A New Class of Multivariate Elliptically Contoured Distributions with Inconsistency Property

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Abstract
We introduce a new class of multivariate elliptically symmetric distributions including elliptically symmetric logistic distributions and Kotz type distributions. We investigate the various probabilistic properties including marginal distributions, conditional distributions, linear transformations, characteristic functions and dependence measure in the perspective of the inconsistency property. In addition, we provide a real data example to show that the new distributions have reasonable flexibility.

Keywords Elliptically contoured distribution · Elliptically symmetric logistic distribution · Kotz type distribution · Inconsistency property · Generalized Hurwitz-Lerch zeta function

Mathematics Subject Classification (2010) 60E · 62F

1 Introduction

The multivariate normal distribution has been widely used in theory and practice because of its tractable statistical features. However, the light tail of the normal distribution can not fit some practical situation well. The elliptically contoured distributions (elliptical distributions), a new family of distributions with similar convenient properties, overcomes the shortcomings of the normal distributions. An n-dimension random vector $\mathbf{X}$ is said to have a multivariate elliptical distribution, written as $\mathbf{X} \sim Ell_n(\mu, \Sigma, \phi)$ if its characteristic function can be expressed as $\psi_{\mathbf{X}}(\mathbf{t}) = \exp(i\mathbf{t}^T \mu)\phi(\mathbf{t}^T \Sigma \mathbf{t})$, where $\mu$ is an n-dimension column vector, $\Sigma$ is an $n \times n$ positive semi-definite matrix, $\phi(\cdot)$ is called characteristic generator. If $\mathbf{X}$ has a probability density function (pdf) $f(\mathbf{x})$, then

$$
 f(\mathbf{x}) = \frac{C_n}{\sqrt{|\Sigma|}} g_n \left( (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right).
$$

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where \( C_n \) is the normalizing constant and \( g_n(\cdot) \) is called density generator (d.g.). The stochastic representation of \( X \) is given by

\[
X = \mu + RA^T U^{(n)},
\]

where \( A \) is a square matrix such that \( A^T A = \Sigma \), \( U^{(n)} \) is uniformly distributed on the unit sphere surface in \( \mathbb{R}^n \), \( R \geq 0 \) is independent of \( U^{(n)} \) and has the pdf given by

\[
f_R(v) = \frac{1}{\int_0^\infty t^{n-1}g(t^2)dt} v^{n-1}g(v^2), \quad v \geq 0.
\]

Many members of the elliptical distributions such as the multivariate normal distributions and student-\( t \) distributions, have been systematically studied. See the books and papers of Cambanis et al. (1981), Fang et al. (1990), Kotz and Ostrovskii (1994), Liang and Bentler (1998) and Nadarajah (2003). Nevertheless, research work on the multivariate symmetric logistic distribution is far less than other members. The elliptically symmetric logistic distribution with density

\[
f(x) = \frac{\Gamma\left(\frac{n}{2}\right) |\Sigma|^{-\frac{1}{2}}}{\pi^{n\frac{n}{2}} \int_0^\infty \frac{\exp(-u)}{(1+\exp(-u))^2} du} \exp\left(-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}\right) \left[1 + \exp\left(-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}\right)\right]^2, \quad x \in \mathbb{R}^n,
\]

was introduced by Jensen (1985) and has been studied by Fang et al. (1990), Kano (1994) and Yin et al. (2019). Several applications of multivariate symmetric logistic distribution in risk management, quantitative finance and actuarial science can be found in various literatures such as Ali et al., (1978), (Landsman and Valdez 2003), Landsman et al. (2016a, b, 2018).

The paper will define a new class of elliptical distributions including the Kotz type distributions and the logistic distributions, give the value of the normalizing constant and study the marginal distributions, conditional distributions, linear transformations, characteristic functions and local dependence functions in perspective of its inconsistency property.

The rest of the paper is organized as follows. In Section 2, we introduce the definition of a new class of multivariate elliptically symmetric distributions which include elliptically symmetric logistic distributions and Kotz type distributions and discuss the expression of the normalizing constant. In Sections 3–7 we study the probabilistic properties of the new class of elliptically distribution including marginal distributions, conditional distributions, linear transformations, characteristic functions in perspective of its inconsistency property. In addition, we give the expression of its local dependence function. In Section 8, we give the data analysis of the new class of elliptically distribution and we conclude in Section 9.

2 Preliminary

We now give the definition of a new class of multivariate elliptically symmetric distributions which include elliptically symmetric logistic distributions and Kotz type distributions.

**Definition 2.1** The \( n \)-dimensional random vector \( X \) is said to have a generalized elliptical logistic (GL) distribution with parameter \( \mu \) (\( n \)-dimensional vector) and \( \Sigma \) (\( n \times n \) matrix with \( \Sigma > 0 \)) if its pdf and density generator have the forms

\[
f(x) = C_n \ |\Sigma|^{-\frac{1}{2}} g \left((x-\mu)^T \Sigma^{-1} (x-\mu)\right), \quad x \in \mathbb{R}^n,
\]
\[ g(t) = \frac{t^{N-1} \exp(-at^{s_1})}{(1 + \exp(-bt^{s_2}))^{2r}}, \quad t > 0, \quad (2.2) \]

respectively, where \(2N + n > 2\), \(a, b, s_1, s_2 > 0\), \(r \geq 0\) are constants. The normalizing constant \(C_n\) will be discussed in Section 2.2.

### 2.1 Special Cases

1) **Generalized logistic distribution** (Yin et al. 2019)

   Setting \(N = 1, s_1 = s_2 = 1\) in Eq. 2.2, we get

   \[ g(t) = \frac{\exp(-at)}{(1 + \exp(-bt))^{2r}}, \quad (2.3) \]

   which is the density generator put forward by Yin et al. (2019).

2) **Multivariate normal distribution**

   Setting \(N = 1, a = \frac{1}{2}, s_1 = 1, r = 0\) in Eq. 2.2, we get \(g(t) = \exp(-\frac{1}{2}t)\), which is the density generator of the multivariate normal distribution.

3) **Multivariate exponential power (Épo) distribution** (Landsman and Valdez 2003)

   For \(N = 1, r = 0\), Eq. 2.2 is the density generator of the multivariate exponential power distribution whose d.g. is usually written as

   \[ g(t) = \exp(-at^{s_1}), \quad a, s_1 > 0. \]

   If \(s_1 = \frac{1}{2}\) and \(a = \sqrt{2}\), we have the d.g. of the double exponential or Laplace distribution defined as

   \[ g(t) = \exp(-\sqrt{2}t). \]

4) **Kotz type (Ko) distribution** (Fang et al. 1990)

   For \(r = 0\), Eq. 2.2 is the density generator of the symmetric Kotz type distribution whose d.g. is usually written as

   \[ g(t) = t^{N-1} \exp(-at^{s_1}), \quad a, s_1 > 0, \quad 2N + n > 2. \quad (2.4) \]

   When \(s_1 = 1\), Eq. 2.4 is the density generator of the original Kotz distribution whose d.g. is written as

   \[ g(t) = t^{N-1} \exp(-at), \quad a > 0, \quad 2N + n > 2. \]

5) **Elliptically symmetric logistic (Lo) distribution** (Fang et al. 1990)

   Setting \(N = 1, a = b = r = 1, s_1 = s_2 = 1\), Eq. 2.2 is the d.g. of the \(n\)-dimensional elliptically symmetric logistic distribution, written as

   \[ g_n(t) = \frac{\exp(-t)}{(1 + \exp(-t))^{2r}}. \quad (2.5) \]

6) **Generalized logistic type I (GLI) distribution** (Balakrishnan and Leung 1988)

   Setting \(N = 1, a = b = 1, s_1 = s_2 = 1\) in Eq. 2.2 gives the density generator of the generalized logistic type I distribution written as

   \[ g(t) = \frac{\exp(-t)}{(1 + \exp(-t))^{2r}}. \quad (2.6) \]

7) **Generalized logistic type III (GLIII) distribution** (Arashi and Nadarajah 2016)

   Setting \(N = 1, b = 1, s_1 = s_2 = 1, r = a\) in Eq. 2.2 gives the density generator of the generalized logistic type III distribution written as

   \[ g(t) = \frac{\exp(-at)}{(1 + \exp(-t))^{2a}}. \quad (2.7) \]
8) **Generalized logistic type IV (GLIV) distribution** (Arashi and Nadarajah 2016)

For \( N = 1, s_1 = s_2 = 1, r = \frac{p+a}{2} \), where \( p > 0 \), Eq. 2.2 is the density generator of the generalized logistic type IV distribution written as

\[
g(t) = \frac{\exp(-at)}{(1 + \exp(-t))^{p+a}}. \tag{2.8}
\]

Setting \( s_1 = s_2 = s \) in Eq. 2.2, we obtain

\[
g_n(t) = \frac{t^{N-1}\exp(-at^s)}{(1 + \exp(-bt^s))^{2r}}. \tag{2.9}
\]

For the sake of simplicity, we discuss probabilistic properties of GL distributions with density generators defined as Eq. 2.9 in following sections.

### 2.2 Normalizing Constant

To calculate the normalizing constant defined in Eq. 2.1, we introduce the Hurwitz-Lerch zeta function. The Hurwitz-Lerch zeta function and its integral representation are respectively defined as

\[
\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad (a \in \mathbb{C} \setminus \mathbb{Z}_0^{-}, s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1),
\]

\[
\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{1 - te^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-(a-1)t}}{e^t - z} dt \quad (\Re(s) > 0, \Re(a) > 0 \text{ when } |z| \leq 1(z \neq 1); \Re(s) > 1 \text{ when } z = 1).
\]

Various generalization and extensions of the Hurwitz-Lerch zeta function \( \Phi(z, s, a) \) have been studied by various researchers. The following expression of generalized Hurwitz-Lerch zeta function which will be used in this paper is defined by (cf. Lin et al. 2006)

\[
\Phi^*_v(z, s, a) = \frac{1}{\Gamma(v)} \sum_{n=0}^{\infty} \frac{\Gamma(v+n)}{n!} \frac{z^n}{(n+a)^s} \quad (v \in \mathbb{C}, a \in \mathbb{C} \setminus \mathbb{Z}_0^{-}, s \in \mathbb{C} \text{ when } |z| < 1; \Re(s - v) > 1 \text{ when } |z| = 1),
\]

\[
\Phi^*_v(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{(1 - ze^{-t})^v} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-(a-v)t}}{(e^t - z)^v} dt \quad (\Re(s) > 0, \Re(a) > 0 \text{ when } |z| \leq 1(z \neq 1); \Re(s) > 1 \text{ when } z = 1).
\]

If \( v = 0 \),

\[
\Phi^*_0(z, s, a) = \Phi^*_0(z, s, a) = \frac{1}{a^s},
\]

thus we denote \( \Phi^*_0(z, s, a) \) by \( \Phi^*_0(s, a) \).

Pointed out by Yin et al. (2019), the normalizing constant of elliptical symmetric logistic distribution suggested by Landsman and Valdez (2003), has no meaning when \( n=1 \) and \( n=2 \). Thus, we calculate the normalizing constant defined in Eq. 2.1 by the generalized Hurwitz-Lerch zeta function. The method was similarly used in Yin et al. (2019).

**Theorem 2.1** Letting \( X \sim GL_n(\mu, \Sigma, g_n) \) where \( g_n \) is defined as Eq. 2.9, then the normalizing constant defined in Eq. 2.1 can be expressed as

\[
C_n = c_n^s(N, b, s) \left[ \Phi^*_2(-1, \frac{1}{3}(N + \frac{n}{2} - 1), \frac{a}{b}) \right]^{-1},
\]

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where
\[ c_n^*(N, b, s) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}(N + \frac{n}{2} - 1)\right)} \frac{b^{\frac{1}{2}(N+n-1)s}}{\pi^{\frac{n}{2}}}, \]
and \( \Phi_{2r}^* \) is the generalized Hurwitz-Lerch zeta function.

**Proof** Since
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x)dx = 1,
\]
where \( f(x) \) is defined in Eq. 2.1 and transformation from the rectangular to polar coordinates. We have
\[
C_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \left[ \int_0^\infty x^{\frac{n}{2}-1} g_n(x)dx \right]^{-1}
= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \left[ \int_0^\infty x^{N+\frac{n}{2}-2} e^{-ax^2}dx \right]^{-1}
= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \frac{b^{\frac{1}{2}(N+n-1)s}}{\Gamma\left(\frac{1}{2}(N + \frac{n}{2} - 1)\right)} \left[ \Phi_{2r}^* \left( -1, \frac{1}{s}(N + \frac{n}{2} - 1), \frac{a}{b} \right) \right]^{-1}
= c_n^*(N, b, s) \left[ \Phi_{2r}^* \left( -1, \frac{1}{s}(N + \frac{n}{2} - 1), \frac{a}{b} \right) \right]^{-1},
\]
where
\[ c_n^*(N, b, s) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}(N + \frac{n}{2} - 1)\right)} \frac{b^{\frac{1}{2}(N+n-1)s}}{\pi^{\frac{n}{2}}}. \]

\[ \square \]

**Corollary 2.1**

1) Supposing \( X \sim \text{Kon}(\mu, \Sigma, g_n) \), where \( g_n \) is defined as Eq. 2.4, then the normalizing constant defined in Eq. 2.1 can be expressed as
\[ C_n = \frac{s_1 \Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}} \Gamma\left(\frac{1}{2s_1}(N + \frac{n}{2} - 1)\right)} \left[ \Phi_0^* \left( \frac{1}{s_1}(N + \frac{n}{2} - 1), a \right) \right]^{-1}. \]

2) Supposing \( X \sim \text{Epo}_n(\mu, \Sigma, g_n) \), where \( g_n \) is
\[ g_n(t) = \exp(-at^{s_1}), \quad a, \quad s_1 > 0, \]
then the normalizing constant defined in Eq. 2.1 can be expressed as
\[ C_n = \frac{s \Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2s_1}\right)} \left[ \Phi_0^* \left( \frac{n}{2s_1}, a \right) \right]^{-1}. \]

3) Supposing \( X \sim \text{GLIn}(\mu, \Sigma, g_n) \), where \( g_n \) is defined as Eq. 2.9, then the normalizing constant defined in Eq. 2.1 can be expressed as \( [\pi^{\frac{n}{2}} \Phi_{2a}^*(-1, \frac{n}{2}, 1)]^{-1} \).

4) Supposing \( X \sim \text{GLIII n}(\mu, \Sigma, g_n) \), where \( g_n \) is defined as Eq. 2.7, then the normalizing constant defined in Eq. 2.1 can be expressed as \( [\pi^{\frac{n}{2}} \Phi_{2a}^*(-1, \frac{n}{2}, a)]^{-1} \).
5) Supposing $X \sim GLIV_n(\mu, \Sigma, g_n)$, where $g_n$ is defined as Eq. 2.8, then the normalizing constant defined in Eq. 2.1 can be expressed as $\left[ \pi^{\frac{n}{2}} \Phi_{p^*}(-1, \frac{n}{2}, a) \right]^{-1}$.

Consider a family of density generators

$$\{ f(u|p) | p \in \mathbb{N} \},$$

where $\mathbb{N}$ denotes the set of all positive integers. According to Kano (1994), we will say that the family in Eq. 3.1 possesses a consistency property if and only if

$$\int_{-\infty}^{\infty} f \left( \sum_{j=1}^{p+1} x_j^2 | p \right) dx_{p+1} = f \left( \sum_{j=1}^{p} x_j^2 | p \right)$$

for any $p \in \mathbb{N}$ and almost all $(x_1, \cdots, x_p) \in \mathbb{R}^p$. We also say that the family is dimension coherent. However, the family of density generators defined in Eq. 2.2 does not satisfy the consistency property i.e. the marginal distributions of the $n$-dimension ($n > 1$) generalized elliptical bimodal logistic distribution — which are elliptically contoured — don’t have Eq. 2.2 as their density generators. In the following sections we will discuss the inconsistency property of the GL distribution and its applications (Figs. 1, 2 and 3).
3 Density Generator

Supposing $g_1(x)$, the d.g. of GL distributed random variable $X$, is defined as follows

$$g_1(t) = \frac{t^{N-1} \exp(-at)}{(1 + \exp(-bt))^{2r}}.$$ 

Before we utilize (cf. Fang et al. 1990)

$$g_m(u) = \int_u^\infty (\omega - u)^{n-m-1} g_n(\omega) d\omega, \quad g_m(u) = \int_u^\infty g_{m+2}(\omega) d\omega, \quad (3.3)$$

to investigate $g_2$, $g_3$, · · · , it is sufficient to verify that $g_1$ is a non-increasing function i.e. $g'_1(u) \leq 0$. Without loss of generality, assume $a = b = 1$, then

$$g'_1(u) = \frac{(N - 1)u^{N-2}e^{-u} - t^{N-1}e^{-u} + (N - 1)u^{N-2}e^{-2u} - t^{N-1}e^{-2u} + 2ru^{N-1}e^{-2u}}{(1 + e^{-u})^{2r+1}}.$$
$g_1'(u) \leq 0$ if and only if

$$
\frac{u^{N-2}e^{-u}}{(1+e^{-u})^{2r+1}}[(N-1)(u+e^{-u})-u(1+e^{-u})+2re^{-u}] \leq 0,
$$

$$
N \leq \frac{u+e^{-u}+ue^{-u}-2re^{-u}}{u+e^{-u}}.
$$

When $u = 0$, $N \leq 1$; $0 < u < 1$, $N \leq 1$; $u \geq 1$, $N \leq 2$. Above all, $g_1$ is a non-increasing function when $N \leq 1$. Therefore, if $N \leq 1$,

$$
\int_{t}^{\infty} g_3(u)du, \ g_2(u) = \int_{t}^{\infty} (u-t)^{-\frac{1}{2}}g_3(u)du.
$$

Then, we have

$$
g_3(u) = -\frac{1}{(1+e^{-bu})^{2r+1}}\left[(N-1)u^{N-2}e^{-au} - au^{N-1}e^{-au} + (N-1)u^{N-2}e^{-(a+b)u}ight],
$$

$$
g_2(u) = ae^{-au} \sum_{j=0}^{N-1} \frac{(N-1)j}{j!} \frac{\Gamma(j+\frac{1}{2})}{b^{j+\frac{1}{2}}} u^{N-1-j} \Phi_{2r+1}(-e^{-bu}, j + \frac{1}{2}, \frac{a}{b})
$$

$$
-(N-1)e^{-au} \sum_{k=0}^{N-2} \frac{(N-2)k}{k!} \frac{\Gamma(k+\frac{1}{2})}{b^{k+\frac{1}{2}}} u^{N-2-k} \Phi_{2r+1}(-e^{-bu}, k + \frac{1}{2}, \frac{a}{b})
$$

$$
-(N-1)e^{-(a+b)u} \sum_{l=0}^{N-2} \frac{(N-2)l}{l!} \frac{\Gamma(l+\frac{1}{2})}{b^{l+\frac{1}{2}}} u^{N-2-l} \Phi_{2r+1}(-e^{-bu}, l + \frac{1}{2}, \frac{a+b}{b})
$$

$$
+ae^{-au} \sum_{v=0}^{N-1} \frac{(N-1)v}{v!} \frac{\Gamma(v+\frac{1}{2})}{b^{v+\frac{1}{2}}} u^{N-1-v} \Phi_{2r+1}(-e^{-bu}, v + \frac{1}{2}, \frac{a+b}{b})
$$

$$
-2re^{-(a+b)u} \sum_{q=0}^{N-1} \frac{(N-1)q}{q!} \frac{\Gamma(q+\frac{1}{2})}{b^{q+\frac{1}{2}}} u^{N-1-q} \Phi_{2r+1}(-e^{-bu}, q + \frac{1}{2}, \frac{a+b}{b}),
$$

**Fig. 3** $a = b = 4$, $r = 1$, $N = 2$, $s_1 = s_2 = 1$, $\rho = 0.5$
where \((x)_n = x(x-1)(x-2)\cdots(x-n+1)\). In the same way, we can obtain \(g_5, g_7, \cdots\), if \(g_1\) is a completely monotone function i.e. \((-1)^n g_1^{(n)}(t) \geq 0\) for every \(n \in \mathbb{N}_0\) and \(t > 0\), where \(\mathbb{N}_0\) denotes the set of non-negative integers. After that we can obtain \(g_4, g_6, \cdots\). If \(X\) has an elliptically symmetric logistic distribution i.e. \(X \sim \text{Lo}_n(\mu, \Sigma, g_n)\), where \(g_n\) is defined as Eq. 2.5, we can obtain

\[
g_2(t) = \sqrt{\pi} (\Phi^*_n(e^{-t}, 1) - \frac{e^{-t}}{\sqrt{2}} (\Phi^*_n(e^{-t}, 1) - 2)),
\]

\[
g_3(t) = \frac{e^{-t} - e^{-2t}}{(1 + e^{-t})^3}.
\]

In the same way, if \(X = (X_1, \cdots, X_n)^T \sim GL_n(\mu, \Sigma, g_n)\), the d.g. of \(X_i\) (i = 1, 2, \cdots, \(n\)) differs from \(g_n\) obviously. We conclude the property in the following theorem.

**Theorem 3.1** Letting \(X = (X^T(m), X^T(n-m))^T \sim GL_n(\mu, \Sigma, g_n)\), where \(g_n(t)\) is defined as Eq. 2.2, \(1 \leq m < n\), \(X^T(m) \in \mathbb{R}^m\) and \(X^T(n-m) \in \mathbb{R}^{n-m}\), then the d.g. of \(X^T(m)\) is

\[
\hat{g}_m(u) = \sum_{j=0}^{N-1} \binom{N-1}{j} u^{N-1-j} \prod_{k=0}^{j} \int_0^\infty \frac{y^{\frac{n-m}{2}+j-1}e^{-\frac{a(y+u)}{2}}}{(1 + e^{-b(y+u)^2})^{2r}} dy,
\]

where \((x)_n = x(x-1)(x-2)\cdots(x-n+1)\) and \(\Phi^*_2\) is the generalized Hurwitz-Lerch zeta function.

**Proof** By formula (3.3), we have

\[
\hat{g}_m(u) = \int_0^\infty (t-u)^{\frac{n-m}{2}-1} g_n(t) dt = \int_0^\infty y^{\frac{n-m}{2}+j-1} (y+u)^{N-1-j} e^{-a(y+u)^2} \frac{dy}{(1 + e^{-b(y+u)^2})^{2r}}.
\]

In addition, setting \(s_1 = s_2 = 1\),

\[
\hat{g}_m(u) = \sum_{j=0}^{N-1} \frac{(N-1)}{j} u^{-N-1-j} \int_0^\infty \frac{y^{\frac{n-m}{2}+j} e^{-ay}}{(1 + e^{-b(y+u)^2})^{2r}} dy
\]

follows.
Setting \( s_1 = s_2 = s, r = 0 \),

\[
\hat{g}_m(u) = \sum_{j=0}^{N-1} \frac{(N-1)^j}{j!} u^{N-1-j} \int_0^\infty y^{\frac{n-m}{2}+j-1} \exp \left( -\sum_{k=0}^{s} \frac{a(s)_k}{k!} y^k u^{-k} \right) dy
\]

\[
= \sum_{j=0}^{N-1} \frac{(N-1)^j}{j!} u^{N-1-j} \int_0^\infty y^{\frac{n-m}{2}+j-1} \exp \left( -au^s - a \sum_{k=1}^{s} \frac{(s)_k}{k!} y^k u^{-k} \right) dy
\]

(\text{letting } x = u^{-k} y)

\[
= \sum_{j=0}^{N-1} \frac{(N-1)^j}{j!} u^{N-1-j} \exp(-au^s) \prod_{k=1}^{s} \int_0^\infty \left( u^{\frac{n-m}{2}+j} \right) \frac{1}{k!} x^{\frac{1}{k}(\frac{n-m}{2}+j)} \exp(-x) \frac{1}{r(s)_k} dx
\]

\[
= \sum_{j=0}^{N-1} \frac{(N-1)^j}{j!} u^{N-1-j} \exp(-au^s) \prod_{k=1}^{s} \frac{1}{k!} u^{\frac{n-m}{2}+j}
\]

\[
\times \left( \frac{k!}{a(s)_k} \right)^{\frac{1}{k}(\frac{n-m}{2}+j)} \Gamma \left( \frac{1}{k} \left( \frac{n-m}{2} + j \right) \right) \]

\[
= \sum_{j=0}^{N-1} \prod_{k=1}^{s} \frac{(N-1)^j}{j!} \left( \frac{k!}{a(s)_k} \right)^{\frac{1}{k}(\frac{n-m}{2}+j)} \frac{1}{k} \Gamma \left( \frac{1}{k} \left( \frac{n-m}{2} + j \right) \right) u^{N-1-j-(\frac{n-m}{2}+j)} \exp(-au^s)
\]

follows.

The theorem above tells us that \( g_n(\cdot) \), \( \hat{g}_m(\cdot) \) are different in form rather than dimension changing. \( \square \)

**Corollary 3.1**

1) \( \text{Let } X = (X_{(m)}^T, X_{(n-m)}^T)^T \sim L_{0n}(\mu, \Sigma, g_n) \), where \( g_n \) is defined as Eq. 2.5 then

\[
\hat{g}_m(u) = \Gamma \left( \frac{n-m}{2} \right) e^{-u} \Phi^*_2(-e^{-u}, \frac{n-m}{2}, 1).
\]

2) \( \text{Let } X = (X_{(m)}^T, X_{(n-m)}^T)^T \sim K_{0n}(\mu, \Sigma, g_n) \), where \( g_n \) is defined as Eq. 2.4 then

\[
\hat{g}_m(u) = \sum_{j=0}^{N-1} \prod_{k=1}^{s_1} \frac{(N-1)^j}{j!} \left( \frac{k!}{a(s_1)_k} \right)^{\frac{1}{k}(\frac{n-m}{2}+j)} \frac{1}{k} \Gamma \left( \frac{1}{k} \left( \frac{n-m}{2} + j \right) \right) u^{N-1-j-(\frac{n-m}{2}+j)} \exp(-au^{s_1}).
\]

**Proof** We use Theorem 3.1 to conclude the density generator of \( X_{(m)} \).
Theorem 4.1 Let 
\[ g_{m+2}(x) = -\frac{1}{\pi} g_m'(x), \ x > 0 \ (a.e.), \] 
where \( g_m'() \) is the derivative of \( g_m() \). If \( N \leq 1 \), Eq. 3.4 can be applied to the density generators of GL distribution.

4 Conditional Distributions

Consider the partitions of \( X \), \( \mu \), \( \Sigma \) as follows:
\[ X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \] 
where \( X^{(1)} \), \( \mu^{(1)} \in \mathbb{R}^m \) \( (m < n) \), \( X^{(2)} \), \( \mu^{(2)} \in \mathbb{R}^{n-m} \), \( \Sigma_{11} \) is an \( m \times m \) matrix, \( \Sigma_{12} \) is an \( m \times (n-m) \) matrix, \( \Sigma_{21} \) is an \( (n-m) \times m \) matrix and \( \Sigma_{22} \) is an \( (n-m) \times (n-m) \) matrix.

Theorem 4.1 Let \( X \sim GL_n(\mu, \Sigma, g_n) \) where \( g_n \) is defined as Eq. 2.9. Conditionally on \( X^{(2)} = x^{(2)} \), we have the conditional distribution of \( X^{(1)} \).

1) \( X^{(1)} \sim ELL_m(\mu_{1,2}, \Sigma_{11,2}, g_{1,2}) \) where
\[ \mu_{1,2} = \mu^{(1)} + \Sigma_{12} \Sigma_{22}^{-1} (x^{(2)} - \mu^{(2)}), \quad \Sigma_{11,2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}. \]

2) The density generator of \( X^{(1)} | X^{(2)} = x^{(2)} \) can be written as
\[ g_{(1,2)}(t) = \frac{(t + q(x^{(2)}))^N - 1}{e^{-a(t + q(x^{(2)}))^{\frac{1}{2}}}} \cdot \frac{N - 1}{(N - 1) \cdot j!} \sum_{j=0}^{N-1} q(x^{(2)})^{N-1-j} \prod_{k=0}^{s_i} I_{k,j}, \]
where \( q(x^{(2)}) = (x^{(2)} - \mu^{(2)})^\top \Sigma_{22}^{-1} (x^{(2)} - \mu^{(2)}), \]
\[ I_{k,j} \triangleq I_{k,j}(m, a, b, s_1, s_2, r) = \int_0^\infty \frac{y^{m+j-1} e^{-\frac{a}{b}(x^{(2)})y^{2} - k^2y^2} \prod_{j=0}^{s_i} q(x^{(2)})^{r-1} y^r}{(1 + e^{-b \sum_{i=0}^{k} q(x^{(2)})^{r-1} y^r})^{2r}} \, dy. \]

Proof

1) The result follows by Fang et al. (1990).

2) Define an \( n \)-dimension vector
\[ Y = \begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix} = DX, \quad D = \begin{pmatrix} I & \Sigma_{12} \Sigma_{22}^{-1} \\ 0 & I \end{pmatrix}, \]
where \( I \) is an \( m \times m \) identity matrix, \( \Sigma_{12}, \Sigma_{22} \) are given in Eq. 4.1, to make \( Y^{(1)} \in \mathbb{R}^m \), \( Y^{(2)} \in \mathbb{R}^{n-m} \) are independent with each other. We have
\[ Y^{(1)} = X^{(1)} - \Sigma_{12} \Sigma_{22}^{-1} X^{(2)}, \quad Y^{(2)} = X^{(2)}. \]
\[
\mu_Y = D\mu = \begin{pmatrix} 1 & \Sigma_{12} \Sigma_{22}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix} = \begin{pmatrix} \tilde{\mu}_{(1,2)} \\ \mu^{(2)} \end{pmatrix}, \quad \tilde{\mu}_{(1,2)} = \mu^{(1)} - \Sigma_{12} \Sigma_{22}^{-1} \mu^{(2)};
\]

\[
\Sigma_Y = D\Sigma_D' = \begin{pmatrix} 1 & \Sigma_{12} \Sigma_{22}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -\Sigma_{12} \Sigma_{22}^{-1} \\ 0 \\ \Sigma_{22} \end{pmatrix} = \begin{pmatrix} \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}.
\]

where \( \Sigma_{11,2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \). Then \( Y \sim GL_n(\mu_Y, \Sigma_Y, g_n) \), the d.g. of \( Y \) is same as \( X \)'s d.g.. Moreover, the density generators of \( Y^{(1)}, Y^{(2)} \) respectively, i.e. \( \hat{g}_X^{(1)}(t) = \hat{g}_Y^{(1)}(t) = \hat{g}_m(t), \hat{g}_X^{(2)}(u) = \hat{g}_Y^{(2)}(u) = \hat{g}_{n-m}(u) \). \( g_n(\cdot), \hat{g}_m(\cdot), \hat{g}_{n-m}(\cdot) \) are pairwise different in form rather than dimension changing.

\[
f_Y = \frac{C_n}{\sqrt{|\Sigma_Y|}} g_n \left( (y - \mu_Y)^T \Sigma_Y^{-1} (y - \mu_Y) \right)
= \frac{C_n}{\sqrt{|\Sigma_Y|}} g_n \left( (Y^{(1)} - \mu_{1,2})^T \Sigma_{11,2}^{-1} (Y^{(1)} - \mu_{1,2}) \right.
+ (Y^{(2)} - \mu^{(2)})^T \Sigma_{22}^{-1} (Y^{(2)} - \mu^{(2)}) \right)
= \frac{C_n}{\sqrt{|\Sigma_Y|}} g_n \left( (X^{(1)} - \Sigma_{12} \Sigma_{22}^{-1} X^{(2)} - \mu_{1,2})^T \Sigma_{11,2}^{-1} (X^{(1)} - \Sigma_{12} \Sigma_{22}^{-1} X^{(2)} - \mu_{1,2}) \right.
+ (X^{(2)} - \mu^{(2)})^T \Sigma_{22}^{-1} (X^{(2)} - \mu^{(2)}) \right)
= \frac{C_n}{\sqrt{|\Sigma_Y|}} g_n \left( (X^{(1)} - \mu_{1,2})^T \Sigma_{11,2}^{-1} (X^{(1)} - \mu_{1,2}) \right.
+ (X^{(2)} - \mu^{(2)})^T \Sigma_{22}^{-1} (X^{(2)} - \mu^{(2)}) \right),
\]

\[
\mu_{1,2} = \mu^{(1)} + \Sigma_{12} \Sigma_{22}^{-1} (X^{(2)} - \mu^{(2)}),
\]

\[
f_{Y^{(2)}} = \frac{\hat{C}_{n-m}}{\sqrt{\Sigma_{22}}} \hat{g}_{n-m} \left( (X^{(2)} - \mu^{(2)})^T \Sigma_{22}^{-1} (X^{(2)} - \mu^{(2)}) \right),
\]

\[
f_{Y^{(1)}|Y^{(2)}} = \frac{\hat{C}_m}{\sqrt{|\Sigma_{11,2}|}} g_{(1,2)} \left( (X^{(1)} - \mu_{1,2})^T \Sigma_{11,2}^{-1} (X^{(1)} - \mu_{1,2}) \right).
\]

Since

\[
f_{Y^{(1)}|Y^{(2)}} = \frac{f_Y}{f_{Y^{(2)}}},
\]

we have

\[
g_{(1,2)} \left( (X^{(1)} - \mu_{1,2})^T \Sigma_{11,2}^{-1} (X^{(1)} - \mu_{1,2}) \right)
= g_n \left( (X^{(1)} - \mu_{1,2})^T \Sigma_{11,2}^{-1} (X^{(1)} - \mu_{1,2}) + q(X^{(2)}) \right) / \hat{g}_{(n-m)}(q(X^{(2)})),
\]
then we have

\[
g_{1.2}(t) = \frac{(t + q(x^{(2)}))^N - 1 e^{-a(t + q(x^{(2)}))t^1}}{(1 + e^{-b(t + q(x^{(2)}))t^2})^{2r} \sum_{j=0}^{N-1} \frac{(N-1)}{j!} q(x^{(2)}) e^{-a(t + q(x^{(2)}))t^1} \prod_{k=0}^{s^1} I_{k,j}},
\]

and

\[
\mu_{1.2} = \mu^{(1)} + \Sigma_{12} \Sigma_{22}^{-1} (x^{(2)} - \mu^{(2)}), \quad \Sigma_{11.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}.
\]

where \( q(x^{(2)}) = (x^{(2)} - \mu^{(2)})^T \Sigma_{22}^{-1} (x^{(2)} - \mu^{(2)}) \),

\[
I_{k,j} \triangleq I_{k,j}(m, a, b, s_1, s_2, r) = \int_0^{\infty} \frac{y^{m+j-1} e^{-a(s_1)y - b(s_2)y - \frac{t^2}{2} - \frac{t^2 y^2}{2}}}{(1 + e^{-b} \sum_{l=0}^{s_2} (\frac{m}{l!} q(x^{(2)})) e^{-a(s_1)y - \frac{t^2}{2} - \frac{t^2 y^2}{2}) \frac{t^2}{2}}} dy.
\]

In addition, setting \( s_1 = s_2 = 1 \),

\[
g_{1.2}(t) = \frac{(t + q(x^{(2)}))^N - 1 e^{-at}}{(1 + e^{-b(t + q(x^{(2)}))t^2})^{2r} \sum_{j=0}^{N-1} \frac{(N-1)}{j!} \frac{1}{b^j + t} (q(x^{(2)})) e^{-a(t + q(x^{(2)}))t^1} \prod_{k=0}^{s^1} I_{k,j}} \end{array}
\]

holds, where \( q(x^{(2)}) = (x^{(2)} - \mu^{(2)})^T \Sigma_{22}^{-1} (x^{(2)} - \mu^{(2)}) \).

**Corollary 4.1**

1) \( \text{Supposing } X \sim L_{n}(\mu, \Sigma, g_n) \) where \( g_n \) is written as Eq. 2.5, the partitions of \( X \), \( \mu \), \( \Sigma \) are same as Eq. 4.1. The density generator of \( X^{(1)} \) conditionally on \( X^{(2)} = x^{(2)} \) is

\[
g_{1.2}(t) = \left[ \Gamma \left( \frac{m}{2} \right) \Phi^*_2 \left( -e^{-q(x^{(2)})}, \frac{m}{2}, 1 \right) \right]^{-1} \frac{e^{-t}}{(1 + e^{-a(t + q(x^{(2))))t^1})^2}.
\]

2) \( \text{Supposing } X \sim K_{n}(\mu, \Sigma, g_n) \) where \( g_n \) is written as Eq. 2.4, the partitions of \( X \), \( \mu \), \( \Sigma \) is same as Eq. 4.1. The density generator of \( X^{(1)} \) conditionally on \( X^{(2)} = x^{(2)} \) is

\[
g_{1.2}(t) = \frac{(t + q(x^{(2)}))^N - 1 e^{-at \pm q(x^{(2)}))t^1}}{\sum_{j=0}^{N-1} \prod_{k=0}^{s^1} I_{k,j} (x^{(2)} - \mu^{(2)})^T \Sigma_{22}^{-1} (x^{(2)} - \mu^{(2)})},
\]

where

\[
\gamma_{x,y} \triangleq \gamma_{x,y}(N, a, s_1, n, m, b) = (N - 1) \frac{y!}{a(s_1)^x} \frac{1}{\frac{n-m}{2} + y} \Gamma \left( \frac{1}{2} \left( \frac{n-m}{2} + y \right) \right) \frac{x!}{a(s_1)^x}.
\]

### 5 Characteristic Functions and Characteristic Generators

**Theorem 5.1** Let \( X \sim G_{n}(\mu, \Sigma, g_n) \) where \( g_n \) is defined as Eq. 2.9. The characteristic function of \( X \) can be expressed as follows.

1) \( \text{If } n = 1 \),

\[
\psi_X(t) = C_1 e^{it\mu} \int_0^{\infty} \frac{x^{N-\frac{3}{2} e^{-ax^3}}}{(1 + e^{-bx^3})^{2r}} \cos(t \sigma \sqrt{x}) dx, \quad t \in (-\infty, \infty).
\]
2) If \( n > 1 \),

\[
\psi_X(t) = e^{itx} \sum_{j=0}^{\infty} \frac{(-1)^j q_j}{(2j)!} \Phi_{2r}^*\left(-1, \frac{1}{2} (\frac{n}{2} + j + N - 1), \frac{q}{n}\right) (t^r \Sigma t)^j,
\]

where \( \Phi_{2r}^* \) is the generalized Hurwitz-Lerch zeta function, \( B(\cdot) \) is the Beta function, \( t = (t_1, t_2, \ldots, t_n)^T \), \( t_i \in (-\infty, \infty) \),

\[
q_s \triangleq q_s(N, n, b, s) = \frac{\Gamma\left(\frac{1}{s} (\frac{n}{2} + x + N - 1)\right)}{b^{\frac{n}{2} + x + N - 1}} \frac{B(n-1, 2x + 1)}{2}.
\]

**Proof** If \( n = 1 \),

\[
\psi_X(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} \frac{C_1}{\sigma} g_1((x - \mu)^2) dx
\]

\[
= \int_{-\infty}^{\infty} C_1 e^{it\sigma x} g_1(x^2) dx = C_1 e^{it\mu} \int_{-\infty}^{\infty} [\cos(t\sigma x) + i \sin(t\sigma x)] g_1(x^2) dx
\]

\[
= C_1 e^{it\mu} \int_{0}^{\infty} \cos(t\sigma \sqrt{y}) y^{-\frac{1}{2}} g_1(y) dy = C_1 e^{it\mu} \int_{0}^{\infty} \frac{\cos(t\sigma \sqrt{y}) y^{N-\frac{1}{2}} e^{-ay^2}}{1 + e^{-by^2}} \frac{dy}{2r} dy.
\]

If \( n > 1 \),

\[
\psi_X(t) = E(e^{itX}) = e^{itx} E(e^{it\tau^T \Sigma t}) = e^{itx} E[\Omega_n(R^{2r} \tau^T \Sigma t)]
\]

\[
= e^{itx} \Omega_n(\tau^T \Sigma t) \frac{1}{\int_{0}^{\infty} t^{\frac{n}{2} - 1} g_n(t) dt} \frac{v^{\frac{n}{2} + N - 1} \exp(-av^2)}{(1 + \exp(-bv^2))^{2r}} dv
\]

\[
= e^{itx} \Omega_1(t^2),
\]

where \( \Omega_n(||t||^2), t \in \mathbb{R}^n \) is the characteristic function of \( U^{(n)} \) (Fang et al. 1990)

\[
\Omega_n(||t||^2) = \frac{1}{B\left(\frac{n-1}{2}, \frac{1}{2}\right)} \int_{0}^{\pi} \exp(i||t||^2 \cos \theta) \sin^{n-2} \theta d\theta,
\]

\[
I_1 = \int_{0}^{\infty} \Omega_n(t^T \Sigma t) \frac{1}{\int_{0}^{\infty} t^{\frac{n}{2} - 1} g_n(t) dt} \frac{v^{\frac{n}{2} + N - 1} \exp(-av^2)}{(1 + \exp(-bv^2))^{2r}} dv
\]

\[
= \int_{0}^{\infty} \frac{1}{B\left(\frac{n-1}{2}, \frac{1}{2}\right)} \left[ \frac{\Gamma\left(\frac{1}{s} (\frac{n}{2} + x + N - 1)\right)}{b^{\frac{n}{2} + x + N - 1}} \Phi_{2r}^*\left(-1, \frac{1}{s} (\frac{n}{2} + N - 1), \frac{a}{b}\right) \right]^{-1}
\]

\[
\times \int_{0}^{\pi} \exp(i \frac{1}{2} \tau^T \Sigma t^2 \cos \theta) \sin^{n-2} \theta d\theta \frac{v^{\frac{n}{2} + N - 2} \exp(-av^2)}{(1 + \exp(-bv^2))^{2r}} dv
\]

\[
= \frac{1}{B\left(\frac{n-1}{2}, \frac{1}{2}\right)} \left[ \frac{\Gamma\left(\frac{1}{s} (\frac{n}{2} + x + N - 1)\right)}{b^{\frac{n}{2} + x + N - 1}} \Phi_{2r}^*\left(-1, \frac{1}{s} (\frac{n}{2} + N - 1), \frac{a}{b}\right) \right]^{-1} \int_{0}^{\pi} I_2 \sin^{n-2} \theta d\theta,
\]

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\[ I_2 = \int_0^\infty v^{\frac{n}{2} + N - 2} \exp(-av^s + i v^{\frac{1}{2}} (t^T \Sigma t)^{\frac{1}{2}} \cos \theta) \frac{dv}{(1 + \exp(-bv^s))^{2r}} \]
\[ = \sum_{l=0}^\infty \frac{i^l (t^T \Sigma t)^{\frac{l}{2}} \cos \theta}{l!} \int_0^\infty \frac{v^{\frac{n}{2} + N - 2} \exp(-av^s)}{(1 + \exp(-bv^s))^{2r}} dv \]
\[ = \sum_{l=0}^\infty \frac{i^l (t^T \Sigma t)^{\frac{l}{2}} \cos \theta}{l!} \int_0^\infty \frac{1 v^{\frac{n}{2} + N - 1 - 1} \exp(-av)}{s} (1 + \exp(-bv^s))^{2r} dv \]
\[ = \sum_{l=0}^\infty \frac{\Gamma\left(\frac{1}{2} (\frac{n}{2} + N - 1)\right) i^l (t^T \Sigma t)^{\frac{l}{2}} \cos \theta}{b^{\frac{l}{2} (\frac{n}{2} + N - 1)} s} \frac{\Phi_{2r}^* (-1, -\frac{1}{2} (\frac{n}{2} + N - 1), \frac{a}{b})}{l!} \Phi_{2r}^* (-1, \frac{n + l}{2} + N - 1, \frac{a}{b}). \]

Then, we have
\[ I_1 = \frac{1}{B(t^T \Sigma t)^{\frac{1}{2} (\frac{n}{2} + N - 1)}} \left[ \frac{\Gamma\left(\frac{1}{2} (\frac{n}{2} + N - 1)\right)}{b^{\frac{1}{2} (\frac{n}{2} + N - 1)} s} \Phi_{2r}^* (-1, -\frac{1}{2} (\frac{n}{2} + N - 1), \frac{a}{b}) \right]^{-1} \]
\[ \times \left( \sum_{l=0}^\infty \frac{\Gamma\left(\frac{1}{2} (\frac{n}{2} + N - 1)\right) i^l (t^T \Sigma t)^{\frac{l}{2}} \cos \theta}{b^{\frac{l}{2} (\frac{n}{2} + N - 1)} s} \Phi_{2r}^* (-1, -\frac{1}{2} (\frac{n}{2} + N - 1), \frac{a}{b}) \right) \]
\[ \times (-1, \frac{n + l}{2} + N - 1, \frac{a}{b}) \int_0^\pi \sin^{n+2} \theta \cos^l \theta d\theta \]
\[ = \frac{1}{B(t^T \Sigma t)^{\frac{1}{2} (\frac{n}{2} + N - 1)}} \left[ \frac{\Gamma\left(\frac{1}{2} (\frac{n}{2} + N - 1)\right)}{b^{\frac{1}{2} (\frac{n}{2} + N - 1)} s} \Phi_{2r}^* (-1, -\frac{1}{2} (\frac{n}{2} + N - 1), \frac{a}{b}) \right]^{-1} \]
\[ \times \left( \sum_{j=0}^\infty \frac{\Gamma\left(\frac{1}{2} (\frac{n}{2} + N - 1)\right)}{b^{\frac{1}{2} (\frac{n}{2} + N - 1)} s} \frac{i^{2j+1} (t^T \Sigma t)^{\frac{2j+1}{2}}}{(2j+1)!} \Phi_{2r}^* (-1, -\frac{1}{2} (\frac{n}{2} + N - 1), \frac{a}{b}) B \right) \]
\[ \times (-1, \frac{n + 2j+1}{2} + N - 1, \frac{a}{b}) \]
\[ + \sum_{j=0}^\infty \frac{\Gamma\left(\frac{1}{2} (\frac{n}{2} + N - 1)\right)}{b^{\frac{1}{2} (\frac{n}{2} + N - 1)} s} \frac{i^{2j} (t^T \Sigma t)^{\frac{2j}{2}}}{(2j)!} \Phi_{2r}^* (-1, -\frac{1}{2} (\frac{n}{2} + N - 1), \frac{a}{b}) \]
\[ \times (-1, \frac{n + 2j+1}{2} + N - 1, \frac{a}{b}) \]
\[ \times \left( \frac{n - 1}{2}, \frac{2j+1}{2} \right) \]
\[ = \left[ B\left(\frac{n}{2} + N - 1\right), \frac{1}{2} \right] \left[ \frac{\Gamma\left(\frac{1}{2} (\frac{n}{2} + N - 1)\right)}{b^{\frac{1}{2} (\frac{n}{2} + N - 1)} s} \Phi_{2r}^* (-1, -\frac{1}{2} (\frac{n}{2} + N - 1), \frac{a}{b}) \right]^{-1} \]
\[ \times \left( \sum_{j=0}^\infty B\left(\frac{n}{2} + N - 1\right), \frac{1}{2} \right) \frac{\Gamma\left(\frac{1}{2} (\frac{n}{2} + N - 1)\right)}{b^{\frac{1}{2} (\frac{n}{2} + N - 1)} s} \frac{(-1)^j (t^T \Sigma t)^{\frac{j}{2}}}{(2j)!} \Phi_{2r}^* (-1, -\frac{1}{2} (\frac{n}{2} + N - 1), \frac{a}{b}) \]
\[ \times (-1, \frac{n + 2j+1}{2} + N - 1, \frac{a}{b}) \]

Finally, we obtain
\[ \psi_X(t) = e^{itT \mu} \sum_{j=0}^\infty \frac{(-1)^j}{(2j)!} q_j \Phi_{2r}^* (-1, -\frac{1}{2} (\frac{n}{2} + j + N - 1), \frac{a}{b}) (t^T \Sigma t)^{\frac{j}{2}}. \]
where
\[ q_x \triangleq q_x(N, n, b, s) = \frac{\Gamma\left(\frac{n}{2} + x + N - 1\right)}{b^{\frac{1}{2}(\frac{n}{2} + N - 1)}} B\left(\frac{n - 1}{2}, \frac{2x + 1}{2}\right). \]

\[ \square \]

**Corollary 5.1** Let \( X \sim GL_n(\mu, \Sigma, g_n) \) where \( g_n \) is defined as Eq. 2.9. The characteristic generator of \( X \) can be expressed as follows.

1) If \( n = 1 \),
\[ \phi_X(t) = C_1 \int_0^\infty x^{N-\frac{1}{2}} e^{-ax^2} \frac{\cos(t \sigma \sqrt{x})}{(1 + e^{-bx^2})^2} dx, \quad t \in (-\infty, \infty). \]

2) If \( n > 1 \), letting \( u_n = (u_1, u_2, \ldots, u_n)^T \),
\[ \phi_X(||u_n||^2) = \sum_{j=0}^\infty (-1)^j q_j \cdot \phi^*_2r(\sum_{k=0}^{\infty} (-1)^k ||u_n||^2)^j, \]
where
\[ q_x \triangleq q_x(N, n, b, s) = \frac{\Gamma\left(\frac{n}{2} + x + N - 1\right)}{b^{\frac{1}{2}(\frac{n}{2} + x + N - 1)}} B\left(\frac{n - 1}{2}, \frac{2x + 1}{2}\right). \]

**Proof** The results directly follow by the definition of characteristic generator. \( \square \)

**Remark 5.1** \( \Omega_n(||t||^2) \) can be expressed in the following alternative forms:
\[ \Omega_n(||t||^2) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}} \sum_{k=0}^\infty (-1)^k ||t||^{2k} \frac{\Gamma(k + \frac{1}{2})}{\Gamma(n + 2k)}, \]
\[ \Omega_n(||t||^2) = 0 F_1\left(\frac{n}{2}; -\frac{1}{4} ||t||^2\right). \]

We can obtain the following equivalent forms of the characteristic functions and characteristic generators with dimension \( n > 1 \).

\[ \psi_X(t) = e^{it^T \mu} \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!} \gamma_k(N, n, b, s) \frac{\Phi^*_{2r}(\sum_{k=0}^{\infty} (-1)^k ||u_n||^2)^k)}{\Phi^*_{2r}(\sum_{k=0}^{\infty} (-1)^k ||u_n||^2)^k) t^T \Sigma t)^k}, \]
\[ \phi_X(||u_n||^2) = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!} \gamma_k(N, n, b, s) \frac{\Phi^*_{2r}(\sum_{k=0}^{\infty} (-1)^k ||u_n||^2)^k)}{\Phi^*_{2r}(\sum_{k=0}^{\infty} (-1)^k ||u_n||^2)^k)} (t^T \Sigma t)^k, \]
\[ \psi_X(t) = e^{it^T \mu} \sum_{k=0}^\infty \frac{\Gamma\left(\frac{n}{2} + k\right)}{\Gamma\left(\frac{n}{2} + N - 1\right) b^{\frac{1}{2}(\frac{n}{2} + N - 1)} 4^k (k)!} \frac{\Phi^*_{2r}(\sum_{k=0}^{\infty} (-1)^k ||u_n||^2)^k)}{\Phi^*_{2r}(\sum_{k=0}^{\infty} (-1)^k ||u_n||^2)^k)} (t^T \Sigma t)^k, \]
\[ \phi_X(||u_n||^2) = \sum_{k=0}^\infty \frac{\Gamma\left(\frac{n}{2} + k\right)}{\Gamma\left(\frac{n}{2} + N - 1\right) b^{\frac{1}{2}(\frac{n}{2} + N - 1)} 4^k (k)!} \frac{\Phi^*_{2r}(\sum_{k=0}^{\infty} (-1)^k ||u_n||^2)^k)}{\Phi^*_{2r}(\sum_{k=0}^{\infty} (-1)^k ||u_n||^2)^k)} (||u_n||^2)^k. \]
Similar as the density generator, the characteristic generator (c.g.) of the class of multivariate elliptically symmetric distribution defined in Definition 2.1 is not dimensionally coherent. In other word, the characteristic function $\psi_X(\cdot)$ and the characteristic generator $\phi_X(\cdot)$ are related to the dimension of $X$. More details will be discussed in Section 7.

### 6 Moments

**Theorem 6.1** Let $X \sim GB_{L_n}(\mu, \Sigma, g_n)$ where $g_n$ is defined as Eq. 2.9.

1) The expectation and the covariance are:

$$E(X) = \mu, \quad Cov(X) = \frac{1}{n} b\frac{1}{s} \Gamma(\frac{1}{s}(N + \frac{n}{2}+\frac{m}{2} - 1))\Phi^s_{2r}(-1, \frac{1}{s}(N + \frac{n}{2}+\frac{m}{2} - 1), \frac{a}{b}) \Sigma;$$

2) For any integers $m_1, \cdots, m_n$, with $m = \sum_{i=1}^n m_i$, the product moments of $Z := \Sigma^{-\frac{1}{2}}(X - \mu)$ are

$$E(\prod_{i=1}^n Z_i^{m_i}) = \frac{1}{n} \Gamma(\frac{1}{s}(N + \frac{n}{2}+\frac{m}{2} - 1))\Phi^s_{2r}(-1, \frac{1}{s}(N + \frac{n}{2}+\frac{m}{2} - 1), \frac{a}{b}) \prod_{i=1}^n (2l_i)! \frac{1}{4l_i!},$$

where $\chi^{[n]} = \chi(x+1) \cdots (x+n-1)$ and $\Phi^s_{2r}$ is the generalized Hurwitz-Lerch zeta function, if $m_i = 2l_i$ are even, $i = 1, \cdots, n, m = 2l$;

$$E(\prod_{i=1}^n Z_i^{m_i}) = 0,$$

if at least one of the $m_i$ is odd.

**Proof** According to Eq. 1.2 we have for real number $p > 0$,

$$E(R^p) = \frac{1}{n} \frac{1}{2s} \Gamma(\frac{1}{s}(N + \frac{n}{2}+\frac{m}{2} - 1))\Phi^s_{2r}(-1, \frac{1}{s}(N + \frac{n}{2}+\frac{m}{2} - 1), \frac{a}{b}) \prod_{i=1}^n (2l_i)! \frac{1}{4l_i!},$$

1) Since $X = \mu + RA^T U^{(n)}$ and $E(U^{(n)}) = 0$, we have

$$E(X) = \mu + E(R)A^T E(U^{(n)}) = \mu,$$
and
\[ \text{Cov}(X) = \text{Cov}(RA^T U^{(n)}) = E(R^2)A^T \text{Cov}(U^{(n)})A = \frac{1}{n} E(R^2) \Sigma \]
\[ = \frac{1}{n} \frac{\Gamma\left(\frac{1}{2} (N + \frac{n}{2})\right) \Phi_{2r}^*(N + \frac{n}{2}, -1, \frac{n}{2}, \frac{1}{4})}{\Gamma\left(\frac{1}{2} (N + \frac{n}{2} - 1)\right) \Phi_{2r}^*(-1, \frac{n}{2}, \frac{1}{4})} \Sigma. \]

2) By Eqs. 2.18 and 3.6 in Fang et al. (1990),
\[ E(\prod_{i=1}^{n} Z_i^m_i) = E(R^m) E(\prod_{i=1}^{n} u_i^m_i), \]
where \( E(\prod_{i=1}^{n} u_i^m_i) = \frac{1}{(\frac{2}{n})^m} \prod_{i=1}^{m} \frac{(2l_i)!}{4^l (l_i)!}, \) if \( m_i = 2l_i (i = 1, \ldots, n) \) are even, \( m = 2l; \)
\[ E(\prod_{i=1}^{n} u_i^m_i) = 0, \] if at least one of the \( m_i \) is odd.

\[ \square \]

**Corollary 6.1**

1) Let \( X \sim \text{Ko}_n(\mu, \Sigma, g_n) \) where \( g_n \) is defined as Eq. 2.4.

(1) The expectation and the covariance are:
\[ E(X) = \mu, \quad \text{Cov}(X) = \frac{1}{n} \frac{\Gamma\left(\frac{1}{2} (N + \frac{n}{2})\right)}{\Gamma\left(\frac{1}{2} (N + \frac{n}{2} - 1)\right)} \Sigma; \]

(2) For any integers \( m_1, \ldots, m_n, \) with \( m = \sum_{i=1}^{n} m_i, \) the product moments of \( Z := \Sigma^{-\frac{1}{2}} (X - \mu) \) are
\[ E(\prod_{i=1}^{n} Z_i^m_i) = \frac{\Gamma\left(\frac{1}{2} (N + \frac{n}{2} + m - 1)\right)}{(\frac{2}{n})^m \Gamma\left(\frac{1}{2} (N + \frac{n}{2} - 1)\right)} \prod_{i=1}^{m} \frac{(2l_i)!}{4^l (l_i)!}, \]
where \( x^{[n]} = x(x + 1) \cdots (x + n - 1), \) \( \Phi_{2r}^* \) is the generalized Hurwitz-Lerch zeta function, if \( m_i = 2l_i (i = 1, \ldots, n) \) are even, \( m = 2l; \)
\[ E(\prod_{i=1}^{n} Z_i^m_i) = 0, \] if at least one of the \( m_i \) is odd.

2) Let \( X \sim \text{Lo}_n(\mu, \Sigma, g_n) \) where \( g_n \) is defined as Eq. 2.5.

(1) The expectation and the covariance are:
\[ E(X) = \mu, \quad \text{Cov}(X) = \frac{1}{n} \frac{\Gamma\left(1 + \frac{n}{2}\right) \Phi_{2r}^*(-1, N + \frac{n}{2}, 1)}{\Gamma\left(\frac{1}{2} \Phi_{2r}^*\right)(-1, n + \frac{n}{2}, 1)} \Sigma; \]

(2) For any integers \( m_1, \ldots, m_n, \) with \( m = \sum_{i=1}^{n} m_i, \) the product moments of \( Z := \Sigma^{-\frac{1}{2}} (X - \mu) \) are
\[ E(\prod_{i=1}^{n} Z_i^m_i) = \frac{\Gamma\left(\frac{n}{2} + \frac{m}{2}\right) \Phi_{2r}^*(-1, \frac{n}{2} + \frac{m}{2}, 1)}{(\frac{2}{n})^m \Gamma\left(\frac{1}{2} \Phi_{2r}^*\right)(-1, n + \frac{n}{2}, 1)} \prod_{i=1}^{m} \frac{(2l_i)!}{4^l (l_i)!}, \]
if \( m_i = 2l_i \) \((i = 1, \cdots, n)\) are even, \( m = 2l; \) \( E(\prod_{i=1}^{n} u_{m_i}) = 0, \) if at least one of the \( m_i \) is odd, where \( x^{[n]} = x(x + 1) \cdots (x + n - 1), \) \( \Phi^{r}_{2r} \) is the generalized Hurwitz-Lerch zeta function.

7 Linear Transform and Marginal Distributions

It has been mentioned in lots of researches that if \( X \sim \text{Ell}_{n}(\mu, \Sigma, \phi), \) \( \text{rank}(\Sigma)= k, \) \( B \) is an \( n \times m \) matrix and \( v \) is an \( m \times 1 \) vector, then

\[
v + B^{T}X \sim \text{Ell}_{m}(v + B^{T}\mu, B^{T}\Sigma B, \phi).
\]

In the theorem above the characteristic generators of elliptical distributions are regarded as unrelated to dimension i.e. the characteristic generator would not change with the dimension during the linear transform. However the dimension coherent property is inapplicable for the characteristic generator of GL distribution. We will demonstrate GL distribution’s linear transform property in the following theorem.

**Theorem 7.1** Assuming \( X \sim GL_{n}(\mu, \Sigma, \phi) \) with stochastic representation \( X = \mu + RA^{T}U^{(n)}, \) \( Y = BX + b, \) where \( B \) is an \( m \times n \) matrix, \( 1 \leq m < n, \) \( \text{rank}(B) = m \) and \( b \in \mathbb{R}^{n}, \) the d.g. of \( X \) is

\[
g_{n}(x) = \frac{x^{N-1} \exp(-ax)}{(1 + \exp(-bx))^{2r}}.
\]

1) \( Y \sim \text{Ell}_{m}(B\mu + b, BS\Sigma B^{T}, g_{(m),y}), \) where

\[
g_{(m),y} = \sum_{j=0}^{N-1} \frac{(N-1)_{j} \Gamma(n-j)}{j!} e^{-a\Phi^{(n-j)}_{2r}(e^{-bu}, \frac{n-m}{2}, \frac{a}{b})}.
\]

2) \( Y \sim \text{Ell}_{m}(B\mu + b, BS\Sigma B^{T}, \phi_{(m),y}), \) where

\[
\phi_{(m),y}(t) = \sum_{j=0}^{N-1} \frac{(N-1)_{j} \Gamma(n-j)}{j!} e^{-a\Phi^{(n-j)}_{2r}(e^{-bu}, \frac{n-1}{2}, \frac{a}{b})} \int_{0}^{\infty} y^{N-\frac{3}{2}} e^{-ay} \cos(\sigma y)dy,
\]

\( t \in (-\infty, \infty), \) if \( m = 1; \)

\[
\phi_{(m),y}(|\xi_{(m)}|^{2}) = \frac{\sum_{j=0}^{N-1} \sum_{l=0}^{\infty} \sum_{o=0}^{\infty} \frac{(-1)^{m} \alpha_{j}^{*}\beta_{j}^{*}q_{0}^{*}(N, m, a, b, j, l)}{\alpha_{j}^{*}\beta_{j}^{*}q_{0}^{*}(N, m, a, b, j, l)} \|\xi_{(m)}\|^{2o}}{\sum_{k=0}^{N-1} \sum_{j=0}^{\infty} \alpha_{j}^{*}\beta_{j}^{*}q_{0}^{*}(N, m, a, b, j, l)}
\]

\( \xi_{i} \in (-\infty, \infty), \)

\[
\alpha_{x}^{*} = \alpha_{x}^{*}(N, n, m, a, b, r) = \frac{(-1)^{r} \Gamma(2r+x)}{x!(x + \frac{a}{b})^{n+N-1}}, \quad \beta_{x}^{*} = \beta_{x}^{*}(N, n, m, b, r)
\]

\[
= \frac{(N-1)_{x} \Gamma(n-x)}{b^{n-m+x} \Gamma(2r)},
\]

\( \xi \) Springer
\[
a_\omega^*(N, m, a, b, j, l) = \frac{\Gamma\left(\frac{m}{2} + \omega + N - j - 1\right) B\left(\frac{m+1}{2} - \frac{2\omega+1}{2}\right)}{b^{m+\omega+j-1}(l + \frac{a}{b})^\omega}, \quad \omega = 0, 1, 2, \cdots,
\]

\[
\xi_{(m)} = (\xi_1, \xi_2, \cdots, \xi_m)^T, \text{ if } m > 1.
\]

**Proof**  
\(Y\) is no longer GL distributed but elliptically distributed still.

1) Applying Theorem 3.1 the result follows.

2) When \(m = 1\),

\[
\psi_Y(t) = E(e^{itY}) = \int_{-\infty}^{\infty} e^{itx} \tilde{C}_1 \tilde{g}_1\left(\frac{y - \mu_Y}{\sigma_Y}\right)^2 dx
\]

\[
= \tilde{C}_1 e^{it\mu_Y} \int_0^\infty \cos(t\sigma_Y \sqrt{y}) \tilde{g}_1(y) y^{-\frac{1}{2}} dy
\]

\[
= \tilde{C}_1 e^{it\mu_Y} \int_0^\infty \cos(t\sigma_Y \sqrt{y}) \sum_{j=0}^{N-1} (N-1)_j \frac{\Gamma\left(\frac{n-1}{2} + j\right)}{b^{n-1+j}} y^{N-j-\frac{3}{2}} e^{-ay} \Phi_2^*
\]

\[
\times (-e^{-by}, \frac{n-1}{2} + j, \frac{a}{b}) dy
\]

\[
= \tilde{C}_1 e^{it\mu_Y} \sum_{j=0}^{N-1} (N-1)_j \frac{\Gamma\left(\frac{n-1}{2} + j\right)}{b^{n-1+j}} \int_0^\infty y^{N-j-\frac{3}{2}} e^{-ay} \cos(t\sigma_Y \sqrt{y}) \Phi_2^* dy
\]

therefore,

\[
\phi_Y(t^2) = \sum_{j=0}^{N-1} C_{N-1}^{j} \frac{\Gamma\left(\frac{n-1}{2} + j\right)}{b^{n-1+j}} \Phi_2^*(-e^{-bu}, \frac{n-1}{2} + j, \frac{a}{b}) \int_0^\infty y^{N-j-\frac{3}{2}} e^{-ay} \cos(t\sigma_Y \sqrt{y}) dy.
\]

When \(m > 1\),

\[
\psi_Y(t) = E(e^{itY}) = e^{it\mu_Y} \frac{1}{\int_{0}^{\infty} t^{n-1} g(t) dt}
\]

\[
\times \int_0^\infty \Omega_n(vt^T \Sigma_Y t) v^{m-1} \sum_{j=0}^{N-1} (N-1)_j \frac{\Gamma\left(\frac{n-m}{2} + j\right)}{b^{n-m+j}} v^{N-1-j} e^{-av} \Phi_2^*
\]

\[
\times (-e^{-bv}, \frac{n-m}{2}, \frac{a}{b}) dv
\]

\[
e^{it\mu_Y} I_{1}^{-1} \frac{1}{B\left(\frac{m-1}{2}, \frac{1}{2}\right)} \int_0^\pi \sin^{m-2} \theta I_2 d\theta,
\]
where

\[ I_{1}^{-1} = \int_{0}^{\infty} t^{-\frac{1}{2}-1} g_m(t) dt \]

\[ = \left[ \int_{0}^{\infty} t^{-\frac{1}{2}} \sum_{k=0}^{N-1} \frac{(N-1)_k}{k!} \frac{\Gamma(\frac{n-m}{2} + k)}{b^{\frac{n-m}{2} + k}} e^{-\frac{t}{b^{\frac{n-m}{2} + k}}} \Phi_{2r}^* (-e^{-bt}, \frac{n-m}{2} + k, \frac{a}{b}) dt \right]^{-1} \]

\[ = \left[ \sum_{k=0}^{N-1} \frac{(N-1)_k}{k!} \frac{\Gamma(\frac{n-m}{2} + k)}{b^{\frac{n-m}{2} + k}} \frac{1}{\Gamma(2r)} \int_{0}^{\infty} t^{\frac{n-m}{2} - k - 2} \Phi_{2r}^* (-e^{-bt}, \frac{n-m}{2} + k, \frac{a}{b}) dt \right]^{-1} \]

\[ = \left[ \sum_{k=0}^{N-1} \beta_k \sum_{j=0}^{\infty} \frac{\alpha_j}{(j + \frac{a}{b})^{\frac{n-m}{2} + k}} \int_{0}^{\infty} t^{\frac{n-m}{2} - k - 2} \Phi_{2r}^* (-e^{-(a+b)t}, \frac{n-m}{2} + k, \frac{a}{b}) dt \right]^{-1} \]

\[ = \left[ \sum_{k=0}^{N-1} \beta_k \sum_{j=0}^{\infty} \alpha_j \beta_j \frac{1}{(j + \frac{a}{b})^{\frac{n-m}{2} + k}} \int_{0}^{\infty} t^{\frac{n-m}{2} - k - 2} \Phi_{2r}^* (-e^{-(a+b)t}, \frac{n-m}{2} + k, \frac{a}{b}) dt \right]^{-1} \]

\[ = \left[ \sum_{k=0}^{N-1} \sum_{j=0}^{\infty} \alpha_j \beta_j \Gamma(\frac{n}{2} + N - k - 1) \right]^{-1} \]

\[ = \left[ \sum_{k=0}^{N-1} \sum_{j=0}^{\infty} \alpha_j \beta_j \Gamma(\frac{n}{2} + N - k - 1) \right]^{-1} \]

\[ \beta_x \triangleq \beta_x(N, n, m, b, r) = \frac{(N-1)_x \Gamma(\frac{n-m}{2} + x)}{x! \frac{n-m}{2} + x \Gamma(2r)}, \quad \alpha_x \triangleq \alpha_x(N, n, a, b, r) = \frac{\alpha_x}{(x + \frac{a}{b})^{\frac{n-m}{2} + N - 1}}. \]

\[ I_2 = \int_{0}^{\infty} \exp\left( v^2 \left( t^\top \Sigma_t \right)^2 \cos \theta \right) \]

\[ \times \sum_{j=0}^{N-1} \frac{(N-1)_j \Gamma(\frac{n-m}{2} + j)}{j! \frac{n-m}{2} + j} \int_{0}^{\infty} v^{\frac{n-m}{2} - j - 2} e^{-av} \Phi_{2r}^* (-e^{av} b^\frac{n-m}{2} + j, \frac{a}{b}) dt \]

\[ = \sum_{j=0}^{N-1} \frac{(N-1)_j \Gamma(\frac{n-m}{2} + j)}{j! \frac{n-m}{2} + j} \int_{0}^{\infty} v^{\frac{n-m}{2} - j - 2} e^{-av} \Phi_{2r}^* (-e^{av} b^\frac{n-m}{2} + j, \frac{a}{b}) dt \]

\[ = \sum_{j=0}^{N-1} \sum_{l=0}^{\infty} \beta_j \frac{1}{(l + \frac{a}{b})^{\frac{n-m}{2} + j}} \int_{0}^{\infty} v^{\frac{n-m}{2} - j - 2} e^{-av} \Phi_{2r}^* (-e^{av} b^\frac{n-m}{2} + j, \frac{a}{b}) dt \]

\[ = \sum_{j=0}^{N-1} \sum_{l=0}^{\infty} \beta_j \frac{1}{(l + \frac{a}{b})^{\frac{n-m}{2} + j}} \int_{0}^{\infty} v^{\frac{n-m}{2} - j - 2} e^{-av} \Phi_{2r}^* (-e^{av} b^\frac{n-m}{2} + j, \frac{a}{b}) dt \]

\[ = \sum_{j=0}^{N-1} \sum_{l=0}^{\infty} \beta_j \frac{1}{(l + \frac{a}{b})^{\frac{n-m}{2} + j}} \int_{0}^{\infty} v^{\frac{n-m}{2} - j - 2} e^{-av} \Phi_{2r}^* (-e^{av} b^\frac{n-m}{2} + j, \frac{a}{b}) dt \]

\[ = \sum_{j=0}^{N-1} \sum_{l=0}^{\infty} \beta_j \frac{1}{(l + \frac{a}{b})^{\frac{n-m}{2} + j}} \int_{0}^{\infty} v^{\frac{n-m}{2} - j - 2} e^{-av} \Phi_{2r}^* (-e^{av} b^\frac{n-m}{2} + j, \frac{a}{b}) dt \]
Then we have
\[
\int_0^\pi \sin^{m-2} \theta I_2 d\theta = \sum_{j=0}^{N-1} \sum_{l=0}^\infty \sum_{q=0}^\infty \alpha_l \beta_j \left( (t^T \Sigma_Y t)^\nu \right) \frac{i^q}{l + \frac{a}{b}} \frac{(a + bl)^{m_2 + N - j - 1}}{(l + a/b)^{m_2 + N - j - 1}} \int_0^\pi \cos^q \theta \sin^{m-2} \theta d\theta
\]
\[
= \sum_{j=0}^{N-1} \sum_{l=0}^\infty \sum_{q=0}^\infty \frac{(-1)^\omega}{(2\omega)!} \alpha_l \beta_j \left( (t^T \Sigma_Y t)^\omega \right) \frac{\Gamma\left( \frac{m_2 + \omega + N - j - 1}{2} \right)}{\beta^{m_2 + N - j - 1}(l + \frac{a}{b})^\omega}.
\]

Therefore,
\[
\psi_Y(t) = e^{it^T \mu_Y} \frac{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \sum_{q=0}^\infty \frac{(-1)^\omega}{(2\omega)!} \alpha_l \beta_j \left( (t^T \Sigma_Y t)^\omega \right) \frac{\Gamma\left( \frac{m_2 + \omega + N - j - 1}{2} \right)}{\beta^{m_2 + N - j - 1}(l + \frac{a}{b})^\omega}}{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \sum_{q=0}^\infty \frac{(-1)^\omega}{(2\omega)!} \alpha_l \beta_j q_\omega(N, m, a, b, j, l)(t^T \Sigma_Y t)^\omega},
\]
where
\[
q_\omega(N, m, a, b, j, l) = \frac{\Gamma\left( \frac{m_2 + \omega + N - j - 1}{2} \right)}{\beta^{m_2 + N - j - 1}(l + \frac{a}{b})^\omega}, \quad \omega = 0, 1, 2, \ldots.
\]

Remark 7.1 Similar as Remark 5.1, we can obtain the following equivalent forms of the characteristic functions and characteristic generators of \( Y = B X + b \) with dimension \( m > 1 \).

\[
\psi_Y(t) = e^{it^T \mu_Y} \frac{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \sum_{p=0}^\infty \frac{(-1)^l}{(2l)!} \alpha_p \beta_k A_l \Gamma\left( \frac{m_2 + N - k + l - 1}{2} \right)}{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \alpha_k \beta_j \Gamma\left( \frac{m_2 + N - j - 1}{2} \right)} (t^T \Sigma_Y t)^l,
\]

\[
\phi_{(m),y}(\|\xi_{(m)}\|^2) = \frac{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \sum_{p=0}^\infty \frac{(-1)^l}{(2l)!} \alpha_p \beta_k A_l \Gamma\left( \frac{m_2 + N - k + l - 1}{2} \right)}{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \alpha_k \beta_j \Gamma\left( \frac{m_2 + N - j - 1}{2} \right)} \|\xi_{(m)}\|^{2l};
\]

\[
\psi_Y(t) = e^{it^T \mu_Y} \frac{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \sum_{k=0}^\infty \frac{(-1)^l}{(2l)!} \alpha_k \beta_j \Phi^*_\omega \left( -e^{-bv}, \frac{n-m-j}{2}, \frac{a}{b} \right)}{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \alpha_k \beta_j \Gamma\left( \frac{m_2 + N - j - 1}{2} \right)} \frac{\Gamma\left( \frac{m_2}{2} \right)}{\pi^{1/2} \Gamma\left( \frac{m_2}{2} \right)^l} (t^T \Sigma_Y t)^l,
\]

\[
\phi_{(m),y}(\|\xi_{(m)}\|^2) = \frac{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \sum_{k=0}^\infty \frac{(-1)^l}{(2l)!} \alpha_k \beta_j \Phi^*_\omega \left( -e^{-bv}, \frac{n-m-j}{2}, \frac{a}{b} \right)}{\sum_{j=0}^{N-1} \sum_{l=0}^\infty \alpha_k \beta_j \Gamma\left( \frac{m_2 + N - j - 1}{2} \right)} \frac{\Gamma\left( \frac{m_2}{2} \right)}{\pi^{1/2} \Gamma\left( \frac{m_2}{2} \right)^l} \|\xi_{(m)}\|^{2l}.
\]
where
\[
A_x \triangleq A_x(N, m, a, b, p, k) = \frac{B(\frac{m}{2}, x + \frac{1}{2})}{B(\frac{m}{2} + x, \frac{1}{2})^{bN-k+x}(p + \frac{a}{b})^{x-m+1}},
\]
\[
\beta_x \triangleq \beta_x(N, n, m, r) = \frac{(N - 1)x}{x!} \sum_{j=0}^{n-m} \frac{\Gamma(\frac{n-m}{2} + j)}{\Gamma(2r)b^{\frac{n-m}{2} + j}} u^{n-1 - j} e^{-au} \Phi_1^*(-e^{-bu}, n - m^2 + 2j, a/b).
\]
\[
\alpha_x \triangleq \alpha_x(r) = \frac{(-1)^{r} \Gamma(2r + x)}{x!}, \quad \alpha_x^* \triangleq \alpha_x^*(N, m, a, b, r) = \frac{\alpha_x}{(x + \frac{a}{b})} e^{\Phi_1^*(n - m - 1, 0, \frac{n}{2} + r - 1, \frac{1}{2})}.
\]

Here \(\Phi_1^*\) is the generalized Hurwitz-Lerch zeta function.

**Corollary 7.1 (Marginal distributions)** Supposing \(X \sim GL_n(\mu, \Sigma, g_n)\), where \(g_n\) is defined as Eq. 2.9 with \(s_1 = s_2 = 1\), the partitions of \(X, \mu, \Sigma\) are given in Eq. 4.1, then
\[
X^{(1)} \sim Ell_m(\mu^{(1)}, \Sigma_{11}, \widehat{g}_m), \quad X^{(2)} \sim Ell_{n-m}(\mu^{(2)}, \Sigma_{22}, \widehat{g}_{n-m}),
\]
where
\[
\widehat{g}_m(u) = \sum_{j=0}^{n-m-1} \frac{(N - 1)x}{j!} \frac{\Gamma(\frac{n-m}{2} + j)}{\Gamma(2r)b^{\frac{n-m}{2} + j}} u^{n-1 - j} e^{-au} \Phi_1^*(-e^{-bu}, n - m^2 + 2j, a/b).
\]
\(\Phi_1^*\) is the generalized Hurwitz-Lerch zeta function.

**Proof** Taking \(B_1 = (I_m, 0_{m \times (n-m)})\), \(B_2 = (0_{m \times (n-m)}, I_{n-m})\), \(X^{(1)} = B_1X\), \(X^{(2)} = B_2X\) in Theorem 4.1 the result follows.

On the basis of Theorem 5.1 with \(s = 1\) and Theorem 7.1, it is clear that the c.g. of GL distributed random vector depends on dimension.

**Example (Local dependence function)** Bairamov et al. (2003) presented the local dependence function denoted by \(H(x, y)\) based on regression concepts as follows:
\[
H(x, y) = \frac{E\{(X - E(X|Y = y))(Y - E(Y|X = x))\}}{\sqrt{E\{(X - E(X|Y = y))^2\}} \sqrt{E\{(Y - E(Y|X = x))^2\}}},
\]

Alternative representations of \(H(x, y)\) are
\[
H(x, y) = \frac{Cov(X, Y) + \xi_Y(x)\xi_X(y)}{\sqrt{Var(X) + \xi_X^2(x)\sqrt{Var(Y) + \xi_Y^2(x)}}},
\]
\[
H(x, y) = \frac{\rho + \phi_X(y)\phi_Y(x)}{\sqrt{1 + \phi_X^2(y)\sqrt{1 + \phi_Y^2(x)}}},
\]
where
\[
\xi_X(y) = E(X|Y = y) - E(X), \quad \xi_Y(x) = E(Y|X = x) - E(Y), \quad \rho = \frac{Cov(X, Y)}{\sqrt{Var(X)\sqrt{Var(Y)}}},
\]
\[
\phi_X(y) = \frac{\xi_X(y)}{\sqrt{Var(X)}}, \quad \phi_Y(x) = \frac{\xi_Y(x)}{\sqrt{Var(Y)}}.
\]
This function can characterize the dependence structure of two random variables \( X, Y \) localized at the fixed point. Suppose \( W = (X, Y)^T \sim GL_2(\mu, \Sigma, g_2) \), where \( g_2 \) is defined as Eq. 2.9 with \( s = 1 \), \( \tilde{I} = \int_0^\infty \tilde{t}^2 \tilde{g}_1(\tilde{t}^2) \, dt \). Without loss of generality, let

\[
\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & \rho' \\ \rho' & 1 \end{pmatrix}.
\]

We have

\[
\text{Cov}(W) = \frac{N}{2b} \Phi_{2r}^*(-1, N + 1, \frac{\alpha}{b}) - \rho',
\]

\[
E(Y|X = x) = \xi_Y(x) = \rho'x, \quad E(X|Y = y) = \xi_X(y) = \rho'y,
\]

respectively.

\[
\text{Var}(X) = \frac{2C_1}{\sqrt{1 - \rho^2}} \int_0^\infty x^2 \tilde{g}_1(x^2) \, dx = 2C_1 \tilde{I} = \text{Var}(Y).
\]

\[
\tilde{I} = \int_0^\infty \tilde{t}^2 \tilde{g}_1(\tilde{t}^2) \, d\tilde{t} = \int_0^\infty \sum_{j=0}^{N-1} \frac{(N-1)_j}{j!} \frac{\Gamma(\frac{j+1}{2})}{b^{j+1}} 2^{(N+1-j)} e^{-at} \Phi_{2r}^*(-e^{-b^2}, \frac{1}{2}, \frac{a}{b}) \, dt
\]

\[
= \sum_{j=0}^{N-1} \frac{(N-1)_j}{j!} \frac{\Gamma(\frac{j+1}{2})}{b^{j+1}} \int_0^\infty t^{2(N+1-j)} e^{-at} \frac{1}{\Gamma(2r)} \sum_{k=0}^{\infty} \frac{\Gamma(2r + k)(-1)^k}{k!} \frac{1}{(k + \frac{a}{b})^{2j+1}} e^{-bkt^2} \, dt
\]

(setting \( y = \tilde{t}^2 \))

\[
= \sum_{j=0}^{N-1} \sum_{k=0}^{\infty} \frac{1}{2} \beta_j \alpha_k \frac{1}{k!} \frac{1}{(k + \frac{a}{b})^{2j+1}} \int_0^\infty y^{N-j-\frac{1}{2}} e^{-ay-bky} \, dy
\]

\[
= \sum_{j=0}^{N-1} \sum_{k=0}^{\infty} \frac{1}{2} \beta_j \alpha_k \frac{1}{(k + \frac{a}{b})^{2j+1}} \frac{\Gamma(N-j+\frac{1}{2})}{(a + bk)^{N-j+\frac{1}{2}}}
\]

\[
= \sum_{j=0}^{N-1} \sum_{k=0}^{\infty} \beta_j^\alpha \alpha_k^\beta,
\]

where

\[
\alpha_k = \frac{\Gamma(2r + k)(-1)^k}{k!}, \quad \alpha_k^* = \frac{\alpha_k}{2(k + \frac{a}{b})^{N+1}}, \quad \beta_j = \frac{1}{\Gamma(2r)} \frac{(N-1)_j}{j!} \frac{\Gamma(\frac{j+1}{2})}{b^{j+1}},
\]

\[
\beta_j^* = \frac{\beta_j \Gamma(N-j+\frac{1}{2})}{b^{N-j+\frac{1}{2}}},
\]

The local dependence function for the elliptically symmetric generalized logistic distribution can be expressed as follows,

\[
H(x, y) = \frac{N}{2b} \Phi_{2r}^*(-1, N+1, \frac{a}{b}) - \rho' + \rho^2 xy \sqrt{\frac{1}{2C_1 \tilde{I} + \rho^2 y^2}},
\]
\[ H(x, y) = \frac{N \Phi_{x,y}^{(-1, N-1, \frac{x}{y})}}{2C_1I} \frac{\rho^r + \rho^{2xy}}{4C_1I} \sqrt{1 + \rho^{2y^2} / 2C_1I} - \frac{1}{1 + \rho^{2y^2} / 2C_1I}, \]

where

\[ \hat{T} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta_j^* \alpha_k^*, \quad \alpha_k^* = \frac{(-1)^k \Gamma(2r+k)}{2k!(k+a)^{N+1}}, \quad \beta_j^* = \frac{(N-1)j \Gamma(\frac{1}{2} + j) \Gamma(N - j + \frac{1}{2})}{j! \Gamma(2r)b^{N+1}}. \]

8 Data Analysis

We provide a numerical illustration for the GL distribution, using data in Table 1 which is concluded by Gupta and Kundu (2010). It represents the strength measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. Table 1 shows the single fibers data set of 10 mm in gauge lengths with sample size 63.

If a random variable X follows a general logistic distribution then its pdf defined as

\[ f(x; \theta) = \frac{e^{x/\theta}}{\theta(1 + e^{x/\theta})^2}, \quad x \in \mathbb{R}. \]

According to Chakraborty et al. (2012), the pdf of the new skew logistic (NSL) distribution is given by

\[ f_{SL}(x; \lambda, \alpha, \beta) = \frac{[1 + \sin(\lambda x/(2\beta))]/(2\beta) e^{-x/\beta}}{\beta [1 + e^{-x/\beta}]^2}, \quad -\infty < x < \infty, \quad \alpha \geq 1, \quad \lambda, \beta \in \mathbb{R}, \quad \beta > 0. \]

The GL distribution whose d.g. is defined as Eq. 2.9, is fitted to the data set and the result is compared with those for the general logistic distribution, the NSL distribution, the skew logistic (SL) distribution, the proportional reversed hazard logistic (PRHL) distribution and the exponentiated-exponential logistic (EEL) distribution. The maximum likelihood estimates, the log-likelihood value, the Akaike information criterion (AIC), the K-S test statistic and its p-value for the fitted distributions are presented in Table 2. The results of the general logistic distribution, the NSL distribution, the SL distribution, the PRHL distribution and the EEL distribution are analyzed by Ghosh and Alzaatreh (2018). Since the data set in Table 1 is widely used for general and generalizations of logistic distributions, we consider the GL distribution with fixed \( N = 1.0000, a = 1.0000, s = 1.0000 \). The results show that the GL distribution with fixed value of \( N, a \) and \( s \) fit data better among provided distributions in terms of Akaike information criterion. However, as for the K-S test statistic, it doesn’t perform well as known distributions.

| Strength (GPA) for single-carbon fibers data | 1.901 | 2.132 | 2.203 | 2.228 | 2.257 | 2.350 | 2.361 | 2.396 | 2.397 | 2.445 |
|--------------------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2.454                                      | 2.474 | 2.518 | 2.522 | 2.525 | 2.532 | 2.575 | 2.614 | 2.616 | 2.618 |       |
| 2.624                                      | 2.659 | 2.675 | 2.738 | 2.740 | 2.856 | 2.917 | 2.928 | 2.937 | 2.937 |       |
| 2.977                                      | 2.996 | 3.030 | 3.125 | 3.139 | 3.145 | 3.220 | 3.223 | 3.235 | 3.243 |       |
| 3.264                                      | 3.272 | 3.294 | 3.332 | 3.346 | 3.377 | 3.408 | 3.435 | 3.493 | 3.501 |       |
| 3.537                                      | 3.554 | 3.562 | 3.628 | 3.852 | 3.871 | 3.886 | 3.971 | 4.024 | 4.027 |       |
| 4.225                                      | 4.395 | 5.020 |       |       |       |       |       |       |       |       |
Table 2  Parameter estimates

| Distribution | Logistic | NSL | SL | PRHL | EEL | GL |
|--------------|----------|-----|----|------|-----|----|
| Parameter estimates $\hat{\theta} = 0.19975$ | $\hat{\alpha} = 1.0000$ | $\hat{a} = 0.5400$ | $\hat{d} = 3.2761$ | $\hat{\alpha} = 218.2300$ | $\hat{N} = 1.0000$ |
| $\hat{\beta} = 1.9975$ | $\hat{\lambda} = 2.6800$ | $\hat{\lambda} = 2.2192$ | $\hat{\lambda} = 0.946$ | $\hat{\alpha} = 1.0000$ |
| $\hat{\lambda} = 1.9694$ | $\hat{\mu} = 2.7400$ | $\hat{\mu} = 2.3369$ | $\hat{\theta} = 0.0486$ | $\hat{\alpha} = 3.2761$ |
| $\hat{\alpha} = 8.7827e + 04$ | $\hat{\alpha} = 3.0593$ | $\hat{\lambda} = 2.2192$ | $\hat{\lambda} = 0.0946$ | $\hat{\alpha} = 218.2300$ |
| $\hat{\lambda} = 4.1739e - 38$ | $\hat{\mu} = 2.7400$ | $\hat{\mu} = 2.3369$ | $\hat{\theta} = 0.0486$ | $\hat{\alpha} = 3.2761$ |
| Log likelihood | $-165.5826$ | $-123.4458$ | $-58.0299$ | $-58.9896$ | $-56.8643$ | $-49.6587$ |
| AIC | 333.1652 | 248.8916 | 122.0597 | 119.9792 | 119.7286 | 107.3174 |
| K-S | 0.7123 | 0.5532 | 0.0918 | 0.0844 | 0.0735 | 0.0987 |
| K-S p-value | 0.0000 | 0.0000 | 0.6632 | 0.7603 | 0.8853 | 0.5714 |

9 Concluding Remarks

This paper defined the generalized logistic distribution whose density generator is defined as

$$g(t) = \frac{t^{N-1} \exp(-a t^{s_1})}{(1 + \exp(-b t^{s_2}))^{2r}}, \quad t > 0, \quad (9.1)$$

where $2N + n > 2$, $a$, $b$, $s_1$, $s_2$ > 0, $r \geq 0$ are constants. By setting different $a$, $b$, $s_1$, $s_2$, $r$, $N$ in (9.1), we obtained various density generators of elliptical distributions, such as the normal distribution, the Kotz type distribution, the exponential power distribution, the symmetric logistic distribution and generalized logistic type I, III, IV distribution, etc. Our interest is to study the inconsistency properties and various probabilistic properties of this distribution including marginal distributions, conditional distributions, linear transformations, characteristic functions. In addition, we gave a data analysis which shows that the GL distributions are more flexible than other distributions. We would give further research on statistic inference of this new kind of elliptical distributions in the subsequent research.

Appendix A: Proofs

A.1. Proof of (5.1)–(5.4)

Proof  When the dimension $n > 1$, $\Omega_n (\| t \|^2)$ defined as

$$\Omega_n (\| t \|^2) = \frac{\Gamma (\frac{n}{2})}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \| t \|^{2k}}{(2k)!} \frac{\Gamma (\frac{2k+1}{2})}{\Gamma (\frac{n+2k}{2})},$$
we have the characteristic function as follows.

\[
\psi_X(t) = E(e^{it^T X}) = e^{it^T \mu} E(e^{it^T R \Sigma^{\frac{1}{2}} U^{(n)}}) = e^{it^T \mu} E[\Omega_n(R^2 t^T \Sigma t)]
\]

\[
= e^{it^T \mu} \int_0^\infty \Omega_n(vt^T \Sigma t) \frac{1}{\int_0^\infty \frac{1}{g_n(t)} dt} v^{\frac{\alpha}{2} - 1} v^{N-1} \exp(-av^s) (1 + \exp(-bv^s))^{-2r} dv
\]

\[
= e^{it^T \mu} \left[ \int_0^\infty \frac{t^{\frac{\alpha}{2} + N - 2} \exp(-at^s)}{(1 + \exp(-bt^s))^{2r}} \right]^{-1} \int_0^\infty \Omega_n(vt^T \Sigma t) v^{\frac{\alpha}{2} + N - 2} \exp(-av^s) (1 + \exp(-bv^s))^{-2r} dv
\]

\[
= e^{it^T \mu} \left[ \frac{\Gamma\left(\frac{\alpha}{2} + N - 1\right)}{b \frac{\Gamma\left(\frac{\alpha}{2} + N - 1\right)}{1 + \exp(-bt^s))^{2r}} \Phi_{2r}^*(-1, \frac{1}{s} + N - 1, a \frac{b}{2}) \right]^{-1}
\]

\[
\times \sum_{k=0}^\infty \frac{\Gamma\left(\frac{\alpha}{2}\right) (-1)^k (t^T \Sigma t)^k v^{k}}{(2k)! \Gamma\left(\frac{\alpha + 2k}{2}\right)} \int_0^\infty v^{\frac{\alpha}{2} + N + k - 2} \exp(-av^s) (1 + \exp(-bv^s))^{-2r} dv
\]

\[
= e^{it^T \mu} \left[ \frac{\Gamma\left(\frac{\alpha}{2} + N - 1\right)}{b \frac{\Gamma\left(\frac{\alpha}{2} + N - 1\right)}{1 + \exp(-bt^s))^{2r}} \Phi_{2r}^*(-1, \frac{1}{s} + N - 1, a \frac{b}{2}) \right]^{-1}
\]

\[
\times \sum_{k=0}^\infty \frac{\Gamma\left(\frac{\alpha}{2}\right) (-1)^k (t^T \Sigma t)^k \Gamma\left(\frac{\alpha + 2k}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)}{b \frac{(\frac{\alpha}{2} + N - 1)!}{\Gamma\left(\frac{\alpha}{2}\right)}} \Phi_{2r}^*(-1, \frac{1}{s} + N - 1, a \frac{b}{2})
\]

\[
= e^{it^T \mu} \sum_{k=0}^\infty \frac{(-1)^k}{(2k)! \Gamma\left(\frac{\alpha}{2}ight)} \frac{\Gamma\left(k + \frac{1}{2}\right) \Gamma\left(\frac{\alpha}{2} + N + k - 1\right)}{\Gamma\left(k + \frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2} + N + k - 1\right)}
\]

\[
\times \Phi_{2r}^*(-1, \frac{1}{s} + N + k - 1, a \frac{b}{2}) \frac{(t^T \Sigma t)^k}{\Phi_{2r}^*(-1, \frac{1}{s} + N - 1, a \frac{b}{2})}
\]

\[
= e^{it^T \mu} \sum_{k=0}^\infty \frac{(-1)^k}{(2k)! \Gamma\left(\frac{\alpha}{2}\right)} \gamma_k(N, n, b, s)
\]

where

\[
\gamma_k(N, n, b, s) = \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\pi \frac{1}{2} \frac{1}{b \frac{\alpha}{2} + N - 1}} \frac{\Gamma\left(k + \frac{1}{2}\right) \Gamma\left(\frac{\alpha}{2} + N + k - 1\right)}{\Gamma\left(k + \frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2} + N + k - 1\right)}
\]

Here \( \Phi_{2r}^* \) is the generalized Hurwitz-Lerch zeta function.

When the dimension \( n > 1 \), \( \Omega_n(||t||^2) \) defined as

\[
\Omega_n(||t||^2) = F_1\left(\frac{n}{2}, -\frac{1}{4} ||t||^2\right),
\]
we have the characteristic function as follows.

\[ \psi_X(t) = e^{it^T \mu} E \{ \Omega_n(R^2 t^T \Sigma t) \} \]

\[ = e^{it^T \mu} \left[ \int_0^{\infty} \frac{t^2 + N - 2}{(1 + \exp(-bt^2))^{2r}} \right] \int_0^{\infty} \Omega_n(vt^T \Sigma t) \frac{v^2 + N - 2}{(1 + \exp(-bv^2))^{2r}} dv \]

\[ = e^{it^T \mu} \left[ \frac{\Gamma(\frac{1}{s} (\frac{n}{2} + N - 1))}{b^{\frac{1}{s} (\frac{n}{2} + N - 1) s}} \Phi_{2r}^*(-1, \frac{n}{2} + N - 1, a \frac{b}{b}) \right]^{-1} \times \int_0^{\infty} 0 F_1 \left( \frac{n}{2} ; \frac{1}{4} vt^T \Sigma t \right) v^2 + N - 2 \exp(-av^2) \frac{dv}{(1 + \exp(-bv^2))^{2r}} \]

\[ = e^{it^T \mu} \left[ \frac{\Gamma(\frac{1}{s} (\frac{n}{2} + N - 1))}{b^{\frac{1}{s} (\frac{n}{2} + N - 1) s}} \Phi_{2r}^*(-1, \frac{n}{2} + N - 1, a \frac{b}{b}) \right]^{-1} \times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{s} (\frac{n}{2} + N - 1))}{\Gamma(\frac{1}{s} (\frac{n}{2} + N - 1)) b^{\frac{k}{s} (\frac{n}{2} + N - 1) s} 4^k (\frac{n}{2})^k k!} \Phi_{2r}^*(-1, \frac{n}{2} + N + k - 1, a \frac{b}{b}) \Phi_{2r}^*(-1, \frac{n}{2} + N + k - 1, a \frac{b}{b})(t^T \Sigma t)^k. \]

Thus, we have characteristic generators as follows:

\[ \phi_X(\|u_n\|^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \gamma_k(N, n, b, s) \Phi_{2r}^*(-1, \frac{n}{2} + N + k - 1, \frac{a}{b}) \Phi_{2r}^*(-1, \frac{n}{2} + N - 1, \frac{a}{b}) (\|u_n\|^2)^k, \]

\[ \phi_X(\|u_n\|^2) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{s} (\frac{n}{2} + N - 1))}{\Gamma(\frac{1}{s} (\frac{n}{2} + N - 1)) b^{\frac{k}{s} (\frac{n}{2} + N - 1) s} 4^k (\frac{n}{2})^k k!} \Phi_{2r}^*(-1, \frac{n}{2} + N + k - 1, \frac{a}{b}) \Phi_{2r}^*(-1, \frac{n}{2} + N + k - 1, \frac{a}{b}) (\|u_n\|^2)^k, \]

where

\[ \gamma_k(N, n, b, s) = \frac{\Gamma(\frac{n}{2})}{\frac{1}{2} \pi b \frac{1}{s} (\frac{n}{2} + N - 1)^{\frac{n}{2}}} \left( \frac{\Gamma(k + \frac{1}{2}) \Gamma(\frac{1}{2} (\frac{n}{2} + N + k - 1))}{\Gamma(k + \frac{1}{2}) \Gamma(\frac{1}{2} (\frac{n}{2} + N - 1))} \right). \]

\[ \square \]

A.2. Proof of (7.1)–(7.4)

Proof When the dimension \( n > 1 \), \( \Omega_n(\|t\|^2) \) defined as

\[ \Omega_n(\|t\|^2) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \|t\|^{2k}}{(2k)!} \frac{\Gamma(\frac{2k+1}{2})}{\Gamma(\frac{n+2k}{2})}, \]

we have the characteristic function as follows.

\[ \psi_Y(t) = E(e^{it^T Y}) = e^{it^T \mu_Y} \frac{1}{\int_0^{\infty} t^2 \sigma_m(t) dt} \times \int_0^{\infty} \Omega_n(vt^T \Sigma t) \sum_{j=0}^{N-1} \frac{(N-1)_j \Gamma(\frac{n+m}{2} + j)}{b^{\frac{n+m}{2} + j}} \right) v^{N + \frac{n}{2} - j - 2 - e^{-av} \frac{n-m}{2} + j, a \frac{b}{b}) dv \]

\[ = e^{it^T \mu_Y} \left[ \sum_{k=0}^{\infty} \sum_{j=0}^{N-1} a_j^k b_j \Gamma(\frac{m}{2} + N - j - 1) \right]^{-1} I_3, \]

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where
\[
\beta_j = \frac{(N - 1)_j}{j!} \frac{\Gamma\left(\frac{n-m}{2} + j\right)}{b^{n-m/2} + 1} \Gamma(2r) \quad \alpha_k = \frac{\Gamma(2r + k)(-1)^k}{k!} \quad \alpha_k^* = \frac{\alpha_k}{(k + \frac{a}{b})^{n/2} + N - 1}.
\]

\[
I_3 = \int_0^\infty \frac{\Gamma\left(\frac{m}{2}\right)}{\pi^{1/2}} \sum_{l=0}^\infty \frac{(N - 1)_k}{k!} \frac{\Gamma\left(\frac{n-m}{2} + k\right)}{b^{n-m/2} + k} v^{N + \frac{m}{2} - k - 1} e^{-av} \Phi_{2r}(-e^{-b\nu}, n-m/2 + k, \frac{a}{b}) dv
\]
\[
= \frac{\Gamma\left(\frac{m}{2}\right)}{\pi^{1/2}} \sum_{l=0}^\infty \frac{(N - 1)_k}{(2l)!\Gamma\left(\frac{n+2l}{2}\right)} (t^T \Sigma_Y t)^l \sum_{k=0}^{N-1} \beta_k \int_0^\infty v^{N + \frac{m}{2} - k + l - 1} e^{-av} \sum_{p=0}^{\infty} \frac{\Gamma(2r + p)}{p!} (-1)^p e^{-\frac{bpv}{b}} \left(p + \frac{a}{b}\right)^{n-m/2 + N + l} dv
\]
\[
\psi_Y(t) = e^{it^T \mu_Y} \sum_{l=0}^\infty \sum_{k=0}^{N-1} \sum_{p=0}^\infty \frac{(-1)^l}{(2l)!\Gamma\left(\frac{m+2l}{2}\right)} \alpha_k^* \beta_k A_I \Gamma\left(\frac{m+m}{2} + N - k - 1\right) \left(t^T \Sigma_Y t\right)^l,
\]
\[
\phi_{(m),y}(\|\xi_{(m)}\|^2) = \frac{\sum_{l=0}^\infty \sum_{k=0}^{N-1} \sum_{p=0}^\infty \frac{(-1)^l}{(2l)!\Gamma\left(\frac{m+2l}{2}\right)} \alpha_k^* \beta_k A_I \Gamma\left(\frac{m+m}{2} + N - k - 1\right)}{\sum_{k=0}^{N-1} \alpha_k^* \beta_k \Gamma\left(\frac{m+m}{2} + N - j - 1\right) \|\xi_{(m)}\|^2},
\]
where
\[
A_x \triangleq A_x(N, m, a, b, p, k) = \frac{B\left(\frac{m}{2}, x + \frac{1}{2}\right)}{B\left(\frac{m}{2} + x, \frac{1}{2}\right) b^{N-k+x} (p + \frac{a}{b})^{m-x} + 1},
\]
\[
\beta_x \triangleq \beta_x(N, n, m, b, r) = \frac{(N - 1)_x \Gamma\left(\frac{n-m}{2} + x\right)}{x! b^{n-m/2 + x} \Gamma(2r)}.
\]
\[
\alpha_x(r) \triangleq \alpha_x(r) = \frac{\Gamma(2r + x)(-1)^x}{x!}, \quad \alpha_x^* \triangleq \alpha_x^* (N, n, a, b, r) = \frac{\alpha_x}{(x + \frac{a}{b})^{n/2} + N - 1}.
\]

When the dimension $n > 1$, $\Omega_n(\|t\|^2)$ defined as
\[
\Omega_n(\|t\|^2) = F_1\left(\frac{n}{2}, -\frac{1}{4}\|t\|^2\right),
\]
we have the characteristic function as follows.

\[
\psi_X(t) = e^{it^T\mu_Y} \frac{1}{t^2 - \frac{1}{2}g_m(t)} \int_0^\infty \Omega_n(vt^T\Sigma_Y t) \\
\times \sum_{j=0}^{N-1} \frac{(N-1)_j \Gamma\left(\frac{n-m}{2} + j\right)}{j! t^{\frac{n+m}{2} + j}} v^{N+m-j-2} e^{-av} \Phi_2^*(e^{-bv}, \frac{n-m}{2} + j, \frac{a}{b}) dv
\]

\[
e^{it^T\mu_Y} \left[ \sum_{k=0}^{\infty} \sum_{j=0}^{N-1} \alpha_k^* \beta_j^* \Gamma\left(\frac{m}{2} + N - j - 1\right) \right]^{-1} \int_0^\infty \frac{\Gamma\left(\frac{m}{2} + N - j - 1\right)}{\pi^\frac{1}{2} 4^l\left(\frac{\pi}{2}\right)^l} \int_0^\infty \left(\frac{t^T\Sigma_Y t}{\pi^\frac{1}{2} 4^l\left(\frac{\pi}{2}\right)^l}\right)^l dv
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{N-1} \alpha_k^* \beta_j^* \Gamma\left(\frac{m}{2} + N - j - 1\right) \frac{\Gamma\left(\frac{m}{2} + N - j - 1\right)}{\pi^\frac{1}{2} 4^l\left(\frac{\pi}{2}\right)^l} \left(\frac{t^T\Sigma_Y t}{\pi^\frac{1}{2} 4^l\left(\frac{\pi}{2}\right)^l}\right)^l
\]

\[
\phi_{(m),y} (\|\xi_{(m)}\|^2) = \sum_{k=0}^{\infty} \sum_{j=0}^{N-1} \alpha_k^* \beta_j^* \Gamma\left(\frac{m}{2} + N - j - 1\right) \frac{\Gamma\left(\frac{m}{2} + N - j - 1\right)}{\pi^\frac{1}{2} 4^l\left(\frac{\pi}{2}\right)^l} \left(\frac{\|\xi_{(m)}\|^2}{\pi^\frac{1}{2} 4^l\left(\frac{\pi}{2}\right)^l}\right)^l
\]

where

\[
\beta_\lambda \triangleq \beta_\lambda (N, n, m, r) = \frac{(N-1)_x \Gamma\left(\frac{n-m}{2} + x\right)}{\Gamma(2r)b^{\frac{n+m}{2} + x}} \quad \beta_\lambda^* \triangleq \beta_\lambda^* (N, n, m, b, r) = \frac{\beta_\lambda \Gamma(N + \frac{m}{2} - x - 1)}{b^{N + \frac{m}{2} - x - 1}}
\]

\[
\alpha_\lambda \triangleq \alpha_\lambda (r) = \frac{(-1)^x \Gamma(2r + x)}{x!} \quad \alpha_\lambda^* \triangleq \alpha_\lambda^* (N, m, a, b, r) = \frac{\alpha_\lambda \Gamma(N + \frac{m}{2} + x + 1)}{(x + \frac{a}{b})^{\frac{m}{2} + x + 1}}
\]

Here \(\Phi_2^*\) is the generalized Hurwitz-Lerch zeta function.

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