Multiscale analysis of re-entrant production lines: An equation-free approach

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Abstract

The computer-assisted modeling of re-entrant production lines, and, in particular, simulation scalability, is attracting a lot of attention due to the importance of such lines in semiconductor manufacturing. Re-entrant flows lead to competition for processing capacity among the items produced, which significantly impacts their throughput time (TPT). Such production models naturally exhibit two time scales: a short one, characteristic of single items processed through individual machines, and a longer one, characteristic of the response time of the entire factory. Coarse-grained partial differential equations for the spatio-temporal evolution of a “phase density” were obtained through a kinetic theory approach in Armbruster \textit{et al.} \cite{2}. We take advantage of the time scale separation to directly solve such coarse-grained equations, even when we cannot derive them explicitly, through an equation-free computational approach. Short bursts of appropriately initialized stochastic fine-scale simulation are used to perform coarse projective integration on the phase density. The key step in this process is \textit{lifting}: the construction of fine-scale, discrete realizations consistent with a given coarse-grained phase density field. We achieve this through computational evaluation of conditional distributions of a “phase velocity” at the limit of large item influxes.

Key words: Production line, Re-entrant, Equation-free, Coarse projective integration

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1 Introduction

Semiconductor production lines are billion dollar investments and their performance characteristics are carefully studied and modelled in detail through discrete event simulations. In such simulations, a network of processors (machines) is set up with connections that describe the product flow through a factory. As the production process of a typical semiconductor goes through several similar layers, machines may process an item several times at different stages in the production process (re-entrancy). The passage of an item through a factory is characterized by its throughput time (TPT), which may vary due to several factors, most importantly the stochastic features of the machines (their failure statistics) and the interaction with human operators. For a typical discrete event model, therefore, the time through a factory process is a random variable sampled from a (given) probability distribution.

A basic model describing the flow of a single type of products through a processing factory is given by

\begin{align}
(a) \quad e_n &= a_n + \tau_n, \\
(b) \quad dP\{\tau_n \leq r\} &= T(r, t = a_n)dr.
\end{align}

where \(a_n\) is the time when item \(n\) enters the factory and \(e_n\) its exit time.

The time interval \(\tau_n\) for which the item stays in the processing factory is sampled from the distribution \(T(r, t = a_n)\), which is determined at the entrance time. In engineering practice, the most significant influence on the form of this distribution comes from the total number of items in progress, i.e., the work in progress (WIP) (see [1,2]). As a result, one rewrites \(T(r, t = a_n)\) in the form of \(T(r, WIP)\), where WIP is the current WIP at the time \(a_n\). When we fix the TPT at the beginning of the simulation, we essentially treat the factory as a single queue whose length at item arrival determines the time that the item needs to get processed. However, for re-entrant flows, significant changes in WIP during the time interval \([a_n, e_n]\) may lead to a change in the TPT. In [2], this situation is treated by introducing the concepts of phase (a scaled position) and phase velocity. The latter is stochastically updated from a distribution that depends on the total WIP in the factory. In this manner, the TPT is a variable that can dynamically change, impacted by later arrivals of new items.

In [2] an advection-diffusion equation for the phase density \(\rho(x, t)\) as a function of the phase, \(x\), and the time \(t\) is derived via a Chapman-Enskog expansion of a Boltzmann equation. This PDE can be solved in a straightforward way allowing us to determine all relevant quantities: total Work in Progress (WIP), the WIP distribution, the throughput and the throughput time. The relevant time scale for the PDE simulation is the time scale of the TPT evolution of the
whole factory. In contrast, for a discrete event simulation model, the relevant
time scale is that of processing by an individual machine, which is at least an
order of magnitude smaller than the TPT. In addition, in order to obtain a
meaningful statistical average, a large ensemble of items must be used. As a
result, the discrete event simulations are time consuming, and the simulation
effort scales with the number of steps in the factory and the number of items
produced. In contrast, the continuous model equation is deterministic and its
simulation is independent of the number of steps and the number of items
produced.

Comparison of a discrete event simulation (DES) model detailed below and
the explicitly derived advection-diffusion equation was performed in [2] and
the results were in reasonable agreement; in that case, the PDE model was
derived “from first principles”. However, the particular discrete event simula-
tions already constituted a significant approximation, as they were written for
particles in a continuum. In the general case of a large scale DES with many
items and many machines, global continuum evolution equations for the item
density cannot easily be analytically derived in a closed form. The question
then arises whether we can still observe the time evolution at the density
level, maintaining the scalability advantages of a density-type model. The re-
cently developed Equation-Free (EF) approach is aimed at precisely this type
of problem: we have a fine-scale (here DES) simulator, we suspect that we can
model its coarse-grained behavior at the level of a density evolution equation,
but do not have this equation explicitly available. The approach attempts to
solve the equation (integrate it, find its stationary solutions and their sta-
bility, etc.) by designing short bursts of appropriately initialized simulations
with the fine-scale (here DES) model. The quantities required for scientific
computation with the unavailable density-level equation are then estimated on demand from these short fine-scale runs [15,10,9]. In this paper we will use
a particular equation-free algorithm (coarse projective integration [4,5,14]) to
illustrate the application of the approach to the re-entrant line problem.

Other computational tasks like coarse-grained fixed point computation, sta-
bility, bifurcation analysis, control, and computation of self-similar solutions,
have been demonstrated in the literature for various types of “inner”, fine-
scale simulators [7,11,12,3,16]. A detailed discussion of the methods can be
found in [10,9]. A key step in EF computations is the so-called lifting: the
construction of fine-scale states consistent with given values of their coarse-

scale observables (the dependent variables in the unavailable coarse equation).
This is required in order to initiate new short bursts of fine-scale evolution.
This is not a deterministic step - many fine-scale configurations share the same
observables. However, the assumption that an evolution equation exists and
meaningfully closes at the level of these coarse observables suggests that the
details of various lifting realizations do not affect the long-time coarse-scale
evolution of the observables (see [7] for detailed discussions). In applying the
EF methods to re-entrant factory production, one may use the joint “phase and phase velocity” density \( f(x, v, t) \) as the observable of choice. However, the work in [2] suggests that at an appropriate (large influx) parameter limit, one can close an equation in terms of a simpler coarse-grained observable: just the phase density \( \rho(x, t) \). While for the former the lifting procedure would be straightforward (just generating two random variables according to their joint probability density function [16]), for the latter care must be taken in generating joint probability densities based on just the phase density \( \rho(x, t) \).

The paper is organized as follows. In Section 2 we briefly describe the phase model of a re-entrant factory discussed in [2]. Section 3 contains our main equation-free results: our lifting procedure, as well as coarse projective integration based on “finite-difference”-type observations of the phase density evolution. We conclude with a brief summary, discussion, and possible extensions of the approach.

2 A Phase Model for a Re-entrant Factory

2.1 The Discrete Model

The phase \( s \) of an item in a given factory of a supply chain is defined as the antiderivative of the phase velocity, \( 1/\tau(t) \), where \( \tau(t) \) may change with time, - in contrast to the constant \( \tau_n \) in Eqn. (1). With a deterministic TPT \( \tau(t) \) the improved model reads:

\[
\begin{align*}
(a) & \quad s = \phi(t), \\
(b) & \quad \frac{d\phi}{dt} = \frac{1}{\tau(t)}, \quad \phi(a_n) = 0, \quad t \geq a_n. 
\end{align*}
\]  

The exit time \( e_n \) is the time at which \( \phi(t) = 1 \). When \( \tau(t) \) is constant, model (2) reduces to model (1). When the TPT is a random variable sampled from a distribution \( T(r, t) \), a discrete form corresponding to (2) must be used (where \( \omega \) is an update frequency):

\[
\begin{align*}
\phi(t + \frac{1}{\omega}) &= \phi(t) + \frac{1}{\omega \tau(t)}, \\
\phi(a_n) &= 0, \quad t \geq a_n, \\
dP{\tau(t) \leq r} &= T(r, t)dr. 
\end{align*}
\]  

Qualitatively, the value of \( \omega \) is influenced by the number of items entering the factory within a characteristic TPT scale.

Given a prescribed TPT distribution function \( T(r, t) \) (which we assume can be expressed in terms of WIP as \( T(r, WIP) \)), an executable algorithm for evolving the phase \( s \) and the real-time TPT \( \tau(t) \) as well as the exit time \( e_n \) is also given in [2], namely,
\[(a) \quad \phi(t + \Delta t) = \phi(t) + \frac{\Delta t}{\tau(t)}, \]
\[
\tau(t + \Delta t) = \kappa(t)\eta(t) + (1 - \kappa(t))\tau(t), \quad t \geq a_n,
\]
\[(b) \quad \mathcal{P}\{\kappa(t) = 1\} = \omega\Delta t, \quad \mathcal{P}\{\kappa(t) = 0\} = 1 - \omega\Delta t,
\]
\[
dP\{\eta(t) \leq r\} = T(r, t)dr,
\]
\[
\phi(a_n) = 0, \quad dP\{\tau(a_n) \leq r\} = T(r, a_n)dr,
\]

where the update frequency \(\omega\) depends on the throughput time \(\tau(t)\) and the time \(t\) itself, i.e., \(\omega = \omega(\tau(t), t)\). The simulation time step \(\Delta t\) is chosen such that \(\Delta t < \frac{1}{\omega}\). For small enough \(\Delta t\) this algorithm can be used to numerically solve model (3).

In [2], \(\omega\) is chosen as
\[
\omega(r, t) = \frac{\lambda(t)}{rT_{-1}},
\]

where \(\lambda(t)\) is the influx of items and \(T_{-1} = \int T(r, t)dr\).

The procedure to compute the phase and throughput time (TPT) of an item in the factory is then as follows:

1. Set the item’s initial phase \(\phi\) to 0 as it enters the factory; increase the WIP \(W(t)\) by one and adjust the distribution of TPT, \(T(r, t)\) accordingly (remember that we have assumed that this distribution can be written as \(T(r, WIP)\)); the item’s initial TPT is then sampled from the udpated \(T(r, WIP)\);
2. Compute \(\omega(\tau(t), t)\) as above, and sample the parameter \(\kappa(t)\) according to its distribution (4)(b);
3. Compute the TPT \(\tau(t + \Delta t)\) and the phase \(\phi(t + \Delta t)\) according to (4)(a);
4. Update the distribution of TPT, \(T(r, t)\), according to current WIP at the time \(t + \Delta t\), and go back to Step (2); repeat this loop until \(\phi = 1\). The time when \(\phi = 1\) is the exit time of the item, \(e_n\).

2.2 Density Equations

Let the joint number density of phase and phase velocity, \(f(x, r, t)\), be defined as
\[
f(x, r, t) = \frac{\partial^2 F(\phi \leq x, \tau \leq r, t)}{\partial \phi \partial \tau},
\]
where \(F(\phi \leq x, \tau \leq r, t)\) is the number of items whose phase \(\phi \leq x\) and TPT \(\tau \leq r\). The PDE for \(f(x, r, t)\) is derived in [2]. It is given by
\[
\frac{\partial f}{\partial t} + \frac{1}{r} \frac{\partial f}{\partial x} = T(r, t) \int \omega(r', t)f(x, r', t)dr' - \omega(r, t)f(x, r, t),
\]
\[
x > 0, \quad t > 0
\]
\[ f(0, r, t) = r\lambda(t)T(r, t), \quad f(x, r, 0) = 0 \] (6)

weakly in \( x, r \) and \( t \), where \( \lambda(t) \) is the influx of items. The chosen form of \( \omega \) in (5) guarantees that the discrete model (1), when there are no temporal changes in influx and TPT, is a Monte Carlo scheme corresponding to a particular solution of (6), and that the instances of random number generation are roughly the same for models (1) and (4).

If we define the number density of phase, \( \rho(x, t) \), as
\[
\rho(x,t) = \int f(x,r,t) \, dr,
\]
then in the limit that \( \lambda_0 \sigma_0^2 T \gg 1 \) (where \( \lambda_0 \) and \( \sigma_0^2 T \) are characteristic scales of influx and standard deviation of TPT, respectively) one can write a closed equation for its evolution as follows:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x} = 0, \quad x, t > 0, \quad F(x, t) = C(t)\rho - D(t)\frac{\partial \rho}{\partial x},
\]
\[
C = \frac{1}{T_1} + \frac{T_{-1}}{\lambda} \frac{1}{T_1} \frac{\partial(T_2/T_1)}{\partial t}, \quad D = \frac{T_{-1}}{\lambda} \frac{T_2 - T_1^2}{T_1^3},
\]
\[
F(0, t) = \lambda(t), \quad \rho(x, 0) = 0
\] (7)

The moments of the probability distribution \( \mathcal{T}(r, t) \) appearing in this formula, \( T_i(t), i = -1, 1, 2 \), are defined by \( T_i(t) = \int r^i \mathcal{T}(r, t) \, dr, i = -1, 1, 2 \).

Based on the proof of Theorem T3 in [2], it can also be deduced that in this limit, the joint number density \( f(x, r, t) \) is controlled by (slaved to) just the number density of phase, \( \rho(x, t) \). Their relationship is given by

\[
f(x, r, t) = \frac{pr\mathcal{T}(r, t)}{T_1}
\]
(8)

Equation (8) gives a simple relationship between the joint number density and the number density of phase. It implies that the conditional number density \( f(r|x, t) (= f(x, r, t)/\rho(x, t)) \) is a constant \( r\mathcal{T}(r, t)/T_1 \) with respect to the phase coordinate \( x \).

3 **Equation-Free Analysis for the Two-scale System**

The Equation-Free approach consists of an ensemble of computational tools that can be used to study the coarse-grained, macroscopic behavior of systems based on their underlying fine-scale, microscopic simulators (see e.g. [4,7,11,11,3,16]). The basic element of these algorithms is the so-called *coarse time-stepper*, whose role is to connect observables across different scales. The coarse time-stepper consists essentially of three components: *lifting* and *micro-simulation* followed by *restriction*. The *lifting* is a procedure that generates micro-scale realizations of a system state consistent with given values of their
macro-scale observables; while restriction is the reverse: obtaining macro-scale observables from the fine-scale system state. Since a fine-scale system state normally possesses far more degrees of freedom than a few macro-scale observables, the lifting procedure is not a one-to-one mapping in general. Care needs to be taken when a lifting algorithm is implemented and tests are usually required to check if the macro-scale evolution is sensitive to the details of a particular lifting.

If we have reason to believe that useful macroscopic equations accurately close at the level of a few macroscopic observables (e.g. in terms of a few moments of microscopically evolving distributions, as is the case in hydrodynamic equation derivation from Boltzmann-level descriptions), equation-free schemes allow us to solve the coarse-grained equations without the explicit closures needed to write them in closed form. The idea is to use short bursts of fine-scale simulation to evaluate the right hand sides of the unavailable closed equations on demand. Traditional continuum numerical analysis is thus transformed into protocols for the design and processing of repeated short bursts of computational experiments with the fine-scale solver. The assumption that coarse-grained equations in principle exist and close in terms of a few coarse-grained observables implicitly suggests that the details of a lifting should be quickly forgotten; the role of the brief fine-scale simulation is to “implement” this loss of memory of initial features of the fine-scale state.

### 3.1 The Lifting Step in the Equation-Free Approach

We are interested in enabling short bursts of simulation with the discrete phase model (4) to numerically analyze the evolution of the number density $\rho(x,t)$ - a coarser-grained observable than the full $f(x,r,t)$. This should only be attempted at conditions when a deterministic equation closes with $\rho(x,t)$, i.e. when the evolution of the full $f(x,r,t)$ is controlled by $\rho(x,t)$. For such a closed equation to exist, it should be possible (possibly after a short initial transient) to express $f(x,r,t)$ in terms of $\rho(x,t)$. This section investigates the effect of the parameter $\lambda_0\sigma_0^T$ on the relationship between $\rho(x,t)$ and $f(x,r,t)$. This is both because $\lambda_0\sigma_0^T$ is the only dimensionless parameter that appears in the discrete model, and also because we are interested in cases of large item influx within the characteristic TPT scale, when $\lambda_0\sigma_0^T \gg 1$.

In what follows, the discrete model (4) is executed with constant influxes and uniform TPT distributions given in Table 1. For all cases, an ensemble of 7,000 realizations is used in order to obtain smooth evolution of the densities $f(x,r,t)$ and $\rho(x,t)$. At time $t=16sec$, the phase $\phi$ and TPT $\tau$ of items existing in the factory (i.e., $0 \leq \phi < 1$) are recorded and used to plot a surface for the conditional density $f(r|x,t)$ (Fig. 1). It is found that when $\lambda_0\sigma_0^T$ is sufficiently large (as in Case 3,5,6,8,9), $f(r|x,t)$ has become independent of the phase coordinate $x$. Actually, for our particular choice of $T(t,r)$ shown in cases...
Case Number | influx, $\lambda(t)$ | PDF of TPT, $T(r, t)$
---|---|---
1 | 0.5 | uniform in $[0.1,2]$ 
2 | 10 | uniform in $[0.1,2]$ 
3 | 20 | uniform in $[0.1,2]$ 
4 | 0.5 | uniform in $[0.1,4]$ 
5 | 10 | uniform in $[0.1,4]$ 
6 | 20 | uniform in $[0.1,4]$ 
7 | 0.5 | uniform in $[0.1,8]$ 
8 | 10 | uniform in $[0.1,8]$ 
9 | 20 | uniform in $[0.1,8]$

Table 1: Cases of influxes and uniform distributions of TPT both constant over the time domain.

3, 5, 6, 8, 9 of Table 1, we find that the conditional density $f(r|\rho, t)$ follows a linear distribution in $r$ passing through the origin (Fig. 2). Clearly, in order to lift effectively, one needs to know the support of the conditional distribution; for our choice of $T(r, t)$ the lower limit of its support in $r$ does not depend on WIP or time, and it makes sense to take this also as the lower limit of support of $f(r|\rho, t)$; we have observed that - to within acceptable error - the upper limit of support of $f(r|\rho, t)$ coincided in our simulations with the upper limit of support of $T(r, t)$. In this case, $f(r|\rho, t)$ can be approximated in the form of $r T(r, t)/C$ where $C$ is a constant independent of $r$. Obviously, $C$ equals $T_1(t)$ since $\int f(r|\rho, t) = 1$. Other choices of $\lambda(t)$ and $T(r, t)$ (e.g., a linearly increasing influx and a linear TPT distribution) give rise to more or less the same relationship between $\rho(\rho, t)$ and $f(x, r, t)$. The “constitutive equation” (8) and the parameter regime under which it is justified were thus found using only short simulations with the discrete fine-level model.

For a prescribed $T(r, t)$, and starting at a given $\rho(x, t)$, the lifting algorithm can now be formulated when $\lambda_0 \sigma^0_T \gg 1$ as follows:

1. Calculate the WIP($t$): $WIP = \int_0^1 \rho(x, t)dx$.
2. Calculate the probability density function (or normalized number density) $f_{\rho}(x, t)$ of the phase coordinate: $f_{\rho}(x, t) = \frac{\rho(x, t)}{WIP}$.
3. Since the number of items generated can only be an integer, we have to systematically generate an ensemble of integers whose mean value equals WIP. Let $a = \text{int}(WIP) + 1 - WIP$, where $\text{int}(WIP)$ is the maximum integer not greater than WIP. Select a random variable $p$ which is uniformly distributed in $[0, 1]$. If $p < a$, then the number of items is $\text{int}(WIP)$. Otherwise it is $\text{int}(WIP) + 1$.
4. Compute the cumulative distribution function (CDF) of the phase coordinate $F_X(x, t) = \int_0^x f_{\rho}(x, t)$ and its inverse $IF_X(F_X, t)$. 

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Fig. 1. Conditional distributions of TPT, $f(r|x,t=16s)$

Fig. 2. Conditional TPT distribution, $f(r|x,t=16s)$. Top: Case 5; Bottom: Case 9

(5) For each realization in the integer ensemble, generate the phase coordinates of items: $\phi_i = IF_X(\zeta_i, t), i = 1, 2, \cdots, n$, where $n$ is the integer number of items and $\zeta_i, i = 1, 2, \cdots, n$ are random numbers uniformly distributed in $[0, 1]$.

(6) Compute the CDF of the TPT $F_R(r,t) = \int_{-\infty}^{r} f(r|x,t)$ and its inverse $IF_R(F_R,t)$. The phase coordinate $x$ is suppressed in $F_R(r,t)$ since it is independent of $x$.

(7) For items in each realization, generate their respective TPT’s: $\tau_i = IF_R(\psi_i, t)$,
\( i = 1, 2, \ldots, n \), where \( \psi_i \) are also random numbers uniformly distributed in \([0, 1]\).

The following test was performed to validate the above procedure. A true trajectory of \( \rho(x, t) \) for Case 9 in Table 1 is computed using 5,000 ensemble realizations in the time domain directly by the discrete phase model (4). Then the trajectory is interrupted at \( t = 10 \); the number density \( \rho(x, t) \) at the interruption is retained, and then lifted as described above to start a new discrete evolution. We observed that the relative difference between the two trajectories (the normally continued one and the one starting from the lifting after the interruption) is quite small (within 2%) even just after the lifting. Another realization of the lifting may be taken which uses a different random number seed. The result shows no significant difference than the previous situation. It can therefore be concluded that the lifting algorithm is effective and particular liftings do not significantly affect subsequent restricted number densities.

### 3.2 Coarse Projective Integration of the Density-level System

Coarse Projective Integration (CPI) is a numerical technique developed in the EF framework to evolve in time the coarse-grained observables of a multiscale system, estimating their temporal derivatives using the underlying fine-scale simulator [4,5,14]. This technique is suitable for systems whose coarse- and fine-level temporal scales are well separated, i.e., the coarse-level observables are smooth over a temporal scale that is significantly larger than the fine-level evolution scales (the scales that it takes for higher-order system observables to become slaved to the slow, “master” ones). In the same spirit with adaptive time-step selection based on error control for standard deterministic integrators [13], real-time computational tests for projective step selection have to be taken to ensure error control in the CPI evolution of macroscopic observables.

We now implement CPI for our two-scale supply chain model. The prescribed influx and TPT distribution are shown in Fig. 3, respectively. The time step \( \Delta t_d \) in the discrete model is chosen as \( \delta t_d = 10^{-3} \text{sec} \). In the following, a finite-difference motivated representation is used to evolve coarse-grained observables, \( \rho(x, t) \), through CPI. If an explicit equation for \( \rho(x, t) \) was discretized in space with finite differences, we would evolve, in time, the values of the field at a number of points – and we would use differences between these values at each moment in time to approximate the right-hand side of the explicit equation. Here, we do not use the finite point representation to approximately evaluate the right-hand side of the equation – we use the representation instead to lift to an item distribution in the entire domain, and run the fine-scale simulator to estimate the left-hand side of the evolution of the representation. It is these values that integration codes need to solve the initial value problem, and the way they process the numbers does not depend on whether they
came from approximating the right-hand side of the equation, or from estimating its left-hand side. We projectively integrate the number density $\rho(x,t)$ at equally spaced phase points $x_j = j/M, j = 0, 1, \ldots, M$; here we choose $M = 8$. (We have also successfully performed coarse projective integration using as variables the projections of the solution on the first few of its empirical orthogonal global basis functions (POD modes) in a POD-assisted approach; these results are not shown because of space limitations). At the starting time $t_0$, the number density $\rho(x_j,t_0), j = 0, 1, \ldots, 8$ is lifted (according to the lifting algorithm in Section 3.1) to initiate the discrete phase model (4). This model is subsequently evolved for 20 discrete time steps. At steps $i, i = 12, 14, \ldots, 20$, phases of items in progress are restricted and binned to obtain number density histories $\rho(x,t_0 + i\delta t_d)$. These number density histories are used to approximate the temporal derivatives $\partial \rho(x,t)/\partial t$ at time $t_0 + 20\delta t_d$ via least-squares fitting. The number density after a coarse-level time interval $\Delta t_c$ can then be obtained based on a simple forward Euler explicit algorithm by projection:

$$\rho(x, t_0 + \Delta t_c) = \rho(x, t_0 + 20\delta t_d) + (\Delta t_c - 20\delta d) \frac{\partial \rho(x, t_0 + 20\delta t_d)}{\partial t}. $$

This explicit Euler scheme illustrates the simplest way to project the number density over a coarse time interval, here with computational savings of roughly $(1 - 20\delta_d/\Delta_c)$. More sophisticated projective algorithms, including projection templated on higher-order continuum integration schemes [14], taking advantage of multiple spectral gaps [6] and even coarse implicit schemes [7], can be utilized as well.

Figure 4 compares trajectories of WIP and outflux in true and CPI evolutions for some selections of the coarse time step $\Delta t_c$. All computations involve an ensemble of 5,000 realizations. It is found that the CPI trajectories of WIP almost coincide as $\Delta t_c = 0.2s$ and $\Delta t_c = 0.3s$, which suggests that the result at $\Delta t_c = 0.2s$ is correct. Comparison also displays that WIP and outflux at $\Delta t_c = 0.2s$ indeed match their true trajectories. Number densities of the true
Fig. 4. True trajectories and some CPI evolutions of WIP and outflux. Left: WIP, clusters of points represent WIP’s at time steps $i \Delta t_d, i = 0, 1, \cdots, 10$ immediately following the lifting; Right: outflux.

Fig. 5. Number densities of the true evolution and CPI evolution with $\Delta t_c = 0.2s$. Left: true evolution; Right: CPI result

evolution and the CPI result with $\Delta t_c = 0.2s$ are also shown in Fig. 5. By using coarse projective integration, the computational load is substantially reduced by 78.3%, taking into account the trial efforts in selecting an appropriate coarse timestep size for projection.

4 Conclusions

We demonstrated certain features of Equation-Free coarse-grained computation for a re-entrant supply-chain model. In cases where an explicit equation for the processed item number density cannot be easily derived, simulations with the fine-scale model can be used to determine the conditions under which such an equation can, in principle, exist. Once these conditions are established, equation-free methods such as coarse projective integration can be used to evolve the coarse-level density through short bursts of appropriately initialized runs with the fine-scale simulator. The lifting process essential to this
EF approach was investigated and a particular governing form establishing a relation between the joint number density $f(x,r,t)$ and the number density $\rho(x,t)$ was found; the form is of course only valid for this model.

EF algorithms based on matrix-free iterative linear algebra (like Newton-Krylov GMRES, [8]) can also be used to effectively implement contraction mappings to find stationary item number densities, if the influx to a factory reaches a stationary state at long times. Additional tasks like continuation, stability and parametric sensitivity analysis, and even control and optimization computations can in principle be implemented in an equation-free framework.

Our illustrative example involved a Monte-Carlo type solution of a Boltzmann equation, which, at a certain limit, approaches a discrete event simulator. In this case, both the Boltzmann equation and the reduced equation for the item number density at the appropriate limit were analytically available; this allowed us to validate our equation-free computations. The real challenge lies in wrapping equation-free algorithms around true discrete event simulations for realistic processing factory configurations. While the derivation of continuum-level equations may be extremely difficult or practically infeasible in such cases, our computer-assisted approach remains essentially the same, independent of the details in the underlying fine-scale simulator. We believe that this approach holds promise for facilitating the extraction of system-level information from complex discrete event simulators.

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