Abstract. We establish a “low rank property” for Sobolev mappings that pointwise solve a first order nonlinear system of PDEs, whose smooth solutions have the so-called “contact property”. As a consequence, Sobolev mappings from an open set of the plane, taking values in the first Heisenberg group $H^1$ and that have almost everywhere maximal rank must have images with positive 3-dimensional Hausdorff measure with respect to the sub-Riemannian distance of $H^1$. This provides a complete solution to a question raised in a paper by Z. M. Balogh, R. Hoefer-Isenegger and J. T. Tyson. Our approach differs from the previous ones. Its technical aspect consists in performing an “exterior differentiation by blow-up”, where the standard distributional exterior differentiation is not possible. This method extends to higher dimensional Sobolev mappings, taking values in higher dimensional Heisenberg groups.

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1. Introduction

A $k$-dimensional tangent distribution in $\mathbb{R}^n$ can be seen as a family of $k$-dimensional planes that are locally spanned by $k$ linearly independent smooth vector fields. When all the tangent spaces of a $k$-dimensional submanifold $\Sigma$ coincide with these $k$-dimensional planes, we say that $\Sigma$ is an integral submanifold of the distribution.

The classical Frobenius theorem characterizes those tangent distributions that give a local foliation of the space into families of integral submanifolds. These special tangent distributions are called involutive, namely, the Lie brackets of those vector fields that linearly span the tangent distribution still belong to the same distribution.

An important example of tangent distribution in $\mathbb{R}^3$ is given by the vector fields

$$X_1 = \partial_{x_1} - x_2 \partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3}.$$  

The vector fields $X_1, X_2$ and $X_3$ generate the Lie algebra of the 3-dimensional Heisenberg group $\mathbb{H}^1$, where $X_3 = \frac{1}{2} [X_1, X_2] = \partial_{x_3}$. For convenience, we mention the group operation

$$x \cdot y = x + y + (0, 0, x_1 y_2 - x_2 y_1),$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$. Precisely, the vector fields $X_1, X_2$ and $X_3$ are left invariant with respect to the group operation (1.2). The tangent distribution defined by (1.1) is the so-called horizontal distribution and it can be identified with the horizontal subbundle $H\mathbb{H}^1$. Integral submanifolds of the horizontal distribution can be equivalently considered to be “tangent” to $H\mathbb{H}^1$. The tangent distribution (1.1) also determines an intrinsic length distance called the sub-Riemannian distance, or, in short, the SR-distance. We refer to [18] for more information on this notion.

Since the horizontal distribution of (1.1) is nowhere involutive, the Frobenius theorem gives the nonexistence of smooth surfaces in $\mathbb{R}^3$ that are tangent to $H\mathbb{H}^1$. However, one may still wonder whether there exist more general “2-dimensional sets” that can be still considered “tangent” to this distribution in a broad sense. This problem is amazingly related to the study of the Hausdorff dimension of sets with respect to the sub-Riemannian distance. In this connection, Z. M. Balogh and J. T. Tyson have constructed an interesting example of “horizontal fractal”, called the Heisenberg square $Q_H$, [3]. The 2-dimensional Hausdorff measure of $Q_H$ with respect to both the SR-distance and the Euclidean distance is finite and positive, see [3, Theorem 1.10]. As proved in [4], it is possible to find a BV function $g : (0, 1)^2 \to \mathbb{R}$, whose graph $G$ is contained in $Q_H$ and satisfies

$$0 < \mathcal{H}_d^2(G) < +\infty.$$  

The symbol $\mathcal{H}_d^2$ denotes the Hausdorff measure with respect to the SR-distance $d$ of $\mathbb{H}^1$. Condition (1.3) never holds for graphs of smooth functions. It can be interpreted as a “metric definition” of horizontality for lower regular sets. In fact, in the general Heisenberg group $\mathbb{H}^n$, represented by $\mathbb{R}^{2n+1}$ equipped with the left invariant vector fields

$$X_i = \partial_{x_i} - x_{i+n} \partial_{x_{2n+1}}, \quad X_{n+i} = \partial_{x_{n+i}} + x_i \partial_{x_{2n+1}}$$

for $i = 1, \ldots, n$, spanning $H\mathbb{H}^n$, every $C^1$ smooth $m$-dimensional submanifold $\Sigma \subset \mathbb{H}^n$ that is everywhere tangent to $H\mathbb{H}^n$ must have the measure $\mathcal{H}_d^m \subset \Sigma$ locally finite. On the other hand, from
Contact Topology, it is well known that not only hypersurfaces but rather all sufficiently smooth submanifolds $\Sigma \subset \mathbb{H}^n$ of dimension $m$, with $n < m \leq 2n$, cannot be everywhere tangent to $H\mathbb{H}^n$, in short $T_x\Sigma \not\subset H_x\mathbb{H}^n$, see for instance [9, Proposition 1.5.12]. Thus, when $m > n$ there must exist at least one point $x \in \Sigma$ such that $T_x\Sigma \not\subset H_x\mathbb{H}^n$.

This fact has an important consequence on the Hausdorff dimension of $\Sigma$ with respect to the SR-distance $d$. In fact, when $\Sigma$ is $C^1$ smooth, in view of a general negligibility result, [13], joined with area-type formulae, [8, 14, 15], the measure $\mathcal{H}_d^{m+1}\llcorner \Sigma$ has an integral representation with respect to $\mathcal{H}_d^m\llcorner \Sigma$ and the integrand is proportional to the length of a suitable $m$-vector. This $m$-vector is the “vertical tangent $m$-vector”, denoted by $\tau_{\Sigma, y}$, and it defined as the projection of the unit tangent $m$-vector of $\Sigma$ onto the orthogonal subspace to the linear space $\Lambda_m(H\mathbb{H}^n)$ of horizontal $m$-vectors, see [14] for more details.

The point is that the vertical tangent $m$-vector does not vanish precisely at all points $x \in \Sigma$ such that $T_x\Sigma \not\subset H_x\mathbb{H}^n$. As a consequence, for each smooth $m$-dimensional submanifold $\Sigma \subset \mathbb{H}^n$ with $m > n$, there holds

$$ (1.5) \quad \mathcal{H}_d^{m+1}(\Sigma) > 0. $$

In the case $n = 1$ and $m = 2$, the non-horizontality condition (1.5) for nonsmooth sets has been shown in [4], where $\Sigma$ is a 2-dimensional Lipschitz graph of $\mathbb{H}^1$. Here the authors raise the interesting question on the existence of horizontal sets in the sense of (1.3) having regularity between Lipschitz and BV. A first answer to this question is given in [16], where it is shown that if $\Sigma$ is either a 2-dimensional $W^{1,1}_{\text{loc}}$-Sobolev graph or the image of a mapping in $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^3)$ with $p \geq 4/3$ and whose weak differential has almost everywhere maximal rank, then (1.5) must hold with $m = 2$.

The starting point is that a smooth parametrization $f : \Omega \to \Sigma$ satisfies the equation

$$ (1.6) \quad df^3 = f^1 df^2 - f^2 df^1 $$

at those points $y \in \Omega$ such that $T_{f(y)}\Sigma \subset H\mathbb{H}^1$, where $\Omega \subset \mathbb{R}^2$ is an open set. In other words, precisely at these points $f$ has the so-called contact property, namely, its differential preserves the “horizontal directions”. In fact, in the source space all directions are horizontal and in the target these directions are spanned by the vector fields (1.1). The differential obstruction in the everywhere validity of equation (1.6) is easily seen by performing its exterior differentiation, since this implies that the rank of the Jacobian matrix of $f$ is everywhere less than two and this conflicts with the starting assumptions.

Once it is proved that (1.6) cannot hold a.e., then the Whitney extension theorem yields a $C^1$ smooth submanifold $\tilde{\Sigma}$ that coincides with the original $\Sigma$ on some measurable subset $A \subset \tilde{\Sigma} \cap \Sigma$ of positive Euclidean surface measure, where in addition $TA \not\subset H\mathbb{H}^1$.

As a consequence, in view of the previous comments on the density of $\mathcal{H}_d^2\llcorner \tilde{\Sigma}$, we achieve

$$ \mathcal{H}_d^3(\Sigma) \geq \mathcal{H}_d^3(A) > 0. $$

In sum, the main technical point consists in being able to accomplish a suitable exterior differentiation of (1.6) under the condition that the regularity of $f$ is as weak as possible. When $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{H}^1)$ an exterior distributional differentiation of (1.6) is possible under the regularity condition $p \geq 4/3$, but this approach does not work for $1 \leq p < 4/3$, where the question was left open. The following theorem answers this question.
Theorem 1.1. Let $\Omega \subset \mathbb{R}^2$ be open, let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^3)$ be such that the Jacobian matrix of $f$ has almost everywhere maximal rank and define $\Sigma = f(\Omega)$. It follows that $H^3(\Sigma) > 0$.

This provides the full answer to the question raised in [4]. Our approach to establish Theorem 1.1 differs from the previous ones and it also works for any higher dimensional Heisenberg group $\mathbb{H}^n$, which we identify with the linear space $\mathbb{R}^{2n+1}$.

Let us summarize the main steps. We consider a mapping $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^{2n+1})$, where $\Omega$ is an open set of $\mathbb{R}^m$, then the contact property of equation (1.6) is replaced by

$$\text{(1.7)} \quad df^{2n+1} = \sum_{j=1}^{n} (f^j df^{j+n} - f^{j+n} df^j).$$

Next, we restrict ourselves to the special case $m = 2$, since it suffices to describe the core of our method. We denote by $Df$ the Jacobian matrix of $f$, whose columns correspond to the weak derivatives of $f$. Then we rescale $f$ at Lebesgue points $z \in \Omega$ of both $f$ and $Df$. Then the rescaled functions $f_{z,\rho}$, introduced in Definition 4.1, are defined on the unit ball $B$ of $\mathbb{R}^2$ for all $\rho > 0$ sufficiently small and converge to the linear mapping $u : y \mapsto Df(z) \cdot y$ in $W^{1,1}(B)$ as $\rho \to 0^+$. If by contradiction we assume that (1.7) holds a.e., then the one-form

$$\text{(1.8)} \quad \sum_{j=1}^{n} \left( f^j_{z,\rho} df^{j+n}_{z,\rho} - f^{j+n}_{z,\rho} df^j_{z,\rho} \right)$$

is “weakly exact” in the sense that it is a.e. equal to $dw_\rho$ for some $w_\rho \in W^{1,1}(B)$, see Lemma 4.1. We exploit this fact by integrating (1.8) on the Euclidean sphere $\partial B(0, r)$ for almost every $r \in (0, 1)$ and pass to the limit with respect to $\rho$ as it goes to zero by a suitable positive infinitesimal sequence ($\rho_k$). Since the limit has the form

$$\sum_{j=1}^{n} \left( u^j du^{j+n} - u^{j+n} du^j \right)$$

with $u(y) = Df(z) \cdot y$ and its oriented integral vanishes on almost every sphere of radius $r \in (0, \rho)$, it follows that

$$\text{(1.9)} \quad \sum_{j=1}^{n} df^j(z) \wedge df^{j+n}(z) = 0,$$

by the classical Stokes theorem. If $m > 2$, then we obtain (1.9) by a slicing argument, so that the whole range $m \geq 2$ is provided. Joining Lemma 6.1 with (1.9), we deduce that the rank of $Df(z)$ is less than $n + 1$. According to Theorem 6.1, this shows that Sobolev mappings that a.e. satisfy the horizontality condition (1.7) must satisfy a.e. a “low rank property”, namely the rank of $Df$ is a.e. less than $n + 1$.

In this connection, we wish to mention that in the first preprint of our paper, [17], we have obtained this low rank result for $W^{1,p(n)}$ functions in Heisenberg groups $\mathbb{H}^n$, where $p(n) = 1$ only for $n = 1$. Then we have been informed that Z. M. Balogh, P. Hajlasz and K. Wildrick have independently obtained this result, with a different approach, [5]. They first observed that the best possible Sobolev class $W^{1,1}$ is available, regardless of the
dimension of $\mathbb{H}^n$. Thanks to their announcement, we immediately realized that a slight modification of our method also gives $p(n) = 1$ for all positive $n \in \mathbb{N}$, as in the current version of Theorem 6.1.

Let us point out that in the case $m \leq n$, the mapping $f$ can be a smooth embedding that is also a contact mapping, namely, $df(y)(\mathbb{R}^m) \subset H_f(y)\mathbb{H}^n$ for every $y \in \Omega$. To see this, it suffices to take local parametrizations of isotropic submanifolds when $\mathbb{H}^n$ is regarded as a contact manifold, [9]. In higher dimensions, the application of Theorem 6.1 appears in the case $n + 1 \leq m \leq 2n$, where it should be seen somehow as a “differential obstruction”. It is worth to compare this obstruction with the “Lipschitz obstructions” appearing in the study of Lipschitz homotopy groups of the Heisenberg group, [6]. Our main application of Theorem 6.1 is the following result.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^m$ be an open set, let $n < m \leq 2n$ and let $f \in W^{1,1}_\text{loc}(\Omega, \mathbb{R}^{2n+1})$. Suppose that the Jacobian matrix $Df$ has rank equal to $m$ almost everywhere and set $\Sigma = f(\Omega)$. Then $\mathcal{H}^{m+1}_d(\Sigma) > 0$.

This theorem also includes Theorem 1.1. In ending, we wish to point out a curious observation on the graph $G$ of the BV function $g$ mentioned above. We can translate the metric horizontality of (1.3) into a “tangential condition”. In fact, one can easily observe that the approximate differential of the graph mapping $f = (x_1, x_2, g)$ must satisfy (1.6) almost everywhere, hence $\text{ap} \nabla g = (-x_2, x_1)$ almost everywhere, see Theorem 6.2. This can be seen as a tangential condition in the sense of Geometric Measure Theory.

Finally, we introduce some notation that will be used throughout the paper. The Euclidean open ball in $\mathbb{R}^m$ with center at $x$ and radius $r$ is denoted by $B(x, r)$. The unit ball $B(0, 1)$ will be simply denoted by $\mathbb{B}$. If $u$ is a Sobolev function on an open set of $\mathbb{R}^m$, then $du$ denotes the measurable one form $\sum_{j=1}^k \partial_{x_j} u \, dx_j$, where every coefficient $\partial_{x_j} u$ is the $j$-th distributional derivative of $u$. The symbol $\nabla u$ denotes the vector of weak derivatives $(\partial_{x_1} u, \ldots, \partial_{x_m} u)$, that a.e. coincides with the vector of approximate partial derivatives. The Jacobi matrix of weak derivatives of a vector-valued function $f$ is denoted by $Df$.

## 2. Slicing

For the reader’s convenience, in this section we recall some well known facts about Sobolev sections, that will be used in the subsequent part of the paper. Let $m$ be a positive integer and denote by $(e_1, \ldots, e_m)$ the canonical basis of $\mathbb{R}^m$. If $\Gamma \subset \{1, \ldots, m\}$ is a set of indices, then $V_\Gamma$ is the linear span of $\{e_j : j \in \Gamma\}$ and $V_\Gamma^\perp$ is the linear span of $\{e_j : j \in \{1, \ldots, m\} \setminus \Gamma\}$. We introduce the orthogonal projections

$$\pi_\Gamma(x) = \sum_{j \in \Gamma} x_j e_j \quad \text{and} \quad \hat{\pi}_\Gamma(x) = x - \pi_\Gamma(x)$$

where $x \in \mathbb{R}^m$, $\pi_\Gamma : \mathbb{R}^m \rightarrow V_\Gamma$ and $\hat{\pi}_\Gamma : \mathbb{R}^m \rightarrow V_\Gamma^\perp$. Let $Q$ be an open $m$-dimensional interval in $\mathbb{R}^m$, namely the product of $m$ open intervals, and fix a nonempty subset $\Gamma \subset \{1, \ldots, m\}$. We define the projected intervals

$$Q_\Gamma = \pi_\Gamma(Q) \quad \text{and} \quad \hat{Q}_\Gamma = \hat{\pi}_\Gamma(Q).$$
If \( u : Q \to \mathbb{R} \) is a function and \( z \in \partial Q \), we define the section \( u^z : Q \to \mathbb{R} \) as
\[
u^z(y) = u(z + y), \quad y \in Q.
\]

**Definition 2.1.** We say that a sequence \( \{u_h\} \) in a Banach space \( (X, \| \cdot \|) \) converges fast to \( u \in X \), or that it is fast convergent, if \( \sum_{h=1}^\infty \|u_h - u\| < \infty \).

We wish to point out that the fast convergence in \( W^{1,1} \) is just the joint fast convergence in \( L^1 \) of functions and their gradients. As a consequence of both Fubini’s theorem and Beppo Levi’s convergence theorem for series, we get the next proposition.

**Proposition 2.1.** Let \( \{u_h\} \subset W^{1,1}(Q) \) be a sequence which converges fast to \( u \in W^{1,1}(Q) \). Then for each \( k = 1, \ldots, m \) and for almost every \( z \in Q \), we have \( u^z, (\partial_y u)^z, u_h^z, (\partial_y u_h)^z \in L^1(Q) \), \( h \in \mathbb{N} \), further, \( \{u_h^z\} \) converges fast to \( u^z \) in \( L^1(Q) \) and \( (\partial_y u_h)^z \) converges fast to \( (\partial_y u)^z \) in \( L^1(Q) \).

Each \( u \in W^{1,1}(Q) \) is a limit of a fast convergent sequence of smooth functions. Applying Proposition 2.1 we obtain the following consequence.

**Proposition 2.2.** Let \( u \in W^{1,1}(Q) \). Then for almost every \( z \in Q \), we have \( u^z \in W^{1,1}(Q) \) and
\[
\partial_y u^z = (\partial_y u)^z \quad a.e. \text{ in } Q, \quad k = 1, \ldots, m.
\]

Summarizing Propositions 2.1 and 2.2 we obtain the following.

**Proposition 2.3.** Let \( \{u_h\} \subset W^{1,1}(Q) \) be a sequence which converges fast to \( u \in W^{1,1}(Q) \). Then for almost every \( z \in Q \), we have \( u^z, u_h^z \in W^{1,1}(Q) \), \( h \in \mathbb{N} \), and \( \{u_h^z\} \) converges fast to \( u^z \) in \( W^{1,1}(Q) \).

### 3. Oriented Integration on the Circle

The idea of slicing can be also applied to study the behavior of Sobolev functions on a.e. sphere. However, for our purposes it is enough to perform this analysis in \( \mathbb{R}^2 \) only, so that we will study Sobolev spaces on circles.

**Definition 3.1** (Function spaces on the circle). Consider the circle \( \partial B(x, r) \) and its parametrization
\[
\psi(t) = (x_1 + r \cos t, x_2 + r \sin t), \quad t \in \mathbb{R}.
\]
We define \( \psi_- = \psi|_{[0, \pi]} \) and \( \psi_+ = \psi|_{(0, 2\pi]} \); hence \((\psi_+, \psi_-)\) is an oriented atlas of \( \partial B(x, r) \). This atlas automatically defines function spaces on \( \partial B(x, r) \). Let \( X \) be a generic function space symbol which may refer e.g. to \( W^{1,p} \), \( L^p \) or \( C \). We say that \( u : \partial B(x, r) \to \mathbb{R} \) belongs to \( X(\partial B(x, r)) \) if \( u \circ \psi_- \) belongs to \( X((-\pi, \pi]) \) and \( u \circ \psi_+ \) belongs to \( X((0, 2\pi]) \).

**Definition 3.2** (Integrable forms on the circle). Let us consider \( u, v : \partial B(x, r) \to \mathbb{R} \). Then the oriented integral of the differential form \( u \, dv \) is defined as follows
\[
\int_{\partial B(x,r)} u \, dv = \int_{-\pi}^{\pi} (u \circ \psi)(t) (v \circ \psi)'(t) \, dt,
\]
whenever this expression makes sense, if e.g. \( u \in L^\infty(\partial B(x, r)) \), \( v \in W^{1,1}(\partial B(x, r)) \) and \((v \circ \psi)'\) is the distributional derivative of \( v \circ \psi \).
The following lemma relates the fast convergence with the convergence of oriented integrals on spherical sections.

**Lemma 3.1.** Let \( u, u_h, v, v_h \in W^{1,1}(B(x, \rho)) \), \( h \in \mathbb{N} \), and suppose that both \( u_h \to u \) and \( v_h \to v \) fast in \( W^{1,1}(B(x, \rho)) \). Then for almost every \( 0 < r < \rho \) the restrictions of \( u, u_h, v, v_h \) to \( \partial B(x, r) \) belong to \( W^{1,1}(\partial B(x, r)) \) and

\[
(3.2) \quad \int_{\partial B(x,r)} u_h \, dv_h \to \int_{\partial B(x,r)} u \, dv.
\]

**Proof.** We use the polar coordinates associated to the mapping \( \Psi \). Fix such a good radius \( r \). Then \( u \circ \Psi^r \), \( u_h \circ \Psi^r \) are absolutely continuous up to a modification on a null set. Using the one-dimensional Sobolev embedding and passing to absolutely continuous representatives, we obtain a uniform convergence \( u_h \circ \Psi^r \to u \circ \Psi^r \) in \( \partial B(x, r) \). By Proposition 2.3, for a.e. \( r \in (\delta, \rho) \) we have that \( u_h \circ \Psi^r, v_h \circ \Psi^r, u \circ \Psi^r, v \circ \Psi^r \in W^{1,1}((-2\pi, 2\pi)) \) and both \( u_h \circ \Psi^r \) and \( v_h \circ \Psi^r \) converge fast in \( W^{1,1}((-2\pi, 2\pi)) \) to \( u \circ \Psi^r \) and \( v \circ \Psi^r \), respectively.

Joining with the \( L^1 \)-convergence \( (v_h \circ \Psi^r)' \to (v \circ \Psi^r)' \) we conclude that

\[
\int_{\partial B(x,r)} u_h \, dv_h = \int_{-\pi}^{\pi} (u_h \circ \Psi^r)(t)(v_h \circ \Psi^r)'(t) \, dt \to \int_{-\pi}^{\pi} (u \circ \Psi^r)(t)(v \circ \Psi^r)'(t) \, dt \to \int_{\partial B(x,r)} u \, dv
\]

as required. By the arbitrary choice of \( \delta > 0 \), we have proved that (3.2) holds for a.e. \( r \in (0, \rho) \). \( \square \)

**Lemma 3.2.** Let \( v \in W^{1,1}(B(x, \rho)) \). For almost every \( r \in (0, \rho) \), the oriented integral

\[
\int_{\partial B(x,r)} dv \text{ is well defined and equal to zero.}
\]

**Proof.** Again, we use the polar coordinates as in the preceding proof. By Proposition 2.2, for a.e. \( r \in (0, \rho) \), the section \( v \circ \Psi^r \) belongs to \( W^{1,1}(-2\pi, 2\pi) \). If \( \tilde{v} \circ \Psi^r \) is the absolutely continuous representative of \( v \circ \Psi^r \), we have

\[
\int_{\partial B(x,r)} dv = \int_{-\pi}^{\pi} (v \circ \Psi^r)'(t) \, dt = \tilde{v} \circ \Psi^r(\pi) - \tilde{v} \circ \Psi^r(-\pi) = 0,
\]

as \( \tilde{v} \circ \Psi^r \) is obviously \( 2\pi \)-periodic. \( \square \)
4. An exterior differentiation by blow up

Throughout this section, we fix an open set $\Omega \subset \mathbb{R}^2$, a mapping $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^{2n+1})$ and a point $z \in \Omega$ that is a Lebesgue point of both $f$ and $Df$. Recall that $z$ is a Lebesgue point for a measurable function $u \in L^1_{\text{loc}}(\Omega)$ if

$$\lim_{r \to 0^+} r^{-2} \int_{B(z,r)} |u(y) - u(z)| \, dy = 0$$

and that almost every point of $\Omega$ is a Lebesgue point of $u$. As already pointed out in the introduction, $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n+1}$ equipped with the vector fields of (1.4). We fix $\rho > 0$ such that $B(z, \rho) \subset \Omega$. $\mathbb{B}$ denotes the open unit ball of $\mathbb{R}^2$ centered at the origin.

**Definition 4.1.** Let $0 < r \leq \rho$ and define the rescaled function $f_{z,r} : \mathbb{B} \to \mathbb{R}^{2n+1}$ as

$$f_{z,r}(y) := \frac{f(z + ry) - f(z)}{r}.$$

Obviously, $f_{z,r} \in W^{1,1}(\mathbb{B}, \mathbb{R}^{2n+1})$ is well defined whenever $0 < r \leq \rho$. We use the assumption that $z$ is a Lebesgue point of both $f$ and $Df$ to conclude that

$$\lim_{r \to 0^+} \int_{\mathbb{B}} |f_{z,r}(y) - Df(z) \cdot y| \, dy = 0,$$

cf. e.g. [19, Theorem 3.4.2]. The next lemma provides us with important information on the rescaled function $f_{z,r}$.

**Lemma 4.1.** If (1.7) holds almost everywhere, then whenever $0 < r < \rho$ there exists $w \in W^{1,1}(\mathbb{B})$ such that

$$dw(y) = \sum_{j=1}^{n} f_{z,r}^j(y) df_{z,r}^{j+n}(y) - f_{z,r}^{j+n}(y) df_{z,r}^j(y) \quad \text{for a.e. } y \in \mathbb{B},$$

where $w = r^{-1} \left( f_{z,r}^{2n+1} - \sum_{j=1}^{n} f^j(z) f_{z,r}^{j+n} - f_{z,r}^{j+n}(z) f_{z,r}^j \right)$.

**Proof.** From the a.e. validity of (1.7) joined with basic properties of weak derivatives of Sobolev functions, we get

$$\nabla f_{z,r}^{2n+1}(y) = \nabla f_{z,r}^{2n+1}(z + ry) = \sum_{j=1}^{n} f^j(z + ry) \nabla f_{z,r}^{j+n}(z + ry) - f_{z,r}^{j+n}(z + ry) \nabla f^j(z + ry)$$

for a.e. $y \in \mathbb{B}$. We add and subtract all terms of the form $f^j(z) \nabla f_{z,r}^{j+n}(z + ry)$, getting

$$\nabla f_{z,r}^{2n+1}(y) = \sum_{j=1}^{n} f^j(z + ry) \nabla f_{z,r}^{j+n}(z + ry) - f_{z,r}^{j+n}(z + ry) \nabla f^j(z + ry)$$

$$= \sum_{j=1}^{n} \left( f^j(z + ry) - f^j(z) \right) \nabla f_{z,r}^{j+n}(z + ry) - \left( f_{z,r}^{j+n}(z + ry) - f_{z,r}^{j+n}(z) \right) \nabla f^j(z + ry)$$

$$+ \sum_{j=1}^{n} f^j(z) \nabla f_{z,r}^{j+n}(z + ry) - f_{z,r}^{j+n}(z) \nabla f^j(z + ry).$$
Dividing by \( r \), we can rewrite the previous equation as follows

\[
\frac{1}{r} \left\{ \nabla f^{j+n+1}_{z,r}(y) - \sum_{j=1}^{n} \left( f^j(z) \nabla f^{j+n}(z+ry) - f^{j+n}(z) \nabla f^j(z+ry) \right) \right\} = \sum_{j=1}^{n} f^j_{z,r}(y) - f^{j+n}_{z,r}(y) \nabla f^j(z+ry).
\]

Since \( Df(z+ry) = Df_{z,r}(y) \), this immediately leads to the conclusion.

Next, we show that, under sufficient integrability conditions, it is possible to take somehow the differential of both sides of (1.7), achieving the following theorem.

**Lemma 4.2.** Let \( f \in W^{1,1}(\Omega, \mathbb{R}^{2n+1}) \) and assume that a.e. we have

\[
(df)^{2n+1} = \sum_{j=1}^{n} \left( f^j df^{j+n} - f^{j+n} df^j \right).
\]

Then \( \sum_{j=1}^{n} df^j \wedge df^{j+n} = 0 \) holds a.e. in \( \Omega \).

**Proof.** Let \( z \in \Omega \) be a Lebesgue point of both \( f \) and \( Df \). We choose \( \rho_h \searrow 0 \) such that \( \rho_1 < \rho \) and set \( u_h = f_{z,\rho_h} \). By Lemma 4.1, there exists \( w_h \in W^{1,1}(\mathbb{B}) \) such that for \( L^2 \)-almost every \( y \in \mathbb{B} \) we have

\[
dw_h(y) = \sum_{j=1}^{n} \left( u^j_h(y) du^{j+n}_h(y) - u^{j+n}_h(y) du^j_h(y) \right).
\]

Furthermore, since \( z \) is a Lebesgue point of both \( f \) and \( Df \), it follows that

\[
u_h \to u \text{ in } W^{1,1}(\mathbb{B}), \text{ where } u(y) = Df(z) \cdot y, \ y \in \mathbb{B}.
\]

We may assume that the sequence \( \rho_h \) is defined in such a way that the convergence in (4.3) is fast. Lemma 3.1 implies that for almost every \( r \in (0, 1) \) the integral

\[
\int_{\partial B(0,r)} \left( \sum_{j=1}^{n} u^j_h du^{j+n}_h - u^{j+n}_h du^j_h \right)
\]

is well defined and equal to \( \int_{\partial B(0,r)} dw_h \). Thus, in view of Lemma 3.2 we have

\[
\int_{\partial B(0,r)} \left( \sum_{j=1}^{n} u^j_h du^{j+n}_h - u^{j+n}_h du^j_h \right) = \int_{\partial B(0,r)} dw_h = 0
\]

for all \( h \) and almost every \( r \in (0, 1) \). Taking into account both (4.3) and Lemma 3.1, for almost every \( r \in (0, 1) \) we have

\[
0 = \int_{\partial B(0,r)} \left( \sum_{j=1}^{n} u^j_h du^{j+n}_h - u^{j+n}_h du^j_h \right) \to \int_{\partial B(0,r)} \left( \sum_{j=1}^{n} u^j dw^{j+n} - u^{j+n} dw^j \right).
\]
It is enough to pick one such a radius, so that by Stokes theorem, we obtain
\begin{equation}
\int_{B(0,r)} \sum_{j=1}^{n} du^j \wedge du^{j+n} = 0.
\end{equation}

The equation (4.4) yields
\begin{equation}
L^2(B(0,r)) \sum_{j=1}^{n} \det(\nabla f^j(z), \nabla f^{j+n}(z)) = 0.
\end{equation}

Thus, we have \( \sum_{j=1}^{n} \det(\nabla f^j(z), \nabla f^{j+n}(z)) = 0 \). Since Lebesgue points of both \( f \) and \( Df \) have full measure in \( \Omega \), our claim follows.

5. The \( m \)-dimensional case

In this section we treat the general case \( m \geq 2 \).

**Theorem 5.1.** Let \( \Omega \subset \mathbb{R}^m \) be open and let \( f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^{2n+1}) \). If (1.7) holds a.e. in \( \Omega \), then
\begin{equation}
\sum_{j=1}^{n} df^j \wedge df^{j+n} = 0
\end{equation}
holds a.e. in \( \Omega \).

**Proof.** It is enough to verify (5.1) on an arbitrary \( m \)-dimensional open cube \( Q \subset \subset \Omega \). Fix \( 1 \leq k < l \leq m \). We set \( \Gamma = \{k,l\} \) and use the notation of Section 2, with the exception that now we use the subscript \( z \) to denote the section \( f_z(y) = f(z+y), \quad y \in Q_\Gamma \).

By Proposition 2.2, for a.e. \( z \in Q_\Gamma \) we have that \( f_z \in W^{1,1}(Q_\Gamma) \) and
\begin{equation}
\frac{\partial f_z}{\partial x_k} = \left( \frac{\partial f}{\partial x_k} \right)_z, \quad \frac{\partial f_z}{\partial x_l} = \left( \frac{\partial f}{\partial x_l} \right)_z \quad \text{a.e. in } Q_\Gamma.
\end{equation}

In particular, we have
\begin{equation}
df^{2n+1}_z = \sum_{j=1}^{n} \left( f^j_z df^{j+n}_z - f^{j+n}_z df^j_z \right) \quad \text{a.e. in } Q_\Gamma.
\end{equation}

Then use Lemma 4.2 on \( Q_\Gamma \) to infer that
\begin{equation}
\sum_{j=1}^{n} df^j_z \wedge df^{j+n}_z = 0 \quad \text{a.e. in } Q_\Gamma.
\end{equation}

Using Fubini’s theorem and (5.2) we obtain that
\begin{equation}
\sum_{j=1}^{n} \det \left( \begin{array}{c}
\frac{\partial f^j}{\partial x_k} \\
\frac{\partial f^{j+n}}{\partial x_k}
\end{array}, \begin{array}{c}
\frac{\partial f^j}{\partial x_l} \\
\frac{\partial f^{j+n}}{\partial x_l}
\end{array} \right) = 0 \quad \text{a.e. in } Q.
\end{equation}

By the arbitrary choice of \( k \) and \( l \), the equality (5.1) holds a.e. in \( Q \). \( \square \)
6. Non-horizontality of higher dimensional Sobolev sets

We begin this section with the following algebraic lemma.

**Lemma 6.1.** Let $m,n$ be positive integers and let $u_1, \ldots, u_{2n} \in \mathbb{R}^m$. If we have

$$\sum_{j=1}^{n} u_j \wedge u_{j+n} = 0,$$

then the matrix $B$ with rows $u_1, \ldots, u_{2n}$ has rank at most $n$.

**Proof.** We denote the inner product in $\mathbb{R}^{2n}$ by $\langle \cdot, \cdot \rangle$. Further, $(e_1, \ldots, e_{2n})$ is the canonical basis of $\mathbb{R}^{2n}$ and $I_n$ is the $n \times n$ identity matrix. We consider the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$ 

Choose $v = (v_1, \ldots, v_m)$, $w = (w_1, \ldots, w_m) \in \mathbb{R}^m$. We have

$$Bw = \sum_{i=1}^{n} \sum_{k=1}^{m} (b^i_k w_k e_i + b^i_{i+n} w_k e_{i+n}), \quad JBv = \sum_{j=1}^{n} \sum_{l=1}^{m} (b^l_j v_l e_{j+n} - b^l_{j+n} v_l e_j)$$

and this implies that

$$\langle Bw, JBv \rangle = \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \langle b^i_k w_k e_i + b^i_{i+n} w_k e_{i+n}, b^l_j v_l e_{j+n} - b^l_{j+n} v_l e_j \rangle.$$ 

The summands are nonzero only for $i = j$, in which case

$$\langle b^i_k w_k e_i + b^i_{i+n} w_k e_{i+n}, b^l_j v_l e_{j+n} - b^l_{j+n} v_l e_j \rangle = w_k v_l \det \begin{pmatrix} b^l_j & b^i_k \\ b^l_{j+n} & b^i_{i+n} \end{pmatrix},$$

so that

$$\langle Bw, JBv \rangle = \sum_{k,l=1}^{m} w_k v_l \sum_{i=1}^{n} \det \begin{pmatrix} b^l_j & b^i_k \\ b^l_{j+n} & b^i_{i+n} \end{pmatrix} = \sum_{k,l=1}^{m} w_k v_l \sum_{i=1}^{n} (u_i \wedge u_{i+n})_{l,k} = 0.$$ 

Then the images of $B$ and of $JB$ are orthogonal subspaces of $\mathbb{R}^{2n}$, having the same dimension, hence the rank of $B$ cannot be greater than $n$. 

**Theorem 6.1.** Let $\Omega \subseteq \mathbb{R}^m$ be an open set and consider $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^{2n+1})$ which satisfies (1.7) a.e. in $\Omega$. Then $Df$ has rank at most $n$ a.e. in $\Omega$.

**Proof.** This is a consequence of Theorem 5.1 and Lemma 6.1. 

By Theorem 6.1, the proof of Theorem 1.2 follows essentially the same lines of [16]. Next, for the sake of the reader, we adapted this proof to our setting.

**Proof of Theorem 1.2.** By Theorem 6.1, the equation (1.7) fails to hold for $f$ on a set $E \subset \Omega$ of positive $\mathcal{L}^m$-measure. We can assume that $E$ is bounded, made of density points, that everywhere on $E$ the approximate differential of $f$ exists and equals its distributional differential and they have everywhere rank equal to $m$. Up to taking a smaller piece of $E$, we can also assume that $f$ is Lipschitz. Then we consider a Lipschitz extension of
\( f \mid_E \) to all of \( \mathbb{R}^m \) and apply Whitney extension theorem, hence finding a subset \( E_0 \) of \( E \)
with positive measure and \( g \in C^1(\mathbb{R}^m, \mathbb{R}^{2n+1}) \) such that \( g \mid_{E_0} = f \mid_{E_0} \) and the approximate differential of \( f \) and the differential of \( g \) coincide on \( E_0 \). We choose \( y_0 \in E_0 \) and notice that for a fixed \( r_0 > 0 \) sufficiently small, we have \( \mathcal{L}^m(B(y_0, r_0) \cap E_0) > 0 \) and \( \Sigma_0 = g(B(y_0, r_0)) \) is an \( m \)-dimensional embedded manifold of \( \mathbb{R}^{2n+1} \). Here \( B(y_0, r_0) \subset \mathbb{R}^m \) denotes the open Euclidean ball of center \( y_0 \) and radius \( r_0 \). By the properties of \( g \) and the classical area formula, we have
\[
\Sigma_1 = f(B(y_0, r_0) \cap E_0) = g(B(y_0, r_0) \cap E_0) \subset \Sigma_0 \cap \Sigma \quad \text{and} \quad \mathcal{H}^m(\Sigma_1) > 0.
\]

Since (1.7) does not hold on \( E_0 \), for any \( y \in B(y_0, r_0) \cap E_0 \), we have \( T_{f(y)} \Sigma_0 \not\subset H_y \mathbb{H}^n \), therefore
\[
\tau_{\Sigma_0, y}(f(y)) \neq 0,
\]
where we have used the notation \( \tau_{\Sigma_0, y}(x) \) with \( x \in \Sigma_0 \) to indicate the vertical tangent \( m \)-vector to \( \Sigma_0 \) at \( x \); see [14, Definition 2.14]. This \( m \)-vector vanishes exactly at those points \( x \) where \( T_{x} \Sigma_0 \subset H_x \mathbb{H}^n \), see [14, Proposition 3.1]. From both [13] and [14], the spherical Hausdorff measure \( \mathcal{S}^{m+1}_d(\Sigma) \) is equivalent, up to geometric constants, to the measure \( |\tau_{\Sigma_0, y}| \mathcal{H}^m \mid_{\Sigma} \), hence in particular \( \mathcal{S}^{m+1}_d(\Sigma_1) > 0 \), therefore
\[
\mathcal{H}^{m+1}_d(\Sigma) \geq \mathcal{H}^{m+1}_d(\Sigma_1) > 0,
\]
so the proof is complete. \( \square \)

6.1. Formal horizontality of some BV graphs. By the arguments in the proof of Theorem 1.2, it is not difficult to establish a kind of “generalized horizontal tangency” of BV functions whose graph satisfies the metric constraint (1.3). In the following theorem, \( \mathbb{H}^1 \) is again identified with \( \mathbb{R}^3 \) equipped with the vector fields (1.1) and \( d \) denotes the corresponding SR-distance.

**Theorem 6.2.** Let \( 2 \leq \alpha < 3 \) and let \( g : (0,1)^2 \to \mathbb{R} \) be a BV function such that its graph
\[
G = \{(x_1, x_2, g(x)) : 0 < x_1, x_2 < 1 \}
\]
satisfies \( \mathcal{H}_d^2(G) < +\infty \).

Then the approximate gradient of \( g \) almost everywhere satisfies
\[
\text{ap} \nabla g(x) = (-x_2, x_1).
\]

**Remark 6.2.** As already mentioned in the introduction, the existence of BV functions that satisfy the assumptions of Theorem 6.2 with \( \alpha = 2 \) has been proved by Z. M. Balogh, R. Hoefer-Isenegger and J. T. Tyson, [4]. The existence of BV functions whose absolutely continuous part of the distributional gradient almost everywhere equals a vector field with nonvanishing curl is a special instance of a general result due to G. Alberti, [1].

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