NOTE ON PARITY AND THE IRREDUCIBLE CHARACTERS OF THE SYMMETRIC GROUP

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Introduction

The object of this short note is to prove a theorem and present a conjecture for the number of even entries in the character table of the symmetric group $S_n$.

Theorem 1. The number of even entries in the character table of $S_n$ is even.

Conjecture 1. The proportion of the character table of $S_n$ covered by even entries tends to 1 as $n \to \infty$.

Theorem 1 is proved in Section 1. Conjecture 1 is discussed in Section 2. To support Conjecture 1 we write down in Table 1 the number of even entries and odd entries in the character table of $S_n$ for $1 \leq n \leq 76$. See Figure 1. Another table (Table 3) in Section 2 suggest a more general phenomenon.

Conjecture 2. The proportion of the character table of $S_n$ covered by entries divisible by a given prime number $p$ tends to 1 as $n \to \infty$.

Figure 1. Proportion of the character table of the symmetric group $S_n$ covered by even entries for $1 \leq n \leq 76$.

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1. Proof of Theorem 1

Let $p_n$ be the number of partitions of $n$. Here a partition of $n$ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n$. The conjugate of $\lambda$ is the partition $\lambda'$ whose parts are $\lambda'_i = \# \{ j : i \leq \lambda_j \}$ for $1 \leq i \leq \lambda_1$. Conjugation is the involution $\lambda \mapsto \lambda'$. The fixed points of this involution are self-conjugate partitions. Self-conjugate partitions $\lambda$ of $n$ are in one-to-one correspondence with partitions $\mu$ of $n$ into odd distinct parts via $\lambda \mapsto \mu$ where $\mu_i = 2(\lambda_i - i) + 1$ for $i$ such that $1 \leq i \leq \lambda_i$.

Proof of Theorem 1. Let $O_n$ be the number of odd entries in the character table of $S_n$. Then

$$O_n \equiv \sum_g \sum_{\chi} \chi(g) \equiv \sum_g \sum_{\chi} \chi(g)^2 \pmod{2} \quad (1)$$

where the two outer sums run over a set of representatives $g$ for the conjugacy classes and the two inner sums run over the irreducible characters $\chi$. Moreover one of the orthogonality relations [1] tells us that

$$\sum_{\chi} \chi(g)^2 = 1^{m_1} m_1! 2^{m_2} m_2! \ldots n^{m_n} m_n! \quad (2)$$

for $m_p$ the number of cycles of period $p$ in the cycle decomposition of $g$. Together (1) and (2) imply

$$O_n \equiv OD_n \pmod{2} \quad (3)$$

where $OD_n$ is the number of partitions of $n$ into odd distinct parts. Let $SC_n$ be the number of self-conjugate partitions of $n$ so that $SC_n = OD_n$ and hence

$$O_n \equiv SC_n \pmod{2}. \quad (4)$$

Let $E_n$ be the number of even entries in the character table of $S_n$. Then

$$O_n + E_n = p_n^2 \equiv p_n \pmod{2}. \quad (5)$$

Together (4) and (5) imply

$$E_n \equiv p_n - SC_n \pmod{2}. \quad (6)$$

But $p_n - SC_n \equiv 0 \pmod{2}$ because conjugation restricts to a fixed-point-free involution on the set of non-self-conjugate partitions of $n$. \qed

2. Remarks and some tables

This section contains some tables and remarks. The main object is Table 1 for the number of even entries in the character table of $S_n$.

2.1. Remarks. Let $\chi(\mu)$ be short for the constant value $\chi(g)$ of the irreducible character $\chi$ of $S_n$ on the class consisting of all permutations $g \in S_n$ for which the periods of the disjoint cycles form the partition $\mu$. 

2.1.1. In terms of the probability that an entry $\chi(\mu)$ is even when chosen uniformly at random from the character table of the symmetric group $S_n$, Conjecture 1 says

$$\text{Prob}(\chi(\mu) \text{ is even}) \to 1 \text{ as } n \to \infty. \quad (7)$$

This parity bias becomes even more striking when compared with the distribution of signs in the character table of $S_n$ (cf. [2, Question 3]). See Figure 2 and Table 2.

**Conjecture 3.** $\text{Prob}(\chi(\mu) > 0 \mid \chi(\mu) \neq 0) \to 1/2 \text{ as } n \to \infty.$

![Figure 2](image2.png)

**Figure 2.** The plot $\bullet$ for $\text{Prob}(\chi(\mu) > 0 \mid \chi(\mu) \neq 0)$ and the plot $\circ$ for $\text{Prob}(\chi(\mu) < 0 \mid \chi(\mu) \neq 0)$ where $1 \leq n \leq 38$.

2.1.2. Conjecture 2 implies that for any integer number $d$ one has

$$\text{Prob}(\chi(\mu) \equiv 0 \text{ (mod } d)) \to 1 \text{ as } n \to \infty. \quad (8)$$

Figure 1 suggests that there is a sharper statement. See for example Figure 3.

![Figure 3](image3.png)

**Figure 3.** The proportion of the character table of $S_n$ covered by even entries for $2 \leq n \leq 76$ and the graph of $2\pi^{-1} \arctan(\sqrt{n/2} - 1)$ for $2 \leq n \leq 76.$
2.2. Tables.

TABLE 1. Number of even entries and number of odd entries in the character table of $S_n$ for $1 \leq n \leq 76$.

| $n$ | no. of evens | no. of odds | $n$ | no. of evens | no. of odds |
|-----|--------------|-------------|-----|--------------|-------------|
| 1   | 0            | 1           | 39  | 799580980   | 172923245   |
| 2   | 0            | 4           | 40  | 1152977342  | 241148902   |
| 3   | 2            | 7           | 41  | 1644080076  | 343563813   |
| 4   | 6            | 19          | 42  | 2352923494  | 474550782   |
| 5   | 16           | 33          | 43  | 3324344208  | 677609913   |
| 6   | 44           | 77          | 44  | 4732761850  | 918518775   |
| 7   | 90           | 135         | 45  | 6639049122  | 1305820834  |
| 8   | 266          | 218         | 46  | 9351080036  | 1791411328  |
| 9   | 508          | 392         | 47  | 13067332410 | 2496282106  |
| 10  | 966          | 798         | 48  | 18309958344 | 3379378185  |
| 11  | 1824         | 1312        | 49  | 25390864566 | 4720061059  |
| 12  | 3548         | 2381        | 50  | 35331180090 | 6377078986  |
| 13  | 6094         | 4107        | 51  | 48786461562 | 8786181687  |
| 14  | 11586        | 6639        | 52  | 67367826002 | 11924538919 |
| 15  | 19254        | 11722       | 53  | 92571070272 | 16283394489 |
| 16  | 37492        | 15869       | 54  | 127268025536| 21847658489 |
| 17  | 61876        | 26333       | 55  | 173744388742| 29905639434 |
| 18  | 103110       | 45115       | 56  | 237567368138| 39975105191 |
| 19  | 170932       | 69168       | 57  | 32002974632 | 54182161084 |
| 20  | 286916       | 106213      | 58  | 439208932802| 72330715598 |
| 21  | 456554       | 170710      | 59  | 594363393060| 97561119340 |
| 22  | 759962       | 244042      | 60  | 804101537262| 129956924827|
| 23  | 1190034      | 384991      | 61  | 1082902860136| 174870604889|
| 24  | 1887766      | 592859      | 62  | 1458789177232| 231616447104|
| 25  | 2937820      | 895944      | 63  | 1956705210484| 309822028517|
| 26  | 4608084      | 1326012     | 64  | 2625259647972| 408015408928|
| 27  | 7004646      | 2055454     | 65  | 3505898738012| 54449063352 |
| 28  | 10938762     | 2884762     | 66  | 4679753246976| 718991943424|
| 29  | 16372732     | 4466493     | 67  | 6226771093726| 953692042995|
| 30  | 24851432     | 6553384     | 68  | 8285512851154| 1248594579071|
| 31  | 37101368     | 9798596     | 69  | 10979998587386| 165336971639|
| 32  | 56368810     | 13336991    | 70  | 14541318538948| 2170163830076|
| 33  | 82688102     | 20192347    | 71  | 19209876952108| 2853857859917|
| 34  | 122855526    | 28680574    | 72  | 25351409083192| 3730699401897|
| 35  | 179808396    | 41695293    | 73  | 33363529811282| 4899218593439|
| 36  | 263406424    | 59766105    | 74  | 43886589872232| 6374420377768|
| 37  | 381814902    | 86344867    | 75  | 57554118617836| 8352091755860|
| 38  | 557951490    | 118828735   | 76  | 75434276878574| 10852934727707|
TABLE 2. Number of positive entries and number of negative entries in the character table of $S_n$ for $1 \leq n \leq 38$.

| $n$ | pos. | neg. |
|-----|------|------|
| 1   | 1    | 0    |
| 2   | 3    | 1    |
| 3   | 6    | 2    |
| 4   | 14   | 7    |
| 5   | 26   | 13   |
| 6   | 58   | 34   |
| 7   | 98   | 72   |
| 8   | 194  | 137  |
| 9   | 344  | 249  |
| 10  | 652  | 524  |
| 11  | 1165 | 953  |
| 12  | 2020 | 1679 |
| 13  | 3552 | 3106 |
| 14  | 6077 | 5270 |
| 15  | 10362| 9398 |
| 16  | 17080| 15666|
| 17  | 28570| 2787256|
| 18  | 46836| 43409|
| 19  | 77045| 72861|

| $n$ | pos. | neg. |
|-----|------|------|
| 20  | 122013| 115940|
| 21  | 198461| 189476|
| 22  | 310602| 297929|
| 23  | 494008| 476904|
| 24  | 767237| 743094|
| 25  | 1205391| 1174624|
| 26  | 1828252| 1782368|
| 27  | 2846995| 2787256|
| 28  | 4277605| 4196505|
| 29  | 6520106| 6413986|
| 30  | 9795470| 9645485|
| 31  | 14738493| 14553197|
| 32  | 21750402| 21483398|
| 33  | 32582580| 32243250|
| 34  | 47614253| 47163539|
| 35  | 70213289| 69606943|
| 36  | 102477724| 101689585|
| 37  | 149340038| 148321445|
| 38  | 215267489| 213892988|

TABLE 3. Number of entries $\equiv 0 \pmod{d}$ in the character table of $S_n$ for $3 \leq d \leq 7$ and $1 \leq n \leq 19$.

| $n$ | $d = 3$ | $d = 4$ | $d = 5$ | $d = 6$ | $d = 7$ |
|-----|---------|---------|---------|---------|---------|
| 1, 2| 0       | 0       | 0       | 0       | 0       |
| 3   | 1       | 1       | 1       | 1       | 1       |
| 4   | 6       | 4       | 4       | 4       | 4       |
| 5   | 11      | 12      | 12      | 11      | 10      |
| 6   | 39      | 30      | 35      | 29      | 29      |
| 7   | 73      | 61      | 64      | 59      | 63      |
| 8   | 181     | 187     | 178     | 163     | 168     |
| 9   | 426     | 368     | 336     | 352     | 339     |
| 10  | 803     | 681     | 726     | 643     | 660     |
| 11  | 1456    | 1272    | 1219    | 1188    | 1147    |
| 12  | 3138    | 2722    | 2668    | 2542    | 2503    |
| 13  | 5289    | 4532    | 4359    | 4135    | 3989    |
| 14  | 9980    | 8443    | 8332    | 8088    | 8031    |
| 15  | 16935   | 14067   | 14173   | 13363   | 13108   |
| 16  | 29669   | 27733   | 25351   | 25171   | 24066   |
| 17  | 49768   | 45156   | 42136   | 42202   | 39316   |
| 18  | 88645   | 77206   | 72601   | 73047   | 68206   |
| 19  | 139983  | 126447  | 115972  | 116635  | 108050  |

References

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