Scheme independent consequence of the NSVZ relation for $\mathcal{N}=1$ SQED with $N_f$ flavors

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Abstract

The exact NSVZ $\beta$-function is obtained for $\mathcal{N}=1$ SQED with $N_f$ flavors in all orders of the perturbation theory, if the renormalization group functions are defined in terms of the bare coupling constant and the theory is regularized by higher derivatives. However, if the renormalization group functions are defined in terms of the renormalized coupling constant, the NSVZ relation between the $\beta$-function and the anomalous dimension of the matter superfields is valid only in a certain (NSVZ) scheme. We prove that for $\mathcal{N}=1$ SQED with $N_f$ flavors the NSVZ relation is valid for the terms proportional to $(N_f)^1$ in an arbitrary subtraction scheme, while the terms proportional to $(N_f)^k$ with $k \geq 2$ are scheme dependent. These results are verified by an explicit calculation of a three-loop $\beta$-function and a two-loop anomalous dimension made with the higher derivative regularization in the NSVZ and MOM subtraction schemes. In this approximation it is verified that in the MOM subtraction scheme the renormalization group functions obtained with the higher derivative regularization and with the dimensional reduction coincide.

Keywords: higher covariant derivative regularization, supersymmetry, $\beta$-function, subtraction scheme.

1 Introduction

The $\beta$-function of $\mathcal{N}=1$ supersymmetric gauge theories is related with the anomalous dimension of the matter superfields. This relation, derived in Refs. [1, 2, 3], is usually called "the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) $\beta$-function". For the $\mathcal{N}=1$ supersymmetric Yang–Mills theory without matter superfields the NSVZ $\beta$-function was obtained in Refs. [1, 4]. In the case of the $\mathcal{N}=1$ supersymmetric electrodynamics (SQED) with $N_f$ flavors, which is considered in this paper, the NSVZ $\beta$-function has the following form [5, 6]:

$$\beta(\alpha_0) = \frac{\alpha_0^2 N_f}{\pi} \left(1 - \gamma(\alpha_0)\right).$$

This relation, derived from general arguments, can be verified by explicit calculations. Usually for calculating quantum corrections SUSY theories are regularized by the dimensional reduction
supplemented by the DR-scheme. However, DRED is not mathematically consistent. As a consequence, supersymmetry can be broken by quantum corrections in higher loops. Explicit calculations made in the DR-scheme in the one- and two-loop approximations agree with the NSVZ $\beta$-function, because a two-loop $\beta$-function and a one-loop anomalous dimension are scheme independent in theories with a single coupling constant. In higher orders the exact NSVZ relation for the renormalization group (RG) functions defined in terms of the renormalized coupling constant can be obtained with the DR-scheme after an additional finite renormalization. This finite renormalization should be fixed in each order of the perturbation theory, starting from the three-loop approximation. However, at present, there are no general prescriptions, how one should construct this finite renormalization using the DR-scheme in all orders.

In the Abelian case the NSVZ $\beta$-function can be obtained in all orders using the Slavnov higher derivative (HD) regularization. This regularization is mathematically consistent and can be formulated in an explicitly supersymmetric way. The HD regularization allows to obtain the NSVZ $\beta$-function for the RG functions defined in terms of the bare coupling constant. The reason for this is that the integrals needed for obtaining such a $\beta$-function in SUSY theories are integrals of total derivatives and even double total derivatives. As a consequence, one of the loop integrals can be calculated analytically, and a $\beta$-function in an $L$-loop approximation can be related with an anomalous dimension of the matter superfields in the $(L-1)$-loop approximation. However, if the RG functions are defined in terms of the renormalized coupling constant, the NSVZ $\beta$-function is obtained only in a special subtraction scheme. This scheme was constructed by imposing the boundary conditions

$$Z_3(\alpha, x_0) = 1; \quad Z(\alpha, x_0) = 1,$$

where $x_0$ is a certain value of $x = \ln \Lambda/\mu$. Without loss of generality it is possible to choose $x_0 = 0$.

In this paper we study $\mathcal{N} = 1$ SQED with $N_f$ flavors. It is shown that the coefficients of the anomalous dimension of the matter superfields proportional to $(N_f)^0$ and the coefficients of the $\beta$-function proportional to $(N_f)^1$ are scheme independent in all orders. As a consequence, they satisfy the NSVZ relation in all orders independently of a choice of a subtraction scheme. In order to verify this result we explicitly calculate a three-loop $\beta$-function and a two-loop anomalous dimension using the HD regularization with different renormalization prescriptions. Also we present the results of a similar calculation which is made using the DR-scheme. Then it is explicitly demonstrated that the terms proportional to $(N_f)^1$ in the $\beta$-function and to $(N_f)^0$ in the anomalous dimension are scheme independent and satisfy the NSVZ relation.

The paper is organized as follows: In Sect. 2 we remind how the NSVZ $\beta$-function can be obtained for $\mathcal{N} = 1$ SQED with $N_f$ flavors using the HD regularization for the RG functions defined in terms of the bare coupling constant. The standard definition of the RG functions (in terms of the renormalized coupling constant) and their scheme dependence are discussed in Sect. 3. In Sect. 4 the results are verified by an explicit three-loop calculation. The two-loop anomalous dimension of the matter superfields and the three-loop $\beta$-function in different subtraction schemes are compared in Sect. 5.

## 2 The NSVZ $\beta$-function for $\mathcal{N} = 1$ SQED with $N_f$ flavors

In terms of $\mathcal{N} = 1$ superfields $\mathcal{N} = 1$ SQED with $N_f$ flavors in the massless limit is described by the action

$$S = \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a W_a + \sum_{i=1}^{N_f} \frac{1}{4} \int d^4x d^4\theta \left( \phi_i^* e^{2V} \phi_i + \tilde{\phi}_i^* e^{-2V} \tilde{\phi}_i \right),$$

(3)
where $\epsilon_0$ is a bare coupling constant. The exact NSVZ $\beta$-function can be naturally obtained for this theory, if the HD method is used for a regularization.

In order to regularize this theory by higher derivatives, it is necessary to insert into the first term of Eq. [3] a regularizing function $R(\partial^2/\Lambda^2)$ such that $R(0) = 1$ and $R(\infty) = \infty$ [20, 21]:

$$
\frac{1}{4\epsilon_0^2} \text{Re} \int d^4x d^2\theta W^a W_a \to \frac{1}{4\epsilon_0^2} \text{Re} \int d^4x d^2\theta W^a R(\partial^2/\Lambda^2) W_a.
$$

(4)

It is convenient to choose $R = 1 + \frac{\partial^2}{\Lambda^{2n}}$, where $\Lambda$ is a dimensionful parameter. Also we should insert into the generating functional the Pauli–Villars determinants, which cancel the remaining one-loop divergences [33, 34]. Then the generating functional can be written in the following form:

$$
Z[J, \tilde{\phi}, \tilde{\phi}] = \int D\phi D\tilde{\phi} \prod_{l=1}^n (\det(V, M_I))^{c_I N_l} \exp \left( i S_{\text{reg}} + i S_{\text{gf}} + i S_{\text{source}} \right),
$$

(5)

where $M_I = m_I \Lambda$ are masses of the Pauli–Villars superfields and the coefficients $m_I$ do not depend on the bare charge. $S_{\text{reg}}$ is the regularized action containing the HD term and $S_{\text{gf}}$ is the gauge fixing term. In the Abelian case it is not necessary to introduce ghost (super)fields. The Pauli–Villars determinants $\det(V, M_I)$ are constructed exactly as in the case $N_f = 1$ (see, e.g. Refs. [24, 25]).

For cancelation of remaining one-loop divergences the coefficients $c_I$ should satisfy the conditions $\sum_I c_I = 1$ and $\sum_I c_I M_I^2 = 0$ [34].

Let us consider a part of the effective action corresponding to the two-point functions of the gauge and matter superfields:

$$
\Gamma^{(2)} - S_{\text{gf}} = -\frac{1}{16\pi} \int \frac{d^4p}{(2\pi)^4} d^2\theta V(\theta, -p) \partial^2 \Pi_{1/2} V(\theta, p) d^{-1}(\alpha_0, \Lambda/p)
+ \frac{1}{4} \sum_{I=1}^{N_f} \int \frac{d^4p}{(2\pi)^4} d^2\theta \left( \phi_i^*(\theta, -p) \phi_i(\theta, p) + \tilde{\phi}_i^*(\theta, -p) \tilde{\phi}_i(\theta, p) \right) G(\alpha_0, \Lambda/p),
$$

(6)

where $\partial^2 \Pi_{1/2} = -D^a D^b D_a / 8$ is the supersymmetric transversal projector. The NSVZ relation is naturally obtained for the RG functions defined according to the following prescriptions:

$$
\beta\left( \alpha_0(\alpha, \Lambda/\mu) \right) \equiv \frac{d \alpha_0(\alpha, \Lambda/\mu)}{d \ln \Lambda} \bigg|_{\alpha = \text{const}};
$$

(7)

$$
\gamma\left( \alpha_0(\alpha, \Lambda/\mu) \right) \equiv -\frac{d \ln Z(\alpha, \Lambda/\mu)}{d \ln \Lambda} \bigg|_{\alpha = \text{const}},
$$

(8)

where $\alpha$ is the renormalized coupling constant and $Z$ is the renormalization constant for the matter superfields. They can be found by requiring finiteness of the functions $d^{-1}(\alpha_0(\alpha, \Lambda/\mu), \Lambda/p)$ and $ZG(\alpha, \Lambda/\mu, \Lambda/p)$ in the limit $\Lambda \to \infty$. Certainly, the renormalized coupling constant $\alpha$ and the renormalization constants $Z$ are not uniquely defined and depend on a choice of a renormalization scheme [35]. However, it is possible to prove (see e.g. [25]) that the RG functions $\beta$ and $\gamma$ are independent of a renormalization prescription.

If the HD method is used for a regularization, the integrals which determine the $\beta$-function [7] are integrals of (double) total derivatives [26, 29, 24]. Therefore, one of the loop integrals can be calculated analytically giving the relation [24]

$$
\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d}{d \ln \Lambda} \left( d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) \bigg|_{\alpha = \text{const}} = \frac{N_f}{\pi} \left( 1 - \frac{d}{d \ln \Lambda} \ln G(\alpha_0, \Lambda/q) \right) \bigg|_{\alpha = \text{const}} = \frac{N_f}{\pi} \left( 1 - \gamma(\alpha_0) \right),
$$

(9)
which is exact in all orders. Thus, the exact NSVZ $\beta$-function \textsuperscript{11} is obtained for the RG functions defined in terms of the bare charge independently of a renormalization prescription.

3 Scheme dependence of the RG functions defined in terms of the renormalized coupling constant

Although the exact NSVZ relation for the considered theory is naturally obtained for the RG functions defined in terms of the bare coupling constant, usually the RG functions are defined in a different way:

\[
\tilde{\beta}(\alpha, \Lambda/\mu) \equiv \frac{d\alpha}{d\ln \mu} \bigg|_{\alpha_0=\text{const}}; \quad (10)
\]

\[
\tilde{\gamma}(\alpha, \Lambda/\mu) \equiv \frac{d}{d\ln \mu} \ln ZG(\alpha, \Lambda/\mu) \bigg|_{\alpha_0=\text{const}} = \frac{d\ln Z(\alpha, \Lambda/\mu)}{d\ln \mu} \bigg|_{\alpha_0=\text{const}}, \quad (11)
\]

where $\alpha_0$ and $\mu$ are considered as independent variables. By definition, these RG functions depend on the renormalized coupling constant. Unlike the RG function \textsuperscript{10} and \textsuperscript{11}, they depend on an arbitrariness of choosing $\alpha$ and $Z$. Therefore, in general, these functions do not satisfy the NSVZ relation, which was originally derived for the bare quantities. Nevertheless, as was shown in Ref. \textsuperscript{[25]}, if there is a point $x_0 = \ln \Lambda/\mu_0$ for which the boundary conditions \textsuperscript{[2]} are valid, the RG functions \textsuperscript{10} and \textsuperscript{11} coincide with the RG functions \textsuperscript{7} and \textsuperscript{8}, respectively, and, as a consequence, satisfy the NSVZ relation.

Under a finite renormalization

\[
\alpha \to \alpha'(\alpha); \quad Z'(\alpha', \Lambda/\mu) = z(\alpha)Z(\alpha, \Lambda/\mu)
\]

the $\beta$-function \textsuperscript{10} and the anomalous dimension \textsuperscript{11} are changed as follows:

\[
\tilde{\beta}'(\alpha') = \frac{d\alpha'}{d\ln \mu} \bigg|_{\alpha_0=\text{const}} = \frac{d\alpha'}{d\alpha} \tilde{\beta}(\alpha); \quad (13)
\]

\[
\tilde{\gamma}'(\alpha') = \frac{d\ln Z'}{d\ln \mu} \bigg|_{\alpha_0=\text{const}} = \frac{d\ln z}{d\alpha} \tilde{\beta}(\alpha) + \tilde{\gamma}(\alpha). \quad (14)
\]

Using these equations it is easy to see that if $\tilde{\beta}(\alpha)$ and $\tilde{\gamma}(\alpha)$ satisfy the NSVZ relation, then

\[
\tilde{\beta}'(\alpha') = \frac{d\alpha'}{d\alpha} \cdot \frac{\alpha^2 N_f}{\pi} \cdot \frac{1 - \tilde{\gamma}'(\alpha')}{1 - \alpha^2 N_f(d\ln z/d\alpha)/\pi} \bigg|_{\alpha=\text{const}}. \quad (15)
\]

This result generalizes a similar equation presented in Ref. \textsuperscript{[25]} for the case $N_f = 1$.

Let us note that quantum corrections to the coupling constant are produced by diagrams which contain at least one loop of the matter superfields. Such a loop gives a factor $N_f$. Thus, it is reasonable to make finite renormalizations of the coupling constant proportional to $N_f$:

\[
\alpha'(\alpha) - \alpha = O(N_f); \quad z(\alpha) = O \left( (N_f)^0 \right). \quad (16)
\]

Then from Eq. \textsuperscript{[17]} we see that all scheme dependent terms in the $\beta$-function are proportional at least to $(N_f)^2$ in all orders of the perturbation theory. Similarly, from Eq. \textsuperscript{[14]} it is evident that the terms proportional to $(N_f)^0$ in the anomalous dimension are scheme independent. Also we know that the NSVZ scheme exists. Therefore, the NSVZ relation is satisfied for terms proportional to $(N_f)^1$ in all orders, while terms proportional to $(N_f)^0$ with $\alpha \geq 2$ are scheme dependent.
4 Scheme dependence in the three-loop approximation

In the case of using the HD regularization with \( R = 1 + \partial^{2n}/\Lambda^{2n} \) for the considered theory the functions \( d^{-1} \) and \( G \) in the three- and two-loop approximations, respectively, are given by the following expressions:

\[
d^{-1}(\alpha_0, \Lambda/p) = \frac{1}{\alpha_0} + \frac{N_f}{\pi} \left( \ln \frac{\Lambda}{p} + d_1 \right) + \frac{\alpha_0 N_f}{\pi^2} \left( \ln \frac{\Lambda}{p} + d_2 \right) + \frac{\alpha^2_0 N_f}{\pi^3} \left( - \frac{N_f}{2} \ln^2 \frac{\Lambda}{p} + d_3 - \ln \frac{\Lambda}{p} \right) \times \left( N_f \sum_{I=1}^{n} c_I \ln a_I + N_f \left( \frac{1}{2} + N_f d_2 \right) \right) \right) + \left( \text{terms vanishing in the limit } \Lambda \to \infty \right) + O(\alpha^3_0); \quad (17)
\]

\[
G(\alpha_0, \Lambda/p) = 1 - \frac{\alpha_0}{\pi} \ln \frac{\Lambda}{p} - \frac{\alpha_0}{2\pi} + \frac{\alpha^2_0 (N_f + 1)}{2\pi^2} \ln^2 \frac{\Lambda}{p} + \frac{\alpha^2_0}{\pi^2} \ln \frac{\Lambda}{p} \left( N_f \sum_{I=1}^{n} c_I \ln a_I + \frac{3N_f}{2} + 1 \right) + \frac{\alpha^2_0}{\pi^2} d_2 + \left( \text{terms vanishing in the limit } \Lambda \to \infty \right) + O(\alpha^3_0), \quad (18)
\]

where \( d_1, d_2, d_3, \) and \( c_2 \) are finite constants, which should be found by explicit calculating Feynman graphs. These equations are derived similar to the case of \( N = 1 \) SQED, which have been described in details in Ref. [25]. The loop integrals which determine these Green functions for \( N_f = 1 \) can be found in Refs. [26] and [30]. The coefficients \( d_1 \) and \( d_2 \), which are needed in this paper, are calculated as follows:

According to Ref. [26] the function \( d^{-1}(\alpha_0, \Lambda/p) \) (obtained with the HD regularization) in the two-loop approximation is given by

\[
d^{-1}(\alpha_0, \Lambda/p) = \frac{1}{\alpha_0} + \frac{N_f}{\pi} \sum_{I=1}^{n} c_I \left( \ln \frac{M_I}{p} + \sqrt{1 + \frac{4M^2_I}{p^2}} \arctanh \sqrt{\frac{p^2}{4M^2_I + p^2}} \right) + \alpha_0 N_f I_2 + O(\alpha^3_0), \quad (19)
\]

where

\[
I_2 \equiv 64\pi^2 \int \frac{d^4q}{(2\pi)^4} \left( \frac{1}{M^2_f} \left( \frac{(k + p + q)^2 + q^2 - k^2 - p^2}{q^2 + (k + q)^2} - \sum_{I=1}^{n} \frac{c_I}{(q^2 + M^2_f)} \right) \right) \times \left( \frac{(k + p + q)^2 + q^2 - k^2 - p^2}{(k + q)^2 + M^2_f} - \frac{4M^2_f}{q^2 + M^2_f} \right). \quad (20)
\]

This expression is written in the Euclidian space after the Wick rotation and \( R_k \equiv R(k^2/\Lambda^2) = 1 + k^{2n}/\Lambda^{2n} \). Subtracting the term proportional to \( \ln \Lambda/p \) and taking the limit \( p \to 0 \), from Eq. (19) we obtain

\[
d_1 = \sum_{I=1}^{n} c_I \ln a_I + 1. \quad (21)
\]

The massive two-loop integrals coming from the Pauli–Villars determinants are finite in the infrared region. As a consequence, calculating their sum (which depends only on \( p/\Lambda \)) it is possible to set \( p = 0 \). Then the corresponding terms in Eq. (20) give the vanishing integral of a total derivative

\[
64\pi^2 \sum_{I=1}^{n} c_I \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{1}{k^2 R_k} \frac{\partial}{\partial q^\mu} \left( \frac{q^\mu}{(q^2 + M^2_f)^2} \right) = 0. \quad (22)
\]
The remaining integral can be rewritten in the following form:

\[ I_2 = 128\pi^2 \int \frac{d^4k \ d^4q}{(2\pi)^4 (2\pi)^4} \frac{q_\mu(q + k + p)_\mu}{k^2 R_k q^2(q + p)^2(k + q + p)^2 + o(1)}, \]

where \( o(1) \) denotes terms vanishing in the limit \( p \to 0 \). Deriving this equation we take into account that the term proportional to \( k_\mu p_\mu \) vanishes, because the sign of this term is inverted after the change of variables \( q_\mu \to q_\mu - p_\mu \) and the subsequent replacement \( p_\mu \to -p_\mu \). In order to calculate the above integral, we add to it

\[ 0 = -128\pi^2 \int \frac{d^4k \ d^4q}{(2\pi)^4 (2\pi)^4} \frac{q_\mu(q + k + p)_\mu}{k^2 R_k q^2(q + p)^2(q + p + k)^2} \left( \frac{\ln \Lambda}{p} + \frac{1}{2} \right) + o(1). \]

The integral over the loop momenta obtained after this procedure is convergent in both ultraviolet and infrared regions and depends only on \( p/\Lambda \). Therefore, its value in the limit \( p \to 0 \) can be found by setting \( \Lambda \to \infty \), so that \( R_k \to 1 \). As a consequence,

\[ d_2 = \frac{1}{2} - 128\pi^2 \int \frac{d^4k \ d^4q}{(2\pi)^4 (2\pi)^4} q^\mu(q + k + p)_\mu \frac{(k^2 + 2k^\alpha q_\alpha)(p^2 + 2p^\beta q_\beta) - 2q^2 k^\alpha p_\alpha}{k^2 q^2(q + k + p)^2(q + p)^2(q + k)^2}. \]

Presenting this integral as a sum of scalar integrals and calculating them using the dimensional regularization in the limit \( d \to 4 \) we obtain

\[ d_2 = \frac{3}{2} \left( 1 - \zeta(3) \right). \]

In this expression the term proportional to \( \zeta(3) \) comes from a certain 2-loop scalar master integral, which has been calculated in Ref. [41] using the Gegenbauer polynomial \( x \)-space technique.

The function \( d^{-1} \) expressed in terms of the renormalized coupling constant \( \alpha \) is finite in the limit \( \Lambda \to \infty \) if \( \alpha \) is related with the bare coupling constant \( \alpha_0 = g_0^2/4\pi \) by the equation

\[ \frac{1}{\alpha_0} - \frac{N_f}{\pi} \left( \ln \frac{\Lambda}{\mu} + b_1 \right) - \frac{\alpha N_f}{\pi^2} \left( \ln \frac{\Lambda}{\mu} + b_2 \right) - \frac{\alpha^2 N_f}{\pi^2} \left( \frac{N_f}{2} \ln^2 \frac{\Lambda}{\mu} - \ln \frac{\Lambda}{\mu} \left( N_f \sum_{l=1}^{n} c_l \ln a_l \right) + N_f + \frac{1}{2} - N_f b_1 \right) + b_3 \right) + O(\alpha^3). \]

In this equation \( b_1, b_2, \) and \( b_3 \) are arbitrary finite constants, which partially define the subtraction scheme. The coefficients \( b_i \) are multiplied by the factor \( N_f \) according to Eq. [41]. Similarly, divergences in the two-point Green function of the matter superfields can be cancelled by multiplying the function \( G(\alpha_0, \Lambda/p) \) by the renormalization constant \( Z \), which is given by

\[ Z = 1 + \frac{\alpha}{\pi} \left( \ln \frac{\Lambda}{\mu} + g_1 \right) + \frac{\alpha^2 (N_f + 1)}{2\pi^2} \ln^2 \frac{\Lambda}{\mu} - \frac{\alpha^2}{\pi^2} \ln \frac{\Lambda}{\mu} \left( N_f \sum_{l=1}^{n} c_l \ln a_l \right) - g_1 \right) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3). \]

Here \( g_1 \) and \( g_2 \) are again finite constants, which (together with \( b_i \)) fix the subtraction scheme in the considered approximation. It is easy to see that for arbitrary values of these constants the function \( ZG \) is finite in the limit \( \Lambda \to \infty \).

The anomalous dimension [50] can be found by differentiating \( \ln Z(\alpha, \Lambda/\mu) \) with respect to \( \ln \Lambda \) and writing the result in terms of \( \alpha_0 \). Then we obtain
\[
\gamma(\alpha_0) = \frac{d \ln Z}{d \ln \Lambda} = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2} \left( N_f \sum_{i=1}^{n} c_I \ln a_I + N_f + \frac{1}{2} \right) + O(\alpha_0^3). \tag{29}
\]

This expression is independent of the finite constants \( g_i \) and \( b_i \), which fix the subtraction scheme.

The anomalous dimension \( \tilde{\gamma}(\alpha) \) defined by Eq. (11) can be constructed similarly. For this purpose we rewrite \( \ln Z \) in terms of \( \alpha_0 \) using Eq. (27) and differentiate the result with respect to \( \ln \mu \). Writing the result in terms of \( \alpha \) we obtain

\[
\tilde{\gamma}(\alpha) = \frac{d \ln Z}{d \ln \mu} = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2} \left( N_f + N_f \sum_{I=1}^{n} c_I \ln a_I - N_f b_1 + N_f g_1 + \frac{1}{2} \right) + O(\alpha^3). \tag{30}
\]

Unlike Eq. (29) this expression depends on the constants \( g_1 \) and \( b_1 \). However, only the terms proportional to \( (N_f)^1 \) depend on these parameters, the terms proportional to \( (N_f)^0 \) being independent of them.

Differentiating Eq. (27) with respect to \( \ln \Lambda \) and writing the result in terms of \( \alpha_0 \) we obtain the \( \beta \)-function defined by Eq. (7):

\[
\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} + \frac{\alpha_0 N_f}{\pi^2} - \frac{\alpha_0^2 N_f}{\pi^3} \left( N_f \sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2} \right) + O(\alpha_0^3). \tag{31}
\]

This \( \beta \)-function does not depend on the finite constants \( g_i \) and \( b_i \) and is related with the anomalous dimension (29) by Eq. (11). The \( \beta \)-function (10) is calculated by re-expressing \( \alpha \) in terms of \( \alpha_0 \) and differentiating the result with respect to \( \ln \mu \):

\[
\frac{\tilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3} \left( N_f \sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2} - N_f b_1 + N_f b_2 \right) + O(\alpha^3). \tag{32}
\]

This equation implies that the terms proportional to \( N_f \) do not depend on the constants \( b_i \) and are, therefore, scheme independent. Moreover, comparing Eqs. (30) and (32) we see that for the terms proportional to \( (N_f)^1 \) the NSVZ relation is satisfied. This result agrees with the general statement presented above, which follows from Eq. (15) in all orders of the perturbation theory.

5 Examples: NSVZ, MOM and DR schemes

Let us compare the results of explicit calculations made with different subtraction schemes, namely, the NSVZ scheme obtained with the HD regularization [25], the MOM scheme, and the DR scheme. Certainly, any pair of these schemes can be related by a finite renormalization [35].

With the HD regularization the NSVZ scheme for the RG function defined in terms renormalized coupling constant is obtained by imposing the boundary conditions (2) on the renormalization constants. Choosing \( x_0 = 0 \), it is easy to see that in this case

\[
g_1 = g_2 = 0; \quad b_1 = b_2 = b_3 = 0 \tag{33}
\]

and, therefore,

\[
\tilde{\gamma}_{NSVZ}(\alpha) = \gamma(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2} \left( \frac{1}{2} + N_f \sum_{I=1}^{n} c_I \ln a_I + N_f \right) + O(\alpha^3); \tag{34}
\]

\[
\tilde{\beta}_{NSVZ}(\alpha) = \beta(\alpha) = \frac{\alpha^2 N_f}{\pi} \left( 1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2} \left( \frac{1}{2} + N_f \sum_{I=1}^{n} c_I \ln a_I + N_f \right) + O(\alpha^3) \right). \tag{35}
\]
Thus, in this scheme the NSVZ relation is valid for terms proportional to both \((N_f)^1\) and \((N_f)^2\). This is in agreement with the general result that in the scheme defined by the conditions (2) the NSVZ \(\beta\)-function is obtained in all orders, if the theory is regularized by HD.

The MOM scheme is defined by the boundary conditions

\[
Z_{\text{MOM}}G(\alpha_{\text{MOM}}, p = \mu) = 1; \quad d^{-1}(\alpha_{\text{MOM}}, p = \mu) = \alpha_{\text{MOM}}^{-1}
\]

imposed on the renormalized Green functions. In this case

\[
g_1 = \frac{1}{2}; \quad g_2 = -c_2 + \frac{1}{4} + \frac{N_f}{2} b_1; \quad b_1 = d_1; \quad b_2 = d_2; \quad b_3 = d_3 + N_f d_1 d_2.
\]

Therefore, in the MOM subtraction scheme the constants \(b_i\) and \(g_i\) are related with the finite parts of the Green functions \((c_i\) and \(d_i\)). Using Eq. (21) and (26) we obtain

\[
\tilde{\gamma}_{\text{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2 (1 + N_f)}{2\pi^2} + O(\alpha^3);
\]

\[
\tilde{\beta}_{\text{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi} \left(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2} \left(1 + 3N_f (1 - \zeta(3))\right) + O(\alpha^3)\right).
\]

Comparing these equations we see that in the MOM scheme only terms proportional to \((N_f)^1\) satisfy the NSVZ relation. Note that the \(\beta\)-function in the MOM subtraction scheme coincides with the Gell-Mann–Low function \[42\] and should not depend on the regularization. The same statement is obvious for the anomalous dimension in the MOM scheme. The RG functions (38) and (39) are obtained using the HD regularization. We have also verified these expressions by the calculation of the anomalous dimension and the \(\beta\)-function in the MOM scheme using the DRED regularization. (In the three-loop approximation we have evaluated only the scheme-dependent terms proportional to \((N_f)^2\).) The results coincide with Eqs. (38) and (39). This confirms the correctness of the calculations made with the HD regularization.

A three-loop \(\beta\)-function and a two-loop anomalous dimension for a general \(\mathcal{N} = 1\) SYM theory with matter in the \(\overline{\text{DR}}\)-scheme have been calculated in Ref. [15].\(^1\) The result has the following form:

\[
\tilde{\gamma}_{\overline{\text{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2 (1 + N_f)}{2\pi^2} + O(\alpha^3);
\]

\[
\tilde{\beta}_{\overline{\text{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi} \left(1 + \frac{\alpha}{\pi} - \frac{\alpha^2 (2 + 3N_f)}{4\pi^2} + O(\alpha^3)\right).
\]

Comparing these RG functions we see that the NSVZ relation is valid for the terms proportional to \((N_f)^1\) and is not satisfied for terms proportional to \((N_f)^2\). All terms proportional to \((N_f)^0\) in different expressions for the anomalous dimension coincide. Similarly, all terms proportional to \((N_f)^1\) in different expressions for the \(\beta\)-function also coincide. This confirms the general conclusions made in this paper. Also we note that \(\zeta(3)\) is present in the three-loop \(\beta\)-function in the MOM scheme and is absent in the expression found with the \(\overline{\text{DR}}\) scheme exactly as in the usual quantum electrodynamics in the MOM and \(\overline{\text{MS}}\) schemes, respectively (see e.g. \[42\]). The reason is that in both theories a certain finite scalar integral proportional to \(\zeta(3)\) \[11\] is essential, if the three-loop \(\beta\)-function is calculated in the MOM scheme. Also it is interesting to note that the anomalous dimension in the MOM scheme coincides with the one in the \(\overline{\text{DR}}\) scheme.

\(^1\)In order to obtain the results of Ref. [15] it is necessary to set \(\alpha = g^2/4\pi\), \(\gamma(\alpha) = 2\gamma(g)\), \(\beta(\alpha) = g\beta(g)/2\pi\).
6 Conclusion

For $N = 1$ SQED with $N_f$ flavors the exact NSVZ $\beta$-function is obtained for the RG functions defined in terms of the bare coupling constant if the theory is regularized by higher derivatives. These RG functions by definition do not depend on a choice of the renormalization scheme. However, the RG functions defined in terms of the renormalized coupling constant depend on a subtraction scheme. In this paper we have demonstrated that the coefficients of the $\beta$-function proportional to $(N_f)^1$ are scheme independent and satisfy the NSVZ relation in all orders. This is explicitly verified by calculating the two-loop anomalous dimension and the three-loop $\beta$-function using different subtraction schemes.

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References

[1] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 229 (1983) 381.

[2] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Lett. B 166 (1986) 329; Sov. J. Nucl. Phys. 43 (1986) 294; [Yad. Fiz. 43 (1986) 459.]

[3] M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B 277 (1986) 456; Sov. Phys. JETP 64 (1986) 428; [Zh. Eksp. Teor. Fiz. 91 (1986) 723.]

[4] D. R. T. Jones, Phys. Lett. B 123 (1983) 45.

[5] A. I. Vainshtein, V. I. Zakharov and M. A. Shifman, JETP Lett. 42 (1985) 224 [Pisma Zh. Eksp. Teor. Fiz. 42 (1985) 182].

[6] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Lett. B 166 (1986) 334.

[7] W. Siegel, Phys. Lett. B 84 (1979) 193.

[8] W. Siegel, Phys. Lett. B 94 (1980) 37.

[9] L. V. Avdeev, Phys. Lett. B 117 (1982) 317.

[10] L. V. Avdeev and A. A. Vladimirov, Nucl. Phys. B 219 (1983) 262.

[11] V. N. Velizhanin, Nucl. Phys. B 818 (2009) 95.

[12] S. Ferrara and B. Zumino, Nucl. Phys. B 79 (1974) 413.

[13] D. R. T. Jones, Nucl. Phys. B 87 (1975) 127.

[14] L. V. Avdeev and O. V. Tarasov, Phys. Lett. B 112 (1982) 356.

[15] I. Jack, D. R. T. Jones and C. G. North, Phys. Lett. B 386 (1996) 138.

[16] I. Jack, D. R. T. Jones and C. G. North, Nucl. Phys. B 486 (1997) 479.
[17] I. Jack, D. R. T. Jones and A. Pickering, Phys. Lett. B 435 (1998) 61.
[18] R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila and M. Steinhauser, JHEP 0612 (2006) 024.
[19] I. Jack, D. R. T. Jones, P. Kant and L. Mihaila, JHEP 0709 (2007) 058.
[20] A. A. Slavnov, Nucl. Phys. B 31 (1971) 301.
[21] A. A. Slavnov, Theor.Math.Phys. 13 (1972) 1064 [Teor. Mat. Fiz. 13 (1972) 174].
[22] V. K. Krivoshchekov, Theor. Math. Phys. 36 (1978) 745 [Teor. Mat. Fiz. 36 (1978) 291].
[23] P. C. West, Nucl. Phys. B 268 (1986) 113.
[24] K. V. Stepanyantz, Nucl. Phys. B 852 (2011) 71.
[25] A. L. Kataev and K. V. Stepanyantz, Nucl. Phys. B 875 (2013) 459.
[26] A. A. Soloshenko and K. V. Stepanyantz, Theor. Math. Phys. 140 (2004) 1264 [Teor. Mat. Fiz. 140 (2004) 430].
[27] A. B. Pimenov, E. S. Shevtsova and K. V. Stepanyantz, Phys. Lett. B 686 (2010) 293.
[28] K. V. Stepanyantz, Phys. Part. Nucl. Lett. 8 (2011) 321.
[29] A. V. Smilga and A. Vainshtein, Nucl. Phys. B 704 (2005) 445.
[30] K. V. Stepanyantz, arXiv:1108.1491 [hep-th].
[31] K. V. Stepanyantz, J. Phys. Conf. Ser. 343 (2012) 012115.
[32] K. V. Stepanyantz, J. Phys. Conf. Ser. 368 (2012) 012052.
[33] L. D. Faddeev and A. A. Slavnov, “Gauge Fields. Introduction To Quantum Theory,” Nauka, Moscow, 1978 and Front. Phys. 50 (1980) 1 [Front. Phys. 83 (1990) 1].
[34] A. A. Slavnov, Theor. Math. Phys. 33 (1977) 977 [Teor. Mat. Fiz. 33 (1977) 210].
[35] A. A. Vladimirov, Theor. Math. Phys. 25 (1976) 1170 [Teor. Mat. Fiz. 25 (1975) 335].
[36] A. A. Soloshenko and K. V. Stepanyants, Theor. Math. Phys. 134 (2003) 377 [Teor. Mat. Fiz. 134 (2003) 430].
[37] G. ’t Hooft and M. J. G. Veltman, Nucl. Phys. B 44 (1972) 189.
[38] C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B 12 (1972) 20.
[39] J. F. Ashmore, Lett. Nuovo Cim. 4 (1972) 289.
[40] G. M. Cicuta and E. Montaldi, Lett. Nuovo Cim. 4 (1972) 329.
[41] K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, Nucl. Phys. B 174 (1980) 345.
[42] S. G. Gorishny, A. L. Kataev, S. A. Larin and L. R. Surguladze, Phys. Lett. B 256 (1991) 81.