Relations for Four Nucleons in a Single $j$ Shell –
Using 9j Symbols to Get Rid of 9j Symbols

L. Zamick$^1$

$^1$Department of Physics and Astronomy, Rutgers University,
New Brunswick, New Jersey 08903, USA

(Dated: September 12, 2018)

Abstract

In previous works we considered the number of pairs of particles with angular momentum $J_{12}$ for systems with total angular momentum zero. Here we extend the work to states of any total angular momentum. We find simplified relations satisfied for a system of four nucleons with isospins zero and two.

PACS numbers: 21.600Fw
I. INTRODUCTION

We previously considered the problem of the number of pairs of nucleons with angular momentum $J_{12}$ for systems with total angular momentum $I=0$ e.g. ground states of even-even Ti isotopes. [1,2,3] We could get general expressions for any $J_{12}$ pairs and any total angular momentum $I$, but we obtained very simplified expressions for the cases where $I$ was equal to zero and $J_{12}$ was also zero i.e. the number of $J_{12}=0$ pairs in the $I=0$ ground states of even-even Ti isotopes. For $^{44}$Ti we went further and obtained simplified expressions for $I=0$ and all even $J_{12}$.

In this work we show that we can get some simplified expressions for all $I$ in many but not all cases. The cases where we succeed include even $J_{12}$ pairs for states of total isospin $T=0$ and $T=2$.

II. METHOD

As in the work of Zamick, Lee and Mekjian [3] we start out by noting that that for the $j^2$ configuration, states of even angular momentum have isospin $T=1$ while states of odd angular momentum have isospin $T=0$. We then consider an interaction which is a constant for all odd $J$ and a different constant for all even $J$.

$$V = a(1 - (-1)^J)/2 + b(1 + (-1)^J)/2 \quad (1)$$

This same interaction can be written using isospin variables.

$$V = a(\frac{1}{4} - t(1) \cdot t(2)) + b(\frac{3}{4} + t(1) \cdot t(2)) \quad (2)$$

The general expression for the matrix element of any interaction for $^{44}$Ti in the single $j$ shell is
\[
< [(j^2)^I_p (j^2)^I_N] V [(j^2)^I_p (j^2)^I_N] > = \]
\[
= (E(J_P) + E(J_N)) \delta_{J_P J_N} \delta_{I_J I_N} + 4 \sum_{J_A J_B} < (j j)^I_P (j j)^I_N | (j j)^J_A (j j)^J_B >^I
\]
\[
< (j j)^I_P (j j)^I_N | (j j)^J_A (j j)^J_B >^I E(J_B)
\]

(3)

Where

\[
E(J_B) = < (j^2)^J_B V (j^2)^J_B >
\]

(4)

The last term in Eq. (3) has a product of two unitary 9j symbols.

Let us now consider the case where b=0 i.e. the T=0 interaction is “a” and the T=1 interaction is zero. Let us take a to be one.

The result for the matrix element of eq. (3) is now

\[
< [(j^2)^I_p (j^2)^I_N] V [(j^2)^I_p (j^2)^I_N] > = 2 \delta_{J_P J'_P} \delta_{J_N J'_N} + 2 < (j j)^I_P (j j)^I_N | (j j)^J_A (j j)^J_B >^I E(J_B)
\]

(5)

We can also evaluate the eigenvalues of this “Hamiltonian” using isospin considerations.

For A nucleons we have

\[
\sum_{i<j} \left( \frac{1}{4} - t(i) \cdot t(j) \right) = \frac{A^2}{8} + \frac{A}{4} - \frac{1}{2} T(T+1)
\]

(6)

For A=4 the eigenvalues are 3, 2, and zero for isospins T=0, 1 and 2 respectively.

A wave function of \( ^{44}_{\text{Ti}} \) can, in the single j shell, be represented as a column vector, one of the entries being \( D^{I \alpha} (J_P, J_N) \) which is the probability amplitude that in a state of total angular momentum I, the protons couple to Jp and the neutrons to Jn.

We have the orthogonality and completeness conditions

\[
\sum_{J_P J_N} D^{I \alpha} (J_P J_N) D^{I \alpha'} (J_P J_N) = \delta_{IJ} \delta_{\alpha \alpha'}
\]

(7)
\[
\sum_\alpha D^{I\alpha}(J_P J_N) D^{I\alpha}(J'_P J'_N) = \delta_{J_P, J'_P} \delta_{J_N, J'_N}
\]  

(8)

From Eq. (5) we now have the following "eigenvalue equation" for a state of total angular momentum I and total isospin T

\[
\sum_{J_P, J'_P} [2\delta_{J_P, J'_P} \delta_{J_N, J'_N} + 2 < jj | J_P (jj) | J_N >^I D^I(J_P J_N) = \lambda^T D^I(J_P J_N)
\]  

(9)

Alternately

\[
2 \sum_{J_P, J'_P} < jj | J_P (jj) | J_N >^I D^I(J_P J_N) = (\delta_{T,0} + 0\delta_{T,1} - 2\delta_{T,2}) D^I(J_P J_N)
\]  

(10)

III. NUMBER OF PROTON-NEUTRON PAIRS WITH ANGULAR MOMENTUM J_{12}

To get the number of neutron-proton pairs with angular momentum J_{12} we refer to Eq (3) where the matrix element of the nuclear Hamiltonian is written as a sum of a neutron-neutron part, a proton-proton part and a proton-neutron part. Clearly the last term is of interest here.

To get the number of p-n pairs of angular momentum J_{12} we set all E(J_b) to zero except for E(J_{12}), which is set equal to one. We then get the expectation value of this Hamiltonian with the wave function of a state of interest.

We get

\[
\# \text{ of p – n pairs } (J_{12}) = \sum_{J_A} |2 \sum_{J_P, J_N} < (j^2)^{J_P} (j^2)^{J_N} | (j^2)^{J_A} (j^2)^{J_{12}} > D^I(J_P J_N) |^2
\]  

(11)

Inside the square of Eq (10) we have something that looks like the left hand of Eq (9b). But we have to be careful. The sum in Eq (10) is over even Jp, Jn – indeed D(Jp,Jn) is defined for only even values of the angular momenta in Eq. (10) however Ja and J_{12} can be either even or odd.

We will now show that in certain cases, despite the above, we can simplify the expressions for the number of p-n pairs. We can only do this for even J_{12}. But still we have to worry
about the fact that \( J_a \) can be odd. We must consider three different isospin cases \( T=0, 1 \) and 2.

For \( T=0 \) and 2 the following is true.

\[
D^I(J_P J_N) = (-1)^I D^I(J_N J_P) \tag{12}
\]

This is a consequence of charge symmetry.

On the other hand for \( T=1 \) we have

\[
D^I(J_P J_N) = (-1)^{(I+1)} D^I(J_N J_P) \tag{13}
\]

Eq 12 insures that terms with \( J_a \) odd will cancel out in the sum over \( J_p, J_n \). We can therefore use Eq (9b) for these two cases. We find \# of p-n pairs (\( J_{12} \) even) \( T=0 \) or 2

\[
= \sum_{J_A} |D^I(J_A J_{12})|^2 (\delta_{T,0} + 4\delta_{T,2}) \tag{14}
\]

We cannot get a correspondingly simple expression for states with isospin \( T=1 \).

By summing Eq. (14) over even \( J_{12} \) we find that for \( T=0 \) states the total number of even \( J_{12} \) pairs is one, and since the total number of \( J_{12} \) pairs is four, the total number of odd \( J_{12} \) pairs is three.

A more general result was previously obtained using isospin considerations, see A(12) and A(13) in ref. [2]. There it is shown that the total number of \( T_{12}=0 \) (and hence odd \( J_{12} \)) pairs is equal to

\[
\frac{A^2}{8} + \frac{A}{4} - \frac{T(T+1)}{2} \tag{15}
\]

The total number of \( T_{12}=1 \) (and hence even \( J_{12} \) pairs), including nn and pp as well as np is equal to

\[
\frac{3A^2}{8} - \frac{3A}{4} + \frac{T(T+1)}{2} \tag{16}
\]

In the above \( A \) is the number of valence nucleons. The above results hold for all total angular momenta \( I \) and for all Ti isotopes, in the single j approximation.

In the single j approximation we get the following results for \( A=4 \) for the number of pairs (nn + pp + np)
(# of odd pairs, # of even pairs), T=0 (3, 3), T=1 (2, 4), T=2 (0, 6).

For T=2 there are no odd $J_{12}$ pairs. This can be understood by the fact that the T=2 state is the double analog of a state in $^{44}$Ca where we are dealing with identical particles i.e. neutrons. Two neutrons in a single $j$ shell must have isospin one and even angular momentum.

The above results are especially simple in the 1s shell. Two neutrons and two protons close this shell so the total isospin must be zero. The even pairs have angular momentum $J_{12} = 0$, while the odd ones have $J_{12} = 1$. Thus there are three $J_{12} = 0$ pairs and three $J_{12} = 1$ pairs in this case.

[0] E. Moya de Guerra, A. A. Raduta, L. Zamick and P. Sarriguren, Nuclear Physics A 727 (2003) 3025.

[0] L. Zamick, A. Escuderos, S. J. Lee, A. Z. Mekjian, E. Moya de Guerra, P. Sarriguren and A. A. Raduta, Phys. Rev. C71 (2005) 034317.

[0] L. Zamick, A. Mekjian and S. J. Lee, Journal of the Korean Physical Society 47 (2005) 18.