On regular hypergraphs of high girth

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Abstract

We give lower bounds on the maximum possible girth of a linear, \( r \)-uniform, \( d \)-regular hypergraph with \( n \) vertices, using the definition of a hypergraph cycle due to Berge. These differ from the trivial upper bound by an absolute constant factor (viz., by a factor of between \( 3/2 + o(1) \) and \( 2 + o(1) \)). We also define a random \( r \)-uniform 'Cayley' hypergraph on \( S_n \) which has girth \( \Omega(\sqrt{\log |S_n|}) \) with high probability, in contrast to random regular \( r \)-uniform hypergraphs, which have constant girth with positive probability.

1 Introduction

The girth of a finite graph \( G \) is the shortest length of a cycle in \( G \). The girth problem asks for the least possible number of vertices \( n(g, d) \) in a \( d \)-regular graph of girth at least \( g \), for any pair of integers \( d, g \geq 3 \). Equivalently, for integers \( n, d \geq 3 \) with \( nd \) even, we denote by \( g_d(n) \) the largest possible girth of an \( n \)-vertex, \( d \)-regular graph.

The girth problem has received much attention for more than half a century. A fairly easy probabilistic argument shows that for any integers \( d, g \geq 3 \), there exist \( d \)-regular graphs with girth at least \( g \). An argument due to Erdős and Sachs [10] then shows that there exists such a graph with at most

\[
2^{\frac{(d-1)^{g-1}-1}{d-2}}
\]

vertices. This implies that

\[
g_d(n) \geq (1 - o(1)) \log_{d-1} n. \tag{1}
\]

(Here, and below, \( o(1) \) stands for a function of \( n \) that tends to zero as \( n \to \infty \).)

On the other hand, if \( G \) is a \( d \)-regular graph of girth at least \( g \), then counting the number of vertices of \( G \) of distance less than \( g/2 \) from a fixed vertex of \( G \) (when \( g \) is odd), or from a fixed edge of \( G \) (when \( G \) is even), immediately shows that

\[
|G| \geq n_0(g, d) := \begin{cases} 
1 + d \sum_{i=0}^{k-1} (d-1)^i & \text{if } g = 2k + 1; \\
2 \sum_{i=0}^{k-1} (d-1)^i & \text{if } g = 2k.
\end{cases}
\]
This is known as the Moore bound. Graphs for which the Moore bound holds with equality are known as Moore graphs (for odd $g$), or generalized polygons (for even $g$). It is known that Moore graphs only exist when $g = 3$ or 5, and generalized polygons only exist when $g = 4, 6, 8$ or 12. It was proved in [1, 5, 16] that

$$n(g, d) \geq n_0(g, d) + 2 \quad \text{for all } g \notin \{3, 4, 5, 6, 8, 12\};$$

even for large values of $g$ and $d$, no improvement on this is known.

A related problem is to give an explicit construction of a $d$-regular graph of girth $g$, with as few vertices as possible. The celebrated Ramanujan graphs of Lubotzky, Phillips and Sarnak [19], Margulis [21] and Morgenstern [22] constituted a breakthrough on both problems, implying that

$$g_d(n) \geq (4/3 - o(1)) \log_{d-1} n$$

via an explicit construction, whenever $d = q + 1$ for some odd prime power $q$.

One can obtain from this a lower bound on $g_d(n)$ for arbitrary $d \geq 3$, by choosing the minimum $d' \geq d$ such that $d' - 1$ is an odd prime power, taking a $d'$-regular Ramanujan graph, and removing $d' - d$ perfect matchings in succession. This yields

$$g_d(n) \geq (4/3 - o(1)) \frac{\log(d - 1)}{\log(d' - 1)} \log_{d-1} n.$$  \hspace{1cm} (3)

By known results on the density of the primes, this bound is always better than (1). Combining it with the Moore bound gives

$$(4/3 - o(1)) \frac{\log(d - 1)}{\log(d' - 1)} \log_{d-1} n \leq g_d(n) \leq (2 + o(1)) \log_{d-1} n.$$  \hspace{1cm} (4)

Improving the constants in (4) seems to be a very hard problem.

In this paper, we investigate an analogue of the girth problem for $r$-uniform hypergraphs, where $r \geq 3$. There are several natural notions of a cycle in a hypergraph, and we refer the reader to Section 4 for a brief discussion of some other interesting notions of girth in hypergraphs, and to [8] for a detailed treatise. Here we consider the least restrictive possible notion, originally due to Berge (see for example [3] and [4]). Namely, if $l \geq 2$, a cycle of length $l$ is a sequence of $l$ edges $(e_1, \ldots, e_l)$ such that there exist $l$ distinct vertices $v_1, \ldots, v_l$ with $v_i \in e_i \cap e_{i+1}$ for all $i$ (where we define $e_{l+1} := e_1$). Observe that two distinct edges $e, f$ with $|e \cap f| \geq 2$ form a cycle of length 2 under this definition, so when considering questions of girth, we may restrict our attention to linear hypergraphs.

We will use the following definitions and notation. A hypergraph $H$ is a pair of finite sets $(V(H), E(H))$, where $E \subset P(V)$. The elements of $V(H)$ are called the vertices of $H$, and the elements of $E(H)$ are called the edges of $H$. A hypergraph is said to be $r$-uniform if all its edges have size $r$. It is said to be $d$-regular if each of its vertices is contained in exactly $d$ edges. It is said to be linear if any two of its edges share at most one vertex.
Let $u$ and $v$ be vertices in a hypergraph $H$. A $u$-$v$ path of length $l$ in $H$ is a sequence of distinct edges $(e_1, \ldots, e_l)$ of $H$, such that $u \in e_1$, $v \in e_l$, $e_i \cap e_{i+1} \neq \emptyset$ for all $i \in \{1, 2, \ldots, l-1\}$, and $e_i \cap e_j = \emptyset$ whenever $j > i + 1$. (Note that some authors call this a geodesic path, and use the term path when non-consecutive edges are allowed to intersect.) The distance from $u$ to $v$ in $H$, denoted $\text{dist}(u, v)$, is the shortest length of a $u$-$v$ path in $H$. The ball of radius $R$ and centre $u$ in $H$ is the set of vertices of $H$ with distance at most $R$ from $u$. The diameter of a hypergraph $H$ is $\max \{\text{dist}(u, v) \mid u, v \in V(H)\}$.

A hypergraph is said to be a cycle if there exists a cyclic ordering of its edges, $(e_1, \ldots, e_l)$ say, such that there exist distinct vertices $v_1, \ldots, v_l$ with $v_i \in e_i \cap e_{i+1}$ for all $i$ (where we define $e_{l+1} := e_1$). As mentioned above, this notion of a hypergraph cycle is originally due to Berge, and is sometimes called a Berge-cycle. The length of a cycle is the number of edges in it. The girth of a hypergraph is the length of the shortest cycle it contains.

Extremal questions concerning Berge-cycles in hypergraphs have been studied by several authors. For example, in [6], Bollobás and Győri prove that an $n$-vertex, 3-uniform hypergraph with no 5-cycle has at most $\sqrt{2n^3/2 + \frac{5}{2}n}$ edges, and they give a construction showing that this is best possible up to a constant factor. In [17], Lazebnik and Verstraëte prove that a 3-uniform hypergraph of girth at least 5 has at most

$$\frac{1}{6n} \sqrt{n - \frac{3}{4}} + \frac{1}{12} n$$

edges, and give a beautiful construction (based on the projective plane $PG(2, q)$) showing that this is sharp whenever $n = q^2$ for an odd prime power $q \geq 27$. Interestingly, neither of these two constructions are regular.

In [13] and [18], Győri and Lemons consider the problem of excluding a cycle of length exactly $k$, for general $k \in \mathbb{N}$. In [13], they prove that an $n$-vertex, 3-uniform hypergraph with no cycle of length $2k+1$ has at most $4k^{2}n^{1+1/k} + O(n)$ edges. In [18], they prove that an $n$-vertex, $r$-uniform hypergraph with no $2k + 1$-cycle has at most $C_{k,r}(n^{1+1/k})$ edges, and furthermore that an $n$-vertex, $r$-uniform hypergraph with no $2k$-cycle has at most $C'_{k,r}(n^{1+1/k})$ edges, where $C_{k,r}, C'_{k,r}$ depend upon $k$ and $r$ alone.

In this paper, we will investigate the maximum possible girth of an $r$-uniform, $d$-regular, $n$-vertex, linear hypergraph, for $r$ and $d$ fixed and $n$ large. If $r \geq 3$ and $d \geq 2$, we let $g_{r,d}(n)$ denote the maximum possible girth of an $r$-uniform, $d$-regular, $n$-vertex, linear hypergraph.

In section 2, we will state upper and lower bounds on the function $g_{r,d}(n)$, which differ by an absolute constant factor. The upper bound is a simple analogue of the Moore bound for graphs, and follows immediately from known results. The lower bound is a hypergraph extension of a similar argument for graphs, due to Erdős and Sachs [10] — not a particularly difficult extension, but still, in our opinion, worth recording.

In section 3, we consider the girth of various kinds of ‘random’ $r$-uniform hypergraphs. We define a random $r$-uniform ‘Cayley’ hypergraph on $S_n$ which has girth $\Omega(\sqrt{\log |S_n|})$ with high probability, in contrast to random regular $r$-
uniform hypergraphs, which have constant girth with positive probability. We conjecture that, in fact, our ‘Cayley’ hypergraph has girth $\Omega(\log |S_n|)$ with high probability. We believe it may find other applications.

2 Upper and lower bounds

In this section, we state upper and lower bounds on the function $g_{r,d}(n)$, which differ by an absolute constant factor.

We first state a very simple analogue of the Moore bound for linear hypergraphs. For completeness, we give the proof, although the result follows immediately from known results, e.g. from Theorem 1 of Hoory [15].

**Lemma 1.** Let $r, d$ and $g$ be integers with $d \geq 2$ and $r, g \geq 3$. Let $H$ be an $r$-uniform, $d$-regular, $n$-vertex, linear hypergraph with girth $g$. If $g = 2k + 1$ is odd, then

$$n \geq 1 + d(r - 1) \sum_{i=0}^{k-1} ((d - 1)(r - 1))^i = 1 + d(r - 1) \frac{(d - 1)^k(r - 1)^k - 1}{(d - 1)(r - 1) - 1}, \quad (5)$$

and if $g = 2k$ is even, then

$$n \geq r \sum_{i=0}^{k-1} ((d - 1)(r - 1))^i = r \frac{(d - 1)^k(r - 1)^k - 1}{(d - 1)(r - 1) - 1}. \quad (6)$$

**Proof.** The right-hand side of (5) is the number of vertices in any ball of radius $k$. The right-hand side of (6) is the number of vertices of distance at most $k - 1$ from any fixed edge $e \in H$. \hfill \Box

The following corollary is immediate.

**Corollary 2.** Let $H$ be a linear, $r$-uniform, $d$-regular hypergraph with $n$ vertices and girth $g$. Then

$$g \leq \frac{2 \log n}{\log(r - 1) + \log(d - 1)} + 2.$$ 

Our aim is now to obtain a hypergraph analogue of the non-constructive lower bound [11]. We first prove the following existence lemma.

**Lemma 3.** For all integers $d \geq 2$ and $r, g \geq 3$, there exists a finite, linear, $r$-uniform, $d$-regular hypergraph with girth at least $g$.

**Proof.** We prove this by induction on $g$, for fixed $r, d$. When $g = 3$, all we need is a linear, $r$-uniform, $d$-regular hypergraph. Let $H$ be the hypergraph on vertex-set $\mathbb{Z}_r^d$, whose edges are all the axis-parallel lines, i.e.

$$E(H) = \{ \{x, x + e_i, x + 2e_i, \ldots, x + (r - 1)e_i\} : x \in \mathbb{Z}_r^d, i \in [d] \}. \quad (Here, e_i denotes the ith standard basis vector in \mathbb{Z}_r^d, i.e. the vector with 1 in the ith coordinate and zero elsewhere.) Clearly, $H$ is linear and $d$-regular.
For \( g \geq 4 \) we do the induction step. We start from a finite, linear, \( r \)-uniform, \( d \)-regular hypergraph \( H \) of girth at least \( g - 1 \). Of all such hypergraphs we consider one with the least possible number of \((g - 1)\)-cycles. Let \( M \) be the number of \((g - 1)\)-cycles in \( H \). We shall prove that \( M = 0 \). If \( M > 0 \), we consider a random 2-cover \( H' \) of \( H \), defined as follows. Its vertex set is \( V(H') = V(H) \times \{0, 1\} \), and its edges are defined as follows. For each \( e \in E(H) \), choose an arbitrary ordering \((v_1, \ldots, v_r)\) of the vertices in \( e \), flip \( r - 1 \) independent fair coins \( c_e^{(1)}, \ldots, c_e^{(r-1)} \in \{0, 1\} \), and include in \( H' \) the two edges

\[
\{(v_1, i), (v_2, i + c_e^{(1)}) \}, \ldots, \{(v_r, i + c_e^{(r-1)})\}
\]

for \( i = 0, 1 \).

Here, \( \oplus \) denotes modulo 2 addition. Do this independently for each edge. Note that \( H' \) is linear and \( d \)-regular, since \( H \) is.

Let \( \pi : V(H') \to V(H) \) be the cover map, defined by \( \pi((v, i)) = v \) for all \( v \in V(H) \) and \( i \in \{0, 1\} \). Since any cycle in \( H' \) is projected to a cycle in \( H \) of the same length, \( H' \) has girth at least \( g - 1 \), and each \((g - 1)\)-cycle in \( H' \) projects to a \((g - 1)\)-cycle in \( H \). Let \( C \) be a \((g - 1)\)-cycle in \( H \). We claim that \( \pi^{-1}(C) \) either consists of two vertex-disjoint \((g - 1)\)-cycles in \( H' \), or a single \( 2(g - 1)\)-cycle in \( H' \), and that the probability of each is \( 1/2 \). To see this, let \((e_1, \ldots, e_{g-1})\) be any cyclic ordering of \( C \); then \( |e_i \cap e_{i+1}| = 1 \) for all \( i \) (since \( H \) is linear). Let \( e_i \cap e_{i+1} = \{w_i\} \) for all \( i \in [g - 1] \). For each \( i \), consider the two edges in \( \pi^{-1}(e_i) \). Either one of the two edges contains \((w_{i-1}, 0)\) and \((w_i, 0)\) and the other contains \((w_{i-1}, 1)\) and \((w_i, 1)\), or one edge contains \((w_{i-1}, 0)\) and \((w_i, 1)\) and the other edge contains \((w_{i-1}, 1)\) and \((w_i, 0)\). Call these two events \( S(e_i) \) and \( D(e_i) \), for ‘same’ and ‘different’. Observe that \( S(e_i) \) and \( D(e_i) \) each occur with probability \( 1/2 \), independently for each edge \( e_i \) in the cycle. Notice that \( \pi^{-1}(C) \) consists of two disjoint \((g - 1)\)-cycles if and only if \( D(e_i) \) occurs an even number of times, and the probability of this is \( 1/2 \), proving the claim.

It follows that the expected number of \((g - 1)\)-cycles in \( H' \) is \( M \). Note that the trivial cover \( H_0 \) of \( H \), which has \( c_e^{(j)} = 0 \) for all \( j \) and \( e \), consists of two vertex-disjoint copies of \( H \), and therefore has \( 2M \) \((g - 1)\)-cycles. It follows that there is at least one 2-cover of \( H \) with fewer than \( M \) \((g - 1)\)-cycles, contradicting the minimality of \( M \). Therefore, \( M = 0 \), so in fact, \( H \) has girth at least \( g \). This completes the proof of the induction step, proving the theorem.

Consider a \( d \)-regular graph with girth at least \( g \), with the smallest possible number of vertices. Erdős and Sachs [10] proved that the diameter of such a graph is at most \( g \). But a \( d \)-regular graph with diameter \( D \) has at most

\[
1 + d \sum_{i=0}^{D-1} (d - 1)^i
\]

vertices (since this is an upper bound on the number of vertices in a ball of radius \( D \)). This yielded the upper bound [11] on the number of vertices in a vertex-minimal \( d \)-regular graph of girth at least \( g \).

We need an analogue of the Erdős-Sachs argument for hypergraphs.
Lemma 4. Let \( r, d \) and \( g \) be integers with \( d \geq 2 \) and \( r, g \geq 3 \). Let \( H \) be a linear, \( r \)-uniform, \( d \)-regular hypergraph with girth at least \( g \), with the smallest number of vertices subject to these conditions. Then \( H \) cannot contain \( r \) vertices every two of which are at distance greater than \( g \) from one another.

Proof. We show that if \( H \) contains \( r \) distinct vertices \( v_1, v_2, \ldots, v_r \) whose pairwise distances are all greater than \( g \), then it is possible to construct a linear, \( r \)-uniform, \( d \)-regular hypergraph with girth at least \( g \), that has fewer vertices than \( H \). Let \( e_i^{(1)}, e_i^{(2)}, \ldots, e_i^{(d)} \) be the edges of \( H \) which contain \( v_i \), for each \( i \in [r] \). Let

\[
W_i = \bigcup_{j=1}^{d} (e_i^{(j)} \setminus \{v_i\})
\]

for each \( i \in [r] \). Notice that \( |W_i| = d(r - 1) \) for each \( i \in [r] \), since the edges \( e_i^{(j)} \) \((j \in [d])\) are disjoint apart from the vertex \( v_i \). Define a new hypergraph \( H' \) by taking \( H \), deleting \( v_1, v_2, \ldots, v_r \) and all the edges containing them, and adding \( d(r - 1) \) pairwise disjoint edges, each of which contains exactly one vertex from \( W_i \) for each \( i \in [r] \). (Note that none of these ‘new’ edges were in the original hypergraph \( H \), otherwise some \( v_a \) and \( v_b \) would have been at distance at most 3 in \( H \), a contradiction.) Clearly, \( H' \) is \( d \)-regular. We claim that it is linear. Indeed, if one of the ‘new’ edges shared two vertices with some edge \( f \in H \) (say it shares \( a \in W_i \) and \( b \in W_j \), where \( i \neq j \)), then there would be a path of length 3 in \( H \) from \( v_i \) to \( v_j \), a contradiction.

We now claim that \( H' \) has girth at least \( g \). Suppose for a contradiction that \( H' \) has girth at most \( g - 1 \). Let \( C \) be a cycle in \( H' \) of length \( l \leq g - 1 \). Let \((f_1, \ldots, f_l)\) be a cyclic ordering of \( C \).

Suppose firstly that \( C \) contains exactly one of the ‘new’ edges (say \( f_1 \) is a ‘new’ edge). Deleting \( f_1 \) from \( C \) produces a path \( P \) of length at most \( g - 2 \) in \( H \). We have \(|f_{i-1} \cap f_i| = |f_i \cap f_{i+1}| = 1\) (let \( f_{i-1} \cap f_i = \{a\} \), and let \( f_i \cap f_{i+1} = \{b\} \)). Note that \( a \neq b \). Suppose that \( a \in W_p \) and \( b \in W_q \). Since \( a \neq b \) and \( a, b \in f_i \), we must have \( p \neq q \), as each ‘new’ edge contains exactly one vertex from each \( W_k \). Let \( e \) be the edge of \( H \) containing both \( v_p \) and \( a \), and let \( e' \) be the edge of \( H \) containing both \( v_q \) and \( b \); adding \( e \) and \( e' \) to the appropriate ends of the path \( P \) produces a path in \( H \) of length at most \( g \) from \( v_p \) to \( v_q \), contradicting the assumption that \( \text{dist}(v_p, v_q) > g \).

Suppose secondly that \( C \) contains more than one of the ‘new’ edges. Choose a minimal sub-path \( P \) of \( C \) which connects two ‘new’ edges. Suppose \( P \) connects the new edges \( f_i \) and \( f_j \), so that \( P = (f_i, f_{i+1}, \ldots, f_{j-1}, f_j) \). Note that \(|i - j| \leq g - 2 \), so \( P \) has length at most \( g - 1 \). Let \( f_i \cap f_{i+1} = \{a\} \), and suppose \( a \in W_p \); let \( f_{j-1} \cap f_j = \{b\} \), and suppose \( b \in W_q \). Let \( e \) be the unique edge of \( H \) which contains both \( v_p \) and \( a \), and let \( e' \) be the unique edge of \( H \) which contains both \( v_q \) and \( b \). If \( p \neq q \), then we can produce a path in \( H \) from \( v_p \) to \( v_q \) by taking \( P \), and replacing \( f_i \) with \( e \) and \( f_j \) with \( e' \); this path has length at most \( g - 1 \), contradicting our assumption that \( d(v_p, v_q) > g \). If \( p = q \), then we can produce a cycle in \( H \) by taking \( P \), removing \( f_i \) and \( f_j \), and adding the edges \( e \) and \( e' \).
(which share the vertex \(v_p\)); this cycle has length at most \(g - 1\), contradicting our assumption that \(H\) has girth at least \(g\).

We may conclude that \(H'\) has girth at least \(g\), as claimed. Clearly, \(H'\) has fewer vertices than \(H\), contradicting the minimality of \(H\), and proving the lemma.

This lemma quickly implies an upper bound on the number of vertices in a vertex-minimal, linear, \(r\)-uniform, \(d\)-regular hypergraph of girth at least \(g\).

**Theorem 5.** Let \(r, d\) and \(g\) be integers with \(d \geq 2\) and \(r, g \geq 3\). There is a linear, \(r\)-uniform, \(d\)-regular hypergraph with girth at least \(g\), and at most

\[
(r - 1) \left( 1 + d(r - 1) \frac{(d - 1)^g(r - 1)^g - 1}{(d - 1)(r - 1) - 1} \right) < 4((d - 1)(r - 1))^{g+1}
\]

vertices.

**Proof.** Let \(H\) be a linear, \(r\)-uniform, \(d\)-regular hypergraph with girth at least \(g\), with the smallest possible number of vertices. Let \(\{v_1, v_2, \ldots, v_k\}\) be a set of vertices of \(H\) whose pairwise distances are all greater than \(g\), with \(k\) maximal subject to this condition. By the previous lemma, we have \(k < r\). Any vertex of \(H\) must have distance at most \(g\) from one of the \(v_i\)'s. For each \(i\), the number of vertices of \(H\) of distance at most \(g\) from \(v_i\) is at most

\[
1 + d(r - 1) \sum_{i=0}^{g-1} (d - 1)(r - 1)^i = 1 + d(r - 1) \frac{(d - 1)^g(r - 1)^g - 1}{(d - 1)(r - 1) - 1},
\]

and therefore the number of vertices of \(H\) is at most

\[
k \left( 1 + d(r - 1) \frac{(d - 1)^g(r - 1)^g - 1}{(d - 1)(r - 1) - 1} \right) \leq (r - 1) \left( 1 + d(r - 1) \frac{(d - 1)^g(r - 1)^g - 1}{(d - 1)(r - 1) - 1} \right).
\]

Crudely, we have

\[
(r - 1) \left( 1 + d(r - 1) \frac{(d - 1)^g(r - 1)^g - 1}{(d - 1)(r - 1) - 1} \right) < 4((d - 1)(r - 1))^{g+1}
\]

for all integers \(r, d\) and \(g\) with \(d \geq 2\) and \(r, g \geq 3\), proving the theorem.

The following corollary is immediate.

**Corollary 6.** Let \(r, d\) and \(g\) be integers with \(d \geq 2\) and \(r, g \geq 3\). Let \(n\) be the smallest number of vertices in a linear, \(r\)-uniform, \(d\)-regular hypergraph with girth at least \(g\). Then

\[
g > \frac{\log n - \log 4}{\log(d - 1) + \log(r - 1)} - 1.
\]
Observe that the lower bound in Corollary 6 differs from the upper bound in Corollary 2 by a factor of (approximately) 2.

To our knowledge, for \( r \geq 3 \), there are no explicit constructions of \( r \)-uniform, \( d \)-regular hypergraphs which improve upon the bound in Corollary 6 for large \( n \). This may be contrasted with the situation for graphs, where the bipartite Ramanujan graphs of Lubotzky, Phillips and Sarnak \[19\] and Morgenstern \[22\] provide \( d \)-regular, \( n \)-vertex graphs of girth at least

\[
(1 - o(1)) \frac{4 \log n}{3 \log(d - 1)}
\]

for infinitely many \( n \), whenever \( d - 1 \) is a prime power:

**Theorem 7** (Lubotzky-Phillips-Sarnak, Margulis, Morgenstern). For any odd prime power \( p \), there exist infinitely many (bipartite) \((p + 1)\)-regular Ramanujan graphs \( X^{p,q} \). The graph \( X^{p,q} \) is a Cayley graph on the group \( \text{PGL}(2,q) \), so has order \( q(q^2 - 1) \). Its girth satisfies

\[
g(X^{p,q}) \geq \frac{4 \log q}{\log p} - \frac{\log 4}{\log p}
\]

We are able to improve upon the lower bound in Corollary 6 when \( r = 3 \) and \( d = 2 \), using the following explicit construction, based upon Ramanujan graphs. Let \( G \) be an \( n \)-vertex, 3-regular graph of girth \( g \). Take any drawing of \( G \) in the plane with straight-line edges, and for each edge \( e \in E(G) \), let \( m(e) \) be its midpoint. Let \( H \) be the 3-uniform hypergraph with

\[
V(H) = \{m(e) : e \in E(G)\},
\]

\[
E(H) = \{\{m(e_1), m(e_2), m(e_3)\} : e_1, e_2, e_3 \text{ are incident to a common vertex of } G\}.
\]

Then the hypergraph \( H \) is 2-regular, and also has girth \( g \). Taking \( G = X^{2,q} \) (the Ramanujan graph of Theorem 7) yields a 3-uniform, 2-regular hypergraph \( H \) with

\[
g(H) = g(X^{2,q}) \geq \frac{4 \log q}{\log 2} - 2 \geq \frac{4 \log n}{3 \log 2} - 2
\]

improving upon the bound in Corollary 6 by a factor of \((1 - o(1))^{1/2}\).

The following explicit construction, also based on Ramanujan graphs, provides \( r \)-uniform, \( d \)-regular hypergraphs of girth approximately \( 2/3 \) of the bound in Corollary 6, whenever \( d \) is a multiple of \( r \). (We thank an anonymous referee of an earlier version of this paper, for pointing out this construction.)

Suppose \( d = rs \) for some \( s \in \mathbb{N} \). Let \( G \) be a \( 2(r - 1)s \)-regular, \( n \) by \( n \) bipartite graph, with vertex-classes \( X \) and \( Y \), and girth \( g \). Then the edge-set
of \( G \) may be partitioned into \((r - 1)\)-edge stars in such a way that each vertex of \( G \) is in exactly \( rs \) of the stars. (Indeed, by Hall’s theorem, we may partition the edge-set of \( G \) into \( 2(r - 1)s \) perfect matchings. First, choose \( r - 1 \) of these matchings, and group the edges of these matchings into \( n \) \((r - 1)\)-edge stars with centres in \( X \). Now choose \( r - 1 \) of the remaining matchings, and group their edges into \( n \) \((r - 1)\)-edge stars with centres in \( Y \). Repeat this process \( s \) times to produce the desired partition of \( E(G) \) into stars.)

Let \( H \) be the \( r \)-uniform hypergraph whose vertex-set is \( X \cup Y \), and whose edge-set is the collection of vertex-sets of these stars; then \( H \) is \((rs)\)-regular, and has girth at least \( g/2 \).

If \( 2(r - 1)s - 1 \) is a prime power, the bipartite Ramanujan graph \( X_{p,q} \) (with \( p = 2(r - 1)s - 1 \)) can be used to supply the graph \( G \). This yields a linear, \( r \)-uniform, \((rs)\)-regular hypergraph with girth \( g(H) \) satisfying

\[
g(H) \geq \frac{1}{2} \left( \frac{4 \log q}{\log(2rs - 2s - 1)} - \frac{\log 4}{\log(2rs - 2s - 1)} \right) \\
\geq \frac{2}{3} \log n \left( \frac{4}{\log(2rs - 2s - 1)} - \frac{\log 2}{\log(2rs - 2s - 1)} \right) \\
= \frac{2}{3} \log \frac{n}{\log(2d - 2d/r - 1)}
\]

where \( d = rs \).

Unfortunately, this lower bound is asymptotically worse than that given by Corollary 3 for all values of \( r \) and \( d \).

### 3 Random ‘Cayley’ hypergraphs

We remark that our existence lemma, Lemma 3, can also be proved by considering a random \( r \)-uniform, \( d \)-regular hypergraph on \( n \) vertices, for \( n \) large. In [7], Cooper, Frieze, Molloy and Reed analyse these using a generalisation of Bollobás’ configuration model for \( d \)-regular graphs. It follows from Lemma 2 in [7] that if \( H \) is chosen uniformly at random from the set of all \( r \)-uniform, \( d \)-regular, \( n \)-vertex, linear hypergraphs, then

\[
\text{Prob}\{\text{girth}(H) \geq g\} = (1 + o(1)) \frac{\exp\left(- \sum_{i=1}^{g-1} \lambda_i \right)}{1 - \exp(- (\lambda_1 + \lambda_2))}
\]

where

\[
\lambda_i = \frac{(r - 1)^i (d - 1)^i}{2i} \quad (i \in \mathbb{N}),
\]

so this event occurs with positive probability for sufficiently large \( n \), giving an alternative proof of Lemma 3.

We now give a construction of random ‘Cayley’ hypergraphs on \( S_n \), which have girth \( \Omega(\sqrt{\log |S_n|}) \) — a much better bound than that given by Lemma 3 or by random regular hypergraphs, though still short of the optimal \( \Theta(\log |V(H)|) \).
The situation is analogous to the graph case, where random \(d\)-regular Cayley graphs on appropriate groups have much higher girth than random \(d\)-regular graphs of the same order (due to the dependency between cycles at different vertices of a Cayley graph).

**Theorem 8.** Let \(r\) and \(n\) be positive integers with \(r \geq 3\) and \(r \mid n\). Let \(X(n, r)\) be the set of permutations in \(S_n\) that consist of \(\frac{1}{n}d\) disjoint \(r\)-cycles. Choose \(d\) permutations \(\tau_1, \tau_2, \ldots, \tau_d\) uniformly at random (with replacement) from \(X(n, r)\), and let \(H\) be the random hypergraph with vertex-set \(S_n\) and edge-set

\[
\{\{\sigma, \sigma \tau_1^2, \ldots, \sigma \tau_i^{r-1}\} : \sigma \in S_n, \ i \in [d]\}.
\]

Then with high probability, \(H\) is a linear, \(r\)-uniform, \(d\)-regular hypergraph with girth at least

\[
c_0 \sqrt{\frac{n \log n}{r(r-1)(\log(d-1) + \log(r-1))}},
\]

for any absolute constant \(c_0\) such that \(0 < c_0 < 1/2\).

**Remark.** Here, ‘with high probability’ means ‘with probability tending to 1 as \(n \to \infty\).’

**Proof.** Note that the edges of the form

\[
\{\sigma, \sigma \tau_1^2, \ldots, \sigma \tau_i^{r-1}\} (\sigma \in S_n)
\]

are simply the left cosets of the cyclic group \(\{\text{Id}, \tau_1, \tau_2, \ldots, \tau_r^{r-1}\}\) in \(S_n\), so they form a partition of \(S_n\). We need two straightforward claims.

**Claim 1.** With high probability, the following condition holds.

\(\tau_1, \ldots, \tau_d\) satisfy \(\tau_i^k \neq \tau_j^l\) for all distinct \(i, j \in [d]\) and all \(k, l \in [r-1]\). (7)

**Proof of claim:** Suppose we have chosen \(\tau_i\), but not \(\tau_j\). Fix \(k, l \in [r-1]\); we shall bound the probability that \(\tau_j^l = \tau_i^k\). Since \(\tau_i\) is a product of \(n/r\) disjoint \(r\)-cycles, \(\tau_i^k\) is a product of \(n/s\) disjoint \(s\)-cycles, for some integer \(s \geq 2\) that is a divisor of \(r\). The set \(X(n, s)\) of permutations which consist of \(n/s\) disjoint \(s\)-cycles has cardinality

\[
\frac{n!}{(n/s)!s^{n/s}} \geq \frac{n!}{(n/2)!2^{n/2}}
\]

(provided \(n \geq 4\)). Notice that \(\tau_i^l\) is uniformly distributed over \(X(n, s)\), for some \(s\) that depends only on \(r\) and \(l\). Therefore,

\[
\text{Prob}\{\tau_i^k = \tau_j^l\} \leq \frac{(n/2)!2^{n/2}}{n!}.
\]

By the union bound,

\[
\text{Prob}\{\tau_i^k = \tau_j^l\ \text{for some} \ i \neq j \ \text{and some} \ k, l \in [r-1]\} \leq (r-1)^2 \binom{d}{2} \frac{(n/2)!2^{n/2}}{n!}
\]

\[
\to 0 \quad \text{as} \ n \to \infty,
\]

proving the claim.
Claim 2. If condition (7) holds, then for all \( i \neq j \) and all \( \sigma, \pi \in S_n \), the following two cosets

\[
\{ \sigma, \sigma \tau_i, \sigma \tau_i^2, \ldots, \sigma \tau_i^{r-1} \} \quad \text{and} \quad \{ \pi, \pi \tau_j, \pi \tau_j^2, \ldots, \pi \tau_j^{r-1} \}
\]

have at most one element in common.

Proof of claim: Suppose for a contradiction that there are two distinct vertices \( v_1, v_2 \) with

\[
v_1, v_2 \in \{ \sigma, \sigma \tau_i, \sigma \tau_i^2, \ldots, \sigma \tau_i^{r-1} \} \cap \{ \pi, \pi \tau_j, \pi \tau_j^2, \ldots, \pi \tau_j^{r-1} \}.
\]

Then \( v_1 = \sigma \tau_i^l = \pi \tau_j^m \) and \( v_2 = \sigma \tau_i^l = \pi \tau_j^m \), where \( l, m, l', m' \in \{0, 1, \ldots, r-1\} \) with \( l \neq m \) and \( l' \neq m' \). Therefore,

\[
v_1^{-1}v_2 = \tau_i^{m-l} = \tau_j^{m'-l'},
\]

contradicting condition (7).

Claim 2 implies that \( H \) is a linear hypergraph, provided condition (7) is satisfied. Moreover, \( H \) is \( d \)-regular: every \( \sigma \in S_n \) is contained in the edges (cosets)

\[
\{ \{ \sigma, \sigma \tau_i, \sigma \tau_i^2, \ldots, \sigma \tau_i^{r-1} \} : i \in [d] \},
\]

and these \( d \) edges are distinct provided condition (7) is satisfied.

Finally, we make the following.

Claim 3. With high probability, \( H \) has girth at least

\[
c_0 \sqrt{\frac{n \log n}{r(r-1)(\log(d-1) + \log(r-1))}},
\]

where \( c_0 \) is any absolute constant such that \( 0 < c_0 < 1/2 \).

Proof of claim: We may assume that condition (7) holds, so that \( H \) is a linear, \( d \)-regular hypergraph. Let \( C \) be a cycle in \( H \) of minimum length, and let \( (e_1, \ldots, e_l) \) be any cyclic ordering of its edges. Then we have \(|e_i \cap e_{i+1}| = 1\) for all \( i \in [l] \) (where we define \( e_{l+1} := e_1 \)), and by minimality, we have \( e_i \cap e_j = \emptyset \) whenever \(|i - j| > 1\). Let \( e_i \cap e_{i+1} = \{w_i\} \) for each \( i \in [l] \). Suppose that \( e_i \) is an edge of the form

\[
\{ \sigma, \sigma \tau_{j_i}, \sigma \tau_{j_i}^2, \ldots, \sigma \tau_{j_i}^{r-1} \}
\]

for each \( i \in [l] \). Since \( e_i \cap e_{i+1} \neq \emptyset \) for each \( i \in [l] \), we must have \( j_i \neq j_{i+1} \) for all \( i \in [l] \) (where we define \( j_{l+1} := j_1 \)). For each \( i \in [l] \), we have \( w_i, w_{i+1} \in e_{i+1}, \) so \( w_i^{-1}w_{i+1} = \tau_{j_i}^{m_i} \) for some \( m_i \in [r-1] \). Therefore,

\[
\text{Id} = (w_1^{-1}w_2)(w_2^{-1}w_3) \ldots (w_{l-1}^{-1}w_l)(w_l^{-1}w_1) = \tau_{j_2}^{m_1} \tau_{j_3}^{m_2} \ldots \tau_{j_l}^{m_{l-1}} \tau_{j_1}^{m_l}.
\]

Since \( j_i \neq j_{i+1} \) for all \( i \in [l] \), the word on the right-hand side of (8) is irreducible. We therefore have an irreducible word in the \( \tau_j \)'s, with length \( L := \sum_{j=1}^l m_i \leq \ldots\)
$(r-1)l$, which evaluates to the identity permutation. We must show that the probability of this tends to zero as $n \to \infty$, for an appropriate choice of $l$. We use an argument similar to that of [11], where it is proved that a random $d$-regular Cayley graph on $S_n$ has girth at least $\Omega(\sqrt{\log d - 1(n!)})$.

Let $W$ be an irreducible word in the $\tau_j$'s, with length $L$. We must bound the probability that $W$ fixes every element of $[n]$. Suppose

$$W = \tau_{j(1)} \tau_{j(2)} \cdots \tau_{j(L)}.$$

Let $x_0 \in [n]$, and define $x_i = \tau_{j(i)}(x_{i-1})$ for each $i \in [L]$, producing a sequence of values $x_0, x_1, x_2, \ldots, x_L \in [n]$; then $W(x_0) = x_L$. We shall bound the probability that $x_L = x_0$. Let us work our way along the sequence, exposing the $r$-cycles of the permutations $\tau_1, \ldots, \tau_d$ only as we need them, so that at stage $i$, the $r$-cycle of $\tau_{j(i)}$ containing the number $x_{i-1}$ is exposed (if it has not already been exposed). If $x_L = x_0$, then (as $j(L) \neq j(1)$), there has to be a first time the sequence returns to $x_0$ via a permutation $\tau \neq \tau_{j(1)}$. Hence, at some stage, we must have exposed an $r$-cycle of $\tau$ containing $x_0$. The probability that, at a stage $i$ where $j(i) \neq j(1)$, we expose an $r$-cycle of $\tau_{j(i)}$ containing $x_0$, is at most

$$\frac{r}{n-(i-2)r} \leq \frac{r}{n-(L-2)r},$$

since a total of at most $i-2$ $r$-cycles of $\tau$ have already been exposed, and the next $r$-cycle exposed is equally likely to be any $r$-element subset of the remaining $n-(i-2)r$ numbers. There are at most $L$ choices for the stage $i$, and therefore

$$\text{Prob}\{W(x_0) = x_0\} \leq L \frac{r}{n-(L-2)r}.$$ 

Suppose we have already verified that $W$ fixes $y_1, y_2, \ldots, y_{m-1}$, by exposing the necessary $r$-cycles. Then we have exposed at most $(m-1)L$ $r$-cycles. As long as $(m-1)Lr < n$, we can choose a number $y_m \in [n]$ such that none of the previously exposed $r$-cycles contains $y_m$. Repeating the above argument yields an upper bound of

$$\frac{Lr}{n-mLr}$$

on the probability that $W$ fixes $y_m$, even when conditioning on the $(m-1)L$ previously exposed $r$-cycles. Therefore,

$$\text{Prob}\{W = \text{Id}\} \leq \left(\frac{Lr}{n-mLr}\right)^m,$$

as long as $mLr < n$. Substituting $m = \left\lceil n/(2Lr) \right\rceil$ yields the bound

$$\text{Prob}\{W = \text{Id}\} \leq \left(\frac{2Lr}{n}\right)^{n/(2Lr)}.$$ 

The number of choices for the word on the right-hand side of (8) is at most $(d-1)^l(r-1)^l$. (By taking a cyclic shift if necessary, we may assume that
$j_2 \neq d$, so there are at most $d - 1$ choices for $j_2$, and at most $d - 1$ choices for all subsequent $j_i$; there are clearly at most $r - 1$ choices for each $m_i$.) Hence, the probability that there exists such a word which evaluates to the identity permutation is at most

$$(d - 1)^l(r - 1)^l \left( \frac{2r(r - 1)l}{n} \right)^{n/(2r(r - 1)l)}.$$ 

To bound the probability that $H$ has a cycle of length less than $g$, we need only sum the above expression over all $l < g$:

$$\text{Prob}\{\text{girth}(H) < g\} \leq \sum_{l=3}^{g-1} (d - 1)^l(r - 1)^l \left( \frac{2r(r - 1)l}{n} \right)^{n/(2r(r - 1)l)} < (d - 1)^g(r - 1)^g \left( \frac{2r(r - 1)g}{n} \right)^{n/(2r(r - 1)g)}.$$ 

In order for the right-hand side to tend to zero as $n \to \infty$, we must choose

$$g = c_0 \sqrt{\frac{n \log n}{r(r - 1)(\log(d - 1) + \log(r - 1))}}$$

for some constant $c_0 < 1/2$; we then have

$$\text{Prob}\{\text{girth}(H) < g\} \leq \exp \left( -\Omega \left( \frac{1}{r} \sqrt{\log(d - 1) + \log(r - 1)(n \log n)} \right) \right).$$

This completes the proof of Claim 3 and thus proves Theorem 8.

4 Conclusion and open problems

Our best (general) upper and lower bounds on the function $g_{r,d}(n)$ differ approximately by a factor of 2:

$$(1 + o(1)) \frac{\log n}{\log(d - 1) + \log(r - 1)} \leq g_{r,d}(n) \leq (2 + o(1)) \frac{\log n}{\log(r - 1) + \log(d - 1)}.$$ 

It would be of interest to narrow the gap, possibly by means of an explicit algebraic construction à la Ramanujan graphs.

In [11], Gamburd, Hoory, Shahshahani, Shalev and Virág conjecture that with high probability, the girth of a random $d$-regular Cayley graph on $S_n$ has girth at least $\Omega(\log |S_n|)$, as opposed to the $\Omega(\sqrt{\log |S_n|})$ which they prove. We believe that the random hypergraph of Theorem 8 also has girth $\Omega(\log |S_n|)$.

In this paper, we considered a very simple and purely combinatorial notion of girth in hypergraphs, but other notions appear in the literature, for example using the language of simplicial topology, such as in [20, 12]. A different
combinatorial definition was introduced by Erdős [9]. Define the \((-2)\)-girth of a 3-uniform hypergraph as the smallest integer \(g \geq 4\) such that there is a set of \(g\) vertices spanning at least \(g - 2\) edges. Erdős conjectured in [9] that there exist Steiner Triple Systems with arbitrarily high \((-2)\)-girth; this question remains wide open (see for example [2]), and seems very hard. In view of this, we raise the following.

Question 9. Is there a constant \(c > 0\) such that there exist \(n\)-vertex 3-uniform hypergraphs with \(cn^2\) edges and arbitrarily high \((-2)\)-girth?

Note that Erdős’ conjecture on Steiner Triple Systems posits a positive answer for every \(c < \frac{1}{6}\). This is clearly tight, since a hypergraph with at least \(n^2/6\) edges cannot be linear, and therefore has \((-2)\)-girth 4.

We turn briefly to some variants of Erdős’ definition. The celebrated \((6, 3)\)-theorem of Ruzsa and Szemerédi [23] states that if \(H\) is an \(n\)-vertex, 3-uniform hypergraph in which no 6 vertices span 3 or more edges, then \(H\) has \(o(n^2)\) edges. Therefore, if we define the \((-3)\)-girth of a 3-uniform hypergraph to be the smallest integer \(g \geq 6\) such that there exists a set of \(g\) vertices spanning at least \(g - 3\) edges, then an \(n\)-vertex, 3-uniform hypergraph with \((-3)\)-girth at least 7 has \(o(n^2)\) edges. Hence, the analogue of Question 9 for \((-3)\)-girth has a negative answer. On the other hand, if we define the \((-1)\)-girth of a 3-uniform hypergraph to be the smallest integer \(g\) such that there exists a set of \(g\) vertices spanning at least \(g - 1\) edges, it can be shown that the maximum number of edges in an \(n\)-vertex, 3-uniform hypergraph with \((-1)\)-girth at least \(g\), is \(n^2 + \Theta(1/g)\).

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\(^1\)The condition \(g \geq 6\) is necessary to avoid triviality: if we replaced it with \(g \geq 5\), then a 3-uniform hypergraph would have \((-3)\)-girth 5 unless it consisted of isolated edges.
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