Calabi-Yau black holes
and
(0,4) sigma models

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Abstract

When an \(M\)-theory fivebrane wraps a holomorphic surface \(P\) in a Calabi-Yau 3-fold \(X\) the low energy dynamics is that of a black string in 5 dimensional \(\mathcal{N} = 1\) supergravity. The infrared dynamics on the string worldsheet is an \(\mathcal{N} = (0, 4)\) 2D conformal field theory. Assuming the 2D CFT can be described as a nonlinear sigma model, we describe the target space geometry of this model in terms of the data of \(X\) and \(P\). Variations of weight two Hodge structures enter the construction of the model in an interesting way.

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1. Introduction

D-brane and M-brane models of black holes have provided an extremely intriguing approach to an understanding of black hole entropy [1] and promise to lead to important insights in other aspects of black hole physics.

The program of Strominger and Vafa is based on mapping the low energy dynamics of certain configurations of branes to the conformal field theory of an effective string. The derivation of this conformal field theory is best understood (and already quite subtle) for black holes in backgrounds preserving 16 supersymmetries, such as IIB compactification on $K3 \times S^1$. In this paper we will investigate an analogous conformal field theory for 4D black holes in backgrounds with 8 unbroken supersymmetries.

Specifically, in this paper we continue the investigation of the microscopic dynamics of wrapped five-branes following the work of Maldacena, Strominger, and Witten [2]. We consider $M$-theory compactifications on $I \mathbb{R}^{1,3} \times S^1 \times X$ where $X$ is a nonsingular compact Calabi-Yau 3-fold. The radius of $S^1$ is taken to be large with respect to the length scale set by $X$, which is in turn large compared to the 11D Planck scale. We usually will take the background 3-form $C^{(3)}$ to vanish. If an $M$5-brane worldvolume $W_6$ wraps $\mathbb{R} \times S^1 \times \mathcal{P}$, where $\mathcal{P}$ is a four-manifold $\mathcal{P} \subset X$ then the resulting object is a string $S$ with worldsheet $W_2$ wrapping $\mathbb{R} \times S^1$ in $\mathbb{R}^{1,3} \times S^1$. At long distances the supergravity background is that of a black hole in $\mathbb{R}^{1,3}$ with 8 unbroken supersymmetries at infinity. The 4-manifold $\mathcal{P}$ must be a holomorphically embedded complex surface to preserve supersymmetry. In this case the low energy dynamics of the string $S$ is described by a $(0, 4)$ CFT and the number of massless boson and fermion degrees of freedom can be expressed purely in terms of the topology of $\mathcal{P}$ and of its embedding into $X$ [2].

Knowing the number of massless degrees of freedom suffices to determine the entropy microscopically, but for many purposes one would certainly like to know the data of the $(0, 4)$ conformal field theory of $S$ in much more detail. The object of this paper is to express this data in terms of the data of the ambient Calabi-Yau geometry and the topology and geometry of $\mathcal{P}$.

In this introduction we summarize the structure of the sigma model that we will find. The detailed justification is described in the subsequent sections. Much of what we say is implicitly (and sometimes explicitly) described in [2]. Let us begin with the overall count of the degrees of freedom. In a supersymmetric configuration the surface $\mathcal{P}$ is a divisor for a holomorphic line bundle $\mathcal{L}$ over $X$. Let $P = [\mathcal{P}] \in H^2(X; \mathbb{Z})$ be the first Chern
class of \(L\). It is Poincaré dual to the 4-cycle defined by \(P\). Macroscopically, in the 5D supergravity obtained from \(M\) theory on \(X\), the string is a black string and \(P\) is the charge of the string. Using index theory and holomorphic geometry \([2]\) computed the left- and right-moving central charges:

\[
\begin{align*}
    c_R &= 6D + \frac{1}{2}c_2 \cdot P \\
    c_L &= 6D + c_2 \cdot P,
\end{align*}
\]

where \(D := \frac{1}{6} \int_X P^3\) and \(c_2 \cdot P := \int_X P c_2(TX)\), verifying microscopically the entropy computed macroscopically in \([3]\). These central charges can also be obtained from the requirement of the complete anomaly cancellation in five dimensions \([4]\).

We now describe some of the local geometry of the target space. This is obtained by considering the collective coordinates of the wrapped 5brane. These include the collective coordinates associated to the 5 scalars \(X^a\) and the chiral two-form \(\beta\) of the 5-brane worldvolume tensormultiplet.

We begin with the collective coordinates \(\varphi\) associated to the five scalars. The space of supersymmetric wrappings of charge \(P\) is the set of divisors in \(X\) in the class \(P\). This is called a “linear system” because all divisors are zero-loci of global holomorphic sections of \(L\), and the latter is a linear space. Of course two sections related by a multiplicative constant have the same divisor, so the linear system is just a projective space

\[
|P| := \mathbb{P}H^0(\mathcal{P}, \mathcal{L}|_p) = \mathbb{CP}^N. \tag{1.2}
\]

Assuming \(\mathcal{P}\) is a smooth ample divisor, as we should to apply classical geometry \([3]\), the Riemann-Roch formula gives the dimension of the linear system \((1.2)\):

\[
N := D + \frac{1}{12}c_2 \cdot P - 1. \tag{1.3}
\]

Taking into account the position in noncompact \(\mathbb{R}^3\), the target space of the scalars is

\[
\varphi : W_2 \to \mathbb{R}^3 \times |P|. \tag{1.4}
\]

In this paper we will make the important restriction that the fivebrane wraps a smooth 4-cycle \(\mathcal{P}\). Thus if \(\mathcal{D}\) is the discriminant locus of singular divisors in the linear system we restrict to maps \(\varphi\) with image in

\[
|P|_s := |P| - \mathcal{D}. \tag{1.5}
\]
Moreover, because of monodromy, we will even restrict attention to maps into a local neighborhood \( U \subset |P|_s \).

Now we consider the collective coordinates arising from the chiral two-form \( \beta \) on worldvolumes of the form \( W_6 = W_2 \times P \). The massless modes are associated with harmonic two-forms on \( P \). Since the form is chiral there are \( b^-_2 := b_2^-(P) \) left-moving and \( b^+_2 := b_2^+(P) \) right-moving chiral bosons. Moreover, as shown in [2], one can express these topological invariants of \( P \) in terms of \( D \) and \( c_2 \cdot P \):

\[
\begin{align*}
b^-_2 &= 4D + \frac{5}{6}c_2 \cdot P - 1 \\
b^+_2 &= 2D + \frac{1}{6}c_2 \cdot P - 1
\end{align*}
\] (1.6)

If \( b_1(P) = 0 \) (which follows if \( b_1(X) = 0 \)) the only fermions are rightmoving. These pair up to form

\[
N = D + \frac{1}{12}c_2 \cdot P
\] (1.7)

\( N = (0,4) \) scalar multiplets with both left- and right-moving scalars. In addition there are real purely leftmoving scalars neutral under supersymmetry. Since

\[
N = D + \frac{1}{12}c_2 \cdot P - 1 = \frac{1}{2}(b^+_2 - 1)
\] (1.8)

there are \( |\sigma(P)| = b_2^-(P) - b_2^+(P) \) such scalars.

Let us now consider the scalars from the chiral two-form in more detail. The splitting into left-movers and right-movers follows from the decomposition

\[
H^2(P; \mathbb{R}) = H^{2,0}(P; \mathbb{R}) \oplus H^{0,2}(P; \mathbb{R})
\] (1.9)

into anti-self-dual and self-dual parts respectively. Since \( P \) is Kähler we may further decompose

\[
H^{2,+}(P; \mathbb{R}) = [H^{2,0}(P) \oplus H^{0,2}(P)]_{\mathbb{R}} \oplus \mathbb{R} \cdot J
\] (1.10)

where \( J \) is the Kahler class of \( P \) induced by that of \( X \), while \( H^{2,-}(P; \mathbb{R}) \) is purely of Hodge type \((1,1)\). A crucial point is that the splitting (1.9)(1.10) depends on \( P \) (i.e., on the values of \( \varphi \)), and hence on the (weight two) Hodge structure of \( H^2(P; \mathbb{Z}) \). Thus, a natural framework for working with the \((0,4)\) model is the theory of variation of Hodge structures. (See the references in [3] for some useful background material.)

1 We thank E. Witten for stressing the importance of the monodromy.
The Hodge structure on $H^2(P)$ decomposes into a “fixed part” and a “variable part” (as functions of $P$):

$$H^2(P; \mathbb{R}) = H^2_f(P; \mathbb{R}) \oplus H^2_v(P; \mathbb{R})$$  \hspace{1cm} (1.11)

where $H^2_v$ is the orthogonal complement of $H^2_f$ in the Hodge metric $(\theta_1, \theta_2) := \int_P \theta_1 \wedge \theta_2$. The “fixed” or “rigid” part is simply the space of 2-forms which extend to $X$. As pointed out in [2], since $P$ is ample the restriction map

$$\iota^* : H^2(X, \mathbb{Z}) \rightarrow H^2(P; \mathbb{Z})$$  \hspace{1cm} (1.12)

is injective so $H^2_f(P; \mathbb{R}) \cong H^2(X; \mathbb{R})$. Physically, the splitting (1.11) means the $(0,4)$ sigma model splits (up to possible discrete identifications by a finite group) into a product of two sigma models which we call the “universal factor” and the “entropic factor.” The terminology refers to the intuition that $P$ should be regarded as large, so that $D$ is a very large positive integer, determining the leading term in the black hole entropy.

The CFT for the universal factor is easily described. It consists of a single $(0,4)$ multiplet with target space $\mathbb{R}^3 \times S^1$ (for left- and right-movers) together with $h^{1,1}(X) - 2$ purely leftmoving bosons. Since $\beta$ can be shifted by large gauge transformations in $H^2(X; \mathbb{Z})$ the universal factor is just a $(0, 4)$ Narain model with leftmoving gauge group of rank $h^{1,1}(X) - 2$. The Narain data is obtained from the projection of $H^2(X; \mathbb{Z}) \otimes \mathbb{R}$ onto the definite signature subspaces of the quadratic form $(\theta_1, \theta_2) = \int_P \theta_1 \wedge \theta_2 = \int_X P \wedge \theta_1 \wedge \theta_2$.

The entropic factor is more subtle and is the focus of much of this paper. In describing this model we will make the important assumption that the model can be described by a geometrical Lagrangian (see footnote 1 above). Roughly speaking, a $(0,4)$ sigma model Lagrangian is determined by a choice of target space $\tilde{M}$ which has a hyper-Kähler connection (with torsion) together with a triholomorphic vector bundle with connection $V \rightarrow \tilde{M}$. The detailed conditions on the sigma model Lagrangian are written in equations 2.13 - 2.20 below.

In the following sections we will argue that in our case the target space $\tilde{M}$ of the entropic factor has a holomorphic projection

$$p : \tilde{M} \rightarrow \mathbb{P}^N$$  \hspace{1cm} (1.13)

\[ 2 \] Except when $h^{1,1}(X) = 1$. In this case the compact scalar in $S^1$ is purely right-moving and there is no left-moving gauge bundle.
where the projective space $\mathbb{P}^N$ is the linear system $|P|$. Physically, the degrees of freedom describing the fibers of $p$ have their origin in those of the self-dual two-form $\beta$ on $W_6$. The fibers are complex tori of complex dimension $N = \frac{1}{2}(b_2^+ - 1)$. Moreover, the vector bundle $\mathcal{V}$ has real rank $|\sigma(P)| - (h^{1,1}(X) - 2) = b_2^-(P) - b_2^+(P) - (h^{1,1}(X) - 2)$, and the connection on $\mathcal{V}$ is vertical, and flat.

In section 5.3 below we argue that the torsion of the connection on $T\widetilde{\mathcal{M}}$ is zero, so that the metric on $\widetilde{\mathcal{M}}$ is hyper-Kähler. This metric may be described as follows. The metric of the ambient Calabi-Yau $X$ induces a metric on the normal bundle of $\mathcal{P} \subset X$, and therefore a metric on the linear system $|P|$. This metric is Kähler, and, by a version of the Calabi ansatz/c-map ([6]/[7]) there is an induced hyper-Kähler metric on $T^*|P|$. Thus, the the local geometry of $\mathcal{M}$ is that of $T^*|P|$ with a hyper-Kähler structure.

Having described the local geometry of the target space we now turn to global issues. There are several interesting issues one should address, but we focus on only one, namely the nature of the fibers of $p$ (working locally in a patch of $|P|$). The derivation of the sigma model Lagrangian uses the chiral fivebrane Lagrangians of [8,9]. Unfortunately, the formalism of these papers does not determine the way in which zeromodes of the $b_2^-$ left-chiral and $b_2^+$ right-chiral scalars are paired with each other. Therefore we must resort to some guess-work.

The key motivation for our guess is that we expect that the target space of the sigma model $\widetilde{\mathcal{M}}$ should be compact. Otherwise it is hard to understand how we could have finite dimensional spaces of BPS states. Since the linear system $|P|$ is compact the question reduces to compactness of the fibers. Therefore, all the $b_2(\mathcal{P})$ modes due to the chiral $\beta$-field should be compact scalars. The most natural way to achieve this is to assume that the fiber above $\mathcal{P}$ is just a conformal field theory on a torus with a flat connection, i.e., a Narain model. The data of the Narain model consists of a choice of lattice $\Gamma$ of signature $(p, q)$ of zeromodes of scalars and an orthogonal projection of $\Gamma \otimes \mathbb{R}$ to the definite signature subspaces defining the spectrum of left- and right-moving parts of the winding/momentum lattice. In our case we will take $\Gamma = H^2(\mathcal{P}; \mathbb{Z})$ and the orthogonal projection is

$$H^2(\mathcal{P}; \mathbb{Z}) \otimes \mathbb{R} \rightarrow H^{2,+}(\mathcal{P}; \mathbb{R}) \perp H^{2,-}(\mathcal{P}; \mathbb{R})$$

induced from the metric on $\mathcal{P}$. Thus, according to our hypothesis, the fibers of the projection in (1.13) are complex tori of dimension $N$, so we have a holomorphic integrable

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3 Warning: the metric on the linear system is not the Fubini-Study metric.
system. Two interesting subtleties in this discussion are, first, there is a nontrivial flat connection on the toroidal fibers, and second, by a mechanism mentioned in [2] most of the charges in $H^2(\mathcal{P}; \mathbb{Z})$ are not conserved (we comment on this briefly in section 6).

Finally we mention another motivation for the present work. D-brane models of black holes appear to have interesting arithmetic properties [10,11], at least for the case of black holes in backgrounds with 16 supersymmetries. One can entertain various conjectures about the arithmetic nature of D-brane black holes in Calabi-Yau compactification and several of these are related to questions about the numbers of BPS states $\dim \mathcal{H}_{BPS}(\gamma)$ for charge $\gamma \in H^{even}(X)$. The results of the present paper might help to elucidate the nature of these BPS degeneracies. Our hope is that these degeneracies can be studied using $(0, 4)$ elliptic genera.

We summarize the remaining sections as follows. In section two we review the general form of $(0, 4)$ Lagrangians. In section three we review the relation between unbroken supersymmetry and holomorphically wrapped 4-cycles. In section four we describe in detail the derivation of the collective coordinates and the relation to variation of Hodge structures. We derive carefully the $(0, 4)$ supermultiplets by reducing the supersymmetry transformations of the 6D tensormultiplet. In section five we use the Kaluza-Klein ansatz of section four in the chiral 5-brane action and derive the geometry of the target space of the $(0, 4)$ model. In section six we comment on some of the global aspects of the target space model. These involve the toroidal fibers of (1.13) and their relation to Narian models. In section seven we describe briefly what we think are some of the most interesting open problems raised by this paper. Many conventions and technical points may be found in the appendices. Appendix D describes the close analogy of the models in this paper with the strings obtained by wrapping D3 branes around holomorphic curves in a K3 surface.

2. Geometrical data for $(0, p)$ $\sigma$-models

In this section we summarize the geometrical data used to construct $(0, 4)$ supersymmetric Lagrangians. This material is standard, and this section follows mostly [12,13].

In two spacetime dimensions the supersymmetry algebra of type $(0, p)$ is carried by $p$ negative-chirality supersymmetries $Q_+^I$, $I = 1, \ldots, p$, obeying:

$$\{Q_+^I, Q_+^J\} = 2\delta^{IJ} P_-$$  \hspace{1cm} (2.1)
In the construction of the \((0,4)\) sigma-models it is convenient to consider a formulation with only \((0,1)\) manifest supersymmetry. \((0,1)\) superspace consists of two Bose coordinates \(x^+, x^-\) and a single negative-chirality anticommuting coordinate \(\theta^-\). Our sigma-model is defined by a map from \((0,1)\) superspace \(\Sigma\) to a \(d\)-dimensional target manifold \(\mathcal{M}\), given by scalar superfields \(\Phi^i(x, \theta^-), i = 1, \ldots d\). In general there can also be another field which is a section of the vector bundle \(S_- \otimes \phi^* \mathcal{V}\) over \(\Sigma\), given by negative-chirality \(4\) spinor superfields \(\Lambda^a_-(x, \theta^-), a = 1, \ldots n\), where \(n\) is the fibre dimension of \(\mathcal{V}\) and \(S_-\) is the spinor bundle over \(\Sigma\). \(\mathcal{V}\) is equipped with a positive definite metric \(h_{ab}\) and a connection \(A_i^a b\) with curvature \(F_{ij}^a b\) valued in some subgroup of \(O(n)\). The requirement of \((0,4)\) SUSY imposes additional constraints which will be analyzed at the end of this section. The superfields have the following expansions:

\[
\Phi^i(x, \theta^-) = \phi(x)^i + i\theta^- \psi^-_i(x) \tag{2.2}
\]

\[
\Lambda^a_+(x, \theta^-) = \lambda^a_+(x) + i\theta^- F^a(x) \tag{2.3}
\]

The action for the model in terms of \((0,1)\) superfields reads:

\[
S = \int d^2 x d\theta^- \left( (g_{ij}(\Phi) + b_{ij}(\Phi)) D_- \Phi^i \partial_+ \Phi^j + i \Lambda^a_+ (D_- \Lambda^b_- + D_- \Phi^i A_i^b c \Lambda^c_+) h_{ab} + imC^a \Lambda^b h_{ab} \right) \tag{2.4}
\]

where

\[
D_- = \frac{\partial}{\partial \theta^-} + i\theta^- \frac{\partial}{\partial x^-} \tag{2.5}
\]

where \(A_i^a b\) is the connection on \(\mathcal{V}\) with a curvature \(F_{ij}^a b\). After eliminating the auxiliary fields and expanding in components, \((2.3)\) reads:

\[
S = \int d^2 x \left[ (g_{ij} + b_{ij}) \partial_+ \phi^i \partial_- \phi^j + ig_{ij} \psi^-_i \nabla^{(\pm)} \psi^-_j - i \lambda^a_+ D_- \lambda^b_- h_{ab} - \frac{1}{2} \lambda^a_+ \lambda^b_- \psi^-_i \psi^-_j F_{ijab} + m \nabla_i C^a \psi^-_i \lambda^b_- h_{ab} - \frac{1}{4} m^2 C^a C^b h_{ab} \right] \tag{2.6}
\]

where \(\nabla^{(\pm)}\) is the covariant derivative with respect to the connection with torsion

\[
\Gamma^{(\pm)}_{jk} = \Gamma^i_{jk} \pm H^i_{jk} \tag{2.7}
\]

\(^4\) the subscript refers to the fact that in a free theory the \(\theta\)-independent component in the expansion of \(\Lambda\) would be a left-moving fermion in the sense that \(\partial_- \lambda^a_+ = 0\)
\[ H_{ijk} = \frac{3}{2} \partial_{[i} b_{jk]} \]  \hspace{1cm} (2.7)

and
\[ D_- \lambda_+ = \partial_- \lambda_+^a + \partial_- \phi^i A_i^a b \lambda_+^b \] \hspace{1cm} (2.8)

The action (2.3) is manifestly invariant under supersymmetry transformations
\[ \delta \Phi^i = i \eta_+ D_- \Phi^i \]
\[ \delta \Lambda_+^a = -i \eta_+ D_- \Lambda_+^a. \] \hspace{1cm} (2.9)

Let us now assume that (2.3) possesses additional supersymmetries, parametrized by anticommuting parameters \( \eta^r_+ \), \( r = 1, \ldots, p - 1 \), of the form
\[ \delta \Phi^i = i \eta^r_+ J^i_r(\Phi) D_- \Phi^j \]
\[ \delta \Lambda_+^a = -\delta \Phi^i A_i^a b \Lambda_+^b + \eta^r_+ J^a_r b(\Phi) S^b - m \eta^r_+ t^a_r(\Phi) \] \hspace{1cm} (2.10)

where
\[ S^a = 2 \nabla_- \Lambda_+^a + m C^a. \] \hspace{1cm} (2.11)

and \( J_r, I_r, t_r \) are to be determined. Invariance of the action under (2.9) and (2.10), and on-shell closure of the supersymmetry algebra are equivalent to the following set of conditions \[13\,14\,15\]
\[ J_r J_s = -\delta_{rs} + f_{rs}^t J_t \] \hspace{1cm} (2.12)
\[ N(J_r, J_s)^i_{jk} = 0 \] \hspace{1cm} (2.13)
\[ F_{ij} J_r^i [k J_s^j l] = F_{kl} \delta_{rs} \] \hspace{1cm} (2.14)
\[ J_r^k (i g_{jk}) = 0 \] \hspace{1cm} (2.15)
\[ \nabla^{(+)} J = 0 \] \hspace{1cm} (2.16)
\[ I_r^c (a h_{bc}) = 0 \] \hspace{1cm} (2.17)
\[ \partial_i (t_r^a C^b h_{ab}) = 0 \] \hspace{1cm} (2.18)
\[ \nabla_i t^a_r = J_r^j i \nabla_j C^a \] \hspace{1cm} (2.19)

Here \( N \) is essentially the Nijenhuis tensor. The SUSY algebra closes on-shell on \( \lambda_+^a \) by virtue of (2.12), (2.13), (2.14), (2.19).

Conditions (2.12) - (2.17) can be summarized as follows \[13\] : \( \mathcal{M} \) admits three complex structures obeying the algebra of quaternions, the metric on \( \mathcal{M} \) is hermitian with respect to all three complex structures and the holonomy of the connection \( \Upsilon^{(+)} \) is a subgroup of \( Sp(d/4) \). The bundle \( \mathcal{V} \otimes \mathbb{C} \) is holomorphic with respect to all three complex structures and carries an Hermitian metric \( h_{ab} \).
3. Supersymmetrically wrapped fivebranes

The low energy states of the black string $S$ are obtained from small deformations of supersymmetrically wrapped cycles. Thanks to the existence of a $\kappa$-symmetric fivebrane action we can analyze which configurations preserve supersymmetry, following the analysis of [16].

The unbroken supersymmetries are the result of combining the supersymmetry $\delta_\epsilon \Theta = \epsilon$ with $\kappa$-symmetry $\delta_\kappa \Theta = 2(1 + \Gamma) \kappa$, where the matrix $\Gamma$ is field dependent (the explicit expression can be found in [17]) and has the property that $(1 \pm \Gamma)$ are projection operators. The condition of unbroken supersymmetry is

$$ (1 - \Gamma) \epsilon \equiv P_+ \epsilon = 0. \quad (3.1) $$

Let the $M5$-brane be stretched in the $X^0 - X^5$ directions, and consider a compactification on a Calabi-Yau threefold along $X^2, ..., X^7$. We consider the wrapping of the fivebrane on a four-cycle $P$ stretched in $X^2 - X^5$ directions. The coordinates on the fivebrane world-sheet are denoted by $\sigma^\alpha$, $\alpha = 0, ..., 5$. It is convenient to choose a gauge such that $\sigma^0 = X^0$, $\sigma^1 = X^1$. Also let $X^m, X^\overline{m}$, $m = 1, 2$ be a complex basis for $X^2 - X^5$ (choosing a gauge such that $dX^6 + idX^7$ is an eigenvector of the complex structure on the Calabi-Yau). An eleven-dimensional spinor $\epsilon$ is decomposed as

$$ \epsilon_\pm = \lambda^{(3)} \otimes \lambda^{(2)}_\pm \otimes \xi^{(6)} \quad (3.2) $$

where $\lambda^{(2)}_\pm$ is a spinor of $Spin(2)_{01}$ of positive (negative) chirality, $\lambda^{(3)}$ is a $Spin(3)_{8910}$ spinor and $\xi^{(6)}$ is the covariantly constant spinor of the Calabi-Yau. The eleven-dimensional $\Gamma$-matrices can be decomposed as follows

$$ \Gamma^{8,9,10} = \gamma^{8,9,10} \otimes \rho^{(2)}_0 \otimes \rho^{(6)} $$

$$ \Gamma^{0,1} = \mathbb{I}_3 \otimes \gamma^{0,1} \otimes \rho^{(6)} $$

$$ \Gamma^{2,3,4,...,7} = \mathbb{I}_5 \otimes \gamma^{2,3,4,...,7} \quad (3.3) $$

where $\rho^{(2)}_0 = i\gamma^0 \gamma^1$ is the chirality operator of $Spin(2)_{01}$ and $\rho^{(6)}$ is the chirality operator acting on the Calabi-Yau spinors. We let $\Gamma_{M_1...M_k} = \Gamma_{[M_1...M_k]}$. Denoting a positive (negative)- chirality spinor on the Calabi-Yau by $\xi^{(6)}_\pm$, we can choose a normalization such that the following identities hold

$$ \gamma_m \xi^{(6)}_+ = 0, \quad \gamma_{mpq} \xi^{(6)}_+ = 2iJ_{p[m} \gamma_{q]} \xi^{(6)}_+ \quad (3.4) $$
where $J$ is the Kähler form for $X$. Also, we have passed to a complex basis for the $\gamma$ matrices such that $\gamma_1^I = \gamma^2 + i\gamma^4 = (\gamma_1^I)^\dagger$ and $\gamma_2^I = \gamma^3 + i\gamma^5 = (\gamma_2^I)^\dagger$. We will omit the subscript on the complex matrices whenever there is no possibility of confusion. Equation (3.4) implies

$$\gamma_{mnpq}^{(6)}\xi^{(6)} = (J_{np}J_{mq} - J_{mp}J_{nq})\xi^{(6)} \quad (3.5)$$

where we used $J_{mn} = ig_{mn}$ and the anticommutation relations. We thus have

$$P_- \epsilon_\pm = \frac{1}{2} \left( 1 \pm \partial_2 X^\overline{m}\partial_3 X^\overline{n}\partial_4 X^\overline{p}\partial_5 X^\overline{q} \frac{1}{4} (J_{\overline{mp}}J_{\overline{nq}} - (\overline{m} \leftrightarrow \overline{n})) \right) \epsilon_\pm \quad (3.6)$$

where $\partial_i = \partial/\partial\sigma^i$, $A_i = 2, ..., 5$, and we have used $\gamma^{\overline{12}} = -4\gamma^{2345}$. As expected, $P_- \epsilon_\pm = 0$ implies

$$(dV)_4 = \pm \epsilon^*(\frac{1}{2} J \wedge J) \quad (3.7)$$

where $dV_4$ is the volume form of the part of the fivebrane wrapping the Calabi-Yau four-cycle and $\epsilon^*(J \wedge J)$ is the pullback of $J \wedge J$ to the fivebrane.

From (3.5), (3.7) it follows that for holomorphic (anti-holomorphic) cycles $P$ only $\epsilon_+ (\epsilon_-)$ can satisfy $P_- \epsilon = 0$. Thus only one eighth of the supersymmetry is preserved and the resulting $\sigma$-model is chiral $(4,0)$ ($(0,4)$)

4. The massless $(0,4)$ supermultiplets

We now turn to the description of the massless degrees of freedom on $S$ arising from small fluctuations around a supersymmetric wrapping of the fivebrane worldvolume on $\mathbb{R} \times S^1 \times P$. Our method will be to reduce the six-dimensional $(2,0)$ multiplet along $P$. Of course, this presupposes some facts about the equations of motion. These will be justified in the section five.

4.1. Reduction of the scalars

The five scalars of the 6D $(2,0)$ tensormultiplet, denoted $X^a$ (cf B.2) parametrize the position of the fivebrane in eleven dimensions. When we wrap an M-theory fivebrane on a real four-cycle $P$ inside a Calabi-Yau manifold, three scalars (call them $X^{8,9,10}$) parametrize the position of the string in the noncompact dimensions, and the remaining two describe the position of the cycle $P$ inside the Calabi-Yau and should therefore be thought of infinitesimally as sections of the normal bundle.
The massless scalars arise from deformations of the position of $W_2 \times \mathcal{P}$ preserving unbroken supersymmetry. The space of deformations of $\mathcal{P}$ as a complex submanifold of $X$ has tangent space

$$T_\mathcal{P}|_\mathcal{P} = H^0(\mathcal{P}, \mathcal{N})$$

(4.1)

where $\mathcal{N}$ is the normal bundle. We may identify $\mathcal{N} \cong \mathcal{L}|_\mathcal{P}$. It follows from the index theorem that if $\mathcal{P}$ is ample the complex dimension of the space of deformations of $\mathcal{P}$ is $N = D + \frac{1}{12} c_2 \mathcal{P} - 1$.

When we consider small fluctuations of the wrapped fivebrane on a cycle $\mathcal{P}$ we are really considering a family of cycles $\mathcal{P}_\varphi$ near a point $\mathcal{P}_{\varphi=0}$ in the moduli space of deformations of $\mathcal{P}$. These fit into a holomorphic fibration

$$\begin{array}{ccc}
X & \xrightarrow{\pi} & \mathcal{U} \\
\downarrow & & \downarrow \\
|\mathcal{P}| & & 
\end{array}$$

(4.2)

where $\mathcal{U} \subset |\mathcal{P}|_s$ is a neighborhood of $\varphi = 0$. The fibers of this family are diffeomorphic to $\mathcal{P}_{\varphi=0}$, but have variable complex structure. In appendix C we show that there is an injection

$$0 \rightarrow H^0(X, \mathcal{L})/\mathbb{C} \cdot s \cong H^0(\mathcal{P}, \mathcal{L}|_\mathcal{P}) \rightarrow H^{0,1}(T\mathcal{P})$$

(4.3)

where $\mathcal{P}$ is the vanishing locus of the section $s$. By Kodaira-Spencer theory $H^{0,1}(T\mathcal{P})$ is the space of infinitesimal deformations of the complex structure of $\mathcal{P}$ [18]. Thus, a first order deformation of $\varphi$ induces a nonzero deformation of complex structure. This will be important in the next section.

Finally, we write out the Kaluza-Klein ansatz for the scalars describing fluctuations around $\mathcal{P}$. Choosing a basis, $\upsilon_I$, for $H^0(\mathcal{P}, \mathcal{L}|_\mathcal{P})$ and a coordinate system on $X$ so that $dX^6 + idX^7$ is normal to $\mathcal{P}$ we may expand, to first order in $\varphi^I$,

$$X^6 + iX^7 = \upsilon_I \varphi^I$$

(4.4)

to obtain complex two-dimensional massless fields $\varphi^I$, $I = 1, ..., N$. These scalars are both left- and right-moving.

4.2. Reduction of the worldvolume two-form

The chiral two-form $\beta$ on $W_6$ reduces to left-moving and right-moving scalars according to the decomposition into self-dual and anti-self dual parts as in equations (1.9) (1.10) of the introduction. As mentioned in the introduction, $H^2(\mathcal{P})$ carries a polarized Hodge structure of weight two. The Hodge decomposition is simply

$$H^2(\mathcal{P}; \mathbb{Z}) \otimes \mathbb{C} = H^{2,0}(\mathcal{P}) \oplus H^{1,1}(\mathcal{P}) \oplus H^{0,2}(\mathcal{P})$$

(4.5)

and the polarization is given by the Hodge metric.
Fig. 1 As we move in the moduli space of deformations, we may use $C^\infty$ diffeomorphisms to define a flat basis for $H^2(P; \mathbb{Z})$ in the fibers. However both the Hodge decomposition and the decomposition into self-dual/antiself-dual parts change.

It is important to bear in mind that the decomposition (4.5) and hence the decomposition of $\beta$ into left- and right-moving scalars depends on $\varphi$. As $\varphi$ changes the Kähler metric and complex structure on $P_\varphi$ varies, as illustrated in Fig. 1. This is the standard geometrical realization of variation of Hodge structures [5]. Since the fibers are all diffeomorphic we can choose a family of $C^\infty$ diffeomorphisms and define a local system (i.e. a flat bundle) with a locally constant basis for $H^2(P; \mathbb{Z})$. Extending by linearity defines the Gauss-Manin connection on the bundle $R^2\pi_* (\mathcal{C})$ over $U$ whose fiber at $P$ is $H^2(P)$. The Gauss-Manin connection allows us to differentiate in the $\varphi^I$ direction, and is crucial in deriving the low-energy $(0, 4)$ Lagrangian.

If we choose a smoothly-varying basis $\omega_I$, $I = 1, \ldots, \frac{1}{2}(b_2^+ - 1)$ of harmonic $(2, 0)$ forms on $P$, and $\omega_{-a}$, $a = 1, \ldots, b_2^-(P)$, of anti-selfdual $(1, 1)$-forms on $P$, then these bases will rotate into one another in accord with Griffiths transversality. That is, if we define the holomorphically varying filtration $H^2(P) = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \mathcal{F}^2$ by

$$
\mathcal{F}^0 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}
$$

$$
\mathcal{F}^1 = H^{2,0} \oplus H^{1,1}
$$

$$
\mathcal{F}^2 = H^{2,0}
$$

then

$$
\nabla : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1(U). \quad (4.7)
$$
Or, in plain English, the connection matrix is upper triangular and increases \( p \) in the decomposition into \((p, q)\) forms by at most one. We can split the Hodge structure into a fixed and a variable part as in (1.11) of the introduction. The Hodge structure \( H_f^2 \) is fixed (as a function of \( P \)) because it is purely of type \((1, 1)\) for all \( P \).

Finally, we write out the Kaluza-Klein ansatz for the chiral two-form \( \beta \) as:

\[
\beta = \rho^a \omega_{-a} + 4(\pi^I \omega_I + c.c.) + u_4 J
\]

The two-dimensional complex scalars \( \pi^I \) and the real scalar \( u_4 \) are right-moving whereas the \( b^- \) real scalars \( \rho^a \) are left-moving.

4.3. Reduction of \((2, 0)\) tensor multiplet fermion fields along \( P \)

We now describe the Kaluza-Klein ansatz for the fermions in the 6D tensor multiplet on \( W_2 \times P \). We will expand these in terms of harmonic 2-forms and 0-forms on \( P \). The conceptual reason we can do this is the following.

The 5-brane breaks local 6D Lorentz symmetry on \( W_6 \) as \( Spin(1, 1) \times Spin(4) \hookrightarrow Spin(1, 5) \) (4.9)

So the fermions are in the \( 4 = (+\frac{1}{2}, 2, 1) \oplus (-\frac{1}{2}, 1, 2) \). Moreover, the tensor multiplet theory has a \( Spin(5) = USp(4) \) \( \mathcal{R} \) symmetry group from local Lorentz rotations in the normal directions (in 11D) to \( W_2 \times P \). The Calabi-Yau background breaks this to \( Spin(3) \times Spin(2) \), where \( Spin(3) \) are rotations in the noncompact normal directions and \( Spin(2) \) is the structure group of the normal bundle \( \mathcal{N} \) for \( P \) in \( X \). The restriction of the spin bundle on \( X \) to \( P \) decomposes as:

\[
S^+ (TX) \cong S^+ (TP) \otimes K^{1/2} \oplus S^- (TP) \otimes K^{-1/2}
\]

but, because \( P \) is Kähler,

\[
S^+ (TP) \otimes K^{1/2} \cong \Omega^{0,0} (P) \oplus \Omega^{2,0} (P).
\]

Hence we expand the zeromodes in terms of 0-forms and 2-forms on \( P \).

In order to reduce the supersymmetry transformations we will need to make (4.11) more explicit. Our conventions for six-dimensional supersymmetry are in Appendix B. We choose a basis for six-dimensional \( \Gamma \) matrices to be

\[
\Gamma^{0,1} = \gamma^{0,1} \otimes \rho^{(4)}
\]
\[
\Gamma^{2,3,4,5} = \mathbb{I}_2 \otimes \gamma^{2,3,4,5}
\]

\( \rho^{(4)} \) are the Pauli matrices. The above basis is equivalent to the usual six-dimensional Pauli matrices \( \Gamma^{ij} \) with \( \mathbb{I}_2 \otimes \mathbb{I}_2 \) replacing \( 1 \).

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where
\[ \gamma^0 = i\sigma^2; \quad \gamma^1 = \sigma^1 \]
\[ \gamma^{2,3,4} = \left( \begin{array}{cc} 0 & \sigma^{1,2,3} \\ \sigma^{1,2,3} & 0 \end{array} \right); \quad \gamma^5 = \left( \begin{array}{cc} 0 & iI_2 \\ -iI_2 & 0 \end{array} \right) \]

(4.13)

We will decompose the covariantly constant spinor on the Calabi-Yau into two- and four-dimensional parts

\[ \xi^{(6)} = \xi^{(2)} \otimes \xi \]

(4.14)

We take \( \xi^{(6)} \) to be anti-chiral in order to conform to the chirality of the tensor multiplet. Note that the \( Spin(2)_{67} \) and \( Spin(4)_{2345} \) spinors \( \xi^{(2)} \) and \( \xi \) are not covariantly constant but only projectively covariantly constant. That is, they are parallel up to a phase, and the phase cancels between \( \xi^{(2)} \) and \( \xi \).

Again, using (4.9), the 6D tensormultiplet spinors \( \psi^{(6)} \) in their turn decompose as

\[ \psi_i^{(6)} = \psi_{i-}^I \otimes \Delta_{(i)}^I + \psi_{i-}^0 \otimes \xi_{(i)}. \]

(4.15)

Here \( i = 1, 2, 3, 4 \) is a \( USp(4) \) \( \mathcal{R} \)-symmetry index (see appendix B) and there is no summation on \( i \). Moreover,

\[ \Delta_{(i)}^I = \begin{cases} \omega_{mn}^{I} \gamma_{mn} \xi & \text{for } i = 1, 3 \\ \omega_{m,n} \gamma_{mn} \xi^* & \text{for } i = 2, 4 \end{cases} \]

(4.16)

and

\[ \xi_{(i)} = \begin{cases} \xi & \text{for } i = 1, 3 \\ \xi^* & \text{for } i = 2, 4 \end{cases} \]

(4.17)

This decomposition is consistent with the symplectic reality condition, which with our conventions reads

\[ \psi_{1-} = i\psi_{2-}^I, \quad \psi_{3-} = -i\psi_{4-}^I \]

(4.18)

Thus the two-dimensional spinors \( \psi_{i-}^I \) correspond to two complex spinor degrees of freedom; they carry a subscript minus since only right-moving spinors survive the wrapping as massless degrees of freedom. Note, for use in section 5.2 that since \( \gamma_m \xi^{(6)} = 0 \) one learns that \( \gamma_m \xi = 0 \).
4.4. Reduction of the supersymmetry transformations

We now come to the reduction of the 6D supersymmetry transformations. In principle, we should take into account all the complications of kappa supersymmetry and the exact superisometries unbroken by the background determined by $X$ and $P$. However, as explained in section 4.6 below, it is sufficient for our purposes to consider the reduction of the supersymmetry transformations from flat space.

As a preliminary to the calculation it will prove very useful to relate the basis $\upsilon_I$ of (4.4) (used for the scalars) to the basis $\omega_I$ of holomorphic $(2,0)$ forms on $P$. Abstractly this is a consequence of

$$H^0(P, \mathcal{L}|_{P}) \cong H^0(P, \Omega^2_P) \cong H^{2,0}(P),$$

which follows from the adjunction formula $(K_X \otimes \mathcal{L})|_P \cong K_P$ and the triviality of the canonical bundle $K_X$ of $X$. More explicitly we have a relation between the basis of the holomorphic two-forms on $P$, $\omega_{I,mn}$, and $\upsilon_I$ given by:

$$\upsilon_I = \xi^I \omega_{I,mn} \gamma^{mn} \xi^*; \quad \overline{\psi}^I = \xi^{Tr} \omega^{I,mn} \gamma_{mn} \xi$$

Conversely, one can write $\iota(\upsilon_I) \Omega^{(3,0)}|_P = \omega^{(2,0)}_I$ where $\iota$ is a contraction and $\Omega^{(3,0)}$ is a nowhere zero holomorphic three-form on $X$.

Substituting the above expansions into the supersymmetry transformations of the tensor multiplet the supersymmetry transformation of the $\beta$-field yields

$$\delta \pi^I = -2(\epsilon_{i+})^\dagger (1 + iT)_{ij} \psi^I_{j-}$$
$$\delta u^4 = 2(\epsilon_{i+})^\dagger T_{ij} \psi^0_{j-}$$
$$\delta \rho^a = 0$$

where $T = \hat{\gamma}^6 \hat{\gamma}^7$ and $\frac{1}{2} (1 + iT)$ is a projection operator. Similarly, the reduction of the susy transformations for the $X^a$ gives

$$\delta \varphi^I = 2(\epsilon_{i+})^\dagger (\hat{\gamma}^6 (1 + iT))_{ij} \psi^I_{j-}$$
$$\delta X^{8,9,10} = 2(\epsilon_{i+})^\dagger \hat{\gamma}^{8,9,10}_{ij} \psi^0_{j-}$$

The second lines in (4.21) and (4.22) can be joined into a single equation

$$\delta u^A = 2(\epsilon_{i+})^\dagger \hat{\gamma}^{A}_{ij} \psi^0_{j-}$$
where \( A = 1, \ldots, 4 \) and we have defined \( \gamma_A^{1,2,3} = \gamma^{8,9,10} \) and \( \gamma_A^{4} = T \), and

\[
\begin{align*}
  u^1 &= X^8, & u^2 &= X^9, & u^3 &= X^{10}
\end{align*}
\]  

Finally, the transformation laws of the fermions reduces to

\[
\delta \psi^I = \frac{1}{2} \partial_- (\varphi^I \tau_{2+} + \pi^I \epsilon_{2+}) ;
\delta \psi^I = \frac{1}{2} \partial_- (-\varphi^I \tau_{2+} + \pi^I \epsilon_{2+})
\]

\[
\delta \psi^0 = \frac{1}{2} \partial_- u_A \gamma_A^{ij} \epsilon_{j+}
\]  

where \( \tau = -i\epsilon^\dagger \); \( \partial_- = -\partial_0 + \partial_1 \). In the first line of (4.25) we have found it more convenient to work explicitly with components. Of course the other two transformations \( (\delta \psi^I, \delta \psi^0) \) are related to the above by the symplectic reality condition (4.18). We will give a considerably more attractive form of this equation in the next section.

4.5. Assembling the multiplets and the Hyper-Kähler structure

We will now summarize the previous sections by describing the \((0,4)\) supermultiplets in terms of the a hyper-Kähler structure.

Note first that from the above supersymmetry transformations the four real scalars \( u_A \) and the four real component spinor \( \psi^0 \) transform amongst themselves. Since this multiplet is present no matter what the topology of \( X \) or \( P \) is, we refer to it as the “universal” multiplet. It has some nice analogies with the universal hypermultiplet of Calabi-Yau compactification of type II strings.

We now cast the susy transformations in quaternionic form. By letting \( w = u_4 - iu_3; \ z = -u_2 + iu_1 \) the bosonic coordinate \( X \) has the form

\[
X = \left( \begin{array}{cl} z & \bar{w} \\ -w & \bar{z} \end{array} \right)
\]  

Moreover if we define the quaternions

\[
\Theta = \left( \begin{array}{cl} \eta & \bar{\eta} \\ -\theta & \bar{\theta} \end{array} \right); \quad \Psi^0 = \left( \begin{array}{cl} \chi & \bar{\psi} \\ -\psi & \bar{\chi} \end{array} \right)
\]  

where we have set \( \eta = i\epsilon_{2+}; \ \theta = -i\epsilon_{4+}; \ \chi = \psi^0_{1-}; \ \psi = \psi^0_{4-}, \) then (4.28) reads

\[
\delta X = -4i \Theta \cdot \Psi.
\]
The corresponding fermionic transformation (the second line of (4.25)) becomes

\[ \delta \Psi^I = -\frac{i}{2} \partial_+ X^I \Theta \]  \hspace{1cm} (4.29)

These are the supersymmetry transformations of the universal superfield in a manifestly (0, 4) invariant form. Geometrically, the four scalars parametrize \( \mathcal{U} = \mathbb{R}^3 \times S^1 \) (the \( u_4 \) scalar coming from tensor field is periodic).

Similarly, if we define

\[ \delta \pi_I := D_I J \delta \pi^J \]  \hspace{1cm} (4.30)
and

\[ D_I J := \int_P \omega_I \wedge \overline{\omega}_J \]  \hspace{1cm} (4.31)

we see from the supersymmetry transformations that the scalars \( \pi^I \) and \( \varphi^I \) mix under \( \delta^2 \).

Proceeding in the same manner we define

\[ \delta X^I = \left( \begin{array}{cc} \delta \varphi^I & \delta \pi_I \\ -\delta \pi^I & \delta \varphi_I \end{array} \right); \quad \Psi^I = \left( \begin{array}{cc} \chi^I & \overline{\psi}_I \\ -\psi^I & \overline{\chi}_I \end{array} \right) \]  \hspace{1cm} (4.32)

and

\[ \Xi = \left( \begin{array}{cc} \zeta & \overline{\lambda} \\ -\lambda & \zeta \end{array} \right) \]  \hspace{1cm} (4.33)

where now \( \zeta = \epsilon_{4+}; \ \lambda = \epsilon_{2+}; \ \chi^I = \psi^I_{2+}; \ \psi^I = \psi^I_{4-}. \) We see that the transformations \( \delta \varphi^I \) and \( \delta \pi_I \) are joined together since the first lines of (4.21) and (4.22) can be expressed compactly as

\[ \delta X^I = -4i \Xi \cdot \Psi^I \]  \hspace{1cm} (4.34)

Similarly the fermionic transformations (the first line of (4.25)) become

\[ \delta \Psi^I = \frac{1}{2} \Xi^j_\dagger \cdot \partial_- X^I \]  \hspace{1cm} (4.35)

And once more the (0, 4) supersymmetry is manifest. It is clear from our construction that \( \pi_I \) play the role of coordinates on the cotangent space. Thus, the target space is locally \( T^*|P| \).
4.6. Comparison with standard (0, 4) models

Now that we have determined the field content and (0, 4) multiplets let us begin to make contact with the general form of the Lagrangian described in section 2. Since the target space is locally \( \mathcal{U} \times T^*|\mathcal{P}| \) there must be \( N + 1 = D + \frac{1}{12} c_2 \cdot P \) scalar multiplets. The scalars in these multiplets have both left- and right-moving degrees of freedom, whereas the complex scalars \( \pi^I \) derived from (4.8) are purely right-moving. On the other hand, the chiral 2-form \( \beta \) also gives \( b^\perp_2 \) real left-moving scalars \( \rho^a \). Since the signature of \( \mathcal{P} \) is negative, we must pair \( b^\perp_2 \) degrees of freedom from the \( \rho^a \) with the rightmovers \( \pi^I \). The remaining \( |\sigma(\mathcal{P})| \) degrees of freedom correspond to the left-moving fermions denoted by \( \lambda \) in section 2. We will discuss this pairing of left- and right-moving degrees of freedom further in section 5 below.

We can also compare with the supersymmetry transformations of section two. Let \( Z^M = (X, \theta) \) be the superembedding coordinates and let \( M = (m, \mu); A = (a, \alpha) \) be “curved” and “inertial” eleven dimensional indices respectively. Lower-case Latin (Greek) letters denote bosonic (fermionic) components. Up to local Lorentz transformations the vielbein transforms under supercoordinate transformations as \( \delta \gamma E^A_M = D_M \gamma^A \) where \( D_M \) is the covariant derivative. We see that the fermionic transformations which preserve a given background are parametrized by covariantly constant spinors \( \epsilon \). The fermionic symmetries of the fivebrane are:

\[
\delta_\epsilon Z^M = E^M_\alpha \epsilon^\alpha; \quad \delta_\kappa Z^M = (1 + \Gamma)_{\beta \alpha \kappa} E^M_\alpha
\] (4.36)

where \( \Gamma \) is as in section 3.

In order to recover the field content of the six dimensional (2, 0) tensor multiplet we need to do some gauge fixing. Upon reducing the resulting equations to two dimensions we get highly non-linear expressions. Ultimately we want to compare with the supersymmetries of the sigma model presented in section 2 (which is quadratic both in derivatives and in the right-moving fermions). Such a truncation brings us back to the supersymmetry transformations of appendix B.2 and their reduction, equations (4.34), (1.35). Comparing with (2.10) we can read off the three complex structures encoded in these equations: Let us choose a real basis \( \delta \phi^I = \delta \phi^{1I} + i \delta \phi^{2I}; \ \delta \pi^I := D_I^J \delta \pi^J = \delta \phi^{3I} + i \delta \phi^{4I} \). Similarly \( \chi^I = \theta^{1I} + i \theta^{2I}; \ \psi^I = -\theta^{4I} + i \theta^{3I} \) and \( \zeta = \epsilon^0 + i \epsilon^1; \ \lambda = \epsilon^2 + i \epsilon^3 \). With these definitions (4.34) takes the form

\[
\delta \phi^{iI} = \epsilon^0 \theta^{iI} + \epsilon^r J_{rI} \theta^{3J}, \quad i = 1, \ldots 4
\] (4.37)
where
\[ J_1 = [\sigma^3 \otimes i\sigma^2]^i_j \delta^I_J, \quad J_2 = [i\sigma^2 \otimes \mathbb{I}_2]^i_j \delta^I_J, \quad J_3 = [\sigma^1 \otimes i\sigma^2]^i_j \delta^I_J \] (4.38)

One can verify that they satisfy \( J_r J_s = -(\delta_{rs} + \varepsilon_{rst} J_t) \).

5. Local target-space geometry

The low-energy two-dimensional Lagrangian encodes the geometry of the target space of the (0, 4) model. We will derive this Lagrangian by Kaluza-Klein reduction of the chiral fivebrane Lagrangian of \([8, 9]\).

5.1. Bosonic Lagrangian

Let us start by reducing the bosonic part of the five-brane action presented in \([8]\). The action possesses manifest general coordinate covariance only along five of the six worldvolume dimensions. Here we will take the distinguished direction to be the spatial direction of the two-dimensional world-sheet \( W_2 \) which is taken to be flat. For our conventions/definitions we refer to appendix A.2.

The action consists of three terms
\[
L_1 = -\sqrt{-\det \left( G_{\hat{\mu}\hat{\nu}} + G_{\hat{\mu}\rho}G_{\hat{\nu}\lambda}\tilde{H}^{\rho\lambda}/\sqrt{-G_5} \right)}
\] (5.1)
\[
L_2 = -\frac{1}{4} \tilde{H}^{\mu\nu} \partial_1 \beta_{\mu\nu}
\] (5.2)
\[
L_3 = \frac{1}{8} \varepsilon_{\mu\nu\rho\kappa\lambda} \frac{G^{1\rho}}{G_{11}} \tilde{H}^{\mu\nu} \tilde{H}^{\kappa\lambda}
\] (5.3)

We will use the “static gauge” \( X^{\hat{\mu}} = \sigma^{\hat{\mu}} \) in which the above expressions simplify and we recover the field content of the (2, 0) six-dimensional multiplet discussed in the appendix B (note however that here we are using a gauge in which the antisymmetric tensor is effectively five-dimensional in the sense that \( \beta_{1\hat{\mu}} = 0 \)). Moreover we will keep only terms at most quadratic in \( \partial X \) and/or \( H \). As explained before, when reducing to \( W_2 \times \mathcal{P} \) the only nonvanishing components of the field \( \beta \) are along \( \mathcal{P} \). Hence (5.1) - (5.3) read
\[
L_1 = \frac{1}{2} g_{ab} \partial_{\hat{\mu}} X^a \partial_{\hat{\nu}} X^b - \frac{1}{4} \tilde{H}^{\mu\nu} \tilde{H}_{\mu\nu} (1 + g_{ab} \partial_{\rho} X^a \partial^\rho X^b) - \frac{1}{2} \tilde{H}^{\mu\kappa} \tilde{H}_{\kappa}^\nu g_{ab} \partial_{\mu} X^a \partial_{\nu} X^b
\] (5.4)
\[
L_2 = -\frac{1}{8} \varepsilon^{ABCD} \partial_0 \beta_{AB} \partial_1 \beta_{CD}
\] (5.5)
\[ L_3 = \frac{1}{4} \tilde{H}^{\mu \nu} H_{\rho \mu \nu} g_{ab} \partial^\rho X^a \partial_1 X^b \]  
(5.6)

where \( g_{ab} \) is the metric on the space transverse to the fivebrane. Keeping only up to two-derivative terms and dropping terms with derivatives along \( \mathcal{P} \) (which are suppressed by the size of \( \mathcal{P} \)) the above expressions simplify further

\[
S = \int_{W_6} dV \left( \frac{1}{2} \left( -\partial_0 X^a \partial_0 X^b + \partial_1 X^a \partial_1 X^b \right) g_{ab} + \frac{1}{4} g^{AC} g^{BD} \partial_0 \beta_{AB} \partial_0 \beta_{CD} 
- \frac{1}{8} \epsilon^{ABCD} \partial_0 \beta_{AB} \partial_1 \beta_{CD} \right),
\]  
(5.7)

Reducing the kinetic term of the scalars in (5.7) using (4.4) we see that it gives rise to a term,

\[
\int_{W_2} d^2 \sigma (\partial_{(+} \varphi^I \partial_{-}) \bar{\varphi}^T G_{I\bar{J}},
\]  
(5.8)

where \( G_{I\bar{J}} = \int_{\mathcal{P}} dV (\mathcal{L}) \omega_I \omega_{\bar{J}} \), and \( G(\mathcal{L}) \) is the hermitian metric on the line bundle \( \mathcal{L}|_\mathcal{P} = N(\mathcal{P} \hookrightarrow X) \) (in complex notation, it has only one component). By using a Fierz identity (see B.1) we can establish the metric on the space of \( \varphi \)'s in terms of the intersection matrix

\[
G_{I\bar{J}} = D_{I\bar{J}} = \int_{\mathcal{P}} \omega_I \wedge \omega_{\bar{J}}.
\]  
(5.9)

As for the chiral two-form, the reduction of (5.7) gives (omitting the universal superfield)

\[
S_0 = \int_{W_2} d^2 \sigma (\partial_{(0} \pi_I \partial_{+)} \pi_{\bar{J}} D^{I\bar{J}} - \partial_0 \rho^a \partial_{-} \rho^b D_{ab}).
\]  
(5.10)

5.2. Fermionic Lagrangian

Equations (5.8), (5.9) and (5.10) contain some of the essential information we need to extract the geometric data for the (0,4) Lagrangian. However, to extract all the data we must consider the quadratic terms in fermions. Therefore, we look at the terms quadratic in the right-moving fermions containing exactly one derivative along \( W_2 \). These come from the reduction to \( W_2 \) of the quadratic Lagrangian for fermions:

\[
\int_{W_2} \int_{\mathcal{P}} dV \left( \frac{1}{2} g^{AC} g^{BD} \partial_0 \beta_{AB} \bar{\theta} \Gamma_0 \Gamma_C D_D \theta + \frac{1}{4} \epsilon^{ABCD} \partial_0 \beta_{AB} \bar{\theta} \Gamma_1 \Gamma_C D_D \theta 
- \bar{\theta} \Gamma^\alpha D_\alpha \theta + \frac{1}{4!} \epsilon^{\alpha \beta} \epsilon^{ABCD} \bar{\theta} \Gamma_{ABCD} \Gamma_\alpha D_\beta \theta \right),
\]  
(5.11)
where \( \theta \) is the eleven-dimensional superspace coordinate (the superpartner of the embedding coordinates \( X^M(\sigma) \)), \( \Gamma \)'s are eleven-dimensional gamma-matrices and \( \mathcal{D} \) is the pullback of the spin connection from the ambient space to the fivebrane worldvolume \( W_6 \). This piece of the action is obtained by gauge fixing the \( \kappa \)-symmetric action of [17] in a general curved background, keeping only the terms quadratic in \( \theta \) which involve exactly one derivative along \( W_2 \), and discarding \( \mathcal{O}((X^a)^2) \) terms. The last term comes from the coupling of the six-form potential to the fivebrane worldvolume.

In reducing (5.11) we can make use of the \( \kappa \)-symmetry to eliminate the unphysical degrees of freedom of \( \theta \) and express it in terms of six-dimensional spinors \( \psi^{-I} \otimes \Delta^I \), \( \chi^{-I} \otimes (\Delta^*)_I \). Here \( \psi^{-I} \), \( \chi^{-I} \) are two-dimensional Weyl spinors and \( \Delta^I \) are four-dimensional Weyl spinors (see section 3.2 and appendix B). The six-dimensional \( \Gamma \)-matrices decompose as in section 3.2. Suppressing internal (along \( P \)) covariant derivatives on the spinors, the reduced action can be cast in the form (the universal superfield is not included)

\[
\int_{W_2} d^2\sigma \partial_+ \varphi^K ( [\psi^{-J} ]^\dagger \psi^{-J} \mathcal{R}^{TI}_K + (\chi^{-J})^\dagger \chi^{-J} \mathcal{R}^{TI}_K ) + \text{c.c.}
\]

where we use the same conventions for the two-dimensional fermions as in section 3 and we have defined

\[
\mathcal{R}^{TI}_K = \int_P dV (\Delta^J)^\dagger \nabla_K \Delta^I; \quad \mathcal{R}^{TI}_K = \int_P dV (\Delta^I)^\dagger \nabla_K (\Delta^*)_I,
\]

where \( \nabla_I \) is the covariant derivative discussed in section 4.2.

We will now analyze the meaning of the “coupling” of (5.13) to extract the target space geometry. As in section 4.1 we consider a family of surfaces \( P_\varphi \) near \( P_\varphi = 0 \). With respect to our basis of holomorphic two-forms \( \nabla \) acts in the following way:

\[
\nabla_I \omega_L = [\Gamma_J]^x_I \omega_x
\]

where \( x \in \{ I, T, a \} \) and \( [\Gamma_J] \) is the \( \varphi \)-dependent Gauss-Manin connection matrix. Therefore

\[
\partial_K D^I_T = \int_P \nabla_K \omega_L \wedge \omega_T = \int_P [\Gamma_K]^L_I \omega_L \wedge \omega_T,
\]

where we have defined \( \partial_I = \frac{\partial}{\partial \varphi^I} \). On the other hand \( \Delta_I = \omega_I(\gamma^{(2)})\xi \), \( \gamma^{(2)} = \gamma^A \gamma^B \frac{\partial}{\partial x^A} \otimes \frac{\partial}{\partial x^B} \) and it’s easy to see that

\[
\mathcal{R}^{TI}_K = \int_P dV \xi^\dagger \omega_T(\gamma^{(2)}) \nabla_K \omega_I(\gamma^{(2)}) \xi
\]

\[
= \int_P dV \xi^\dagger \gamma^{mn} \omega_T \omega_I [\Gamma_K]^x_I \omega_{xAB} \gamma^{AB} \xi = \partial_K D^I_T.
\]

For the last step we have used (5.15), the fact that \( \gamma^{mn} \xi = 0 \) (as in 4.3) and a little bit of gamma-matrix algebra. We conclude that \( \mathcal{R}^{TI}_K \) is just the Christoffel symbol of the manifold \( |P| \) with Kähler metric (5.9).
5.3. Comparison to the standard \((0, 4)\) Lagrangian

Let us now assemble the data we have gathered and compare to the standard Lagrangian spelled out in section two. The part of (the bosonized version of) the \((0, 4)\) action \((2.5)\) containing all one-derivative terms quadratic in fermions is

\[
\psi^i_+ \psi^j_- [F_{ij\hat{a}} \partial_+ \rho^{\hat{a}} + \Upsilon_{ijk}^{(+) \partial_+ \phi^k}].
\]

(5.17)

where \(\rho^{\hat{a}}, \hat{a} = 1, \ldots, b^-_2 - b^+_2\) is the set of purely left-moving scalar fields.

Comparison to \((5.12)\) implies that the \(b\)-field and the gauge connection of the vector bundle over the target-space are flat, and that the metric on the target-space is

\[
ds^2 = D(\varphi, \overline{\varphi}) I J d\varphi^I d\overline{\varphi}^J + D(\varphi, \overline{\varphi}) I J (\nabla \pi)_I (\nabla \pi)_J
\]

(5.18)

where \(D_{IJ}\) is the intersection pairing defined in \((5.9)\). Since the connection on \(T\overline{\mathcal{M}}\) has no torsion supersymmetry requires that the metric \((5.18)\) is hyper-Kähler, and indeed, gives a way to prove the hyperkähler property of the metric. Using the supersymmetry transformations \((4.37)\) and the fact that \((0, 4)\) symmetry is unbroken, it follows that the 3 complex structures in \((4.38)\) are covariantly constant. \(^5\)

It is interesting to compare the metric with the c-map construction \([7]\). The metric there reads

\[
ds^2 = D_{IJ} d\varphi^I \otimes d\overline{\varphi}^J + D_I^J (\nabla \pi)_I \otimes (\nabla \pi)_J
\]

(5.19)

(note that in this case \(D_{IJ} = D_{JI}\)) One has three closed two-forms \(\omega^1, \omega^2, \omega^3\) where \(\omega^1\) is the associated \((1, 1)\) form and \(\omega^{1,2}\) are the real, imaginary parts of \(d\varphi^I \wedge (\nabla \pi)_I\).

Setting \(d\varphi^I = d\phi^{1I} + id\phi^{2I}; (\nabla \pi)_I := D_{I J} (\nabla \pi)_J = d\phi^{3I} + id\phi^{4I}\), the metric takes the form \(ds^2 = G_{iI,jJ} d\phi^{iI} \otimes d\phi^{jJ}\), where \(G_{iI,jJ} = D_{IJ} \delta_{ij}\). The three two-forms can be used to construct three complex structures \(J_{rI}^{iI,jJ} = C_{iIn,jK}^{iI} kK \omega^r_{kK,iJ}\) which in components read:

\(J_1 = [\sigma^3 \otimes i\sigma^2]^i_j \delta^I_j, J_2 = [i\sigma^2 \otimes \mathbb{1}_2]^i_j \delta^I_j, J_3 = [\sigma^1 \otimes i\sigma^2]^i_j \delta^I_j\). These are exactly the same as in \((4.38)\);

\(^5\) It would be desirable to have a more direct, and more standard proof of this fact. This is being investigated in \([19]\).
6. Comments on the global structure of the target space

6.1. Narain theory in the entropic factor

We now consider to the periodicities of $\beta$, needed to determine the global structure of the fibers of $p$ in (1.13). As we have stressed in the introduction, we expect the target space to be compact. Therefore, while the local target manifold is $\mathbb{R}^4 \times T^*|P|$, the fibers should be compactified, maintaining the hyper-Kähler property. The most natural (perhaps the only) way to do this is to take a quotient by a lattice in the fiber of $T^*|P|$ so that the fibers of $\tilde{\mathcal{M}}$ are complex tori.

Passing to a real basis $\{\omega_I, I = 1, \ldots b_2^+\}$ of self-dual two forms on $\mathcal{P}$ we can expand \( \beta = \pi^I \omega_I + \rho^a \omega_a \quad (6.1) \) and reexpress (5.10) as \[
S_0 = \int_{W_2} d^2 \sigma (\partial_0 \pi^I \partial_+ \pi^J D_{I,J} - \partial_0 \rho^a \partial_- \rho^b D_{ab}). \quad (6.2)
\]
where $D_{I,J} = \int_P \omega_I \wedge \omega_J$. Thus the metric is diagonal on left- and right-movers. However, the information of how to “combine” the zero-modes of the left and a right-moving bosons to define the statespace of the full conformal field theory is not contained in (6.2). This has to be imposed ad hoc.

Our ansatz is that the Lagrangian in (6.2) is a Narain $\sigma$ - model with non-trivial Narain data: a constant metric, a constant torsion, and Wilson lines. Thus, we take the periodicities \( \beta \to \beta + n^x U_x, \ n^x \in \mathbb{Z} \quad (6.3) \) where we have introduced a basis $\{U_x; \ x = 1, \ldots b_2\}$ of $H^2(\mathcal{P}, \mathbb{Z})$. The data for the zero-modes of the scalars are encoded in the projections onto the definite signature subspaces:

\[
P : H^2(\mathcal{P}; \mathbb{Z}) \otimes \mathbb{R} \to H^2^- (\mathcal{P}; \mathbb{R}) \perp H^2^+ (\mathcal{P}; \mathbb{R}). \quad (6.4)
\]
In particular, the left and right-moving momenta are just $p = (F^a_{,x} n^x e_a; f_\hat{I}^I n^x e_{\hat{I}})$, where $F^a_{,x} = \int_P \omega_a \wedge U_x$; $f_\hat{I}^I = \int_P \omega_{\hat{I}} \wedge U_x$, and $e_a$ is the vielbein for the metric $D_{ab}$ and $e_{\hat{I}}$ is the vielbein for $D_{\hat{I},\hat{J}}$. 

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6.2. Charge violation by instantons: The “MSW effect”

The Narain model of the previous section is somewhat peculiar because the conserved $U(1)$ charges coupling to the string are in the lattice $H^2(X; \mathbb{Z})$ which is a (small!) sublattice of $H^2(\mathcal{P}; \mathbb{Z})$. This puzzle was resolved in [2] as follows. The charges $H^2(\mathcal{P}; \mathbb{Z})$ are conserved in the $(0, 4)$ sigma model studied in this paper, but they are violated by membrane instanton processes in the full $M$-theory. As mentioned in [2] if a state in the $(0, 4)$ CFT is charged under an element in $H^2(\mathcal{P}; \mathbb{Z})$ which is not in $H^2(X; \mathbb{Z})$ it will decay to a state charged in $H^2(X; \mathbb{Z})$. Indeed, since the map (1.12) is injective the dual map:

$$H^2(\mathcal{P}; \mathbb{Z}) \xrightarrow{\iota^*} H^2(X; \mathbb{Z}) \to 0$$

is surjective, and hence has a large kernel. Elements of the kernel are nontrivial surfaces $[\Sigma] \in H^2(\mathcal{P}; \mathbb{Z})$ which bound a 3-ball in $X$, $\Sigma = \partial B$. It is possible to have a membrane instanton whose worldvolume is $B$ because the equation

$$dH = -Q(M2)\delta(\Sigma \hookrightarrow W_6),$$

where $H = d\beta$ and $Q(M2)$ is the membrane charge, allows membranes to end on five-branes [20]. Since this process uses $M$-theory instantons, it will only be important near degenerations of $\mathcal{P}$.

One interesting question raised by this “MSW effect” is whether states on the 5brane can carry torsion charges. The kernel of $\iota_*$ is a sublattice of $H^2(\mathcal{P}; \mathbb{Z})$. We claim that under Poincaré duality $PD : H^2(\mathcal{P}; \mathbb{Z}) \to H^2(\mathcal{P}; \mathbb{Z})$, we have $PD(\ker \iota_*) = \iota^* (H^2(X; \mathbb{Z}))^\perp$ where the orthogonal complement is in the Hodge metric of $\mathcal{P}$. To prove this note that if $[\Sigma] \in \ker \iota_*$ then its Poincaré dual form $\eta_\Sigma \in H^2(\mathcal{P}; \mathbb{Z})$ satisfies

$$\int_{\mathcal{P}} \eta_\Sigma \wedge \iota^*(\theta) = \int_{\iota_*(\Sigma)} \theta$$

for all $\theta \in H^2(X; \mathbb{Z})$. Since $\iota^*(H^2(X; \mathbb{Z}))$ is not unimodular while $H^2(\mathcal{P}; \mathbb{Z})$ is unimodular, the sublattice $\iota^*(H^2(X; \mathbb{Z})) \oplus \iota^*(H^2(X; \mathbb{Z}))^{\perp}$ will have finite index in $H^2(\mathcal{P}; \mathbb{Z})$. The quotient group is a (large) group of potential torsion charges. We say “potential” because we do not fully understand the model globally on $|\mathcal{P}|$. It would be interesting to understand how the above torsion charges can be understood in the framework of the K-theory interpretation of D-brane charges [21][22].

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6 We thank J. Maldacena and E. Witten for important clarifying explanations about this process.
6.3. Narain data for the universal factor

It is possible to be much more explicit about the Lagrangian for the universal multiplet. Just as for the rest of the fields, its action follows from the reduction of (5.1) and (5.4).

Let \{J, \theta; \Lambda = 1, \ldots h^{1,1}(X) \} be a basis of \( H^{1,1}(X, \mathbb{R}) \) such that \( \theta_\Lambda \) restricts to a basis of anti-self-dual forms on \( \mathcal{P} \). Moreover, let \( \{Y_w; w = 0, \ldots h^{1,1}(X) \} \) be a basis of \( H^{1,1}(X, \mathbb{Z}) \). The part \( \beta_u \) of the chiral 2-form contributing to the “universal” multiplet is expanded as

\[
\beta_u = u_4 J + \rho^\Lambda \theta_\Lambda \tag{6.8}
\]

with the periodicities

\[
\beta_u \rightarrow \beta_u + n^w Y_w; \quad n^w \in \mathbb{Z}. \tag{6.9}
\]

The universal multiplet is governed by the action

\[
S_{un} = \int_{W_2} d^2 \sigma \{ D_{00} \partial_0 u_4 \partial_+ u_4 - D_{\Lambda \Lambda'} \partial_0 \rho^\Lambda \partial_- \rho^{\Lambda'} \} \tag{6.10}
\]

where \( D_{\Lambda \Lambda'} := \int_{\mathcal{P}} \theta_\Lambda \wedge \theta_{\Lambda'} = \int_X P \wedge \theta_\Lambda \wedge \theta_{\Lambda'}; \quad D_{00} := \int_{\mathcal{P}} J \wedge J = 2 \text{Vol}(\mathcal{P}) \).

Repeating the analysis of section 6.1 we see that the left/right-moving momenta are given by

\[
P = (F^\Lambda_w n^w e_\Lambda; f^0_w n^w e_0) \tag{6.11}
\]

where \( F^\Lambda_w := \int_{\mathcal{P}} \theta_\Lambda \wedge Y_w; \quad f^0_w := \int_{\mathcal{P}} J \wedge Y_w \) are the projections

\[
P: H^2(X, \mathbb{Z}) \otimes \mathbb{R} \rightarrow H^{2+}(X, \mathbb{R}) \oplus H^{2-}(X, \mathbb{R}), \tag{6.12}
\]

and \( e_\Lambda (e_0) \) is a vielbein for the metric \( D_{\Lambda \Lambda'} (D_{00}) \). The self-dual (right-moving) piece is generated by \( J \). The the radius \( R \) of the \( S^1 \) in the target space is given by \( R^2 = \frac{1}{2 \pi^2} \text{Vol}(\mathcal{P}) \), when \( h^{1,1}(X) = 1 \), and by more complicated formulae in general.

6.4. Effects of the \( M \)-theory 3-form

Finally, let us comment on two effects that happen when we turn on the \( C_3 \) field of the eleven-dimensional supergravity.

First, in the Kaluza-Klein reduction of \( M \) theory on \( X \) the field \( C_3 \) gives rise to \( h^{1,1}(X) \) five-dimensional vectors (together with KK modes from the metric these form the
gravity multiplet and $h^{1,1}(X) - 1$ vector multiplets). The coupling of $C_3$ to the fivebrane worldvolume induces string couplings to the background gauge fields

$$\int_{W_2} d^2\sigma \{ A_+^\Lambda \partial_- \rho ^N D_{\Lambda \Lambda'} + A_0^0 \partial_+ u_4 D_{00} \} \quad (6.13)$$

where $A^\Lambda$ are the abelian vector fields and $A^0$ is the graviphoton. Such couplings are also important for cancellation of anomalies in the gauge transformations in the presence of the string $[23]$. Since the projections in (6.12) already encode Narain data, including the flat connection on the gauge bundle, we see that turning on $C_3$ just shifts the gauge fields.

Second, the 5brane action consists of a Dirac part and a WZ part. In the Dirac part the fieldstrength of the chiral 2-form enters through $H = d\beta - C_3$. In the Kaluza-Klein reduction this leads to shifts of the periodicities of the chiral scalars, for example, $\partial_+ \pi^I \to \partial_+ \pi^I + C^I$. If $C_3 = dX^1 \wedge \theta$, with $\theta \in H^2(X; \mathbb{R})$ then the Narain vectors are shifted by $p \to p + \theta$, leading to a shift in $L_0 - \tilde{L}_0$. If we consider the corresponding IIA picture this is in accord with the Witten-effect shifting of the D0 charge:

$$\Delta(L_0 - \tilde{L}_0) = \int_{\mathcal{P}} \left( p \wedge \theta + \frac{1}{2} \theta \wedge \theta \right). \quad (6.14)$$

where we have identified $p$ with the first Chern class of the Chan-Paton bundle on the D4 brane.

7. Conclusion: 5 problems on 5 branes

First and foremost it would be good to extend the discussion in this paper to understand the global geometry on $|P|$. This consists of at least two important sub-problems. First, we have restricted to an open neighborhood in $|P|_s$. It would be interesting to take into account the effects of monodromy. Second, the 4-cycle $\mathcal{P}$ will degenerate on a codimension one discriminant locus $\mathcal{D} = |P| - |P|_s$ of the linear system. The generic singularity will be a rational double point. Many interesting and important questions depend crucially on understanding what happens to the $(0, 4)$ model when the fivebrane degenerates. In $[24]$ a drastic degeneration with $D$ points of self-intersection was successfully used to count black hole entropy at leading order in large charges.

Second, as mentioned in the introduction, one of the original motivations for this work was to find a state-counting formula for BPS states in $M$-theory compactifications which are macroscopically 4d black holes with 8 supersymmetries. We believe that combining
the elliptic genus of $(0,4)$ models studied in \[25\] with the results of this paper one can derive formulae for the BPS degeneracies. This idea is currently under investigation.

Third, it would be nice to clarify the status of the above model as a CFT. Since the $\sigma$-model described in this paper is rather elaborate, it would be nice to have a clear understanding of whether the entropic factor is, in fact, a conformal field theory (and if not, what it flows to). Moreover, it might be useful to find a linear sigma model which renormalizes to the above nonlinear model. This would be possible if the metric on $T^*|P|$ were given by a hyper-Kähler quotient. Thus, an interesting question raised by this work is whether there is a sense in which the metric on $T^*|P|$ induced by the Calabi-Yau metric becomes the hyperkähler quotient metric in the limit of large $P$.

Fourth, it would be nice to extend the discussion to fivebranes with even less supersymmetry, leaving a $(0,2)$ string. Such configurations would appear if the $M5$ worldvolume is near a boundary, as in the Horava-Witten picture. At a formal level, much of the above discussion generalizes to the $(0,2)$ case. However quantum corrections are expected to be much more important here.

Fifth, if the $M$-theory compactification has a heterotic dual then there must be a description of the same strings in the heterotic picture. Indeed, in the case $X = K3 \times T^2$ with $P = K3$ one reproduces the heterotic string itself [26,27]. However, in the case of $P$ defined by a class $P$ with $P$ large there will be a large number of left- and right-moving degrees of freedom. Because of the MSW effect it is not obvious that these charges should really be visible. We think this is worth understanding better.

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Appendix A. List of some notation

A.1. General notation

$A_{[1\ldots k]} = A_{1\ldots k}$ for a $k$-form $A$

$\alpha = 0, 1$: the directions along the string world-sheet

$a = 6, \ldots 10$: the directions transverse to $W_6$;

$(a = 1, \ldots b^-_2$ also enumerates the basis vectors of $H^{2-}(\mathcal{P}))$

$\hat{a} = 1, \ldots b^-_2 - b^+_2$

$A, B = 2, \ldots 5$: the (real) directions along $P$.

$\beta$ The chiral 2-form of the 5brane 6D tensor multiplet.

$\gamma^\mu, \Gamma^M, \hat{\gamma}^a$: Gamma matrices defined in sections 3, 4.3 and B1.

$\hat{\gamma}^1 - \hat{\gamma}^4$: matrices defined in section 4.4 (below (4.23)).

$D_{IJ} \equiv \int_P \omega_I \wedge \omega_J$

$D_{ab} \equiv \int_P \omega_a \wedge \omega_b$

$H^{1,1}(X)^\perp$: The subspace of $H^{1,1}(X)$ “orthogonal” to $J$, see section 4.1.

$H^{2\pm}(\mathcal{P})$: the spaces of self-dual, antself-dual 2-forms on $\mathcal{P}$.

$\theta_A, \Lambda = 1, \ldots h^{1,1}(X) - 1$: a basis of $H^{1,1}(X)^\perp$.

$i = 1, \ldots 4$: a $USp(4)$ index, except in section 2.

$I = 1, \ldots \frac{1}{2}(b^+_2 + 1)$, except in section 2.

$\hat{I} = 1, \ldots b^+_2$

$J$: the Kähler form on $\mathcal{P}$ and on $X$

$\mathcal{L}$: the holomorphic line bundle over $X$, associated to the divisor $\mathcal{P}$.

$M = 0, \ldots 10$: the spacetime index

$\mu, \nu = 0, \ldots 5$: the directions along $W_6$.

$m, \overline{m} = 1, 2$: the (complex) directions along $\mathcal{P}$.

$N$ The number of $(0,4)$ multiplets. Defined in (1.3).

$\omega_{-a}, a = 1, \ldots b^-_2$: a basis of $H^{2-}(\mathcal{P})$.

$\omega_I (\omega_J), I = 1, \ldots \frac{1}{2}(b^+_2 + 1)$: A basis of $H^{(2,0)}(\mathcal{P})$ ($H^{(0,2)}(\mathcal{P})$)

$\omega_i, \hat{I} = 1, \ldots b^+_2$: a basis of $H^{2+}(\mathcal{P}, \mathbb{R})$

$\mathcal{P}$: A generic smooth holomorphic surface inside $X$.

$|P|_s$ The locus of smooth divisors in the linear system $|P|$.

$\mathcal{P}$: The cohomology class in $H^2(X; \mathbb{Z})$ dual to the 4-cycle $\mathcal{P}$.

$\sigma^0 - \sigma^5$: the coordinates on $W_6$

$U_x, x = 1, \ldots b_2(\mathcal{P})$: a basis of $H^2(\mathcal{P}, \mathbb{Z})$
Y_w, w = 1, \ldots, h^{1,1}(X): a basis of H^{1,1}(X, \mathbb{Z})

X A Calabi-Yau 3-fold, used for compactifying M-theory.

X^M(\sigma): the embedding of W_6 to the eleven-dimensional spacetime.

\xi^{(6)}: the covariantly constant spinor of the Calabi-Yau.

\xi: the component of \xi^{(6)} along P (in a local decomposition).

A.2. Conventions for Section 5.1

\hat{\mu}, \hat{\nu} = 0, \ldots, 5: the directions along W_6.

\mu, \nu = 0, 2, \ldots, 5: omitting the "distinguished" direction.

\sigma^\hat{\mu}: the coordinates on W_6.

G_{\hat{\mu}\hat{\nu}} = \eta_{MN} \partial_{\hat{\mu}} X^M \partial_{\hat{\nu}} X^N

G_5 = \det(G_{\mu\nu})

H_{\mu\nu\rho} = 3 \partial_{[\mu} \beta_{\nu\rho]}\]

\tilde{H}^{\mu\nu} = \frac{1}{6} \varepsilon^{\mu\nu\rho\kappa\lambda} H_{\rho\kappa\lambda}

Appendix B. (2,0) tensor multiplet

B.1. The conventions

In this section we work in six-dimensional Minkowski space. The R-symmetry group for the theory with sixteen real supercharges is SO(5). Let a = 1, \ldots, 5 (the index a is SO(5) Euclidean) and \mu = 0, 1, \ldots, 5. A basis of gamma-matrices \hat{\gamma}^a in five-dimensional Euclidean space can be constructed as follows:

\hat{\gamma}^{6,7,8} = \begin{pmatrix} 0 & \sigma^{1,2,3} \\ \sigma^{1,2,3} & 0 \end{pmatrix}; \quad \hat{\gamma}^{9} = \begin{pmatrix} 0 & i\mathbb{I}_2 \\ -i\mathbb{I}_2 & 0 \end{pmatrix}; \quad \hat{\gamma}^{10} = \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \quad \text{(B.1)}

In checking the susy transformations of the (2,0) multiplet of B.2 it is more convenient to work in a slightly different basis than the one we used in 4.3 for gamma-matrices \Gamma^\mu in six-dimensional Minkowski space:

\Gamma^\mu = \begin{pmatrix} 0 & \hat{\gamma}^\mu \\ \hat{\gamma}^\mu & 0 \end{pmatrix} \quad \text{(B.2)}

where

\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix}; \quad \gamma^{1,2,3} = \begin{pmatrix} 0 & \sigma^{1,2,3} \\ \sigma^{1,2,3} & 0 \end{pmatrix}; \quad \gamma^4 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \quad \text{(B.3)}

\tilde{\gamma}^{0,4} = \gamma^{0,4}; \quad \tilde{\gamma}^5 = -\gamma^5 = -i\mathbb{I}_4

\text{29}
In this basis the charge-conjugation and the chirality matrices are
\[
C^{(6)} = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}; \quad \rho^{(6)} = i^2 \Gamma^1 \ldots \Gamma^5 \Gamma^0 = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}
\] (B.4)
where
\[
c = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}
\] (B.5)
with \(\epsilon\) given by
\[
\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2; \quad \epsilon^{AB} = -\epsilon_{AB}.
\] (B.6)

The real, antisymmetric tensor of \(USp(4)\) obeys
\[
\Omega = -\Omega^{-1}; \quad \Omega(\gamma^a)^T r = \gamma^a \Omega
\] (B.7)
We can write \(\Omega\) in this basis explicitly
\[
\Omega = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}
\] (B.8)

For an (anti)chiral Spin(1,5) spinor \(\theta_i, i = 1, \ldots 4\) transforming in the 4 of \(USp(4)\) \((i\) is a \(USp(4)\) index) the symplectic-reality condition reads
\[
\theta_i = \Omega_{ij} \theta^T_j r; \quad \bar{\theta}_i = -\Omega_{ij} \bar{\theta}^T_j c
\] (B.9)
where \(\bar{\theta} = \theta^+ \gamma^0\).

B.2. 6D supersymmetry

The 6D \((2, 0)\) multiplet consists of a self-dual (on-shell) antisymmetric two-form \(\beta_{\mu \nu}\), four six-dimensional Weyl spinors \(\{\psi_i, i = 1 \ldots 4\}\) obeying the symplectic reality condition (B.9), and five scalars \(\{X^a, a = 6, \ldots 10\}\). In other words, under the little group \(Spin(4) \times USp(4)\), \(\beta_{\mu \nu}, \psi, X^a\) transform in the \((\mathbf{3}, 1); 1), (\mathbf{4}; 4), ((\mathbf{1}, 1); \mathbf{5})\) respectively. After the elimination of the auxiliary field introduced in the covariant formulation of the fivebrane of \([\mathbb{L}7]\), supersymmetry closes on-shell.

The susy transformations are (suppressing the USp(4) index on the fermions):
\[
\delta X^a = -2\epsilon \gamma^a \psi
\]
\[
\delta \psi = \left(\frac{1}{2} \gamma^\mu \partial_\mu X^a \gamma_a + \frac{1}{8} \gamma^{\mu \nu \rho} H_{\mu \nu \rho}\right) \epsilon
\]
\[
\delta \beta_{\mu \nu} = -2\epsilon \gamma_{\mu \nu} \psi
\] (B.10)
where \(H_{\mu \nu \rho} = \partial_{[\mu} \beta_{\nu \rho]}\). Using these equations one checks that the algebra closes on shell.
Appendix C. Some remarks on Kodaira-Spencer theory

In this appendix we show (4.3). Let \( X \) be a complex manifold with a divisor \( P \). There are two exact sheaf sequences which we are going to use

\[
0 \to \mathcal{O}(TP) \to \mathcal{O}(TX|_P) \to \mathcal{O}(L|_P) \to 0,
\]

which is a sequence over \( P \), and

\[
0 \to \mathcal{O}(TX \otimes [-P]) \to \mathcal{O}(TX) \to \mathcal{O}(TX|_P) \to 0,
\]

which is a sequence over \( X \). From the long exact sheaf-cohomology sequence associated to (C.1) we obtain:

\[
\cdots \to H^0(P, \mathcal{O}(TX|_P)) \to H^0(P, \mathcal{O}(L)) \to H^1(P, \mathcal{O}(TP)) \to H^1(P, \mathcal{O}(TX|_P)) \to \cdots
\]

(C.3)

Since \( H^1(P, \mathcal{O}(TP)) \cong H^{0,1}(TP) \), in order to show that the mapping (4.3) is injective, it suffices to show that \( H^0(P, \mathcal{O}(TX|_P)) = 0 \). For this we will use the following part of the exact long sheaf-cohomology sequence associated to (C.2):

\[
\cdots \to H^0(X, \mathcal{O}(TX)) \to H^0(P, \mathcal{O}(TX|_P)) \to H^1(X, \mathcal{O}(TX \otimes [-P])) \to \cdots
\]

where we noted that \( H^*(X, \mathcal{O}(TX|_P)) = H^*(P, \mathcal{O}(TX|_P)) \). However \( H^0(X, \mathcal{O}(TX)) \cong H^{0,0}(TX) \cong H^2(X) = 0 \) (where the last equivalence can be seen using the existence of a unique nowhere-vanishing holomorphic three-form on \( X \)). Moreover using Kodaira-Serre duality we have

\[
H^q(X, \mathcal{O}(TX \otimes [-P])) \cong H^{3-q}(X, \Omega^1(L))^* = 0; \quad q = 0, 1, 2,
\]

(C.5)

where the last equality is due to the fact that \( L \) is associated to the very ample divisor \( P \), and we can take \( c_1(L) \) to be arbitrarily large. Thus it immediately follows from (C.4) that \( H^0(P, \mathcal{O}(TX|_P)) = 0 \) and hence (4.3) is indeed injective. One can show that \( H^1(P, \mathcal{O}(TX|_P)) \cong H^1(X, \mathcal{O}(TX)) \cong H^{0,1}(TX) \cong H^{2,1}(X) \neq 0 \) so we cannot conclude from (C.3) that (4.3) is surjective.

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Appendix D. D3 on K3 and the (4, 4) σ-model

Although outside the main line of development of this paper, it is worthwhile discussing the properties of (4, 4) models within the framework of this paper. For a recent account see [28]. Here we address some complementary issues. To obtain a (4, 4) model we will consider a D3 wrapped on a holomorphic two-cycle (a Riemann surface) $\mathcal{P}$ inside $X = K3$ (whenever it doesn’t lead to confusion, we will keep the same notation as for the corresponding discussion in the case where $X$ is a Calabi-Yau three-fold). This is a much simpler system to analyze, since all the scalars coming from the reduction to the string world-sheet of the D3-brane low-energy lagrangian, are non-chiral. The number of left and right-movers is given by a formula similar to the one for the fivebrane:

$$N_L^B = N_R^B = d_P + b_1(\mathcal{P}) + 4$$  \hspace{1cm} (D.1)

The (bosonic part of the) gauge theory on the worldvolume involves a vector field $A_\mu$ and six scalars. When wrapped on the two-cycle, two scalars $X^4, X^5$ will parametrise the deformations, yielding $d_P$ scalars on the string worldvolume, while the other four $X^6 - X^9$ will form the universal superfield (here, in analogy to the discussion for the M5, we consider the D3-brane to be along $X^0 - X^3$ while $X$ is taken along $X^2 - X^5$). In this case, the universal superfield does not contain compact scalars, and is given simply by $\mathbb{R}^4$. In its turn, the vector field gives rise to $b_1(\mathcal{P})$ scalars. Note that the Kähler form no longer appears in our analysis of the scalar spectrum and the structure of the universal superfield is considerably simpler. More precisely, the counting goes as follows

$$\chi(\mathcal{P}) = \int_{c_1(\mathcal{P})} c_1(\mathcal{P}) = -\int_X P^2 = 2 - b_1(\mathcal{P}),$$  \hspace{1cm} (D.2)

where as before we have set $P = c_1(\mathcal{L}) = -c_1(\mathcal{P})$. On the other hand

$$\chi(\mathcal{L}) = \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{L}) = h^0(X, \mathcal{L}), \quad h^i(X, \mathcal{L}) \equiv \dim_{\mathbb{C}} H^i(X, \mathcal{L})$$  \hspace{1cm} (D.3)

where the last equality follows from the fact that $\mathcal{P}$ is very ample, and

$$h^0(X, \mathcal{L}) = \int_X e^P Td(X) = \int_X \left( \frac{1}{2} P^2 + \frac{1}{12} c_2(X) \right).$$  \hspace{1cm} (D.4)

Taking (D.2) into account and the fact that $\chi(X) = \int_X c_2(X) = 24$ for $X = K3$, we finally get

$$d_P = b_1(\mathcal{P}) = 2D + 2,$$  \hspace{1cm} (D.5)
where again $d_P$ stands for the real dimension of $H^0(X, \mathcal{L})$ and $D = \frac{1}{2} \int_X P^2$ as before. The $(4,4)$ action follows from the reduction of the Born-Infeld action for $D3$

$$L = -\frac{1}{4} F^2 + \sum_{a=4}^9 \partial_a X^a \partial^\alpha X^a + \ldots$$  \hspace{1cm} (D.6)

Let $\{\omega_I(\varphi), I = 1, \ldots, \frac{1}{2} b_1\}$ be a basis of holomorphic one-forms on $\mathcal{P}_\varphi$. We can write the gauge field in terms of this basis

$$A^m = \pi_I \omega^I,^m$$  \hspace{1cm} (D.7)

Moreover, to the first order in $\varphi$ we expand

$$X^4 + iX^5 = \varphi^I v_I,$$  \hspace{1cm} (D.8)

with $\{v_I\}$ a basis of holomorphic sections of $\mathcal{L}_P$.

The $c$-map works in the same way as in the case of [7], and we obtain the result that all the terms in the reduction of (D.6) come from a Kähler potential of the form $\overset{\sim}{\kappa}$

$$\overset{\sim}{\kappa} = \kappa(\varphi, \overline{\varphi}) + D_I \overline{\pi}_I \overline{\pi}_J; \quad D_I = \partial_I \overline{\partial}_J \kappa,$$  \hspace{1cm} (D.9)

Again supersymmetry mixes the scalars effectively doubling the coordinates and yielding a target space of real dimension $2b_1 = 4D + 4$.

Finally, the coupling of $D3$ to background RR fields gives rise to the $\sigma$-model $b$-field. In particular we have $\int_{D3} \phi^{RR} F \wedge F$, where the RR scalar has to be kept fixed in the $D3$ background. The discussion of the compactness of the target space as well as the dependence of the $\sigma$-model data in terms of $K3$ geometry follows the generic Calabi-Yau constructions.

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7 To prove this, one should use $\omega_I = \iota(v_I) \Omega|^{(2,0)}_P$. 

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