Geometric measure of entanglement of multi-qubit graph states and its detection on a quantum computer

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received 19 June 2021; accepted in final form 9 December 2021
published online 9 March 2022

Abstract – Multi-qubit graph states generated by the action of controlled phase shift operators on a separable quantum state of a system, in which all the qubits are in arbitrary identical states, are examined. The geometric measure of entanglement of a qubit with other qubits is found for the graph states represented by arbitrary graphs. The entanglement depends on the degree of the vertex representing the qubit, the absolute values of the parameter of the phase shift gate and the parameter of state the gate is acting on. Also the geometric measure of entanglement of the graph states is quantified on the quantum computer ibm-qathens. The results obtained on the quantum device are in good agreement with analytical ones.

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Introduction. – Studies of entanglement of quantum states and its quantifying on a quantum computer have received much attention (see, for instance, [1–14] and references therein). Entanglement corresponds to non-classical correlations between the subsystems and presupposes that the state of a system cannot be factorized [1]. This physical phenomenon plays a critical role in quantum information in particular, in quantum cryptography, quantum teleportation (see, for example, [1,15–27]).

The geometric measure of entanglement proposed by Shimony [2] is defined as a minimal squared Fubini-Study distance $d^2_{FS} = 1 − | ⟨ ψ | ψ_s ⟩ |^2$ between an entangled state $| ψ ⟩$ and a set of separable pure states $| ψ_s ⟩$. It reads

$$E(|ψ⟩) = \min_{|ψ_s⟩} (1 − | ⟨ ψ | ψ_s ⟩ |^2).$$

(1)

The authors of paper [8] showed that the geometric measure of entanglement of a spin one-half (or qubit) with a quantum system in a pure state $|ψ⟩$ is entirely determined by the mean spin in this state. Namely the following relation is satisfied:

$$E(|ψ⟩) = \frac{1}{2} (1 − | ⟨ σ | ⟩),$$

(2)

$$| ⟨ σ | ⟩ = \sqrt{(σ^z)^2 + (σ^y)^2 + (σ^x)^2},$$

here $σ^x$, $σ^y$, $σ^z$ are the Pauli matrices, $⟨ σ ⟩ = ⟨ ψ | σ | ψ ⟩$. Therefore, in order to quantify the geometric measure of entanglement the mean values of the Pauli matrices have to be calculated. Quantum protocol for detecting $⟨ σ^x ⟩$, $⟨ σ^y ⟩$, $⟨ σ^z ⟩$ on a quantum computer is presented in [9]. Mean value $⟨ σ^z ⟩$ can be represented as $⟨ σ^z ⟩ = ⟨ ψ | σ^z | ψ ⟩ = ⟨ ψ^y | σ^y | ψ^y ⟩ = ⟨ ψ^y | 0 ⟩^2 − ⟨ [ψ^y | 1 ⟩^2$, where $⟨ ψ^y ⟩ = \exp(\pm iπσ^y/4)|ψ⟩$ and $|⟨ ψ^y | 0 ⟩|^2$, $|⟨ ψ^y | 1 ⟩|^2$ are probabilities that define the result of measurement in the standard basis. Thus, to quantify $⟨ σ^z ⟩$ the $RY(−π/2)$ gate has to be applied to the state of a qubit before conducting the measurement in the standard basis (the state of the qubit has to be rotated around the y-axis by $π/2$. Similarly, to detect $⟨ σ^y ⟩$ one has to apply the $RX(π/2)$ gate and then measure the qubit in the standard basis $⟨ σ^y ⟩ = ⟨ ψ | σ^y | ψ ⟩ = ⟨ ψ^x | σ^x | ψ^x ⟩ = ⟨ [ψ^x | 0 ⟩^2 − ⟨ [ψ^x | 1 ⟩^2$, here $⟨ ψ^x ⟩ = \exp(−iπσ^x/4)|ψ⟩$. Lastly, for $σ^z$ we have $⟨ σ^z ⟩ = ⟨ ψ | σ^z | ψ ⟩ = ⟨ ⟨ [ψ | 0 ⟩^2 − ⟨ [ψ | 1 ⟩^2$. This way one can obtain the values $⟨ σ^x ⟩$, $⟨ σ^y ⟩$, $⟨ σ^z ⟩$ on the basis of the results of measurement of the qubit in the standard basis [9].

The authors of paper [28] proposed a method to derive experimentally accessible lower bounds for measures of genuine multipartite entanglement and coherence on the basis of measuring the fidelity between the target state and a pure state, which is chosen to have high overlap with the target state. The lower bounds were quantified for several real experimental states.

Graph states are quantum states that can be represented by graphs [13,14,29–36]. These states have
various applications in quantum information, for instance in quantum error correction [29–31], quantum cryptography [32,33] and practical quantum metrology in the presence of noise [34]. Much attention has been devoted to examining multi-qubit graph states generated by 2-qubit controlled-Z operators acting on a separable quantum state of the system, in which all qubits are in the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ (see, for example, [13,14,37–41] and references therein). The state represented by undirected graph $G(V,E)$ $(V,E)$ denote vertices and edges of the graph, respectively reads

$$|\psi_G\rangle = \prod_{(i,j)\in E} CZ_{ij}^{} |+\rangle^\otimes V. \tag{3}$$

Here $CZ_{ij}$ is the controlled-Z gate acting on the states of qubits $q[i], q[j]$.

The authors of paper [13] studied graph states (3) corresponding to rings on IBM’s quantum computer ibmQx5 and showed that the 16-qubit quantum computer can be fully entangled. In [14] the entanglement of graph states was examined on the basis of calculations on the quantum computer IBM Q Poughkeepsie. Entanglement of graph states of spin systems generated by the operator of evolution with Ising Hamiltonian was examined analytically and on the 5-qubit quantum computer IBM Q Valencia in [11]. It was shown that the entanglement of a spin with other spins in the graph state is related to the degree of the vertex representing the spin [11].

In the present paper we study graph states obtained as a result of action of controlled phase shift operators on a separable quantum state of the system, in which all the qubits are in arbitrary identical states

$$|\psi_G(\phi,\alpha,\theta)\rangle = \prod_{(i,j)\in E} CP_{ij}(\phi)|\psi(\alpha,\theta)\rangle^\otimes V, \tag{4}$$

where $CP_{ij}(\phi)$ is the controlled phase shift gate that acts on the qubits $q[i], q[j]$. This gate acting on $q[i]$ as the control qubit and $q[j]$ as the target qubit is defined as $CP_{ij}(\phi) = |0\rangle_i|0\rangle_j + |1\rangle_i|1\rangle_j P_j(\phi)$. Here $P_j(\phi)$ is the phase gate $P_j(\phi) = |0\rangle_j|0\rangle_j + e^{i\phi}|1\rangle_j|1\rangle_j$ acting on the state of the qubit $q[j]$.

State (5) is an arbitrary one-qubit state, $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$. In particular case $\phi = \pi$, $\alpha = 0$, $\theta = \pi/2$ state (4) coincides with (3). We find the expression for the entanglement of a qubit with other qubits in graph state (4) represented by an arbitrary graph. It is shown that the entanglement is determined by the absolute values of the parameters $\phi$ and $\theta$ as well as the degree of the vertex representing the qubit in the graph. The entanglement of graph states is also studied on IBM’s quantum computer ibmQathens.

The paper is organized as follows. In the next section an expression for the geometric measure of entanglement of a qubit with other qubits in graph state (4) is found.

In the third section we present results of quantum computations on ibmQathens for the entanglement of the graph states (4) represented by the chain, the claw and the complete graphs. Conclusions are made in the last section.

**Geometric measure of entanglement of multi-qubit graph states.** – According to the result (2) in order to find the geometric measure of entanglement of the qubit $q[l]$ with other qubits in graph state (4) one has to calculate $\langle \sigma_l \rangle = \langle \psi_G(\phi,\alpha,\theta)|\psi_G(\phi,\alpha,\theta)\rangle$. Note that accurate to the phase factor the quantum state $|\psi(\alpha,\theta)\rangle$ can be prepared by the action of the rotation operators $RZ(\alpha), RY(\theta)$ on $|0\rangle$. We have

$$|\psi(\alpha,\theta)\rangle = e^{i\frac{\alpha}{2}RZ(\alpha)}RY(\theta)|0\rangle = e^{\frac{i\alpha}{2}}e^{-i\frac{\alpha}{2}\sigma^y}e^{-i\frac{\theta}{2}\sigma^z}|0\rangle. \tag{6}$$

The controlled phase shift gate can be represented as

$$CP_{ij}(\phi) = e^{i\phi}RZ_1(\phi)RZ_2(\phi)|0\rangle, \tag{7}$$

where $RZ_i$ is the unit operator.

Thus, for $\langle \sigma_l \rangle$ we obtain

$$\langle \sigma_l \rangle = \langle \psi_0 \rangle \prod_{q\in V} e^{i\frac{\alpha}{2}RZ_q(\phi)RZ_q(\phi)^\dagger} \prod_{(j,k)\in E} (CP_{jk}(\phi))^\dagger \times \sigma_l^q \prod_{(m,n)\in E} CP_{mn}(\phi) \prod_{p\in V} e^{-i\frac{\theta}{2}\sigma^z_p}e^{-i\frac{\alpha}{2}\sigma^y_p}|\psi_0\rangle, \tag{8}$$

where we use notation $\psi_0 = |0\rangle^\otimes V$. Taking into account that operators $\sigma^x_l, \sigma^z_l$ anticommute ($\{\sigma^x_l, \sigma^z_l\} = 0$), we can write

$$\langle \sigma_l \rangle = \langle \psi_0 \rangle \prod_{(j,k)\in E} (CP_{jk}(\phi))^\dagger \prod_{(m,n)\in E} CP_{mn}(\phi)$$

$$\times \prod_{(m,n)\in E} e^{-i\frac{\theta}{2}(1-\sigma^x_l)(1-\sigma^x_n)} \times \prod_{(m,n)\in E} e^{i\frac{\alpha}{2}(1-\sigma^y_m)(1-\sigma^y_n)}$$

$$= e^{i\frac{\alpha}{2}n_l}e^{-i\frac{\theta}{2}\sum_{l\in N_G(l)}(1-\sigma^x_l)(1-\sigma^x_l)}$$

$$\times e^{i\frac{\alpha}{2}n_l}e^{-i\frac{\theta}{2}\sum_{l\in N_G(l)}(1-\sigma^y_l)(1-\sigma^y_l)}$$

$$= e^z \sin \theta \Re z, \tag{10}$$

where $N_G(l)$ is a closed neighbourhood of the vertex $l$ (a set of vertices adjacent to the vertex $l$ and the vertex $l$).

We use notation $z$ for complex number

$$z = e^{-i(\phi + \frac{\pi}{2}n_l)} \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \right)^{n_z}.$$
Similarly, for $\langle \sigma_i^\dagger \rangle$ we find
\[
\langle \sigma_i^\dagger \rangle = \langle \psi_G | \sigma_i^\dagger | \psi_G \rangle = \langle \psi_0 | \prod_{p \in N_G(i)} e^{i/2 \sigma_p^\dagger} e^{-i/2 \sigma_p^\dagger} | \psi_0 \rangle = -\sin \theta \Im z,
\]
number $z$ is given by (11). Mean value $\langle \sigma_i^\dagger \rangle$ reads
\[
\langle \sigma_i^\dagger \rangle = \langle \psi_G | \sigma_i^\dagger | \psi_G \rangle = \langle \psi_0 | \prod_{p \in N_G(i)} e^{i/2 \sigma_p^\dagger} \sigma_i^\dagger e^{-i/2 \sigma_i^\dagger} | \psi_0 \rangle = \cos \theta.
\]

Finally, on the basis of (2) for the geometric measure of entanglement of the qubit $q[l]$ with other qubits in graph state (4) we obtain the following expression:
\[
E_l = \frac{1}{2} \left( 1 - \sqrt{\sin^2 \theta |z|^2 + \cos^2 \theta} \right) = \frac{1}{2} - \frac{1}{2} \sqrt{\sin^2 \theta \left( \cos^2 \left( \frac{\phi}{2} \right) + \sin^2 \left( \frac{\phi}{2} \cos \theta \right) \right) + \cos^2 \theta}.
\]

Note that the geometric measure of entanglement of the qubit $q[l]$ with other qubits in graph state (4) depends on the degree of the vertex $n_l$ representing $q[l]$ in the graph, the absolute values of the parameter of the controlled phase gate $\phi$ and the parameter of state (5) $\theta$. It does not depend on the value of $\alpha$.

Preparation of multi-qubit graph states and detection of their entanglement on a quantum computer. — We quantify the geometric measure of entanglement of graph states (4) on IBM’s 5-qubit quantum computer ibmq_athens [42]. The structure of the quantum computer is presented in fig. 1, where the arrows link qubits to which the CNOT gate can be directly applied.

We consider graph state (4) corresponding to the graph with the structure of ibmq_athens. It reads
\[
|\psi_G^{(1)}(\phi, \alpha, \theta)\rangle = CP_{01}(\phi)CP_{12}(\phi)CP_{23}(\phi)CP_{34}(\phi)|\psi(\alpha, \theta)\rangle \otimes^5.
\]

Degrees of vertices in the graph (see fig. 2(a)) corresponding to state (15) are the following: $\deg(V_0) = \deg(V_4) = 1$, $\deg(V_1) = \deg(V_2) = \deg(V_3) = 2$, where $V_i$ is the vertex representing the qubit $q[i]$. We also study graph states (4) associated with the claw and the complete graphs (see fig. 2(b), (c)) and determine the geometric measure of entanglement of qubits represented by vertices with degrees 3 and 4. These graph states are defined as follows:
\[
|\psi_G^{(2)}(\phi, \alpha, \theta)\rangle = CP_{10}(\phi)CP_{12}(\phi)CP_{13}(\phi)|\psi(\alpha, \theta)\rangle \otimes^4,
\]
\[
|\psi_G^{(3)}(\phi, \alpha, \theta)\rangle = \prod_{i<j} CP_{ij}(\phi)|\psi(\alpha, \theta)\rangle \otimes^5.
\]

State (16) corresponds to the claw graph (see fig. 2(b)), which is the most simple graph with maximal vertex degree $\deg(V_1) = 3$. State (17) can be represented by the complete graph (see fig. 2(c)). In this case $\deg(V_i) = 4$, $i = \{0, \ldots, 4\}$.

To quantify dependence of the geometric measure of entanglement of a graph state on the angle $\phi$ we fix parameter $\theta$ as $\theta = \pi/2$. Taking into account that expression (14) obtained in the previous section does not depend on the value of $\alpha$, for convenience we set $\alpha = 0$. In this case from (5) we obtain $|\psi(0, \pi/2)\rangle = |+\rangle$ and the graph state reads
\[
|\psi_G(\phi, 0, \pi/2)\rangle = \prod_{(i,j) \in E} CP_{ij}(\phi)|+\rangle \otimes^V.
\]

Note that for $\phi = \pi$ state (18) coincides with (3).

In case of fixing parameter $\phi$ as $\phi = \pi$ for $\alpha = 0$ graph state (4) transforms to
\[
|\psi_G(\pi, 0, \theta)\rangle = \prod_{(i,j) \in E} CZ_{ij}|\psi(0, \theta)\rangle \otimes^V.
\]

Quantum protocols for preparing graph states $|\psi_G(\phi, 0, \pi/2)\rangle$, $|\psi_G(\pi, 0, \theta)\rangle$ corresponding to the chain, the claw and the complete graphs (see (15), (16), (17), respectively) are presented in figs. 3, 4.

Two-qubit gates $CZ_{ij}$ and $CP_{ij}(\phi)$ can be represented as $CZ_{ij} = H_iCNOT_{ij}H_j$, $CP_{ij}(\phi) = RZ_i(\phi/2)CNOT_{ij}RZ_j(-\phi/2)CNOT_{ij}RZ_iRZ_j(\phi/2)$, where $H_i$ is the Hadamard gate acting on the qubit $q[j]$, gate $RZ_j(\pi/2)$ rotates the state of the qubit $q[j]$ around the $z$ axis by $\pi/2$. Note that in the quantum protocols, fig. 3(b), (c), fig. 4(b), (c), gates $CZ_{ij}$, $CP_{ij}(\phi)$ and therefore $CNOT_{ij}$ have to be applied to qubits $q[i], q[j]$ that are not connected by arrows in the scheme, fig. 1.

To apply these gates in the transpiled circuits the states of $q[i], q[j]$ are transferred by SWAP gates to the qubits to which $CNOT$ can be directly applied according to the connectivity map.
We detect entanglement of qubits corresponding to vertices with degrees 1, 2, 3, 4 in graph states $|\psi_G^1(\phi, 0, \pi/2)\rangle$ or $|\psi_G(\phi, 0, \pi/2)\rangle$ we prepare these states on the quantum device with the help of protocols presented in figs. 3, 4. Then to obtain $|\sigma_f^q\rangle$ we apply $RY(-\pi/2)$ gate to the qubit $q[l]$ before measuring it in the standard basis. Using the results of measurements, namely knowing the frequencies $p_0, p_1$ of outcomes $|0\rangle_i, |1\rangle_i$ we calculate $|\sigma_f^q\rangle = p_0 - p_1$. Similarly, to detect $|\sigma_f^p\rangle$ in a graph state we prepare this state and apply $RX(\pi/2)$ gate to the qubit $q[l]$ before measuring it in the standard basis. Using the results of measurements we calculate $|\sigma_f^p\rangle = p_0 - p_1$ (here $p_0, p_1$ are frequencies of outcomes $|0\rangle_i, |1\rangle_i$). For quantifying $|\sigma_f^q\rangle$ in a graph state we prepare this graph state, perform measurements of the state of the qubit $q[l]$ in the standard basis and calculate $|\sigma_f^q\rangle = p_0 - p_1$ (here $p_0, p_1$ are frequencies of outcomes $|0\rangle_i, |1\rangle_i$). Finally, on the basis of expression (2) we determine the geometric measure of entanglement of the qubit $q[l]$ in the graph state.

We quantify geometric measure of entanglement of qubit $q[0]$ corresponding to the vertex with degree 1 with other qubits in the states $|\psi_G(\phi, 0, \pi/2)\rangle$ for different values of $\phi$ (see fig. 6(a)). In addition, the entanglement of the qubit $q[1]$ corresponding to the vertex with degree 2 was quantified in the states $|\psi_G^1(\phi, 0, \pi/2)\rangle$ (see fig. 5(b), fig. 6(b)). In order to detect the geometric measure of entanglement of qubits represented by vertices with the same degrees in the graph state $|\psi_G^1(\phi, 0, \pi/2)\rangle$ (see table 1).

The geometric measure of entanglement of the qubit $q[1]$ corresponding to the vertex with degree 3 in the claw graph with other qubits in the states $|\psi_G^2(\phi, 0, \theta)\rangle, |\psi_G^2(\phi, 0, \pi/2)\rangle$ was also calculated (see fig. 5(c), fig. 6(c)). To quantify the geometric measure of entanglement in the case of $n_t = 4$ graph states $|\psi_G^3(\phi, 0, \pi/2)\rangle$ represented by the complete graph were prepared using protocols (see fig. 3(c), fig. 4(c)) and the geometric measure of entanglement of the qubit $q[0]$ with other qubits was quantified (see fig. 5(d), fig. 6(d)).

The results for the geometric measure of entanglement obtained on the quantum computer for qubits in the graph states $|\psi_G^1(\phi, 0, \pi/2)\rangle, |\psi_G^1(\phi, 0, \pi/2)\rangle$ (see fig. 5(a), (b), fig. 6(a), (b)) are in good agreement with theoretical ones. In the case of graph states $|\psi_G^2(\phi, 0, \pi/2)\rangle, |\psi_G^2(\phi, 0, \pi/2)\rangle, |\psi_G^3(\phi, 0, \pi/2)\rangle, |\psi_G^3(\phi, 0, \pi/2)\rangle$ the results for the entanglement are not in so good agreement with analytical results because to prepare these states two-qubit gates have to be applied to qubits that are not connected directly according to connectivity map of IBM’s quantum computer ibmq_athens (see fig. 1). Also, quantum protocols

Fig. 3: Quantum protocols for preparing graph states $|\psi_G^1(\phi, 0, \pi/2)\rangle$ (15) (a), $|\psi_G^2(\phi, 0, \pi/2)\rangle$ (16) (b), $|\psi_G^3(\phi, 0, \pi/2)\rangle$ (17) (c).

Fig. 4: Quantum protocols for preparing graph states $|\psi_G^2(\pi, 0, \theta)\rangle$ (15) (a), $|\psi_G^2(\pi, 0, \theta)\rangle$ (16) (b), $|\psi_G^2(\pi, 0, \theta)\rangle$ (17) (c).
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Fig. 5: Results of detecting geometric measure of entanglement on ibmqathens quantum computer (marked by crosses) and analytical results (line) for qubit $q[0]$ (a) and qubit $q[1]$ (b) with other qubits in state $|\psi_{G}^{(1)}(\pi, 0, \theta)\rangle$ (15) for different values of $\theta$, for qubit $q[1]$ with other qubits in state $|\psi_{G}^{(2)}(\pi, 0, \theta)\rangle$ (16) (c) and for qubit $q[0]$ with other qubits in state $|\psi_{G}^{(3)}(\pi, 0, \theta)\rangle$ (17) (d) for different values of $\theta$.

Fig. 6: Results of detecting geometric measure of entanglement on ibmqathens quantum computer (marked by crosses) and analytical results (line) for qubit $q[0]$ (a) and qubit $q[1]$ (b) with other qubits in state $|\psi_{G}^{(1)}(\phi, 0, \pi/2)\rangle$ (15) for different values of $\phi$, for qubit $q[1]$ with other qubits in state $|\psi_{G}^{(2)}(\phi, 0, \pi/2)\rangle$ (16) (c) and for qubit $q[0]$ with other qubits in state $|\psi_{G}^{(3)}(\phi, 0, \pi/2)\rangle$ (17) (d) for different values of $\phi$. 
Table 1: The calibration parameters of IBM’s quantum computer ibmq_athens on 9 June 2021 [42].

|                  | $Q_0$ | $Q_1$ | $Q_2$ |
|------------------|-------|-------|-------|
| Readout error ($10^{-2}$) | 1.07  | 1.30  | 1.70  |
| Gate error ($10^{-4}$)     | 2.98  | 3.16  | 5.26  |
| $Q_3$                | $Q_4$ |
| Readout error ($10^{-2}$) | 1.31  | 2.00  |
| Gate error ($10^{-4}$)     | 2.54  | 2.89  |
| CNOT error ($10^{-3}$)    | CX0,1 | CX1,0 | CX1,2 |
|                        | 12.04 | 12.04 | 11.13 |
|                        | CX2,1 | CX2,3 | CX3,2 |
|                        | 11.13 | 18.50 | 18.50 |
|                        | CX3,4 | CX4,3 | 6.80  |

for preparing graph states represented by the complete graph (see fig. 3(c), fig. 4(c)) contain more gates than those for preparing graph states corresponding to the chain and the claw (see fig. 3(a), (b), fig. 4(a), (b)). This leads to accumulation of errors.

Conclusion. — In the paper we have studied the geometric measure of entanglement of graph states generated by the action of controlled phase shift operators on a separable quantum state of the system, in which all the qubits are in arbitrary identical states (4). The expression for the geometric measure of entanglement of a qubit with other qubits in graph state (4) represented by an arbitrary graph has been found (14). We have concluded that the entanglement depends on the absolute values of the parameter of the phase shift operator $\phi$ and the parameter $\theta$ of states (5) as well as on the degree of the vertex representing the qubit in the graph.

The geometric measure of entanglement has also been calculated on the 5-qubit IBM’s quantum computer ibmq_athens. Graph states corresponding to the graph with the structure of quantum computer ibmq_athens, the claw and the complete graphs have been prepared and the geometric measure of entanglement of qubits represented by vertices with degrees 1, 2, 3, 4 has been found. Fixing parameter $\phi = \pi$ we have prepared graph states (15)–(17) and quantified their entanglement for different values of the parameter of the controlled phase gate $\phi$ in the case of graph states (15)–(17) (see fig. 6). The results obtained with quantum calculations on the quantum device ibmq_athens are in good agreement with theoretical ones (see figs. 5, 6).

The authors thank Prof. V. M. Tkachuk for great support and useful comments during the research studies. This work was supported by Project 2020.02/0196 (No. 0120U104801) from National Research Foundation of Ukraine.

Data availability statement: All data that support the findings of this study are included within the article (and any supplementary files).

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