Admissible operators and $\mathcal{H}_\infty$ calculus

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Abstract

Given a Hilbert space and the generator $A$ of a strongly continuous, exponentially stable, semigroup on this Hilbert space. For any $g(-s) \in \mathcal{H}_\infty$ we show that there exists an infinite-time admissible output operator $g(A)$. If $g$ is rational, then this operator is bounded, and equals the “normal” definition of $g(A)$. In particular, when $g(s) = 1/(s + \alpha)$, $\alpha \in \mathbb{C}_0^+$, then this admissible output operator equals $(\alpha I - A)^{-1}$.

Although in general $g(A)$ may be unbounded, we always have that $g(A)$ multiplied by the semigroup is a bounded operator for every (strictly) positive time instant. Furthermore, when there exists an admissible output operator $C$ such that $(C, A)$ is exactly observable, then $g(A)$ is bounded for all $g$’s with $g(-s) \in \mathcal{H}_\infty$, i.e., there exists a bounded $\mathcal{H}_\infty$-calculus. Moreover, we rediscover some well-known classes of generators also having a bounded $\mathcal{H}_\infty$-calculus.

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1 Introduction

Functional calculus is a sub-field of mathematics with a long history. It started in the thirties of the last century with the work by von Neumann for self-adjoint operators [III], and was further extended by many researchers,
see e.g. [8] and [3]. For an overview, see the book by Markus Haase, [7]. The basic idea behind functional calculus for the operator $A$ is to construct a mapping from an algebra of (scalar) functions to the class of (bounded) operators, such that

- The function identically equals to one is mapped to the identity operator;
- If $f(s) = (s - a)^{-1}$, then $f(A) = (sI - A)^{-1}$;
- Furthermore, the operator associated to $f_1 \cdot f_2$ equals $f(A)f_2(A)$.

Before we explain the contribution of this paper, we introduce some notation. By $X$ we denote separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and by $A$ we denote an unbounded operator from its domain $D(A) \subset X$ to $X$. We assume that $A$ generates an exponentially stable semigroup on $X$, which we denote by $(T(t))_{t \geq 0}$.

By $\mathcal{H}_\infty$ we denote the space of all bounded, analytic functions defined on the half-plane $\mathbb{C}^- := \{ s \in \mathbb{C} \mid \text{Re}(s) < 0 \}$. It is clear that this function class is an algebra under pointwise multiplication and addition. Hence this could serve as a class for which one could build a functional calculus. However, it is known that there exists a generator of exponential stable semigroup, which does not have a functional calculus with respect to $\mathcal{H}_\infty$. For proof of this and many more we refer to [1], [7], and the references therein. Although a bounded functional calculus is not possible, an unbounded functional calculus is always possible.

**Theorem 1.1** Under the assumptions stated above, we have that for all $g \in \mathcal{H}_\infty$ there exists an operator $g(A)$ which is bounded from the domain of $A$ to $X$, and which is admissible, i.e.,

$$
\int_0^\infty \| g(A)T(t)x_0 \|^2 dt \leq \gamma_A \| g \|_\infty^2 \| x_0 \|^2, \quad x_0 \in X.
$$

The mapping $g \mapsto g(A)$ satisfies the conditions of a functional calculus. Furthermore, for all $t > 0$, we have that $g(A)T(t)$ can be extended to a bounded operator, and

$$
\| g(A)T(t) \| \leq \frac{\gamma}{\sqrt{t}}.
$$
Apart from proving this theorem, we shall also rediscover some classes of generators for which \( g(A) \) is bounded for all \( g \in \mathcal{H}_\infty^- \), i.e., for which there is a bounded functional calculus.

For the proof of the above result, we need beside the Hardy space \( \mathcal{H}_\infty^- \) also the Hardy spaces \( \mathcal{H}_2(X) \) and \( \mathcal{H}_2^+ (X) \).

\( \mathcal{H}_2(X) \) and \( \mathcal{H}_2^+ (X) \) denote the Laplace transform, \( \mathcal{L} \), of functions in \( L^2((0, \infty), X) \) and \( L^2((\infty, 0), X) \), respectively. It is known that this transformation is an isometry. Every function in \( \mathcal{H}_\infty^-, \mathcal{H}_2(X) \) and \( \mathcal{H}_2^+ (X) \) has a unique extension to the imaginary axis on which these functions are bounded, and square integrable, respectively. Furthermore, the norm of \( g \in \mathcal{H}_\infty^- \) equals the (essential) supremum over the imaginary axis of the boundary function.

Let \( f(t) \) be a function in \( L^2((0, \infty), X) \) with Laplace transform \( F(s) \), and let \( f_{\text{ext}}(t) \) be the function in \( L^2((\infty, \infty), X) \) defined by

\[
    f_{\text{ext}}(t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}
\]

Then the Fourier transform \( \hat{f}_{\text{ext}} \) of \( f_{\text{ext}}(t) \) satisfies \( \hat{f}_{\text{ext}}(\omega) = F(i\omega) \), for almost all \( \omega \in \mathbb{R} \). Here \( F(i\cdot) \) denote the boundary function of the Laplace transform \( F(s) \).

We define the following Toeplitz operator on \( L^2((0, \infty); X) \)

**Definition 1.2** Let \( g \) be an element of \( \mathcal{H}_\infty^- \). Associated to this function we define the mapping \( M_g \) as

\[
    M_g f = \mathcal{L}^{-1} (\Pi (gF)) , \quad f \in L^2((0, \infty), X),
\]

where \( F \) denotes the Laplace transform of \( f \). \( \Pi \) denotes the projection onto \( \mathcal{H}_2(X) \).

It is clear that this is a linear bounded map from \( L^2((0, \infty); X) \) into itself, and

\[
    \| M_g \| \leq \| g \|_{\infty}.
\]

Furthermore, it follows easily from (1) that if \( K \) is a bounded mapping on \( X \), then its commutes with \( M_g \), i.e.,

\[
    KM_g = M_g K.
\]

It is easy to see that \( \mathcal{H}_\infty^- \) is an algebra under the multiplication and addition. In particular \( g_1g_2 \in \mathcal{H}_\infty^- \) whenever \( g_1, g_2 \in \mathcal{H}_\infty^- \). Furthermore, we have the following result.
Lemma 1.3 Let $g_1$ and $g_2$ be elements of $\mathcal{H}_\infty$. Then
\[ M_{g_1 g_2} = M_{g_1} M_{g_2}. \] (4)

In particular, if $g$ is invertible in $\mathcal{H}_\infty$, then $M_g$ is (boundedly) invertible and $(M_g)^{-1} = M_{g^{-1}}$.

Proof We use the fact that any $g \in \mathcal{H}_\infty$ maps $\mathcal{H}_2^\perp$ into $\mathcal{H}_2^\perp$.

\[ M_{g_1} M_{g_2} f = \mathcal{L}^{-1} (\Pi g_1 (\Pi (g_2 F))) \]
\[ = \mathcal{L}^{-1} (\Pi (g_1 g_2 F)) + \mathcal{L}^{-1} (\Pi (g_1 (I - \Pi) (g_2 F))) \]
\[ = \mathcal{L}^{-1} (\Pi (g_1 g_2 F)) + 0, \]

where we have used the above mentioned fact that $g_1 (I - \Pi)$ maps into $\mathcal{H}_2^\perp$, and so $\Pi g_1 (I - \Pi) = 0$. Since by definition $\mathcal{L}^{-1} (\Pi (g_1 g_2 F))$ equals $M_{g_1 g_2} f$, we have proved the first assertion.

The last assertion follows directly, since $M_1 = I$. $\square$

By $\sigma_\tau$ we denote the shift with $\tau \geq 0$, i.e.,
\[ (\sigma_\tau f)(t) = f(t + \tau), \quad t \geq 0. \] (5)

This is also a linear bounded map from $L^2((0, \infty); X)$ into itself. This mapping commutes with $M_g$ as is shown next.

Lemma 1.4 For all $\tau > 0$ and all $g$ in $\mathcal{H}_\infty$, we have that
\[ \sigma_\tau (M_g f) = M_g (\sigma_\tau f), \quad f \in L^2((0, \infty), X). \] (6)

Proof We use the following well-known equality. If $h$ is Fourier transformable, then the Fourier transform of $h(\cdot + \tau)$ equals $e^{i \omega \tau} \hat{h}(\omega)$, where $\hat{h}$ denotes the Fourier transform of $h$.

Let $h \in L^2((0, \infty); X)$, then
\[ \mathcal{L}(\sigma_\tau h) = \overline{(\sigma_\tau h)} = \overline{\sigma_\tau h} - \hat{q} = e^{i \omega \tau} \overline{\hat{h}} - \hat{q} = e^{i \omega \tau} \mathcal{L}(h) - \hat{q}, \] (7)
with $q \in L^2((-\infty, 0); X)$. In particular, we find for every $h \in L^2(0, \infty); X)$ that
\[ \mathcal{L}(\sigma_\tau h) = \Pi (\mathcal{L}(\sigma_\tau h)) = \Pi (e^{i \omega \tau} \mathcal{L}(h)) = 0 = \mathcal{L}(M_{e^{i \omega \tau}} h), \] (8)
where we have used that $e^{i \omega \tau}$ is the boundary function corresponding to $e^{i \omega \tau} \in \mathcal{H}_\infty$. 

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Using (7) we see that
\[ M_g(\sigma \tau f) = L^{-1} \left( \Pi \left( ge^{i\tau L(f)} \right) \right) - L^{-1} \left( \Pi \left( g\hat{q} \right) \right) = L^{-1} \left( \Pi \left( ge^{i\tau L(f)} \right) \right) , \quad (9) \]
since \( \hat{q} \in H^1_2(X) \), and since \( g \in H^\infty \). Using Lemma [1.3] we find that
\[ M_g(\sigma \tau f) = L^{-1} \left( \Pi \left( ge^{i\tau L(f)} \right) \right) = M_{e^{i\tau}g} f = M_{e^{i\tau}M_g f}. \quad (10) \]
Now using (8), we see that
\[ M_g(\sigma \tau f) = \sigma \tau (M_g f). \quad (11) \]
\[ \square \]

2 Output maps and admissible output operators

In this section we study admissible operators which commute with the semigroup. We begin by defining well-posed output maps.

**Definition 2.1** Let \( (T(t))_{t \geq 0} \) be a strongly continuous semigroup on the Hilbert space \( X \), and let \( Y \) be another Hilbert space. We say that the mapping \( O \) is a well-posed (infinite-time) output map if

- \( O \) is a bounded linear mapping from \( X \) into \( L^2((0, \infty); Y) \), and
- For all \( \tau \geq 0 \) and all \( x_0 \in X \), we have that \( \sigma_\tau O x_0 = O(T(\tau)x_0) \).

Closely related to well-posed output mappings are admissible operators, which are defined next.

**Definition 2.2** Let \( (T(t))_{t \geq 0} \) be a strongly continuous semigroup on the Hilbert space \( X \). Let \( D(A) \) be the domain of its generator \( A \). A linear mapping \( C \) from \( D(A) \) to \( Y \), another Hilbert space, is said to be an (infinite-time) admissible output operator for \( (T(t))_{t \geq 0} \) if \( CT(\cdot)x_0 \in L^2((0, \infty), Y) \) for all \( x_0 \in D(A) \) and there exists an \( m \) independent of \( x_0 \) such that
\[ \int_0^\infty \|CT(t)x_0\|^2_Y dt \leq m\|x_0\|^2_X. \quad (12) \]
If $C$ is (infinite-time) admissible, then for all $x_0 \in X$ we can uniquely define an $L^2((0, \infty), X)$-function. We denote this function by $CT(\cdot)x_0$. Hence $O : X \to L^2((0, \infty); Y)$ defined by $Ox_0 = CT(\cdot)x_0$ is a well-posed output map. From [12] we know that the converse holds as well.

**Lemma 2.3** If $O$ is a well-posed output mapping, then there exists a (unique) linear bounded mapping from $D(A)$ to $Y$, $C$, such that $Ox_0 = CT(\cdot)x_0$ for all $x_0$.

In the sequel of this section we concentrate on admissible output operators which commute with the semigroup, i.e., $C$ a linear operator from $D(A)$ to $X$ and

$$CT(t)x_0 = T(t)Cx_0 \quad \text{for all } t \geq 0 \text{ and } x_0 \in D(A). \quad (13)$$

For these operators we have the following results.

**Lemma 2.4** Let $C$ be the admissible output operator associated with the well-posed output map $O$. Then (13) holds if and only if for all $t \geq 0$ there holds $OT(t) = T(t)O$.

**Theorem 2.5** Let $C$ be a bounded linear operator from $D(A)$ to $X$, which is admissible for the exponentially stable semigroup $(T(t))_{t \geq 0}$ and which commutes with this semigroup. Then the following holds

1. For all $x_0 \in D(A)$, we have that $CA^{-1}x_0 = A^{-1}Cx_0$.

2. For all $t > 0$, the operator $CT(t) : D(A) \to X$ can be extended to a bounded operator on $X$. Furthermore, $\|CT(t)\| \leq \gamma t^{-1/2}$ for some $\gamma$ independent of $t$.

**Proof** The first assertion follows easily from (13) by using Laplace transforms. We concentrate on the second assertion.
Let $x_0 \in D(A)$ and $x_1 \in X$, then for $t > 0$ we have that
\[
t\langle x_1, CT(t)x_0 \rangle = \int_0^t \langle x_1, CT(t)x_0 \rangle d\tau = \int_0^t \langle x_1, CT(\tau)T(t-\tau)x_0 \rangle d\tau = \int_0^t \langle x_1, T(\tau)CT(t-\tau)x_0 \rangle d\tau = \int_0^t \langle T(\tau)^*x_1, CT(t-\tau)x_0 \rangle d\tau \leq \sqrt{\int_0^t \|T(\tau)^*x_1\|^2 d\tau} \sqrt{\int_0^t \|CT(t-\tau)x_0\|^2 d\tau}.
\]

Using the fact that the semigroup, and hence its adjoint, are uniformly bounded, and the fact that $C$ is (infinite-time) admissible, we find that
\[
t\langle x_1, CT(t)x_0 \rangle \leq \sqrt{tM\|x_1\|m\|x_0\|}.
\]

Since this holds for all $x_1 \in X$, we conclude that
\[
t\|CT(t)x_0\| \leq \sqrt{tmM\|x_0\|}.
\]

This inequality holds for all $x_0 \in D(A)$. The domain of a generator is dense, and hence we have proved the second assertion. \(\square\)

From Theorem 2.5 it is clear that if the semigroup is surjective, then any admissible $C$ which commutes with the semigroup is bounded. However, this does not hold for a general semigroup as is shown in the following example. Furthermore, this example also shows that the estimate in the previous theorem cannot be improved.

**Example 2.6** Let $\{\phi_n, n \in \mathbb{N}\}$ be an orthonormal basis of $X$, and define for $t \geq 0$ the operator
\[
T(t) \sum_{n=1}^N \alpha_n \phi_n = \sum_{n=1}^N e^{-n^2t} \alpha_n \phi_n.
\]

It is not hard to show that this defines an exponentially stable $C_0$-semigroup on $X$. The infinitesimal generator $A$ is given by
\[
A \sum_{n=1}^N \alpha_n \phi_n = \sum_{n=1}^N -n^2 \alpha_n \phi_n.
\]
with domain

\[ D(A) = \{ x = \sum_{n=1}^{\infty} \alpha_n \phi_n \in X \mid \sum_{n=1}^{\infty} |n^2 \alpha_n|^2 < \infty \} \]

We define \( C \) as the square root of \(-A\), i.e.

\[ C \sum_{n=1}^{N} \alpha_n \phi_n = \sum_{n=1}^{N} n \alpha_n \phi_n \quad (15) \]

with domain

\[ D(C) = \{ x = \sum_{n=1}^{\infty} \alpha_n \phi_n \in X \mid \sum_{n=1}^{\infty} |n \alpha_n|^2 < \infty \} \]

A straightforward calculation gives that for \( x_0 = \sum_{n=1}^{N} \alpha_n \phi_n \), we have that

\[ \int_{0}^{\infty} \|CT(t)x_0\|^2 dt = \int_{0}^{N} \sum_{n=1}^{N} |ne^{-n^2 t} \alpha_n|^2 dt = \frac{1}{2} \sum_{n=1}^{N} |\alpha_n|^2 = \frac{1}{2} \|x_0\|^2. \]

Since the finite sums lie dense, we conclude that \( C \) is admissible. It is easy to see that \( C \) commutes with the semigroup, and thus from Theorem 2.3 we have that

\[ \|CT(t)\| \leq \frac{\gamma}{\sqrt{t}}. \quad (16) \]

for some \( \gamma \) independent of \( t \).

Next choose \( x_0 = \phi_n \) and \( t = n^{-2} \). Using (14) and (15) we see that

\[ CT(t)x_0 = ne^{-1} \phi_n = e^{-1} \frac{1}{\sqrt{t}} x_0, \]

and thus the estimate (16) cannot be improved.

The Lebesgue extension of an admissible operator is defined by

\[ C_L x = \lim_{t \to 0} \frac{1}{t} C \int_{0}^{t} T(\tau)xd\tau, \]

where

\[ D(C_L) = \{ x \in X \mid \text{limit exists} \}. \]
A similar extension can be define using the resolvent. The Lambda extension of an admissible operator is defined by

\[ C_\Lambda x = \lim_{\lambda \to \infty} \lambda C(\lambda I - A)^{-1}x, \]

where

\[ D(C_\Lambda) = \{ x \in X \mid \text{limit exists} \}. \]

The relation between these extension is still not completely understood, but for admissible operators which commute with the semigroup, we have that both extensions are closed operators.

**Lemma 2.7** Let \( C \) be an admissible operator which commutes with the semigroup, then the same holds for its Lebesgue and Lambda extension. Furthermore, these extensions are closed operators.

**Proof** Since \( A^{-1} \) and \( CA^{-1} \) are bounded, we find for \( x_0 \in D(C_L) \)

\[
A^{-1}C_Lx_0 = A^{-1}\lim_{t \downarrow 0} \frac{1}{t} \int_0^t T(\tau)x_0 d\tau = \lim_{t \downarrow 0} \frac{1}{t} A^{-1}C \int_0^t T(\tau)x_0 d\tau
\]

\[
= \lim_{t \downarrow 0} \frac{1}{t} CA^{-1} \int_0^t T(\tau)x_0 d\tau = CA^{-1} \lim_{t \downarrow 0} \frac{1}{t} \int_0^t T(\tau)x_0 d\tau
\]

\[ = CA^{-1}x_0 = C_LA^{-1}x_0, \]

where we have used that \( \int_0^t T(\tau)x_0 d\tau \in D(A) \) and \( C \) commutes with \( A^{-1} \). This proves the first assertion.

Using once more that \( CA^{-1} \) and \( A^{-1} \) are bounded, we have for \( x_0 \in D(C_L) \)

\[
CA^{-1} \int_0^t T(\tau)x_0 d\tau = \int_0^t CA^{-1}T(\tau)x_0 d\tau
\]

\[
= \int_0^t T(\tau)CA^{-1}x_0 d\tau
\]

\[ = \int_0^t T(\tau)A^{-1}C_Lx_0 d\tau = A^{-1} \int_0^t T(\tau)C_Lx_0 d\tau. \]

Let \( x_n \) be a sequence in \( D(C_L) \) which converges to \( x \in X \), such that \( C_Lx_n \) converges to \( z \in X \). Then by the above we find that

\[ CA^{-1} \int_0^t T(\tau)x d\tau = A^{-1} \int_0^t T(\tau)z d\tau \quad (17) \]
Since \( \int_0^T \tau \, d\tau \in D(A) \), we find that
\[
A^{-1} \int_0^t T(\tau) z \, d\tau = CA^{-1} \int_0^t T(\tau) x \, d\tau = A^{-1} C \int_0^t T(\tau) x \, d\tau.
\] (18)

Hence we have that
\[
\int_0^t T(\tau) z \, d\tau = C \int_0^t T(\tau) x \, d\tau.
\]

Since \( t^{-1} \int_0^t T(\tau) z \, d\tau \) converges to \( z \) for \( t \downarrow 0 \), we conclude from the above equality that \( x \in D(C_L) \) and \( C_L x = z \).

The proof for \( C_{\lambda} \) goes very similarly. Basically in the above proof, \( \int_0^t T(\tau) x \, d\tau \) is replaced by \( (\lambda I - A)^{-1} x \).

By Weiss [14] we have that \( C_{\lambda} \) is an extension of \( C_L \). We claim that for admissible \( C \)'s which commute with the semigroup they are equal.

3 \( \mathcal{H}_\infty \)-calculus

For \( g \in \mathcal{H}_\infty \) we define the following mapping from \( X \) to \( L^2((0, \infty); X) \)
\[
\mathcal{D}_g x_0 = M_g (T(t) x_0).
\] (19)

Hence we have taken in Definition [1,2] \( f(t) = T(t) x_0 \).

It is clear that \( \mathcal{D}_g \) is a linear bounded operator from \( X \) into \( L^2((0, \infty); X) \). Furthermore, from (9) we have that
\[
\sigma (\mathcal{D}_g x_0) = M_g (\sigma (T(t) x_0)) = M_g T(t+\tau) x_0 = \mathcal{D}_g (T(\tau) x_0),
\] (20)

where we have used the semigroup property. Hence \( \mathcal{D}_g \) is a well-posed output map, and so by Lemma [2,3] we conclude that \( \mathcal{D}_g \) can be written as
\[
\mathcal{D}_g x_0 = g(A) T(t) x_0
\] (21)

for some infinite-time admissible operator \( g(A) \) which is bounded from the domain of \( A \) to \( X \).

Since for all \( t, \tau \in [0, \infty) \) there holds \( T(\tau) T(t) = T(t) T(\tau) \), we conclude from (19) and (3) that
\[
\mathcal{D}_g T(t) = T(t) \mathcal{D}_g, \quad t \geq 0.
\]
Hence by (21), we see that $g(A)$ is an admissible operator which commutes with the semigroup. Theorem 2.5 implies that for $t > 0$, $g(A)T(t)$ can be extended to a bounded operator and

$$\|g(A)T(t)\| \leq \frac{\gamma}{\sqrt{t}}. \quad (22)$$

Note that for $t \in [0, 1]$ this $\gamma$ can be chosen as $\sup_{t \in [0,1]} \|T(t)\| \cdot \|g\|_{\infty}$.

The Laplace transform of $D_g$ equals $g(A)(sI - A)^{-1}$. Combining this with the definition of $D_g$, implies that

$$\|g(A)(sI - A)^{-1}\| \leq \frac{\|g\|_{\infty}}{\sqrt{\text{Re}(s)}} \|x_0\|, \quad (23)$$

where we have taken the norm in $X$, see also Weiss [13].

Since we have written this admissible operator as the function $g$ working on the operator $A$, there is likely to be a relation with functional calculus. This is shown next.

**Lemma 3.1** If $g \in H_{\infty}$ is the inverse Fourier transform of the function $h$, with $h \in L^1(-\infty, \infty)$ with support in $(-\infty, 0)$, then $g(A)$ is bounded

$$g(A)x_0 = \int_0^\infty T(t)h(-t)x_0 dt, \quad (24)$$

and so $g(A)$ corresponds to the classical definition of the function of an operator.

So if $g$ is the Fourier transform of an absolutely integrable function, then $g(A)$ is bounded. We would like to know when it is bounded for every $g$. For this, we extend the definition of $D_g$.

Let $C$ be an admissible output operator for the semigroup $(T(t))_{t \geq 0}$. By definition, we know that $CT(\cdot)x_0 \in L^2((0, \infty); Y)$ for all $x_0 \in X$. We define

$$(C \circ D_g)x_0 = M_g(CT(t)x_0) \quad (25)$$

It is clear that this is a bounded mapping from $X$ to $L^2((0, \infty); Y)$.

As before we have that

$$\sigma_{\tau}(C \circ D_g) (x_0) = (C \circ D_g) (T(\tau)x_0). \quad (26)$$

And so we can write $(C \circ D_g)x_0$ as $\tilde{C}_g T(\cdot)x_0$ for some infinite-time admissible $\tilde{C}_g$. We have that
Lemma 3.2 The infinite-time admissible operator \( \tilde{C}_g \) satisfies
\[
\tilde{C}_g x_0 = C g(A) x_0, \quad \text{for } x_0 \in D(A^2).
\]  

Proof For \( x_0 \in D(A^2) \), we introduce \( x_1 = A x_0 \). Then the following equalities hold in \( L^2((0, \infty); Y) \).
\[
\tilde{C}_g T(t)x_0 = (C \circ \Omega_g)x_0 = M_g (CT(t)x_0) = M_g (CT(t)A^{-1}x_1) = M_g (CA^{-1}T(t)x_1) = CA^{-1}g(A)T(t)x_1 = \tilde{C}_g A^{-1}g(A)T(t)x_1 = C g(A)T(t)x_0,
\]
where we have used (3). Since both functions are continuous at zero, we find that (27) holds. □

Based on this result, we denote \( \tilde{C}_g \) by \( C \circ g(A) \).

Using this, we can prove the following theorems.

Theorem 3.3 The mapping \( g \mapsto g(A) \) forms a (unbounded) \( \mathcal{H}_\infty \)-calculus.

Proof It only remains to show that \((g_1 g_2)(A) = g_1(A) g_2(A)\). By Lemma 1.3 we have that
\[
\Omega_{g_1 g_2} x_0 = M_{g_1 g_2} (T(t)x_0) = M_{g_1} M_{g_2} (T(t)x_0).
\]
For \( x_0 \in D(A) \) the last expression equals \( M_{g_1} (g_2(A)T(t)x_0) \), see (21). Since \( g_2(A) \) commutes with the semigroup, we find that
\[
\Omega_{g_1 g_2} x_0 = M_{g_1} (T(t)g_2(A)x_0).
\]
Using (21) twice, we obtain
\[
(g_1 g_2)(A)T(t)x_0 = \Omega_{g_1 g_2} x_0 = g_1(A)T(t)g_2(A)x_0
\]
This is an equality in \( L^2((0, \infty); X) \). However, if we take \( x_0 \in D(A^2) \), then this holds point-wise, and so for \( x_0 \in D(A^2) \).
\[
(g_1 g_2)(A)x_0 = g_1(A) g_2(A)x_0
\]
This concludes the proof. □
Theorem 3.4 If there exists an admissible $C$ such that $(C,A)$ is exactly observable, i.e., there exists an $m_1 > 0$ such that for all $x_0 \in X$ there holds

$$\int_0^\infty \|CT(t)x_0\|^2 dt \geq m_1 \|x_0\|^2$$

then $g(A)$ is bounded for every $g \in \mathcal{H}_\infty^-$. Furthermore, if $m_2$ is the admissibility constant, see equation (12), then

$$\|g(A)\| \leq \sqrt{\frac{m_2}{m_1}} \|g\|_\infty.$$  \hspace{1cm} (28)

Proof Let $x_0 \in D(A^2)$

$$m_1 \|g(A)x_0\|^2 \leq \|CT(t)g(A)x_0\|_{L^2((0,\infty);Y)}^2 \leq \|g\|_\infty^2 \|CT(t)x_0\|_{L^2((0,\infty);Y)}^2 \leq m_2 \|g\|_\infty^2 \|x_0\|^2.$$

Since $D(A^2)$ is dense, we obtain the result. \hfill \Box

As a corollary we obtain the well-known von Neumann inequality. Recall that the operator $A$ is dissipative if

$$\langle x_0, Ax_0 \rangle + \langle Ax_0, x_0 \rangle \leq 0 \quad \text{for all } x_0 \in D(A).$$  \hspace{1cm} (29)

Corollary 3.5 If $A$ is a dissipative operator and its corresponding semigroup is exponentially stable, then $A$ has a bounded $\mathcal{H}_\infty^-$ calculus and for all $g \in \mathcal{H}_\infty^-$

$$\|g(A)\| \leq \|g\|_\infty.$$  \hspace{1cm} (30)

Proof Since $A$ is dissipative and since its semigroups is exponentially stable, we have that $A^{-1}$ is bounded and dissipative. We define $Q$ via

$$\langle x_1, Qx_2 \rangle = -\langle A^{-1}x_1, x_2 \rangle - \langle x_1, A^{-1}x_2 \rangle, \quad x_1, x_2 \in X.$$  \hspace{1cm} (31)
It is easy to see that $Q$ is bounded, self-adjoint and by the dissipativity of $A^{-1}$ we have that $Q \geq 0$. Define on the domain of $A$ the operator $C$ as $C = \sqrt{QA}$, then from (31) we find that

$$-\langle Cx_1, Cx_2 \rangle = \langle x_1, Ax_2 \rangle + \langle Ax_1, x_2 \rangle, \quad x_1, x_2 \in D(A). \quad (32)$$

Combining this Lyapunov equation with the exponential stability, gives that for all $x_0 \in D(A)$

$$\int_0^\infty \|CT(t)x_0\|^2 dt = \|x_0\|^2. \quad (33)$$

Thus we see that the constants $m_1$ and $m_2$ in Theorem 3.4 can be chosen to be one, and so (28) gives the results. $\square$

If $A$ generates an exponentially stable semigroup and if there exists an admissible $C$ for which $(C, A)$ is exactly observable, then it is not hard to show that the semigroup is similar to a contraction semigroup. Using this, one can also obtain the above result by Theorem G of [1]. The following result has been proved by McIntosh in [10].

**Theorem 3.6** Assume that $A$ generates an exponentially stable semigroup. If $(-A)^{\frac{1}{2}}$ is admissible for $(T(t))_{t \geq 0}$ and $(-A^*)^{\frac{1}{2}}$ is admissible for the adjoint semigroup $(T(t^*))_{t \geq 0}$, then $g(A)$ is bounded for every $g \in \mathcal{H}_{-\infty}$. Thus this semigroup has a bounded $\mathcal{H}_{-\infty}$-calculus.

**Proof** Since $A^{1/2}$ is admissible, Lemma 3.2 gives that $A^{1/2} \circ g(A)$ is also
admissible. Consider for \( x_1 \in D(A^*) \) and \( x_0 \in D(A^2) \) the following

\[
\langle x_1, g(A)x_0 \rangle - \langle x_1, g(A)T(t)x_0 \rangle
\]

\[
= \int_0^t \langle x_1, (-A)T(\tau)g(A)x_0 \rangle d\tau
\]

\[
= \int_0^t \langle (-A^*)^{\frac{1}{2}}x_1, (-A)^{\frac{1}{2}}g(A)T(\tau)x_0 \rangle d\tau
\]

\[
= \int_0^t \langle (-A^*)^{\frac{1}{2}}T(\frac{T}{2})^*x_1, g(A)(-A)^{\frac{1}{2}}T(\frac{T}{2})x_0 \rangle d\tau
\]

\[
\leq \sqrt{\int_0^t \| (-A^*)^{\frac{1}{2}}T(\frac{T}{2})^*x_1 \|^2 d\tau} \sqrt{\int_0^t \| g(A)(-A)^{\frac{1}{2}}T(\frac{T}{2})x_0 \|^2 d\tau}
\]

\[
\leq \sqrt{\int_0^t \| (-A^*)^{\frac{1}{2}}T(\frac{T}{2})^*x_1 \|^2 d\tau} \| g \|_\infty \sqrt{\int_0^\infty \| (-A)^{\frac{1}{2}}T(\frac{T}{2})x_0 \|^2 d\tau}
\]

\[
\leq m_1 \| x_1 \| m_2 \| g \|_\infty \| x_0 \|
\]

where \( m_1 \) and \( m_0 \) are the admissibility constant of \((-A^*)^{\frac{1}{2}}\) and \((-A^*)^{\frac{1}{2}}\), respectively. Furthermore, we used (2).

Since the sets \( D(A^*) \) and \( D(A^2) \) are dense in \( X \), we obtain that

\[
\| g(A) \| \leq m_1 m_2 \| g \|_\infty + \| g(A)T(t) \|. \tag{34}
\]

By Theorem 2.5 we know that \( g(A)T(t) \) is bounded, and so we conclude that \((T(t))_{t \geq 0}\) has a bounded \( H_\infty \)-calculus.

In McIntosh [10] the above theorem was proved using square function estimates. The admissibility of \((-A)^{\frac{1}{2}}\) can be written as

\[
m \| x_0 \|^2 \geq \int_0^\infty \| (-A)^{\frac{1}{2}}T(t)x_0 \|^2 dt = \int_0^\infty \| (-tA)^{\frac{1}{2}}T(t)x_0 \|^2 \frac{dt}{t}.
\]

The latter is the “square function estimate” for \( \psi(s) = (-s)^{\frac{1}{2}}e^s \), and so the admissibility condition can be seen as a square function estimate. The other condition used in [10] is that the operator \( A \) is sectorial on a sector larger than the sector on which the scalar functions are defined. Since we have as function class \( \mathcal{H}_\infty \) and since our operators \( A \) are assumed to generate an
exponential semigroup, this condition seems not to satisfied. However, the admissibility assumptions made in the theorem imply that $A$ generates a bounded analytic semigroup, and so the condition of McIntosh is satisfied.

**Lemma 3.7** Let $A$ generate an exponentially stable semigroup and let $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ be admissible operators for for $(T(t))_{t \geq 0}$ and $(T(t^*))_{t \geq 0}$, respectively. Then $A$ generates a bounded analytic semigroup.

**Proof** The proof is similar to the proof of Theorem 2.3. Let $x_1 \in D(A^*)$ and $x_0 \in D(A)$. Then for $t > 0$ we find

$$t \langle x_1, AT(t)x_0 \rangle = \int_0^t \langle x_1, AT(t)x_0 \rangle d\tau$$

$$= - \int_0^t \langle (-A^*)^{\frac{1}{2}} x_1, (-A)^{\frac{1}{2}} T(t)x_0 \rangle d\tau$$

$$= - \int_0^t \langle (-A^*)^{\frac{1}{2}} T(\tau^*) x_1, (-A)^{\frac{1}{2}} T(t - \tau)x_0 \rangle d\tau$$

$$\leq \sqrt{\int_0^t \| (-A^*)^{\frac{1}{2}} T(\tau^*) x_1 \|^2 d\tau} \sqrt{\int_0^t \| (-A)^{\frac{1}{2}} T(t - \tau)x_0 \|^2 d\tau}$$

$$\leq m_1 \| x_1 \| m_2 \| x_0 \|,$$

where we used that $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ are admissible. Since the domain of $A^*$ and $A$ are dense, we obtain that

$$\| AT(t) \| \leq \frac{M}{t}, \quad t > 0$$

By Theorem II.4.6 of [4], we conclude that generates a bounded analytic semigroup. □

From [10] we know that if the conditions of Theorem 3.6 hold, then is the semigroup similar to a contraction (or $(-A)^{\frac{1}{2}}$ is exactly observable). We show this next.

**Lemma 3.8** Under the condition of Theorem 3.6 we have that $(-A)^{\frac{1}{2}}$ is exactly observable, and thus $(T(t))_{t \geq 0}$ is similar to a contraction.
Proof In idea the proof is the same as that of Theorem 3.6. Let $x_1 \in D(A^*)$ and $x_0 \in D(A)$ We have that

$$\langle x_1, x_0 \rangle = \int_0^\infty \langle x_1, (-A)T(\tau)x_0 \rangle d\tau$$

$$= \int_0^\infty \langle (-A^*)^{\frac{1}{2}} x_1, (-A)^{\frac{1}{2}} T(\tau)x_0 \rangle d\tau$$

$$= \int_0^\infty \langle (-A^*)^{\frac{1}{2}} T(\frac{\tau}{2})^* x_1, (-A)^{\frac{1}{2}} T(\frac{\tau}{2}) x_0 \rangle d\tau.$$ (35)

Hence

$$|\langle x_1, x_0 \rangle| \leq \sqrt{\int_0^\infty \| (-A^*)^{\frac{1}{2}} T(\frac{\tau}{2})^* x_1 \|^2 d\tau \int_0^\infty \| (-A)^{\frac{1}{2}} T(\frac{\tau}{2}) x_0 \|^2 d\tau }$$

$$\leq m_1 \| x_1 \| \sqrt{\int_0^\infty \| (-A)^{\frac{1}{2}} T(\frac{\tau}{2}) x_0 \|^2 d\tau }.$$ (36)

Since the domain of $A^*$ is dense we conclude that

$$\| x_0 \| = \sup_{x_1 \neq 0} \frac{|\langle x_1, x_0 \rangle|}{\| x_1 \|} \leq m_1 \sqrt{\int_0^\infty \| (-A)^{\frac{1}{2}} T(\frac{\tau}{2}) x_0 \|^2 d\tau }.$$ (36)

Thus $(-A)^{\frac{1}{2}}$ is exactly observable. $\square$

We remark that with the above result, Theorem 3.6 follows also from Theorem 3.4. However, we decided to present this independent proof.

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