Projective Geometric Algebra as a Subalgebra of Conformal Geometric algebra

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Communicated by Dietmar Hildenbrand

Abstract. We show that if Projective Geometric Algebra (PGA), i.e. the geometric algebra with degenerate signature \((n, 0, 1)\), is understood as a subalgebra of Conformal Geometric Algebra (CGA) in a mathematically correct sense, then flat primitives share the same representation in PGA and CGA. Particularly, we treat duality in PGA in the framework of CGA. This leads to unification of PGA and CGA primitives which is important especially for software implementation and symbolic calculations.

Mathematics Subject Classification. Primary 15A66, Secondary 51N25.

Keywords. Conformal geometric algebra, Projective geometric algebra, Euclidean geometry.

1. Introduction

Projective geometric algebra (PGA) is a model for Euclidean geometry and computations with flat primitives. We use “PGA” to refer to geometric algebras with degenerate signature \((n, 0, 1)\), in particular it covers both Euclidean PGA and dual Euclidean PGA, see [4, 5]. Conformal geometric algebra (CGA) defined by nondegenerate signature \((n + 1, 1)\) contains the same model and, moreover, allows Euclidean transformations of round primitives and dilation (conformal geometry), see [8–11]. Clearly, PGA is a subalgebra of CGA but the representation of Euclidean geometry looks very different at the first sight. PGA representation of a point is a multivector of grade \(n - 1\) while a CGA point is of grade 1. This indicates that we have to think dually, or in other words in a complementary way. In what follows, we clarify how PGA can be viewed in CGA. We note that such an inclusion has been introduced

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in [12], yet we propose that in our notation it is clear that the flat primitives precisely coincide in PGA and CGA. By a choice of basis and signs we show that PGA duality can be completely described on the elements of CGA using the CGA duality. Therefore our notation slightly differs compared to [6] and Table 1. Consequently, a unified approach to both algebras is introduced. We treat the case \( n = 3 \) in this paper, however, the results hold for arbitrary dimension \( n \), in particular also for \( n = 2 \). Our observation reads that if one needs to use calculations in PGA, it is enough to implement CGA only and therefore there is no need to implement a structure with degenerate metric.

First, we briefly introduce the frameworks of CGA and PGA as models of Euclidean geometry and we summarize basic formulae in Sect. 2. In Sect. 3, we show that there are two naturally related copies of PGA in CGA, see Proposition 3.1. After the identification of the two copies, duality in PGA is obtained in terms of CGA operations, see Proposition 3.2. The duality directly describes the correspondence between flat primitives and versors for Euclidean transformations in CGA, and the objects and versors in PGA, see Proposition 3.5. Basic ideas are then demonstrated on a simple example.

### 1.1. Notation

We denote elements of geometric algebras by bold letters - capitals for general multivectors and lower case letters for vectors. We also set the notation such that we can easily distinguish the elements of different algebras. Namely, \( A, B, \ldots \) will denote multivectors in PGA, \( A_c, B_c, \ldots \) will denote multivectors in CGA and \( A_E, B_E, \ldots \) will denote the elements of \( \mathbb{G}_3 \), i.e. the geometric algebra of three dimensional space. Similar notation will be used for objects and transformations of algebras, namely \( P, \ell, p \) will denote a point, line and plane in PGA, respectively, while \( P_c, \ell_c, p_c \) will denote direct representations of the respective objects in CGA. The corresponding dual representations will be accented by star superscript. By \( P_E \) we mean the representation of a Euclidean point in \( \mathbb{G}_3 \). The versors for rotations and translations in PGA and CGA will be denoted by \( R, T \) and \( R_c, T_c \), respectively. In order to distinguish among dualities in different algebras and to be consistent with the notation in the literature we denote the duality in CGA by an ordinary star symbol, by \( *_p \) the duality in PGA and by \( *_E \) the duality in \( \mathbb{G}_3 \), respectively.

### 2. Geometric Algebras for Euclidean Geometry

The geometric algebra \( \mathbb{G}_3 \) can describe vector geometry and rotations, see e.g. [1] for the geometric background. If we want to describe Euclidean geometry, we need to add a null vector in order to represent translations. The minimal way is to raise the dimension by one and add a one-dimensional space of null vectors. This model is known as the projective geometric algebra (PGA). However, this procedure returns a Clifford algebra with a degenerate quadratic form. Alternatively, we may raise the dimension by two and add a symplectic two-dimensional vector space. Then the quadratic form is nondegenerate with indefinite signature and we have even two linearly independent
null vectors. The resulting model is known as conformal geometric algebra (CGA). We list the basic facts and formulae of both models.

2.1. Euclidean Geometry in CGA

From the algebraic point of view, CGA for dimension 3 Euclidean space is a Clifford algebra defined by a nondegenerate quadratic form of signature \((4, 1, 0)\). Vectors \(e_0, e_1, e_2, e_3, e_{\infty}\) denote an orthogonal basis of the generating vector space \(\mathbb{R}^{4,1}\) with inner product given by the quadratic form

\[
B = \begin{pmatrix}
0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]  

(1)

Hence \(e_0, e_{\infty}\) are null vectors and \(e_0 \cdot e_{\infty} = e_{\infty} \cdot e_0 = -1\). The duality in CGA is defined by

\[
A_c^* = A_c I_c^{-1} = A_c \cdot I_c^{-1}, \tag{2}
\]

where \(I_c = e_{0123\infty}\) is the conformal pseudoscalar, \(I_c^{-1} = -I_c\). The geometry in CGA is defined by the following embedding of a Euclidean point \(P_E\) into the geometric algebra \(\mathbb{G}_3\), particularly onto an element \(P_c\) of the form

\[
P_c = e_0 + P_E + \frac{1}{2}(P_E \cdot P_E)e_{\infty}, \tag{3}
\]

where \(P_E \cdot P_E\) coincides with the square of the Euclidean norm. In coordinates, if \(P_E = xe_1 + ye_2 + ze_3\), then we get the well known formula

\[
P_c = e_0 + xe_1 + ye_2 + ze_3 + \frac{1}{2}(x^2 + y^2 + z^2)e_{\infty} \tag{4}
\]

together with the standard property \(P_c \cdot P_c = 0\). The nondegeneracy of the quadratic form (1) implies that we have two mutually dual representations of geometric primitives in CGA. Namely, a multivector \(A_c\) is the direct representation (also called Outer Product Null Space (OPNS) representation) of an object in CGA if and only if the object is formed exactly by points \(P_c\) satisfying

\[
P_c \wedge A_c = 0. \tag{5}
\]

The duality in CGA given by (2) defines a dual representation (or IPNS representation). Namely, the same object can be also represented by \(A_c^*\) in the sense that it is formed exactly by points \(P_c\) satisfying

\[
P_c \cdot A_c^* = 0, \tag{6}
\]

where the dot denotes the inner product. Note that this duality of representations follows from the duality between the inner and outer product. The direct representation is useful for constructing geometric primitives as a join of points while the advantage of the dual representation is that one can easily read off the internal parameters of the primitives and find intersection of spheres.

Taking outer products of points in CGA we get representatives of general spheres spanned by these points, i.e. point pairs, circles, spheres and also flat primitives if one of the points lies at infinity. Thus for a flat point \(FP_c\), a line
\( \ell_c \) spanned by points \( P_{1c}, P_{2c} \) and a plane \( p_c \) spanned by points \( P_{1c}, P_{2c}, P_{3c} \) we have the following respective representations

\[
FP_c = P_c \wedge e_\infty, \quad (7) \\
\ell_c = P_{1c} \wedge P_{2c} \wedge e_\infty, \quad (8) \\
p_c = P_{1c} \wedge P_{2c} \wedge P_{3c} \wedge e_\infty. \quad (9)
\]

Let us also recall that Euclidean transformations are represented by versors which act on objects by conjugation. The versor for translation by vector \( \vec{t} \), which we identify with \( t_E \in G_3 \), is given by

\[
T_c = e^{-\frac{1}{2} t_E e_\infty} = 1 - \frac{1}{2} t_E e_\infty. \quad (10)
\]

The versor for rotation by an angle \( \alpha \) and with normalised dual representation of the rotation axis \( \ell^* \), i.e. \( \ell^* \cdot \ell^* = -1 \), is represented by CGA element

\[
R_c = e^{\frac{1}{2} \alpha \ell^*} = \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \ell^*. \quad (11)
\]

**Remark 2.1.** For general dimension \( n \), we consider the orthogonal basis \( e_0, e_1, \ldots, e_n, e_\infty \in \mathbb{R}^{n+1} \) with inner product given by matrix (1), where the central block is an \( n \times n \) identity matrix. The duality prescription (2) remains, however, the pseudoscalar satisfies

\[
I_c^{-1} = (-1)^{(n-1)/2} I_c, \quad (12)
\]

thus the duality map is either an involution or an anti–involution depending on the dimension. Equation (3) defining a CGA point is independent of the dimension as well as formulae (10) and (11) for CGA transformations. The representations of Euclidean primitives (by which we understand flat primitives only) in CGA are \( P_{1c} \wedge \cdots \wedge P_{kc} \wedge e_\infty \) for any \( 1 \leq k \leq n \).

### 2.2. Euclidean Geometry in PGA

We shall use conventions and notation as close to [4,5] and [6] as possible, however, we modify the sign conventions for the duality in PGA slightly. We want to stress that these changes do not influence the validity of general formulae in [6] for incidence relations, projections, rejections, etc. We also note that in [3] the author strictly distinguishes geometric algebras generated by \( \mathbb{R}^{3,0,1}_* \) and \( \mathbb{R}^{3,0,1} \), i.e. plane-based and point-based models, respectively. This is correct but algebraically these are isomorphic vector spaces and as algebras, \((G_{3,0,1}, \lor) \cong (G_{3,0,1}^*, \land)\) via Poincare isomorphism. Indeed, the identification of regressive product and wedge product via this duality allows us to use one algebra with both operations. This concept allows representatives of both points and planes to be elements of a single algebra. Algebraically, PGA for 3D Euclidean space is a Clifford algebra generated by a degenerate quadratic form of signature \( (3,0,1) \). We consider a basis of the generating vector space \( \mathbb{R}^{3,0,1} \) in which the quadratic form is given by the matrix

\[
B = \begin{pmatrix} 0 & 0 \\ 0 & 1_{3\times3} \end{pmatrix}, \quad (13)
\]

and we denote this basis by \( e_0, e_1, e_2, e_3 \), i.e. \( e_0 \) is a null vector. Note that the symbols for basis elements are the same as symbols for the corresponding...
basis elements in CGA in this notation, which is usual in the literature. Indeed, PGA can be viewed as a subalgebra of CGA spanned by these elements. However for relating the geometry of PGA and CGA, the \( e_0 \) from PGA plays rather the role of the element \( e_\infty \) in CGA notation. This relation is discussed in detail in the following sections.

The duality in PGA cannot be obtained in the same way as the duality in CGA, by division with the pseudoscalar, because the quadratic form (13) defining PGA is degenerate. One can use Hodge duality approach, [13], i.e. that the dual to a basis element \( A \) can be defined as the complement to the projective pseudoscalar, \( A \wedge A^* = 1 \) or in the reverse order. However, such a map is neither involutive nor anti–involutive in general and the signs coming from such a definition are not compatible with the CGA duality. Therefore we use the idea of a Poincare duality approach, [4,5], which is a mapping between an algebra and its dual. In our concept, both these algebras can be understood as isomorphic subalgebras of CGA which leads to the Definition 3.2, see the next section. Note that similar concept of duality has been introduced in [12].

The representation of the Euclidean geometry in PGA is given by the embedding of a point. In coordinates, a Euclidean point \( P_E = x e_1 + y e_2 + z e_3 \) is represented in PGA by a multivector of grade three

\[
P = x e_{032} + y e_{013} + z e_{021} + e_{123}.
\]

The formula for a point in PGA can be written in a coordinate free way using the Euclidean duality \( \ast_E \), given by the division with the Euclidean pseudoscalar \( I_E = e_{123} \), particularly

\[
P = e_0 P_E^* + I_E,
\]

which can be rewritten as

\[
P = I_E + P_E^{\ast E} e_0,
\]

since the grade (width of the basis blades) of \( P_E^{\ast E} \) is two. The degeneracy of the PGA quadratic form (13) causes us to have only one representation of geometric primitives in PGA. Since the grade of points is three, therefore sub–maximal, the only way to represent primitives is as the null space of the regressive product (RPNS representation). Recall that the regressive product is dual to the outer product, i.e. \( (A \vee B)^\ast = A^\ast \wedge B^\ast \). Hence a point represented by \( P \) belongs to an object represented by \( A \) if and only if

\[
P \vee A = 0.
\]

By regressive products of respective points in PGA we get representatives of flat primitives. Thus for a line \( \ell \) spanned by points \( P_1, P_2 \) and a plane \( p \) spanned by points \( P_1, P_2, P_3 \) we have the representations

\[
\ell = P_1 \vee P_2,
\]

\[
p = P_1 \vee P_2 \vee P_3.
\]

One can find several formulae for PGA representations of Euclidean transformations in [4], however a direct formula for translation by vector \( t_E \) is missing. If one reads between the lines, the versor for translation in PGA is given by

\[
T = e^{-\frac{1}{2} e_0 t_E} = 1 - \frac{1}{2} e_0 t_E.
\]
Note that this formula is a direct consequence of formula (10) for the translator in CGA and Proposition 3.5 which will be proved later. Rotation is realized by (11), the same form of a versor as in CGA, i.e. $R_c$. However, the dual line $\ell^*$ in the formula must be replaced by $\ell$ given by (17) in this case.

Remark 2.2. For general dimension $n$, we consider the orthogonal basis $e_0, e_1, \ldots, e_n \in \mathbb{R}^{n,0,1}$ with the inner product given by matrix (13), where the lower–right block is an $n \times n$ identity matrix. Equation (15) defines a PGA point in any dimension. The representations of transformations are independent of the dimension and Euclidean primitives are given by $P_1 \lor \cdots \lor P_k$ for any $1 \leq k \leq n$. The duality in $n$D PGA is discussed in the next Section, see Remark 3.4.

3. PGA in CGA

Our main finding will be that the usual choice of basis and inner product in CGA, gives two distinct subalgebras of CGA which are both algebraically isomorphic to PGA and also an involutive isomorphism on CGA that relates these two subalgebras. In such identification, the duality in PGA can be seen as a “twisted” CGA duality. We note that this concept has been already published in [12]. Indeed, the author represents the projectivised exterior algebras of $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1*}$ as subalgebras of CGA and defines the duality by means of two different pseudoscalars of these spaces by assertion (18.37). The form of this duality corresponds to our concept up to the sign. In the notation of [12], $\hat{W}$ corresponds to CGA$_0$ and $\hat{W}^*$ to CGA$_\infty$. Furthermore, $\hat{J}$ corresponds to $\sharp$. We used notation that is more familiar to mathematicians and provide connections with CGA to clarify transitions between algebras.

3.1. Duality in PGA

In CGA, the subalgebra formed by elements which do not contain $e_\infty$ is present as well as the subalgebra formed by elements which do not contain $e_0$. The former is generated by $\{e_0, e_1, e_2, e_3\}$ and will be denoted by CGA$_0$ and the latter is generated by $\{e_1, e_2, e_3, e_\infty\}$ and will be denoted by CGA$_\infty$. The notation accents the null vector present and stresses the fact that both are subalgebras of CGA.

The choice of CGA basis and the inner product defines also a distinct involution in CGA which relates subalgebras CGA$_0$ and CGA$_\infty$. Namely, the choice defines an isomorphism between the two vector spaces, $\mathbb{R}^{4,1}$ and its dual, by taking the usual dual basis. The quadratic form (1) defines another isomorphism between these spaces which is known as the musical isomorphism in the literature. The composition of these two isomorphisms is a bijective linear map of $\mathbb{R}^{4,1}$ onto itself, for $i = 1, 2, 3$ given by

$$\sharp : e_i \mapsto e_i, e_0 \mapsto -e_\infty, e_\infty \mapsto -e_0,$$

where we used the notation of musical isomorphism in order to distinguish between the duality on vector space $\mathbb{R}^{4,1}$ and the usual CGA duality. The minus signs correspond to the choice of inner product (1), i.e. $e_0 \cdot e_\infty = -1$. This map is a linear involution and preserves the quadratic form in
Figure 1. Two copies of PGA inside CGA

CGA, thus it defines a unique extension to CGA as a homomorphism of Clifford algebras. In the following, we will use the symbol ♯ also for this extension. Using this notation we can summarize the above observations into the following statement.

**Proposition 3.1.** The choice of basis of null vectors \(e_0, e_\infty\) in CGA defines two subalgebras \(CGA_0\) and \(CGA_\infty\) both isomorphic to PGA and an involutive isomorphism ♯ between them which acts by replacing \(e_0\) with \(-e_\infty\) and vice versa in each basis blade.

Let us describe the structure of subalgebras \(CGA_0\) and \(CGA_\infty\) and the isomorphism ♯ in a more detailed way. The intersection of these subalgebras is generated by \(\langle e_1, e_2, e_3 \rangle\) with inner product given by the identity matrix, thus it forms the algebra \(G_3\). The map ♯ that switches between the two subalgebras acts as identity on this intersection. On the complement of the union of the subalgebras to CGA, it acts as minus identity. Note that the union of subalgebras contains the elements with either \(e_0\) or \(e_\infty\) only, therefore the complement to CGA is not empty containing elements with both \(e_0\) and \(e_\infty\). Schematically, this is depicted in Fig. 1.

We stress that ♯ is an isomorphism of Clifford algebras and thus it preserves all products in the algebra.

Once we know how to identify PGA with any of the two subalgebras in CGA, it is easy to understand duality in PGA. It can be defined in the same way as duality in CGA, by multiplication with a suitable pseudoscalar inverse. But we have to multiply by the inner product and also the pseudoscalar inverse must be taken with respect to the inner product. Such an inversion exists but it lies in the other subalgebra. Indeed, it is easy to see that \(I \cdot I^\sharp = 1\) and thus for \(I = e_{0123} \in CGA_0\), the inner product inverse is \(I^\sharp = e_{123\infty} \in CGA_\infty\) and vice versa: for \(I^\sharp = e_{123\infty} \in CGA_\infty\), the inner product inverse is \((I^\sharp)^\sharp = I\).

**Definition 3.2.** Understanding PGA as a subalgebra of CGA, **PGA duality** is an involution defined for each \(A \in PGA\) by

\[
A^{\ast \rho} = (A \cdot I^\sharp)^\sharp = A^\sharp \cdot I.
\]
Note that the equality in the definition follows from the fact that $\sharp$ preserves the inner product. In the diagram visualisation, this can be described by the commutative diagram in Fig. 2. The duality in PGA is the composition from left to right, or equivalently from right to left, since it is an involution.

We also note that using a pseudoscalar in another subspace to move between subspaces is not new, see e.g. [2,3,12].

Having the duality in PGA defined according to the Definition 3.2, it is easy to relate it to the standard duality in CGA. Namely, it is a CGA duality twisted by a null vector as follows.

**Proposition 3.3.** For $A \in \text{PGA}$ the following identities hold

$$A^*P = (A \wedge e_\infty)^\sharp = (A^2 \wedge e_0)^\star.$$  \hspace{1cm} (22)

**Proof.** By definition of PGA duality (21), we need to show $(A \wedge e_\infty)^* = A \cdot I^\sharp$ in order to prove the first identity. Indeed, we compute

$$(A \wedge e_\infty)^* = (A \wedge e_\infty) \cdot I_{c}^{-1} = A \cdot (e_\infty \cdot I_{c}^{-1}) = A \cdot I^\sharp.$$  

The first equality is the CGA duality in terms of the inner product, see (2). The inner product coincides with the left contraction in this case since $I_{c}^{-1}$ is of the highest grade. Then the second equality is a general property of left contraction. The third equality follows from the fact that $e_\infty \cdot I_{c}^{-1} = e_\infty \cdot e_{0321\infty} = (e_\infty \cdot e_0)e_{321\infty} = -e_{321\infty} = I^\sharp$. The second identity in the proposition follows from the definition of isomorphism $\sharp$. Namely, $(A \wedge e_\infty)^\sharp = -A^2 \wedge e_0$ and $(I_{c}^{-1})^\sharp = -I_{c}^{-1}$.

We also get a sort of PGA duality between the inner product and outer product similar to the duality between these products in CGA. Namely, supposing the grade of a blade $A$ is less or equal to the grade of a blade $B$, we have

$$(A \wedge B)^*P = A^2 \cdot B^*P.$$  \hspace{1cm} (23)

This formula holds for general multivectors $A, B$ if we replace the inner product on the right-hand side by left contraction. It is worth to look also at the relation between the duality in PGA and the usual Euclidean duality. For that we need to express a multivector $A \in \text{PGA}$ as a sum $C_E + D_E \wedge e_0$, where $C_E, D_E \in \mathbb{G}_3$. Then we compute

$$A^*P = (C_E + D_E \wedge e_0)^*P = -D_E^*E + C_E^*E \wedge e_0.$$  \hspace{1cm} (24)
Table 1. PGA duality

| 1 | e0 | e1 | e2 | e3 | e01 | e02 | e03 | e12 | e13 | e23 | e012 | e013 | e023 | e123 | I |
|---|---|---|---|---|----|----|----|----|----|----|-----|-----|-----|-----|---|
| A | a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p |
| A^P | p | o | -n | m | -l | -k | j | -i | -h | g | -f | -e | d | -c | b | a |

This formula is particularly convenient for an explicit computation of dual basis blade coefficients, see Table 1.

**Remark 3.4.** The definition of the isomorphism \( \sharp \) is independent of the dimension. However, for general dimension \( n \) we have

\[
\mathbf{I} \cdot \mathbf{I}^\sharp = (-1)^{n(n+1)/2},
\]

thus the inner product inversion to \( \mathbf{I} \) gains this sign and it should also enter in formula (21) for the definition of the PGA duality. Note that PGA duality is an involution or anti–involution depending on the dimension. Formulae (22) and (23) hold in any dimension. Indeed, the only change in the proof of Proposition 3.3 is that \( e_\infty \cdot I_{\mathbf{c}}^{-1} = (-1)^{n(n+1)/2} I^\sharp \). Equation (24) changes in \( nD \) to

\[
(C_E + D_E \wedge e_0)^*P = (-1)^n D_E^*E + C_E^*E \wedge e_0.
\]

### 3.2. Geometry

All geometric primitives in PGA are constructed by means of the regressive product which is dual to the outer product, c.f. (17) and (18). Let us start by computing the projective dual of a point. By applying formula (21) for the PGA duality to the PGA point (15) we get

\[
P^*P = e_0 + P_E.
\]

Hence a point in PGA is the dual to its usual homogeneous representation in \( \mathbb{R}^4 \). Now we compute the representation of a PGA point in the subalgebra CGA_\( \infty \). Applying the isomorphism \( \sharp \) to formula (15) for a point we get

\[
P^\sharp = I_E - P_E^*E e_\infty.
\]

We will show in the following Proposition 3.5 that this is a formula for the dual representation of a flat point in CGA. Indeed, looking at the direct representations of flat primitives and Euclidean transformations in CGA we observe that they all lie in the subalgebra CGA_\( \infty \). Hence this subalgebra is the suitable copy of PGA in CGA for geometric purposes and the map \( \sharp \) gives a geometric embedding in the following sense.

**Proposition 3.5.** Let \( P, \ell, p \) be representations of a point, a line and a plane in PGA, respectively. Then \( P^\sharp, \ell^\sharp, p^\sharp \) are the dual (IPNS) representations of the same point (viewed as a flat point), line and plane in CGA. Moreover, if \( V \) is a versor for a Euclidean transformation in PGA, then \( V^\sharp \) is the versor for the same transformation in CGA.
Proof. At first we prove the correspondence of points. We need to show that (27) is the form of the dual flat point $FP^*_c = (P_c \wedge e_\infty)^*$. Indeed, we compute

$$FP^*_c = (e_0 \infty + P_E e_\infty)^* = I_E - P^{*E} e_\infty = P^\sharp. \quad (28)$$

For the proof of the correspondence between other geometric primitives we use the formula for dual point in terms of a conformal point. Since the dual point (26) actually is the conformal representation of a point without the quadratic part, we can write

$$P^*P = P_c + (e_0 \cdot P_c)e_\infty \quad (29)$$

Now we apply this formula to a PGA line $\ell = P_1 \vee P_2$. By the definition of the regressive product and formula (22) for the PGA duality we get

$$\ell^\sharp = [(P_{1c} + (e_0 \cdot P_{1c})e_\infty) \wedge (P_{2c} + (e_0 \cdot P_{2c})e_\infty) \wedge e_\infty]^* = (P_{1c} \wedge P_{2c} \wedge e_\infty)^* = \ell^*_c. \quad (29)$$

Similarly, for a plane $p = P_1 \vee P_2 \vee P_3$ we get

$$p^\sharp = (P_{1c} \wedge P_{2c} \wedge P_{3c} \wedge e_\infty)^* = p^*_c,$$

thus the representation of any object in PGA coincides with its dual representation in CGA. The claim about versors follows from the correspondence between objects and the fact that $\sharp$ preserves the geometric product and that it commutes with the reversion operation.

Remark 3.6. The duality in PGA is defined in such way that dual PGA points always coincide with their homogeneous representations, i.e. the formula (26) holds for arbitrary dimension. On the other hand, the correspondence in Proposition 3.5 changes sign according to the sign of (25). Namely, in dimension $n$ we have

$$P^\sharp = (-1)^{n(n+1)/2}(P_c \wedge e_\infty)$$

and also the remaining objects are endowed with the same sign.

3.3. Example

It is obvious that we do not need the quadratic parts of points in CGA when we deal with flat primitives only. PGA is certainly more efficient in that case. However, we do not need to abandon the CGA concept while using all the amazing formulae from PGA. We only have to keep in mind that points correspond to flat points and that the PGA duality differs from conformal duality, see (21) and (22).

Let us consider an elementary example to demonstrate a convenient way of using PGA inside CGA. We have a point $P$ lying on a line $\ell$ which intersects a plane $p$ and we have a sphere $s$ which also intersects the plane, see Fig. 3.

Then we can use formulae from PGA to compute the intersection of the line and plane $\ell \wedge p$ and the orthogonal projection of the point to the plane $(P \cdot p)p$, the orthogonal projection of the line to the plane $(\ell \cdot p)p$. If we need to calculate with the sphere, we only need to replace $e_0$ by $-e_\infty$ in the representations of flat primitives, which is realized by the map $\sharp$, and then we can use all formulae known in CGA. For instance, the intersection of the
Figure 3. Intersections and projections of flats and rounds

sphere and the plane is a dual circle \( \mathbf{p}^\# \wedge \mathbf{s}^\star \), the orthogonal projection of the point to the sphere is \( (\mathbf{p}^\# \cdot \mathbf{s}^\star)\mathbf{s}^\star \) and the orthogonal projection of the line to the sphere is a circle \( (\ell^\# \cdot \mathbf{s}^\star)\mathbf{s}^\star \).

Figure 3 was created by the web-based experimental platform of Ganja.js \([7]\). The corresponding full code follows.

```javascript
// Create a Clifford Algebra with 4,1 metric for 3D CGA.
Algebra(4,1,()=>{
    // We start by defining a null basis, and upcasting for points
    // which we need for rounds only
    var ni = 1e4+1e5, no = .5e5-.5e4;
    var up = (x)= > no + x + .5*x*x*ni;

    // Sharp map in both directions in terms of CGA products
    var IN = (x)= > x + ni*(no<<x) + no*(no<<x);
    var NI = (x)= > x + ni*(ni<<x) + no*(ni<<x);

    // PGA pseudoscalar
    var I = no`1e1`1e2`1e3;

    // PGA duality and upcasting to PGA
    var dual = (x)= > NI(x)<<I;
    var upP = (x)= > dual(no+x);

    // Definition of regressive product for 2 and three inputs
    var reg = (x,y)= > dual(dual(x)`dual(y));
    var reg3 = (x,y,z)= > dual(dual(x)`dual(y)`dual(z));

    // Formulas from PGA can be used in CGA

    // We define 4 points
    var P = upP(0.5e1-1.5e3);
    var P1 = upP(1e1), P2 = upP(1e2), P3 = upP(-1e3);
```
4. Conclusion

We introduced a notation for PGA primitives that is compatible with their CGA description, once PGA is understood as a subalgebra of CGA. We also solved the issues with the noninvertibility of the PGA pseudoscalar in computing duality and showed the exact forms of dual counterparts to geometric primitives in PGA using another copy of PGA in CGA. This has great potential for symbolic calculations and their software implementation, because we can flexibly switch between PGA notation which is efficient for flat object manipulation and CGA operations on round elements in our problems. The next step is to show the advantages of this approach in various computational platforms, together with code optimisation and applications. Generally, the idea is that it is enough to implement CGA only and there is no need to implement an extra structure with degenerate metric.

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Received: April 24, 2020.
Accepted: January 13, 2021.