UNCONDITIONALLY SATURATED BANACH SPACE WITH THE SCALAR-PLUS-COMPACT PROPERTY

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Abstract. We construct a Bourgain-Delbaen $L_\infty$-space $X_{Kus}$ with strongly heterogenous structure: any bounded operator on $X_{Kus}$ is a compact perturbation of a multiple of the identity, whereas the space $X_{Kus}$ is saturated with unconditional basic sequences.

1. Introduction

J. Bourgain and F. Delbaen presented in [8] a brilliant method of constructing $L_\infty$-spaces with a peculiar structure. Their method relies on a careful choice of an increasing sequence of finite dimensional subspaces $F_n$ of $\ell_\infty(\Gamma)$, with infinite countable $\Gamma$ and each $F_n$ uniformly isomorphic to $\ell^\dim F_n$. A suitable choice of $(F_n)_n$ guarantees that the space $\bigcup_{n\in\N} F_n$ is an $L_\infty$-space with no unconditional basis. The Bourgain-Delbaen example contains no isomorphic copy of $c_0$, answering a long-open problem in the theory of $L_\infty$-spaces. Later R. Haydon [16] proved that this space is saturated with reflexive $\ell_p$ and introduced the notation used nowadays. The Bourgain-Delbaen method was used to construct Banach spaces that solved other several long-standing conjectures on the structure of Banach spaces and showed that one may not hope for an ordinary classification of $L_\infty$-spaces as it happens in the $C(K)$-spaces case, see [1], [2], [3], [11]. We refer to [7] and [8] for the properties of the classical Bourgain-Delbaen spaces.

In [2] a general Bourgain-Delbaen-$L_\infty$-space is defined and the authors show a remarkable fact that up to isomorphism any separable $L_\infty$-space is isomorphic to such a space. We recall from [2] that a BD-$L_\infty$-space is a space $X \subseteq \ell_\infty(\Gamma)$, with $\Gamma$ countable, associated to a sequence $(\Gamma_q, i_q)_{q \in \N}$, where $(\Gamma_q)$ is an increasing sequence of finite sets with $\Gamma = \bigcup_{q \in \N} \Gamma_q$ and $(i_q)_q$ are uniformly bounded compatible extension operators $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$, i.e. $i_q(x)\big|_{\Gamma_q} = x$ and $i_q(x) = i_p(i_q(x)\big|_{\Gamma_p})$ for any $q < p$ and $x \in \ell_\infty(\Gamma_q)$. The space $X = X_{(\Gamma_q, i_q)_q}$ is defined as $X = \langle d_\gamma : \gamma \in \Gamma \rangle$, with $d_\gamma$ given by $d_\gamma = i_q(e_\gamma)$, with $q$ chosen so that $\gamma \in \Gamma_q \setminus \Gamma_{q-1}$. An efficient method of defining particular examples of BD-$L_\infty$-spaces as quotients of canonical BD-$L_\infty$-spaces was given in [3]. The authors proved that given a BD-$L_\infty$-space $X \subseteq \ell_\infty(\Gamma)$ any so-called self-determined set $\Gamma' \subseteq \Gamma$ produces further $L_\infty$-space $Y = \langle d_\gamma : \gamma \in \Gamma \setminus \Gamma' \rangle$ and a BD-$L_\infty$-space $X/Y$, with the quotient map defined by the restriction of $\Gamma$ to $\Gamma'$.

S.A. Argyros and R. Haydon in [3] used the BD-method in order to produce an $L_\infty$-space $X_{AH}$ which is hereditary indecomposable (III) i.e. contains no infinitely dimensional subspace which is a direct sum of further two infinitely dimensional subspaces (in particular the space $X_{AH}$ admits no unconditional basic sequence), and with dual isomorphic to $\ell_1$. Moreover, using in essential way the local unconditional structure imposed by the $\ell^\dim F_n$-spaces they proved that the space $X_{AH}$ has the scalar-plus-compact property i.e. every bounded operator on the space is of the form $\lambda I + K$, with $K$ compact and $\lambda$ scalar.

Although it readily follows that there does not exist a Banach space with an unconditional basis and the scalar-plus-compact property, the latter property does not exclude rich unconditional structure inside the space. This is witnessed in [1], where it was shown that, among other spaces, any separable and uniformly convex Banach space embeds into an $L_\infty$-space with the scalar-plus-compact property. Therefore, a natural question on a Banach space with the scalar-plus-compact property that is saturated with unconditional basic sequences arises.
Recall here that the first example of a space with an unconditional basis and a small family of operators is due to W.T. Gowers, who ”unconditionalized” in [13] the famous Gowers-Maurey space, producing a space $X_G$ with unconditional basis that solved the hyperplane problem. Afterwards, W.T. Gowers and B. Maurey, [15], proved that any bounded operator on the space $X_G$ is of the form $D + S$, with $D$ diagonal and $S$ strictly singular. Gowers asked if an analogous property holds for the operators defined on subspaces of $X_G$ and if such property characterises a class of so-called tight by support Banach spaces, as it is in the case of complex HI space according to [9]. This question was answered negatively by the first two named authors [17].

An example of a space with rich unconditional structure and a small family of bounded operators of a different type was presented in [4], where the authors build a Banach space saturated with unconditional sequences and satisfying the following property: any bounded operator on the space is a strictly singular perturbation of a multiple of identity (recall that an operator is strictly singular provided none of its restriction to an infinitely dimensional subspace is an isomorphism onto its range). The construction used the saturated norms technique in mixed Tsirelson space setting.

In this paper we continue the study of Banach spaces with a small family of operators by showing the existence of a Banach space with a strongly heterogeneous structure. More precisely we construct a BD-$\ell_\infty$-space $\mathfrak{Kus}$ with a basis satisfying the following properties:

1. Any bounded operator $T : \mathfrak{Kus} \to \mathfrak{Kus}$ is of the form $T = \lambda Id_{\mathfrak{Kus}} + K$, with $K$ compact and $\lambda$ scalar,
2. The space $\mathfrak{Kus}$ is saturated with unconditional basic sequences,
3. The space $\mathfrak{Kus}$ is tight by range, i.e. no two subspaces spanned by block sequences with pairwise disjoint ranges are comparable,
4. The dual space to $\mathfrak{Kus}$ is isomorphic to $\ell_1$.

The structure of the space of bounded operators $B(\mathfrak{Kus})$ implies that the space $\mathfrak{Kus}$ is indecomposable, however, as unconditionally saturated, it fails to have any HI structure. The space $\mathfrak{Kus}$ is the first example of Banach space with the scalar-plus-compact property failing to have any HI structure. Let us recall that M. Tarbard in [18] constructed an indecomposable BD-$\ell_\infty$-space $\mathfrak{Kus}$, that is not HI BD-space, but the Calkin algebra $B(\mathfrak{K})/K(\mathfrak{K})$ is isomorphic to $\ell_1(\mathbb{N})$.

In order to build $\mathfrak{Kus}$ we adapt the idea of a construction of a Banach space $X_{ius}$ of [4] to the scheme of Argyros-Haydon construction of Bourgain-Delbaen spaces [8]. This framework allows to pass from strictly singular operators to compact ones, however, in order to profit from this key property of Argyros-Haydon construction we need to strengthen some results of [4] in the following way: we prove that if a bounded operator on the space converges to zero on the basis, then it converges to zero on any element of a special class of basic sequences, called RIS, not only on a saturating family of RIS (Prop. [7.2]). In order to avoid technical inductive construction of the space $\mathfrak{Kus}$ we follow the scheme of [5], defining $\mathfrak{Kus}$ as a suitable quotient of some variation of the canonical BD-$\ell_\infty$-space $\mathfrak{B}_{mT}$ defined in [3].

The balance between unconditional saturation and restricted form of bounded operators on the whole space in the case of $X_{ius}$ was guaranteed by the form of so-called special functionals - the major tool in construction of saturated norms. Any special functional in the norming set of $X_{ius}$ is an average of a sequence of functionals, where the odd parts are averages of the basis. Roughly speaking, the choice of the next functional of the average is determined by the previously chosen odd parts and supports of the even parts. The freedom on the side of even parts allows changing signs of parts of even functionals of the average, which in turn provides saturation by unconditional sequences. On the other hand, the control over the supports of the even parts guarantees the typical property of such construction, i.e. in our case given two RIS $(x_n)$ and $(y_n)$ with pairwise disjoint ranges and $\epsilon > 0$ one is able to built on $(y_n)$ an average $\sum_n a_n y_n$ of norm 1, such that $||\sum_n a_n x_n|| < \epsilon$. This last property is crucial for proving the form of a bounded operator on a space.

The direct translation of the special functionals described above into the setting of BD spaces is not possible, as any change of signs of a part of a norming functional changes its support. In
order to overcome this obstacle we use in the definition of functionals on the space $X_{Kus}$ projections on finite intervals instead of projections on right intervals of the form $[p, \infty)$ (Section 2.1) and substitute the equality of supports of even parts of special functionals by tight relation between tree-analysis of even parts (definition of special nodes, Section 5). The latter notion in the setting of Argyros-Haydon construction comes from [12] and proves to be a very efficient tool in our case.

The paper is organized as follows: in Section 2, we describe the construction of the general space we shall use, including different kinds of analysis of norming functionals. Section 3 is devoted to the properties the basis, including the notion of neighbour nodes, within the general framework. In Section 4, we give the definition of $X_{Kus}$. In Section 5. and 6, we study the rapidly increasing sequences (RIS) and the dependent sequences respectively. Section 7. contains the results on bounded operators on the space, whereas Section 8 - the proof of unconditional saturation.

We are grateful to Spiros Argyros and Pavlos Motakis for suggesting using the approach to defining BD-$\mathcal{L}_\infty$-spaces of [5] which greatly simplified presentation of the definition of the space $X_{Kus}$.

2. THE BASE BD-$\mathcal{L}_\infty$-SPACE $X_\Gamma$

We present in this section a BD-$\mathcal{L}_\infty$-space $X_\Gamma$, which is a a minor modification of the space $\mathfrak{B}_{mT}$ defined in [3]. We shall define later the space $X_{Kus}$ as determined by some set $\Gamma \subset \Gamma$ following the general scheme of [5].

2.1. Definition. Pick $(m_k)_k, (n_k)_k, (l_k)_k \nearrow +\infty$ such that $m_1 = 4, n_1 = 4$ and

\[(2.1)\quad m_k m_{k-1} \leq ml_{k-1} \leq \frac{n_k}{m_{k-1}m_k}.
\]

For example take $(2^{2k})_k, (2^{2k^2})_k, (2^k)_k$.

Following [3] we shall define recursively finite sets of nodes $\bar{\Delta}_q$ and $\bar{\Gamma}_q = \bar{\Delta}_1 \cup \cdots \cup \bar{\Delta}_q$, $q \in \mathbb{N}$. Along with each set $\bar{\Delta}_q$ we define functionals $(\bar{c}_\gamma^*)_{\gamma \in \bar{\Delta}_q} \subset \ell_1(\bar{\Gamma}_q)$ and further $(\bar{d}_\gamma^*)_{\gamma \in \bar{\Delta}_q} \subset \ell_1(\bar{\Gamma}_q)$ as $\bar{d}_\gamma^* = c_\gamma^* - \bar{c}_\gamma$. Having defined all sets $\bar{\Delta}_q$, $q \in \mathbb{N}$, we let $\bar{\Gamma} = \bigcup_q \bar{\Gamma}_q$.

We proceed now to the inductive construction. We let $\bar{\Delta}_1 = \{1\}$, $e_1^* = 0$ and thus $\bar{d}_1^* = e_1^*$.

Assume we have defined sets $\bar{\Delta}_1, \ldots, \bar{\Delta}_q$. By $(e_\gamma^*)_{\gamma \in \bar{\Gamma}_q}$ we denote the standard unit vector basis of $\ell_1(\bar{\Gamma}_q)$. We enumerate set $\bar{\Delta}_q$ using $\{\#\bar{\Gamma}_q, \#\bar{\Gamma}_q - 1, \ldots, \#\bar{\Gamma}_q\}$ as the index set and in the set $\bar{\Gamma}_q$ we consider the corresponding enumeration. Thus we can regard sets $\bar{\Delta}_q$ and $\bar{\Gamma}_q$ as intervals in $\mathbb{N}$.

For any interval $I \subset \bar{\Gamma}_q$ let $\bar{P}_I^*$ be the projection onto $\langle \bar{d}_\gamma^* : \gamma \in I \rangle$. For simplicity for any $n \in \mathbb{N}$ by $\bar{P}_n^*$ we denote the projection $\bar{P}_{\{0,n\}}^*$.

For each $q \in \mathbb{N}$ let $\text{Net}_{1,q}$ be a finite symmetric $1/4n^2$-net of $[-1,1]$ containing $\pm 1$. We set

$$B_{p,q} = \{\lambda e_{\eta}^* : \lambda \in \text{Net}_{1,q}, \eta \in \bar{\Gamma}_q \setminus \bar{\Gamma}_p\}$$

where for $p = 0$ we let $\bar{\Gamma}_0 = \emptyset$. For simplicity we write $B_q = B_{0,q}, q \in \mathbb{N}$.

The set $\bar{\Delta}_{q+1}$ is defined to be the set of nodes

$$\bar{\Delta}_{q+1} = \bigcup_{j=1}^q \{(q+1, 0, m_j, I, \epsilon, b^*) : I \text{ interval } \subset \bar{\Gamma}_q, \epsilon \in \{-1, 1\}, b^* \in B_q \text{ and } \bar{P}_I^* b^* \neq 0\}$$

$$\bigcup \bigcup_{1 \leq p < q} \{(q+1, \xi, m_j, I, \epsilon, b^*) : \xi \in \bar{\Delta}_p, w(\xi) = m_j^{-1}, \text{age}(\xi) < n_j, \epsilon \in \{-1, 1\}, b^* \in B_{p,q}, I \text{ interval } \subset \bar{\Gamma}_q \setminus \bar{\Gamma}_p, \bar{P}_I^* b^* \neq 0\}.$$
For any $\gamma \in \bar{\Delta}$ we define $\tilde{c}_\gamma^*$ as follows.

\begin{equation}
\tilde{c}_\gamma^* = \begin{cases} 
\frac{1}{m_j} \epsilon \tilde{P}_j^* b^* & \text{for } \gamma = (q+1, 0, m_j, I, \epsilon, b^*) \\
\epsilon \xi^* + \frac{1}{m_j} \epsilon \tilde{P}_j^* b^* & \text{for } \gamma = (q+1, \xi, m_j, I, \epsilon, b^*)
\end{cases}
\end{equation}

We let also $\tilde{d}_\gamma^* = e_\gamma^* - \tilde{c}_\gamma^*$.

**Notation 1.** For any $\gamma = (q+1, 0, m_j, I, \epsilon, b^*)$ we define $\operatorname{age}(\gamma) = 1$ and for $\gamma = (q+1, \xi, m_j, I, \epsilon, b^*)$ we define $\operatorname{age}(\gamma) = \operatorname{age}(\xi) + 1$. For any $\gamma = (q+1, 0, m_j, I, \epsilon, b^*)$ or $\gamma = (q+1, \xi, m_j, I, \epsilon, b^*)$ we define $\operatorname{rank}(\gamma) = q + 1$ and weight $w(\gamma) = m_j^\epsilon$.

**Remark 2.1.** The main difference with the construction from [3] is that in the $q$-th step instead of taking $b^*$ from the net of the unit ball of the suitable $\ell_1(\bar{\Gamma}_q \setminus \Gamma_p)$, we take $b^*$ only of the form $\epsilon \lambda e^*_n$, where $\epsilon = \pm 1$, $\lambda$ belongs to the suitable net of $[-1, 1]$, and $\eta \in \bar{\Gamma}_q \setminus \Gamma_p$. Moreover we allow projections on all intervals $I \subset \bar{\Gamma}_q \setminus \Gamma_p$, while in [3] the allowable intervals are of the form $I = \bar{\Gamma}_q \setminus \Gamma_p$.

Adapting the reasoning of [3] we obtain the following two lemmas.

**Lemma 2.2.** $(\tilde{d}_n^* : i \leq n) = (\epsilon_n^* : i \leq n)$ for every $n \in \mathbb{N}$.

**Lemma 2.3.** $\|\tilde{P}_m^*\| \leq \frac{m_i}{m_{i-2}} = 2$ for every $m \in \mathbb{N}$.

The above lemma yields that $(\tilde{d}_n^* \gamma_n)_{n \in \mathbb{N}}$ is a triangular basis of $\ell_1(\Gamma)$ (in the sense of [3], Def. 3.1). Let $(d_{\gamma_n})_{n \in \mathbb{N}}$ be its biorthogonal sequence. Regarding each projection $\tilde{P}_m^*$ as an operator $\ell_1(\Gamma) \to \ell_1^n$ we consider the dual operator $i_n : \ell_1^n \to \ell_\infty(\Gamma)$, which is an isomorphic embedding satisfying $\|i_n\| \leq 2$. We are ready to define the following.

**Definition 2.4.** Let $\mathcal{X}_\Gamma = (d_n^* : n \in \mathbb{N}) \subset \ell_\infty(\Gamma)$.

Repeating the results of [3] in our setting we obtain the following.

**Theorem 2.5.** The space $\mathcal{X}_\Gamma$ is a BD-$\mathcal{L}_\infty$-space defined by the sequence $(\bar{\Gamma}_q, \tilde{i}_q)_q$.

**Notation 2.** For any interval $I \subset \mathbb{N}$ we denote by $\tilde{P}_I$ the canonical projection $\tilde{P}_I : \mathcal{X}_\Gamma \to (d_n^* : i \in I)$. In case $I = \{1, \ldots, n\}$, $n \in \mathbb{N}$, we write simply $\tilde{P}_n$.

Given any $q \in \mathbb{N}$ we let $M_q = i_{\max \bar{\Delta}_q} \| \ell_\infty(\Delta_q) \|$. In the rest of the paper we shall consider supports and ranges of vectors, thus also block sequences, with respect both to the basis $(d_n^* \gamma_n)_{n \in \mathbb{N}}$ of $\mathcal{X}_\Gamma$ and to the FDD $(M_q)_{q \in \mathbb{N}}$ of $\mathcal{X}_\Gamma$. In the first case we shall use for any $x \in \mathcal{X}_\Gamma$ the notation $\operatorname{supp} x$, $\operatorname{rng} x$, whereas in the second we write $\supp_{\text{FDD}} x$ and $\text{rng}_{\text{FDD}} x$.

**Definition 2.6.** We say that a block sequence $(x_n) \subset \mathcal{X}_\Gamma$ is skipped provided $\max \text{rng}_{\text{FDD}} x_n + 1 < \min \text{rng}_{\text{FDD}} x_{n+1}$ for each $n$.

### 2.2. The analysis of nodes.

We introduce different types of analysis of a node following [3] and [12], adjusting their scheme to our situation.

**The evaluation analysis of $e^*_\gamma$.**

First we notice that every $\gamma \in \bar{\Gamma}$ admits a unique analysis as follows (Prop. 4.6 [3]). Let $w(\gamma) = m_j^\epsilon$. Then using backwards induction we determine a sequence of sets $(I_i, \epsilon_i, b^*_i, \xi_i)_{i=1}^a$ so that $\xi_a = \gamma$, $\xi_1 = (q+1, 0, m_j, I_1, \epsilon_1, b^*_1)$ and $\xi_i = (q_i + 1, \xi_{i-1}, m_j, I_i, \epsilon_i, b^*_i)$ for every $1 < i \leq a$, where $b^*_i = 1_{\lambda_i} \epsilon^*_i$ for some $\lambda_i \in \text{Net}_{1, q_i}$.

Repeating the reasoning of [3], as $\epsilon^*_\xi = \tilde{d}_\xi^* + e^*_\xi$ for each $\xi \in \Gamma$, with the above notation we have

$$e^*_\gamma = \sum_{i=1}^a \tilde{d}_i^* + m_j^{-1} \sum_{i=1}^a \epsilon_i \tilde{P}_i^* b^*_i = \sum_{i=1}^a \tilde{d}_i^* + m_j^{-1} \sum_{i=1}^a \epsilon_i \lambda_i \tilde{P}_i^* e^*_\xi$$
Definition 2.7. Let $\gamma \in \tilde{\Gamma}$. Then the sequence $(I_i, \varepsilon_i, \lambda_i, \eta_i, \xi_i)_{i=1}^a$ satisfying all the above properties will be called the evaluation analysis of $\gamma$.

We define the bd-part and mt-part of $e_\gamma^*$ as

$$\text{bd}(e_\gamma^*) = \sum_{i=1}^a d_{\xi_i}^*, \quad \text{mt}(e_\gamma^*) = m_j^{-1} \sum_{i=1}^a \varepsilon_i \lambda_i \bar{P}_I e_\eta_i^*.$$  

Remark 2.9. For any $\xi \in \Gamma_q$ we have $\bar{P}_{\Delta_{\text{rank}}(\xi)}^* e_\gamma^* = \bar{d}_\xi^*.$

The $I$ (interval)-analysis of a functional $e_\gamma^*$.

Let $I \subset \mathbb{N}$ and $\gamma \in \Gamma$ with $\bar{P}_I^* e_\gamma^* \neq 0$. Let $w(\gamma) = m_j^{-1}, a \leq n_j$ and $(I_i, \varepsilon_i, \lambda_i, \eta_i, \xi_i)_{i=1}^a$ the evaluation analysis of $\gamma$. We define the $I$-analysis of $e_\gamma^*$ as follows:

(a) If there is at least one $i$ we have $\bar{P}_{I\cap I_i} e_\eta_i^* \neq 0$, then the $I$-analysis of $e_\gamma^*$ is of the following form

$$(I_i \cap I, \varepsilon_i, \lambda_i, \eta_i, \xi_i)_{i \in A_I},$$

where $A_I = \{i : \bar{P}_{I\cap I_i} e_\eta_i^* \neq 0\}$. In this case we say that $e_\gamma^*$ is $I$-decomposable.

(b) If $\bar{P}_{I\cap I_i} e_\eta_i^* = 0$ for all $i = 1, \ldots, a$, then we assign no $I$-analysis to $e_\gamma^*$ and we say that $e_\gamma^*$ is $I$-indecomposable.

Remark 2.9. Notice that in the second case above, as $I$ is interval and $\bar{P}_I^* e_\gamma^* \neq 0$, $\bar{P}_I^* e_\gamma^* = d_{\xi_0}^*$ for some $i_0 \in \{1, \ldots, a\}$. In other words, $e_\gamma^*$ is $I$-indecomposable iff $\bar{P}_I^* e_\gamma^* = d_{\xi_0}^*$ for some element $d_{\xi_0}^*$ of the bd-part of $e_\gamma^*$.

Now we introduce the tree-analysis of $e_\gamma^*$ analogous to the tree-analysis of a functional in a mixed Tsirelson space (see [10] Chapter II.1).

We start with some notation. We denote by $(T, \leq)$ a finite partially ordered set which is a tree. Its elements are finite sequences of natural numbers ordered by the initial segment partial order. For every $t \in T$ we denote by $S_t$ the set of immediate successors of $t$.

Let $\{I_t\}_{t \in T}$ be a tree of intervals of $\mathbb{N}$ such that $t \leq s$ if $I_t \supset I_s$ and $t, s$ are incomparable iff $I_t \cap I_s = \emptyset$. For such a family $\{I_t\}_{t \in T}$ and $t, s$ incomparable we write $t < s$ iff $I_t < I_s$ (i.e. $\max I_t < \min I_s$).

The tree-analysis of a functional $e_\gamma^*$.

Let $\gamma \in \tilde{\Gamma}$. A family of the form $(I_t, \varepsilon_t, \eta_t)_{t \in T}$ is called the tree-analysis of $e_\gamma^*$ if the following are satisfied:

1. $T$ is a finite tree with a unique root denoted by $\emptyset$.
2. We set $\eta_\emptyset = \gamma$, $I_\emptyset = (1, \max \Delta_{\text{rank}} \gamma)$, $\varepsilon_\emptyset = 1$ and let $(I_t, \varepsilon_t, \lambda_t, \eta_t, \xi_t)_{t \in T}$ be the evaluation analysis of $\eta_t$. Set $S_\emptyset = \{(1), (2), \ldots, (a)\}$ and for every $s = (i) \in S_\emptyset$, $(I_s, \varepsilon_s, \eta_s) = (I_i, \varepsilon_i, \eta_i)$.
3. Assume that for $t \in T$ the set $(I_t, \varepsilon_t, \eta_t)$ has been defined. Let $(I_t, \varepsilon_t, \lambda_t, \eta_t, \xi_t)$ be the evaluation analysis $e_\eta_t^*$. There are two cases:
   (a) If $e_\eta_t^*$ is $I_t$-decomposable, let $(I_t, \varepsilon_t, \lambda_t, \eta_t, \xi_t)_{t \in I_t}$ be the $I_t$-analysis of $e_\eta_t^*$. We set $S_t = \{(t^- i) : i \in A_t\}$. For every $s = (t^- i) \in S_t$, we set $(I_s, \varepsilon_s, \eta_s) = (I_i, \varepsilon_i, \eta_i)$.
   (b) If $e_\eta_t^*$ is $I_t$-indecomposable, then $t$ is a terminal node of the tree-analysis.

Definition 2.10. Given any $\gamma \in \Gamma$, in notation of Remark 2.9 let

$$\text{mt-sup} e_\gamma^* = \{\xi_t : t \in T, t \text{ terminal}\} = \{\xi_t : t \in T, \bar{P}_t e_\gamma^* = \bar{d}_t^*\}$$

and $\text{bd-sup} e_\gamma^* = \text{sup} e_\gamma^* \setminus \text{mt-sup} e_\gamma^*$.

3. Properties of the basis $(\bar{d}_\gamma^*)$.

We present here estimates on the averages of the basis $(\bar{d}_\gamma^*)_{\gamma \in \mathbb{N}}$. 

3.1. Neighbours nodes. The result of this section is crucial for the estimates in the sequel.

**Definition 3.1.** We shall call two nodes $\xi_1, \xi_2$ neighbours if there exists $\gamma \in \Gamma$ with $\text{bd}(e^*_\gamma) = \sum_{j=1}^a d^*_\gamma$ such that $\xi_i = \zeta_j$, for some $j_1 < j_2$.

Note that from the definitions it follows that if $\xi_1, \xi_2$ are neighbours then $w(\xi_1) = w(\xi_2)$.

**Lemma 3.2.** Let $(\bar{d}_{\gamma_n})_{n \in N}$ be a subsequence of the basis. Then there exists infinite $M \subset N$ such that no two nodes $\gamma_n, \gamma_m, n, m \in M$, are neighbours.

The proof is based on the fact that the age is uniquely determined for each node.

**Proof.** If there are infinitely many nodes with different weights we are done. So assume that for all but finite nodes it holds $w(\gamma_n) = m_k^{-1}$.

Applying Ramsey theorem we obtain an infinite set such that either no two nodes from this set are neighbours or any two are neighbours.

In the first case we are done. Otherwise passing to a further subsequence we may assume that $\text{rank}(\gamma_n) < \text{rng}(\gamma_{n+1})$ for every $n$.

Since we have that $\gamma_j, \gamma_j+1$ are neighbours it follows by a simple induction that

$$\text{age}(\gamma_{j+1}) \geq \text{age}(\gamma_j) + 1 \geq j + 1.$$ 

Take $j = n_k + 1$ and pick $e^*_\gamma$ of the form

$$e^*_\gamma = \sum_{r=1}^a d^*_\gamma + m_k^{-1} \sum_{r=1}^a \epsilon_r \lambda_r e^*_n P_{i_r}$$

with $d^*_{\gamma_k+1} = d^*_\gamma$, for some $r$. Then we have $w(\epsilon_r) \leq n_k$ which yields a contradiction and ends the proof. \hfill \square

3.2. Estimates on some averages of the basis. In [3] it is proved that the sequence $\sum_{\xi \in \Delta_n} d^*_\xi$ generates an $\ell_1$-spreading model in the space $X_K$. We show that if we take $y = n_j^{-1} \sum_{\xi \in \mathcal{F}} d^*_\xi$, where $\xi$’s are pairwise not neighbours the norm of such a vector is determined by the mt-part of the nodes.

In the sequel we shall use basic properties of mixed Tsirelson spaces. Recall that a mixed Tsirelson space $T[(A_{n_k}, m_k^{-1})_{k \in \mathbb{N}}]$ is the completion of $c_{00}$ with the norm defined by a norming set $D$ that contains the unit vectors $\{\pm e_n\}$ and satisfies for any $k \in \mathbb{N}$ the following condition: for any block sequences $f_1 < \cdots < f_d$, $d \leq n_k$, of elements of $D$ also the weighted average $m_k^{-1}(f_1 + \cdots + f_d)$ belongs to $D$. For further details see [5].

**Lemma 3.3.** Let $x = n_j^{-1} \sum_{i \in G} d^*_\xi$, be such that no two $\xi$’s are neighbours and $\#G \leq n_j$. Then for any $\gamma \in \Gamma$ with $w(e^*_\gamma) = m_k^{-1}$ we have the following

$$|e^*_\gamma(x)| \leq \begin{cases} \frac{1}{n_j} + \frac{2}{m_k} & \text{if } k \geq j \\ \frac{7}{m_k m_j} & \text{if } k < j \end{cases},$$

In particular

$$\|n_j^{-1} \sum_{i=1}^{n_j} d^*_\xi\| \leq 7m_j^{-1}.$$ 

**Proof.** We shall construct functionals $\phi_\gamma$ in the norming set of the mixed Tsirelson space $X_{aux} = T[(A_{n_k}, m_k^{-1})_{k \in \mathbb{N}}]$ such that

$$|e^*_\gamma(x)| \leq \phi_\gamma(y) + \frac{2}{m_j m_{j-1}}$$

where $y = 2 \sum_{k \in \mathbb{N}} e_k/n_j \in c_{00}(\mathbb{N})$.

Let $\gamma \in \Gamma$ with $e^*_\gamma = \sum_{r=1}^a d^*_\gamma + m_k^{-1} \sum_{r=1}^a \epsilon_r \lambda_r e^*_n P_{i_r}$. Let $g_\gamma = \text{bd}(e^*_\gamma)$ and $f_\gamma = \text{mt}(e^*_\gamma)$. 

We shall consider two cases.

**Case 1.** \( w(\gamma) \leq m_j^{-1} \).

Since the nodes \((\xi_i)_i\) are pairwise no neighbours and \((\beta_i)_i\) are pairwise neighbours it follows that

\[
|g_\gamma(x)| \leq n_j^{-1}.
\]

Also for every \( r \leq a \) using that \( |e^*_\eta P_{I_r}(x)| \leq 2 \) for all \( \zeta, \beta \), we get

\[
|e^*_\eta P_{I_r}(x)| \leq 2 \# \{ i : \text{rng}(\xi_i) \in I_r \} \frac{n_j}{n_j}.
\]

It follows from (3.1), (3.2), using that \( \eta \) and has room for \( \eta \) and that \( \lambda_r \leq 1 \) for every \( r \), that

\[
|e^*_\gamma(x)| \leq \frac{1}{n_j} + 2m_k^{-1} \sum_{r=1}^{a} \# \{ i : \text{rng}(\xi_i) \in I_r \} \frac{n_j}{n_j} \leq \frac{1}{n_j} + \frac{2}{m_k}.
\]

Taking \( \phi_\gamma = m_k^{-1} \sum_{n \in F} \eta_n^* \) where \( F = \bigcup_{r \leq a} \{ n \mid \gamma_n = \xi, \text{rng}(\xi_i) \in I_r \text{ for some } i \in G \} \) it follows that \( \# F \leq n_j \leq n_k \) and \( \phi_\gamma \) belongs to the norming set of the mixed Tsirelson space \( X_{aux} \).

From (3.3) we get

\[
|e^*_\gamma(x)| \leq \frac{1}{n_j} + 2m_k^{-1} \sum_{n \in F} \eta_n^* \frac{n_j}{n_j} = \frac{1}{n_j} + \phi_\gamma(y).
\]

**Case 2.** \( w(\gamma) = m_k^{-1} > m_j^{-1} \).

As in the previous case we get

\[
|g_\gamma(x)| \leq n_j^{-1}.
\]

Using that \( e^*_\gamma = g_\gamma + f_\gamma \) and \( \lambda_r \leq 1 \) for every \( r \), we get

\[
|e^*_\gamma(x)| \leq n_j^{-1} + |f_\gamma(x)| \leq n_j^{-1} + m_k^{-1} \sum_{r=1}^{a} |e^*_\eta P_{I_r}(x)|.
\]

We shall split now the successors \( e^*_\eta \) of \( e^*_\gamma \) into those with weight smaller or equal to \( m_j^{-1} \) and those with weight bigger that \( m_j^{-1} \). For a node \( \gamma \) we set

\[
S_{\gamma,1} = \{ r \in S_\gamma : w(\eta_r) \leq m_j^{-1} \} \quad \text{and} \quad S_{\gamma,2} = S_\gamma \setminus S_{\gamma,1}.
\]

From (3.6) we get

\[
|e^*_\gamma(x)| \leq n_j^{-1} + m_k^{-1} \left( \sum_{r \in S_{\gamma,1}} |e^*_\eta P_{I_r}(x)| + \sum_{r \in S_{\gamma,2}} |e^*_\eta P_{I_r}(x)| \right).
\]

Using (3.4) for the \( r \in S_{\gamma,1} \), (3.6) for the \( r \in S_{\gamma,2} \) and that \( \# S_{\gamma,1} + \# S_{\gamma,2} \leq n_k, k < j \), we get the following

\[
|e^*_\gamma(x)| \leq n_j^{-1} + \frac{n_k}{m_k n_j} + \frac{1}{m_k} \left( \sum_{r \in S_{\gamma,1}} \phi_r(y) + \sum_{r \in S_{\gamma,2}} w(\eta_r) \sum_{s \in S_r} |e^*_\eta P_{I_s}(x)| \right)
\]

\[
\leq \frac{1}{n_j} (1 + \frac{n_j^{-1}}{m_j^{-1}}) + \frac{1}{m_k} \left( \sum_{r \in S_{\gamma,1}} \phi_r(y) + \sum_{r \in S_{\gamma,2}} w(\eta_r) \sum_{s \in S_r} |e^*_\eta P_{I_s}(x)| \right).
\]

Note that the functional \( m_k^{-1} \left( \sum_{r \in S_{\gamma,1}} \phi_r \right) \) belongs to the norming set of the mixed Tsirelson space \( X_{aux} \) and has room for \( \# S_{\gamma,2} \) more functionals.
We shall replay the above splitting for every $e_{n_j}^* P_{I_s}$. To avoid complicated notation we shall set $n_s = \#S_s$ and $m_s^{-1} = w(e_{n_j}^*)$. From (3.7) using $e_{n_j}^* P_{I_s}$ in the place of $e^*_\gamma$ we get

\begin{equation}
|e_{n_j}^* P_{I_s}(x)| \leq \frac{1}{n_j} (1 + \frac{n_j - 1}{m_j - 1}) + m_s^{-1} \left( \sum_{t \in S_{s,1}} \phi_t(y) + \sum_{t \in S_{s,2}} m_t^{-1} \sum_{u \in S_t} |e_{n_j}^* P_{I_u}(x)| \right).
\end{equation}

It follows that

\begin{equation}
\sum_{r \in S_{s,2}} w(e_{n_r}) \sum_{s \in S_r} |e_{n_r}^* P_{I_s}(x)| \leq \sum_{r \in S_{s,2}} m_r^{-1} \sum_{s \in S_r} \frac{1}{n_j} (1 + \frac{n_j - 1}{m_j - 1})
\begin{align*}
&+ \sum_{r \in S_{s,2}} m_r^{-1} \sum_{s \in S_r} m_s^{-1} \left( \sum_{t \in S_{s,1}} \phi_t(y) + \sum_{t \in S_{s,2}} m_t^{-1} \sum_{u \in S_t} |e_{n_r}^* P_{I_u}(x)| \right) \\
&\leq n_k \frac{n_r}{m_r} \frac{1}{n_j} (1 + \frac{n_j - 1}{m_j - 1}) \text{ since } \#S_{s,2} \leq n_k \text{ and } \#S_r \leq n_r \\
&+ \sum_{r \in S_{s,2}} m_r^{-1} \sum_{s \in S_r} m_s^{-1} \left( \sum_{t \in S_{s,1}} \phi_t(y) + \sum_{t \in S_{s,2}} m_t^{-1} \sum_{u \in S_t} |e_{n_r}^* P_{I_u}(x)| \right).
\end{align*}
\end{equation}

By (3.8) and (3.9), using that $\frac{m_r}{m_r - 1} \leq \frac{n_r}{m_j - 1}$ we get

\begin{equation}
|e^*_\gamma(x)| \leq \frac{1}{n_j} (1 + \frac{n_j - 1}{m_j - 1}) + (\frac{n_j - 1}{m_j - 1})^2 + (\frac{n_j - 1}{m_j - 1})^3
\begin{align*}
&+ \frac{1}{n_k} \left( \sum_{r \in S_{s,1}} \phi_r(y) + \sum_{r \in S_{s,2}} m_r^{-1} \sum_{s \in S_r} m_s^{-1} \left( \sum_{t \in S_{s,1}} \phi_t(y) + \sum_{t \in S_{s,2}} m_t^{-1} \sum_{u \in S_t} |e_{n_r}^* P_{I_u}(x)| \right) \right).
\end{align*}
\end{equation}

Note that the functional

$$
\phi_\gamma = \frac{1}{n_k} \left( \sum_{r \in S_{s,1}} \phi_r(y) + \sum_{r \in S_{s,2}} m_r^{-1} \sum_{s \in S_r} m_s^{-1} \sum_{t \in S_{s,1}} \phi_t(y) \right)
$$

belongs to the norming set of the mixed Tsirelson space $X_{aux}$ and the functional $m_s^{-1} \sum_{t \in S_{s,1}} \phi_t$ has room for $\#S_{s,2}$ more functionals.

We continue this splitting at most $l_j$ times, see (2.1) for the choice of $l_j$, or till $S_{s,2} = \emptyset$ i.e. we do not have nodes with weight $> m_j^{-1}$.

If we stop before the $l_j$-step we get that $|e^*_\gamma(x)|$ is dominated by $\phi_\gamma(y)$ plus the errors in (3.10), where the sum end to the $l_j$-power of $n_j - 1/m_j - 1$. Since $\phi_\gamma$ belongs to the norming set of the mixed Tsirelson space $X_{aux}$ it follows from [6], Lemma II.9, that

$$
\phi_\gamma(y) \leq 4m_k^{-1}m_j^{-1}.
$$

If we continue the splitting $l_j$-times, then there exists some node with $w(\gamma_t) > m_j^{-1}$. For every such node we have

$$
\left( \prod_{s < t} w(e_{n_s}) \right) |e^*_\gamma(x)| \leq \left( \frac{1}{m_1} \right)^{l_j} |e^*_\gamma(x)| \leq 2m_k^{-1}m_j^{-1} \#\{i : \text{rng}(\xi_i) \in I_t \cap G\}
$$

since $m_1^{-l_j} \leq (m_j m_j - 1)^{-1}$, see (2.1).

Summing the estimation of all those nodes we get upper estimate equal to $2\#G / m_n m_n \leq 2/m_n m_j$.

The remaining nodes provide us with a functional in the norming set of the mixed Tsirelson space $X_{aux}$. By [6] it is bounded by $4m_k^{-1}m_j^{-1}$. 
Corollary 3.5. Let \( x = m_j n_j^{-1} \sum_{i=1}^{n_j} d_{xi} \) such that no two \( \xi_i \)'s are neighbours. Let \( i < j, (e^*_n)_p \) nodes such that \( w(e^*_n) = m_t \neq m_j \) and \( m_t < m_{t+1} \) for all \( p \leq n_i \). Then

\[
\sum_{p=1}^{n_i} |e^*_n P_{tp}(x)| \leq 14 \frac{1}{m_{p1}}.
\]

Proof. From (3.3) we get

\[
\sum_{p=1}^{n_i} |e^*_n P_{tp}(x)| \leq \sum_{p \not< j} \frac{7}{m_p} + \sum_{p \not> j} \frac{1}{n_j} + \frac{2m_j}{m_p} \\
\leq \sum_{p \not< j} \frac{7}{m_p} + \sum_{p \not> j} \frac{n_i}{n_j} + \frac{2}{m_p-1} \leq 14 \frac{1}{m_{p1}}.
\]

\[\square\]

4. The space \( X_{Kus} \)

In this section we define the space \( X_{Kus} \). We shall need the following notion from [5].

**Definition 4.1.** Let \( X \) be a BD-\( L_\infty \)-subspace of \( \ell_\infty(\Gamma) \). A subset \( \Gamma \) of \( \Gamma \) is called self-determined provided \( \{ \langle d^*_\gamma : \gamma \in \Gamma \rangle \} = \{ \langle e^*_\gamma : \gamma \in \Gamma \rangle \} \), where \( (d^*_\gamma)_{\gamma \in \Gamma} \) denotes the biorthogonal sequence to the basis \( (\tilde{d}^*_\gamma)_{\gamma \in \Gamma} \) and for \( \gamma \in \Gamma \), \( e^*_\gamma \) denotes the element \( e_\gamma \) of \( \ell_1(\Gamma) \) restricted to \( X \).

Now we proceed to the choice of a self-determined subset \( \Gamma \) of \( \tilde{\Gamma} \) which will determine the space \( X_{Kus} \). This set will consist of regular and special nodes.

We introduce first the notion which will describe the "freedom" in choosing special nodes. For any \( \gamma \in \tilde{\Gamma} \) we write \( \text{rank}(\text{bd}(e^*_\gamma)) = \{ \text{rank} \xi_i, i \in A \} \), where \( \text{bd}(e^*_\gamma) = \sum_{i \in A} d^*_\xi_i \).

**Definition 4.2.** We say that the functionals \( e^*_\gamma, e^*_\tilde{\gamma} \), \( \gamma, \tilde{\gamma} \in \tilde{\Gamma} \), have compatible tree-analyses if

\begin{enumerate}[label=(CT\arabic*), align=left]
  \item \( e^*_\gamma, e^*_\tilde{\gamma} \) have tree-analyses \( (I_t, e_t, \tilde{\eta}_t)_{t \in \mathcal{T}} \), \( (I_t, \tilde{\xi}_t, \tilde{\eta}_t)_{t \in \mathcal{T}} \) respectively,
  \item \( w(\eta_t) = w(\tilde{\eta}_t) \) for any \( t \in \mathcal{T} \),
  \item \( \text{mt-sup} e^*_\eta = \text{mt-sup} e^*_\tilde{\eta} \) for any \( t \in \mathcal{T} \),
  \item \( \text{rank}(\eta_t) = \text{rank}(\tilde{\eta}_t) \) for any \( t \in \mathcal{T} \),
  \item \( \text{rank}(\text{bd}(e^*_\eta)) = \text{rank}(\text{bd}(e^*_\tilde{\eta})) \) for any \( t \in \mathcal{T} \).
\end{enumerate}

For every \( \gamma = (q + 1, \xi, m_k, \epsilon, I, e^*_\theta) \in \tilde{\Gamma} \) and \( x \in \tilde{X}_\Gamma \) we set

\[
\lambda_{\gamma,x} = \begin{cases}
  e^*_\eta(x) & \text{if } e^*_\eta(x) \neq 0 \\
  e_n^{-2} & \text{otherwise.}
\end{cases}
\]

For every \( q \in \mathbb{N} \) let \( \text{Net}_{2,q} \) be a \( 1/n^2_q \) net in the unit ball of \( \{ \langle \tilde{d}_\gamma : \gamma \in \tilde{\Gamma}_q \rangle \} \) containing all averages \( \frac{1}{n_r} \sum_{s=1}^{n_r} \tilde{d}_{\gamma_s} \) with \( r \leq q \), and \( \gamma_s \in \tilde{\Gamma}_q \) for \( s = 1, \ldots, n_r \).
Definition 4.3 (The tree of the special sequences). We denote by $\mathcal{Q}$ the set of all finite sequences of pairs $\{(\zeta_1, \bar{x}_1), \ldots, (\zeta_k, \bar{x}_k)\}$ satisfying the following:

(i) $\zeta_i \in \bar{\Gamma}$ with $\text{rank}(\zeta_i) = q_i \geq \min \text{rng}_{FDD} \bar{x}_i$ for $i = 1, \ldots, k$

(ii) $\bar{x}_i \in \text{Net}_{\ell, q_i}$ are successive with respect to the FDD $(M_q)_q$.

We choose a one-to-one function $\sigma: \mathcal{Q} \rightarrow \mathbb{N}$, called the coding function, so that

\[
\sigma(\{(\zeta_1, \bar{x}_1), \ldots, (\zeta_k, \bar{x}_k)\}) \geq w(\bar{x}_k)^{-1} \max \text{supp}_{FDD} \bar{x}_k \ \forall \{(\zeta_1, \bar{x}_1), \ldots, (\zeta_k, \bar{x}_k)\} \in \mathcal{Q}.
\]

Definition 4.4. A finite sequence $(\zeta_1, \bar{x}_1)^{d-1}_{i=1} \in \mathcal{Q}$ is called a $j$-special sequence, $j \in \mathbb{N}$, if $d \leq n_{2j-1}$ and the following conditions are satisfied.

(i) $\zeta_1 = (q_1 + 1, m_{2j-1}, I_1, e_1)\|e^*_n)$ and $\zeta_i = (q_i + 1, \zeta_{i-1}, m_{2j-1}, I_i, \epsilon_i, \lambda_i e^*_n)$ for every $i \leq d$,

(ii) $w(\eta_1) = m_4^{-1} - n_{2j-2}$ and $w(\eta_1) = m_4^{i-1}$ for $i = 2, \ldots, d$.

(iii) if $i$ is odd then $\lambda_i = 1$,

(iv) if $i$ is even then $\epsilon_i = 1$.

We denote by $\mathcal{U}$ the tree of all special sequences, endowed with the natural ordering “$\subseteq$” of initial segments.

Fix $\Gamma = \cup q \Gamma_q$, $\Gamma_q \subset \bar{\Gamma}_q$. A $j$-special sequence $(\zeta_1, \bar{x}_1)$, with $\zeta_1 = (q + 1, m_{2j-1}, I_1, \epsilon_1)\|e^*_n)$ is called $(\Gamma, j)$-special if $\eta \in \Gamma_q$. A $j$-special sequence $(\zeta_1, \bar{x}_1)^{d-1}_{i=1}, d \leq n_{2j-1}$, with $\zeta_i = (q_i + 1, \zeta_{i-1}, m_{2j-1}, I_i, \epsilon_i, \lambda_i e^*_n)$ is called $(\Gamma, j)$-special if $\eta_d \in \Gamma_q \setminus \Gamma_{q-1}$, $\zeta_d \in \Gamma_{q-1}$ and $(\zeta_1, \bar{x}_1)^{d-1}_{i=1}$ is a $(\Gamma_q, j)$-special sequence.

Now we are ready to define inductively on $q \in \mathbb{N}$ the families of nodes $(\Delta_q)_q$ and $(\Gamma_q)_q$ satisfying $\Delta_q \subset \bar{\Delta}_q$ and $\Gamma_q = \cup_{p=1}^{\tilde{q}} \Delta_p$ for any $q \in \mathbb{N}$.

Set $\Gamma_1 = \Delta_1 = \bar{\Delta}_1$. Fix $q \in \mathbb{N}$ and assume we have defined all objects up to $q$-th level.

The set of regular nodes is defined as

\[
\Delta_{q+1}^{reg} = \bigcup_{j=1}^{\lfloor (q+1)/2 \rfloor} \{(q + 1, m_{2j}, I, \epsilon, e^*_n) \in \bar{\Delta}_{q+1} : \eta \in \Gamma_q\}

\cup \bigcup_{1 \leq p < q} \bigcup_{j=1}^{\lfloor (q+1)/2 \rfloor} \{(q + 1, \xi, m_{2j}, I, \epsilon, e^*_n) \in \bar{\Delta}_{q+1} : \xi \in \Delta_p, \eta \in \Gamma_q \setminus \Gamma_p\}
\]

Now we define the special nodes, i.e. the nodes compatible to the special sequences defined above (counterparts of special functionals in [1]).

Definition 4.5. We say that a node $\gamma = (q + 1, m_{2j-1}, I, \epsilon, e^*_n) \in \bar{\Delta}_{q+1}$ is compatible with a $(\Gamma_q, j)$-special sequence $(\zeta_1, \bar{x}_1)$, where $\zeta_1 = (q + 1, m_{2j-1}, I, \epsilon_0, e^*_n)$, if $\eta \in \Gamma_q$ and $\eta, \eta_0$ have compatible tree-analyses.

We say that a node $\gamma = (q + 1, \xi, m_{2j-1}, I, \epsilon, \lambda e^*_n) \in \bar{\Delta}_{q+1}$ is compatible with a $(\Gamma_q, j)$-special sequence $(\zeta_i, \bar{x}_i)_{i=1}^{q\text{age}(\gamma)}$, where $\zeta_{\text{age}(\gamma)} = (q + 1, \zeta_{\text{age}(\gamma)-1}, m_{2j-1}, I_{\text{age}(\gamma)}, 1, \lambda_{\text{age}(\gamma)} e^*_n_{\text{age}(\gamma)})$, provided

1. $\eta, \xi \in \Gamma_q$,
Remark 4.8. This follows readily from the definition of $\bar{c}$.

Remark 4.9. According to Proposition 1.5 \cite{5} it is enough to show that for every $q \in \mathbb{N}$ and $i \in \Gamma$ we let $\gamma = (q + 1, 0, m_{2j-1}, I, \epsilon, e^*_\eta) \in \Delta_{q+1} : \gamma$ is compatible with some

$$
\text{(4.3)}
\Delta^{sp}_{q+1} = \bigcup_{j=1}^{\lfloor (q+1)/2 \rfloor} \{ \gamma = (q + 1, 0, m_{2j-1}, I, \epsilon, e^*_\eta) \in \Delta_{q+1} : \gamma \text{ is compatible with some} \}
$$

$$(\Gamma_q, j)-\text{special sequence } (\xi_1, x_1) \}$$

$$
\cup \bigcup_{p=1}^{q} \bigcup_{j=1}^{\lfloor (q+1)/2 \rfloor} \{ \gamma = (q + 1, \xi, m_{2j-1}, I, \epsilon, \lambda e^*_\eta) \in \Delta_{q+1} : \gamma \text{ is compatible with some} \}
$$

$$(\Gamma_q, j)-\text{special sequence } (\xi_i, x_i)_{i=1}^{\text{age(\gamma)}} \} .$$

Finally we set

$$
\Delta_{q+1} = \Delta^{reg}_{q+1} \cup \Delta^{sp}_{q+1} \text{ and } \Gamma_{q+1} = \Gamma_q \cup \Delta_{q+1}
$$

Obviously $\Delta_q \subset \Delta_q$ for any $q \in \mathbb{N}$. We set $\Gamma = \cup_{q} \Gamma_q$. Following \cite{5} we denote by $R$ the restriction on $\mathcal{X}_\Gamma$ of the restriction operator $\ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ and for any $q \in \mathbb{N}$ we let $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$ be defined by $i_q(x) = R(i_q(x))$ for any $x$. Given any $q \in \mathbb{N}$ we let $M_q = \max_{1 \leq i \leq \text{max} \Gamma_q} |\ell_\infty(\Gamma_q)|$.

Proposition 4.6. The set $\Gamma$ is a self-determined subset of $\Gamma$, hence it defines a BD-$\mathcal{L}_\infty$-space $\mathcal{X}(\Gamma_q, i_q)_{q \in \mathbb{N}}$.

Moreover, the restriction $R : \mathcal{X}_\Gamma \to \mathcal{X}(\Gamma_q, i_q)_{q \in \mathbb{N}}$ is a well-defined operator of norm at most 1 inducing the isomorphism between $\mathcal{X}(\Gamma_q, i_q)_{q \in \mathbb{N}}$ and $\mathcal{X}_\Gamma/Y$, where $Y = \langle \{ d_\gamma : \gamma \in \Gamma \setminus \Gamma \} \rangle$.

Proof. According to Proposition 1.5 \cite{5} it is enough to show that for every $\gamma \in \Delta_{q+1}$ it holds

$$
\bar{c}_\gamma^* \in \{ e^*_\gamma \circ P_E : \gamma \in \Gamma_q, E \subset \mathbb{N} \cup \{0\} \}
$$

This follows readily from the definition of $\bar{c}_\gamma^*$, see \cite{2.2}, using that $\hat{d}_\gamma^* = e^*_\gamma \circ P_{\{\text{rank(\gamma)}\}}$.

The second part of Proposition follows by Proposition 1.9 \cite{3}.

Definition 4.7. We let $\mathcal{X}_{Kus} = \mathcal{X}(\Gamma_q, i_q)_{q \in \mathbb{N}}$.

In the sequel we shall use the usual notation $e^*_\gamma, d^*_\gamma, \bar{d}^*_\gamma, \bar{d}_\gamma$ etc for the objects in the space $\mathcal{X}_{Kus}$.

Remark 4.8. Notice that all the results from Section 3 are valid also in the space $\mathcal{X}_{Kus}$, as $d_\gamma = Rd_\gamma, R^*e^*_\gamma = e^*_\gamma, \gamma \in \Gamma$ by Remark 1.11 \cite{5} and $\|R\| = 1$.

By Proposition 1.13 \cite{4} we can use the analysis of nodes introduced in Section 2 in the space $\mathcal{X}_{Kus}$ and write the precise form of each $e^*_\gamma$ depending on the type of the node $\gamma \in \Gamma$.

Remark 4.9. Let a node $\gamma$ has the evaluation analysis $(I_i, \epsilon_i, e^*_\eta_i, \xi_i)_{i=1}^a$. Then

1. if $w(\gamma) = m_{2j}^{-1}$ then

$$
e^*_\gamma = \sum_{i=1}^a d^*_\xi_i + \frac{1}{m_{2j}} \sum_{i=1}^a \epsilon_i P^*_I \bar{e}^*_\eta_i,$$
Remark 4.10. (1) Fix \((\eta_s)_{s=1,\ldots,a}\), with \(a \leq n_{2j} - 2j \leq q_1\), \(\eta_s \in \Gamma_{q_s} \setminus \Gamma_{p_s}\), \(s = 1, \ldots, a\), \(p_1 < q_1 < p_2 < q_2 < \ldots\), and \((I_s)_{s=1}^a\) with \(I_s \subset \Gamma_{q_s} \setminus \Gamma_{p_s}\), \(P_{I_s}^* e_{\eta_s}^* \neq 0\) and \((\varepsilon_s)_{s=1,\ldots,a} \subset \{\pm 1\}\).

Then the formulas \(\xi_1 = (q_1 + 1, m_{2j}, I_1, \varepsilon_1, e_{\eta_1}^*)\) and \(\xi_s = (q_s + 1, \xi_{s-1}, m_{2j}, I_s, \varepsilon_s, e_{\eta_s}^*)\) for any \(s \leq a\) give well-defined regular nodes.

It follows that for any functional \(e_\gamma^*\) given by a regular node \(\gamma\) with

\[
\text{mt}(e_\gamma^*) = \frac{1}{m_{2j}} \sum_{i=1}^a \varepsilon_i e_{\eta_i}^* P_{I_i}
\]

and any \((\tilde{\varepsilon}_i)_{i\leq a} \subset \{\pm 1\}\) and any \((\tilde{\eta}_i)_{i\leq a}\) with \(\text{rank}(\tilde{\eta}_i) = \text{rank}(\eta_i)\) and \(P_{I_i}^* e_{\eta_i}^* \neq 0\) there is a regular node \(\tilde{\gamma}\) with

\[
\text{mt}(e_{\tilde{\gamma}}^*) = \frac{1}{m_{2j}} \sum_{i=1}^a \tilde{\varepsilon}_i e_{\tilde{\eta}_i}^* P_{I_i} \quad \text{and} \quad \text{rank}(\tilde{\gamma}) = \text{rank}(\gamma).
\]

(2) Take a functional \(e_\gamma^*\) where \(\gamma\) which is compatible with a \(j\)-special sequence, with

\[
\text{mt}(e_\gamma^*) = \frac{1}{m_{2j-1}} \left( \sum_{i=1}^{[a/2]} (\varepsilon_{2i-1} e_{\eta_{2i-1}}^* P_{I_{2i-1}} + \lambda_{2i} e_{\eta_{2i}}^* P_{I_{2i}}) + [\varepsilon a e_{\eta_a}^* P_{I_a}] \right),
\]

evaluation analysis \((I_i, \varepsilon_i, e_{\eta_i}^*, \xi_i)_{i=1}^a\) and weight \(w(\gamma) = m_{2j-1}^{-1}\). Let \((\tilde{\varepsilon}_i)_{i=1}^a\) and \((\tilde{\eta}_i)_{i=1}^a\) satisfy the following:

(i) if \(i\) is even then \(\tilde{\varepsilon}_i = 1\), \(\tilde{\eta}_i = \eta_i\),

(ii) if \(i\) is odd then \(\tilde{\eta}_i\) has compatible tree-analysis with \(\eta_i\).

Then the formulas \(\tilde{\xi}_1 = (q_1 + 1, m_{2j-1}, I_1, \tilde{\varepsilon}_1, \lambda_1 e_{\eta_1}^*)\) and \(\tilde{\xi}_i = (q_i + 1, \tilde{\xi}_{i-1}, m_{2j-1}, I_i, \tilde{\varepsilon}_i, \tilde{\lambda}_i e_{\tilde{\eta}_i}^*)\), \(i \leq a\), give well-defined special nodes. Indeed, it follows from direct application of the definition of a special node. It follows that the node \(e_{\tilde{\gamma}}^*\) with

\[
\text{mt}(e_{\tilde{\gamma}}^*) = \frac{1}{m_{2j-1}} \left( \sum_{i=1}^{[a/2]} (\tilde{\varepsilon}_{2i-1} e_{\tilde{\eta}_{2i-1}}^* P_{I_{2i-1}} + \tilde{\lambda}_{2i} e_{\tilde{\eta}_{2i}}^* P_{I_{2i}}) + [\varepsilon a e_{\eta_a}^* P_{I_a}] \right)
\]

where \(\tilde{\lambda}_{2i}\) are chosen according to Def. 4.10(4), is compatible with the \(j\)-special sequence which \(e_\gamma^*\) is compatible and hence is a special node with the same rank with \(e_{\tilde{\gamma}}^*\).

Remark 4.11. Notice that by the definition of \(\Gamma\), for any \(\gamma \in \Gamma\) with a tree-analysis \((I_t, \varepsilon_t, \eta_t)_{t \in \mathcal{T}}\) we have \(\eta_t \in \Gamma\) and \(\text{supp bd}(\eta_t) \subset \Gamma\) for any \(t \in \mathcal{T}\).
5. **Rapidly Increasing Sequences**

From now on we work in the space $\mathcal{X}_{Kus}$. In this section we introduce the basic classical tool, i.e. Rapidly Increasing Sequences and state their properties, in particular the fundamental property of Bourgain-Delbaen spaces in the Argyros-Haydon setting that allows to pass from strictly singular operators to compact ones. As the proofs of all the results stated here follows directly the reasoning of [3], we do not present them here.

Recall that skipped block sequences are defined with respect to the FDD $(M_q)_{q\in\mathbb{N}}$.

**Definition 5.1.** Let $I$ be an interval in $\mathbb{N}$ and $(x_k)_{k\in I} \subset \mathcal{X}_{Kus}$ be a skipped block sequence. We shall say that $(x_k)_{k\in I}$ is a Rapidly Increasing Sequence with constant $C > 0$ (C-RIS) if there exists an increasing sequence $(j_k)_{n\in I} \subset \mathbb{N}$ such that

1. $\|x_k\| \leq C$ for all $k \in I$,
2. $\text{rng}_{FDD} x_k \subset j_{k+1}$,
3. $|x_k(\gamma)| \leq C m_i^{-1}$ for all $\gamma$ with $w(\gamma) = m_i^{-1}$ and $i < j_k$.

**Lemma 5.2** (Proposition 5.6 [3]). Let $(x_k)_{k=1}^{n_j}$ be a C-RIS and $s \in \mathbb{N}$.

1. If $\gamma \in \Gamma$ and $w(\gamma) = m_i^{-1}$ then

\[
|c^*_\gamma P_{(s, +\infty)} \left( \frac{1}{n_{j_0}} \sum_{k=1}^{n_j} x_k \right) | \leq \begin{cases} 
16C m_i^{-1} m_{j_0}^{-1} & \text{if } i < j_0 \\
5C n_i^{-1} + 6C m_i^{-1} & \text{if } i \geq j_0.
\end{cases}
\]

In particular for $i > j_0$ we have

\[
|c^*_\gamma \left( \frac{1}{n_{j_0}} \sum_{k=1}^{n_j} x_k \right) | \leq 10C m_{j_0}^{-2}
\]

and also

\[
\| \frac{1}{n_{j_0}} \sum_{k=1}^{n_j} x_k \| \leq 10C m_{j_0}^{-1}.
\]

2. If $\lambda_k, 1 \leq k \leq n_{j_0}$ are scalars with $|\lambda_k| \leq 1$, satisfying the property

\[
|c^*_\gamma \left( \sum_{k \in J} \lambda_k x_k \right) | \leq C \max_{k \in J} |\lambda_k|
\]

for every $\gamma \in \Gamma$ with $w(\gamma) = m_{j_0}^{-1}$ and every interval $J \subset \{1, \ldots, n_{j_0}\}$ then we have

\[
\| \frac{1}{n_{j_0}} \sum_{k=1}^{n_j} \lambda_k x_k \| \leq \frac{10C}{m_{j_0}^2}.
\]

**Corollary 5.3.** Let $i < j \in \mathbb{N}$, $(x_k)_{k=1}^{n_j}$ be a C-RIS, $x = \frac{m_i}{n_j} \sum_{k=1}^{n_j} x_k$ and $(e^*_{\eta_p})_{p=1}^{n_i}$ be nodes such that $w(e^*_{\eta_p}) = m_{i_p}^{-1}$ and $m_{i_p} \neq m_j, m_{i_p} < m_{i_{p+1}}$ for all $p \leq n_i$. Then for every choice of intervals $I_p, p \leq n_i$, we have

\[
\sum_{p=1}^{n_i} |e^*_{\eta_p} (P_p x) | \leq 64C/m_{p1}.
\]

**Lemma 5.4** (Corollary 8.5 [3]). For every block subspace $Y \subset \mathcal{X}_{Kus}$, $C > 2$ and every interval $J \subset \mathbb{N}$ there exists a normalized C-RIS $(x_k)_{k \in J}$ in $Y$. Moreover, for any $\varepsilon > 0$ and $C > 2$ the sequence $(x_k)_{k \in J}$ can be chosen to satisfy $|d^*_\gamma (x_k) | < \varepsilon$ for any $k \in J$ and $\gamma \in \Gamma$. 

Notice that if \( x \in \oplus_{n=1}^{q} M_n \) with \( q \) minimal then there exists a unique \( u \in \ell_{\infty}(\Gamma_q) \) such that \( i_q(u) = x \). The local support of \( x \) is defined to be the set \( \{ \gamma \in \Gamma_q \mid u(\gamma) \neq 0 \} \). Next results are again quoted from [3].

**Lemma 5.5** (Lemma 5.7 [3]). Let \( \gamma \in \Gamma \) be of weight \( m^{-1}_h \) and assume that \( w(\xi) \neq m^{-1}_h \) for all \( \xi \) in the local support of \( x \). Then \( |x(\gamma)| \leq 4m^{-1}_h \|x\| \).

We recall the two classes of block sequences, characterised by the weights of the elements of the local support.

**Definition 5.6** (Definition 5.8 [3]). We say that a block sequence \((x_k)_{k \in \mathbb{N}}\) in \( \mathcal{X}_{Kus} \) has bounded local weight if there exists some \( j_1 \) such that \( w(\gamma) \geq m^{-1}_{j_1} \) for all \( \gamma \) in the local support of \( x_k \), and all values of \( k \). We say that \((x_k)_{k \in \mathbb{N}}\) has rapidly increasing local weight if, for each \( k \) and each \( \gamma \) in the local support of \( x_{k+1} \), we have \( w(\gamma) < m^{-1}_{i_k} \) where \( i_k = \max \text{rng}_{FDD} x_k \).

**Proposition 5.7** (Prop. 5.9 [3]). Let \((x_k)_{k \in \mathbb{N}} \subset \mathcal{X}_{Kus} \) be a bounded block sequence. If either \((x_k)\) has bounded local weight, or \((x_k)\) has rapidly increasing local weight, then the sequence \((x_k)\) is a RIS.

**Proposition 5.8** (Proof of Prop. 5.10 [3]). Any bounded block sequence \((x_n) \subset \mathcal{X}_{Kus} \) has a subsequence of the form \( x_{k_n} = y_n + z_n \), \( n \in \mathbb{N} \), where both \((y_n)\) and \((z_n)\) are RIS.

**Corollary 5.9** (Prop. 5.10 [3]). Let \( Y \) be any Banach space and \( T : \mathcal{X}_{Kus} \to Y \) be a bounded linear operator. If \( \|Tx_k\| \to 0 \) for every RIS \((x_k)\) in \( \mathcal{X}_{Kus} \) then \( \|Tx_k\| \to 0 \) for every bounded block sequence \((x_k)\) in \( \mathcal{X}_{Kus} \).

**Corollary 5.10** (Prop. 5.11 [3]). The basis \( (d^{\ast}_{\gamma_n})_n \) is shrinking. It follows that the dual space to \( \mathcal{X}_{Kus} \) is isomorphic to \( \ell_1(\Gamma) \).

### 6. Dependent sequences

In this section we introduce the classical tools in the study of spaces defined with the use of saturated norms.

**Lemma 6.1.** a) Let \( j \in \mathbb{N} \) and \( k \leq n_{2j} \). Let also \((x_k)_{k \in \mathcal{X}_{Kus}} \) be a normalized skipped block sequence such that \( \text{rng}_{FDD}(x_k) = (p_{k-1},p_k) \) for some strictly increasing \((p_k)\). Then there exists a node \( \gamma \in \Gamma \) such that

\[
epsilon^{\ast}_\gamma = \sum_{k=1}^{n_{2j}} d^{\ast}_{\xi_k} + m^{-1}_{2j} \sum_{k=1}^{n_{2j}} \epsilon_k e^{\ast}_{\eta_k} P_k
\]

with the following properties

(i) \( \text{rank}(\xi_k) = p_k + 1 \) for each \( k \),
(ii) \( \epsilon_k e^{\ast}_{\eta_k} P_k (x_k) \geq 1/2 \) and \( \eta_k \in \Gamma_{p_k} \setminus \Gamma_{p_k-1} \) for each \( k \),
(iii) \( \epsilon^{\ast}_\gamma (\sum_{k=1}^{n_{2j}} x_k) \geq m_{n_{2j}} \).

b) Let \( (d^{\ast}_{\xi_i})_{i=1}^{n_{2j}} \) be a finite subsequence of the basis such that \( \text{rank}(\xi_i) + 1 < \text{rank}(\xi_{i+1}) \) for every \( i \).

Then the node

\[
epsilon^{\ast}_{\xi} = \sum_{i=1}^{n_{2j}} d^{\ast}_{\xi_i} + m^{-1}_{2j} \sum_{i=1}^{n_{2j}} d^{\ast}_{\xi_i}
\]

with \( \text{rank}(\xi) = \text{rank}(\xi_k) + 1 \) is a regular node and \( e^{\ast}_{\xi} (\sum_{i=1}^{n_{2j}} d_{\xi}) = \frac{n_{2j}}{m_{2j}} \).

**Proof.** a) (see [3], Proposition 4.8) Let \( x_k = i_k(u_k) \) where \( u_k \in \Gamma_{p_k} \setminus \Gamma_{p_k-1} \) is the restriction of \( x_k \) on \( \Gamma_{p_k} \). Since

\[2\|u_k\| \geq \|i_{p_k}(u_k)\| = \|x_k\| = 1\]
Moreover, we say that a sequence (6.3) 

Lemma 6.4. Let $x_k$ such that 

Passing to a further subsequence we may assume that 

Proof.

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\[ \|e^{\ast}_{\eta_k}(x_k)\| = \varepsilon_k e^{\ast}_{\eta_k}(x_k) = \varepsilon_k e^{\ast}_{\eta_k}(u_k) \geq 1/2. \]

The nodes $\gamma_k = (p_k + 1, \gamma_{k-1}, m_{2j}, I_k, \varepsilon_k, e^{\ast}_{\eta_k}), \gamma_0 = 0, k = 1, \ldots, n_{2j}$ give the node $\gamma = \gamma_{n_{2j}}$ with the properties (i)-(iii).

b) Take the nodes $\zeta_i = (\text{rank}(\xi_i) + 1, \zeta_{i-1}, m_{2j}, I_i, 1, e^{\ast}_{\xi_i}), \zeta_0 = 0$, where $I_i = \Delta\text{rank}(\xi_i)$.

Definition 6.2. Let $d \leq n_{2j-1}$ and $(\gamma_k, x_k)_{k=1}^{d}$ be a $(\Gamma, j)$-special sequence. 

A sequence $(\gamma_k, x_k)_{k=1}^{d}$ with $x_k \in \mathbb{X}_{Kus}$ and $\gamma_k = (q_k + 1, \gamma_{k-1}, m_{2j-1}, I_k, 1, e^{\ast}_{\eta_k})$ for each $k$, where $\gamma_0 = 0$, is called a $j$-dependent sequence of length $d$ with respect to $(\gamma_k, x_k)_{k=1}^{d}$ if the following conditions are satisfied for some fixed $C > 1$.

1) if $k$ is even then $x_k = R\bar{x}_k$,

2) if $k$ is odd then $x_k = \frac{c_k m_k}{n_k} \sum_{l=1}^{n_k} x_{k,l}$, where $(x_{k,j})_l$ is a normalized skipped block sequence which is a $C$-RIS of length $n_k$, where $m_k = w(\eta_k), \|x_k\| = 1/2$, and $e^{\ast}_{\eta_k}(x_k) \geq 1/20C$.

3) $|e^{\ast}_{\eta_k}(x_k) - e^{\ast}_{\eta_k}(x_k)| < 1/m^2_{\eta_k}$ for every $\gamma \in \Gamma$ and every $k$,

4) $(\gamma_k, x_k)_{k=1}^{d}$ is $j$-dependent of length $d-1$ with respect to the $(\Gamma, j)$-special sequence $(\gamma_k, x_k)_{k=1}^{d-1}$.

Moreover, we say that a sequence $(\gamma_k, x_k)_{k=1}^{d}$ is a $j$-dependent sequence of length $d$, if it is $j$-dependent with respect to some $(\Gamma, j)$-special sequence.

Remark 6.3. Take $(x_{k,j})_l$ as in (2) of Definition 6.2. Then Lemmas 5.2a) and 6.1a) yield that there is a node $\eta_k \in \Gamma$ such that

\[ \frac{1}{2} \leq e^{\ast}_{\eta_k}(x_{k,j}) \leq \|x_{k,j}\| \leq 10C. \]

Therefore $c_k$ in Definition 6.2 satisfies $1/20C \leq c_k < 2$.

Moreover, the last condition in the property (2) of Definition 6.2 i.e. $e^{\ast}_{\eta_k}(x_k) \geq 1/20C$, follows from (6.3) using the lower bound of $c_k$.

Lemma 6.4. Let $(\gamma_k)_{k=1}^{q}$ be a normalized block sequence in $\mathbb{X}_{Kus}$ and $(d_{\xi_n})_{n=1}^{M}$ be a subsequence of the basis. Then for every $j \in \mathbb{N}$ there exists a $j$-dependent sequence of length $n_{2j-1}$, $(\gamma_i, x_i)_{i\leq n_{2j-1}}$, such that $x_{2i-1} \in (\gamma_k, x_k)_{k=1}^{d}$ and $x_{2i} \in (d_{\xi_n} : n \in M)$.

Proof. Passing to a further subsequence we may assume that

(6.4)

$d_{\xi_n}$ are pairwise no neighbours and $\text{rank}(\xi_n) + 1 < \text{rank}(\xi_{n+1})$.

Let $j_1$ be such that $m_{4j_1-2} > n^2_{2j-1}$ and choose $q_1$ big enough to guarantee that $4j_1 - 2 < q_1$ and $2^{-q_1} \leq 1/m^2_{4j_1-2}$. Let $(x_{1,k})_{k=1}^{n_{4j_1-2}}$ be a normalized skipped block sequence of $(z_l : l \geq q)$ which is a $C$-RIS. Setting

\[ x_1 = c_1 m_{4j_1-2} \sum_{k=1}^{n_{4j_1-2}} x_{1,k} \quad \text{with} \quad \|x_1\| = 1/2 \]

from Remark 6.3 we get $1/20C \leq c_1 \leq 2$ and that there exists a node $\eta_1 \in \Gamma$ with $w(\eta_1) = m_{4j_1-2}$ such that

\[ e^{\ast}_{\eta_1} P_{I_1}(x_1) \geq \frac{1}{40C}, \]

where $I_1 = \bigcup \{ \Delta_p : p \in \text{rng}_{FDD}(x_1) \}$.

Using that $R$ is a quotient operator of norm 1 take a block $\bar{y}_1 \in \mathbb{X}_\Gamma$ such that $x_1 = R(\bar{y}_1)$ and $\|\bar{y}_1\| < 1$. Then choose $\bar{x}_1 \in \text{Net}_{2,q_1}$ such that $\|\bar{x}_1 - \bar{y}_1\|_{\mathbb{X}_\Gamma} < 1/4m^2_{\eta_1}.}$
Note that $R(\bar{x}) = R(\bar{x}_1 - \bar{y}_1) + R(\bar{y}_1) = R(\bar{x}_1 - \bar{y}_1) + x_1$ and hence for every $\gamma \in \Gamma$,

$$|e^*_\gamma(\bar{x}_1) - e^*_\gamma(x_1)| = |e^*_\gamma R(\bar{x}_1) - e^*_\gamma R(x_1)| \leq \|e^*_\gamma \circ R\| \|\bar{x}_1 - \bar{y}_1\|_F \leq \frac{1}{4n_{q_1}^2}.$$

We take $\gamma_1$ to be the node

$$\gamma_1 = (q_1 + 1, 0, m_{2j-1}, I_1, 1, e^*_{q_1}).$$

From the above we get that $(\gamma_1, x_1)$ is a $j$-dependent couple of length 1 with respect to the $(\Gamma, j)$-special sequence $(\gamma_1, \bar{x}_1)$.

Set $j_2 = \sigma(\gamma_1, \bar{x}_1)$ and choose $x_2, e^*_{q_2}$ such that

$$x_2 = m_{4j_2}n_{4j_2}^{-1} \sum_{k \in F_2} d_{2\xi_k} \in X_{Kus} \quad \text{and} \quad mt(e^*_{q_2}) = m_{4j_2}^{-1} \sum_{k \in F_2} d_{2\xi_k}^2,$$

where $|F_2| = n_{4j_2}$ and $q_1 + 2 < \min \text{rng}_{FDD}(x_2)$. Such a node exists by Lemma 6.1(b) since $\text{rank}(\xi_n) + 1 < \text{rank}(\xi_{n+1})$. We also take the node

$$\gamma_2 = (q_2 + 1, \gamma_1, m_{2j-1}, I_2, 1, \lambda_2 e^*_{q_2}) \in \Gamma$$

where $I_2 = [p_2, q_2]$ is the range of $x_2$ with respect to the basis and $x_2 \in \text{Net}_{1,q_1}$ is chosen such that

$$|\lambda_2 - e^*_{q_2}(y_1)| \leq \frac{1}{4n_{q_1}^2}.$$ 

From the above equation and (6.5) we get

$$|\lambda_2 - e^*_{q_2}(x_1)| \leq \frac{1}{2n_{q_1}^2} \Rightarrow \lambda_2 \geq e^*_{q_2}(x_1) - \frac{1}{2n_{q_1}^2} \geq \frac{c_1}{2} - \frac{1}{2n_{q_1}^2} \geq \frac{1}{45c}.$$

Pick $\bar{x}_2$ so that $x_2 = R\bar{x}_2$ (recall that $d_\gamma = R\bar{d}_\gamma$ for each $\gamma \in \Gamma$). Then we get that $(\gamma_i, x_i)_{i=1}^2$ is a $j$-dependent length 2 with respect to the $(\Gamma, j)$-special sequence $(\gamma_i, \bar{x}_i)_{i=1}^2$.

Set $j_3 = \sigma(\gamma_2, \bar{x}_i)_i^2$. We continue to choose $x_3, e^*_{q_3}, x_4, e^*_{q_4}$ in the same way we have chosen $x_1, e^*_{q_1}, x_2, e^*_{q_2}$ taking care that $x_1, x_2, x_3, x_4$ is a skipped block sequence (with respect to the FDD).

The functional $e^*_{q_2} \in X_{Ku_1}$ gives that

$$e^*_{q_2} \left( \frac{m_{2j-1}}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}} x_i \right) \geq \frac{1}{n_{2j-1}} \left( \sum_{i=1}^{n_{2j-1}/2} e^*_{\eta_{2i-1}} P_{I_{2i-1}}(x_{2i-1}) + \lambda_2 e^*_{\eta_{2i}}(x_{2i}) \right)$$

$$\geq \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} \left( c_{2i-1}/2 + c_{2i-1}/2 - \frac{1}{2n_{q_1}^2} \right) \geq \frac{1}{45c},$$

using that $c_{2i-1} \geq 1/20c$ and $n_{q_1}^2 > 200C$ for $q_1$ large.

**Lemma 6.5.** Let $(\gamma_i, x_i)_{i \leq n_{2j-1}}$ be a $j$-dependent sequence as in the previous lemma, with $n_{2j+1} > 200C$. Then

$$\| \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}} (-1)^{i+1} x_i \| \leq \frac{250}{m_{2j-1}}.$$

**Proof.** Let $J$ be an interval of $\{1, \ldots, n_{2j-1}\}$ and $z = \sum_{i \in J} (-1)^{i+1} x_i$. We shall verify the assumption of (2) in Lemma 6.2 for $j_0 = 2j - 1$.

Let $(\gamma_k, \bar{x}_k)_{k=1}^{n_{2j-1}}$ be the special sequence associated with the dependent sequence $(\gamma_k, x_k)_{k \leq n_{2j-1}}$; $\gamma_k = (q_k + 1, \gamma_{k-1}, m_{2j-1}, I_k, \delta_k, \lambda_k e^*_{k})$ for each $k$, where $\gamma_0 = 0$.

Consider a node $\beta$ with the evaluation analysis

$$e^*_{\beta} = \sum_{i=1}^{n_{2j-1}} d_{\xi_i}^2 + m_{2j-1}^{-1} \sum_{i=1}^{n_{2j-1}/2} (e_1 e^*_{\delta_{2i-1}} P_{I_{2i-1}} + \lambda_2 e^*_{\eta_{2i}} P_{I_{2i}}).$$
which is produced from a \((\Gamma, j)\)-special sequence \((\zeta_k, \bar{z}_k)_{k \leq n_{2j-1}}\). Let
\[
k_0 = \min\{k \leq n_{2j-1} : (\gamma_k, \bar{x}_k) \neq (\zeta_k, \bar{z}_k)\}
\]
if such an \(i\) exists. We estimate separately \(|e^*_{\beta k_0 - 1}(z)|\) and \(|(e^*_\beta - e^*_{\beta k_0 - 1})(z)|\).

We start with \(|e^*_{\beta k_0 - 1}(z)|\). Notice that \(e^*_{\beta k_0 - 1}\), if \(k_0 > 1\), has the following evaluation analysis
\[
e^*_{\beta k_0 - 1} = \sum_{i=1}^{k_0 - 1} d^*_\xi_i + m_{2j-1}^{-1} \sum_{i=1}^{((k_0-1)/2)} (e^*_{\delta i_1} P_i \lambda_{2i} + \lambda_{2i} e^*_{\eta i_2} P_i) + [e^*_{k_0 - 1} - e^*_{\delta k_0 - 1} P_1].
\]
where \(e^*_{\delta i_1}\) have compatible tree analysis with \(e^*_{\eta i_1}\) and the last term in square brackets appears if \(k_0 - 1\) is odd. By the definition of nodes we have \(\text{rank}(\xi_i) = \text{rank}(\gamma_i) \in (\max \text{rgs}_{\text{FDD}}(x_i), \min \text{rgs}_{\text{FDD}}(x_{i+1}))\) for every \(i < k_0\). Therefore
\[
(6.6) \quad \left(\sum_{i=1}^{k_0 - 1} d^*_\xi_i\right) \sum_{i} (-1)^{i+1} x_i = 0.
\]
We partition the indices \(P = \{1, 2, \ldots, ((k_0 - 1)/2)\}\) into the sets \(A = \{i \in P : e^*_{\delta i_1} (\bar{x}_{2i-1}) \neq 0\}\) and its complement \(B\).

For every \(i \in A\) from the choice of \(\lambda_{2i}\) and (3) of Def. \(6.2\) we have
\[
(6.7) \quad |\lambda_{2i} - e_{\delta i_1}^* (\bar{x}_{2i-1})| \leq \frac{1}{4n_{2j-1}^2} \quad \text{and} \quad |e^*_{\delta i_1} (\bar{x}_{2i-1}) - e^*_{\delta i_1} (x_{2i-1})| \leq \frac{1}{4n_{2j-1}^2}.
\]
It follows that
\[
(6.8) \quad |e_i e^*_{\delta i_1} (x_{2i-1}) + \lambda_{2i} e^*_{\eta i_2} (x_{2i})| = |e_i e^*_{\delta i_1} (x_{2i-1}) - \lambda_{2i}| \leq \frac{1}{2n_{2j-1}^2} \quad \text{by (6.7)}.
\]
Similarly for every \(i \in B\),
\[
(6.9) \quad |e_i e^*_{\delta i_1} (x_{2i-1}) + \lambda_{2i} e^*_{\eta i_2} (x_{2i})| = |e_i e^*_{\delta i_1} (x_{2i-1}) - \lambda_{2i}| \leq 2n_{2j-1}^{-2}.
\]
Using that \(\|x_{2i-1}\| \leq 1, \|x_{2i}\| \leq 7\), the inequalities \((6.6),(6.9)\) and considering the cases \(I = [l, m]\), where \(l, m\) are odd or even we get
\[
|e^*_{\beta k_0 - 1} \left(\sum_{i \in J} (-1)^{i+1} x_i\right)| \leq 10.
\]
Now we proceed to estimate \(|(e^*_\beta - e^*_{\beta k_0 - 1})(z)|\).

Observe that as \(x_{2l-1}\) is a normalized C-RIS of length \(n_{2l-1}\) we have
\[
(6.10) \quad \left|\left(\sum_{i=k_0}^{n_{2j-1}} d^*_\xi_i\right) (x_{2l-1})\right| \leq 3n_{2j-1} c_{2l-1} C m_{2j-1} \leq 2m_{2j-1}^{-2} < n_{2j-1}^{-3} \quad \forall l.
\]
The same inequality holds also for the averages of the basis i.e.
\[
(6.11) \quad \left|\sum_{i=k_0}^{n_{2j-1}} d^*_\xi_i (x_{2l})\right| \leq n_{2j-1}^{-3} \frac{m_{2j}}{n_{2j}} \leq m_{2j}^{-3} < n_{2j-1}^{-3} \quad \forall l.
\]
We shall distinguish the cases \(k_0\) is odd or even. Assume first that \(k_0 = 2i_0 - 1\) for some \(i_0\).

Then for every \(i < i_0\) and every \(k > k_0\),
\[
(e_i e^*_{\delta i_1} P_i \lambda_{2i} + \lambda_{2i} e^*_{\delta i_1} P_i)(x_k) = 0.
\]
From the injectivity of $\sigma$ it follows that $w(e_{\delta_{i-1}}^i), w(e_{\delta_i}^i) \neq \{w(e_{\alpha}^i) \mid i' > i_0\}$ for every $i > i_0$.

Hence by Corollary \ref{cor:injectivity}, using that $|\lambda_{2i}| \leq 1$ and $c_k \leq 2$, we get for every odd $k > k_0$ the following

$$| \sum_{i \geq i_0} (\epsilon_i e_{\delta_{2i-1}}^i + \lambda_{2i} e_{\delta_{2i}}^i)(x_k) | \leq 64 c_k C w(\delta_1) \leq 128 C n_{2j-1}^{-2}. \quad (6.12)$$

Also from Corollary \ref{cor:injectivity} we obtain for every even $k > k_0$ the following

$$| \sum_{i \geq i_0} (\epsilon_i e_{\delta_{2i-1}}^i + \lambda_{2i} e_{\delta_{2i}}^i)(x_k) | \leq 14 n_{2j-1}^{-2}. \quad (6.13)$$

For $x_{k_0}$ we also obtain the following

$$| \sum_{i \geq i_0} (\epsilon_i e_{\delta_{2i-1}}^i P_{I_{2i-1}} + \lambda_{2i} e_{\delta_{2i}}^i P_{I_{2i}})(x_{k_0}) | \leq | e_{k_0}^* P_{I_{k_0}}(x_{k_0}) | + \left( \sum_{i > i_0} (\epsilon_i e_{\delta_{2i-1}}^i P_{I_{2i-1}} + \lambda_{2i} e_{\delta_{2i}}^i P_{I_{2i}})(x_{k_0}) \right) \leq 4 + 128 C n_{2j-1}^{-2}, \quad (6.14)$$

using that $\|x_{k_0}\| \leq 1$ and $\|e_{k_0}^* \circ P_{I}\| \leq \|P_{I}\| \leq 4$ while for the second term we get an upper bound as in \eqref{6.12}.

The case where $k_0$ is even is similar, except that $| e_{k_0}^* P_{I_{k_0}}(x_{k_0}) | \leq 7$.

Splitting $J$ to $J_1 = J \cap [1, i_0], J_2 = J \cap (i_0, n_{2j-1})$ and considering the cases $\min J_1$ is odd or even we get $|(\epsilon_i e_{\delta_{k_0 i}}^i - e_{\delta_{k_0 i}}^i)(\sum_{i \in J} (-1)^{i+1} x_i) | \leq 15$, using that $n_{2j+1} > 200 C$. \hfill \Box

The lemmas above imply the following.

**Proposition 6.6.** Let $M \subset \mathbb{N}$ be infinite and $(y_k)_k \subset \mathcal{X}_{Kus}$ be a normalized block sequence. Then

$\inf \{ \|x - y\| : x \in \langle d_{\gamma_n} : n \in M \rangle, y \in \langle y_k : k \in \mathbb{N} \rangle, \|x\| = \|y\| = 1 \} = 0.$

7. Bounded operators on the space $\mathcal{X}_{Kus}$

In this section we show that the space $\mathcal{X}_{Kus}$ has the scalar-plus-compact property, as well as a small family of isomorphisms inside the space, which means here tightness by range.

**Proposition 7.1.** Let $T : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ be a bounded operator and $(\gamma_n)_{n \in M}$ be a subsequence of the basis. Then

$$\lim_{M \ni n \to +\infty} \text{dist}(Td_{\gamma_n}, \mathbb{R}d_{\gamma_n}) = 0.$$

**Proof.** Assume that $\text{dist}(Td_{\gamma_n}, \mathbb{R}d_{\gamma_n}) > 4 \delta$ for infinitely many $n \in M$ and some $\delta > 0$.

By Corollary \ref{cor:injectivity} and Lemma \ref{lem:injectivity} passing to a further subsequence and admitting a small perturbation we may assume that

(P1) $(Td_{\gamma_n})_{n \in M}$ is a skipped block sequence and setting $R_n$ to be the minimal interval containing $\text{rng}(Td_{\gamma_n})$ and $\{n\}$ we have

$$\max \text{rank}(R_n) + 2 < \min \text{rank}(R_{n+1}).$$

(P2) no two elements of $(d_{\gamma_n})_{n \in M}$ are neighbours.
By the assumption that dist($T d_{\gamma n}, \mathbb{R} d_{\gamma n}$) > $4 \delta$ it follows that either
\[ \| P_{n-1} T d_{\gamma n} \| \geq 2 \delta \text{ or } \| (I - P_n) T d_{\gamma n} \| \geq 2 \delta \]
(recall that $P_m$ denotes the canonical projection onto $\langle d_{\gamma n} : i \leq m \rangle$, $m \in \mathbb{N}$).

Passing to a further subsequence we may assume that one of the two alternatives holds for any $n \in \mathbb{N}$. Let
\[ q_n = \begin{cases} \max \text{rank}(P_{n-1} T d_{\gamma n}) & \text{in the first case} \\ \max \text{rank}((I - P_n) T d_{\gamma n}) & \text{in the second case}. \end{cases} \]
In the first case we take $I_n = [\min \text{rng}(T d_{\gamma n}), n - 1]$. Also $P_{n-1} T d_{\gamma n} = i_{q_n}(u_n)$ where $u_n = r_{q_n}(P_{n-1} T d_{\gamma n})$ and hence we may choose $\epsilon_n \in \{-1, 1\}$ and $\eta_n \in \Gamma_{q_n} \setminus \Gamma_{\max \text{rank}(R_{n-1})+1}$ such that
\[ \epsilon_n \epsilon_{q_n}^* P_{I_n}(T d_{\gamma n}) = \epsilon_n \epsilon_{\eta_n}^* (P_{k_n-1} T d_{\gamma n}) = \epsilon_n \epsilon_{\eta_n}^* (u_n) \geq \delta \]
using that $2 \delta \leq \| i_{q_n}(u_n) \| \leq 2 \| u_n \|$.

In the second case we take $I_n = [n + 1, \max \text{rng}(T d_{\gamma n})]$. Also since $(I - P_n) T d_{\gamma n} = i_{q_n}(u_n)$ where $u_n = r_{q_n}((I - P_n) T d_{\gamma n})$ we get $\epsilon_n \in \{-1, 1\}$, $\eta_n \in \Gamma_{q_n} \setminus \Gamma_{\max \text{rng}(R_{n-1})+1}$ such that
\[ \epsilon_n \epsilon_{q_n}^* P_{I_n}(T d_{\gamma n}) = \epsilon_n \epsilon_{\eta_n}^* ((I - P_n) T d_{\gamma n}) = \epsilon_n \epsilon_{\eta_n}^* (u_n) \geq \delta. \]

Assume the first case holds. The second case will follow analogously. Notice that by (P1) for any $i \in \mathbb{N}$ and $A \subset M$ with $\# A = n_{2i}$ there is a functional $e_{\psi_i}^*$ associated to a regular node of the form
\[ e_{\psi_i}^* = \sum_{n \in A} d_{\xi_n}^* + \frac{1}{m_{2i}} \sum_{n \in A} \epsilon_n \epsilon_{\eta_n}^* P_{I_n}. \]
with rank($\xi_n$) = max rank($R_n$) + 1 for each $n \in A$. Let $x = m_{2i} n_{2i-1}^{-1} \sum_{n \in A} d_{\gamma n}$. It follows that
\[ \| T x \| \geq e_{\psi_i}^*(T x) = \left( \sum_{n \in A} d_{\xi_n}^* + \frac{1}{m_{2i}} \sum_{n \in A} \epsilon_n \epsilon_{\eta_n}^* P_{I_n} \right) \left( \frac{m_{2i}}{n_{2i}} \sum_{n \in A} T d_{\gamma n} \right) \]
\[ = m_{2i} n_{2i-1}^{-1} \sum_{n \in A} d_{\xi_n}^*(T d_{\gamma n}) + \frac{1}{n_{2i}} \sum_{n \in A} \epsilon_n \epsilon_{\eta_n}^* P_{I_n}(T d_{\gamma n}) \]
\[ = \frac{1}{n_{2i}} \sum_{n \in A} \epsilon_n \epsilon_{\eta_n}^* P_{k_{n-1}-1}(T d_{\gamma n}) \geq \delta. \]

Fix $j \in \mathbb{N}$ and choose inductively a $j$-dependent sequence $(\zeta_i, x_i)$, $\zeta_i = (q_{i+1}, \zeta_{i-1}, m_{2j-1}, j_i, 1, \psi_i)$, $i = 1, \ldots, n_{2j-1}$, with $\zeta_0 = 0$, with respect to a $(\Gamma, j)$-special sequence $(\zeta_i, x_i)$, so that it satisfies for any $i$ the following
\[ e_{\psi_{2i-1}}^* = \sum_{n \in A_i} d_{\xi_n}^* + \frac{1}{m_{2j-1}} \sum_{n \in A_i} \epsilon_n \epsilon_{\eta_n}^* P_{I_n}, \quad x_{2i-1} = \frac{c_{2i-1} m_{2j-1}}{n_{2j-1}} \sum_{n \in A_i} d_{\gamma n}, \quad \| x_{2i-1} \| = 1/2 \]
with rank($\xi_n$) = max rank($R_n$) + 1 for each $n \in \cup A_i$. Lemma 3.3 yields that $1/4 \leq c_{2i-1} \leq 1$. Recall that by definition each vector $\tilde{x}_{2i-1} \in \text{Net}_{2,q}$ satisfies
\[ \| e_{\psi_{2i-1}}^*(\tilde{x}_{2i-1}) - e_{\psi_{2i-1}}^*(x_{2i-1}) \| \leq 4 n_{2i-1}^{-2} \quad \forall \gamma \in \Gamma. \]
For any $i$ let $J_{2i-1} = \text{rng}(e_{\psi_{2i-1}}^*)$. We demand also that supp $e_{\psi_{2i}}^* \cap \text{supp} x_{2k-1} = \emptyset$ for any $i, k$, thus the even parts of the chosen special functional play no role in the estimates on the averages of $(x_{2i-1})$. We assume also $m_{2j}/m_{2j+1} \leq 1/n_{2j-1}^2$.

By the previous remark we have for each $i$ the following
\[ e_{\psi_{2i-1}}^*(T x_{2i-1}) \geq \delta/14. \]
Let
\[ y = \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} x_{2i-1} = \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} c_{2i-1} \frac{m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n \in A_i} d_n \]
and consider the functional associated to the special node \( \zeta_{n_{2j-1}} \), i.e., of the form
\[ e_{\zeta_{n_{2j-1}}}^* = \sum_{i=1}^{n_{2j-1}} d_{\zeta_i}^* + \frac{1}{m_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} (e_{\psi_{2i-1}}^* P_{J_{2i-1}} + \lambda_2 e_{\psi_{2i}}^* P_{J_{2i}}). \]
Then
\[ \|Ty\| \geq e_{\zeta_{n_{2j-1}}}^* (Ty) \]
\[ = \left( \sum_{i=1}^{n_{2j-1}} d_{\zeta_i}^* + \frac{1}{m_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} (e_{\psi_{2i-1}}^* P_{J_{2i-1}} + \lambda_2 e_{\psi_{2i}}^* P_{J_{2i}}) \right) \left( \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} T x_{2i-1} \right) \]
\[ = \left( \sum_{i=1}^{n_{2j-1}} d_{\zeta_i}^* \right) \left( \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} T x_{2i-1} \right) + \frac{1}{n_{2j-1}m_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} e_{\psi_{2i-1}}^* P_{J_{2i-1}} (T x_{2i-1}) \]
where in the last line the first sum disappears by the choice of \((q_{2i-1})\), as \( \text{rank}(\text{bd}(e_{\zeta_{n_{2j-1}}}^*)) \cap \text{rank}(T x_{2i-1}) = \emptyset \) for any \( i \). Therefore we have
\[ \|Ty\| \geq \frac{\delta}{28m_{2j-1}}. \]

On the other hand we estimate \( \|y\| \). By (P2) and Lemma 5.3, we get that \((x_i)\) is 7-RIS. By Lemma 5.2 it is enough to estimate \( |e_\beta^*(z)| \), where \( e_\beta^* \) is associated to a \((\Gamma, j)\)-special sequence \((\delta_i, z_i)_{i=1}^a\), and \( z = \sum_{i \in J} x_{2i-1} \) for some interval \( J \subseteq \{1, \ldots, n_{2j-1}\} \).

Let \( e_\beta^* \) has the following form
\[ e_\beta^* = \sum_{i=1}^a d_{\zeta_i}^* + \frac{1}{m_{2j-1}} \sum_{i=1}^{a/2} (\tilde{e}_{2i-1} e_{\psi_{2i-1}}^* P_{J_{2i-1}} + \tilde{\lambda}_2 e_{\psi_{2i}}^* P_{J_{2i}}) \]
with \( a \leq n_{2j-1} \). Let \( i_0 = \max\{i \leq a : (\zeta_i, x_i) = (\delta_i, z_i)\} \) if such \( i \) exists. We estimate \( |e_\beta^*(z)| \)
assuming \( i_0 \) is well-defined. We estimate separately \( |\sum_{i=1}^a d_{\zeta_i}^* (z)| \), \( |\text{mt}(e_{\zeta_0}^*)(z)| \) and \( |(\text{mt}(e_\beta^*) - \text{mt}(e_{\zeta_0}^*))(z)| \).

First notice that taking into account coordinates of \( z \) with respect to the basis \((d_n)\) and that \( c_{2i-1} \leq 1 \), we have
\[ \left| \sum_{i=1}^a d_{\zeta_i}^* (z) \right| \leq n_{2j-1} \frac{m_{j_{2i-1}}}{n_{j_{2i-1}}} \cdot \]
Now consider the tree-analysis of \( e_{\zeta_0}^* \), recall that it is compatible with the tree-analysis of \( e_{\zeta_{i_0}}^* \). Then by the definition of a special node we have
\[ \text{mt}(e_{\zeta_{i_0}}^*) = \begin{cases} \frac{1}{m_{2j-1}} \sum_{i=1}^{i_0/2} (\tilde{e}_{2i-1} e_{\psi_{2i-1}}^* P_{J_{2i-1}} + \tilde{\lambda}_2 e_{\psi_{2i}}^* P_{J_{2i}}) & \text{if } i_0 \text{ even} \\ \frac{1}{m_{2j-1}} \sum_{i=1}^{i_0/2} (\tilde{e}_{2i-1} e_{\psi_{2i-1}}^* P_{J_{2i-1}} + \tilde{\lambda}_2 e_{\psi_{2i}}^* P_{J_{2i}}) + \tilde{e}_{i_0} e_{\psi_{i_0}}^* P_{J_{i_0}} & \text{if } i_0 \text{ odd} \end{cases} \]
where for each \( 2i - 1 \leq i_0 \) we have
\[ e_{\psi_{2i-1}}^* = \sum_{n \in A_i} d_{\zeta_n}^* + \frac{1}{m_{j_{2i-1}}} \sum_{n \in A_i} \tilde{e}_{n} e_{\zeta_n}^* P_{J_{n}}. \]
Notice that as $M \cap I_n = \emptyset$ for any $n$ and by the choice of $e_{\psi_i}^*$ and ranks of $\xi_n$, thus also ranks of $\tilde{\xi}_n$, we get, assuming that $i_0$ is even,

\[
\begin{align*}
\frac{1}{m_{2j-1}} \sum_{i=1}^{i_0/2} (\tilde{e}_{2i-1} e_{\psi_{2i-1}}^* P_{j_2i-1} + \tilde{\lambda}_{2i} e_{\psi_{2i}}^* P_{j_{2i}})(z) & \leqslant |(1) \frac{1}{m_{2j-1}} \sum_{i=1}^{i_0/2} \sum_{n \in A_i} e_{\psi_{2i}}^* P_{j_{2i}}(x_{2i-1})| \\
& = |\left( \frac{1}{m_{2j-1}} \sum_{i=1}^{i_0/2} \sum_{n \in A_i} e_{\psi_{2i}}^* P_{j_{2i}}(x_{2i-1}) \right) \sum_{i=1}^{i_0/2} \sum_{n \in A_i} d^*_{\xi_n} | = 0.
\end{align*}
\]

The same holds if $i_0$ is odd.

Now consider $m_{(e_{\beta})^*} - m_{(e_{\xi_0}^*)}$ assuming that $i_0 < a$. Notice that

1. $w(\psi_s) \neq w(\tilde{\psi}_s)$ for each $s, i > i_0$ provided at least one of the indices $s, i$ is bigger than $i_0 + 1$,
2. $(m_{(e_{\beta})^*} - m_{(e_{\xi_0}^*)})(x_{2k-1}) = 0$ for any $2k - 1 \leq i_0$.

Using Corollary 3.5 for the terms $\sum_{i=i_0+1}^{a} e_{\psi_{i}}^* P_{j_{i}}(x_{2k-1})$ and that $|e_{\psi_{i_0+1}}^* P_{j_{i_0+1}}(x_{i_0+1})| \leq 4$, it follows that

\[
(7.7) \quad |(m_{(e_{\beta})^*} - m_{(e_{\xi_0}^*)})(z)| \leq \frac{1}{m_{2j-1}} \sum_{i=i_0+1}^{a} \sum_{n \in A_i} |e_{\psi_{i}}^* P_{j_{i}}(x_{2k-1})| \leq \frac{4}{m_{2j-1}} + \frac{1}{m_{2j-1}} \sum_{i=i_0+1}^{a} \sum_{n \in A_i} \frac{14}{m_{j_{i_0+1}}} \leq \frac{5}{m_{2j-1}}.
\]

Therefore by (7.5), (7.6), (7.7) and the choice of $j_1$ we have $|e_{\beta}^*(z)| \leq 6/m_{2j-1}$, thus we can apply Lemma 5.2 obtaining that $\|y\| \leq 60 \cdot 7/m_{2j-1}$. For sufficiently big $j$ we obtain contradiction with (7.1) and boundedness of $T$.

**Proposition 7.2.** Let $T : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ be a bounded operator. If $T d_{\gamma_n} \to 0$, then $Ty_n \to 0$ for every RIS $(y_n)_n$.

**Proof.** Take $T : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ with $T d_{\gamma_n} \to 0$ and suppose there are a normalized $C$-RIS $(y_n)_n$ and $\delta > 0$ such that $\|Ty_n\| > \delta$ for all $n \in \mathbb{N}$. Passing to a subsequence we may assume as in the proof of Prop. 7.1 that

$$\max \text{ rank } R_n + 2 < \min \text{ rank } R_{n+1} \quad \text{where } R_n = \text{rng}(Ty_n) \cup \text{rng}(y_n).$$

Pick $(\mu_n) \subset \{\pm 1\}$ and nodes $(\psi_n)$ with $\mu_n e_{\psi_n}^*(Ty_n) > \delta$.

Below we shall construct dependent sequences. In order to avoid the complexity of approximating the vectors of the dependent sequence by elements of the nets $\text{Net}_{2,q}$, we shall assume the vectors of the dependent sequence are in the nets $\text{Net}_{2,q}$. The general proof follows by slight and obvious modifications.

**Case 1.** There exist a constant $c > 0$, an infinite set $M \subset \mathbb{N}$ and nodes $(\varphi_n)_{n \in M}$ such that $|e_{\varphi_n}^*(y_n)| > c$ and $e_{\varphi_n}^*, e_{\psi_n}^*$ have compatible tree-analyses.

Pick signs $(\nu_n)_{n \in M}$ with $\nu_n e_{\varphi_n}^*(y_n) = |e_{\varphi_n}^*(y_n)| > c$ for each $n$. Passing to a subsequence we can assume that $\|T d_{\gamma_n}\| \leq 2^{-n}$ for all $n$. For a fixed $j \in \mathbb{N}$, $n_{2j+1} > 200C$, we pick a $j$-dependent sequence $(\zeta_i, x_i)$ where $\zeta_i = (q_1 + 1, \zeta_{i-1}, m_{2j-1}, J_i, 1, \gamma_i), i = 1, \ldots, n_{2j-1}$, with $\zeta_0 = 0$, satisfies

$$\begin{align*}
\frac{1}{m_{2j-1}} \sum_{n \in A_{2j-1}} \nu_n e_{\varphi_n}^* P_{j_n}, \quad x_{2i-1} & = \frac{c_{2i-1} m_{2j-1}}{n_{2j-1}} \sum_{n \in A_{2j-1}} y_n, \quad \|x_{2i-1}\| = \frac{1}{2},
\end{align*}$$

\]
where \( I_n = [\min \{ R_n, \max R_n \} \), so that the functional associated to the special node \( \zeta_{n_{2j+1}} \) with \( \text{mt}-\)part of the form

\[
\text{mt}(e^*_{\zeta_{n_{2j-1}}}) = \frac{1}{m_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} \left( e^*_{\gamma_{2i-1}} P_{J_{2i-1}} + \lambda_{2i} e^*_{\gamma_{2i}} P_{J_{2i}} \right),
\]

satisfies \( J_{2i-1} \cap \text{rng}(Tx_{2i-1}) = \emptyset \) and \( \text{rank}(\text{bd}(e^*_{\zeta_{n_{2j+1}}})) \cap \text{rng}(Tx_{2i-1}) = \emptyset \) for any \( i, k \).

From Remark 6.3 we get

\[
1/20C \leq c_{2i-1} \leq 2.
\]

Using gaps between sets \( R_n \) we pick nodes \( (\xi_{2i-1})_{2i-1 \leq n_{2j+1}} \), with

\[
\text{mt}(e^*_{\xi_{2i-1}}) = \frac{1}{m_{2j-1}} \sum_{n \in A_{2i-1}} \mu_n e^*_{\psi_n} P_{I_n}.
\]

It follows that \( e^*_{\xi_{2i-1}}(Tx_{2i-1}) > \delta/20C \) for each \( i \).

Notice also that for \( x_{2i} = \frac{m_{2j}}{n_{2j}} \sum_{n \in A_{2i}} d_{n} \), by the condition on \( (Td_{\gamma_n}) \) we have \( \|Tx_{2i}\| < \frac{m_{2j}}{n_{2j}} < 2^{-i} \) for each \( i \).

Let \( x = \frac{m_{2j-1}}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} x_{2i-1} \) and \( d = \frac{m_{2j-1}}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} x_{2i} \). We have

\[
\|Tx\| \leq \frac{m_{2j-1}}{n_{2j-1}}
\]

and by Lemma 6.5

\[
\|x - d\| \leq \frac{250}{m_{2j-1}}.
\]

On the other hand by the choice of \( (\varphi_n) \) and \( (\psi_n) \) there is a well-defined special node \( \beta \), associated to the same \( j \)-special sequence as \( \zeta_{n_{2j+1}} \) with

\[
\text{mt}(e^*_{\beta}) = \frac{1}{m_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} \left( e^*_{\xi_{2i-1}} P_{J_{2i-1}} + \lambda_{2i} e^*_{\gamma_{2i}} P_{J_{2i}} \right),
\]

so that \( \text{rank}(\text{bd}(e^*_{\beta})) \cap \text{rng}(Tx_{2i-1}) = \emptyset \) for any \( i \). Thus

\[
\|Tx\| \geq e^*_{\beta}(Tx) \geq \frac{\delta}{40C}
\]

which contradicts (7.8) and (7.9) for sufficiently big \( j \) as \( T \) is bounded.

**Case 2.** Case 1 does not hold.

Applying this assumption for \( c = n_{2j-1}^{-1} m_{k}^{-1} \), \( k \in \mathbb{N} \), we pick inductively an increasing sequence \( (p_k) \subset \mathbb{N} \) such that for any node \( \varphi \) and \( n > p_k \) so that \( e^*_{\varphi}, e^*_{\psi} \) have compatible tree-analyses we have \( |e^*_{\varphi}(y_n)| \leq n_{2j-1}^{-1} m_{k}^{-1} \). Let \( M = (p_k)_k \).

Now we repeat the proof of Prop. 7.1 using \( (y_n) \) instead of \( (d_{\gamma_n}) \). For a fixed \( j \in \mathbb{N} \) we pick a \( j \)-dependent sequence \( (\zeta_i, x_i) \), \( \zeta_i = (g_{i+1}, \zeta_{i-1}, m_{2j-1}, J_i, 1, \gamma_i), i = 1, \ldots, n_{2j-1}, \) with \( \zeta_0 = 0 \), such that for each \( i \) we have

\[
\text{mt}(e^*_{\gamma_{2i-1}}) = \frac{1}{m_{2j-1}} \sum_{n \in A_i} \mu_n e^*_{\psi_n} P_{I_n}, \quad x_{2i-1} = \frac{c_{2i-1} m_{2j-1}}{n_{2j-1}} \sum_{n \in A_i} y_n, \quad \|x_{2i-1}\| = 1/2,
\]

with \( A_i \subset M, \#A_i = n_{2j-1}, J_{2i-1} = \text{rng}(e^*_{\gamma_{2i-1}}), J_{2i} \cap \text{supp} x_{2k-1} = \emptyset \) for any \( i, k \), \( I_n = [\min R_n, \max R_n] \) and \( \text{rank}(\xi_n) = \max \text{rng} R_n + 1 \) for any \( n \). As in the previous case, \( 1/20C \leq c_{2i-1} \leq 2 \). Pick \( j_1 \) with
$m_{j_1}/m_{j_1+1} \leq 1/n_{2j-1}^2$ and let

$$y = \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} x_{2i-1}$$

As in the proof of Prop. 4.1 it follows that

$$\|Ty\| \geq e_{\xi_{n_{2j-1}}}^* (y) \geq \frac{1}{m_{2j-1}n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} \frac{\delta}{2} e_{2i-1} \geq \frac{\delta}{80C m_{2j-1}}.$$  

We shall estimate now $\|y\|$. As before we consider a special node $\beta$ which is compatible with a $(\Gamma, j)$-special sequence $(\delta_i, \tilde{z}_i)_{i=1}^a$, $a \leq n_{2j-1}$, and estimate $|e_{\xi_i}^*(z)|$ where $z = \sum_{i \in J} x_{2i-1}$ for some interval $J \subset \{1, \ldots, n_{2j-1}\}$. Writing

$$e_{\beta}^* = \sum_{i=1}^{a} d_{\xi_i}^* + \frac{1}{m_{2j-1}} \sum_{i=1}^{a/2} (e_{2i-1} e_{\tilde{z}_{2i-1}}^* P_{J_{2i-1}} + \tilde{\lambda}_2 e_{\tilde{z}_{2i}}^* P_{J_{2i}})$$

with $a \leq n_{2j-1}$ we pick as before $i_0 = \min\{i \leq a : (\xi_i, x_i) \neq (\delta_i, \tilde{z}_i)\}$ (if such $i$ exists) and estimate separately $|\sum_{i=1}^{a} d_{\xi_i}^*(w)|$, $|\mt(e_{\xi_i}^*)(w)|$ and $|\mt(e_{\beta}^*) - \mt(e_{\xi_i}^*)|(w)|$.

Repeating the reasoning of the proof of Prop. 4.1 as $(y_n)$ have norm bounded by 1 and all $\|d_{\xi_i}^*\| \leq 3$, we obtain

$$\left| \sum_{i=1}^{a} d_{\xi_i}^*(z) \right| \leq 3 \cdot 2n_{2j-1} - \frac{m_{j_1}}{m_{j_1}} \leq \frac{1}{m_{2j-1}}.$$  

Using Corollary 5.3 and the fact that $|e_{\xi}^* P_{J}(x_{i_0}+1)| \leq 4$ we obtain that

$$\left| (\mt(e_{\beta}^*) - \mt(e_{\xi_i}^*)) (z) \right| \leq \frac{4}{m_{2j-1}} + \frac{2}{m_{2j-1}} \frac{n_{2j-1}}{m_{j_0+1}} \frac{64C}{m_{2j-1}} \leq \frac{5}{m_{2j-1}}$$

using that $m_{j_1}^{-1} < n_{2j-1}^2$ and $n_{2j+1} > 200C$.

Now consider $e_{\xi_i}^*$, recall this functional has the tree-analysis compatible with the tree-analysis of $e_{\xi_i}^*$. Therefore we have

$$\mt(e_{\xi_i}^*) = \begin{cases} \frac{1}{m_{2j-1}} \sum_{i=1}^{a/2} (e_{2i-1} e_{\tilde{z}_{2i-1}}^* P_{J_{2i-1}} + \tilde{\lambda}_2 e_{\tilde{z}_{2i}}^* P_{J_{2i}}) & \text{if } i_0 \text{ even} \\ \frac{1}{m_{2j-1}} \sum_{i=1}^{a/2} (e_{2i-1} e_{\tilde{z}_{2i-1}}^* P_{J_{2i-1}} + \tilde{\lambda}_2 e_{\tilde{z}_{2i}}^* P_{J_{2i}}) + e_{i_0} e_{\tilde{z}_{i_0}}^* P_{J_{i_0}} & \text{if } i_0 \text{ odd} \end{cases}$$

where for each for each $2i - 1 \leq i_0$ we have

$$e_{\xi_{2i-1}}^* = \sum_{n \in A_i} d_{\xi_n}^* + \frac{1}{m_{j_2i-1}} \sum_{n \in A_i} \tilde{e}_n e_{\varphi_n}^* P_{I_n}.$$  

By choice of the objects above we have

$$\left| \mt(e_{\xi_i}^*) (z) \right| \leq \frac{1}{m_{2j-1}} \left( \sum_{i=1}^{n_{2j-1}/2} \sum_{n \in A_i} d_{\xi_n}^* \right) \left( \sum_{2i-1 \in J} \frac{c_{2i-1} m_{2j-1}}{n_{j_2i-1}} \sum_{n \in A_i} y_n \right)$$

+ \frac{1}{m_{2j-1}} \sum_{2i-1 \in J} \frac{c_{2i-1} m_{2j-1}}{n_{j_2i-1}} \sum_{n \in A_i} |e_{\varphi_n}^* (y_n)|.$$

As for each $n$ the nodes $\psi_n, \varphi_n$ have compatible tree-analyses the last sum can be estimated by $2m_{2j-1}^{-1}$. The first sum equals 0 by the condition on ranks of $\xi_n$, thus also $\xi_n$. Therefore we have

$$\left| \mt(e_{\xi_i}^*) (z) \right| \leq \frac{2}{m_{2j-1}}.$$  

(7.13)
As before by (7.11), (7.12), (7.13) we have $|e_β^*(z)| \leq 8/m2j−1$, thus we can apply Lemma 5.2 obtaining that $\|y\| \leq 80C/m^2j−1$. For sufficiently big $j$ we obtain contradiction with (7.10) and boundedness of $T$.

\[ \square \]

**Theorem 7.3.** Let $T : X_{Kus} \to X_{Kus}$ be a bounded operator. Then there exist a compact operator $K : X_{Kus} \to X_{Kus}$ and a scalar $\lambda$ such that $T = \lambda Id + K$.

**Proof.** By Prop. 7.1 any $(d_\gamma)_{n \in N}$ has a further subsequence $(d_\gamma)_{n \in M}$ such that $Td_\gamma - \lambda d_\gamma \to 0$ as $M \ni n \to \infty$, for some $\lambda$. By Prop. 6.6 there is a universal $\lambda$ so that $Td_\gamma - \lambda d_\gamma \to 0$ as $n \to \infty$. Applying Prop. 7.2 to the operator $T - \lambda Id$ we get that $Ty_n - \lambda y_n \to 0$ for any RIS $(y_n)$ and thus, by Prop. 5.9 for any bounded block sequence $(y_n)$. It follows that the operator $T - \lambda Id$ is compact.

\[ \square \]

The above theorem implies immediately the following.

**Corollary 7.4.** The space $X_{Kus}$ is indecomposable, i.e. it is not a direct sum of two its infinitely dimensional closed subspaces.

We conclude this section with a result showing that even though $X_{Kus}$ admits rich unconditional structure, it has a small family of isomorphisms between subspaces. Recall that two Banach spaces are comparable if one of them embeds isomorphically into the other.

**Definition 7.5 (10).** A Banach space $X$ with a basis is called tight by range if no two block subspaces spanned by infinite block sequences with pairwise disjoint ranges are comparable.

**Theorem 7.6.** The space $X_{Kus}$ it tight by range.

**Proof.** We shall prove a stronger statement, namely the following: for any bounded block sequence $(w_n)$ and any bounded operator $T : \langle w_n : n \in N \rangle \to X_{Kus}$ with $\text{rng}(Tw_n) \cap \text{rng}(w_n) = \emptyset$, we have $Tw_n \to 0$.

The proof follows the reasoning proving Prop. 7.1. Pick a block sequence $(w_n)$ and a bounded operator $T : \langle w_n : n \in N \rangle \to X_{Kus}$ as above and assume $\limsup_n \|Tw_n\| = 2\delta > 0$. Passing to a subsequence, applying Prop. 6.4 and admitting small perturbation we can assume that

$$\max \text{rank } R_n + 2 < \max \text{rank } R_{n+1},$$

where $R_n = \text{rng}(Tw_n) \cup \text{rng}(w_n)$.

and $w_n = y_n + z_n$, $n \in N$, for some $C$-RIS $(y_n)$, $(z_n)$, $C > 1$.

It follows that for any $n \in N$ either

$$\|P_{\min \text{rng}(w_n)−1}Tw_n\| \geq \delta \text{ or } \|(I - P_{\max \text{rng}(w_n)})Tw_n\| \geq \delta.$$

We repeat now the first part of the proof of Prop. 7.1. Assume that the first alternative holds for infinitely many $n \in N$ (if the second alternative holds we proceed analogously) and for such $n$’s let $I_n = [\min \text{rng}(Tw_n), \min \text{rng}(w_n) - 1]$ and pick $(\mu_n) \subset \{\pm 1\}$ and nodes $(\psi_n)$ with

$$\mu_n e_\psi^*(P_{\min \text{rng}(w_n)−1}Tw_n) > \delta/2.$$ 

Notice that for any $\phi_n$ with $e_\phi^*, \delta_\psi^*$ having compatible tree-analyses we have $e_\phi^*(w_n) = 0$.

Now we repeat the reasoning of Case 2 of the proof of Prop. 7.2. The crucial fact here is that $w_n = y_n + z_n$, $n \in N$, with $(y_n)$ and $(z_n)$ $C$-RIS. It follows by (5.3) that for suitably chosen $A \subset N$, $\#A = n_j$, any semi-normalized vector of the form $\frac{c_{n_j}}{n_j} \sum_n w_n$ can be used to define depended sequences. This fact allows us to continue with the reasoning of Case 2 in the proof of Prop. 7.2 producing for any sufficiently large $j \in N$ a vector $y$ with $\|y\| \leq 80C/m^2j−1$ and $\|Ty\| \geq \delta/80Cm^2j−1$, which for sufficiently big $j$ yields a contradiction with boundedness of $T$.

\[ \square \]
Remark 7.7. One should notice that in the above proof, similarly to proofs of Prop. 7.2 and Prop. 7.4, Case 2, we do not use the full strength of the saturated norms technique, i.e., properties of dependent sequences, but we need only the estimates on the norm of averages of RIS provided by the so-called basic inequality. This is the reason why the statement from the beginning of the proof is true, contrary to the case of bounded operators on subspaces in the original Argyros-Haydon space: there is a strictly singular non-compact operator on a subspace of Argyros-Haydon space \[3\]. Recall that dependent sequences form an important tool in proving the scalar-plus-compact property in Argyros-Haydon space.

8. Unconditional saturation of the space \(X_{Kus}\)

This section is devoted to the proof of saturation of the space \(X_{Kus}\) by unconditional basic sequences. We follow the idea of the proof of the corresponding fact from \[3\] with additional work in order to control the bd-parts of norming functionals. Below we present a construction of unconditional sequences in \(X_{Kus}\).

Fix a block subspace \(Y \subset X_{Kus}\) and pick sequences \(j_k < j_{k,1} < j_{k,2} < \cdots < j_{k,n_k}, k \in \mathbb{N}\), with \((j_k)\) increasing, and a block sequence \((x_k)_k \subset Y\), with \(x_k = \frac{m_k}{n_k} \sum_{i=1}^{n_k} x_{k,i}\) where for some fixed \(C > 2\) and for each \(k \in \mathbb{N}\) the sequence \((x_{k,i})_i \subset Y\) is a \(C\)-RIS with parameters \((j_{k,i},i)\) chosen according to Lemma 5.4 to satisfy \(|d_{\gamma}(x_{k,i})| < 1/n_{j_k}^2\) for any \(i \leq n_{j_k}\) and \(\gamma \in \Gamma\). Therefore

\[
|d_{\gamma}(x_k)| < C/n_{j_k}^2 \quad \text{for any } k \in \mathbb{N}, \gamma \in \Gamma.
\]

The sequence \((x_k)\) is fixed for the sequel.

Let \(\gamma \in \Gamma\) and let \((I_t, \epsilon_t, \eta_t)_{t \in \mathcal{T}}\) be the tree-analysis of \(\gamma\). Recall that \(S_t\) denotes the set of immediate successors of \(t\) in the tree \(\mathcal{T}\). We order the sets \(S_t\) with the order on \(\{I_s\}_{s \in S_t}\) and we write \(s_\gamma\) for the immediate predecessor of \(s\).

Definition 8.1. Let \(\gamma \in \Gamma\) and let \((I_t, \epsilon_t, \eta_t)_{t \in \mathcal{T}}\) be the tree-analysis of \(\gamma\). A couple of nodes \((\eta_{s_\gamma}, \eta_s)\) is called a dependent couple with respect to \(\gamma\) if \(s_\gamma, s \in S_t\) and \(w(\eta_t) = m_{2j+1}^{-1}\) for some \(j \in \mathbb{N}\) and \(s\) is at the even position in the mt-part of \(e_{\eta_t}^s\).

Let \(\mathcal{E}_\gamma = \{s \in \mathcal{T} : (\eta_{s_\gamma}, \eta_s)\) is a dependent couple with respect to \(\gamma\}\).

Definition 8.2. Let \(\gamma \in \Gamma\) and let \((I_t, \epsilon_t, \eta_t)_{t \in \mathcal{T}}\) be the tree-analysis of \(\gamma\). For \(k \in \mathbb{N}\) a couple of nodes \((\eta_{s_\gamma}, \eta_s)\) is called a dependent couple with respect to \(\gamma\) and \(x_k\) if \((\eta_{s}, \eta_s)\) is a dependent couple with respect to \(\gamma\) and moreover

\[
\min \operatorname{supp}(x_{k+1}) > \max(e_{s}^* P_{I_s}) \geq \min \operatorname{supp}(x_k),
\]

\[
\max \operatorname{supp}(x_{k-1}) \geq \min(e_{s}^* P_{I_{s_\gamma}}).
\]

Remark 8.3. Note that if \((s_\gamma, s), (t_\gamma, t)\) are dependent couples it holds that \(t, s\) are incomparable nodes.

Let \(\mathcal{F}_\gamma = \{s \in \mathcal{T} : (\eta_{s_\gamma}, \eta_s)\) is a dependent couple with respect to \(x_k\) for some \(k\) and \(\gamma\}\) and let \(Q_\gamma = \sum_{s \in \mathcal{F}_\gamma} P_{I_s}\).

Then we define \(y_k = Q_\gamma x_k\) and \(x'_k = x_k - y_k\). As our basis \((d_{\gamma})_{\gamma \in \Gamma}\) is not unconditional, the projections \((Q_\gamma)_\gamma\) are not uniformly bounded. However, we have the following lemma repeating the proof line from \[3\].

Lemma 8.4. (i) For every \(k \in \mathbb{N}\) and \(t \in \mathcal{T}\) we have \(|e_{\eta_t}^s P_{I_t}(y_k)| \leq 10C/m_{j_k}\),

(ii) For every \(k \in \mathbb{N}\) and \(t \in \mathcal{T}\) with \(w(\eta_t) < m_{j_k}^{-1}\) we have \(|e_{\eta_t}^s P_{I_t}(x'_k)| \leq 11C/m_{j_k}\).
Proof. Concerning (i), notice first that for any \( s \in F_\gamma \) we have \( |e^*_n P_i(x_k)| \leq 20C/m_{jk} \). Indeed, for \( w(\eta_s) = m_{2j} \) for some \( j \), we consider the following two cases. If \( m_{2j}^{-1} \leq m_{jk}^{-1} \) then the estimate follows by \( \text{(5.1)} \). If \( m_{2j}^{-1} \geq m_{jk}^{-1} \), then by the form of \( e^*_n \) and \( \text{(8.1)} \) we have

\[
|e^*_n P_i(x_k)| \leq 2m_{2j} \max_{\gamma \in \Gamma} |d^*_\gamma(x_k)| \leq 2C/n_{jk}
\]

Now, as each of the sets \( \{ s \in F_\gamma | |s| = i, \text{rng}(x_k) \cap I_s \neq \emptyset \} \), \( i \in \mathbb{N} \), has at most two elements, we have

\[
|e^*_n P_i(y_k)| \leq \sum_{s \in F_\gamma} \left( \prod_{t < u < s} w(\eta_u) \right) |e^*_n P_i(x_k)| = \sum_{i \in \mathbb{N}} \sum_{s \in F_\gamma, |s| = i} \left( \prod_{t < u < s} w(\eta_u) \right) |e^*_n P_i(x_k)| \leq \frac{20C}{m_{jk}} \sum_{i} \frac{1}{m_i^1} = \frac{10C}{m_{jk}}
\]

Condition (ii) follows from Lemma \( \text{(5.2)} \) and (i).

**Lemma 8.5.** For \( \gamma \in \Gamma \) and every choice of signs \( (\delta_k) \) there exists a node \( \tilde{\gamma}_i \in \Gamma \) such that \( Q_\gamma = Q_{\tilde{\gamma}_i} \) and \( \epsilon \in \{ \pm 1 \} \) so that

\[
|e^*_\gamma(x_k^i) - ee^*_\gamma(\delta_k x_k^i)| \leq \frac{6C}{m_{jk}} \text{ for any } k \in \mathbb{N}.
\]

Proof. Define

\[
D = \{ t \in T | \text{rng}(x_k) \cap \text{rng}(e^*_t P_i) \neq \emptyset \text{ for at most one } k \}
\]

and if \( t \in S_\gamma \) then \( \text{rng}(x_i) \cap \text{rng}(e^*_u P_i) \neq \emptyset \) for at least two \( i \).

Since for every branch \( b \) of \( T \) the set \( b \cap D \) has exactly one element we can define a subtree \( T' \) of \( T \) such that \( D \) is the set of terminal nodes for \( T' \). Observe that for \( (T \setminus T') \cap F_\gamma = \emptyset \).

If \( \gamma \in D \), then we pick the unique \( k_0 \) with \( \text{rng}(e^*_t) \cap \text{rng}(x_{k_0}) \neq \emptyset \) (as \( I_0 = [1, \max \Delta_{\text{rank}(\gamma)}) \) and let \( \tilde{\gamma} = \gamma \) and \( \epsilon = \delta_{k_0} \). Then we have the estimate in the lemma for any \( k \in \mathbb{N} \).

Assume that \( \gamma \not\in D \). Using backward induction on \( T' \) we shall define a node \( \tilde{\gamma}_i \) with a tree-analysis \( (I_t, \tilde{\epsilon}_t, \tilde{\eta}_t) \in T' \), by modifying the nodes \( (I_t, \epsilon_t, \eta_t) \in T \), starting from elements of \( D \) such that

\[
\begin{align*}
(T1) & \; e^*_\gamma, e^*_i \text{ have compatible tree-analyses}, \\
(T2) & \; F_{\tilde{\eta}_t} = F_{\eta_t} \text{ for any } t \in T', \\
(T3) & \; \tilde{\epsilon}_t e^*_n P_i(\delta_k x_k^i) = \epsilon_t e^*_n P_i(x_k^i) \text{ for any } t \in T \setminus E_\gamma \text{ and } k, \\
(T4) & \; \tilde{\lambda}_t e^*_n P_i(\delta_k x_k^i) = \lambda_t e^*_n P_i(x_k^i) \text{ for any } t \in T \setminus E_\gamma \text{ and } k
\end{align*}
\]

Roughly speaking we need to modify only \( \epsilon_t, t \in D \), changing signs of some of them. These changes determines changes in the rest of the tree, i.e. \( \eta_u, u \in T' \setminus D \) according to the rules of producing nodes and Remark \( \text{(4.10)} \).

**Step 1.** Take \( t \in D \).

**Case 1a.** \( t \not\in E_\gamma \cup \bigcup_{u \in E_\gamma} S_u \). We set \( \tilde{\eta}_t = \eta_t \) and \( \tilde{\epsilon}_t = \delta_k \epsilon_t \) if \( \text{rng}(e^*_t P_i) \) intersects \( \text{rng}(x_k) \) for some (unique) \( k \), otherwise \( \tilde{\epsilon}_t = \delta_m \epsilon_t \) where \( m = \min \{ i : \text{rng}(e^*_n P_i) \leq \text{rng}(x_i) \} \).

The condition (T3) follows straightforward.

**Case 1b.** \( t \in E_\gamma \cup \bigcup_{u \in E_\gamma} S_u \). In this case we set \( \tilde{\eta}_t = \eta_t \) and \( \tilde{\epsilon}_t = \epsilon_t (= 1) \). Moreover, for \( t \in E_\gamma \) we set \( \tilde{\lambda}_t = \delta_k \lambda_t \). Such choice is possible since \( Net_{1,q} \) is symmetric and we have

\[
|\tilde{\lambda}_t - \epsilon_t e^*_{\eta_t} (y_{2i-1})| = |\delta_k \lambda_t - \delta_k \epsilon_t e^*_{\eta_t} (y_{2i-1})| = |\lambda_t - \epsilon_t e^*_{\eta_t} (y_{2i-1})|
\]

where \( (y_i^j) \) are the vectors of the suitable special sequences.

Notice that...
(1) if \( t \in \mathcal{F}_{\gamma} \) or \( t \in S_u \) for some \( u \in \mathcal{E}_{\gamma} \), then \( u \in \mathcal{F}_{\gamma} \), so \( \text{rng}(e^*_u P_t) \cap \text{rng} x'_k = \emptyset \) for any \( k \) by def. of \( x'_k \), so we have (T3).

(2) if on the other hand \( t \in \mathcal{E}_{\gamma} \setminus \mathcal{F}_{\gamma} \) and \( \text{rng}(e^*_u P_t) \cap \text{rng} x'_k \neq \emptyset \) for some \( k \) then \( e^*_u \in D \) as well and moreover \( \text{rng}(e^*_{\eta_u} P_t) \) either intersects only \( \text{rng} x_k \) or intersects no \( \text{rng} x_i \). In both cases \( \bar{\varepsilon}_{t-} = \delta_k \varepsilon_{t-} \) and so \( \lambda_t = \delta_k \lambda_t \) and (T3) holds.

Notice that in either case conditions (T1)-(T2) and (T4) are straightforward satisfied.

**Step 2.** Now we define inductively nodes in \( t \in \mathcal{T}' \setminus D \). Take \( t \in \mathcal{T}' \setminus D \) and assume we have defined \((\bar{\varepsilon}_s, \eta_s, I_s)_{s \in S_1}\) satisfying (T1)-(T4). In all cases we let \( \bar{\varepsilon}_t = \varepsilon_t \), thus (T4) is satisfied. Notice that \( t \not\in \bigcup_{u \in \mathcal{E}_{\gamma}} S_u \).

**Case 2a.** \( t \in \mathcal{E}_{\gamma} \). In this case we set \( \bar{\eta}_t = \eta_t \). Obviously we have (T1)-(T2).

**Case 2b.** \( t \not\in \mathcal{E}_{\gamma} \), \( w(\eta_t) = m_{2j}^{-1} \). Then using Remark 4.10 (1) we define \( \bar{\eta}_t \) so that

\[
\text{mt}(e^*_\bar{\eta}_t) = \frac{1}{m_{2j}} \sum_{s \in S_1} \bar{\varepsilon}_s e^*_{\eta_s} P_{I_s}.
\]

By definition we have (T1)-(T2).

**Case 2c.** \( w(\eta_t) = m_{2j+1}^{-1} \), with \( \eta_t \) compatible with a \((\Gamma, j)\)-special sequence \((\bar{x}_s, \bar{\eta}_t)\). Then using Remark 4.10 (2) we define a special node \( \bar{\eta}_t \) which is compatible with the same \((\Gamma, j)\)-special sequence \((\bar{x}_s, \bar{\eta}_t)\) so that

\[
\text{mt}(e^*_\bar{\eta}_t) = \frac{1}{m_{2j+1}} \sum_{s \in S_1 \cap \mathcal{E}_{\gamma}} (\bar{\varepsilon}_s e^*_{\eta_s} P_{I_s} + \bar{\lambda}_s e^*_{\eta_s} P_{I_s}).
\]

By definition we have (T1)-(T2).

Let \( \bar{\gamma} = \bar{\eta}_t \). Notice that by conditions (T1)-(T2) we have \( Q_{\bar{\gamma}} = Q_{\gamma} \).

Now we proceed to show the estimate part of the lemma. Fix \( k \in \mathbb{N} \). For any not terminal \( u \in \mathcal{T} \) let

\[
S_{u,k} := \{ s \in S_u \mid \text{rng}(x_k) \cap \text{rng}(e^*_u P_t) \neq \emptyset \}.
\]

Let \( G \) be the set of minimal nodes \( u \) of \( \mathcal{T}' \) with \( u \in D \) or \( w(\eta_u) < m_{jk}^{-1} \). By \( T'' \) denote the subtree of \( T' \) with the terminal nodes in \( G \).

We shall prove by induction starting from \( G \) that for any \( u \in T'' \) we have

\[
|e_u e^*_{\eta_u} P_{I_u}(x'_k) - \bar{\varepsilon}_u e^*_{\eta_u} P_{I_u}(\delta_k x'_k)| \leq \frac{22C}{m_{jk}}.
\]

This will end the proof as it follows by (T4) that \( |e_{\emptyset(x'_k)} e^*_{\eta_{\emptyset(x'_k)}} (\delta_k x'_k)| = |e^*_\gamma(x'_k) - e^*_\gamma(\delta_k x'_k)| \).

Thus taking \( \varepsilon = 1 \) we obtain the estimate of the lemma.

**Step 1.** \( u \in G \). If \( w(\eta_u) < m_{jk}^{-1} \) then the estimate (8.2) holds true by Lemma 8.3 (2). If \( u \not\in D \) then the estimate (8.2) holds true by (T3).

**Step 2.** \( u \in T'' \setminus G \). In particular \( w(\eta_u) \geq m_{jk}^{-1} \). Obviously \( S_u \subset T'' \).

**Case 2a.** \( w(\eta_u) = m_{2j}^{-1} \).
We estimate using (T3)

\[
|e^*_{\eta_u} P_{I_u}(\delta_k x'_k) - e^*_{\eta_u} P_{I_u}(x'_k)| = \left| \left( \sum_{s \in S_u} d_{\xi_k}^s + \frac{1}{m_{2j}} \sum_{s \in S_u \cap D} \tilde{\epsilon}_{s} e^*_{\eta_u} P_{I_s} + \frac{1}{m_{2j}} \sum_{s \in S_{u,k} \setminus D} \tilde{\epsilon}_{s} e^*_{\eta_u} P_{I_s} \right) (\delta_k x'_k) \right|
\]

\[
- \left( \sum_{s \in S_u} d_{\xi_k}^s + \frac{1}{m_{2j}} \sum_{s \in S_{u,k} \cap D} \epsilon_{s} e^*_{\eta_u} P_{I_s} + \frac{1}{m_{2j}} \sum_{s \in S_{u,k} \setminus D} \epsilon_{s} e^*_{\eta_u} P_{I_s} \right) |x'_k|\]

\[
\leq \left| \sum_{s \in S_u} d_{\xi_k}^s (\delta_k x'_k) \right| + \left| \sum_{s \in S_u} d_{\xi_k}^s (x'_k) \right| + \frac{1}{m_{2j}} \sum_{s \in S_{u,k} \setminus D} |\epsilon_{s} e^*_{\eta_u} P_{I_s}(\delta_k x'_k) - \epsilon_{s} e^*_{\eta_u} P_{I_s}(x'_k)|
\]

\[\leq \ldots\]

The first two sums estimate using (8.1) and \#\(S_u \leq m_{2j} \leq n_{jk}\), for the third element use the inductive hypothesis and the fact that \(#(S_{u,k} \setminus D) \leq 2\), obtaining the following

\[
\ldots \leq 2n_{2j} \frac{C}{n_{jk}} + \frac{2}{m_{2j}} \frac{22C}{m_{jk}} \leq \frac{22C}{m_{jk}}.
\]

**Case 2b.** \(w(\eta_u) = m_{2j+1}\).

Recall that by (T3) we have \(\epsilon_{s} e^*_{\eta_u} P_{I_{s,\gamma}}(x'_k) = \tilde{\epsilon}_{s} e^*_{\eta_u} I_{s,\gamma}(\delta_k x'_k)\) for any \(s \in S_u \cap \mathcal{E}_{\gamma}\) with \(s \in D\) and \(\lambda_s e^*_{\eta_u} P_{I_s}(x'_k) = \tilde{\lambda}_s e^*_{\eta_u} I_s(\delta_k x'_k)\) for any \(s \in S_u \cap \mathcal{E}_{\gamma} \cap D\). Moreover \(\mathcal{E}_{\gamma} \setminus D \subset \mathcal{F}_{\gamma}\) thus \(e^*_{\eta_u} P_{I_s}(x'_k) = 0 = e^*_{\eta_u} P_{I_s}(\delta_k x'_k)\) for any \(s \in (S_u \cap \mathcal{E}_{\gamma}) \setminus D\). Therefore we have

\[
|e^*_{\eta_u} P_{I_u}(\delta_k x'_k) - e^*_{\eta_u} P_{I_u}(x'_k)|
\]

\[= \left| \left( \sum_{s \in S_u} d_{\xi_k}^s + \frac{1}{m_{2j+1}} \sum_{s \in S_{u,k} \cap \mathcal{E}_{\gamma}} \tilde{\epsilon}_{s} e^*_{\eta_u} P_{I_s} + \frac{1}{m_{2j+1}} \sum_{s \in S_{u,k} \setminus \mathcal{E}_{\gamma}} \tilde{\lambda}_s e^*_{\eta_u} P_{I_s} \right) (\delta_k x'_k) \right|
\]

\[\leq \ldots\]

Proceeding as in Case 2a we obtain

\[
\ldots \leq 2n_{2j+1} \frac{C}{n_{jk}} + \frac{2}{m_{2j+1}} \frac{22C}{m_{jk}} \leq \frac{22C}{m_{jk}}.
\]

**Theorem 8.6.** The space \(X_{K_{us}}\) is unconditionally saturated.
Proof. In every block subspace of $X_{Kus}$ pick a sequence $(x_k)_k$ as above with $m_{j_1} > 100C$. We claim that such a sequence is unconditional. To this end consider finite sequence of scalars $(a_k)$ with $\|\sum_k a_k x_k\| = 1$ and $(\delta_k) \subset \{\pm 1\}$. We want to estimate the norm of the vector $\sum_k \delta_k a_k x_k$. Take $\gamma \in \Gamma$ with $e^{\gamma}_* (\sum_k a_k x_k) = 1$. Define $Q_\gamma$, $(y_k)$ and $(x'_k)$ and consider $\tilde{\gamma}$ and $\epsilon$ provided by Lemma 8.5. Notice that as $Q_{\tilde{\gamma}} = Q_\gamma$, the projection $Q_\gamma$ defines also $(y_k)$ and $(x'_k)$. Estimate, applying Lemma 8.5 and Lemma 8.3 (1) both for $\gamma$ and $\tilde{\gamma}$, as follows

$$|e^{\gamma}_* (\sum_k a_k x_k) - e^{\tilde{\gamma}}_* (\sum_k \delta_k a_k x_k)|$$

$$\leq |e^{\gamma}_* (\sum_k a_k x'_k) - e^{\tilde{\gamma}}_* (\sum_k \delta_k a_k x'_k)| + |e^{\gamma}_* (\sum_k a_k y_k)| + |e^{\tilde{\gamma}}_* (\sum_k \delta_k y_k)|$$

$$\leq \sum_k |a_k| |e^{\gamma}_* (x'_k) - e^{\tilde{\gamma}}_* (\delta_k x'_k)| + \sum_k |a_k| |e^{\gamma}_* (y_k)| + \sum_k |\delta_k| |e^{\tilde{\gamma}}_* (y_k)|$$

$$\leq 24C \sum_k m_{j_k}^{-1} \leq 48C m_{j_1}^{-1} \leq 1/2.$$

Therefore $\|\sum_k \delta_k a_k x_k\| \geq |e^{\gamma}_* (\sum_k \delta_k a_k x_k)| \geq 1/2$, which ends the proof. \[\square\]

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