Spline-backfitted kernel forecasting for functional-coefficient autoregressive models

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Abstract

We propose three methods for forecasting a time series modeled using a functional coefficient autoregressive model (FCAR) fit via spline-backfitted local linear (SBLL) smoothing. The three methods are a “naive” plug-in method, a bootstrap method, and a multistage method. We present asymptotic results of the SBLL estimation method for FCAR models and show the estimators are oracally efficient. The three forecasting methods are compared through simulation. We find that the naive method performs just as well as the multistage method and even outperforms it in some situations. We apply the naive and multistage methods to solar irradiance data and compare forecasts based on our method to those of a linear AR model, the model most commonly applied in the solar energy literature.

Keywords: Functional-coefficient autoregressive model, Spline smoothing, Local linear estimation, Oracle smoothing, Bootstrap forecast, Multistage forecast
1. Introduction

There are many useful time series models that lie between the class of linear, fully parametric models, and nonlinear nonparametric models. One such model is the functional coefficient autoregressive (FCAR) model, defined as

\[ X_t = \sum_{\alpha=1}^{p} m_{\alpha} (U_{t-d}) X_{t-\alpha} + \sigma (V_t, X_t) \varepsilon_t, \quad t = 1, \ldots, n, \quad (1) \]

where \( p \) is a positive integer, \( m_{\alpha} (U_t) \) is a measurable function of \( U_t \), for \( \alpha = 1, \ldots, p \), \( \sigma^2 (V_t, X_t) \) is a variance function dependent on \( V_t = (X_1, \ldots, X_{n-d})' \) and \( X_t = (X_1, \ldots, X_n)' \), and \( \{\varepsilon_t\} \) is a sequence of i.i.d. random variables with mean 0 and variance \( \sigma^2 < \infty \). Usually the variable \( U_t \) is taken to be a lagged value of the series; i.e., \( U_t = X_{t-d} \), where \( d \) is a positive integer. Although the FCAR model imposes an autoregressive structure, its flexibility lies in allowing the autoregressive coefficients \( m_{\alpha} \), to vary as a function of \( U_t \). While reducing the size of the class of nonlinear models, the class of FCAR models is broad enough to include some common nonlinear time series models as specific cases. Among these are the threshold autoregressive (TAR) model of Tong (1983), the exponential autoregressive (EXPAR) model of Haggan and Ozaki (1981), and the smooth transition autoregressive (STAR) model of Chan and Tong (1986).

Chen and Tsay (1993) introduced the FCAR model and proposed a procedure for building the model based on arranged local autoregression which constructs estimators based on an iterative recursive formula that resembles local constant smoothing. They compared the proposed model building procedure to threshold and linear time series models through multi-step forecasts. The FCAR model performed much better than the other two models in terms of bias. However, the FCAR model only performed better for short term forecasts in terms of mean square error (MSE). For long term forecasts, the linear model performed the best in MSE.

Cai, Fan and Yao (2000) used a local linear fitting method to estimate \( m_{\alpha} (\cdot) \) in (1). They used the method on simulated data from an EXPAR model and assessed the fit by calculating the square root of the average squared errors (RASE). The performance of the method was gauged by comparing the RASE to the standard deviation of the time series. Their results showed the local linear method provided an adequate fit of the model with the RASE well below the standard deviation of the series. Real data examples were used to assess the post sample forecasting performance of the local
linear method. The two examples were the Canadian lynx data set (Tong, 1990, p. 377) and the Wolf’s annual sunspot numbers data set (Tong, 1990, p. 420). The local linear method was compared with the linear AR model, the TAR model, and the arranged local regression procedure of Chen and Tsay (1993) using a one-step ahead and a iterative two-step ahead forecast. In terms of average absolute predictive errors, the local linear method had much better performance than both the linear AR model and the TAR model in the Canadian Lynx example and performed just as well as the other two models in the sunspot numbers example.

Huang and Shen (2004) propose a global smoothing procedure based on polynomial splines for estimating FCAR models. The authors note that the spline method yields a fitted model with a parsimonious explicit expression which is an advantage over the local polynomial method. This feature allows one to produce multi-step ahead forecasts conveniently. Additionally, their spline method is less computationally intensive than the local polynomial method.

Forecasting for FCAR models was discussed in Harvill and Ray (2005). They compared three methods, the first of which is the naive forecasts presented in Fan and Yao (2003). Another method was the multistage method of Chen (1996). This method was developed for a general non-linear AR model and was adapted for the FCAR model by Harvill and Ray (2005). The last method was the bootstrapping method of Huang and Shen (2004). In that paper, bootstrapped residuals are added to the forecasted values after fitting the model with splines. Harvill and Ray (2005) used bootstrapped residuals after fitting with the local linear method. A comparison of the three methods showed that the bootstrap method outperforms the other two methods for non-linear forecasting and performs well for forecasting a linear process.

A recent development in estimating nonlinear time series data is the spline-backfitted kernel (SBK) method of Wang and Yang (2007). This method combines the computational speed of splines with the asymptotic properties of kernel smoothing. To estimate a component function in the model, all other component functions are “pre-estimated” with splines and then the difference is taken of the observed time series and the pre-estimates. This difference is then used as pseudo-responses for which kernel smoothing is used to estimate the function of interest. By constructing the estimates in this way, the method does not suffer from the “curse of dimensionality.” The SBK method is adapted for i.i.d. data in Wang and Yang (2009), to generalized additive models in Liu, Yang and Härdle (2011), and to partially linear
additive models in Ma and Yang (2011). In Song and Yang (2010), a spline-backfitted spline (SBS) procedure is proposed. Liu and Yang (2010) develops the SBK method for additive coefficient models which are generalized forms of FCAR models.

Using SBK in forecasting algorithms has not been introduced, and in particular, not in forecasting with the FCAR model. In this paper, we develop new forecasting algorithms that make use of the SBK method, and compare the performance of the new method to that of the naive, the bootstrap, and the multistage methods for the FCAR model.

In Section 2, the SBK method for estimating the functional coefficients is given. Section 3 presents three forecasting methods that employ the SBK estimates. In Section 4, simulation results are used to compare the forecasting methods and the methods are applied to solar irradiance data in Section 5. We conclude with a discussion in Section 6.

2. Estimation of Functional Coefficients

To estimate the functional coefficients \( m_{\alpha}(U_t), \alpha = 1, 2, \ldots, p \) in (1), we use the oracle smoothing of Linton (1997) and Wang and Yang (2007). For the remainder of this paper, we take the variable \( U_t \) in equation (1) to be a lagged value of the series; that is, \( U_t = X_{t-d} \), where \( d \) is a positive integer. Consider a fixed integer \( \gamma, 1 \leq \gamma \leq p \). If the coefficient functions \( m_{\alpha}(X_{t-d}) \), \( \alpha = 1, \ldots, p \), for \( \alpha \neq \gamma \), are known by “oracle,” then we can construct \( \{X_{t-d}, X_{t-\gamma}, Y_{\gamma,t}\}_{t=1}^{n} \), where

\[
Y_{\gamma,t} = m_{\gamma}(X_{t-d}) X_{t-\gamma} + \sigma(X_t) \varepsilon_t = X_t - \sum_{\alpha=1,\alpha \neq \gamma}^{p} m_{\alpha}(X_{t-d}) X_{t-\alpha},
\]

from which we can estimate the unknown \( m_{\gamma}(X_{t-d}) \). This oracle smoother removes the “curse of dimensionality,” since there is only one function being estimated. Clearly, the coefficient functions, \( m_{\alpha}(X_{t-d}) \), \( \alpha = 1, \ldots, p \), \( \alpha \neq \gamma \), are not known and must be estimated. For additive models, Linton (1997) used marginal integration kernel estimates to estimate the functions and Wang and Yang (2007) used an under-smoothed spline procedure. We now adapt the procedure of Wang and Yang (2007) to estimate the FCAR model.

We first “pre-estimate” the coefficient functions with a constant spline procedure. Let \( U_t = X_{t-d} \) be distributed on the compact interval \([a, b]\) and denote the knots as \( a = \kappa_0 < \kappa_1 < \cdots < \kappa_N < \kappa_{N+1} = b \) where the number of
interior knots are \( N \sim n^{1/4} \ln n \). The B-spline basis functions are determined on the \( N + 1 \) equally spaced intervals with length \( (b - a) (N + 1)^{-1} \). The basis function are defined as

\[
B_J(u) = \begin{cases} 
1, & \kappa_J \leq x < \kappa_{J+1}, \\
0, & \text{otherwise},
\end{cases} \quad J = 0, \ldots, N.
\]

The pre-estimates are defined as

\[
\hat{m}_\alpha(u) = \sum_{J=0}^{N} \hat{\lambda}_{(N+1)(\alpha-1)+J} B_J(u), \quad \alpha = 1, \ldots, p,
\]

where the coefficients \( \hat{\lambda}_{0}, \ldots, \hat{\lambda}_{p(N+1)-1} \) are solutions to the least squares problem

\[
\left\{ \hat{\lambda}_{0}, \ldots, \hat{\lambda}_{p(N+1)-1} \right\} = \arg \min_{\mathbb{R}^{p(N+1)-1}} \sum_{t=1}^{n} \left\{ X_t - \sum_{\alpha=1}^{p} \left( \sum_{J=0}^{N} \lambda_{(N+1)(\alpha-1)+J} B_J(U_t) \right) X_{t-\alpha} \right\}^2.
\] (2)

Define \( X_t = (X_1, \ldots, X_n)' \), \( X_\alpha = (X_{1-\alpha}, \ldots, X_{n-\alpha})' \), \( U_t = (U_1, \ldots, U_n)' \), \( \hat{\lambda} = (\hat{\lambda}_{0}, \ldots, \hat{\lambda}_{p(N+1)-1})' \),

\[
B = \begin{bmatrix}
B_0(U_1) & B_1(U_1) & \cdots & B_N(U_1) \\
B_0(U_2) & B_1(U_2) & \cdots & B_N(U_2) \\
\vdots & \vdots & \ddots & \vdots \\
B_0(U_n) & B_1(U_n) & \cdots & B_N(U_n)
\end{bmatrix},
\]

and \( Z = (B \odot \bar{X}_1, B \odot \bar{X}_2, \ldots, B \odot \bar{X}_p) \) where \( \odot \) denotes the Hadamard product and \( \bar{X}_\alpha \) is a \( n \times (N + 1) \) matrix with \( X_\alpha \) for each column. In matrix notation, the least squares estimates are

\[
\hat{\lambda} = (Z'Z)^{-1} Z'X_t,
\]

and the pre-estimates are

\[
\hat{m}_\alpha(U_t) = B \left[ \hat{\lambda}_{(N+1)(\alpha-1)}, \ldots, \hat{\lambda}_{\alpha(N+1)-1} \right]', \quad \alpha = 1, \ldots, p.
\]
We now define the “pseudo-responses” as

\[ \hat{Y}_{\gamma,t} = X_t - \sum_{\alpha=1,\alpha \neq \gamma}^p \hat{m}_{\alpha} (U_t) X_{t-\alpha}, \quad t = 1, \ldots, n. \]

Define the vector of pseudo-responses as \( \hat{Y}_\gamma = (\hat{Y}_{\gamma,1}, \ldots, \hat{Y}_{\gamma,n})' \). The spline-backfitted local linear (SBLL) estimate for the coefficient function \( m_\gamma (u) \) is

\[ \hat{m}_{\text{SBLL},\gamma} (u) = (1, 0) (V'WV)^{-1} V'W\hat{Y}_\gamma, \] (3)

where

\[ V = \begin{bmatrix} X_{p+1-\gamma} & X_{p+1-\gamma} (U_{p+1} - u) \\ \vdots & \vdots \\ X_{n-\gamma} & X_{n-\gamma} (U_n - u) \end{bmatrix}, \]

\[ W = \text{diag} \{ K_h (U_{p+1} - u), \ldots, K_h (U_n - u) \}, \]

\[ K_h (u) = h^{-1} \frac{15}{16} \left\{ 1 - \left( \frac{u}{h} \right)^2 \right\}^2 I_{\{|u/h| \leq 1\}}, \]

\( I_A \) is an indicator variable equal to one if condition \( A \) is true, and is zero otherwise; \( h \) is a bandwidth selected by the rule of thumb criterion of [Fan and Gijbels (1996)]. Likewise, we define the oracle local linear smoother as

\[ \hat{m}_{O,\gamma} (u) = (V'WV)^{-1} V'WY_\gamma, \]

[Wang and Yang (2007)] first proposed the method using a Nadaraya-Watson smoother in the last step and then proposed using a local linear smoother. We use only the local linear smoother in this paper.

2.1. Asymptotic properties

The following assumptions are necessary for the data generating process to be geometric ergodic and for the asymptotic properties of the SBLL method. For the interval \([a, b]\) and functions \( m \), let \( C^{(2)} [a, b] = \{ m|m'' \in C [a, b] \} \) denote the space of second-order continuous smooth functions and let

\[ \text{Lip} ([a, b], C) = \{ m \mid |m (u) - m (v)| \leq C |u - v|, \forall u, v \in [a, b] \} \]

denote the class of Lipschitz continuous functions for any fixed constant \( C > 0 \).
(A1) The coefficient function \( m_\gamma (u) \in C^{(2)} [a, b] \) and \( m_\alpha (u) \in \text{Lip} ([a, b], C_\infty) \), \( \alpha = 1, \ldots, p, \alpha \neq \gamma \), for some constant \( 0 < C_\infty < \infty \).

(A2) For the process \( \{ \theta_t = (U_t, X_t, X_{t-1}, \ldots, X_{t-p}, \varepsilon_t) \}_{t=1}^n \), there exist positive constants \( K_0 \) and \( \lambda_0 \) such that \( \alpha (k) \leq K_0 e^{-\lambda_0 k} \) holds for all \( k \), with the \( \alpha \)-mixing coefficients for \( \{ \theta_t \}_{t=1}^n \) defined as

\[
\alpha (k) = \sup_{B \in \sigma \{ \theta_s, s \leq t \}, C \in \sigma \{ \theta_s, s \geq t+k \}} \left| P \left( B \cap C \right) - P \left( B \right) P \left( C \right) \right|, \quad k \geq 1.
\]

(A3) The conditional variance function \( \sigma^2 (X_t) \) is measurable and bounded. The noise \( \varepsilon_t \) satisfies \( E (\varepsilon_t | X_t) = 0, E (\varepsilon_t^2 | X_t) = 1 \) and \( E \left( |\varepsilon_t|^{2+\delta} | X_t \right) < M_\delta \) for some \( \delta > 1/2 \) and a finite positive \( M_\delta \).

(A4) The delay variable \( U_t \) has a continuous probability density function \( f (u) \) that satisfies

\[
0 < c_f \leq \inf_{u \in [a, b]} f (u) \leq \sup_{u \in [a, b]} f (u) \leq C_f < \infty,
\]

for some constants \( c_f \) and \( C_f \), and has continuous derivatives on \([a, b]\).

(A5) The kernel function \( K \in \text{Lip} \left([\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}], C_k \right) \) for some constant \( C_k > 0 \), and is bounded, nonnegative, symmetric and supported on \([-1, 1]\). The bandwidth is \( c_h n^{-1/5} \leq h \leq C_h n^{-1/5} \) for some positive constants \( c_h \) and \( C_h \).

(A6) The number of interior knots is \( c_N n^{1/4} \log n \leq N \leq C_N n^{1/4} \log n \) for some positive constants \( c_N \) and \( C_N \).

Assumptions (A1)-(A5) are common in the nonparametric literature; see, for example Fan and Gijbels (1996). We apply the SBLL method to a TAR model in section 4, thus relaxing the smoothness assumption in assumption (A1). Theorem 1.1 in Chen and Tsay (1993) gives sufficient conditions for FCAR models to be geometrically ergodic, which implies \( \alpha \)-mixing, thus satisfying Assumption (A2). Assumption (A6) is used in the pre-estimate stage in which we under smooth with the splines to reduce the bias. The increase in variance resulting from the pre-estimate stage is reduced in the local linear regression stage where bandwidth \( h \) is of the order found in assumption (A5). When implementing the method, we impose an additional constraint on \( N \) such that

\[
N = \min \left\{ n^{1/4} \log n, \left( \frac{n}{2p} \right) - 1 \right\}.
\]
This additional constraint ensures that the number of terms in the least squares problem (2) is no greater than \(n/2\).

To simplify the notation, we denote

\[
\mu_j = \int_{-\infty}^{\infty} x^j K(x) \, dx, \quad \nu_j = \int_{-\infty}^{\infty} x^j K^2(x) \, dx, \quad \Omega_\gamma(u) = E \left( X_\gamma X'_\gamma \mid U_t = u \right),
\]

and

\[
\Omega^*_\gamma(u) = E \left( (X_\gamma X'_\gamma) \sigma^2(U_t, X_t) \mid U_t = u \right).
\]

Under assumptions (A1)-(A5), it is straightforward from the results of Cai et al. (2000) to verify that, as \(n \to \infty\),

\[
\sqrt{nh} \left\{ \tilde{m}_{\Omega,\gamma}(u) - m_\gamma(u) - b_\gamma(u) h^2 \right\} \to N \left( 0, \nu_\gamma(u) \right).
\]

where

\[
b_\gamma(u) = \frac{h^2}{2} \mu_2 m''_\gamma(u), \quad \nu_\gamma(u) = \frac{\nu_0}{f'(u)} \mathbf{e}'_{\gamma,p} \Omega^{-1}(u) \Omega^*(u) \Omega^{-1}(u) \mathbf{e}_{\gamma,p},
\]

and \(\mathbf{e}_{j,p}\) is a \(p \times 1\) vector with 1 in the \(j\)th position. Furthermore, from the results of Liu and Yang (2010) it can be shown that as \(n \to \infty\), the oracle smoother satisfies

\[
\sup_{u \in [a+h,b-h]} | \tilde{m}_{\Omega,\gamma}(u) - m_\gamma(u) | = O_p \left( \frac{\log n}{\sqrt{nh}} \right).
\]

The following theorem gives the asymptotic uniform magnitude of the difference between \(\tilde{m}_{SBLL,\gamma}(u)\) and \(\tilde{m}_{O,\gamma}(u)\).

**Theorem 2.1.** Under assumptions (A1)-(A6), as \(n \to \infty\), the SBLL estimator \(\tilde{m}_{SBLL,\gamma}(u)\) given in (3) satisfies

\[
\sup_{u \in [a,b]} | \tilde{m}_{SBLL,\gamma}(u) - \tilde{m}_{O,\gamma}(u) | = o_p \left( n^{-2/5} \right).
\]

Theorem 2.1 states that the distance \(\tilde{m}_{SBLL,\gamma}(u) - \tilde{m}_{O,\gamma}(u)\) is of the order \(o_p \left( n^{-2/5} \right)\), which is dominated by the asymptotic size of \(\tilde{m}_{O,\gamma}(u) - m_\gamma(u)\). This implies that \(\tilde{m}_{SBLL,\gamma}(u)\) will have the same asymptotic distribution as \(\tilde{m}_{O,\gamma}(u)\) which results in the following theorem.
Theorem 2.2. Under assumptions (A1)-(A6), as \( n \to \infty \), with \( b_\gamma (u) \) and \( v_\gamma^2 (u) \) defined in (4) and (5),

\[
\sqrt{nh} \left\{ \tilde{m}_{SBLL,\gamma} (u) - m_\gamma (u) - b_\gamma (u) h^2 \right\} \to N \left( 0, v_\gamma^2 (u) \right).
\]

The proofs of Theorems 2.1 and 2.2 follow from the proofs of Liu and Yang (2010).

To compare the performance of \( \tilde{m}_{SBLL,\gamma} (u) \) to \( \tilde{m}_{O,\gamma} (u) \), we use the oracle efficiencies of Wang and Yang (2007) which are defined as

\[
eff_\gamma = \left[ \frac{\sum_{t=1}^{n} \{ \tilde{m}_{O,\gamma} (u_t) - m_\gamma (u_t) \}^2}{\sum_{t=1}^{n} \{ \tilde{m}_{SBLL,\gamma} (u_t) - m_\gamma (u_t) \}^2} \right]^{1/2}.
\]  

(6)

By Theorems 2.1 and 2.2, \( \eff_\gamma \) should approach 1 as \( n \to \infty \) for all \( \gamma = 1, \ldots, p \). We demonstrate this result via simulation in Section 4.

3. Forecasting Methods

We now present the three forecasting methods discussed in Harvill and Ray (2005) and adapt them to be used with the SBLL method. Assuming \( m_\alpha (U) \) is known and \( U \) is exogenous, we want to find an estimator of the conditional expectation

\[
E [X_{n+M}|X_n, \ldots, X_{n-p}] = E \left[ \sum_{\alpha=1}^{p} m_\alpha (U_{n+M}) X_{n+M-\alpha}|X_n, \ldots, X_{n-p} \right] \]

(7)

\[
= \sum_{\alpha=1}^{p} m_\alpha (U_{n+M}) E [X_{n+M-\alpha}|X_n, \ldots, X_{n-p}]
\]

\[
= \sum_{\alpha=1}^{p} m_\alpha (U_{n+M}) \hat{X}_{n+M-\alpha}.
\]

The expectation in (7) is no longer a simple linear operation when \( U_t = X_{t-d} \) for some positive constant \( d \). The three forecasting methods described below deal with this expectation in a different way.
**Naive predictor**

The naive approach simply ignores the fact the expectation in (7) is not a linear function of $X_{t+M-\alpha}$ and substitutes $\hat{X}_{t+M-\alpha}$ into the forecast equation. We estimate the coefficient function only using the within-sample series values. The naive predictor is defined as

$$\hat{X}_{n+M} = \sum_{\alpha=1}^{p} \hat{m}_\alpha (\hat{X}_{n+M-d}) \hat{X}_{n+M-\alpha},$$

where $\hat{X}_t = X_t, t \leq n$. For the SBLL estimator, $\hat{m}_\alpha (\cdot)$ is the value obtained by the general form of (3). Thus, the spline pre-estimate is not computed for each value of $\hat{X}_{n+M-d}$, but the local linear estimation is computed.

**Bootstrap predictor**

The bootstrap predictor is like the naive predictor in that it estimates the functional coefficients using only the within-sample values. However, we bootstrap the within-sample residuals from the estimated model and find the predicted value as

$$\hat{X}_{n+M} = \sum_{\alpha=1}^{p} \hat{m}_\alpha (\hat{X}_{n+M-d}) \hat{X}_{n+M-\alpha} + \epsilon^b,$$

where $\epsilon^b$ is the bootstrapped residual. We obtain bootstrapped forecasts for $b = 1, \ldots, B$, and use the average of these values as the $M$-step ahead forecast. For the SBLL method, we estimate $\hat{m}_\alpha (\cdot)$ as with the naive predictor. An advantage of using the bootstrap values is that the set of all values allows us to estimate the predictive density of $X_{t+M}$. A disadvantage is that the estimated functional coefficients may become unreliable when $\hat{X}_{t+M-d}$ is outside or near the boundary of the range of the original $X_{t-d}$. This disadvantage was first noted by Huang and Shen (2004) and reiterated by Harvill and Ray (2005).

**Multistage predictor**

Another way to handle the expectation in (7) is to incorporate the information from $X_t$ encoded in the predicted response at time $n + j, j = 1, \ldots, M - 1$. This is accomplished by updating the functional coefficients at each step and obtaining the forecast by

$$\hat{X}_{n+M} = \sum_{\alpha=1}^{p} \hat{m}_\alpha^M (\hat{X}_{n+M-d}) \hat{X}_{n+M-\alpha},$$
where $\hat{X}_t = X_t$, $t \leq n$. The functional coefficient $\hat{m}_M^\alpha(\cdot)$ is estimated by the SBLL method at each step. That is, we include the predicted values $\hat{X}_t$, $t = n + 1, \ldots, M - 1$, with the original values $X_t$, $t = 1, \ldots, n$, and then re-estimate the functional coefficient with the SBLL method using the new set of data. Clearly, this method is more computationally intensive than the naive predictor and possibly the bootstrap predictor, dependent on the number of bootstraps taken.

4. Simulation

In the following, we investigate the performance of the three forecasting methods via simulation. We use three parametric models, described in more detail in each of the three examples. An FCAR model is fit to data from the three models using the method described in Section 2, and forecasts obtained using the methods explained in Section 3. The models in the first two examples satisfy all assumptions provided in Section 2.1. However, the model in the third example is a self-exciting threshold autoregressive model, and so the continuity assumption (A1) is not satisfied. For all three models, we provide empirical efficiencies as defined in equation (6). We also provide the root mean square prediction error (RMPE) for the three forecasting methods. For each model, we ran 500 Monte Carlo iterations for series lengths $n = 75, 150, 250$ and 500, and orders $p = 4, 10$.

Example 4.1. We first consider the model

$$X_t = \sum_{\alpha=1}^{p} a_{\alpha} \sin (\omega \pi X_{t-d}) X_{t-\alpha} + \sigma (X_t) \varepsilon_t, \quad \varepsilon_t \sim N(0, 1),$$

where

$$\sigma (X_t) = 0.1 \left( \frac{\sqrt{p}}{2} \right) \frac{5 - \exp \left( \sum_{\alpha=1}^{p} |X_{t-\alpha}| / p \right)}{5 - \exp \left( \sum_{\alpha=1}^{p} |X_{t-\alpha}| / p \right)}. \quad (8)$$

In both cases, we set the delay to $d = 2$. The term $\sigma (X_t)$ ensures the model is heteroscedastic with the variance roughly proportional to $p$. For $p = 4$, we used the parameters

$$a = (0.5, -0.5, 0.5, -0.5)^t$$

and $\omega = 4.5$. For $p = 10$, we used

$$a = (0.5, -0.5, 0.5, -0.5, 0.5, -0.5, 0.5, -0.5, 0.5, -0.5)^t$$
and $\omega = 1.5$. These choices for $a$ make the bounds of the coefficient functions $\pm |a_\alpha|$ so that the roots of the characteristic polynomial

$$\lambda^p - a_1\lambda^{p-1} - \cdots - a_p = 0$$

are inside the unit circle, thus ensuring the process is geometrically ergodic (see Chen and Tsay, 1993).

Figures 1(a)–1(d) show the estimated coefficient function using the SBLL method and the oracle estimate for $m_\alpha (X_{t-2})$ with $p = 4, 10$ and $n = 75, 500$. The densities of the empirical efficiencies are shown in Figures 1(e) and 1(f). We can see that the mode of the densities tend to be closer to one as $n$ increases confirming the convergence results of Theorem 2.1.

For comparing the three forecasting methods, we calculate the root mean prediction error (RMPE),

$$\text{RMPE}_M = \left\{ \frac{1}{500} \sum_{i=1}^{500} \left( \hat{X}_{n+M,i} - X_{n+M,i} \right)^2 \right\}^{1/2} \quad M = 1, \ldots, 10, \quad (9)$$

where $\hat{X}_{n+M}$ is the forecast at time $n + M$ for iteration $i$, and $X_{n+M}$ is the value at time $n + M$ of the $i$th simulated process. The RMPE is calculated for each of the three forecasting methods. Figure 2 shows the RMPE$_M$, $M = 1, \ldots, 10$, for Example 4.1 for three series lengths. For $p = 4, n = 75$ (Figure 2(a)), we can see that the multistage method has lower RMPE for $M = 1, \ldots, 7$, but has higher RMPE than the naive method for $M = 8, 9, 10$. For $p = 10, n = 75$ (Figure 2(b)), the naive and multistage methods are much closer at the lower values of $M$ with the naive method performing better for the larger values of $M$. For $p = 4$, the larger series lengths show the naive method having RMPE just as low, if not lower, than the other two methods (Figures 2(c) and 2(e)). For $p = 10$, the larger series lengths show the multistage method having the lowest RMPE for some smaller values of $M$ with the naive method being lower for the larger values of $M$ (Figures 2(d) and 2(f)). The simulation results show that the naive method performs just as well, if not better, than both the bootstrap and multistage methods. An additional advantage in using the naive method over the other two is that it is computationally faster.
Example 4.2. The next model we consider is the EXPAR model

$$X_t = \sum_{\alpha=1}^{p} \left( a_{\alpha} + b_{\alpha} \exp \{-\delta X_{t-\alpha}\} \right) X_{t-\alpha} + \sigma \{X_t\} \varepsilon_t, \quad \varepsilon_t \sim N(0, 1),$$

where $\sigma \{X_t\}$ is defined in (8). We use

$$a = (0.3, -0.35, 0.1, -0.2)',$$

$$b = (0.2, -0.15, 0.4, -0.3)'$$

and $\delta = 25$ for $p = 4$, and

$$a = (0.3, -0.35, 0.1, -0.2, 0.35, -0.1, 2, -0.3, 0.25, -0.25)'$$

$$b = (0.2, -0.15, 0.4, -0.3, 0.15, -0.4, 0.3, -0.2, 0.25, -0.25)'$$

and $\delta = 5$ for $p = 10$. For both cases, we set the delay variable to $X_{t-2}$. As was the case in Example 4.1, the values of $a$ and $b$ are determined so that the bounds of the coefficient functions ensure the process is geometrically ergodic.

The estimated coefficient function using the SBLL method and the oracle estimate for $m_\alpha(X_{t-2})$ with $p = 4, 10$ and $n = 75, 500$ are shown in Figures 3(a) – 3(d). The densities of the empirical efficiencies are shown in Figures 3(e) and 3(f). For this example, the efficiencies are shown for $\alpha = 4$ and $\alpha = 10$ for $p = 4$ and $p = 10$, respectively. As we see in Example 4.1, the mode of the densities tend to be closer to one as $n$ increases.

The RMPEs for the three forecasting methods are shown in Figure 4. For this example, we see the multistage method having the lowest RMPE for $p = 4$ for most values of $M$ (Figures 4(a), 4(c) and 4(e)). Note the difference between the multistage and the naive methods are small, particularly for $n = 500$ (Figure 4(e)). For $p = 10$, we see the naive method have the smallest RMPE for most values of $M$ except for the series length $n = 500$ (Figure 4(f)). For $n = 500$, the multistage method tends to have lower RMPE. Again, the differences between the methods are small.
Example 4.3. For the last example, we relax the continuity Assumption (A1) and apply the estimation and forecasting methods to the self-exciting threshold autoregressive model (SETAR)

\[ X_t = \sum_{\alpha=1}^{p} \phi_{\alpha} (X_{t-d}) X_{t-\alpha} + \varepsilon_t, \quad \varepsilon_t \sim N(0, 1), \]

where

\[ \phi_{\alpha} (X_{t-d}) = \begin{cases} 
  a_{\alpha} & \text{if } X_{t-d} < r_{\alpha} \\
  b_{\alpha} & \text{if } X_{t-d} \geq r_{\alpha} 
\end{cases} \]

For \( p = 4 \), we used

\[ a = (0.5, 0.2, 0.1, -0.4)', b = (0.4, -0.5, 0.5, -0.5)', \]

and

\[ r = (0, -0.1, -0.2, 0)' \]

For \( p = 10 \), we used

\[ a = (0.5, 0.2, 0.1, -0.4, 0.4, -0.1, -0.2, -0.5, -0.25, 0.25)', b = (0.4, -0.5, 0.5, -0.5, 0.5, -0.5, 0.5, -0.4, 0.5, -0.5)', \]

and

\[ r = (0, -0.1, -0.2, 0.1, 0.2, 0.3, -0.3, 0, 0.1, 0.2)' \]

For both cases, we used the delay variable \( X_{t-1} \). We chose the values for \( a, b, \) and \( r \) to ensure the process is geometrically ergodic as was we did in Examples 4.1 and 4.2.

The estimated coefficient function using the SBLL method and the oracle estimate for \( m_{\alpha} (X_{t-1}) \) with \( p = 4, 10 \) and \( n = 75, 500 \) are shown in Figures 5(a) - 5(d). The densities of the empirical efficiencies are shown in Figures 5(e) and 5(f). For this example, the efficiencies are shown for \( \alpha = 3 \) for both \( p = 4 \) and \( p = 10 \). The estimated coefficient functions looks to be biased at the value of the regime \( r_2 = -0.1 \). However, the mode of the efficiencies still tend to be closer to 1 as the series lengths increase indicating that relaxing Assumption (A1) does not affect the asymptotic behavior of the SBLL method, or, at least, it affects the behavior in the same manner as it affects the behavior of the oracle estimator.
The RMPEs for the three forecasting methods are shown in Figure 6. From Figures 6(a) and 6(b), we see that the multistage method tends to have the highest RMPE for the larger values of $M$ when the series lengths are $n = 75$. This inflated RMPE indicates that the SBLL method is not updating the coefficient functions well in the multistage method. As the series lengths increase (Figures 6(c) - 6(f)), the multistage method becomes less inflated in RMPE. For series length $n = 500$, the multistage method has the lowest RMPE for most values of $M$, although, the naive method is close just as it was for the previous two examples.

5. Application to Solar Irradiance

We now apply the SBLL method to solar irradiance data taken from a sensor located in Ashland, OR, as part of the University of Oregon Solar Radiation Monitoring Laboratory. Many variables affect solar irradiance. However, one variable that is more influential than others is the amount of cloud cover. At a typical site, a day (or a period of time during a day) is classified as "clear sky,” "partly cloudy”, or "overcast.” In the solar energy literature, methodology used for fitting this type of time series includes ARIMA models (Martín et al., 2010), regression analysis (Reikard, 2009), $k$-nearest neighbors (Paoli et al., 2010), and Bayesian models (Paoli et al., 2010). More recently, in Patrick et al. (2015), an FCAR model is used to fit solar irradiance, and is shown to be superior to existing methods in the solar energy literature.

The data in this paper contains measured irradiance in W/m$^2$ at five minute intervals throughout the day. Figure 7(a) contains a plot of the data (solid line), with a clear sky model (dotted line) superimposed. In the measured irradiance, a diurnal trend is clearly present. In accordance with the methods described in Section 2 and 3, we fit an FCAR model to data taken on November 11, 2013 (a mostly cloudy day) using the SBLL method and forecast ten time points beginning at 1:35 PM (all times are PST).

We begin by removing the diurnal trend in the measured irradiance. Figure 7(a) contains a plot of the measured irradiance which are affected by the cloud cover and a theoretical clear sky model; that is, the expected measured...
irradiance if there were no clouds present in the sky. Even though the data we are using is from a mostly cloudy day, we can still see a diurnal trend that must be removed. A clear sky model is used to remove this trend. A discussion of clear sky irradiance models can be found in Reno et al. (2012). For our application, we used the Ineichen clear sky model (Ineichen and Perez, 2002) and calculated the clear sky index with is defined as the ratio of the measured irradiance to the clear sky model. Figure 7(b) shows the clear sky index for the time interval starting at 8:00 AM ($t = 96$) and ending at 4:00 PM ($t = 192$).

Figure 7 about here.

The fit of the FCAR model can be seen in Figure 7(c). We found that $p = 2$ and $d = 5$ gave the best fit. We also fit a linear AR model of order 4 which we determined by minimizing the AIC. The MSE’s of the fitted models were 0.0009 for the FCAR model and 0.0020 for the linear AR model.

We forecast for $M = 1, \ldots, 10$ using the naive and multistage methods and compared them to forecasts using the linear AR model. Figure 7(d) shows the forecasts and the observed values starting at 1:35 PM ($t = 163$). The root squared prediction errors (RSPE) for the three methods are given in Table 1. For $M = 1, \ldots, 5$, the FCAR forecasting methods had lower RSPEs with the naive method having the lowest at $M = 1, 3, 4, 5$ and the multistage method have the lowest at $M = 2$. At $M = 6, \ldots, 10$, the linear AR model had the lowest RSPE. For the smaller values of $M$, these results agree with the simulation results in that the naive method performs just as well if not better than the multistage method.

Table 1 about here.

6. Discussion

We have adapted the SBLL method to FCAR models and shown that these estimators are oracally efficient. We examined the performance of naive, bootstrap, and multistage forecasting methods with a model estimated with the SBLL method. By estimating the model in this way, we have shown through simulation results that the bootstrap method did not perform as well as the other two methods. We have also shown that the naive method performs just as well as the multistage method and even outperforms it in
some situations. The main advantage to using the naive method is that it is much faster computationally than the other two methods.

For a real world example, we showed the naive and multistage methods performing better than a linear AR model when applied to solar irradiance data for $M = 1, \ldots, 6$. The day we selected for our application was mostly cloudy throughout the day. Future research will examine the fit of an FCAR model to irradiance data when the cloud cover conditions change during the day. This model will need to incorporate a covariate for the cloud cover. In this paper, we have shown that the SBLL method is adequate for fitting the model and forecasting in the absence of a covariate as long as the amount of cloud cover is constant throughout the day.

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Table 1: Root squared prediction error for $M = 1, \ldots, 10$ of the naive, multistage, and linear AR forecasts. Values in bold indicate the smallest prediction error for that value of $M$.

|       | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|-------|------|------|------|------|------|------|------|------|------|------|
| Naive | **0.001** | 0.017 | **0.013** | **0.014** | **0.190** | 0.171 | 0.175 | 0.185 | 0.206 | 0.231 |
| Multistage | **0.001** | **0.014** | 0.026 | 0.064 | 0.201 | 0.154 | 0.162 | 0.186 | 0.209 | 0.222 |
| Linear AR | 0.203 | 0.210 | 0.227 | 0.266 | 0.204 | **0.111** | **0.080** | **0.034** | **0.006** | **0.027** |
Figure 1: Plots (a) – (d) are graphs of the true coefficient function (solid line), the SBLL estimate (dashed line), and the oracle estimate (dotted line) for Example 4.1: (a) $p = 4$, $n = 75$, (b) $p = 10$, $n = 75$, (c) $p = 4$, $n = 500$, (d) $p = 10$, $n = 500$. Plots (e) and (f) contain the empirical efficiency densities for Example 4.1 with series lengths $n = 75$ (thin solid line), 150 (dashed line), 250 (dotted line), and 500 (thick solid line) for $p = 4$ (e) and $p = 10$ (f).
Figure 2: Plots of the RMPE for Example 4.1. For each of the plots, the solid line represents the naive forecast, the dashed line represents the bootstrap forecast, and the dotted line represents the multistage forecast. (a) $p = 4$, $n = 75$, (b) $p = 10$, $n = 75$, (c) $p = 4$, $n = 250$, (d) $p = 10$, $n = 250$, (e) $p = 4$, $n = 500$, (f) $p = 10$, $n = 500$. 
Figure 3: Plots (a) – (d) are graphs of the true coefficient function (solid line), the SBLL estimate (dashed line), and the oracle estimate (dotted line) for Example 4.2: (a) $p = 4, n = 75$, (b) $p = 10, n = 75$, (c) $p = 4, n = 500$, (d) $p = 10, n = 500$. Plots (e) and (f) contain the empirical efficiency densities for Example 4.2 with series lengths $n = 75$ (thin solid line), 150 (dashed line), 250 (dotted line), and 500 (thick solid line) for $p = 4$ (e) and $p = 10$ (f).
Figure 4: Plots of the RMPE for Example 4.2. For each of the plots, the solid line represents the naive forecast, the dotted line represents the bootstrap forecast, and the dashed line represents the multistage forecast. (a) $p = 4$, $n = 75$, (b) $p = 10$, $n = 75$, (c) $p = 4$, $n = 250$, (d) $p = 10$, $n = 250$, (e) $p = 4$, $n = 500$, (f) $p = 10$, $n = 500$. 

25
Figure 5: Plots (a) – (d) are graphs of the true coefficient function (solid line), the SBLL estimate (dashed line), and the oracle estimate (dotted line) for Example 4.3: (a) $p = 4$, $n = 75$, (b) $p = 10$, $n = 75$, (c) $p = 4$, $n = 500$, (d) $p = 10$, $n = 500$. Plots (e) and (f) contain the empirical efficiency densities for Example 4.1 with series lengths $n = 75$ (thin solid line), 150 (dashed line), 250 (dotted line), and 500 (thick solid line) for $p = 4$ (e) and $p = 10$ (f).
Figure 6: Plots of the RMPE for Example 4.3. For each of the plots, the solid line represents the naive forecast, the dotted line represents the bootstrap forecast, and the dashed line represents the multistage forecast. (a) $p = 4$, $n = 75$, (b) $p = 10$, $n = 75$, (c) $p = 4$, $n = 250$, (d) $p = 10$, $n = 250$, (e) $p = 4$, $n = 500$, (f) $p = 10$, $n = 500$. 
Figure 7: Plots of (a) measured irradiance (solid line) and clear sky model (dotted line) for November 11, 2013, (b) the clear sky irradiance transformation between 8:00 AM to 4:00 PM, (c) the observed transformed data (solid line) fitted FCAR model using the SBK method (dashed line) and of a linear AR(4) model (dot-dash line), (d) observed transformed data (thin solid line) forecasts for $M = 1, \ldots, 10$ starting at 1:30 PM using the naive method (thick solid line), the multistage method (dashed line), and a linear AR(4) (dot-dash line) model.