IS THE MINIMUM VALUE OF AN OPTION ON VARIANCE GENERATED BY LOCAL VOLATILITY?

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Abstract. We discuss the possibility of obtaining model-free bounds on volatility derivatives, given present market data in the form of a calibrated local volatility model. A counter-example to a wide-spread conjecture is given.

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1. Introduction

"... it has been conjectured that the minimum possible value of an option on variance is the one generated from a local volatility model fitted to the volatility surface."; Gatheral [Gat06, page 155].

Leaving precise definitions to below, let us clarify that an option on variance refers to a derivative whose payoff is a convex function of total realized variance. Turning from convex to concave, this conjecture, if true, would also imply that the maximum possible value of a volatility swap \( f(x) = x^{1/2} \) is the one generated from a local volatility model fitted to the volatility surface. Given the well-documented model-risk in pricing volatility swaps, such bounds are of immediate practical interest.

The mathematics of local volatility theory (à la Dupire, Derman, Kani, ...) is intimately related to the following

**Theorem 1** (Gyöngy [Gyö86]). Assume \( dY_t = \mu(t, \omega) dt + \sigma(t, \omega) dB_t \) is a multi-dimensional Itô-process where \( B \) is a multi-dimensional Brownian motion, \( \mu, \sigma \) are progressively measurable, bounded and \( \sigma\sigma^T \geq \varepsilon^2 I \) for some \( \varepsilon > 0 \) (\( \sigma^T \) denotes the transpose of \( \sigma \)). Then

\[
\begin{align*}
\text{d} \tilde{Y}_t &= \mu_{\text{loc}}(t, \tilde{Y}_t) \text{d}t + \sigma_{\text{loc}}(t, \tilde{Y}_t) \text{d}\tilde{B}_t, \\
\tilde{Y}_0 &= Y_0,
\end{align*}
\]

\( \mu_{\text{loc}}(t, y) = E[\mu(t, \omega) | Y_t = y] \),

\( \sigma_{\text{loc}}(t, y) = E[\sigma(t, \omega) \sigma^T(t, \omega) | Y_t = y]^{1/2} \),

(where the power \( \frac{1}{2} \) denotes the positive square root of a positive definite matrix) has a weak solution \( \tilde{Y}_t \) such that \( \tilde{Y}_t \) \( \text{law} \) \( = Y_t \) for all fixed \( t \).

We will apply theorem 1 only in the simple one dimensional (resp. two dimensional in section 4 below) setting where it is well known that the solution to (1.1) is unique (cf. [Kry67] or [SV06, Chapter 7]).

A generic stochastic volatility model (already written under the appropriate equivalent martingale measure and with suitable choice of numéraire) is of the form \( dS = S \sigma dB \) where \( \sigma = \sigma(t, \omega) \) is the (progressively measurable) instantaneous volatility process. It will suffice for our application to assume \( \sigma \) to be bounded

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from above and below by positive constants.) Arguing on log-price \( X = \log S \) rather than \( S \),
\[
(1.2) \quad dX_t = \sigma(t, \omega) \, dB_t - (\sigma^2(t, \omega)/2) \, dt,
\]
a classical application of theorem [1] yields the following Markovian projection result, the (weak) solution to
\[
(1.3) \quad d\tilde{X}_t = \sigma_{loc}(t) \, d\tilde{B}_t - \left( \sigma_{loc}^2(t) / 2 \right) \, dt, \quad \tilde{X}_0 = X_0,
\]
has the one-dimensional marginals of the original process \( X_t \). Equivalently, the process \( \tilde{S} = \exp \tilde{X} \),
\[
d\tilde{S}_t = \sigma_{loc}(t) \tilde{S}_t \, dB_t,
\]
known as (Dupire’s) local volatility model, gives rise to identical prices of all European call options \( C(T, K) \)\(^4\). It easily follows that \( \sigma_{loc}^2(t, \tilde{S}) \) is given by Dupire’s formula
\[
(1.4) \quad \sigma_{loc}^2(T, \tilde{S})|_{\tilde{S}=K} = 2 \frac{\partial_T C}{K^2 \partial_{KK} C}.
\]
Volatility derivatives are options on realized variance; that is, the payoff is given by some function \( f \) of realized variance. The latter is given by
\[
V_T := \langle \log S \rangle_T = \langle X \rangle_T = \int_0^T \sigma^2(t, \omega) \, dt;
\]
in the model \( dS = \sigma(t, \omega) S \, dB \) and by
\[
\tilde{V}_T := \langle \log \tilde{S} \rangle_T = \langle \tilde{X} \rangle_T = \int_0^T \sigma_{loc}^2(t, \tilde{X}_t) \, dt
\]
in the corresponding local volatility model.

Common choices of \( f \) are \( f(x) = x \), the variance swap, \( f(x) = x^{1/2} \), the volatility swap, or simply \( f(x) = (x - K)^+ \), a call-option on realized variance. See [FG05] for instance. As is well-known, see e.g. [Gat06], the pricing of a variance swap, assuming continuous dynamics of \( S \) such as those specified above, is model free in the sense that it can be priced in terms of a log-contract; that is, a European option with payoff \( \log S_T \). In particular, it follows that
\[
E \left[ \tilde{V}_T \right] = E \left[ V_T \right].
\]
Of course this can also be seen from [13], after exchanging \( E \) and integration over \([0, T]\). Passing from \( V_T \) to \( f(V_T) \) for general \( f \) this is not true, and the resulting differences are known in the industry as convexity adjustment. We can now formalize the conjecture given in the first lines of the introduction\(^3\).

**Conjecture 1.** For any convex \( f \) one has \( E[f(V_T)] \leq E[f(\tilde{V}_T)] \).

Our contribution is twofold: first we discuss a simple (toy) example which provides a counterexample to the above conjecture; secondly we refine our example using a 2-dimensional Markovian projection (which may be interesting in its own right) and thus construct a perfectly sensible Markovian stochastic volatility model in which the conjectured result fails. All this narrows the class of possible dynamics

\(^{1}\)Let us quickly remark that Markovian projection techniques have led recently to a number of new applications (see [Pit06], for instance).

\(^{2}\)The abuse of notation, by writing both \( \sigma_{loc}(t, \tilde{X}) \) and \( \sigma_{loc}(t, \tilde{S}) \), will not cause confusion.

\(^{3}\)We emphasize that \( C(T, K) \) denotes the price at time \( t = 0 \) of European call with maturity \( T \) and strike \( K \).

\(^{4}\)It is tacitly assumed that \( f(V_T), f(\tilde{V}_T) \) are integrable.
for $S$ for which the conjecture can hold true and so should be a useful step towards positive answers.

2. Idea and numerical evidence

Example 2. Consider a Black–Scholes “mixing” model $dS = S \sigma dB$, $S_0 = 1$ with time horizon $T = 3$ in which $\sigma^2(t, \omega)$ is given by $\sigma^2_1(t)$ or $\sigma^2_2(t)$,

\[
\sigma^2_1(t) := \begin{cases} 2 & \text{if } t \in [0, 1], \\ 3 & \text{if } t \in [1, 2], \\ 1 & \text{if } t \in [2, 3], \end{cases} \quad \sigma^2_2(t) := \begin{cases} 2 & \text{if } t \in [0, 1], \\ 1 & \text{if } t \in [1, 2], \\ 3 & \text{if } t \in [2, 3], \end{cases}
\]

depending on a fair coin flip $\epsilon = \pm 1$ (independent of $B$). Obviously $V = V_3 = \int_0^3 \sigma^2 dt \equiv 6$ in this example, hence $E[(V - 6)^+] = (V - 6)^+ = 0$. On the other hand, the local volatility is explicitly computable (cf. the following section) and one can see from simple Monte Carlo simulations that for $V = V_3$

\[
E \left[ (\tilde{V} - 6)^+ \right] \approx 0.026 > 0
\]

thereby (numerically) contradicting conjecture [7] with $f(x) = (x - 6)^+$.

Our analysis of this toy model is simple enough: in section 3 below we prove that $P[\tilde{V} = 6] \neq 1$. Since $E[\tilde{V}] = E[V] = 6$ and $(x - 6)^+$ is strictly convex at $x = 6$, Jensen’s inequality then tells us that $E[(\tilde{V} - 6)^+] > 0 = E[(V - 6)^+]$.

The reader may note that an even simpler construction would be possible, i.e. one could simply leave out the interval $[0, 1]$ where $\sigma^2_1$ and $\sigma^2_2$ coincide. We decided not to do so for two reasons. First, insisting on $\sigma^2(t) = \sigma^2(t)$ for $t \in [0, 1]$ leads to well behaved coefficients of the SDE describing the local volatility model. Second, we will use the present setup to obtain a complete model contradicting conjecture [7] at the end of the next section.

3. Analysis of the toy example

We recall that it suffices to show that $\tilde{V} = \int_0^3 \sigma^2_{loc}(t, \tilde{X}_t) dt$ is not a.s. equal to $V \equiv 6$. The distribution of $X_t$ is simply the mixture of two normal distributions. More explicitly, $X_t = I_{\{\epsilon = +1\}} X_{t, +} + I_{\{\epsilon = -1\}} X_{t, -}$,

\[
X_{t, \pm} = \int_0^t \sigma_{\pm}(s) dB_s - \frac{1}{2} \int_0^t \sigma^2_{\pm}(s) ds \sim N \left( \frac{1}{2} \Sigma_{\pm}(t), \Sigma_{\pm}(t) \right),
\]

where $\Sigma_{\pm}(t) := \int_0^t \sigma^2_{\pm}(s) ds$. Thus $\sigma^2_{loc}(t, x) = E[\sigma^2(t, \omega)|X_t = x]$ is given by

\[
(3.1) \quad \sigma^2_{loc}(t, x) = \frac{\sigma^2_1(t)}{\sqrt{\Sigma_{+}(t)}} \exp \left[ -\frac{(x+\Sigma_{+}(t)/2)^2}{2\Sigma_{+}(t)} \right] + \frac{\sigma^2_2(t)}{\sqrt{\Sigma_{-}(t)}} \exp \left[ -\frac{(x+\Sigma_{-}(t)/2)^2}{2\Sigma_{-}(t)} \right].
\]

Since $\sigma_{loc} = \sigma_{loc}(s, x)$ is bounded, measurable in $t$ and Lipschitz in $x$ (uniformly w.r.t. $t$) and bounded away from zero it follows from [SV06] Theorem 5.1.1 that the SDE

\[
d\tilde{X}_t = \sigma_{loc} \left( t, \tilde{X}_t \right) dB_t - \frac{1}{2} \sigma^2_{loc} \left( t, \tilde{X}_t \right) dt
\]

5More general expression for local volatility are found in [BM06] Chapter 4 and [Lee01, HL09]. Note the necessity to keep $\sigma^2(\cdot, \omega)$ constant on some interval $[0, \epsilon]$, for otherwise the local volatility surface is not Lipschitz in $x$, uniformly as $t \to 0$. 

has a unique strong solution (started from \( \tilde{X}_0 = 0 \), say). Since \( \sigma_{\text{loc}} \) is uniformly bounded away from 0 it follows that the process \( (\tilde{X}_t) \) has full support, i.e. for every continuous \( \varphi : [0, T] \to \mathbb{R} \), \( \varphi(0) = 0 \) and every \( \varepsilon > 0 \)

\[
P[\|\tilde{X}_t - \varphi(t)\|_{[0,T]} \leq \varepsilon] > 0.
\]

Indeed, there are various ways to see this: one can apply Stroock–Varadhan’s support theorem, in the form of [Pin95, Theorem 6.3] (several simplifications arise in the proof thanks to the one-dimensionality of the present problem); alternatively, one can employ localized lower heat kernel bounds (à la Fabes–Stroock [FS86]) or exploit that the Itô-map is continuous here (thanks to Doss–Sussman, see for instance [RW00, page 180]) and deduce the support statement from the full support of \( B \).

![Figure 1. Time evolution of local variance \( \sigma_{\text{loc}}^2(t, x) \) in dependence of log-moneyness. The bright strip indicates a set of paths with realized variance strictly larger than 6.](image)

Figure 1 illustrates the dependence of \( \sigma_{\text{loc}}^2(t, x) \) on time \( t \) and log-moneyness \( x \). To gain our end of proving that \( \tilde{V}(\omega) = \int_0^3 \sigma_{\text{loc}}^2(t, \tilde{X}_t) \, dt \) is not constantly equal 6, we can determine a set of paths \( (\tilde{X}_t(\omega)) \) for which \( \tilde{V} \) is strictly larger than 6. In view of Figure 1 it is natural to consider paths which are large, i.e. \( \tilde{X}_t(\omega) \in [8, 10] \), for \( t \in ]1, 2 - \frac{1}{10} \) and small, i.e. \( |\tilde{X}_t(\omega)| \leq 1 \), on the interval \( ]2, 3] \). A short mathematica-calculation reveals that \( \tilde{V}(\omega) \geq 6.65 > 6 \) for each such path and according to the full-support statement the set of all such paths has positive probability, hence \( \tilde{V} \) is indeed not deterministic.

Using elementary analysis it is not difficult to turn numerical evidence into rigorous mathematics. Making (3.1) explicit yields that \( \sigma_{\text{loc}}^2(t, x) \equiv 2 \) for \( t \in [0, 1] \) and that

\[
\sigma_{\text{loc}}^2(t + 1, x) = \frac{3}{\sqrt{2+3t}} e^{(2x+2+3t)^2/8(2+3t)} + \frac{1}{\sqrt{2+3t}} e^{(2x+2+1)^2/8(2+1)}
\]

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\[
\sigma_{\text{loc}}^2(t + 1, x) = \frac{1}{\sqrt{2+3t}} e^{-(2x+2+3t)^2/8(2+3t)} + \frac{1}{\sqrt{2+3t}} e^{-(2x+2+1)^2/8(2+1)}
\]
whereafter elliptic regularization. That is, we consider \( (\text{non deterministic}) \hat{\sigma} \) by positive constants. We would like to apply theorem 1 to the 2D diffusion \((X_t, V_t)\) for every path \( \hat{X}_t(\omega) \) satisfying \( \hat{X}_t(\omega) > \frac{1}{\varepsilon} \) for \( t \in [1 + \varepsilon, 2 - \varepsilon] \) and \( |\hat{X}_t(\omega)| < \delta \) for \( t \in [2, 3] \). This set of paths \( \hat{X}_t(\omega) \) has positive probability and the quantity on the right side of (3.4) is strictly larger than 6 provided that \( \varepsilon \) was chosen sufficiently small. Hence we find that \( \hat{V} \) is not constantly equal to 6 as required.

For what it’s worth, the example can be modified such that volatility is adapted to the filtration of the driving Brownian motion.

The trick is to choose a random sign \( \hat{\varepsilon} \), \( P(\hat{\varepsilon} = +1) = P(\hat{\varepsilon} = -1) = \frac{1}{2} \) depending solely on the behavior of \((B_t)_{0 \leq t \leq 1}\) and in such a way that \( S_1 \) is independent of \( \hat{\varepsilon} \). For instance, if we let \( m(s) = \frac{1}{2} \) for \( s \) satisfying \( P(S_{t/2} > m(s)) S_1 = s \) for \( S_{t/2} \) and \( S_1 \) are equivalent in law. It follows that \( \hat{\varepsilon} := +1 \) if \( S_{t/2} > m(S_1) \) and \( \hat{\varepsilon} := -1 \) otherwise.

We then leave the stock price process unchanged on \([0, 1]\), i.e. we define \( \hat{\sigma}^2(t) := \sigma^2(t) = 2 \) and \( \hat{S}_{t} = \hat{S}_{t} \) for \( t \in [0, 1] \). On \([1, 2] \) resp. \([2, 3] \) we set \( \hat{\sigma}^2(t) := 2 + \hat{\varepsilon} \) resp. \( \hat{\sigma}^2(t) := 2 - \hat{\varepsilon} \) and define \( \hat{S}_{t}, t \in [1, 3] \) as the solution of the SDE

\[
\begin{align*}
\frac{d\hat{S}_t}{\hat{S}_t} &= \hat{\sigma}(t, \omega) \frac{dB_t}{\hat{S}_t}, \quad \hat{S}_1 = 1.
\end{align*}
\]

Here (3.5) depends only on \( \hat{S}_1 \) and the process \((B_t - B_1)_{1 \leq t \leq 3}\); since both are independent of \( \hat{\varepsilon} \), we obtain that \((\hat{S}_t)_{1 \leq t \leq 3}\) and \((S_t)_{1 \leq t \leq 3}\) are equivalent in law. It follows that \( \hat{V} = \int_0^3 \hat{\sigma}^2(t, \omega) dt \equiv 6 \) and since \( \hat{S}_t \) and \( S_t \) have the same law for each \( t \in [0, 3] \), they induce the same local volatility model and in particular the same (non deterministic) \( \hat{V} \).

4. **Counterexample for a Markovian stochastic volatility model**

Recall that \( X \) denotes the log-price process of a general stochastic volatility model;

\[
\begin{align*}
\frac{dX_t}{X_t} &= \sigma(t, \omega) \frac{dB_t - \langle \sigma^2(t, \omega) / 2 \rangle dt}{X_t},
\end{align*}
\]

where \( \sigma = \sigma(t, \omega) \) is the (progressively measurable) instantaneous volatility process. Recall also our standing assumption that \( \sigma \) is bounded from above and below by positive constants. We would like to apply theorem 1 to the 2D diffusion \((X_t, V_t)\) where \( dV = \sigma^2 dt \) keeps track of the running realized variance. We can only do so after elliptic regularization. That is, we consider

\[
\begin{align*}
\frac{dX_t}{X_t} &= \sigma(t, \omega) \frac{dB_t - \langle \sigma^2(t, \omega) / 2 \rangle dt}{X_t},
\end{align*}
\]

\[
\begin{align*}
\frac{d\sigma_t}{\sigma_t} &= \sigma^2(t, \omega) dt + \varepsilon^{1/2} dZ_t.
\end{align*}
\]

\[\text{We note that we intend to insert later the volatility of example } \]_6\_ but so far our considerations hold in general.

\[\text{In other words, } \int_0^T \sigma^2(t, \omega) dt, \text{ if } V_0 = 0 \text{ which we shall assume from here on.} \]_7\_
where $Z$ is a Brownian motion, independent of of the filtration generated by $B$ and $\sigma$. It follows that the following “double-local” volatility model
\[
\begin{align*}
d\tilde{X}^\varepsilon_t &= \sigma^2_{\text{loc}}(t, \tilde{X}^\varepsilon_t, \tilde{a}^\varepsilon_t) d\tilde{B}_t - \left(\sigma^2_{\text{loc}}(t, \tilde{X}^\varepsilon_t, \tilde{a}^\varepsilon_t)/2\right) dt, \\
d\tilde{a}^\varepsilon_t &= \sigma^2_{\text{loc}}(t, \tilde{X}^\varepsilon_t, \tilde{a}^\varepsilon_t) dt + \varepsilon^{1/2} d\tilde{Z}_t,
\end{align*}
\]
(with $\sigma^2_{\text{loc}}(t, x, a) = E[\sigma^2(t, \omega) | X_t = x, a^2 = a]$) has the one-dimensional marginals of the original process $(X_t, a^2_t)$. That is, for all fixed $t$ and $\varepsilon$,
\[
X_t \overset{\text{law}}{=} \tilde{X}_t^\varepsilon \quad \text{and} \quad \tilde{a}_t^\varepsilon \overset{\text{law}}{=} a^2_t.
\]
Let us also note that the law of $a^2_t$ is the law of $V_t = a^2_0$ convolved with a standard Gaussian of mean 0 and variance $\varepsilon$. The log-price processes $X$ and $\tilde{X}^\varepsilon$ induce the same local volatility surface. To this end, just observe that $X_t \overset{\text{law}}{=} \tilde{X}_t^\varepsilon$ implies identical call option prices for all strikes and maturities and hence (by Dupire’s formula) the same local volatility:
\[
\sigma^2_{\text{loc}}(t, x) = E[\sigma^2(t, \omega) | X_t = x] = E[\sigma^2_{\text{loc}}(t, \tilde{X}^\varepsilon_t, \tilde{a}^\varepsilon_t) | \tilde{X}^\varepsilon_t = x].
\]
Since the law of a time inhomogeneous Markov process is fully specified by its generator, it follows that the law of the local volatility process associated to ($X$) has the same law as the local volatility process associated to ($\tilde{X}^\varepsilon$).

We apply this to the toy model discussed earlier. Recall that in this example, with $T = 3$
\[
V_T = \int_0^T \sigma^2(t, \omega) dt = 6
\]
whereas realized variance under the corresponding local volatility model,
\[
\tilde{V}_T = \int_0^T \sigma^2_{\text{loc}}(t, \tilde{X}_t) dt
\]
was seen to be random (but with mean $V_T$, thanks to the matching variance swap prices). As a particular consequence, using Jensen
\[
E\left( \int_0^T \sigma^2_{\text{loc}}(t, \tilde{X}_t) dt - 6 \right)^+ > \left( E \int_0^T \sigma^2_{\text{loc}}(t, \tilde{X}_t) dt - 6 \right)^+ = \left( E \int_0^T \sigma^2(t, \omega) dt - 6 \right)^+ = (V_T - 6)^+ = 0.
\]
We claim that this persists when replacing the abstract stochastic volatility model ($X$) by ($\tilde{X}^\varepsilon$), the first component of a 2D Markov diffusion, for any $\varepsilon > 0$. Indeed, thanks to the identical laws of the respective local volatility processes the left-hand side above does not change when replacing ($\tilde{X}$) by the local volatility process associated to ($\tilde{X}^\varepsilon$). On the other hand
\[
E \int_0^T \sigma^2_{\text{loc}}(t, \tilde{X}^\varepsilon_t, \tilde{a}^\varepsilon_t) dt = E(\tilde{a}^\varepsilon_T - \varepsilon^{1/2} Z_T)
\]
\[
= E(\tilde{a}^\varepsilon_T) = E(a^2_T) = E(V_T + \varepsilon^{1/2} Z_T) = V_T.
\]
Thus, insisting again that the process $\tilde{X}$ is (in law) the local volatility model associated to the double local volatility model ($\tilde{X}^\varepsilon, \tilde{a}^\varepsilon$) we see that
\[
c = E\left( \int_0^T \sigma^2_{\text{loc}}(t, \tilde{X}_t) dt - 6 \right)^+ > \left( E \int_0^T \sigma^2_{\text{loc}}(t, \tilde{X}^\varepsilon_t, \tilde{a}^\varepsilon_t) dt - 6 \right)^+ = 0.
\]
(Observe that $c > 0$ is independent of $\epsilon$.) Using the Lipschitz property of the hockeystick function, again Gyöngy and the fact that $a_T^\epsilon$ is normally distributed with mean $V_T$ and variance $\epsilon T$ we can conclude that

$$E\left( \int_0^T \sigma^2_{\text{dloc}}(t, \tilde{X}_t^\epsilon, \tilde{a}_t^\epsilon) dt - 6 \right) = E\left( \int_0^T \sigma^2_{\text{dloc}}(t, \tilde{X}_t^\epsilon, \tilde{a}_t^\epsilon) dt + \epsilon^{1/2} Z_T - 6 - \epsilon^{1/2} Z_T \right)$$

$$\leq E\left( (\tilde{a}_T^\epsilon - 6)^+ + |\epsilon^{1/2} Z_T| \right)$$

$$= E(\tilde{a}_T^\epsilon - 6)^+ + E[\epsilon^{1/2} Z_T] = 3\sqrt{\epsilon T / 2\pi}.$$  

Now we choose $\epsilon$ small enough such that $3\sqrt{\epsilon T / 2\pi} < c$, whence the conjecture fails to hold true in the double local volatility model for $\epsilon > 0$ small enough.

5. Conclusion

Summing up, the double-local volatility model constitutes an example of a continuous 2D Markovian stochastic volatility model, where stochastic volatility is a function of both state variables, in which conjecture fails, i.e. in which the minimal possible value of a call option is not generated by a local volatility model.

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