Hilbert space representation of higher-dimensional minimum-length deformed uncertainty relation and some of its implications for ADD model

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We construct a Hilbert space representation of minimum-length deformed uncertainty relation in presence of extra dimensions. Following this construction, we study corrections to the gravitational potential (back reaction on gravity) with the use of correspondingly modified propagator in presence of two (spatial) extra dimensions. Interestingly enough, for $r \to 0$ the gravitational force approaches zero and the horizon for modified Schwarzschild-Tangherlini space-time disappears when the mass approaches quantum-gravity energy scale. This result points out to the existence of zero-temperature black hole remnants in ADD brane-world model.

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I. INTRODUCTION

String theory has inspired a number of ideas which, have had a significant impact on particle physics model building and quantum gravity phenomenology [1–3]. One of the interesting predictions of string theory is that several extra dimensions must exist. Using the idea of large extra dimensions, Arkani-Hamed, Dimopoulos, and Dvali (ADD) suggested a phenomenological brane-world scenario for unifying the elementary particle forces with gravity near the electroweak energy scale [4, 5]. String theory, suggesting an unified description of gravity and elementary particles, becomes usually relevant only at very short distances - of the order of Planck length $l_P \approx 10^{-33}$ cm [6], while ADD model opens the door for detecting some of the string theory ideas at distances comparable to the $\sim 10^{-17}$ cm [7].

The other idea inspired by the string theory that received appreciable attention for studying the quantum gravity effects is the minimum-length deformed position-momentum uncertainty relation [8, 10].

$$\delta X \delta P \geq \frac{\hbar}{2} + \beta G_N \delta P^2,$$

(1)

where $\beta$ is a numerical factor of order unity and $G_N$ is the Newtonian coupling (from the very outset we assume $\hbar = 1$). A profound feature of Eq. (1) is that it exhibits a lower bound on the position uncertainty of the order of $l_P = \sqrt{\hbar G_N}$. Apart from the string theory, the Eq. (1) has a strong motivation from the black hole physics [11, 12]. Let us notice that the concept of minimum length in quantum gravity [13–15] also hints at the modification of position-momentum uncertainty relation in such a way as to have the lower bound on position uncertainty.

In what follows we will define both ingredients the brane-world and the minimum-length deformed position-momentum uncertainty relation in a purely phenomenological framework.

In ADD model [4, 5] the extra dimensions are compactified on $n$-torus. Matter fields are localized on the 3 dimensional surface referred to as a (mem)brane and gravity is allowed to live in the whole space. The relation between the four and higher dimensional Newtonian couplings is

Volume of extra space $\times G_N = \mathcal{G}_N$.

Thus, the quantum gravity energy scale

$$E_P = \hbar^{(n+1)/(n+2)} G_N^{-1/(n+2)},$$

can be lowered at the expense of the volume of extra space. In such a scenario gravity becomes strong at the energy scale $E_P$. Roughly, the gravitational potential on the ADD brane behaves as

$$V(r) \approx \begin{cases} -G_N r^{-(1+n)} , & \text{if } r < l_{ex} , \\ -G_N r^{-1} , & \text{if } r \geq l_{ex} , \end{cases}$$

where $l_{ex}$ denotes the size of extra dimensions. In what follows we will restrict ourselves to this semi-qualitative picture. The results obtained below are immediately applicable to the ADD model whenever the characteristic length scale of the process under consideration is $\ll l_{ex}$.

The ultimate goal of the present paper is to estimate the back-reaction on gravity due to minimum-length deformed quantum mechanics in presence of extra spatial dimensions and see how the Schwarzschild-Tangherlini space-time gets modified.

II. DEFORMED QUANTUM MECHANICS IN $4+n$ DIMENSIONAL SPACE-TIME

On dimensional grounds one can consider various gravitational corrections to the Heisenberg uncertainty relation that results in the lower bound on position uncertainty ($c = 1$, that is, $[\hbar] = g$ cm, $[G_N] = \text{cm}^{n+1}/g$).
\[ \delta X \delta P \geq \frac{\hbar}{2} + \beta \hbar^{(n-1)/\alpha} G_N^{-1/\alpha(n+1)} \delta P^{(n+2)/(n+1)}, \]

where \( \beta \) is a numerical factor of order unity. In order to have a lower bound on position uncertainty, one should require

\[ \alpha \leq \frac{n+2}{n+1}. \]

On the other hand, it to be possible to switch off quantum mechanics \( \hbar \to 0 \), one has to require \( \alpha \geq 1 \). The choice \( \alpha = 1 \) is unique in that in this case the correction does not depend on \( \hbar \) and therefore survives even when \( \hbar \to 0 \). By taking this specific choice one arrives at Eq. (1) in absence of extra dimensions. So, the generalization of Eq. (1) to the higher dimensional space takes the form

\[ \delta X \delta P \geq \frac{1}{2} + \beta G_N^{\frac{n}{2}} \delta P^{\frac{n+2}{n+1}}, \]

(2)

(from now on we will adopt system of units \( \hbar = c = 1 \)).

The correction term in Eq. (2) makes sense even when \( \delta P \gtrsim G_N^{-1/2} \). In this case it is motivated by the fact that in high center of mass energy scattering, \( \sqrt{s} \gtrsim G_N^{-1/2} \), the production of black holes dominates all perturbative processes [16, 17], thus limiting the ability to probe short distances. (It is important to notice that at high energies, \( \sqrt{s} \gg G_N^{-1/2} \), the black hole production is increasingly a long-distance, semi-classical process). To make the point clearer, the refined measurement of particle’s position requires large energy transfer during a scattering process used for the measurement. But when the gravitational radius associated with this energy transfer \( \sim G_N \sqrt{s} \) becomes grater than the impact parameter, the black hole will form and what one can say about the particle’s position is that it was somewhere within the region \( \sim G_N \sqrt{s} \). The gravitational radius of the black hole formed in the scattering process grows with energy as \( r_g \sim G_N \sqrt{s} \) determining therefore high energy behavior of the position uncertainty.

Similar reasoning is valid in higher-dimensional case as well. The \( 4+n \) dimensional spherically symmetric gravitational field is described by the Schwarzschild-Tangherlini solution [18, 19]

\[ ds^2 = \left[ 1 - \left( \frac{r_g}{r} \right)^{n+1} \right] dt^2 - \left[ 1 - \left( \frac{r_g}{r} \right)^{n+1} \right]^{-1} dr^2 - r^2 dO_{n+2}^2, \]

(3)

where \( dO_{n+2}^2 \) is a line element of a \( 2+n \) dimensional unit sphere and the gravitational radius reads

\[ r_g(m) = (G_N m)^{\frac{1}{n+1}} \left[ \frac{16 \pi}{(n+2) \text{Vol}(S^{n+2})} \right]^{\frac{1}{n+1}}. \]

(4)

Thus we arrive at Eq. (2); somewhat similar argument was used in [20] for deriving this equation.

III. HILBERT SPACE REPRESENTATION OF UNCERTAINTY RELATION (2)

Here we closely follow the paper [21]. To find a concrete representation of \( \hat{X}, \hat{P} \) operators that follow from uncertainty relation (2); we define \( \delta P^{(n+2)/(n+1)} \) as

\[ \delta P^{\frac{n+2}{n+1}} = \left\langle P - \langle P \rangle \right\rangle^{\frac{n+2}{n+1}}, \]

which implies that

\[ \left\langle |P|^{\frac{n+2}{n+1}} \right\rangle \geq \delta P^{\frac{n+2}{n+1}}. \]

Thus, the deformed QM that follows from Eq. (2) takes the form

\[ \left[ \hat{X}, \hat{P} \right] = i \left( 1 + \beta G_N^{\frac{n}{2}} \hat{P}^{\frac{n+2}{n+1}} \right), \]

(5)

(numerical factors of order unity are absorbed in \( \beta \)).

A particular representation of \( \hat{X}, \hat{P} \) operators for a multidimensional generalization of Eq. (5)

\[ \left[ \hat{X}_i, \hat{X}_j \right] = 0, \quad \left[ \hat{P}_i, \hat{P}_j \right] = 0, \quad \left[ \hat{X}_i, \hat{P}_j \right] = i \left\{ \Xi \left( \hat{P}^2 \right) \delta_{ij} + \Theta \left( \hat{P}^2 \right) \hat{P}_i \hat{P}_j \right\}, \]

(6)

can be constructed in terms of the standard \( \hat{x}, \hat{p} \) operators as

\[ \hat{X}_i = \hat{x}_i = i \frac{\partial}{\partial p^i}, \quad \hat{P}_j = \hat{p}_j \left( \hat{p}^2 \right) = p_j \left( \hat{p}^2 \right). \]

(7)

The simplest ansatz would be to take

\[ \Theta = \frac{2 \beta G_N^{\frac{n}{2}}}{\hat{P}^{\frac{n+2}{n+1}}}, \]

thus from Eqs. (6, 7) we get

\[ \left( \frac{\partial}{\partial p^i} p_j \left( \hat{p}^2 \right) - p_j \left( \hat{p}^2 \right) \frac{\partial}{\partial p^i} \right) \psi(p) = \left( \frac{\sigma_{ij}}{\hat{p}^2} \right) \delta_{ij} + 2 \beta G_N^{\frac{n}{2}} \frac{p_j p_i}{\hat{p}^{\frac{n+2}{n+1}}} \psi(p), \]

that is,
\[
\frac{d\xi(p^2)}{dp^2} = \beta \frac{p^{n+2}}{2n+4} \xi_p^{n+1}/p^{n+1}, \quad \Rightarrow \\
\xi(p^2) = \left(1 - \frac{2\beta}{n+2} \frac{p^{n+2}}{2n+4}\right)^{(n+1)}.
\]

Simplifying our notations by replacing
\[
\frac{2\beta}{n+2} \times \frac{p^{n+2}}{2n+4} \rightarrow \beta,
\]
the above constructed representation takes the form
\[
\hat{X}_i = \hat{x}_i, \\
\hat{P}_j = \hat{p}_j \left(1 - \beta \hat{p}^{n+2}/(2n+4)\right)^{(n+1)},
\]
or in the eigen-representation of operator \(\hat{p}\)
\[
\hat{X}_j = i\frac{\partial}{\partial p_j}, \\
\hat{P}_j = p_j \left(1 - \beta p^{n+2}/(2n+4)\right)^{(n+1)},
\]
with the scalar product
\[
\langle \psi_1|\psi_2 \rangle = \int d^{3+n}p \: \psi_1^\dagger(p)\psi_2(p).
\]

In the case \(n = 0\) one recovers the result of [21].

Let us notice that the cutoff \(p^{(2+n)/(n+1)} < \beta^{-1}\) arises merely from the fact that when \(p\) runs over the region \(p < \beta^{-(n+1)/(n+2)}\), \(P\) covers the whole region from \(0\) to \(\infty\); see Eqs. (9) [12]. It might be instructive to look at this cutoff from the standpoint of Fourier transform. The standard uncertainty relation can be understood on the basis of Fourier transform since the spatial and momentum wave functions are related through it. The Fourier transform has the property that the more tightly localized the spatial wave function is, the less tightly localized the momentum function must be; and vice versa. Consequently, for in the minimum-length deformed quantum theory the momentum wave function can not spread beyond the region \(\sim \beta^{-(n+1)/(n+2)}\), the spatial wave function can not be localized beneath the region \(\sim \beta^{(n+1)/(n+2)}\).

### IV. IMPRINTS ON FIELD THEORY

For simplicity let us consider a neutral scalar field. The modified field theory takes the form
\[
\mathcal{A}[\Phi] = -\int d^{3+n}x \: \frac{1}{2} \left[\phi \partial^2 \Phi + \hat{\Phi}^2 \phi + m^2 \Phi^2\right],
\]
that results in the equation of motion
\[
\left(\partial^2 + \hat{\Phi}^2 + m^2\right)\Phi = 0.
\]
Substituting a plane wave solution \(\propto \exp(\im\mathbf{p}\cdot \mathbf{x} - \varepsilon\mathbf{p}t)\) in Eq. (12), one finds the dispersion relation
\[
\varepsilon^2 = \mathbf{p}^2 + m^2 = \frac{p^2}{(1 - \beta p^{n+2}/(2n+4))^{2(n+1)}} + m^2 = \\
\sum_{n=0}^{\infty} (1+n)\beta^n p^{2(n+1)} + m^2.
\]
In order to express \(p\) in terms of \(\mathbf{P}\) one has to solve the equation
\[
p^2 = \frac{p^2}{(1 - \beta p^{n+2}/(2n+4))^{2(n+1)}},
\]
for \(p\) and substitute it in
\[
\mathbf{p} = \mathbf{P} \left[1 - \beta p^{n+2}/(2n+4)\right]^{n+1}.
\]
The cut-off \(p < \beta^{-(1+n)/(2+n)}\) readily indicates that the wave-length
\[
\lambda = \frac{2\pi}{p(P)},
\]
is bounded from below \(\lambda \geq 2\pi\beta^{(1+n)/(2+n)}\) no matter what the value of \(P\) is.

The field operator takes the form
\[
\Phi(t, \mathbf{x}) = \int \frac{d^{3+n}p}{(2\pi)^{3+n}2\varepsilon_p} \left[e^{i(\mathbf{p}\cdot \mathbf{x} - \varepsilon_p t)}a(\mathbf{p}) + e^{-i(\mathbf{p}\cdot \mathbf{x} - \varepsilon_p t)}a^\dagger(\mathbf{p})\right],
\]
\begin{align*}
\Phi(t, \mathbf{x}) &= \int \frac{d^{3+n}p}{(2\pi)^{3+n}2\varepsilon_p} \\
&\times \left[e^{i(\mathbf{p}\cdot \mathbf{x} - \varepsilon_p t)}a(\mathbf{p}) + e^{-i(\mathbf{p}\cdot \mathbf{x} - \varepsilon_p t)}a^\dagger(\mathbf{p})\right].
\end{align*}
where
\[
\epsilon_p = \left( \frac{p^2}{1 - \beta p^{2+\omega}} \right)^{2(n+1)} + m^2.
\]

From Eqs. (11, 14) one gets
\[
\mathcal{H} = \frac{1}{2} \int d^{3+n}x \left[ H^2 + \Phi \bar{P} \Phi + m^2 \Phi^2 \right] = \int \left. d^{3+n}p \frac{\epsilon_p}{2} \left[ a^+ (p) a (p) + a (p) a^+ (p) \right] \right|_{p^{2+n}/(1+n) \lessgtr \beta^{-1}}.
\]

The field quantization condition
\[
[H (x), \Phi (y)] = -i \int \frac{d^{3+n}p}{p^{2+n}/(1+n) \lessgtr \beta^{-1}} e^{i p (x-y)} \left( \frac{1}{2 k^{(n+1)}} - \beta k \right)^{2(n+1)} e^{ikr} = \frac{\text{Vol} (S^{n+2})}{(2 \pi)^{3+n}} \int \left. d^{3+n}k \right|_{k^{2+n}/(n+1) \lessgtr \beta^{-1}} \frac{1}{k^2} \left[ 2 (n+1) \beta \right. + (n+1)(2n+1) \beta^2 k^{2/(n+1)} + \ldots - 2 (n+1) \beta^{2n+1} k^{(2n^2+3n)/(n+1)} + \beta^{2(n+1)} k^{2(n+1)} \left. \right] e^{ikr}, \tag{15}
\]
results in the modified Schwarzschild-Tangherlini spacetime
\[
ds^2 = \left[ 1 - r_g^{n+1} V (r) \right] dt^2 - \left[ 1 - r_g^{n+1} V (r) \right]^{-1} dr^2 - r^2 d\Omega_{n+2}^2, \tag{16}
\]
where \( r_g \) is given by Eq. (14).

The corrected potential (calculated by the modified propagator with respect to Eq. (13))
\[
V (r) = \frac{\text{Vol} (S^{n+2})}{(2 \pi)^{3+n}} \int \left. d^{3+n}k \right|_{k^{2+n}/(n+1) \lessgtr \beta^{-1}} \frac{1}{k^2} \left[ 2 (n+1) \beta \right. + (n+1)(2n+1) \beta^2 k^{2/(n+1)} + \ldots - 2 (n+1) \beta^{2n+1} k^{(2n^2+3n)/(n+1)} + \beta^{2(n+1)} k^{2(n+1)} \left. \right] e^{ikr}, \tag{15}
\]

\[
\left( \tilde{k} = k^{3/4}, \tilde{r} = \beta^{-3/4} \right) \text{ with the asymptotic behavior (see Eq. (24) in the Appendix)}
\]
\[
V \left( r \ll \beta^{3/4} \right) = \frac{2}{3 \pi \beta^{3/4}} \left[ 0.00295112 + 0.0000393787 \frac{r^2}{\beta^{3/2}} + 3.0709 \times 10^{-7} \frac{r^4}{\beta^2} + 1.65633 \times 10^{-9} \frac{r^6}{\beta^3} + \ldots \right]. \tag{18}
\]

The equation for the gravitational radius now reads
\[
\frac{\beta^{9/4}}{r_g^3} = \tilde{V} (r), \tag{19}
\]
where $\tilde{V}(r) = \beta^{3/4} V(r)$. Observing that $\tilde{V}(r)$ is a monotonically decreasing function, see Fig.1, with its maximum $\tilde{V}(0) = 0.00196741/\pi$, one infers that the solution of Eq.(19) approaches zero when the black hole mass approaches $m^{\text{remnant}} = \frac{\pi^2 \beta^{9/4}}{0.00295112 \times G_N} \rightarrow \frac{\pi^2 \beta^{9/4} G_N^{-1/4}}{2^{9/4} \times 0.00295112} \propto \beta^{3/4}$ (20)

where we have used Eqs.(17, 8). That is, in the last expression of Eq.(20) $\beta$ is a dimensionless parameter of order unity. Thus, when black hole evaporates to the mass, $m^{\text{remnant}}$, the horizon disappears. For masses smaller than $m^{\text{remnant}}$, the Eq.(19) does not have any solution.

Recalling that Hawking temperature is proportional to the surface gravity, one infers that black hole emission temperature approaches zero when black hole evaporates down to the $m^{\text{remnant}}$ (we use Eq.(13))

$$T_H(m^{\text{remnant}}) \propto \frac{dV(r)}{dr} \bigg|_{r=0} = 0 \quad (21)$$

VI. DISCUSSION

We have constructed the Hilbert space representation for a higher-dimensional minimum-length deformed position-momentum uncertainty relation, which was originally suggested in [20]. In absence of extra dimensions one recovers well known example studied in [21].

The minimum-length deformed quantum mechanics implies corrections for the both sides of the Einstein equation

$$R_{\mu\nu} - \frac{g_{\mu\nu} R}{2} = 8\pi G_N T_{\mu\nu} \quad (22)$$

The energy-momentum tensor of the matter fields is now understood to be modified with respect to the minimum-length deformed field theory (modifications arise both at first- and second-quantization levels, for details see [29, 31]). It is hard to imagine what might be the analogue of the minimum-length deformed field theory for the gravitational field, to figure out the corresponding corrections to the right-hand side of Eq.(22). Nevertheless, one might proceed with a semi-classical description; the disturbance (graviton field) can be separated off from the background space-time for which the minimum-length deformed field formalism can be applied immediately. Let us notice that this kind of approach is most straightforward for formulating a quantum theory of gravity [31] (in this paper, the expansion of the gravitational field around the flat space-time was used that allows one to view gravity as QFT for self-interacting spin-2 field). It is also worth noticing that despite the non-renormalizability, the method of expansion of the gravitational field around the fixed background enables to make meaningful predictions for graviton radiative corrections in the framework of an effective field theory approach [32, 33].

Two sources of the corrections to the Newtonian potential can be identified. The first concerns the existence of maximally localized states in the framework of minimum-length deformed quantum mechanics that naturally replaces the delta-function distribution for a point-like particle and alters the Poisson equation (see [38]). Correspondingly, one can take some sort of smeared-out delta function instead of delta-function distribution, which describes the point-like source and finds its gravitational field to get some idea about the minimum-length modified black holes [39, 41].

Another source altering the Newtonian potential is the modified dispersion relation that enters the propagator and thus provides corrections to the right-hand side of Eq.(22) (in the linearized theory). Following this way, as in the four-dimensional case [38], the modified gravity shows up the following interesting features, gravitational force vanishes when $r \rightarrow 0$ (see Eq.(15)); the horizon disappears when the black hole evaporates down to the Planck mass (see Eqs.(19, 20)) and the corresponding Hawking temperature vanishes (see Eq.(21)).

The question of black hole remnants in the framework of generalized uncertainty relation was first addressed in a heuristic way in [42]. It was observed that the temperature of black hole radiation obtained heuristically from the generalized uncertainty relation becomes imaginary when the black hole mass drops below the Planck scale (that was interpreted as the reason for ceasing the emission). Following this paper, the extra-dimensional (and braneworld) corrections to the black hole radiation due to Eq.(41) have been studied in many papers; see some of them: [43, 52].

Finally let us note that the above constructed representation for minimum-length deformed quantum mechanics in presence of extra dimensions can be used for estimating corrections to various quantities and processes in
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To estimate the integral

$$\int d^{3+n}k \left[ \frac{1}{k^2} - \frac{2(n+1)\beta}{k^{n/(n+1)}} + (n+1)(2n+1)\beta^2 k^{2/(n+1)} + \ldots + \beta^{2(n+1)} k^{2(n+1)} \right] e^{ikr}, \quad (23)$$

let us choose the axis $x_{3+n}$ along $k$ and introduce spherical coordinates in the momentum space

$$k_1 = k \sin \varphi \prod_{j=1}^{n+1} \sin \theta_j, \quad k_2 = k \cos \varphi \prod_{j=1}^{n+1} \sin \theta_j, \quad k_{i+2} = k \cos \theta_i \prod_{j=1}^{n+1} \sin \theta_j, \quad k_{3+n} = k \cos \theta_{n+1},$$

where $i = 1, \ldots, n$; $k \geq 0$, $0 \leq \varphi < 2\pi$, $0 \leq \theta_j \leq \pi$. Thus, we get $kr = kx_{3+n} \cos \theta_{n+1}$, $d^{3+n}k = k^{2+n} dk d\varphi \prod_{j=1}^{n+1} \sin^j \theta_j d\theta_j$, and the integral (23) reduces to

$$\text{Vol} \left( S^{n+1} \right) \int_0^{\pi} d\theta_{n+1} \sin^{n+1} \theta_{n+1} e^{ikx_{3+n} \cos \theta_{n+1}}.$$
\[ V(r) = \frac{\text{Vol}(S^4) \text{Vol}(S^3)}{(2\pi)^5} \left[ \frac{\sin(kr)}{(kr)^3} - \frac{\cos(kr)}{(kr)^2} \right] \]

where the dimensionless quantities \( \tilde{k}, \tilde{r} \) are defined as \( k = \tilde{k} \beta^{-3/4}, r = \tilde{r} \beta^{3/4} \). The behavior of this potential for distances \( r \ll \beta^{3/4} \) can be obtained by using the asymptotic expression

\[
\frac{\sin(x)}{x^3} - \frac{\cos(x)}{x^2} = \frac{1}{3} - \frac{x^2}{5 \cdot 3!} + \frac{x^4}{7 \cdot 5!} - \frac{x^6}{9 \cdot 7!} + \cdots,
\]

that gives

\[
V \left( r \ll \beta^{3/4} \right) = \frac{4 \text{Vol}(S^4) \text{Vol}(S^3)}{(2\pi)^5 \beta^{9/4}} \left[ 0.00295112 + 0.000393787 \frac{r^2}{\beta^{3/2}} + 3.0709 \times 10^{-7} \frac{r^4}{\beta^3} + 1.65633 \times 10^{-9} \frac{r^6}{\beta^9/2} + \cdots \right]. \quad (24)
\]
