DEFECTS, DUALITIES AND THE GEOMETRY OF STRINGS VIA GERBES
I. DUALITIES AND STATE FUSION THROUGH DEFECTS

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ABSTRACT. This is the first of a series of papers discussing canonical aspects of the two-dimensional non-linear sigma model in the presence of conformal defects on the world-sheet in the framework of gerbe theory. In the paper, the basic tools of the state-space analysis are introduced, such as the symplectic structure on the state space of the sigma model and the pre-quantum bundle over it, and a relation between the defects and dualities of the sigma model is established. Also, a state-space description of the splitting-joining interaction of the string across the defect is presented, leading to an interpretation of the geometric data associated to junctions of defect lines in terms of intertwiners between the incoming and outgoing sectors of the theory in interaction.

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1. INTRODUCTION

The study of symmetries of a physical model and of dualities that relate it to other points in the relevant moduli space has often proved an indispensable source of knowledge on the structure of the physical system of interest – suffice it to invoke the use of Ward–Takahashi identities in the derivation of correlation functions of a quantum field theory, the ‘duality net’ of consistent (super)string backgrounds, or the AdS/CFT duality that has gained us insights into a strongly coupled QCD-type field theory through the study of a weakly coupled string theory. This by now well-ingrained and widely exploited constatation forms the base of a series of papers, of which the present one is the first, discussing the role played by world-sheet defects in the description of symmetries and dualities of a large...
The more general physical meaning of the full-blown 2-categorical structure associated with bundle gerbes and its naturality in the context of the two-dimensional field theory have been brought to the fore by the construction of the multi-phase σ-model in Ref. [RS09a]. In this construction, the two-dimensional spacetime \( \Sigma \) (or its euclidean version) is split into a collection of domains \( \Sigma_i, i \in \mathbb{N} \), each carrying its own phase of the full theory (i.e. a choice \( M_i \) of a connected component of the target space with the attendant structures of the metric and the gerbe) and separated from adjacent domains by lines of discontinuity of the embedding field \( X \), termed defect lines and mapped by \( X \) into a correspondence space \( Q \) called the bi-brane world-volume. Defect lines, in turn, intersect at the so-called defect junctions, sent by \( X \) into another correspondence space \( T \), dubbed the inter-bi-brane world-volume. The constitutive elements of the construction of Ref. [RS09a] and the ensuing definition of the action functional of the σ-model are recalled in Section 2.

An important source of inspiration for the construction, with its assignment of distinguished 1-cells of \( \mathcal{B} \mathcal{G} \mathcal{E} \mathcal{B} \mathcal{B} \mathcal{T}(Q) \) to defect lines and 2-cells of \( \mathcal{B} \mathcal{G} \mathcal{E} \mathcal{B} \mathcal{B} \mathcal{T}(T) \) to defect junctions, were the earlier findings of Refs. [FW99] [Kap00] [CJM04] [GR02] [Gaw05] and Ref. [FSW08] in which the relevant cohomological structures had been identified over the submanifolds of \( M \) and \( M_i \times M_j \), respectively, defining the

\[
\tau^\text{Zumino–Witten (WZW)} \text{realised already in Ref. [Gaw88] and subsequently employed in Ref. [FGK88] in the setting of the Wess–}
\]

\[
\text{geometric definition followed in Refs. [Mur96, MS00]. Gerbes are geometric structures representing}
\]

\[
\text{the classical action functional – indeed, it gives rise to a natural classification scheme}
\]

\[
\text{for \( \sigma \)-models on a given target space in terms of equivalence classes of gerbes, which – for a given}
\]

\[
\text{curvature – span the sheaf-cohomology group \( H^2(M, U(1)) \), and it canonically defines the pre-quantum}
\]

\[
\text{bundle of the theory, thus establishing the basis of the geometric quantisation scheme. This was}
\]

\[
\text{realised already in Ref. [Gaw88] and subsequently employed in Ref. [FGK88] in the setting of the Wess–}
\]

\[
\text{Zumino–Witten (WZW) \( \sigma \)-model with a compact simple connected Lie group as the target space.}
\]

\[
\text{Over and above these, gerbe theory provides us with concrete tools for constructing new theories}
\]

\[
\text{from the existing ones by way of gauging subgroups \( K \) of the isometry group of \( (M, g) \) through}
\]

\[
\text{orientifolding, both procedures being founded on the notion of an equivariant (resp. twisted-equivariant,}
\]

\[
\text{cp. Refs. [SSW07, GSW08a]) structure on the gerbe. The structure uses distinguished 1- and 2-cells of the 2-category of bundle gerbes over the nerve of the action groupoid \( K \setminus M \), and it descends to a}
\]

\[
\text{gerbe structure on the quotient space \( M/K \) under suitable conditions. Since, furthermore, all gerbes}
\]

\[
\text{on \( M/K \) can be obtained in this manner, cp. Ref. [Gaw88], it gives us a classification of \( \sigma \)-models on}
\]

\[
\text{the quotient space (resp. that of \( \sigma \)-models for unorientable world-sheets in the case of orientifolding).}
\]

\[
\text{The existence and uniqueness theorems established in Refs. [GR02, GR03] and Ref. [Gaw88] for the}
\]

\[
\text{orbifolded resp. orientifolded variants of the WZW model, for which there exists an explicit construction}
\]

\[
\text{of the gerbe (worked out, in steps, in Refs. [Gaw88, Cha98, Mei03], are in perfect agreement with}
\]

\[
\text{known results of the structure-heavy Conformal Field-Theory (CFT) analyses of modular invariants}
\]

\[
\text{from Refs. [FGK88, KS94, BH04] and those of the categorial quantisation of the \( \sigma \)-model, reported}
\]

\[
\text{in Ref. [FRS04]. Analogous statements from Ref. [Gaw88] pertaining to the case of continuous group}
\]

\[
\text{actions, the latter presenting an additional complication due to the coupling between the gerbe and the}
\]

\[
\text{principal \( K \)-connection on a (generically non-trivial) principal \( K \)-bundle over \( \Sigma \) which may fail}
\]

\[
\text{because of global gauge anomalies, go far beyond the long-established results of both the geometric}
\]

\[
\text{and algebraic discussion of Refs. [HS89, HS91, FOS, FOS94], and the (conformal) field-theoretic analysis}
\]

\[
\text{of Refs. [GK085, GK90, Hor96, FSS96].}
\]
codomain (the so-called D-brane or $G$-brane world-volume) of the restriction of $X$ to connected components of the boundary of $\Sigma$ in the former case, and determining the discontinuity of $X$ along a defect line homeomorphic to $S^1$ in the latter case. These cohomological structures, corresponding to vector bundles twisted by the gerbe in a well-defined manner, contribute their own part to the classification scheme of consistent $\sigma$-models on world-sheets with defects and straightforwardly accommodate a variant of the orientifolding and gauging constructions set up for the bulk gerbes, as demonstrated in Refs. [GR02, Gaw05, GSW08a, GSW]. It well deserves to be pointed out that the ensuing explicit constructions of orbifold and orientifold $G$-branes in the controlled setting of the WZW model, presented in Refs. [Gaw05, GSW11], indicate the presence of an essentially new species of $G$-brane, dubbed the non-abelian brane in the original Ref. [Gaw05], over those conjugacy classes in the target Lie group which are invariant under the action of non-cyclic components of $K$. These $G$-branes have properties suggestive of an interpretation in terms of irreversible stacks of fixed-point fractional branes of Ref. [DDG98]. Their existence, peculiar to the maximal orbifold $\text{Spin}(4n)/((\mathbb{Z}_2 \times \mathbb{Z}_2)$ and ubiquitous on proper orientifolds of group manifolds, had not been predicted by the standard CFT methods, and so they provide a tangible example of a novel string-theoretic insight gained by purely gerbe-theoretic methods. The intriguing internal (open-string) dynamics of these branes still awaits an in-depth treatment.

A piece of motivation that is more immediately related to the subject matter of the present paper, and also of a more field-theoretic flavour (as seen from the two-dimensional perspective), comes from the CFT studies of conformal interfaces (i.e. defect lines transmissive to that half of the conformal symmetries of either of the two phases of the theory supported over the two domains of the world-sheet separated by the defect line which preserve that line). The concept originated from the condensed-matter considerations of Ref. [OA97] and was later transplanted into the string-theoretic domain in Ref. [PZ01] (in a purely operator-algebraic language) and in Ref. [BBDO02] (in a more geometric world-sheet terms). Subsequent studies have diverged into a variety of specialised directions, including the classificatory analysis and specific constructions of Refs. [FRS02, QSO2, FGRS07, FSW08, BM10, GSW] for various distinguished classes of CFT (such as, e.g., the free boson, the (gauged) WZW model and, more generally, an arbitrary rational CFT), the discussion of the fusion of conformal interfaces in the quantum regime in Refs. [PZ01, FRS02, BB08], alongside a description of perturbed defect CFTs and Renormalisation-Group (RG) flows in their presence advanced in Refs. [EG04, AM07, Rum07, KRW09, BM10], and related to certain integrable structures of CFT in Refs. [Rum08, MR10]. Analogous results have also been obtained in the context of supersymmetric two-dimensional field theories, cf., e.g., Refs. [BR07, BJR09, BRR10, BR10]. The studies carried out to date, and in particular those reported in Refs. [FPRS04, FFRS07, ST07, RS09b, SS08, Bac09], bear ample evidence of a prominent rôle played by conformal interfaces in establishing correspondences between phases of CFT, in encoding order-disorder dualities among various CFTs, and in mapping into one another their RG flows as well as UV and IR fixed points of the latter. Finally, they can be associated with the so-called spectrum-generating symmetries of string theory, relating – via fusion with boundary states (a process that has not been fully understood up to now) – the D-brane categories of a dual pair of CFTs. All this leads to a natural question as to a state-space interpretation of the conformal world-sheet defects of Ref. [RS09a] and the attendant cohomological structures on the codomain of $\sigma$-model fields, encompassing the data carried by both the defect lines and their junctions. This question is at the core of the present paper, and in our search for an answer, we shall be guided by insights inferred from the detailed treatment of the maximally symmetric WZW defects in Refs. [FSW08, RS09b, RS09a, RS]. The latter provide an excellent setting in which to look, in particular, for conditions necessary and sufficient for the correspondence between the phases of the $\sigma$-model determined by the defect to be compatible with the module structure on the respective state spaces with respect to the action of an extended current symmetry algebra. This issue is put in a wider generalised-geometric context and subsequently elaborated at great length in Ref. [Sus11], forming the second part of the series opened by the present article.

A natural framework for establishing the sought-after state-space interpretation of the conformal defects is provided by the canonical description of the $\sigma$-model. The description can be derived in the so-called covariant (or first-order) formalism of Refs. [Gaw72, Kij73, Kij74, KS76, Sz76, KT79] which leads to a systematic reconstruction of the symplectic structure on the state space of the two-dimensional field theory of interest. The basic tools of the formalism are introduced in Section 3.1. These are subsequently applied, in the remainder of Section 3, to the two qualitatively distinct sectors of string theory present on a generic world-sheet with an embedded defect, that is the untwisted sector, composed of strings represented by smooth loops embedded in the respective connected components of
the target space, and the twisted sector, with states represented by piecewise smooth maps from the unit circle into the target space, with point-like discontinuities which can be understood as resulting from transversal intersections with defect lines. A prototypical example of the latter sector is provided by the twisted sector of string theory on an orbifold of a smooth target space, first discussed in Refs. [DHVW85, DHVW86], in which case the discontinuities are determined by elements of the orbifold group. The upshot of the analysis carried out in Section 3 is a full-fledged canonical description of the classical (bosonic) string with a multi-phase world-sheet.

The key advantage of working with the global geometric structures from \( \mathcal{B} \mathfrak{S} \mathfrak{h} \mathcal{V}(M \cup Q \cup T) \) in the classical setting is that they actually afford inroads into the quantum régime of the theory. As mentioned already in the opening paragraph of the present section, this fact has been known ever since the introduction of the hypercohomological language into rigorous studies of the two-dimensional \( \sigma \)-model in Ref. [Gaw88], which is where the transgression map was defined. The latter is a cohomology map canonically assigning to the 1-isomorphism class of the gerbe of the \( \sigma \)-model with target space \( M \) the isomorphism class of a circle bundle over its configuration space \( \mathcal{L} \equiv C^\infty(S^1, M) \) (the free-loop space of \( M \)) with a connection whose curvature yields, upon pullback to the state space \( \mathcal{P} \sigma \cong T^* \mathcal{L} \) of the theory and correction by a canonical (and topologically trivial) term, the symplectic form of the (defect-free) \( \sigma \)-model. In other words, the gerbe determines a pre-quantum bundle \( \mathcal{L} \sigma \rightarrow \mathcal{P} \sigma \) of the closed string, a prerequisite of its geometric quantisation, cp., e.g., Ref. [Woo92]. The important novel result for strings with multi-phase world-sheets, anticipated by the findings of Refs. [GR02, Gaw05] and derived in Section 3 in analogy with the original result for the untwisted sector, is the existence of a straightforward generalisation of the transgression map to the twisted sector of the string. This generalised cohomology map canonically assigns the isomorphism class of a pre-quantum bundle over the space of twisted states to the equivalence class of a coupled pair consisting of the \( \sigma \)-model gerbe and the associated bi-brane, a fact following directly from Theorem 3.18.

The canonical formalism thus reconstructed constitutes an excellent basis for phrasing the question about the rôle of defects in a rigorous manner. Guided by the simple geometric intuition conveyed by the world-sheet picture of the cross-defect identification of states effected by the propagation of the closed string, as illustrated in Figure 1, we are led to investigate conditions under which the data of

![Figure 1](image_url)

**Figure 1.** The correspondence between states mediated by the defect line: the state \( \psi_1 \) from the phase CFT\(_1\) is transferred to the state \( \psi_2 \) from the phase CFT\(_2\) across the defect line \( \ell_{1,2} \). The arrow above the picture represents the world-sheet time direction.

the defect (including the gluing condition to be imposed on the lagrangean fields of the model) define a duality between the phases of the \( \sigma \)-model separated by the defect. The point of departure for these investigations is the identification of a (pre-quantum) duality with an isomorphism

\[
\text{pr}_1^* \mathcal{L} \sigma \mid \mathcal{J}_\sigma \cong \text{pr}_2^* \mathcal{L} \sigma \mid \mathcal{J}_\sigma
\]

of the pullbacks of the pre-quantum bundle along the canonical projections \( \text{pr}_\alpha : \mathcal{P} \sigma \rightarrow \mathcal{P} \sigma \), \( \alpha \in \{1, 2\} \) over the graph \( \mathcal{J}_\sigma \) of a symplectomorphism that preserves the hamiltonian density of the \( \sigma \)-model. The relevant conditions are stated in Theorem 4.9 and it is worth underlining that they single
out bi-brane world-volumes $Q$ that are surjectively submersed onto the target space $M$, in keeping with the results of Ref. [FNSW09]. The reverse question as to the circumstances under which a duality gives rise to consistent defect data is subsequently examined for an important class of dualities in the remainder of Section 4, culminating in Theorems 4.14 and 4.15. Altogether, the findings of Section 4 establish a rather strong and general correspondence between the so-called topological defects and dualities of string theory, the latter including, in particular, symmetries of a single phase induced from distinguished isometries of the target space and the proper (T-)duality between string models on topologically non-equivalent principal torus bundles, of the kind originally discussed in Refs. [Bus88, Bus87].

Just as defect-line data constrain the ‘tunelling’ of the closed string between images (with respect to the embedding map) of the supports of adjacent phases of the theory, those carried by defect junctions are of relevance to the stringy interaction processes, represented by regions in the world-sheet homeomorphic to a sphere with (at least) three punctures and an embedded defect subgraph. The simple world-sheet intuition behind this statement is depicted in Figure 2. It can readily be formalised in the canonical language, in which one is led to expect the emergence of symplectomorphic identifications among multi-string states in interaction, lifting to isomorphisms of the associated pullback pre-quantum bundles over an interaction subspace spanned by these states. Theorems 5.5 and 5.8 confirm these expectations independently for each of the two sectors of the state space of the $\sigma$-model, that is for the untwisted sector and the twisted sector, altogether giving rise to a canonical picture of the cross-defect splitting-joining interaction of the string. In this picture, the data of the 2-isomorphism associated with a defect junction are shown to transgress, in a canonical manner, to local data of the expected isomorphism of the pullback pre-quantum bundles. This result can be viewed as a logical completion of the transgression scheme for $\mathcal{BGrb}^\nabla(M \cup Q \cup T)$ anticipated by the findings of Ref. [Gaw88]. A natural, if also merely implicit in the present treatment, consequence of the existence of cross-defect identifications among multi-string states is an interpretation of the defect-junction data in terms of intertwiners between representations of (current) symmetry algebras furnished by the multi-string state spaces that enter the definition of the interaction subspace. This interpretation is substantiated in Ref. [Sus11].

The clear-cut canonical interpretation of world-sheet defects, and – in particular – the relation between the latter and dualities of string theory, in conjunction with the by now rich knowledge – gathered in Refs. [GR03, Gaw05, SSW07, GSW08a, GSW08b] and further enhanced in Ref. [GSW10] through the study of continuous group actions – on the gerbe-theoretic structure of multi-phase world-sheets and on their rôle in defining string theory on quotient target spaces suggest a natural generalisation of the notion of a smooth (pseudo-)riemannian manifold with extra (cohomological) structure. Inspired by the pioneering work [Hal05], but also taking into account the findings of Ref. [JK06], where multi-phase world-sheets were considered from the point of view of a consistent interaction of the closed string on the orbifolded target space, we may conceive a situation in which the string world-sheet is mapped

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The basic $2 \to 1$ splitting-joining interaction across the three-valent defect junction. The arrow to the right of the picture represents the time direction.}
\end{figure}
into a target space modelled on a (pseudo-)riemannian manifold with a gerbe over it only \textit{locally}, with geometric data (those of the metric and of the gerbe) from local charts glued together by means of local data of the bi-brane implementing \textit{bona fide} $\sigma$-model dualities. This ultimately leads to the concept of a non-geometric background with the structure of a duality ‘quotient’ (provided that the latter can be defined in a meaningful manner), or a D(uality)-fold, generalising the idea of a T-fold based on the T-duality group. The correspondence between gerbes on K-spaces and those on K-quotients of the latter is a strong indication that string theory on a would-be D-fold prerequires a self-consistent hierarchy of cohomological structures, to wit, the bulk gerbe, the duality bi-brane and the basic inter-bi-brane (for three-valent defect junctions) from which all components of the inter-bi-brane associated with defect junctions of valence higher that 3 could be induced in a well-defined and physically intuitive manner.

The relevant intuition derives from the observation that an insertion of a local defect field for a defect junction of valence $n > 3$ can be generated in a suitably regularised limiting procedure of bringing together a number of insertions of local defect fields for defect junctions for valence 3. The existence of such a hierarchy was first discussed (and illustrated with the explicit example of the central-jump WZW defect) in \textcite{RS09b} under the name of defect-junction data with induction. In Remark 5.6, the original discussion is extended and rephrased in the language of simplicial objects in the category of differentiable manifolds, inspired by the study of equivariant structures and purely physical considerations, whereupon the notion of a simplicial string background is introduced. This is then conjectured to be the point of departure in any consistent construction of a D-fold, to which we are hoping to return in the future.

Prior to concluding this introductory section, let us add a few more comments on the structure of the present paper. First of all, we have decided to organise the discourse into a collection of definitions, propositions and theorems, interspersed with examples and occasional remarks of a looser nature. Secondly, the more technical proofs have been relegated to the appendices. Finally, several open questions have been collected in the closing Section 6, alongside a brief recapitulation of the results.

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2. \textbf{Defects in the Lagrangean picture}

In the present paper, we shall be concerned with a theory of bosonic fields on an oriented two-dimensional space-time in the presence of domain walls that split the space-time into domains supporting the respective phases of the field theory. Fields of the theory take values in differentiable manifolds with additional geometric structure, captured neatly by gerbe theory, and a working knowledge of the rudiments thereof is assumed throughout this paper. The reader unfamiliar with the theory is referred to, e.g., the literature cited in the Introduction. Thus, let us begin with

\textbf{Definition 2.1.} \textbf{A string background} is a triple $\mathcal{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$ composed of the following geometric structures:

- the \textbf{target} $\mathcal{M} = (M, g, \mathcal{G})$ consisting of a manifold $M$, termed the \textbf{target space}, with a metric $g$, a closed 3-form $\mathcal{H}$ and an abelian gerbe $\mathcal{G}$ (with connection) of curvature $\mathcal{H}$;
- the \textbf{$\mathcal{G}$-bi-brane} $\mathcal{B} = (Q, \iota_\alpha, \omega, \Phi \mid \alpha \in \{1, 2\})$ consisting of a manifold $Q$, termed the \textbf{$\mathcal{G}$-bi-brane world-volume}, with a 2-form $\omega$, termed the \textbf{$\mathcal{G}$-bi-brane curvature}, and a pair of smooth maps $\iota_\alpha : Q \to M$, $\alpha \in \{1, 2\}$, and of a gerbe 1-isomorphism (a $(\iota_1^*\mathcal{G}, \iota_2^*\mathcal{G})$-bi-module)

$$\Phi : \iota_1^*\mathcal{G} \cong \iota_2^*\mathcal{G} \otimes I_\omega,$$

written in terms of a trivial gerbe $I_\omega$ with curving $\omega$, obeying the identity

$$\Delta_\mathcal{H} \omega = -d\omega, \quad \Delta_\mathcal{G} := \iota_2^* - \iota_1^*$$;

- the \textbf{$(\mathcal{G}, \mathcal{B})$-inter-bi-brane} $\mathcal{J} = \left(T_n, (\varepsilon_n^{k,k+1}, \sigma_n^{k,k+1} \mid k \in \mathbb{N}_3, \varphi_n \mid n \in \mathbb{N}_3), \right)$ consisting of a disjoint sum of manifolds $\bigcup_{n \in \mathbb{N}_3} T_n =: T$, termed the \textbf{$(\mathcal{G}, \mathcal{B})$-inter-bi-brane world-volume}, with a collection of orientation maps $\varepsilon_n^{k,k+1} : T_n \to \{-1, +1\}$


6
and smooth maps \( \pi_{n}^{k,k+1} : T_{n} \to Q \) subject to the constraints
\[
\tilde{t}_{i_{2}}^{k,k} \circ \pi_{n}^{k-1,k} = \tilde{t}_{i_{1}}^{k,k+1} \circ \pi_{n}^{k,k+1}, \quad k \in \overline{1,n},
\]
with \((t_{1}^{1}, t_{2}^{1}) = (t_{1}, t_{2})\) and \((t_{1}^{1}, t_{2}^{2}) = (t_{2}, t_{1})\), and of distinguished gerbe 2-isomorphisms
\[
\mathcal{G}_{n}^{3} \otimes I_{\omega_{n}^{2} + \omega_{n}^{1} + \omega_{n}^{1}}
\]
written in terms of 1-isomorphisms \( \Phi_{n}^{k,k+1} = \pi_{n}^{k,k+1} \mathcal{E}_{k,k+1} \), with \( \Phi^{+1} = \Phi \) and \( \Phi^{-1} = \Phi^{-} \) (the dual 1-isomorphism), between gerbes \( \mathcal{G}_{n}^{k} = (\epsilon_{n}^{-} \circ \pi_{n}^{k,k+1})^{\ast} \mathcal{G} \), and the trivial gerbes with global curvings \( \omega_{n}^{k,k+1} = \epsilon_{n}^{-} \circ \pi_{n}^{k,k+1} \ast \omega \). The latter satisfy the Defect-Junction Identity (DJI)
\[
\Delta_{T_{n}} \omega = 0, \quad \Delta_{T_{n}} = \sum_{k=1}^{n} \epsilon_{n}^{-} \circ \pi_{n}^{k,k+1} \ast \omega.
\]

In the subsequent sections, we shall oftentimes have a need for a more explicit description of the gerbe-theoretic concepts invoked in Definition 2.1. For this reason, we recall

**Definition 2.2.** Let \( \mathcal{M} \) be a differentiable manifold, and let \( S_{\mathcal{M}}^{q} \), \( q \in \overline{0, \dim \mathcal{M}} \) be the following sheaves over \( \mathcal{M} \):
- \( S_{\mathcal{M}}^{0} := \text{U}(1) \) the sheaf of locally smooth \( \text{U}(1) \)-valued maps on \( \mathcal{M} \);
- \( S_{\mathcal{M}}^{q} := \Omega^{q}(\mathcal{M}) \), \( q > 0 \), the sheaf of locally smooth (real) \( q \)-forms on \( \mathcal{M} \).

Given the differential Deligne complex
\[
D(n)^{\ast}_{\mathcal{M}} : S_{\mathcal{M}}^{0} \xrightarrow{\partial(0) = \frac{1}{\varepsilon} d \log} S_{\mathcal{M}}^{1} \xrightarrow{\partial(1) = d} S_{\mathcal{M}}^{2} \xrightarrow{\partial(2) = d} S_{\mathcal{M}}^{3} \xrightarrow{\partial(n-1) = d} S_{\mathcal{M}}^{n},
\]
denote by \( A^{\ast} \left( O_{\mathcal{M}}, D(n)^{\ast}_{\mathcal{M}} \right) \) the diagonal sub-complex of the Čech–Deligne double complex \( \tilde{C}^{\ast}(O_{\mathcal{M}}, D(n)^{\ast}_{\mathcal{M}}) \) obtained, for a given choice \( O_{\mathcal{M}} = (O_{\mathcal{M}}^{i})_{i \in \mathcal{I}} \) of a good open cover of \( \mathcal{M} \) (with non-empty multiple intersections of its elements denoted as \( O_{\mathcal{M}}^{i1} \cap O_{\mathcal{M}}^{i2} \cap \ldots \cap O_{\mathcal{M}}^{i_{n}} =: O_{\mathcal{M}}^{i_{1} \ldots i_{n}} \) and assumed contractible), by extending \( D(n)^{\ast}_{\mathcal{M}} \) through the Čech complexes
\[
\tilde{C}^{0}(O_{\mathcal{M}}, S_{\mathcal{M}}^{q}) \xrightarrow{\tilde{d}(0)} \tilde{C}^{1}(O_{\mathcal{M}}, S_{\mathcal{M}}^{q}) \xrightarrow{\tilde{d}(1)} \tilde{C}^{2}(O_{\mathcal{M}}, S_{\mathcal{M}}^{q}) \xrightarrow{\tilde{d}(2)} \ldots
\]
associated to \( O_{\mathcal{M}} \), and with the standard Čech coboundary operators
\[
\tilde{d}^{(p)} : \tilde{C}^{p}(O_{\mathcal{M}}, S_{\mathcal{M}}^{q}) \to \tilde{C}^{p+1}(O_{\mathcal{M}}, S_{\mathcal{M}}^{q})
\]
\[
:= (s_{i_{0}i_{1} \ldots i_{p}}) \mapsto \left( (\delta^{(p)} s)_{i_{0}i_{1} \ldots i_{p}} := \sum_{k=0}^{p+1} (-1)^{k} s_{i_{0}i_{1} \ldots i_{k}i_{k+1} \ldots i_{p+1}} \right)_{O_{\mathcal{M}}^{i_{0}i_{1} \ldots i_{p+1}}}
\]

The above is written in the additive notation for local sections \( s_{i_{0}i_{1} \ldots i_{p}} \in S_{\mathcal{M}}^{q}(O_{\mathcal{M}}^{i_{0}i_{1} \ldots i_{p}}) \) of the sheaves \( S_{\mathcal{M}}^{q} \), in which “+” stands for multiplication of sections if \( q = 0 \) and for addition of sections otherwise, and in which multiplication of a section by a real number \( c \) stands for the raising of the section to the power \( c \) if \( q = 0 \) and for the multiplying of the section by \( c \) otherwise. This notation shall be used throughout the paper. Finally, we write as \( -\#D(r) \) the Deligne differential defined component-wise as
\[
-\#D(r) : A^{n,r}(O_{\mathcal{M}}) \to A^{n,r+1}(O_{\mathcal{M}}), \quad -\#D(r)|\tilde{C}^{p}(O_{\mathcal{M}}, S_{\mathcal{M}}^{q}) = \partial^{(q)} + (-1)^{q+1} \tilde{d}^{(p)},
\]
with the corresponding Deligne (hyper-)cohomology groups denoted as \( \mathbb{H}^{r}(\mathcal{M}, D(n)^{\ast}_{\mathcal{M}}) \). A local presentation of string background \( \mathcal{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J}) \) consists of the following data.
for the gerbe $G$ over the target space $M$, a Čech–Deligne cochain

$$G \xrightarrow{\text{loc}} (B_i, A_{ij}, g_{ijk}) =: b \in \mathcal{A}^{3,2}(\mathcal{O}_M)$$

with curvings $B_i$, connections $A_{ij}$ and transition functions $g_{ijk}$, satisfying the cohomological identity

$$MD_{(2)} b = (\Pi|_{\mathcal{O}_M}, 0, 0, 1) ;$$

the local data $b$ are determined up to gauge transformations

$$b \mapsto b + MD_{(1)} \pi , \quad \pi := (\Pi_i, \chi_{ij}) \in \mathcal{A}^{3,1}(\mathcal{O}_M) ;$$

thus, gauge equivalence classes of local data correspond to elements of $H^2(M, \mathcal{D}(2)^\bullet_M)$;

- for the $G$-bi-brane 1-isomorphism $\Phi$, a Čech–Deligne cochain

$$\Phi \xrightarrow{\text{loc}} (P_i, K_{ij}) =: p \in \mathcal{A}^{2,1}(\mathcal{O}_Q)$$

satisfying the cohomological identity

$$QD_{(1)} p = \Delta Q b + \bar{\omega},$$

in which $\bar{\omega} = (\omega, 0, 0, 1)$ are local data of the trivial gerbe $I_\omega$, and $\Delta Q := i_{\pi}^* - i_1^*$ for the Čech-extended $G$-bi-brane maps $i_\pi = (i_\alpha, \phi_\alpha)$ with index maps $\phi_\alpha : \mathcal{I}_Q \to \mathcal{I}_M$ covering the respective manifold maps $i_\alpha : Q \to M$ as per

$$i_\alpha (\mathcal{O}_Q^I) \subset \mathcal{O}^M_{\phi_\alpha(i)},$$

and in which we use the shorthand notation

$$i_\alpha (B_i, A_{ij}, g_{ijk}) := i^*_\alpha (B_{\phi_\alpha(i)}, A_{\phi_\alpha(i)\phi_\alpha(j)}, g_{\phi_\alpha(i)\phi_\alpha(j)});$$

the local data $p$ are determined up to $G$-twisted gauge transformations

$$p \mapsto p + \Delta Q \pi - Q D_{(0)} w, \quad w := (W_i) \in \mathcal{A}^{2,0}(\mathcal{O}_Q),$$

with the $G$-twist $\Delta Q \pi$ ensuring that the defining identity (2.5) is preserved under a gauge transformation (2.4);

- for the $(G, \mathcal{B})$-inter-bi-brane 2-isomorphisms $\varphi_n$, Čech–Deligne cochains

$$\varphi_n \xrightarrow{\text{loc}} (f_{n,i}) =: F_n \in \mathcal{A}^{1,0}(\mathcal{O}_{T_n})$$

satisfying the cohomological identities

$$T_n D_{(0)} F_n = - \Delta_{T_n} p ,$$

in which $\Delta_{T_n} = \sum_{k=1}^{n+1} \pi^{k,k+1}_n \tilde{\omega}^{k,k+1}_n$ for the Čech-extended $(G, \mathcal{B})$-inter-bi-brane maps $\pi^{k,k+1}_n = (\pi^{k,k+1}_n, \psi^{k,k+1}_n)$ with index maps $\psi^{k,k+1}_n : \mathcal{I}_{T_n} \to \mathcal{I}_Q$ covering the respective manifold maps $\pi^{k,k+1}_n : T_n \to Q$ as per

$$\pi^{k,k+1}_n (\mathcal{O}^Q_{T_n}) \subset \mathcal{O}^M_{\psi^{k,k+1}_n(i)} ,$$

and in which we use the shorthand notation

$$\tilde{\pi}^{k,k+1}_n (P_i, K_{ij}) := \pi^{k,k+1}_n (P_{\psi^{k,k+1}_n(i)}, K_{\psi^{k,k+1}_n(i)});$$

the local data $F_n$ undergo a compensating gauge transformation

$$F_n \mapsto F_n + \Delta_{T_n} w$$

under a $G$-twisted gauge transformation (2.6), ensuring that the defining identity (2.7) is preserved.

The reader is urged to consult Ref. [Bry93] for a thorough introduction to the cohomological constructs used in the above definition. Here, we merely point out an important consequence of the cohomological description of gerbes and 1- and 2-isomorphisms, which provides us with a natural classification scheme of string backgrounds.

**Proposition 2.3.** The set of 1-isomorphism classes of gerbes with a given curvature over a manifold $\mathcal{M}$ is a torsor under a natural action of the sheaf-cohomology group $H^2(\mathcal{M}, \mathbb{U}(1))$.

**Proposition 2.4.** The set of 2-isomorphism classes of 1-isomorphisms between two given gerbes over a manifold $\mathcal{M}$ is a torsor under a natural action of the sheaf-cohomology group $H^1(\mathcal{M}, \mathbb{U}(1))$.

\[ \checkmark \]
**Proposition 2.5.** The set of inequivalent 2-isomorphisms between two given 1-isomorphisms of gerbes over a manifold $\mathcal{M}$ with $|\pi_0(\mathcal{M})|$ connected components is a torsor under a natural action of the sheaf-cohomology group $H^0(\mathcal{M}, U(1)) \cong U(1)^{|\pi_0(\mathcal{M})|}$.

All three statements are simple corollaries of the relation between the Deligne hypercohomology and sheaf cohomology, taken in conjunction with the contents of Definition 2.2, cf., e.g., [Bry93, GR02, Gom05].

Another auxiliary concept of use in the sequel is introduced in the following

**Definition 2.6.** Let $\Sigma$ be a closed oriented two-dimensional manifold with an intrinsic metric $\gamma$ of a lorentzian signature $(-, +)$, termed the world-sheet and split into patches $\wp$, forming the patch set $\wp_\Sigma$, by an embedded oriented graph $\Gamma$, to be termed the defect quiver. The graph is composed of a collection of oriented lines $\ell$, termed defect lines, forming the edge set $\mathcal{E}_\Gamma$ of $\Gamma$ and intersecting at a number of points $\upsilon$, termed defect junctions and forming the vertex set $\wp_\Gamma$ of $\Gamma$. Furthermore, let $\mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$ be a string background as in Definition 2.1. A network-field configuration $(X|\Gamma)$ in string background $\mathfrak{B}$ on world-sheet $(\Sigma, \gamma)$ with defect quiver $\Gamma$ is a pair composed of the defect quiver $\Gamma$ embedded in the world-sheet $\Sigma$, together with a map $X : \Sigma \to M \cup Q \cup T$ such that

- $X$ restricts to a once differentiable map $\Sigma \setminus \Gamma \to M$, a once differentiable map $\Gamma \setminus \wp_\Gamma \to Q$, and it sends $\wp_\Gamma \to T$ in such a manner that a defect junction $\upsilon$ of valence $n_{\upsilon}$ is mapped to $T_{n_{\upsilon}}$;
- for every $p \in \Gamma \setminus \wp_\Gamma$ and $U \subset \Sigma$ a small neighbourhood of $p$ split into subsets $U_\alpha$, $\alpha \in \{1, 2\}$ by $\Gamma$ so that the vector $\vec{n}$ normal to $\Gamma$ at $p$ and pointing towards $U_2$ together with the vector $\vec{T}$ tangent to $\Gamma$ at $p$ and determining its orientation define a right-handed basis $(\vec{n}, \vec{T})$ of $T_p\Sigma$ as in Figure 3. the map $X|_\alpha$ admits a differentiable extension $X|_\alpha : \overline{U}_\alpha \to M$ to the closure of $U_\alpha$, with the property $X|_\alpha (\overline{p}) = \iota_\alpha \circ X(p)$;
- the Defect Gluing Condition (DGC)

$$DGC_{\mathfrak{B}}(\psi|_1, \psi|_2, X) \equiv p|_1 \circ \iota_1 \ast - p|_2 \circ \iota_2 \ast - X|_\lambda \mathcal{J}(X) = 0, \quad \psi|_{\alpha} = (X|_{\alpha}, p|_{\alpha}),$$

(2.8)

---

Note that – unlike Ref. [RS09b] – we are dealing with the lorentzian version of the world-sheet theory here as we intend to discuss its canonical structure. It is a classic result in topology, cf., e.g., Ref. [Ste54, Thm. 40.10], that a global lorentzian structure can exist on $\Sigma$ if $\Sigma$ is non-compact or $\Sigma$ is homeomorphic with a torus or a Klein bottle. In what follows, we shall mainly be interested in $\Sigma \cong \mathbb{R} \times S^1$ (an infinite cylinder) with a view to a canonical interpretation of defects. In the case of the trinion (also known as “pair-of-pants”) geometry representing the basic splitting-joining interaction of strings, we shall disregard the signature problem, with the implicit understanding that a proper treatment of the world-sheet metric may require passing to the euclidean version of the theory, accompanied by the complexification of the field space. These manipulations are not going to invalidate our conclusions.
is satisfied at each \( p \in \ell \in \mathcal{C}_T \) for a vector \( \ell \) tangent to \( \Gamma \) at \( p \) and determining its orientation, and for

\[
p_{|\sigma} = g(X_{|\sigma})(X_{|\sigma} \cdot \vec{n}, \cdot)
\]

with a vector \( \vec{n} = \gamma^{-1}(\ell \cdot \text{Vol}(\Sigma, \gamma), \cdot) \) written in terms of the metric volume form \( \text{Vol}(\Sigma, \gamma) \) on \( \Sigma \) and defining \( X_{|\sigma} \vec{n} \) in terms of a (one-sided) derivative:

- for \( j \in \mathcal{G}_\Gamma \) an \( n_j \)-valent defect junction and \( \ell_{k,k+1} \) a defect line converging at \( j \), the map \( X|_{\ell_{k,k+1} \cap \mathcal{G}_\Gamma} \) admits a differentiable extension \( X_{k,k+1} : \ell_{k,k+1} \to Q \) such that \( X_{k,k+1}(j) = \pi_{n_j}^{k,k+1} \circ X(j) \);
- for \((j, \ell_{k,k+1})\) as above, the orientation map takes the value \( \varepsilon_{n_j}^{k,k+1}(X(j)) = +1 \) if \( \ell_{k,k+1} \) is oriented towards \( j \) (an incoming defect line), and the opposite value \( \varepsilon_{n_j}^{k,k+1}(X(j)) = -1 \) otherwise.

\[\checkmark\]

We may now give a precise description of the main object of our study.

**Definition 2.7.** Let \((X|\Gamma)\) be a network-field configuration in string background \( \mathcal{B} \) with field space \( M \cup Q \cup T \) on world-sheet \((\Sigma, \gamma)\) with defect quiver \( \Gamma \), and choose a local presentation of \( \mathcal{B} \) with respect to good open covers \( \mathcal{O}_{\mathcal{M}} \). \( \mathcal{M} \in \{M, Q, T\} \). Furthermore, let \( \Delta(\Sigma) \) be a **triangulation of \( \Sigma \)** subordinate to \( \mathcal{O}_{\mathcal{M}} \). \( \mathcal{M} \in \{M, Q, T\} \) with respect to \((X|\Gamma)\) in the following sense:

- \( \Delta(\Sigma) \) induces a triangulation \( \Delta(\Sigma) \subset \Delta(\Sigma) \) of the defect quiver in such a manner that each defect line \( \ell \in \mathcal{C}_\Gamma \) is covered by the edges \( e \in \Delta(\Gamma) \) and \( \mathcal{G}_\Gamma \subset \Delta(\Gamma) \);
- for each plaquette \( p \in \Delta(\Sigma) \), there exists a \( \check{C}ech \) index \( i_p \in \mathcal{I}_M \) of \( \mathcal{O}_M \) such that \( X(p) \in \mathcal{O}_{i_p} \), which we fix;
- for each defect edge \( e \in \Delta(\Gamma) \), there exists a \( \check{C}ech \) index \( i_e \in \mathcal{I}_Q \) of \( \mathcal{O}_Q \) such that \( X(e) \in \mathcal{O}_{i_e} \), which we fix;
- for each defect vertex \( j \in \Gamma \) of valence \( n_j \), we fix a \( \check{C}ech \) index \( i_j \in \mathcal{I}_{T_{n_j}} \) of \( T_{n_j} \) such that \( X(j) \in \mathcal{T}_{i_j} \).

The **two-dimensional** non-linear \( \sigma \)-model for network-field configurations \((X|\Gamma)\) in string background \( \mathcal{B} \) on world-sheet \((\Sigma, \gamma)\) with defect quiver \( \Gamma \) is a theory of continuously differentiable maps \( X : \Sigma \to M \cup Q \cup T \), determined by the principle of least action applied to the action functional

\[
S_{\sigma}(X|\Gamma; \gamma) = -\frac{1}{2} \int_{\Sigma} g_X(dX^{\ast} + \gamma \cdot dX) + S_{\text{top}}[(X|\Gamma)],
\]

in which

- \( dX \equiv \partial_\sigma X^\mu \sigma^\alpha \otimes \partial_\mu \), in local coordinates \( \{\sigma^\alpha\}_{\alpha \in \{1,2\}} \) on \( \Sigma \) and \( \{X^\mu\}_{\mu \in \text{dim}M} \) on \( M \), and the target-space metric is assumed to act on the second factor of the tensor product;
- \( + \gamma \cdot \) is the Hodge operator on \( \Gamma(\wedge^\ast \mathcal{T}^{\ast} \Sigma) \) determined by the world-sheet metric \( \gamma \);
- the topological term

\[
S_{\text{top}}[(X|\Gamma)] = -i \log \text{Hol}_{\mathcal{B}}(X|\Gamma)
\]

is given by the generalised surface holonomy \( \text{Hol}_{\mathcal{B}}(X|\Gamma) \) for the network-field configuration \((X|\Gamma)\), which, in a triangulation of \( \Sigma \) subordinate to the good open covers \( \mathcal{O}_{\mathcal{M}} \), \( \mathcal{M} \in \{M, Q, T\} \) with respect to \((X|\Gamma)\) (consisting of plaquettes \( p \), edges \( e \) and vertices \( v \)) and a local presentation of \( \mathcal{B} \) as described in Definition 2.2, takes the form

\[
-i \log \text{Hol}_{\mathcal{B}}(X|\Gamma) = \sum_{p \in \mathcal{G}_\Sigma} \left( \int_p X^p B^p + \sum_{e \in p} \left( \int_e X^e A^e_{p|e} - i \sum_{v \in e} \log X^e g^e_{p|e,v} (v) \right) \right)
+ \sum_{e \in \Delta(\mathcal{G}_\Gamma)} \left( \int_e X^e P_e - i \sum_{v \in e} \log X^e K^e_{v} (v) \right)
-i \sum_{j \in \mathcal{G}_\Gamma} \log X^j f_{n_j,j}(j).
\]

\[\checkmark\]

An extensive discussion of the various components of the string background \( \mathcal{B} \) was presented, alongside a derivation of the local formula for \( S_{\text{top}}[(X|\Gamma)] \), in Ref. [RS09b], to which we refer the reader for details. Here, we merely point out the statement of consistency:
Proposition 2.8. [RS09b] Sec. 2.7] The topological term $S_{\text{top}}(X|\Gamma)$ of the action functional of the non-linear $\sigma$-model for network-field configurations $(X|\Gamma)$ in string background $\mathcal{B}$ on world-sheet $(\Sigma, \gamma)$ with defect quiver $\Gamma$ is independent of the choice $\triangle(\Sigma)$ of the triangulation and invariant under gauge transformations of the local presentation of the string background $\mathcal{B}$, as described in Definition 2.3.

And the statement of symmetry:

Theorem 2.9. [RS09b] Sec. 2.9] The non-linear $\sigma$-model for network-field configurations $(X|\Gamma)$ in string background $\mathcal{B}$ on world-sheet $(\Sigma, \gamma)$ with defect quiver $\Gamma$ of Definition 2.7 is invariant with respect to arbitrary (gauge) transformations

$$X \mapsto X \circ D, \quad \gamma \mapsto D^* \gamma, \quad D \in \mathcal{Diff}_+^+ (\Sigma),$$

$$\gamma \mapsto e^{2w \cdot \gamma}, \quad e^{2w} \in \text{Weyl}(\gamma)$$

from the semidirect product $\mathcal{Diff}_+^+ (\Sigma) \ltimes \text{Weyl}(\gamma)$ of the group $\mathcal{Diff}_+^+ (\Sigma)$ of those (orientation-preserving) diffeomorphisms of $\Sigma$ that preserve $\Gamma$, with the group $\text{Weyl}(\gamma)$ of Weyl rescalings of the metric $\gamma$.

Remark 2.10. It deserves to be emphasised that a generic string background $\mathcal{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$ does not lead to a consistent quantum field theory with a non-anomalous realisation of the gauge symmetries of the classical action functional. Such a realisation prerequires that the Weyl anomaly of the theory vanish, which – in turn – imposes constraints on the various components of a consistent string background, cf., e.g., Refs. [Fri85, BCZ85, Gaw].

Prior to finishing this introductory section by presenting a couple of examples of bi-branes, let us add the following preparatory

Remark 2.11. The world-sheet $\Sigma$ with an embedded defect quiver $\Gamma$ can be understood as a model of multiple phases, in coexistence and undergoing transitions, of the underlying CFT, in which the particular phases are represented by the patches $\rho \in \mathcal{P}_\Sigma$. From this point of view, it is natural to set up the canonical description of the theory in each patch independently, and only upon completing the task, examine the relations between the ensuing phase-restricted state spaces imposed by the defects that separate the phases. In this picture, the defect lines $\ell \in \mathcal{E}_\Gamma$ appear as space-like domain walls of the two-dimensional field theory carrying the geometric data that effect the transition. This is the basic setting in which we shall carry out our analysis in the next section, phrasing our considerations in terms of Cauchy data of the dynamical evolution, to be localised on a space-like Cauchy contour $\mathcal{E}$. In principle, we might attach the phase (patch) label to the dynamical objects defined over a given patch but we choose, instead, to shift the dependence on the phase of the underlying CFT to the definition of the embedding map $X$, along the lines of Ref. [RS09b], which enables us to develop a
unified treatment of all admissible phases at the same time. The modular invariance of the quantised (euclidean version of the) CFT leads us to consider a dual of the picture described in which time-like and space-like contours are swapped. Having set out with space-like defect lines, we thus end up with time-like ones, and the obvious question arises as to the nature of the canonical description of this dual CFT. Motivated by the distinguished example of boundary defects and the associated $G$-branes of string theory, we should be inclined to formulate our description in terms of Cauchy data localised on open segments stretched transversally between defect lines, each contained in a single patch of the world-sheet. The problem with this description, masked by the triviality of the patch data for the patch ‘behind’ the defect line in the boundary case, is that consistency of its formulation in a single patch prerequires the knowledge of the field configuration across the defect lines to which the open segment is attached, cf. Eq. (2.8), written in terms of the standard left-invariant Maurer–Cartan 1-form $\theta_L(g) = g^{-1} dg \in \Gamma(T^*G) \otimes g$ on $G$;\footnote{Meinrenken’s construction of the basic gerbe for a general (compact simple 1-connected) Lie group was preceded by that of Ref. \cite{Gaw88} for $SU(2)$ and that of Refs. \cite{Cha08, HIT01} which works for $SU(N)$. The non-simply connected case was worked out in Refs. \cite{CHR02, CHR03}.}

Given an arbitrary target $M = (M, g, G)$, there always exists the trivial $G$-bi-brane $B_{\text{triv}} = (M, \text{id}_M, \text{id}_M, 0, \text{id}_G)$, and the attendant trivial $(G, B_{\text{triv}})$-inter-bi-brane $J_{\text{triv}} = \left( \bigoplus_{n \geq 3} \text{M, } \left( \varepsilon_n^{k+1}, \text{id}_M \mid k \in \mathbb{N} \right), \text{id}_G \mid n \in \mathbb{N}_{\geq 3} \right)$.

Example 2.12. The trivial (inter-)bi-brane.

Example 2.13. The maximally symmetric WZW defects.

The target. We consider here a distinguished class of defects in the Wess–Zumino–Witten (WZW) $\sigma$-model of Ref. \cite{Wir84}, with target $M_4 = (G, g_k, G_k)$, where

(T.i) $G$ is the group manifold of an arbitrary compact simple 1-connected Lie group, with Lie algebra $g$ and a trace $\text{tr}_g$ on $g$ normalised such that the equality

$$\text{tr}_g(t_A t_B) = -\frac{1}{2} \delta_{AB}$$

holds for generators $t_A$ of $g$, the latter satisfying the defining commutation relations

$$[t_A, t_B] = f_{ABC} t_C,$$

with $f_{ABC}$ the structure constants of $g$ – this prescription yields the standard matrix trace for, e.g., $G = SU(2)$;

(T.ii) $g_k$ is the Cartesian–Killing metric

$$g_k = \frac{k}{4 \pi} \text{tr}_g \left( \theta_L \otimes \theta_L \right), \quad k \in \mathbb{Z}_{>0},$$

written in terms of the standard left-invariant Maurer–Cartan 1-form $\theta_L(g) = g^{-1} dg \in \Gamma(T^*G) \otimes g$ on $G$;

(T.iii) $G_k = G^{[k]}$ is the $k$-th power of the basic gerbe $G_1$ of Ref. \cite{Mei03}, with curvature equal to the Cartan 3-form

$$H_k = \frac{k}{12 \pi} \text{tr}_g (\theta_L \wedge \theta_L \wedge \theta_L)$$

whose cohomology class is the generator of $H^3(G) \cong \mathbb{Z}$.
The action functional for a defect-free world-sheet $\Sigma$ is given by

$$S_{WZW,k}[g] = \frac{k}{8\pi} \int_{\Sigma} \text{tr}_g(\theta_L(g) \wedge \ast_R H_L(g)) - i \log \text{Hol}_{\sigma}(g),$$

and the constant $k \in \mathbb{Z}_{>0}$ is called the level of the WZW model. The field equations of the model have the compact form

$$\left(\eta^{ab} + \epsilon^{ab}\right) \partial_a \left( g^{-1} \partial_b g \right) = 0,$$

which can further be rewritten, using the light-cone coordinates $\sigma^\pm = \sigma^2 \pm \sigma^1$ and the attendant derivatives $\partial_\pm = \frac{\partial}{\partial \sigma^\pm}$, as

$$\partial_\pm \left( g^{-1} \partial_\pm g \right) = 0.$$ 

A general solution to this equation factorises as

$$g(\sigma) = g_L(\sigma^+) \cdot g_R(\sigma^-)^{-1},$$

for independent $G$-valued maps $g_L$ and $g_R$ on $\mathbb{R}$ with equal monodromies (when viewed as maps on $\mathbb{R}/2\pi \mathbb{Z}$), cf. Ref. [CTTNB04]. In addition to the standard conformal symmetry of Theorem 2.9 realised, on the infinitesimal level, by two (chiral) copies of the Lie algebra of diffeomorphisms of the circle, the bulk theory enjoys a level-$k$ Kač–Moody symmetry, realised on fields by the chiral currents

$$J_L(\sigma) = \frac{k}{2\pi} g(\sigma) \partial_\sigma g(\sigma)^{-1}, \quad J_R(\sigma) = \frac{k}{2\pi} g(\sigma)^{-1} \partial_\sigma g(\sigma)$$

that generate the centrally extended current algebra $\widehat{\mathfrak{g}}_k \oplus \widehat{\mathfrak{g}}_R$. Note that the currents become functions of the respective light-cone coordinates $\sigma^\pm$ upon using the field equations of the $\sigma$-model. Their (infinitesimal) action integrates to

$$g(\sigma) \rightarrow h_L(\sigma^+) \cdot g(\sigma) \cdot h_R(\sigma^-)^{-1},$$

with independent chiral transformation maps $h_L, h_R \in LG$ from the loop group $LG \equiv C^\infty(S^1, G)$.

**The boundary $G_k$-bi-brane.** We shall first consider defects describing boundary maximally symmetric $G_k$-bi-branes of the WZW model, or maximally symmetric $G_k$-branes for short. To this end, we focus on (the vicinity of) a connected component $\ell$ of the edge set $\mathcal{E}_\Gamma$ of $\Gamma$, which we take to be an oriented circle embedded in $\Sigma$ at $t = 0$ in a local coordinate system $(t, \varphi)$ described on p. 11. As argued in Ref. [RS09b, p. 12], the corresponding string background $\mathfrak{B}^0_k = (\mathcal{M}_k, \mathcal{B}_k, \nu)$ consists of

(TD) the target $\mathcal{M}_k = \mathcal{M}_k \cup \{\bullet\}$ given by the disjoint union of the bulk target $\mathcal{M}_k = (G, g_k, \mathcal{G}_k)$ and a single point $\{\bullet\}$ with no structure over it;

(D) the $G_k$-bi-brane $\mathcal{B}_k = (Q_k, \iota_{Q_k}, \mathcal{G}_k, \mathcal{G}_0)$, with

(D.i) the world-volume

$$Q_k^0 = \bigsqcup_{x \in P^0_k(g)} \mathcal{G}_\lambda, \quad \mathcal{G}_\lambda = \left\{ \text{Ad}_x e_\lambda \mid x \in G \right\}$$

given by the disjoint sum of the conjugacy classes of Cartan elements $e_\lambda = e^{2\pi i \lambda} \in G$ labelled by weights $\lambda$ from the fundamental affine Weyl alcove at level $k$,

$$P_k^0(g) = k \mathcal{A}_W(g) \cap P(g),$$

the latter being the intersection of the weight lattice $P(g)$ of $\mathfrak{g}$ with its $k$-inflated Weyl alcove $k \mathcal{A}_W(g)$, i.e. a subset

$$\mathcal{A}_W(g) = \left\{ \lambda \in \mathfrak{t} \mid \text{tr}_g(\lambda \cdot \theta) \leq 1 \wedge \text{tr}_g(\lambda \cdot \alpha_i) \geq 0, \ i \in \overline{1, \text{rank} \mathfrak{g}} \right\}$$

of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, defined in terms of the simple roots $\alpha_i$, $i \in \overline{1, \text{rank} \mathfrak{g}}$ of $\mathfrak{g}$ and its longest root $\theta$;

(D.ii) the $G_k$-bi-brane maps, defined component-wise by the embedding $\iota_{Q_k^0} \iota_{\mathcal{E}_\lambda} \equiv \iota_{\mathcal{E}_\lambda} : \mathcal{G}_\lambda \hookrightarrow G$ of the conjugacy class $\mathcal{G}_\lambda$ in the group manifold, and the constant map $\bullet : Q_k^0 \rightarrow \{\bullet\}$;

(D.iii) the curvature, also defined component-wise as

$$\omega^0_{k,\lambda} = \frac{k}{8\pi} \iota^*_\lambda \text{tr}_g \left( \theta_L \wedge \frac{\text{id}_x + \text{Ad}_x}{\text{id}_x - \text{Ad}_x} \theta_L \right) = \omega^0_{k,\lambda}.$$
(D.iv) the $G_k$-bi-brane 1-isomorphism given on each component $\mathcal{C}_\lambda$ of $Q_k^\partial$ by the corresponding trivialisation

$$\Phi_k^\partial|_{\mathcal{C}_\lambda} =: \Phi_k^\partial : \mathcal{G}_k \xrightarrow{\sim} I_{\omega_k^\partial}$$

of the restricted gerbe $\mathcal{G}_k$.

The DGC obtained by plugging the above data into Eq. (2.8) is the familiar statement

$$(J_L - J_R)|_\ell = 0$$

of maximal symmetry of the boundary defect, the symmetry being determined by a single copy of the level-$k$ Kač–Moody algebra $\mathfrak{g}_k$ embedded diagonally in the current-symmetry algebra $\mathfrak{g}^L_k \oplus \mathfrak{g}^R_k$ of the bulk theory.

It is vital to note that – as was argued for $G = SU(N)$ in Ref. [GR02 Sec. 8.1] and for an arbitrary compact simple 1-connected Lie group $G$ in Ref. [Gaw03 Sec. 5.1] – the stable isomorphisms $\Phi_k^\partial$ exist for $\lambda \in P^\pm_k(g)$ exclusively, and so they single out a subset of conjugacy classes in $G$ which coincides with the set of world-volumes of stable (untwisted) maximally symmetric D-branes of the WZW model at level $k$, cf., e.g., Ref. [FFS00b], which – in turn – are in a one-to-one correspondence with the (untwisted) maximally symmetric boundary states of the associated BCFT (ib.).

**The non-boundary $G_k$-bi-brane.** The next type of maximally symmetric WZW defects that we want to discuss are those implementing jumps by elements of the target Lie group in the sense that the limiting values attained at a point $p$ on the defect circle $\ell$ by the one-sided local extensions $(g_{1\ell}, g_{2\ell})$ of the embedding map $g : \Sigma \setminus \Gamma \to G$ to the defect line $\ell$, described in Definition 2.6, are two generically distinct points in the group manifold. A special class of such defects – the central-jump defects at which $g_{2\ell} = z \cdot g_{1\ell}$ for $z$ from the centre $Z(G)$ of $G$ – were studied in Ref. [RS09a]. The more general jump defects, with – as above – the jump given by $g_{1\ell} \cdot g_{2\ell} \in G$, were first studied in Ref. [FSW08], where the notion of a bi-brane was introduced. They shall be expanded upon in Ref. [RS]. In the conventions of the latter paper, the string background $\mathfrak{B}_k = (\mathcal{M}_k, \mathcal{B}_k, \cdot)$ for these jump defects consists of

(TB) the target $\mathcal{M}_k = (G, g_k, \mathcal{G}_k)$ of the defect-free model;

(B) the $G_k$-bi-brane $\mathcal{B}_k = (Q_k, d_1, d_0, \omega_k, \Phi_k)$, with

(B.i) the world-volume

$$Q_k = G \times Q_k^\partial;$$

(B.ii) the $G_k$-bi-brane maps, defined explicitly as

$$d_0(g, h) = g \cdot h, \quad d_1(g, h) = g;$$

(B.iii) the curvature, defined component-wise as

$$\omega_k|_{G \times \mathcal{C}_\lambda} = -pr_2^\omega \omega_k^\partial + \rho_k =: \omega_k^\lambda, \quad \rho_k = \frac{k}{4\pi} tr_0 (pr_1^* \theta_L \wedge pr_2^* \theta_R)$$

in terms of the canonical projections $pr_\alpha : G \times G \to G$, $\alpha \in \{1, 2\}$;

(B.iv) the $G_k$-bi-brane 1-isomorphism with a component-wise definition

$$\Phi_k|_{G \times \mathcal{C}_\lambda} =: \Phi_k, \quad \mathcal{G}_k = \mathfrak{g}_k \oplus I_{pr_2^\omega \omega_k^\partial} \oplus I_{-pr_1^\omega \omega_k^\partial} \xrightarrow{id_{id_{\mathfrak{g}^L_k}} \Phi_k \otimes id_{-pr_2^\omega \omega_k^\partial}} pr_1^* \mathfrak{g}^L_k \otimes pr_2^* \mathfrak{g}^L_k \otimes I_{-pr_2^\omega \omega_k^\partial}$$

by the corresponding 1-isomorphism

$$\mathcal{M}_k : pr_1^* \mathfrak{g}^L_k \otimes pr_2^* \mathfrak{g}^R_k \xrightarrow{\sim} m^* \mathfrak{g}_k \otimes I_{\rho_k}, \quad m : G \times G \to G : (g, h) \mapsto g \cdot h$$

of the multiplicative structure on $\mathfrak{g}_k$, as introduced in Ref. [CJM+05] and developed in Refs. [Wall09 GW09].

The DGC is, once more, the statement of maximal symmetry,

$$(J_L^1 - J_L^2)|_\ell = 0 = (J_R^1 - J_R^2)|_\ell,$$

the symmetry being determined by the full bi-chiral level-$k$ Kač–Moody algebra $\mathfrak{g}^L_k \oplus \mathfrak{g}^R_k$ of the bulk theory. The chiral currents $J_L^1, J_R^1$ are defined as previously but using the respective one-sided local extensions $g_{1\ell}$ of the patch component of the σ-model field. Owing to the form of the energy-momentum
tensor, given by a sum of terms quadratic in the chiral currents, the continuity of the latter across the defect line ensures the topologicality of the defect associated to \( B_k \), as defined in Ref. [RS09b].

**Remark 2.14.** It ought to be emphasised that the existence and uniqueness (up to a 2-isomorphism) of the 1-isomorphism \( M_k \) on a compact simple 1-connected Lie group \( G \) implies, via a simple topological argument (cf. Ref. [RS]), that the maximally symmetric WZW \( G_k \)-bi-brane \( B_k \) has precisely as many connected components (labelled by weights \( \lambda \in P^k_+ (g) \)) as its boundary analogon \( B_k^0 \).

* * * * *

**Remark 2.15.** The world-volume \( Q_k \) of \( B_k \) is \( G \times G \)-equivariantly isomorphic to the disjoint union of the bi-conjugacy classes

\[
(2.13) \quad \mathcal{B}_{(t, e)} = \{ (x \cdot t \cdot y^{-1}, x \cdot y^{-1}) \mid x, y \in G \},
\]

of Ref. [FSW08] for the pairs of group elements \((t, e) \in \mathcal{C} \times \{ e \} \subset G \times G\).

* * * * *

3. The canonical structure and pre-quantisation of the \( \sigma \)-model

Having written out the \( \sigma \)-model action functional of interest in Eq. (2.9), we may now analyse the symplectic structure on its state space, a task best completed within the framework of covariant classical field theory (or first-order formalism) of Refs. [Gaw72, Kij73, Kij74, KS76, Szc76, KT79], cf. also Ref. [GM] for an exhaustive exposition of the modern approach and a comprehensive list of references. This formalism enables us to interpret the (inter-)bi-brane data in terms of the canonical structure of the underlying two-dimensional field theory, whereupon a clear-cut field-theoretic statement can be made in regard to the relation between defects and dualities of the \( \sigma \)-model, in the spirit of, e.g., Ref. [FFRS07]. We begin by briefly reviewing those elements of the general formalism that are instrumental in the subsequent analysis of the physical system of interest.

3.1. Elements of the covariant formalism. Let us first introduce some basic notions.

**Definition 3.1.** Let \( \pi : \mathcal{F} \rightarrow \mathcal{M} \) be a fibre bundle over a (pseudo-)riemannian base \( (\mathcal{M}, g) \) of dimension \( \text{dim} \mathcal{M} = d \), and let \( J^1 \mathcal{F} \rightarrow \mathcal{M} \) be the first-jet bundle of \( \mathcal{F} \), with local coordinates \((x^\mu, \phi^A, \xi^B_\nu) \), \( \mu, \nu \in \Gamma, d, A, B \in \Gamma, N \), where \( N \) is the dimension of the typical fibre of \( \mathcal{F} \). Consider an \( \mathcal{F} \)-field theory \( \mathcal{F} \) on \( \mathcal{M} \), i.e. a theory of continuously differentiable sections \((\phi^A)^{A \in \Gamma, N} \) of the bundle \( \mathcal{F} \) (termed the covariant configuration bundle of \( \mathcal{F} \)-field theory \( \mathcal{F} \) in this context), determined by the principle of least action applied to the action functional

\[
(3.1) \quad S_{\mathcal{F}} [\phi^A] = \int_\mathcal{M} \mathcal{L}_{\mathcal{F}} (x^\mu, \phi^A, \xi^B_\nu | \xi^B_\nu = \partial_{\nu} \phi^B, d^d x),
\]

in which \( d^d x = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_d \in \Gamma (d^d \mathcal{T}^* \mathcal{M}) \) is the volume form in local coordinates \( x^\mu \) on \( \mathcal{M} \), \( \partial_{\mu} = \frac{\partial}{\partial x^\mu} \) are the associated partial derivatives, and the map \( \mathcal{L}_{\mathcal{F}} \) on \( J^1 \mathcal{F} \) with values in the space of scalar densities (of weight 0) on \( \mathcal{M} \) is termed the lagrangean (density) of \( \mathcal{F} \)-field theory \( \mathcal{F} \) and considered regular iff the matrix functional \( \delta^2 \mathcal{L}_{\mathcal{F}} / \delta \xi^B_\nu \delta \xi^B_{\nu'} \) (with a multi-index \( \nu' \)) is invertible on sections of \( \mathcal{F} \) that extremise \( S_{\mathcal{F}} \). The Cartan form \( \Theta_{\mathcal{F}} \) of \( \mathcal{F} \)-field theory \( \mathcal{F} \) is the \( d \)-form on \( J^1 \mathcal{F} \) given by the formula

\[
\Theta_{\mathcal{F}} (x^\mu, \phi^A, \xi^B_\nu) = \left( \mathcal{L} - \zeta^C_\lambda \frac{\delta \mathcal{L}_{\mathcal{F}}}{\delta \xi^C} \right) (x^\mu, \phi^A, \xi^B_\nu) d^d x + \frac{\delta \mathcal{L}_{\mathcal{F}}}{\delta \xi^B_\nu} (x^\mu, \phi^A, \xi^B_\nu) \delta \phi^B \wedge (\partial_\lambda + d^d x).
\]

Above, and in what follows, we use the symbol \( \delta \) to distinguish differentation in the direction of the fibre of \( J^1 \mathcal{F} \), termed \( \mathcal{F} \)-vertical, from that along the base \( \mathcal{M} \), e.g., \( \delta \phi^A \) and \( \frac{\delta \mathcal{L}_{\mathcal{F}}}{\delta \xi^B_\nu} \) vs. \( dx^\mu \) and \( \frac{\partial}{\partial x^\nu} \).}

The significance of the Cartan form rests on the following

**Proposition 3.2.** [Gaw91] Let \( \mathcal{F} \) be an \( \mathcal{F} \)-field theory on a (pseudo-)riemannian manifold \( (\mathcal{M}, g) \), determined by a regular lagrangean density \( \mathcal{L}_{\mathcal{F}} \), with a covariant configuration bundle \( \pi : \mathcal{F} \rightarrow \mathcal{M} \), the attendant first-jet bundle \( J^1 \mathcal{F} \rightarrow \mathcal{M} \), and the Cartan form \( \Theta_{\mathcal{F}} \) on the latter. Then,

1) the principle of least action applied to the functional

\[
S_{\Theta_{\mathcal{F}}} [\Psi] = \int_\mathcal{M} \Psi^* \Theta_{\mathcal{F}}, \quad \Psi \in \Gamma (J^1 \mathcal{F})
\]
yields, as the Euler–Lagrange equations, the field equations of \( \mathcal{F} \), that is the Euler–Lagrange equations of the action functional (3.1), and – for \( \mathcal{M} \) with a non-empty boundary – also the boundary conditions of \( \mathcal{F} \); the equations follow from the condition

\[
0 = \mathcal{V} \cup \delta S_{\mathcal{F}}[\Psi_{\text{cl}}],
\]

to be satisfied by the extremal (or classical) sections \( \Psi_{\text{cl}} \) of \( J^1 \mathcal{F} \) for an arbitrary \( \mathcal{F} \)-vertical vector field \( \mathcal{V} \), i.e. for \( \mathcal{V} \in \Gamma(TJ^1 \mathcal{F}) \cap \ker \pi_{\mathcal{F}} = \Gamma((TJ^1 \mathcal{F})^\perp) \) (in particular, the boundary conditions of \( \mathcal{F} \) are implied by the vanishing of the boundary term on the right-hand side of Eq. (3.2));

i) \( \Theta_{\mathcal{F}} \) canonically determines a closed 2-form \( \Omega_{\mathcal{F}} \) on the space \( \mathcal{P}_{\mathcal{F}} \) of extremal sections of \( J^1 \mathcal{F} \).

A complete proof can be extracted from the original paper. However, the proof being constructive in nature, it appears useful to give at least an idea thereof – after Ref. [Gaw07] – to prepare the reader for the subsequent considerations.

Ad i) The key point is to note that the Euler–Lagrange equations obtained for the distinguished choice \( \mathcal{V} = \delta \frac{\delta S_{\mathcal{F}}}{\delta \xi^A} \) of a \( \mathcal{F} \)-vertical vector field on \( J^1 \mathcal{F} \) read

\[
\frac{\delta^2 \mathcal{L}}{\delta \xi^B \delta \xi^A} (\mathcal{V}^\mu, \phi^A, \xi^B) (\xi^D_D - \partial_A \phi^D) = 0,
\]

and so, assuming regularity of \( \mathcal{L}_{\mathcal{F}} \), we conclude that the equality

\[
\xi^D_A = \partial_A \phi^A
\]

holds true on \( \mathcal{P}_{\mathcal{F}} \). The Euler–Lagrange equations of the action functional (3.1) then follow straightforwardly.

Ad ii) Assume \( \partial \mathcal{M} = \emptyset \). Pick up a pair \( C_\alpha, \alpha \in \{1,2\} \) of Cauchy hypersurfaces in \( \mathcal{M} \) and cut out a region \( \mathcal{M}_{1,2} \subset \mathcal{M} \) such that \( \partial \mathcal{M}_{1,2} = C_1 \cup (-C_2) \), where the minus in front of \( C_2 \) represents the reversal of the orientation on \( C_2 \) induced from that on \( \mathcal{M} \), and such that the two hypersurfaces can be homotopically transformed into one another across \( \mathcal{M}_{1,2} \). Write

\[
S_{1,2}[\Psi_{\text{cl}}] := \int_{\mathcal{M}_{1,2}} (\Psi_{\text{cl}}|_{\mathcal{M}_{1,2}})^* \Theta_{\mathcal{F}}.
\]

Using Eq. (3.2), we find

\[
\delta S_{1,2}[\Psi_{\text{cl}}] = \int_{C_1} (\Psi_{\text{cl}}|_{C_1})^* \Theta_{\mathcal{F}} - \int_{C_2} (\Psi_{\text{cl}}|_{C_2})^* \Theta_{\mathcal{F}},
\]

and therefore conclude that the 2-form

\[
\Omega_{\mathcal{F}}[\Psi_{\text{cl}}] := \int_{\mathcal{F}} (\Psi_{\text{cl}}|_{\mathcal{F}})^* \delta \Theta_{\mathcal{F}},
\]

written for an arbitrary Cauchy hypersurface \( \mathcal{C} \), is manifestly closed (and independent of the choice of \( \mathcal{C} \)) and hence defines a presymplectic form on \( \mathcal{P}_{\mathcal{F}} \). The proof proceeds analogously for \( \partial \mathcal{M} \neq \emptyset \), and the analysis below (for \( \mathcal{M} = \Sigma \) with domain walls) is readily seen to cover that case. The sole difference is the appearance of a more complicated expression

\[
\delta S_{1,2}[\Psi_{\text{cl}}] = \Xi_{C_1}[\Psi_{\text{cl}}|_{C_1}] - \Xi_{C_2}[\Psi_{\text{cl}}|_{C_1}],
\]

with the two functional 1-form contributions once more localised on the two Cauchy hypersurfaces.

Remark 3.3. Vector fields that span the kernel of the presymplectic form \( \Omega_{\mathcal{F}} \) are identified with generators of infinitesimal gauge transformations of \( \mathcal{F} \). Upon performing the standard symplectic reduction on the state space \( \mathcal{P}_{\mathcal{F}} \) with respect to the characteristic distribution \( K_{\mathcal{F}} \) of \( \Omega_{\mathcal{F}} \) (assumed reducible) and subsequently restricting \( \Omega_{\mathcal{F}} \) to the space \( \overline{\mathcal{P}}_{\mathcal{F}} = \mathcal{P}_{\mathcal{F}}//K_{\mathcal{F}} \) of leaves of this distribution, we ultimately obtain a canonical form \( \overline{\Omega}_{\mathcal{F}} \) on the physical (reduced) state space \( \mathcal{P}_{\mathcal{F}} \), alongside a Poisson bracket

\[
\{ \mathcal{G}_1, \mathcal{G}_2 \}_{\mathcal{F}}[\Psi_{\text{cl}}] = \overline{\Omega}_{\mathcal{F}}[\Psi_{\text{cl}}](X_{\mathcal{G}_1}, X_{\mathcal{G}_2})
\]

of hamiltonian functions \( \mathcal{G}_i \), i.e. functionals on the reduced state space which generate hamiltonian vector fields associated with \( \mathcal{G}_i \) as per

\[
\delta \mathcal{G}_i = - \mathcal{F}_{\mathcal{G}_i} \cup \overline{\Omega}_{\mathcal{F}}[\Psi_{\text{cl}}].
\]

Here, \( X_{\mathcal{G}_i} \) are vectors tangent to \( \mathcal{P}_{\mathcal{F}} \) at the state \( \Psi_{\text{cl}} \). They are defined by the corresponding \( \mathcal{F} \)-vertical vector fields \( \mathcal{F}_{\mathcal{G}_i} \) tangent to \( J^1 \mathcal{F} \) and satisfying the linearised variant of the field equations of
\( \mathcal{F} \). In the case of a space-time with a non-empty boundary (or domain walls), the \( \mathcal{F}_{\partial} \) are additionally required to obey the linearised version of the boundary (resp. domain-wall gluing) conditions of \( \mathcal{F} \).

* * *

The reconstruction of the symplectic form on the state space of the \( \mathcal{F} \)-field theory \( \mathcal{F} \) is the first step towards a geometric quantisation of the latter as it induces, iff \( \frac{1}{\Omega_{\mathcal{F}}} \) has integral periods over 2-cycles of \( \mathcal{P}_{\mathcal{F}} \), a circle bundle over \( \mathcal{P}_{\mathcal{F}} \) whose space of sections, when suitably polarised, can be identified with the Hilbert space of \( \mathcal{F} \), cf., e.g., Ref. [Woo92]. In the present paper, we do not address the question of the choice of the polarisation, and so we content ourselves with the following

**Definition 3.4.** Let \( \mathcal{F} \) be an \( \mathcal{F} \)-field theory on a (pseudo-)riemannian manifold \((\mathcal{M}, g)\) with a covariant configuration bundle \( \pi_{\mathcal{F}} : \mathcal{F} \to \mathcal{M} \), and let \((\mathcal{P}_{\mathcal{F}}, \Omega_{\mathcal{F}})\) be the symplectic space of extremal sections of the first-jet bundle \( J^1\mathcal{F} \) of \( \mathcal{F} \), equipped with a symplectic form \( \Omega_{\mathcal{F}} \) of Proposition 3.2.

The **pre-quantum bundle** \( \pi_{\mathcal{L}_{\mathcal{F}}} : \mathcal{L}_{\mathcal{F}} \to \mathcal{P}_{\mathcal{F}} \) of \( \mathcal{F} \)-field theory \( \mathcal{F} \) is a circle bundle over \( \mathcal{P}_{\mathcal{F}} \) with connection \( \nabla_{\mathcal{L}_{\mathcal{F}}} \) of curvature

\[
\text{curv}(\nabla_{\mathcal{L}_{\mathcal{F}}}) = \pi^*_{\mathcal{L}_{\mathcal{F}}} \Omega_{\mathcal{F}}. 
\]

Fix a choice \( \mathcal{O}_{\mathcal{P}_{\mathcal{F}}} = \{ O^i_{\mathcal{P}_{\mathcal{F}}} \}_{i, \ell, \mathcal{P}_{\mathcal{F}}} \) of an open cover of \( \mathcal{P}_{\mathcal{F}} \) and a local presentation, in the sense of Definition 2.2, of the pre-quantum bundle in terms of its \( \tilde{\text{C}} \)-\text{ech}–\text{De}l\text{igne} data \( \mathcal{L}_{\mathcal{F}} \xrightarrow{\text{loc.}} (\theta_{\mathcal{F}}, \gamma_{\mathcal{F}ij}) \in \mathcal{A}^{2,1}(\mathcal{O}_{\mathcal{P}_{\mathcal{F}}}) \) associated with \( \mathcal{O}_{\mathcal{P}_{\mathcal{F}}} \) and subject to the cohomological constraints

\[
D(\theta_{\mathcal{F}}, \gamma_{\mathcal{F}ij}) = (\Omega_{\mathcal{F}}|_{\mathcal{O}_{\mathcal{P}_{\mathcal{F}}}}, 0, 1). 
\]

A **pre-quantisation of \( \mathcal{F} \)-field theory** \( \mathcal{F} \), understood in the sense of, e.g., Ref. [Woo92], is an assignment, to every smooth function \( h \in C^\infty(\mathcal{P}_{\mathcal{F}}, \mathbb{R}) \) and to the associated (global) **hamiltonian vector field** \( \mathcal{X}_h \) on \( \mathcal{P}_{\mathcal{F}} \), determined by the relation

\[
\mathcal{X}_h \triangleright \Omega_{\mathcal{F}} = -\delta h, 
\]

of a collection \( \tilde{\mathcal{O}}_h := (\tilde{h}_i)_{i, \ell, \mathcal{P}_{\mathcal{F}}} \) of local linear operators

\[
(3.8) \quad \tilde{h}_i := -\mathcal{L}_{(\theta_{\mathcal{F}}, \gamma_{\mathcal{F}ij})} \mathcal{X}_h - \mathcal{X}_\theta_{\mathcal{F}i} + h|_{\mathcal{O}_{\mathcal{P}_{\mathcal{F}}}} 
\]

on the space \( \Gamma(\mathcal{L}_{\mathcal{F}}) \) of sections of the pre-quantum bundle, the latter being regarded as the **pre-quantisation Hilbert space**. The collection \( \tilde{\mathcal{O}}_h \) shall be termed the **pre-quantum hamiltonian** for \( h \). By the very construction, the commutator of a pair \( \tilde{\mathcal{O}}_{h_i}, \alpha \in \{1, 2\} \) of pre-quantum hamiltonians takes the canonical form

\[
(3.9) \quad [\tilde{\mathcal{O}}_{h_1}, \tilde{\mathcal{O}}_{h_2}] = -i\tilde{\mathcal{O}}_{(h_1, h_2)_{\mathcal{P}_{\mathcal{F}}}}. 
\]

3.2. **The covariant formalism for the \( \sigma \)-model.** Specialisation of the above general discussion to the non-linear \( \sigma \)-model of Eq. 2.9 prerequires a number of modifications, which – while preserving the basic conceptual framework – serve to adapt the tools introduced to the setting in hand, in which forms on the space-time \( \Sigma \) are replaced by locally smooth forms associated with a given triangulation \( \Delta(\Sigma) \), and in which the space-time itself is split into domains, supporting the respective phases of the two-dimensional field theory. As for the latter point, the reader is advised to acquaint herself or himself, by way of a warm-up, with the treatment of the world-sheet with a non-empty boundary in Ref. [GTTNB04].

The first modification consists in replacing the covariant configuration bundle \( \mathcal{F} \) with

**Definition 3.5.** The **covariant configuration bundles** \( \pi_{\mathcal{F}_\sigma} : \mathcal{F}_\sigma \to \Sigma \) of the non-linear \( \sigma \)-model for network-field configurations \((X | \Gamma)\) in string background \( \mathcal{B} \) on world-sheet \((\Sigma, \gamma)\) with **defect quiver** \( \Gamma \) are given by a disjoint sum of fibre bundles over the respective components of the disjoint union of elements of \( \mathfrak{B} \Sigma, \mathfrak{E}_\Gamma \) and \( \mathfrak{M}_\Gamma \) with restrictions

\[
\mathcal{F}_\sigma|_{\mathfrak{B}_\Sigma} := \mathfrak{B} \times \mathcal{M} \to \mathfrak{B}, \quad \mathcal{F}_\sigma|_{\mathfrak{E}_\Gamma} := \mathfrak{E} \times Q \to \mathfrak{E}, \quad \mathcal{F}_\sigma|_{\mathfrak{M}_\Gamma} := \mathfrak{M} \times T_{n_j} \to \mathfrak{M}.
\]

The associated first-jet bundles, \( J^1\mathcal{F}_\sigma \to \Sigma \), admit local coordinates

- (\( \sigma^a, X^\mu, \xi^\nu_{\gamma} \)) over \( \mathfrak{B}_\Sigma \), where \( \sigma^a \) are local coordinates on the patch \( \mathfrak{B} \), and \( X^\mu \) are local coordinates on \( \mathfrak{M} \);
- (\( \varphi, X^A, \xi^\nu_{\gamma} \)) over \( \mathfrak{E}_\Gamma \), where \( \varphi \) is a local coordinate on the defect line \( \mathfrak{E} \), and \( X^A \) are local coordinates on \( Q \);
• \((\sigma_j, X^i)\) over \(j \in \mathcal{V}_\Gamma\), where \(\sigma_j\) are the coordinates of the defect junction \(j\) within \(\Sigma\), and \(X^i\) are local coordinates on \(T_n\); (the first-jet extension is trivial over the point \(j\)).

The action functional \(\mathcal{L}_p(\sigma, X, J)\) being defined in terms of local expressions sourced by plaquettes of a triangulation \(\Delta(\Sigma)\) of the world-sheet (and their lower-dimensional submanifolds), the Cartan form naturally splits into a sum over terms supported by the particular plaquettes. Below, we define the Cartan form as an object glued up from these contributions. Our point of departure is the local expression

\[
\mathcal{L}_p(\sigma, X, \xi) d^2\sigma = \frac{1}{2} g_{\mu\nu}(X) * \eta(\xi^\mu \wedge * \eta \xi^\nu) d^2\sigma - B_{\nu \mu \lambda}(X) * \eta(\xi^\mu \wedge \xi^\nu) d^2\sigma + \sum_{e \in p} \left( A_{\mu \nu \lambda}(X) \xi_\mu \wedge \delta_e - \sum_{v \in v} \delta_{\nu v} \log g^{\nu \mu}_{\mu \nu \lambda}(X) d^2\sigma \right) + \sum_{e \in p^p} \left( P_{\nu \lambda}(X) \xi_\lambda - \sum_{v \in v} \delta_{\nu v} \log K^{-\epsilon_{\nu \lambda}}(X) d^2\sigma \right) - i \sum_{p \in p \cap \mathcal{V}_\Gamma} \delta_j \log f_{n_j, i_j}(X) d^2\sigma,
\]

with

\[
\delta \mathcal{L}_p(\sigma, X, \xi) = -g_{\mu \nu}(\eta a b - 2B_{\nu \mu \lambda}(\eta a b))(X) = -L^{ab}_{\nu \mu \lambda}(X),
\]

all written in terms of local coordinates \(\sigma^a\), \(a \in \{1, 2\}\) on \(p\), with \(d^2\sigma = d\sigma^1 \wedge d\sigma^2\), alongside objects \(\xi = \xi_\nu d\sigma^\nu\), and the Dirac distributions \(\delta_e = g^2(\sigma - \sigma_e)\) on \(\Sigma\), as well as the singular (Dirac-type) currents \(\delta_e\) supported over \(e \subset p\), with the defining property

\[
\forall_{x \in \Omega_\Gamma(p)} : \int_p \theta \wedge \delta_e = \int_{i_e} e \theta,
\]

where \(e : e \mapsto p\) is the embedding map. In the minkowskian gauge, we have

\[
* \eta = 1 = d^2\sigma, \quad * \eta d\sigma = -d\sigma = -d\sigma, \quad * \eta d^2\sigma = -d\sigma = -d\sigma.
\]

This yields

**Definition 3.6.** Let \(\mathcal{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})\) be a string background of Definition [2.1] The Cartan form of the non-linear \(\sigma\)-model for network-field configurations \((X | \Gamma)\) in string background \(\mathcal{B}\) on world-sheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\) is a 2-form \(\Theta_\sigma\) on the first-jet bundles \(J^1\mathcal{F}_\sigma\) of the covariant configuration bundles \(\mathcal{F}_\sigma\) of the \(\sigma\)-model, given in terms of its restrictions \(\Theta_\sigma|_p = \Theta_p\) to patches \(p \in \Delta(\Sigma)\) of a triangulation \(\Delta(\Sigma)\) of \(\Sigma\) subordinate to \(\mathcal{O}_\#\), \(\mathcal{M} \in \{M, Q, T\}\) with respect to \((X | \Gamma)\) that take the form

\[
\Theta_\sigma(\sigma, X, \xi) = \frac{1}{2} g_{\mu \nu}(X) \xi^\mu \wedge * \eta \xi^\nu - g_{\mu \nu}(X) \delta X^\mu \wedge * \eta \xi^\nu - B_{\nu \mu \lambda}(X) \xi^\nu \wedge * \eta \xi^\mu + \sum_{e \in p} \left( A_{\mu \nu \lambda}(X) \xi_\mu \wedge \delta_e - \sum_{v \in v} \delta_{\nu v} \log g^{\nu \mu}_{\mu \nu \lambda}(X) d^2\sigma \right) + \sum_{e \in p^p} \left( P_{\nu \lambda}(X) \xi_\lambda - \sum_{v \in v} \delta_{\nu v} \log K^{-\epsilon_{\nu \lambda}}(X) d^2\sigma \right) - i \sum_{p \in p \cap \mathcal{V}_\Gamma} \delta_j \log f_{n_j, i_j}(X) d^2\sigma.
\]

**Remark 3.7.** Consider a generic \(\mathcal{F}_\sigma\)-vertical vector field \(\mathcal{V}\) on \(J^1\mathcal{F}_\sigma\) with restrictions

\[
\mathcal{V}|_p = V^\mu \frac{\delta}{\delta \sigma^\mu} + V^\mu \frac{\delta}{\delta \xi^\nu}, \quad \mathcal{V}|_c = V^A \frac{\delta}{\delta X^A} + V^A \frac{\delta}{\delta \xi^B}, \quad \mathcal{V}|_{\mathcal{V}_\Gamma} = V^i \frac{\delta}{\delta X^i},
\]

where the various components are constrained as per

\[
(3.10) \quad V^A \frac{\partial \pi_{\alpha}^k}{\partial X^\alpha} = V^\alpha \circ \pi_{\alpha}^k, \quad V^i \frac{\partial \pi_{\alpha}^{k+1}}{\partial X^\alpha} = V^A \circ \pi_{\alpha}^{k}.
\]
The requirement that $\mathcal{V}$ obey the linearised version of Eq. (3.3) is tantamount to the imposition of the relation

$$V^\mu_a = \partial_a V^\mu.$$  

Hence, a vector field tangent to the space of extremal sections at a section $\Psi_{\sigma,cl}$ is necessarily of the form

$$\mathcal{V}|_{\mathcal{V}} = V^\mu \frac{\delta}{\delta X^\mu} + \partial_a V^\mu \frac{\delta}{\delta X^a},$$  

where the various components are related as in Eq. (3.10), and where the $V^\mu$ satisfy the linearised version of Eq. (2.10).

* * *

We have

**Proposition 3.8.** Let $\Theta_\sigma$ be the Cartan form of the non-linear $\sigma$-model for network-field configurations $(X|\Gamma)$ in string background $\mathfrak{B}$ on world-sheet $(\Sigma, \gamma)$ with defect quiver $\Gamma$, explicit in Def. 3.6. Given a section $\Psi_\sigma \in \Gamma(\mathcal{J}^1 \mathcal{F}_\sigma)$ of the first-jet bundles $\mathcal{J}^1 \mathcal{F}_\sigma \to \Sigma$ of the covariant configuration bundles for the $\sigma$-model, write

$$S_{\Theta_\sigma}[\Psi_\sigma] := \int_\Sigma \Psi^* \Theta_\sigma.$$  

The principle of least action applied to the functional $S_{\Theta_\sigma}$ as per

$$\mathcal{V} \cdot \delta S_{\Theta_\sigma}[\Psi_{\sigma,cl}] = 0,$$  

with $\mathcal{V} \in \Gamma(T \mathcal{J}^1 \mathcal{F}_\sigma)^{1+\gamma}$ an arbitrary $\mathcal{F}_\sigma$-vertical vector field on $\mathcal{J}^1 \mathcal{F}_\sigma$, yields the field equations (2.10) alongside the Defect Gluing Condition 2.8 for classical sections $\Psi_{\sigma,cl}$ of the $\sigma$-model.

As the proof of the proposition is rather technical, it has been relegated to Appendix A

**Definition 3.9.** Let $\mathfrak{B}$ be a string background with target space $M$. The untwisted state space $P_{\sigma,\mathfrak{B}}$ of the non-linear $\sigma$-model for network-field configurations $(X|\Gamma)$ in string background $\mathfrak{B}$ on world-sheet $(\Sigma, \gamma)$ with defect quiver $\Gamma$ is given by the cotangent bundle over the free-loop space $LM = C^\infty(S^1, M)$ of the target space $M$ of the $\sigma$-model,

$$P_{\sigma,\mathfrak{B}} = T^* LM.$$  

It has local coordinates $(X^\mu, p_\nu)$, where $X : S^1 \to M$ is a smooth loop in $M$ and $p = p_\mu \delta X^\mu$ is a normal covector field on $X^\mu$.  

The twisted counterpart is introduced in

**Definition 3.10.** Let $\mathfrak{B}$ be a string background with target $\mathcal{M} = (M, g, \mathcal{G})$ and bi-brane $\mathfrak{B} = (Q, \iota_\alpha, \omega, \Phi | \alpha \in \{1, 2\})$, and let $\Gamma$ be a defect quiver embedded in a world-sheet $(\Sigma, \gamma)$ in such a manner that there exists a closed space-like curve $\mathcal{C} \cong S^1 \subset \Sigma$ that intersects $\Gamma \in \mathbb{N}_0$ defect lines $\ell_k \in \Gamma_\ell$, $k \in \Gamma_\ell$ of $\Gamma$ at the respective points $\sigma_k$ so that the tangent vectors $\tilde{\ell}_k$ of the defect lines $\ell_k$ are all time-like or anti-time-like at the $\sigma_k$, with $X_\ell \tilde{\ell}_k = V_k \in T_{q_k} Q$. Write $\varepsilon_k = +1$ if $\tilde{\ell}_k$ is time-like, and $\varepsilon_k = -1$ if $\tilde{\ell}_k$ is anti-time-like at $\sigma_k$. Fix a collection of points $\{P_k\}_{k \in \Gamma_\ell} \in S^1$, write $S^1_{\{P_k\}} := S^1 \setminus \{P_k\}_{k \in \Gamma_\ell}$ and define the space of smooth maps

$$L^\infty(\{P_k, \varepsilon_k\}) = \{ (X, q_k | k \in \Gamma_\ell) \in C^\infty(S^1_{\{P_k\}}), M) \times Q^+ | \lim_{\varepsilon_k \to 0^+} X(P_k + (-1)^{\alpha + 1} \varepsilon_k \epsilon) = \iota_\alpha(q_k) \}.$$  

Denote as $\mathcal{P}_1(\varepsilon) := \lim_{\varepsilon \to 0^+} X_\ell(P_k + (-1)^{\alpha + 1} \varepsilon_k \epsilon)$ the (one-sided) limiting values of the push-forward of the tangent vector field $\tilde{\ell}(\cdot)$ on $S^1_{\{P_k\}}$ along $X_\ell$, and write $(i_1^{1+}, i_1^{1-}) := (\iota_1, \iota_2)$ and $(i_2^{1+}, i_2^{1-}) := (\iota_2, \iota_1)$. The $k$-twisted state space $P_{\sigma,\mathfrak{B}[\{P_k, \varepsilon_k\}]}$ of the non-linear $\sigma$-model for network-field
configurations \((X|\Gamma)\) in string background \(\mathcal{B}\) on worldsheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\) is naturally identified with the space

\[
P_{\sigma,\mathcal{B}((p_k,\varepsilon_k))} = \left\{ (X,p = p_\mu \delta X^\mu, q_k, V_k \mid k \in \Gamma, \mathcal{T}) \in T^* C^\infty(S^1_{(p_k)}, M) \times TQ^\times \right\}
\]

\[
\wedge \left\{ \lim_{\epsilon \to 0} p(P_k + (-1)^{\alpha+1} \epsilon) = g(\xi_k^\alpha(q_k))(\varepsilon_k \xi_k^\alpha, V_k, \cdot) \right\}.
\]

The space \(P_{\sigma,\mathcal{B}((p_k,\varepsilon_k))}\) shall be described in terms of its local coordinates \((X^\mu, p_\mu, q_k, V_k \mid k \in \Gamma, \mathcal{T})\).

We may now formulate the following fundamental statements:

**Proposition 3.11.** Let \(\mathcal{B}\) be a string background with target \(\mathcal{M} = (M, g, \mathcal{G})\), and let \(P_{\sigma,\mathcal{B}}\) be the untwisted state space of the non-linear \(\sigma\)-model for network-field configurations \((X|\Gamma)\) in string background \(\mathcal{B}\) on worldsheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\). Denote by

\[
ev_M : LM \times S^1 \to M
\]

the canonical evaluation map. The Cartan form \(\Theta_\sigma\) of the \(\sigma\)-model from Definition 3.6 canonically defines a closed 2-form on \(P_{\sigma,\mathcal{B}}\), given by the formula

\[
\Omega_{\sigma,\mathcal{B}} = \delta \theta_{TM} + \pi_{TM} \int_{S^1} ev^*_M H,
\]

in which

\[
\theta_{TM}[(X,p)] = \int_{S^1} \text{Vol}(S^1) \wedge p, \quad p = p_\mu \delta X^\mu
\]

is the canonical 1-form on the total space of the cotangent bundle \(\pi_{TM} : T^* LM \to LM\), written using the volume form \(\text{Vol}(S^1)\) on \(S^1\), and \(H = \text{curv}(\mathcal{G})\). The 2-form is to be evaluated on an arbitrary classical section \(\Psi_{\sigma,cl} \in \Gamma(j^1 F_\sigma)\) of the first-jet bundles of the covariant configuration bundles \(F_\sigma\) of the \(\sigma\)-model. The section (state) is represented by its Cauchy data \((X^\mu, \rho_\nu) \in P_{\sigma,\mathcal{B}}\) localised on an arbitrary untwisted Cauchy contour \(\mathcal{C} \cong S^1\).

A proof of the proposition is given in Appendix 3.

**Proposition 3.12.** Let \(\mathcal{B}\) be a string background with target \(\mathcal{M} = (M, g, \mathcal{G})\) and \(G\)-bi-brane \(\mathcal{B} = (Q, \iota_\alpha, \omega, \Phi \mid \alpha \in \{1,2\})\), and let \(P_{\sigma,\mathcal{B}((p_k,\varepsilon_k))}\) be the \(k\)-twisted state space of the non-linear \(\sigma\)-model for network-field configurations \((X|\Gamma)\) in string background \(\mathcal{B}\) on worldsheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\). Write \(S^1_{(p_k)} = S^1 \setminus \{p_k\}_{k \in \Gamma, \mathcal{T}}\) and denote by

\[
ev_{M,(p_k)} : C^\infty(S^1_{(p_k)}, M) \times S^1_{(p_k)} \to M
\]

the canonical evaluation map, and by \(\text{pr}_{T^* C^\infty(S^1_{(p_k)}, M)} : P_{\sigma,\mathcal{B}((p_k,\varepsilon_k))} \to T^* C^\infty(S^1_{(p_k)}, M)\) and \(\text{pr}_{Q,k} : P_{\sigma,\mathcal{B}((p_k,\varepsilon_k))} \to Q\) the canonical projections, the latter having the \(k\)-th cartesian factor as the codomain. The Cartan form \(\Theta_\sigma\) of the \(\sigma\)-model from Definition 3.6 canonically defines a closed 2-form on \(P_{\sigma,\mathcal{B}((p_k,\varepsilon_k))}\), given by the formula

\[
\Omega_{\sigma,\mathcal{B}((p_k,\varepsilon_k))} = \text{pr}_{T^* C^\infty(S^1_{(p_k)}, M)}(\delta \theta_{TM} + \pi_{TM} \int_{S^1} ev^*_M (p_k) H) + \sum_{k=1}^I \varepsilon_k \text{pr}_{Q,k}^* \omega,
\]

in which

\[
\theta_{TM}(S^1_{(p_k), M})[(X,p)] = \int_{S^1_{(p_k)}} \text{Vol}(S^1_{(p_k)}) \wedge p, \quad p = p_\mu \delta X^\mu
\]

is the canonical 1-form on the total space of the cotangent bundle \(\pi_{TM} : T^* C^\infty(S^1_{(p_k)}, M) \to C^\infty(S^1_{(p_k)}, M)\), written using the volume form \(\text{Vol}(S^1_{(p_k)})\) on \(S^1_{(p_k)}\). The 2-form is to be evaluated on an arbitrary classical section \(\Psi_{\sigma,cl} \in \Gamma(j^1 F_\sigma)\) of the first-jet bundles of the covariant configuration bundles \(F_\sigma\) of the \(\sigma\)-model. The section (state) is represented by its Cauchy data \((X^\mu, \rho_\nu, q_k, V_k \mid k \in \Gamma, \mathcal{T}) \in P_{\sigma,\mathcal{B}((p_k,\varepsilon_k))}\) localised on an arbitrary twisted Cauchy contour \(\mathcal{C} \cong S^1_{(p_k)}\), as in Definition 3.10.
A proof of the proposition is given in Appendix C.

In order to give an explicit description of the pre-quantum bundles for the two types of the state space of the $\sigma$-model, we should first recall the necessary facts about the (Fréchet) manifold $LM$, as defined in Ref. [Ham22]. We use, after Ref. [Gaw88], the straightforward

**Proposition 3.13.** Let $LM = C^\infty(S^1, M)$ be the free-loop space of a manifold $M$, the latter coming with a choice $O_M = \{O^M_i\}_{i \in I}$ of an open cover. Consider the non-empty open set $O_1 = \{X \in LM \mid \forall e, v \in \triangle(S^1) : X(e) \in O^M_i \land X(v) \in O^M_i\}$, with the index $i$ given by a pair $(\triangle(S^1), \phi)$ consisting of a choice of triangulation of the unit circle, with its edges $e$ and vertices $v$, and a choice $\phi : \triangle(S^1) \to I, \phi \mapsto i_{f}$ of the assignment of indices of $O_M$ to elements of $\triangle(S^1)$. By varying these two choices arbitrarily, whereby an index set $\mathcal{F}_O_M$ is formed, all of $LM$ is covered, thus yielding an open cover $O_{LM} = \{O_1\}_{i \in I}$ of free-loop space $LM$.

Similarly,

**Proposition 3.14.** Let $B$ be a string background with target $M = (M, g, G)$ and $G$-bi-brane $B = (Q, \iota, \omega, \Phi \mid \alpha \in \{1, 2\})$. Given a collection $\{P_k\}_{k \in \mathbb{Z}}$ of points on the unit circle $S^1$, and a collection $\{\xi_k\}_{k \in \mathbb{Z}}$ of $I$ elements of $(-1, 1)$, let $L_{Q \{Q_{\iota, \omega, \Phi}^1\}^{1, 2}} M$ be the space introduced in Definition 3.16. Fix a choice $O_M = \{O^M_i\}_{i \in I}$ of an open cover of $M$, and a choice $O_Q = \{O_Q^M\}_{i \in I}$ of an open cover of $Q$, for which there exist Čech-extended $G$-bi-brane maps $(\iota, \omega, \Phi), \alpha \in \{1, 2\}$, as described in Definition 2.2. Consider the non-empty open sets $O_{\{(P_k, \iota, \omega, \Phi)\}_{i \in I}} = \{\{X, q_k \mid k \in \mathbb{Z}\} \in L_{Q \{Q_{\iota, \omega, \Phi}^1\}^{1, 2}} M \mid (X(e) \in O^M_i \land X(v) \in O^M_i) \land q_k \in O^M_{1/2}\}$, with the index $i$ given by a triple $(\triangle(P_k), \phi^1, \phi^2)$ consisting of a choice of triangulation of the unit circle, with its edges $e$ and vertices $v$ of which $I$ are fixed at the $P_k$, $k \in \mathbb{Z}$, and choices $\phi : (\triangle(P_k) \setminus \{P_k\}_{k \in \mathbb{Z}}) \to I, \phi \mapsto i_{f}$ and $\phi^2 : (P_k)_{k \in \mathbb{Z}} \to I, P_k \mapsto \iota_{P_k}^2$ of the assignment of indices of the open covers $O_M$ and $O_Q$ to elements of $\triangle(S^1)$. By varying these choices arbitrarily, whereby an index set $\mathcal{F}_{O_Q \{Q\}}^{M}$ is formed, all of $L_{Q \{Q_{\iota, \omega, \Phi}^1\}^{1, 2}} M$ is covered, thus yielding an open cover $O_{L_{Q \{Q_{\iota, \omega, \Phi}^1\}^{1, 2}} M} = \{O_{\{(P_k, \iota, \omega, \Phi)\}_{i \in I}} \}_{i \in I}$ of space $L_{Q \{Q_{\iota, \omega, \Phi}^1\}^{1, 2}} M$.

**Remark 3.15.** It is straightforward to describe intersections of elements of the open cover $O_{LM}$. In so doing, we follow Ref. [Gaw88] once more. Given a pair $O^n, n \in \{1, 2\}$ with the respective triangulations $\triangle_n(S^1)$ (consisting of edges $e_n$ and vertices $v_n$) and index assignments $(e_n, v_n) \mapsto \{i_{e_n}^n, \iota_{v_n}^n\}$, we consider the triangulation $\bar{\triangle}(S^1)$ obtained by intersecting $\triangle_1(S^1)$ with $\triangle_2(S^1)$, by which we mean that the edges $\bar{e}$ of $\bar{\triangle}(S^1)$ are the edges of the $\triangle_n(S^1)$, and its vertices $\bar{v}$ are taken from the set-theoretic union of the two vertex sets. As previously, the incoming (resp. outgoing) edge of $\bar{\triangle}(S^1)$ at the vertex $\bar{v}$ is denoted by $\bar{v}_i(\bar{v})$ (resp. $\bar{v}_o(\bar{v})$). A non-empty double intersection $O_{1/2} \cap O_{1/2} = O_{1/2}$ is then labelled by the triangulation $\bar{\triangle}(S^1)$, taken together with the indexing convention such that $i_{e_n, i_{e_n}}^n$ is the Čech index assigned via $i_n$ to the edge of $\triangle_n(S^1)$ containing $\bar{e} \in \bar{\triangle}(S^1)$, and $\iota_{v_n}^n$ is the Čech index assigned via $i_n$ to $\bar{v}$ if $\bar{v} \in \triangle_n(S^1)$, or the Čech index assigned via $i_n$ to the edge of $\triangle_n(S^1)$ containing $\bar{v}$ otherwise. Analogous remarks apply to $O_{L_{Q \{Q_{\iota, \omega, \Phi}^1\}^{1, 2}} M}$. * * * * * 

We have the fundamental result:

**Theorem 3.16.** [Gaw88] Let $M$ be a manifold with a gerbe $G$ of curvature $\text{curv}(G) = H$ over it, and denote by $\text{ev}_{M} : L_M \times S^1 \to M$ the canonical evaluation map for the free-loop space $LM = C^\infty(S^1, M)$ of $M$. The gerbe $G$ canonically defines a circle bundle $\pi_{L_G} : L_M \to LM$ with connection $\nabla_{L_G}$ of curvature $\text{curv}(\nabla_{L_G}) = \int_{S^1} \text{ev}_{M}^* H$. 

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6 The free-loop space $LM$ is equipped with the compact-open topology.
Theorem 3.18. Gawędzki's construction is readily verified to generalise to the twisted case.

(3.18)

3.16 and of the trivial circle bundle $T$ given by the tensor product of the pullback, along $\sigma$ canonical map from the total space of the cotangent bundle over the free-loop space $L$ of $E$, terms of its local data to be termed the $LH$ Beviausal transformation

Let $\sigma$ induce gauge transformations $\phi$.

Under gauge transformations (2.4) of the local data of $G$, the local symplectic potentials undergo induced gauge transformations

(3.17) $E_i \rightarrow E_i - i \delta \log H_i,$

where $H_i[X] = \prod_{e \in \Delta(S^1)} e^{\int_e X^*_e e^{i \chi^*_e(v)}} \prod_{v \in \Delta(S^1)} X^*_e X^*_v i_e(v).$

The physical significance of the last proposition can be phrased as

**Corollary 3.17.** Let $B$ be a string background with target $M = (M, g, G)$, and let $(P_{\sigma, g}, \Omega_{\sigma, g})$ be the unit twisted state space of the non-linear $\sigma$-model for network-field configurations $(X | \Gamma)$ in string background $B$ on world-sheet $(\Sigma, \gamma)$ with defect quiver $\Gamma$. Denote by $\pi^{*}\text{LM} : T^{*}LM \rightarrow LM$ the canonical map from the total space of the cotangent bundle over the free-loop space $LM = C^{\infty}(S^1, M)$ of $M$ onto its base. The pre-quantum bundle $\pi_{L, g} : L_{\sigma, g} \rightarrow P_{\sigma, g}$ for the untwisted sector of the $\sigma$-model is the circle bundle

$L_{\sigma, g} := \pi^{*}_{L, g} L_{\sigma} \otimes (T^{*}LM \times S^1) \rightarrow T^{*}LM \cong P_{\sigma, g}\$

given by the tensor product of the pullback, along $\pi^{*}_{L, g}$, of the transgression bundle $L_{\sigma}$ of Theorem 3.16 and of the trivial circle bundle $T^{*}LM \times S^1 \rightarrow T^{*}LM$ with a global connection 1-form equal to the canonical 1-form $\theta_{L^{*}LM}$ on $T^{*}LM$, explicited in Proposition 3.11. In particular, given the open cover $O_{LM} = \{O_i\}_{i \in A_{LM}}$ of $LM$ defined in Proposition 3.13, local data of $L_{\sigma, g}$ associated with the induced open cover $O_{P_{\sigma, g}} = \{O^*_i\}_{i \in A_{LM}}, O^*_i := \pi^{*}_{L, g} (O_i)$ of $P_{\sigma, g}$ can be expressed in terms of the local data $(E_i, G_{ii})$ of the bundle $L_{\sigma}$ from Theorem 3.16 as

(3.18) $\theta_{\sigma, g_i} = \theta_{L^{*}LM} O^*_i + \pi^{*}_{L, g} E_i$, $\gamma_{\sigma, g_i} = \pi^{*}_{L, g} G_{ii}.$

Gawędzki's construction is readily verified to generalise to the twisted case.

**Theorem 3.18.** Let $B$ be a string background with target $M = (M, g, G)$ and $G$-bi-brane $B = (Q, \iota, \iota_1, \Phi | \alpha \in \{1, 2\})$, and let $L_{Q|(P_{\sigma, g})} M$ be the space introduced in Definition 3.10. Write $S^1(P_k) = S^1 \setminus \{P_k\}$ and denote by

$ev_{M,(P_k)} : C^{\infty}(S^1(P_k), M) \times S^1(P_k) \rightarrow M$

the canonical evaluation map. The pair $(G, B)$ canonically defines a circle bundle $\pi_{L(G, B)|(P_{\sigma, g})} : L(G, B)|(P_{\sigma, g}) \rightarrow L_{Q|(P_{\sigma, g})} M$ with connection $\nabla_{L(G, B)|(P_{\sigma, g})}$ of curvature

$\text{curv}(\nabla_{L(G, B)|(P_{\sigma, g})}) = \pi^{*}_{L(G, B)|(P_{\sigma, g})} (pr_{C^{\infty}(S^1(P_k), M)} \int_{S^1(P_k)} ev^{*}_{M,(P_k)} H + \sum_{k=1}^{j} \varepsilon_k pr^{*}_{Q,k}(\omega)),$

written in terms of the canonical projections $pr_{C^{\infty}(S^1(P_k), M)} : L_{Q|(P_{\sigma, g})} M \rightarrow C^{\infty}(S^1(P_k), M)$ and $pr_{Q,k} : L_{Q|(P_{\sigma, g})} M \rightarrow Q$, the latter having the k-th cartesian factor as the codomain. Under the
assignment \( (\mathcal{G}, \mathcal{B}) \rightarrow L((\mathcal{G}, \mathcal{B})) \), (gauge-)equivalence classes of pairs \((\mathcal{G}, \mathcal{B})\), as described in Definition 2.2, are mapped to isomorphism classes of bundles with connection.

**Proof:** We define \( L((\mathcal{G}, \mathcal{B})) \) explicitly in terms of its local data \((E_i((p_k, \varepsilon_k))_{1\leq i \leq k}, G_i((p_k, \varepsilon_k)))_{1\leq i \leq k} \in \mathcal{A}^2(\mathcal{O}_{L((p_k, \varepsilon_k))})^M\) associated with the open cover \( \mathcal{O}_{L((p_k, \varepsilon_k))}^M \) from Proposition 3.14 and determined by local data of \((\mathcal{G}, \mathcal{B})\), as written out in Definition 2.2. Write \((X^0, q_k | k \in \{1, \ldots, T\}) \equiv (X, \{q_k\})\). It is easy to check that the objects

\[
E_i((p_k, \varepsilon_k))_i[(X, \{q_k\})] = - \sum_{e \in \Delta(p_k)} \int_{X^e} B_{ie} - \sum_{v \in (\mathcal{P}_k)_{1\leq e \leq k, 1\leq i \leq T}} X^e A_{i_{e, (\varepsilon)}(1)}(v)
\]

\[
+ \sum_{k=1}^T \varepsilon_k (1)^2 A_{1_{\varepsilon_k}}(p_k, \phi_1(i_{1_k}^2) + \phi_2(i_{2_k}^2) + P_{i_{2_k}^2}) (q_k),
\]

\[
G_i((p_k, \varepsilon_k))_i[(X, \{q_k\})] = \prod_{e \in \Delta(p_k)} e^{-1} \int_{X^e} X^e A_{i_{e, (\varepsilon)}}(v)
\]

\[
\cdot \prod_{v \in (\mathcal{P}_k)_{1\leq e \leq k, 1\leq i \leq T}} X^e (g_{i_{e, (\varepsilon)}(1), (\varepsilon)}(v), g_{i_{e, (\varepsilon)}(1), (\varepsilon)}^{-1}(v))
\]

\[
\cdot \prod_{k=1}^T \left( (1)^2 (g_{i_{e_k}^2(p_k), \phi_2(i_{1_k}^2)}(1)^2, g_{i_{e_k}^2(p_k), \phi_2(i_{1_k}^2)}^{-1}(1)^2), K_{i_{e_k}^2}^{-1}(1)^2 \right)(q_k),
\]

written in terms of \((\sigma_k, \bar{\sigma}_k) = (+, -)\) if \(\varepsilon_k = +1\), and \((\sigma_k, \bar{\sigma}_k) = (-, +)\) otherwise, obey the required cohomological constraints. Similarly, one verifies through inspection that under a gauge transformation \((2.4)\) of the local data of \(\mathcal{G}\), accompanied by the \(\mathcal{G}\)-twisted gauge transformation \((2.6)\) of the local data of the \(\mathcal{G}\)-bi-brane 1-isomorphism, the pair \((E_i((p_k, \varepsilon_k))_i, G_i((p_k, \varepsilon_k))_i)\) undergoes induced gauge transformation

\[
(E_i((p_k, \varepsilon_k))_i, G_i((p_k, \varepsilon_k))_i) \rightarrow (E_i((p_k, \varepsilon_k))_i, G_i((p_k, \varepsilon_k))_i) + D(0)(H((p_k, \varepsilon_k))_i),
\]

with

\[
H((p_k, \varepsilon_k))_i[(X, \{q_k\})] = \prod_{e \in \Delta(p_k)} e^{-1} \int_{X^e} X^e (1)^{-1}_{i_{e, (\varepsilon)}}(v)
\]

\[
\cdot \prod_{v \in (\mathcal{P}_k)_{1\leq e \leq k, 1\leq i \leq T}} X^e (1)^{-1}_{i_{e, (\varepsilon)}}(v), \exp(i_{e_k}^2 - i_{e_k}^2)(q_k).
\]

The physical content of the above result is summarised in

**Corollary 3.19.** Let \( \mathcal{B} \) be a string background with target \( \mathcal{M} = (M, g, \mathcal{G}) \) and \( \mathcal{G}\)-bi-brane \( \mathcal{B} = (Q, \iota, \omega, \Phi \mid \alpha \in \{1, 2\}) \), and let \((\mathcal{P}_\mathcal{B}((p_k, \varepsilon_k)), L, \mathcal{B}((p_k, \varepsilon_k)))_{1\leq i \leq k} \) be the \(\kappa\)-twisted state space of the non-linear \(\sigma\)-model for network-field configurations \((X \mid \Gamma)\) in string background \(\mathcal{B}\) on world-sheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\). Denote by \(pr_{L((p_k, \varepsilon_k))}^M : \mathcal{P}_\mathcal{B}((p_k, \varepsilon_k)) \rightarrow L((p_k, \varepsilon_k))^M\) the canonical projection from the \(\kappa\)-twisted state space to the space \(L((p_k, \varepsilon_k))^M\) from Definition 3.14. The pre-quantum bundle \(\pi_{\mathcal{P}_\mathcal{B}((p_k, \varepsilon_k))} : L, \mathcal{B}((p_k, \varepsilon_k)) \rightarrow \mathcal{P}_\mathcal{B}((p_k, \varepsilon_k))\) for the twisted sector of the \(\sigma\)-model is the circle bundle

\[
L, \mathcal{B}((p_k, \varepsilon_k)) := pr_{L((p_k, \varepsilon_k))}^M L((\mathcal{G}, \mathcal{B}))((p_k, \varepsilon_k)) \times \mathbb{S}^1 \rightarrow \mathcal{P}_\mathcal{B}((p_k, \varepsilon_k))
\]

given by the tensor product of the pullback, along \(pr_{L((p_k, \varepsilon_k))}^M\), of the bundle \(L((\mathcal{G}, \mathcal{B}))((p_k, \varepsilon_k))\) of Theorem 3.13 and of the trivial circle bundle \(\mathcal{P}_\mathcal{B}((p_k, \varepsilon_k)) \times \mathbb{S}^1 \rightarrow \mathcal{P}_\mathcal{B}((p_k, \varepsilon_k))\) with a global connection 1-form equal to the pullback, along the canonical projection \(pr_{T^\ast C^\infty(S^1(p_k), M)} : \mathcal{P}_\mathcal{B}((p_k, \varepsilon_k)) \rightarrow T^\ast C^\infty(S^1(p_k), M)\), of the canonical 1-form \(\theta_{T^\ast C^\infty(S^1(p_k), M)} \) on \(T^\ast C^\infty(S^1(p_k), M)\), explicit in Proposition 3.12. In particular, given the open cover \(\mathcal{O}_{L((p_k, \varepsilon_k))}^M = \{ pr_{L((p_k, \varepsilon_k))}^M(\mathcal{O}_{L((p_k, \varepsilon_k))}) \}_{1\leq i \leq k} \) of \(L((p_k, \varepsilon_k))^M\) defined in Proposition 3.14, local data of \(L, \mathcal{B}((p_k, \varepsilon_k))\) associated with the induced open cover \(\mathcal{O}_{\mathcal{P}_\mathcal{B}((p_k, \varepsilon_k))} = \{ pr_{L((p_k, \varepsilon_k))}^M(\mathcal{O}_{\mathcal{P}_\mathcal{B}((p_k, \varepsilon_k))}) \}_{1\leq i \leq k} \) of \(\mathcal{P}_\mathcal{B}((p_k, \varepsilon_k))\) can be expressed in terms of the local data \((E_i((p_k, \varepsilon_k))_i, G_i((p_k, \varepsilon_k))_i)\) of the bundle \(L((\mathcal{G}, \mathcal{B}))((p_k, \varepsilon_k))\) from Theorem 3.13 as

\[
\theta_{\mathcal{P}_\mathcal{B}((p_k, \varepsilon_k))} = pr_{T^\ast C^\infty(S^1(p_k), M)}(\theta_{T^\ast C^\infty(S^1(p_k), M)}) + pr_{L((p_k, \varepsilon_k))}^M E_i,
\]
Prior to passing to the subsequent sections, in which we exploit the knowledge, gained heretofore, of the canonical and pre-quantum structure on the space of states of the \( \sigma \)-model in the presence of defects, we pause to briefly discuss a simple application of our results, of particular relevance to the study of the concept of ‘emergent geometry of string theory’. 

**Remark 3.20.** The non-commutative geometry of the bi-brane world-volume. The presence of the defect-line contributions
\[
\omega(q_k) = \omega_{AB}(q_k) \delta X^A \land \delta X^B, \quad q_k \in Q
\]
in Eq. (3.15) is a clear-cut indication that the quantisation of the defect \( \sigma \)-model yields a non-commutative deformation of the algebra of functions on the bi-brane world-volume, the latter being generated by the coordinate functions \( X^A \). This is a bi-brane variant of the long-known phenomenon of the (gerbe-induced) non-commutativity of the D-brane geometry in the so-called ‘stringy régime’, first discussed in Ref. [DH98]. The actual form of the non-commutativity of the quantum position operators depends strongly on the choice of the quantisation scheme, as indicated in Ref. [SW99]. However, under certain circumstances, one can get some insight into the matter already on the (semi)classical level. Indeed, assume that the string background \( M \) comes with a small dimensionless parameter \( \epsilon \) (derived, e.g., from a common length scale for \( M \) and \( Q \)), and that there exists a geometric régime, to be referred to as a decoupling régime (e.g., a vicinity of a distinguished point in the large target space)
\[
X^\mu, X^A = O(\epsilon^d), \quad d_X \in \mathbb{N}
\]
in which the target-space metric behaves as
\[
g = O(\epsilon^{d_g}), \quad d_g \in \mathbb{N}.
\]
The condition of a vanishing Weyl anomaly, mentioned in Remark [2.10] then fixes the scaling behaviour of the gerbe curvature to be
\[
H = O(\epsilon^{d_H}), \quad d_H \geq d_g
\]
(consistently with the field equations (2.10)), and so the sum of the first two terms in \( \Omega_{\sigma, B|((P_k, \epsilon_k))} \) scales as \( O(\epsilon^{d_g}) \). The bi-brane curvature, on the other hand, need not decrease at the same rate since the relation
\[
(i_4^*H - i_3^*H) = d\omega
\]
that follows from Eqs. (2.3)-(2.5) and determines the scaling behaviour of the defect-line terms in the symplectic structure contains the difference of the pullbacks along the two bi-brane maps, which may affect the value of the critical exponent \( d_\omega \) in
\[
\omega = O(\epsilon^{d_\omega}).
\]
Thus, whenever \( d_g - d_\omega > 0 \), we may, in the decoupling régime described, approximate the symplectic structure as
\[
\Omega_{\sigma, B|((q_k, \epsilon_k))} = \sum_{k=1}^l \epsilon_k \text{pr}^*_Q \circ \omega (1 + O(\epsilon^{d_g - d_\omega})).
\]
From now onwards, we restrict our attention to the case of \( k = 1 \), with \( P_1 = P, \quad q_1 = q \) and \( \epsilon_1 = +1 \).

If the bi-brane curvature is invertible in the decoupling régime, with the inverse defining a Poisson bivector
\[
\Pi = \Pi^{AB} \partial_A \land \partial_B, \quad \Pi^{AB} = -\frac{1}{4} (\omega^{-1})^{AB},
\]
we find natural defect observables \( X^A(P) \equiv X^A \) represented by the hamiltonian vector fields
\[
\mathcal{H}^A = 2\Pi^{AB}(X) \frac{\delta}{\delta X^B}.
\]
Canonical quantisation of their Poisson bracket
\[
\{X^A, X^B\}_{\Omega_{\sigma, B|((P, +1))}} = 2\Pi^{AB}(X)
\]

---

\(^7\) For an earlier account of the phenomenon, exhibiting its rich mathematical structure in the closed-string sector, cf. Ref. [FG94] and Refs. [FGR98b, FGR98a, FGR99]. In the more recent Ref. [RS98], the gerbe-related description of the open string in the WZW model was worked out along the lines of these earlier papers.
gives a non-commutative algebra of stringy coordinates on $Q$, as claimed. By the usual argument, the closedness of $\omega$ ensures the vanishing of the jacobian of the Poisson bracket thus defined, necessary for the associativity of the non-commutative deformation of the algebra of functions on the bi-brane.

By way of illustration of the general phenomenon, we treat in some detail the case of the maximally symmetric WZW $G_k$-bi-branes from Example 2.13. The WZW model for a compact Lie group $G$ comes with a natural parameter $\epsilon_k = \frac{1}{k}$ that sets – through the Cartan–Killing metric $g_k$ (and in conjunction with the string tension, which we suppressed in the present notation) – the characteristic length scale of the string background and plays the rôle of Planck’s constant $\hbar$ and a parameter of the non-commutative deformation of the algebra of functions on $G$ (or a submanifold thereof) determined by the operator content of the quantised $\sigma$-model, cf., e.g., Refs. [FG94] and [RS08]. Here, very large but finite values of $k$ give access to a semiclassical approximation of the quantum geometry of the WZW string\textsuperscript{8}. More specifically, let us write elements of $G$, and hence also fields of the WZW model, in terms of the canonical (Riemann normal) coordinates $X^A = \epsilon_k \tilde{X}_A$ on the group manifold, suitably rescaled, to wit,

$$g = e^{\epsilon_k \tilde{X}_A} t_A,$$

and subsequently pass to the decoupling régime

$$\tilde{X}^A = O(1), \quad \epsilon_k \ll 1$$

of Ref. [ARS99]. Having thus restricted our analysis to world-sheets embedded in an immediate vicinity of the group unit in a large group manifold, we readily establish the equalities

$$d_X = 1, \quad d_{sg} = 1, \quad d_{Hh} = 2,$$

and – for $\lambda \in P^k_+ (g)$ small and for values of the $G_k$-bi-brane field $(g, h_\lambda)$ restricted to a small neighbourhood of $(e, e) \in G \times G$ –

$$\omega^\partial_{k, \lambda} = O(1), \quad \omega_{k, \lambda} = -\text{pr}_2^* \omega^\partial_{k, \lambda} + O(\epsilon_k).$$

This follows straightforwardly from the relations

$$\theta_L = O(\epsilon_k), \quad \text{id}_g - \text{Ad}_g = -\epsilon_k \text{ad}_g + O(\epsilon^2).$$

Consequently, the deformation of the commutative algebra of functions on both the boundary and the non-boundary maximally symmetric WZW $G_k$-bi-brane, as encoded by the decoupling limit of the respective symplectic forms, is determined by the properties of the 2-form

$$\omega^\partial_{k, \lambda}(\tilde{X}) = -\frac{1}{8\pi} \text{tr}_g (\tau_A \text{ad}^3 \tau_B) \delta \tilde{X}^A \wedge \delta \tilde{X}^B + O(\epsilon_k), \quad \tilde{X} = \tilde{X}^A t_A$$

which coincides with the Kirillov–Kostant–Souriau symplectic form on the coadjoint orbit $\mathcal{O}_\lambda \cong G/G_\lambda$ (for $G_\lambda$ the Ad-stabiliser of $\lambda$ in $G$) of Refs. [Kos70, Sou70, Kir75]. Equivalently, the deformation is characterised by the associated Poisson bivector

$$\Pi^\partial_{k, \lambda}(\tilde{X}) = 4\pi f_{ABC} \tilde{X}^A \delta \delta \wedge \delta + O(\epsilon_k).$$

Taking the latter as the germ of a deformation quantisation of the smooth geometry of conjugacy classes $\mathcal{O}_\lambda$ close to the group unit leads to the emergence of the so-called fuzzy conjugacy classes, analogous to the fuzzy sphere of Refs. [Hop82, Mad92]. These are precisely the non-commutative geometries emerging from perturbative calculations of the quantised WZW model. They were first explored in the present context in Refs. [ARS99, ARS00, ARS01], cf. also Ref. [RS08] for a gerbe-related discussion based on the spectral data of the supersymmetric extension of the boundary WZW model.

* * *

4. State-space isotropics from defects, and $\sigma$-model dualities

Now that we have developed a symplectic formalism for the description of the two-dimensional field theory in hand, we may apply it to study defects. Thus, motivated by the discussion, presented in Refs. [FPRS04, FPRS07, Bac09], of the rôle that defects play in mediating dualities of the underlying two-dimensional field theory, and also by the study of chosen examples of dualities in the gerbe-theoretic context in Refs. [ST07, RSS06, SS08], we seek to establish an appropriate rigorous result with the canonical framework, to wit, we want to describe the symplectic relations within the untwisted state space $(P_{\sigma, Z} = T^* L M, \Omega_{\sigma, Z})$ of the $\sigma$-model inhabiting two adjacent patches $\sigma_\alpha, \alpha \in \{1, 2\}$ separated by

\footnotetext[8]{For a proposal of an algebraic description of that geometry in the quantum régime, consistent with the quantum-group symmetries of the rational conformal field theory of the WZW model, cf. Refs. [PS01, PS02, PS03, PS06].}
a space-like connected component \( \ell \cong S^1 \) of the defect quiver \( \Gamma \) that are induced by the intermediary \( G \)-bi-brane structure \( \mathcal{B} \). To this end, using independence of \( \Omega_{\sigma,\mathcal{G}} \) over each of the two patches of the choice of the Cauchy contour used to define it, we push the respective Cauchy contours, \( C_1 \) and \( C_2 \), to \( \ell \). Following this simple prescription, we find

**Proposition 4.1.** Let \( \mathcal{B} \) be a string background with target \( \mathcal{M} = (M,g,G) \) and \( G \)-bi-brane \( \mathcal{B} = (\mathcal{Q},t_\alpha,\omega,\Phi \mid \alpha \in \{1,2\}) \), and let \( (P_{\sigma,\mathcal{G}}\equiv \mathcal{T}^\Gamma L,M,\Omega_{\sigma,\mathcal{G}}) \) be the untwisted state space of the non-linear \( \sigma \)-model for network-field configurations \( (X\mid \Gamma) \) in string background \( \mathcal{B} \) on world-sheet \( (\Sigma,\gamma) \) with defect quiver \( \Gamma \). Consider the symplectic structure on the product space \( P_{\sigma,\mathcal{G}} \times P_{\sigma,\mathcal{G}} \equiv P_{\sigma,\mathcal{G}}^2 \) determined by the ‘difference’ symplectic form

\[
\Omega_{\sigma,\mathcal{G}} = pr_1^*\Omega_{\sigma,\mathcal{G}} - pr_2^*\Omega_{\sigma,\mathcal{G}}.
\]

The \( G \)-bi-brane \( \mathcal{B} \) together with the Defect Gluing Condition (2.8) canonically defines an isotropic submanifold in \((P_{\sigma,\mathcal{G}}^2,\Omega_{\sigma,\mathcal{G}})\), given by

\[
\mathcal{J}_\sigma(\mathcal{B}) = \{ (\psi_1,\psi_2) \in P_{\sigma}^2, \psi_\alpha = (X_\alpha,p_\alpha), \alpha \in \{1,2\} \mid (X_1,X_2) \in (\zeta_1,\zeta_2)(LQ) \}
\]

and

\[
\exists X_\varepsilon((\zeta_1,\zeta_2)^{-1}(X_1,X_2)) : DGC\left((\psi_1,\psi_2,X) = 0\right)
\]

in terms of Cauchy data \( \psi_\alpha \). The latter subspace is a fibration over the free-loop space \( LQ \), and we shall identify it with the corresponding subspace in \( P_{\sigma,\mathcal{G}}^2 \times LQ \) in what follows.

**Proof:** Take a pair of states \((\psi_1,\psi_2) \in \mathcal{J}_\sigma(\mathcal{B})\) with \((X_1,X_2) = (\zeta_1(X),\zeta_2(X))\), satisfying the DGC (2.8), i.e.

\[
p_{\mu,\nu} W_{\nu}^\mu - p_{1,\mu} W_{1}^\mu = -2(X,\tilde{\Gamma})^A \omega_{AB} W_B,
\]

for \( W = W^A \frac{\delta}{\delta X^\alpha} \) an arbitrary vector field from \( \Gamma(TQ) \) restricted to \( X \), the latter being modelled on \( S^1 \) with a normalised tangent vector field \( \tilde{\tau} \), and for \( W_\alpha = i_{\tau_\alpha} W \). We want to consider the distinguished subspace \( \mathfrak{T}_\sigma(\mathcal{B}) \) of \( \Gamma(TP_{\sigma,\mathcal{G}}^2) \) over \( \mathcal{J}_\sigma(\mathcal{B}) \) spanned by vector fields

\[
\tilde{\nu} = \tilde{\tau}_1 \oplus \tilde{\tau}_2,
\]

\[
\tilde{\nu}_\alpha = \nu^\mu \frac{\delta}{\delta X^\alpha} + \nu_{\mu} \frac{\delta}{\delta p_{\alpha,\mu}},
\]

describing tangential deformations \( \nu_\alpha \equiv \nu^\mu \frac{\delta}{\delta X^\alpha} = i_{\tau_\alpha} \nu^A \frac{\delta}{\delta X^A} \) of the \( X_\alpha \) induced by deformations \( \nu = \nu^A \frac{\delta}{\delta X^A} \) of the defect loop \( X \), and augmented by deformations \( \nu_{\mu} \frac{\delta}{\delta p_{\alpha,\mu}} \) of the normal covector fields satisfying a linearised version of Eq. (4.1):

\[
\sum_{\alpha=1,2} (-1)^\alpha \left[ \nu_{\mu} \frac{\delta}{\delta X^\alpha} + p_{\alpha,\mu} \nu^A \left( W^B \frac{\partial B}{\partial B} \nu_{\alpha} + \frac{\partial A}{\partial B} W_B \right) \right]
\]

\[
= -2(X,\tilde{\Gamma})^A \left[ (\partial B \omega_{AC} W_C + \omega_{AC} \partial B W_C) + \omega_{BC} W_C \partial A W_B \right],
\]

where \( \partial A = \frac{\partial}{\partial X^\alpha} \) and where all fields are implicitly functions on \( S^1 \). Clearly, \( \nu \) is a vector field on \( Q \) and the pair \( W_\alpha, \alpha \in \{1,2\} \) can be completed to another admissible deformation of the \( X_\alpha \),

\[
\tilde{\nu} = \tilde{\nu}_1 \oplus \tilde{\nu}_2,
\]

\[
\tilde{\nu}_\alpha = \nu^\mu \frac{\delta}{\delta X^\alpha} + \nu_{\mu} \frac{\delta}{\delta p_{\alpha,\mu}},
\]

by the addition of deformations \( \nu_{\mu} \frac{\delta}{\delta p_{\alpha,\mu}} \) of the normal covector fields satisfying an analog of Eq. (4.2). Upon subtracting the resulting equation from Eq. (4.2) and using Eq. (2.8) in conjunction with the closedness of \( \Gamma(TQ) \) under the Lie bracket, we obtain the relation

\[
\sum_{\alpha=1,2} (-1)^\alpha \left( \nu_{\mu} \frac{\delta}{\delta X^\alpha} - \nu_{\mu} \frac{\delta}{\delta X^\alpha} \right) = -W \cdot \nu \cdot X \cdot \Gamma \cdot \omega + X \cdot \Gamma \cdot d\left( W \cdot \nu \cdot \omega \right).
\]

We may now evaluate \( \Omega_{\sigma,\mathcal{G}} \), taken at an extremal section \((\Psi_1^\sigma,\Psi_2^\sigma)\) determined by the Cauchy data \((\psi_1,\psi_2) \in \mathcal{J}_\sigma(\mathcal{B} ; X)\), on a pair \( (\tilde{V},\tilde{W}) \) of vectors obtained by evaluating the pair \( (\tilde{V},\tilde{W}) \) of vector fields from \( \mathfrak{T}_\sigma(\mathcal{B}) \) at \((\psi_1,\psi_2)\). Putting Eqs. (4.3) and (4.4) together and employing Eq. (3.2), we readily verify the desired identity

\[
\Omega_{\sigma,\mathcal{G}}[(\psi_1,\psi_2)](\tilde{V},\tilde{W}) = 0.
\]

It leads us to conclude that the defect defines an isotropic subspace \( \mathfrak{T}_\sigma(\mathcal{B}) \) within \( \Gamma(TP_{\sigma,\mathcal{G}}^2) \mathcal{J}_\sigma(\mathcal{B}) \) of the distinguished submanifold \( \mathcal{J}_\sigma(\mathcal{B}) \). \( \square \)
Whenever the subspace $\mathcal{J}_\sigma(\mathcal{B})$ with an isotropic tangent $\mathfrak{S}_\sigma(\mathcal{B})$ is actually a graph, the $\mathcal{G}$-bi-brane defines a symplectomorphism of the untwisted state space\textsuperscript{9} which can be understood as an identification between states, chosen arbitrarily, incident on the defect carrying the data of the $\mathcal{G}$-bi-brane $\mathcal{B}$ from one side of the defect line and those emerging from it on the other side. However, it is to be kept in mind that states on either side of the defect line carry charges of the symmetries of the untwisted $\sigma$-model, notably, the energy and, in the case of extended (internal) symmetry, additional charges to which the symmetry currents couple. Thus, for the defect to describe a duality\textsuperscript{10} of the untwisted theory, we should demand that the charges of the two states identified with one another at the defect match, or, equivalently, that the corresponding symmetry currents be continuous at the defect. Whereas, in concrete examples, one may wish to impose weaker correspondence constraints, allowing for a partial breakdown of some internal symmetries, the gauge symmetry of the $\sigma$-model, that is the conformal symmetry, should always be preserved. As was shown in Ref. [RS09], one linear combination of the conformal currents, to wit, the one that generates diffeomorphisms preserving the defect line, $T_{++} - T_{--}$, is automatically preserved at the defect by virtue of the DGC – this is the content of Theorem 2.9. The last property identifies the defects considered as conformal in the sense of Ref. [OA97].

For a space-like defect line, which is what we have been considering in the present section, it is the other linear combination,

$$T_{++} + T_{--} \equiv \frac{1}{2} \left( \left( g^{-1} \right)^{\mu \nu} (X) p_\mu (X) p_\nu (X) + g_{\mu \nu} (X) (X_\sigma t)^{\mu} (X_\tau t)^{\nu} \right) = \mathcal{H}_\sigma,$$

that gives the hamiltonian density $\mathcal{H}_\sigma$ of the $\sigma$-model. It generates conformal transformations on the world-sheet which deform the defect line, and hence the continuity of $\mathcal{H}_\sigma$ at the latter is related to the extendibility of the defect, as made precise in

**Definition 4.2.** Let $\mathfrak{B}$ be a string background with target $\mathcal{M} = (M,g,\mathcal{G})$ and $\mathcal{G}$-bi-brane $\mathcal{B} = (Q, \iota_\sigma, \omega, \Phi | \alpha \in \{1,2\})$, and let $(X | \Gamma)$ be a network-field configuration in string background $\mathfrak{B}$ on world-sheets $(\Sigma, \gamma)$ with defect quiver $\Gamma$. Denote by $U$ a tubular neighbourhood of an edge $\ell$ of $\Gamma$ within $\Sigma$, with the property $U \cap \Gamma = \ell$. The neighbourhood $U$ is split by the oriented line $\ell$ into subsets $U_\alpha$, $\alpha \in \{1,2\}$ as in Definition 2.6. An extension $\tilde{X}$ of network-field configuration $(X | \Gamma)$ on neighbourhood $U$ of defect line $\ell$ is a map $\tilde{X} : U \to Q$, such that

$$\tilde{X}|_{\ell} = X, \quad \iota_\alpha \circ \tilde{X}|_{U_\alpha} = X|_{U_\alpha},$$

and such that the relation

$$\iota_1^* \mathcal{L}_\mathfrak{B}(\tilde{X}(p)) \left( \tilde{X}_*, \tilde{u}^\perp, \ldots \right) - \iota_2^* \mathcal{L}_\mathfrak{B}(\check{X}(p)) \left( \check{X}_*, \check{u}^\perp, \ldots \right) - \tilde{X}_* \check{u} \omega (\check{X}(p)) = 0, \quad \check{u}^\perp = \gamma^{-1} (\check{u} \cdot \text{Vol}(\Sigma, \gamma), \cdot)$$

is satisfied at any point $p \in U$ and for an arbitrary vector $\check{u} \in T_p \Sigma$. A defect $\ell$ of a network field configuration that admits an extension on a neighbourhood of $\ell$ shall be termed extendible.

The less restrictive condition of continuity of the conformal current gives rise to

**Definition 4.3.** Let $\mathfrak{B}$ be a string background with target $\mathcal{M} = (M,g,\mathcal{G})$, and let $\Gamma$ be a defect quiver embedded in a world-sheets $\Sigma$. Consider the non-linear $\sigma$-model for network-field configurations $(X | \Gamma)$ in string background $\mathfrak{B}$ on world-sheets $(\Sigma, \gamma)$ with defect quiver $\Gamma$. Choose a local coordinate system $\{ \sigma^a \}_{a \in \{1,2\}}$ in the neighbourhood of a point in $\Gamma$. The defect $\Gamma$ is called topological iff the conformal current $T$, with local components

$$T^{ab} = \frac{2}{\sqrt{\det \gamma}} \frac{\delta S_\mathcal{G}_{\gamma}}{\delta \gamma_{ab}},$$

is continuous across $\Gamma$.

**Remark 4.4.** The notion of topologicality can be regarded as a classical counterpart of the quantum concept introduced in Ref. [PZ01].

* * *

Extendible defects have the desired property of topologicality, as stated in

\textsuperscript{9}A symplectomorphism can be viewed as a maximal isotropic in $TP^2$, cf., e.g., Ref. [Woo92].

\textsuperscript{10}Note that the introduction of the bi-brane, whose world-volume is a priori not related to the target space, allows for a unified treatment of symmetries which do not leave a connected component of the target space (and solely put in correspondence extremal sections that map the world-sheet into different regions thereof) and proper dualities which act between different connected components of the target space, oftentimes of inequivalent topology. A notable example of the latter type is the T-duality between principal torus bundles, cf. Example 4.16.
Theorem 4.5. [RS09b, Sec. 2.9] The non-linear \( \sigma \)-model of Definition 2.7 for network-field configurations \((\Sigma, X)\) in string background \(\mathcal{B}\) on world-sheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\) composed of extendible defects is invariant with respect to arbitrary (gauge) transformations

\[
X \mapsto X \circ D, \quad \gamma \mapsto D^* \gamma, \quad D \in Diff^*(\Sigma),
\]

\[
\gamma \mapsto e^{2w} \cdot \gamma, \quad e^{2w} \in \text{Weyl}(\gamma)
\]

from the semidirect product \(Diff^*(\Sigma) \rtimes \text{Weyl}(\gamma)\) of the group \(Diff^*(\Sigma)\) of (orientation-preserving) diffeomorphisms of \(\Sigma\) with the group \(\text{Weyl}(\gamma)\) of Weyl rescalings of the metric \(\gamma\). All components of the conformal current are continuous across the defect lines of \(\Gamma\), that is the defect is \textit{topological} in the sense of Definition 4.3.

Remark 4.6. From the point of view of the categorial quantisation of the \(\sigma\)-model in the presence of defects, as discussed, e.g., in Ref. [RS09b], it is natural to expect a topological defect to be deformable, and hence necessarily extendible. However, the statement in the quantum theory is usually formulated in terms of correlation functions assigned to (decorated) world-sheets with embedded defect quivers, and so – in the present setting – the issue of finding its proper classical counterpart is obscured by aspects of the quantisation procedure such as the choice of the renormalisation scheme that affects the field-theoretic functionals entering the DGC, cf., e.g., Refs. [BG04, AM07, BM10] for an illustration. As we are not addressing here the issue of quantisation beyond the construction of the pre-quantum bundle, we shall restrict ourselves to topological rather than extendible defects in what follows.

\* \* \*

Thus, it is amidst topological defects that we should look for those that describe dualities of the untwisted \(\sigma\)-model. Before we do that, however, let us make the very notion of duality precise, using the various field-theoretic constructs introduced hitherto.

Definition 4.7. Let \(\mathcal{B}\) be a string background with target \(\mathcal{M} = (M, g, \mathcal{G})\), and let \((P_{\sigma, \omega} = T^*LM, \Omega_{\sigma, \omega})\) be the untwisted state space of the non-linear \(\sigma\)-model for network-field configurations \((X | \Gamma)\) in string background \(\mathcal{B}\) on world-sheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\). Furthermore, let \(\pi_{\mathcal{L}, \omega} : \mathcal{L}_{\sigma, \omega} \to P_{\sigma, \omega}\) be the pre-quantum bundle for the untwisted sector of the \(\sigma\)-model, constructed in Corollary 3.17. A pre-quantum duality of the untwisted sector of the non-linear \(\sigma\)-model for network-field configurations \((X | \Gamma)\) in string background \(\mathcal{B}\) on world-sheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\) is a pair \((\mathcal{J}_\sigma, \mathcal{D}_\sigma)\) which consists of

- a graph \(\mathcal{J}_\sigma \subset P_{\sigma, \omega}^2\), isotropic with respect to the ‘difference’ symplectic form \(\Omega^*_{\sigma, \omega}\) of Proposition 4.1, and having the property that the difference

\[
\mathcal{H}_\sigma^- = pr_1^* \mathcal{H}_\sigma - pr_2^* \mathcal{H}_\sigma
\]

of the pullbacks, along the canonical projections \(pr_\alpha : P_{\sigma, \omega} \times P_{\sigma, \omega} \to P_{\sigma, \omega}\), \(\alpha \in \{1, 2\}\), of the hamiltonian density \(\mathcal{H}_\sigma\) of the \(\sigma\)-model, as given in Eq. 4.4, vanishes identically on restriction to \(\mathcal{J}_\sigma\);

- a bundle isomorphism

\[
\mathcal{D}_\sigma : pr_1^* \mathcal{L}_{\sigma, \omega}\big|_{\mathcal{J}_\sigma} \xrightarrow{\cong} pr_2^* \mathcal{L}_{\sigma, \omega}\big|_{\mathcal{J}_\sigma},
\]

between the restrictions to \(\mathcal{J}_\sigma\) of the pullbacks of \(\mathcal{L}_{\sigma, \omega}\) along the canonical projections \(pr_\alpha\).

\* \* \*

Remark 4.8. Pre-quantum dualities of the \(\sigma\)-model which are consistent with a given choice of the polarisation of the pre-quantum bundle defining the Hilbert space of the theory give rise to bona fide dualities of the quantised \(\sigma\)-model.

\* \* \*

The first relation between conformal defects and \(\sigma\)-model dualities is established in the following

Theorem 4.9. Let \(\mathcal{B}\) be a string background with target \(\mathcal{M} = (M, g, \mathcal{G})\) and \(\mathcal{G}\)-bi-brane \(\mathcal{B} = (Q, \iota_\alpha, \omega, \Phi | \alpha \in \{1, 2\})\), and consider the non-linear \(\sigma\)-model for network-field configurations \((X | \Gamma)\) in string background \(\mathcal{B}\) on world-sheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\). The \(\mathcal{G}\)-bi-brane \(\mathcal{B}\) together with the Defect Glueing Condition (2.8) canonically defines a pre-quantum duality of the untwisted sector of the \(\sigma\)-model iff the following conditions are satisfied:
Recall that, in virtue of Proposition 4.1, we still have to require that upon choosing a specific parent loop $\sigma$, for arbitrary $i\nu$ we have a common triangulation of both loops in $(\psi_1, \psi_2) \in \mathcal{J}_\sigma(B)$ coming from a triangulation of the parent loop $X \in \mathcal{L}$. Thus, in particular, at each point $(\psi_1, \psi_2) \in \mathcal{J}_\sigma(B)$, we have a common triangulation $\Delta(S^1)$, with edges $e$ and vertices $v$, of the unit circle parameterising $X$ and $X_\alpha = \tau_\alpha \circ X$, and, for each element $f \in \Delta(S^1)$, a triple of indices $(i^1, i^2, i^3) \in \mathcal{L} \times \mathcal{L} \times \mathcal{Q}$, related as per

$$i^2 = \phi_\alpha(i^1, i^2).$$

Next, fix a local presentation of $\mathcal{L}$ associated with this choice of covers as in Definition 2.2. It is then a matter of a simple calculation to verify that the local data $(\theta_{\sigma, \alpha \nu}, \gamma_{\sigma, \alpha \nu})$ of the pre-quantum bundle $\mathcal{L}_{\sigma, \alpha}$ associated – as in Corollary 5.17 – with the open cover of a cartesian factor in $\mathcal{J}_\sigma(B)$ (induced as in Proposition 4.13) satisfy the identities

$$p_{\sigma, \alpha}^\star \theta_{\sigma, \sigma \nu} = -id \log f_{\sigma, B}(i^1, i^2),$$

$$p_{\sigma, \alpha}^\star \gamma_{\sigma, \alpha \nu} = f_{\sigma, B}(i^1, i^2) \cdot p_{\sigma, \alpha}^\star \theta_{\sigma, \sigma \nu} \cdot f_{\sigma, B}(i^1, i^2)^{-1},$$

written in terms of the canonical projections $p_{\sigma, \alpha} : \mathcal{J}_\sigma(B) \to P_{\sigma, \sigma}$, $\alpha \in \{1, 2\}$ and of the $U(1)$-valued functionals

$$f_{\sigma, B}(i^1, i^2)[(\psi_1, \psi_2)] = \prod_{v \in \Delta(S^1)} e^{\int_{\tau_\alpha \circ X_\alpha}(\psi_1, \psi_2)} X^\star_{\tau_\alpha \circ X_\alpha}(v)$$

on $P_{\sigma, \alpha} \times P_{\sigma, \alpha} \subset \mathcal{J}_\sigma(B)$, where $P_{\sigma, \alpha} = \pi_{\tau_\alpha \circ X_\alpha}^{-1}(\mathcal{L})$. Hence, the $f_{\sigma, B}(i^1, i^2)$ can be identified with local data of an isomorphism $\mathcal{D}_{\sigma}(B)$ from Definition 4.7. It remains to establish the conditions under which the isotropic submanifold $\mathcal{J}_\sigma(B) \subset P_{\sigma, \sigma}$ becomes a graph.

For this to be the case, it is necessary that the two maps $\tau_\alpha$ be surjective so that any loop in $M$ can be descended from a loop in $Q$. Having thus established a correspondence, fibred over $Q$, between loops in either cartesian factor of $P_{\sigma, \sigma}$, or – in the world-sheet picture – on either side of the defect line, we still have to require that upon choosing a specific parent loop $X \in \mathcal{L}$ and thus picking up a pair $(X_1, X_2)$ of loops from $\mathcal{L}$ and putting them in correspondence, and upon determining either of the
loop momenta, \( p_1 \) or \( p_2 \), the other loop momentum is already fixed uniquely by the DGC. Inspection of the latter,
\[
p_1 \circ \tilde{\tau}_1 - p_2 \circ \tilde{\tau}_2 - X_1 \tilde{\tau} \omega = 0,
\]
reveals that for this to hold, also the tangent maps \( \tilde{\tau}_\alpha \) must admit local right inverses. That is, altogether, the \( \tilde{\tau}_\alpha \) should be surjective submersions, with smooth local sections \( s_{i_{1,\alpha}}: O_{i_{\alpha}} \to LQ \) satisfying the identities
\[
\tilde{\tau}_\alpha \circ s_{i_{1,\alpha}} = \text{id}_{O_{i_{\alpha}}}, \quad \tilde{\tau}_{\alpha^*} \circ s_{i_{1,\alpha^*}} = \text{id}_{\Gamma(TM|O_{i_{\alpha}})}.
\]
Indeed, for \( X_\alpha \in O_{i_{\alpha}}, \ X \in (\iota_1 \times \iota_2)^{-1}\{(X_1, X_2)\} \) and a pair of sections \( (s_{i_1,1}, s_{i_2,2}) \) such that
\[
(4.13) \quad X = s_{i_{1,\alpha}}(X_\alpha),
\]
there arise functional relations
\[
(4.14) \quad X_2(X_1) = \tilde{\tau}_2 \circ s_{i_{1,1}}(X_1), \quad X_1(X_2) = \tilde{\tau}_1 \circ s_{i_{2,2}}(X_2)
\]
between the loop coordinates, alongside the functional relations
\[
(4.15) \begin{align*}
p_2(p_1, X_1) &= p_1 \circ \tilde{\tau}_1 \circ s_{i_{1,2}} + (X_2 \tilde{\tau} \omega(X)) \circ s_{i_{1,2}}, \\
p_1(p_2, X_2) &= p_2 \circ \tilde{\tau}_2 \circ s_{i_{1,1}} + (X_1 \tilde{\tau} \omega(X)) \circ s_{i_{1,1}}
\end{align*}
\]
between the loop momentum coordinates on \( \mathcal{J}_\sigma(B) \). The dependence of the loop momentum \( p_\alpha \) on the loop coordinate \( X_{3-\alpha} \) is given by Eq. (4.13). The above are statements valid at every point along the loop, and the pushforward operators \( s_{i_{1,\alpha}} \) are to be understood as characterising the local sections of the \( \tilde{\tau}_\alpha \) that enter the definition of the \( s_{i_{1,\alpha}} \). The relations define a graph if they agree for any two choices \( s_{i_{1,\alpha}}^1, s_{i_{1,\alpha}}^2 \) of sections (i.e. for any two choices \( X_\alpha \in LQ, \ n \in \{1, 2\} \) of the parent loop) corresponding to a given pair \( X_\alpha \) of loops in \( M \). This is tantamount to imposing conditions
\[
(4.7) \quad \text{which ensure that arbitrary curves of loops in } L.M \text{ are mapped into one another in a unique manner},
\]
\[
\text{together with}
\]
\[
\left(X_1 \tilde{\tau} \omega(X^1)\right) \circ s_{i_{1,\alpha^*}^1} = \left(X_2 \tilde{\tau} \omega(X^2)\right) \circ s_{i_{1,\alpha^*}^2},
\]
or – equivalently –
\[
X_\alpha \tilde{\tau} \left(s_{i_{1,\alpha^*}^1} \omega - s_{i_{1,\alpha^*}^2} \omega\right)(X_\alpha) = 0.
\]

The corresponding local statement, at a given point \( X_\alpha(\varphi) \in O_{i_{\alpha}}^M \) along the loop in \( M \), yields Eq. (4.8) by virtue of the arbitrariness of the vector \( X_\alpha \tilde{\tau} \).

Finally, the identity from point iv) is a simple rewrite of the condition \( \mathcal{H}_{\sigma_{\| \mathcal{E}_\tau}} = 0 \) taking into account the relations (4.13) and (4.15).

**Remark 4.10.** It deserves to be noted that the requirement that the state correspondence engendered by the defect be independent of the choice of the local section of the surjective submersion \( \tilde{\tau}_\alpha : LQ \to LM \) is automatically satisfied in the (physically) most natural setting, which is that of \( Q \) being a submanifold within \( M \times M \) projecting surjectively on both cartesian factors.

**Remark 4.11.** Our discussion indicates that surjective submersions play a prominent rôle in the canonical description of dualities of the \( \sigma \)-model on world-sheets with defect quivers. This is to be compared with the categorial treatment of gerbes and gerbe bi-modules in Ref. [FNSW09] which also appears to distinguish maps of this kind, albeit in a more formal manner.

Once the conditions for the defect to describe a duality of the untwisted sector of the \( \sigma \)-model have been established, it is tempting to reverse the question and enquire as to the necessary conditions for a duality to define a bi-brane that can subsequently be put over a defect line. General as it stands, the question falls beyond the compass of the present paper. We may, nonetheless, try to draw useful insights from the study of a wide class of dualities for which there exists a concise explicit description in terms of canonical transformations on the state space of the untwisted sector of the \( \sigma \)-model, determined by generating functionals of a restricted ‘linear’ form, to be described below. Dualities of this type, including abelian and non-abelian dualities, as well as the Poisson–Lie T-duality of the WZW model,
were examined in a series of papers by Alvarez, Refs. [Alv00a, Alv00b], from which we borrow some of our conventions and a number of observations.

The chief idea of the approach advertised above consists in explicitly enforcing the isotropy of the space \( \mathfrak{I}_\sigma \) of sections of the tangent bundle of a graph \( \mathcal{I}_\sigma \) in \( \mathcal{P}_\sigma \) representing the duality by trivialising the symplectic potential of \( \Omega_{\sigma, \xi} \) with the help of a generating functional of a canonical transformation \( \psi_1 \mapsto \psi_2 \) determined by the graph \( \mathcal{I}_\sigma \ni (\psi_1, \psi_2) \), \( \psi_\alpha = (X_\alpha, p_\alpha) \), \( \alpha \in \{1, 2\} \). In so doing, the generating functional is chosen such that the transformation between the two sets of variables: \( (X_1, \mathcal{I}_1, p_1) \) and \( (X_2, \mathcal{I}_2, p_2) \) induced by the canonical transformation is invertible and preserves the hamiltonian density \( 4.4 \).

The latter condition, in conjunction with the distinguished form of the hamiltonian density (a sum of terms quadratic in \( p \) and \( X, \mathcal{I} \), respectively), was used in Ref. [Alv00a] to restrict the choice of the generating functional, for a specific trivialisation of \( \Omega_{\sigma, \xi} \), to those depending linearly on the \( X, \mathcal{I} \) and further constrained by the requirement of orthogonality with respect to the metric \( (g, g^{-1}) \) entering the definition of the hamiltonian density.

An obvious problem with the above description of a canonical transformation lies with the lack of a global definition of the symplectic potential of \( \Omega_{\sigma, \xi} \) in general, a simple variation on the theme of the lack of a global definition of the topological term in the \( \sigma \)-model action functional, only transferred one degree lower in cohomology and – simultaneously – from the target space to its free-loop space. Below, we resolve this problem by considering the full structure of the pre-quantum bundle over the state space of the untwisted sector of the \( \sigma \)-model, in a local presentation suggested by Corollary 3.17 in conjunction with Proposition 3.13. Moreover, we extend the analysis to a larger class of trivialisations of \( \Omega_{\sigma, \xi} \), thereby gaining access to a canonical description of geometric symmetries of the \( \sigma \)-model.

With view towards organising the discussion of our results, we begin by providing a precise description of the class of dualities to be considered in the sequel.

**Definition 4.12.** Let \( \mathcal{B} \) be a string background with target \( \mathcal{M} = (M, g, \mathcal{G}) \), and let \( (\mathcal{I}_\sigma, \mathcal{D}_\sigma) \) be a pre-quantum duality of the untwisted sector of the non-linear \( \sigma \)-model for network-field configurations \( (X, \mathcal{G}) \) in string background \( \mathcal{B} \) on world-sheet \( (\Sigma, \gamma) \) with defect quiver \( \mathcal{G} \). Assume that \( (\mathcal{I}_\sigma, \mathcal{D}_\sigma) \) is determined by a generating functional \( \Phi_\sigma \) of a canonical transformation \( \mathcal{D}_\sigma : \mathcal{P}_{\sigma, \xi} \rightarrow \mathcal{P}_{\mathcal{I}^* \sigma, \xi} \), given as a collection of smooth real-valued functionals \( \Phi_\sigma \) on elements of an open cover \( \mathcal{O}_{\mathcal{I}^* \sigma} = \{ O_{\mathcal{I}^* \sigma} \}_{i \in \mathcal{I}_\sigma} \) of \( \mathcal{I}_\sigma \), i.e. \( \mathcal{I}_\sigma \) is the graph of \( \mathcal{D}_\sigma \) and the local data of \( \mathcal{D}_\sigma \) yield a local presentation of the bundle isomorphism \( \mathcal{D}_\sigma \). Assume further that the \( \Phi_\sigma \) depend at most linearly on the variables \( (X_\alpha, \mathcal{I}_\sigma, p_\alpha) \), \( \alpha \in \{1, 2\} \). Fix \( \mathcal{O}_{\mathcal{I}^* \sigma} \) to be the open cover induced from the open covers \( \mathcal{O}_{\mathcal{M}} \) of the free-loop space \( \mathcal{M} = C^\infty(\mathcal{S}^1, M) \) of the target space \( M \) from Proposition 3.13 on the cartesian factors of \( \mathcal{I}_\sigma \), coming from a sufficiently fine good open cover \( \mathcal{O}_M \) of \( M \), so that elements of \( \mathcal{O}_{\mathcal{I}^* \sigma} \) are of the form

\[
\mathcal{O}_{\mathcal{I}^* \sigma} = \mathcal{O}_M^\alpha \times \mathcal{O}_M^2
\]

for \( \mathcal{O}_M^\alpha \) as defined in Corollary 3.17. We call \( (\mathcal{I}_\sigma, \mathcal{D}_\sigma) \) a pre-quantum duality of type \( T \) of the untwisted sector of the non-linear \( \sigma \)-model for network-field configurations \( (X, \mathcal{G}) \) in string background \( \mathcal{B} \) on world-sheet \( (\Sigma, \gamma) \) with defect quiver \( \mathcal{G} \) iff its local data can be put in the form

\[
\Phi_\sigma[(\psi_1, \psi_2)] = \sum_{\mathcal{I} \in \Delta(\mathcal{I}^*)} \left\{ \int (X_1 e_i, X_2 e_j) \right\} \mathcal{P}_{(i^1, i^2)} + i \sum_{\mathcal{I} \in \Delta(\mathcal{I}^*)} (X_1, X_2) \log K_{(i^1, \nu_1^1, i^2, \nu_1^2)} (i^1, \nu_1^1, i^2, \nu_1^2) (v)
\]

\[4.16\]

for \( (\psi_1, \psi_2) \in \mathcal{O}_{\mathcal{I}^* \sigma} \) with \( \psi_\alpha = (X_\alpha, p_\alpha) \), some smooth 1-forms \( \mathcal{P}_{(i^1, i^2)} \) on \( \mathcal{O}_M^\alpha \times \mathcal{O}_M^2 \) and some smooth U(1)-valued maps \( K_{(i^1, i^2), (j^1, j^2)} = K_{(j^1, j^2), (i^1, i^2)} \) on \( \mathcal{O}_M^1 \times \mathcal{O}_M^2 \). The data are required to satisfy the identities

\[
pr_1^* (\theta_{\mathcal{I}_\mathcal{M}} + \pi_{\mathcal{I}_\mathcal{I}_\mathcal{M}} E_{i^1}) - pr_2^* (\theta_{\mathcal{I}_\mathcal{I}_\mathcal{M}} + \pi_{\mathcal{I}_\mathcal{I}_\mathcal{M}} E_{i^2}) = -i \delta \log f_{\sigma[(i^1, i^2)]},
\]

\[4.17\]

\[
pr_1^* \pi_{\mathcal{I}_\mathcal{I}_\mathcal{M}} G_{1^1, 1^2} \cdot pr_2^* \pi_{\mathcal{I}_\mathcal{I}_\mathcal{M}} G_{1^2, 1^1} = f_{\sigma[(i^1, i^2)]} \cdot f_{\sigma[(i^1, i^2)]}^{-1}
\]

written in terms of the smooth U(1)-valued functionals

\[4.18\]

\[
f_{\sigma[(i^1, i^2)]} = e^{-\Phi_\sigma[(i^1, i^2)]}
\]

and of the canonical projections \( pr_\alpha : \mathcal{P}_{\mathcal{I}^* \sigma} \times \mathcal{P}_{\mathcal{I}^* \sigma} \rightarrow \mathcal{P}_{\mathcal{I}^* \sigma}, \alpha \in \{1, 2\} \), the canonical 1-form \( \theta_{\mathcal{I}_\mathcal{I}_\mathcal{M}} \) on the total space of the cotangent bundle \( \pi_{\mathcal{I}_\mathcal{I}_\mathcal{M}} : \mathcal{T}^* \mathcal{M} \rightarrow \mathcal{M} \) from Proposition 3.11 and the local data \( (E_i, G_{ij}) \) of the transgression bundle \( \mathcal{L}_{\mathcal{G}} \rightarrow \mathcal{M} \) from Theorem 3.16.

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Analogously, we call \((J, \mathcal{D}_\sigma)\) a pre-quantum duality of type \(N\) of the untwisted sector of the non-linear \(\sigma\)-model for network-field configurations \((X \mid \Gamma)\) in string background \(\mathcal{B}\) on world-sheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\) iff, in the same notation, its local data can be put in the form
\[
\Phi_{\sigma}(\psi_1, \psi_2) = -\int_{\mathcal{S}_1} \text{Vol}(\mathcal{S}_1) \ p_2 \mu \ F^\mu(X_1) + W_{(i^1_i, i^2_i)}(\psi_1, \psi_2)
\]
for an arbitrary smooth map
\[
F : M \to M,
\]
and for smooth real-valued functionals \(W_{(i^1_i, i^2_i)}\) on the \(\mathcal{O}_{(i^1_i, i^2_i)}\) of the form
\[
W_{(i^1_i, i^2_i)} = \sum_{e \in \Delta(\mathcal{S}_1)} \int_{\Sigma} (X_{1e}, X_{2e}) \ P_{(i^1_i, i^2_i)} + \sum_{v \in \Delta(\mathcal{S}_1)} (X_1, X_2) \ \log K_{(i^1_i, i^2_i)}(\psi_1, \psi_2) \ .
\]
(4.20)

\[
\text{with } P_{(i^1_i, i^2_i)} \in \Omega^1(\mathcal{O}_1 \times \mathcal{O}_2) \text{ and } K_{(i^1_i, i^2_i)}(\psi_1, \psi_2) \in U(1) \mathcal{O}_1 \times \mathcal{O}_2 . \ (\text{It is understood that there is no dependence on the } p_{\alpha} \text{ in } W_{(i^1_i, i^2_i)}.) \text{ Here, the identities to be satisfied by } f_{\sigma}(i^1_i, i^2_i) \text{ as in Eq. (4.18) read}
\]
\[
\begin{align*}
\text{pr}_1^* \left( \theta_{\Gamma, \mathcal{L}_M} + \pi_1 \mathcal{L}_M \mathcal{E}_1 \right) & - \text{pr}_2^* \left( \theta_{\Gamma, \mathcal{L}_M} + \pi_2 \mathcal{L}_M \mathcal{E}_2 \right) = -i \delta f_{\sigma}(i^1_i, i^2_i), \\
\text{pr}_1^* \pi_1 \mathcal{L}_M G_{i^1_i} \cdot \text{pr}_2^* \pi_2 \mathcal{L}_M G_{i^2_i}^{-1} & = f_{\sigma}(i^1_i, i^2_i) \cdot f_{\sigma}(i^1_i, i^2_i),
\end{align*}
\]
(4.21)

where
\[
\theta_{\Gamma, \mathcal{L}_M} = -\int_{\mathcal{S}_1} \text{Vol}(\mathcal{S}_1) \wedge X^\mu \delta p_\mu .
\]

**Remark 4.13.** The form of the edge terms in the definition (4.16) of the local data of the generating functional of the duality of type \(T\) is dictated by the requirement that the ensuing canonical transformation induce a linear map \((X_1, \mathcal{T}, p_1) \to (X_2, \mathcal{T}, p_2)\), as discussed earlier in this section and in Alvarez’s papers, and the vertex corrections are perfectly consistent with this requirement in the local description of the generating functional.

The expression (4.19) defining the generating functional of the duality of type \(N\), on the other hand, should be regarded as a natural local deformation of the global generating functional
\[
\Phi_{\text{id}}[(\psi_1, \psi_2)] = -\int_{\mathcal{S}_1} \text{Vol}(\mathcal{S}_1) \ p_2 \mu X_1^\mu ,
\]
readily verified to yield the identity canonical transformation on \(\mathcal{P}_{\sigma, \mathcal{B}}\). Thus, unlike dualities of type \(T\), dualities of type \(N\) are continuously deformable to the trivial duality (i.e. to the identity symplectomorphism).

\[
* \quad * \quad *
\]

We are now ready to present our findings which can be summarised as follows

**Theorem 4.14.** Let \(\mathcal{B}\) be a string background with target \(\mathcal{M} = (M, g, \mathcal{G})\). Consider the non-linear \(\sigma\)-model for network-field configurations \((X \mid \Gamma)\) in string background \(\mathcal{B}\) on world-sheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\). For every duality \((J, \mathcal{D}_\sigma)\) of type \(T\) of the \(\sigma\)-model, there exists a topological defect with a \(\mathcal{G}\)-bi-brane \(\mathcal{B}_{\mathcal{D}_\sigma} = (Q, \iota_{\alpha}, \omega, \Phi \mid \alpha \in \{1, 2\})\) over it with the following properties:

i) the world-volume \(Q\) is a submanifold of the cartesian square \(M \times M\) of the target space \(M\);

ii) the \(\mathcal{G}\)-bi-brane maps are given by the canonical projections \(\iota_{\alpha} : Q \rightarrow M, \alpha \in \{1, 2\}\);

iii) \(Q\) carries a symplectic form
\[
\Omega_{\mathcal{D}_\sigma}(X_1, X_2) = \omega^{i^2_1}_{\mu \nu}(X_1, X_2) \ dX_1^\mu \wedge dX_2^\nu , \quad (X_1, X_2) \in Q
\]
defined by the curvature (no summation over the repeated indices)
\[
\omega(X_1, X_2) = \sum_{0 < i < j < 3} (-1)^{i+1} \omega_{i \alpha \beta} , \quad \omega_{i \alpha \beta} = \omega^{i \alpha \beta}_{\mu \nu}(X_1, X_2) \ dX_1^\mu \wedge dX_2^\nu
\]
of \(\mathcal{B}_{\mathcal{D}_\sigma}\), the latter being given in terms of globally smooth maps \(\omega^{i \alpha \beta}_{\mu \nu} \in C^\infty(Q, \mathbb{R})\).
iv) the duality background

\[ \mathcal{B}_{D_s} = (M, \mathcal{B}_{D_s}) \]

satisfies the duality-background constraints

\[ E_2 = -\omega_{1\Delta} \circ \nu_1^{-1} \circ \omega_{1\Delta}, \]

written in terms of the background operators

\[ E_\alpha := g_\alpha + \omega_{\alpha\alpha} : \Gamma(TQ) \to \Gamma(T^*Q) : \gamma \mapsto E_\alpha(\gamma \cdot \cdot \cdot), \quad g_\alpha = pr_{\alpha}^*g, \quad \alpha \in \{1, 2\}. \]

Proof: Let us adopt the notation of Definition 4.12. Using the identity

\[ \delta \int_C \eta^* \eta = - \int_C \eta^* \delta \eta + X^* \eta|_{|C}, \]

valid for an arbitrary edge \( e \in \Delta(S^1) \) and for any \( \eta \in \Omega^1(X(e)) \), we readily extract from the first of identities \((4.17)\) the relations

\[ p_1 - p_2 = (X_1, T, X_2, T) \quad \big( \text{pr}_1^*B_1 - \text{pr}_2^*B_2 = \text{pr}_1^*B_1 + \text{pr}_2^*B_2 \big) \]

\[ p_1 - p_2 = \big( X_1, T, X_2, T \big) \quad \big( \text{pr}_1^*A_{ij} + pr_2^*A_{ij} + P_{ij, j^2} - P_{ij, i^2} - i \log K_{ij, j^2} \big) = 0, \]

implied by the requirement that both the edge term and the vertex term of the identity vanish independently. The relations are to be satisfied on the submanifold \( Q \subset M \times M \) obtained by taking the set of all pairs of points in \( M \) intersected by pairs of loops from \((\pi_T, M, \pi_T, M)(\Sigma, \pi, \alpha)\). The manifold \( Q \) canonically projects onto \( M \). Taking the exterior derivative of both sides of Eq. \((4.25)\) and, subsequently, using Eq. \((2.3)\), we obtain the equality

\[ \text{pr}_1^*B_1 - \text{pr}_2^*B_1 + dP_{ij, i^2} = \Gamma(\wedge^2 T^*Q). \]

This is in keeping with Eq. \((4.24)\) as the latter requires that the expression on the right-hand side be a smooth 1-form. We also note that it yields a relation

\[ K_{ij, j^2}(k^1, k^2) \cdot K_{ij, j^2}(k^1, k^2) \cdot K_{ij, j^2}(k^1, k^2) \cdot K_{ij, j^2}(k^1, k^2) \cdot \text{pr}_2^*g_{ij, j^2} \cdot \text{pr}_1^*g_{ij, j^2} =: C_{ij, j^2}(k^1, k^2), \]

\[ (4.27) \]

in which \((C_{ij, j^2}(k^1, k^2))\) is a locally constant \( U(1) \)-valued Čech 2-cochain on \( Q \). Clearly,

\[ (\hat{\delta}(2) C)_{ij, j^2}(k^1, k^2)_{ij, \Sigma} = 1, \]

and the class \([\hat{\delta}(2) C]_{ij, j^2}(k^1, k^2)_{ij, \Sigma} \in H^2(Q, U(1))\) is readily seen to define the obstruction to the existence of a \( G \)-bi-brane \((Q, \omega, \text{pr}_1, \text{pr}_2, \Phi)\) with 1-isomorphism \( \Phi : \text{pr}_1^*G \to \text{pr}_2^*G \otimes I_\omega \) with local data \((P_{ij, i^2}, K_{ij, i^2}(j^1, j^2)). \) Indeed, Eqs. \((4.25), (4.26)\) and \((4.27)\) can be rewritten concisely in the familiar form

\[ \text{pr}_1^*B_1 - \text{pr}_2^*B_1 + dP_{ij, i^2} = \Gamma(\wedge^2 T^*Q), \]

from which we infer the existence of a globally defined 2-form

\[ \omega = \text{pr}_1^*B_1 - \text{pr}_2^*B_1 + dP_{ij, i^2} \in \Gamma(\wedge^2 T^*Q). \]

and it is immediately clear that a pair of 2-cochains \((C_{ij, j^2}(k^1, k^2))\) and \((C'_{ij, j^2}(k^1, k^2))\) cohomologous as per \((C'_{ij, j^2}(k^1, k^2)) = (C_{ij, j^2}(k^1, k^2)) \cdot \hat{\delta}(1) c\) for some (locally constant) 1-cochain \( c \) corresponds to a pair of 1-cochains \((K_{ij, j^2}(k^1, k^2))\) and \((K'_{ij, j^2}(k^1, k^2))\) related by the shift \((K'_{ij, j^2}(k^1, k^2)) = (K_{ij, j^2}(k^1, k^2)) \cdot c^{-1} \).

Finally, Eq. \((4.24)\) rewrites as

\[ p_1 - p_2 = (X_1, T, X_2, T) \quad \big( \text{pr}_1^*A_{ij} + pr_2^*A_{ij} + P_{ij, j^2} - P_{ij, i^2} - i \log K_{ij, j^2} \big) = 0, \]

and so we recover the complete description of a conformal defect up to the obstruction \([\hat{\delta}(2) C]_{ij, j^2}(k^1, k^2)\). The latter is removed on taking into account the second of identities \((4.17)\). Indeed, using Eqs. \((4.25)\) and \((4.27)\), we readily cast the above relation in the compact form

\[ \prod_{\gamma \in \Delta(S^1)} \big( X_1, X_2 \big) \quad \big( \frac{C_{ij, j^2}(\gamma)}{C'_{ij, j^2}(\gamma)} \big) \quad \big( \frac{C_{ij, j^2}(\gamma)}{C'_{ij, j^2}(\gamma)} \big) \quad \big( \frac{C_{ij, j^2}(\gamma)}{C'_{ij, j^2}(\gamma)} \big) \quad \big( \frac{C_{ij, j^2}(\gamma)}{C'_{ij, j^2}(\gamma)} \big) \big( \gamma \big) = 1, \]
which – in view of the arbitrariness of $(X_1,X_2)(\mathbf{v})$ and of the triangulation used – requires

$$C(i^{1},i^{2})(j^{1},j^{2})(k^{1},k^{2}) = C(i^{1},i^{2})(j^{1},j^{2})$$

(4.30)

for any pair of quadruples $(i^{\alpha},j^{\alpha},k^{\alpha},l^{\alpha}) \in \mathcal{J}_{M}^4$, $\alpha \in \{1,2\}$ such that $\mathcal{O}_{i^{\alpha},j^{\alpha},k^{\alpha},l^{\alpha}} \neq \emptyset$. Hence,

$$C(i^{1},i^{2})(j^{1},j^{2})(k^{1},k^{2}) = C(i^{1},i^{2})(j^{1},j^{2}) = \tilde{C}(i^{1},i^{2})(j^{1},j^{2})$$

and the newly defined maps $\tilde{C}(i^{1},i^{2})(j^{1},j^{2})$, with values in the set $\{-1,1\}$, form a locally constant 2-cochain – in particular,

$$\tilde{C}(j^{1},j^{2})(i^{1},i^{2}) = \tilde{C}^{-1}(i^{1},i^{2})(j^{1},j^{2})$$

Using Eqs. (4.28) and (4.30), we then find

$$C(i^{1},i^{2})(j^{1},j^{2})(k^{1},k^{2}) = C(i^{1},i^{2})(j^{1},j^{2}) \cdot C^{-1}(i^{1},i^{2})(j^{1},j^{2}) \cdot C(i^{1},i^{2})(j^{1},j^{2})(i^{1},i^{2})$$

$$= \tilde{C}(i^{1},i^{2})(j^{1},j^{2}) \cdot \tilde{C}^{-1}(i^{1},i^{2})(j^{1},j^{2}) \cdot \tilde{C}(i^{1},i^{2})(j^{1},j^{2})(i^{1},i^{2})$$

Clearly, the Čech cohomology class of this 2-cochain is trivial and it can be absorbed into a redefinition of the local data of the 1-isomorphism $\Phi$, cf. Eq. (4.27). This leaves us with statements iii) and iv) of the theorem to demonstrate.

The remainder of the proof uses solely elementary analysis of canonical transformations defined in terms of generating functions, cf., e.g., Ref. [MR94, Sec. 6.5]. Thus, upon recalling that the space $\mathcal{J}_{\sigma}$ is – by assumption – diffeomorphic to $T^*\mathbb{L}M$, we can choose the loop variables $(X_1,X_2)$ as independent local coordinates on $\mathcal{J}_{\sigma}$, which has the following two consequences: First of all, Eq. (4.26) yields three independent relations:

$$\omega_{1\Lambda 1} = \text{pr}_{1}^{*}B_{i^{1}} + \left[dP_{(i^{1},i^{2})}\right]_{1\Lambda 1},$$

(4.31)

$$\omega_{2\Lambda 2} = \text{pr}_{2}^{*}B_{i^{2}} - \left[dP_{(i^{1},i^{2})}\right]_{2\Lambda 2},$$

(4.32)

$$\omega_{1\Lambda 2} = \left[dP_{(i^{1},i^{2})}\right]_{1\Lambda 2},$$

(4.33)

written in terms of the components $\omega_{i,j}$ of $\omega$ from Eq. (4.22) and those of $dP_{(i^{1},i^{2})}$, defined analogously. Secondly, we may extract from Eq. (4.29) a pair of coupled equations

$$\begin{pmatrix} -\omega_{1\Lambda 1} & \text{id}_{\Gamma(T^*\mathbb{L}M)} \\ -\omega_{1\Lambda 2} & 0 \end{pmatrix} \begin{pmatrix} X_{1,\Lambda} \tilde{T} \\ \text{pr}_{1} \end{pmatrix} = \begin{pmatrix} \omega_{1\Lambda 2} & 0 \\ -\omega_{2\Lambda 2} & \text{id}_{\Gamma(T^*\mathbb{L}M)} \end{pmatrix} \begin{pmatrix} X_{2,\Lambda} \tilde{T} \\ \text{pr}_{2} \end{pmatrix},$$

(4.34)

to be understood as representing the action of linear operators on sections of $\mathcal{T}Q \oplus T^{*}\mathbb{L}Q$, with 2-form fields acting on vector fields through contraction, i.e. as per $\omega_{i,j} \triangleright X_{\alpha,\Lambda} \tilde{T} = X_{\alpha,\Lambda} \tilde{T} \cdot \omega_{i,j}$. Clearly, for the transformation between the two pairs $(X_{\alpha,\Lambda} \tilde{T},\alpha_{0})$, $\alpha \in \{1,2\}$ thus defined to be invertible, we have to demand that $\omega_{1\Lambda 2}$, regarded as a map from $\Gamma(TQ)$ to $\Gamma(T^{*}Q)$, possess an inverse,

$$\omega_{1\Lambda 2}^{-1} = \frac{1}{2} \left( (\omega^{1\Lambda 2})^{-1} \right)^{\mu
u} \partial_{\mu} \wedge \partial_{\nu},$$

(4.35)

$$\left( (\omega^{1\Lambda 2})^{-1} \frac{\lambda^{\mu} \omega_{1\Lambda 2}^{\lambda \nu}}{\delta^{\lambda \nu}} \right)$$

acting on 1-forms as $\omega_{1\Lambda 2}^{-1} \triangleright (\eta_{\mu} \text{d}X^{\nu}) := \frac{1}{2} \left( (\omega^{1\Lambda 2})^{-1} \right)^{\mu\nu} \eta_{\mu} \partial_{\nu}$. This proves statement iii) of the theorem. Having ensured the invertibility of $\omega_{1\Lambda 2}$, we may express $(X_{\alpha,\Lambda} \tilde{T},\alpha_{0})$ through $(X_{3-\alpha,\Lambda} \tilde{T},\alpha_{3-\alpha})$. Demanding that $\mathcal{H}_{\sigma}$ of Eq. (4.5) vanish identically on $\mathcal{J}_{\sigma}$ then produces a relation

$$M_{\omega} \circ \bar{g}_{1} \circ M_{\omega} = \bar{g}_{2},$$

written in terms of the operators

$$\bar{g}_{\alpha} = \begin{pmatrix} g_{\alpha} & 0 \\ 0 & g_{\alpha}^{-1} \end{pmatrix}, \quad \alpha \in \{1,2\}$$

and

$$M_{\omega} = \begin{pmatrix} \omega_{1\Lambda 2}^{-1} \circ \omega_{2\Lambda 2} & -\omega_{1\Lambda 2}^{-1} \\ \omega_{1\Lambda 2} + \omega_{1\Lambda 1} \circ \omega_{1\Lambda 2}^{-1} \circ \omega_{2\Lambda 2} & -\omega_{1\Lambda 1} \circ \omega_{1\Lambda 2}^{-1} \end{pmatrix},$$

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and of the transpose of the latter,

\[ M_T = \begin{pmatrix} \omega_{2 \alpha 2} \circ \omega_{1 \alpha 2}^{-1} & -\omega_{1 \alpha 2} - \omega_{2 \alpha 2} \circ \omega_{1 \alpha 2}^{-1} \circ \omega_{1 \alpha 1} \\ -\omega_{1 \alpha 2}^{-1} & \omega_{1 \alpha 2}^{-1} \circ \omega_{1 \alpha 1} \end{pmatrix}. \]

We shall next demonstrate that Eq. (4.34) is equivalent to the duality-background constraints (4.23).

To this end, we first note that the latter actually encodes a pair of independent relations for the symmetric and antisymmetric component of the background operator \( E \). Explicitly,

\[ g_2 = -\frac{1}{2} \omega_{1 \alpha 2} \circ (E_1^{-1} + E_1^{-T}) \circ \omega_{1 \alpha 2} \equiv -\frac{1}{2} \omega_{1 \alpha 2} \circ \left( (g_1 + \omega_{1 \alpha 1})^{-1} \circ (g_1 - \omega_{1 \alpha 1})^{-1} \circ (g_1 + \omega_{1 \alpha 1}) \circ (g_1 - \omega_{1 \alpha 1})^{-1} \circ (g_1 + \omega_{1 \alpha 1})^{-1} \circ (g_1 - \omega_{1 \alpha 1})^{-1} \circ \omega_{1 \alpha 2} \right) + (g_1 + \omega_{1 \alpha 1})^{-1} \circ (g_1 + \omega_{1 \alpha 1}) \circ (g_1 - \omega_{1 \alpha 1})^{-1} \circ \omega_{1 \alpha 2} = -\omega_{1 \alpha 2} \circ \left( (g_1 - \omega_{1 \alpha 1}) \circ g_1^{-1} \circ (g_1 + \omega_{1 \alpha 1})^{-1} \circ \omega_{1 \alpha 2} \equiv -\omega_{1 \alpha 2} \circ (g_1 - \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 1})^{-1} \circ \omega_{1 \alpha 2} \right), \]

and, analogously,

\[ \omega_{2 \alpha 2} = \omega_{1 \alpha 2} \circ g_1^{-1} \circ \omega_{1 \alpha 1} \circ \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 1} \circ \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 2} \circ \omega_{2 \alpha 2}. \]

The above are to be compared with the independent relations determined by the continuity constraint (4.34). These are easily found to be

\[ g_2^{-1} = \omega_{1 \alpha 2}^{-1} \circ (g_1^{-1} \circ g_1 \circ g_1^{-1} \circ g_1) \circ \omega_{1 \alpha 2}^{-1}, \]

\[ g_2 = \omega_{2 \alpha 2} \circ \omega_{1 \alpha 2} \circ g_1 \circ \omega_{1 \alpha 2} \circ \omega_{1 \alpha 2} - \left( (g_1 + \omega_{1 \alpha 1} \circ \omega_{1 \alpha 2} \circ \omega_{1 \alpha 2} \circ g_2 \circ \omega_{1 \alpha 2}) \circ g_1^{-1} \circ \left( \omega_{1 \alpha 2} + \omega_{1 \alpha 1} \circ \omega_{1 \alpha 2} \circ \omega_{1 \alpha 2} \circ g_2 \right) \right) \circ g_1^{-1} \circ \left( \omega_{1 \alpha 2} + \omega_{1 \alpha 1} \circ \omega_{1 \alpha 2} \circ \omega_{1 \alpha 2} \circ g_2 \right), \]

\[ 0 = \omega_{1 \alpha 2} \circ g_1 \circ \omega_{1 \alpha 2} \circ \omega_{2 \alpha 2} - \omega_{1 \alpha 2} \circ \omega_{1 \alpha 1} \circ g_1^{-1} \circ \left( \omega_{1 \alpha 2} + \omega_{1 \alpha 1} \circ \omega_{1 \alpha 2} \circ \omega_{1 \alpha 2} \circ g_2 \right). \]

The remaining relation is a transpose of the bottom one. Evidently, the top one is an inverse of the symmetric component of Eq. (4.23), and so we are left with the other two to examine.

Upon using the bottom relation in the middle one, we reduce the latter to the form

\[ g_2 = -\omega_{1 \alpha 2} \circ g_1^{-1} \circ \left( \omega_{1 \alpha 2} + \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 2} \circ g_2 \right), \]

which can be combined with (the inverse of) the top one and subsequently substituted back into the bottom relation to yield

\[ \omega_{2 \alpha 2} = \omega_{1 \alpha 2} \circ g_1^{-1} \circ \omega_{1 \alpha 1} \circ g_1^{-1} \circ \left( \omega_{1 \alpha 2} + \omega_{1 \alpha 1} \circ \omega_{1 \alpha 2} \circ g_2 \right) = -\omega_{1 \alpha 2} \circ g_1^{-1} \circ \omega_{1 \alpha 1} \circ \omega_{1 \alpha 2} \circ g_2 \]

which is the desired form of the antisymmetric component of Eq. (4.23). At this stage, it remains to verify that the two components found hitherto ensure that the remaining relation (4.35) is satisfied identically. With \( \omega_{2 \alpha 2} \) as above, its right-hand side takes the form

\[ g_2 = -\omega_{1 \alpha 2} \circ g_1^{-1} \circ \left[ \omega_{1 \alpha 2} - \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 1} \circ \left( \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 2} \right) \right], \]

and so we must show the identity

\[ -\omega_{1 \alpha 2} \circ g_1^{-1} \circ \left[ \omega_{1 \alpha 2} - \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 1} \circ \left( \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 2} \right) \right] = \omega_{1 \alpha 2} \circ \left( \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 1} \circ g_1^{-1} \circ \omega_{1 \alpha 2} \right), \]

which follows straightforwardly upon regrouping its terms. \( \square \)

A similar result can be established for dualities of type \( N \), namely,

**Theorem 4.15.** Let \( \mathcal{B} \) be a string background with target \( M = (M, g, \mathcal{G}) \). Consider the non-linear \( \sigma \)-model for network-field configurations \( (X, \Gamma) \) in string background \( \mathcal{B} \) on world-sheet \( (\Sigma, \gamma) \) with defect quiver \( \Gamma \). To every duality \( (\mathcal{Q}, \mathcal{D}_\alpha) \) of type \( N \) of the \( \sigma \)-model, there is associated a topological defect with a \( \mathcal{G} \)-bi-brane \( \mathcal{B}_{\mathcal{D}} = (Q, i_\alpha, \omega, \Phi \mid \alpha \in \{1, 2\}) \) over it with the following properties:

i) the world-volume \( Q \) is a submanifold \( Q = (\text{id}_M \times F)(M) \subset M \times M \) of the cartesian square \( M \times M \) of the target space \( M \);

ii) \( F \) is an isometry of the metric manifold \( (M, g) \);

iii) the \( \mathcal{G} \)-bi-brane maps are given by the canonical projections \( i_\alpha = \text{pr}_\alpha : Q \to M, \alpha \in \{1, 2\} \);

iv) the curvature \( \omega \) vanishes identically;
v) the pullback of the 1-isomorphism $\Phi$ along the isomorphism $\text{id}_M \times F : M \xrightarrow{\sim} Q$ is of the form (4.36) 
\[(\text{id}_M \times F)^*\Phi : \mathcal{G} \xrightarrow{\sim} F^*\mathcal{G}.
\]

Proof: We use the notation of Definition 4.12 and reasoning as in the proof of Theorem 4.14 – choose $(X_1, p_2)$ as independent local coordinates on $\mathcal{G}$, which leads to the relation (4.37)
\[X_2 = F[X_1],\]
extracted from the first of identities (4.17). Clearly, for the generating functional (4.19) to define a duality of the $\sigma$-model, $F$ has to induce an invertible map on $M$. The remaining relation encoded by the first of identities (4.17) reads
\[\int_{\mathbb{S}^1} \text{Vol}(\mathbb{S}^1) \wedge (p_1 - \tilde{\Phi} \circ p_2) = E_{i\bar{z}}[X_2] - E_{i\bar{z}}[X_1] - \delta W_{i\bar{z}1}\{\psi, \psi_2\}],\]
where we introduced the operator $\tilde{\Phi} = \frac{\delta F^u}{\delta X^1_1} \otimes \delta X^1_2$, acting on $p_2 = p_{2\mu} \delta X^\mu_2$ through contraction,
\[\tilde{\Phi} \circ \delta X^\mu_2 = \frac{\delta F^\mu}{\delta X^1_1} \delta X^\nu_1.
\]
The left-hand side of Eq. (4.38) being globally defined, so must be its right-hand side, hence
\[E_{i\bar{z}}[X_2] - E_{i\bar{z}}[X_1] - \delta W_{i\bar{z}1}\{\psi, \psi_2\} = : O[X_1],\]
for some $O \in \Gamma(\wedge^1 TM)$ induced by a global 2-form on $M$ as per
\[O = \int_{\mathbb{S}^1} \text{ev}_M^*\omega,
\]
where $\text{ev}_M : LM \times \mathbb{S}^1 \to M$ is the canonical evaluation map. Here, we made explicit use of relation (4.37) to express the combination of local objects on the left-hand side of Eq. (4.39) as a functional of the independent variable $X_1$ exclusively. Substituting formula (4.39) back into Eq. (4.38), we now establish a linear transformation between the pairs $(X_1, \tilde{T}, p_1)$ and $(X_2, \tilde{T}, p_2)$ which reads
\[
\begin{pmatrix}
\tilde{F}^* & 0 \\
-\omega & \text{id}_{\Gamma(TM)}
\end{pmatrix}
\begin{pmatrix}
X_1, \tilde{T} \\
p_1
\end{pmatrix}
= 
\begin{pmatrix}
\text{id}_{\Gamma(TM)} & 0 \\
0 & \tilde{F}^*
\end{pmatrix}
\begin{pmatrix}
X_2, \tilde{T} \\
p_2
\end{pmatrix},
\]
with $\tilde{F}^* = \frac{\delta F^\mu}{\delta X^1_1} \delta X^\nu_1 \otimes \frac{\delta}{\delta X^2_1}$ acting on $X_1, \tilde{T}$ via contraction,
\[\tilde{F}^* \circ X_1, \tilde{T} = (X_1, \tilde{T})^\nu \frac{\delta F^\mu}{\delta X^1_1} \frac{\delta}{\delta X^2_1}.
\]
The invertibility of the transformation thus defined necessitates the existence of an inverse of the tangent map $F^*$, which identifies $F$ as a $(\mathcal{C}^1)$-diffeomorphism of $M$. Demanding, furthermore, that the transformation preserve the Hamiltonian density yields the constraints (4.40)
\[\omega = 0, \quad F^* g = g,
\]
and so $F$ is a $(\mathcal{C}^1)$-isometry of $(M, g)$.

Finally, taking into account the assumed form of the functionals $W_{i\bar{z}1}$, we readily establish – reasoning along the same lines as in the proof of Theorem 4.14 – that local data $(P_{i\bar{z}1}, K_{i\bar{z}1})$ define a 1-isomorphism
\[\Phi : \text{pr}_1^*\mathcal{G} \xrightarrow{\sim} \text{pr}_2^*\mathcal{G}
\]
over the manifold $(\text{id}_M \times F)(M) \cong M$, with a pullback along $\text{id}_M \times F$ as claimed in the thesis of the theorem. 

Prior to passing to the discussion of the canonical interpretation of defect junctions, we pause to present a couple of examples that give some flesh to the abstract constructions of the present section.

Example 4.16. Duality of type $T$ from the T-duality defect.

An important example of a proper duality that (generically) involves a non-trivial change of the topology of the connected component of the target space is provided by T-duality, generalising the duality between the $\sigma$-model with target space $\mathbb{S}^1_R$, i.e. a circle of radius $R$, and that with target space $\mathbb{S}^1_{\frac{R}{2}}$, i.e. a circle of the (T-)dual radius $\frac{1}{R}$ (in certain natural units). In the latter case, translational charges of the string are interchanged with the winding charges under the duality. A local description of the algebraic relations between the various components of the background established by the duality was first
worked out in Refs. [Bus88, Bus87], whence they are called the Buscher rules, cf. also Ref. [GPR94] for a review of the early studies of the subject, and Refs. [Alv00a, Alv00b] for an analysis carried out in the canonical framework. Global issues were attacked in Ref. [AAGBL94] and, more recently, in Refs. [BHM07], where the important concept of a correspondence space was introduced and the topological transitions accompanying T-dualisation in the presence of non-trivial backgrounds were studied in a systematic manner (cf. also Ref. [BHM07] for an attempt at a full-fledged gerbe-theoretic formulation). The duality was also studied in the context of the lagrangean description of the string in the presence of world-sheet defects in Ref. [SS08].

The string background $\mathcal{B}_T = (\mathcal{M}_T, \mathcal{B}_T, \cdot)$ for the T-duality defect that we want to consider here consists of

(TT) the target $\mathcal{M}_T = (M_T, g_T, \mathcal{G}_T)$ with the target space

$$M_T = T^1 \cup T^2$$

given by the disjoint union of a pair of $n$-dimensional tori, with the metric $g_T$ of constant restrictions

$$g_T|_{T^n} = g_\alpha,$$

and the gerbe $\mathcal{G}_T$ of trivial restrictions

$$\mathcal{G}_T|_{T^n} = I_{B_\alpha},$$

with constant curvings $B_\alpha \in \Gamma(\wedge^2 T^* T^n)$;

(T) the $\mathcal{G}_T$-bi-brane $B_T = (Q_T, pr_1, pr_2, \omega_T, \Phi_T)$, with

(T.i) the world-volume

$$Q_T = T^1 \times T^2 \subset M_T \times M_T;$$

(T.ii) the $\mathcal{G}_T$-bi-brane maps, given by the canonical projections

$$\iota_\alpha = pr_\alpha : T^1 \times T^2 \rightarrow T^n \subset M_T;$$

(T.iii) the closed curvature $\omega_T$, with components

$$\omega_T \alpha \wedge \alpha = pr^*_\alpha B_\alpha, \quad \omega_T 1 \wedge 2 = F_P,$$

given in terms of the curvings $B_\alpha$ and of the curvature 2-form $\pi_{\mu \nu}^\alpha B_\alpha \in \Gamma(\wedge^2 T^* T^n)$

of a connection $\nabla_{\pi_{\mu \nu}^\alpha B_\alpha}$ on the Poincaré bundle $\pi_{\mu \nu}^\alpha B_\alpha : P_{T^1 \times T^2} \rightarrow T^1 \times T^2$ over the double torus $T^1 \times T^2$;

(T.iv) the $\mathcal{G}_T$-bi-brane 1-isomorphism

$$\Phi_T : I_{pr_1^* B_1} \rightarrow I_{pr_2 B_2 + \omega_T},$$

induced (e.g., on the level of the local data) by the Poincaré bundle $P_{T^1 \times T^2}$. Given these background data, the DGC [2,8] produces the compact formulæ

$$\pi_2 = -X_1 \wedge F_P, \quad \pi_1 = X_2 \wedge F_P$$

defining an isotropic graph $\mathfrak{I}_T \subset P^*_{\sigma, \varnothing}$ and written here in terms of the canonical momentum fields

$$\pi_\alpha = p_\alpha - X_\alpha \wedge B_\alpha.$$ In local angle coordinates $\theta_\alpha^\mu$, $\mu \in \Gamma, n$ on $T^n$, we have a simple expression for the curvature 2-form of the Poincaré bundle:

$$F_P = \frac{1}{\pi} \delta_{\mu \nu} \wedge \dd\theta_\alpha^\mu \wedge \dd\theta_\alpha^n.$$ This form ensures the required symplecticity of $(Q, F_P)$. The duality-background constraints [4,23], on the other hand, are identical with the Buscher rules of Refs. [Bus88, Bus87], relating components of the T-dual pairs $(g_\alpha, B_\alpha)$ as per

$$g_2 = -F_P \circ (g_1 - B_1 \circ g_1^{-1} \circ B_1)^{-1} \circ F_P, \quad B_2 = -F_P \circ g_1^{-1} \circ B_1 \circ F_P \circ g_2.$$
Example 4.17. Duality of type $N$ from the central-jump WZW defect.

An example of a geometric duality engendered by an extendible defect associated with an isometry of the target is provided by the central-jump WZW defect – a subdefect of the non-boundary maximally symmetric WZW defect at which the discontinuity of the $G$-valued lagrangean field $g : \Sigma \to G$ of the $\sigma$-model is constrained to take values in the disjoint union of the distinguished point-like conjugacy classes $C_{\Lambda_k} = \{z\}$ of elements $z \in Z(G)$ of the centre $Z(G)$ of the target Lie group $G$,

$$g_{2} = z \cdot g_{1}.$$  

The world-volume of the associated $G_k$-bi-brane, equipped with a $Z(G)$-invariant (Cartan–Killing) metric and of a vanishing curvature, all in conformity with Eq. (4.40), is identified with $G \times Z(G)$, and the (pullback) $G_k$-bi-brane 1-isomorphisms of Eq. (4.36),

$$A_{k,z} : G_k \xrightarrow{\sim} (z^{-1})^* G_k,$$

one for each element of $Z(G)$, form part of the data of the $Z(G)$-equivariant structure on $G_k$ constructed explicitly in Ref. [GR03]. The extendibility of the defect was verified in Ref. [RS09b], where the defect data were subsequently shown to encode a piece of the Moore–Seiberg data of the WZW model, to wit, the fusing matrix restricted to the simple-current sector of the quantised CFT.

5. Fusion of states through defect junctions

Hereunder, we continue to unravel, in the canonical framework adopted in the present paper, the physical contents of the gluing conditions satisfied by the $\sigma$-model field and components of the string background at the defect quiver, this time focusing on the DJI

$$\Delta_{T_{n_j}} \omega = 0,$$

to be imposed at any defect junction $j \in 2\mathfrak{P}$ of valence $n_j$. From the point of view of the underlying gerbe theory, the identity expresses a consistency condition for the trivialising 2-isomorphism $\varphi_{n_j}$ assigned to $j$, cf. Eq. (2.7). Much in the same fashion as the DGC (2.8) constrains propagation of states in the world-sheet with an embedded defect quiver by determining which states of the untwisted sector of the theory are transmitted through the defect line, the DJI turns out to be associated intimately with the natural geometric splitting-joining interactions of the string in that it restricts the spectrum of states emerging from a collision taking place at the defect quiver with defect junctions.\[11\] Thus, in particular, it will be shown, in the companion paper [Sus11], to define an intertwiner for a representation of the symmetry algebra of the $\sigma$-model on the space of multi-string states associated with an interaction vertex decorated with a defect junction, cf. the recent findings of Refs. [RS09a, RS] to this effect. This result can be regarded as a straightforward completion of the chain of results: the old one, reported in Ref. [GawSS], which shows that the $\sigma$-model gerbe transgresses to a circle bundle over the configuration space of the untwisted sector of the theory and thus defines a pre-quantum bundle of the theory, and the novel one, presented in the previous section, which demonstrates that the bi-brane, considered together with the attendant DGC, on one hand transgresses to an isomorphism of the pre-quantum bundle over an isotropic submanifold in the space of two-string states, and on the other hand canonically defines a pre-quantum bundle of the twisted sector of the theory.

In order to illustrate our point and – in so doing – introduce convenient means of description, let us consider the following (simplest possible)

Example 5.1. The splitting-joining interaction in the absence of defects.

Let $I = [0, \pi]$ denote the closed $\pi$-unit interval, and write

$$\varsigma_1 = \text{id}_{S^1}, \quad \varsigma_2 : \varphi \mapsto 2\pi - \varphi, \quad \tau : \varphi \mapsto \varphi + \pi, \quad \varphi \in S^1$$

for the identity map, the standard parity-reversal map and the $\pi$-shift map on the unit circle, respectively. We shall think of $I$ as a submanifold of the unit circle $S^1$, and so, in particular, $\varsigma_2(I) = [-\pi, 2\pi]$ (the minus denotes the orientation reversal) and $\tau(I) = [\pi, 2\pi]$. We then take the cartesian product $P_{\sigma,\emptyset}^{\otimes 2}$ of two copies of the untwisted state space $P_{\sigma,\emptyset} = T^*LM$, and, for an arbitrarily chosen free open path $Y_{1,2} \in C^\infty(1, M) \equiv \text{IM}$ in $M$, define a subspace

$$P_{\sigma,\emptyset}^{\otimes (\mathfrak{T}_{\text{vir}}; Y_{1,2})} = \left\{ (\psi_1, \psi_2) \in P_{\sigma,\emptyset}^{\otimes 2}, \quad \psi_\alpha = (X_\alpha, p_\alpha), \quad \alpha \in \{1, 2\} \right\}$$

$$\left\{ \begin{array}{l}
X_\alpha|_{c_{\alpha}(I)} = Y_{1,2} \\
Y_{1,2} = p_2|_{\varsigma_2(I)}
\end{array} \right\},$$

\[11\] Throughout the present section, one ought to keep in mind the contents of the clarifying footnote \[4\]
where \( B_{\text{triv}} \) stands for the trivial \( G \)-bi-brane from Example 2.12. The gluing condition for the loop momenta of the two states can be thought of as a trivial instance of the DGC (2.8) imposed along the half-loop interval \( [1, \ell] \). Clearly, elements of \( P_{\sigma,\Omega}^{(B_{\text{triv}}, X_{1,2})} \) are generic states assigned to the two incoming legs of the standard stringy ‘pair-of-pants’ diagram with the contour \( \ell \equiv 1 \), which carries no extra string-background data, fixed (arbitrarily) within the world-sheet \( \Sigma \) as in Fig. 4. Upon varying the half-loop \( Y_{1,2} \), we obtain a subspace

\[
\mathcal{P}^{B_{\text{triv}}} = \bigcup_{Y_{1,2} \in \mathcal{M}} \mathcal{P}_{\sigma,\Omega}^{(B_{\text{triv}}, Y_{1,2})} \subset \mathcal{P}^{\Sigma^2}
\]

in the space of untwisted two-string states, which, for the reason just named and also for other reasons that shall become clear shortly when we come to discuss less trivial examples, we choose to call the \( B_{\text{triv}} \)-fusion subspace of the untwisted string.

We may next consider a mapping

\[
i_{\sigma,\Omega}(B_{\text{triv}}, Y_{1,2}) : \mathcal{P}_{\sigma,\Omega}^{B_{\text{triv}}} \to \mathcal{P}_{\sigma,\Omega},
\]

labelled by the trivial inter-bi-brane of Example 2.12 which assigns to a pair of states \((\psi_1, \psi_2)\) a third state \(\psi_3\) with the loop embedding field satisfying a pair of ‘half-loop’ gluing conditions

\[
X_2|_{\ell} = X_3|_{\ell}, \quad X_1|_{\tau(t)} = X_3|_{\tau(t)},
\]

and with the loop momentum field constrained analogously as per

\[
p_2|_{\ell} = p_3|_{\ell}, \quad p_1|_{\tau(t)} = p_3|_{\tau(t)},
\]

across – in the simple case in hand – the distinguished defect \((B_{\text{triv}}, X_3)\). The conditions identify \(\psi_3\) as a generic state to be placed around the ‘waist’ in the ‘pair-of-pants’ diagram of Fig. 4. Accordingly, we call \(i_{\sigma,\Omega}(B_{\text{triv}}, Y_{1,2}; B_{\text{triv}})\) the \(2 \to 1\) cross-\((B_{\text{triv}}, J_{\text{triv}})\) interaction of the untwisted string. Due to

\[
\text{Figure 4. A canonical description of the splitting-joining interaction. (a) Fusion of the states } \psi_1 \text{ and } \psi_2 \text{ along a half-loop } \ell \subset \Sigma, \ell \equiv 1. \text{ (b) The interaction } i_{\sigma,\Omega}(B_{\text{triv}}, J_{\text{triv}}; B_{\text{triv}}) \text{ sends a two-string state } (\psi_1, \psi_2) \text{ from the } B_{\text{triv}} \text{-fusion subspace into an emergent state } \psi_3 \text{ across the loose half-loops.}
\]

the trivial character of the gluing conditions, the interaction \(i_{\sigma,\Omega}(B_{\text{triv}}, J_{\text{triv}}; B_{\text{triv}})\) is manifestly surjective and many-to-one (pairs of loops differing by the choice of the differentiable extension of the given ‘loose’ half-loop embedding field and an extension of the attendant momentum field to the fused half-loop all map to the same loop in \( \mathcal{P}_{\sigma,\Omega} \)). It leads us to

**Definition 5.2.** Let \( \mathcal{B} \) be a string background with target \( \mathcal{M} = (M, g, G) \), and let \( (\mathcal{P}_{\sigma,\Omega}, \Omega, \sigma, \omega) \) be the untwisted state space of the non-linear \( \sigma \)-model for network-field configurations \( (X | \Gamma) \) in string background \( \mathcal{B} \) on world-sheet \( (\Sigma, \gamma) \) with defect quiver \( \Gamma \). Furthermore, let \( \mathcal{P}_{\sigma,\Omega}^{B_{\text{triv}}} \) be the \( B_{\text{triv}} \)-fusion subspace in \( \mathcal{P}_{\sigma,\Omega} = \mathcal{P}_{\sigma,\Omega} \times \mathcal{P}_{\sigma,\Omega} \) given in Eqs. (5.3)-(5.4), and \( i_{\sigma,\Omega}(B_{\text{triv}}, J_{\text{triv}}; B_{\text{triv}}) \) the \(2 \to 1\)

\[\text{cross-} (B_{\text{triv}}, J_{\text{triv}}) \text{ interaction of the untwisted string.} \]

\[\text{Due to the trivial character of the gluing conditions, the interaction } i_{\sigma,\Omega}(B_{\text{triv}}, J_{\text{triv}}; B_{\text{triv}}) \text{ is manifestly surjective and many-to-one (pairs of loops differing by the choice of the differentiable extension of the given ‘loose’ half-loop embedding field and an extension of the attendant momentum field to the fused half-loop all map to the same loop in } \mathcal{P}_{\sigma,\Omega}. \text{ It leads us to} \]

\[\textbf{Definition 5.2.} \text{ Let } \mathcal{B} \text{ be a string background with target } \mathcal{M} = (M, g, G), \text{ and let } (\mathcal{P}_{\sigma,\Omega}, \Omega, \sigma, \omega) \text{ be the untwisted state space of the non-linear } \sigma \text{-model for network-field configurations } (X | \Gamma) \text{ in string background } \mathcal{B} \text{ on world-sheet } (\Sigma, \gamma) \text{ with defect quiver } \Gamma. \text{ Furthermore, let } \mathcal{P}_{\sigma,\Omega}^{B_{\text{triv}}} \text{ be the } B_{\text{triv}} \text{-fusion subspace in } \mathcal{P}_{\sigma,\Omega} = \mathcal{P}_{\sigma,\Omega} \times \mathcal{P}_{\sigma,\Omega} \text{ given in Eqs. (5.3)-(5.4), and } i_{\sigma,\Omega}(B_{\text{triv}}, J_{\text{triv}}; B_{\text{triv}}) \text{ the } 2 \to 1 \]

\[\text{cross-} (B_{\text{triv}}, J_{\text{triv}}) \text{ interaction of the untwisted string.} \]

\[\text{Due to the trivial character of the gluing conditions, the interaction } i_{\sigma,\Omega}(B_{\text{triv}}, J_{\text{triv}}; B_{\text{triv}}) \text{ is manifestly surjective and many-to-one (pairs of loops differing by the choice of the differentiable extension of the given ‘loose’ half-loop embedding field and an extension of the attendant momentum field to the fused half-loop all map to the same loop in } \mathcal{P}_{\sigma,\Omega}. \text{ It leads us to}\]
cross-\((\mathcal{B}_{\text{triv}}, \mathcal{J}_{\text{triv}})\) interaction defined by Eqs. \((5.5)-(5.7)\). The \(2 \to 1\) cross-\((\mathcal{B}_{\text{triv}}, \mathcal{J}_{\text{triv}})\) interaction subspace of the untwisted string is the space

\[
\mathcal{I}_\sigma(\# \mathcal{B}_{\text{triv}} : \mathcal{J}_{\text{triv}} : \mathcal{B}_{\text{triv}}) = \left\{ (\psi_1, \psi_2, \psi_3) \in \mathcal{P} \otimes \mathcal{I}_{\text{triv}} \times \mathcal{P} \otimes \mathcal{I}_{\text{triv}} \mid \psi_3 = i_{\sigma}(\# \mathcal{B}_{\text{triv}} : \mathcal{J}_{\text{triv}} : \mathcal{B}_{\text{triv}})(\psi_1, \psi_2) \right\}.
\]

It is physically pertinent to enquire as to the distinctive features of \(\mathcal{I}_\sigma(\# \mathcal{B}_{\text{triv}} : \mathcal{J}_{\text{triv}} : \mathcal{B}_{\text{triv}})\), the latter viewed as a subspace in a symplectic space. These could then be interpreted as a canonical manifestation of the basic interaction process in the string theory in hand. The answer is contained in the following

**Proposition 5.3.** Let \(\mathcal{B}\) be a string background with target \(\mathcal{M} = (M, g, \mathcal{G})\), and let \((\mathcal{P}_\sigma \otimes \Omega_\sigma, \mathcal{L}_\sigma)\) be the untwisted state space of the non-linear \(\sigma\)-model for network-field configurations \((X | \Gamma)\) in string background \(\mathcal{B}\) on world-sheet \((\Sigma, \gamma)\) with defect quiver \(\Gamma\). Consider the symplectic manifold \((\mathcal{P}^{\times 3}_\sigma \otimes \Omega^{\times 3}_\sigma)\) defined as

\[
\mathcal{P}^{\times 3}_\sigma := \mathcal{P}_\sigma \times \mathcal{P}_\sigma \times \mathcal{P}_\sigma, \quad \Omega^{\times 3}_\sigma := \mathcal{P}_\sigma \times \mathcal{P}_\sigma \times \mathcal{P}_\sigma
\]

in terms of the canonical projections \(\mathcal{P}_\sigma \otimes \Omega_\sigma \to \mathcal{P}_\sigma \otimes \Omega_\sigma\). Furthermore, let \(\pi_\mathcal{L}_\sigma : \mathcal{L}_\sigma \to \mathcal{P}_\sigma \otimes \Omega_\sigma\) be the pre-quantum bundle for the untwisted sector of the \(\sigma\)-model. Then, the \(2 \to 1\) cross-\((\mathcal{B}_{\text{triv}}, \mathcal{J}_{\text{triv}})\) interaction subspace \(\mathcal{I}_\sigma(\# \mathcal{B}_{\text{triv}} : \mathcal{J}_{\text{triv}} : \mathcal{B}_{\text{triv}})\) is an isotropic submanifold in \((\mathcal{P}^{\times 3}_\sigma \otimes \Omega^{\times 3}_\sigma)\) and there exists a canonical bundle isomorphism

\[
\mathcal{I}_\sigma(\# \mathcal{B}_{\text{triv}} : \mathcal{J}_{\text{triv}} : \mathcal{B}_{\text{triv}}) \simeq \mathcal{P}_\sigma \otimes \Omega_\sigma \otimes \mathcal{P}_\sigma \otimes \Omega_\sigma \otimes \mathcal{P}_\sigma \otimes \Omega_\sigma
\]

between the restrictions to \(\mathcal{I}_\sigma(\# \mathcal{B}_{\text{triv}} : \mathcal{J}_{\text{triv}} : \mathcal{B}_{\text{triv}})\) of the (tensor) pullback bundles.

A proof of the proposition can be obtained through specialisation of the proof of Theorem 5.3 upon setting \(\mathcal{B} = \mathcal{B}_{\text{triv}}\) and \(\mathcal{J} = \mathcal{J}_{\text{triv}}\), the latter two being as in Example 2.12.

It is owing to the purely geometric nature of the field theory under consideration that we obtain a simple yet structured representation of the interaction in the canonical description, which – as is obvious from the hitherto discussion – generalises straightforwardly to higher-rank fusion and interaction subspaces.

We are now ready to go directly to the main point of interest of this section, which is a canonical interpretation of the DJI for world-sheets decorated with non-trivial defect quivers. In order to describe these in a fashion suggested by the above example, we shall have to modify our construction of the fusion subspace and that of the cross-defect interaction suitably.

5.1. **Interactions in the untwisted sector.** We begin with the canonical analysis of the splitting-joining interaction of untwisted states across a non-trivial defect (sub-)quiver. While reading the formal definitions and mathematical expressions appearing in this part of the section, it is good to keep in mind the physical situation depicted in Fig. 5 that is being modelled by them.

![Figure 5](image-url)

**Figure 5.** The splitting-joining interaction mediated by defects, crossing at a pair of defect junctions \(j\) and \(j'\). Fusion of the states \(\psi_1\) and \(\psi_2\) along the defect \((\mathcal{B}_1 Y_{1,2})\) produces an emergent state \(\psi_3\) via the \(2 \to 1\) cross-\((\mathcal{B}, \mathcal{J})\) interaction.
Definition 5.4. Let \( \mathcal{B} \) be a string background with target \( \mathcal{M} = (M, g, G) \), \( G \)-bi-brane \( \mathcal{B} = (Q, \iota_\alpha, \omega, \Phi | \alpha \in \{1, 2\}) \) and \( (G, B) \)-inter-bi-brane \( \mathcal{J} = (T_n, \{ \psi^{k+1}_n, \pi^{k+1}_n \} | \psi, \pi \in \mathbb{G}_n \) and let \( (P_{\sigma, \Omega}, \Omega_{\sigma, \Omega}) \) be the untwisted state space of the non-linear \( \sigma \)-model for network-field configurations \((X | \Gamma)\) in string background \( \mathcal{B} \) on world-sheet \((\Sigma, \gamma)\) with defect quiver \( \Gamma \). For \( I = [0, \pi] \), denote by \( \text{IQ} = C^\infty(1, Q) \) the free open-path space of \( Q \). The \( B \)-fusion subspace of the untwisted string is the subset of \( P_{\sigma, \Omega}^2 = P_{\sigma, \Omega} \times P_{\sigma, \Omega} \) given by the formula

\[
P_{\sigma, \Omega}^B = \{ (\psi_1, \psi_2) \in P_{\sigma}^2, \quad \psi_\alpha = (X_\alpha, p_\alpha), \quad \alpha \in \{1, 2\} \mid (X_1)_{(1)}(X_2)_{(1)}(1) \}\]

\[
(5.8) \quad \exists Y_{1,2} \in \{ (X_1, X_2) : DGC_B(\psi_1, \psi_2_{(1)}, Y_{1,2}) = 0 \}.
\]

It is a fibration over the free-path space \( \text{IQ} \), and we shall identify it with the corresponding subspace in \( P_{\sigma, \Omega}^2 \times \text{IQ} \) in what follows. Consider a map \( i^{2-1}`_{B,B,J,I} \) to be termed the 2 \( \rightarrow \) 1 \textbf{cross-(B,J) interaction of the untwisted string}, which assigns to pairs of states from \( P_{\sigma, \Omega}^B \) subsets of \( P_{\sigma, \Omega} \) such that a pair \((\psi_1, \psi_2)\) fused along a free open path \( Y_{1,2} \in \text{IQ} \) is mapped to the set of all those states \( \psi_3 = (X_3, p_3) \) that satisfy the relations

\[
X_{2}\mid = \iota_1 \circ Y_{2,3}, \quad \pi_{3} = \iota_2 \circ Y_{2,3}, \quad (X_1)_{(1)} = \iota_1 \circ Y_{1,3}, \quad (X_3)_{(1)} = \iota_2 \circ Y_{1,3},
\]

\[
(5.9) \quad Y_{1,3}(\iota) = \pi_{3} I_{J} \circ Z, \quad (I, J) \in \{(1, 2), (2, 3), (1, 3)\},
\]

\[
(5.10) \quad (\psi_1, \psi_2) = 0, \quad (\psi_1, \psi_2) = \psi_3 \in i^{2-1}`_{B,B,J,I}(B,B,J,I)(\psi_1, \psi_2).
\]

Once again, the latter subspace is a fibration over the cartesian cube \( \text{IQ}^3 \), and we shall identify it with the corresponding subspace in \( P_{\sigma}^3 \times \text{IQ}^3 \) in what follows.

We have

Theorem 5.5. Let \( \mathcal{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J}) \) be a string background, and let \( (P_{\sigma, \Omega}, \Omega_{\sigma, \Omega}) \) be the untwisted state space of the non-linear \( \sigma \)-model for network-field configurations \((X | \Gamma)\) in string background \( \mathcal{B} \) on world-sheet \((\Sigma, \gamma)\) with defect quiver \( \Gamma \), with the pre-quantum bundle for the untwisted sector of the \( \sigma \)-model over it, \( \pi_{\mathcal{L}_{\sigma, \Omega}} : \mathcal{L}_{\sigma, \Omega} \rightarrow P_{\sigma, \Omega} \). Furthermore, let \( (P_{\sigma, \Omega}^{\sigma, \Omega} : \Omega_{\sigma, \Omega}^{\sigma, \Omega}) \) be the symplectic manifold defined in Proposition 5.3. Then, the following statements hold true:

i) the 2 \( \rightarrow \) 1 \textbf{cross-B interaction subspace of the untwisted string}, \( \mathcal{I}_B(B, J, B) \), constructed in \( \text{Definition 5.4} \), is an isotropic submanifold of \( (P_{\sigma, \Omega}^{\sigma, \Omega}, \Omega_{\sigma, \Omega}^{\sigma, \Omega}) \);

ii) the background \( \mathcal{B} \) canonically induces a bundle isomorphism

\[
\mathcal{I}_B(B, J, B) : (\text{fr}_{1} \mathcal{L}_{\sigma, \Omega} \otimes \text{fr}_{2} \mathcal{L}_{\sigma, \Omega}) | \mathcal{I}_{\sigma}(B, J, B) \stackrel{\sim}{\rightarrow} \text{fr}_{1} \mathcal{L}_{\sigma, \Omega} | \mathcal{I}_{\sigma}(B, J, B)
\]

between the restrictions to \( \mathcal{I}_{\sigma}(B, J, B) \) of the (tensor) pullback bundles.

A proof of the theorem is given in Appendix D.

Remark 5.6. The relation between defects and dualities of the \( \sigma \)-model worked out in the previous section distinguished those bi-branes whose maps \( \iota_{\alpha} : Q \rightarrow M \) are surjective submersions. The canonical analysis of the splitting-joining interaction of the untwisted string immediately leads to similar conclusions for the inter-bi-brane. Indeed, it is clear that for a given interaction vertex of Fig. 5 to allow the appearance of arbitrary outgoing and incoming states, i.e. for the interaction subspace to project onto each cartesian component \( P_{\sigma, \Omega} \subset P_{\sigma, \Omega}^\sigma \) the inter-bi-brane maps \( \pi^{k+1}_n : \mathcal{B} \rightarrow Q \) should all be surjective. Taking into account the additional requirement of topologicality of the defect, we note that – at least in the case of extendible defects – the inter-bi-brane maps should, moreover, be submersions, so that, once more, surjective submersions become singled out. These are particularly interesting in the case of inter-bi-branes admitting induction, as introduced in Ref. RS09b Sec. 2.8.
The latter is motivated by the physical observation that a defect junction $j$ of valence $n_j > 3$, represented by a defect-field insertion in the underlying CFT, can be regarded as a product of a stepwise limiting procedure in which a collection of 'elementary' 3-valent vertices are merged by sending the lengths of the interconnecting defect lines to zero, whereby the associated defect fields of the CFT may have to be renormalised in order to remove the ensuing divergencies. In what follows, we briefly recall the idea of induction in restriction to the component of the background obtained by fixing the values of the orientation maps $\varepsilon^{k,k+1}_n$ to be all +1 except for $\varepsilon^{n,1}_n = -1$ for all $n \in \mathbb{N}_{\geq 3}$ and taking the inter-bi-brane 2-isomorphisms $\varphi_n$ restricted to the corresponding submanifolds $T_{n,+++}$ of $T_n$ (the remaining components of the background are left unrestricted). As the very construction of the $\sigma$-model in string background $\mathcal{B}$ clearly indicates, there are no additional geometric insights to gain from considering the more general case. Indeed, as long as we are concerned with a single defect junction (which is, in particular, all we need to determine the associated field-space data), we are at liberty to choose an arbitrary relative-orientation pattern for the defect lines converging at that junction.

Denote the inter-bi-brane maps $\pi^{k,k+1}_n$, $k \in \{1, 2, 3\}$ as $\pi^{1,2}_3 = d^{(2)}_0$, $\pi^{2,3}_3 = d^{(2)}_2$, $\pi^{3,1}_3 = d^{(2)}_1$. The background $\mathcal{B}$ shall be termed a string background with induction iff, for each $n \geq 3$, there exist smooth maps $d^{(n)}_i : T_{n+1,+++} \rightarrow T_{n,+++}$, $i \in \overline{0,n}$ satisfying the identities

$$d^{(n-1)}_i \circ d^{(n)}_j = d^{(n-1)}_j \circ d^{(n)}_i$$

for $i < j$, and such that the inter-bi-brane 2-isomorphisms $\varphi_n$ are induced from $\varphi_3$ in a natural manner illustrated in Ref. [RS09b, Sec. 2.8] on an explicit example amenable to a straightforward generalisation (which we leave out here for the sake of conciseness). Identities (5.12) arise as simple consistency conditions to be imposed on the limiting values attained by the $\sigma$-model field at the defect junctions of an embedded defect quiver as we decompose the defect quiver at these junctions into clusters of defect junctions of lower valence bridged by intermediate defect lines, prior to passing to the limit of the vanishing length of the intermediate defect lines, cf. Fig. 6. On the other hand, it is tempting to view them as the simplicial identities obeyed by the face maps of a simplicial space composed by the family of manifolds $\{M, Q, T_3, T_4, T_5, \ldots\}$. That this is the proper manner of thinking of the string background with induction can be seen as follows: First of all, the bi-brane maps provide a natural completion of the family $\{d^{(n)}_i | i \in \overline{0,n}, n \in \mathbb{N}_{\geq 2}\}$ of smooth maps interrelated as per Eq. (5.12), which can readily be seen upon setting $d^{(1)}_0 := \iota_1$, $d^{(1)}_1 := \iota_2$ and recalling relations (2.1). Furthermore, it is natural, from the point of view of the associated $\sigma$-model, to incorporate the trivial defect into the formal definition of the string background by allowing degenerate defect quivers in which some defect lines carry the (trivial) data of the trivial defect. Indeed,
we should always be able to insert a circular trivial defect into the world-sheet, or attach a trivial-defect line to a given defect junction, between any two of its defect lines, whereby the valence of the defect junction is increased by 1. It is easy to see, going through similar consistency checks of the limiting values of the \( \sigma \)-model fields as those used in the derivation of Eq. (5.12), that this can be formalised as a requirement of the existence of distinguished sections

\[
\begin{align*}
\sigma_i^{(n-1)} & : T_{n,+++\ldots} \rightarrow T_{n+1,+++\ldots}, \quad i \in \overline{0,n-1}, \\
\sigma_j^{(1)} & : Q \rightarrow T_{3,+++\ldots}, \quad j \in \{0,1\}, \quad \sigma_0^{(0)} : M \rightarrow Q
\end{align*}
\]

of the respective surjective submersions, satisfying the identities

\[
\begin{align*}
\sigma_j^{(n+1)} \circ \sigma_i^{(n)} &= \sigma_j^{(n+1)} \circ \sigma_i^{(n)} \quad \text{if } i \leq j, \\
d_i^{(n+1)} \circ \sigma_j^{(n)} &= \begin{cases} 
\sigma_j^{(n)} \circ d_i^{(n)} & \text{if } i < j \\
\id_{T_{n+1,+++\ldots}} & \text{if } i = j \vee i = j + 1 \\
\sigma_j^{(n-1)} \circ d_{i-1}^{(n)} & \text{if } i > j + 1
\end{cases}
\end{align*}
\]

Here, \( \sigma_0^{(0)} \) puts the image, with respect to \( X : \Sigma \setminus \Gamma \rightarrow M, \) of an arbitrary point from the interior of a world-sheet patch on the world-volume of the trivial defect. Similarly, \( \sigma_j^{(1)} \) expresses the possibility of viewing the image, with respect to \( X : \Gamma \setminus \mathcal{E}_T \rightarrow Q, \) of a point from the interior of a defect line embedded in \( \Sigma \) as the image of the degenerate 3-valent defect junction, with the trivial-defect line joining the original one from the side of \( U_1 \) (for \( j = 1 \)) or \( U_2 \) (for \( j = 0 \)) in the notation of Definition 2.6. Finally, the maps \( \sigma_i^{(n-1)} \) represent the process of increasing the valence of a given defect junction through attachment of a trivial-defect line between the neighbouring defect lines \( \ell_{n-i,1,n-i} \) and \( \ell_{n-i,n-i+1} \) that converge at this junction (with the usual convention \( \ell_{0,1} = \ell_1 \)).

Altogether, Eqs. (5.12)-(5.15) reproduce the full set of simplicial identities for the face maps \( d_i^{(n)} \) and degeneracy maps \( \sigma_i^{(n)} \) of a simplicial space

\[\cdots \xrightarrow{d_4^{(4)}} T_4 \xrightarrow{d_3^{(3)}} T_3 \xrightarrow{d_2^{(2)}} Q \xrightarrow{d_1^{(1)}} M\]

A simplicial string background \( \mathcal{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J}) \), i.e. a string background with induction, equipped with the family (5.13) of sections, can be regarded as a straightforward generalisation of a simplicial background describing a proper (geometric) symmetry of the \( \sigma \)-model with target \( \mathcal{M} = (M, g, \mathcal{G}) \) endowed with structure of a \( K \)-space for some group \( K \), acting on \( (M, g) \) by isometries

\[\ell : K \times M \rightarrow M : (g, m) \mapsto \ell_g(m) = : g.m, \quad \ell_g^*g = g\]

that lift to the gerbe \( \mathcal{G} \) in the sense made precise in Ref. [GSW10]. The relevant simplicial space is given by the nerve

\[N(K \ltimes M)^\bullet : \cdots \xrightarrow{\mathcal{G}_d^{(4)}} K^3 \times M \xrightarrow{\mathcal{G}_d^{(3)}} K^2 \times M \xrightarrow{\mathcal{G}_d^{(2)}} K \times M \xrightarrow{\mathcal{G}_d^{(1)}} M\]

of the action groupoid

\[K \ltimes M : K \times M \xrightarrow{pr_2 \circ \sigma_0^{(1)}} M, \]

written in terms of the action \( \ell \) and of the canonical projection \( pr_2 \). The action groupoid is understood as the small category with object and morphism sets

\[\text{Ob}(K \ltimes M) = M, \quad \text{Mor}(K \ltimes M) = K \times M,\]

with the identity morphism (\( e \) is the group unit)

\[\id_m = (e, m),\]

and with source and target maps

\[s(g, m) = m, \quad t(g, m) = g.m,\]
which, altogether, lead to a natural identification of the spaces of $n$-tuples of composable morphisms (i.e. the remaining members of the family $N(K \times M)^\bullet$ of spaces) with the respective product spaces $K^n \times M$. The (inter-)bi-brane data compose a $K$-equivariant structure on $G$ as in Ref. [GSW10]. This structure was shown to be a prerequisite of the gauging of the internal (rigid) $K$-symmetry of the $\sigma$-model (the latter being obtained as a lift of the geometric action $\ell$ to the phase space of the $\sigma$-model) and it is readily proven necessary for the gauged $\sigma$-model to descend to the coset $M/K$. Reasoning by analogy, we conjecture that the existence of a simplicial string background associated with a given $\sigma$-model duality is a necessary ingredient of a consistent formulation of string theory on a ‘quotient’ background of a $\sigma$-model descended from the original one upon ‘gauging’ the duality group, whenever such a group and the attendant ‘quotient’ can be defined. Here, the submersive surjectivity of the face maps is necessary to establish duality equivalences on the entire phase space and to ensure that all states are transmitted by any defect quiver carrying the duality data. More specifically, the bi-brane face maps $d^{(1)}_i : Q \to M$ encode an element-wise presentation of the set of duality transformations on the space of states, the 3-valent inter-bi-brane face maps $d^{(2)}_i : T_3 \to Q$ render the presentation distributive with respect to the group operation on the set of dualities, and the requirement that the 4-valent inter-bi-brane structure induced from the 3-valent one by means of the face maps $d^{(3)}_i : T_4 \to T_3$ be independent of the choice of the defining simplicial move enforce the associativity of the presentation. Finally, the existence of an induced inter-bi-brane structure on the component world-volumes $T_n$ of valence $n \geq 5$, independent of the choice of defining simplicial moves, guarantees that the associative presentation of the duality group carries over to arbitrary interaction schemes (i.e. to arbitrary defect quivers). As the treatment of coset $\sigma$-models in Ref. [GSW10] suggests, there are, generically, extra constraints to be imposed on the thus obtained ‘duality-equivariant’ structure for the $\sigma$-model to descend to the duality quotient. A motivating explicit instantiation of this idea (if also far from being understood rigorously to date) is the notion of a T-fold, advanced in Ref. [Hal05], in which the target is described in terms of local charts (carrying local metric and gerbe data) patched together using T-duality transformations. Let us also point out that the above duality scheme bears a deep affinity with the categorial descent scheme discussed in Ref. [FNSW09]. We hope to return to these issues in the near future.

5.2. Interactions in the twisted sector. Another class of string processes in which the inter-bi-brane and the associated DJI are naturally expected to transgress to the canonical description is the splitting-joining interaction of twisted states. When assembling the necessary formal ingredients, we are guided by the depiction of the corresponding network-field configuration on the world-sheet, the simplest of its kind, presented in Fig. [7]. Thus, we consider three defect lines, $\ell_{1,2},\ell_{2,3}$ and $\ell_{3,1}$, converging at a defect junction $\jmath$. To each defect line, we attach the corresponding Cauchy contour $C_{I,J}$, $(I,J) \in \{(1,2),(2,3),(3,1)\}$, oriented as the ones drawn in the figure and crossing the respective defect lines transversally, each at a single point. The contours are next pushed towards one another in such a manner that they overlap pairwise along open arcs, all crossing at a pair of points in $\Sigma$, the defect junction $\jmath$ being one of them.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig7.png}
\caption{The splitting-joining interaction of a triple of 1-twisted states. The Cauchy contours representing the states are drawn in black.}
\end{figure}
Our first task consists in identifying the various subspaces within the cartesian square and the cartesian cube of the 1-twisted state space of the $\sigma$-model, of relevance to the problem in hand.

**Definition 5.7.** Let $\mathcal{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$ be a string background as in Definition 2.1. Fix $p \in \mathbb{S}^1$ and $\varepsilon \in \{-1, +1\}$, and let $(P_{\sigma, \mathcal{B}(p, \varepsilon)}, \Omega_{\sigma, \mathcal{B}(p, \varepsilon)})$ be the 1-twisted state space of the non-linear $\sigma$-model for network-field configurations $(X | \Gamma)$ in string background $\mathcal{B}$ on world-sheet $(\Sigma, \gamma)$ with defect quiver $\Gamma$. The $B_{\text{triv}}$-fusion subspace of the 1-twisted string is a subspace in

$$P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)} := P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)} \times P_{\sigma, \mathcal{B}(\varepsilon_2, \varepsilon_3)}$$

given by the formula

$$P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)} = \{(\psi_1, \psi_2) \in P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)) : \psi_\alpha = (X, p, q_\alpha, V_\alpha), \alpha \in \{1, 2\}$$

$$\mid X_{1|1} = X_{2|3|0} \land p_{1|1} = p_{2|3|0} \}.$$  

Clearly, the specific choice $P_1 = \pi = P_2$ of the intersection point is immaterial to the outcome of our analysis. Consider a map $i^{2-1}_{\sigma, (\mathcal{B}_{\text{triv}}, \mathcal{J}; \mathcal{B}_{\text{triv}})}$, to be termed the $2 \to 1$ cross-$B_{\text{triv}}$ interaction of the untwisted string, which assigns to pairs of states from $P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)}$ subsets of $P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)}$ such that a pair $(\psi_1, \psi_2)$ is mapped to the set of all those states $\psi_3 = (X, p, q_3, V_3)$ that satisfy the relations

$$X_{2|1} = X_{3|1}, \quad X_{1|3|0} = X_{3|3|0},$$

$$p_{2|1} = p_{3|1}, \quad p_{1|3|0} = p_{3|3|0},$$

augmented with the constraints

$$q_k = \pi^{k,k+1}(t_3), \quad k \in \{1, 2, 3\},$$

the latter being expressed in terms of a fixed point $t_3 \in T_{3,++,} \subset T$ from the component $T_{3,++}$ of $T_3 \subset T$ corresponding to the values $\varepsilon_3 = +1 = \varepsilon_2 = -\varepsilon_1$. Our first task consists in identifying the various subspaces within the cartesian square and the cartesian cube of the 1-twisted state space of the $\sigma$-model over it, $\pi_{\mathcal{B}(\varepsilon_1, \varepsilon_2)} : \mathcal{L}_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)} \to P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)}$. Endow the space

$$P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2, \varepsilon_3)} := P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)} \times P_{\sigma, \mathcal{B}(\varepsilon_2, \varepsilon_3)}$$

with the symplectic structure defined by the 2-form

$$\Omega_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2, \varepsilon_3)} := pr_1^*\Omega_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)} + pr_2^*\Omega_{\sigma, \mathcal{B}(\varepsilon_2, \varepsilon_3)} - pr_3^*\Omega_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_3)}$$

in terms of the canonical projections $pr_k : P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2, \varepsilon_3)} \to P_{\sigma, \mathcal{B}(\varepsilon_k, \varepsilon_3)}$, $k \in \{1, 2, 3\}$, and consider the pullback circle bundle

$$\mathcal{L}_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2, \varepsilon_3)}^\perp := P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2, \varepsilon_3)}^\perp.$$

Then, the following statements hold true:

1. the 2 to 1 cross-$B_{\text{triv}}$ interaction subspace $\mathcal{J}_{\sigma}(\mathcal{B}_{\text{triv}}, \mathcal{J}; \mathcal{B}_{\text{triv}})$ of the 1-twisted string is an isotropic submanifold in the symplectic manifold $(P_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2, \varepsilon_3)}, \Omega_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2, \varepsilon_3)});$  
2. the background $\mathcal{B}$ canonically induces a trivialisation

$$\mathcal{J}_{\sigma}(\mathcal{B}_{\text{triv}}, \mathcal{J}; \mathcal{B}_{\text{triv}}) \to (pr_1^*\mathcal{L}_{\sigma, \mathcal{B}(\varepsilon_1, \varepsilon_2)} \otimes pr_2^*\mathcal{L}_{\sigma, \mathcal{B}(\varepsilon_2, \varepsilon_3)}) |\mathcal{J}_{\sigma}(\mathcal{B}_{\text{triv}}, \mathcal{J}; \mathcal{B}_{\text{triv}}) \otimes \mathcal{B}_{\text{triv}}) \to pr_3^*\mathcal{L}_{\sigma, \mathcal{B}(\varepsilon_3, \varepsilon_3)} |\mathcal{J}_{\sigma}(\mathcal{B}_{\text{triv}}, \mathcal{J}; \mathcal{B}_{\text{triv}}) \otimes \mathcal{B}_{\text{triv}}),$$

between the restrictions to $\mathcal{J}_{\sigma}(\mathcal{B}_{\text{triv}}, \mathcal{J}; \mathcal{B}_{\text{triv}}) \otimes \mathcal{B}_{\text{triv}}$ of the (tensor) pullback bundles.

A proof of the theorem is given in Appendix E.
**Remark 5.9.** A generic interaction process on a world-sheet with an arbitrary embedded defect quiver is a combination of the two `pure' types considered in detail above, with higher-rank fusion and interaction subspaces involved. Our conclusions are readily seen to generalise to arbitrary such processes.

![Diagram](image)

**Example 5.10.** The WZW fusion ring and the maximally symmetric inter-bi-brane.

The correspondence between inter-bi-brane 2-isomorphisms and interaction subspaces in the space of states of the untwisted string may, in fact, carry over to the quantised theory, as demonstrated in Refs. [RS09a, RS], where the inter-bi-brane for the non-boundary maximally symmetric WZW bi-brane of Example 2.13 was reconstructed following the general scheme laid out in Ref. [RS09b].

The point of departure of the reconstruction scheme proposed in Ref. [RS] is the simplicial $G \times G$-space given by the nerve

$$N(G \times G)^{\ast} : 
\begin{array}{cccc}
G^4 & \rightarrow & G^3 & \rightarrow & G^2 & \rightarrow & G \\
\vdots & & \vdots & & \vdots & & \vdots \\
\end{array}$$

of the action groupoid

$$G \times G : 
\begin{array}{ccc}
G^2 & \rightarrow & G \\
pr_2 \circ G_0^{(1)} & \rightarrow & \varrho = G_1^{(1)} \\
\end{array}$$

defined in terms of the right regular action

$$\varrho : G \times G \rightarrow G : (g,h) \mapsto g \cdot h = \varrho_h(g) .$$

Note that it is the first factor in each component $G^n$ of the nerve that plays the rôle of the $G$-space from Remark 5.6. Apart from that, the construction follows that of the nerve of the group (viewed as a small category) first presented in Ref. [Seg68]. The component inter-bi-brane world-volumes $T_n$, $n \in \mathbb{N}_{\geq 3}$ were assumed to be composed of full orbits under the $G \times G$-action on $G^n$ intertwined, by the manifestly $G \times G$-equivariant face maps of $N(G \times G)^{\ast}$, with the standard action of $G \times G$ on $G \equiv N(G \times G)^{(0)}$ by left and right regular translations. They were then shown to split into disjoint unions of such orbits, for example:

$$T_{n,+\ldots,+} = G \times \bigsqcup_{1 \leq i_1 < \ldots < i_n \leq \lambda_n} T_{i_1,\ldots,i_n} ,$$

where $T_{i_1,\ldots,i_n}$ is the stabiliser of the Cartan element $e_{i_1}$ (cf. Ref. [RS] for a general definition). The proof of the splitting of the inter-bi-brane world-volume into a disjoint union of diagonal $\text{Ad}_e$-orbits, each multiplied with the `reference' factor $G$, rests upon the identification of the sign-weighted sum of pullbacks, along the inter-bi-brane maps, of the bi-brane world-volume $\omega_k$ appearing in the DJI 5.1 (to be satisfied necessarily by all vector fields tangent to the inter-bi-brane world-volume) as a pre-symplectic form on partially symplectically reduced, in a manner detailed in Ref. [AM95], space of classical field configurations of the level-$k$ Chern–Simons theory with gauge group $G_{\mathbb{R} \times \mathbb{C} P^1}$ on $\mathbb{R} \times \mathbb{C} P^1$ (with $\mathbb{R}$ playing the rôle of the time axis) in the presence of $n-1$ vertical time-like Wilson lines of holonomies fixed to lie in the respective conjugacy classes $C_{\lambda_{i-1}}$, $i \in \mathbb{T}_n \setminus \mathbb{T}$, and a single vertical anti-time-like Wilson line of a holonomy constrained to lie in $C_{\lambda_n}$. The quantisation of the weight labels, all taken from the discrete set $P_k^{\pm}(g)$ of Eq. (2.11), expresses the requirement that there exist component 1-isomorphisms $\Phi_{k,\lambda_i}$, $i \in \mathbb{T}_n$, entering the definition of the inter-bi-brane 2-isomorphism, cf. Diagram (2.2).

Finally, the set of admissible components of the inter-bi-brane world-volume is restricted, as indicated by the tilde over the symbol of the disjoint union over $P_k^{\pm}(g)$ in Eqs. (5.20) and (5.21), to those which support a non-vanishing inter-bi-brane 2-isomorphism (conjectured to correspond to the non-vanishing Verlinde fusion coefficients). The existence of the latter is topologically obstructed on a generic space $G \times T_{\lambda_1,\ldots,\lambda_{n-1}}$ owing to, in particular, the non-simple connectedness of that space, cf. Proposition 2.4.
In the presence of the multiplicative structure on $G_k$, mentioned in Example 2.13 cf. also Remark 2.14 the question of existence of the inter-bi-brane 2-isomorphism over $G \times T_{\lambda_1, \lambda_2, \ldots, \lambda_n}$ reduces to the same question for a so-called fusion 2-isomorphism over $T_{\lambda_1, \lambda_2, \ldots, \lambda_n}$. Thus, for instance, in the case of the elementary inter-bi-brane $T_{3,4}$, it boils down to constructing a 2-isomorphism

$$\left.\left(\begin{array}{c}
pr_1^*G_k \otimes pr_2^*G_k
\end{array}\right)\right|_{T_{\lambda_1, \lambda_2}} \xrightarrow{\mathcal{M}_k|_{T_{\lambda_1, \lambda_2}}} \left.\left(\begin{array}{c}
m^*G_k \otimes I_{\rho_k}
\end{array}\right)\right|_{T_{\lambda_1, \lambda_2}}$$

whose definition involves, beside the 1-isomorphism $\mathcal{M}_k$ of the multiplicative structure, the component 1-isomorphisms $\Phi_{\lambda_1, \lambda_2}^i$, $i \in \{1, 2, 3\}$ of the boundary maximally symmetric WZW bi-brane described in Example 2.13. The task in hand was explicitly carried out for $G = SU(2)$ in Ref. [RS09], whereby it was shown that the inter-bi-brane 2-isomorphism exists iff the Verlinde fusion coefficient $N_{\lambda_1, \lambda_2}$ for the triple of chiral sectors of the quantised WZW model labelled by $\lambda_1, \lambda_2, \lambda_3$ does not vanish, in accord with the prediction of the categorial quantisation of the WZW model elaborated in Refs. [FRS05, FFRS07]. This result constitutes a quantum variant of the pre-quantum result discussed in the present section.

**Remark 5.11.** Incidentally, the last example indicates the possibility of adding further structure to the correspondence between the inter-bi-brane and the interaction subspace, valid whenever the corresponding defect preserves some of the symmetry of the untwisted sector of the $\sigma$-model. The structure in question is that of an intertwiner of the representations of the symmetry algebra associated with the states undergoing the splitting-joining interaction mediated by the defect quiver. Making this observation rigorous calls for a detailed study of the issue of symmetry transmission across the defect, which shall be addressed in the framework of generalised geometry in the companion paper [Sus11].

**6. Conclusions and outlook**

At the focus of our interest in the present paper lay the state-space interpretation of conformal defects in the classical and (pre-)quantum formulation of the two-dimensional (bosonic) non-linear $\sigma$-model on an oriented multi-phase world-sheet. The latter theory is defined in terms of cohomological structures, termed the gerbe $G$, the $G$-bi-brane $B$ and the $(G, B)$-inter-bi-brane $J$, coming from the 2-category $\mathcal{B}$ of bundle gerbes (with connection) over the codomain $M \sqcup Q \sqcup T$ of the $\sigma$-fields fields. The issue of interest was addressed in the canonical framework of description of the state space of the theory, reconstructed in the classical régime using the techniques of covariant classical field theory (Propositions 3.11 and 3.12) and subsequently extended to the quantum régime by means of transgression maps (Theorems 3.16 and 3.18), derived along the lines of the long-known explicit construction for the closed string with a mono-phase world-sheet. The transgression maps were employed to induce a pre-quantum bundle over the state space of the $\sigma$-model from gerbe and bi-brane data, in both the untwisted sector and the twisted sector of that space (Corollaries 3.17 and 3.19). In the presence of the pre-quantum bundles, the notion of a pre-quantum duality of the $\sigma$-model was formalised, whereupon a precise correspondence was established between pre-quantum dualities and (non-intersecting) topological world-sheet defects. The correspondence associates a duality to a topological defect with bi-brane maps given by surjective submersions satisfying some additional technical conditions (Theorem 4.14), and conversely identifies the bi-brane encoded by the data of a duality of one of the two distinguished types: type $T$ (Theorem 4.14) and type $N$ (Theorem 4.15). From a further extension of the canonical framework to the interacting multi-string state space, both classical and (pre-)quantum, an intuitive picture was shown to emerge of the defect junctions and the attendant geometric data from $\mathcal{B}$ of $M \sqcup Q \sqcup T$ acting as intertwiners between representations of the symmetry algebra realised on the spaces of states of the string in interaction, be it untwisted or twisted (Theorems 5.5 and 5.8). In the minimal scenario, the symmetry algebra in question is the Virasoro algebra of the conformal group in two dimensions. The case in which this algebra is extended by some Kac–Moody algebra induced from distinguished isometries of the (pseudo-)riemannian target space of the $\sigma$-model is elaborated in Ref. [Sus11]. As a by-product of our analysis of conformal defects, the
notion of a simplicial string background was introduced (Remark 5.6) and a conjectural statement was made with regard to its rôle in defining string theory on (generically non-geometric) ‘duality quotients’, generalising the concept of a T-fold in a manner dictated by a duality scheme. The scheme is largely motivated by the idea of categorial descent for the 2-category of bundle gerbes and the gerbe-theoretic construction of the gauged σ-model.

There are two main inferences that can be drawn from our findings recapitulated above: The first is the manifest naturality, i.e. completeness and minimality of the full-blown 2-categorical structure \( \mathcal{B}Grb (M \sqcup Q \sqcup T) \) viewed as a scheme of description of the two-dimensional dynamics (including the purely geometric interactions) of the field theory in hand, and of its modifications obtained through gauging and orientifolding. Additional evidence in favour of the said naturality can be extracted from the analysis of the algebraic structure on the set of continuous internal symmetries of the multi-phase σ-model. This issue is examined at length in the companion paper [Sus11], whereby the concept of generalised geometry twisted by the background \( \mathcal{B} \) is seen to arise, encompassing the familiar construction of Refs. [Hit03, Gua03] as a special case. The second basic inference is the fundamental rôle of world-sheet defects (and so also of the attendant cohomological structures over \( M \sqcup Q \sqcup T \)) in probing the ‘topography’ of the moduli space of two-dimensional (bosonic) non-linear σ-models via the associated dualities. Taken in conjunction, the two offer insights into the very deep structure underlying the lagrangean formulation of (critical) string theory, and that from the level of readily tractable geometric constructs from the smooth category. This alone provides strong motivation for further work in the directions suggested by the hitherto results.

An outstanding problem from this last category is the precise relation between the 2-category \( \mathcal{B}Grb (M \sqcup Q \sqcup T) \), considered together with the transgressed structures over the state space of the σ-model, and elements of the categorial quantisation scheme thereof (in the sense of Segal, cp. Ref. [Seg88]). There is ample and highly non-trivial evidence indicating that certain topologically protected (and quantised) results of the gerbe-theoretic approach, such as, e.g., the existence and uniqueness results for the σ-model in a given string background, its quotients and orientifolds, as well as the cohomological data describing the fusion of topological defects (e.g., the conditions of existence of inter-bi-brane 2-isomorphisms, admitting a straightforward interpretation in terms of spaces of conformal blocks, cp. Ref. [RS], and the recoupling coefficients for simple associator moves on topological defect quivers embedded in the world-sheet, related to the fusing matrices of Moore and Seiberg, cp. Ref. [RS09]), carry over unaltered to the quantum régime, as defined rigorously by the categorial quantisation (or by the operator-algebraic quantisation, for that matter). It therefore seems apposite to enquire whether the cohomological structure of the gerbe and its 2-categorial descendants is sufficiently rigid to encode (still more) essential non-perturbative data of the quantised σ-model, also for string backgrounds which – unlike the previously examined cases of canonical gerbes on compact Lie groups and their maximally symmetric (inter-)bi-branes – are devoid of a rich symmetry that could independently constrain the quatisation procedure. It stands to reason that our understanding in this matter can be furthered by a search for a direct ‘holographic’ relation between the 2-category \( \mathcal{B}Grb (M \sqcup Q \sqcup T) \) for the (rational) two-dimensional σ-model and the higher categorial structure behind a three-dimensional Topological Field Theory (TFT) that defines the categorial quantisation scheme of the σ-model in a manner detailed in Refs. [FFFS01, FFFS02]. A link between the two structures was established in the largely tractable WZW setting (in which the relevant TFT is the Chern–Simons theory with the gauge group given by the target Lie group of the WZW model) in Ref. [CJM05] but even in this highly symmetric example an exhaustive analysis of the relation between the two-dimensional CFT on a multi-phase world-surface and the corresponding three-dimensional TFT coupled to a collection of intersecting Wilson lines is lacking to date.

In an attempt at gauging the actual extent to which gerbe theory is an intrinsic element of a CFT description of string theory, one could be even more audacious and explore string backgrounds away from criticality (at which the Weyl anomaly vanishes), implicitly chosen as the basis (or completion) of the σ-model discussion. One possible way of grappling with the issue in hand might be the concept of a generalised Ricci flow, winning an ever increasing popularity of late. Here, the hope would be that the ideas of Refs. [Str07, You08], originally applied to (principal) fibre bundles, could be successfully adapted to handle gerbes over (pseudo-)riemannian bases. A novel alternative for this line of development appears to be offered by the study of the so-called String structures of Ref. [Kil87], cp. Ref. [Wal09] for a modern treatment in the higher-categorical language.
Another important question merely touched upon by hitherto gerbe-theoretic analyses carried out with reference to world-sheet defects takes its origin in the concept of a non-geometric string background. While the general notion of a simplicial string background forwarded in the present paper seems to be perfectly tailored to describe the latter, specific conditions for the existence of a ‘duality quotient’ and extra constraints prerequisite for gauging a ‘duality group’ should be worked out, and explicit examples of those intricate stringy (non-)geometries should be found. An obvious point of departure for these general considerations is an in-depth understanding of the cohomological duality between principal torus bundles with gerbes, or T-duality. Already this outwardly (physically) well-studied subject offers interesting conceptual challenges such as, e.g., the geometric description of the procedure of descending the gauged σ-model from the correspondence space (i.e. from the intermediary bi-toroidal fibration linking the dual backgrounds) to the T-dual toroidal fibration via elimination of the gauge field and symplectic reduction. Its peculiarity consists in that it mixes various tensorial objects defining the string background, as accounted for, e.g., by the Buscher rules of Example 4.16 (or, more generally, by the duality-background constraints (4.23)). This prompts to conceive a significant departure from the established mode of description of geometric constructs such as the metric structure, with its global tensorial representation by the metric field, and the gerbe, with its local differential-geometric presentation. Such a departure, capable of incorporating also the dilaton field, should lead to the emergence of a unified geometric treatment of the various components of the full multiplet of massless closed-string excitations. It is worth pointing out that a possible first step towards such a unified treatment is Hitchin’s construction, advanced in Ref. [HR09], of a generalised metric on the generalised tangent bundle with a torsion-full metric connection, combining, in a most natural manner, the metric tensor and local gerbe data.

Last but not least, given the prominent rôle played by $\mathcal{BGrb}(M \cup Q \cup T)$ in the geometric quantisation of the σ-model, it is tempting to investigate the conditions of compatibility of a choice of polarisation of the pre-quantised theory with structures carried by the conformal defect, and the ensuing constraints on the admissible quantum dualities. The passage from the classical to the quantum régime is bound to result in a renormalisation of the functional relations determining the behaviour of the σ-model fields at the defect (cp., e.g., Refs. [BG04] [AM07]), and it would be desirable to attain a good understanding of these quantum effects.

Thus, altogether, it seems fair to conclude the present paper with the constatation that the hitherto incursions into the physics of conformal defects of the two-dimensional non-linear σ-regime is bound to result in a renormalisation of the functional relations determining the bahaviour of the string background, as accounted for, e.g., by the Buscher rules of Example 4.16 (or, more generally, by the duality-background constraints (4.23)). This prompts to conceive a significant departure from the established mode of description of geometric constructs such as the metric structure, with its global tensorial representation by the metric field, and the gerbe, with its local differential-geometric presentation. Such a departure, capable of incorporating also the dilaton field, should lead to the emergence of a unified geometric treatment of the various components of the full multiplet of massless closed-string excitations. It is worth pointing out that a possible first step towards such a unified treatment is Hitchin’s construction, advanced in Ref. [HR09], of a generalised metric on the generalised tangent bundle with a torsion-full metric connection, combining, in a most natural manner, the metric tensor and local gerbe data.

**APPENDIX A. A proof of Proposition 3.8**

Take an arbitrary $\mathcal{F}_\sigma$-vertical vector field $\mathcal{V}$ on $J^1\mathcal{F}_\sigma$ with restrictions

\begin{equation}
(\mathcal{V}|_\mathcal{F})_\varphi = V^\mu \frac{\delta}{\delta X^\mu} + V_a \frac{\delta}{\delta \xi_a}, \quad (\mathcal{V}|_\mathcal{F})_\varphi = V^A \frac{\delta}{\delta X^A} + V_a \frac{\delta}{\delta \xi_a}, \quad (\mathcal{V}|_\mathcal{F})_\varphi = V^i \frac{\delta}{\delta \xi^i},
\end{equation}

with components constrained as in Eq. (3.10), and denote by $\mathcal{F}$ a vector field on the field space $\mathcal{F} = M \cup Q \cup T$ with restrictions

\begin{equation}
(\mathcal{F}|_\mathcal{F})_M = V^\mu \frac{\delta}{\delta X^\mu}, \quad (\mathcal{F}|_\mathcal{F})_Q = V^A \frac{\delta}{\delta X^A}, \quad (\mathcal{F}|_\mathcal{T})_T = V^i \frac{\delta}{\delta \xi^i}.
\end{equation}

Using the identity

$$\ast \eta \, d\sigma^a = \eta^{ab} \varepsilon_{bc} \, d\sigma^c$$

in conjunction with the relation

$$\delta \int_e X^*_e \eta = - \int_e X^*_e \delta \eta + X^* \eta|_{\partial e},$$

valid for any edge $e \in \Delta(\Sigma)$, an arbitrary 1-form $\eta \in \Omega^1(X(e))$ and for $X_e = X|_e$, we may express the Lie derivative of the functional $S_{\Theta_{\sigma}}$ along $\mathcal{V}$ as

\begin{equation}
(\mathcal{V} \mathcal{\cdot} \delta S_{\Theta_{\sigma}})[\Psi] = \sum_{p \in \Delta(\Sigma)} \left\{ \int_p \Psi \left[ (d^2 \sigma \xi^\mu_a \left( \frac{1}{2} \xi_b^\nu \delta \xi^\nu_a - V_b^\nu \delta X^\mu_b - \xi_b^\nu \delta X^\mu_b \delta \lambda_a) \right) \right] \right\}
\end{equation}

\[ \left[ L_{V^\mu, \mu} \right] \]
in terms of the restrictions \( \Psi_f := \Psi|_f \), \( f \in \Delta(\Sigma) \) and the standard antisymmetrizer \( V^{[\mu} W^{\nu]} := V^{\mu} W^{\nu} - V^{\nu} W^{\mu} \).

We begin by considering the distinguished \( \mathcal{F}_\sigma \)-vertical vector field \( \frac{\delta}{\delta \xi^\mu} \), for which

\[
0 = \frac{\delta}{\delta \xi^\mu} \delta S_{\Theta_\sigma}[\Psi_{\sigma,cl}] = \sum_{p \in \Delta(\Sigma)} \int_p \Psi_{\sigma,cl}^* ( (d^2 \sigma \xi^\mu - d\sigma^c \circ \varepsilon_{ca} \delta X^\mu ) L_{p,\mu}^{ab} V_b^{\nu} )
\]

whence the classical relation

\[
\xi^\mu_a = \partial_\alpha X^\mu.
\]

Upon taking the latter into account, invoking the identity

\[
L_{p,\mu}^{ab} = \Gamma_{p,\mu}^{ab}
\]

as well as Eqs. (2.3), (2.5) and (2.7), and – finally – denoting by \( \text{Vol}(e) \) and \( \hat{\tau} \) the volume form and the versor tangent to \( e \in \mathfrak{C}_\Gamma \), respectively, we readily reduce Eq. (3.12) to the simpler form

(A.4) \[
\Psi^*(\mathcal{F}_\sigma \triangledown \delta S_{\Theta_\sigma}[\Psi_{\sigma,cl}])
\]

\[
= \sum_{p \in \Delta(\Sigma)} \left[ - \int_p d^2 \sigma \Psi_{\sigma,cl}^* \left( [V^{\mu} (\xi_{\mu} \eta^{ab} \partial_\alpha \partial_\beta X^\nu + \frac{1}{2} (\partial_\mu L_{p,\mu \rho \sigma}^{ab} - \partial_\rho L_{p,\mu \sigma \rho}^{ab} - \partial_\sigma L_{p,\mu \rho \mu}^{ab}) \partial_\alpha X^\rho \partial_\beta X^\sigma)]
\right) + \sum_{e \in \mathfrak{C}_\Gamma} \left[ \int_e \Psi_{\sigma,cl}^*(\mathcal{F}_\sigma \triangledown B_{\mu \rho}) - \sum_{\nu \in \mathfrak{V}} \varepsilon_{pev} \Psi^*(\mathcal{F}_\sigma \triangledown A_{\nu,\nu})(v) \right)
\right] + \sum_{\epsilon \in \mathfrak{V}} \varepsilon_{ev} \Psi^*(\mathcal{F}_\sigma \triangledown (e_2 A_{\nu,\nu} - e_1 A_{\nu,\nu} + P_{\nu,\nu} + i \delta \log K_{\nu,\nu}))(e) \right] - i \sum_{j \in \mathfrak{P} \cap \mathfrak{G}_\Gamma} \Psi^*(\mathcal{F}_\sigma \triangledown \delta \log f_{n_j,i_j})(j) \right]
\]

\[
= \sum_{p \in \Delta(\Sigma)} \left[ - \int_p d^2 \sigma \Psi_{\sigma,cl}^* \left( [V^{\mu} (\xi_{\mu} \eta^{ab} \partial_\alpha \partial_\beta X^\nu + \frac{1}{2} (\partial_\mu L_{p,\mu \rho \sigma}^{ab} - \partial_\rho L_{p,\mu \sigma \rho}^{ab} - \partial_\sigma L_{p,\mu \rho \mu}^{ab}) \partial_\alpha X^\rho \partial_\beta X^\sigma)]
\right) + \sum_{e \in \mathfrak{C}_\Gamma} \left[ \int_e \Psi_{\sigma,cl}^*(\mathcal{F}_\sigma \triangledown (e_2 A_{\nu,\nu} - e_1 A_{\nu,\nu} + P_{\nu,\nu} + i \delta \log K_{\nu,\nu}))(e) \right] + \sum_{\epsilon \in \mathfrak{V}} \varepsilon_{ev} \Psi^*(\mathcal{F}_\sigma \triangledown (e_2 A_{\nu,\nu} - e_1 A_{\nu,\nu} + P_{\nu,\nu} + i \delta \log K_{\nu,\nu}))(e) \right] - i \sum_{j \in \mathfrak{P} \cap \mathfrak{G}_\Gamma} \Psi^*(\mathcal{F}_\sigma \triangledown \delta \log f_{n_j,i_j})(j) \right]
\]

\[
= \sum_{p \in \Delta(\Sigma)} \left[ - \int_p d^2 \sigma \Psi_{\sigma,cl}^* \left( [V^{\mu} (\xi_{\mu} \eta^{ab} \partial_\alpha \partial_\beta X^\nu + \frac{1}{2} (\partial_\mu L_{p,\mu \rho \sigma}^{ab} - \partial_\rho L_{p,\mu \sigma \rho}^{ab} - \partial_\sigma L_{p,\mu \rho \mu}^{ab}) \partial_\alpha X^\rho \partial_\beta X^\sigma)]
\right) + \sum_{e \in \mathfrak{C}_\Gamma} \left[ \int_e \Psi_{\sigma,cl}^*(\mathcal{F}_\sigma \triangledown (e_2 A_{\nu,\nu} - e_1 A_{\nu,\nu} + P_{\nu,\nu} + i \delta \log K_{\nu,\nu}))(e) \right] + \sum_{\epsilon \in \mathfrak{V}} \varepsilon_{ev} \Psi^*(\mathcal{F}_\sigma \triangledown (e_2 A_{\nu,\nu} - e_1 A_{\nu,\nu} + P_{\nu,\nu} + i \delta \log K_{\nu,\nu}))(e) \right] - i \sum_{j \in \mathfrak{P} \cap \mathfrak{G}_\Gamma} \Psi^*(\mathcal{F}_\sigma \triangledown \delta \log f_{n_j,i_j})(j) \right]
\]

It is now a matter of a simple check to verify that the upper integrand coincides with the pullback of the contraction of the field equation (2.10) with the arbitrary vector \( \mathcal{F}_\sigma \), whereas the bottom one is the pullback of the contraction of the DGC (2.8) with the same vector field. Thus, demanding that the variation vanish for all \( \mathcal{F}_\sigma \) is tantamount to imposing the field equations and the Defect Gluing Condition of the \( \sigma \)-model given by the action functional (2.9).

\[ \square \]

**APPENDIX B. A PROOF OF PROPOSITION 3.11**

The proof is an adaptation of the constructive proof of Proposition 3.2 to the circumstances in hand. Thus, we consider a region \( \Sigma_{1,2} \) in the world-sheet \( \Sigma \) bounded, as \( \partial \Sigma_{1,2} = C_2 \cup (-C_1) \), by a
pair of Cauchy contours $C_A$, $A \in \{1, 2\}$ with orientation as in Fig. 8. Choose a triangulation $\Delta(\Sigma)$ of

\[ \Sigma \text{ subordinate to } \mathcal{O}_M, \quad M \in \{M, Q, T\} \quad \text{with respect to } (X|\Gamma) \text{ such that it induces a triangulation } \Delta(\Sigma_{1,2}) \text{ of } \Sigma_{1,2}, \text{ and so also triangulations } \Delta(C_A) \text{ of the two Cauchy contours, the latter consisting of the respective edges } e_A \text{ and vertices } v_A \text{ (this can always be achieved via refinement of a given triangulation of the world-sheet). Define}
\]

\[ S_{1,2}[\Psi_{\sigma,cl}] = \int_{\Sigma_{1,2}} (\Psi_{\sigma,cl}|_{\Sigma_{1,2}})^* \Theta_\sigma \]

and use the previous result (A.3) alongside the first three lines of the computation (A.4) to write, for $\mathcal{V}$ tangent to $P_{\sigma, \mathcal{V}}$, and $\mathcal{V}$ as in Eqs. (A.1) and (A.2), respectively,

\[ \mathcal{V} \cdot \delta S_{1,2}[\Psi_{\sigma,cl}] = \sum_{A=1}^{2} (-1)^{A} \left( \int_{C_A} \text{Vol}(C_A) \left( \Psi_{\sigma,cl}|_{C_A} \right)^* (\mathcal{V} \cdot \mathcal{P}) + \sum_{e_A \in \Delta(C_A)} \int_{e_A} \Psi_{\sigma,cl}^* e_A (\mathcal{V} \cdot B_{v_A}) \right) \]

(B.1)

\[ \quad - \sum_{v_A \in \Delta(C_A)} \mathcal{V} \cdot (A_{v_A}(v_A) e_{v_A} - i \delta \log \frac{g_{i_{v_A}(v_A)} e_{v_A}}{g_{i_{v_A}(v_A)} e_{v_A}}) (X(v_A)) \]

where $\text{Vol}(C_A)$ is a volume form on $C_A$ and $e_{+}(v_A)$ (resp. $e_{-}(v_A)$) denotes the incoming (resp. outgoing) edge at $v_A$, and where

\[ \int_{e_A} \Psi_{\sigma,cl}^* e_A (\mathcal{V} \cdot B_{v_A}) \equiv \int_{e_A} \text{Vol}(e_A) \Psi_{\sigma,cl}^* e_A (\mathcal{V} \cdot B_{v_A}) = \int_{e_A} \text{Vol}(e_A) \Psi_{\sigma,cl}^* e_A \left( \mathcal{V} \cdot (\tilde{T}_{C_A} \cdot B_{v_A}) \right) \]

for $\tilde{T}_{C_A}$ the tangent vector field on $C_A$. The last equality enables us to derive, from Eq. (B.1), the sought-after expression

\[ \delta S_{1,2}[\Psi_{\sigma,cl}] = \sum_{A=1}^{2} (-1)^{A} \left( \int_{C_A} \text{Vol}(C_A) \wedge \left( \Psi_{\sigma,cl}|_{C_A} \right)^* \mathcal{P} - \sum_{e_A \in \Delta(C_A)} \int_{e_A} \Psi_{\sigma,cl}^* e_A B_{v_A} \right) \]

(B.2)

\[ \quad - \sum_{v_A \in \Delta(C_A)} \Psi_{\sigma,cl}^* (A_{v_A}(v_A) e_{v_A} - i \delta \log \frac{g_{i_{v_A}(v_A)} e_{v_A}}{g_{i_{v_A}(v_A)} e_{v_A}}) (v_A) \]

whence the thesis of the proposition follows straightforwardly upon defining

\[ \Omega_{\sigma,\emptyset}[\Psi_{\sigma,cl}] := \delta \left( \int_{\emptyset} \text{Vol}(\emptyset) \wedge \left( \Psi_{\sigma,cl}|_{\emptyset} \right)^* \mathcal{P} - \sum_{e \in \Delta(\emptyset)} \int_{e} \Psi_{\sigma,cl}^* e B_{v} \right) \]

\[ \quad - \sum_{v \in \Delta(\emptyset)} \Psi_{\sigma,cl}^* (A_{v}(v) e_{v} - i \delta \log \frac{g_{i_{v}(v)} e_{v}}{g_{i_{v}(v)} e_{v}}) (v) \]
(manifestly independent of the choice of the Cauchy contour $C$) and upon employing Eq. (2.3) and the identity

\[ \delta \int_{e} \Psi_{\sigma,cl}^{*} e B_{i_{e}} = - \int_{e} \Psi_{\sigma,cl}^{*} e \delta B_{i_{e}} + \Psi_{\sigma,cl}^{*} B_{i_{e}} |_{\partial e}. \]

\[ \Box \]

**Appendix C. A proof of Proposition 3.12**

The proof develops essentially along the lines of the constructive proof of Proposition 3.2. Here, we consider a region $\Sigma_{1,2}$ in the world-sheet $\Sigma$ bounded, as $\partial \Sigma_{1,2} = C_{2} \cup (C_{1})$, by a pair of twisted Cauchy contours $C_{A}, A \in \{1, 2\}$ with orientation as in Fig. 9 each intersecting a family of $I \in \mathbb{N}_{\geq 0}$ defect lines $\ell_{k}, k \in \Gamma, I$. Once again, we choose a triangulation $\Delta(\Sigma)$ of $\Sigma$ subordinate to $O_{def}, M \in \{M, Q, T\}$.

![Figure 9. A pair of disjoint twisted Cauchy contours, $C_{1}$ and $C_{2}$, intersecting the defect quiver $\Gamma$ at points $v_{A}^{k}, k \in \Gamma, I$ from the respective (anti-)time-like defect lines $\ell_{k}$ (time runs radially, from $C_{1}$ towards $C_{2}$). The intermediate region $\Sigma_{1,2}$, cut out from the world-sheet $\Sigma$ by the two contours, is free of defect junctions, whereas its complement $\Sigma \setminus \Sigma_{1,2}$ contains an arbitrary sub-graph of $\Gamma$ with $I$ free legs, embedded in an arbitrary world-sheet topology. The arrows represent the orientation of the various curves.](image)

with respect to $(X|\Gamma)$ which induces a triangulation $\Delta(\Sigma_{1,2})$ of $\Sigma_{1,2}$, and so also triangulations $\Delta(C_{A})$ of the two Cauchy contours (superimposed upon that of the segments of $\ell_{k}$ cut out by the two Cauchy contours), the latter consisting of the respective edges $e_{A}$ and vertices $v_{A}$, with the distinguished vertices $v_{A}^{k}$ at intersections $\ell_{k} \cap C_{A}$. Define

\[ S_{1,2}[\Psi_{\sigma,cl}] = \int_{\Sigma_{1,2}} (\Psi_{\sigma,cl}|\Sigma_{1,2})^{*} \Theta_{\sigma}. \]

The only difference with respect to the previous result, Eq. (B.2), appears at the $v_{A}^{k}$ and yields

\[ \delta S_{1,2}[\Psi_{\sigma,cl}] = \sum_{A=1}^{2} (-1)^{A} \left( \int_{C_{A}} \text{Vol}(C_{A}) \wedge (\Psi_{\sigma,cl}|C_{A})^{*} p - \sum_{e_{A} \in \Delta(C_{A})} \int_{e_{A}} \Psi_{\sigma,cl}^{*} e_{A} B_{i_{e_{A}}} \\
- \sum_{v_{A} \in \Delta(C_{A}) \cap \Gamma} \Psi_{\sigma,cl}^{*}(A_{i_{e_{A}}(v_{A})}i_{e_{A}(v_{A})} - i \delta \log g_{i_{e_{A}}(v_{A})i_{e_{A}}(v_{A})}^{A}(v_{A})) \\
+ \sum_{k=1}^{I} \Psi_{\sigma,cl}^{*}(\ell_{1}^{k} A_{i_{e_{A}}(v_{A})}^{\phi_{1}^{k}}(i_{A}^{k}) - \ell_{2}^{k} A_{i_{e_{A}}(v_{A})}^{\phi_{2}^{k}}(i_{A}^{k}) + \varepsilon_{k} P_{i_{A}^{k}})(v_{A}^{k})) \right), \]

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where \( \varepsilon_k = +1 \) if \( \ell_k \) is time-like, \( \varepsilon_k = -1 \) if \( \ell_k \) is anti-time-like, and where \((\phi_1^{+1}, \phi_2^{+1}) = (\phi_1, \phi_2) \) and \((\phi_1^{-1}, \phi_2^{-1}) = (\phi_2, \phi_1) \). The thesis of the proposition is demonstrated by defining a 2-form

\[
\Omega_{\sigma,B}\{(P_\varepsilon,\varepsilon_k)\}[\Psi_{\sigma,cl}] := \left( \int_\mathcal{C} \text{Vol}(\mathcal{C}) \wedge (\Psi_{\sigma,cl}|_{\mathcal{C}})^* \varphi - \sum_{v \in \Delta(\mathcal{C})} \int_\tau \Psi_{\sigma,cl}^*(A_{i(v)}(\sigma_{i(v)}) - i \delta \log g_{i(v)}(\sigma_{i(v)})) (v) \right) - \sum_{v \in \Delta(\mathcal{C}) \setminus \{P_\varepsilon\}} \Psi_{\sigma,cl}^*(A_{i(v)}(\sigma_{i(v)}) - i \delta \log g_{i(v)}(\sigma_{i(v)}) (v)) + \sum_{k=1}^{I} \int_{\pi^{-1}(\psi_k)} \left( i_{\varepsilon_k}^* A_{i(v)}(\psi_k) - i_{\varepsilon_k}^* A_{i(v)}(\psi_k) \right) d\tau \Omega_{\sigma,cl} (P_\varepsilon) \right) ,
\]

manifestly independent of the choice of the Cauchy contour \( \mathcal{C} \). The 2-form acquires the desired form after a simple computation using Eqs. (2.3), (2.5) and (B.3).

\[\square\]

**APPENDIX D. A PROOF OF THEOREM 5.5**

Ad i) First, through direct inspection of Eqs. (4.3) and (3.21), we establish that the ‘sum’ symplectic form

\[
\Omega^+_{\sigma,\mathcal{B}} = \text{pr}^+_{1, \sigma, \mathcal{B}} \Omega_{\sigma, \mathcal{B}} + \text{pr}^+_{2, \sigma, \mathcal{B}} \Omega_{\sigma, \mathcal{B}}
\]

on \( P_{\sigma, \mathcal{B}} \), the latter space being considered with the two canonical projections \( \text{pr}_\alpha : P_{\sigma, \mathcal{B}} \to P_{\sigma, \mathcal{B}} \), \( \alpha \in \{1,2\} \), restricts – the restriction being marked by the bar over \( \Omega^+_{\sigma, \mathcal{B}} \) – as

\[
\Omega^+_{\sigma, \mathcal{B}}[\{\psi_1, \psi_2\}] = \int \frac{d\varphi}{2}(\Psi_{\sigma,cl}^+)^*(\delta \varphi_2 + \partial_\varphi X_2 \cdot \mathcal{H}) + \int \frac{d\varphi}{2}(\Psi_{\sigma,cl}^-)^*(\delta \varphi_1 + \partial_\varphi X_1 \cdot \mathcal{H})
\]

(D.1)

\[+ Y^{\mathcal{B}}_{\gamma, \omega}(0) + Y^{\mathcal{B}}_{\gamma, \omega}(0) \]

to the tangent \( T\mathcal{P}_{\sigma, \mathcal{B}} \) of the \( \mathcal{B} \)-fusion subspace \( \mathcal{P}_{\sigma, \mathcal{B}} \) within \( \Gamma(\mathcal{TP}_{\sigma, \mathcal{B}}|_{\mathcal{P}_{\sigma, \mathcal{B}}}) \). Here, \( (\Psi_{\sigma,cl}^+, \Psi_{\sigma,cl}^-) \) is the pair of extremal sections represented by the Cauchy data \( (\psi_1, \psi_2) \), with \( \psi_\alpha = (X_\alpha, p_\alpha) \), \( \alpha \in \{1,2\} \). Given the above result, we readily check, using Eq. (4.3), that \( \Omega^+_{\sigma, \mathcal{B}} \) restricts to the subspace \( T\mathcal{J}_\sigma(\mathcal{B} : \mathcal{J} : \mathcal{B}) \) within \( \Gamma(\mathcal{TP}_{\sigma, \mathcal{B}}|_{\mathcal{J}_\sigma(\mathcal{B} : \mathcal{J} : \mathcal{B})}) \) spanned by vector fields tangent to \( \mathcal{J}_\sigma(\mathcal{B} : \mathcal{J} : \mathcal{B}) \) as

\[
\Omega^+_{\sigma, \mathcal{B}}[\{\psi_1, \psi_2, \psi_3\}] = (\pi_3^{1,2,\omega} + \pi_3^{2,3,\omega} - \pi_3^{3,1,\omega}) \in Z_{10}^\pi = 0,
\]

which proves the first statement of the relevant DJI.

Ad ii) We begin by describing a circle bundle \( \mathcal{L}_{\sigma,\mathcal{B}} : P_{\sigma, \mathcal{B}} \to \mathcal{P}_{\sigma, \mathcal{B}} \) over the \( \mathcal{B} \)-fusion subspace of the untwisted string, with curvature equal to the restriction \( \Omega^+_{\sigma, \mathcal{B}} \) of Eq. (D.1). The bundle, given by the restriction to \( P_{\sigma, \mathcal{B}} \) of the the tensor product

\[\mathcal{L}_{\sigma,\mathcal{B}} := (\text{pr}^+_{1, \sigma, \mathcal{B}} \mathcal{L}_{\sigma, \mathcal{B}} \otimes \text{pr}^+_{2, \sigma, \mathcal{B}} \mathcal{L}_{\sigma, \mathcal{B}})|_{\mathcal{P}_{\sigma, \mathcal{B}}} \]

of the pullbacks of the pre-quantum bundle for the untwisted sector of the \( \sigma \)-model along the canonical projections \( \text{pr}_\alpha \), provides – for a choice of the polarisation – a definition of the untwisted two-string Hilbert space. Writing out the local data of the bundle, which we shall need in subsequent computations, precludes fixing an open cover of \( P_{\sigma, \mathcal{B}} \). Similarly the proof of Theorem 4.9 we induce it from the open cover of the loop-space basis \((\mathcal{P} \tau_{\mathcal{L}M}, \mathcal{P} \tau_{\mathcal{L}M'}) \) of the fusion space, obtained by varying triangulations of the two loops in \((\psi_1, \psi_2) \in P_{\sigma, \mathcal{B}} \) of the following form, cf. Fig. 10. Along the two half-loops \( X_{\alpha}|_{\alpha (=)} \), \( \alpha \in \{1,2\} \), both triangulations come from a triangulation of the free open path \( Y_{1,2} \), and so, for a choice \( \Delta(1) = \Delta_{1,2} \) of a triangulation, with edges \( e \) and vertices \( v \), of the \( \pi \)-unit interval \( I \) parameterising \( Y_{1,2} \) and \( X_{\alpha}|_{\alpha (=)} \), we have an assignment of indices \( f \mapsto (i_f^{1,1}, i_f^{2,1}, i_f^{2,2}) \in \mathcal{J}_\mathcal{M} \times \mathcal{J}_\mathcal{M} \times \mathcal{J}_\mathcal{Q} \) to every element \( f \in \Delta_{1,2} \), related as per Eq. (4.9). Note that this entails fixing a pair of vertices \( v_+, v_- \) of the triangulation corresponding to the two boundary points of \( I \). The triangulation of each of the two loops \( X_\alpha \) is next completed by specifying an arbitrary triangulation of the interval parameterising the free half-loop, to wit, \( \Delta(\tau(1)) = \Delta_1 \) for \( \alpha = 1 \) and \( \Delta(\tau(1)) = \Delta_2 \) for \( \alpha = 2 \), together with the respective index assignments.

Thus, let the pair Čech index \( i_\alpha \) of the open cover of \( \mathcal{L}M \), as described previously, encode the choice \( (\Delta_{1,2}, \Delta_\alpha) \) of the triangulation of the two half-loops that compose \( X_\alpha \), together with the choice of the indexing maps \( \Delta_{1,2} \to \mathcal{J}_\mathcal{M} \) and \( \Delta_\alpha \to \mathcal{J}_\mathcal{M} \), of which the former is induced, as described, by an indexing map \( \Delta_{1,2} \to \mathcal{J}_\mathcal{Q} \) and Čech extensions \( \phi_\alpha \), \( \alpha \in \{1,2\} \) of
the $t_\alpha$. Last, introduce the shorthand notation $\Delta_\alpha$ and $\Delta_{1,2}$ for the set of edges and vertices of the triangulations $\Delta_\alpha$ and $\Delta_{1,2}$, respectively, with the boundary vertices $v_k$ removed, and fix a local presentation of $\mathcal{B}$ associated with the ensuing choice of open covers of the $\mathcal{B}$-fusion subspace, in conformity with Definition 2.2. After a tedious but otherwise completely straightforward calculation, we find the explicit expressions

$$
\theta_{\sigma, \mathcal{B}((i_1,i_2), (j_1,j_2))}((\psi_1, \psi_2)) = \int \text{d}\varphi \wedge (\Psi_0^2 |\varphi_1|) \ast_{\varphi} + \int \text{d}\varphi \wedge (\Psi_0^2 |\varphi_1|) \ast_{\varphi} - \sum_{e \in \Delta_2} \int e^* A_{2} \cdot B_{e^*} - \sum_{e \in \Delta_1} \int e^* A_{1} \cdot B_{e^*}

- \sum_{e \in \Delta_2} X_{2}^* A_{e^*}^2(v) - \sum_{e \in \Delta_1} X_{1}^* A_{e^*}^1(v)

- (X_{2}^* A_{e^*}^2 - X_{1}^* A_{e^*}^1)(v^+)- Y_{1,2}^* P_{e^*}^{-1}(v^+)

+ (X_{2}^* A_{e^*}^2 - X_{1}^* A_{e^*}^1)(v^-) - Y_{1,2}^* P_{e^*}^{-1}(v^-)

+ i\delta \log f_{\sigma, \mathcal{B}, i_1, i_2}((\psi_1, \psi_2))

and (defining the $\overline{\Delta}_\alpha$ in analogy with the $\overline{\Delta}_\alpha$)

$$
\gamma_{\sigma, \mathcal{B}((i_1,i_2), (j_1,j_2))}((\psi_1, \psi_2)) = \prod_{e \in \overline{\Delta}_2} e^{-i \int_{e} X_{2}^* A_{e} \cdot B_{e^*}} \cdot \prod_{e \in \overline{\Delta}_1} e^{-i \int_{e} X_{1}^* A_{e^*} \cdot B_{e^*}}

\cdot \prod_{e \in \overline{\Delta}_2} X_{2}^* (g_{1,2}^{-1} (g_{2,1}^{j_1} g_{1,2}^{j_2} g_{2,1}^{j_1} g_{1,2}^{j_2}))(v^+)

\cdot \prod_{e \in \overline{\Delta}_1} X_{1}^* (g_{1,2}^{-1} (g_{2,1}^{j_1} g_{1,2}^{j_2} g_{2,1}^{j_1} g_{1,2}^{j_2}))(v^+)

\cdot [Y_{1,2}^* K_{j_1}^{i_1} j_2^{i_2} X_{1}^* (g_{2,1}^{-1} g_{1,2}^{j_1} g_{2,1}^{j_1} g_{1,2}^{j_2}))(v^+)

\cdot X_{2}^* (g_{2,1}^{-1} g_{1,2}^{j_1} g_{2,1}^{j_1} g_{1,2}^{j_2}))(v^+)

\cdot [Y_{1,2}^* K_{j_1}^{i_1} j_2^{i_2} X_{1}^* (g_{2,1}^{-1} g_{1,2}^{j_1} g_{2,1}^{j_1} g_{1,2}^{j_2}))(v^-)

\cdot X_{2}^* (g_{2,1}^{-1} g_{1,2}^{j_1} g_{2,1}^{j_1} g_{1,2}^{j_2}))(v^-)

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\[
(f_{\sigma_1;\sigma_2;\iota_{1,2}})\psi_1,\psi_2 \rightleftharpoons \frac{1}{\Delta_{\iota_{1,2}}} f_{\sigma_1;\sigma_2;\iota_{1,2}}^{-1} \psi_1,\psi_2
\]

for the local connection 1-forms

\[
\theta_{\sigma_1;\sigma_2;\iota_{1,2}} = (pr_1^* \theta_{\sigma_1;\iota_{1,2}} + pr_2^* \theta_{\sigma_2;\iota_{1,2}})_{|p_{ss}}
\]

and the local transition functions

\[
\gamma_{\sigma_1;\sigma_2;\iota_{1,2}} = (pr_1^* \gamma_{\sigma_1;\iota_{1,2}} + pr_2^* \gamma_{\sigma_2;\iota_{1,2}})_{|p_{ss}}
\]

of the bundle \(L_{\sigma_1;\sigma_2}\), defined in terms of the connection 1-forms and transition functions of \(L_{\sigma_1;\iota_{1,2}}\) as well as the relations

\[
\psi_{1,2}^{1,2} = \prod_{(i),\epsilon \Delta_{\iota_{1,2}}} e^{i \int_{\Delta_{\iota_{1,2}}} Y_{1,2}^{1,2} P_{1,2}^{1,2}}
\]

and \(\psi_{1,2}^{1,2} = \prod_{(i),\epsilon \Delta_{\iota_{1,2}}} e^{i \int_{\Delta_{\iota_{1,2}}} Y_{1,2}^{1,2} P_{1,2}^{1,2}}\) the local transition functions that satisfy at the triple junctions and written in terms of the index maps \(\psi_{1,2}^{1,2}\) and \(\psi_{1,2}^{1,2} = \prod_{(i),\epsilon \Delta_{\iota_{1,2}}} e^{i \int_{\Delta_{\iota_{1,2}}} Y_{1,2}^{1,2} P_{1,2}^{1,2}}\). In so doing, we assume – as under the same restriction, demonstrated in the proof of statement i) of the present theorem, \(\psi_{1,2}^{1,2} = \prod_{(i),\epsilon \Delta_{\iota_{1,2}}} e^{i \int_{\Delta_{\iota_{1,2}}} Y_{1,2}^{1,2} P_{1,2}^{1,2}}\) that this is the case through a direct computation employing Definition 5.4 and Eqs. (5.9)-(5.11) which describe the base of the bundles of interest locally. In so doing, we assume – as previously – the triangulations of the various half-loops of the untwisted states in interaction to be induced by the triangulations of the respective parent half-loops \(Y_{1,2}^{1,2} = \prod_{(i),\epsilon \Delta_{\iota_{1,2}}} e^{i \int_{\Delta_{\iota_{1,2}}} Y_{1,2}^{1,2} P_{1,2}^{1,2}}\) across which the states are identified. This entails, in particular, fixing a pair of vertices \(v_*\) and \(v_*\) in each of the triangulations, corresponding to the pair \(j,\jmath\) of defect junctions in Fig. A lengthy yet direct computation, invoking the defining relations (2.3)-(2.7), yields the anticipated identities

\[
(pr_1 \times pr_2)^* \theta_{\sigma_1;\sigma_2;\iota_{1,2}} = \prod_{(i),\epsilon \Delta_{\iota_{1,2}}} e^{i \int_{\Delta_{\iota_{1,2}}} Y_{1,2}^{1,2} P_{1,2}^{1,2}}
\]

Here, the index \(i^3\) represents a composite triangulation of the circle that parameterises the end-state loop \(X_3\), induced from those of the parent half-loops \(Y_{1,3}\) and \(Y_{2,3}\) together with the corresponding indexing maps. The latter are subject to some obvious relations

\[
i^1 = \phi_1(1^2), \quad i^2 = \phi_2(1^2) \quad \text{along } Y_{1,2},
\]

\[
i^1 = \phi_1(1^3), \quad i^3 = \phi_2(1^3) \quad \text{along } Y_{1,3},
\]

\[
i^2 = \phi_1(2^3), \quad i^3 = \phi_2(2^3) \quad \text{along } Y_{2,3},
\]

as well as the relations

\[
i^{1,3} = \psi^{1,3}_3(i^{1,2})
\]

satisfied at the triple junctions and written in terms of the index maps \(\psi^{1,3}_3: \mathcal{I}_{\mathcal{O}_3} \rightarrow \mathcal{I}_{\mathcal{O}_3}\) covering the \(\pi_{3,1}\) (for \(\psi_{3,1} = \psi_{3,1}\) and \(\pi_{3,1} = \pi_{3,1}\)). For completeness, we also give the local data of the isomorphism – they read (the \(\Delta_{1,2}\) are defined in analogy with \(\Delta_{1,2}\))

\[
f_{\sigma_1;\sigma_2;\iota_{1,2}}^{-1}[(\psi_1, \psi_2)] = \prod_{(i,j) \in \{(1,2),(2,3),(1,3)\}} \prod_{(i),\epsilon \Delta_{1,2}} e^{i \int_{\Delta_{1,2}} Y_{1,2}^{1,2} P_{1,2}^{1,2}}
\]
\[
\prod_{v(1)=v(2)} Y_{\ast,j}^{\ast,j} K_{\ast,j}^{\ast,j} (\epsilon(v(2),c_{(i,j)}) \cdot t_{v_{c_{(i,j)}}} (v(1),j)) \\
\cdot (X_{1} g_{1}^{i(1),j} (v_{c_{(i,j)}}) X_{2} g_{2}^{i(2),j} (v_{c_{(i,j)}}) X_{3} g_{3}^{i(3),j} (v_{c_{(i,j)}})) \\
\cdot Y_{1,2}^{i(1),j} K_{1(1),j}^{i(2),j} \cdot Y_{2,3}^{i(1),j} K_{2(1),j}^{i(2),j} \cdot Y_{1,3}^{i(1),j} K_{1(3),j}^{i(2),j} (Z f_{3,1,1}^{i(2),j} (v_{c_{(i,j)}})) \\
\cdot (X_{1} g_{1}^{i(1),j} (v_{c_{(i,j)}}) X_{2} g_{2}^{i(2),j} (v_{c_{(i,j)}}) X_{3} g_{3}^{i(3),j} (v_{c_{(i,j)}})) \\
\cdot Y_{1,2}^{i(1),j} K_{1(1),j}^{i(2),j} \cdot Y_{2,3}^{i(1),j} K_{2(1),j}^{i(2),j} \cdot Y_{1,3}^{i(1),j} K_{1(3),j}^{i(2),j} (Z f_{3,1,1}^{i(2),j} (v_{c_{(i,j)}})).
\]

(D.4)

This completes the proof of statement ii), and so also the proof of the theorem. □

APPENDIX E. A PROOF OF THEOREM 5.5

Ad i) Take the manifold \( P^{\times 2}_{\sigma,B(\epsilon,\epsilon)} \) of Eq. (5.16), together with the ‘sum’ symplectic form

\[
\Omega_{\ast}^{\sigma,B(\epsilon,\epsilon)} = \operatorname{pr}_{1}^{\ast} \Omega_{\sigma,B(\epsilon,\epsilon)} + \operatorname{pr}_{2}^{\ast} \Omega_{\sigma,B(\epsilon,\epsilon)}
\]

on it, the latter being expressed in terms of the canonical projections \( \operatorname{pr}_{\alpha} : P^{\times 2}_{\sigma,B(\epsilon,\epsilon)} \rightarrow P_{\sigma,B(\epsilon,\epsilon)} \), \( \alpha \in \{1,2\} \). The symplectic form \( \Omega_{\ast}^{\sigma,B(\epsilon,\epsilon)} \) is readily checked to restrict – with the restriction marked by the bar over \( \Omega_{\sigma,B(\epsilon,\epsilon)}^{\ast} \) – to the tangent \( \mathbb{T} P^{\times 2}_{\sigma,B(\epsilon,\epsilon)} \) of the \( B_{\text{triv}} \)-fusion subspace \( P^{\times 2}_{\sigma,B(\epsilon,\epsilon)} \) of the \( (\mathbb{T} P^{\times 2}_{\sigma,B(\epsilon,\epsilon)} |_{\sigma,B(\epsilon,\epsilon)} ) \) as

\[
\mathbb{T}^{\ast} \sigma_{B(\epsilon,\epsilon)}^{\ast}((\psi_{1},\psi_{2})) = \int_{1} d\phi (\Psi_{\sigma,B(\epsilon,\epsilon)}^{\ast}) \ast (\delta p_{1} + \partial_{\phi} X_{1} \cdot H) + \int_{\tau(t)} d\phi (\Psi_{\sigma,B(\epsilon,\epsilon)}^{\ast}) \ast (\delta p_{1} + \partial_{\phi} X_{1} \cdot H) + \varepsilon_{1} \omega(q_{1}) + \varepsilon_{2} \omega(q_{2}).
\]

Here, \( (\Psi_{\sigma,B(\epsilon,\epsilon)}^{\ast}) \) is a pair of extremal sections represented by the Cauchy data \( (\psi_{1},\psi_{2}) \) with \( \psi_{\alpha} = (\epsilon_{\alpha})_{\pi_{a}(\epsilon,\epsilon)} \), \( \alpha \in \{1,2\} \). Using this, we readily check, through inspection, that the symplectic form \( \Omega_{\ast}^{\sigma,B(\epsilon,\epsilon)} \) vanishes identically when restricted to the subspace \( \mathbb{T} \sigma_{B(\epsilon,\epsilon)}^{\ast} : J = B_{\text{triv}} |_{\mathbb{T} P^{\times 2}_{\sigma,B(\epsilon,\epsilon)} |_{\sigma,B(\epsilon,\epsilon)}} \subset \Gamma(\mathbb{T} P^{\times 2}_{\sigma,B(\epsilon,\epsilon)} |_{\sigma,B(\epsilon,\epsilon)})) \) tangent to \( J_{\sigma}(\mathbb{B}_{\text{triv}} : J = B_{\text{triv}} |_{\mathbb{T} P^{\times 2}_{\sigma,B(\epsilon,\epsilon)} |_{\sigma,B(\epsilon,\epsilon)}}) \), as per

\[
\mathbb{T}^{\ast} \sigma_{B(\epsilon,\epsilon)}^{\ast}((\psi_{1},\psi_{2},\psi_{3})) = \sum_{k=1}^{3} \varepsilon_{k} \omega(q_{k}) = \sum_{k=1}^{3} \varepsilon_{k} \pi_{3} \omega(t_{3}) = 0.
\]

This proves statement i).

Ad ii) We begin by reconstructing local data of the bundle

\[
\mathcal{L}^{\beta(\epsilon,\epsilon)}_{\sigma_{B(\epsilon,\epsilon)}} := (\operatorname{pr}_{1}^{\ast} \mathcal{L}_{\sigma,B(\epsilon,\epsilon)} |_{\sigma_{B(\epsilon,\epsilon)}} \otimes \operatorname{pr}_{2}^{\ast} \mathcal{L}_{\sigma,B(\epsilon,\epsilon)} |_{\sigma_{B(\epsilon,\epsilon)}}) \big|_{B_{\text{triv}}},
\]

produced by restricting, to the \( B_{\text{triv}} \)-fusion subspace, the tensor product of the pullbacks of the pre-quantum bundles for the 1-twisted sector of the \( \sigma \)-model along the canonical projections \( \operatorname{pr}_{\alpha} \). To this end, we make a convenient choice of an open cover of the base of the bundle, consistent with the various half-loop identifications present in the definition of \( P^{\beta_{B(\epsilon,\epsilon)}}_{\sigma,B(\epsilon,\epsilon)} \).

As the construction parallels that carried out in the proof of statement ii) of Theorem 5.5, we restrict here to naming the differences. Thus, we fix a common triangulation \( \Delta_{1,2} \) of the parameterising \( \pi \)-unit interval \( I \) and a common index assignment for the two half-loops \( X_{\alpha}(\epsilon_{\alpha}) \), \( \alpha \in \{1,2\} \) (with the common vertex \( v_{+} \) removed), the sole difference with respect to the situation depicted in Fig. 1, being: beside the appearance of the two intersection points \( q_{\alpha} \) – the equality of the indices

\[
i_{f(1,2)}^{1} = i_{f(1,2)}^{2} = i_{f(1,2)}^{(1,2)} = i_{f(1,2)}^{(1,2)} \in \Delta_{1,2},
\]

and the presence of two independent Čech indices from \( \mathcal{J}_{Q} \) assigned to the distinguished common vertex \( v_{+} \) (corresponding to the boundary point \( \pi \) in \( l \)), to be mapped to the defect junction \( J \), namely \( i_{v_{+}}^{1,1,2} \) (for \( X_{1} \)) and \( i_{v_{+}}^{2,1,2} \) (for \( X_{2} \)). They give rise, through the index maps \( \phi_{\alpha} \) covering the \( i_{\alpha} \), to a triple of Čech indices from \( \mathcal{J}_{M} \), to wit,

\[
i_{v_{+}}^{1} = \phi_{1}(i_{v_{+}}^{1,1,2}), \quad i_{v_{+}}^{(1,2)} = \phi_{2}(i_{v_{+}}^{1,1,2}) = \phi_{1}(i_{v_{+}}^{2,1,2}), \quad i_{v_{+}}^{2} = \phi_{2}(i_{v_{+}}^{2,1,2}).
\]
The other boundary vertex of the triangulation of the $\pi$-unit interval $I$, corresponding to the value of the angular parameter $\varphi = 0$, shall be denoted by $v_-$. The local data of the bundle $L_{\pi, B}^\gamma(\epsilon_1, \epsilon_2)$ associated with the ensuing open cover of its base $P_{\pi, B}^\gamma(\epsilon_1, \epsilon_2)$ take the form

$$
\theta_{\delta, \epsilon_1, \epsilon_2} = \int d\varphi \wedge (\Psi_{\delta, \epsilon_1, \epsilon_2})^* p_2 + \int d\varphi \wedge (\Psi_{\delta, \epsilon_1, \epsilon_2})^* p_1 - \sum_{\alpha \in \Sigma} \int \, X_{\alpha}(\epsilon_1, \epsilon_2) B_{\alpha} \delta \int \, X_{\alpha}(\epsilon_1, \epsilon_2) B_{\alpha} \delta
$$

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$$

Note that by virtue of the gluing conditions assumed, we can interchange the fields $X_\alpha$, $\alpha \in \{1, 2\}$ at $v_-$ – this removes the apparent asymmetry in the above formulæ due to the presence of $X_1$.

The last step in our analysis consists in explicitly establishing an isomorphism between the pullback bundles $(\pi_1 \times \pi_2)^* L_{\pi, B}^\gamma(\epsilon_1, \epsilon_2)$ and $\pi_1^* L_{\pi, B}^\gamma(\pi_1, \epsilon_2)$ over the interaction subspace of the 1-twisted string. The demonstration of its anticipated existence proceeds along similar lines as in the proof of Theorem 5.5, with additional simplifications of the triangulations and indexing involved, peculiar to the trivial gluing conditions imposed, namely: We choose a common triangulation and indexing for the pairs $(X_2, X_3)|_{\{1\}}$ and $(X_1, X_3)|_{\{2\}}$. At the distinguished common vertex $v_-$, we now have, in addition to relations (E.1), those obeyed by the extra indices $i^2_{v_+}$, to wit,

$$
i^1_{v_+} = \psi_3^1(1,2,3), \quad i^2_{v_+} = \psi_3^2(1,2,3), \quad i^3_{v_+} = \psi_3^3(1,2,3).
$$

It is now completely straightforward to verify, using Eq. (2.7), the identities

$$
(\pi_1 \times \pi_2)^* \theta_{\delta, \epsilon_1, \epsilon_2} = -i d\log f_{\sigma}^{+*}(1,2,3),
$$

expressed in terms of the local functionals

$$
f_{\sigma}^{+*}(1,2,3)(\psi_1, \psi_2, \psi_3) = X_1^l g_{\sigma, \epsilon_1, \epsilon_2}^{(1,2)}(1,2,3) (v_-) \cdot f_{\sigma}^{+*}(1,2,3)(v_-).
$$

The latter define the isomorphism sought after. This concludes the proof of the theorem. □
