Full tomography from compatible measurements

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We put forward a reconstruction scheme prompted by the relation between a von Neumann measurement and the corresponding informationally complete measurement induced in a relevant reconstruction subspace. This method is specially suited for the full tomography of complex quantum systems, where the intrincacies of the detection part of the experiment can be greatly reduced provided some prior information is available. In broader terms this shows the importance of this often-disregarded prior information in quantum theory. The proposed technique is illustrated with an experimental tomography of photonic vortices of moderate dimension.

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Introduction. The quantum state is a mathematical object that encodes complete information about a system [1]: once it is known, the outcomes of any possible measurement can be predicted. Apart from fundamental reasons, acquiring the system state is invaluable for verifying and optimizing experimental setups. For instance, in some protocols of quantum key distribution, the knowledge of the entangled state distributed between the parties greatly limits the ability of a third party to eavesdrop on the communication channel [2].

The reconstruction of the unknown state from a suitable set of measurements is called quantum tomography [3]. Over the past years, this technique has evolved from the first theoretical [4] and experimental [5] concepts to a widely acknowledged and fairly standard method extensively used for both discrete [6, 7] and continuous [8] variables.

In this work, we focus on measurement strategies for the tomographic reconstruction, leaving aside data post-processing issues. In practice, a sufficient number of independent observations must be included in the set of measurements so that all physical aspects of the measured system are addressed. When dealing with complicated systems, such measurements may be difficult to implement in the laboratory due to various physical and technical limitations on the available controlled interactions between the system and the meter.

The goal of this Letter is to present a method of generating a tomographically complete measurement set from a simple von Neumann measurement that is readily implemented in the laboratory. Obviously, a von Neumann measurement is not complete, as all the measured projections are compatible and hence provide information only about the same aspects. However, as we shall show here, things are radically different when only a part of the full Hilbert space is of interest: In this subspace, even a simple von Neumann projection may become informationally complete. This should not be taken as an approximation, in the sense that some accuracy is traded for experimental feasibility. First of all, the energy of any system is always bounded, so one can restrict the attention to the subspace spanned by low-energy states. Second, due to the finite resources, all quantum systems are de facto discrete and may be represented by a relatively small number of parameters. In that case, there is no necessity of sophisticated measurements that are informationally complete in the original large Hilbert space: since only a small subset is accessible, even much simpler observations are able to supply the information needed. This is the main idea behind the present contribution.

Quantum tomography. Let us consider a density matrix $\rho$ describing a $d$-dimensional quantum system. A convenient representation of $\rho$ can be obtained with the help of a traceless Hermitian operator basis $\{\Gamma_i\}$, satisfying $\text{Tr}(\Gamma_i) = 0$ and $\text{Tr}(\Gamma_i\Gamma_j) = \delta_{ij}$ [4]:

$$\rho = \frac{1}{d} + \sum_{i=1}^{d^2-1} a_i \Gamma_i,$$

where $\{a_i\}$ are real numbers. The set $\{\Gamma_i\}$ coincides with the orthogonal generators of $\text{SU}(d)$, which is the associated symmetry algebra.

In general, the measurements performed on the system are described by positive operator-valued measures (POVMs), which are a set of operators $\{\Pi_j\}$ (with $\Pi_j \geq 0$ and $\sum_j \Pi_j = 1$), such that each POVM element represents a single output channel of the measuring apparatus. The probability of detecting the $j$th output is given by a generalized projection postulate $p_j = \text{Tr}(\rho \Pi_j)$.

By decomposing the POVM elements in the same basis $\{\Gamma_i\}$, we get

$$\Pi_j = b_j + \sum_{i=1}^{d^2-1} c_{ji} \Gamma_i,$$

where $\{b_j\}$ are again known real numbers and $C = \{c_{ji}\}$ is a real matrix.

Informational completeness. A set of measurements will be called informationally complete if any quantum state $\rho$ is unambiguously assigned to the corresponding theoretical probabilities $p_j$. Since the projection postulate can be rewritten as

$$p_j - b_j = \sum_{i} c_{ji} a_i,$$

informational completeness requires the matrix $C$ to have at least $d^2 - 1$ linearly independent rows. Numerically, this can be easily verified by calculating the rank of $C$, given by the
number of nonzero singular values. These are readily computed from the singular value decomposition of $C$. Thus, a set of measurements is informationally complete provided

$$\text{rank } C \geq d^2 - 1. \quad (4)$$

For example, a light mode can be treated as a harmonic oscillator. The eigenstates of the rotated quadrature operators $Q(\theta) = x \cos \theta + p \sin \theta$ comprise an informationally complete POVM. Naturally, only a finite set of projections can be done, so that a truncation of the original infinite-dimensional Hilbert space is necessary \([10, 11]\). In consequence, consider a von Neumann projection defined in the infinite-dimensional Hilbert space $\mathcal{H}$: $\sum_{k=0}^{\infty} |k\rangle\langle k| = \mathbb{1}$, where $|k\rangle$ is an orthonormal basis. Experimentally such measurements do not pose any difficulty: all that has to be done is to determine the spectrum of a single observable. Nevertheless, this simple von Neumann measurement is not informationally complete in $\mathcal{H}$, for all the observations are in this case mutually compatible and consequently no information about any of the existing complementary observables is available.

**Generating informationally complete measurements.** As we will now show, this interpretation no longer holds when only a subspace $S$ of $\mathcal{H}$ is considered. Let us specify $S$ by introducing the projector $P_S = \sum_{s=1}^{S} |s\rangle\langle s|$, where $|s\rangle$ are eigenstates of $P_S$ and $S$ is the dimension. By projecting the original measurement on $S$, a POVM is induced in this subspace, namely

$$\sum_k \Pi_k = \sum_k P_S |k\rangle\langle k| P_S = \mathbb{1}_S, \quad (5)$$

whose elements, in general, no longer commute $[\Pi_k, \Pi_{k'}] \neq 0$. Indeed, since the original commuting projections have different overlaps with the subspace $S$, their mutual properties (commutators) are not preserved. In this way, an informationally complete POVM may be generated. Obviously, this observation has many potential applications beyond tomography, although, due to strict space limitation only that topic will be discussed.

The protocol we propose consists of the following steps: (i) A reconstruction subspace $S$ is selected according to the particular experiment, in such a way that all the relevant states are included. (ii) An experimentally feasible von Neumann projection is chosen. (iii) The effective POVM induced in $S$, as given by Eq. (5), is calculated and its informational completeness is checked with the help of condition $[4]$. If the induced POVM is informationally complete, the task is finished, otherwise the whole procedure is repeated with different choices of either the von Neumann projection or the reconstruction subspace or both.

Before we proceed further, let us comment on the differences between our protocol and the Naimark extension $[12]$, which is another way of representing POVMs by projective measurements. This extension works by enlarging the Hilbert space with an ancilla, so the projective measurement acts on the product space of the system and ancilla. In our approach, the possibility of representing a tomographical scheme by a projective measurement stems from the available prior information. In fact, the unpopulated states or unused range of variables play the role of ancilla here and, consequently, the measurement acts on a sum rather than a product space.

**Optical vortices.** As a relevant example, we use our protocol for the tomography of optical vortices. As the wave function (or density matrix) in quantum theory, any transverse distribution of complex amplitude (or coherence matrix) can be decomposed in a complete basis; the Laguerre-Gauss modes being a very convenient one

$$\text{LG}_{p}^{\ell}(x, y) = (x, y|\ell, p) \propto r^{\ell} L_{p}^{\ell}(2r^2)e^{-r^2}e^{i\ell\phi}, \quad (6)$$

where $r^2 = x^2 + y^2$ and $\phi = \arctan(y/x)$ are polar coordinates in the transverse plane and $L_{p}^{\ell}$ is a generalized Laguerre polynomial. It is well known $[13]$ that $L_{p}^{\ell}$ beams exhibit helicoidal wavefronts that induce a vortex structure and carry orbital angular momentum of $\hbar \ell$ per photon. Suppose a photon has been emitted into a superposition of modes, and we need to identify the resulting state. In general, this is an involved task $[14, 15, 16]$ requiring the use of complicated optical devices. However, provided that only beams with bounded vorticities (i.e., values of $|\ell|$) are considered, as it is usually the case, our protocol can be employed and an informationally complete measurement can be generated from a very basic one, such as a single transverse intensity scan that is easy to record. In the language of quantum theory, this intensity scan is just $I(x, y) \propto \text{Tr}(g(x, y)|\ell, p)$, where $x$ and $y$ denote now the coordinates of a given pixel of the position-sensitive detector. Although detections in any pair of pixels are always compatible, in a subspace with bounded vorticities noncommuting POVM elements can be induced. This is illustrated in Fig. 1 which shows the noncommutativity (incompatibility) corresponding to the positions $(x, y) = (0, 0), (0, 1)$.
Experimental setup of vortex tomography by means of compatible observations.

Truncating the Hilbert space at smaller vorticities typically leads to stronger noncommutativity, although some nonmonotonicity is also observed as oscillations of gray shades appearing from the top-right to the bottom-left corner.

**Experiment.** To demonstrate the potential of the procedure, a full tomography of an optical vortex field from a single intensity scan has been performed in a controlled experiment. The experimental scheme is shown in Fig. 2. The beam generated by a He-Ne laser is spatially filtered by a microscope objective and a pinhole. After the beam is expanded and collimated by a lens, it impinges on an amplitude spatial light modulator (CRL Opto, 1024 × 768 pixels) displaying a hologram computed as an interference pattern of the required light and the inclined reference plane wave.

Light behind the hologram consists of three diffraction orders (-1, 0, +1), which can be separated and Fourier filtered by means of the 4f optical system consisting of the lenses L_1 and L_2, and an iris diaphragm. The undesired 0th and -1st orders are removed by an aperture placed at the back focal plane of the lens L_1. This completes the preparation of a given state of light.

Finally, a collimated beam with the required complex amplitude profile is obtained at the back focal plane of the second Fourier lens L_2, where a transverse intensity scan I(x, y) is acquired by a CCD camera. In the image plane, each pixel detection can be approximated by a projection on the position eigenstates |x, y⟩⟨x, y|. As it has been shown above, while such detections are compatible in the full infinite-dimensional Hilbert space, an informationally complete POVM is induced in a subspace of truncated vorticities.

In our experiment the superposition

\[
|\Psi⟩ = \frac{1}{\sqrt{2}}(|\ell = 1, p = 0⟩ + |\ell = 2, p = 0⟩)
\]  

(7)

was prepared by letting an amplitude spatial light modulator to display an interference pattern of the transverse amplitude ⟨x, y|Ψ⟩ and a reference plane wave, as mentioned above. Results for this state are shown in Fig. 3. The ideal intensity distribution in the detection plane I(x, y) ∝ |⟨x, y|Ψ⟩|^2 is shown in the left panel. This should be compared to the corresponding noisy recorded image shown in the middle panel. Finally, the right panel shows the best fit obtained with a maximum-likelihood algorithm [17] in the subspace p = 0 and ℓ = 0, . . . , 4. The reconstructed 5-dimensional density matrix is shown in Fig. 4. Notice that, due to experimental imperfections (such as a discrete structure of the spatial light modulator, detection noise, etc.), the reconstructed state slightly differs from the ideal one (typical fidelities in our experiment are F ≈ 96%). In view of the complexity of the system and the simplicity of the experiment, we consider this to be a very good result.

Given the promising performance of the proposed scheme in this proof-of-principle experiment, the natural question is whether an experimentally feasible von Neumann measurement (such as a single intensity scan by a CCD camera with possibly very fine resolution) would furnish an informationally complete measurement for any reconstruction subspace. To get some insights into this problem, we consider two different scenarios related to the experiment above (see Fig. 5).

In the first case, only photons with nonnegative vorticities are considered: the full tomography from a single intensity scan is always possible. In the second case, both positive and negative vorticities are allowed. Here a single intensity scan fails to provide complete information. It is easy to see why: since different scenarios related to the experiment above (see Fig. 5).

The crosses in the plots for the same truncation ℓ_{cutoff}, as it is also obvious from the fact that the phases exp(iℓφ) and exp(−iℓφ)

![Image of experimental setup](image1)

**FIG. 2:** Experimental setup of vortex tomography by means of compatible observations.

![Image of experimental tomography](image2)

**FIG. 3:** Experimental tomography of optical vortex fields. From left: ideal intensity distribution, measured intensity distribution, and the corresponding best theoretical fit of measured data.

![Image of reconstructed density matrix](image3)

**FIG. 4:** Real (on the left) and imaginary (on the right) elements of the reconstructed density matrix.
in superpositions like $LG_0^\ell + LG_0^{-\ell}$ and $LG_0^\ell + LG_0^{-\ell}$ can be distinguished via interference with the other mode. This partial information is however not sufficient for the full characterization of this part of the reconstruction subspace. Provided one wants to keep the simple intensity detection, it is always possible to use a fixed unitary transformation prior to detection to optimize the scheme. For instance, by increasing angular momentum of the measured beam by $\ell_{\text{cutoff}}$ (using, e.g., a charged fork-like hologram) the reconstruction subspace can be moved inside the nonnegative part of the angular momentum spectrum. This example nicely illustrates the role of prior information in experimental quantum tomography.

**Conclusions.** We have shown that simple compatible observations may provide full information about the measured system when some prior information is available. This prior information does not only bring about a quantitative improvement of our knowledge, but may also make feasible a no-go task. Based on this observation, an efficient protocol was sketched providing the full characterization of complex systems from simple measurements. This was demonstrated in an experiment with photonic vortices. In our opinion, this constitutes an improvement that will have a significant benefit in the number of different physical architectures where quantum information experiments are being performed.

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