Junction equations for two spherically symmetric spacetimes and the distributional method

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Abstract
Applying the distributional formalism to study the dynamics of thin shells in general relativity, we regain the junction equations for matching of two spherically symmetric spacetimes separated by a singular hypersurface. In particular, we have shown how to define and insert the relevant sign functions in the junction equations corresponding to the signs of the extrinsic curvature tensor occurred in the Darmois–Israel method.
1 Introduction

Recently a distributional method has been developed to solve directly the Einstein’s field equations for thin shells embedded in an arbitrary space–time[1]. This method requires construction of space–time coordinates that match continuously on the shell and in which the four–metric based on the Lichnerowicz condition [2], is continuous, but has a finite jump in its first derivatives on the shell, so that its curvature tensor will contain a Dirac delta function.

So far the spherical thin shells in static spherically symmetric spacetimes and the case of cylindrically symmetric thin layers have been solved by this method[3,4]. In addition, an explicit formulation of distributional method for handling nonlightlike surface layers and its equivalence to the jump conditions of Darmois–Israel method through the analysis of the Bianchi identities has been represented [1].

On the other hand, it has been shown in the last years that in the framework of Darmois–Israel formalism, the sign of the angular component of the extrinsic curvature tensor of the shell, namely \( K^\theta_\theta \), play a key role in the classification of the global spherically symmetric space–time structures, as mentioned by Sato [5], Berezin, Kuzmin, Tkachev [6] and Sakai and Maeda [7]. This sign function gives us information related to increase or decrease of the relevant coordinate in the direction normal to the shell. In the distributional method of Mansouri–Khorrami mentioned above this sign function, in spite of its importance, has not yet been introduced. The aim of this paper is to show how one can generate the necessary sign functions in the framework of Mansouri–Khorrami distributional method. In this way we will show the full equivalence of the sign functions introduced and their role in the junction equations with the corresponding signs of the extrinsic curvature \( K^\theta_\theta \) on the both sides of the shell in the Darmois–Israel approach.

The paper is arranged as follows. In section 2 we review shortly the distributional method for thin shells due to Mansouri and Khorrami. In section 3 this distributional formalism is applied to the junction of two Schwarzchild-de Sitter space–times through a timelike spherical thin shell. Section 4 is devoted to the junction equations for two FRW space-times bounded by a nonlightlike spherical thin shell.

Conventions and definitions: We use the signature \((+−−−)\) and put \( c = 1 \). Greek indices run from 0 to 3. Overdot denotes differentiation with respect to the shell proper time \( \tau \). The square brackets, \([F]\), are used to indicate the jump of any quantity \( F \) across the shell. As we are going to work with distributional valued tensors, there may be terms in a tensor quantity \( F \) proportional to some \( \delta \)–function. These terms are indicated by \( \tilde{F} \).

2 Distributional Method

Consider a space–time manifold \( M \) consisting of overlapping domains \( M_+ \) and \( M_- \) with metrics \( g^{\alpha\beta}_+(x^\mu_+) \) and \( g^{\alpha\beta}_-(x^\mu_-) \) in terms of independent disconnected charts \( x^\mu_+ \) and \( x^\mu_- \), respectively. The common boundary of the domains is denoted by \( \Sigma \). In other words, the Manifolds \( M_+ \) and \( M_- \) are glued together along the common boundary \( \Sigma \). The equation
of $\Sigma$ is written as $\phi(x^\mu) = 0$, where $\phi$ is a smooth function, and $x^\mu$ is a single chart called admissible coordinate system (e.g., skew-Gaussian coordinates attached to geodesics) that covers the overlap and reaches into both domains. The domains of $M$ in which $\phi$ is positive or negative are contained in $M_+$ or $M_-$, respectively [8]. By applying the coordinate transformations $x_\pm^\mu = x_\pm^\mu(x^\nu)$ on the corresponding domains, a pair of metrics $g_{\alpha\beta}^+(x^\mu)$ and $g_{\alpha\beta}^-(x^\mu)$ is formed over $M_+$ and $M_-$ respectively, each suitably smooth (say $C^3$).

The main step in the distributional approach is the definition of a hybrid metric $g_{\alpha\beta}(x^\mu)$ over $M$ which glues the metrics $g_{\alpha\beta}^+(x^\mu)$ and $g_{\alpha\beta}^-(x^\mu)$ together continuously on $\Sigma$:

$$g_{\alpha\beta} = g_{\alpha\beta}^+\theta(\phi) + g_{\alpha\beta}^-\theta(-\phi),$$

(1)

where $\theta$ is the Heaviside step function and

$$[g_{\alpha\beta}(x^\mu)] = 0.$$  

(2)

We expect on $\Sigma$ the curvature and Ricci tensor to be proportional to $\delta$ function. It follows from (1) and (2) that the first derivative of $g_{\alpha\beta}$ is proportional to the step function. The $\delta$ distribution can only occur in the second derivative of the metric which enters linearly in the expressions for curvature and Ricci tensor. So the only relevant terms in the Ricci tensor are[1]

$$\tilde{R}_{\mu\nu} = \tilde{\Gamma}_{\rho\mu,\nu} - \tilde{\Gamma}_{\mu\nu,\rho}.  

(3)$$

Using the metric in the form (1), we finally arrive at the following expression for the components of the Ricci tensor proportional to $\delta$ distribution [1, see also 8 and 9]

$$\tilde{R}_{\mu\nu} = \left(\frac{1}{2} g [g_{,\nu} \partial_\mu \phi - [\Gamma^\rho_{\mu\nu}] \partial_\rho \phi] - \delta(\phi),

(4)$$

where $g$ is the determinant of the metric and the partial derivatives are done with respect to the coordinates $x^\mu$. The Einstein’s equations for the dynamics of the singular hypersurface or thin shell $\Sigma$ is then conveniently written as

$$\tilde{R}_{\mu\nu} = -8\pi G (\tilde{T}_{\mu\nu} - \frac{1}{2} \tilde{T} g_{\mu\nu}).$$

(5)

The energy–momentum tensor of the shell $\tilde{T}_{\mu\nu}$, considered as a distribution, is given by [1,8]

$$\tilde{T}_{\mu\nu} = |\alpha| S_{\mu\nu} \delta(\phi),$$

(6)

where $S_{\mu\nu}$ is the surface 4-tensor of energy-momentum of the shell, and $\alpha$ is related to the unit normal four-vector $n^\mu$ of the shell:

$$n^\mu = \alpha^{-1}\partial_\mu \phi,$$

(7)
with
\[ \alpha = \pm \sqrt{|g^{\nu\sigma} \partial_\nu \phi \partial_\sigma \phi|}. \] (8)

It is convenient to choose the negative (positive) sign in (8) for time–(space–)like \( \Sigma \) [1,6,8].

In this way the unit normal vector \( n_\mu \) is always directed from \( M^- \) to \( M^+ \). Now, we require that the admissible coordinates be such that the vector \( n_\mu \) is in the direction of increasing a space– or time–like admissible coordinate \( x^\mu \) corresponding to time– or space–like \( \Sigma \).

This is no restriction to the choice of the admissible coordinates, as we are always free to choose \( M^+ \) instead of \( M^- \) or vise versa. Note that in general, \( n^\mu \) may point towards greater or smaller values of a space– or time–like coordinate \( x^\mu \), which is the case in some of the spherically symmetric examples we are going to consider. There we will see the crucial role of these definitions and their leading to the sign function needed to glue different manifolds.

In the following we restrict ourselves to timelike hypersurfaces. The case of spacelike hypersurfaces is very similar and will be treated in the appendix. Having a consistent definition of admissible coordinates and the direction of \( n^\mu \), we will look for a sign function similar to that in the Darmois–Israel method and crucial for the topological classification of manifolds to be glued together. This can best be seen in the case of spherically symmetric space times we are going to consider in this paper. Assume a congruence of timelike hypersurfaces parallel to \( \Sigma \). Let \( \bar{R}_\pm \) be the physical radius of the 2–D spheres parallel to \( \Sigma \) in \( M^\pm \). Moving in the direction of \( n^\mu \), each of the \( \bar{R}_\pm \) may either increase or decrease, depending on the non–staticity and topology of \( M^\pm \). Therefore, we may define the following sign functions:
\[ \epsilon_\pm = \text{sgn} \left( n^\mu \partial_\mu \bar{R}_\pm \right) \bigg|_\Sigma, \] (9)

where \( \epsilon_\pm \) take the values +1 or −1 accordingly as \( \bar{R}_\pm \) increase or decrease along the normal vector \( n^\mu \) directed from \( M^- \) to \( M^+ \). We will see in the next sections that this sign functions correspond to that introduced in Darmois–Israel method related to the sign of \( K_\theta^\pm \) [7, 8, 10].

Now, the boundary \( \Sigma \) maybe just a singular hypersurface or a thin layer. In general, we may therefore assume for the boundary a surface energy–momentum tensor \( S^{\mu\nu} \) of a perfect fluid type given by
\[ S^{\mu\nu} = \sigma u^\mu u^\nu + w (h^{\mu\nu} - u^\mu u^\nu), \] (10)

where \( u^\mu \) is the unit four-velocity of any observer whose world line lies within the shell; \( \sigma \) and \( \omega \) are respectively the surface-energy density and tension measured by that observer. \( h_{\mu\nu} \) denotes the induced three-metric on \( \Sigma \) and is written as
\[ h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \] (11)

We are now ready to apply the above formalism to obtain the junction conditions in spherically symmetric spacetimes, bounded by a timelike shell. The corresponding junction equations for spacelike shells will be given in the Appendix.
3 Junction equations in the static spherically symmetric space-times

Consider a spherical thin shell with a (2+1)-dimensional timelike history $\Sigma$ in static spherically symmetric spacetimes on both sides described by the Schwarzschild-de Sitter metrics given in the from

$$ds^2|_\pm = f_\pm(r_\pm)dt_\pm^2 - f_\pm^{-1}(r_\pm)dr_\pm^2 - r_\pm^2d\Omega^2,$$

(12)

with

$$f_\pm(r_\pm) = 1 - \frac{2Gm_\pm}{r_\pm} - \frac{\Lambda_\pm r_\pm^2}{3},$$

(13)

where $m_\pm$ and $\Lambda_\pm$ are the mass parameters and the cosmological constants associated with $M_\pm$. The equation of the spherical timelike shell can be represented as

$$\phi(x^\mu) = r - R(t) = 0,$$

(14)

where $R(t)$ is the radius of the shell as a function of the relevant timelike coordinate $t$.

Now, we apply the following transformations to make the metric continuous on the shell

$$r_+ = A(r, t), \quad r_- = C(r, t),$$

$$t_+ = B(r, t), \quad t_- = D(r, t).$$

(15)

Carrying out the transformations and requiring the continuity of the metric on $\Sigma$ according to (2), we obtain

$$\begin{align*}
U &\equiv f_+B_r^2 - f_+^{-1}A_r^2 \overset{\Sigma}{=} f_-D_r^2 - f_-^{-1}C_r^2, \\
V &\equiv f_+B_rB_t - f_+^{-1}A_rA_t \overset{\Sigma}{=} f_-D_tD_t - f_-^{-1}C_tC_t, \\
W &\equiv f_+B_r^2 - f_+^{-1}A_r^2 \overset{\Sigma}{=} f_-D_r^2 - f_-^{-1}C_r^2, \\
A(R(t), t) &= C(R(t), t) = R(t),
\end{align*}$$

(16)

where $\overset{\Sigma}{=} \Sigma$ means that both sides of the equality are evaluated on $\Sigma$. The sign functions $\epsilon_+$ and $\epsilon_-$ as defined by (9) take the forms

$$\begin{align*}
\epsilon_+ &= \text{sgn}(n^\mu \partial_\mu A)\big|_{\Sigma}, \\
\epsilon_- &= \text{sgn}(n^\mu \partial_\mu C)\big|_{\Sigma},
\end{align*}$$

(17)

where $n_\mu$ defined by (7) is given by

$$n_\mu = |Y|(\dot{R}, -\dot{t}, 0, 0)\big|_{\Sigma},$$

(18)
with
\[ Y^2 = V^2 - UW. \] (19)

Using (16) and (18) we obtain finally
\[
\begin{align*}
\epsilon_+ &= \zeta_+ \text{sgn}(f_+ \dot{B})_{\Sigma'}, \\
\epsilon_- &= \zeta_- \text{sgn}(f_- \dot{D})_{\Sigma'},
\end{align*}
\] (20, 21)

where \( \zeta_{\pm} \) are
\[
\begin{align*}
\zeta_+ &= \text{sgn}(B_t A_r - B_r A_t)_{\Sigma'}, \\
\zeta_- &= \text{sgn}(D_t C_r - D_r C_t)_{\Sigma'}.
\end{align*}
\] (22, 23)

Comparing (20) and (21) with the signs of \( K_\theta^{\theta\pm} \) obtained by the Darmois-Israel method, we can see that \( \zeta_{\pm} \) generated by the transformations (15) in the distributional method, are equivalent to the undetermined signs of the normals \( n_{\mu}^{\pm} \) defined in the Darmois-Israel approach [6,7,10]. It is assumed that \( t_{\pm} \) and \( \tau \), the shell proper time, are future directed, so that \( \dot{B} \) and \( \dot{D} \) are positive. Then in the \( R \) region mentioned by Berezin et al [6,11], where \( f_\pm |_{\Sigma} > 0 \), the sign factors \( \zeta_{\pm} \) differentiate the interior-exterior characters of \( M_{\pm} \) [7,10], and the sign functions \( \epsilon_{\pm} \) are just determined by \( \zeta_{\pm} \) according to (20) and (21). Thus the global topology of static spherically symmetric space-times is determined by \( \epsilon_{\pm} \): if \( \epsilon_+ = \epsilon_- \), then we have an ordinary centered shell (black hole type matching); if \( \epsilon_- = -1 \) and \( \epsilon_+ = +1 \), then we have a shell in a space-time with no center (wormhole matching); if \( \epsilon_- = +1 \) and \( \epsilon_+ = -1 \), then the shell is in a space-time with two centers (anti-wormhole matching)[10–13]. Note that these matchings can best be visualized in terms of Kruskal coordinates. Section I of Kruskal diagram corresponds to \( \epsilon = +1 \) and section IV corresponds to \( \epsilon = -1 \). Therefore an ordinary centered shell corresponds to matching of section I or two section IV manifolds. It is also possible to glue different part of section I to IV or vise versa which leads to other possibilities discussed above [11]. We restrict now, without loss of generality of the matching, the transformation (15) on the chart \( x^\mu_- \) to the conditions
\[
t_- = t, \quad C_r |_{\Sigma} = \zeta_-.
\] (24)

This allows us to generate the sign function \( \zeta \) properly. Note that in the original distributional formalism of Mansouri-Khorrami it was set \( \zeta_- = 1 \), which means no coordinate transformation of the chart \( x^\mu_- \) [3, 4]. This was a restriction to the topology of the manifold \( M^- \). Using (24), the set of equations (16) can be solved for the unknown functions in
terms of \( \dot{B}, i \) and \( \dot{R} \):

\[
\begin{align*}
C_{,t} |_{\Sigma} &= \frac{\dot{R}}{t} (1 - \zeta_-) |_{\Sigma}, \\
A_{,r} |_{\Sigma} &= -\zeta_- f_-^{-1} \dot{R}^2 + \zeta_+ f_+ \dot{B} i |_{\Sigma}, \\
A_{,t} |_{\Sigma} &= \frac{\dot{R}}{t} + \zeta_- f_-^{-1} \frac{\dot{R}^2}{t^2} - \zeta_+ f_+ \dot{B} \dot{R} |_{\Sigma}, \\
B_{,r} |_{\Sigma} &= -\zeta_- f_-^{-1} \dot{B} \dot{R} + \zeta_+ f_+^{-1} \dot{R} i |_{\Sigma}, \\
B_{,t} |_{\Sigma} &= \frac{\dot{R}}{t} + \zeta_- \frac{\dot{B} \dot{R}}{f_-} - \zeta_+ f_+^{-1} \dot{R}^2 |_{\Sigma}.
\end{align*}
\]  

(25)

where for a given \( \zeta_- \), the same sign factor \( \zeta_+ = \pm 1 \) takes care of the two possible solutions for \( A_{,r}, A_{,t}, B_{,r} \) and \( B_{,t} \) in (16). It is easily seen from (25) that the sign of \( (B_{,t} A_{,r} - B_{,r} A_{,t}) |_{\Sigma} \) is determined by \( \zeta_+ \) independent of the sign factor \( \zeta_- \), in accordance with (22). It is also seen that the metric \( g_{\alpha \beta}^{\pm}(x^\nu) \) on \( \Sigma \) takes a diagonal form \( (C_{,t} = 0) \) for static shells \( (\dot{R} = 0) \) or when \( M_\perp \) has a center \( (\zeta_- = +1) \).

We would like to note that in principle one may choose the restriction \( r_- = r \) and \( D_{,t} |_{\Sigma} = \zeta_- \) with \( \dot{D} = i \) instead of (24). This, however, leads to \( D_{,r} |_{\Sigma} = \frac{i}{R} (1 - \zeta_-) |_{\Sigma} \) which becomes divergent for static shells \( (\dot{R} = 0) \).

To proceed further, we need the derivatives of the time coordinates on both sides of the shell with respect to its proper time, \( \tau \). It is easily seen that

\[
\dot{B} = f_+^{-1} \sqrt{f_+ + \dot{R}^2} |_{\Sigma}, \quad i = f_-^{-1} \sqrt{f_- + \dot{R}^2} |_{\Sigma}.
\]  

(26)

\( \text{From (4) we obtain the nonzero components of Ricci tensor:} \)

\[
\begin{align*}
\dot{R}_{22} &= R \left( \zeta_- \frac{\dot{R}}{t f_-} A_{,t} + f_- A_{,r} - \zeta_- f_- - \frac{\dot{R}^2}{t^2 f_-} A_{,r} (1 - \zeta_-) \right) |_{\Sigma}, \\
\dot{R}_{00} &= 2 \frac{\dot{R}^2}{R^2} \dot{R}_{22} - \frac{U}{2 \dot{R} t} \left( f_- t i^2 - f_-^{-1} \dot{R}^2 - 2 \frac{\dot{R}}{f_-} \frac{d C_{,t}}{dt} - f_+ i \dot{B}^2 + f_+ i \dot{R}^2 - 2 f_+ B \frac{dB_{,t}}{dt} + \frac{2 \dot{R}^2}{f_+} \frac{d A_{,t}}{dt} \right) |_{\Sigma}, \\
\dot{R}_{10} &= \dot{R}_{01} = \frac{V}{U} \dot{R}_{00} - 2 \frac{\dot{R}^2}{R^2} \left( \dot{R} i + \frac{V}{U} \dot{R}^2 \right) \dot{R}_{22}, \\
\dot{R}_{11} &= \frac{W}{U} \dot{R}_{00} + 2 \frac{\dot{R}^2}{R^2} \left( i^2 - \frac{W}{U} \dot{R}^2 \right) \dot{R}_{22}.
\end{align*}
\]  

(27)-(30)
\[ R_{33} = \sin^2 \theta R_{22}. \]  

(31)

Using the definition (9) and the unit timelike vector field \( u^\mu \) on \( \Sigma \),

\[ u^\mu = (t, \dot{R}, 0, 0), \]

(32)

we obtain \( \tilde{T}_{\mu\nu} \) for the nondiagonal metric \( g^{\pm}_{\mu\nu}(x^\sigma) \) on the shell:

\[
\tilde{T}_{\mu\nu} = \begin{bmatrix}
\frac{\sigma(Ui+V\dot{R})^2}{t} & \frac{\sigma(Ui+V\dot{R})(Vi+W\dot{R})}{t} & 0 & 0 \\
\frac{\sigma(Vi+W\dot{R})^2}{t} & \frac{\sigma(Vi+W\dot{R})}{t} & 0 & 0 \\
0 & 0 & -\omega R^2 & 0 \\
0 & 0 & 0 & -\frac{wR^2\sin^2 \theta}{t}
\end{bmatrix} \bigg|_\Sigma. \]

(33)

The junction equations and dynamics of the singular hypersurface is given by the equation (5). In our case, this reduces to the following two independent equations:

\[
\tilde{R}_{22} = -8\pi G \tilde{P}_{22},
\]

(34)

\[
\tilde{R}_{00} - \frac{2\dot{R}^2}{r_-^2} \bigg|_\Sigma \tilde{R}_{22} = -8\pi G \left( \tilde{P}_{00} - \frac{2\dot{R}^2}{r_-^2} \bigg|_\Sigma \tilde{P}_{22} \right),
\]

(35)

where \( \tilde{P}_{\mu\nu} \) denotes the right hand side of (5):

\[
\tilde{P}_{\mu\nu} = \tilde{T}_{\mu\nu} - \frac{1}{2} \tilde{T} g_{\mu\nu}.
\]

These are equivalent to the set of equations (2.59a) and (2.59b) in Ref. [6]. Making use of relations obtained so far, we obtain after some manipulations the junction equations in the final form:

\[
\epsilon_- \sqrt{f_- + \dot{R}^2} - \epsilon_+ \sqrt{f_+ + \dot{R}^2} \equiv 4\pi G\sigma R,
\]

(36)

\[
\frac{d}{d\tau} \left( \epsilon_+ \sqrt{f_+ + \dot{R}^2} - \epsilon_- \sqrt{f_- + \dot{R}^2} \right) \equiv 8\pi G \dot{R} \left( \frac{\sigma}{2} - w \right),
\]

(37)

where we have inserted the sign functions \( \epsilon_\pm \) instead of \( \zeta_\pm \) according to (20) and (21) for static spherically symmetric spacetimes. These junction equations are equivalent to those obtained by Berezin et al [6].

### 4 Junction equations for FRW space-times

Consider a spherical thin shell with timelike history \( \Sigma \) as the boundary of two Friedmann-Robertson-Walker spacetimes with metrics given by

\[
\left. ds^2 \right|_\pm = dt^2 \pm a^2_\pm(t_\pm) \left[ d\chi^2_\pm + r^2_\pm(\chi_\pm) d\Omega^2 \right],
\]

(38)
where

\[
\begin{align*}
  r(\chi) &= \begin{cases} 
    \sin \chi & (k = +1, \text{ closed universe}), \\
    \chi & (k = 0, \text{ flat universe}), \\
    \sinh \chi & (k = -1, \text{ open universe}).
  \end{cases}
\end{align*}
\]

We now apply the following transformations to make the four–metric continuous on \( \Sigma \):

\[
\begin{align*}
  \chi_+ &= A(r, t), & \chi_- &= C(r, t), \\
  t_+ &= B(r, t), & t_- &= D(r, t).
\end{align*}
\]

According to (2), we get

\[
\begin{align*}
  \left\{ \begin{array}{l}
    B_+^2 - a_+^2 A_+^2 \equiv D_+^2 - a_+^2 C_+^2, \\
    B_- B_+ - a_+^2 A_+ A_- \equiv D_- D_+ - a_+^2 C_+ C_-,, \\
    B_-^2 - a_-^2 A_-^2 \equiv D_-^2 - a_-^2 C_-^2, \\
    l \equiv a_+(B(R(t, t))r_+(A(R(t, t)) = a_- (D(R(t, t))r_- (C(R(t, t)),
  \end{array} \right.
\]

where \( \phi(x^\mu) = r - R(t) = 0 \) is the equation of the shell and \( l \) represents the physical radius of it.

The sign functions \( \epsilon_+ \) and \( \epsilon_- \) as defined by (9) are given by

\[
\begin{align*}
  \epsilon_+ &= sgn \left( n^\mu \partial_\mu (a_+ r_+) \right) |_{\Sigma}, \\
  \epsilon_- &= sgn \left( n^\mu \partial_\mu (a_- r_-) \right) |_{\Sigma}.
\end{align*}
\]

Using \( n_\mu \) as obtained in (18) we obtain after some manipulations

\[
\begin{align*}
  \epsilon_+ &= \zeta_+ sgn \left( \frac{dr_+}{d\chi_+} + lH_+ \frac{a_+ \dot{A}}{B} \right) |_{\Sigma}, \\
  \epsilon_- &= \zeta_- sgn \left( \frac{dr_-}{d\chi_-} + lH_- \frac{a_- \dot{C}}{D} \right) |_{\Sigma},
\end{align*}
\]

where \( H_\pm \) are the Hubble parameters for \( M_\pm \), and the sign factors \( \zeta_+ \) and \( \zeta_- \) are the same as that given by (22) and (23) respectively, differentiating the interior-exterior characters of a flat or open universe (for a closed universe one can change the signs of \( \zeta_\pm \) by the coordinate transformations \( \chi_\pm \rightarrow \pi - \chi_\pm \), see Ref.[10]). Note that these relations correspond to Eq.(5) in Ref. [7].

Similarly, we have assumed \( t_\pm \) and \( \tau \) to be future directed. For given \( \zeta_\pm \) and \( k_\pm \), one can determine the topology of FRW space times regardless of the signs of \( \epsilon_\pm \). (A list of possible topology types has been given in Ref. [7]). In other words, we may have any sign of \( \epsilon_\pm \) for any known value of \( \zeta_\pm \) and \( k_\pm \). Particularly for \( k_+ \leq 0 \) and \( \zeta_+ > 0 \), the sign of \( \epsilon_+ \) can be negative, if the comoving radius of the shell decreases in time, namely if \( \dot{A} < 0 \) or equivalently, if the peculiar velocity of the shell \( v_{pe}^+ = a_+ \frac{\dot{A}}{B} \mid_{\Sigma} \) observed in \( M_+ \) is negative. Therefore, when the second term in the expression within the brackets in (43),...
Again, without loss of generality, we may restrict the transformations (40) on the chart \( x^\mu \), by requiring
\[
t_- = t, \quad C_{,r}|_\Sigma = \zeta_-, \quad C_{,t}|_\Sigma = 0. \tag{45}
\]
Using these restrictions we obtain
\[
\begin{align*}
\dot{\zeta} &= \zeta_- \dot{\zeta}_- |_\Sigma, \\
A_{,r}|_\Sigma &= -\zeta_- a_-^2 \dot{C} \dot{A} + \zeta_+ \frac{a_-}{a_+} \dot{B} \dot{i} |_\Sigma, \\
A_{,t}|_\Sigma &= \dot{A} i - \zeta_- \zeta_+ \frac{a_-}{a_+} \dot{B} \dot{C} |_\Sigma, \\
B_{,r}|_\Sigma &= -\zeta_- a_-^2 \dot{B} \dot{C} + \zeta_+ a_- a_+ \dot{A} \dot{t} |_\Sigma, \\
B_{,t}|_\Sigma &= \dot{B} i - \zeta_- \zeta_+ a_- a_+ \dot{A} \dot{C} |_\Sigma,
\end{align*}
\tag{46}
\]
where for given \( \zeta_- \), the same sign factor \( \zeta_+ = \pm 1 \) takes care of the two possible solutions for \( A_{,r}, A_{,t}, B_{,r} \) and \( B_{,t} \) in (41). Remarkably, from (46) we may show that sign of \( \dot{B}_{,r}A_{,r} - B_{,t}A_{,t} \)|\( _\Sigma \) in accordance with (22) is fixed by \( \zeta_+ \) regardless of \( \zeta_- \).

We were free to choose instead of the restriction (45) the transformation
\[
t_- = t, \quad C_{,r}|_\Sigma = \zeta_-, \quad \dot{C} = \dot{R}
\]
This would have the disadvantage of leading to a nondiagonal continuous metric on \( \Sigma \). Following relations between \( B, A, C, t, \) and \( l \) is obtained by the requirement that \( \tau \) is the proper time on \( \Sigma \) seen from \( M_- \) or \( M_+ \):
\[
\dot{B}^2 - a_+^2 \dot{A}^2 |_\Sigma = 1, \quad \dot{B}^2 - a_-^2 \dot{C}^2 |_\Sigma = 1. \tag{47}
\]
Differentiating the fourth equation in (41) with respect to the shell proper time \( \tau \) leads to
\[
\dot{l} = a_- \frac{d r_-}{d \chi_-} \dot{C} + l H_- \dot{i} |_\Sigma, \quad \dot{i} = a_+ \frac{d r_+}{d \chi_+} \dot{A} + l H_+ \dot{B} |_\Sigma. \tag{48}
\]
Solving (47) and (48) for \( \dot{B} \) and \( \dot{i} \), we obtain
\[
\dot{B} = -li H_+ + \sqrt{(1 + l^2 - \frac{8 \pi G}{3} \rho_+ l^2)(1 + l^2 H_+^2 - \frac{8 \pi G}{3} \rho_+ l^2)} |_\Sigma,
\]
\[
\dot{i} = a_+ \frac{d r_+}{d \chi_+} \dot{A} + l H_+ \dot{B} |_\Sigma.
\]
\[ \dot{t} = -\dot{l}H_+ + \sqrt{\left(1 + \dot{l}^2 - \frac{8\pi G}{3} \rho_\pm \dot{l}^2\right)} \left(1 + \dot{l}^2 H_+ - \frac{8\pi G}{3} \rho_\pm \dot{l}^2\right) \bigg|_\Sigma, \]  

(49)

where we have used the following Friedmann equation for \( M_\pm \):

\[ H_\pm^2 + \frac{k_\pm}{a_\pm^2} = \frac{8\pi G}{3} \rho_\pm, \]  

(50)

with \( \rho_\pm(t_\pm) \) being the energy density in \( M_\pm \). \( \dot{B} \) and \( \dot{t} \) are positive in \( R \) or \( T \) regions where the denominators in (49) are positive or negative, respectively [6].

The nonzero components of Ricci tensor computed from (4) are

\[ \check{R}_{00} = \frac{2\check{R}^2a_\pm^2}{l^2} \check{R}_{22} + \frac{1}{a_\pm^2 \check{R}t} \left( \frac{\check{B}_\pm dB_\pm}{\check{d}t} - a_\pm^2 \check{A}\check{A}_t - a_\pm^2 H_\pm \check{A}^2 B_\pm + a_\pm^2 \check{R}^2 H_\pm \right) \bigg|_\Sigma, \]  

(51)

\[ \check{R}_{10} = \check{R}_{01} = -\frac{2a_\pm^2 \check{R}t}{l^2} \check{R}_{22} \bigg|_\Sigma, \]  

(52)

\[ \check{R}_{11} = -a_\pm^2 \check{R}_{00} + \frac{2a_\pm^2}{l^2} (\dot{l}^2 + a_\pm^2 \check{R}^2) \check{R}_{22} \bigg|_\Sigma, \]  

(53)

\[ \check{R}_{22} = \frac{l^2}{\check{R}t a_\pm^2} \left( \check{H}_- - H_+ \check{B}_t - \frac{a_+}{l} \frac{\check{d}r_+}{\check{d}t} \check{A}_t \right) \bigg|_\Sigma, \]  

(54)

\[ \check{R}_{33} = \sin^2 \theta \check{R}_{22}. \]  

(55)

Likewise, we obtain \( \check{T}_{\mu\nu} \) for the diagonal metric \( g_{\mu\nu}(x^\sigma) \) on \( \Sigma \):

\[ \check{T}_{\mu\nu} = \frac{1}{ia_-} \left[ \begin{array}{cccc} \sigma \dot{l}^2 & -\sigma a_\pm^2 \check{R}t & 0 & 0 \\ -\sigma a_\pm^2 \check{R}t & \sigma a_\pm^2 \check{R}^2 & 0 & 0 \\ 0 & 0 & -wl^2 & 0 \\ 0 & 0 & 0 & -wl^2 \sin^2 \theta \end{array} \right] \bigg|_\Sigma. \]  

(56)

Here again we are faced with two independent Einstein equations (34) and (35) on the shell. The junction equations of FRW space-times are then obtained after some manipulations:

\[ \zeta_+ \dot{t} \left( \frac{\check{d}r_-}{\check{d}x_-} + \frac{\dot{l}H_- a_- \dot{C}_-}{t} \right) - \zeta_+ \check{B} \left( \frac{\check{d}r_+}{\check{d}x_+} + \frac{\dot{l}H_+ a_+ \dot{A}}{B} \right) \equiv 4\pi G l \sigma, \]  

(57)

\[ \zeta_- \frac{\dot{B}}{a_+ \dot{A}} + a_+ \check{A} H_+ - \zeta_- \left( \frac{i}{a_- \dot{C}} + a_- \dot{C} H_- \right) \equiv 8\pi G (\frac{\sigma}{2} - \omega). \]  

(58)
In the framework of Darmois-Israel method, the left hand sides of (57) and (58) show the jump of extrinsic curvatures $K_\theta^\theta$ and $K_\tau^\tau$ across the shell, respectively [7,10]. Now, using (43) and (44) for the sign functions $\epsilon_\pm$, the junction equation (57) can be written in the following final from:

$$\epsilon_- \sqrt{1 + \dot{l}^2 - \frac{8\pi G}{3} \rho_- l^2} - \epsilon_+ \sqrt{1 + \dot{l}^2 - \frac{8\pi G}{3} \rho_+ l^2} \equiv 4\pi G l \sigma,$$

(59)

which is the same as that obtained by Berezin et al [6]. To eliminate the metric functions from the second junction equation, we apply the second Friedmann equation:

$$\frac{1}{a_\pm} \frac{da_\pm^2}{dt^2} = -\frac{4\pi G}{3} (\rho_\pm + 3p_\pm),$$

(60)

where $p_\pm$ denote the pressures of the matter in $M_\pm$. Finally, after some lengthy manipulations, the equation (58) can be written as follows

$$\zeta_+ \sqrt{1 + \dot{l}^2 - \frac{8\pi G}{3} \rho_+ l^2} \left\{ \dot{l} - \frac{8\pi G}{3} \rho_+ l + 4\pi G l (\rho_+ + p_+) \right\}$$

$$\cdot \left( 1 + \frac{2l^2 H_+^2}{\Delta_+^2} \right) + \frac{1}{\Delta_+} \left( \dot{l}^2 + l^2 H_+^2 - \frac{2l^2 H_+}{\Delta_+} \sqrt{l^2 \dot{l}^2 H_+^2 + \Delta_+^2} \right)$$

$$\cdot \left( 1 + \frac{2l^2 \Delta_+^2}{\Delta_+^2} \right) + \frac{1}{\Delta_+} \left( \dot{l}^2 + l^2 H_+^2 - \frac{2l^2 H_+}{\Delta_+} \sqrt{l^2 \dot{l}^2 H_+^2 + \Delta_+^2} \right)$$

$$\equiv 8\pi G (\frac{\sigma}{2} - w),$$

(61)

where $\Delta_\pm$ is given by

$$\Delta_\pm = 1 - \frac{8\pi G}{3} \rho_\pm l^2 \bigg|_{\Sigma}.$$  

(62)

The same result has been obtained by Berezin et al [6].

5 Conclusion

In gluing different manifolds together, the global characterization of the glued manifold is of utmost importance. This characterization depends basically on the signs of $K_\theta^\theta$, which is the central quantity in the Darmois–Israel approach. So far such global classifications had been neglected within the distributional approach of Mansouri–Khorrami.
Now, we have been able to define the relevant sign functions $\epsilon_{\pm}$ properly by the equation (9), and to insert them in the junction equations, so that the global classification of spherically symmetric space-times glued together can also be done within this distributional formalism. This makes the Mansouri–Khorrami formalism as general as possible and, in some cases, more suitable to apply than Darmois–Israel method. It has been successfully applied to solve special problems like the dynamics of two shells[14] or thick shells [15] within general relativity.

We have also seen, given the unique normal $n_{\mu}$ in the distributional formalism, that the sign factors $\zeta_+$ and $\zeta_-$ are independently generated by applying the transformations $x_{\mu}^+=x_{\mu}^+(x^\nu)$ and $x_{\mu}^-=x_{\mu}^-(x^\nu)$ on $M_+$ and $M_-$ respectively. This is in contrast to the Darmois–Israel method in which the sign factors are inserted in front of the normal vectors $n_{\pm\mu}$. Our sign factors are naturally related to the interior-exterior characterizations of $M_{\pm}$[7,10]. In addition, we have seen that to include the most general case of glueing manifolds in the distributional formalism along the singular hypersurface it is not enough to make a coordinate transformation on just one of the manifolds to make the four–metric continuous. We have to apply the nontrivial transformations $x_{\mu}^\pm=x_{\mu}^\pm(x^\nu)$ on both sides of the shell to generate the sign factors $\zeta_{\pm}$. These generalizations had been neglected in the papers [3,4]. It should also be noted that admissible coordinates have to be such that the relevant coordinate along the vector $n_{\mu}$ is increasing.

**Appendix: Junction equations in the spherically symmetric space–times separated by a spacelike shell**

The following junction equations for FRW space times bounded by a space–like shell can be obtained from (59) and (61) by means of the substitutions $\tau \to i\xi$, $\sigma \to -i\sigma$ and $\omega \to -i\omega$, where $\xi$ is the distance from the center of the sphere of the radius $R(\xi)$:

$$\epsilon_-\sqrt{l^2 - 1 + \frac{8\pi G}{3} \rho_- l^2} - \epsilon_+\sqrt{l^2 - 1 + \frac{8\pi G}{3} \rho_+ l^2} \equiv -4\pi G l \sigma, \quad (A1)$$

$$\frac{\zeta_+}{\sqrt{l^2 - 1 + \frac{8\pi G}{3} \rho_+ l^2}} \left\{ -l'' - \frac{8\pi G}{3} \rho_+ l + 4\pi G l (\rho_+ + p_+) \right\} \left(1 - \frac{2l^2 H^2_+}{\Delta_+} + \frac{1}{\Delta_+} \left( -l'^2 + l^2 H^2_+ - \frac{2ll'H_+}{\Delta_+} \sqrt{l^2 l^2 H^2_+ - \Delta^2_+} \right) \right)$$

$$- \frac{\zeta_-}{\sqrt{l^2 - 1 + \frac{8\pi G}{3} \rho_- l^2}} \left\{ -l'' - \frac{8\pi G}{3} \rho_- l + 4\pi G l (\rho_- + p_-) \right\} \left(1 - \frac{2l^2 H^2_-}{\Delta_-} + \frac{1}{\Delta_-} \left( -l'^2 + l^2 H^2_- - \frac{2ll'H_-}{\Delta_-} \sqrt{l^2 l^2 H^2_- - \Delta^2_-} \right) \right)$$

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\[ \Sigma = -8\pi G \left( \frac{\sigma}{2} - w \right), \]  \hspace{1cm} (A2)

where \( l' \equiv \frac{dl}{d\xi} \).

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