Work relation in instantaneous-equilibrium transition of forward and reverse processes

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Abstract
Realizing quasistatic processes in finite times requires additional control parameters to keep the system in instantaneous equilibrium (ieq). However, the finite-rate ieq transition of the reverse process is not just the time-reversal of the ieq forward process due to the odd-parity of controlling parameters. We theoretically show a work relation that the dissipated work of the ieq transition of the forward process is the same as that of the corresponding reverse process. The work relation can be interpreted as a generalization of equilibrium (quasistatic) processes. The work relation and the associated statistics of nonequilibrium work are experimentally confirmed in three different Brownian systems: the time-varying harmonic and non-harmonic potentials and the Brownian harmonic transport, which are manipulated by the optical feedback trap.

1. Introduction

The second law of thermodynamics has been the cornerstone concept of the classical thermodynamics for last two centuries, saying that the entropy production cannot be negative, i.e., the free energy difference cannot be larger than work applied to the system during the transition between two equilibrium states [1]. However, in the microscopic system where thermal fluctuations play a dominant role, the systems in such transition are always out of equilibrium so that it is hard to determine the free energy difference, for example, in the conformational change of proteins or DNAs. For the last three decades, there have been intensive studies on the theoretical framework of nonequilibrium statistical physics to describe the dynamics in small systems [2–5]. In particular, Jarzynski equality (JE) [6] and Crooks theorem (CT) [7] have been used to estimate the free energy difference even in systems far from equilibrium. These theories have been successfully demonstrated in various experimental systems, such as colloidal systems [8], heat engines [9, 10], biological systems [11–13], and quantum systems [14, 15].

The positive entropy production for a transition between two equilibrium states implies the process is in general irreversible. And for a reversible process, the transition path must be quasi-static, i.e., infinitely slow such that equilibrium is achieved at every moment during the transition. Speeding a system’s transition up from one equilibrium state to another in a time much shorter than the intrinsic relaxation time and thus reproduce the same output as a quasi-static process, has been a challenging issue, which requires complicated protocols achieved by controlling one or more parameters [16]. For the past decade, this idea has attracted attention in the quantum system [17–20], known as the shortcuts-to-adiabaticity. In the stochastic system, usually described by the overdamped Langevin equation of a colloidal system, several theoretical works have been proposed [21, 22] and experimentally confirmed in Brownian systems in time-dependent harmonic potentials (HPs) [23–26] and barrier escape [27]. Of particular interest, the shortcuts-to-isothermality (Sci) protocol provides a unified framework to achieve a finite-rate isothermal process while keeping the system in instantaneous equilibrium (ieq) during the transition [22], and was demonstrated experimentally in the system of the transport in a moving HP [23]. Previous research in stochastic systems have been mainly focused on how to achieve a finite-rate transition [24, 26], how much
dissipated work is required [23], and how to reduce the dissipated work [25] for this process, but few studies have been devoted to the investigation on the properties of the work involved in these processes [23, 26].

In this paper, we investigate work statistics for the ScI protocols of the ieq forward (ScIf) and reverse (ScIr) processes with a Brownian particle confined in different trapping potentials. During our previous study on the ScI protocols of the compression and expansion processes of the HP [26], we recognized that the dissipated work done for the former is the same as that of the latter for a given driving time. That is, here, it is generalized for any ScI protocols of the forward (ScIf) and reverse (ScIr) processes, referred to as a work relation (hereinafter, WR). We also investigate the connection of the JE and CT to the ScIf and ScIr processes. To experimentally test the fluctuation theorems and the WR, we perform the experiment of the overdamped dynamics of a Brownian particle in a time-varying HP, time-varying non-harmonic potential (NHP), and a Brownian harmonic transport (BHT) with the corresponding auxiliary potentials under ScI protocols. The latter is realized using moving optical tweezers while the two former protocols are created and manipulated by the optical feedback trap (OFT) we recently developed [26, 28].

2. Work relation for ScI protocol

Let us consider a particle trapped in a one-dimensional time-dependent potential, $U_{\text{RAMP}}(x, \lambda(t))$, with the protocol $\lambda(t)$, shown schematically as the red dashed arrow for the forward process (RAMPf) in figure 1. The ‘RAMP’ represents the original (ramped) protocol for the transition from one equilibrium state to another. The transition is performed in a duration $\tau$ and is assumed to be smooth at the start and end of the driving process, i.e., $\dot{\lambda}(0) = \dot{\lambda}(\tau) = 0$. The reverse process of the particle under the protocol $\lambda_R(t) = \lambda(\tau - t)$, RAMPr, and under the potential $U_{\text{RAMP}}(x, \lambda_R(t))$ is also depicted by the blue dashed arrow. Figure 2(a) shows the time-varying stiffness of forward and reverse RAMP protocols with a transition duration $\tau$ in the HP experiment. Although the protocol ends, the system is still in a nonequilibrium state, and the extra time beyond the relaxation time is required for the system to equilibrate.

To achieve ieq for the ScIf [22], $U_{\text{RAMP}}(x, \lambda(t))$ is escorted by an auxiliary potential of the form $\dot{\lambda}(t)f(x, \lambda(t))$ (the dot above a variable represents its time derivative) for some function $f$ (which can be
determined uniquely, see appendix A for details) so that the particle experiences a total potential

\[ U_{\text{ScIf}} = U_{\text{RAMP}}(x, \lambda(t)) + \dot{\lambda}(t)f(x, \lambda(t)). \]  

From the work production rate \( \dot{W} = \dot{U}_{\text{ScIf}} \), the mean work produced for \( 0 < t < \tau \) under ScIf is given by

\[ \langle W \rangle = \Delta F + \int_0^\tau dt \left[ \dot{\lambda}(f(x, \lambda)) + \dot{\lambda}^2 \left\langle \frac{\partial f}{\partial \lambda} \right\rangle \right], \]

where \( \Delta F \) is the free energy change and \( \langle ... \rangle \) is the average over the Boltzmann distribution associated with \( U_{\text{RAMP}}(x, \lambda(t)) \). Under the ScI of the reverse protocol (ScIr) \( \lambda_R(t) \), the particle will experience a total potential of

\[ U_{\text{ScIr}} = U_{\text{RAMP}}(x, \lambda_R(t)) + \dot{\lambda}_R(t)f(x, \lambda_R(t)). \]

The mean work under ScI of the reverse process is calculated in a similar way, but the average denoted by \( \langle ... \rangle_R \) is over the Boltzmann distribution associated with \( U_{\text{RAMP}}(x, \lambda_R(t)) \). We get

\[ \langle W_R \rangle_R = -\Delta F + \int_0^\tau dt \left[ \dot{\lambda}_R(f(x, \lambda)) + \dot{\lambda}_R^2 \left\langle \frac{\partial f}{\partial \lambda} \right\rangle \right], \]

where we use the work production rate \( \dot{W}_R = \dot{U}_{\text{ScIr}} \) for the ScIr protocol, and the integral is rewritten by changing variable as \( \tau - t \rightarrow t \) together with the properties \( \dot{\lambda}_R(t) = -\dot{\lambda}(\tau - t) \), \( \dot{\lambda}_R(t) = \dot{\lambda}(\tau - t) \). Hence we get the WR for finite-rate ScI transitions of the forward and reverse processes as follows,

\[ \langle W \rangle = \langle W_R \rangle + 2\Delta F, \]

which is the main finding of our work. The subscript of the average of the ScI of a reverse process, \( \langle ... \rangle_R \), can be dropped if no confusion arises. For quasi-static processes, equation (5) also holds but trivially with \( \langle W \rangle = -\langle W_R \rangle = \Delta F \). Thus the WR for ScI can be interpreted as a generalization of equilibrium (quasi-static) processes echoing the ieq nature of ScI. To highlight the novelty of the ScI WR, it should be noted that the mean works for the forward and reverse RAMP processes, in general, do not satisfy equation (5).

3. Experimental protocol

To experimentally confirm the WR, we performed three different protocols, HP, NHP, and BHT, as shown in table 1. We will present the results of the HP protocol in the main text and others in appendix A.
1 μm particle is immersed in water and confined in a time-dependent HP $U_{\text{RAMP}}(x, \lambda(t)) = \frac{1}{2} \lambda(t)x^2$ with $\lambda(t) = k_{\min} \left[ 2 - \cos(\pi t / \tau) \right]$ (denoted by the red dashed curves in figure 2(a)), where $\tau$ and $k_{\min}$ is the process time and the initial (also the minimal) stiffness of the HP for the ScIf protocol, respectively. For the RAMP protocol that begins from $t = 0$ and ends at $t = \tau$ smoothly, the conditions $\dot{\lambda}(0) = \lambda(\tau)$ hold, and the free energy change in the process is

$$\Delta F = \frac{1}{2\beta} \ln \frac{\lambda(\tau)}{\lambda(0)} = \frac{1}{2\beta} \ln 3,$$

where $\lambda(0) = k_{\min} = 5 \text{ pN} \text{ μm}^{-1}$ and $\lambda(\tau) = k_{\max} = 15 \text{ pN} \text{ μm}^{-1}$. Under the RAMP protocol, the particle distribution does not follow the Boltzmann distribution due to the time-dependent variation of the potential stiffness that yields a typical non-equilibrium behavior, while the particle distribution function is still Gaussian [23, 24].

The potential for the ScIf protocol is given by [22]

$$U_{\text{ScIf}}(x, \lambda(t)) = \frac{1}{2} \dot{\lambda}(t)x^2,$$

$$\dot{\lambda}(t) \equiv \lambda(t) + \frac{\gamma}{2} \dot{\dot{\lambda}}(t),$$

where $\dot{\lambda}(0) = k_{\min}$ and $\dot{\lambda}(\tau) = k_{\max}$. $\dot{\lambda}(t)$ is presented as the red solid curve in figure 2(a). The potential for the ScIr protocol is given similarly as

$$U_{\text{ScIr}}(x, \lambda(t)) = \frac{1}{2} \ddot{\lambda}_{\text{Ir}}(t)x^2,$$

$$\ddot{\lambda}_{\text{Ir}}(t) \equiv \lambda(t) - \frac{\gamma}{2} \dot{\dot{\lambda}}_{\text{Ir}}(t),$$

where $\ddot{\lambda}_{\text{Ir}}(0) = k_{\max}$ and $\ddot{\lambda}_{\text{Ir}}(\tau) = k_{\min}$. This protocol is depicted by the blue solid curve in figure 2(a).

Interestingly, to perform the ScIr protocol, the stiffness of the potential must be negative for a certain period as shadowed in yellow color in figure 2(a), which can be implemented using the OFT technique [26]. We note that the intrinsic relaxation time of the HP for the expansion process, $\tau_{R} = \gamma / k_{\min} = 1.88 \text{ ms}$. Figure 2(b) shows the variance evolution of position trajectories of the ScIf (solid curves) and ScIr (dashed curves) for various process times ranging from 1 ms to 100 ms. The ScIr protocol keeps the system in |eq, and the corresponding variances of the position fluctuations follow well with the theoretical expectations, $1/[\beta \lambda(t)]$ and $1/[\beta \lambda(t - \tau - t)]$ for ScIf and ScIr, respectively.

### 4. Results

The works produced in the ScIf and ScIr processes are calculated from the trajectory of a single particle using the stochastic thermodynamic framework [29], as $W = \frac{1}{2} \int_{0}^{\tau_R} dt \dot{\lambda}(t) x(t)^2$ and $W_{R} = \frac{1}{2} \int_{0}^{\tau_{R}} dt \dot{\lambda}_{R}(t) x(t)^2$, respectively. In these protocols, each work can be viewed as a weighted sum of strongly correlated random variable $x^2$, and thus one would expect the corresponding distribution to be non-Gaussian. Figure 3(a) shows the PDFs of the measured work, $P(W)$ for the ScIf (filled circles) and $P_{R}(-W)$ for the ScIr (open circles) for $\tau = 2 \text{ ms}$ and 20 ms (more PDFs in figure D1(a)). As predicted, they are non-Gaussian, and their variances reduce with increasing $\tau$. (See figure D1(b) for the work variances as a function of $\tau$, which agree well with the theoretical prediction equations (D.8) and (D.9)) figure 3(b) shows $W$ for the ScIf process (circles) and $-\langle W_{R} \rangle$ for the ScIr process (squares) are shown as functions of $\tau$. From the WR, the mean of the two curves is expected to be $\Delta F$, which agrees well with the experimental data. It also shows
that the dissipated works, \(\langle W \rangle - \Delta F\) for the ScIf process and \(\langle W_R \rangle + \Delta F\) for the ScIr process, are large for small \(\tau\) and approaches to zero as \(\tau\) increases. In the quasi-static limit, the work done on the system becomes equal to the free energy difference. The mean work can be analytically computed to give

\[
\beta \langle W \rangle = \frac{1}{2} \ln 3 + \frac{\pi^2}{8} \frac{1}{3^{1/2}} \frac{\tau_R}{\tau}, \tag{11}
\]

\[
\beta \langle W_R \rangle = -\frac{1}{2} \ln 3 + \frac{\pi^2}{8} \frac{1}{3^{1/2}} \frac{\tau_R}{\tau}, \tag{12}
\]

which are plotted as solid curves in figure 3(b) showing excellent agreement with the measured values.

We next check the validity of the JE \(\phi \equiv \langle \exp(-\beta(W - \Delta F)) \rangle = 1\) and the CT \(P(W)/P_R(-W) = \exp(\beta(W - \Delta F))\), for the ScI of forward and reverse processes. As shown in figure 3(c), both of the ScIf and ScIr processes satisfy the JE. On the contrary, the CT does not generally hold for the ScI protocols since \(\tilde{\lambda}_R(t)\) and \(\tilde{\lambda}(t)\) are not time-reversed protocols to each other, i.e. \(\tilde{\lambda}_R(t) \neq \tilde{\lambda} (\tau - t)\) due to the presence of an odd-parity term \([30]\) proportional to \(\dot{\lambda}\) in equations (1) and (10). CT predicts the two PDFs, \(P(W)\) and \(P_R(-W)\), should intersect at \(W = \Delta F\). Figure 3(a) shows that for \(\tau = 2\) ms and 20 ms, the point of the intersection of the two PDFs (indicated by arrows) is away from the free energy difference, \(\Delta F\) (denoted by the vertical dashed lines). We note that the CT holds for the special case of the BHT as in figure C1(c) of appendix, which is theoretically proved in appendix F.

To further test the validity and the universality of the WR for the ScI processes given in equation (5), we plot \((W)/\Delta F\) as a function of \((W_R)/\Delta F\). As shown in figure 4, for the HP protocol (red circles), \((W)/\Delta F\) is positive for \(\tau < \tau_R\), and approaches to \((1, -1)\) in the quasistatic limit as \(\tau \to \infty\), which is denoted the red arrow. For the NHP protocol (squares, also in the inset), the dissipated work, \(\beta \langle W_{\text{diss}} \rangle\) is about 0.041 for ScIf and 0.038 for ScIr at \(\tau = 1\) ms as shown in figure E1 in appendix. For the
Figure 4. The relation between the work done during the ScI protocols of the forward and reverse processes. The circles, squares, and triangles represent the works in HP, NHP, and BHT, respectively. The gray solid straight line is the theoretical prediction of $\langle W \rangle / \Delta F = \langle W_R \rangle / \Delta F + 2$. The dashed lines indicate the asymptotic value $(-1, 1)$ in the quasistatic process. The right inset is the magnification of the yellow box. The arrows indicate the direction of increasing process time $\tau$.

BHT protocol (triangles), $W / \Delta F$ is negative because $\Delta F < 0$, and $(\langle W / \Delta F \rangle, \langle W_R / \Delta F \rangle)$ saturates to the quasistatic values of $(1, -1)$. The gray solid line is the theoretical prediction of equation (5), where all data points collapsed, highlighting the universal validity of the WR.

5. Conclusion and outlook

We find a new WR in this study for the ScI protocols of the forward and reverse processes as given in equation (5), which holds trivially for quasi-static processes: $\langle W \rangle = \Delta F = -\langle W_R \rangle$. Furthermore, the WR gives $\langle W \rangle - \Delta F = \langle W_R \rangle + \Delta F$, implying the mean dissipated works of the ScI of forward and reverse processes are identical. It can be interpreted as a generalization of the equilibrium (quasi-static) processes echoing the ieq nature of ScI. In addition, equation (5) can be re-written into the following form

$\Delta F = \frac{1}{2} (\langle W \rangle - \langle W_R \rangle)$

suggesting that the equilibrium free energy difference can be obtained by measuring the difference of the mean (highly non-equilibrium) works between the ScI forward and reverse processes. With the recently proposed variational approach, one can obtain an accurate approximation of the auxiliary potential for realizing shortcuts to isothermality [31]. This enables the WR to be used to measure the free energy difference. When compared with the traditional JE that the average is taken over the exponential of the non-equilibrium work, one anticipates that $\Delta F$ deduced from WR by measuring the difference of the ScI mean works will be more efficient and with better accuracy.

The WR can also be shown to hold theoretically for the underdamped case following the same steps as in appendix A. Though the case of the overdamped situation is investigated experimentally here, it is experimentally very challenging to realize the underdamped situation in a Brownian colloidal system, but may be possible in an RCL electric circuit under Johnson–Nyquist noises [32].

We also investigate the fluctuation theorem specifically for three ScI protocols for forward and reverse processes: BHT, time-varying HP and NHPs. The ScI protocol is characterized by the time-derivative of the RAMP protocol, $\dot{\lambda}(t)$, which has odd-parity in the reverse process. As expected, the Jazynski WR holds for all cases, while the Crook theorem does not hold in general due to the odd-parity nature of the ScI protocol. However, the BHT is an exception in which the CT can be shown to hold theoretically (appendix F) and also confirmed in experiments (see figure C1(c)), the details will be presented elsewhere.

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Appendix A. Theoretical background of Sci and derivation of the work relation of Sci of the forward and reverse processes

Here we consider an overdamped Brownian particle with drag coefficient \( \gamma \) under the time-dependent potential \( U_{RAMP}(x, \lambda(t)) \), where \( \lambda(t) \) is the time-dependent protocol for the non-equilibrium transition process (the RAMP protocol) of duration \( \tau \) for \( 0 \leq t \leq \tau \), which is assumed to be smooth in the start and end of the driving process, i.e. \( \lambda(0) = \lambda(\tau) = 0 \). The forward RAMP process, RAMPf, is specified by the potential \( U_{RAMP}(x, \lambda(t)) \) whereas its corresponding reverse process, RAMPr, is under \( U_{RAMP}(x, \lambda_R(t)) \) where \( \lambda_R(t) = \lambda(\tau - t) \).

A.1. Summary of Sci protocol and the auxiliary escort potential

Under the Sci recipe [22] to achieve ieq, the RAMP potential \( U_{RAMP}(x, \lambda(t)) \) is escorted by an auxiliary potential \( U_{aux}(x, t) \) so that the particle experiences a total potential

\[
U_{Sci} = U_{RAMP}(x, \lambda(t)) + U_{aux}(x, t) \tag{A.1}
\]

such that the Brownian particle will be at ieq that follows the Boltzmann distribution \( \rho_{eq}(x, \lambda) = e^{-\beta(F(\lambda) - U_{RAMP}(x, \lambda)))} \) where \( \beta = 1/(k_B T) \) and \( F(\lambda) = -k_B T \ln \int e^{-\beta U_{RAMP}(x, \lambda(t))} \) is the free energy of the RAMP(original) system at equilibrium for some given value of \( \lambda \) during the transition process. It was shown in [22] that \( U_{aux}(x, t) \) is of the form \( \lambda(t)f(x, \lambda(t)) \) where the function \( f \) can be determined for given RAMP potential \( U_{RAMP}(x, \lambda(t)) \) and protocol \( \lambda(t) \) by

\[
f(x, \lambda(t)) = \beta \gamma \int_x^\infty \frac{dz h(z, \lambda(t))}{\rho_{eq}(z, \lambda)} \tag{A.2}
\]

\[
h(z, \lambda) = \left( \frac{dF}{d\lambda} - \frac{\partial U_{RAMP}}{\partial \lambda} \right) \rho_{eq}(z, \lambda). \tag{A.3}
\]

Hence under the Sci, the Brownian particle will experience the potential

\[
U_{Sci} = U_{RAMP}(x, \lambda(t)) + \lambda(t)f(x, \lambda(t)), \tag{A.4}
\]

and will be at ieq and obey the Boltzmann distribution \( \rho_{eq}(x, \lambda(t)) \) at any time \( (0 \leq t \leq \tau) \) during the transition process. The subscript ScIf (Sci forward) is used to differentiate it from the ScIr (Sci reverse) protocol.

A.2. Derivation of the work relation for the Sci processes: \( \langle W \rangle = \langle W_R \rangle + 2\Delta F \)

Here we present the proof of the Sci WR for forward and reverse processes for general protocols. Under Sci protocol [22], the RAMP potential \( U_{RAMP}(x, \lambda(t)) \) is escorted by an auxiliary potential so that the particle experiences a total potential given by \( \langle A.4 \rangle \). Since the work rate under the Sci protocol is \( W = \partial U_{Sci}/\partial t \), from \( \langle A.4 \rangle \) the mean work under Sci is given by

\[
\langle W \rangle = \int_0^\tau dt \left( \frac{\partial U_{Sci}}{\partial t} \right) = \int_0^\tau dt \left[ \dot{\lambda} \left( \frac{\partial U_{RAMP}}{\partial \lambda} \right) + \dot{\lambda} \dot{f}(x, \lambda) + \dot{\lambda}^2 \left( \frac{\partial f}{\partial \lambda} \right) \right] = \Delta F + \int_0^\tau dt \left[ \dot{\lambda} \dot{f}(x, \lambda) + \dot{\lambda}^2 \left( \frac{\partial f}{\partial \lambda} \right) \right]. \tag{A.5}
\]

where \( \langle \ldots \rangle \) is the average over the equilibrium Boltzmann distribution \( e^{\beta(F(\lambda) - U_{RAMP}(x, \lambda(t)))} \) due to the ieq nature of Sci. Under the Sci of the reverse protocol, \( \lambda_R(t) = \lambda(\tau - t) \), the particle will experience a total potential of

\[
U_{Sci} = U_{RAMP}(x, \lambda_R(t)) + \dot{\lambda}_R(t)f(x, \lambda_R(t)). \tag{A.6}
\]

Under the Sci of the reverse process, the work rate is \( \dot{W}_R = \partial U_{Sci}/\partial t \). Then, the mean work is calculated in a similar way, but the average is over the equilibrium Boltzmann distribution of the reverse protocol \( e^{\beta(F(\lambda_R) - U_{RAMP}(x, \lambda_R(t)))} \) denoted by \( \langle \ldots \rangle_R \):
similar steps above, one case easily sees that the WR (A.8) also holds for the underdamped case. The subscript of the average of the ScI of reverse process, \( \langle W \rangle_R \) is also shown (horizontal dot-dashed line). (a) \( U_{\text{RAMP}} + 4 \lambda(t) x^2 \), \( \tau_R \equiv \gamma/k_0 \). The ScI results are calculated from (D.6) and (D.7). The RAMP results are given by (D.3) by evaluating the integral numerically in (D.4). (b) \( U_{\text{RAMP}} = \frac{1}{2} \lambda(t)x^2 \), \( \tau_R \equiv \gamma/\sqrt{k_0} \). The ScI results are calculated from (E.9) and (E.10) and the RAMP results are measured from simulations of Langevin dynamics.

\[
\langle W \rangle_R = \int_0^\tau dt \left( \frac{\partial U_{\text{ScI}}}{\partial \lambda} \right)_R
\]

\[
= \int_0^\tau dt \left[ \dot{\lambda}_R \left( \frac{\partial U_{\text{RAMP}}}{\partial \lambda} \right)_{\lambda(t)} + \ddot{\lambda}_R(t) f(x, \lambda_R(t)) \right]_R + \ddot{\lambda}_R \left( \frac{\partial f}{\partial \lambda} \right)_{\lambda(t)}_R
\]

\[
= \int_0^\tau dt \left[ -\dot{\lambda}(\tau - t) \left( \frac{\partial U_{\text{RAMP}}}{\partial \lambda} \right)_{\lambda(t)} + \ddot{\lambda}(\tau - t) f(x, \lambda_R(t)) \right]_R + \ddot{\lambda}(\tau - t)^2 \left( \frac{\partial f}{\partial \lambda} \right)_{\lambda(t)}_R
\]

\[
= -\Delta F + \int_0^\tau dt \left[ \dot{\lambda}(f(x, \lambda)) + \lambda^2 \left( \frac{\partial f}{\partial \lambda} \right) \right],
\]

where the relations \( \dot{\lambda}_R(t) = -\dot{\lambda}(\tau - t) \) and \( \ddot{\lambda}_R(t) = \ddot{\lambda}(\tau - t) \) are used. Subtracting (A.5) from (A.7) gives the WR

\[
\langle W \rangle = \langle W \rangle_R + 2\Delta F.
\]

The subscript of the average of the ScI of reverse process, \( \langle \ldots \rangle_R \), can be dropped if no confusion arises. It is worth to note that ScI can also be implemented for the underdamped situation [22] with the auxiliary potential also given by (A.4) with \( f(x, \lambda(t)) \) replace by a momentum dependent \( f(x, \dot{p}, \lambda(t)) \). Following similar steps above, one case easily sees that the WR (A.8) also holds for the underdamped case.

Figure A1 shows the results of \( \langle W \rangle - \langle W \rangle_R \) for the RAMP and ScI processes for the case of time-dependent stiffness protocol, by evaluating the integrals numerically for the mean works in (D.3) and (D.5). It is shown that the ScI WR (A.8) holds perfectly for the ScI processes, but significant deviations from (A.8) occur for the RAMP processes. Only under quasi-static conditions (\( \tau \gg \tau_R \)) will (A.8) hold asymptotically.

### Appendix B. Optical feedback trap and non-harmonic potential

To realize the ScI protocol for the time-varying HP and NHP, the traditional tools such as the optical tweezers and the magnetic tweezers are not appropriate because they basically manipulate nearly HPs. Even for the HP, the stiffness of the trap cannot be changed promptly enough to follow the time-variation of the ScI protocol which is sometimes needed to have negative values [26]. The OFT we developed is suitable for the ScI experiments thanks to the capability for the creation of arbitrary shape of spatiotemporal potential [28]. Basically, the OFT can create the virtual potential by repeating the feedback cycles: (i) to determine the location of the particle, (ii) to calculate the force by the virtual potential based on the particle position, and (iii) to apply the same amount of the real force as that of the virtual potential. During this feedback cycle, the particle behaves as if it is in the genuine potential.

The OFT is implemented in the experiment, as shown in figure B1. It consists of the microscope, the lasers, and the feedback system. A trapping laser (L1, Cobolt Rumba) with 1064 nm wavelength is used for trapping and manipulating a Brownian particle. The trapping laser beam is incident on the acousto-optic deflector (AOD, Gooch and Housego, AODF 4090-6) to shift the trap’s center position and thus to create the virtual potential. The beam steerer is placed before the objective lens (Olympus, UPLFN100XO) for the fine control of the focused laser beam. The tracking laser (L2) of the wavelength 980 nm also focuses on the same image plane as the trapping laser and reaches the position sensitive device (PSD, Pacific Silicon Sensor, Pacific Silicon Sensor).
Figure B1. The schematic drawing of the experimental setup of the OFT.

Figure B2. (a) The basic protocol to create the arbitrary-shaped virtual potential, for instance, $U = \frac{1}{2}\lambda v x^4$ (black dashed line). (b) the probability distribution of the particle’s position and (c) the corresponding virtual potential. The colors are for different stiffness with $\lambda v = 1 \times 10^9 \text{ N m}^{-3}$ (red), $\lambda v = 1.5 \times 10^9 \text{ N m}^{-3}$ (green), and $\lambda v = 2 \times 10^9 \text{ N m}^{-3}$ (blue).

DL100-7-PCBA3). The signal acquired by PSD is then sampled at the rate of 100 kHz by a field programming gate array (FPGA) data acquisition board (National Instruments, NI PCIe-7851R). The particle position and control of AOD is done by a home-made software using LabVIEW programmed on the FPGA target. The particle position, $x$, is acquired at every $10 \mu$s, and $10^5$ position data stored in the FPGA are sent to the computer at every second for further analysis. The 1 $\mu$m-diameter polystyrene particles are immersed in the deionized water. The parameter values used in the experiment are the stiffness of the optical tweezers, $k_{ot} = 10 \text{ pN} \mu\text{m}^{-1}$ and temperature, $T = 300 \text{ K}$. The stiffness of the optical tweezers is calibrated using two methods based on equipartition theorem and power spectrum [33].

The protocol for a virtual HP of the desired stiffness is discussed in previous work [28], and here, we briefly describe the basic operation of how to create the non-HP: $U(x, t) = 1/2\lambda v x^4$ as outlined in figure B2(a). Here, $\lambda v$ is the stiffness of the potential with the unit of N m$^{-3}$. Let us imagine that a freely-moving particle (gray circle) is trapped in the virtual non-HP (black dashed line) centered at $x = 0$, and experiences the restoring force $f_v = -2\lambda v x^3$ (green arrow). We address that there is no genuine spatial potential. However, the particle can behave as if it is in the potential when the optical tweezers exert the equal amount of physical force $f_{ot} = -k_{ot}(x - x_L)$ (blue arrow) to the particle by shifting the laser center position $x_L$ instantaneously by an amount

$$x_L(t) = x(t - t_d) - \frac{2\lambda v}{k_{ot}} x^3(t - t_d), \quad (B.1)$$

where $k_{ot}$ is the stiffness of the optical tweezers and $t_d$ is the force delay time after position measurement.

Figure B2(b) shows the probability distribution function (PDF) of the particle’s position for three different stiffness of the NHP, $\lambda v = 1 \times 10^9 \text{ N m}^{-3}$ (red), $1.5 \times 10^9 \text{ N m}^{-3}$ (green), and $2 \times 10^9 \text{ N m}^{-3}$ (blue). It shows the behavior of the particle under the NHP where it will stay more around the center of the potential. Figure B2(c) shows the potentials corresponding to the position PDF that is assumed to be a Boltzmann distribution. The NHP generated by the OFT agrees well with the desired stiffness (solid lines).
Appendix C. Brownian harmonic transport (BHT)

The detailed description for the BHT is presented in reference [23]. A particle trapped in a one-dimensional HP is subject to a time-dependent protocol of the form

$$U_{\text{RAMP}}(x, t) = \frac{1}{2} kx^2 - \lambda(t)x.$$  \hfill (C.1)

where \(k\) is the stiffness of the potential and \(\lambda(t)\) is the driving protocol.

The ScI protocol to maintain the evolution of the system always in ieq with the original potential \(U_{\text{RAMP}}\) is given as following:

$$U_{\text{ScIf}}(x, \lambda(t)) = \frac{1}{2} kx^2 - \tilde{\lambda}(t)x, \quad \tilde{\lambda}(t) \equiv \lambda(t) + \frac{\gamma \dot{\lambda}(t)}{k}. \hfill (C.2)$$

To ensure the ScIf protocol reduces smoothly to the RAMP potential at the beginning and end of the ScI process, the conditions \(\dot{\lambda}(0) = \dot{\lambda}(\tau) = 0\) are imposed, and \(\lambda(t)\) is chosen as:

$$\lambda(t) = a \left[ 1 - \cos \left( \frac{\pi t}{\tau} \right) \right] k \sigma,$$  \hfill (C.3)

where \(a\) is the strength of the auxiliary force, \(\sigma = \sqrt{1/k\beta}\) is the width of the particle fluctuation in the HP, and \(\tau\) is the transport time. In contrast, the ScI for the reverse process is given as follows:

$$U_{\text{ScIr}}(x, \lambda(t)) = \frac{1}{2} kx^2 - \tilde{\lambda}_R(t)x,$$

$$\tilde{\lambda}_R(t) \equiv \lambda(\tau - t) - \frac{\gamma \dot{\lambda}(\tau - t)}{k}. \hfill (C.4)$$

The ScIf and ScIr protocols are depicted as the red and blue solid curves in figure C1(a), respectively.

The experiment consists of 3 steps: (i) the particle starts at the initial equilibrium state for 80 ms. (ii) The ScI protocol is then implemented at various driving times \(\tau = 1\) ms to 75 ms. Lastly, (iii) the particle will then be allowed to settle at the final equilibrium state for 80 ms. The experiment is repeated for about 10 000 cycles to achieve good statistics. Figure C1(b) presents the rescaled mean trajectories, \(\tilde{x}_{\text{ScI}}\), of
the particle for the ScI protocols for various values of \( \tau \), which are expected to be equal to \( \lambda/k \) and \( \lambda_R/k \) from the ieq nature, respectively for the forward and reverse processes, given as

\[
\dot{x}_{\text{ScI}}(t) = a\sigma \left[ 1 - \cos \left( \frac{\pi t}{\tau} \right) \right] \tag{C.5}
\]

\[
\dot{x}_{\text{ScR}}(t) = a\sigma \left[ 1 - \cos \left( \frac{\pi(\tau - t)}{\tau} \right) \right] = a\sigma \left[ 1 + \cos \left( \frac{\pi t}{\tau} \right) \right]. \tag{C.6}
\]

For each measurement, we rescaled the mean particle trajectories by the final equilibrium position, \( \bar{x}_0(\tau) = 2a\sigma \). For the short \( \tau < 20 \text{ ms} \), we observed that relaxation to the final equilibrium state for the RAMP protocol takes 20 ms, while for the ScI transport, it is equal to the transition time \( \tau \) because the mean trajectories accord with the expected equilibrium trajectories due to the RAMP protocol. It is much faster compared to that for the RAMP protocol. For the 1 ms transition time, the relaxation time to the equilibrium state for ScI process is 1 ms, which is 20 times faster than that for the RAMP protocol. As \( \tau \) increases, the difference between two mean trajectories decreases, and for \( \tau > 20 \text{ ms} \), two trajectories overlap each other, indicating that such a slow transition is close to the quasi-static process.

The measured work done during a single process of transport can be determined using the discretized Sekimoto formula \( W = \int_0^\tau (\partial U / \partial t)dt \) [29]:

\[
W = \Delta t \sum_{i=1}^N \left[ -a \sin \left( \frac{\pi t_i}{\tau} \right) \frac{\pi k\sigma}{\tau} x(t_i) + \Delta t \sum_{i=1}^N \left[ -a\gamma \cos \left( \frac{\pi t_i}{\tau} \right) \frac{\pi^2\sigma}{\tau^2} x(t_i) \right] \right] \tag{C.7}
\]

\[
W_R = \Delta t \sum_{i=1}^N \left[ -a \sin \left( \frac{\pi(\tau - t_i)}{\tau} \right) \frac{\pi k\sigma}{\tau} x(t_i) + \Delta t \sum_{i=1}^N \left[ -a\gamma \cos \left( \frac{\pi(\tau - t_i)}{\tau} \right) \frac{\pi^2\sigma}{\tau^2} x(t_i) \right] \right]. \tag{C.8}
\]

Here \( N = \tau / \Delta t, \Delta t = 10 \mu s, t_i = i\Delta t \) is the time step with \( i = 0, 1, \ldots, N \). Figure C1(c) shows the measured PDFs (various symbols) of the work done for various transport times \( \tau = 1, 2, \) and 3 ms. The solid and open symbols represent the work distributions \( P(\beta W) \) and \( P(-\beta W_R) \), respectively. The vertical line indicates the theoretical free energy difference \( \beta \Delta F = -2a^2 = -0.125 \). The work PDFs for three different values of \( \tau \) are well described by Gaussian distributions: \( P(W) = 1/\sqrt{2\pi \sigma_W} \exp(-\frac{(W - \langle W \rangle)^2}{2\sigma_W^2}) \), which are shown by the solid curves in figure C1(c).

Figure C1(d) shows the mean work as a function of \( \tau \) for the ScI of the forward (red circle), and reverse (blue circle). The averaged works for ScI and ScR for the BHT are computed by averaging equations (C.7) and (C.8) and using \( \langle x(t) \rangle = \lambda(t)/k \) and \( \langle x(t) \rangle_R = \lambda(\tau - t)/k \), given as

\[
\beta \langle W \rangle = \frac{k^2}{\tau} a^2 - 2a^2, \quad \tau_c = k \tag{C.9}
\]

\[
\beta \langle W_R \rangle = \frac{k^2}{\tau} a^2 + 2a^2, \tag{C.10}
\]

where \( \beta \Delta F = -2a^2 \). Finally, from the difference of (C.9) and (C.10), the WR of ScI is explicitly verified for the BHT,

\[
\beta \langle W \rangle - \beta \langle W_R \rangle = 2\Delta F, \tag{C.11}
\]

as shown in figure 5 in the main text.

**Appendix D. Time-varying harmonic potential (HP)**

As a second example, we study the ScI protocol in time-varying HP whose details are presented in the main text. Here, we assume that a particle is trapped and manipulated by the time varying HP: \( U_{\text{RAMP}} = \frac{1}{2} \lambda(t)x^2 \). Then, the driving protocols for ScI and ScR are given as follow [22, 26]:

\[
U_{\text{ScI}}(x, \lambda(t)) = \frac{1}{2} \dot{x}(t)x^2, \quad \lambda(t) = \lambda(t) + \frac{\gamma \lambda(t)}{2} \frac{\dot{x}(t)}{\lambda(t)} \tag{D.1}
\]

\[
U_{\text{ScR}}(x, \lambda(t)) = \frac{1}{2} \dot{x}(t)x^2, \quad \lambda(t) = \lambda(\tau - t) - \frac{\gamma \lambda(\tau - t)}{2} \frac{\dot{x}(t)}{\lambda(\tau - t)}. \tag{D.2}
\]

where \( \dot{x}(0) = k_{\text{min}} \) and \( \lambda(t) = k_{\text{max}} \) are the minimal and maximal stiffness, respectively.
Here we consider a virtual time-dependent NHP given by

\[ U_{RAMP}(t) = \frac{1}{2} \lambda(t)x^4, \]  

\[ \text{(E.1)} \]

Appendix E. Time-varying non-harmonic potential (NHP)

Here we consider a virtual time-dependent NHP given by

\[ U_{RAMP}(t) = \frac{1}{2} \lambda(t)x^4, \]

\[ \text{(E.1)} \]

Figure D1. (a) The probability distribution of measured works of ScIf (filled symbols) and ScIr (open symbols) for 1, 5, 60 and 100 ms. (b) The work variances for the ScIf (circle) and ScIr (square). The solid curves are the theoretical predictions (D.8) and (D.9).
where $\lambda(t)$ is the time-dependent ‘stiffness’. The ScI protocol can be computed from (A.3) and (A.4) to give

$$U_{\text{Scf}}(t) = \frac{1}{2} \lambda(t)x^4 + \frac{\gamma}{8 \lambda(t)} \ddot{x}^2,$$

(E.2)

where $\gamma$ is the friction coefficient. In the experiment, we employ

$$\lambda(t) = \lambda_0 \left[ \frac{5}{4} - \frac{1}{4} \cos \frac{\pi t}{\tau} \right],$$

(E.3)

where $\lambda_0$ is the reference stiffness. The ScI protocol for the corresponding reverse process is then

$$U_{\text{Scr}}(t) = \frac{1}{2} \lambda(\tau - t)x^4 - \frac{\gamma}{8 \lambda(\tau - t)} \ddot{x}^2,$$

(E.4)

$$\lambda(\tau - t) = \lambda_0 \left[ \frac{5}{4} + \frac{1}{4} \cos \frac{\pi t}{\tau} \right].$$

The ScI protocols for the forward and reverse processes are implemented on a Brownian particle via the OFT, that is described in section 2.

The ieq property of ScI guarantees that the position distribution is Boltzmann with the original $U_{\text{RAMP}}$ and is given by

$$p(x, t) = e^{\beta [F(\lambda(t)) - \frac{1}{4} \beta \dot{\lambda}^2]},$$

(E.5)

Direct calculation using $\bar{p}(x, t)$ gives

$$\langle x^4 \rangle = \frac{2}{4 \beta \lambda(t)},$$

(E.6)

$$\langle x^2 \rangle = \left( \frac{2}{\beta \lambda(t)} \right)^{1/2} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \approx 0.34 \left( \frac{2}{\beta \lambda(t)} \right)^{1/2},$$

(E.7)

where $\Gamma(x)$ is the gamma function.

In the experiments, the transition is from the minimal stiffness $\lambda(0) = 4.5 \times 10^9 \text{ N m}^{-3}$ to the maximal value $\lambda(\tau) = 6.75 \times 10^9 \text{ N m}^{-3}$. To increase the characteristic time $\tau_R$ of the NHP, we increase the viscosity of the surrounding medium by adding glycerol, so the viscosity is $\eta = 3.3 \text{ Pa s}$. The free energy difference $\Delta F$ is then

$$\beta \Delta F = \frac{1}{4} \ln \left[ \frac{\lambda(\tau)}{\lambda(0)} \right] = \frac{1}{4} \ln \frac{3}{2} \approx 0.10.$$

(E.8)

Figure E1(a) shows the time evolution of the variance of the particle’s position rescaled by the minimum variance for the ScIf (max $\rightarrow$ min) and the ScIr (min $\rightarrow$ max) for $\tau = 1$ ms (red), 3 ms (yellow), 10 ms (green), 35 ms (blue), and 100 ms (purple): $\langle x^2 \rangle_{\text{max}} / \langle x^2 \rangle_{\text{min}} = \sqrt{\lambda_{\text{max}} / \lambda_{\text{min}}} = \sqrt{3/2} \approx 1.22$. The black curves are the theoretical predictions for ScIf (solid) and ScIr (dashed) represented by equation (E.7). The variance of the particle’s trajectories (solid curves for the forward process and dashed curves for the reverse process) stops changing immediately after the driving time (at the end of the shaded region). It means that the ScI protocol completes the transition immediately right after the driving time.

We calculate the work done during the ScI processes. The mean work for the ScIf process is given by

$$\beta \langle W \rangle = \frac{1}{2} \int_0^\tau dt \ddot{x}^2 + \frac{\gamma}{8} \int_0^\tau dt \left( \frac{d}{dt} \frac{\dot{x}^2}{\lambda(t)} \right),$$

$$= \frac{1}{4} \ln \frac{\lambda(\tau)}{\lambda(0)} + \beta \gamma \frac{2}{\beta} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \int_0^\tau \frac{d}{dt} \frac{\dot{x}^2}{\lambda(t)^{2+\frac{1}{4}}},$$

$$= \Delta F + \frac{0.04}{\tau},$$

(E.9)

and the mean work for the ScIr process is given in similar way by

$$\beta \langle W_R \rangle = \frac{1}{4} \ln \frac{\lambda(\tau)}{\lambda(0)} + \beta \gamma \frac{2}{\beta} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \int_0^\tau \frac{d}{dt} \frac{\dot{x}^2}{\lambda(t)^{2+\frac{1}{4}}},$$

$$= -\Delta F + \frac{0.04}{\tau}.$$
Appendix F. Proof of validity of the Crooks theorem for the Brownian harmonic transport protocol

The Langevin equation for the ScI protocol is

$$\dot{x}(t) = \frac{1}{\gamma}(-kx + \lambda(t)) + \xi(t), \quad (F.1)$$

where $\gamma$ is the drag coefficient and $\xi$ is a zero-mean white noise with $\langle \xi(t)\xi(t') \rangle = \frac{2kT}{\gamma} \delta(t-t')$. It is convenient to separate the motion of the particle into a deterministic part, $X(t) \equiv \langle x(t) \rangle$, and a stochastic part, $z(t)$, with $x = X + z$, and (F.1) becomes

$$\dot{X} = -\frac{k}{\gamma}X + \frac{1}{\gamma} \dot{\lambda}(t), \quad (F.2)$$

$$\dot{z} = -\frac{k}{\gamma}z + \xi(t), \quad (F.3)$$

whose solutions are

$$X(t) = \frac{1}{\gamma} \int_0^t dt' e^{-\frac{k}{\gamma}(t-t')} \dot{\lambda}(t') + X(0) e^{-\frac{k}{\gamma}t} \quad (F.4)$$

$$z(t) = \frac{1}{\gamma} \int_0^t dt' e^{-\frac{k}{\gamma}(t-t')} \xi(t') + z(0) e^{-\frac{k}{\gamma}t} \quad (F.5)$$

Figure E1(b) shows that the dissipated works, $\langle W \rangle = \Delta F$ for the forward process and $\langle W_R \rangle + \Delta F$ for the reverse process, are large at $\tau \ll \tau_R \equiv \gamma \sqrt{\beta/\lambda_0} \approx 7.2$ ms, and asymptotically approaches to zero. The solid curves are $\langle W \rangle$ and $\langle W_R \rangle$ for ScIf and ScIr, respectively, which scale as $1/\tau$, as shown in equations (E.9) and (E.10).

Figure E1(c) shows the validity of the JE $\phi \equiv \langle \exp(-\beta(W - \Delta F)) \rangle = 1$ for the ScIf and ScIr of the NHP. The ScIf (circle) and ScIr (square) are very close to the unity value, indicating that both satisfy the JE. However, the CT is not expected to hold for any pairs of the ScIf and ScIr processes due to odd-parity $\lambda$ in the protocol. This can be revealed from the intersection of the two PDFs, $P(W)$ and $P_R(-W)$ in figures E1(d)–(f) for the NHP. The intersection of the two PDFs is away from $W = \Delta F$, predicted by the CT, (marked by the vertical dashed line) for $\tau = 1$ ms. The two PDFs become close to each other in the quasistatic limit for large $\tau = 100$ ms.
Inserting the expression of $\tilde{\lambda}(t)$ from (C.2) into (F.4) and integrating by parts, one has

$$X(t) = \frac{\lambda(t)}{k} + (X(0) - \frac{\lambda(0)}{k})e^{-kt}. \quad (F.6)$$

Using the condition of the protocol that initially the system is at equilibrium in a harmonic well centered at $\lambda(0)/k$, therefore $X(0) = \langle \chi(0) \rangle = \lambda(0)/k$ and hence $X(t) = \langle \chi(t) \rangle = \lambda(t)/k$. Furthermore, (F.4) can be rewritten (integrating by parts) as

$$X(t) = \frac{\lambda(t)}{k} - \frac{1}{k} \int_0^t dt' \tilde{\lambda}(t')e^{-k(t-t')} \quad (F.7)$$

To investigate the CT in ScI protocols of forward and reverse processes, we define the generating function as follows:

$$g(\alpha) \equiv \langle e^{-\alpha W} \rangle, \quad P(W) = \frac{1}{2\pi} \int d\alpha \, e^{i\alpha W} g(i\alpha). \quad (E.8)$$

For ScIf given by (C.2), $g(\alpha)$ is calculated using (E.7) and by cumulant expansion as follows:

$$\ln g(\alpha) = \ln \langle e^{-\alpha W} \rangle = \ln \langle e^{\alpha W \tilde{\lambda}(t)|X(t) + \tilde{\lambda}(t)|} \rangle$$

$$= \frac{\beta \alpha (1 - \alpha)}{k} \int_0^T dt \tilde{\lambda}(t) \int_0^T e^{\frac{\alpha}{k}(t-t')} \tilde{\lambda}(t') + \alpha \beta \Delta F$$

$$\Delta F \equiv -\frac{1}{2k}[\tilde{\lambda}(\tau)^2 - \tilde{\lambda}(0)^2]. \quad (F.9)$$

Using integration by parts the double integral in (F.9) can be further simplified to $\frac{\gamma}{k} \int_0^T dt \lambda(t)^2$, and one finally has

$$\ln g(\alpha) = -\beta \alpha \left[ \frac{(1 - \alpha)\gamma}{k^2} \int_0^T dt \lambda(t)^2 + \Delta F \right]. \quad (E.10)$$

The reverse protocol is given by $\lambda_R(t) = \lambda(\tau - t)$ and thus the ScIr of the BHT is given by

$$\tilde{\lambda}_R(t) = \lambda_R(t) + \frac{\gamma}{k} \tilde{\lambda}_R(t) = \lambda(\tau - t) - \frac{\gamma}{k} \tilde{\lambda}(\tau - t). \quad (E.11)$$

Thus the ScIr for the reverse protocol is NOT the same as the reverse protocol of the ScI of the forward process: $\tilde{\lambda}_R(t) \neq \tilde{\lambda}(\tau - t)$. Hence one would not expect Cohen–Gallavotti (CG) symmetry (and hence CT) to hold for the ScI and ScIr trajectories. Remarkably, as we will show below, that indeed CG symmetry is restored for ScI processes with $\lambda(0) = \tilde{\lambda}(\tau) = 0$ for the BHT protocol considered explicitly here.

The generating function of the ScIr can be similarly calculated as in (E.10)

$$\ln g_R(\alpha) = -\beta \alpha \left[ \frac{(1 - \alpha)\gamma}{k^2} \int_0^T dt \lambda_R(t)^2 - \Delta F \right] \quad (F.12)$$

Thus from (E.11) and (F.12), we have

$$\frac{g(\alpha)}{g_R(1 - \alpha)} = e^{-\beta \Delta F} \quad (F.13)$$

and CG symmetry is obeyed. Putting $\alpha = 1$ in (F.13), one has

$$\langle e^{-\beta W} \rangle = g(1) = g_R(0)e^{-\beta \Delta F} = e^{-\beta \Delta F} \quad (E.14)$$

and hence JE holds for the ScI BHT processes. Furthermore, the work distribution function for the ScI of the reverse process can be obtained from $g_R(\alpha)$ and CG symmetry in (F.13) leads to
\[
\frac{P(W)}{P(-W)} = e^{\beta(W - \Delta F)},
\]
and hence CT holds between Scf and Scr of the BHT protocols.

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