The action and the physical scale of field theory

Yuri Vladimirovich Gusev

Lebedev Research Center in Physics,
Leninsky Prospekt 53, str. 11,
Moscow 119991, Russia

Email: yuri.v.gussev@gmail.com

(Dated: June 1, 2016)

Abstract

The action of the field theory is a generating functional for the equations of physics. It is introduced axiomatically by the kernel of the evolution equation. Mathematical consistency requires the lower limit of the defining proper time integral have an arbitrary positive value. This limit introduces into the dimensionless field theory the physical scale (the characteristic length), which relates geometrical orders of the action. The action is finite in all orders and nonlocal starting from the second order. Its two lowest (local) orders correspond to the cosmological constant term and the Einstein-Hilbert action of gravitation that confirm the Sakharov-DeWitt mechanism of the induced gravity. No absolute values of physical quantities are special in the field theory, therefore, the Planck scale (the Planck length etc) has no physical significance, as elucidated by the New SI of physical units. It is shown that the dimensional regularization technique is an ill-defined procedure, while regularization and renormalization are not needed for phenomenological physical theories.
I. GEOMETRICAL FORMALISM FOR THE FIELD THEORY

Certainly P.A.M. Dirac was not the first scientist to use the mathematical reasoning for building a physical theory, when he worked on quantum mechanics. Nevertheless, he was one of the first to explain how the axiomatic method for theory starts with a branch of mathematics, which is then developed in the simplest form, as far as possible before making comparison with experiments, as those maybe unavailable or erroneous. Quantum field theory developed by this method is usually associated with the constructive (axiomatic) quantum field theory. It employs the mathematical language of probability and statistics as well as the physical language of particles. However, if the aim is not describing properties of individual particles, e.g. observed in the scattering experiments, but deriving the equations of physics, quantum field theory can be built entirely by the means of geometry and field theory. As a result of such development, the field theory has lost its features pertinent to quantum field theory and become a part of geometric analysis. Furthermore, it has become meaningless to separate physics into classical and quantum domains because the classical actions of gravity and electromagnetism are derived together with their modifications, which were formerly considered as their ‘quantum corrections’.

The technical subject of the present paper is the geometrical (used to be called Schwinger-DeWitt) formalism for quantum field theory. The geometrical formalism is derived with help of the evolution equation on a manifold with the metric and the vector (gauge field) connection. Its main functional, the covariant action, is expressed in terms of geometrical characteristics of the base manifold and the field strengths (tensors of phenomenological fields). The covariant perturbation theory was proposed in, developed in a series of papers and later used to derive a theory for the study of vacuum radiation. This mathematical theory naturally evolved to its present form, which does not need the notion of quantization, if the action is introduced axiomatically. The procedure of ‘quantization’ is not even defined in this field theory, but it is not needed, because specific particles (quanta) are not studied. Some form of quantum theory might still be needed to study discrete physical events. Neither the Planck constant, nor the Planck length can serve as small parameters in the field theory, since the former one has a different dimensionality and the latter one is an arbitrary parameter. Then, we are forced to abandon the view on the ‘one-loop covariant effective action’ as a quantum correction. Its form and physical content
compel us to affirm that this ‘effective action’ is the universal action of physical fields.

The debate about the scales in physics has been going on for a long time. However, it is usually limited to the discussion of very large (small) physical parameters (constants) and possible mechanisms for their generation. However, the true problem of theoretical physics is: Where does the physical scale come from, if the field theory is dimensionless? We suggest there is only one way to gain the characteristic length, which is the only physical quantity in geometry: the axiomatic principle of the action.

The present paper is based on the analysis of dimensionality of functionals of physical fields, in relation to the spacetime dimension. In Appendix A we clarify that the Planck values are not physically significant because they are arbitrary physical values. Therefore, the Planck length cannot serve as a physical scale. In Appendix B we discard the technique of dimensional regularization as erroneous. Phenomenological theories cannot contain divergences, thus, regularization and renormalization are not even needed. Then we revisit the covariant perturbation theory and find that if defined axiomatically the covariant effective action can be understood as the universal action of physical fields. From this definition the unique length scale emerges throughout the action. Two lowest order in the curvature terms of the evolution kernel (‘the trace of the heat kernel’) are also present in it. These are the Einstein-Hilbert gravity action and the cosmological constant term that appear from the heat kernel, as was proposed in quantum field theory independently by A.D. Sakharov and B.S. DeWitt fifty years ago. After these corrections, the action method may have many more applications in physics than its initial goal of the black hole physics and the gauge field theory.

II. EVOLUTION KERNEL IN THE COVARIANT PERTURBATION THEORY

Let us begin with standard notations and definitions [5, 9, 10]. The spacetime (or rather space) has the Euclidean metric signature [15] and the dimension $D$. Most physically important field theories belong to the class determined by the generic differential operator [5],

$$
\hat{F}(\nabla) = \Box \hat{1} + \hat{P} - \frac{1}{6} R \hat{1}.
$$  \hspace{1cm} (1)

The Laplace-Beltrami operator (Laplacian) is constructed of the covariant derivatives,

$$
\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu,
$$  \hspace{1cm} (2)
whose indexes are contracted by the matrix, $g^{\mu \nu}$, the inverse of which is the metric of the Riemannian manifold (for the details and conditions, see [5]). The potential term $\hat{P}$ in (1) can be an arbitrary local function of (mean) fields. The 'hat' notation stands for the indexes of internal degrees of freedom, i.e. non-Abelian field structures.

The covariant derivatives in (2) contain both metric and gauge field connections, which are characterized by the commutator curvature,

$$ (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \varphi = \hat{R}_{\mu \nu} \varphi, $$

where the field $\varphi$ can have arbitrary spin-tensor properties. The field tensor $\hat{R}_{\mu \nu}$ for the Abelian gauge fields is proportional to the Maxwell’s electromagnetic tensor [13, 14, 16]. The commutator curvature $\hat{R}_{\mu \nu}$ is one type of the field strength tensors, the Ricci tensor $R_{\mu \nu}$ and the potential $\hat{P}$ are two others; the full set of these curvatures is denoted as,

$$ \mathfrak{R} \equiv (R_{\mu \nu}, \hat{R}_{\mu \nu}, \hat{P}). $$

We seek the kernel, $\hat{K}(s|x, x')$, of the evolution equation (commonly called the heat equation) [3, 4, 10],

$$ \frac{\partial}{\partial s} \hat{K}(s|x, x') = \hat{F}(\nabla x) \hat{K}(s|x, x'), $$

with the initial condition,

$$ \hat{K}(s|x, x') = \delta(x, x'), \sigma(x, x')/s \gg 1, $$

where $s$ is a parameter with the dimensionality $m^{-2}$ called the proper time, and $\sigma(x, x')$ is the Ruse-Synge world function [18, 19]. In the Cartesian coordinates, it is half a square of the geodesic distance between spacetime points $x$ and $x'$ [4]. The meaning of the initial condition (6) of the evolution equation (5) is different from the previously accepted [10, 20]. The proper time is a dimensionful parameter, therefore, the asymptotics, $s \to 0$, is not mathematically defined. It can only be compared with the other available scale, the length defined by the world function.

Indeed, the evolution equation (5) with the initial condition (6) gives the zeroth order of the heat kernel as the covariant delta function [4, 10],

$$ \hat{K}_0(s|x, x') = \frac{g^{1/4}(x)g^{1/4}(x')}{(4\pi s)^{D/2}} \exp \left( -\frac{\sigma(x, x')}{2s} \right) \delta_0(x, x'), $$

where $\delta_0(x, x')$ is the Dirac delta function.
where \( \hat{a}_0(x, x') \) is the propagator of parallel transport \cite{3, 5, 10}. The delta function in the \( D \)-dimensional spacetime \cite{6} does not exist for \( s = 0 \), while the asymptotics, \( \sigma(x, x')/s \to \infty \), explicitly forbids the zero value of the proper time. In the heat kernel \( \hat{K}(s|x, x') \), the positive proper time \( s \) is a parameter external to the spacetime variables, and the evolution equation \( \hat{K}(s|x, x') \) is different from the similarly looking equations of diffusion and heat propagation \cite{21}.

In order to obtain the action, we only need the functional trace of the heat kernel,

\[
\text{Tr} K(s) = \int d^Dx \text{tr} \hat{K}(s|x, x),
\]

which, beside the matrix trace over the field indexes, \( \text{tr} \), assumes the coincidence of spacetime points and the integral over the whole domain of spacetime, \( \mathbb{R}^D \). The metric’s determinant, \( g^{1/2}(x) \), is included in \( K(s|x, x) \). The functional trace of the heat kernel, \( \text{Tr} K(s) \), is a dimensionless functional, in contrast to the heat kernel, \( K(s|x, x) \), whose dimensionality coincides with the delta-function’s and depends on the spacetime dimension, \( D \).

The algorithms of the covariant perturbation theory (CPT) developed in \cite{10} were used for computing the heat kernel. The CPT is the covariant expansion of the heat kernel, in asymptotically flat curved spacetime, in orders of the curvatures that are expressed as nonlocal tensor invariants \cite{10, 22}. The CPT heat kernel contains an infinite number of covariant derivatives acting on the curvatures, thus, it is a nonlocal expression \cite{10, 23}. The zeroth and first orders of the trace of the heat kernel are local expressions \cite{10}. Starting from the second order, the trace of the heat kernel is nonlocal, and its full form, up to the third order in the curvatures (for \( D \leq 5 \)) is \cite{12, 22},

\[
\text{Tr} K(s) = \frac{1}{(4\pi s)^{D/2}} \int d^Dx g^{1/2}(x) \text{tr} \left\{ \hat{1} + s\hat{P} + s^2 \sum_{i=1}^{5} f_i(s, \Box_1, \Box_2) \mathcal{R}_1 \mathcal{R}_2(i) \\
+ s^3 \sum_{i=1}^{29} F_i(s, \Box_1, \Box_2, \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i) + O[\mathcal{R}^4] \right\}.
\]

The five structures quadratic in the curvatures were derived in \cite{10}, they are reviewed below. The third order in the curvatures contains 29 cubic structures that were computed and studied in \cite{12, 22, 23}. The CPT calculations are carried out with the accuracy \( O[\mathcal{R}^n] \), so that the highest computed order contains terms of up to the \((n-1)\)-order in the curvatures \textit{explicitly}. Therefore, the validity of the covariant perturbation theory obeys the condition,

\[
\nabla \nabla \mathcal{R} \gg \mathcal{R}^2.
\]
Only the denominator of the heat kernel’s prefactor $9$ depends explicitly on the spacetime dimension, which greatly simplifies the field theory calculations.

The explicit expressions for the form factors of $9$ can be found in $10, 12$. The form factors, $f_i$, as well as the third order ones, $F_i$, are analytic functions of the dimensionless operator-valued variables,

$$\xi_i = -s \Box_i.$$  

They act on tensors of the nonlocal invariants, and the index of the Laplacian in (11) indicates the local curvature it is acting on, i.e., $\Box_2 R_1 \hat{P}_2 \equiv R(x)(\Box \hat{P}(x))$. The expressions resulting from these operations are taken at the coincident space time point, $x$, which is a variable of the spacetime integral $9$. The heat kernel with the coincident points, $\hat{K}(s|x, x)$, up to the second order in the curvatures, is similar in the structure to $9$ and can be derived from it $20$. All the second order form factors in $9$ are expressed via the basic form factor,

$$f(-s \Box) \equiv \int_0^1 d\alpha \exp(-\alpha(1 - \alpha)(-s \Box)).$$  

Two form factors with subtractions are given by the integrals,

$$f(-s \Box) - 1 = \int_0^1 d\alpha_1 \alpha_1 (1 - \alpha_1) \int_0^1 d\alpha_2 \exp\left(-\alpha_2 \alpha_1 (1 - \alpha_1) (-s \Box)\right),$$  

$$f(-s \Box) - 1 - \frac{1}{6}(-s \Box)^2 = \int_0^1 d\alpha_1 \alpha_1^2 (1 - \alpha_1)^2 \int_0^1 d\alpha_2 \alpha_2 \int_0^1 d\alpha_3 \exp\left(-\alpha_3 \alpha_2 \alpha_1 (1 - \alpha_1) (-s \Box)\right).$$

Let us now use the method of $24$ and replace the tensor basis for $\text{Tr}K(s)$ of $10$ with the Ricci tensor by the basis based on the Weyl tensor, $C_{\alpha \mu \beta \nu}$ (from this point all expressions are in four dimensions). Namely, a new tensor quantity was defined,

$$C_{\mu \nu} \equiv \frac{2}{\Box} \nabla^\beta \nabla^\alpha C_{\alpha \mu \beta \nu},$$

which is related to the Ricci tensor, $R_{\mu \nu}$, via the perturbative expression $24$,

$$R_{\mu \nu} = C_{\mu \nu} + \frac{1}{3} \nabla_\mu \nabla_\nu \frac{1}{\Box} R + \frac{1}{6} g_{\mu \nu} R + O(\mathcal{R}^2).$$

Thus, the new basis differs from the old one $10$ only by the first term,

$$\mathcal{R}_1 \mathcal{R}_2(1) = C_{1 \mu \nu} C_{2}^{\mu \nu} \hat{1},$$

$$\mathcal{R}_1 \mathcal{R}_2(2) = R_1 R_2 \hat{1},$$

6
\[ \mathcal{R}_1 \mathcal{R}_2 (3) = \hat{P}_1 R_2, \] (19)

\[ \mathcal{R}_1 \mathcal{R}_2 (4) = \hat{P}_1 \hat{P}_2, \] (20)

\[ \mathcal{R}_1 \mathcal{R}_2 (5) = \hat{R}_{1 \mu \nu} \hat{R}_2^{\mu \nu}. \] (21)

In the \( C_{\mu \nu} \) tensor basis, only form factors of the pure gravity terms change,

\[ \tilde{f}_1 (\xi) = \frac{1}{2} f_1 (\xi), \] (22)

\[ \tilde{f}_2 (\xi) = f_2 (\xi) + \frac{1}{3} f_1 (\xi), \] (23)

where \( f_i (\xi) \) are the original form factors of \[10\]. The new set of form factors becomes,

\[ \tilde{f}_1 (-s \Box) = \frac{f(-s \Box) - 1 - \frac{1}{6} s \Box}{(s \Box)^2}, \] (24)

\[ \tilde{f}_2 (-s \Box) = \frac{1}{24} \left[ \frac{1}{12} f(-s \Box) - \frac{f(-s \Box) - 1 - \frac{1}{6} s \Box}{s \Box} \right], \] (25)

\[ \tilde{f}_3 (-s \Box) = f_3 = \frac{1}{12} f(-s \Box) - \frac{1}{2} \frac{f(-s \Box) - 1}{s \Box}, \] (26)

\[ \tilde{f}_4 (-s \Box) = f_4 = \frac{1}{2} f(-s \Box), \] (27)

\[ \tilde{f}_5 (-s \Box) = f_5 = \frac{1}{2} \frac{f(-s \Box) - 1}{s \Box}. \] (28)

The short and large proper time asymptotics of the second order form factors can be found in \[10\]. The new basis leads to simplifications in the third order form factors \[24\], which we do not consider here.

III. COVARIANT ACTION: LOCAL AND NONLOCAL TERMS

The effective action was introduced to quantum field theory by J.S. Schwinger \[25\], who also first proposed to use the Euclidean spacetime \[15\]. The covariant effective action was invented by B.S. DeWitt in order to apply this method to gravity and gauge field theories \[3\]. It is expressed in terms of phenomenological fields, also called the 'expectation value' fields \[13\], and serves as a generating functional of the field theory currents, including the energy-momentum tensor that enters the effective equations \[14\]. "The effective action [below] does not refer even to quantum field theory. It is an action for the observable field, and its implications may be valid irrespective of the underlying fundamental theory" \[13\], p. 759.
The Schwinger-DeWitt (geometrical) formalism of the effective action is entirely different from the Feynman’s path integral. It is a differential technique in contrast to the integral technique of R.P. Feynman. Nonetheless, the effective action is often assumed to be derived from the functional integral, even though it cannot be because the Planck constant is not compensated by another dimensionful quantity. This contradiction can be removed, if we assume that the action is defined rather than derived (see below). This is it, the action is not the first order term in an expansion of some functional by the orders of the Planck constant, therefore, it is not ‘a quantum correction’ to some classical action. In fact, the action in the field theory is not the action by the physical dimensionality, therefore, the Planck constant that does appear in quantum mechanics (and the path integral technique) cannot be used as a physical parameter in the field theory.

Thus, axiomatically the covariant action is the following proper time integral of the evolution kernel,

$$\int_{1/\mu^2}^{\infty} \frac{ds}{s} \text{Tr}K(s).$$

Because the evolution kernel does not exist for $s = 0$ there must be a positive lower limit of the proper integral. It is arbitrary finite and denoted traditionally as the inverse parameter, $1/\mu^2$ with the dimensionality $m^2$. This limit is not required to be small, because there is no other quantity in this integral to compare with $1/\mu^2$. The solution for the evolution kernel (heat kernel trace) is substituted into (29). The action functional $W$ is dimensionless, because both the evolution kernel, $\text{Tr}K(s)$, and the proper time integration measure (29) are dimensionless.

After the proper time integration, the series in the curvatures of the evolution kernel leads to the corresponding series of the covariant action, which depends on the $\mu^2$-scale,

$$\sum_{n=0}^{\infty} (\mu^2)^{(2-n)} W_n(\mu^2).$$

The action in four dimensions, including the third order in the curvatures, was computed in (10, 12, 28). That expression changes now by the addition of two local terms that before were eliminated by the dimensional regularization (5, 10) (Appendix B),

$$\frac{1}{(4\pi)^2} \int d^4x \frac{1}{2} \text{Tr} \left\{ \mu^4 \frac{1}{2} + \mu^2 \hat{P} \right\}
+ \mu^0 \sum_{i=1}^{5} \gamma_i \left( -\Box_2/\mu^2 \right) R_1 R_2(i) +$$
\[ + \mu^{-2} \sum_{i=1}^{29} G_i(-\Box_1/\mu^2, -\Box_2/\mu^2, -\Box_3/\mu^2) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i) + O[\mathcal{R}^4]. \] (31)

The third order form factors of the action (31) are now dimensionless (as different from the representations of [12]) and contain the \(1/\mu^2\) factor. The third order in the curvatures gains therefore the overall factor \(1/\mu^2\) (30). The functional \(W\) is defined up to a multiplier (the calibration constant), which should be determined with experiment. The value of the scaling constant \(\mu^2\) is not fixed a priori, however, the relations between different orders of the covariant action are already established by the existing physical laws and constants, which could give a value of \(\mu^2\).

Let us now re-analyze the second order form factors of (31). In this order of the action, the integral (29) of the basic form factor (12) has the form,

\[ \gamma(-\Box/\mu^2) \equiv \int_{1/\mu^2}^{\infty} \frac{ds}{s} \int_{1}^{0} d\alpha \exp \left( -\alpha(1-\alpha) s(-\Box) \right). \] (32)

It can be expressed after changing the order of integrals and introducing the dimensionless parameter, \(t\), as,

\[ \gamma(-\Box/\mu^2) = \int_{0}^{1} d\alpha \int_{z}^{\infty} \frac{dt}{t} \exp(-t), \] (33)

where the exponential integral has the lower limit,

\[ z = \alpha(1-\alpha)(-\Box/\mu^2). \] (34)

It can be solved by standard mathematical tools [29],

\[ \int_{z}^{\infty} \frac{dt}{t} \exp(-t) = -\mathcal{C} - \ln(z) - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n \cdot n!}, \] (35)

where the Euler constant is \(\mathcal{C} \approx 0.577216\). Substituting this back to (33) and doing the \(\alpha\)-integral gives the following expression,

\[ \gamma(-\Box/\mu^2) = -\ln \left( \frac{\Box}{\mu^2} \right) - \mathcal{C} + 2 - \frac{1}{6} \frac{\Box}{\mu^2} + O\left[(-\Box/\mu^2)^2\right]. \] (36)

In (36) we keep the first term of the sum in (35), because it adds to the third order of the covariant action [12]. In this truncated form, the expression (36) is valid as the asymptotic,

\[ (-\Box/\mu^2) \ll 1, \] (37)

while the expression (35) holds for arbitrary \((-\Box/\mu^2)\).
We use the integral representations (13)-(14) to compute two other integrals for the basic form factors with subtractions,

$$\eta(-\Box/\mu^2) = \int_{1/\mu^2}^{\infty} \frac{ds}{s} \frac{f(-s\Box) - 1}{s\Box},$$

$$\theta(-\Box/\mu^2) = \int_{1/\mu^2}^{\infty} \frac{ds}{s} \frac{f(-s\Box) - 1 - 1/6(-s\Box)}{(-s\Box)^2}. \tag{38}$$

Their asymptotic expansions are given by,

$$\eta(-\Box/\mu^2) = \frac{1}{6} \left[ \ln \left( \frac{-\Box}{\mu^2} \right) + C \right] + \frac{4}{9} - \frac{1}{60} \frac{\Box}{\mu^2} + O\left[ (-\Box/\mu^2)^2 \right], \tag{40}$$

$$\theta(-\Box/\mu^2) = \frac{1}{60} \left[ \ln \left( \frac{-\Box}{\mu^2} \right) + C \right] + \frac{23}{450} - \frac{1}{840} \frac{\Box}{\mu^2} + O\left[ (-\Box/\mu^2)^2 \right]. \tag{41}$$

The full list of the second order form factors in the action (31) is,

$$\gamma_1(-\Box/\mu^2) = -\frac{1}{60} \left[ \ln \left( \frac{-\Box}{\mu^2} \right) + C \right] + \frac{23}{450} - \frac{1}{840} \frac{\Box}{\mu^2}, \tag{42}$$

$$\gamma_2(-\Box/\mu^2) = -\frac{1}{1080} - \frac{1}{7560} \frac{\Box}{\mu^2}; \tag{43}$$

$$\gamma_3(-\Box/\mu^2) = -\frac{1}{18} - \frac{1}{180} \frac{\Box}{\mu^2}; \tag{44}$$

$$\gamma_4(-\Box/\mu^2) = -\frac{1}{2} \left[ \ln \left( \frac{-\Box}{\mu^2} \right) + C \right] + 1 - \frac{1}{12} \frac{\Box}{\mu^2}; \tag{45}$$

$$\gamma_5(-\Box/\mu^2) = -\frac{1}{12} \left[ \ln \left( \frac{-\Box}{\mu^2} \right) + C \right] + \frac{2}{9} - \frac{1}{120} \frac{\Box}{\mu^2}. \tag{46}$$

The second order of the action (31), given by these expressions (42)-(46) together with the nonlocal tensor invariants (17)-(21), reproduces the result of [24]. The difference is that the expressions above are only the asymptotics \((-\Box/\mu^2) \ll 1\) of the exact form factor (33)-(35). Let us note that neither \(\mu^2 \rightarrow \infty\), nor \(\Box \rightarrow 0\) are correct forms of this asymptotics. However, the asymptotics (37) was denoted as \(\Box \rightarrow 0\) in previous works on the covariant perturbation theory. The opposite limit, \((-\Box/\mu^2) \gg 1\), might also be worth studying.

In the new tensor basis (17)-(21), the second order terms with the scalar Ricci tensor in the action (31) are local as seen from (43)-(44) (the local term \(\hat{P}R\) is known since Ref. [10]).

The properties of the covariant action related to the scalar Ricci tensor are studied in [24].

**IV. THE PHYSICAL SCALE AND THE GRAVITY ACTION**

The content of the preceding section belongs to the operator methods of geometrical analysis. Let us connect these mathematical expressions with physics starting with the
lowest orders in the curvature. The first local term of the covariant action (31) is trivially universal for any field model. The second local term is defined by the potential matrix, \( \hat{P} \). Therefore, this order is zero, when \( \text{tr} \hat{P} = 0 \). This condition is true only for the conformal invariant scalar field model. The operator for these models (11) contains the Ricci scalar term with the coefficient \((-1/6)\) (this was the reason it is singled out in (11)) and no potential term. Its contribution to the first order is identically zero, while the contribution to the zeroth order is not; this creates an apparent physical discord. The field theory that does not generate the gravity action (as explained below) is unphysical.

However, in the modern theory of the interactions of fundamental \textit{particles}, quantum chromodynamics, basic fields are massless spinors [30, 31] that can be viewed as a spin manifold in the mathematical formalism of R. Penrose and W. Rindler [32]. For spinor fields, the action is \((-1)\) times the calculated result (31), but the overall factor is not important. However, physically important is the fact that the Ricci scalar term in the operator (11) has the different numerical factor, \((-1/4)\). The operator (11) with \((-R/4)\) instead of \((-R/6)\) is called the Lichnerowicz-DeWitt operator [3, 33]. To obtain such a form of the operator \( \hat{F}(\nabla) \) by the algorithmic rules of the action [5], the potential term, \( \hat{P} \) (it also includes the gauge field tensor [13, 16], but in the first order in the curvature the trace of the gauge field tensor is zero) must be,

\[
\text{tr} \hat{P} = -\frac{1}{12} R \text{tr} \hat{1}.
\] (47)

With the substitution (47), two local terms of the covariant action have now opposite signs. The Einstein-Hilbert theory of gravitation was developed with the linear term (the scalar Ricci curvature) without any dimensionful factor [34]. This scaling can be achieved by multiplying the whole action by \( \mu^{-2} \). Since the action is found up to an unknown factor, this operation does not change physics. Thus, the action, normalized by unessential factors as,

\[
\bar{W}(\mu^2) \equiv (4\pi)^2 \frac{12}{\mu^2} W(\mu^2).
\] (48)

has the final form,

\[
\bar{W}(\mu^2) = \int dx^4 g^{1/2} \left\{ \text{tr} \hat{1} (6\mu^2 - R) + \mu^{-2} O[\mathcal{R}^2] \right\}.
\] (49)

The zeroth order term of \( W \) naturally corresponds to the cosmological constant, and the first order term presents the Einstein-Hilbert action of gravity. They emerge from the
covariant action, which previously was deemed existing only from the second order, because
the dimensional regularization (see Appendix B) eliminated the power law terms \[5\], which
are the lowest order terms of \(W\).

It is well known that the Einstein-Hilbert gravitation action cannot be added (in models
for the study of black hole or early Universe physics) to the action without a dimensionful
factor, because they have different dimensionalities: the action is dimensionless, \(m^0\), while
the dimensionality of the gravity action is \(m^2\) (this adds the dimensional factor to the gravity
action in \[14\], which was not used yet for physics applications any way). The Planck length
is usually employed as a needed dimensionful parameter. However, we argue (see Appendix
A) that the Planck length is an arbitrarily chosen value. The scale \(\mu^2\) appears from the
mathematical solution, even though its physical meaning is yet to be studied. Absolute
numerical coefficients of these two terms are not defined, because the value of the scale
parameter \(\mu^2\) can only be measured, but not derived. The local orders as well as all pure
gravity terms of the higher orders of the action \[29\] have the same relative coefficients for
a field theory with any spin group, because the trace of the unity matrix, \(tr \hat{1}\), factorizes out
in those terms.

The large number of extensions of Einstein’s general relativity have been proposed and
tested with astrophysical observations \[35\]. Some of the most natural candidates for the
gravity theory modifications are the ones whose Einstein-Hilbert action is supplemented
with the higher orders in curvatures or with the nonlocal terms. Different mechanisms
were proposed for the appearance of such modifications. For example, Planck Collaboration
studied the limits imposed by its observations on the modified gravity models \[37\] that were
derived within the ’effective field theory’. These gravity modifications contain up to nine
free parameters \[37\]. The modification suggested in the present paper \[31\] is axiomatically
derived, therefore, it can deliver numerical coefficients of the local and nonlocal terms of the
higher orders in the curvatures, which makes it possible to develop an axiomatic cosmological
theory. This is important in view of the fact that the dominating paradigm of modern
cosmology, the \(\Lambda\)CDM (‘Cosmological Constant with Cold Dark Matter’) model \[36\], is
handicapped by unresolved problems and paradoxes.

The expression \[49\] makes it possible to obtain the value of the universal physical scale
from the measured cosmological constant $\Lambda$,

$$6\mu^2 = \Lambda. \quad (50)$$

The currently accepted value of the cosmological constant is approximately $10^{-52} \text{ m}^{-2}$ [36]. Then, by Eq. (50) the length constant $1/\mu$ equals to $8.8 \cdot 10^{26} \text{ m}$, which is the size of the observable Universe (‘the comoving angular diameter to the last scattering surface’) as derived from the cosmic microwave background (CMB) observations by the Planck collaboration [36]. It seems quite natural that the physical scale of the field theory is supplied by the largest length in Nature, which makes the theory of observed physical phenomena closed. Assigning this small fixed value to the constant $\mu$ does not change the computation of the form factor in the lowest order in the curvature [17].

These physical conjectures naturally follow from the mathematical derivations, however, they should be further studied before attempting to develop physical theories based on them.

V. SUMMARY

In the present paper the dimensional analysis is applied to the covariant perturbation theory with the intention to extend its range of possible applications in physics. Let us summarize the main points.

- Planck values (Planck length, Planck mass, etc) have no physical significance.
- Dimensional regularization technique is not applicable to the derivation of the action.
- The positive lower limit of the proper time integral defining the action introduces the physical scale to the dimensionless field theory.
- This scale is present at all orders of the covariant action, which is finite and nonlocal.
- The covariant action contains the cosmological constant term and the action of gravitation, confirming the Sakharov-DeWitt mechanism, together with their nonlocal, higher order in the curvatures, contributions.
- The physical scale value can be determined by the cosmological constant and vice versa.
VI. DISCUSSION

The 'classical' action in quantum field theory does not really have the dimensionality of the action (as it does in quantum mechanics), contrary it is \( \text{meter}^2 \). The dimensionality of the Planck constant is \( \text{Joule} \cdot \text{second} \), but nor time, neither energy appear yet in the \textit{geometry} of Euclidean spacetime. Therefore, we are left without both traditional small physical parameters, because the Planck length is not relevant to physics as well, see Appendix A. However, through the action definition (29) we gain another physical scale with the correct dimensionality. This means that likely all mathematical derivations of the covariant perturbation theory would remain unchanged.

The action, obtained by solving the evolution equation for the spinor field Laplacian, is a phenomenological functional of the gauge field and gravity strength tensors. Its different orders in the curvatures (meaning the different geometrical orders) are dimensionless, but they are related (scaled) to each other by the unique length scale, which dynamically arises from the theory. According to the conjecture about the universal nature of the covariant action, terms that were formerly considered the 'classical' actions (the Ricci scalar term for gravity theory and the Maxwell action for electrodynamics) arise as the lower order contributions in the action. The higher order contributions that were previously deemed to be 'quantum' actions add up in (31) with the scaling factor, which is the new unique characteristic parameter of the theory.

The proper time method was discovered and developed in parallel in physics and mathematics during the 20th century. The early work in physics was done by V. Fock [38], then the proper time method became popular in quantum field theory after the Schwinger's paper [25], but it also appeared in the concurrent works of S. Tomonaga, Y. Nambu, and R. Feynman. The 'heat kernel' is reviewed in mathematical terms in the recent monograph [39], with applications in mathematical finance. However, even though the evolution equation (5) is traditionally called the 'heat equation', this name creates confusion with the Fourier equation of heat conductivity [21], and the proper time - with the time coordinate. Let us recommend to borrow mathematical terminology and use the term 'evolution kernel' for the trace of the heat kernel (9). The \textit{evolution equation} is widely studied in mathematics literature, where it is used to develop the theory of geometrical flows. Its first applications in geometry via the Ricci flows were studied by R. Hamilton [40] (cf. [22]).
The first order term of the action \( (29) \) is suggested to be the Einstein-Hilbert action of gravitation. The emergence of the gravity action in quantum field theory was first suggested by A.D. Sakharov \([41]\) in the proposal known as the *induced gravity*. Even though Sakharov’s work used mathematics similar to ours (the heat kernel from the mathematics literature), it could not be closed without replacing the particle view on physics by the field view. In the field theory approach, the dimensionful factor present in the action is fundamentally physical, it is not an auxiliary parameter (regulator). The appearance of the cosmological constant and the Einstein’s gravity action from the evolution kernel was also suggested in the 1964 Les Houches lectures of B.S. DeWitt \([3]\). Thus, this idea was contemporary to Sakharov’s work, although the misconceptions of divergences and renormalization have survived to the present time \([4]\). We suggest to call this generation of gravity the *Sakharov-DeWitt mechanism*. A large number of papers have been published on this subject that either attempt to remove divergences, which do not appear in this problem, or struggle with the Planck length discrepancy, which does not belong to physics.

As obvious from the second order of the action \( (31) \), \( (21) \) and the commutator curvature \([4]\) for the Abelian gauge fields, this result contains electrodynamics modified by nonlocal (and higher order) contributions. This fact is neither surprising nor new. The generation of the Maxwell electrodynamics within quantum field theory was proposed by Ya.B. Zeldovich \([42]\), who immediately applied Sakharov’s induced gravity proposal. However the proper implementation of this idea could not have been worked out at that time, because one has to abandon the concepts of quantum vacuum and particles to achieve the mathematically acceptable solution. It goes without saying that modifications (both nonlocal and non-perturbative) of electrodynamics have been proposed in the past, but this subject and its relation to \([14]\) lies beyond the scope of the present paper.

The only thing that could be said about the physical scale \( \mu^2 \) is its measured value. Its possible relation to the cosmological constant is discussed above, but this relies on the validity of the current cosmological model and should be re-analyzed, if this model were to change. In the dimensionless field theory we have yet to gain the notion of mass, therefore, no comparison with other physical constants can be made yet. The length scale, \( 1/\mu \), which enters the form factors of the nonlocal covariant action, could specify the assumed 'massless' approximation \([13, 14]\) and apparently does not change the mathematical derivations.

The covariant action is computed in spacetime with the Euclidean metric signature. The
transfer to the Minkowski spacetime is trivial because the local terms of (31) are insensitive to the metric signature. The procedure to do this transfer in the higher order, nonlocal, terms was derived in [9] and it consists of 1) doing the variation over the metric and 2) replacing all Euclidean Green functions in the obtained energy-momentum tensor with the retarded ones.

The action could produce the amplitudes, if applied to the scattering problems. This was designed originally [3] and explains why the action was at first considered the generating functional of all one-particle irreducible (1PI) Feynman diagrams [4]. However, the action was invented in order to avoid complicated computations encountered in the gauge field and gravity theory that used to be impossible to perform before the advent of modern software. Nowadays such computer symbolic manipulations created an active research field and thereby expelled the generating functional method. In the present meaning, the action should not be expanded into the diagram series, because there are no Feynman diagrams in the covariant perturbation theory. There is only the 'loop' of the heat kernel, i.e. its functional trace (8).

About forty years ago Schwinger began publishing the three volume book “Particles, sources, and fields” that presented the 'source theory', "to which the concept of renormalization is foreign" [43], because this was a phenomenological theory of elementary particles without divergences. In general, phenomenological physical theories do need renormalization. The development of the covariant perturbation theory [9–12] began at about the same time with a different goal of dealing with the (hypothetical so far) effects of particle creation by the electromagnetic ('Schwinger effect' [25]) and gravitational ('Hawking radiation' [13]) fields. The history of the form factors of the nonlocal action can be traced as far back as to the 19th century, when the operator methods were invented and developed by O. Heaviside [44]. Eventually Heaviside’s physical mathematics was incorporated into mathematics [45].

We hope this paper will help promoting the use of the evolution equation’s solutions in physics and motivating its study in mathematics.

Acknowledgments

I thank Max Planck Institute for Gravitational Physics (Albert Einstein Institute) at Potsdam-Golm, Germany for hospitality and support during many research visits.
Appendix A. The Planck values and the New SI of physical units

In this paper we use the term of physical *dimensionality* to avoid confusion with the spacetime dimension. Dimensionality appears from physical experiments that measure a physical quantity by comparing it with the reference quantity, a physical unit (etalon). The operation of measurement brings the scale to physics. Physical theory unifies the phenomena by its laws that relate different physical quantities to each other. As a result, the number of independent physical quantities is rather small, while the rest of them are derivative. The International Committee of Weights and Measures (*Bureau international des poids et mesures, BIPM*) agreed that the minimal number of physical quantities (units) is seven. BIPM recommends to use these SI (Système International) physical units [46] in physical and engineering sciences.

However, in the near future science and technology will have to switch to the New SI of physical units [47, 48]. The New SI will have seven physical constants [49] with the exact numerical values, while physical units will be known with some uncertainty. The General Conference on Weights and Measures (CGPM) proposed that the kilogram, ampere, kelvin and mole will have relative uncertainties. These physical units will be defined by exact values (yet to be determined) of the Planck constant, the Boltzmann constant, the Avogadro constant, the elementary charge. Three other physical constants, including the speed of light in vacuum, are already exact and determine the second (the unit which does not depend on any other), metre and candela [48]. The fundamental physical constants will be called the *defining* constants to reflect their nature. This system to be adopted in 2018 will be similar to the system of physical units (called the natural or Planck system) proposed by Max Planck in his paper on thermal radiation [50], Sect. 26. Since this topic is related to physics of the present paper, let us briefly review it.

The system of natural physical units is also presented in M. Planck’s book “The theory of heat radiation” [51], Sect. 164 “Natural units”. Planck explains that physical units used to be chosen ad hoc and based on material artifacts that are special or relevant to the existing intelligent life, in the given conditions. However, “with the aid of the two constants $h$ and $k$ which appear in the universal law of radiation, we have the means of establishing units of length, mass, time, and temperature, which are independent of special bodies or substances, which necessarily retain their significance for all times and for all environments, terrestrial
and human or otherwise, and which may, therefore, be described as “natural units”. In other words, with two new constants, the system of physical units becomes coherent, as called in the modern metrology literature, because the number of units equals the number of constants. Then, a particular choice of absolute numerical values is not relevant to physics, only relations among them, expressed as physics laws, are.

Planck suggested to use this arbitrariness by selecting values of the fundamental physical constants in some predetermined way. The simplest choice seemed to be assigning values 1 to four fundamental constants. As a side effect, physical units of this new system would have unusual numerical values, when expressed via existing units (the SI units now). For example, the proposed unit of length

\[ l_{\text{Planck}} = \sqrt{\frac{G_N}{(hc^3)}}, \]  

where \( G_N \) is the Newton’s constant of gravitation. These values are commonly referred to as the Planck values (quantities). The only physical meaning of the Planck length is a value of the unit of length in the Planck system expressed by the SI metre,

\[ 1 \text{ unit of length (Planck system)} \approx 3.99 \cdot 10^{-35} \text{ meter (SI)}. \]  

However, the only reason to assign one to fundamental physical constants instead of any other numbers is that 1 is the smallest number in any numeral system, which is clearly not a physical reason. Selecting any other numbers would arbitrarily change the Planck values.

M. Planck writes further, "These quantities retain their natural meaning as long as the law of gravitation and that of the propagation of light in a vacuum and the two principles of thermodynamics remain valid; they therefore must be found always the same, when measured by the most widely differing intelligences according to the most widely differing methods" [51]. Under the words “terrestrial and human or otherwise” above, emphasized by the phrase “the most widely differing intelligences” in this paragraph, Planck apparently meant that any human society or, perhaps, extraterrestrial intelligent life that developed a system of physical units in the simplest possible way, i.e. by assigning ones to the fundamental physical constants, would find the same physical etalons used by any other intelligence, when they compare the implementations of their physical units (like the metal bar of one metre that used to be an etalon of SI).

In fact, with the Planck system one could trade very large (small) values of physical constants for very large (small) values of physical units. The existing system SI [46] is set
up according to our historical conventions. Obviously, it could be inconvenient to use very small (large) units. Therefore, historical conventions will be approximately kept in the New SI. For example, in the New SI units, the r.h.s. of (52) is already made close (but not exact) to 1 SI metre, by fixing (making exact) the value of the speed of light in vacuum. The principal difference of the new SI is that the values of physical units (meter, second, etc.) will be known only *approximately*, while the physical constants will always be exact; the opposite agreement currently holds in the SI.

Summarizing, in his works Max Planck did not declare that the numerical values of new physical units in the proposed natural system expressed via the physical units of the traditional system would have special physical meanings. The Planck length, mass, etc. are just a curiosity: changing the scale of the system of physical units does not change physics. Assigning any physical significance to the Planck values contradicts the core idea of the Planck system, its modern-day implementation the New SI, and the scale-free nature of field theory. Therefore, the paradox of the observed cosmological constant vs. its quantum field theory inspired value does not exist.

**Appendix B. Dimensional regularization and the action**

When singularities (called the ultraviolet divergences) are encountered in quantum field theory, the methods of *regularization* are used to explore them by introducing an auxiliary parameter. In the 'effective action' method, the *dimensional* regularization used to be employed. However, in the covariant perturbation theory divergences are an artifact of improper computations and should not appear at all.

The origin of the dimensional regularization is phase space integrals of the Hamiltonian formalism of the elementary particle theory. The Feynman’s path integral formalism uses spacetime coordinates and particle momenta as variables. The phase space integrals explicitly depend on the spacetime dimension $D$. After making the spacetime dimension an arbitrary (complex) parameter and computing the $D$-dimensional integrals, physical spacetime dimension $D = 4$ is restored.

In the covariant perturbation theory, which uses mathematics of the evolution kernel, variables are the world function’s covariant derivatives, and the action is a functional of the nonlocal tensor invariants. The singular behaviour of the second order
form factors is caused with the pole of the exponential integral (33). The proper time integral can be computed [29], while the spacetime integral is defined in four dimensions. As we have seen above, the resulting covariant action is finite and nonlocal.

As matter of fact, the dimensional regularization is not consistent with the method of derivations. In the traditional definition, e.g. [4, 5, 10] and all other works, when the lower limit of the proper time integral (29) is taken $s = 0$, the form factor in $D$ spacetime dimensions admits the solutions [10],

$$
\gamma(-\Box) = (-\Box)^{D/2-2} \frac{\Gamma(2-D/2)\Gamma(D/2-1)^2}{\Gamma(D-2)},
$$

(53)

where $\Gamma$ is the gamma function. The form factor (53) has the dimensionality $m^{4-D}$, which is compensated with the dimensionality of the spacetime integration measure $dx^D$. They make, together with the tensor structures (17), a dimensionless functional. The expansion of (53) in the four-dimensional asymptotic, $\epsilon \equiv 2 - D/2 \ll 1$, is then taken. However, an expansion of the power law function used to derive the action’s form factor is not valid at the assumed conditions. This discrepancy is more general than this specific problem, it also appears in the potential theory in lower dimensions, e.g. [54], Chap. 9.

Let us re-consider the derivation of the basic form factor at $D = 4$ [10], where the following expansion in $\epsilon = D/2 - 2$ appears,

$$
\left(\frac{-\Box}{\mu^2}\right)^{D/2-2} \equiv \left(\frac{-\Box}{\mu^2}\right)^{-\epsilon} = \exp\left(-\epsilon \ln \left(\frac{-\Box}{\mu^2}\right)\right) = 1 - \epsilon \ln \left(\frac{-\Box}{\mu^2}\right) + O[\epsilon^2], \epsilon \ll 1.
$$

(54)

Its product with the gamma functions expansions gave the final result [10],

$$
\gamma \left(\frac{-\Box}{\mu^2}\right) = \frac{1}{\epsilon} - \ln \left(\frac{-\Box}{\mu^2}\right) - C + 2 + O[\epsilon], \epsilon \ll 1,
$$

where we introduced the missing parameter $\mu^2$ into the logarithm. This expression differs from the correct one (36) by the additional divergent term. However, the resulting spacetime integrals in the action are taken at fixed $D = 4$, while the divergence term, $1/(2 - D/2)$ (which was assumed to identify the local divergences), has $D \neq 4$, therefore, it cannot be present in the four-dimensional integral. One can see that the expansion (54) and the resulting form factor are valid only in the limit, $(-\Box/\mu^2) = O[1]$, not for arbitrary values of its argument (35). The first term of (54) is the redundant divergence, $1/\epsilon$, because the form factor’s singular behaviour is already expressed by the logarithmic function. These problems are avoided if the dimensional regularization were not used at all, as suggested in Sec. III.20.
The discussed technical problems were already corrected in final expressions for the energy-momentum tensor in the covariant perturbation theory \[55\]: the $1/\epsilon$ divergences were discarded as local contributions, and the logarithmic form factors of nonlocal contributions gained the $\mu^2$-parameter. Nevertheless, they prevented the appearance of two lowest order terms in the covariant action, because all power law divergences that would correspond to the first two terms of \[31\] were deemed to be nil in the dimensional regularization.

[1] Dirac PAM. 1940 The relation between mathematics and physics. Proc. Royal Soc. Edinburgh 59 (1), 122-129. (doi:10.1017/S0370164600012207)

[2] Bogoliubov NN, Logunov AA, Todorov IT. 1975 Introduction to Axiomatic Quantum Field Theory. Reading, MA: A. Benjamin, Inc.

[3] DeWitt BS. 1965 Dynamical Theory of Groups and Fields. New York, NY: Gordon and Breach. Also in DeWitt C. and DeWitt B. (eds.) 1964 Relativity, Groups and Topology, pp. 587-822. New York, NY: Gordon and Breach.

[4] DeWitt BS. 2003 Global Approach to Quantum Field Theory. Vols. 1 and 2. Oxford, UK: Oxford University Press.

[5] Barvinsky AO and Vilkovisky GA. 1985 The generalized Schwinger-Dewitt technique in gauge theories and quantum gravity. Phys. Rep. 119, 1-74. (10.1016/0370-1573(85)90148-6)

[6] Vilkovisky GA. 1984 The Gospel according to DeWitt. In S.M. Christensen (ed.), Quantum Theory of Gravity, pp. 169-209. Bristol, UK: Adam Hilger.

[7] Vilkovisky GA. 1992 Heat kernel: Recontre entre physiciens et mathématiciens. In Publication de l’Institut de Recherche Mathématique Avancée, R.C.P. 25, 43, 203. Strasbourg, France: IRMA. (CERN-TH-6392-92)

[8] Vilkovisky GA. 1992 Effective ection in quantum gravity. Class. Quant. Grav. 9, 895-903. (doi:10.1088/0264-9381/9/4/008)

[9] Barvinsky AO and Vilkovisky GA. 1987 Beyond the Schwinger-Dewitt technique: Converting loops into trees and in-in currents. Nucl. Phys. B 282, 163-188. (10.1016/0550-3213(87)90681-X)

[10] Barvinsky AO and Vilkovisky GA. 1990 Covariant perturbation theory. 2: Second order in the curvature. General algorithms. Nucl. Phys. B 333, 471-511. (10.1016/0550-3213(90)90047-H)
[11] Barvinsky AO and Vilkovisky GA. 1990 Covariant perturbation theory. 3: Spectral representations of the third order form-factors. *Nucl. Phys. B* **333**, 512-524. (doi:10.1016/0550-3213(90)90048-I)

[12] Barvinsky AO, Gusev YuV, Zhytnikov VV and Vilkovisky GA. 1993 Covariant Perturbation Theory (IV). Third order in the curvature Winnipeg, MB: U. Manitoba. Preprint SPIRES-HEP: PRINT-93-0274 (MANITOBA). (arXiv:0911.1168)

[13] Vilkovisky GA. 2008 Expectation values and vacuum currents of quantum fields. *Lect. Notes Phys.* **737**, 729-784. (doi:10.1007/978-3-540-74233-3_23)

[14] Mirzabekian AG and Vilkovisky GA. 1998 Particle creation in the effective action method. *Ann. Phys. (N.Y.)* **270**, 391-496. (doi:10.1006/aphy.1998.5860)

[15] Schwinger J. 1959 Euclidean quantum electrodynamics. *Phys. Rev.* **115**, 721-731. (doi:10.1103/PhysRev.115.721)

[16] Shore GM. 2002 A local effective action for photon gravity interactions. *Nucl. Phys. B* **646**, 281-300. (10.1016/S0550-3213(02)00833-7)

[17] Mirzabekyan AG. 1994 Vacuum radiation in the effective equations of quantum gravity with arbitrary form factors. *JETP* **79** (1), 1-16. (www.jetp.ac.ru)

[18] Ruse HS. 1931 Taylor’s theorem in the tensor calculus. *Proc. London Math. Soc.* **32**, 87-92. (10.1112/plms/s2-32.1.87)

[19] Synge JL. 1960 *Relativity. The General Theory*. Amsterdam, Netherlands: North Holland.

[20] Gusev YuV. 2009 Heat kernel expansion in the covariant perturbation theory. *Nucl. Phys. B* **807**, 566-590. (doi:10.1016/j.nuclphysb.2008.08.008)

[21] Vladimirov VS. 1971 *Equations of Mathematical Physics*. New York, NY: Marcel Dekker Inc.

[22] Barvinsky AO, Gusev YuV, Vilkovisky GA and Zhytnikov VV. 1994 The basis of nonlocal curvature invariants in quantum gravity theory. (Third order). *J. Math. Phys.* **35**, 3525-3542. (doi:10.1063/1.530427)

[23] Barvinsky AO, Gusev YuV, Vilkovisky GA and Zhytnikov VV. 1994 Asymptotic behaviors of the heat kernel in covariant perturbation theory. *J. Math. Phys.* **35**, 3543-3559. (doi:10.1063/1.530428)

[24] Mirzabekian AG, Vilkovisky GA and Zhytnikov VV. 1996 Partial summation of the nonlocal expansion for the gravitational effective action in four-dimensions. *Phys. Lett. B* **369**, 215-220. (doi:10.1016/0370-2693(95)01527-2)
[25] Schwinger J. 1951 On gauge invariance and vacuum polarization. Phys. Rev. 82, 664-679. (doi:10.1103/PhysRev.82.664)

[26] Schwinger J. 1973 A report on quantum electrodynamics. In Mehra J. (ed.). The Physicist’s Conception of Nature, pp. 413-429. Dordrecht, Netherlands: D. Reidel Publ. Co.

[27] Kleinert H. 2015 Particles and Quantum Fields. Singapore: World Scientific.

[28] Barvinsky AO, Gusev YuV, Vilkovisky GA and Zhytnikov VV. 1995 The one-loop effective action and trace anomaly in four-dimensions. Nucl. Phys. B 439, 561-582. (doi:10.1016/0550-3213(94)00585-3)

[29] Olver FWJ. 1974 Asymptotics and Special Functions. New York, NY: Academic Press.

[30] Olive KA et al (Particle Data Group). 2014 Review of particle physics. Chin. Phys. C 38 (9), 090001. (doi:10.1088/1674-1137/38/9/090001)

[31] Álvarez-Gaumé L and Vázquez-Mozo Á. 2012 An Invitation to Quantum Field Theory. Berlin: Springer-Verlag.

[32] Penrose R and Rindler W. 1984 Spinors and space-time. Vol. 1. Two-spinor calculus and relativistic fields. Cambridge: Cambridge University Press.

[33] Lichnerowicz A. 1964 Propagateurs, commutateurs et anticommutateurs en relativite generale. In DeWitt C and DeWitt B (eds.). Relativity, Groups and Topology, pp. 821-861. New York, NY: Gordon and Breach.

[34] Landau LD and Lifshitz EM. 1996 The Classical Theory of Fields. Oxford, UK: Butterworth-Heinemann.

[35] Berti E et al. 2015 Testing general relativity with present and future astrophysical observations. Class. Quantum Grav. 32, 243001 (179pp). (doi:10.1088/0264-9381/32/24/24300)

[36] Planck Collaboration. 2016 Planck 2015 results. XIII. Cosmological parameters. Astron. & Astrophys. 594, A13. (doi:10.1051/0004-6361/201525830)

[37] Planck Collaboration. 2016 Planck 2015 results. XIV. Dark energy and modified gravity. Astron. & Astrophys. 594, A14. (doi:10.1051/0004-6361/201525814)

[38] Fock V. 1937 Proper time in classical and quantum mechanics. Izvestia AN 4-5, 551. English trans. in Fock VA. 2004 Selected Works. Quantum Mechanics and Quantum Field Theory, pp. 421-439. Boca Raton, FL: Chapman & Hall/CRC.

[39] Avramidi IG. 2015 Heat Kernel Method and Its Applications. Cham, Switzerland: Birkhauser.

[40] Hamilton R. 1993 The Harnack estimate for the Ricci flow. J. Diff. Geom. 37 (1), 225-243.
[41] Sakharov AD. 1967 Vacuum quantum fluctuations in curved space and the theory of grav-
titation. *Dokl. Akad. Nauk Ser. Fiz.* 177, 70-71. Reprint: 1991 *Sov. Phys. Uspekhi* 34 (5),
394-394. (doi:10.1070/PU1991v034n05ABEH002498)

[42] Zeldovich YaB. 1967 Interpretation of electrodynamics as a consequence of the quantum the-
ory. *JETP Letters* 6 (10), 345-347. (www.jetpletters.ac.ru/ps/1674/article_25534.shtml)

[43] Schwinger JS. 1989 *Particles, Sources, and Fields*. Vol. 3 Reading, MA: Perseus Books.

[44] Heaviside O. 1892 On operators in physical mathematics. Part I. *Proc. Royal Soc. London* 52,
504-529. (doi:10.1098/rspl.1892.0093)

[45] Erdelyi A. 1962 *Operational Calculus and Generalised Functions*. New York, NY: Holt, Rine-
hart and Winston, Inc.

[46] Bureau International des Poids et Mesures. 2014 *The International System of Units (SI)*. 8th ed.
Sèvres, France: BIPM. (http://www.bipm.org/en/publications/si-brochure/)

[47] Mills IM, Mohr PJ, Quinn TJ, Taylor BN and Williams ER. 2011 Adapting the Interna-
tional System of Units to the twenty-first century. *Phil. Trans. R. Soc. A* 369, 3907-3924.
(doi:10.1098/rsta.2011.0180)

[48] Stenger J and Ullrich JH. 2016 Units based on constants. Redefinition of the International System of units. *
Ann. Rev. Cond. Matter Phys.* 7 (1), 35-59. (doi:10.1146/annurev-conmatphys-031115-011311)

[49] Mohr PJ, Newell DB, Taylor BN. 2014 CODATA recommended values of the fundamental physical constants: 2014. *
Rev. Mod. Phys.* 88, 035009. (doi:10.1103/RevModPhys.88.035009)

[50] Planck M. 1900 Über irreversible Strahlungsvorgänge [About irreversible radiation processes]. *
Ann. Phys. (Berlin)* 306 (1), 69-122. (doi:10.1002/andp.1900306010)

[51] Planck M and Masius M. 1914 *The Theory of Heat Radiation*. Philadelphia, PA: P. Blakin-
ston’s Son Inc. (The Project Gutenberg EBook 40030 (2012))

[52] Zeldovich YaB. 1981 The theory of vacuum may solve the enigma of cosmology. *Sov. Phys. Usp.* 24 (3), 216-230. (doi:10.1070/PU1981v024n03ABEH004772)

[53] Leibbrandt G. 1975 Introduction to the technique of dimensional regularization. *Rev. Mod. Phys.* 47, 849-876. (doi:10.1103/RevModPhys.47.849)

[54] Arnold VI. 2004 *Lectures on the Partial Differential Equations*. Berlin, Germany: Springer-
Verlag.
[55] Mirzabekian AG and Vilkovisky GA. 1993 The model-independent approach to quantum gravity: Implications of asymptotic flatness. *Phys. Lett. B* 317, 517-522. (doi:10.1016/0370-2693(93)91365-T)