Linear Quadratic Stochastic Optimal Control Problems with Operator Coefficients: Open-Loop Solutions

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Abstract: An optimal control problem is considered for linear stochastic differential equations with quadratic cost functional. The coefficients of the state equation and the weights in the cost functional are bounded operators on the spaces of square integrable random variables. The main motivation of our study is linear quadratic optimal control problems for mean-field stochastic differential equations. Open-loop solvability of the problem is investigated, which is characterized as the solvability of a system of linear coupled forward-backward stochastic differential equations (FBSDE, for short) with operator coefficients. Under proper conditions, the well-posedness of such an FBSDE is established, which leads to the existence of an open-loop optimal control. Finally, as an application of our main results, a general mean-field linear quadratic control problem in the open-loop case is solved.

Keywords: linear stochastic differential equation with operator coefficients, open-loop solvability, forward-backward stochastic differential equations, mean-field linear quadratic control problem.

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1 Introduction

Let \((\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})\) be a complete filtered probability space on which a standard one-dimensional Brownian motion \(\{W(t), t \geq 0\}\) is defined such that \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\) is the natural filtration of \(W(\cdot)\) augmented by all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\). Consider the following controlled linear (forward) stochastic differential equation (FSDE, for short) on \([t,T]\\):

\[
\begin{aligned}
    dX(s) &= [\mathcal{A}(s)X(s) + \mathcal{B}(s)u(s) + b(s)]ds + [\mathcal{C}(s)X(s) + \mathcal{D}(s)u(s) + \sigma(s)]dW(s), \quad s \in [t,T], \\
    X(t) &= x.
\end{aligned}
\]

In the above, \(X(\cdot)\) is called the state process taking values in the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\\); \(u(\cdot)\) is called the control process taking values in \(\mathbb{R}^m\\); \((t,x)\) is called an initial pair with \(t \in [0,T)\) and \(x\) being a square integrable \(\mathbb{R}^n\)-valued \(\mathcal{F}_t\)-measurable random variable; \(b(\cdot)\) and \(\sigma(\cdot)\) are called non-homogeneous terms.

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To explain the coefficients of the system, we first recall the following spaces: For any $t \in [0, T],$
\[
L_{F_t}^2(\Omega; \mathbb{R}^n) = \{ \xi : \Omega \rightarrow \mathbb{R}^n \mid \xi \text{ is } F_t\text{-measurable and } \|\xi\|_2 \equiv \left(\mathbb{E}||\xi|^2\right)^{\frac{1}{2}} < \infty \}, \quad L^2(\Omega; \mathbb{R}^n) = L_{F_T}^2(\Omega; \mathbb{R}^n),
\]

$L_{F_t}^2(t, T; \mathbb{R}^n) = \{ \varphi : [t, T] \times \Omega \rightarrow \mathbb{R}^n \mid \varphi(\cdot) \text{ is } F\text{-progressively measurable, } \mathbb{E}\int_t^T |\varphi(s)|^2 ds < \infty \},$

$L_{F_t}(\Omega; L^1([t, T]; \mathbb{R}^n)) = \{ \varphi : [t, T] \times \Omega \rightarrow \mathbb{R}^n \mid \varphi(\cdot) \text{ is } F\text{-progressively measurable, } \left\| \int_t^T |\varphi(r)| dr \right\|_2 < \infty \},$

$L_{F_t}^2(\Omega; C([t, T]; \mathbb{R}^n)) = \{ \varphi : [t, T] \times \Omega \rightarrow \mathbb{R}^n \mid \varphi(\cdot) \text{ is } F\text{-progressively measurable, } s \mapsto \varphi(s, \omega) \text{ is continuous almost surely, } \mathbb{E}\left[ \sup_{s \in [t, T]} |\varphi(s)|^2 \right] < \infty \}.$

For any Banach spaces $X$ and $Y,$ we let $\mathcal{L}(X; Y)$ be the set of all linear bounded operators from $X$ to $Y,$ and denote $\mathcal{L}(X; X) = \mathcal{L}(X).$ Also, when $X$ is a Hilbert space, we let $\mathcal{S}(X)$ be the set of all bounded self-adjoint operators on $X.$ In the state equation (1.1), we assume that

\[
(1.2) \quad \mathcal{A}(s), \mathcal{C}(s) \in \mathcal{L}\left(L_{F_t}^2(\Omega; \mathbb{R}^n)\right), \quad \mathcal{B}(s), \mathcal{D}(s) \in \mathcal{L}\left(L_{F_t}^2(\Omega; \mathbb{R}^m); L_{F_t}^2(\Omega; \mathbb{R}^n)\right), \quad \forall s \in [0, T].
\]

More precisely, for any $\xi \in L_{F_t}^2(\Omega; \mathbb{R}^n),$ $\mathcal{A}(s)\xi \in L_{F_t}^2(\Omega; \mathbb{R}^n),$ with
\[
\left(\mathbb{E}|\mathcal{A}(s)\xi|^2\right)^{\frac{1}{2}} \equiv \|\mathcal{A}(s)\xi\|_2 \leq \|\mathcal{A}(s)\| \|\xi\|_2,
\]
where
\[
\|\mathcal{A}(s)\| = \sup \left\{ \|\mathcal{A}(s)\xi\|_2 \mid \xi \in L_{F_t}^2(\Omega; \mathbb{R}^n), \|\xi\|_2 = 1 \right\}.
\]

Some additional conditions will be assumed for $\|\mathcal{A}(\cdot)\|,$ $\|\mathcal{C}(\cdot)\|,$ $\|\mathcal{B}(\cdot)\|$ and $\|\mathcal{D}(\cdot)\|$ later. In what follows, the set of all initial pairs is denoted by
\[
\mathcal{D} = \left\{ (t, x) \mid t \in [0, T], \ x \in L_{F_t}^2(\Omega; \mathbb{R}^n) \right\},
\]
and the set of all admissible controls on $[t, T]$ is denoted by $\mathcal{U}[t, T] = L_{F_t}^2(t, T; \mathbb{R}^m).$

One can show that under certain conditions, for any initial pair $(t, x) \in \mathcal{D}$ and control $u(\cdot) \in \mathcal{U}[t, T],$ the state equation (1.1) admits a unique strong solution $X(\cdot) \equiv X(\cdot; t, x, u(\cdot)) \in L_{F_t}^2(\Omega; C([t, T]; \mathbb{R}^n)).$ The performance of the control process is measured by the following cost functional:

\[
J(t, x; u(\cdot)) = \mathbb{E}\left[ (\mathcal{G}X(T), X(T)) + 2(g, X(T)) + \int_t^T \left( (\mathcal{Q}(s)X(s), X(s)) + 2(\mathcal{S}(s)X(s), u(s)) + (\mathcal{R}(s)u(s), u(s)) \right) ds \right],
\]

where

\[
(1.3) \quad \mathcal{G} \in \mathcal{S}\left(L_{F_t}^2(\Omega; \mathbb{R}^n)\right), \quad \mathcal{Q}(s) \in \mathcal{S}\left(L_{F_t}^2(\Omega; \mathbb{R}^n)\right), \quad \mathcal{S}(s) \in \mathcal{S}\left(L_{F_t}^2(\Omega; \mathbb{R}^m); L_{F_t}^2(\Omega; \mathbb{R}^n)\right), \quad \mathcal{R}(s) \in \mathcal{S}\left(L_{F_t}^2(\Omega; \mathbb{R}^m)\right), \quad \forall s \in [0, T],
\]

with certain additional conditions, and $g \in L^2(\Omega; \mathbb{R}^n),$ $q(\cdot) \in L_{F_t}^2(\Omega; L^1(0, T; \mathbb{R}^n)),$ $\rho(\cdot) \in L_{F_t}^2(0, T; \mathbb{R}^m).$

Our optimal control problem can be stated as follows.

**Problem (OLQ).** For given $(t, x) \in \mathcal{D},$ find a $\bar{u}(\cdot) \in \mathcal{U}[t, T],$ called an open-loop optimal control, such that

\[
J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)).
\]

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The above Problem (OLQ) clearly includes the classical stochastic linear quadratic (LQ, for short) optimal control problem for which all the coefficients and quadratic weighting operators in the cost functional are matrix-valued processes \([26, 21, 22]\). On the other hand, by allowing the coefficients of the state equation and the quadratic weighting operators in the cost functional to be linear bounded operators between Hilbert spaces of square integrable random variables, our problem will cover stochastic LQ optimal control problem for mean-field forward stochastic differential equations (FSDEs, for short) with cost functionals also involving mean-field terms (such problems are referred to as MF-LQ problems). In \([24]\), for a simple MF-LQ problem (with deterministic coefficients), under proper conditions, optimal control is obtained via the solution to a system of Riccati equations. See \([9, 11, 25]\) for some follow-up works.

There are a plenty of literatures on the classical LQ optimal control problems and two-person differential games. See \([1, 3, 5, 7, 12, 14, 17, 18, 19, 20, 23, 31]\) for deterministic coefficient cases, and \([4, 6, 10, 21, 22, 29]\) for random coefficient cases. In \([18]\) (see also, \([19, 20]\)) open-loop and closed-loop solvabilities/saddle points were introduced, and the following interesting equivalent relations were established for LQ optimal control problems with deterministic coefficients: The open-loop solvability of the LQ problem is equivalent to the solvability of an FBSDE, and the closed-loop solvability of the LQ problem is equivalent to the solvability of the corresponding Riccati equation. For two-person differential games, similar results are also valid. In the current paper, we focus on the open-loop solvability of our Problem (OLQ). The studies of closed-loop case and differential game problems will be carried out in our future publications. For the solvability of FBSDEs or Riccati equations arising in the classical LQ stochastic optimal control problems and stochastic differential game problems, one is referred to \([16, 8, 28, 30]\). We may regard the current work as a continuation of \([24, 9, 11]\) and \([18, 19, 20]\).

Several contributions have been made in this work. Firstly, all the involved coefficients are operator-valued processes or operator-valued variables. The appearance of the operator coefficients in the state equation and the cost functional prompts us to develop some new methods and techniques. Actually, our results on the FSDEs and backward stochastic differential equations (BSDEs, for short) with operator coefficients are of independent interests themselves. Secondly, we establish the equivalence between the open-loop solvability of Problem (OLQ) and the well-posedness of a coupled FBSDE with operator coefficients. Thirdly, under some conditions, the well-posedness of the relevant FBSDE with operator coefficients is established by the method of continuation. Fourthly, as an application of our general abstract results, we present the solution to the mean-field LQ problem (which is a major motivation of the current work). The open-loop optimal control for the MF-LQ control problem is characterized by the solution of a Fredholm type integral equation of the second kind.

The rest of this paper is organized as follows. Some motivations of Problem (OLQ) are carefully presented in Section 2. We also develop some general results for FSDEs and BSDEs with operator coefficients. Section 3 is concerned with Problem (OLQ). Open-loop optimal controls are characterized, and the solvability of the relevant coupled FBSDEs with operator coefficients is established by the method of continuation. In Section 4, an MF-LQ optimal control problem is worked out. Finally, we wind up this paper in Section 5.

2 Preliminaries

2.1 Motivations

In this subsection, we look at some motivations of our Problem (OLQ). First of all, for the state equation (1.1), let us look at some special cases.

- The classical linear SDE:
  \[
  \begin{cases}
  A(s)\xi = A(s)\xi, & C(s)\xi = C(s)\xi, \quad \forall \xi \in L^2_F(\Omega; \mathbb{R}^n), \\
  B(s)\eta = B(s)\eta, & D(s)\eta = D(s)\eta, \quad \forall \eta \in L^2_F(\Omega; \mathbb{R}^m),
  \end{cases}
  \]
with $A(\cdot), C(\cdot), B(\cdot), D(\cdot)$ being some matrix-valued processes.

- The case of simple mean-field SDE (MF-SDE):

$$\begin{align*}
A(s)\xi &= A(s)\xi + \bar{A}(s)\mathbb{E}[\bar{A}(s)\xi], \\
C(s)\xi &= C(s)\xi + \bar{C}(s)\mathbb{E}[\bar{C}(s)\xi], \\
B(s)\eta &= B(s)\eta + \bar{B}(s)\mathbb{E}[\bar{B}(s)\eta], \\
D(s)\eta &= D(s)\eta + \bar{D}(s)\mathbb{E}[\bar{D}(s)\eta],
\end{align*}$$

with $A(s), C(s), B(s), D(s)$ being matrix-valued processes that are deterministic in $s$. The above equation is supported by $\lambda \in \mathbb{R}$, and $\mathbb{E}[\cdot]$ being a Borel measure on $\mathbb{R}$. Some conditions are needed in order the above make sense. A special case of the above is the following (with $\mu(\cdot)$ supported at $\{1, 2, 3, \cdots\}$):

$$\begin{align*}
A(s)\xi &= A(s)\xi + \sum_{k \geq 1} \bar{A}_k(s)\mathbb{E}[\bar{A}_k(s)\xi] \\&= A(s)\xi + \bar{A}(s)\mathbb{E}[\bar{A}(s)\xi], \\
C(s)\xi &= C(s)\xi + \sum_{k \geq 1} \bar{C}_k(s)\mathbb{E}[\bar{C}_k(s)\xi] \\&= C(s)\xi + \bar{C}(s)\mathbb{E}[\bar{C}(s)\xi], \\
B(s)\eta &= B(s)\eta + \sum_{k \geq 1} \bar{B}_k(s)\mathbb{E}[\bar{B}_k(s)\eta] \\&= B(s)\eta + \bar{B}(s)\mathbb{E}[\bar{B}(s)\eta], \\
D(s)\eta &= D(s)\eta + \sum_{k \geq 1} \bar{D}_k(s)\mathbb{E}[\bar{D}_k(s)\eta] \\&= D(s)\eta + \bar{D}(s)\mathbb{E}[\bar{D}(s)\eta],
\end{align*}$$

with

$$\begin{align*}
\bar{A}(s) = (\bar{A}_1(s), \bar{A}_2(s), \cdots), \\
\bar{A}(s) = (\bar{A}_1(s)^T, \bar{A}_2(s)^T, \cdots), \\
\bar{B}(s) = (\bar{B}_1(s), \bar{B}_2(s), \cdots), \\
\bar{B}(s) = (\bar{B}_1(s)^T, \bar{B}_2(s)^T, \cdots), \\
\bar{D}(s) = (\bar{D}_1(s), \bar{D}_2(s), \cdots), \\
\bar{D}(s) = (\bar{D}_1(s)^T, \bar{D}_2(s)^T, \cdots),
\end{align*}$$

for some matrix-valued processes $\bar{A}_k(\cdot), \bar{A}_k(\cdot), \bar{C}_k(\cdot), \bar{B}_k(\cdot), \bar{B}_k(\cdot), \bar{D}_k(\cdot), \bar{D}_k(\cdot), k = 1, 2, \cdots$. We will look at the above case in details in Section 4.

From the above, we see that by allowing $A(\cdot), C(\cdot), B(\cdot), D(\cdot)$ to be operator-valued processes (not just matrix-valued processes), our state equation can cover a very big class of stochastic linear systems.
Next, we look at the cost functional. To get some feeling about the operators in the cost functional, let us look at the case compatible with (2.2)–(2.3). Let

$$\tilde{X}(T) = \begin{pmatrix} X(T) \\ E[\tilde{G}X(T)] \end{pmatrix}, \quad X(s) = \begin{pmatrix} X(s) \\ E[\tilde{Q}(s)X(s)] \end{pmatrix}, \quad u(s) = \begin{pmatrix} u(s) \\ E[\tilde{R}(s)u(s)] \end{pmatrix},$$

for some

$$\tilde{G} = \begin{pmatrix} \tilde{G}_1 \\ \tilde{G}_2 \\ \vdots \end{pmatrix}, \quad \tilde{Q}(s) = \begin{pmatrix} \tilde{Q}_1(s) \\ \tilde{Q}_2(s) \\ \vdots \end{pmatrix}, \quad \tilde{R}(s) = \begin{pmatrix} \tilde{R}_1(s) \\ \tilde{R}_2(s) \\ \vdots \end{pmatrix}. \quad \quad (2.4)$$

For the terminal cost, we propose the quadratic term as

$$E(\tilde{G}\tilde{X}(T), \tilde{X}(T)) = E\left( \begin{pmatrix} G & G^T \\ G & \tilde{G} \end{pmatrix} \begin{pmatrix} X(T) \\ E[GX(T)] \end{pmatrix}, \begin{pmatrix} X(T) \\ E[GX(T)] \end{pmatrix} \right)$$

$$= E\left[ (GX(T), X(T)) + (G^T E[\tilde{G}X(T)], X(T)) + (GX(T), E[\tilde{G}X(T)]) + \langle \tilde{G} E[\tilde{G}X(T)], E[\tilde{G}X(T)] \rangle \right]$$

$$= E(GX(T) + \sum_{k \geq 1} \tilde{G}_k E[\tilde{G}_k X(T)] + \tilde{G} E[\tilde{G}_k X(T)]) + \sum_{i,j \geq 1} \tilde{G}_{ij} E[\tilde{G}_i X(T), X(T)]$$

$$\equiv E(GX(T) + G^T E[\tilde{G}X(T)] + \tilde{G}^T E[\tilde{G}X(T)] + \tilde{G}^T E[\tilde{G}] E[\tilde{G}X(T)], X(T)) \equiv E(GX(T), X(T)),$$

and the linear term as

$$E(g, \tilde{X}(T)) = E\left( \begin{pmatrix} g_0 \\ g \end{pmatrix}, \begin{pmatrix} X(T) \\ E[\tilde{G}X(T)] \end{pmatrix} \right) = E\left[ \langle g_0, X(T) \rangle + \langle g, E[\tilde{G}X(T)] \rangle \right]$$

$$= E\left[ \langle g_0, X(T) \rangle + \sum_{k \geq 1} \langle g_k, E[\tilde{G}_k X(T)] \rangle \right] = E\langle g_0 \rangle + \sum_{k \geq 1} \tilde{G}_k E[\tilde{G}_k X(T)]$$

$$= E\langle g_0 \rangle + \tilde{G}^T E[\tilde{G}], X(T)) \equiv E(g, X(T)),$$

for some

$$G^T = G, \quad \tilde{G} = \begin{pmatrix} \tilde{G}_1 \\ \tilde{G}_2 \\ \vdots \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} & \cdots \\ \tilde{G}_{21} & \tilde{G}_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \tilde{G}^T = \tilde{G}, \quad g = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \end{pmatrix}. \quad \quad (2.5)$$

For the running cost, we propose the quadratic terms as

$$E(Q(s)X(s), X(s)) = E\left( \begin{pmatrix} Q(s) \\ Q(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ E[Q(s)X(s)] \end{pmatrix} \right), \quad \tilde{Q}(s)X(s)) = E\left[ \langle Q(s), X(s) \rangle + \langle Q(s) \rangle E[Q(s)X(s)], X(s) \rangle + \langle Q(s), X(s) \rangle, E[Q(s)X(s)] \rangle$$

$$\quad + \langle Q(s) \rangle E[Q(s)X(s)], E[Q(s)X(s)] \rangle$$

$$\equiv E(Q(s)X(s) + \tilde{Q}(s)E[\tilde{Q}(s)X(s)] + \tilde{Q}(s)E[\tilde{Q}(s)X(s)], X(s))$$

$$\equiv E(Q(s)X(s), X(s)),$$
and similarly,

\[
\mathbb{E}(R(s)u(s), u(s)) = \mathbb{E}\left( \begin{pmatrix} R(s) & \tilde{R}(s) \end{pmatrix} \begin{pmatrix} u(s) \\ \mathbb{E}[R(s)u(s)] \end{pmatrix} \right) \cdot \begin{pmatrix} u(s) \\ \mathbb{E}[\tilde{R}(s)u(s)] \end{pmatrix} 
\]

\[
= \mathbb{E}\left[ (R(s)u(s), u(s)) + \langle \tilde{R}(s)^\top \mathbb{E}[\tilde{R}(s)u(s)], u(s) \rangle + \langle R(s)u(s), \mathbb{E}[\tilde{R}(s)u(s)] \rangle \right] 
\]

\[
+ \langle \tilde{R}(s)\mathbb{E}[\tilde{R}(s)u(s)], \mathbb{E}[\tilde{R}(s)u(s)] \rangle 
\]

\[
= \mathbb{E}\left[ (R(s)u(s) + \tilde{R}(s)^\top \mathbb{E}[\tilde{R}(s)u(s)]) + \tilde{R}(s)^\top \mathbb{E}[\tilde{R}(s)u(s)] \right] 
\]

\[
= \mathbb{E} \mathbb{E}(R(s)u(s), u(s)),
\]

and the linear terms as

\[
\mathbb{E}(q(s), X(s)) = \mathbb{E}\left( \begin{pmatrix} q_0(s) \\ q(s) \end{pmatrix} , \begin{pmatrix} X(s) \\ \mathbb{E}[Q(s)X(s)] \end{pmatrix} \right) = \mathbb{E}\left[ (q_0(s), X(s)) + \langle q(s), \mathbb{E}[\tilde{Q}(s)X(s)] \rangle \right]
\]

\[
= \mathbb{E}\langle q_0(s), X(s) \rangle + \sum_{k \geq 1} \mathbb{E}(\tilde{Q}_k(s), X(s)) \equiv \mathbb{E}(q(s), X(s))
\]

\[
\mathbb{E}(\rho(s), u(s)) = \mathbb{E}\left( \begin{pmatrix} \rho_0(s) \\ \rho(s) \end{pmatrix} , \begin{pmatrix} u(s) \\ \mathbb{E}[R(s)u(s)] \end{pmatrix} \right) = \mathbb{E}\left[ (\rho_0(s), u(s)) + \langle \rho(s), \mathbb{E}[\tilde{R}(s)u(s)] \rangle \right]
\]

\[
= \mathbb{E}\langle \rho_0(s), X(s) \rangle + \sum_{k \geq 1} \mathbb{E}(\tilde{R}_k(s), u(s)) \equiv \mathbb{E}(\rho(s), u(s))
\]

for some

\[
(2.6) \quad Q(s)^\top = Q(s), \quad \tilde{Q}(s) = \begin{pmatrix} Q_1(s) \\ Q_2(s) \\ \vdots \end{pmatrix}, \quad \tilde{Q}(s) = \begin{pmatrix} \tilde{Q}_{11}(s) \\ \tilde{Q}_{12}(s) \\ \vdots \end{pmatrix}, \quad \tilde{Q}(s)^\top = \tilde{Q}(s),
\]

\[
(2.7) \quad R(s)^\top = R(s), \quad \tilde{R}(s) = \begin{pmatrix} R_1(s) \\ R_2(s) \\ \vdots \end{pmatrix}, \quad \tilde{R}(s) = \begin{pmatrix} \tilde{R}_{11}(s) \\ \tilde{R}_{12}(s) \\ \vdots \end{pmatrix}, \quad \tilde{R}(s)^\top = \tilde{R}(s).
\]
When all the weighting functions in the cost functional are deterministic, the above will be reduced to the following: (see [24])

\[
\begin{aligned}
G(s) &= G(s) + 
\begin{bmatrix}
G_{s+} E[G_{s+}] + G_{s+}^\top E[G_{s+}] + \tilde{G}_{s+}^\top E[\tilde{G}_{s+}] \& \xi \in L^2(\Omega; \mathbb{R}^n), \\
Q(s) &= Q(s) + Q(s) E[Q(s)] + Q(s) E[\tilde{Q}(s)] \& \xi \in L^2(\Omega, \mathbb{R}^n) \\
S(s) &= S(s) + S(s) E[S(s)] \& \xi \in L^2(\Omega, \mathbb{R}^n), \\
\end{aligned}
\]

We see that, in the above case,

\[
\begin{aligned}
G(s) &= G(s) + \tilde{G}^\top E[\tilde{G}], \quad \xi \in L^2(\Omega; \mathbb{R}^n), \\
Q(s) &= Q(s) + \tilde{Q}^\top E[\tilde{Q}], \quad S(s) = S(s) + \tilde{S}^\top E[\tilde{S}], \quad \xi \in L^2(\Omega; \mathbb{R}^n), \\
R(s) &= R(s) + \tilde{R}^\top E[\tilde{R}], \quad q(s) = q(s) + \tilde{q}^\top E[\tilde{q}], \quad \rho(s) = \rho(s) + \tilde{\rho}^\top E[\tilde{\rho}].
\end{aligned}
\]

The above suggests that if the coefficients of the state equation are given by (2.1), the corresponding operators in the cost functional could look like the following:

\[
\begin{aligned}
G(s) &= G(s) + \int \left( \tilde{G}_{s+}^\top E[\tilde{G}_{s+}] + \tilde{G}_{s+}^\top E[\tilde{G}_{s+}] \right) \mu(d\lambda) + \int \tilde{G}_{s+} E[\tilde{G}_{s+}] \mu(d\lambda), \quad \xi \in L^2(\Omega; \mathbb{R}^n), \\
Q(s) &= Q(s) + \int \left( \tilde{Q}_{s+}^\top E[\tilde{Q}_{s+}] + \tilde{Q}_{s+}^\top E[\tilde{Q}_{s+}] \right) \mu(d\lambda) + \int \tilde{Q}_{s+} E[\tilde{Q}_{s+}] \mu(d\lambda), \quad \xi \in L^2(\Omega; \mathbb{R}^n), \\
S(s) &= S(s) + \int \left( \tilde{S}_{s+}^\top E[\tilde{S}_{s+}] + \tilde{S}_{s+}^\top E[\tilde{S}_{s+}] \right) \mu(d\lambda) + \int \tilde{S}_{s+} E[\tilde{S}_{s+}] \mu(d\lambda), \quad \xi \in L^2(\Omega; \mathbb{R}^n), \\
\end{aligned}
\]

In the above, \( \tilde{G}_{s+}, \tilde{Q}_{s+}, \tilde{S}_{s+}, \tilde{R}_{s+} \) are deterministic, and

\[
\begin{aligned}
\tilde{G}_{s+}^\top &= \tilde{G}_{s+}, \\
\tilde{Q}_{s+}^\top &= \tilde{Q}_{s+}, \\
\tilde{R}_{s+}^\top &= \tilde{R}_{s+}, \\
\forall s \in [0, T] \text{ and } \lambda, \nu \in \mathbb{R}.
\end{aligned}
\]

The above shows that our framework can cover many problems involving mean fields.

### 2.2 The state equation and the cost functional

We now return to our state equation (1.1) and cost functional (1.3). Recall the Hilbert space

\[
L^2(\Omega; \mathbb{R}^n) \subseteq L^2(\Omega; \mathbb{R}^n) \equiv L^2(\Omega; \mathbb{R}^n), \quad s \in [0, T],
\]
with the norm:

\[ \|\xi\|_2 = \left( \mathbb{E}[|\xi|^2] \right)^{\frac{1}{2}}, \quad \forall \xi \in L^2_{\mathbb{F}}(\Omega; \mathbb{R}^n). \]

Also, we recall the spaces \( L^p_{\mathbb{F}}(\Omega; L^1(t, T; \mathbb{R}^n)) \) and \( L^p_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n)) \) introduced in the first section. We now introduce the following definition concerning the operator-valued processes.

**Definition 2.1.** An operator-valued process \( B : [0, T] \to \mathcal{L}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right) \) is said to be **strongly \( \mathbb{F} \)-progressively measurable** if

\[ B(s) \in \mathcal{L}\left(L^2_{\mathbb{F}}(\Omega; \mathbb{R}^m); L^2_{\mathbb{F}}(\Omega; \mathbb{R}^n)\right), \quad \forall s \in [0, T], \]

and for any \( \eta(\cdot) \in L^p_{\mathbb{F}}(0, T; \mathbb{R}^m) \), \( B(\cdot)\eta(\cdot) \) is \( \mathbb{F} \)-progressively measurable. The set of all strongly \( \mathbb{F} \)-progressively measurable operator-valued processes valued in \( \mathcal{L}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right) \) is denoted by \( \mathcal{L}_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right) \), and denote

\[ \mathcal{L}_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right) = \mathcal{L}_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^n)\right). \]

Further, for any \( p \in [1, \infty] \), we denote

\[ \mathcal{L}^p_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right) = \left\{ B(\cdot) \in \mathcal{L}_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right) \mid \|B(\cdot)\|_p < \infty \right\}, \]

where

\[ \|B(\cdot)\|_p = \left\{ \left( \int_0^T \|B(t)\|^p dt \right)^{\frac{1}{p}}, \quad p \in [1, \infty), \right. \]

\[ \left. \sup_{t \in [0, T]} \|B(t)\|, \quad p = \infty, \right\} \]

with \( \|B(t)\| \) being the operator norm of \( B(t) \) for given \( t \in [0, T] \). Also, we denote

\[ \mathcal{L}^p_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right) = \mathcal{L}^p_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^n)\right), \]

and

\[ \mathcal{F}^p\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right) = \left\{ B(\cdot) \in \mathcal{L}^p_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^n)\right) \mid B(s) \in \mathcal{F}\left(L^2_{\mathbb{F}}(\Omega; \mathbb{R}^n)\right), \forall s \in [0, T] \right\}. \]

Strong measurability for operator-valued functions can be found in [27]. Our operator-valued processes have an additional feature of \( \mathbb{F} \)-progressive measurability. Therefore, the above definition is necessary.

Now, for state equation (1.1), we introduce the following hypothesis.

**(H1)** Let

\[ A(\cdot) \in \mathcal{L}^1_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^n)\right), \quad B(\cdot) \in \mathcal{L}^2_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right), \]

\[ C(\cdot) \in \mathcal{L}^2_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right), \quad D(\cdot) \in \mathcal{L}^\infty_{\mathbb{F}}\left(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\right), \]

\[ b(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0, T; \mathbb{R}^n)), \quad \sigma(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n). \]

The following result gives the well-posedness of (1.1).

**Proposition 2.2.** Let (H1) hold. Then for any \((t, x) \in \mathcal{D} \), and \( u(\cdot) \in U[t, T] \), there exists a unique solution \( X(\cdot) \equiv X(\cdot; t, x) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n)) \) to (1.1) and the following estimate holds:

\[ \mathbb{E}\left[ \sup_{s \in [t, T]} |X(s)|^2 \right] \leq K \left[ \|x\|^2_2 + \left( \|B(\cdot)\|^2_2 + \|D(\cdot)\|^2_{\infty} \right) \int_t^T \|u(r)\|^2_2 dr \right. \]

\[ + \left. \int_t^T |b(r)| dr \right] + \int_t^T \|\sigma(r)\|^2_2 dr, \]

for some constant \( K > 0 \) depending on \( \|A(\cdot)\|_1 \) and \( \|C(\cdot)\|_2 \).

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Proof. Let \((t, x) \in \mathcal{D}\) be fixed and \(u(\cdot) \in \mathcal{U}[t, T]\) be given. For any \(\hat{X}(\cdot) \in L^2_2(\Omega; C([t, T]; \mathbb{R}^n))\), we define the process \(X(\cdot)\) by
\[
X(s) = x + \int_t^s \left( A(r) \hat{X}(r) + B(r)u(r) + b(r) \right) dr + \int_t^s \left( C(r) \hat{X}(r) + D(r)u(r) + \sigma(r) \right) dW(r), \quad s \in [t, T].
\]
For any \(t \leq t_1 < t_2 \leq T\) and any \(s \in [t_1, t_2]\),
\[
|X(s)|^2 \leq 3 \left| |X(t_1)|^2 + \left( \int_{t_1}^s |A(r)\hat{X}(r) + B(r)u(r) + b(r)| dr \right)^2 \right.
\[
+ \left. \left( \int_{t_1}^s |C(r)\hat{X}(r) + D(r)u(r) + \sigma(r)| dW(r) \right)^2 \right\}. \]

By Minkowski’s integral inequality and the Burkholder–Davis–Gundy inequality, we have
\[
\mathbb{E} \left[ \sup_{\tau \in [t_1, s]} |X(\tau)|^2 \right] \leq 3 \left\{ \mathbb{E} \left[ \sup_{t \leq \tau \leq T} |X(t)|^2 \right] + \mathbb{E} \left[ \left( \int_{t_1}^s |A(r)\hat{X}(r) + B(r)u(r) + b(r)| dr \right)^2 \right] \right.
\[
+ \left. \mathbb{E} \left[ \left( \int_{t_1}^s |C(r)\hat{X}(r) + D(r)u(r) + \sigma(r)| dW(r) \right)^2 \right] \right\}
\]
\[
\leq 3 \left\{ \left( \int_{t_1}^s |A(r)\hat{X}(r)| dr \right)^2 + \left( \int_{t_1}^s |B(r)u(r)| dr \right)^2 + \left( \int_{t_1}^s |C(r)\hat{X}(r)| dr \right)^2 + \left( \int_{t_1}^s |D(r)u(r)| dr \right)^2 + \left( \int_{t_1}^s |\sigma(r)| dW(r) \right)^2 \right\}
\]
\[
\leq 3 \left\{ \left( \int_{t_1}^s |A(r)| dr \right)^2 + \left( \int_{t_1}^s |B(r)| dr \right)^2 + \left( \int_{t_1}^s |C(r)| dr \right)^2 + \left( \int_{t_1}^s |D(r)| dr \right)^2 \right\}
\]
where \(c_2 > 0\) is the constant in the Burkholder–Davis–Gundy inequality. Thus, it follows that for some generic constant \(K > 0\), (hereafter, \(K > 0\) will stand for a generic constant which could be different from line to line)
\[
\mathbb{E} \left[ \sup_{\tau \in [t_1, s]} |X(\tau)|^2 \right] \leq K \left\{ \left( \int_{t_1}^s |A(r)| dr \right)^2 + \left( \int_{t_1}^s |B(r)| dr \right)^2 + \left( \int_{t_1}^s |C(r)| dr \right)^2 + \left( \int_{t_1}^s |D(r)| dr \right)^2 \right\}
\]
\[
\leq K \left\{ \left( \int_{t_1}^s |A(r)| dr \right)^2 + \left( \int_{t_1}^s |B(r)| dr \right)^2 + \left( \int_{t_1}^s |C(r)| dr \right)^2 + \left( \int_{t_1}^s |D(r)| dr \right)^2 \right\} \mathbb{E} \left[ \sup_{\tau \in [t_1, s]} |X(\tau)|^2 \right].
\]

For \(\delta > 0\) small enough, we have
\[
K \left( \int_{t_1}^{t+\delta} |A(r)| dr \right)^2 + \int_{t_1}^{t+\delta} |C(r)|^2 dr < 1.
\]
Thus, the map \(\hat{X}(\cdot) \mapsto X(\cdot)\) is a contraction from \(L^2_2(\Omega; C([t, t + \delta]; \mathbb{R}^n))\) into itself. Therefore, it admits a unique fixed point which is a solution to the state equation on \([t, t + \delta]\). Repeating the same argument, we can obtain the unique existence of the solution \(X(\cdot) \in L^2_2(\Omega; C([t, T]; \mathbb{R}^n))\) to the state equation (1.1).
Moreover, for the solution $X(\cdot)$, from the above, we have

$$
\mathbb{E}\left[ \sup_{\tau \in [t,s]} |X(\tau)|^2 \right] \leq K \left\{ |x|^2 + \mathbb{E} \left[ \int_t^s |b(r)|dr \right]^2 + \mathbb{E} \left[ \int_t^s |\sigma(r)|^2dr + \left( \|\mathcal{B}(\cdot)\|_2^2 + \|\mathcal{D}(\cdot)\|_2^2 \right) \right] \int_t^s \|u(r)\|^2dr \right\} 
+ K \left( \left( \int_t^s \|A(r)\|dr \right)^2 + \int_t^s \|C(r)\|^2dr \right) \mathbb{E}\left[ \sup_{\tau \in [t,s]} |X(\tau)|^2 \right].
$$

Then by a simple iteration argument, we obtain (2.13).

Now, for the cost functional, we introduce the following hypothesis.

**H2** The operator $\mathcal{G} \in \mathcal{S}\left( L^2(\Omega; \mathbb{R}^n) \right)$ and the operator-valued processes

$$
\mathcal{Q}(\cdot) \in \mathcal{S}\left( L^2(\Omega; \mathbb{R}^n) \right), \quad \mathcal{S}(\cdot) \in \mathcal{L}\left( L^2(\Omega; \mathbb{R}^n); L^2(\Omega; \mathbb{R}^m) \right), \quad \mathcal{R}(\cdot) \in \mathcal{S}\left( L^2(\Omega; \mathbb{R}^m) \right).
$$

Also,

$$
g \in L^2(\Omega; \mathbb{R}^n), \quad g(\cdot) \in L^2_0(\Omega; L^1(0,T; \mathbb{R}^n)), \quad \rho(\cdot) \in L^2_0(0,T; \mathbb{R}^m).
$$

We have the following result.

**Proposition 2.3.** Let (H1)–(H2) hold. Then for any $(t, x) \in \mathcal{D}$ and any $u(\cdot) \in \mathcal{U}[t, T]$, the cost functional $J(t, x; u(\cdot))$ is well-defined.

**Proof.** First of all, Proposition 2.2 implies that the state equation (1.1) admits a unique state process $X(\cdot) \equiv X(\cdot; t, x, u(\cdot)) \in L^2_0(\Omega; C([t, T]; \mathbb{R}^n))$. Let us observe the following estimates:

$$
\mathbb{E}[|\mathcal{G}X(T)|^2] \leq \|\mathcal{G}\|_2 \|X(T)\|_2^2, \quad \mathbb{E}[|g, X(T)|] \leq \|g\|_2 \|X(T)\|_2;
$$

$$
\int_t^T \mathbb{E}[\|\mathcal{R}(s)u(s), u(s)\|]ds \leq \int_t^T \|\mathcal{R}(s)\| \|u(s)\|_2^2 ds \leq \|\mathcal{R}(\cdot)\|_\infty \int_t^T \|u(s)\|_2^2 ds;
$$

$$
\int_t^T \mathbb{E}[\|\mathcal{S}(s)X(s), u(s)\|] ds \leq \left( \int_t^T \|\mathcal{S}(s)X(s)\|_2^2 ds \right)^\frac{1}{2} \left( \int_t^T \|u(s)\|_2^2 ds \right)^\frac{1}{2};
$$

$$
\leq \|\mathcal{S}(\cdot)\|_2 \left[ \mathbb{E}\left[ \sup_{s \in [t,T]} |X(s)|^2 \right] \right]^{\frac{1}{2}} \left( \int_t^T \|u(s)\|_2^2 ds \right)^\frac{1}{2};
$$

$$
\mathbb{E}[\|\mathcal{Q}(s)X(s), X(s)\|] ds \leq \int_t^T \|\mathcal{Q}(s)\|_1 \|X(s)\|_2^2 ds \leq \|\mathcal{Q}(\cdot)\|_1 \mathbb{E}\left[ \sup_{s \in [t,T]} |X(s)|^2 \right];
$$

$$
\int_t^T \mathbb{E}[|q(s), X(s)|] ds \leq \mathbb{E} \left[ \left( \int_t^T |q(s)| ds \right) \sup_{s \in [t,T]} |X(s)| \right] \leq \left( \int_t^T |q(s)| ds \right) \left[ \mathbb{E}\left[ \sup_{s \in [t,T]} |X(s)|^2 \right] \right]^{\frac{1}{2}};
$$

$$
\int_t^T \mathbb{E}[|\rho(s), u(s)|] ds \leq \left( \int_t^T |\rho(s)|_2^2 ds \right)^{\frac{1}{2}} \left( \int_t^T \|u(s)\|_2^2 ds \right)^{\frac{1}{2}}.
$$

This implies that the cost functional $J(t, x; u(\cdot))$ is well-defined.

Next we look at the quadratic form of the cost functional (1.3), from which we will get some abstract results for Problem (OLQ).

It is clear that for given $t \in [0, T)$, $(x, u(\cdot)) \mapsto X(\cdot; t, x, u(\cdot))$ is affine. Therefore, we may write

$$
X(\cdot; t, x, u(\cdot)) = [F_1(t)u(\cdot)](\cdot) + [F_0(t)x](\cdot) + f_0(t, \cdot),
$$

where

$$
F_1(t) \in \mathcal{L}\left( \mathcal{U}[t, T]; L^2_0(t, T; \mathbb{R}^n) \right), \quad F_0(t) \in \mathcal{L}\left( L^2_0(\Omega; \mathbb{R}^n); L^2_0(t, T; \mathbb{R}^n) \right), \quad f_0(t, \cdot) \in L^2_0(t, T; \mathbb{R}^n).
$$

Let

$$
\bar{F}_1(t)u(\cdot) = [F_1(t)u(\cdot)](T), \quad \bar{F}_0(t)x = [F_0(t)x](T), \quad \bar{f}_0(t) = f_0(t, T).
$$
Consequently,

\[ J(t, x; u(\cdot)) = \mathbb{E} \left[ G \{ \hat{F}_1(t)u(\cdot) + \hat{F}_0(t)x + \hat{f}_0(t) \} + \hat{F}_1(t)u(\cdot) + \hat{F}_0(t)x + \hat{f}_0(t) \} \right] + \int_t^T \langle \left\{ [F_1(t)u(\cdot)](s) + [F_0(t)x(s) + f_0(t,s)] \} + [F_1(t)u(\cdot)](s) + [F_0(t)x(s) + f_0(t,s)] \right\} + 2\langle [F_1(t)u(\cdot)](s) + [F_0(t)x(s) + f_0(t,s), u(s)] + \langle R(s)u(s), u(s) \right\} ds \]

\[ \equiv \langle G[\hat{F}_1 u + \hat{F}_0 x + \hat{f}_0], \hat{F}_1 u + \hat{F}_0 x + \hat{f}_0 \rangle + 2\langle g, \hat{F}_1 u + \hat{F}_0 x + \hat{f}_0 \rangle + 2\langle [F_1 u + F_0 x + f_0, u] + \langle R u, u \right\rangle + 2\langle q, 1 u + F_0 x + f_0 \rangle + 2\langle q, F_0 x + f_0 \rangle + 2\langle q, F_0 x + f_0 \rangle \]

\[ = \langle \Phi_2 u(\cdot), u(\cdot) \rangle + 2\langle u(\cdot), \varphi_1 \rangle + \varphi_0, \]

with

\[
\begin{aligned}
\Phi_2 &= \hat{F}_1^* G \hat{F}_1 + F_1^* Q F_1 + S F_1 + F_1^* S^* + R, \\
\varphi_1 &= \hat{F}_1^* \left[ G(\hat{F}_0 x + f_0) + g \right] + F_1^* \left[ Q(\hat{F}_0 x + f_0) + q \right] + S(\hat{F}_0 x + f_0) + \rho, \\
\varphi_0 &= \langle G(\hat{F}_0 x + f_0), \hat{F}_0 x + f_0 \rangle + 2\langle g, \hat{F}_0 x + f_0 \rangle + \langle Q(\hat{F}_0 x + f_0), \hat{F}_0 x + f_0 \rangle + 2\langle q, \hat{F}_0 x + f_0 \rangle.
\end{aligned}
\]

Clearly,

\[ \Phi_2 \in \mathcal{S}(U[t, T]), \quad \varphi_1 \in U[t, T], \quad \varphi_0 \in \mathbb{R}. \]

Therefore, according to [15], we have the following result.

**Proposition 2.4.** For any \((t, x) \in \mathcal{S}\), the map \(u(\cdot) \mapsto J(t, x; u(\cdot))\) admits a minimum in \(U[t, T]\) if and only if

\[ \Phi_2 \geq 0, \quad \varphi_1 \in \mathcal{S}(\Phi_2) \equiv \text{the range of } \Phi_2. \]

In particular, if the following holds:

\[ \Phi_2 \geq \delta I, \]

for some \(\delta > 0\), then (2.15) holds and the map \(u(\cdot) \mapsto J(t, x; u(\cdot))\) admits a unique minimum given by the following:

\[ \bar{u}(\cdot) = -\Phi_2^{-1} \varphi_1(\cdot). \]

It is clear that if

\[ G \geq 0, \quad Q(\cdot) - S(\cdot)^* R(\cdot)^{-1} S(\cdot) \geq 0, \quad R(\cdot) \geq \delta I, \]

then (2.16) holds. Thus, under (H1)–(H2) and (2.16), Problem (OLQ) admits a unique open-loop optimal control.

If we denote

\[ J(t, x; u(\cdot)) = J(t, x; u(\cdot); b(\cdot), \sigma(\cdot), g, q(\cdot), \rho(\cdot)), \]

indicating the dependence on the nonhomogeneous terms \(b(\cdot), \sigma(\cdot)\) in the state equation and the linear weighting coefficients \(g, q(\cdot), \rho(\cdot)\) in the cost functional, then we define

\[ J^0(t; u(\cdot)) = J(t, 0; u(\cdot); 0, 0, 0, 0, 0) = \langle \Phi_2 u(\cdot), u(\cdot) \rangle, \quad \forall u(\cdot) \in U[t, T]. \]

Hence, we see that the following is true.
Proposition 2.5. Let (H1)–(H2) hold. Then the following are equivalent:

(i) $\Phi_2 \geq 0$;
(ii) $J^0(t; u(\cdot)) \geq 0$ for all $u(\cdot) \in \mathcal{U}[t, T]$;
(iii) $u(\cdot) \mapsto J(t, x; u(\cdot))$ is convex.

2.3 BSDEs with operator coefficients

In this subsection, let us consider the following BSDE with operator coefficients:

\[
\begin{aligned}
\frac{dY(s)}{ds} &= -\left( A(s)^* Y(s) + C(s)^* Z(s) + \varphi(s) \right) ds + Z(s) dW(s), \quad s \in [t, T], \\
Y(T) &= \xi \in L^2(\Omega; \mathbb{R}^n).
\end{aligned}
\]

A pair $(Y(\cdot), Z(\cdot))$ of adapted processes is called an adapted solution if the above is satisfied in the usual Itô’s sense. We have the following well-posedness and regularity result for the above BSDE.

Proposition 2.6. Suppose (H1) holds and $\varphi(\cdot) \in L_2^2(\Omega; L^1(t, T; \mathbb{R}^n))$. Then (2.20) admits a unique adapted solution $(Y(\cdot), Z(\cdot)) \in L_2^2(\Omega; C([t, T]; \mathbb{R}^n)) \times L_2^2(t, T; \mathbb{R}^n)$, and the following estimate holds:

\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |Y(s)|^2 + \int_t^T |Z(s)|^2 ds \right] \leq K \left[ \|\xi\|^2_2 + \int_t^T \|\varphi(s)\| ds \right]^2,
\]

where $K > 0$ is a constant depending on $\|A(\cdot)\|_1$ and $\|C(\cdot)\|_2$.

Proof. Denote

\[
\mathcal{M}[t, T] = L_2^2(\Omega; C([t, T]; \mathbb{R}^n)) \times L_2^2(t, T; \mathbb{R}^n),
\]

and

\[
\|(Y(\cdot), Z(\cdot))\|_{\mathcal{M}[t, T]} = \mathbb{E} \left[ \sup_{s \in [t, T]} |Y(s)|^2 + \int_t^T |Z(s)|^2 ds \right].
\]

It is clear that $\|\cdot\|_{\mathcal{M}[t, T]}$ is a norm under which $\mathcal{M}[t, T]$ is a Banach space. We introduce a map $\mathcal{F} : \mathcal{M}[t, T] \to \mathcal{M}[t, T]$ by the following:

\[
\mathcal{F}(y(\cdot), z(\cdot)) = (Y(\cdot), Z(\cdot)),
\]

where $(y(\cdot), z(\cdot)) \in \mathcal{M}[t, T]$ and $(Y(\cdot), Z(\cdot))$ is the adapted solution to the following BSDE:

\[
Y(s) = \xi + \int_s^T \left( A(r)^* y(r) + C(r)^* z(r) + \varphi(r) \right) dr - \int_s^T Z(r) dW(r), \quad s \in [t, T].
\]

Then a standard result for BSDEs leads to the following estimate:

\[
\mathbb{E} \left[ \sup_{r \in [s, T]} |y(r)|^2 + \int_s^T |Z(r)|^2 dr \right] \leq K \left[ \|\xi\|^2_2 + \int_s^T \left( \|A(r)^* y(r)\| + \|C(r)^* z(r)\| + \|\varphi(r)\| \right) dr \right]^2.
\]

The above estimate implies that the map $(y(\cdot), z(\cdot)) \mapsto (Y(\cdot), Z(\cdot))$ is a contraction on $\mathcal{M}[T - \delta, T]$ for some small $\delta > 0$. Then a routine argument gives the well-posedness of BSDE (2.20) on $[t, T]$, and estimate (2.21) holds.
3 Open-Loop Optimal Controls and FBSDEs

3.1 Solvability of Problem (OLQ)

Let us first give the following definition.

**Definition 3.1.** A control process $\bar{u}(\cdot) \in U[t, T]$ is called an *open-loop optimal control* of Problem (OLQ) at $(t, x) \in \mathcal{D}$ if

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in U[t, T]} J(t, x; u(\cdot)).$$

If $\bar{u}(\cdot) \in U[t, T]$ exists satisfying (3.1), we say that Problem (OLQ) is *open-loop solvable* at $(t, x) \in \mathcal{D}$, and $\bar{X}(\cdot) \equiv X^{\bar{u}(\cdot)}(\cdot)$ is called the *open-loop optimal state process*.

The following gives a characterization of optimal open-loop control of Problem (OLQ).

**Theorem 3.2.** Let (H1)–(H2) hold. Given $(t, x) \in \mathcal{D}$. Then $\bar{u}(\cdot) \in U[t, T]$ is an open-loop optimal control of Problem (OLQ) at $(t, x) \in \mathcal{D}$ with $\bar{X}(\cdot)$ being the corresponding open-loop optimal process if and only if $u(\cdot) \mapsto J(t, x; u(\cdot))$ is convex and the following FBSDE with operator coefficients:

$$
\begin{cases}
  d\bar{X}(s) = \left( A(s)\bar{X}(s) + B(s)\bar{u}(s) + b(s) \right) ds + \left( C(s)\bar{X}(s) + D(s)\bar{u}(s) + \sigma(s) \right) dW(s), & s \in [t, T], \\
  d\bar{Y}(s) = -\left( A(s)^*\bar{Y}(s) + C(s)^*\bar{Z}(s) + Q(s)\bar{X}(s) + S(s)^*\bar{u}(s) + q(s) \right) ds + \bar{Z}(s) dW(s), \\
  \bar{X}(t) = x, \quad \bar{Y}(T) = \mathcal{G}\bar{X}(T) + g,
\end{cases}
$$

admits an adapted solution $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$ such that the following stationarity condition holds:

$$\mathcal{R}(s)\bar{u}(s) + B(s)^*\bar{Y}(s) + D(s)^*\bar{Z}(s) + S(s)\bar{X}(s) + \rho(s) = 0, \quad s \in [t, T], \text{ a.s.}$$

**Proof.** For $(t, x) \in \mathcal{D}$ and $u(\cdot), \bar{u}(\cdot) \in U[t, T]$, let $X(\cdot) = X(\cdot; t, x, u(\cdot))$ and $\bar{X}(\cdot) = X(\cdot; t, x, \bar{u}(\cdot))$ be the state process (1.1) corresponding to $u(\cdot)$ and $\bar{u}(\cdot)$, respectively. Denote

$$\tilde{X}(\cdot) = X(\cdot) - \bar{X}(\cdot), \quad \tilde{u}(\cdot) = u(\cdot) - \bar{u}(\cdot).$$

Then $\tilde{X}(\cdot)$ satisfies the following FSDE:

$$
\begin{cases}
  d\tilde{X}(s) = \left( A(s)\tilde{X}(s) + B(s)\tilde{u}(s) \right) ds + \left( C(s)\tilde{X}(s) + D(s)\tilde{u}(s) \right) dW(s), & s \in [t, T], \\
  \tilde{X}(t) = 0.
\end{cases}
$$

Now, let $(\bar{Y}(\cdot), \bar{Z}(\cdot))$ be the adapted solution to the BSDE in (3.2). Then we have the following duality:

\[
\mathbb{E}(\mathcal{G}\bar{X}(T) + g, \bar{X}(T)) = \mathbb{E}(\bar{X}(T), \bar{Y}(T))
\]

\[
= \mathbb{E} \int_t^T \left( \langle A(s)\bar{X}(s) + B(s)\bar{u}(s), \bar{Y}(s) \rangle + \langle C(s)\bar{X}(s) + D(s)\bar{u}(s), \bar{Z}(s) \rangle - \langle \tilde{X}(s), A(s)^*\bar{Y}(s) + C(s)^*\bar{Z}(s) + Q(s)\bar{X}(s) + S(s)^*\bar{u}(s) + q(s) \rangle \right) ds
\]

\[
= \mathbb{E} \int_t^T \left( -\langle \tilde{X}(s), Q(s)\bar{X}(s) + S(s)^*\bar{u}(s) + q(s) \rangle + \langle \tilde{u}(s), B(s)^*\bar{Y}(s) + D(s)^*\bar{Z}(s) \rangle \right) ds.
\]
Thus, if constraint (3.3) holds then by letting \( \alpha \to \infty \) in the above, we see that \( u(\cdot) \mapsto J(t; x; u(\cdot)) \) is convex and the constraint (3.3) holds.

We note that (3.2)–(3.3) is a coupled linear FBSDE with operator coefficients. The above theorem tells us that the open-loop solvability of Problem (OLQ) is equivalent to the solvability of an FBSDE with operator coefficients. A similar result for LQ problem with constant (matrix) coefficients were established in [19]. The proof presented above is similar to that found in [19], with some simplified arguments.

### 3.2 Well-posedness of FBSDEs with operator coefficients

We now look at the solvability of FBSDE (3.2)–(3.3). To abbreviate the notations, we drop the bars in \( \bar{X}, \bar{Y}, \bar{Z} \) of (3.2) and (3.3), that is, we consider the following:

\[
\begin{align*}
\left\{
\begin{array}{l}
\frac{dX(s)}{ds} = \left( A(s)X(s) + B(s)u(s) + b(s) \right) ds + \left( C(s)X(s) + D(s)u(s) + \sigma(s) \right) dW(s), \\
nY(s) = \left( A(s)^*Y(s) + C(s)^*Z(s) + Q(s)X(s) + S(s)^*u(s) + q(s) \right) ds + Z(s)dW(s), \\
X(t) = x, \quad Y(T) = \mathcal{G}X(T) + g, \\
\mathcal{R}(s)u(s) + B(s)^*Y(s) + D(s)^*Z(s) + S(s)X(s) + \rho(s) = 0.
\end{array}
\right.
\end{align*}
\]

For the well-posedness of the above equation, we need the following hypothesis.
(H3) For some $\delta > 0$,

$$G \geq 0, \quad Q(s) - S(s)\mathcal{R}(s)^{-1}S(s) \geq 0, \quad \mathcal{R}(s) \geq \delta I, \quad s \in [0,T],$$

where $I$ in the above is the identity operator on $L^2(\Omega; \mathbb{R}^m)$.

Note that the last condition in (H3) ensures the existence of the inverse $\mathcal{R}^{-1}$ of operator $\mathcal{R}$, and

$$(3.5) \quad \mathbb{E}\mathcal{R}(s)^{-1}u, u \geq \frac{\delta}{\|\mathcal{R}(s)\|^2} \mathbb{E}|u|^2,$$

for any $u \in L^2_{\mathbb{F}}(\Omega; \mathbb{R}^m)$, a.e., $s \in [t, T]$.

On the other hand, by Schur’s Lemma, one sees that that (H3) is equivalent to the following:

$$(3.6) \quad u(s) = -\mathcal{R}(s)^{-1}\left(B(s)^*Y(s) + D(s)^*Z(s) + S(s)X(s) + \rho(s)\right), \quad s \in [t, T].$$

By substituting it into the first two equations of (3.4), the system becomes a coupled FBSDEs with operator coefficients as follows (suppressing s):

$$dX = \left((A - B\mathcal{R}^{-1}S)X - B\mathcal{R}^{-1}B^*Y - B\mathcal{R}^{-1}D^*Z - B\mathcal{R}^{-1}_\rho + b\right)ds$$

$$+ \left((C - D\mathcal{R}^{-1}S)X - D\mathcal{R}^{-1}B^*Y - D\mathcal{R}^{-1}D^*Z - D\mathcal{R}^{-1}_\rho + \sigma\right)dW(s),$$

$$dY = -\left((Q - S^*\mathcal{R}^{-1}S)X + (A^* - S^*\mathcal{R}^{-1}B^*)Y + (C^* - S^*\mathcal{R}^{-1}D^*)Z\right)$$

$$\quad + (Q - S^*\mathcal{R}^{-1}S)\rho + \gamma)ds + ZdW(s),$$

$$X(t) = x, \quad Y(T) = \mathcal{G}X(T) + g.$$

**Remark 3.3.** From (H1), (H2), it is easy to check that

$$A(\cdot) - B(\cdot)\mathcal{R}(\cdot)^{-1}S(\cdot) \in \mathcal{L}^2_{\mathbb{F}}(L^2(\Omega; \mathbb{R}^m)),$$

$$C(\cdot) - D(\cdot)\mathcal{R}(\cdot)^{-1}S(\cdot) \in \mathcal{L}^2_{\mathbb{F}}(L^2(\Omega; \mathbb{R}^m)),$$

$$Q(\cdot) - S(\cdot)^*\mathcal{R}(\cdot)^{-1}S(\cdot) \in \mathcal{L}^2_{\mathbb{F}}(L^2(\Omega; \mathbb{R}^m)).$$

For example,

$$\int_0^T \|A(s) - B(s)\mathcal{R}(s)^{-1}S(s)\|ds \leq \int_0^T \left(\|A(s)\| + \|B(s)\mathcal{R}(s)^{-1}S(s)\|\right)ds$$

$$\leq \int_0^T \|A(s)\|ds + \left(\int_0^T \|B(s)\|^2ds\right)^{\frac{1}{2}} \sup_{s \in [0,T]} \|\mathcal{R}(s)^{-1}\| \left(\int_0^T \|S(s)\|^2ds\right)^{\frac{1}{2}} < \infty.$$

The remaining two can be verified similarly.

The following is the main result of this subsection.

**Theorem 3.4.** Let (H1)–(H3) hold. Then there exists a unique adapted solution $(X(\cdot), Y(\cdot), Z(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^m))^2 \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$ to the coupled FBSDE (3.7) with operator coefficients, and the following estimate holds:

$$(3.8) \quad \mathbb{E}\left[\sup_{s \in [t, T]} |X(s)|^2 + \sup_{s \in [t, T]} |Y(s)|^2 + \int_t^T |Z(s)|^2ds\right]$$

$$\leq K\left\{\|x\|^2_2 + \|g\|^2_2 + \int_t^T \|b(s)\|ds_2^2 + \int_t^T \|q(s)\|ds_2^2 + \int_t^T \|\sigma(s)\|^2ds + \int_t^T \|\rho(s)\|^2ds\right\},$$

where $K > 0$ is a constant depending on $\|A(\cdot)\|_1$, $\|B(\cdot)\|_2$, $\|C(\cdot)\|_2$, $\|D(\cdot)\|_\infty$, $\|Q(\cdot)\|_1$, $\|S(\cdot)\|_2$, $\|\mathcal{R}(\cdot)^{-1}\|_\infty$ and $\|G\|$. 

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Before presenting a proof of Theorem 3.4, we introduce the following auxiliary FBSDE with operator coefficients parameterized by $\alpha \in [0, 1]$:  

$$
\begin{aligned}
\left\{ 
\begin{array}{l}
\frac{dX^\alpha}{dt} = \left( \alpha(A - BR^{-1}S)X^\alpha - BR^{-1}B^*Y^\alpha - BR^{-1}D^*Z^\alpha + \varphi \right)ds \\
\quad + \left( \alpha(C - DR^{-1}S)X^\alpha - DR^{-1}B^*Y^\alpha - DR^{-1}D^*Z^\alpha \right) + \psi \right) dW(s), \\
\frac{dY^\alpha}{dt} = -\left( \alpha(Q - S^*R^{-1}S)Y^\alpha + \alpha(A^* - S^*R^{-1}B^*)Y^\alpha \right)ds \\
\quad + \alpha(C^* - S^*R^{-1}D^*)Z^\alpha + \gamma ds + Z^\alpha dW(s), \\
X^\alpha(t) = \xi, \quad Y^\alpha(T) = \alpha G^\alpha(X^\alpha(T)) + \eta,
\end{array}
\right. \\

s \in [t, T],
\end{aligned}
$$

(3.9)$_\alpha$

where 

$$(\xi, \varphi, \psi, \gamma, \eta) \in M[t, T] = L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; L^1(t, T; \mathbb{R}^n)) \times L^2(t, T; \mathbb{R}^n) \times L^2(\Omega; L^1(t, T; \mathbb{R}^n)) \times L^2(\Omega; \mathbb{R}^n).$$

The following gives an a priori estimate of the adapted solution $(X^\alpha(\cdot), Y^\alpha(\cdot), Z^\alpha(\cdot))$ to the parameterized equation (3.9)$_\alpha$.  

**Proposition 3.5.** Let (H1)–(H3) hold and $\alpha \in [0, 1]$ be given. Let $(X_1, Y_1, Z_1)$ and $(X_2, Y_2, Z_2)$ be the solutions to (3.9)$_\alpha$, corresponding to the different $(\xi_1, \varphi_1, \psi_1, \gamma_1, \eta_1)$, $(\xi_2, \varphi_2, \psi_2, \gamma_2, \eta_2) \in M[t, T]$, respectively. Then the following estimate holds:  

$$
\begin{aligned}
\mathbb{E}\left[ \sup_{s \in [t, T]} |X_1(s) - X_2(s)|^2 + \sup_{s \in [t, T]} |Y_1(s) - Y_2(s)|^2 + \int_t^T |Z_1(s) - Z_2(s)|^2 ds \right] \\
\leq K \left\{ \|\xi_1 - \xi_2\|^2_2 + \|\eta_1 - \eta_2\|^2_2 + \int_t^T \|\varphi_1(s) - \varphi_2(s)\|_2 ds + \int_t^T \|\psi_1(s) - \psi_2(s)\|_2 ds \right\}
\end{aligned}
$$

(3.10)

where $K > 0$ is a constant depending on $\|A(\cdot) - B(\cdot)\mathcal{R}(\cdot)^{-1}\mathcal{S}(\cdot)\|_1$, $\|C(\cdot) - D(\cdot)\mathcal{R}(\cdot)^{-1}\mathcal{S}(\cdot)\|_2$, $\|Q(\cdot) - S(\cdot)^*\mathcal{R}(\cdot)^{-1}\mathcal{S}(\cdot)\|_1$, $\|B(\cdot)\|_2$, $\|D(\cdot)\|_\infty$, $\|\mathcal{R}(\cdot)^{-1}\|_\infty$, $\|\mathcal{G}\|$, and independent of $\alpha \in [0, 1]$.  

**Proof.** For convenience, we denote $(\tilde{X}, \tilde{Y}, \tilde{Z}) := (X_1 - X_2, Y_1 - Y_2, Z_1 - Z_2)$, which satisfies  

$$
\begin{aligned}
\left\{ 
\begin{array}{l}
\frac{d\tilde{X}}{dt} = \left( \alpha(A - BR^{-1}S)\tilde{X} - BR^{-1}B^*\tilde{Y} - BR^{-1}D^*\tilde{Z} + \varphi \right)ds \\
\quad + \left( \alpha(C - DR^{-1}S)\tilde{X} - DR^{-1}B^*\tilde{Y} - DR^{-1}D^*\tilde{Z} + \psi \right) dW(s), \\
\frac{d\tilde{Y}}{dt} = -\left( \alpha(Q - S^*R^{-1}S)\tilde{X} + \alpha(A^* - S^*R^{-1}B^*)\tilde{Y} \right)ds \\
\quad + \alpha(C^* - S^*R^{-1}D^*)\tilde{Z} + \gamma ds + \tilde{Z} dW(s), \\
\tilde{X}(t) = \xi, \quad \tilde{Y}(T) = \alpha G^{\alpha}(\tilde{X}(T)) + \tilde{\eta},
\end{array}
\right. \\

s \in [t, T],
\end{aligned}
$$

(3.11)

with $(\tilde{\xi}, \tilde{\varphi}, \tilde{\psi}, \tilde{\gamma}, \tilde{\eta}) = (\xi_1 - \xi_2, \varphi_1 - \varphi_2, \psi_1 - \psi_2, \gamma_1 - \gamma_2, \eta_1 - \eta_2)$.  

For the forward equation, by applying Proposition 2.2, we have  

$$
\begin{aligned}
\mathbb{E}\left[ \sup_{s \in [t, T]} |\tilde{X}(s)|^2 \right] \\
\leq K \left\{ \|	ilde{\xi}\|^2_2 + \int_t^T |BR^{-1}(B^*\tilde{Y} + D^*\tilde{Z}) + \varphi| ds \right\}^2 + \left\{ \int_t^T -DR^{-1}(B^*\tilde{Y} + D^*\tilde{Z}) + \psi \right\}^2 ds \right\} \\
\leq K \left\{ \|	ilde{\xi}\|^2_2 + \int_t^T |\tilde{\varphi}| ds \right\}^2 + \int_t^T \|	ilde{\psi}\|^2 ds \right\} + \left\{ \|B(\cdot)\|^2_2 + \|D(\cdot)\|^2_\infty \right\} \|\mathcal{R}(\cdot)^{-1}\|_{\infty} \int_t^T \|B^*\tilde{Y} + D^*\tilde{Z}\|^2 ds \right\}
\end{aligned}
$$

(3.12)

$$
\begin{aligned}
\leq K \left\{ \|	ilde{\xi}\|^2_2 + \int_t^T |\tilde{\varphi}| ds \right\}^2 + \int_t^T \|	ilde{\psi}\|^2 ds + \int_t^T \|B^*\tilde{Y} + D^*\tilde{Z}\|^2 ds \right\},
\end{aligned}
$$

(3.13)

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where $K$ depends on $\|A(\cdot) - B(\cdot)R(\cdot)^{-1}S(\cdot)\|_1$, $\|C(\cdot) - D(\cdot)R(\cdot)^{-1}S(\cdot)\|_2$, $\|B(\cdot)\|_2$, $\|D(\cdot)\|_\infty$, $\|\mathcal{R}(\cdot)^{-1}\|_\infty$, independent of $\alpha$.

Similarly, for the backward equation, Proposition 2.6 leads to

$$
\mathbb{E}\left[ \sup_{s \in [t,T]} |\tilde{Y}(s)|^2 + \int_t^T |\tilde{Z}(s)|^2 ds \right] 
\leq K \left\{ \|\alpha G \tilde{X}(T) + \tilde{\eta}\|_2^2 + \int_t^T |\alpha (Q - S^*R^{-1}S)\tilde{X} + \tilde{\gamma}|ds \right\}_2^2 
\leq K \left\{ \|\tilde{\eta}\|_2^2 + \int_t^T |\tilde{\gamma}|ds \right\}_2^2 + \left( \|G\|_2^2 + \|Q(\cdot) - S(\cdot)^*R(\cdot)^{-1}S(\cdot)\|_1^2 \right) \mathbb{E}\left[ \sup_{s \in [t,T]} |\tilde{X}(s)|^2 \right] 
\leq K \left\{ \|\tilde{\eta}\|_2^2 + \int_t^T |\tilde{\gamma}|ds \right\}_2^2 + \mathbb{E}\left[ \sup_{s \in [t,T]} |\tilde{X}(s)|^2 \right],
$$

where $K$ depends on $\|A(\cdot) - B(\cdot)R(\cdot)^{-1}S(\cdot)\|_1$, $\|C(\cdot) - D(\cdot)R(\cdot)^{-1}S(\cdot)\|_2$, $\|Q(\cdot) - S(\cdot)^*R(\cdot)^{-1}S(\cdot)\|_1$, $\|B(\cdot)\|_2$, $\|D(\cdot)\|_\infty$, $\|\mathcal{R}(\cdot)^{-1}\|_\infty$, $\|\mathcal{G}\|$, independent of $\alpha$. Then, we get

$$
\mathbb{E}\left[ \sup_{s \in [t,T]} |\tilde{X}(s)|^2 + \sup_{s \in [t,T]} |\tilde{Y}(s)|^2 + \int_t^T |\tilde{Z}(s)|^2 ds \right] 
\leq K \left\{ \|\tilde{\xi}\|_2^2 + \|\tilde{\eta}\|_2^2 + \int_t^T |\tilde{\varphi}|ds \right\}_2^2 + \int_t^T \|\tilde{\psi}\|_2^2 ds + \int_t^T \|\tilde{\gamma}\|ds \right\}_2^2 + \int_t^T \|B^*\tilde{Y} + D^*\tilde{Z}\|_2^2 dr.
$$

with $K$ depending on $\|A(\cdot) - B(\cdot)R(\cdot)^{-1}S(\cdot)\|_1$, $\|C(\cdot) - D(\cdot)R(\cdot)^{-1}S(\cdot)\|_2$, $\|Q(\cdot) - S(\cdot)^*R(\cdot)^{-1}S(\cdot)\|_1$, $\|B(\cdot)\|_2$, $\|D(\cdot)\|_\infty$, $\|\mathcal{R}(\cdot)^{-1}\|_\infty$, $\|\mathcal{G}\|$, independent of $\alpha$.

On the other hand, by applying Itô’s formula to $\langle \tilde{X}(\cdot), \tilde{Y}(\cdot) \rangle$, we have

$$
\mathbb{E}\left\{ \langle \tilde{X}(T), \alpha G \tilde{X}(T) + \tilde{\eta} \rangle - \langle \tilde{X}(t), \tilde{Y}(t) \rangle \right\} = \mathbb{E}\left\{ \langle \tilde{X}(T), \tilde{Y}(T) \rangle - \langle \tilde{X}(t), \tilde{Y}(t) \rangle \right\} 
= \mathbb{E}\left\{ \int_t^T \left\{ \langle \alpha (A - BR^{-1}S)\tilde{X} - BR^{-1}B^*\tilde{Y} - BR^{-1}D^*\tilde{Z} + \tilde{\varphi}, \tilde{Y} \rangle - \langle \tilde{X}, \alpha (Q - S^*R^{-1}S)\tilde{X} + \alpha (A^* - S^*R^{-1}B^*)\tilde{Y} + \alpha (C^* - S^*R^{-1}D^*)\tilde{Z} + \tilde{\gamma} \rangle \right\} ds \right\} 
= \mathbb{E}\left\{ \int_t^T \left\{ \langle -BR^{-1}B^*\tilde{Y} - BR^{-1}D^*\tilde{Z}, \tilde{Y} \rangle - \alpha (\tilde{X}, (Q - S^*R^{-1}S)\tilde{X}) \right\} ds \right\} 
\leq \mathbb{E}\left\{ \int_t^T \left\{ \langle -BR^{-1}B^*\tilde{Y} - BR^{-1}D^*\tilde{Z}, \tilde{Y} \rangle - \alpha (\tilde{X}, (Q - S^*R^{-1}S)\tilde{X}) \right\} ds \right\} 
= -\mathbb{E}\left\{ \int_t^T \left\{ \langle R^{-1}(B^*\tilde{Y} + D^*\tilde{Z}), B^*\tilde{Y} + D^*\tilde{Z} \rangle - \alpha (\tilde{X}, (Q - S^*R^{-1}S)\tilde{X}) \right\} ds \right\}.
$$

Hence,

$$
\mathbb{E}\left\{ \int_t^T \langle R^{-1}(B^*\tilde{Y} + D^*\tilde{Z}), B^*\tilde{Y} + D^*\tilde{Z} \rangle dr \right\} 
= -\mathbb{E}\left\{ \langle G \tilde{X}(T), \tilde{X}(T) \rangle + \int_t^T \langle (Q - S^*R^{-1}S)\tilde{X}, \tilde{X} \rangle dr \right\} 
+ \mathbb{E}\left\{ \langle \tilde{\xi}, \tilde{Y}(t) \rangle - \langle \tilde{X}(T), \tilde{Y}(t) \rangle + \int_t^T \left[ \langle \tilde{\varphi}, \tilde{Y} \rangle + \langle \tilde{\psi}, \tilde{Z} \rangle - \langle \tilde{\gamma}, \tilde{X} \rangle \right] dr \right\}.
$$

By (H3), we know

$$
\mathbb{E}\left\{ \int_t^T |B^*\tilde{Y} + D^*\tilde{Z}|^2 ds \right\} \leq \frac{\|R\|^2}{\delta} \mathbb{E}\left\{ \langle \tilde{\xi}, \tilde{Y}(t) \rangle - \langle \tilde{\eta}, \tilde{X}(T) \rangle + \int_t^T \left[ \langle \tilde{\varphi}, \tilde{Y} \rangle + \langle \tilde{\psi}, \tilde{Z} \rangle - \langle \tilde{\gamma}, \tilde{X} \rangle \right] dr \right\}.
$$

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Then, for any \( \varepsilon > 0 \), we have

\[
E \int_t^T |B^s \tilde{Y} + D^s \tilde{Z}|^2 \, ds \leq \varepsilon E \left[ \sup_{s \in [t, T]} |\tilde{X}(s)|^2 + \sup_{s \in [t, T]} |\tilde{Y}(s)|^2 + \int_t^T |\tilde{Z}(s)|^2 \, ds \right] \\
+ \frac{\|R\|}{4\varepsilon} \left\{ \|\tilde{\xi}\|_2^2 + \|\tilde{\eta}\|_2^2 + \left\| \int_t^T |\tilde{\varphi}| \, dr \right\|_2^2 + \left\| \int_t^T |\tilde{\gamma}| \, dr \right\|_2^2 + \int_t^T \|\tilde{\psi}\|_2^2 \, dr \right\}.
\]  

(3.14)

By selecting \( \varepsilon = 1/(2K) \) (\( K \) is the one in (3.12)), we get the desired result (3.10) from (3.12) and (3.14).

**Remark 3.6.** It is known that, for general coupled FBSDEs, monotonicity condition leads to the solvability ([13]). Our condition (H3) is basically the monotonicity condition for linear FBSDEs with operator coefficients.

Next lemma gives the method of continuation.

**Lemma 3.7.** Let (H1)–(H3) hold. Then there exists a constant \( \varepsilon_0 > 0 \) such that for any given \( \alpha_0 \in [0, 1) \), if FBSDE (3.9)\( _{\alpha_0} \) admits a unique adapted solution for any \( (\xi, \varphi, \psi, \gamma, \eta) \in M[t, T] \), then for \( \varepsilon = \alpha_0 + \varepsilon \) with \( \varepsilon \in (0, \varepsilon_0], \alpha_0 + \varepsilon \leq 1 \), (3.9)\( _{\alpha} \) also admits a unique adapted solution for any \( (\xi, \varphi, \psi, \gamma, \eta) \in M[t, T] \).

**Proof.** Let \( \varepsilon > 0 \) be undetermined, and \( \varepsilon \in (0, \varepsilon_0] \). We focus on the following FBSDE:

\[
\begin{aligned}
dX &= \left( \alpha_0(A - BR^{-1}S)X - BR^{-1}(B^*Y + D^*Z) + \varepsilon(A - BR^{-1}S)X + \varphi \right) \, ds + \left( \alpha_0(C - DR^{-1}S)X - DR^{-1}(B^*Y + D^*Z) + \varepsilon(C - DR^{-1}S)X + \psi \right) \, dW(s), \\
dY &= -\left( \alpha_0(A^* - S^*R^{-1}B^*)Y + \alpha_0(C^* - S^*R^{-1}D^*)Z + \alpha_0(Q - S^*R^{-1}S)X \right) \, ds \\
&\quad + \varepsilon(C^* - S^*R^{-1}D^*)Z + \varepsilon(Q - S^*R^{-1}S)X + \gamma \right) \, ds \\
+ Z \, dW(s), \\
X(t) &= \xi, \quad Y(T) = \alpha_0 G X(T) + \varepsilon G X(T) + \eta,
\end{aligned}
\]

(3.15)

where \((X, Y, Z) \in [L_2^2(\Omega; C([t, T]; \mathbb{R}^n))]^2 \times L_2^2(t, T; \mathbb{R}^n)\) is arbitrarily chosen.

Our assumption ensures the solvability of the above equation. Therefore, we can define a mapping \( \mathcal{L}_{\alpha_0 + \varepsilon} \) from the space \([L_2^2(\Omega; C([t, T]; \mathbb{R}^n))]^2 \times L_2^2(t, T; \mathbb{R}^n)\) into itself as follows:

\[
(X(\cdot), Y(\cdot), Z(\cdot)) = \mathcal{L}_{\alpha_0 + \varepsilon}(X(\cdot), Y(\cdot), Z(\cdot)).
\]

For another given \((\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot)) \in [L_2^2(\Omega; C([t, T]; \mathbb{R}^n))]^2 \times L_2^2(t, T; \mathbb{R}^n)\), let

\[
(X(\cdot), Y(\cdot), Z(\cdot)) = \mathcal{L}_{\alpha_0 + \varepsilon}(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot)).
\]
Denote $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot)) = (X(\cdot) - \bar{X}(\cdot), Y(\cdot) - \bar{Y}(\cdot), Z(\cdot) - \bar{Z}(\cdot))$. From Proposition 3.5, we have

\[
\mathbb{E} \left[ \sup_{s \in [t,T]} |\tilde{X}(s)|^2 + \sup_{s \in [t,T]} |\tilde{Y}(s)|^2 + \int_t^T |\tilde{Z}(s)|^2 ds \right] \\
\leq K \left\{ \left\| \mathcal{G} \tilde{X}(T) \right\|^2_T + \left( \int_t^T \left\| (A - BR^{-1}S)\tilde{X} \right\|_2^2 dr \right)^2 + \int_t^T \left\| C - DR^{-1}S \right\|_2^2 dr \\
+ \left( \int_t^T \left\| (\mathcal{A}^* - S^*R^{-1}B^*)\tilde{Y} + \left( C^* - S^*R^{-1}D^* \right)\tilde{Z} + (Q - S^*R^{-1}S)\tilde{Y} \right\|_2^2 dr \right)^2 \right\} \\
\leq K \left\{ \left\| \mathcal{G} \tilde{X}(T) \right\|^2_T + \left( \left\| \mathcal{A}(\cdot) - B(\cdot)R(\cdot)^{-1}S(\cdot) \right\|_2^2 + \left\| C(\cdot) - D(\cdot)R(\cdot)^{-1}S(\cdot) \right\|_2^2 \right)^2 + \left\| Q(\cdot) - S(\cdot)^*R(\cdot)^{-1}S(\cdot) \right\|_2^2 \right\} \\
+ \left[ \sup_{s \in [t,T]} |\tilde{Y}(s)|^2 \right] \right\} \\
+ \left[ \sup_{s \in [t,T]} \left| \tilde{Z}(s) \right|^2 + \int_t^T \left| \tilde{Z}(s) \right|^2 ds \right],
\]

where $K > 0$ is a constant independent of $\alpha_0$ and $\varepsilon$. Therefore, we can choose $\varepsilon_0 > 0$, such that $K \varepsilon_0^2 \leq 1/4$.

Then, for any $\varepsilon \in [0, \varepsilon_0]$, $\mathcal{L}_{\alpha_0 + \varepsilon}$ is a contraction map. Consequently, there exists a unique fixed point for the mapping $\mathcal{L}_{\alpha_0 + \varepsilon}$ which is just the unique solution to FBSDE (3.9)$_{\alpha}$ with $\alpha = \alpha_0 + \varepsilon$.

Now we present a proof of Theorem 3.4.

Proof of Theorem 3.4. When $\alpha = 0$, (3.9)$_0$ becomes

\[
\begin{align*}
\begin{cases}
dX^0 = \left( -BR^{-1}(B^*Y^0 + D^*Z^0) + \varphi \right) ds + \left( -DR^{-1}(B^*Y^0 + D^*Z^0) + \psi \right) dW(s), \\
\int_t^T |\tilde{Z}(s)|^2 ds \\leq \left\| \mathcal{G} \tilde{X}(T) \right\|^2_T \\
\int_t^T \left\| \tilde{Y}(s) \right|^2 ds \\
\int_t^T \left\| \tilde{Z}(s) \right|^2 ds \\
\end{cases}
\end{align*}
\]

whose solvability is clear. In fact, one can first solve the BSDE to obtain $(Y^0, Z^0)$, then solve the FSDE to get $X^0$.

Next, by Lemma 3.7, for any $(\xi, \varphi, \psi, \gamma, \eta) \in M[t,T]$, and any $\alpha \in [0,1]$, (3.9)$_{\alpha}$ is uniquely solvable. In particular, when $\alpha = 1$, (3.9)$_1$ with

(3.16) $\xi = x, \quad \varphi = -BR^{-1}b + b, \quad \psi = -DR^{-1}b + b, \quad \gamma = -SR^{-1}b + q, \quad \eta = g$

becomes (3.4) which is also uniquely solvable.

Let $\alpha = 1$, $(\xi, \varphi, \psi, \gamma, \eta)$ is given by (3.16), $(\xi_2, \varphi_2, \psi_2, \gamma_2, \eta_2) = (0, 0, 0, 0, 0)$ (the corresponding solution
4 Mean-Field LQ Control Problem

We have mentioned that a major motivation of this work is the study of stochastic LQ problem of mean-field FSDE with cost functional also involving mean-field terms. In this section, we will carry out some details to FBSDE is \((0,0,0)\). By the a priori estimate \((3.10)\), we have

\[
E\left[ \sup_{t \in [0,T]} |X(t)|^2 + \sup_{t \in [0,T]} |Y(t)|^2 + \int_0^T |Z(s)|^2 ds \right] 
\leq K \left\{ \|x\|^2 + \|g\|^2 + \int_0^T |B(s)R^{-1}(s)\rho(s) + b(s)| ds \right\}^2
\]

\[
+ \int_0^T \| - D(s)R^{-1}(s) \sigma(s) + \sigma(s) \|^2 ds + \int_0^T \| - S(s)R^{-1}(s) \rho(s) + q(s) \|^2 ds \right\}
\]

We obtain \((3.8)\).

**Remark 3.8.** Note that, in our setting, the norms \(\|A(\cdot) - B(\cdot)R^{-1}(\cdot)S(\cdot)\|, \|Q(\cdot) - S(\cdot)^*R^{-1}(\cdot)S(\cdot)\|, \|B(\cdot)R^{-1}(\cdot)B(\cdot)^*\| \) of coefficients are only required to belong to \(L^1(t,T;\mathbb{R})\), and \(\|C(\cdot) - D(\cdot)R^{-1}(\cdot)S(\cdot)\|, \|D(\cdot)R^{-1}(\cdot)B(\cdot)^*\| \) are not necessarily bounded.

**Corollary 3.9.** Under \((H1)-(H3)\), Problem \((OLQ)\) admits a unique open-loop optimal control given by \((3.6)\), where \((X(\cdot), Y(\cdot), Z(\cdot))\) is the unique solution to FBSDE \((3.7)\).

**Proof.** Let us denote by \(J^u(t; u(\cdot))\) the cost functional \((1.3)\) when \(x, b(\cdot), \sigma(\cdot), g, q(\cdot), \rho(\cdot)\) are all 0. Obviously, Assumption \((H3)-(ii)\) implies \(J^u(t; u(\cdot)) \geq 0\) for all \(u(\cdot) \in \mathcal{U}[t,T]\). By Proposition 2.5, we know \(u(\cdot) \mapsto J(t,x,u(\cdot))\) is convex. Moreover, by Theorem 3.4 and the expression \((3.6)\), there exists a unique \((X(\cdot), Y(\cdot), Z(\cdot))\) satisfying \((3.2)\) and \((3.3)\). Thanks to Theorem 3.2, we obtain the result.
with the above coefficients given by (2.3)-(2.9).

We introduce the following hypothesis for the involved coefficients in the state equation.

(H4) For \( k \geq 1 \), let

\[
A, \tilde{A}_k, \bar{A}_k, C, \bar{C}_k, \tilde{C}_k : [0, T] \times \Omega \to \mathbb{R}^{n \times n}, \quad B, \tilde{B}_k, \bar{B}_k, D, \bar{D}_k, \tilde{D}_k : [0, T] \times \Omega \to \mathbb{R}^{n \times m},
\]

be \( F \)-progressively measurable processes satisfying

\[
\begin{array}{l}
\text{(i)} \quad \int_0^T \left[ \sup_{\omega \in \Omega} |A(s, \omega)| + \left( \sum_{k \geq 1} \mathbb{E}[|\tilde{A}_k(s)|^2] \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}[|\bar{A}_k(s)|^2] \right)^{\frac{1}{2}} \right] ds < \infty, \\
\text{(ii)} \quad \int_0^T \left[ \sup_{\omega \in \Omega} |B(s, \omega)|^2 + \left( \sum_{k \geq 1} \mathbb{E}[|\tilde{B}_k(s)|^2] \right) \left( \sum_{k \geq 1} \mathbb{E}[|\bar{B}_k(s)|^2] \right) \right] ds < \infty, \\
\text{(iii)} \quad \int_0^T \left[ \sup_{\omega \in \Omega} |C(s, \omega)|^2 + \left( \sum_{k \geq 1} \mathbb{E}[|\tilde{C}_k(s)|^2] \right) \left( \sum_{k \geq 1} \mathbb{E}[|\bar{C}_k(s)|^2] \right) \right] ds < \infty, \\
\text{(iv)} \quad \sup_{s \in [0, T]} \left[ \sup_{\omega \in \Omega} |D(s, \omega)| + \left( \sum_{k \geq 1} \mathbb{E}[|\tilde{D}_k(s)|^2] \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}[|\bar{D}_k(s)|^2] \right)^{\frac{1}{2}} \right] < \infty,
\end{array}
\]

and \( b(\cdot) \in L^2_\mathcal{F}(\Omega; L^1(0, T; \mathbb{R}^n)), \sigma(\cdot) \in L^2_\mathcal{F}(0, T; \mathbb{R}^n) \).

Note that (H4) ensures the operators \( \mathcal{A}(\cdot), B(\cdot), C(\cdot), D(\cdot) \) satisfy (H1). In fact, take any \( \xi \in L^2_{\mathcal{F}}(\Omega; \mathbb{R}^n) \), we have

\[
\|A(s)\xi\|_2 = \left( \mathbb{E}|A(s)\xi|^2 \right)^{\frac{1}{2}} = \left( \mathbb{E}|A(s)\xi| + \sum_{k \geq 1} \mathbb{E}|\tilde{A}_k(s)| |\bar{A}_k(s)\xi| \right)^{\frac{1}{2}} \leq \left\{ \sup_{\omega \in \Omega} |A(s, \omega)| + \left( \mathbb{E}\left| \sum_{k \geq 1} |\tilde{A}_k(s)| (|\bar{A}_k(s)|^2)^{\frac{1}{2}} \|\xi\|_2 \right|^2 \right)^{\frac{1}{2}} \right\} \|\xi\|_2.
\]

Then, (4.3)-(i) implies that

\[
\int_0^T \|A(s)\| ds \leq \int_0^T \left[ \sup_{\omega \in \Omega} |A(s, \omega)| + \left( \sum_{k \geq 1} \mathbb{E}[|\tilde{A}_k(s)|^2] \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}[|\bar{A}_k(s)|^2] \right)^{\frac{1}{2}} \right] ds < \infty.
\]

Therefore, \( \mathcal{A}(\cdot) \in \mathcal{L}^2_\mathcal{F}(L^2(\Omega; \mathbb{R}^n)) \).

Similarly, (4.3)-(ii), (iii), (iv) imply that \( B(\cdot) \in \mathcal{L}^2_\mathcal{F}(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)), C(\cdot) \in \mathcal{L}^2_\mathcal{F}(L^2(\Omega; \mathbb{R}^n)), D(\cdot) \in \mathcal{L}^\infty_\mathcal{F}(L^2(\Omega; \mathbb{R}^n); L^2(\Omega; \mathbb{R}^n)) \), respectively.

Next, for the cost functional, we introduce the following hypothesis:

(H5) For \( i, j, k \geq 1 \), let

\[
G, \tilde{G}_k, \bar{G}_k, \tilde{G}_{ij} : \Omega \to \mathbb{R}^{n \times n}, \quad G^\top = G, \quad \tilde{G}^\top_{ij} = \tilde{G}_{ji},
\]

be \( \mathcal{F}_T \)-measurable and satisfy

\[
\mathbb{E}\sup_{\omega \in \Omega} |G(\omega)| + \sum_{k \geq 1} \mathbb{E} \left( |\tilde{G}_k|^2 + |\bar{G}_k|^2 \right) + \sum_{i,j \geq 1} \mathbb{E} |\tilde{G}_{ij}|^2 < \infty.
\]
and for $i,j,k \geq 1$, let

\begin{align*}
Q, \tilde{Q}_k, \bar{Q}_k, \tilde{Q}_{ij} : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, \quad Q(\cdot)^\top = Q(\cdot), \quad \tilde{Q}_{ij}(\cdot)^\top = \tilde{Q}_{ij}(\cdot), \\
R, \tilde{R}_k, \bar{R}_k, \tilde{R}_{ij} : [0, T] \times \Omega \rightarrow \mathbb{R}^{m \times m}, \quad R(\cdot)^\top = R(\cdot), \quad \tilde{R}_{ij}(\cdot)^\top = \tilde{R}_{ij}(\cdot), \\
S, \tilde{S}_k, \bar{S}_k, \tilde{S}_{ij} : [0, T] \times \Omega \rightarrow \mathbb{R}^{m \times n}, \quad \bar{S}_k : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times m},
\end{align*}

be $\mathbb{F}$-progressively measurable processes satisfying

\begin{equation}
\begin{aligned}
&\text{(i)} \quad \int_0^T \left\{ \sup_{\omega \in \Omega} |Q(s, \omega)| + \sum_{k \geq 1} \mathbb{E}\left( |\tilde{Q}_k(s)|^2 + |Q_k(s)|^2 \right) \\
&\quad + \left( \sum_{i,j \geq 1} |\mathbb{E}[\tilde{Q}_{ij}(s)]|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} |\mathbb{E}[\tilde{Q}_k(s)]|^2 \right) \right\} ds < \infty, \\
&\text{(ii)} \quad \int_0^T \left\{ \sup_{(s, \omega) \in [0, T] \times \Omega} |R(s, \omega)| + \sup_{s \in [0, T]} \left[ \sum_{k \geq 1} \mathbb{E}\left( |\tilde{R}_k(s)|^2 + |R_k(s)|^2 \right) + \sum_{i,j \geq 1} |\mathbb{E}[\tilde{R}_{ij}(s)]|^2 \right] < \infty, \\
&\text{(iii)} \quad \int_0^T \left\{ \sup_{\omega \in \Omega} |S(s, \omega)|^2 + \left[ \sum_{k \geq 1} \mathbb{E}\left( |\tilde{S}_k(s)|^2 + |S_k(s)|^2 + |\tilde{S}_{ij}(s)|^2 + |\bar{S}_k(s)|^2 \right) \right]^2 \\
&\quad + \left( \sum_{i,j \geq 1} |\mathbb{E}[\tilde{S}_{ij}(s)]|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} |\mathbb{E}[\tilde{S}_k(s)]|^2 \right)^{\frac{1}{2}} \right\} ds < \infty.
\end{aligned}
\end{equation}

Also, $g_0 \in L^2(\Omega; \mathbb{R}^n)$, $q_0(\cdot) \in L^2(\Omega; L^1(0, T; \mathbb{R}^n))$, $\rho_0(\cdot) \in L^2(0, T; \mathbb{R}^m)$, and $\bar{q}_k : \Omega \times [0, T] \rightarrow \mathbb{R}^n$, $\bar{\rho}_k : \Omega \times [0, T] \rightarrow \mathbb{R}^m$ are $\mathbb{F}$-progressively measurable, and

\begin{equation}
\sum_{k \geq 1} |\mathbb{E}[\bar{q}_k(s)]|^2 + \int_0^T \sum_{k \geq 1} \left( |\mathbb{E}[\bar{q}_k(s)]|^2 + |\mathbb{E}[\bar{\rho}_k(s)]|^2 \right) ds < \infty.
\end{equation}

Under (H5), one has all the conditions in (H2) are satisfied. Detailed calculations are collected in the appendix.

Now, for any $(t, x) \in \mathbb{D}$, our state equation is given by

\begin{equation}
\begin{aligned}
&dX(s) = \left( A(s)X(s) + \tilde{A}(s)^\top \mathbb{E}[\tilde{A}(s)X(s)] + B(s)u(s) + \bar{B}(s)^\top \mathbb{E}[\bar{B}(s)u(s)] + b(s) \right) ds \\
&+ \left( C(s)X(s) + \tilde{C}(s)^\top \mathbb{E}[\tilde{C}(s)X(s)] + D(s)u(s) + \bar{D}(s)^\top \mathbb{E}[\bar{D}(s)u(s)] + \sigma(s) \right) dW(s), \quad s \in [t, T], \\
X(t) = x,
\end{aligned}
\end{equation}

and the quadratic cost functional is

\begin{equation}
\begin{aligned}
J(t, x; u(\cdot)) = \mathbb{E}\left\{ \langle GX(T), X(T) \rangle + 2\langle G^\top X(T), \mathbb{E}[\tilde{G}X(T)] \rangle + \langle \tilde{G}^\top \mathbb{E}[\tilde{G}X(T)], \mathbb{E}[\tilde{G}X(T)] \rangle \right. \\
+ 2\langle g_0, X(T) \rangle + 2\langle g, \mathbb{E}[\tilde{G}X(T)] \rangle + \int_t^T \left[ \langle QX(s), X(s) \rangle + 2\langle QX, \mathbb{E}[\tilde{Q}X] \rangle + \langle \tilde{Q}\mathbb{E}[\tilde{Q}X], \mathbb{E}[\tilde{Q}X] \rangle \right] \\
+ 2\langle SX, u \rangle + 2\langle \tilde{S}X, \mathbb{E}[\tilde{R}u] \rangle + 2\langle \mathbb{E}[\tilde{S}X], \mathbb{E}[\tilde{R}u] \rangle + \langle \tilde{S}\mathbb{E}[\tilde{Q}X], \mathbb{E}[\tilde{R}u] \rangle + \langle \tilde{R}u, \mathbb{E}[\tilde{R}u] \rangle \right. \\
+ \left. \langle \tilde{R}\mathbb{E}[\tilde{R}u], \mathbb{E}[\tilde{R}u] \rangle + 2\langle \bar{q}_0, X \rangle + 2\langle \bar{q}, \mathbb{E}[\tilde{Q}X] \rangle + 2\langle \bar{q}_0, u \rangle + 2\langle \bar{\rho}, \mathbb{E}[\tilde{R}u] \rangle \right\} ds.
\end{aligned}
\end{equation}

The argument $s$ is suppressed in the above functional.

According to a standard well-posedness result of MF-SDE (refer to [2]), for any $u(\cdot) \in L^2_2(t, T; \mathbb{R}^m)$, there exists a unique strong solution $X(\cdot) = X(\cdot; t, x, u(\cdot))$ to (4.7), and the cost functional $J(t, x; u(\cdot))$ is
well-defined. Further, the cost functional can be rewritten as follows:

\[
J(t, x; u(\cdot)) = E\left\{ \langle G\tilde{X}(T), \tilde{X}(T) \rangle + 2\langle g, \tilde{X}(T) \rangle + \int_{t}^{T} \left[ \langle Q(s)X(s), X(s) \rangle + 2\langle S(s)X(s), u(s) \rangle + \langle R(s)u(s), u(s) \rangle \right] ds \right\},
\]

where, for any \( s \in [t, T] \), the above used notations (introduced in Subsection 2.1) are repeated again as

\[
\tilde{X}(T) = \begin{pmatrix} X(T) \\ E[GX(T)] \end{pmatrix}, \quad X(s) = \begin{pmatrix} X(s) \\ E[Q(s)X(s)] \end{pmatrix}, \quad u(s) = \begin{pmatrix} u(s) \\ E[R(s)u(s)] \end{pmatrix},
\]

\[
G = \begin{pmatrix} G & G^\top \\ G & G \end{pmatrix}, \quad Q(s) = \begin{pmatrix} Q(s) & Q(s)^\top \\ Q(s)^\top & \tilde{Q}(s) \end{pmatrix}, \quad S(s) = \begin{pmatrix} S(s) & \tilde{S}(s) \\ \tilde{S}(s)^\top & \tilde{S}(s) \end{pmatrix},
\]

\[
R(s) = \begin{pmatrix} R(s) & R(s)^\top \\ R(s)^\top & \tilde{R}(s) \end{pmatrix}, \quad g = \begin{pmatrix} g_0 \\ g \end{pmatrix}, \quad q(s) = \begin{pmatrix} q_0(s) \\ q(s) \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} \rho_0(s) \\ \rho(s) \end{pmatrix},
\]

where \( G, Q, S, R \) are assumed to satisfy

\[
(4.10) \quad G \succeq 0, \quad \begin{pmatrix} Q(\cdot) & S(\cdot)^\top \\ S(\cdot) & R(\cdot) \end{pmatrix} \succeq 0, \quad R \succeq \delta \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.
\]

Now, we propose the MF-LQ stochastic optimal control problem as follows:

**Problem (MF-LQ).** For any given \( (t, x) \in \mathcal{D} \), find an admissible control \( \tilde{u}(\cdot) \in \mathcal{U}[t, T] \) such that

\[
(4.11) \quad J(t, x; \tilde{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)).
\]

In the above \( \tilde{u}(\cdot) \) is called an *open-loop optimal control* of Problem (MF-LQ), and the corresponding *optimal state trajectory* \( X^u(\cdot) \) is denoted by \( \tilde{X}(\cdot) \).

The following result characterizes the optimal control \( \tilde{u}(\cdot) \) of Problem (MF-LQ).

**Theorem 4.1.** Under (H4)–(H5). For any \( (t, x) \in \mathcal{D} \), \( \tilde{u}(\cdot) \in \mathcal{U}[t, T] \) is an open-loop optimal control of Problem (MF-LQ) at \( (t, x) \) with \( \tilde{X}(\cdot) \) being the corresponding open-loop optimal state process, if and only if \( u(\cdot) \rightarrow J(t, x; u(\cdot)) \) is convex and \( \{\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot), \tilde{u}(\cdot)\} \in [L^2_2(\Omega; \mathbb{C}([t, T]; \mathbb{R}^n))]^2 \times L^2_2(t, T; \mathbb{R}^n) \times \mathcal{U}[t, T] \) solves the following system (the argument \( s \) is suppressed)

\[
(4.12) \quad \begin{cases}
    d\tilde{X} = \left( A\tilde{X} + A^\top E[\tilde{A}\tilde{X}] + Bu + B^\top E[\tilde{B}u] + b \right) ds \\
    \quad + \left( C\tilde{X} + C^\top E[\tilde{C}\tilde{X}] + Du + D^\top E[\tilde{D}u] + \sigma \right) dW, \quad s \in [t, T], \\
    d\tilde{Y} = -\left( A^\top \tilde{Y} + A^\top E[\tilde{A}\tilde{Y}] + C^\top \tilde{Z} + C^\top E[\tilde{C}\tilde{Z}] + Q\tilde{X} + Q^\top E[\tilde{Q}\tilde{X}] + Q^\top E[\tilde{Q}\tilde{X}] \right) \\
    \quad + \tilde{Q}^\top E[\tilde{Q}] + \tilde{S}^\top \tilde{u} + \tilde{S}^\top E[\tilde{R} \tilde{u}] + \tilde{Q}^\top E[\tilde{S}] E[\tilde{R} \tilde{u}] + q ds \\
    \quad + \tilde{Z} dW, \quad s \in [t, T], \\
    \tilde{X}(t) = x, \\
    \tilde{Y}(T) = G \tilde{X}(T) + \tilde{G}^\top E[G\tilde{X}(T)] + \tilde{G}^\top E[G\tilde{X}(T)] + \tilde{G}^\top E[\tilde{G}\tilde{X}(T)] + g_0 + \tilde{G}^\top E[g], \\
    R \tilde{u} + \tilde{R}^\top E[\tilde{R} \tilde{u}] + \left( \tilde{R}^\top + \tilde{R}^\top E[\tilde{R}] \right) E[\tilde{R} \tilde{u}] + B^\top \tilde{Y} + B^\top E[\tilde{B}\tilde{Y}] + D^\top \tilde{Z} + D^\top E[\tilde{D}\tilde{Z}] \\
    \quad + S \tilde{X} + \tilde{S} E[\tilde{S}] + \tilde{Q}^\top E[\tilde{Q}] + \tilde{R}^\top E[\tilde{R}] E[\tilde{Q}] + \rho = 0, \quad s \in [t, T].
\end{cases}
\]

Moreover, if (4.10) holds true, then the above system (4.12) admits a unique solution, and Problem (MF-LQ) admits a unique optimal open-loop control \( \tilde{u}(\cdot) \).
Proof. According to the results obtained in Section 3 (Theorems 3.2 and 3.4, and Corollary 3.9), we only need to verify the operators in (4.1) and (4.2) satisfy Assumptions (H1), (H2) and (H3). By the calculations collected in the appendix, we know that (H4)—(H5) imply (H1)—(H2).

Next we will show (4.10) implies Assumption (H3). Firstly, for any $\xi \in L^2(\Omega; \mathbb{R}^n)$,

$$E(G\xi, \xi) = E(G\xi + \tilde{G}^T E[G\xi] + \tilde{G}^T E[G\xi] + \tilde{G}^T E[G\xi], \xi)$$

$$= E(G\xi, \xi) + 2E(G\xi, E[\tilde{G}\xi]) + E(GE[\tilde{G}\xi], E[\tilde{G}\xi]) = E(G\xi, \xi) \geq 0,$$

where $\tilde{\xi} \equiv (\xi^T, (E[\tilde{G}\xi])^T)$.

Secondly, for any $s \in [t, T]$, any $\xi \in L^2(\Omega; \mathbb{R}^n)$, any $\eta \in L^2(\Omega; \mathbb{R}^m)$, we have (the argument $s$ is suppressed for simplicity):

$$E\left( \begin{pmatrix} Q & S^T \\ S & R \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right) = E(Q\xi, \xi) + 2E(S\xi, \eta) + E(R\eta, \eta)$$

$$= \left\{ E(Q\xi, \xi) + 2E(Q\xi, E[Q\xi]) + E(QE[Q\xi], E[Q\xi]) \right\}$$

$$+ 2\left\{ E(S\xi, \eta) + E(S\xi, E[R\eta]) + E(SE[Q\xi], \eta) + E(SEQ[Q\xi], E[R\eta]) \right\}$$

$$+ \left\{ E(R\eta, \eta) + 2E(R\eta, E[R\eta]) + E(RE[R\eta], E[R\eta]) \right\}$$

$$= E(Q\xi, \xi) + 2E(S\xi, \eta) + E(R\eta, \eta) = E\left( \begin{pmatrix} Q & S^T \\ S & R \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right) \geq 0,$$

where $\tilde{E} \equiv (\xi^T, (E[Q\xi])^T)$ and $\tilde{\eta} \equiv (\eta^T, (E[R\eta])^T)$.

Thirdly, for any $s \in [t, T]$, any $\eta \in L^2(\Omega; \mathbb{R}^m)$,

$$E(R\eta, \eta) - \delta E|\eta|^2 = E(R\eta, \eta) - \delta E\left( \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \eta, \eta \right) = E\left( R - \delta \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right) \eta, \eta \geq 0.$$

Then our conclusion follows from the results in Section 3.

Remark 4.2. From the above proof, under (4.10), the inverse of operator $R(\cdot)$ exists. In other words, $\bar{u}(\cdot)$ can be solved from the algebraic equation (the last two lines) in the system (4.12), in terms of $(X(\cdot), Y(\cdot), Z(\cdot))$. Therefore, system (4.12) is a coupled mean-field FBSDE.

Next, we try to characterize the optimal control $\bar{u}(\cdot)$ by the solution of a Fredholm type integral equation of the second kind.

Proposition 4.3. Let (H4)–(H5) and (4.10) hold. Then for any given $(t, x) \in \mathcal{D}$, Problem (MF-LQ) admits a unique open-loop optimal control $\bar{u}(\cdot)$. Let $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$ be the corresponding adapted solution to the FBSDE (4.12), and let

$$\begin{cases}
\bar{\theta}(s, \omega) = -R(s, \omega)^{-1} \left( B(s, \omega)^T \bar{Y}(s, \omega) + \tilde{B}(s, \omega)^T E[B(s)\bar{Y}(s)] + D(s, \omega)^T \bar{Z}(s, \omega) \\
+ D(s, \omega)^T E[D(s)\bar{Z}(s)] + S(s, \omega)X(s, \omega) + \bar{R}(s, \omega)^T E[S(s)X(s)] \\
+ \tilde{S}(s, \omega)E[\tilde{Q}(s)X(s)] + \tilde{R}(s, \omega)^T E[\tilde{S}(s)\tilde{Q}(s)X(s)] + \rho(s, \omega), \quad \omega \in \Omega, \ s \in [t, T],
\end{cases}$$

$$\Gamma(s, \omega, \omega') = -R(s, \omega)^{-1} \left[ \tilde{R}(s, \omega)^T \bar{R}(s, \omega') + \left( \bar{R}(s, \omega)^T + \tilde{R}(s, \omega)^T E[\tilde{R}(s)] \right) \bar{R}(s, \omega') \right],$$

$$\omega, \omega' \in \Omega, \ s \in [t, T].$$

Then the following integral equation

$$u(s, \omega) = \bar{\theta}(s, \omega) + \int_{\Omega} \Gamma(s, \omega, \omega') u(s, \omega') dP(\omega'), \quad \omega \in \Omega, \ s \in [t, T],$$

(4.13)
admits a unique solution which is the open-loop optimal control \( \bar{u}(\cdot) \). Moreover,

\[
(4.14) \quad \bar{u}(s, \omega) = \tilde{\theta}(s, \omega) + \int_\Omega \Phi(s, \omega, \omega') \tilde{\theta}(s, \omega') d\mathbb{P}(\omega'), \quad (s, \omega) \in [t, T] \times \Omega,
\]

with \( \Phi(s, \omega, \omega') \) being the unique solution to the following equation:

\[
(4.15) \quad \Phi(s, \omega, \omega') = \Gamma(s, \omega, \omega') + \int_\Omega \Gamma'(s, \omega, \omega) \Phi(s, \omega, \omega') d\mathbb{P}(\omega), \quad (s, \omega, \omega') \in [t, T] \times \Omega \times \Omega,
\]

**Proof.** Note that

\[
\mathbb{E}(\mathcal{R}u, u) = \mathbb{E}\left( \begin{pmatrix} \bar{R} & \bar{R}^T \\ \bar{R} & \bar{R}^T \end{pmatrix} \begin{pmatrix} u \\ \mathbb{E}[\mathcal{R}u] \end{pmatrix}, \begin{pmatrix} u \\ \mathbb{E}[\mathcal{R}u] \end{pmatrix} \right) \geq \delta \mathbb{E}[u]^2.
\]

Thus, \( \mathcal{R} \) is invertible. This means that for any \( v \in \mathcal{U}[t, T] \),

\[
\mathcal{R}u = v
\]

admits a unique solution \( u \in \mathcal{U}[t, T] \).

We now write the equality (in (4.12)) which \( \tilde{u}(\cdot) \) satisfies more carefully as follows:

\[
(4.16) \quad R(t)u(t) + \bar{R}(t)^T \mathbb{E}[\bar{R}(t)u(t)] + \left( \bar{R}(t)^T + \bar{R}(t)^T \mathbb{E}[\bar{R}(t)] \right) \mathbb{E}[\bar{R}(t)u(t)] = \theta(t),
\]

with

\[
\theta(t) = -\left( B(t)^T \tilde{Y}(t) + \tilde{B}(t)^T \mathbb{E}[B(t)\tilde{Y}(t)] + D(t)^T \tilde{Z}(t) + \tilde{D}(t)^T \mathbb{E}[D(t)\tilde{Z}(t)] + S(t)\tilde{X}(t) + \bar{R}(t)^T \mathbb{E}[\tilde{S}(t)\tilde{X}(t)] + \bar{R}(t)^T \mathbb{E}[\tilde{Q}(t)\tilde{X}(t)] + \rho(t) \right).
\]

Note that (4.16) is indeed a Fredholm type integral equation of the second kind:

\[
R(t, \omega)u(t, \omega) + \int_\Omega \left[ \bar{R}(t, \omega)^T \bar{R}(t, \omega') + \bar{R}(t, \omega)^T \mathbb{E}[\bar{R}(t)] \right] u(t, \omega') d\mathbb{P}(\omega') = \theta(t, \omega),
\]

which can be rewritten into the following standard form of Fredholm integral equation of the second kind:

\[
(4.17) \quad u(t, \omega) = \tilde{\theta}(t, \omega) + \int_\Omega \Gamma(t, \omega, \omega') u(t, \omega') d\mathbb{P}(\omega'), \quad \omega \in \Omega, \; t \in [0, T],
\]

where

\[
\tilde{\theta}(t, \omega) = R(t, \omega)^{-1} \theta(t, \omega), \quad \Gamma(t, \omega, \omega') = -R(t, \omega)^{-1} \left[ \bar{R}(t, \omega)^T \bar{R}(t, \omega') + \left( \bar{R}(t, \omega)^T + \bar{R}(t, \omega)^T \mathbb{E}[\bar{R}(t)] \right) \bar{R}(t, \omega') \right].
\]

Our assumption has guaranteed the well-posedness of the above equation. If we define \( \Phi(t, \omega, \omega') \) to be the unique solution to the equation (4.15), and let \( \bar{u}(\cdot) \) be defined by (4.14). Then

\[
\int_\Omega \Gamma(s, \omega, \omega') \bar{u}(s, \omega') d\mathbb{P}(\omega') = \int_\Omega \Gamma(s, \omega, \omega') \tilde{\theta}(s, \omega') + \int_\Omega \Phi(s, \omega, \omega') \bar{u}(s, \omega') d\mathbb{P}(\omega') = \int_\Omega \Gamma(s, \omega, \omega') \bar{u}(s, \omega') + \int_\Omega \left[ \int_\Omega \Gamma(s, \omega, \omega') \Phi(s, \omega, \omega') d\mathbb{P}(\omega') \right] \tilde{\theta}(s, \omega') d\mathbb{P}(\omega) + \int_\Omega \left[ \Phi(s, \omega, \omega') - \Gamma(s, \omega, \omega') \right] \bar{u}(s, \omega') d\mathbb{P}(\omega') = \int_\Omega \Phi(s, \omega, \omega') \bar{u}(s, \omega') d\mathbb{P}(\omega') = \bar{u}(s, \omega) - \tilde{\theta}(s, \omega).
\]

Thus, \( \bar{u}(\cdot) \) defined by (4.14) is the solution to the integral equation (4.13).
5 Conclusion

It is the mean-field LQ Problem (MF-LQ) that inspires us to study the LQ optimal control problem with operator coefficients (i.e., Problem (OLQ)). As we known, (4.7)–(4.8) is a new form of mean-field LQ problems. Besides, all the coefficients are allowed to be random in our study. As a start, we only study the open-loop case. The closed-loop case of the control problems, as well as differential games are under our investigation.

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6 Appendix.

We collect some detailed computations of Section 4 below. Recall that

\[

g \xi = G \xi + \Gamma^T E [G \xi] + \Gamma^T E [\Gamma \xi], \quad \xi \in L^2(\Omega; \mathbb{R}^n),
\]

\[
Q(s) \xi = Q(s) \xi + \Gamma(s)^T E [Q(s) \xi] + \Gamma(s)^T E [\Gamma(s) \xi] + \Gamma(s)^T E [Q(s) \xi] + \Gamma(s)^T E [\Gamma(s) \xi], \quad \xi \in L^2_F(\Omega; \mathbb{R}^n),
\]

\[
S(s) \xi = S(s) \xi + R(s)^T E [S(s) \xi] + \Gamma(s)^T E [Q(s) \xi] + \Gamma(s)^T E [\Gamma(s) \xi] + \Gamma(s)^T E [Q(s) \xi] + \Gamma(s)^T E [\Gamma(s) \xi], \quad \xi \in L^2_F(\Omega; \mathbb{R}^n),
\]

\[
\mathcal{R}(s) \eta = R(s) \eta + \Gamma(s)^T E [\mathcal{R}(s) \eta] + \Gamma(s)^T E [R(s) \eta] + \Gamma(s)^T E [\Gamma(s) \eta] + \Gamma(s)^T E [R(s) \eta] + \Gamma(s)^T E [\Gamma(s) \eta], \quad \eta \in L^2_F(\Omega; \mathbb{R}^m),
\]

\[
g = g_0 + \Gamma^T E [g], \quad q(s) = q_0(s) + \Gamma(s)^T E [q(s)], \quad \rho(s) = \rho_0(s) + \Gamma(s)^T E [\rho(s)].
\]
This implies that
\[
\| G \|_2 \leq \| G \|_2 + \| G^T \mathbb{E}[G] \|_2 + \| G^T \mathbb{E}[G] \|_2 + \| G^T \mathbb{E}[G] \|_2
\]
\[
\leq \left( \mathbb{E}[\| G \|^2] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \sum_{k \geq 1} |G_k| \left( \mathbb{E}[G_k^2] \right)^{\frac{1}{2}} \| \xi \|_2^2 \right] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \sum_{k \geq 1} |G_k| \left( \mathbb{E}[G_k^2] \right)^{\frac{1}{2}} \| \xi \|_2^2 \right] \right)^{\frac{1}{2}}
\]
\[
+ \left( \mathbb{E} \left[ \sum_{i,j \geq 1} |G_{ij}| \left( \mathbb{E}[G_{ij}^2] \right)^{\frac{1}{2}} \| \xi \|_2^2 \right] \right)^{\frac{1}{2}}
\]
\[
\leq \left\{ \text{esssup}_{\omega \in \Omega} |G(\omega)| + \left( \sum_{k \geq 1} \mathbb{E}[G_k^2] \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}[G_k^2] \right)^{\frac{1}{2}} + \left( \sum_{i,j \geq 1} \mathbb{E}[G_{ij}^2] \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}[G_k^2] \right)^{\frac{1}{2}}
\]
\[
+ \left( \sum_{i \geq 1} \mathbb{E}[G_{i}^2] \right)^{\frac{1}{2}} \left( \sum_{j \geq 1} \mathbb{E}[G_{j}^2] \right)^{\frac{1}{2}} \right\} \| \xi \|_2
\]
\[
\leq \left\{ \text{esssup}_{\omega \in \Omega} |G(\omega)| + \sum_{k \geq 1} \mathbb{E}[G_k^2] + \sum_{k \geq 1} \mathbb{E}[\tilde{G}_k^2] + \left( \sum_{i,j \geq 1} \mathbb{E}[\tilde{G}_{ij}^2] \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}[\tilde{G}_k^2] \right)^{\frac{1}{2}} \right\} \| \xi \|_2.
\]

With the same idea, we have
\[
\| Q(\omega, \xi) \|_2 \leq \| Q(\omega, \xi) \|_2 + \| Q(\omega) \|_2 + \| Q(\omega) \|_2 + \| Q(\omega) \|_2 + \| Q(\omega) \|_2 + \| Q(\omega) \|_2
\]
\[
= \left( \mathbb{E}[Q(\omega, \xi)^2] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \sum_{k \geq 1} |Q_k(\omega)| \left( \mathbb{E}[Q_k(\omega)]^2 \right)^{\frac{1}{2}} \| \xi \|_2^2 \right] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \sum_{k \geq 1} |Q_k(\omega)| \left( \mathbb{E}[Q_k(\omega)]^2 \right)^{\frac{1}{2}} \| \xi \|_2^2 \right] \right)^{\frac{1}{2}}
\]
\[
+ \left( \mathbb{E} \left[ \sum_{i \geq 1} \left( \sum_{j \geq 1} |Q_{ij}(\omega)| \left( \mathbb{E}[Q_{ij}(\omega)]^2 \right)^{\frac{1}{2}} \| \xi \|_2^2 \right] \right)^{\frac{1}{2}}
\]
\[
\leq \left\{ \text{esssup}_{\omega \in \Omega} |Q(\omega, \xi)| + \left( \sum_{k \geq 1} \mathbb{E}[Q_k(\omega)] \left( \mathbb{E}[Q_k(\omega)]^2 \right)^{\frac{1}{2}} \| \xi \|_2^2 \right] \right\} \| \xi \|_2
\]
\[
\leq \left\{ \text{esssup}_{\omega \in \Omega} |Q(\omega, \xi)| + \sum_{k \geq 1} \mathbb{E}[\tilde{Q}_k(\omega)]^2 + \sum_{k \geq 1} \mathbb{E}[Q_k(\omega)]^2 + \left( \sum_{i,j \geq 1} \mathbb{E}[\tilde{Q}_{ij}(\omega)]^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}[Q_k(\omega)]^2 \right)^{\frac{1}{2}} \right\} \| \xi \|_2.
\]

which leads to
\[
\| Q(\omega, \xi) \| \leq \text{esssup}_{\omega \in \Omega} |Q(\omega, \xi)| + \sum_{k \geq 1} \mathbb{E}[\tilde{Q}_k(\omega)]^2 + \sum_{k \geq 1} \mathbb{E}[Q_k(\omega)]^2 + \left( \sum_{i,j \geq 1} \mathbb{E}[\tilde{Q}_{ij}(\omega)]^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}[Q_k(\omega)]^2 \right)^{\frac{1}{2}}.
\]
Next,
\[
\| \mathcal{R}(s) \eta \|_2 \leq \| R(s) \eta \|_2 + \| \mathcal{R}(s)^\top \mathbb{E}[\mathcal{R}(s) \eta] \|_2 + \| \mathcal{R}(s)^\top \mathbb{E}[\mathcal{R}(s) \eta] \|_2 + \| \mathcal{R}(s)^\top \mathbb{E}[\mathcal{R}(s)] \mathbb{E}[\mathcal{R}(s) \eta] \|_2 \\
= \left( \mathbb{E}[\| R(s) \eta \|_2^2] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \sum_{k \geq 1} \tilde{R}_k(s)^\top \mathbb{E}[\tilde{R}_k(s) \eta] \right] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \sum_{k \geq 1} \tilde{R}_k(s)^\top \mathbb{E}[\tilde{R}_k(s) \eta] \right] \right)^{\frac{1}{2}} \\
+ \left( \mathbb{E} \left[ \sum_{i,j \geq 1} \tilde{R}_i(s)^\top \mathbb{E}[\tilde{R}_j(s)] \mathbb{E}[\tilde{R}_j(s) \eta] \right] \right)^{\frac{1}{2}} \\
\leq \text{esssup}_{\omega \in \Omega} |R(s, \omega)| \| \eta \|_2 + \left[ \mathbb{E} \left( \sum_{k \geq 1} |\tilde{R}_k(s)| (\mathbb{E}[|\tilde{R}_k(s)|^2]^2 \| \eta \|_2^2)^2 \right)^{\frac{1}{2}} + \left[ \mathbb{E} \left( \sum_{k \geq 1} |\tilde{R}_k(s)| (\mathbb{E}[|\tilde{R}_k(s)|^2]^2 \| \eta \|_2^2)^2 \right)^{\frac{1}{2}} \right]^2 \\
+ \left( \sum_{i,j \geq 1} \mathbb{E}[\tilde{R}_i(s)^2] \sum_{j \geq 1} \left( \sum_{j \geq 1} \mathbb{E}[\tilde{R}_j(s)] (\mathbb{E}[|\tilde{R}_j(s)|^2]^2)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \| \eta \|_2 \\
\leq \text{esssup}_{\omega \in \Omega} |R(s, \omega)| + \sum_{k \geq 1} \mathbb{E}[|\tilde{R}_k(s)|^2] + \sum_{k \geq 1} \mathbb{E}[|\tilde{R}_k(s)|^2] + \left( \sum_{i,j \geq 1} \mathbb{E}[|\tilde{R}_i(s)|^2] \left( \sum_{k \geq 1} \mathbb{E}[|\tilde{R}_k(s)|^2] \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \| \eta \|_2,
\]
which leads to
\[
\| \mathcal{R}(s) \| \leq \text{esssup}_{\omega \in \Omega} |R(s, \omega)| + \sum_{k \geq 1} \mathbb{E}[|\tilde{R}_k(s)|^2] + \sum_{k \geq 1} \mathbb{E}[|\tilde{R}_k(s)|^2] + \left( \sum_{i,j \geq 1} \mathbb{E}[|\tilde{R}_i(s)|^2] \left( \sum_{k \geq 1} \mathbb{E}[|\tilde{R}_k(s)|^2] \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \| \eta \|_2.
\]
Further,
\[
|S(s) \xi |_2 \leq |S(s) \xi |_2 + |\mathcal{R}(s)^\top \mathbb{E}[\mathcal{S}(s) \xi] |_2 + |\mathcal{S}(s)^\top \mathbb{E}[\mathcal{Q}(s) \xi] |_2 + |\mathcal{R}(s)^\top \mathbb{E}[\mathcal{S}(s)] \mathbb{E}[\mathcal{Q}(s) \xi] |_2 \\
= \left( \mathbb{E}[|S(s) \xi |_2^2] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \sum_{k \geq 1} |\tilde{R}_k(s)| (\mathbb{E}[|\tilde{R}_k(s)|^2]^2 \| \xi \|_2^2)^2 \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \sum_{k \geq 1} |\tilde{S}_k(s)| (\mathbb{E}[|\tilde{Q}_k(s)|^2]^2 \| \xi \|_2^2)^2 \right)^{\frac{1}{2}} \\
+ \left( \mathbb{E} \left[ \sum_{i,j \geq 1} |\tilde{R}_i(s)|^2 \sum_{j \geq 1} \left( \sum_{j \geq 1} |\tilde{S}_j(s)| (\mathbb{E}[|\tilde{Q}_j(s)|^2]^2)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \| \xi \|_2 \\
\leq \text{esssup}_{\omega \in \Omega} |S(s, \omega)| \| \xi \|_2 + \left[ \mathbb{E} \left( \sum_{k \geq 1} |\tilde{R}_k(s)| (\mathbb{E}[|\tilde{R}_k(s)|^2]^2 \| \xi \|_2^2)^2 \right)^{\frac{1}{2}} + \left( \mathbb{E} \left( \sum_{k \geq 1} |\tilde{S}_k(s)| (\mathbb{E}[|\tilde{Q}_k(s)|^2]^2 \| \xi \|_2^2)^2 \right)^{\frac{1}{2}} \right]^2 \\
+ \left( \sum_{i,j \geq 1} \mathbb{E}[|\tilde{R}_i(s)|^2] \sum_{j \geq 1} \left( \sum_{j \geq 1} \mathbb{E}[|\tilde{S}_j(s)|] (\mathbb{E}[|\tilde{Q}_j(s)|^2]^2)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \| \xi \|_2 \\
\leq \text{esssup}_{\omega \in \Omega} |S(s, \omega)| + \sum_{k \geq 1} \mathbb{E}[|\tilde{R}_k(s)|^2] + \sum_{k \geq 1} \mathbb{E}[|\tilde{S}_k(s)|^2] + \sum_{k \geq 1} \mathbb{E}[|\tilde{R}_k(s)|^2] + \left( \sum_{i,j \geq 1} \mathbb{E}[|\tilde{S}_i(s)|^2] \left( \sum_{k \geq 1} \mathbb{E}[|\tilde{Q}_k(s)|^2] \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \| \xi \|_2,
\]
which leads to

$$\|S(s)\| \leq \text{esssup}_{\omega \in \Omega} |S(s, \omega)| + \sum_{k \geq 1} \mathbb{E}|\tilde{R}_k(s)|^2 + \sum_{k \geq 1} \mathbb{E}|\tilde{S}_k(s)|^2 + \sum_{k \geq 1} \mathbb{E}|\tilde{Q}_k(s)|^2$$

$$+ \left( \sum_{i,j \geq 1} \mathbb{E}|\tilde{S}_{ij}(s)|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}|\tilde{R}_k(s)|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}|\tilde{Q}_k(s)|^2 \right)^{\frac{1}{2}}.$$

Finally,

$$\|g\|_2 \leq \|g_0\|_2 + \|\mathbf{G}^\top \mathbb{E}[g]\|_2 = \|g_0\|_2 + \sqrt{\mathbb{E} \left[ \sum_{k \geq 1} \tilde{G}_k^\top \mathbb{E}[\tilde{g}_k] \right]^2}$$

$$\leq \|g_0\|_2 + \left( \sum_{k \geq 1} \mathbb{E}|\tilde{G}_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}|\tilde{g}_k|^2 \right)^{\frac{1}{2}} \leq \|g_0\|_2 + \sum_{k \geq 1} \mathbb{E}|\tilde{G}_k|^2 + \sum_{k \geq 1} \mathbb{E}|\tilde{g}_k|^2;$$

$$\left\| \int_0^T |g(s)|ds \right\|_2 \leq \left\| \int_0^T |q_0(s)|ds \right\|_2 + \int_0^T \|\tilde{Q}(s)^\top \mathbb{E}[\tilde{q}(s)]\|_2 ds$$

$$= \left\| \int_0^T |q_0(s)|ds \right\|_2 + \int_0^T \left[ \mathbb{E} \left[ \sum_{k \geq 1} \tilde{Q}_k(s)^\top \mathbb{E}[\tilde{q}_k(s)] \right]^2 \right]^{\frac{1}{2}} ds$$

$$\leq \left\| \int_0^T |q_0(s)|ds \right\|_2 + \int_0^T \left( \sum_{k \geq 1} \mathbb{E}|\tilde{Q}_k(s)|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}|\tilde{q}_k(s)|^2 \right)^{\frac{1}{2}} ds$$

$$\leq \left\| \int_0^T |q_0(s)|ds \right\|_2 + \int_0^T \left( \sum_{k \geq 1} \mathbb{E}|\tilde{Q}_k(s)|^2 \right) ds + \int_0^T \left( \sum_{k \geq 1} \mathbb{E}|\tilde{q}_k(s)|^2 \right) ds;$$

$$\left\| \rho(s) \right\|_2 \leq \left\| \rho_0(s) \right\|_2 + \left\| \tilde{R}(s)^\top \mathbb{E}[\tilde{\rho}(s)] \right\|_2 = \left\| \rho_0(s) \right\|_2 + \left[ \mathbb{E} \left[ \sum_{k \geq 1} \tilde{R}_k(s)^\top \mathbb{E}[\tilde{\rho}_k(s)] \right]^2 \right]^{\frac{1}{2}}$$

$$\leq \left\| \rho_0(s) \right\|_2 + \left( \sum_{k \geq 1} \mathbb{E}|\tilde{R}_k(s)|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 1} \mathbb{E}|\tilde{\rho}_k(s)|^2 \right)^{\frac{1}{2}} \leq \left\| \rho_0(s) \right\|_2 + \sum_{k \geq 1} \mathbb{E}|\tilde{R}_k(s)|^2 + \sum_{k \geq 1} \mathbb{E}|\tilde{\rho}_k(s)|^2.$$