ON THE HISTORY OF LIE BRACKETS, CROSSED MODULES, AND LIE-RINEHART ALGEBRAS

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Abstract. The aim here is to sketch the development of ideas related to brackets and similar concepts: Some purely group theoretical combinatorics due to Ph. Hall led to a proof of the Jacobi identity for the Whitehead product in homotopy theory. Whitehead introduced crossed modules to characterize a second relative homotopy group; guided by combinatorial group theory considerations, Reidemeister and Peiffer explored this kind of structure to develop normal forms for the decomposition of a 3-manifold; but crossed modules are also lurking behind a forgotten approach of Turing to the extension problem for groups: Turing concocted the obstruction 3-cocycle isolated later by Eilenberg-Mac Lane and already proved the Eilenberg-Mac Lane theorem to the effect that the vanishing of the class of that cocycle is equivalent to the existence of a solution for the corresponding extension problem. This Turing cocycle is related to what has come to be known as Teichmüller cocycle. There was a parallel development for Lie algebras including a forgotten paper by Goldberg and, likewise, for Lie-Rinehart algebras and Lie algebroids. Versions of Turing’s theorem were discovered several times under such circumstances, and there is rarely a hint at the mutual relationship. Also, Lie-Rinehart algebras have for long occurred in the literature on differential algebra, at least implicitly.

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1. Introduction

Lie brackets and variants thereof have a history. Here I attempt to give an account of the evolution of some ideas intricately relating groups and Lie algebras. The story, not complete, somewhat reflects the accursed first half of the 20th century, however. The reader can imagine what the outcome of interaction or collaboration among, e.g., Baer, Reidemeister, Turing, Whitehead, Teichmüller, Hochschild, Magnus, Witt, etc. might have been!

Section 2 recalls how some combinatorics due to P. Hall [Hal34], at first purely group theoretical, eventually led to a proof of the graded Jacobi identity of the Whitehead product.

Section 3 discusses how the crossed module concept evolved out of considerations of Turing, Whitehead, Reidemeister and Peiffer. In particular, Turing’s forgotten paper [Tur38] about the extension problem for groups implicitly contains the idea of a (free) crossed module. Furthermore, it develops the group cohomology 3-cocycle which encapsulates the obstruction.

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for an abstract kernel to be extendible (i.e., realizable by a group extension, see Section 3), and Tur38 Theorem 4 p. 365] establishes the fact that the vanishing of the cohomology class is equivalent to extendibility of the abstract kernel under discussion. In the rest of the paper I refer to this theorem as “Turing’s theorem”. That obstruction 3-cocycle was later identified in EM47b together with a proof of Turing’s theorem. From Section 3 on, Turing’s theorem is a kind of Ariane’s thread: I illustrate how variants of this theorem were discovered several times in the literature, mostly without notice of the mutual relationship. To this end, I spell out various situations where crossed modules provide conceptual means to simplify otherwise complicated calculations. Also, these conceptual means provide new insight; for example, from crossed modules one can built a cohomology theory for topological groups not covered by (global) cocycle descriptions.

Section 4 discusses the relationship between the Teichmüller cocycle and crossed modules and the eventual outcome thereof.

Section 5 recalls the corresponding development in the Lie algebra case. I show in particular that the forgotten paper Gol53 implicitly develops a Lie algebra variant of Turing’s theorem in terms of crossed 2-fold extensions of Lie algebras.

Section 6 illustrates how Lie-Rinehart algebras, a far reaching algebraic generalization of Lie algebroids, have for long occurred in the literature on differential algebra, at least implicitly. A MR or Zentralblatt search does not fully reveal historical details of the kind described below nor does Google scholar, and it proved indispensable to look up the papers themselves and to try to assimilate the content; indeed at times those bibliographical tools do not even find relevant references quoted in the papers themselves. A bibliographic metric fails to unveil the origins of the mathematical ideas I describe below; see the penultimate paragraph of Section 3 for details. A historical irony is the fact that Turing and Teichmüller, both forerunners regarding the third group cohomology group, worked as WWII codebreakers on opposite sides, cf. Hil00, HHK+92, p. 12. How the second group cohomology group evolved out of work of Brauer Bra32, Noether Noe29, Noe30 and Baer Bae34 is not an issue in this paper, and the reader should not confuse crossed modules and crossed products.

I hope my considerations help some of the presently young mathematicians a bit to maintain contact with the past. Various of the items I discuss in this paper were among K. Mackenzie’s research interests.

2. Graded Lie algebras in the early days of algebraic topology: From P. Hall’s “collecting process” to the Jacobi identity for the Whitehead product

The paper Mag37 starts with the sentence: “P. Hall Hal34 established a wide system of relations among higher commutators.” Hal50 Theorem 4.1] says: The standard commutators of weight n are a basis of \( F_n/F_{n+1} \). (Here \( F_n \) is the group generated by commutators of weight n and higher in a free group \( F \).) Thereafter: “... the standard commutators are precisely those which arise in Philip Hall’s collecting process given in Hal34.”

In Wh41, J.H.C. Whitehead introduced the operation of what later has come to be known as \textit{Whitehead product}, an operation of the kind \( \pi_q \times \pi_r \to \pi_{q+r-1} \) involving the homotopy groups \( \pi_* \) of a space. For \( * \geq 2 \), this operation yields a graded Lie algebra (the homotopy groups being regraded down by 1) and, for \( r \geq 0 \), the operation \( \pi_1 \times \pi_r \to \pi_r \) gives the classical action of \( \pi_1 \) on \( \pi_* \) (\( \pi_0 \) being only a set). In his MR review of Samelson’s

* Turing, Hilton and J.H.C. Whitehead worked as WWII codebreakers at Bletchley Park (UK) Hil00.
paper [Sam53], P. Hilton wrote: “The object of this note is to establish a formula which is an important step towards proving the conjectured Jacobi identity for Whitehead products.” Thus in the mid 1950s the idea of a graded Lie algebra was well understood among topologists. The papers [Cha54], [Hil55], [NT54], [Suz54], [UM57], [Whi54] offer proofs of the Jacobi identity for the Whitehead product. In his MR review on [Suz54], J. Moore wrote “This identity has been proved recently by others including Hilton and Serre, Massey and Uehara, G.W. Whitehead, Nakaoka, and Toda.” A footnote to [UM57] says: “The authors have been informed that independent proofs of this Jacobi identity have recently been found by H. Toda and M. Nakaoka in Japan, by Serre, Hilton, and Green in Europe, and by G. Whitehead in this country.” I am indebted to Serre for having helped me to clarify his unpublished contribution and that of Green (“Sandy Green”): At the end of the introduction of [Hil55], Hilton wrote: “The author wishes to acknowledge the decisive contributions made by J.-P. Serre and J. A. Green; the fundamental idea in the proof of Theorem A (identifying the homotopy group $\pi_n$ ($n \geq 2$) of a one point union of finitely many spheres with a direct sum of $n$th homotopy groups of spheres arising from systematically exploiting a family of Whitehead products) is due to Serre, and the necessary extension of the algebraic methods is due essentially to Green.” In a letter to Hilton, Serre had explained the idea that the homotopy groups of a one point union of finitely many spheres recover all multivariable operations among homotopy groups of spheres. Using the graded analogue of Witt’s result [Wit37] establishing the link between free Lie algebras and free associative algebras (in modern terminology: the universal algebra associated with a free Lie algebra is the free associative algebra on the very same generators), Hilton managed to render Serre’s idea rigorous. To this end, he introduced “basic commutators” imitating Philipp Hall’s basic commutators in group theory; these are the standard commutators in [Hal50], see also [MKS66, Theorem 5.13. A p. 343] and the literature there. The corresponding somewhat more ring theoretic analogue is [MKS66, Theorem 5.8 p. 323], first proved in [Mag37], a paper in turn quoted by Hilton. It is presumably at this point where Green worked out “the necessary extension of the algebraic method”: In a footnote on p. 164 of [Hil55], Hilton wrote: “The ensuing calculation is implicit in the last remark of [Sam53].” Thus, commutator calculus known from group theory, in particular from [Hal34], applied to the loop space, close to a group, indeed, an H-space, properly interpreted, apparently enabled Hilton to prove his Theorems A and B (Jacobi identity for Whitehead products). In a footnote to [Hil55], Hilton noted that Hurewicz established the Jacobi identity under discussion as well, but I could not find any trace in the literature to this effect. Also, in [Wit37], Witt acknowledged that a talk of Magnus about the results in [Mag37] prompted Witt to write his paper.

As for the terminology graded Lie algebra: In [Car59] (seminar delivered in May 1955, typed manuscript dated “Juillet 1957”), Cartier considered the notion of a graded Lie algebra (algèbre de Lie graduée). Actually, Cartier explored a differential graded Lie algebra without saying so, the universal algebra associated with that differential graded Lie algebra yields a variant of the description in [CE99, Ch. XIII, Ex. 14, p. 287] of the standard resolution of the ground field nowadays familiar in Lie algebra theory, and the map $\varphi$ on the bottom of p. 287 of [CE99] renders the two descriptions under discussion explicit. For a given Lie algebra $\mathfrak{g}$, that variant of the standard resolution of the ground field is written in [CE99, Ch. XIII, Ex. 14, p. 287] as $W(\mathfrak{g})$—coincidentally, in [Car51a, Car51b], H. Cartan used the letter $W$ for the Weil algebra. Among other things, [CE99, Ch. XIII, Ex. 15, p. 287] describes the structure of $W(\mathfrak{g})$ as a “graded differential algebra”. Thus Cartan and Eilenberg had understood, at least implicitly, the idea of a differential graded Lie algebra in September 1953, date of the preface...
of their book [CE99]. This book is quoted in [Car59]. In his MR review on [Hil56], J. Moore wrote “If one defines ..., one obtains a ring satisfying the usual identities for a graded Lie ring.” In a footnote to [Hil56], Hilton wrote: “Cartier has considered the notion of a graded Lie algebra in [Car59].” Thus at the time, the terminology graded Lie ring and graded Lie algebra was lingua franca in algebraic topology.

3. From Schreier’s extension theory to crossed modules

To explore group extensions, Schreier introduced factor sets and studied their properties [Sch26]; see [Kur70, §48] for an account of Schreier’s theory. Baer [Bae34] elaborated on [Sch26] in terms of an additional ingredient, the idea of an abstract kernel: For a group $Q$, an abstract $Q$-kernel or simply $Q$-kernel (terminology due to [EM47b]) is a group $N$ together with a homomorphism $Q \to \operatorname{Out}(N) = \operatorname{Aut}(N) / \operatorname{InAut}(N)$ (“Kollektivcharacter” in the terminology of [Bae34]). An extension $N \to E \to Q$ determines, via conjugation in $E$, an abstract $Q$-kernel structure $Q \to \operatorname{Out}(N)$. However, not every abstract kernel is realizable in this manner; already [Bae34] exhibits a counterexample.

To understand a relative version of the product operation discussed in the previous section, Whitehead isolated the concept of a crossed module; the idea occurs in [Whi41], is made precise in [Whi46], and the crossed module terminology occurs in [Whi49] §2 p. 453; the paper [Whi41] quotes [Rei32, Rei34] and [Whi49] quotes [Rei38]. Here is the definition:

A crossed module consists of groups $G$ and $\Gamma$, an action of $G$ on $\Gamma$ from the left which we here write as $G \times \Gamma \to \Gamma$, $(x, y) \mapsto x y$, and a $G$-homomorphism $\partial : \Gamma \to G$, the action of $G$ on itself being by conjugation, subject to

$$(\partial(x)) x y^{-1} x^{-1} = 1, \quad x, y \in \Gamma.$$  \hfill (3.1)

The paper [Mac49] also introduces crossed modules (without a name), hypothesis (i) in I.2 (p. 738) of this paper encapsulates the crossed module axiom (3.1), and Theorem 3.1 (p. 742) establishes the equivalence of abstract kernels and crossed modules. Likewise, Footnote 2 of [Whi49] says: “Anne Cobbe has pointed out to me that a crossed $(\gamma, d)$-module determines a $Q$-kernel ... and that any $Q$-kernel has a representation as a crossed module.”

The paper [Pei49] (Ph.D. thesis supervised by Reidemeister, submitted for publication in 1944) arose out of a combinatorial study of 3-manifolds and also isolated the identities (3.1) defining a crossed module; these identities have since [Lyn50] come to be known as Peiffer identities. See [Rei49] for a survey. But there is no hint of Whitehead’s crossed modules, nor had Whitehead recognized the Peiffer-Reidemeister theory. In the notation of [Rei49] §1, $\mathcal{R}/\mathcal{P} \to \mathcal{G}$ is a crossed module (here $\mathcal{P}$ refers to the corresponding Peiffer elements) and, furthermore, in [Rei49] §2, $\mathcal{R}/\mathcal{P} \to \mathcal{G}$ is the free crossed module associated with the presentation of the group $\mathcal{G}$ under discussion. A footnote on p. 379 of [Smi51] hints at the relationship and [Pap63] Section 2, penultimate paragraph on p. 207] explicitly recognizes the equivalence of the two approaches. In response to a letter, R. Peiffer revealed she was not familiar with the notion of crossed module, however. There is no mention of Reidemeister or Peiffer in [Sie76], no mention of Whitehead in [Met79]. At the end of the 1970s, the relationship became more widely known [Hue79a, Hue79b].

In [Whi41] p. 423, in the course of the proof that the second relative homotopy group arising from attaching 2-cells is a free crossed module, i.e., characterized merely by the identities (3.1), Whitehead gives credit to [Rei26]. A footnote says: “This fact is, so to speak, half the
content of the proof.” Here Whitehead interprets the identities (3.1) in terms of the Wirtinger relations defining the fundamental group of the exterior of a (tame) knot in the 3-sphere. The paper Bro80 offers a modern account of these ideas. An extension thereof is in Hue12, the main result of this paper says that, for \( n \geq 3 \), Artin’s braid group \( B_n \), as a crossed module over itself, has a single generator, which can be taken to be any of the Artin generators, and that the kernel of the surjection from the corresponding free crossed module is the second homology group \( H_2(B_n) \), well known to be cyclic of order 2 when \( n \geq 4 \) and trivial for \( n = 2 \) and \( n = 3 \). This includes an interpretation of the Artin relations in terms of Peiffer identities.

To delve briefly into Tur38 and fulfill the promise in the introduction, sticking to Turing’s notation (to some extent the same as that in Bae34), let \( \langle e_1, \ldots, e_n; r_1, \ldots, r_l \rangle \) be a presentation of a group \( \mathcal{G} \). Turing gives credit to Rei26 for the following requisite combinatorial group theory: Let \( \mathfrak{F} \) denote the free group on \( e_1, \ldots, e_n \) and \( \mathfrak{R} \) the normal closure of \( r_1, \ldots, r_l \) in \( \mathfrak{F} \). Let \( \Phi \) be the free \( \mathfrak{F} \)-operator group on \( r_1, \ldots, r_l \) and let \( \mathbf{P} \) denote the kernel of the obvious epimorphism \( \tau : \Phi \to \mathfrak{R} \). Turing refers to the members of \( \mathbf{P} \) as “relations between relations” (in his Zbl review of Turing’s paper, Baer attributes the idea of exploring such relations between relations to Rei26); these are the identities among relations in Pei49 and Rei49, see BHS2 for a modern account (but at the time of writing we were unaware of Turing’s contributions). Write the action as

\[
\Phi \times \mathfrak{F} \to \Phi, \quad (y, w) \mapsto y^w, \quad y \in \Phi, \quad w \in \mathfrak{F},
\]

so that \( \tau(y^w) = w^{-1}\tau(y)w \), for \( y \in \Phi \) and \( w \in \mathfrak{F} \). Accordingly, write the elements Tur38 (9) p. 363 of \( \mathbf{P} \) in the form

\[
y^\tau(x) = y^x, \quad y = r_i^a, \quad x = r_j^b, \quad a, b \in \mathfrak{F}, \quad i, j = 1, \ldots, l.
\]

With the requisite modifications, since we are now working with a right action, these elements formally recover those on the left-hand side of (3.1). The notation in Tur38 is \( E_i, a \) for \( r_i^a \in \Phi \) (\( a \in \mathfrak{F} \)), and its is useful to remember that, as a group, \( \Phi \) is freely generated by the \( E_i, a \) (\( i = 1, \ldots, l \)), as \( a \in \mathfrak{F} \).

Now Turing proceeds as follows to construct a family of members of \( \mathbf{P} \) which, together with (3.3), generate \( \mathbf{P} \): Choose a “function” \( v : \mathfrak{F} \to \mathfrak{F} \) that factors through the canonical epimorphism can: \( \mathfrak{F} \to \mathcal{G} \) as \( \mathfrak{F} \to \mathcal{G} \to \mathfrak{F} \) and whose composite with the canonical epimorphism coincides with that epimorphism. Thus \( v \) encapsulates a section, not necessarily a homomorphism, for that epimorphism. For \( a \in \mathfrak{F} \), write \( v(a) \), so that \( r_a = v^{-1}a \in \mathfrak{R} \); then, for \( b \in \mathfrak{F} \), necessarily \( v_{r_i, r_j} = r_i^{-1} v_{r_i, r_j} r_i = r_i \), since \( v_{r_i, r_j} = v_{b} \). By means of a recursive procedure, Turing then defines corresponding members \( R_{v, r_i} \) of \( \Phi \), and Tur38 Theorem 3 p. 364 says the following: The members (3.3) and

\[
(R_{v, r_i} E_i^{-1})^a, \quad a, b \in \mathfrak{F}, \quad i = 1, \ldots, l,
\]

generate the kernel \( \mathbf{P} \) of \( \tau : \Phi \to \mathfrak{R} \). Turing applies these constructions to an abstract kernel \( \langle \mathfrak{R}, X : \mathcal{G} \to \text{Out} (\mathfrak{R}) \rangle \) (the notation \( X \) is due to Bae34), together with a lift \( \chi : \mathfrak{F} \to \text{Aut} (\mathfrak{R}) \) of \( X \). Tur38 Theorem 4 p. 365 says the following (in somewhat more modern terminology): There is a group extension \( \mathfrak{R} \to \mathcal{G} \to \mathcal{G} \) realizing the abstract kernel \( \langle \mathfrak{R}, X : \mathcal{G} \to \mathcal{G} \rangle \).

\(^{†}\)Turing’s notation is \( \mathcal{G}' \) rather than \( \mathcal{G} \) but, in modern notation, this is misleading since \( \mathcal{G}' \) is standard notation for \( [\mathcal{G}, \mathcal{G}] \).
Out($\mathfrak{N}$) if and only if the pair $(\chi, X)$ extends to a commutative diagram

$$
\begin{array}{c}
\Phi \\
\downarrow \phi \\
\mathfrak{N}
\end{array}
\xrightarrow{\tau}
\begin{array}{c}
\mathfrak{F} \\
\downarrow x
\end{array}
\xrightarrow{\text{can}}
\begin{array}{c}
\mathfrak{G} \\
\downarrow \chi
\end{array}
\xrightarrow{1}
\begin{array}{c}
\mathfrak{N}
\end{array}
$$

with exact rows such that $\vartheta$ is trivial on the elements (3.4). Here “can” refers to the canonical maps, and Turing’s notation is $\mathfrak{G}$ for $\mathfrak{C}$ and $\mathfrak{G}'$ for $\mathfrak{E}$. Needless to point out, neither the idea of an exact sequence nor that of a commutative diagram was available to Turing.

Let $T$ (here $T$ stands for “Turing”) denote the normal closure in $\Phi$ of the elements (3.3)—the various groups $\mathfrak{T}$, $\Phi$, $\mathfrak{F}$ correspond precisely to the groups, respectively, $\mathfrak{P}$, $\mathfrak{R}$, $\mathfrak{E}$, in [Rei49 §2]. The images of the members (3.1) generate the kernel $\pi_2$ of the induced epimorphism $\Phi/T \to \mathfrak{R}$. Here the notation $\pi_2$ serves here as a mnemonic for the fact that this kernel recovers the second homotopy group of the geometric realization of the presentation (2-dimensional CW-complex realizing the presentation) under discussion, see, e.g., [BH82]. With hindsight we understand that the quotient group $\Phi/T$ is the free crossed $\mathfrak{F}$-module on $r_1, \ldots, r_l$ and that, by virtue of the interpretation of the third group cohomology group in terms of “crossed 2-fold extensions” (details below), the restriction to $\pi_2$ of the homomorphism $\tilde{\vartheta}: \Phi/T \to \mathfrak{R}$ which $\vartheta$ induces recovers a version of the Eilenberg-Mac Lane 3-cocycle [EM47b] associated with the abstract kernel under discussion. Thus we can interpret Turing’s theorem [Tur38, Theorem 4 p. 365] as saying that an abstract kernel is extendible if and only if the associated 3-cohomology class vanishes, a result established nine years later as [EM47b, Theorem 8.1 p. 33].

The paper [Mac79] discusses the history of the interpretation, for $n \geq 1$, of the group cohomology group $H^{n+1}(Q, M)$ for a group $Q$ and a $Q$-module $M$ in terms of crossed $n$-fold extensions [Ger66, Hol79, Hue77, Hue80, Lod78, Rat77]; see also the appendix. Suffice it to point out the following: A crossed 2-fold extension of a group $Q$ by a $Q$-module $\pi$ is an exact sequence

$$
0 \to \pi \to C \xrightarrow{\partial} G \to Q \to 1
$$

(3.6)
of groups having $\partial : C \to G$ a crossed module and $\pi \to C$ a morphism of $G$-groups, the action of $G$ on $\pi$ being through $G \to Q$. Congruence classes of crossed 2-fold extensions of $Q$ by $\pi$ constitute the cohomology group $H^3(Q, \pi)$. To reconcile this with Turing’s theorem, consider the pullback group $G$ characterized by requiring that

$$
\begin{array}{c}
G
\end{array}
\xrightarrow{1}
\begin{array}{c}
\mathfrak{G}
\end{array}
\xrightarrow{\text{can}}
\begin{array}{c}
\Aut(\mathfrak{N})
\end{array}
\xrightarrow{\text{can}}
\begin{array}{c}
\Out(\mathfrak{N})
\end{array}
$$

(3.7)

be a pullback diagram, and let $Z$ be the center of $\mathfrak{N}$. With the obvious structure, $\mathfrak{N} \to \Aut(\mathfrak{N})$ is a crossed module whence it is immediate that the homomorphism $\vartheta$ in (3.5) above is trivial.

\[\text{in the terminology of [Bae34, p. 378 ff] “Auflösung des Kollektivcharakters”}\]
on $T$. Thus the diagram \( \begin{array}{ccc} 0 & \rightarrow & \pi_2 \rightarrow \Phi/T \rightarrow \tilde{\phi} \rightarrow \mathcal{G} \rightarrow 1 \\ \downarrow \tilde{\vartheta} & & \downarrow \tilde{\chi} \end{array} \) (3.8)

of crossed 2-fold extensions, and the restriction of $\tilde{\vartheta}$ to $\pi_2$ recovers a version of the Eilenberg-Mac Lane 3-cocycle $\E{3}$ of $\mathcal{G}$ with values in the center $Z$ of $\mathcal{G}$.

The sequence \([\text{Mac49]} \ (2.1) \ p. 738\) is a crossed 2-fold extension. The terminology “crossed sequence” is in \([\text{MW50}]\). The interpretation of the third group cohomology group in terms of crossed 2-fold extensions goes back to \([\text{Ger66}]\), without usage of the crossed module terminology, however: In this paper, a *category of interest* is defined to be a subcategory $\mathcal{C}$ of the category of all rings (associative or not) closed under kernels, cokernels and fibered products, and an introductory remark says: “Had we considered categories of interest inside *almost abelian categories* (in the sense of J. Moore), then what follows could also have included the cases of groups and schemes.” Now, consider two objects $A$ and $C$ of $\mathcal{C}$. An $A$-structure on $C$ is a split extension $C \rightarrow B \rightarrow A$ together with a choice of splitting. For an $A$-structure on $C$, in the terminology of \([\text{Ger66}]\), a morphism $\varphi: C \rightarrow A$ is *conform* if the $C$-structure on $C$ which $\varphi$ induces coincides with canonical $C$-structure of $C$ on itself \([\text{Ger66}] \ \text{Section 1 p. 3}\).

To fulfill the need for a *cohomology theory that is satisfactory for deformation theory* \((\text{Ger66} \ \text{Section 1 p. 3})\), for objects $A$ and $M$ of $\mathcal{C}$ with an $A$-module structure on $M$, Gerstenhaber then introduces the cohomology group $\E{3}(A, M)$ in terms of congruence classes of *admissible* exact sequences in $\mathcal{C}$ of the kind

\[
0 \rightarrow M \rightarrow N \rightarrow B \rightarrow A \rightarrow 0;
\]

(3.9)

here “admissible” signifies that $N$ carries a $B$-structure, that $N \rightarrow B$ conforms, and that $M \rightarrow N$ is a morphism of $B$-structures, that on $M$ being through $B \rightarrow A$ via the $A$-structure on $M$. Theorem 6 (3.) in \([\text{Ger66}] \ p. 7\) includes (among other valuable insight not relevant here except for Lie algebras, see Section \([\text{5}]\) below), after the circumspect hint “had we given the details, the category of groups” (i.e., $\mathcal{C}$), a natural isomorphism $\E{3}(A, M) \rightarrow H^3_{\mathcal{C}}(A, M)$. When $\mathcal{C}$ is the category of groups, the conformity constraint comes down to (3.1) and $\E{3}(A, M)$ recovers, for a group $A$ and an $A$-module $M$, the interpretation of $H^3(A, M)$ in terms of crossed 2-fold extensions. In this case, \([\text{Ger66}] \ \text{Theorem 4 p. 5}\) is a version of Turing’s theorem.

The description of group cohomology in terms of crossed $n$-fold extensions ($n \geq 1$) is susceptible to generalizations where cocycles are not necessarily available. This observation reflects the search in \([\text{Ger66} \ \text{p. 3}]\) for a cohomology theory that is “satisfactory for deformation theory”. For example, for an extension of a topological group $G$ by a continuous $G$-module whose underlying bundle is non-trivial, (global) continuous cocycles are not available, and there are exact sequences relating the cocycle part of classifying in the trivial bundle case and a classification of the underlying bundle, see, e.g., \([\text{Sta78}]\) and the literature there.

For a compact semisimple Lie group $G$, the 4-term sequence

\[
1 \rightarrow S^1 \rightarrow \hat{\Omega}G \rightarrow PG \rightarrow G \rightarrow 1
\]

(3.10)

involving the universal central extension $\hat{\Omega}G$ of the group $\Omega G$ of based loops in $G$ by the circle group $S^1$ and the group $PG$ of based paths in $G$ acquires the structure of a crossed two-fold extension \([\text{BSCS07}] \ \text{Prop. 3.1 p. 115}\) and hence represents a class in a differentiable or continuous third cohomology group $\E{3}(G, S^1)$ that does not admit a description in terms of
global continuous cocycles—the corresponding cohomology being trivial in higher dimensions for a compact Lie group. Also, $S^1$ is not a $G$-representation, and an exact sequence of the kind mentioned in the previous paragraph does not apply. One can view that crossed 2-fold extension as a geometric object realizing the first Pontryagin class of the classifying space $BG$ or, equivalently, the 3-cohomology class associated with $G$ resulting from the 3-form which E. Cartan exhibits in [Car28, p. 197] to prove that the third Betti number of $G$ is non-zero. With hindsight one could say this class is an early instance of a 3-dimensional group cohomology class. That crossed 2-fold extension is equivalent to but essentially different from the corresponding gerbe, cf. [Bry93]. Without reference to the “crossed”-terminology, a crossed 2-fold extension occurs in [Gro85, p. 97], written there as $A \rightarrow N \rightarrow E \rightarrow G$ together with the claim that a 3-cocycle associated with it leads to a representation of the ‘t Hooft commutation relations. It seems to me [Bry93] renders some of the claims in [Gro85] mathematically rigorous.

The crossed module concept, variants, and generalizations thereof are nowadays very lively in mathematics; see, e.g., [BHS11, Fio07, Nor90] and the references there. The equivalence of crossed modules and 2-groups, observed by the Grothendieck school in the mid 1960s (unpublished), see [BHS11, I.1.8 p. 29], [BS76, Fio07, Theorem 5.13 p. 153], is relevant in string theory [BSC07].

Turing’s contributions in [Tur38] have been completely ignored—perhaps none except Turing himself ever understood them. A present day MR citation search finds a single paper, a ZBL citation search finds four papers but two of them do not quote Turing’s paper under discussion here (they deal with Turing machines and cite his corresponding articles). Google scholar finds 34 citations, and I know a few more references which none of these bibliographical tools find. I checked the references available to me and, as far as I can see, Turing’s paper is only cited, without any understanding of its mathematical content. Thus Turing would have slipped through an evaluation system based on bibliographic metrics.

The paper [Nee07] is the only one which a present day MR citation search finds for the citations of [Tur38]. That paper has its own version of Turing’s theorem, [Nee07, Theorem 3.8 p. 251]. In terms of the notation of [Nee07], lurking behind [Nee07, Theorem 3.8 p. 251] is the commutative diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & \hat{N} & \rightarrow & \hat{G} & \rightarrow & G/N & \rightarrow & 1 \\
| & | & | & | & | & | \\
1 & \rightarrow & Z & \rightarrow & \hat{N} & \rightarrow & G & \rightarrow & G/N & \rightarrow & 1
\end{array}
\]

(3.11)

encapsulating the extendibility of the abstract kernel $(\hat{N}, G/N \rightarrow \text{Out}(\hat{N}))$ associated with the crossed module $\hat{N} \rightarrow G$ such that the cohomology class in $H^3(G/N, Z)$ (suitably interpreted) which the bottom row of (3.11) represents is zero. Local cocycle calculations establish [Nee07, Theorem 3.8 p. 251]. Perhaps consideration of crossed 2-fold extensions renders such calculations obsolete. Besides this paper, a ZBL search also finds [Cal51].

4. FROM THE TEICHMÜLLER COCYCLE TO CROSSED MODULES

The paper [Tei40] develops a group 3-cocycle (not spelled out in this language) over the Galois group (associated with the data there); the cohomology class thereof recovers the obstruction for the Brauer class of a central simple algebra, normal in the sense of Galois
theory, to be trivially normal in the sense that the Brauer class arises by extension of scalars from the fixed field. This is another instance of Turing’s theorem discussed in Section 3 above.

Eilenberg-Mac Lane [EM48] rework this approach in terms of ordinary group cohomology. Theorem 10.1 (p. 13) is this paper recovers the version of Turing’s theorem under discussion, as does the exactness at $H^2(K, L^*)$ of the exact sequence in [HS53] Section 5 p. 130]. Section 15 p. 19 ff of [EM48] also develops an interpretation in terms of abstract kernels, and the interpretation of Turing’s theorem in Section 3 above sheds new light on the remark in [Mac67, IV.11 p. 137] saying “The 3-dimensional cohomology groups of a group were first considered by Teichmüller [Tei40].”

A crossed 2-fold extension description yields the generalized “Teichmüller cocycle” map [Hue18a, (8.1) p. 58]; this map includes the map $\rho$ in [FW00, Theorem 3.4 (i)], given there by a cocycle description. The exactness at $XB(S, Q)$ of [Hue18a] (9.1) p. 64] can be seen as an instance of Turing’s theorem under the circumstances there, as can the exactness at $XB(T|S; G, Q)$ of the exact sequence [Hue18b] (16.1) p. 104]. The same kind of remark applies to the exact sequences in [FW00] Theorem 4.2 involving $QB_0(R, \Gamma)$ and $QB(R, \Gamma)$, constructed there by cocycle consideration. These exact sequences come down to that in [HS53] Section 5 p. 130] under the circumstances of that paper.

5. Lie algebra crossed modules

The axiom (3.1) makes perfect sense for Lie algebras, and the interpretation of the $(n+1)$th Lie algebra cohomology group in terms of crossed $n$-fold extensions ($n \geq 1$) is available. Also the abstract kernel concept extends to Lie algebras in an obvious manner, as does the equivalence between abstract kernels and crossed modules. The following quote from [Hoc54, p. 698], given there as ordinary text prose, not as a formal statement, spells out the Lie algebra analogue of Turing’s theorem discussed in Section 3 above: “With every $L$-kernel $M$, one can associate a certain 3-dimensional cohomology class for $L$ in the center of $M$, and the $L$-kernel is extendible if and only if this cohomology class is 0. This is the precise analogue of a result of Eilenberg-Mac Lane [EM47b], for the case of groups, and can easily be proved in the same way, mutatis mutandis.” The reader will notice that the sentence “the $L$-kernel is extendible if and only if this cohomology class is 0” is the corresponding analogue of Turing’s theorem.

The paper [Gol53] characterizes the third Lie algebra cohomology group of a Lie algebra in terms of its nonabelian extensions by cocycle identities which, according to the author’s claims, are analogous to the identities in Teichmüller’s theory, cf. Section 3 above. In the notation of [Gol53], the data in that paper fit into a crossed 2-fold extension

$$0 \rightarrow W \rightarrow U \rightarrow L^+ \rightarrow L \rightarrow 0$$  \hspace{1cm} (5.1)

of Lie algebras, and [Gol53] Theorem 4.9 p. 476] admits the following interpretation: There is an extension $U \rightarrow L^{++} \rightarrow L$ of Lie algebras together with an epimorphism $L^{++} \rightarrow L^+$ of Lie algebras rendering commutative the diagram

$$\begin{array}{cccccc}
0 & \rightarrow & W & \rightarrow & U & \rightarrow & L^+ & \rightarrow & L & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & U & \rightarrow & L^{++} & \rightarrow & L & \rightarrow & 0
\end{array}$$  \hspace{1cm} (5.2)

§In [Tei40] p. 148], the precise statement reads: “$\xi_{u, \mu, \nu}$ zerfällt dann und nur dann, wenn die Algebrenklasse von $A$ durch Erweiterung einer Algebrenklasse über $P$ entsteht.”
if and only if the class in $H^3(L,W)$ which (5.1) represents is zero. The reader will notice
in the MR review of [Gol53], the cocycle identities in this paper are analogous to those in
Teichmüller’s theory. The Lie algebra version of Turing’s theorem in terms of Lie algebra
abstract kernels is in [Hoc54], with a special emphasis on the restricted Lie algebra case,
which exhibits subtleties, as well as in [Mor53, Lemma 5 p. 177].

The interpretation of the third Lie algebra cohomology group in terms of crossed 2-fold
extensions (without usage of the crossed module terminology) likewise goes back to [Ger66]:
The category of Lie algebras is a category of interest in the sense of [Ger66], and an
admissible sequence is, then, precisely a crossed 2-fold extension of Lie algebras. Theorem 6
(3.) in [Ger66] p. 7] includes, for a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module $M$, a natural isomorphism
$E^3_C(\mathfrak{g}, M) \to H^3_C(\mathfrak{g}, M)$. Over a field or, more generally, when $\mathcal{C}$ is the category of Lie algebras
over a commutative ring $R$ with unit having underlying $R$-module projective, this comes
down to an isomorphism $E^3_C(\mathfrak{g}, M) \to H^3(\mathfrak{g}, M)$ onto the ordinary 3rd Lie algebra cohomology
group. (In the general case, the isomorphism $E^3_C(\mathfrak{g}, M) \to H^3_C(\mathfrak{g}, M)$ recovers the corresponding
relative Lie algebra cohomology group.) In the Lie algebra case, [Ger66, Theorem 4 p. 5]
is a Lie algebra version of Turing’s theorem.

As far as I know, [KL82] is the first paper to use the terminology “Lie algebra crossed
module”. The results in [Hue81a, Hue81b] extend to the Lie algebra case in a straightforard
manner. Hochschild’s wording at the end of the first paragraph of the present section applies
here.

Crossed Lie algebras occur as infinitesimal objects associated with 2-groups. For a semisim-
ple Lie algebra, a purely algebraic construction (i.e., not involving a loop Lie algebra) yields
a crossed 2-fold extension representing the infinitesimal version of E. Cartan’s 3-cohomology
class mentioned in Section 3. Such an extension represents the infinitesimal object as-
associated with the class represented by a crossed 2-fold extension of the kind (3.10), viz.
$S^1 \to \Omega G \to PG \to G$.

6. LIE-RINEHART ALGEBRAS VS LIE ALGEBROIDS

K. Mackenzie studied abstract kernels of Lie algebroids [Mac87, p. 221 ff]—his terminology
is *coupling*—and his Cor. 3.22 (p. 225) is a Lie algebroid version of Turing’s theorem discussed
in Section 2. In the setting of holomorphic Lie algebroids, [BMRT15, Theorem 3.2 (i)] is the
appropriate variant of Turing’s theorem. Crossed modules of Lie algebroids occur in [Mac05,
pp. 309, 332].

Let $R$ be a commutative ring and $A$ a commutative $R$-algebra. An $(R, A)$-Lie algebra
[Lin53] is an $R$-Lie algebra $(L, [\cdot, \cdot])$ together with an action of $L$ on $A$ by derivations and a
left $A$-module structure on $L$ that satisfy the two axioms

$$[\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta], \ a \in A, \ \alpha, \beta \in L,$$

$$\ (aa)b = a(\alpha(b)), \ a, b \in A, \ \alpha \in L.$$  

The basic example is the pair $(A, \text{Der}(A))$, with the obvious structure of mutual interaction.
The concept of an $(R, A)$-Lie algebra is the algebraic analogue of a Lie algebroid: Indeed, a
standard example is the pair $(C^\infty(M), \Gamma(\lambda))$ that consists of the smooth functions
$C^\infty(M)$ and smooth sections $\Gamma(\lambda)$ of a Lie algebroid $\lambda: E \to M$ on a smooth manifold $M$, with
the obvious structure of mutual interaction. However, there are $(R, A)$-Lie algebras that do not
arise from a Lie algebroid. For example, the $(R, A)$-Lie algebra structure on the $A$-module
of formal differentials of $A$ associated with a Poisson structure on $A$ \cite{Hue90} does not in general arise from a Lie algebroid. Other examples follow below. The $R$-algebra $A$ being fixed, $(R, A)$-Lie algebras constitute a category. When we let the algebra variable vary, we also obtain a category. For a good notion of morphism in this case see \cite{HM93}. To cope with that situation, in \cite{Hue90}, I introduced the terminology \textit{Lie-Rinehart algebra} for $(R, A)$-Lie algebras when the variable $A$ is allowed to vary. Unfortunately, in the subsequent literature, the two notions Lie-Rinehart algebra and $(R, A)$-Lie algebra became confused.

Lie-Rinehart algebras have played a major role in differential algebra and differential Galois theory for long, even though (it seems to me) the structure was not explicitly recognized. Indeed, a differential field $K$ with field of constants $k$ is tantamount to the $(k, K)$-Lie algebra $(K, \text{Der}_k(K))$. Suffice it to mention the following: Lie-Rinehart algebras occur in \cite{Hoc55} and \cite{Jac44} though without the name and yield a crucial tool in \cite{NW82} (the terminology there is $K[k]$-Lie algebra) for classifying the Lie algebras that arise as Lie algebras of differential formal groups of Ritt: There are two cases, finite-dimensional simple Lie algebras and Lie algebras of Cartan type (associated with pseudogroups of transformations—the concept of a pseudogroup of transformations evolved out of Lie’s ideas). In the latter case, the classification is somewhat delicate and leads to a mathematically exceedingly interesting theory involving linear compactness. In his MR review on \cite{NW82}, A. M. Vinogradov wrote: “The authors begin their paper with the sentence: ‘This is an attempt to understand the last four papers of J. Ritt.’ Their work is undoubtedly an important contribution to this area. But this paper itself requires similar ‘attempts to understand’. In the reviewer’s opinion, the use of the language of differential calculus in commutative algebras could be very helpful for this purpose.” The four papers under discussion are \cite{Rit38} \cite{Rit50a} \cite{Rit50c} \cite{Rit50b}. The Lie-Rinehart technology provides some such differential calculus.

A very large number of authors developed the idea of a Lie-Rinehart algebra, before it was so named, most of whom independently proposed their own terminology; see \cite{Hue04} p. 305 (incomplete list, compiled with the help of K. Mackenzie) as well as \cite{Mac95} p. 100, the terminology there being \textit{Lie pseudoalgebra}. This is also the terminology in \cite{Her53a, Her53b}; these describe the structure in a form which renders its generality clear. The Lie pseudoalgebra concept in these two papers is not equivalent to that of a Lie-Rinehart algebra, however, and there is no mention of the difference in the subsequent literature: For a field $K$, not necessarily commutative, \cite{Her53a} defines a $K$-Lie pseudoalgebra (pseudo-alg`ebre de Lie) to be a $K$-vector space $E$ together with a Lie bracket over the integers, subject to a variant \cite{Her53a, III} of (6.1) with $K$ substituted for $A$, spelled out in terms of the operation $K \times E \to E$ of scalar multiplication. There then results a $\mathbb{Z}$-linear map $E \to \text{Der}(K)$. For a proper skew field $K$, the commutator operation turns $K$ into a $K$-Lie pseudoalgebra, and an observation in \cite{Her53b} Section 4 says that, for a $K$-vector space $E$ and a $K$-linear map $\varphi: E \to K$, setting $u(\lambda) = \varphi(u)\lambda - \lambda\varphi(u)$ ($u \in E$, $\lambda \in K$) yields a $K$-Lie pseudoalgebra in such a way that $\varphi$ is a morphism of $K$-Lie pseudoalgebras and that every $K$-Lie pseudoalgebra arises in this manner. In particular, a $K$-Lie pseudoalgebra is then an ordinary Lie algebra over the center of $K$. For a commutative field $K$, \cite{Her53b} Section 5 contains the observation that a $K$-Lie pseudoalgebra $E$ of $K$-dimension at least equal to 2 necessarily satisfies the Lie-Rinehart axiom (6.2) and is hence an ordinary $(K, k)$-Lie algebra where $k \subseteq K$ refers to the fixed field. Nowadays there are researchers trying to develop the Lie-Rinehart concept relative to a not necessarily commutative algebra. Also, \cite{Nom54} (1.1) p. 35 isolates axiom (6.1) to explore connections on a Lie group and on a homogeneous space. I chose the terminology \textit{Lie-Rinehart algebra} in \cite{Hue90} for the following reason: The paper \cite{Rin63} goes beyond the evident formal
similarity of the Cartan-Chevalley-Eilenberg and de Rham complexes noted, e.g., in [Pal61] but presumably folk-lore at the time, establishes the Poincaré-Birkhoff-Witt theorem for these objects and thereby subsumes ordinary Lie algebra and de Rham cohomology under a single theory, that of derived functors. The paper [Rin63] paves, furthermore, the way for the subsequent interplay between Lie-Rinehart algebras and Lie algebroids in the literature.

The Lie algebroid terminology goes back to [Pra67] and Lie-Rinehart algebras occur there as Lie pseudoalgebras, with reference to earlier consideration of the structure. On p. 246 the reader finds a hint that the idea of a Lie algebroid underlies earlier work of Ehresmann, Libermann, and Rodrigues, and a remark on p. 101 of [Mac95] says that the Lie algebroid concept is implicit in Ehresmann’s work on higher order connections and prolongations of structures on manifolds at the early 1950s.

Lie-Rinehart algebras play a major role in a number of areas in mathematics and physics. See [Hue04] for an overview. Here are a few remarks: Lie-Rinehart algebras underly a program aimed at developing quantization in the presence of singularities, see [Hue11] and the literature there. The Schouten-Nijenhuis bracket makes sense for Lie-Rinehart algebras [Hue98a] and thereby explains various structures that have arisen in the literature, see also [Kos85], and so does the Frélicher-Nijenhuis bracket. The formalism of connection and curvature extends to Lie-Rinehart algebras [Hue90], see also [Kos86] (first published in 1965). Differential graded Lie-Rinehart algebras [Hue98b, Hue00] underly Hodge theory, the Dolbeault complex, the BRST-complex [Hue91], explain Kodaira-Spencer deformation theory, etc.

7. My scientific relationship with K. Mackenzie

In 1989 or so, I established contact with K. Mackenzie and invited him for a stay in Heidelberg. Also we met in England at various occasions and we intensely discussed mathematics by email. He had a great interest in work related to Lie-Rinehart algebras, see, e.g., [HM93], [Hue00], [Mac95]. We refereed each other’s work. Once I received a request to referee a book project. I wrote my review, sent it in with a negative recommendation, and also, at the editor’s request, suggested K. Mackenzie as a possible referee. Later I inquired and learnt from Kirill he had refereed that project in the first place, sent in a negative review, and suggested me as an alternate reviewer. The book project was realized, however. Once I talked to Anne Kostant (then Editorial Director, Springer Mathematics) about this story, and here laconic reaction was: “The book project had been refereed.”

I was saddened to hear we had lost Kirill.

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Appendix

Headings (1)-(6) below are a few observations related to [Mac79], (2)-(6) complementing the historical remarks in [BHS11]; in none of the papers mentioned in (2)-(5) does the crossed module terminology occur.
(1) The paper [Ger64] establishes a description of the cohomology group $H^{n+1}(Q, M)$ ($n \geq 2$) for a group $Q$ and a $Q$-module $M$ in terms of $n$-fold extensions of the kind

$$
0 \rightarrow M \rightarrow M_n \rightarrow \ldots \rightarrow M_2 \rightarrow G \rightarrow Q \rightarrow 1 \quad (7.1)
$$

having $M_2, \ldots, M_n$ ordinary $Q$-modules, that is, do not properly involve crossed modules. Such an $n$-fold extension is a crossed $n$-fold extension but not the most general one (and there is no mention of crossed modules in [Ger64]); cf. the “Note added in proof” in [Mac79].

(2) No structure equivalent to a crossed module does occur in [Wu78], at least not explicitly.

(3) The first satellite in [Rin69, Prop. 4.1 p. 313] involves, in the notation of this paper, the crossed 2-fold extension

$$
0 \rightarrow Z \rightarrow E \rightarrow B_0 \rightarrow B_1 \rightarrow 1. \quad (7.2)
$$

The construction of this satellite relies on the group case in [Ger66].

(4) In [LGM76, II.2 p. 131 ff],

$$
0 \rightarrow B \rightarrow K \rightarrow D \rightarrow G \rightarrow 1 \quad (7.3)
$$

is a crossed 2-fold extension. Here the authors elaborate on (the group case in) [Ger66, Rin69]. The authors’ aim is to study group cohomology in a variety of groups. Such a cohomology theory is not necessarily accessible in terms of cocycles. This is related to Gerstenhaber’s search in [Ger66, p. 3] for a cohomology theory that is “satisfactory for deformation theory” already hinted at in Section 3.

(5) Section 4 p. 60 ff of [Dus75], discusses objects equivalent to crossed $n$-fold extension; see in particular p. 71 l. -8/-7: “... classifies certain kinds of $n$-fold extensions of $X$ by the $X$-module II.” Such an $n$-fold extension arises as the Moore complex of a $K(\pi, n)$-torsor.

(6) In [Jon79] the interpretation of the third group cohomology group in terms of crossed 2-fold extensions leads to a structural result in the theory of von Neumann algebras.

(7) References [Bae38, EM42, EM47a, EM47b, Kur70, Kur72, Mac49] quote [Tur38] as does [BHS11, p. 437] with the remark “the use of identities among relations for discussing nonabelian extensions was given in [Tur38]”. Neither MR nor Zentralblatt find those quotes. Google scholar finds them except [Kur70, Kur72] and finds some others as well.

(8) Lie-Rinehart algebras are lurking behind $D$-modules.

(9) There is also a literature on crossed modules of Lie-Rinehart algebras.

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