Weak measurement with orthogonal pre-selection and post-selection

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Weak measurement is a novel quantum measurement scheme, which is usually characterized by the weak value formalism. To guarantee the validity of the weak value formalism, the fidelity between the pre-selection and the post-selection should not be too small generally. In this work, we study the weak measurement on a qubit system with exactly or asymptotically orthogonal pre- and post-selections. We shall establish a general rigorous framework for the weak measurement beyond the weak value formalism, and obtain the average output of a weak measurement when the pre- and post-selections are exactly orthogonal. We shall also study the asymptotic behavior of a weak measurement in the limiting process that the pre- and post-selections tend to be orthogonal.

I. INTRODUCTION

In the conventional quantum theory, a quantum measurement consists of a set of probabilistic orthogonal projections onto the eigenstates (or eigenspaces if any degeneracy) of an observable [1]. The output of a quantum measurement ranges between the minimal eigenvalue and the maximal eigenvalue of the observable. Such an ideal quantum measurement can be realized if the spread of the wave function of the probe is sufficiently sharp.

In 1988, Aharonov, Albert, and Vaidman (AAV) proposed a novel quantum measurement scheme called weak measurement [2]. A weak measurement involves three stages generally: first, the system to be measured and a measuring probe are prepared in initial states; then the system and the probe are coupled by such a weak interaction that the state of the system remains almost undisturbed after the interaction; lastly, the system is observed, and if it is found to be in a specific final state \(|\psi_f\rangle\), the pointer shift of the probe is recorded, otherwise, discarded. Usually, the initial state of the system in the first stage is called pre-selection, and the specific choice of the final state of the system in the last stage is called post-selection. At the beginning, the new concept of weak measurement aroused some controversy, but soon its physical significance was clarified by [3].

In contrast to the conventional orthogonal projective measurement, the spread of the initial wave function of the probe is usually chosen to be quite wide in a weak measurement in order that its output can be far beyond the range of the eigenvalues of the observable on the system. The magic large pointer shift of the probe in a weak measurement is attributed to the interference in the superposition of several slightly different probe states after the post-selection on the system.

Since the birth of weak measurement, a lot of research has been devoted to this interesting field, including weak measurement with arbitrary probe [4], weak measurement involving the contribution of probe dynamics [2], weak measurement with entangled probes [6], weak measurement with a qubit probe [7], weak measurement with a probe in a mixed state [8], geometric phase in weak measurement [9,10], continuous quantum measurement of coherent oscillations between two quantum states of an individual two-state system [11], and so on. Moreover, weak measurement has been found universal to implement any general quantum measurement [12].

In recent years, weak measurement has been realized in experiment [13]. Besides, weak measurement has also been used to experimentally examine quantum paradoxes [14] and for experimental feedback control of quantum systems in the presence of noise [15]. And more experiment protocols using weak measurements have been proposed [16–21].

An important physical quantity in weak measurement is the weak value introduced in [2], and it is defined as

\[ A_w = \frac{\langle \psi_f | A | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle}, \]  

(1)

where \(A\) is an observable acting on the system, and \(|\psi_i\rangle, |\psi_f\rangle\) are the pre-selection and post-selection of the system state respectively. Roughly speaking, weak value characterizes the pointer shift of the probe after a weak measurement. Weak value has also received intensive study [22,27].

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It can be seen that when the fidelity between the pre- and post-selections $\langle \psi_f | \psi_i \rangle$ is small in (1), the weak value $A_w$ can be very large, indicating a large position shift of the probe. This important characteristic of weak measurement is found useful to amplify and measure tiny physical quantities which are difficult to detect by conventional techniques in experiments. For example, weak measurement has been used to observe the spin Hall effect of light [28], optical beam deflection [29, 30] and optical frequency change [31], etc.

However, a necessary condition for the validity of the weak value formalism is that the fidelity between the pre- and post-selections should not be too small [2, 3]. Then, some interesting questions arise naturally: what is the result of a weak measurement when the fidelity between the pre- and post-selections is very small, or even zero? And is it possible that a weak measurement has larger, or even infinitely large amplification effect in such cases?

In the literature, it has been noticed that the original AAV’s weak value formalism breaks down in a weak measurement with nearly orthogonal pre- and post-selections. Specific cases such as a Sagnac interferometer with continuous phase amplification [32] and the original Stern-Gerlach setup [33] were studied in such situations. In [34], some general results of weak measurement have been derived, and orthogonal weak value was introduced as well. In [35], the maximum position shift of a Gaussian pointer is obtained for a qubit system and the signal-to-noise ratio is studied in detail.

The purpose of this article is to provide a more rigorous treatment of the weak measurement for a qubit system interacting with a continuous probe, when the pre- and post-selections are exactly or asymptotically orthogonal. We shall derive rigorous general results without any approximations, which can be readily applied to cases with specified pre- and post-selections, and we shall also derive the asymptotic behavior of the output for a weak measurement with pre- and post-selections approaching orthogonality in various cases.

The structure of this paper is as follows. We shall first review the weak value formalism in Sec. II, then establish a general framework for weak measurement which is suitable for studying a weak measurement with variable fidelity between the pre- and post-selections in Sec. III. In that section, we shall also prove that the amplification effect of a weak measurement cannot be infinitely large, and obtain the analytical average output of a weak measurement when the pre- and post-selections are exactly orthogonal. In Sec. IV, we shall discuss the case with asymptotically orthogonal pre- and post-selections in detail.

\section{Preliminary}

In a weak measurement, a typical Hamiltonian in the interaction picture is

$$H_{int} = g(t)\hat{A} \otimes \hat{p}, \quad g(t) = g\delta(t - t_0),$$

where $g$ is a small coupling constant and $\hat{p}$ is the momentum operator on the probe conjugate to the position operator $\hat{q}$. The time factor $\delta(t - t_0)$ means that the weak interaction happens only for a very short instant.

Suppose the initial states of the system and the probe are $|\psi_i \rangle$ and $|\phi \rangle$ respectively. When the weak interaction is finished, the total state of the system and the probe evolves to

$$e^{-ig\hat{A}\otimes\hat{p}}|\psi_i \rangle \otimes |\phi \rangle.$$  \hfill (3)

When the post-selection on the system is $|\psi_f \rangle$, the state of the probe collapses to

$$\frac{\langle \psi_f | e^{-ig\hat{A}\otimes\hat{p}} |\psi_i \rangle \otimes |\phi \rangle}{\langle \psi_f | e^{-ig\hat{A}\otimes\hat{p}} |\psi_i \rangle}.$$  \hfill (4)

When $g$ is sufficiently small, the position shift of the probe is roughly proportional to the weak value $A_w$ defined in (1). It should be noted that in [2] the Hamiltonian was $g(t)\hat{A} \otimes \hat{q}$ and the weak value actually characterized the momentum shift of the probe.

The weak value $A_w$ can often be complex, and it has a beautiful physical interpretation given by [24]: the real part and imaginary part of the weak value are responsible for the position shift and momentum shift of the probe state induced by the weak measurement respectively. Specifically, it was shown in [24] that

$$\langle \hat{q} \rangle_f = \langle \hat{q} \rangle_i + g \text{Re} A_w + g \text{Im} A_w \frac{d}{dt} \text{Var}_\hat{q},$$

$$\langle \hat{p} \rangle_f = \langle \hat{p} \rangle_i + 2g \text{Im} A_w \text{Var}_\hat{p},$$

where $\text{Var}_\hat{q}, \text{Var}_\hat{p}$ are the square deviations of position and momentum of the initial probe state $|\phi \rangle$. 

From Eq. (3), it can be seen that when the fidelity between the pre- and post-selections is small, the position shift and momentum shift of the probe state can be very large. And this phenomenon has been used to magnify and observe small physical quantities in experiments, as referred to in the last section.

However, the weak value formalism could not be applied to the situation \( \langle \psi_f | \psi_i \rangle \to 0 \) because \( A_w \to \infty \) and thus \( \langle \hat{q} \rangle_f, \langle \hat{p} \rangle_f \to \infty \) in this situation. Such a divergence is obviously non-physical, and it results from the derivation of the weak value, in which only the terms of \( g \) up to the first order were considered in the expansion of the Hamiltonian \( A \) \( \hat{p} \). That approximation would be no longer valid if \( \langle \psi_f | \psi_i \rangle \to 0 \), therefore, in order to study the average output from the probe in a weak measurement when the pre- and post-selections are exactly or nearly orthogonal, such an approximation should be avoided.

In the following sections, we shall establish a new rigorous framework for a general weak measurement on a qubit system with a continuous probe beyond the weak value formalism, and derive the results of the weak measurement when \( \langle \psi_f | \psi_i \rangle = 0 \) or \( \langle \psi_f | \psi_i \rangle \to 0 \).

III. WEAK MEASUREMENT WITH EXACTLY ORTHOGONAL PRE- AND POST-SELECTIONS

A. General result

In this section, we shall first derive a general exact formula for the average pointer reading of the probe after the post-selection, then obtain an analytical result for the situation that the pre- and post-selections are exactly orthogonal. In particular, the case that the observable on the probe is the position operator \( \hat{q} \) or the momentum operator \( \hat{p} \) will be studied respectively in detail. The dimension of the system will be assumed to be two throughout this paper.

Suppose the initial state of the probe is \( |\psi_i\rangle \), and the post-selection and pre-selection satisfy

\[
|\psi_f\rangle = \alpha |\psi_i\rangle + \sqrt{1 - \alpha^2} |\psi_f^\perp\rangle, \quad \alpha \geq 0.
\]

Then, the fidelity between the post-selection and pre-selection is

\[
\langle |\psi_f| |\psi_i\rangle = \alpha.
\]

The total state of the system and the probe after the unitary evolution is

\[
e^{-ig\hat{A}\otimes\hat{p}}|\psi_i\rangle \otimes |\phi\rangle,
\]

so the probe state after the post-selection is

\[
\frac{\langle \psi_f | e^{-ig\hat{A}\otimes\hat{p}} |\psi_i\rangle \otimes |\phi\rangle}{\langle \psi_f | e^{-ig\hat{A}\otimes\hat{p}} |\psi_i\rangle \otimes |\phi\rangle}.
\]

Suppose the observable on the probe to be observed after the post-selection is \( \hat{M} \), then the average reading from the probe is

\[
\langle \hat{M} \rangle = \frac{\langle \psi_f | \otimes \langle \phi | e^{ig\hat{A}\otimes\hat{p}} |\psi_f\rangle M(|\psi_f| e^{-ig\hat{A}\otimes\hat{p}} |\psi_i\rangle \otimes |\phi\rangle)\langle \psi_f | e^{-ig\hat{A}\otimes\hat{p}} |\psi_i\rangle \otimes |\phi\rangle}{\langle \psi_f | e^{-ig\hat{A}\otimes\hat{p}} |\psi_f\rangle \otimes |\phi\rangle}.
\]
As the phases of $|a_1\rangle$ and $|a_2\rangle$ can be arbitrary, we can assume that $\langle a_1 | ψ_i \rangle \geq 0$, $\langle a_2 | ψ_i \rangle \geq 0$, and

$$\langle a_1 | ψ_i \rangle = x, \quad \langle a_2 | ψ_i \rangle = \sqrt{1 - x^2}, \quad x \geq 0.$$  \hspace{1cm} (14)

Note that

$$\langle ψ_i | ψ_i^{\dagger} \rangle = \langle ψ_i | (|a_1\rangle + |a_2\rangle) | ψ_i^{\dagger} \rangle = x\sqrt{1 - x^2}(e^{iθ} + e^{iθ'}) = 0,$$  \hspace{1cm} (15)

so

$$θ - θ' = π.$$  \hspace{1cm} (16)

Plugging Eq. (14) into (13), we have

$$\langle ψ_1 | e^{-iθ}\hat{A}_{\psi} | ψ_i \rangle = α(x^2e^{-iga_1\hat{p}} + (1 - x^2)e^{-iga_2\hat{p}}) + \sqrt{1 - α^2}x\sqrt{1 - x^2}e^{-iθ}(e^{-iga_1\hat{p}} - e^{-iga_2\hat{p}}).$$  \hspace{1cm} (17)

Now, we define some notations for convenience. Let

$$W_{ij} = \langle φ | e^{iga_i\hat{p}} \hat{M} e^{-iga_j\hat{p}} | φ \rangle = \int \phi^* (q - ga_i) \hat{M} ϕ(q - ga_j) dq,$$

$$Y_{ij} = \langle φ | e^{iφ(a_i - a_j)} | φ \rangle = \int \phi^* (q - ga_i) | φ(q - ga_j) dq.$$  \hspace{1cm} (18)

Then, by plugging Eq. (17) into (19) with the notations (18), we can get the expectation value of the observable $\hat{M}$ on the probe after post-selection. To simplify the calculation, let us define

$$β = \frac{α}{\sqrt{1 - α^2}} \quad y = \frac{x}{\sqrt{1 - x^2}}$$  \hspace{1cm} (19)

then it can be worked out that

$$\langle \hat{M} \rangle = \frac{2β^2 y^2(W_{11} + W_{22} + 2y^2 ReW_{12}) + y^2(W_{11} + W_{22} - 2ReW_{12})}{β^2(y^4 + 1 + 2y^2 ReY_{12}) + 2y^2(1 - ReY_{12}) + 2β^2(y^2 - 1)(1 - ReY_{12}) \cos θ - (y^2 + 1) ImY_{12} \sin θ}.$$  \hspace{1cm} (20)

Eq. (20) is a general exact formula for the expectation value of an observable $\hat{M}$ on the probe after a weak measurement, and it is the starting point of our further study on the asymptotic property of $\langle \hat{M} \rangle$ when $\langle ψ_1 | ψ_i \rangle \to 0$. In particular, when $|\langle ψ_1 | ψ_1 \rangle| = 0$, i.e. $β = 0$, we can immediately obtain

$$\langle \hat{M} \rangle = \frac{W_{11} + W_{22} - 2ReW_{12}}{2(1 - ReY_{12})}.$$  \hspace{1cm} (21)

from Eq. (20).

From (13), $ReY_{12} < 1$, so $\langle \hat{M} \rangle$ can never be infinity when $|\langle ψ_1 | ψ_1 \rangle| = 0$, in sharp contrast to (11) and (10) which indicate that $\langle \hat{M} \rangle$ diverges if $|\langle ψ_1 | ψ_1 \rangle| = 0$.

And it can also be proved that the amplification effect of a weak measurement cannot be infinitely large when $|\langle ψ_1 | ψ_i \rangle| ≠ 0$. From Eq. (20), it can be seen that $\langle \hat{M} \rangle$ can be infinity only when the denominator is zero. Since the denominator of (20) is a quadratic polynomial of $β$, the necessary condition that the denominator can be zero is that its discriminant is non-negative. However, by some calculation, it turns out that

$$Δ = 4y[(y - 1)(1 - ReY_{12}) \cos θ - (y + 1)ImY_{12} \sin θ]^2 - 8y(1 - ReY_{12})(y^2 + 1 + 2yReY_{12})$$

$$≤ 4y[(y - 1)^2(1 - ReY_{12})^2 + (y + 1)^2(ImY_{12})^2] - 8y(1 - ReY_{12})(y^2 + 1 + 2yReY_{12})$$

$$= -4y(y + 1)^2(1 - |Y_{12}|^2) < 0,$$

so the denominator of (20) cannot be zero and $\langle \hat{M} \rangle$ can never be infinity then.

An interesting thing to note in (21) is that $\langle \hat{M} \rangle$ only depends on the initial state of the probe $|φ\rangle$ and the coupling constant $g$ when the pre- and post-selections are orthogonal.
B. Examples

In this subsection, we give some examples to illustrate the results obtained in the last subsection. We shall calculate the expectation value of the position and momentum of the probe after post-selection respectively, in particular when the pre- and post-selections are orthogonal.

1. \(\hat{M} = \hat{q}\)

If \(\hat{M} = \hat{q}\), it can be easily verified that

\[
\begin{align*}
W_{11} &= \langle \hat{q} \rangle_i + g a_1, \\
W_{22} &= \langle \hat{q} \rangle_i + g a_2, \\
W_{12} &= \int \phi^*(q - ga_1)q\phi(q - ga_2) dq,
\end{align*}
\]

where \(\langle \hat{q} \rangle_i = \int \phi^*(q)q\phi(q) dq\) is the expectation value of the probe position before the weak interaction.

Plugging the above equations into (20), we can get the mean position shift of the probe after a weak measurement. Particularly, when \(|\langle \psi_f | \psi_i \rangle| = 0\), i.e. \(\alpha = 0\),

\[
\langle \hat{q} \rangle = \frac{\langle \hat{q} \rangle_i + g(a_1 + a_2)/2 - \text{Re}W_{12}}{1 - \text{Re}W_{12}}.
\]

Provided that \(q\phi^{(n)}(q) \to 0 (n = 0, 1, 2 \cdots)\) as \(q \to \pm \infty\) (this can be satisfied by most wave functions like the Gaussian-type wave functions, but not by some wave functions which oscillate extremely quickly as \(q \to \pm \infty\), it can be worked out that

\[
\begin{align*}
\text{Re}Y_{12} &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} g^{2n}(a_1 - a_2)^2 \int |\phi^{(n)}(q)|^2 dq, \\
\text{Re}W_{12} &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} g^{2n}(a_1 - a_2)^2 \int q|\phi^{(n)}(q)|^2 dq + \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} g^{2n+1}(a_1 + a_2)(a_1 - a_2)^2 \int |\phi^{(n)}(q)|^2 dq.
\end{align*}
\]

The details of calculation are presented in the appendix.

Plugging Eq. (25) into (24), we get

\[
\langle \hat{q} \rangle = \frac{1}{2} g(a_1 + a_2) + \frac{\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n + 2)!} g^{2n+1}(a_1 - a_2)^2 \int q|\phi^{(n+1)}(q)|^2 dq}{\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n + 2)!} g^{2n}(a_1 - a_2)^2 \int |\phi^{(n+1)}(q)|^2 dq}.
\]

A straightforward conclusion from Eq. (26) is that if the initial wave function \(\phi(q)\) of the probe is symmetric or anti-symmetric,

\[
\langle \hat{q} \rangle = \frac{1}{2} g(a_1 + a_2)
\]

when \(|\langle \psi_f | \psi_i \rangle| = 0\). This has been verified by the case of \(\phi(q)\) being a Gaussian state [35].

In addition, when \(g\) is small, only low orders of \(g\) need to be remained in Eq. (26), which can simplify the calculation of \(\langle \hat{q} \rangle\).

2. \(\hat{M} = \hat{p}\)

If \(\hat{M} = \hat{p}\), then

\[
\begin{align*}
W_{11} &= W_{22} = \langle \hat{p} \rangle_i = 2 \int \text{Re}\phi(q)\text{Im}\hat{\phi}(q) dq; \\
W_{12} &= \int \phi^*(q - ga_1)\hat{p}\phi(q - ga_2) dq = -i \int \phi^*(q - ga_1)\hat{\phi}(q - ga_2) dq.
\end{align*}
\]
By plugging (28) into (20), one can get the mean momentum shift of the probe after a weak measurement. When the pre- and post-selections are orthogonal, i.e. \( \alpha = 0 \),

\[
\langle \hat{p} \rangle = \frac{\langle \hat{p} \rangle_1 - \text{Re}W_{12}}{1 - \text{Re}Y_{12}}.
\]

(29)

Provided that \( q\phi^{(n)}(q) \to 0 \) (\( n = 0, 1, 2 \cdots \)) as \( q \to \pm \infty \), detailed calculation shows

\[
\text{Re}W_{12} = 2 \sum_{n=0}^{+\infty} \frac{(-1)^n q^{2n}}{(2n)!} (a_1 - a_2)^{2n} \int (\text{Re}\phi(q))^{(n)} (\text{Im}\phi(q))^{(n+1)} dq.
\]

(30)

The details of calculation are given in the appendix.

Therefore, when \( |\langle \psi_f | \psi_i \rangle| = 0 \),

\[
\langle \hat{p} \rangle = \frac{\sum_{n=0}^{+\infty} (-1)^n g^{2n} (a_1 - a_2)^{2n} \int (\text{Re}\phi(q))^{(n)} (\text{Im}\phi(q))^{(n+2)} dq}{\sum_{n=0}^{+\infty} (-1)^n g^{2n} (a_1 - a_2)^{2n} \int |\phi^{(n+1)}(q)|^2 dq}.
\]

(31)

If the initial wave function \( \phi(q) \) of the probe is symmetric or anti-symmetric, it can be inferred from Eq. (31) that

\[
\langle \hat{p} \rangle = 0
\]

(32)

when \( |\langle \psi_f | \psi_i \rangle| = 0 \).

**IV. WEAK MEASUREMENT WITH ASYMPTOTICALLY ORTHOGONAL PRE- AND POST-SELECTIONS**

In the last section, we gave a general rigorous framework for the weak measurement on a qubit system with a continuous probe, and applied it to the case of exactly orthogonal pre- and post-selections. In this section, we study the weak measurement with asymptotically orthogonal pre- and post-selections.

It is known that, when the pre- and post-selections are not extremely orthogonal, the average output of a weak measurement \( \langle \hat{M} \rangle \) can be characterized by the weak value \( \langle \psi_f | \psi_i \rangle \) and it is roughly proportional to the reciprocal of the fidelity between the pre- and post-selections. However, when the pre- and post-selections tend to be orthogonal, the weak value would be no longer valid because it may diverge. In the last section, it was shown that when the pre- and post-selections are exactly orthogonal, the average output of a weak measurement \( \langle \hat{M} \rangle \) is still finite. That gives a hint that when the pre- and post-selections tend to be orthogonal, \( \langle \hat{M} \rangle \) does not vary as the reciprocal of the fidelity between them actually. So, it is interesting to study how the weak measurement behaves in the limiting process that the pre- and post-selections tend to be orthogonal, and this is what we shall focus on in this section.

Let \( |\psi_i\rangle \) and \( |\psi_f\rangle \) denote the pre- and post-selection of the system as before. Note that in a limiting process \( \langle \psi_f | \psi_i \rangle \to 0 \), the fidelity between the pre-selection and one of the eigenstates of the observable \( \hat{A} \) on the system, either \( x_1 \) or \( x_2 \), can also vary. If \( x_1, x_2 \to 0 \) or \( 1 \), then \( \langle \psi_f | \psi_i \rangle \) dominates the asymptotic behavior of \( \langle \hat{M} \rangle \). However, if \( x \to 0 \) or \( 1 \) as \( \langle \psi_f | \psi_i \rangle \to 0 \), i.e. the pre-selection tends towards one of the eigenstates of the observable \( \hat{A} \), the problem becomes more complex, because \( x_1 \) or \( x_2 \) may give considerable contribution to \( \langle \hat{M} \rangle \) and there will be competition between \( \langle \psi_f | \psi_i \rangle \) and \( x_1 \) or \( x_2 \). So, whether \( x_1, x_2 \to 0 \) or \( 1 \) needs be taking into account in studying the asymptotic behavior of \( \langle \hat{M} \rangle \).

We first consider the relatively simpler case that \( |\psi_i \rangle \to |a_1 \rangle \) or \( |a_2 \rangle \), i.e. \( x_1, x_2 \to 0 \) or \( 1 \) as \( \langle \psi_f | \psi_i \rangle \to 0 \). In this case, there is no competition between \( \langle \psi_f | \psi_i \rangle \) and \( x_1 \) or \( x_2 \), and only \( \langle \psi_f | \psi_i \rangle \) dominates \( \langle \hat{M} \rangle \) when \( \langle \psi_f | \psi_i \rangle \to 0 \). Thus, the quadratic and higher order terms of \( \beta \) can be omitted in (20), and (20) can be simplified to

\[
\langle \hat{M} \rangle \approx \frac{y(W_{11} + W_{22} - 2\text{Re}W_{12})/2 + \beta y^2 ((W_{11} - \text{Re}W_{12}) \cos \theta - \text{Im}W_{12} \sin \theta) + (\text{Re}W_{12} - W_{22}) \cos \theta - \text{Im}W_{12} \sin \theta]}{y(1 - \text{Re}Y_{12}) + \beta ((y^2 - 1)(1 - \text{Re}Y_{12}) \cos \theta - (y^2 + 1) \text{Im}Y_{12} \sin \theta)}.
\]

(33)
Considering
\[ \frac{a_1 z + b_1}{a_2 z + b_2} \approx \frac{b_1}{b_2} + \left( \frac{a_1}{b_2} - \frac{a_2 b_1}{b_2^2} \right) z, \quad z \to 0, \]  
we have
\[ \langle \hat{M} \rangle \approx \frac{W_{11} + W_{22} - 2\text{Re}W_{12}}{2(1 - \text{Re}Y_{12})} + \frac{y^2((W_{11} - \text{Re}W_{12}) \cos \theta - \text{Im}W_{12} \sin \theta) + (\text{Re}W_{12} - W_{22}) \cos \theta - \text{Im}W_{12} \sin \theta}{y(1 - \text{Re}Y_{12})} \]
\[ - \frac{(y^2 - 1)(1 - \text{Re}Y_{12}) \cos \theta - (y^2 + 1)\text{Im}W_{12} \sin \theta)(W_{11} + W_{22} - 2\text{Re}W_{12})}{2y(1 - \text{Re}Y_{12})^2} |\alpha\rangle. \]  

Eq. (35) shows that the average output of a weak measurement, $\langle \hat{M} \rangle$, goes linearly as the fidelity $\alpha$ between the pre- and post-selections, when $\alpha \to 0$ and the pre-selection $|\psi_i\rangle \to |a_1\rangle$ or $|a_2\rangle$.

Now, let us consider the more complex case: if $|\psi_i\rangle \to |a_1\rangle$ or $|a_2\rangle$ very fast when $\langle \psi_f | \psi_i \rangle \to 0$, then the contribution of $x$ must be considered. In this case, how fast $|\psi_i\rangle \to |a_1\rangle$ or $|a_2\rangle$ is in comparison with $\langle \psi_f | \psi_i \rangle \to 0$, plays a critical role in the asymptotic behavior of $\langle \hat{M} \rangle$, and dominates the competition between $\langle \psi_f | \psi_i \rangle$ and $x_1$ or $x_2$.

Suppose $|\psi_i\rangle \to |a_2\rangle$, i.e. $y \to 0$. To characterize the speed of $|\psi_i\rangle \to |a_2\rangle$, we assume that
\[ \beta^s \propto y, \quad s > 0, \]  
as $\alpha \to 0$. The exponent $s$ in $\beta^s$ characterizes the relative speed that the pre-selection $|\psi_i\rangle$ tends towards the eigenstate $|a_2\rangle$ of $\hat{A}$ compared with $\langle \psi_f | \psi_i \rangle \to 0$, and determines how fierce the competition between $\langle \psi_f | \psi_i \rangle$ and $x$ is. It will be shown explicitly below how $s$ affects the asymptotic behavior of $\langle \hat{M} \rangle$.

Now, by plugging Eq. (36) into (21), one can get
\[ \langle \hat{M} \rangle = \frac{\beta^2(\beta^{4s}W_{11} + W_{22} + 2\beta^{2s}\text{Re}W_{12}) + \beta^{2s}(W_{11} + W_{22} - 2\text{Re}W_{12})}{\beta^2(\beta^{4s} + 1 + 2\beta^{2s}\text{Re}Y_{12}) + 2\beta^{2s}(1 - \text{Re}Y_{12})} \]
\[ + 2\beta^{s+1}[\beta^{2s}((1 - \text{Re}Y_{12}) \cos \theta - \text{Im}Y_{12} \sin \theta) + ((\text{Re}Y_{12} - 1) \cos \theta - \text{Im}Y_{12} \sin \theta)] \]
Six different powers of $y$ occur in the above equation:
\[ \beta^{2+4s}, \beta^2, \beta^{2+2s}, \beta^{2s}, \beta^{1+3s}, \beta^{1+s}. \]  

When $\langle \psi_f | \psi_i \rangle \to 0$, the two terms with lowest orders of $\beta$ dominate $\langle \hat{M} \rangle$. Since the orders of $\beta$ depend on $s$, the asymptotic behavior of $\langle \hat{M} \rangle$ also depends on $s$, thus in the study of the asymptotic behavior of $\langle \hat{M} \rangle$ below, the range of $s$ will be taken into account.

1. $s < 1$. For this range of $s$, $\beta^{2s}$ and $\beta^{1+s}$ are the lowest order terms, so
\[ \langle \hat{M} \rangle \approx \frac{(W_{11} + W_{22} - 2\text{Re}W_{12})/2 + \beta^{1-s}((\text{Re}W_{12} - W_{22}) \cos \theta - \text{Im}W_{12} \sin \theta)}{1 - \text{Re}Y_{12} + \beta^{1-s}((\text{Re}Y_{12} - 1) \cos \theta - \text{Im}Y_{12} \sin \theta)} \approx \frac{W_{11} + W_{22} - 2\text{Re}W_{12} + \beta^{1-s}((\text{Re}W_{12} - W_{22}) \cos \theta - \text{Im}W_{12} \sin \theta)}{2(1 - \text{Re}Y_{12})} \]
\[ - \frac{((\text{Re}Y_{12} - 1) \cos \theta - \text{Im}Y_{12} \sin \theta)(W_{11} + W_{22} - 2\text{Re}W_{12})}{2(1 - \text{Re}Y_{12})^2}, \]  
and the speed that $\langle \hat{M} \rangle$ varies is
\[ \frac{d\langle \hat{M} \rangle}{d\beta} \propto \beta^{-s}. \]  

Eq. (39) shows that the limit of $\langle \hat{M} \rangle$ is (21), and this means that in the range $0 < s < 1$, $y$ only affects the speed that $\langle \hat{M} \rangle$ converges to (21) but cannot change the limit of $\langle \hat{M} \rangle$ as $\alpha \to 0$, and the fidelity between the pre-selection and the post-selection $\langle \psi_f | \psi_i \rangle$ is still dominant in its competition with $x$.

2. $s = 1$. In this case, $\beta^{2s}$, $\beta^{1+s}$ and $\beta^2$ have the same order, so
\[ \langle \hat{M} \rangle \approx \frac{W_{11}/2 + W_{22} - \text{Re}W_{12} + ((\text{Re}W_{12} - W_{22}) \cos \theta - \text{Im}W_{12} \sin \theta)}{3/2 - \text{Re}Y_{12} + ((\text{Re}Y_{12} - 1) \cos \theta - \text{Im}Y_{12} \sin \theta),} \]
and

\[ \frac{d\langle \hat{M} \rangle}{d\beta} \approx 0. \]  

(42)

So, \( \langle \psi_f | \psi_i \rangle \) and \( x \) contribute almost equally to \( \langle \hat{M} \rangle \) as \( \alpha \to 0 \) when \( s = 1 \). Intuitively, in this case the contribution from \( x \) “stops” \( \langle \hat{M} \rangle \) running to its limit \( (21) \) as \( \alpha \to 0 \) and keeps it almost stationary.

3. \( s > 1 \). In this case, \( \beta^{1+s} \) and \( \beta^2 \) are the lowest order terms, so

\[
\langle \hat{M} \rangle \approx \frac{W_{22} + 2\beta^{s} - 1 ((\text{Re}W_{12} - W_{22}) \cos \theta - \text{Im}W_{12} \sin \theta)}{1 + 2\beta^{s} - 1 ((\text{Re}Y_{12} - 1) \cos \theta - \text{Im}Y_{12} \sin \theta)} \\
\approx W_{22} + 2\beta^{s} - 1 ((\text{Re}W_{12} - W_{22}) \cos \theta - \text{Im}W_{12} \sin \theta) - W_{22}((\text{Re}Y_{12} - 1) \cos \theta - \text{Im}Y_{12} \sin \theta)),
\]

and

\[ \frac{d\langle \hat{M} \rangle}{d\beta} \propto \beta^{s-2}. \]  

(43)

(44)

Eq. \( (43) \) shows that the limit of \( \langle \hat{M} \rangle \) is \( W_{11} \), and this means that in the range \( s > 1 \), the contribution from \( x \) exceeds that from \( \langle \psi_f | \psi_i \rangle \) as \( \alpha \to 0 \), and \( x \) becomes dominant in its competition with \( \langle \psi_f | \psi_i \rangle \). Note that \( W_{22} \) is the result of a conventional projective measurement, so in this case, the weak measurement turns to behave like a conventional projective measurement in the limit \( \alpha \to 0 \).

When \( \langle \psi_f | \psi_i \rangle \to 0 \) and \( s < 0 \), i.e. when the pre-selection tends to \( |q_1 \rangle \) as \( \alpha \to 0 \), similar results as \( (39)-(43) \) can be obtained:

1. \( -1 < s < 0 \): only the terms \( \beta^{1+3s} \) and \( \beta^{2s} \) should be remained, so

\[
\langle \hat{M} \rangle \approx \frac{(W_{11} + W_{22} - 2\text{Re}W_{12})/2 + \beta^{1+s}((W_{11} - \text{Re}W_{12}) \cos \theta - \text{Im}W_{12} \sin \theta)}{1 - \text{Re}Y_{12} + \beta^{1+s}((1 - \text{Re}Y_{12}) \cos \theta - \text{Im}Y_{12} \sin \theta)} \\
\approx \frac{W_{11} + W_{22} - 2\text{Re}W_{12}}{2(1 - \text{Re}Y_{12})} + \beta^{1+s}[(W_{11} - \text{Re}W_{12}) \cos \theta - \text{Im}W_{12} \sin \theta] \\
- \frac{(1 - \text{Re}Y_{12}) \cos \theta - \text{Im}Y_{12} \sin \theta)(W_{11} + W_{22} - 2\text{Re}W_{12})}{2(1 - \text{Re}Y_{12})^2},
\]

and

\[ \frac{d\langle \hat{M} \rangle}{d\beta} \propto \beta^{s}. \]  

(45)

(46)

2. \( s = -1 \): the terms \( \beta^{1+3s} \), \( \beta^{2+4s} \) and \( \beta^{2s} \) should be remained, so

\[ \langle \hat{M} \rangle \approx \frac{W_{11} + W_{22}/2 - \text{Re}W_{12} + (W_{11} - \text{Re}W_{12}) \cos \theta - \text{Im}W_{12} \sin \theta}{3/2 - \text{Re}Y_{12} + (1 - \text{Re}Y_{12}) \cos \theta - \text{Im}Y_{12} \sin \theta}, \]

(47)

and

\[ \frac{d\langle \hat{M} \rangle}{d\beta} \approx 0. \]  

(48)

3. \( s < -1 \): only the terms \( \beta^{1+3s} \) and \( \beta^{2+4s} \) should be remained, and

\[
\langle \hat{M} \rangle \approx \frac{W_{11} + 2\beta^{-(1+s)}((W_{11} - \text{Re}W_{12}) \cos \theta - \text{Im}W_{12} \sin \theta)}{1 + 2\beta^{-(1+s)}((1 - \text{Re}Y_{12}) \cos \theta - \text{Im}Y_{12} \sin \theta)} \\
\approx W_{11} + 2\beta^{-(1+s)}[(W_{11} - \text{Re}W_{12}) \cos \theta - \text{Im}W_{12} \sin \theta - W_{11}(1 - \text{Re}Y_{12}) \cos \theta - \text{Im}Y_{12} \sin \theta)],
\]

and

\[ \frac{d\langle \hat{M} \rangle}{d\beta} \propto \beta^{-(1+s)}. \]  

(49)

(50)
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Range of \(s\) & \(\langle \hat{M} \rangle\) & Convergence Speed of \(\langle \hat{M} \rangle\) \\
\hline
\(0 < |s| < 1\) & \(\beta^{1-|s|}\) & \(\beta^{-|s|}\) \\
\hline
\(|s| = 1\) & constant & 0 \\
\hline
\(|s| > 1\) & \(\beta^{|s|-1}\) & \(\beta^{|s|-2}\) \\
\hline
\(|\psi_i\rangle \rightarrow |a_1\rangle \) or \( |a_2\rangle \) & \(\beta\) & 1 \\
\hline
\end{tabular}
\end{center}

Table I: Asymptotic behavior of \(\langle \hat{M} \rangle\) with \(\beta^* \sim y\) in the limit \(\langle \psi_f | \psi_i \rangle \rightarrow 0\).

Eq. (39)-(49) together with (35) give the asymptotic value of \(\langle \hat{M} \rangle\) when \(|\langle \psi_f | \psi_i \rangle| \rightarrow 0\).

The table I summarizes the results of \(\langle \hat{M} \rangle\) as \(|\langle \psi_f | \psi_i \rangle|\) tends towards zero asymptotically, involving the competition from the limiting process that the pre-selection approaches one of the eigenstates of the observable \(A\). The coefficients and constant terms are omitted in the table because they do not characterize the asymptotic behavior of \(\langle \hat{M} \rangle\). In contrast to the prediction by the weak value (1), \(\langle \hat{M} \rangle\) actually converges as a polynomial of \(|\langle \psi_f | \psi_i \rangle|\) but not the reciprocal of \(|\langle \psi_f | \psi_i \rangle|\) as \(|\langle \psi_f | \psi_i \rangle| \rightarrow 0\).

An interesting thing to note in the table I is that the asymptotic value of \(\langle \hat{M} \rangle\) transits continuously between different regions of \(s\). In the region \(0 < s < 1\) and \(-1 < s < 0\), when \(s \rightarrow 0\), \(\langle \hat{M} \rangle \rightarrow \beta\) (the coefficients and the constant term have been omitted), which is exactly the asymptotic value of \(\langle \hat{M} \rangle\) when \(|\psi_i\rangle \not\rightarrow |a_1\rangle \) or \(|a_2\rangle\). In the region \(s \neq \pm 1\), when \(s \rightarrow \pm 1\), \(\langle \hat{M} \rangle \rightarrow\) constant, the same as \(s = \pm 1\).

V. CONCLUSION

Weak measurement is an interesting quantum measurement scheme with the ability to amplify tiny physical quantity. The weak value formalism is valid mainly when the fidelity between the pre- and post-selections is not too small, and the case that the pre- and post-selections tend to be orthogonal is beyond the weak value formalism because the weak value diverges in this case. Our work bridged this gap.

In the first half of this paper, a general rigorous framework for the weak measurement on a two-dimensional system with a continuous probe was established, and it was shown that however small the fidelity between the pre- and post-selections was, the output of a weak measurement would be always finite. That result set a limit for the amplification ability of a weak measurement actually. Also two typical examples were calculated in detail.

In the second half of the paper, the asymptotic behavior of a weak measurement was considered when the pre- and post-selections tended to be orthogonal. Generally, the asymptotic behavior of a weak measurement is dominated by the fidelity between pre- and post-selections which tends to be zero in the limiting process. But when the pre-selection tends towards an eigenstate of the observable on the system sufficiently fast at the same time, the contribution from the fidelity between the pre-selection and that eigenstate of the observable will be so prominent that it will enter a competition with the fidelity between the pre- and post-selections. In the second half of the paper, we gave a simple model to characterize the speed of the pre-selection approaching an eigenstate of the observable and analyzed the competition in detail. The result showed explicitly when the fidelity between the pre- and post-selections prevailed in the competition and when the fidelity between the pre-selection and an eigenstate of the observable did.

We hope this work can contribute to the further understanding and application of weak measurement.

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Appendix

A. Derivation of (25)

It is explicit that the real parts of $W_{12}$ and $Y_{12}$ are

$$\text{Re}W_{12} = \int q \text{Re}(q - ga_1) \text{Re}(q - ga_2) dq + \int q \text{Im}(q - ga_1) \text{Im}(q - ga_2) dq,$$

$$\text{Re}Y_{12} = \int \text{Re}(q - ga_1) \text{Re}(q - ga_2) dq + \int \text{Im}(q - ga_1) \text{Im}(q - ga_2) dq. \quad (51)$$

Provided that $q\phi^{(n)}(q) \to 0 \ (n = 0, 1, 2 \cdots)$ as $q \to \pm \infty$, it can be straightforwardly verified that

$$\int (\text{Re}(q))^{(k)} (\text{Re}(q))^{(2n-k)} dq = (-1)^{n-k} \int ((\text{Re}(q))^{(n)})^2 dq,$$

$$\int (\text{Re}(q))^{(k)} (\text{Re}(q))^{(2n+1-k)} dq = 0,$$

$$\int q (\text{Re}(q))^{(k)} (\text{Re}(q))^{(2n-k)} dq = (-1)^{n-k} \int q ((\text{Re}(q))^{(n)})^2 dq,$$

$$\int q (\text{Re}(q))^{(k)} (\text{Re}(q))^{(2n+1-k)} dq = (-1)^{n-k+1} (n-k + \frac{1}{2}) \int ((\text{Re}(q))^{(n)})^2 dq, \quad (52)$$

using the method of integration by parts repeatedly. Similar integral results can be obtained for the imaginary part of $\phi(q)$.

With the Taylor’s expansion of $\phi(q - ga_i)$, i.e.

$$\phi(q - ga_i) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \phi^{(n)}(q) g^n a_i^n, \ i = 1, 2, \quad (53)$$

and the integral formulae (52), one can get

$$\text{Re}Y_{12} = \sum_{n=0}^{+\infty} g^{2n} \sum_{k=0}^{2n} \frac{(-1)^k (1-2n-k)}{k!(2n-k)!} d_1^k d_2^{2n-k} \int [((\text{Re}(q))^{(k)} (\text{Re}(q))^{(2n-k)} + ((\text{Im}(q))^{(k)} (\text{Im}(q))^{(2n-k)}) dq$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} g^{2n} (a_1 - a_2)^2n \int [((\text{Re}(q))^{(n)})^2 + ((\text{Im}(q))^{(n)})^2)] dq$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} g^{2n} (a_1 - a_2)^2n \int |\phi^{(n)}(q)|^2 dq,$$

$$\text{Re}W_{12} = \sum_{n=0}^{+\infty} g^{2n} \sum_{k=0}^{2n} \frac{(-1)^k (1-2n-k)}{k!(2n-k)!} d_1^k d_2^{2n-k} \int q [((\text{Re}(q))^{(k)} (\text{Re}(q))^{(2n-k)} + ((\text{Im}(q))^{(k)} (\text{Im}(q))^{(2n-k)}) dq$$

$$+ \sum_{n=0}^{+\infty} g^{2n+1} \sum_{k=0}^{2n+1} \frac{(-1)^k (1-2n+1-k)}{k!(2n+1-k)!} d_1^k d_2^{2n+1-k} \int q [((\text{Re}(q))^{(k)} (\text{Re}(q))^{(2n+1-k)} + ((\text{Im}(q))^{(k)} (\text{Im}(q))^{(2n+1-k)}) dq$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} g^{2n} (a_1 - a_2)^2n \int |\phi^{(n)}(q)|^2 dq + \frac{1}{2} \sum_{n=0}^{+\infty} g^{2n+1} \sum_{k=0}^{2n+1} \frac{(-1)^{n-k} a_2^k a_1^{2n-k}}{k!(2n-k)!} d_1^k d_2^{2n-k}$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} g^{2n} (a_1 - a_2)^2n \int |\phi^{(n)}(q)|^2 dq + \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(-1)^n a_1 a_2}{(k-1)!(2n+1-k)!} d_1^{k-1} d_2^{2n-k} \int |\phi^{(n)}(q)|^2 dq,$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} g^{2n} (a_1 - a_2)^2n \int |\phi^{(n)}(q)|^2 dq + \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(-1)^n g^{2n+1} (a_1 + a_2)(a_1 - a_2)^{2n} \int |\phi^{(n)}(q)|^2 dq, \quad (54)$$

where we have used the binomial formula

$$(a + b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}. \quad (55)$$
B. Derivation of (30)

Note that

\[ \text{Re} W_{12} = \int \text{Re} \phi(q - ga_1) \text{Im} \phi(q - ga_2) dq - \int \text{Im} \phi(q - ga_1) \text{Re} \phi(q - ga_2) dq. \quad (56) \]

Provided that \( g^{(n)}(q) \to 0 \) \((n = 0, 1, 2 \cdots)\) as \( q \to \pm \infty \), it can be straightforwardly verified that

\[
\int (\text{Re} \phi(q))^{(k)} (\text{Im} \phi(q))^{(2n-k)} dq = (-1)^{n-k} \int (\text{Re} \phi(q))^{(n)} (\text{Im} \phi(q))^{(n)} dq,
\]

\[
\int (\text{Re} \phi(q))^{(k)} (\text{Im} \phi(q))^{(2n+1-k)} dq = (-1)^{n-k} \int (\text{Re} \phi(q))^{(n)} (\text{Im} \phi(q))^{(n+1)} dq,
\]

\[
\int (\text{Im} \phi(q))^{(k)} (\text{Re} \phi(q))^{(2n+1-k)} dq = (-1)^{n-k} \int (\text{Im} \phi(q))^{(n)} (\text{Re} \phi(q))^{(n+1)} dq,
\]

using the method of integration by parts repeatedly.

Plug the Taylor’s expansion of \( \phi(q - ga) \) into (56), it follows that

\[
\text{Re} W_{12} = \sum_{n=0}^{\infty} g^{2n} \sum_{k=0}^{2n} \frac{(-1)^k (-1)^{2n-k} a_k^2 a_k^{2n-k}}{k!(2n-k)!} (-1)^{n-k} \int (\text{Re} \phi(q))^{(n)} (\text{Im} \phi(q))^{(n+1)} - (\text{Im} \phi(q))^{(n)} (\text{Re} \phi(q))^{(n+1)} dq
\]

\[
+ \sum_{n=0}^{\infty} g^{2n+1} \sum_{k=0}^{2n+1} \frac{(-1)^k (-1)^{2n+1-k} a_k^2 a_k^{2n+1-k}}{k!(2n+1-k)!} (-1)^{n+1-k} \int (\text{Re} \phi(q))^{(n+1)} (\text{Im} \phi(q))^{(n+1)} dq
\]

\[
= 2 \sum_{n=0}^{\infty} \frac{(-1)^n g^{2n}}{(2n)!} (a_1 - a_2)^2 n \int (\text{Re} \phi(q))^{(n)} (\text{Im} \phi(q))^{(n+1)} dq,
\]

where the binomial formula (55) has been used.
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