Abstract. Consider a metric measure space with non-negative Ricci curvature in the sense of Lott, Sturm and Villani. We prove a sharp upper bound on the diameter of any subset whose boundary has a positive lower bound on its generalized mean curvature. This provides a nonsmooth analog to a result of Kasue (1983) and Li (2014). We also prove a stability statement concerning such bounds.

1. Introduction

Kasue proved a sharp estimate for the boundary distance function \( d_{\partial M} \) of a smooth, \( n \)-dimensional Riemannian manifold \( M \) with nonnegative Ricci curvature and smooth boundary \( \partial M \) whose mean curvature is bounded from below by \( n - 1 \). More precisely, he concluded \( d_{\partial M} \leq 1 \) [Kas83]. This result was also rediscovered by Li [Li14] and extended for Bakry-Emery curvature bounds by Sakurai [Sak19]. Their result can be seen as a Riemannian analog of the Hawking singularity theorem from general relativity [Haw66]. There has been considerable interest in generalizing Hawking’s result to a nonsmooth setting [KSSV15, LMO19, Gra19]. Motivated in part by this goal, we give a generalization of Kasue’s result which is interesting in itself and can serve as a model for the Lorentzian case. Independently and simultaneously, Cavalletti and Mondino have proposed a synthetic new framework for Lorentzian geometry (also under investigation by one of us independently [McC18]) in which they establish an analogue of the Hawking result [CM20].

In this short note we generalize Kasue and Li’s estimate to subsets \( \Omega \) of a (potentially nonsmooth) space \( X \) satisfying a curvature dimension condition \( CD(K, N) \) with \( K \) near zero, provided the topological boundary \( \partial \Omega \) has a positive lower bound on its inner mean curvature in the sense of [Ket19]. The notion of inner mean curvature in [Ket19] is defined by means of the \( 1D \)-localisation (needle decomposition) technique of Cavalletti and Mondino [CM17] and coincides with the classical mean curvature of a submanifold in the smooth context. We also assume that the boundary \( \partial \Omega \) satisfies a measure theoretic regularity condition that is implied by an exterior ball condition. Hence, our result not only covers Kasue’s theorem but also holds for a large class of domains in Alexandrov spaces or in Finsler manifolds. Kasue (and Li) were also able to prove a rigidity result: namely that, among smooth manifolds, their diameter bound is obtained precisely by the Euclidean unit ball. In the nonsmooth case there are also truncated cones that attain the maximal diameter, and we have not ruled out the possibility of other nonsmooth optimizers.

Our main theorem reads as follows.
Theorem 1.1 (Diameter bounds for metric measure spaces). Let $X$ be an essentially nonbranching $CD(K', N)$ space with $K' \in \mathbb{R}$, $N \in (1, \infty)$ and $\text{spt} \, m = X$. Let $\Omega \subset X$ be open and relatively compact with $m(\partial \Omega) = 0$ such that $\Omega$ satisfies the restricted curvature-dimension condition $CD_r(K, N)$ for $K \in \mathbb{R}$ (Definition 2.3) and $\partial \Omega = S$ has inner mean curvature $H_S$ and finite inner curvature. Then the following hold:

1. If $K' \geq 0 = K$ and $H_S \geq N - 1$ $m_S$-a.e., then $d_{\partial \Omega} \leq 1$ where $\Omega' = X \setminus \Omega$.
2. If $K' \geq 0$ and $K = \kappa(N - 1) > 0$ and $H_S \geq 0$ $m_S$-a.e., then $d_{\partial \Omega} \leq \pi/(4K)^{1/2}$.
3. For $\epsilon > 0$ there exists $\delta > 0$ such that: If $K' \geq -\delta$ and $\partial \Omega$ has finite inner curvature with $H_S \geq (1 - \delta)(N - 1)$ $m_S$-a.e., then $d_{\partial \Omega} \leq 1 + \epsilon$.

Remark 1.2 (Definitions). (1) The curvature-dimension conditions $CD(K, N)$ and the restricted curvature-dimension condition $CD_r(K, N)$ are defined in Definition 2.3. If $X$ satisfies the condition $CD(K, N)$ then $\Omega \neq \emptyset$ trivially satisfies $CD_r(K, N)$ for the same $K$.
(2) The property “having finite inner curvature” (Definition 2.17) is implied by an exterior ball condition for $\Omega$ (Lemma 2.21). The surface measure $m_S$ is defined in Definition 2.15.
(3) For $S$ with finite inner curvature, the definition of generalized inner mean curvature $H_S$ is given in Definition 2.17. Let us briefly sketch the idea. Using a needle decomposition associated to the signed distance function $d_S := d_{\Omega} - d_{\Omega^c}$, one can disintegrate the reference measure $m_X$ with conditional measures $m_t$ that are supported on curves $\gamma$ of maximal slope w.r.t. $d_S$, the so-called needles. For almost every curve $\gamma$ there exists a density $h_\gamma$ of $m_\gamma$ w.r.t. the 1-dimensional Hausdorff measure $H^1$. Then the inner mean curvature for $m_\gamma$-a.e. $p = \gamma(t_0) \in S$ is defined as $\frac{d}{dt} \log h_\gamma(t_0) = H_S(p)$. We postpone details to the Sections 2.3 and 2.4. In the case $X$ is a Riemannian manifold and $\partial \Omega$ is a submanifold the inner mean curvature coincides with the classical mean curvature.
(4) In [CM20] a notion of mean curvature bounded from below was introduced that does not require the assumption of finite inner curvature but that the boundary measure $m_S$ is a Radon measure. We can phrase and prove our theorem also for this setting (see Appendix).
In any case some additional regularity for $\Omega$ is necessary.
(5) Our assumptions cover the case of a Riemannian manifold with boundary: If $M$ is an $n$-dimensional Riemannian manifold with boundary and $\text{ric}_M \geq K$, then one can always construct a geodesically convex, $n$-dimensional Riemannian manifold with boundary $\tilde{M}$ such that $M$ isometrically embeds into $\tilde{M}$, and such that $\text{ric}_{\tilde{M}} \geq K'$ [Won08]. In particular, one can consider $M$ as a $CD_r(K, N)$ space that is a subset of the $CD(K', N)$ space $(\tilde{M}, d_{\tilde{M}}, \text{vol}_{\tilde{M}})$ (Remark 5.8 in [Ket19]).

1.1. **Truncated euclidean cone and spherical suspension.** Let $X$ be a metric measure space. The euclidean $N$-cone over $X$ is defined as the metric measure space 

$$(0, \infty) \times X/ \sim, d_C, m_C^N := [0, \infty) \times^N X$$

where the equivalence relation $\sim$ is defined by $(0, x) \sim (0, y)$ $\forall x, y \in X$, and $(t, x) \sim (t, y)$ for $t > 0$ iff $x = y$. $o$ denotes the tip of the cone. The distance $d_C$ is defined by

$$d_C^2((t, x), (s, y)) = t^2 + s^2 - 2ts \cos [d_X(x, y) \wedge \pi].$$

and the measure $m_C^N$ is given by $r^N \text{dr} \otimes \text{d}m_X$. The truncated cone $[0, 1] \times^N X$ is defined as the closed ball of radius 1 at $o$.

Similar, the spherical $N$-suspension over $X$ is defined as the metric space 

$$(0, \pi) \times X/ \sim, d_S, m_S^N := [0, \pi] \times^N X$$

where the equivalence relation $\sim$ is defined by $(0, x) \sim (0, y)$ $\forall x, y \in X$, $(\pi, x) \sim (\pi, y)$ $\forall x, y \in X$ and $(t, x) \sim (t, y)$ for $t \in (0, \pi)$ iff $x = y$. The distance $d_S$ is defined by

$$\cos d_S((t, x), (s, y)) = \cos t \cos s + \sin t \sin s \cos [d_X(x, y) \wedge \pi]$$

and the measure $m_S^N$ is given by $\sin^N \text{tdt} \otimes \text{d}m_X$. 


We make the following conjecture.

**Conjecture 1.3 (Rigidity).** Let $X$ be $\text{RCD}(K', N)$ and let $\Omega$ be as in the previous theorem satisfying $\text{CD}_{r}(0, N)$. There exists $x \in \text{spt } m_X$ such that $d_{\Omega^V}(x) = 1$ if and only if there exists an $\text{RCD}(N - 2, N - 1)$ space $Y$ such that $(\Omega, d_{\Omega}, m|_{\Omega})$ is isomorphic to the truncated euclidean cone $[0,1] \times_{\tau}^{N-1} Y$ where $d_{\Omega}$ denotes the induced intrinsic distance of $\Omega$.

The Riemannian curvature-dimension condition $\text{RCD}(K, N)$ (Definition 2.5) is a strengthening of the curvature-dimension condition that rules out Finsler manifolds and allows to prove rigidity theorems for metric measure spaces.

**Remark 1.4.** In the previous conjecture one direction is obvious.

Let $Y$ be an $\text{RCD}(N - 2, N - 1)$ space. Then the truncated euclidean cone $[0,1] \times_{\tau}^{N-1} Y \equiv \Omega$ is geodesically convex and satisfies $\text{RCD}(0, N)$ [Ket13]. The signed distance function $d_{S}$ for $S = \partial \Omega$ in $X$ restricted to $\Omega$ is given by $d_{\Omega^V}(t, x) = 1 - t$. In particular, $d_{\Omega^V}(0, x) = d_{\Omega^V}(\partial) = 1$.

Moreover, $S$ has (inner) mean curvature equal to $N - 1$ in the sense of Definition 2.17 in $[0, \infty) \times_{\tau}^{N-1} Y$. Indeed, we can see that points $(x, s)$ and $(y, t)$ in $\Omega$ lie on the same needle iff $x = y$ or either $x$ or $y$ is 0. Hence, the needles in $\Omega$ for the corresponding 1D-localization are $t \in (0, 1) \mapsto \gamma(t) = (1 - t, x)$, $x \in Y$. One can also easily check that $h_{t}^{1/N-1}(t) = t$ for all needles $\gamma$ in the corresponding disintegration of $m|_{\Omega}$. Hence $H_{\Omega^V}^{N-1} \equiv N - 1$.

A similar conjecture for $K = (N - 1) > 0$ and $H \geq 0$ is the following one.

**Conjecture 1.5 (Spherical Rigidity).** Let $X$ be $\text{RCD}(K', N)$ and let $\Omega$ be as in the previous theorem satisfying $\text{CD}_{r}(N - 1, N)$ and $\text{H}^{N}_{\Omega} \geq 0$. There exists $x \in \text{spt } m_X$ such that $d_{\Omega^V}(x) = \pi/2$ if and only if there exists an $\text{RCD}(N - 2, N - 1)$ space $Y$ such that $(\Omega, d_{\Omega}, m|_{\Omega})$ is isomorphic to the truncated spherical $(N - 1)$-cone $[0, \pi/2] \times_{\tau}^{N-1} Y$ where $d_{\Omega}$ denotes the induced intrinsic distance of $\Omega$.

2. Preliminaries.

2.1 Curvature-dimension condition. Let $(X, d)$ be a complete and separable metric space and let $m$ be a locally finite Borel measure. We call $(X, d, m)$ a metric measure space. We always assume $\text{spt } m = X$.

A geodesic is a length minimizing curve $\gamma : [a, b] \to X$. We denote the set of constant speed geodesics $\gamma : [0, 1] \to X$ with $\mathcal{G}(X)$; these are characterized by the identity

$$ d(\gamma_{s}, \gamma_{t}) = (t - s)d(\gamma_{0}, \gamma_{1}) $$

for all $0 \leq s \leq t \leq 1$. For $t \in [0, 1]$ let $e_{t} : \gamma \in \mathcal{G}(X) \mapsto \gamma(t)$ be the evaluation map. A subset of geodesics $F \subset \mathcal{G}(X)$ is said to be non-branching if for any two geodesics $\gamma, \tilde{\gamma} \in F$ such that there exists $\epsilon \in (0, 1)$ with $|\gamma|_{(0, \epsilon)} = |\tilde{\gamma}|_{(0, \epsilon)}$, it follows $\gamma = \tilde{\gamma}$.

**Example 2.1 (Euclidean geodesics).** When $X \subset \mathbb{R}^{n}$ is convex and $d(x, y) = |x - y|$ then $\mathcal{G}(X)$ consists of the affine maps $\gamma : [0, 1] \to X$.

The set of (Borel) probability measures on $X$ is denoted with $\mathcal{P}(X)$, the subset of probability measures with finite second moment is $\mathcal{P}^{2}(X)$, the set of probability measures in $\mathcal{P}^{2}(X)$ that are $m$-absolutely continuous is denoted with $\mathcal{P}^{2}(X, m)$ and the subset of measures in $\mathcal{P}^{2}(X, m)$ with bounded support is denoted with $\mathcal{P}^{2}_{b}(X, m)$.

The space $\mathcal{P}^{2}(X)$ is equipped with the $L^{2}$-Wasserstein distance $W_{2}$, e.g. [Vil09]. A dynamical optimal coupling is a probability measure $\Pi \in \mathcal{P}(\mathcal{G}(X))$ such that $t \in [0, 1] \mapsto (e_{t})_{\#}\Pi$ is a $W_{2}$-geodesic in $\mathcal{P}^{2}(X)$. The set of dynamical optimal couplings $\Pi \in \mathcal{P}(\mathcal{G}(X))$ between $\mu_{0}, \mu_{1} \in \mathcal{P}^{2}(X)$ is denoted with $\text{OptGeo}(\mu_{0}, \mu_{1})$.

A metric measure space $(X, d, m)$ is called essentially nonbranching if for any pair $\mu_{0}, \mu_{1} \in \mathcal{P}^{2}(X, m)$ any $\Pi \in \text{OptGeo}(\mu_{0}, \mu_{1})$ is concentrated on a set of nonbranching geodesics.
For $\kappa \in \mathbb{R}$ we define $\cos_\kappa : [0, \infty) \to \mathbb{R}$ as the solution of
\begin{equation}
\label{eq:cos_def}
v'' + \kappa v = 0 \quad v(0) = 1 \quad \text{and} \quad v'(0) = 0.
\end{equation}
$\sin_\kappa$ is defined as solution of the same ODE with initial value $v(0) = 0$ and $v'(0) = 1$.

**Definition 2.2 (Distortion coefficients).** For $K \in \mathbb{R}$, $N \in (0, \infty)$ and $\theta \geq 0$ we define the distortion coefficient as
\[ t \in [0, 1] \mapsto \sigma_{K,N}^{(t)}(\theta) := \begin{cases} 
\frac{\sin_{K/N}(\theta)}{\sin_{K/N}(\theta)} & \text{if } \theta \in [0, \pi_{K/N}), \\
\infty & \text{otherwise},
\end{cases}\]
where $\pi_{K} := \infty$ if $\kappa \leq 0$ and $\pi_{K} := \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$. Note that $\sigma_{K,N}(0) = t$. Moreover, for $K \in \mathbb{R}$, $N \in [1, \infty)$ and $\theta \geq 0$ the modified distortion coefficient is defined as
\[ t \in [0, 1] \mapsto \tau_{K,N}^{(t)}(\theta) := \begin{cases} 
\theta \cdot \infty & \text{if } K > 0 \text{ and } N = 1, \\
\frac{1}{t^*} \left[ \sigma_{K,N-1}^{(t)}(\theta) \right]^{1 - \frac{1}{N}} & \text{otherwise},
\end{cases}\]
where our conventions are $0 \cdot \infty =: 0$ and $\infty^0 =: 1$.

**Definition 2.3 (Curvature-dimension conditions [Stu06, LV09]).** An essentially non-branching metric measure space $(X, d, m)$ satisfies the curvature-dimension condition $CD(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty)$ if for every $\mu_0, \mu_1 \in \mathcal{P}^2_b(X, m)$ there exists a dynamical optimal coupling $\Pi$ between $\mu_0$ and $\mu_1$ such that for all $t \in (0, 1)$
\begin{equation}
\rho_t(\gamma) - \frac{1}{N} \geq t^{1 - \frac{1}{N}}(d(\gamma_0, \gamma_1))\rho_0(\gamma_0) - \frac{1}{N} + \tau_{K,N}^{(t)}(d(\gamma_0, \gamma_1))\rho_1(\gamma_1) - \frac{1}{N}
\end{equation}
for $\Pi$-a.e. $\gamma \in \mathcal{G}(X)$ and for all $t \in [0, 1]$ where $(e_t)_\# \Pi = \rho_t \mu$.

We say that $\Omega \subset X$ with $m(\Omega) > 0$ for an essentially non-branching metric measure space $(X, d, m)$ satisfies the restricted curvature-dimension condition $CD_r(K, N)$ if for every dynamical optimal coupling $\Pi$ between $\mu_0, \mu_1 \in \mathcal{P}^2_b(X, m)$ with $(e_t)_\# \Pi(\Omega) = 1$ for all $t \in [0, 1]$.

**Remark 2.4 (Consequences).** A $CD(K, N)$ space $X$ for $N \in [1, \infty)$ is geodesic and locally compact.

We recall briefly the Riemannian curvature-dimension condition that is a strengthening of the $CD(K, N)$ condition and the result of the combined efforts by several authors [AGS14b, Gig15, EKS15, AGMR15, AMS19].

The Cheeger energy $\text{Ch} : L^2(m_X) \to [0, \infty]$ of metric measure space $X$ is defined as
\[ 2 \text{Ch}(f) = \liminf_{\text{Lip}(X) \ni u_n \searrow f} \int (\text{Lip} u_n)^2 d m_X \]
where $\text{Lip}(X)$ is the space of Lipschitz functions on $X$ and $\text{Lip} u(x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}$ is the local slope of $u \in \text{Lip}(X)$. The $L^2$-Sobolev space is defined as $W^{1,2}(X) = \{ f \in L^2(m) : \text{Ch}(f) < \infty \}$ and equipped with the norm $\| f \|^2 := \| f \|^2_{L^2(m)} + 2 \text{Ch}(f)$ [AGS13, AGS14a].

**Definition 2.5.** A metric measure space $X$ satisfies the Riemannian curvature-dimension condition $RCD(K, N)$ if $X$ satisfies the condition $CD(K, N)$ and $W^{1,2}(X)$ is a Hilbert space.

2.2. Disintegration of measures. For further details about the content of this section we refer to [Fre06, Section 452].

Let $(R, \mathcal{R})$ be a measurable space, and let $\Omega : R \to Q$ be a map for a set $Q$. One can equip $Q$ with the $\sigma$-algebra $\mathcal{Q}$ that is induced by $\Omega$ where $B \in \mathcal{Q}$ if $\Omega^{-1}(B) \in \mathcal{R}$. Given a probability measure $m$ on $(R, \mathcal{R})$, one can define a probability measure $q$ on $Q$ via the pushforward $\Omega_* m =: q$.

**Definition 2.6.** A disintegration of $m$ that is consistent with $\Omega$ is a map $(B, \alpha) \in \mathcal{R} \times Q \mapsto m_\alpha(B) \in [0, 1]$ such that it follows
- $m_\alpha$ is a probability measure on $(R, \mathcal{R})$ for every $\alpha \in Q$, 
- $m_\alpha = \int Q \cdot q$ for $\alpha \in Q$.

Note that the disintegration of a probability measure is not unique in general.
• \( \alpha \rightarrow m_\alpha(B) \) is \( q \)-measurable for every \( B \in \mathcal{R} \),
and for all \( B \in \mathcal{R} \) and \( C \in \mathcal{Q} \) the consistency condition
\[
m(B \cap \Omega^{-1}(C)) = \int_C m_\alpha(B)q(\alpha)
\]
holds. We use the notation \( \{m_\alpha\}_{\alpha \in \mathcal{Q}} \) for such a disintegration. We call the measures \( m_\alpha \) conditional probability measures.

A disintegration \( \{m_\alpha\}_{\alpha \in \mathcal{Q}} \) is called strongly consistent with respect to \( \{\Omega^{-1}(\alpha)\}_{\alpha \in \mathcal{Q}} \) if for \( q \)-a.e. \( \alpha \) we have \( m_\alpha(\Omega^{-1}(\alpha)) = 1 \).

**Theorem 2.7.** Assume that \((R, \mathcal{R}, m)\) is a countably generated probability space and \( R = \bigcup_{\alpha \in \mathcal{Q}} R_\alpha \) is a partition of \( R \). Let \( \Omega : R \to \mathcal{Q} \) be the quotient map associated to this partition, that is \( \alpha = \Omega(x) \) if and only if \( x \in R_\alpha \) and assume the corresponding quotient space \((\mathcal{Q}, \mathcal{Q})\) is a Polish space.

Then, there exists a strongly consistent disintegration \( \{m_\alpha\}_{\alpha \in \mathcal{Q}} \) of \( m \) w.r.t. \( \Omega : R \to \mathcal{Q} \) that is unique in the following sense: if \( \{m'_\alpha\}_{\alpha \in \mathcal{Q}} \) is another consistent disintegration of \( m \) w.r.t. \( \mathcal{Q} \) then \( m_\alpha = m'_\alpha \) for \( q \)-a.e. \( \alpha \in \mathcal{Q} \).

2.3. 1D-localization. In this section we will recall the localization technique introduced by Cavalletti and Mondino for 1-Lipschitz functions as a nonsmooth analogue of Klartag’s needle decomposition; here needle refers to any geodesic along which the Lipschitz function attains its maximum slope that Klartag also calls transport rays [Kln17]. The presentation follows Section 3 and 4 in [CM17]. We assume familiarity with basic concepts in optimal transport (for instance [Vil09]).

Let \((X, d, m)\) be a locally compact metric measure space that is essentially nonbranching. We assume that \( \text{spt} \ m = X \).

Let \( u : X \to \mathbb{R} \) be a 1-Lipschitz function. Then
\[
\Gamma_u := \{(x, y) \in X \times X : u(y) - u(x) = d(x, y)\}
\]
is a \( d \)-cyclically monotone set, and one defines \( \Gamma_u^{-1} = \{(x, y) \in X \times X : (y, x) \in \Gamma_u\} \).

Note that we switch orientation in comparison to [CM17] where Cavalletti and Mondino define \( \Gamma_u \) as \( \Gamma_u^{-1} \).

The union \( \Gamma_u \cup \Gamma_u^{-1} \) defines a relation \( R_u \) on \( X \times X \), and \( R_u \) induces the transport set with endpoint and branching points
\[
\mathcal{T}_{u,e} := P_1(R_u \setminus \{(x, y) : x = y \in X\}) \subset X
\]
where \( P_1 \) is the projection onto the first coordinate. For \( x \in \mathcal{T}_{u,e} \) one defines \( \Gamma_u(x) := \{y \in X : (x, y) \in \Gamma_u\} \), and similarly \( \Gamma_u^{-1}(x) \) and \( R_u(x) \). Since \( u \) is 1-Lipschitz, \( \Gamma_u, \Gamma_u^{-1} \) and \( R_u \) are closed as well as \( \Gamma_u(x), \Gamma_u^{-1}(x) \) and \( R_u(x) \).

The forward and backward branching points are defined respectively as
\[
A_+ := \{x \in \mathcal{T}_{u,e} : \exists z, w \in \Gamma_u(x) \& (z, w) \notin R_u\}, \quad A_- := \{x \in \mathcal{T}_{u,e} : \exists z, w \in \Gamma_u(x)^{-1} \& (z, w) \notin R_u\}.
\]
Then one considers the (nonbranched) transport set as \( \mathcal{T}_u := \mathcal{T}_{u,e} \setminus (A_+ \cup A_-) \) and the (nonbranched) transport relation as the restriction of \( R_u \) to \( \mathcal{T}_u \times \mathcal{T}_u \).

As showed in [CM17] \( \mathcal{T}_{u,e}, A_+ \) and \( A_- \) are \( \sigma \)-compact, and \( \mathcal{T}_u \) is a Borel set. In [Cav14] Cavalletti shows that the restriction of \( R_u \) to \( \mathcal{T}_u \times \mathcal{T}_u \) is an equivalence relation. Hence, from \( R_u \) one obtains a partition of \( \mathcal{T}_u \) into a disjoint family of equivalence classes \( \{X_\alpha\}_{\alpha \in \mathcal{Q}} \). There exists a measurable section \( s : \mathcal{T}_u \to \mathcal{T}_u \), that is \( s(x) \in R_u(x) \), and \( Q \) can be identified with the image of \( \mathcal{T}_u \) under \( s \). Every \( X_\alpha \) is isometric to an interval \( I_\alpha \subset \mathbb{R} \) via an isometry \( \gamma_\alpha : I_\alpha \to X_\alpha \) where \( \gamma_\alpha \) is parametrized such that \( d(\gamma_\alpha(t), s(\gamma_\alpha(t))) = sgn(\gamma_\alpha(t))t, t \in I_\alpha \), where \( sgn \) is the sign of \( u(x) - u(s(x)) \). The map \( \gamma_\alpha : I_\alpha \to X \) extends to a geodesic also denoted \( \gamma_\alpha \) and defined on the closure \( I_\alpha \) of \( I_\alpha \). We set \( \mathcal{T}_\alpha = [a(X_\alpha), b(X_\alpha)] \).

Then, the quotient map \( \Omega : \mathcal{T}_u \to \mathcal{Q} \) is measurable, and we set \( q := \Omega \# m_{|\mathcal{T}_u} \).

**Theorem 2.8** (Disintegration [CM17]). Let \((X, d, m)\) be a compact geodesic metric measure space with \( \text{spt} \ m = X \) and \( m(X) < \infty \). Let \( u : X \to \mathbb{R} \) be a 1-Lipschitz function, let \( \{X_\alpha\}_{\alpha \in \mathcal{Q}} \) be the induced partition of \( \mathcal{T}_u \) via \( R_u \), and let \( \Omega : \mathcal{T}_u \to \mathcal{Q} \) be the induced quotient map as above. Then, there exists a unique strongly consistent disintegration \( \{m_\alpha\}_{\alpha \in \mathcal{Q}} \) of \( m_{|\mathcal{T}_u} \) w.r.t. \( \mathcal{Q} \).
Now, we assume that \((X, d, m)\) is an essentially non-branching \(CD(K, N)\) space for \(K \in \mathbb{R}\) and \(N > 1\). The following lemma is Theorem 3.4 in [CM17].

**Lemma 2.9** (Negligibility of branching points). Let \((X, d, m)\) be an essentially non-branching \(CD(K, N)\) space for \(K \in \mathbb{R}\) and \(N \in (1, \infty)\) with \(\text{spt} \ m = X\) and \(m(X) < \infty\). Then, for any 1-Lipschitz function \(u : X \to \mathbb{R}\), it follows \(m(\mathcal{T}_{u,e} \setminus \mathcal{T}_u) = 0\).

The initial and final points are defined as follows
\[
a := \{x \in \mathcal{T}_{u,e} : \Gamma_u(x) = \{x\}\}, \quad b := \{x \in \mathcal{T}_{u,e} : \Gamma_u(x) = \{x\}\}.
\]

In [CM16, Theorem 7.10] it was proved that under the assumption of the previous lemma there exists \(\hat{Q} \subset Q\) with \(q(\hat{Q} \setminus Q) = 0\) such that for \(\alpha \in \hat{Q}\) one has \(\mathcal{X}_\alpha \setminus \mathcal{T}_u \subset a \cup b\). In particular, for \(\alpha \in \hat{Q}\) we have
\[
R_u(x) = \mathcal{X}_\alpha \supset X_\alpha \supset (R_u(x))^\circ \forall x \in \Omega^{-1}(\alpha) \subset \mathcal{T}_u.
\]

where \((R_u(x))^\circ\) denotes the relative interior of the closed set \(R_u(x)\).

**Theorem 2.10** (Factor measures inherit curvature-dimension bounds [CM17]). Let \((X, d, m)\) be an essentially non-branching \(CD(K, N)\) space with \(\text{spt} \ m = X\), \(m(X) < \infty\), \(K \in \mathbb{R}\) and \(N \in (1, \infty)\).

Then, for any 1-Lipschitz function \(u : X \to \mathbb{R}\) that is strongly consistent with \((X, d, m)\), it was proved that under the assumption of the previous lemma there exists \(\hat{Q} \subset Q\) with \(q(\hat{Q} \setminus Q) = 0\) such that for \(\alpha \in \hat{Q}\) one has \(\mathcal{X}_\alpha \setminus \mathcal{T}_u \subset a \cup b\). In particular, for \(\alpha \in \hat{Q}\) we have
\[
R_u(x) = \mathcal{X}_\alpha \supset X_\alpha \supset (R_u(x))^\circ \forall x \in \Omega^{-1}(\alpha) \subset \mathcal{T}_u.
\]

where \((R_u(x))^\circ\) denotes the relative interior of the closed set \(R_u(x)\).

**Lemma 2.9** (Negligibility of branching points). Let \((X, d, m)\) be an essentially non-branching \(CD(K, N)\) space for \(K \in \mathbb{R}\) and \(N \in (1, \infty)\) with \(\text{spt} \ m = X\) and \(m(X) < \infty\). Then, for any 1-Lipschitz function \(u : X \to \mathbb{R}\), it follows \(m(\mathcal{T}_{u,e} \setminus \mathcal{T}_u) = 0\).

The initial and final points are defined as follows
\[
a := \{x \in \mathcal{T}_{u,e} : \Gamma_u(x) = \{x\}\}, \quad b := \{x \in \mathcal{T}_{u,e} : \Gamma_u(x) = \{x\}\}.
\]

In [CM16, Theorem 7.10] it was proved that under the assumption of the previous lemma there exists \(\hat{Q} \subset Q\) with \(q(\hat{Q} \setminus Q) = 0\) such that for \(\alpha \in \hat{Q}\) one has \(\mathcal{X}_\alpha \setminus \mathcal{T}_u \subset a \cup b\). In particular, for \(\alpha \in \hat{Q}\) we have
\[
R_u(x) = \mathcal{X}_\alpha \supset X_\alpha \supset (R_u(x))^\circ \forall x \in \Omega^{-1}(\alpha) \subset \mathcal{T}_u.
\]

where \((R_u(x))^\circ\) denotes the relative interior of the closed set \(R_u(x)\).

**Theorem 2.10** (Factor measures inherit curvature-dimension bounds [CM17]). Let \((X, d, m)\) be an essentially non-branching \(CD(K, N)\) space with \(\text{spt} \ m = X\), \(m(X) < \infty\), \(K \in \mathbb{R}\) and \(N \in (1, \infty)\).

Then, for any 1-Lipschitz function \(u : X \to \mathbb{R}\) that is strongly consistent with \((X, d, m)\), it was proved that under the assumption of the previous lemma there exists \(\hat{Q} \subset Q\) with \(q(\hat{Q} \setminus Q) = 0\) such that for \(\alpha \in \hat{Q}\) one has \(\mathcal{X}_\alpha \setminus \mathcal{T}_u \subset a \cup b\). In particular, for \(\alpha \in \hat{Q}\) we have
\[
R_u(x) = \mathcal{X}_\alpha \supset X_\alpha \supset (R_u(x))^\circ \forall x \in \Omega^{-1}(\alpha) \subset \mathcal{T}_u.
\]

where \((R_u(x))^\circ\) denotes the relative interior of the closed set \(R_u(x)\).
2.4. **Generalized mean curvature.** Let \((X,d,m)\) be a metric measure space as in Theorem 2.10. Let \(\Omega \subset X\) be a closed subset, and let \(S = \partial \Omega\) such that \(m(S) = 0\). The function \(d_\Omega : X \to \mathbb{R}\) is given by

\[
\inf_{y \in \Omega} d(x,y) =: d_\Omega(x).
\]

Let us also define \(d^*_\Omega := d_\Omega^{-1}\). The signed distance function \(d_S\) for \(S\) is given by

\[
d_S = d_\Omega - d^*_\Omega : X \to \mathbb{R}.
\]

It follows that \(d_S(x) = 0\) if and only if \(x \in S\), \(d_S \leq 0\) if \(x \in \Omega\) and \(d_S \geq 0\) if \(x \in \Omega^c\). It is clear that \(d_S|_S = -d^*_\Omega\) and \(d_S|_{\Omega^c} = d_\Omega\). Setting \(v = d_S\) we can also write

\[
d_S(x) = \text{sign}(v(x))d(\{v = 0\},x), \forall x \in X.
\]

Then \(d_S\) is 1-Lipschitz. Let \(\Omega^c\) denote the topological interior of \(\Omega\).

Let \(\mathcal{T}_{d,e}\) be the transport set of \(d_S\) with end- and branching points. We have \(\mathcal{T}_{d,e} \supset X \setminus S\). In particular, we have \(m(X \setminus \mathcal{T}_{d,e}) = 0\) by Lemma 2.9.

Therefore, the 1-Lipschitz function \(d_S\) induces a partition \(\{X_\alpha\}_{\alpha \in Q}\) of \(X\) up to a set of measure zero for a measurable quotient space \(Q\), and a disintegration \(\{m_\alpha\}_{\alpha \in Q}\) that is strongly consistent with the partition. The subset \(X_\alpha, \alpha \in Q\), is the image of a geodesic \(\gamma_\alpha : I_\alpha \to X\).

We consider \(Q^\dagger \subset Q\) as in Remark 2.13. One has the representation

\[
(5) \quad m(B) = \int_Q m_\alpha(B)d\varrho(\alpha) = \int_{Q^\dagger} \int_{\gamma_\alpha^{-1}(B)} h_\alpha(r)d\varrho(\alpha)
\]

for all Borel \(B \subset X\). For any transport ray \(X_\alpha, \alpha \in Q^\dagger\), it follows that \(d_S(\gamma_\alpha(b(X_\alpha))) \geq 0\) and \(d_S(\gamma_\alpha(a(X_\alpha))) \leq 0\).

**Remark 2.14 (Measurability and zero-level selection).** It is easy to see that \(A := \Omega^{-1}(Q(S \cap \mathcal{T}_{d})) \subset \mathcal{T}_{d}\) is a measurable subset. The set \(A \subset \mathcal{T}_{d}\) is defined such that \(\forall \alpha \in \Omega(A)\) we have \(X_\alpha \cap S = \{\gamma(t_\alpha)\} \neq \emptyset\) for a unique \(t_\alpha \in I_\alpha\). Then, the map \(\hat{s} : \gamma(t) \in A \mapsto \gamma(t_\alpha) \in S \cap \mathcal{T}_{d}\) is a measurable section on \(A \subset \mathcal{T}_{d}\), one can identify the measurable set \(\Omega(A) \subset Q\) with \(A \cap S\) and one can parameterize \(\gamma_\alpha\) such that \(t_\alpha = 0\).

This measurable section \(\hat{s}\) is fixed for the rest of the paper.

Moreover, we define

\[
A \cap \mathcal{T}_{d}^\dagger =: A^\dagger \quad \text{and} \quad \bigcup_{x \in A^\dagger} R_\alpha(x) =: A^\dagger_e.
\]

The sets \(A^\dagger\) and \(A^\dagger_e\) are measurable, and also

\[
B^\dagger_{in} := \Omega^c \cap \mathcal{T}_{d}^\dagger \setminus A^\dagger \subset \mathcal{T}_{d}\quad \text{and} \quad B^\dagger_{out} := \Omega^c \cap \mathcal{T}_{d}^\dagger \setminus A^\dagger \subset \mathcal{T}_{d}\n\]

as well as \(\bigcup_{x \in B^\dagger_{in}} R_\alpha(x) =: B^\dagger_{in,e}\) and \(\bigcup_{x \in B^\dagger_{out}} R_\alpha(x) =: B^\dagger_{out,e}\) are measurable.

Let us phrase the construction again in words. The set \(A\) is the union of all disjoint needles that intersect with \(\partial \Omega\), \(B_{in}\) is the union of all needles inside \(\Omega\) and \(B_{out}\) is the union of all needles inside \(\Omega^c\). The superscript \(\dagger\) indicates the intersection with \(\mathcal{T}_{d}\).

Moreover, considering \(S_t = \partial \Omega_t\) where \(\Omega_t = B_t(\Omega)\) for \(t > 0\) and \(\Omega_t = B_t(\Omega^c)\) for \(t < 0\) we define \(A_t := \Omega^{-1}(Q(S_t \cap \mathcal{T}_{d})) \subset \mathcal{T}_{d}\) and \(A^\dagger_t\) and \(A^\dagger_{t,e}\) accordingly. We note that \(A_t = \emptyset\) if \(|t| > \text{diam} X\).

The map \(\alpha \in \Omega(A^\dagger) \mapsto h_\alpha(\gamma_\alpha(0)) \in \mathbb{R}\) is measurable (see [CM16, Proposition 10.4]).

**Definition 2.15 (Surface measures).** Taking \(S = \partial \Omega\) as above, we use the disintegration of Remark 2.14 to define the surface measure \(m_S\) via

\[
\int \phi(x) d m_S(x) := \int_{\Omega(A^\dagger)} \phi(\gamma_\alpha(0))h_\alpha(0)d\varrho(\alpha)
\]

for any continuous function \(\phi : X \to \mathbb{R}\). That is \(m_S\) is the pushforward of \(h_\alpha(\gamma_\alpha(0))\varrho(\alpha)\) under the map \(\gamma \in \Omega(A^\dagger) \mapsto \gamma(0)\).
Similarly we define a one parameter family of surface measure $m_S$, via
\[ \int \phi(x) d m_S(x) := \int_{\Omega(A^t)} \phi(\gamma_\alpha(0)) h_\alpha(t) dq(\alpha). \]
The measure $m_S$ corresponds to the measure $H_\alpha$ in Cavalletti-Mondino [CM20] and is the pushforward of the measure $h_\alpha(\gamma_\alpha(t)) q(\alpha)$ under the map $\gamma \in \Omega(A^t) \mapsto \gamma(0)$.

Note that the measure $m_S$ (the measure $m_S$) vanishes outside $S \cap A^t$.

Let us recall another result of Cavalletti-Mondino.

**Theorem 2.16** (Nonsmooth Laplacian [CM18]). Let $(X,d,m)$ be a CD($K,N$) space, and $\Omega$ and $S = \partial \Omega$ as above. Then $d_S \in D(\Delta, X \setminus S)$, and one element of $\Delta d_S|_{X \setminus S}$ that we denote with $\Delta d_S|_{X \setminus S}$ is the Radon functional on $X \setminus S$ given by the representation formula
\[ \Delta d_S|_{X \setminus S} = (\log h_\alpha)' m|_{X \setminus S} + \int_Q (h_\alpha \delta_\alpha(x,\cdot) \cap \{ d_S > 0 \} - h_\alpha \delta_\alpha(x,\cdot) \cap \{ d_S < 0 \}) dq(\alpha). \]

We note that the Radon functional $\Delta d_S|_{X \setminus S}$ can be represented as the difference of two measures $[\Delta d_S]^+$ and $[\Delta d_S]|_{X \setminus S}^-$ such that
\[ [\Delta d_S]|_{X \setminus S}^+ - [\Delta d_S]|_{X \setminus S}^- = (\log h_\alpha)' \text{ m - a.e.} \]
where $[\Delta d_S]|_{X \setminus S}^\pm$ denotes the m-absolutely continuous part in the Lebesgue decomposition of $[\Delta d_S]|_{X \setminus S}^\mp$. In particular, $-(\log h_\alpha)'$ coincides with a measurable function $m$-a.e.

**Definition 2.17** (Inner mean curvature). Set $S = \partial \Omega$ and let $\{X_q\}_{q \in \Omega}$ be the disintegration induced by $u : = d_S$. Recalling (6), we say that $S$ has finite inner (respectively outer) curvature if $m(B_{in}^t) = 0$ (respectively $m(B_{out}^t) = 0$), and $S$ has finite curvature if $m(B_{out}^t \cup B_{in}^t) = 0$. If $S$ has finite inner curvature we define the inner mean curvature of $S m_S$ almost everywhere as
\[ p \in S \mapsto H^-(p) := \left\{ \begin{array}{ll} \frac{d}{dr} \log h_\alpha(\gamma_\alpha(0)) & \text{if } p = \gamma_\alpha(0) \in S \cap A^t \\ \infty & \text{if } p \in B_{out,e} \cap S. \end{array} \right. \]
Here, we set $\frac{d}{dr} \log(0) = - \infty$.

**Remark 2.18.** We point out two differences in comparison to [Ket19]. For the definition of $A^t$ we do not remove points that lie in $a$ and $b$.

We switched signs in the definition of inner mean curvature. This allows us to work with mean curvature bounded below instead of bounded above.

**Remark 2.19** (Smooth case). Let us briefly address the case of a Riemannian manifold $(M,g)$ equipped with a measure of the form $m = \Psi \ vol_g$ for $\Psi \in C^\infty(M)$ and $\Omega$ with a boundary $S$ which is a smooth compact submanifold. For every $x \in S$ there exist $a_x < 0$ and $b_x > 0$ such that $\gamma_x(r) = \exp_x(r\nabla d_S(x))$ is a minimal geodesic on $(a_x,b_x) \subset \mathbb{R}$, and we define
\[ U = \{(x,r) \in S \times \mathbb{R} : r \in (a_x,b_x)\} \subset S \times \mathbb{R} \]
and the map $T : U \to M$ via $T(x,r) = \gamma_x(r)$. The map $T$ is a diffeomorphism on $U$, with $\text{vol}_g(M \setminus T(U)) = 0$ and the integrals can be computed effectively by the following formula:
\[ \int g dm = \int_S \int_{a_x}^{b_x} g \circ T(x,r) \det DT(x,r)|_{T_x S} \Psi \circ T(x,r) dr d\text{vol}_S(x) \]
where $\text{vol}_S$ is the induced Riemannian surface measure on $S$. By comparison with the needle technique disintegration it is not difficult to see that $d m_S = \Psi \ vol_S$. Moreover, the open needles w.r.t. $d_S$ are the geodesics $\gamma_x : (a_x,b_x) \to M$ and the densities $h_x(r)$ are given by $c(x) \ det DT(x,r) \Psi \circ T(x,r)$ for some normalization constant $c(x)$, $x \in S$.

A direct computation then yields
\[ \frac{d}{dr} \log h_x(0) = H_S(x) + \langle \nabla d_S(x), \nabla \log \Psi \rangle(x), \ \forall x \in S. \]
where $H_S$ is the standard mean curvature of $S$.

**Definition 2.20** (Exterior ball condition). Let $\Omega \subset X$ and $\partial \Omega = S$. Then $S$ satisfies the exterior ball condition if for all $x \in S$ there exists $r > 0$ and $p_x \in \Omega^c$ such that $d(x, p_x) = r$ and $B_r(p_x) \subset \Omega^c$.

**Lemma 2.21** (Exterior ball criterion for finite inner curvature). Let $\Omega \subset X$. If $S = \partial \Omega$ satisfies the exterior ball condition, then $S$ has finite inner curvature.

**Proof.** Let $S$ satisfy the exterior ball condition. Then for every $x \in S$ there exists a point $p_x \in \Omega^c$ and a geodesic $\gamma_x : [0, r_x] \to \Omega^c$ from $x$ to $p_x$ such that $L(\gamma_x) = d(x, p_x) = r_x$ and $d(p_x, y) > r_x$ for any $y \in S \setminus \{x\}$. Hence, $d_S(p_x) = r_x$ and the image of $\gamma_x$ is contained in $R_{d_S}(x)$.

Recall the definition of $Q^\dagger \subset Q$ (Remark 2.13). Since $Q^\dagger$ has full $\mathfrak{q}$-measure, it is enough to show that for all $\alpha \in Q^\dagger$ the endpoint $b(X_\alpha) > 0$. Then also $B_{R_S}(0) = \emptyset$. Assume the contrary. Let $\alpha' \in Q^\dagger$ and let $\gamma' := \gamma_\alpha$ be the corresponding geodesic such that $b(X_{\alpha'}) = 0$, that is $\text{Im}(\gamma'(0)) \subset \Omega$. The concatenation $\gamma'' : (a(X_{\alpha'}), r_x) \to \gamma'$ with $\gamma''(0)$ satisfies $\gamma''(0) = x$ and

$$d(\gamma''(s), \gamma''(t)) = d(\gamma''(s), x) + d(x, \gamma''(t)) = d_S(\gamma''(t)) - d_S(\gamma''(s)).$$

Hence, $\text{Im}(\gamma'') \subset R_{d_S}(\gamma'(s))$, the points that are $R_{d_S}$-related to $\gamma'(s)$. These are exactly the points $y$ that satisfy (8) with $\gamma''(t)$ replaced by $y$. But this contradicts $\overline{\gamma'} = R(\gamma'(s))$ that is required by definition of $Q^\dagger$. □

**Remark 2.22.** We note that the previous notion of mean curvature under the assumption that $S$ has finite inner curvature, allows to assign to any point $p \in S \cap A^+$ a number that is the mean curvature of $S$ in $p$. This was useful for proving the Heintze-Karcher inequality in [Ket19].

If one is just interested in lower bounds for the mean curvature, one can adapt a definition of Cavalletti-Mondino [CM20]. They define achronal FTC Borel subsets in a Lorentz length space having forward mean curvature bounded below. We will not recall their definition for Lorentz length spaces but we give a corresponding definition for $CD(K,N)$ metric measure spaces in the appendix of this article and outline how our result also follows for this notion of lower mean curvature bounds.

### 3. Proof of the main theorem

To prove our main theorem requires one more fact from [Ket19].

**Definition 3.1** (Jacobian). Let $K \in \mathbb{R}$, $H \in (-\infty, \infty)$, $N > 1$. The Jacobian function

$$r \in \mathbb{R} \mapsto J_{H,K,N}(r) := \left( \frac{\cos_K((N-1)r)}{N-1} + \frac{H}{N-1} \sin_K((N-1)r) \right)^{N-1}_+.$$ is defined using the solutions $\sin_K$ and $\cos_K$ of (1) where $(a)_+ := \max\{a,0\}$ for $a \in \mathbb{R}$. Here $J_{H,K,N}$ is pointwise monotone non-decreasing in $H$ and $K$, and monotone non-increasing in $N$.

**Lemma 3.2** (Comparison inequality). Let $h : [a, b] \to (0, \infty)$ be continuous such that $a \leq 0 < b$, $h(0) > 0$ and every affine map $\gamma : [0,1] \to [a, b]$ satisfies

$$h(\gamma) \frac{1}{N-1} \geq \sigma_{K/N-1}^{(1-N)}(\gamma) h(\gamma_0) + \sigma_{K/N-1}^{(1)}(\gamma) h(\gamma_1) \frac{N-1}{N-1}.$$ Then $h(r)h(0)^{-1} \leq J_{H,K,N}(r)$ for $r \in (0, b)$ where $H = \frac{d}{dr} \log h(0)$.

**Proof.** If $a < 0$, the lemma is exactly the statement of Corollary 4.3 in [Ket19]. For the case $a = 0$ we first recall that the right derivative $\frac{d}{dr}h$ at 0 exists in $(-\infty, \infty)$. If the right derivative is $\infty$, the inequality trivially holds. Otherwise one can find an extension $\tilde{h} : [-\delta, b] \to [0, \infty)$ of $h$ with $\tilde{h}|_{[0,0]} = h$ for some $\delta > 0$ that also satisfies (9). For this we also recall $h(0) > 0$. Then, the claim follows again by Corollary 4.3 in [Ket19]. □
Proof of the Theorem 1.1. Let $X$ be $CD(K',N)$ and consider $\Omega \subset X$ satisfying $CD_t(K,N)$ as in the main theorem. Let $u = d_\gamma$ be corresponding signed distance function. Let $(\gamma, x) \in Q$ be the decomposition of $\mathcal{T}_u$ and $\int m_\gamma d\mathcal{Q}(\gamma)$ be the disintegration of $m$ given by Theorem 2.8 and Remark 2.14. There exists $Q^1$ of full $\mathcal{Q}$-measure such that $m_\gamma = h_\gamma \mathcal{H}^1, X_{\gamma,e} = X_\gamma$ and $h_\gamma$ satisfies

$$ (h_\gamma^N)^{\alpha} + \frac{K}{N-1}h_\gamma^{N-1} \leq 0 \text{ on } (a_\gamma, 0) $$

in distributional sense. Note Remark 2.11. For $\tilde{h}_\gamma(r) := h_\gamma(-r)$, (10) still is true.

1. Assume $K = 0$ and $\tilde{H}_\gamma \geq N - 1 m_{S}$-a.e. . In particular, $h_{a_\gamma}(0) > 0$ for $q$-a.e. $\alpha$. Recall that

$$ H_\gamma(\gamma(0)) = \frac{d^-}{dr} \log h_\gamma(0) = -\frac{d^+}{dr} \log \tilde{h}_\gamma(0). $$

By Lemma 3.2 it follows that $r \in [0, -a_\gamma] \mapsto \tilde{h}(r)\tilde{h}(0)^{-1}$ is bounded from above by $[1 - r]^{N-1}$. Hence $a_\gamma \leq 1$ for all $\gamma \in Q^1$.

To produce a contradiction, assume there exists $x \in \Omega \cap \text{spt } m$ such that $d_\gamma(x) > 1$. Then, there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap B_1(S) = \emptyset$ but $m(B_\epsilon(x)) > 0$. Hence, there exists $\gamma \in Q^1$ and $r_0 \in (a_\gamma, 0)$ with $-r_0 = d_{\gamma}(\gamma(r_0)) > 1$. This is a contradiction.

2. Now, assume $K > 0$ and $\tilde{H}_\gamma \geq N - 1 m_{S}$-a.e. . Again by Lemma 3.2 it follows $r \in [0, -a_\gamma] \mapsto \tilde{h}(r)\tilde{h}(0)^{-1}$ is bounded above by $[\cos_{2\pi}(r)]^{N-1}$.

We can argue as before to conclude that $d_\gamma(x) \leq \frac{1}{2}\pi K/(N-1)$ for every $x \in \Omega \cap \text{spt } m$.

3. Assume $K = -\delta(N-1) < 0$ and $\tilde{H}_\gamma \geq (1 - \delta)(N-1) m_{S}$-a.e. . Arguing as before yields $r \in [0, -a_\gamma] \mapsto \tilde{h}(r)\tilde{h}(0)^{-1}$ is bounded from above by $\cos_{-\delta}(r) - (1 - \delta) \sin_{-\delta}(r)$. The first 0 of the latter appears at $r(\delta) = \frac{\pi}{\sqrt{\delta}} \text{arctanh}(\sqrt{\frac{\pi}{\delta}})$ and $r(\delta) \to 1$ if $\delta \to 0$.

It follows that for any $\epsilon > 0$ there exists $\delta > 0$ such that $a_\gamma \leq 1 + \epsilon$ for any $\gamma \in Q^1$. We conclude by arguing as before.

4. Appendix

Definition 4.1 (Backward mean curvature bounded below). Recall the family of measures $m_{S_t}, t \in \mathbb{R}\setminus\{0\}$ and $m_{S}$. The boundary $S = \partial \Omega$ has backward mean curvature bounded from below by $H \in \mathbb{R}$ if the surface measure $m_{S}$ is a Radon measure on $X$ and

$$ \limsup_{t \uparrow 0} \frac{1}{t} \left( \int \phi d m_{S_t} - \int \phi d m_{S} \right) \geq H \int \phi d m_{S} $$

for a bounded and continuous function $\phi : X \to [0, \infty)$.

Remark 4.2. Since it is not assumed that $m_{S_t}$ for $t < 0$ is a Radon measure, the integral $\int \phi d m_{S_t}$ can be infinite.

Lemma 4.3. Assume $S = \partial \Omega$ for some Borel set $\Omega$ has finite inner curvature in the sense of Definition 2.17 and backward mean curvature bounded below by $H \in \mathbb{R}$. Then $m_{S}$-almost everywhere, the pointwise inner mean curvature of $S$ is bounded from below by $H$.

Proof. By the definition of $m_{S_t}$ and $m_{S}$ we can compute the following for $t < 0$.

$$ \frac{1}{t} \left( \int \phi d m_{S_t} - \int \phi d m_{S} \right) = \frac{1}{t} \left( \int_{\Omega(A^\epsilon_t)} \phi(\gamma_\alpha(0)) h_\alpha(\gamma_\alpha(t)) d\alpha(\alpha) - \int_{\Omega(A^\epsilon_t)} \phi(\gamma_\alpha(0)) d\alpha(\alpha) \right) $$

$$ = \frac{1}{t} \int \phi_\alpha(\gamma_\alpha(0)) \left( \int_{\Omega(A^\epsilon_t)} (\gamma_\alpha(0)) h_\alpha(\gamma_\alpha(t)) - 1_{\Omega(A^\epsilon_t)}(\gamma_\alpha(t))h_\alpha(\gamma_\alpha(t)) \right) d\alpha(\alpha). $$
Then by the backward mean curvature condition together with the lemma of Fatou it follows

\[ H \int \phi d m_S = H \int_{\Omega(A^\perp)} \phi(\gamma(0)) h_\alpha(\gamma(0))d\alpha q(\alpha) \leq \limsup_{t \to 0} \frac{1}{t} \left( \int \phi d m_{S_t} - \int \phi d m_S \right) \]

\[ \leq \int \phi(\gamma(0)) \limsup_{t \to 0} \frac{1}{t} \left( 1_{\Omega(A^\perp)}(\alpha) h_\alpha(\gamma(t)) - 1_{\Omega(A^\perp)}(\alpha) h_\alpha(\gamma(0)) \right) d\alpha q(\alpha) \]

\[ \leq \int \phi(\gamma(0)) \lim_{t \to 0} \frac{1}{t} \left( 1_{\Omega(A^\perp) \cap A^\perp}(\alpha) h_\alpha(\gamma(t)) - 1_{\Omega(A^\perp)}(\alpha) h_\alpha(\gamma(0)) \right) d\alpha q(\alpha) \]

\[ = \int_{\Omega(A^\perp)} \phi(\gamma(0)) \frac{d^-}{dt} \log h_\alpha(\gamma(0)) h_\alpha(0) d\alpha q(\alpha) \]

\[ = \int \phi \frac{d^-}{dt} \log h_\alpha d m_S = \int \phi H_S^2 d m_S . \]

Since φ was arbitrary, it follows that the inner mean curvature is bounded from below by K mS-almost everywhere. □

4.1. Proof under backward mean curvature bounded below. First, we note that it follows directly from Theorem 2.10 that

\[ h_\alpha(\gamma(0)) \frac{d^-}{dt} \geq \sigma_{K/\alpha}^{(\alpha)}(a(X_\alpha)) h_\alpha(\gamma(0)) \]

for an \( t \in (a(X_\alpha), 0) \) and any \( \alpha \in \mathbb{Q}^1 \). Therefore, it follows

\[ \liminf_{t \to 0} \frac{1}{t} (h_\alpha(\gamma(0)) - h_\alpha(\gamma(0))) \leq \frac{d^-}{dt} \left| \int_{t=0}^{(\alpha\alpha^{-1})} (a(X_\alpha))^{N-1} h_\alpha(\gamma(0)) \right| \]

Assuming the backward mean curvature is bounded below by H it follows with the lemma of Fatou that

\[ H \int \phi d m_S = H \int \phi(\gamma(0)) h_\alpha(0) d\alpha q(\alpha) \leq \int_{\Omega(A^\perp)} \phi(\gamma(0)) \frac{d^-}{dt} \left| \int_{t=0}^{(\alpha\alpha^{-1})} (a(X_\alpha))^{N-1} h_\alpha(0) d\alpha q(\alpha) \right| \]

\[ = \int_{\Omega(A^\perp)} \phi(\gamma(0)) (N-1) \frac{\cos_{\alpha}^{K/(N-1)}(a(X_\alpha))}{\sin_{\alpha}^{K/(N-1)}(a(X_\alpha))} h_\alpha(0) d\alpha q(\alpha) \]

Hence \( \frac{H}{N-1} \leq \frac{\cos_{\alpha}^{K/(N-1)}(a(X_\alpha))}{\sin_{\alpha}^{K/(N-1)}(a(X_\alpha))} \) for q-a.e. \( \alpha \in \mathbb{Q} \).

1. Assume \( K = 0 \) and \( H \geq N - 1 \). Then it follows \( \sin_{\alpha}^{K/(N-1)}(r) = r \) and \( \cos_{\alpha}^{K/(N-1)}(r) = 1 \), and hence \( 1 \geq \alpha_\alpha \) for q-a.e. \( \alpha \).

2. If \( K > 0 \) and \( H \geq 0 \), it follows \( 0 \leq \frac{\cos_{\alpha}^{K/(N-1)}(a(X_\alpha))}{\sin_{\alpha}^{K/(N-1)}(a(X_\alpha))} \), and hence \( \alpha_\alpha \leq \frac{\pi K/(N-1)}{2} \) for q-a.e. \( \alpha \in \mathbb{Q} \).

3. Similar.

Now, for each case we can finish the proof as in the proof of Theorem 1.1. □

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Department of Mathematics, Radboud University, PO Box 9010, Postvak 59, 6500 GL Nijmegen, The Netherlands
E-mail address: burtscher@math.ru.nl

Department of Mathematics, University of Toronto, 40 St George St, Toronto Ontario, Canada M5S 2E4
E-mail address: ckettere@math.toronto.edu

Department of Mathematics, University of Toronto, 40 St George St, Toronto Ontario, Canada M5S 2E4
E-mail address: mccann@math.toronto.edu

Department of Mathematical and Statistical Sciences and Theoretical Physics Institute, University of Alberta, Edmonton AB, Canada T6G 2G1
E-mail address: ewoolgar@ualberta.ca