Local Estimates for Some Fully Nonlinear Elliptic Equations

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Abstract

We present a method to derive local estimates for some classes of fully nonlinear elliptic equations. The advantage of our method is that we derive Hessian estimates directly from $C^0$ estimates. Also, the method is flexible and can be applied to a large class of equations.

Let $(M, g)$ be a smooth Riemannian manifold of dimension $n \geq 2$. We are interested in a priori estimates for solutions of some classes of fully nonlinear elliptic equations on $(M, g)$. These kinds of equations arise naturally from geometry and other fields of analysis and share structures similar to those of the Monge-Ampere equations.

Regularity problems are studied by people in different fields separately. One would like to ask if it is possible to give a unified proof and to generalize further to a large class of equations. The answer is affirmative provided the equations satisfy some algebraic structures which can induce the cancellation phenomenon. We will see how this phenomenon helps us to get the Hessian bound directly. One of the interesting cases is the Schouten tensor equation arising from conformal geometry:

$$\sigma_k^1(g^{-1}((\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g)) = f(x)e^{-2u}$$

where $\sigma_k$ is the kth elementary symmetric function. Local $C^2$ estimates are proved for this equation by Chang, Gursky, and Yang [2] ($k = 2, n = 4$) and by P.Guan and G.Wang [7] for all $k \leq n$. The same results with specific dependence on the radius of the domain are established by Gursky and Viaclovsky [10]. P.Guan and G.Wang [8] also prove the local estimates for quotients of the elementary symmetric functions. Other related works in this direction include [12], [11] and [14].

Another interesting case is the following equation in optics geometry:

$$\det(g_c^{-1}(\nabla^2 v - \frac{\nabla v|^2}{2v} g_c + \frac{v}{2} g_c)) = \left(\frac{\nabla v|^2 + v^2}{2v}\right)^n \nu(x)\phi(S(x, v, \nabla v)).$$
Interior $C^2$ estimates are proved by X.Wang [11] for $n = 2$, while local $C^2$ estimates for all $n \geq 2$ are by P.Guan and X.Wang [9].

It turns out that in getting local $C^2$ estimates for nonlinear equations as described above, the coefficient in front of the gradient square term plays an important role. For arbitrary coefficients, in general it is not true that we have local estimates. See [14] for a counterexample.

In the degenerate case, when the gradient square term disappears, one can have maximal principles for second derivatives, which means the Hessian bound over $\overline{\Omega}$ is less than or equal to that on $\partial \Omega$. Examples are the general Monge-Ampere equations. In particular, the Gauss curvature equation in a domain $\Omega$ in $\mathbb{R}^n$,

$$\frac{1}{n} \det (\nabla^2 u) = \kappa_1^n (x) (1 + |\nabla u|^2)^{\frac{n+2}{2n}},$$

is of this type. Another relevant equation is the Gauss curvature equation for a radial graph in a domain $\Omega$ in $S^n$:

$$\det (g_c^{-1}(\nabla^2 v + v g_c)) = \kappa_1^n (x) \left( \frac{v^2 + |\nabla v|^2}{g_c} \right)^{\frac{n+2}{2n}}.$$

Maximal principles for the first equation is studied in Caffarelli, Nirenberg, and Spruck [1], and for the latter by B.Guan and J. Spruck [5]. See also [6], [16] and [4] for related works.

In this paper, we consider more general operators, which in particular include the equations discussed above. We will derive local $C^2$ estimates directly from $C^0$ bounds and also prove the maximal principles for second derivatives. For the reader who is more interested in the aforementioned equations, he or she can jump directly to Section 2 where a brief explanation and statements of results about them are given.

Now we turn to the equations we are going to discuss. Let

$$W = \nabla^2 u + a(x) du \otimes du + b(x)|\nabla u|^2 g + B(x)$$

be a $(0,2)$ tensor on a Riemannian manifold $(M^n, g)$. The derivatives are covariant derivatives with respect to the metric $g$. Consider the equation

$$F(g^{-1}W) = f(x, u) h(x, \nabla u)$$

where $F$ satisfies some fundamental structure conditions listed later and $g^{-1}$ is the induced inverse tensor of metric tensor $g$. Equation (2) means that we apply $F$ to the eigenvalues of matrix (or $(1,1)$ tensor) $g^{-1}W$. When the manifold is flat (e.g., the Euclidean case), we have $g_{ij} = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta. In this case, we drop $g^{-1}$ and simply write $F(W) = f(x, u) h(x, \nabla u)$.

We now describe the fundamental structure conditions for $F$. Let $\Gamma$ be an open convex cone with vertex at the origin satisfying $\{\lambda : \lambda_i > 0, \forall i\} \subset \Gamma \subset \{\lambda : \sum_i \lambda_i >$
0\}$. Suppose that $F(\lambda)$ is a homogeneous symmetric function of degree one in $\Gamma$ normalized with $F(e) = F((1, \ldots, 1)) = 1$. Moreover, $F$ satisfies the following in $\Gamma$:

(S0) $F$ is positive.
(S1) $F$ is concave. (i.e., $\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}$ is negative semi-definite.)
(S2) $F$ is monotone. (i.e., $\frac{\partial F}{\partial \lambda_i}$ is positive.)

In some cases, we need an additional condition:

(A) $\sum_i \frac{\partial F}{\partial \lambda_i} \geq \mu_0 \left( \frac{\sum \lambda_i}{F} \right)^{\mu_1}$, for some $\mu_0, \mu_1 > 0$.

An easy example is $F = \lambda_1 + \cdots + \lambda_n$ with $\Gamma = \{\lambda : \lambda_1 + \cdots + \lambda_n > 0\}$. Then $F(g^{-1}W) = tr_g W = \Delta u + (a(x) + nb(x))|\nabla u|^2 + tr_g B(x)$ is just the Laplace-Beltrami operator plus some lower order terms, where $tr_g$ is the trace with respect to $g$. More interesting examples are discussed in Section 1. Condition (S1) is necessary in most elliptic theories. Condition (S2) is the actual ellipticity. It is an elementary fact that if $F$ is a symmetric function of eigenvalues, then $\frac{\partial F}{\partial \lambda_i} > 0$ for all $i$ if and only if $F^{ij} := \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}$ is positive definite. Condition (A) is used previously in [9]. There is one more key point. In general, we do not have uniform ellipticity for fully nonlinear elliptic equations. This is because $F^{ij}$ involves $\nabla^2 u$ whose a priori estimates need to be derived.

A natural question is whether we can consider the tensor in forms other than (1). It turns out that for some equations coming from geometry, they can be formulated in the form of (1) after a wise choice of the function $u(x)$. We will see in Section 2 that it is certainly the case for geometric optics equations and Gauss curvature equations on spheres.

Before stating the theorems, we introduce the following notations. Let $f(x, z) : M^n \times \mathbb{R} \to \mathbb{R}$ and $h(x, p) : M^n \times \mathbb{R}^n \to \mathbb{R}$ be two given positive functions. Let $u = u(x) : M^n \to \mathbb{R}$ be a solution to (2). We define

\[
\begin{align*}
\alpha_{inf} & = \inf_{x \in M} f(x, u), \\
\alpha_{sup} & = \sup_{x \in M} (f + |\nabla_x f(x, u)| + |f_2(x, u)| + |\nabla^2_x f(x, u)| + |\nabla_x f_2(x, u)| + |f_{zz}(x, u)|), \\
\epsilon_{sup} & = \sup_{x \in M} (h(x, \nabla u) + |\nabla_x h(x, \nabla u)| + |\nabla_p h(x, \nabla u)| + |\nabla^2_p h(x, \nabla u)| \\
& \quad + |\nabla_x \nabla_p h(x, \nabla u)| + |\nabla^2_x h(x, \nabla u)|).
\end{align*}
\]

If we restrict $x$ to a local ball $B_r$, we use the corresponding notations $\alpha_{inf}(r), \alpha_{sup}(r)$ and $\epsilon_{sup}(r)$. We also use the convention that a $(0, 2)$ tensor $T_{ij} \geq g_{ij}$ means that $T(v, v) \geq g(v, v)$ for all vectors $v$ as a bilinear form.

In the following theorem, cases (a) and (b) show how relations of $a(x), b(x)$ and $h(x, \nabla u)$ give us Hessian estimates directly. However, if we have gradient bounds already, then $h$ is bounded. Case (c) shows more general results.
Consider a special type of equation in geometry; an example of case (b) and (c) is the geometric optics equation. If the manifold has enough symmetry, say of constant sectional curvature, where $a = 0$ and $b = 0$ (e.g., Euclidean space), this is the Monge-Ampere type equation. An example of case (a) is the Schouten tensor equation arising from conformal geometry; an example of case (b) and (c) is the geometric optics equation. For the degenerate case $b = 0$, we do not in general have local estimates. However, if the manifold has enough symmetry, say of constant sectional curvature $K$, we may consider a special type of equation

$$F(g^{-1}(\nabla^2 u + adu \otimes du + Kg)) = f(x, u)h(\nabla u)$$

where $a$ is a constant. Note that when $a = 0$ and $K = 0$ (e.g., Euclidean space), this is the Monge-Ampere type equation.

**Theorem 1.** (Local estimates) Let $F$ satisfy the structure conditions (S0)-(S2) in a corresponding cone $\Gamma$ and $u(x)$ be a $C^4$ solution to (3) in a local geodesic ball $B_r$. Suppose that $b(x) < -\delta_1$ and $a(x) + nb(x) < -\delta_2$.

- **case (a):** $h = h_0$ is a positive constant. Then
  $$\sup_{x \in B_r} (|\nabla^2 u| + |\nabla u|^2) \leq C_1,$$
  where $C_1 = C_1(n, r, \|a\|_{C^2}, \|b\|_{C^2}, \|B\|_{C^2}, \|g\|_{C^3}, h_0, \delta_1, \delta_2, c_{sup}(r))$ but is independent of $c_{inf}(r)$.

- **case (b):** Suppose $h = h(\nabla u)$ and $f = f(x)$. Let $\Lambda(p)$ be a positive function such that $h_{p,p_j} \geq \Lambda(p)g_{ij}$. If there exists some number $M > 0$ such that
  $$h \leq M\Lambda(p)(1 + |p|)^2 \quad \text{and} \quad |\nabla_p h| \leq M\Lambda(p)(1 + |p|),$$
  then
  $$\sup_{x \in B_r} (|\nabla^2 u| + |\nabla u|^2) \leq C_2,$$
  where $C_2 = C_2(n, r, \|a\|_{C^2}, \|b\|_{C^2}, \|B\|_{C^2}, \|g\|_{C^3}, \delta_1, \delta_2, M, \sup_p \Lambda(p), c_{sup}(r), c_{inf}(r)).$

- **case (c):** Suppose that $F$ satisfies the additional condition (A) and that $\Gamma^+_2 \subset \Gamma$. (See Section 7 for the definition of $\Gamma^+_2$.) Then
  $$\sup_{x \in B_r} |\nabla^2 u| \leq C_3,$$
  where $C_3 = C_3(\mu_0, \mu_1, n, r, \|a\|_{C^2}, \|b\|_{C^2}, \|B\|_{C^2}, \|g\|_{C^3}, \delta_1, \delta_2, c_{sup}(r), c_{sup}(r), c_{sup}(r), c_{inf}(r), \sup_{B_r} |\nabla u|)$.

Theorem 2. (Maximum Principle) Let $F$ satisfy the structure conditions (S0)-(S2) in a corresponding cone $\Gamma$. Suppose that $(M, g)$ is of nonnegative constant sectional curvature $K$ and that $h_{p,p_j} \geq \epsilon \delta_{ij}$ for some positive $\epsilon$. Let $u(x)$ be a $C^4$ solution to (3) in a bounded domain $\Omega \subset M$. Then

$$\sup_{x \in \Omega} |\nabla^2 u| \leq C_4$$

where $C_4 = C_4(n, a, K, \epsilon, c_{sup}, c_{inf}, \|u\|_{C^3(\Omega)}, \sup_{\partial \Omega} |\nabla^2 u|)$. 

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Examples of Theorem 2 are the Gauss curvature equations on a domain in $\mathbb{R}^n$ and in $S^n$.

This paper is organized as follows. We start with some background in Section 1. In Section 2, we discuss applications and give the statements of results. The proofs of Theorems 1 and 2 are in Sections 3 and 4, respectively.

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1 Background

First, we give some basic facts about homogeneous symmetric functions.

Lemma 1. Let $\Gamma$ be an open convex cone with vertex at the origin satisfying $\{\lambda : \lambda_i > 0, \forall i\} \subset \Gamma$ and $e = (1, \ldots, 1)$ be the identity. Suppose that $F$ is a homogeneous symmetric function of degree one normalized with $F(e) = 1$, and that $F$ is concave in $\Gamma$. Then the following are true:

(a) $\sum_i \lambda_i \frac{\partial F(\lambda)}{\partial \lambda_i} = F(\lambda)$, for $\lambda \in \Gamma$.

(b) $\sum_i \frac{\partial F(\lambda)}{\partial \lambda_i} \geq F(e) = 1$, for $\lambda \in \Gamma$.

Proof. (a) By homogeneity, $F(\theta \lambda) = \theta F(\lambda)$. Let $\theta$ be some positive number. Since $F$ is concave in $\Gamma$, then

$$(\theta - 1)F(\lambda) = F(\theta \lambda) - F(\lambda) \leq \sum_i (\theta \lambda_i - \lambda_i) \frac{\partial F(\lambda)}{\partial \lambda_i}.$$ 

Choose some $\theta < 1$ and some $\theta > 1$ and cancel out the factor $(\theta - 1)$, which gives the result.

(b) $\Gamma$ contains the identity $e$ and since $F$ is concave in $\Gamma$, we have

$$F(e) - F(\lambda) \leq \sum_i (1 - \lambda_i) \frac{\partial F(\lambda)}{\partial \lambda_i} = \sum_i \frac{\partial F(\lambda)}{\partial \lambda_i} - F(\lambda)$$

where the equality holds by (a). Cancelling out $F(\lambda)$, we prove (b). \qed

Now we focus on elementary symmetric functions because most interesting cases are related to them.

Definition 1. Let $W$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\sigma_k(\lambda(W)) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$ for $k \leq n$ is called the $k$th elementary symmetric function of the eigenvalues of $W$. We denote $\sigma_0 = 1$. For examples, $\sigma_1 = \lambda_1 + \cdots + \lambda_n = tr W$ and $\sigma_n = \lambda_1 \cdots \lambda_n = det W$. 

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The elementary symmetric functions are examples of hyperbolic polynomials introduced by Garding [3], which have nice properties in the associated cones.

**Definition 2.** The set $\Gamma^+_k = \{ \text{the connected component of } \sigma_k(\lambda) > 0 \text{ which contains the identity} \}$ is called the positive $k$-cone. Equivalently, it is shown in [3] that for $k > 0$, $\Gamma^+_k = \{ \lambda : \sigma_i(\lambda) > 0, 1 \leq i \leq k \}$ is an open convex cone with vertex at the origin, e.g., $\Gamma^+_1 = \{ \lambda : \lambda_1 + \cdots + \lambda_n > 0 \}$ and $\Gamma^+_n = \{ \lambda : \lambda_i > 0, 1 \leq i \leq n \}$. We also have the nested relation

$$\Gamma^+_1 \supset \Gamma^+_2 \supset \cdots \supset \Gamma^+_n.$$ We say that $W \in \Gamma^+_k$ if the eigenvalues $\lambda(W) \in \Gamma^+_k$.

We list some basic properties of elementary symmetric functions.

**Lemma 2.** (see [3], [13] and [15] for the proof) Let $G = \left( \frac{\sigma_k}{\sigma_l} \right)^{1/k-1}, 0 \leq l < k \leq n$.

(a) $G$ is positive and concave in $\Gamma^+_k$.

(b) $G$ is monotone in $\Gamma^+_k$, i.e., the matrix $G^{ij} = \frac{\partial G}{\partial W_{ij}}$ is positive definite.

(c) For $1 \leq m < k \leq n$, we have the Newton-MacLaurin inequality

$$k(n - m + 1)\sigma_{m-1}\sigma_k \leq m(n - k + 1)\sigma_m\sigma_{k-1}.$$ Therefore, $F = \left( \binom{n}{k}^{-\frac{1}{k-1}} \right)^{1/k-1} G$ satisfies the structure conditions (S0)-(S2) in $\Gamma^+_k$. We further show that if $l = 0$ and $k \geq 2$, then $F = \left( \binom{n}{k}^{-\frac{1}{k-1}} \right)^{1/k-1} \sigma_k$ satisfies (A) with $\mu_0 = n^{-\frac{1}{k-1}}$ and $\mu_1 = \frac{1}{k-1}$.

By Lemma 2(c), for $1 \leq m \leq k - 1$, we have the recursive formula

$$\sigma_m \geq \frac{k(n - m + 1)}{m(n - k + 1)} \left( \frac{\sigma_k}{\sigma_{k-1}} \right) \sigma_{m-1}.$$ Then

$$\sigma_{k-1} \geq \frac{k^{k-2}(n - k + 2) \cdots (n - 1)}{(n - k + 1)^{k-2}(k-1)!} \left( \frac{\sigma_k}{\sigma_{k-1}} \right)^{k-2} \sigma_1 = \left( \binom{n}{k}^{-\frac{1}{k-1}} \right)^{k-1} \frac{(n^k)^{k-1}}{n} \left( \frac{\sigma_k}{\sigma_{k-1}} \right)^{k-2} \sigma_1,$$

which implies

$$\sum_i \frac{\partial F}{\partial \lambda_i} = \left( \binom{n}{k}^{-\frac{1}{k-1}} \right)^{k-1} \frac{n - k + 1}{k} \frac{1}{\sigma_{k-1}\sigma_k} \geq n^{-\frac{1}{k-1}} \left( \frac{\sigma_1}{F} \right)^{k-1}.$$ Another useful function, which is also a variant of the elementary symmetric functions, is

$$\frac{1}{t} \sigma_k^\frac{1}{t} (t\lambda + s\sigma_1(\lambda)e).$$ Suppose $t, s \geq 0$ with $t + s \geq 1$. Let $\Gamma = \{ \lambda : t\lambda + s\sigma_1(\lambda)e \in \Gamma^+_k \}$. Then $\Gamma$ is an open convex cone with vertex at the origin. Let $F = \frac{1}{t+s} \binom{n}{k}^{-\frac{1}{k-1}} \sigma_k^\frac{1}{t} (t\lambda + s\sigma_1(\lambda)e)$. It
is easy to see that $F$ is a homogeneous symmetric function of degree one. Moreover, it is shown in [12] that $F$ is concave in $\Gamma$.

Since we consider equations on manifolds, all derivatives are the covariant derivatives with respect to the metric $g$. Let $u$ be a function on a manifold. Recall that $u_{ij} = u_{ji}$. However, when we consider higher order derivatives, we should get some curvature terms if we change the order of differentiations. We denote the Riemannian, Ricci, and scalar curvature by $R_{ijkl}, R_{ij}$ and $R$, respectively. The following formulae are very useful. We remind the readers that we assume $g_{ij}(0) = \delta_{ij}$ without loss of generality:

\[
\begin{align*}
    u_{kij} &= u_{ijk} + R_{mikj}u_m \\
    u_{ijkl} &= u_{ijlk} + R_{mjkl}u_{mi} + R_{mikl}u_{mj} \\
    u_{kkij} &= u_{ijkk} + 2R_{mikj}u_{mk} - R_{mj}u_{mi} - R_{mi,j}u_m + R_{mikj,k}u_m
\end{align*}
\]

Hence,

\[
\begin{align*}
    u_{kij} &= u_{ijk} + O(|\nabla u|) \\
    u_{kkij} &= u_{ijkk} + O(|\nabla^2 u| + |\nabla u|).
\end{align*}
\]

2 Applications

In this section, we will list examples where Theorem 1 and Theorem 2 can be applied.

1. Schouten tensor and conformal geometry

Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$. The Schouten tensor of $g$ is defined as

\[
A_g = \frac{1}{n-2}(Ric - \frac{R}{2(n-1)} g).
\]

Under the conformal change $g_u = e^{-2u}g$, the tensor $A_{g_u}$ satisfies

\[
A_{g_u} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g.
\]

We consider the equation ($0 \leq l < k \leq n$)

\[
(\frac{\sigma_k}{\sigma_l})^{k-l} (g^{-1}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g)) = f(x) e^{-2u}. \tag{4}
\]

Local estimates are proved by Chang,Gursky, and Yang [2] ($k = 2, l = 0$ or $1, n = 4$), Guan-Wang [7] ($l = 0$), and Guan-Wang [8] with the additional assumption $(n - k + 1)(n - l + 1) > 2(n + 1)$. As a corollary of Theorem II (a), we prove local $C^2$ estimates for all $0 \leq l < k \leq n$ with specific dependence on the radius. The following argument is a modification of that in Gursky and Viaclovsky [10] where the case $l = 0$ is proved.

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Corollary 1. Let $u(x)$ be a $C^4$ solution to (4) with $A_{gu} \in \Gamma_k^+$ in a geodesic ball $B_r$ in $(M, g)$, $0 \leq l < k \leq n$. Suppose that $f$ is positive. Then

$$\sup_{B_{\frac{r}{2}}} (|\nabla u(x)|^2 + |\nabla^2 u(x)|) \leq C(r^{-2} + \sup_{x \in B_r} e^{-2u})$$

(5)

where $C$ depends on $n, k, l, \|g\|_{C^4}, \|f\|_{C^2}$ but does not depend on $\inf f$

Proof. In Section 1, we showed that $F = (\frac{r}{t})^{\frac{1}{k-l}} (\frac{n}{l})^{\frac{1}{k-l}} (\frac{2n}{\sigma_1})^{\frac{1}{k-l}}$ satisfies the structure conditions (S0)-(S2). Let us check the conditions in Theorem 1 (a). In this case, $a = 1, b = -\frac{1}{t}$ and $h_0 = 1$. Therefore, $\delta_1 = \frac{1}{k}, \delta_2 = \frac{n-2}{k}$ and $B = A_{\tilde{g}}$ whose $C^2$ norm depends on $\|g\|_{C^4}$. Finally,

$$c_{\sup} = \left(\frac{n}{k}\right)^{\frac{1}{k-l}} \left(\frac{1}{l}\right)^{\frac{1}{k-l}} \sup_{x \in B(1)} (4f + |2f_i| + |f_{ij}|) e^{-2u} \leq c \sup_{x \in B(1)} e^{-2u}$$

where $c$ depends on $n, k, l, \|f\|_{C^2}$ but does not depend on $\inf f$. Hence, we prove the case for $r = 1$. As for general $r$, without loss of generality, we may assume the injectivity radius $\iota$ is greater or equal to one and $r < 1$. Define the mapping

$$E(y) : B_1 \subset \mathbb{R}^n \rightarrow B_r \subset M^n$$

$$y \rightarrow \exp(ry) = x$$

where $\exp$ is the exponential map. On $B(1)$, define the metric $\tilde{g} = r^{-2} E^* g$ and the function $\tilde{u}(y) = u(E(y)) - \ln r$. Then $\tilde{u}$ satisfies

$$\frac{\sigma_k}{\sigma_l} r^{\frac{1}{k-l}} (\tilde{g}^{-1}(\nabla^2 \tilde{u} + d\tilde{u} \otimes d\tilde{u} - \frac{1}{2} \nabla \tilde{u} |^2 \tilde{g} + A_{\tilde{g}})) = f(E(x)) e^{-2\tilde{u}}$$

on $B_1$. By the estimates we obtained for $r = 1$, we get

$$\sup_{B_{\frac{r}{2}}} (|\nabla \tilde{u}|^2 + |\nabla^2 \tilde{u}|)(y) \leq C(1 + \sup_{y \in B_1} e^{-2\tilde{u}}).$$

Now by the definitions of $E, \tilde{g}$ and $\tilde{u}$, it is not hard to see $|\nabla u(x)|^2 + |\nabla^2 u(x)| = r^{-2}(|\nabla \tilde{u}|^2 + |\nabla^2 \tilde{u}|)(y)$ and $e^{-2\tilde{u}} = r^2 e^{-2u}$. It remains to verify the conditions on the constant $C$. Since $r < 1$, we have $\|\tilde{g}\|_{C^4(B_1)} \leq \|g\|_{C^4(B_r)}$ and $\|E^* f\|_{C^2(B_1)} \leq \|f\|_{C^2(B_r)}$. $\square$

In [12] and [11], they consider the following equations

$$\sigma_k^t (t\lambda(A_{gu}) + s\sigma_1(\lambda(A_{gu}))g) = f(x, u)$$

for $f(x, u) = f_0(x)e^{-2u}$ and $f(x, u) = f_0(x)e^{2u}$, respectively, with $t\lambda + s\sigma_1(\lambda)g \in \Gamma_k^+, t, s \geq 0$ and $t + s \geq 1$. The local estimates are derived in [12] and [11] accordingly. We reprove these results as a corollary.
Corollary 2. Let \( f_1(x,u) = f_0(x)e^{-2u} \) and \( f_2(x,u) = f_0(x)e^{2u} \). Suppose \( u_i(x) \) is a \( C^4 \) solution of the following equations in a geodesic ball \( B_r \):

\[
\sigma_k^{\frac{1}{2}}(t\lambda(A_{g_{u_i}}) + s\sigma_1(\lambda(A_{g_{u_i}}))g) = f_i(x, u_i)
\]

for \( i = 1 \) or \( 2 \), \( t\lambda + s\sigma_1(\lambda)g \in \Gamma_k^+ \), \( t + s \geq 1 \) and \( t + ns \leq c_0 \). Then

\[
\sup_{B_S^r} (|\nabla u_1(x)|^2 + |\nabla^2 u_1(x)|) \leq C(1 + \sup_{x \in B_r} e^{-2u_1})
\]

and

\[
\sup_{B_S^r} (|\nabla u_2(x)|^2 + |\nabla^2 u_2(x)|) \leq C(1 + \sup_{x \in B_r} e^{2u_2})
\]

where \( C = C(n, k, r, \|g\|_{C^4}, \|f_0\|_{C^2}) \) but is independent of \( t, s \) and \( \inf f_0 \).

Proof. The proof is similar to that of Corollary 1. Let \( F(\lambda) = \frac{1}{1+ns}(\frac{n}{k})^{-\frac{1}{2}}\sigma_k^{\frac{1}{2}}(t\lambda + s\sigma_1(\lambda)g) \), so \( F \) satisfies (S0)-(S2). \( \square \)

2. Optics Geometry

Let \((S^2, g_c)\) be the standard 2-sphere. Suppose there is a point source light at the origin with the density function \( \nu(x), x \in S^2 \) and the light reflects according to the geometric optics. Given domains \( \Omega, D \subset S^2 \), we are asked to find a star-shaped surface \( \Sigma \subset \mathbb{R}^3 \) whose projection to \( S^2 \) is \( \Omega \) such that the light reflected from \( \Sigma \) travels in directions in \( D \) with density \( \phi^{-1}(x), x \in D \). This is related to the reflector antenna design problem. Mathematically, it means to find a positive solution \( v \) of the fully nonlinear elliptic equation

\[
\det(g_c^{-1}(\nabla^2 v - \frac{\nabla v}{2g_c}g_c + \frac{v}{2}g_c)) = \left( \frac{\|\nabla v\|^2 + v^2}{2v} \right)^2 \nu(x)\phi(S(x, v, \nabla v))
\]

where \( S(x, v, \nabla v) = -\frac{2v\nabla v + (v^2 - \|\nabla v\|^2)N(x)}{\|\nabla v\|^2 + v^2} \) and \( N(x) \) is the unit vector pointing to \( x \in S^2 \). (For background and derivation of the equation, see \[18\], \[17\] and \[9\].) Let us consider the general equation on \( S^m \):

\[
\det(g_c^{-1}(\nabla^2 v - \frac{\nabla v}{2g_c}g_c + \frac{v}{2}g_c)) = \left( \frac{\|\nabla v\|^2 + v^2}{2v} \right)^n \nu(x)\phi(S(x, v, \nabla v)).
\]

The tensor inside \( \det \) is not in the form of (1). However, since \( v \) is positive, let \( u = \ln v \). The equation becomes

\[
\det(g_c^{-1}(\nabla^2 u + du \otimes du - \frac{1}{2}\|\nabla u\|^2 g_c + \frac{g_c}{2})) = \left( \frac{\|\nabla u\|^2 + 1}{2} \right)^n \nu(x)\phi(T(x, \nabla u))
\]

where \( T(x, u) = -\frac{2v\nabla u + (1-\|\nabla u\|^2)N(x)}{1+\|\nabla u\|^2} \). In \[9\], local \( C^2 \) estimates are proved. As a corollary of Theorem 1(c), we prove the following.
Corollary 3. Let $u(x)$ be a $C^4$ solution to (8) in a geodesic ball $B_r$ with
\[ \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g_c + \frac{1}{2}g_c \in \Gamma^+_n. \]
Then
\[ \sup_{B_{\frac{r}{2}}} |\nabla^2 u(x)| \leq C \]
where $C$ depends on $n, r, \|\nu\|_{C^2}, \|\phi\|_{C^2}, \sup_{B_r} |\nabla u|$ but does not depend on $\inf \nu$ and $\inf \phi$.

Proof. In Section 1, we showed that $F = \sigma_{\frac{1}{n}}^1$ satisfies (S0)-(S2) and (A) with $\mu_0 = n^{-\frac{1}{n-1}}$ and $\mu_1 = \frac{1}{n-1}$. Besides, in our case, $f = \nu^{\frac{1}{n}}(x)$ and $h = \frac{1}{2}\phi^{\frac{1}{n}}(x, \nabla u)(1 + |\nabla u|^2)$.

For a special case when $\phi$ is a positive constant, we can prove local $C^2$ estimates without using the gradient bound. This is a corollary of Theorem 1(b).

Corollary 4. Suppose that $\phi = \phi_0$ is a positive constant. Let $u(x)$ be a $C^4$ solution to (8) in a geodesic ball $B_r$ with $\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g_c + \frac{1}{2}g_c \in \Gamma^+_n$. Then
\[ \sup_{B_{\frac{r}{2}}} (|\nabla u|^2 + |\nabla^2 u(x)|) \leq C \]
where $C$ depends on $n, r, \phi_0, \|\nu\|_{C^2}$ and $\inf \nu$.

Proof. Let $f = \phi^{\frac{1}{n}}\nu^{\frac{1}{n}}(x)$ and $h = \frac{1}{2}(1 + |\nabla u|^2)$. We only need to check the conditions on $h$. Choose $\Lambda = 1$ and $M = 1$. It is easy to see that they satisfy the required conditions.

3. Convex Hypersurface and Gauss Curvature Equation

Given a closed smooth embedded $(n-1)$-dimensional submanifold $\Sigma$ in $\mathbb{R}^{n+1}$, we are asked whether there exists a hypersurface of constant Gauss-Kronecker curvature in $\mathbb{R}^{n+1}$ with $\Sigma$ as its boundary. Locally this problem is reduced to some Monge-Ampere type equation. If a hypersurface is locally strictly convex, we can express it as a graph $(x, u(x))$ for $x \in \Omega \subset \mathbb{R}^n$ which satisfies
\[ \det \left( \frac{1}{n} (\nabla^2 u) \right) = \kappa^{\frac{1}{n}}(x)(1 + |\nabla u|^2)^{\frac{n+2}{2n}} \]  
(9)
where $\kappa$ is the Gauss curvature which is positive. In Caffarelli-Nirenberg-Spruck [1], this type of Monge-Ampere equation is studied in a strictly convex domain $\Omega$. On the other hand, Guan-Spruck [2] consider star-shaped regions, i.e., radial graphs over a domain $\Omega \subset S^n$. In this setting, the problem becomes finding a positive solution $v(x)$ of the following equation in $\Omega \subset S^n$:
\[ \det \left( \frac{1}{n} g^{-1}_c(\nabla^2 v + vg_c) \right) = \kappa^{\frac{1}{n}}(x) \left( \frac{v^2 + |\nabla v|^2}{v^2} \right)^{\frac{n+2}{2n}} \]
where $g_c$ is the standard metric on $S^n$. Since $v(x)$ is positive, let $u(x) = \ln v(x)$. The equation becomes
\[
\det \left( g_c^{-1}(\nabla^2 u + du \otimes du + g_c) \right) = \kappa_n^1(x)(1 + |\nabla u|^2)^{n+2 \over 2n} e^{-u}.
\]
(10)

Equation (10) is in the form of (3) now. It is proved in Guan-Spruck [5] that the Hessian bound of the solution $u(x)$ to (10) over $\Omega$ is less than or equal to that on $\partial \Omega$. Here, as a corollary of Theorem 2, we have the following.

**Corollary 5.** Let $u(x)$ be a $C^4$ solution to (9) (or (10)) in a bounded domain $\Omega \subset \mathbb{R}^n$ (or $S^n$, respectively) with $\nabla^2 u \in \Gamma_n^+$ (or $\nabla^2 u + du \otimes du + g_c \in \Gamma_n^+$, respectively). Suppose that $\kappa$ is positive. Then
\[
\sup_{\Omega} |\nabla^2 u| < C
\]
where $C$ depends on $n, \|\kappa\|_{C^2}, \|u\|_{C^4(\Omega)}, \sup_{\partial \Omega} |\nabla^2 u|$ and $\inf \kappa$.

**Proof.** For equation (9), $K = a = 0$, and for equation (10), $K = a = 1$. In both cases, $h = (1 + |p|^2)^{n-2 \over 2n}$. We only need to check the convexity condition on $h$. If $n = 2$, then $h_{p,p} = 2\delta_{ij}$. When $n > 2$,
\[
h_{p,p} = \frac{n+2}{n} (1 + |p|^2)^{-n-2 \over 2n} \left( \frac{-n-2}{2n} \frac{p_i p_j}{1 + |p|^2} + \delta_{ij} \right)
\geq \frac{n+2}{n} (1 + |p|^2)^{-n-2 \over 2n} \left( \frac{1 + \frac{2}{n} |p|^2}{1 + |p|^2} \delta_{ij} \right) \geq \frac{n+2}{n} (1 + |p|^2)^{-n-2 \over 2n} \delta_{ij}
\]
Hence, $h_{p,p} > \epsilon \delta_{ij}$ with $\epsilon$ depending on $\sup |\nabla u|$.

3 **Proof of Theorem [1]**

**Proof.** We always assume $g_{ij} = \delta_{ij}$ at the point we are evaluating. Let $W = \nabla^2 u + a(x)du \otimes du + b(x)|\nabla u|^2 g + B(x)$. We will show that $\Delta u$ is bounded. By the condition $\Gamma \subset \Gamma_1^+$, we have
\[
0 < tr_g W = \Delta u + (a(x) + nb(x))|\nabla u|^2 + tr_g B(x).
\]
Since $a(x) + nb(x) < -\delta_2$, the Laplacian $\Delta u$ has lower bound and
\[
|\nabla u|^2 < C(\Delta u + 1).
\]
(11)
Therefore, we may assume $\Delta u$ is positive. Let $H = \eta(\Delta u + a(x)|\nabla u|^2) = \eta L$ where $0 \leq \eta \leq 1$ is a cutoff function such that $\eta = 1$ in $B_{\frac{1}{2}}$ and $\eta = 0$ outside $B_r$, and also
Then \(|\nabla \eta| < C \frac{\sqrt{\eta}}{r^2}\) and \(|\nabla^2 \eta| < \frac{C}{r^2}\). Without loss of generality, we assume \(r = 1\) since for general \(r\), the proof is similar.

Now by the condition \(\Gamma \subset \Gamma_1^+\) again, we get
\[
L > -nb|\nabla u|^2 - tr_g B \geq \delta_1 n|\nabla u|^2 - tr_g B > -C.
\]
Hence, \(L\) is lower bounded and we only need to get the upper bound of \(L\). Suppose \(x_0\) is the maximal point of \(H\). At \(x_0\), we have
\[
H_i = \eta_i L + \eta L_i = \eta_i(\Delta u + a(x)|\nabla u|^2) + \eta(u_{kki} + a_i|\nabla u|^2 + 2au_k u_{ki}) = 0,
\]
and
\[
H_{ij} = \eta_{ij} L + \eta_i L_j + \eta_j L_i + \eta_{ij} = (\eta_{ij} - \frac{2\eta_i \eta_j}{\eta})L + \eta L_{ij}
\]
is negative semi-definite where in the second equality we have used (13). Moreover,
\[
L_{ij} = u_{kki} + a_{ij}|\nabla u|^2 + 2a_i u_k u_{kj} + 2a_j u_k u_{ki} + 2au_k u_{ki} + 2au_k u_{kj}.
\]
Using the positivity of \(F^{ij}\) and the condition on \(\eta\), we get
\[
0 \geq F^{ij} H_{ij} = F^{ij}(\eta_{ij} - \frac{2\eta_i \eta_j}{\eta})L + \eta L_{ij} \geq -C \sum_i F^{ii} L + \eta F^{ij} L_{ij}. \tag{14}
\]
Now to compute \(F^{ij} L_{ij}\), we note that \(F^{ij}(2a_i u_k u_{kj}) = F^{ij}(2a_j u_k u_{ki})\) because \(F^{ij}\) is symmetric. Thus, we obtain
\[
F^{ij} L_{ij} = F^{ij}(u_{kki} + a_{ij}|\nabla u|^2 + 4a_i u_k u_{kj} + 2au_k u_{ki} + 2au_k u_{kj}).
\]
Changing the order of the covariant differentiations and using (11) give
\[
F^{ij} L_{ij} \geq F^{ij} u_{ijkk} + F^{ij}(2au_k u_{ki} + 2au_k u_{ij}) - C \sum_i F^{ii}(1 + |\nabla^2 u|^2)
= I + II - C \sum_i F^{ii}(1 + |\nabla^2 u|^2).
\]
To compute \(I\), notice that
\[
W_{ij,kk} = u_{ij,kk} + \Delta a_{ij} u_j + 2a_k u_{ik} u_j + 2a_k u_{jk} u_i + a(u_{ikk} u_j + u_{ik} u_{jk} + u_{ik} u_{kk})
+ (\Delta b|\nabla u|^2 + 4b_k u_{ti} u_i + 2b|\nabla^2 u|^2 + 2bu_{ikk}) \delta_{ij} + B_{ij,kk}.
\]
Then
\[
I = F^{ij} u_{ij,kk} = F^{ij}(W_{ij,kk} - \Delta u_{ij} u_j - 4a_k u_{ik} u_j - 2a(u_{ikk} u_j + u_{ik} u_{jk})
- (\Delta b|\nabla u|^2 + 4b_k u_{ki} u_i + 2b|\nabla^2 u|^2 + 2bu_{ikk}) \delta_{ij} - B_{ij,kk})
\geq F^{ij} W_{ij,kk} + F^{ij}(-2a(u_{ikk} u_j + u_{ik} u_{jk}) - 2b(|\nabla^2 u|^2 + u_{ikk} \delta_{ij})
- C \sum_i F^{ii}(1 + |\nabla^2 u|^2).
\]
Changing the order of covariant differentiations again yields

\[ I \geq F^{ij}W_{ij,ik} + F^{ij}(-2a(u_{kki}u_j + u_{ik}u_{jk}) - 2b(\nabla^2 u)^2 + u_iu_{kkl})\delta_{ij} - C\sum_i F^{ii}(1 + |\nabla^2 u|^\frac{2}{3}). \]

Now we replace the terms \( u_{kki} \) and \( u_{kkl} \) by (13) to get

\[ I \geq F^{ij}W_{ij,ik} + F^{ij}(2au_j(a_i|\nabla u|^2 + 2au_ku_{ki} + \frac{\eta}{\eta} L) - 2a_ku_ju_{ik} - 2b|\nabla^2 u|^2\delta_{ij}
+ 2bu_i(a_i|\nabla u|^2 + 2au_ku_{kl} + \frac{\eta}{\eta} L)\delta_{ij} - C\sum_i F^{ii}(1 + |\nabla^2 u|^\frac{2}{3}). \]

Using (13) again and the condition on \( \eta \), we have

\[ I \geq F^{ij}W_{ij,ik} + F^{ij}(4a^2 u_{kki}u_j - 2au_ku_ju_{ik} - 2b|\nabla^2 u|^2\delta_{ij} + 4abu_iu_{ikl}\delta_{ij})
- C\eta^{-\frac{1}{2}} \sum_i F^{ii}|\nabla u|L - C\sum_i F^{ii}(1 + |\nabla^2 u|^\frac{2}{3}). \]

For II, we use the formula

\[ W_{ij,k} = u_{ij,k} + a_ku_iu_j + au_{ik}u_j + au_{jk}u_i + b_k|\nabla u|^2\delta_{ij} + 2bu_lu_{kl}\delta_{ij} + B_{ij,k} \]

to obtain

\[ II = F^{ij}(2au_{kl}u_{kj} + 2au_ku_{ij}) = F^{ij}(2au_{kl}u_{kj} + 2au_kW_{ij,k}
+ 2au_k(-a_ku_iu_j - 2au_ku_j - b_k|\nabla u|^2\delta_{ij} - 2bu_lu_k\delta_{ij} - B_{ij,k})) \geq 2au_kF^{ij}W_{ij,k} + F^{ij}(2au_{kl}u_{kj} - 4a^2 u_{kki}u_{ij} - 4abu_iu_{ikl}\delta_{ij})
- C\sum_i F^{ii}(1 + |\nabla^2 u|^\frac{3}{2}). \]

Combining I and II together, we find that

\[ F^{ij}L_{ij} \geq I + II - C\sum_i F^{ii}(1 + |\nabla^2 u|^\frac{2}{3}) \]

\[ \geq F^{ij}W_{ij,ik} + 2au_kF^{ij}W_{ij,k} + F^{ij}(4a^2 u_{kki}u_j - 2au_ku_ju_{ik} - 2b|\nabla^2 u|^2\delta_{ij}
+ 4abu_lu_k\delta_{ij}) + F^{ij}(2au_{kl}u_{kj} - 4a^2 u_{kki}u_{ij} - 4abu_iu_{ikl}\delta_{ij})
- C\eta^{-\frac{1}{2}} \sum_i F^{ii}|\nabla u|L - C\sum_i F^{ii}(1 + |\nabla^2 u|^\frac{3}{2}). \]

After the cancellations, finally we arrive at

\[ F^{ij}L_{ij} \geq F^{ij}W_{ij,ik} + 2au_kF^{ij}W_{ij,k} - 2b\sum_i F^{ii}|\nabla^2 u|^2
- C\eta^{-\frac{1}{2}} \sum_i F^{ii}|\nabla u|L - C\sum_i F^{ii}(1 + |\nabla^2 u|^\frac{3}{2}) \]
Now returning to \[ (14) \] and applying \( \eta \) on both sides produces

\[
0 \geq \eta F^{ij} H_{ij} \geq -C \eta \sum_i F^{ii} L + \eta^2 F^{ij} L_{ij}
\]

\[
\geq \eta^2 F^{ij} W_{ij,kk} + 2a \eta^2 u_k F^{ij} W_{ij,k} - 2b \eta^2 \sum_i F^{ii} |\nabla^2 u|^2 - C \eta^2 \sum_i F^{ii} |\nabla u|^2
\]

\[
-C \eta \sum_i F^{ii} L - C \eta^2 \sum_i F^{ii} (1 + |\nabla^2 u|^\frac{3}{2})
\]

\[
\geq \eta^2 F^{ij} W_{ij,kk} + 2a \eta^2 u_k F^{ij} W_{ij,k} - 2b \eta^2 \sum_i F^{ii} |\nabla^2 u|^2
\]

\[
-C \sum_i F^{ii} (1 + \eta |\nabla^2 u| + (\eta |\nabla^2 u|)^\frac{3}{2}).
\]

By the concavity of \( F \) and Lemma 11 (a), we have \( F^{ij} W_{ij,kk} \geq (F^{ij} W_{ij})_{kk} = (f(x, u) h(x, \nabla u))_{kk} \). Hence,

\[
0 \geq \eta^2 \left( f(x, u) h(x, \nabla u) \right)_{kk} + 2a \eta^2 u_k (f(x, u) h(x, \nabla u))_k - 2b \eta^2 \sum_i F^{ii} |\nabla^2 u|^2
\]

\[
-C \sum_i F^{ii} (1 + \eta |\nabla^2 u| + (\eta |\nabla^2 u|)^\frac{3}{2}).
\]

\( \text{case(a): } h \text{ is a positive constant. By Lemma 11(b), } \sum_i F^{ii} \geq F(e) = 1, \text{ hence} \)

\[
0 \geq \eta^2 \frac{\partial^2 f(x, u(x))}{\partial x_k^2} h + 2a \eta^2 u_k \frac{\partial f(x, u(x))}{\partial x_k} h - 2b \eta^2 \sum_i F^{ii} |\nabla^2 u|^2
\]

\[
-C \sum_i F^{ii} (1 + \eta |\nabla^2 u| + (\eta |\nabla^2 u|)^\frac{3}{2})
\]

\[
\geq -2b \eta^2 \sum_i F^{ii} |\nabla^2 u|^2 - C \sum_i F^{ii} (1 + \eta |\nabla^2 u| + (\eta |\nabla^2 u|)^\frac{3}{2}).
\]

By the condition on \( b \), finally we arrive at

\[
0 \geq \sum_i F^{ii} (2 \delta (\eta |\nabla^2 u|)^2 - C(\eta |\nabla^2 u|) - C(\eta |\nabla^2 u|)^\frac{3}{2} - C).
\]

This gives \( (\eta |\nabla^2 u|)(x_0) \leq C \) and hence \( H(x) = \eta(\Delta u + a|\nabla u|^2) = (\Delta u + a|\nabla u|^2) = L \) is bounded for \( x \in B_{\frac{1}{2}} \). Now by \( \delta_{1} n |\nabla u|^2 \leq L + tr_{y} B \leq C \), which implies \( |\nabla u|^2 \) is bounded. And then \( \Delta u = L - a|\nabla u|^2 \) is bounded.

\( \text{case(b): } h = h(\nabla u) \text{ and } f = f(x) \). First we perform some computations:

\[
(f(x)h(\nabla u))_{kk} = f_{kk} h + 2 f_{k} h_{p} u_{ik} + f h_{p} u_{ik} u_{jk} + f h_{p} u_{ikk}
\]

\[
\geq -C \Lambda (1 + |\nabla^2 u|^\frac{3}{2}) + f \Lambda |\nabla^2 u|^2 + f h_{p} u_{ikk}
\]

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where we have used the conditions on $h$. Now changing the order of differentiations of $u_{kkk}$ and using (13) to replace $u_{kki}$ gives

$$(f(x)h(\nabla u))_{kk} \geq -CA(1 + |\nabla^2 u|^2) + fA|\nabla^2 u|^2 - f h_{pi}(a_i |\nabla u|^2 + 2au_k u_{ki} + \frac{\eta k}{\eta} L)
$$

$$\geq -CA(1 + |\nabla^2 u|^2) + fA|\nabla^2 u|^2 - 2af h_{pi} u_k u_{ki} - \frac{C}{\sqrt{\eta}} f|\nabla p h| L.$$  

On the other hand, we have

$$2au_k(f(x)h(\nabla u))_k = 2af_k h u_k + 2af h_{pi} u_{ik} u_k \geq -CA(1 + |\nabla^2 u|^2) + 2af h_{pi} u_{ik} u_k.$$ Therefore, returning to (15), we get

$$0 \geq -CA(1 + |\nabla^2 u|^2) + fA|\nabla^2 u|^2 - 2an\eta f h_{pi} u_k u_{ki} - C|\nabla p h| L + 2an\eta f h_{pi} u_{ik} u_k - 2\eta C \sum_i F_{ii} (1 + \eta |\nabla^2 u|) + (\eta |\nabla^2 u|)^\frac{3}{2}) \geq -CA(1 + |\nabla^2 u|^2) + fA|\nabla^2 u|^2 - 2bn\eta \sum_i F_{ii} |\nabla^2 u|^2 - C \sum_i F_{ii} (1 + \eta |\nabla^2 u|) + (\eta |\nabla^2 u|)^\frac{3}{2}).$$

Applying the conditions $b$ and using Lemma (b) to obtain

$$0 \geq C \sum_i F_{ii} (1 + \eta |\nabla^2 u|) \geq C \sum_i F_{ii} (1 + \eta |\nabla^2 u|) \geq C \sum_i F_{ii} (\delta_i \eta^2 |\nabla^2 u|^2 - C) \geq C \sum_i F_{ii} (\delta_i \eta^2 |\nabla^2 u|^2 - C).$$

This gives $(\eta|\nabla^2 u|)(x_0) \leq C$ and then $H \leq C$. Therefore, $\Delta u$ and $|\nabla u|^2$ are all bounded.

**case(c):** $|\nabla u|$ is bounded and thus $h$ is bounded. This gives

$$(f(x, u)h(\nabla x, \nabla u))_{kk} + 2au_k(f(x, u)h(\nabla x, \nabla u))_k \geq -CA(1 + |\nabla^2 u|^2) + f h_{pi} u_{kk}.$$ We change the order of differentiations of third derivative terms and use (13) to replace $u_{kki}$:

$$(f(x, u)h(\nabla x, \nabla u))_{kk} + 2au_k(f(x, u)h(\nabla x, \nabla u))_k \geq -CA(1 + |\nabla^2 u|^2) - \frac{C}{\sqrt{\eta}} (1 + |\nabla^2 u|).$$

Hence, (13) becomes

$$0 \geq -CA(1 + |\nabla^2 u|^2) - C\eta^2 (1 + |\nabla^2 u|) + \sum_i F_{ii} (-2b\eta^2 |\nabla^2 u|^2 - C - C\eta |\nabla^2 u|)
$$

$$\geq -C - C\eta^2 |\nabla^2 u|^2 + \sum_i F_{ii} (-2b\eta^2 |\nabla^2 u|^2 - C - C\eta |\nabla^2 u|).$$
By (A) and condition on $b$, we see that
\[ 0 \geq -C - C\eta^2 |\nabla^2 u|^2 + \sum_i F^{ii} (2\delta_1 \eta^2 |\nabla^2 u|^2 - C) \eta u_{ii} \]
\[ \geq -C - C\eta^2 |\nabla^2 u|^2 + \mu_0 (\frac{\Gamma}{F})^{\mu_1} (2\delta_1 \eta^2 |\nabla^2 u|^2 - C). \]

Apply $(\eta F)^{\mu_1}$ on both sides and note that $\sigma_1 = \Delta u + (a(x) + nb(x))|\nabla u|^2 + tr B(x) \geq \Delta u - C$, so we have
\[ 0 \geq -C - C\eta^2 |\nabla^2 u|^2 + \mu_0 \sigma_1^{\mu_1} (2\delta_1 \eta^2 |\nabla^2 u|^2 - C \eta^{\mu_1}) \]
\[ \geq -C - C\eta^2 |\nabla^2 u|^2 + 2\delta_1 \mu_0 \sigma_1^{\mu_1} (\Delta u)^{\mu_1} |\nabla^2 u|^2 - C \eta^{\mu_1} (\Delta u)^{\mu_1}. \]

This gives $(\eta \Delta u)(x_0) \leq C$, and consequently $\Delta u$ is bounded.

Once $\Delta u$ is bounded, to get the Hessian bounds for cases (a) and (b), we simply consider the maximum of the tensor $\eta (\nabla^2 u + au \otimes du)$ over the set $(x, \xi) \in (B_1, S^n)$. As for case (c), we use the basic fact that if $\Gamma_2 \subset \Gamma$, then $-n^{-2} \sigma_1 \leq \lambda_i \leq \sigma_1$ for $\lambda \in \Gamma$. □

4 Proof of Theorem 2

Proof. We assume $g_{ij} = \delta_{ij}$ at the point we are evaluating. Now we start with some computations on curvatures. It is known that the Riemannian curvature has the decomposition

\[ R_{ijkl} = W_{ijkl} + (A_{ik} g_{jl} + A_{jl} g_{ik} - A_{il} g_{jk} - A_{jk} g_{il}) \]

where $W$ is the Weyl tensor and $A$ is the Schouten tensor. If $g$ is of constant sectional curvature $K$, then $W$ is zero, $Ric = (n-1)Kg$ and $R = n(n-1)K$. Hence we have

\[ R_{ijkl} = K (g_{ik} g_{jl} - g_{il} g_{jk}). \]

Let $W = \nabla^2 u + adu \otimes du + K g$. By $\Gamma \subset \Gamma_1^+$, we get $\Delta u + a|\nabla u|^2 + nK > 0$. Thus $\Delta u$ is lower bounded and we only need to get the upper bound. Let $H = \Delta u + a|\nabla u|^2$. We may assume $H$ is large and suppose $x_0$ is the maximal point of $H$. At $x_0$, we have

\[ H_i = u_{kki} + 2 au_k u_{ki} = 0, \quad (16) \]

and

\[ H_{ij} = u_{kki} + 2 au_k u_{kj} + 2 au_k u_{kij} \]

is negative semi-definite. Using the positivity of $F^{ij}$, we get

\[ 0 \geq F^{ij} H_{ij} = F^{ij} u_{kki} + F^{ij} (2au_k u_{kj} + 2au_k u_{kij}) = I + II. \]
Before computing I and II, we examine carefully the formulae at the end of Section

\[ u_{kij} = u_{ijk} + R_{mjk}u_m = u_{ijk} + K(g_{ij}u_k - g_{ik}u_j), \]
\[ u_{kkj} = u_{ikj} + 2R_{mjk}u_m - R_{mj}u_m - R_{mi}u_{mj} = u_{ikj} + 2K\Delta u_{ij} - 2Kn u_{ij}. \]

Thus I becomes

\[ I = F^{ij}u_{kkj} = F^{ij}(u_{ikj} + 2K\Delta u_{ij} - 2Kn u_{ij}). \]

Now use the formula

\[ W_{ij,kk} = u_{ikj} + a(u_{ikk}u_j + 2u_{ik}u_j + u_{ij}u_{kk}) \]

to get

\[ I = F^{ij}(W_{ij,kk} - 2a(u_{ikk}u_j + u_{ik}u_{jk}) + 2K\Delta u_{ij} - 2Kn u_{ij}), \]

where we have used \( F^{ij}a u_{ikk}u_j = F^{ij}a u_{ikj} \) because \( F^{ij} \) is symmetric. Changing the order of the differentiations of \( u_{ikk} \) and replacing it by \( [16] \) gives

\[ I = F^{ij}(W_{ij,kk} - 2a(u_{kki} + (n - 1)Ku_j)u_j - 2au_{ik}u_{jk} + 2K\Delta u_{ij} - 2Kn u_{ij}) \]
\[ = F^{ij}(W_{ij,kk} + 4a^2u_{ik}u_{kj} - 2a(n - 1)Ku_ju_j - 2au_{ik}u_{jk} + 2K\Delta u_{ij} - 2Kn u_{ij}). \]

For II, we first change the order of differentiations of \( u_{kij} \) and then replace \( u_{ijk} \) by \( W_{ij,k} - au_1u_{jk} - au_ju_k \) to get

\[ II = F^{ij}(2au_{ik}u_{kj} + 2au_{ik}u_{kij}) = F^{ij}(2au_{ik}u_{kj} + 2au_{ik}u_{ijk} + 2aKu_k(g_{ij}u_k - g_{ik}u_j)) \]
\[ = F^{ij}(2au_{ki}u_{kj} + 2au_kW_{ij,k} - 4a^2u_{ik}u_{kj} + 2aK|\nabla u|^2g_{ij} - 2aKu_iu_j). \]

We combine I and II, and note the cancellation. We obtain

\[ I + II = F^{ij}W_{ij,kk} + 2au_kF^{ij}W_{ij,k} + F^{ij}(-2aKau_iu_j - 2Kn u_{ij} + 2aK|\nabla u|^2g_{ij} + 2K\Delta u_{ij}). \]

Replacing \( u_{ij} \) by \( W_{ij} - au_1u_{ij} - Kg_{ij} \) and using the concavity of \( F \), we get

\[ 0 \geq I + II = F^{ij}W_{ij,kk} + 2au_kF^{ij}W_{ij,k} + F^{ij}(-2Kn W_{ij} + 2K^2ng_{ij} + 2KHg_{ij}) \]
\[ \geq (f(x,u)h(\nabla u))_{kk} + 2au_k(f(x,u)h(\nabla u))_k - 2Kn f(x,u)h(\nabla u) \]
\[ + 2K \sum_i F^{ii}(Kn + H). \]

Since we have \( C^1 \) bounds and nonnegative \( K \), we obtain

\[ 0 \geq (f(x,u)h(\nabla u))_{kk} + 2au_k(f(x,u)h(\nabla u))_k - 2Kn f(x,u)h(\nabla u) \]
\[ \geq -C - C|\nabla^2 u| + fh_{p,p}u_{ik}u_{jk} + fh_{p,u}u_{ikk}. \]
Changing the order of the differentiations of $u_{i\ell k}$ and replacing it by (16) again, produce

$$0 \geq -C - C|\nabla^2 u| + fh_{p;p,u_{ik}u_{jk}} + fh_p(-2au_{ki}u_{ki} + (n-1)Ku_i)$$

$$\geq -C - C|\nabla^2 u| + fh_{p;p,u_{ik}u_{jk}}.$$  

Then by the convexity of $h$, we arrive at

$$0 \geq -C - C|\nabla^2 u| + f\epsilon|\nabla^2 u| \geq -C - C|\nabla^2 u| + \epsilon c_{inf}|\nabla^2 u|^2.$$

This gives $|\nabla^2 u(x_0)| < C$ and hence $H < C$. Finally, to get the Hessian bounds, we consider the tensor $\nabla^2 u + adu \otimes du$ over the set $(x, \xi) \in (\Omega, S^n)$.

\[\square\]

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