Monotonic Mechanisms for Selling Multiple Goods*

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Abstract

Maximizing the revenue from selling two or more goods has been shown to require the use of nonmonotonic mechanisms, where a higher-valuation buyer may pay less than a lower-valuation one. Here we show that the restriction to monotonic mechanisms may not just lower the revenue, but may in fact yield only a negligible fraction of the maximal revenue; more precisely, the revenue from monotonic mechanisms is no more than \(k\) times the simple revenue obtainable by selling the goods separately, or bundled (where \(k\) is the number of goods), whereas the maximal revenue may be arbitrarily larger. We then study the class of monotonic mechanisms and its subclass of allocation-monotonic mechanisms, and obtain useful characterizations and revenue bounds.

Contents

1 Introduction 2

2 Preliminaries 8

2.1 The Model .......................................................... 8
2.2 Pricing Functions ................................................ 10
2.3 The Canonical Pricing Function ................................. 11
2.4 Favorable Tie Breaking ........................................... 12

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1 Introduction

Consider the basic problem of a single seller—a monopolist—who is maximizing his revenue when selling *multiple* goods to a single buyer, in the standard Bayesian setup where the seller knows only the distribution of the buyer’s valuation of the goods (with arbitrary correlation between the goods). Even in the simplest case where the buyer’s valuation
is additive over sets of goods, the seller has neither cost nor value for the goods, and both seller and buyer have quasilinear risk-neutral utilities, the problem turns out to be extremely complex—unlike the single-good case, where it is optimal to simply set up a price for the good (a “take-it-or-leave-it offer,” Myerson 1981). For two or more goods, optimal mechanisms are structurally and conceptually complex (McAfee and McMillan 1988; Thanassoulis 2004; Maneli and Vincent 2006; Alaei, Fu, Haghtpanah, Hartline, and Malekian 2012; Daskalakis, Deckelbaum, and Tzamos 2017); they may require the use of randomizations, i.e., the buyer is offered to buy lottery tickets (Manelli and Vincent 2007, 2012; Chawla, Malec, and Sivan 2010; Daskalakis, Deckelbaum, and Tzamos 2014); there may be arbitrarily many, even infinitely many, outcomes (Daskalakis, Deckelbaum, and Tzamos 2013); and no simple mechanism can guarantee a positive fraction of the optimal revenue (Briest, Chawla, Kleinberg, and Weinberg 2010/2015; Hart and Nisan 2013/2019; Dughmi, Han, and Nisan 2014).

One surprising feature is the nonmonotonicity of the maximal revenue (Hart and Reny 2015), where the buyer is willing to pay more for the goods, yet ends up paying less! That is, there are situations where the buyer’s valuation of each good (his “willingness to pay”) goes up, and yet the seller’s optimal revenue goes down. This cannot happen when there is a single good, where incentive compatibility implies that a buyer with a higher valuation must get more of the good and pay more for it. For two or more goods, this is however no longer true. Consider the following example, from Hart and Reny (2015).

**Example 1.1 Nonmonotonic mechanism** (see Figure 1, top left). Consider the deterministic mechanism for two goods where the price of the first good is 1, the price of the second good is 2, and the price of the bundle of both goods is 4. A buyer who values the first good at 1 and the second good at 2.3 will buy the second good and pay 2 (he prefers getting 2.3 − 2 = 0.3 from the second good to 1 − 1 = 0 from the first good), whereas a buyer with the higher valuations of 2 for the first good and 2.7 for the second will buy the first good instead and pay only 1 (he prefers 2 − 1 = 1 to 2.7 − 2 = 0.7). This mechanism is nonmonotonic: a higher valuation pays less than a lower valuation (here: (1, 2.3) pays 2, and (2, 2.7) pays 1).

The fact that such a mechanism exists is in itself not surprising. What is surprising is that nonmonotonic mechanisms are needed in order to maximize the seller’s revenue. As a consequence, in such cases the seller’s revenue goes down as the buyer’s valuations increase.

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1. Despite the risk neutrality of both seller and buyer; in the risk-averse case, lotteries are needed already in the single-good case (Siedner 2019).
2. By “2010/2015” we mean “conference proceeding in 2010 and journal publication in 2015.”
3. In the first-order dominance sense; i.e., the probability of higher valuations increases.
4. I.e., a higher allocation—which means a higher probability of getting the good in the indivisible case, and a higher fraction of the good in the divisible case.
5. The other options are not better: they yield 0 (when getting nothing) or 1 + 2.3 − 4 = −0.7 (when buying both goods); similarly for the other valuation (2, 2.7).
Figure 1: Deterministic mechanisms for two goods ($q$ denotes the allocation of the two goods, and $s$ denotes the payment; see Section 2.1)
go up (just move a small probability mass from a low valuation that pays more to a high valuation that pays less).

The first result of the present paper is that the restriction to using only monotonic mechanisms may not just decrease the revenue, but may in fact yield only a negligible portion of the optimal revenue. Indeed, we have (see Theorem \ref{thm:main} in Section \ref{sec:main}):

\begin{itemize}
  \item \textbf{Result A.} For every \( k \)-good valuation \( X \), the maximal revenue \( \text{MonRev}(X) \) that is obtained by monotonic mechanisms is no more than \( k \) times the revenue \( \text{SRev}(X) \) that is obtained by selling the goods separately, or the revenue \( \text{BRev}(X) \) that is obtained by selling the goods as a single bundle:

\[
\text{MonRev}(X) \leq k \cdot \text{SRev}(X) \quad \text{and} \quad \text{MonRev}(X) \leq k \cdot \text{BRev}(X).
\]

\end{itemize}

Since the separate and bundled revenues (as well as the revenue from any other simple mechanisms) can be arbitrarily small relative to the optimal revenue (Briest et al. 2015 for \( k \geq 3 \), and Hart and Nisan 2019 for \( k \geq 2 \)), from Result A we immediately get (see Theorem \ref{thm:main2} in Section \ref{sec:main}):

\begin{itemize}
  \item \textbf{Result B.} For every number of goods \( k \geq 2 \) there exist \( k \)-good valuations \( X \) whose optimal revenue \( \text{Rev}(X) \) is infinite, yet all monotonic mechanisms yield a bounded revenue:

\[
\text{Rev}(X) = \infty \quad \text{and} \quad \text{MonRev}(X) = 1
\]

(for bounded valuations, for every \( \varepsilon > 0 \) there are \( X \) with values in, say, \([0, 1]^k\), such that \( \text{MonRev}(X) < \varepsilon \cdot \text{Rev}(X) \)).

The bound \( k \) in Result A is tight relative to the bundled revenue \( \text{BRev} \) (since selling separately—which is a monotonic mechanism—may yield \( k \) times \( \text{BRev} \), already in the i.i.d. case), while relative to the separate revenue \( \text{SRev} \) we do not know whether it is so: the largest gap that we have obtained is only of the order of \( \log k \).

We thus study the class of monotonic mechanisms.\footnote{See Rubinstein and Weinberg (2015) and Yao (2018) for some studies of monotonic mechanisms.} Two classes of monotonic mechanisms have been identified in Hart and Reny (2015): the class of symmetric deterministic mechanisms, and the class of submodular mechanisms (i.e., those where the price increase due to increasing the quantity of one good is lower when the quantities of other goods are higher). The latter mechanisms turn out to satisfy a stronger form of monotonicity, namely, allocation monotonicity, which requires the allocation of goods to be a nondecreasing function of the valuation. This is easily seen to imply the monotonicity of the mechanism (i.e., the payment being a nondecreasing function of the valuation; see Proposition \ref{prop:allocation_monotonicity}). An example of such a mechanism:
**Example 1.2** *Monotonic (and thus allocation-monotonic) mechanism* (see Figure 1, bottom). Consider the submodular deterministic mechanism for two goods where the price of good 1 is 1.5, the price of good 2 is 2, and the price of the bundle is 3 (which is less than 1.5 + 2).

Allocation monotonicity is a strictly stronger requirement than monotonicity; for example:

**Example 1.3** *Monotonic but not allocation-monotonic mechanism* (see Figure 1, top right). Consider the symmetric deterministic mechanism for two goods where the price of each single good is 1 and the price of the bundle is 4 (which is more than 1 + 1, and so the mechanism is supermodular). When the valuation increases from, say, (1, 2, 3) to (2, 1, 7) the allocation changes from buying good 2 to buying good 1; this contradicts allocation monotonicity, since the higher valuation (2, 1, 7) gets less of good 2 (it does not contradict monotonicity, since the payment stays the same, at 1).

Our next result is a characterization of allocation monotonicity for deterministic mechanisms (Theorem C (iv) in Section 4.1; note that characterization results require some “regularity” when breaking ties):

- **Result C1.** A deterministic mechanism is allocation monotonic if and only if it is submodular.

This does not however hold for general, probabilistic mechanisms, where submodularity implies allocation monotonicity (see Theorem C (ii) in Section 4.1), but the converse is not true: see Example 5.2, which belongs to the interesting class of “quadratic mechanisms,” introduced and studied in Section 5. In fact, allocation-monotonic mechanisms satisfy a strict weakening of submodularity, namely, *separable subadditivity*: buying a bundle of goods, whether deterministic or probabilistic, costs no more than buying the goods separately (one may refer to this as “subadditivity across goods”; it is in general weaker than subadditivity, where the same requirement applies also to different quantities of the same good—see Section 2.5). We have (see Theorem C (ii)–(iii) in Section 4.1):

- **Result C2.** Every submodular mechanism is allocation monotonic, and every allocation-monotonic mechanism is separably subadditive.

(Neither one of these implications is an equivalence when there are multiple goods.)

The property of separable subadditivity is closely related to the notion of “sybil-proofness” of Chawla, Teng, and Tzamos (2022). We appeal to their Theorem 1.3 (see our Appendix A.2) and obtain (see Theorem D in Section 4.2):
• **Result D.** For every $k$-good valuation $X$, the maximal revenue $\text{AMonRev}(X)$ that is obtained by allocation-monotonic mechanisms is no more than $O(\log k)$ times the revenue $\text{SRev}(X)$ that is obtained by selling the goods separately:

$$\text{AMonRev}(X) \leq O(\log k) \cdot \text{SRev}(X).$$

Next, we deal with the other class of monotonic mechanisms, namely, the symmetric deterministic mechanisms, for which we obtain (see Theorem 6.2 in Section 6.2):

• **Result E.** For every $k$-good valuation $X$, the maximal revenue $\text{SymDRev}(X)$ that is obtained by symmetric deterministic mechanisms (which are monotonic) is no more than $O(\log^2 k)$ times the revenue $\text{SRev}(X)$ that is obtained by selling the goods separately:

$$\text{SymDRev}(X) \leq O(\log^2 k) \cdot \text{SRev}(X).$$

We get this result by first studying the revenue of *supermodular* symmetric deterministic mechanisms, for which we obtain a tight bound of $\ln k$ relative to $\text{SRev}$, and then extend it to the entire class, again by the Chawla, Teng, and Tzamos (2022) result. The above class of supermodular mechanisms also provides a gap of $\ln k$ between the revenue of allocation-monotonic mechanisms $\text{AMonRev}$ and that of monotonic mechanisms $\text{MonRev}$ (see Example 6.2 and Corollary 6.3).

The paper is organized as follows. Section 2 presents the basic model, concepts, notation, and preliminaries, including in particular the useful notion of the canonical pricing function in Section 2.3. In Section 3 we deal with the monotonic revenue, and obtain Results A and B. Allocation-monotonic mechanisms—their characterizations (Results C1 and C2) and revenue bound (Result D)—are studied in Section 4. The class of quadratic mechanisms is introduced in Section 5.1, providing, in Section 5.1, an example of an allocation-monotonic mechanism that is not submodular. Section 6 is devoted to the class of symmetric deterministic mechanisms—which are monotonic—and provides Result E. Characterizations of monotonicity for general (non-symmetric) deterministic mechanisms are presented in Section 7 and we conclude with a number of problems that have remained open. The appendices contain several complements and proofs; in particular, in Appendix A.9 we study general deterministic mechanisms (which need not be monotonic), extending the analysis of the symmetric case of Section 6.
2 Preliminaries

2.1 The Model

The notation follows Hart and Nisan (2017, 2019) and Hart and Reny (2015).

One seller (a monopolist) is selling a number $k \geq 1$ of indivisible goods (or items, objects, etc.) to one buyer; let $K := \{1, 2, \ldots, k\}$ denote the set of goods. The goods have no cost or value to the seller, and their values to the buyer are $x_1, x_2, \ldots, x_k \geq 0$. The value of getting a set of goods is *additive*: each subset $I \subseteq K$ of goods is worth $x(I) := \sum_{i \in I} x_i$ to the buyer (and so, in particular, the buyer’s demand is not restricted to one good only). The valuation of the goods is given by a random variable $X$ to the buyer (and so, in particular, the buyer’s demand is not restricted to one good only). The realization $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}_+^k$ of $X$ is known to the buyer, but not to the seller, who knows only the distribution $F$ of $X$ (which may be viewed as the seller’s belief); we refer to a buyer with valuation $x$ also as a buyer of *type* $x$. The buyer and the seller are assumed to be risk neutral and to have quasilinear utilities.

The objective is to *maximize* the seller’s (expected) *revenue*.

As was well established by the so-called Revelation Principle (starting with Myerson 1981; see for instance the book of Krishna 2010), we can restrict ourselves to “direct mechanisms” and “truthful equilibria.” A direct mechanism $\mu$ consists of a pair of functions $(q, s)$, where $q = (q_1, q_2, \ldots, q_k) : \mathbb{R}_+^k \to [0, 1]^k$ and $s : \mathbb{R}_+^k \to \mathbb{R}$, which prescribe the allocation and the payment, respectively. Specifically, if the buyer reports a valuation vector $x \in \mathbb{R}_+^k$, then $q_i(x) \in [0, 1]$ is the probability that the buyer receives good $i$ (for all $i = 1, 2, \ldots, k$), and $s(x)$ is the payment that the seller receives from the buyer. When the buyer of type $x$ reports his type truthfully, his payoff is $b(x) = \sum_{i=1}^k q_i(x) x_i - s(x) = q(x) \cdot x - s(x)$, and the seller’s payoff is $s(x)$.

The mechanism $\mu = (q, s)$ satisfies individual rationality (IR) if $b(x) \geq 0$ for every $x \in \mathbb{R}_+^k$; it satisfies incentive compatibility (IC) if $b(x) \geq q(y) \cdot x - s(y)$ for every alternative report $y \in \mathbb{R}_+^k$ of the buyer when his value is $x$, for every $x \in \mathbb{R}_+^k$.

The (expected) revenue of a mechanism $\mu = (q, s)$ from a buyer with random valuation $X$, which we denote by $R(\mu; X)$, is the expectation of the payment received by the seller, i.e., $R(\mu; X) = \mathbb{E}[s(X)]$. We now define

- **REV($X$)**, the *optimal revenue*, is the maximal revenue that can be obtained: $\text{REV}(X) = \sup_\mu R(\mu; X)$, where the supremum is taken over the class of all IC and IR mechanisms $\mu$.

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7For vectors $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}_+^k$, we write $x \geq 0$ when $x_i \geq 0$ for all $i$, and $x \gg 0$ when $x_i > 0$ for all $i$. The nonnegative orthant is $\mathbb{R}_+^k = \{x \in \mathbb{R}_+^k : x \geq 0\}$, and $x \cdot y = \sum_{i=1}^k x_i y_i$ is the scalar product of $x$ and $y$ in $\mathbb{R}_+^k$.

8All functions are assumed to be Borel-measurable; see Hart and Reny (2015), footnotes 10 and 48.

9An alternative interpretation of the model: the goods are infinitely divisible and the valuation is linear in quantity, in which case $q_i$ is the quantity (i.e., fraction) of good $i$ that the buyer gets.
When there is only one good, i.e., when \( k = 1 \), Myerson’s (1981) result is that

\[
\text{Rev}(X) = \sup_{t \geq 0} t \cdot (1 - F(t)), \tag{1}
\]

where \( F(t) = \mathbb{P}[X \leq t] \) is the cumulative distribution function of \( X \). Thus, it is optimal for the seller to “post” a price \( p \), and then the buyer buys the good for the price \( p \) whenever his value is at least \( p \); in other words, the seller makes the buyer a “take-it-or-leave-it” offer to buy the good at price \( p \).

Besides the maximal revenue \( \text{Rev}(X) \), we are also interested in what can be obtained from certain classes of mechanisms. For any class \( \mathcal{N} \) of IC and IR mechanisms we denote

- \( \mathcal{N}\text{-Rev}(X) := \sup_{\mu \in \mathcal{N}} R(\mu; X) \), the maximal revenue over the class \( \mathcal{N} \).

In particular:

- \( \text{SRev}(X) \), the separate revenue, is the maximal revenue that can be obtained by selling each good separately. Thus

\[
\text{SRev}(X) = \text{Rev}(X_1) + \text{Rev}(X_2) + \ldots + \text{Rev}(X_k).
\]

- \( \text{BRev}(X) \), the bundling revenue, is the maximal revenue that can be obtained by selling all the goods together in one “bundle.” Thus

\[
\text{BRev}(X) = \text{Rev}(X_1 + X_2 + \ldots + X_k).
\]

- \( \text{DRev}(X) \), the deterministic revenue, is the maximal revenue that can be obtained by deterministic mechanisms; these are the mechanisms in which every good \( i = 1, 2, \ldots, k \) is either fully allocated or not at all, i.e., \( q_i(x) \in \{0, 1\} \) for all valuations \( x \in \mathbb{R}_+^k \) (rather than \( q_i(x) \in [0, 1] \)).

As seen in Hart and Nisan (2017, Proposition 6), when maximizing revenue we can limit ourselves without loss of generality to those IC and IR mechanisms that satisfy in addition the no positive transfer (NPT) property, namely, \( s(x) \geq 0 \) for every \( x \in \mathbb{R}_+^k \), from which it follows that \( s(0) = b(0) = 0 \), where \( 0 = (0, 0, \ldots, 0) \in \mathbb{R}_+^k \).

From now on we will assume that all mechanisms \( \mu \) are \( k \)-good mechanisms that are given in direct form and satisfy IC, IR, and NPT; thus, \( \mu = (q, s) : \mathbb{R}_+^k \to [0, 1]^k \times \mathbb{R}_+ \) and \( b : \mathbb{R}_+^k \to \mathbb{R}_+ \), and \( s(0) = b(0) = 0 \).

We conclude with a standard technical result. The function \( b \) is nonexpansive if \( b(y) - b(x) \leq \sum_{i=1}^k (y_i - x_i) \) for all \( x \leq y \) in \( \mathbb{R}_+^k \); a vector \( g \in \mathbb{R}^k \) is a subgradient of the function \( b \) at the point \( x \in \mathbb{R}_+^k \) if \( b(y) \geq b(x) + g \cdot (y - x) \) for every \( y \in \mathbb{R}_+^k \); the set of subgradients of \( b \) at \( x \) is denoted by \( \partial b(x) \).

\[10\]See also Riley and Samuelson (1981) and Riley and Zeckhauser (1983).

\[11\]For some of the results only IC is needed.
Proposition 2.1 A function \( b : \mathbb{R}^k_+ \to \mathbb{R}_+ \) is a buyer payoff function of some mechanism if and only if \( b \) is continuous, convex, nondecreasing, and nonexpansive. In this case, \( b \) is obtained from the mechanism \( \mu = (q, s) \) if and only if \( q(x) \in \partial b(x) := \{g \in \partial b(x) : g \geq 0\} \) and \( s(x) = q(x) \cdot x - b(x) \) for every \( x \in \mathbb{R}^k_+ \).

See Appendix A.1 in Hart and Reny (2015) for details. The set \( \partial b(x)^{-} \) differs from \( \partial b(x) \) only at boundary points \( x \) of \( \mathbb{R}^k_+ \): if \( x_i = 0 \) then the \( i \)-th coordinate \( g_i \) of any subgradient \( g \in \partial b(x) \) can be lowered arbitrarily, and so be negative (see Theorem 25.6 in Rockafellar 1970).

2.2 Pricing Functions

An equivalent description of mechanisms is by means of pricing functions.

A pricing function \( p : [0,1]^k \to \mathbb{R} \cup \{\infty\} \) assigns to each allocation \( g \in [0,1]^k \) a price \( p(g) \), which may be infinite; it generates the set of choices (a “potential menu”) \( \mathcal{M}_p := \{(g,p(g)) : g \in [0,1]^k\} \). Given a mechanism \( \mu = (q, s) \), we will say that \( p \) is a pricing function of \( \mu \) if for every buyer valuation \( x \in \mathbb{R}^k_+ \) the choice \( (q(x), s(x)) \) of \( \mu \) is an optimal choice from the set \( \mathcal{M}_p \). This means, first, that \( p(q(x)) = s(x) \), and second, that \( g \cdot x - p(g) \leq q(x) \cdot x - p(q(x)) = b(x) \) for all \( g \in [0,1]^k \), where \( b \) is the buyer payoff function of \( \mu \). Thus, the price of the allocation \( q(x) \) equals the corresponding payment \( s(x) \) (this is well defined since \( q(x) = q(x') \) implies that \( s(x) = s(x') \) by IC), and if one can choose any allocation \( g \) in \([0,1]^k\) for the corresponding price \( p(g) \) then \( q(x) \) is an optimal choice for a buyer with valuation \( x \). Let \( Q \equiv Q_\mu := q(\mathbb{R}^k_+) \equiv \{q(x) : x \in \mathbb{R}^k_+\} \subseteq [0,1]^k \) denote the range of allocations of \( \mu \); while the price of any \( g \in Q \) (i.e., any allocation \( g \) that is chosen at some valuation) is well defined, the prices of all \( g \notin Q \) are not: they just need to be high enough so that these \( g \) are never chosen (which is easily achieved, for instance, by making the price of each \( g \notin Q \) infinite). To summarize: \( p \) is a pricing function of the mechanism \( \mu \) with buyer payoff function \( b \) if and only if

\[
\text{for every } x \in \mathbb{R}^k_+: \text{note that } b(0) = 0 \text{ implies that } p \text{ cannot have negative values, and so the range of } p \text{ is } \mathbb{R}_+ \cup \{\infty\} = [0, \infty].
\]

When \( \mu \) is a deterministic mechanism, i.e., \( Q \subseteq \{0,1\}^k \), it is sometimes simpler and more convenient to consider deterministic pricing functions \( p \equiv p^0 \), which are defined only on the set \( \{0,1\}^k \) of deterministic allocations. In this case we identify \( \{0,1\}^k \) with

\[12\]In Hart and Reny (2015) the function \( b \) is extended to a convex function on the entire space \( \mathbb{R}^k \), and then \( b \) is continuous and \( \partial b(x) \subseteq [0,1]^k \) everywhere, and there is no need for \( \partial b(x)^{-} \). Here we have found it more natural to keep \( \mathbb{R}^k_+ \) as the domain of \( b \) (and so \( b(x) = \infty \) for \( x \notin \mathbb{R}^k_+ \)).

\[13\]Infinite prices must be allowed. For a simple example, take two goods \( k = 2 \), and let \( \mu \) sell the first good at price 1 (i.e., \( p(1,0) = 1 \); any price function \( p \) of \( \mu \) must then put an infinite price on the second good (i.e., \( p(0,1) = \infty \)).
the set of subsets \(2^K\) of \(K = \{1, \ldots, k\}\), and write \(p(A)\) instead of \(p(1_A)\) (where \(1_A\) denotes the indicator vector of \(A\), i.e., \((1_A)_i = 1\) for \(i \in A\) and \((1_A)_i = 0\) for \(i \notin A\)); thus, \(p \equiv p^D: \{0, 1\}^k \sim 2^K \to [0, \infty]\).

In general, each mechanism has at least one pricing function—perhaps more than one, when not all allocations are chosen—and each pricing function generates at least one mechanism—perhaps more than one, depending on the way that the buyer breaks ties when indifferent. In the next two sections we deal with these multiplicities and provide useful selections that simplify the analysis.

### 2.3 The Canonical Pricing Function

Among all pricing functions of a given mechanism there is a “canonical” one that turns out to be particularly convenient and useful: the price of any unused allocation \(g\) is set to be the smallest price that ensures that \(g\) is never strictly optimal (setting it to be the largest price, i.e., infinite, is unwieldy: for instance, it may yield a pricing function that is at times decreasing). Since \(p(g)\) must satisfy the inequalities \(g \cdot x - p(g) \geq b(x)\), i.e., \(p(g) \geq g \cdot x - b(x)\), for all \(x\), the smallest such price is \(\sup_x (g \cdot x - b(x))\).

Following Hart and Reny (2015, Appendix A.2), we thus define the canonical pricing function \(p_0\) of the mechanism \(\mu\), with buyer payoff function \(b\), by

\[
p_0(g) := \sup_{x \in \mathbb{R}^+_k} (g \cdot x - b(x)) \tag{3}
\]

for every \(g \in [0, 1]^k\). As seen above, \(p\) is a pricing function of \(\mu\) if and only if \(p_0(g) \leq p(g) \leq \infty\) for every \(g \in [0, 1]^k\), and \(p(g) = p_0(g)\) for every \(g \in Q\), and so \(p_0\) is the minimal pricing function of \(\mu\). The function \(p_0\) is nondecreasing, convex, and closed (i.e., lower semicontinuous\(^{14}\)), because it is the supremum of such functions; also, \(p_0(0) = 0\) (because \(b(0) = 0\)). In Proposition A.1 in Appendix A.1 we will show that the canonical pricing function is in fact the unique pricing function that is nondecreasing, convex, and closed. Moreover, the buyer payoff function \(b\) and the canonical pricing function \(p_0\) are Fenchel conjugates; see (2) for \(p = p_0\) and (3).

When \(\mu\) is a deterministic mechanism, i.e., \(Q \subseteq \{0, 1\}^k\), the restriction to \(\{0, 1\}^k\) of the canonical pricing function \(p_0\) will be called the canonical deterministic pricing function of \(\mu\), and denoted by \(p^D_0\); thus, \(p^D_0: \{0, 1\}^k \to [0, \infty]\) is given by (3) for every \(g \in \{0, 1\}^k\). When \(\{0, 1\}^k\) is identified with \(2^K\), the set of subsets of \(K = \{1, \ldots, k\}\), (3) becomes

\[
p^D_0(A) := \sup_{x \in \mathbb{R}^+_k} (x(A) - b(x)),
\]

\(^{14}\)The function \(p_0\) is lower semicontinuous if \(\lim_{h \to 0} p_0(h) \geq p_0(g)\) for every \(g \in [0, 1]^k\). At relatively interior points \(g\) of \(\text{dom} \ p_0 = \{ g: p_0(g) < \infty \}\) the function \(p_0\) is in fact continuous, since it is a convex; see Rockafellar (1970), Theorems 9.4 and 10.2.
for every $A \subseteq K$, where $x(A) = \sum_{i \in A} x_i$. From the properties of $p_0$ on $[0,1]^k$ it follows that $p_0^D$ is a nondecreasing function, i.e., if $A \subseteq B \subseteq K$ then $p_0^D(A) \leq p_0^D(B)$ (convexity and closedness are irrelevant here); in fact, as we will now show, $p_0^D$ is the unique pricing function that is nondecreasing. Let $p_Q : Q \to \mathbb{R}_+$ denote the common restriction to $Q$ of all pricing functions of $\mu$.

**Proposition 2.2** Let $\mu$ be a deterministic mechanism. Then the canonical deterministic pricing function $p_0^D$ of $\mu$ is the unique deterministic pricing function of $\mu$ that is nondecreasing, and it is given by

$$p_0^D(A) = \inf\{p_Q(B) : B \in Q, \ B \supseteq A\} \tag{4}$$

for every $A \subseteq K$.

**Proof.** Let $p_2(A)$ denote the right-hand side of (4). When $A \subseteq B \in Q$ we get $p_0(A) \leq p_0(B) = p_Q(B)$ (because $p_0$ is nondecreasing), and then the infimum over $B \in Q$ yields $p_0(A) \leq p_2(A)$, with equality when $A \in Q$ (because then $B = A$ is included). Thus $p_2$ is indeed a pricing function of $\mu$.

Assume by way of contradiction that $p_0(A) < p_2(A)$ for some $A \notin Q$. Consider the valuation $x = M1_A$ for large $M > 0$ (specifically, $M > \max_{B \in Q} p_Q(B)$). We claim that $(A,p_0(A))$ is strictly better at $x$ than $(B,p_0(B))$ for any $B \in Q$; indeed, if $B \supseteq A$ then $p_0(A) < p_2(A) \leq p_Q(B)$, and so $x(B) - p_Q(B) = M |A| - p_Q(B) < M |A| - p_0(A) = x(A) - p_0(A)$, and if $B \not\supseteq A$ then at least one coordinate of $A$ is not included in $B$, and so $x(A) - x(B) \geq M$, which, for large enough $M$, yields $x(B) - p_Q(B) < x(A) - p_0(A)$. Thus $(A,p_0(A))$ is the unique optimal choice at $x$, in contradiction to $A \notin Q$.

Uniqueness: let $p$ be any nondecreasing pricing function of $\mu$; since $p$ coincides with $p_Q$ on $Q$ and is nondecreasing, it follows that $p \leq p_2$ (because $p(A) \leq p(B) = p_Q(B)$ for every $A \subseteq B \in Q$, and so $p(A) \leq p_2(A)$); now $p_2 = p_0$, the minimal pricing function (proved above), and so $p \leq p_2 = p_0$ yields $p = p_0$.

### 2.4 Favorable Tie Breaking

A pricing function $p$ does not determine the mechanism in case of ties, i.e., when two or more allocations provide the same maximal payoff for some buyer valuation: $b(x) = g \cdot x - p(g) = g' \cdot x - p(g')$ for some $x$ and distinct $g$ and $g'$. More generally, ties occur at points $x$ where the buyer payoff function $b$ (which is uniquely determined by $p$; see (2)) is not differentiable and there are multiple subgradients in $\partial b(x)^+$ (see Proposition 2.1). In order to fully specify the mechanism one needs to specify how ties are broken in such cases. The “favorable” tie-breaking rules below—which maximize the payment and the allocation—are particularly convenient, because they simplify the analysis while affecting neither the maximization of revenue nor the relevant mechanism properties (such
as monotonicity and allocation monotonicity). Moreover, these properties could well fail if one were to break ties arbitrarily; see Hart and Reny (2015) (the reason that we need to go beyond their seller favorability is that for allocation monotonicity the choice of allocation matters as well).

We will say that a mechanism \( \mu = (q, s) \), with buyer payoff function \( b \), is

- **seller favorable** (Hart and Reny 2015) if for every \( x \in \mathbb{R}^k_+ \) the payment \( s(x) \) is maximal; i.e., there is no \( g \in \partial b(x)^+ \) such that \( g \cdot x - b(x) > s(x) \);

- **buyer favorable** if for every \( x \in \mathbb{R}^k_+ \) the allocation \( q(x) \) is maximal; i.e., there is no \( g \in \partial b(x)^+ \) such that \( g \geq q(x) \) and \( g \neq q(x) \);

- **tie favorable** if it is both seller and buyer favorable.

We will at times refer to a mechanism \( \mu' \) with the same buyer payoff function \( b \) as a (tie-breaking) version of the mechanism \( \mu \). Since the canonical pricing function \( p_0 \) is determined by \( b \), all tie-breaking versions of \( \mu \) have the same canonical pricing function.

The seller-favorability condition is equivalent to the subgradient \( q(x) \in \partial b(x)^+ \) being maximal in the direction \( x \), i.e., \( q(x) \in \partial b(x)^+_x := \arg \max_{g \in \partial b(x)^+} g \cdot x \), and then \( s(x) = b'(x; x) - b(x) \), where \( b'(y; z) \) denotes the derivative of \( b \) at \( y \) in the direction \( z \); see Hart and Reny 2015, Appendix A.1. Tie favorability requires in addition that \( q(x) \) be a maximal element of the set \( \partial b(x)^+_x \) (and thus of \( \partial b(x)^+ \) as well; at interior points \( x \gg 0 \) any seller-favorable choice is buyer favorable, and hence tie favorable). Since there are always such choices (the sets \( \partial b(x)^+ \) and \( \partial b(x)^+_x \) are nonempty, compact, convex, and \( \subseteq [0, 1]^k \)), there always exist seller-, buyer-, and tie-favorable versions of any mechanism.

The restriction to seller-favorable mechanisms is without loss of generality when maximizing revenue, because these have the highest payments and revenues; moreover, all seller-favorable versions of the mechanism with buyer payoff function \( b \) have identical payment functions \( s \) (given by the above formula), and so they always yield identical revenues. Therefore the further restriction to the subclass of tie-favorable mechanisms is also without loss of generality when maximizing revenue.

Finally, as we will show in Proposition A.3 and Corollary 4.2, monotonicity and allocation monotonicity are preserved when one considers tie-favorable versions of such mechanisms. This of course applies also to any property (such as submodularity; see Section 2.5) that depends only on the buyer payoff function \( b \) or only on the canonical pricing function \( p_0 \), which are common to all versions of a mechanism.

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\[ ^{15} \] What is needed is a certain consistency when breaking ties. Indeed, some of the characterization results below hold under other tie-breaking rules, such as always choosing a minimal allocation; we focus on tie-favorable mechanisms because these are the ones that maximize revenue.
2.5 Submodularity and Supermodularity

Notions of submodularity and supermodularity will play an important role in our analysis and results. We will use these notions for functions that are defined not only on discrete domains (such as deterministic pricing functions on \( \{0, 1\}^k \sim 2^k \)) but also on continuous domains (such as general pricing functions on \([0, 1]^k\), and buyer payoff functions on \(\mathbb{R}^k\)).

Let \( \mathcal{L} \subseteq \mathbb{R}^k_+ \) be a lattice with respect to coordinatewise maximum and minimum; i.e., for any \( x = (x_i)_{i=1,...,k} \) and \( y = (y_i)_{i=1,...,k} \) in \( \mathcal{L} \) the vectors \( x \lor y := (\max\{x_i, y_i\})_{i=1,...,k} \) and \( x \land y := (\min\{x_i, y_i\})_{i=1,...,k} \) belong to \( \mathcal{L} \) as well; in our use, \( \mathcal{L} \) will be \( \mathbb{R}^k_+, [0, 1]^k, \) or \( \{0, 1\}^k \). The vectors \( x, y \geq 0 \) are orthogonal if \( x \cdot y = 0 \), which is the same as \( x \land y = 0 \); i.e., \( x \) and \( y \) are positive on disjoint sets of coordinates.

Let \( f : \mathcal{L} \to \mathbb{R} \cup \{\infty\} \) be a function (such as a buyer payoff function \( b \), or a pricing function \( p \); the value \( \infty \) is thus allowed).

- \( f \) is **submodular** if for every \( x, y \in \mathcal{L} \),
  \[
  f(x) + f(y) \geq f(x \lor y) + f(x \land y). \tag{5}
  \]

- \( f \) is **supermodular** if for every \( x, y \in \mathcal{L} \),
  \[
  f(x) + f(y) \leq f(x \lor y) + f(x \land y) \tag{6}
  \]
  (for convex functions, supermodularity is equivalent to the stronger requirement of **ultramodularity**; see the discussion below).

- \( f \) is **separably subadditive** if for every \( x, y, x+y \in \mathcal{L} \) such that \( x \) and \( y \) are orthogonal,
  \[
  f(x+y) \leq f(x) + f(y). \tag{7}
  \]

- \( f \) is **separably superadditive** if for every \( x, y, x+y \in \mathcal{L} \) such that \( x \) and \( y \) are orthogonal,
  \[
  f(x+y) \geq f(x) + f(y). \tag{8}
  \]

A number of comments:

**Submodularity and supermodularity.** The submodularity condition (5) can be rewritten as follows: put \( d^1 := x \land y \geq 0 \) and \( d^2 := y \land y \geq 0 \); then \( d^1 \) and \( d^2 \) are orthogonal (because \( d^1_i > 0 \) when \( x_i > y_i \), and \( d^2_i > 0 \) when \( y_i > x_i \)), and replacing \( x \land y \) with \( x \) yields \( f(x + d^1) + f(x + d^2) \geq f(x + d^1 + d^2) + f(x) \). Thus: \( f \) is submodular if and only if

\[
  f(x + d^2) - f(x) \geq f(x + d^1 + d^2) - f(x + d^1) \tag{9}
  \]

for every \( x \in \mathcal{L} \) and \( d^1, d^2 \geq 0 \) such that \( d^1 \) and \( d^2 \) are orthogonal (assume that all vectors are in the domain where \( f \) is finite). The interpretation of (9) is that the contribution
of $d^2$ to $f$ can only decrease as we increase $x$ by $d^1$; since $d^1$ and $d^2$ are orthogonal, this means that the change in $f$ when we increase some coordinates can only be smaller when other coordinates are larger. (This equivalent formulation shows that (5) is the appropriate definition of submodularity for pricing functions; cf. Babaioff, Nisan, and Rubinstein 2018).

When $f$ is twice differentiable, it is thus submodular if and only if its second-order partial derivatives satisfy

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \leq 0 \text{ for all } i \neq j;$$

i.e., all off-diagonal elements of the Hessian matrix $\nabla^2 f(x)$ are nonpositive: the $i$-th partial derivative $\partial f(x)/\partial x_i$ at $x$ can only decrease when some coordinate $x_j$ with $j$ different from $i$ increases. We emphasize that submodularity does not entail (9) for all $d^1, d^2 \geq 0$, but only for orthogonal $d^1, d^2 \geq 0$; thus, there is no requirement on the diagonal elements of the Hessian matrix. In fact, the function $f$ may well be convex, in which case the opposite inequality holds for $d^1 = d^2$, and the diagonal elements of the Hessian matrix are nonnegative, i.e., $\partial^2 f(x)/\partial x_i^2 \geq 0$ for all $i$.

Similarly, supermodularity is equivalent to

$$f(x + d^2) - f(x) \leq f(x + d^1 + d^2) - f(x + d^1)$$

for orthogonal $d^1, d^2 \geq 0$, which means that the marginals of $f$ can only increase when other coordinates increase (and so $\partial^2 f(x)/\partial x_i \partial x_j \geq 0$ for all $i \neq j$ in the twice-differentiable case).

**Ultramodularity.** A function $f$ is ultramodular if (11) holds for all $d^1, d^2 \geq 0$ (whether or not they are orthogonal); when $f$ is twice differentiable, this translates to all the elements of the Hessian matrix $\nabla^2 f(x)$ being nonnegative: $\partial^2 f(x)/\partial x_i \partial x_j \geq 0$ for all $i, j$. For convex functions, supermodularity is equivalent to the stronger condition of ultramodularity (because (11) holds for $i = j$ by convexity; see Corollary 4.1 in Marinacci and Montrucchio 2005).

**Separable subadditivity and superadditivity.** First, we note that separable subadditivity is weaker than subadditivity, which requires $f(x + y) \leq f(x) + f(y)$ to hold not only for orthogonal $x$ and $y$, but for all $x$ and $y$ (the two properties are equivalent when the domain $L$ is $\{0, 1\}^k$, because then $x + y \in L$ if and only if $x$ and $y$ are orthogonal). The same applies to separable superadditivity vs. superadditivity. For functions $f$ that satisfy $f(0) = 0$ supermodularity implies separable superadditivity, and submodularity implies separable subadditivity (because for orthogonal vectors $x, y$ we have $x \vee y = x + y$ and $x \land y = 0$).

Next, the separable subadditivity condition (7) can be equivalently expressed as fol-

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16For example, the function $f(x) = (x_1 - x_2)^2$ is submodular and convex.
lows. Given a partition of $K$ into two disjoint sets $K'$ and $K''$, every vector $x \in \mathbb{R}^k_+$ can be expressed as the sum $x = x' + x''$ of the orthogonal vectors $x'$ and $x''$ with supports $K'$ and $K''$, respectively (put $x'_i = x_i$ and $x''_i = 0$ for $i \in K'$, and $x'_i = 0$ and $x''_i = x_i$ for $i \in K''$).\footnote{When $\mathcal{L}$ is $\mathbb{R}^k_+$, $[0, 1]^k$, or $\{0, 1\}^k$, if $x \in \mathcal{L}$ then $x', x'' \in \mathcal{L}$.} Then $f$ is separably subadditive if and only if
\[ f(x) \leq f(x') + f(x'') \]
for every $x$ and every partition $K = K' \cup K''$. Applying this repeatedly yields
\[ f(x) \leq \sum_{i=1}^k f(x_ie^i), \]
where $e^i \in \mathbb{R}^k_+$ denotes the $i$-th unit vector (and so $x = \sum_i x_ie^i$). The same applies to separable superadditivity, with all inequalities reversed.

Finally, if $f$ is both submodular and supermodular, i.e., (5) and (6) hold as equalities, then (7) and (8) hold as equalities, and $f$ is separably additive (or “additive across coordinates”), i.e., $f(x) = \sum_{i=1}^k f(x_ie^i)$ (where $e^i$ is the $i$-th unit vector).

Turning now to mechanisms, we define:

- A mechanism $\mu$ is submodular / supermodular / separably subadditive / separably superadditive if its canonical pricing function $p : [0, 1]^k \rightarrow [0, \infty]$ is, respectively, submodular / supermodular / separably subadditive / separably superadditive.

Thus, $\mu$ is submodular if
\[ p(g) + p(h) \geq p(g \lor h) + p(g \land h) \quad (12) \]
holds for its canonical pricing function $p$ for every $g, h \in [0, 1]^k$; i.e., (see (9) and (10)), the marginal price of good $i$ can only decrease as the quantity of good $j$ increases. For deterministic mechanisms it suffices that (12) holds for deterministic allocations, i.e., for sets of goods:
\[ p(A) + p(B) \geq p(A \cup B) + p(A \cap B) \quad (13) \]
for all $A, B \subseteq K$ (which is easily seen to be equivalent to
\[ p(A \cup \{i\}) - p(A) \geq p(A \cup \{i, j\}) - p(A \cup \{j\}) \quad (14) \]
for all $A \subset K$ and $i \neq j$ not in $A$); Proposition A.5 in Appendix A.5 will show that it suffices that (13) holds only for $A$ and $B$ in the range of allocations $Q$ of the mechanism and, moreover, for some pricing function $p$, not necessarily the canonical one.
2.6 Monotonicity and Allocation Monotonicity

Let $\mu = (q, s)$ be a mechanism.

- $\mu$ is monotonic (Hart and Reny 2015) if its payment function $s$ is nondecreasing, i.e., $s(x) \leq s(y)$ for every two valuations $x \leq y$ in $\mathbb{R}_+^k$.

- $\mu$ is allocation monotonic if its allocation function $q$ is nondecreasing, i.e., $q(x) \leq q(y)$ for every two valuations $x \leq y$ in $\mathbb{R}_+^k$.

Let $\text{MonRev}(X)$ and $\text{AMonRev}(X)$ denote the maximal revenue that can be achieved by monotonic and allocation-monotonic mechanisms, respectively, for a $k$-good random valuation $X$. In the case of one good, i.e., when $k = 1$, every IC mechanism is monotonic and allocation monotonic (this follows from the IC conditions: see the proof of Proposition 2 in Hart and Reny 2015), and so $\text{AMonRev}(X) = \text{MonRev}(X) = \text{Rev}(X)$.

Two immediate observations: allocation monotonicity is a stronger requirement than monotonicity, and both $\text{MonRev}$ and $\text{AMonRev}$—unlike $\text{Rev}$—are nondecreasing with respect to first-order stochastic dominance: they can only increase when the buyer valuations, i.e., his “willingness to pay,” increase.

**Proposition 2.3** If the mechanism $\mu$ is allocation monotonic then it is monotonic.

**Proof.** If $q(y) \geq q(x)$ but $s(y) < s(x)$ then $(q(y), s(y))$ is always strictly better than $(q(x), s(x))$ (a higher allocation at a strictly cheaper payment), and so the latter cannot be the optimal choice at $x$. ■

**Proposition 2.4** Let $X$ and $Y$ be random valuations. If $Y$ first-order stochastically dominates $X$ then

$$\text{MonRev}(Y) \geq \text{MonRev}(X) \quad \text{and} \quad \text{AMonRev}(Y) \geq \text{AMonRev}(X).$$

**Proof.** Without loss of generality assume that the random variables $X$ and $Y$ are “coupled,” i.e., that they are defined on the same probability space and $X \leq Y$ pointwise. Then, for every (allocation-)monotonic mechanism $\mu = (q, s)$ we have $s(X) \leq s(Y)$ (because $s$ is nondecreasing), and so $R(\mu; X) = \mathbb{E}[s(X)] \leq \mathbb{E}[s(Y)] = R(\mu; Y)$. Taking the supremum over all (allocation-)monotonic mechanisms $\mu$ yields the result. ■

Two classes of monotonic mechanisms have been identified in Hart and Reny (2015): the class of deterministic symmetric mechanisms, and the class of submodular mechanisms (Propositions 5 and 8 there). We will consider these two classes in Sections 6 and 4 below.

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18 Adding the IC inequalities $q(x) \cdot x - s(x) \geq q(y) \cdot x - s(y)$ (at $x$) and $q(y) \cdot y - s(y) \geq q(x) \cdot y - s(x)$ (at $y$) gives $(q(y) - q(x)) \cdot (y - x) \geq 0$, which does not yield $q(y) \geq q(x)$ when $y \geq x$ unless we are in the one-dimensional case of $k = 1$ (see Hart and Reny 2015).
While the mechanisms in the first class need not be allocation monotonic—see Example 1.3 in the Introduction—those in the second class are allocation monotonic (the argument that proves Proposition 8 in Hart and Reny 2015 can be easily adapted to show this; see also Section 4 below).

3 Revenue of Monotonic Mechanisms

In this section we prove that monotonic mechanisms for \( k \) goods cannot extract more than \( k \) times the separate revenue, and also no more than \( k \) times the bundled revenue (Theorem A), which is very different from general—and hence necessarily nonmonotonic—mechanisms for \( k \geq 2 \), which can extract an arbitrarily large multiple of these (and any other simple) revenues (Theorem B).

**Theorem A** Let \( X \) be a \( k \)-good random valuation. Then

\[
\text{MonRev}(X) \leq k \cdot \min\{\text{SRev}(X), \text{BRev}(X)\}.
\]

Since selling separately and selling the bundle of all goods are monotonic mechanisms, we trivially have

\[
\text{MonRev}(X) \geq \max\{\text{SRev}(X), \text{BRev}(X)\}. \quad (15)
\]

An immediate implication of Theorem A and the results of Hart and Nisan (2019)\(^ {19} \) is that for two or more goods monotonic mechanisms cannot guarantee any positive fraction of the optimal revenue. The Guaranteed Fraction of Optimal Revenue (GFOR) for monotonic mechanisms is thus zero, the same as that of any class of “simple mechanisms,” and the revenue is mostly obtained from nonmonotonic mechanisms. Theorem A implies that monotonic mechanisms are no better, in terms of revenue, than mechanisms with a menu size that is at most the number of goods. Formally:

**Theorem B** For every \( k \geq 2 \):

(i) There exists a \( k \)-good random valuation \( X \) such that

\[
\text{MonRev}(X) = 1 \quad \text{and} \quad \text{Rev}(X) = \infty.
\]

(ii) For every \( \varepsilon > 0 \) there exists a \( k \)-good random valuation with bounded values (in, say, \([0,1]^k\)) such that

\[
\text{MonRev}(X) < \varepsilon \cdot \text{Rev}(X).
\]

(iii) There exists a \( k \)-good random valuation \( X \) such that

\[
\text{MonRev}(X) \leq \frac{k^2}{2^k - 1} \cdot \text{DRev}(X).
\]

\(^{19}\)See Briest et al. (2015) for \( k \geq 3 \).
Proof. Use \( \text{MonRev}(X) \leq k \cdot \text{BRev}(X) \) and Theorems A and D in Hart and Nisan (2019).

To prove Theorem \( \text{A} \) we start with two preliminary results. A \( k \)-good valuation \( X \) all of whose coordinates are equal, i.e., with support included in the set \( \{ ye : y \geq 0 \} \subset \mathbb{R}_+^k \), where \( e := (1, ..., 1) \in \mathbb{R}_+^k \), is called a diagonal valuation. A diagonal valuation is thus

\[
(Y, ..., Y) \equiv Ye,
\]

where \( Y \) is a one-good random valuation.

Lemma 3.1 Let \( ye \) be a \( k \)-good diagonal valuation. Then

\[
\text{Rev}(Ye) = k \cdot \text{Rev}(Y).
\]

Proof. The restriction of any (IC and IR) \( k \)-good mechanism \( \mu = (q, s) \) to the diagonal \( \{ ye : y \geq 0 \} \subset \mathbb{R}_+^k \) yields an (IC and IR) one-good mechanism \( \nu = (\hat{q}, \hat{s}) \) given by

\[
\hat{q}(y) := (1/k) \sum_{i=1}^k q_i(ye) \quad \text{and} \quad \hat{s}(y) := (1/k)s(ye)
\]

for every \( y \geq 0 \). Conversely, any (IC and IR) one-good mechanism \( \nu = (\hat{q}, \hat{s}) \) yields an (IC and IR) \( k \)-good mechanism \( \mu = (q, s) \) given by

\[
q(x) := (\hat{q}(\bar{x}), ..., \hat{q}(\bar{x})) \in [0,1]^k \quad \text{and} \quad s(x) := k \hat{s}(\bar{x}),
\]

where \( \bar{x} := (1/k) \sum_{i=1}^k x_i \), for every \( x \in \mathbb{R}_+^k \). For the revenue, we have \( R(\mu; Ye) = k \cdot R(\nu; Y) \).

The following result may be of independent interest.

Proposition 3.2 Let \( X_1, ..., X_n \) be one-good random valuations. Then

\[
\text{Rev} \left( \max_{1 \leq i \leq n} X_i \right) \leq \min \left\{ \sum_{i=1}^n \text{Rev}(X_i), \, \text{Rev} \left( \sum_{i=1}^n X_i \right) \right\}.
\]

Proof. For SRev: for every \( t \geq 0 \) we have

\[
\mathbb{P} \left[ \max_{1 \leq i \leq n} X_i \geq t \right] = \mathbb{P} \left[ \bigcup_{1 \leq i \leq n} \{ X_i \geq t \} \right] \leq \sum_{i=1}^n \mathbb{P} \left[ X_i \geq t \right],
\]

and thus

\[
t \cdot \mathbb{P} \left[ \max_{1 \leq i \leq n} X_i \geq t \right] \leq \sum_{i=1}^n t \cdot \mathbb{P} \left[ X_i \geq t \right] \leq \sum_{i=1}^n \text{Rev}(X_i).
\]

Taking the supremum over \( t \geq 0 \) yields \( \text{Rev}(\max_i X_i) \leq \sum_i \text{Rev}(X_i) \) by the Myerson result for one good (II).

For BREV: since \( \max_i X_i \leq \sum_i X_i \) and the revenue is monotonic for one good (Proposition 11 in Hart and Nisan 2017) we get \( \text{Rev}(\max_i X_i) \leq \text{Rev}(\sum_i X_i) \).
Remarks. (a) The result of Proposition 3.2 is tight: in the following example we have $\text{REV}(\max_i X_i) = \sum_i \text{REV}(X_i) = \text{REV}(\sum_i X_i)$. Let $(X_1, \ldots, X_n)$ take as values the $n$ unit vectors in $\mathbb{R}^n_+$, with probability $1/n$ each; then $\max_i X_i = \sum_i X_i = 1$ and so $\text{REV}(\max_i X_i) = \text{REV}(\sum_i X_i) = 1$; and, for each $i$ we have $\text{REV}(X_i) = 1/n$ (obtained by selling at price 1), and so $\sum_i \text{REV}(X_i) = 1$.

(b) Let $\text{SYM} \text{REV}(X)$ denote the maximal revenue that is obtained by symmetric separate mechanisms; by Myerson’s result for one good (1), this is achieved by selling each one of the goods at the same price $t$, i.e.,

$$\text{SYM} \text{REV}(X) = \sup_{t \geq 0} \sum_{i=1}^n t \cdot \mathbb{P}[X_i \geq t]$$

(wheras $\text{REV}(X) = \sum_{i=1}^n \sup_{t \geq 0} t \cdot \mathbb{P}[X_i \geq t]$). The proof of (i) above shows that

$$\text{REV} \left( \max_{1 \leq i \leq n} X_i \right) \leq \min \{ \text{SYM} \text{REV}(X_1, \ldots, X_n), \text{BREV}(X_1, \ldots, X_n) \}.$$ 

This tightening of $\text{REV}$ to $\text{SYM} \text{REV}$ applies to Theorem A as well.

21 Another instance where one can replace $\text{REV}$ with $\text{SYM} \text{REV}$ is in Theorem A.7 in Hart and Nisan (2019), whose proof shows that $\int 1/w(t) \, dt$ is in fact a bound on the multiple of the symmetric separate revenue.

We can now prove Theorem A. See Appendix A.4 for an easy generalization.

Proof of Theorem A. Let $X$ be a $k$-good random valuation, and put $Y := \max_{1 \leq i \leq k} X_i$. Since $X \leq Ye$, for any monotonic mechanism $\mu = (q, s)$ we have $s(X) \leq s(Ye)$, and so, taking expectation, $R(\mu; X) \leq R(\mu; Ye)$. Taking the supremum over all monotonic $\mu$ yields $\text{MONREV}(X) \leq \text{REV}(Ye)$, which equals $k \cdot \text{REV}(Ye) = k \cdot \text{REV}(\max_i X_i)$ by Lemma 3.1 thus,

$$\text{MONREV}(X) \leq k \cdot \text{REV} \left( \max_{1 \leq i \leq n} X_i \right).$$

Proposition 3.2 completes the proof. ■

Remarks. (a) The multiple $k$ is tight for $\text{BREV}$ in Theorem A: Example 27 in Hart and Nisan (2017) shows that for every $\varepsilon > 0$ there are $k$ independent goods such that

$$\text{MONREV}(X) \geq \text{SREV}(X) \geq (k - \varepsilon) \cdot \text{BREV}(X)$$

(the first inequality is due to selling separately being a monotonic mechanism).

(b) When $k = 2$ the multiple $k$ is tight for $\text{SREV}$ in Theorem A: Example A.3 in Hart and Nisan (2019) yields $\text{MONREV}(X) = \text{BREV}(X) = 2 \cdot \text{SREV}(X)$.

When $k \geq 3$ the best-known lower bound is of the order of $\log k$: Proposition 25 in
Hart and Nisan (2017) shows that for $k$ independent ER goods we have

$$\text{MonRev}(X) \geq \text{BRev}(X) \geq \Omega(\log k) \cdot \text{SRev}(X)$$ (17)

(the second inequality cannot be improved because $\text{BRev}(X) \leq O(\log k) \cdot \text{SRev}(X)$ for any $X$ by Proposition A.4 in Hart and Nisan 2019). See also Example 6.2 below, where $\text{MonRev}(X) \geq \ln k \cdot \text{SRev}(X)$.

4 Allocation-Monotonic Mechanisms

In this section we provide characterizations of allocation-monotonic mechanisms (Theorem C), and then show that these mechanisms can yield at most an $O(\log k)$ multiple of the separate revenue (Theorem D).

4.1 Characterization of Allocation-Monotonic Mechanisms

Hart and Reny (2015) proved that (seller-favorable) submodular mechanisms are monotonic; the proof there shows that they are in fact allocation monotonic. In this section we will show that for deterministic mechanisms the converse is true, i.e., allocation monotonicity is equivalent to submodularity; however, this equivalence does not hold for probabilistic mechanisms (see Section 5.1; the correct characterization of allocation monotonicity is the supermodularity of the Fenchel conjugate of the canonical pricing function, which is the buyer payoff function). Since these results may not hold for arbitrary tie-breaking rules, we consider buyer-favorable mechanisms, which suffice when maximizing revenue (see Section 2.4), and preserve allocation monotonicity, as we show in Corollary 4.2 below.

We now state the characterization result.

**Theorem C** Let $\mu$ be a buyer-favorable mechanism. Then:

(i) $\mu$ is allocation monotonic if and only if the buyer payoff function $b$ of $\mu$ is supermodular.

(ii) If $\mu$ is submodular then $\mu$ is allocation monotonic.

(iii) If $\mu$ is allocation monotonic then $\mu$ is separably subadditive.

(iv) When $\mu$ is a deterministic mechanism: $\mu$ is allocation monotonic if and only if $\mu$ is submodular.

When there are multiple goods, the converse of (ii) is not true for general, non-deterministic, mechanisms: see Example 5.2 in Section 5.1; also, the converse of (iii)
is not true: see Appendix A.7. Thus, the class of buyer-favorable allocation-monotonic mechanisms lies between the strictly smaller class of submodular mechanisms and the strictly larger class of separably subadditive mechanisms.

The proof of Theorem C consists of showing the connections between properties of a mechanism \( \mu \) and those of its buyer utility function \( b \) and its canonical pricing function \( p \). These are summarized in the two tables below, one for general, probabilistic, mechanisms, and the other for deterministic mechanisms. The vertical relations are proved in the three Propositions 4.1, 4.3, and 4.4 that follow, and the horizontal implications are immediate (because \( b(0) = 0 \) and \( p(0) = 0 \); see Section 2.5).

| General mechanisms                  | Deterministic mechanisms                  |
|-------------------------------------|-------------------------------------------|
| \( \mu \) allocation monotonic     | \( \mu \) allocation monotonic            |
| \( \Downarrow [P4.1] \)             | \( \Downarrow [P4.1] \)                   |
| \( b \) supermodular \( \Rightarrow \) \( b \) separably superadditive | \( b \) supermodular \( \Rightarrow \) \( p \) submodular |
| \( \Uparrow [P4.3] \)             | \( \Uparrow [P4.3] \)                   |
| \( p \) submodular \( \Rightarrow \) \( p \) separably subadditive | \( p \) submodular |
| (\( \mu \) submodular)             | (\( \mu \) submodular)                   |

The first proposition shows the equivalence between allocation monotonicity and supermodularity of the buyer payoff function \( b \). This is easy to see when \( b \) is twice differentiable: allocation monotonicity means that \( q_i(x) = \partial b(x)/\partial x_i \) is nondecreasing in \( x \) for all \( i \), i.e., \( \partial^2 b(x)/\partial x_i \partial x_j \geq 0 \) for all \( i, j \); this is equivalent to the ultramodularity of \( b \), which, since \( b \) is a convex function, is the same as the supermodularity of \( b \).

**Proposition 4.1** Let \( \mu \) be a mechanism, with buyer payoff function \( b \). Then:

(i) If \( \mu \) is allocation monotonic then \( b \) is supermodular.

(ii) If \( b \) is supermodular then the unique buyer-favorable (and thus tie-favorable) version \( \tilde{\mu} \) of \( \mu \) is allocation monotonic.

**Proof.** (i) Assume that the function \( q \) is nondecreasing; we will show that \( b \) is ultramodular, i.e., \( b(x + d^2) - b(x) \leq b(x + d^1 + d^2) - b(x + d^1) \) for every \( x \) in \( \mathbb{R}_+^k \) and every \( d^1, d^2 \geq 0 \).

First, we claim that for every \( x \in \mathbb{R}_+^k \) and \( d \geq 0 \) we have

\[
b(x + d) - b(x) = d \cdot \int_0^1 q(x + td) \, dt.
\]  

(18)

Indeed, let \( \beta(t) := b(x + td) \) for \( t \geq 0 \); then \( \beta \) is a convex function of \( t \), and the IC inequality\(^{22}\)

\[
b(x + (t + \delta)d) - b(x + td) \geq q(x + td) \cdot \delta d
\]

\(^{22}\)Use \( b(x) = q(x) \cdot x - s(x) \) and \( b(x + \delta d) \geq q(x) \cdot (x + \delta d) - s(x) \).
yields, after we divide by \( \delta \) and take the limit as \( \delta \to 0^- \) and \( \delta \to 0^+ \),

\[
\beta'_-(t) \leq q(x + td) \cdot d \leq \beta'_+(t)
\]

for every \( t > 0 \); applying Theorem 24.2 and Corollary 24.2.1 in Rockafellar (1970) proves (18).

Next, for every \( x \in \mathbb{R}^k_+ \) and every \( d^1, d^2 \geq 0 \), we have

\[
b(x + d^2) - b(x) = d^2 \cdot \int_0^1 q(x + td^2) \, dt \leq d^2 \cdot \int_0^1 q(x + d^1 + td^2) \, dt = b(x + d^1 + d^2) - b(x + d^1),
\]

where the inequality is by \( d^1 \geq 0 \) (since \( q \) is nondecreasing) and \( d^2 \geq 0 \), and the equalities are by (18). Therefore \( b \) is ultramodular.

(ii) If \( b \) is ultramodular then \( \nabla^+ b(x) := (b'(x; e^i))_{i=1, \ldots, k} \) is the maximal subgradient of \( b \) at \( x \), for every \( x \in \mathbb{R}^k_+ \), and \( \nabla^+ b(x) \) is nondecreasing in \( x \) (by Lemma 5.1 and Theorem 5.4 (iii) in Marinacci and Motrucchio 2005)\(^{23}\). Thus \( \tilde{q}(x) = \nabla^+ b(x) \) is the unique buyer-favorable choice at \( x \), and \( \tilde{q} \) is nondecreasing. \( \blacksquare \)

This immediately implies that buyer favorability, and hence tie favorability, preserves allocation monotonicity; when maximizing revenue, the restriction to tie-favorable versions of allocation-monotonic mechanisms is thus without loss of generality.

**Corollary 4.2** Let \( \mu \) be a mechanism, with buyer payoff function \( b \), and let \( \tilde{\mu} \) be a buyer-favorable mechanism with the same buyer payoff function \( b \). If \( \mu \) is allocation monotonic then \( \tilde{\mu} \) is allocation monotonic.

**Proof.** If \( \mu \) is allocation monotonic then \( b \) is supermodular by Proposition 4.1 (i), and then the buyer-favorable mechanism \( \tilde{\mu} \) with the same \( b \) is allocation monotonic by (ii) of the same proposition. \( \blacksquare \)

The next proposition provides the relation between the supermodularity of \( b \) and the submodularity of its Fenchel conjugate \( p \).

**Proposition 4.3** Let \( \mu \) be a mechanism, with buyer payoff function \( b \) and canonical pricing function \( p \).

(i) If \( p \) is submodular then \( b \) is supermodular.

(ii) When \( \mu \) is a deterministic mechanism: \( p \) is submodular if and only if \( b \) is supermodular.

\(^{23}\)Marinacci and Montrucchio (2005) prove these claims only for interior points \( x \); the proof extends to boundary points as well, as shown in Proposition A.6 in Appendix A.6.
The converse of (i) does not hold in the general, non-deterministic case: see Section 5.1 below. Assuming differentiability (and ignoring technicalities), \( p \) is submodular if and only if the off-diagonal entries of the Hessian matrix \( \nabla^2 p \) are \( \leq 0 \); this implies that the off-diagonal entries of its inverse \( (\nabla^2 p)^{-1} \), which is the Hessian matrix \( \nabla^2 b \) of \( p \)'s Fenchel conjugate\(^{24}\), are \( \geq 0 \), i.e., \( b \) is supermodular; the converse is however not true in general (cf. Proposition 5.1 and Example 5.2).

**Proof.** (i) Since for any \( x, y \in \mathbb{R}_+^k \) and \( g, h \in Q \) we have

\[
g \cdot x + h \cdot y \leq (g \lor h) \cdot (x \lor y) + (g \land h) \cdot (x \land y)
\]

(because\(^{25}\) the difference between the right-hand side and the left-hand side is \( (g - g \land h) \cdot (y - x \land y) + (h - h \land h) \cdot (x - x \land y) \geq 0 \)), subtracting the submodularity inequality\(^{12}\), i.e., \( p(g) + p(h) \geq p(g \lor h) + p(g \land h) \), gives

\[
[g \cdot x - p(g)] + [h \cdot y - p(h)] \leq [(g \lor h) \cdot (x \lor y) - p(g \lor h)] + [(g \land h) \cdot (x \land y) - p(g \land h)]
\]

\[
\leq b(x \lor y) + b(x \land y)
\]

(the inequality on the second line because \( p \) is a pricing function; see \(^2\)). Taking the supremum over \( g \in Q \) and \( h \in Q \) yields, again by \(^2\), \( b(x) + b(y) \leq b(x \lor y) + b(x \land y) \), and so \( b \) is supermodular.

(ii) Let \( \mu \) be a deterministic mechanism, with range of allocations \( Q \subseteq 2^K \), and let \( p : 2^K \to [0, \infty) \) be its canonical deterministic pricing function. Thus, \( b(x) = \max_{A \subseteq K} (x(A) - p(A)) = \max_{A \subseteq Q} (x(A) - p(A)) \) for every \( x \in \mathbb{R}_+^k \), and \( p(A) = \sup_{x \in \mathbb{R}_+^k} (x(A) - b(x)) \) for every \( A \subseteq K \).

Fix a large \( M > 0 \) (specifically, \( M \geq \max_{A \subseteq Q} p(A) \)); for \( x = M1_A \) (with \( A \subseteq K \)) we have

\[
b(x) \geq x(A) - p(A) = M |A| - p(A).
\]

Moreover, if \( A \in Q \) then

\[
b(x) = x(A) - p(A) = M |A| - p(A)
\]

(i.e., \( A \) is optimal at \( x \)). Indeed, \( x(A) - p(A) \geq x(B) - p(B) \) for every \( B \subseteq K \), because if \( B \supseteq A \) then \( x(B) = x(A) \) and \( p(A) \leq p(B) \), and if \( B \nsubseteq A \) then \( x(B) = x(A) - M |A \setminus B| \leq x(A) - M \) (since \( A \setminus B \neq \emptyset \)) and \( p(A) - p(B) \leq p(A) \leq M \) (since \( A \in Q \)).

Let \( A, B \in Q \). Take \( x = M1_A \) and \( y = M1_B \); then \( x \lor y = M1_{A \cup B} \) and \( x \land y = M1_{A \cap B} \).

\(^{24}\)See Crouzeix (1977).

\(^{25}\)This is the ultramodularity of the function \( (x, g) \mapsto g \cdot x = \sum_i g_i x_i \) (all its second-order partial derivatives are either 1 or 0).
which yields

\[ b(x) = M |A| - p(A), \]
\[ b(y) = M |B| - p(B), \]
\[ b(x \lor y) \geq M |A \cup B| - p(A \cup B), \]
\[ b(x \land y) \geq M |A \cap B| - p(A \cap B) \]

(the last two are inequalities because \( A \cup B \) and \( A \cap B \) need not be in \( Q \)). Since \(|A| + |B| = |A \cup B| + |A \cap B|\), we get

\[ b(x) + b(y) - b(x \lor y) - b(x \land y) \leq p(A \cup B) + p(A \cap B) - p(A) - p(B). \]

If \( b \) is supermodular then the left-hand side is \( \geq 0 \), and so the right-hand side is \( \geq 0 \), which yields the submodularity inequality (13) for all \( A, B \in Q \); this extends to all \( A, B \subseteq K \) by Proposition A.5 in Appendix A.5.

The third proposition shows the equivalence between the separable superadditivity of \( b \) and the separable subadditivity of its Fenchel conjugate \( p \) (unlike super/submodularity, where only one direction holds in general, and there is equivalence only in the deterministic case; see Proposition 4.3).

**Proposition 4.4** Let \( \mu \) be a mechanism, with buyer payoff function \( b \) and canonical pricing function \( p \). Then \( b \) is separably superadditive if and only if \( p \) is separably subadditive.

**Proof.** (i) Assume that \( b \) is separably superadditive. Take a partition \( K = K' \cup K'' \). For every \( g \in [0, 1]^k \) and \( x \in \mathbb{R}_+^k \) we then have \( g \cdot x = g' \cdot x' + g'' \cdot x'' \) (where \( x = x' + x'' \) and \( g = g' + g'' \) are the corresponding decompositions, as in Section 2.5), and \( b(x) \geq b(x') + b(x'') \) (by the separable superadditivity of \( b \)), which yields

\[
g \cdot x - b(x) \leq (g' \cdot x' + g'' \cdot x'') - (b(x') + b(x'')) = g' \cdot x' - b(x') + g'' \cdot x'' - b(x'') \leq p(g') + p(g'').
\]

Taking the supremum over \( x \) gives \( p(g) \leq p(g') + p(g'') \).

(ii) Assume that \( p \) is separably subadditive. Let \( x \in \mathbb{R}_+^k \) and take a partition \( K = K' \cup K'' \), resulting in vectors \( x' \) and \( x'' \) as above. Let \( g' := q(x') \wedge 1_{K'} \) and \( g'' := q(x'') \wedge 1_{K''} \). Then \( g' \cdot x' = q(x') \cdot x' \), which implies that \( p(g') \geq p(q(x')) \) by IC at \( x' \); but \( g' \leq q(x') \) and \( p \) is nondecreasing, and so we have equality, i.e., \( p(g') = p(q(x')) \), which yields \( b(x') = g' \cdot x' - p(g') \). Similarly, \( b(x'') = g'' \cdot x'' - p(g'') \). Put \( g := g' + g'' \) then \( g \in [0, 1]^k \) (because \( g' \) and \( g'' \) are orthogonal) and \( g \cdot x = g' \cdot x' + g'' \cdot x'' \), and so we get

\[
b(x) \geq g \cdot x - p(g) \geq (g' \cdot x' + g'' \cdot x'') - (p(g') + p(g'')) = (g' \cdot x' - p(g')) + (g'' \cdot x'' - p(g'')) = b(x') + b(x''),
\]
where the second inequality is by the separable subadditivity of $p$. ■

The three Propositions 4.1, 4.3, and 4.4 prove Theorem C (see the implications in the above tables).

### 4.2 Revenue of Allocation-Monotonic Mechanisms

In this section we prove that allocation-monotonic mechanisms yield a multiple of at most $O(\log k)$ of the separate revenue (Theorem D); this follows from the separable subadditivity property of allocation-monotonic mechanisms (Theorem C (iii)), by applying a result of Chawla, Teng, and Tzamos (2022).

**Theorem D** Let $X$ be a $k$-good random valuation. Then

$$\text{AMonRev}(X) \leq 2 \ln(2k) \cdot \text{SRev}(X).$$

**Proof.** Let $\mu$ be a tie-favorable allocation-monotonic mechanism (recall Corollary 4.2), with buyer payoff function $b$ and canonical pricing function $p$. Define a new pricing function $p' : [0, 1]^k \to [0, \infty]$ by

$$p'(g) := \sum_{i=1}^{k} p(g_i e^i);$$

thus $p'$ is a separably additive function, and thus it yields a separable mechanism (namely, take for each good $i$ the one-good mechanism with pricing function $p_i(g_i) := p(g_i e^i)$).

The function $p$ is separably subadditive by Theorem C (iii), and so

$$p(g) \leq p'(g) \quad \text{(19)}$$

for every $g$ (see Section 2.5). Because $p$ is nondecreasing we have $p(g) \geq p(g_i e^i)$ for all $i$, and so $p(g)$ is at least as large as their average, i.e.,

$$p(g) \geq \frac{1}{k} p'(g). \quad \text{(20)}$$

Since $p$ is a nondecreasing and closed function, so is the derived function $p'$. We can therefore apply the result of Chawla, Teng, and Tzamos (2022) (see Theorem A.2 in Appendix A.2): $\text{AMonRev}(X)$, which is the revenue obtainable from any pricing function $p$ as above, is, by (19) and (20), at most $2 \ln(2k)$ times the revenue obtainable from any derived pricing function $p'$; the pricing functions $p'$ are separable, and so their revenue is at most $\text{SRev}(X)$. ■
5 Quadratic Mechanisms

In this section we introduce a useful class of mechanisms that have a simple representation, and are thus amenable to an easier analysis: mechanisms where the relevant functions—specifically, the payment function $s$, the buyer payoff function $b$, and the pricing function $p$—are all convex quadratic functions (in appropriate domains). In particular, this will allow the construction in Section 5.1 of an allocation-monotonic mechanism that does not have a submodular pricing function.

In the single good case, let $a > 0$ be a parameter, and consider a mechanism $\mu$ with $q(x) = ax$ and $s(x) = \frac{1}{2}ax^2$, and thus $b(x) = ax \cdot x - \frac{1}{2}ax^2 = \frac{1}{2}ax^2$, for $x$ in a set $V \subseteq [0, a^{-1}]$ (so that $q(x) \leq 1$; the IC conditions hold because $b'(x) = ax = q(x)$). The corresponding pricing function is then $p(g) = s(a^{-1}g) = \frac{1}{2}a^{-1}g^2$ for $g = ax$ (with $x \in V$), and so the functions $s, b,$ and $p$ are convex quadratics (in certain domains).

In the $k$-good case, let $A$ be a positive definite (and thus symmetric and invertible) $k \times k$ matrix, and let $V \subseteq \{ x \in \mathbb{R}^k_+ : Ax \in [0, 1]^k \}$ be a domain of valuations (in this section all vectors should be understood as column vectors, even when written as, say, $x = (x_1, \ldots, x_k)$; the transpose of $x$, which is a row vector, is denoted by $x^\top$). For convenience we assume that $V$ is a closed convex set with a nonempty interior that contains $0$. We define:

- A mechanism $\mu = (q, s)$ is quadratic with matrix $A$ and domain $V$ as above if

$$q(x) = Ax \quad \text{and} \quad s(x) = \frac{1}{2}x^\top Ax$$

for every $x$ in $V$

(thus, there are no restrictions outside $V$). For every $x$ in $V$ we then have $b(x) = q(x) \cdot x - s(x) = Ax \cdot x - \frac{1}{2}x^\top Ax = \frac{1}{2}x^\top Ax$, i.e.,

$$b(x) = \frac{1}{2}x^\top Ax,$$

and so the function $b$ is a convex quadratic function on $V$, and $q(x) = Ax$ is a subgradient of $b$ at $x \in V$ (for $x$ in the interior of $V$, this is the gradient); see Proposition 2.1.

Do quadratic mechanisms exist? Yes: for every $A$ and $V$ let $\mu$ be the IC mechanism with menu $\{(Ax, \frac{1}{2}x^\top Ax) : x \in V\}$. Then for each $x \in V$ the choice $(Ax, \frac{1}{2}x^\top Ax)$ is optimal at $x$, because $Ax \cdot x - \frac{1}{2}x^\top Ax \geq Ay \cdot y - \frac{1}{2}y^\top Ay$ for all $y \in V$ (this inequality, which is equivalent to $f(y) \geq f(x) + (y - x) \cdot \nabla f(x)$ for the function $f(x) = \frac{1}{2}x^\top Ax$, holds for all $x, y \in \mathbb{R}^k$ by the convexity of $f$). Therefore $\mu$ satisfies (21) (and $0 \in V$ gives $b(0) = s(0) = 0$, i.e., IR and NPT).

Let $Q_V \subseteq Q$ be the range of allocations for valuations in $V$, i.e., $Q_V := \{q(x) : x \in V\} \subseteq [0, 1]^k$. Each $g \in Q_V$ is obtained as $q(x)$ for a unique $x \in V$, namely, $x = A^{-1}g$, and
so $Q_V = A^{-1}(V)$ is a closed convex set with a nonempty interior that contains $0$ (because $V$ is such a set, and $A^{-1}$ is regular). The restriction to $Q_V$ of any pricing function $p$ of $\mu$ then satisfies
\[
p(g) = s(A^{-1}g) = \frac{1}{2}(A^{-1}g)^\top A(A^{-1}g) = \frac{1}{2}g^\top A^{-1}g
\]for every $g \in Q_V$, and so $p$ is a convex quadratic function on $Q_V$.

5.1 Allocation Monotonicity without Submodularity

For quadratic mechanisms there are simple necessary conditions for allocation monotonicity and for submodularity.

**Proposition 5.1** Let $\mu$ be a quadratic mechanism with matrix $A$ and domain $V$. Then:

(i) If $\mu$ is allocation monotonic then the off-diagonal entries of $A$ are all nonnegative: $(A)_{ij} \geq 0$ for all $i \neq j$.

(ii) If $\mu$ is submodular then the off-diagonal entries of $A^{-1}$ are all nonpositive: $(A^{-1})_{ij} \leq 0$ for all $i \neq j$.

For the diagonal entries we have $(A)_{ii} \geq 0$ and $(A^{-1})_{ii} \geq 0$ for all $i$, by the convexity of the functions $b$ and $p$ (in fact, $(A)_{ii} > 0$ and $(A^{-1})_{ii} > 0$ by the positive definiteness of $A$ and $A^{-1}$).

**Proof.** (i) The function $q(x) = Ax$ is nondecreasing on the nonempty open set $\text{int} V$ if and only if $(A)_{ij} \geq 0$ for all $i, j$ (alternatively: $b(x) = \frac{1}{2}x^\top Ax$ is supermodular if and only if $(A)_{ij} \geq 0$ for all $i \neq j$; see Proposition 4.1).

(ii) The function $p(g) = \frac{1}{2}g^\top A^{-1}g$ is submodular on the nonempty open set $\text{int} Q_V$ if and only if $(A^{-1})_{ij} \leq 0$ for all $i \neq j$; see (10). □

For positive definite matrices $A$ and $A^{-1}$, the condition on $A^{-1}$ in (ii) of nonpositive nondiagonal entries implies the condition on $A$ in (i) of nonnegative nondiagonal entries (see Plemmons 1977, Theorem 1, C9 $\iff$ F15, applied to $A^{-1}$, and Proposition 4.3 (i)). While the converse is easily shown to be true when $k = 2$, it is not true in general, when $k \geq 3$. This allows the construction of an example of a quadratic mechanism that is allocation monotonic but **not** submodular.

**Example 5.2** Let $k = 3$ and take
\[
A = \begin{bmatrix}
6 & 3 & 1 \\
3 & 6 & 3 \\
1 & 3 & 6
\end{bmatrix};
\]
then $A$ is a positive definite matrix, and its inverse is the positive definite matrix

$$A^{-1} = \frac{1}{120} \begin{bmatrix} 27 & -15 & 3 \\ -15 & 35 & -15 \\ 3 & -15 & 27 \end{bmatrix}.$$ 

We will construct a quadratic mechanism $\mu$ with matrix $A$ and domain $V$ that is allocation monotonic (on the whole space $\mathbb{R}^k_+$, not just on $V$). However, $\mu$ is not submodular, since $A^{-1}$ has positive off-diagonal entries, namely, $(A^{-1})_{13} = (A^{-1})_{31} = 3 > 0$, and so, by Proposition 5.1, the pricing function $p$ is not submodular (already on $Q_V$).

To construct the mechanism $\mu$, take $v \gg 0$ such that $Av \leq e$ (e.g., $v = (1/15)e$), and put $V = \{x \in \mathbb{R}^3 : 0 \leq x \leq v\}$; then $0 \leq Ax \leq Av \leq e$ for every $x \in V$ (because $A \geq 0$). For every $x \in \mathbb{R}^3_+$ denote $\hat{x} := x \land v$, and let

$$b(x) = \frac{1}{2} \hat{x}^T A \hat{x} + e \cdot (x - \hat{x}) = \frac{1}{2} \hat{x}^T A \hat{x} + \sum_{i=1}^k [x_i - v_i]_+$$

(thus, $b(x) = \frac{1}{2} x^T A x$ for every $x \in V$). We claim that $b$ is a continuous, nondecreasing, nonexpansive, convex, and supermodular function on $\mathbb{R}^3_+$. Indeed, the derivative of $b$ in direction $e^i$ (the $i$-th unit vector) is $b'(x; e^i) = (A \hat{x})_i$ if $x_i < v_i$, and $b'(x; e^i) = 1$ if $x_i \geq v_i$. Since $(A \hat{x})_i$ is nondecreasing in $x$ (because the mapping $x \to \hat{x}$ is nondecreasing, and $A \geq 0$) and $(A \hat{x})_i \leq (Av)_i \leq 1$, the function $b'(x; e^i)$ is nondecreasing in $x$. Therefore $b$ is an ultramodular function (by Theorem 5.5 (i) in Marinacci and Montrucchio 2005), since $0 \leq b'(x; e^i) \leq 1$, it is nondecreasing and nonexpansive. Finally, to see that $b$ is convex, represent it as

$$b(x) = \max_{I \subseteq K} \left\{ b_0(x_{K \setminus I}, v_I) + \sum_{i \in I} (x_i - v_i) \right\}$$

where $b_0(x) := \frac{1}{2} x^T A x$. Indeed: (i) the nonexpansiveness of $b_0$ yields $b_0(y_{-i}, x_i) \leq b_0(y_{-i}, v_i) + x_i - v_i$ when $x_i \geq v_i$, and $b_0(y_{-i}, v_i) + x_i - v_i \leq b_0(y_{-i}, x_i)$ when $x_i \leq v_i$, and so the maximum in (24) is attained at $I = \{i : x_i \geq v_i\}$, i.e., when $(x_{K \setminus I}, v_I) = x \land v = \hat{x}$ and $b(x) = b_0(\hat{x}) + e \cdot (x - \hat{x})$; (ii) for each $I$ the corresponding function in (24) is convex, because $b_0$ is convex, and so their maximum is convex as well.

Therefore $b$ is a buyer payoff function (by Proposition 2.1) that is supermodular. By Proposition 4.1 the corresponding buyer-favorable (and tie-favorable) mechanism $\mu = (q, s)$, whose allocation function is $q_i(x) = b'(x; e^i)$ (i.e., $q_i(x) = (A \hat{x})_i$ if $x_i < v_i$ and $q_i(x) = 1$ if $x_i \geq v_i$) for $i = 1, ..., k$, is allocation monotonic.

---

26The standard extension that uses the menu $\{(Ax, \frac{1}{2}x^T A x ) : x \in V\}$, as in the previous section, is not allocation monotonic, and so we construct a different extension.

27Which yields ultramodularity of $b$ on the interior of $\mathbb{R}^k_+$; since $b$ is continuous, it extends to $\mathbb{R}^k_+$ (see also Proposition A.6 in Appendix A.6).
6 Revenue of Symmetric Deterministic Mechanisms

In this section we study the class of symmetric deterministic mechanisms, which are monotonic (by Theorem 4 and Proposition 5 in Hart and Reny 2015), but in general not allocation monotonic (see Example 1.3 in the Introduction). Perhaps surprisingly, here we start with the supermodular case, where we show that the revenue is at most an \( O(\log k) \) multiple of the separate revenue (Theorem 6.1), from which we obtain an \( O(\log^2 k) \) multiple in the general case (Theorem E).

Let \( \mu \) be a symmetric deterministic mechanism. Its canonical deterministic pricing function \( p \) is then also symmetric: the price of a set \( A \subseteq K \) of goods depends only on the size \( |A| \) of \( A \). We thus write \( p(|A|) \) instead of \( p(A) \), where \( p : \{0, 1, \ldots, k\} \to [0, \infty] \) is a nondecreasing function with \( p(0) = 0 \).

6.1 The Supermodular Case

The symmetric deterministic mechanism \( \mu \) is supermodular if

\[
p(|A|) + p(|B|) \leq p(|A \cup B|) + p(|A \cap B|)
\]

for every \( A, B \subseteq K \). This is easily seen to be equivalent to the sequence of price differences

\[
d(m) := p(m) - p(m - 1)
\]

being nondecreasing, i.e.,

\[
d(m) \geq d(m - 1)
\]

for all \( 1 \leq m \leq k \) (when \( p(m) = \infty \) put \( d(m) = \infty \). Let \( \text{SUPERMODSYM}D\text{REV} \) denote the maximal revenue achievable by supermodular symmetric deterministic mechanisms.

For every \( k \geq 1 \) let

\[
H(k) := 1 + \frac{1}{2} + \ldots + \frac{1}{k}
\]

denote the harmonic sum up to \( k \); thus, \( \ln k \leq H(k) \leq \ln k + 1 \), and \( H(k) - \ln k \to \gamma \equiv 0.577\ldots \) as \( k \to \infty \).

Theorem 6.1 Let \( X \) be a \( k \)-good random valuation. Then

\[
\text{SUPERMODSYM}D\text{REV}(X) \leq H(k) \cdot \text{SYMSREV}(X) \leq O(\log k) \cdot \text{SREV}(X).
\]

Proof. Let \( \mu = (q, s) \) be a symmetric deterministic mechanism, whose canonical deterministic pricing function \( p : \{0, 1, \ldots, k\} \to [0, \infty] \) is supermodular. Let \( k_0 \leq k \) be the maximal size for which the price is finite, i.e., \( p(m) < \infty \) if and only if \( m \leq k_0 \). We will show that

\[
R(\mu; X) \leq H(k_0) \cdot \text{SYMSREV}(X),
\]

which yields the result.

Given a random valuation \( X \), for each subset size \( 0 \leq m \leq k \) let \( \beta_m := \mathbb{P} [|q(X)| = m] \) be the probability that \( \mu \) allocates exactly \( m \) goods; then \( \beta_m = 0 \) for \( m > k_0 \) (because

\[28\] Such functions are the restriction to \( \{0, 1, \ldots, k\} \) of convex functions on the interval \( [0, k] \).
the price there is infinite), and

$$R(\mu; X) = \sum_{m=1}^{k_0} \beta_m p(m)$$  \hspace{1cm} (26)

(the sum starts at 1 since \(p(0) = 0\)).

We claim that for every \(n = 1, \ldots, k_0\),

$$\sum_{i=1}^{k_0} \mathbb{P}[X_i \geq d(n)] \geq \sum_{m=n}^{k_0} m \beta_m.$$  \hspace{1cm} (27)

Indeed, if \(q(x)\) is a set of size at least \(n\), say \(q(x) = A\) with \(|A| = m \geq n\), then for each one of the \(m\) elements \(i\) of \(A\) we have \(b(x) = x(A) - p(m) \geq x(A\{i\}) - p(m - 1)\) (by (2)), and so \(x_i \geq p(m) - p(m - 1) = d(m) \geq d(n)\) (by supermodularity). Therefore \(A\) contributes to the left-hand sum at least \(m\) times \(\mathbb{P}[q(X) = A]\). Summing over all \(A\) with \(|A| = m \geq n\) (the events \(\{q(X) = A\}\) are disjoint for different sets \(A\)) yields the inequality (27).

Therefore

$$\text{SYMSREV}(X) \geq d(n) \sum_{i=1}^{k_0} \mathbb{P}[X_i \geq d(n)] \geq d(n) \sum_{m=n}^{k_0} m \beta_m \geq d(n)n \sum_{m=n}^{k_0} \beta_m$$

(the second inequality is (27)), and so

$$d(n) \sum_{m=n}^{k_0} \beta_m \leq \frac{1}{n} \text{SYMSREV}(X).$$

Summing over \(n = 1, \ldots, k_0\) yields (25), since

$$\sum_{n=1}^{k_0} d(n) \sum_{m=n}^{k_0} \beta_m = \sum_{m=1}^{k_0} \beta_m \sum_{n=1}^{m} d(n) = \sum_{m=1}^{k_0} \beta_m p(m),$$

which is \(R(\mu; X)\) by (26). \(\square\)

Remarks. (a) The bound in (25) is tight for each \(k \geq 1\): see Example 6.2 below.

(b) The bound in (25) need not hold when \(p\) is not supermodular: see Example 6.4 below.

The following example shows the tightness of (25); in addition, it provides a gap of \(\ln k\) between \(\text{AMONREV}\) and \(\text{MONREV}\).
Example 6.2 Let $M > 1$ be large, and define for each $m \geq 1$

$$z_m := (M^m, ..., M^m, 0, ..., 0) \in \mathbb{R}_+^k$$

and

$$\beta_m := \frac{1}{mM^m},$$

and put $z_0 := 0$ and $\beta_0 := 1 - \sum_{m=1}^{k} \beta_m \geq 0$. Let the random valuation $X$ take as values the $\binom{k}{m}$ permutations of (the coordinates of) $z_m$ with probability $\beta_m/\binom{k}{m}$ each, for all $m = 0, ..., k$; thus, for every $A \subseteq K$ we have $\mathbb{P}[X = M^m 1_A] = \beta_m/\binom{k}{m}$, where $m := |A|$.

We claim that, as $M \to \infty$,

1. SupermodSymDRev(X) and Rev(X) converge to $H(k)$, and thus so do SymDRev(X), DRev(X), and MonRev(X) (which all lie between the above two revenues).

2. SRev(X) (which is the same as SymSRev(X)) and AMonRev(X) converge to 1.

The proof is divided into four parts, (i)–(iv).

(i) SupermodSymDRev(X) $\geq H(k)$.

Proof. Take the symmetric deterministic mechanism $\mu = (q, s)$ with supermodular pricing function $p(m) = M^m$ for all $m = 1, ..., k$ (and $p(0) = 0$); we will show that $R(\mu; X) = H(k)$. Indeed, for each permutation $z_m'$ of $z_m$ we have $q(z_m') = \{i \in K : z_m', i = M^m\}$ (e.g., $q(z_m) = \{1, ..., m\}$) and $s(z_m') = p(m)$. To see why this is so, consider $z_m$: the IC and IR conditions hold at $z_m$ because $q(z_m) \cdot z_m - s(z_m) = mM^m - p(m) = (m - 1)M^m$, whereas for each $z_n'$ (some permutation of $z_n$) that is different from $z_m$, if $n \leq m$ then $q(z_n') \cdot z_m - s(z_n') \leq nM^m - M^n < (m - 1)M^m$ (because the set $q(z_n')$ contains at most $n$ elements from the set $\{1, ..., m\}$; when $n = m$ use $z_n' \neq z_m$), and if $n > m$ then $q(z_n') \cdot z_m - s(z_n') \leq mM^m - M^n < (m - 1)M^m$. The same holds for any permutation $z_m'$ of $z_m$. Therefore $R(\mu; X) = \sum_{m=1}^{k} \beta_mp(m) = \sum_{m=1}^{k} 1/m = H(k)$. □

(ii) Rev(X) $\to H(k)$ as $M \to \infty$.

Proof. Let $\mu = (q, s)$ be a mechanism. Since $X$ is symmetric we can assume without loss of generality that $\mu$ is symmetric, in the sense that if the coordinates of $x'$ are a permutation of the coordinates of $x$ then $q(x')$ is the corresponding permutation $q(x)'$ of the coordinates of $q(x)$, and $s(x') = s(x)$ (indeed, the average of all “coordinate permutations” of $\mu$ yields the same revenue from $X$ as the original $\mu$). Therefore the

\footnote{\text{BRev(X) converges to 1 as well, because $mM^m \cdot P[\sum_{i} X_i \geq mM^m] \to 1$ for all $m \geq 1$ (cf. the proof of (ii)).}}
first \( m \) coordinates of \( q(z_m) \) are all equal, say, \( q_i(z_m) = \gamma_m \) for some \( 0 \leq \gamma_m \leq 1 \), and 
\( s(z_m') = s(z_m) \) for every permutation \( z_m' \) of \( z_m \), which yields

\[
R(\mu; X) = \sum_{m=1}^{k} \frac{1}{MM^m} s(z_m).
\]

For every \( m \geq 1 \), by IC at \( z_m \) vs. \( z_{m-1} \), we have

\[
m\gamma_m M^m - s(z_m) \geq (m - 1)\gamma_{m-1} M^m - s(z_{m-1}).
\]

By IR at \( z_{m-1} \) we have \( s(z_{m-1}) \leq (m - 1)M^{m-1} \leq mM^{m-1} \), and so dividing by \( MM^m \) yields

\[
\frac{1}{MM^m} s(z_m) \leq \gamma_m \frac{m - 1}{m} \gamma_{m-1} + \frac{1}{M}.
\]

Summing (28) over \( m = 1, \ldots, k \) gives

\[
R(\mu; X) \leq \sum_{m=1}^{k} \frac{1}{m} \gamma_{m-1} + \frac{k}{M} \leq \sum_{m=1}^{k} \frac{1}{m} + \frac{k}{M} \to H(k),
\]

where the second inequality is by \( \gamma_m \leq 1 \) for all \( m \). Part (i) completes the proof. □

(iii) \( SREV(X) = SYMSREV(X) \to 1 \) as \( M \to \infty \).

Proof. \( X_i \) takes the value \( M^m \) in the fraction \( m/k \) of the permutations of \( z_m \), and so \( P[X_i = M^m] = (m/k)\beta_m = 1/(kM^m) \), which yields

\[
M^m \cdot P[X_i \geq M^m] = M^m \cdot \sum_{n \geq m} \frac{1}{kM^n} \to \frac{1}{k}
\]

as \( M \to \infty \). Therefore \( REV(X_i) \to 1/k \) for each \( i \), and so \( SYMSREV(X) = SREV(X) \to 1 \). □

(iv) \( AMONREV(X) \to 1 \) as \( M \to \infty \).

Proof. Let \( \mu = (q, s) \) be an allocation-monotonic mechanism; as in the the proof of (ii), we assume without loss of generality that \( \mu \) is symmetric, because the symmetrization preserves allocation monotonicity (an average of allocation-monotonic mechanisms is clearly allocation monotonic). Define \( \gamma_m \) as in (ii).

For each \( m \geq 1 \), let \( y_m \) have its first \( m \) coordinates equal to \( M^{m-1} \) and the rest equal to 0; then \( y_m \geq z_{m-1} \) and \( y_m \geq z_{m-1}^\# \), where \( z_{m-1}^\# \) is the permutation of \( z_{m-1} \) for which coordinates \( 2, \ldots, m \) are equal to \( M^{m-1} \); by allocation monotonicity we thus have \( q(y_m) \geq q(z_{m-1}) \) and \( q(y_m) \geq q(z_{m-1})^\# \), and so each one of the first \( m \) coordinates of \( q(y_m) \) is \( \geq \gamma_{m-1} \). By IR at \( y_m \) we have \( s(y_m) \leq q(y_m) \cdot y_m \leq mM^{m-1} \), and then by IC at \( z_m \) vs. \( y_m \) we get

\[
m\gamma_m M^m - s(z_m) \geq q(y_m) \cdot z_m - s(y_m) \geq mM_{m-1} M^m - mM^{m-1}.
\]
Dividing by $m M^m$ yields
\[
\frac{1}{m M^m p_m} \leq \gamma_m - \gamma_{m-1} + \frac{1}{M},
\]
and then summing over $m = 1, \ldots, k$ gives
\[
R(\mu; X) \leq \gamma_k + \frac{k}{M} \leq 1 + \frac{k}{M} \to 1.
\]
Part (iii) completes the proof, since selling separately is an allocation-monotonic mechanism. □

We thus have a random valuation where monotonic mechanisms yield a revenue that is $\ln k$ times higher than the revenue of allocation-monotonic mechanisms (use $H(k) > \ln k$ for $k \geq 2$).

**Corollary 6.3** For every $k \geq 2$ there exists a $k$-good random valuation $X$ such that
\[
\text{MonRev}(X) \geq \ln k \cdot \text{AMonRev}(X).
\]

Theorem 6 implies that $\text{MonRev}(X) \leq k \cdot \text{AMonRev}(X)$ (because selling separately, or bundled, is allocation monotonic). We do not know what is the correct bound here (between $\ln k$ and $k$).

The next example shows that the bound in (25) need not hold for symmetric deterministic mechanisms that are *not* supermodular.

**Example 6.4** Let $k = 3$, and take $p(0) = 0$, $p(1) = p(2) = 1$, and $p(3) = M$, where $M > 1$ is a large number; then $p$ is *not* supermodular, because $p(2) - p(1) < p(1) - p(0)$.

We will construct a random valuation $X$ such that, by letting $B_m$ be the event that $|q(X)| = m$, and $\beta_m := P[B_m]$ its probability, we have $\beta_1 = 0$, $\beta_2 = 1/2$, $\beta_3 = 1/(12M)$, and $\beta_0 = 1-1/2-1/(12M)$. Let the random valuation $X = (X_1, X_2, X_3)$ be exchangeable (i.e., permuting the coordinates does not change the distribution of $X$), as follows: the set $B_2$ consists of the valuations $(U, 1-U, 0)$ and their permutations, where $U$ is uniformly chosen from the interval $(0,1)$; the set $B_3$ consists of the single valuation $(M, M, M)$; and $B_0$ consists of $0 = (0, 0, 0)$ (the set $B_1$ is empty). The IC and IR conditions are easily checked. Then $R(\mu; X) = \sum_m \beta_m p_m = 1/2 \cdot 1 + 1/(12M) \cdot M = 7/12$, whereas for each $i$ we have $\text{Rev}(X_i) = \sup t \cdot P[X_i \geq t] \sim 1/12$ (attained at $t \sim 1/2$), yielding $\text{SymSRev}(X) = \text{SRev}(X) \sim 1/4$ and thus $R(\mu; X)/\text{SRev}(X) \sim (7/12)/(1/4) = 7/3 > 11/6 = 1 + 1/2 + 1/3 = H(3)$.

---

30. We write $a \sim b$ to mean $a/b \to 1$ as $M \to \infty$.
31. For $t = M$ we have $M \cdot \beta_3 = 1/2$, and for $0 \leq t \leq 1$ we have $t \cdot ((1-t)(1/3)+1/(12M)) \sim t(1-t)/3$, whose maximum, at $t = 1/2$, equals $1/12$. 

34
6.2 The General Case

We can now bound the maximal revenue obtainable by symmetric deterministic mechanisms, which we denote by $\text{SymDRev}$, relative to the (symmetric) separate revenue.

**Theorem E** Let $X$ be a $k$-good random valuation. Then

$$\text{SymDRev}(X) \leq 2 \ln(2k) H(k) \cdot \text{SymSRev}(X) \leq O(\log^2 k) \cdot \text{SRev}(X).$$

**Proof.** We will show that any symmetric deterministic $k$-good pricing function can be bounded within a factor of $k$ by a supermodular one, which yields the first $\log k$ factor; the second $\log k$ factor comes from Theorem 6.1.

Let $\mu$ be a symmetric deterministic mechanism; let $p$ be its canonical deterministic pricing function, with $p(0) = 0$. Let $d(m) := p(m) - p(m-1)$ be the price differences; then $0 \leq d(m) \leq p(m)$ (because $p$ is nondecreasing and $p(0) = 0$). Define $d'(m) := \max_{1 \leq n \leq m} d(n)$ and $p'(m) := \sum_{\ell=1}^{m} d'(\ell)$ for all $m \geq 1$, and $p'(0) := p(0) = 0$. We have: $p' \geq p$ (because $d' \geq d$); $p'$ is nondecreasing (because $d' \geq d \geq 0$); $p'$ is supermodular (because $d'$ is nondecreasing); and, for each $m \geq 1$, if $d'(m) = d(n)$ for some $1 \leq n \leq m$ then $p'(m) \leq md(n)$ (because $d'(\ell) \leq d(n)$ for all $1 \leq \ell \leq m$), and so $p'(m) \leq kp(m)$ (because $d(n) \leq p(n) \leq p(m)$ and $m \leq k$). Altogether, we have

$$\frac{1}{k} p'(m) \leq p(m) \leq p'(m)$$

for all $m$, with $p'$ supermodular. By the result of Chawla, Teng, and Tzamos (2022) (Theorem A.2), it follows that

$$\text{SymDRev}(X) \leq 2 \ln(2k) \cdot \text{SupermodSymDRev}(X);$$

together with Theorem 6.1, this yields the result. 

7 Monotonicity of Deterministic Mechanisms

In this section we obtain conditions on a deterministic pricing function to yield a monotonic mechanism. These conditions are more complex than the submodularity condition for allocation monotonicity (see Theorem C (iv) in Section 4.1). A mechanism $\mu = (q, s)$ is tie consistent if the buyer breaks ties in a consistent manner, i.e., in the same way at all valuations. This means that if the same two distinct choices are optimal both at $x$ and at $y$, then $\mu$ cannot choose one at $x$ and the other at $y$. For example, if the price of each single good is 1, we cannot have good 1 allocated at $x = (2, 2, 0)$ and good 2 allocated at $y = (2, 2, 1)$. Formally, tie consistency requires that if $q(x)$ is optimal at $y$ (which means

32Recall that we put $d(m) = \infty$ when $p(m) = \infty$. 

35
that $q(y) \cdot x - s(y) = b(x) = q(x) \cdot x - s(x)$ and $q(y)$ is optimal at $x$ then $q(x) = q(y)$. Choosing among the seller-favorable choices the set $A$ that maximizes, say, $\sum_{i \in A} i$ yields a mechanism that is tie favorable and tie consistent.

**Theorem 7.1** Let $\mu$ be a deterministic buyer-favorable tie-consistent mechanism, with a nondecreasing pricing function $p : 2^K \to \mathbb{R}_+$. A necessary and sufficient condition for $\mu$ to be monotonic is: if

$$p(A) > p(B)$$

for some $A, B \subseteq K$ then there is no $z \in \mathbb{R}^{A \setminus B}$ satisfying

$$p(A) - p(A \setminus C) \leq \sum_{i \in C} z_i < p(B \cup C) - p(B) \text{ for all } \emptyset \neq C \subseteq A \setminus B. \quad (29)$$

To clarify: such a $z$ would need to satisfy simultaneously all $2 \cdot (2^{A \setminus B} - 1)$ inequalities in (29).

**Proof.** Let $x \in \mathbb{R}^k_+$, and let $A := q(x)$ be the set of goods allocated to $x$. If we increase coordinates in $A$ and decrease coordinates outside $A$, i.e., if $x' \in \mathbb{R}^k_+$ satisfies $x'_i \geq x_i$ for all $i \in A$ and $x'_i \leq x_i$ for all $i \notin A$, then $q(x') = A$ as well. This is easily seen since $[x(A) - p(A)] - [x(B) - p(B)] \geq [x'(A) - p(A)] - [x'(B) - p(B)]$ for every $B$, which implies that, first, $A$ is optimal at $x'$, and second, any $B$ that is optimal at $x'$ is also optimal at $x$; by tie consistency, $A$ must therefore be chosen at $x'$ too.

Next, the monotonicity of $\mu$ is equivalent to: if $p(A) > p(B)$ then there are no $x \leq y$ such that $q(x) = A$ and $q(y) = B$ (because then $s(x) = p(A) > p(B) = s(y)$). Applying the observation of the previous paragraph to both $x$ and $y$, proceed as follows while keeping the inequality $x \leq y$: decrease $x_i$ and $y_i$ to $0$ for $i \notin A \cup B$; increase $x_i$ and $y_i$ to a large number $M$ for $i \in A \cap B$; decrease $x_i$ to $0$ and increase $y_i$ to $M$ for $i \in B \setminus A$; and, finally, decrease $y_i$ to $x_i$ for $i \in A \setminus B$. All these changes therefore do not affect the allocation, and so, by letting $z \in \mathbb{R}^{A \setminus B}$ be the restriction of $x$ to $A \setminus B$, we have without loss of generality that $q(x) = A$ and $q(y) = B$ for $x$ and $y$ of the form

|       | $A \setminus B$ | $A \cap B$ | $B \setminus A$ | $K \setminus (A \cup B)$ |
|-------|----------------|-----------|-----------------|--------------------------|
| $x$   | $z$            | $(M, \ldots, M)$ | $(0, \ldots, 0)$ | $(0, \ldots, 0)$       |
| $y$   | $z$            | $(M, \ldots, M)$ | $(M, \ldots, M)$ | $(0, \ldots, 0)$       |

For large $M$, the conditions that $q(x) = A$, namely $x(A) - p(A) \geq x(A') - p(A')$ for all $A'$, reduce to

$$z(C) \geq p(A) - p(A \setminus C)$$

for all $C \subseteq A \setminus B$ (indeed, only sets $A'$ such that $A \cap B \subseteq A' \subseteq A$, i.e., $A' = A \setminus C$, matter).

\[33\] Assume for simplicity that all prices are finite; $p$ need not be the canonical pricing function.
Similarly, the conditions for $q(y) = B$ reduce to
\[ z(C) < p(B \cup C) - p(B) \]
for all $C \subseteq A \setminus B$ (indeed, only sets $B'$ such that $B \subsetneq B' \subseteq B \cup (A \setminus B)$, i.e., $B' = B \cup C$, matter; the inequalities here are strict since otherwise $B'$ would be chosen at $y$ instead of $B$, by buyer favorability).

Remark. From the proof it follows that it suffices to consider sets $A, B$ that are in the range of allocations $Q = q(\mathbb{R}^k_+)$ of $\mu$, and, moreover, incomparable (i.e., $A \setminus B$ and $B \setminus A$ are both nonempty; indeed, if $A \subseteq B$ then we cannot have $p(A) > p(B)$, and if $B \not\subseteq A$ then condition (29) for $C = A \setminus B$ becomes $p(A) - p(B) \leq z(A \setminus B) < p(A) - p(B)$, a contradiction that shows that there can be no such $z$). This applies to the two corollaries below as well.

The nonexistence of $z$ satisfying (29) is a somewhat unwieldy condition; we obtain from it two simpler conditions, one that is sufficient and one that is necessary.

**Corollary 7.2** Let $\mu$ be a deterministic buyer-favorable tie-consistent mechanism. Then $\mu$ is monotonic if
\[ p(A) + p(B) \geq p(A \cup B) + p(A \cap B) \]
(30)
for all $A, B \subseteq K$ with $p(A) \neq p(B)$.

This is a restricted version of submodularity, where inequality (30) is not required when $p(A) = p(B)$.

**Proof.** If $p(A) > p(B)$ then there can be no $z$ satisfying (29), because for $C = A \setminus B$ it would require that $p(A) - p(A \cap B) \leq z(A \setminus B) < p(A \cup B) - p(B)$, which contradicts (30). ■

**Corollary 7.3** Let $\mu$ be a deterministic buyer-favorable tie-consistent mechanism. If $\mu$ is monotonic then
\[ p(A \cup \{i\}) + p(A \cup J) \geq p(A \cup J \cup \{i\}) + p(A) \]
for all pairwise disjoint sets $A, \{i\}, J \subseteq K$ for which
\[ p(A \cup \{i\}) > p(A \cup J). \]

**Proof.** When $A \setminus B$ is a singleton, say $A \setminus B = \{i\}$, condition (29) becomes $p(A) - p(A \setminus \{i\}) \leq z_i < p(B \cup \{i\}) - p(B)$, and so the nonexistence of $z$ is equivalent to $p(A) - p(A \setminus \{i\}) \geq p(B \cup \{i\}) - p(B)$. Put $J := B \setminus A$ and replace $A \cap B$ by $A$. ■

Since (30) is symmetric in $A$ and $B$, requiring it when $p(A) > p(B)$ is the same as requiring it when $p(A) \neq p(B)$. 34
For disjoint sets $A, I \subset K$, define the discrete derivative of $p$ at $A$ in the direction $I$ by
\[
p'(A; I) := p(A \cup I) - p(A);
\]
this is the marginal price of the set of goods $I$, i.e., the change in price due to adding $I$. With this notation, the condition of Corollary 7.3 becomes:
\[
\text{if } p'(A; \{i\}) > p'(A; J) \text{ then } p'(A; \{i\}) \geq p'(A \cup J; \{i\}),
\]
which is a “decreasing marginal price” condition. Similarly, putting $I := A \setminus B$ and $J := B \setminus A$, and replacing $A \cap B$ by $A$, the condition of Corollary 7.2 becomes:
\[
\text{if } p'(A; I) > p'(A; J) \text{ then } p'(A; I) \geq p'(A \cup J; I).
\]
(31)
Thus, a sufficient condition for monotonicity is “(31) for all pairwise disjoint $A, I, J$,” while a necessary condition for monotonicity is “(31) for all pairwise disjoint $A, \{i\}, J$” (i.e., singleton sets $I$).

Additional conditions for monotonicity are provided in Appendix A.8.

8 Open Problems

We list a number of issues that remain open, specifying in each case the best that we know.

1. The main open problem is to find useful characterizations of the monotonicity of mechanisms; see Section 7 and Appendix A.8 for the deterministic case, and the last paragraph in Appendix A.3. This should also help to address some of the other open problems below.

2. The bound on the ratio between $\text{MonRev}$ and $\text{SRev}$: it is at most $k$ by Theorem A and at least $\Omega(\log k)$, obtained by $\text{BRev}$ (see (17)) and Example 6.2.

3. The bound on the ratio between $\text{MonRev}$ and $\text{AMonRev}$: it is at most $k$, as implied by Theorem A (because $\text{SRev} \leq \text{AMonRev}$), and at least $\Omega(\log k)$ by Example 6.2 (see Corollary 6.3).

4. The bound on the ratio between $\text{SymDRev}$ and $\text{SRev}$: it is at most $O(\log^2 k)$ by Theorem E and at least $\Omega(\log k)$, obtained by $\text{BRev}$ (see (17)).

5. The bound on the ratio between $\text{MonRev}$ and $\text{MondRev}$, the revenue from monotonic deterministic mechanisms: it is at most $k$, as implied by Theorem A (because $\text{SRev} \leq \text{MondRev}$), and, of course, at least 1.
A Appendices

A.1 Appendix: The Canonical Pricing Function

The canonical pricing function is nondecreasing, convex, and closed (see Section 2.3); we show here that it is the unique such pricing function. This generalizes the result of Proposition 2.2 in the deterministic case (where only the property of being nondecreasing matters). Recall that \( p_Q : Q \rightarrow \mathbb{R}_+ \) denotes the common restriction of all pricing functions to \( Q \).

**Proposition A.1** Let \( \mu \) be a mechanism, with range of allocations \( Q \subseteq [0, 1]^k \).

(i) The canonical pricing function \( p_0 \) of \( \mu \) is the unique pricing function of \( \mu \) that is nondecreasing, convex, and closed, and is defined as follows: for every \( g \in [0, 1]^k \), let

\[
p_1(g) := \inf \sum_{j=1}^{J} \lambda^j p_Q(g^j),
\]

where the infimum is taken over all convex combinations of elements of \( Q \) that are no less than \( g \)—i.e., \( \sum_{j=1}^{J} \lambda^j g^j \geq g \), where \( \sum_{j=1}^{J} \lambda^j = 1 \), \( \lambda^j \geq 0 \) and \( g^j \in Q \) for every \( j = 1, ..., J \); then

\[
p_0(g) = (\text{cl } p_1)(g) := \lim_{h \to g} p_1(h).
\]

(ii) If the set \( Q \) is closed then the canonical pricing function \( p_0 \) of \( \mu \) is the unique pricing function of \( \mu \) that is nondecreasing and convex; it is the function \( p_1 \) given by (32) (i.e., in this case \( p_1 \) is closed, and \( p_0 = \text{cl } p_1 = p_1 \)).

**Proof.** (i) For each \( g \in [0, 1]^k \) let

\[
\phi_g(x) := \begin{cases} 
g \cdot x - p_Q(g), & \text{if } x \geq 0, \\
\infty, & \text{otherwise.}
\end{cases}
\]

Then \( \phi_g \) is a convex function, and \( b = \sup \{ \phi_g : g \in Q \} \). The Fenchel conjugate \( \phi_g^* \) of \( \phi_g \) is

\[
\phi_g^*(h) = \sup_x (h \cdot x - \phi_g(x)) = \sup_{x \geq 0} (h \cdot x - g \cdot x + p_Q(g))
\]

\[
= \begin{cases} 
p_Q(g), & \text{if } h \leq g, \\
\infty, & \text{otherwise.}
\end{cases}
\]

By Theorem 16.5 in Rockafellar (1970), the Fenchel conjugate \( p_0 \) of \( b \) equals \( \text{cl}(\text{conv}\{\phi_g^*: g \in Q\}) \). By definition, \( \text{conv}\{\phi_g^*: g \in Q\}(h) \) is the infimum of all convex combinations \( \alpha = \sum_j \lambda^j \phi_g^*(h^j) \), where \( \sum_j \lambda^j h^j = h \) and \( g^j \in Q \) for all \( j \). Now the
expression α is finite only if \( h^j \leq g^j \) (otherwise \( \phi^*_g(h^j) \) is infinite) for all \( j \), in which case we get \( \alpha = \sum_j \lambda^j p_Q(g^j) \) and \( \sum_j \lambda^j g^j \geq \sum_j \lambda^j h^j = h \); conversely, given \( \sum_j \lambda^j g^j \geq h \) we can always find \( h^j \leq g^j \) for all \( j \) such that \( \sum_j \lambda^j h^j = h \). Therefore \( \text{conv}\{\phi^*_g : g \in Q\}(h) \) is the infimum of \( \sum_j \lambda^j p_Q(g^j) \) over all convex combinations \( \sum_j \lambda^j g^j \geq h \) with \( g^j \in Q \) for all \( j \), which is precisely \( p_1(h) \), and so \( p_0 = \text{cl} \{ \text{conv}\{\phi^*_g : g \in Q\} \} = \text{cl} p_1 \).

Uniqueness: let \( p \) be any nondecreasing, convex, and closed pricing function of \( \mu \); since \( p \) coincides with \( p_Q \) on \( Q \), it follows that \( p \leq p_1 \) (because \( p \) is nondecreasing and convex), and thus \( p \leq \text{cl} p_1 = p_0 \) (because \( p \) is closed). Now \( p \geq p_0 \) (because \( p_0 \) is the minimal pricing function of \( \mu \)), and so \( p = p_0 \).

(ii) When \( Q \subseteq [0,1]^k \) is a closed set (and thus compact) the function \( p_1 \) is already closed. Indeed, let \( g(n) \to g \), and for each \( n \) take a convex combination \( \sum_{j=1}^J \lambda^j_{(n)} g^j_{(n)} \geq g(n) \) (i.e., \( \lambda^j_{(n)} \geq 0 \) and \( \sum_{j=1}^J \lambda^j_{(n)} = 1 \)) with \( \sum_{j=1}^J \lambda^j_{(n)} p_Q(g^j_{(n)}) \leq p_1(g(n)) + 1/n \), where all \( g^j_{(n)} \in Q \) and \( J \leq k + 2 \) (by Carathéodory’s theorem, since \( (h, p_Q(h)) \in \mathbb{R}^{k+1} \)). The compactness of \( Q \) implies that there is a subsequence, without loss of generality the original sequence, such that \( \lambda^j_{(n)} \to \lambda^j \) and \( g^j_{(n)} \to g^j \) for every \( j \). Since \( Q \) is closed and \( g^j_{(n)} \in Q \) we have \( g^j \in Q \), and so \( \lim_n p_Q(g^j_{(n)}) \geq p_Q(g^j) \) (because \( p_Q = p_0 \) on \( Q \) and \( p_0 \) is lower semicontinuous); therefore in the limit we get a convex combination \( \sum_{j=1}^J \lambda^j g^j \geq g \) (i.e., \( \lambda^j \geq 0 \) and \( \sum_{j=1}^J \lambda^j = 1 \)) with

\[
\sum_{j=1}^J \lambda^j p_Q(g^j) \leq \lim_n \sum_{j=1}^J \lambda^j_{(n)} p_Q(g^j_{(n)}) \leq \lim_n p_1(g(n)).
\]

By the definition of \( p_1(g) \) we thus have \( p_1(g) \leq \sum_{j=1}^J \lambda^j p_Q(g^j) \), and so \( p_1(g) \leq \lim_n p_1(g(n)) \), as claimed. ■

A slightly different way of viewing this characterization of \( p_0 \) is similar to the notion of the “convexification” of a function \( f \), which is the maximal convex function that is \( \leq f \) (such a function always exists, as the supremum of all such functions is a convex function \( \leq f \)). Proposition \ref{prop:closed_convexification} (i) says that \( p_0 \) is the nondecreasing closed convexification of \( p_Q \), i.e., the maximal function \( p_1 \) that is nondecreasing, closed, and \( \leq p_Q \) on \( Q \); again, this is the supremum of all such functions (because the properties are preserved when we take the supremum). Since \( p_Q \) satisfies these properties on \( Q \) (because it is the restriction of \( p_0 \) to \( Q \)), we may also characterize \( p_0 \) as the maximal extension of \( p_Q \) that is nondecreasing, convex, and closed. The requirement reduces to being a nondecreasing and convex function when \( Q \) is a closed set (see (ii)), and to being a nondecreasing function when we consider deterministic pricing of a deterministic mechanism (see Proposition \ref{prop:convexification}).

The following examples show that both convexification and closure are indeed needed.\footnote{Maximal among all functions, not only pricing functions.}
Example. The need of convexification. Let $k = 2$, and consider the separate selling of each good at price 1. Thus, $Q = \{0,1\}^2$, and $p_Q(0,0) = 0$, $p_Q(1,0) = p_Q(0,1) = 1$, $p_Q(1,1) = 2$. Let $g = (1/2,1/2)$; then the minimal nondecreasing pricing function $p_2$ satisfies $p_2(g) = 2$ (since the only element of $Q$ that is $\geq g$ is $(1,1)$). However, $p_0(g) = p_1(g) = 1$ (use $g = 1/2(1,0) + 1/2(0,1)$).

Example. The need of closure. Let $k = 1$, and let $\mu = (q,s)$ be the mechanism whose menu consists of paying $2 - 2\delta$ for an allocation of $1 - \delta^2$ of the good, for all $\delta \in (0,1]$. Then it is straightforward to verify that every valuation $x \leq 1$ chooses $\delta = 1$ (i.e., $(q(x),s(x)) = (0,0)$), and every valuation $x \geq 1$ chooses $\delta = 1/x$ (i.e., $(q(x),s(x)) = (1 - 1/x^2,2 - 2/x)$). The buyer payoff function is thus $b(x) = 0$ for $x \leq 1$ and $b(x) = x + 1/x - 2$ for $x \geq 1$; note that the limit option of paying 2 for an allocation of 1 (i.e., $\delta = 0$) is therefore never optimal: $x - 2 < b(x)$. Thus, $Q = [0,1)$ and $p_0(g) = 2 - 2\sqrt{1-g}$ for every $g \in [0,1)$, which implies that $p_0(1) \equiv \lim_{g \to 1} p_0(g) = 2$ (the inequality $\geq$ is because $p_0$ is nondecreasing, and the inequality $\leq$ is because $p_0$ is closed). However, $p_1(1) = \infty$, because there is no $g' \in \text{conv } Q$ such that $g' \geq 1$.

To avoid infinite prices, we embed this into a two-good example: let $k = 2$, and let the menu consist of $((1-\delta^2,0),2-2\delta)$ for all $\delta \in (0,1]$, together with $((1,1),3)$ (and so there is no need to include the limit option of $(1,0),2$), since for every $x \in \mathbb{R}_+^k$ with $x_1 \geq 2$ the option $((1 - 1/x_1^2,0),2 - 2/x_1)$ yields a strictly better payoff: $x_1 + 1/x_1 - 2 > x_1 - 2$.

Thus $Q = ([0,1) \times \{0\}) \cup \{(1,1)\}$ and $p_Q(g_1,0) = 2 - 2\sqrt{1-g_1}$ for all $g_1 \in [0,1)$, and $p_Q(1,1) = 3$, which yields $p_0(1,0) = 2$ (because $p_0$ is nondecreasing and closed), whereas $p_1(1,0) = 3$ (because the only element of conv $Q$ that is $\geq (1,0)$ is $(1,1)$).

A.2 Appendix: Pricing Approximation

We provide here the result of Chawla, Teng, and Tzamos (2022) on comparing revenues by pricing functions, which we use in Sections 4.2 and 6.2 and Appendix A.9.1. Let $\mathcal{P}$ and $\mathcal{P}'$ be two classes of $k$-good pricing functions, defined on $G = \{0,1\}^k$ (in the deterministic case) or on $G = [0,1]^k$ (in the general case); the pricing functions need not be canonical. The maximal revenues that are obtainable by mechanisms with pricing functions in $\mathcal{P}$ and $\mathcal{P}'$ are denoted by $\mathcal{P}$-REV and $\mathcal{P}'$-REV, respectively. A set $Z$ is a cone if $z \in Z$ implies $\alpha z \in Z$ for every scalar $\alpha \geq 0$.

**Theorem A.2 (Chawla, Teng, and Tzamos)** Let $\mathcal{P}'$ be a cone of nondecreasing and closed $k$-good pricing functions. Assume that there are constants $0 < c_1 < c_2 < \infty$ such that for every $p \in \mathcal{P}$ there is $p' \in \mathcal{P}'$ satisfying

$$c_1 p'(g) \leq p(g) \leq c_2 p'(g)$$

(33)
for every $g$ in $G$; then

$$\mathcal{P}\text{-Rev}(X) \leq 2 \ln \left( \frac{c_2}{c_1} \right) \cdot \mathcal{P}'\text{-Rev}(X)$$

for every $k$-good random valuation $X$.

We have added the technical assumptions that the pricing functions are nondecreasing and closed (conditions that are always satisfied by canonical pricing functions, and also by the separable functions that we generate from them). These assumptions are used to fill in some missing details in the proof of Lemma 3.1 of Chawla, Teng, and Tzamos (2022). Specifically, they yield (in their notation) the existence of a maximal $\lambda_n$, and allow the application of the envelope theorem (Milgrom and Segal 2002, Theorem 2): the function $u_o$ is Lipschitz, and hence absolutely continuous.

### A.3 Appendix: Monotonic Mechanisms

In this appendix we show, first, that if a mechanism is monotonic, then so are its seller-favorable, and thus tie-favorable, versions.

Let $\mu = (q, s)$ be a mechanism, and let $\tilde{\mu} = (\tilde{q}, \tilde{s})$ be a seller-favorable version of $\mu$; thus, $\mu$ and $\tilde{\mu}$ have the same buyer payoff function $b$, and $\tilde{s}(x) = b'(x; x) - b(x)$ for every $x$. Denote by $\mathcal{D} \equiv \mathcal{D}_b \subseteq \mathbb{R}^+_k$ the set of points $x$ in the interior of $\mathbb{R}^+_k$ where the convex function $b$ is differentiable, i.e., $\partial b(x) = \{\nabla b(x)\}$ (where $\nabla b(x)$ denotes the gradient of $b$ at $x$); the set $\mathcal{D}$ is dense in $\mathbb{R}^+_k$, and hence in $\mathbb{R}^+_k$, and its complement $\mathbb{R}^+_k \setminus \mathcal{D}$ has Lebesgue measure zero (by Theorem 25.5 in Rockafellar 1970). The mechanisms $\mu$ and $\tilde{\mu}$ may differ only on $\mathbb{R}^+_k \setminus \mathcal{D}$: at every $x \in \mathcal{D}$ we have $q(x) = \tilde{q}(x) = \nabla b(x)$ and $s(x) = \tilde{s}(x) = b'(x; x) - b(x) = \nabla b(x) \cdot x - b(x)$.

**Proposition A.3** Let $\mu$ be a mechanism with buyer payoff function $b$, and let $\tilde{\mu}$ be a seller-favorable mechanism with the same buyer payoff function $b$. If $\mu$ is monotonic then $\tilde{\mu}$ is monotonic.

**Proof.** First, we claim that for every $x \in \mathbb{R}^+_k$ there is a sequence $\{x_n\}_{n=1}^\infty \subset \mathcal{D}$ such that $x_n \to x$ and $\tilde{s}(x_n) \to \tilde{s}(x)$. Indeed, for $x \neq 0$ let $\{x_n\}_n$ be a sequence of points in $\mathcal{D}$ such that $x_n \to x$ from the direction $x$, i.e., $\|x_n - x\|^{-1}(x_n - x) \to \|x\|^{-1} x$ (such a sequence exists since $\mathcal{D}$ is dense in $\mathbb{R}^+_k$). For example, take $x_n \in \mathcal{D}$ to be within $1/n^2$ of $(1 + 1/n)x$. By Theorem 24.6 in Rockafellar (1970), every limit point of the sequence $\{\nabla b(x_n)\}_n$ belongs to $\partial b(x)$, and so $\tilde{s}(x_n) = \nabla b(x_n) \cdot x_n - b(x_n) \to b'(x; x) - b(x) = \tilde{s}(x)$. For $x = 0$, for any sequence $\{x_n\}_n \in \mathcal{D}$ with $x_n \to 0$ we have $\tilde{s}(x_n) = \nabla b(x_n) \cdot x_n - b(x_n) \to 0 - b(0) = b'(0; 0) - b(0) = \tilde{s}(0)$.

Second, we claim that if $\tilde{s}$ is nondecreasing on $\mathcal{D}$ then $\tilde{s}$ is nondecreasing on $\mathbb{R}^+_k$. Indeed, take $x \leq y$ to be two points in $\mathbb{R}^+_k$. Take $x_n \in \mathcal{D}$ such that $x_n \to x$ and
\[ s(x_n) \to \tilde{s}(x); \] put \( y_n := x_n + y - x \geq x_n, \) and let \( y'_n \in D \) be such that \( y'_n \geq y_n \) and \( \|y'_n - y_n\| \leq 1/n \) (the existence of \( y'_n \) is by the density of \( D \) in \( \mathbb{R}^k_+ \)). Thus, \( x_n \leq y'_n \) and \( y'_n \to y \), and so \( \tilde{s}(x_n) \leq \tilde{s}(y'_n) \) (because \( \tilde{s} \) is nondecreasing on \( D \), which contains both \( x_n \) and \( y'_n \)), and \( \lim_n \tilde{s}(y'_n) \leq \tilde{s}(y) \) (because \( \tilde{s} \) is upper semicontinuous \(^{36}\)).

This completes the proof: if \( s \) is nondecreasing on \( \mathbb{R}^k_+ \) then \( \tilde{s} \) is nondecreasing on \( D \) (where it coincides with \( s \)), and so it is nondecreasing on \( \mathbb{R}^k_+ \). \( \blacksquare \)

Second, we provide a formulation of monotonicity in terms of the allocation function \( q \). Assume for simplicity that the function \( b \) is twice differentiable \(^{37}\) then we have \( \nabla s(x) = \nabla (\nabla b(x) \cdot x - b(x)) = \nabla^2 b(x) x \), where \( \nabla^2 b(x) x \) is the product of the \( k \times k \) matrix \( \nabla^2 b(x) \) (the Hessian of \( b \) at \( x \)) and the column vector \( x \). Consider \( q(tx) = \nabla b(tx) \) as a function of \( t \geq 0 \); its derivative is \( dq(tx)/dt = \nabla^2 b(tx) x = (1/t)\nabla s(tx) \). The monotonicity of the mechanism is equivalent to \( \nabla s(x) \geq 0 \) for all \( x \), and so it is equivalent also to the function \( q(tx) \) being nondecreasing in \( t \) for every \( x \), i.e., to the allocation function \( q \) being nondecreasing along any ray from the origin (cf. allocation monotonicity, where \( q \) is nondecreasing everywhere).

### A.4 Appendix: Revenue of Monotonic Mechanisms

We generalize the result of Theorem \( \ref{thm:allocation} \) (see Section \( \ref{sec:monotonic} \)). A **bundling partition** of the goods is \( \Pi = \{K_j\}_{j \in J} \), where \( \bigcup_{j \in J} K_j = K = \{1, \ldots, k\} \) and the sets \( K_j \) are disjoint (i.e., \( K_j \cap K_{j'} = \emptyset \) for \( j \neq j' \)). The corresponding revenue \( \Pi-\text{REV} \) is defined by

\[
\Pi-\text{REV}(X) := \sum_{j \in J} \text{REV}\left( \sum_{i \in K_j} X_i \right).
\]

This is the maximal revenue obtained by selling each bundle of goods \( K_j \) separately, i.e. (by Myerson’s single-good result \( \ref{thm:myerson} \)), by setting a price \( p_j \) for each bundle \( K_j \). The partition into singletons, \( \Pi_S = \{\{1\}, \ldots, \{k\}\} \), yields the separate revenue \( S\text{REV} \), and the singleton partition, i.e., \( \Pi_B = \{K\} \), yields the bundling revenue \( B\text{REV} \). Since all bundling-partition mechanisms are clearly monotonic, it follows that \( \Pi-\text{REV}(X) \leq \text{MONREV}(X) \) (cf. \( \ref{thm:monotonic} \)).

Theorem \( \ref{thm:allocation} \) easily generalizes to:

**Proposition A.4** Let \( X \) be a \( k \)-good random valuation. Then

\[
\text{MONREV}(X) \leq k \cdot \min_{\Pi} \Pi-\text{REV}(X),
\]

---

\(^{36}\)A real function \( f \) is **upper semicontinuous** if \( \lim_{y \to x} f(y) \leq f(x) \) for every \( x \); equivalently, the set \( \{x : f(x) \geq t\} \) is closed for every real \( t \).

\(^{37}\)And hence differentiable, which implies that there is a unique mechanism with buyer payoff function \( b \).
where the minimum is taken over all partitions $\Pi$ of the set of goods.

**Proof.** Let $\Pi = (K_j)_{j \in J}$ be a partition of $K$. For each $j \in J$ put $Y_j := \max_{i \in K_j} X_i$; then $\max_{i \in K} X_i = \max_{j \in J} Y_j$, and so

\[
\text{REV} \left( \max_{i \in K} X_i \right) = \text{REV} \left( \max_{j \in J} Y_j \right) \leq \sum_{j \in J} \text{REV} (Y_j)
\]

\[
\leq \sum_{j \in J} \text{REV} \left( \sum_{i \in K_j} X_i \right) = \Pi - \text{REV}(X),
\]

where we have used Proposition 3.2 twice, in its $S\text{REV}$ version for the first inequality, and in its $B\text{REV}$ version for the second. Apply (16). □

---

### A.5 Appendix: Submodular Deterministic Mechanisms

A deterministic mechanism is submodular if its canonical deterministic pricing function $p^D_0$ is submodular (see Section 2.5); we show here that it suffices that *some* pricing function is submodular, and only on the range of allocations $Q$.

**Proposition A.5** Let $\mu$ be a deterministic mechanism with range of allocations $Q \subseteq 2^K$. If some deterministic pricing function $p$ of $\mu$ satisfies the submodularity inequality (13) for every $A, B \in Q$, then the canonical deterministic pricing function $p^D_0$ of $\mu$ satisfies (13) for every $A, B \subseteq K$.

**Proof.** Let $A, B$ be two subsets of $K$ with $p_0(A), p_0(B) < \infty$ (otherwise the inequality below is trivial); by Proposition 2.2 there are $A', B' \in Q$ with $A' \supseteq A$ and $B' \supseteq B$ such that $p_0(A) = p_q(A') = p(A')$ and $p_0(B) = p_q(B') = p(B')$, and so

\[
p_0(A) + p_0(B) = p(A') + p(B') \geq p(A' \cup B') + p(A' \cap B')
\]

\[
\geq p_0(A' \cup B') + p_0(A' \cap B') \geq p_0(A \cup B) + p_0(A \cap B),
\]

where the first inequality is by the submodularity of $p$ on $Q$, the second because $p \geq p_0$, and the third because $p_0$ is nondecreasing. □

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### A.6 Appendix: Ultramodular Functions

The following result is used in the proof of part (ii) of Proposition 4.1 in Section 4.1. Marinacci and Montrucchio (2005, Lemma 5.1 and Theorem 5.4) prove it for interior points; for completeness, we provide here a short version of their proof that applies to boundary points as well. Recall the notation $\nabla^+ f(x) := (f'(x; e^i))_{i=1,...,k}$.

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\[38\] Whether $A \cup B$ and $A \cap B$ are in $Q$ or not.
**Proposition A.6** Let \( f: \mathbb{R}^k_+ \to \mathbb{R} \) be an ultramodular function. Then the vector \( \nabla^+ f(x) \) is the maximal subgradient of \( f \) at \( x \) for every \( x \in \mathbb{R}^k_+ \), and the function \( x \mapsto \nabla^+ f(x) \) is nondecreasing in \( x \) (i.e., \( \nabla^+ f(y) \geq \nabla^+ f(x) \) for every \( y \geq x \)).

**Proof.** (i) Let \( g \in \partial f(x) \); then
\[
(f(x + \delta e_i) - f(x)) / \delta \geq g_i \quad \text{for every } \delta > 0
\]
by the subgradient inequality, and so, by dividing by \( \delta \) and taking the limit as \( \delta \to 0^+ \), we get
\[
f'(x; e_i) \geq g_i.
\]
Thus,
\[
\partial f(x) \subseteq \{ g : g \leq \nabla^+ f(x) \}.
\]

(ii) Let \( \delta > 0 \); we have
\[
f(x + \delta e) - f(x) = [f(x_1 + \delta, ..., x_k + \delta) - f(x_1, x_2 + \delta, ..., x_k + \delta)] + [f(x_1, x_2 + \delta, ..., x_k + \delta) - f(x_1, x_2, x_3 + \delta, ..., x_k + \delta)] + ... + [f(x_1, ..., x_{k-1}, x_k + \delta) - f(x_1, ..., x_k)]
\]
where the inequality is by supermodularity (term by term, since \( \delta > 0 \)). Dividing by \( \delta \) and taking the limit as \( \delta \to 0^+ \) yields
\[
f'(x; e) = \sum_{i=1}^k f'(x; e_i) = e \cdot \nabla^+ f(x).
\]

(iii) By (i) we have \( f'(x; e) = \max_{g \in \partial f(x)} e \cdot g \leq e \cdot \nabla^+ f(x) \), which together with (ii) yields equality. Therefore \( \nabla^+ f(x) \) must belong to the closed set \( \partial f(x) \) (otherwise the maximum would be strictly smaller), and is thus its maximal element.

(iv) Let \( y \geq x \); then \( f(y + \delta e_i) - f(y) \geq f(x + \delta e_i) - f(x) \) for every \( \delta > 0 \) by ultramodularity, and so, by dividing by \( \delta \) and taking the limit as \( \delta \to 0^+ \), we get
\[
f'(y; e_i) \geq f'(x; e_i).
\]

A.7 Appendix: Separable Subadditivity without Allocation Monotonicity

Here we show that the converse of Theorem C (iii) in Section 4.1 does not hold for two or more goods: separably subadditive pricing may not suffice for allocation monotonicity. This is easy to see, already in the class of symmetric deterministic mechanisms (Example A.7). However, it requires at least 3 goods, and so we provide another example for 2 goods (Example A.8).
Example A.7 Let $k \geq 3$, and let $p$ be the symmetric deterministic pricing function $p(m) = m$ for every $m \neq 1$ and $p(1) = 2$. Then $p$ is separably subadditive, because $p(m) + p(n) - p(m + n)$ equals 0, 1, or 2 (according to how many of $m, n$ equal 1). But $p$ is not submodular, because $p(2) + p(2) < p(3) + p(1)$ (e.g., $p\{1, 2\} + p\{1, 3\} < p\{1, 2, 3\}$), and thus the corresponding deterministic mechanism is not allocation monotonic by Theorem C(iii).

Example A.8 Let $k = 2$. We first construct such a function $b$ on a bounded domain, specifically $D = [0, 1]^2$, and then show how to extend it to all $\mathbb{R}^2_+$. Put $b(x) := [f(x)]_+$ for $x \in [0, 1]^2$, where $f$ is the convex function $f(x_1, x_2) := (1/3)(x_1^2 + x_2^2 + x_1 + x_2 - x_1x_2 - 2)$. It is easy to see that $b = 0$ on the convex set $D_0 := \{x \in [0, 1]^2 : f(x) \leq 0\}$, whose boundary goes through the three points $(1, 0), (\sqrt{3} - 1, \sqrt{3} - 1)$, and $(0, 1)$, and the gradient of $b$ is $(0, 0)$ for $x$ in the interior of $D_0$, and $(1/3)(2x_1 + 1 - x_2, 2x_2 + 1 - x_1)$ for $x$ in the interior of $D \setminus D_0$, and so all subgradients of $b$ lie in $[0, 1]^2$ (use $\nabla f(x) \geq 0$ when $f(x) \geq 0$); thus, $b$ is continuous, nondecreasing, nonexpansive, and convex on $D$. Moreover, it is separably superadditive there, because $f(x_1, 0)$ and $f(0, x_2)$ are $\leq 0$ for $0 \leq x_i \leq 1$, and so $b(x_1, 0) + b(0, x_2) = 0 + 0 \leq b(x)$. However, it is not supermodular because, for interior points of $D \setminus D_0$ (in particular, for $x \ll (1, 1)$ close to $(1, 1)$) we have $\partial^2 b/\partial x_1 \partial x_2 = -1/3 < 0$.

To extend $b$ to all $\mathbb{R}^2_+$ while keeping all the above properties, we define $b(x) := b(\hat{x}) + [x_1 - 1]_+ + [x_2 - 1]_+$, where $\hat{x}_i := \min\{x_i, 1\}$ (cf. Example 5.2 for a similar extension). Thus, there are four regions, $C_{00} := D = [0, 1]^2$, $C_{10} := [1, \infty) \times [0, 1)$, $C_{01} := [0, 1] \times [1, \infty)$, and $C_{11} := [1, \infty)^2$ (the subscript gives the coordinates of the minimal point in each region), where the function $b$ is, respectively, $b_{00}(x) := [f(x)]_+$, $b_{10}(x) := (1/3)x_1^2 + x_2 - 1$, $b_{01}(x) := (1/3)x_1^2 + x_2 - 1$, and $b_{11}(x) := x_1 + x_2 - 5/3$. We will see below that $b$ is a convex function. Since all its gradients lie in $[0, 1]^2$, it is nondecreasing and nonexpansive. It is also separably superadditive, because

$$b(x) - b(x_1, 0) - b(0, x_2) = (b(\hat{x}) + [x_1 - 1]_+ + [x_2 - 1]_+)$$

$$- (b(\hat{x}_1, 0) + [x_1 - 1]_+) - (b(0, \hat{x}_2) + [x_2 - 1]_+)$$

$$= b(\hat{x}) - b(\hat{x}_1, 0) - b(0, \hat{x}_2) \geq 0$$

because $\hat{x} \in D$. However, as we have already seen, $b$ is not supermodular (already on $D$).

It remains to show that $b$ is a convex function. Since this is equivalent to convexity along any straight-line segment, and $b$ is convex in each region, we have to show that at every boundary point $x^*$, for example $x^* \in C_{00} \cap C_{10}$, as we cross from $C_{00}$ to $C_{10}$ in the direction $z$, we have $b_{00}(x^*; z) \leq b_{10}(x^*; z)$, i.e., $(\nabla b_{10}(x^*) - \nabla b_{00}(x^*)) \cdot z \geq 0$. Indeed: $x_1^* = 1$ and $0 \leq x_2^* \leq 1$; since we go from $\{x_1 < 1\}$ to $\{x_1 > 1\}$, we have $z_1 \geq 0$ (with no restriction on $z_2$); thus $\nabla b_{00}(x^*) = (1/3)(2x_1^* + 1 - x_2^*, 2x_2^* + 1 - x_1^*) = (1-(1/3)x_2^*, (2/3)x_2^*)$.
and \( \nabla b_{10}(x^*) = (1, (2/3)x_2^*) \), and the desired inequality holds (because \( x_2^* \geq 0 \) and \( z_1 \geq 0 \)). Another case: for \( x^* \in C_{01} \cap C_{11} \) and \( z_1 \geq 0 \), we have \( x_1^* = 1 \), \( \nabla b_{01}(x^*) = (2/3, 1) \), and \( \nabla b_{11}(x^*) = (1, 1) \), and so again \( (\nabla b_{11}(x^*) - \nabla b_{01}(x^*)) \cdot z \geq 0 \). The other cases are similar.

### A.8 Appendix: Monotonicity of Deterministic Mechanisms

Here we obtain further conditions for the monotonicity of deterministic mechanisms (see Section 7).

We start with a sufficient condition that is weaker than that of Corollary 7.2, and is easily obtained from Theorem 7.1.

**Corollary A.9** Let \( \mu \) be a deterministic buyer-favorable tie-consistent mechanism, with a nondecreasing pricing function \( p : 2^K \to \mathbb{R}_+ \). A sufficient condition for \( \mu \) to be monotonic is: if

\[
p(A) > p(B)
\]

for some \( A, B \subseteq K \), then there is \( \emptyset \neq C \subseteq A \setminus B \) such that

\[
p(A) - p(A \setminus C) \geq p(B \cup C) - p(B).
\]

**Proof.** Condition (34) contradicts condition (29).

Putting \( I := A \setminus B \), \( J := B \setminus A \), and replacing \( A \cap B \) with \( A \), this can rewritten as follows (cf. (31)): if

\[
p'(A; I) > p'(A; J)
\]

then there is \( \emptyset \neq \tilde{I} \subseteq I \) such that

\[
p'(A \cup I \setminus \tilde{I}; \tilde{I}) \geq p'(A \cup J; \tilde{I}).
\]

Proposition 5 in Hart and Reny (2015) shows that seller-favorable deterministic symmetric mechanisms are monotonic; the same holds for buyer-favorable (tie-consistent) mechanisms.

**Corollary A.10** Let \( \mu \) be a deterministic symmetric buyer-favorable tie-consistent mechanism. Then \( \mu \) is monotonic.

**Proof.** If \( p(A) > p(B) \) then \( |A| > |B| \) (because \( p \) is nondecreasing), and so taking a set \( C \subseteq A \setminus B \) with of size \( |C| = |A| - |B| \) (such a \( C \) exists since \( |A \setminus B| \geq |A| - |B| \)) yields \( p(A) - p(A \setminus C) = p(B \cup C) - p(B) \) (because \( |A| = |B \cup C| \) and \( |A \setminus C| = |B| \)), which is (34). □

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39The condition for convexity is thus \( \partial b_{00}/\partial x_1 \leq \partial b_{10}/\partial x_1 \) and \( \partial b_{00}/\partial x_2 = \partial b_{10}/\partial x_2 \) at each \( x^* \in C_{00} \cap C_{10} \).
Using a “Theorem of the Alternative” yields another characterization of monotonicity for deterministic mechanisms. The next proposition gives the technical result; we denote by $\mathcal{P}(I)$ the set of nonempty subsets of a finite set $I$.

**Proposition A.11** Let $I$ be a nonempty finite set, and let $v, w : \mathcal{P}(I) \to \mathbb{R}$. The following two statements are equivalent:

(I) There is no $z \in \mathbb{R}^I$ such that

$$v(C) \leq z(C) < w(C) \quad \text{for all } C \in \mathcal{P}(I).$$

(II) There are $\lambda, \mu \in \mathbb{R}^{\mathcal{P}(I)} \setminus \{0\}$ such that

$$\sum_{C \in \mathcal{P}(I)} \lambda_C \mathbf{1}_C = \sum_{C \in \mathcal{P}(I)} \mu_C \mathbf{1}_C \quad \text{and}$$

$$\sum_{C \in \mathcal{P}(I)} \lambda_C v(C) \geq \sum_{C \in \mathcal{P}(I)} \mu_C w(C).$$

**Proof.** Let $M$ denote the $(2^{|I|} - 1) \times |I|$ matrix whose rows are the characteristic vectors $\mathbf{1}_C \in \mathbb{R}^I$ of all $C \in \mathcal{P}(I)$. Writing $v$ for the vector $(v(C))_{C \in \mathcal{P}(I)}$ and $w$ for the vector $(w(C))_{C \in \mathcal{P}(I)}$, (I) is equivalent to the system of inequalities

$$Mz - v\zeta \geq 0$$

$$-Mz + w\zeta \gg 0$$

$$\zeta > 0$$

not having a solution $z \in \mathbb{R}^I$, $\zeta \in \mathbb{R}$ (indeed, for one direction take $\zeta = 1$, and for the converse replace $z$ with $(1/\zeta)z$). By Motzkin’s Theorem of the Alternative (see, e.g., Mangasarian 1994), this is equivalent to the system

$$\lambda^T M + \mu^T (-M) = 0$$

$$\lambda^T (-v) + \mu^T w + \eta = 0$$

$$\lambda, \mu, \eta \geq 0$$

$$(\mu, \eta) \neq 0$$

having a solution $\lambda, \mu \in \mathbb{R}^I, \eta \in \mathbb{R}$. Now, if $\mu = 0$ then $\lambda = 0$ (because $\lambda^T M = \mu^T M = 0$ implies $\lambda = 0$), and then $\eta = 0$ (because $\eta = \lambda^T v - \mu^T w$), contradicting $(\mu, \eta) \neq 0$. Therefore $\mu \neq 0$, and so $\lambda \neq 0$, and we get

$$\lambda^T M = \mu^T M$$

$$\lambda^T v \geq \mu^T w,$$

which is (II). $\blacksquare$
Applying this equivalence to the result of Theorem 7.1 yields

**Theorem A.12** Let $\mu$ be a deterministic buyer-favorable tie-consistent mechanism, with a nondecreasing pricing function $p : 2^K \rightarrow \mathbb{R}_+$. A necessary and sufficient condition for $\mu$ to be monotonic is: if

$$p(A) > p(B)$$

for some $A, B \subseteq K$, then there are $\lambda, \mu \in \mathbb{R}^{P(A \setminus B) \setminus \{0\}}$ such that

$$\sum_{C \in P(A \setminus B)} \lambda_C 1_C = \sum_{C \in P(A \setminus B)} \mu_C 1_C \quad \text{and} \quad \sum_{C \in P(A \setminus B)} \lambda_C (p(A) - p(A \setminus C)) \geq \sum_{C \in P(A \setminus B)} \mu_C (p(B \cup C) - p(B)).$$

### A.9 Appendix: Revenue of Deterministic Mechanisms

In this appendix we generalize the line of proof of Section 6 from symmetric deterministic mechanisms (which are monotonic) to general deterministic mechanisms (which need not be monotonic). This yields, in particular, an improvement of an $O(\log k)$ factor to the result of Corollary A.5 in Hart and Nisan (2019), on the ratio between the deterministic and separate revenues. We consider first the supermodular case, and then obtain the general result by pricing function approximation.

#### A.9.1 The Supermodular Case

A deterministic pricing function $p : 2^K \rightarrow [0, \infty]$ is supermodular if $p(A) + p(B) \leq p(A \cup B) + p(A \cap B)$ for every $A, B \subseteq K$; let $\text{SUPERMODDRev}$ denote the maximal revenue achievable by supermodular deterministic mechanisms.

**Proposition A.13** Let $X$ be a $k$-good random valuation. Then

$$\text{SUPERMODDRev}(X) \leq \frac{1}{k} (2^k - 1) \cdot \text{SRev}(X).$$

**Proof.** Let $\mu = (q, s)$ be a $k$-good deterministic mechanism with canonical deterministic pricing function $p$ that is supermodular. Without loss of generality assume that $p(A)$ is finite for all $A$; otherwise, replacing each $p(B) = \infty$ by, say, $p'(B) = |B|M$, where $M > \max\{p(A) : p(A) < \infty\} \geq \max\{s(x) : x \in \mathbb{R}_+^k\}$, keeps the pricing function nondecreasing and supermodular, and can only increase the revenue (because any switch to a new price yields a higher payment).

All sets $A, B$ below should be understood as subsets of $K$, and we will write $p_A$ instead of $p(A)$. Given a random allocation $X$, for each set $A$ let $\beta_A := \mathbb{P}[q(X) = A]$ be the probability that the allocation is the set $A$; then

$$R(\mu; X) = \sum_{A \neq \emptyset} \beta_A p_A$$
(because $p_\emptyset = 0$). Consider the expression

$$Z := \sum_{A \neq \emptyset} \frac{1}{|A|} \sum_{i \in A} (p_A - p_{A \setminus \{i\}}) \sum_{B \supseteq A} \beta_B.$$ 

In the steps below we will prove the two inequalities

$$R(\mu; X) \leq Z \leq \frac{2^k - 1}{k} \cdot \text{SRev}(X),$$

which yield the result.

(i) For every set $A \neq \emptyset$ with $p(A) < \infty$ and every $i \in A$ we have

$$(p_A - p_{A \setminus \{i\}}) \sum_{B \supseteq A} \beta_B \leq \text{REV}(X_i).$$

**Proof.** If $q(x) = A$ then $x(A) - p_A \geq x(A \setminus \{i\}) - p_{A \setminus \{i\}}$ by IC, and so $x_i \geq p_A - p_{A \setminus \{i\}}$. Since $p_B - p_{B \setminus \{i\}} \geq p_A - p_{A \setminus \{i\}}$ for every $B \supseteq A$ by the supermodularity of $p$, we get

$$\mathbb{P} [X_i \geq p_A - p_{A \setminus \{i\}}] \geq \sum_{B \supseteq A} \mathbb{P} [q(X) = B] = \sum_{B \supseteq A} \beta_B;$$

multiplying by $p_A - p_{A \setminus \{i\}}$ completes the proof. 

(ii)

$$Z \leq \frac{2^k - 1}{k} \cdot \text{SRev}(X).$$

**Proof.** By (i) we have

$$Z \leq \sum_{A \neq \emptyset} \frac{1}{|A|} \sum_{i \in A} \text{REV}(X_i) = \sum_{i=1}^{k} \text{REV}(X_i) \sum_{A \ni i} \frac{1}{|A|}.$$ 

For each $i \in K$ there are $\binom{k-1}{\ell-1}$ sets $A$ of size $\ell$ that contain $i$, and so

$$\sum_{A \ni i} \frac{1}{|A|} = \sum_{\ell=1}^{k} \frac{1}{\ell} \binom{k-1}{\ell-1} = \sum_{\ell=1}^{k} \frac{1}{\ell} \frac{k}{\ell} = \frac{1}{k} (2^k - 1).$$

(iii)

$$Z = R(\mu; X) + \sum_{B \neq \emptyset} \beta_B \sum_{A \ni B} p_A \frac{2^k |A| + 1 - |B|}{|A| + 1}.$$ 

**Proof.** Fix $B$, and consider all the terms of $Z$ that include $\beta_B$. For each $A \subseteq B$ the term $\frac{1}{|A|} p_A \beta_B$ appears $|A|$ times (once for each $i \in A$), yielding $p_A \beta_B$ in total; and for each $A \nsubseteq B$ the term $-(|A| + 1)^{-1} p_A \beta_B$ appears $|B| - |A| |A|$ times (once for each $A' \subseteq B$ such
that $A = A \setminus \{i\}$, i.e., once for each $i \in B \setminus A$, yielding $-(|B| - |A|)(|A| + 1)^{-1}p_A\beta_B$ in total. Thus,

$$Z = \sum_{B \neq \emptyset} \beta_B \sum_{A \subseteq B} p_A \left(1 - \frac{|B| - |A|}{|A| + 1}\right)$$

$$= \sum_{B \neq \emptyset} \beta_B p_B + \sum_{B \neq \emptyset} \beta_B \sum_{A \subseteq B} p_A \left(1 - \frac{|B| - |A|}{|A| + 1}\right),$$

where in the second line we have split the interior sum into $A = B$ and $A \subsetneq B$; this completes the proof. □

(iv) For every $B \neq \emptyset$ we have

$$\sum_{A \subseteq B} p_A \frac{2|A| + 1 - |B|}{|A| + 1} \geq 0.$$  

Proof. Let $n := |B|$; for each $m = 1, \ldots, n - 1$ put

$$\pi_m := \left(\frac{n}{m}\right)^{-1} \sum_{A \subseteq B: |A| = m} p_A \quad \text{and}$$

$$\lambda_m := \frac{2m + 1 - n}{m + 1} \left(\frac{n}{m}\right);$$

we need to show that

$$\sum_{m=1}^{n-1} \lambda_m \pi_m \geq 0.$$  

We will show that each term $\lambda_m \pi_m$ whose coefficient $\lambda_m$ is negative (i.e., when $m < (n - 1)/2$) is “covered” by the corresponding term $\lambda_{n-m-1} \pi_{n-m-1}$ (whose coefficient is positive); i.e.,

$$\lambda_m \pi_m + \lambda_{n-m-1} \pi_{n-m-1} \geq 0. \quad (36)$$

Indeed, for each $m < (n - 1)/2$ we have

$$|\lambda_m| = \frac{n - 2m - 1}{m + 1} \left(\frac{n}{m}\right) = \frac{n - 2m - 1}{m} \left(\frac{n}{n - m}\right) = \lambda_{n-m-1};$$

together with $\pi_m \leq \pi_{n-1-m}$, which we will prove in (v) below, this yields (36). Summing over all $m < (n - 1)/2$ completes the proof. □

(v) Let $N$ be a set of size $n$, and for each $m = 0, 1, \ldots, n$ let $\pi_m$ be the average price of subsets of $N$ of size $m$, i.e.,

$$\pi_m := \left(\frac{n}{m}\right)^{-1} \sum_{C \subseteq N: |C| = m} p_C.$$  

Then $\pi_m$ is a nondecreasing function of $m$.  

51
Proof. Let $m < n$; we will show that $\pi_m \leq \pi_{m+1}$. The function $p$ is nondecreasing and so

$$\sum_C \sum_D p_C \leq \sum_C \sum_D p_D,$$

where both sums range over all pairs $(C, D)$ such that $|C| = m$, $|D| = m + 1$, and $C \subset D$. Each $p_C$ appears in the left-hand sum $n - m$ times (once for each $i \in N \setminus C$), and each $p_D$ appears in the right-hand sum $m + 1$ times (once for each $i \in D$). Therefore

$$(n - m) \binom{n}{m} \pi_m \leq (m + 1) \binom{n}{m+1} \pi_{m+1}.$$ 

Since $(n - m) \binom{n}{m} = (m + 1) \binom{n}{m+1}$, we get $\pi_m \leq \pi_{m+1}$. □

**Remark.** The result of Proposition A.13 is tight: the pricing functions in the proof of Proposition 7.1 in Hart and Nisan (2019)—which are used in Proposition A.10 there to get the ratio $DRev (X)/SRev (X)$ close to $(2^k - 1)/k$—are supermodular. Indeed, the sequence $t_n$ constructed there increases exponentially, i.e., $t_n \leq \varepsilon \cdot t_{n+1}$ for all $n$, where $\varepsilon > 0$ is small; hence, by taking $\varepsilon \leq 1/2$, in the nontrivial case where $A, B \not\subseteq A \cup B$ we get $p(A), p(B) \leq (1/2)p(A \cup B)$, and so $p(A) + p(B) \leq p(A \cup B) \leq p(A \cup B) + p(A \cap B)$.

**A.9.2 The General Case**

We approximate any deterministic pricing function by a supermodular one.

**Proposition A.14** For every deterministic pricing function $p$ on $k$ goods there exists a supermodular pricing function $p'$ such that

$$p(A) \leq p'(A) \leq 2^{k-1}p(A)$$

for every $A \subseteq K$.

**Proof.** The minimal supermodular function $p'$ that majorizes $p$ (i.e., $p' \geq p$) is constructed inductively, starting with $p'(\emptyset) := 0$ and $p'({i}) := p({i})$ for every $i \in K$, by letting

$$p'(A) := \max\{p(A), \hat{p}(A)\}$$

where

$$\hat{p}(A) := \max_{i,j \in A, i \neq j} \{p'(A \setminus {i}) + p'(A \setminus {j}) - p'(A \setminus {i, j})\}$$

for every $A \subseteq K$ with $|A| \geq 2$ (indeed, the function $p'$ is clearly supermodular, and $p' \geq p$; it is straightforward to show inductively that any supermodular $p''$ such that $p'' \geq p$ must satisfy $p'' \geq p'$).

40 This fact by itself does not yield the result immediately; recall Simpson’s paradox.
We claim that
\[ p'(A) \leq 2^{|A|-1}p(A) \] (37)
for all \( A \not= \emptyset \). We prove this by induction. For singleton sets \( A = \{i\} \) we have equality. Take \( A \) with \(|A| \geq 2\), and let the maximum in the definition of \( \hat{p}(A) \) be attained at a certain pair \( i \neq j \); then
\[
\hat{p}(A) \leq p'(A\setminus\{i\}) + p'(A\setminus\{j\}) \leq 2^{|A|-2}p(A\setminus\{i\}) + 2^{|A|-2}p(A\setminus\{j\}) \\
\leq 2^{|A|-2}p(A) + 2^{|A|-2}p(A) = 2^{|A|-1}p(A)
\]
(where we have used: \( p \geq 0 \); (37) for the smaller sets \( A\setminus\{i\} \) and \( A\setminus\{j\} \); and \( p \) being nondecreasing). This yields (37) for \( A \). \( \blacksquare \)

The bound of \( 2^{|k|} \) is tight.

**Example.** Let \( k \) be even, say \( k = 2m \), and partition the \( k \) goods into the \( m \) disjoint pairs \( K_1 = \{1, 2\}, K_2 = \{3, 4\}, \ldots, K_m = \{2m-1, 2m\} \). Let \( p(A) = 1 \) if \( A \) contains at least one element of each \( K_\ell \) (i.e., \(|A \cap K_\ell| \geq 1 \) for all \( \ell = 1, \ldots, m \)), and 0 otherwise. We claim that the minimal supermodular \( p' \) that majorizes \( p \) is given by
\[
p'(A) := \prod_{\ell=1}^m |A \cap K_\ell|,
\] (38)
and so, in particular,
\[
p'(K) = 2^m = 2^{k/2}p(K).
\]

To show this, let \( p'' \) be the minimal supermodular function that majorizes \( p \). We will show that \( p'' \geq p' \); since \( p' \geq p \) and \( p' \) is supermodular (as is easy to verify), it follows that \( p'' = p' \).

For every set \( A \subseteq K \) put \( a_\ell := |A \cap K_\ell| \), and let \( a := (a_1, \ldots, a_m) \in \{0, 1, 2\}^m \) be the “profile” of \( A \). The pricing functions \( p \), and thus \( p'' \) as well as \( p' \), depend only on the profile \( a \); we will abuse notation and write \( p(a) \) instead of \( p(A) \), and similarly for \( p' \) and \( p'' \).

In the domain where at least one coordinate of \( a \) is 0 the function \( p \) vanishes, and so it is supermodular, which yields \( p''(a) = 0 = p'(a) \).

In the remaining domain, where \( a \geq (1, \ldots, 1) \), the function \( p \) is the constant function 1. Starting with \( p''(1, \ldots, 1) = p(1, \ldots, 1) = 1 \), we proceed by induction on \( n := |\{\ell : a_\ell = 2\}| \), the number of coordinates in \( a \) that equal 2. The supermodularity condition yields
\[
p''(2, \ast) + p''(0, \ast) \geq 2p''(1, \ast)
\]
for every completion \( \ast \) (for example: \( p''(B \cup \{1, 2\}) + p''(B) \geq p''(B \cup \{1\}) + p''(B \cup \{2\}) \) for every \( B \subseteq \{3, \ldots, 2m\} \)). Since \( p''(0, \ast) = p(0, \ast) = 0 \), we get \( p''(2, \ast) \geq 2p''(1, \ast) \), and thus, by induction, \( p''(a) \geq 2^n p''(1, \ldots, 1) = 2^n = p'(a) \) (recall that \( n \) is the number of 2’s
Thus $p'' \geq p'$ as claimed.

Corollary A.5 of Hart and Nisan (2019) yields $D\text{Rev}(X) \leq O(2^{k \log k}) \cdot S\text{Rev}(X)$; we improve this by a factor of $\log k$.

**Proposition A.15** Let $X$ be a $k$-good random allocation. Then

$$D\text{Rev}(X) \leq \ln 4 \cdot (2^k - 1) \cdot S\text{Rev}(X).$$

**Proof.** Proposition A.14 yields, by the result of Chawla, Teng, and Tzamos (2022) (Theorem A.2), an approximation factor of $2 \ln(2 \cdot 2^k - 1) = \ln 4 \cdot k$, namely,

$$D\text{Rev}(X) \leq \ln 4 \cdot k \cdot \text{SupermodDRev}(X)$$

for all $X$. Combine this with the result of Proposition A.13.

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