Level One Representations of Quantum Affine Algebras $U_q(C_n^{(1)})$

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Abstract

We give explicit constructions of quantum symplectic affine algebras at level 1 using vertex operators.

1 Introduction

Vertex operator constructions of affine Lie algebras started with the work of Lepowsky-Wilson [14] in the principal picture. The homogeneous construction was given by Frenkel-Kac [3] and Segal [13] for the simply laced types. The non-simply laced cases were provided in [2] and [8]. These constructions led to many applications in mathematics and physics.

The quantization of these constructions for the quantum affine algebras started with the work of Frenkel-Jing [4], where the simply laced types were constructed. Subsequently the twisted types $(ADE)^{(r)}$ [2], type $B_n^{(1)}$ [11] and $G_2^{(1)}$ [8] at level 1 were constructed. Very recently we have provided explicit realizations of the quantum symplectic affine algebra $U_q(C_n^{(1)})$ at level $-1/2$ in [12], which are the so-called admissible representations (with rational levels). The bosonic realizations of $U_q(C_n^{(1)})$ at level $-1/2$ were
obtained by implicitly quantizing some $\beta\gamma$-system. All these constructions have been used in obtaining $q$-vertex operators, which in turn provide solutions to associated quantum Knizhnik-Zamolodchikov equations.

In this paper we give an explicit construction of $U_q(C_n^{(1)})$ at level 1. Our constructions can be viewed as a quantization of the constructions in [2] and [6]. We consider some auxiliary bosonic fields to build the Fock space. The action of the quantum affine algebra on the Fock space is given via vertex operators. We hope that our paper also further clarifies and simplifies the classical constructions.

The idea is to represent the vertex operator as a sum of two fields, where each one resembles to the original field operator. This idea was already present in the classical cases, but one needs new techniques to be able to quantize the construction.

The realization of level $-1/2$ is easier than the level 1 case due to the simpler character formulas and $\beta\gamma$-system. As we saw in [12] the level $-1/2$ construction is also completely free from cocycle consideration, which is critical in almost all level one cases.

It turns out that there is really a subtlety about the sign changes in the construction. Besides the usual cocycle construction, we have to incorporate further sign factors $(-1)^{(1-\epsilon)\partial_i} (\epsilon = \pm 1)$ to simplify the original construction of [2]. It is fair to say that the correct construction depends on whether we completely understand the behavior of the sign factors, which may partly explain why our simple construction is found now.

In all rigorous constructions of quantum affine algebras, the Serre relations are always the most complicated relations to prove. As explained by Jing in [7] and more generally in [11] one has to be able to prove some combinatorial identities (cf. (21, 22)). We use the ideas of [7] and [11] to prove all Serre relations.

The paper is organized as follows. In section two we give the basic notations and preliminaries. In the next section the main results are presented. The proof of the main theorem occupies section four. Our representation is a reducible representation, and we show that our space contains submodules with all level one dominant highest weights for $U_q(C_n^{(1)})$.

2 Quantum affine algebras $U_q(C_n^{(1)})$

Let $\alpha_i = e_i - e_{i+1}$ ($i = 1, \cdots, n-1$) and $\alpha_n = 2e_n$ be the simple roots of the simple Lie algebra $sp_{2n}$, and $\lambda_i = e_1 + \cdots + e_i$ ($i = 1, \cdots, n$) be the
fundamental weights. Let \( P = \mathbb{Z}c_1 + \cdots + \mathbb{Z}c_n \) and \( Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n \) be the weight and root lattices. We then let \( \Lambda_i \), \( i = 0, \ldots, n \) be the fundamental weights for the affine Lie algebra \( \widehat{sp}_{2n} \), here \( \Lambda_i = \lambda_i + \Lambda_0 \). We still use \( \alpha_i \) \((i = 1, \ldots, n)\) together with \( \alpha_0 \) as the simple roots for the affine Lie algebra \( \widehat{sp}_{2n} \). The nondegenerate symmetric bilinear form \(( \cdot, \cdot )\) on \( h^* \), the dual Cartan subalgebra of \( \widehat{sp}_{2n} \), satisfies that
\[
(e_i | e_j) = \frac{1}{2} \delta_{ij}, \quad (\delta | \alpha_i) = (\delta | \delta) = 0 \quad \text{for all } i, j \in I, \tag{1}
\]
and then the generalized Cartan matrix \((A_{ij})\) of \( C^{(1)}_n \) is given by \( A_{ij} = \frac{\langle \alpha_i, \alpha_j \rangle}{d_i} \), where \( d_i = \sum_{\alpha \in \Delta} \frac{\langle \alpha, \alpha \rangle}{2} \) and \((d_0, \ldots, d_n) = (1, 1/2, \ldots, 1/2, 1)\).

For \( q \) generic (not a root of unity) let \( q_i = q^{d_i}, i \in I \). The quantum affine algebra \( U_q(C^{(1)}_n) \) is the associative algebra with 1 over \( C(q^{1/2}) \) generated by the elements \( x_{ik}^+, a_{ij}, K_i^\pm, \gamma^\pm/2, q^{\pm d} \ (i = 1, 2, \ldots, n, k \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\}) \) with the following defining relations:
\[
[\gamma^\pm/2, u] = 0 \quad \text{for all } u \in U_q(C^{(1)}_n), \tag{2}
\]
\[
[a_{ik}, a_{jl}] = \delta_{k+l,0} \left[ \frac{\langle \alpha_i, \alpha_j \rangle \gamma^k - \gamma^{-k}}{q^k - q^{-k}} \right], \tag{3}
\]
\[
[a_{ik}, K_j^\pm] = [q^{\pm d}, K_j^\pm] = 0, \tag{4}
\]
\[
q^{d_{ik}q^{-d}} = q^{k} x_{ik}^+q^{-d} = q^{d} a_{id}q^{-d} = q^{l} a_{il}, \tag{5}
\]
\[
K_i x_{jk}^+ K_i^{-1} = q^{\pm \langle \alpha_i, \alpha_j \rangle} x_{jk}^+, \tag{6}
\]
\[
[a_{ik}, x_{jl}^\pm] = \pm \frac{\langle \alpha_i, \alpha_j \rangle \gamma^{\mp |k|/2}}{k} x_{jk+l}^\pm, \tag{7}
\]
\[
(z - q^{\pm \langle \alpha_i, \alpha_j \rangle} w) X_{i}^\pm(z) X_{j}^\mp(w) + (w - q^{\pm \langle \alpha_i, \alpha_j \rangle} z) X_{j}^\pm(w) X_{i}^\mp(z) = 0 \tag{8}
\]
\[
[X_{i}^+(z), X_{j}^-(w)] = \frac{\delta_{ij}}{(q_i - q_i^{-1})z w} \left( \psi_i(w^{\gamma^1/2}) \delta(w^{\gamma^{-1}}) \right)
- \varphi_i(w^{\gamma^{-1}/2}) \delta(w^{\gamma-1}) \tag{9}
\]
where \( X_i^\pm(z) = \sum_{m \in \mathbb{Z}} x_{i,m}^\pm z^{-m-1} \), \( \psi_{im} \) and \( \varphi_{im} \ (m \in \mathbb{Z}_{\geq 0}) \) are defined by
\[
\psi_i(z) = \sum_{m=0}^{\infty} \psi_{im} z^{-m} = K_i e^{\exp \left( (q - q^{-1}) \sum_{k=1}^{\infty} a_{ik} z^{-k} \right)},
\]
\[
\varphi_i(z) = \sum_{m=0}^{\infty} \varphi_{im} z^{m} = K_i^{-1} e^{\exp \left( -(q - q^{-1}) \sum_{k=1}^{\infty} a_{i,-k} z^{k} \right)},
\]
where \([m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})\).

### 3 The main results

Let \(m \in \mathbb{Z}\). Let \(a_i(m)\) \((i = 1, \ldots, n)\) and \(b_i(m)\) \((i = 1, \ldots, n - 1)\) be two sets of independent operators satisfying the following Heisenberg relations:

\[
\begin{align*}
[a_i(m), a_j(l)] &= \delta_{m+l,0} \frac{[\alpha_i, \alpha_j]}{m}[m], \\
[b_i(m), b_j(l)] &= \delta_{m+l,0} \frac{[\alpha_i, \alpha_j]}{m}[m], \\
[a_i(m), b_j(l)] &= 0, \\
[a_i(0), a_j(m)] = [b_i(0), b_j(m)] &= 0
\end{align*}
\]

In order to construct the Fock space we introduce an identical copy of the root lattice of \(A_{n-1}\) as the sublattice \(\tilde{Q} = Q[A_{n-1}]\) of short roots of \(Q\). The basis of \(\tilde{Q}\) will be denoted by \(\tilde{\alpha}_i, i = 1, \ldots, n - 1\). Thus

\[
(\tilde{\alpha}_i|\tilde{\alpha}_j) = (\alpha_i|\alpha_j) = \delta_{ij} - \frac{1}{2} \delta_{|i-j|,1}.
\]

We also consider the associated weight lattice \(\tilde{P} = P[A_{n-1}]\) defined by the inner product.

The Fock space \(\mathcal{V}\) is defined to be the tensor product of the symmetric algebra generated by \(a_i(-m), b_i(-m)\) and the group algebra generated by \(e^\lambda \otimes e^{\tilde{\lambda}}\) such that \((\tilde{\alpha}_i|\lambda) \pm (\tilde{\alpha}_i|\tilde{\lambda}) \in \mathbb{Z}\) for each \(i \in \{1, \ldots, n\}\), where \(\lambda \in P\) and \(\tilde{\lambda} \in \tilde{P}\). Note that we treat \(\tilde{\alpha}_n = 0\).

The action of \(a_i(m)\) and \(b_i(m)\) with \((m \neq 0)\) on \(\mathcal{V}\) is obtained by considering the Fock space \(\mathcal{V}\) as some quotient space of the Heisenberg algebra tensored with the group algebras of \(P\) and \(\tilde{P}\). The operators \(a_i(0), b_i(0), e^\alpha, e^{\tilde{\alpha}}\) act on \(\mathcal{V}\) by

\[
\begin{align*}
a_i(0)e^\lambda e^{\tilde{\lambda}} &= (\alpha_i|\lambda)e^\lambda e^{\tilde{\lambda}}, \\
b_i(0)e^\lambda e^{\tilde{\lambda}} &= (\tilde{\alpha}_i|\tilde{\lambda})e^\lambda e^{\tilde{\lambda}}, \\
e^\alpha e^\lambda e^{\tilde{\lambda}} &= e^{\alpha+\lambda}e^{\tilde{\lambda}}, \\
e^{\tilde{\alpha}}e^\lambda e^{\tilde{\lambda}} &= e^{\lambda+\tilde{\alpha}}e^{\tilde{\lambda}}.
\end{align*}
\]

The normal product : is defined as usual:

\[
:a_i(m)a_j(l) := a_i(m)a_j(l) (m \leq l), \text{ or } a_j(l)a_i(m) (m > l),
\]
\[ e^\alpha a_i(0) := a_i(0)e^\alpha := e^\alpha a_i(0), \]
\[ e^\alpha b_i(0) := b_i(0)e^\alpha := e^\alpha b_i(0). \]

and similarly for product involving the \( b_i(m) \).

The degree operator \( d \) is defined by
\[ d.v = (m_1 + \cdots + m_s + l_1 + \cdots + l_t + (\lambda|\lambda) + (\tilde{\lambda}|\tilde{\lambda}))v, \]
where \( v = a_i_1(m_1) \cdots a_i_s(m_s)b_j_1(l_1) \cdots b_j_t(l_t)e^\lambda e^{\tilde{\lambda}} \) is a basis element in \( V \).

It is easy to see that \( a_i(m), b_j(l), e^\alpha, e^{\tilde{\alpha}} \) commute with each other except that
\[ [a_i(0), e^{\alpha_j}] = (\alpha_i|\alpha_j)e^{\alpha_j}, \quad [b_i(0), e^{\tilde{\alpha}_j}] = (\tilde{\alpha}_i|\tilde{\alpha}_j)e^{\tilde{\alpha}_j}. \]

Let \( \varepsilon(\ , \ ) : P \times P \rightarrow \pm 1 \) be the quasi-cocycle such that
\[ \varepsilon(\alpha, \beta + \theta) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \theta), \]
\[ \varepsilon(\alpha + \beta, \theta) = \varepsilon(\alpha, \theta)\varepsilon(\beta, \theta)(-1)^{(\alpha+\beta|\lambda-\tilde{\lambda})}, \]
\[ \varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{((\alpha|\beta)+(\tilde{\alpha}|\tilde{\beta})}, \]
\[ \varepsilon(\alpha, \beta + \theta)\varepsilon(\beta, \theta) = \varepsilon(\alpha, \beta)\varepsilon(\alpha + \beta, \theta)(-1)^{(\alpha+\beta|\lambda-\tilde{\lambda})}. \]

where the \( - \) is the projection from \( P \) to \( \tilde{P} \) defined by
\[ \alpha = \sum_{i=1}^{n} m_i\alpha_i \in P \mapsto \overline{\alpha} = \sum_{i=1}^{n-1} \overline{m_i}\alpha_i, \quad m_i = m_i(mod 2). \]

We construct such a cocycle directly by
\[ \varepsilon(\alpha_i, \alpha_j) = \begin{cases} 
-1, & \text{if } i = j \\
1, & \text{if } i < j \\
(-1)^{\lambda_{ij}}, & \text{if } i > j 
\end{cases} \quad (12) \]
and it is easy to verify that the quasi-cocycle satisfies all the defining relations. In particular, we have
\[ \varepsilon(\alpha_i, \alpha_j)\varepsilon(\alpha_j, \alpha_i) = \begin{cases} 
(-1)^{2(\alpha_i|\alpha_j)}, & \text{if } 1 \leq i, j \leq n - 1 \\
(-1)^{\alpha_i|\alpha_j}, & \text{otherwise} \end{cases} \quad (13) \]

For \( \alpha \in P \) we define the operators \( \varepsilon_\alpha \) on \( V \) such that
\[ \varepsilon_\alpha e^\lambda e^{\tilde{\lambda}} = \varepsilon(\alpha, \lambda)e^\lambda e^{\tilde{\lambda}}. \quad (14) \]
then
\[ \varepsilon_{\alpha} \varepsilon_{\beta} = \varepsilon(\alpha, \beta) \varepsilon(\beta, \alpha) \varepsilon_{\beta} \varepsilon_{\alpha}. \] (15)

We denote \( \varepsilon_i = \varepsilon_{\alpha_i} \) for \( i = 1, \cdots, n \). In particular, we have:
\[
\varepsilon_i \varepsilon_j = (-1)^{2(\alpha_i | \alpha_j)} \varepsilon_j \varepsilon_i, \quad 1 \leq i, j \leq n - 1,
\]
\[
\varepsilon_i \varepsilon_n = (-1)^{(\alpha_i | \alpha_n)} \varepsilon_n \varepsilon_i, \quad i = 1, \cdots, n.
\]

We can now introduce the main vertex operators.
\[
Y^\pm_i(z) = \exp(\pm \sum_{k=1}^{\infty} a_i(\varepsilon^k z^k))
\]
\[
\times \exp(\mp \sum_{k=1}^{\infty} b_j(\varepsilon^k z^k)) \exp(- \sum_{k=1}^{\infty} a_i(\varepsilon^k z^{-k})e^{\pm \alpha_i z^{\pm a_j(0)} \varepsilon_i},
\]
\[
U_j(z) = \exp(\sum_{k=1}^{\infty} b_j(\varepsilon^k z^k)) \exp(- \sum_{k=1}^{\infty} b_j(\varepsilon^k z^{-k})e^{\pm \alpha_i z^{\pm a_j(0)} \varepsilon_i},
\]
\[
U^*_j(z) = \exp(- \sum_{k=1}^{\infty} b_j(\varepsilon^k z^k)) \exp(\sum_{k=1}^{\infty} b_j(\varepsilon^{-k} z^{-k})e^{\pm \alpha_i z^{\pm a_j(0)} \varepsilon_i},
\]
\[
Z^\pm_j(z) = U_j(q^{\pm 1/2} z) + (-1)^{2a_j(0)} U^*_j(q^{\pm 1/2} z),
\]
where \( i \in \{1, \cdots, n\}, j \in \{1, \cdots, n-1\} \). For simplicity we define \( Z^\pm_n(z) = 1 \).

**Theorem 3.1** The Fock space \( \mathcal{V} \) is a \( U_q \)-module of level 1 under the action defined by
\[
K_i \mapsto q^{a_i(0)}, \quad q^d \mapsto q^d,
\]
\[
a_i m \mapsto a_i(m), \quad \gamma \mapsto q,
\]
\[
X^\pm_i(z) \mapsto Y^\pm_i(z)Z^\pm_i(z), \quad i = 1, \cdots, n.
\]

The module \( \mathcal{V} \) contains submodules generated by the highest weight vectors \( e^{\lambda_i} e^{\bar{\lambda}_i} \) with weight \( \Lambda_i \) where \( i = 0, \cdots, n \). Here we denote \( \lambda_0 = \bar{\lambda}_0 = 0 \).

**4 Proof of the main theorem**

In this section we prove theorem [3.1] by vertex operator techniques. Since the operator \( a_i(m) \) commutes with \( b_j(m) \) (or the operator \( Y^\pm_i(z) \) commute with \( U_j(w) \) and \( U^*_j(w) \)) it is clear that relations [3.4] satisfy the Drinfeld relations.
Following [9] we use basic hypergeometric series to simplify the presentation.

For \( a \in \mathbb{R} \) we define
\[
(1 - z)^a_{q^2} := \frac{(zq^{-a+1}; q^2)_\infty}{(zq^{a+1}; q^2)_\infty} = \exp(- \sum_{n \geq 1} \frac{[an]}{n} z^n)
\]
where \((w; q^2)_\infty = \prod_{n=0}^{\infty} (1 - wq^{2n})\) is the usual \(q\)-number. Note that these \(q\)-series are defined as power series in \(w\). The first few examples of \(q\)-polynomials are listed in the following.

\[
(1 - z)^{\pm 1}_{q^2} = (1 - z)^{\pm 1},
\]
\[
(1 - z)^2_{q^2} = (1 - qz)(1 - q^{-1}z),
\]
\[
(1 - z)^{1/2}_{q^2} = \frac{(zq^{1/2}; q^2)_\infty}{(zq^{3/2}; q^2)_\infty}.
\]

It is clear that
\[
(1 - z)^{-a}_{q^2} = \frac{1}{(1 - z)^a_{q^2}}.
\]

For \( \epsilon, \epsilon' \in \{ \pm \} \), the operator product expansions (OPE) are given by:
\[
Y^\epsilon_i(z)Y^{\epsilon'}_j(w) = : Y^\epsilon_i(z)Y^{\epsilon'}_j(w) : (1 - q^{-(\epsilon + \epsilon')/2}w/z)^{\epsilon\epsilon'(\alpha_i|\alpha_j)}
\]
\[
U_i(z)U_j(w) = : U_i(z)U_j(w) : (1 - w/z)^{(\alpha_i|\alpha_j)}z^{(\alpha_i|\alpha_j)},
\]
\[
U_i(z)U^*_j(w) = : U_i(z)U^*_j(w) : (1 - w/z)^{-(\alpha_i|\alpha_j)}z^{-(\alpha_i|\alpha_j)}.
\]

and the OPE’s among \(U^*_i(z)\) are the same as \((17)\).

**Proof of relation (8).** There are four cases to be considered: \((\alpha_i|\alpha_j) = -1/2, (\alpha_i|\alpha_j) = -1, (\alpha_i|\alpha_j) = 1,\) and \((\alpha_i|\alpha_j) = 2.\) Note that the construction of \(X^\pm_n(z)\) implies immediately that the last case holds because they are the same as the simply laced cases. Since the other verifications are similar, we just show some computations in the following.

First let us consider \((\alpha_i|\alpha_j) = -1:\)
\[
(z - qw)X^\pm_{n-1}(z)X_n^\pm(w) = : X^\pm_{n-1}(z)X_n^\pm(w) : \text{ using (14)}
\]
\[
= (qz - w)X^\pm_{n-1}(w)X_n^\pm(z)
\]
For simplicity in the following we will usually write
\[(z - w)^a q^b = (1 - w/z)^a z^a,\]
which is considered as a power series in \(w/z\).

For \(i = 1, \cdots, n - 2\) we have that
\[
X_i^+(z)X_{i+1}^+(w) = \varepsilon_i \varepsilon_{i+1} : Y_i^+ Y_{i+1}^+ (w) : U_i U_{i+1} (w - q^{-1/2} w^{-1} q^{1/4}) + (-1)^{2a_i} U_i U_i^+ : z - q^{1/2} w - q^{-1/2} q^{1/4}
\]
\[
+ (-1)^{2a_i+1} : U_i U_i^+ : z - q^{1/2} w - q^{-1/2} q^{1/4}
\]
Using \((z - w)^{-1/2}(z - q^{-1/2} w)^{-1/2} = (z - q^{-1/2} w)^{-1}\) we have
\[
X_i^+(z)X_{i+1}^+(w) = \varepsilon_i \varepsilon_{i+1} : Y_i^+ Y_{i+1}^+ : U_i U_{i+1} (w - q^{-1/2} w^{-1} q^{1/4}) + (-1)^{2a_i} U_i U_i^+ : z - q^{1/2} w - q^{-1/2} q^{1/4}
\]
\[
+ (-1)^{2a_i+1} : U_i U_i^+ : z - q^{1/2} w - q^{-1/2} q^{1/4}
\]
\[
\text{from which it follows that}
\]
\[
(z - q^{-1/2} w)X_i^+(z)X_{i+1}^+(w) + (w - q^{-1/2} w)X_i^+(z)X_{i+1}^+(w) = 0.
\]

**Proof of relation (9).** Again we need to consider the following four cases:
\((\alpha_i | \alpha_j) = -1/2, (\alpha_i | \alpha_j) = -1, (\alpha_i | \alpha_j) = 1, \text{and } (\alpha_i | \alpha_j) = 2\). We only give the proof for \(i = j = 1, \cdots, n - 1\), since the other cases are either immediate or similar to our previous considerations.

\(\zeta\) From (10) it follows that
\[
X_i^+(z)X_i^-(w) = \varepsilon_i^2 : Y_i^+ Y_i^- (w) :
\]
\[
(U_i(q^{1/2} z))U_i(q^{1/2} w) : q^{1/2}
\]
\[+(-1)^{2a_i(0)} : U_i(q^{1/2}z)U_i^*(q^{-1/2}w) : \frac{1}{(q^{1/2}z - q^{-1/2}w)(z - w)}\]
\[+(-1)^{2a_i(0)} : U_i^*(q^{-1/2}z)U_i(q^{1/2}w) : \frac{1}{(q^{-1/2}z - q^{1/2}w)(z - w)}\]
\[+ (-1)^{4a_i(0)} : U_i^*(q^{-1/2}z)U_i^*(q^{-1/2}w) : q^{-1/2}\]

Then we have

\[\begin{align*}
[X_i^+(z), X_i^-(w)] &= \varepsilon_i^2 : U_iU_i^*Y_i^+Y_i^- : \frac{1}{(q^{1/2}z - q^{-1/2}w)(z - w)} \\
&\quad - \frac{1}{(q^{-1/2}w - q^{1/2}z)(w - z)} \\
+ \varepsilon_i^2 : U_i^*U_iY_i^+Y_i^- : \frac{1}{(q^{-1/2}z - q^{1/2}w)(z - w)} \\
&\quad - \frac{1}{(q^{1/2}w - q^{-1/2}z)(w - z)} \\
&= \varepsilon_i^2 : U_iU_i^*Y_i^+Y_i^- : \frac{1}{(q^{1/2} - q^{-1/2})zw} \left( \delta \left( \frac{w}{z} \right) - \delta \left( \frac{q^{-1}w}{z} \right) \right) \\
&\quad + \varepsilon_i^2 : U_i^*U_iY_i^+Y_i^- : \frac{1}{(q^{1/2} - q^{-1/2})zw} \left( \delta \left( \frac{q^2w}{z} \right) - \delta \left( \frac{w}{z} \right) \right) \\
&= \frac{1}{(q_i - q_i^{-1})zw} \left( \varepsilon_i(qw^{1/2})\delta \left( \frac{qw}{z} \right) - \varphi_i(qw^{-1/2})\delta \left( \frac{qw^{-1}}{z} \right) \right).
\end{align*}\]

**Proof of Serre relations** (10). For \( i = 1, \ldots, n - 1 \), let us write the operator \( X_i^{\pm}(z) \) as a sum of two terms:

\[X_i^{\pm}(z) = \sum_{\epsilon = \pm} Y_i^\epsilon(z)U_i^\epsilon(zq^{\pm\epsilon/2})(-1)^{(1-\epsilon)a_i(0)}\]

where we identify \( U_i^+(z) = U_i(z) \) and \( U_i^-(z) = U_i^*(z) \).

From the OPE (15, 16) it follows that

\[X_{i_{\epsilon_1}}^+(z_1)X_{i_{\epsilon_2}}^+(z_2)X_{i_{\epsilon_{1+1}}}^-(w) = X_{i_{\epsilon_1}}^+(z_1)X_{i_{\epsilon_2}}^+(z_2)X_{i_{\epsilon_{1+1}}}^+(w) : (-1)^{(\epsilon_1-\epsilon_2)}/2\]

\[\times (z_1 - q^{-1}z_2)(q^{1/2}z_1 - q^{-1/2}z_2)^{\epsilon_1\epsilon_2}q^{\epsilon_2/2} \]

\[\times \frac{z_2 - q^{1/2}w}{z_1 - q^{-1/2}w} \frac{z_2 - q^{-1/2}w}{z_1 - q^{1/2}w}\]

\[9\]
where we include the sign factor \((-1)^{(1-\epsilon)a_i(0)}\) and \(\epsilon_i\) in the normal ordered product. Similar normal product computation gives that

\[
X_{i\epsilon_1}^+(z_1)X_{i\epsilon_2}^+(z_2)X_{i+1,\epsilon}^+(w) - (q^{1/2} + q^{-1/2})X_{i\epsilon_1}^+(z_1)X_{i+1,\epsilon}^+(w)X_{i\epsilon_2}^+(z_2) + X_{i+1,\epsilon}(w)X_{i\epsilon_1}^+(z_1)X_{i\epsilon_2}^+(z_2)
\]

\[= X_{i\epsilon_1}^+(z_1)X_{i\epsilon_2}^+(z_2)X_{i+1,\epsilon}^+(w) : (z_1 - q^{-1/2}z_2)(q^{\epsilon_1/2}z_1 - q^{\epsilon_2/2}z_2)^{\epsilon_1\epsilon_2}q^{-\epsilon/2} \cdot (-1)^{(\epsilon_1-\epsilon_2)/2} \frac{(z_1 - q^{\epsilon/2}w)^{|\epsilon_1-e_2|/2}(z_2 - q^{\epsilon/2}w)^{|\epsilon_1-e_2|/2}}{(z_1 - q^{-1/2}w)(z_2 - q^{-1/2}w)}
\]

\[
+ [2] q^{1/2} (-1)^{(\epsilon_1-\epsilon_2)/2} \frac{(z_1 - q^{\epsilon/2}w)^{|\epsilon_1-e_2|/2}(q^{\epsilon/2}w - z_2)^{|\epsilon_1-e_2|/2}}{(z_1 - q^{-1/2}w)(w - q^{-1/2}z_2)}
\]

\[
+ \frac{(q^{\epsilon/2}w - z_1)^{|\epsilon_1-e_2|/2}(q^{\epsilon/2}w - z_2)^{|\epsilon_1-e_2|/2}}{(w - q^{-1/2}z_1)(w - q^{-1/2}z_2)} \quad (19)
\]

where each term in the parentheses corresponds to the OPE’s for the three normal products, and we have also used the relation:

\[
: X_{i\epsilon_1}^+(z_1)X_{i\epsilon_2}^+(z_2)X_{i+1,\epsilon}^+(w) := - X_{i\epsilon_1}^+(z_1)X_{i+1,\epsilon}^+(w)X_{i\epsilon_2}^+(z_2) : .
\]

We first claim that for \(\epsilon_1 \neq \epsilon_2\) we have

\[
X_{i\epsilon_1}^+(z_1)X_{i\epsilon_2}^+(z_2)X_{i+1,\epsilon}^+(w) - (q^{1/2} + q^{-1/2})X_{i\epsilon_1}^+(z_1)X_{i+1,\epsilon}^+(w)X_{i\epsilon_2}^+(z_2) + X_{i+1,\epsilon}(w)X_{i\epsilon_1}^+(z_1)X_{i\epsilon_2}^+(z_2) + (z_1 \leftrightarrow z_2, \epsilon_1 \leftrightarrow \epsilon_2) = 0 \quad (20)
\]

The claim is verified by checking 4 cases for \(\epsilon, \epsilon_i\), which are all similar and relied upon the following important identity \[13\]:

\[
(z_1 - aw)(z_2 - aw) + (a + a^{-1})(z_1 - aw)(w - az_2) + (w - az_1)(w - az_2) = (a^{-1} - a)w(z_1 - a^2z_2) \quad (21)
\]

for any \(a \in \mathbb{C}\).

In fact for \(\epsilon = 1, \epsilon_1 = -\epsilon_2 = 1\), the parentheses in \((19)\) is simplified to the following expression times \(q^{-1/2} \prod_i (z_i - q^{-1/2}w)^{-1} (w - q^{-1/2}z_2)^{-1} \).

\[
q^{-1/2} \left( (z_1 - q^{1/2}w)(z_2 - q^{1/2}w) + [2] (z_1 - q^{1/2}w)(w - q^{1/2}z_2) + (w - q^{1/2}z_1)(w - q^{1/2}z_2) \right)
\]

\[
= (q^{-1/2} - q)w(z_1 - q^{-1}z_2).
\]
Under the symmetry \((z_1, \epsilon_1) \leftrightarrow (z_2, \epsilon_2)\) it follows that the claim holds due to
\[
w(z_1 - q^{-1}z_2) + w(q^{-1}z_2 - z_1) = 0.
\]

We now turn to the other 4 cases with \(\epsilon_1 = \epsilon_2\). The 4 cases are also similar. Take the case \(\epsilon = -\epsilon_1 = -\epsilon_2 = 1\) for example. Using the identity \((21)\) again to simplify the parentheses in \((19)\), the contraction function in the Serre relation becomes
\[
q^{-1}(z_1 - q^{-1}z_2)(z_1 - z_2) \left(\frac{(z_1 - q^{1/2}w)(z_2 - q^{1/2}w)}{(z_1 - q^{-1/2}w)(z_2 - q^{-1/2}w)}\right)
- [2]_z \frac{z_1 - q^{1/2}w}{z_1 - q^{-1/2}w} q^{1/2} + q
= \frac{q^{-1}(q^{-1/2} - q^{1/2})w(z_1 - z_2)(z_1 - q^{-1}z_2)(z_1 - qz_2)}{(z_1 - q^{-1/2}w)(z_2 - q^{-1/2}w)}
\]
which is anti-symmetric under \((z_1 \leftrightarrow z_2)\), hence the sub-Serre relation is proved in this case. That is,
\[
X_{\epsilon_1}^+(z_1)X_{\epsilon_1}^+(z_2)X_{\epsilon_1}^+(w) - (q^{1/2} + q^{-1/2})X_{\epsilon_1}^+(z_1)X_{\epsilon_1}^+(w)X_{\epsilon_1}^+(z_2)
+ X_{\epsilon_1}^+(w)X_{\epsilon_1}^+(z_1)X_{\epsilon_1}^+(z_2) + (z_1 \leftrightarrow z_2) = 0
\]
Combining this sub-Serre relation with \((20)\) we prove the Serre relation for \(A_{i,i+1} = A_{i+1,i} = -1\).

We remark that the case \(A_{n-1,n} = -1\) is easily proved by using the identity \((21)\) with \(a = q\).

Finally let’s show the fourth order Serre relation with \(A_{n,n-1} = -2\).

\[
Sym_{z_1, z_2, z_3}(X_{\epsilon_1}^+(z_1)X_{\epsilon_1}^+(z_2)X_{\epsilon_1}^+(z_3)X_{\epsilon_1}^+(w))
- [3]q^{1/2}X_{\epsilon_1}^+(z_1)X_{\epsilon_1}^+(z_2)X_{\epsilon_1}^+(w)X_{\epsilon_1}^+(z_3)
+ [3]q^{1/2}X_{\epsilon_1}^+(z_1)X_{\epsilon_1}^+(w)X_{\epsilon_1}^+(z_2)X_{\epsilon_1}^+(z_3)
- X_{\epsilon_1}^+(w)X_{\epsilon_1}^+(z_1)X_{\epsilon_1}^+(z_2)X_{\epsilon_1}^+(z_3)) = 0
\]
First we have
\[
X_{n-1, \epsilon_1}^+(z_1)X_{n-1, \epsilon_2}^+(z_2)X_{n-1, \epsilon_3}^+(z_3)X_{n, \epsilon}^+(w)
= : X_{n-1, \epsilon_1}^+(z_1) \cdots : \prod_{1 \leq i < j < 3}(z_i - q^{-1}z_j)(q^{\epsilon_i/2}z_i - q^{\epsilon_j/2}z_j)^{\epsilon_i \epsilon_j}
\prod_{i=1}^{3}(z_i - q^{-1}w)(q^{\epsilon_i/2}z_i - q^{\epsilon_i/2}w)^{\epsilon_i \epsilon}
\]
Pulling out the common normal product we have
\[
X_{n-1,e_1}(z_1)X_{n-1,e_2}(z_2)X_{n-1,e_3}(z_3)X_{n,e}(w)
- [3]_{q^{1/2}}X_{n-1,e_1}(z_1)X_{n-1,e_2}(z_2)X_{n,e}(w)X_{n-1,e_3}(z_3)
+ [3]_{q^{1/2}}X_{n-1,e_1}(z_1)X_{n,e}(w)X_{n-1,e_2}(z_2)X_{n-1,e_3}(z_3)
- X_{n,e}(w)X_{n-1,e_1}(z_1)X_{n-1,e_2}(z_2)X_{n-1,e_3}(z_3)X_{n,e}(w)
\]
\[= X_{n-1,e_1}(z_1) \cdots \prod_{1 \leq i < j \leq 3} q^{-3}(z_i - q^{-1}z_j) (q^{i/2}z_i - q^{j/2}z_j)^{e_1 e_j} \cdot \left( (z_1 - qw)(z_2 - qw)(z_3 - qw) + [3]_{q^{1/2}}(z_1 - qw)(z_2 - qw)(w - qz_3) + [3]_{q^{1/2}}(w - qz_1)(w - qz_2)(w - qz_3) \right) \]

The Serre relation is then equivalent to the following combinatorial identity (cf. (11)):
\[
\sum_{\sigma \in S_3} (-1)^{||\sigma||} \sigma \cdot \{(z_1 - qw)(z_2 - qw)(z_3 - qw)
+ [3]_{q^{1/2}}(z_1 - qw)(z_2 - qw)(w - qz_3)
+ [3]_{q^{1/2}}(w - qz_1)(w - qz_2)(w - qz_3)\} \prod_{1 \leq i < j \leq 3} (z_i - q^{-1}z_j) = 0 \quad (22)
\]
where the symmetric group \(S_3\) acts on the ring of functions in \(z_i (i = 1, 2, 3)\) in the natural way: \(\sigma.z_i = z_{\sigma(i)}\). Note that the expression in the parenthesis can be simplified to
\[
(q^{-1} - q) \left( w^2(z_1 - (q + q^{-1})z_2 + q^3z_3) + w(z_1z_2 - (q + q^{-1})z_1z_3 + q^3z_2z_3) \right)
\]
Hence the identity (22) is equivalent to the following.
\[
\sum_{\sigma \in S_3} sgn(\sigma) \sigma\cdot(z_1 - (q + q^2)z_2 + q^3z_3) \prod_{i < j} (qz_i - z_j) = 0 \quad (23)
\]
which was already proved in [12]. The proof of the Serre relation is thus completed.

**Highest weight vectors.** To calculate the highest weight vectors we need the exact isomorphism [11] between Drinfeld realizations and the Drinfeld-Jimbo definition of quantum affine algebras. From [11] we have
\[
\begin{align*}
e_0 &= [X_1^{-}(0), \cdots, X_n^{-}(0)], \\
X_{n-1}^{-}(0), \cdots, X_1^{-}(1)]_{q^{-1/2}, \cdots, q^{-1/2}, q^{-1/2}, \cdots, q^{-1/2}} \gamma K_0^{-1} \\
e_i &= X_i^+(0)
\end{align*}
\]
where $K_\theta = K_1^2 \cdots K_{n-1}^2 K_n$. The $q$-multibracket is defined inductively by

\[
[a_1, a_2]_v = a_1 a_2 - va_2 a_1 \\
[a_1, a_2, \cdots, a_n]_{v_1, \cdots, v_{n-1}} = [a_1, [a_2, \cdots, a_n]_{v_1, \cdots, v_{n-2}]}_{v_{n-1}}
\]

Standard calculation of contour integrals of vertex operators will immediately give the following result.

**Lemma 4.1** (i) Let $\lambda \in P$ and $\tilde{\lambda} \in \tilde{P}$ be dominant with $(\lambda, \alpha_i) = (\tilde{\lambda}, \tilde{\alpha}_i)$, then for any $j = 1, \cdots, n$ and $n \geq 0$ we have

\[
X_j^+(n) e^\lambda e^\tilde{\lambda} = 0.
\]

(ii) For the fundamental weights $\lambda_i \in P$, $\tilde{\lambda}_i \in \tilde{P}$ we have for $j = 1, \cdots, n$

\[
X_j^-(0) e^\lambda e^\tilde{\lambda} = -\epsilon(\alpha_i, \lambda_i) e^{\lambda_i - \alpha_i} e^{\tilde{\lambda}_i - \tilde{\alpha}_i} \delta_{ij}, \quad X_j^-(1) e^\lambda e^\tilde{\lambda} = 0.
\]

Using this lemma and the isomorphism of $e_0$ and $e_i$ it is easy to see that $e^\lambda_i e^{\tilde{\lambda}_i} (i = 0, \cdots, n)$ are indeed highest weight vectors contained in the module $\mathcal{V}$.

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