HYPERPLANE SECTIONS OF HYPERSURFACES

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Abstract. We compute some numerical invariants of the lines on hyperplane sections of a smooth cubic threefold over complex numbers. We also prove that for any smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d$ over an algebraically closed field of characteristic zero, if $d > n > 1$ and $(n, d) \neq (2, 3), (3, 4)$, then a general hyperplane section only admits finitely many others which are isomorphic to it.

This paper consists of two parts. In the first section, we study the lines on hyperplane sections of a smooth cubic threefold $X \subset \mathbb{P}^4$ over $\mathbb{C}$. All hyperplane sections provide a family $\mathcal{X} \to \mathbb{P}^4^*$ whose fibers are cubic surfaces. Let $U \subset \mathbb{P}^4^*$ be the dense open set parametrizing smooth intersections, and $\mathcal{X}_U \to U$ be the restriction on $U$. Then, the relative Fano variety of lines

$$F(\mathcal{X}_U/U) \to U$$

is a finite étale morphism of degree 27. We would like to compute several numerical invariants of it, especially the monodromy group. Similarly and more classically, let $\mathcal{Y} \to \mathbb{P}^N$ be the universal family of cubic surfaces in $\mathbb{P}^3$ and $\mathcal{Y}_V \to V$ be the restriction on the smooth part. Then,

$$F(\mathcal{Y}_V/V) \to V$$

is also a finite étale morphism of degree 27. The monodromy group of $F(\mathcal{Y}_V/V) \to V$ is well-known as $W(E_6)$. However, it is not properly stated in any literature. We collect the proof in Section 1.2 mainly from [Dem80] and [Har79]. Then, we go back to study the family $F(\mathcal{X}_U/U) \to U$. The monodromy group of $F(\mathcal{X}_U/U) \to U$ turns out to be $W(E_6)$ as well, which is stated in [ADF+15] without proof. After some basic properties in Section 1.2, we give a proof in Section 1.3 of the following theorem:

**Theorem 0.1.** The monodromy group of the finite étale morphism

$$F(\mathcal{X}_U/U) \to U$$

is $W(E_6)$, that is, the same as the monodromy group of the $F(\mathcal{Y}_V/V) \to V$.

The proof is via studying the monodromy action on the middle cohomology and identify some elements in the cohomology group as lines. In the computation of action on the middle cohomology, we follow a lot from [Voi03, Ch. 1 to Ch. 3]. This approach works equally for $F(\mathcal{Y}_V/V) \to V$.

In the second section, we study the variation of hyperplane sections of a smooth hypersurface over an algebraically closed field of characteristic zero. Given a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d$, we consider the intersection $X_H$ of $X$ with a hyperplane $H$. One may ask that, as $H$ varies, how do the moduli of $X_H$ vary? Will it always vary maximally for general $n, d$? This is mainly motivated by the question in the previous section: if we take $(n, d) = (3, 3)$, the hyperplanes are parametrized by $\mathbb{P}^4^*$, and the
moduli space of cubic surfaces is also of dimension 4. Therefore, if $X_H$ varies maximally in the moduli space, a general cubic surface can be obtained by taking a hyperplane section of a fixed smooth cubic threefold, and this might provide some connection between the monodromy group of $F(X_U/U) \rightarrow U$ and of $F(Y_V/V) \rightarrow V$. For the Fermat (hence a general) cubic threefold, this is a corollary of the fact that a general cubic form can be written as a sum of 5 cubes of linear forms, cf. [Dol12, Thm. 9.4.1 and Cor.9.4.2]. But it does not imply that this is true for any smooth cubic threefold. For a general $(n, d)$, this was mainly studied via the tangent maps. For example, in [Bea90] it proves that the $X_H$ cannot be constant in moduli, and in [HMP98] it proves that $X_H$ will vary maximally when the degree $d$ is sufficiently low. Again, it would be rather easy to show $X_H$ will vary maximally for a particular (hence a general) hypersurface $X$, but we did not know if it is true for any smooth one. We will give a new approach using the geometry of $X$, and prove the following theorem:

**Theorem 0.2.** Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d$, where $d > n > 1$ and $(n, d) \neq (2, 3), (3, 4)$. Let $M_{d,n-1} := |\mathcal{O}_{\mathbb{P}^n}(d)|^* / \text{SL}(n + 1)$ be the moduli space of hypersurfaces of degree $d$ in $\mathbb{P}^n$, and

$$\Phi: \mathbb{P}^{(n+1)*} - X^* \rightarrow M_{d,n-1}$$

be the natural morphism defined by taking smooth hyperplane sections. Then the image of $\Phi$ is always of dimension $n + 1$.

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1. **Lines on Cubic Threefolds**

Throughout this section, we work over complex numbers unless otherwise stated, although most arguments still hold for an arbitrary algebraically closed field of characteristic zero.

1.1. **Lines on cubic surfaces.** We briefly collect some basic facts about lines on cubic surfaces for later use.

Let $S$ be a smooth cubic surface, then the configuration of the 27 lines is known. Note that

$$H^2(S, \mathbb{Z}) \simeq \text{Pic}(S) \simeq \mathbb{I}_{1,6}.$$  

We denote the basis by $\{e_i\}_{0 \leq i \leq 6}$, where

$$\langle e_i, e_j \rangle = \begin{cases} 
1, & \text{if } i = j = 0; \\
-1, & \text{if } i = j \neq 0; \\
0, & \text{if } i \neq j,
\end{cases}$$

and the hyperplane class is given by

$$h = \mathcal{O}_S(1) = \omega_S^* = 3e_0 - e_1 - \cdots - e_6.$$
Then, the solutions to the equations

$$\langle \theta, \theta \rangle = -1, \; \langle \theta, h \rangle = 1$$

for $\theta \in I_{1,6}$ correspond to the 27 lines. They are explicitly given by

$$a_i = e_i, \; 1 \leq i \leq 6;$$
$$b_{ij} = e_0 - e_i - e_j, \; 1 < i \leq j < 6;$$
$$c_i = 2e_0 - e_1 - \cdots - e_6 + e_i, \; 1 \leq i \leq 6.$$  \hspace{1cm} (1.2)

We denote the set by $L_6$. Similarly, we consider the equations

$$\langle \theta, \theta \rangle = -2, \; \langle \theta, h \rangle = 0.$$  \hspace{1cm} (1.3)

The solution set is denoted by $R_6$, whose 72 elements are called roots. Indeed, they form a root system $E_6$, and a base can be chosen as

$$\alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6$$

where

$$\alpha_1 = e_0 - e_1 - e_2 - e_3;$$
$$\alpha_i = e_i - e_{i-1}, \; 2 \leq i \leq 6.$$ 

One can check that they span the lattice $h^\perp$, hence $h^\perp \simeq E_6(-1)$.

More generally, for every $2 \leq r \leq 6$, we consider the lattice $I_{1,r}$ as a sub-lattice of $I_{1,6}$, spanned by $\{e_i\}_{0 \leq i \leq r}$. We denote the set of solutions for $\theta \in I_{1,r}$ for the line equation (1.1) by $L_r$ and the solution for the root equation (1.3) by $R_r$. By definition we have

$$L_r = L_6 \cap I_{1,r}$$

and

$$R_r = R_6 \cap I_{1,r}.$$ 

Let

$$s_\alpha : I_{1,6} \rightarrow I_{1,6}$$

$$\theta \mapsto \theta - 2\frac{\langle \theta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

be the reflection associate with $\alpha$. If $\alpha$ is a root, this takes the simple form

$$s_\alpha(\theta) = \theta + \langle \theta, \alpha \rangle \alpha.$$ 

Consider the Weyl group $W_r$ generated by reflections of all roots $R_r$. Note that for $3 \leq r \leq 6$, the root system $R_r$ has a base $\{\alpha_i\}_{1 \leq i \leq r}$, hence $W_r$ is generated by $\{\alpha_i\}_{1 \leq i \leq r}$. Clearly, elements in $W_r$ preserve the intersection form and the hyperplane class $h$. We will show $W_r$ contains all such automorphisms. Firstly, we prove that:

**Lemma 1.1.** For $1 \leq r \leq 6$ and $r \neq 2$, the Weyl group $W_r$ acts transitively on $L_r$.\[\]
Proof. For the case \( r = 1 \), \( L_1 = \{e_1\} \), hence the action is trivially transitive. For \( 3 \leq r \leq 6 \), we only need to prove the case \( r = 6 \), as the other case is simply by restriction. Note that for \( i = 2, \ldots, 6 \) the reflection \( s_{\alpha_i} \) will permute \( a_i \) with \( a_{i-1} \), and the \( s_{\alpha_1} \) will map \((a_0, \ldots, a_6)\) to 
\[
(a_0 + \theta, \alpha_1), a_1 - \theta, a_2 - \theta, \alpha_1), a_3 - \theta, \alpha_1), a_4, a_5, a_6).
\]
Consider the orbit of \( a_1 = e_1 \). By \( \{s_{\alpha_i}\}_{2 \leq i \leq 6} \), every \( a_i \) lies in this orbit. Apply \( s_{\alpha_1} \) to \( a_1 \) we see \( b_{23} = e_0 - e_2 - e_3 \) is in this orbit, and then apply \( \{s_{\alpha_i}\}_{2 \leq i \leq 6} \), every \( b_{ij} \) is in this orbit. Similarly, apply \( s_{\alpha_1} \) to \( b_{45} \) we get \( c_6 \), and then use \( \{s_{\alpha_i}\}_{2 \leq i \leq 6} \) again we get all the \( c_i \)'s.

\[ \square \]

Remark 1.2. When \( r = 2 \), the case is a little different, as the root vector \( \alpha_1 = e_0 - e_1 - e_2 - e_3 \) does not exist in \( I_{1.2} \). In this case, the group \( W_2 \) only consists of one nontrivial element \( s_{\alpha_2} \), and the action has two orbits: one is \( \{a_1, a_2\} \), and the other is \( \{e_0 - e_1 - e_2\} \).

More generally, we study how the Weyl group acts on pairwise disjoint lines. Let \( L_r^{(s)} \) be the set of all \( s \) pairwise disjoint lines in \( L_r \), i.e.
\[
L_r^{(s)} := \\{ \{l_1, \ldots, l_s\} \mid l_i \cdot l_j = 0 \text{ for any } i \neq j \}.
\]
Then we have:

Proposition 1.3. The Weyl group \( W_r \) acts transitively on \( L_r^{(s)} \) for \( s \neq r - 1 \).

Proof. This is equivalent to show that, for arbitrary \( s \) \((s \neq r - 1)\) pairwise disjoint lines \( l_1, \ldots, l_s \), there exists a \( w \in W_r \) such that \( w(l_i) = e_i \). We prove this by induction on \( s \). As discussed above, for the case \( s = 1 \), it is true when \( r \neq 2 \). Assume we have proved the statement for \( s = k - 1 \). Let \( l_1, \ldots, l_k \) be \( k \) disjoint lines in \( I_{1.r} \). We want to find an element in \( W_r \) sending \( l_i \) to \( e_i \). Firstly, we can find a \( w \) such that \( w(l_k) = e_r \). Since \( w(l_1), \ldots, w(l_k) \) are pairwise disjoint, we have \( w(l_i) \cdot e_r = 0 \), hence \( w(l_i) \in I_{1,r-1} \). Therefore, \( w(l_1), \ldots, w(l_{k-1}) \) are \( k - 1 \) disjoint lines in \( I_{1,r-1} \), and then it is done by inductive hypothesis.

\[ \square \]

Corollary 1.4. On a smooth cubic surface \( S \), every \( k \) disjoint lines are alike, except for \( k = 5 \).

Corollary 1.5. The automorphism group of \( I_{1.6} \) (as a lattice) preserving the hyperplane class \( h \) is \( W(E_6) \).

Proof. Because every such automorphism will send 6 disjoint lines to 6 disjoint lines and is determined by this.

\[ \square \]

The 27 lines also record the information of an automorphism of \((I_{1.6}, h)\) faithfully:

Proposition 1.6. The natural map
\[
\text{Aut}(I_{1.6}, h) \to \text{Aut}(L_6)
\]
by restriction is an isomorphism.

Proof. Clearly, an automorphism of \((I_{1.6}, h)\) induces an automorphism of \( L_6 \) by restriction. On the other hand, let \( \sigma \) be such an automorphism of lines. It will extend to an automorphism \( \tilde{\sigma} \) of \( I_{1.6} \) since the 27 lines span the whole \( I_{1.6} \).
Note that the hyperplane class $h$ is the sum of any 3 mutually intersecting lines, so $\sigma$ will also preserve $h$. Easy to check the two compositions are identities.  

**Corollary 1.7.** The automorphism group of the 27 lines on $S$ preserving the intersection relations are $W(E_6)$. 

By the same argument in the proof of Lemma 1.1, one can easily prove the following lemma which will be used in the following.

**Lemma 1.8.** The Weyl group $W_6$ acts transitively on $R_6$. 

Let $\mathcal{V}_V \to V$ be the universal family of smooth cubic surfaces, and $F(\mathcal{V}_V/V) \to V$ be the relatively Fano variety of lines. The latter is a finite étale morphism, so one can ask about the monodromy group. A priori, it would be a subgroup of $W(E_6)$. We claim it is the full $W(E_6)$. The key is the following proposition given by [Har79, Prop. on page 717].

**Proposition 1.9.** Let $\sigma$ be an automorphism of the 27 lines preserving the intersection relations. Then, we can find a loop on $V$ such that the induced action on the 27 lines is $\sigma$. 

**Proof.** We claim that any automorphism of configuration of 27 lines will contains an invariant set of 6 disjoint lines, i.e. sends $l_i$ to $l_{\tau(i)}$ for $\{l_i\}_{1 \leq i \leq 6}$ disjoint lines and $\tau \in S_6$ some permutation. By Corollary 1.1, we only need to prove for $\sigma = s_{\alpha_i}$, as $W(E_6)$ is generated by such reflections. And by Lemma 1.8 we may assume $\sigma = s_{\alpha_2}$. It will send $(e_1, e_2, e_3, e_4, e_5, e_6)$ to $(e_2, e_1, e_3, e_4, e_5, e_6)$. In particular, the set of lines $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is invariant. This proves the claim.

Then, we show that we can always realize the wanted diffeomorphism by moving the 6 blow-up points on $\mathbb{P}^2$, once we have such an invariant set. Recall that any smooth cubic surface $S \subset \mathbb{P}^3$ can be realized as $\mathbb{P}^2$ blowing up 6 points, together with a global frame $\{s_1, \ldots, s_4\}$ in $\omega_S^*$, where $\omega_S$ is the canonical bundle. Say $e_i$ is the exceptional line corresponding to the point $p_i$ on $\mathbb{P}^2$. Let $p_i(t)$ be a path on $\mathbb{P}^2$ such that $p_i(0) = p_i$ and $p_i(1) = p_{\tau(i)}$. We may assume that for each $t$ the six points $p_1(t), \ldots, p_6(t)$ are in general position, hence for each $t$ we get a smooth cubic surface $S_t$. This is not only an abstract surface, but with the embedding in $\mathbb{P}^3$, as we can take a frame $\{s_1(t), \ldots, s_4(t)\}$ in $\omega_{S_t}^*$. If each $s_i(t)$ varies continuously, the $S_t \subset \mathbb{P}^3$ also varies continuously, thus induces a path on $V$. Since the general line group is connected, we can choose the frame $\{s_i(t)\}$ such that $S_0$ and $S_1$ coincide in $\mathbb{P}^3$, hence the path becomes a loop. Consider the monodromy action by this loop, and clearly it will effects the permutation $\sigma$ on the lines. □

**Corollary 1.10.** The monodromy group of $F(\mathcal{V}_V/V) \to V$ is $W(E_6)$. 

However, this approach to computing the monodromy group does not work for $F(\mathcal{X}_U/U) \to U$. We will discuss that monodromy group later.

Now we consider the lines on singular cubic surfaces.

**Lemma 1.11.** A general singular cubic surface contains 6 double lines and 15 reduced lines.
Lemma 1.12. Let $D$ hold for arbitrary $t$. Therefore, we have natural isomorphism $H^0(O_X(1)) \sim H^0(O_X(1))$, hence we can (and will) identify hyperplanes with hyperplane sections

1A priori we do not know it is a divisor, but the following discussion shows it is.

2Note that by $0 \to O_{P^4}(-3) \to O_{P^4} \to O_X \to 0$ we have natural isomorphism $H^0(O_{P^4}(1)) \sim H^0(O_X(1))$, hence we can (and will) identify hyperplanes with hyperplane sections.
Basic properties of $C$ in this subsection. Most results here are also stated in [ADF+15], and we just write the proofs as exercises.

Firstly, we note $C$ is a smooth and connected curve.

**Proposition 1.13.** The curve $C$ as defined above is smooth and irreducible.

**Proof.** Firstly, consider the universal hyperplane sections

$$X = \{(H, x) \in \mathbb{P}^{4*} \times X \mid H \ni x\}$$

(1.4)

and the relative Fano variety of lines (w.r.t. the first projection)

$$F(X) = \{(H, l) \in \mathbb{P}^{4*} \times F(X) \mid H \ni l\},$$

where $F(X)$ is the Fano variety of lines on $X$. It has two natural projections: the first projection $\text{pr}_1: F(X) \to \mathbb{P}^{4*}$ is generically finite of degree 27, and the second projection $\text{pr}_2: F(X) \to F(X)$ is a projective bundle. Since $F(X)$ is smooth and irreducible, so is $F(X)$. Then, note that $\pi: C \to \mathbb{P}^{1*}$ is the pullback of $\text{pr}_1: F(X) \to \mathbb{P}^{4*}$ along a general $\mathbb{P}^{1*} \to \mathbb{P}^{4*}$, and a general $\mathbb{P}^{1*}$ is just the intersection of 3 hyperplanes in $\mathbb{P}^{4*}$. Consider the complete linear system $|O_{\mathbb{P}^{4*}}(1)|$ on $\mathbb{P}^{4*}$. By pull-back to $F(X)$ we obtain a linear system $\text{pr}_1^*|O_{\mathbb{P}^{4*}}(1)|$, whose element is of the form $\text{pr}_1^*H$ where $H$ is a divisor in $|O_{\mathbb{P}^{4*}}(1)|$. If we take 3 general divisors $H_1, H_2$ and $H_3$ in $|O_{\mathbb{P}^{4*}}(1)|$, then

$$C \cong \text{pr}_1^*H_1 \cap \text{pr}_1^*H_2 \cap \text{pr}_1^*H_3,$$

Since $|O_{\mathbb{P}^{4*}}(1)|$ is base-point-free, so is $\text{pr}_1^*|O_{\mathbb{P}^{4*}}(1)|$. By Bertini’s theorem with base point, cf. [Har77, III. Rem. 10.9.2], a general divisor in $\text{pr}_1^*|O_{\mathbb{P}^{4*}}(1)|$ is smooth and irreducible. Repeat it twice more, we see $C$ is smooth and irreducible. (Note that the dimension condition for irreducibility always holds.)

Now we study the ramification of $\pi: C \to \mathbb{P}^{1*}$. The following proposition shows how it ramifies locally.

**Proposition 1.14.** Let $t_0$ be a branch value of $\pi: C \to \mathbb{P}^{1*}$, and $\alpha$ be a simple loop which encloses no other branch values except for $t_0$, with branch point $t$. Then the 27 points in the fiber $\pi^{-1}(t)$ can be divided into 6 pairs of points

$$[l'_1], [l''_1], \ldots, [l'_6], [l''_6]$$

and 15 single points

$$[l_{12}], [l_{13}], \ldots, [l_{56}]$$

such that the monodromy action of $\alpha$ will switch the 6 pairs and remain the 15 single points, i.e.

$$\rho_\alpha: [l'_i] \mapsto [l''_i]$$

$$[l''_i] \mapsto [l'_i]$$

$$[l_{ij}] \mapsto [l_{ij}].$$

**Proof.** By Lemma [1.11] $\pi^{-1}(t_0)$ consists of 6 double points and 15 reduced points. Let $\gamma$ be a path joining $t_0$ to $t$. This will divide the 27 reduced points $\pi^{-1}(t)$ into 6 pairs (split from $[l_i]$) and 15 single points (come from $[l_{ij}]$). Since $\alpha$ encloses no other branch values, we can assume $\alpha$ very close to $t_0$. Then by continuity the monodromy action of $\alpha$ will send each $[l_{ij}]$
to $[l_j]$, and $[l'_j]$ to $[l''_j]$ a priori. But since $C$ is smooth and $[l_j]$ is a double point, the Riemann’s existence theorem says $[l'_j]$ can only be sent to $[l''_j]$. The same for $[l''_j]$. □

**Corollary 1.15.** The branched cover $\pi: C \to \mathbb{P}^{1*}$ is not Galois.

To know how $\pi: C \to \mathbb{P}^{1*}$ is ramified globally, we need to know the number of branch values. Let $\overline{D} \subseteq \mathbb{P}^{4*}$ be the discriminant divisor of the family $pr_1: F(\mathcal{X}) \to \mathbb{P}^{4*}$, and $D$ be the discriminant divisor of the family $\mathcal{X} \to \mathbb{P}^{4*}$. By Lemma [EH16, Prop. 2.9] we see $\overline{D} = D$. Then the number of branch values of $\pi: C \to \mathbb{P}^{1*}$ is just the intersection number of the dual hypersurface $\mathcal{X} \times \mathcal{X}$ with a general line $\mathbb{P}^{1*} \subseteq \mathbb{P}^{4*}$, and by [EH16, Prop. 2.9] we can compute $\deg(\mathcal{X}) = 3 \cdot 2^3 = 24$.

Combining with Proposition [1.14] we have the following result:

**Proposition 1.16.** The branched cover $\pi: C \to \mathbb{P}^{1*}$ has precisely 24 branch values, each has 6 branch points with multiplicity 2. In particular, the Riemann–Hurwitz formula reads $e(C) = 27 \cdot 2 - 24 \cdot 6 = -90$, hence $g(C) = 46$.

### 1.3. Monodromy group

Now we want to compute the monodromy group of $\pi: C \to \mathbb{P}^{1*}$. This proceeds in four steps:

(i) Study the monodromy action on (middle) cohomology and link it with the action on lines.

(ii) Reduce the case to a Lefschetz pencil $\{X_t\}_{t \in \mathbb{P}^{1*}}$ of $X$, and for the embedding $X_0 \subseteq X$, we link the primitive cohomology with the vanishing cohomology.

(iii) Note that the embedding $X_0 \to \tilde{X} - X_\infty$ is just the embedding of some fiber in a fibration, and around each singular fiber is a Lefschetz degeneration, hence the monodromy action can be formulated via Picard–Lefschetz formula. And we find the monodromy group is generated by vectors in $\{v_i\}$ with a special configuration.

(iv) Prove that the reflections associated with vectors in $\{v_i\}$ is the Weyl group $W(E_6)$.

Recall that $\mathcal{X} \to \mathbb{P}^{4*}$ is the universal family hyperplane sections defined in [1.1]. Let $U \subseteq \mathbb{P}^{4*}$ be the open subset parametrizing the smooth ones, and restrict to $U$ we get $\mathcal{X}_U \to U$. By Ehresmann’s fibration theorem, $\mathcal{X}_U \to U$ is locally free, so any path $\gamma: [0, 1] \to U$ will induce a diffeomorphism $\varphi_\gamma: f^{-1}(\gamma(0)) \to f^{-1}(\gamma(1))$. The $\varphi_\gamma$ is not canonical, but different choices are isotopic to each other, so

$$\varphi^*_\gamma: H^i(f^{-1}(\gamma(1)), \mathbb{Z}) \to H^i(f^{-1}(\gamma(0)), \mathbb{Z})$$

is well-defined. If $\gamma$ is a loop, and denote $X_0 = f^{-1}(\gamma(0))$, then $\varphi^*_\gamma$ is an element in $\text{Aut}(H^i(X_0, \mathbb{Z}))$, thus defines the monodromy action. Since a diffeomorphism preserves the intersection relations and the canonical bundle $\omega_{X_0}$, this is an automorphism of lattices preserving the hyperplane class $h = \omega^*_X_{X_0}$. Note that the 27 lines correspond to the 27 vectors in $\{l_j\}$. By Proposition [1.6] we can study the monodromy action of lines by the action of $H^2(X_0, \mathbb{Z})$, and the monodromy groups will be isomorphic.

Then, we note that $\mathbb{P}^{1*} \cap U \to U$ will induce a surjection on $\pi_1$, cf. [Voi03, Thm. 3.22]. So it will be enough to reduce $\mathcal{X} \to \mathbb{P}^{4*}$ to $\mathcal{X}_{\mathbb{P}^{1*}} \to \mathbb{P}^{1*}$. Let
\{X_t\}_{t \in \mathbb{P}^1_s}$ be a Lefschetz pencil of hyperplane sections on $X$ with base locus $X_0 \cap X_\infty$, and let $\tilde{X}$ be its total space. It is known that $\tilde{X}$ is the blow-up of $X$ along $X_0 \cap X_\infty$. Without loss of generality we may assume $X_0$ and $X_\infty$ are both smooth. Let

$$i: X_0 \hookrightarrow \tilde{X}, \quad i': X_0 \hookrightarrow \tilde{X} - X_\infty, \quad j: X_0 \hookrightarrow X$$

be the inclusions. Since $X_0$ comes from hyperplane section, we have

$$[X_0]|_{X_0 \cup X_\infty} = j^* \circ j_*: H^2(X_0, \mathbb{Z}) \xrightarrow{i_*} H^4(X, \mathbb{Z}) \xrightarrow{j_*} H^4(X_0, \mathbb{Z})$$

where $[X_0]$ is the cohomology class associated with $X_0$ in $X$, and $j_*$ is the Poincaré dual of pushforward of cycles, cf. [Voi03, Sec. 1.2.2].

Note that the Lefschetz hyperplane theorem says this $j^*$ here is an isomorphism, and the restriction of the hyperplane class $[X_0]$ to $X_0$ is just the hyperplane class on $X_0$. Therefore, the primitive cohomology $\text{Ker}([X_0]|_{X_0 \cup X_\infty})$ equals the vanishing cohomology $\text{Ker}(j_*).$

So we need to consider the vanishing cohomology. It is more clear to deal with the dual homology case

$$H_2(X_0)_{\text{van}} = \text{Ker}(H_2(X_0) \xrightarrow{j_*} H_2(X)).$$

We compare it with

$$\text{Ker}(H_2(X_0) \xrightarrow{i'_*} H_2(\tilde{X} - X_\infty)).$$

Note there is a map

$$\tilde{X} - X_\infty \xrightarrow{i} \tilde{X} \xrightarrow{\pi} X$$

hence a priori $\text{Ker}(j_* \supset \text{Ker}(i'_*)).$ By [Voi03, Cor. 2.23] we know that $\iota_* \supset j_*$ is injective, and note the blow-down map $\pi$ will contract the exceptional divisor $X_0 \cap X_\infty$ which is missed by the image of $\iota$, so the composition $\pi \circ \iota_*$ is also injective. Therefore,

$$H_2(X_0)_{\text{van}} = \text{Ker}(H_2(X_0) \xrightarrow{i'_*} H_2(\tilde{X} - X_\infty)).$$

So, $i_*, i'_*$ and $j_*$ have the same kernel on $H_2(X_0)$, which is isomorphic to the primitive homology $H_2(X_0)_{\text{prim}}.$ And from Section 1.1 we see it is isomorphic to $E_6(-1).$

Note the map $i': X_0 \to \tilde{X} - X_\infty$ is the inclusion of some fiber in the fibration $\tilde{X} - X_\infty \to \mathbb{C} \simeq \mathbb{P}^{1s} - \infty$, and each singular fiber contains precisely one ordinary double point. We call the local fibration around each singular fiber with this property a Lefschetz degeneration. For a Lefschetz degeneration, the singular fiber has the same homotopy type as the smooth fiber with a 3-dimensional ball attached along the so-called vanishing sphere. In our case, let $0_1, \ldots, 0_24$ be the critical values, then there exists a small disk $\Delta_i$ around each $0_i$ and a $t_i \in \Delta_i$ such that $\tilde{X}_{\Delta_i}$ can be retracted by deformation onto the union of $\tilde{X}_{t_i} = X_{t_i}$ with a 3-dimensional ball glued to $X_{t_i}$ along the vanishing sphere $S_i^2.$ Let $\gamma_i$ be the paths in $\mathbb{C} - \cup \Delta_i$ joining 0 to $t_i.$ Then $\gamma_i$ will induce a diffeomorphism from $X_0$ to $X_{t_i}.$ We denote the element in $H_2(X_0, \mathbb{Z})$ corresponding to $[S_i^2]$ by $\delta_i.$ Easy to see $\mathbb{C} \simeq \mathbb{P}^{1s} - \infty$ admits a retraction by deformation onto the union of the disks $\Delta_i$ with the paths $\gamma_i$. Since $\tilde{X} - X_\infty - \{X_i\} \to \mathbb{C} - \{0_i\}$ is a fibration, the retraction
by deformation of $\mathbb{C}$ onto $\bigcup_i(\gamma_i \cup \Delta_i)$ will induce a retraction by deformation of $\bar{X} - X_\infty$ onto $\bigcup_i(\bar{X}_{\gamma_i} \cup \bar{X}_{\Delta_i})$, which has the homotopy type of the union of $X_0$ with some 3-balls attached along some spheres corresponding to $\delta_i$’s. This shows $\delta_i$’s generate $\text{Ker}(H_2(X_0, \mathbb{Z}) \xrightarrow{\sim} H_2(\bar{X} - X_\infty, \mathbb{Z}))$, hence the primitive cohomology $H_2(X_0)_{\text{prim}} \simeq E_6(-1)$. Therefore, all the $\delta_i$’s generate this $E_6(-1)$.

Then we compute the monodromy group. Let $\alpha_i$ be a simple loop around $0_i$, which lies in $\Delta_i$ and with base point $t_i$. By Picard–Lefschetz formula (cf. [Voi03, Thm.3.16]) the monodromy action of the loop $\alpha_i$ is

$$H^2(X_{t_i}) \to H^2(X_{t_i})$$

$$\theta \mapsto \theta + \langle \theta, [S_i^2] \rangle [S_i^2]$$

hence the monodromy action of the loop $\gamma_i \ast \alpha_i \ast \gamma_i^{-1}$ is of the form

$$s_{\delta_i}: H^2(X_0) \to H^2(X_0)$$

$$\theta \mapsto \theta + \langle \theta, \delta_i \rangle \delta_i$$

which turns out to be the reflection associated with $\delta_i$, i.e. the reflection in the hyperplane $\delta_i^\perp$. Also note that diffeomorphisms will preserve intersection form, hence

$$\langle \theta, \theta \rangle = \langle \theta + \langle \theta, \delta_i \rangle \delta_i, \theta + \langle \theta, \delta_i \rangle \delta_i \rangle$$

which leads to

$$\langle \delta_i, \delta_i \rangle = -2,$$

hence each $\delta_i$ is of minimal nontrivial norm in the lattice, that is, a root considered in (1.3).

Since $[\gamma_i \ast \alpha_i \ast \gamma_i^{-1}]$’s generates $\pi_1(\mathbb{C} - \{0_1, \ldots, 0_{24}\})$, $s_{\delta_i}$’s generate the monodromy group, say $W$. Therefore, $W$ is generated by the reflections associated with vectors in $\{\delta_i\}$, such that

(i) The set of vectors $\{\delta_i\}$ spans the full lattice $E_6(-1)$ as a $\mathbb{Z}$-module.

(ii) Each $\delta_i$ of of minimal nontrivial norm in the lattice $E_6(-1)$.

We will show that reflections associated with vectors in $\{v_i\}$ satisfying the special configuration described above will generate the full Weyl group of $W(E_6)$. This is guaranteed by the following proposition about any simply-laced root system.

**Proposition 1.17.** Let $R$ be a simply-laced root system, i.e. of type ADE. Let $\Lambda$ be the lattice spanned by $R$, and $\{v_i\}$ be a set a vectors satisfying

(i) The set of vectors $\{v_i\}$ spans the full lattice $\Lambda$ as a $\mathbb{Z}$-module.

(ii) Each $v_i$ of of minimal nontrivial norm (i.e. $\sqrt{2}$ if the root system is normalized).

Then the reflections associated with these vectors in $\{v_i\}$ generate the full Weyl group $W(R)$.

---

3There should be some classical result which can cover this, but I did not find it. I am aware of the fact that reflections associated with a base (or simple system in some literature) of a root system will generate the Weyl group, but the conditions here are not enough to ensure $\{v_i\}$ a base, even if we add some minus signs.
Remark 1.18. We will not essentially use root systems, but only the following properties of a subset $R$ of any lattice $\Lambda$ in some (finite-dimensional) Euclidean space:

(i) The lattice $\Lambda$ is generated by $R$ as a $\mathbb{Z}$-module.

(ii) The vectors in $R$ are precisely the vectors in $\Lambda$ with minimal nontrivial norms.

(iii) The angle between any two vectors in $R$ can only be $0, \pi/3, \pi/2, 2\pi/3$ and $\pi$.

They will be automatically satisfied if $R$ is a simply-laced root system.

Proof of Proposition 1.17. By (ii) in Remark 1.18 we know $\{v_i\} \subset R$.

Now a key observation is, for two vectors $v, w$ with the same norms, if the angle between them is $2\pi/3$, then the reflection associated with $v + w$, denoted by $s_{v+w}$, can be represented by the reflections associated with $v$ and $w$, namely

$$s_{v+w} = s_v \circ s_w \circ s_v = s_w \circ s_v \circ s_w.$$  

Again by (ii) in Remark 1.18 if we want $v + w \in R$ for any $v, w \in R$, then the degree between $v$ and $w$ can only be $2\pi/3$. By (i) in Remark 1.18 for every vector $u \in R$ we can find a sequence of vectors $v^{(i)}$ such that $u = \sum_{i=1}^{n} v^{(i)}$ (allow repetition), where $v^{(i)} \in \pm\{v_i\}$. So, if the sequence satisfies that for every $1 \leq k \leq n$, the sum $\sum_{i=1}^{k} v^{(i)}$ lies in $R$, then we can apply $2\pi/3$-sum formula again and again for every $u \in R$, hence the reflections associated with these $v_i$ will generate the full Weyl group. A sequence with this property is called a good sequence. So it only remains to show a good sequence always exists. This is true by the following lemma. □

Lemma 1.19. Let $\{v_i\} \subset R \subset \Lambda$ be as above. Then every vector $u \in R$ admits a good sequence.

Proof. Let $U \subset R$ be the set of all root vectors which do not have a good sequence. For each $u \in U$, we define its length to be the minimal $n$ such that $u = \sum_{i=1}^{n} v^{(i)}$, where $v^{(i)} \in \pm\{v_i\}$. Since $U$ is a finite set, we can find $u_0 \in U$ with the minimal length.

Now set $R_{>0} \subset R$ be the set of root vectors whose inner products with $u_0$ are positive. Since it will have contribution on the direction of $u_0$, $R_{>0} \cap \{v^{(i)}\}$ cannot be empty. Let us take $v \in R_{>0} \cap \{v^{(i)}\}$. By (iii) in Remark 1.18 the angle between $u_0$ and $v$ can only be $0$ or $\pi/3$, and the case $0$ is not possible because we can take the trivial good sequence $u_0 = v$, so we can assume the angle is $\pi/3$. In this case $u_0 - v \in R$, and it is an element in $U$ with the length at most $n - 1$, contradicted to the minimal length assumption. □

Take $v_i$ to be the $\delta_i$ in the previous discussion, we have already proved:

Proposition 1.20. The monodromy group on $H^2(X_0, \mathbb{Z})$ of $\tilde{X} \to \mathbb{P}^1$ is $W(E_6)$.

Take the relative Fano variety of lines, we get:

Corollary 1.21. The monodromy group of $\pi: C \to \mathbb{P}^1$ is $W(E_6)$.

By the preceding discussion, we have already proved:
Theorem 1.22. The monodromy group of the finite \'{e}tale morphism

\[ F(\mathcal{X}_U/U) \to U \]

is \( W(E_6) \), that is, the same as the monodromy group of the \( F(\mathcal{Y}_V/V) \to V \).

Remark 1.23. This approach applies equally to the universal family of smooth cubic surfaces \( Y \subset \mathbb{P}^3 \), say \( Y \to \mathbb{P}^N \). One also reduces it to \( Y_{\mathbb{P}^1} \to \mathbb{P}^1 \), and then one can prove \( Y_{\mathbb{P}^1} \) is isomorphic to the blow-up of \( \mathbb{P}^3 \) along the base curve \( Y_0 \cap Y_\infty \). Then, one still has the isomorphism of primitive cohomology and vanishing cohomology, and the monodromy group will be generated by the reflections to the set of elements described in Proposition 1.17, hence is also \( W(E_6) \).

2. Variation of Hyperplane Sections

Throughout this section, we assume the base field is algebraically closed and of characteristic zero. We denote \( M_{d,n-1} := |\mathcal{O}_{\mathbb{P}^n}(d)|^{ss}/SL(n+1) \) to be the projective moduli space of (semi-stable) hypersurfaces of degree \( d > 2 \) in \( \mathbb{P}^n \). Here we assume \((n,d) \neq (2,3), (3,4)\) to ensure every automorphism is linear. We also need \( n > 1 \) to ensure a general one has trivial automorphism group. For any smooth hypersurface \( X \subset \mathbb{P}^{n+1} \) of degree \( d \), we define the natural morphism

\[ \Phi: \mathbb{P}^{(n+1)*} - X^* \to M_{d,n-1} \]

by taking smooth hyperplane sections. Since \( M_{d,n-1} \) is complete, we can extend \( \Phi \) to the maximal open set to get

\[ \Phi_{\text{max}}: \mathbb{P}^{(n+1)*} - I \to M_{d,n-1} \]

where \( I \subset \mathbb{P}^{(n+1)*} \) the indeterminacy locus of codimension at least 2. This induces a dominant morphism

\[ \Phi_0: \mathbb{P}^{(n+1)*} - I \to B \]  \hspace{1cm} (2.1)

for \( B \subset M_{d,n-1} \) also projective. We will prove that \( \Phi_0 \) is generically finite general \( n,d \).

The strategy of the proof is as follows. Let \( \mathcal{X} \) be the universal family of hyperplane sections, i.e. the natural hypersurface of bidegree \((1,1)\) in \( \mathbb{P}^{(n+1)*} \times X \). If a general fiber of \( \Phi_0 \) is of positive dimension, one can find a curve \( T_0 \) on \( \mathbb{P}^{(n+1)*} \), such that

(i) \( \Phi_0 \) will send the generic point of \( T_0 \) to a closed point, which represents a smooth hypersurface with a trivial automorphism group.

(ii) \( \mathcal{X}_T \) is smooth, where \( \mathcal{X}_T := \mathcal{X} \times_{\mathbb{P}^{(n+1)*}} T \) and \( T \) is the normalization of \( T_0 \).

Both (i) and (ii) will lead to some restrictions on the geometry of \( \mathcal{X}_T \) with quite different natures. This will provide a contradiction for general \( n,d \).

\[ H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathcal{O}_X(1)) \]

is always isomorphic, so we may identify hyperplanes with hyperplane sections.
2.1. Automorphism groups of hyperplane sections. We study the automorphism group of a hyperplane section $X_H$. Assume that $X$ admits a nontrivial linear automorphism $\sigma$. A linear automorphism $\sigma : \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ sends hyperplanes to hyperplanes, hence in this way induces a linear automorphism on the dual spaces, say $\sigma^* : \mathbb{P}^{(n+1)*} \to \mathbb{P}^{(n+1)*}$. A fixed point of $\sigma^*$ corresponds to an invariant hyperplane, say $H$. Then the restriction $\sigma|_H$ will be a linear automorphism of $X_H$.

**Definition 2.1.** An automorphism $\sigma_H \in \text{Aut}(X_H)$ is *extendable* if it is the restriction of some linear automorphism of $X$ to an invariant hyperplane $H$.

The fixed points of $\sigma^*$ form a disjoint union of projective $\mathbb{P}^{r_i}$ in $\mathbb{P}^{(n+1)*}$ with $r_i \leq n$, as $\sigma$ is nontrivial. Hence the $X_H$ which admits an automorphism extending to $\sigma$ is at most parametrized by a space of dimension $n$.

Since $\text{Aut}(X)$ is finite, we can control the dimension of hyperplane sections which admit nontrivial extendable automorphisms.

**Lemma 2.2.** Let $E \subset \mathbb{P}^{(n+1)*}$ be the subset parametrizing the hyperplane sections which admit nontrivial extendable automorphisms. Then $\dim(E) \leq n$.

Using this, we can control the automorphism group of a general hyperplane section.

**Proposition 2.3.** Let $X \subset \mathbb{P}^{n+1}$ be as above. Then, for a general hyperplane section $X_H$, the automorphism group $\text{Aut}(X_H)$ is trivial.

Before the proof, we state some general results which will be used.

**Lemma 2.4.** Let $X \subset \mathbb{P}^{n+1}$ be a hypersurfaces of degree $d$, and let $X \to \mathbb{P}^{(n+1)*}$ be the universal family of hyperplane sections of $X$. Let $\mathcal{Y} \to \mathbb{P}^N$ be the universal family of hypersurfaces of degree $d$ in $\mathbb{P}^n$. Then there exists a cover $\{U_i\}$ of $\mathbb{P}^{(n+1)*}$, such that over each $U_i$ the $X_{U_i} \to U_i$ is a pull-back of $\mathcal{Y} \to \mathbb{P}^N$.

**Proof.** Let $\mathcal{H} \to \mathbb{P}^{(n+1)*}$ be the universal hyperplanes in $\mathbb{P}^{n+1}$, where $\mathcal{H}$ is the natural divisor of bidegree $(1,1)$ in $\mathbb{P}^{(n+1)*} \times \mathbb{P}^{(n+1)}$. By definition, $\mathcal{H} \to \mathbb{P}^{(n+1)*}$ is a projective bundle, so we can find a cover $\{U_i\}$ of $\mathbb{P}^{(n+1)*}$ such that each $\mathcal{H}_{U_i} \to U_i$ is a trivial bundle. In particular, there exists a morphism $U_i \to \mathbb{P}^N$ such that $\mathcal{X}_{U_i} = \mathcal{Y} \times_{\mathbb{P}^N} U_i$. □

**Lemma 2.5.** Let $\mathcal{Y}_W \to W$ be the universal family of stable hypersurfaces of degree $d$ in $\mathbb{P}^n$ where $W \subset \mathbb{P}^N$ is the dense open set parametrizing the stable ones. Then $\text{Aut}(\mathcal{Y}_W/W, \mathcal{O}_{\mathcal{Y}_W}(1)) \to W$ is finite.

**Proof.** Consider

$$\text{PGL}(n+1) \times W \to W \times W$$

$$(g, X) \mapsto (g \cdot X, X)$$

which is proper by definition. Since $\text{PGL}(n+1)$ is affine, the morphism is also finite. Note that the restriction to the diagonal $\Delta \subset W \times W$ is just $\text{Aut}(\mathcal{Y}_W/W, \mathcal{O}_{\mathcal{Y}_W}(1)) \to W$, so the latter is finite as well. □
Consider

$$\text{Aut} := \text{Aut}(\mathcal{X}/\mathbb{P}^{(n+1)*}, \mathcal{O}_\mathcal{X}(1)) \rightarrow \mathbb{P}^{(n+1)*}.$$ 

For a Lefschetz pencil $\mathbb{P}^1 \subset \mathbb{P}^{(n+1)*}$, by restriction we get

$$\text{Aut}_{\mathbb{P}^1} \rightarrow \mathbb{P}^1*.$$ 

**Lemma 2.6.** The $\text{Aut}_{\mathbb{P}^1*} \rightarrow \mathbb{P}^1*$ defined above is finite étale.

**Proof.** By Hilbert–Mumford numerical criterion, if $n(d-2) > 2$, any hypersurfaces of degree $d$ in $\mathbb{P}^n$ with at worst ordinary double point singularities is stable. In particular, any fiber of $\mathcal{X}_{\mathbb{P}^1*} \rightarrow \mathbb{P}^1*$ is a stable hypersurface. By Lemma 2.4, locally $\mathcal{X}_{\mathbb{P}^1*} \rightarrow \mathbb{P}^1*$ is a pull-back from $Y_W \rightarrow W$, and by Lemma 2.5 we get the finiteness. It remains to show it is not ramified. Note that $H^0(X_H, T_{X_H}) \cong T_{\text{id}} \text{Aut}(X_H) = T_{\text{id}} \text{Aut}(X_H, \mathcal{O}_{X_H}(1)) = 0$.

Therefore, $\text{Aut}_{\mathbb{P}^1*} \rightarrow \mathbb{P}^1*$ is finite étale. □

**Proof of Proposition 2.3.** For a general $H$, let $\mathbb{P}^1* \subset \mathbb{P}^{(n+1)*}$ be a Lefschetz pencil passing through it. By Lemma 2.6, $\text{Aut}_{\mathbb{P}^1*} \rightarrow \mathbb{P}^1*$ is finite étale. Note that the projective line does not admit any nontrivial unramified cover, so $\text{Aut}_{\mathbb{P}^1*}$ is the disjoint union of some (say $r$) projective lines and some points. If $\text{Aut}(X_H)$ is not trivial for a general $H$, then $r > 1$, so there exists at least one nontrivial section. This section defines a rational map $\sigma: \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n+1}$ which sends $X$ to $X$. Note that $\sigma|_H$ is linear on one $H$ implies $\sigma$ is also linear, hence $\sigma \in \text{Aut}(X, \mathcal{O}_X(1))$. Therefore, a general $X_H$ admits a nontrivial extendable automorphism, contradicted to Lemma 2.2. □

**Corollary 2.7.** Let $\Phi_0: \mathbb{P}^{(n+1)*} \rightarrow B$ be defined in (2.1). Then, a general point in a general fiber of $\Phi_0$ represents some smooth hypersurface with trivial automorphism group.

### 2.2. Generic smoothness.

In this subsection, we will study for which curve $T_0$ on $\mathbb{P}^{(n+1)*}$, the total space $\mathcal{X}_T$ over the normalization of $T_0$ will be smooth. If $\Phi_0$ is not generically finite, we will show that such $T_0$ exists in a general fiber. This is the following proposition.

**Proposition 2.8.** Let $\Phi_0: \mathbb{P}^{(n+1)*} \rightarrow B$ be defined in (2.1), and assume a general fiber of $\Phi_0$ is of positive dimension. Then for a general fiber $P_0$, we can find a curve $T_0 \subset P_0$ such that $\mathcal{X}_T$ is smooth, where $T \rightarrow T_0$ is the normalization.

Firstly, we notice that a hyperplane section of a smooth hypersurface will only contain finitely many singular points.

**Proposition 2.9.** Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$. Then for any hyperplane section $\mathcal{X}_H$, it contains at most finitely many singular points.

**Proof.** This is equivalent to the twisted normal bundle $\mathcal{N}_{\mathcal{X}/\mathbb{P}^{n+1}}(-1)$ being ample, cf. [Laz13, Prop. 6.3.5]. And $\mathcal{N}_{\mathcal{X}/\mathbb{P}^{n+1}} \cong \mathcal{O}_X(d)$. □
Remark 2.10. Although Proposition 2.9 suggests that the singularity of \(X_H\) is not that bad, we cannot expect \(X_H\) to be semi-stable. For example, let \(X \subset \mathbb{P}^{n+1}\) be defined by
\[
x_0^d + x_1 x_2^{d-1} + x_2^d + \cdots + x_{n+1}^d = 0.
\]
It is smooth, but if we take \(H\) corresponding to \(x_0 = 0\), the \(X_H\) will be unstable for general \(n, d\).

Now, we study for which curve \(T_0 \subset \mathbb{P}^{n+1}\) the corresponding \(X_{T_0}\) will be smooth. The smoothness of \(X_T\) away from the discriminant divisor is clear, so we only need to care about the case near \(X^\ast\). Note that if \(L\) is the tangent line at some smooth point of \(T_0\), say \(H\), then \(X_{T_0}\) is smooth at \(X_H\) if and only if \(X_L\) is. Therefore, we only need to study in which direction to approach a point will lead to a smooth total space. It turns out that any direction not tangent to \(X^\ast\) will lead to a smooth total space.

Proposition 2.11. The family \(\mathcal{X} \to \mathbb{P}^{(n+1)\ast}\) satisfies:

(i) Each \(H \in X^\ast_{\text{sing}}\) corresponds to the case that \(H\) tangent to \(r > 1\) points in \(X\). So, one can define the tangent space \(T_H X^\ast\) at \(H\) to be the union of \(r\) hyperplanes.

(ii) For each \(H \in \mathbb{P}^{(n+1)\ast}\) and a line \(P_{1\ast} \subset \mathbb{P}^{(n+1)\ast}\) passing through \(H\), \(X_{P_{1\ast}}\) is smooth at \(X_H\) if and only if \(P_{1\ast} \not\subset T_H X^\ast\).

Proof. The fact (i) follows from that \(X^\ast\) is the image of the Gauss map, or also from the following discussion. For (ii), we test it at any \(H \in \mathbb{P}^{(n+1)\ast}\). For each line \(P_{1\ast} \subset \mathbb{P}^{(n+1)\ast}\), \(X_{P_{1\ast}}\) is the blow-up of \(X\) along the base locus, which is just \(X_H \cap H'\) with \(H' \simeq \mathbb{P}^{n-1}\) a hyperplane in \(H\). Note the natural map
\[
\{P_{1\ast} \subset \mathbb{P}^{(n+1)\ast} \mid H \in P_{1\ast}\} \to \{H' \in |O_H(1)|\}
\]
defined by sending the base locus of a pencil (of the corresponding hyperplanes) is a one-to-one correspondence. By Proposition 2.9, \(X_H = X_H\) contains finitely many singular points, and \(H \in X^\ast\) (respectively, \(H \in X^\ast_{\text{sing}}\)) corresponds to that \(X_H\) contains at least one (respectively, two) singular point(s). Then, \(T_H X^\ast\) corresponds to \(H'\) which contains at least one singular point. When \(H'\) does not contain any singular point, the blow-up would be smooth near \(X_H\), since it is isomorphic to the smooth \(X\) away from the base locus.

One can also prove it by writing the equations: let \(X\) be defined by
\[
F(x_0, \ldots, x_{n+1}) = 0
\]
and \(X_H\) by
\[
F(0, x_1, \ldots, x_{n+1}) = 0.
\]
Then, a line \(P_{1\ast} \supset H\) corresponds to a linear form \(l = l(x_1, \ldots, x_{n+1})\) up to scaling, and the corresponding \(X_{P_{1\ast}} \subset P_{1\ast} \times \mathbb{P}^n\) near \(H\) is defined by
\[
G(\epsilon, x_1, \ldots, x_n) := F(\epsilon l, x_1, \ldots, x_{n+1}) = 0.
\]
Then,
\[
\frac{\partial G}{\partial x_i}(0, x_1, \ldots, x_{n+1}) = \frac{\partial F}{\partial x_i}(0, x_1, \ldots, x_{n+1}),
\]
\[
\frac{\partial G}{\partial \epsilon}(0, x_1, \ldots, x_{n+1}) = \frac{\partial F}{\partial x_0}(0, x_1, \ldots, x_{n+1})l(x_1, \ldots, x_{n+1}).
\]

If \( \overline{x} = [0 : \overline{x}_1 : \cdots : \overline{x}_{n+1}] \in X_H \) is a singular point of \( X_{\mathbb{P}^1*} \), \( \frac{\partial G}{\partial \epsilon}(\overline{x}) \) has to be zero. But \( \frac{\partial F}{\partial x_0}(\overline{x}) \) cannot be zero, since otherwise \( X \) will be singular at \( \overline{x} \). So \( l(\overline{x}) = 0 \). Therefore, \( X_{\mathbb{P}^1*} \) is smooth at \( X_H \) if and only if the linear form does not vanish at each singular point \( \overline{x} \), which corresponds to the directions out of \( T_H X \).

\( \square \)

**Remark 2.12.** For general \( n, d \), the universal family of hypersurfaces \( \mathcal{Y} \to \mathbb{P}^N \) does not enjoy this proposition. This is the very reason why the argument cannot apply to the GIT-quotient \( \mathbb{P}^N - D \to M_{d,n-1} \).

**Proof of Proposition 2.8.** We may assume \( P_0 \) is of dimension \( k = 1 \). For the case \( k > 1 \), one can refine the fibers by taking \( k - 1 \) projective lines \( L_i \simeq \mathbb{P}^1 \subset \mathbb{P}^{n+1} \) in general position and replacing \( \Phi_0 \) by

\[
\Phi_{k-1}: \mathbb{P}^{(n+1)*} - I \to B \subset M_{d,n-1} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad H \mapsto (\Phi_0(H), H \cap L_1, \ldots, H \cap L_{k-1}).
\]

Then we simply take \( T_0 = P_0 \) to be a general fiber of \( \Phi_0 \). Let \( T \to \overline{T}_0 \) be the normalization. We claim that a general such \( T \) would not tangent to \( X^* \) at \( I \). For each component \( I^a \) of \( I \), let \( r \) be its pure dimension. For each point \( H \in I^a \), the codimension of \( I^a \) in \( X^* \) is \( n - r \), so those \( T \)’s which are tangent to \( X^* \) at \( H \) are at most parametrized by a space of dimension \( n - r - 1 \). Hence, those \( T \)’s which are tangent to \( X^* \) at \( I^a \) are at most parametrized by a space of dimension \( n - 1 \). Since \( T \) is parametrized by \( B \) which is of dimension \( n \) by the assumption, a general \( T \) would not tangent to \( X^* \) at \( I \). Therefore, by Proposition 2.11 \( \lambda_{X_T}^I \) would be smooth at \( \lambda_{X_T}^I \) for a general \( T \). Here we identify \( T_0 \) with its inverse along the normalization.

It remains to show the smoothness of \( \lambda_{X_T}^I \). By generic smoothness of morphisms, cf. [Har77, III Cor. 10.7], a general fiber \( T_0 \) of \( \Phi_0 \) would be smooth, hence the normalization is an identity on it. Then, we tautologically extend \( \Phi_0 \) to

\[
\tilde{\Phi}_0: \mathcal{X} - X_I \to M_{d,n-1}
\]

\[
(H, x) \mapsto \Phi_0(X_H)
\]

where \( \mathcal{X} - X_I \) will be smooth because \( \mathcal{X} \) is a projective bundle over \( X \). Again, the generic smoothness of \( \tilde{\Phi}_0 \) implies the smoothness of \( \mathcal{X}_{T_0}^I \) for a general \( T_0 \).\footnote{This discussion might be not necessary because if a fiber \( T_0 \) represents some smooth hypersurface, it would not intersect with \( X^* \). However, we define \( \Phi_0 \) by extending \( \Phi \), so a priori \( \Phi_0 \) may define on some points that represent unstable hyperplane sections. We use this generic smoothness argument to avoid that discussion.}
2.3. Comparing Hodge numbers. So far, we have shown that if a general
nonempty fiber of \( \Phi_0 \) has positive dimension, one can find an (irreducible)
curve \( T_0 \) on \( \mathbb{P}^{(n+1)*} \), such that

(i) \( \Phi_0 \) will send the generic point of \( T_0 \) to a closed point, which represents
a smooth hypersurface with a trivial automorphism group. (Corollary 2.7)

(ii) \( \mathcal{X}_T \) is smooth, where \( \mathcal{X}_T := \mathcal{X} \times \mathbb{P}^{(n+1)*}, T \) and \( T \) is the normalization of
\( T_0 \). (Proposition 2.8)

We will show these conditions are naturally exclusive to each other.

Let \( X_H = \mathcal{X}_H \) be a general smooth fiber of \( \mathcal{X}_T \to T \). We compare \( \mathcal{X}_T \to T \)
with \( T \times X_H \to T \) by considering the Isom scheme \( \text{Isom}_T(\mathcal{X}_T, T \times X_H) \to T \).
By (i), it is one-to-one over a dense open subset of \( T \). In particular, \( \mathcal{X}_T \) is birational to \( T \times X_H \). Therefore,

\[
H^0(\Omega_{\mathcal{X}_T}^{n-1}) \simeq H^0(\Omega_{T \times X_H}^{n-1}) \simeq H^0(\Omega_T^1) \otimes H^0(\Omega_{X_H}^{n-2}) \oplus H^0(\Omega_X^{n-1}) \tag{2.2}
\]

On the other hand, by (ii), \( \mathcal{X}_T \subseteq T \times X \) is a smooth ample divisor of \( T \times X \).
We shall denote \( \mathcal{O}_{T \times X}(\mathcal{X}_T) \) by \( \mathcal{O}_{T \times X}(1, 1) \) where the bidegree \( (1, 1) \) comes
from \( T \times X \to \mathcal{O}_0 \times X \subseteq \mathbb{P}^{(n+1)*} \times \mathbb{P}^{n+1} \). Then we have the short exact sequence

\[
0 \to \mathcal{O}_{T \times X}(-1, -1) \to \mathcal{O}_{T \times X} \to \mathcal{O}_{\mathcal{X}_T} \to 0
\]

and hence

\[
0 \to \Omega_{T \times X}^p(-1, -1) \to \Omega_{T \times X}^p \to \Omega_{\mathcal{X}_T}^p |_{\mathcal{X}_T} \to 0. \tag{2.3}
\]

Also we have the conormal bundle sequence for \( \mathcal{X}_T \subseteq T \times X \)

\[
0 \to \mathcal{O}_{\mathcal{X}_T}(-1, -1) \to \Omega_{T \times X} |_{\mathcal{X}_T} \to \Omega_{\mathcal{X}_T} \to 0
\]

and the \( p \)-th exterior power of it

\[
0 \to \Omega_{\mathcal{X}_T}^{p-1}(-1, -1) \to \Omega_{T \times X}^p |_{\mathcal{X}_T} \to \Omega_{\mathcal{X}_T}^p \to 0. \tag{2.4}
\]

Apply Kodaira vanishing to the long exact sequences of (2.3) and (2.4), we get the composition of

\[
H^q(\Omega_{\mathcal{X}_T}^p) \to H^q(\Omega_{T \times X}^p |_{\mathcal{X}_T}) \to H^q(\Omega_{\mathcal{X}_T}^p)
\]

is an isomorphism for \( p + q < n \). In particular,

\[
H^0(\Omega_{\mathcal{X}_T}^{n-1}) \simeq H^0(\Omega_{T \times X}^{n-1}) \simeq H^0(\Omega_T^1) \otimes H^0(\Omega_X^{n-2}) \oplus H^0(\Omega_X^{n-1}). \tag{2.5}
\]

Compare (2.2) with (2.5), we get

\[
H^0(\Omega_{X_H}^{n-1}) \simeq H^0(\Omega_X^{n-1}).
\]

The right hand side will vanish since \( n > 1 \), while by the adjunction formula

\[
\omega_{X_H} = \mathcal{O}_{X_H}(d - n - 1)
\]

the left hand side will never vanish if \( d > n \). Therefore, we already proved our main theorem.

\footnote{This might be a bad notation as the induced map on Picard groups may not be injective, but it reminds us it is always ample.}
Theorem 2.13. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d$, where $d > n > 1$ and $(n,d) \neq (2,3),(3,4)$. Let $M_{d,n-1} := |O_{\mathbb{P}^n}(d)|^{ss}/\text{SL}(n+1)$ be the moduli space of hypersurfaces of degree $d$ in $\mathbb{P}^n$, and

$$\Phi: \mathbb{P}^{(n+1)*} - X^* \to M_{d,n-1}$$

be the natural morphism defined by taking smooth hyperplane sections. Then the image of $\Phi$ is always of dimension $n+1$.

I believe the condition $d > n$ will not be necessary. For example, I am wondering if the following is true.

Conjecture 2.14. Let $S \to T$ be a family of hypersurfaces of degree $d > 2$. If $S$ is smooth, and it is a trivial bundle over $T_0 := T - \{\text{a closed point}\}$, then $S \to T$ itself has to be a trivial bundle as well.

If this is true, the above theorem can be generalized to the following.

Conjecture 2.15. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d$, where $n > 1$, $d > 2$, and $(n,d) \neq (2,3),(3,4)$. Let

$$\Phi: \mathbb{P}^{(n+1)*} - X^* \to M_{d,n-1}$$

be the natural morphism defined above. Then the image of $\Phi$ is always of dimension $n+1$.

References

[ADF+15] Valery Alexeev, Ron Donagi, Gavril Farkas, Elham Izadi, and Angela Ortega. The uniformization of the moduli space of principally polarized abelian 6-folds. J. Reine Angew. Math., 2015.

[Bea90] Arnaud Beauville. Sur les hypersurfaces dont les sections hyperplanes sont à module constant. In The Grothendieck Festschrift, Volume I, volume 86 of Progress in Mathematics, pages 121–133. Birkhäuser, Boston, 1990.

[Dem80] Michel Demazure. Surfaces de Del Pezzo - II. Eclater n points dans $\mathbb{P}^2$. In Séminaire sur les Singularités des Surfaces, volume 777 of Lecture Notes in Mathematics, pages 23–35. Springer, Berlin-Heidelberg-New York, 1980.

[Dol12] Igor Dolgachev. Classical Algebraic Geometry: a modern view. Cambridge University Press, Cambridge, 2012.

[EH16] David Eisenbud and Joe Harris. 3624 and All That. Cambridge University Press, Cambridge, 2016.

[Har77] Robin Hartshorne. Algebraic Geometry, volume 52 of Graduate Texts in Mathematics. Springer, New York, 1977.

[Har79] Joe Harris. Galois groups of enumerative problems. Duke Math. J., 46(4):685–724, 1979.

[HMP98] Joe Harris, Barry Mazur, and Rahul Pandharipande. Hypersurfaces of low degree. Duke Math. J., 95(1):125–160, 1998.

[Laz03] Robert Lazarsfeld. Positivity in Algebraic Geometry, II, Positivity for Vector Bundles, and Multiplier Ideals. Springer, New York, 2003.

[Voi03] Claire Voisin. Hodge Theory and Complex Algebraic Geometry II. Cambridge University Press, Cambridge, 2003.

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