Sequential identification of nonignorable missing data mechanisms

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Abstract

With nonignorable missing data, likelihood-based inference should be based on the joint
distribution of the study variables and their missingness indicators. These joint models cannot
be estimated from the data alone, thus requiring the analyst to impose restrictions that make the
models uniquely obtainable from the distribution of the observed data. We present an approach
for constructing classes of identifiable nonignorable missing data models. The main idea is to use
a sequence of carefully set up identifying assumptions, whereby we specify potentially different
missingness mechanisms for different blocks of variables. We show that the procedure results in
models with the desirable property of being non-parametric saturated.

Key words and phrases: Identification; Non-parametric saturated; Missing not at random;
Partial ignorability; Sensitivity analysis.

1 Introduction

When data are missing not at random (MNAR) (Rubin (1976)), appropriate likelihood-based in-
ference requires explicit models for the full-data distribution, i.e., the joint distribution of the study
variables and their missingness indicators. Because of the missing data, this distribution is not
uniquely identified from the observed data alone (Little and Rubin (2002)). To enable inference,
analysts must impose restrictions on the full-data distribution. Such assumptions generally are
untestable; however, a minimum desideratum is that they result in a unique full-data distribution
for the observed-data distribution at hand, i.e., the distribution that can be identified from the
incomplete data.

We present a strategy for constructing identifiable full-data distributions with nonignorable
missing data. In its most general form, the strategy is to expand the observed-data distribution
sequentially by identifying parts of the full-data distribution associated with blocks of variables, one
block at a time. This partitioning of the variables allows analysts to specify different missingness
mechanisms in the different blocks; for example, use the missing at random (MAR, Rubin (1976))
assumption for some variables and a nonignorable missingness assumption for the rest to obtain a
partially ignorable mechanism (Harel and Schafer (2009)). We ensure that the resulting full-data
distributions are non-parametric saturated (NPS, Robins (1997)), that is, their implied observed-data distribution matches the actual observed-data distribution, as detailed in Section 2.2.

Related approaches to partitioning variables with missing data have appeared previously in the literature. Zhou et al. (2010) proposed to model blocks of study variables and their missingness indicators in a sequential manner; however, their approach does not guarantee identifiability of the full-data distribution. Harel and Schafer (2009) mentioned the possibility of treating the missingness in blocks of variables differently, but they do not provide results on identification. Robins (1997) proposed the group permutation missingness mechanism, which assumes MAR sequentially for blocks of variables and results in a NPS model. This is a particular case of our more general procedure, as we describe in Section 3.4.

The remainder of the article is organized as follows. In Section 2, we describe notation and provide more details on the NPS property. In Section 3, we introduce our strategy for identifying a full-data distribution in a sequential manner. In Section 4 we present some examples of how to use this strategy for the case of two categorical study variables, for the construction of partially ignorable mechanisms, and for sensitivity analyses. Finally, in Section 5 we discuss possible future uses of our identifying approach.

2 Notation and Background

2.1 Notation

Let \( X = (X_1, \ldots, X_p) \) denote \( p \) random variables taking values on a sample space \( \mathcal{X} \). Let \( M_j \) be the missingness indicator for variable \( j \), where \( M_j = 1 \) when \( X_j \) is missing and \( M_j = 0 \) when \( X_j \) is observed. Let \( M = (M_1, \ldots, M_p) \), which takes values on \( \{0, 1\}^p \). Let \( \mu \) be a dominating measure for the distribution of \( X \), and let \( \nu \) represent the product measure between \( \mu \) and the counting measure on \( \{0, 1\}^p \). The full-data distribution is the joint distribution of \( X \) and \( M \). We call its density \( f \) with respect to \( \nu \) the full-data density. In practice, the full-data distribution cannot be recovered from sampled data, even with an infinite sample size.

An element \( m = (m_1, \ldots, m_p) \in \{0, 1\}^p \) is called a missingness pattern. Given \( m \in \{0, 1\}^p \) we define \( \tilde{m} = 1_p - m \) to be the indicator vector of observed variables, where \( 1_p \) is a vector of ones of length \( p \). For each \( m \), we define \( X_m = (X_j : m_j = 1) \) to be the missing variables and \( X_{\tilde{m}} = (X_j : \tilde{m}_j = 1) \) to be the observed variables, which have sample spaces \( \mathcal{X}_m \) and \( \mathcal{X}_{\tilde{m}} \), respectively. The observed-data distribution is the distribution involving the observed variables and the missingness indicators, which has density \( f(X_{\tilde{m}} = x_{\tilde{m}}, M = m) = \int_{\mathcal{X}_m} f(X = x, M = m) \mu(dx_m) \), where \( x \in \mathcal{X} \) represents a generic element of the sample space, and we define \( x_m \) and \( x_{\tilde{m}} \) similarly as with the random vectors.

An alternative way of representing the observed-data distribution is by introducing the mate-
rialized variables $X^* = (X^*_1, \ldots, X^*_p)$, where

$$X^*_j = \begin{cases} X_j, & \text{if } M_j = 0; \\ *, & \text{if } M_j = 1; \end{cases}$$

and “*” is a placeholder for missingness. The sample space $X^*_j$ of each $X^*_j$ is the union of {$*$} and the sample space $X_j$ of $X_j$. The materialized variables contain all the observed information: if $X^*_j = *$ then $X_j$ was not observed, and if $X^*_j = x_j$ for any value $x_j \neq *$ then $X_j$ was observed and $X_j = x_j$. Given $m \in \{0,1\}^p$ and $x_m \in X_m$, we define $x^* \equiv x^*(m,x_m)$, such that $x^*_m = x_m$ and $x^*_j = *$, where * is a vector with the appropriate number of * symbols. For example, if $m = (1,1,0)$ and $x_m = x_3$, then $x^* = (*,*,x_3)$. The event $X^* = x^*(m,x_m)$ is equivalent to $M = m$ and $X^*_m = x_m$, which implies that the distribution of $X^*$ is equivalent to the observed-data distribution. Therefore, with some abuse of notation, the observed-data density can be written in terms of $X^*$, that is $f\{X^* = x^*(m,x_m)\} \equiv f(X^*_m = x_m, M = m)$. When there is no need to refer to the $m$ and $x_m$ that define $x^*$, we simply write $f(X^* = x^*)$ to denote the observed-data density evaluated at an arbitrary point.

In what follows we often write $f(X = x, M = m)$ simply as $f(X, M)$, $f(X^* = x^*)$ as $f(X^*)$, and likewise for other expressions, provided that there is no ambiguity. For the sake of simplicity, we use “f” for technically different functions, but their actual interpretations should be clear from the arguments passed to them. For example, we denote the missingness mechanism as $f(M = m|X = x)$, or simply $f(M|X)$.

2.2 Non-Parametric Saturated Modeling

Since the true joint distribution of $X$ and $M$ cannot be identified from observed data alone, we need to work under the assumption that the full-data distribution falls within a class defined by a set of restrictions.

**Definition 1 (Identifiability).** Consider a class of full-data distributions $\mathcal{F}_A$ defined by a set of restrictions $A$. We say that the class $\mathcal{F}_A$ is identifiable if there is a mapping from the set of observed-data distributions to $\mathcal{F}_A$.

If we only require identifiability from a set of full-data distributions, two different observed-data distributions could map to the same full-data distribution. Robins (1997) introduced the stricter concept of a class of full-data distributions being non-parametric saturated — also called non-parametric identified (Vansteelandt et al. (2006); Daniels and Hogan (2008)).

**Definition 2 (Non-parametric Saturation).** Consider a class of full-data distributions $\mathcal{F}_A$ defined by a set of restrictions $A$. We say that the class $\mathcal{F}_A$ is non-parametric saturated if there is a one-to-one mapping from the set of observed-data distributions to $\mathcal{F}_A$.

The set $A$ of restrictions, or identifiability assumptions, that define a NPS class allow us to build a full-data distribution, say with density $f_A(X = x, M = m)$, from an observed-data distribution
with density $f(X_m = x_m, M = m)$, so that $f_A(X_m = x_m, M = m) = f(X_m = x_m, M = m)$, where by definition $f_A(X_m = x_m, M = m) = \int_{X_m} f_A(X = x, M = m) \mu(dx_m)$. In terms of $X^*$, the NPS property is expressed as $f_A(X^*) = f(X^*)$.

NPS is a desirable property, particularly for comparing inferences under different approaches to handling nonignorable missing data. When two missing data models satisfy NPS, we can be sure that any discrepancies in inferences are due entirely to the different assumptions on the non-identifiable parts of the full-data distribution. In contrast, without NPS, it can be difficult to disentangle what parts of the discrepancies are due to the identifying assumptions and what parts are due to differing constraints on the observed-data distribution. Thus, NPS greatly facilitates sensitivity analysis (Robins (1997)).

For a given $m$, we refer to the conditional distribution of the missing study variables given the observed data as the missing-data distribution—also known as the extrapolation distribution (Daniels and Hogan (2008))—with density $f(X_m = x_m | X \bar{m} = \bar{x}_m, M = m)$. These distributions correspond to the non-identifiable parts of the full-data distribution. A NPS approach is equivalent to a recipe for building these distributions from the observed-data distribution without imposing constraints on the latter.

NPS models can be constructed in many ways. For example, in pattern mixture models, one can use the complete-case missing-variable restriction (Little (1993)), which sets $f(X_m = x_m | X \bar{m} = \bar{x}_m, M = m) = f(X_m = x_m | X \bar{m} = \bar{x}_m, M = 0_p)$, for all $m \in \{0, 1\}^p$. Although Little (1993) considered parametric models for each $f(X_m = x_m | M = m)$, this does not have to be the case, and therefore pattern mixture models can be NPS. Another example is the permutation missingness model of Robins (1997), which for a specific ordering of the study variables assumes that the probability of observing the $k$th variable depends on the previous study variables and the subsequent observed variables. The group permutation missingness model of Robins (1997) is an analog of the latter for groups of variables and is also NPS. Sadinle and Reiter (2017) introduced a missingness mechanism where each variable and its missingness indicator are conditionally independent given the remaining variables and missingness indicators, which leads to a NPS model. Tchetgen Tchetgen et al. (2016) proposed a NPS approach based on discrete choice models. Finally, we note that MAR models also can be NPS, as shown by Gill et al. (1997).

### 3 Sequential Identification Strategy

We consider the $p$ variables as divided into $K$ blocks, $X = (X_1, \ldots, X_K)$, where $X_k = (X_{t_k-1+1}, \ldots, X_{t_k})$, which contains $t_k - t_{k-1}$ variables. As our results only concern the identification of full-data distributions starting from an observed-data distribution, we assume that $f(X^*) = f(X_1^*, \ldots, X_K^*)$ is known. The identification strategy consists of specifying a sequence of assumptions $A_1, \ldots, A_K$, one for each block of variables, where each $A_k$ allows us to identify the conditional distribution of $X_k$ and $M_k$ given $X_{<k} \equiv (X_1, \ldots, X_{k-1})$, $X_{>k}^* \equiv (X_{k+1}^*, \ldots, X_K^*)$, and a carefully chosen subset of the missingness indicators $M_{<k} \equiv (M_1, \ldots, M_{k-1})$ described below. We first provide a general
description of how $A_1, \ldots, A_K$ allow us to identify parts of the full-data distribution in a sequential manner, and then in Theorem 1 present the formal identification result.

3.1 Description

We now present the steps needed to implement the identification strategy. A graphical summary of the procedure is provided in Figure 1.

**Step 1.** Write $f(X^*) = f(X_1^*|X_{>1}^*)f(X_{>1}^*)$. Consider an identifiability assumption $A_1$ on the distribution of $X_1$ and $M_1$ given $X_{>1}$ that allows us to obtain a distribution with density $f_{A_1}(X_1, M_1|X_{>1})$ with the NPS property $f_{A_1}(X_1^*|X_{>1}^*) = f(X_1^*|X_{>1})$. From this we can define $f_{A_1}(X_1, M_1, X_{>1}) \equiv f_{A_1}(X_1, M_1|X_{>1})f(X_{>1})$.

**Step 2.** Suppose we divide the $t_1$ variables in $X_1$ into two sets indexed by $R_1$ and $S_1$, where $R_1 \cup S_1 = \{1, \ldots, t_1\}$ and $R_1 \cap S_1 = \emptyset$; see Remark 1 below for discussion of why one might want to do so. Let $M_{R_1}$ and $M_{S_1}$ be the corresponding missingness indicators. We can write $f_{A_1}(M_{S_1}|X_1, M_{R_1}, X_{>1})f_{A_1}(X_1, M_{R_1}, X_{>1})$, where $f_{A_1}(X_1, M_{R_1}, X_{>1}) = f_{A_1}(X_2^*|X_1, M_{R_1}, X_{>2})f_{A_1}(X_1, M_{R_1}, X_{>2})$. Consider an identifiability assumption $A_2$ on the distribution of $X_2$ and $M_2$ given $X_1$, $M_{R_1}$, and $X_{>2}$ that allows us to obtain a distribution with density $f_{A_2}(X_2, M_2|X_1, M_{R_1}, X_{>2})$ with the NPS property $f_{A_2}(X_2^*|X_1, M_{R_1}, X_{>2}) = f_{A_1}(X_2^*|X_1, M_{R_1}, X_{>2})$. From this we can define

$$f_{A_2}(X_{<2}, M_{R_1}, M_2, X_{>2}^* \equiv f_{A_2}(X_2, M_2|X_1, M_{R_1}, X_{>2}^*)f_{A_1}(X_1, M_{R_1}, X_{>2}^*)$$

The notation $f_{A_2}$ emphasizes that the distribution relies on $A_1$ and $A_2$.

**Step $k+1$.** At the end of the $k$th step we have $f_{A_{<k}}(X_{<k}, M_{R_{k-1}}, M_k, X_{>k}^*)$. Let $R_k \cup S_k = \{t_{k-1}+1, \ldots, t_k\} \cup R_{k-1}$, $R_k \cap S_k = \emptyset$, and $M_{R_k}$ and $M_{S_k}$ be the corresponding missingness indicators. We can write $f_{A_{<k}}(X_{<k}, M_{R_{k-1}}, M_k, X_{>k}^*) = f_{A_{<k}}(M_{S_k}|X_{<k}, M_{R_k}, X_{>k}^*)f_{A_{<k}}(X_{<k}, M_{R_k}, X_{>k}^*)$, where $f_{A_{<k}}(X_{<k}, M_{R_k}, X_{>k}^*) = f_{A_{<k}}(X_{<k+1}^*|X_{<k}, M_{R_k}, X_{>k+1}^*)f_{A_{<k}}(X_{<k}, M_{R_k}, X_{>k+1}^*)$. Now, consider an identifiability assumption $A_{k+1}$ on the distribution of $X_{k+1}$ and $M_{k+1}$ given $X_{<k}$, $M_{R_k}$ and $X_{>k+1}^*$ that allows to obtain a distribution with density $f_{A_{<k+1}}(X_{<k+1}^*|X_{<k}, M_{R_k}, X_{>k+1}^*)$ with the NPS property $f_{A_{<k+1}}(X_{<k+1}^*|X_{<k}, M_{R_k}, X_{>k+1}^*) = f_{A_{<k}}(X_{<k}^*|X_{<k}, M_{R_k}, X_{>k+1}^*)$. From this we can define

$$f_{A_{<k+1}}(X_{<k+1}, M_{R_k}, M_{k+1}, X_{>k+1}^*) \equiv f_{A_{<k+1}}(X_{<k+1}, M_{k+1}|X_{<k}, M_{R_k}, X_{>k+1}^*)f_{A_{<k}}(X_{<k}, M_{R_k}, X_{>k+1}^*)$$

**Step $K$.** For the final step, assumption $A_K$ is on the distribution of $X_K$ and $M_K$ given $X_{<K}$ and $M_{R_{K-1}}$. Following the previous generic identifying step, we obtain $f_{A_{<K}}(X, M_{R_{K-1}}, M_K) \equiv f_{A_{<K}}(X_K, M_K|X_{<K}, M_{R_{K-1}})f_{A_{<K}}(X_{<K}, M_{R_{K-1}})$. We can obtain the implied distribution of the study variables, with density $f_{A_{<K}}(X)$, from this last equation.

**Remark 1.** The main characteristic of the $R_k$ subsets is that if an index does not appear in $R_{k-1}$, then it cannot appear in $R_k$, unless it is one of $t_{k-1}+1, \ldots, t_k$. The flexibility in the choosing of
these subsets gives flexibility in the setting up of the identifiability assumptions: different versions of our identification approach can be obtained by making assumptions conditioning on different subsets of the missingness indicators. As long as the $R_k$ subsets satisfy $R_k \subseteq \{t_{k-1} + 1, \ldots, t_k\} \cup R_{k-1}$, Theorem 1 guarantees that the final full-data distribution is NPS.

Remark 2. The sequence for $A_1, \ldots, A_K$ follows the order of the blocks $X_1, \ldots, X_K$. In many cases these blocks may not have a natural order. Different orderings of the blocks lead to different sets of assumptions, thereby leading to different final full-data distributions and implied distributions of the study variables. To clarify this point, suppose that we have three blocks of variables: demographic variables $X_D$, income variables $X_I$, and health-related variables $X_H$. When $X_D$ is first in the order, $A_1$ concerns the distribution of $X_D$ and $M_D$ given $X_I^*$ and $X_H^*$; likewise, when $X_I$ is first in the order, $A_1$ concerns the distribution of $X_I$ and $M_I$ given $X_D^*$ and $X_H^*$. Similarly, $A_2$ and $A_3$ also will change depending on the order of the variables, thereby implying changes in the final full-data distribution.
3.2 Non-Parametric Saturation

The previous presentation makes it clear that the identifying assumptions $A_1, \ldots, A_K$ allow us to identify $f_{A \leq K}(X, M_{R_{K-1}}, M_K)$, and furthermore, $f_{A \leq k}(M_{S_k}|X_{\leq k}, M_{R_k}, X^*_{>k})$ for each $k < K$, although each of these conditional densities remains unused after step $k$ in the procedure. A full-data distribution with density $\tilde{f}_{A \leq K}(X, M)$ that encodes $A_1, \ldots, A_K$ can be expressed as

$$\tilde{f}_{A \leq K}(X, M) = f_{A \leq K}(X, M_{R_{K-1}}, M_K) \tilde{f}_{A \leq K}(M_{S_1}, \ldots, M_{S_{K-1}}|X, M_{R_{K-1}}, M_K),$$

where the second factor can be written as $\prod_{k=1}^{K-1} \tilde{f}_{A \leq K}(M_{S_k}|X, M_{S_{>k}}, M_{R_{K-1}}, M_K)$, with $S_{>k} = S_{k+1} \cup \cdots \cup S_{K-1}$, and $M_{S_{>k}} \equiv (M_{S_{k+1}}, \ldots, M_{S_{K-1}})$. From the definition of the sets $S_k$ and $R_k$, it is easy to see that $S_{>k} \cup R_{K-1} = R_k \cup \{t_k + 1, \ldots, t_{K-1}\}$, and therefore we can rewrite $\tilde{f}_{A \leq K}(M_{S_k}|X, M_{R_{k}}, M_{>k})$.

The sequential identification procedure does not identify any $\tilde{f}_{A \leq K}(M_{S_k}|X, M_{R_k}, M_{>k})$, but only $f_{A \leq k}(M_{S_k}|X_{\leq k}, M_{R_k}, X^*_{>k})$, that is, it identifies the distribution of $M_{S_k}$ given the variables $X_{\leq k}$, the missingness indicators $M_{R_k}$, and the materialized variables $X^*_{>k}$, but not given the missing variables among $X_{>k}$ according to $M_{>k}$. Nevertheless, the full specification of $\tilde{f}_{A \leq K}(M_{S_k}|X, M_{R_k}, M_{>k})$ is irrelevant given that any such conditional distribution that agrees with $f_{A \leq k}(M_{S_k}|X_{\leq k}, M_{R_k}, X^*_{>k})$ would lead to the same $f_{A \leq k}(X)$. One such distribution is one where

$$\tilde{f}_{A \leq k}(M_{S_k}|X, M_{R_k}, M_{>k}) = f_{A \leq k}(M_{S_k}|X_{\leq k}, M_{R_k}, X^*_{>k}), \quad k < K,$$

that is, where the conditional distribution of $M_{S_k}$ given $X, M_{R_k}$ and $M_{>k}$ does not depend on the missing variables among $X_{>k}$ according to $M_{>k}$. This guarantees the existence of a full-data distribution with density

$$\tilde{f}_{A \leq K}(X, M) = f_{A \leq K}(X, M_{R_{K-1}}, M_K) \prod_{k=1}^{K-1} f_{A \leq k}(M_{S_k}|X_{\leq k}, M_{R_k}, X^*_{>k}),$$

which encodes the assumptions $A_1, \ldots, A_K$. Theorem 1 guarantees that this construction leads to NPS full-data distributions.

**Theorem 1.** Let $R_1, \ldots, R_{K-1}$ be a sequence of subsets such that $R_k \subseteq \{t_k-1 + 1, \ldots, t_k\} \cup R_{k-1}$. Let $A_1, \ldots, A_K$ be a sequence of identifying assumptions, with each $A_k$ being an assumption on the conditional distribution of $X_k$ and $M_k$ given $X_{<k}, M_{R_{k-1}}$, and $X^*_{>k}$, such that for a given density $g(X^*_{>k}|X_{<k}, M_{R_{k-1}}, X^*_{>k})$, it allows the construction of a density $g_{A_k}(X_k, M_k|X_{<k}, M_{R_{k-1}}, X^*_{>k})$ with the NPS property $g_{A_k}(X^*_{>k}|X_{<k}, M_{R_{k-1}}, X^*_{>k}) = g(X^*_{>k}|X_{<k}, M_{R_{k-1}}, X^*_{>k})$. Then, given an observed-data density $f(X^*)$, there exists a full-data density $\tilde{f}_{A \leq K}(X, M)$ that encodes the assumptions $A_1, \ldots, A_K$ and satisfies the NPS property $\tilde{f}_{A \leq K}(X^*) = f(X^*)$.

**Proof.** We explained how assumptions $A_1, \ldots, A_K$ along with the extra assumption in (1) lead to the full-data density in (2). We now show the NPS property of (2). To start, we integrate (2) over the
missing variables in $X_K$ according to $M_K$. Since none of the factors in $\prod_{k=1}^{K-1} f_{A_{\leq k}}(M_{S_k}|X_{\leq k}, M_{R_k}, X_{>k}^*)$ depend on these missing variables, we obtain

$$f_{A_{\leq K-1}}(X_{\leq K-1}, M_{R_{K-1}}, X_K^*) \prod_{k=1}^{K-1} f_{A_{\leq k}}(M_{S_k}|X_{\leq k}, M_{R_k}, X_{>k}^*)$$

$$= f_{A_{\leq K-1}}(X_{\leq K-1}, M_{R_{K-1}}, M_{S_{K-1}}, X_K^*) \prod_{k=1}^{K-2} f_{A_{\leq k}}(M_{S_k}|X_{\leq k}, M_{R_k}, X_{>k}^*)$$

$$= f_{A_{\leq K-1}}(X_{\leq K-1}, M_{R_{K-2}}, M_{K-1}, X_K^*) \prod_{k=1}^{K-2} f_{A_{\leq k}}(M_{S_k}|X_{\leq k}, M_{R_k}, X_{>k}^*). \quad (3)$$

Similarly, we now integrate [3] over the missing variables in $X_{K-1}$ according to $M_{K-1}$. Given that none of the factors in $\prod_{k=1}^{K-2} f_{A_{\leq k}}(M_{S_k}|X_{\leq k}, M_{R_k}, X_{>k}^*)$, depend on these missing variables, and given the way $f_{A_{\leq K-1}}(X_{\leq K-1}, M_{R_{K-2}}, M_{K-1}, X_K^*)$ is constructed (see generic step $k + 1$ in Section 3.1), we obtain

$$f_{A_{\leq K-2}}(X_{\leq K-2}, M_{R_{K-2}}, X_{>K-2}) \prod_{k=1}^{K-2} f_{A_{\leq k}}(M_{S_k}|X_{\leq k}, M_{R_k}, X_{>k}^*).$$

These arguments and process can be repeated, sequentially integrating over the missing variables in $X_k$ according to $M_k$, $k = K - 2, \ldots, 1$, finally obtaining the observed-data density $f(X^*)$. \[ \square \]

### 3.3 Special Cases

It is worth describing two special sequential identification schemes that can be derived from our general presentation. One is obtained when we take all $R_k = \{t_k-1 + 1, \ldots, t_k\} \cup R_{k-1}$, $S_k = \emptyset$, and therefore $M_{R_k} = M_{\leq k}$. In this case, each $A_{k+1}$ is on the distribution of $X_{k+1}$ and $M_{k+1}$ given $X_{\leq k}$, $M_{\leq k}$ and $X_{>k+1}^*$, that is, the assumption conditions on the whole set of missingness indicators $M_{\leq k}$ and not just on a subset of these. The other is obtained when we take all $R_k = \emptyset$, $S_k = \{t_k-1 + 1, \ldots, t_k\}$, and therefore $M_{S_k} = M_K$. In this case, each $A_{k+1}$ is on the distribution of $X_{k+1}$ and $M_{k+1}$ given $X_{\leq k}$ and $X_{>k+1}^*$, that is, each assumption conditions on none of the missingness indicators $M_{\leq k}$.

### 3.4 Connection with the Mechanisms of Robins (1997)

An important particular case of our sequential identification strategy is obtained when all $M_{S_k} = M_k$ and each $A_k$ is taken to be a conditional MAR assumption, that is, when we assume that $f(M_k|X_{\leq k-1}, X_k, X_{>k}^*) = f(M_k|X_{\leq k-1}, X_k^*, X_{>k}^*)$. Along with [1], this leads to the combined assumption

$$f(M_k|X, M_{\geq k}) = f(M_k|X_{\leq k}, X_{>k}^*). \quad (4)$$
The missingness mechanism derived from this approach corresponds to the group permutation missingness of Robins (1997). When each block contains only one variable, it corresponds to the permutation missingness mechanism of Robins (1997). If the ordering of the variables or blocks of variables is regarded as temporal, as in a longitudinal study or a survey that asks questions in a fixed sequence, Robins (1997) interpreted (4) as follows: the nonresponse propensity at the current time period depends on the values of study variables in the previous time periods, whether observed or not, but not on what is missing in the present and future time periods.

If the order of the blocks of variables was reversed, that is, if \( A_1 \) was on the distribution of \( X_K \) and \( M_K \) given \( X_{<K}^* \), \( A_2 \) was on the distribution of \( X_{K-1} \) and \( M_{K-1} \) given \( X_{<K-1}^* \) and \( X_K \), and so on, then we would have the following interpretation: the nonresponse propensity at the current time period depends on the values of study variables in the future time periods, whether observed or not, but not on what is missing in the present and past time periods. This interpretation is arguably easier to explain in the context of respondents answering a questionnaire. The nonresponse propensity for question \( t \) can depend on the respondent’s answers to questions that appear later in the questionnaire and to questions that she has already answered, but not on the information that she has not revealed.

4 Applications

4.1 Sequential Identification for Two Categorical Variables

Consider two categorical random variables \( X_1 \in \mathcal{X}_1 = \{1, \ldots, I\} \) and \( X_2 \in \mathcal{X}_2 = \{1, \ldots, J\} \). Let \( M_1 \) and \( M_2 \) be their missingness indicators. Let \( \mathbb{P} \) denote the joint distribution of \((X_1, X_2, M_1, M_2)\). The observed-data distribution corresponds to the probabilities

\[
\begin{align*}
\pi_{ij00} & \equiv \mathbb{P}(X_1^* = i, X_2^* = j) = \mathbb{P}(X_1 = i, X_2 = j, M_1 = 0, M_2 = 0), \\
\pi_{i+01} & \equiv \mathbb{P}(X_1^* = i, X_2^* = *) = \mathbb{P}(X_1 = i, M_1 = 0, M_2 = 1), \\
\pi_{+j10} & \equiv \mathbb{P}(X_1^* = *, X_2^* = j) = \mathbb{P}(X_2 = j, M_1 = 1, M_2 = 0), \\
\pi_{++11} & \equiv \mathbb{P}(X_1^* = *, X_2^* = *) = \mathbb{P}(M_1 = 1, M_2 = 1),
\end{align*}
\]

for \( i \in \mathcal{X}_1, j \in \mathcal{X}_2 \). We seek to construct a full-data distribution \( \mathbb{P}_{A \subseteq 2}(X_1, X_2, M_1, M_2) \) from the observed-data distribution \( \mathbb{P}(X_1^*, X_2^*) \) by imposing some assumptions \( A_1 \) and \( A_2 \). The goal is a full-data distribution such that \( \mathbb{P}_{A \subseteq 2}(X_1^*, X_2^*) = \mathbb{P}(X_1^*, X_2^*) \), that is, we want \( \mathbb{P}_{A \subseteq 2} \) to be NPS.

To use the general identification strategy presented in Section 3 we define each variable as its own block. With only two variables, set \( R_1 \) can be either \( R_1 = \{1\} \) or \( R_1 = \emptyset \). We present two examples below corresponding to these two options.

Example 1. We first consider \( R_1 = \{1\} \), \( S_1 = \emptyset \), and the following identifying assumptions: \( A_1 : X_1 \perp M_1 |X_2^* ; \) and \( A_2 : X_2 \perp M_2 |M_1, X_1 \).

Under \( A_1 \), \( \mathbb{P}_{A_1}(X_1, M_1|X_2^*) = \mathbb{P}_{A_1}(X_1|X_2^*)\mathbb{P}_{A_1}(M_1|X_2^*) = \mathbb{P}(X_1|X_2^*, M_1 = 0)\mathbb{P}(M_1|X_2^*) \), where
\( \mathbb{P}(X_1 | X_2^*, M_1 = 0) \) and \( \mathbb{P}(M_1 | X_2^*) \) are identified from the observed data distribution. When \( X_2^* \neq \ast \), \( \mathbb{P}(X_1 = i | X_2 = j, M_1 = 0, M_2 = 0) = \pi_{ij00}/\pi_{i+00} \), and \( \mathbb{P}(M_1 = m_1 | X_2 = j) = \mathbb{P}(M_1 = m_1 | X_2 = j, M_2 = 0) = \pi_{jm00}/\pi_{j+00} \). Similarly, when \( X_2^* = \ast \) we find \( \mathbb{P}(X_1 = i | X_2 = \ast, M_1 = 0) = \pi_{i+01}/\pi_{i+00} \) and \( \mathbb{P}(M_1 = m_1 | X_2 = \ast) = \pi_{i+m1}/\pi_{i+m0} \). Since \( \mathbb{P}(X_2^*) \) can be obtained from the observed-data distribution as \( \mathbb{P}(X_2 = j, M_2 = 0) = \pi_{i+j0} \) when \( X_2^* = j \neq \ast \), and as \( \mathbb{P}(M_2 = 1) = \pi_{i++1} \) when \( X_2^* = \ast \), using \( \mathbb{P}_{A_1}(X_1, M_1 | X_2^*) \) we obtain a joint distribution for \( (X_1, M_1, X_2^*) \) that relies on \( A_1 \), defined as \( \mathbb{P}_{A_1}(X_1, M_1, X_2^*) = \mathbb{P}_{A_1}(X_1, M_1 | X_2^*) \mathbb{P}(X_2^*) \). Note that \( \mathbb{P}_{A_1} \) can be written as an explicit function of the observed-data distribution.

We now use \( \mathbb{P}_{A_1} \) and identifying assumption \( A_2 \) to obtain \( \mathbb{P}_{A_{\leq 2}}(X_2, M_2 | X_1, M_1) \). From the definition of \( X_2^* \), \( \mathbb{P}_{A_1}(X_1, M_1, X_2^*) \) can be written as \( \mathbb{P}_{A_1}(X_1, M_1, X_2, M_2 = 0) \) when \( X_2^* \neq \ast \) and \( \mathbb{P}_{A_1}(X_1, M_1, M_2 = 1) \) when \( X_2^* = \ast \). From this we can obtain

\[
\mathbb{P}_{A_1}(M_2 = 1 | X_1, M_1) = \frac{\mathbb{P}_{A_1}(X_1, M_1, M_2 = 1)}{\mathbb{P}_{A_1}(X_1, M_1, M_2 = 1) + \sum_{x_2 \in X_2} \mathbb{P}_{A_1}(X_1, M_1, X_2 = x_2, M_2 = 0)},
\]

and

\[
\mathbb{P}_{A_1}(X_2 | X_1, M_1, M_2 = 0) = \frac{\mathbb{P}_{A_1}(X_1, M_1, X_2, M_2 = 0)}{\sum_{x_2 \in X_2} \mathbb{P}_{A_1}(X_1, M_1, X_2 = x_2, M_2 = 0)}.
\]

We then obtain \( \mathbb{P}_{A_{\leq 2}}(X_2, M_2 | X_1, M_1) = \mathbb{P}_{A_{\leq 2}}(X_2 | X_1, M_1) \mathbb{P}_{A_{\leq 2}}(M_2 | X_1, M_1) = \mathbb{P}_{A_1}(X_2 | X_1, M_1, M_2 = 0) \mathbb{P}_{A_1}(M_2 | X_1, M_1) \), which gives us a way to obtain \( \mathbb{P}_{A_{\leq 2}}(X_2, M_2 | X_1, M_1) \) as a function of the distribution \( \mathbb{P}_{A_1} \), which in turn is a function of the observed-data distribution. The final full-data distribution is obtained as \( \mathbb{P}_{A_{\leq 2}}(X_1, M_1, X_2, M_2) = \mathbb{P}_{A_{\leq 2}}(X_2, M_2 | X_1, M_1) \mathbb{P}_{A_1}(X_1, M_1) \), where \( \mathbb{P}_{A_1}(X_1, M_1) \) can be obtained from \( \mathbb{P}_{A_1} \). After some algebra we find

\[
\mathbb{P}_{A_{\leq 2}}(X_1 = i, X_2 = j, M_1 = m_1, M_2 = m_2) = \frac{\pi_{ij00} \pi_{j+m10}}{\sum_{l} \pi_{il00} \pi_{l+m10}^{m_2}} \left( \frac{\pi_{i+01} \pi_{++m1}}{\pi_{++01}} \right)^{m_2}.
\]

It is easy to see that \( \mathbb{P}_{A_{\leq 2}} \) is NPS, that is \( \mathbb{P}_{A_{\leq 2}}(X_1^*, X_2^*) = \mathbb{P}(X_1^*, X_2^*) \). From the final distribution \( \mathbb{P}_{A_{\leq 2}}(X_1, X_2, M_1, M_2) \) we can now obtain

\[
\mathbb{P}_{A_{\leq 2}}(X_1 = i, X_2 = j) = \pi_{ij00} + \pi_{i+01} \pi_{j00} + \pi_{i+j00} + \pi_{i+j10} + \pi_{i+01} \pi_{i+j00} + \pi_{i+j00} \sum_{l} \pi_{il00} \pi_{i+j10},
\]

which is the distribution of inferential interest.

In closing this example, we stress that the final full-data distribution is not invariant to the order in which the blocks of variables appear in the sequence of assumptions. From expression (5) it is clear that the final distribution of the study variables would be different had we identified a distribution for \( (X_1^*, X_2, M_2) \) first. Indeed, if we were to follow the steps in the previous example but reversing the order of the variables, then we would be assuming that \( X_2 \perp \!\!\!\!\perp M_2 | X_1^* \) and \( X_1 \perp \!\!\!\!\perp M_1 | M_2, X_2 \), which are different from \( A_1 \) and \( A_2 \) in this example.
**Example 2.** We now consider $R_1 = \emptyset$, $S_1 = \{1\}$, and the identifying assumptions $B_1 : X_1 \perp M_1|X_2^*$, and $B_2 : X_2 \perp M_2|X_1$.

Assumption $B_1$ is the same as $A_1$ in Example 1, and so $\mathbb{P}_{B_1}(X_1, M_1, X_2^*) = \mathbb{P}_{A_1}(X_1, M_1, X_2^*)$. Assumption $B_2$ is made conditioning only on $X_1$, so we need to marginalize over $M_1$ to obtain $\mathbb{P}_{B_1}(X_1, X_2^*) = \mathbb{P}_{B_1}(X_1, M_1 = 0, X_2^*) + \mathbb{P}_{B_1}(X_1, M_1 = 1, X_2^*)$:

$$\mathbb{P}_{B_1}(X_1 = i, X_2^* = j) = \mathbb{P}_{B_1}(X_1 = i, X_2 = j, M_2 = 0) = \frac{\pi_{ij00}}{\pi_{+j00}} \pi_{+j01},$$

$$\mathbb{P}_{B_1}(X_1 = i, X_2^* = *) = \mathbb{P}_{B_1}(X_1 = i, M_2 = 1) = \frac{\pi_{i001}}{\pi_{++01}} \pi_{++11}.$$

From this we can obtain

$$\mathbb{P}_{B_1}(M_2 = 1|X_1) = \frac{\mathbb{P}_{B_1}(X_1, M_2 = 1)}{\mathbb{P}_{B_1}(X_1, M_2 = 1) + \sum_{x_2 \in X_2} \mathbb{P}_{B_1}(X_1, X_2 = x_2, M_2 = 0)},$$

and

$$\mathbb{P}_{B_1}(X_2|X_1, M_2 = 0) = \frac{\mathbb{P}_{B_1}(X_1, X_2, M_2 = 0)}{\sum_{x_2 \in X_2} \mathbb{P}_{B_1}(X_1, X_2 = x_2, M_2 = 0)}.$$

Using assumption $B_2$, we obtain $\mathbb{P}_{B_{\leq 2}}(X_2, M_2|X_1) = \mathbb{P}_{B_{\leq 2}}(X_2|X_1)\mathbb{P}_{B_{\leq 2}}(M_2|X_1) = \mathbb{P}_{B_1}(X_2|X_1, M_2 = 0)\mathbb{P}_{B_1}(M_2|X_1)$. From this we obtain $\mathbb{P}_{B_{\leq 2}}(X_1, X_2, M_2) \equiv \mathbb{P}_{B_1}(X_1)\mathbb{P}_{B_{\leq 2}}(X_2, M_2|X_1)$ as

$$\mathbb{P}_{B_{\leq 2}}(X_1 = i, X_2 = j, M_2 = m_2) = \frac{\pi_{ij00} \pi_{+j00} \pi_{+j0+}}{\left(\sum_{l} \pi_{lj00} \pi_{+lj0+}\right)^{m_2}} \left(\frac{\pi_{i001}}{\pi_{++01}} \pi_{++11}\right)^{m_2}.$$

Marginalizing over $M_2$, we get

$$\mathbb{P}_{B_{\leq 2}}(X_1 = i, X_2 = j) = \pi_{+j00} \pi_{+j0+} + \sum_{l} \pi_{lj00} \pi_{+lj0+} \pi_{i001} \pi_{++01}\pi_{++11}\pi_{++1+}.$$

Assumptions $B_1$ and $B_2$ are enough to identify $\mathbb{P}_{B_{\leq 2}}(X_1, X_2, M_2)$, and thereby a distribution of the study variables $\mathbb{P}_{B_{\leq 2}}(X_1, X_2)$. Although irrelevant for obtaining the distribution of the study variables, it is worth noticing that $B_1$ and $B_2$ do not allow us to fully identify $\mathbb{P}_{B_{\leq 2}}(M_1|X_1, X_2, M_2)$. From $\mathbb{P}_{B_1}(X_1, M_1, X_2^*)$ we have $\mathbb{P}_{B_1}(M_1|X_1, X_2, M_2 = 0) = \mathbb{P}(M_1|X_2, M_2 = 0)$ and $\mathbb{P}_{B_1}(M_1|X_1, M_2 = 1) = \mathbb{P}(M_1|M_2 = 1)$, but $\mathbb{P}_{B_{\leq 2}}(M_1|X_1, X_2, M_2 = 1)$ remains unidentified. A full-data distribution $\mathbb{P}_{B_{\leq 2}}$ becomes identified under the extra assumption $\mathbb{P}_{B_{\leq 2}}(M_1|X_1, X_2, M_2 = 1) = \mathbb{P}_{B_{\leq 2}}(M_1|X_1, M_2 = 1)$, which corresponds to the extra assumption in [1].

The set of assumptions that we used in this example can be summarized in terms of the missingness mechanism $\mathbb{P}_{B_{\leq 2}}(M_1, M_2|X_1, X_2) = \mathbb{P}_{B_{\leq 2}}(M_1|X_1, X_2, M_2)\mathbb{P}_{B_{\leq 2}}(M_2|X_1, X_2)$, where $\mathbb{P}_{B_{\leq 2}}(M_1|X_1, X_2, M_2 = 1) = \mathbb{P}(M_1|M_2 = 1)$, $\mathbb{P}_{B_{\leq 2}}(M_1|X_1, X_2, M_2 = 0) = \mathbb{P}(M_1|X_2, M_2 = 0)$, and $\mathbb{P}_{B_{\leq 2}}(M_2|X_1, X_2) = \mathbb{P}_{B_1}(M_2|X_1)$. This corresponds to the permutation missingness (PM) mechanism of [Robins 1997].

As in Example 1, the full-data distribution changes when we modify the order in which the

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blocks of variables appear in the identifying assumptions. Changing the order of the variables in this example would correspond to making the assumptions \(X_2 \perp M_2|X_1^*\) and \(X_1 \perp M_1|X_2\).

### 4.2 Sequential Identification for Partially Ignorable Mechanisms

[Harel and Schafer (2009)] introduced different notions of the missing data being partially ignorable. In particular, in some scenarios one may think that the missingness is ignorable for some, but not for all the variables. For example, consider a survey with two blocks of items \(X_S\) and \(X_N\), which contain responses to sensitive and non-sensitive questions, respectively. Given the nature of these variables, one may think that the missingness among the \(X_N\) variables could be ignored, but not among \(X_S\). Our sequential identification procedure can be used to guarantee identifiability under such partially ignorable mechanisms. Our goal here is to show that we can identify a full-data distribution \(\tilde{f}_{A_{\leq 2}}(X_S, X_N, M_S, M_N)\) with the property that the missingness mechanism for \(X_N\) is partially MAR given \(M_S\) ([Harel and Schafer (2009)]), that is,

\[
\tilde{f}_{A_{\leq 2}}(M_N|X_N, X_S, M_S) = f_{A_{\leq 2}}(M_N|X_N^*, X_S^*),
\]

while \(\tilde{f}_{A_{\leq 2}}(M_S|X_N, X_S)\) is determined by some nonignorable assumption. As before, we consider \(f(X_S^*, X_N^*)\) to be known.

Following our sequential identification procedure, we first consider an identifying assumption for the distribution of \(X_N\) and \(M_N\) given \(X_S^*\). We use the conditional MAR assumption:

\[
A_1: \quad f(M_N|X_N, X_S^*) = f(M_N|X_N^*, X_S^*). \tag{7}
\]

This assumption guarantees the existence of a distribution of the variables \(X_N, M_N,\) and \(X_S^*\) with density \(f_{A_1}(X_N, M_N, X_S^*) = f_{A_1}(M_N|X_N, X_S^*) f_{A_1}(X_N, X_S^*) = f(M_N|X_N^*, X_S^*) f_{A_1}(X_N|X_S^*) f(X_S^*)\), where \(f_{A_1}(X_N|X_S^*)\) can be obtained from \(f(X_N|X_S^*)\) as described in page 28 of [Robins (1997)].

Taking \(R_1 = \emptyset\) in our identification procedure, we can now consider any identifying assumption, say \(A_2\), for the distribution of \(X_S\) and \(M_S\) given \(X_N\) that allows us to obtain \(f_{A_{\leq 2}}(X_S, M_S|X_N)\) with the NPS property \(f_{A_{\leq 2}}(X_S^*|X_N) = f_{A_1}(X_S^*|X_N)\). For example, \(A_2\) could come from one of the approaches mentioned in Section 2.2. We then define \(f_{A_{\leq 2}}(X_N, X_S, M_S) \equiv f_{A_{\leq 2}}(X_S, M_S|X_N) f_{A_1}(X_N)\).

To fully identify a full-data distribution \(\tilde{f}_{A_{\leq 2}}(X_S, X_N, M_S, M_N)\) that encodes assumptions \(A_1\) and \(A_2\), we further require the conditional missingness mechanism \(\tilde{f}_{A_{\leq 2}}(M_N|X_N, X_S, M_S)\). Under the extra assumption

\[
\tilde{f}_{A_{\leq 2}}(M_N|X_N, X_S, M_S) = \tilde{f}_{A_{\leq 2}}(M_N|X_N, X_S^*), \tag{8}
\]

and then using \(A_1\) we have identified a full-data distribution with density \(f(M_N|X_N^*, X_S^*) f_{A_{\leq 2}}(X_N, X_S, M_S)\). The NPS property of this distribution is guaranteed by Theorem 1.

A possibility for the \(A_2\) assumption could come from the itemwise conditionally independent nonresponse (ICIN) mechanism of [Sadinle and Reiter (2017)], which is NPS. Denoting \(X_S =
From this we can define \( f_{NPS\ full-data\ distribution}. \) The observed-data density can be written as the product of the density of the observed variables given each missingness pattern times the probability of the missingness pattern, that is \( f(X_m,M = m) = f_m(X_m)\pi_m, \) which for three variables is given by \( f_{000}(X_1, X_2, X_3)\pi_{000}, f_{100}(X_2, X_3)\pi_{100}, f_{010}(X_1, X_3)\pi_{010}, \ldots, f_{011}(X_1)\pi_{011}, \) and \( \pi_{111}. \) Assumption A1 in (7) in this case becomes \( A_1: X_1 \perp M_1|X_2^*, X_3^*, \) which for all \( x_2 \in X_2 \) and \( x_3 \in X_3 \) can be expanded as \( X_1 \perp M_1|M_2 = 0, M_3 = 0, X_2 = x_2, X_3 = x_3; X_1 \perp M_1|M_2 = 1, M_3 = 0, X_3 = x_3; X_1 \perp M_1|M_2 = 0, M_3 = 1, X_2 = x_2; \) and \( X_1 \perp M_1|M_2 = 1, M_3 = 1. \) Using A1 and the observed-data distribution we can obtain \( f_{A_1}(X_1, M_1|X_2^*, X_3^*) = f_{A_1}(X_1|X_2^*, X_3^*)f_{A_1}(M_1|X_2^*, X_3^*) = f(X_1|M_1 = 0, X_2^*, X_3^*)f(M_1|X_2^*, X_3^*), \) where \( f(X_1|M_1 = 0, X_2^*, X_3^*) \) is obtained from

\[
f(X_1|M_1 = 0, M_2 = 0, M_3 = 0, X_2, X_3) = f_{000}(X_1, X_2, X_3)/f_{000}(X_2, X_3),
\]
\[
f(X_1|M_1 = 0, M_2 = 1, M_3 = 0, X_3) = f_{010}(X_1, X_3)/f_{010}(X_3),
\]
\[
f(X_1|M_1 = 0, M_2 = 0, M_3 = 1, X_2) = f_{001}(X_1, X_2)/f_{001}(X_2),
\]
\[
f(X_1|M_1 = 0, M_2 = 1, M_3 = 1) = f_{011}(X_1); \text{ and } f(M_1|X_2^*, X_3^*) \text{ from }
\]

\[
f(M_1 = m_1|M_2 = M_3 = 0, X_2, X_3) \propto [f_{000}(X_2, X_3)\pi_{000}]^I(m_1=0)[f_{100}(X_2, X_3)\pi_{100}]^I(m_1=1),
\]
\[
f(M_1 = m_1|M_2 = 1, M_3 = 0, X_3) \propto [f_{010}(X_3)\pi_{010}]^I(m_1=0)[f_{110}(X_3)\pi_{110}]^I(m_1=1),
\]
\[
f(M_1 = m_1|M_2 = 0, M_3 = 1, X_2) \propto [f_{001}(X_2)\pi_{001}]^I(m_1=0)[f_{101}(X_2)\pi_{101}]^I(m_1=1),
\]
\[
f(M_1 = m_1|M_2 = 1, M_3 = 1) \propto [\pi_{011}]^I(m_1=0)[\pi_{111}]^I(m_1=1).
\]

From this we can define \( f_{A_1}(X_1, M_1|X_2^*, X_3^*) = f_{A_1}(X_1, M_1|X_2^*, X_3^*)f(X_2^*, X_3^*), \) where \( f(X_2^*, X_3^*) \) is obtained from \( f(M_2 = 0, M_3 = 0, X_2, X_3) = f_{000}(X_2, X_3)\pi_{000} + f_{010}(X_2, X_3)\pi_{100}, f(M_2 = 1, M_3 = 0, X_3) = f_{010}(X_3)\pi_{010} + f_{110}(X_3)\pi_{110}, f(M_2 = 0, M_3 = 1, X_2) = f_{001}(X_2)\pi_{001} + f_{101}(X_2)\pi_{101}, \) and \( f(M_2 = 1, M_3 = 1) = \pi_{011} + \pi_{111}. \)

We now incorporate the ICIN assumption for the distribution of \( (X_2, X_3, M_2, M_3) \) given \( X_1. \) We have \( A_2: \{X_2 \perp M_2|X_3, M_3, X_1; \text{ and } X_3 \perp M_3|X_2, M_2, X_1\}. \) The identification results of [Sadine and Reiter (2017)](https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6134547/) guarantee that assumption \( A_2 \) leads to a conditional distribution \( f_{A_{\leq 2}}(X_2, X_3, M_2, M_3|X_1) \) with the NPS property \( f_{A_{\leq 2}}(X_2^*, X_3^*|X_1) = f_{A_1}(X_2^*, X_3^*|X_1), \) where \( f_{A_1}(X_2^*, X_3^*|X_1) \) can be obtained easily from \( f_{A_1}(X_1, M_1, X_2^*, X_3^*). \) Section 5.1 of [Sadine and Reiter (2017)](https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6134547/) provides explicit formulae for the full-data distribution under the ICIN assumption as a function.
of the observed-data distribution, in the case of two variables. We can use those formulae here
with \( f_{A_1}(X_2^*, X_3^*|X_1) \) to obtain conditional ICIN full-data distributions that depend on \( X_1 \). To
simplify the notation below we replace \( f_{A_1} \) by \( g \), and \( f_{A_2} \) by \( h \), and we denote \( g_{m_2m_3}(X_2, X_3|X_1) \equiv g(X_2, X_3|X_1, M_2 = m_2, M_3 = m_3) \) and \( h_{m_2m_3}(X_2, X_3|X_1) \equiv h(X_2, X_3|X_1, M_2 = m_2, M_3 = m_3) \).
Following the formulae of [Sadinle and Reiter (2017)] we obtain
\[ h_{01}(X_2, X_3|X_1) = g_{01}(X_2, X_3|X_1)/g_{00}(X_2|X_1), \]
\[ h_{10}(X_2, X_3|X_1) = g_{10}(X_2, X_3|X_1)/g_{00}(X_3|X_1), \]
and \( h(M_2 = m_2, M_3 = m_3|X_1) = g(M_2 = m_2, M_3 = m_3|X_1) \).
Putting everything together, we obtain
\[ f_{A_{\leq 2}}(X_1, X_2, X_3, M_2 = m_2, M_3 = m_3) = h_{m_2m_3}(X_2, X_3|X_1)g(M_2 = m_2, M_3 = m_3, X_1), \]
from which we can obtain the distribution of the study variables \( f_{A_{\leq 2}}(X_1, X_2, X_3) \). A full-data
density \( \tilde{f}_{A_{\leq 2}}(X_1, X_2, X_3, M_1, M_2, M_3) \) becomes identified under the extra assumption \[ 8 \]. This
distribution therefore encodes the partial ignorability assumption \[ 6 \] for the missingness in \( X_1 \) and
the ICIN assumption \[ 9 \] for \( (X_2, X_3, M_2, M_3) \) given \( X_1 \).

4.3 Usage in Sensitivity Analysis

To illustrate how this approach could be used for sensitivity analysis, we use data related to the
1991 plebiscite where Slovenians voted for independence from Yugoslavia ([Rubin et al. (1995)]). The
data come from the Slovenian public opinion survey, which contained the questions: \( X_I \): are you
in favor of Slovenia’s independence? \( X_S \): are you in favor of Slovenia’s secession from Yugoslavia?
\( X_A \): will you attend the plebiscite? We call these the Independence, Secession, and Attendance
questions, respectively. The possible responses to each of these were YES, NO, and DON’T KNOW.
We follow [Rubin et al. (1995)] in treating DON’T KNOW as missing data.

We use the missingness mechanism presented in Example 3 and compare it with an ignorable
approach, a pattern mixture model (PMM) under the complete-case missing-variable restriction
([Little (1993)]), and the ICIN approach of [Sadinle and Reiter (2017)] which here corresponds to
assuming \( X_j \perp M_j|X_{-j}, M_{-j}; j = I, S, A \). The Attendance question is arguably the less sensitive
of the three questions studied here, so it seems reasonable to consider a partially ignorable mecha-
nism where the nonresponse for \( X_A \) is ignorable given \( M_I \) and \( M_S \), as in \[ 9 \], and the nonresponse
for \( X_I \) and \( X_S \) satisfy the ICIN assumption conditioning on \( X_A \), as in \[ 9 \] in Example 3. Nothing
prevents us from using this approach exchanging the roles of the variables, so we also consider two
other partially ignorable missingness mechanisms, depending on whether we take the nonresponse
for \( X_I \) or for \( X_S \) as ignorable.

To implement these approaches, we first use a Bayesian approach to estimate the observed-
data distribution. The observed data can be organized in a three-way contingency table with cells corresponding to each element of \{YES, NO, DON’T KNOW\}³, as presented in [Rubin et al., 1995]. Treating these data as a random sample from a multinomial distribution, we take a conjugate prior distribution for the cell probabilities: symmetric Dirichlet with parameter 1/27. We take 5,000 draws from the posterior distribution of the observed-data distribution, and for each of these we apply the formulae presented in Example 3 to obtain posterior draws of the full-data distribution under each of the three partially ignorable mechanisms. We use a similar approach to obtain posterior draws of the full-data distribution under ICIN, PMM, and ignorability. For each of the approaches we then obtain draws of the implied probabilities for the items.

Figure 2 displays 5,000 draws from the joint posterior distribution of \( P(\text{Independence} = \text{YES}, \text{Attendance} = \text{YES}) \) and \( P(\text{Attendance} = \text{NO}) \) under each of the six missingness mechanisms considered here. Despite the fact that all of these approaches agree in their fit to the observed data, we obtain quite different inferences under each assumption. When inferences are so sensitive to the identifying assumptions, perhaps the most honest way to proceed is to report all the results under all assumptions deemed plausible given the context.

Figure 2: Samples from joint posterior distributions of \( P(\text{Independence} = \text{YES}, \text{Attendance} = \text{YES}) \) and \( P(\text{Attendance} = \text{NO}) \). Pattern mixture model under the complete-case missing-variable restriction. The three partially ignorable missingness (PIM) mechanisms correspond to which variable we take as having ignorable missingness. The plebiscite results are represented by ⬤. These are shown to illustrate differences between approaches and not to declare better vs worse assumptions for these data.
5 Discussion

The sequential identification procedure can be set up in many different ways, leading to different possibilities for constructing nonignorable missingness mechanisms. The main differences among these possibilities lie in the assumptions about how missingness from any one block of variables affects missingness in other blocks, as illustrated in the examples of Section 3.3 and Section 4. In general, the procedure allows for different levels of dependence on missing variables while ensuring non-parametric saturated models, which provides a useful framework for sensitivity analysis.

Although we presented our identification strategy for arbitrary $K$ blocks of variables, we expect that in practice most analysts would use $K = 2$ blocks when the variables do not naturally fall into ordered blocks of variables. For example, analysts may want to partition variables into one group that requires careful assessment of sensitivity to various missingness mechanisms, such as outcome variables in regression modeling with high fractions of missingness, and a second group that can be treated with generic missingness mechanisms like conditional MAR, such as covariates with low fractions of missingness. These cases require partially ignorable mechanisms like those in Section 4.2. Another scenario where two blocks naturally might arise is when analysts have prior information on how the missingness occurs for a set of variables but not for the rest. Related, analysts might have auxiliary information on the marginal distribution of a few variables, perhaps from a census or other surveys, that enable the identification of mechanisms where the probability of nonresponse for a variable depends explicitly on the variable itself (Hirano et al. (2001); Deng et al. (2013)).

Our sequential identification procedure provides a constructive way of obtaining estimated full-data distributions from estimated observed-data distributions while ensuring non-parametric saturated models. To implement these approaches in practice, one needs sufficient numbers of observations for each missing data pattern, so as to allow accurate non-parametric estimation of the observed-data distribution. This can be challenging in modest-sized samples with large numbers of variables. Of course, this is the case with most methods for handling missing data, including pattern mixture models. In such cases, one may have to sacrifice non-parametric saturated modeling of the observed data in favor of parametric models.

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