BROWNIAN MOTION WITH GENERAL DRIFT

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Abstract. We construct and study the weak solution to stochastic differential equation \( dX(t) = -b(X(t))dt + \sqrt{2}dW(t) \), \( X_0 = x \), for every \( x \in \mathbb{R}^d \), \( d \geq 3 \), with \( b \) in the class of weakly form-bounded vector fields, containing, as proper subclasses, a sub-critical class \( [L^d + L^\infty]^d \), as well as critical classes such as weak \( L^d \) class, Kato class, Campanato-Morrey class, Chang-Wilson-T. Wolff class.

Let \( \mathcal{L}^d \) be the Lebesgue measure on \( \mathbb{R}^d \), \( L^p = L^p(\mathbb{R}^d, \mathcal{L}^d) \) the standard (real) Lebesgue spaces, \( C_b = C_b(\mathbb{R}^d) \) the space of bounded continuous functions endowed with the sup-norm, \( C_\infty \subset C_b \) the closed subspace of functions vanishing at infinity. We denote by \( \mathcal{B}(X, Y) \) the space of bounded linear operators between Banach spaces \( X \to Y \), endowed with the operator norm \( \| \cdot \|_X \to Y \); \( \mathcal{B}(X) := \mathcal{B}(X, X) \). Put \( \| \cdot \|_{p \to q} := \| \cdot \|_{L^p \to L^q} \).

1. Let \( d \geq 3 \), \( b : \mathbb{R}^d \to \mathbb{R}^d \). The problem of existence and uniqueness of a weak solution to the stochastic differential equation (SDE)

\[
X(t) = x - \int_0^t b(X(s))ds + \sqrt{2}W(t), \quad t \geq 0, \quad x \in \mathbb{R}^d,
\]

with a locally unbounded vector field \( b \), has been investigated by many authors. The first principal result is due to \( \text{[Po]} \): if \( b \in [L^p + L^\infty]^d \), \( p > d \), then there exists a unique in law weak solution to (1).

By the results in \( \text{[CW]} \), a unique in law weak solution to (1) exists for \( b \) from the Kato class \( \mathcal{K}_0^{d+1} \).

(Recall that a \( b : \mathbb{R}^d \to \mathbb{R}^d \) belongs to the Kato class \( \mathcal{K}_0^{d+1} \), \( 0 < \delta < 1 \), if \( |b| \in L^1_{\text{loc}} \) and there exists \( \lambda = \lambda_\delta > 0 \) such that

\[
\|b| (\lambda - \Delta)^{-\frac{1}{2}} \|_{1 \to 1} \leq \delta,
\]

and \( \mathcal{K}_0^{d+1} := \cap_{\delta > 0} \mathcal{K}_0^{d+1} (\sup_{[L^p + L^\infty]}^d, p > d) \).

Definition. Fix \( \delta \in [0, 1] \). A \( b : \mathbb{R}^d \to \mathbb{R}^d \) belongs to \( \mathcal{F}_\delta^{1/2} \), the class of weakly form-bounded vector fields, if \( |b| \in L^1_{\text{loc}} \) and there exists \( \lambda = \lambda_\delta > 0 \) such that

\[
\|b| (\lambda - \Delta)^{-\frac{1}{2}} \|_{2 \to 2} \leq \sqrt{\delta}.
\]

The class \( \mathcal{F}_\delta^{1/2} \) contains, as proper subclasses, a sub-critical class \( [L^d + L^\infty]^d \) (\( \sup_{[L^d + L^\infty]}^d := \cap_{\delta > 0} \mathcal{F}_\delta^{1/2} \)), as well as critical classes such as the Kato class \( \mathcal{K}_0^{d+1} \), the weak \( L^d \) class, the Campanato-Morrey class, the Chang-Wilson-T. Wolff class, see \( \text{[KiS]} \) sect. 4.

Set \( m_\delta := \pi^{\frac{d}{2}} (2e)^{-\frac{1}{2}} d^2 (d-1)^{\frac{d+1}{2}} \). Let \( b \in \mathcal{F}_\delta^{1/2} \) for some \( \delta \) such that \( m_\delta \delta < \frac{4(d-2)}{(d-1)^2} \).
Assume that $\{b_n\} \subset [L^\infty \cap C^1]^d \cap F_{\delta_1}^{1/2}$, $m_d\delta_1 < 4(d-2)/(d-1)^2$, $b_n \to b$ strongly in $[L^1_{\text{loc}}]^d$. Then [K1, Theorem 2], [KGS, Theorem 4.4]

$$s-C_\infty \lim_n e^{-t\Lambda_{C_\infty}(b_n)}$$

exists uniformly in $t \in [0,1]$, and hence determines a positivity preserving $L^\infty$ contraction $C_0$ semigroup $e^{-t\Lambda_{C_\infty}(b)}$ (Feller semigroup).

Here $\Lambda_{C_\infty}(b_n) := -\Delta + b_n \cdot \nabla$ of domain $(1 - \Delta)^{-1}C_\infty(R^d)$.

For instance, one can take

$$b_n := \gamma_{\varepsilon_n} \ast 1_n b, \quad n = 1, 2, \ldots,$$

where $1_n$ is the indicator of $\{x \in R^d : |x| \leq n, |b(x)| \leq n\}$ and $\gamma_\varepsilon(x) := \frac{1}{\varepsilon^d} \gamma \left( \frac{x}{\varepsilon} \right)$ is the K. Friedrichs mollifier, i.e. $\gamma(x) := c \exp \left( \frac{1}{|x|^2 - 1} \right)1_{|x| < 1}$ with the constant $c$ adjusted to $\int_{R^d} \gamma(x)dx = 1$, for appropriate $\varepsilon_n \downarrow 0$.

2. The space $D([0, \infty], R^d)$ is defined to be the set of all right-continuous functions $X : [0, \infty] \to R^d$ (here and elsewhere, $R^d := R^d \cup \{\infty\}$ is the one-point compactification of $R^d$) having the left limits, such that $X(t) = \infty$, $t > s$, whenever $X(s) = \infty$ or $X(s-) = \infty$.

By $F_t \equiv \sigma \{X(s) \mid 0 \leq s \leq t, X \in D([0, \infty], R^d)\}$ denote the minimal $\sigma$-algebra containing all cylindrical sets $\{X \in D([0, \infty], R^d) \mid (X(s_1), \ldots, X(s_n)) \in A, A \subset (R^d)^n \text{ is open}\}$.

By a classical result, for a given Feller semigroup $T^t$ on $C_\infty(R^d)$, there exist probability measures $\{P_x\}_{x \in R^d}$ on $F_\infty \equiv \sigma \{X(s) \mid 0 \leq s < \infty, X \in D([0, \infty], R^d)\}$ such that $(D([0, \infty], R^d), F_t, F_\infty, P_x)$ is a Markov process (strong Markov after completing the filtration) and

$$E_{P_x}[f(X(t))] = T^t f(x), \quad X \in D([0, \infty], R^d), \quad f \in C_\infty, \quad x \in R^d.$$ 

The space $C([0, \infty], R^d)$ is defined to be the set of all continuous functions $X : [0, \infty] \to R^d$.

Set $G_t := \sigma \{X(s) \mid 0 \leq s \leq t, X \in C([0, \infty], R^d)\}$, $G_\infty := \sigma \{X(s) \mid 0 \leq s < \infty, X \in C([0, \infty], R^d)\}$.

**Theorem 1** (Main result). Let $d \geq 3$, $b \in F_{\delta}^{1/2}$, $m_d\delta_1 < 4d^2/(d-1)^2$.

Let $(D([0, \infty], R^d), F_t, F_\infty, P_x)$ be the Markov process determined by $T^t = e^{-t\Lambda_{C_\infty}(b)}$. The following is true for every $x \in R^d$:

(i) The trajectories of the process are $P_x$ a.s. finite and continuous on $0 \leq t < \infty$.

We denote $P_x \uparrow (C([0, \infty], R^d), G_\infty)$ again by $P_x$.

(ii) $E_{P_x} \int_0^t |b(X(s))|ds < \infty, \quad X \in C([0, \infty], R^d)$.

(iii) There exists a $d$-dimensional Brownian motion $W(t)$ on $(C([0, \infty], R^d), G_t, P_x)$ such that $P_x$ a.s.

$$X(t) = x - \int_0^t b(X(s))ds + \sqrt{2}W(t), \quad t \geq 0,$$

i.e. $(X(t), W(t)), (C([0, \infty], R^d), G_t, G_\infty, P_x)$ is a weak solution to the SDE (3).

**Remark 1.** One can show, using the methods of this paper, that if $\{Q_x\}_{x \in R^d}$ is another weak solution to (3) such that

$$Q_x = w\lim \lim P_x(b_n) \quad \text{for every } x \in R^d,$$

where $\{b_n\} \subset F_{\delta_1}^{1/2}$, $m_d\delta_1 < 4d^2/(d-1)^2$, then $\{Q_x\}_{x \in R^d} = \{P_x\}_{x \in R^d}$.

**Theorem 1** covers critical-order singularities of $b$, as the following example shows.

**Example 1.** Consider the vector field $(d \geq 3)$

$$b(x) := c|x|^{-2}x, \quad c > 0,$$

then $b \in F_{\delta}^{1/2}$, $c = \frac{d-2}{2} \sqrt{d}$. 
1) If $c < 2m^{-1}_d(d - 2)^2(d - 1)^{-2}$, then by Theorem [1] the SDE
\[ X(t) = -\int_0^t b(X(s))ds + \sqrt{2}W(t), \quad t \geq 0. \]
has a weak solution. (For this particular vector field the result is, in fact, stronger, see Remark [2] below.)

2) If $c \geq d$, then the SDE doesn't have a weak solution.

Indeed, following [CE Example 1.17], suppose by contradiction that there is a weak solution to the SDE if $c \geq d$, i.e. there are a continuous process $X(t)$ and a Brownian motion $W(t)$ on a probability space $(\Upsilon, \mathcal{F}_t, \mathbb{Q})$ such that $\int_0^t |b(X(s))|ds < \infty$ and the SDE holds $\mathbb{Q}$ a.s. Then $X(t) = (X_1(t), \ldots, X_d(t))$ is a continuous semimartingale with cross-variation $[X_i, X_k]_t = 2\delta_{ik}t$. By Itô's formula,
\[ |X(t)|^2 = -2\int_0^t X(s) \cdot b(X(s))ds + 2\sqrt{2}\int_0^t X(s)dW(s) + 2\int_0^t d[W, W], \]
i.e.
\[ |X(t)|^2 = -2c\int_0^t 1_{X(s) \neq 0}ds + 2\sqrt{2}\int_0^t X(s)dW(s) + 2td. \]
If we accept that $\int_0^t 1_{X(s) = 0}ds = 0$ a.s., then, clearly,
\[ |X(t)|^2 = 2(d - c)\int_0^t 1_{X(s) \neq 0}ds + 2\sqrt{2}\int_0^t X(s)dW(s) \quad \text{a.s.} \]
Therefore, $|X(t)|^2 \geq 0$ is a local supermartingale if $c > d$ and is a local martingale if $c = d$. Then a.s. $|X(0)| = 0 \Rightarrow X(t) = 0$, which contradicts to $[X_1, X_1]_t = 2t$.

It remains to prove that $\int_0^t 1_{X(s) = 0}ds = 0$ a.s. It suffices to show that $\int_0^t 1_{X_1(s) = 0}ds = 0$ a.s. Since $X_1(t)$ is a continuous semimartingale, $[X_1, X_1]_t = 2t$, by the occupation times formula
\[ \int_0^t 1_{X_1(s) = 0}d[X_1, X_1]_s = \int_{-\infty}^\infty 1_{a=0}L_{a, X_1}^0(t)da = 0 \quad \text{a.s., where } L_{a, X_1}^0(t) \text{ is the local time of } X_1 \text{ at } a \text{ on } [0, t]. \]

**Remark 2.** Recall the following

**Definition.** A $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{F}_\delta$, the class of form-bounded vector fields, if $|b| \in L^2_{\text{loc}}$ and there exists $\lambda = \lambda_\delta > 0$ such that
\[ \|b|\lambda - \Delta|^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta}. \]

Note that $\mathbf{F}_{\delta_1} \subseteq \mathbf{F}_{\delta_1}^{1/2}$ for $\delta = \sqrt{\delta_1}$.

For $b \in \mathbf{F}_{\delta_1}$, the constraint $m_d\sqrt{\delta_1} < 4\frac{d^2-2}{(d-1)^2}$ in Theorem [1] can be relaxed to $\delta_1 \leq 1 + (\frac{2}{d-2})^2$. The proof of Theorem [1] extends to such $b$ after replacing Lemma A below by evident modifications of [KSi Lemma 5], [KiS Theorem 3.7].

For $b(x) := c|x|^{-2}x \in \mathbf{F}_{\delta_1}$, $\delta_1 := c^2\frac{4}{(d-2)^2}$, the result is even stronger: $-1 < c < \frac{1}{2}$ if $d = 3$, $-\infty < c < 1$ if $d = 4$, $-\infty < c < (d - 3)/2$ if $d \geq 5$ (after replacing Lemma A by evident modifications of Theorems 3.8, 3.9 in [KSi]).

We refer to [KiS] for a more detailed discussion on classes $\mathbf{F}_{\delta_1}, \mathbf{F}_{\delta_1}^{1/2}$.

**1. Preliminaries**

Denote by $C^{0,\alpha} = C^{0,\alpha}(\mathbb{R}^d)$ the space of Hölder continuous functions $(0 < \alpha < 1)$, $\mathbf{S}$ the L. Schwartz space of test functions, $W^{k,p} = W^{k,p}(\mathbb{R}^d, \mathcal{L}^d)$ $(k = 1, 2)$ the standard Sobolev spaces, $\mathcal{W}^\alpha,p, \alpha > 0$, the Bessel potential space endowed with norm $\|u\|_{p,\alpha} := \|g\|_p, u = (1 - \Delta)^{-\frac{\alpha}{2}}g, g \in L^p$, and $\mathcal{W}^{-\alpha,p'}, p' = p/(p - 1)$, the anti-dual of $\mathcal{W}^\alpha,p$. 


The proof of Theorem 1 is based on the following analytic results [Ki Theorems 1, 2], [KiS Theorems 4.3, 4.4].

Set
\[ I_s := \left[ \frac{2}{1 + \sqrt{1 - m_d \delta}}, \frac{2}{1 - \sqrt{1 - m_d \delta}} \right]. \]

For every \( p \in I_s \), there exists a holomorphic semigroup \( e^{-t \Lambda_p(b)} \) on \( L^p \) such that the resolvent set of \( -\Lambda_p(b) \) contains the half-plane \( \mathcal{O} := \{ \zeta \in \mathbb{C} : \text{Re}\zeta \geq \kappa_d \lambda \} \),

\[
(\zeta + \Lambda_p(b))^{-1} = (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{1}{2}} \frac{1}{2\pi} Q_p(q)(1 + T_p)^{-1} G_p(r)(\zeta - \Delta)^{-\frac{1}{2}}, \quad \zeta \in \mathcal{O},
\]

where \( 1 \leq r < q, \kappa_d := \frac{d - 1}{d - 1}, Q_p(q), G_p(r), T_p \in B(L^p), \)

\[
G_p(r) := b_r^\frac{1}{2} \cdot \nabla(\zeta - \Delta)^{-\frac{1}{2}} \frac{1}{2}, \quad b_r := |b|^{-\frac{1}{2}},
\]

\( Q_p(q), T(p) \) are the extensions by continuity of densely defined on \( \mathcal{E} := \bigcup_{\varepsilon > 0} e^{-\varepsilon |b|} L^p \) operators

\[
Q_p(q) \upharpoonright \mathcal{E} := (\zeta - \Delta)^{-\frac{1}{2}} |b|^{-\frac{1}{2}}, \quad T_p \upharpoonright \mathcal{E} := b_r^\frac{1}{2} \cdot \nabla(\zeta - \Delta)^{-\frac{1}{2}} |b|^{-\frac{1}{2}},
\]

\[
|T_p|_{p \rightarrow p} \leq m_d c_p \delta, \quad c_p := \frac{pp}{4}, \quad m_d c_p \delta < 1 \quad (\Leftrightarrow p \in I_s).
\]

By (1),

\[
(\zeta + \Lambda_p(b))^{-1} \in \mathcal{B}(W^{-\frac{1}{2}} L^p, W^{1 + \frac{1}{2}} L^p).
\]

Fix numbers \( p \in I_s, p > d - 1 \) and \( q \) sufficiently close to \( p \). By (1) and the Sobolev Embedding Theorem, \( (\zeta + \Lambda_p(b))^{-1} |L^p| \subset C^{\alpha, \alpha}, \alpha < 1 - \frac{1}{d - 1} \). Define \( \Lambda_{C_\infty}(b) \) by

\[
(\mu + \Lambda_{C_\infty}(b))^{-1} := \left( (\mu + \Lambda_p(b))^{-1} \upharpoonright L^p \cap C_\infty \right)^{cl_{C_\infty}}_{C_\infty}, \quad \mu \geq \kappa_d \lambda.
\]

Then

\[
e^{-t \Lambda_{C_\infty}(b)} \upharpoonright L^p \cap C_\infty \in \mathcal{B}(L^p, C_\infty), \quad p \in \left] d - 1, \frac{2}{1 - \sqrt{1 - m_d \delta}} \right[ , \quad t > 0.
\]

By (1),

\[
e^{-t \Lambda_{C_\infty}(b)} = s^{-C_\infty} \lim_{n \rightarrow \infty} e^{-t \Lambda_{C_\infty}(b_n)} \quad (\text{uniformly on every compact interval of } t \geq 0),
\]

where \( D(\Lambda_{C_\infty}(b_n)) = (1 - \Delta)^{-1} C_\infty \).

The following estimates are direct consequences of (1): There exist constants \( C_i = C_i(\delta, p), i = 1, 2 \), such that, for all \( h \in C_\infty \) and \( \mu \geq \kappa_d \lambda \delta, \)

\[
\left| (\mu + \Lambda_{C_\infty}(b))^{-1} |b_m| h \right|_{\infty} \leq C_1 |b_m| \frac{1}{p} h \|_p, \quad (7)
\]

\[
\left| (\mu + \Lambda_{C_\infty}(b))^{-1} |b_m - b_n| h \right|_{\infty} \leq C_2 |b_m - b_n| \frac{1}{p} h \|_p. \quad (8)
\]

Our proof of Theorem 1 employs also the following weighted estimates. Set

\[
\rho(y) \equiv \rho(y) := (1 + t |y|^2)^{-\nu}, \quad \nu > \frac{d}{2p} + 1, \quad l > 0, \quad y \in \mathbb{R}^d.
\]

**Lemma A.** Fix \( p \in I_s, p > d - 1 \). There exist constants \( K_i = K_i(\delta, p), i = 1, 2 \) such that, for all \( h \in C_\infty(\mathbb{R}^d) \), \( \mu \geq \kappa_d \lambda \delta \), and all sufficiently small \( l = l(\delta, p) > 0, \)

\[
\left| \rho(\mu + \Lambda_{C_\infty}(b_n))^{-1} h \right|_{\infty} \leq K_1 \rho h \|_p, \quad (E_1)
\]

\[
\left| \rho(\mu + \Lambda_{C_\infty}(b_n))^{-1} |b_m| h \right|_{\infty} \leq K_2 |b_m| \frac{1}{p} \rho h \|_p. \quad (E_2)
\]

Since \( m_d \delta < \frac{d - 2}{(d - 1)p} \), such \( p \) exists.
This technical lemma is proven in the appendix.

2. Proof of Theorem

Lemma 1. For every \( x \in \mathbb{R}^d \) and \( t > 0 \), \( b_n(X(t)) \to b(X(t)) \) \( \mathbb{P}_x \) a.s. as \( n \to \infty \).

Proof. By (5) and the Dominated Convergence Theorem, for any \( \mathcal{L}^d \)-measure zero set \( G \subset \mathbb{R}^d \) and every \( t > 0 \), \( \mathbb{P}_x[X(t) \in G] = 0 \). Since \( b_n \to b \) pointwise in \( \mathbb{R}^d \) outside of an \( \mathcal{L}^d \)-measure zero set, we have the required.

Let \( \mathbb{P}_x^n \) be the probability measures associated with \( e^{-t\Lambda C_{\infty}(b_n)} \), \( n = 1, 2, \ldots \) Set \( \mathbb{E}_x := \mathbb{E}_{\mathbb{P}_x} \), and \( \mathbb{E}_x^n := \mathbb{E}_{\mathbb{P}_x^n} \).

Fix a \( u \in C([0, \infty]) \), \( u(s) = 1 \) if \( 0 \leq s \leq 1 \), \( u(s) = 0 \) if \( s \geq 2 \). Set
\[
\xi_k(y) := \left\{ \begin{array}{ll}
  u(|y| - 1 - k) & |y| \geq k, \\
  1 & |y| < k.
\end{array} \right.
\]
(9)

Lemma 2. For every \( x \in \mathbb{R}^d \) and \( t > 0 \), \( \mathbb{P}_x[X(t) = \infty] = 0 \).

Proof. First, let us show that for every \( \mu \geq \kappa_d \lambda_d \),
\[
\int_0^\infty e^{-\mu t} \mathbb{E}_x^n[\xi_k(X(t))] dt \to \frac{1}{\mu} \quad \text{as } k \to \infty \text{ uniformly in } n.
\]
(10)

Since \( \int_0^\infty e^{-\mu t} \mathbb{E}_x^n[1_{\mathbb{R}_d}(X(t))] dt = \frac{1}{\mu} \), (10) is equivalent to \( \int_0^\infty e^{-\mu t} \mathbb{E}_x^n[(1_{\mathbb{R}_d} - \xi_k)(X(t))] dt \to 0 \) as \( k \to \infty \) uniformly in \( n \). We have
\[
\int_0^\infty e^{-\mu t} \mathbb{E}_x^n[(1_{\mathbb{R}_d} - \xi_k)(X(t))] dt
\]
(we use the Dominated Convergence Theorem)
\[
= \lim_{r \to \infty} \int_0^\infty e^{-\mu t} \mathbb{E}_x^n[\xi_r(1 - \xi_k)(X(t))] dt
\]
\[
= \lim_{r \to \infty} (\mu + \Lambda C_{\infty}(b_n))^{-1} \int_0^\infty \mathbb{E}_x^n[\xi_r(1 - \xi_k)](x)
\]
(we apply crucially (E.1))
\[
\leq \rho(x)^{-1} K_1 \lim_{r \to \infty} \| \rho \xi_r(1 - \xi_k) \|_p \leq \rho(x)^{-1} K_1 \| \rho(1 - \xi_k) \|_p \to 0 \quad \text{as } k \to \infty,
\]
which yields (10).

Now, since \( \mathbb{E}_x[\xi_k(X(t))] = \lim_n \mathbb{E}_x^n[\xi_k(X(t))] \) uniformly on every compact interval of \( t \geq 0 \), see (5), it follows from (10) that
\[
\int_0^\infty e^{-\mu t} \mathbb{E}_x[\xi_k(X(t))] dt \to \frac{1}{\mu} \quad \text{as } k \to \infty.
\]

Finally, suppose that \( \mathbb{P}_x[X(t) = \infty] \) is strictly positive for some \( t > 0 \). By the construction of \( \mathbb{P}_x \), \( t \mapsto \mathbb{P}_x[X(t) = \infty] \) is non-decreasing, and so \( \kappa := \int_0^\infty e^{-\mu t} \mathbb{E}_x[1_{X(t) = \infty}] dt > 0 \). Now,
\[
\frac{1}{\mu} = \int_0^\infty e^{-\mu t} \mathbb{E}_x[1_{\mathbb{R}_d}(X(t))] dt \geq \kappa + \int_0^\infty e^{-\mu t} \mathbb{E}_x[\xi_k(X(t))] dt.
\]
Selecting \( k \) sufficiently large, we arrive at contradiction. □

The space \( D([0, \infty[, \mathbb{R}^d) \) is defined to be the subspace of \( D([0, \infty[, \mathbb{R}^d) \) consisting of the trajectories \( X(t) \neq \infty, 0 \leq t < \infty \). Let \( \mathcal{F}^t := \sigma(X(s) \mid 0 \leq s \leq t, X \in \mathcal{D}) \), \( \mathcal{F}_{\infty} := \sigma(X(s) \mid 0 \leq s < \infty, X \in \mathcal{D}) \).

By Lemma 2, \( (D([0, \infty[, \mathbb{R}^d), \mathcal{F}_{\infty}) \) has full \( \mathbb{P}_x \)-measure in \( (D([0, \infty[, \mathbb{R}^d), \mathcal{F}_{\infty}) \). We denote the restriction of \( \mathbb{P}_x \) from \( (D([0, \infty[, \mathbb{R}^d), \mathcal{F}_{\infty}) \) to \( (D([0, \infty[, \mathbb{R}^d), \mathcal{F}_{\infty}') \) again by \( \mathbb{P}_x \).
Lemma 3. For every $x \in \mathbb{R}^d$ and $g \in C_c^\infty(\mathbb{R}^d)$,
\[
g(X(t)) - g(x) + \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds,
\]
is a martingale relative to $(D([0, \infty[, \mathbb{R}^d), F_t, \mathbb{P})$.

Proof. Fix $\mu \geq \kappa_d \lambda_d$. In what follows, $0 < t \leq T < \infty$.

(a) $\mathbb{E}_x \int_0^t |b \cdot \nabla g|(X(s))ds < \infty$. Indeed,
\[
\mathbb{E}_x \int_0^t |b \cdot \nabla g|(X(s))ds = \liminf_{n \to \infty} \mathbb{E}_x \int_0^t e^{-s \Lambda C_\infty(b)}|b_n \cdot \nabla g|(x)ds
\]
\[
\leq \mathbb{E}_x \int_0^t e^{\mu T} e^{-\mu s}e^{-s \Lambda C_\infty(b)}|b_n \cdot \nabla g|(x)ds
\]
\[
\leq \mathbb{E}_x \int_0^t e^{\mu T} \mathbb{E}_x \int_0^t (|b||\nabla g|^p)^{\frac{1}{p}} ds < \infty.
\]

(b) $\mathbb{E}_x^n [g(X(t))] \to \mathbb{E}_x[g(X(t))]$, $\mathbb{E}_x^n \int_0^t (b \cdot \nabla g)(X(s))ds \to \mathbb{E}_x \int_0^t (b \cdot \nabla g)(X(s))ds$, and also, for $h \in C_c^\infty$, $\mathbb{E}_x^n \int_0^t (h|b_n)(X(s))ds \to \mathbb{E}_x \int_0^t (|b|h)(X(s))ds$ as $n \to \infty$. Indeed, the first convergence follows from (a), and the third one from $\mathbb{E}_x \int_0^t (|b|h)(X(s))ds < \infty$, a straightforward modification of (a).

(c) $\mathbb{E}_x \int_0^t (b \cdot \nabla g)(X(s))ds - \mathbb{E}_x^n \int_0^t (b \cdot \nabla g)(X(s))ds \to 0$. We have:
\[
\mathbb{E}_x \int_0^t (b \cdot \nabla g)(X(s))ds - \mathbb{E}_x^n \int_0^t (b \cdot \nabla g)(X(s))ds
\]
\[
= \int_0^t \left( e^{-s \Lambda C_\infty(b)} - e^{-s \Lambda C_\infty(b)} \right) (b \cdot \nabla g)(x)ds
\]
\[
= \int_0^t \left( e^{-s \Lambda C_\infty(b)} - e^{-s \Lambda C_\infty(b)} \right) ((b_n - b_m) \cdot \nabla g)(x)ds
\]
\[
+ \int_0^t \left( e^{-s \Lambda C_\infty(b)} - e^{-s \Lambda C_\infty(b)} \right) (b_m \cdot \nabla g)(x)ds =: S_1 + S_2,
\]
where $m$ is to be chosen. Arguing as in the proof of (a), we obtain:
\[
S_1(x) \leq e^{\mu T} (\mu + \Lambda C_\infty(b))^{-1}(|b_n - b_m|) \cdot \nabla g)(x) + e^{\mu T} (\mu + \Lambda C_\infty(b))^{-1}(|b_n - b_m|) \cdot \nabla g)(x).
\]
Since $b_n - b_m \to 0$ in $L^1_{\text{loc}}$ as $n, m \uparrow \infty$, (a) yields $S_1 \to 0$ as $n, m \uparrow \infty$. Now, fix a sufficiently large $m$. Since $e^{-s \Lambda C_\infty(b)} = s^{-C_\infty} \lim_{s \to 0} e^{-s \Lambda C_\infty(b)}$ uniformly in $0 \leq s \leq T$, cf. (a), we have $S_2 \to 0$ as $n \to \infty$. The proof of (c) is completed.

Now we are in position to complete the proof of Lemma 3. Since $b_n \in [C_c^\infty(\mathbb{R}^d)]^d$,
\[
g(X(t)) - g(x) + \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds
\]
is a martingale under $\mathbb{P}^n$.
so the function
\[ x \mapsto \mathbb{E}_x^n [g(X(t))] - g(x) + \mathbb{E}_x^n \int_0^t (-\Delta g + b_n \cdot \nabla g)(X(s))ds \] is identically zero in \( \mathbb{R}^d \).

Thus by (b), the function
\[ x \mapsto \mathbb{E}_x[g(X(t))] - g(x) + \mathbb{E}_x \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds \] is identically zero in \( \mathbb{R}^d \), i.e. \( g(X(t)) - g(x) + \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds \) is a martingale under \( \mathbb{P}_x \). \( \square \)

**Lemma 4.** For \( x \in \mathbb{R}^d \), \( C([0, \infty[; \mathbb{R}^d) \) has full \( \mathbb{P}_x \)-measure in \( D([0, \infty[, \mathbb{R}^d) \).

**Proof.** Let \( A, B \) be arbitrarily bounded closed sets in \( \mathbb{R}^d \), \( \text{dist}(A, B) > 0 \). Fix \( g \in C^\infty_c(\mathbb{R}^d) \) such that \( g = 0 \) on \( A \), \( g = 1 \) on \( B \). Set \( (X \in D([0, \infty[, \mathbb{R}^d)) \)

\[ M^g(t) := g(X(t)) - g(x) + \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds, \quad K^g(t) := \int_0^t 1_A(X(s-))dM^g(s), \]

then

\[ K^g(t) = \sum_{s \leq t} 1_A(X(s-))g(X(s)) + \int_0^t 1_A(X(s-))(-\Delta g + b \cdot \nabla g)(X(s))ds. \]

By Lemma 3, \( M^g(t) \) is a martingale, and hence so is \( K^g(t) \). Thus, \( \mathbb{E}_x \left[ \sum_{s \leq t} 1_A(X(s-))g(X(s)) \right] = 0. \) Using the Dominated Convergence Theorem, we obtain \( \mathbb{E}_x \left[ \sum_{s \leq t} 1_A(X(s-))1_B(X(s)) \right] = 0. \) The proof of Lemma 3 is completed. \( \square \)

We denote the restriction of \( \mathbb{P}_x \) from \( (D([0, \infty[, \mathbb{R}^d), F_t^\infty) \) to \( (C([0, \infty[, \mathbb{R}^d), G_t^\infty) \) again by \( \mathbb{P}_x. \) Lemma 3 and Lemma 4 combined yield

**Lemma 5.** For every \( x \in \mathbb{R}^d \) and \( g \in C^\infty_c(\mathbb{R}^d) \),

\[ g(X(t)) - g(x) + \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds, \quad X \in C([0, \infty[, \mathbb{R}^d), \]

is a continuous martingale relative to \( (C([0, \infty[, \mathbb{R}^d), G_t, \mathbb{P}_x) \).

**Lemma 6.** For every \( x \in \mathbb{R}^d \) and \( t > 0, \mathbb{E}_xf^t_0 |b(X(s))|ds < \infty \), and, for \( f(y) = y_i \) or \( f(y) = y_\iota y_j \), \( 1 \leq i, j \leq d, \)

\[ f(X(t)) - f(x) + \int_0^t (-\Delta f + b \cdot \nabla f)(X(s))ds, \quad X \in C([0, \infty[, \mathbb{R}^d), \]

is a continuous martingale relative to \( (C([0, \infty[, \mathbb{R}^d), G_t, \mathbb{P}_x) \).

**Proof.** Define \( f_k := \xi_k f \in C^\infty_c(\mathbb{R}^d) \) (see 12 for the definition of \( \xi_k \)). Set \( \alpha := ||\nabla \xi_k||_\infty, \beta := ||\Delta \xi_k||_\infty \) (\( \alpha, \beta \) don’t depend on \( k \)). Fix \( 0 < T < \infty \). In what follows, \( 0 < t < T \).

(a) \( \mathbb{E}_x \int_0^t |\left|b(\nabla f') + \alpha |f'|\right|)(X(s))ds < \infty. \)

Indeed, set \( \varphi := |\nabla f' + \alpha |f'| \in C \cap W^{1,2}_{loc}, \varphi_k := \xi_k \varphi_k \in C \cap W^{1,2}_{loc} \). First, let us prove that

\[ \mathbb{E}_x^n \int_0^t \left|b_n \varphi_k\right|(X(s))ds \leq \text{const independent of } n, k. \]
Fix $p \in ]d-1, \frac{2}{1-d}]$. Then $\sqrt{(\rho \varphi)}^p \in W^{1,2}$ (recall that $\rho(x) := (1 + |x|^2)^{-\nu}, \nu > \frac{d}{2p} + 1$.) We have
\[
E^n_x \int_0^t (|b_n| \varphi_k)(X(s))ds = \int_0^t e^{-s \Lambda C_\infty(b_n)}|b_n| \varphi_k(x)ds \\
\leq e^{\mu T} (\mu + \Lambda C_\infty(b_n))^{-1}|b_n| \varphi_k(x) \\
(\text{we apply (E2)}) \\
\leq e^{\mu T} \rho(x)^{-1}K_2 (|b_n|(\rho \varphi_k)^p)^{\frac{1}{p}} \leq e^{\mu T} \rho(x)^{-1}K_2 (|b_n|(\rho \varphi_k)^p)^{\frac{1}{p}} \\
\left(\text{we use } b_n \in \mathbb{P}^{1/2}, m_d \delta_1 < 4 \frac{d-1}{(d-2)^2}\right) \\
\leq e^{\mu T} \rho(x)^{-1}K_2 \delta_1 \| (\lambda - \Delta)^{\frac{1}{2}} \sqrt{(\rho \varphi)}^p \|_2^2 < \infty.
\]
By step (b) in the proof of Lemma 3, $E^n_x \int_0^t (|b_n| \varphi_k)(X(s))ds \to E_x \int_0^t (|b| \varphi_k)(X(s))ds$ as $n \uparrow \infty$. Therefore, $E_x \int_0^t (|b| \varphi_k)(X(s))ds \leq C$ implies $E_x \int_0^t (|b| \varphi_k)(X(s))ds \leq C (C \neq C(k))$. Now, Fatou’s Lemma yields the required.

(b) For every $t > 0$, $E_x \int_0^t (|\Delta f| + 2\alpha |\nabla f| + \beta |f|)(X(t))ds < \infty$.

The proof is similar to the proof of (a) (use (E1) instead of (E2)).

(c) For every $t > 0$, $E_x[|f(X(t))|] < \infty$.

Indeed, set $g(y) := 1 + |y|^2$, $y \in \mathbb{R}^d$. Since $|f| \leq g$, it suffices to show that $E_x[g(X(t))] < \infty$. Set $g_k(y) := \xi_k(y)g(y)$. By Lemma 5,
\[
E_x[g_k(X(t))] = g_k(x) - E_x \int_0^t (-\Delta g_k)(X(s))ds - E_x \int_0^t (b \cdot \nabla g_k)(X(s))ds.
\]
Note that
\[
\sup_k E_x \int_0^t (|b| |g_k|)(X(s))ds < \infty, \quad \sup_k E_x \int_0^t |\Delta g_k||X(s))ds < \infty
\]
for, arguing as in the proofs of (a) and (b), we have:
\[
E_x \int_0^t (|b||\nabla g| + \alpha |g|)(X(s))ds < \infty, \quad E_x \int_0^t (|\Delta g| + 2\alpha |\nabla g| + \beta |g|)(X(t))ds < \infty.
\]
Therefore, $\sup_k E_x[g_k(X(t))] < \infty$, and so, by the Monotone Convergence Theorem, $E_x[g(X(t))] < \infty$. This completes the proof of (c).

Let us complete the proof of Lemma 6. By (a), $E_x \int_0^t |b(X(s))|ds < \infty$. By (a)-(c),
\[
M^f(t) := f(X(t)) - f(x) + \int_0^t (-\Delta f + b \cdot \nabla f)(X(s))ds, \quad t > 0,
\]
satisfies $E_x[|M^f(t)|] < \infty$ for all $t > 0$. By Lemma 7 for every $k$, $M^f_k(t)$ is a martingale relative to $(\mathcal{C}([0, \infty[, \mathbb{R}^d), \mathcal{G}_t, \mathbb{P}_x)$. By (a) and the Dominated Convergence Theorem, since $|\nabla f_k| \leq |\nabla f| + \alpha |f|$ for all $k$, we have $E_x \int_0^t (b \cdot \nabla f_k)(X(s))ds \to E_x \int_0^t (b \cdot \nabla f)(X(s))ds$. By (b), $E_x \int_0^t (-\Delta f_k)(X(s))ds \to E_x \int_0^t (-\Delta f)(X(s))ds$. By (c), $E_x[f_k(X(t))] \to E_x[f(X(t))]$. So, $M^f(t)$ is also a martingale on $(\mathcal{C}([0, \infty[, \mathbb{R}^d), \mathcal{G}_t, \mathbb{P}_x)$. The proof of Lemma 6 is completed.

We are in position to complete the proof of Theorem 1. Lemma 4 yields (i). Lemma 6 yields (ii). By classical results, Lemma 6 yields existence of a $d$-dimensional Brownian motion $W(t)$ on $(\mathcal{C}([0, \infty[, \mathbb{R}^d), \mathcal{G}_t, \mathbb{P}_x)$ such that $X(t) = x - \int_0^t b(X(s))ds + \sqrt{2}W(t), 0 \leq t < \infty, \mathbb{P}_x \text{ a.s.}$ (iii). The proof of Theorem 1 is completed.
The proofs of (\( E_1 \)) and (\( E_2 \)) are similar. For instance, let us prove (\( E_1 \)).

We will use the bounds:

\[
\|(\mu - \Delta)^{-\frac{1}{2}}|b|^{\frac{3}{p}}\|_{p \to p} \leq C_{p,\delta} < \infty, \quad \|b|^\frac{1}{p}(\mu - \Delta)^{-\frac{2}{p}}\|_{p \to p} \leq C_{p',\delta} < \infty \quad \text{(by duality)}
\]

(11)

(for \( \|Q_p(q)\|_{p \to p} \leq C_{p,q,\delta} < \infty \), see section \( \text{I} \)).

By the definition of \( \rho \),

\[
|\nabla \rho| \leq \nu \sqrt{I_0} \equiv C_1 \sqrt{I_0}, \quad |\Delta \rho| \leq 2\nu(2\nu + d + 2)l \rho \equiv C_2 \rho.
\]

(\( \star \star \star \))

Set \( u = (\mu - \Delta)^{-1}f, \ f \in C_c(\mathbb{R}^d) \). We have \( (\mu - \Delta)\rho u = - (\Delta \rho)u - 2\nabla \rho \cdot \nabla u + \rho(\mu - \Delta)u \), and so

\[
\rho u = - (\mu - \Delta)^{-1}(\Delta \rho)u - 2(\mu - \Delta)^{-1}\nabla \rho \cdot \nabla u + (\mu - \Delta)^{-1}\rho(\mu - \Delta)u.
\]

Thus,

\[
\rho(\mu - \Delta)^{-1}f = - (\mu - \Delta)^{-1}(\Delta \rho)(\mu - \Delta)^{-1}f
\]

\[
- 2(\mu - \Delta)^{-1}\nabla \rho \cdot \nabla (\mu - \Delta)^{-1}f
\]

\[
+ (\mu - \Delta)^{-1}\rho f.
\]

(\( \star \star \star \star \))

We obtain from (\( \star \star \star \star \)):

\[
\rho \nabla (\mu - \Delta)^{-1}f = - (\nabla \rho)(\mu - \Delta)^{-1}f
\]

\[
- \nabla (\mu - \Delta)^{-1}(\Delta \rho)(\mu - \Delta)^{-1}f
\]

\[
- 2\nabla (\mu - \Delta)^{-1}\nabla \rho \cdot \nabla (\mu - \Delta)^{-1}f
\]

\[
+ \nabla (\mu - \Delta)^{-1}\rho f.
\]

Then

\[
I_0 := \|\rho(|b_n|^{\frac{1}{p}} + 1)\nabla (\mu - \Delta)^{-1}f\|_p
\]

\[
\leq C_1 \sqrt{I_0} \|(|b_n|^{\frac{1}{p}} + 1)\rho(\mu - \Delta)^{-1}f\|_p
\]

\[
+ C_2 lm_d \|(|b_n|^{\frac{1}{p}} + 1)(\kappa^{-\frac{1}{2}}_{d} - (\mu - \Delta)^{-\frac{1}{2}})\rho(|\mu - \Delta|)^{-1}f\|_p
\]

\[
+ 2C_1 \sqrt{I_0} \|(|b_n|^{\frac{1}{p}} + 1)(\kappa^{-\frac{1}{2}}_{d} - (\mu - \Delta)^{-\frac{1}{2}})\rho(\nabla (\mu - \Delta)^{-1}f\|_p
\]

\[
+ \|(|b_n|^{\frac{1}{p}} + 1)\nabla (\mu - \Delta)^{-1}\rho f\|_p
\]

\[
=: C_1 \sqrt{I_1} + C_2 lm_d I_2 + 2C_1 \sqrt{I_0} I_3 + \|(|b_n|^{\frac{1}{p}} + 1)\nabla (\mu - \Delta)^{-1}\rho f\|_p.
\]

We have:

\[
I_3 \leq \|(|b_n|^{\frac{1}{p}} + 1)(\kappa^{-\frac{1}{2}}_{d} - (\mu - \Delta)^{-\frac{1}{2}})\|_{p \to p}\|\rho(\nabla (\mu - \Delta)^{-1}f\|_p
\]

(we use (11))

\[
\leq c\|\rho \nabla (\mu - \Delta)^{-1}f\|_p \leq cI_0.
\]

We estimate \( I_1 \) using (\( \star \star \star \star \)) and (\( \star \star \star \)):

\[
I_1 \leq C_2 \|(|b_n|^{\frac{1}{p}} + 1)(\mu - \Delta)^{-1}\rho(\mu - \Delta)^{-1}f\|_p
\]

\[
+ 2C_1 \sqrt{I_0} \|(|b_n|^{\frac{1}{p}} + 1)(\mu - \Delta)^{-1}\|_{p \to p}\|\rho \nabla (\mu - \Delta)^{-1}f\|_p
\]

\[
+ \|(|b_n|^{\frac{1}{p}} + 1)(\mu - \Delta)^{-1}\rho f\|_p,
\]

and so

\[
I_1 \leq C_2 I_1 + 2C_1 \sqrt{I_0} I_3 + \|(|b_n|^{\frac{1}{p}} + 1)(\mu - \Delta)^{-1}\rho f\|_p.
\]
We estimate $I_2$ again using (2) and (3): 
\[
I_2 \leq C_2 l \|((b_n)_{\|\rho\|_p} + 1)(\kappa_1^{-1} \mu - \Delta)^{-\frac{1}{2}} (\mu - \Delta)^{-1} \rho |(\mu - \Delta)^{-1} f\|_p \\
+ 2C_1 \sqrt{I} \|((b_n)_{\|\rho\|_p} + 1)(\kappa_1^{-1} \mu - \Delta)^{-\frac{1}{2}} (\mu - \Delta)^{-1} \rho |\nabla(\mu - \Delta)^{-1} f\|_p \\
+ \|((b_n)_{\|\rho\|_p} + 1)(\kappa_1^{-1} \mu - \Delta)^{-\frac{1}{2}} \| (\mu - \Delta)^{-1} \rho f\|_p ,
\]
and so $I_2 \leq C_2 c' I_1 + 2C_1 c' \sqrt{I_3} + \|((b_n)_{\|\rho\|_p} + 1)(\kappa_1^{-1} \mu - \Delta)^{-\frac{1}{2}} (\mu - \Delta)^{-1} \rho f\|_p$.

Assembling the above estimates, we conclude that there exists a constant $C > 0$ such that, for any $\varepsilon_0 > 0$, there exists a sufficiently small $l > 0$ such that 
\[
(1 - \varepsilon_0) l_0 \leq \|((b_n)_{\|\rho\|_p} + 1) (\mu - \Delta)^{-1} \rho f\|_p \\
+ C\varepsilon_0 \left[ \|((b_n)_{\|\rho\|_p} + 1)(\mu - \Delta)^{-1} \rho f\|_p + \|((b_n)_{\|\rho\|_p} + 1)(\kappa_1^{-1} \mu - \Delta)^{-\frac{1}{2}} (\mu - \Delta)^{-1} \rho f\|_p \right].
\]

Put $f := (b_n)_{\|\rho\|_p} h$, $h \in C_0$, Then, using $\|T_p(b_n)\|_{p \to \rho} \leq m_d c_p \delta$ (cf. section 11, and applying (11) to the terms in brackets [ ], we obtain: For any $\varepsilon > 0$ there exists $l > 0$ such that, uniformly in $n$, 
\[
\|\rho((b_n)_{\|\rho\|_p} + 1) (\mu - \Delta)^{-1} |b_n|_{\|\rho\|_p} h\|_p < (1 + \varepsilon) m_d c_p \delta \|\rho h\|_p , \tag{12}
\]

so 
\[
\|\rho T_p(b_n) h\|_p \leq (1 + \varepsilon) m_d c_p \delta \|\rho h\|_p . \tag{13}
\]

We select $\varepsilon > 0$ so that $(1 + \varepsilon) m_d c_p \delta < 1$. (Recall that $m_d c_p \delta < 1$.)

Arguing as in the proof of (12) but taking $f := h$ we find a constant $M_1 < \infty$ such that 
\[
\|\rho |b_n|^{\frac{1}{2}} \nabla (\mu - \Delta)^{-1} h\|_p \leq M_1 \|\rho h\|_p , \tag{14}
\]

Also, we find a constant $M_2 < \infty$ such that 
\[
\|\rho (\mu - \Delta)^{-1} |b_n|^{\frac{1}{2}} h\|_{\infty} \leq M_2 \|\rho h\|_p , \tag{15}
\]

Indeed, using (3) with $f := (b_n)_{\|\rho\|_p} h$, we obtain 
\[
\|\rho (\mu - \Delta)^{-1} |b_n|^{\frac{1}{2}} h\|_{\infty} \leq C_2 \|\rho |b_n|^{\frac{1}{2}} h\|_{\infty} \|\rho (\mu - \Delta)^{-1} |b_n|^{\frac{1}{2}} h\|_{\infty} \\
+ 2C_1 \sqrt{I} \|\rho (\mu - \Delta)^{-1} \|_{p \to \rho} \|\rho \nabla(\mu - \Delta)^{-1} |b_n|^{\frac{1}{2}} h\|_p \\
+ \|\rho (\mu - \Delta)^{-\frac{1}{2}} \|_{p \to \rho} \|Q_p(q) \rho h\|_p ,
\]

where $\|\rho \nabla(\mu - \Delta)^{-1} |b_n|^{\frac{1}{2}} h\|_p \leq (1 + \varepsilon) m_d c_p \delta \|\rho h\|_p$ by (12), and $\| (\mu - \Delta)^{-\frac{1}{2}} \|_{p \to \rho} < \infty$ because $p > d - 1$ and $q$ can be chosen arbitrarily close to $p$. Select $l > 0$ so that $C_2 l \mu^{-1} < 1$. (15) follows.

Now, (11) combined with (13)-(15) yields (1).

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