Abstract

Starting from a microscopic approach, we develop a covariant formalism to describe a set of interacting gases. For that purpose, we model the collision term entering the Boltzmann equation for a class of interactions and then integrate this equation to obtain an effective macroscopic description. This formalism will be useful to study the cosmic microwave background non-perturbatively in inhomogeneous cosmologies. It should also be useful for the study of the dynamics of the early universe and can be applied, if one considers fluids of galaxies, to the study of structure formation.

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1 Introduction

The formation and evolution of cosmological perturbations, leading for instance to the cosmic microwave background fluctuations, are usually studied in the framework of cosmological perturbations about a Friedmann-Lemaître spacetime [1]. In such a framework, the matter is described by a set of decoupled species including cold dark matter, neutrinos and a baryon-photon fluid (since one must consider the coupling between these two species via Compton scattering). In the latter case, the interaction between the two coupled components (i.e. photons and baryons) is studied via the linearised Boltzmann equation. A first generalisation that comes to mind is to develop a framework to model this interaction in a covariant way, i.e. without specifying the spacetime geometry. Such an approach will be useful for a study of the cosmic microwave background in inhomogeneous cosmologies [2, 3]. Earlier in the history of the universe all these fluids were coupled through scattering and/or annihilation process [4]. This leads to the second generalisation which is to model this coupling in a very general way for a wide class of interactions. Such a formalism will then be adapted to study the dynamics during the early universe, independently of the spacetime geometry. Moreover, it can be applied to structure formation, since it can describe fluids of galaxies.

The Einstein equations and the Bianchi identities lead to the fact that the total energy-momentum tensor of matter is conserved

$$\nabla_\mu T^{\mu\nu} = 0.$$  (1)

When the matter content of the universe is a single perfect fluid, this equation splits into two equations, the Euler equation and the matter conservation equation. The latter can be reset as

$$\nabla_\mu n^\mu = 0,$$  (2)
meaning that the number flux vector, $n^\mu$, is conserved.

If one now considers a system composed of an arbitrary number of fluids, this property applies to the “global fluid” and to each fluid only if they are interacting by gravitational interaction (i.e. if they are decoupled). If we want to study the evolution of one particular fluid of a system of interacting fluids, we have to take into account a force and the preceding law can be rewritten as

$$\nabla_\mu T^\mu_\nu = F^\nu_\nu.$$  \hfill (3)

Furthermore, if the collisions are not elastic, then the total number of each species will not be conserved and we will have to take into account a source term

$$\nabla_\mu n^\mu_i = \varepsilon_i.$$  \hfill (4)

Such a splitting of the global energy-momentum tensor ($T^{\mu\nu} = \sum T^{\mu\nu}_i$) and of the number flux vector ($n^\mu = \sum n^\mu_i$) is not straightforward. In fact, for a general multi-constituent fluid the lagrangian of the system, $\mathcal{L}$ say, will not split in the sum of individual lagrangians describing the dynamics of each fluid. Since the energy-momentum tensor is related to this lagrangian via

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L} \sqrt{-g}}{\delta g^{\mu\nu}},$$

one cannot define the energy-momentum tensor of a single component of the fluid system. However, as long as we are dealing with gases, such a splitting will be possible. Indeed, each fluid is then conveniently described by its distribution function, $f_i$ say, from which one can define an energy-momentum tensor (see &2 and &3 for the complete definitions) as

$$T^\mu_\nu_i = \int p^\mu p^\nu f_i(x, p) \pi_+(p).$$

This standard kinetic theory approach will be valid if the particles interact weakly, i.e. for systems which are not too dense. As explained in the section, such a description is adapted to cosmology.

A way to model the interacting forces, $F^\mu_\nu$, and the source terms, $\varepsilon_i$, is to relate them to the collision term entering the equation of evolution of the distribution function. We will then start from the kinetic theory, try to give a general form of the collision term for a class of interactions, compute the forces and the source terms, and end up with a macroscopic description of the dynamics of interacting fluids. It will be non perturbative, in the sense that we will not linearise the metric, and will not assume anything on the form of the fluid (e.g. perfect fluid).

Both the kinetic theory and the theory of fluid dynamics have been studied in the context of general relativity. On the one hand, a general covariant formulation of kinetic theory in general relativity was first developed by Tauber and Weinberg and independently by Chernikov (see also Marle and Israel and Stewart). These authors were mostly concerned with relativistic generalisations of classical gas theory (proof of the H-theorem, equilibrium configurations etc.). Many authors have also studied the coupled Einstein-Boltzmann equation and solved it in some cases. It is not our goal to solve such a general problem here, since we do not want to assume anything concerning the symmetries and/or the geometry of the spacetime.

On the other hand, relativistic fluid dynamics has been studied in detail by many authors (see e.g. Carter), who usually do not deal with a system of interacting fluids even if they present a general formalism for a multi-constituent fluid. In fact, as will be explained later, the formalism developed by Carter for multi-constituent perfect fluids does not overlap with the description which we shall give here.
In this article, we stand in between the kinetic and the fluid descriptions. Note that even if such attempts have already been made, they relied on a different approach to the problem. For instance, Lindquist [14] studied the diffusion of photons under the transport approximation (i.e. he studied the evolution of particles flowing through an emitting and absorbing medium described in macroscopic terms). Most of the studies of the transfer equation are also based on moment methods [8], on the “grad’s method of moment” [15] or on a spherical harmonic analysis [16]. We will not use such expansion methods here.

We will first use the microscopic approach (§2) to relate the interacting force and the source term to the collision term of the Boltzmann equation (§3). In §4, this result is used to compute the force on a system of conducting fluids. We then turn to the computation of the force for elastic collisions in §5, where we discuss the general form of the force and then compute it for cases of cosmological interest (e.g. photon-fermion (using results on the photon-electron collision [17]) and fermion-fermion scattering). §6 is devoted to the computation of the general form of the source term and of the force for inelastic collisions, namely fusion and fission, and we use it in the case of photon (bremstrahlung) of fermion-antifermion annihilation and of recombination (which is very important for a study of decoupling in cosmology).

This general formalism is then used in to establish the equation of evolution of a Compton scattering coupled photon-matter system in Cosmology to linear order (§7). All the theory of multifluid linear cosmological perturbations [18] can be recovered from our formalism which is non perturbative.

2 Definition - notations - microscopic quantities

The goal of this section is to introduce the distribution function and its equation of evolution. For that purpose we need to introduce the space on which such a function is defined. We finish by the description of the 3+1 splitting with respect to an arbitrary vector field.

2.1 Distribution function

Let us first consider a single test particle with mass \( m \) which moves in a gravitational field. Its motion is determined by the geodesic equation

\[
p^\mu = \frac{dx^\mu}{d\lambda}, \quad Dp^\mu = \frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}p^\nu p^\rho = 0,
\]

where \( \lambda \) is an affine parameter defined by the requirement that \( p^\mu \) be the 4-momentum. Hereafter, \( \nabla_\mu \) denotes the covariant derivative associated to the metric \( g_{\mu\nu} \), whose Christoffel symbols are \( \Gamma^\mu_{\nu\rho} \). Let us note that if there are non gravitational forces (e.g. electromagnetic forces) then we have to modify this equation (see & [4]).

The rest mass of the particle is defined as

\[
m^2 = -p^\mu p_\mu.
\]

Thus, according to (3), the state of the particle is determined by the couple \((x^\mu, p^\mu)\) and the phase space is then the tangent bundle over the spacetime manifold, i.e.

\[
\mathcal{T} = \{(x, p), x \in \mathcal{M}, p \in T_x \}, \quad (7)
\]

where \( \mathcal{M} \) is the space-time and \( T_x \) is the tangent space to \( \mathcal{M} \) at \( x \). From now on, we will use the bold style to denote quadri-vectors when components notations are not needed (hence \( \mathbf{p} \equiv p^\mu \) and \( \mathbf{p}^2 \equiv p^\mu p_\mu \) and greek indices run from 0 to 3).
The volume element on $T_x$ supported by the displacements $dp_1, dp_2, dp_3, dp_4$ (with components $dp_1^\alpha$ etc.) is
\[ \pi(p) = \epsilon_{\alpha\beta\gamma\delta} dp_1^\alpha dp_2^\beta dp_3^\gamma dp_4^\delta, \] (8)
where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric tensor such that $\epsilon_{0123} = \sqrt{-g}$.

We also define $\pi^+(p)$, the volume element corresponding to the subspace of $T_x$ such that $p^\mu$ is non-spacelike and future directed,
\[ \pi^+(p) = H(-p_\mu u^\mu)H(-p^2)\pi(p), \] (9)
where $H$ is the heavyside function (i.e. $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x > 0$) and $u^\mu$ an arbitrary timelike vector field.

$T_x$ is sliced in hypersurfaces, $P_m$, of constant $m$ called the mass-shell, and defined by
\[ P_m(x) = \{ p \in T_x, p_\mu p^\mu = -m^2, p_\mu u^\mu > 0 \}. \] (10)

The volume element (8) on $T$ can then be decomposed on a volume element, $m\pi_m$, on $P_m$ by
\[ \pi^+(p) = m\pi_m(p)dm. \] (11)

The factor $m$ allows one to include particles of zero rest mass (see Ehlers [6]). This defines the induced volume element $m\pi_m(p)$ on $P_m$.

If we introduce an arbitrary future directed unit timelike vector $u^\mu$ (i.e. satifying $u_\mu u^\mu = -1$), the 3-volume supported by the three displacements $dx_1, dx_2, dx_3$ (with components $dx_1^\alpha$ etc.) in the hypersurface perpendicular to $u^\mu$ is
\[ dV(u) = \epsilon_{\alpha\beta\gamma} u^\lambda dx_1^\alpha dx_2^\beta dx_3^\gamma. \] (12)

We now consider a single fluid composed of particles which mass are a priori different. The distribution function, $f(x,p)$ will be defined as the mean number of particles (on a statistical set) in a volume $dV$ around $x$ and $\pi(p)$ around $p$ measured by an observer with 4-velocity $u^\mu$,
\[ dN(x,p) = f(x,p)(-p_\mu u_\mu)dV(u)\pi(p). \] (13)

The assumptions involved in its existence have been discussed in details by Ehlers [3]. Synge [20] has demonstrated that $(-p_\mu u_\mu)dV(u)$ is independent of $u^\mu$, which implies that the distribution function is a scalar. Moreover, $f(x,p) \geq 0$ for all $x^\mu$ and all allowed $p^\mu$.

For a gas, $dN$ is the number of particles in a volume $dV\pi(p)$ thus the smoothness of $f$ depends on the existence of a sufficient number of particles.

**2.2 Equation of evolution**

The equations of motion (5) define on $T$ an operator called the Liouville operator (see e.g. [22]) which reads
\[ \mathcal{L} = p^\mu \frac{\partial}{\partial x^\mu} + \frac{dp^\mu}{d\lambda} \frac{\partial}{\partial p^\mu} = \frac{d}{d\lambda}, \] (14)
which characterises the rate of change of $f$ along the particle worldlines. Using (3), this operator can be rewritten as
\[ \mathcal{L}[f] = p^\mu \partial_\mu f - \Gamma^\mu_{\nu\rho} p^\nu p^\rho \frac{\partial}{\partial p^\mu} f. \] (15)

The fact that the mass $m$ of the particle as defined in (6) is a scalar function which is constant on each phase orbit leads to
\[ \mathcal{L}[m^2] = 0. \] (16)
The Boltzmann equation states that this rate of change is equal to the rate of change due to collisions, i.e.

$$\mathcal{L}[f] = C[f].$$

(17)

$C[f]$ is the collision term and encodes the information about the interactions between the particles of the fluid.

If we know consider a system of $N$ fluids (labelled by $i, j...$), each of which is described by its distribution function $f_i(x, p)$, the Boltzmann equation for a given fluid $i$ becomes

$$\mathcal{L}[f_i] = \sum_j C_j[f_i, f_j] \equiv C_i[f_i],$$

(18)

$C_j[f_i, f_j]$ is the collision term describing the interaction between the fluid $i$ and the fluid $j$. For elastic collisions, it must satisfy the symmetry

$$C_j[f_i, f_j] = C_i[f_i, f_j],$$

(19)

which means that in a collision between $i$ and $j$ the two distribution functions undergo the same change. Following Israel and Stewart [10], we will require that $C_j[f_i, f_j]$ is a local function of the “$f_i$” (i.e independent of their derivatives).

### 2.3 3+1 splitting

We perform a 3+1 splitting with respect to an arbitrary timelike unit vector field $u^\mu$ (we discuss which vectors to use according to the problem at hand in §3.2). The projection tensor into the “rest-space” of an observer moving with this 4-velocity is defined by

$$\perp_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu.$$  

(20)

Any vector $p^\mu$ can be decomposed with respect to $u^\mu$ as

$$p^\mu = \lambda e^\mu + Eu^\mu,$$

(21)

where the energy ($E$), the norm of the particles’ 3-momentum ($\lambda$) and the direction with respect to $u^\mu$ are

$$E \equiv (-p^\mu u_\mu), \quad \lambda^2 \equiv \perp_{\mu\nu} p^\mu p^\nu = E^2 - m^2 \quad \text{and} \quad e_\mu \equiv \frac{\perp_{\mu\nu} p^\nu}{\sqrt{\perp_{\mu\nu} p^\mu p^\nu}},$$

(22)

so that $e^\mu$ and $u^\mu$ satisfy

$$u^\mu e_\mu = 0, \quad e_\mu e^\mu = 1.$$  

(23)

We have dropped the $u$-dependence since there is no ambiguity. However we will restore it if necessary (e.g we will write $\perp_{\mu\nu}^u$). Here, we have split the 4-momentum $p^\mu$ but any other vector can be split according to the same procedure (e.g. in equation 38). Note that for a zero-mass particle these relations lead to $\lambda = E$.

From these definitions and equations (8) and (11), we can reexpress the volume elements on $T_x$ and on the shell-mass as (see §3, §10 for details)

$$\pi_+(p) = m d\lambda dE d\Omega \iff \pi_m(p) = \lambda dE d\Omega,$$

(24)

where $d\Omega$ is the solid angle spanned by three independent $e^\mu$. The passage from (8) to (24) can be seen as the passage from a cartesian coordinate system to a spherical coordinate system (i.e.
we replaced the integrations on $dp^\alpha$ by an integration on the norm and the angles.)

We also introduce the symmetric traceless tensor

$$\Delta_{\mu \nu} = \epsilon_\mu \epsilon_\nu - \frac{1}{3} \perp_{\mu \nu}$$

which verifies

$$\Delta_{\mu \nu} \perp_{\mu \nu} = 0, \quad \Delta_{\mu \nu} g^{\mu \nu} = 0, \quad \Delta_{\mu \nu} u^\mu = 0 \quad \text{and} \quad \Delta_{\mu \nu} e^\mu = \frac{2}{3} e_\nu.$$  \hspace{1cm} (26)

To finish, we give the following useful integrals (see e.g. \textsuperscript{16}),

$$\int e^{\mu_1} ... e^{\mu_n} \frac{d\Omega}{4\pi} = \begin{cases} 0 \\ \frac{1}{n+1} \left( \perp (\mu_1 \mu_2 ... \perp \mu_{n-1} \mu_n) \right) \end{cases} \begin{cases} n = 2p+1 \\ n = 2p, \end{cases} \hspace{1cm} (27)$$

where $e^\mu$ satisfies equation (23). In particular we have

$$\int \frac{d\Omega}{4\pi} = 1, \hspace{1cm} (28)$$

$$\int e^\nu e^\mu \frac{d\Omega}{4\pi} = \frac{1}{3} \perp_{\mu \nu}, \hspace{1cm} (29)$$

$$\int \Delta^\alpha_{\mu} \Delta^\beta_{\nu} \frac{d\Omega}{4\pi} = \frac{1}{45} \left( -2 \perp \alpha \perp \gamma \delta + 3 \perp \alpha \gamma \perp \beta \delta + 3 \perp \alpha \delta \perp \alpha \gamma \right). \hspace{1cm} (30)$$

We also choose the following conventions of symmetrisation and anti-symmetrisation

$$A_{(x_1,..,x_n)} = \begin{cases} \frac{1}{n!} \sum_{\sigma \in \text{perm}(1..n)} A_{\sigma (1) ... \sigma (n)} \end{cases} \quad \text{and} \quad A_{[x_1,..,x_n]} = \begin{cases} \frac{1}{n!} \sum_{\sigma \in \text{perm}(1..n)} \epsilon (\sigma) A_{\sigma (1) ... \sigma (n)} \end{cases},$$

where $\epsilon (\sigma)$ is the signature of the permutation.

### 3 Macroscopic quantities

In the previous paragraph, we have given a microscopic description of a set of interacting gases. The goal of this section is to define a set of macroscopic quantities from the distribution function and the collision term and then find the relations between these quantities.

#### 3.1 Definition

At any point $x$, one can introduce, following Ellis et al. \textsuperscript{16}, given a distribution function $f_i$, a set of macroscopic quantities associated with each fluid $i$ by

$$X_{i,a}^{\mu_1 ... \mu_n} (x) = \int_{T_x} (-p_\mu p^\mu)^{\alpha / 2} p^{\mu_1} ... p^{\mu_n} f_i (x, p) \pi_+ (p) = \int_{P_m} m^{\alpha} p^{\mu_1} ... p^{\mu_n} f_i (x, p) m \pi_+ (p) dm,$$

and

$$Y_{i,a}^{\mu_1 ... \mu_n} (x) = \int_{T_x} (-p_\mu p^\mu)^{\alpha / 2} p^{\mu_1} ... p^{\mu_n} C [f_i] (x, p) \pi_+ (p) = \int_{P_m} m^{\alpha} p^{\mu_1} ... p^{\mu_n} C [f_i] (x, p) m \pi_+ (p) dm,$$

where $m$ is the mass of the particles (defined in (6)) and $a$ an integer. The particles of a given fluid can have different rest mass (this is the case e.g. when one is dealing with a fluid of stars
The quantities \( \rho, P, q \) are timelike eigenvectors (see e.g. \cite{8}). Whatever the timelike unit vector field \( \mathbf{n} \), we can split the energy-momentum tensor under the general form

\[
\partial \mu X^{\mu_1 \ldots \mu_n \nu} (x) = -X^{\mu_1 \ldots \mu_n - 2\mu} (x).
\]

The quantities \( \rho, n, P \) are related by

\[
\rho = \rho n u^\mu + P - \mu \nu + 2q(\mu u^\mu) + \pi_{\mu \nu},
\]

and

\[
q_{\mu \nu} = \pi_{\mu \nu} = \mu \nu u^\mu = 0.
\]

By using the definition of \( T_{\mu \nu} \) from the distribution function (equation (31)) and performing the splitting (32), a simple identification with equation (39) easily shows that

\[
\rho (x) = \int_0^\infty dm \int_m^\infty dE \int_{\Omega} f(x, p) E^2 \lambda d\Omega,
\]

\[
P (x) = \frac{1}{3} \int_0^\infty dm \int_m^\infty dE \int_{\Omega} f(x, p) \lambda^3 d\Omega,
\]

\[
q_{\mu} (x) = \int_0^\infty dm \int_m^\infty dE \int_{\Omega} f(x, p) E \lambda^2 e_{\mu} d\Omega,
\]

\[
\pi_{\mu \nu} (x) = \int_0^\infty dm \int_m^\infty dE \int_{\Omega} f(x, p) \lambda^3 \Delta_{\mu \nu} d\Omega.
\]
All these quantities depend intrinsically on the choice of the vector field $u^\mu$ via the splitting defined in equation (21). Thus, $\rho$, $P$, $q^\mu$ and $\pi_{\mu\nu}$ will respectively be the energy density, the pressure, the energy flux and the anisotropic stress measured by an observer comoving with $u^\mu$.

### 3.2 Macroscopic fluid dynamics

Using the equation (34) as well as the definitions (31-32) and (35), we can relate the force $F^\mu_i$ and the source term $\epsilon_i$ to the collision term defined in (3-4) by

$$F^\mu_i(x) \equiv Y^\mu_{i,0} = \int_{T_x} p^\mu C_i[f_i](x,p) \pi_+(p),$$

and

$$\epsilon_i(x) \equiv Y_{i,0} = \int_{T_x} C_i[f_i](x,p) \pi_+(p).$$

The Bianchi identities state that the total energy-momentum tensor is conserved, which implies that we must have the usual action-reaction law, i.e. that

$$\sum_i F^\mu_i = 0,$$

as long as there is no long range external force (such a force can only be of electromagnetic origin [5]; this will be studied in §4).

Most of the time we will have to pick up a special frame to compute the collision term. Some choices are possible even if there are not compulsory.

- When a massless particle is interacting with a massive particle we will choose the rest frame of the massive particle and thus

$$\bar{u}^\mu = \frac{p^\mu}{m}.$$  

- When one has an elastic collision of two massive particles we can use the center of mass rest frame defined by

$$P^\mu = p^\mu_1 + p^\mu_2 \quad \text{and} \quad U^\mu = \frac{P^\mu}{\sqrt{-P^\mu P_\mu}}.$$  

Besides these two velocities, we have seen that there exist some preferred timelike vector fields associated with the motion of the matter (e.g. $v^\mu$, $V^\mu$).

One could add to this dynamical description a thermodynamical description. We have to emphasize here that we must give an equation of state to close the system. This comes from the fact that when developed into moments, at a given order the Boltzmann equation involves multipoles of higher orders [15]. To close the system one has either to truncate the system at a given order (with all the arbitrariness it implies) or give an (or more if needed) equation of state for the fluid. Even if we do not specify it, we assume that such an equation can be given for concrete applications (see §7).

The goal of the following section is to compute explicitly the quantities (46-47) in terms of the macroscopic variables defined in section 3.1.
4 Conducting fluid

The easiest case where one can compute the force acting on a fluid is the case of a conducting fluid in an electromagnetic field. This has been computed for a single fluid by many authors (see [13] for a review concerning electrodynamics in continuum media). We will just make the link between the microscopic and macroscopic approaches and show how useful the latter can be. Let us start from the usual approach with a single fluid.

Since the total energy-momentum tensor, $T_{em}^{\mu\nu}$, is conserved, we have

$$\nabla_\nu T_{em}^{\mu\nu} = -\nabla_\nu T_{em}^{\mu\nu}, \quad (51)$$

the electromagnetic field energy-momentum tensor being defined by

$$T_{em}^{\mu\nu} = F^{\mu\lambda} F^{\nu}_\lambda - \frac{1}{4} g^{\mu\nu} F^2, \quad (52)$$

where $F^{\mu\nu}$ is the electromagnetic tensor. Using Maxwell’s equations,

$$\nabla_\nu F^{\mu\nu} = j^\mu, \quad (53)$$

$j^\mu$ being the current density, the electromagnetic force on the fluid is

$$F_B^{\mu} \equiv -\nabla_\nu T_{em}^{\mu\nu} = F^{\mu}_\alpha j^\alpha. \quad (54)$$

However in the case of a multi-fluid system, the only force that can be computed with this method is the global electromagnetic force on the system of fluids and not the force on each fluid.

If we turn to the microscopic approach, then we have to take into account the fact that, because of the Lorentz force, the particles do not follow a geodesic between two collisions. Their equation of motion is then given by

$$p_i^\mu \nabla_\mu p_i^\nu = e_i F^{\mu\nu} p_\nu \iff \frac{dp_i^\mu}{d\tau} + \Gamma_{\nu\rho}^\mu p_i^\rho p_i^\nu = e_i F^{\mu\nu} p_\nu, \quad (55)$$

where $e_i$ is the charge of the particles.

If we cast this relation in the Boltzmann equation as we did in the section 2.1, it can be written

$$\mathcal{L}[f] = C_B[f] \equiv -e_i p_\mu F^{\mu\nu} \frac{\partial f}{\partial p_\nu}. \quad (56)$$

Hence the electromagnetic force acts as a collision term. In the previous equation $\mathcal{L}$ stands for the Liouville operator with no electromagnetic field, as defined in section 2.1.

We can now compute the force coming from this collision term,

$$F_{B \rightarrow i}^\mu = - e_i \int p^\alpha p_\mu F^{\alpha\beta}_\alpha \frac{\partial f}{\partial p_\beta} \pi_+(p),$$

$$= e_i F^{\mu}_\alpha n_i^\alpha = F^{\mu}_\alpha j^\alpha, \quad (57)$$

where we have performed an integration by part. Thus, the force acting on a single fluid can be expressed in terms of macroscopic quantities, namely the electromagnetic tensor and the current density.

If we now compute the total electromagnetic force on all the fluids we get

$$F_B^\mu = \sum_i F_{B \rightarrow i}^\mu = F^{\mu}_\alpha j^\alpha = F^{\mu}_\alpha j^\alpha,$$

which is the result we obtain with the macroscopic approach.
5 Elastic collisions

In this paragraph, we will compute the force $F^\mu$ for elastic collisions. We begin by a derivation for binary collisions of classical (in the sense that they are non quantum) particles, trying to stay as general as possible. We then turn to the case of the Compton scattering. In the Thomson limit, even if electrons and protons are not classical particles, the computation of the force turns out to be an application of the general case. However, in this case, we can give the form of the corrections coming from the quantum statistics.

We finish by a discussion on the range of validity of these computations and determine the domain of applicability (range of temperature) in which they can be used and then discuss the pertinence of this formalism for cosmology.

5.1 general case

We will give the most general form for the collision term for the process

$$A(p) + B(q) \rightarrow A(p') + B(q').$$

Before computing the force, we will study its general symmetries, which starts by the study of the kinematics of such a collision. During the computation, we will have to break this symmetry by working in the rest frame of one of the particles, and restore it at the end.

We first work in the rest frame of the center of mass and thus use the vector field $U^\mu$ defined in (50) to perform the splitting. We then have from equation (22)

$$E_A = \frac{m_A^2 - p^\mu q_\mu}{P} \quad \text{and} \quad E_B = \frac{m_B^2 - p^\mu q_\mu}{P} \quad \text{with} \quad P = (m_A^2 + m_B^2 - 2p^\mu q_\mu)^{1/2},$$

$$\lambda^2 \equiv \lambda_A^2 = \lambda_B^2 = \frac{(p^\mu q_\mu)^2 - m_A^2 m_B^2}{P^2}, \quad \text{and} \quad (p^\mu q_\mu) = E_A E_B - \lambda_A \lambda_B.$$

$E_A$ and $E_B$ are the energies of the two particles $A$ and $B$ in the center of mass rest frame, $\lambda$ is the amplitude of the particles’ 3-momentum in this frame.

The vectors $e_A^\mu$ and $e_B^\mu$, the direction of the ingoing particles, are given by

$$e_A^\mu = -e_B^\mu = \frac{1}{\lambda P^2} \left\{ (m_B^2 - p^\nu q_\nu)p^\mu - (m_A^2 - p^\nu q_\nu)q^\mu \right\}$$

$$= \frac{1}{\lambda P} \left\{ E_B p^\mu - E_A q^\mu \right\}. \quad (60)$$

The general form for the collision term for binary collisions of uncharged classical particle is

$$C[f_A](p) = \int (-p^\nu q_\nu) [f_A(p') f_B(q') - f_A(p) f_B(q)] W(p, q, p', q') \pi_+(q) \pi_+(q') \pi_+(p'), \quad (61)$$

where $W(p, q, p', q')$ is the probability of a collision $(p, q) \rightarrow (p', q')$. General expressions for processes including electromagnetic effects and quantum statistical effects have been proposed [6, 19]. We will take such effects into account in the next section (see equation (95)). Here, we want to be as general as possible and try not to describe the interaction in detail but give the general form of the force. The microscopic reversibility of the collision imposes the symmetry

$$W(p, q, p', q') = W(p', q', p, q). \quad (62)$$
Before we estimate \( W(p, q, p', q') \), let us study the symmetry of the force

\[
F_{B \rightarrow A}^\mu = \int (p' q_\mu) \left\{ f_A(p') f_B(q') - f_A(p) f_B(q) \right\} p'^\mu W(p, q, p', q') \pi_+(q') \pi_+(p') \pi_+(p).
\]

(63)

If we sum the two forces (i.e. \( F_{B \rightarrow A}^\mu \) and \( F_{A \rightarrow B}^\mu \)) and set \( W(p, q, p', q') = \delta^{(4)}(P - P') R(p, q, p', q') \), where \( R \) and \( W \) have the same symmetry, in order to make the conservation of energy-momentum explicit, then since \( P^2 = P'^2 \) implies that \(( -pq ) = ( -p' q' )\), the integrand of \( F_{B \rightarrow A}^\mu + F_{A \rightarrow B}^\mu \) will be antisymmetric in the transformation \(( p, q ) \rightarrow ( p', q' )\) and thus

\[
F_{B \rightarrow A}^\mu + F_{A \rightarrow B}^\mu = 0.
\]

(64)

This was expected by construction from (68) but had to be checked on the general form.

To compute the force, we must relate the \( W(p, q, p', q') \) probability to the differential cross section \( d\sigma_{pq \rightarrow p'q'} \) defined as

\[
W(p, q, p', q') \pi_+(q') \pi_+(p') = (-p^\mu q_\mu) d\sigma_{pq \rightarrow p'q'}.
\]

(65)

The cross section can be decomposed as the product of a matrix element \( M^{kp \rightarrow k'p'} \) and of a two body phase space element \( (D^{kp \rightarrow k'p'})^2 \) as

\[
d\sigma_{pq \rightarrow p'q'} = \frac{1}{4} \frac{1}{(-p^\mu q_\mu)} |M^{kp \rightarrow k'p'}|^2 (D^{kp \rightarrow k'p'})^2,
\]

(66)

with

\[
(D^{kp \rightarrow k'p'})^2 = (2\pi)^4 \delta^{(4)}(p + q - p' - q') \pi_+(q') \pi_+(p').
\]

(67)

If we consider massive particles with respective rest mass \( m_A \) and \( m_B \), one can convince oneself that the matrix element has to be of the form

\[
|M^{kp \rightarrow k'p'}|^2 = \sigma \Upsilon(m_A^2, m_B^2, p^\mu q_\mu, e^\mu, e'^\mu),
\]

(68)

where \( \sigma \) is the scalar cross section and the function \( \Upsilon \) can be decomposed as

\[
\Upsilon = \alpha + \beta e^\mu e'^\mu + \gamma \Delta^{\mu\nu} \Delta'^{\mu\nu} + ..., \quad (69)
\]

where the coefficients \( \alpha, \beta, \gamma \) depend on \( m_A^2, m_B^2 \) and \( p_\mu q^\mu \) and where \( e^\mu \) and \( e'^\mu \) are the directions of the ingoing and outgoing particles. This form comes from a multipole expansion of the matrix element in which the coefficients depend only on the scalar invariants of the collision (an example of such a function \( \Upsilon \) is given in equation (69)). As will be shown below, these are the only relevant terms to compute the force since higher multipoles will not contribute.

We will now assume that the particles are either non-relativistic in the center of mass rest frame or that one of the two particles is massless. The first approximation can be stated by assuming that the 3-momentum \( \lambda \) defined in (59) is small compared to the total energy \( E_A \) and/or \( E_B \) which is of same order of magnitude that the rest mass \( m_A \) and/or \( m_B \), i.e. that

\[
\lambda \ll E \sim m.
\]

Since \( \Upsilon(m_A^2, m_B^2, p^\mu q_\mu) = \Upsilon(m_A^2, m_B^2, \lambda, E_A, E_B) \), in this approximation we can Taylor expand each coefficient of equation (69) in power of \( \lambda/m_{AB} \), with \( m_{AB} = \min(m_A, m_B) \) as (on the example of \( \alpha \))

\[
\alpha(m_A^2, m_B^2, p^\mu q_\mu) = \alpha_{NR} + \frac{\lambda}{m_{AB}} \tilde{\alpha} + O \left( \left( \frac{\lambda}{m_{AB}} \right)^2 \right).
\]
On the other hand if we assume that one of the particle is ultra-relativistic, then we can make a “Thomson-like” approximation, which says that the energy of the zero-mass particle is small compared with the rest mass of the particle it scatters with (i.e. \( p^\mu q_\mu \ll m^2_{AB} \) with \( m_{AB} = \text{sup}(m_A, m_B) \)). Each coefficient of equation (30) can then be expanded as

\[
\alpha(m_A^2, m_B^2, p^\mu q_\mu) = \alpha_{UR} - \frac{p^\mu q_\mu}{m_{AB}^2} \Delta + O\left(\frac{(p^\mu q_\mu)^2}{m_{AB}^4}\right).
\]

Technically these two approximations reduce the knowledge of the function \( \Upsilon \) to a set of scalars \((\alpha_{NR}, \beta_{NR}, \gamma_{NR}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})\) in the first approximation and \((\alpha_{UR}, \beta_{UR}, \gamma_{UR}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})\) in the second one.

Since at zeroth order the two approximations lead to the same form of the force, we will first compute this term and then evaluate the first order correction in the two approximations. Note that even if the form of the force is the same at zeroth order, the force itself will be different because \( \alpha_{NR} \) and \( \alpha_{UR} \) are not the same constants. We have

\[
\alpha_{NR} = \alpha(m_A^2, m_B^2, E_A = m_A^2, E_B = m_B^2, \lambda = 0),
\]

\[
\alpha_{UR} = \alpha(m_A^2, m_B^2, \mu, p^\mu q_\mu = 0).
\]

For the sake of simplicity we set \( \alpha = (\alpha_{NR}, \alpha_{UR}) \).

We will compute \( F^\mu_{AB} \) in the rest frame of \( B \). As explained at the beginning of this section, this breaks the symmetry between \( A \) and \( B \) (see (64)). Therefore, we will have to restore this “hidden” or “lost” symmetry at the end of the computation. Another solution would have been to split the force into two halves and compute one part in the rest frame of \( A \) and the other half in the rest frame of \( B \). Let us note that this symmetry can be restored only in the first approximation where we can compute the force either in the rest frame of \( A \) or the rest frame of \( B \). In the second approximation, this is no longer possible and the force will not be obviously symmetric. We define \( F^\mu_{AB} \) by

\[
F^\mu_{AB} = \int \left( \{ p^\nu q_\nu \} \{ f_A(p') - f_A(p) \} f_B(q) p^\mu \Upsilon(m_A^2, m_B^2, p^\mu q_\mu, \mu, e^\mu, e'\mu) \frac{d\Omega'}{4\pi} \pi_+(q) \pi_+(-q) \right)_{rfB},
\]

where “\( rfB \)” means that the quantities are evaluated in the rest frame of \( B \). From equation (64), it follows that in the first approximation (i.e. the non-relativistic approximation) we have

\[
F^\mu_{A\rightarrow B} = F^\mu_{AB},
\]

(\( [AB] \) means that we anti-symmetrise on \( A \) and \( B \) the expression (70)) and in the second approximation (the ultra-relativistic approximation)

\[
F^\mu_{A\rightarrow B} = F^\mu_{AB}, \quad F^\mu_{B\rightarrow A} = -F^\mu_{AB}.
\]

We perform the splitting (23) of \( p^\mu \) with respect to \( u^\mu \) defined by \( u^\mu = \frac{\xi^\mu}{m_B} \), and inject in the integral (70). For the sake of clarity we will split \( F^\mu_{AB} \) in \( F^{(0)\mu}_{AB} + F^{(1)\mu}_{AB} \) where (0) and (1) refer to the zeroth and first orders in the expansion of \( \Upsilon \) either in “\( \lambda/m_{AB} \)” or in “\( (-pq)/m_{AB}^2 \)”.

Thus, taking into account the fact that \( q^\mu e_\mu = 0 \) and that \( \int e^\mu d\Omega = \int \Delta^{\mu\nu} d\Omega = 0 \), it follows

\[
F^{(0)\mu}_{AB} = \sigma \alpha \int p^\mu p^\nu q_\nu f_A(p)f_B(q)\pi_+(q)\pi_+(p) + \sigma \left( E^2 q^\mu + E \lambda m_B e^\mu \right) f_B(q)\pi_+(q)
\]

\[
\left\{ \alpha \int f_A(p') \frac{d\Omega'}{4\pi} + \beta e^\alpha \int f_A(p') e_\alpha \frac{d\Omega'}{4\pi} + \gamma \Delta^{\alpha\beta} \int f_A(p') \Delta_{\alpha\beta} \frac{d\Omega'}{4\pi} \right\} \lambda dE d\Omega.
\]

(71)
Using the definitions \( \{ 12-15 \} \) and the integrals \( \{ 28 \} \), this reduces to

\[
F_{AB}^{(0)\mu} = \sigma \left\{ \alpha m_B u_A^{\mu} + \alpha \rho \nu n_B^{\mu} + \beta T_{\nu}^{\mu} \right\}.
\]  

(72)

Let us recall that \( q_A^{\mu} \) is the energy flux with respect to \( n_B^{\mu} \), i.e. with respect to the unit vector \( u_B^{\mu} \) colinear to \( n_B^{\mu} \),

\[
q_A^{\mu} = - \left\{ T_A^{\mu\nu} u_B^{\nu} + T_A^{\alpha\beta} u_B^{\alpha} u_B^{\beta} \right\}.
\]

Thus, we have obtained the expression of the force on the fluid composed of particles \( A \) in terms of the macroscopic quantities describing the two fluids and of the coefficients \( (\alpha, \beta, \gamma, \sigma) \) describing the collision.

Let us now turn to the evaluation of the first order corrections. We begin by the “non-relativistic” approximation,

\[
F_{AB}^{(1)\mu} = \frac{\sigma}{m_{AB}} \tilde{\alpha} \int p^\mu p^\nu q_{\nu} f_A(p) f_B(q) \lambda \pi(p) \pi(q) + \frac{\sigma}{m_{AB}} \int \left( E^2 q^{\mu} + E \lambda m_B e^{\mu} \right) f_B(q) \pi(q)
\]

\[
\left\{ \tilde{\alpha} \int f_A(p') d\Omega + \tilde{\beta} e^{\alpha} \int f_A(p') e' d\Omega + \tilde{\gamma} \Delta^{\alpha\beta} \int f_A(p') \Delta^{\alpha\beta} d\Omega \right\} \lambda^2 \nu \partial \nu.
\]

Using the definition of \( \lambda \) \( \{ 22 \} \), and the set of new quantities defined by

\[
J_A = \int_{m}^{\infty} \int_{\Omega} f_A(x, p) \lambda^2 E^2 d\Omega d\Omega,
\]

(74)

\[
J_A^{\alpha} = \int_{m}^{\infty} \int_{\Omega} f_A(x, p) \lambda^2 E e^\alpha d\Omega d\Omega,
\]

(75)

\[
J_A^{\alpha\beta} = \int p^\alpha p^\beta p^\gamma e^\delta f_A(x, p) \pi(p),
\]

(76)

and the definitions \( \{ 12-13 \} \) and integrals \( \{ 28 \} \), we obtain

\[
F_{AB}^{(1)\mu} = \frac{\sigma}{m_{AB}} \left\{ \tilde{\alpha} n_B^{\nu} J_A^{\nu\mu} + \tilde{\alpha} J_A m_B^{\mu} + \tilde{\beta} T_{\nu}^{\mu} J_A^{\nu} \right\}.
\]

(77)

We will now turn to the second approximation (i.e. \( p^\mu q_{\mu} \ll m_{AB}^2 \)) and evaluate the first order contribution. It reads

\[
F_{AB}^{(1)\mu} = \frac{\tilde{\sigma}}{m_{AB}^2} \alpha \int p^\mu p^\nu q_{\nu} q_{\lambda} f_A(p) f_B(q) \pi(q) \pi(p) + \frac{\sigma}{m_{AB}^2} \int \left( m_B E^2 q^{\mu} + E^2 \lambda m_B^2 e^{\mu} \right) f_B(q) \pi(q)
\]

\[
\left\{ \tilde{\alpha} \int f_A(p') d\Omega + \tilde{\beta} e^{\alpha} \int f_A(p') e' d\Omega + \tilde{\gamma} \Delta^{\alpha\beta} \int f_A(p') \Delta^{\alpha\beta} d\Omega \right\} \lambda^2 \nu \partial \nu.
\]

(78)

and thus

\[
F_{AB}^{(1)\mu} = \frac{\sigma}{m_{AB}} \left\{ \tilde{\alpha} n_B^{\nu} I_A^{\nu\mu} + \frac{\tilde{\beta}}{3} \perp_{B}^{\mu} T_{B}^{\mu} I_{A}^{\alpha} - \tilde{\alpha} T_{B}^{\mu} X_{0A}^{\mu\lambda} \right\},
\]

(79)

where we have introduced the two macroscopic quantities

\[
I_A = \int_{m}^{\infty} \int_{\Omega} f_A(x, p) \lambda E^2 d\Omega d\Omega,
\]

(80)

\[
J_A^{\alpha} = \int_{m}^{\infty} \int_{\Omega} f_A(x, p) \lambda^2 E e^\alpha d\Omega d\Omega,
\]

and where the indice \( B \) in \( \perp_{B}^{\mu} \) means that we used the 4-velocity \( u_B^{\mu} \). \( X_{0A}^{\mu\lambda} \) is defined in \( \{ 31 \} \).
In conclusion the force is given by

\[ F_{B 	o A}^\mu = 2 \left( F_{[AB]}^{(0)\mu} + F_{[AB]}^{(1)\mu} \right) \]
\[ F_{B 	o A}^\mu = F_{[AB]}^{(0)\mu} + F_{[AB]}^{(1)\mu}, \tag{81} \]

according to the first or the second approximation. \( F^{(0)} \) is given by (72) an \( F^{(1)} \) either by (77) or (73).

Let us stress that we could go on in the expansion of \( \Upsilon \) and compute corrections to the force at different orders. This would however involve the introduction of new macroscopic quantities (like e.g. \( J_A, J_A^\alpha \)) and of tensors of higher rank (like e.g. \( X_{\mu\nu}^{\lambda} \)). This is an example of what we mentioned in \& 3.2 since \( F^{(0)} \) involves \( q^\mu \) and \( F^{(1)}, X_{\mu\nu}^{\lambda} \).

In the case of quantum particles, we have to take into account quantum statistics effects which will be evaluated in the next section.

### 5.2 Photon-electron scattering

We will focus here on the elastic Compton scattering between electrons and photons,

\[ e^- (p) + \gamma (k) \to e^- (p') + \gamma (k'). \]

We will try to follow the general computation that we have developed in the former paragraph. However we have to take into account quantum statistics. The general form of the collision term (see e.g. [24, 25, 26]) is

\[ C[f_\gamma](x, k) = \int (-p^\mu k_\nu) \pi_+ (p) \left\{ f_e (p') f_\gamma (k') \left( 1 + \frac{f_e (k)}{2} \right) \left( 1 - \frac{f_e (p)}{2} \right) - f_e (p) f_\gamma (k) \left( 1 + \frac{f_e (k)}{2} \right) \left( 1 - \frac{f_e (p')}{2} \right) \right\} \, d\sigma^{kp \to k'p'}, \tag{82} \]

where the factors \( \left( 1 + \frac{f_e (k)}{2} \right) \) and \( \left( 1 - \frac{f_e (p)}{2} \right) \) are terms coming from the Bose-Einstein and Pauli statistics for the photon and electron respectively [22]. The differential cross section for the Compton scattering is given by [23]

\[ d\sigma^{kp \to k'p'} = \frac{1}{4} \left( -p^\mu k_\nu \right) |M_{kp \to k'p'}|^2 \left( D_{kp \to k'p'} \right)^2, \tag{83} \]

where \( |M_{kp \to k'p'}| \) and \( \left( D_{kp \to k'p'} \right)^2 \) are respectively the matrix element and the two bodies phase space element. If we now work in the reference frame of the electron, we have to choose

\[ u^\mu = \frac{p^\mu}{m_e}, \tag{84} \]

to perform the splitting (21), from which it follows that the coefficients \( |M_{kp \to k'p'}|^2 \) and \( \left( D_{kp \to k'p'} \right)^2 \) are respectively given by

\[ \left( D_{kp \to k'p'} \right)^2 = (2\pi)^4 \delta^{(4)}(p + k - p' - k') \pi_+ (p') \pi_+ (k') = \frac{1}{4} (2\pi)^{-2} \frac{E}{m_e} \left( \frac{E'}{E} \right)^2 \, dQ', \tag{85} \]

and

\[ |M_{kp \to k'p'}|^2 = 16 (2\pi)^2 m_e^2 \left( \frac{E}{E'} \right)^2 \left\{ 1 + \Delta_{\mu\nu} \Delta^{\mu\nu} + \frac{3}{4} \left( \frac{E}{E'} + \frac{E'}{E} - 2 \right) \right\} \frac{\sigma_T}{4\pi}. \tag{86} \]
\( \sigma_T \) is the Thomson scattering cross section. The differential cross section is then

\[
d\sigma^{kp\rightarrow k'p'} = \left\{ 1 + \frac{3}{4} \Delta_{\mu\nu} \Delta_{\mu'\nu'} + \frac{3}{4} \left( \frac{E'}{E} + \frac{E}{E'} - 2 \right) \right\} \frac{\sigma_T}{4\pi} d\Omega'. \tag{87}
\]

If we use the conjugate process (i.e. \( kp' \rightarrow k'p \)), we can factorise \( f_e(p) \).

We will also make two following approximations

- we will take the Thomson limit of the Compton scattering, which implies that
  \[
  \frac{E}{m_e} \sim \frac{E'}{m_e} \ll 1, \tag{88}
  \]

- and we will neglect the quantum statistics.

With these approximations the collision term can be reset as

\[
C_T[f_\gamma](x, k) = \sigma_T \int (-p'^{\mu} k^\mu) \pi_+(p) f_e(p) \left( 1 + \frac{3}{4} \Delta_{\mu\nu} \Delta_{\mu'\nu'} \right) (f_\gamma(k') - f_\gamma(k)) \frac{d\Omega'}{4\pi}, \tag{89}
\]

from which it follows that the force is

\[
F^\nu(x^\mu) = \int C_T[f_\gamma](x, k) k^\nu \pi_+(k). \tag{90}
\]

We can compute this force by using the integrals (28-30) given in \& 2 and the macroscopic quantities (12-15), we obtain

\[
F^\nu_{e\rightarrow \gamma} = \sigma_T \left( n_e^{\mu\nu} T^{\mu\nu}_\gamma + \rho_{\gamma} n^\nu_e \right). \tag{91}
\]

Note that this result could have been obtained from the general derivation of the section 5.1 in the second approximation (i.e. by making \( m_B = m_e \rightarrow +\infty \) in (72)). The force on the electron is given by

\[
F^\nu_{\gamma\rightarrow e} = -F^\nu_{e\rightarrow \gamma}. \tag{92}
\]

We now need to compute the corrections to this force. They are of two origins, the corrections coming from the fact that we do not take the Thomson limit (i.e. corrections in \( E/m_e \) and \( E'/m_e \)) and corrections coming from quantum effects (i.e. terms in \( f f \)). The first one are of the same kind that the one for the diffusion (section 5.3). Equation (87) tells us that \( \hat{\beta} = 0, \hat{\gamma} = 0, \hat{\alpha} = 3/2 \) and thus, according to (77), the correction is given by

\[
F^{(1)\mu}_{e\rightarrow \gamma} = \frac{3}{2} \frac{\sigma_T}{m_e^2} \left\{ N_\gamma I_\gamma - T_{\mu\nu}\Sigma_{\mu\nu} \right\} \tag{93}
\]

where we have introduced the macroscopic quantity

\[
I_\gamma = \int_m^\infty \int_{\Omega} f_A E^4 dE d\Omega, \tag{94}
\]

We will now deal with the quantum statistics corrections in the Thomson limit. Using the general expression of the collision term we have

\[
F^{(quant)\mu}_{e\rightarrow \gamma} = \frac{\sigma_T}{2} \int (-p'^{\nu} k^\nu) \left\{ f_e(p) (f_e(p') f_\gamma(k) - f_\gamma(k') f_e(k)) + f_e(p') f_\gamma(k') f_e(k) \right\} \phi^{(4)}(p + k - p' - k') \pi_+(p) \pi_+(p') \pi_+(k) \pi_+(k'). \tag{95}
\]

To compute this integral, we need to define a whole set of macroscopic quantities related to the moments of \( "f^2" \). We will not go further here, but we see that such a computation is possible.
5.3 Photon-baryon scattering

If we still work in the Thomson limit, which is equivalent to assume that the electron mass is infinite, the collision term coming from the scattering of photons by baryons will be the same and thus

\[ F_{p\to\gamma}^\nu = \sigma_T \left( n_{p\mu} T^{\mu\nu} + \rho_\gamma n_p^\nu \right). \]  

(96)

If we have other fermions in the problem, all interaction between photons and these fermions will be described by the same force since we stay in the Thomson limit. The first order term will differ from one fermion to the other since it is proportional to \( m_f^2 \) as seen on (93).

5.4 Discussion of the domain of validity

Three approximations have to be discussed, namely

- \((-p^\mu q_\mu) \ll m_{AB}^2 \) or \( \lambda \ll m_{AB} \), i.e. the fact that the particles are treated either as non relativistic or ultra relativistic,
- the classical approximation, i.e. the fact that we have neglected the quantum factors coming from Bose-Einstein and Fermi-Dirac statistics,
- the gas approximation.

We have assumed that the particles were non relativistic in the rest frame of the center of mass, which amounts to assuming that the temperature was not too high since we described the constituents by massive particles. For a given element this gives a maximum temperature

\[ \Theta_i < \Theta_{is} = \frac{m_i c^2}{k_B}, \]

\( k_B \) being the Boltzmann constant. The index \( i \) emphasizes the fact that the fluids do not have to be in thermal equilibrium and can have different temperature (moreover this temperature is defined as the statistical temperature). For electrons, we have \( \Theta_{is} = 6.10^9 K \). However, this limit can be relaxed if we take into account the corrections in “\( \lambda/m \)”. For higher temperature, electrons can be treated like a radiation fluid, which means that we then used the “ultra-relativistic” approximation.

If we turn to quantum effects, the temperature must not be too low, in order that the medium be non degenerate. For fermions of rest mass \( m \) and of spin \( s \), this condition (see e.g. [22]) is satisfied if

\[ T > \frac{2\pi \hbar^2}{k_B m} \left( \frac{n}{2s+1} \right)^{2/3}. \]

In a Friedmann-Lemaître universe with \( \Omega_0 = 1 \), the density is equal to the critical density \( \rho_{c0} \sim 10^{-29} h^2 g.cm^{-3} \), where \( h \) is related to the Hubble constant via \( H_0 = 100 h km.s^{-1}.Mpc^{-1} \). This leads to an average particles density of \( n_0 \sim 1 \) particle.cm\(^{-3} \) for the matter. Thus, for electrons \((s = 1/2, m = m_e)\) the former condition and the assumption that \( n_{e0} \sim n_0 \) gives

\[ \left( \frac{T}{T_0} \right) > 10^{-16} \left( \frac{\rho_e}{\rho_{e0}} \right)^{2/3}, \]

where \( T_0 \equiv 2.7K \) and where we have assumed \( h \sim 1 \). Assuming that the electrons behave like matter, so that they scale like \( a^3 \), \( a \) being the scale factor of the universe, this can be rewritten in term of redshift as

\[ z < 10^{16}. \]
For this range of redshifts the gas of photons (or electrons) will behave like a classical perfect gas. We see that we have to treat electrons like radiation before we have to take into account the quantum effects. The limit for bosons is of the same order of magnitude.

Since we have computed the general force between uncharged classical particles (section [5.1]), the formalism can be applied, for instance, to fluid of galaxies in a cluster or of stars in galaxies etc.. In such a case one cannot assume that the fluid is composed of particles with the same rest mass and each fluid will have a mass spectrum. Since we perform all the integrations on $T_x$ instead of the mass shell, all the previous results still stand. Note that the approximation of non relativistic relative speed is a very good approximation for this kind of fluids.

The last point we need to discuss concerns the gas approximation, i.e. of the splitting of the total fluid (i.e. of energy-momentum tensor) into individual fluids. For that purpose we will compare the mean free path, $l_c$ say, and the average distance between two particles, $d$ say, for the electrons. On the one hand,

$$l_c \sim \frac{1}{n\sigma T} \sim \frac{1.5 \times 10^{24} \text{cm}}{(1+z)^3},$$

where $\sigma T$ is the Thomson scattering cross section ($\sigma T \sim 6.65 \times 10^{-25} \text{cm}^{-2}$). On the other hand, since $d \sim n_e^{1/3}$, using the previous approximate value of $n_e$, we have

$$d \sim \frac{1 \text{cm}}{1+z}.$$

The gas approximation is valid if

$$l_c > d \iff z < 1.22 \times 10^{12}.$$ 

Thus, from the time of electron-positron annihilation ($T \sim 10^{10} \text{K}$, i.e. $z \sim 3 \times 10^9$) until the time of recombination of hydrogen, it is a very good approximation [4] to treat the content of the universe as a nonrelativistic gas plus blackbody electromagnetic radiation.

Moreover, we know from the theory of the strong interaction that thanks to the “asymptotic freedom”, the concept of weakly interacting particles is appropriate for very dense systems [23]. Thus, beyond its conventional range of applicability discussed just above, the hypothesis of weakly interacting particles and thus their description by the kinetic theory may be extended to early universe.

As we see on this discussion, it is a crucial point of the theory to assume that kinetic theory can be applied and this will have to be checked on any particular case.

6 Inelastic collisions

Source terms can arise from many phenomena. For instance if there are some unstable particles in the problem, we must take into account decay, fission and fusion. In the early universe matter and anti-matter coexisted, which drives us to consider fermion-anti-fermion annihilation. When one turns to photons, one major effect has to be considered, namely the Bremstrahlung and the recombination. We finish by relating our approach to other work on particles production.

6.1 General case of fission

We will try to give the most general term of the source term for fission

$$A(p) \rightarrow B(p') + C(q').$$
As for the scattering, we start by the kinematics. We work in the rest frame of the decaying particle and thus
\[ u^\mu = \frac{p^\mu}{m_A}. \] 

(97)

It is obvious to see that \( E_A = m_A^2 \) and \( \lambda_A = 0 \). This implies that the vector \( e^\mu \equiv e_A^\mu \) is an arbitrary unit vector. The vectors \( p'^\mu \) and \( q'^\mu \) are then given by
\[
p'^\mu = \frac{1}{2} \left( 1 + \delta_{ABC} \right) p^\mu + \frac{1}{2} \left( 1 - \alpha_{ABC} + \delta_{ABC} \right)^{1/2} e^\mu,
\] 

(98)
\[
q'^\mu = \frac{1}{2} \left( 1 - \delta_{ABC} \right) p^\mu - \frac{1}{2} \left( 1 - \alpha_{ABC} + \delta_{ABC} \right)^{1/2} e^\mu,
\] 

(99)

with \( \delta_{ABC} \equiv (m_B^2 - m_C^2)/m_A^2 \) and \( \alpha_{ABC} \equiv (m_B^2 + m_C^2)/m_A^2 \). The fission is possible if and only if \( m_A \geq m_B + m_C \).

In fact fission is the easiest case of non elastic collision. Since \( p^\mu \) is the only vector of the problem and the emission is isotropic in the rest frame of \( A \), it is obvious that
\[
C[f_A] = C(-p^2) = -\tau_A^{-1},
\] 

(100)

where \( \tau_A \) is a constant representing the lifetime of \( A \). Thus, \( C[f_A] \) represents the probability of decay of \( A \) per unit time and the Boltzmann equation with such a collision term (\( \mathcal{L}[f] = -f/\tau \)) describes the relaxation toward the equilibrium solution \( f_{eq} = 0 \).

The source term is then given by
\[
\epsilon_A = \int C[f_A] \pi_+(p) = -\frac{n_A}{\tau_A}.
\] 

(101)

The production of \( B \) and \( C \) are related to \( \epsilon_A \) by
\[
\epsilon_B = \epsilon_C = -\epsilon_A.
\] 

(102)

The force is
\[
F^\mu_{\rightarrow A} = \int C[f_A] p^\mu \pi_+(p) = -\frac{n^\mu_A}{\tau_A}.
\] 

(103)

It can be understood as a “rubbing” coming from the decay. The force on \( B \) and \( C \) is given by
\[
F^\mu_{\rightarrow B} = \int \frac{p^\mu f_A(p)}{\tau_A} \pi_+(p'),
\] 

(104)

where \( p' \) and \( p \) are related by the kinematic relations. If we cast (98) in the equation of \( F^\mu_{\rightarrow B} \) and use the fact that the emission is isotropic in the rest frame of \( A \) (since \( e^\mu \) is arbitrary), we get that
\[
F^\mu_{\rightarrow B} = \frac{1}{2} \left( 1 + \delta_{ABC} \right) n^\mu_A/\tau_A \quad \text{and} \quad F^\mu_{\rightarrow C} = \frac{1}{2} \left( 1 - \delta_{ABC} \right) n^\mu_A/\tau_A.
\] 

(105)

It can be checked that, as expected,
\[
F^\mu_{\rightarrow A} + F^\mu_{\rightarrow B} + F^\mu_{\rightarrow C} = 0.
\]

We must stress that this source term and these forces were computed without any approximation.
6.2 General case of fusion

We will try to give the most general term of the source term for fusion

$$A(p) + B(q) \rightarrow C(q').$$

The kinematics of the fusion is analogous to the one of the fission. If we work in the rest frame of particle $C$, we have only to relabel the variables as

$$A \leftrightarrow C, \quad p^\mu \leftrightarrow p'^\mu, \quad q^\nu \rightarrow q'^\nu.$$  

The collision term can be written as

$$C[f_A] = -\int (-p^\mu q_\mu)f_A(p)f_B(q)W(p, q, q')\pi_+(q_\pi_+(q').$$

Using the same decomposition that in the section 5.2, it follows that its general form is

$$C[f_A] = -\int T(m_A^2, m_B^2, p^\mu q_\mu, e^\mu, e'^\mu)(-p^\mu q_\mu)f_A(p)f_B(q)\frac{d\Omega'}{4\pi}\pi_+(q). \quad (106)$$

Using the same decomposition that in the section 5.2, it follows that its general form is

$$C[f_A] = -\int \alpha(m_A^2, m_B^2, p^\mu q_\mu)\epsilon^\mu f_A(p)f_B(q)\pi_+(q). \quad (107)$$

Since $e'^\mu$ is arbitrary, the integration over $\Omega'$ is straightforward and thus

$$C[f_A] = -\int \alpha(m_A^2, m_B^2, p^\mu q_\mu)\epsilon^\mu f_A(p)f_B(q)\pi_+(q). \quad (108)$$

We do now the “Thomson” approximation consisting in neglecting ($p^\mu q_\mu$) compared with $m_A^2/\rho_B$. In that limit the source term is given by

$$\epsilon_A^{(0)} = \sigma \alpha(m_A^2, m_B^2)\int (-p^\mu q_\mu)f_A(p)f_B(q)\pi_+(q)\pi_+(p) = \sigma_{ABC} n_B n_{A\mu}, \quad (109)$$

which means that the reaction rate is proportional to the number of the two reactive constituents and to their relative speed. Because of the symmetries we have $\epsilon_A = \epsilon_B = -\epsilon_C$. Refinements can be included if there are any stoichiometric coefficients.

Let us now turn to the force,

$$F_{\rightarrow A}^{(0)} = \sigma \alpha(m_A^2, m_B^2)\int (-p^\mu q_\mu)f_A(p)f_B(q)\pi_+(q)\pi_+(p) = \sigma_{ABC} n_B T_{A\mu}. \quad (110)$$

If we use the kinematic relations, we can check that

$$(1 + \delta_{ABC})F_{\rightarrow A}^{(0)} = (1 - \delta_{ABC})F_{\rightarrow B}^{(0)}; \quad (111)$$

which can be understood on two particular cases. If $m_A = m_B$ then $\delta_{ABC} = 0$ and $F_{\rightarrow A}^{(0)} = F_{\rightarrow B}^{(0)}$, i.e. each fluid undergoes the same force because of the symmetry. If $m_A \gg m_B$ then $\delta_{ABC} \sim 0$ and $F_{\rightarrow A}^{(0)} \sim 0$, i.e. the variation of impulsion is very small for the fluid $A$ when it merges with the fluid $B$.

Furthermore, we have

$$F_{\rightarrow A}^{(0)} + F_{\rightarrow B}^{(0)} = \sigma \alpha(m_A^2, m_B^2)\int (-p^\mu q_\mu)f_A(p)f_B(q)(p^\mu + q^\mu)\pi_+(q)\pi_+(p) \equiv -F_{\rightarrow C}^{(0)}, \quad (112)$$
because of the kinematics relations and thus
\[ F^{(0)\mu} A + F^{(0)\mu} B + F^{(0)\mu} C = 0. \]  
(113)

We can give the correction to the source terms and the force coming from the \((p^\mu q_\mu)\)-dependence in \( \Upsilon \), which we develop as in the section 5.1. This implies a correction to the collision term, \( C_A^{(1)} \), say, which is given by
\[ C_A^{(1)} = -\frac{\sigma A}{m_{AB}} \int p^\mu q_\mu f_A(p) f_B(q) \lambda \pi_+(p), \]
(114)
from which we can compute the correction to the source term, \( \delta \epsilon \), say,
\[ \delta \epsilon_A = -\frac{\sigma A}{m_{AB}} n_B K^\mu_A, \quad \text{with} \quad K^\mu_A = \int f_A(x, p) p^\nu p^\nu \epsilon_\nu \pi_+(p). \]
(115)

The correction to the force, \( F_A^{(1)\lambda} \), is given by
\[ \delta F_A^{(1)\mu} = -\frac{\sigma A}{m_{AB}} \int p^\mu p'^\nu q_\nu \lambda f_A(p) f_B(q) \pi_+(p) = -\frac{\sigma A}{m_{AB}} J_A^{\mu} T_B^{\mu\nu}, \]
(116)
where \( J_A^\nu \) is defined in (73).

### 6.3 General case of annihilation

A general annihilation can be written under the form
\[ A(p) + \bar{A}(q) \rightarrow \gamma(p') + \gamma(q'), \]
with \( A \) and \( \bar{A} \) being two fermions.

An easy way to compute the source term and the force for such a mechanism is to use the work we have done on fusion and fission and assume that the annihilation can be seen as
\[ A(p) + \bar{A}(q) \rightarrow B \rightarrow \gamma(p') + \gamma(q'), \]
with the decay time of \( B \) being zero (i.e. \( \tau_B \rightarrow 0 \)).

Hence the source terms are
\[ \epsilon_A = \epsilon_{\bar{A}} = -\epsilon_\gamma / 2 = \sigma_A n_A n_{\bar{A}}. \]
(117)

The force on \( A \) and \( \bar{A} \) is the same that the force computed for fusion. The case of the photons is a little bit more tedious. A solution will be to compute it from the integral (46) as we did in the former sections. We can however quote that we will have to sum on two photons which are travelling in opposite directions and which have the same energy. Thus, by symmetry, the total force will be the same as if we had only one particle at the end of the annihilation with the 4-momentum
\[ P^\mu = p'^\mu + q'^\mu, \]
and with the rest mass
\[ M^2 = -P^2. \]

This is exactly the situation we have during a fusion and thus,
\[ F_{\rightarrow A}^\mu = F_{\rightarrow A}^\mu = \sigma_A n_A (A T_{\bar{A}})^{\mu\nu}, \quad F_{\rightarrow \gamma}^\mu = -2\sigma_A n_A (A T_{\bar{A}})^{\mu\nu}. \]
(118)
6.4 Photons

Photons can be produced via Bremstrahlung which can be formally written as
\[ e(p) \rightarrow e(p') + \gamma(k). \]

If we follow Thorne \[15]\), the general collision term for such a collision can be written as
\[ C[f_\gamma] = E \chi_\gamma(E) \{ G_\gamma(E) - f_\gamma(k) \} n_e, \tag{119} \]
where \( E = -w^\mu_k k_\mu \) and \( G_\gamma = \eta_\gamma/(E^3 \chi_\gamma) \). \( \eta_\gamma \) and \( \chi_\gamma \) are the standard emission and absorption coefficients. If we are in a local thermodynamical equilibrium, then Kirchoff’s law holds and
\[ G_\gamma = 2 \left( e^{E/T} - 1 \right)^{-1}. \tag{120} \]

In general, the force on the photons can be written as
\[
F^\mu_{\rightarrow \gamma} = \int \frac{E^3}{m_e} \chi_\gamma(E) G_\gamma(E) (u^\mu + e^\mu) n_e dE d\Omega \\
- \int \frac{E^4}{m_e} (u^\mu + e^\mu) \chi_\gamma(E) f_\gamma(k) n_e dE d\Omega. \tag{121}
\]
This can be computed only if we know the functions \( \chi_\gamma(E) \) and \( G_\gamma(E) \). However we can give its general form, which must be
\[
F^\mu_{\rightarrow \gamma} = \frac{1}{m_e} \left\{ U(\Theta_\gamma) n_e^\mu - q^\nu_t V(\Theta_\gamma) \right\}, \tag{122}
\]
where \( U(\Theta) \) and \( V(\Theta) \) are two coefficients which depend on the photon temperature, \( \Theta_\gamma \). We cannot go further if we want to remain general. However the form has the advantage to be covariant and flexible.

The situation is similar for the source term but, as for the fusion which is in a way very similar, we can model it as
\[
\epsilon_\gamma = \frac{n_e}{\tau_\epsilon(\Theta_\gamma)} \quad \text{and} \quad \epsilon_e = 0, \tag{123}
\]
where the rate of emission depends on the temperature.

If we have other fermions in the problem then the force and the source term for the Bremstrahlung induced by these fermions will be alike, with coefficients \( U_f(\Theta) \), \( V_f(\Theta) \) and \( \tau_f(\Theta_\gamma) \).

6.5 Recombination

A special case of interest in Cosmology is the recombination of electrons and protons in hydrogen which occurs at the last scattering surface \[27]\). It is however different from the general case of fusion since the inverse process is possible, the rate of each process depending on the temperature.
\[ e(p) + p(q) \Rightarrow H(q') + \gamma(k'). \]

From our study of fusion and fission, it is clear that the source term will take the general form (to lowest order),
\[
\epsilon_H = \epsilon_\gamma = -\epsilon_e = -\epsilon_p
= \sigma_{H\gamma} n_e n_H - \sigma_{ep} n_e n_p, \tag{124}
\]
The two coefficients, \( \sigma_{H\gamma} \) and \( \sigma_{ep} \), are all functions of the temperature. Details on the ionisation and recombination processes are needed to have their exact value. Examples of such functions can be found in \[28, 29]\).
6.6 Relation with quantum particle creation

We want to emphasize that, in our framework, even if there is particle creation via unelastic collisions the total energy-momentum tensor is conserved. The situation is then different from the one studied in cosmology where the particles production arises from some quantum process \[30\].

Such phenomena were phenomenologically described by an effective viscous pressure \[31\] and a microscopic justification of such an approach was proposed in \[32\]. Their work is based on the introduction of a source term in the Boltzmann equation beside the usual collision term. Unfortunately this source term gives also birth to a force acting on the fluid (they only have one fluid).

If one wants to take into account such quantum creation of particles, this can be achieved by using a linear coupling (see \[32\] section (3-1)

\[
C[f_i] = \left(-\frac{u^\mu p_\mu}{\tau(x)} + \nu(x)\right) f_i(x, p),
\]

where \(u^\mu\) is, as usual, an arbitrary vector field used to perform the splitting.

Using the same methods than before it will lead to a source term and a force given by

\[
e(x) = -\frac{u^\mu n_\mu(x)}{\tau(x)} + \nu(x)n(x),
\]

\[
F^\mu(x) = -\frac{u^\nu T^\mu_{\nu}(x)}{\tau(x)} + \nu(x)n^\mu(x).
\]

In such a situation the global energy-momentum will not be conserved (see \[32\] for a detailed discussion of this problem). This is beyond the scope of this article.

6.7 Recapitulation

Before we give some applications of our formalism, we will sum up the different source terms and forces we have computed either in the non-relativistic limit or in the Thomson limit. We do not give the corrections than have been computed. In fact, we computed them to show how one can give a general form. However, as seen on equations (77), (79), (93), (115) and (116), they imply new macroscopic quantities.

For any applications, we need as input the cross sections of all the collisions between the species we are considering and the lifetime of all the unstable particles (i.e of “\(\sigma\)”, “\(\alpha\)”, “\(\beta\)”, “\(\tau\)”, “\(U\)” and “\(V\)”).

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Each fluid has an energy-momentum tensor given by

\[ \Pi^\mu_\nu = \frac{1}{2} \mathcal{F}^\mu_\nu = \frac{\mathcal{F}^{(0)}_\alpha^\mu A^\alpha}{\mathcal{F}^\gamma_\gamma} + \rho_\alpha A^\mu A^\nu + \frac{\beta}{3m_B} T^\mu_\nu \]

and

\[ F^\mu_\nu = -\frac{\rho_\alpha}{\mathcal{F}^\gamma_\gamma} - \frac{\beta}{3m_B} T^\mu_\nu \]

7 Application to the theory of cosmological perturbations

We give here a straightforward application of this general formalism by considering the evolution of a gas of photons and electrons interacting via Compton scattering in a Friedmann-Lemaître universe at linear order. The study of the formation and evolution of the cosmic microwave background fluctuations begins usually with the study of the radiative transport which is then integrated. This has been done by many authors (e.g. [18, 27]) in a perturbed Friedmann-Lemaître universe.

The metric of space-time is given by

\[ ds^2 = a(\eta)^2 \left( -d\tau^2 + 2D_4B dx^i dx^j + (\gamma_{ij} + h_{ij}) dx^i dx^j \right) = (\bar{g}_{\mu\nu} + \delta \bar{g}_{\mu\nu}) dx^\mu dx^\nu, \]

if we focus on scalar perturbations. \( \gamma_{ij} \) is the metric of the unperturbed \( \{ t = constant \} \)-hypersurfaces, \( D_i \) is the covariant derivative with respect to \( \gamma_{ij}, \eta \) is the conformal time, \( \alpha \) the scale factor and a prime denotes a derivative with respect to \( \eta \). We also introduce \( h = h_{ij} \gamma^{ij} \).

Each fluid has an energy-momentum tensor given by

\[ T^\mu_\nu = (P + \rho) u^\mu u_\nu + P(\delta g^\mu_\nu + \delta \bar{g}^\mu_\nu) + (\delta P + \delta \rho) u^\mu u_\nu + \delta P \bar{g}^\mu_\nu + 2(P + \rho) u_i^{(\mu} \delta u^{\nu)} + a^2 \Pi^\mu_\nu, \]

where \( P \) and \( \rho \) are the pressure and density of the fluid in the background and \( \delta P, \delta \rho, \delta u^\mu \) and \( \Pi^\mu_\nu \) are respectively the density perturbation, the pressure perturbation, the velocity perturbation and the anisotropic stress tensor. Since \( u^\mu u_\mu = -1 \) then \( \delta u_0 = 0 \) and we can write \( \delta u_k \) as

\[ \delta u_k = a(D_k B + v_k). \]

We also introduce \( c_s^2 = dP_m/d\rho_m \), the sound speed of the matter and \( \delta \equiv \delta \rho/\rho \), the density contrast. The law of evolution of the two energy-momentum tensors are, according to equation [46]

\[ \nabla_\nu T^{\mu_\nu} = F^\mu_\nu \rightarrow \gamma, \]

\[ \nabla_\nu T^\mu_\nu = F^\mu_\nu \rightarrow m, \]
can be expanded to linear order. Since $u^\mu u_{\gamma \mu} = -1$ to first order, the force on the matter (96) reduces to
\[ F^\mu_{m \rightarrow \gamma} = \frac{4}{3} \sigma T n_e \rho_\gamma \left( u^\mu_m - u^\mu_\gamma \right). \] (130)

Using the metric (125) and the above expression of the force, the equation of conservation (128-129) read
\[ \delta'_\gamma + \frac{4}{3} \left( \Delta v_\gamma + \frac{1}{2} H' \right) = 0 \] (131)
\[ \delta'_m + (1 + \omega_m) \left[ \Delta v_m + \frac{1}{2} H' \right] = 0 \] (132)
\[ (v^i_\gamma + B^i)' + D^i A + \frac{1}{4} D^i \delta_\gamma + \frac{1}{4} D_j \Pi^{ij}_\gamma = \frac{4}{3} a \sigma T n_e (v^i_b - v^i_\gamma) \] (133)
\[ (v_m + B)' + \mathcal{H}(1 - 3c_m^2) (v_m + B) - A = \frac{4}{3} \rho_m a \sigma T n_e (v_\gamma - v_m), \] (134)
which is the usual result (see e.g. [18]). Let us stress that since the Compton scattering is elastic, the number of photons and electrons are conserved, which can be checked on the first two equations.

If we now turn to the inelastic collision
\[ A + B \iff C + D \]
within the standard model of Cosmology, the equation of conservation for $A$ reads
\[ \nabla_{\mu} n^\mu_A = -\sigma_{AB} n^\mu_A n_{B \mu} + \sigma_{CD} n^\mu_C n_{D \mu}. \] (135)

If we study a small deviation of $n^\mu_A$ from the equilibrium, then
\[ \nabla_{\mu} n^\mu_A = -\sigma_{AB} n_{Beq} \left( n^\mu_A - n^\mu_{Aeq} \right), \] (136)
the subscript “eq” meaning at equilibrium. In Friedmann-Lemaître, this becomes
\[ \dot{n}_A = -3H n_A - \gamma_A (n_A - n_{Aeq}), \] (137)
with $\gamma_A = \sigma_{AB} n_{Beq}$. We recover the general phenomenon of “freezing”. If $3H < \gamma_A$ then the system will relax toward the equilibrium, but, if $3H > \gamma_A$, the system is decoupled since the interaction rate is not sufficient to compensate the expansion rate.

Our formalism, being fully covariant, can be applied to study the same problems in more general situations, for instance if we turn to inhomogeneous cosmologies.

8 Conclusion

We have developed a fully covariant framework for a system of interacting fluids. We have computed general form of the source terms $\epsilon_i$, and of the forces $F_{\mu i}$, related to the number flux vectors $n^\mu_i$ and the energy-momentum tensors $T^\mu_\nu_i$ by
\[ \nabla_{\mu} n^\mu_i = \epsilon_i \quad \text{and} \quad \nabla_{\nu} T^\mu_\nu_i = F^\mu_i. \]

We have considered the cases of elastic collisions (and the example of the photon-electron scattering), of inelastic collisions (including fission, fusion, bremsstrahlung and recombination) and
we have included a possible magnetic force (see table in section 6.7 for a summary of the results for all these cases). This computation first required the modelisation of the collision term that enters the Boltzmann equation and then the integration of this equation.

All the quantities have been computed for situations of cosmological relevance (Compton scattering, recombination...) and can be used in a wide range of redshift (e.g. $0 < z < 10^{12}$ if we just consider photons-electrons and baryons). Since we have the general force for fluids constituted of massive particles, this formalism is also suited for fluids of stars, galaxies...Moreover, thanks to the “asymptotic freedom”, this formalism of interacting gases can be hoped to apply to very dense systems, and thus during the early universe. However one has to be careful in such an application and must check that the gas approximation holds.

This formalism is non perturbative, in the sense that we do not expand the geometry around a background spacetime. It is of course perturbative in the sense that we expand and integrate the Boltzmann equation to a given order. When expanded to first order (in the perturbation of the metric), it reduces to the equation of cosmological linear perturbations. But, our formalism can be applied to study the dynamics of a system of gases in a more general context. It will be useful, for example, to study the microwave background in inhomogeneous cosmologies or compute the sound speed of the photon-baryon fluid during the decoupling.

Let us stress that we do not assume anything on the fluids but the fact that they are gases, i.e. that they can be described by a distribution function. When they are perfect fluids, the system of equations is closed. Otherwise, we need either an equation of state or an equation of evolution (e.g. for $\Pi$) which can be obtained from higher moments of the Boltzmann equation.

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