A modular type formula for Euler infinite product 
\((1 - x)(1 - xq)(1 - xq^2)(1 - xq^3)\cdots\)

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Abstract

The main goal of this paper is to give a modular type representation for the infinite product 
\((1 - x)(1 - xq)(1 - xq^2)(1 - xq^3)\cdots\). It is shown that this representation essentially contains the well-known modular formulae either for Dedekind’s eta function, Jacobi theta function or for certain Lambert series. Thus a new and unified approach is outlined for the study of elliptic and modular functions and related series.

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Introduction

There are more and more studies on $q$-series and related topics, not only in traditional themes, but also in more recent branches, such as quantum physics, random matrices. A first non-trivial example of $q$-series may be the infinite product $(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots$, that is considered in Euler [10, Chap. XVI] and then is revisited by many of his successors, particularly intensively by Hardy and Ramanujan [13, p. 238-241; p. 276-309; p. 310-321]. Beautiful formulae are numerous and motivations are often various: elliptic and modular functions theory, number and partition theory, orthogonal polynomials theory, etc. Concerning this wonderful history, one may think of Euler's pentagonal number theorem [4, p. 30], Jacobi's triple product identity [5, §10.4, p. 499-501], Dedekind modular eta function [20, (44), p. 154], to quote only some examples of important masterpieces.

However, the infinite product $(1-x)(1-xq)(1-xq^2)(1-xq^3)\cdots$, already appearing as initial model in the same work [10, Chap. XVI] of Euler, receives less attention although it always plays a remarkable role in all above-mentioned subjects: several constants in the elliptic integral theory, Gauss' binomial formula [5, §10.2, p. 487-491], generalized Lambert series, etc. Indeed, rapidly the situation may become more complicated and, what is really important, the modular relation does not occur for generic values of $x$.

In the present paper, we shall point out how, up to an explicit part, the function defined by the product $(1-x)(1-xq)(1-xq^2)(1-xq^3)\cdots$ can be seen somewhat modular. This non-modular part can be considered as being represented by a divergent but Borel-summable power series on variable $\log q$ near zero, that is, when $q$ tends toward the unit value. This result, subject of Theorem 1.1, gives rise to new and unified approaches to treat Jacobi theta function, Lambert series, . . . .

The paper is organized as follows. Section 1 is devoted to sight-read certain terms contained in Theorem 1.1 of §1.1. Firstly, in §1.2 we will give two equivalent formulations of Theorem 1.1, one of which will be used in complex plane in §1.6. In §1.3 and §1.4 we will observe that the modular relation remains almost valid but a perturbation term exists. In §1.5 we deal with the remainder term of the Stirling asymptotic formula for $\Gamma$-function. Theorem 1.9, given in §1.6, is another equivalent version of Theorem 1.1 and will be used in Section 2 as
it is formulated in terms of complex variables. Relation (38) shows that the above-mentioned non-modular part can be expressed in terms of the quotient of two Barnes’ double Gamma functions.

In Section 2, we will explain how to utilize Theorem 1.1 to get the classical modular formula for eta or theta function. In §2.1, it will be merely observed that the so-called non-modular part identically vanishes; in the theta function case (§2.2), two non-modular parts are of opposite sign and then cancel each other out. In §2.3, a second proof will be delivered to θ-modular equation from the point view of q-difference equations. In §2.4, we consider the first order derivatives of \((1 - x)(1 - xq)(1 - xq^2)(1 - xq^3)\) ... and then get some results for two families of q-series, including Lambert series as special cases that will be treated in §2.5. In §2.6, we will give some remarks about the limit behavior when \(q\) goes to one by real values.

In Section 3, we give a complete proof of our main Theorem. To do this, we need several elementary but somewhat technical calculations, that will be formulated in terms of various lemmas.

We are interested in studying the analytical theory of differential, difference and q-difference equations, à la Birkhoff [8]; see [9], [18], [27], [28], [30]. The elliptic functions and one variable modular functions can be seen as specific solutions of certain particular difference or q-difference equations; in this line, we shall give a proof on Theta function modular equation in §2.3. We believe that a good understanding of singularities structure, that is, Stokes analysis [17] as well as other geometric tools, often permits a lot more of comprehension about certain magical formulas or, say, some Ramanujan’s dream.

Let us mention that in [14], Theorem 1.1 is employed to the study of Jackson Gamma function.

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1 Modular type expansion of \((x; q)_\infty\)

Let \(q\) and \(x\) be complex numbers; if \(|q| < 1\), we let

\[(x; q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n).\]

In §1.1 and §1.2, we will suppose that \(q \in (0, 1)\) and \(x \in (0, 1)\), so that the infinite product \((x; q)_\infty\) converges in \((0, 1)\); therefore, one can take the logarithm of this function. From §1.3, we will work with complex variables. In §1.6, a modular type expansion for \((x; q)_\infty\) will be given in complex plane. As usual, log will stand for the principal branch of the logarithmic function over its Riemann surface denoted by \(\mathbb{C}^*\) and, in the meantime, the broken plane \(\mathbb{C} \setminus (\infty, 0]\) will be identified to a part of \(\mathbb{C}^*\).
1.1 Main Theorem

The main result of our paper is the following statement.

**Theorem 1.1.** Let \( q = e^{-2\pi \alpha} \), \( x = e^{-2\pi (1 + \xi) \alpha} \) and suppose \( \alpha > 0 \) and \( \xi > -1 \). The following relation holds:

\[
\log(x; q)_{\infty} = -\frac{\pi}{12 \alpha} + \log \frac{\sqrt{2\pi}}{\Gamma(\xi + 1)} + \frac{\pi}{12} \alpha - \left( \xi + \frac{1}{2} \right) \log \frac{1 - e^{-2\pi \xi \alpha}}{\xi} \\
+ \int_{0}^{\xi} \frac{2\pi \alpha t}{e^{2\pi \alpha t} - 1} \, dt + M(\alpha, \xi),
\]

where

\[
M(\alpha, \xi) = -\sum_{n=1}^{\infty} \frac{\cos 2n\pi \xi}{n(e^{2n\pi \alpha} - 1)} - \frac{2}{\pi} \text{PV} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{\sin 2n\xi \pi t}{n(e^{2n\pi \alpha} - 1)} \, dt 1 - t^2.
\]

In the above, \( \text{PV} \int \) stands for the principal value of a singular integral in the Cauchy’s sense; see [25, §6.23, p. 117] or the corresponding definition recalled later in §3.4. We will leave the proof until Section 3.

Before extending the main theorem to the complex plane (§1.6), we first give some equivalent statements.

1.2 Variants of Theorem 1.1

Throughout all the paper, we let

\[
B(t) = \frac{1}{e^{2\pi t} - 1} - \frac{1}{2\pi t} + \frac{1}{2};
\]

therefore, Theorem 1.1 can be stated as follows.

**Theorem 1.2.** Let \( q, x, \alpha, \xi \) and \( M(\alpha, \xi) \) be as given in Theorem 1.1. Then the following relation holds:

\[
\log(x; q)_{\infty} = -\frac{\pi}{12 \alpha} - \left( \xi + \frac{1}{2} \right) \log 2\pi \alpha + \log \frac{\sqrt{2\pi}}{\Gamma(\xi + 1)} + \frac{\pi}{2} \left( \xi + 1 \right) \xi \alpha \\
+ \frac{\pi}{12} \alpha + 2\pi \alpha \int_{0}^{\xi} \left( t - \xi - \frac{1}{2} \right) B(\alpha t) \, dt + M(\alpha, \xi),
\]

where \( B \) is defined in (3).

**Proof.** It suffices to notice the following elementary integral: for any real numbers \( \lambda \) and \( \mu \),

\[
\int_{0}^{\lambda} B(\mu t) \, dt = \frac{1}{2\pi \mu} \log \frac{1 - e^{-2\pi \lambda \mu}}{2\pi \lambda \mu} + \frac{\lambda}{2}.
\]
As usual, let $\text{Li}_2$ denote the dilogarithm function; recall $\text{Li}_2$ can be defined as follows [5, (2.6.1-2), p. 102]:

$$\text{Li}_2(x) = - \int_0^x \frac{\log(1-t)}{t} dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2}. \quad (6)$$

**Theorem 1.3.** The following relation holds for any $q \in (0,1)$ and $x \in (0,1)$:

$$\log(x; q)_\infty = \frac{1}{\log q} \text{Li}_2(x) + \log \sqrt{1-x} - \frac{\log q}{24} - \log(1-e^{-2\pi \alpha}) + \int_0^\infty B\left(-\frac{\log q}{2\pi t}\right) x^t \frac{dt}{t} + M\left(-\frac{\log q}{2\pi}, \log_q x\right), \quad (7)$$

where $B$ denotes the function given by (3).

**Proof.** By the first Binet integral representation stated in [5, Theorem 1.6.3 (i), p. 28] for $\log \Gamma$, Theorem 1.1 can be formulated as follows:

$$\log(x; q)_\infty = \frac{\pi^2}{12\alpha} + \frac{\pi}{12} \alpha - I_\Gamma(\xi) - \frac{1}{2} \log(1 - e^{-2\pi \alpha}) - \int_0^\xi \log(1 - e^{-2\pi t}) dt + M(\alpha, \xi), \quad (8)$$

where $\xi = \log_q (x/q)$ and $I_\Gamma$ denotes the corresponding Binet integral in term of the function $B$ defined by (3):

$$I_\Gamma(\xi) = \log \Gamma(\xi + 1) - (\xi + \log \xi + \log \sqrt{2\pi}) = \int_0^\infty B(t) e^{-2\pi \xi t} \frac{dt}{t}. \quad (9)$$

Write $\log(x; q)_\infty = \log(1 - x/q) + \log(x; q)_\infty$, and substitute $q$ by $e^{-2\pi \alpha}$ and $x/q = e^{-2\pi \xi} = q^\xi$ by $x$ in (3), respectively; we arrive at once at the following expression:

$$\log(x; q)_\infty = \frac{\pi^2}{6 \log q} + \frac{\log q}{24} + \log \sqrt{1-x} - \int_0^{\log_q x} \log(1 - q^t) dt - \int_0^\infty B\left(-\frac{\log q}{2\pi t}\right) x^t \frac{dt}{t} + M\left(-\frac{\log q}{2\pi}, \log_q x\right),$$

from which, using (6), we easily deduce the expected formula (7), for $\text{Li}_2(1) = \frac{\pi^2}{6}$.

### 1.3 Almost modular term $M$

We shall write the singular integral part in [2] by means of contour integration in the complex plane, as explained in [25, §6.23, p. 117]. Fix a real $r \in (0,1)$ and let $\ell^r_-$ (resp. $\ell^r_+$) denote the path that goes along the positive axis from the
origin \( t = 0 \) to infinity via the half circle starting from \( t = 1 - r \) to \( 1 + r \) below (resp. over) its center point \( t = 1 \). Define \( P^\pm(\alpha, \xi) \) as follows:

\[
P^\pm(\alpha, \xi) := -\frac{2}{\pi} \int_{\ell^\pm} \sum_{n=1}^{\infty} \frac{\sin 2n\xi \pi t}{n(e^{2n\pi \alpha} - 1)} \frac{dt}{1 - t^2},
\]

where \( \alpha > 0 \) and where \( \xi \) may be an arbitrary real number.

Observe that the integral on the right hand side of (10) is independent of the choice of \( r \in (0, 1) \), so that we leave out the parameter \( r \) from \( P^\pm \). Moreover, the principal value of the singular integral considered in (2) is merely the average of \( P^+ \) and \( P^- \), that is to say:

\[
P^-(\alpha, \xi) + P^+(\alpha, \xi) = -\frac{4}{\pi} \text{PV} \int_0^\infty \sum_{n=1}^{\infty} \frac{\sin 2n\xi \pi t}{n(e^{2n\pi \alpha} - 1)} \frac{dt}{1 - t^2}.
\]

(11)

By the residues Theorem, we find:

\[
P^-(\alpha, \xi) - P^+(\alpha, \xi) = 2i \sum_{n=1}^{\infty} \frac{\sin 2n\xi \pi}{n(e^{2n\pi \alpha} - 1)},
\]

(12)

from which we arrive at the following expression:

\[
M(\alpha, \xi) = P^-(\alpha, \xi) - \sum_{n=1}^{\infty} \frac{e^{2n\pi \xi i}}{n(e^{2n\pi \alpha} - 1)}.
\]

(13)

**Theorem 1.4.** Let \( M \) be as in Theorem 1.1 and let \( P^- \) be as in (10). For any \( \xi \in \mathbb{R} \) and \( \alpha > 0 \), the following relation holds:

\[
M(\alpha, \xi) = \log(e^{2\pi \xi i - 2\pi / \alpha}; e^{-2\pi / \alpha})_{\infty} + P^-(\alpha, \xi),
\]

(14)

where \( \log \) denotes the principal branch of the logarithmic function over its Riemann surface.

**Proof.** Relation (14) follows directly from (13). Indeed, for the last series of (13), one can expand each fraction \( (e^{2n\pi \alpha} - 1)^{-1} \) as power series in \( e^{-2n\pi \alpha} \) and then permute the summation order inside the obtained double series, due to absolute convergence. \( \square \)

Consequently, the term \( M \) appearing in Theorem 1.1 can be considered as being an *almost modular term* of \( \log(x; q)_{\infty} \); the correction term \( P^- \) given by (14) will be called *disruptive factor or perturbation term.*

### 1.4 Perturbation term \( P \)

In view of the classical relation

\[
\cot \frac{t}{2} = \frac{2}{t} - \sum_{n=1}^{\infty} \frac{4t}{4\pi^2 n^2 - t^2},
\]

(15)
from (10) one can obtain the following expression:

\[
P^{-}(\alpha, \xi) = \int_{\ell^{-\ast}} \frac{\sin \xi t}{e^{t/\alpha} - 1} \left( \cot \frac{t}{2} - \frac{2}{t} \right) \frac{dt}{t}.
\]  

(16)

In the last integral (16), \( r \in (0, 1) \) and

\[
\ell^{-\ast} = (0, 1 - r) \cup \left( \cup_{n \geq 1} \left( C_{n,r}^{-} \cup (n + r, n + 1 - r) \right) \right)
\]

where for any positive integer \( n \), \( C_{n,r}^{-} \) denotes the half circle passing from \( n - r \) to \( n + r \) by the right hand side.

One may replace the integration path \( \ell^{-\ast} \) by any half line from origin to infinity which does not meet the real axis. In view of what follows in matter of complex extension considered in §1.6, let us first introduce the following modified complex version of \( P^{-} \): for any \( d \in (-\pi, 0) \), let

\[
P^{d}(\tau, \nu) = \int_{0}^{\infty} \frac{\sin \frac{\pi}{2} t}{e^{it/\tau} - 1} \left( \cot \frac{t}{2} - \frac{2}{t} \right) \frac{dt}{t},
\]  

(17)

the path of integration being the half line starting from origin to infinity with argument \( d \).

From then on, if we let \( \tilde{C}^{\ast} \) to denote the Riemann surface of the logarithm, we will define

\[
S(a, b) := \{ z \in \tilde{C}^{\ast} : \arg z \in (a, b) \}
\]  

(18)

for any pair of real numbers \( a < b \); notice that the Poincaré’s half-plane \( \mathcal{H} \) will be identified to \( S(0, \pi) \) while the broken plane \( \mathbb{C} \setminus (-\infty, 0] \) will be seen as the subset \( S(-\pi, \pi) \subset \tilde{C}^{\ast} \).

**Lemma 1.5.** The family of functions \( \{ P^{d} \}_{d \in (-\pi, 0)} \) given by (17) gives rise to an analytical function over the domain

\[
\Omega_{-} := S(-\pi, \pi) \times \left( \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)) \right) \subset \mathbb{C}^{2}.
\]  

(19)

Moreover, if we denote this function by \( P_{-}(\tau, \nu) \), then the following relation holds for all \( \alpha > 0 \) and \( \xi \in \mathbb{R} \):

\[
P_{-}(\alpha i, \xi \alpha i) = P^{-}(\alpha, \xi).
\]  

(20)

**Proof.** Let \( B \) be as in (3); from the relation

\[
\cot \frac{t}{2} - \frac{2}{t} = 2i B\left( \frac{it}{2\pi} \right),
\]  

(21)

it follows that the function \( P^{d} \) given by (17) is well defined and analytic at \( (\tau, \nu) = (\tau_{0}, \nu_{0}) \) whenever the corresponding integral converges absolutely, that is, when the following condition is satisfied:

\[
\left| \Re\left( \frac{e^{id}}{\tau_{0} \nu_{0} i} \right) \right| < \left| \Re\left( \frac{e^{id}}{\tau_{0} i} \right) \right|.
\]
Therefore, \( P_d \) is analytic over the domain \( \Omega_d \) if we set
\[
\Omega_d = \cup_{\sigma \in (0, \pi)} (0, \infty e^{i(d+\sigma)}) \times \{ \nu \in \mathbb{C} : |\Im(\nu e^{-i\sigma})| < \sin \sigma \}.
\] (22)

Thus we get the analyticity domain \( \Omega_+ \) and also relation (20) by using the standard argument of analytic continuation.

Let us give some precision about the above-employed continuation procedure, which is really a radial continuation. In fact, for any pair of directions of arguments \( d_1, d_2 \in (-\pi, 0) \), say \( d_1 < d_2 \), the common domain \( \Omega_{d_1} \cap \Omega_{d_2} \) contains a (product) disk \( D(\tau_0; r) \times D(0; r) \) for certain \( \tau_0 \in S(d_2, d_1 + \pi) \) and some radius \( r > 0 \), and all points in both \( \Omega_{d_1} \) and \( \Omega_{d_2} \) can be almost radially joined to this disk.

On the other hand, if we take the arguments \( d \in (0, \pi) \) instead of \( d \in (-\pi, 0) \) in (17), we can get an analytical function, say \( P_+ \), defined over
\[
\Omega_+ := S(0, 2\pi) \times \left( \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)) \right)
\]
and such that, for all \( \alpha > 0 \) and \( \xi \in \mathbb{R} \):
\[
P_+(\alpha i, \xi i) = P^+(\alpha, \xi).
\] (23)

Therefore, the Stokes relation (12) can be extended in the following manner.

**Theorem 1.6.** For any \( \tau \in \mathcal{H} \), the relation
\[
P_-(\tau, \nu) - P_+(\tau, \nu) = 2i \sum_{n=1}^{\infty} \frac{\sin \left( \frac{2\nu \pi}{\tau} \right)}{n(\sin \frac{2\nu \pi}{\tau} - 1)}
\] (24)
holds provided that \( |\Im(\nu/\tau)| < -\Im(1/\tau) \).

**Proof.** In view of (20) and (23), one may observe that the expected relation (24) really reduces to (12) when \( \tau = \alpha i, \nu = \xi i, \alpha > 0 \) and \( \xi \in \mathbb{R} \). Thus one can get (24) by an analytical continuation argument. Another way to arrive at the result is to directly use the residues theorem.

Using (21), one can write (17) as follows:
\[
P^d(\tau, \nu) = 2i \int_0^{\infty e^{i\theta}} \left( B\left( \frac{t}{\tau} i \right) - \frac{\tau i}{2\pi t} \right) B(it) \sin \left( \frac{2\pi \nu t}{\tau} \right) \frac{dt}{t},
\]
where \( B \) denotes the odd function given by (3). We guess that this expression contains some *modular* information about the perturbation term!
1.5 Remainder term relating Stirling’s formula

Let us consider the integral term involving the function $B$ in formula (7) of Theorem 1.3, which is, up to the sign, the remainder term $I_\Gamma$ appearing in the Stirling’s formula; see [10]. So, we introduce the following family of associated functions: for any $d \neq \frac{\pi}{2} \mod \pi$, define

$$g^d(z) = -\int_0^{\infty} B(t) e^{-2\pi z t} \frac{dt}{t}.$$  \hspace{1cm} (25)

It is obvious that $g^d$ is analytic over the half plane $S(-\frac{\pi}{2} - d, \frac{\pi}{2} - d)$, where $S(a, b)$ is in the sense of [18]. By usual analytic continuation, each of the families of functions $\{g^d\}_{d \in (-\frac{\pi}{2}, \frac{\pi}{2})}$ and $\{g^d\}_{d \in (\frac{\pi}{2}, 3\frac{\pi}{2})}$ will give rise to a function that we denote by $g^+$ and $g^-$ respectively; that is, $g^+$ is defined and analytical over the domain $S(-\pi, \pi)$ while $g^-$, over $S(-2\pi, 0)$. Since $B(-t) = -B(t)$, it follows that

$$g^+(z) = -g^-(e^{-\pi i} z)$$  \hspace{1cm} (26)

for any $z \in S(-\pi, \pi)$. Moreover, if $z \in S(-\pi, 0)$, one can choose a small $\epsilon > 0$ such that $g^\pm(z) = g^d(z)$, $d = \pi/2 \mp \epsilon$; by applying the residues theorem to the following contour integral

$$\left( \int_0^{\infty} e^{i(\pi/2 + \epsilon)} - \int_0^{\infty} e^{i(\pi/2 - \epsilon)} \right) B(t) e^{-2\pi z t} \frac{dt}{t},$$

we find:

$$g^+(z) - g^-(z) = -2\pi i \sum_{n \geq 1} \frac{e^{-2\pi z (ni)}}{2\pi ni} = \log(1 - e^{-2\pi i z}).$$  \hspace{1cm} (27)

**Lemma 1.7.** The following relations hold: for any $z \in S(-\pi, 0)$,

$$g^+(z) + g^+(e^{\pi i} z) = \log(1 - e^{-2\pi i z});$$

for any $z \in H = S(0, \pi)$,

$$g^+(z) + g^+(e^{-\pi i} z) = \log(1 - e^{2\pi i z}).$$

**Proof.** The result follows immediately from (26) and (27). \hfill \square

Lemma 1.7 is essentially the Euler’s reflection formula on $\Gamma$-function, as it is easy to see that, from [9], $I_\Gamma(z) = -g^+(z)$. If we set $G(\tau, \nu) = g^+(\frac{\nu}{\tau})$, that is to say:

$$G(\tau, \nu) = -\log \Gamma\left(\frac{\nu}{\tau} + 1\right) + \left(\frac{\nu}{\tau} + \frac{1}{2}\right) \log \frac{\nu}{\tau} - \frac{\nu}{\tau} + \log \sqrt{2\pi},$$  \hspace{1cm} (28)

then $G(\tau, \nu)$ is well defined and analytic over the domain $U^+$ given below:

$$U^+ := \{(\tau, \nu) \in \mathbb{C}^* \times \mathbb{C}^* : \nu/\tau \notin (-\infty, 0)\}.$$  \hspace{1cm} (29)
Proposition 1.8. Let $G$ be as in (28). Then, for any $(\tau, \nu) \in U^+$,
\[ G(\tau, \nu) + G(\tau, -\nu) = \log(1 - e^{\mp 2\pi i \nu / \tau}) \tag{30} \]
according to $\frac{\nu}{\tau} \in S(-\pi, 0)$ or $S(0, \pi)$, respectively.

Proof. The statement comes from Lemma 1.7. \qed

1.6 Modular type expansion of $(x; q)_\infty$

We shall discuss how to understand Theorem 1.3 in the complex plane, for both complex $q$ and complex $x$. As before, let $\hat{C}^*$ be the Riemann surface of the logarithm function. Let $M$ be the automorphism of the 2-dimensional complex manifold $\hat{C}^* \times \hat{C}^*$ given as follows:

\[ M: (q, x) \mapsto M(q, x) = (\iota(q), \iota q(x)), \]

where
\[ \iota(q) = q^* := e^{4\pi^2 / \log q}, \quad \iota q(x) = x^* := e^{2\pi i \log x / \log q}. \tag{31} \]

In the following, we will use the notations $q^*$ and $x^*$ instead of $\iota(q)$ and $\iota q(x)$ each time when any confusion does not occur.

If we let $\hat{D}^* = \exp (iH) \subset \hat{C}^*$, then $M$ induces an automorphism over the sub-manifold $\hat{D}^* \times \hat{C}^*$. From then now, we always write $q = e^{2\pi i \tau}$, $x = e^{2\pi i \nu}$ and suppose $\tau \in H$, so that $0 < |q| < 1$. Sometimes we shall use the pairs of modular variables $(\tau^*, \nu^*)$ as follows:

\[ i(\tau) = \tau^* := -1/\tau, \quad i_+(\nu) = \nu^* := \nu / \tau, \tag{32} \]

so that we can continue to write $q^* = e^{2\pi i \tau^*}$ and $x^* = e^{2\pi i \nu^*}$.

Theorem 1.9. Let $q = e^{2\pi i \tau}$, $x = e^{2\pi i \nu}$ and let $q^*$, $x^*$ as in (31). The following relation holds for any $\tau \in H$ and $\nu \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ such that $\nu / \tau \notin (-\infty, 0)$:

\[ (x; q)_\infty = q^{-1/24} \sqrt{1 - x} \left( x^* q^*; q^* \right)_\infty \times \exp \left( \frac{\text{Li}_2(x)}{\log q} + G(\tau, \nu) + P(\tau, \nu) \right), \tag{33} \]

where $\sqrt{1 - x}$ stands for the principal branch of $e^{\frac{1}{2} \log(1 - x)}$, $\text{Li}_2$ denotes the dilogarithm recalled in (6), $G$ is given by (28) and where $P$ denotes the function $P_-$ defined in Lemma 1.5.

Proof. By Theorem 1.4 and relation (20), we arrive at the expression

\[ M \left( -\frac{\log q}{2\pi} \frac{\log x}{\log q} \right) = \log(x^* q^*; q^*)_\infty + P(\tau, \nu); \]

making then suitable variable change in (7) allows one to arrive at (33), by taking into account the standard analytic continuation argument. \qed
If we denote by $G^*$ the anti-symmetrization of $G$ given by

$$G^*(\tau, \nu) = \frac{1}{2} (G(\tau, \nu) - G(\tau, -\nu)),$$

then, according to relation (30), we may rewrite (33) as follows:

$$\left( x; q \right)_\infty = q^{-1/24} \left( \frac{1-x}{1-x^*} \right) (x^*; q^*_\infty)$$

$$\times \exp \left( \frac{\text{Li}_2(x)}{\log q} + G^*(\tau, \nu) + P(\tau, \nu) \right)$$

(34)

if $\nu \in \tau \mathcal{H}$, and

$$\left( x; q \right)_\infty = q^{-1/24} \left( \frac{1-x(1-1/x^*)}{1-x^*} \right) (x^*; q^*_\infty)$$

$$\times \exp \left( \frac{\text{Li}_2(x)}{\log q} + G^*(\tau, \nu) + P(\tau, \nu) \right)$$

(35)

if $\nu \in -\tau \mathcal{H}$, that is, if $\frac{\nu}{\tau} \in S(-\pi, 0)$.

In the above, $G^*$ and $P$ are odd functions on the variable $\nu$:

$$G^*(\tau, -\nu) = -G^*(\tau, \nu), \quad P(\tau, -\nu) = -P(\tau, \nu);$$

(36)

$\text{Li}_2$ satisfies the so-called Landen’s transformation [4, Theorem 2.6.1, p. 103]:

$$\text{Li}_2(1-x) + \text{Li}_2(1-x^{-1}) = -\frac{1}{2} \left( \log x \right)^2.$$  

(37)

Finally, if we write $\vec{\omega} = (\omega_1, \omega_2) = (1, \tau)$ and denote by $\Gamma_2(z, \vec{\omega})$ the Barnes’ double Gamma function associated to the double period $\vec{\omega}$ [6], then Thoerme 1.9 and Proposition 5 of [21] imply that

$$\frac{\Gamma_2(1+\tau-\nu, \vec{\omega})}{\Gamma_2(\nu, \vec{\omega})} = \sqrt{\tau} \sqrt{1-x} \exp \left( \frac{\pi i}{12\tau} + \frac{\pi i}{2} \left( \frac{\nu^2}{\tau} - (1+\frac{1}{\tau})\nu \right) \right.
$$

$$+ \frac{\text{Li}_2(x)}{\log q} + G(\tau, \nu) + P(\tau, \nu) \right)
$$

$$= \sqrt{2} \sin \pi \nu \exp \left( \frac{\pi i}{12\tau} + \frac{\nu(\nu - 1)\pi i}{2\tau} \right)
$$

$$+ \frac{\text{Li}_2(e^{2\pi i \nu})}{2\pi i \tau} + G(\tau, \nu) + P(\tau, \nu) \right).$$

(38)

2 Dedekind $\eta$-function, Jacobi $\theta$-function and Lambert series

In the following, we will first see in what manner Theorem 1.9 essentially contains the modular equations known for $\eta$ and $\theta$-functions; see Theorems 2.11.
and In §2.4 we will consider two families of series, called $L_1$ and $L_2$, that can be obtained as logarithmic derivatives of the infinite product $(x; q)_\infty$; some modular type relations will be given in Theorem 2.3. In §2.5 classical Lambert series will be viewed as particular cases of the previous series $L_1$ and $L_2$.

2.1 Dedekind $\eta$-function

Let us mention a first application of Theorem 1.9 as follows.

**Theorem 2.1.** Let $q = e^{2\pi i \tau}$, $\tau \in \mathcal{H}$ and let $q^* = e^{-2\pi i / \tau}$. Then

$$ (q; q)_\infty = q^{-1/24} \sqrt{\frac{\tau}{\pi}} (q^*)^{1/24} (q^*; q^*_\infty). $$

**Proof.** If we set

$$ G_0(\tau, \nu) = G(\tau, \nu) - \log \sqrt{\frac{2\pi \nu}{\tau}} $$

$$ = \log \Gamma(\nu^* + 1) + \nu^* \log \nu^* - \nu^* $$

we can write relation (39) of Theorem 1.9 as follows:

$$ (xq; q)_\infty = q^{-1/24} \sqrt{\frac{2\pi \nu}{(1 - e^{2\pi i \nu})\tau}} (x^* q^*; q^*_\infty) $$

$$ \times \exp \left( \frac{\text{Li}_2(x)}{\log q} + G_0(\tau, \nu) + P(\tau, \nu) \right), $$

where $x = e^{2\pi \nu}$. Suppose $\nu \to 0$, so that $x \to 1$, $\nu^* \to 0$ and $x^* \to 1$; from (28), it follows:

$$ \lim_{\nu \to 0} G_0(\tau, \nu) = 0; $$

therefore, one easily gets relation (39), remembering that $\text{Li}_2(1) = \frac{\pi^2}{6}$ and that $P(\tau, 0) = 0$ as is said in (36).

The function $(q; q)_\infty$ plays a very important role in number theory and it is really linked with the well-known Dedekind $\eta$-function:

$$ \eta(\tau) = e^{\pi i / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) = q^{1/24} (q; q)_\infty, $$

where $\tau \in \mathcal{H}$. For instance, see [13, Lectures VI, VIII] and [5, Chapters 10, 11]. The modular relation (39), written as

$$ \eta\left(\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{\pi}} \eta(\tau), $$

is traditionally obtained as consequence of Poisson’s summation formula (cf. [11, p. 597-599]) or that of Mellin transform of some Dirichlet series (cf [5, p. 538-540]); see also [22], for a simple proof.
2.2 Modular relation on Jacobi theta function

In order to get the modular equation for Jacobi theta function, we first mention the following relation for any \( x \in \mathbb{C} \setminus (-\infty,0] \):

\[
\text{Li}_2(-x) + \text{Li}_2\left(-\frac{1}{x}\right) = -\frac{\pi^2}{6} - \frac{1}{2} (\log x)^2 ,
\]

which can be deduced directly from the definition \([4]\) of \( \text{Li}_2 \), for

\[
\frac{d}{dx} \left( \text{Li}_2(-x) + \text{Li}_2\left(-\frac{1}{x}\right) \right) = \frac{\log(1 + x)}{-x} - \frac{\log(1 + \frac{1}{x})}{-x} = \frac{\log x}{-x} .
\]

One can also check (41) by making use of a suitable variable change and considering both the Landen’s transformation \([37]\) and formula \([5]\ (2.6.6), p. 104]\):

\[
\text{Li}_2(x) + \text{Li}_2(1-x) = \frac{\pi^2}{6} - \frac{1}{2} (\log x)(\log(1-x)) ;
\]

see \([26]\) for more information.

For any fixed \( q = e^{2\pi i \tau}, \tau \in \mathcal{H} \), the modular variable transformation \( x \mapsto \tau_q(x) = x^* \) introduced in \([20]\) defines an automorphism of the Riemann surface \( \tilde{\mathbb{C}}^* \) of the logarithm and satisfies the following relations (\( q^* = \tau(q) = e^{4\pi^2i/\log q} \)):

\[
\tau_q(xy) = \tau_q(x) \tau_q(y) ; \quad \tau_q(e^{2\pi i k}) = (q^*)^{-k} , \quad \tau_q(q^k) = e^{2\pi i k} \]

for all \( k \in \mathbb{R} \). In particular, one finds:

\[
\tau_q(\sqrt{q} x) = e^{\pi i} \tau_q(x) , \quad \tau_q(x e^{\pi i}) = \tau_q(x)/\sqrt{q} .
\]

As usual, for any \( m \) given complex numbers \( a_1, \ldots, a_m \), let

\[
(a_1, \ldots, a_m ; q)_\infty = \prod_{k=1}^{m} (a_k; q)_\infty .
\]

**Theorem 2.2.** Let \( q = e^{2\pi i \tau} \) and \( x = e^{2\pi i \nu} \) and let

\[
\theta(q,x) = (q, -\sqrt{q} x, -\sqrt{q}/x ; q)_\infty .
\]

Then, the following relation holds for any \( \tau \in \mathcal{H} \) and any \( \nu \) of the Riemann surface of the logarithm:

\[
\theta(q,x) = q^{1/8} \sqrt{\frac{1}{\tau x}} \exp\left(-\frac{(\log \frac{1}{\sqrt{q} x})^2}{2\log q}\right) \theta(q^*, x^*) .
\]

**Proof.** First, suppose \( \nu \in \tau \mathcal{H} \) and write \( (x; q)_\infty \) and \( (1/x; q)_\infty \) by means of \([31]\) and \([35]\), respectively. By taking into account relation \([36]\) about the parity of \( G^* \) and \( P \), we find:

\[
(x ; \frac{1}{x}; q)_\infty = q^{-1/12} \frac{1-x}{1-1/x^*} \sqrt{-\frac{1}{x}} (x^* ; \frac{1}{x^*}; q^*)_\infty \exp\left(\frac{1}{\log q} \left(\text{Li}_2(x) + \text{Li}_2\left(\frac{1}{x}\right)\right)\right) ,
\]

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where we used the relation $1/x^* = (1/x)^*$, deduced from (12). Thus, it follows that

$$(xq, x; q)_\infty = q^{-1/12} \sqrt{-\frac{1}{x}} (x^*, q^*; q)_\infty \exp \left( \frac{1}{\log q} \left( \text{Li}_2(x) + \text{Li}_2 \left( \frac{1}{x} \right) \right) \right),$$

Change $x$ by $e^{-\pi i} x/\sqrt{q}$ and make use of (43) and (41); we get:

$$(-\sqrt{q} x, -\sqrt{q} x; q)_\infty = q^{-1/12} \sqrt{-\frac{\sqrt{q}}{x}} (\sqrt{q}^* x, -\sqrt{q}^* x; q^*)_\infty$$

$$\times \exp \left( \frac{1}{\log q} \left( -\frac{\pi^2}{6} - \frac{1}{2} (\log \frac{x}{\sqrt{q}})^2 \right) \right)$$

for $x \neq (\infty, 0]$. By the modular equation (39) for $\eta$-function, we arrive at the expected formula (45).

Finally we end the proof of the Theorem by the standard analytic continuation argument. □

Known as the modular formula on Jacobi’s theta function, relation (45) can be written as follows:

$$\theta(q, x) = \sqrt{1 \tau} \exp \left( -\frac{(\log x \sqrt{q})^2}{2 \log q} \right) \theta(q^*, x^*),$$

which has a very long history, and is attached to Gauss, Jacobi, Dedekind, Hermite, etc.... It is generally obtained by applying Poisson’s summation formula to the Laurent series expansion:

$$(q, -\sqrt{q} x, -\sqrt{q} x; q)_\infty = \sum_{n \in \mathbb{Z}} q^{x^n} x^n,$$

which is the so-called Jacobi’s triple product formula; for instance, see [5, §10.4, p. 496-501].

### 2.3 Another proof of Theta modular equation

As what is pointed out in [29, p. 214-215], formula (45) can be interpreted in term of $q$-difference equations. We shall elaborate on this idea and give a simple proof for (45).

For any fix $q$ such that $0 < |q| < 1$, let

$$f_1(x) = \theta(q, x), \quad f_2(x) = \sqrt{1 \tau} \exp \left( -\frac{(\log x \sqrt{q})^2}{2 \log q} \right)$$

and

$$f(x) = \frac{f_1(x)}{f_2(x)} = g(x^*),$$
where $x^*$ is given by (31). As $f_1$ and $f_2$ are solutions of the same first order linear equation
\[ y(qx) = \frac{1}{\sqrt{q} x} y(x), \]
f is a $q$-constant, that means, $f(qx) = f(x)$; equivalently, $g$ is uniform on the variable $x^*$, for $qx$ is translated into $x^* e^{2\pi i}$ by (42). On the other hand, it is easy to check the following relation:
\[ f(x e^{-2\pi i}) = e^{-2\pi i \frac{\log q}{\sqrt{q}}} f(x) = \frac{1}{\sqrt{q}^*} x^* f(x^*), \]
so that, using (42), we find:
\[ g(q^* x^*) = \frac{1}{\sqrt{q}^*} x^* g(x^*). \]

Summarizing, $g$ is a uniform solution of $y(q^* x^*) = y(x^*)/(\sqrt{q^*} x^*)$ and vanishes over the $q^*$-spiral $-\sqrt{q}^* q^* Z$ of the $x^*$-Riemann surface of the logarithm; it follows that there exists a constant $C$ such that $g(x^*) = C \theta(q^*, x^*)$ for all $x^* \in \mathbb{C}^*$. Write
\[ C = \frac{\theta(q, x)}{\theta(q^*, x^*)} \sqrt{q} \exp \left( \frac{(\log \frac{x}{\sqrt{q}})^2}{2 \log q} \right) \]
and let $x \to e^{\pi i}/\sqrt{q}$, so that $x^* \to e^{\pi i}/\sqrt{q}$. Since
\[ \frac{\theta(q, x)}{1 + \sqrt{q} x} \to (q^* q)^3, \quad \frac{\theta(q^*, x^*)}{1 + \sqrt{q^*} x^*} \to (q^* q^*)^3, \]
and
\[ \lim_{x \to e^{\pi i}/\sqrt{q}} \frac{1}{1 + \sqrt{q} x} = \frac{\sqrt{q}}{q^*} \frac{\sqrt{q}}{x} \frac{dx}{dx} \bigg|_{x = e^{\pi i}/\sqrt{q}} = \frac{\pi}{2}, \]
where $q = e^{2\pi i \tau}$, we get the following expression, deduced from $\eta$-modular equation (39):
\[ C = q^{1/4} e^{-\frac{\pi^2}{12} \frac{q^3}{(q^* q^*)^3} \frac{\tau}{q}} = q^{1/8} \sqrt{\frac{\tau}{\pi}}, \]
and we end the proof of (45).

One key point of the previous proof is to use the dual variables $q^*$ and $x^*$; the underlying idea is really linked with the concept of local monodromy group of linear $q$-difference equations \cite{19} §2.2.3, Théorème 2.2.3.5. In fact, as there exists two generators for the fundamental group of the elliptic curve $\mathbb{C}^*/q^* Z$, one needs to consider the “monodromy operators” in two directions or “two periods”, $x \mapsto x e^{2\pi i}$ and $x \mapsto x q$, which exactly correspond to $x^* \mapsto x^* q^*$ and $x^* \mapsto x^* e^{-2\pi i}$, in view of (42).
2.4 Generalized Lambert series $L_1$ and $L_2$

As before, let $q = e^{2\pi i \tau}$, $x = e^{2\pi i \nu}$ and suppose $\tau \in \mathcal{H}$. Consider the following series, which can be considered as generalized Lambert series:

$$L_1(\tau, \nu) = \sum_{n=0}^{\infty} \frac{x q^n}{1 - x q^n}, \quad L_2(\tau, \nu) = \sum_{n=0}^{\infty} \frac{(n+1) x q^n}{1 - x q^n},$$

that are both absolutely convergent for any $x \in \mathbb{C} \setminus q^{-N}$, due to the fact $|q| < 1$.

By expanding each term $(1 - xq^n)^{-1}$ into geometric series, one easily finds:

$$L_1(\tau, \nu) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{1 - q^{n+1}}, \quad L_2(\tau, \nu) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(1 - q^{n+1})^2},$$

where convergence requires $x$ to be inside the unit circle $|x| < 1$ of $x$-plane.

Observe that

$$L_1(\tau, \nu + \tau) - L_1(\tau, \nu) = -\frac{x}{1 - x}$$

and

$$L_2(\tau, \nu + \tau) - L_2(\tau, \nu) = -L_1(\tau, \nu).$$

In this way, one may guess how to define more series such as $L_3$, $L_4$, etc...

A direct computation yields the following formulas:

$$L_1(\tau, \nu) = -x \frac{\partial}{\partial x} \log(x; q)_{\infty},$$

$$L_2(\tau, \nu) = -q \frac{\partial}{\partial q} \log(x; q)_{\infty} + L_1(\tau, \nu);$$

$$x \frac{\partial x^*}{\partial x} = \frac{x^*}{\tau}, \quad x \frac{\partial \nu}{\partial x} = \frac{1}{2\pi i},$$

and

$$q \frac{\partial x^*}{\partial q} = -\frac{\nu}{\tau^2} x^*, \quad q \frac{\partial q^*}{\partial q} = \frac{1}{\tau^2} q^*, \quad q \frac{\partial \tau}{\partial q} = \frac{1}{2\pi i}.$$  

Here and in the following, $q$ and $x$ are considered as independent variables as well as the pair $(\tau, \nu)$ or their modular versions $(q^*, x^*)$ and $(\tau^*, \nu^*)$.

**Theorem 2.3.** Let $q = e^{2\pi i \tau}$, $x = e^{2\pi i \nu}$ and let $q^*$, $x^*$, $\tau^*$ and $\nu^*$ be as in (31) and (32). If $\tau \in \mathcal{H}$ and $\nu/\tau \notin (-\infty, 0]$, then the following relations hold:

$$L_1(\tau, \nu + \tau) = \frac{\log(1 - e^{2\pi i \nu})}{2\pi i} + \frac{e^{2\pi i \nu}}{2(1 - e^{2\pi i \nu})} + L_1(\tau^*, \nu^* + \tau^*) \frac{1}{\tau} + \frac{1}{2\pi \tau i} \left( \frac{\Gamma'(e^{2\pi i \nu} + 1)}{\Gamma(e^{2\pi i \nu} + 1)} - \log \frac{\nu}{\tau} - \frac{\tau}{2\nu} - \tau \frac{\partial}{\partial \nu} P(\tau, \nu) \right)$$

(54)
and

\[
L_2(\tau, \nu + \tau) = \frac{1}{24} - \frac{\text{Li}_2(e^{2\pi i \nu})}{4\pi^2} \frac{1}{\tau^2} - L_1(\tau^*, \nu^* + \tau^*) \frac{\nu}{\tau^2} \\
+ L_2(\tau^*, \nu^* + \tau^*) \frac{1}{\tau^2} - \frac{\nu}{2\pi i \tau^2} \left( \frac{\Gamma'(\frac{\nu^*}{\tau^*} + 1)}{\Gamma(\frac{\nu^*}{\tau^*} + 1)} \right) \\
- \log \frac{\nu}{\tau} - \frac{\tau}{2\nu} + \frac{\nu}{\tau} \frac{\partial}{\partial \tau} P(\tau, \nu),
\]

(55)

Proof. By taking the logarithmic derivative with respect to the variable \(x\) in (33) and in view of (52), we find:

\[
x \frac{\partial}{\partial x} \log(x; q)_\infty = -\frac{x}{2(1-x)} + \frac{x^*}{1-x^*} \left( 1 + \frac{1}{\tau} \frac{x^*}{\partial x^*} \log(x^*; q^*)_\infty \right) \\
- \log_q(1-x) + \frac{1}{2\pi i} \left( \frac{\partial}{\partial \nu} G(\tau, \nu) + \frac{\partial}{\partial \nu} P(\tau, \nu) \right),
\]

so that, by (50), we arrive at the following expression:

\[
L_1(\tau, \nu) = \log_q(1-x) + \frac{x}{2(1-x)} + \left( L_1(\tau^*, \nu^*) - \frac{x^*}{1-x^*} \right) \frac{1}{\tau} \\
- \frac{1}{2\pi i} \left( \frac{\partial}{\partial \nu} G(\tau, \nu) + \frac{\partial}{\partial \nu} P(\tau, \nu) \right).
\]

From (28), it follows that

\[
\tau \frac{\partial}{\partial \nu} G(\tau, \nu) = -\frac{\Gamma'(\frac{\nu}{\tau} + 1)}{\Gamma(\frac{\nu}{\tau} + 1)} + \log \frac{\nu}{\tau} + \frac{\tau}{2\nu},
\]

(56)

that leads to the wanted relation (54), using (48).

On the other hand, using (53), a direct computation shows that (33) implies the following expression:

\[
q \frac{\partial}{\partial q} \log(x; q)_\infty = -\frac{1}{24} + \frac{\nu}{\tau^2} \frac{\partial}{\partial x^*} \log(1-x^*) - \frac{\nu}{\tau^2} \frac{x^*}{\partial x^*} \log(x^*; q^*)_\infty \\
+ \frac{1}{\tau} \frac{\partial}{\partial q^*} \log(x^*; q^*)_\infty - \frac{\text{Li}_2(x)}{(\log q)^2} \\
+ \frac{1}{2\pi i} \left( \frac{\partial}{\partial \tau} G(\tau, \nu) + \frac{\partial}{\partial \tau} P(\tau, \nu) \right),
\]

or, by taking into account (51):

\[
q \frac{\partial}{\partial q} \log(x; q)_\infty = -\frac{1}{24} + \frac{\text{Li}_2(x)}{4\pi^2} \frac{1}{\tau^2} - \frac{x^*}{1-x^*} \frac{\nu}{\tau^2} + L_1(\tau^*, \nu^*) \frac{\nu + 1}{\tau^2} \\
- L_2(\tau^*, \nu^*) \frac{1}{\tau^2} + \frac{1}{2\pi i} \left( \frac{\partial}{\partial \tau} G(\tau, \nu) + \frac{\partial}{\partial \tau} P(\tau, \nu) \right).
\]

(57)

Since

\[
\tau \frac{\partial}{\partial \tau} G(\tau, \nu) = -\nu \frac{\partial}{\partial \nu} G(\tau, \nu),
\]

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by putting together (51), (54), (56) with the last relation (57), we get the following relation:

\[
L_2(\tau, \nu) = \frac{1}{24} \frac{L_2(x)}{x^2 \tau^2} + \frac{x^*}{1-x^*} \frac{\nu - \tau}{\tau^2} - L_1(\tau^*, \nu^*) \frac{\nu + 1}{\tau^2} \\
+ L_1(\tau, \nu) + L_2(\tau^*, \nu^*) \frac{1}{\tau^2} - \frac{\nu}{2 \pi i \tau^2} \left( \frac{\Gamma(\frac{\nu}{\tau} + 1)}{\Gamma(\frac{\nu}{\tau} + 1)} \right) \\
- \log \frac{\nu}{\tau} - \frac{\tau}{2 \nu^*} + \frac{\tau^2}{\nu} \frac{\partial}{\partial \nu} P(\tau, \nu),
\]

so that we arrive at the wanted formula (55) by applying (49).

2.5 Classical Lambert series

If we let \( \nu = \tau \), we reduce series \( L_1 \) and \( L_2 \) to the following classical Lambert series:

\[
L_1(\tau, \tau) = \sum_{n=0}^{\infty} \frac{q^{n+1}}{1 - q^{n+1}}
\]

and

\[
L_2(\tau, \tau) = \sum_{n=0}^{\infty} \frac{(n+1)q^{n+1}}{1 - q^{n+1}} = \sum_{n=0}^{\infty} \frac{q^{n+1}}{(1 - q^{n+1})^2}.
\]

By considering the limit case \( \nu = 0 \) in (54) and (55) of Theorem 2.3, we arrive at the following statement.

**Theorem 2.4.** For all \( \tau \in \mathcal{H} \), the following relations hold:

\[
L_1(\tau, \tau) = \frac{\log(-2\pi i \tau)}{2\pi i \tau} + \frac{1}{4} - \frac{\gamma}{2\pi i} \frac{\partial}{\partial \nu} P(\tau, 0) + L_1(\tau^*, \tau^*) \frac{1}{\tau},
\]

(58)

where \( \gamma \) denotes Euler’s constant, and

\[
L_2(\tau, \tau) = \frac{1}{24} + \frac{1}{4\pi i \tau} - \frac{1}{24 \tau^2} + L_2(\tau^*, \tau^*) \frac{1}{\tau^2}.
\]

(59)

**Proof.** If we set

\[
A(\tau, \nu) = \frac{1}{2\pi i \tau} \log \frac{1 - e^{2\pi i \nu}}{\nu / \tau} - \frac{1}{2} \left( \frac{e^{2\pi i \nu}}{1 - e^{2\pi i \nu}} + \frac{1}{2 \pi i \nu} \right),
\]

we can write (54) as follows:

\[
L_1(\tau, \nu + \tau) = A(\tau, \nu) + L_1(\tau^*, \nu^* + \tau^*) \frac{1}{\tau} \\
+ \frac{1}{2\pi i} \left( \frac{\Gamma(\frac{\nu}{\tau} + 1)}{\Gamma(\frac{\nu}{\tau} + 1)} - \frac{\partial}{\partial \nu} P(\tau, \nu) \right).
\]
so that, remembering $\gamma = -\Gamma'(1)$, we get (58), as it is easy to see that

$$\lim_{\nu \to 0} A(\tau, \nu) = \log\left(\frac{-2\pi i\tau}{2\pi i\tau} + 1\right).$$

In the same time, putting $\nu = 0$ in (55) allows one to obtain relation (59), for $P(\tau, 0) = 0$ for all $\tau \in \mathcal{H}$ implies $\frac{\partial}{\partial \tau} P(\tau, 0) = 0$ identically.

Formula (58) has been known since Schl"omilch; see Stieltjes [23, (84), p. 54]. Relation (59) is really a modular relation and is traditionally obtained by taking derivative with respect to the variable $\tau$ in modular formula (39); see [1, Exercises 6 and 7, p. 71].

**2.6 Some remarks when $q$ tends toward one**

For the sake of simplicity, we will limit ourself to the real case and we suppose $q \to 1^-$ by real values in $(0, 1)$, so that one can let $\tau = i\alpha$, $\alpha \to 0^+$. As $\tau^* = -1/\tau = i/\alpha$, one may observe that $\Im(\tau^*) \to +\infty$ and therefore $q^* \to 0^+$ rapidly or, exactly saying, exponentially with respect to the variable $1/\alpha$. The relation

$$|x^*| = e^{2\pi \nu / \alpha} = e^{2\pi \Re(\nu) / \alpha}$$

shows that, as $\alpha \to 0^+$, the modular variable $x^*$ belongs to the unit circle if and only if the initial variable $x$ takes a real value; otherwise, $x^*$ goes rapidly to $\infty$ or 0 according to the sign of $\Re \nu$.

Since $e^{2\pi i(\nu + 1)} = e^{2\pi i \nu}$ in $x$-plane, one can always suppose that $\Re(\nu) \in [0, 1)$; in this way, it follows that

$$(x^* q^*; q^*)_{\infty} = 1 + O\left(e^{-(1-\Re(\nu))/\alpha}\right) \sim 1.$$ 

**Lemma 2.5.** Let $\tau = i\alpha$, $\alpha > 0$ and let $G, P$ as in Theorem (1.9). Then the following limits hold: for any fix $\nu \in (0, 1)$,

$$\lim_{\alpha \to 0^+} P(\tau, \nu) = \lim_{\alpha \to 0^+} G(\tau, \nu) = 0.$$

**Proof.** The null limit of $G$ is just a consequence of the Stirling’s asymptotic formula on $\log \Gamma$. The reader may complete the proof by direct estimates. \qed

Thus, from Theorem (1.9) we find:

$$\log(x; q)_{\infty} \sim \frac{\log(1 - x)}{2} - \frac{\text{Li}_2(x)}{2\pi \alpha}$$

when $q = e^{-2\pi \alpha} \to 1^-$. 

Our final remark concerns the limit behavior for generalized series $L_1$ and $L_2$. From Theorem (2.3) it is easy to see that, naturally,

$$2\pi \alpha L_1(\alpha i, \nu) \sim \log(1 - x), \quad 4\pi^2 \alpha^2 L_2(\alpha i, \nu) \sim \text{Li}_2(x)$$
if $\alpha \to 0^+$. In a forthcoming paper [14], we shall give a compactly uniform Gevrey asymptotic expansion for $(x; q)_\infty$ when $q \to 1$ inside the unit disc, $x$ being a complex parameter; see [15 §1.4.1, p. 84-86] for Gevrey asymptotic expansion with parameters.

3 Proof of Theorem 1.1

In all this section, we let

$$q = e^{-a} = e^{-2\pi \alpha}, \quad x = e^{-(1+\xi)a} = q^{1+\xi}$$

and suppose

$$a = 2\pi \alpha > 0, \quad 0 < q < 1, \quad \xi > -1, \quad 0 < x < 1,$$

For any positive integer $N$, define

$$V_N(a, \xi) := \sum_{n=1}^{N} \log(1 - e^{-(n+\xi)a}). \quad (60)$$

It is easy to see that

$$\log (x; q)_\infty = \lim_{N \to \infty} V_N(a, \xi).$$

We shall prove Theorem 1.1 in several steps, and our approach is well inspired by Stieltjes’ work “Étude de la fonction $P(a) = \sum_{1}^{\infty} \frac{1}{e^\pi - 1}$ that one can find in his Thesis [23, p. 57-62]. The starting point is to use the fact that

$$\frac{1}{e^{\sqrt{2\pi}u} - 1} - \frac{1}{\sqrt{2\pi} u}$$

is a self-reciprocal function with respect to Fourier sine transform [24 (7.2.2), p. 179], so that one may write each finite sum $V_N$ by four or five appropriate sine or cosine integrals depending of $N$ and make then estimation over these integrals.

3.1 Some preparatory formulas

We are going to use the following formulas:

$$\int_{0}^{\infty} \frac{\sin \lambda u}{e^{2\pi u} - 1} du = \frac{1}{4} + \frac{1}{2} \left( \frac{1}{e^\lambda - 1} - \frac{1}{\lambda} \right), \quad (61)$$

and

$$\int_{0}^{\infty} \frac{1 - \cos \lambda u}{e^{2\pi u} - 1} \frac{du}{u} = \frac{\lambda}{4} + \frac{1}{2} \log \frac{1 - e^{-\lambda}}{\lambda}. \quad (62)$$

where $\lambda$ is assumed to be a real or complex number such that $|3\lambda| < 2\pi$; notice that the second formula can be deduced from the first one by integrating on $\lambda$. 20
For instance, see [23, (82) & (83), p. 57], [24, (7.2.2), p. 179] or [25, Example 2, p. 122].

From (62), it follows that

\[
V_N(a, \xi) = \sum_{n=1}^{N} \log (n + \xi) - \frac{N}{2} \xi a - \frac{N(N + 1)}{4} a + N \log a + R_N(a, \xi),
\]

where

\[
R_N(a, \xi) = \int_{0}^{\infty} \frac{h_N(au, \xi)}{e^{2\pi u} - 1} \frac{du}{u}
\]

and

\[
h_N(t, \xi) = 2N - 2 \sum_{n=1}^{N} \cos(n + \xi)t.
\]

By using the elementary relations

\[
2 \sum_{n=1}^{N} \cos nt = \cos Nt + \sin Nt \cot \frac{t}{2} - 1
\]

and

\[
2 \sum_{n=1}^{N} \sin nt = \sin Nt + (1 - \cos Nt) \cot \frac{t}{2},
\]

we obtain:

\[
h_N(t, \xi) = 2N + \cos \xi t - \cos(N + \xi)t + (\sin \xi t - \sin(N + \xi)t) \cot \frac{t}{2}.
\]

Let us define the following integrals:

\[
R^{(1)}_N(a, \xi) = \int_{0}^{\infty} \frac{\cos \xi au - \cos(N + \xi)au}{e^{2\pi u} - 1} \frac{du}{u}
\]

and

\[
R^{(2)}_N(a, \xi) = \int_{0}^{\infty} \frac{2N + (\sin \xi au - \sin(N + \xi)au) \cot \frac{\pi u}{2}}{e^{2\pi u} - 1} \frac{du}{u},
\]

so that

\[
R_N(a, \xi) = R^{(1)}_N(a, \xi) + R^{(2)}_N(a, \xi).
\]

We will look for the limits of \(R^{(1)}_N\) and \(R^{(2)}_N\) while \(N\) becomes indefinitely large. To simplify, we will write \(a_N \sim b_N\) if the quantity \((a_N - b_N)\) tends to zero as \(N \to \infty\). From (62), we first observe the following result.

**Lemma 3.1.** The following relation holds:

\[
R^{(1)}_N(a, \xi) \sim_N \frac{N}{4} a - \frac{1}{2} \log N - \frac{1}{2} \log \frac{1 - e^{-\xi a}}{\xi}.
\]
Proof. Applying (62) to the following integrals
\[
\int_0^{\infty} \frac{1 - \cos \xi au}{e^{2\pi u} - 1} \frac{du}{u}, \quad \int_0^{\infty} \frac{1 - \cos(N + \xi)au}{e^{2\pi u} - 1} \frac{du}{u}
\]
implies directly Lemma 3.1

The following well-known result, due to Riemann, will be often taken into account in the course of the proof.

**Lemma 3.2.** Let \( f \) be a continuous and integrable function on a finite or infinite closed interval \([\alpha, \beta] \subset \mathbb{R}\). Then the following relations hold:

\[
\int_{\alpha}^{\beta} f(t) \sin Nt \, dt \sim N \quad 0, \quad \int_{\alpha}^{\beta} f(t) \cos Nt \, dt \sim N \quad 0.
\]

### 3.2 First part of \( R^{(2)}_N \)

The integral (65) of \( R^{(2)}_N \) seems more complicated than \( R^{(1)}_N \), because of the simple poles at \( u = \frac{2}{a \pi}, \frac{4}{a \pi}, \frac{6}{a \pi}, \text{ etc} \), \( \cdots \) that the function \( \cot \frac{au}{2} \) admits on \((0, +\infty)\). In such a situation, one very classical technique may consist in replacing the function by its decomposition in simple parts as given in (15). By considering \( \frac{2}{au} \) instead of \( \cot \frac{au}{2} \) in (65), we are led to the following integral:

\[
R^{(21)}_N(a, \xi) = \frac{2}{a} \int_0^{\infty} \frac{Nau + \sin \xi au - \sin(N + \xi)au}{e^{2\pi u} - 1} \frac{du}{u^2}, \quad (67)
\]

if we set

\[
R^{(22)}_N(a, \xi) = 4a \int_0^{\infty} \sum_{n=1}^{\infty} \frac{\sin(N + \xi)au - \sin \xi au}{4\pi^2 n^2 - a^2 u^2} \frac{du}{e^{2\pi u} - 1}, \quad (68)
\]

then, in view of (15) we obtain the following equality:

\[
R^{(2)}_N(a, \xi) = R^{(21)}_N(a, \xi) + R^{(22)}_N(a, \xi).
\]

**Lemma 3.3.** The following relation holds for \( a > 0 \) and \( \xi > 0 \):

\[
R^{(21)}_N(a, \xi) \sim_N \frac{N(N + 2\xi)}{4} a - (N + \xi) \log(N + \xi) + N(1 - \log a) - \frac{\pi^2}{6a} + \xi \log \xi - \frac{1}{a} \int_0^{\xi a} \log(1 - e^{-t}) \, dt. \quad (69)
\]

**Proof.** For any pair \((N, \xi) \in \mathbb{N} \times \mathbb{R}\), let \( f_N(a, \xi) = \frac{a}{2} R^{(21)}_N(a, \xi) \); it is easy to see that \( a \mapsto f_N(a, \xi) \) represents an odd analytic function at the origin 0 of the real axis, for merely

\[
f_N(a, \xi) = \int_0^{\infty} \frac{Nau + \sin \xi au - \sin(N + \xi)au}{e^{2\pi u} - 1} \frac{du}{u^2}.
\]
Let $f'_N(a, x)$ denote the derivative of $f_N(a, \xi)$ with respect to the variable $a$. It follows that

$$f'_N(a, \xi) = \int_0^\infty \frac{(N + \xi)(1 - \cos(N + \xi)au) - \xi(1 - \cos \xi au)}{e^{2\pi u} - 1} \, du,$$

so that applying (62) gives rise to the following relation:

$$f'_N(a, \xi) = \frac{N(N + 2\xi)}{4} a + \frac{N + \xi}{2} \log 1 - e^{-(N + \xi)a} - \frac{N}{2} \log a - \frac{\xi}{2} \log 1 - e^{-\xi a}.$$

To come back to $f_N(a, x)$, we integrate $f'_N(t, \xi)$ over the interval $(0, a)$ and remark that $f_N(0, \xi) = 0$; it follows that

$$f_N(a, \xi) = \frac{N(N + 2\xi)}{8} a^2 - \frac{N + \xi}{2} a \log(N + \xi) - \frac{N}{2} (\log a - 1)a + \frac{\xi}{2} (\log \xi) a + \frac{1}{2} I(a, N + \xi) - \frac{1}{2} I(a, \xi), \quad (70)$$

where

$$I(a, \delta) = \int_0^{3a} \log(1 - e^{-t}) \, dt.$$ 

Now we suppose $a > 0$ and let $N \to +\infty$. Noticing that

$$I(a, N + \xi) \sim_N \int_0^\infty \log(1 - e^{-t}) \, dt = -\text{Li}_2(1) = -\frac{\pi^2}{6}, \quad (71)$$

we get immediately (70) from (70). \hfill \Box

The term $-\frac{\pi^2}{6a}$ included in expression (71) plays a most important role for understanding the asymptotic behavior of $\log(x; q)_\infty$ as $q \to 1^-$, that is, $a \to 0^+$. The crucial point is formula (71), that remains valid for all complex numbers $a$ such that $\Re a > 0$.

### 3.3 Intermediate part in $R_N^{(2)}$

Now consider $R_N^{(22)}(a, \xi)$ of (68); then

$$R_N^{(22)}(a, \xi) = \frac{2}{\pi} \left( I_N(a, \xi) - J_N(a, \xi) \right), \quad (72)$$

if we set

$$I_N(a, \xi) = \int_0^\infty \sum_{n=1}^\infty \frac{\sin 2nN \pi t \cos 2n\xi \pi t}{n(e^{4n\pi^2 t/a} - 1)} \frac{dt}{1 - t^2}, \quad (73)$$

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and
\[
J_N(a, \xi) = \int_0^\infty \sum_{n=1}^\infty \frac{\sin 2n\xi \pi t (1 - \cos 2nN \pi t)}{n(e^{4n\pi^2 t/a} - 1)} \frac{dt}{1 - t^2}, \tag{74}
\]

Here, each series under the integral converges absolutely to an integrable function over \((0, \infty)\) except near zero and \(t = 1\). Lemma 3.4 given below will tell us how to regularize the situation at origin; notice also that these integrals behave more convergent at \(t = 1\) than at 0, due to big factors \((e^{4n\pi^2 t/a} - 1)\).

**Lemma 3.4.** Let \(\delta \in (0, 1), \lambda > 0\) and let \(\{h_{n,N}\}_{n,N \in \mathbb{N}}\) be a uniformly bounded family of continuous functions on \([0, \delta]\). For any positive integer \(M\), let \(A_M(N)\) denote the integral given by
\[
A_M(N) = \int_0^{\delta} \sum_{n=1}^M \frac{h_{n,N}(t)}{n} \left( \frac{1}{e^{n\lambda t} - 1} - \frac{1}{n\lambda t} \right) \frac{dt}{1 - t^2}.
\]

Then, as \(M \to \infty\), the sequence \(\{A_M(N)\}\) converges uniformly for \(N \in \mathbb{N}\).

**Proof.** We suppose \(\lambda = 1\), the general case being analogous; thus, one can write \(A_M(N)\) as follows:
\[
A_M(N) = \sum_{n=1}^M \int_0^{\delta} h_{n,N}(t/n) \left( \frac{1}{e^{t/n} - 1} - \frac{1}{t/n} \right) \frac{dt}{t/n^2 - t^2}.
\]

Observe that the function \((e^{t} - 1)^{-1} - t^{-1}\) increases rapidly from \(-1/2\) toward zero when \(t\) tends to infinity by positive values; indeed, \((e^{t} - 1)^{-1} - t^{-1} = O(t^{-1})\) for \(t \to +\infty\). Therefore, if we make use of the relation \(\int_0^{\delta} = \int_0^{\sqrt{\pi} \delta} + \int_{\sqrt{\pi} \delta}^{\delta} \) and let \(n \to +\infty\), we find:
\[
\left| \int_0^{\delta} h_{n,N}(t/n) \left( \frac{1}{e^{t/n} - 1} - \frac{1}{t/n} \right) \frac{dt}{t/n^2 - t^2} \right| \leq C n^{-3/2},
\]

where \(C\) denotes a suitable positive constant independent of \(N\) and \(n\); this ends the proof of Lemma 3.4. \(\square\)

We come back to the integral \(I_N\) given in (73).

**Lemma 3.5.** The following relation holds:
\[
I_N(a, \xi) \sim_N \frac{\pi}{48} a - \frac{\pi}{2} \sum_{n=1}^\infty \frac{\cos 2n\pi \xi}{n(e^{4n\pi^2 t/a} - 1)}.
\]

**Proof.** We fix a small \(\delta > 0\), cut off the interval \((0, \infty)\) into four parts \((0, \delta), (\delta, 1 - \delta), (1 - \delta, 1 + \delta)\) and \((1 + \delta, \infty)\), and the corresponding integrals will be
denoted by $I_N^{0\delta}$, $I_N^{\delta+}$, $I_N^{1+\delta}$ and $I_N^{\delta\infty}$, respectively. According to Lemma 3.2 we find:

$$I_N^{\delta+}(a, \xi) \sim_N 0, \quad I_N^{\delta\infty}(a, \xi) \sim_N 0,$$

for

$$\sum_{n=M}^{\infty} \left( \int_{1-\delta}^{1+\delta} \sin 2nN \pi t \cos 2n\xi \pi t \, dt \right) \to 0$$

when $M \to \infty$. In the same way, we may observe that

$$I_N^{0\delta}(a, \xi) \sim_N \frac{a}{4\pi^2} \int_0^{\delta} \sum_{n=1}^{\infty} \frac{\sin 2nN \pi t \cos 2n\xi \pi t}{n^2(1-t^2)} \, dt$$

$$\sim_N \frac{a}{4\pi^2} \int_0^{\delta} \sum_{n=1}^{\infty} \frac{\sin 2nN \pi t \, dt}{n^2}$$

$$= \frac{a}{4\pi^2} \sum_{n=1}^{\infty} \int_0^{\delta} \frac{\sin 2nN \pi t \, dt}{n^2},$$

where the first approximation relation is essentially obtained from Lemma 3.4, combining together with Lemma 3.2. Since

$$\int_0^{\infty} \frac{\sin t \, dt}{t} = \frac{\pi}{2},$$

from (77) we deduce the following relation:

$$I_N^{0\delta}(a, \xi) \sim_N \frac{a}{8\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{48} a.$$

A similar analysis allows one to write the following relations for the remaining integral $I_N^{1+\delta}$:

$$I_N^{1+\delta}(a, \xi) \sim_N \frac{1}{2} \int_{1-\delta}^{1+\delta} \sum_{n=1}^{\infty} \frac{\sin 2nN \pi t \cos 2n\xi \pi t}{n(e^{4n\pi^2 t/a} - 1)} \, dt$$

$$\sim_N \frac{1}{2} \int_{1-\delta}^{1+\delta} \sum_{n=1}^{\infty} \frac{\cos 2n\xi \pi}{n(e^{4n\pi^2 t/a} - 1)} \frac{2nN \pi t}{1-t} \, dt$$

$$\sim_N \frac{1}{2} \sum_{n=1}^{\infty} \frac{\cos 2n\xi \pi}{n(e^{4n\pi^2 t/a} - 1)} \int_{-\infty}^{+\infty} \frac{\sin t}{t} \, dt$$

$$= -\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\cos 2n\xi \pi}{n(e^{4n\pi^2 t/a} - 1)},$$

where the last equality comes from (78).

Accordingly, we obtain the wanted expression (75) by putting together the estimates (76), (79) and (80) and thus the proof is complete.
3.4 Singular integral as limit part of $R_N^{(2)}$

In order to give estimates for $J_N(a, \xi)$ of (74), we shall make use of the Cauchy principal value of a singular integral. The situation we have to consider is the following [25, §6.23, p. 117]: $f$ be a continuous function over $(0, 1) \cup (1, +\infty)$ such that, for any $\epsilon > 0$, $f$ is integrable over both intervals $(0, 1 - \epsilon)$ and $(1 + \epsilon, +\infty)$; one defines

$$\mathcal{P} \int_0^\infty f(t) dt = \lim_{\epsilon \to 0^+} \left( \int_0^{1-\epsilon} f(t) dt + \int_{1+\epsilon}^\infty f(t) dt \right)$$

whenever the last limit exists.

**Lemma 3.6.** The following relation holds:

$$J_N(a, \xi) \sim_N \mathcal{P} \int_0^\infty \sum_{n=1}^{\infty} \frac{\sin 2n\xi \pi t}{n(e^{4n\pi^2t/a} - 1)} \frac{dt}{1 - t^2}.$$  \hspace{1cm} (81)

**Proof.** For any given number $\epsilon \in (0, 1)$, let

$$J_N^{(1+\epsilon)}(a, \xi) = \int_{1-\epsilon}^{1+\epsilon} \sum_{n=1}^{\infty} \frac{\sin 2n\xi \pi t(1 - \cos 2nN\pi t)}{n(e^{4n\pi^2t/a} - 1)} \frac{dt}{1 - t^2};$$

Thanks to suitable variable change, we can get the following expression:

$$J_N^{(1+\epsilon)}(a, \xi) = \int_0^{\epsilon} \sum_{n=1}^{\infty} \left( \frac{h_n(a, \xi, t)}{2 - t} - \frac{h_n(a, \xi, -t)}{2 + t} \right) \frac{1 - \cos 2nN\pi t}{nt} dt,$$

where

$$h_n(a, \xi, t) = \frac{\sin 2n\xi \pi (1 - t)}{e^{4n\pi^2(1-t)/a} - 1}.$$

From Lemma 3.2 one deduces:

$$J_N^{(1+\epsilon)}(a, \xi) \sim_N \int_0^{\epsilon} \sum_{n=1}^{\infty} \left( \frac{h_n(a, \xi, t)}{2 - t} - \frac{h_n(a, \xi, -t)}{2 + t} \right) dt.$$  \hspace{1cm} (82)

Again applying Lemma 3.2 implies that

$$J_N(a, \xi) - J_N^{(1+\epsilon)}(a, \xi) \sim_N \left( \int_0^{1-\epsilon} + \int_{1+\epsilon}^\infty \right) \sum_{n=1}^{\infty} \frac{\sin 2n\xi \pi t}{n(e^{4n\pi^2t/a} - 1)} \frac{dt}{1 - t^2},$$

which, using (82), permits to conclude, as it is clear that

$$\lim_{\epsilon \to 0^+} \int_0^{\epsilon} \sum_{n=1}^{\infty} \left( \frac{h_n(a, \xi, t)}{2 - t} - \frac{h_n(a, \xi, -t)}{2 + t} \right) dt = 0.$$
3.5 End of the proof of Theorem 1.1

Proof. Consider the functions $V_N(a, \xi)$ given in (60) and recall that $\log(x; q)_\infty$ is the limit of $V_N(a, \xi)$ when $N$ goes to infinity; so we need to know the limit behavior of the right hand side of (63) for infinitely large $N$.

Letting

$$G_N(a, \xi) = \sum_{n=1}^{N} \log(n + \xi) - \frac{N}{2} \xi a - \frac{N(N + 1)}{4} a + N \log a,$$

it follows that

$$V_N(a, \xi) = G_N(a, \xi) + R_N^{(1)}(a, \xi) + R_N^{(21)}(a, \xi) + \frac{2}{\pi} (I_N(a, \xi) - J_N(a, \xi)),$$

where $R_N^{(1)}$, $R_N^{(21)}$, $I_N$ and $J_N$ are considered in Lemmas 3.1, 3.3, 3.5 and 3.6 respectively. From Stirling’s asymptotic formula [5, Theorem 1.4.1, page 18], one easily gets:

$$\sum_{n=1}^{N} \log(n + \xi) = \log \Gamma(N + \xi + 1) - \log \Gamma(\xi + 1) 
\sim_N \log \sqrt{2\pi} + (N + \xi + \frac{1}{2}) \log(N + \xi + 1) 
- (N + \xi + 1) - \log \Gamma(\xi + 1).$$

Thanks to (69) of Lemma 3.3 one finds:

$$G_N(a, \xi) + R_N^{(21)}(a, \xi) \sim_N - \frac{N}{4} a + \frac{1}{2} \log N - \log \Gamma(\xi + 1) - \frac{\pi^2}{6a} 
+ \log \sqrt{2\pi} - \xi + \xi \log \xi - \frac{1}{a} \int_{0}^{\xi a} \log(1 - e^{-t}) dt.$$

Thus, using (64) of Lemma 3.1 one can deduce the following expression:

$$G_N(a, \xi) + R_N^{(1)}(a, \xi) + R_N^{(21)}(a, \xi) 
\sim_N - \frac{\pi^2}{6a} + \log \sqrt{2\pi} - \xi - \log \Gamma(\xi + 1) 
- (\xi + \frac{1}{2}) \log \frac{1 - e^{-\xi a}}{\xi} + \frac{1}{a} \int_{0}^{\xi a} \frac{t}{e^t - 1} dt,$$

which implies the starting formula (1) of our paper with the help of Lemmas 3.5 and 3.6 replacing all $a$ by $2\pi \alpha$. \qed

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