mass matrices

In this section, we give a brief review of cascade hierarchies for mass matrices of the fermions. First we discuss the cascade textures for quark and lepton sectors independently. The study of cascades for the lepton sector has been discussed in [4]. Then a possible expansion of the study to quark sector was presented in [5], which was considered in a SUSY $SU(5)$ GUT. The work [4] has pointed out that the neutrino Dirac mass matrix of a cascade form can lead to the tri-bimaximal mixing at the leading order in the framework of type I seesaw mechanism. Since the tri-bimaximal structure can be almost induced from the neutrino sector, the mixing angles from the charged lepton sectors should be small [5]. This means that the form of charged lepton mass matrix can be taken as a cascade or H.C. because mixing angles for those textures are small enough. How about the quark sector? The CKM structure is almost determined by a structure of down quark mass matrix because of large mass hierarchies among up-type quarks. And it is known that the down quark mass matrix of a H.C. form can give the realistic CKM structure (e.g. see [4, 5]). The contributions from up-quark sector to the CKM mixing are automatically tiny. This means that the form of up quark mass matrix can be taken as a cascade or H.C.. Finally, we comment on the structure of right-handed Majorana mass matrix. The contribution from the right-handed Majorana mass matrix should be also small because a nearly tri-bimaximal mixing are almost induced from the neutrino Dirac mass matrix with the seesaw mechanism, which means it is possible to take the right-handed Majorana mass matrix as a cascade or H.C.. More detailed explanations about the above points including mass eigenvalues given from each mass matrix of cascade and H.C. is given in [5]. Here, we summarize the above discussions about possible structures of mass matrices of the fermions as,

\begin{align}
M_u &= \text{cascade or H.C. or small mixing matrix,} \\
M_d &= \text{H.C.,} \\
M_{\nu_D} &= \text{cascade,} \\
M_e &= \text{cascade or H.C. or small mixing matrix,} \\
M_R &= \text{cascade or H.C. or small mixing matrix,}
\end{align}
where \( M_u, M_d, M_{\nu D}, M_e, \) and \( M_R \) are mass matrices of up- and down-type quarks, neutrino Dirac, charged lepton, and right-handed neutrino, respectively.

Next, we comment on the cascade textures in a \( SU(5) \) case. The \( SU(5) \) GUT predicts a relation between mass matrices for the down-type quark and charged lepton,

\[
M_e \simeq M_d^T,
\]

(2.6)
due to an unification of matter contents. As discussed above, since only mass matrix of a H.C. form are allowed for \( M_d \) in the study of cascade texture, the mass matrix for charged lepton should also have the H.C. from. On the other hand, some hierarchical structure of the mass matrices for the up-type quark and right-handed neutrino are allowed as long as induced mixing angles from these matrices can be treated as collections for the CKM and PMNS structures, respectively. Therefore, we can parametrize the mass matrices of the cascade or H.C. form for the fermions as

\[
M_u \simeq \begin{pmatrix}
\epsilon_u & \delta_u & \delta_u \\
\delta_u & \lambda_u & \lambda_u \\
\delta_u & \lambda_u & 1
\end{pmatrix} v_u, \quad \text{with } \begin{cases}
|\epsilon_u| = |\delta_u| \ll |\lambda_u| \ll 1 : \text{cascade}, \\
|\epsilon_u| \ll |\delta_u| \ll |\lambda_u| \ll 1 : \text{H.C.},
\end{cases}
\]

(2.7)

\[
M_d \simeq \begin{pmatrix}
\epsilon_d & \delta_d & \delta_d \\
\delta_d & \lambda_d & \lambda_d \\
\delta_d & \lambda_d & 1
\end{pmatrix} \xi_d v_d, \quad \text{with } |\epsilon_d| \ll |\delta_d| \ll |\lambda_d| \ll 1 : \text{H.C.},
\]

(2.8)

\[
M_{\nu D} \simeq \begin{pmatrix}
\delta_{\nu} & \delta_{\nu} & \delta_{\nu} \\
\delta_{\nu} & \lambda_{\nu} & \lambda_{\nu} \\
\delta_{\nu} & \lambda_{\nu} & 1
\end{pmatrix} \xi_{\nu} v_u, \quad \text{with } |\delta_{\nu}| \ll |\lambda_{\nu}| \ll 1 : \text{cascade},
\]

(2.9)

\[
M_e \simeq \begin{pmatrix}
\epsilon_d & \delta_d & \delta_d \\
\delta_d & -3\lambda_d & \lambda_d \\
\delta_d & \lambda_d & 1
\end{pmatrix} \xi_d v_d, \quad \text{with } |\epsilon_d| \ll |\delta_d| \ll |\lambda_d| \ll 1 : \text{H.C.},
\]

(2.10)

without \( O(1) \) coefficients for all elements. Here \( v_u \) and \( v_d \) are vacuum expectation values (VEVs) of up- and down-type Higgs fields in a supersymmetric scenario, and the overall factor \( \xi_d \) and \( \xi_{\nu} \) could be small. We also notice that the Georgi-Jarlskog (GJ) factor \[6\] is introduced to mass ratio between the down-type quarks and charged leptons for each generation,

\[
\left( \frac{m_\tau}{m_b}, \frac{m_\mu}{m_s}, \frac{m_\tau}{m_d} \right) \sim \left( 1, \frac{1}{3} \right).
\]

(2.11)

3 Cascade hierarchies in \( SO(10) \) GUT

3.1 A SUSY \( SO(10) \) Model

We consider about embedding the (hybrid) cascade hierarchical mass matrices into \( SO(10) \) GUT in this paper. A simple \( SO(10) \) GUT predicts relations between mass matrices the
up-type quark and neutrino Dirac,

\[ M_u \simeq M_{\nu D}, \] (3.1)

in addition to the relation (2.6). As discussed above, since only the mass matrix of the H.C. form is allowed for \( M_d \), the mass matrix for the charged lepton should also have the H.C. form like in \( SU(5) \) case. For the up-type quark sector, a simple \( SO(10) \) case predicts a GUT relation of the mass matrices \( M_u \simeq M_{\nu D} \), and the up-type quark mass matrix \( M_u \) should be restricted to a cascade form because the cascade form of neutrino Dirac mass matrix is needed for generating the tri-bimaximal neutrino mixing at the leading order. For the structures of right-handed neutrino mass matrix, some arbitrary matrices are allowed as long as induced mixing angles can be treated as collections for the PMNS matrix.

To demonstrate the idea, we consider a simple SUSY \( SO(10) \) model, which the Standard Model (SM) fermions with the right-handed neutrino are included into the spinor 16-dimensional representation, \( \psi \). To give suitable fermion masses, we introduce the Higgs fields, i.e. two Higgs 10-plets, \( H_{1,2} \) and two Higgs 126-plets, \( \Delta_{1,2} \). There are several ways to break \( SO(10) \) down to the SM. Here, we consider a minimal framework where the breaking of \( SO(10) \) is achieved by the Higgs 210-plet \[ \Phi \], which breaks \( SO(10) \) down to Pati-Salam group: \( SU(4)_C \times SU(2)_L \times SU(2)_R \). We choose that the Pati-Salam group is broken further down to the SM via the VEV of the SM singlet component in \( \Delta_2 \) and the VEV also gives Majorana masses for the right-handed neutrinos. Since this singlet VEV gives the non-vanishing contribution to D-term in the superpotential resulting in the unwanted source of SUSY breaking at high energy (close to the GUT scale), we introduce a 126-plet, \( \Delta \), whose SM singlet component obtains the VEV to cancel the D-term contribution (for instance, see \[9, 10\]). Here we include two 10-plets, \( H_{1,2} \) because the mass matrices of the up-type quark and down-type quark have to be different in order to predict the correct CKM mixing angles, as well motivated from the previous discussion. Moreover, we also need one 126-plet, \( \Delta_1 \), in order to achieve the GJ relations (2.11), that is, to give the factor of \(-3\) in the (2,2), (2,3), and (3,2) elements of the charged lepton mass matrix with respect to that of the down quark mass matrix. Another 126-plet, \( \Delta_2 \), is introduced to generate the different texture for the right-handed neutrino masses and also break the Pati-Salam group to the SM. In our setup, there are six pairs of Higgs doublets, \( \phi_u = (H_{1,u}, H_{2,u'}, \Delta_{1,u}, \Delta_{2,u'}, \Delta_u, \Phi_u)^T \) and \( \phi_d = (H_{1,d}, H_{2,d'}, \Delta_{1,d}, \Delta_{2,d'}, \Delta_d, \Phi_d)^T \) with the mass term \( \phi_u M_H \phi_d^T \). Note the label \( u,d \) refer to the \( SU(2)_L \) doublet component with hypercharge \( \pm 1/2 \) within the GUT multiplet. The mass matrix \( M_H \) can be diagonalized by \( U_{\phi_u}^T M_H U_{\phi_d} \), which \( U_{\phi_u}, U_{\phi_d} \) are unitaty matrices acting on \( \phi_u \) and \( \phi_d \) respectively. In the diagonal basis, the Higgs fields are given by \( (\phi'_u)_\alpha = (U_{\phi_u}^*)_{\beta \alpha} (\phi_u)_{\beta} \) and \( (\phi'_d)_\alpha = (U_{\phi_d}^*)_{\beta \alpha} (\phi_d)_{\beta} \).

\* We note that the factor of \(-3\) can be obtained by the coupling of the Higgs 120 or 126, see for instance [11].
For the sake of the study, we will not specify how $SO(10)$ is broken in detail, but by some doublet-triplet splitting mechanism (for instance see [12, 13, 14]) we will assume that $H_u = (\phi'_u)_1$ and $H_d = (\phi'_d)_1$ have mass at the electroweak scale while the others are so heavy and decoupled from the low energy theory. The Higgs fields, $H_{u,d}$ are the two Higgs doublets of the Minimal Supersymmetric Standard Model (MSSM).

The superpotential of the model is given by

$$W_Y = \tilde{Y}_1^{10}\psi H_1 \psi + \tilde{Y}_2^{10}\psi H_2 \psi + \tilde{Y}_1^{126}\psi \bar{\Delta}_1 \psi + \tilde{Y}_2^{126}\psi \bar{\Delta}_2 \psi ,$$  \hspace{1cm} (3.2)

which can be written in terms of the SM components as follows [15]:

$$W_Y \supset Q(\tilde{Y}_1^{10}H_{1,u} + \tilde{Y}_2^{10}H_{2,u} + \tilde{Y}_1^{126}\Delta_{1,u} + \tilde{Y}_2^{126}\Delta_{2,u})U^c$$
$$+ L(\tilde{Y}_1^{10}H_{1,d} + \tilde{Y}_2^{10}H_{2,d} + \tilde{Y}_1^{126}\Delta_{1,d} + \tilde{Y}_2^{126}\Delta_{2,d})N$$
$$+ (\tilde{Y}_1^{10}H_{1,T} + \tilde{Y}_2^{10}H_{2,T})E^c$$
$$+ L(\tilde{Y}_1^{10}H_{1,T} + \tilde{Y}_2^{10}H_{2,T} - 3\tilde{Y}_1^{126}\Delta_{1,T} - 3\tilde{Y}_2^{126}\Delta_{2,T})E^c ,$$  \hspace{1cm} (3.3)

where the doublet component in the GUT multiplet can be written in term of the MSSM Higgs doublets as $(\phi_u)_\alpha = (U_{\phi_u})_\alpha H_u$ and $(\phi_d)_\alpha = (U_{\phi_d})_\alpha H_d$.

For the neutrino sector, we assume that the $SU(2)_L$ triplet component, $\bar{\Delta}_{2,T}$, and the SM singlet component, $\bar{\Delta}_{2,S}$, in $\bar{\Delta}_2$, give tiny Majorana masses for the left-handed neutrinos and the heavy Majorana masses for the right-handed neutrinos respectively. This results in the seesaw formula as follow:

$$M_\nu = M_{LL} - M_{\nu D}M_R^{-1}M_{\nu D} ,$$  \hspace{1cm} (3.4)

where $M_{LL} = \tilde{Y}_2^{126}\langle \bar{\Delta}_{2,T} \rangle = \tilde{Y}_2^{126}v_L$, $M_R = \tilde{Y}_2^{126}\langle \bar{\Delta}_{2,S} \rangle = \tilde{Y}_2^{126}v_R$ and $M_{\nu D}$ is the Dirac mass term whose structure will be discussed below. Since the triplet VEV $\langle \bar{\Delta}_{2,T} \rangle = v_L$ depends on parameters in Higgs superpotential (for instance, see [9]), we assume that the VEV is tiny such that the second term in Eq (3.4) dominates, resulting in the type I seesaw dominance. Note that the singlet VEV, $\langle \bar{\Delta}_{2,S} \rangle = v_R$, is of order $10^{16}$ GeV.

After the electroweak symmetry is broken via the doublet VEVs, $\langle H_{u,d} \rangle = v_{u,d}$, the fermion masses are given by

$$M_u \simeq (U_{\phi_u})_{11}\tilde{Y}_1^{10}v_u = Y_1^{10}v_u$$
$$M_{\nu D} \simeq (U_{\phi_u})_{11}\tilde{Y}_1^{10}v_u = Y_1^{10}v_u$$
$$M_d \simeq ((U_{\phi_d})_{21}\tilde{Y}_2^{10} + (U_{\phi_d})_{31}\tilde{Y}_1^{126})v_d = (Y_2^{10} + Y_1^{126})v_d$$
$$M_e \simeq ((U_{\phi_d})_{21}\tilde{Y}_2^{10} - 3(U_{\phi_d})_{31}\tilde{Y}_1^{126})v_d = (Y_2^{10} - 3Y_1^{126})v_d ,$$  \hspace{1cm} (3.5) \hspace{1cm} (3.6) \hspace{1cm} (3.7) \hspace{1cm} (3.8)

where we assume that the main contribution for the up-type quark (Dirac neutrino) masses comes from the coupling to $H_1$ while for the down-type quark (charged lepton) masses
they arise from the $H_2$ and $\bar{\Delta}_1$ couplings. These can be achieved through the following assumptions: $(U_{\phi_u})_{11} \gg (U_{\phi_u})_{i1}$ and $(U_{\phi_d})_{21}, (U_{\phi_d})_{31} \gg (U_{\phi_d})_{i1}$. The Yukawa couplings are defined as $Y_{10}^{10} = (U_{\phi_u})_{11} \tilde{Y}_{10}^{10}$, $Y_{10}^{126} = (U_{\phi_d})_{21} \tilde{Y}_{126}^{10}$ and $Y_{126}^{126} = (U_{\phi_d})_{31} \tilde{Y}_{126}^{126}$. We impose the hierarchical forms to the Yukawa couplings,

$$Y_{10}^{10} \approx \begin{pmatrix} \delta_u & \delta_u & \delta_u \\ \delta_u & \lambda_u & \lambda_u \\ \delta_u & \lambda_u & 1 \end{pmatrix}, \quad \text{with } |\delta_u| \ll |\lambda_u| \ll 1,$$

$$Y_{10}^{126} \approx \begin{pmatrix} \epsilon_d & \delta_d & \delta_d \\ \delta_d & \delta_d & \delta_d \\ \delta_d & \delta_d & 1 \end{pmatrix}, $$

$$Y_{126}^{126} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_d & \lambda_d \\ 0 & \lambda_d & \lambda_d \end{pmatrix}, \quad \text{with } |\epsilon_d| \ll |\delta_d| \ll |\lambda_d| \ll 1.$$  

(3.9) (3.10) (3.11)

The structure of $\tilde{Y}_{2}^{126}$ will be discussed in the following sections in term of the right-handed neutrino mass matrix, $M_R = \tilde{Y}_{2}^{126} v_R$. These lead to the fermion mass matrices at the GUT scale as

$$M_u \approx \begin{pmatrix} \delta_u & \delta_u & \delta_u \\ \delta_u & \lambda_u & \lambda_u \\ \delta_u & \lambda_u & 1 \end{pmatrix} v_u,$$

$$M_{\nu D} \approx \begin{pmatrix} \delta_u & \delta_u & \delta_u \\ \delta_u & \lambda_u & \lambda_u \\ \delta_u & \lambda_u & 1 \end{pmatrix} v_u, \quad \text{with } |\delta_u| \ll |\lambda_u| \ll 1 : \text{cascade},$$

$$M_d \approx \begin{pmatrix} \epsilon_d & \delta_d & \delta_d \\ \delta_d & \lambda_d & \lambda_d \\ \delta_d & \lambda_d & 1 \end{pmatrix} \xi_d v_d,$$

$$M_e \approx \begin{pmatrix} \epsilon_d & \delta_d & \delta_d \\ \delta_d & -3\lambda_d & -3\lambda_d \\ \delta_d & -3\lambda_d & 1 \end{pmatrix} \xi_d v_d, \quad \text{with } |\epsilon_d| \ll |\delta_d| \ll |\lambda_d| \ll 1 : \text{H.C.},$$

(3.12) (3.13) (3.14) (3.15)

where $O(1)$ coefficients for all elements have been dropped.

### 3.2 Cabibbo fitting of cascade mass matrices

The cascade hierarchical parameters are determined by observed values. It is naturally expected that such hierarchies are originated from a symmetry and/or some dynamics in a high energy regime rather than solely determined by the magnitudes of Yukawa couplings. Although the origin of the hierarchies is not specified in the analysis, one can estimate and study the relative magnitudes of the hierarchies introducing a small parameter in the mass matrices. In the following, we choose the Cabibbo angle, $\sin \theta_c \approx \lambda = 0.227$, as a fitting parameter, and study significant implications of the cascade $SO(10)$ scenario. Then we
have

$$\lambda_u \simeq 0.87 \times \lambda^4, \quad \delta_u \simeq 0.85 \times \lambda^8,$$

(3.16)

for up-quark mass matrix of the cascade form and

$$\lambda_d \simeq 0.35 \times \lambda^2, \quad \delta_d \simeq 0.35 \times \lambda^3,$$

(3.17)

for down-quark one of the H.C. form at GUT scale, where we utilized values of quark masses listed in [16].

Notice that the $\xi_d$ is a parameter, which determines a ratio between (3,3) element of Yukawa matrices for up- and down-type quarks, and thus, it is correlated with the tan $\beta$ as,

$$\tan \beta = \frac{v_u}{v_d} \simeq \begin{cases} m_t/m_b \sim \mathcal{O}(50) & \text{for } \xi_d \sim \lambda^0 \text{ [large]} \\ \lambda m_t/m_b \sim \mathcal{O}(10) & \text{for } \xi_d \sim \lambda^1 \text{ [moderate]} \\ \lambda^2 m_t/m_b \sim \mathcal{O}(1) & \text{for } \xi_d \sim \lambda^2 \text{ [small]} \end{cases}.$$

(3.18)

As the results we can write cascading textures at GUT scale as

$$M_u \simeq \begin{pmatrix} \lambda^8 & \lambda^8 & \lambda^8 \\ \lambda^8 & \lambda^4 & \lambda^4 \\ \lambda^8 & \lambda^4 & 1 \end{pmatrix} v_u,$$

(3.19)

$$M_d \simeq \begin{cases} \lambda^{k_d+3} & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \\ \lambda^{k_d+4} & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^4 & \lambda^3 & \lambda \\ \lambda^{k_d+5} & \lambda^5 & \lambda^5 \\ \lambda^5 & \lambda^4 & \lambda^4 \\ \lambda^5 & \lambda^4 & \lambda^2 \end{cases} v_d \text{ [large tan } \beta\text{]},$$

(3.20)

$$\text{[moderate tan } \beta\text{]},$$

where $k_d \geq 1$ is needed to obtain suitable mass eigenvalues after diagonalizing these matrices. It should be remembered that $M_e \sim M_d$ but the additional GJ factor $-3$ is multiplied to the (2,2), (2,3), and (3,2) elements of $M_e$ as discussed in the previous section.

### 3.3 Neutrino sector

Next, we consider the structure of neutrino mass matrices. In the cascade model [4, 5], cascade parameters are constrained as

$$\left| \frac{\delta_\nu}{\lambda_\nu} \right|^2 \ll \frac{\Delta m^2_{21}}{\Delta m^2_{31}} \simeq 3.19 \times 10^{-2} < \lambda^2,$$

(3.21)
in order to preserve the tri-bimaximal mixing at the leading order with
\[
\Delta m_{21}^2 = (7.695 \pm 0.645) \times 10^{-5} \text{ eV}^2, \quad (3.22)
\]
\[
|\Delta m_{31}^2| = 2.40^{+0.12}_{-0.11} \times 10^{-3} \text{ eV}^2. \quad (3.23)
\]

at the 3σ level [1]. Due to the \( SO(10) \) GUT relation \( M_{\nu D} \approx M_u \), the neutrino mass matrix can be parametrized as
\[
M_{\nu D} \approx \begin{pmatrix}
\lambda^8 & \lambda^8 & \lambda^8 \\
\lambda^8 & \lambda^4 & -\lambda^4 \\
\lambda^8 & -\lambda^4 & 1
\end{pmatrix} v_u, \quad (3.24)
\]
where we note that an opposite sign between (2,2) and (2,3) elements is experimentally required to obtain the tri-bimaximal mixing as commented in [4].

### 3.3.1 Diagonal \( M_R \) case

Let us discuss the case of a diagonal Majorana mass matrix of the right-handed neutrinos, \( M_R = \text{Diag}[\lambda^{x_1}, \lambda^{x_2}, 1]M \), where \( x_1 \geq x_2 \geq 0 \). The cascade model requires the normal mass hierarchy of light neutrino mass spectrum in order to realize a nearly tri-bimaximal mixing [4]. The mass eigenvalues can be estimated as
\[
m_1 \approx \frac{v_u^2}{6M} \equiv \bar{m}_1, \quad (3.25)
\]
\[
m_2 \approx \left( 3\lambda^{16-x_1} + \frac{1}{3} \right) \frac{v_u^2}{M} \equiv \bar{m}_2 + 2\bar{m}_1, \quad (3.26)
\]
\[
m_3 \approx \left( 2\lambda^{8-x_2} + \frac{1}{2} \right) \frac{v_u^2}{M} \equiv \bar{m}_3 + 3\bar{m}_3, \quad (3.27)
\]
with a leading order corrections of \( \mathcal{O}(\bar{m}_1) \). In order to understand the hierarchical structure of the mass matrix and the constraints on the cascade parameters, we write down the effective neutrino mass matrix as
\[
M_{\nu} \approx \frac{v_u^2}{M} \begin{pmatrix}
4 & -2 & -2 \\
-2 & 1 & 1 \\
-2 & 1 & 1
\end{pmatrix} + \frac{\lambda^{16-x_1} v_u^2}{M} \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} + \frac{\lambda^{8-x_2} v_u^2}{M} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{pmatrix}
\]
\[
+ \frac{v_u^2}{M} \begin{pmatrix}
-4 + \lambda^{16} & 2 - \lambda^{12} & 2 + \lambda^8 \\
2 - \lambda^{12} & -1 + \lambda^8 & -1 - \lambda^4 \\
2 + \lambda^8 & -1 - \lambda^4 & 0
\end{pmatrix} + \frac{\lambda^{8-x_2} v_u^2}{M} \begin{pmatrix}
\lambda^8 & \lambda^4 & -\lambda^4 \\
\lambda^4 & 0 & 0 \\
\lambda^4 & 0 & 0
\end{pmatrix}. \quad (3.28)
\]

We find that if the terms in the first and second lines are leading contributions, the tri-bimaximal mixing can be realized at the leading order. In order that the first term in the

\[†\] Since the Dirac mass matrix and the up-type quark mass matrix are constrained to have the same structure, this opposite sign is also imposed on the up-type quark mass matrix (3.19).
second line does not spoil the structures given in the first line, \( m_1 \ll m_2, m_3 \) is required. That is the reason why the neutrino mass spectrum in the cascade model should be the normal hierarchy. In the case, we can well approximated as

\[
m_2 \simeq \sqrt{\Delta m^2_{21}} \quad \text{and} \quad m_3 \simeq \sqrt{|\Delta m^2_{31}|}.
\] (3.29)

Now we can obtain the following four constraints on the cascade parameters: (i) The neutrino masses should satisfy \( m_1 \ll m_2 \). This means that \( x_1 \geq 17 \) for the parameters by utilizing (3.25) and (3.26). This constraint leads to small mass of the lightest right-handed neutrino as shown later. (ii) In order to be consistent with a experimental data for the neutrino mass squared difference as

\[
r \equiv \frac{\sqrt{\Delta m^2_{21}}}{\sqrt{|\Delta m^2_{31}|}} \simeq 0.18,
\] (3.30)

one should have a relation among the cascade parameters as \( x_1 - x_2 = 7 \) or 8, where we use the fact that \( \lambda \sim r \). (iii) We have a relation among the cascade parameters, light and heavy neutrino mass scales, that is,

\[
M \simeq \frac{\lambda^{8-x_2} v_u^2}{\sqrt{|\Delta m^2_{31}|}},
\] (3.31)

where \( m_3 \simeq \sqrt{|\Delta m^2_{31}|} \) is taken. (iv) The hierarchy \( m_2 \gg m_3 \lambda^4 \) is required in order that the second term in the last line of (3.28) does not spoil the democratic structure in the first line. This gives a constraint \( x_1 - x_2 \geq 5 \). The above four constraints restrict the neutrino Dirac mass matrix of the cascade form and the right-handed one of the diagonal form to textures presented in Tab.1 and 2. We find that the minimal model for the neutrino mass matrices is described by \((x_1, x_2) = (17, 10)\) given in Tab.1. In this case, mass spectrum of the right-handed neutrinos is estimated as

\[
(M_1, M_2, M_3) \sim (10^5, 10^{10}, 10^{16}) \text{ GeV}.
\] (3.32)

Here we comment on the predicted mixing angles from cascade model. The mixing angles of the cascade model deviate from the exact tri-bimaximal mixing angles even if
Table 1: The textures of the neutrino Dirac mass matrix of the cascade form and the right-handed neutrino Majorana one of the diagonal from constrained by the experimentally observed values of the neutrino masses with the condition $x_1 - x_2 = 7$.

| $x_1$ | $x_2$ | $M_{\nu D}/v_u$ | $M_R/M$ |
|-------|-------|-----------------|---------|
| 17    | 10    | $\begin{pmatrix} \lambda^8 & \lambda^8 & \lambda^8 \\ \lambda^8 & \lambda^4 & -\lambda^4 \\ \lambda^8 & -\lambda^4 & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^{17} & 0 & 0 \\ 0 & \lambda^{10} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| 18    | 11    | $\begin{pmatrix} \lambda^8 & \lambda^8 & \lambda^8 \\ \lambda^8 & \lambda^4 & -\lambda^4 \\ \lambda^8 & -\lambda^4 & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^{18} & 0 & 0 \\ 0 & \lambda^{11} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

The right-handed neutrino mass matrix is diagonal. The mixing angles can be estimated as

$$\sin^2 \theta_{12} \approx \left| \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}} \theta_{12}^{(1)} \right|^2$$

(3.33)

$$\sin^2 \theta_{23} \approx \left| -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \theta_{13}^{(1)} + \frac{1}{\sqrt{3}} \theta_{23}^{(1)} \right|^2$$

(3.34)

$$\sin^2 \theta_{13} \approx \left| \frac{2}{\sqrt{6}} \theta_{13}^{(1)} + \frac{1}{\sqrt{3}} \theta_{23}^{(1)} \right|^2$$

(3.35)

$$\sin^2 \theta_{13} \approx \left| \frac{\lambda^4}{\sqrt{2}} \frac{\bar{m}_3 - \frac{2}{3} \bar{m}_2}{\bar{m}_3 - \bar{m}_2} + \frac{\sqrt{2} \bar{m}_1 \bar{m}_2}{\bar{m}_3 (\bar{m}_3 - \bar{m}_2)} \right|^2$$

(3.36)

in a perturbative method,\(^{\ddagger}\) where parameters $\theta_{ij}^{(1)}$ indicate deviations from the exact tribimaximal mixing angles. These are elements of the following mixing matrix,

$$V^{(1)} \sim \begin{pmatrix} 1 & \theta_{12}^{(1)} & \theta_{13}^{(1)} \\ -\theta_{12}^{(1)} & 1 & \theta_{23}^{(1)} \\ -\theta_{13}^{(1)} & -\theta_{23}^{(1)} & 1 \end{pmatrix}.$$  

(3.39)

In our notation, the experimentally observed PMNS mixing matrix is given by $V_{\text{PMNS}} \approx V_{TB} V^{(1)} P_M$, where the $P_M$ is a diagonal phase matrix.

### 3.3.2 Non-diagonal $M_R$ case

We discuss the case of non-diagonal $M_R$, which is generically allowed in the context of the cascade textures. First, we define the diagonalized right-handed neutrino mass matrix,\(^{\ddagger}\)

\(^{\ddagger}\)See [5] for a detailed derivation.
Table 2: The textures of the neutrino Dirac mass matrix of the cascade form and the right-handed neutrino Majorana one of the diagonal from constrained by the experimentally observed values of the neutrino masses with the condition $x_1 - x_2 = 8$.

\[ D_R \equiv U_{\nu R}^T M_R U_{\nu R} \equiv \begin{pmatrix} \lambda^{x_1} & 0 & 0 \\ 0 & \lambda^{x_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} M \text{ with } x_1 \geq x_2 \geq 0, \quad (3.40) \]

where $M_R$ is a non-diagonal mass matrix for the right-handed neutrinos but mixing angles among each generation are assumed to be small in order to preserve the tri-bimaximal mixing. If the mixing angles among each generation of the right-handed neutrino are small enough, $U_{\nu R}$ can be written by

\[ U_{\nu R} \simeq \begin{pmatrix} 1 & \theta_{R,12} & \theta_{R,13} \\ -\theta_{R,12} & 1 & \theta_{R,23} \\ -\theta_{R,13} & -\theta_{R,23} & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & \lambda_{q_{12}} & \lambda_{q_{13}} \\ -\lambda_{q_{12}} & 1 & \lambda_{q_{23}} \\ -\lambda_{q_{13}} & -\lambda_{q_{23}} & 1 \end{pmatrix} \text{ with } q_{ij} \geq 1, \quad (3.41) \]
up to the first order of $\theta_{R,ij}$ ($i, j = 1 \sim 3$). After the seesaw mechanism, we obtain the Majorana mass matrix of light neutrinos in low-energy as,

$$M_\nu \simeq M_{\nu_D}^T M_R^{-1} M_{\nu_D}$$

$$\simeq \left[ \begin{array}{ccc}
\lambda^{16}(M^{-1}_R)_{11} & \lambda^8(M^{-1}_R)_{22} \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \right] + \left[ \begin{array}{ccc}
\lambda^8 & \lambda^4 & \lambda^4 \\
\lambda^4 & 0 & 0 \\
\lambda^4 & 0 & 0 \\
\end{array} \right] + \lambda^8(M^{-1}_R)_{22}$$

$$+ \left[ \begin{array}{ccc}
\lambda^{16} & -\lambda^{12} & \lambda^8 \\
-\lambda^{12} & \lambda^8 & -\lambda^4 \\
\lambda^8 & -\lambda^4 & 1 \\
\end{array} \right] + \left[ \begin{array}{ccc}
2\lambda^{16} & 0 & \lambda^8(1-\lambda^4) \\
0 & -2\lambda^8 & \lambda^4(1+\lambda^4) \\
\lambda^8(1-\lambda^4) & \lambda^4(1+\lambda^4) & -2\lambda^4 \\
\end{array} \right]$$

$$+ \lambda^8(M^{-1}_R)_{12} \left[ \begin{array}{ccc}
2\lambda^8 & \lambda^8+\lambda^4 & \lambda^8-\lambda^4 \\
\lambda^8+\lambda^4 & 2\lambda^4 & 0 \\
\lambda^8-\lambda^4 & 0 & -2\lambda^4 \\
\end{array} \right] + \lambda^8(M^{-1}_R)_{13} \left[ \begin{array}{ccc}
2\lambda^8 & \lambda^8-\lambda^4 & \lambda^8+1 \\
\lambda^8-\lambda^4 & -2\lambda^4 & 1-\lambda^4 \\
\lambda^8+1 & 1-\lambda^4 & 2 \\
\end{array} \right] \nu_u^2. \quad (3.42)
When we operate the $V_{\text{TB}}$ to $M_\nu$ as $V_{\text{TB}}^T M_\nu V_{\text{TB}}$, the neutrino mass matrix is

$$
\mathcal{M} \equiv V_{\text{TB}}^T M_\nu V_{\text{TB}} \\
\simeq 3\lambda^6 (M_R^{-1})_{11} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2\lambda^8 (M_R^{-1})_{22} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
+ \frac{\lambda^8 (M_R^{-1})_{22}}{3} \begin{pmatrix} 2\lambda^8 & \sqrt{2}\lambda^8 & -2\sqrt{3}\lambda^4 \\ \sqrt{2}\lambda^8 & \lambda^8 & -\sqrt{6}\lambda^4 \\ -2\sqrt{3}\lambda^4 & -\sqrt{6}\lambda^4 & 0 \end{pmatrix} \\
+ \frac{(M_R^{-1})_{33}}{6} \begin{pmatrix} c_1^2 & -\sqrt{2}c_1c_2 & -\sqrt{3}c_1c_+ \\ -\sqrt{2}c_1c_2 & 2c_2^2 & \sqrt{6}c_2c_+ \\ -\sqrt{3}c_1c_+ & \sqrt{6}c_2c_+ & 3c_+^2 \end{pmatrix} \\
+ \frac{(M_R^{-1})_{23}}{3\sqrt{2}} \begin{pmatrix} -2\sqrt{2}c_1\lambda^8 & c_3\lambda^8 & \sqrt{6}c_- (\lambda^4 + \lambda^8) \\ c_3\lambda^8 & 2\sqrt{2}c_2\lambda^8 & -\sqrt{3}c_- (2\lambda^4 - \lambda^8) \\ \sqrt{6}c_- (\lambda^4 + \lambda^8) & -\sqrt{3}c_- (2\lambda^4 - \lambda^8) & -6\sqrt{2}c_+\lambda^4 \end{pmatrix} \\
+ \frac{\lambda^8 (M_R^{-1})_{12}}{2} \begin{pmatrix} 0 & \sqrt{2}\lambda^8 & 0 \\ \sqrt{2}\lambda^8 & 2\lambda^8 & -\sqrt{6}\lambda^4 \\ 0 & -\sqrt{6}\lambda^4 & 0 \end{pmatrix} \\
+ \frac{\lambda^8 (M_R^{-1})_{13}}{\sqrt{2}} \begin{pmatrix} -c_1 & 0 & 0 \\ 0 & 2\sqrt{2}c_2 & \sqrt{3}c_+ \\ 0 & \sqrt{3}c_+ & 0 \end{pmatrix} \right] v_u^2,
$$

(3.43)

where

$$
c_1 \equiv 1 - \lambda^4 - 2\lambda^8, \quad c_2 \equiv 1 - \lambda^4 + \lambda^8, \quad c_3 \equiv 1 - \lambda^4 + 4\lambda^8, \quad c_\pm \equiv 1 \pm \lambda^4.
$$

(3.44)

This mass matrix can be rewritten by

$$
\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_{\text{off}} \equiv \mathcal{M}_0 + \begin{pmatrix} m^R_1 & m^R_{12} & m^R_{13} \\ m^R_{12} & m^R_2 & m^R_{23} \\ m^R_{13} & m^R_{23} & m^R_3 \end{pmatrix},
$$

(3.45)

where $\mathcal{M}_0$ comes from the diagonal elements of $M_R$, which is given by

$$
\mathcal{M}_0 \simeq \begin{pmatrix} \bar{m}_1 + \frac{\lambda^8}{\sqrt{6}} \bar{m}_3 & -\sqrt{2}\bar{m}_1 + \frac{\lambda^8}{\sqrt{6}} \bar{m}_3 & -\sqrt{3}\bar{m}_1 - \frac{\lambda^4}{\sqrt{6}} \bar{m}_3 \\ -\sqrt{2}\bar{m}_1 + \frac{\lambda^8}{\sqrt{6}} \bar{m}_3 & \bar{m}_2 + 2\bar{m}_1 + \frac{\lambda^8}{\sqrt{6}} \bar{m}_3 & \sqrt{6}\bar{m}_1 - \frac{\lambda^4}{\sqrt{6}} \bar{m}_3 \\ -\sqrt{3}\bar{m}_1 - \frac{\lambda^4}{\sqrt{3}} \bar{m}_3 & \sqrt{6}\bar{m}_1 - \frac{\lambda^4}{\sqrt{6}} \bar{m}_3 & \bar{m}_3 + 3\bar{m}_1 \end{pmatrix}.
$$

(3.46)

In (3.45) $\mathcal{M}_{\text{off}}$ has effects from the off-diagonal elements of $M_R$. In order to obtain experimentally accepted mixing structure without unnatural cancellations, we focus only on a case that the collections from the off-diagonal elements of $M_R$ are small enough not to spoil the nearly tri-bimaximal mixing constructed by the cascade neutrino Dirac mass matrix. This means that the resultant structure of neutrino mass matrix given in (3.43).
Table 3: The textures of the neutrino Dirac mass matrix of the cascade form and the right-handed neutrino Majorana one of the non-diagonal form. The mat rices are constrained by the experimentally observed values of the neutrino masses with the condition $x_1 - x_2 = 7$.

\[
\begin{array}{ccc}
 x_1 & x_2 & M_{\nu D}/v_u \\
 \hline
 17 & 10 & \begin{pmatrix}
 \lambda^8 & \lambda^8 & \lambda^8 \\
 \lambda^8 & -\lambda^4 & -\lambda^4 \\
 \lambda^8 & -\lambda^4 & 1
\end{pmatrix} \\
 18 & 11 & \begin{pmatrix}
 \lambda^8 & \lambda^8 & \lambda^8 \\
 \lambda^8 & -\lambda^4 & -\lambda^4 \\
 \lambda^8 & -\lambda^4 & 1
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{ccc}
 M_R/M \\
 \hline
 \begin{pmatrix}
 \lambda^7 & \lambda^{17} & \lambda^{10} \\
 \lambda^{17} & \lambda^{10} & \lambda^7 \\
 \lambda^{10} & \lambda^7 & 1
\end{pmatrix} \\
 \begin{pmatrix}
 \lambda^{18} & \lambda^{19} & \lambda^{11} \\
 \lambda^{19} & \lambda^{11} & \lambda^8 \\
 \lambda^{11} & \lambda^8 & 1
\end{pmatrix}
\end{array}
\]

should not be drastically differed from the (3.40), and thus the magnitude of neutrino mass eigenvalues (3.25)–(3.27) and the above four constraints should be satisfied at the leading order even in non-diagonal $M_R$ case. These discussions give the following neutrino mass eigenvalues up to the next leading order,

\[
m_1 \simeq \frac{v_u^2}{6M} + m_1^R = \bar{m}_1 + m_1^R, \quad (3.47)
\]

\[
m_2 \simeq \left(3\lambda^{16-x_1} + \frac{1}{3}\right) \frac{v_u^2}{M} + m_2^R = \bar{m}_2 + m_2^R + 2\bar{m}_1, \quad (3.48)
\]

\[
m_3 \simeq \left(2\lambda^{8-x_2} + \frac{1}{2}\right) \frac{v_u^2}{M} + m_3^R = \bar{m}_3 + m_3^R + 3\bar{m}_3, \quad (3.49)
\]

where $m_i^R$ include effects from the off-diagonal element of $M_R$ described by

\[
m_1^R \equiv \frac{v_u^2}{6M} \lambda^{-x_1} \theta_{R,23}, \quad (3.50)
\]

\[
m_2^R \equiv \frac{v_u^2}{M} (\lambda^{-x_2} \theta_{R,23}^2 - 2\lambda^{8-x_1} \theta_{R,13}), \quad (3.51)
\]

\[
m_3^R \equiv \frac{v_u^2}{2M} \lambda^{-x_1} (2\lambda^{4}\theta_{R,12} - \theta_{R,13})^2. \quad (3.52)
\]

Typical textures of non-diagonal $M_R$ are presented in Tabs. 3 and 4. The collections to the generation mixing angles are also estimated as

\[
\theta_{12}^{(1)} \simeq -\frac{\sqrt{2}\bar{m}_1 + m_{12}^R}{\bar{m}_2 + m_2^R}, \quad (3.53)
\]

\[
\theta_{23}^{(1)} \simeq -\frac{\sqrt{6}\bar{m}_1 - \lambda^4 \bar{m}_3 + m_{23}^R}{(\bar{m}_3 + m_3^R) - (\bar{m}_2 + m_2^R)}, \quad (3.54)
\]

\[
\theta_{13}^{(1)} \simeq -\frac{\sqrt{3}\bar{m}_1 - \lambda^4 \bar{m}_3 + m_{13}^R}{\bar{m}_3 + m_3^R}, \quad (3.55)
\]

\footnote{Detailed discussions is given in the Appendix.}
where
\[
m_{12}^R \simeq -\frac{1}{6\sqrt{2}}[\lambda^{-8}\theta_{R,23,3}^2\bar{m}_3 - 2(2\theta_{R,12} - \lambda^{-8}\theta_{R,13})\bar{m}_2],
\]
\[
m_{23}^R \simeq \frac{1}{\sqrt{6}}[\theta_{R,23,3}\bar{m}_3 + \lambda^{-8}(2\lambda^{d2}\theta_{R,12} - \theta_{R,13})(1 - \lambda^{-8}\theta_{R,13})\bar{m}_2],
\]
\[
m_{13}^R \simeq -\frac{1}{2\sqrt{3}} \left[ \frac{\lambda^{-4}}{2} \frac{(2 + \lambda^{-4}\theta_{R,23})\theta_{R,23,3}\bar{m}_3 - \frac{4}{3}\lambda^{-4}\theta_{R,12,3}\bar{m}_2}{\bar{m}_3(m_2 + m_2^R)} \right].
\]

Finally, the PMNS mixing angles including collections from the off-diagonal elements are

\[
\sin\theta_{12} \simeq \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}} \frac{-\bar{m}_1 + m_{12}^R}{\bar{m}_2 + m_{22}^R},
\]
\[
\sin\theta_{23} \simeq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{\bar{m}_1[3(\bar{m}_3 + m_{33}^R) - (\bar{m}_2 + m_{22}^R)]}{\sqrt{6} \bar{m}_3(m_2 + m_2^R)} + \frac{1}{\sqrt{3}} \frac{m_{13}^R}{m_{23}^R},
\]
\[
\sin\theta_{13} \simeq \frac{1}{\sqrt{2}} \frac{\bar{m}_3[(\bar{m}_3 + m_{33}^R) - \frac{2}{3}(\bar{m}_2 + m_{22}^R)]}{\sqrt{6} \bar{m}_3(m_2 + m_2^R)} + \frac{1}{\sqrt{3}} \frac{m_{13}^R}{m_{23}^R},
\]

Table 4: The textures of the neutrino Dirac mass matrix of the cascade form and the right-handed neutrino Majorana one of the diagonal from. The matrices constrained by the experimentally observed values of the neutrino masses with the condition \(x_1 - x_2 = 8\).

| \(x_1\) | \(x_2\) | \(M_{\nu D}/v_u\) | \(M_R/M\) |
|---|---|---|---|
| 17 | 9 | \(\begin{pmatrix} \lambda^8 & \lambda^8 & \lambda^8 \\ \lambda^8 & \lambda^4 & -\lambda^4 \\ \lambda^8 & -\lambda^4 & 1 \end{pmatrix}\) | \(\begin{pmatrix} \lambda^1 & \lambda^4 & \lambda^{10} \\ \lambda^1 & \lambda^9 & \lambda^6 \\ \lambda^{10} & \lambda^6 & 1 \end{pmatrix}\) |
| 18 | 10 | \(\begin{pmatrix} \lambda^8 & \lambda^8 & \lambda^8 \\ \lambda^8 & \lambda^4 & -\lambda^4 \\ \lambda^8 & -\lambda^4 & 1 \end{pmatrix}\) | \(\begin{pmatrix} \lambda^{18} & \lambda^{18} & \lambda^{10} \\ \lambda^{18} & \lambda^{10} & \lambda^7 \\ \lambda^{10} & \lambda^7 & 1 \end{pmatrix}\) |

3.4 Charged lepton and quark sectors

At the end of this section, we study the charged lepton and quark sectors. Under the condition of the \(SO(10)\) scenario, we examine quantitative features of the masses and mixing angles.
We take the charged lepton mass matrix as
\[
M_e \simeq \begin{pmatrix}
\epsilon_d & \delta_d & \delta_d \\
\delta_d & -3\lambda_d & -3\lambda_d \\
\delta_d & -3\lambda_d & 1
\end{pmatrix} \xi_d v_d,
\] (3.62)
in our study. The magnitudes of cascade parameters can be partially evaluated from the observed values of charged lepton masses, and are given by \(|\lambda_d| \simeq m_\mu/(3m_\tau)|d_\mu| \simeq 3\sqrt{m_e m_\mu/m_\tau}|d_\mu| \simeq 3\sqrt{m_e m_\mu/m_\tau}$. We find that the corrections from the charged lepton sector are generally small; the total leptonic mixing angles can be written as

\[
\sin \theta_{12} \simeq \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}} \frac{-\bar{m}_1 + m_{12}^R}{\bar{m}_2 + m_{22}^R} + \frac{3m_e}{m_\mu},
\] (3.63)
\[
\sin \theta_{23} \simeq -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{\bar{m}_1 [3(\bar{m}_3 + m_{33}^R) - (\bar{m}_2 + m_{22}^R)]}{\lambda_{d_1-d_2} \bar{m}_3 (\bar{m}_2 + m_{22}^R)} - \frac{1}{\sqrt{6}} \frac{m_{13}^R}{\bar{m}_3 + m_{33}^R} + \frac{1}{\sqrt{3}} \frac{m_{23}^R}{\bar{m}_3 + m_{33}^R} - \frac{m_{13}^R}{3m_\tau},
\] (3.64)
\[
\sin \theta_{13} \simeq -\frac{1}{\sqrt{2}} \frac{\bar{m}_1 [3(\bar{m}_3 + m_{33}^R) - (\bar{m}_2 + m_{22}^R)]}{\sqrt{2} \bar{m}_3 (\bar{m}_3 + m_{33}^R) - (\bar{m}_2 + m_{22}^R)} + \frac{2}{\sqrt{6}} \frac{m_{13}^R}{\bar{m}_3 + m_{33}^R} + \frac{1}{\sqrt{3}} \frac{m_{23}^R}{\bar{m}_3 + m_{33}^R} + \frac{2}{\sqrt{2}} \frac{m_e}{m_\mu},
\] (3.65)
at the first order of perturbations.

Next, we comment on the quark sector. One must remember that the mass matrix of the H.C. form is motivated for the mass spectra and mixing angles of quark sector. The mixing matrices for the up- and down-sector are given by the cascading mass matrices (3.19) and (3.20). From the mass matrices, one can estimate the following mixing angles

\[
V_d = \begin{pmatrix}
\mathcal{O}(1) & \mathcal{O}(\lambda) & \mathcal{O}(\lambda^3) \\
\mathcal{O}(\lambda) & \mathcal{O}(1) & \mathcal{O}(\lambda^2) \\
\mathcal{O}(\lambda^3) & \mathcal{O}(\lambda^2) & \mathcal{O}(1)
\end{pmatrix}, \quad V_u = \begin{pmatrix}
\mathcal{O}(1) & \mathcal{O}(\lambda^4) & \mathcal{O}(\lambda^8) \\
\mathcal{O}(\lambda^4) & \mathcal{O}(1) & \mathcal{O}(\lambda^4) \\
\mathcal{O}(\lambda^8) & \mathcal{O}(\lambda^4) & \mathcal{O}(1)
\end{pmatrix},
\] (3.66)
where \(V_d\) and \(V_u\) are unitary mixing matrices determined by \(M_u\) and \(M_d\). It can be easily seen from the structure of \(V_d\) that the experimentally observed values of CKM matrix can be realized at the leading order and the collections from the \(V_u\) are generally small. Detailed numerical calculations are given in the next section.
4 Phenomenologies

In this section, we perform the numerical study of phenomenologies based on the above analyses of the cascade textures for the quark and lepton sectors: the PMNS mixing angles, lepton flavor violation (LFV), baryon asymmetry of the Universe (BAU) via thermal leptogenesis.

4.1 PMNS mixing angles

Firstly, we show numerical analyses of the generation mixing angles of the quark and lepton sectors predicted from the cascade model. Here, we investigate two typical types of minimal texture for the neutrino Dirac and right-handed Majorana neutrino mass matrices,

\[
\text{Model I : } M_{\nu D} = \begin{pmatrix} c_{\nu} \lambda^8 & c_{\nu} \lambda^4 & c_{\nu} \lambda^8 \\ c_{\nu} \lambda^8 & b_{\nu} \lambda^4 & -b_{\nu} \lambda^4 \\ c_{\nu} \lambda^8 & -b_{\nu} \lambda^4 & a_{\nu} \end{pmatrix} v_u,
\]

\[
M_R = \begin{pmatrix} f_R \lambda^{17} & e_R \lambda^{17} & d_R \lambda^{10} \\ e_R \lambda^{17} & c_R \lambda^{10} & b_R \lambda^7 \\ d_R \lambda^{10} & b_R \lambda^7 & a_R \end{pmatrix} M,
\]

and

\[
\text{Model II : } M_{\nu D} = \begin{pmatrix} c_{\nu} \lambda^8 & c_{\nu} \lambda^8 & c_{\nu} \lambda^8 \\ c_{\nu} \lambda^8 & b_{\nu} \lambda^4 & -b_{\nu} \lambda^4 \\ c_{\nu} \lambda^8 & -b_{\nu} \lambda^4 & a_{\nu} \end{pmatrix} v_u,
\]

\[
M_R = \begin{pmatrix} f_R \lambda^{17} & e_R \lambda^{17} & d_R \lambda^{10} \\ e_R \lambda^{17} & c_R \lambda^{10} & b_R \lambda^6 \\ d_R \lambda^{10} & b_R \lambda^6 & a_R \end{pmatrix} M,
\]

for the cases of the condition \( x_1 - x_2 = 7 \). Here \( a_{\nu}, b_{\nu}, c_{\nu}, \) and \( a_R, \cdots, f_R \) are complex numbers whose absolute values are taken as \( 0.4 \sim 1.4 \). In both models, the following charged lepton, up and down quark mass matrices are utilized

\[
M_e = \begin{pmatrix} 0 & e_e \lambda^3 & d_e \lambda^3 \\ e_e \lambda^3 & -3c_e \lambda^2 & -3b_e \lambda^2 \\ d_e \lambda^3 & -3b_e \lambda^2 & a_e \end{pmatrix} \lambda v_d,
\]

and

\[
M_u = \begin{pmatrix} c_{\nu} \lambda^8 & c_{\nu} \lambda^8 & c_{\nu} \lambda^8 \\ c_{\nu} \lambda^8 & b_{\nu} \lambda^4 & -b_{\nu} \lambda^4 \\ c_{\nu} \lambda^8 & -b_{\nu} \lambda^4 & a_{\nu} \end{pmatrix} v_u, \quad M_d = \begin{pmatrix} 0 & e_e \lambda^3 & d_e \lambda^3 \\ e_e \lambda^3 & c_e \lambda^2 & b_e \lambda^2 \\ d_e \lambda^3 & b_e \lambda^2 & a_e \end{pmatrix} \lambda v_d,
\]

\*We focus on only the \( x_1 - x_2 = 7 \) case given in Tab. 3 as a typical example (Model I). In order to see effects from off-diagonal elements of right-handed neutrino mass matrix to the PMNS mixing angles, we also analyse a slightly different model for \( M_R \) as a comparison (Model II).
where \( a_e, \cdots, e_e \) are also complex values whose range are the same as \( a_\nu, b_\nu, c_\nu, \) and \( a_R, \cdots, f_R \). Moreover, notice that the mass matrices of down and up quarks have the same structures of ones of charged lepton, except for the GJ factor, and neutrino respectively because of the mass relations of \( SO(10) \) model at the GUT scale.

The results of numerical calculation for the PMNS mixing angles in Model I and II are given in Figs. 1. The all plots can fit the data of quark masses, CKM mixing angles, charged lepton masses, and the mass ratio of two mass squared differences of neutrino. These are

\[
\begin{align*}
\frac{m_u}{m_c} &= 0.0026(6), & \frac{m_c}{m_t} &= 0.0023(2), & y_t &= 0.51(2), \\
\frac{m_d}{m_s} &= 0.051(7), & \frac{m_s}{m_b} &= 0.018(2), & y_b &= 0.34(3), \\
A &= 0.73(3), & \lambda_c &= 0.227(1), & \bar{\rho} &= 0.22(6), & \bar{\eta} &= 0.33(4),
\end{align*}
\]

at the GUT scale, where the numbers in parentheses mean an uncertainty in last digit [16].

\[\text{We have taken } a_\nu \text{ and } a_e \text{ as } 0.49 \leq a_\nu \leq 0.53 \text{ and } 0.31 \leq a_e \lambda \leq 0.37 \text{ to fit } y_t \text{ and } y_b \text{ in our numerical}\]
Here $A$, $\lambda$, $\bar{\rho}$, and $\bar{\eta}$ are the Wolfenstein parameters which correspond to three mixing angles and one phase in the CKM matrix; $y_t$ and $y_b$ represent largest eigenvalues of Yukawa matrices for the up-type and down-type quarks, respectively. These values are calculated with the 2-loop gauge coupling and 2-loop Yukawa coupling renormalization equation taking $\tan \beta$, threshold corrections $\gamma_{t,b,d}$, and an effective SUSY scale as $\tan \beta = 38$, $\gamma_b = -0.22$, $\gamma_d = -0.21$, $\gamma_t = 0$, and $m_{\text{SUSY}} = 500$ GeV, respectively. The threshold corrections are approximated by

$$
\gamma_t \sim \frac{y_t^2 \tan \beta \mu A_t}{32\pi^2 m_t^2}, \quad \gamma_u \sim 0, \quad \gamma_b \sim \frac{4}{3} g_3^2 \tan \beta \mu M_3}{16\pi^2 m_b^2}, \quad \gamma_d \sim \frac{4}{3} g_3^2 \tan \beta \mu M_3}{16\pi^2 m_d^2},
$$

(4.10)

where $\mu$, $A_t$, $m_{\tilde{q}}$, $g_3$, and $M_3$ indicate supersymmetric Higgs mass $\mu H_u H_d$, soft top quark tri-linear coupling, mass of the squark $\tilde{q}$, strong coupling, and gaugino soft breaking mass, respectively [17]. The utilized values of SUSY can lead to the relations at the GUT scale,

$$
\frac{m_b}{m_\tau} \frac{3m_s}{m_\mu} = \frac{m_d}{3m_e} = 1,
$$

(4.11)

in a good accuracy. Therefore, the above values can automatically reproduce the experimental observed charged lepton masses at the low energy.

The numerical calculations of Model I suggest that the predicted region of solar angle covers the experimental upper bound but the model gives a predicted lower bound around $0.29 \lesssim \sin^2 \theta_{12}$. On the other hand, a constrained region is predicted for the reactor angle as $0.002 \lesssim \sin^2 \theta_{13} \lesssim 0.007$. This result is one of important predictions of the present cascade textures. It might be checked by the upcoming DoubleChooz [18], RENO [19], and DayaBay [20] experiments as the reactor experiments in addition to the accelerator experiments such as T2K [21] and NO$\nu$A [22]. Finally, the atmospheric angle covers the current experimentally allowed region. However, there is a relatively clear correlation between the magnitudes of reactor and atmospheric angles in Fig. 1. On the other hand, the result for Model II can cover the experimental allowed region because of largeness of corrections from the right-handed neutrino mass matrix but there is an upper bound of $\sin^2 \theta_{13}$ which is $\sin^2 \theta_{13} \lesssim 0.015$. Therefore, we can conclude that the minimal cascade textures in the context of SUSY SO(10) (Model I) can lead to clear predictions, which are $0.29 \lesssim \sin^2 \theta_{12}$ and $0.002 \lesssim \sin^2 \theta_{13} \lesssim 0.007$, and relatively sharp correlations between the reactor and atmospheric angles. In the next-to minimal cascade model (Model II) lead to only the upper bound of the reactor angle while the model can explain about almost ranges of PMNS mixing angles. The minimal realization of cascade model is predictive and interesting in the framework with the cascade hierarchies in SUSY SO(10) GUT, and

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*See [23] for an excellent review of sensitivities of the upcoming experiments.*
thus it might be checked by the future experiments. It would be too difficult to distinguish other cascade textures even if the future precision data of neutrino oscillation experiment could be used. It is worth studying a new method to check the models.

At the end of this subsection, we give a brief comparison between our results and ones from a similar $SO(10)$ approach, which utilizes type II seesaw mechanism and a simple ansatz such that the dominant Yukawa matrix has rank one $^{24}$†. The work gives some predicted regions for leptonic mixing angles based on three typical models in this approach:

(A) $V_\nu = 1$ case where $V_{PMNS} = V_e V_{TB} V_\nu^\dagger$, and $V_e$ and $V_\nu$ are diagonalizing matrices of Yukawa matrix for the charged lepton and neutrino mass matrix for light neutrinos in a tetrahedral coordinate, respectively, (B1) $V_\nu \neq 1$ and $(f^{\text{tetra}})_{12} = (f^{\text{tetra}})_{21} = (f^{\text{tetra}})_{13} = (f^{\text{tetra}})_{31} = 0$ where $f^{\text{tetra}}$ is a coupling to 126+126 Higgs in tetrahedral coordinate, and (B2) $V_\nu \neq 1$ and $(f^{\text{tetra}})_{12} = (f^{\text{tetra}})_{21} = (f^{\text{tetra}})_{23} = (f^{\text{tetra}})_{32} = 0$. The numerical calculations predict $\sin^2 \theta_{12} \simeq 0.28$ and $0.006 \lesssim \sin^2 \theta_{13} \lesssim 0.008$ in the model A, $0.32 \lesssim \sin^2 \theta_{12} \lesssim 0.33$, $0.003 \lesssim \sin^2 \theta_{13} \lesssim 0.006$, and $0.59 \lesssim \sin^2 \theta_{23} \lesssim 0.60$ in the model B1, and $\sin^2 \theta_{12} \simeq 0.30$, $0.04 \lesssim \sin^2 \theta_{13} \lesssim 0.06$, and $0.62 \lesssim \sin^2 \theta_{23} \lesssim 0.63$ in the model B2 by using the data $\Delta m^2_{21}/\Delta m^2_{31} = 0.027 - 0.038$ at 2$\sigma$ level. These can be compared with our results and might be also checked by the future neutrino experiments.

### 4.2 Lepton flavor violation

Next, we investigate the branching ratios of LFV process $l_i \rightarrow l_j \gamma$, in our cascade lepton mass matrices. We suppose that soft SUSY breaking masses of sleptons are universal at the GUT scale, $\Lambda_{\text{GUT}}$, for simplicity. In the case, the off-diagonal matrix elements are generated by radiative corrections from the Yukawa couplings of neutrinos $^{25}$. The one-loop renormalization group evolution gives the left-handed slepton masses. The leading contribution is estimated by

\[
(m^2_{ij})_{ij} \sim \frac{3m_0^2 + |a_0|^2}{8\pi^2 v^2 \sin^2 \beta} \sum_k (M_{\nu D}^f)_{ik} (M_{\nu D})_{kj} \ln \left( \frac{|M_k|}{\Lambda_{\text{GUT}}} \right) \quad \text{for } i \neq j, \tag{4.12}
\]

where $m_0$ and $a_0$ are the universal SUSY breaking mass and three-point coupling of scalar superpartners given at the GUT scale. The branching fractions of each LFV process are roughly given by

\[
\text{Br}(l_i \rightarrow l_j \gamma) \simeq \frac{3\alpha}{2\pi} \frac{|(m^2_{ij})_{ij}|^2 M^4_{W}}{m^8_{\text{SUSY}}} \tan^2 \beta, \tag{4.13}
\]

$^{†}$The paper by Dutta, et al. in refs. $^{3}$ presents an $S_4$ flavor model, which is one of realizations of the rank I approach by flavor symmetry. Here we focus on the general results of this approach given in $^{24}$. 
where $\alpha$, $M_W$, and $m_{\text{SUSY}}$ are the fine structure constant, $W$ boson mass, and a typical mass scale of superparticles, respectively. These branching ratios are estimated by

$$
\text{Br}(\mu \to e\gamma) \simeq \frac{3\alpha}{8\pi^5} B \left[ \lambda^{16} \ln \left( \frac{|M_1|}{\Lambda_{\text{GUT}}} \right) + \lambda^{12} \ln \left( \frac{|M_2|}{\Lambda_{\text{GUT}}} \right) - \lambda^{12} \ln \left( \frac{|M_3|}{\Lambda_{\text{GUT}}} \right) \right]^2,
$$

for both Model I and II. Here we define $B \equiv (M_W/m_{\text{SUSY}})^4 \tan^2 \beta$ and take $m_0 = |a_0| = m_{\text{SUSY}}$. Typical magnitudes of the branching ratios are shown in Tab. 5. In these analyses, $\Lambda_{\text{GUT}} = 2 \times 10^{16}$ GeV is taken. These results are compared with the current experimental upper bounds at 90% confidence level [26, 27]:

$$
\text{Br}(\mu \to e\gamma) \leq 1.2 \times 10^{-11}, \quad \text{Br}(\tau \to e\gamma) \leq 1.2 \times 10^{-7}, \quad \text{Br}(\tau \to \mu\gamma) \leq 4.5 \times 10^{-8}.
$$

The magnitudes of the branching ratios for the lepton flavor violating process in the model with the applicable heaviest right-handed Majorana neutrino mass are far below the experimental limit. Once one fix the value of $\tan \beta$, the current experimental limit gives lower bound on $m_{\text{SUSY}}$, which is e.g. $m_{\text{SUSY}} \geq 41.6$ GeV from $\text{Br}(\tau \to \mu\gamma) \leq 4.5 \times 10^{-8}$ with $\tan \beta = 38$. In the present case, the bound can be easily satisfied.

### 4.3 Leptogenesis

At the end of this section, we examine whether the thermal leptogenesis [28] works in our model. The CP asymmetry parameter in the decay process of right-handed neutrino, $R_i$, is given by

$$
\epsilon_i = \frac{\sum_j \Gamma(R_i \to L_j H) - \sum_j \Gamma(R_i \to L_j H^\dagger)}{\sum_j \Gamma(R_i \to L_j H) + \sum_j \Gamma(R_i \to L_j H^\dagger)},
$$

where $L_i$ and $H$ denote the left-handed lepton and Higgs fields. An approximation for $\epsilon_i$ at low temperature is estimated by [29], $\epsilon_1 = \frac{1}{8\pi} \sum_{i \neq 1} \text{Im}[A_{ii}]^2 F(r_i)/|A_{ii}|$, where $r_i \equiv$
\[
|M_i/M_1|^2, \quad A \equiv (D M_{\nu D} M_{\nu D}^D D^1)/v_\nu^2, \quad \text{and} \quad D \text{ being a diagonal phase matrix to make the eigenvalues } M_i \text{ real and positive.}
\]
The function \( F \) describes contributions from the one-loop vertex and self-energy corrections,
\[
F(x) = \sqrt{x} \left[ \frac{2}{1-x} - \ln \left( 1 + \frac{1}{x} \right) \right].
\]
We here define the resultant CP asymmetry, \( \eta_{\text{CP}} \), as the ratio of the lepton asymmetry to the photon number density \( n_\gamma \),
\[
\eta_{\text{CP}} = \frac{135 \zeta(3) \kappa \epsilon_1}{(4 \pi^4 g_* n_\gamma)}. \tag{4.19}
\]
In the equation, \( \kappa, s, \) and \( g_* \) are the efficiency factor, entropy density, and the effective number of degrees of freedom in thermal equilibrium. They are given by \( \text{[30]} \), \( s = 7.04 n_\gamma \), \( g_* = 228.75 \), and
\[
\kappa^{-1} \simeq \frac{3.3 \times 10^{-3} \text{ eV}}{m_{\text{eff}}} + \left( \frac{m_{\text{eff}}}{5.5 \times 10^{-4} \text{ eV}} \right)^{1.16}. \tag{4.20}
\]
The \( m_{\text{eff}} \) is the effective light neutrino mass defined as \( m_{\text{eff}} \equiv |(M_{\nu D}^D M_{\nu D})_{11}/M_1| \). The BAU, \( \eta_B \), is transferred via spharelron interactions as \( \eta_B = -8 \eta_{\text{CP}}/23 \). Finally, the baryon asymmetry in our model is predicted as \( \eta_B \sim 10^{-23} \sin \theta_B \), where \( \theta_B \equiv \theta_3 - \theta_1 \) and \( \theta_i = \arg(M_i) \). These results are compared with the current observational data at 68% confidence level from the WMAP 7-year resulting in the standard \( \Lambda \text{CDM} \) model \( \text{[31]} \). We can see that the baryon asymmetry generated through the leptogenesis is too small to explain the BAU. This is because the hierarchy in the Dirac neutrino mass matrix in the \( SO(10) \) model is determined by the up-type quark mass matrix. Therefore, there is no freedom to adjust the Dirac neutrino mass matrix such that the BAU can be generated in the present model. It is expected that enough BAU might be realized by extending our cascade model to the inverse seesaw case, see \( \text{[32]} \). In the case, the structure of effective light neutrino mass matrix is slightly changed but the realistic PMNS mixing angels would be obtained.

5 Discussion

At the end of the paper, we give a comment on phenomenological aspects of proton decay. In general for SUSY GUTs, there are three sources that mediate the proton decays. The first one comes from the dimension-6 operators, arising from the exchanging of the heavy gauge bosons. Note that this type of operators exists in both non-SUSY and SUSY GUTs. These operators are significantly suppressed by \( 1/\Lambda_{\text{GUT}}^2 \), therefore, we have no problem with the proton decay from these operators if \( \Lambda_{\text{GUT}} \) is large enough, i.e. \( \Lambda_{\text{GUT}} \geq 10^{16} \text{ GeV} \). The second source is from the dimension-5 operators, arising from the exchanging of color triplet Higgsino fields. In this case, the proton decay contribution is suppressed by \( 1/M_H \), where \( M_H \) is the mass of the color triplet Higgsinos. To suppress the proton decay contributions, the mass \( M_H \) has to be very heavy, which can achieved by some doublet-triplet splitting
mechanism \[12, 13, 14\]. The third contribution arises from the dimension-4 operators, which are not suppressed by the GUT scale, however, these operators are eliminated by the $R-$parity. In the class of $SO(10)$ models, which do not contain the spinor 16 or $\bar{16}$ as the Higgs fields, the $R-$parity is conserved. Otherwise, in order to avoid the proton decay contributions from these operators, the $R-$parity has to be introduced by hands. Since we consider an $SO(10)$ model without using the spinor Higgs representations, we have no problem with these dimension-4 operators \[33, 34\].

6 Summary

We have done texture analyses of cascade model in supersymmetric $SO(10)$ model. The neutrino Dirac mass matrix of a cascade form can realize the tri-bimaximal mixing at the leading order while the down-quark one of a H.C. form can lead to realistic structure of CKM mixing. This fact gives us a strong motivation to study cascade hierarchical textures in a grand unified theory.

We analytically clarified possible structures of neutrino Dirac, charged lepton, quark, and right-handed neutrino mass matrices by estimating collection from them to the tri-bimaximal mixing. The numerical analyses based on two typical models have been also presented. The minimal cascade texture in the context of SUSY $SO(10)$ GUT can lead to clear predictions for the PMNS mixing angles, which are $0.29 \lesssim \sin^2 \theta_{12}$ and $0.002 \lesssim \sin^2 \theta_{13} \lesssim 0.007$, and relatively sharp correlations between the reactor and atmospheric angles. This result is a hot topic for the upcoming experiments of the reactor neutrino mixing angle. It might be checked by such future experiments. We have also shown that our typical cascade models can pass the constraints from the lepton flavor violation searches. For generating the BAU, we cannot generate enough asymmetry through the thermal leptogenesis mechanism in our model. Therefore, we need other mechanisms to generate the BAU.

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A Constraints on structure of non-diagonal $M_R$ case

The constraints on the structure of non-diagonal $M_R$ are presented in this Appendix. We defined the diagonalized mass matrix of the right-handed neutrino in (3.40) and an unitary matrix which diagonalizes the $M_R$. The neutrino mass after the seesaw mechanism and
operating the $V_{\text{TB}}$ is given in [3,42]. All matrix elements are given by

$$M_{11} \simeq \frac{v_u^2}{6M} \left[ 1 + 4\lambda^{16-x_2} + 4\lambda^8\theta_{R,23} + \lambda^{-x_1}\theta_{R,23}^2 \right.$$  

$$+ \lambda^{-x_1}(4\lambda^{16}\theta_{R,12} - 4\lambda^8\theta_{R,12}\theta_{R,13} + \theta_{R,13}^2)], \quad (A.1)$$

$$M_{22} \simeq \frac{v_u^2}{M} \left[ 3\lambda^{16-x_1} + \frac{1}{3} + \frac{\lambda^{16-x_2}}{3} + \lambda^{-x_2}( - \frac{2\lambda^8}{3}\theta_{R,23} + \theta_{R,23}^2 ) + \lambda^{-x_1}( -2\lambda^8\theta_{R,13} + \theta_{R,13}^2 ) \right], \quad (A.2)$$

$$M_{33} \simeq \frac{v_u^2}{12M} \left[ 2\lambda^{8-x_2} + \frac{1}{2} + \lambda^{-x_2}(2\lambda^4\theta_{R,23} + \frac{\theta_{R,23}^2}{2}) - 2\lambda^{4-x_1}\theta_{R,12}\theta_{R,13} \right.$$  

$$+ \lambda^{-x_1}(2\lambda^8\theta_{R,12}^2 + \frac{1}{2}\theta_{R,13}^2)], \quad (A.3)$$

$$M_{12} \simeq -\frac{v_u^2}{3\sqrt{2}M} \left[ 1 + \lambda^{8-x_2}\theta_{R,23} + 3\lambda^{-x_1}(-2\lambda^{16}\theta_{R,12} + \lambda^8\theta_{R,13}) \right], \quad (A.4)$$

$$M_{23} \simeq \frac{v_u^2}{6M} \left[ 1 - 2\lambda^{12-x_2} + 2\lambda^{4-x_2}\theta_{R,23} \right.$$  

$$+ 3\lambda^{-x_1}(2\lambda^{12}\theta_{R,12} - \lambda^8\theta_{R,13} - 2\lambda^4\theta_{R,13}\theta_{R,12} + \theta_{R,13}^2)], \quad (A.5)$$

$$M_{13} \simeq -\frac{v_u^2}{2\sqrt{3}M} \left[ 1 + \lambda^{-x_2}(4\lambda^{12} + 2\lambda^4\theta_{R,23} + \theta_{R,23}^2) \right.$$  

$$- \lambda^{-x_1}(2\lambda^4\theta_{R,12}\theta_{R,13} - 4\lambda^{12}\theta_{R,12}^2 - \theta_{R,13}^2)]. \quad (A.6)$$

We require that the magnitudes of leading order of each term in this mass matrix are the same one as in the case of diagonal $M_R$ case because the tri-bimaximal mixing can be already realized at the leading order by neutrino Dirac mass matrix. It leads to constraints on the mixing angles as follows:

$$\theta_{R,13} < \frac{3}{2}\lambda^8, \frac{1}{3}\lambda^{-8+x_1}, 2\lambda^{4+(x_1-x_2)/2}, \frac{1}{\sqrt{3}}\lambda^{x_1/2}, \quad (A.7)$$

$$\theta_{R,23} < \lambda^4, \frac{1}{2}\lambda^{-4+x_2}, \lambda^{x_2/2} \quad (A.8)$$

$$\theta_{R,12} < \frac{1}{6}\lambda^{-12+x_1}, \lambda^{(x_1-x_2)/2}, \frac{1}{2}\lambda^{-6+x_1/2}, \quad (A.9)$$

$$\theta_{R,12}\theta_{R,13} < \lambda^{4+x_1-x_2}, \frac{1}{6}\lambda^{-4+x_1}. \quad (A.10)$$

After fixing the values of $(d_1, d_2, x_1, x_2)$ so that they must satisfy the four conditions, one can obtain the structure leading to maximal collections to the PMNS mixing angles and neutrino mass spectra as shown in Tabs. [3] and [4].

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