ENTANGLED GRAPHS

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Abstract. In this paper we prove a separability criterion for mixed states in $\mathbb{C}^p \otimes \mathbb{C}^q$. We also show that the density matrix of a graph with only one entangled edge is entangled.

INTRODUCTION

One of the major problems in Quantum Mechanics is to characterize entangled states of a quantum system. There are several partial criteria for entanglement of mixed states [6], [8], [10], but there is not yet a general criterion. Entanglement is connected to the important concept of non-locality in Quantum Mechanics. Entangled states are also useful in quantum cryptography and other quantum information processing tasks [1],[9]. A mixed quantum state is separable (and entangled, otherwise) if it can be written as a convex combination of pure separable states. Solving the quantum separability problem simply means determining whether a given quantum state is entangled or separable.

Following [2], a class of states that are represented by the density matrices of graphs are considered. It is shown in [2] that certain classes of graphs always represent entangled (separable) states. Also they have shown that a number of considered states have an exactly fractional value of their concurrence, a measure of entanglement of formation in small quantum systems.

A graph $G = (V, E)$ is a pair of a non-empty and finite set $V$ (or $V(G)$) whose elements are called vertices; and a non-empty set of unordered pairs of vertices $E$ (or $E(G))$, whose elements are called edges. A loop is an edge of the form $\{v_i, v_i\}$, for some vertex $v_i$. We assume that $E(G)$ does not contain only loops [2]. Two distinct vertices $v_i$ and $v_j$ are adjacent if $\{v_i, v_j\} \in E(G)$. The adjacency matrix of a graph on $n$ vertices $G$ is an $n \times n$ matrix $M(G)$, having rows and columns labelled by the vertices of $G$, and $ij$-th entry defined to be 1 if $v_i$ and $v_j$ are adjacent; and 0, otherwise. The degree of a vertex $v_i$ is the number $d_G(v_i)$ of edges adjacent to $v_i$. The degree of $G$ is defined by $d_G = \sum_{i=1}^{n} d_G(v_i)$. Note that $d_G = 2|E(G)|$. The degree matrix of $G$ is an $n \times n$ matrix $A(G)$, having $ij$-th entry $d_G(v_i)$ if $i = j$; and 0, otherwise. The Laplacian matrix of a graph $G$ is the matrix $L(G) = A(G) - M(G)$. Note that $L(G)$ does not change if one adds loops to or deletes loops from $G$. The density matrix of a graph $G$ is the matrix $\sigma(G) = \frac{1}{d_G} L(G)$. A graph $G$ has $k$ components, $G_1, G_2, \ldots, G_k$, if there

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is an ordering of $V(G)$, such that $M(G) = \sum_{i=1}^{k} M(G_i)$. In this case we write $G = G_1 \oplus G \oplus \cdots \oplus G_k$. When no such decomposition exists except for $k = 1$, $G$ is called connected. We refer the reader to [2] for examples and more details.

Let $tr(A)$ be the trace of a matrix $A$. A density matrix $\rho$ is said to be pure if $tr(\rho^2) = 1$, and mixed, otherwise. [2, Theorem 2.4] gives a necessary and sufficient condition on a graph $G$ for $\sigma(G)$ to be pure.

If $A$ is an $n \times n$ matrix, decomposed into $p^2$ blocks:

$$A = \begin{bmatrix}
A_{1,1}^{1,1} & A_{1,1}^{1,2} & \cdots & A_{1,1}^{1,p} \\
A_{2,1}^{2,1} & A_{2,1}^{2,2} & \cdots & A_{2,1}^{2,p} \\
\vdots & \vdots & \ddots & \vdots \\
A_{p,1}^{p,1} & A_{p,1}^{p,2} & \cdots & A_{p,1}^{p,p}
\end{bmatrix},$$

where each $A_{ij}$ is a $q \times q$ matrix and $n = pq$, then $(p,q)$-partial transpose $A^{T_B}$ is given by:

$$A^{T_B} = \begin{bmatrix}
(A_{1,1}^{1,1})^T & (A_{1,2}^{1,2})^T & \cdots & (A_{1,p}^{1,p})^T \\
(A_{2,1}^{2,1})^T & (A_{2,2}^{2,2})^T & \cdots & (A_{2,p}^{2,p})^T \\
\vdots & \vdots & \ddots & \vdots \\
(A_{p,1}^{p,1})^T & (A_{p,2}^{p,2})^T & \cdots & (A_{p,p}^{p,p})^T
\end{bmatrix}.$$

**RESULTS**

Next two results positively answer two open problems raised in [2, Conjecture 6.5]. A more general result in this direction is obtained by Wu [11], but our method of proof is direct and gives a better intuition in this special case.

**Theorem 1.** Let $G$ be a graph $(|V| = pq)$. If $G$ has only one entangled edge, then $\sigma(G)$ is entangled.

**Proof.** Let $\{ij \mid | \sigma(ij) - |st)\}$ be the only entangled edge of $G$ such that $1 \leq i, s \leq p$, $1 \leq j, t \leq q$ and let $G$ have all the possible edges. We show that $G$ is entangled. To prove this, we use the separability's necessary condition (If $\sigma(G)$ is separable, then $(\sigma(G))^{T_B} \geq 0$).

We look for a vector such that as $X$ with $X^T \sigma(G)^{T_B} X < 0$. Consider the following

$$x = \left[ \frac{1}{2} \frac{1}{2} \ldots \frac{1}{2} \frac{1}{2} \frac{p+q-1}{2} \ldots \frac{1}{2} \frac{p+q-1}{2} \frac{1}{2} \ldots \frac{1}{2} \right]^T,$$

Now if we compute $X^T \sigma(G)^{T_B} X$ then after simplification and using the fact that all the edges which are not connected to vertices $x_{ij}$ and $x_{st}$ in the sum $X^T \sigma(G)^{T_B} X$ arise to the terms in the form $x^2 + y^2 - 2xy(x = y = 1/2)$, which is equal to zero, and the sum of the other terms in $X^T \sigma(G)^{T_B} X$ for those edges that are connected with $x_{ij}$ or $x_{st}$ may be written as:

$$(p + q - 1)(x_{ij}^2 + x_{st}^2) + \sum_{k=1,k\neq j}^{q} x_{ik}^2 + \sum_{k=1,k\neq i}^{p} x_{kj}^2 + \sum_{k=1,k\neq i,j}^{q} x_{sk}^2 + \sum_{k=1,k\neq s,i}^{p} x_{kt}^2.$$
\[-2x_{ij} \left( \sum_{k=1, k \neq j}^q x_{ik} + \sum_{k=1, k \neq i}^p x_{kj} \right) - 2x_{st} \left( \sum_{k=1, k \neq t}^q x_{sk} + \sum_{k=1, k \neq s}^p x_{kt} \right). \]

Now since \( x_{ij} = x_{st} = \frac{p + q - 1}{2(p + q)} \) and the remaining \( x_{mn} \)s \( ((m, n) \neq (i, j), (s, t)) \) are \( \frac{1}{2} \) after substituting we have:

\[
2(p + q - 1)\left( \frac{p + q - 1}{2(p + q)} \right)^2 + 2(p + q - 3) \frac{1}{4} - 4(p + q - 2)\left( \frac{p + q - 1}{4(p + q)} \right) = -\frac{p + q - 1}{2(p + q)^2},
\]

so \( \sigma(G)^{T_B} \) is not positive semi-definite, and therefore \( G \) is entangled.

Next we suppose that there is a separable edge such as \( P[\frac{1}{\sqrt{2}} (|kl| - |mn|)] \) that is not contained in \( G \). If one of vertices of mentioned edge is \( x_{ij} \) or \( x_{st} \), then, in the sum \( X^T \sigma(G)^{T_B} X \), the term \( x_{mn}^2 + x_{kl}^2 - 2x_{mn}x_{kl} \) appears that after substituting, we get

\[
\left( \frac{p + q - 1}{2(p + q)} \right)^2 + \frac{1}{4} - \frac{p + q - 1}{2(p + q)} = \frac{1}{4(p + q)^2},
\]

Now this expression is positive even if the edge is not in \( G \), and the proof goes as before. If the edge is not involving the vertices \( x_{ij} \) or \( x_{st} \), then

\[
x_{mn}^2 + x_{kl}^2 - 2x_{mn}x_{kl} = \frac{1}{4} + \frac{1}{4} - \frac{1}{2},
\]

and again we are done. To complete the proof we need to prove our claim also for the following simple cases:

1. If the graph is just one edge that is entangled, trivially it is entangled.
2. If all of separable edges of the graph are not connected with vertices \( x_{ij} \) and \( x_{st} \), then for the vector \( X \) above, in the sum \( X^T \sigma(G)^{T_B} X \), only the expression \( x_{ij}^2 + x_{st}^2 - 2x_{ij}x_{st} \) remains that after substitution, it becomes

\[
2(p + q - 1)\left( \frac{p + q - 1}{2(p + q)} \right)^2 - 2(p + q - 1)\left( \frac{1}{2} \right) = -\frac{p + q - 1}{2(p + q)^2} < 0.
\]

Therefore all of the possible cases are considered, and we are done.

\[\Box\]

**Theorem 2.** If all the entangled edges of graph \( G \) are incident to the same vertex, then \( G \) is entangled.

**Proof.** We use Theorem 1. Let \( G \) have all the possible separable edges and the edge \( P[\frac{1}{\sqrt{2}} (|ij| - |st|)] \) be one of the entangled edges and the vertex \( x_{ij} \), be the common vertex of the entangled edges. We prove that \( \sigma(G)^{T_B} \) is not positive semi-definite.

We omit all the entangled edges of graph \( G \) except \( P[\frac{1}{\sqrt{2}} (|ij| - |st|)] \) and call the resulting graph \( H \). We consider the vertex \( X \) as in Theorem 1.

From the proof of Theorem 1, the sum \( X^T \sigma(H)^{T_B} X \) is negative. Now if another edge of \( G \) such as \( P[\frac{1}{\sqrt{2}} (|ij| - |mn|)] \) is added to \( H \), the expression

\[
x_{ij}^2 + x_{mn}^2 - 2x_{ij}x_{mn},
\]

appears in the sum \( X^T \sigma(H)^{T_B} X \), which after substitution gives

\[
\left( \frac{p + q - 1}{2(p + q)} \right)^2 + \frac{1}{4} - \frac{1}{2} = -\frac{2(p + q) + 1}{4(p + q)^2} < 0.
\]
Since the above expression is negative, if all of the omitted entangled edges of $G$ are added to $H$, the sum $X^T\sigma(H)^TbX$ remains negative and therefore $G$ is entangled. Similar to the proof of Theorem 1, one can show that the hypothesis that $G$ contains all the possible separable edges could be removed, and we are done. □

**Definition 3.** A matrix is line sum symmetric if the $i$-th column sum is equal to the $i$-th row sum for each $i$.

We may use our technique combined with results of [11] to give simpler proofs of some of the results proved in [2] with a different method. The next three results are of this kind. For the rest of the paper, $p$ and $q$ denote two arbitrary natural numbers.

**Theorem 4.** The density matrix of the tensor product of two graphs on $p$ and $q$ vertices is separable in $C^p \otimes C^q$.

**Proof.** Let $G$ be a graph on $p$ vertices and $H$ be a graph on $q$ vertices, with density matrices $\sigma(G)$ and $\sigma(H)$ respectively. By Theorem 8 of [11], it is enough to prove that matrices $A^{ij}$ of $\sigma(G \otimes H)$ are line sum symmetric. Any matrix $A^{ij}$ is symmetric and so is sum line symmetric. Clearly we need only to show that the matrices $M^{ij}$ of $M(G \otimes H)$ are line sum symmetric, for $i \neq j$. Since $M(G \otimes H) = M(G) \otimes M(H)$, so each $M^{ij}$ are equal to a multiplier of $M(H)$. Since $M(H)$ is symmetric, we are done. □

Next we can decide on separability of the density matrix of two special graphs, namely the complete graph $K_n$ on $n$ vertices, and the star graph $K_{1,n-1}$ (see [2] for details).

**Proposition 5.** (i) For $n = pq$, the density matrix $\sigma(K_n)$ is separable in $C^p \otimes C^q$.

(ii) The density matrix of the star graph $K_{1,n-1}$ on $n = pq \geq 4$ vertices is entangled in $C^p \otimes C^q$.

**Proof.** (i) Again note that for each of matrices $A^{ij}$ of $\sigma(K_n)$, the $l$-th row sum is equal to the $l$-th column sum, for $l = 1, \ldots, n$. Indeed, since the graph $K_n$ is complete, all elements in $\sigma(K_n)$, except for the diagonal elements, are equal to -1, and we are one.

(ii) Let $G = K_{1,n-1}$. It is obvious that in $\sigma(G)$, for $i, j = 2, \ldots, p, A^{ij}$ is equal to $I_q,$ if $i = j,$ and 0, otherwise. By Theorem 3 of [11], it is enough to show that there exists a row in $\sigma(G)^Tb$ with a nonzero row sum. Consider $(p - 1)q + 1$-th row. But this row sum is clearly the summation of the first row sums of matrices $(A^{pq})^T$, for $j = 1, \ldots, p$. This last sum is now easily seen to be equal to $-q + 1 < 0$, therefore, for $pq \geq 4$, $G = K_{1,n-1}$ is entangled in $C^p \otimes C^q$. □

**Definition 6.** An e-matching is a matching having all edges entangled [2]. Each vertex of an e-matching on $n = pq$ vertex can be labelled by an ordered pairs $(i, j)$, where $1 \leq i \leq p$ and $i \leq j \leq q$. A pe-matching of a graph $G$ is an e-matching spanning $V(G)$.

**Theorem 7.** Let $G$ be a graph on $n = 2p$ vertices. If all the entangled edges of $G$ belong to the same pe-matching, then $\sigma(G)$ is separable in $C^2 \otimes C^p$.

**Proof.** Let $G$ be as above, we may divide $G$ into two graphs, consisting of all separable edges and all entangled edges, respectively. Let’s call the second graph $H$. It is enough to show that $H$ is separable.
The density matrix $\sigma(H)$ contains matrices $A^{11} = A^{22} = I_q$, $A^{12}$ and $A^{21}$. Since the entangled edges of $G$ form a pe-matching, so each row or column of matrices $A^{12}$ and $(A^{21})$ has one $-1$ and all others zero. By Theorem 7 of [11], $H$ is separable in $C^2 \otimes C^p$, and we are done. □

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