Modeling scalar fields consistent with positive mass

Tetsuya Shiromizu
Department of Physics, Kyoto University

With Masato Nozawa (KEK)

Nozawa and Shiromizu, Physical Review D89, 023011 (2014)
1. Introduction
2. Positive mass theorem
3. Einstein-scalar system
4. Future issues
1. Introduction
Positive mass theorem

~ Positive mass theorem

Schoen&Yau 1981, Witten 1981, Gibbons et al 1983,…

\[ M \geq 0 \]

\[ M = 0 \iff \text{Minkowski/anti-deSitter} \]

for GR, SUGRA, regular spacetimes, energy condition, …

The existence of ground state
Restriction on theories

Scalar potentials consistent with positive mass

Boucher 1984, Townsend 1985

\[ L = R - \frac{1}{2} (\nabla \phi)^2 - U(\phi) \]

\[ U(\phi) = 8 \left( \frac{dW(\phi)}{d\phi} \right)^2 - 12(W(\phi))^2 \]

Cf) SUGRA, W superpotential
Summary of our work

The cases consistent with positive mass are

(i) \[ K = X - U(\phi) = X - 8 \left( \frac{dW(\phi)}{d\phi} \right)^2 + 12(W(\phi))^2 \]

(ii) \[ K = 4\sqrt{2} \frac{dW(\phi)}{d\phi} (-X)^{1/2} + 12(W(\phi))^2 \]

- **strong restriction**
- **classical stability is automatically guaranteed**

No cosmological solution

Nozawa & Shiromizu 2014

\[ S = \int d^4 x \sqrt{-g} \left[ R + 2K(\phi, X) + L_{\text{matter}} \right] \]

\[ X = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \]
2. Positive mass theorem

Back to Witten 1981
Positivity: essence

\[ \gamma^i \nabla_i \varepsilon = 0 \quad \varepsilon: \text{spinor} \]

\[ M \sim \int_{S_\infty} \varepsilon^+ \nabla_i \varepsilon dS^i \]

\[ = \int_\Sigma \nabla^i (\varepsilon^+ \nabla_i \varepsilon) d\Sigma \]

\[ = \int_\Sigma \left( | \nabla \varepsilon |^2 + \varepsilon^+ \nabla^2 \varepsilon \right) d\Sigma \]

\[ \sim \int_\Sigma \left( | \nabla \varepsilon |^2 + T_{00} | \varepsilon |^2 \right) d\Sigma \geq 0 \]

If the energy-momentum tensor satisfies the energy condition, we can prove the positivity of mass.
Rigidity

\[ M \sim \int_{\Sigma} \left( \left| \nabla \varepsilon \right|^2 + T_{00} \left| \varepsilon \right|^2 \right) d\Sigma = 0 \]

⇒ \nabla \varepsilon = 0

⇒ \ R_{abcd} = 0

Minkowski spacetime
Precisely

\[ N^{\mu\nu} := -i\left( \bar{\epsilon} \gamma^{\mu\nu\rho} \nabla_\rho \epsilon - \overline{\nabla_\rho \epsilon} \gamma^{\mu\nu\rho} \epsilon \right) \quad \bar{\epsilon} = i \epsilon^+ \gamma^0 \]

\[ \frac{1}{2} \int_{\partial \Sigma} N_{\mu\nu} dS^{\mu\nu} = -\int_\Sigma \nabla_\nu N^{\mu\nu} u_\mu d\Sigma \]

\[ u^\mu : \text{future directed unit normal vector to } \Sigma \]

\[ \nabla_\nu N^{\mu\nu} = 2i \nabla_\rho \epsilon \gamma^{\mu\nu\rho} \nabla_\nu \epsilon - G_\nu^{\mu} V^\nu \]

\[ V^\mu := i \bar{\epsilon} \gamma^\mu \epsilon \]

\[ \gamma^i \nabla_i \epsilon = 0 \]

\[ \Rightarrow 8\pi GM = \frac{1}{2} \int_{\partial \Sigma} N_{\mu\nu} dS^{\mu\nu} = \int_\Sigma \left( 2 |\nabla_i \epsilon|^2 - 8\pi G T_\mu^0 V^\mu \right) d\Sigma \geq 0 \]

\( \geq 0 \) (energy condition)
3. Einstein-scalar system

Nozawa & Shiromizu 2014
action

\[ S = \int d^4 x \sqrt{-g} \left[ R + 2K(\phi, X) + 2L_{\text{matter}} \right] \]

\[ X = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \]

\[ G_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(\text{matter})}_{\mu\nu} \]

\[ T^{(\phi)}_{\mu\nu} = \partial_\mu K \nabla_\nu \phi + Kg_{\mu\nu} \]

does not satisfy the (dominant) energy condition in general
\[ \hat{\nabla}_\mu \varepsilon = (\nabla_\mu + A_\mu) \varepsilon, \quad \gamma^i \hat{\nabla}_i \varepsilon = 0 \]

\[ 8\pi GM = \int \Sigma \left[ 2i \hat{\nabla}_\rho \varepsilon \gamma^{\mu \nu \rho} \hat{\nabla}_\nu \varepsilon - G^\mu_\nu V^\nu + S^\mu \right] \mu \mu \]

\[
\begin{cases}
G^\mu_\nu := R^\mu_\nu - \frac{1}{2} g^\mu_\nu R \\
V^\mu = i \bar{\varepsilon} \gamma^\mu \varepsilon \\
S^\mu := -i \bar{\varepsilon} \gamma^{\mu \nu \rho} F^\nu_\mu \varepsilon \\
F^\mu_\nu = 2(\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]} )
\end{cases}
\]
We imposed

\[ \overline{A}_\nu \gamma^{\mu \nu \rho} = \gamma^{\mu \nu \rho} A_\nu \]

Otherwise, non-controllable terms appear

\[ \nabla_\nu \hat{N}^{\mu \nu} = 2i \hat{\nabla}_\rho \varepsilon \gamma^{\mu \nu \rho} \hat{\nabla}_\nu \varepsilon - G^{\mu \nu} \varepsilon - \frac{i}{2} \varepsilon (\overline{F}_{\nu \rho} \gamma^{\mu \nu \rho} + \gamma^{\mu \nu \rho} F_{\nu \rho}) \varepsilon \]

\[ -i \bar{\varepsilon} (\overline{A}_\nu \gamma^{\mu \nu \rho} - \gamma^{\mu \nu \rho} A_\nu) \hat{\nabla}_\rho \varepsilon + i \hat{\nabla}_\rho \varepsilon (\overline{A}_\nu \gamma^{\mu \nu \rho} - \gamma^{\mu \nu \rho} A_\nu) \varepsilon \]

\[ F_{\mu \nu} = 2(\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]}) \]
\[ \gamma^i \hat{\nabla}_i \varepsilon = 0 \quad \varepsilon : \text{spinor} \]

\[ M \sim \int_{S_\infty} dS_i \varepsilon^+ \hat{\nabla}^i \varepsilon = \int_{\Sigma} \nabla_i (\varepsilon^+ \hat{\nabla}^i \varepsilon) d\Sigma \]

\[ T^{(\phi)}_{\mu \nu} = \partial_X K \nabla_\mu \phi \nabla_\nu \phi + K g_{\mu \nu} \]

\[ = \int_{\Sigma} \left( \| \hat{\nabla} \varepsilon \|^2 + T_{00}^{\text{matter}} + T^{(\phi)}_{00} + S^0 \right) d\Sigma \]

Einstein eq.

Look for the theory for scalar field to have the form \[ \left| \delta \lambda \right|^2 \] for
\[ A_\mu = W(\phi) \gamma_\mu \]

\[
S^\mu = -i \bar{\epsilon} \gamma^{\mu \nu \rho} F_{\mu \nu} \epsilon \\
= -4i \bar{\epsilon} \gamma^{\mu \nu} \epsilon \nabla_\nu \phi \partial_\phi W + 12 V^\mu W^2 \\
= i \bar{\delta} \lambda \gamma^\mu \partial_\phi + V^\nu \left[ f^2 \nabla^\mu \phi \nabla_\nu \phi + \left( -\frac{1}{2} f^2 (\nabla \phi)^2 - 8 f^{-2} (\partial_\phi W)^2 + 12 W^2 \right) \right] \\
\]

\[ \delta \lambda := \frac{1}{\sqrt{2}} \left( f(\phi, X) \gamma^\mu \nabla_\mu \phi - 4 f^{-1}(\phi, X) \frac{dW(\phi)}{d\phi} \right) \epsilon \]

\[ \hat{\nabla}_\mu \epsilon = (\nabla_\mu + A_\mu) \epsilon \]

\[ 8\pi G M = \int_\Sigma d\Sigma \left[ 2i \hat{\nabla}_\rho \epsilon \gamma^{\mu \nu \rho} \hat{\nabla}_\nu \epsilon - G^\mu_\nu V^\nu + S^\mu_\mu \right] \]

\[ F_{\mu \nu} = 2(\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]} ) \]

\[ \hat{\nabla}_\mu \epsilon = (\nabla_\mu + A_\mu) \epsilon \]

\[ \delta \lambda := \frac{1}{\sqrt{2}} \left( f(\phi, X) \gamma^\mu \nabla_\mu \phi - 4 f^{-1}(\phi, X) \frac{dW(\phi)}{d\phi} \right) \epsilon \]

\[ S^\mu = i \bar{\delta} \lambda \gamma^\mu \partial_\phi + T_{\nu}^{(\phi) \mu} V^\nu \]

If

\[
\begin{cases}
\partial_x K = f^2 \\
K = f^2 X - 8 f^{-2} \left( \frac{dW(\phi)}{d\phi} \right)^2 + 12 (W(\phi))^2
\end{cases}
\]
Then

\[
\begin{aligned}
\partial_x K &= f^2 \\
K &= f^2 X - 8f^{-2} \left( \frac{dW(\phi)}{d\phi} \right)^2 + 12(W(\phi))^2 \\
XK_x - K - \frac{8W^2_\phi}{K_x} &= -12W(\phi)^2 \\
\partial_x \left( XK_x - K - \frac{8W^2_\phi}{K_x} \right) &= K_{xx} \left( X + \frac{8W^2_\phi}{K_x^2} \right) = 0
\end{aligned}
\]

(i) \( K_{xx} = 0 \) \hspace{1cm} (ii) \( X + \frac{8W^2_\phi}{K_x^2} = 0 \)

\[
\begin{aligned}
K &= X - U(\phi) = X - 8 \left( \frac{dW(\phi)}{d\phi} \right)^2 + 12(W(\phi))^2 \\
K &= 4\sqrt{2} \left( \frac{dW(\phi)}{d\phi} \right)(-X)^{1/2} + 12(W(\phi))^2
\end{aligned}
\]
Case (ii)

\[ K = 4\sqrt{2} \frac{dW(\phi)}{d\phi} (-X)^{1/2} + 12(W(\phi))^2 \]

For homogeneous-isotropic spacetimes,

\[ \phi = \phi(t) \implies X = \dot{\phi}^2 / 2 > 0 \]

Due to the factor of \((-X)^{1/2}\), the case (ii) does not work
Summary

\[ S = \int d^4 x \sqrt{-g} \left[ R + 2K(\phi, X) + L_{\text{matter}} \right] \]

\[ X = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \]

(i)

\[ K = X - U(\phi) = X - 8 \left( \frac{dW(\phi)}{d\phi} \right)^2 + 12(W(\phi))^2 \]

Canonical form with “superpotential”

(ii)

\[ K = 4\sqrt{2} \frac{dW(\phi)}{d\phi} (-X)^{1/2} + 12(W(\phi))^2 \]

No cosmological solution

Nozawa & Shiromizu 2014
4. Future issues
Future issues

\[ \bar{A}_\nu \gamma^{\mu \nu \rho} = \gamma^{\mu \nu \rho} A_\nu \]  
\[ A_\mu = W(\phi) \gamma_\mu \]  

general enough?

Extension to more general cases/modified gravity?
Some basics
Local Lorentz transformation \( \varepsilon^{\hat{\alpha}\hat{\nu}} \)

\[ \psi \rightarrow \psi + \delta \psi, \quad \delta \psi = \frac{1}{4} \varepsilon^{\hat{\mu}\hat{\nu}} \gamma_{\hat{\mu}\hat{\nu}} \psi \]

\[ \nabla_\mu \psi = \left( \partial_\mu + \frac{1}{4} \omega^{\hat{\alpha}\hat{\beta}}_\mu \gamma_{\hat{\alpha}\hat{\beta}} \right) \psi \]

\( \omega^{\hat{\alpha}\hat{\beta}}_\mu \rightarrow \omega^{\hat{\alpha}\hat{\beta}}_\mu + \delta \omega^{\hat{\alpha}\hat{\beta}}_\mu \)

\[ \delta \omega^{\hat{\alpha}\hat{\beta}}_\mu = \varepsilon^{\hat{\alpha}}_\gamma \omega^{\hat{\beta}}_\mu + \varepsilon^{\hat{\beta}}_\gamma \omega^{\hat{\alpha}}_\mu - \partial_\mu \varepsilon^{\hat{\alpha}\hat{\beta}} \]

\[ \Rightarrow \delta(\nabla_\mu \psi) = \frac{1}{4} \varepsilon^{\hat{\alpha}\hat{\beta}} \gamma_{\hat{\alpha}\hat{\beta}} \nabla_\mu \psi \]

\[ D_\mu e^{\hat{\alpha} \nu} := \partial_\mu e^{\hat{\alpha} \nu} + \omega_{\hat{\alpha} \beta \mu} e^{\hat{\beta} \nu} - \Gamma^{\rho}_{\mu \nu} e^{\hat{\alpha} \rho} = \nabla_\mu e^{\hat{\alpha} \nu} + \omega_{\hat{\alpha} \beta \mu} e^{\hat{\beta} \nu} = 0 \]

\[ \Rightarrow D_\mu g_{\alpha \beta} = 0 \]
Witten spinor

\[ \gamma^i D_i \varepsilon = 0 \]  
(Witten equation)

\[ D_i \varepsilon = (\partial_i + (n-1) \Gamma_i) \varepsilon, \]
\[ (n-1) \Gamma_i = -\frac{1}{8} (e^k)^j D_i (e^l)^j [\gamma_i, \gamma_k] \]
\[ g_{kl} (e^i)^k (e^j)^l = \delta_{ij} \]

We have solutions which are asymptotically approaches a constant spinor

\[ \varepsilon \xrightarrow{r \to \infty} \varepsilon_0 \]

\((\Sigma, q)\): (n - 1) - dim. spacelike hypersurface

\[ S_\infty \]
Proof

\[ \frac{1}{2} D^i D_i |\varepsilon|^2 = |D\varepsilon|^2 + \frac{1}{4} (n-1) R |\varepsilon|^2 \]

\[ 8 \pi M_{ADM} |\varepsilon_0|^2 = \frac{1}{2} \int dS_i (\varepsilon^+ D^i \varepsilon + \text{c.c.}) = \int \sum \left[ |D\varepsilon|^2 + \frac{1}{4} (n-1) R |\varepsilon|^2 \right] \]

\[ (n-1) R \geq 0 \quad \Rightarrow \quad M_{ADM} \geq 0 \]

\[ M_{ADM} = 0 \quad \Rightarrow \quad D_i \varepsilon = 0 \quad \Rightarrow \quad [D_i, D_j] \varepsilon \propto (n-1) R_{ijkl} [\gamma^k, \gamma^l] \varepsilon = 0 \]

\[ \sum \text{ is flat space} \]
\[
\varepsilon^+ D_i \varepsilon = \varepsilon_0^+ \partial_i \varepsilon + \varepsilon_0^{+ (n-1)} \Gamma_1 \varepsilon_0
\]

\[
\frac{1}{2} \int dS_1 (\varepsilon^+ D^1 \varepsilon + c.c.) = \int dS^1 \varepsilon_0^+ (\Gamma_1 - \gamma_i \gamma^{j (n-1)} \Gamma_j) \varepsilon_0 - \int dS^i \varepsilon_0 \gamma^A \partial_A \varepsilon
\]

\[
= \frac{1}{4} \int dS^i \varepsilon_0^+ (\partial_i h_j^i - \partial_j h_i^j) \varepsilon_0
\]

\[
(\Gamma_i = \frac{1}{16} (\partial^j h^k_i - \partial^k h^j_i) [\gamma_k, \gamma_j] + O \left( \frac{1}{r^{n-1}} \right))
\]

\[
g_{ij} = \delta_{ij} + h_{ij}, \quad (e^i)_j = (0^i)^i_j + \frac{1}{2} h_{kj} (0^i)^k_j
\]