ON SOME NESTED FLOOR FUNCTIONS AND THEIR JUMP DISCONTINUITIES

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ABSTRACT. This paper investigates some particular limits involving nested floor functions. We’ll prove some cases and then we’ll show a more general result. Then we’ll count the discontinuity points of those functions, and we’ll prove a method to find them all. Surprisingly the set of the jump discontinuities of \( f_n \) is a subset of the set of the jump discontinuities of \( f_{n+1} \), \( \forall n \in \mathbb{Z}^+ \) where:

\[
fn(x) = \left\lfloor \left\lfloor \left\lfloor \cdots \left\lfloor x \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor \quad \text{n times}
\]

Furthermore we’ll give some generalizations of the result and lots of considerations; for example we’ll prove that the cardinality of the set of the discontinuities of \( f_n \) in a given limited interval approaches infinity as \( n \to \infty \).

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1. INTRODUCTION

Definition. In mathematics and computer science, the floor function is the function that takes as input a real number \( x \), and gives as output the greatest integer less than or equal to \( x \), denoted \( \lfloor x \rfloor \).

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1.1. First problem. Let \( k \in \mathbb{Z}^+ \setminus \{1\} \) and \( n \in \mathbb{Z}^+ \); we want to compute the following limit:

\[
\lim_{x \to k^-} \left[ \underbrace{x \lfloor x \lfloor \ldots \lfloor}_{n \text{ times}} \right]
\]

by varying \( k, n \).

1.2. Example. Let \( k = 3 \) and \( n = 2 \), we want to compute:

\[
\lim_{x \to 3^-} \left[ x \lfloor x \right]
\]

The answer of this case is 5, and it’s possible to prove it using the definition of limit: 

\[
\forall \varepsilon > 0 \exists \delta > 0 \text{ such that: } \left\| \left[ x \lfloor x \right] - 5 \right\| < \varepsilon \forall x \in (3 - \delta, 3)
\]

Using \( \delta = \frac{1}{2} \), we have \( x \in (\frac{5}{2}, 3) \). So:

\[
\frac{5}{2} < x < 3 \implies \lfloor x \rfloor = 2
\]

And then, by replacying it in our definition we have:

\[
\left\| 2x - 5 \right\| < \varepsilon
\]

But furthermore since \( \frac{5}{2} < x < 3 \) we know that:

\[
\frac{5}{2} < x < 3 \implies 5 < 2x < 6 \implies \lfloor 2x \rfloor = 5
\]

and finally:

\[
|5 - 5| = 0 < \varepsilon \forall \varepsilon \in \mathbb{R}^+
\]

So:

\[
\lim_{x \to 3^-} \left[ x \lfloor x \right] = 5
\]

2. First generalization

In this section we will generalize the example 1.2

**Theorem 1.** Let \( k \in \mathbb{Z}^+ \setminus \{1\} \) and \( n \in \mathbb{Z}^+ \):

\[
\lim_{x \to k^-} \left[ \underbrace{x \lfloor x \lfloor \ldots \lfloor}_{n \text{ times}} \right] = \frac{1}{k - 1} \left[ (k - 2) \cdot k^n + 1 \right]
\]

**Proof.** It is sufficient to prove that \( \forall k \in \mathbb{Z}^+ \setminus \{1\} \) and \( \forall n \in \mathbb{Z}^+ \exists \delta_n = \frac{k - 1}{(k - 2)k^{n - 1}} \) such that

\[
\left\| \underbrace{x \lfloor x \lfloor \ldots \lfloor}_{n \text{ times}} \right\| = \frac{1}{k - 1} \left[ (k - 2) \cdot k^n + 1 \right] \forall x \in (k - \delta_n, k)
\]

The base case is when \( n = 1 \). We have that:

\[
|x| = k - 1 \forall x \in (k - \frac{1}{k - 1}, k)
\]
and this identity is true because \( k \) is a positive integer.
Suppose that this identity holds for \( n \) and we’ll prove it right for \( n + 1 \).
Since \( k^n > k^{n-1} \forall n \in \mathbb{Z}^+ \forall k \in \mathbb{Z}^+ \setminus \{1\} \), consider:

\[
\delta_{n+1} = \frac{k-1}{(k-2)k^n+1} < \frac{k-1}{(k-2)k^{n-1}+1} = \delta_n
\]

we want to show that:

\[
\left\lfloor \frac{1}{(k-2)k^{n+1}+1} \right\rfloor \forall x \in (k - \delta_{n+1}, k)
\]

but since \( \delta_{n+1} < \delta_n \) we have that:

\[
\left\lfloor \frac{1}{(k-2)k^{n+1}+1} \right\rfloor = \left\lfloor \frac{k-1}{1} \left( (k-2) \cdot k^n + 1 \right) x \right\rfloor \forall x \in I = (k - \delta_{n+1}, k) \subset (k - \delta_n, k)
\]

Furthermore:

\[
\left\lfloor \frac{1}{(k-2)k^{n+1}+1} \right\rfloor = \left\lfloor \frac{k-1}{1} \left( (k-2) \cdot k^n + 1 \right) x \right\rfloor \forall x \in I = (k - \delta_{n+1}, k)
\]

This is true in fact:

\[
I = (k - \delta_{n+1}, k) = \left( \frac{k^{n+2} - 2k^{n+1} + 1}{k^{n+1} - 2k^n + 1}, k \right)
\]

So:

\[
\frac{k^{n+2} - 2k^{n+1} + 1}{k^{n+1} - 2k^n + 1} < x < k \Rightarrow \frac{k^{n+2} - 2k^{n+1} + 1}{k^{n+1} - 2k^n + 1} < (k^{n+1} - 2k^n + 1)x < k^{n+2} - 2k^{n+1} + k
\]

\[
(k - 2)k^{n+1} + 1 < (k - 2)k^n + 1 < (k - 2)k^{n+1} + k
\]

And dividing all by \( \frac{1}{k-1} \), we’ll get:

\[
\frac{(k - 2)k^{n+1}}{k-1} + \frac{1}{k-1} < \frac{1}{k-1} \left( (k - 2)k^n + 1 \right) x < \frac{(k - 2)k^{n+1}}{k-1} + \frac{k}{k-1}
\]

Call \( n_1 \) the left hand side of the inequality and \( n_2 \) the right hand side. We’ll prove that they are in fact natural numbers in the next section. Note that:

\[
0 < n_2 - n_1 = 1
\]

So the number \( \frac{1}{k-1} \left( (k - 2)k^n + 1 \right) x \) is strictly between \( n_1, n_2 \), which are positive integers whose distance between each others is equal to 1. So the floor of

\[
\frac{1}{k-1} \left( (k - 2)k^n + 1 \right) x
\]

must be equal to \( n_1 \) (which is the nearest integer less than or equal to \( \frac{1}{k-1} \left( (k - 2)k^n + 1 \right) x \)).
Let $\delta_n = \frac{k-1}{(k-2)k^n+1}$. We want to show that:

$$\left\lfloor x \left\lfloor x \left\lfloor \ldots \right\rfloor \right\rfloor \right\rfloor - \frac{1}{k-1} (k-2) \cdot k^n + 1 \right\rfloor < \varepsilon \forall x \in (k-\delta_n, k)$$

But now this is obvious because:

$$\left\lfloor x \left\lfloor x \left\lfloor \ldots \right\rfloor \right\rfloor \right\rfloor = \frac{1}{k-1} (k-2) \cdot k^n + 1 \right\rfloor \forall x \in (k-\delta_n, k)$$

And finally $\forall \varepsilon > 0$:

$$\left\lfloor x \left\lfloor x \left\lfloor \ldots \right\rfloor \right\rfloor \right\rfloor - \frac{1}{k-1} (k-2) \cdot k^n + 1 \right\rfloor = 0 < \varepsilon \forall x \in (k-\delta_n, k)$$

And the thesis follows from the limit definition. □

**Lemma 2.** $n_1$ and $n_2$ defined in Theorem 1 are positive integers.

We want to prove that:

$$(k-2)k^{n+1} + 1 \equiv 0 \pmod{(k-1)}$$

$$(k^{n+2} - 2k^{n+1} + 1 \equiv 0 \pmod{(k-1)}$$

$\forall k \in \mathbb{Z}^+ \setminus \{1\}$ and $n \in \mathbb{Z}^+$.

**Proof.** Note that:

$$k^{n+2} - 2k^{n+1} + 1 = (k-1)(k^{n+1} - k^n - k^{n-1} - \ldots - k - 1)$$

So the numerator of $n_1$ is a multiple of $(k-1)$. Since the numerator of $n_2$ is equal to the numerator of $n_2$ plus $k - 1$ we conclude that also $n_2$ is an integer (because its numerator is again a multiple of $k - 1$). □

### 3. Jump Discontinuities

Let $f(x)$ be the function:

$$f_n(x) = \left\lfloor x \left\lfloor x \left\lfloor \ldots \right\rfloor \right\rfloor \right\rfloor$$

Note that:

$$\lim_{x \to k^-} f_n(x) = \frac{1}{k-1} (k-2) \cdot k^n + 1 \right\rfloor \text{ while } \lim_{x \to k^+} f_n(x) = k^n \forall k \in \mathbb{Z}^+$$

From these relations we know that $x = k$ is a point of discontinuity of the first kind $\forall k \in \mathbb{Z}^+$ with jump’s length equal to:

$$|J(k, f_n)| = k^n - \frac{1}{k-1} (k-2) \cdot k^n + 1 \right\rfloor = \frac{k^n - 1}{k-1}$$

But the function $f_n(x)$ has more jump discontinuities than these. For example:

$$f_2(x) = \left\lfloor x \right\rfloor$$
has a jump discontinuity in $x = \frac{10}{3}$, in fact:

$$\lim_{x \to \frac{10}{3}^-} \lfloor x \rfloor = 9 \text{ while } \lim_{x \to \frac{10}{3}^+} \lfloor x \rfloor = 10$$

As you can see from these graphs:

![Graphs of $f_2(x)$](image)

**Figure 1.** Graph of $f_2(x)$

**Remark 3.** Let $f_1(x) = \lfloor x \rfloor$, $P(a, b, f_1)$ the set of discontinuity points of $f_1(x)$ in the interval $[a, b)$ and $P(f_1)$ the set of all discontinuity points over the domain $x \geq 1$. Then:

$$P(f_1) = \mathbb{Z}^+$$

In fact every $x = k$ (where $k \in \mathbb{Z}^+$) is a jump discontinuity for $f_1$, where:

$$J(k, f_1) = 1 \ \forall k \in P(f_1)$$

**Theorem 4.** Let $f_2(x)$ be defined as before, $P(a, b, f_2)$ the set of discontinuity points of $f_2(x)$ in the open interval $[a, b)$ and $P(f_2)$ the set of all discontinuity points over the domain $x \geq 1$. Then:

$$P(f_2) = \bigcup_{k=1}^{+\infty} P(k, k+1, f_2) = \bigcup_{k=1}^{+\infty} \left\{ k, k + \frac{1}{k}, k + \frac{2}{k}, \ldots, k + \frac{k-1}{k} \right\}$$

Or:

$$P(f_2) = \bigcup_{k=1}^{+\infty} \bigcup_{r=0}^{k-1} \left\{ k + \frac{r}{k} \right\}$$

where:

$$|J(k, f_2)| = \frac{k^2 - 1}{k - 1} = k + 1 \ \forall k \in \mathbb{Z}^+$$

and:

$$\left| J\left(k + \frac{r}{k}, f_2\right) \right| = 1 \ \forall k \in \mathbb{Z}^+ \text{ where } r \in \{1, 2, \ldots, k - 1\}$$

**3.0.1. Examples.** For example, consider the function $f_2(x)$ and the interval $[4, 5)$. Assuming true Theorem 2 we know that:

$$P(4, 5, f_2) = \left\{ 4, \frac{17}{4}, \frac{9}{2}, \frac{19}{4} \right\}$$

In fact:

$$\lim_{x \to 4^-} f_2(x) = 11 \land \lim_{x \to 4^+} f_2(x) = 16$$

and:

$$|J(4, f_2)| = 4 + 1 = 5$$

the jump is in fact: $16 - 11$
While:
\[
\lim_{x \to \frac{17}{4}^-} f_2(x) = 16 \land \lim_{x \to \frac{17}{4}^+} f_2(x) = 17 \\
\lim_{x \to \frac{9}{2}^-} f_2(x) = 17 \land \lim_{x \to \frac{9}{2}^+} f_2(x) = 18 \\
\lim_{x \to \frac{19}{4}^-} f_2(x) = 18 \land \lim_{x \to \frac{19}{4}^+} f_2(x) = 19
\]

So:
\[
\left| J\left(\frac{17}{4}, f_2\right) \right| = \left| J\left(\frac{9}{2}, f_2\right) \right| = \left| J\left(\frac{19}{4}, f_2\right) \right| = 1
\]

As you can see from this image:

![Figure 2. Graph of $f_2(x)$ in $I = [4, 5)$](image)

Proof. We've already proved that $x = k$ is a discontinuity point $\forall k \in \mathbb{Z}^+$, we should prove that:
\[
\left\{ k + \frac{1}{k}, \ldots, k + \frac{k-1}{k} \right\}_{k=1}^{k=+\infty}
\]
are the only others discontinuity points of $f_2$

So we're proving that:
\[
\bigcup_{k=1}^{+\infty}\left\{ k, k + \frac{1}{k}, k + \frac{2}{k}, \ldots, k + \frac{k-1}{k} \right\} \subseteq P(f_2) \land P(f_2) \subseteq \bigcup_{k=1}^{+\infty}\left\{ k, k + \frac{1}{k}, k + \frac{2}{k}, \ldots, k + \frac{k-1}{k} \right\}
\]

Let:
\[
\lim_{x \to \left( k + \frac{r}{k}\right)^-} f_2(x) = L(k, r)^- \text{ and } \lim_{x \to \left( k + \frac{r}{k}\right)^+} f_2(x) = L(k, r)^+
\]

Then we’ll prove that $\forall r \in \{1, 2, \ldots, k-1\}$:
\[
L(k, r)^- = k^2 + r - 1 \land L(k, r)^+ = k^2 + r
\]

Using $\delta = \frac{1}{k}$ we have that:
\[
\left| \lfloor x \rfloor - (k^2 + r - 1) \right| < \varepsilon \ \forall x \in (k + \frac{r-1}{k}, k + \frac{r}{k})
\]

We know that:
\[
k + \frac{r-1}{k} < x < k + \frac{r}{k} \Rightarrow \lfloor x \rfloor = k \ \forall x \in (k + \frac{r-1}{k}, k + \frac{r}{k})
\]

And:
\[
k + \frac{r-1}{k} < x < k + \frac{r}{k} \Rightarrow k^2 + r - 1 < kx < k^2 + r \Rightarrow \lfloor kx \rfloor = k^2 + r - 1
\]

So:
\[
\left| \lfloor x \rfloor - (k^2 + r - 1) \right| = \left| \lfloor kx \rfloor - (k^2 + r - 1) \right| = \left| k^2 + r - 1 - (k^2 + r - 1) \right| = 0 < \varepsilon
\]
Similarly, using $\delta = \frac{1}{k}$ again it’s possible to prove with the same technique that:
\[
\left| f(x) - (k^2 + r) \right| < \varepsilon \forall x \in (k + \frac{r}{k}, k + \frac{r+1}{k})
\]
We prove the first inequality. In order to prove the second inequality it’s sufficient to note that $f_2$ is a monotone increasing function over its domain and that $f_2(x) \in \mathbb{N}, \forall x \geq 1$.
Furthermore:
\[
|J\left( k + \frac{r}{k}, f_2 \right)| = 1 \forall k \in \mathbb{Z}^+ \text{ where } r \in \{1, 2, \ldots, k - 1\}
\]
Since:
\[
L(k, r)^- = k^2 + r - 1 \land L(k, r)^+ = k^2 + r
\]
So we have that:
\[
k^2 + r - 1 \leq f_2(x) < k^2 + r \forall x \in \left[k + \frac{r-1}{k}, k + \frac{r}{k}\right]
\]
But $f_2(x) \in \mathbb{N}$, so the function in that interval is constant, and is equal to:
\[
f_2(x) = k^2 + r - 1 \forall x \in \left[k + \frac{r-1}{k}, k + \frac{r}{k}\right]
\]
Finally we prove that every discontinuity point of $f_2$ are elements of the following set:
\[
P(f_2) = \bigcup_{k=1}^{+\infty} \left\{ k + \frac{1}{k}, k + \frac{2}{k}, \ldots, k + \frac{k-1}{k} \right\}
\]
Or:
\[
P(f_2) = \bigcup_{k=1}^{+\infty} \bigcup_{r=0}^{k-1} \left\{ k + \frac{r}{k} \right\}
\]
\吸入{-}\]

### 3.1. First considerations.
The set of discontinuity points of $f_2(x)$ is a countable set. In fact it’s the countable union of countable sets. \吸入{-}\]
So:
\[
|P(f_2)| = |\mathbb{N}| = \aleph_0
\]
Let $h \in \mathbb{Z}^+ \setminus \{1\}$, the cardinality of the finite set defined as:
\[
|P(1, h, f_2)| = \left| \bigcup_{k=1}^{h-1} \left\{ k + \frac{1}{k}, k + \frac{2}{k}, \ldots, k + \frac{k-1}{k} \right\} \right|
\]
is equal to:
\[
|P(1, h, f_2)| = 1 + 2 + 3 + \cdots + h - 1 = \frac{h(h - 1)}{2}
\]

### 3.2. Generalizations.
If we consider the functions $f_3, f_4, f_5, \ldots, f_n$ it’s easy to see that there are more and more discontinuity points as $n$ increases. Back to the $f_2$ case, it’s possible to construct a partition of a generic interval $I = [a, b]$ , made of the discontinuity points of $f_2$ in that interval. For example, let $I = [3, 4)$, then:
\[
P(3, 4, f_2) = \left\{ 3, 3 + \frac{1}{3}, 3 + \frac{2}{3} \right\}
\]
while we’ll prove that:
\[
P(3, 4, f_3) = \left\{ 3, 3 + \frac{1}{9}, 3 + \frac{2}{9}, 3 + \frac{3}{3}, 3 + \frac{2}{5}, 3 + \frac{1}{2}, 3 + \frac{3}{5}, 3 + \frac{2}{3}, 3 + \frac{8}{11}, 3 + \frac{9}{11}, 3 + \frac{10}{11} \right\}
\]
Note that:
\[ P(3, 4, f_2) \subset P(3, 4, f_3) \land |P(3, 4, f_2)| < |P(3, 4, f_3)| \]

**Theorem 5.** Let \( P(f_n) \) denotes the set of the discontinuity points of the function \( f_n \). Then:
\[ P(f_1) \subset P(f_2) \subset \cdots \subset P(f_n) \forall n \in \mathbb{Z}^+ \]

**Proof.** We’ll prove this result by induction on \( n \).
The base case \((n = 2)\) has been already proved before. So we know that 
\( P(f_1) \subset P(f_2) \).
Suppose that \( P(f_1) \subset P(f_2) \subset \cdots \subset P(f_{n-1}) \). We’ll prove that:
\[ P(f_{n-1}) \subset P(f_n) \]
Let \( d \) be an element of \( P_{n-1} \); consider the following limits:
\[ \lim_{x \to d^-} f_{n-1}(x) = L_d^- \land \lim_{x \to d^+} f_{n-1}(x) = L_d^+ \]
From the induction hypothesis we know that:
\[ L_d^- \neq L_d^+ \]
So \( \exists \delta^-, \delta^+ > 0 \) such that:
\[ |f_{n-1}(x) - L_d^-| = 0 \forall x \in (d - \delta^-, d) \]
\[ |f_{n-1}(x) - L_d^+| = 0 \forall x \in (d, d + \delta^+) \]
But from the definition of \( f_n \) we have that:
\[ f_n(x) = [x \cdot f_{n-1}(x)] \]
So:
\[ \lim_{x \to d^-} f_n(x) = \lim_{x \to d^-} [x \cdot f_{n-1}(x)] \]
\[ \lim_{x \to d^+} f_n(x) = \lim_{x \to d^+} [x \cdot f_{n-1}(x)] \]
But \( f_{n-1} = L_d^- \forall x \in (d - \delta^-, d) \) and \( f_{n-1} = L_d^+ \forall x \in (d, d + \delta^+) \), so substituing in we will obtein:
\[ \lim_{x \to d^-} f_n(x) = \lim_{x \to d^-} [x \cdot L_d^-] \]
\[ \lim_{x \to d^+} f_n(x) = \lim_{x \to d^+} [x \cdot L_d^+] \]
This last step is motivated by using the limit definition.
But we know that \( L_d^-, L_d^+ \in \mathbb{Z}^+ \), and since \( f_n \) is monotone increasing and \( L_d^- \neq L_d^+ \) we can conclude that:
\[ L_d^+ > L_d^- \implies L_d^+ = L_d^- + k \text{ for some } k \in \mathbb{Z}^+ \]
So:
\[ \lim_{x \to d^-} f_n(x) = \lim_{x \to d^-} [x \cdot L_d^-] \]
\[ \lim_{x \to d^+} f_n(x) = \lim_{x \to d^+} [x \cdot L_d^- + kx] \]
But using \( \delta = \min\{\delta^-, \delta^+\} \) we have:
\[ d < x < d + \delta \implies kd < kx < kd + k\delta \]
And finally we have that $kx$ in this interval is bigger than $kd$ (which is a positive rational number bigger than or equal to 1). So:

$$\lim_{x \to d^+} f_n(x) = \lim_{x \to d^+} |x \cdot L_d^- + kx| = \lim_{x \to d^+} [x \cdot L_d^- + kx - \lfloor kx \rfloor + \lfloor kx \rfloor] \geq 0$$

and combining all the inequalities we’ll get:

$$\lim_{x \to d^+} f_n(x) = \lim_{x \to d^+} [x \cdot L_d^- + kx - \lfloor kx \rfloor + \lfloor kx \rfloor] \geq \lim_{x \to d^-} [x \cdot L_d^- + |kd|] > \lim_{x \to d^-} f_n(x)$$

\[ \square \]

**Theorem 6.** Given the interval $[k, k + 1)$ where $k \in \mathbb{Z}^+$, then:

$$\lim_{n \to \infty} |P(k, k + 1, f_n)| = \infty$$

Where $|P(k, k + 1, f_n)|$ represents the cardinality of the set of all the discontinuity points of $f_n$ in the interval $[k, k + 1)$.

**Proof.** Since from Theorem 5 we know that:

$P(f_1) \subset P(f_2) \subset \cdots \subset P(f_n) \implies P(k, k+1, f_1) \subset P(k, k+1, f_2) \subset \cdots \subset P(k, k+1, f_n)$

it’s sufficient to show that $\forall n \geq 2, \exists d \in P(k, k+1, f_n) \setminus P(k, k+1, f_{n-1})$. In fact we’ll show that:

$$d = k + \frac{1}{k^{n-1}} \in P(k, k+1, f_n) \setminus P(k, k+1, f_{n-1})$$

First we want to prove that:

$$k + \frac{1}{k^n} \in P(k, k+1, f_{n+1})$$

Or, using the definition of this set:

$$\lim_{x \to (k + \frac{1}{k^n})^-} f_{n+1}(x) \neq \lim_{x \to (k + \frac{1}{k^n})^+} f_{n+1}(x)$$

In fact we’ll prove by induction on $n \geq 2$ that using $\delta = \frac{1}{k^n}$:

$$f_{n+1}(x) = k^{n+1} \forall x \in \left(k + \frac{1}{k^n} - \delta, k + \frac{1}{k^n}\right) \text{ while } f_{n+1}(x) = k^{n+1} + 1 \forall x \in \left(k + \frac{1}{k^n}, k + \frac{1}{k^n} + \delta\right)$$

Assuming that true we will have:

$$\lim_{x \to (k + \frac{1}{k^n})^-} f_{n+1}(x) = k^{n+1} \text{ while } \lim_{x \to (k + \frac{1}{k^n})^+} f_{n+1}(x) = k^{n+1} + 1$$

which is our thesis.

By the induction hypothesis:

$$f_n(x) = k^n \forall x \in \left(k, k + \frac{1}{k^n}\right)$$

$$f_n(x) = k^{n+1} + 1 \forall x \in \left(k + \frac{1}{k^n}, k + \frac{2}{k^n}\right)$$

But using the definition of $f_{n+1}$ with $\delta = \frac{1}{k^n} < \frac{1}{k^{n-1}}$ we’ll have:

$$f_{n+1} = [x \cdot f_n(x)] = [x \cdot k^n] \forall x \in \left(k, k + \frac{1}{k^n}\right) \subset \left(k, k + \frac{1}{k^{n-1}}\right)$$

$$f_{n+1} = [x \cdot f_n(x)] = [x \cdot k^n] \forall x \in \left(k + \frac{1}{k^n}, k + \frac{2}{k^n}\right) \subset \left(k, k + \frac{1}{k^{n-1}}\right)$$
But furthermore:

\[ k < x < k + \frac{1}{k^n} \implies k^{n+1} < k^n x < k^{n+1} + 1 \]
\[ k + \frac{1}{k^n} < x < k + \frac{2}{k^n} \implies k^{n+1} + 1 < k^n x < k^{n+1} + 2 \]

And finally:

\[ f_{n+1} = \lfloor x \cdot k^n \rfloor = k^{n+1} \forall x \in \left(k, k + \frac{1}{k^n}\right) \]
\[ f_{n+1} = \lfloor x \cdot k^n \rfloor = k^{n+1} + 1 \forall x \in \left(k + \frac{1}{k^n}, k + \frac{2}{k^n}\right) \]

Now we know that \( k + \frac{1}{k^n}, k + \frac{2}{k^n} \in P(k, k + 1, f_{n+1}) \), and we want to prove that \( + \frac{1}{k^n} \leq P(k, k + 1, f_n) \). This is true in fact:

\[
\lim_{x \to (k + \frac{1}{k^n})^-} f_n(x) = \lim_{x \to (k + \frac{1}{k^n})^+} f_n(x) = k^n
\]

In order to prove it, as seen before, we know that:

\[ f_n(x) = k^n \forall x \in \left(k, k + \frac{1}{k^n}\right) \]

But then:

\[ f_n(x) = k^n \forall x \in \left(k, k + \frac{1}{k^n}\right) \subset \left(k + \frac{1}{k^n}, k + \frac{2}{k^n}\right) \]
\[ f_n(x) = k^n \forall x \in \left(k + \frac{1}{k^n}, k + \frac{2}{k^n}\right) \subset \left(k + \frac{2}{k^n}, \ldots, k + \frac{1}{k^n}\right) \]

\[ \forall k \geq 2, \forall n \geq 1. \text{ So we’ll have:} \]

\[ \{k + \frac{1}{k}, \ldots\} \subset \{k + \frac{1}{k}, k + \frac{1}{k^2}, \ldots\} \subset \ldots \subset \{k + \frac{1}{k}, k + \frac{1}{k^2}, \ldots, k + \frac{1}{k^n}, \ldots\} \]

\[ P(k, k + 1, f_2) \subset P(k, k + 1, f_3) \subset \ldots \subset P(k, k + 1, f_{n+1}) \]

Where \( P(k, k + 1, f_{n+1}) \) has \( n \) terms of the form \( k + \frac{1}{k^r} \), where \( r \in \{1, \ldots, n\} \).

\[ \square \]

3.3. **Script in Mathematica language.** In order to compute discontinuity points of the function \( f_n \) in a given interval, is possible to use this script: [2]

**Listing 1.** To compute discontinuity points of \( f_3 \) in \( I = (2, 3) \)

```mathematica
FunctionDomain[{Floor[x*Floor[x-Floor[x]]], x}, 2 < x < 3, x]
```

which gives:

\[
\text{Always } 2 < x < 9/4 \parallel 9/4 < x < 5/2 \parallel 5/2 < x < 13/5 \parallel 13/5 < x < 14/5 \parallel 14/5 < x < 3+
\]

where \( \frac{9}{4}, \frac{5}{2}, \frac{13}{5}, \frac{14}{5} \) are in fact all the discontinuity points in that interval.

**Conjecture 7.** Let \( f_3(x) \) be defined as before, \( P(a, b, f_3) \) the set of discontinuity points of \( f_3(x) \) in the open interval \( [a, b) \) and \( P(f_3) \) the set of all discontinuity points over the domain \( x \geq 1 \). Then:

\[ P(f_3) = P(f_3) \cup P(f_2) \cup \bigcup_{k=1}^{+\infty} \bigcup_{i=0}^{k-1} \left\{ k + \frac{(k+1)i}{k^2+i} \right\} \]

Or:

\[ P(f_3) = P(f_3) \cup P(f_2) \cup \bigcup_{k=1}^{+\infty} \bigcup_{i=0}^{k-1} \bigcup_{p=0}^{k-1} \left\{ k + \frac{(k+1)i + p}{k^2+i} \right\} \]
where:
\[ |J(k, f_3)| = \frac{k^3 - 1}{k - 1} = k^2 + k + 1 \forall k \in \mathbb{Z}^+ \]

and:
\[ |J(k + \frac{r}{k}, f_3)| = k + 1 \forall k \in \mathbb{Z}^+ \text{ where } r \in \{1, 2, \ldots, k - 1\} \]

and:
\[ |J(p, f_3)| = 1 \forall p \in P(f_3) \setminus P(f_2) \cap P(f_1) \]

For example, consider the interval \([3, 4)\). We’ve already said that:
\[ P(3, 4, f_3) = \left\{ 3, 3 + \frac{1}{9}, 3 + \frac{2}{9}, 3 + \frac{3}{9}, 3 + \frac{4}{9}, 3 + \frac{5}{9}, 3 + \frac{6}{9}, 3 + \frac{7}{9}, 3 + \frac{8}{9}, 3 + \frac{9}{9} \right\} \]

This should represent this set:
\[ P(3, 4, f_1) \cup P(3, 4, f_2) \cup \bigcup_{i=0}^{2} \bigcup_{p=0}^{2} \left\{ 3 + \frac{4i + p}{9 + i} \right\} (k = 3 \text{ in the formula above}) \]

In fact:
\[ P(3, 4, f_3) = \left\{ 3 \right\} \cup \left\{ 3 + \frac{1}{3}, 3 + \frac{2}{3} \right\} \cup \left\{ 3, 3 + \frac{1}{9}, 3 + \frac{2}{9}, 3 + \frac{4}{9}, 3 + \frac{5}{9}, 3 + \frac{6}{9}, 3 + \frac{7}{9}, 3 + \frac{8}{9}, 3 + \frac{9}{9} \right\} \]

3.4. Considerations of conjecture \[7\] As in Theorem \[4\] we would like to prove our result using double inclusion of sets. But from Theorem \[5\] we know that:
\[ P(f_1) \subset P(f_2) \subset P(f_3) \]

So it’s to sufficient to prove that all the elements in the set \( P(f_3) \setminus P(f_1) \cup P(f_2) \) are jump discontinuities for \( f_3 \) (because we already known that the elements in \( P(f_1) \cup P(f_2) \) are jump discontinuities for \( f_3 \)). In general, the set: \( P(f_3) \setminus P(f_1) \cup P(f_2) \) is given by:
\[ P(f_3) \setminus P(f_1) \cup P(f_2) = \bigcup_{k=1}^{+\infty} \bigcup_{i=0}^{k-1} \bigcup_{p=0}^{k-1} \left\{ k + \frac{(k + 1)i + p}{k^2 + i} \right\} \setminus \{k\} \]

We would like to show that:
\[ \lim_{x \to (k + \frac{(k + 1)i + p}{k^2 + i})^-} f_3(x) = k^3 + 2ik + i + p - 1 \]

While:
\[ \lim_{x \to (k + \frac{(k + 1)i + p}{k^2 + i})^+} f_3(x) = k^3 + 2ik + i + p \]

For example:
\[ \lim_{x \to (3 + \frac{4}{3})^-} f_3(x) = \lim_{x \to \frac{11}{3}} f_3(x) = 3^3 + 2 \cdot 1 \cdot 3 + 0 - 1 = 33 \]
4. OTHER GENERALIZATIONS

Given the result obtained before, it’s trivial to prove that under the same conditions:

\[ \forall m \in \mathbb{Z}^+ \]

\[
\lim_{x \to k^-} \left( x^m \left\lfloor x^m \right\rfloor \ldots \right) = \frac{1}{k^m - 1} \left[ (k^m - 2) \cdot k^{m-n} + 1 \right]
\]

For example, for \( k = 4, n = 3, m = 2 \) we have that:

\[
\lim_{x \to 4^-} \left( x^2 \left\lfloor x^2 \right\rfloor \ldots \right) = 3823 = \frac{1}{4^2 - 1} \left[ (4^2 - 2) \cdot 4^{2-3} + 1 \right] = \frac{57345}{15}
\]

REFERENCES

[1] Union of countably many countable sets is countable. (2021). MathStackExchange.math.stackexchange.com

[2] Jump discontinuities in Mathematica. (2022). MathematicaStackExchange.mathematica.stackexchange.com. Available online at: https://mathematica.stackexchange.com/questions/264829/jump-discontinuities-in-mathematica