CONFIGURATION LIE GROUPOIDS AND ORBIFOLD BRAID GROUPS

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ABSTRACT. We propose two definitions of configuration Lie groupoids and in both the cases we prove a Fadell-Neuwirth type fibration theorem for a class of Lie groupoids. We show that this is the best possible extension, in the sense that, for the class of Lie groupoids corresponding to global quotient orbifolds with nonempty singular set, the fibration theorems do not hold. Secondly, we prove a short exact sequence of fundamental groups (called pure orbifold braid groups) of one of the configuration Lie groupoids of the Lie groupoid corresponding to the punctured complex plane with cone points. This shows the possibility of a quasifibration type Fadell-Neuwirth theorem for Lie groupoids.

As consequences, first we see that the pure orbifold braid groups have poly-virtually free structure, which generalizes the classical braid group case. We also provide an explicit set of generators of the pure orbifold braid groups. Secondly, we prove that a class of affine and finite complex Artin groups are virtually poly-free, which partially answers the question if all Artin groups are virtually poly-free ([4], Question 2). Finally, combining this poly-virtually free structure and a recent result ([5]), we deduce the Farrell-Jones isomorphism conjecture for the above class of orbifold braid groups. This also implies the conjecture for the case of the affine Artin group of type \( \tilde{B}_n \), which was left open in ([24], Problem).

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1. Introduction

For a topological space $X$ and a positive integer $n$, the configuration space of $n$-tuple of distinct points of $X$ is defined by the following.

$$PB_n(X) := X^n - \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } i, j \in \{1, 2, \ldots, n\}\}.$$ 

This is also called the pure braid space of $X$.

Let $M$ be a connected manifold of dimension $\geq 2$. Then, the Fadell-Neuwirth fibration theorem ([14], Theorem 3) says that the projection map $M^n \rightarrow M^{n-1}$ to the first $n-1$ coordinates defines a locally trivial fibration $f : PB_n(M) \rightarrow PB_{n-1}(M)$, with fiber homeomorphic to $M - \{(n-1) \text{ points}\}$.

It is an important and classical subject to study the topology and geometry of the configuration space of 2-dimensional manifolds. This has applications in different areas of Mathematics and Physics. [3], [14] and [8] are among some of the fundamental works on this subject. The Fadell-Neuwirth fibration theorem is the key fundamental result, to understand the homotopy theory of the configuration spaces of manifolds. The work in [2] turned the attention to consider 2-dimensional orbifolds, its configuration orbifold and introduced orbifold braid groups. This has connection with some of the Artin groups (Theorem 5.4). On the other hand, a systematic and categorical way to study orbifolds is via Lie groupoids ([22]).

In this paper we define two notions of a fibration between Lie groupoids, generalizing Hurewicz fibration (Definition 2.4). We also formulate two definitions of configuration Lie groupoids of a Lie groupoid (Definitions 2.6 and 2.8). Then, we prove a Fadell-Neuwirth type fibration theorem (Theorem 2.10) for a class of Lie groupoids (Definition 2.1) with respect to both the definitions. We also show that this is the best possible class of Lie groupoids to which the fibration theorem can be extended. This is in the sense that the fibration theorem fails, with respect to our definitions of a fibration, for all Lie groupoids corresponding to global quotient orbifolds with nonempty singular set (Proposition 2.11).

Finally, we study the fundamental groups (called pure orbifold braid group) of the configuration Lie groupoids (with respect to one definition) of Lie groupoids arising from the class $\mathcal{S}$ of genus zero 2-dimensional orbifolds with cone points and at least one puncture. We prove a short exact sequence of pure orbifold braid groups (Theorem 2.14). This short exact sequence is similar to the one which results from the long exact sequence of homotopy groups of the Fadell-Neuwirth fibration $f$ for the complex plane. That is, for any member of $\mathcal{S}$, $f$ shows a quasifibration type property in low degree. In fact, for some members of $\mathcal{S}$, $f$ can be shown to be a quasifibration. This indicates the possibility of a quasifibration type Fadell-Neuwirth theorem for Lie groupoids. See [25] for some more work in this direction.

By [2], elements of the pure orbifold braid group have pictorial braid representation, which is our main motivation for this work. We develop a simple method of stretching a string of a braid to understand the pure orbifold braid groups. See Lemma 4.4 and Proposition 4.6.

There are now multiple consequences of the above study of the pure orbifold braid groups.

- On the way of proving the above short exact sequence, we needed to find an explicit set of generators of the pure orbifold braid group (Lemma 4.1). Note that, finding an explicit set of generators is an important problem in group theory. For
the classical braid groups a set of generators was given in [3]. This gave us the crucial ingredient to find the generators of the pure orbifold braid groups.

- From the proof of Theorem 2.14, as a particular case, we provide an explicit set of generators of the kernel of the homomorphism $f_*$ (Corollary 4.7). Giving an explicit set of generators of this group as a subgroup of the classical pure braid group, seems to be new.

- We see that the pure orbifold braid group, has a poly-virtually free structure, as well as the structure of being an iterated semi-direct product of virtually finitely generated free groups (Corollary 2.18). The classical pure braid group has such a structure but with finitely generated free groups ([10]).

- The poly-virtually free structure of the pure orbifold braid group is also used in Theorem 2.19, to prove that a class of affine and finite complex Artin groups are virtually poly-free. This gives a partial answer to the question, if all Artin groups are virtually poly-free ([4], Question 2). A recent work in this direction is [6], also see [30] for a simple proof of the result of [6]. Poly-free groups have nice properties like locally indicable and right orderable. Also, an inductive argument using ([13], Theorem 2.3) shows that a virtually poly-free group has finite asymptotic dimension.

- Our final application of the poly-virtually free structure of the pure orbifold braid group, is to prove the Farrell-Jones isomorphism conjecture for any group which contains the pure orbifold braid group (Theorem 2.20), as a subgroup of finite index. Consequently, this settles the case of the Artin group of type $\tilde{D}_n$, left open in [[24], Problem]. There is much interest in proving the Farrell-Jones conjecture for Artin groups in recent times. Also see [17] and [30].

We conclude the introduction with a few words on the organization of the paper. In the next section we state our main results, and also recall some background required for the statements. Section 3 contains some more background materials on Lie groupoids and proofs of some lemmas left in Section 2. In this section we also formalize the results we have proved, in the category of orbifold groupoids. In Section 4, we give a detailed introduction to orbifold braid groups and use the stretching technique to prove some crucial results. The proofs of the main results are given in Section 5.

2. Statements of results

We start with some conventions we will follow. All manifolds are assumed to be smooth, Hausdorff and paracompact, and by a ‘map’ we will always mean a smooth map. Also by a ‘fibration’ we will mean a Hurewicz fibration.

Let $\mathcal{G}$ be a Lie groupoid with object space $\mathcal{G}_0$, and morphism space $\mathcal{G}_1 = \bigcup_{x,y \in \mathcal{G}_0} \text{mor}_{\mathcal{G}}(x,y)$. $s,t : \mathcal{G}_1 \to \mathcal{G}_0$ are the source and the target maps, that is for a morphism $\alpha \in \text{mor}_{\mathcal{G}}(x,y)$, $s(\alpha) = x$ and $t(\alpha) = y$. $s$ and $t$ are assumed to be smooth submersions. A homomorphism $f$ between two Lie groupoids is a smooth functor, and we denote the object level map by $f_0$, and the morphism level map by $f_1$. For any $x \in \mathcal{G}_0$, the set $t(s^{-1}(x))$ of all points which are the targets of morphisms emanating from $x$, is called the orbit of $x$. The space (with quotient topology) of all orbits is called the base space, and is denoted by $|\mathcal{G}|$.

**Definition 2.1.** A Lie groupoid $\mathcal{G}$ is called a c-groupoid, if the above quotient map $\kappa : \mathcal{G}_0 \to |\mathcal{G}|$ is a covering map. $\mathcal{G}$ is said to be Hausdorff if the space $|\mathcal{G}|$ is Hausdorff. Clearly, a c-groupoid is Hausdorff.
For $x \in \mathcal{G}_0$, the star $St_x$ at $x$ is defined by $s^{-1}(x) = \cup_{y \in \mathcal{G}_0} \text{mor}_{\mathcal{G}}(x, y)$. $St_x$ is a submanifold of $\mathcal{G}_1$, since $s$ is a submersion. The dimension of the manifold $\mathcal{G}_0$ is called the dimension of the Lie groupoid $\mathcal{G}$. The isotropy group at $x \in \mathcal{G}_0$ is defined by $\mathcal{G}_x := \text{mor}_{\mathcal{G}}(x, x)$. $\mathcal{G}_x$ is a Lie group.

Our main examples of Lie groupoids for this work is the following.

**Example 2.2.** Let $M$ be a manifold and a Lie group $H$ acting on $M$ smoothly. We construct a Lie groupoid $\mathcal{G}(M, H)$, called the translation groupoid, out of this information. Define $\mathcal{G}(M, H)_0 = M$, $\mathcal{G}(M, H)_1 = H \times M$, $s(h, x) = x$, $t(h, x) = h(x)$, $u(x) = (1, x)$, $i(h, x) = (h^{-1}, x)$ and $(h', h(x)) \circ (h, x) = (h'h, x)$. When $H$ is the trivial group then $\mathcal{G}(M, H)$ is called the unit groupoid, and is identified with $M$. If $H$ is discrete and acts on $M$ freely and properly discontinuously, then $\mathcal{G}(M, H)$ is a $c$-groupoid. If the action is effective and properly discontinuous then, $M/H$ is an orbifold and hence $\mathcal{G}(M, H)$ is Hausdorff ([[29], Proposition 5.2.6]).

### 2.1. Fibrations of Lie groupoids and the Fadell-Neuwirth fibration theorem.

Defining a covering map between orbifolds is well-known, it satisfies properties parallel to an ordinary covering map of spaces (see [[29], Definition 5.3.1]). In [[29], Definition 5.11.1] a definition of an orbifold fiber bundle was given but with fiber a manifold. There are other definitions of an orbifold fibration in the literature, but those are with assumptions suitable for some specific purposes. For example, these definitions either give only ‘submersion’ or ‘path lifting property’ when restricted to manifolds.

To define a morphism between orbifolds, independent of atlases, we need to first look at orbifolds as Lie groupoids. This problem was solved in [22], by considering the category of Lie groupoids and generalized maps. A generalized map is a certain equivalence class of homomorphisms. In the next section, we will describe the concept of ‘orbifold groupoids’ which helps in accommodating orbifolds in the category of Lie groupoids.

We now proceed to give two definitions of a fibration for orbifolds, in the set up of Lie groupoids, which restrict to Hurewicz fibration when there is no singularity in the orbifolds, that is, for manifolds.

To define one of the notions of a fibration between Lie groupoids, we need to generalize the construction of a Lie groupoid from a Lie group action on a manifold as in Example 2.2, to a Lie groupoid action on a manifold.

**Definition 2.3.** Given a Lie groupoid $\mathcal{G}$ and a manifold $M$, $M$ is called a (left) $\mathcal{G}$-space if there is a smooth map $\pi : M \to \mathcal{G}_0$, and an action map

$$\mu : \mathcal{G}_1 \times_{\mathcal{G}_0} M \to M$$

satisfying the following properties. For $\alpha \in \mathcal{G}_1$ and $x \in M$ with $\pi(x) = s(\alpha)$ (defining the fibered product), $\pi(\mu(\alpha, x)) = t(\alpha)$. Finally, if $\pi(x) = y$ then $\mu(1_y, x) = x$ and $\mu(\alpha, \mu(\beta, x)) = \mu(\alpha\beta, x)$ for $\beta \in \text{mor}_{\mathcal{G}}(y, w)$ and $\alpha \in \text{mor}_{\mathcal{G}}(w, z)$.

**Definition 2.4.** Let $\mathcal{H}$ and $\mathcal{G}$ be two Lie groupoids and $f : \mathcal{H} \to \mathcal{G}$ a homomorphism, so that $f_0 : \mathcal{H}_0 \to \mathcal{G}_0$ is a submersion fibration.

- $f$ is called an a-fibration if $\mathcal{H}_0$ is a $\mathcal{G}$-space with $f_0 = \pi$, $\mathcal{H}_1 = \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0$ and $f_1$ is the first projection. The source and the target maps of $\mathcal{H}$ are respectively the second projection and the action map.
- $f$ is called a b-fibration if the map $\mathcal{H}_1 \to \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0$ defined by $\alpha \mapsto (f_1(\alpha), s(\alpha))$ is a surjective submersion.
Remark 2.5. The idea of an $a$-fibration comes from the way covering Lie groupoid of a Lie groupoid is defined. See [[22], §5.3]. The same framework works to define a fiber bundle or a vector bundle on a Lie groupoid also. And the $b$-fibration definition is primarily motivated from [16]. It can be shown that the second condition in the definition of a $b$-fibration is equivalent to demanding that $f_1|_{St_x} : St_x \rightarrow St_{f_0(x)}$ is a submersion for all $x \in \mathcal{H}_0$.

Now, we are in a position to define the configuration Lie groupoids of a Lie groupoid.

The first definition is given below, which facilitates to prove an $a$-fibration type Fadell-Neuwirth fibration theorem.

Definition 2.6. Let $\mathcal{G}$ be a Hausdorff Lie groupoid. The $a$-configuration Lie groupoid $PB_n^a(\mathcal{G})$ of $\mathcal{G}$ has the object space defined as follows.

$$PB_n(\mathcal{G})_0 := \mathcal{G}_0^n - \{(x_1, x_2, \ldots, x_n) \in \mathcal{G}_0^n \mid t(s^{-1}(x_i)) = t(s^{-1}(x_j))$$

for some $i, j \in \{1, 2, \ldots, n\}$.}

Since $\mathcal{G}$ is Hausdorff, $PB_n^a(\mathcal{G})_0$ is an open set in $\mathcal{G}_0^n$. This follows from the fact that $(\alpha^n)^{-1}(PB_n(\mathcal{G})) = PB_n^a(\mathcal{G})_0$ and $PB_n(\mathcal{G})$ is open in $|\mathcal{G}|^n$, as $|\mathcal{G}|$ is Hausdorff. Therefore, $PB_n^a(\mathcal{G})_0$ is a smooth manifold.

The morphism space, the source and the target maps are defined inductively. We denote these maps for $PB_n^a(\mathcal{G})$ with a subscript ‘$n$’, with the understanding that the corresponding maps with subscript ‘1’ are those of $\mathcal{G}$.

For $n = 1$, by convention $PB_1^a(\mathcal{G})_0 = \mathcal{G}_0$. Therefore, define $PB_1^a(\mathcal{G})_1 = \mathcal{G}_1$. Hence, $PB_1^a(\mathcal{G})$ is a Lie groupoid.

Assume, we have defined $PB_n^a(\mathcal{G})_1$ and the structure maps so that $PB_n^a(\mathcal{G})_1$ is a Lie groupoid.

Define

$$PB_n^a(\mathcal{G})_1 := PB_n^{a-1}  \times_{PB_n^{a-1}(\mathcal{G})_0} PB_n^a(\mathcal{G})_0,$$

which is the fibered product of the source map $s_{n-1} : PB_n^{a-1}(\mathcal{G})_1 \rightarrow PB_n^{a-1}(\mathcal{G})_0$ and the projection $PB_n^a(\mathcal{G})_0 \rightarrow PB_n^{a-1}(\mathcal{G})_0$ to the first $n - 1$ coordinates. Note that, $PB_n^a(\mathcal{G})_1$ is a smooth manifold because the projection map and the source map are submersions.

Let $\alpha_n = (\alpha_{n-1}, x) \in PB_n^a(\mathcal{G})_1$, then $t_n$ and $s_n$ are defined as follows.

$$t_n(\alpha_n) = (t_{n-1}(\alpha_{n-1}), x_n), s_n(\alpha_{n-1}, x) = x,$$

here $x = (x_1, x_2, \ldots, x_n) \in PB_n(\mathcal{G})_0$. Clearly, $t_n$ and $s_n$ are both smooth, but we still have to check that $t_n(\alpha_n)$ lies in the appropriate space. That is, $x_n$ and the coordinates of $t_{n-1}(\alpha_{n-1})$ have distinct orbits. Note that, by induction,

$$\alpha_{n-1} = (\alpha_1, (x_1, x_2), (x_1, x_2, x_3), \ldots, (x_1, x_2, \ldots, x_{n-1})),$$

for some $\alpha_1 \in \mathcal{G}_1$ with $s_1(\alpha_1) = x_1$. Therefore, $t_n(\alpha_n) = (t_1(\alpha_1), x_2, \ldots, x_{n-1}, x_n)$. We have to check $t_1(\alpha_1)$ and $x_n$ have distinct orbits, which is clear, since $x_1$ and $t_1(\alpha_1)$ have the same orbit and $x_1$ and $x_n$ have distinct orbits. This completes the definition.

The following lemma gives the homomorphism which we need to show to be an $a$-fibration.
Lemma 2.7. The projection $PB_n(G)_0 \to PB_{n-1}(G)_0$ to the first $n-1$ coordinates and the projection $PB_n(G)_1 \to PB_{n-1}(G)_1$ to the first coordinate define a homomorphism

$$F^a : PB_n(G) \to PB_{n-1}(G)$$

of Lie groupoids.

Now, we give the second definition of the configuration Lie groupoid, which will give a $b$-fibration type Fadell-Neuwirth fibration theorem.

Definition 2.8. We define the $b$-configuration Lie groupoid $PB_n^b(G)$ of a Hausdorff Lie groupoid $G$ as follows. The object space is the same as that of an $a$-configuration Lie groupoid, that is $PB_n(G)_0$. Define the morphism space by

$$PB_n^b(G)_1 := (s^n, t^n)^{-1}(PB_n(G)_0 \times PB_n(G)_0).$$

Here $(s^n, t^n) : G^n_1 \to G^n_0 \times G^n_0$ is defined coordinate-wise, that is

$$(s^n, t^n)(\alpha_1, \alpha_2, \ldots, \alpha_n) = ((s(\alpha_1), s(\alpha_2), \ldots, s(\alpha_n)), (t(\alpha_1), t(\alpha_2), \ldots, t(\alpha_n))).$$

Next, we have to give the homomorphism in this context, which we need to show to be a $b$-fibration.

Lemma 2.9. The projection maps $PB_n(G)_0 \to PB_{n-1}(G)_0$ and $PB_n^b(G)_1 \to PB_{n-1}^b(G)_1$ both to the first $n-1$ coordinates, define a homomorphism

$$F^b : PB_n^b(G) \to PB_{n-1}^b(G)$$

of Lie groupoids.

Proofs of Lemmas 2.7 and 2.9 will be given in the next section.

Now, we are in a position to state our first theorem. Recall the definition of a $c$-groupoid from Definition 2.1.

Theorem 2.10. Let $G$ be a $c$-groupoid of dimension $\geq 2$. Then, the homomorphism $F^* : PB_n^c(G) \to PB_{n-1}^c(G)$ is an $*$-fibration. Here, $F^*$ is defined in Lemmas 2.7 and 2.9, for $* = a, b$, respectively.

2.2. Counter examples and a short exact sequence. Recall that an orbifold is called good if its universal cover is a manifold. An orbifold $S$ is called a global quotient if there is a finite group $H$ acting effectively on a connected manifold $M$, such that $S = M/H$. That is, $M \to S$ is an orbifold covering map. For example, any 2-dimensional good orbifold with finitely generated orbifold fundamental group is a global quotient ([26], p. 426]). Note that, an orbifold is Hausdorff, and hence if $S$ is a global quotient as above, we can construct the configuration Lie groupoids of the Lie groupoid $G(M, H)$. See [[29], §5.2] for more on orbifolds.

The following Proposition shows that the Fadell-Neuwirth fibration theorem is not extendable to all Hausdorff Lie groupoids. The $c$-groupoids are the best possible cases after the unit Lie groupoids. The main obstruction in these examples is that the quotient map $G(M, H)_0 \to |G(M, H)|$ is not a genuine covering map.

Proposition 2.11. Let $S$ be a global quotient orbifold of dimension $\geq 2$, with at least one singular point. Let $M$ and $H$ be as defined above. Then, the homomorphism

$$F^* : PB_n^c(G(M, H)) \to PB_{n-1}^c(G(M, H)),$$

is not a $*$-fibration of Lie groupoids. In fact, the object level map $F^*_0$ is not even a quasi-fibration. Here, for $* = a, b$, $F^*$ is defined in Lemmas 2.7 and 2.9, respectively.
We now recall the definition of the homotopy groups of a Lie groupoid.

Given a Lie groupoid $G$, consider the following $n$-times iterated fibered product manifold

$$G_n = G_1 \times_{\varphi_0} G_1 \times_{\varphi_0} \cdots \times_{\varphi_0} G_1.$$ 

Then $G_n$ defines a simplicial manifold and its geometric realization is defined as the classifying space $BG$ of $G$. Up to weak homotopy equivalence this space is unique in the equivalence class (see [[22], §2.4, §4.3]) of $G$. The homotopy groups of $G$ are then defined as follows. See [[22], §4.3].

$$\pi_k(G, *) := \pi_k(BG, *).$$

An $a$ or a $b$-fibration between Lie groupoids induces a quasifibration on their classifying spaces ([[20], Theorem 12.7]). Recall that, a quasifibration is a map which induces a long exact sequence of homotopy groups, similar to the one induced by a (Serre) fibration, equivalently, for path connected base, if the homotopy fiber is weak homotopy equivalent to a fiber. Hence, for $c$-groupoids, we can calculate the homotopy groups of the $a$ or the $b$-configuration Lie groupoids inductively using the long exact sequence of homotopy groups.

**Remark 2.12.** Let $f : \mathcal{H} \to \mathcal{G}$ be an $a$ or a $b$-fibration between Lie groupoids. The fiber over $x \in G_0$ is the Lie groupoid $F$ whose object space is the manifold $F_0 = f_0^{-1}(x)$ (since $f_0$ is a submersion), and the morphism space is $F_1 = f_1^{-1}(G_x)$. Since $f_1$ is a submersion, $F_1$ is a manifold.

The long exact sequence of homotopy groups of an $a$ or a $b$-fibration, mentioned above, then involves the homotopy groups of this fiber Lie groupoid.

Next, we give an infinite series of examples of 2-dimensional Hausdorff Lie groupoids $G$ which are not $c$-groupoids, but still we can deduce a short exact sequence connecting the fundamental groups of $PB_n^a(G)$, $PB_n^b(G)$, and the fundamental group of the fiber of the homomorphism $F^b : PB_n^b(G) \to PB_{n-1}^b(G)$.

**Example 2.13.** We denote by $C(k, m; q)$ the complex plane with $k$ punctures at the points $p_1, p_2, \ldots, p_k \in \mathbb{C}$, $m$ marked points (called cone points) $x_1, x_2, \ldots, x_m \in \mathbb{C} \setminus \{p_1, p_2, \ldots, p_k\}$, and an integer $q_i > 1$ attached to $x_i$ for $i = 1, 2, \ldots, m$. $q_i$ is called the order of the cone point $x_i$. Here $q$ denote the $m$-tuple $(q_1, q_2, \ldots, q_m)$. Note that, $C(k, m; q)$ is a 2-dimensional orbifold. In the Introduction we denoted this class of orbifolds by $S$. We now show that $C(k, m; q)$ is a good orbifold. Let $B$ be a big enough closed disk in $\mathbb{C}$, which contains the punctures and cone points of $C(k, m; q)$ in its interior. Remove small disjoint open disks (contained in the interior of $B \setminus \{x_1, x_2, \ldots, x_m\}$) around the puncture points, namely the resulting space $B'$. Let $DB'$ be the double of $B'$. Then $DB'$ is a closed and good 2-dimensional orbifold. This follows from [[29], p. 237]. Hence $DB'$ has a finite sheeted orbifold covering which is a manifold. Therefore, we conclude that $C(k, m; q)$ also has a finite sheeted orbifold covering which is a manifold. Hence, there is a 2-manifold $S(k, m; q)$ and a finite group $H(k, m; q)$ acting effectively on $S(k, m; q)$, with quotient $C(k, m; q)$. Hence, by Example 2.2, $C(k, m; q)$ is realized as the Lie groupoid $G(S(k, m; q), H(k, m; q))$. Note that, the Lie groupoid $G(S(k, m; q), H(k, m; q))$ is Hausdorff. Therefore, we can consider the configuration Lie groupoids of this Lie groupoid.

Now, we recall some consequences of the Fadell-Neuwirth fibration theorem for 2-manifolds, to motivate our next theorem.
Theorem 2.14. There is a split exact sequence of fundamental groups of Lie groupoids as follows.

\[ 1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(PB_n(M)) \xrightarrow{f_*} \pi_1(PB_{n-1}(M)) \longrightarrow 1. \]

Here \( F = M - \{(n - 1) - \text{points}\} \). The second surjective homomorphism is induced by the fibration map \( f : PB_n(M) \to PB_{n-1}(M) \). We will give an explicit pictorial description of this homomorphism in terms of braids, when \( M = \mathbb{C} \).

The above exact sequence also gives an interesting, and useful poly-free as well as ‘iterated semi-direct product of finitely generated free groups structure’ on the pure braid group \( \pi_1(PB_n(M)) \). See [10] for more on this subject.

Proposition 2.11 says that the Lie groupoids \( G(S(k, m; q), H(k, m; q)) \), for \( m \geq 1 \), do not have a Fadell-Neuwirth type fibration. Nevertheless, we can still prove a short exact sequence similar to the one above, in the following theorem, directly.

This theorem shows that \( F^b \) is a kind of ‘quasifibration’ in low degree.

Theorem 2.14. There is a split exact sequence of fundamental groups of Lie groupoids as follows.

\[ 1 \longrightarrow K \longrightarrow \pi_1(PB^b_n(\mathcal{X})) \xrightarrow{F^b_*} \pi_1(PB^b_{n-1}(\mathcal{X})) \longrightarrow 1. \]

Here, \( K \) is isomorphic to \( \pi_1(F) \),

\[ \mathcal{X} = G(S(k, m; q), H(k, m; q)) \]

and

\[ F = G(S(k + n - 1, m; q), H(k + n - 1, m; q)). \]

Remark 2.15. The orbifold fundamental group of an orbifold, as defined in [[29], Definition 5.3.5], is identified with the fundamental group of an associated orbifold groupoid (see Example 3.4). We just need to note here that, \( \mathcal{X} \) and \( PB^b_n(\mathcal{X}) \) are examples of orbifold groupoids. Furthermore, the \( b \)-configuration Lie groupoid of \( \mathcal{X} \) is the correct model of a Lie groupoid inducing the orbifold structure on \( PB_n(\mathbb{C}(k, m; q)) \). Hence, \( \pi_1(PB^b_n(\mathcal{X})) \) is isomorphic to \( \pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))) \), since \( PB_n(\mathcal{G}(M, H)) = G(PB_n(M), H^n) \). See Example 2.2 for notation.

Hence, the above exact sequence in Theorem 2.14 is equivalent to the following.

\[ 1 \longrightarrow K \longrightarrow \pi_1^{orb}(PB_n(S)) \longrightarrow \pi_1^{orb}(PB_{n-1}(S)) \longrightarrow 1. \]

Here, \( K \) is isomorphic to \( \pi_1^{orb}(F) \), \( S = \mathbb{C}(k, m; q) \) and \( F = S - \{(n - 1) - \text{regular points}\} \). By regular points we mean points which are not singular points in an orbifold, that is, in this case these are points in \( \mathbb{C} - \{x_1, \ldots, x_m, p_1, \ldots, p_k\} \). That is, \( F = \mathbb{C}(k + n - 1, m; q) \). In fact, we will give the proof of this exact sequence. This exact sequence can also be obtained for the genus zero 2-dimensional orbifold \( S \) with countably infinite number of punctures and cone points. This can be done by writing \( S \) as an infinite increasing union of orbifolds of the form \( \mathbb{C}(k, m; q) \), and then taking a direct limit, since direct limit of a directed system of exact sequences is exact again.

Consider the free action of the symmetric group \( S_n \) on \( PB_n(\mathbb{C}(k, m; q)) \) by permuting the coordinates. The quotient orbifold is denoted by \( B_n(\mathbb{C}(k, m; q)) = PB_n(\mathbb{C}(k, m; q))/S_n \).
Definition 2.16. The group $\pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))))$ is called the pure orbifold braid group of the orbifold $\mathbb{C}(k, m; q)$, and $\pi_1^{orb}(B_n(\mathbb{C}(k, m; q)))$ is called its orbifold braid group.

We get the following consequence on the structure of the pure orbifold braid group, from Remark 2.15.

Before that we recall the following definition.

Definition 2.17. Let $\mathcal{F}$ and $\mathcal{VF}$ denote the class of free groups and virtually-free groups, respectively. Let $\mathcal{C}$ be either $\mathcal{F}$ or $\mathcal{VF}$. A group $G$ is called poly-$\mathcal{C}$, if $G$ admits a normal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G$, such that $G_{i+1}/G_i \in \mathcal{C}$, for $i = 0, 1, \ldots, n - 1$. The minimum such $n$ is called the length of the poly-$\mathcal{C}$ structure. $G$ is called virtually poly-$\mathcal{C}$ if $G$ contains a finite index poly-$\mathcal{C}$ subgroup.

Note that, a subgroup of a poly-$\mathcal{C}$ group is poly-$\mathcal{C}$ and an extension of a poly-$\mathcal{C}$ group by a poly-$\mathcal{C}$ group is poly-$\mathcal{C}$. Also, the class of virtually-$\mathcal{F}$ groups is closed under taking finite free products.

Corollary 2.18. The group $\pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))))$ has a poly-$\mathcal{VF}$ structure, consisting of finitely presented subgroups in a normal series. Furthermore, it has an iterated semi-direct product of virtually finitely generated free group structure.

Proof. Note that, for all $k, m$ and $q$, $\pi_1^{orb}(\mathbb{C}(k, m; q)))$ is isomorphic to the free product of the finite cyclic groups of order $q$, for $i = 1, 2, \ldots, m$, and a free group on $k$ generators. The Corollary follows from Remark 2.15 and by induction on $n$. \qed

2.3. Virtual poly-freeness of affine Artin groups. We first recall some basics related to Artin groups. For details on this subject see [18], [7] and [8].

Let $K = \{s_1, s_2, \ldots, s_k\}$ be a finite set, and $m : K \times K \to \{1, 2, \ldots, \infty\}$ be a map with the property that $m(s, s) = 1$, and $m(s', s) = m(s, s') \geq 2$ for $s \neq s'$. The Coxeter group associated to the pair $(K, m)$ is by definition the following group.

$$W_{(K, m)} = \langle K \mid (ss')^{m(s, s')} = 1, \; s, s' \in S \text{ and } m(s, s') < \infty \rangle.$$  

A complete classification of finite, irreducible Coxeter groups is known ([12]). A Coxeter group is called irreducible if it is not the direct product of two non-trivial Coxeter groups. Finite Coxeter groups are exactly the finite reflection groups. Also, there are infinite Coxeter groups which are affine reflection groups ([18]).

The Artin group associated to the Coxeter group $W_{(K, m)}$ is, by definition,

$$A_{(K, m)} = \langle K \mid ss'ss' \cdots = s'ss's \cdots; \; s, s' \in K \rangle.$$  

Here, the number of times the factors in $ss'ss' \cdots$ appear is $m(s, s')$: e.g., if $m(s, s') = 3$, then the relation is $ss's = s'ss'$. $A_{(K, m)}$ is called the Artin group of type $W_{(K, m)}$.

A finite type or an affine type Artin group is by definition the Artin group corresponding to a finite or affine type Coxeter group, respectively. There are complex type Artin groups also, which are the Artin groups whose corresponding Coxeter group is generated by reflections along complex hyperplanes in some complex space.

It is still an open question if all Artin groups are virtually poly-$\mathcal{F}$. See [4], Question 2. Among the finite type Artin groups, the groups of types $A_n, B_n (= C_n), D_n, F_4, G_2$ and $I_2(p)$ are already known to be virtually poly-$\mathcal{F}$ ([8]).

Here, we extend this class and prove the following theorem.
Theorem 2.19. Let $A$ be an Artin group of the affine type $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$ or of the finite complex type $G(de,e,r)$ ($d,r \geq 2$). Then, $A$ is virtually poly-$F$.

Recently, it was shown in [6] that the even Artin groups (that is when $m(s,s') = 2$ for all $s \neq s'$) of $FC$-types (certain amalgamation of finite type even Artin groups) are poly-$F$. A simple proof of this result of [6] is given in [30].

2.4. The Farrell-Jones isomorphism conjecture. Using Corollary 2.18 and a recent result ([5]), we prove the following theorem. Before we give the statement of the result, we recall that the Farrell-Jones isomorphism conjecture is an important conjecture in Geometry and Topology, and much works have been done in recent times. The conjecture implies some of the classical conjectures in Topology, like Borel and Novikov conjectures, and provides a better understanding of the $K$- and $L$-theory of a group.

Theorem 2.20. The Farrell-Jones isomorphism conjecture with coefficients and finite wreath product, is true for the orbifold braid group of the orbifold $\mathbb{C}(k,m,q)$. Consequently, it is true for the Artin group of type $\tilde{D}_n$.

We recall that the case of the Artin group of type $\tilde{D}_n$ was left open in [24]. See Problem at the end of [24].

The proof of Theorem 2.20 is short and does not require the exact statement of the conjecture, and some well-known results in this area. Therefore, we do not state the conjecture, and refer the reader to [19] or [23] for more on this subject.

3. Some basics

In this section we recall some more basics on Lie groupoids and complete some checking, left in the last section, including the proofs of Lemmas 2.7 and 2.9.

3.1. Lie groupoids. For the material recalled in this subsection see [1] or [22].

A groupoid is a small category $\mathcal{G}$ with all morphisms invertible. We denote the object set by $G_0$ and the union of all morphism sets by $G_1 := \bigcup_{x,y \in G_0} mor_\mathcal{G}(x,y)$. There are the following structure maps to define a groupoid.

(ST), $s,t : G_1 \to G_0$ are defined by $s(\alpha) = x, t(\alpha) = y$, if $\alpha \in mor_\mathcal{G}(x,y)$. $s$ is written as $\alpha : x \to y$. These are respectively called the source and target maps.

(I). $i : G_1 \to G_1$ defined by $i(\alpha) := \alpha^{-1} \in mor_\mathcal{G}(y,x)$ if $\alpha \in mor_\mathcal{G}(x,y)$. $i$ is called the inverse map.

(M). $m : G_1 \times_{G_0} G_1 \to G_1$ is denoted by $m(\alpha, \beta) := \beta \circ \alpha \in mor_\mathcal{G}(x,z)$ if $\alpha \in mor_\mathcal{G}(x,y)$ and $\beta \in mor_\mathcal{G}(y,z)$. This is called the multiplication or composition map. Here, $G_1 \times_{G_0} G_1 = \{(\alpha,\beta) \in G_1 \times G_1 \mid t(\alpha) = s(\beta)\}$.

(U). $u : G_0 \to G_1$ defined by $u(x) = id_x \in mor_\mathcal{G}(x,x)$, called the unit map.

These maps should satisfy the following.

(C). The multiplication is associative, that is, $f \circ (g \circ h) = (f \circ g) \circ h$ whenever they are defined. The unit map is a two-sided unit of the composition, which means for all $x,y \in G_0$ and $\alpha : x \to y$, $s(u(x)) = x = t(u(x))$ and $\alpha \circ (u(x)) = u(y) \circ \alpha$. Finally, $\alpha^{-1}$ is a two-sided inverse of $\alpha$. That is $\alpha \circ \alpha^{-1} = u(y)$ and $\alpha^{-1} \circ \alpha = u(x)$.

Definition 3.1. A groupoid $\mathcal{G}$ is called a Lie groupoid if $G_0$ and $G_1$ are smooth manifolds, all the structure maps are smooth, and in addition $s$ and $t$ are submersions. The last condition is necessary to make sure that the fiber product $G_1 \times_{G_0} G_1$ is a smooth manifold.
Now, recall that if a discrete group $H$ acts on a manifold $M$ effectively and properly discontinuously, then the quotient $M/H$ has an orbifold structure. In Example 2.2 we have seen how to associate a Lie groupoid to this data, in a more general setting.

**Definition 3.2.** ([22], §1.5) A Lie groupoid is called proper if the map $(s, t): G_1 \rightarrow G_0 \times G_0$ is a proper map. Consequently, for a proper Lie groupoid $G$, $[G]$ is compact, for all $x \in G_0$. $G$ is called a foliation groupoid, if $G_x$ is discrete for all $x \in G_0$. An orbifold groupoid is by definition a proper foliation groupoid. Hence, an orbifold groupoid has finite isotropy groups. For an orbifold groupoid $G$, $[G]$ is the orbifold, on which $G$ is the Lie groupoid structure.

**Remark 3.3.** An orbifold groupoid is Hausdorff since it is proper.

**Example 3.4.** If a discrete group $H$ acts on a manifold $M$ effectively and properly discontinuously, then $G(M, H)$ is an orbifold groupoid, and gives an orbifold structure on $M/H$. Conversely, given an orbifold, there is a translation Lie groupoid, namely the frame bundle manifold with the orthogonal group action, which induces the orbifold structure on the orbifold. Note that, any two orbifold groupoids have finite isotropy groups. For an orbifold groupoid $G$, $[G]$ is the orbifold, on which $G$ is the Lie groupoid structure.

In the language of orbifold groupoids, we can now use the above remarks to formalize the main results we have proved.

**Theorem 3.5.** Let $G$ be a connected Hausdorff Lie groupoid.

(Theorem 2.10) Assume that $G_0 \rightarrow [G]$ is a covering map, that is, the orbifold $[G]$ does not have any singular point, consequently, it is a manifold. Then $F^a(F^b)$ is an $a/b$-fibration.

Now let $G$ be an orbifold groupoid.

(Theorem 2.14) Assume that $[G]$ is equivalent to $\mathbb{C}(k,m;q)$ as an orbifold, then there is an exact sequence of fundamental groups as follows.

$$1 \rightarrow \pi_1(F) \rightarrow \pi_1(PB^b_\infty(G)) \rightarrow \pi_1(PB^b_{n-1}(G)) \rightarrow 1.$$

Here $F$ is a fiber of the homomorphism $F^b$.

(Proposition 2.11) Assume that $G_0 \rightarrow [G]$ is an orbifold covering map, $[G]$ is a global quotient and has a nonempty singular set. Then, we can find another orbifold groupoid $H$, Morita equivalent to $G$, such that $F^a(F^b)$ is not an $a/b$-fibration for $H$. In fact, the object level map $F^a_0(F^b_0)$ for $H$ is not even a quasifibration.

3.2. $PB^a_n(G)$, $PB^b_n(G)$, $F^a$, and $F^b$. We have already defined the source $(s_n)$ and the target $(t_n)$ maps for $PB^a_n(G)$ in Section 2, and observed they are smooth. Now, we define the other structure maps and show that they are smooth and satisfy the conditions in C of the definition of a groupoid. Recall that

$$PB^a_n(G)_1 := PB^a_{n-1}(G)_1 \times_{PB^b_{n-1}(G)_0} PB^b_n(G)_0.$$

Since $PB^a_n(G)_1 = G_1$, we again use induction to define the other maps. So, assume we have defined the inverse, multiplication and the unit maps for $PB^a_{n-1}(G)_1$ and they are smooth, and satisfies the conditions in C.
Let $\alpha_n = (\alpha_{n-1}, x) \in PB^s_n(\mathcal{G})_1$. Define $i(\alpha_n) = (\alpha_{n-1}^{-1}, x)$. Next, define $u(x) = (id_{(x_1, x_2, \ldots, x_{n-1}, x)} x)$. Hence, $u(x) \in PB^s_n(\mathcal{G})_0$.

Recall that $s_n(\alpha_n) = x$ and $t_n(\alpha_n) = (t_{n-1}(\alpha_{n-1}), x_n)$. Let $\alpha'_n \in PB^s_n(\mathcal{G})_1$, such that $t_n(\alpha_n) = s_n(\alpha'_n)$. This implies $(t_{n-1}(\alpha_{n-1}), x_n) = x' = (x_1', x_2', \ldots, x'_n)$. Therefore, $t_{n-1}(\alpha_{n-1}) = (x_1', x_2', \ldots, x'_{n-1}) = s_{n-1}(\alpha'_{n-1})$. By induction, we define $\alpha'_n \circ \alpha_n = (\alpha'_{n-1} \circ \alpha_{n-1}, x)$.

Clearly, all the maps defined above are smooth. The checking of the conditions in $\mathcal{C}$ are straightforward.

Now we consider the case $PB^b_n(\mathcal{G})$. Recall that,

$$PB^b_n(\mathcal{G})_1 := (s^n, t^n)^{-1}(PB^s_n(\mathcal{G})_0 \times PB^s_n(\mathcal{G})_0).$$

The structure maps in this case are easily defined. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in PB^b_n(\mathcal{G})_1$, and $x = (x_1, x_2, \ldots, x_n) \in PB^s_n(\mathcal{G})_0$, then, define the source, target, inverse, unit and multiplication maps as follows.

$$s(\alpha) = (s(\alpha_1), s(\alpha_2), \ldots, s(\alpha_n)), \ t(\alpha) = (t(\alpha_1), t(\alpha_2), \ldots, t(\alpha_n)),

i(\alpha) = (\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1}), \ u(x) = (id_{x_1}, id_{x_2}, \ldots, id_{x_n}).$$

If $\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_n) \in PB^b_n(\mathcal{G})_1$ with $t(\alpha) = s(\alpha')$ then $t(\alpha_i) = s(\alpha'_i)$, for all $i = 1, 2, \ldots, n$ and hence we can define the multiplication as follows.

$$\alpha' \circ \alpha = (\alpha'_1 \circ \alpha_1, \alpha'_2 \circ \alpha_2, \ldots, \alpha'_n \circ \alpha_n).$$

Since $s(\alpha'_i \circ \alpha_i) = s(\alpha_i)$ and $t(\alpha'_i \circ \alpha_i) = t(\alpha'_i)$, for all $i = 1, 2, \ldots, n$, and no two of $s(\alpha_i)$ or of $t(\alpha'_i)$ have the same orbit, $\alpha' \circ \alpha$ is well-defined.

Finally, all of these maps are smooth and satisfy the conditions in $\mathcal{C}$.

Next, we prove Lemmas 2.7 and 2.9.

Proofs of Lemmas 2.7 and 2.9. Recall that we need to prove that $F^a$ and $F^b$ are homomorphisms of Lie groupoids. That is, they are smooth functors and commute with the source, target, inverse, unit and multiplication maps of the domain and range groupoids. This means the following.

Let $f : \mathcal{K} \to \mathcal{L}$ be a smooth functor between two Lie groupoids. Assume $f$ is defined by two maps $f_0 : K_0 \to L_0$ and $f_1 : K_1 \to L_1$ on the object and morphism spaces. We denote the structure maps of $\mathcal{K}$ and $\mathcal{L}$ by the same notations. $f$ is called a homomorphism if $f_0$ and $f_1$ commute with the structure maps on the domain and the range Lie groupoids. That is, the following are satisfied.

$$(a). \ s \circ f_1 = f_0 \circ s, (b). \ t \circ f_1 = f_0 \circ t, (c). \ f_1 \circ u = u \circ f_0,$$

$$(d). \ f_1 \circ i = i \circ f_1, (e). \ f_1 \circ m = m \circ (f_1 \times f_1).$$

Here, $f_1 \times f_1$ denotes the induced map $K_1 \times_{K_0} K_1 \to L_1 \times_{L_0} L_1$, using (a) and (b).

Clearly, $F^a$ and $F^b$ are smooth functors. Showing the properties in the above display, for them are straightforward, nevertheless, we check it for $F^a$ and leave the $F^b$ case for the reader.

We check the equations (a) to (e) for $F^a$. Recall that we denoted the source and the target maps of $PB^b_n(\mathcal{G})$ by $s_n$ and $t_n$ respectively. Let $n \geq 2$. Let $\alpha_n = (\alpha_{n-1}, x) \in PB^b_n(\mathcal{G})_1$. Then $F^a_0(\alpha_n) = \alpha_{n-1}$, $t_n(\alpha_n) = (t_{n-1}(\alpha_{n-1}), x_n)$, $s_n(\alpha_n) = x_n$, $u(x) = id_x = (id_{(x_1, x_2, \ldots, x_{n-1}, x)}, x)$, $i(\alpha_n) = (\alpha_{n-1}, x)$.

$$(a). \text{We have } (s_{n-1} \circ F^a_1)(\alpha_n) = s_{n-1}(\alpha_{n-1}) = (x_1, x_2, \ldots, x_{n-1}), \text{ and } (F^a_0 \circ s_n)(\alpha_n) = F^a_0(x) = (x_1, x_2, \ldots, x_{n-1}).$$

$$(b). \text{ Note that, } (t_{n-1} \circ F^a_1)(\alpha_n) = t_{n-1}(\alpha_{n-1}). \text{ On the other hand } (F^a_0 \circ t_n)(\alpha_n) = F^a_0(t_{n-1}(\alpha_{n-1}), x_n) = t_{n-1}(\alpha_{n-1}).$$
(c). \((F^n \circ u)(x) = F^n(id_{x_1,x_2,..., x_n}) = F^n(id_{x_1,x_2,..., x_{n-1}}, x) = id_{x_1,x_2,..., x_{n-1}}\),
and \((u \circ F^n)(x) = u(x_1, x_2, ..., x_n) = \text{id}_{x_1,x_2,..., x_{n-1}}\).

(d). \((F^n_1 \circ i)(\alpha_n) = F^n_1(\alpha_{n-1}^{-1}, x) = \alpha_{n-1}^{-1}. \) Next, \((i \circ F^n_1)(\alpha_n) = i(\alpha_{n-1}) = \alpha_{n-1}^{-1}. \)

(e). Let \(\alpha_n' \in PB^n_1(G)\) such that \(t_n(\alpha_n') = s_n(\alpha_n), \) that is \((t_{n-1}(\alpha_{n-1}), x_n) = x'. \)

Then,
\[
(m \circ (F^n_1 \times F^n_1))(\alpha_n, \alpha_n') = m(\alpha_{n-1}, \alpha_n') = (\alpha_{n-2} \circ \alpha_{n-2}, (x_1, x_2, ..., x_{n-1}))
\]

and
\[
(F^n_1 \circ m)(\alpha_n, \alpha_n') = F^n_1(\alpha_{n-1}' \circ \alpha_{n-1}, x) = \alpha_{n-1}' \circ \alpha_{n-1}
\]

\[
= (\alpha_{n-2} \circ \alpha_{n-2}, (x_1, x_2, ..., x_{n-1})).
\]

This completes the proof that \(F^n\) is a homomorphism, that is, the proof of Lemma 2.7 is complete. As we mentioned before that the proof of Lemma 2.9 is similar.

\(\square\)

Example 3.6. If \(\mathcal{G}\) is an orbifold groupoid, then the configuration Lie groupoids \(PB^n_1(\mathcal{G})\) and \(PB^n_0(\mathcal{G})\) are also orbifold groupoids. But for the same orbifold groupoid \(\mathcal{G}\), the above two configuration Lie groupoids, although have the same object space, they define orbifold groupoid structures on different orbifolds. \(|PB^n_1(\mathcal{G})|\) is a larger space than \(|PB^n_0(\mathcal{G})|\). In fact, there is a homomorphism \(PB^n_1(\mathcal{G}) \rightarrow PB^n_0(\mathcal{G})\) which is identity on the object space and sends \((\alpha, (x_1, x_2), ..., (x_1, x_2, ..., x_n)) \in PB^n_1(\mathcal{G})\) to \((\alpha, \text{id}_{x_1}, \text{id}_{x_2}, ..., \text{id}_{x_n}) \in PB^n_0(\mathcal{G})\).

4. ORBIFOLD BRAID GROUPS

In this section we give a short introduction to orbifold braid groups. We also use a stretching technique and prove few basic results on the orbifold braid group of the orbifold \(\mathbb{C}(k, m; q)\), which are needed to prove Theorem 2.14.

We have already defined the pure orbifold braid group of \(n\) strings of \(\mathbb{C}(k, m; q)\), as the orbifold fundamental group of the configuration orbifold (Definition 2.16). Since the underlying space of \(\mathbb{C}(k, m; q)\) is an open subset of \(\mathbb{C}\), there is one more way one can define the (pure) orbifold braid group, which give the same result (see [2], p. 3). It is the pictorial way as in the classical braid group case.

The later pictorial definition is relevant for us. We describe it now from [2].

Recall that any element of the classical braid group \(\pi_1(B_n(\mathbb{C}))\) is identified with an equivalence class of a braid. An example of a braid is given in the first picture of Figure 1. Two braids are called equivalent, if one can be obtained from the other by moving the strings, fixing the end points, such that, in the process no two strings touch or cross each other. Juxtaposing one braid over another gives the group operation. And, the identity element is the braid which joins the vertex \(j\) to \(j\), for \(j = 1, 2, ..., n\), and no two strings entangle with each other, as in the second picture of Figure 1. See [3].
Consider the complex plane with only one cone point, that is \( \mathbb{C}(0, 1; q) \). The underlying topological space of \( \mathbb{C}(0, 1; q) \) is nothing but the complex plane. Therefore, \( B_n(\mathbb{C}(0, 1; q)) \) is an orbifold when we consider the orbifold structure of \( \mathbb{C}(0, 1; q) \), otherwise it is the classical braid space \( B_n(\mathbb{C}) \).

Therefore, although the fundamental group of the underlying topological space of \( B_n(\mathbb{C}(0, 1; q)) \) has the classical braid representation as above, the orbifold fundamental group of \( B_n(\mathbb{C}(0, 1; q)) \) needs a different treatment. We point out here a similar braid representation of the orbifold fundamental group of \( B_n(\mathbb{C}(0, 1; q)) \) from [2]. The pictures in Figure 2 above shows the case of one cone point \( x \). The thick line represents \( x \times I \). Here \( S = \mathbb{C}(0, 1; q) \).

Note that both the braids in Figure 2 represent the same element in \( \pi_1(B_n(\mathbb{C})) \) (in this case there is no thick line), but different in \( \pi_{orb}^1(B_n(\mathbb{C}(0, 1; q))) \), depending on the order of the cone point. We describe it below.

One has to define new relations among braids, respecting the cone points of the orbifold. We produce one situation to see how this is done. The second picture in Figure 3 represents part of a typical element.

Now, if a string in the braid wraps the thick line \( x \times I \), \( q \) times (that is, \( 2q \) crossings), then it is equal to the third picture. This is because, if a loop circles \( q \) times around the cone point \( x \), then the loop gives the trivial element in the orbifold fundamental group of \( \mathbb{C}(0, 1; q) \). Therefore, both braids represent the same element in the orbifold fundamental group of \( B_n(\mathbb{C}(0, 1; q)) \). Furthermore, if the string wraps the thick line, not in a multiple of \( q \) number of times, then it is not equal to the unwrapped braid. For more details see [2].

Let us now consider the orbifold \( \mathbb{C}(1, 0; q) \), that is the punctured complex plane. When there is a puncture \( p \), then the braids will have to satisfy a similar property, but in this case if any of the string wraps \( p \times I \) at least once, then the braid will have infinite order. See the first picture in Figure 3. One can also think of \( p \times I \) as a fixed string (see Remark 4.3).
We now consider the general case. Recall that \( \mathbb{C}(k, m; q) \) is the orbifold, whose underlying space is \( \mathbb{C} - \{p_1, p_2, \ldots, p_k\} \), with cone points at \( x_i \in \mathbb{C} - \{p_1, p_2, \ldots, p_k\} \) of order \( q_i \) for \( i = 1, 2, \ldots, m \). Then, we have the following exact sequence, since the quotient map \( PB_n(\mathbb{C}(k, m; q)) \to B_n(\mathbb{C}(k, m; q)) \) is an orbifold covering map, with \( S_n \) as the group of covering transformation.

\[
1 \longrightarrow \pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))) \longrightarrow \pi_1^{orb}(B_n(\mathbb{C}(k, m; q))) \longrightarrow S_n \longrightarrow 1.
\]

Therefore, the elements of \( \pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))) \) are braids, where the strings join \( j \) to \( j \) for \( j = 1, 2, \ldots, n \).

A typical element of \( \pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))) \) looks like \( A \) as in the following figure. Here, note that we have placed the punctures and cone points conveniently on the right hand side. This does not change the group, because the topology of the orbifold \( \mathbb{C}(k, m; q) \) is independent of the position of the punctures or the cone points. The group operation is again given by juxtaposing one orbifold braid onto another, and the identity element is also obvious.

\[\text{Figure 4: An element } A \text{ in } \pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))).\]

Now, we describe a set of generators for \( \pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))) \).

Recall from \([3]\), that \( \pi_1(B_n(\mathbb{C})) \) is generated by \( \sigma_i, \ i = 1, 2, \ldots, n - 1 \), and \( \pi_1(PB_n(\mathbb{C})) \) is generated by the braids \( B^{(n)}_{ij}, \ i < j \) as shown below. Here, the string from \( i \) to \( i \) is going below all the strings up to the string \( j - 1 \) to \( j - 1 \).

\[\text{Figure 5: The generator } \sigma_i \text{ of } \pi_1(B_n(\mathbb{C})) \text{ and } B^{(n)}_{ij} \text{ of } \pi_1(PB_n(\mathbb{C})).\]

A quick drawing shows the following.

\[
B^{(n)}_{ij} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-2} \sigma_{j-1}^{-1}.
\]

**Lemma 4.1.** A set of generators for \( \pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))) \) is given in Figures 6, 7, and 8.

\[\text{Figure 6: The generator } X_{ij}\]

\[\text{Figure 7: The generator } \sigma_{ij}.\]
Proof. Consider the classical pure braid group $\pi_1(PB_{n+k+m}(C))$ and its generators $B_{ij}^{(n+k+m)}$, $i = 1, 2, \cdots, n + k + m - 1$; $j = 2, \cdots, n + k + m, i < j$, as recalled above. Let $G$ be the subgroup of $\pi_1(PB_{n+k+m}(C))$ generated by $B_{ij}^{(n+k+m)}$, $i = 1, 2, \cdots, n$; $j = 2, \cdots, n + k + m$; $i < j$. Clearly, any braid $B$ representing an element of $G$ is equivalent to a braid, whose all the last $k + m$ strings are vertically straight and not entangling with each other.

Now, we replace these last $k + m$ straight strings by dotted lines and denote the corresponding braid by $\bar{B}$. That is, we introduced $k + m$ punctures in $C$. Then, clearly we have an isomorphism from $G$ onto $\pi_1(PB_n(C(k + m, 0; q)))$, sending $B$ to $\bar{B}$ (see [[15], page 26]). Since, the compositions in $G$ and in $\pi_1(PB_n(C(k + m, 0; q)))$ are same, that is, juxtaposition of braids. Furthermore, considering the group operations in $G$ and in $\pi_1(PB_n(C(k + m, 0; q)))$, there is no difference between a dotted line and a straight string.

Hence, the relations in a presentation of $G$ with respect to the above set of generators, and the relations in a presentation of $\pi_1(PB_n(C(k + m, 0; q)))$, in terms of the generators $\bar{B}_{ij}^{(n+k+m)}$, $i = 1, 2, \cdots, n$; $j = 2, \cdots, n + k + m$; $i < j$ are identical, except with a bar.

Next, we replace the last $m$ dotted lines in $\bar{B}$, by thick lines, and denote it by $\bar{\bar{B}}$. That is, we introduced the last $m$ punctures by orbifold points of order $q_i$, $i = 1, 2, \cdots, m$. Then, the element $\bar{B}_{ij}^{(n+k+m)}$, $j = n + k + 1, \cdots, n + k + m$ has order $q_j - n - k$, as we described the orbifold braid groups before, from [2]. Hence, we get back the group $\pi_1(PB_n(C(k, m; q)))$, and its generators are as described in the statement of the Lemma. Since, clearly any element of $\pi_1(PB_n(C(k, m; q)))$ is of the form $B$, for some $B \in G$. □

Remark 4.2. We denote by

$$O_n : G \simeq \pi_1(PB_n(C(k + m, 0; q))) \rightarrow \pi_1^{orb}(PB_n(C(k, m; q))),$$

the surjective homomorphism sending $B$ to $\bar{\bar{B}}$.

Remark 4.3. From the above proof we also get that the pure orbifold braid group of $n$ strings of $C(k, m; q)$, can be embedded into the pure orbifold braid group of $n + k$ strings of $C(0, m; q)$. For $k = m = 1$, this embedding was proved in [[24], Proposition 4.1].

Now we come to the crucial lemma, which is the main ingredient for this paper.

Lemma 4.4. Let $A \in \pi_1^{orb}(PB_n(C(k, m; q)))$ satisfying the following property.

- $A$ is equivalent to a braid which has all the first $n - 1$ strings not entangling with each other, they are vertically straight and only the string from $n$ to $n$ (say $N$) is entangling with some (or all) of the first $n - 1$ strings or the dotted or the thick lines.

Then, $A$ is equivalent to a juxtaposition of the following braids or their inverses.
Remark 4.5. Clearly, the braid $A$ in Lemma 4.4 lies in the kernel of the homomorphism $\Delta : \pi_{1}^{orb}(PB_{n}(C(k,m; q))) \to \pi_{1}^{orb}(PB_{n-1}(C(k,m; q)))$, where $\Delta$ sends a braid of $n$ strings to the braid of $n-1$ strings, after removing the string from $n$ to $n$. We will be using the lemma in cases where $\phi$ is easily seen by some simple movements of a strings.

Proof of Lemma 4.4. The proof is basically an application of Lemma 4.1 and Remark 4.3.

First consider the punctured complex plane $C(k + m + n - 1,0;0)$. Then by Lemma 4.1 the free group

$$\pi_{1}(C(k + m + n - 1,0;0)) = \pi_{1}(PB_{1}(C(k + m + n - 1,0;0)))$$

is generated by the braids of one string as in the first picture of Figure 9. Since positions of the punctures do not affect the fundamental group, we move the first $n-1$ dotted lines to the left of the string as in the second picture in Figure 9. The generators which wraps these first $n-1$ dotted lines also are similarly drawn using allowable moves.

$$\pi_{1}(C(k + m + n - 1,0;0)) \to \pi_{1}(PB_{n}(C(k,m; q)))$$

by replacing the first $n-1$ dotted lines with straight strings. Next, by replacing the last $m$ dotted lines with thick lines, that is, by filling the last $m$ punctures with cone points of order $q_1, q_2, \ldots, q_m$, we get a homomorphism

$$Q : \pi_{1}(PB_{1}(C(k + m + n - 1,0;0))) \to \pi_{1}^{orb}(PB_{n}(C(k,m; q)))$$

Note that, the image of this homomorphism consists of braids exactly of the type $A$, satisfying $\phi$. Choose $\tilde{A} \in Q^{-1}(A)$. Then $\tilde{A}$ is a juxtaposition of generators of the type as in Figure 9 and their inverses. Now, take the image of these generators which appear in a decomposition of $\tilde{A}$, under $Q$, to get the desired braids as in the statement of the lemma, to construct $A$.

This completes the proof of Lemma 4.4.

Next, we apply Lemma 4.4 to prove the following proposition, crucial for this paper.

Proposition 4.6. The subgroup $H$ generated by the following set of braids is normal in $\pi_{1}^{orb}(PB_{n}(C(k,m; q)))$.

$$N = \{X_{nr}, r = 1,2,\ldots,m; P_{ns}, s = 1,2,\ldots,k; B_{in}, i = 1,2,\ldots,n-1\}.$$
Proof. Recall that, $\pi_1^{orb}(PB_n(C(k, m; q)))$ is generated by the braids as described in Lemma 4.1. Let us denote this set of braids by $\mathcal{P}$.

Therefore, it is enough to prove that $ZYZ^{-1} \in H$, for all $Z \in \mathcal{P}$ and $Y \in \mathcal{N}$. That is, we need to show that $ZYZ^{-1}$ is a juxtaposition of elements from $\mathcal{N}$ or their inverses. We only have to consider the following nine cases.

\begin{align*}
&1. X_{kl}X_{nr}X_{kl}^{-1}, \quad 2. X_{kl}P_{ns}X_{kl}^{-1}, \quad 3. X_{kl}B_{in}X_{kl}^{-1}, \\
&4. P_{kl}X_{nr}P_{kl}^{-1}, \quad 5. P_{kl}P_{ns}P_{kl}^{-1}, \quad 6. P_{kl}B_{in}P_{kl}^{-1}, \\
&7. B_{lk}X_{nr}B_{lk}^{-1}, \quad 8. B_{lk}P_{ns}B_{lk}^{-1}, \quad 9. B_{lk}B_{in}B_{lk}^{-1}.
\end{align*}

Therefore, it is enough to show that all the above nine cases satisfy the condition $\diamond$ of Lemma 4.4. As there are only three braids in each case, we draw their picture and observe that condition $\diamond$ is satisfied. We recall that, condition $\diamond$ says that the braid under study is equivalent to a braid, where all the first $n - 1$ strings are vertically straight. The method we use is simply stretch the string $N$, so that the other strings (maximum three, depending on the case) become straight.

After this pictorial proof, we will observe a general pattern in the stretching method. Then we will give an argument using isotopy, which will be applicable in all the nine cases.

We can partition the above nine cases as follows, so that the cases in each partition class need similar treatment.

\[
A = \{(1), (2), (4), (5)\}, \quad B = \{(3), (6)\}, \quad C = \{(7), (8)\}, \quad D = \{(9)\}.
\]

Therefore, we give the proof for one case from each class, and then describe the modifications required in the proof to prove the other cases.

First, we note that for $k = n$, there is nothing to prove in all the nine cases. So, we can assume that $k \neq n$.  

**Class A.** We give the proof for the Case (1). There are three possibilities to consider; $l = r$, $l < r$ and $l > r$.

This deduction is shown in Figure 10.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Case (1), $X_{kl}X_{nr}X_{kl}^{-1}$ in Class A.}
\end{figure}
Now, we describe the other cases in Class A. Case (5) is exactly the same as Case (1), we only have to replace the thick lines by dotted ones in Figure 10. Next, note that in our presentation of the braids, the dotted lines appear before the thick lines, and hence, for Case (2), only the third column in Figure 10 is needed with the first thick line (corresponding to $r$) converted into a dotted line. Similarly, for Case (4), the middle column in Figure 10 gives the proof, with the first thick line (corresponding to $l$) converted into a dotted line.

**Class B.** We consider the Case (3) first. In this case also we have to consider three possibilities: $i = k$, $i < k$ and $i > k$. This stretching procedure is shown in Figure 11.

The treatment for Case (6) in Class B is exactly the same as Case (3) as in Figure 11, but we have to replace the thick lines by dotted ones.

**Class C.** Cases (7) and (8) are the simplest of all the cases. Case (7) is shown below in Figure 12. Case (8) needs the same treatment as Case (7), with the thick line replaced by a dotted one.

**Class D.** Case (9) has several possibilities depending on where $i$ lies: $k < i$, $i < l$, $i = k$, $i = l$ and $l < i < k$. The details are shown in Figure 13.

One point to note that, for clarity, in the pictures we did not show the other strings, the dotted lines or the thick lines. Since, the movements of the strings are taking place under all the other strings, the dotted lines or the thick lines.

Now, we make a general argument which is applicable to all the above cases.
An observation and the general argument. Let $B \in \ker(\Delta)$. First note that, then there is an isotopy of $C(k+m,0;0)$ which sends $\Delta(B)$ to the identity braid of $PB_{n-1}(C(k,m;q))$. But some string of $\Delta(B)$ might have had entanglement with the thick lines, where the movement as in Figure 3 were used to get the triviality.

But in our nine cases above, it is immediate that if we remove the string $N$ from any of the orbifold braids, then the other strings (maximum number is three) become straight after an isotopy. That is, the strings other than $N$, are not entangled with any of the thick lines, that is why we did not need to invoke Figure 3 movement here, which might have complicated matter during the isotopy. Therefore, all that
we need to do, when \( B \) is one of the nine orbifold braids, is that, we choose an isotopy \( T \) to straighten up the other strings in \( B \), and during the isotopy we do not interfere with any of the other strings except \( N \). Now, during the isotopy the string \( N \) might have intersected with the other strings. Note that, we can make sure that there are only finitely many such intersections. At each intersection point we make one of the two following local isotopies. During the isotopy \( T \), if the string \( N \) intersected any other string, say \( N' \), then there are two possibilities. Just before \( N \) intersected \( N' \), (a) \( N \) was above \( N' \) or (b) \( N \) was below \( N' \). In case (a), we make the simple local isotopy by moving \( N \) away from \( N' \) in the opposite direction, so that there is no intersection in the neighbourhood of the intersection point. Similarly in case (b) we make a similar local isotopy. The local isotopies are shown in Figure 14. The idea is not to change the equivalence class of \( B \) during the local isotopies. Hence, \( T \) together with these two local isotopies, give the isotopy which sends the orbifold braid \( B \) to an orbifold braid, satisfying \( \circ \) of Lemma 4.4.

Therefore, the subgroup \( H \) generated by the set of braids \( N \), is a normal subgroup of \( \pi_1^{orb}(PB_n(\mathbb{C}(k, m; q))) \).

This completes the proof of the proposition. \( \square \)

We now have an immediate consequence of Proposition 4.6 when restricted to the case \( C \). This gives another proof of the fact that the kernel of \( f^* \) is a free group, which is a consequence of the Fadell-Neuwirth fibration theorem. But our proof also produce an explicit set of generators of this free group as a subgroup of the classical pure braid group.

**Corollary 4.7.** The homomorphism \( f^* : \pi_1(PB_n(\mathbb{C})) \to \pi_1(PB_{n-1}(\mathbb{C})) \), induced by the projection to the first \( n-1 \) coordinates, \( f : PB_n(\mathbb{C}) \to PB_{n-1}(\mathbb{C}) \), has kernel freely generated by the elements \( B_{in}^{(n)} \) for \( i = 1, 2, \cdots, n-1 \).

**Proof.** From Case (9) in the proof of Proposition 4.6 and Lemma 4.4, the corollary follows. We need to specialize to the case \( C \) only, that is, put \( k = m = 0 \). Then note that the group generated by \( B_{in}^{(n)} \) for \( i = 1, 2, \cdots, n-1 \) is isomorphic to the fundamental group of \( PB_1(\mathbb{C}(n-1, 0; 0)) = \mathbb{C}(n-1, 0; 0) \), which is free. \( \square \)

5. Proofs

We start this section with the proof of the Fadell-Neuwirth type fibration theorem for c-groupoids.

We need the following lemma.

**Lemma 5.1.** Let \( f : M \to N \) and \( g : N \to L \) be surjective maps between manifolds. If \( g \) is a covering map and \( g \circ f \) is a fibration, then \( f \) is also a fibration.

**Proof.** By [27], p. 95-96, Theorems 12, 13], it is enough to prove that \( f \) is a local fibration, that is, \( N \) has a numerable covering by open sets \( \{U_i\}_{i \in I} \), so that \( f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i \) is a fibration for all \( i \in I \).

Since \( N \) is paracompact, every open covering has a numerable refinement. See [27], p. 95-96. Therefore, since \( g \) is a covering map, there is a numerable open covering \( \{U_i\}_{i \in I} \), such that for each \( i \in I \), \( g(U_i) \) is connected, evenly covered by \( g \) and \( g|_{U_i} : U_i \to g(U_i) \) is a diffeomorphism.

Now, showing that \( f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i \) is a fibration is straight forward. \( \square \)
Proof of Theorem 2.10. Let $\mathcal{G}$ be a c-groupoid of dimension $\geq 2$. That is, the quotient map $f : \mathcal{G}_0 \to |\mathcal{G}|$ is a covering map, and hence $|\mathcal{G}|$ is again a manifold of dimension $\geq 2$. We have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{G}_0^n & \longrightarrow & \mathcal{G}_0^{n-1} \\
|\mathcal{G}|^n & \downarrow & |\mathcal{G}|^{n-1} \\
\end{array}
$$

Here, the horizontal maps are projections and the vertical maps are covering maps. Note that, the diagram induces the following.

By the Fadell-Neuwirth fibration theorem, since $|\mathcal{G}|$ is a manifold of dimension $\geq 2$, the lower horizontal map is a locally trivial fibration. Since the two vertical maps are covering maps, the top horizontal map is a fibration of manifolds by Lemma 5.1.

Now, we check the other conditions of a- and b-fibrations.

**F^a.** To prove that $F^a$ is an a-fibration, we only have to show that $PB_n(\mathcal{G})_0$ is a (left) $PB_{n-1}(\mathcal{G})$-space, with the structure maps induced from the structure maps of $PB_{n-1}(\mathcal{G})$.

We recall the target map from Definition 2.6,

$$
\mu : PB_n^a(\mathcal{G})_1 = PB_{n-1}^a(\mathcal{G})_1 \times_{PB_{n-1}(\mathcal{G})_0} PB_n(\mathcal{G})_0 \to PB_n(\mathcal{G})_0
$$

of the Lie groupoid $PB_n^a(\mathcal{G})$. This is defined as follows.

Let $(x_1, x_2, \ldots , x_n) \in PB_n(\mathcal{G})_0$ and

$$(\alpha_1, (x_1, x_2), \ldots , (x_1, x_2, \ldots , x_{n-1})) \in PB_{n-1}^a(\mathcal{G})_1.$$

Note that $s(\alpha_1) = x_1$, then define

$$
\mu((\alpha_1, (x_1, x_2), \ldots , (x_1, x_2, \ldots , x_{n-1})), (x_1, x_2, \ldots , x_n)) = (t(\alpha_1), x_2, \ldots , x_{n-1}, x_n).
$$

The action map is nothing but the target map of $PB_n^a(\mathcal{G})$.

The remaining properties (see Definition 2.3) are easy to check.

Therefore, the homomorphism $F^a : PB_n^a(\mathcal{G}) \to PB_{n-1}^a(\mathcal{G})$ is an a-fibration of Lie groupoids.

**F^b.** We now prove that the homomorphism $F^b$ is a b-fibration. For this we only have to check that the map

$$
F^b : PB_n^b(\mathcal{G})_1 \to PB_{n-1}^b(\mathcal{G})_1 \times_{PB_{n-1}(\mathcal{G})_0} PB_n(\mathcal{G})_0
$$

is a surjective submersion. But this follows from the following observations about the different maps involved in the above display.

(i) Let $(\alpha_1, \alpha_2, \ldots , \alpha_n) \in PB_n^b(\mathcal{G})_1$. Then, this element under the above map goes to

$$
((\alpha_1, \alpha_2, \ldots , \alpha_{n-1}), (s(\alpha_1), s(\alpha_2), \ldots , s(\alpha_n))).
$$

Hence, the map $F^b$ is obviously surjective, since $s(x_n) = x_n$, and we only need that the last coordinate in the range has a preimage in $\mathcal{G}_1$ under $s$. 

(ii) Next, note that the spaces involved are open sets in the product space \( G_0^n \) or \( G_1^n \) and the maps are either projections or the map \( s \) at the coordinate level. Furthermore, \( s \) is a submersion by assumption. Therefore, the map \( f^b \) is a submersion.

This completes the proof of the theorem. \( \square \)

**Remark 5.2.** A statement similar to Theorem 2.10 also holds by taking projection to the first \( k \) coordinates, from \( PB_n(G) \) to \( PB_k(G) \), and by suitably modifying Lemmas 2.7 and 2.9.

We now prove Proposition 2.11, which gives the counter examples to the Fadell-Neuwirth fibration theorem, in the context of Lie groupoids. That is, we show that Theorem 2.10 is not true in general.

**Proof of Proposition 2.11.** Recall that \( S = M/H \), where \( H \) is a finite group acting effectively on a connected manifold \( M \) of dimension \( \geq 2 \).

By definition the object space of the Lie groupoid \( PB_n^*(G(M, H)) \) is as follows. Here \( \ast = a \) or \( b \).

\[
PB_n^*(G(M, H)) = \{(x_1, x_2, \ldots, x_n) \in M^n \mid Hx_i \neq Hx_j, \text{ for } i \neq j \}.
\]

Next, consider the projection \( f : PB_n(G(M, H))_0 \rightarrow PB_{n-1}(G(M, H))_0 \) to the first \( n-1 \) coordinates. Note that \( f = F_0^b = F_0^a \). We will show that \( f \) is not a fibration.

Since \( S \) has at least one singular point, there is a point \( s \in M \), so that the isotropy group \( H_s = \{ h \in H \mid hs = s \} \) is nontrivial. There is a neighbourhood \( U_s \subset M \) of \( s \) preserved by \( H_s \) and \( hU_s \cap U_s = \emptyset \) for all \( h \in H \setminus H_s \). Such a neighbourhood exists, see the proof of \([29], \text{Proposition 5.2.6}\). Since regular points are dense in \( S \), there is a point \( s' \in U_s \) which has trivial isotropy group. That is, \( s' \) corresponds to a regular point and \( s \) corresponds to a singular point on \( S \).

Choose a point \( p = (s, x_2, \ldots, x_{n-1}) \in PB_{n-1}(G(M, H))_0 \), such that \( Hx_i \neq Hs' \) for all \( i = 2, 3, \ldots, n-1 \). Let \( p' = (s', x_2, \ldots, x_{n-1}) \). Note that \( |Hs'| < |Hs| \). Then it follows that \( f^{-1}(p) \) and \( f^{-1}(p') \) are obtained from \( M \), by removing different number of points. Hence, they are not homotopy equivalent. Therefore, \( f \) is not a fibration.

In fact, we have proven that \( f \) is not even a quasifibration. Since a quasifibration with a path connected base space demands that any two fibers should have isomorphic homotopy groups.

This proves Proposition 2.11. \( \square \)

Next, we give the proof of the short exact sequence of orbifold fundamental groups of the configuration orbifold of the orbifold \( \mathbb{C}(k, m; q) \). This proof is the crucial application of the explicit set of generators of the pure orbifold braid group, we constructed it in Lemma 4.1.

**Proof of Theorem 2.14.** By Remark 2.15, we have to prove the exactness of the following sequence.

\[
\begin{array}{ccc}
1 & \longrightarrow & K \\
\downarrow & & \Delta \downarrow \\
\pi_1^{orb}(PB_n(S)) & \longrightarrow & \Delta \pi_1^{orb}(PB_{n-1}(S)) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \Delta \pi_1^{orb}(F)
\end{array}
\]

Where, \( S = \mathbb{C}(k, m; q) \) and \( F = S - \{(n-1)\text{-regular points}\} \) and \( K \) is isomorphic to \( \pi_1^{orb}(F) \).

First, let us recall that, when \( S = \mathbb{C} \) then the above exact sequence follows from Corollary 4.7.
The map $\pi_1(PB_n(\mathbb{C})) \to \pi_1(PB_{n-1}(\mathbb{C}))$ is obtained by removing the last string in a braid, representing an element of $\pi_1(PB_n(\mathbb{C}))$. This is the main idea to understand the homomorphism $\Delta$ in terms of braid pictures, and to prove the above exact sequence.

Now, following the braid presentation of elements of $\pi_1^{orb}(B_n(\mathbb{C}(k, m; q)))$ as was done in Section 4, we know that a typical element of $\pi_1^{orb}(PB_n(\mathbb{C}(k, m; q)))$ looks like $A$ as in Figure 4.

So, we send $A$ to the braid $\Delta(A) \in \pi_1^{orb}(PB_{n-1}(\mathbb{C}(k, m; q)))$ after removing the last string from $n$ to $n$, as is shown in Figure 15.

![Figure 15: The braid $\Delta(A)$ in $\pi_1^{orb}(PB_{n-1}(\mathbb{C}(k, m; q)))$.](image)

Clearly, $\Delta$ is a surjective homomorphism, as the law of composition is juxtaposition of braids. This also follows from the fact that, by adding the extra relations $X_{nr}, r = 1, 2, \ldots, m; P_{ns}, s = 1, 2, \ldots, k$ and $B_{in}, i = 1, 2, \ldots, n-1$ in the presentation of $\pi_1^{orb}(PB_n(\mathbb{C}(k, m; q)))$, we get the presentation of $\pi_1^{orb}(PB_{n-1}(\mathbb{C}(k, m; q)))$.

Since, any element of $\pi_1^{orb}(PB_n(\mathbb{C}(k, m; q)))$ is a juxtaposition of the generators as in Lemma 4.1, therefore, removing the string from $n$ to $n$ is equivalent to attaching the extra relations as above.

Recall that, we denoted the above set of braids by $\mathcal{N}$ in Proposition 4.6. A splitting map is defined by sending $\Delta(A)$ to an element $A$, with the string from $n$ to $n$ not entangling with any other strings, the dotted or the thick lines and going over all of them, as shown in Figure 16. It is again easy to see that this splitting is a homomorphism.

![Figure 16: The braid $\tilde{A}$ in $\pi_1^{orb}(PB_n(\mathbb{C}(k, m; q)))$.](image)

Clearly, the braids in the set $\mathcal{N}$ belong to the kernel of $\Delta$.

Let $\Delta(A)$ be the trivial element in $\pi_1^{orb}(PB_{n-1}(\mathbb{C}(k, m; q)))$.

We want to prove that, $A$ is equivalent to a juxtaposition of finitely many members of $\mathcal{N}$ or its inverses. Equivalently, we have to show that the braids in $\mathcal{N}$ generate a subgroup, which is normal in $\pi_1^{orb}(PB_n(\mathbb{C}(k, m; q)))$. Hence, we apply Proposition 4.6, to see that the kernel of $\Delta$ is generated by $\mathcal{N}$.

Now, recall that $X_{nr}$ is of order $q_r$, and the remaining two sets of generators are of infinite order.
We will now show that the elements in $\mathcal{N}$ freely generate the kernel of $\Delta$, except the relation, $X_{nt}^{gr}, r = 1, 2, \ldots, m$. That is, the kernel of $\Delta$ is isomorphic to

$$C_{q_1} \ast C_{q_2} \ast \cdots \ast C_{q_m} \ast C \ast \cdots \ast C \simeq \pi_1^{orb}(F).$$

Here, $C_{q_i}$ is the cyclic group of order $q_i$, $C$ is infinite cyclic and there are $k + n - 1$ number of factors of $C$ in the above display.

First, recall that by Corollary 4.7, the kernel of $f_* : PB_n(C) \to PB_{n-1}(C)$ is a free group on $n - 1$ generators, and it is generated by $B_{in}^{(n)}$ for $i = 1, 2, \ldots, n - 1$.

Now, we replace the dotted and the thick lines in the generators $X_{nt}, P_{ns}$ and $B_{in}$ of the kernel of $\Delta$ by straight strings, number the strings and denote them by

$$\tilde{X}_{nt}, r = 1, 2, \ldots, m; \quad \tilde{P}_{ns}, s = 1, 2, \ldots, k; \quad \tilde{B}_{in}, i = 1, 2, \ldots, n - 1.$$

These braids look as follows.

![Figure 17: $\tilde{X}_{nt}$](image1)

![Figure 18: $\tilde{P}_{ns}$](image2)

![Figure 19: $\tilde{B}_{in}$](image3)

Then, $\tilde{\mathcal{N}} = \{\tilde{X}_{nt}, \tilde{P}_{ns}, \tilde{B}_{in}\}$ generates a free group, since it is easily identified (by permuting the coordinates using the transposition $(n, n + k + m)$) with the kernel of

$$\pi_1(PB_{m+k+n}(C)) \to \pi_1(PB_{m+k+n-1}(C)),$$

generated by

$$\{B_{in}^{(n+k+m)}, B_{nj}^{(n+k+m)}, \text{ for } j = n + 1, \ldots, n + k + m; \text{ and } i < n\},$$

and we know that the above kernel is a free group by Corollary 4.7.

Next, we put back the dotted and the thick lines (using the map $O_{n+k+m}$ in Remark 4.2), then, it is clear from Section 4, that we only have to put the relations $X_{nt}^{gr}, r = 1, 2, \ldots, m$ to get a complete set of relations in the presentation of the kernel of $\Delta$. Since, there is no relations in a presentation of the subgroup generated by $\tilde{\mathcal{N}}$ in $\pi_1(PB_{m+k+n}(C))$, we get that after removing the $\tilde{\gamma}$, there is no relations in a presentation of the subgroup generated by $\mathcal{N}$ in $\pi_1^{orb}(PB_n(C(k, m; q)))$, except the above finite order relations. This is because $O_{n+k+m}$ is a homomorphism. Therefore, the kernel of $\Delta$ is isomorphic to $\pi_1^{orb}(F)$.

This completes the proof of the theorem.
Now, we are in a position to give the proof of Theorem 2.19, giving virtual poly-$\mathcal{F}$ structure on some affine and finite complex Artin groups.

To prove the theorem, furthermore, we need the following two results.

**Theorem 5.3.** ([21]) All affine type Artin groups are torsion free.

**Theorem 5.4.** ([2]) Let $A$ be an Artin group, and $O$ be an orbifold as described in the following table. Then, $A$ can be embedded as a normal subgroup in $\pi_{orb}^1(B_n(O))$. The third column gives the quotient group $\pi_{orb}^1(B_n(O))/A$.

| Artin group of type $\tilde{A}$ | Orbifold $O$ | Quotient group $n$ |
|-------------------------------|-------------|------------------|
| $B_n$                         | $C(1,0)$    | $<1>$            |
| $\tilde{A}_{n-1}$            | $C(1,0)$    | $\mathbb{Z}$     |
| $\tilde{B}_n$                | $C(1,1; (2))$ | $\mathbb{Z}/2$  |
| $\tilde{C}_n$                | $C(2,0)$    | $<1>$            |
| $\tilde{D}_n$                | $C(0,2; (2,2))$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |

Table: Embedding Artin groups in orbifold braid groups

We also need the following.

**Proposition 5.5.** If a torsion free, finitely presented poly-$\mathcal{V}\mathcal{F}$ group has a normal series with finitely presented subgroups, then the group is virtually poly-$\mathcal{F}$.

**Proof.** Let $H$ be a poly-$\mathcal{V}\mathcal{F}$ group of length $n$ satisfying the hypothesis of the statement. The proof is by induction on $n$. If $n = 1$, then $H$ is virtually free, and hence free, since it is torsion free ([28]). Therefore, assume that the lemma is true for all poly-$\mathcal{V}\mathcal{F}$ groups of length $\leq n - 1$ satisfying the hypothesis. Consider a finitely presented normal series for $H$ giving the poly-$\mathcal{V}\mathcal{F}$ structure. Then, $H_{n-1}$ is finitely presented, torsion free, and has a poly-$\mathcal{V}\mathcal{F}$ structure of length $n - 1$. Hence, by the induction hypothesis, there is a finite index subgroup $K \leq H_{n-1}$ and $K$ is poly-$\mathcal{F}$. Since $H_{n-1}$ is finitely presented, we can find a finite index subgroup $K'$ of $K$ which is also a characteristic subgroup of $H_{n-1}$. Hence, $K'$ is a poly-$\mathcal{F}$ group, and is a normal subgroup of $H = H_n$ with quotient virtually free. Let $q : H \to H/K'$ be the quotient map. Choose a free subgroup $L$ of $H/K'$ of finite index, then $q^{-1}(L)$ is a finite index poly-$\mathcal{F}$ subgroup of $H$.

This proves the Proposition.

Now, we are ready to prove Theorem 2.19.

**Proof of Theorem 2.19.** From the Table in Theorem 5.4, we see that the Artin group of type $\tilde{A}_n$ is a subgroup of the finite type Artin group of type $B_n$. Hence, by [8], the Artin group of type $\tilde{A}_n$ is virtually poly-$\mathcal{F}$.

Now, let $A$ be an Artin group of type $B_n$, $C_n$ or $D_n$. Then, by Theorem 5.4, $A$ can be embedded as a normal subgroup in $\pi_{orb}^1(B_n(C(k,m;q)))$ of finite index, for some suitable $k$, $m$, $q$ and $n$. We know by Corollary 2.18 that, $\pi_{orb}^1(PB_n(C(k,m;q)))$ is poly-$\mathcal{V}\mathcal{F}$ by a normal series consisting of finitely presented subgroups. Therefore, the same is true for $\pi_{orb}^1(B_n(C(k,m;q)))$, since the pure orbifold braid group is
embedded as a finite index normal subgroup in the orbifold braid group. Next, since \( \mathcal{A} \) is a subgroup of finite index in \( \pi_1^{orb}(B_n(\mathbb{C}(k,m;q))) \), it follows that \( \mathcal{A} \) is also poly-\( F \), by a normal series consisting of finitely presented subgroups. But by Theorem 5.3, \( \mathcal{A} \) is also torsion free. Hence, by Proposition 5.5 \( \mathcal{A} \) is virtually poly-\( F \).

The \( G(\text{de}, e, r) \) \((d, r \geq 2)\) type case is easily deduced from the fact that, this Artin group can be embedded as a subgroup in the finite type Artin group of type \( B_r \). See [[11], Proposition 4.1].

Therefore, we have completed the proof of Theorem 2.19. □

We now give the proof of Theorem 2.20, that is, the proof of the isomorphism conjecture for the orbifold braid group of the orbifold \( \mathbb{C}(k,m;q) \).

**Proof of Theorem 2.20.** We start with the following definition of a class of groups, which contains the class of groups \( \mathcal{C} \) defined in [[24], Definition 3.1].

**Definition 5.6.** Let \( \mathcal{D} \) denote the smallest class of groups satisfying the following conditions.

1. The fundamental group of any connected manifold of dimension \( \leq 3 \) belongs to \( \mathcal{D} \).
2. If \( H \) is a subgroup of a group \( G \), then \( G \in \mathcal{D} \) implies \( H \in \mathcal{D} \). This reverse implication is also true if \( H \) is of finite index in \( G \).
3. If \( G_1, G_2 \in \mathcal{D} \) then \( G_1 \times G_2 \in \mathcal{D} \).
4. If \( \{ G_i \}_{i \in I} \) is a directed system of groups and \( G_i \in \mathcal{D} \) for each \( i \in I \), then the \( \lim_{i \in I} G_i \in \mathcal{D} \).
5. Let

\[
1 \longrightarrow K \longrightarrow G \xrightarrow{p} H \longrightarrow 1
\]

be a short exact sequence of groups. If \( K, H \) and \( p^{-1}(C) \), for any infinite cyclic subgroup \( C \) of \( H \), belong to \( \mathcal{D} \) then \( G \) also belongs to \( \mathcal{D} \).

6. If a group \( G \) has a normal subgroup \( H \), so that \( H \) is free and \( G/H \) is infinite cyclic, then \( G \) belongs to \( \mathcal{D} \).

Note that, [[24], Definition 3.1] did not have the condition 6. In [[24], Theorem 3.3] we noted that the Farrell-Jones isomorphism conjecture with coefficients, and finite wreath product is true for any group belonging to the class \( \mathcal{C} \). Two recent papers ([5] and [9]) help us to conclude that the conjecture is true for any member of \( \mathcal{D} \). In 6, when \( H \) is finitely generated, then the conjecture is proved in [5], and it is generalized for arbitrary free group case in [9].

We will now prove that, the orbifold braid groups of the orbifold \( \mathbb{C}(k,m;q) \) belong to \( \mathcal{D} \).

Recall the following exact sequence from Remark 2.15. Here, \( S \) is the orbifold \( \mathbb{C}(k,m;q) \).

\[
1 \longrightarrow \pi_1^{orb}(F) \longrightarrow \pi_1^{orb}(PB_n(S)) \longrightarrow \pi_1^{orb}(PB_{n-1}(S)) \longrightarrow 1.
\]

Where \( F = S - \{(n - 1) - \text{regular points}\} \).

Note that, \( \pi_1^{orb}(B_n(\mathbb{C}(k,m;q))) \) contains \( \pi_1^{orb}(PB_n(\mathbb{C}(k,m;q))) \) as a subgroup of finite index, and hence, by 2, it is enough to prove that \( \pi_1^{orb}(PB_n(\mathbb{C}(k,m;q))) \in \mathcal{D} \).

The proof is by induction on \( n \). Note that, for \( n = 1, \pi_1^{orb}(PB_n(\mathbb{C}(k,m;q))) \simeq \pi_1^{orb}(\mathbb{C}(k,m;q)) \), which is virtually finitely generated free (see Corollary 2.18), and hence by 1 and 2, \( \pi_1^{orb}(\mathbb{C}(k,m;q)) \in \mathcal{D} \). By the same argument \( \pi_1^{orb}(F) \in \mathcal{D} \). We would like to use 5 now. Therefore, we assume \( \pi_1^{orb}(PB_{n-1}(\mathbb{C}(k,m;q))) \in \mathcal{D} \) and let \( C \) be an infinite cyclic subgroup of \( \pi_1^{orb}(PB_{n-1}(\mathbb{C}(k,m;q))) \). The inverse image
of $C$ under the surjective homomorphism in the above display is an extension of \( \pi_{1}^{\text{orb}}(F) \) by $C$, that is, isomorphic to the semi-direct product $\pi_{1}^{\text{orb}}(F) \rtimes C$. Since, $\pi_{1}^{\text{orb}}(F)$ has a finitely generated free subgroup of finite index, it is easy to deduce that it has a finitely generated free characteristic subgroup $K$ (say) of finite index. Therefore, the action of $C$ on $\pi_{1}^{\text{orb}}(F)$ preserves $K$ and hence, $K \rtimes C$ is a subgroup of $\pi_{1}^{\text{orb}}(F) \rtimes C$ of finite index. Therefore, $\pi_{1}^{\text{orb}}(F) \rtimes C \in D$ using 6 and 2. Hence, $\pi_{1}^{\text{orb}}(\mathcal{P}B_{n}(C(k, m; q))) \in D$ by 5.

Now, we see using the Table in Theorem 5.4, that the Artin group of type $\tilde{D}_{n}$ is a subgroup of the orbifold braid group of $C(0, 2; (2, 2))$. Therefore, the Artin group of type $\tilde{D}_{n}$ belongs to $D$, using 2.

This completes the proof of the theorem. \qed
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