Finite temperature correlation function for one-dimensional Quantum Ising model: 
the virial expansion.

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We rewrite the exact expression for the finite temperature two-point correlation function for the magnetization as a partition function of some field theory. This removes singularities and provides a convenient form to develop a virial expansion (the expansion in powers of soliton density).

To calculate correlation functions in strongly correlated systems is not an easy task, even if the corresponding models happen to be integrable. For models with dynamically generated spectral gaps the most powerful technique is the formfactor approach pioneered by Karowski et. al. and perfected by Smirnov. This approach works wonderfully for zero temperature, but encounters difficulties at \(T \neq 0\). These difficulties are related to singularities in the operator matrix elements (formfactors). These singularities exist for operators nonlocal with respect to solitons, they originate from forward scattering processes and their treatment requires careful infrared regularization. Despite long efforts a correct regularization has not been yet found. However, for models of free fermions (such as the XY model or the Quantum Ising model), there are alternative means to calculate the correlation functions which allow to bypass the above problems. These alternative approaches include the determinant representation of the correlation functions and the semiclassical method (which may have much wider application, see [3]). For these results to have a greater use one has to establish their relationship with the formfactor approach. A step in this direction was made in [4] where the semiclassical results were reproduced by summing up the leading singularities in the formfactor expansion. Such summation was restricted to the leading order in the soliton density \(n \sim \exp(-M/T)\) (\(M\) is the spectral gap). In this paper we describe a formfactor-based representation for correlation functions which, though in its present form is valid only for models of free fermions, is rather suggestive and may give rise to useful generalizations in the future. For the Quantum Ising (QI) model, which is the main object of this paper, this procedure naturally gives rise to a virial expansion of the dynamical spin susceptibility.

The QI model Hamiltonian is

\[
H = \sum_n [-J \sigma^n_x \sigma^{n+1}_x + h \sigma^n_z] \tag{1}
\]

where \(s^x, s^z\) are the Pauli matrices. By the Jordan-Wigner transformation this Hamiltonian can be transformed into the Hamiltonian of non-interacting fermions:

\[
H = \sum_p \epsilon(p) F_p^+ F_p; \quad \epsilon(p) = \sqrt{(J - h)^2 + 4Jh \sin^2(p/2)} \tag{2}
\]

Hamiltonian (1) possesses a property of self-duality: the transformation \(\rho_{n+1/2}^x = \prod_{j<n} \sigma_j^x, \rho_{n+1/2}^z = \sigma_n^z \rho_{n+1}^z\) preserves both the commutation relations and the form of the Hamiltonian:

\[
H = \sum_n [-h \rho_{n-1/2}^z \rho_{n+1/2}^z + J \rho_{n+1/2}^x] \tag{3}
\]

As follows from the form of the dispersion law (2), \(h = J\) is the critical point. It is easy to see that at \(T = 0\) it separates the regions where \(\langle \sigma^z \rangle \neq 0(J > h)\) and \(\langle \rho_{n+1/2}^z \rangle \neq 0(J < h)\). Operators \(\sigma^z\) and \(\mu^z\) are respectively called order and disorder parameter operators. The duality allows to study the correlation functions at one side of the transition only. For instance, a correlation function of \(\sigma^z\) at \(J > h\) coincides with the correlation function for \(\mu^z\) with \(J, h\) interchanged.

Though one cannot observe a single fermion, the fermionic statistics can be indirectly tested by measuring the correlation functions of \(\sigma^z\). The operator \(\sigma^z\) (as well as \(\mu^z\)) is local in fermions:

\[
\sigma^z(x) = \sum_k e^{-i \epsilon x \gamma} \gamma(k - q) \hat{F}_k \hat{F}_{q-k}, \quad \gamma(k) = \sqrt{1 + k^2/\epsilon(k)}. \tag{4}
\]

Its finite temperature correlation functions are just polarization loops; they clearly contain the Fermi distribution functions of the fermion fields \(F\). Since the order \((\sigma^z)\) and disorder parameter \((\mu^z)\) fields are nonlocal in terms of fermions, their correlation functions are more complicated.

The duality allows us to consider \(h > J\) or \(h < J\) phase only. In this paper we will study the two-point correlation functions of the \(\sigma^z\) and \(\mu^z\) operators in the “ordered” phase \(h < J\). For technical reasons it will be convenient to work in the limit \(|J - h| < J\) when the spectral gap is much smaller than the bandwidth and one can formulate a continuous description. In this limit the excitation spectrum is relativistic \(\epsilon(p) = \sqrt{c^2 p^2 + M^2}, c^2 = Jh\).
Energy and momentum of a quasi-particle are conveniently parameterized by a rapidity, \( \theta \), \((cp = M \sinh \theta)\). Then the eigenstates of Hamiltonian \( H \) are labeled by sets of rapidities, \( \{ \theta_i \} \), such that the energy and momentum of the system are equal to

\[
E = M \sum_{i=1}^{n} \cosh \theta_i, \quad P = e^{-1}M \sum_{i=1}^{n} \sinh \theta_i.
\]  

Below we set \( c = 1 \).

A convenient finite temperature expression for the two point correlation functions of \( \sigma \) and \( \mu \) was derived by Bourgij and Lisovyy. This expression for the Matsubara time correlation function is manifestly free of singularities and has the following form:

\[
\langle \sigma(\tau, x) \sigma(0, 0) \rangle = CM^{1/4} e^{-|x|/\Delta(T)} \times \sum_{N=0}^{\infty} \frac{T^{2N}}{(2N)!} \prod_{q_{1}, \ldots, q_{2N}} \epsilon_i \prod_{i \neq j} (q_i - q_j)^2, \tag{6}
\]

where \( \tau \) is imaginary time. The same expression holds for \( \mu^2 \), but with \( 2N \) replaced by \( 2N + 1 \). \( q = 2\pi T n \) \((n \text{ integer})\), and \( \epsilon(q) = \sqrt{M^2 + q^2} \). The term in the exponent is \((\beta = 1/T)\)

\[
\eta(q) = \frac{2\epsilon(q)}{\pi} \int_{0}^{\infty} \frac{dx}{e^{2q(x)} + x^2} \ln \cosh[\beta \epsilon(x)/2]. \tag{7}
\]

\[
\sum_{n=1}^{2N} \frac{1}{n!(2N-n)!} \int \prod_{i=1}^{n} \frac{d\theta_i}{2\pi} f^{(+)}(\theta_i) e^{\tau_{e_i} + i\epsilon_{p_i}} \prod_{j=1}^{2N-n} \frac{d\theta'_j}{2\pi} f^{(-)}(\theta'_j) e^{-\tau_{e_j} - i\epsilon_{p_j}} \prod_{i>j} \pi \tan^2[\theta_i - \theta_j]/2 \prod_{i \neq p} \tan^2[(\theta_i - \theta'_p + i\epsilon)/2] \tag{9}
\]

are numbers of particles and antiparticles. Now we rearrange the double sum \((N)\) in such a way that we first sum all terms which contain a fixed difference between numbers of particles and antiparticles \(2N - 2n = 2k\). Then such term in Eq. (9) can be represented as an integral of the correlation function of a Gaussian field theory:

\[
\Delta(T) = \int_{-\infty}^{\infty} \frac{dp}{\pi} \ln \cosh[\beta \epsilon(p)/2]. \tag{8}
\]

The symmetry breaking transition at \( T = 0 \) leads to a finite magnetization, \( (\sigma) = \pm [CM^{1/4}]^{1/2} \). This is reflected in the zeroth order term in Eq. (6).

\[
FIG. 1: A graphic representation of Eq. (9). The ellipses are formfactors of \( \sigma \) operator. Lines with left arrows are \( f^{(+)}(\theta) \exp[i\epsilon(\theta) + i\epsilon p(\theta)] \), lines with the right arrows are \( f^{(-)}(\theta) \exp[-i\epsilon(\theta) - i\epsilon p(\theta)] \).
\]

In Eq. (5) was rewritten in the form which allowed an analytic continuation for real time. The \( N \)-th term in the square brackets of Eq. (6) is given as \( \eta(q) \)

\[
(\Phi(\theta_1) \Phi(\theta_2)) \equiv G_0(\theta_{12}) = -\ln \left[ \tanh^2(\theta_{12}) + a_0^2 \right] \tag{12}
\]
theory with the action
\[ S = \frac{1}{2} \int d\theta_1 d\theta_2 \Phi(\theta_1)G_0^{-1}(\theta_{12})\Phi(\theta_2) + \int \frac{d\theta}{2\pi a_0} V[\theta, \Phi(\theta)] \tag{13} \]
\[ V = \frac{1}{2} f^{(+)}(\theta)e^{-\beta M} \cosh^{\theta+1} + M \sinh^{\theta} e^{i\Phi(\theta+1\alpha)} + \frac{1}{2} f^{(-)}(\theta)e^{-\beta M} \cosh^{\theta-1} + M \sinh^{\theta} e^{-i\Phi(\theta-1\alpha)} \tag{14} \]

In this theory \( x, t \) are external parameters. The field \( \Phi \) lives on an infinite line in \( \theta \) space. It is easy to see that the entire correlation function can be written as
\[ \langle \sigma(\tau, x)\sigma(0, 0) \rangle = \lim_{\Delta(T) \to \infty} \sum_{k=-\infty}^{\infty} \left( e^{2i\Phi(R)} \right)_V a_0^{4k^2} \]
\[ Z_0 = \int D\Phi e^{-S[\Phi]} / \int D\Phi e^{-S_0[\Phi]} \]

All transformations so far have been exact. Now we would like to concentrate on the causal Green’s functions. To obtain them one has to include \( \tau \) with it in \( \Phi \). Assuming that \( T \ll M \), we consider the region of frequencies \( |\omega| < 2M \), where the only terms of the expansion contributing to the spectral function are those which contain equal number of particles and antiparticles. As we shall demonstrate, all formfactor singularities are contained in the first term of expansion \( \Phi \). Since \( |f^{(+)}| \sim \exp(-\beta M) << 1 \), \( |f^{(-)}| \sim 1 \), this expansion is in powers of soliton density \( \exp(-\beta M) \). The first term is given by
\[ -F^{(1)}(t, x) = \frac{1}{4\pi^2} \int d\theta_1 d\theta_2 f^{(+)}(\theta_1) f^{(-)}(\theta_2) \frac{\exp \{ i[t\epsilon(\theta_1) - \epsilon(\theta_2)] + x[p(\theta_1) - p(\theta_2)] \} \tanh^2(\theta_{12} + i\theta) / 2}{(\theta_{12} + i\theta)^2} \tag{18} \]
\[ \approx \frac{1}{2\pi^2} \int d\theta d\nu f^{(+)}(\theta + v) f^{(-)}(\theta - v) \frac{\exp \{ i\nu M(\tau \sinh \theta + x \cosh \theta) \} / (v + i\theta)^2}{v + i\theta} \approx (t - |x|) \left\{ \frac{1}{\pi} \int_{\text{tanh} \theta < -|x|/t} d\theta f^{(+)}(\theta) f^{(-)}(\theta) M \left[ (t \sinh \theta + x \cosh \theta) \right] + \frac{2i}{\pi} g[\theta = \text{tanh}^{-1}(x/t)] \right\} \]

where \( g(\theta) = f^{(+)}(\theta) - f^{(-)}(\theta) \) and, as follows from \( f^{(+)}(\theta) f^{(-)}(\theta) = \{ 4 \cosh^2(\beta \omega / 2) \}^{-1} \), we assumed that \( M, (t, T)^{1/2} \gg 1 - (x/t)^2 \). The higher order cumulants contain higher powers of \( \exp(-\beta M) \) and also do not contain positive powers of \( t \). This justifies keeping \((...)_V \) stands for averaging with action \( \Phi \).
weak magnetic field the XY model spectrum is a gapless, 

so that the ground state becomes ferromagnetic. How-

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response function in the XY model obtained in [3]. This 

⟨(x, t)σ(0, 0)⟩T = CM^{1/4} \exp(-\delta \Delta|x|) \exp \left\{ -\frac{1}{4 \pi} \int dp \frac{|tv(p) - x|}{\cosh^2(\beta \epsilon/2)} - \frac{4 i}{\pi} \exp[-\beta M/\sqrt{1 - (x/t)^2}] \right\}, t > |x|

⟨(x, t)σ(0, 0)⟩T = CM^{1/4} \exp(-\Delta |x|), \quad |x| > t \tag{19}

⟨(x, t)σ(0, 0)⟩T = ⟨(x, t)σ(0, 0)⟩_{T=0} \tag{20}

δ\Delta = \frac{1}{\pi} \int dp \left[ \ln \coth(\beta \epsilon/2) - \frac{1}{2 \cosh^2(\beta \epsilon/2)} \right] \sim \exp(-3\beta M)

The imaginary part of (19) in the time-like domain t > |x| reflects a quantum nature of the excitations. For T = 0 such imaginary part was first found in [11]. In the leading order in \exp(-\beta M) Eq. (20) coincides with the one found in [3]. At x = 0 we have

⟨(x = 0, t > 0)σ(0, 0)⟩ \approx CM^{1/4} \exp \left[ -t/\tau_0 + \frac{4 i}{\pi} e^{-\beta M} + O(e^{-2\beta M}) \right] \tag{21}

where \tau_0^{-1} = \frac{2 T}{\pi \sinh(\beta M)}. Since single solitons are not directly observable, it is rather interesting to note that \tau_0 contains the distribution function of a single soliton. It would be very interesting to see what happens in correlation functions in models containing particles with fractional statistics.

Since our approach shares certain common features with the Fredholm determinant representation introduced by Korepin et. al. [3, 4, 12, 13], we feel obliged to comment on the subject. The main difference is that we do not represent the correlation functions as determinants though in certain limits this is possible. For instance, if one adopts the nonrelativistic limit \theta << 1 in action (13), it can be fermionized and rewritten as a theory of free fermions. Then by integrating over fermions one obtains the determinant representation. However, we would like to point out that such representation is not a goal in itself. By representing correlation functions as partition functions of some field theory one already achieves a lot since now one can concentrate on connected diagrams where it is easier to keep track of singularities. It is possible that acting along the lines of [14] one can obtain such representations for interacting models.

We also would like to warn against the direct comparison of Eq. (19) with a similar equation for the \langle \sigma - \sigma^+ \rangle correlation function in the XY model obtained in [3]. This warning is necessary because the XY model in magnetic field is rather similar to the QI model; the similarity increases when the magnetic field exceeds the band width so that the ground state becomes ferromagnetic. However, as was explained to us by Korepin (private communication), the formulae of (3) were obtained for the case of weak magnetic field the XY model spectrum is a gapless, which explains the difference in the final results.

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