Hermite-Hadamard-Fejér Type Inequalities for Harmonically Quasi-convex Functions via Fractional Integrals

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Abstract. In this paper, some Hermite-Hadamard-Fejér type integral inequalities for harmonically quasi-convex functions in fractional integral forms have been obtained.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The inequality

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) \, dx
\]

is well known in the literature as Hermite-Hadamard’s inequality [5].

The most well-known inequalities related to the integral mean of a convex function \( f \) are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [4], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

**Theorem 1.1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex function. Then the inequality

\[
f \left( \frac{a + b}{2} \right) \int_a^b g(x) \, dx \leq \int_a^b f(x) g(x) \, dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) \, dx
\]

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Received July 26, 2015; revised March 15, 2016; accepted March 16, 2016.
2010 Mathematics Subject Classification: 26A51, 26A33, 26D10.
Key words and phrases: Hermite-Hadamard inequality, Hermite-Hadamard-Fejér inequality, Riemann-Liouville fractional integral, harmonically quasi-convex function.
holds, where \( g : [a, b] \to \mathbb{R} \) is nonnegative, integrable and symmetric to \( a + b/2 \).

For some results which generalize, improve and extend the inequalities (1.1) and (1.2) see [1, 6, 7, 16, 18].

We recall the following inequality and special functions which are known as Beta and hypergeometric function respectively:

\[
\beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad x, y > 0,
\]

\[
2F_1 (a; b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt,
\]

\( c > b > 0, |z| < 1 \) (see [13]).

**Lemma 1.2.** ([15, 20]) For \( 0 < \alpha \leq 1 \) and \( 0 \leq a < b \) we have \( |a^\alpha - b^\alpha| \leq (b - a)^\alpha \).

The following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

**Definition 1.3.** ([13]) Let \( f \in L[a, b] \). The Riemann-Liouville integrals \( J_{a^+}^\alpha f \) and \( J_{b^-}^\alpha f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > a.
\]

and

\[
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt, \quad x < b
\]

respectively, where \( \Gamma(\alpha) \) is the Gamma function defined by \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt \) and \( J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x) \).

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [3, 8, 9, 17, 19, 20].

**Definition 1.4.** ([21]) A function \( f : I \subseteq (0, \infty) \to [0, \infty) \) is said to be harmonically quasi-convex, if

\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq \sup \{ f(x), f(y) \}
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

In [11], İscan defined the so-called harmonically convex functions and established following Hermite-Hadamard type inequality for them as follows:
**Definition 1.5.** Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

\begin{equation}
(1.3)
f \left( \frac{xy}{tx + (1-t)y} \right) \leq tf(y) + (1-t)f(x)
\end{equation}

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.3) is reversed, then $f$ is said to be harmonically concave.

**Theorem 1.6.** ([11]) Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities holds:

\begin{equation}
(1.4)
f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}
\end{equation}

In [10], İşcan and Wu presented a Hermite-Hadamard type inequality for harmonically convex functions in fractional integral forms as follows:

**Theorem 1.7.** Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If $f$ is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals holds:

\begin{equation}
(1.5)
f \left( \frac{2ab}{a+b} \right) \leq \frac{\Gamma (\alpha + 1)}{2} \left( \frac{ab}{b-a} \right)^\alpha \left\{ J_{1/a}^\alpha (f \circ h)(1/b) + J_{1/b}^\alpha (f \circ h)(1/a) \right\}
\end{equation}

\begin{equation}
\leq \frac{f(a) + f(b)}{2}
\end{equation}

with $\alpha > 0$ and $h(x) = 1/x$.

In [14] Latif et al. gave the following definition:

**Definition 1.8.** A function $g : [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $2ab/a + b$, if

\[ g(x) = g \left( \frac{1}{\frac{1}{a} + \frac{1}{b} - \frac{1}{x}} \right) \]

holds for all $x \in [a, b]$.

In [2] Chan and Wu presented a Hermite-Hadamard-Fejér inequality for harmonically convex functions as follows:

**Theorem 1.9.** Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ and $g : [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a + b$, then

\begin{equation}
(1.6)
f \left( \frac{2ab}{a+b} \right) \int_a^b \frac{g(x)}{x^2} dx \leq \int_a^b \frac{f(x) g(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx.
\end{equation}
In [12] İşcan and Kunt presented a Hermite–Hadamard-Fejér type inequality for harmonically convex functions in fractional integral forms and established following identity as follows:

**Theorem 1.10.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a harmonically convex function with $a < b$ and $f \in L^1[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a+b$, then the following inequalities for fractional integrals holds:

\[
\begin{align*}
\int_{t_1}^{t_2} f(t) \, dt & \leq \left[ J_{t_1/t_2}^\alpha (g \circ h) (1/a) + J_{t_2/t_2}^\alpha (g \circ h) (1/b) \right] \\
& \leq \frac{f(a) + f(b)}{2} \left[ J_{t_1/t_2}^\alpha (g \circ h) (1/a) + J_{t_2/t_2}^\alpha (g \circ h) (1/b) \right]
\end{align*}
\]

with $\alpha > 0$ and $h(x) = 1/x, x \in \left[ \frac{1}{b}, \frac{1}{a} \right]$.

**Lemma 1.11.** ([12]) Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L^1[a, b]$, where $a, b \in I$ and $a < b$. If $g : [a, b] \rightarrow \mathbb{R}$ is integrable and harmonically symmetric with respect to $2ab/a+b$, then the following equality for fractional integrals holds:

\[
\begin{align*}
\int_{t_1}^{t_2} f(t) \, dt & = \frac{f(a) + f(b)}{2} \left[ J_{t_1/t_2}^\alpha (g \circ h) (1/a) + J_{t_2/t_2}^\alpha (g \circ h) (1/b) \right] \\
& \leq \frac{f(a) + f(b)}{2} \left[ J_{t_1/t_2}^\alpha (g \circ h) (1/a) + J_{t_2/t_2}^\alpha (g \circ h) (1/b) \right]
\end{align*}
\]

with $\alpha > 0$ and $h(x) = 1/x, x \in \left[ \frac{1}{b}, \frac{1}{a} \right]$.

In this paper, we give some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for harmonically quasi-convex function in fractional integrals.

**2. Main Results**

Throughout this section, we write $\|g\|_\infty = \sup_{t \in [a, b]} |g(t)|$, for the continuous function $g : [a, b] \rightarrow \mathbb{R}$.

**Theorem 2.1.** Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L^1[a, b]$, where $a, b \in I$ and $a < b$. If $f'$ is harmonically quasi-convex on $[a, b]$, $g : [a, b] \rightarrow \mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$, 

\[
\int_{t_1}^{t_2} f(t) \, dt \leq \frac{f(a) + f(b)}{2} \left[ J_{t_1/t_2}^\alpha (g \circ h) (1/a) + J_{t_2/t_2}^\alpha (g \circ h) (1/b) \right]
\]

with $\alpha > 0$ and $h(x) = 1/x, x \in \left[ \frac{1}{b}, \frac{1}{a} \right]$.
then the following inequality for fractional integrals holds:

\[
(2.1) \quad \left| \frac{f(a)+f(b)}{2} \left[ J_{1/b^+}^\alpha (g \circ h) (1/a) + J_{1/a^+}^\alpha (g \circ h) (1/b) \right] - \left[ J_{1/b^+}^\alpha (fg \circ h) (1/a) + J_{1/a^+}^\alpha (fg \circ h) (1/b) \right] \right| \leq \frac{\|g\|_\infty ab (b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^\alpha C_1(\alpha) \sup \{|f'(a)|, |f'(b)|\}
\]

where

\[
C_1(\alpha) = \begin{cases} 
\frac{b^{-2}}{\alpha^2} 2F_1 \left( 2, 1; \alpha + 2; 1 - \frac{a}{b} \right) \\
- \frac{b^{-2}}{\alpha^2+1} 2F_1 \left( 2, 1 + \alpha; 2; 1 - \frac{a}{b} \right) \\
+ \frac{b^{-2}}{\alpha^2} 2F_1 \left( 2, \alpha + 2; 2 + 1 - \frac{a}{b} \right) 
\end{cases}
\]

with \(0 < \alpha \leq 1\) and \(h(x) = 1/x\), \(x \in \left[ \frac{1}{b}, \frac{1}{a} \right]\).

Proof. From Lemma 1.11 we have

\[
(2.2) \quad \frac{f(a)+f(b)}{2} \left[ J_{1/b^+}^\alpha (g \circ h) (1/a) + J_{1/a^+}^\alpha (g \circ h) (1/b) \right] - \left[ J_{1/b^+}^\alpha (fg \circ h) (1/a) + J_{1/a^+}^\alpha (fg \circ h) (1/b) \right] \leq \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{b}}^{\frac{1}{a}} \left| f(t) \left( \frac{1}{a} - s \right)^{\alpha-1} (g \circ h) (s) ds \right| dt.
\]

Since \(g\) is harmonically symmetric with respect to \(2ab/a + b\), using Definition 1.8 we have \(g \left( \frac{1}{x} \right) = g \left( \frac{1}{(\frac{b}{a})^x} \right)\) for all \(x \in \left[ \frac{1}{b}, \frac{1}{a} \right]\).

\[
(2.3) \quad \left| \int_{\frac{1}{a}}^{\frac{1}{b}} \left( \frac{1}{a} - s \right)^{\alpha-1} (g \circ h) (s) ds - \int_{\frac{1}{b}}^{\frac{1}{a}} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds \right| = \left| \int_{\frac{1}{a}}^{\frac{1}{b}} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds + \int_{\frac{1}{b}}^{\frac{1}{a}} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds \right| = \left| \int_{\frac{1}{a}}^{\frac{1}{b}} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds \right| \leq \left\{ \begin{array}{ll} 
\int_{\frac{1}{a}}^{\frac{1}{b}} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds & , \quad t \in \left[ \frac{1}{b}, \frac{a+b}{2ab} \right] \\
\int_{\frac{1}{a}}^{\frac{1}{b}} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds & , \quad t \in \left[ \frac{a+b}{2ab}, \frac{1}{a} \right]. 
\end{array} \right.
\]
If we use (2.3) in (2.2), we have

\[
\left| \frac{f(a)+f(b)}{2} \right| \left[ J_{\alpha}^\alpha (g \circ h) (1/a) + J_{\alpha}^\alpha (g \circ h) (1/b) \right] \\
- \left[ J_{\alpha}^\alpha (fg \circ h) (1/a) + J_{\alpha}^\alpha (fg \circ h) (1/b) \right] \\
\leq \frac{1}{\Gamma(\alpha)} \left[ \int_{\frac{a}{b}}^{\frac{b}{a}} \int_{\frac{s}{b}}^{\frac{s}{a}} \frac{1}{\Gamma(\alpha)} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) \, ds \left( f \circ h \right)' (t) \, dt \right] \\
+ \frac{1}{\Gamma(\alpha)} \left[ \int_{\frac{a}{b}}^{\frac{b}{a}} \int_{\frac{s}{b}}^{\frac{s}{a}} \frac{1}{\Gamma(\alpha)} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) \, ds \left( f \circ h \right)' (t) \, dt \right] \\
\leq \frac{\|g\|_{\infty}}{\Gamma(\alpha)} \left[ \int_{\frac{a}{b}}^{\frac{b}{a}} \int_{\frac{s}{b}}^{\frac{s}{a}} \frac{1}{\Gamma(\alpha)} \left( s - \frac{1}{b} \right)^{\alpha-1} ds \right] \frac{1}{\Gamma(\alpha)} \left( f \circ h \right)' (t) \, dt \\
+ \frac{1}{\Gamma(\alpha)} \left[ \int_{\frac{a}{b}}^{\frac{b}{a}} \int_{\frac{s}{b}}^{\frac{s}{a}} \frac{1}{\Gamma(\alpha)} \left( s - \frac{1}{b} \right)^{\alpha-1} ds \right] \frac{1}{\Gamma(\alpha)} \left( f \circ h \right)' (t) \, dt \\
\leq \frac{\|g\|_{\infty}}{\Gamma(\alpha)} \left[ \int_{\frac{a}{b}}^{\frac{b}{a}} \int_{\frac{s}{b}}^{\frac{s}{a}} \frac{1}{\Gamma(\alpha)} \left( s - \frac{1}{b} \right)^{\alpha-1} ds \right] \frac{1}{\Gamma(\alpha)} \left( f \circ h \right)' (t) \, dt \\
\leq \left\| g \right\|_{\infty} \frac{ab (b - a)}{1 - (1 - u) a} \sup \{|f'(a)|, |f'(b)|\}.
\]

Since \( f' \) is harmonically quasi-convex on \([a, b]\), we have

\[
\left| f'\left( \frac{ab}{ub + (1-u)a} \right) \right| \leq \sup \{|f'(a)|, |f'(b)|\}.
\]

If we use (2.5) in (2.4), we have

\[
\left| f(a)+f(b) \right| \left[ J_{\alpha}^\alpha (g \circ h) (1/a) + J_{\alpha}^\alpha (g \circ h) (1/b) \right] \\
- \left[ J_{\alpha}^\alpha (fg \circ h) (1/a) + J_{\alpha}^\alpha (fg \circ h) (1/b) \right] \\
\leq \frac{\|g\|_{\infty}}{\Gamma(\alpha)} \frac{ab (b - a)}{1 - (1 - u) a} \sup \{|f'(a)|, |f'(b)|\} \\
\times \left[ \int_{0}^{\frac{1}{2}} \frac{(1 - u)^{\alpha} - u^{\alpha}}{(ub + (1-u)a)^2} du + \int_{\frac{1}{2}}^{1} \frac{u^{\alpha} - (1-u)^{\alpha}}{(ub + (1-u)a)^2} du \right].
\]
Using Lemma 1.2, we have

\begin{equation}
(2.7) \int_{0}^{1} \frac{(1-u)^{\alpha} - u^{\alpha}}{(ub + (1-u)a)^{2}} du + \int_{0}^{1} \frac{u^{\alpha} - (1-u)^{\alpha}}{(ub + (1-u)a)^{2}} du
\end{equation}

\begin{equation}
= \int_{0}^{1} \frac{u^{\alpha} - (1-u)^{\alpha}}{(ub + (1-u)a)^{2}} du + 2 \int_{0}^{1} \frac{(1-u)^{\alpha} - u^{\alpha}}{(ub + (1-u)a)^{2}} du
\end{equation}

\begin{equation}
= \int_{0}^{1} \frac{u^{\alpha}}{(ub + (1-u)a)^{2}} du - \int_{0}^{1} \frac{(1-u)^{\alpha}}{(ub + (1-u)a)^{2}} du + 2 \int_{0}^{1} \frac{(1-u)^{\alpha} - u^{\alpha}}{(ub + (1-u)a)^{2}} du
\end{equation}

Calculated the following integrals, we have

\begin{equation}
(2.8) \int_{0}^{1} \frac{u^{\alpha}}{(ub + (1-u)a)^{2}} du - \int_{0}^{1} \frac{(1-u)^{\alpha}}{(ub + (1-u)a)^{2}} du + 2 \int_{0}^{1} \frac{(1-u)^{\alpha} - u^{\alpha}}{(ub + (1-u)a)^{2}} du
\end{equation}

\begin{equation}
= \int_{0}^{1} \frac{(1-u)^{\alpha} b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)^{-2} du - \int_{0}^{1} u^{\alpha} b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)^{-2} du
\end{equation}

\begin{equation}
+ \int_{0}^{1} \nu^{\alpha} \left(\frac{a + b}{2}\right)^{-2} \left(1 - v \left(\frac{b - a}{b + a}\right)^{-2} dv
\end{equation}

\begin{equation}
= \left[\frac{b^{-2}}{\alpha + 1} 2F_{1} \left(2, 1; \alpha + 2; 1 - \frac{a}{b}\right)\right] \left[\frac{\nu^{\alpha}}{\alpha + 1} 2F_{1} \left(2, 1; \alpha + 2; 1 - \frac{a}{b}\right)\right]
\end{equation}

\begin{equation}
= C_{1}(\nu) .
\end{equation}

If we use (2.7) and (2.8) in (2.6), we have (2.1). This completes the proof. □

**Corollary 2.2.** In Theorem 2.1:

(1) If we take \(\alpha = 1\) we have the following Hermite-Hadamard-Fejér inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.6):

\begin{equation}
\left| f(a) + f(b) \int_{a}^{b} \frac{g(x) x^{2}}{x^{2}} dx - \int_{a}^{b} \frac{f(x) g(x) x^{2}}{x^{2}} dx \right|
\end{equation}

\begin{equation}
\leq \left\| g \right\|_{\infty} \frac{\left(\frac{b-a}{2}\right)^{2}}{2} C_{1}(1) \sup \{|f'(a)|, |f'(b)|\} .
\end{equation}

(2) If we take \(g(x) = 1\) we have following Hermite-Hadamard type inequality for harmonically quasi-convex functions in fractional integral forms which is related
to the right-hand side of (1.5):

\[
\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2} \left( \frac{ab}{b-a} \right)^{\alpha} \left( J_{1/a-}^\alpha (f \circ h) (1/b) + J_{1/a+}^\alpha (f \circ h) (1/a) \right) 
\leq \frac{ab(b-a)}{2} C_1(\alpha) \sup \{ |f'(a)|, |f'(b)| \}
\]

(3) If we take \( \alpha = 1 \) and \( g(x) = 1 \) we have the following Hermite-Hadamard type inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.4):

\[
\frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b f(x) \, dx 
\leq \frac{ab(b-a)}{2} C_1(1) \sup \{ |f'(a)|, |f'(b)| \}
\]

**Theorem 2.3.** Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^\circ \) such that \( f' \in L^q[a,b] \), where \( a, b \in I \) and \( a < b \). If \( |f'|^q, q \geq 1 \), is harmonically quasi-convex on \([a,b], g : [a,b] \to \mathbb{R} \) is continuous and harmonically symmetric with respect to \( 2ab = a + b \), then the following inequality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b f(x) \, dx 
\leq \left\| g \right\|_\infty \frac{ab(b-a)}{\Gamma(\alpha + 1)} \left( \frac{b-a}{ab} \right)^{\alpha} C_2(\alpha) \left[ \sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{\frac{1}{q}}
\]

where

\[
C_2(\alpha) = \frac{b^{-2}}{\alpha + 1} 2F_1 \left( 2, 1; \alpha + 2; \frac{b-a}{b+a} \right) - \frac{b^{-2}}{\alpha + 1} 2F_1 \left( 2, \alpha + 1; \alpha + 2; \frac{b-a}{b+a} \right)
\]

\[+ \frac{4(a+b)^{-2}}{(\alpha + 1)} 2F_1 \left( 2, \alpha + 1; \alpha + 2; \frac{b-a}{b+a} \right),
\]

with \( 0 < \alpha \leq 1 \) and \( h(x) = 1/x, x \in \left[ \frac{1}{b}, \frac{1}{a} \right] \).

**Proof.** Using (2.4), power mean inequality and the harmonically quasi-convexity of
Using Lemma 1.2, we have

\begin{equation}
(2.11) \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub + (1-u)a)^2} du + \int_0^{\frac{1}{2}} \frac{u^\alpha - (1-u)^\alpha}{(ub + (1-u)a)^2} du
\end{equation}
For the appearing integrals, we have

\[
\begin{align*}
(2.12) & \int_0^1 \frac{u^\alpha}{(ub + (1 - u)a)^2} du - \int_0^1 \frac{(1 - u)^\alpha}{(ub + (1 - u)a)^2} du + 2 \int_0^1 \frac{(1 - 2u)^\alpha}{(ub + (1 - u)a)^2} du \\
& = \int_0^1 (1 - u)^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du - \int_0^1 u^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du \\
& \quad + \int_0^1 \frac{(1 - u)^\alpha}{\left(\frac{u}{b} + (1 - \frac{u}{b})\right)^2} du \\
& = \int_0^1 (1 - u)^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du - \int_0^1 u^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du \\
& \quad + \int_0^1 \frac{v^\alpha (a + b)^{-2}}{v \left(1 - v \left(\frac{b - a}{b + a}\right)\right)^{-2}} dv \\
& = \frac{b^{-2}}{\alpha + 1} 2F_1 \left(2, 1; \alpha + 2; \frac{b - a}{b + a}\right) - \frac{b^{-2}}{\alpha + 1} 2F_1 \left(2, \alpha + 1; \alpha + 2; \frac{b - a}{b + a}\right) \\
& \quad + \frac{4(a + b)^{-2}}{(\alpha + 1)} 2F_1 \left(2, \alpha + 1; \alpha + 2; \frac{b - a}{b + a}\right) \\
& = C_2(\alpha). 
\end{align*}
\]

If we use (2.11) and (2.12) in (2.10), we have (2.9). This completes the proof.

\[\square\]

**Corollary 2.4.** In Theorem 2.3:

1. If we take \( \alpha = 1 \) we have the following Hermite-Hadamard-Fejér inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.6):

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{x^2} \int_a^b f(x) g(x) \, dx \right| \leq \frac{\|g\|_\infty (b - a)^2}{2} C_2(1) \left[ \sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{\frac{1}{q}},
\]

2. If we take \( g(x) = 1 \) we have following Hermite-Hadamard type inequality for harmonically quasi-convex functions in fractional integral forms which is related to the right-hand side of (1.5):

\[
\left| \frac{f(a) + f(b)}{2} - \Gamma(\alpha + 1) \left( \frac{ab}{b - a} \right)^\alpha \left\{ J_{1/a}^\alpha (f \circ h) (1/b) + J_{1/b}^\alpha (f \circ h) (1/a) \right\} \right| \leq \frac{ab(b - a)^2}{2} C_2(\alpha) \left[ \sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{\frac{1}{q}},
\]
(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard type inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.4):

$$
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b - a)}{2} C_2(1) \left[ \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right]^{\frac{1}{q}}.
$$

We can state another inequality for $q > 1$ as follows:

**Theorem 2.5.** Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I$ such that $f' \in L[a, b]$ where $a, b \in I$ and $a < b$. If $|f'|^q, q > 1$, is harmonically quasi-convex on $[a, b]$, $g : [a, b] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a + b$, then the following inequality for fractional integrals holds:

$$
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \|g\|_\infty ab \left( \frac{b - a}{ab} \right)^\alpha \left[ C_3^\frac{1}{q} + C_4^\frac{1}{q} \right]
$$

where

$$
C_3(\alpha) = \left( \frac{a + b}{2} \right)^{-2p} \frac{1}{2(\alpha p + 1)} 2F_1 \left( 2p, \alpha p + 1; \alpha p + 2; \frac{b - a}{b + a} \right),
$$

$$
C_4(\alpha) = b^{-2p} \frac{1}{2(\alpha p + 1)} 2F_1 \left( 2p, 1; \alpha p + 2; \frac{1}{2} (1 - \frac{a}{b}) \right),
$$

with $0 < \alpha \leq 1$, $h(x) = 1/x$, $x \in \left[ \frac{1}{b}, \frac{1}{a} \right]$ and $1/p + 1/q = 1$.

**Proof.** Using (2.4), Hölder’s inequality and the harmonically quasi-convexity of $|f'|^q$, it follows that...
Using Lemma 1.2, we have

\begin{equation}
\int_0^{\frac{1}{2}} \frac{\left[(1-u)^{\alpha}-u^\alpha\right]^p}{\left(ab+(1-u)a\right)^{2p}} du \leq \int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha p}}{(ub+(1-u)a)^{2p}} du
\end{equation}

and

\begin{equation}
\int_0^{\frac{1}{2}} \frac{\left[u^\alpha-(1-u)^\alpha\right]^p}{\left(ab+(1-u)a\right)^{2p}} du \leq \int_0^{\frac{1}{2}} \frac{(2u-1)^{\alpha p}}{(ub+(1-u)a)^{2p}} du.
\end{equation}
For the appearing integrals, we have

\begin{align*}
(2.17) \quad & \int_{0}^{1} \frac{(1-2u)^{op}}{(ub + (1-u)a)^{2p}} du \\
& = \frac{1}{2} \int_{0}^{1} \frac{(1-u)^{op}}{\left(\frac{7}{2} b + (1 - \frac{v}{2})a\right)^{2p}} du \\
& = \frac{1}{2} \int_{0}^{1} v^{op} \left(\frac{a + b}{2}\right)^{-2p} \left[1 - v \left(\frac{b - a}{b + a}\right)\right]^{-2p} dv \\
& = \left(\frac{a + b}{2}\right)^{-2p} \frac{1}{2(\alpha p + 1)} F_{1}(2p, \alpha p + 1; \alpha p + 2; \frac{b-a}{b+a}) \\
& = C_{3}(1)
\end{align*}

and

\begin{align*}
(2.18) \quad & \int_{\frac{1}{2}}^{1} \frac{(2u - 1)^{op}}{(ua + (1-u)b)^{2p}} du \\
& = \int_{0}^{1} \frac{(1-2u)^{op}}{(ua + (1-u)b)^{2p}} du \\
& = \frac{1}{2} \int_{0}^{1} \frac{(1-u)^{op}}{\left(\frac{7}{2} a + (1 - \frac{v}{2})b\right)^{2p}} dv \\
& = \frac{1}{2} \int_{0}^{1} (1-v)^{op} b^{-2p} \left(1 - v \left(\frac{1 - \frac{a}{b}}{1}\right)\right)^{-2p} dv \\
& = b^{-2p} \frac{1}{2(\alpha p + 1)} F_{1}(2p, 1; \alpha p + 2; \frac{1}{2} (1 - \frac{a}{b})) \\
& = C_{4}(1).
\end{align*}

If we use (2.15), (2.16), (2.17) and (2.18) in (2.14), we have (2.13). This completes the proof.

\begin{corollary}
In Theorem 2.5:

(1) If we take \( \alpha = 1 \) we have the following Hermite-Hadamard-Fejér inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.6):

\[
\begin{align*}
& \left| \frac{f(a) + f(b)}{2} \int_{a}^{b} \frac{g(x)}{x^2} dx - \int_{a}^{b} \frac{f(x) g(x)}{x^2} dx \right| \\
\leq & \frac{\|g\|_{\infty} (b-a)^2}{2^{\frac{q}{q+1}}} \left( \sup \{ |f'(a)|^q, |f'(b)|^q \} \right)^{\frac{1}{q}} \left[ C_{3}^1(1) + C_{4}^1(1) \right],
\end{align*}
\end{corollary}
If we take \( g(x) = 1 \) we have following Hermite-Hadamard type inequality for harmonically quasi-convex functions in fractional integral forms which is related to the right-hand side of (1.5):

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(a + 1)}{2} \left( \frac{ab}{b-a} \right)^{\alpha} \left( J_{1/a}^{1} (f \circ h)(1/b) + J_{1/b}^{1} (f \circ h)(1/a) \right) \right| \\
\leq \frac{ab(b-a)}{2^{\frac{1}{q}+1}} \left( \sup \{ |f'(a)|^q, |f'(b)|^q \} \right)^{\frac{1}{q}} \left[ C_{3}^{\frac{1}{q}} (a) + C_{4}^{\frac{1}{q}} (a) \right],
\]

If we take \( \alpha = 1 \) and \( g(x) = 1 \) we have the following Hermite-Hadamard type inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.4):

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_{a}^{b} \frac{f(x)}{x^2} dx \right| \\
\leq \frac{ab(b-a)}{2^{\frac{1}{q}+1}} \left( \sup \{ |f'(a)|^q, |f'(b)|^q \} \right)^{\frac{1}{q}} \left[ C_{5}^{\frac{1}{q}} (1) + C_{6}^{\frac{1}{q}} (1) \right].
\]

Acknowledgments. The authors are very grateful to the referee for helpful comments and valuable suggestions.

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