Two spaces that already found their geometer in the thirties

Gavriel Segre

Abstract

Giorgio Parisi’s recent speculations on the concept of continuous dimension are compared with Von Neumann’s serious work

*URL: http://www.gavrielsegre.com; Electronic address: info@gavrielsegre.com
On a paper appeared recently at the Arxiv [1] Giorgio Parisi expresses his opinion that some geometer has to mathematically formalize the concepts of non-integer-dimensional vector spaces and of matrix-action on them, euristically occurring many times in Physics, e.g. in the dimensional regularization usual in Quantum Field Theory [2] or in the $\epsilon$-expansion of Statistical Mechanics [3] and claimed to be important for Spin Glass Theory (perhaps also as to the serious literature [4]).

As any serious physicist, mathematician or mathematical-physicist should know such a geometer has a name: John Von Neumann, who realized what Parisi is looking for yet in the thirties, in a way I will briefly review demanding to some serious literature for details [5], [6], [7], [8], [9], [10], [11], [12], [13].

Given a D-dimensional linear space $V$ on the field $K$ let us define its projective geometry $PG(V)$ as the set of all its linear subspaces:

$$PG(V) := \bigcup_{k=0}^{D} G_{k,D}(K)$$

where $G_{k,D}(V)$ is the $k^{th}$ Grassmannian of $V$, namely the set of all the $k$-dimensional linear subspaces of $V$.

Introduced on $PG(V)$ the partial ordering relation:

$$a \preceq b := a \subseteq b \ , \ a, b \in PG(V)$$

the partially-ordered-set $(PG(V), \preceq)$ is an atomic lattice, with:

$$a \wedge b := a \cap b$$

$$a \vee b := a \oplus b$$

whose atoms are the one-dimensional subspaces, namely the elements of $G_{1,D}(K)$, said the points of the projective geometry of $V$.

It may be easily verified that the following map:

$$d(a) := \dim(a) \ , \ a \in PG(V)$$

is a dimension function for the lattice $(PG(V), \preceq)$.

Since:

$$\infty \notin \text{Range}(d) = \{0, 1, \cdots, D\}$$

the lattice $(PG(V), \preceq)$ admits a finite dimension function and is, consequentially, modular.

Indeed eq. (6) completely characterizes the nature of the analyzed projective geometry:
• the discrete nature of Range(d), encoding the **atomicity** of PG(V), tells us that that it doesn’t admit ”intermediate subspaces” lying between points and lines, between lines and planes and so on

• the finite nature of Range(d), encoding the **modularity** of PG(V), tells us that that it doesn’t admit subspaces of arbitrary large dimensionality

Let us now suppose that V is an infinite-dimensional, separable vector space. 

\((PG(V), \preceq)\) is again a lattice on which the map d defined in eq.5 is again a dimension function. In this case, anyway, such a map is no more finite:

\[ \infty \in \text{Range}(d) = \{0, 1, \cdots, \infty\} \quad (7) \]

and the lattice \((PG(V), \preceq)\) is no more modular.

Indeed, again, eq.5 completely characterizes the nature of the analyzed projective geometry:

• the discrete nature of Range(d), encoding the **atomicity** of PG(V), tells us that that it doesn’t admit ”intermediate subspaces” lying between points and lines, between lines and planes and so on

• the nonfinite nature of Range(d), encoding the **nonmodularity** of PG(V), tells us that it admits subspaces of arbitrary large dimensionality

Let us observe, at this point, that in both the analyzed cases the structure of the projective geometry PG(V) rules the Representation Theory of Lie groups on G, as may be easily understood looking at reduction of representations.

The correct way of introducing ”intermediate subspaces” of suitable projective geometries lied at the heart of Von Neumann’s classification of factors (completed by Alain Connes):

the key ingredient is the notion of **relative dimension**:

given a **noncommutative space**, namely a Von Neumann algebra A acting on an Hilbert space \(\mathcal{H}\), two projectors \(p_1\) and \(p_2\) belonging to A are called **equivalent w.r.t. A** (this fact being denoted as \(p_1 \sim_A p_2\)) if there exist in A a partial isometry between \(\text{Range}(p_1)\) and \(\text{Range}(p_2)\), a condition that may be intuitively seen as the requirement that, ”**from the viewpoint of A**”, the dimensionality of the spaces on which \(p_1\) and \(p_2\) project are equal.
Such a notion of relative equivalence of projections immediately induces a partial ordering $\leq_A$ on the projections of $A$, according to which the projection $p_1$ is considered less or equal to another projection $p_2$ w.r.t. $A$ if $p_1$ has a dimension w.r.t. $A$ less or equal to that of $p_2$.

The resulting notion of relative dimension $d_A$ w.r.t. a noncommutative space $A$ allowed Von Neumann to introduce his celebrated classification of factors:

- A is of type n-dimensional, discrete ($\text{Type}(A) = I_n$) iff $\text{Range}(d_A) = \{0, 1, \cdots n\}$
- A is of type infinite, discrete ($\text{Type}(A) = I_\infty$) iff $\text{Range}(d_A) = \{0, 1, \cdots \infty\}$
- A is of type finite, continuous ($\text{Type}(A) = II_1$) $\text{Range}(d_A) = [0, 1]$
- A is of type infinite, continuous ($\text{Type}(A) = II_\infty$) iff $\text{Range}(d_A) = [0, \infty]$
- A is of type purely infinite ($\text{Type}(A) = III$) iff $\text{Range}(d_A) = \{0, \infty\}$

As it has been strongly emphasized by Miklos Redei, Von Neumann explicitly realized that what he was doing was nothing but Theory of Noncommutative Cardinality, i.e. the theory of noncommutative cardinal numbers describing the ”sizes” (and ”infinity’s degrees”) of noncommutative sets:

"... the whole algorithm of Cantor theory is such that the most of it goes over in this case. One can prove various theorems on the additivity of equivalence and the transitivity of equivalence, which one would normally expect, so that one can introduce a theory of alephs here, just as in set theory. ... I may call this dimension since for all matrices of the ordinary space, is nothing else but dimension” (Unpublished, cited in [3])

"One can prove most of the Cantoreal properties of finite and infinite, and, finally, one can prove that given a Hilbert space and a ring in it , a simple ring in it, either all linear sets except the null sets are infinite (in which case this concept of alephs gives you nothing new), or else the dimensions, the equivalence classes, behave exactly like numbers and there are two qualitatively different cases. The dimensions either behave like integers, or else they behave like all real numbers. There are two subcases, namely there is either a finite top or there is not” (Unpublished, cited in [3])
This fact has induced me to suggest a notation remarking it by the explicit introduction of the following notion of noncommutative cardinality:

\[ \text{card}_{\text{NC}}(A) := \int_{\mathbb{Z}(A)}^\otimes \text{card}_{\text{NC}}(A_\lambda) \ d\nu(\lambda) \]

where:

\[ A = \int_{\mathbb{Z}(A)}^\otimes A_\lambda d\nu(\lambda), \ \mathbb{Z}(A_\lambda) = \{ C I \} \ \forall \lambda \in \mathbb{Z}(A) \]

and:

- **A HAS NONCOMMUTATIVE CARDINALITY EQUAL TO \( n \in \mathbb{N} \):**

  \[ \text{cardinality}_{\text{NC}}(A) = n := \text{Type}(A) = I_n \]  \hspace{1cm} (8)

- **A HAS NONCOMMUTATIVE CARDINALITY EQUAL TO \( \aleph_0 \):**

  \[ \text{cardinality}_{\text{NC}}(A) = \aleph_0 := \text{Type}(A) = I_\infty \]  \hspace{1cm} (9)

- **A HAS NONCOMMUTATIVE CARDINALITY EQUAL TO \( \aleph_1 \):**

  \[ \text{cardinality}_{\text{NC}}(A) = \aleph_1 := \text{Type}(A) \in \{ II_1, II_\infty \} \]  \hspace{1cm} (10)

- **A HAS NONCOMMUTATIVE CARDINALITY EQUAL TO \( \aleph_2 \):**

  \[ \text{cardinality}_{\text{NC}}(A) = \aleph_2 := \text{Type}(A) = III \]  \hspace{1cm} (11)

Let us consider now a noncommutative space \( A \) of noncommutative cardinality \( \aleph_1 \):

all the category-equivalence’s theorems giving foundations to Noncommutative Geometry allow to look at the lattice of projections \( P(A) \) as the **projective geometry** of \( A \).

\( P(A) \) is, anyway, **nonatomic**, i.e. it is a projective geometry **without points**, explicitly manifesting the phenomenon of **continuous geometry**: according to the basic "noncommutative metaphore" (formalized by the mentioned category-equivalence’s theorems) looking at \( A \) "as if it was an algebra of functions on a new kind of space" we can infer that, in this case "such a strange kind of space" has subspaces of any integer dimension \( \epsilon \) between zero and one.

If \( A \) is finite, furthermore, the geometric structure of these "strange kind of subspaces of noninteger dimension" is astonishing: considered a subfactor \( B \) of \( A \) the ratio between the
relative dimension w.r.t. \( B \) and the relative dimension w.r.t. \( A \) allows to define the Jones’ index of \( B \) w.r.t. \( A \) whose role in Knot Theory realizes a link to Algebraic Topology that has led to some of the more extraordinary results of the last decade both in Mathematics and in Physics \([14], [15], [16], [17]\).

As to the second mathematical object Parisi is looking for, namely a mathematical formalization of the action of matrices on noninteger-dimensional linear spaces, let us observe that the structure of the projective geometry \( \text{PG}(A) \) rules the Representation Theory of \( G \) by automorphisms.

The usual application of Noncommutative Geometry as to Serious Quantum Statistical Mechanics and Serious Quantum Ergodic Theory \([18], [19], [20], [21], [22], [23]\) could lead to think erroneously that the only role Noncommutative Geometry plays in Physics concerns its role as to Quantum Mechanics:

this is not, anyway, true since Noncommutative Geometry may be used also to analyze many mathematical structures appearing in Classical (i.e. non quantum) Physics, among which it has certainly to be mentioned the other great mathematical adventure concerning spaces with non-integer dimension: Fractal Geometry \([24]\):

as an example, introduced the spectral triple \( (A, \mathcal{H}, D) \) of 1-dimensional noncommutative calculus:

- \( \mathcal{H} := L^2(S^{(1)}, d\bar{x}_{\text{Lebesgue}}) \) (12)

- \( A := L^\infty(S^{(1)}, d\bar{x}_{\text{Lebesgue}}) \) (13)

where a function \( f \in A \) is seen as a multiplication operator:

\( (f\psi)(t) := f(t)\psi(t) \quad f \in A, \psi \in \mathcal{H} \) (14)

- \( D \) is the linear operator on \( \mathcal{H} \) defined by:

\[ De_n := \text{sign}(n)e_n, \quad e_n(\theta) := e^{in\theta} \quad \forall \theta \in S^{(1)} \] (15)
and denoted by $D$ the Hausdorff dimension of the Julia set $J[p_c(z)]$:

$$J[p_c(z)] = \partial \{ z \in \mathbb{C} : \sup_{n \in \mathbb{N}} |p_c^{(n)}(z)| < \infty \}$$

(16)

of the quadratic maps on the complex plane $p_c(z) := z^2 + c$, Alain Connes proved that:

1. $|d_{NC}Z|$ is an infinitesimal of order $\frac{1}{D}$

2. 

$$\exists \lambda > 0 \ : \ (\int_{J[p_c(z)]} f d\Lambda_D) = \lambda \int_{NC} f(Z)|D|^{-1}|d_{NC}Z|^D \ \forall f \in C(J[p_c(z)])$$

(17)

where $d\Lambda_D$ is the Hausdorff measure on $J[p_c(z)]$.

The eq.(17) tells us that the integral w.r.t. the Hausdorff measure of continuous functions over the Julia set $J[p_c(z)]$ may be computed as a noncommutative integral in the spectral triple ($A$, $\mathcal{H}$, $D$).

Since the Mandelbrot’s set $\mathcal{M}$ is linked to the family of Julia sets $J[p_c(z)]$ by the condition:

$$\mathcal{M} = \{ c \in \mathbb{C} : J[p_c(z)] \text{ is connected} \}$$

(18)

eq.(17) could be useful to investigate some of the still unknown properties of one of the most precious diamonds of Fractal Geometry: Mandelbrot’s set $\mathcal{M}$.

Beside all the mentioned reasons to think that the geometer Parisi is looking for was John Von Neumann in the thirties (together with his most prominent successor Alain Connes), let us observe that the notion of non-integer dimensional vector space obtained through Noncommutative Geometry strongly differs from the strategy Parisi himself proposes in order of obtaining a definition of a non-integral dimensional vector space, strategy consisting in:

1. considering a suitable family $\{S_i\}$ of submanifolds of the D-dimensional euclidean space ($\mathbb{R}^D$, $\delta = \delta_{\mu\nu} dx^\mu \otimes dx^\nu$)

2. making the analytical continuation in D of the formulas:

$$\mu(S_i) = f_i(D)$$

(19)

expressing the (induced)-measure of each submanifold $S_i$. 

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It is sufficient, anyway, to think just some minute to realize how sick is Parisi’s illusion that such a strategy could lead to characterize in a meaningful way a notion of non-integer dimensional linear space:

which is the family \( \{S_i\} \) to be considered?

and why, even before of taking any analytical continuation in \( D \), one should think that their measures encode all the geometrical structure of \((\mathbb{R}^D, \delta = \delta_{\mu\nu} dx^\mu \otimes dx^\nu)\) such as, for example, the measure of a submanifold not belonging to the chosen family?
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