Realization of quantum gates with multiple control qubits or multiple target qubits in a cavity

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In this paper, we propose a scheme to realize three-qubit controlled phase gate and multiqubit controlled-NOT gate of one qubit simultaneously controlling n target qubits with four level quantum system in a cavity. Adjustment of level spacing during the gates implementation is not required. Implementation time for multiqubit controlled NOT gate is independent of number of qubit. Three-qubit phase gate is generalized to n-qubit phase gate with the number of steps (complexity) reduces linearly as compare to conventional gate decomposition method. Our scheme can be applied to various types of physical systems such as superconducting qubits coupled to resonator and trapped atoms in a cavity. Experimental possibility of our approach is also presented.

Key words: multiqubit quantum gates, superconducting quantum interference devices (SQUIDs), superconducting resonator.

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I. INTRODUCTION

Quantum computing has the potential ability to carry out certain computations much faster than classical computing. For example factorization of a large number via Shor’s algorithm [1] and the search of an item in an unsorted database containing N elements [2]. Two-qubit gates and one-qubit gates are the building blocks for quantum computing networks [3]. Many physical systems have been proposed as candidates for implementation of quantum information processing like atom in cavity quantum electrodynamics (QED) and nuclear magnetic resonance (NMR). Among them cavity QED analogs with superconducting qubit systems are getting favorable attention [4]. A two-qubit gate was experimentally realized using superconducting qubit systems coupled through capacitors [5, 6], mutual inductance [7], or cavities [8].

Multiqubit quantum gates play a significance role in quantum information processing. Experimentally, a three-qubit controlled NOT gate has been demonstrated with trapped ions [9]. Multiqubit gates constructed through conventional gate decomposition method [10], usually makes the procedure complicated for the case of large number of qubits (for example see Sec. III.A). This is because the number of single qubit gate and two qubit gates required for the implementation heavily depends on number n of the qubit. The purpose of this work is to realize three-qubit controlled phase gate and multiqubit controlled NOT-gate of one qubit simultaneously controlling n qubits (which we will denote as NTCNOT-gate) in a cavity QED. Let’s first introduce these gate below.

A. Two kind of multiqubit quantum gates

In three-qubit quantum controlled phase gates when two control qubits $|q_1\rangle$ and $|q_2\rangle$ are in state $|1\rangle$, phase shift $e^{i\delta}$ induces to the state $|1\rangle$ of the target qubit $|q_3\rangle$. When control qubits are in state $|0\rangle$ nothing happens to target qubit. This transformation can be written as [11]

$$U_\delta |q_1, q_2, q_3\rangle = e^{i\delta_{q_1,1}\delta_{q_2,1}\delta_{q_3,1}} |q_1, q_2, q_3\rangle. \quad (1)$$

Here, $\delta_{q_1,1}$, $\delta_{q_2,1}$, and $\delta_{q_3,1}$ are the standard Kronecker delta functions and $|q_1, 0, q_2\rangle$ and $|0, q_1, q_2\rangle$ stand for basis states $|0\rangle$ or $|1\rangle$ for qubit 1, 2 and 3. Circuit for three-qubit controlled phase gate is shown in Fig. (Ia). Thus three-qubit quantum phase gate introduces a phase $\eta$ only when the input state of all three qubits in $|1\rangle$. In this proposal, we discuss the implementation of three-qubit quantum phase gate with $\eta = \pi$. It may be mentioned that three-qubit controlled-NOT gate (known as Toffoli gate) can also be achieved using present proposal. Toffoli gate is equivalent to three-qubit controlled phase gate plus two Hadamard gates on target qubit as shown in Fig. (Ib).

Here, we consider NTCNOT-gate consist of control qubit 1 and n target qubits labeled by 2, 3, ..., n as shown in Fig. (IIa). We define control qubit in $|0\rangle$, $|1\rangle$ basis and each target qubit in $|+\rangle$, $|-\rangle$ basis. Thus input state can be written as

$$|\psi\rangle_i = |0\rangle \prod_{k=2}^{n} (|+\rangle_k + |-\rangle_k) + |1\rangle \prod_{k=2}^{n} (|+\rangle_k + |-\rangle_k). \quad (2)$$

When NTCNOT-gate is applied to above state, we obtained final state given by

$$|\psi\rangle_f = |0\rangle \prod_{k=2}^{n} (|+\rangle_k + |-\rangle_k) + |1\rangle \prod_{k=2}^{n} (|-\rangle_k + |+\rangle_k). \quad (3)$$
It is clear from Eq. 2 and 3 that when control qubit is in state $|1\rangle$ then state at $\oplus$ is flipped as $|1\rangle \rightarrow |0\rangle$ and $|0\rangle \rightarrow |1\rangle$. However, when control qubits 1 and 2 are in state $|0\rangle$ then state at $\oplus$ remain unchanged. For right side of (b) portion inclosed in dashed box represents three-qubit controlled phase gate. The element $H$ is called Hadamard gate and leads the transformation $|0\rangle \rightarrow |+\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ and $|1\rangle \rightarrow |\rangle = (1/\sqrt{2})(|0\rangle - |1\rangle)$.

**B. Motivation and advantages**

Multiqubit quantum controlled phase gate shown in Fig. 1 play a key role in the realization of quantum error correction and implementation of Grover’s algorithm for eight objects. Quantum gate with multiple target qubits shown in Fig. 2 are of great importance in the realization of entanglement preparation, error correction, discrete cosine transform, and quantum cloning.

Some interesting schemes for the realization of multiqubit quantum gates have been proposed. In an earlier studies, J. T Chang et al. presented three-qubit quantum phase gate with four-level atom in a cascade configuration initially in its ground state through a three-mode optical cavity. C. P. Yang et al. presented an n-qubit controlled phase gate with superconducting quantum-interference devices (SQUIDs) by coupling them to a superconducting resonator. Recently, some schemes are proposed for the realization of multiqubit phase gate with a fixed phase-shift of $\pi$ on each target qubit and multiqubit phase gate with random phase-shift on each target qubit.

Our goal of this work is to realize three qubit controlled phase gate shown in Fig. 1(a) and NTCNOT-gate shown
in Fig. 2(a) with four level quantum system in cavity or coupled to superconducting resonator. Our proposal have several advantages: (i) Decoherence due to spontaneous decay of level $|3\rangle$ is suppressed because the excited level $|3\rangle$ is unpopulated during the gates operation. (ii) The level spacing of the qubit system during the gates operations which may cause decoherence is not needed. (iii) Operation time for the realization of NTCNOT-gate is independent of the number of qubits. (iv) In case of flux (SQUID) qubit system each qubit can have much longer storage time as discussed in Sec. IV. (v) We do not require identical coupling constants of each qubit system with cavity mode. Similarly, the detuning of the cavity mode with the transition of the relevant levels in every target qubit system is not identical, therefore our scheme is tolerable to inevitable non-uniformity in device parameters. (vi) Finite second order detuning $\delta = \Delta_\mu - \Delta_\nu$ is not required which improves the gate speed by one order. (vii) Three qubit controlled phase gate shown in Fig. 1 is generalized to $n$-qubit quantum gate with multiple control qubits. Interesetly, complexity (number of operations) reduces linearly as compare to conventional gate decomposition method. In addition, our proposal is general and can be applied to various kind four level physical system like superconducting devices coupled to superconducting resonator and trapped atoms in a cavity.

II. SYSTEM DYNAMICS

We consider here four level qubit system which could be either natural atoms or artificial atoms as shown in Fig. 3. It may be mentioned that Fig. 3 (a) applies to superconducting charged qubit, 3(b) Fig. 3 (b) applies to phase qubit system, 3(c) Fig. 3 (c) applies to flux qubit system and 3(d) Fig. 3 (d) applies to superconducting quantum interference devices (SQUIDs). Four level structure shown in Fig. 3 (b) could also be applied to atoms.

A. System-cavity-pulse resonance Raman coupling

We consider four- level qubit system 1 and 2 coupled to single-mode cavity field and driven by classical microwave pulse as shown in Fig. 1 (a) and (b). Consider qubit system 1 for which cavity mode is coupled to $|2\rangle_1 \leftrightarrow |3\rangle_1$ transition but highly decoupled from the transition between any other two levels. In addition, microwave pulse is applied which is coupled to $|1\rangle_1 \leftrightarrow |3\rangle_1$ transition but highly decouple from the transition between any other two levels as shown in Fig. 1(a). Thus, Hamiltonian of the system can be written as

$$H = \hbar \omega_1 a^\dagger a + \sum_{n=1}^{3} E_n |n\rangle_1 \langle n| + \hbar g_1 (a^\dagger |2\rangle_1 \langle 3| + H.c.)$$

$$+ \hbar \Omega_{13} (e^{i\omega_\mu t} |1\rangle_1 \langle 3| + H.c.),$$  (4)

where $a^\dagger (a)$ is the photon creation (annihilation) operator of the cavity mode with frequency $\omega_c$, $g_1$ is the coupling constant of the between the cavity mode and $|2\rangle_1 \leftrightarrow |3\rangle_1$ transition of qubit system 1. Rabi frequency of pulse is $\Omega_{13}$ and the frequency of the pulse is $\omega_\mu$.

We assume that cavity mode is off-resonance with $|2\rangle_1 \leftrightarrow |3\rangle_1$ transition of the qubit system 1 (i.e., $\Delta_\nu = \omega_{32} - \omega_\nu >> g_1$). Here, $\Delta_\nu$ is the detuning between $|2\rangle_1 \leftrightarrow |3\rangle_1$ transition frequency $\omega_{32}$ of the qubit system 1 and frequency of cavity field $\omega_\nu$. Microwave pulse is off-resonance with $|1\rangle_1 \leftrightarrow |3\rangle_1$ transition of the qubit system 1 (i.e., $\Delta_\mu = \omega_{13} - \omega_\mu >> \Omega_{13}$). Here, $\Delta_\mu$ is the detuning between $|1\rangle_1 \leftrightarrow |3\rangle_1$ transition frequency $\omega_{13}$ of the qubit system 1 and frequency of pulse $\omega_\mu$. The level $|3\rangle_1$ can be eliminated adiabatically. Thus, for case $\Delta_\mu = \Delta_\nu$, effective Hamiltonian in interaction picture (assuming $\hbar = 1$) can be written as 27.

FIG. 3: Desire four level qubit system with four energy level $|0\rangle , |1\rangle , |2\rangle , and |3\rangle$. (a) Represents charged qubit system and transition frequencies between the levels satisfies the condition $\nu_{21} > \nu_{10}, \nu_{32}$ and $\nu_{23} < \nu_{10}$. (b) Represents phase qubit system and transition frequencies between the levels satisfies the condition the condition $\nu_{10} > \nu_{21} > \nu_{32}$. (c) Represents flux qubit system and transition frequencies between the levels satisfies the condition satisfies the condition $\nu_{21} > \nu_{10}, \nu_{32}$ and $\nu_{23} > \nu_{10}$. (d) Represents the SQUIDS qubit system and transition frequencies between the levels satisfies the condition satisfies the condition $\nu_{32} < \nu_{21} < \nu_{20} < \nu_{11} < \nu_{30}$. The level $|0\rangle$ and $|2\rangle$ lies in right well of SQUID while level $|1\rangle$ lies in left well of SQUID. Their is potential barrier between these two wells.
\[ H_1 = -\frac{\Omega_{13}^2}{\Delta_c}[1]_1 \langle 1 | + \frac{g_1^2}{\Delta_c}a^\dagger a [2]_1 \langle 2 | + \frac{\Omega_{13} g_1}{\Delta_c}(a^\dagger [2]_1 \langle 1 | + H.c.) \]  

(5)

The last two terms describes resonance Raman coupling between levels \( |1\rangle_c \) and \( |2\rangle_c \). For the case \( \Omega_{13} = g_1 \), initial state \( |2\rangle_c \) and \( |1\rangle_c \) of the qubit system 1, under the Hamiltonian we can be written as

\[ |1\rangle \langle 0|_c \rightarrow e^{i\theta}(\cos(\theta) |1\rangle \langle 0|_c - i \sin(\theta) |2\rangle \langle 1|_c), \]
\[ |2\rangle \langle 1|_c \rightarrow e^{i\theta}(\cos(\theta) |2\rangle \langle 1|_c - i \sin(\theta) |1\rangle \langle 0|_c). \]  

(6)

Here, \( \theta = \frac{g_1 t}{\Delta_c} \) and \( |0\rangle_c \langle 1|_c \) are the vacuum state (single photon state) of the cavity field. The state \( |0\rangle_1 \langle 0| \) remain unchanged under the Hamiltonian. For the pulse duration \( t_1 = \pi \Delta_c/(2g_1^2) \) (i.e., \( \theta = \pi/2 \)) we obtained the transformation \( |1\rangle_1 \rightarrow |2\rangle_1 \) and \( |2\rangle_1 \rightarrow |1\rangle_1 \) for qubit system 1 and cavity field. We will denote this transformation as \( G_1 \).

In the case of qubit system 2, for notation convenience we denote ground state (first excited state) as level \( |0\rangle_2 \) \( (|0\rangle_3 \) as shown in Fig. 4 (b). Cavity mode is coupled to \( |0\rangle_2 \leftrightarrow |3\rangle_2 \) transition and microwave pulse is coupled to \( |0\rangle_3 \leftrightarrow |3\rangle_3 \) transition of qubit system 2 as shown in Fig. 4 (b). In similar fashion, for pulse duration \( t_2 = \pi \Delta_c/(2g_2^2) \) we obtained the transformation \( |0\rangle_2 \rightarrow |2\rangle_2 \) and \( |2\rangle_2 \rightarrow |0\rangle_2 \) for qubit system 2 and cavity field. We will denote this transformation as \( G_2 \). However, state \( |1\rangle_1 \langle 0|_c \) and \( |1\rangle_2 \langle 1|_c \) of qubit system remain unchanged under transformation \( G_2 \).

**B. System-cavity off-resonant interaction**

Consider qubit system \( k \), for which cavity field interacts off-resonantly with \( |2\rangle_k \leftrightarrow |3\rangle_k \) transition (i.e., \( \Delta_{c,k} = \omega_{c} - \omega_{32} \gg g_k \)) while decoupled from any transition between the other levels as shown in Fig. 4 (c). Here, \( \Delta_{c,k} \) is the detuning between \( |2\rangle_k \leftrightarrow |3\rangle_k \) transition frequency \( \omega_{32} \) of qubit system \( k \) and frequency of cavity field \( \omega_{c} \) while \( g_k \) is the coupling constant between resonator mode and \( |2\rangle_k \leftrightarrow |3\rangle_k \) transition. The effective Hamiltonian of system in interaction picture can be written as

\[ H_1 = -\frac{\hbar g_k^2}{\Delta_{c,k}}(3) \langle 3 | - |2\rangle \langle 2 | a^\dagger a. \]  

(7)

In the presence of single photon in cavity, the evolution of initial state \( |2\rangle_1 \) \( \langle 1|_c \) and \( |3\rangle_1 \) \( \langle 1|_c \) is given by

\[ |2\rangle_k \langle 1|_c \rightarrow e^{i\Omega_{13}^2 t/\Delta_{c,k}} |2\rangle_k \langle 1|_c, \]
\[ |3\rangle_k \langle 1|_c \rightarrow e^{-i\Omega_{13}^2 t/\Delta_{c,k}} |3\rangle_k \langle 1|_c. \]  

(8)

It is clear that phase shift of \( e^{i\Omega_{13}^2 t/\Delta_{c,k}} \) \( (e^{-i\Omega_{13}^2 t/\Delta_{c,k}}) \) is induced to the state \( |2\rangle_k \langle 1|_c \langle 3\rangle_k \langle 1|_c \) of qubit system \( k \). However, states \( |2\rangle_k \langle 0|_c \) and \( |3\rangle_k \langle 0|_c \) remain unchanged.

**C. System-pulse resonant interaction**

Assuming that microwave pulse resonant to \( |j\rangle \rightarrow |2\rangle \) transition of each qubit system is applied. Here, \( j = 1 \) for qubit system 1 and \( k \), while \( j = 0 \) for qubit system 2. Then, the evaluation of state is given by

\[ |j\rangle \rightarrow \cos(\Omega_{13} \tau) |j\rangle - i e^{i\varphi} \sin(\Omega_{13} \tau) |2\rangle, \]
\[ |2\rangle \rightarrow \cos(\Omega_{13} \tau) |2\rangle - i e^{i\varphi} \sin(\Omega_{13} \tau) |j\rangle, \]  

(9)

where \( \Omega_{13} \) is the Rabi frequency between two levels \( |j\rangle \) and \( |2\rangle \). Where as \( \tau \) represents interaction time of qubit system with microwave pulse and \( \varphi \) is the phase associated with microwave pulse. For pulse duration \( \tau = \pi/2(2\Omega_{13}) \) and phase \( \varphi = \pi/2 \), transformation \( |j\rangle \rightarrow |2\rangle \rightarrow |j\rangle \rightarrow |2\rangle \) is obtained which is denoted by \( R \). For the case of phase \( \varphi = -\pi/2 \), we obtained the transformation \( |j\rangle \rightarrow |2\rangle \rightarrow |j\rangle \rightarrow |2\rangle \) \( R^1 \). It may be mentioned that resonant interaction of microwave pulse with qubit system can be carried out in a very short time by increasing the Rabi frequency of pulse.
III. IMPLEMENTATION OF MULTIQUBIT CONTROLLED GATE

The goal of this section to demonstrate how three-qubit quantum phase gate and NTCNOT-gate can be realized based on system dynamics described in Sec. II.

A. Three-qubit controlled phase gate

We consider qubit system 1, 2 and \( k \) (with \( k = 3 \)) for the implementation of three-qubit controlled phase gate shown in Fig. 1. For each qubit system two lowest energy levels \(|0\rangle\) and \(|1\rangle\) represents logical state of each qubit while other higher energy levels \(|2\rangle\) and \(|3\rangle\) are utilized for gate realization. We assume that cavity is initially in vacuum state \(|0\rangle_c\). Three-qubit controlled phase gate is realized in the following steps.

Step (i): Apply the transformation \( G_1 \) to qubit system 1 for time \( t_1 \). Namely, when qubit 1 is initially in state \(|0\rangle_1\), a photon is emitted in cavity. However, the state \(|0\rangle_1 \langle 0|_c\) remains unchanged under transformation \( G_1 \).

Step (ii): Apply transformation \( R \) to qubit system 1 and \( R^\dag \) to qubit system 2, simultaneously. In this step we set \( \tau = \pi/(2\Omega_0) = \pi/(2\Omega_1) \) by adjusting the intensities of two pulses.

Step (iii): After the above operations level \(|2\rangle_1\) of qubit system 1 is unpopulated. The level \(|0\rangle_2\) of qubit system 2 transform to level \(|2\rangle_2\). Apply transformation \( G_2 \) (for time duration \( t_2 \)) to qubit system 2 which will absorb the single photon from the cavity. However, if qubit system 2 is in state \(|1\rangle\) single photon from the cavity will not be absorbed.

Step (iv): Apply transformation \( R^\dag \) (for time duration \( \tau \)) to qubit system \( k = 3 \). After this operation, when cavity is in photon state level \(|2\rangle\) of both qubit system 1 and 2 are unpopulated. Under this condition cavity field interacts off-resonantly to \(|2\rangle_3\) transition of qubit system 3. It is clear from Eq. \( \ref{eq:delta_c3} \) that for \( t_k = (\pi\Delta_{c,3})/\Omega_3^2 \) the state \(|2\rangle_3 |1\rangle_c\) of qubit system 3 changes to \(|2\rangle_3 |1\rangle_c\). However, the states \(|0\rangle_3 \langle 0|_c\), \(|0\rangle_3 |1\rangle_c\) and \(|2\rangle_3 |0\rangle_c\) of qubit system 3 remain unchanged. Finally, apply transformation \( R \) (for time duration \( \tau \)) to qubit system 3.

Step (v): Apply transformation \( G_2 \) (for time duration \( t_2 \)) to qubit system 2.

Step (vi): Apply transformation \( R^\dag \) to qubit system 1 and \( R \) to qubit system 2, simultaneously for time duration \( \tau \).

Step (vii): Apply the transformation \( G_1 \) to qubit system 1 for time \( t_1 \). As a result, qubit 1 is transformed back to state \(|1\rangle_1\) and cavity field returns to its original vacuum state.

Above all operations are schematically presented in Fig. 5. The states of the whole system after the above operations are summarized below

\[
\begin{align*}
& |100\rangle_1 |0\rangle_c & \rightarrow & |200\rangle_1 |1\rangle_c & \rightarrow & |120\rangle_1 |1\rangle_c \\
& |101\rangle_1 |0\rangle_c & \rightarrow & |201\rangle_1 |1\rangle_c & \rightarrow & |121\rangle_1 |1\rangle_c \\
& |110\rangle_1 |0\rangle_c & \rightarrow & |210\rangle_1 |1\rangle_c & \rightarrow & |110\rangle_1 |1\rangle_c \\
& |111\rangle_1 |0\rangle_c & \rightarrow & |211\rangle_1 |1\rangle_c & \rightarrow & |111\rangle_1 |1\rangle_c \\
& |100\rangle_2 |0\rangle_c & \rightarrow & |100\rangle_2 |0\rangle_c & \rightarrow & |120\rangle_2 |1\rangle_c \\
& |101\rangle_2 |0\rangle_c & \rightarrow & |101\rangle_2 |0\rangle_c & \rightarrow & |121\rangle_2 |1\rangle_c \\
& |110\rangle_2 |1\rangle_c & \rightarrow & |110\rangle_2 |1\rangle_c & \rightarrow & |110\rangle_2 |1\rangle_c \\
& |111\rangle_2 |1\rangle_c & \rightarrow & |111\rangle_2 |1\rangle_c & \rightarrow & |111\rangle_2 |1\rangle_c.
\end{align*}
\]  

(10)

Here, state \(|abc\rangle\) is the abbreviation for the states \(|a\rangle_1, |b\rangle_2\) and \(|c\rangle_c\) of qubit \((1, 2, 3)\) with \(a, b, c \in \{0, 1, 2\}\).

On the other hand, states \(|000\rangle_1 |0\rangle_c, |001\rangle_1 |0\rangle_c, |010\rangle_1 |0\rangle_c, |011\rangle_1 |0\rangle_c, \) and \(|111\rangle_1 |0\rangle_c\) remain unchanged. This is because state \(|0\rangle_1\) of the qubit system 1 is not affected by the application of transformation \( G_1 \). Hence, it is clear from Eq. \( \ref{eq:delta_c3} \) that three-qubit controlled phase gate was achieved with three qubits (i.e., control qubit 1 and 2, as well as target qubit 3) after the above operations.

Present proposal provide simple way to realize the Toffoli gate shown in Fig. 1(b). It is well known that at least six qubit controlled-NOT-gates and ten single-qubit gates (i.e., two Hadamard, one phase , and seven \( \pi/8 \) gates) are required to construct a Toffoli gate by conven-
tional gates decomposition methods [30]. The two qubit CNOT-gate is equivalent to two Hadamard gate and single two-qubit phase gate. If we assume that realization of single-qubit gate and two-qubit phase gate requires only one step operation, than, using conventional gate decomposition method, at least 28 steps will be required to realize Toffoli gate. However, present proposal requires only 9 steps i.e., 7 steps for three-qubit phase gate plus two steps operations for two Hadamard gate.

Our scheme can easily be generalized to n-qubit controlled phase gate with multiple control qubits. For this purpose, (i) Apply transformation $G_1$ and $R$ to qubit 1. (ii) Apply transformation $R^\dagger$ and $G_2$ to qubit system 2, 3, ..., $n-1$. (iii) Apply transformation $R^\dagger$, $G_\pi$, and $R$ to last qubit. (iv) Apply transformation $G_2$ and $R$ to qubit system 2, 3, ..., $n-1$. (v) Apply transformation $R^\dagger$ and $G_1$ to qubit 1. Hence, n-qubit controlled phase gate can be achieved by a sequence of operations which are summarized as

$$U_\eta = G_1 \otimes R^\dagger \otimes \prod_{i=2}^{n-1} [R \otimes G_2] \otimes (R \otimes G_\pi) \otimes R^\dagger \otimes \prod_{i=2}^{n-1} [G_2 \otimes R] \otimes R \otimes G_1,$$  

(11)

where $\prod_{i=2}^{n-1} G = G \otimes G \otimes ... \otimes G$, while $\prod_{i=2}^{n-1} G = G \otimes G \otimes ... \otimes G$. Total number of steps for n-qubit quantum gate with multiple control qubits will be $4n-5$. Assuming that realization of basic gates requires only one steps operation, than, using conventional gate decomposition method, required number of steps will be $22n-75$ [30]. Hence, our scheme reduces the number of steps (complexity) linearly as for as number $n$ of the qubit is concerned, as compare to conventional gate decomposition method.

B. NTCNOT-gate

In order to implement NTCNOT-gate we consider qubit system 1 (as shown in Fig. 4 (a)) initially prepared in state $(|0\rangle_1 + |1\rangle_1) / \sqrt{2}$. In this case we consider $n - 1$ qubit system of type $k$ shown in Fig. 4 (c) with $k = 2, 3, ..., n$. Each qubit system $k$ is initially prepared in state $|0\rangle_k$. In the new rotated basis for qubit system $k$, the state of the whole system can be written as

$$|\psi\rangle = \frac{1}{2}(|0\rangle_1 + |1\rangle_1) \otimes \prod_{k=2}^{n} (|+\rangle_k + |-\rangle_k),$$  

(12)

where, $|\pm\rangle_k = 1/\sqrt{2}(|0\rangle_k \pm |1\rangle_k)$. Operations for realizing NTCNOT-gate are described as follow.

**Step (i):** Apply the transformation $G_1$ to qubit system 1 for time $t_1$. Namely, when qubit 1 is initially in state $|1\rangle_1$, a photon is emitted in cavity. However, the state $|0\rangle_1|0\rangle_c$ remain unchanged under transformation $G_1$.

**Step (ii):** Apply transformation $R$ to qubit system 1 and $R^\dagger$ to each qubit system $k$ for time duration $\tau$, simultaneously. As a result transformation $|+\rangle_k (|-\rangle_k) \rightarrow |a\rangle_k (|b\rangle_k)$ is obtained for each qubit system $k$. Here, $|a\rangle_k = 1/\sqrt{2}(|0\rangle_k + |2\rangle_k)$ and $|b\rangle_k = 1/\sqrt{2}(|0\rangle_k - |2\rangle_k)$.

**Step (iii):** After above operations, when cavity is in single photon state level $|2\rangle_c$ and level $|3\rangle_c$ of qubit system 1 is unpopulated. Under this condition cavity field interacts off-resonantly to $|2\rangle_c \rightarrow |3\rangle_c$ transition of each qubit system $k$. It is clear from Eq. (8) that for $t_k = (\pi \Delta_{c,k})/g_{k}^2$ the state $|2\rangle_c |1\rangle_k$, of each qubit system $k$ changes to $|2\rangle_c |1\rangle_k$. That is in the presence of single photon in cavity the state $|a\rangle_k |1\rangle_c$ of each qubit system $k$ changes to $|b\rangle_k |1\rangle_c$, while $|b\rangle_k |1\rangle_c$ of each qubit system $k$ changes to $|a\rangle_k |1\rangle_c$. However, the states $|a\rangle_2 |0\rangle_c$ and $|b\rangle_2 |0\rangle_c$ remained unchanged.

**Step (iv):** Apply transformation $R^\dagger$ to qubit system 1 and $R$ to each qubit system $k$ for time duration $\tau$, simultaneously.

**Step (v):** Apply the transformation $G_1$ to qubit system 1 for time $t_1$. As a result, qubit 1 is transformed back to state $|1\rangle_1$ and cavity field returns to its original vacuum state.

After the above operations, one can easily see that controlled-NOT gate of one qubit simultaneously controlling $n$ qubits described by Eq. (2) and Eq. (3) was achieved with $n$ qubit system (i.e., control qubit 1 and target qubit systems $k = 2, 3, ..., n$).

In order to understand better realization we consider example of three-qubit case. It can be checked that states of the whole system after the above operations are summarized below

$$
\begin{align*}
|1 + +\rangle |0\rangle_c & \rightarrow |2 + +\rangle |1\rangle_c \\
|1 + -\rangle |0\rangle_c & \rightarrow |2 - +\rangle |1\rangle_c \\
|1 - +\rangle |0\rangle_c & \rightarrow |2 + -\rangle |1\rangle_c \\
|1 - -\rangle |0\rangle_c & \rightarrow |2 - -\rangle |1\rangle_c \\
|1bb\rangle |1\rangle_c & \rightarrow |2 - -\rangle |1\rangle_c \\
|1ba\rangle |1\rangle_c & \rightarrow |2 + -\rangle |1\rangle_c \\
|1ab\rangle |1\rangle_c & \rightarrow |2 + +\rangle |1\rangle_c \\
|1aa\rangle |1\rangle_c & \rightarrow |2 + +\rangle |1\rangle_c \\
\end{align*}
$$  

(13)

Hence, it can be concluded from Eq. (13) that three-qubit controlled-NOT gate of one qubit simultaneously controlling 2 qubits with $k = 2, 3$ was achieved with 3 qubit system (i.e., control qubit 1 and two target qubit systems 2 and 3). For $k = 2$ in Eq. (2) and Eq. (3) our scheme reduces to two-qubit controlled-NOT gate which can be used to implement two qubit Deutsch-Jozsa algorithm as described below.

1. DEUTSCH-JOZSA ALGORITHM

Deutsch-Jozsa algorithm is designed to distinguished between the constant and balanced functions on $2^n$ in-
puts [31]. For constant function, the function \( f(x) = \text{constant} \) for all \( 2^n \) inputs. For the balanced function, the function \( f(x) = 0 \) for half of all possible inputs, and \( f(x) = 1 \) for other half. A classical algorithm will need \( 2^n/2 + 1 \) queries to determine whether function is constant or balanced since there may be \( 2^n/2 \) 0’s before finally a 1 appears, showing that function is balanced. In contrast, the Deutsch-Jozsa algorithm requires only one query.

Next, we discuss the scheme to implement two-qubit Deutsch-Jozsa algorithm using four-level qubit system shown in Fig. 3 coupled to cavity or resonator. The qubit system 1 shown in Fig. 4(a) represents query qubit while qubit system \( k = 2 \) shown in Fig. 4(c) represents auxiliary qubit. We prepare the two qubit system in the state \( |\psi\rangle = 1/\sqrt{2}(|0\rangle_1 + |1\rangle_1) \otimes |1\rangle_2 \) which can be written in rotating basis for qubit system \( k = 2 \) as

\[ |\psi\rangle = \frac{1}{2}(|0\rangle_1 + |1\rangle_1) \otimes (|+\rangle_2 - |-\rangle_2), \quad (14) \]

The function \( f(x) \) is characterized by the unitary mapping transformation \( U_f,1 \) and \( |x, y\rangle \rightarrow |x, y \oplus f(x)\rangle \), where \( \oplus \) is a symbol of addition modulo 2. After unitary transformation \( U_f,1 \), initial state of the system changes to

\[ \frac{1}{2}(-1)^{f(0)}|0\rangle_1 + (-1)^{f(1)}|1\rangle_1 \otimes (|+\rangle_2 - |-\rangle_2). \quad (15) \]

There are four possible transformations: (i) \( U_{f,1} \) corresponding to \( f(0) = f(1) = 0 \); (ii) \( U_{f,2} \) corresponding to \( f(0) = f(1) = 1 \); (iii) \( U_{f,3} \) corresponding to \( f(0) = 0 \) and \( f(1) = 1 \); (iv) \( U_{f,4} \) corresponding to \( f(0) = 1 \) and \( f(1) = 0 \). Then Hadamard gate applied on query qubit. As a result, the state of query qubit becomes \( |f(0) + f(1)\rangle \). If \( f(x) \) is constant, the state of query qubit becomes \( |0\rangle_1 \). On other hand, if \( f(x) \) is balanced, the state of the query qubit becomes \( |1\rangle_1 \). Therefore, a measurement on query qubit provides the desire information whether the function \( f(x) \) is constant or balanced. The \( U_{f,n} \) operations are applied to the state \( |\psi\rangle \) as follow

**\( U_{f,1} \) operation.** This is an identity operation. Both qubit system are kept far off with the cavity field and microwave puls. As a result system remain in the state \( |\psi\rangle \).

**\( U_{f,2} \) operation.** We first apply two-qubit controlled NOT-gate as described above. Next, we apply single-qubit rotation \( |0\rangle \rightarrow |1\rangle \) and \( |1\rangle \rightarrow -|0\rangle \) on qubit system 1. Then we repeat two-qubit controlled NOT operation and perform the single-qubit rotation \( |0\rangle \rightarrow -|1\rangle \) and \( |1\rangle \rightarrow |0\rangle \) on qubit system 1. Finally, we obtained

\[ |\psi_2\rangle = \frac{1}{2}(-|0\rangle_1 - |1\rangle_1) \otimes (|+\rangle_2 - |-\rangle_2). \quad (16) \]

**\( U_{f,3} \) operation.** We apply two-qubit controlled-NOT operation. Then state of system evolved to

\[ |\psi_3\rangle = \frac{1}{2}(|0\rangle_1 - |1\rangle_1) \otimes (|+\rangle_2 - |-\rangle_2). \quad (17) \]

**\( U_{f,4} \) operation.** We apply single-qubit rotation \( |0\rangle \rightarrow |1\rangle \) and \( |1\rangle \rightarrow -|0\rangle \) on qubit system 1. Then we perform controlled-NOT operation. Finally, we apply single-qubit rotation \( |0\rangle \rightarrow -|1\rangle \) and \( |1\rangle \rightarrow |0\rangle \) on qubit system 1. The resultant state becomes

\[ |\psi_d\rangle = \frac{1}{2}(-|0\rangle_1 + |1\rangle_1) \otimes (|+\rangle_2 - |-\rangle_2). \quad (18) \]

In this way we obtained the unitary mapping transformation \( U_f \) of Eq. (15). After Hadamard transformation on qubit system 1, if state of qubit system 1 becomes \( |0\rangle_1 \), the function \( f(x) \) is constant. On other hand, if state of qubit system 1 becomes \( |1\rangle_1 \), the function \( f(x) \) is balanced.

### IV. POSSIBLE EXPERIMENTAL IMPLEMENTATIONS

In this section, we give a discussion on experimental possibilities of three-qubit controlled phase gate and NTCNOT-gate described in Sec. III. Total operation time of three-qubit controlled phase gate is given by

\[ \tau_{3cp} = 2\tau_1 + 2\tau_2 + \tau_k + 4\tau \]

\[ = 2\left(\frac{\pi \Delta c}{2g_1^2}\right) + 2\left(\frac{\pi \Delta k}{2g_2^2}\right) + \left(\frac{\pi \Delta c,k}{g_k^2}\right) + 4\left(\frac{\pi}{2\Omega_{12}}\right). \quad (19) \]

Secondly, total operation time for NTCNOT-gate is given by

\[ \tau_{ntcnot} = 2\tau_1 + 2\tau + \tau_k = 2\left(\frac{\pi \Delta c}{2g_1^2}\right) + 2\left(\frac{\pi}{2\Omega_{12}}\right) + \left(\frac{\pi \Delta c,k}{g_k^2}\right). \quad (20) \]

The operation time \( \tau_{3cp} \) and \( \tau_{ntcnot} \) should be shorter than (i) energy relaxation time \( \gamma_2^{-1} \) of level \( 2 \) (it may be mentioned that level \( 3 \) is unpopulated during the entire operations), and (ii) the life time of the cavity mode \( \kappa_1^{-1} = Q/2\pi \nu_c \), where, \( Q \) is loaded quality factor of the cavity and \( \nu_c \) is the resonator frequency. In principle, these requirements can be achieved through (i) reducing operation time by increasing coupling constants and Rabi frequencies, (ii) increasing \( \kappa_1^{-1} \) by employing high \( Q \) cavity or resonator, and (iii) choosing qubit system (e.g., atoms) or designing qubits (e.g., superconducting devices) such that the energy relaxation time \( \gamma_2^{-1} \) of level \( 2 \) is sufficiently long.

Here, we consider without loss of generality \( g_1 \sim g_2 \sim g_k \sim g \). On choose \( \Delta_c \sim \Delta_{c,3} \sim \Delta_{c,k} \sim 10g \), and \( \Omega_{12} \sim 10g \), total operation time required for the gates implementation would be \( \tau_{3cp} \sim 30.2\pi/g \) and \( \tau_{ntcnot} \sim 20\pi/g \). For rough estimate, we assume \( g/\pi \sim 440MHz \), which could be achieved for superconducting qubits coupled to a one-dimensional standing-wave coplanar wave guide (CPW) transmission resonator [32]. For \( g \) chosen here, we have \( \tau_{3cp} \sim 0.068\mu s \) and \( \tau_{ntcnot} \sim 0.045\mu s \), much shorter than \( \gamma_2^{-1} \sim 1\mu s \) [33]. In addition, consider
resonator with frequency $\nu_c \sim 3GHz$ and $Q \sim 10^5$ [3], we have $\kappa^{-1} \sim 5.3\mu s$ which is much longer than operation times $\tau_{cp}$ and $\tau_{ntcnot}$. It may be mentioned that superconducting coplanar wave guide resonator with a (loaded) quality factor $Q \sim 10^6$ have been experimentally demonstrated [34].

As a quantitative example of our technique, consider a superconducting quantum interference devices (SQUIDs) as qubit system. For SQUID, level structure is straightforward to implement by changing external control parameters e.g., magnetic flux $\phi_c$ [27]. For example, consider rf SQUID shown in Fig. 6 with junction capacitance $C=90fF$, loop inductance $L=100pH$, junction’s damping resistance $R \sim 1G\Omega$, potential shape parameter $\beta_L = 1.12$, and external flux $\phi_x = 0.4995\phi_0$. Here, $\phi_0 = \hbar/2e$ is flux quantum. It may be mentioned that SQUIDs with these parameters are available currently [27]. With these choices, decay time of level $|2\rangle$ would be $\tau_2^{-1} \sim 100\mu s$, the $|2\rangle \rightarrow |3\rangle$ coupling matrix element is $\phi_{32} \sim 7.8 \times 10^{-2}$, and $|2\rangle \rightarrow |3\rangle$ transition frequency is $\nu_{32} \sim 4.9GHz$. We choose cavity mode frequency $\nu_c = \omega_c/(2\pi) = 3.6GHz$ and $Q \sim 10^5$, we have $\kappa^{-1} \sim 4.42\mu s$. The SQUID-cavity coupling constant for $|2\rangle \rightarrow |3\rangle$ transition is given by $g = (1/L)\sqrt{\omega_c/2\mu_0\phi_{32}\phi_0} \int_S \mathbf{B}_s(r)dS$. Here, $S$ is the surface bounded by the SQUID ring and $\mathbf{B}_s(r)$ is the magnetic component of cavity mode in the SQUID loop. For standing wave cavity, $\mathbf{B}_s(z) = \mu_0 \sqrt{2/V} \cos kz$, where $k$, $V$, and $z$ are wave number, cavity volume, and the cavity axis, respectively. For $10 \times 1 \times 1mm^3$ standing-wave cavity and a SQUID with $40 \times 40\mu m^2$ loop located at antinode of the cavity mode. Simple calculation shows that $g \sim 4.3 \times 10^8s^{-1}$ for which time required for (i) three-qubit phase gate would be $\tau_{cp} \sim 0.219\mu s$ and (ii) for NTCNOT-gate would be $\tau_{ntcnot} \sim 0.146\mu s$. These implementation times are much shorter than $\gamma_2^{-1}$ and $\kappa^{-1}$. Moreover, we have an additional advantage in case of flux-qubit system that is tunneling between the levels $|1\rangle$ and $|0\rangle$ is not needed during the gates operation. Therefore, potential barrier between levels $|1\rangle$ and $|0\rangle$ can be adjusted priori such that decay from level $|1\rangle$ becomes negligibly small [24,25]. As a result, each qubit can have much longer storage time.

During the operation of step (ii) or (iv) for three-qubit phase gate and of step (iii) or (iv) for NTCNOT-gate, a single photon is populated in the cavity mode and state $|2\rangle$ of each qubit system is occupied. The unwanted system-cavity-pulse resonance Raman coupling and system-cavity off-resonant interaction between resonator mode and $|2\rangle \rightarrow |3\rangle$ transition, is suppressed as accumulated phase shift to state $|2\rangle$ of each qubit system, which would effect the desire gate performance. However, when $\tau << \tau_1, \tau_2, \tau_k$ this unwanted phase shift is sufficiently small and can be neglected. Note that for $\Omega_{12} = \Omega_{02}$, we have $\tau = \pi/(2\Omega_{12})$, $\tau_1 = \pi\Delta_c/(2g_1^2)$, $\tau_2 = \pi\Delta_c/(2g_2^2)$, and $\tau_k = \pi\Delta_c/(2g_k^2)$. Thus condition turns into $\Omega_{12} >> 2g_1^2/\Delta_c, 2g_2^2/\Delta_c, g_k^2/\Delta_c$, which can be met by increasing the Rabi frequency of pulse (i.e., by increasing the intensity of resonant pulse). For $\Delta_c > 10g_k$, the occupation probability of level $|3\rangle$ of target qubits is approximately 0.04 which reduces the gate error [27].

V. CONCLUSION

Before conclusion, we should mention some interesting previous proposals on multiqubit gates [13,21]. However, the present alternative approach is different from these previous proposals. The proposal [13] employs three-mode cavity. The present scheme requires only single mode cavity or resonator which makes the technical requirements comparatively easy. As compare to the proposal [19], decoherence due to adjustment of level spacing (which is also undesirable in experiment) is avoided. In our previous proposal [20] level $|3\rangle$ is populated during the gate operations, the spontaneous decay from this level may cause decoherence. In the present scheme, level $|3\rangle$ is unpopulated during the entire operations, hence suppressed the spontaneous decay of this level. The previous proposal [21] requires 8 steps operations to implement NTCNOT-gate, assuming that Hadamard gate can be implemented in single steps and also required finite second order detuning $\delta = \Delta_c - \Delta_m$. However, using present proposal NTCNOT-gate can be implemented in 5 steps without Hadamard gate and finite second order detuning is not required which improves the gate speed by one order.

We have proposed a scheme for realizing three-qubit
controlled phase gate and NTCNOT-gate as shown in Fig. 1 and Fig. 2 with three types of interactions. These interactions are system-cavity-pulse resonance Raman coupling, system-cavity off-resonant interaction, and system-pulse resonant interaction. Our proposal can be applied to various kind of physical system with four-level configuration. For different systems, frequency regimes of cavity mode could be different, e.g., optical cavities in case of atoms and microwave cavities in case of superconducting qubits. However working of these gate in light of experimental errors requires a rather lengthy and complex analysis which should be further investigated.

We have shown that our proposal have several advantages: (i) Decoherence due to spontaneous decay of level $|3\rangle$ is suppressed because the excited level $|3\rangle$ is unpopulated during the gates operation. (ii) The level spacing of the qubit system during the gates operations which may cause decoherence, is not needed. (iii) For the quantum gate with multiple control qubit, the number of steps (complexity) reduces linearly for number $n$ of the qubit, as compare to conventional gate decomposition method. (iv) Operation time for the realization of NTCNOT-gate is independent of the number of qubits.

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