Discrete orthogonality of hypergeometric polynomial sequences on linear and quadratic lattices

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Abstract

We present a method to obtain weight functions associated with linear and quadratic lattices that yield discrete orthogonality with respect to a quasi-definite moment functional of the orthogonal polynomial sequences in the Askey scheme, with the exception of the Jacobi, Bessel, Laguerre, and Hermite polynomials.

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1. Introduction

The families of hypergeometric orthogonal polynomial sequences included in the Askey scheme [7] are some of the most important and have been studied for a long time. Some of those families have an orthogonality determined by a weight function \( w(x_k) \) defined on a finite or infinite set of points \( x_k \) that belong to a linear or quadratic lattice. In such cases the orthogonality of a polynomial sequence \( \{u_n(t) : n \in \mathbb{N}\} \) is given by

\[
\sum_{k \in M} u_n(x_k)u_m(x_k)w(x_k) = K_n\delta_{n,m}, \quad n, m \in M,
\]

where \( M = \{0, 1, 2, \ldots, N\} \), for some positive integer \( N \), or \( M = \mathbb{N} \), and \( K_n \) is a nonzero constant for \( n \in M \).

The polynomial families that satisfy certain types of differential or difference equations and have a discrete orthogonality with positive weights \( w(x_k) \) have been characterized and studied in great detail. The main references for discrete orthogonal polynomials are the books [9] and [7]. See also [1], [4], [5], and [10].

In the present paper we present a method to find a weight function for each polynomial family in a subset \( \mathcal{B} \) of the Askey scheme that determines a discrete orthogonality with respect to a quasi-definite moment functional. The families that are not included in \( \mathcal{B} \) are the ones of Jacobi, Bessel, Laguerre, and Hermite polynomials. The weights \( w(x_k) \)

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are the values at $t = 1$ of a sequence of hypergeometric functions $f_k(t)$ that are of type $3F_2$ or type $2F_1$. The parameters of $f_k(t)$ depend on the parameters that determine the coefficients in the three-term recurrence relation of the corresponding polynomial family. We show that the hypergeometric functions $f_k(t)$ are convergent at $t = 1$ and that they are obtained from $f_0(t)$ by repeated differentiation and multiplication by certain factors. In this paper we do not deal with the problem of characterizing the cases for which the weights are positive.

For the classical families of discrete orthogonal polynomials of the Hahn, Meixner, Krawtchouk, and Charlier polynomials, the orthogonality that we obtain coincides with the classical ones presented in [7], with a different normalization in some cases.

We obtain our results using the linear algebraic approach of our previous papers [12], [13], [14], and [15]. In [15] we presented a unified construction of all the hypergeometric and basic hypergeometric orthogonal polynomial sequences that uses three linearly recurrent sequences of numbers that satisfy certain difference equation of order three, with constant coefficients. The initial terms of such numerical sequences determine the sequences of orthogonal polynomials and provide us with a uniform parametrization of all the hypergeometric and basic hypergeometric sequences. In the present paper we used several results from [15].

In Section 2 we present some results related with the construction of the hypergeometric orthogonal polynomials and their uniform parametrization from [15]. In Section 3 we present the main result, whose proof uses infinite systems of linear equations and properties of hypergeometric functions. In Section 4 we find the weight functions for some families of orthogonal polynomials from the Askey scheme. Analogous results for the families of basic hypergeometric polynomial families in the $q$-Askey scheme will be presented elsewhere.

2. The class $\mathcal{H}_1$ of hypergeometric orthogonal polynomial sequences

In this section we present some results about the class $\mathcal{H}_1$ of the hypergeometric orthogonal polynomial sequences that were obtained in [15].

Consider the homogeneous difference equation

$$s_{k+3} = 3(s_{k+2} - s_{k+1}) + s_k, \quad k \geq -1. \quad (2)$$

The characteristic polynomial of this equation has 1 as a root of multiplicity 3, and therefore the general solution is a quadratic polynomial in $k$. Let $x_k, h_k,$ and $e_k$ be solutions of (2). Then

$$x_k = b_0 + b_1 k + b_2 k^2, \quad h_k = a_0 + a_1 k + a_2 k^2, \quad e_k = d_0 + d_1 k + d_2 k^2. \quad (3)$$

The sequence $x_k$ determines the Newtonian basis $\{v_n(t) : n \geq 0\}$ of the complex vector space of polynomials in $t$, defined by

$$v_n(t) = (t - x_0)(t - x_1) \cdots (t - x_{n-1}), \quad n \geq 1, \quad (4)$$
and \( v_0(t) = 1 \). We define the sequence \( g_k \) by
\[
g_k = x_{k-1}(h_k - h_0) + e_k, \quad k \geq 1,
\]
and \( g_0 = 0 \). Therefore we must have \( e_0 = d_0 = 0 \). In addition, we suppose that \( h_k \neq h_j \) if \( k \neq j \), and \( g_k \neq 0 \) for \( k \geq 1 \).

Let \( D \) be the linear operator on the space of polynomials defined by
\[
Dv_k = h_kv_k + g_kv_{k-1}, \quad k \geq 0.
\]
Since \( g_0 = 0 \) we see that \( D^m = h_n t^n + \text{polynomial of lower degree} \). For \( n \geq 0 \) let \( u_n \) be a monic polynomial of degree \( n \) which is an eigenfunction of \( D \) with eigenvalue \( h_n \). That is
\[
Du_k = h_ku_k, \quad k \geq 0.
\]

The operator \( D \) is a generalized difference operator, which in concrete examples becomes a second order differential operator or a difference operator on a linear or quadratic lattice. In [15] we showed that
\[
u_n(t) = \sum_{k=0}^{n} c_{n,k}v_k(t), \quad n \geq 0,
\]
where the coefficients \( c_{n,k} \) are given by
\[
c_{n,k} = \prod_{j=k}^{n-1} \frac{g_{j+1}}{h_n - h_j}, \quad 0 \leq k \leq n - 1,
\]
and \( c_{n,n} = 1 \) for \( n \geq 0 \). This expression for \( u_n(t) \) was also obtained by Vinet and Zhedanov in [16] using a different approach. The idea of representing orthogonal polynomials in terms of a Newtonian basis was introduced by Geronimus in [6].

The matrix of coefficients \( C = [c_{n,k}] \) is an infinite lower triangular matrix and the coefficients of \( u_n \) appear in the \( n \)-th row of \( C \). Since \( c_{n,n} = 1 \) for \( n \geq 0 \), \( C \) is invertible. Let \( C^{-1} = [\hat{c}_{n,k}] \) and define the polynomials
\[
w_{n,k}(t) = \prod_{j=k}^{n} (t - h_j), \quad 0 \leq k \leq n.
\]
Using divided differences we obtain
\[
\hat{c}_{n,k} = \frac{\prod_{j=k+1}^{n} g_j}{w'_{n+1,k}(h_k)} = \prod_{j=k+1}^{n} \frac{g_j}{h_k - h_j}, \quad 0 \leq k \leq n - 1,
\]
and \( \hat{c}_{n,n} = 1 \) for \( n \geq 0 \).
The entries in the 0-th column of $C^{-1}$ are given by

$$
\hat{c}_{k,0} = \prod_{j=1}^{k} \frac{g_j}{h_0 - h_j}, \quad k \geq 1,
$$

(12)

and $\hat{c}_{0,0} = 1$. We denote them by $m_k = \hat{c}_{k,0}$ for $k \geq 0$. They satisfy $m_0 = 1$ and

$$
\sum_{k=0}^{n} c_{n,k} m_k = 0, \quad n \geq 1.
$$

(13)

Note that the sequence $m_n$ satisfies a recurrence relation of order one.

In [15] we also proved that the polynomial sequence $u_n(t)$ satisfies a three-term recurrence relation of the form

$$
u_{n+1}(t) = (t - \beta_n)u_n(t) - \alpha_n u_{n-1}(t), \quad n \geq 1,
$$

(14)

where the coefficients are given by

$$
\beta_n = x_n + \frac{g_{n+1}}{h_{n} - h_{n+1}} - \frac{g_n}{h_{n-1} - h_{n}},
$$

(15)

and

$$
\alpha_n = \frac{g_n}{h_{n-1} - h_{n}} \left( \frac{g_{n-1}}{h_{n-2} - h_{n}} - \frac{g_n}{h_{n-1} - h_{n}} + \frac{g_{n+1}}{h_{n-1} - h_{n+1}} + x_n - x_{n-1} \right).
$$

(16)

By Favard’s theorem [3, Thm. 4.4], [8], if all the $\alpha_n$ are positive and the $\beta_n$ are real then $\{u_n\}$ is orthogonal with respect to a positive-definite moment functional, and if all the $\alpha_n$ are nonzero then $\{u_n\}$ is orthogonal with respect to a quasi-definite moment functional.

Let $B$ be the class of all the families of hypergeometric orthogonal polynomial sequences whose recurrence coefficients are determined by a sequence of nodes $x_k = b_0 + b_1 k + b_2 k^2$, where at least one of $b_1$ or $b_2$ is nonzero. The families in the Askey scheme that are not included in $B$ are the families of Jacobi, Bessel, Laguerre, and Hermite. For these families $x_k = x_0$ for $k \geq 0$.

3. Generalized moments and discrete orthogonality

Let us suppose that the parameters that determine the sequences $x_k, h_k$, and $e_k$ are such that $h_k \neq h_j$ if $k \neq j$, and the coefficients $\alpha_n$, given by (16), are nonzero for $n \geq 1$. Then, by Favard’s theorem there exists a unique quasi-definite moment functional $\tau$ on the space of polynomials such that the polynomial sequence $\{u_n(t) : n \geq 0\}$ is orthogonal with respect to $\tau$, that is,

$$
\tau(u_n(t)u_m(t)) = K_n \delta_{n,m}, \quad K_n \neq 0, \quad n, m \in \mathbb{N}.
$$

(17)
Since $u_0(t) = 1$ we have
\[ \tau(u_n(t)) = \sum_{k=0}^{n} c_{n,k} \tau(v_k(t)) = 0, \quad n \geq 1, \tag{18} \]
and by the uniqueness of the inverse of $C$ we must have $\tau(v_k(t)) = \hat{c}_{k,0} = m_k$ for $k \geq 0$. This means that the numbers $m_k$ are the generalized moments of $\tau$ with respect to the Newtonian basis $\{v_k(t) : k \geq 0\}$. Note that $\tau(1) = m_0 = 1$.

**Theorem 3.1.** Suppose that at least one of $b_1$ and $b_2$ is nonzero, at least one of $a_1$ and $a_2$ is nonzero, and $\alpha_n \neq 0$ for $n \geq 1$. Then there exists a unique weight function $w$, defined on the nodes $x_k$, such that for every polynomial $p(t)$ we have
\[ \tau(p(t)) = \sum_{k=0}^{\infty} p(x_k)w(x_k). \tag{19} \]

**Proof:** From (3) we see that the hypothesis implies that $\{x_k\}$ and $\{h_k\}$ are sequences of pairwise distinct numbers, and that the polynomial sequence $\{u_n(t) : n \geq 0\}$ is orthogonal with respect to the quasi-definite moment functional $\tau$.

We will write $r_j = w(x_j)$ for $j \geq 0$. Since $\tau(v_k(t)) = m_k$ for $k \geq 0$, the numbers $r_j$ that we want to find must satisfy
\[ m_k = \tau(v_k(t)) = \sum_{j=0}^{\infty} v_k(x_j) r_j, \quad k \geq 0. \tag{20} \]
This is an infinite system of linear equations. Let us denote by $P$ the matrix of coefficients. Then $P = [v_k(x_j)]$, where $j$ is the index for rows, $k$ is the index for columns and $j$ and $k$ are non-negative integers. From the definition of the polynomials $v_k(t)$ in (4) we can see that $P$ is an infinite lower triangular matrix and its entries in the main diagonal are
\[ v_k(x_k) = (x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1}), \]
and they are nonzero. Therefore $P$ is invertible.

Using basic properties of divided differences with respect to the nodes $x_j$ it is easy to show that $P^{-1}$ is the lower triangular matrix whose $(j, k)$ entry equals $1/v'_{j+1}(x_k)$. Therefore, from the system (20) we obtain
\[ r_k = \sum_{j=k}^{\infty} \frac{m_j}{v'_{j+1}(x_k)}, \quad k \geq 0. \tag{21} \]
We will show that if the parameters $a_1, a_2, b_1, b_2, d_1, d_2$ satisfy certain conditions, then all the series in (21) are convergent (or terminating) hypergeometric series.

Let us define the power series
\[ f_k(t) = \sum_{j=k}^{\infty} \frac{m_j}{v'_{j+1}(x_k)} t^j, \quad k \geq 0. \tag{22} \]
The denominators in the previous equation are

\[ v'_{j+1}(x_k) = \prod_{i=0, i \neq k}^j (x_k - x_i), \quad k \geq j \geq 0. \]  

(23)

By substitution of (23) and (12) in (22), writing all the sequences using the formulas (3) and (5), we find an expression for \( f_k(t) \) in terms of the parameters \( a_1, a_2, b_0, b_1, b_2, d_1, d_2 \). For example, the first 3 terms of \( f_0(t) \) are

\[
f_0(t) = 1 + \frac{a_1 b_0 + a_2 b_0 + d_1 + d_2}{(a_1 + a_2)(b_1 + b_2)} t + \frac{(a_1 b_0 + a_2 b_0 + d_1 + d_2)((b_0 + b_1 + b_2)(a_1 + 2a_2) + d_1 + 2d_2)}{(a_1 + a_2)(a_1 + 2a_2)(b_1 + b_2)(b_1 + 2b_2)} t^2 + \cdots
\]

(24)

Note that \( a_0 \) does not appear here because the sequence of eigenvalues \( h_k \) enters only trough the differences \( h_k - h_j \). Therefore the value of \( a_0 \) is not relevant and we can suppose that \( a_0 = 0 \).

The series in (24) looks like a hypergeometric function. We can confirm that it is indeed a hypergeometric function through the changes of parameters that we define next. We will consider four cases, corresponding to whether \( a_2 \) and \( b_2 \) are zero or nonzero.

**Case 1.** If \( a_2 \neq 0 \) and \( b_2 \neq 0 \) we introduce the parameters \( p, r, y_1, y_2 \) as follows

\[
\begin{align*}
a_1 &= (r - 1)a_2, \\
b_1 &= (p - 1)b_2, \\
d_1 &= a_2(b_2((y_1 - 1)(y_2 - 1)(p + r - y_1 - y_2 - 2) + (r - 1)(p - 2)) - (r - 1)b_0), \\
d_2 &= a_2(b_2((y_1 + y_2)(p + r - y_1 - y_2 - 1) + y_1y_2 - r(p - 1)) - b_0).
\end{align*}
\]

(25)

The new parameters \( y_1 \) and \( y_2 \) can be expressed in terms of the original parameters by solving for \( y_1 \) and \( y_2 \) the last two equations in (25).

Substitution of \( a_1, b_1, d_1, d_2 \) in \( f_0(t) \) using (25) gives us

\[ f_0(t) = F(y_1, y_2, p + r - y_1 - y_2 - 1; r, p; t), \]

(26)

where \( F \) is the hypergeometric function of type 3F2 defined by

\[ F(y_1, y_2, y_3; r, p; t) = \sum_{k=0}^{\infty} \frac{(y_1)_k(y_2)_k(y_3)_k}{(r)_k(p)_k} \frac{t^k}{k!}, \]

(27)

where \((a)_k\) denotes the shifted factorial, defined by \((a)_0 = 1\) and

\[ (a)_k = a(a + 1)(a + 2) \cdots (a + k - 1), \quad a \in \mathbb{C}, \quad k \geq 1. \]

For any values of the parameters \( p, r, y_1 \) and \( y_2 \) the parametric excess (the sum of the lower parameters minus the sum of the upper parameters) of the hypergeometric function
in (26) is equal to one. Therefore it is convergent at \( t = 1 \), since a hypergeometric function converges at \( t = 1 \) if the real part of the parametric excess is positive. See [2, Thm. 2.1.2, p.62] or [11, p.45].

The functions \( f_k(t) \), for \( k \geq 1 \), can also be expressed in terms of hypergeometric functions using the parameters \( p, r, y_1 \) and \( y_2 \). In those functions the factors of the form \( p + i \) in the denominators appear in a less regular way than in \( f_0(t) \). This is so because they come from the derivatives \( v'_{j+1}(x_k) \) and this produces jumps and shifts in the index \( i \) of the factors \( p + i \).

We can verify by direct computations that

\[
f_k(t) = (-1)^k \left( \frac{p + 2k - 1}{p + k - 1} \right) \frac{t^k}{k!} \times D_t^k F(y_1, y_2, r + p - y_1 - y_2 - 1; r, p + k; t), \quad k \geq 0,
\]

where \( D_t \) denotes differentiation with respect to \( t \).

For \( k \geq 0 \) the parametric excess of \( F(y_1, y_2, r + p - y_1 - y_2 - 1; r, p + k; t) \) is \( k + 1 \), and thus the function converges at \( t = 1 \). Therefore \( f_k(t) \) converges at \( t = 1 \) and the weights \( r_k = f_k(1) \) are well defined for \( k \geq 0 \).

The functions \( f_k(t) \) can also be expressed as follows

\[
f_k(t) = (-1)^k \frac{(y_1)_k(y_2)_k(p + r - y_1 - y_2 - 1)_k}{(r)_k(p - 1 + k)_k} \frac{t^k}{k!} \times F(y_1 + k, y_2 + k, r + p - y_1 - y_2 - 1 + k; r + k, p + 2k; t).
\]

The sequence of weights \( r_k = f_k(1) \) that we have found satisfies (20) for \( k \geq 0 \), and since the set \( \{ v_k(t) : k \geq 0 \} \) is a basis for the space of all polynomials, we conclude that \( \tau(p(t)) = \sum_{k=0}^{\infty} p(x_k) r_k \) for every polynomial \( p(t) \).

**Case 2.** We consider now the case with \( b_2 = 0 \) and \( a_2 \neq 0 \). We use here the substitutions

\[
b_2 = 0,
\]
\[
a_1 = (r - 1)a_2,
\]
\[
d_1 = a_2(b_1(r - 1 + (y_1 - 1)(y_2 - 1)) - b_0(r - 1)),
\]
\[
d_2 = a_2(b_1(y_1 + y_2 - r) - b_0),
\]

and obtain

\[
f_0(t) = F_2(y_1, y_2; r; t),
\]

where \( F_2 \) is the hypergeometric function of type \( {}_2F_1 \) defined by

\[
F_2(y_1, y_2; r; t) = \sum_{k=0}^{\infty} \frac{(y_1)_k(y_2)_k t^k}{(r)_k k!}.
\]
From (30) we can obtain $y_1$ and $y_2$ by solving a system of equations. They satisfy

$$y_1 + y_2 = \frac{(b_1 r + b_0) a_2 + d_2}{b_1 a_2},$$

and therefore the function (31) converges at $t = 1$ if the real part of $-(a_2 b_0 + d_2)/(a_2 b_1)$ is positive.

The substitution of (30) in $f_k(t)$ gives us

$$f_k(t) = (-1)^k t^k D_t^k F_2(y_1, y_2; r; t), \quad k \geq 0. \quad (33)$$

These functions are also convergent at $t = 1$ if the real part of $-(a_2 b_0 + d_2)/(a_2 b_1)$ is positive. We can also write the functions $f_k(t)$ as

$$f_k(t) = (-1)^k \frac{(y_1)_k (y_2)_k}{(r)_k} \frac{t^k}{k!} F_2(y_1 + k, y_2 + k; r + k; t), \quad k \geq 1. \quad (34)$$

**Case 3.** Suppose now that $a_2 = 0$ and $b_2 \neq 0$. In this case we see from (24) that $f_0(t)$ becomes a simpler hypergeometric function. The parameter substitutions

$$a_2 = 0, \quad b_1 = (p - 1) b_2, \quad d_1 = a_1 b_2 ((y_1 - 1)(y_2 - 1) + p - 2) - b_0, \quad d_2 = - a_1 b_2 (p - y_1 - y_2 - 1), \quad (35)$$

in $f_0(t)$ give us

$$f_0(t) = F_2(y_1, y_2; p; t). \quad (36)$$

The parameters $y_1$ and $y_2$ can be obtained from (35) by solving a system of equations. They satisfy

$$y_1 + y_2 = p - 1 + \frac{d_2}{a_1 b_2}. \quad (37)$$

Therefore the parametric excess of the function $F_2(y_1, y_2; p; t)$ in (36) is $p - y_1 - y_2 = 1 - \frac{d_2}{a_1 b_2}$, and thus $f_0(t)$ converges at $t = 1$ if

$$\text{Re} \left(1 - \frac{d_2}{a_1 b_2}\right) > 0. \quad (38)$$

In this case, applying the substitutions of parameters (35) to the series $f_k(t)$, we obtain hypergeometric functions similar to $F_2(y_1, y_2; p; t)$. By a straightforward computation we can verify that

$$f_k(t) = (-1)^k \left(\frac{p + 2k - 1}{p + k - 1}\right) \frac{t^k}{k!} D_t^k F_2(y_1, y_2; p + k; t), \quad k \geq 0. \quad (39)$$
Note that all the functions $f_k(t)$ are convergent at $t = 1$ if (38) holds. Another expression for $f_k(t)$ in this case is the following

$$f_k(t) = (-1)^k \frac{(y_1)_k(y_2)_k}{(p - 1 + k)_k} \frac{t^k}{k!} F_2(y_1 + k, y_2 + k; p + 2k; t).$$

(40)

**Case 4.** In this case we have $a_2 = 0$ and $b_2 = 0$ and $f_0(t)$ has the form

$$f_0(t) = 1 + \frac{a_1b_0 + d_1 + d_2}{a_1b_1} t + \frac{(a_1b_0 + d_1 + d_2)(a_1b_0 + a_1b_1 + d_1 + 2d_2)}{a_1^2b_1^2} \frac{t^2}{2!} + \cdots$$

(41)

The substitutions

$$a_2 = 0, \quad b_2 = 0,$$

$$d_1 = a_1(b_1(1 + (y - 1)z) - b_0),$$

$$d_2 = (z - 1)a_1b_1,$$

(42)

give us

$$f_0(t) = F_2(y, 1; 1; zt).$$

(43)

By the ratio test we see that $F_2(y, 1; 1; zt)$ converges at $t = 1$ if $|z| < 1$. With the change of parameters (42) the series $f_k(t)$ become the hypergeometric functions

$$f_k(t) = (-1)^k \frac{t^k}{k!} D^k(F_2(y, 1; 1; zt)), \quad k \geq 1,$$

(44)

where $D_t$ denotes differentiation with respect to $t$. It is clear that $f_k(t)$ converges at $t = 1$ if $|z| < 1$.

From (42) we can solve for $y$ and $z$ in terms of the other parameters and we obtain

$$y = \frac{a_1b_0 + d_1 + d_2}{a_1b_1 + d_2}, \quad z = 1 + \frac{d_2}{a_1b_1}.$$

(45)

Therefore the functions $f_k(t)$ are convergent at $t = 1$ if $|1 + \frac{d_2}{a_1b_1}| < 1$.

We have proved that in each of the four cases the weights $r_k = f_k(1)$ are well defined if the parameters satisfy certain conditions.

It is easy to verify that in the four cases considered in the previous theorem we have

$$\sum_{k=0}^{n} \text{coefficient of } t^n \text{ in } f_k(t) = 0, \quad n \geq 1,$$

(46)

and therefore

$$\sum_{k=0}^{\infty} f_k(t) = 1,$$

(47)

for every $t$ in the region of convergence of $f_0(t)$, and in particular $\sum_{k=0}^{\infty} r_k = 1$. This property of the weights also follows from $\tau(v_0) = \tau(1) = m_0 = 1$.

Let us note that in the four cases the hypergeometric function $f_0(t)$ determines to a large extent the sequence of functions $f_k(t)$ and their region of convergence.
Corollary 3.1. With the hypothesis of the previous theorem, if the parameters satisfy the conditions of the corresponding case then the polynomials \( u_k(t) \) satisfy the discrete orthogonality

\[
\tau(u_n(t)u_m(t)) = \sum_{k=0}^{\infty} u_n(x_k)u_m(x_k)r_k = K_n\delta_{n,m},
\]

where the constants \( K_n = \alpha_1\alpha_2 \cdots \alpha_n \) are nonzero for \( n \geq 0 \).

4. Some examples of the weights \( r_k \)

In this section we find the weights \( r_k \) that correspond to some families of orthogonal polynomial sequences from the Askey scheme [7].

For each of the four cases we can find the coefficients of the three-term recurrence relation of the corresponding polynomial family by applying to the general formulas for \( \beta_n \) and \( \alpha_n \), given in (15) and (16), the parameter substitutions that we used in the proof of Theorem 3.1.

Case 1. In the first case the parameter substitution (25) applied to \( \alpha_n \) yields

\[
\alpha_n = \frac{b_2^2n(n+y_1-1)(n+y_2-1)(n-p+y_1+y_2)}{(2n+r-3)(2n+r-2)(2n+r-1)} \times
\]

\[
(n-2+r)(n+r-y_1-1)(n+r-y_2-1)(n+p+r-y_1-y_2-2).
\]

The change of parameters

\[
r = a+b+c+d, \quad p = 2c+1, \quad y_1 = c+d, \quad y_2 = b+c,
\]

transforms (49) into the sequence \( \alpha_n \) that corresponds to the Wilson polynomials written in terms of the parameters \( a, b, c, d \) used in [7, eq. 9.1.5]. The analogous result for the sequence \( \beta_n \) of the Wilson polynomials holds. Therefore the weights \( r_k \) for the Wilson polynomials are the values at \( t = 1 \) of the functions \( f_k(t) \) of (28) with the parameters changed by the substitutions (50). We have

\[
f_k(t) = (-1)^k \frac{(2c+2k)}{(2c+k)} \frac{t^k}{k!} \times \]

\[
D^k F(c+d+b+c, a+c; a+b+c+d, 2c+1+k; t).
\]

For any values of \( a, b, c, d \) the parametric excess of the hypergeometric function in the previous equation is equal to \( 1+k \) and therefore \( f_k(t) \) converges at \( t = 1 \) for \( k \geq 0 \).

Note that (50) allows us to express the parameters \( a, b, c, d \) in terms of \( p, r, y_1, y_2 \). From (29) we obtain another expression for the functions \( f_k(t) \) of the Wilson polynomials.

\[
f_k(t) = (-1)^k k!(c+d)_k(b+c)_k(a+c)_k \frac{t^k}{(a+b+c+d)_k(2c+k)_k} \times
\]

\[
F(c+d+k, b+c+k, a+c+k; a+b+c+d+k, 2c+1+2k; t).
\]
If $y_1$ or $y_2$ is a negative integer then the hypergeometric functions $f_k(t)$ defined in (28) are terminating. In that case the corresponding set of orthogonal polynomials is finite. This is the case for the Racah polynomial family.

**Case 2.** We present next examples that correspond to the second case in the proof of Theorem 3.1, that is, with $b_2 = 0$ and $a_2$ nonzero.

The substitution (30) applied to the general formula for $\alpha_n$ gives us

$$\alpha_n = -\frac{b_1^2 n(n + r - 2)(n + y_1 - 1)(n + y_2 - 1)(n + r - y_1 - 1)(n + r - y_2 - 1)}{(2n + r - 2)^2(2n + r - 3)((2n + r - 1)\times}$

$$= \frac{n(n + a + b + c + d - 2)(n + a + c - 1)}{(2n + a + b + c + d - 2)^2(2n + a + b + c + d - 1)^2(n + a + d - 1)(n + b + c - 1)(n + b + d - 1)}.$$

This $\alpha_n$ corresponds to the continuous Hahn polynomials [7, Eq. 9.4.4].

Applying the change of parameters (54) to the functions $f_k(t)$ in (33) we obtain

$$f_k(t) = (-1)^k \frac{k!}{k!} D_k^k F_2(a + c, a + d; a + b + c + d; t), \quad k \geq 0. \quad (56)$$

These hypergeometric functions converge at $t = 1$ if the real part of $b - a$ is positive.

The Hahn polynomials [7, Eq.9.5.4], with parameters $\alpha, \beta, N$, are obtained from the general formulas for $\alpha_n$ and $\beta_n$ with the substitutions

$$b_2 = 0, \quad b_1 = 1, \quad b_0 = 0,$$

$$a_1 = (\alpha + \beta + 1)a_2,$$

$$d_1 = -a_2(N\alpha - \beta - 1),$$

$$d_2 = -a_2(N + \beta + 1). \quad (57)$$

With these values of the parameters we obtain $f_0(t) = F_2(-N, \alpha + 1; \alpha + \beta + 2; t)$, and this terminating hypergeometric function determines the discrete orthogonality [7, Eq. 9.5.2] of the Hahn polynomials.

**Case 3.** We consider next the case with $a_2 = 0$ and $b_2 \neq 0$. The continuous dual Hahn polynomials, with parameters $a, b, c$ [7, Eq.9.3,5], are obtained from the general formulas taking

$$a_2 = 0, \quad a_1 = 1,$$

$$b_2 = 1, \quad b_1 = 2c, \quad b_0 = c^2,$$

$$d_2 = a + b, \quad d_1 = ab + ac + bc - a - b. \quad (58)$$
Combining these equations with (35) we obtain
\[ p = 2c + 1, \quad y_1 = a + c, \quad y_2 = b + c. \] (59)

By substitution in (39) we obtain
\[ f_k(t) = (-1)^k \frac{2c + 2k}{2c + k} \frac{t^k}{k!} D_t^k F_2(a + c, b + c; 2c + 1 + k; t), \quad k \geq 0. \] (60)

These functions are convergent at \( t = 1 \) if the real part of \( 1 - a - b \) is positive. The numbers \( r_k = f_k(1) \) are the weights for the continuous dual Hahn polynomials.

**Case 4.** We consider here examples with \( a_2 = 0 \) and \( b_2 = 0 \). Some of the classical discrete orthogonal polynomials fall in this case.

If \( a_2 = 0 \) and \( b_2 = 0 \) the functions \( f_k(t) \) are given in (44), which is
\[ f_k(t) = (-1)^k \frac{t^k}{k!} D_t^k F_2(y, 1; 1; zt), \quad k \geq 0, \] (61)
where \( y \) and \( z \) are given in (45), and all the \( f_k(t) \) are convergent at \( t = 1 \) if \( |z| < 1 \). It is clear from (61) that \( f_0(t) \) determines all the functions \( f_k(t) \), and the weights \( r_k = f_k(1) \).

The Krawtchouk polynomials [7, Eq. 9.11.4], with parameters \( p \) and \( N \), are obtained with
\[ a_2 = 0, \quad b_2 = 0, \quad b_1 = 1, \]
\[ d_2 = (p - 1)a_1, \quad d_1 = -a_1(Np + p + b_0 - 1), \] (62)
and these equations give \( f_0(t) = F_2(-N, 1; 1; pt) \), which yields the discrete orthogonality in [7, Eq. 9.11.2].

The Meixner polynomials, with parameters \( c \) and \( \beta \) [7, Eq. 9.10.4], are obtained with
\[ a_2 = 0, \quad b_2 = 0, \quad b_1 = 1, \]
\[ d_2 = \frac{a_1}{c - 1}, \quad d_1 = \frac{a_1(\beta - b_0)c + b_0 - 1}{c - 1}, \] (63)
and these parameters give us \( f_0(t) = F_2(\beta, 1; 1; \frac{a_t}{c - 1}) \), which produces the discrete orthogonality described in [7, Eq. 9.10.2].

The Charlier polynomials, with parameter \( a \) [7, Eq. 9.14.4], are obtained with
\[ a_2 = 0, \quad b_2 = 0, \quad b_1 = 1, \]
\[ d_2 = -a_1, \quad d_1 = (1 - a - b_0)a_1, \] (64)
which give \( f_0(t) = \exp(-at) \). This function produces the discrete orthogonality [7, Eq. 9.14.2].

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We have shown that all the hypergeometric orthogonal polynomial sequences in the
Askey scheme, other than the very classical families (Jacobi, Bessel, Laguerre, and
Hermite), satisfy a discrete orthogonality on a sequence of distinct nodes \( x_k \) in the
complex plane. The nodes are of the form \( x_k = b_0 + b_1 k + b_2 k^2 \), where at least one of
\( b_1 \) and \( b_2 \) is nonzero. The weights \( r_k \) associated with the nodes \( x_k \) are the values at
\( t = 1 \) of a sequence of hypergeometric functions. We have not dealt with the problem
of characterizing the cases for which the weights \( r_k \) are positive.

The approach used in this paper can be used to obtain discrete orthogonality for some
of the families of basic hypergeometric orthogonal polynomials in the \( q \)-Askey scheme.

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