The dynamical twisting and nondynamical r-matrix structure of elliptic Ruijsenaars-Schneider model

Bo-yu Hou\textsuperscript{a,b} and Wen-Li Yang\textsuperscript{a,b} \textsuperscript{*\dagger}

\textsuperscript{a} CCAST (World Laboratory), P.O.Box 8730, Beijing 100080, China
\textsuperscript{b} Institute of Modern Physics, Northwest University, Xian 710069, China \textsuperscript{‡}

Abstract

From the dynamical twisting of the classical r-matrix, we obtain a new Lax operator for the elliptic Ruijsenaars-Schneider model (cf. Ruijsenaars'). The corresponding r-matrix is shown to be the classical $Z_n$-symmetric elliptic r-matrix, which is the same as that obtained in the study of the nonrelativistic version—the $A_{n-1}$ Calogero-Moser model.

Mathematics Subject Classification : 70F10, 70H33, 81U10.

I Introduction

Following the successes of the Calogero-Moser (CM) models \cite{1,2}, a relativistic generalization of the CM models—the so-called Ruijsenaars-Schneider (RS) models have been proposed \cite{3}, which the intergrability has been conserved. The RS model describes a completely integrable system of n one-dimensional interacting relativistic particles. Its importance lies in the fact that it is related to the dynamics of solitons in some integrable relativistic field theories\cite{4,5} and its discrete-time version has been connected with the Bethe ansatz equation of the solvable lattice statistical model \cite{6}. Recent development was shown that it can be obtained by a Hamiltonian reduction of the cotangent bundle of some Lie group \cite{7}, and can also be considered as the gauged WZW theory \cite{8}. The study of RS model would play an universal role in study of completely integrable multi-particle systems. Among all type RS models, the elliptic RS model is the most general one and the other type such as the rational, hyperbolic and trigonometric type is just the various

\textsuperscript{*} e-mail : wlyang@phy.nwu.edu.cn
\textsuperscript{†} Fax : 86-29-8302331
\textsuperscript{‡} Mailing address
degenerations of the elliptic one. In this paper, we shall study the elliptic $A_{n-1}$ type RS model with generic $n$ ($n > 2$).

The Lax representation and its corresponding r-matrix structure for rational, hyperbolic and trigonometric $A_{n-1}$ type RS models were constructed by Avan et. al [9]. The Lax representation for the elliptic RS models was constructed by Ruijsenaars [10], and the corresponding r-matrix structure was given by Nijhoff et al [11] and Suris [12]. It turns out that the r-matrix structure of the RS model is given in terms of a quadratic Poisson-Lie bracket with dynamical r-matrices (i.e. the r-matrix depends upon the dynamical variables). Particularly, in contrast with the dynamical Yang-Baxter equation of the r-matrix structure of the CM model, the generalized Yang-Baxter relations for the quadratic Poisson-Lie bracket with a dynamical r-matrix is still an open problem [11]. Since the Poisson bracket of the Lax operator is no longer closed, the quantum version of such classical L-operator has not been able to be constructed.

It is well-known that the Lax representation for a completely integrable models is not unique. It has been recognized [13, 14] that the r-matrix of a model can be changed drastically by the choice of Lax representation. In our former work [14], we succeeded in constructing a new Lax operator (cf. Krichever’s [15]) for the elliptic $A_{n-1}$ CM model and showing that the corresponding r-matrix is a nondynamical one, which is the classical $Z_n$-symmetric elliptic r-matrix [16, 14]. Very recently, we found a “good” Lax operator for the elliptic RS model with a very special case $n=2$ [17]. In present paper, extending our former work in [17], we construct a “good” Lax operator (in such a sense that it has a nondynamical r-matrix structure) for the elliptic RS model with a general $n$ ($n > 2$).

The paper is organized as follows. In section 2, we construct the dynamical twisting relations of the classical r-matrix for the quadratic Poisson-Lie bracket. The condition that the “good” Lax representation could exist is found. In section 3, some briefly reviews of Nijhoff et al.’s work on dynamical r-matrix of the elliptic RS model was given. In section 4, we construct the “good” Lax representation for elliptic RS model with generic $n$, and obtain the corresponding nondynamical r-matrix structure. The quantum version L-operator of the Lax operator is constructed in section 5. Finally, we give summary and discussions. Appendix contains some detailed calculations.

II The dynamical twisting of classical r-matrix

In this section we will give some general theories of the completely integrable finite particles systems.

A Lax pair $(L, M)$ consists of two functions on the phase space of the system with values in some Lie algebra $g$, such that the evolution equations may be written in the following form

$$\frac{dL}{dt} = [L, M], \quad (\text{II.1})$$

where $[,]$ denotes the bracket in the Lie algebra $g$. The interest in the existence of such a pair lies in the fact that it allows for an easy construction of conserved quantities (integrals of motion). It follows that the adjoint-invariant quantities $trL^l$ ($l = 1, ..., n$) are the integrals of the motion. In order to implement Liouville theorem onto this set of possible action variables we need them to be Poisson-commuting. As shown in [13], the commutativity of the integrals $trL^l$ of the Lax operator can be deduced from that the fundamental Poisson bracket $\{L_1(u), L_2(v)\}$ could be represented in the linear commutator
form
\[ \{L_1(u), L_2(v)\} = [r_{12}(u, v), L_1(u)] - [r_{21}(v, u), L_2(v)], \] (II.2)
or quadratic form \[ \{L_1(u), L_2(v)\} = L_1(u)L_2(v)r_{12}^-(u, v) - r_{21}^+(v, u)L_1(u)L_2(v) + L_1(u)s_{12}^+(u, v)L_2(v) - L_2(v)s_{12}^-(u, v)L_1(u), \] (II.3)
where we have used the notation
\[ L_1 \equiv L \otimes 1, \quad L_2 \equiv 1 \otimes L, \quad a_{21} = Pa_{12}P, \]
and \( P \) is the permutation operator such that \( Px \otimes y = y \otimes x \).

The dynamical twisting of the linear Poisson-Lie bracket (II.2) was studied in the [14] (we refer therein) and also studied by Babelon et al [13]. We are to investigate the general dynamical twisting of the quadratic Poisson-Lie bracket (II.3).

In order to define a consistent Poisson bracket, one should impose some constraints on the r-matrices. The skew-symmetry of Poisson bracket require that
\[ r_{21}^+(v, u) = -r_{12}^+(u, v), \quad s_{21}^+(v, u) = s_{12}^-(u, v), \] (II.4)
\[ r_{12}^+(u, v) - s_{12}^+(u, v) = r_{12}^-(u, v) - s_{12}^-(u, v). \] (II.5)
As for the numerical r-matrices \( r^\pm(u, v), s^\pm(u, v) \) case, some constraints condition (sufficient condition) imposed on the r-matrices to make Jacobi identity satisfied, was given by Freidel et al [18]. However, generally speaking, the Jacobi identity for the dynamical r-matrices \( r^\pm(u, v), s^\pm(u, v) \) would take a very complicated form.

It should be remarked that such a classification (from dynamical and nondynamical r-matrix structure) is by no means unique, which drastically depend on the Lax representation which one choose for a system. Therefore, there is no one-to-one correspondence between a given dynamical system and a defined r-matrix. The same dynamical system may have several Lax representations and several r-matrix [14]. The different Lax representation of a system is conjugated each other. Namely, if \( (\tilde{L}, \tilde{M}) \) is one of other Lax pair of the same dynamical system conjugated with the old one \( (L, M) \), it means that
\[ \tilde{L}(u) = g(u)L(u)g^{-1}(u), \quad \tilde{M}(u) = g(u)M(u)g^{-1}(u) - \left( \frac{d}{dt}g(u) \right)g^{-1}(u), \] (II.6)
where \( g(u) \in G \) whose Lie algebra is \( g \). Then, we have

**Proposition 1.** The Lax pair \( (\tilde{L}, \tilde{M}) \) has the following r-matrix structure
\[ \{\tilde{L}_1(u), \tilde{L}_2(v)\} = \tilde{L}_1(u)\tilde{L}_2(v)\tilde{r}_{12}^-(u, v) - \tilde{r}_{21}^+(v, u)\tilde{L}_1(u)\tilde{L}_2(v) \]
\[ + \tilde{L}_1(u)s_{12}^+(u, v)\tilde{L}_2(v) - \tilde{L}_2(v)s_{12}^-(u, v)\tilde{L}_1(u), \] (II.7)
where
\begin{align*}
\tilde{r}_{12}^-(u, v) &= g_1(u)g_2(v)r_{12}^-(u, v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12}(u, v) + \tilde{\Delta}_{21}(u, v), \\
\tilde{r}_{12}^+(u, v) &= g_1(u)g_2(v)r_{12}^+(u, v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12}^{(1)}(u, v) + \tilde{\Delta}_{21}^{(1)}(u, v), \\
\tilde{s}_{12}^+(u, v) &= g_1(u)g_2(v)s_{12}^+(u, v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12}(u, v) - \tilde{\Delta}_{12}^{(1)}(u, v), \\
\tilde{s}_{12}^-(u, v) &= g_1(u)g_2(v)s_{12}^-(u, v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12}(u, v) - \tilde{\Delta}_{12}^{(1)}(v, u), \\
\tilde{\Delta}_{12}(u, v) &= \tilde{\Delta}_{12}^{(1)}(v, u) = \frac{1}{2}[g_1(u), g_2(v)]g_1^{-1}(u)g_2^{-1}(v), \\
\Delta_{12}(u, v) &= \frac{1}{2}[g_1(u), g_2(v)]g_1^{-1}(u)g_2^{-1}(v), \\
\end{align*}

\[ \times g_2(v)[g_1(u), L_2(v)]g_1^{-1}(u)g_2^{-1}(v) \]
and the properties of (III.4) and (II.3) are conserved
\[
\tilde{r}_{21}^\pm (v, u) = -\tilde{r}_{12}^\pm (u, v), \quad \tilde{s}_{21}^+ (v, u) = \tilde{s}_{12}^- (u, v), \\
\tilde{r}_{12}^- (u, v) - \tilde{s}_{12}^- (u, v) = \tilde{r}_{12}^- (u, v) - \tilde{s}_{12}^- (u, v).
\]

**Proof:** The proof is direct substituting (II.6) into the fundamental Poisson bracket (II.3) and use the following identity
\[
[[a_{12}, L_1], L_2] = [[a_{12}, L_2], L_1].
\]
where \(a_{12}\) is any matrix on \(g \otimes g\).

It can be seen that: (I). The Lax operator \(L(u)\) is transferred as a similarity transformation from the different Lax representation; (II). The corresponding \(M\) is undergone the usual gauge transformation; (III). The \(r\)-matrices are transferred as some generalized gauge transformation, which can be considered as the generalized classical version of the dynamically twisting relation of the quantum R-matrix [19]. Therefore, it is of great value to find a "good" Lax representation for a system if it exists, in which the corresponding \(r\)-matrices are all nondynamical ones and \(r_{12}^+(u, v) = r_{12}^-(u, v), \quad s_{12}^+(u, v) = 0\). In this special case, the corresponding Poisson-Lie bracket becomes the Sklyanin bracket and the well-studied theories [20, 21] can be directly applied in the system.

**Corollary 1.** For given Lax pair \((L, M)\) and the corresponding \(r\)-matrices, if there exist \(g(u)\) satisfied
\[
\begin{align*}
g_1(u)g_2(v)s_{12}^+(u, v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{21} (v, u) - \tilde{\Delta}_{12}^{(1)} (u, v) & = 0, \\
g_1(u)g_2(v)s_{12}^-(u, v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12} (u, v) - \tilde{\Delta}_{21}^{(1)} (v, u) & = 0, \quad (II.8) \\
\partial_u h_{12} = \partial_v h_{12} & = 0, \quad (II.9)
\end{align*}
\]
where
\[
\begin{align*}
h_{12}(u, v) & = g_1(u)g_2(v)r_{12}^-(u, v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12} (u, v) + \tilde{\Delta}_{21} (v, u) \\
& \equiv g_1(u)g_2(v)r_{12}^+(u, v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12}^{(1)} (u, v) + \tilde{\Delta}_{21}^{(1)} (v, u), \quad (II.10)
\end{align*}
\]
the nondynamical Lax representation with Sklyanin Poisson-Lie Bracket of the system would exist.

The main purpose of this paper is to find a “good” Lax representation for the elliptic RS model with generic \(n (n > 2)\).

### III Review of elliptic RS model

We first define some elliptic functions
\[
\begin{align*}
\theta^{(j)}(u) & = \theta \left[ \frac{1}{2} - \frac{i}{n} \right] (u, n\tau), \quad \sigma(u) = \theta \left[ \frac{1}{2} \right] (u, \tau), \\
\theta \left[ \frac{a}{b} \right] (u, \tau) & = \sum_{m=-\infty}^{\infty} \exp \{ \sqrt{-1} \pi [(m + a)^2 \tau + 2(m + a)(z + b)] \}, \\
\theta^{(j)}(u) & = \partial_a \{ \theta^{(j)}(u) \}, \quad \sigma'(u) = \partial_a \{ \sigma(u) \}, \quad \xi(u) = \partial_a \{ \ln \sigma(u) \}, \quad (III.11)
\end{align*}
\]

where \( \tau \) is a complex number with \( \text{Im}(\tau) > 0 \).

The Ruijsenaars-Schneider model is the system of \( n \) one-dimensional relativistic particles interacting by the two-body potential. In terms of the canonical variables \( p_i, q_i \ (i = 1, \ldots, n) \) enjoying in the canonical Poisson bracket

\[
\{p_i, p_j\} = 0, \quad \{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij},
\]

the Hamiltonian of the system is expressed as [10]

\[
H = mc^2 \sum_{i=1}^{n} \cosh p_i \prod_{k \neq j} \left\{ \frac{\sigma(q_{jk} + \gamma)\sigma(q_{jk} - \gamma)}{\sigma^2(q_{jk})} \right\}^{\frac{1}{2}}, \quad q_{jk} = q_j - q_k. \quad (\text{III.13})
\]

Here, \( m \) denotes the particle mass, \( c \) denotes the speed of light, \( \gamma \) is the coupling constant. The Hamiltonian [10] is known to be completely integrable. The most effective way to show its integrability is to construct the Lax representation for the system (namely, to find the classical Lax operator). One L-operator for the elliptic RS model was given by Ruijsenaars [10]

\[
L_R(u)^i_j = \frac{\epsilon^{pj} \sigma(\gamma + u + q_{ji})}{\sigma(\gamma + q_{ji})\sigma(u)} \prod_{k \neq j} \left\{ \frac{\sigma(q_{jk} + \gamma)\sigma(q_{jk} - \gamma)}{\sigma^2(q_{jk})} \right\}^{\frac{1}{2}}, \quad i, j = 1, \ldots, n. \quad (\text{III.14})
\]

Alternatively, we adopt another Lax operator \( \tilde{L}_R \), which is similar to that of Nijhoff et al in [11]

\[
\tilde{L}_R(u)^i_j = \frac{\epsilon^{pj} \sigma(\gamma + u + q_{ji})}{\sigma(u)\sigma(\gamma + q_{ji})} \prod_{k \neq j} \frac{\sigma(\gamma + q_{jk})}{\sigma(q_{jk})}. \quad (\text{III.15})
\]

The relation of \( \tilde{L}_R \) with the standard Ruijsenaars’ \( L_R(u) \) can be obtained from a Poisson map (or a canonical transformation)

\[
q_i \rightarrow \tilde{q}_i, \quad p_i \rightarrow p_i + \frac{1}{2} \ln \prod_{k \neq i} \frac{\sigma(q_{ik} + \gamma)}{\sigma(q_{ik} - \gamma)}. \quad (\text{III.16})
\]

**Proposition 2.** The map defined in (III.16) is a Poisson map.

**Proof:** The proposition 2 can be proven from considering the symplectic two-form

\[
\sum_i dp_i \wedge dq_i \left( \frac{1}{2} \ln \prod_{k \neq i} \frac{\sigma(q_{ik} + \gamma)}{\sigma(q_{ik} - \gamma)} \right) \wedge dq_i = \sum_i dp_i \wedge dq_i - \frac{1}{2} \sum_{k \neq i} \left( \frac{\sigma'(q_{ik} + \gamma)}{\sigma(q_{ik} + \gamma)} - \frac{\sigma'(q_{ik} - \gamma)}{\sigma(q_{ik} - \gamma)} \right) dq_k \wedge dq_i = \sum_i dp_i \wedge dq_i - \frac{1}{2} \sum_{k < i} \left( \frac{\sigma'(q_{ik} + \gamma)}{\sigma(q_{ik} + \gamma)} + \frac{\sigma'(q_{ik} - \gamma)}{\sigma(q_{ik} - \gamma)} - \frac{\sigma'(q_{ik} - \gamma)}{\sigma(q_{ik} - \gamma)} - \frac{\sigma'(q_{ik} + \gamma)}{\sigma(q_{ik} + \gamma)} \right) dq_k \wedge dq_i = \sum_i dp_i \wedge dq_i,
\]

where we have used the property that the elliptic function \( \sigma(u) \) is an odd function with regard to argument \( u \). \( \square \)

It is well-known that the Poisson bracket is invariant under the Poisson map. Hence the study of the r-matrix structure for the standard Ruijsenaars Lax operator \( L_R(u) \) is equivalent to that of Lax operator \( \tilde{L}_R(u) \).
Following the work of Nijhoff et al [11], the fundamental Poisson bracket of the Lax operator \( \tilde{L}_R (u) \) can be given in the following quadratic r-matrix form with a dynamical \( r \)-matrices

\[
\{ \tilde{L}_R (u)_1, \tilde{L}_R (v)_2 \} = \tilde{L}_R (u)_1 \tilde{L}_R (v)_2 r_{12}^-(u,v) - r_{21}^+(v,u) \tilde{L}_R (u)_1 \tilde{L}_R (v)_2 + \tilde{L}_R (u)_1 s_{12}^+(u,v) \tilde{L}_R (v)_2 - \tilde{L}_R (v)_2 s_{12}^-(u,v) \tilde{L}_R (u)_1,
\]

where

\[
r_{12}^-(u,v) = a_{12}(u,v) - s_{12}(u) + s_{21}(v), \quad r_{12}^+(u,v) = a_{12}(u,v) + u_{12}^+ + u_{12}^-,
\]

\[
s_{12}^+(u,v) = s_{12}(u) + u_{12}^+, \quad s_{12}^-(u,v) = s_{21}(v) - u_{12}^-,
\]

and

\[
a_{12}(u,v) = r_{12}^0(u,v) + \sum_{i=1}^{\infty} \xi(u-v)e_{ii} \otimes e_{ii} + \sum_{i \neq j} \xi(q_{ij}) e_{ii} \otimes e_{jj},
\]

\[
r_{12}^0(u,v) = \sum_{i \neq j} \frac{\sigma(q_{ij} + u-v)}{\sigma(q_{ij}) \sigma(u-v)} e_{ij} \otimes e_{ji}, \quad s_{12}(u) = \sum_{i,j} \left( \tilde{L}_R (u) \partial_{\gamma} \tilde{L}_R (u) \right)^i_j e_{ij} \otimes e_{jj},
\]

\[
u_{12}^+ = \sum_{i,j} \xi(q_{ji} \pm \gamma) e_{ii} \otimes e_{jj}.
\]

The matrix element of \( e_{ij} \) is equal to \( (e_{ij})^l_k = \delta_{il} \delta_{jk} \). It can be checked that the following symmetric condition hold for the r-matrices \( r_{12}^\pm(u,v) \) and \( s_{12}^\pm(u,v) \)

\[
r_{21}^-(v,u) = -r_{12}^-(u,v), \quad s_{21}^+(v,u) = s_{12}^+(u,v),
\]

\[
r_{12}^+(u,v) - s_{12}^+(u,v) = r_{12}^-(u,v) - s_{12}^-(u,v).
\]

The classical r-matrices \( r_{12}^\pm(u,v), s_{12}^\pm(u,v) \) are of dynamical ones (i.e the matrix element of theirs do depend upon the dynamical variables \( q_i \)). The quadratic Poisson bracket \( (\text{III.17}) \) and the symmetric conditions of \( (\text{III.18})-(\text{III.19}) \) lead to the evolution integrals \( tr(\tilde{L}_R (u))^l \).

Due to the r-matrices depending on the dynamical variables, the Poisson bracket of \( \tilde{L}_R (u) \) is no longer closed. The complexity of the r-matrices \( (\text{III.17}) \) results in that it is still an open problem to check the generalized Yang-Baxter relations for the RS model. Moreover, the quantum version of the algebraic relation \( (\text{III.17}) \) is still not found. The same situation also occurs for the standard Lax operator \( L_R(u) \), and the corresponding r-matrices was given by Suris [12].

**IV** The “good” Lax representaion of elliptic RS model and its r-matrix

The L-operator of the elliptic RS model given by Ruijsenaars \( L_R(u) \) in \( (\text{III.14}) \) (or its Poisson equivalent counterpart \( \tilde{L}_R (u) \) in \( (\text{III.13}) \)) and corresponding r-matrix \( r_{12}(u,v) \) given by Suris [12] (or given by Nijhoff et al [11] ) leads to some difficulties in the investigation of the RS model. This motivates us to find a “good” Lax representation of the RS model. As see from proposition 1 and corollary 1 in section II, this means to find \( g(u) \) which satisfies \( (\text{III.8})-(\text{III.9}) \). In our former work [17], we have succeeded in find such a \( g(u) \) for the elliptic RS model with a special case \( n = 2 \). Fortunately, we could also find such a \( g(u) \)
for the elliptic RS model with a generic \( n \) \((n > 2)\) (This kind L-operator does not always exist for general completely integrable system). The fundamental Poisson bracket of this new L-operator \( L(u) \) would be expressed in the Sklyanin Poisson-Lie bracket form with a numeric r-matrix. The corresponding r-matrix is the classical \( Z_n \)-symmetric r-matrix in [14]. Namely, the elliptic RS and the corresponding non-relativistic version—the elliptic \( A_{n-1} \) CM model [14] are governed by the exact same r-matrix (cf. [12]) in some gauge. In order to compare with the L-operator given by Ruijsenaars \( R(u) \) and its Poisson equivalence \( \sim L_R (u) \), we call this L-operator as the new Lax operator (alternatively, a “good” Lax operator).

Set an \( n \otimes n \) matrix \( A(u; q) \)

\[
A(u; q)^i_j \equiv A(u; q_1, q_2, ..., q_n)^i_j = \theta^{(i)}(u + nq_j - \sum_{k=1}^{n} q_k + \frac{n - 1}{2}).
\]

(IV.20)

We remark that \( A(u; q)^i_j \) correspond to the interwiner function \( \varphi^{(i)}_j \) between the \( Z_n \)-symmetric Belavin model and the \( A^{(1)}_{n-1} \) face model [22] in [23].

Define

\[
g(u) = A(u; q) \Lambda(q), \quad \Lambda(q)^i_j = h_i(q) \delta^i_j,
\]

\[
h_i(q) \equiv h_i(q_1, ..., q_n) = \frac{1}{\prod_{l \neq i} \sigma(q_{il})}.
\]

Let us construct the new Lax operator \( L(u) \)

\[
L(u) = g(u) \tilde{L}_R (u) g^{-1}(u).
\]

(IV.21)

It will turn out that such a Lax operator \( L(u) \) give a “good” Lax representation for the elliptic RS model. This is our main results of this paper. To recover this, let us express the “good” Lax operator \( L(u) \) more explicitly.

**Proposition 3.** The Lax operator \( L(u) \) can be rewritten in the factorized form

\[
L(u)^i_j = \frac{1}{\sigma(\gamma)} \sum_{k=1}^{n} A(u + n\gamma; q)^i_k A^{-1}(u; q)^k_j e^{p_k}, \quad i, j = 1, 2, ..., n.
\]

(IV.22)

**Proof:** First, let us introduce a matrix \( T(u) \) with matrix elements

\[
T(u)^i_j = \sum_k e^{p_j} A^{-1}(u; q)^i_k A(u + n\gamma; q)^k_j.
\]

From the definition of \( A(u; q)^i_j \) and the determinant formula of Vandermonde type [23]

\[
det[\theta^{(i)}(u_k)] = \text{Const.} \times \sigma(\frac{1}{n} \sum_k u_k - \frac{n - 1}{2}) \prod_{1 \leq j < k \leq n} \sigma(\frac{u_k - u_j}{n}),
\]

(IV.23)

where the Const. does not depend upon \( \{u_k\} \), we have

\[
\sum_k A^{-1}(u; q)^i_k A(u + n\gamma; q)^k_j = \frac{\sigma(\gamma + u + q_{ij})}{\sigma(u)} \prod_{k \neq i} \frac{\sigma(\gamma + q_{jk})}{\sigma(q_{ik})}.
\]
Namely,
\[
T(u)^i_j = \frac{e^{p_j} \sigma(\gamma + u + q_{ji})}{\sigma(u)} \prod_{k \neq i} \frac{\sigma(\gamma + q_{jk})}{\sigma(q_{ik})}
\]
\[
= \frac{1}{\prod_{k \neq i} \sigma(q_{ik})} \left\{ \frac{e^{p_j} \sigma(\gamma + u + q_{ji}) \sigma(\gamma)}{\sigma(u) \sigma(\gamma + q_{ji})} \prod_{k \neq j} \frac{\sigma(\gamma + q_{jk})}{\sigma(q_{jk})} \right\} \prod_{k \neq j} \sigma(q_{jk}).
\]

Then, we obtain
\[
\frac{1}{\sigma(\gamma)} \sum_k A(u + n\gamma; q) \sigma^k \sigma(u) \sigma(q) \prod_{k \neq l} \sigma(q_{lk})
\]
\[
= \frac{1}{\sigma(\gamma)} \sum_{m}\sum_i A(u; q)^i_m T^m_l(u) A^{-1}(u; q)^l_j
\]
\[
= \sum_{m,l} \frac{A(u; q)^i_m}{\prod_{k \neq m} \sigma(q_{mk})} \left\{ \frac{e^{-p_i} \sigma(\gamma + u + q_{im})}{\sigma(u) \sigma(\gamma + q_{im})} \prod_{k \neq l} \frac{\sigma(\gamma + q_{lk})}{\sigma(q_{lk})} \right\} A^{-1}(u; q)^l_j \prod_{k \neq l} \sigma(q_{lk})
\]
\[
= \sum_{m,l} g(u)^i_m \tilde{L}_R (u)^m_l g^{-1}(u)^l_j \equiv L(u)^i_j
\]
\[\square\]

Let us consider the non-relativistic limit of our Lax operator \(L(u)\). First, rescale the monenta \(\{p_i\}\), the coupling constant \(\gamma\) and the Lax operator \(L(u)\) as follows \[11\]
\[
p_i := -\beta p'_i, \quad n\gamma := \beta s, \quad L(u) := \sigma(\frac{\beta s}{n}) L'(u), \quad (IV.24)
\]
where \(p'_i\) is the conjugated momenta of \(q_i\) in the CM model.

Then the non-relativistic limit is obtained by taking \(\beta \to 0\), we have the following asymptotic properties
\[
L'(u)^i_j = \delta^i_j - \beta (\sum_k A(u; q)^i_k A^{-1}(u; q)^j_k p'_k - s \partial_{q_i} (A(u; q)^i_j A^{-1}(u; q)^j_k)) + O(\beta^2).
\]

If we make the canonical transformation
\[
p'_i \to p'_i - \frac{s}{n} \frac{\partial}{\partial q_i} \ln M(q), \quad M(q) = \prod_{i < j} \sigma(q_{ij}),
\]
we obtain the “good” Lax operator of the elliptic \(A_{n-1}\) CM model in \[14\]
\[
L_{CM}(u)^i_j = - \lim_{\beta \to 0} \frac{L'(u)^i_j - \delta^i_j}{\beta} |_{p'_i \to p'_i - \frac{s}{n} \frac{\partial}{\partial q_i} \ln M(q)}. \quad (IV.25)
\]

Now, we have a position to calculate the r-matrix structure of the “good” Lax operator \(L(u)\) for the elliptic RS model. From proposition 3 and through the straightforward calculation, we have the main theorem of this paper:

**Theorem 1. (Main Theorem)** The fundamental Poisson bracket of \(L(u)\) can be given in the quadratic Poisson-Lie form with a nondynamical r-matrix (or Sklyanin bracket)
\[
\{L_1(u), L_2(v)\} = [r_{12}(u - v), L_1(u)L_2(v)], \quad (IV.26)
\]
where the numeric $r$-matrix $r_{12}(u)$ is the classical $Z_n$-symmetric $r$-matrix \[ [4]

\[
r_{ij}^{hk}(v) = \begin{cases} 
(1 - \delta_i^j) \frac{\theta(j-1)(v)\theta(i)(v)}{\theta(i)(v)\theta(j)(v)} + \delta_i^j \frac{\theta(i-1)(v)\theta(j)(v)}{\theta(i)(v)\theta(j)(v)} - \delta_i^j \frac{\sigma(v)}{\sigma(v)} & \text{if } i + j = l + k \text{ mod } n \\
0 & \text{otherwise}
\end{cases}
\] (IV.27)

**Remark:** I. The elliptic RS and CM model are governed by the exact same nondynamical $r$-matrix in the special Lax representation.

II. It was shown in \[14\] that such a $Z_n$-symmetric $r$-matrix satisfies the nondynamical classical Yang-Baxter equation

\[ [r_{12}(v_1 - v_2), r_{13}(v_1 - v_3)] + [r_{12}(v_1 - v_2), r_{23}(v_2 - v_3)] + [r_{13}(v_1 - v_3), r_{23}(v_2 - v_3)] = 0, \]

and enjoys in the antisymmetric properties

\[-r_{21}(-v) = r_{12}(v). \quad (IV.28)\]

Moreover, the $r$-matrix $r_{12}(u)$ also enjoys in the $Z_n \otimes Z_n$ symmetry

\[ r_{12}(v) = (a \otimes a)r_{12}(v)(a \otimes a)^{-1}, \quad \text{for } a = g, h, \quad (IV.29) \]

where the $n \times n$ matrices $h, g$ are defined in section 5.

**Corollary 2.** The Lax operator $L_{CM}(u)$ of the elliptic $A_{n-1}$ CM model in (IV.23) satisfies the nondynamical linear Poisson-Lie bracket

\[ \{L_{CM}(u)_1, L_{CM}(v)_2\} = [r_{12}(u - v), L_{CM}(u)_1 + L_{CM}(v)_2]. \quad (IV.30) \]

The direct proof that such a “good” (classical) Lax operator $L_{CM}(u)$ of the elliptic $A_{n-1}$ CM model satisfies (IV.30) was given in \[14\].

**V The quantum L-operator for the elliptic quantum RS model**

In this section, we will construct the quantum L-operator for the quantum elliptic RS model, which satisfies the nondynamical “RLL=LLR” relation.

We first introduce the elliptic $Z_n$-symmetric quantum R-matrix related to $Z_n$-symmetric Belavin model, which is the quantum version of the classical $Z_n$-symmetric $r$-matrix defined in (IV.27).

We define $n \times n$ matrices $h, g$ and $I_\alpha$ by

\[ h_{ij} = \delta_{i+1,j \mod n}, \quad g_{ij} = \omega^i \delta_{i,j}, \quad I_{\alpha_1, \alpha_2} = I_\alpha = g^{\alpha_2} h^{\alpha_1}, \]

where $\alpha_1, \alpha_2 \in Z_n$ and $\omega = exp(2\pi \sqrt{-1}/n)$. Define the $Z_n$-symmetric Belavin’s R-matrix \[24, 22, 23\]

\[ R_{ij}^{hk}(v) = \begin{cases} 
\frac{\theta(j)(v)\theta(i)(v)}{\theta(i)(v)\theta(j)(v)} & \text{if } i + j = l + k \mod n \\
0 & \text{otherwise}
\end{cases} \]

(V.31)

where $\hbar$ is the Planck’s constant and $\sqrt{-1}\hbar$ is usually called as the crossing parameter of the R-matrix. We remark that our R-matrix coincide with the usual one in \[23\] up
to a scalar factor $\frac{\theta^{(0)}(0)\sigma(v)}{\sigma^{(0)}(0)\theta^{(0)}(v)} \prod_{j=1}^{n-1} \frac{\theta^{(j)}(v)}{\theta^{(j)}(0)}$, which is to make (V.34) satisfied. The R-matrix satisfies quantum Yang-Baxter equation (QYBE)

$$R_{12}(v_1 - v_2)R_{13}(v_1 - v_3)R_{23}(v_2 - v_3) = R_{23}(v_2 - v_3)R_{13}(v_1 - v_3)R_{12}(v_1 - v_2).$$

Moreover, the R-matrix enjoys in following $Z_n \otimes Z_n$ symmetric properties

$$R_{12}(v) = (a \otimes a) R_{12}(v)(a \otimes a)^{-1}, \quad \text{for} \quad a = g, h.$$ (V.33)

The $Z_n$-symmetric r-matrix has the following relation with its quantum counterpart

$$R_{12}(v)|_{\hbar=0} = 1 \otimes 1,$$

$$R_{12}(v) = 1 \otimes 1 + \sqrt{-1}\hbar r_{12}(v) + O(\hbar^2), \quad \text{when} \quad \hbar \rightarrow 0.$$ (V.34)

Now, we construct the quantum version of L-operator $L(u)$. The usual canonical quantization procedure reads

$$p_j \rightarrow \hat{p}_j = -\sqrt{-1}\hbar \frac{\partial}{\partial q_j}, \quad q_j \rightarrow q_j, \quad j = 1, \ldots, n.$$

Then, the corresponding quantum L-operator $\hat{L}(u)$ consequently reads

$$\hat{L}(u)_{l}^{m} = \frac{1}{\sigma(\gamma)} \sum_{k=1}^{n} A(u + n\gamma; q)_{k}^{m} A^{-1}(u; q)_{k}^{l} e^{\hat{p}_{k}}$$

$$= \frac{1}{\sigma(\gamma)} \sum_{k=1}^{n} A(u + n\gamma; q)_{k}^{m} A^{-1}(u; q)_{k}^{l} e^{-\sqrt{-1} \hbar \frac{\partial}{\partial q_{k}}}.$$ (V.35)

It should be remarked that such a quantum L-operator is just the factorized difference representation for the elliptic L-operator [23]. So, we have

**Theorem 2.** ([23, 26, 25]) The quantum L-operator $\hat{L}(u)$ defined in (V.33) satisfies

$$R_{12}(u-v)\hat{L}_{1}(u)\hat{L}_{2}(v) = \hat{L}_{2}(v)\hat{L}_{1}(u)R_{12}(u-v),$$

and $R_{12}(u)$ is the $Z_n$-symmetric R-matrix.

The proof of Theorem 2. was given by Hou et al in [23], by Quano et al in [26], by Hasegawa in [25], through the face-vertex corresponding relations independently. The direct proof was also given in [27].

From the quantum L-operator $\hat{L}(u)$ and the fundamental relation $RLL = LLR$, Hasegawa constructed the skew-symmetric fusion of $\hat{L}(u)$ and succeeded in relating them with the elliptic type Macdonald operator in [23], which is actually equivalent to the quantum Ruijsenaar’s operators.

**VI Discussions**

In this paper, we only consider the most general RS model—the elliptic RS model. Such a nondynamical r-matrix structure should exist for the degenerated case: the rational, hyperbolic and trigonometric RS model.

From the results of the [23, 28], when the coupling constant $\frac{n}{\sqrt{-\hbar}} = \text{nonegative integer}$, the corresponding quantum L-operator $\hat{L}(u)$ have finite dimensional representation. This means that the states of quantum RS model should degenerate in this special case.
Acknowledgements.

This work has been financially supported by National Natural Science Foundation of China. We would like to thank Heng Fan for a careful reading of the manuscript and many helpful comments. W.L.Yang was also partially supported by the grant of Northwest University.

Appendix. The proof of Theorem 1.

In this appendix, we give the proof of Theorem 1, which is the main result of this paper.

**Lemma 1.** The classical L-operator $L(u)$ for the elliptic RS model satisfies the following algebraic relations

$$[r_{12}(u-v), L_1(u)L_2(v)]_{\alpha\beta}^{\rho\delta} = \sum_{i,j} \{A(u + n\gamma; q)_i^\rho A^{-1}(v; q)_i^\delta e_{\rho i} \partial_{q_i} (A(v + n\gamma; q)_j^\delta A^{-1}(v; q)_j^\rho) e_{\rho j} - A(v + n\gamma; q)_i^\rho A^{-1}(v; q)_i^\delta e_{\rho i} \partial_{q_i} (A(u + n\gamma; q)_j^\delta A^{-1}(u; q)_j^\rho) e_{\rho j}\}.$$  

**Proof:** Let us introduce the difference operators $\{\hat{D}_j\}$

$$\hat{D}_j = e^{-\sqrt{-1}\hbar \frac{\partial}{\partial q_j}} \quad \text{and} \quad \hat{D}_j f(q) = f(q_1, ..., q_{j-1}, q_j - \sqrt{-1}\hbar, q_{j+1}, ..., q_n).$$

Define

$$T(i, j)_{\alpha\beta}^{\rho\delta} = \begin{cases} \sum_{\rho', \delta'} R(u-v)_{\rho' \delta'}^{\rho \delta} A(u + n\gamma; q)_i^{\rho'} A^{-1}(u; q)_i^\delta \hat{D}_i (A(v + n\gamma; q)_i^\rho A^{-1}(v; q)_i^\delta), & \text{if } i = j \\ \sum_{\rho', \delta'} R(u-v)_{\rho' \delta'}^{\rho \delta} \{A(u + n\gamma; q)_i^{\rho'} A^{-1}(u; q)_i^\delta \hat{D}_i (A(v + n\gamma; q)_j^\delta A^{-1}(v; q)_j^\rho) + A(u + n\gamma; q)_j^\delta A^{-1}(u; q)_j^\rho \hat{D}_j (A(v + n\gamma; q)_i^\rho A^{-1}(v; q)_i^\delta)\} & \text{if } i \neq j \end{cases},$$

and

$$G(i, j)_{\alpha\beta}^{\rho\delta} = \begin{cases} \sum_{\rho', \delta'} R(u-v)_{\rho' \delta'}^{\rho \delta} A(v + n\gamma; q)_i^{\rho'} A^{-1}(v; q)_i^\delta \hat{D}_i (A(u + n\gamma; q)_j^\rho A^{-1}(u; q)_j^\delta), & \text{if } i = j \\ \sum_{\rho', \delta'} R(u-v)_{\rho' \delta'}^{\rho \delta} \{A(v + n\gamma; q)_i^{\rho'} A^{-1}(v; q)_i^\delta \hat{D}_i (A(u + n\gamma; q)_j^\rho A^{-1}(u; q)_j^\delta) + A(v + n\gamma; q)_j^\rho A^{-1}(v; q)_j^\delta \hat{D}_j (A(u + n\gamma; q)_i^\rho A^{-1}(u; q)_i^\delta)\} & \text{if } i \neq j \end{cases}.$$  

The quantum L-operator $\hat{L}(u)$ satisfying the “RLL = LLR” relation results in

$$T(i, j)_{\alpha\beta}^{\rho\delta} = G(i, j)_{\alpha\beta}^{\rho\delta}. \quad \text{(VI.37)}$$

Considering the asymptotic properties when $\hbar \to 0$

$$R_{12}(u) = 1 + \sqrt{-1}\hbar r_{12}(u) + 0(h^2),$$

$$\hat{D}_j = 1 - \sqrt{-1}\hbar \frac{\partial}{\partial q_j} + 0(h^2),$$

we have
I. if $i = j$
\[
T(i, j)_{\alpha\beta} = T^{(0)}(i, j)_{\alpha\beta} + \sqrt{-1}hT^{(1)}(i, j)_{\alpha\beta} + O(h^2).
\]
\[
= A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} A(v + n\gamma; q)^{\delta} A^{-1}(v; q)^{\gamma}
\]
\[
+ \sqrt{-1}h \sum_{\rho', \delta'} r(u - v)^{\rho'} \frac{\partial}{\partial y_{\rho'}} (A(u + n\gamma; q)^{\delta} A^{-1}(u; q)^{\gamma})
\]
\[
+ \sqrt{-1}h A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} \frac{\partial}{\partial q_i} (A(v + n\gamma; q)^{\delta} A^{-1}(v; q)^{\gamma}) + O(h^2).
\]

II. if $i \neq j$
\[
T(i, j)_{\alpha\beta} = T^{(0)}(i, j)_{\alpha\beta} + \sqrt{-1}hT^{(1)}(i, j)_{\alpha\beta} + O(h^2)
\]
\[
= A(u + n\gamma; q)^{\alpha} i A(v + n\gamma; q)^{\delta} A^{-1}(v; q)^{\beta}
\]
\[
+ A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} A(v + n\gamma; q)^{\delta} A^{-1}(v; q)^{\gamma}
\]
\[
+ \sqrt{-1}h \sum_{\rho', \delta'} r(u - v)^{\rho'} \frac{\partial}{\partial y_{\rho'}} (A(u + n\gamma; q)^{\delta} A^{-1}(u; q)^{\gamma})
\]
\[
+ A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} \frac{\partial}{\partial q_i} (A(v + n\gamma; q)^{\delta} A^{-1}(v; q)^{\gamma})
\]
\[
+ A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} \frac{\partial}{\partial q_i} (A(v + n\gamma; q)^{\delta} A^{-1}(v; q)^{\gamma}) + O(h^2).
\]

III. if $i = j$
\[
G(i, j)_{\alpha\beta} = G^{(0)}(i, j)_{\alpha\beta} + \sqrt{-1}hG^{(1)}(i, j)_{\alpha\beta} + O(h^2)
\]
\[
= A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} A(v + n\gamma; q)^{\delta} A^{-1}(v; q)^{\gamma}
\]
\[
+ \sqrt{-1}h \sum_{\rho', \delta'} r(u - v)^{\rho'} \frac{\partial}{\partial y_{\rho'}} (A(u + n\gamma; q)^{\delta} A^{-1}(u; q)^{\gamma})
\]
\[
+ A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} \frac{\partial}{\partial q_i} (A(u + n\gamma; q)^{\delta} A^{-1}(u; q)^{\gamma})
\]
\[
+ A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} \frac{\partial}{\partial q_i} (A(v + n\gamma; q)^{\delta} A^{-1}(v; q)^{\gamma}) + O(h^2).
\]

IV. if $i \neq j$
\[
G(i, j)_{\alpha\beta} = G^{(0)}(i, j)_{\alpha\beta} + \sqrt{-1}hG^{(1)}(i, j)_{\alpha\beta} + O(h^2)
\]
\[
= A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} A(v + n\gamma; q)^{\delta} A^{-1}(v; q)^{\gamma}
\]
\[
+ \sqrt{-1}h \sum_{\rho', \delta'} r(u - v)^{\rho'} \frac{\partial}{\partial y_{\rho'}} (A(u + n\gamma; q)^{\delta} A^{-1}(u; q)^{\gamma})
\]
\[
+ A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} \frac{\partial}{\partial q_i} (A(u + n\gamma; q)^{\delta} A^{-1}(u; q)^{\gamma})
\]
\[
+ A(u + n\gamma; q)^{\alpha} i A^{-1}(u; q)^{\beta} \frac{\partial}{\partial q_i} (A(u + n\gamma; q)^{\delta} A^{-1}(u; q)^{\gamma}) + O(h^2).
\]

Noting (VI.37) and considering the term of the first order with regard to $\hbar$, we have
\[
T^{(1)}(i, j)_{\alpha\beta} = G^{(1)}(i, j)_{\alpha\beta}.
\]
Multiplying by $e^{p_i+p_j}$ from the both sider of \((\ref{V.38})\) and sum up for $i$ and $j$, we have

$$\sum_{i,j} T^{(1)}(i,j)_{\alpha\beta}^\delta e^{p_i} e^{p_j} = \sum_{i,j} G^{(1)}(i,j)_{\alpha\beta}^\delta e^{p_i} e^{p_j}. $$

Due to the commutativity of $\{e^{p_j}\}$, we obtain

$$\sum_{\rho',\delta',i,j} \{r(u-v)_{\rho'\delta'} \{A(u+n\gamma;q)_{\rho'}^\alpha A^{-1}(u;q)_{\delta'}^j A^{-1}(v;q)_j^\alpha e^{p_i} A(v+n\gamma;q)_{\delta'}^j A^{-1}(v;q)_j^\alpha e^{p_j} \\
- r(u-v)_{\alpha\beta}^\delta A(v+n\gamma;q)_i^\alpha A^{-1}(v;q)_j^\alpha e^{p_i} A(u+n\gamma;q)_j^\alpha A^{-1}(u;q)_\rho^\beta e^{p_j} \}
= \sum_{i,j} \{A(u+n\gamma;q)_i^\alpha A^{-1}(v;q)_j^\alpha e^{p_i} \frac{\partial}{\partial q_i}(A(v+n\gamma;q)_j^\alpha A^{-1}(v;q)_\rho^\beta e^{p_j} \\
- A(v+n\gamma;q)_i^\alpha A^{-1}(v;q)_j^\alpha e^{p_i} \frac{\partial}{\partial q_i}(A(u+n\gamma;q)_j^\alpha A^{-1}(u;q)_\rho^\beta e^{p_j} \}
.$$ 

Namely, we have

$$[r_{12}(u-v), L_1(u)L_2(v)]_{\alpha\beta}^\delta \\
= \sum_{i,j} \{A(u+n\gamma;q)_i^\alpha A^{-1}(v;q)_j^\alpha e^{p_i} \frac{\partial}{\partial q_i}(A(v+n\gamma;q)_j^\alpha A^{-1}(v;q)_\rho^\beta e^{p_j} \\
- A(v+n\gamma;q)_i^\alpha A^{-1}(v;q)_j^\alpha e^{p_i} \frac{\partial}{\partial q_i}(A(u+n\gamma;q)_j^\alpha A^{-1}(u;q)_\rho^\beta e^{p_j} \}
.$$ 

Now, we have a position to calculate the fundamental Poisson bracket of $L(u)$

$$\{L_1(u), L_2(v)\}_{\alpha\beta}^\delta = \{L(u)_\alpha^\delta, L(v)_\beta^\delta \}
= \sum_i A(u+n\gamma;q)_i^\alpha A^{-1}(u;q)_\alpha^i e^{p_i}, \sum_j A(v+n\gamma;q)_j^\alpha A^{-1}(v;q)_\beta^j e^{p_j} \}
= \sum_{i,j} \{A(u+n\gamma;q)_i^\alpha A^{-1}(u;q)_\alpha^i e^{p_i} \frac{\partial}{\partial q_i}(A(v+n\gamma;q)_j^\alpha A^{-1}(v;q)_\beta^j e^{p_j} \\
- A(v+n\gamma;q)_i^\alpha A^{-1}(v;q)_j^\alpha e^{p_i} \frac{\partial}{\partial q_i}(A(u+n\gamma;q)_j^\alpha A^{-1}(u;q)_\beta^j e^{p_j} \}
= [r_{12}(u-v), L_1(u)L_2(v)]_{\alpha\beta}^\delta.$$ 

We have used the Lemma 1 in the last equation. Thus, we have

$$\{L_1(u), L_2(v)\} = [r_{12}(u-v), L_1(u)L_2(v)].$$

References

[1] F. Calogero, \textit{Lett. Nuovo. Cim.} \textbf{13}, 411 (1975); \textit{Lett.Nuovo. Cim.} \textbf{16}, 77 (1976).

[2] J. Moser, \textit{Adv. Math.} \textbf{16}, 1 (1975).

[3] S.N.M. Ruijsenaars, H. Schneider, \textit{Ann. Phys. (NY) Vol.170}, 370 (1986).

[4] O. Babelon, D. Bernard, \textit{Phys. Lett.} \textbf{B317}, 363 (1993).

[5] H.W. Braden, R. Sasaki, \textit{Prog. Theor. Phys.} \textbf{97}, 1003 (1997).
[6] F.W. Nijhoff, O. Ragnisco, V.B. Kuznetsov, *Comm. Math. Phys.* **176**, 681 (1996).

[7] G.E. Arutyunov, S.A. Frolov, P.B. Medredev, e-print hep-th/9607170.

[8] A. Gorsky, N. Nekrasov, *Nucl. Phys.* **B414**, 213 (1994); *Nucl. Phys.* **B436**, 582 (1995).

[9] J. Avan, T. Talon, *Phys. Lett.* **B303**, 33 (1993).

[10] S.N.M. Ruijsenaars, *Comm. Math. Phys.* Vol.110, 191 (1987).

[11] F.W. Nijhoff, V.B. Kuznetsov, E.K. Sklyanin, e-print solv-int/9603006.

[12] Yuri B. Suris, *Phys. Lett.* **A225**, 253 (1997).

[13] O. Babelon, C.M. Viallet, *Phys. Lett.* **B237**, 411 (1989).

[14] B.Y. Hou, W.L. Yang, *Lett. Math. Phys.* Vol. **44**, No.1, 35 (1998); J. Phy. **A32**, 1475 (1999).

[15] I.M. Krichever, *Funct. Anal. Appl.* **14**, 282 (1980).

[16] B.Y. Hou, H. Wei, *J. Math. Phys.* **30**, 2750 (1989).

[17] B.Y.Hou, W.-L.Yang, e-print solv-int/9802015, *Comm. Theor. Phys.*, in press.

[18] L. Freidel, J.-M. Maillet, *Phys. Lett.* **B262**, 278 (1991).

[19] J. Avan, O. Babelon, E. Billey, *Comm. Math. Phys.* **178**, 281 (1996).

[20] A.A. Belavin, V.G. Drinfeld, *Soviet Sci. reviews*, Sect.C **4**, 93 (1984).

[21] L.D. Faddeev, L. Takhtajan, Hamiltonian methods in the theory of solitons, Springer Verlag (1987).

[22] M. Jimbo, T. Miwa, M. Okado, *Nucl. Phys.* **B300**, 74 (1988).

[23] B.Y. Hou, K.J. Shi, Z.X. Yang, J. Phys. **A26**, 4951 (1993).

[24] M.P. Richey, C.A. Tracy, *J. Stat. Phys.* bf 42, 311 (1986).

[25] K. Hasegawa, *Comm. Math. Phys.* **187**, 289 (1997); *Jour. Math. Phys.* **35**, 6158 (1994).

[26] Y.H. Quano, A. Fujii, *Mod. Phys. Lett.* **A8**, 1585 (1993).

[27] B.Y. Hou, K.J. Shi, W.L. Yang, Z.X. Yang, *Phys. Lett.* **A 178**, 73 (1993).

[28] B.Y. Hou, K.J. Shi, W.L. Yang, Z.X. Yang, S.Y. Zhou, *Int. Jour. Mod. Phys.* **A Vol.12, No.16**, 2927 (1997).