Stabilization for equations of one-dimensional viscous compressible heat-conducting media with nonmonotone equation of state

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Abstract

We consider the Navier-Stokes system describing motions of viscous compressible heat-conducting and “self-gravitating” media. We use the state function of the form $p(\eta, \theta) = p_0(\eta) + p_1(\eta)\theta$ linear with respect to the temperature $\theta$, but we admit rather general nonmonotone functions $p_0$ and $p_1$ of $\eta$, which allows us to treat various physical models of nuclear fluids (for which $p$ and $\eta$ are the pressure and specific volume) or thermoviscoelastic solids. For an associated initial-boundary value problem with “fixed-free” boundary conditions and possibly large data, we prove a collection of estimates independent of time interval for solutions, including two-sided bounds for $\eta$, together with its asymptotic behaviour as $t \to \infty$. Namely, we establish the stabilization pointwise and in $L^q$ for $\eta$, in $L^2$ for $\theta$, and in $L^q$ for $v$ (the velocity), for any $q \in [2, \infty)$.

1 Introduction

The problem of large-time behaviour of solutions to equations of a 1d-flow of viscous compressible heat-conducting fluids (or gases) with large data was studied in a lot of papers including [6] [17] [13] [19]. All these papers deal with the case of particular (polytropic gas) or general pressure law $p(\eta, \theta)$ but always monotone with respect to the variable $\eta$ (here $\eta$ and $\theta$ are the specific volume and the absolute temperature). It is well known that this monotonicity is not valid in a number of physical situations. In particular, the case of the two-term pressure

$$p(\eta, \theta) = p_0(\eta) + p_1(\eta)\theta,$$

which is linear in $\theta$ but with complicated nonmonotone $p_0(\eta)$ is of importance for nuclear fluid models, see [8] [9] and references therein.

The case of the two-term function (1) with other properties of $p_0$ and $p_1$, and nonmonotone $p_1$ is also interesting in a completely different physical context, namely for thermoviscoelastic solids (shape memory alloys), see [20] [12] [21] and references therein. In these papers, for models with essentially simplified forms of the viscosity term and heat flux in the equations, the stabilization of solutions was studied but for $\eta$ it was proved only in the case $p_0 = 0$.

We also mention papers concerning stabilization in nonmonotone barotropic case (where $p = p(\eta)$) for fluids [13] [15] [10] [11] and for viscoelastic solids [3] [18].

Notice that nonmonotonicity of $p$ complicates in an essential way the problem of stabilization. In particular, the stationary specific volume becomes nonunique and can be discontinuous.
In this paper, we consider the pressure law (1) with rather general nonmonotone $p_0$ and $p_1$ and we study both the cases of nuclear fluids and of thermoviscoelastic solids (without the aforementioned simplification in the viscosity term and the heat flux). Moreover a large external force of “self-gravitation” type is also taken into consideration. For an initial-boundary value problem with “fixed-free” boundary conditions and large initial data, we prove a collection estimates independent of time interval for solutions, including two-sided bounds for the specific volume $\eta$. Moreover we establish the pointwise and $L^q$-stabilization for $\eta$, $L^2$-stabilization for the temperature $\theta$ and the pressure $p$, and $\tilde{L}^q$-stabilization for the velocity for any $q \in [2, \infty)$, as time tends to infinity. In the nuclear fluid case, we also justify the sharpness of the main condition on the “self-gravitating” force.

2 Statement of the problem and main results

We consider the following system of quasilinear differential equations for 1d-motions of viscous compressible heat-conducting media

$$
\begin{aligned}
\eta_t &= v_x, \\
v_t &= \sigma_x + g, \\
e[\eta, \theta]_t &= \sigma v_x + \pi_x,
\end{aligned}
$$

where $(x, t) \in Q = \Omega \times \mathbb{R}^+ = (0, M) \times (0, +\infty)$ are the Lagrangian mass coordinates, with $M$ being the total mass of the medium.

The unknown quantities $\eta > 0$, $v$, and $\theta > 0$ are the specific volume, the velocity, and the absolute temperature. We also denote by $\rho = \frac{1}{\eta}$ the density, $\sigma = \nu v_x - p[\eta, \theta]$ the stress, $e(\eta, \theta)$ the internal energy, and $-\pi = -\kappa[\eta, \theta] \rho \theta_x$ the heat flux.

In all the paper, the notation $\mu[\eta, \theta](x, t) = \mu(\eta(x, t), \theta(x, t))$, for $\mu = e, p, \kappa$, etc. is adopted.

In order to fix the state functions $p(\eta, \theta)$ and $e(\eta, \theta)$, we define the Helmholtz free energy

$$
\Psi(\eta, \theta) = -c_V \theta \log \theta - P_0(\eta) - P_1(\eta) \theta,
$$

where $c_V = const > 0$. Then thermodynamics tells us that

$$
p(\eta, \theta) = -\Psi_\eta(\eta, \theta) = p_0(\eta) + p_1(\eta) \theta,
$$

with $p_0 = P_0'$ and $p_1 = P_1'$, as well as

$$
e(\eta, \theta) = \Psi(\eta, \theta) - \theta \Psi_\theta(\eta, \theta) = -P_0(\eta) + c_V \theta,
$$

where $\Psi_\eta = \frac{\partial \Psi}{\partial \eta}$ and $\Psi_\theta = \frac{\partial \Psi}{\partial \theta}$.

First, we consider the more difficult case of the nuclear fluid. We suppose that the functions $p_0, p_1 \in C^1(\mathbb{R}^+)$ are such that $\lim_{\eta \to 0^+} p_0(\eta) = +\infty$, $\lim_{\eta \to \infty} p_0(\eta) = 0$, $p_1(\eta) \geq 0$, $\eta \eta p_1(\eta) = O(1)$ as $\eta \to \infty$.

Suppose also that the viscosity and heat conductivity coefficients are such that $\nu = const > 0$ and $\kappa \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$, with $0 < \underline{\kappa} \leq \kappa(\eta, \theta) \leq \overline{\kappa}$, where $\underline{\kappa}$ and $\overline{\kappa}$ are given constants.

Let the so-called “self-gravitation force” $g \in L^1(\Omega)$. In fact, this name does not correspond exactly to the physical situation, as, at least in the nuclear fluid case, the corresponding “physical” force is the Coulomb force between charged particles, which contrary to the Newton gravitational force, is attractive. Although the distinction Coulomb-Newton is of utmost importance in multidimensional problems, it is harmless in the 1d-context.

\footnote{Note that $C^1(\mathbb{R}^+)$ stands for the space of continuously differentiable functions on $\mathbb{R}^+$, but not necessarily bounded. The spaces $C^1(\mathbb{R}^+ \times \mathbb{R}^+), C(\mathbb{R}^+), C(\mathbb{R})$, etc. used below are understood similarly.}
Let us supplement equations (3) with the following boundary and initial conditions

\[
v|_{x=0} = 0, \quad \sigma|_{x=M} = -p_T, \quad \theta|_{x=0} = \theta_T, \quad \pi|_{x=M} = 0,
\]

\[
\eta|_{t=0} = \eta^0(x), \quad v|_{t=0} = v^0(x), \quad \theta|_{t=0} = \theta^0(x),
\]

with an outer pressure \(p_T = const\) and a given temperature \(\theta_T = const > 0\).

Throughout the paper, we use the classical Lebesgue spaces \(L^q(G)\) together with their anisotropic version \(L^q(\Gamma)\), for \(q, r \in [1, +\infty]\), and we denote the associated norm by \(\| \cdot \|_{L^q(\Gamma)}\). In Section 2, we also use the abbreviate notation \(\| \cdot \|_{L^2(\Gamma)}\).

Let also \(V_2(Q)\) be the standard space of functions \(w\) having finite (parabolic) energy \(\|w\|_{V_2(Q)} = \|w\|_{L^2(Q)} + \|w_x\|_{L^2(Q)}\). We denote by \(H^1(\Omega)\) (resp. \(H^{2,1}(Q_T)\)) the standard Sobolev space equipped with the norm \(\|w\|_{H^1(\Omega)} = \|w\|_{L^2(Q)} + \|w_x\|_{L^2(Q)}\) (resp. \(\|w\|_{H^{2,1}(Q_T)} = \|w\|_{L^2(Q_T)} + \|w_x\|_{L^2(Q_T)}\)). Hereafter \(Q_T = \Omega \times (0, T)\).

In Section 2, we shall also exploit the integration operators \(I^s\phi(x) = \int_0^M \phi(x) \, d\xi\), for \(\phi \in L^1(\Omega)\), and \(I_0a(t) = \int_0^t a(\tau) \, d\tau\), for \(a \in L^1(0, T)\).

Suppose that the initial data are such that \(\eta_0 \in L^\infty(\Omega)\) with \(\ess\inf_\Omega \eta_0 > 0\), \(v^0 \in L^4(\Omega)\), \(\theta^0 \in L^2(\Omega)\), \(\log \theta^0 \in L^1(\Omega)\) with \(\theta^0 > 0\).

Though it is possible to establish our main results for weak solutions \([3]\), to simplify the presentation, we limit ourselves to the case of so-called regular weak solutions \([6]\) such that existence of the latter solutions in Appendix. Throughout the paper, we use the classical Lebesgue spaces \(L^p(\Omega)\) for any \(p \in (0, \infty)\), which can also depend on \(M, \nu, \kappa\), etc, but neither on the initial data nor on \(g\).

**Theorem 1**. Suppose that the initial data, \(p_T\), and \(g\) are such that

\[
N^{-1} \leq \eta_0 \leq N, \quad \|v^0\|_{L^4(\Omega)} + \|\log \theta^0\|_{L^1(\Omega)} + \|\theta^0\|_{L^2(\Omega)} \leq N,
\]

\[
\|g\|_{L^1(\Omega)} \leq N, \quad N^{-1} \leq \underline{p}_S.
\]

Then the following estimates in \(Q\) together with \(L^2(\Omega)\)-stabilization property hold

\[
0 < K^{1-1}_{\eta} = \underline{\eta} \leq \eta(x, t) \leq \overline{\eta} = K_2 \quad \text{in } Q_T,
\]

\[
\|v\|_{V_2(Q)} + \|v^2\|_{V_2(Q)} + \|\log \theta\|_{L^1(\Omega)} + \|(\log \theta)_x\|_{L^2(\Omega)} + \|\theta - \theta_T\|_{V_2(\Omega)} \leq K_3,
\]

\[
\|p|_{\eta, \theta} - p_S\|_{L^2(Q)} \leq K_4,
\]

\[
\|v^2(\cdot, t)\|_{L^2(\Omega)} + \|\theta(\cdot, t) - \theta_T\|_{L^2(\Omega)} + \|p|_{\eta, \theta}(\cdot, t) - p_S(\cdot)\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.
\]

2. Suppose that \(p|_{\eta, \theta}\) satisfies the following additional condition:

For any \(c \in [\underline{p}_S, \overline{p}_S]\), there exists no interval \((\eta_1, \eta_2)\) such that \(p|_{\eta, \theta_T} \equiv c\) on \((\eta_1, \eta_2)\).
Then the following pointwise stabilization property holds for \( \eta \): there exists a function \( \eta_S \in L^\infty(\Omega) \) satisfying
\[ p(\eta_S(x), \theta_T) = p_S(x) \quad \text{and} \quad \underline{\eta} \leq \eta_S(x) \leq \overline{\eta} \text{ on } \overline{\Omega}, \] (14)
such that
\[ \eta(x,t) \to \eta_S(x) \quad \text{as } t \to \infty, \quad \text{for all } x \in \overline{\Omega}. \] (15)
and consequently \( \|\eta(\cdot,t) - \eta_S(\cdot)\|_{L^\infty(\Omega)} \to 0 \) as \( t \to \infty \), for any \( q \in [1, \infty). \)

3. Suppose that, additionally to the hypotheses of Claim 1, \( \|v^0\|_{L^q(\Omega)} \leq N \), for some \( q \in (4, \infty) \). Then the following estimate in \( Q \) together with \( L^q(\Omega) \)-stabilization property hold
\[ \|v\|_{L^{\infty}(Q)} + \|v\|_{L^q(\Omega)} \leq K_S q, \]
\[ \|v(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty, \]
where \( K_S \) does not depend on \( q \).

Remarks:
1. An elementary but important consequence of Claim 2 is that \( V(t) := \int_{\Omega} \eta(x,t) \, dx \to V_S > 0 \) as \( t \to \infty \), where \( V(t) \) is the volume of the fluid (or in other words, the Eulerian position of the free boundary).

2. For nonmonotone \( p(\eta, \theta_T) \), if there exist two points \( 0 < \eta^{(1)} < \eta^{(2)} \) such that
\[ \underline{p}_S < \underline{p}^{(1)} := p(\eta^{(1)}, \theta_T) < \underline{p}^{(2)} := p(\eta^{(2)}, \theta_T) < \overline{\nu}_S, \]
and such that, moreover
\[ \begin{aligned}
    p^{(1)}(\eta, \theta_T) &\leq p(\eta, \theta_T), & \text{for } 0 < \eta \leq \eta^{(1)}, \\
    p^{(1)}(\eta, \theta_T) &\leq p^{(2)}(\eta, \theta_T), & \text{for } \eta^{(1)} < \eta < \eta^{(2)}, \\
    p(\eta, \theta_T) &\leq p^{(2)}(\eta, \theta_T), & \text{for } \eta^{(2)} \leq \eta,
\end{aligned} \]
then, necessarily \( \eta_S \notin C(\Omega) \). Moreover, consequently, the convergence in (13) cannot be uniform in \( x \). In fact, even for \( g \equiv 0 \), if the equation \( p(\eta, \theta_T) = p_T \) has more than one solution, then \( \eta_S \) can be discontinuous in \( \overline{\Omega} \). Namely, if this equation has exactly \( k \) solutions \( \eta^{(1)} < \ldots < \eta^{(k)} \), then the function \( \eta_S \) can be written as
\[ \eta_S = \sum_{j=1}^{k} \chi(E_j) \eta^{(j)}, \]
where \( E_j, 1 \leq j \leq k, \) are any measurable nonintersecting subsets of \( \overline{\Omega} \) (some of them may be empty) such that \( \bigcup_{j=1}^{k} E_j = \overline{\Omega} \), and \( \chi(E_j) \) are their characteristic functions. Unfortunately, we cannot assert more about \( \eta_S \).

Let us justify that the second condition (10) is essential in Theorem 1. Set \( m(\theta_T) := \inf_{\eta > 0} p(\eta, \theta_T) \). Obviously \( m(\theta_T) \leq 0 \), and if \( p_0 > -p_1 \theta_T \), then \( m(\theta_T) = 0 \).

Proposition 1 Let the hypotheses of theorem 1, Claim 1, be valid, but suppose that \( \underline{p}_S < m(\theta_T) \), instead of \( N^{-1} \leq \underline{p}_S \). Then
\[ \limsup_{t \to \infty} V(t) = \infty. \] (16)

This property means that the upper bound for \( \eta \) in (11) is violated and physically, that the fluid can asymptotically expand in the whole halfspace.

Let us also consider the borderline case \( \underline{p}_S = m(\theta_T) \).
Proposition 2 Let the hypotheses of theorem 1, Claim 1, be valid and \( p(\eta, \theta \Gamma) > m(\theta \Gamma) = 0 \), but \( p_S(0) = 0 \) instead of \( N^{-1} \leq p_S \). Then at least one of the following properties holds:

\[
\limsup_{t \to \infty} \left| \int_\Omega v(x,t) \, dx \right| = \infty, \tag{17}
\]

\[
\lim_{t \to \infty} \eta(0,t) = \infty. \tag{18}
\]

If in addition \( \int_1^\infty p(\eta, \theta \Gamma) \, d\eta < \infty \) and \( p_S = p_S(0) = 0 \), then \( \|v\|_{L^\infty(Q)} \leq K_S \), but property (18) holds.

Properties (17) and (18) mean that estimate (12) and the upper bound for \( \eta \) in (11) are violated respectively.

Note that propositions 1 and 2 go back to results of [25] where the barotropic case was studied.

Finally, we consider the case of thermoviscoelastic solids. Let \( p_S \leq p_S \) be fixed. Suppose that, instead of (5) and (6), the following conditions hold

\[
p_S \leq p_0(\eta) \text{ and } 0 \leq p_1(\eta) \text{ for } 0 < \eta \leq \hat{\eta}, \tag{19}
\]

\[
p_0(\eta) \leq p_S \text{ and } p_1(\eta) \leq 0 \text{ for } 0 \leq \hat{\eta} < \eta, \tag{20}
\]

for some \( 0 < \hat{\eta} \leq \hat{\eta} < \infty \). The conditions of such kind are of standard type for the thermoviscoelastic case.

Theorem 2 All the Claims 1-3 of theorem 1 remain valid under conditions (19) and (20), and without the condition \( N^{-1} \leq p_S \).

Remark:

We could consider the viscosity coefficient \( \nu = \nu(\eta) \geq \nu_0 > 0 \), \( \nu \in C^1(\mathbb{R}^+) \) as well as body force and boundary data in the form \( g(x,t) = g_S(x) + \Delta g(x,t) \), \( p_T(t) = p_{T,S} + \Delta p_T(t) \), and \( \theta_T(t) = \theta_{T,S} + \Delta \theta_T(t) \), with perturbations \( \Delta g \), \( \Delta p_T \), and \( \Delta \theta_T \) tending to zero as \( t \to \infty \) in some weak sense (compare with the barotropic case [25] [11]). To simplify the presentation of the results and their proof, we do not realize this possibility in the paper.

3 Proof of the results

We begin with the proof of theorem 1 which follows from a lengthy series of lemmas, providing necessary a priori estimates and stabilization properties: Claims 1, 2, and 3 will be proved respectively in lemmas 1-9, lemmas 10 and 11, and lemmas 12 and 13.

Then we proceed with the proofs of propositions 1 and 2 and theorem 2.

3.1 A priori estimates and proof of theorem 1

Lemma 1 The following energy estimates hold

\[
\|\eta\|_{L^1,\infty(Q)} + \|v\|_{L^2,\infty(Q)} + \|\theta\|_{L^1,\infty(Q)} + \|\log \theta\|_{L^1,\infty(Q)} \leq K^{(1)}, \tag{21}
\]

\[
\|\sqrt{\frac{\rho}{\theta}} v_x\|_{Q} + \|\sqrt{\frac{\rho}{\theta}} \theta_x\|_{Q} \leq K^{(2)}. \tag{22}
\]
Hereafter we use the notation where

\[ \hat{\theta} z \]

\[ \check{\theta} z \]

1

\[ \check{\theta} z \]

where the number \( \check{\theta} z \) is such that \( \frac{1}{2} \alpha \leq \alpha - \log \alpha + \log 2 - 1 \) is taken into account.

The following auxiliary result on ordinary differential inequalities is useful to prove lower and upper bounds for the specific volume \( \eta \) in various situations.

**Lemma 2** Let \( N_0 \geq 0 \), \( N_1 \geq 0 \), and \( \varepsilon_0 > 0 \) be three parameters.

Let \( f \in C(\mathbb{R}) \) and \( y, b \in W^{1,1}(0, T) \), for any \( T > 0 \). The following claims are valid:

1. if
   \[
   \frac{dy}{dt} \geq f(y) + \frac{db}{dt} \quad \text{on } \mathbb{R}^+,
   \]
   where \( f(-\infty) = +\infty \) and \( b(t) - b(\tau) \geq -N_0 - N_1(t - \tau) \), for any \( 0 \leq \tau \leq t \), then the uniform lower bound holds:
   \[
   \min \{ y(0), \check{z} \} - N_0 \leq y(t) \quad \text{on } \mathbb{R}^+,
   \]
   where the number \( \check{z} = \check{z}(N_1) \) is such that \( f(z) \geq N_1 \), for \( z \leq \check{z} \);

2. if
   \[
   \frac{dy}{dt} \leq f(y) + \frac{db}{dt} \quad \text{on } \mathbb{R}^+,
   \]
   where \( \limsup_{z \to +\infty} f(z) \leq 0 \), and \( b(t) - b(\tau) \leq N_0 - \varepsilon_0(t - \tau) \), for any \( 0 \leq \tau \leq t \), then the uniform upper bound holds:
   \[
   y(t) \leq \max \{ y(0), \check{z} \} + N_0 \quad \text{on } \mathbb{R}^+,
   \]
   where the number \( \check{z} = \check{z}(\varepsilon_0) \) is such that \( f(z) \leq \varepsilon_0 \), for \( z \geq \check{z} \).
Remark:

In lemma 2, one can drop the conditions \( f(-\infty) = +\infty \) and \( \limsup_{z \to +\infty} f(z) \leq 0 \), take \( f \in C(\mathbb{R} \times \mathbb{R}^+) \) and replace \( f(y) \) by \( f(y, t) \). Then Claim 1 remains valid if, for a fixed \( N_1 \), there exists \( \tilde{z} \) such that \( f(z, t) \geq N_1 \), for \( z \leq \tilde{z} \) and \( t \geq 0 \). Similarly, Claim 2 remains valid if, for a fixed \( \varepsilon_0 \geq 0 \), there exists \( \tilde{z} \) such that \( f(z, t) \leq \varepsilon_0 \), for \( z \geq \tilde{z} \) and \( t \geq 0 \).

Lemma 2 is borrowed from [24], where in both claims, differential equalities are used, but one checks easily that the proof remains valid for inequalities; the similar conclusion is valid concerning the above remark. The statements of the type specified in this remark are well known in viscoelastic and thermo-viscoelastic contexts.

**Lemma 3** For \( \eta \), the uniform lower bound holds
\[
0 < \eta = \left(K(3)\right)^{-1} \leq \eta(x, t) \text{ in } \overline{Q}.
\]

**Proof:** The action of the operator \( I^* \) on the second equation (2) gives the equation
\[
I^*v_t = -\nu \rho v_x + p[\eta, \theta] - p_S,
\]
which together with the relation \( \rho v_x = (\log \eta)_t \) lead to the another important equation
\[
(\nu \log \eta)_t = p[\eta, \theta] - p_S - I^*v_t. \tag{27}
\]
By putting \( y := \nu \log \eta \), exploiting the property \( p_1[\eta, \theta] \geq 0 \), and fixing any \( x \in \Omega \), we get
\[
\frac{dy}{dt} \geq p_0 \left( \exp \left(\frac{y}{\nu}\right) \right) - p_S - \frac{d}{dt}I^*v.
\]
The function \( f(z) := p_0 \left( \exp \left(\frac{z}{\nu}\right) \right) - p_S \) satisfies the property \( f(-\infty) = +\infty \) (see (5)). Moreover, due to the energy estimate (21)
\[
\left| I^*v \right| \leq 2 \sup_{\overline{Q}} |I^*v| \leq 2M^{1/2} \|v\|_{L^2,\infty(Q)} \leq K_0. \tag{28}
\]
Now Claim 1 in lemma 2 (with \( N_1 = 0 \)) implies the estimate
\[
\min \{\nu \log \eta_0(x), \nu \log \tilde{\eta} \} - K_0 \leq y(x, t),
\]
with a number \( \tilde{\eta} \) such that \( p_0(\tilde{\eta}) - p_S \geq 0 \), for any \( 0 < \eta \leq \tilde{\eta} \). Then:
\[
\eta := \min \{N^{-1}, \tilde{\eta} \} \exp \left( -\frac{K_0}{\nu} \right) \leq \eta(x, t) \text{ in } \overline{Q}. \tag*{\Box}
\]

The next auxiliary result on ordinary integral inequality is useful to deduce a uniform upper bound for \( \eta \).

**Lemma 4** Let \( b \) be a nondecreasing function on \([0, T]\) with \( b(0) \geq 0 \), and let \( a \in L^1(0, T) \) be a nonnegative function. If \( z \in L^\infty(0, T) \), \( z \geq 0 \) satisfies
\[
z(t) \leq b(t) + \int_0^t a(\tau)z(\tau) d\tau \text{ on } (0, T),
\]
then the upper bound holds:
\[
z(t) \leq b(t) \exp \left( \int_0^t a(\tau) d\tau \right) \leq b(t) \exp \left( \|a\|_{L^1(0, T)} \right) \text{ on } (0, T).
\]
The result follows immediately from the integral Gronwall’s lemma (for example see [3]) if one takes into account that
\[ z(s) \leq b(t) + \int_0^s a(\tau)z(\tau)\,d\tau, \text{ for } 0 < s \leq t < T. \]

**Lemma 5** For \( \eta \), the uniform upper bound holds
\[ \eta(x,t) \leq \eta = K^{(4)} \text{ in } Q. \]

**Proof:** Let us rewrite the first equation (2) as follows
\[ \eta_t = \frac{1}{\nu}(\sigma + \delta) \eta + \frac{1}{\nu} \eta (p[\eta,\theta] - \delta), \]
where \( \delta \) is a parameter. We consider this equation as an ordinary differential equation with respect to \( \eta \) and obtain the formula
\[ \eta = \exp \left( \frac{1}{\nu} I_0(\sigma + \delta) \right) \left\{ \eta^0 + \frac{1}{\nu} I_0 \left[ \exp \left( -\frac{1}{\nu} I_0(\sigma + \delta) \right) \eta (p[\eta,\theta] - \delta) \right] \right\}. \] (29)

By applying the operator \( I_0 \) to equation (26), we find
\[ I_0 \sigma = -p \sigma t - I^* (v - v^0). \]

So by choosing \( \delta := \frac{1}{\nu} p S \) and using estimate (28), we get
\[ \frac{1}{\nu} I_0(\sigma + \delta)^\nu = -\frac{1}{\nu} (p \sigma - \delta) (t - \tau) - \frac{1}{\nu} I^* v \leq -\alpha(t - \tau) + K_1 \text{ on } \Omega, \text{ for all } 0 \leq \tau \leq t \]
with \( \alpha := \frac{1}{2\nu p S} > 0 \). Conditions (3) and (4) on \( p_0 \) and \( p_1 \) together with the lower bound \( \eta \leq \eta \) give
\[ \eta (p[\eta,\theta] - \delta) \leq \eta \max \{p_0[\eta] - \delta, 0\} + \eta (p_1)[\eta] \theta \leq K_2 + K_3 \theta. \]

Therefore formula (29) implies the estimate
\[ \hat{\eta}(t) := \|\eta(\cdot,t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha t} \left[ 1 + \int_0^t e^{\alpha \tau} \left( 1 + \|\theta(\cdot,\tau)\|_{L^\infty(\Omega)} \right) d\tau \right]. \] (30)

Set \( a := \left\| \frac{\partial^2 \varphi}{\partial \theta_x^2} \right\|^2_{\Omega} \). It is well known [1] [2] that the inequalities
\[ \|\theta\|_{L^\infty(\Omega)} \leq \frac{1}{M} \|\theta\|_{L^1(\Omega)} + \|\theta\|_{L^1(\Omega)} \leq \frac{1}{M} \|\theta\|_{L^1(\Omega)} + (a \|\theta\|_{L^1(\Omega)} \|\theta\|_{L^\infty(\Omega)} \hat{\eta})^{1/2} \]
\[ \leq \varepsilon \|\theta\|_{L^\infty(\Omega)} + \frac{1}{M} \|\theta\|_{L^1(\Omega)} + \frac{1}{4\varepsilon} a \|\theta\|_{L^1(\Omega)} \hat{\eta}, \forall \varepsilon > 0 \]
together with the estimate \( \|\theta\|_{L^\infty(Q)} \leq K^{(1)} \) imply
\[ \|\theta\|_{L^\infty(\Omega)} \leq K_5 (1 + a \hat{\eta}). \]

So by using estimate (30), the function \( z(t) := e^{\alpha t} \hat{\eta}(t) \) satisfies
\[ z(t) \leq K_6 \left( e^{\alpha t} + \int_0^t a(\tau)z(\tau)\,d\tau \right) \text{ on } \mathbb{R}^+. \]

As \( \|a\|_{L^1(\mathbb{R}^+)} \leq (K^{(2)})^2 \) according to lemma 4, by using lemma 5
\[ z(t) \leq K_6 \exp (\alpha t + K_6 (K^{(2)})^2) = K_7 e^{\alpha t} \text{ on } \mathbb{R}^+. \]

This means that \( \eta \leq \hat{\eta} \leq \eta := K_7 \text{ in } Q. \)  \( \square \)
Corollary 1 For \( v \), the following estimate holds
\[
\frac{1}{\sqrt{M}} \| v \|_{Q} \leq \| v \|_{L^{\infty}(Q)} \leq (K^{(1)})^{1/2} \left\| \frac{v_{x}}{\sqrt{\theta}} \right\|_{Q} \leq K^{(5)}.
\]

Proof: In fact, by using lemma 1, we have
\[
\| v \|_{C(\Omega)} \leq \| v_{x} \|_{L^{1}(\Omega)} \leq \| \theta \|_{L^{1}(\Omega)}^{1/2} \left\| \frac{v_{x}}{\sqrt{\theta}} \right\|_{\Omega} \leq (K^{(1)})^{1/2} \left\| \frac{v_{x}}{\sqrt{\theta}} \right\|_{\Omega},
\]
and
\[
\left\| \frac{v_{x}}{\sqrt{\theta}} \right\|_{Q} \leq \left\| \frac{v_{x}}{\sqrt{\theta}} \right\|_{\Omega} \leq \eta^{1/2} \left\| \frac{v_{x}}{\sqrt{\theta}} \right\|_{Q} \leq \eta^{1/2} K^{(2)}. \quad \square
\]

Note that similarly \( \| (\log \theta)_{x} \|_{Q} \leq \eta^{1/2} K^{(2)} \).

The following auxiliary result on ordinary differential inequalities will be exploited when proving \( V_{2}(Q) \)-estimates and \( L^{2}(\Omega) \)-stabilization for \( v^{2} \) and \( \theta - \theta_{T} \).

Lemma 6 Let \( a_{0} = \text{const} > 0 \) and \( a, h \in L^{1}(\mathbb{R}^{+}) \). If a function \( y \geq 0 \) on \( \mathbb{R}^{+} \) satisfies \( y \in W^{1,1}(0,T) \) for any \( T > 0 \) and
\[
\frac{dy}{dt} + (a_{0} + a)y \leq h \quad \text{on} \quad \mathbb{R}^{+},
\]
then the following upper bound together with stabilization property hold:
\[
y(t) \leq (y(0) + \| h \|_{L^{1}(\mathbb{R}^{+})}) \quad \exp (\| a \|_{L^{1}(\mathbb{R}^{+})}) \quad \text{on} \quad \mathbb{R}^{+},
\]
\[
y(t) \to 0 \quad \text{as} \quad t \to \infty.
\]

It is easy to derive this simple known result by multiplying (32) by \( \exp \int_{0}^{t} (a_{0} + a) \) and integrating the result; of course, estimate (33) holds also for \( a_{0} = 0 \). Note that more general result can be found in [22], lemma 2.1.

Lemma 7 For \( v^{2} \) and \( \theta - \theta_{T} \), the following estimate with the stabilization property hold
\[
\| v^{2} \|_{V_{3}(Q)} + \| \theta - \theta_{T} \|_{V_{3}(Q)} \leq K^{(6)},
\]
\[
\| v^{2}(\cdot,t) \|_{\Omega} + \| \theta(\cdot,t) - \theta_{T} \|_{\Omega} \to 0 \quad \text{as} \quad t \to \infty.
\]

Proof: By rewriting equation (33) as follows
\[
\left( \frac{1}{2} v^{2} + c_{V}(\theta - \theta_{T}) \right) = (\sigma v + \pi)_{x} + p_{0}[\eta]v_{x} + gv
\]
and taking \( L^{2}(\Omega) \)-inner product with \( \frac{1}{2} v^{2} + c_{V}(\theta - \theta_{T}) \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} v^{2} + c_{V}(\theta - \theta_{T}) \right) dx + \int_{\Omega} \left( \nu \rho v_{x} - p[\eta, \theta] \right) v + \kappa [\eta, \theta ] \rho \theta_{x} ( v v_{x} + c_{V} \theta_{x} ) dx
\]
\[
= \int_{\Omega} (p_{0}[\eta]v_{x} + gv) \left( \frac{1}{2} v^{2} + c_{V}(\theta - \theta_{T}) \right) dx - p_{T} \left( v \left( \frac{1}{2} v^{2} + c_{V}(\theta - \theta_{T}) \right) \right) \bigg|_{x=M}.
\]
We also take $L^2(\Omega)$-inner product of the second equation (3) with $v^3$:

$$\frac{1}{4} \frac{d}{dt} \int_\Omega \nu \nu v_x^2 dx + 3 \int_\Omega (\nu \nu v_x - \nu \nu \nu \theta \theta) v^2 v_x dx = \int_\Omega g v^3 dx - p_{\Gamma} v^3|_{x=M}.$$ 

By summing up equality (33) and the latter one multiplied by a parameter $\delta \geq 1$, we get

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \left( \left( \frac{1}{2} v^2 + c v (\theta - \theta_\Gamma) \right)^2 + \frac{\delta}{2} v^4 \right) dx + \int_\Omega \left[ (1 + 3\delta) \nu \nu v^2 v_x^2 + c v \kappa [\eta, \theta] \rho \theta_x^2 \right] dx$$

$$= - \int_\Omega (\nu \nu \nu + \kappa [\eta, \theta]) \rho \nu v_x \theta_x dx$$

$$+ \int_\Omega [p_0 [\eta] v_x \left( \frac{1}{2} v^2 + c v (\theta - \theta_\Gamma) \right) + p [\eta, \theta] \left( (1 + 3\delta) \nu \nu v_x^2 + c v \nu \theta_x \right)] dx$$

$$+ \int_\Omega g v \left( \frac{1}{2} + \delta \right) v^2 + c v (\theta - \theta_\Gamma) \right) dx - p_{\Gamma} \left( v \left( \frac{1}{2} + \delta \right) v^2 + c v (\theta - \theta_\Gamma) \right) \bigg|_{x=M}$$

$$=: I_1 + I_2 + I_3 + I_4.$$ 

Let us estimate the summands in the last equality. First, by using the two-sided bounds $\underline{\eta} \leq \eta \leq \overline{\eta}$ and $\underline{\kappa} \leq \kappa \leq \overline{\kappa}$, we deduce

$$K_1^{-1} (\delta \|v v_x\|^2_{L^2} + \|\theta_x\|^2_{L^2}) \leq \int_\Omega \left[ (1 + 3\delta) \nu \nu v^2 v_x^2 + c v \kappa [\eta, \theta] \rho \theta_x^2 \right] dx,$$

and

$$|I_1| \leq K_2 \|v v_x\|_{L^2} \|\theta_x\|_{L^2} \leq \frac{K_2^2}{4\varepsilon} \|v v_x\|^2_{L^2} + \varepsilon \|\theta_x\|^2_{L^2}, \quad \forall \varepsilon > 0.$$ 

Second, by using the estimates $|p_0 [\eta]| \leq K_3$ and

$$|p [\eta, \theta]| = |p [\eta, \theta_T] + p_1 [\eta] (\theta - \theta_T)| \leq K_4 \left(1 + |\theta - \theta_\Gamma| \right),$$

we have

$$|I_2| \leq K_5 \left[ \int_\Omega (\delta \nu^2 |v_x| + |\nu \theta_x|) dx + \int_\Omega |\theta - \theta_\Gamma| \left( |v_x| + \delta \nu^2 |v_x| + |\nu \theta_x| \right) dx \right] =: K_5 (I_{21} + I_{22}).$$

Furthermore the following estimates hold, for any $\varepsilon > 0$:

$$I_{21} \leq \delta \|v v_x\|_{L^2} \|v\|_{L^2} + \delta \|\theta_x\|_{L^2} \leq \varepsilon \left( \delta \|v v_x\|^2_{L^2} + \|\theta_x\|^2_{L^2} \right) + \frac{\delta^2 + 1}{4\varepsilon} \|v\|^2_{L^2},$$

and

$$I_{22} \leq \|\theta - \theta_\Gamma\|_{L^\infty(\Omega)} \left\| \frac{v_x}{\sqrt{\theta}} \right\|_{L^1(\Omega)} + \|\theta - \theta_\Gamma\|_{L^\infty(\Omega)} \left( \|v v_x\|_{L^\infty(\Omega)} + \|\theta_x\|_{L^\infty(\Omega)} \right)$$

$$\leq \varepsilon \left( \frac{\delta}{2} \|v v_x\|^2_{L^2} + \|\theta_x\|^2_{L^2} \right) + \frac{M K_1 (1)}{2\varepsilon} \left\| \frac{v_x}{\sqrt{\theta}} \right\|_{L^2(\Omega)} + \frac{\delta}{\varepsilon} \|v\|_{L^\infty(\Omega)}^2 \|\theta - \theta_\Gamma\|^2_{L^2(\Omega)}.$$

Third, we obtain

$$|I_3| + |I_4| \leq (\|g\|_{L^1(\Omega)} + p_{\Gamma}) \|v\|_{C_1(\Omega)} M^{1/2} \left(1 + 2\delta\right) \nu \nu v_x + c v \theta_x \|_{L^2}$$

$$\leq \varepsilon \left( \delta \|v v_x\|^2_{L^2} + \|\theta_x\|^2_{L^2} \right) + \frac{K_5 \delta}{\varepsilon} \|v\|^2_{C_1(\Omega)},$$

where $M$ is the maximum value of $\|g\|_{L^1(\Omega)} + p_{\Gamma}$ on $\Omega$. 

where all the above quantities $K_i, 1 \leq i \leq 6$, do not depend on $\delta$ and $\varepsilon$.

Now, by choosing $\varepsilon := K^{-1} T$ small enough and then $\delta := K\varepsilon^{-1}$ large enough, and setting

$$y := \int_{\Omega} \left[ \left( \frac{1}{2} v^2 + c_v (\theta - \theta_T) \right)^2 + \frac{\delta}{2} v^4 \right] \, dx,$$

we get

$$\frac{dy}{dt} + K_9^{-1} \left( \|v v_x\|_{\Omega}^2 + \|\theta - \theta_T\|_{\Omega}^2 \right) \leq K_{10} (\alpha y + h),$$

with $a := \|v\|_{L^\infty(\Omega)}^2$ and $h := \left\| \frac{v}{\sqrt{\eta}} \right\|_{\Omega}^2$ (see (31)); moreover

$$K_{11}^{-1} \left( \frac{1}{2} \|v_x\|_{\Omega}^2 + \|\theta - \theta_T\|_{\Omega}^2 \right) \leq y \leq K_{11} \left( \frac{1}{2} \|v_x\|_{\Omega}^2 + \|\theta - \theta_T\|_{\Omega}^2 \right).$$

It is clear that

$$\frac{dy}{dt} + K_{12}^{-1} y \leq K_{10} (\alpha y + h),$$

with $K_{12} := K_9 K_{11} M^2$. By corollary 3 we have $\|a\|_{L^1(\mathbb{R}^+)} \leq K(1) \|h\|_{L^1(\mathbb{R}^+)} \leq (K(5))^2$, so lemma 6 implies

$$y(t) \leq K_{13} \text{ on } \mathbb{R}^+, \text{ and } y(t) \to 0 \text{ as } t \to \infty.$$

By integrating inequality (36) over $\mathbb{R}^+$, we also obtain

$$K_{14}^{-1} \left( \|v v_x\|_{\Omega}^2 + \|\theta - \theta_T\|_{\Omega}^2 \right) \leq y(0) + K_{10} \left( \|a\|_{L^1(\mathbb{R}^+)} \sup_{\mathbb{R}^+} y + \|h\|_{L^1(\mathbb{R}^+)} \right),$$

so that $\|v v_x\|_{\Omega} + \|\theta - \theta_T\|_{\Omega} \leq K_{14}$, and the lemma is proved. □

Let us now estimate $v_x$ in $L^2(Q)$.

**Lemma 8** The following estimate holds

$$\|v_x\|_{Q} \leq K(7).$$

**Proof:** By taking $L^2(\Omega)$-inner product of the second equation (2) with $v$, we get the equality (compare with (23))

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} v^2 + p_S \eta - p_\eta \theta + p_\eta \theta_T \right) \, dx + \int_{\Omega} \nu \rho v_x^2 \, dx = \int_{\Omega} p_\eta \theta - \theta_T \, v_x \, dx.$$

By integrating it over $(0, T)$ and exploiting the bounds $\eta \leq \eta \leq T$, we get

$$\|v_x\|_{Q_T}^2 \leq K_1 \left( 1 + \|\theta - \theta_T\|_{Q_T} \|v_x\|_{Q_T} \right).$$

So $\|v_x|_{Q_T} \leq K_2^{1/2} + K_1^{1/2} \|\theta - \theta_T|_{Q_T} \leq K_2^{1/2} + K_1 M \|v_x|_{Q_T}$, for any $T > 0$, and the result follows from the previous lemma. □

Now we prove additional properties of $p[\eta, \theta] - p_S$.

**Lemma 9** For $p[\eta, \theta] - p_S$, the following estimate together with stabilization property hold

$$\|p[\eta, \theta] - p_S\|_\Omega \leq K(8),$$

$$\|p[\eta, \theta](\cdot, t) - p_S(\cdot)\|_\Omega \to 0 \text{ as } t \to \infty. \quad (37)$$

$$\|p[\eta, \theta](\cdot, t) - p_S(\cdot)\|_\Omega \to 0 \text{ as } t \to \infty. \quad (38)$$
Proof:

1. Equation (Q1) implies the following equality, for any $T > 0$

$$
\|p[\eta, \theta] - p_S\|_{Q_T}^2 + \|I^* v_t\|_{Q_T}^2 = \|\nu p v_x\|_{Q_T}^2 + 2 \int_{Q_T} (p[\eta, \theta] - p_S) I^* v_t \, dx \, dt.
$$

Elementary transformations and the bounds $\underline{\eta} \leq \eta \leq \overline{\eta}$ give

$$
\int_{Q_T} (p[\eta, \theta] - p_S) I^* v_t \, dx \, dt = \int_{Q_T} (p[\eta, \theta_T] - p_S) I^* v_t \, dx \, dt + \int_{Q_T} p[\eta](\theta - \theta_T) I^* v_t \, dx \, dt
$$

$$
= \int (p[\eta, \theta_T] - p_S) I^* v_t \Big|_0^T - \int_{Q_T} p[\eta, \theta_T] \eta I^* v_t \, dx \, dt + \int_{Q_T} p[\eta](\theta - \theta_T) I^* v_t \, dx \, dt
$$

$$
\leq K_1 (\|v(\cdot, T)\|_{\Omega} + \|v^0\|_{\Omega} + \|v_x\|_{Q_T} \|v\|_{Q_T} + \|\theta - \theta_T\|_{Q_T} \|I^* v_t\|_{Q_T}).
$$

Therefore

$$
\|p[\eta, \theta] - p_S\|_{Q_T}^2 + \frac{1}{2} \|I^* v_t\|_{Q_T}^2 \leq \nu \underline{\eta}^{-2} \|v_x\|_{Q_T}^2 + K_1 (\|v(\cdot, T)\|_{\Omega} + N + M \|v_x\|_{Q_T}^2) + (K_1 M)^2 \|v_x\|_{Q_T}^2,
$$

so estimate (Q3) follows from lemmas (Q1) and (Q2).

2. First, instead of property (Q3), let us prove that

$$
\|p[\eta, \theta_T](\cdot, t) - p_S(\cdot)\|_{\Omega} \to 0 \text{ as } t \to \infty. \tag{39}
$$

By using the estimates $\underline{\eta} \leq \eta$, (Q7), and $\|\theta_x\|_Q \leq K(\theta)$, we have

$$
\|p[\eta, \theta_T] - p_S\|_Q \leq \|p[\eta, \theta] - p_S\|_Q + \|p[\eta]\|_{L^\infty(Q)} \|\theta - \theta_T\|_Q \leq K_2. \tag{40}
$$

Then also

$$
\int_0^\infty \left| \frac{d}{dt} (\|p[\eta, \theta_T] - p_S\|_{\Omega}^2) \right| \, dt = 2 \int_0^\infty \left| \int_{\Omega} p[\eta, \theta_T] \eta (p[\eta, \theta_T] - p_S) \, dx \right| \, dt
$$

$$
\leq 2 \|p[\eta, \theta_T]\|_{L^\infty(Q)} \|v_x\|_Q \|p[\eta, \theta_T] - p_S\|_Q \leq K_3. \tag{41}
$$

Estimates (41) and (41) imply property (Q3).

But by the bounds $\underline{\eta} \leq \eta \leq \overline{\eta}$ the stabilization property (Q3) we get

$$
\left[ \|p[\eta, \theta] - p_S\|_{\Omega}^2 - \|p[\eta, \theta_T] - p_S\|_{\Omega}^2 \right]
$$

$$
\leq 2M^{1/2} (\|p[\eta, \theta_T]\|_{L^\infty(\Omega)} + \overline{\eta} P) + \|p[\eta]\|_{L^\infty(\Omega)} \|\theta - \theta_T\|_\Omega \|p[\eta]\|_{L^\infty(\Omega)} \|\theta - \theta_T\|_\Omega
$$

$$
\leq K_4 (1 + \|\theta - \theta_T\|_\Omega) \|\theta - \theta_T\|_\Omega \to 0 \text{ as } t \to \infty,
$$

so that (39) implies (Q3). □

To establish the pointwise convergence of the specific volume $\eta(x, t)$ as $t \to \infty$, we need a modification of the Ball-Pego lemma (Q8) concerning “almost autonomous” ordinary differential equations.

**Lemma 10** Let $f \in C(\mathbf{R})$ be such that, for a given constant $f_S$, there exists no interval $(z_1, z_2)$ such that $f(z) \equiv f_S$ on $(z_1, z_2)$. Let also $\alpha, \beta \in C(\mathbf{R}^+)$ be two functions such that $\alpha(t) \to 0$ and $\beta(t) \to 0$ as $t \to \infty$, as well as $a \in L^1(\mathbf{R}^+)$. If a function $y(t)$ satisfies $\sup_y |y(t)| < \infty$, $y \in W^{1, 1}(0, T)$ for all $T > 0$, and

$$
\frac{dy}{dt} = f(y + a) - f_S + a + \beta \text{ on } \mathbf{R}^+,
$$

then

$$
y(t) \to y_S \text{ as } t \to \infty, \text{ and } f(y_S) = f_S.
$$
The result remains valid if one sets $\beta = 0$ and replaces the condition $a \in L^1(\mathbb{R}^+)$ by the following ones

$$|a| \leq |a_1| + |\beta_1|, \ a, a_1, \beta_1 \in C(\mathbb{R}^+), \ a_1 \in L^1(\mathbb{R}^+), \mbox{ and } \beta_1(t) \to 0 \mbox{ as } t \to \infty.$$  

**Proof:** We set $A(t) := \int_t^\infty a(\tau) \, d\tau$ and, for $z := y - A$, we get

$$\frac{dz}{dt} = f(z + \tilde{\alpha}) - f_S + \beta, \tag{42}$$

where $\tilde{\alpha} := \alpha + A \in C(\mathbb{R}^+)$ and $\tilde{\alpha}(t) \to 0$ as $t \to \infty$. Note that $z \in C^1(\mathbb{R}^+)$, in virtue of equation (42), and

$$\sup_{\mathbb{R}^+} |z(t)| \leq \sup_{\mathbb{R}^+} |y(t)| + \|a\|_{L^1(\mathbb{R}^+)}.$$

Suppose that $z_1 := \liminf_{t \to \infty} z(t) < z_2 := \limsup_{t \to \infty} z(t)$. Then for any $z_0 \in (z_1, z_2)$, there exist two sequences $\{t_{1k}\}$ and $\{t_{2k}\}$ such that

$$z(t_{1k}) = z(t_{2k}) = z_0, \ \frac{dz}{dt}(t_{1k}) \geq 0, \ \frac{dz}{dt}(t_{2k}) \leq 0.$$  

Equation (42) applied for $t = t_{1k}$ and $t = t_{2k}$ as $k \to \infty$ implies that $f(z_0) - f_S = 0$. So by contradiction with the condition on $f$, $z_1 = z_2 = z := \lim_{t \to \infty} z(t)$.

By integrating equation (42) over the interval $(k-1, k)$ and passing to the limit as $k \to \infty$, we obtain: $f(z) - f_S = 0$. It remains to use the equality $\lim_{t \to \infty} y(t) = \lim_{t \to \infty} z(t)$ to obtain the required result.

To prove the last part of the lemma, it suffices to apply the decomposition $a = \tilde{a} + \beta$, with $\tilde{a} := \frac{a}{|a_1| + \beta_0} |a_1|, \ \beta := \frac{a}{|a_1| + \beta_0} \beta_1$ and $\tilde{\beta}_1(t) := |\beta(t)| + \frac{1}{t + 1}$; here $\tilde{a} \in L^1(\mathbb{R}^+), \ \beta \in C(\mathbb{R}^+)$, and $\tilde{\beta}_1(t) \to 0$ as $t \to \infty$ (since $|\tilde{a}| \leq |a_1|$ and $|\tilde{\beta}| \leq |\beta(t)| + \frac{1}{t + 1}$). $\Box$

**Lemma 11** Let condition (13) be satisfied. Then the following pointwise stabilization property holds for the specific volume $\eta$: there exists a function $\eta_S \in L^\infty(\Omega)$ satisfying (44) such that

$$\eta(x, t) \to \eta_S(x) \mbox{ as } t \to \infty, \mbox{ for all } x \in \overline{\Omega}.$$  

**Proof:** For any fixed $x \in \overline{\Omega}$, we rewrite equation (43) in the following form

$$\frac{dy}{dt} = f(y + \alpha) - ps + p_1[\eta](\theta - \theta_T), \tag{43}$$

with $y := \nu \log \eta - \alpha$, $\alpha := -I^*v$, and $f(z) := p(\exp(\frac{z}{p}), \theta_T)$. Property (13) yields the corresponding property of $f$ in lemma (11), for any $fs = ps$.

By using the bounds $\eta \leq \eta \leq \overline{\eta}$ and the stabilization property (34) we get

$$\sup_{t \geq 0} |y(t)| \leq \nu \max\{|\log \eta|, |\log \overline{\eta}|\} + M^{1/2} \|v\|_{L^{2}\infty(Q)} \leq K_1,$$

$$|\alpha(t)| \leq M^{1/2} \|v(\cdot, t)\|_\Omega \to 0 \mbox{ as } t \to \infty.$$  

We also have, by the Hölder inequality for numbers

$$|p_1[\eta](\theta - \theta_T)| \leq \|p_1[\eta]\|_{L^\infty(Q)} \|\theta - \theta_T\|_{C(\overline{\Omega})} \leq K_2 \|\theta_T\|_{\overline{\Omega}}^{-1/2} \|\theta - \theta_T\|_{\overline{\Omega}}^{1/2} \leq \|\theta_T\|_\Omega^2 + K_2^{4/3} \|\theta - \theta_T\|_{\overline{\Omega}}^{2/3} =: a_1 + \beta_1.$$  

13
The functions \(a(\cdot, t) := p_1[\eta](\cdot, t)(\theta(\cdot, t) - \Theta)\) and \(a_1, b_1\) satisfy the conditions of the final part of lemma 11 by virtue of lemma 8 (together with the properties \(\eta(\cdot, t), \theta(\cdot, t), \|\theta_x(\cdot, t)\|_\Omega \in C(\mathbb{R}^+)\)).

So by condition (13) and lemma 11, there exists
\[
\lim_{t \to \infty} y(t) = y_S, \quad \text{with } f(y_S) = p_S,
\]
i.e. \(\eta(\cdot, t) \to \eta_S(\cdot) = \exp(y_S/\nu)\) as \(t \to \infty\) and \(p(\eta_S(\cdot), \theta_T) = p_S(\cdot)\). The bounds \(\eta \leq \eta \leq \eta\) and the measurability of \(\eta(\cdot, t)\) on \(\Omega\) imply the bounds \(\eta \leq \eta \leq \eta\) and the measurability of \(\eta_S \) on \(\Omega\). □

Note that the Lebesgue dominated convergence theorem immediately gives
\[
\|\eta(\cdot, t) - \eta_S(\cdot)\|_{L^q(\Omega)} \to 0 \quad \text{as } t \to \infty, \quad \text{for any } q \in [1, \infty).
\]

To prove the stabilization for \(v\) in \(L^q(\Omega)\), we turn to the auxiliary linear parabolic problem
\[
\begin{aligned}
\left\{
\begin{array}{l}
u_t = (\mu u_x - \phi)_x + g \quad \text{in } Q, \\
x \big|_{x=0} = 0, \quad (\mu u_x - \phi)|_{x=M} = -p_T(t), \quad u|_{t=0} = u^0(x).
\end{array}
\right.
\end{aligned}
\]

(44)

Suppose that \(\mu \in L^\infty(Q_T)\) and \(\mu_0 \in L^2(Q_T)\) for any \(T > 0\), with \(0 < \mu \leq \mu_0 \in Q\). Suppose also that \(\phi \in L^2,\infty(Q), \ g \in L^1,\infty(Q), \ p_T \in L^\infty(\mathbb{R}_+)\), and that \(v^0 \in H^1(\Omega), \) with \(v^0(0) = 0\).

Set \(\|\|u\|\|_q := \|u\|_{L^q(\Omega)} + \|u\|_{L^\infty(\Omega)}\) to shorten the notation.

**Lemma 12** Let \(u \in H^1(Q_T) \cap L^\infty(Q_T)\) for any \(T > 0\) be a weak solution to problem (44) such that \(\|\|u\|\|_2 < \infty\). Then, for any \(q \in [2, \infty)\), the following estimate together with stabilization property hold
\[
\|\|u\|\|_q \leq C \left[\|u^0\|_{L^q(\Omega)} + q \left(\|\phi\|_{L^2,\infty(Q)} + \|g\|_{L^1,\infty(Q)} + \|p_T\|_{L^\infty(\mathbb{R}_+)} + \||u||_2\right)\right],
\]
\[
\|u(\cdot, t)\|_{L^q(\Omega)} \to 0 \quad \text{as } t \to \infty,
\]
where \(C\) depends only on \(\mu\) and \(M\).

More general assertions of such kind (together with applications to barotropic fluid equations) were given in [23], [25], [24], and the lemma follows from these assertions.

**Lemma 13** Let \(\|v^0\|_{L^q(\Omega)} \leq N, \) for some \(q \in (4, \infty)\). For \(v\), the following estimate together with stabilization property hold
\[
\|\|v\|\|_q \leq qK^{(9)},
\]
\[
\|v(\cdot, t)\|_{L^q(\Omega)} \to 0 \quad \text{as } t \to \infty,
\]
where \(K^{(9)}\) is independent of \(q\).

**Proof:** We consider \(v\) as the solution to problem (14) with given \(\mu := \nu\rho, \ \phi := p[\eta, \theta]\). By the bounds \(\eta \leq \eta \leq \eta\) and lemma 8, the following estimates are valid
\[
K_{\mu}^{-1} \leq \mu,
\]
\[
\|\phi\|_{L^2,\infty(Q)} \leq M^{1/2}\|p[\eta, \theta_T]\|_{L^\infty(Q)} + \|p_1[\eta]\|_{L^\infty(Q)} \|\theta - \theta_T\|_{L^2,\infty(Q)} \leq K_2,
\]
\[
\|v\|_{L^2(\Omega)} \leq K_3,
\]
and the result is proved, by applying the previous lemma 12. □

By collecting all of the results of the above lemmas the proof of theorem 3 is complete. □
3.2 Proof of proposition 1

Note that condition $N^{-1} \leq p_S$ has been used above in lemma 1, but not in lemma 2.

Let us turn to the proof of lemma 1, supposing that in contrast to (16), we have

$$V := \sup_{t \geq 0} V(t) < \infty.$$  \hspace{1cm} (45)

By using the formula $p_S \eta = \varepsilon \eta + (p_S - \varepsilon) \eta$ and the estimate

$$\left| \int (p_S - \varepsilon) \eta dx \right| \leq (|p_T| + \|g\|_{L^1(\Omega)} + \varepsilon) V \forall \varepsilon > 0,$$

we see that lemma 1 remains valid and consequently lemma 3 is also valid. The quantities $K^{(1)} - K^{(3)}$ now depend on $V$ as well.

Consider equation (43). By applying the operator $I_0$ to it and exploiting the bound $\eta \leq \eta$, we get

$$\nu \log \eta \geq \nu \log \eta^0 - I^*(v - v_0) + I_0(p[\eta, \theta_T] - p_S) - K_1 I_0 \max\{\theta_T - \theta, 0\}$$

as $p_1[\eta] \leq K_1$. Let us introduce the set $E_t := \{x \in \Omega : \theta(x, t) \leq \theta_T\}$. Then

$$\|\max\{\theta_T - \theta(\cdot, t), 0\}\|_{C(\Omega)} \leq \|\theta_T(\cdot, t)\|_{L^1(\Omega)} \leq \|\theta_T\theta(\cdot, t)\|_{L^1(E_t)}$$

$$\leq \|\theta_T\theta(\cdot, t)\|_{L^1(\Omega)} \leq \theta_T V^{1/2}\|\theta_T\theta(\cdot, t)\|_{\Omega}.$$  \hspace{1cm} (46)

By using estimates (22) and (48) imply

$$I_0 \max\{\theta_T - \theta(\cdot, t), 0\} \leq \theta_T V^{1/2} K^{(2)} t^{1/2}.$$  \hspace{1cm} (47)

This estimate together with (28) imply

$$\nu \log \eta \geq -\frac{1}{\varepsilon} K_2 - \varepsilon t + I_0(p[\eta, \theta_T] - p_S), \quad \forall \varepsilon \in (0, 1).$$

As $p_S < m(\theta_T)$, for some $x_0$ and for $\varepsilon_0 > 0$ and $\delta > 0$, both small enough, we have

$$p_S(x) \leq m(\theta_T) - \varepsilon_0, \quad \text{for } x \in [x_0, x_0 + \delta] \subset \Omega.$$  \hspace{1cm} (48)

By choosing $\varepsilon := \varepsilon_0/2$, estimate (40) gives

$$\nu \log \eta \geq \frac{1}{2} \varepsilon_0 t - \frac{2}{\varepsilon_0} K_2 \quad \text{on } [x_0, x_0 + \delta] \times \overline{\Omega}.$$  \hspace{1cm} (49)

But then

$$V(t) \geq K_3 \delta \exp\left(\frac{\varepsilon_0}{2\nu} t\right) \to \infty \text{ as } t \to \infty,$$

with $K_3 := \exp\left(-\frac{2}{\nu \varepsilon_0} K_2\right)$, which clearly contradicts (43). \hspace{1cm} $\square$

3.3 Proof of proposition 2

Suppose that in contrast to (17)

$$\sup_{t \geq 0} \left| \int_\Omega v(x, t) dx \right| \leq C_1 < \infty.$$  \hspace{1cm} (47)

Set $\eta_0(t) := \eta(0, t)$, consider equation (27) for $x = 0$ and integrate it in $t$:

$$\nu \log \eta_0(t) = \nu \log \eta^0(0) + \int_\Omega (v^0(x) - v(x, t)) dx + \int_0^t p(\eta_0(\tau), \theta_T) d\tau$$  \hspace{1cm} (48)
as $\theta|_{x=0} = \theta_1$ and $p_S(0) = 0$. It is straightforward that (see (18) and (19))

$$\nu \log \eta^0(0) + \int_{\Omega} (v^0(x) - v(x, t)) \, dx \leq K_1 + C_1. \quad (49)$$

Now set $b(t) := \int_0^t p(\eta_0(\tau), \theta_T) \, d\tau$. As $p(\eta, \theta_T) > m(\theta_T) = 0$, the function $b$ is increasing and positive on $\mathbb{R}^+$. Let us show the property

$$b(t) \to \infty \text{ as } t \to \infty. \quad (50)$$

Indeed if, in contrast to this property, $0 < b(t) \leq C_2$ on $\mathbb{R}^+$, then according to (18) and (19)

$$0 < \eta_0(t) \leq C_3 \text{ on } \mathbb{R}^+. \quad (51)$$

This estimate implies $p(\eta_0(t), \theta_T) \geq \varepsilon_0 > 0$ on $\mathbb{R}^+$ and so $b(t) \geq \varepsilon_0 t$ on $\mathbb{R}^+$. This contradiction proves (50).

Property (18) immediately follows from (18) and (19).

Let us justify the last part of proposition 2. By the conditions on $p_S$ and $p(\eta, \theta_T)$, we can consider

$$0 \leq p_S \eta, -P(\eta, \theta_T) = \int_\eta^\infty p(\zeta, \theta_T) \, d\zeta > 0.$$ 

So if we turn to the proof of lemma 1, we see that it remains valid but only the first summand in (21) should be dropped. In particular $\|v\|_{L^2(Q)} \leq K(1)$, consequently property (17) holds, and by the first part of the proof so does property (18).

### 3.4 Proof of theorem 2

Properties (19) and (20) imply the following estimates

$$-(P(\eta, \theta_T) - P(\hat{\eta}, \theta_T)) = \int_\eta^{\hat{\eta}} (p(\zeta) + p(\zeta, \theta_T)) \, d\zeta \geq p_S(\hat{\eta} - \eta) \quad \text{for } 0 < \eta \leq \hat{\eta},$$

$$P(\eta, \theta_T) - P(\hat{\eta}, \theta_T) = \int_\eta^{\hat{\eta}} (p(\zeta) + p(\zeta, \theta_T)) \, d\zeta \leq p_S(\hat{\eta} - \eta) \quad \text{for } \hat{\eta} \leq \eta.$$ 

Therefore

$$p_S \eta - P(\eta, \theta_T) \geq C := \min\{ -P(\hat{\eta}, \theta_T) + p_S \hat{\eta}, -\max_{\eta \leq \eta \leq \hat{\eta}} P(\eta, \theta_T), -P(\hat{\eta}, \theta_T) + p_S \hat{\eta} \} \quad \text{for all } \eta > 0.$$ 

This means that lemma 1 remains valid but only the first summand in (21) should be dropped.

In order to check the bounds in lemmas 3 and 3, we can use the properties, respectively

$$p(\eta, \theta) - p_S(\eta) \geq 0 \quad \text{for } 0 < \eta \leq \hat{\eta}, 0 < \theta, \text{ and } x \in \overline{\Omega},$$ 

$$p(\eta, \theta) - p_S(\eta) \leq 0 \quad \text{for } \hat{\eta} \leq \eta \leq \eta, 0 < \theta, \text{ and } x \in \overline{\Omega}.$$ 

(see properties (19) and (20)). But by using equation (27), estimate (28), and the remark after lemma 2 (with $N_1 = 0$ and $\varepsilon_0 = 0$), the uniform bounds $\eta \leq \eta(x, t)$ and $\eta(x, t) \leq \overline{\eta}$ in $Q$ hold.

After the bounds $\overline{\eta} \leq \eta \leq \overline{\eta}$, in fact, the rest of the proof of theorem 1 remains unchanged.

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Appendix

This appendix is devoted to the proof of the existence of a regular weak solution to the problem (2), (7), and (8).

Proposition 3 Suppose that either conditions (5), (6), and (10) are valid. Suppose also that \( \kappa \eta \in C(\mathbb{R}^+ \times \mathbb{R}^+) \) and \( \nu, v, \theta \in H^1(\Omega), \ g \in L^2(\Omega) \) with
\[
\| \eta \|_{H^1(\Omega)} + \| v \|_{H^1(\Omega)} + \| \theta \|_{H^1(\Omega)} \leq N, \quad \| g \|_{L^2(\Omega)} \leq N,
\]
\( N^{-1} \leq \eta, \quad N^{-1} \leq \theta, \quad v(0) = 0, \quad \theta(0) = \theta_T. \)

Then for any \( T > 0 \), the problem (5), (6), and (8) admits a unique regular weak solution, and it satisfies the following estimates
\[
\| \eta_x \|_{L^2,\infty(Q_T)} + \| \eta \|_{L^2,\infty(Q_T)} + \| v \|_{H^2,1(Q_T)} + \| \theta \|_{H^2,1(Q_T)} \leq K(10),
\]
\[0 < \underline{\eta} \leq \eta(x, t) \leq \overline{\eta}, \quad 0 < \underline{\theta} := (K(11))^{-1} \leq \theta(x, t) \in \overline{Q_T}.
\]

Hereafter, the quantities \( K_i \) and \( K^{(i)} \) may depend also on \( T \).

Proof: We shall exploit a priori estimates given in theorems 1 and 2 and derive additional estimates in \( Q_T \) in several steps. We shall finish by the proof of a local (in time) existence theorem.

1. We set \( w := \nu(\log \eta)x - v \) and rewrite the second equation (3) as follows
\[w_t = (p_{x0}[\eta] + p_{1\eta}[\eta] \theta) \eta_x + p_{1}[\eta] \theta_x - g .\]

By taking \( L^2(\Omega) \)-inner product with \( w \), using the formula \( \eta_x = \frac{1}{\theta} \eta(w + v) \) and the bounds \( \underline{\eta} \leq \eta \leq \overline{\eta} \), we obtain the inequality
\[
\frac{d}{dt} \| w \|_{\Omega}^2 \leq K_1 \left[ (1 + \| \theta \|_{L^\infty(\Omega)}) (\| w \|_{\Omega}^2 + \| v \|_{\Omega}^2) + \| \theta_x \|_{\Omega}^2 + \| \theta \|_{\Omega}^2 \right].
\]
The estimates \( \| \theta \|_{L^\infty(\Omega)} \leq \theta_T + \sqrt{M} \| \theta_x \|_{\Omega}, \| \theta_x \|_{\Omega} \leq K(6) \), and \( \| \nu(\log \eta \theta)_x - v \|_{\Omega} \leq K_2 \), together with the Gronwall lemma imply the bound \( \| w \|_{L^2,\infty(Q_T)} \leq K_3 \) and therefore
\[
\| \eta_x \|_{L^2,\infty(Q_T)} \leq K(12) .
\]

Consequently, the function \( \rho \) is a Hölder continuous one on \( \overline{Q_T} \).

2. The function \( u := I^* v \) satisfies the nondivergent parabolic problem (see (24) and (3))
\[\begin{cases}
u u = \nu \rho u_x + p[\eta, \theta] - p_S \text{ in } Q, \\
ux|_{x=0} = 0, \quad ux|_{x=M} = 0, \quad u|_{t=0} = I^* v_0(x).
\end{cases}
\]
The standard parabolic \( H^{2,1,2}(Q_T) \)-estimates (15) together with the bounds \( \underline{\eta} \leq \eta \leq \overline{\eta}, \| \theta \|_{L^\infty(Q_T)} \leq K_1 \) lead to the estimate
\[
\| v_x \|_{L^4(Q_T)} = \| u_{xx} \|_{L^4(Q_T)} \leq K_2 \left( \| p[\eta, \theta] - p_S \|_{L^2(Q_T)} + \| v_0 \|_{L^2(\Omega)} \right) \leq K(13).
\]

3. We also consider the second equation (3) as a linear parabolic equation
\[v_t = (\nu \rho v_x - p[\eta, \theta])_x + g ,
\]
with corresponding boundary and initial conditions (see (3) and (8)). After the bounds \( \underline{\eta} \leq \eta \leq \overline{\eta}, \) (3), and (55), we have \( \| \rho x \|_{L^2,\infty(Q_T)} \leq K_1 \) and
\[
\| p[\eta, \theta] \|_{Q_T} \leq K_2 \left[ (1 + \| \theta \|_{L^\infty(Q_T)}) \| \eta_x \|_{L^2,\infty(Q_T)} + \| \theta_x \|_{Q_T} \right] \leq K_3 .
\]
The following formulas hold

$$K[\eta, \theta]_t = K[\eta, \theta]_\tau v_x + A \theta_t, \quad K[\eta, \theta]_x = K[\eta, \theta]_\eta \eta_x + \pi,$$

$$K[\eta, \theta]_{xt} = (K[\eta, \theta]_\eta v_x + K[\eta, \theta]_\theta \eta_t) \eta_x + K[\eta, \theta]_\eta \eta_{xx} + \pi_t.$$
By using the bounds $\eta \leq \eta \leq \eta$ together with $\theta \leq \theta \leq K^{(14)}$ (see (50) and (61)) we have

$$|\mathcal{K}_\eta[\eta, \theta] + |\mathcal{K}_{\eta \theta}[\eta, \theta] + |\mathcal{K}_{\eta \theta}[\eta, \theta]| \leq K_0.$$  

Now from equality (62) it follows that

$$K_1^{-1}\||\theta_t||^2_{Q_T} + \frac{1}{2}\||\pi||^2_{\Omega|_0} \leq K_2 \int_{Q_T} \||\theta_t|| |v_x| + \|\pi| \frac{(|v_x| + |\theta_t|)|\eta_x| + |v_{xx}|)}{+ |F| (|v_x| + |\theta_t|)} \, dx \, dt \leq K_2 \left(\||\theta_t||_{Q_T} \||v_x||_{Q_T} + \||\pi||_{L^\infty(\Omega)} \right) \left(\||v_x||_{Q_T} + \||\theta_t||_{Q_T} \right) \right) \||\eta_x||_{L^2(\Omega)} \right) + \|\pi||v_{xx}||_{Q_T} + \|F||Q_T \||v_x||_{Q_T} + \||\theta_t||_{Q_T} \right) \right).$$

Let us use the estimates $\||v_x||_{Q_T} \leq K^{(6)}$, $\||\pi||_{Q_T} \leq K_3$ as well as (53), (56), and (58), for $\eta_x, v, F$. By applying also the estimate $\||\pi||_{L^\infty(Q_T)} \leq \sqrt{2} \||\pi||^{1/2}_{Q_T} \||\pi||^{1/2}_{Q_T} \leq \sqrt{2} K_3 \||\pi||^{1/2}_{Q_T}$, we get

$$\||\theta_t||_{Q_T} \||v_x||_{Q_T} + \||\pi||_{L^2(\Omega)} \leq K_4 (1 + \||\pi||_{Q_T} + \||\theta_t||_{Q_T} \right).$$

By combining this estimate and the trivial one $\||\pi_x||_{Q_T} \leq cV \||\theta_t||_{Q_T} + \|F||Q_T$ (see (57)), we obtain

$$\||\theta_t||_{Q_T} + \||\pi||_{v_\tau(Q_T)} \leq K_5.$$  

In particular $\||\theta_t||_{L^\infty(Q_T)} \leq K_6$ and $\||\pi||_{L^\infty(Q_T)} \leq \sqrt{M} K_5$.

Therefore by using the formula

$$\theta_{xx} = \left(\Lambda^{-1}\pi\right)_x = \left(\kappa\eta[\eta, \theta]_\eta + \kappa\theta[\eta, \theta]_\theta \right) \pi + \kappa[\eta, \theta] \pi_x,$$

with $\kappa(\eta, \theta) := \frac{\eta}{\kappa(\eta, \theta)_x}$, we also get

$$\||\theta_{xx}||_{Q_T} \leq K_7 \left(\||\eta_x||_{L^\infty(Q_T)} + \||\theta_x||_{L^\infty(Q_T)} \right) \||\pi||_{L^\infty(Q_T)} \leq K_8.$$  

So the estimate $\||\theta||_{H^1(Q_T)} \leq K^{(15)}$ is proved. As a consequence $\||v||_{H^1(Q_T)} \leq K^{(16)}$ (see (56)). This completes the proof of all the a priori estimates (53) and (54).

It is not difficult to verify the uniqueness of a regular weak solution similarly to [4].

7. Now we briefly describe the proof of a local existence theorem. Let us fix the data satisfying the hypotheses and the additional conditions

$$p_0, \eta, \tau, \eta \in C(R^+, \, \eta \in L^2(\Omega), \, v \in L^2(\Omega).$$

We define the Banach space $B_\tau, 0 < \tau \leq T$, of triples $z = (\eta, v, \theta)$ equipped with the norm $\||z||_B = \||z||_{Q_T} + \||z||_{L^\infty(\tau)} + \||\theta||_{Q_T}$ and the bounded closed convex set

$$S_\tau = \{ z \in B_\tau : \||z||_{L^\infty(\tau)} + \||\theta||_{Q_T} \leq N_1, (2N)^{-1} \leq \eta \leq 2c_0 N, (2N)^{-1} \leq \theta \leq 2c_0 N, v|_{\tau=0} = 0 \},$$

where $N_1 > 0$ and $c_0$ is such that $\eta^0 \leq c_0, \eta^0 \leq c_0 N$.

We introduce also the nonlinear operator $A : S_\tau \rightarrow B_\tau$ such that $A(\eta, v, \theta) = (\eta, v, \theta)$, where $\theta$ and $v$ satisfy the linear parabolic equations

$$c V \theta_t = (\kappa \eta, \theta) \theta_\theta) \pi_x + (\nu \pi v_x - p_1 \theta \theta) \pi_x \in Q_T,$$ (64)

$$v_t = (\nu \pi v_x - p \theta \theta) \pi_x + g \in Q_T,$$ (65)

with $\tilde{\rho} = \tilde{\eta}^{-1}$, and $\eta > 0$ satisfies the ordinary differential equation

$$(\nu \log \eta) = p[\eta, \theta] - p_1 - I^* v_t \in Q_T,$$ (66)
together with the boundary conditions

\[ \theta|_{x=0} = \theta_0, \quad (\kappa[\bar{\eta}, \bar{\theta}]\bar{\rho}\theta_x)|_{x=M} = 0, \]

\[ v|_{x=0} = 0, \quad (\nu\bar{\rho}v_x - p[\bar{\eta}, \bar{\theta}]|_{x=M} = -p_0, \]

and the initial conditions \[ \text{for all} \quad \tau \]

Problems (64) and (67); (65) and (68); and (66), with the initial conditions (8), can be solved sequentially. By the linear parabolic equation theory there exist unique solutions \( \theta, v \in H^{2,1}(Q_T) \) to the first and second problems, and they satisfy the estimates

\[ \|\theta\|_{H^{2,1}(Q_T)} \leq K_1 \exp \left( K_2 \| (\kappa[\bar{\eta}, \bar{\theta}]\bar{\rho})_x \|_{L^1(Q_T)} \right) \left( 1 + \|\bar{v}_x\|_{L^2(Q_T)} \right) \leq K_3, \]

\[ \|v\|_{H^{2,1}(Q_T)} \leq K_4 \exp \left( K_5 (1 + \|\bar{v}\|_{L^1(Q_T)}) \right) \left( 1 + \|p[\bar{\eta}, \bar{\theta}]\|_{H^{1}(Q_T)} \right) \leq K_6, \]

compare with above items 3 and 6. Hereafter the quantities \( K_i \) (excluding \( K_1, K_2 \) and \( K_4, K_5 \)) depend also on \( N_1 \).

The following inequalities hold

\[ \|\varphi\|_{L^4(Q_T)} \leq c(M, T) \tau^{1/12} \|\varphi\|_{V_2(Q_T)}, \quad \forall \varphi \in V_2(Q_T), \]

\[ \|\varphi - \varphi_{t=0}\|_{C(T_0^\infty)} \leq c_2(M) \tau^{1/4} \|\varphi\|_{H^{2,1}(Q_T)} \quad \forall \varphi \in H^{2,1}(Q_T) \]

which follow from the Hölder inequality, the embedding \( V_2(Q_T) \subset L^6(Q_T) \), and the inequality \( \|\phi\|_{C(T_0^\infty)} \leq c_3(M) \|\phi\|_{H^1(T_0^\infty)}^{1/2} \|\phi\|_{H^1(T_0^\infty)}^{1/2} \). Thus, for \( 0 < \tau \leq \tau_1 \) small enough,

\[ \|\theta_x\|_{L^4(Q_T)} + \|v_x\|_{L^4(Q_T)} \leq N_1/2, \quad (2N)^{-1} \leq \theta \leq 2c_0 N \quad \text{in} \quad \overline{Q}_T. \]

We rewrite the problem for \( \eta \) as the integral equation

\[ \nu \log \eta = \nu \log \eta^0 + I_0(p[\eta, \theta] - p_0) - I^*(v - v^0). \]

For \( 0 < \tau \leq \tau_2 \) small enough, this equation has a unique solution \( \eta \in C(T_0^\infty) \), \( \eta > 0 \), and it satisfies the bounds

\[ (2N)^{-1} \leq \eta \leq 2c_0 N \quad \text{in} \quad \overline{Q}_T. \]

Moreover, from (66) and (74) it follows that \( \eta_t \in V_2(Q_T) \), \( \eta \in H^{2,1}(Q_T) \), and

\[ \|\eta_t\|_{V_2(Q_T)} \leq K_7, \quad \|\eta_t\|_{L^2(\infty, Q_T)} \leq K_8, \quad \|\eta_{xx}\|_{L^2(\infty, Q_T)} \leq K_9 \]

for the last estimate we use conditions (68). So by applying estimate (71), for \( 0 < \tau \leq \tau_3 \) small enough,

\[ \|\eta_t\|_{L^4(Q_T)} + \|\eta_t\|_{Q_T} \leq N_1/2. \]

In addition, the following estimate holds

\[ \sup_{0 < \tau < \tau} \gamma^{-1/2} \|\Delta \gamma\eta_t\|_{Q_{\tau}} \leq K_{10} \]

with \( \Delta \gamma \varphi(x, t) = \varphi(x, t + \gamma) - \varphi(x, t) \). This estimate is valid in virtue of the equation

\[ \nu(\log \eta)_t = p[\eta, \theta] - p[\bar{\eta}, \bar{\theta}] + \nu \bar{\rho}v_x \]

(where equation (53) is used) and the known estimate \( \sup_{0 < \tau < \tau} \gamma^{-1/2} \|\Delta \gamma \varphi_x\|_{Q_{\tau}} \leq c_4(M, T) \|\varphi\|_{H^{2,1}(Q_T)} \) for all \( \varphi \in H^{2,1}(Q_T) \).

Thus, for \( \tau = \min \{ \tau_1, \tau_3 \} \), the operator \( A \) is well defined and \( A(S_{\varphi}) \subset S_{\varphi} \), see (73), (75), and (77). Moreover estimates (60), (70), (73), and (77) imply that the set \( A(S_{\varphi}) \) is precompact in \( B_{\varphi} \).
To prove the continuity of $A$, take a sequence $\{\tilde{z}_n\} \subset S$, $\|\tilde{z}_n - \tilde{z}\|_{B} \to 0$ as $n \to \infty$ and set $z_n = (\eta_n, v_n, \theta_n) := Az_n$ and $z = (\eta, v, \theta) := A\tilde{z}$. By considering problems for $\theta - \theta_n$ and $v - v_n$, applying the standard parabolic energy estimate and estimates (69), (70), we obtain

$$
\|\theta - \theta_n\|_{V^2(Q)} \leq K_{11} \|\tilde{z} - \tilde{z}_n\|_{B} \to 0,
$$
$$
\|v - v_n\|_{V^2(Q)} \leq K_{12} (\|\tilde{z} - \tilde{z}_n\|_{B} + \|\theta - \theta_n\|_{Q}) \to 0.
$$

Considering the difference of equation (74) for $\eta$ and the similar one for $\eta_n$, we also obtain

$$
\|\eta - \eta_n\|_{L^2,Q} \leq K_{13} (\|\theta - \theta_n\|_{Q} + \|v - v_n\|_{L^2,Q}) \to 0.
$$

As the set $A(S)$ is precompact, the last three limiting properties imply that $\|z - z_n\|_{B} \to 0$.

Combining all the properties of $S$ and $A$, by the classical Schauder theorem, we establish that $A$ has a fixed point in $S$. Evidently this fixed point serves as a regular weak solution to the original problem.

Remark:
In the case $\kappa = \kappa(\eta)$, the existence of $\kappa_{\eta \eta} \in C(R^+)$ is not required and the proof can be simplified in an essential manner. Namely, the standard parabolic $H^{2,1}(Q_T)$-estimates imply $\|\theta\|_{H^{2,1}(Q_T)} \leq K^{15}$ in step 3, and estimate (79) in step 4 together with the main part of step 6 can be omitted.

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