VACUUM POLARISATION AND THE BLACK HOLE SINGULARITY

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In order to investigate the effects of vacuum polarisation on mass inflation singularities, we study a simple toy model of a charged black hole with cross flowing radial null dust which is homogeneous in the black hole interior. In the region $r^2 \ll e^2$ we find an approximate analytic solution to the classical field equations. The renormalized stress-energy tensor is evaluated on this background and we find the vacuum polarisation backreaction corrections to the mass function $m(r)$. Asymptotic analysis of the semiclassical mass function shows that the mass inflation singularity is much stronger in the presence of vacuum polarisation than in the classical case.

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1. Introduction

The causal structure of charged or rotating black holes in General Relativity (GR) suggests the possibility of travelling through such a hole to another universe. However a closer look soon reveals a difficulty which was first pointed out by Penrose [1]; the inner (Cauchy) horizon is unstable due to the infinite blueshift of infalling radiation. In a generic collapse such radiation will always be present, due to the tail of backscattered radiation from the body which produced the black hole. The divergence of linear perturbations on this horizon has been verified by several authors [2], and it was recently shown that this blueshifted radiation combined with outflow from the star produces a scalar curvature singularity. Since quantitative results exist only in the spherical case (i.e. for charged black holes), attention is restricted to these spacetimes in what follows.

Hiscock [4] has modelled the backscattered radiation as a null dust and shown that this results in an observer dependent singularity at the inner horizon. This is a “whimper” singularity, in the classification scheme of Ellis and King [3], and none of the curvature invariants diverge. However, whimper singularities are generally believed to be unstable to perturbations [5]. One such perturbation to Hiscock’s model, which preserves the spherical symmetry, is the addition of a radial outflowing stream of null dust (which provides a crude way to model radiation from the collapsing star). Poisson and Israel [6] have shown that for such a crossflow, a phenomenon which has come to be known as “mass inflation” occurs. The mass function of the spherical black hole becomes infinite on the inner horizon causing a scalar curvature singularity to occur.

More recently, Ori [7] has introduced a model in which the outflow is compressed into a thin null shell, for which he finds an exact mass inflation solution to the Einstein-Maxwell equations. His analysis of the resulting mass inflation singularity leads him to conclude that, although it is stronger than the “whimper” singularity of the purely infalling null radiation, it is still rather weak in the sense that a physical body approaching it will experience only a finite deformation due to gravitational tidal forces. This has lead to intense debate about
the possible continuation of spacetime beyond this null singularity [8].

It is now generally believed, however, that the appearance of singularities in classical GR signal regimes in which a quantum theory of gravity is needed for a correct physical description. Although there exists no satisfactory quantum theory of gravity at present, a semiclassical approximation may be useful for investigating the behavior of spacetimes in regions of high curvature. Our purpose in this paper is to investigate the mass inflation singularity using the machinery of semiclassical gravity.

For simplicity, we will use as our classical mass inflation background a toy model developed by Page and independently by Ori [9]. This model (henceforth known as the homogeneous mass inflation or HMI model) consists of a spherically symmetric charged black hole interior with crossflowing null radiation which is homogeneous in the region between the inner and outer horizons. The HMI model has the advantage that the unknown metric functions depend only on the radial coordinate $r$. Furthermore, it is important to note that Page has shown that in this model the mass inflation singularity is not null but rather spacelike, and it is tempting to speculate that this will also be true of part of the singularity in more realistic models.

In section II we will present a classical solution for the HMI model in the region $r^2 \ll e^2$ where $e$ is the charge of the black hole. This particular solution did not explicitly appear in the work of Page and Ori [1], although it is implicit therein. In Section III, we will find the renormalised stress-energy tensor for a conformally coupled massless scalar field on the classical HMI background, and we argue that the dominant terms should be insensitive to the quantum state we use. The backreaction equations are then solved approximately, to find vacuum polarisation corrections to the classical background. In particular it is shown that the correction to the mass function diverges more strongly than in the classical case.

II. THE HMI BACKGROUND

The spherically symmetric line element may always be written in the form
\[ ds^2 = \left( \frac{r^2}{e^2} \right) [d\sigma^2 + e^2 d\Omega^2], \quad d\sigma^2 = \gamma_{ab} dx^a dx^b, \] (2.1)

where \( d\Omega^2 \) is the line element on a unit sphere, and \( e \) is a constant with the dimensions of length. Latin indices \( a, b, \ldots \) range over \((0,1)\). Thus the problem reduces to finding the two dimensional metric \( \gamma_{ab} \), and the scalar function \( r(x^a) \) which is defined so that the area of a two sphere is \( 4\pi r^2 \). We introduce the scalar invariant

\[ \gamma^{ab} r_a r_b = -e^{-2} B(r) \equiv -e^{-2} (2m(x^a) r - r^2 - e^2), \] (2.2)

where \( m(x^a) \) is the mass of the black hole interior. It is convenient to use \( r(x^a) \) as a coordinate so that \( d\sigma^2 = -e^{-2}dr^2/B(r) + g(r)dt^2 \).

The Einstein field equations can now be written in the covariant form

\[ 2e^2 r \gamma^{ab} r_{ab} = 4m(r)r - 4e^2 + 8\pi r^2 e^2 \gamma^{ab} T_{ab}, \] (2.3)

\[ m_a = 4\pi e^2 \left[ \gamma^{cb} T_{ac} - \delta^b_a \gamma^{ed} T_{ed} \right] r_{\dot{b}}, \] (2.4)

where the semicolon (;) denotes the covariant derivative associated with \( d\sigma^2 \), and \( T_{ab} \) is the \( 2 \times 2 \) submatrix of the stress-energy tensor.

The stress-energy tensor for a homogeneous crossflow of null radiation takes the form

\[ -T^t_t = T^r_r = \rho(r), \quad T_{tr} = X(r). \] (2.5)

Covariant conservation of the stress tensor requires that \( \rho(r) = -C^2/(g(r)r^4) \) and \( X(r) = D(r^4 B(r)g(r))^{-1/2} \). Without loss of generality, we will take the constants of integration to be \( C^2 = -|e|D/2 = e^2/4\pi \), since any arbitrariness can in fact be absorbed into the coordinate \( t \).

We are now in a position to solve the classical equations for which \( T^a_a = 0 \) and \( T^t_t = e^2/(4\pi g(r)r^4) \). We can reduce the system (2.3) and (2.4) to a single integral equation for the mass function

\[ \frac{dm(r)}{dr} = \frac{r^2}{B(r)} \int_0^r \left( \frac{d m(r)}{dr} \right)^2 dr. \] (2.6)
Since we are interested in the behavior near the mass inflation singularity, which Page has shown to be at $r = 0$, we will consider only the region $r^2 \ll e^2$, so that $B(r) \approx 2m(r) r - e^2$.

With this simplifying assumption it is easily verified that

$$m(r) = \frac{e^2}{r}, \quad g(r) = 1,$$

is a solution to the field equations. Thus we have the HMI background metric

$$ds^2 = \frac{r^2}{e^2}[-dr^2 + dt^2 + e^2d\Omega^2],$$

from which it is clear that the singularity is spacelike. This simple form will allow us to analyse the vacuum polarisation in some detail. Note that this is not the general solution to the equations, about which Page and Ori were able to obtain qualitative information. However the main feature, that the mass function inflates to infinity like $1/r$, is captured in our solution.

### III. VACUUM POLARISATION

Next, we wish to consider semiclassical corrections to the metric functions, due to the presence of a conformally coupled massless scalar field $\phi$ obeying

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi - \frac{1}{6} R \phi = 0.$$  \hspace{1cm} (3.1)

Later we will argue that our conclusions should be insensitive to the quantum state in which we evaluate the stress energy tensor. For now we ignore this problem and present a derivation for states possessing the natural symmetries of the manifold; $R^2 \times S^2$. Our method is to calculate the $\zeta$-function and effective potential for $d\tilde{s}^2 = -dr^2 + dt^2 + e^2d\Omega^2$, and then to use Page’s generalisation of a result due to Brown and Cassidy to obtain the stress energy tensor in the physical spacetime.

Some detail is included below for completeness. The $\zeta$-function is obtained from the heat kernel for this space (which is homogeneous) and is defined via the Mellin transform.
\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}K(0,0;t)dt. \] (3.2)

Here

\[ K(0,0;t) = \left( (4\pi e^2 t)^{-1} \sum_{l=0}^\infty (2l+1) \exp[-t(12(l+1/2)^2 + 1)/12e^2] \right) \] (3.3)

is the heat kernel for \( ds^2 \) at \( x = x' = 0 \). After analytically continuing this \( \zeta \)-function to a meromorphic function in the complex \( s \)-plane we find

\[ \zeta(s) = \frac{2e^{2s-4}}{(4\pi)^2} \sum_{k=0}^\infty \frac{\Gamma(s+k-1)(-1)^k}{\Gamma(s)k!12^k} \zeta_R(2s+2k-3, 1/2) \] (3.4)

where \( \zeta_R(z,q) \) is the Riemann-Hurwitz zeta function \([10]\). The effective potential is formally given by \( V = (-1/2)[\zeta'(0) + \log(\mu^2)\zeta(0)] \) \([11]\) and is now easily evaluated using (3.4). It is

\[ V = -\left( \frac{A}{16\pi e^4} + \frac{T}{2} \ln(e^2\mu^2) \right), \] (3.5)
\[ A = \sum_{k=3}^\infty \frac{(-1)^k}{k(k-1)12^k} \zeta_R(2k-3, 1/2) - 2\zeta'_R(-3, 1/2) \]
\[ -\frac{1}{6}\zeta'_R(-1, 1/2) + \frac{7}{960} - \frac{1}{288}\psi(1/2), \] (3.6)

where \( \mu \) is an arbitrary mass scale, \( T = (1440\pi^2 e^4)^{-1} \) is the trace anomaly, \( \zeta'_R(z,q) \equiv \partial \zeta_R(z,q)/\partial z \), and \( \psi(x) = \Gamma'(x)/\Gamma(x) \). By performing the usual variation of the effective action we obtain the stress tensor in this static space

\[ \langle T^\alpha_\beta \rangle_{ren} = \frac{2e^{\gamma\mu}}{\sqrt{g}} \frac{\delta}{\delta g^{\alpha\beta}} \frac{\delta}{\delta \gamma} V = \text{diag}[-V, -V, V + \frac{T}{2}, V + \frac{T}{2}]. \] (3.7)

In fact, one may evaluate the above stress-energy tensor for massless conformally coupled fields of other spins. This only changes the numerical values of the \( T \) and \( A \) above. For spin-1/2 (two-component) theory the trace anomaly is \( T = (480\pi^2 e^4)^{-1} \) and

\[ A = 4\zeta_R(-3) + \frac{1}{60}, \] (3.8)

where \( \zeta_R(z) \) is the Riemann zeta function \([10]\). For the Maxwell field we find \( T = (120\pi^2 e^4)^{-1} \) and
\[ A = \sum_{k=3}^{\infty} \frac{1}{k(k-1)2^{2k-1}} \zeta_R(2k-3,3/2) - 4\zeta_R'(-3,3/2) \]
\[ + \zeta_R'(-1,3/2) + \frac{127}{480} - \frac{1}{16}\psi(3/2). \tag{3.9} \]

We now invoke a result due to Page [12]. He finds that the renormalised stress-energy tensor for a conformally coupled massless scalar field transforms conformally as

\[ T_{\mu\nu} = \Omega^{-4}T_{\mu\nu} - 8\alpha\Omega^{-4}[\nabla_{\alpha}\nabla^{\beta}(C^{\alpha\mu}_{\beta\nu} \ln \Omega) + \frac{1}{2}R_{\alpha}^{\beta}C^{\alpha\mu}_{\beta\nu} \ln \Omega] \]
\[ + \beta[(R_{\alpha}^{\beta}C^{\alpha\mu}_{\beta\nu} - 2H_{\mu\nu}) - \Omega^{-4}(R_{\alpha}^{\beta}C^{\alpha\mu}_{\beta\nu} - 2H_{\mu\nu})] \]
\[ - \frac{1}{6}\gamma[I_{\mu\nu} - \Omega^{-4}T_{\mu\nu}], \tag{3.10} \]

where

\[ H_{\mu\nu} \equiv -R^\alpha_{\mu}R_{\alpha\nu} + \frac{2}{3}RR_{\mu\nu} + \left(\frac{1}{2}R^\alpha_{\beta}R^\beta_{\alpha} - \frac{1}{4}R^2\right)g_{\mu\nu}, \tag{3.11} \]
\[ I_{\mu\nu} \equiv 2R_{\mu\nu} - 2RR_{\mu\nu} + \left(\frac{1}{2}R^2 - 2R_{(\alpha)}g_{\mu\nu}, \tag{3.12} \right. \]

and, in the case of a scalar field, \( \alpha = 12/(2^945\pi^2) \), \( \beta = -4/(2^945\pi^2) \), \( \gamma = 8/(2^945\pi^2) \). The barred tensors are evaluated in \( d\vec{s}^2 = \Omega^{-2}ds^2 \), and in our case \( \Omega^2 = r^2/e^2 \). It is now a straightforward matter to obtain \( \langle T^{\alpha\beta}_{\mu\nu} \rangle_{\text{ren}} \) for the metric (2.8). We are interested, however, only in the effect of \( \langle T^{\alpha\beta}_{\mu\nu} \rangle_{\text{ren}} \) near the mass inflation singularity. Thus, we present here only the dominant contributions in the asymptotic limit \( r \to 0 \). The relevant contributions are

\[ \langle T^t_t \rangle_{\text{ren}} \sim -10\beta e^4r^{-8}, \tag{3.13} \]
\[ \langle T^r_r \rangle_{\text{ren}} \sim 6\beta e^4r^{-8}. \tag{3.14} \]

It is worth noting that for the spin 1/2 and spin 1 fields the dominant contributions are identical apart from a positive multiplicative constant.

Before examining the backreaction, some comments are in order about the quantum state in which this stress energy tensor is evaluated. The boundary conditions defining the state in which our system was prepared should appear in the derivation of \( \langle T^{\alpha\beta}_{\mu\nu} \rangle \). However, it is not an easy task to impose these in this case, since we only have an asymptotic classical
solution for \( r^2 \ll e^2 \). Instead we suggest that the asymptotic behavior will be insensitive to the initial conditions. In particular, since the singularity is spacelike, one may ignore the quantum influx due to Hawking radiation as it will most likely be swamped by the classical radiation. Other non-local effects are expected to contribute to the full stress-energy tensor as \( f \sqrt{-g(\text{curvature})^2} dr \) [13]; thus these effects will behave at most like \( r^{-3} \). For these reasons we feel that the dominant behavior is captured with our present model.

IV. BACKREACTION

To solve for the backreaction of the vacuum polarisation due to (3.13) and (3.14) on the HMI background, we consider the following expansions:

\[
\begin{align*}
m(r) &= m^{(0)}(r) + \hbar m^{(1)}(r) + O(\hbar^2), \\
g(r) &= g^{(0)}(r) + \hbar g^{(1)}(r) + O(\hbar^2), \\
T_{ab} &= T_{ab}^{(0)} + \hbar \langle T_{ab}\rangle_{\text{ren}} + O(\hbar^2)
\end{align*}
\]

(4.1) \quad (4.2) \quad (4.3)

where \( m^{(0)}(r) \), \( g^{(0)}(r) \), and \( T_{ab}^{(0)} \) are the classical metric functions and stress-energy tensor given by (2.7) and (2.5) respectively.

Substituting (4.1) - (4.3) into the field equations (2.3) and (2.4) and keeping only terms of less than \( O(\hbar^2) \) we obtain the backreaction equations for the terms of \( O(\hbar) \)

\[
\begin{align*}
e^2 \frac{dg^{(1)}}{dr} + 2r \frac{dm^{(1)}}{dr} + 6m^{(1)} &= -8\pi r^3 \langle T_{a}^{a}\rangle_{\text{ren}}, \\
-e^2 g^{(1)} + r^2 \frac{dm^{(1)}}{dr} &= -4\pi r^4 \langle T_{t}^{t}\rangle_{\text{ren}}.
\end{align*}
\]

(4.4) \quad (4.5)

Eliminating \( g^{(1)} \) from these equations and inserting only the dominant terms (3.13) and (3.14) for the \( \langle T_{a}^{a}\rangle_{\text{ren}} \) contributions we obtain

\[
\begin{align*}
r^2 \frac{d^2 m^{(1)}}{dr^2} + 4r \frac{dm^{(1)}}{dr} + 6m^{(1)} &\simeq -\frac{128\pi \beta e^4}{r^5},
\end{align*}
\]

(4.6)

which is valid as \( r \to 0 \). A particular solution to this equation is

\[
m^{(1)} \simeq -\frac{8\pi \beta e^4}{r^5}.
\]

(4.7)
Recalling that $\beta$ is negative, we see that $m^{(1)}$ diverges even more strongly than the classical mass function ($m^{(0)} \sim r^{-1}$).

V. CONCLUSION

It is widely believed that singularities in GR indicate regimes where a quantum treatment is necessary for a correct physical description. We have attempted a semiclassical analysis of the region near the mass inflation singularity for the HMI model. This is only a toy model which is useful as a tool to begin our investigations. The main feature is the spacelike singularity which is present, in contrast to the other mass inflation models which have a null singularity. In our opinion, it is entirely possible that the singularity inside a generic black hole is spacelike, however, this present model is too simple to be considered as realistic enough to describe such a situation. Our findings indicate that in the region of the mass inflation singularity, vacuum polarisation effects lead to a worsening of the singularity, rather than having a regulating effect as might be hoped. If this remains true in more realistic analyses, we will have to wait for a theory of quantum gravity to be able to resolve the singularity problem.

Along similar lines, Balbinot and Poisson [14] have estimated the quantum stress-energy tensor from the conservation law and the trace anomaly, on the Ori background [8]. However, the sign of the coefficient of the dominant term is ambiguous and does not allow a firm conclusion. Their results suggest that quantum effects may either act to curb or intensify mass inflation.

Without doubt the most serious drawback in our work is the imposition (or not) of boundary conditions. It may, in fact, be possible to obtain some of the nonlocal terms by examining the solution for $r^2 \gg \varepsilon^2$ and matching the solutions near $r = |\varepsilon|$. Nevertheless, it is difficult to see how this will alter our conclusions and for this reason such an effort does not seem to us worthwhile in this case. A more complete analysis in the original mass inflation scenario [9] is under way, and some of the deficiencies of this model are dealt with
in more detail there [15].

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