Nonlinear two-dimensional frequency- and temperature-dependent vortex dynamics in a tilted washboard pinning potential

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Abstract. The Langevin equation for a two-dimensional nonlinear guided vortex motion in a tilted cosine pinning potential in the presence of an ac current is exactly solved in terms of a matrix continued fraction at arbitrary value of the Hall effect. The influence of an ac current of arbitrary amplitude and frequency on the dc and ac magnetoresistivity tensors is theoretically considered. Experimental realization of this model in thin-film geometry opens up a possibility for a variety of experimental studies of directed motion of vortices under (dc + ac)-driving simply by measuring longitudinal and transverse voltages.

1. Introduction

As the pinning force in a washboard planar pinning potential (WPPP) is directed perpendicular to the washboard channels, the vortices generally tend to move along these channels. Such a guided motion of vortices in the presence of the Hall effect produces anisotropic transport behaviour for which even (+) and odd (−) (with respect to the magnetic field reversal) longitudinal (∥) and transverse (⊥) dc nonlinear magnetoresistivities \( \rho_\parallel, \rho_\perp \) depend nontrivially on the angle \( \alpha \) between the dc current density vector \( j \) and the direction of the WPPP channels ("guiding direction"). The dc-current nonlinear guiding problem recently has been extensively studied both theoretically [1-3] and experimentally [4-6]; it was exactly solved for the washboard WPPP within the framework of the two-dimensional (2D) single-vortex stochastic model of anisotropic pinning based on the Fokker-Planck equation and rather simple formulas were derived for the dc magnetoresistivities \( \rho_\parallel, \rho_\perp \) [2,3].

On the other hand, in contrast to the dc current case, the temperature-dependent ac-driven vortex motion in theory has been exactly solved only for one-dimensional (1D) nontilted cosine pinning potential at a small oscillation amplitude of the vortices [7-8]. At the same time it is evident that the examination of a strong nonlinear ac-driven response of the vortices in the presence of arbitrary dc bias will be interesting both for theory and for different high-frequency or microwave applications.

In the following we study 2D vortex dynamics in the presence of (dc + ac)-driven Lorentz forces of arbitrary magnitude. For the exact solution of this problem we used the matrix continued fraction technique earlier suggested and later extensively employed for calculation of 1D nonlinear (ac + dc)-driven response of overdamped Josephson junction with noise in [9,10].
As a result, two groups of new findings were obtained. First, for the previously solved in [1,2] 2D dc-problem the influence of ac current on the overall shape and appearance of the Shapiro steps on the anisotropic dc $\rho_{||,\perp}^+\,$- CVC’s was calculated and analyzed. Second, for the ac current at a frequency $\omega$ plus dc bias the 2D nonlinear time-dependent stationary $\rho_{||,\perp}^{ac\pm}\,$ ac-response on the frequency $\omega$ in terms of nonlinear impedance tensor $\hat{Z}$ and a nonlinear ac response at $\omega$-harmonics was studied. Below we present a short derivation of main analytical results that allows to consider these findings in the future publications.

2. Formulation of the problem

The Langevin equation for a vortex moving with velocity $v$ in a magnetic field $B = nB \ (B \equiv |B|, \ n = nz, \ z$ is the unit vector in the $z$ direction and $n = \pm 1)$ has the form

$$\eta \dot{v} + n\alpha_H v \times z = F_L + F_p + F_{th}, \tag{1}$$

where $F_L = n(\Phi_0/c)j \times z$ is the Lorentz force ($\Phi_0$ is the magnetic flux quantum, $c$ is the speed of light), $j = j(t) = j^{dc} + j^{ac} \cos \omega t$, where $j^{dc}$ and $j^{ac}$ are the dc and ac current density amplitudes and $\omega$ is the angular frequency, $F_{th}$ is the thermal fluctuation force represented by a Gaussian white noise with zero mean [1,2,7,8], $\eta$ is the vortex viscosity, $\alpha_H$ is the Hall constant, $F_p = -\nabla U_p(x)$ is the anisotropic pinning force and we assume, as usual [1,7] an $a$-periodic pinning potential of the form $U_p(x) = (U_p/2)(1-\cos kx)$ where $k = 2\pi/a$.

The main quantity of physical interest in our problem is the average electric field, induced by the moving vortex system, which is given by

$$\langle E \rangle = \langle n/c \rangle B \times \langle v \rangle = n(B/c)(-\langle v_y \rangle x + \langle v_x \rangle y), \tag{2}$$

where $x$ and $y$ are the unit vectors in the $x$ and $y$ directions, respectively. As follows from Eq. (1) $\langle v_y \rangle = F_{ly}/\eta + \delta \langle v_x \rangle$ where $\delta \equiv ne, \ \epsilon \equiv \alpha_H/\eta$, and so for determination of $\langle E \rangle$ from Eq. (2) it is sufficient to calculate the $\langle v_x \rangle$ from Eq. (1). This calculation gives

$$\langle v_x \rangle(t) = (\Phi_0 j_0/\epsilon \eta D)[j^{dc} + j^{ac} \cos \omega t - \langle \sin x \rangle(t)], \tag{3}$$

where $D = 1 + \delta^2, \ j^{dc} = n(j_y^{dc} + \delta j_x^{dc})/j_c, \ j^{ac} = n(j_y^{ac} + \delta j_x^{ac})/j_c, \ j_c = eF_p/\Phi_0, \ j_y = j \cos \alpha, \ j_x = j \sin \alpha, \ (\sin x \langle t \rangle = (i/2)[(r(t) - r^{-1}(t)], \ r(t) = \exp[-ix(t)], \ \text{and} \ x = kx$.

Looking for only the stationary ac response, which is independent of the initial condition, one may seek all the $\langle r^m \rangle(t)$ in the form [9]

$$\langle r^m \rangle(t) = \sum_{k=-\infty}^{\infty} F_k^m(\omega)e^{ik\omega t}. \tag{4}$$

On substituting Eq. (4) into Eq. (3) we obtain recurrence equations for the $(j^{ac}, j^{dc}, \Omega, g)$-dependent Fourier amplitudes $F_k^m$, where $g = (U_p/2T)$, $T$ is the temperature in energy units, $\Omega = \omega \bar{\tau}$ is the dimensionless frequency and $\bar{\tau} \equiv \eta D/kF_p$ is the relaxation time. From the solution of these recurrence equations in terms of matrix continued fraction (see details in Ref. [9]) we can find the dimensionless average pinning force $\langle F_{px} \rangle(t)$ which is the main anisotropic nonlinear component of our theory

$$\langle \dot{F}_{px} \rangle(t) = -\langle \sin x \rangle(t) = \sum_{k=0}^{\infty} \text{Im}(\psi_k e^{ik\omega t}), \tag{5}$$

where $\psi_0 \equiv F_0^1(\omega)$ and for $k \geq 1$ we have $\psi_k \equiv F_k^1(\omega) - F_k^{-1}(\omega)$. From Eq. (5) we can decompose average pinning force $\langle \dot{F}_{px} \rangle(t)$ into three components

$$\langle \dot{F}_{px} \rangle(t) = \langle \dot{F}_{px} \rangle_0^1 + \langle \dot{F}_{px} \rangle_{t1} + \langle \dot{F}_{px} \rangle_{t2}^1. \tag{6}$$
In Eq. (6) \( \langle \hat{F}_{px} \rangle_0^\nu \equiv -\langle \sin x \rangle_0^\nu = \text{Im} \psi_0 \) is the time independent (but frequency dependent) static average pinning force; \( \langle \hat{F}_{px} \rangle_t^{11} \equiv -\langle \sin x \rangle_t^{11} = \text{Im}(\psi_1 e^{i\omega t}) \) is the time-dependent dynamic average pinning force; \( \langle \hat{F}_{px} \rangle_t^{k>1} \equiv -\langle \sin x \rangle_t^{k>1} = \text{Im}(\psi_k e^{ik\omega t}) \) describes a contribution of the harmonics with \( k > 1 \) into the dynamic average pinning force.

3. \( \Omega \)-dependent nonlinear dc resistivity response
In order to derive the \( \omega \)-dependent nonlinear dc resistivity and conductivity tensors we first express (see Eq. (2)) the time independent part of \( \langle E_y \rangle(t) \) and \( \langle E_z \rangle(t) \) as

\[
\langle E_y \rangle_0^\nu = (\rho_f / D) \nu_0^{dc} (j_y^{dc} + \delta j_x^{dc}),
\]

\[
\langle E_z \rangle_0^\nu = (\rho_f / D) [j_x^{dc} (1 + \delta^2 (1 - \nu_0^{dc})) - \delta j_y^{dc}].
\]

Here \( \rho_f \equiv B \Phi_0 / \eta c^2 \) is the flux-flow resistivity, \( \nu_0^{dc} \equiv 1 - \langle \sin x \rangle_0^{dc} / j^{dc} = 1 + \langle \hat{F}_{px} \rangle_0^\nu / j^{dc} \), where \( \nu_0^{dc} \) at \( j^{ac} = 0 \) can be considered as the probability of vortex hopping over the pinning potential barrier under the influence of the dimensionless generalized moving force \( \hat{F}_{Lx} = j^{dc} \) in the \( x \) direction [2].

From Eqs. (7) we find the magnetoresistivity tensor for the dc nonlinear law \( \langle \mathbf{E}_{0}^\omega (\omega) \rangle = \hat{\rho}_0^{\omega \nu dc} \) as

\[
\hat{\rho}_0^{\omega \nu dc} = \begin{pmatrix}
\rho_{xx}^{dc} & \rho_{xy}^{dc} & \rho_{yz}^{dc} \\
\rho_{yx}^{dc} & \rho_{yy}^{dc} & \rho_{yz}^{dc} \\
\rho_{zx}^{dc} & \rho_{zy}^{dc} & \rho_{zz}^{dc}
\end{pmatrix} = \frac{\rho_f}{D} \begin{pmatrix} D - \delta^2 \nu_0^{\omega} & -\delta \nu_0^{\omega} & -\delta \nu_0^{\omega} \\
\delta \nu_0^{\omega} & -\delta \nu_0^{\omega} & -\delta \nu_0^{\omega} \\
\delta \nu_0^{\omega} & -\delta \nu_0^{\omega} & -\delta \nu_0^{\omega}
\end{pmatrix}.
\]

The experimentally observable longitudinal and transverse (with respect to the \( j \) direction) dc magnetoresistivities \( \rho_{||} = E_{||} / j \) and \( \rho_{\perp} = E_{\perp} / j \) (where \( j \) is the dc current density \( j^2 = j_x^2 + j_y^2 \)) have the form

\[
\rho_{||} = (\rho_f / D) [(D - \delta^2 \nu_0^{\omega}) \sin^2 \alpha + \nu_0^{\omega} \cos^2 \alpha],
\]

\[
\rho_{\perp} = (\rho_f / D) [\delta \nu_0^{\omega} - D(1 - \nu_0^{\omega}) \cos \alpha \sin \alpha],
\]

where in order to separate the even and odd components of \( \rho_{||,\perp} \) we should use \( \nu_0^{\omega \pm}(n) = [\nu_0^{\omega}(n) \pm \nu_0^{\omega}(-n)]/2 \) which are the even and odd components (relative to the magnetic field inversion) of the function \( \nu_0^{\omega}(n) \) (compare with Eqs. (13)-(14) in Ref. [3]).

4. Nonlinear stationary ac response
Using Eq. (2) we determine nonlinear stationary ac response as

\[
\langle \mathbf{E} \rangle_t \equiv \langle \langle \mathbf{E} \rangle(t) - \langle \mathbf{E} \rangle_0^\nu \rangle = (nB/c) \{ v_x \mathbf{y} - \langle v_y \rangle \mathbf{x} \},
\]

where \( \langle \mathbf{E} \rangle_0^\nu \) is the time-independent part of \( \langle \mathbf{E} \rangle(t) \) (see also Eqs. (7)), whereas \( \langle v_y \rangle_t \) and \( \langle v_x \rangle_t \) are time-dependent periodic parts of \( \langle v_y \rangle(t) \) and \( \langle v_x \rangle(t) \). From Eqs. (10) we have

\[
\langle E_y \rangle_t = (n \rho_f j_c / D) \sum_{k=1}^{\infty} (j^{ac} k)^k \text{Re} \{ Z_k(\omega) e^{ik\omega t} \},
\]

where \( Z_k(\omega) = \delta_{1,k} - i \psi_k(\omega) / (j^{ac})^k \) and \( \delta_{1,k} \) is Kronecker’s delta. The dimensionless transformation coefficients \( Z_k \) in Eq. (11) have a physical meaning of the \( k \)-th harmonic with frequency \( \Omega_k \equiv k \omega \) in the \( ac \) nonlinear \( \langle E_y \rangle_t \) response. \( Z_k(\omega) \) for \( k = 1 \) yields the dimensionless nonlinear impedance \( Z_1 \) as

\[
Z_1 = 1 - i \psi_1 / j^{ac} = \rho_1 - i \zeta_1
\]
where $\rho_1$ and $\zeta_1$ are the dynamic resistivity and the reactivity, respectively. Now using Eqs. (10) and (11) we can express the nonlinear stationary ac responses $E_{y1}^{ac}$ and $E_{x1}^{ac}$ on the frequency $\omega$ in terms of $Z_1$ as

$$E_{y1}^{ac} = (\rho_f/D)(\delta_{x}^{ac} + j\delta_{y}^{ac})\text{Re}\{Z_1 e^{i\omega t}\},$$

$$E_{x1}^{ac} = (\rho_f/D)\text{Re}\{e^{i\omega t}[(D - \delta^2 Z_1)j_{x}^{ac} - \delta Z_1 j_{y}^{ac}]\}. \quad (13)$$

From Eqs. (13) follows that the complex amplitudes of the electric field $E_1$ and the current density $J = j^{ac} e^{i\omega t}$ are connected by the relation $E_1 = \tilde{Z} J$, where $\tilde{Z}$ is the frequency-dependent nonlinear impedance tensor

$$\tilde{Z}(\omega) = \left( \begin{array}{cc} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{array} \right) = \frac{\rho_f}{D} \left( \begin{array}{cc} D - \delta^2 Z_1 & -\delta Z_1 \\ \delta Z_1 & Z_1 \end{array} \right). \quad (14)$$

It is relevant to remark the similarity of Eq. (8) and Eq. (14) from which follows that for the ac response $Z_1$ plays the same role as $\nu_0^s$ for the dc response. It is easy to show that $\langle F_{xx}\rangle_t = \text{Re}\{e^{i\omega t}[j^{ac}(Z_1 - 1)]\}$ which gives a physical interpretation of the $Z_1$ impedance.

The real quantities $E_1 = \text{Re}E_1$ and $j^{ac} = \text{Re}J$ are connected by the relation $E_1 = \hat{\rho}^{ac} J$, where the ac-response resistivity tensor is

$$\hat{\rho}^{ac} = \left( \begin{array}{cc} \rho_{xx}^{ac} & \rho_{xy}^{ac} \\ \rho_{yx}^{ac} & \rho_{yy}^{ac} \end{array} \right) = \text{Re}\{\tilde{Z}(\omega)e^{i\omega t}\}. \quad (15)$$

The longitudinal and transverse (with respect to the ac current direction which is the same, for simplicity, as the dc current direction) ac magnetoresistivities are

$$\rho_{||}^{ac} = \text{Re}\{Z_{||} e^{i\omega t}\}, \quad \rho_{\perp}^{ac} = \text{Re}\{Z_{\perp} e^{i\omega t}\}, \quad (16)$$

where the equations for $Z_{||}$ and $Z_{\perp}$ are similar to Eqs. (9) with the only difference in a change of $\nu_0^s$ for $Z_1$ and $\rho_{||,\perp}$ for $Z_{||,\perp}$; note also that for $\alpha \neq 0, \pi/2$ even and odd components of $Z_1$ and $Z_{||,\perp}$ should be considered.

In order to calculate the power absorbed per unit volume $\mathcal{P}$ (and averaged over the period of an ac cycle) we use the standard relation $\mathcal{P} = (1/2)\text{Re}\{E_1 \cdot J\}$ where $E_1$ and $J$ are the complex amplitudes of the ac electric field and current density, respectively. Then we can show that

$$\mathcal{P} = (j^2/2)\bar{\rho} \equiv (j^2/2)\text{Re}Z_{||}, \quad \bar{\rho} = (\rho_f/D)[D \sin^2 \alpha + (1 - D \sin^2 \alpha)\text{Re}Z_1].$$

In conclusion, the considered exactly solvable two-dimensional model of the vortex dynamics is of great interest since a very rich physics is expected from combination of a strong dc and ac driving, arbitrary value of the Hall effect, and the low temperature-mediated vortex hopping (or running) in a washboard pinning potential. Experimental realization of this model in thin-film HTSCs (which demonstrate WPPP in $c$-direction) opens up a possibility for a variety of experimental studies of directed motion of vortices under (dc + ac)-driving simply by measuring longitudinal and transverse voltages.

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