New nonultralocal quantum integrable models through gauge transformation

Anjan Kundu *
Saha Institute of Nuclear Physics, Theory Group
1/AF Bidhan Nagar, Calcutta 700 064, India.

March 27, 2022

Abstract

One of the few schemes for obtaining an integrable nonultralocal quantum model is its possible generation from an ultralocal model by a suitable gauge transformation. Applying this scheme we discover two new nonultralocal models, which fit well into the braided Yang-Baxter relations ensuring their quantum integrability. Our first model is generated from a lattice Liouville-like system, while the second one which is an exact lattice version of the light-cone sine-Gordon is gauge transformed from a model, which gives also the quantum mKdV for a different gauge choice.

PACS numbers 02.30.Ik, 02.20.Uw, 11.10.Lm 03.65.Fd

1. Introduction

Extension from the classical to the quantum domain and exact solution of a number of quantum models are undoubtedly a major achievement in the theory of integrable systems. However, truly speaking inspite of an impressive list of such systems the success is limited mostly to a class of models known as ultralocal (UL) models, for which the representative Lax operators at different lattice points must commute: $L_{2k}^{ul}(v)L_{1j}^{ul}(u) = L_{1j}^{ul}(u)L_{2k}^{ul}(v)$, for $j \neq k$. In formulating their quantum integrability this ultralocality plays a crucial role, since only under such constraint the quantum Yang-Baxter equation (YBE)

$$ R_{12}(u - v) L_{1j}^{ul}(u) L_{2j}^{ul}(v) = L_{2j}^{ul}(v) L_{1j}^{ul}(u) R_{12}(u - v), \quad j = 1, \ldots, N $$ (1)

can be lifted for the monodromy matrix $T_{a}^{ul}(u) = L_{a N}^{ul}(u) \ldots L_{a 1}^{ul}(u)$, to its global form

$$ R_{12}(u - v) T_{1}^{ul}(u) T_{2}^{ul}(v) = T_{2}^{ul}(v) T_{1}^{ul}(u) R_{12}(u - v). \quad (2) $$

*email: anjan@tnp.saha.ernet.in
Defining $\tau(u) = tr(T(u))$ therefore the trace identity: $[\tau(u), \tau(v)] = 0$ can be proved, which is equivalent to the integrability condition $[c_n, c_m] = 0$, for the set of conserved quantities $c_n, n = 1, 2, \ldots$.

On the other hand there exists a rich class of classical integrable models, e.g. mKdV, KdV, light-cone sine-Gordon, complex sine-Gordon, derivative NLS, nonlinear $\sigma$ model etc. [3], which violate ultralocality condition and therefore make their quantum description through standard YBE formulation difficult. Nevertheless starting from eighties to early nineties a number of nonultralocal (NUL) systems were discovered showing quantum integrability [1, 2, 3, 4, 5, 6]. Thereafter except [10, 11] and few others there were not much serious attempts for a considerable period of time to develop this important theory, until probably [12] where a braided YBE formulation was developed for NUL models and applied successfully to bring the mKdV model into this quantum integrable class [13]. The situation however became more discouraging in recent years with almost no new discoveries of such models made and no new theoretical achievements taking place, inspite of an urgent need for such developments in this subject. Only very recently a mixed left-right component quantum integrable mKdV model has been introduced with its possible connection to perturbed CFT [14].

In such a scenario therefore it is highly desirable to apply some systematic scheme for discovering new quantum integrable NUL systems which can enrich this important class of models. In fact nonultralocalization of an ultralocal model may serve as such a technique, where the quantum Lax operator of a NUL model can be constructed from that of an integrable UL model by a suitable operator dependent local gauge transformation. This method, which is capable not only to derive new models but also to reveal their direct relation with an UL model, was introduced first possibly in [4]. In subsequent works this scheme was implemented implicitly in [2], incorporated in a general framework in [12] and used explicitly in [13]. Our aim here is to apply the same scheme for generating new integrable models, which make valuable additions to the existing list of such quantum integrable NUL systems. In particular we discover two models; the first one being a new discrete quantum NUL system constructed by a simple gauge transformation from a lattice Liouville-like model. Our second model is an exact lattice version of the light-cone sine-Gordon (LCSG), which is also new as a quantum NUL model. Interestingly the LCSG is constructed through gauge transformation from the same UL model, which yields the quantum mKdV [15], but for a different gauge choice.

We find that the quantum NUL models we obtain fit well into the braided YBE formulation [12] and can be solved exactly through algebraic Bethe ansatz modified for the NUL models [13, 14].

2. The scheme:

The idea of the scheme is to start from the Lax operator $L_{ul}^j$ of an UL model and applying a local gauge transformation like

$$D_{j+1}^{-1}L_{ul}^j D_j = L_{null}^j$$  \hspace{1cm} (3)

construct the Lax operator of a NUL model. However to be a proper integrable system the representative Lax operator of the NUL model must satisfy the braided YBE [12]

$$R_{12} (u-v)L_{1j}^{null} (u)L_{2j}^{null} (v) Z_{12}^{-1} = L_{2j}^{null} (v)L_{1j}^{null} (u) Z_{21}^{-1} R_{12} (u-v).$$  \hspace{1cm} (4)
together with the nonultralocal braiding relation

\[ L_{2j+1}^{nul}(v)Z_{12}^{-1}L_{1j}^{nul}(u) = L_{ij}^{nul}(u)L_{2j+1}^{nul}(v). \]  

(5)

It should be noted that compared to the general relations introduced in [12] we have restricted only to the nearest neighbour braiding and taken the equations in a conjugate form to emphasize the fact that these relations, though look differently are in fact equivalent and lead to the same integrability for the NUL systems. It is obvious that at the trivial limit of \( Z = 1 \) the braided YBE reduces to the standard YBE and the braiding relation turns into the usual ultralocality condition.

The success of the gauge transformation would depend naturally on the suitable choice of the gauge operators, which must ensure the above braided extensions. In fact using the inverse of transformation (3) directly in the YBE (1) for \( L^{ul} \), it can be shown through some simple algebraic manipulations that in the simplest case the gauge operator \( D_m \) that transforms (1) into the braided YBE (4) for \( L^{nul} \) must satisfy the conditions

\[ D_{aj}L_{bj}^{nul}D_{aj}^{-1} = L_{bj}^{nul}Z_{ab}^{-1}, \quad [R_{12}, \ D_{1j}D_{2j}] = 0. \]  

(6)

Interestingly the ultralocality condition is transformed automatically to the braiding relation (3), whenever the first of the conditions (6) holds. Note that the corresponding monodromy matrix for the periodic discrete chain of size \( N \) would also be gauge related through (3) as \( T^{nul} = L_0^{nul}(u) \ldots L_1^{nul}(u) = D_{N+1}^{-1}T^{nul}D_1 = D_1^{-1}T^{nul}D_1 \) due to \( D_{N+1} = D_1 \). Therefore, YBE (3) for \( T^{ul} \) due to \( D_{a1}T_{bj}^{nul}D_{a1}^{-1} = T_{bj}^{nul}Z_{ab}^{-1} \), induced by (3), yields the global braided YBE

\[ R_{12}(u - v)T_1^{nul}(u)Z_{21}^{-1}T_2^{nul}(v)Z_{12}^{-1} = T_2^{nul}(v)Z_{12}^{-1}T_1^{nul}(u)Z_{21}^{-1}R_{12}(u - v). \]  

(7)

associated with the NUL models. In [12] the structure of the braiding matrices \( Z \) for which (3) leads to the integrability condition has been analyzed. In the present applications however we show the integrability by directly using in (4) the explicit form of the braiding matrices, which are much simpler in our case.

Since the NUL model must share the same \( R \)-matrix with the UL model, from which it is to be generated, we search for such a suitable source model associated with the well known \( 4 \times 4 \) trigonometric \( R \)-matrix, the nontrivial elements of which are

\[ R_{11}^{11} = R_{22}^{22} = \sin \alpha(u + 1), \quad R_{12}^{12} = R_{21}^{21} = \sin \alpha u, \quad R_{21}^{12} = R_{12}^{21} = \sin \alpha. \]  

(8)

The general discrete Lax operator of the UL systems related to (3) as shown in [16] may be given by

\[ L_k(u) = \begin{pmatrix} c_+^+\xi q^{S_k^3} + c_+^1\xi q^{-S_k^3} & 2\sin \alpha S_k^- \\ 2\sin \alpha S_k^+ & c_+^-\xi q^{-S_k^3} + c_+^1\xi q^{S_k^3} \end{pmatrix}, \quad q = e^{i\alpha}, \quad \xi = e^{iu} \]  

(9)

linked with the underlying generalized quantum algebra

\[ [S_k^3, S_i^+] = \pm\delta_{ki}S_k^\pm, \quad [S_k^+, S_i^-] = \delta_{ki}(M^+ [2S^3_k]_q + M^-[2S^3_k]_q), \quad [M^\pm, \cdot] = 0. \]  

(10)
Here \([x]_q \equiv \frac{\sin(\alpha x)}{\sin \alpha} \), \([x]_q^\pm \equiv \frac{\cos(\alpha x)}{\sin \alpha} \), and the central elements \(M^\pm\) are related to the other set of such elements appearing in the \(L\)-operator as \(M^\pm = \pm \frac{1}{2} \sqrt{\pm 1} (c^+_p + c^-_p \pm c^-_p c^+_p)\). The ultralocality condition for (3) holds naturally due to the algebraic relations (10), which is valid only at the same lattice sites. It is important to note that since condition (6) is a linear relation for the \(L\)-operator, the same condition must hold also for the UL source model, provided the braiding matrix \(Z\) commutes with the gauge operator \(D_j\). Therefore our strategy is to seek for the suitable structure of an UL Lax operator as some realization of (9), such that it also satisfies (6), crucial for generating NUL models. We are able to find two such structures for two different sets of choice of the central elements in (9).

3. NUL model from the Liouville model:

Lattice Liouville model (LLM) [17] can be derived from the general Lax operator (9) for the choice \(c^-_+ = c^+_+ = 0, \ c^-_+ = c^-_+ = 1\), which reduces the second algebraic relation of (10) to

\[
[S^+_k, S^-_l] = -2i\delta_{kl} e^{2i\alpha S^3_k} \sin \alpha
\]

and allows a realization of the generators as

\[
S^+_k = e^{-ipk} g(u_k), \quad S^-_k = g(u_k) e^{ipk}, \quad S^3_k = u_k, \quad g(u) = (\kappa + \Delta e^{i\alpha(u + \frac{1}{2})})^{\frac{1}{2}}
\]

in canonical variables \([u_k, p_l] = i\delta_{kl}\). Note that for any arbitrary value of \(\kappa\) and under any canonical transformation (12) remains a proper realization of (11) and therefore retains the integrability of the model. The well known LLM is obtained at \(\kappa = 1\), while for the generation of our new model we choose \(\kappa = 0\) along with a trivial canonical transformation \(p \to \alpha u, \ \alpha u \to -p\). This gives the Lax operator of our LLM as a realization of (9) in the form

\[
L_k(u) = \left( \begin{array}{cc} \xi e^{-ipk} & e^{i(\alpha u_k - pk)} \\ e^{-i(\alpha u_k + pk)} & \frac{1}{\xi} e^{-ipk} \end{array} \right),
\]

which would serve now as our source UL model. Since NUL Lax operators usually involve derivatives of the canonical variables or operators with current-like commutation relations \([v(x), v(y)] = \pm i\alpha \delta'(x - y)\), we introduce for our lattice construction the discrete version of such fields \(v^\pm_k\), defining their commutators in the form

\[
[v^+_k, v^+_l] = \pm i \frac{\alpha}{2} (\delta_{k+1,l} - \delta_{k,l+1}).
\]

The fact that these nonultralocal fields can be realized through canonical variables as

\[
v^+_k = -\frac{\alpha}{2}(u_{k+1} - u_k) - p_k, \quad v^-_k = \frac{\alpha}{2}(u_{k+1} - u_k) - p_k
\]

would play a significant role in our construction.

Remarkably, the structure of (13), more specifically the same exponential dependence as \(e^{-pk}\) for all its elements permits us to choose a simple gauge operator in the form \(D_j = e^{-i\frac{\alpha}{2}u_j \sigma^3}\). Applying this matrix gauge operator in transformation (3) on the LLM-like Lax operator (13) and changing from canonical to current-like variables as in (13) we can generate finally the Lax operator of our quantum integrable NUL model.
Straightforward calculations yield the explicit form of this \( L \)-operator as

\[
L_k^{(1)}(u) = \begin{pmatrix}
\xi W_k^+ & W_k^+
\frac{1}{\xi} W_k^-
\end{pmatrix}, \quad W_k^\pm = e^{iu_k^\pm}, \quad \xi = e^{iu},
\]

expressed completely through variables \( u_k^\pm \) having nonultralocal commutation relations (14).

With all required objects in hand we must check now that the essential conditions (3) are fulfilled in our construction. Indeed we find that the first of these conditions is satisfied by both the Lax operators (13) and (14) yielding the explicit form of the braiding matrix \( Z \) as

\[
Z_{12} = e^{i\frac{\alpha}{2}\sigma_3}.
\]

The second condition on the other hand holds due to the symmetry of the trigonometric \( R \)-matrix (8) and the specific form of our gauge operator. Consequently, as shown above, Lax operator (16) should also satisfy the braided YBE relations (4), (5) and (7). The associated \( R \)-matrix is given by the same (8) and the braiding matrix as in (17). Thus we conclude that the Lax operator (16) we have generated from a LLM-like model (13) represents a new integrable quantum NUL model. The full set of conserved quantities for this integrable periodic model may be obtained directly from (7) through diagonal \( \{L_k^{(1)}(u)\} \), the braiding matrix as in (17). For finding the dynamical equations however we have to use the noncanonical commutation relations (14) together with

\[
[v_k^+, v_l^-] = i\frac{\alpha}{2}(\delta_{k+1,l} - 2\delta_{k,l} + \delta_{k,l+1}),
\]

which is also compatible with the realization (15). Note the intriguing fact that the noncommutativity of the operators (14) and (18) at different sites induce nearest-neighbour interaction in the model.

4. Light-cone sine-Gordon as NUL model:

For constructing our next model we choose \( c_+^+ = c_- = 0, c_+^- = c_-^+ = \Delta \) or its complementary set \( c_+^- = c_-^- = \Delta, c_+^+ = c_-^+ = 0 \), where \( \Delta \) is the lattice constant. Evidently both of these choices give \( M^\pm = 0 \) and reduce (10) to a simplified algebra

\[
[S_k^+, S_l^-] = 0, \quad [S_k^3, S_l^\pm] = \pm \delta_{k,l} S_k^\pm,
\]

which may be realized as \( S_k^+ = (S_k^-)^{-1} = e^{-ip_k}, \alpha S_k^3 = \alpha u_k \mp p_k \). The Lax operators with the above two sets of choices for the central elements and using (19) therefore can be realized from (8) in the form

\[
L^{(-)}_k(u) = \begin{pmatrix}
\Delta \xi e^{i(\alpha u_k - p_k)} & e^{ip_k} \\
e^{-ip_k} & \Delta \xi e^{-i(\alpha u_k - p_k)}
\end{pmatrix}, \quad L^{(+)}_k(u) = \begin{pmatrix}
\Delta \xi e^{-i(\alpha u_k + p_k)} & e^{ip_k} \\
e^{-ip_k} & \Delta \xi e^{i(\alpha u_k + p_k)}
\end{pmatrix},
\]

representing two quantum integrable UL lattice models, which we intend to use as our source model for constructing NUL systems. Note that the right operator \( L^{(+)}_k(u) \) can be obtained formally from the left one \( L^{(-)}_k(u) \) through a simple mapping \( \xi \to \frac{1}{\xi}, \alpha \to -\alpha \). Therefore we deal explicitly with the left case only and recover the results related to the complementary right case through the above
mapping. Interestingly, one can choose the gauge operator \( \tilde{D}_j \) in the present case as the *square* of that considered above for the LLM, i.e. \( \tilde{D}_j(\alpha) = D_j^2 = e^{-i\alpha\sigma^3} \) for the left model and applying gauge transformation (3) as \( q^{-\frac{1}{2}}\tilde{D}_{j+1}(\alpha) \left( L_{j}^{(-)}(u)^{\sigma^1} \right) \tilde{D}_{j}^{-1}(\alpha) = L_{j}^{(-)lcsg}(u) \) construct the Lax operator

\[
L_{j}^{(-)lcsg}(u) = \begin{pmatrix}
  e^{i(p_j-\alpha\nabla u_j)} & \Delta \xi e^{-i(p_j+\alpha u_{j+1})} \\
  \Delta \xi e^{i(p_j+\alpha u_{j+1})} & e^{-i(p_j-\alpha\nabla u_j)}
\end{pmatrix}, \quad \nabla u_j \equiv u_{j+1} - u_j
\]

with \( \xi = e^{iu} \). Note that (21) is the Lax operator of an exact lattice version of the LCSG model and represents indeed a new quantum NUL model. We can check directly that both the source UL model \( L_{k}^{(-)}(u) \) in (20) and the L-operator of the NUL model (21) respect conditions (3) required for the scheme, giving the braiding matrix in the explicit form

\[
Z_{12}^{(-)lcsg} = e^{i\alpha\sigma^3 \otimes \sigma^3}.
\]

Therefore the Lax operator (21) satisfies also the braided YBE (2), (7) and the nonultralocal relation (3) with R-matrix (8) and Z-matrix (22), which establishes the quantum integrability of the NUL model we have generated. The significance of the model is revealed at the continuum limit \( \Delta \to 0 \), when it reduces to the light-cone sine-Gordon field model. At this field limit we have \( p_j \to \Delta \partial_x u(x) \), \( \alpha u_j \to u(x) \), \( \alpha u_{j+1} \to u(x) + \Delta \partial_x u(x) \). Defining therefore \( \partial_x u \pm \partial_x \alpha = \frac{1}{2} \partial_x \alpha \) in the light-cone coordinates it is not difficult to show that the lattice LCSG (22) reduces to the well known field operator form

\[
L_{j}^{(-)lcsg}(u) \to I + \Delta U_-(x), \text{ where } U_-(x) = \begin{pmatrix} i\frac{1}{2} \partial_x u(x) & \xi e^{-iu(x)} \\
 \xi e^{iu(x)} & -i\frac{1}{2} \partial_x u(x) \end{pmatrix}, \text{ yields one of the well known Lax pair: } \partial_x \Phi = U_\Phi.
\]

It is intriguing to follow that a complementary *right* LCSG similarly may be generated from \( L_{k}^{(+)}(u) \) in (20) as \( q^{-\frac{1}{2}}\tilde{D}_{j+1}(-\alpha) \left( L_{j}^{(+)}(u)^{\sigma^1} \right) \tilde{D}_{j}^{-1}(-\alpha) = L_{j}^{(+)lcsg}(u) \), which represents again a new quantum integrable NUL model associated with the same R-matrix, while the braiding matrix is given by \( Z_{12}^{(+)lcsg} = e^{-i\alpha\sigma^3 \otimes \sigma^3} \). Note that the NUL model \( L_{j}^{(+)lcsg}(u) \) can be obtained formally from \( L_{j}^{(-)lcsg}(u) \) by using the same mapping \( \xi \to \frac{1}{\xi}, \alpha \to -\alpha \) and yields at the continuum limit \( L_{j}^{(+)lcsg}(u) \to I - \Delta U_+(x) \), with \( U_+(x) \) being the other component of the Lax pair. Zero curvature condition: \( \partial U_+ - \partial U_+ + [U_+, U_-] = 0 \) involving both Lax operators gives the well known form of the sine-Gordon field equation in light-cone coordinates: \( \partial^2_{-} u = 2\sin 2u \).

As it has been shown recently (13), the NUL quantum mKdV(\( \pm \)) models can also be obtained by gauge transforming some UL models. It is remarkable that these UL models may be given exactly by the same source models \( L_{k}^{(\pm)}(u) \) (21) found here for the LCSG. However the gauge operator required for constructing the mKdV should be given by \( D_j(\alpha) = (\tilde{D}_j(\alpha))^{\pm} = e^{-i\frac{1}{2} \alpha \sigma^3} \) and therefore the braiding matrix for the mKdV is related similarly to that of the LCSG as \( Z^{(\pm)mkdv} = (Z^{(\pm)lcsg})^{\pm} = q^{-\frac{1}{2} \sigma^3 \otimes \sigma^3} \), as can be confirmed by comparing (13) with our result. It is intriguing to note that, the gauge operator \( D_j(\alpha) \) for the mKdV model coincides on the other hand with that for our new nonultralocal LLM-type model, we have found above.

For exact solution of the eigenvalue problem of the Hamiltonian and higher conserved operators for quantum NUL systems, one needs modification of the algebraic Bethe ansatz (ABA). Such a formulation for the NUL mKdV has already been developed in (13, 14). Since the quantum NUL
systems that we have discovered here are very close in structure to the quantum mKdV, we can apply successfully the modified ABA to the present models following closely the steps of the quantum mKdV and therefore we omit them here.

5. Conclusion

It seems that the gauge generation scheme adopted here for constructing new quantum integrable NUL models starting from ultralocal ones is far more promising than might have been expected and applied for, so far. Our success encourages us to look for the explicit applicability of this scheme to the well known NUL models, e.g. quantum mapping, Coulomb-gas CFT related models, WZWN model etc. It would certainly be desirable to exploit this scheme for solving the challenging problem of establishing the braided YBE formulation for NUL models like nonlinear σ-model, derivative NLS, complex sine-Gordon model etc.

References

[1] L. D. Faddeev, Sov. Sc. Rev. C1 (1980) 107.
[2] P. Kulish and E. K. Sklyanin, Lect. Notes in Phys. (ed. J. Hietarinta et al, Springer,Berlin, 1982) vol. 151 p. 61.
[3] S. A. Tsyplyaev, Teor. Mat. Fiz. 46 (1981) 24
  J.M. Maillet, Phys. Lett. 162 B (1985) 137; Nucl. Phys. B269 (1986) 54 ; Phys. Lett. 167 B (1986) 401.
  M. Semenov-Tian-Shansky, Funct. Anal. Appl. 17 (1983) 259
[4] V. E. Korepin, J. Sov. Math. 23 (1983) 2429.
[5] F.W. Nijhoff, H.W. Capel and V.G. Papageorgiou, Integrable quantum mappings, Clarkson Univ. preprint INS 168/91 (1991)
  F.W. Nijhoff and H.W. Capel, Integrable quantum mapping and nonultralocal Yang-Baxter structure, Clarkson Univ. preprint INS 171/91 (1991).
[6] O. Babelon and L. Bonora, Phys. Lett. 253 B (1991) 365 ; O. Babelon, Comm. Math. Phys. 139 (1991) 619 ; L. Bonora and V. Bonservizi, Nucl. Phys. B 390 (1993) 205.
[7] A. Alekseev, L.D. Faddeev, M. Semenov-Tian-Shansky Comm. Math. Phys. 149 (1992) 335 ; L. D. Faddeev, Comm. Math. Phys. 132 (1990) 131 ; A. Alekseev, S. Statashvili, Comm. Math. Phys. 133 (1990) 353 ; B. Blok, Phys. Lett. 233B (1989) 359
[8] N. Yu. Reshetekhin and M. Semenov-Tian-Shansky , Lett. Math. Phys. 19 (1990) 133
[9] M. Chu, P. Goddard, I. Halliday, D. Olive and A. Schwimmer, Phys. Lett. 266 B (1991) 71.
[10] L. Freidel and J.M. Maillet, Phys. Lett. 262 B (1991) 278 ; 263 B (1991) 403
[11] L. Hlavaty, J. Math. Phys. 35 (1994) 2560
[12] L. Hlavaty and Anjan Kundu Int. J. Mod. Phys. A11 (1996) 2143
[13] Anjan Kundu, Mod. Phys. Lett. A10 (1995) 2955,
[14] D. Fioravanti and M. Rossi, J. Phys. A 35 (2002) 3649
[15] D. Fioravanti and M. Rossi, J. Phys. A 34 (2001) L567
[16] Anjan Kundu, Phys. Rev. Lett. 82 (1999) 3936
[17] L.D. Faddeev, O. Tirkkonen, Nucl. Phys. B453 (1995) 647
[18] Anjan Kundu, under preperation