ON KNOTS WITH INFINITE SMOOTH CONCORDANCE ORDER

ADAM SIMON LEVINE

ABSTRACT. We use the Heegaard Floer obstructions defined by Grigsby, Ruberman, and Strle to show that forty-six of the sixty-seven knots through eleven crossings whose concordance orders were previously unknown have infinite concordance order.

Let $K$ be an oriented knot in $S^3$. If $K$ bounds a smoothly embedded disk in $D^4$, we say that $K$ is (smoothly) slice. Two knots $K, K'$ are said to be (smoothly) concordant if $K \# K'$ is slice, where $K'$ denotes the mirror of $K$. The set of concordance classes of knots forms a group $C_1$ under the connect sum operation with identity the unknot. The concordance order of a knot $K$ is the order of $K$ in $C_1$. The structure of the torsion in $C_1$ is of considerable interest; see, for instance, Livingston-Naik [6, 7] and Jabuka-Naik [4].

Let $Y_K = \Sigma_2(K)$ be the double branched cover of $K$, and let $\tilde{K}$ be the inverse image of $K$ in $Y$. Grigsby, Ruberman, and Strle [3] defined numerical invariants $D_n(K)$ and $T_n(K)$ ($n \in \mathbb{N}$) coming from the Heegaard Floer homology of $Y_K$ and the knot Floer homology of $\tilde{K}$. They proved:

**Theorem 1.** Let $K$ be a knot in $S^3$. Let $p$ be prime, and suppose that $p^m$ is the largest power of $p$ that divides $\det(K)$. If $K$ has finite concordance order, then for each integer $0 \leq e \leq \left\lfloor \frac{m+1}{2} \right\rfloor$, we have $D_{pe}(K) = T_{pe}(K) = 0$.

In practice, we are usually interested in $D_p(K)$ and $T_p(K)$, where $p$ is 1 or a prime that divides $\det(K)$, so we restrict our discussion to this case.

According to Livingston's database KnotInfo [2], the smooth concordance orders of sixty-seven knots with up to eleven crossings, listed in Table[1] were previously unknown. We show here that forty-six of these knots, listed in Tables[2] and[3] have at least one nonzero $D_p$ invariant and hence have infinite concordance order. For the remaining knots, all of the relevant $D_p$ invariants vanish, so the concordance orders of these knots remains unknown. The $T_p$ invariants for several of these knots
can be obtained using the author’s computations of \( \widehat{\text{HFK}}(Y_K, \tilde{K}) \) \[5\], but we do not obtain any new concordance information in this manner.

For the remainder of this paper, we describe the techniques used to compute the \( D_p \) and \( T_p \) invariants for the knots considered here.

Let us briefly recall the definition of these invariants in the case where \( H^2(Y_K; \mathbb{Z}) \) is cyclic. (For the general case, see \[3\, \text{Definition 4.1}]\.) Let \( s_0 \in \text{Spin}^c(Y_K) \) be the so-called canonical spin\(^c\) structure on \( Y_K \), uniquely characterized by the property that \( c_1(s_0) = 0 \). Recall that \( \text{Spin}^c(Y_K) \) is an affine space for \( H^2(Y_K; \mathbb{Z}) \), so we may identify \( \text{Spin}^c(Y_K) \) with \( H^2(Y_K; \mathbb{Z}) \) via the identification \( s \mapsto c_1(s) \). Let \( G_m \) be the unique order-\( p \) subgroup of \( H^2(Y_K; \mathbb{Z}) \). The invariants \( D_p(K) \) and \( T_p(K) \) are then defined as

\[
D_p(K) = \sum_{s \in s_0 + G_p} d(Y_K, s)
\]

\[
T_p(K) = \sum_{s \in s_0 + G_p} \tau(Y_K, \tilde{K}, s).
\]

Here \( d(Y_K, s) \) is the correction term for \( \text{HF}^+(Y_K, s) \), and \( \tau(Y_K, \tilde{K}, s) \) is the \( \tau \)-invariant for \( \widehat{\text{HFK}}(Y_K, \tilde{K}, s) \). (See Ozsváth-Szabó \[8\, \text{[11]}\] for the definitions of \( d \) and \( \tau \).

In many cases, the results of Ozsváth and Szabó \[9\, 12\, 13\] may be used to compute the correction terms \( d(Y, s) \) combinatorially. Given a projection of \( K \), let \( G \) be its Goeritz matrix (defined in \[12\, \text{section 3}]\). Let \( |G| \) denote the rank of \( G \). The double cover \( Y_K \) bounds a 4-manifold \( X_G \) whose intersection form on \( H_2 \), \( Q = Q_{X_G} \), is given by \( G \) (with respect to a basis of spheres). Let \( \text{Char}(G) \subset H^2(X_G; \mathbb{Z}) \) denote the set of characteristic vectors for \( Q \), i.e., vectors \( \alpha \in H^2(X_G; \mathbb{Z}) \) such that

| 9_{30} | 9_{33} | 9_{44} | 10_{58} | 10_{60} | 10_{91} | 10_{102} | 10_{119} |
| 10_{135} | 10_{158} | 10_{164} | 11_{a_4} | 11_{a_5} | 11_{a_8} | 11_{a_{11}} | 11_{a_{24}} |
| 11_{a_26} | 11_{a_30} | 11_{a_{38}} | 11_{a_{44}} | 11_{a_{47}} | 11_{a_{52}} | 11_{a_{56}} | 11_{a_{67}} |
| 11_{a_{72}} | 11_{a_{76}} | 11_{a_{80}} | 11_{a_{88}} | 11_{a_{98}} | 11_{a_{104}} | 11_{a_{109}} | 11_{a_{112}} |
| 11_{a_{126}} | 11_{a_{135}} | 11_{a_{160}} | 11_{a_{167}} | 11_{a_{168}} | 11_{a_{170}} | 11_{a_{187}} | 11_{a_{189}} |
| 11_{a_{233}} | 11_{a_{249}} | 11_{a_{257}} | 11_{a_{265}} | 11_{a_{270}} | 11_{a_{272}} | 11_{a_{287}} | 11_{a_{288}} |
| 11_{a_{289}} | 11_{a_{300}} | 11_{a_{303}} | 11_{a_{315}} | 11_{a_{350}} | 11_{n_{12}} | 11_{n_{34}} | 11_{n_{45}} |
| 11_{n_{48}} | 11_{n_{53}} | 11_{n_{55}} | 11_{n_{85}} | 11_{n_{100}} | 11_{n_{110}} | 11_{n_{114}} | 11_{n_{130}} |
| 11_{n_{145}} | 11_{n_{157}} | 11_{n_{165}} |

\text{Table 1. Knots through eleven crossings with unknown concordance order.}
| Knot $K$ | $\det(K)$ | Nonzero GRS invariants |
|---------|-----------|------------------------|
| $9_{30}$ | 53        | $D_{53} = 4$            |
| $9_{33}$ | 61        | $D_{61} = 4$            |
| $10_{58}$| 65        | $D_{13} = 4$            |
| $10_{60}$| 85        | $D_{17} = 4$            |
| $10_{102}$| 73        | $D_{73} = -12$          |
| $10_{119}$| 101      | $D_{101} = -16$         |
| $11a_{4}$ | 97        | $D_{97} = -24$          |
| $11a_{8}$ | 117       | $D_{13} = -4$           |
| $11a_{11}$| 113       | $D_{113} = 12$          |
| $11a_{24}$| 157       | $D_{157} = 12$          |
| $11a_{26}$| 157       | $D_{157} = 12$          |
| $11a_{30}$| 149       | $D_{149} = 12$          |
| $11a_{52}$| 137       | $D_{137} = 16$          |
| $11a_{56}$| 109       | $D_{109} = -8$          |
| $11a_{67}$| 125       | $D_{25} = -4$           |
| $11a_{76}$| 145       | $D_{29} = -4$           |
| $11a_{80}$| 137       | $D_{137} = -12$         |
| $11a_{88}$| 101       | $D_{101} = -8$          |
| $11a_{126}$| 145      | $D_{5} = 4, D_{29} = 4$ |
| $11a_{160}$| 145      | $D_{29} = -4$           |
| $11a_{167}$| 113      | $D_{113} = 12$          |
| $11a_{170}$| 185      | $D_{37} = -4$           |
| $11a_{189}$| 149      | $D_{149} = -12$         |
| $11a_{233}$| 173      | $D_{101} = 16$          |
| $11a_{249}$| 117      | $D_{13} = -4$           |
| $11a_{257}$| 97       | $D_{97} = -8$           |
| $11a_{265}$| 109      | $D_{109} = 24$          |
| $11a_{270}$| 137      | $D_{137} = 12$          |
| $11a_{272}$| 149      | $D_{149} = 12$          |
| $11a_{287}$| 181      | $D_{181} = -12$         |
| $11a_{288}$| 205      | $D_{5} = 4, D_{41} = 4$ |
| $11a_{289}$| 145      | $D_{29} = 4$            |
| $11a_{300}$| 153      | $D_{17} = -4$           |
| $11a_{303}$| 149      | $D_{149} = 36$          |
| $11a_{315}$| 157      | $D_{157} = 12$          |
| $11a_{350}$| 185      | $D_{5} = 4, D_{37} = 4$ |

Table 2. Alternating knots with non-vanishing Grigsby-Ruberman-Strle $D_p$ invariants.
Knot \( K \) | \( \det(K) \) | Nonzero GRS invariants
--- | --- | ---
9_{44} | 17 | \( D_{17} = 4 \)
10_{135} | 135 | \( D_{37} = 4 \)
11_{n_{12}} | 13 | \( D_{13} = -8 \)
11_{n_{48}} | 29 | \( D_{29} = -8 \)
11_{n_{53}} | 37 | \( D_{37} = -8 \)
11_{n_{55}} | 61 | \( D_{61} = 12 \)
11_{n_{110}} | 41 | \( D_{41} = -12 \)
11_{n_{114}} | 53 | \( D_{53} = -4 \)
11_{n_{130}} | 53 | \( D_{53} = 12 \)
11_{n_{165}} | 85 | \( D_{17} = -4 \)

**Table 3.** Non-alternating knots with non-vanishing Grigsby-Ruberman-Strle \( D_p \) invariants.

\[ \langle \alpha, v \rangle \equiv Q(v, v) \pmod{2} \] for every \( v \in H_2(X_G; \mathbb{Z}) \). The restriction map \( i^*: H^2(X_G) \to H^2(Y_K) \) partitions \( \text{Char}(G) \) into equivalence classes \( \text{Char}(G, s) \) corresponding to the spin\(^c\) structures on \( Y_K \). Given certain hypotheses on \( G \), including that \( G \) is negative-definite, Ozsváth and Szabó [9, Corollary 1.5] proved that the correction terms for HFK\(^+\)(\( Y_K \)) are given by the formula

\[
(1) \quad d(Y_K, s) = \max_{\alpha \in \text{Char}(G, s)} \frac{\alpha^2 + |G|}{4}.
\]

Ozsváth and Szabó provide an algorithm for finding the vectors in each equivalence class that realize this maximum. Moreover, since \( H^2(Y_K; \mathbb{Z}) \cong \text{coker}(G) \), we may easily identify the group structure on \( \text{Spin}^c(Y_K) \) (specifically, which spin\(^c\) structures are in the special subgroup \( G_p \)) using the Smith normal form for \( G \).

As shown in [12], Equation (1) holds whenever \( G \) is computed from an alternating projection. More generally, if \( K \) admits a projection that is alternating except in a region that consists of left-handed twists, Ozsváth and Szabó [13] show how to use Kirby calculus on \( X_G \) to obtain a matrix \( \tilde{G} \) for \( Q \) that satisfies the correct hypotheses. (See also Jabuka-Naik [4] for a concise explanation.) All of the non-alternating knots in Table 3 satisfy this hypothesis, so we may compute the \( D_p \) invariants as described above.

Finally, to compute the \( T_p \) invariants of a knot, one must compute the integers \( \tau(Y_K, \tilde{K}, s) \) associated to the spectral sequence from

\(^1\)That \( K = 11_{n_{12}} \) has infinite concordance order also follows from the simpler fact that \( \tau(S^3, K) = 1 \), as was computed by Baldwin and Gillam [1].
When \( \widehat{\text{HFK}}(Y_K, \tilde{K}, s) \) and \( \widehat{\text{HF}}(Y_K, s) \) are sufficiently simple, one can sometimes determine \( \tau \) without knowing all the differentials in the spectral sequence. For instance, if \( \widehat{\text{HFK}}(Y_K, \tilde{K}, s) \) has rank 1 and \( \widehat{\text{HFK}}(Y_K, \tilde{K}, s) \) is supported on a single diagonal, \( \tau(Y_K, \tilde{K}, s) \) is equal to the Alexander grading of the nonzero group in Maslov grading \( d(Y_K, s) \). The author [5] has shown how to compute \( \text{HFK}(Y_K, \tilde{K}) \) (with coefficients in \( \mathbb{Z}/2 \)) for any knot \( K \) using grid diagrams and has computed the values of \( \tau \) for several of the non-alternating knots considered here (9\( _{44} \), 10\( _{135} \), 10\( _{158} \), 10\( _{164} \), 11\( _{100} \), and 11\( _{145} \)). However, the \( T_p \) invariants all vanish in these cases, so we do not obtain any new concordance information.

REFERENCES

[1] J. A. Baldwin and W. D. Gillam, Computations of Heegaard Floer knot homology, preprint, math/0610167.
[2] J. C. Cha and C. Livingston, KnotInfo: an online table of knot invariants, www.indiana.edu/~knotinfo.
[3] J. E. Grigsby, D. Ruberman, and S. Strle, Knot concordance and Heegaard Floer homology invariants in branched covers, preprint, math/0701460.
[4] S. Jabuka and S. Naik, Order in the concordance group and Heegaard Floer homology, Geom. Topol. 11 (2007), 979–994.
[5] A. S. Levine, Computing knot Floer homology in branched double covers, preprint to appear in Alg. Geom. Topol., math/0709.1427.
[6] C. Livingston and S. Naik, Obstructing four-torsion in the classical knot concordance group, J. Diff. Geom. 51 (1999), 1-12.
[7] C. Livingston and S. Naik, Knot concordance and torsion, Asian J. Math. 5 (2001), 161–168.
[8] P. S. Ozsváth and Z. Szabó, Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary, Adv. Math. 173 (2003) 179–261.
[9] P. S. Ozsváth and Z. Szabó, On the Floer homology of plumbed three-manifolds, Geom. Topol. 7 (2003), 185-224.
[10] P. S. Ozsváth and Z. Szabó, Heegaard Floer homology and alternating knots, Geom. Topol. 7 (2003), 225-254.
[11] P. S. Ozsváth and Z. Szabó, Holomorphic disks and knot invariants, Adv. Math. 186 (2004), 58–116.
[12] P. S. Ozsváth and Z. Szabó, On the Heegaard Floer homology of branched double-covers, Adv. Math. 194 (2005), 1-33.
[13] P. S. Ozsváth and Z. Szabó, Knots with unknotting number one and Heegaard Floer homology, Topol. 44 (2005), 705-745.

Department of Mathematics, Columbia University, New York, NY 10027

E-mail address: alevine@math.columbia.edu