GRADED LIE ALGEBRAS, FOURIER TRANSFORM AND PRIMITIVE PAIRS

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Abstract. In this paper we study the Fourier transform on graded Lie algebras. Let $G$ be a complex, connected, reductive, algebraic group, and $\chi : \mathbb{C}^\times \to G$ be a fixed cocharacter that defines a grading on $\mathfrak{g}$, the Lie algebra of $G$. Let $G_0$ be the centralizer of $\chi(\mathbb{C}^\times)$. Here under some assumptions on the field $k$ and also assuming two conjectures for the group $G$, we prove that the Fourier transform sends parity complexes to parity complexes. Primitive pairs have played an important role in Lusztig’s paper [Lu] to prove a block decomposition in the graded setting. A long term goal of this project is to prove a similar block decomposition in positive characteristic.

In this paper we have tried to understand the primitive pair and its relation with the Fourier transform.

1. Introduction

In this paper, we study the Fourier transform in the context of graded Lie algebras in positive characteristic. Geometry of the $\mathbb{Z}$-graded Lie algebra has been studied by Lusztig [Lu] in characteristic 0. The main goal of this paper is to extend some results of Lusztig related to Fourier transform on the graded Lie algebras to positive characteristic. To understand the representation theory of the Weyl group better, one of the main goal of the generalized Springer correspondence was to find a block decomposition of the collection of all pairs with nilpotent orbit and irreducible local system on that nilpotent orbit. This was studied in [Lu3], [Lu4], and [Lu5] for characteristic 0 and for positive characteristic in [AHJR1], [AHJR2], [AHJR3]. Following that motivation, a long term goal of this project is to find a similar block decomposition in graded setting for positive characteristic.

Let $G$ be a complex, connected, reductive, algebraic group and $\mathfrak{g}$ be the Lie algebra of $G$. We fix a cocharacter $\chi : \mathbb{C}^\times \to G$. The group $\mathbb{C}^\times$ acts on $\mathfrak{g}$ through the adjoint action. The space $\mathfrak{g}_n$ denotes the $n$-th root space of this action. This defines a grading on $\mathfrak{g}$. The centralizer $G_0$ of $\chi(\mathbb{C}^\times)$ acts on $\mathfrak{g}_n$ through the adjoint action. The $G_0$-equivariant derived category of sheaves, $\mathcal{D}_G^b(\mathfrak{g}_n)$ has been studied in [Lu] with sheaf coefficients in a field of characteristic 0 and in [Ch] when the characteristic is positive. Parity sheaves are important objects in $\mathcal{D}_G^b(\mathfrak{g}_n)$ [JMW]. For any pair $(O, L)$, where $O$ is $G_0$-orbit in $\mathfrak{g}_n$ and $L$ is a $G_0$-equivariant irreducible local system on $O$, there exists at-most one indecomposable parity sheaf up-to shift, which will be denoted by $\mathcal{E}(O, L)$. We will denote the collection of pairs $(O, L)$ as $\mathcal{F}(\mathfrak{g}_n)$. Similarly, on nilpotent cone for each the pair $(C, F)$, where $C$ is a $G$-orbit in the nilpotent cone $\mathcal{N}$ and $F$ is a $G$-equivariant irreducible local system on $C$, there exists at-most one parity sheaf, denote it by $\mathcal{E}(C, F)$. We will denote this collection of pairs $(C, F)$ by $\mathcal{F}(G)$. For the group $GL_n$, the existence of parity sheaves on the nilpotent cone has been proved in [JMW]. For a parabolic subgroup $P$ with Levi $L$ containing $\chi(\mathbb{C}^\times)$, we define two functors on $\mathcal{D}_G^b(\mathfrak{g}_n)$ [Ch],

$$\text{Ind}_P^L : \mathcal{D}_{G_0}^b(\mathfrak{g}_n) \to \mathcal{D}_G^b(\mathfrak{g}_n),$$

Date: November 28, 2022.
and
\[ \text{Res}_g^p : D^b_{L_0}(\mathfrak{g}_n) \to D^b_{L_0}(I_n). \]

Similar to the nilpotent cone, these two functors, induction and restriction play important roles in the study of the equivariant derived category in the graded setting. When the field characteristic satisfies some assumptions (Assumption 2.5) it has been proved in [Ch] that for each pair \((\mathcal{O}, \mathcal{L})\) as mentioned above, \(\mathcal{E}(\mathcal{O}, \mathcal{L})\) exists and \(\text{Ind}_g^p\) sends parity to parity. In this paper the first result that we prove is that,

- \(\text{Res}_g^p\) sends parity complexes to parity complexes with the same assumptions on the field.

Fourier transform has been studied in [HK], [Lu4], [Mi] for characteristic 0 and for positive characteristic in [Ju]. Some of the literatures including [AHJR5] and [AM] consider the functor \(\mathcal{R}\) from \(D^b_G(N, k)\) to \(D^b_G(N, k)\) given by the Fourier transform on \(g\) followed by the restriction on \(N\). It has been proved in [AHJR5] that for arbitrary characteristic in the context of Springer sheaf, \(\mathcal{R}\) corresponds to the tensoring by the sign character of the Weyl group. In [AM], it has been shown that \(\mathcal{R}\) is an auto-equivalence. These reprove an already well-known result in characteristic 0. The result is that the map between the group algebra of the Weyl group \(W\) and the endomorphism ring of the Springer sheaf is an isomorphism through the restriction functor in arbitrary characteristic. Moreover for \(GL_n\), \(\mathcal{R}\) is the geometric version of Ringel duality [AM].

Fourier transform on the graded Lie algebras in characteristic 0 has been studied by Lusztig [Lu]. He has proved that Fourier transform sends cuspidal pairs [Ch, Definition 2.7] to cuspidal pairs and semisimple complexes to semisimple complexes. The goal of this paper is to find the appropriate substitution of these statements in positive characteristic. Motivated by the work that has been already done in modular representation theory both on nilpotent cone and graded Lie algebras we make some assumptions on the field characteristic (Assumption 2.5). Under the assumptions we prove,

- In positive characteristic Fourier transform sends cuspidal pairs to cuspidal pairs.
- Fourier transform sends parity complexes to parity complexes.

Another explored topic in Lusztig’s paper is the primitive pair both on nilpotent cone and graded pieces (Definition 5.1, Definition 5.2), which plays an important role to define a block decomposition,
\[ \mathcal{I}(\mathfrak{g}_n) = \sqcup \mathcal{I}(C, F), \]
where \((C, F) \in \mathcal{I}(G)\) is a primitive pair. We have tried to recover some of the statements about primitive pair in this paper in positive characteristic. A long term goal is to find out if that same block decomposition is still valid in positive characteristic.

1.1. Outline. In section 2 we build the necessary background, assumptions and notations. In section 3 we define \(\text{Res}_g^p\) in the graded setting and prove that it sends parity complexes to parity complexes. In section 4 we introduce Fourier transform for the graded Lie algebras. Here we prove that the Fourier transform sends cuspidal pairs to cuspidal pairs and parity complexes to parity complexes. In section 5 we define primitive pairs both on the nilpotent cone and the graded Lie algebras. We prove some statements about the primitive pair in positive characteristic that was known in characteristic 0. In section 6 we introduce quasi-monomials and good pairs and reprove some statements in the modular setting that already exists in characteristic 0.

1.2. Acknowledgement. The author would like to thank her PhD advisor Pramod N. Achar for suggesting the thesis problem, whose continuation is this paper.) The author is also thankful to Daniel Juteau for giving many useful advice and insights both for the present work and future
motivation. The author thanks Daniel Nakano and her post-doc mentor Laura Rider for their support and advices.

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2. Background

In this section we go over some definitions, notations and some already known results. Let \( k \) be a field of characteristic \( \ell \geq 0 \). We consider sheaves with coefficients in \( k \). The varieties we work on will be over \( \mathbb{C} \). Let \( H \) be a linear algebraic group and \( X \) be a \( H \)-variety. We denote by \( D^b_H(X, k) \) or \( D^b_H(X) \), the \( H \)-equivariant derived category of constructible sheaves, which is defined in [BL]. Let \( \text{Perv}_H(X, k) \) be the full subcategory of \( H \)-equivariant perverse \( k \)-sheaves. The constant sheaf on \( X \) is denoted by \( \mathcal{V}_X \) or more simply by \( \mathcal{V} \), where \( V \) is a \( k \)-vector space.

Let \( G \) be a connected, reductive, algebraic group over \( \mathbb{C} \) and \( g \) be the Lie algebra of \( G \). We fix a cocharacter map, \( \chi : \mathbb{C}^\times \to G \) and define,

\[
G_0 = \{ g \in G | g_\chi(t) = \chi(t)g, \forall t \in \mathbb{C}^\times \}.
\]

For \( n \in \mathbb{Z} \), define,

\[
g_n = \{ x \in g | \text{Ad}(\chi(t))x = t^n x, \forall t \in \mathbb{C}^\times \}.
\]

This defines a grading on \( g \),

\[
g = \bigoplus_{n \in \mathbb{Z}} g_n.
\]

Clearly, \( g_0 = \text{Lie}(G_0) \) and \( G_0 \) acts on \( g_n \). For \( n \neq 0 \), \( G_0 \) acts on \( g_n \) with only finitely many orbits. In [Ch] we studied the \( G_0 \)-equivariant bounded derived category of sheaves, \( D^b_{G_0}(g_n, k) \) with some restriction on the field characteristic of \( k \), which will be discussed later in this section.
Here we will review some notations that already have been introduced in [Ch].
Recall that \( sl_2 \) is the Lie algebra of \( SL_2 \) generated by,
\[
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
Let \( J_n = \{ \phi : sl_2 \to g | \phi(e) \in g_n, \phi(f) \in g_{-n}, \phi(h) \in g_0 \} \). We have an action of \( G_0 \) on \( J_n \) by \((g, \phi) \to \text{Ad}(g) \circ \phi \). It is easy to check that this action is well-defined.

The map from the set of \( G_0 \)-orbits on \( J_n \) to the set of \( G_0 \)-orbits on \( g_n \), defined by \( \phi \to \phi(e) \) is a bijection. The proof is in [Lu] Prop 3.3.

2.1. The set \( \mathcal{I}(G, k) \), \( \mathcal{I}(g_n, k) \) and cuspidal pairs. The set \( \mathcal{I}(G, k) \) is the set of pairs \((C, E)\) where \( C \subset \mathcal{N}_G \) is a nilpotent \( G \)-orbit in \( g \) and \( E \) is an irreducible \( G \)-equivariant \( k \)-local system on \( C \) (up to isomorphism). The number of \( G \)-orbits in \( \mathcal{N}_G \) is finite and irreducible local systems on \( C \) are in bijection with the irreducible representation of the component group \( A_G(x) := G^x/(G^x)^0 \) on \( k \)-vector spaces, where \( x \) is in \( C \) and
\[
(G^x)^0 \text{ is the identity component of } G^x. \text{ Hence the set } \mathcal{I}(G, k) \text{ is finite. Sometimes when there is no confusion about the field of coefficients then we will just write } \mathcal{I}(G).
\]
Let \( \mathcal{I}(g_n, k) \) or \( \mathcal{I}(g_n) \) be the set of all pairs \((O, L)\) where \( O \) is a \( G_0 \)-orbit in \( g_n \) and \( L \) is an irreducible, \( G_0 \)-equivariant \( k \)-local system on \( O \) (up to isomorphism). Following the similar reasoning as for \( \mathcal{I}(G) \), \( \mathcal{I}(g_n) \) is finite.

Let \( P \) be a parabolic subgroup of \( G \) with unipotent radical \( U_P \) and let \( L \subset P \) be a Levi factor of \( P \). One can identify \( L \) with \( P/U_P \) through the natural morphism, \( L \hookrightarrow P \to P/U_P \). As defined in [AHJR], we have a functor \( \text{Res}^G_P : D^b_G(\mathcal{N}_G) \to D^b_L(\mathcal{N}_L) \). A simple object \( F \) in \( \text{Perv}_G(\mathcal{N}_G, k) \) is called cuspidal if \( \text{Res}^G_P(F) = 0 \), for any proper parabolic \( P \) and Levi factor \( L \subset P \). A pair \((C, E) \in \mathcal{I}(G)\), is called cuspidal if the corresponding simple perverse sheaf \( \mathcal{I}(C, E) \) is cuspidal. We will denote the collection of all cuspidal pairs on \( \mathcal{N}_G \) by \( \mathcal{I}(G)\)\text{cusp}. Using modular reduction map [AHJR] 2.3, we can define 0-cuspidal and we will denote this collection by \( \mathcal{I}(G)\)\text{0-cusp} and \( \mathcal{I}(G)\)\text{0-cusp} \( \subset \mathcal{I}(G)\)\text{cusp} [AHJR] Lemma 2.3. A brief discussion on cuspidal pairs and 0-cuspidal pairs can also be found in [Ch] Section 2.

Definition 2.1. A pair \((O, L) \in \mathcal{I}(g_n, k)\) will be called cuspidal if there exists a pair \((C, E) \in \mathcal{I}(G)\)\text{cusp}, such that \( C \cap g_n = O \) and \( L = E|_{O} \). We will denote the set of all cuspidal pairs on \( g_n \) by \( \mathcal{I}(g_n)\)\text{cusp}.

Under the assumptions on the field characteristic, one of the main theorems in [Ch] is the following.

Theorem 2.2. Any cuspidal pair \((O, L) \in \mathcal{I}(g_n)\)\text{cusp} is clean and therefore parity exists for cuspidals.

A discussion on cleanness can be found in [Ch] 2.2.

2.2. Induction and restriction. Induction and restriction are two important functors both on nilpotent cone and graded Lie algebras which allow us to go from Levi to the actual group or from the group to a Levi subgroup. Let \( P \) be a parabolic subgroup of \( G \) and \( L \) be a Levi factor in \( P \) with \( U_P \), the unipotent radical. For parabolic induction and restriction on nilpotent cone we use the following diagram,
\[
\mathcal{N}_L \xleftarrow{\pi_p} \mathcal{N}_L + u_P \xrightarrow{\iota_p} G \times_P (\mathcal{N}_L + u_P) \xrightarrow{\mu_p} \mathcal{N}_G.
\]
Here \( u_P = \text{Lie}(U_P), \pi_P, e_P \) are the obvious maps and \( \mu_P(g, x) = \text{Ad}(g)x \). Let
\[
i_P = \mu_P \circ e_P : N_L + u_P \to N_G.
\]

The parabolic restriction functor,
\[
\text{Res}^G_P : D^b_G(N_G, k) \to D^b_L(N_L, k)
\]
is defined by \( \text{Res}^G_P(\mathcal{F}) = \pi_P^* e_P^* \mu_P^* \mathcal{F} \). Here
\[
\text{For}^G_L : D^b_G(N_G, k) \to D^b_L(N_L, k)
\]
is the forgetful functor. The parabolic induction comes from the same diagram above,
\[
\text{Ind}^G_P : D^b_L(N_L, k) \to D^b_G(N_G, k),
\]
and is defined by \( \text{Ind}^G_P(\mathcal{F}) := \mu_P^! (e_P^! \text{For}^G_L(\mathcal{F}))^{-1} \pi_P^* (\mathcal{F}) \). Here again \( \text{For}^G_L \) denotes the forgetful functor and \( e_P^* \) the induction equivalence map.

Here we make the assumption that \( L \) defined above contains \( \chi(\mathbb{C}^*) \). Let \( p, l, n \) be the Lie algebras of \( P, L, U_P \) respectively. Then \( p, l, n \) inherit grading from \( g \).

For induction and restriction on graded Lie algebras we use the diagram below,
\[
\begin{align*}
\text{In}_n & \leftarrow p_n \xrightarrow{\pi} G_0 \times P_n p_n \xrightarrow{i} g_n \\
\text{Res}^G_p : D^b_{G_0}(g_n, k) & \to D^b_{L_0}(I_n, k) \\
\text{Ind}^G_p : D^b_{I_0}(I_n, k) & \to D^b_{G_0}(g_n, k)
\end{align*}
\]

All the maps here have the same meaning as in the diagram above. The restriction,
\[
\text{Res}^G_p : D^b_{G_0}(g_n, k) \to D^b_{L_0}(I_n, k)
\]
is defined by \( \text{Res}^G_p(\mathcal{F}) = \pi e^* \mu^* \text{For}^G_{L_0}(\mathcal{F}) = \pi i^* \text{For}^G_{I_0}(\mathcal{F}) \) with \( i : p_n \to g_n \) is the inclusion map. Here the parabolic induction comes from the same diagram above,
\[
\text{Ind}^G_p : D^b_{I_0}(I_n, k) \to D^b_{G_0}(g_n, k)
\]
and is defined by \( \text{Ind}^G_p(\mathcal{F}) := \mu^! (e^* \text{For}^G_{I_0})^{-1} \pi^* (\mathcal{F}) \).

In this paper we will use a different notation of induction and restriction when we deal with the Fourier transform. The notations will be \( \mathcal{F} \mathcal{I}_n \) for induction , and \( \mathcal{F} \mathcal{R}_n \) for the restriction on the \( n \)-th graded setting. This notation will be used in Section 4 and later on. The reason behind this is that Fourier transform sends sheaf complexes on \( n \)-th graded pieces to sheaf complexes on \((-n)\)-th graded pieces.

2.3. Parity sheaves. Parity sheaves were first introduced by Juteau, Mautner and Williamson [JMWW]. They are constructible complexes on some stratified space, where the stratas satisfy some cohomology vanishing properties. Once these conditions are there then for any stratified space \( X \) with strata \( X_\lambda \) and local system \( \mathcal{L} \) on \( X_\lambda \), there exists at-most one parity sheaf, we denote this by \( \mathcal{E}(X_\lambda, \mathcal{L}) \). A detailed discussion is in [JMWW] and a summarized version can be found in [Ch Subsection 2.5]. The cohomology vanishing property for our case is the following theorem that has been proved in [Ch Th. 6.4].

**Theorem 2.3.** With the assumptions on the field charateristic, for any pair \((O, \mathcal{L}) \in \mathcal{F}(g_n)\) we have,
\[
H^*_G(g_n)(\mathcal{L}) = 0, \text{ for } * \text{ odd}.
\]

The next theorem provides the cohomology vanishing condition for the nilpotent cone.
Theorem 2.4. With the assumptions on the field characteristic, for any pair $(C, \mathcal{F}) \in \mathcal{I}(G)$ we have,
\[ H^*_G(\mathcal{F}) = 0, \text{ for } * \text{ odd}. \]

This theorem has been proved in [JMW]. Once we have these theorems we can talk about parity sheaves on both nilpotent cone and graded pieces. But unlike IC's, the existence of parity is not automatic. The existence for nilpotent cone has been discussed in [JMW 4.3]. The existence for graded Lie algebras has been proved in [Ch Th. 6.11]. The assumptions that were needed in [Ch] were the followings,

Assumption 2.5. (1) The characteristic $l$ of $\mathbb{k}$ is a pretty good prime for $G$ [Ch 2.6].
(2) The field $\mathbb{k}$ is big enough for $G$; i.e., for every Levi subgroup $L$ of $G$ and pair $(C_L, \mathcal{E}_L) \in \mathcal{I}(L)$, the irreducible $L$-equivariant $k$-local system $\mathcal{E}_L$ is absolutely irreducible.
(3) We assume the Mautner’s cleanness conjecture is true in our setting, that is every 0-cuspidal in $\mathcal{I}(G)$ is l-clean [Ch Conjecture 2.10].
(4) For any parabolic $P$ and Levi subgroup $L \subset P$ with $(C, \mathcal{F}) \in \mathcal{I}(G)\text{cusp}$, $\text{Ind}^G_P \mathcal{E}(C, \mathcal{F})$ is parity [Ch Conjecture 2.28].

We make these same assumptions in this paper. The cause is that the results of this paper are dependent on the the results in [Ch].

3. Restriction

Recall from the modular springer correspondence [AHJR1], [AHJR2], [AHJR3] the induction and restriction play an important role. Same is true in the graded setting. In [Ch] it has been proved that the induction functor sends parity complexes to parity complexes. In this section our main goal is to prove restriction sends parity complexes to parity complexes.

Let $P$ be a parabolic containing a Levi $L$ and $(L', \mathcal{L}') \in \mathcal{I}(l_n)\text{cusp}$. Let $P'$ be another parabolic containing a Levi $L'$. In this section we will study $\text{Res}^P_{P'} \text{Ind}^G_P \mathcal{E}(O', L')$.

Recall the induction diagram for cuspidal pair $(O', \mathcal{L}') \in \mathcal{I}(l_n)\text{cusp}$ from [Ch] Lemma 5.2,

\[
\begin{align*}
O' & \xleftarrow{\pi} O' + u_n \xrightarrow{e} G_0 \times P_0 (O' + u_n) \xrightarrow{\mu} g_n, \\
\end{align*}
\]

where $\pi$ and $e$ are obvious maps and $\mu(g, x) = \text{Ad}(g)x$. The induction space, $G_0 \times P_0 (O' + u_n)$ can be identified with

\[
\{(gP_0, x) \in G_0/P_0 \times g_n|\text{Ad}(g^{-1})x \in \pi^{-1}(O')\}.
\]

Under this identification the map $\mu$ simply becomes the projection on $g_n$.

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
O' \xleftarrow{\pi} O' + u_n \xrightarrow{e} G_0 \times P_0 (O' + u_n) \xrightarrow{\mu} g_n \\
\end{array} \\
\end{array}
\end{align*}
\]

Here $\mu^{-1}(p'_n) = \{(gP_0, x) \in G_0/P_0 \times p'_n|\text{Ad}(g^{-1})x \in \pi^{-1}(O')\}$ and the map $\sigma$, defined in the above diagram, sends $(gP_0, x) \mapsto \pi'(x)$. Then, \(\text{Res}^P_{P'} \text{Ind}^G_P \mathcal{E}(O', L') \cong \sigma((e^* \text{ For}^G_0)_{-1} \pi^* \mathcal{L}'[\dim O'])_{-1}(p'_n)).\)
Let $\Omega$ be a $(P'_0, P_0)$-double coset of $G_0$. We define the closed subvariety,

$$p_{n,\Omega} = \{(gP_0, x) \in \Omega / P_0 \times p'_n | Ad(g^{-1})x \in \pi^{-1}(O')\}.$$ 

Therefore $\mu^{-1}(p'_n) = \bigcup p_{n,\Omega}$. Define $\sigma_\Omega := \sigma|_{p_{n,\Omega}}$. We can choose the collection of all $(P'_0, P_0)$-double cosets $\Omega_0, \ldots, \Omega_m$, so that for all $j \in [0, m]$, $\Omega_0 \cup \cdots \cup \Omega_j$ is closed and $\mu^{-1}(p'_n) = \bigcup_{j=0}^m p_{n,\Omega_j}$.

**Definition 3.1.**

1. We define $\Omega$ to be good if there exists $g_0 \in \Omega$ so that, $g_0P_0^{-1} \cap P'$ contains a Levi subgroup of $g_0P_0^{-1}$. We will say $\Omega$ is bad if it is not good.

2. We define $K_\Omega = \sigma_\Omega((c^* P_0^\circ)^{-1} \pi^* L'[\dim O'])|_{p_{n,\Omega}}$.

### 3.1. $\Omega$ good

Let $j \in [0, m]$, call $\Omega_j = \Omega$. Let $(gP_0, z) \in p_{n,\Omega}$ and $g_0 \in \Omega$. Then $g$ can be written as $h g_0$, where $h \in P'_0$. Therefore,

$$p_{n,\Omega} = \{(hQ_0, x) \in P'_0Q_0/Q_0 \times p'_n | Ad(h^{-1})x \in Ad(g_0)\pi^{-1}(O')\},$$

where $Q = g_0P_0^{-1}$ is a parabolic subgroup. Clearly $Q$ contains $\chi(C^\times)$. Let $M$ be the Levi subgroup of $Q$ containing $\chi(C^\times)$ and, $\pi_0 : q_n \to m_n$ be the projection with $v$ being the nil-radical of $q$. The above can be again identified with,

$$\{(h(P'_0 \cap Q_0), x) \in P'_0/P'_0 \cap Q_0 \times p'_n | Ad(h^{-1})x \in O'' + v_n\},$$

where $O''$ is the translation of $O'$ by $Ad(g_0)$. The map $\sigma_\Omega$ becomes $(h(P'_0 \cap Q_0), x) \to \pi'(x)$.

Let $L$ and $\hat{M}$ be two Levi subgroups of $P'$ and $Q$ respectively sharing a common maximal torus $T$ containing $\chi(C^\times)$. Their Lie algebras can be denoted by $\hat{l}$ and $\hat{m}$ respectively. Under the projection map, $\pi' : \hat{m} \to \hat{l}$, $\hat{I}$ can be identified with $l'$. Similarly $\hat{m}$ can be identified with $m$ by the projection $\pi : q \to m$.

### Theorem 3.2

For $\Omega$ good, there exists a parabolic $P''$ with Levi $L''$ and $(O'', L'') \in \mathcal{F}(l''_n)^{cusp}$ such that $K_\Omega \cong Ind_{\hat{l}''}^{l''} \mathcal{E}(O'', L'')$ up to some shift.

**Proof.** If $\Omega$ is good then there exists $g_0$, such that $P'$ contains a Levi $\hat{M}'$ of $Q = g_0P_0^{-1}$. Let $T'$ be the maximal torus inside $\hat{M}'$. As $T$ and $T'$ both are maximal inside $P' \cap Q$, therefore there exists $c \in P' \cap Q$, such that $T = cT'c^{-1}$. Then we can consider $c\hat{M}'c^{-1}$ to be the new $\hat{M}$ which contains $T$ as maximal torus. This implies $\hat{M}' = \hat{M}$. So $\hat{M}$ is a Levi subgroup of the parabolic $L \cap Q$ inside $\hat{L}$. Consider the induction diagram for $\hat{L}$ with parabolic $L \cap Q$ and Levi $\hat{M}$ with the identification of $\hat{m}$ with $m$,

$$O'' \xleftarrow{\pi} O'' + (l''_n \cap v_n) \xrightarrow{c} \hat{L}_0 \times \hat{L}_0 \cap Q_0 (O'' + (l''_n \cap v_n)) \xrightarrow{\mu} \hat{l}_n,$$

where $\hat{L}_0 \times \hat{L}_0 \cap Q_0 (O'' + (l''_n \cap v_n))$ can be identified with,

$$\{(l|\hat{L}_0 \cap Q_0), \zeta \in \hat{L}_0 / (\hat{L}_0 \cap Q_0) \times l''_n | Ad(l^{-1}) \zeta \in O'' + (l''_n \cap v_n))\}.$$

If $(h(P'_0 \cap Q_0), x)$ belongs to $p_{n,\Omega}$, this implies $Ad(h^{-1})x \in O'' + v_n$. But also $h \in P'_0$ and $x \in p'_n$, implies $Ad(h^{-1})x \in p'_n$. Therefore $Ad(h^{-1})x \in (O'' + v_n) \cap p'_n$. Now we define a map

$$d : p_{n,\Omega} \to \hat{L}_0 \times \hat{L}_0 \cap Q_0 (O'' + (l''_n \cap v_n))$$

by,

$$d(h(P'_0 \cap Q_0), x) = (l|\hat{L}_0 \cap Q_0), \zeta).$$

Here $l$ is the image of $h$ under the projection $P'_0 \to L_0$ and $\zeta$ is the image of $x$ under the projection $p'_n \to l'_n$. This map is well-defined as $Ad(l^{-1}) \zeta = \pi'(Ad(h^{-1})x) \in O'' + (l''_n \cap v_n)$. Our claim is that
this is a vector bundle. Recall from the definition and the above fact $p_{n, \Omega}$ is same as $P'_0 \times P'_0 \cap Q_0 (O'' + u_n \cap p'_n)$. Then we define a vector bundle map $P'_0 \times P'_0 \cap Q_0 (O'' + u_n \cap p'_n) \rightarrow P'_0 \times P'_0 \cap Q_0 (O'' + u_n \cap l'_n)$, through the map $\pi' : p'_n \rightarrow l'_n$. The fiber of this map is $u'_n \cap v_n$, which is clearly a vector space. Now consider the quotient map $P'_0 \times P'_0 \cap Q_0 (O'' + u_n \cap l'_n) \rightarrow P'_0 \times U_0 (P'_0 \cap Q_0)$, whose fiber is $U_0 (P'_0 \cap Q_0)/(P_0' \cap Q_0)$ which is isomorphic to $U_0'(U_0' \cap Q_0) \cong U_0'(U_0' \cap Q_0)$, which is again a vector space. Now the space $P'_0 \times U_0 (P'_0 \cap Q_0) (O'' + u_n \cap l'_n)$ is isomorphic to $L_0' \times (L_0 \cap Q_0) (O'' + u_n \cap l'_n)$. The map $d$ is combinations of all these maps,

$$
\begin{align*}
P'_0 \times P'_0 \cap Q_0 (O'' + u_n \cap p'_n) & \xrightarrow{\pi'} P'_0 \times P'_0 \cap Q_0 (O'' + u_n \cap l'_n) \\
P'_0 \times U_0 (P'_0 \cap Q_0) (O'' + u_n \cap l'_n) & \cong L_0' \times (L_0 \cap Q_0) (O'' + u_n \cap l'_n)
\end{align*}
$$

where the first and the second maps are vector bundles defined above. Therefore $d$ is a vector bundle of rank $\dim(u'_n \cap v_n) + \dim U_0'/(U_0' \cap Q_0)$.

consider the diagram below,

$$
\begin{array}{ccc}
O' & \xrightarrow{\pi} & O' + u_n \\
\downarrow & & \downarrow \text{Ad}(g_0) & \\
O'' + v_n & \xrightarrow{e} & G_0 \times P_0 (O'' + u_n) \\
\downarrow \text{Ad}(g_0) & & \downarrow \pi' & \\
O'' + v_n & \xrightarrow{\mu_L'} & L_0 \times (L_0 \cap Q_0) (O'' + (l'_n \cap v_n)) & \xrightarrow{d} & p_{n, \Omega}
\end{array}
$$

We are trying to calculate $K_\Omega$, which is by definition $\sigma_\Omega((e^* \text{For}_{L_0}^{G_0})^{-1} \pi_* L''[\dim O''])[p_{n, \Omega}]$. Now consider $\text{Ind}_{L_0 \cap Q_0}^L \mathcal{E}(O'', L'')$, where $L''$ is obtained from $L'$ through the translation by $\text{Ad}(g_0)$, therefore is cuspidal.

As $d$ is a vector bundle, we have,

$$
((e^* \text{For}_{L_0}^{G_0})^{-1} \pi_* L''[\dim O''])[m] \cong dd^*((e^* \text{For}_{L_0}^{G_0})^{-1} \pi_* L''[\dim O'']),
$$

where $m$ is the rank of $d$. Therefore,

$$
\text{Ind}_{L_0 \cap Q_0}^L \mathcal{E}(O'', L'') \cong \mu_L dd^*((e^* \text{For}_{L_0}^{G_0})^{-1} \pi_* L''[\dim O'' - m])
$$

$$
\cong \sigma_\Omega dd^*((e^* \text{For}_{L_0}^{G_0})^{-1} \pi_* L''[\dim O'' - m])
$$

From the commutativity of the above diagram, we get,

$$
\sigma_\Omega dd^*((e^* \text{For}_{L_0}^{G_0})^{-1} \pi_* L''[\dim O'' - m]) \cong \sigma_\Omega dd^* \text{Ad}(g_0)^* (e^* \text{For}_{P_0}^{G_0})^{-1} \pi_* L''[\dim O' - m].
$$
Now from the commutativity of the right square this is same as,
\[ \sigma_\Omega((e^* \text{For}_{P_0}^G)^{-1}\pi^* \mathcal{L}'|\dim \mathcal{O}' - m)|_{p_n,\nu}) \cong K_\Omega[-m]. \]

Therefore we can clearly see, \( K_\Omega \cong \text{Ind}_{\Omega}^1 \mathcal{E}(\mathcal{O}', \mathcal{L}'')|_m. \)

\[ \square \]

**Theorem 3.3.** For \( \Omega \) bad, \( K_\Omega = 0. \)

**Proof.** Recall,
\[ p_{n,\Omega} = \{(h(P_0' \cap Q_0), x) \in P_0'/P_0' \cap Q_0 \times p_n'|\text{Ad}(h^{-1})x \in \mathcal{O}'' + v_n\}, \]
and \( \sigma_\Omega : p_{n,\Omega} \to l'_n \) is defined by \( (h(P_0' \cap Q_0), x) \to \pi'(x). \) Let \( y \in l'_n \) and we want to show, \( (K_\Omega)y = 0. \) Which is by the following cartesian diagram,
\[ \begin{array}{ccc}
p_{n,\Omega} & \xrightarrow{\sigma_\Omega} & l'_n \\
\downarrow & & \downarrow \\
\sigma_\Omega^{-1}(y) & \xrightarrow{\sigma_\Omega} & y \\
\end{array} \]
\[ R\Gamma(((e^* \text{For}_{P_0}^G)^{-1}\pi^* \mathcal{L}'|\dim \mathcal{O}'|_{\sigma_\Omega^{-1}(y)})) = 0. \]

Which is again same as,
\[ H^*_{\mathcal{O},\nu}(\sigma_\Omega^{-1}(y), ((e^* \text{For}_{P_0}^G)^{-1}\pi^* \mathcal{L}'|\dim \mathcal{O}'|_{p_n,\Omega})) = 0. \]

consider the diagram below again,
\[ \begin{array}{ccc}
pr_1^{-1}(h(P_0' \cap Q_0)) & \xrightarrow{\sigma_\Omega^{-1}(y)} & y \\
\downarrow pr_1 & & \downarrow pr_1 \\
\{h(P_0' \cap Q_0)\} & \xrightarrow{pr_1} & P_0'/(P_0' \cap Q_0) \rightarrow y \\
\end{array} \]
where \( pr_1 \) is the projection on the first coordinate. We have,
\[ R\Gamma(((e^* \text{For}_{P_0}^G)^{-1}\pi^* \mathcal{L}'|\dim \mathcal{O}'|_{\sigma_\Omega^{-1}(y)})) \cong R\Gamma(pr_1^{-1}((e^* \text{For}_{P_0}^G)^{-1}\pi^* \mathcal{L}'|\dim \mathcal{O}'|_{\sigma_\Omega^{-1}(y)})) \]
\[ \cong R\Gamma((((e^* \text{For}_{P_0}^G)^{-1}\pi^* \mathcal{L}'|\dim \mathcal{O}'|_{\sigma_\Omega^{-1}(y)})|_{pr_1^{-1}(h(P_0' \cap Q_0)))}. \]

where \( pr_1^{-1}(h(P_0' \cap Q_0)) = \{z \in p'_n|\text{Ad}(h^{-1})z \in \mathcal{O}'' + v_n, \pi'(z) = y\}. \) Now \( \text{Ad}(h^{-1})z \) also belongs to \( p'_n \) as \( h \in P_0' \) and \( z \in p'_n. \) So \( \text{Ad}(h^{-1})z \in p'_n \cap q_n. \) The intersection \( p'_n \cap q_n \) can be written as the following direct sums,
\[ p'_n \cap q_n = (l'_n \cap m_n) \oplus (u'_n \cap m_n) \oplus (p'_n \cap v_n), \]
\[ (l'_n \cap m_n) \oplus (l'_n \cap v_n) \oplus (u'_n \cap q_n). \]
In the former case \( \text{Ad}(h^{-1})z \) can be written as \( \gamma + \nu + \mu \) and in the latter case \( \gamma + \nu' + \mu', \) where \( \gamma \in (l'_n \cap m_n), \nu \in (u'_n \cap m_n), \mu \in (p'_n \cap v_n), \nu' \in (l'_n \cap v_n) \) and \( \mu' \in (u'_n \cap q_n). \) From the fact
\[ \gamma + \nu + \mu = \gamma + \nu' + \mu', \]
we get \( \mu' - \nu = \mu - \nu' \in (u'_n \cap v_n) \), call it \( \tilde{\mu} \). Then \( \operatorname{Ad}(h^{-1})z = \gamma + \nu + \nu' + \tilde{\mu} \). Now as \( \gamma + \nu \in (p'_n \cap m_n) \), hence \( \gamma + \nu \in \mathcal{O}' \). As \( \pi' = y \in \mathcal{P}' \) is fixed, therefore \( \gamma \) and \( \nu' \) are being uniquely determined by the condition \( \gamma + \nu' \in \mathcal{P}' \) and \( \operatorname{Ad}(h^{-1})y - (\gamma + \nu') \in u'_n \). So,

\[
\text{(3.1) } p_{r^{-1}}(h(P'_0 \cap Q_0)) \cong \mathcal{O}' \cap (\gamma + (u'_n \cap m_n)) \times (u'_n \cap v_n).
\]

Now consider the following diagram,

\[
\begin{array}{ccc}
\mathcal{O}' & \xrightarrow{\pi} & \mathcal{O}' + u'_n & \xrightarrow{G_0 \times P_0} (\mathcal{O}' + u_n) \\
\downarrow \mathcal{O}'' & & \downarrow \sigma_{\Omega}^{-1}(y) & \downarrow \mathcal{O}'' \\
\mathcal{O}'' \cap (\gamma + (u'_n \cap m_n)) & \xrightarrow{pr_2} & \mathcal{O}'' \cap (\gamma + (u'_n \cap m_n)) \times (u'_n \cap v_n) & \xrightarrow{y} \{pt\} \\
\{pt\} & & & \xrightarrow{u'_n \cap v_n} y
\end{array}
\]

From (3.1), proving

\[
\Gamma((c^* \operatorname{Res}^G_{P_0})^{-1} \pi^* \mathcal{L}[\dim \mathcal{O}]|_{\sigma_{\Omega}^{-1}(y)}) = 0
\]

is same as proving,

\[
\Gamma((c^* \operatorname{Res}^G_{P_0})^{-1} \pi^* \mathcal{L}'[\dim \mathcal{O}]|_{\sigma'' \cap (\gamma + (u'_n \cap m_n))}) = 0.
\]

Which is from the above diagram is same as proving,

\[
\Gamma(\mathcal{L}''|_{\sigma'' \cap (\gamma + (u'_n \cap m_n))}) = 0.
\]

But the last term is isomorphic to the stalk of

\[
\operatorname{Res}^m_{C_{\Gamma}} \mathcal{E}((\mathcal{O}''), \mathcal{L}'').
\]

Which is indeed 0 as \( (\mathcal{O}'', \mathcal{L}'') \in \mathcal{J}(m_n) \).

\[\square\]

**Theorem 3.4.** For any pair \( (\mathcal{O}', \mathcal{L}') \in \mathcal{J}(l_n) \), \( \operatorname{Res}^m_p \operatorname{Ind}^p \mathcal{E}(\mathcal{O}', \mathcal{L}') \) is a parity complex.

**Proof.** We have already seen that \( \operatorname{Res}^m_p \operatorname{Ind}^p \mathcal{E}(\mathcal{O}', \mathcal{L}') \cong \sigma_{\Omega}((c^* \operatorname{Res}^G_{P_0})^{-1} \pi^* \mathcal{L}'[\dim \mathcal{O}]|_{\mu^{-1}(p_n)}) \), where \( \sigma : p_n, \Omega \to \mathcal{P}' \) and \( \Omega \) is a \( (P'_0, P_0) \)-double coset. We can choose \( (P'_0, P_0) \)-double coset \( \Omega_0, \ldots, \Omega_m \), such that \( p_n, \Omega = \cup_{i=0}^m p_n, \Omega_i \). Also assume for any \( j \in [0, m] \), \( p_n, \Omega_0 \cup \ldots, \cup p_n, \Omega_j \) is closed in \( p_n, \Omega \). Let \( \sigma_j \) be the map \( p_n, \Omega_0 \cup \ldots, \cup p_n, \Omega_j \to \mathcal{P}' \) defined by various \( \sigma_{\Omega_i} \). By induction we will show for any \( j \), \( \sigma_j((c^* \operatorname{Res}^G_{P_0})^{-1} \pi^* \mathcal{L}'[\dim \mathcal{O}]|_{\mu^{-1}(p_n)}) \) is parity. Which will imply,

\[
\sigma_{\Omega_j}((c^* \operatorname{Res}^G_{P_0})^{-1} \pi^* \mathcal{L}'[\dim \mathcal{O}]|_{\mu^{-1}(p_n)}) \cong \sigma_1((c^* \operatorname{Res}^G_{P_0})^{-1} \pi^* \mathcal{L}'[\dim \mathcal{O}]|_{\mu^{-1}(p_n)})
\]

is parity. For the base case of induction, we get \( K_{\Omega} \) which is either 0 or \( \operatorname{Ind}^1 \mathcal{E}(\mathcal{O}'', \mathcal{L}'') \), which means parity. We assume the statement is true for \( p_n, \Omega_0 \cup \ldots, \cup p_n, \Omega_j \). Now we have the following open closed triangle,

\[
p_n, \Omega_0 \cup \ldots, \cup p_n, \Omega_j \xrightarrow{i} p_n, \Omega_0 \cup \ldots, \cup p_n, \Omega_{j+1} \xrightarrow{j} p_n, \Omega_{j+1}.
\]

From that we deduce the following distinguished triangle,

\[
i_i \sigma_{j+1, \Omega}(c^* \operatorname{Res}^G_{P_0})^{-1} \pi^* \mathcal{L}'[\dim \mathcal{O}]|_{\mu^{-1}(p_n)} \to \sigma_{j+1}(c^* \operatorname{Res}^G_{P_0})^{-1} \pi^* \mathcal{L}'[\dim \mathcal{O}]|_{\mu^{-1}(p_n)} \to j_* K_{\Omega} \to
\]
Now the LHS is parity by induction and RHS is parity by Theorem 3.2 and Theorem 3.3. Therefore the middle term is parity.

**Corollary 3.5.** For any parabolic $P$ with Levi $L$ containing $\chi(\mathbb{C}^\times)$, $\text{Res}_P^g$ sends parity complexes to parity complexes.

**Proof.** We know that for any pair $(\mathcal{O}, \mathcal{L}) \in \mathcal{F}(\mathfrak{g}_n)$, there exists a parabolic subgroup $P'$ of $G$ with Levi $L'$ and $(\mathcal{O}_L, \mathcal{E}_L) \in \mathcal{F}(\mathfrak{l}_n)^{\text{cusp}}$ such that $\mathcal{E}(\mathcal{O}, \mathcal{L})$ appears as direct summand of $\text{Ind}_{P'}^G \mathcal{E}(\mathcal{O}_L, \mathcal{E}_L)$ [Ch.]. Therefore $\text{Res}_P^g \mathcal{E}(\mathcal{O}, \mathcal{L})$ appears as direct summand of $\text{Res}_P^g \text{Ind}_{P'}^G \mathcal{E}(\mathcal{O}_L, \mathcal{E}_L)$, which is a parity complex by Theorem 3.4. Hence $\text{Res}_P^g \mathcal{E}(\mathcal{O}, \mathcal{L})$ is parity. □

## 4. Fourier Sato transform

Fourier transform was studied by Hotta-Kashiwara [HK] for $D$-modules and later by Brylinski [Bry] in characteristic 0. Study of Fourier transform in the context of Springer sheaf in positive characteristic has been initiated by Juteau [Ju]. As $\mathcal{N}$ is not a vector space, Fourier transform in the usual sense does not make sense. In [AM] a new functor $\mathcal{R}$ from $D_G^b(\mathcal{N}, k)$ to $D_G^b(\mathcal{N}, k)$ has been introduced, which is given by the Fourier transform on $\mathfrak{g}$ followed by the restriction on the nilpotent cone. This is an auto-equivalence [AM]. Later in [AHJR] it has been proved that the Fourier transform of the Springer sheaf is associated to the sign change representation of the Weyl group. In this section we will recall the Fourier-Sato transform [AHJR] and prove that it sends cuspidal pairs to cuspidal pairs and parity complexes to parity complexes.

### 4.1. Fourier-Sato transform on vector spaces

Let $V$ be a complex vector space and $H$ be an algebraic group acting linearly on $V$. An object $\mathcal{F}$ in the $H$-equivariant bounded derived category of $V$, $D_H^b(V, k)$ will be called conic if for any $v \in V$ and $i \in \mathbb{Z}$, the sheaf $H^i(\mathcal{F})|_{\mathbb{C}^\times = 0}$ is locally constant. Or we can say a sheaf complex is conic if it is constructible with respect to the $\mathbb{C}^\times$ orbits in $V$. $D_H^b(V, k)$ is the full-subcategory of conic objects in $D_H^b(V, k)$. We will denote the dual of $V$ by $V^\ast$.

The Fourier-Sato transform is a functor,

$$
\Phi_V : D_{H,\text{con}}^b(V, k) \to D_{H,\text{con}}^b(V^\ast, k).
$$

This functor was initially introduced in [KS, 3.7] and modified in [AM] with a shift of $[\text{dim } V]$ so that the functor is $t$-exact for the perverse $t$-structure [KS Prop. 10.3.8]. This functor is an equivalence of categories with the following properties,

1. $\Phi_V(\delta_V) = \mathbb{K}_{V^\ast}[\text{dim } V^\ast]$, where $\delta_V$ is the skyscraper sheaf on $V$.
2. It commutes with the external tensor product, that is for $V = V_1 \times V_2$,

$$
\Phi_V(M_1 \boxtimes M_2) \cong \Phi_{V_1}(M_1) \boxtimes \Phi_{V_2}(M_2),
$$

with $M_1 \in D_{H,\text{con}}^b(V_1)$ and $M_2 \in D_{H,\text{con}}^b(V_2)$.

### 4.2. Fourier transform on Nilpotent cone

Fourier transform on nilpotent cone has been studied by Achar and Mautner [AM]. As the nilpotent cone is not a vector space the Fourier-Sato transform does not make sense there. So they define the functor in the following way. We fix a $G$-equivariant isomorphism, $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^\ast$, which allows us to consider the Fourier-Sato tranformation $\Phi_\mathfrak{g}$ as a functor,

$$
D_{G,\text{con}}^b(\mathfrak{g}, k) \to D_{G,\text{con}}^b(\mathfrak{g}, k).
$$

Let $i : \mathcal{N}_G \to \mathfrak{g}$ be the inclusion map. The Fourier transform on $\mathcal{N}_G$ is defined by,
\[ \mathcal{R} : D^b_G(N_G, k) \to D^b_G(N_G, k), \]

with \( \mathcal{R} := i^* \Phi_g i_*[\dim N_G - \dim \mathfrak{g}] \). \( i_* \) takes values in \( D^b_{G,\text{con}}(\mathfrak{g}, k) \), therefore \( \mathcal{R} \) makes sense. The functor \( \mathcal{R} \) is an auto-equivalence with the inverse \( \mathcal{R}' \), which is defined as,

\[ \mathcal{R}' := i^! \Phi_g' i_*[\dim \mathfrak{g} - \dim N_G]. \]

Here \( \Phi_g' \) is the inverse of \( \Phi_g \). One important property of the functor \( \mathcal{R} \) is that for \( GL_n \) it is a geometric version of Ringel duality [AM].

### 4.3. Fourier transform on graded Lie algebras

As being a vector space, we can talk about Fourier-Sato transform on \( \mathfrak{g}_n \). Also as \( G_0 \) commutes with \( \mathbb{C}^\times \) action on \( \mathfrak{g}_n \), \( D^b_{G_0,\text{con}}(\mathfrak{g}_n) \) is same as \( D^b_{G_0}(\mathfrak{g}_n) \). We fix a \( G \)-equivariant isomorphism, \( \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^* \) as before. But in the graded case the dual of \( \mathfrak{g}_n \) is \( \mathfrak{g}_{-n} \). Therefore we have,

\[ \Phi_{\mathfrak{g}_n} : D^b_{G_0}(\mathfrak{g}_n) \to D^b_{G_0}(\mathfrak{g}_{-n}). \]

To deal with this grading change, in this section particularly we will use the notations,

\[ {}_{\mathfrak{g}}^0 \mathcal{I}_n : D^b_{L_0}(I_n) \to D^b_{G_0}(\mathfrak{g}_n) \]

and

\[ {}_{\mathfrak{g}}^0 \mathcal{R}_n : D^b_{G_0}(\mathfrak{g}_n) \to D^b_{L_0}(I_n), \]

for induction and restriction respectively.

**Theorem 4.1.** For \( P \subset G \) with \( \chi(\mathbb{C}^\times) \subset P \), \( \Phi_{\mathfrak{g}_n} {}_{\mathfrak{g}}^0 \mathcal{R}_n \cong {}_{\mathfrak{g}}^0 \mathcal{R}_{-n} \Phi_{\mathfrak{g}_n}. \)

**Proof.** Let \( \bar{\mathfrak{p}} \) be the opposite Lie algebra of \( \mathfrak{p} \), which means \( \bar{\mathfrak{p}} = l \oplus \bar{\mathfrak{u}} \). Then using [MI] we get the following commutative diagram,

\[
\begin{array}{ccc}
\mathfrak{p}_n & \xrightarrow{\pi} & I_n \\
\downarrow j & & \downarrow j' \\
\mathfrak{g}_n & \xrightarrow{\tau} & \bar{\mathfrak{p}}_n
\end{array}
\]

where \( j \) is inclusion and \( \tau \) is projection maps. Therefore, for \( \mathcal{F} \in D^b_{G_0}(\mathfrak{g}_n) \) we have,

\[ {}_{\mathfrak{g}}^0 \mathcal{R}_n(\mathcal{F}) = \text{For}_{L_0} \pi_! i^*(\mathcal{F}) \cong \text{For}_{L_0} j^! \tau_*(\mathcal{F}). \]

The Lie algebras, \( \mathfrak{p}_n \) and \( \bar{\mathfrak{p}}_{-n} \) are dual to each other, and the inclusion \( i : \mathfrak{p}_n \to \mathfrak{g}_n \) has dual map \( \tau : \mathfrak{g}_{-n} \to \bar{\mathfrak{p}}_{-n} \). Hence by [Ac] prop. 6.9.13],

\[ \tau_* \Phi_{\mathfrak{g}_n} \mathcal{F} \cong \Phi_{\mathfrak{p}_n} j^! \mathcal{F}[\dim \mathfrak{g}_n - \dim \mathfrak{p}_n]. \]

Similarly, the map \( j : I_{-n} \to \bar{\mathfrak{p}}_{-n} \) is dual to the map \( \pi : \mathfrak{p}_n \to I_n \). Hence by the similar argument as above we have,

\[ j^! \Phi_{\mathfrak{p}_n} \cong \Phi_{I_n} \pi_*[\dim I_n - \dim \mathfrak{p}_n]. \]

Therefore we get,

\[ {}_{\mathfrak{g}}^0 \mathcal{R}_{-n} \Phi_{\mathfrak{g}_n} \mathcal{F} = \text{For}_{L_0} j^! \tau_* \Phi_{\mathfrak{g}_n} \mathcal{F} \]

\[ \cong \text{For}_{L_0} j^! \Phi_{\mathfrak{p}_n} i^! \mathcal{F}[\dim \mathfrak{g}_n - \dim \mathfrak{p}_n] \]

\[ \cong \text{For}_{L_0} \Phi_{I_n} \pi_* i^! \mathcal{F}[\dim I_n + \dim \mathfrak{g}_n] \]

\[ \cong \Phi_{I_n} \text{For}_{L_0} \pi_* i^! \mathcal{F}[\dim I_n + \dim \mathfrak{g}_n], \text{ as } \Phi_{\mathfrak{g}_n} \text{ commutes with forgetful functor.} \]

\[ \cong \Phi_{I_n} {}_{\mathfrak{g}}^0 \mathcal{R}_n \mathcal{F}. \]
Corollary 4.2. For $P \subset G$ with $\chi(C^x) \subset P$, $\Phi_g \circ \Delta \cong \Phi_{g_n} \circ \Delta$.  

Proof. The proof follows from the fact that the restriction is left adjoint to the induction.  

Theorem 4.3. The Fourier Sato transform sends cuspidal pairs to cuspidal pairs.  

Proof. $\Phi_g : \mathcal{O}_{\mathcal{C}}(\mathfrak{g}_n) \rightarrow \mathcal{O}_{\mathcal{C}}(\mathfrak{g}_n)$ sends simple perverse sheaves to simple perverse sheaves. Now our claim is that $\mathcal{O}_{\mathcal{C}}(\mathfrak{g}_n)$ sends non-cuspidal pairs to non-cuspidal pairs, which will be enough to prove that $\Phi_g$ sends cuspidal pairs to cuspidal pairs as it is an equivalence of categories. We will prove this by induction on the support. For the base case,  

$\Phi_g(\mathcal{O}_{\mathcal{C}}(\mathfrak{g}_n)) = \mathcal{I}C(0, \mathcal{E}_{\mathcal{C}})$.  

We know $\mathcal{E}_{\mathcal{C}}$ is non-clean, hence non-cuspidal. Therefore the base case is proved. Now we want to show that for the non-cuspidal pair $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}C(\mathfrak{g}_n)$, $\Phi_g(\mathcal{I}C(\mathcal{O}, \mathcal{L}))$ is non-cuspidal. By [Ch 6.11], $\mathcal{E}(\mathcal{O}, \mathcal{L})$ appears as direct summand of $\mathcal{I}C(\mathcal{O}_L, \mathcal{E}_L)$, where $P$ is a parabolic containing a Levi $L$ with $\chi(C^x) \subset L$ and $(\mathcal{O}_L, \mathcal{E}_L) \in \mathcal{F}(\mathcal{L})$. By Corollary 4.2, $\Phi_g$ commutes with induction, so $\Phi_g(\mathcal{I}C(\mathcal{O}, \mathcal{L}))$ is a direct summand of $\mathcal{I}C(\mathcal{O}_L, \mathcal{E}_L)$. By induction, $\Phi_g, \mathcal{I}C(\mathcal{O}_L, \mathcal{E}_L)$ is cuspidal. This implies $\mathcal{E}(\mathcal{O}, \mathcal{L})$ is non-cuspidal. The triangulated category generated by $\mathcal{E}(\mathcal{O}, \mathcal{L})$ and all the $\mathcal{I}C$’s coming from smaller orbits is same as the triangulated category generated by $\mathcal{I}C(\mathcal{O}, \mathcal{L})$ and every $\mathcal{I}C$’s coming from smaller orbits. But from the induction and the above argument the former triangulated category is subset of the Fourier-Sato transform of non-cuspidal pairs. Therefore same is true for the later triangulated category. Hence $\Phi_g, \mathcal{I}C(\mathcal{O}, \mathcal{L})$ is non-cuspidal.  

Corollary 4.4. $\Phi_g$ sends parity complexes to parity complexes.  

Proof. Let $(\mathcal{O}, \mathcal{L}) \in \mathcal{F}(\mathfrak{g}_n)$ and $\mathcal{E}(\mathcal{O}, \mathcal{L})$ be the associated parity sheaf. By [Ch 6.11], there exists parabolic subgroup $P$ with a Levi $L$ containing $\chi(C^x)$ and $(\mathcal{O}_L, \mathcal{E}_L) \in \mathcal{F}(\mathcal{L})$ such that, so $\mathcal{E}(\mathcal{O}, \mathcal{L})$ is direct summand of $\mathcal{I}C(\mathcal{O}_L, \mathcal{E}_L)$. By Corollary 4.2, $\Phi_g, \mathcal{I}C(\mathcal{O}, \mathcal{L})$ is direct summand of $\mathcal{I}C(\mathcal{O}_L, \mathcal{E}_L)$. By Theorem 4.3, $\Phi_g, \mathcal{I}C(\mathcal{O}_L, \mathcal{E}_L)$ is cuspidal. By [Ch 6.12], $\mathcal{I}C(\mathcal{O}_L, \mathcal{E}_L)$ is parity. Therefore $\Phi_g, \mathcal{I}C(\mathcal{O}, \mathcal{L})$ is parity.  

5. Primitive Pair  

Primitive pairs have been introduced by Lusztig [Lu] both on the nilpotent cone and the graded Lie algebra. One of the main goal of that paper was to define a block decomposition of $\mathcal{F}(\mathfrak{g}_n)$ using the primitive pairs on the nilpotent cone. In this section we will try to prove some of the results of [Lu] in positive characteristic. A long term goal will be to study the block decomposition of $\mathcal{F}(\mathfrak{g}_n)$ in positive characteristic.  

First we will introduce the primitive pair on the nilpotent cone.  

Definition 5.1. A pair $(\mathcal{C}, \mathcal{E}) \in \mathcal{F}(G)$ is said to be primitive if there exists a parabolic subgroup $P$ with Levi $\mathcal{L}$ and a cuspidal pair $(\mathcal{C}, \mathcal{E}) \in \mathcal{F}(\mathcal{L})^0$ such that the following holds,  

1. There exists a $P$-orbit $C'$ in $\mathcal{N}_P$, so that for all $x \in C'$, $\pi(x) \in C_L$, where $\pi : \mathfrak{p} \rightarrow \mathfrak{l}$.  

2. For any $x \in C'$, we have,  

$L^x(x)/(L^x(x))^0 \cong P^x/(P^x)^0 \cong G^x/(G^x)^0$.  

3. $\mathcal{E}_{|C'} \cong (\pi|_{C'})^* \mathcal{E}_L$.  

□
The definition of the primitive pair on \( \mathfrak{g}_n \) is similar to the above definition on the nilpotent cone.

**Definition 5.2.** A pair \((\mathcal{O}, L) \in \mathcal{I}(\mathfrak{g}_n)\) is said to be primitive if there exists a parabolic subgroup \( P \) containing the image of \( \chi \) with \( \text{Levi} \) \( L \) and a cuspidal pair \((\mathcal{O}_L, \mathcal{E}_L) \in \mathcal{I}(\mathfrak{g}_n)_{\text{cusp}}\) such that the following holds,

1. There exists a \( P_0 \)-orbit \( \mathcal{O}' \) in \( \mathfrak{p}_n \), so that for all \( x \in \mathcal{O}' \), \( \pi(x) \in \mathcal{O}_L \), where \( \pi : \mathfrak{p}_n \to \mathfrak{l}_n \).
2. For any \( x \in \mathcal{O}' \), we have,

\[
L^\pi/(L^\pi)^{\circ} \cong P^\pi/(P^\pi)^{\circ} \cong G^\pi/(G^\pi)^{\circ}.
\]

3. \( \mathcal{L}|_{\mathcal{O}'} \cong (\pi|_{\mathcal{O}'})^* \mathcal{E}_L \).

**Theorem 5.3.** Assume \((C, E) \in \mathcal{I}(G)\) be a primitive pair with \( C \cap \mathfrak{g}_n \neq \emptyset \) associated to the parabolic subgroup \( P \) containing a Levi \( L \) and \( (C_L, E_L) \in \mathcal{I}(L)^{0-\text{cusp}} \) with \( \chi(C^\times) \subset L \) and \( C_L \cap \mathfrak{l}_n \neq \emptyset \). Then \((C, E) \in \mathcal{I}(G)\) gives rise to a primitive pair \((\mathcal{O}, L) \in \mathcal{I}(\mathfrak{g}_n)\).

**Proof.** Let \( C' \) be the \( P \)-orbit in \( \mathcal{N}_P \) which satisfies the conditions in Definition 5.1. Let \( \mathcal{O}_L = C_L \cap \mathfrak{l}_n \) and \( \mathcal{F}_L = E_L|_{\mathcal{O}_L} \). Then by the definition of cuspidal pair in the graded setting, \((\mathcal{O}_L, \mathcal{F}_L) \in \mathcal{I}(\mathfrak{g}_n)_{\text{cusp}}\). We define,

\[
\mathcal{O}' = \{ x \in \mathfrak{p}_n \mid \text{there exists a Levi } M' \subset P, \text{ with } \chi(C^\times) \subset M', x \in m'_n, \text{ and } \pi(x) \in \mathcal{O}_L \}
\]

As all the Levi inside \( P \) are \( P_0 \)-conjugate to each other, therefore \( \mathcal{O}' \) is a single \( P_0 \)-orbit in \( \mathfrak{p}_n \). Let \( \mathcal{O} \) be the unique \( G_0 \)-orbit in \( \mathfrak{g}_n \) containing \( \mathcal{O}' \) and \( L = \mathcal{E}|_{\mathcal{O}} \). Our claim is that \((\mathcal{O}, L) \) is primitive. By the construction of \( \mathcal{O}' \), it satisfies the first condition in Definition 5.2. As \((C, E) \in \mathcal{I}(G)\) is primitive, for \( x \in \mathcal{O}' \), we have,

\[
L^\pi/(L^\pi)^{\circ} \cong P^\pi/(P^\pi)^{\circ} \cong G^\pi/(G^\pi)^{\circ}.
\]

First we want to show \( \mathcal{L} \) is irreducible. For that it is enough to show that \( G_0^\pi/(G_0^\pi)^{\circ} \to G^\pi/(G^\pi)^{\circ} \) is surjective.

We have the following commutative diagram,

\[
\begin{array}{ccc}
L^\pi/(L^\pi)^{\circ} & \leftrightarrow & P^\pi/(P^\pi)^{\circ} \\
\downarrow & & \downarrow \\
L_0^\pi/(L_0^\pi)^{\circ} & \leftrightarrow & P_0^\pi/(P_0^\pi)^{\circ} \\
& & \downarrow \\
& & G_0^\pi/(G_0^\pi)^{\circ}
\end{array}
\]

The lower horizontal maps are all isomorphisms by the definition of the primitive pair on the nilpotent cone. For the map \( P_0^\pi \to L_0^\pi \), we have connected kernel, hence induces isomorphism on the component groups. By [Lu Prop. 4.3], \((L, \chi)\) is \( n \)-rigid and therefore by [Lu Prop. 4.2(c)],

\[
L_0^\pi/(L_0^\pi)^{\circ} \cong L_0^\pi/(L_0^\pi)^{\circ}.
\]

Hence the middle vertical arrow is an isomorphism. Which tells us that in the right square the composition of top horizontal with the right vertical map is an isomorphism. This implies the map \( G_0^\pi/(G_0^\pi)^{\circ} \to G^\pi/(G^\pi)^{\circ} \) is at least surjective. So we are done with the claim that \( \mathcal{L} \) is irreducible.

But to show that \((\mathcal{O}, L) \) is primitive, we need isomorphisms for the upper horizontal maps.

By the definition of \( \mathcal{O}' \), there exists \( M' \), a Levi subgroup of \( P \) with \( \chi(C^\times) \subset M', x \in m'_n \) and \( \pi(x) \in \mathcal{O}_L \). Now we can find a Lie algebra homomorphism \( \phi : \mathfrak{sl}_2 \to m' \) with \( \phi(x) = x \), \( \phi(f) \in \mathfrak{g}_{-n} \).
and \( \phi(h) \in g_0 \), where \( \{e, h, f\} \) is the standard triple in \( \mathfrak{sl}_2 \). Let \( \tilde{\phi} : SL_2 \to M' \) be such that \( d\tilde{\phi} = \phi \) and we define \( \chi' : \mathbb{C}^\times \to M' \) by,

\[
\chi'(a) = \tilde{\phi} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.
\]

Then \( \phi \) satisfies the condition in [Ch, Definition 6.1] with respect to \( \chi : \mathbb{C}^\times \to M' \). Considering \( \chi \) as a map to \( G \), we use the construction [Ch, 6.2] to define the parabolic and Levi subalgebras \( \mathfrak{q}, \mathfrak{m} \).

It is not hard to see that \( x \) is in \( \mathfrak{m} \). By [Ch, Th. 6.3], \((M, \chi)\) is \( n \)-rigid and hence,

\[
M^x/(M^x)^o \cong M_0^x/(M_0^x)^o, \quad \text{by [Lu] Prop. 4.2(c)}.
\]

Also by [Lu] Prop. 5.8,

\[
M_0^x/(M_0^x)^o \cong G_0^x/(G_0^x)^o.
\]

As \( M' \) is a Levi subgroup of \( P \) conjugate to \( L \), we have from Definition 5.1,

\[
M^x/(M^x)^o \cong G^x/(G^x)^o.
\]

Now as \( \phi : \mathfrak{sl}_2 \to \mathfrak{m}' \) is \( n \)-adapted to \( \chi \), we have,

\[
m \mathfrak{m}' = \mathfrak{m}' \mathfrak{m} \quad \text{if} \quad mn/2 \in \mathbb{Z}.
\]

By the construction [Ch, 6.2], \( \mathfrak{m} = \bigoplus \{m \mathfrak{g} \cap \mathfrak{g}_{mn/2}\} \), when \( mn/2 \in \mathbb{Z} \). Therefore \( \mathfrak{m}' \subset \mathfrak{m} \). Hence the triple \((M, M', x)\) plays the same role as \((G, M', x)\). Hence we have,

\[
M^x/(M^x)^o \cong M_0^x/(M_0^x)^o.
\]

Combining (5.3), (5.4), (5.5), we get,

\[
G_0^x/(G_0^x)^o \cong G^x/(G^x)^o.
\]

In the commutative diagram above, we already know the top left horizontal map is isomorphism. We want to show the right top horizontal map is isomorphism too. As we have already seen the left and right vertical and the lower horizontal maps are isomorphisms in the right square, hence the top horizontal map is isomorphism too.

Now we have from the following commutative diagram,

\[
\begin{array}{ccc}
C_L & \leftarrow & C' \\
\uparrow & & \uparrow \\
O_L & \leftarrow & O'
\end{array}
\]

\[\mathcal{L}|_{O'} \cong (\mathcal{E}|_{O'})|_{O'} \] by the right commutative square.

\[\cong ((\pi|_{O'})^*\mathcal{E})|_{O'} \] by the condition of primitive pair on \( G \).

\[\cong (\pi|_{O'})^*\mathcal{E}|_L \] by the left commutative square.

\[\cong (\pi|_{O'})^*\mathcal{F}_L.\]

Hence we are done with the third condition in Definition 5.2. \( \square \)

The next theorem appeared in [Lu]. The proof is similar in all characteristics.

**Lemma 5.4.** Assume \((\mathcal{O}, \mathcal{L}) \in \mathcal{J}(\mathfrak{g}_n)\) to be a primitive pair. \( P, L, (\mathcal{O}_L, \mathcal{L}_L) \in \mathcal{J}(\mathfrak{l}_n)^{cusp} \), \( P', L', (\mathcal{O}_{L'}, \mathcal{L}_{L'}) \in \mathcal{J}(\mathfrak{l}_n)^{cusp} \) be the two sets of parabolics, Levis and cuspidal pairs associated to the primitive pair. Then there exists \( g \in G_0 \) such that \( P', L', (\mathcal{O}_{L'}, \mathcal{L}_{L'}) \in \mathcal{J}(\mathfrak{l}_n)^{cusp} \) is \( g \) conjugate to \( P, L, (\mathcal{O}_L, \mathcal{L}_L) \in \mathcal{J}(\mathfrak{l}_n)^{cusp} \).
The proof follows from [Lu, 11.4].

5.1. More on primitive pairs. Now we fix the following assumptions,
- A primitive pair $(C, F) \in \mathcal{I}(G)$ such that $C \cap g_n \neq \emptyset$.
- A $G_0$-orbit $O \subset g_n$ such that $O \subset C$.
- $y \in O$ and $\phi \in J_n$ such that $\phi(e) = y$.
- A maximal torus $S$ of $(G_0^\phi)\circ$ where $G_0^\phi = \{g \in G_0|\text{Ad}(g)x = x, \text{for all } x \in \text{Im} \phi\}$.

Let $L = F|_O$ and $L = Z_G(S)$. This implies $\chi(C^\times) \subset L$.

Proposition 5.5. With the above assumptions, the followings are true,
1. $L$ is irreducible local system.
2. Let $O_L$ be the unique open $L_0$-orbit in $O_n$., then $O_L \subset O$ and $(O_L, L|_{O_L}) \in \mathcal{I}(I_n)_{\text{cusp}}$.
3. For any parabolic $P$ containing $L$, $(O, L)$ is the primitive pair associated to $(P, L, O|_{O_L})$.
4. If there exists parabolic $Q$ containing Levi $M$ with cuspidal $(O_M, L_M) \in \mathcal{I}(I_n)_{\text{cusp}}$ such that $(O, L)$ is the primitive pair, then $(Q, M, O_M, L_M)$ is $G_0$-conjugate to $(P, L, O|_{O_L})$.

Proof. (1) Follows from Theorem 5.3.
(2) $S$ is a maximal torus in $(G_0^\phi)\circ$. This means $S$ is a torus in $(G_0^\phi)\circ$. Our claim is that $S$ is maximal in $(G_0^\phi)\circ$. Which is true because for any maximal torus $T$ in $(G_0^\phi)\circ$, it commutes with $\chi(C^\times)$ as $\chi(C^\times)$ normalizes $(G_0^\phi)\circ$. Therefore $T$ is a torus in $(G_0^\phi)\circ$ and hence $T \subset S$.

Now by [Lu 4.2(d)], $(G_0^\phi)\circ$ admits $(G_0^\phi)\circ$ as Levi factor. Hence $S$ is also a maximal torus inside $(G_0^\phi)\circ$. Now let $Q$ be the cuspidal subgroup of $M$ be the associated Levi and $(C_M, F_M) \in \mathcal{I}(M)_{\text{cusp}}$ be the corresponding cuspidal pair involved with the primitive pair $(C, F)$. Recall $y$ is in $O$, hence in $C$. By conjugating $Q$ by an element of $G$, we can assume $y \in M' for an Levi subgroup $M'$ of $Q$ and $\pi(y) \in C_M$. By [Lu 2.7(f)], $M'$ is the centralizer of a maximal torus $T$ of $(G_0^\phi)\circ$. Therefore $T$ is conjugate to $S$ by an element of $(G_0^\phi)\circ$. Now we conjugate $Q$, $M'$ by that element of $(G_0^\phi)\circ$ so that we get $S = T$. As $L$ is also the centralizer of $S$, we get $L = M'$. Therefore $L$ is conjugate to $M$ in $P$ and let $(C_L, F_L) \in \mathcal{I}(L)_{\text{cusp}}$ be the cuspidal pair corresponding to $(C_M, F_M) \in \mathcal{I}(M)_{\text{cusp}}$.

(3) This follows from Theorem 5.3.
(4) This is true by Lemma 5.4.

5.2. $n$-rigid. Recall the definition of $n$-rigidity from [Ch] 6.1. In this subsection we keep all the assumptions from subsection 5.1. Additionally we assume $(G, \chi)$ is $n$-rigid and the unique open $G_0$ orbit $O_n$ in $g_n$ is contained inside $C$. Then by [Lu 4.2(f)], $C \cap g_n = O_n$. Therefore $O_n = O$. Hence $\phi(e) = y$ is in $O_n$ [Lu 4.2(a)]. Now as $(G, \chi)$ is $n$-rigid, then $\phi$ is $n$-adapted to $\chi$. Hence by [Lu 4.2(c)], $G_0^\phi = G^\phi$. Therefore as $G_0^\phi \subset G^\phi$, hence $G^\phi \subset G_0^\phi$. So we get $G^\phi = G_0^\phi$.

Lemma 5.6. For $P, L, U$ as defined above, the $P_0$-orbit of $y$ is same as $O_L + u_n$. 

The proof follows from [Lu, Prop. 11.10].

**Lemma 5.7.** For \( x \in \mathcal{O}_L \), we have the following isomorphism,
\[
(G_0^0)^\circ / (P_0^y)^\circ \cong \mu^{-1}(x) \cap G_0 \times F_0 (\mathcal{O}_L + u_n),
\]
where \( \mu : G_0 \times F_0 (\mathcal{O}_L + u_n) \to g_n \).

**Proof.** The map,
\[
(G_0^0)^\circ / (P_0^y)^\circ \to \mu^{-1}(x) \cap G_0 \times F_0 (\mathcal{O}_L + u_n),
\]
defined by \( g(P_0^y) \to (gP_0, x) \) is well-defined. Recall from Proposition 5.5(2), \( \mathcal{O}_L \subset \mathcal{O} \). Hence if we can prove the statement for \( y \in \mathcal{O}_L \), then it is true for any \( x \in \mathcal{O}_L \). The injectivity of the above map is obvious. Let \( g \in G_0 \) such that \( \text{Ad}(g^{-1})y \in \mathcal{O}_L + u_n \). By Lemma 5.6 there exists \( p \in P_0 \) such that,
\[
\text{Ad}(g^{-1})y = \text{Ad}(p)g.
\]
This implies,
\[
\text{Ad}(gp)y = y.
\]
We can replace \( g \) by \( gp \). Then we see \( g \in G_0^y \). Hence the above mentioned map is an isomorphism. Because \( (\mathcal{O}, \mathcal{L}) \) is primitive with respect to \( (P, \mathcal{L}, \mathcal{O} \wedge \mathcal{L}) \) and \( y \in \mathcal{O}_L \), then by the definition of primitive pair,
\[
(G_0^0)^\circ / (P_0^y)^\circ \cong (G_0^y)^\circ / (P_0^y)^\circ.
\]
Therefore we get,
\[
G_0^y / (P_0^y)^\circ \cong (G_0^0)^\circ / (P_0^0)^\circ
\]
and, we are done. \( \square \)

**Remark 5.8.** As we explained at the beginning of this subsection that \( (P_0^y)^\circ \) is a Borel subgroup of \( (G_0^y)^\circ \), hence \( (G_0^0)^\circ / (P_0^0)^\circ \) is a flag variety. Therefore there exist \( s_1, \ldots, s_n \in \mathbb{Z} \) such that,
\[
\text{RT}(\mathcal{L}^{-1} y \cap G_0 \times F_0 (\mathcal{O}_L + u_n)) \cong \bigoplus_{j=1}^n \mathbb{R}[2s_j].
\]

Before the next theorem we make the assumption that the \( (P_0^0)^\circ \)-orbits in \( G_0^0 / P_0^0 \) are all \( P_0^y \)-stable.

**Theorem 5.9.** \( 1 \) \( H^i(\text{Ind}_p^g \mathcal{E}(\mathcal{O}_L, \mathcal{L} \wedge \mathcal{L} \wedge \mathcal{O}_L) | \mathcal{O} | ) \cong \begin{cases} 0, \text{ if } i \text{ is odd, } \\ \oplus_j \mathcal{L}[-2s_j] \text{ for } i = -\dim \mathcal{O}. \end{cases} \)

\( 2 \) \( \text{Ind}_p^g \mathcal{E}(\mathcal{O}_L, \mathcal{L} \wedge \mathcal{O}_L) |_{\mathcal{O}} = 0. \)

\( 3 \) \( \text{Ind}_p^g \mathcal{E}(\mathcal{O}_L, \mathcal{L} \wedge \mathcal{O}_L) \cong \oplus \mathcal{E}(\mathcal{O}, \mathcal{L})[-2s_j]. \)

\( 4 \) \( \mathcal{E}(\mathcal{O}, \mathcal{L}) |_{\mathcal{O}} = 0. \)

**Proof.** \( 1 \) By [CH, Cor. 4.5], \( \mathcal{E}(\mathcal{O}_L, \mathcal{L} \wedge \mathcal{O}_L) \) exists. If we can show \( \text{Ind}_p^g \mathcal{E}(\mathcal{O}_L, \mathcal{L} \wedge \mathcal{O}_L) |_{\mathcal{O}_L} \cong \oplus_j \mathcal{L}[-2s_j] \) then by using the isomorphism,
\[
L_0^{\pi(x)} / (L_0^{\pi(x)})^\circ \cong (G_0^0)^\circ / (G_0^0)^\circ,
\]
for any \( x \in \mathcal{O}_L + u_n \), we will be done. Our first claim is that the image of the map \( \mu : G_0 \times F_0 (\mathcal{O}_L + u_n) \to g_n \) is \( \mathcal{O} \).

Now \( G_0 \times F_0 (\mathcal{O}_L + u_n) \) is isomorphic to
\[
\{(gP_0, x) \in G_0 / P_0 \times g_n | \ \text{Ad}(g^{-1})x \in \mathcal{O}_L + u_n, \}
\]
and the map \( \mu \) becomes the projection on the second coordinate. If \( x \) is in the image of \( \mu \) that means there exists \( g \in G_0 \) such that \( \text{Ad}(g^{-1})x \in \mathcal{O}_L + u_n \). But by Lemma 5.6 \( \mathcal{O}_L + u_n \),
is the $P_0$-orbit of $y$, call it $O_P$. This implies $\text{Ad}(g^{-1})x \in O_P$. But $O_P \subset O$. This implies $\text{Ad}(g^{-1})x \in O$, which means $x \in O$.

For the fixed $y \in O_L$, we have the following commutative diagram,

$$
\begin{array}{ccc}
O_L & \xrightarrow{\pi} & O_L + u_n \\
& & e \\
& & G_0 \times P_0 (O_L + u_n) \\
& & \mu \rightarrow O \\
\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n) & & y
\end{array}
$$

Our next claim is that,

$$
G_0 \times P_0 (O_L + u_n) \cong G_0 \times P_0 (\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n)).
$$

We define a map,

$$
G_0 \times (\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n)) \rightarrow G_0 \times P_0 (O_L + u_n),
$$

by

$$(g, (hP_0, y)) \rightarrow (ghP_0, \text{Ad}(g)y).$$

Recall $O_L + u_n$ is a $P_0$-orbit. Hence $G_0 \times P_0 (O_L + u_n)$ is isomorphic to a $G_0$-orbit. Therefore any element in $G_0 \times P_0 (O_L + u_n)$ is of the form $(gP_0, y)$, whose inverse image under the above map is $(g, (P_0, y))$. Hence the above map is surjective. If $(ghP_0, \text{Ad}(g)y) = (P_0, y)$, then $\text{Ad}(g)y = y$. This implies $g \in G_0$. So we get a bijection between $G_0 \times P_0 (\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n))$ and $G_0 \times P_0 (O_L + u_n)$. Now $G_0 \times P_0 (O_L + u_n)$ is smooth as $O_L + u_n$ is a $P_0$-orbit. Also by Lemma 5.7, we have $\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n) \cong (G_0^y)^q / (P_0^y)^q$. Recall $(P_0^y)^q / (P_0^y)^q$ is a flag variety, therefore smooth. Bijection between two smooth varieties induces isomorphism of varieties. Hence we are done with our claim. This means the map

$$
\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n) \rightarrow G_0 \times P_0 (O_L + u_n)
$$

induces induction equivalence.

The projection,

$$
G_0 \times P_0 (\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n)) \rightarrow G_0 / G_0^y \cong O
$$

is a fiber bundle with the fiber $\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n)$. But being a flag variety, $\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n)$ is simply connected. Hence we get the following isomorphism of the fundamental groups,

$$
\pi_1(O) \cong \pi_1(G_0 \times P_0 (O_L + u_n)).
$$

The $G_0$-equivariant local systems on $O$, denoted by $\text{Loc}_{G_0}(O)$ is isomorphic to the representation of the fundamental group, $\text{Rep}(\pi_1(O))$. Now we have,

$$
\text{Loc}_{G_0}(O) \cong \text{Rep}(\pi_1(O))
$$

$$
\cong \text{Rep}(\pi_1(G_0 \times P_0 (O_L + u_n)))
$$

$$
\cong \text{Loc}_{G_0^y}(\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n)), \text{ by induction equivalence.}
$$

On the other hand, by induction equivalence,

$$
\text{Loc}_{G_0}(O) \cong \text{Loc}_{G_0^y}(\{y\}).
$$

Combining all these we get,

$$
\text{Loc}_{G_0^y}(\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n)) \cong \text{Loc}_{G_0^y}(\{y\}).$$
Hence pullback along the map,
\[ \mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n) \to \{ y \} \]
induces an equivalence of categories.

Now our claim is that,
\[ \mu^*(\mathcal{L}) \cong (e^* \text{For}^{G_0}_{P_0})^{-1} \pi^*(\mathcal{L}|_{O_L}). \]

From the diagram below,
\[
\begin{array}{ccc}
O_L & \xrightarrow{i} & O_L + u_n \\
| & | & | \\
\downarrow \pi & & \downarrow \pi \\
G_0 \times P_0 (O_L + u_n) & \xrightarrow{\mu} & O
\end{array}
\]

Let \( \mathcal{G} \) be a local system on \( G_0 \times P_0 (O_L + u_n) \), uniquely determined by,
\[ (e^* \text{For}^{G_0}_{P_0}) \mathcal{G} \cong \pi^*(\mathcal{L}|_{O_L}). \]

By Definition 5.2,
\[ \text{Loc}_{P_0}(O_L) \cong \text{Loc}_{P_0}(O_L + u_n), \]
and \( i^* \) is the inverse of \( \pi^* \). So we can say, \( \mathcal{G} \) is uniquely determined by,
\[ i^*(e^* \text{For}^{G_0}_{P_0})(\mathcal{G}) \cong \mathcal{L}|_{O_L}. \]

From the above diagram, it is clear that,
\[ i^*(e^* \text{For}^{G_0}_{P_0}) \mu^* \mathcal{L} \cong \mathcal{L}|_{O_L}. \]

Hence \( \mathcal{G} \cong \mu^* \mathcal{L}. \) Therefore \( \text{Ind}^G_y \mathcal{E}(O_L, \mathcal{L}|_{O_L}) \) becomes \( \mu \mu^* \mathcal{L}. \) But now,
\[ \mu \mu^* \mathcal{L} \cong \mu(\mathfrak{k} \otimes L \mu^* \mathcal{L}) \]
\[ \cong \mu(\mathfrak{k}) \otimes L \mathcal{L}. \]

We have the following diagram,
\[
\begin{array}{ccc}
D^b_{G_0}(G_0 \times P_0 (O_L + u_n)) & \xrightarrow{\mu^*} & D^b_{G_0}(O) \\
| \cong | & | \cong | \\
D^b_{G_0}(\mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n)) & \cong & D^b_{G_0}(y).
\end{array}
\]

Hence \( \mathcal{L} \) can be considered as a \( G_0^\circ \)-equivariant local system on \( \{ y \} \) and \( \mu \mu^* \mathcal{L} \) can considered as the composition of the pullback and push-forward along the map,
\[ \mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n) \to \{ y \}. \]

With an abuse of notation, we also call the above map to be \( \mu \). As \( \mu^{-1}(y) \cap G_0 \times P_0 (O_L + u_n) \cong G_0^\circ / P_0^\circ \), if we can show \( G_0^\circ / (G_0^\circ)^{0} \) acts on \( H^*_G(G_0^\circ / P_0^\circ) \) trivially, then by the discussion in Remark 5.8 we will be done. This is same as showing the constant local system along the map,
\[ D^b_{G_0}(G_0^\circ / P_0^\circ) \to D^b_{G_0}(\{ y \}) \]
gets sent to a sheaf complex whose cohomologies are constant local systems. By Definition \ref{def:localization_of_cohomologies}
\[ \text{Loc}_{G_0}(\{y\}) \cong \text{Loc}_{P_0}(\{y\}). \]

Therefore Proposition \ref{prop:lifting_of_cohomologies} reduces to showing, the map below,
\[ D^b_{P_0}(G_0/P_0) \to D^b_{P_0}(\{y\}) \]
sends constant local system to a sheaf complex whose cohomologies are constant local systems. Now \( G_0/P_0 \cong (G_0^{\mu})^{\circ}/(P_0^{\mu})^{\circ} \), which is a flag variety stratified by the \((P_0^{\mu})^{\circ}\)-orbits, which are Schubert cells, hence affine. By our assumption, \((P_0^{\mu})^{\circ}\)-orbits are stable under \( P_0^{\mu} \) in \( G_0^{\mu}/P_0^{\mu} \). Hence our claim further reduces to showing that along the map,
\[ D^b_{P_0}(\Lambda^n) \to D^b_{P_0}(\{y\}) \]
constant local system gets sent to the constant local system, where \( \Lambda^n \) is an affine space. Which is indeed true. So we are done. (2) As we have shown in (1) that the image of \( \mu : G_0 \times P_0 (\mathcal{O}_L + u_\mu) \to g_0 \) is \( \mathcal{O} \), so we are done. (3) By [Ch. Th. 5.3], \( \text{Ind}_0^{\mu} \mathcal{E}(\mathcal{O}_L, \mathcal{L}|_{\mathcal{O}_L}) \) is parity. Therefore by (1) we can conclude (3). (4) It follows from (1), (2) and (3).

5.3. Primitive pair and Fourier transform. The goal of this section is to study the Fourier transform of the primitive pairs. We make the assumption that \( (G, \chi) \) is \( n \)-rigid.

Proposition 5.10. Assume \( (\mathcal{O}, \mathcal{L}) \in \mathcal{J}(g_\mu) \) be a primitive pair associated to the parabolic \( P \) containing a Levi \( L \) with \( (\mathcal{O}_L, \mathcal{E}_L) \in \mathcal{J}(l_n)^{\text{cusp}} \) and \( \pi : p \to 1 \) be the projection map. Also assume \( \mathcal{O}_n, \mathcal{O}_{-n} \) be the unique open \( G_0 \) orbits in \( g_\mu \) and \( g_{-n} \) respectively and, \( \mathcal{O}_L' \) be the unique open \( L_0 \)-orbit in \( l_{-n} \). If,
\[ \mathcal{O}_n \cap \pi^{-1}(\mathcal{O}_L) \neq \emptyset \] and \( \mathcal{O}_{-n} \cap \pi^{-1}(\mathcal{O}_L') \neq \emptyset \),
then \( \mathcal{O} = \mathcal{O}_n \).

The above proposition has been proved in [Lu] Prop. 12.2].

Proposition 5.11. Assume \( (\mathcal{O}, \mathcal{L}) \in \mathcal{J}(g_\mu) \) with \( \mathcal{O} \) being the unique open \( G_0 \)-orbit in \( g_\mu \). Also assume \( \phi_n \mathcal{E}(\mathcal{O}, \mathcal{L})[\dim g_\mu] = \mathcal{E}(\mathcal{O}', \mathcal{L}')[\dim g_{-n}] \) with \( (\mathcal{O}', \mathcal{L}') \in \mathcal{J}(g_{-n}) \) and \( \mathcal{O}' \) being the unique open \( G_0 \)-orbit in \( g_{-n} \). Then \( (\mathcal{O}, \mathcal{L}) \) is a primitive pair.

Proof. By [Ch. Th. 6.11], \( \mathcal{E}(\mathcal{O}, \mathcal{L}) \) is normal. Therefore there exists a parabolic \( P \) with Levi \( L \) and \( (\mathcal{O}_L, \mathcal{E}_L) \in \mathcal{J}(l_n)^{\text{cusp}} \) such that \( \mathcal{E}(\mathcal{O}, \mathcal{L}) \) appears as a direct summand of \( \text{Ind}_0^{\mu} \mathcal{E}(\mathcal{O}_L, \mathcal{E}_L) \). Let \( (\mathcal{O}', \mathcal{L}') \in \mathcal{J}(g_{-n}) \) be the associated primitive pair to \( P, L \) and \( (\mathcal{O}_L, \mathcal{E}_L) \in \mathcal{J}(l_n)^{\text{cusp}} \). By the Corollary 4.12 induction commutes with Fourier transform. Hence \( \Phi_n \mathcal{E}(\mathcal{O}, \mathcal{L}) \) appears as direct summand of \( \Phi_{l_n} \mathcal{E}(\mathcal{O}_L, \mathcal{E}_L) \). We know from the definition of induction for cuspidal pair [Ch. Lemma 5.2] and our assumption that there exists \( x \in \mathcal{O} \) and \( g \in G_0 \) such that \( \text{Ad}(g^{-1})x \in \mathcal{O}_L + u_\mu \). Also the assumption that \( \Phi_n \mathcal{E}(\mathcal{O}, \mathcal{L}) = \mathcal{E}(\mathcal{O}', \mathcal{L}') \) implies \( \mathcal{E}(\mathcal{O}', \mathcal{L}') \) appears as a direct summand of \( \Phi_{l_n} \mathcal{E}(\mathcal{O}_L, \mathcal{E}_L) \). But Fourier transform sends cuspidal pairs to cuspidal pairs by Theorem 4.3. Hence there exists \( (\mathcal{O}_L', \mathcal{E}_L') \in \mathcal{J}(l_{-n})^{\text{cusp}} \) such that \( \Phi_{l_n} \mathcal{E}(\mathcal{O}_L, \mathcal{E}_L) = \mathcal{E}(\mathcal{O}_L', \mathcal{E}_L') \). Which implies \( \mathcal{O}_L' \) is the unique open \( L_0 \)-orbit in \( l_{-n} \). Hence there exists \( y \in \mathcal{O}' \) and \( g' \in G_0 \) such that \( \text{Ad}(g'^{-1})y \in \mathcal{O}_L' + u_{-n} \). If we replace \( x \) by \( \text{Ad}(g^{-1})x \) and \( g \) by 1, then we see the new \( x \) belongs to both \( \mathcal{O} \) and \( \mathcal{O}_L + u_\mu \). Therefore,
\[ \mathcal{O} \cap \pi^{-1}(\mathcal{O}_L) \neq \emptyset. \]
Similarly we can define a new $y$ which belongs to both $O'$ and $O'_L + u_n$ and hence,

$$O' \cap \pi^{-1}(O'_L) \neq \emptyset.$$  

Therefore by Proposition 5.10, $O = \tilde{O}$. Now we can apply Theorem 5.9 a), which says that $\text{Ind}^p E(O_L, E_L)$ is direct sum of copies of $E(O, \tilde{L})$ up to shift. Combining our assumption we get, $E(O, \tilde{L}) = E(O, L)$, which implies $L = \tilde{L}$.  

\section{Quasi-monomial}

In this section we introduce good pairs, which can be thought as the more generalization of the normal complex, that has been introduced in [Lu] and [Ch].

**Definition 6.1.** A pair $(O, L) \in \mathcal{I}(g_n)$ is called **quasi-monomial** if there exists a parabolic subgroup $P$ with Levi $L$ and the following condition holds,

1. $(L, \chi)$ is $n$-rigid.
2. There exists a primitive pair $(O_L, E_L) \in \mathcal{I}(l_n)$ such that $E(O, L)$ is isomorphic to $\text{Ind}^p E(O_L, E_L)$ up to shift. We will call $E(O, L)$ to be a quasi-monomial object.

**Remark 6.2.** A quasi-monomial object will be called **proper quasi-monomial** if $P$ is a proper parabolic subgroup of $G$.

**Definition 6.3.** A pair $(O, L) \in \mathcal{I}(g_n)$ is called **good pair** if $E(O, L)$ is direct sum of quasi-monomial objects. It will be called **proper** if all the quasi-monomial objects appearing in the direct sum are proper. We will call $E(O, L)$ to be a good object.

**Theorem 6.4.** Assume $(O, L) \in \mathcal{I}(g_n)$. Then there exists good pairs $(O', L')$ and $(O'', L'')$ in $\mathcal{I}(g_n)$ such that, $E(O, L) \oplus E(O', L') \cong E(O'', L'')$.

**Proof.** In this proof we will use induction on the reductive quotient of proper parabolic subgroups containing $\chi(C^\times)$. We will prove the above theorem in several steps.

1. First we want to prove that if $(O, L) \in \mathcal{I}(g_n)$ with either $(G, \chi)$ is not $n$-rigid or $O$ is not the unique open $G_0$-orbit in $g_n$, then the statement of the theorem is true. We will proceed with induction on the dimension of $O$. Let $x \in O$. We can construct $P, L, U, p, n, l$ as in [Ch 6.2]. Then $P \neq G$. By [Ch 6.3], $x$ is contained in the open $L_0$-orbit $O_L \subset O$ in $l_n$ and $L|_{O_L}$ is an irreducible local system, we called it $L'$. Now by [Ch Th 6.5] $\text{Ind}^p E(O_L, L')$ has support $\tilde{O}$ with,

$$\text{Ind}^p E(O_L, L')|_{\tilde{O}} = L[\dim O_L].$$

Therefore combining the fact that $\text{Ind}^p E(O_L, L')$ is a parity complex [Ch Th 6.12], we get,

$$\text{Ind}^p E(O_L, L') = E(O, L) \oplus E',$$

where $E'$ is a parity complex supported on $O - \tilde{O}$. Now by the induction, there exists good pairs $(O'_L, E'_L)$ and $(O''_L, E''_L)$ in $\mathcal{I}(l_n)$ such that,

$$E(O_L, L') \oplus E(O'_L, E'_L) = E(O''_L, E''_L).$$

Hence we get,

$$\text{Ind}^p E(O_L, L') \oplus \text{Ind}^p E(O'_L, E'_L) = \text{Ind}^p E(O''_L, E''_L).$$

Combining (6.8) and (6.9) we get,
Now we are ready to prove the theorem. By (1) if \((G, \chi)\) is not \(-n\)-rigid or \(O\) is not the unique open \(G_0\)-orbit then the statement of the theorem is true. Hence we assume \((G, \chi)\) is \(-n\)-rigid and the support of \(\Phi_{g_n} E(O, L)\) is not \(-n\)-rigid or \(O\) is not the unique open \(G_0\)-orbit in \(\mathfrak{g}_{-n}\) then the statement of the theorem is true. The proof will be similar to (1).

Now our claim is that if \((O, L) \in \mathcal{J}(\mathfrak{g}_{-n})\) is a proper pair then there exists good pairs \((O', L')\) and \((O'', L'')\) in \(\mathcal{J}(\mathfrak{g}_n)\) such that,

\[
\Phi_{g_n} E(O, L) \oplus E(O', L') = E(O'', L'').
\]

As \(E(O, L)\) is proper, so we can assume \(E(O, L) = \mathfrak{g}_{e-n} \mathcal{I}_n E(O_M, E_M)\) where \(Q\) is a proper parabolic subgroup of \(G\) containing \(\chi(C^\infty)\) and \((O_M, E_M) \in \mathcal{J}(\mathfrak{m}_{-n})\) is a primitive pair with \(M\) being the Levi inside \(Q\). Now as induction commutes with the Fourier transform we have,

\[
\Phi_{g_{-n}} E(O, L) = \mathfrak{g}_{-n} \mathcal{I}_n \Phi_{m_{-n}} E(O_M, E_M).
\]

Now by induction there exists good objects \(E_M'\) and \(E''_M\) in \(\mathcal{J}(\mathfrak{m}_n)\) such that,

\[
\Phi_{m_{-n}} E(O_M, E_M) \oplus E_M' = E''_M.
\]

Therefore

\[
\Phi_{g_{-n}} E(O, L) \oplus \mathfrak{g}_{-n} \mathcal{I}_n E_M' = \mathfrak{g}_{-n} \mathcal{I}_n E''_M.
\]

Now the above statement follows by the transitivity of induction.

Now we claim if \((O, L) \in \mathcal{J}(\mathfrak{g}_n)\) with \(\mathcal{O} = \mathfrak{g}_n\) and the support of \(\Phi_{g_n} E(O, L)\) is distinct from \(\mathfrak{g}_{-n}\), then the statement of the theorem is true. By (2), there exist proper pairs \((O', L')\) and \((O'', L'')\) in \(\mathcal{J}(\mathfrak{g}_n)\) such that,

\[
\Phi_{g_n} E(O, L) \oplus E(O', L') = E(O'', L'').
\]

Therefore we have,

\[
E(O, L) \oplus \Phi_{g_n} E(O', L') = \Phi_{g_n} E(O'', L'').
\]

We can apply (3) to the proper pairs \((O', L')\) and \((O'', L'')\) which proves our claim.

Now we are ready to prove the theorem. By (1) if \((G, \chi)\) is not \(-n\)-rigid or \(O\) is not the unique open \(G_0\)-orbit then the statement of the theorem is true. Hence we assume \((G, \chi)\) is \(-n\)-rigid. If the condition of (4) holds then also the statement is true. Therefore we assume \(\mathcal{O} = \mathfrak{g}_n\) and the support of \(\Phi_{g_n} E(O, L)\) is \(\mathfrak{g}_{-n}\). Then by Proposition (5.11) \((O, L)\) is primitive, in particular good.

\[\square\]
Remark 6.5. We can prove the same statement as in Theorem 6.4 for \( g_{-n} \) and proof will follow the same steps.

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