Concepts of trace, determinant and inverse of Clifford algebra elements

D. S. Shirokov

Steklov Mathematical Institute
Gubkin St.8, 119991 Moscow, Russia

email: shirokov@mi.ras.ru

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Abstract

In our paper we consider the notion of determinant of Clifford algebra elements. We present some new formulas for determinant of Clifford algebra elements for the cases of dimension 4 and 5. Also we consider the notion of trace of Clifford algebra elements. We use the generalization of the Pauli’s theorem for 2 sets of elements that satisfy the main anticommutation conditions of Clifford algebra.

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1 Introduction

The notion of determinant of Clifford algebra elements was considered in \cite{3}. In our work we present some new formulas for determinant of Clifford algebra elements for the cases of dimension $n = 4$ and $5$. Also we consider the notion of trace of Clifford algebra elements. We use the generalization of the Pauli’s theorem for 2 sets of elements that satisfy the main anticommutation conditions of Clifford algebra.

After writing this paper author found the article \cite{4} on the subject that is close to the subject of this paper. In particular, the article \cite{4} contains the formulas that are similar to the formulas for the determinant in this paper. However, note that for the first time most of these formulas ($n = 1, 2, 3$) were introduced in \cite{3}.

2 Complex Clifford algebras

Let $p$ and $q$ be nonnegative integers such that $p + q = n \geq 1$. We consider complex Clifford algebra $\mathcal{C}(p, q)$. The construction of Clifford algebra $\mathcal{C}(p, q)$ is discussed in details in \cite{2}.

Generators $e^1, e^2, \ldots, e^n$ satisfy the following conditions

$$e^a e^b + e^b e^a = 2 \eta^{ab},$$

where $\eta = ||\eta^{ab}||$ is the diagonal matrix whose diagonal contains $p$ elements equal to $+1$ and $q$ elements equal to $-1$.

The elements

$$e^{a_1} \ldots e^{a_k} = e^{a_1 \ldots a_k}, \quad 1 \leq a_1 < \ldots < a_k \leq n, \quad k = 1, 2, \ldots n$$

together with the identity element $e$ form a basis of Clifford algebra $\mathcal{C}(p, q)$. The number of basis elements equals to $2^n$.

Any Clifford algebra element $U \in \mathcal{C}(p, q)$ can be written in the following form

$$U = u e + u_a e^a + \sum_{a_1 < a_2} u_{a_1 a_2} e^{a_1 a_2} + \ldots + u_{1 \ldots n} e^{1 \ldots n},$$ \hspace{1cm} (1)

where $u, u_a, u_{a_1 a_2}, \ldots u_{1 \ldots n}$ are complex constants.
We denote the vector subspaces spanned by the elements $e^{a_1\ldots a_k}$ enumerated by the ordered multi-indices of length $k$ by $\mathcal{C}_k(p,q)$. The elements of the subspace $\mathcal{C}_k(p,q)$ are denoted by $\bar{U}$ and called elements of rank $k$. We have

$$\mathcal{C}(p,q) = \bigoplus_{k=0}^{n} \mathcal{C}_k(p,q). \quad (2)$$

Clifford algebra $\mathcal{C}(p,q)$ is a superalgebra, so we have even and odd subspaces:

$$\mathcal{C}(p,q) = \mathcal{C}_{\text{Even}}(p,q) \oplus \mathcal{C}_{\text{Odd}}(p,q). \quad (3)$$

where

$$\mathcal{C}_{\text{Even}}(p,q) = \mathcal{C}_0(p,q) \oplus \mathcal{C}_2(p,q) \oplus \mathcal{C}_4(p,q) \oplus \ldots,$$

$$\mathcal{C}_{\text{Odd}}(p,q) = \mathcal{C}_1(p,q) \oplus \mathcal{C}_3(p,q) \oplus \mathcal{C}_5(p,q) \oplus \ldots$$

3 Operations of conjugation

Let denote complex conjugation of matrix by $\overline{A}$, transpose matrix by $A^T$, Hermitian conjugate matrix (composition of these 2 operations) by $A^\dagger$.

Now let define some operations on Clifford algebra elements.

**Complex conjugation.** Operation of complex conjugation $U \to \bar{U}$ acts in the following way

$$\bar{U} = \overline{\bar{u} e + \bar{u}_a e^a} + \sum_{a_1 < a_2} \overline{\bar{u}_{a_1 a_2}} e^{a_1 a_2} + \sum_{a_1 < a_2 < a_3} \overline{\bar{u}_{a_1 a_2 a_3}} e^{a_1 a_2 a_3} + \ldots \quad (4)$$

We have

$$e^a = e^a, \quad a = 1, \ldots, n, \quad \overline{U} = U, \quad (\overline{U V}) = \overline{U} \overline{V}, \quad (\overline{U + V}) = \overline{U} + \overline{V},$$

$$\overline{(\lambda U)} = \overline{\lambda} \overline{U}, \quad \forall U, V \in \mathcal{C}(p,q), \quad \lambda \in \mathbb{C}.$$ 

**Reverse.** Let define operation reverse for $U \in \mathcal{C}(p,q)$ in the following way

$$U^\sim = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \bar{U}.$$
We have

\[(e^a)^\sim = e^a, \quad a = 1, \ldots, n, \quad U^\sim = U, \quad (UV)^\sim = V^\sim U^\sim,\]

\[(U + V)^\sim = U^\sim + V^\sim, \quad (\lambda U)^\sim = \lambda U^\sim.\]

**Pseudo-Hermitian conjugation.** Let define Pseudo-Hermitian conjugation as composition of reverse and complex conjugation:

\[U^\dagger = \bar{U}^\sim.\]

We have

\[(e^a)^\dagger = e^a, \quad a = 1, \ldots, n, \quad U^{\dagger\dagger} = U, \quad (UV)^\dagger = V^\dagger U^\dagger,\]

\[(U + V)^\dagger = U^\dagger + V^\dagger, \quad (\lambda U)^\dagger = \lambda U^\dagger.\]

**Grade involution.** Let define operation of grade involution \(U \to U^\wedge\) in the following way

\[U^\wedge = \sum_{k=0}^{n} (-1)^k \frac{k}{2} U.\]

We have

\[(e^a)^\wedge = -e^a, \quad a = 1, \ldots, n, \quad U^{\wedge\wedge} = U, \quad (UV)^\wedge = U^\wedge V^\wedge,\]

\[(U + V)^\wedge = U^\wedge + V^\wedge, \quad (\lambda U)^\wedge = \lambda U^\wedge.\]

**Clifford conjugation.** Let define Clifford conjugation as composition of grade involution and reverse \(U \to U^{\wedge\sim}\):

\[U^{\wedge\sim} = \sum_{k=0}^{n} (-1)^{\frac{k(k+1)}{2}} \frac{k}{2} U.\]

We have

\[(e^a)^{\wedge\sim} = -e^a, \quad a = 1, \ldots, n, \quad U^{\wedge\sim\wedge\sim} = U, \quad (UV)^{\wedge\sim} = V^{\wedge\sim} U^{\wedge\sim},\]

\[(U + V)^{\wedge\sim} = U^{\wedge\sim} + V^{\wedge\sim}, \quad (\lambda U)^{\wedge\sim} = \lambda U^{\wedge\sim}.\]
**Hermitian conjugation.** In [2] we consider operation of Hermitian conjugation. We have the following formulas for these operation:

\[ U^\dagger = (e^1 \ldots p)^{-1} U^\dagger e^1 \ldots p, \quad \text{if } p \text{ - odd}, \]

\[ U^\dagger = (e^1 \ldots p)^{-1} U^\dagger e^1 \ldots p, \quad \text{if } p \text{ - even}, \] \hspace{1cm} (5)

\[ U^\dagger = (e^{p+1} \ldots n)^{-1} U^\dagger e^{p+1} \ldots n, \quad \text{if } q \text{ - even}, \]

\[ U^\dagger = (e^{p+1} \ldots n)^{-1} U^\dagger e^{p+1} \ldots n, \quad \text{if } q \text{ - odd}, \]

We have

\[(e^a)^\dagger = (e^a)^{-1}, \quad a = 1, \ldots, n, \quad U^\dagger U^\dagger = U, \quad (UV)^\dagger = V^\dagger U^\dagger, \]

\[(U + V)^\dagger = U^\dagger + V^\dagger, \quad (\lambda U)^\dagger = \bar{\bar{\lambda}} U^\dagger. \]

### 4 Matrix representations of Clifford algebra elements, recurrent method.

Complex Clifford algebras \(\mathcal{C}(p, q)\) of dimension \(n\) and different signatures \((p, q), p + q = n\) are isomorphic. Clifford algebras \(\mathcal{C}(p, q)\) are isomorphic to the matrix algebras of complex matrices. In the case of even \(n\) these matrices are of order \(2^\frac{n}{2}\). In the case of odd \(n\) these matrices are block diagonal of order \(2^\frac{n+1}{2}\) with 2 blocks of order \(2^\frac{n-1}{2}\).

Consider the following matrix representations of Clifford algebra elements.

Identity element \(e\) of Clifford algebra \(\mathcal{C}(p, q)\) maps to identity matrix of corresponding order: \(e \rightarrow 1\).

For \(\mathcal{C}(1, 0)\) element \(e^1\) maps to the following matrix

\[ e^1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

For \(\mathcal{C}(2, 0)\) we have

\[ e^1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^2 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Further, suppose we have a matrix representation for \(\mathcal{C}(2k, 0), n = 2k:\)

\[ e^1, \ldots, e^n \rightarrow \gamma^1, \ldots, \gamma^n. \]
Then, for Clifford algebra $C_\ell(2k + 1, 0)$ we have

$$ e^a \rightarrow \begin{pmatrix} \gamma^a & 0 \\ 0 & -\gamma^a \end{pmatrix}, \quad a = 1, \ldots, n, \quad e^{n+1} \rightarrow \begin{pmatrix} i^k \gamma^1 \cdots \gamma^n & 0 \\ 0 & -i^k \gamma^1 \cdots \gamma^n \end{pmatrix}. $$

For Clifford algebra $C_\ell(2k + 2, 0)$ we have the same matrices for $e^a, a = 1, \ldots, n + 1$ as in the previous case $n = 2k + 1$ and for $e^{n+2}$ we have

$$ e^{n+2} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. $$

So, we have matrix representation for all Clifford algebras $C_\ell(n, 0)$. In the cases of other signatures elements $e^a, a > p$ maps to the same matrices as in signature $(n, 0)$ but with multiplication by imaginary unit $i$.

For example, we have the following matrix representations for Clifford algebras $C_\ell(3, 0), C_\ell(4, 0)$ and $C_\ell(1, 3)$.

$C_\ell(3, 0)$:

$$ e^1 \rightarrow \gamma^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^2 \rightarrow \gamma^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, $$

$$ e^3 \rightarrow \gamma^3 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. $$

$C_\ell(4, 0)$:

$$ e^1 \rightarrow \gamma^1, \quad e^2 \rightarrow \gamma^2, \quad e^3 \rightarrow \gamma^3, \quad e^4 \rightarrow \gamma^4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. $$

$C_\ell(1, 3)$:

$$ e^1 \rightarrow \gamma^1, \quad e^2 \rightarrow i\gamma^2, \quad e^3 \rightarrow i\gamma^3, \quad e^4 \rightarrow i\gamma^4. $$
5 Operation of trace of Clifford algebra elements

Consider complex Clifford algebra $\mathcal{C}(p, q)$ and introduce the operation of trace of Clifford algebra element $U \in \mathcal{C}(p, q)$ as the following operation of projection onto subspace $\mathcal{C}_0(p, q)$:

$$\text{Tr}(U) = \langle U \rangle_0|_{e \rightarrow 1}. \quad (6)$$

For arbitrary element $U \in \mathcal{C}(p, q)$ in the form (1) we have

$$\text{Tr}(ue + u_a e^a + \ldots) = u.$$

**Theorem 1.** Operation trace (6) of Clifford algebra element $U \in \mathcal{C}(p, q)$ has the following properties:

- **linearity:**
  $$\text{Tr}(U + V) = \text{Tr}(U) + \text{Tr}(V), \quad \text{Tr}(\alpha U) = \alpha \text{Tr}(U)$$
  $$\forall U, V \in \mathcal{C}(p, q), \quad \forall \alpha \in \mathbb{C},$$

- **cyclic recurrence:**
  $$\text{Tr}(UV) = \text{Tr}(VU), \quad \text{Tr}(UVW) = \text{Tr}(VWU) = \text{Tr}(WUV)$$
  $$\forall U, V, W \in \mathcal{C}(p, q),$$

  but, in general:
  $$\text{Tr}(UVW) \neq \text{Tr}(UWV).$$

- **invariance under similarity transformation:**
  $$\text{Tr}(U^{-1}VU) = \text{Tr}(V) \quad \forall V \in \mathcal{C}(p, q), U \in \mathcal{C}^\times(p, q),$$

  where $\mathcal{C}^\times(p, q)$ is the set of all invertible Clifford algebra elements.

- **invariance under conjugations:**
  $$\text{Tr}(U) = \text{Tr}(U^\dagger) = \text{Tr}(U^\sim) = \text{Tr}(U_{\sim}) = \text{Tr}(U^{\perp}) = \text{Tr}(U^{\parallel}).$$
Proof. Linearity follows from the definition \((6)\).

We have
\[
\text{Tr}(UV) = \text{Tr}(VU)
\]
because \(\text{Tr}([U, V]) = 0\) for \(k, l = 0, \ldots, n\) (see \([2]\)). Then we obtain cyclic recurrence for 3 elements. We obtain invariance under similarity transformation as a simple consequence of cyclic recurrence. Last properties follow from properties of conjugations. ■

There is a relation between operation trace \(\text{Tr}\) of Clifford algebra element \(U \in \mathcal{A}(p, q)\) and operation trace \(\text{tr}\) of quadratic matrix. To obtain this relation, at first, we will prove the following statement.

Lemma 1. Consider recurrent matrix representation of Clifford algebra \(\mathcal{A}(p, q)\) (see above). For this representation \(U \rightarrow \underline{U}\) we have
\[
\text{tr}(\underline{U}) = 2^{\left\lceil \frac{n+1}{2} \right\rceil} \text{Tr}(U), \quad \text{tr}(\underline{U}^\wedge) = \text{tr}(\underline{U}).
\]

Proof. Coefficient \(2^{\left\lceil \frac{n+1}{2} \right\rceil}\) equals to the order of corresponding matrices. It is not difficult to see that trace of almost all matrices that correspond to basis elements equals to zero
\[
\text{tr}(\underline{e}^A) = 0, \quad \text{where } A - \text{any multi-index except empty.}
\]
The only exception is identity element \(e\), which corresponds to the identity matrix. In this case we have \(\text{tr}(e) = 2^{\left\lceil \frac{n+1}{2} \right\rceil}\). Further we use linearity of trace and obtain
\[
\text{tr}(\underline{U}) = 2^{\left\lceil \frac{n+1}{2} \right\rceil} u = 2^{\left\lceil \frac{n+1}{2} \right\rceil} \text{Tr}(U).
\]
The second property is a simple consequence of the first property, because
\[
\text{tr}(\underline{U}^\wedge) = 2^{\left\lceil \frac{n+1}{2} \right\rceil} \text{Tr}(U^\wedge) = 2^{\left\lceil \frac{n+1}{2} \right\rceil} \text{Tr}(U) = \text{tr}(\underline{U}).
\]
■

Theorem 2. Consider complex Clifford algebra \(\mathcal{C}(p, q)\) and operation trace \(\text{Tr}\). Then
\[
\text{Tr}(U) = \frac{1}{2^{\left\lceil \frac{n+1}{2} \right\rceil}} \text{tr}(\gamma(U)), \quad (7)
\]
where \(\gamma(U)\) - any matrix representation of Clifford algebra \(\mathcal{A}(p, q)\) of minimal dimension. Moreover, this definition of trace \((7)\) is equivalent to the definition \((6)\). New definition is well-defined because it doesn’t depend on the choice of matrix representation.
Proof. This property proved in the previous statement for the recurrent matrix representation. Let we have besides recurrent matrix representation

\[ U = U|_{e^a \rightarrow \gamma^a} \]

another matrix representation

\[ U = U|_{e^a \rightarrow \beta^a}. \]

Then, by Pauli’s theorem in Clifford algebra of even dimension \( n \) there exists matrix \( T \) such that

\[ \beta^a = T^{-1}\gamma^a T, \quad a = 1, \ldots, n. \]

Then, we have

\[ U = T^{-1}UT \]

and

\[ \text{tr}(U) = \text{tr}(T^{-1}UT) = \text{tr}(U). \]

In the case of odd \( n \) we can have also another case (by Pauli’s theorem), when two sets of matrices relate in the following way

\[ \beta^a = -T^{-1}\gamma^a T, \quad a = 1, \ldots, n. \]

In this case we have

\[ U = T^{-1}U^\wedge T. \]

From \( \text{tr}(U^\wedge) = \text{tr}(U) \) (see Lemma 1) we obtain

\[ \text{tr}(U) = \text{tr}(T^{-1}U^\wedge T) = \text{tr}(U^\wedge) = \text{tr}(U). \]

\[ \blacksquare \]

6 Determinant of Clifford algebra elements

Determinant of Clifford algebra element \( U \in \mathcal{C}(p, q) \) is a complex number

\[ \text{Det}U = \det(U), \quad (8) \]

which is a determinant of any matrix representation \( U \) of minimal dimension.

Now we want to show that this definition is well-defined. Let prove the following Lemma.
Lemma 2. Consider the recurrent matrix representation (see above) of Clifford algebra $\mathcal{C}l(p, q)$. For this representation $U \rightarrow U^\wedge$ we have
\[
\det(U^\wedge) = \det(U).
\]

Proof. In the case of Clifford algebra of even dimension $n$ we have
\[
U^\wedge = (e^{1\ldots n})^{-1}Ue^{1\ldots n}.
\]
So, we obtain
\[
det(U^\wedge) = det((e^{1\ldots n})^{-1}Ue^{1\ldots n}) = det((e^{1\ldots n})^{-1}) det(U) det(e^{1\ldots n}) = det(U).
\]
In the case of Clifford algebra of odd dimension generators maps to the block diagonal matrices and blocks are identical up to the sign:
\[
e^a \rightarrow \begin{pmatrix}
\gamma^a & 0 \\
0 & -\gamma^a
\end{pmatrix}.
\]
Then for elements of the rank 2 we obtain
\[
e^{ab} \rightarrow \begin{pmatrix}
\gamma^a \gamma^b & 0 \\
0 & \gamma^a \gamma^b
\end{pmatrix}.
\]
It is not difficult to see that even part $U_{\text{Even}}$ of arbitrary element $U = U_{\text{Even}} + U_{\text{Odd}}$ maps to the matrix with identical blocks, and odd part $U_{\text{Odd}}$ of the element $U$ maps to the matrix with the blocks differing in sign:
\[
U_{\text{Even}} \rightarrow \begin{pmatrix}
A & 0 \\
0 & A
\end{pmatrix}, \quad U_{\text{Odd}} \rightarrow \begin{pmatrix}
B & 0 \\
0 & -B
\end{pmatrix}.
\]
Then we have
\[
U \rightarrow \begin{pmatrix}
A + B & 0 \\
0 & A - B
\end{pmatrix}, \quad U^\wedge \rightarrow \begin{pmatrix}
A - B & 0 \\
0 & A + B
\end{pmatrix}
\]
and
\[
det(U) = (A - B)(A + B) = det(U^\wedge).
\]

Theorem 3. Definition (8) is well-defined, i.e. it doesn’t depend on the matrix representation.
Proof. Consider the recurrent matrix representation

\[ U = U|_{e^a \rightarrow \gamma^a}. \]

The statement for this representation proved in the previous lemma. Let we have another matrix representation

\[ \underline{U} = U|_{e^a \rightarrow \beta^a}. \]

Then, by Pauli’s theorem in Clifford algebra of even dimension \( n \) there exists a matrix \( T \) such that

\[ \beta^a = T^{-1}\gamma^a T, \quad a = 1, \ldots, n. \]

Then we have

\[ \underline{U} = T^{-1}U^T \]

and obtain

\[ \det(\underline{U}) = \det(T^{-1}U^T) = \det(T^{-1})\det(U)\det(T) = \det(U). \]

In the case of odd \( n \), by Pauli’s theorem we also have another case, where 2 sets of matrices relate in the following way

\[ \beta^a = -T^{-1}\gamma^a T, \quad a = 1, \ldots, n. \]

In this case we have

\[ \underline{U} = T^{-1}U^\perp T. \]

From \( \det(U^\perp) = \det(U) \) (see Lemma [2]) we obtain

\[ \det(\underline{U}) = \det(T^{-1}U^\perp T) = \det(U^\perp) = \det(U). \]

\[ \blacksquare \]

Let formulate some properties of operation determinant of Clifford algebra element.

**Theorem 4.** Operation determinant \( \Det \) of Clifford algebra element \( U \in \mathcal{A}(p,q) \) has the following properties

\[ \Det(UV) = \Det(U)\Det(V), \quad \Det(\alpha U) = \alpha^{\frac{n+1}{2}}\Det(U) \quad (9) \]

\[ \forall U, V \in \mathcal{A}(p,q), \quad \forall \alpha \in \mathbb{C}. \]
• Arbitrary element $U \in \mathcal{C}(p, q)$ is invertible if and only if $\text{Det}U \neq 0$.

• For any invertible element $U \in \mathcal{C}(p, q)$

\[
\text{Det}(U^{-1}) = (\text{Det}U)^{-1}. \tag{10}
\]

• Invariance under similarity transformation:

\[
\text{Det}(U^{-1}VU) = \text{Det}(V) \quad \forall V \in \mathcal{C}(p, q), \quad U \in \mathcal{C}^\times(p, q),
\]

where $\mathcal{C}^\times(p, q)$ is set of all invertible Clifford algebra elements.

• Invariance under conjugations:

\[
\text{Det}(U) = \text{Det}(U^\dagger) = \text{Det}(U^\sim) = \text{Det}(U^\ invit) = \text{Det}(U^\sym).
\]

**Proof.** The first 4 properties are simple and follow from the definition of determinant (8).

In lemma 2 we have the property $\text{Det}(U) = \text{Det}(U^\dagger)$ for the recurrent matrix representation. But it is also valid for other matrix representations because of independence on the choice of representations (see Theorem 3).

It is known that operation $\sim$ relates to the operation of matrix transpose (as similarity transformation) and also we have $\text{det}(U) = \text{det}(U^T)$. Analogously we can consider another operations of conjugation. $
$  

Definition (8) of determinant of Clifford algebra element $U \in \mathcal{C}(p, q)$ is connected with its matrix representation. We have shown that this definition doesn’t depend on matrix representation. So, determinant is a function of complex coefficients $u_{a_1...a_k}$ located before basis elements $e^{a_1...a_k}$ in (11). In the cases of small dimensions $n \leq 5$ we give expressions for determinant of Clifford algebra elements that doesn’t relate to the matrix representation.

Now we need also 2 another operations of conjugations $\nabla, \Delta$:

\[
(U + U + U + U + U)^\nabla = U + U + U + U - U, \quad n = 4,
\]

\[
(U + U + U + U + U + U)^\nabla = U + U + U + U - U - U, \quad n = 5,
\]

\[
(U + U + U + U + U)^\nabla = U + U + U + U + U - U, \quad n = 5.
\]
Theorem 5. We have the following formulas for the determinant of Clifford algebra element $U \in \mathcal{Cl}(p, q)$:

$$
\text{Det } U = \begin{cases} 
U, & n = 0; \\
UU^\wedge, & n = 1; \\
UU\sim^\wedge, & n = 2; \\
UU\sim U^\sim U^\sim, & n = 3; \\
UU\sim(U^\sim U^\sim)^\bigtriangledown = UU\sim(U^\sim U^\sim)^\bigtriangledown, & n = 4; \\
UU\sim(U^\sim U^\sim)^\bigtriangledown(UU\sim(U^\sim U^\sim)^\bigtriangledown)^\bigtriangleup, & n = 5.
\end{cases}
$$

(11)

Note, that these expressions are Clifford algebra elements of the rank 0. In this case we identify them with the constants: $ue \equiv u$.

Proof. The proof is by direct calculation. ■.

Note, that properties (9) and (10) for small dimensions also can be proved with the formulas from Theorem 5. For example, in the case $n = 3$ we have

$$
\text{Det}(UV) = UV(\text{UV})^\sim(UV)^\sim(\text{UV})^\sim = UVV^\sim \text{U}^\sim \text{U}^\sim V^\sim V^\sim U^\sim =
$$

$$
= UU\sim \text{U}^\sim \text{U}^\sim VV^\sim \text{V}^\sim \text{V}^\sim = \text{Det}(U)\text{Det}(V)
$$

We used the fact that $VV^\sim$ and $V^\sim V^\sim = (VV^\sim)^\sim$ are in Clifford algebra center $\mathcal{C}_0(p, q) \oplus \mathcal{C}_3(p, q)$ and commute with all elements.

Theorem 5 give us explicit formulas for inverse in $\mathcal{Cl}(p, q)$. We have the following theorem.

Theorem 6. Let $U$ be invertible element of Clifford algebra $\mathcal{Cl}(p, q)$. Then we have the following expressions for $U^{-1}$:

$$
(U)^{-1} = \begin{cases} 
\hat{u}, & n = 0; \\
\frac{U^\wedge}{UU^\sim}, & n = 1; \\
\frac{U^\sim^\wedge}{UU\sim^\wedge}, & n = 2; \\
\frac{U^\sim U^\sim U^\sim}{UU\sim U^\sim U^\sim}, & n = 3; \\
\frac{U^\sim(U^\sim U^\sim)^\bigtriangledown}{UU(U^\sim U^\sim)^\bigtriangledown}, & n = 4; \\
\frac{U^\sim(U^\sim U^\sim)^\bigtriangledown(UU(U^\sim U^\sim)^\bigtriangledown)^\bigtriangleup}{UU(U^\sim U^\sim)^\bigtriangledown(UU(U^\sim U^\sim)^\bigtriangledown)^\bigtriangleup}, & n = 5.
\end{cases}
$$

(12)
Note, that in denominators we have Clifford algebra elements of the rank 0. We identify them with the constants: $ue \equiv u$.

**Proof.** Statement follows from Theorem 5. ■

Note, that formulas for determinant in Theorem 5 are not unique. For example, in the case of $n = 4$ we can use the following formulas, but only for even and odd Clifford algebra elements $U \in \mathcal{C}_{\text{Even}}(p, q) \cup \mathcal{C}_{\text{Odd}}(p, q)$.

Consider operation $+$, that acts on the even elements such that it changes the sign before the basis elements that anticommutes with $e^1$. For example, elements $e, e^{23}, e^{24}, e^{34}$ maps under $+$ into themselves, and elements $e^{12}, e^{13}, e^{14}, e^{1234}$ change the sign.

**Theorem 7.** Let $U \in \mathcal{C}_{\text{Even}}(p, q), n = p + q = 4$. Then

$$\det U = UU^\sim U^{\sim +} U^+. \quad (13)$$

Let $U \in \mathcal{C}_{\text{Odd}}(p, q), n = p + q = 4$. Then

$$\det U = UU^\sim U^\sim U. \quad (14)$$

**Proof.** The proof is by direct calculation. ■

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