STABILITY OF SOLUTION FOR DYNAMIC GEODETIC NETWORK ADJUSTMENT

XU Caijun
XI Changyuan

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ABSTRACT This paper discusses theoretically the stability of solutions for dynamic geodetic network adjustment, Kalman filtering and sequence adjustment, and two examples are given. The solution for dynamic geodetic network adjustment is stable if the dynamic geodetic network is a classical network. There is not rank deficit in datum, or else the solution is not stable, which will depend on the initial value.

1 Introduction

Usually, the solution of the non-linear inversion problem relies strongly on initial values of the parameter vector, and at present, there is no general rule for selecting the initial values to solve the non-linear inversion problem. From the point of view of the algorithm, all methods currently used search for the solution in the whole parameter space, but for a practical inversion problem, some properties of the parameter vector may have been known before the inversion, which is referred to as "a prior information" of the parameter vector. In this case, the domain of the initial values can be reduced to a relatively feasible domain by using this information, and therefore, the convergence of the non-linear inversion problem will be improved.

Given a set of observations and a functional model, we are faced with estimating accurately the model parameters by solving, for instance, $L_1$, $L_2$ or $L_{oo}$ norm minimization problem, which is an inversion problem in nature. Solutions to linear complementarity problems and stability analysis of algorithms are discussed (Xu P L, et al., 1999). We discuss the stability of solution for dynamic geodetic network adjustment in this paper. A key problem on dynamic geodetic network adjustment is to determine the time series of position changes of surveying sites, and the Kalman filtering method and sequence adjustment method are two most common methods.

2 The concept of stability of solution

There is a state equation and an observation equation for a linear random control system:

$$
\begin{align*}
X_{k+1} & = \Phi_k X_k + \Gamma_k Q_k \\
L_k & = H_k X_k + \Delta_k
\end{align*}
$$

If there is a positive integer $N$ and positive real numbers $a, \beta$, which lead that Eq. (2) is true for all numbers $K \geq N$, then the system (1) is defined as a totally consistent controllable system.

$$
\begin{align*}
C_k & = \sum_{i=1}^{k} \Phi_k \Gamma_{i-1} G_{i-1} \Gamma_{i-1}^T \Phi_k^T > 0 \\
E \leq C_k & \leq \beta E
\end{align*}
$$

Where $E$ is a unit matrix, $C_k > 0$ means that $C_k$ is a positive definite matrix.

If there is a $K > N$ and above $N$, $a, \beta$ for Eq. (1), Eq. (3) is true, then the system (1) is defined as a totally consistent observable system.
For a normal constant system:
\[ \begin{align*}
X_{k+1} &= \mathcal{Q}X_k + \Delta_k \\
L_k &= HX_k + \Delta_k
\end{align*} \tag{4} \]

Conditions for the totally consistent controllable system and the totally consistent observable system are the same as Eq. (5) and Eq. (6), respectively.
\[ C_k = \sum_{j=0}^{\infty} \mathcal{Q}^{j}T^{T}(\mathcal{Q}^{j})^{T} > 0 \] \tag{5}
\[ O_k = \sum_{j=0}^{\infty} (\mathcal{Q}^{j})^{T}H^{T}H \mathcal{Q}^{j} > 0 \] \tag{6}

Where \( n \) is the dimensions for a state vector.

3 Stability analyses for dynamics leveling network adjustment

Usually, a leveling elevation in leveling network is a time series of benchmark (BM) position changes. It is to say that BM is a moving site along vertical direction. This repeated surveying leveling network is called dynamic leveling network. The Kalman filtering method is often used to adjust the dynamic leveling network.

If a leveling elevation of a BM is denoted as \( X_i \), and its moving rate is denoted as \( S_i \), changes of \( S_i \) are denoted as \( O_i \), which is noise dynamic, we can have the state equation and observation equation as follows.
\[ \begin{align*}
X_k &= \mathcal{Q}X_{k-1} + \Gamma \Omega_k \\
L_k &= HX_k + \Delta_k
\end{align*} \tag{7} \]

and \( X_k = (x_1 S_1 X_2 S_2 \cdots X_n S_n)^T \)

\[ \mathcal{Q} = \begin{bmatrix} B_1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

\[ T = \begin{bmatrix} 1 & T & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]

The form of the coefficient matrix \( H \) of observation equation is the same as static state leveling network. For observation \( L_{ij} \) of the \( K \) term, coefficient elements for \( X_j, X_i \) are +1, -1, respectively, and the other elements are zero.
\[ C_k = \sum_{j=0}^{\infty} \mathcal{Q}^{j}T^{T}(\mathcal{Q}^{j})^{T} \]

\[ TT^{T} = \begin{bmatrix} A_1 \\
\vdots \\
A_2 \end{bmatrix} \]

\[ \mathcal{Q}^i = \begin{bmatrix} A_1 \\
\vdots \\
A_2 \end{bmatrix} \]

Where
\[ A_1 = \begin{bmatrix} \frac{1}{4} T^4 & \frac{1}{2} T^3 \\
\frac{1}{2} T^3 & T^2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & iT \\
0 & 1 \end{bmatrix} \]

\[ C = \begin{bmatrix} \frac{1}{12} n(2n - 1)(2n - 3) T^4 & \frac{1}{2} n^2 T^3 \\
\frac{1}{2} n^2 T^3 & n T^3 \end{bmatrix} \]

Then \( C_k \) is positive matrix, that is to say, the system (7) is totally consistent controllable.

To discuss the character of the totally consistent observable system, we change the order of the state vector like this:
\[ X_k = (x_1 x_2 x_3 \cdots x_n S_1 S_2 S_3 \cdots S_n)^T \]

Then
\[ \mathcal{Q} = \begin{bmatrix} E & TE \\
0 & E \end{bmatrix}, \quad T = \begin{bmatrix} \frac{1}{2} T^2 E \\
TE \end{bmatrix} \]

Thus the observation equation can be expressed as follows.
\[ L_k = (H_0 X_k + \Delta_k \]

\[ O_k = \sum_{i=0}^{\infty} \left[ E \begin{bmatrix} iT E \end{bmatrix}^T \begin{bmatrix} H_1 H_1 \end{bmatrix} \right] \begin{bmatrix} E \begin{bmatrix} iT E \end{bmatrix} \end{bmatrix} = \sum_{i=0}^{\infty} \begin{bmatrix} H_1 H_1 \begin{bmatrix} i^2 T^2 H_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} i^2 T^2 H_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} T^2 \\
0 \end{bmatrix} \begin{bmatrix} 0 \\
0 \end{bmatrix} \]
\[
\begin{bmatrix}
  nH_1^T H_1 & n(n-1)/2 \cdot TH_1^T H_1 \\
  n(n-1)/2 \cdot TH_1^T H_1 & n(n-1)(2n-1)/6 \cdot T^2 H_1^T H_1
\end{bmatrix}
\]

(8)

\( O_k \) is positive definite matrix when \( H_1^T H_1 \) is positive definite matrix, then the system (7) is totally consistent observable, and \( O_k \) is not positive definite matrix when \( H_1^T H_1 \) is not positive definite matrix.

\( O_k \) is not positive definite matrix if a dynamic leveling network is rank deficit free network, then the dynamic leveling network does not satisfy the condition of totally consistent observation. However, \( O_k \) is positive definite matrix if a dynamic leveling network is a traditional network, and the dynamic leveling network satisfy the condition of totally consistent observation. It should be noted that the condition of totally consistent observation is only matrix \( H_1 \), but matrix \( H \) belonging to row full rank. Ulteriorly, the system (7) is stable \( C_k \) is positive definite matrix and \( O_k \) is positive definite matrix as well.

4 Stability analyses for sequence adjustment

For sequence adjustment, the linear system (1) can be expressed as follows.

\[
\begin{align*}
X_k &= X_{k-1} \\
L_k &= HX_k + \Delta_k
\end{align*}
\]

(9)

According to the method of sequence adjustment, if we get an estimated value for \( X_{k-1} \) by using measurements \( L_1, L_2, \ldots, L_{k-1} \), then the adjustment model for \( L_k, X_k \) can be expressed as follows:

\[
\begin{align*}
V_k &= HX_k - L_k \\
V_{sk} &= X_k
\end{align*}
\]

(10)

It is easy to prove that the system (10) is totally consistent controllable. Now we discuss its character for totally consistent observation.

\( O_k = E \)

\( \therefore \)

\( O_k = \sum_{j=0}^{n-1} \langle \varphi^j \rangle T^j H^j = nH^TH \)

(11)

If \( H^TH \) is or not positive definite matrix, that is to say, the solution for sequence adjustment is or not stable, depending on the network type of observation adjustment. If observations belong to tradition network, \( O_k \) is positive definite matrix, then the system (9) is stable, else the system (9) is not stable.

5 Some examples and discussion

For further discussing the stability of solutions to dynamic network adjustment, we calculate using Kalman filtering method and sequence adjustment method respectively with practical data from a dynamic leveling network in North China (Fig. 1); there is 9 terms of observation between 1967 and 1975. The results are listed in Table 1, 2 and shown in Fig. 2, 3.

![Fig. 1 A sketch map of leveling network](image-url)
Table 1  Adjustment results for state vector and its variance in Kalman filtering

| Datum        | The number Of the site | Initial value | Adjust value in surveying year |
|--------------|------------------------|---------------|-------------------------------|
|              |                        | State Var.    | 69 | 72 | 75 | State Var.    | 69 | 72 | 75 |
|              |                        | $H$ $D_{69}$  | $H$ $D_{69}$  | $H$ $D_{69}$  | $H$ $D_{69}$  | $H$ $D_{69}$  | $H$ $D_{69}$  |
| Bascenter    | 2                      | 7.261 6 1.000 0 | 7.251 1 2.352 8 | 7.253 1 7.008 2 | 7.263 0 15.848 3 |
|              |                        | 0.000 0 1.000 0 | -0.002 9 0.453 4 | -0.000 5 0.421 5 | 0.002 2 0.464 0 |
|              |                        | 7.161 6 1.500 0 | 6.850 6 3.113 6 | 6.553 2 9.660 2 | 6.263 1 21.707 0 |
|              |                        | -0.100 0 1.500 0 | -0.103 0 0.557 4 | -0.100 4 0.494 7 | -0.097 7 0.335 6 |
| Fixed site 1 | 2                      | 7.261 6 1.000 0 | 7.258 9 1.270 8 | 7.261 8 1.235 1 | 7.269 7 1.168 6 |
|              |                        | 0.000 0 1.000 0 | -0.000 6 0.354 2 | 0.000 6 0.247 3 | 0.002 6 0.244 4 |
|              |                        | 7.161 6 1.500 0 | 7.149 8 1.392 0 | 7.233 5 1.266 1 | 7.271 6 1.175 1 |
|              |                        | -0.100 0 1.500 0 | -0.015 3 0.399 1 | 0.008 0 0.250 9 | 0.010 3 0.245 0 |

Fig. 2  The bias for initial vector affect the bias for state vector solution for Kalman filtering
Table 2  Adjustment results for state vector and its variance in sequence adjusting

| Datum       | No. | State Var. | 69 | 72 | 75 |
|-------------|-----|------------|----|----|----|
| Barycenter  | 2   | 7.2616     | 1.000 0 | 7.2568 | 0.4051 | 7.2563 | 0.3058 | 7.2580 | 0.2623 |
|             | 5   | 6.8151     | 1.000 0 | 6.8006 | 0.5136 | 6.8000 | 0.4211 | 6.8019 | 0.3010 |
|             | 2   | 7.2616     | 1.000 0 | 7.2604 | 0.5948 | 7.2588 | 0.3585 | 7.2612 | 0.2366 |
| Fixed site 1| 5   | 6.8151     | 1.000 0 | 6.8170 | 0.5866 | 6.8167 | 0.3710 | 6.8156 | 0.2711 |

According to Table 1 and the theorem, there is no Li’s constant satisfying Eq. (9) to rank deficit dynamic leveling network. If there is a bias for initial vector, the bias for state vector solution will become more and more when iterative degree increases, e.g. the bias for Site 2’s elevation increases from 0.1 (in 1967) to 1.000 1 (in 1975), and its variance bias increases from 0.5 (in 1967) to 5.86 (in 1975). This iterative processing is emanative, and its solution is not stable. However, there is a Li’s constant satisfying Eq. (9) to full rank dynamic leveling network (fixed Site 1). If there is a bias for initial vector, its value will become smaller and smaller when iterative degree increases, and finally converges to exact solution, e.g. the bias for Site 2’s elevation decreases from 0.1 (in 1967) to 0.002 (in 1975), and its variance bias decreases from 0.5 (in 1967) to 0.006 (in 1975). This iterative processing is convergent, and its solution is stable. This result proofs the conclusion in Part 3.

Form Table 2, when No. 2 site has elevation bias of about 0.5 m and variance bias of 1.0 m for initial value at the datum of barycenter (rank deficit dynamic network), the sequence adjustment for every term has about pendular bias of 0.5 m in elevation and pendular variance bias of about 0.17 m. there is not Li’s constant. The iterative processing is emanative, and its solution depends on the initial value. It is not stable. No. 5 is in the similar case. However, if we fixed Site 1, when No. 2 has elevation bias of 0.5 m and variance bias of 1.0 m for initial value, the bias will become smaller and small-
er when iterative item increases, and the elevation bias and its variance bias separately becomes 0.05 m and 0.03 m in 1975. No. 5 has elevation bias of 0.10 m and variance bias of 0.5 m in 1967 (for initial value), which became separately 0.035 m and 0.01 m in 1975. They all converge to the exact solutions, which are stable. Its result proofs the conclusion in Part 4.

6 Conclusion

If a dynamic leveling network is a classical network which will be a totally consistent controllable and totally consistent observable system, it belongs to a stable system and its solution is stable or else its solution depends on the initial values of parameters. The solution for sequence adjustment is or not stable it depends on the network type of observation adjustment. If observations belong to tradition network, then adjustment system is stable or else the system is not stable. A prior information of the parameter vector will improve the stability of solutions if a system is not stable, and it is very useful in the geodetic inversion problems.

References

1 Xu P L, Cannon E, Lachapelle G. Stabilizing ill-conditioned linear complementarity problems. Journal of Geodesy, 1999, 73(7):204~213
2 Rice J R. Numerical methods, software, and analysis. New York, 1983
3 Aki K, Richards P G. Quantitative seismology theory and methods. San Francisco; W. H. Freeman and Company, 1980
4 Tarantola A. Inversion problem theory. Elsevier, 1994