ENumeration of Singular Varieties with Tangency Conditions

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Abstract. We construct the algebraic cobordism theory of bundles and divisors on varieties. It has a simple basis (over \( \mathbb{Q} \)) from projective spaces and its rank is equal to the number of Chern numbers. An application of this algebraic cobordism theory is the enumeration of singular subvarieties with give tangent conditions with a fixed smooth divisor, where the subvariety is the zero locus of a section of a vector bundle. We prove that the generating series of numbers of such subvarieties gives a homomorphism from the algebraic cobordism group to the power series ring. This implies that the enumeration of singular subvarieties with tangency conditions is governed by universal polynomials of Chern numbers, when the vector bundle is sufficiently ample. This result combines and generalizes the Caporaso-Harris recursive formula, Göttsche’s conjecture, classical De Jonquièere’s Formula and node polynomials from tropical geometry.

1. Introduction

The number of (possibly reduced) \( \delta \)-nodal degree \( d \) curves passing through general \( \frac{d(d+3)}{2} \) points on the projective plane is classically known as the Severi degree \( N^{d,\delta} \), because it is the degree of the Severi variety in \( |O(d)| \). They remained unknown for a very long period until Caporaso and Harris [6] found a recursive formula to determine all \( N^{d,\delta} \). This recursive formula contains not only the Severi degrees but also the number of \( \delta \)-nodal curves with tangency condition at assigned points (denoted by \( \alpha \)) and unassigned points (denoted by \( \beta \)), which is called the Caporaso-Harris invariants \( N^{d,\delta}(\alpha, \beta) \).

Di Francesco and C. Itzykson [7] conjectured that for any fixed \( \delta \), the Severi degrees \( N^{d,\delta} \) is a polynomial in \( d \) if \( d \) is large enough. This conjecture was proven by Fomin and Mikhalkin [8] in 2009. They used methods from tropical geometry and reduced the problem to counting certain combinatorial diagrams.

For arbitrary smooth surfaces \( S \) and line bundles \( L \), Göttsche [10] conjectured that the number of \( \delta \)-nodal curves in the linear system of a sufficiently ample line bundle is given by a universal polynomial of degree \( \delta \) in \( L^2, LK_S, c_1(S)^2 \) and \( c_2(S) \). The case of \( \delta \leq 3 \) can be computed directly by standard intersection theory and was known in the 19th century. In 1994, Vainsencher [28] proved and computed the universal polynomials for \( \delta \leq 6 \) and later Kleiman and Piene [14] improved the result to \( \delta \leq 8 \). In 2000, A.K. Liu [22, 23] proposed a proof of existence of universal polynomials for all \( \delta \) using sympletic geometry.

The first complete proof of Göttsche’s conjecture was achieved by showing the number of nodal curves are invariants of the algebraic cobordism group of surfaces and line

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bundles by the author [27]. Kool, Shende and Thomas [16] gave another proof by different techniques in algebraic geometry.

1.1. Tangency conditions. On the projective plane, general curves intersect a fixed line at isolated points and the tangency conditions at those points can be complete classified by the intersection multiplicities, which are nonnegative integers. Therefore the tangent conditions of a plane curve with a fixed line can be recorded by two sequences of nonnegative integers

\[ \alpha = (\alpha_1, \alpha_2, \ldots) \quad \text{and} \quad \beta = (\beta_1, \beta_2, \ldots), \]

which indicate tangency multiplicity \( i \) at \( \alpha_i \) assigned points and \( \beta_i \) unassigned points. The numbers of \( \delta \)-nodal degree \( d \) curves which having tangency type \((\alpha, \beta)\) with a fixed line are Caporaso-Harris invariants \( N^{d,\delta}(\alpha, \beta) \).

Caporaso and Harris’ recursive formula [6] completely determines all Caporaso-Harris invariants, but it is hard to see other structures from it. Inspired by Fomin and Mikhalkin’s methods, Block [4] showed that \( N^{d,\delta}(\alpha, \beta) \) are polynomials in the components of \( \alpha \) and \( \beta \) multiplied by some natural coefficients if \( \sum \beta_i \geq \delta \). He called them node polynomials and gave the formula when \( \delta \leq 6 \).

In higher dimensions, the tangency conditions of a subvariety with a divisor are much more complicated. If the subvarieties are vanishing locus of sections of vector bundles, then their intersections with a divisor generically have the expected dimension and may have isolated singularities. If so, the singularity must be isolated complete intersection singularities (ICIS). For this reason, in this article we only impose tangency conditions at isolated points so that generically the subvarieties and divisor still intersect transversally of expected codimension. The tangency types are recorded by two finite collections of ICIS: \( \alpha \) (at assigned points) and \( \beta \) (at unassigned points). Caporaso-Harris invariants can be recovered by imposing suitable ICIS. Some cases we do not discuss include: a sphere touching a plane only at a point, or the parabolic cylinder \( V(z = x^2) \) intersecting the \( xy \)-plane at a double line in 3-space.

Definition 1.1. Let \( X \) be a subvariety and \( D \) be a smooth divisor of a variety \( Y \). We say \( X \) is tangent to \( D \) at \( p \) if the scheme-theoretic intersection \( X \cap D \) is singular at \( p \). The tangency type of \( X \) with \( D \) is the singularity type of \( X \cap D \).

1.2. Non-induced singular type. In [20], Li and the author generalized Göttsche’s conjecture to curves with arbitrary singularities (also independently by [25]). The key idea is showing the number of such varieties is an invariant of the algebraic cobordism group of bundles on varieties \( \omega_{2,1}(\mathbb{C}) \).

For Caporaso-Harris invariants, general nodal curves are tangent to the fixed line at smooth points of the curves. However, in higher dimension certain tangency conditions may only occur at singular points of subvarieties. For example, consider an ICIS defined by \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0), f(x, y) = (x^2, y^3) \). Let \( C \) be a curve in \( \mathbb{C}^3 \) defined by \( g_1 = g_2 = 0 \). If the intersection of \( C \) and the \( xy \)-plane has this ICIS at a point \( p \), then the Jacobian matrix of \( g_1 \) and \( g_2 \) at \( p \) has rank at most one. Therefore \( C \) must be singular at \( p \).

In general, consider an ICIS defined by \( r \) equations. If the Jacobian matrix of those \( r \) equations has rank less than \( r - 1 \), then any codimension \( r \) subvariety with tangency type with a divisor of this ICIS must be singular at the point, since the Jacobian matrix of the defining equations has rank less than \( r \). Then this point can be counted as a tangency point or a singular point, and there are many possible singularity types. The following definition is made to avoid the ambiguity.
Definition 1.2. Let $X$ be a subvariety and $D$ be a smooth divisor of a variety $Y$. A non-induced singularity point of $X$ is a singular point of $X$ which is not on $D$. The non-induced singular type of $X$ is the collection of ICIS of $X$ at its non-induced singular points.

If $X$ is singular at $p \in X \cap D$ and $X \cap D$ has codimension 1 in $X$. Then a computation of tangent spaces shows $X \cap D$ must be singular at $p$. As a result, $X$ is always tangent to $D$ at singular points of $X$, if the points lie on $D$. Therefore the singular points of $X$ are either on $D$ or non-induced.

1.3. Main results. If $\delta$ is a collection of ICIS, denote the number of elements in $\delta$ by $|\delta|$. A vector bundle $E$ on $Y$ is $k$-very ample if for every zero-dimensional scheme $Z$ of length $k+1$ in $Y$, the natural restriction map $H^0(E) \to H^0(E \otimes O_Z)$ is surjective.

Theorem 1.1. Fix two positive integers $n$ and $r$ and collections of dimension $n - r$ ICIS $\delta$ and dimension $n - r - 1$ ICIS $\beta$, there is a universal polynomial $T_{\beta,\delta}$ of degree $|\beta| + |\delta|$ with the following property. For any smooth variety $Y$ of dimension $n$, any smooth divisor $D \subset Y$ and any $(N(\beta) + N(\delta) + n - r + 1)$-very ample vector bundle $E$ of rank $r$, the number of subvarieties in a $(\tau(\delta) + \tau(\beta))$-dimensional general linear subspace $V \subset \mathbb{P}^0(H^0(E))$ with non-induced singularity type $\delta$ and tangency type $\beta$ with a fixed smooth divisor $D$ at unassigned points is given by plugging in $T_{\beta,\delta}$ with the Chern numbers of $Y$, $D$, and $E$.

Remark. In the case of counting hypersurfaces, if $X$ is defined by $f(x_1, x_2, \ldots, x_n)$ and $p$ is a given point on $D := (x_1 = 0)$ near $p$, then $X$ is tangent to $D$ at $p$ means $f(p) = \frac{\partial f}{\partial x_i}(p) = 0$ for $i = 2, \ldots, n$. The conditions for $p$ to be a singular point of $X$ is $f(p) = \frac{\partial f}{\partial x_i}(p) = 0$ for $i = 1, \ldots, n$. Since the codimension of the former is $n$ and the latter is $n + 1$, in the generic case (which can be guaranteed if the line bundle is sufficient ample), we can drop the “non-induced” assumption in the theorem. In other words, for general linear subspace $V \subset \mathbb{P}(H^0(E))$, the subvarieties enumerated by Theorem 1.1 is tangent to $D$ at smooth points of $X$ and the non-induced singular type equals the singular type of $X$.

Since the degrees of universal polynomials are known, their coefficients can be computed by special cases. Consider the generating series 

$$T_{\beta}(Y, D, E) = \sum_{\beta,\delta} T_{\beta,\delta}(\text{Chern numbers of } Y, D, E) y^\beta z^\delta.$$ 

The following theorem gives the structure of the generating series. It gives many relations between universal polynomials.

Theorem 1.2. If $\{\Theta_1, \Theta_2, \ldots, \Theta_m\}$ forms a basis of the finite-dimensional $\mathbb{Q}$-vector space of graded degree $n$ polynomials in the Chern classes of 

$$\{c_1(T_Y)\}_{i=0}^n, c_1(O(D)), \{c_1(E)\}_{i=0}^r.$$ 

Then there exist power series $A_1, A_2, \ldots, A_m$ in $\mathbb{Q}[[y^\beta, z^\delta]]$ such that the generating series has the form 

$$T_{\beta}(Y, D, E) = \prod_{j=1}^m A_j^{\Theta_j(c_1(T_Y), c_1(D), c_1(E))}.$$
The theorems above are proven by showing $T_\emptyset$ (and therefore its coefficients) are invariants of the algebraic cobordism group of bundles and divisors of varieties.

1.4. An Outline of the paper. Algebraic cobordism theory of bundles on varieties is reviewed in Section 2 and we generalize it to arbitrary number of bundles. Algebraic cobordism theory of bundles and divisors on varieties is studied in Section 3. We prove that there is a natural isomorphism between these two algebraic cobordism theories and determine a natural basis. Our next goal is the enumeration of singular varieties of a fixed tangency type with a divisor. We begin with a brief review about facts of isolated complete intersection singularities and notations in Section 4. Then in Section 5, the number of singular varieties is expressed as an intersection number on Hilbert schemes of points and a degeneration formula is derived. The degeneration formula combined with the structure of algebraic cobordism group lead us to the proof of main theorems in Section 6. In the last section, we provide a list of current developments of the enumeration of singular varieties with tangency conditions.

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2. Algebraic cobordism of bundles on varieties

2.1. Algebraic cobordism ring of varieties. In this section, we will study the algebraic cobordism theory of arbitrary number of bundles on varieties. First we give an exposition of the algebraic cobordism theory of varieties in [18] and the algebraic cobordism theory of a pair of variety and bundle in [17]. Then we show how these theories can be (rather straightforwardly) generalized to arbitrary number of bundles over varieties following the approach in [17].

Let $k$ be a field of characteristic 0, $\text{Sch}_k$ be the category of separated schemes of finite type over $k$, and $\text{Sm}_k$ be the full subcategory of smooth quasi-projective $k$-schemes.

For $X \in \text{Sch}_k$, let $\mathcal{M}(X)$ be the set of the isomorphism classes over $X$ of projective morphisms $f : M \to X$ with $M \in \text{Sm}_k$. The set $\mathcal{M}(X)$ is graded by the dimension of $M$ and it has the structure of a monoid under disjoint of domains. Let $\mathcal{M}^+_k(X)$ be the graded group completion of $\mathcal{M}(X)$ and $[f : M \to X]$ be the element in $\mathcal{M}^+_k(X)$ for $f$.

Consider projective morphisms

$$\pi : Y \to X \times \mathbb{P}^1$$

and composition

$$\pi_2 = p_2 \circ \pi : Y \to \mathbb{P}^1$$

where $p_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection to the second factor, $Y \in \text{Sm}_k$ and $X \in \text{Sch}_k$. We say $\pi_2$ is a double point degeneration\(^1\) over $\infty \in \mathbb{P}^1$ if $Y_0 := \pi_2^{-1}(0)$ is a smooth fiber over 0, and $\pi_2^{-1}(\infty)$ can be written as $Y_1 \cup Y_2$ where $Y_1$ and $Y_2$ are smooth Cartier divisors intersecting transversely along a smooth divisor $B = Y_1 \cap Y_2$. Let

$$Y_3 := \mathbb{P}(\mathcal{O}_B \otimes N_{Y_1/B}) \cong \mathbb{P}(N_{Y_2/B} \otimes \mathcal{O}_B)$$

\(^1\)The point 0 can be replaced by any regular value of $\pi$. 
is a $\mathbb{P}^1$ bundle over $B$. Then the double point relation over $X$ defined by $\pi$ is
\begin{equation}
\{Y_0 \to X\} - \{Y_1 \to X\} - \{Y_2 \to X\} + \{Y_3 \to X\}.
\end{equation}
Denote the subgroup of $\mathcal{M}_n(X)^+$ generated by all double point relations over $X$ by $\mathcal{R}_n(X)$. The algebraic cobordism theory $\omega_*^n$ is defined by
\[
\omega_*(X) := \mathcal{M}_n(X)^+ / \mathcal{R}_n(X).
\]
Since the double point relation is homogeneous, $\omega_*(X)$ is graded by the dimension of the domain.

The first algebraic cobordism theory $\Omega_*$ was constructed by Morel and Levine [19] to be the universal oriented Borel-Moore homology theory of schemes. A key theorem of [18] is a canonical isomorphism between the two theories $\omega_*$ and $\Omega_*$. Morel and Levine showed that $\Omega_*(k)$ is isomorphic to the Lazard ring $L_*$, which implies $L_*$ is also isomorphic to $\omega_*(k)$. It is well known that $L_* \otimes_{\mathbb{Z}} \mathbb{Q}$ has a basis formed by products of projective spaces, hence
\begin{equation}
\omega_*^n(k) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{\lambda = (\lambda_1, \ldots, \lambda_r)} \mathbb{Q}[\mathbb{P}^{\lambda_1} \times \cdots \times \mathbb{P}^{\lambda_r}]
\end{equation}
where $\lambda$ runs over all partitions of nonnegative integers.

2.2. Bundles on varieties. The construction of algebraic cobordism theory was first generalized to pairs $[S, L]$ for a smooth surface $S$ and a line bundle $L$. The resulting algebraic cobordism group $\omega_2,1(\mathbb{C})$ played a central role in the proofs of the formula of generating series of Donaldson-Thomas invariants in [18] and Göttsche’s conjecture in [27]. More generally, Lee and Pandharipande in [17] constructed the algebraic cobordism theory of a pair of smooth variety and vector bundle. We first recall their definitions and main results.

For $X \in \text{Sch}_k$, let $\mathcal{M}_{n,r}(X)$ be the set of isomorphism classes $[f : M \to X, E]$ with $M \in \text{Sm}_k$ of dimension $n$, $E$ a vector bundle on $M$ of rank $r$, and $f$ projective. Let $\mathcal{M}_{n,r}^+(X)$ denote the group completion of $\mathcal{M}_{n,r}(X)$.

Suppose the projective morphism $\pi : Y \to X \times \mathbb{P}^1$ together with $\pi_2 = p_2 \circ \pi : Y \to \mathbb{P}^1$ give a double point degeneration over $X$ and $E$ is a vector bundle over $Y$. Denote the restriction of $E$ to $Y_i$ by $E_i$ for $i = 0, 1, 2$ and the pullback of the restriction of $E$ to $B$ via the morphism $X_3 \to B$ by $E_3$. The double point relation of bundles over varieties defined by $\pi$ and $E$ over $X$ is
\[
\{Y_0 \to X, E_0\} - \{Y_1 \to X, E_1\} - \{Y_2 \to X, E_2\} + \{Y_3 \to X, E_3\}.
\]
Denote $\mathcal{R}_{n,r}(X) \subset \mathcal{M}_{n,r}(X)^+$ to be subgroup generated by all double point relations over $X$. The algebraic cobordism group of bundles of rank $r$ over varieties of dimension $n$ is defined by the quotient
\[
\omega_{n,r}(X) := \mathcal{M}_{n,r}(X)^+ / \mathcal{R}_{n,r}(X).
\]
The graded sum
\[
\omega_{*,r}(X) := \bigoplus_{n=0}^{\infty} \omega_{n,r}(X)
\]
is an $\omega_s(k)$-module via product (and pullback). It is also a $\omega_s(X)$-module if $X \in \text{Sm}_k$.

The main result of [17] is a natural basis of $\omega_{n,r}(k)$ and the dimension of $\omega_{n,r}(k) \otimes \mathbb{Z} \mathbb{Q}$.

A partition pair of size $n$ and type $r$ is a pair $(\lambda, \mu)$ where

1. $\lambda$ is a partition of $n$,
2. $\mu$ is a subpartition of $\lambda$ of length $l(\mu) \leq r$.

The second condition means $\mu$ is a obtained by deleting some parts of $\lambda$. In particular, $\mu$ is allowed to be empty or equal to $\lambda$ if $l(\lambda) \leq r$. Two subpartitions are equivalent if they only differ by permuting equal parts of $\lambda$.

Let $\mathcal{P}_{n,r}$ be the set of equivalent classes of all partition pairs of size $n$ and type $r$. To each $(\lambda, \mu) \in \mathcal{P}_{n,r}$ we can associate an element $\phi(\lambda, \mu) \in \omega_{n,r}(k)$ as follows. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$, let $\mathbb{P}^{\lambda} = \mathbb{P}^{\lambda_1} \times \cdots \times \mathbb{P}^{\lambda_{l(\lambda)}}$. To each part $m$ of $\mu$, let $L_m$ be the line bundle on $\mathbb{P}^{\lambda}$ obtained by pulling back $O_{\mathbb{P}^{m}}(1)$ via the projection $P^\lambda \to \mathbb{P}^m$. Define

$$\phi(\lambda, \mu) = \left[ \mathbb{P}^{\lambda}, \mathcal{O}_{\mathbb{P}^{\lambda}}^{r-l(\mu)} \oplus \left( \bigoplus_{m \in \mu} L_m \right) \right].$$

Lee and Pandharipande [17, Theorem 1] proved that for any nonnegative integer $n$ and $r$,

$$\omega_{n,r}(k) \otimes \mathbb{Z} \mathbb{Q} = \bigoplus_{(\lambda, \mu) \in \mathcal{P}_{n,r}} \mathbb{Q} \cdot \phi(\lambda, \mu). \tag{2.3}$$

This result means $\{\phi(\lambda, \mu) | (\lambda, \mu) \in \mathcal{P}_{n,r}\}$ form a basis of $\omega_{n,r}(k) \otimes \mathbb{Z} \mathbb{Q}$ and the dimension of $\omega_{n,r}(k) \otimes \mathbb{Z} \mathbb{Q}$ is the cardinality of $\mathcal{P}_{n,r}$. The dimension of $\omega_{n,r}(k) \otimes \mathbb{Z} \mathbb{Q}$ is also equal to total number of degree $n$ monomials in Chern classes of dimension $n$ varieties and rank $r$ vector bundles because the Chern numbers given by these polynomials are the only invariants of $\omega_{n,r}(k) \otimes \mathbb{Z} \mathbb{Q}$ [17, Theorem 4]. The special case of $r = 0$ reduces to algebraic cobordism ring of varieties in (2.2).

Moreover, $\omega_{s,r}(k)$ is a free $\omega_s(k)$-module with basis

$$\omega_{s,r}(k) = \bigoplus_{\lambda} \omega_s(k) \cdot \phi(\lambda, \lambda)$$

where the sum is over all partitions $\lambda$ of length at most $r$ [17, Theorem 2].

Once we understand the structure of $\omega_{s,r}(k)$, $\omega_{s,r}(X)$ is simply an extension of scalars of $\omega_s$ because

$$\omega_s(X) \otimes \omega_s(k) \omega_{s,r}(k) \to \omega_{s,r}(X)$$

is an isomorphism [17, Theorem 3].

### 2.3. Lists of bundles

To prove (2.3), Lee and Pandharipande defined the algebraic cobordism group of lists of line bundles $\omega_{n,1^r}(X)$. The construction of $\omega_{n,1^r}(X)$ is similar to $\omega_{n,1^r}(X)$ with only two exceptions. First, the elements in $\omega_{n,1^r}(X)$ are isomorphism classes of the form $[f: M \to X, L_1, \ldots, L_r]$, where $L_1, \ldots, L_r$ is an ordered list of line bundles on $M$. Second, the double point relation

$$[Y_0 \to X, \{L_{0,j}\}_{j=1}^r] - [Y_1 \to X, \{L_{1,1}\}_{j=1}^r] - [Y_2 \to X, \{L_{2,1}\}_{j=1}^r] + [Y_3 \to X, \{L_{3,1}\}_{j=1}^r] \tag{2.4}$$
is given by a double point degeneration \( \pi: Y \to X \times \mathbb{P}^1 \) and an ordered list of line bundles \( L_1, \ldots, L_r \) on \( Y \) such that \( L_{i,j} \) are line bundles on \( Y_j \) from \( L_j \) in the same manner of Section 2.2.

More generally, one can analogously define the theory of algebraic cobordism of lists of vector bundles on varieties to be

\[
\omega_{n,r_1,\ldots,r_k}(X) = \mathcal{M}_{n,r_1,\ldots,r_k}(X)^+ / \mathcal{R}_{n,r_1,\ldots,r_k}(X).
\]

Here \( \mathcal{M}_{n,r_1,\ldots,r_k}(X)^+ \) is the free abelian group over \( \mathbb{Z} \) generated by isomorphism classes of \( [f: M \to X, E_1, \ldots, E_k] \) with rank \( E_i = r_i \). The double point relation is the same as (2.1) except now \( E_i \) and \( E_{ij} \) are vector bundles. \( \mathcal{R}_{n,r_1,\ldots,r_k}(X) \) is the subgroup generated by double point relations. Therefore \( \omega_{n,1^k}(X) \) is a special case of \( \omega_{n,r_1,\ldots,r_k}(X) \) with all \( r_i = 1 \).

The graded sum

\[
\omega_{*,r_1,\ldots,r_k}(X) := \bigoplus_{n=0}^{\infty} \omega_{n,r_1,\ldots,r_k}(X)
\]

is an \( \omega(k) \)-module via product (and pullback). If \( X \in \text{Sm}_k \), then \( \omega_{*,r_1,\ldots,r_k}(X) \) is also a module over the ring \( \omega_*(X) \).

**Definition 2.1.** A partition list of size \( n \) and type \( (r_1, \ldots, r_k) \) is an ordered list \( (\lambda, \mu_1, \ldots, \mu_k) \) where

1. \( \lambda \) is a partition of \( n \),
2. \( \mu_i \) has length \( l(\mu_i) \leq r_i \),
3. the union of \( \mu_1, \ldots, \mu_k \) is a subpartition of \( \lambda \).

Two partition lists \( (\lambda, \mu_1, \ldots, \mu_k) \) and \( (\lambda', \mu'_1, \ldots, \mu'_k) \) are *equivalent* if permuting equal parts of \( \lambda \) makes \( \mu'_i \) become \( \mu_i \). Let \( \mathcal{P}_{n,r_1,\ldots,r_k} \) be the set of partition lists of size \( n \) and type \( (r_1, \ldots, r_k) \). For example,

\[
\mathcal{P}_{3,2,1} = \begin{cases}
(3, \emptyset, \emptyset), (3, 3, \emptyset), (3, \emptyset, 3), \\
(21, \emptyset, \emptyset), (21, 2, \emptyset), (21, 1, \emptyset), (21, \emptyset, 2), (21, \emptyset, 1), \\
(21, 21, \emptyset), (21, 2, 1), (21, 1, 2) \\
(111, \emptyset, \emptyset), (111, 1, \emptyset), (111, \emptyset, 1), (111, 1, 1), (111, 11, 1), (111, 11, 1) 
\end{cases}
\]

To each \( (\lambda, \mu_1, \ldots, \mu_k) \in \mathcal{P}_{n,r_1,\ldots,r_k} \), we associate an element

\[
\phi(\lambda, \mu_1, \ldots, \mu_k) = \left[ \mathbb{P}^\lambda \mathcal{O}_{\mathbb{P}^\lambda}^{r_1-\ell(\mu_1)} \bigoplus \left( \bigoplus_{m \in \mu_1} L_m \right), \ldots, \mathbb{O}_{\mathbb{P}^\lambda}^{r_k-\ell(\mu_k)} \bigoplus \left( \bigoplus_{m \in \mu_k} L_m \right) \right]
\]

in \( \omega_{n,r_1,\ldots,r_k}(k) \) (recall \( L_m \) be the line bundle on \( \mathbb{P}^\lambda \) obtained by pulling back \( \mathcal{O}_{\mathbb{P}^m}(1) \) via the projection \( \mathbb{P}^\lambda \to \mathbb{P}^m \)). Since \( \cup \mu_i \) is a subpartition of \( \lambda \), the line bundles \( L_m \) in \( \phi(\lambda, \mu_1, \ldots, \mu_k) \) all come from distinct factors of \( \mathbb{P}^\lambda \).

The following theorem describes a basis of \( \omega_{n,r_1,\ldots,r_k}(k) \otimes_{\mathbb{Z}} \mathbb{Q} \).

**Theorem 2.1.** For \( n, r_1, \ldots, r_k \geq 0 \), we have

\[
\omega_{n,r_1,\ldots,r_k}(k) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{(\lambda, \mu_1, \ldots, \mu_k) \in \mathcal{P}_{n,r_1,\ldots,r_k}} \mathbb{Q} \cdot \phi(\lambda, \mu_1, \ldots, \mu_k).
\]
Proof. Let \( m \subset \omega_\ast(k) \) be the ideal generated by varieties of positive dimension. Define the graded quotient
\[
\tilde{\omega}_{s,r_1,\ldots,r_k}(k) = \omega_{s,r_1,\ldots,r_k}(k)/m \cdot \omega_{s,r_1,\ldots,r_k}(k), \quad \tilde{\omega}_{s,r_1,\ldots,r_k}(k) = \bigoplus_{n=0}^{\infty} \omega_{n,r_1,\ldots,r_k}(k)
\]
from the \( \omega_\ast(k) \)-module structure of \( \omega_{n,r_1,\ldots,r_k}(k) \). Lemma 13 in [17] and the discussion after it imply that for any smooth variety \( Y \) and vector bundle \( E \) of rank \( r \) on \( Y \), there exists a birational morphism \( \tilde{Y} \rightarrow Y \) such that
\[
[Y \rightarrow Y, E] = [\tilde{Y} \rightarrow Y, L_1 \oplus \cdots \oplus L_r]
\]
in \( \tilde{\omega}_{s,r}(Y) \) for some line bundles \( L_1, \ldots, L_r \) on \( \tilde{Y} \).

If we apply this fact \( r \) times, every
\[
[Y \rightarrow Y, E_1, \ldots, E_k] = [\tilde{Y} \rightarrow Y, \oplus_{j=1}^{r_1} L_{1j}, \ldots, \oplus_{j=1}^{r_k} L_{kj}]
\]
in \( \tilde{\omega}_{s,r_1,\ldots,r_k}(Y) \) for some \( \tilde{Y} \) and line bundles \( L_{ij} \) on \( \tilde{Y} \). After pushing forward to \( X \),
\[
[Y \rightarrow X, E_1, \ldots, E_k] = [\tilde{Y} \rightarrow X, \oplus_{j=1}^{r_1} L_{1j}, \ldots, \oplus_{j=1}^{r_k} L_{kj}]
\]
in \( \tilde{\omega}_{s,r_1,\ldots,r_k}(k) \).

The double point relation for the list of line bundles \([M \rightarrow X, L_{11}, \ldots, L_{kr_k}]\) in \( \omega_{n,1^k}(X) \) implies a double point relation \([M \rightarrow X, \oplus_{j=1}^{r_1} L_{1j}, \ldots, \oplus_{j=1}^{r_k} L_{kj}]\) by taking direct sums of line bundles. Therefore there is a group homomorphism from \( \omega_{n,1^k}(X) \) to \( \tilde{\omega}_{n,r_1,\ldots,r_k}(k) \) which also a homomorphism of \( \omega_\ast(k) \)-modules. Moreover, the discussion above shows this morphism is surjective and [17] Proposition 10 implies \( \omega_{n,r_1,\ldots,r_k}(k) \) is generated by
\[
\{ \phi(\lambda, \mu_1, \ldots, \mu_k) \mid (\lambda, \mu_1, \ldots, \mu_k) \in \mathcal{P}_{n,r_1,\ldots,r_k}, \cup_i \mu_i = \lambda \}
\]
over \( \mathbb{Z} \).

We now prove the result by induction on \( n \). The \( n = 0 \) case is trivial. Assume the result is true for all \( n' < n \), then \( \bigoplus_{n'<n} (\omega_{n-n'}(k) \cdot \omega_{n',r_1,\ldots,r_k}(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \) is spanned by
\[
\{ \phi(\lambda, \mu_1, \ldots, \mu_k) \mid (\lambda, \mu_1, \ldots, \mu_k) \in \mathcal{P}_{n,r_1,\ldots,r_k}, \cup_i \mu_i \varsubsetneq \lambda \}
\]
over \( \mathbb{Q} \). Since \( \omega_{n,r_1,\ldots,r_k}(k) \otimes_{\mathbb{Z}} \mathbb{Q} \) is spanned by generators of \( \omega_{n-n'}(k) \cdot \omega_{n',r_1,\ldots,r_k}(k) \) for \( n' < n \), and \( \omega_{n,r_1,\ldots,r_k}(k) \), it is generated by
\[
\{ \phi(\lambda, \mu_1, \ldots, \mu_k) \mid (\lambda, \mu_1, \ldots, \mu_k) \in \mathcal{P}_{n,r_1,\ldots,r_k} \}
\]
As a result, the dimension of \( \omega_{n,r_1,\ldots,r_k}(k) \otimes_{\mathbb{Z}} \mathbb{Q} \) is less or equal to the size of \( \mathcal{P}_{n,r_1,\ldots,r_k} \).

We will see \( \dim \omega_{n,r_1,\ldots,r_k}(k) \otimes_{\mathbb{Z}} \mathbb{Q} \geq |\mathcal{P}_{n,r_1,\ldots,r_k}| \) in the next Theorem so we conclude these generators form a basis of \( \omega_{n,r_1,\ldots,r_k}(k) \otimes_{\mathbb{Z}} \mathbb{Q} \).

Let \( \mathcal{C}_{n,r_1,\ldots,r_k} \) be the finite-dimensional \( \mathbb{Q} \)-vector space of graded degree \( n \) polynomials in the Chern classes of
\[
\{ c_i(T_X) \}_{i=1}^{n}, \{ c_i(E_1) \}_{i=1}^{r_1}, \ldots, \{ c_i(E_k) \}_{i=1}^{r_k}.
\]

**Theorem 2.2.** For all \( \Theta \in \mathcal{C}_{n,r_1,\ldots,r_k} \), the Chern invariants of the list \([M, E_1, \ldots, E_k] \)
\[
\int_Y \Theta(c_1(T_M), \ldots, c_n(T_M), c_1(E_1), \ldots, c_{r_1}(E_1), c_1(E_2), \ldots, c_{r_k}(E_k))
\]
Proof. The proof of [17, Proposition 5] can be applied directly to show the Chern invariants respect algebraic cobordism by a splitting argument for all \( E_i \). Therefore the integration map descends to a well-defined bilinear map

\[
\rho : (\omega_{s,r_1,\ldots,r_k}(k) \otimes_{\mathbb{Z}} \mathbb{Q}) \times C^*_{n,r_1,\ldots,r_k} \to \mathbb{Q}.
\]

To prove the map is an isomorphism, we follow the same line in [17, Section 1]. Let \( Q_{n,r_1,\ldots,r_k} \) be the set of partition lists \((\nu, \mu_1, \ldots, \mu_k)\) where

1. every \( \mu_i \) is a partition of size \(|\mu_i|\) and \( \sum_{i=1}^{k} |\mu_i| \leq n \),
2. the largest part of \( \mu_i \) is at most \( r_i \),
3. \( \nu \) is a partition of \( n - \sum |\mu_i| \).

Taking the number of parts equal to \( m \) in \( \mu_i \) to the power of \( c_{m}(E_i) \) defines a bijective correspondence between \( Q_{n,r_1,\ldots,r_k} \) and the monomials in \( C_{n,r_1,\ldots,r_k} \). Let \( C(\nu, \mu_1, \ldots, \mu_k) \) be the image of \((\nu, \mu_1, \ldots, \mu_k)\) in \( C_{n,r_1,\ldots,r_k} \). On the other hand, there is a natural bijection \( \epsilon : Q_{n,r_1,\ldots,r_k} \to P_{n,r_1,\ldots,r_k} \) defined by \( \epsilon(\nu, \mu_1, \ldots, \mu_k) = (\nu \cup (\cup \mu_i), \mu_1', \ldots, \mu_k') \) where \( \mu_i' \) is the partition obtained by transposing the Young diagram of \( \mu_i \). Hence \( \mu_i' \) has length at most \( r_i \). In particular \( |P_{n,r_1,\ldots,r_k}| = \dim C_{n,r_1,\ldots,r_k} \).

Now we define a partial ordering on the monomials in \( C_{n,r_1,\ldots,r_k} \) (and on \( Q_{n,r_1,\ldots,r_k} \) via the bijection). If \( F \) and \( G \) are two monomials in \( C_{n,r_1,\ldots,r_k} \) and

\[
c_1(E_k)^{a_1} \cdots c_{r_k}(E_k)^{a_{r_k}} \quad \text{and} \quad c_1(E_k)^{b_1} \cdots c_{r_k}(E_k)^{b_{r_k}}
\]

are the factors of \( \{c_i(E_k)\}_{i=0}^{r_k} \) in \( F \) and \( G \). We say \( F < G \) if \( a_{r_k} < b_{r_k} \) or \( a_j = b_j \) for all \( j > i \) and \( a_i < b_i \). This partial ordering agrees with the one defined on \( C_{n,r} \) in [17, Section 1.2.2].

If \( M_{n,r_1,\ldots,r_k} \) is the matrix with rows and columns indexed by \( Q_{n,r_1,\ldots,r_k} \) and has entries

\[
M_{n,r_1,\ldots,r_k}[(\nu, \mu_1, \ldots, \mu_k), (\nu', \mu'_1, \ldots, \mu'_k)] = \rho(\phi(\epsilon(\nu, \mu_1, \ldots, \mu_k), C(\nu', \mu'_1, \ldots, \mu'_k))).
\]

The proof of Lemma 7 in [17] can be applied to show

\[
M_{n,r_1,\ldots,r_k}[(\nu, \mu_1, \ldots, \mu_k), (\nu', \mu'_1, \ldots, \mu'_k)] = 0
\]

if \((\nu, \mu_1, \ldots, \mu_k) < (\nu', \mu'_1, \ldots, \mu'_k)\). As a result, \( M_{n,r_1,\ldots,r_k} \) is a block lower triangular matrix and the block are determined by all \((\nu, \mu_1, \ldots, \mu_k)\) with the same \( \mu_k \). As in the proof of [17, Proposition 8], the block corresponding to \( \mu_k \) is the matrix \( M_{n-|\mu_k|,r_1,\ldots,r_{k-1}} \). Since the base case \( k = 1, 0 \) is already established there, an inductive argument shows \( M_{n,r_1,\ldots,r_k} \) is a nonsingular matrix and \( \rho \) is an isomorphism. \( \square \)

The structure of \( \omega_{s,r_1,\ldots,r_k}(k) \) over \( \mathbb{Z} \) is determined by:

**Theorem 2.3.** For \( r_i \geq 0 \), \( \omega_{s,r_1,\ldots,r_k}(k) \) is a free \( \omega_s(k) \)-module with basis

\[
\omega_{s,r_1,\ldots,r_k}(k) = \bigoplus_{(\mu_1, \ldots, \mu_k)} \omega_s(k) \cdot \phi(\cup \mu_i, \mu_1, \ldots, \mu_k)
\]

where the sum is over all partitions \((\mu_1, \ldots, \mu_k)\) with length \( l(\mu_i) \leq r_i \).
Proof. In the proof of Theorem 2.1 we have seen \( \tilde{\omega}_{n,r_1,...,r_k}(k) \) can be generated by
\[
\{ \phi(\lambda, \mu_1, \ldots, \mu_k) \mid (\lambda, \mu_1, \ldots, \mu_k) \in P_{n,r_1,...,r_k}, \cup_i \mu_i = \lambda \}
\]
over \( \mathbb{Z} \). Hence \( \omega_{n,r_1,...,r_k}(k) \) is generated by the same set and the subgroups
\[
\omega_n(k) \cdot \omega_{0,r_1,...,r_k}(k), \ldots, \omega_1(k) \cdot \omega_{n-1,r_1,...,r_k}(k).
\]
Since \( \omega_s(k) \cong \mathbb{L} \) and it is well known that \( \mathbb{L} \) is a free \( \mathbb{Z} \)-module of rank equal to the number of partitions of \( i \). Using induction we can choose \( |P_{n,r_1,...,r_k}| \) elements to generate \( \omega_{s,r_1,...,r_k}(k) \) over \( \mathbb{Z} \). From Theorem 2.1 \( \omega_{s,r_1,...,r_k}(k) \otimes \mathbb{Q} \) has dimension \( |P_{n,r_1,...,r_k}| \) thus there are no relations among these generators.

In fact, \( \omega_{n,r_1,...,r_k}(k) \) determines \( \omega_n(X) \) for general \( X \in \text{Sch}_k \).

**Theorem 2.4.** The algebraic cobordism theory of bundles on varieties \( \omega_{s,r_1,...,r_k} \) is an extension of scalars of the original algebraic cobordism theory \( \omega_s(X) \), i.e.
\[
\omega_s(X) \otimes \omega_s(k) \omega_{s,r_1,...,r_k}(k) \to \omega_{s,r_1,...,r_k}(X)
\]
is an isomorphism.

**Proof.** Consider an element \( [Y \to X, E_1, \ldots, E_k] \). We have proved
\[
[Y \to Y, E_1, \ldots, E_k] = [\tilde{Y} \to Y, \oplus_{j=1}^{r_1} L_{1j}, \ldots, \oplus_{j=1}^{r_k} L_{kj}]
\]
in \( \tilde{\omega}_{s,r_1,...,r_k}(Y) \) for some \( \tilde{Y} \) and line bundles \( L_{ij} \) on \( \tilde{Y} \). Taking suitable direct sum of line bundles in the surjectivity argument and applying the pairing \( C_\psi \) for every bundle in the injectivity argument of [17, Section 4] will prove the result.

3. Algebraic Cobordism Group of Bundles and Divisors on Varieties

Since our goal is to count singular subvarieties with tangency conditions with a smooth divisor by deriving a degeneration formula, in this section we will study the algebraic cobordism group of bundles and divisors on varieties.

3.1. **Settings.** Let \( D \) be a reduced effective Cartier divisor (including \( D = 0 \)) on a smooth \( k \)-scheme \( Y \) with irreducible components \( D_1, \ldots, D_s \). For \( I \subset \{1, \ldots, s\} \), we set 
\[ D_I = \cap_{i \in I} D_i. \]
Recall that \( D \) is a strict normal crossing divisor if for each \( I, D_I \) is smooth over \( k \) and the codimension of \( D_I \) in \( Y \) is \( |I| \).

For \( X \in \text{Sch}_k \), let \( N_{n,1^*,r_1,...,r_k}(X) \) be the set of the isomorphism classes of lists \([f: M \to X, D_1, \ldots, D_s, E_1, \ldots, E_k] \), where \( f: M \to X \) is a projective morphism with \( M \in \text{Sm}_k \) of dimension \( n \), \( E_i \) are vector bundles of rank \( r_i \) on \( M \), and \( \sum_{i=1}^s D_i \) is a strict normal crossing divisor in \( M \). The set \( N_{n,1^*,r_1,...,r_k}(X) \) is graded by the dimension of \( M \) and let \( N^+_{n,1^*,r_1,...,r_k}(X) \) be the group completion of \( N_{n,1^*,r_1,...,r_k}(X) \) over \( \mathbb{Z} \).

Suppose a projective morphism \( \pi: Y \to X \times \mathbb{P}^1 \) together with \( \pi_2 = p_2 \circ \pi: Y \to \mathbb{P}^1 \) give a double point relation over \( X \). Recall \( Y_0 := \pi^{-1}_2(0) \) is the smooth fiber over 0, and \( \pi_2^{-1}(\infty) = Y_1 \cup Y_2 \) and \( B = Y_1 \cap Y_2 \).

Let \( \{E^k_i\}_{i=1}^k \) be vector bundles on \( Y \) and \( \{D_j\}_{j=1}^s \) be smooth effective divisors on \( Y \).
Assume every \( D_j \) intersects \( Y_i \) and \( B \) transversally along smooth divisors and \( \sum_{j=1}^s D_j \) is a strict normal crossing divisor in \( Y \). Denote the restriction of \( E_j \) to \( Y_i \) by \( E_{ij} \) for \( i = 0, 1, 2 \) and call \( E_{3j} \) the pullback of the restriction of \( E_j \) to \( B \) via the morphism
Y_3 \to B$. Denote the intersection $D_j \cap Y_i$ by $D_{ij}$ for $i = 0, 1, 2$ and call $D_{3j}$ the inverse image of $D_j \cap B$ via the morphism $Y_3 \to B$.

Suppose $\sum_{j=1}^k D_{ij}$ is a strict normal crossing divisor on $Y_i$ for all $i$. The double point relation given by $(\pi, \{D_i\}_{i=1}^k, \{E_i\}_{i=1}^k)$ is

$$[Y_0 \to X, \{D_{0j}\}, \{E_{0j}\}] - [Y_1 \to X, \{D_{1j}\}, \{E_{1j}\}]$$
$$- [Y_2 \to X, \{D_{2j}\}, \{E_{2j}\}] + [Y_3 \to X, \{D_{3j}\}, \{E_{3j}\}].$$

(3.1)

Denote the subgroup of $N_{n,1^{s},r_1,\ldots,r_k}(X)$ generated by double point relations (3.1) by $Q_{n,1^{s},r_1,\ldots,r_k}(X)$. The algebraic cobordism group of divisors and bundles over varieties is defined by

$$\nu_{n,1^{s},r_1,\ldots,r_k}(X) = N_{n,1^{s},r_1,\ldots,r_k}(X)/Q_{n,1^{s},r_1,\ldots,r_k}(X).$$

The graded sum

$$\nu_{s,1^{s},r_1,\ldots,r_k}(X) := \bigoplus_{n=0}^\infty \nu_{n,1^{s},r_1,\ldots,r_k}(X)$$

is an $\omega_s(k)$-module via product (and pullback).

For simplicity we usually write the list $[f : M \to X, D_1, \ldots, D_s, E_1, \ldots, E_k]$ by $[M \to X, D_1, E_1]$ and $[f : M \to \text{Spec } k, D_1, \ldots, D_s, E_1, \ldots, E_k]$ by $[M, D_1, E_1]$.

Let

$$P'_{n,s,r_1,\ldots,r_k} = \{ (\lambda, \pi_1, \ldots, \pi_s, \mu_1, \ldots, \mu_k) | (\lambda, \pi_1, \ldots, \pi_s, \mu_1, \ldots, \mu_k) \in P_{n,1^{s},r_1,\ldots,r_k} \}$$

Consider an element $(\lambda; \pi_1, \ldots, \pi_s; \mu_1, \ldots, \mu_k) \in P'_{n,s,r_1,\ldots,r_k}$.

Since $\pi_i$ is a partition of length at most 1, it is either a positive integer $m$ or the empty set $\emptyset$. If $\pi_i = \emptyset$, set $H_i$ to be the divisor 0. If $\pi_i = (m)$, let $H_i$ be the inverse image of a hyperplane of $\mathbb{P}^m$ via the projection map $\mathbb{P}^\lambda \to \mathbb{P}^{m}$.

Define

$$\psi : P'_{n,s,r_1,\ldots,r_k} \to \nu_{n,1^{s},r_1,\ldots,r_k}(k)$$

by

$$\psi(\lambda; \pi_i; \mu_j) = \left[ \mathbb{P}^\lambda, H_1, \ldots, H_s, \mathcal{O}_{\mathbb{P}^\lambda}^{l(\mu_1)} \oplus \bigoplus_{m \in \mu_1} L_m, \ldots, \mathcal{O}_{\mathbb{P}^\lambda}^{l(\mu_k)} \oplus \bigoplus_{m \in \mu_k} L_m \right].$$

**Theorem 3.1.** The natural map

$$[f : M \to X, D_i, E_j] \to [f : M \to X, \mathcal{O}(D_i), E_j]$$

induces an isomorphism

$$\nu_{n,1^{s},r_1,\ldots,r_k}(X) \to \omega_{n,1^{s},r_1,\ldots,r_k}(X)$$

as abelian groups or as $\omega_s(k)$-modules.

It is easy to see the map sends $\psi(\lambda; \pi_i; \mu_j)$ to $\phi(\lambda, \pi_i, \mu_j)$. Theorem 3.1 together with Theorem 2.1 and Theorem 2.3 give a natural basis of $\nu_{n,1^{s},r_1,\ldots,r_k}(k)$.

---

2 We will abbreviate $(\lambda; \pi_1, \ldots, \pi_s; \mu_1, \ldots, \mu_k)$ by $(\lambda; \pi_i; \mu_j)$ in the following.
Corollary 3.2. The dimension of $\nu_{n,1^{s},r_{1},...,r_{k}}(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the number of monomials of degree $n$ in the Chern classes of $T_{M}$, $\mathcal{O}(D_{i})$ and $E_{j}$. Furthermore, for $s, r_{i} \geq 0$

$$\nu_{n,1^{s},r_{1},...,r_{k}}(k) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{(\lambda; \pi_{1},...,\pi_{s};\mu_{1},...,\mu_{k}) \in \mathbb{P}^{s}_{n,s^{r_{1}}...r_{k}}} \mathbb{Q} \cdot \psi(\lambda; \pi_{1},...,\pi_{s};\mu_{1},...,\mu_{k}).$$

The $\omega_{*}(k)$-module $\nu_{*,1^{s},r_{1},...,r_{k}}(k)$ is free with basis

$$\nu_{*,1^{s},r_{1},...,r_{k}}(k) = \bigoplus_{(\pi_{1},...,\pi_{s};\mu_{1},...,\mu_{k})} \omega_{*}(k) \cdot \psi(\cup_{i} \cup \mu_{i}; \pi_{1},...,\pi_{s};\mu_{1},...,\mu_{k})$$

where the sum is over partitions $(\pi_{1},...,\pi_{s},\mu_{1},...,\mu_{k})$ with length $l(\pi_{i}) \leq 1$, $l(\mu_{i}) \leq r_{i}$.

3.2. Proof of Theorem 3.1

Lemma 3.3. The map $[f : M \to X, D_{i}, E_{j}]$ to $[f : M \to X, \mathcal{O}(D_{i}), E_{j}]$ sends double point relations to double point relations. Hence

$$\nu_{n,1^{s},r_{1},...,r_{k}}(X) \to \omega_{n,1^{s},r_{1},...,r_{k}}(X)$$

is a well-defined morphism of abelian groups or $\omega_{*}(k)$-modules.

Proof. If $(\pi, D_{j}, E_{j})$ gives a double point relation $[3.1]$, then $(\pi, \mathcal{O}(D_{i}), E_{j})$ gives a double point relation in $\omega_{n,1^{s},r_{1},...,r_{k}}(X)$. This is because $D_{j}$ intersect transversally with $Y_{1}$ and $B$ along smooth divisors and $\mathcal{O}(D_{j})|_{Y_{1}} = \mathcal{O}(D_{ij}), \mathcal{O}(D_{i})|_{B} = \mathcal{O}(D_{i} \cap B)$. The addition and $\omega_{*}(k)$-module structure on both sides are compatible so the map induces a morphism of abelian groups or $\omega_{*}(k)$-modules from $\nu_{n,1^{s},r_{1},...,r_{k}}(X) \to \omega_{n,1^{s},r_{1},...,r_{k}}(X)$.

Lemma 3.4. The morphism $\nu_{n,1^{s},r_{1},...,r_{k}}(k) \to \omega_{n,1^{s},r_{1},...,r_{k}}(k)$ is surjective.

Proof. From Theorem 2.2, $\omega_{n,1^{s},r_{1},...,r_{k}}(k)$ is generated by $\omega_{*}(k) \cdot \phi(\lambda, \pi_{i}, \mu_{j})$ over $\mathbb{Z}$. Since $\phi(\lambda, \pi_{i}, \mu_{j})$ is the image of $\psi(\lambda; \pi_{i}; \mu_{j})$ the map is surjective.

Proposition 3.5. For $s, r_{i} \geq 0$, $\nu_{s,1^{s},r_{1},...,r_{k}}(k)$ is generated by

$$\{ \omega_{*}(k) \cdot \psi(\cup \pi_{i} \cup \mu_{j}; \pi_{1},...,\pi_{s};\mu_{1},...,\mu_{k}) \mid l(\pi_{i}) \leq 1, l(\mu_{j}) \leq r_{j} \}$$

over $\mathbb{Z}$.

Proof. Consider an element $[M, \{D_{i}\}_{i=1}^{s}, \{E_{j}\}_{j=1}^{k}]$ in $\nu_{n,1^{s},r_{1},...,r_{k}}(k)$. If $D_{s} \neq 0$, the degeneration to the normal cone of $D_{s}$ yields a double point relation

$$[M, \{D_{i}\}_{i=1}^{s}, \{E_{j}\}_{j=1}^{k}] - [M, \{D_{i}\}_{i=1}^{s-1}, 0, \{E_{j}\}_{j=1}^{k}]$$

$$- [\mathbb{P}_{D_{s}}, \{p^{*}(D_{i} \cap D_{s})\}_{i=1}^{s-1}, D_{\infty}, \{p^{*}E_{j}|_{D_{i} \cap D_{s}}\}_{i=1}^{k}] + [\mathbb{P}_{D_{s}}, \{p^{*}(D_{i} \cap D_{s})\}_{i=1}^{s-1}, 0, \{p^{*}E_{j}|_{D_{s}}\}_{j=1}^{k}]$$

where $p : \mathbb{P}_{D_{s}} := \mathbb{P}(O \otimes N_{M/D_{s}}) \to D_{s}$ and $D_{\infty} \cong D_{s}$ is the section of $\mathbb{P}_{D_{s}}$ at infinity. If we call $P$ to be the subgroup of $\nu_{1^{s},r_{1},...,r_{k}}(k)$ generated by elements of the form $[\mathbb{P}_{V}(O \oplus N), \{p^{-1}D_{i}\}_{i=1}^{s-1}, s, p^{*}E_{j}]$ where $V$ is a smooth variety of dimension $n-1$, $N$ is a line bundle and $E_{j}$ are vector bundles of rank $r_{j}$ on $V$, $\cup D_{i}$ is a strictly normal crossing divisors in $V$, and $s$ is a section of $p : \mathbb{P}_{V}(O \oplus N) \to V$. Then $\nu_{n,1^{s},r_{1},...,r_{k}}(k)$ is generated by elements with $D_{s} = 0$ and $P$.

Now we treat elements in $P$. Suppose

$$[V_{0}, D_{00}, N_{0}, E_{0j}] - [V_{1}, D_{11}, N_{1}, E_{1j}] - [V_{2}, D_{2j}, N_{2}, E_{2j}] + [V_{3}, D_{3j}, N_{3}, E_{3j}]$$


is a double point relation in $N_{n-1,1,s-1,1,r_1,...,r_k}$ given by a double point degeneration $\pi : \mathcal{V} \to \mathbb{P}^1$, divisors $D_i$ of $\mathcal{V}$, a line bundle $\mathcal{N}$ and vector bundles $E_j$ on $\mathcal{V}$. Then $\mathbb{P}_\mathcal{V}(\mathcal{O} \oplus N) \to \mathbb{P}^1$, the section $s = \mathbb{P}_\mathcal{V}(\mathcal{O}_\mathcal{V})$ and the pullback of $D_i$ and $E_j$ yield a double point relation

$$\begin{align*}
[\mathbb{P}_\mathcal{V}(\mathcal{O} \oplus N_0), p^{-1}D_0, s_0, p^*(E_{0j})] - [\mathbb{P}_\mathcal{V}(\mathcal{O} \oplus N_1), p^{-1}D_1, s_1, p^*(E_{1j})] & = [\mathbb{P}_\mathcal{V}(\mathcal{O} \oplus N_2), p^{-1}D_2, s_2, p^*(E_{2j})] + [\mathbb{P}_\mathcal{V}(\mathcal{O} \oplus N_3), p^{-1}D_3, s_3, p^*(E_{3j})].
\end{align*}$$

As a result, $P$ naturally inherits double point relations from $N_{n-1,1,s-1,1,r_1,...,r_k}$ so that the map

$$[\mathcal{V}, \{D_i\}_{i=1}^{s-1}, N, E_j] \to [\mathbb{P}_\mathcal{V}(\mathcal{O} \oplus N), \{p^{-1}D_i\}_{i=1}^{s-1}, s, p^*E_j].$$

defines a surjective homomorphism from $\nu_{n-1,1,s-1,1,r_1,...,r_k}$ to $P$.

By induction on $s$, $\nu_{n-1,1,s-1,1,r_1,...,r_k}$ is generated by $\omega_s(k) \cdot \psi(\cup \pi_i \cup \mu_j; \pi_i; \pi_i; \mu_j)$ which has the form $[M] \cdot [\mathbb{P}^{\cup \pi_i \cup \mu_j}, H_i, L_s, E_j]$. If $L_s$ is trivial, $[M] \cdot [\mathbb{P}^{\cup \pi_i \cup \mu_j}, H_i, L_s, E_j]$ maps to

$$[M] \cdot [\mathbb{P}^{\cup \pi_i \cup \mu_j} \times \mathbb{P}^1, p^{-1}H_i, \mathbb{P}^{\cup \pi_i \cup \mu_j} \times 0, p^*E_j].$$

If $L_s$ is not trivial, i.e. when $\pi_s = m > 0$, $[M] \cdot [\mathbb{P}^{\cup \pi_i \cup \mu_j}, H_i, L_s, E_j]$ maps to

$$[M] \cdot [\mathbb{P}^{\cup \pi_i \cup \mu_j}(\mathcal{O} \oplus L_m), p^{-1}H_i, s, p^*E_j].$$

In fact $\mathbb{P}_{\mathcal{V}}^{\cup \pi_i \cup \mu_j}(\mathcal{O} \oplus L_m)$ is the product of $\mathbb{P}^\lambda$ and $\mathbb{P}_{\mathcal{V}}(\mathcal{O} \oplus \mathcal{O}_{\mathcal{V}}(1))$ where $\lambda$ is the union of all $\pi_i$ and $\mu_j$ except $\pi_s$. The degeneration to the normal cone of a hyperplane $H_{m+1}$ in $\mathbb{P}^{m+1}$ yields the double point relation

$$[\mathbb{P}^{m+1}, H_{m+1} = [\mathbb{P}^{m+1}, 0] - [\mathbb{P}_{\mathcal{V}}(\mathcal{O} \oplus \mathcal{O}_{\mathcal{V}}(1)), s] + [\mathbb{P}_{\mathcal{V}}(\mathcal{O} \oplus \mathcal{O}_{\mathcal{V}}(1)), 0].$$

Multiplying the family by $M$ and $\mathbb{P}^\lambda$ and adding $H_i$, $E_j$ give a double point relation which expresses $[M] \cdot [\mathbb{P}_{\mathcal{V}}^{\cup \pi_i \cup \mu_j}(\mathcal{O} \oplus L_m), p^{-1}H_i, s, p^*E_j]$ as the sum of elements with $D_s = 0$ or in $\{ \omega_s(k) \cdot \psi(\cup \pi_i \cup \mu_j; \pi_i; \pi_i; \mu_j) \mid (\cup \pi_i \cup \mu_j; \pi_i; \mu_j) \in \mathcal{P}_s, s, r_1,...,r_k \}$. Therefore $P$ is spanned by these elements over $\mathbb{Z}$.

It is easy to see

$$[M, \{D_i\}_{i=1}^{s-1}, \{E_j\}_{j=1}^{r_j}] \to [M, \{D_i\}_{i=1}^{s}, \{E_j\}_{j=1}^{r_j}].$$

defines an isomorphism from $\nu_{n,1,s-1,...,r_k}(k)$ to the subgroup of $\nu_{n,1,s-1,...,r_k}(k)$ with $D_s = 0$. By induction on $s$, $\nu_{n,1,s-1,...,r_k}(k)$ is generated by $\omega_s(k) \cdot \psi(\cup \pi_i \cup \mu_j; \pi_i; \pi_i; \mu_j)$ which are mapped to $\omega_s(k) \cdot \psi(\cup \pi_i \cup \mu_j; \pi_i; \mu_j).$ They generate the subgroup of $\nu_{n,1,s-1,...,r_k}(k)$ with $D_s = 0$ and this completes our proof.

**Proposition 3.6.** The morphism $\nu_{n,1,s-1,...,r_k}(k) \to \omega_{n,1,s-1,...,r_k}(k)$ is injective.

**Proof.** $\nu_{n,1,s-1,...,r_k}(k)$ is generated by

$$\{ \omega_s(k) \cdot \psi(\cup \pi_i \cup \mu_j; \pi_i; \mu_j) \mid l(\pi_i) \leq 1, l(\mu_j) \leq r_j \}$$

over $\mathbb{Z}$. The morphism sends $\{ \omega_s(k) \cdot \psi(\cup \pi_i \cup \mu_j; \pi_i; \mu_j) \}$ to $\{ \omega_s(k) \cdot \phi(\cup \pi_i \cup \mu_j, \pi_i, \mu_j) \}$ and the latter by Theorem 2.3 has no nontrivial relation in $\omega_{n,1,s-1,...,r_k}(k)$. $\square$

Now we have proved $\nu_{n,1,s-1,...,r_k}(k) \cong \omega_{n,1,s-1,...,r_k}(k)$. The proof of Theorem 3.1 will be complete by the following theorem and Theorem 2.4.
**Theorem 3.7.** The algebraic cobordism theory of bundles and divisors on varieties $\nu_{s,1^{s},r_{1},\ldots,r_{k}}$ is an extension of scalars of the algebraic cobordism theory $\omega_{s}$, i.e.

$$
\gamma_{X} : \omega_{s}(X) \otimes_{\omega_{s}(k)} \nu_{s,1^{s},r_{1},\ldots,r_{k}}(k) \rightarrow \nu_{s,1^{s},r_{1},\ldots,r_{k}}(X)
$$

is an isomorphism.

**Proof.** The morphism

$$
\omega_{s}(X) \otimes_{\omega_{s}(k)} \nu_{s,1^{s},r_{1},\ldots,r_{k}}(k) \equiv \omega_{s}(X) \otimes_{\omega_{s}(k)} \omega_{s,1^{s},r_{1},\ldots,r_{k}}(k) \equiv \nu_{s,1^{s},r_{1},\ldots,r_{k}}(X)
$$

equals the composition of

$$
\omega_{s}(X) \otimes_{\omega_{s}(k)} \nu_{s,1^{s},r_{1},\ldots,r_{k}}(k) \xrightarrow{\gamma_{X}} \nu_{s,1^{s},r_{1},\ldots,r_{k}}(X) \rightarrow \omega_{s,1^{s},r_{1},\ldots,r_{k}}(X)
$$

thus $\gamma_{X}$ is injective.

Proposition 3.5 still holds when every scheme is over base $X$. So $\nu_{s,1^{s},r_{1},\ldots,r_{k}}(X)$ is generated by elements of the form

$$
\bigg[ \mathbb{P}^{\lambda} \rightarrow X, H_{1}, \ldots, H_{k}, O_{\mathbb{P}^{\lambda}}^{r_{1}-(\mu_{1})} \oplus \bigg( \bigoplus_{m \in \mu_{1}} L_{m} \bigg), \ldots, O_{\mathbb{P}^{\lambda}}^{r_{k}-(\mu_{k})} \oplus \bigg( \bigoplus_{m \in \mu_{k}} L_{m} \bigg) \bigg]
$$

The line bundles $O(H_{i})$ and each factor $L_{m}$ defines a projective morphism

$$
g : \mathbb{P}^{\lambda} \rightarrow X \times \mathbb{P}^{d_{1}} \times \ldots \mathbb{P}^{d_{k}}
$$

so that every $O(H_{i})$ and $L_{m}$ is $g^{*}(O_{\mathbb{P}^{\lambda}(1)})$. Then the discussion in the beginning of Section 4 in [17] shows $\nu_{n,1^{s},r_{1},\ldots,r_{k}}(X)/\mathfrak{m} \cdot \nu_{n,1^{s},r_{1},\ldots,r_{k}}(X)$ is generated by $\omega_{s}(X)$ and $\nu_{n-\delta,1^{s},r_{1},\ldots,r_{k}}(k)$. It follows that $\gamma_{X}$ is surjective by the proof of [17] Proposition 14].

\[ \square \]

4. Notations for singularity and tangency conditions

4.1. **Singularities.** First, we recall some results in singularity theory from [24]. Let $(X, x)$ be an isolated complete intersection singularity (ICIS) of dimension $p$. Suppose $(X, x)$ can be locally embedded in $\mathbb{C}^{N}$ so that $X$ is locally defined by common zeros of $f = (f_{1}, \ldots, f_{N-p}) : (\mathbb{C}^{N}, 0) \rightarrow (\mathbb{C}^{N-p}, 0)$. The miniversal deformation space of the ICIS $(X, x)$ exists and its dimension is equal to the Tjurina number

$$
\tau(f) := \dim_{\mathbb{C}} O_{\mathbb{C}^{N}, 0}^{N-p}(Df \cdot O_{\mathbb{C}^{N}, 0}^{N} + \langle f_{1}, f_{2}, \ldots, f_{N-p} \rangle O_{\mathbb{C}^{N}, 0}^{N-p}).
$$

Two map-germs $f, g : (\mathbb{C}^{N}, 0) \rightarrow (\mathbb{C}^{N-p}, 0)$ define the same ICIS if they are $\mathcal{X}$-equivalent, i.e. there exists a commutative diagram

$$
\begin{array}{ccc}
(\mathbb{C}^{N}, 0) & \xrightarrow{f} & (\mathbb{C}^{N-p}, 0) \\
\downarrow & & \downarrow \\
(\mathbb{C}^{N}, 0) & \xrightarrow{g} & (\mathbb{C}^{N-p}, 0)
\end{array}
$$

in which the vertical maps are biholomorphisms.

It is well known [31] that every ICIS is finitely determined, which means if $f$ defines an ICIS then there exists a finite integer $k$ such that any map-germ $g$ with the same $k$-jet defines the same ICIS. If this holds then we say the ICIS is $k$-determined.
Definition 4.1. Let \( \delta = (X, x) \) be an ICIS and \( \mathfrak{m}_{X,x} \) be the maximal ideal in the local ring \( O_{X,x} \). Let

\[
    k(\delta) = \min_k \{ k \mid \delta \text{ is } k\text{-determined} \} \quad \text{(the degree of determinacy)},
\]

\[
    \xi(\delta) = O_{X,x}/\mathfrak{m}_{X,x}^{k(\delta)+1},
\]

and \( N(\delta) \) be the length of the zero-dimensional scheme \( \xi(\delta) \). It is easy to see the isomorphism class of \( \xi(\delta) \), \( k(\delta) \) and \( N(\delta) \) can be computed from any representation of \( \delta \).

Let \( \delta = (\delta_1, \delta_2, \ldots, \delta_{|\delta|}) \) be a collection of ICIS. A variety \( X \) has singularity type \( \delta \) if \( X \) is singular at exactly \( |\delta| \) points \( \{x_1, x_2, \ldots, x_{|\delta|}\} \) and the singularity at \( x_i \) is exactly \( \delta_i \). Such variety must contain a subscheme isomorphic to \( \prod_{i=1}^{|\delta|} \xi(\delta_i) \).

Define

\[
    N(\delta) = \sum_{i=1}^{|\delta|} N(\delta_i), \quad \text{and} \quad \tau(\delta) = \sum_{i=1}^{|\delta|} \tau(\delta_i).
\]

The number \( \tau(\delta) \) is the expected number of conditions for a subvariety to have singularity type \( \delta \).

4.2. Tangency conditions. Let \( Y \) be a smooth variety of dimension \( n \), \( E \) be a vector bundle of rank \( r \) and \( D \) is a smooth divisor in \( Y \). If \( E \) is globally generated, the zeros of generic sections of \( E \) will be dimension \( n - r \). Recall in introduction we explained why the tangency conditions are recorded by two collections of ICIS \( \alpha \) (at assigned points) and \( \beta \) (at unassigned points). Here we further assume the dimensions of ICIS in \( \alpha \) and \( \beta \) are \( n - r - 1 \).

Denote the numbers of ICIS in \( \alpha \) and \( \beta \) by \( |\alpha| \) and \( |\beta| \) and index them by

\[
    \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{|\alpha|}), \beta = (\beta_1, \beta_2, \ldots, \beta_{|\beta|}).
\]

Fix a collection of distinct points

\[
    \Omega = \{p_i\}_{1 \leq i \leq |\alpha|} \subset D
\]

of \( |\alpha| \) distinct points on \( D \). If \( s \) is a section of \( E \), assume its zero locus \( X \) is a reduced subvariety of dimension \( n - r \) and denote the smooth locus of \( X \) by \( X^{sm} \). For any sequence \( \alpha \) and \( \beta \), we say the subvariety \( X \) has tangency type \( (\alpha, \beta) \) with \( D \) if there are \( |\beta| \) distinct points \( \{q_j\}_{1 \leq j \leq |\beta|} \) in \( D \setminus \Omega \), such that the scheme-theoretical intersection of \( D \) and \( X \) has singularity type \( \alpha_i \) at \( p_i \) and \( \beta_j \) at \( q_j \).

The condition \( D \cap X \) has a given ICIS at some point \( p \in D \) is equivalent to say the restriction of \( s \) on \( D \) defines the ICIS at \( p \). Since \( D \) is smooth everywhere, the expected codimension of this condition is the Tjurina number \( \tau \) of this ICIS. If the point \( p \) is given, then the expected codimension becomes \( \tau + n - 1 \). As a result, the expected codimension of tangency type \( (\alpha, \beta) \) is

\[
    \text{codim}(\alpha, \beta) := (n - 1)|\alpha| + \tau(\alpha) + \tau(\beta).
\]

Example 4.1. If \( X \) is a curve, \( X \) is tangent to \( D \) of multiplicity \( k \) at a smooth point if and only if \( X \cap D \) has the ICIS \( (\text{Spec} \mathbb{C}[t]/t^k, 0) \). The expected codimension for \( X \) to be tangent to \( D \) of multiplicity \( k \) at an unassigned point is \( k - 1 \). If tangency multiplicity \( k \) at an assigned point is required, then the expected codimension is \( n - k - 2 \).
5. Degeneration formula

In this section we show the numbers of singular subvarieties with tangency conditions can be expressed as an intersection numbers on Hilbert schemes of points under ampleness assumptions (Proposition 5.3). Then we show the intersection numbers behave nicely under double point relations by proving a degeneration formula (Corollary 5.6, 5.7). The degeneration formula combined with the basis of the algebraic cobordism groups discussed in Section 4 will lead us to the existence of universal polynomials and the formula for generating series in Section 6.

5.1. Hilbert schemes. Recall $Y$ is a smooth variety of dimension $n$, $D$ is a smooth divisor in $Y$, and $\alpha$, $\beta$, $\delta$ are three collections of ICIS.

Fix a set of $|\alpha|$ distinct points $\Omega = \{p_i\}_{1 \leq i \leq |\alpha|}$ in $D$ and let $Y^{[m]}$ be the Hilbert scheme of $m$ points on $Y$. Let

$$N(\alpha, \beta, \delta) = N(\alpha) + N(\beta) + N(\delta)$$

which is the total length of the zero-dimensional schemes $\xi(\alpha)$, $\xi(\beta)$ and $\xi(\delta)$. Define $Y^{0}_{\Omega}(\alpha, \beta, \delta) \subset Y^{[N(\alpha, \beta, \delta)]}$ to be the set of subschemes $\coprod_{k=1}^{[\alpha]+|\beta|+|\delta|} \eta_k$ satisfying the following conditions:

1. If $D$ is the zero divisor, then $Y^{0}_{\Omega}(\alpha, \beta, \delta)$ is the empty set except for the case $\alpha = \beta$ are both the empty set $\emptyset$.
2. If $\alpha$, $\beta$ and $\delta$ are all empty sets, let $Y^{0}_{\Omega}(\alpha, \beta, \delta)$ be $\text{Spec} \mathbb{C}$.
3. Every $\eta_k$ is supported on distinct points of $Y$.
4. For $1 \leq i \leq |\alpha|$, $\eta_i$ is a subscheme of $D$ support at $p_i$ and $\eta_i \cong \xi(\alpha_i)$ as schemes.
5. For $1 \leq i \leq |\beta|$, $\eta_{|\alpha|+i}$ is a subscheme of $D$ and $\eta_{|\alpha|+i} \cong \xi(\beta_i)$ as schemes.
6. For $1 \leq i \leq |\delta|$, $\eta_{|\alpha|+|\beta|+i}$ is isomorphic to $\xi(\delta_i)$.

Take $Y_{\Omega}(\alpha, \beta, \delta)$ to be the closure of $Y^{0}_{\Omega}(\alpha, \beta, \delta)$ in $Y^{[N(\alpha, \beta, \delta)]}$ with reduced induced scheme structure. For every $m \in \mathbb{N}$, let $Z_m \subset Y \times Y^{[m]}$ be the universal closed subscheme with projections $p : Z_m \to Y$, $q : Z_m \to Y^{[m]}$.

**Definition 5.1.** If $E$ is a vector bundle of rank $r$ on $Y$, define $E^{[m]} = q_*p^*E$. Because $q$ is finite and flat, $E^{[m]}$ is a vector bundle of rank $rm$ on $Y^{[m]}$ and it is called the tautological bundle of $E$.

We call $E$ $k$-very ample if for every zero-dimensional scheme $Z$ of length $k + 1$ in $Y$, the natural restriction map $H^0(E) \to H^0(E \otimes O_Z)$ is surjective.

For simplicity, let $N = N(\alpha, \beta, \delta)$, $c = \text{codim}(\alpha, \beta) + \tau(\delta)$ be the expected codimension for having non-induced singularity $\delta$ and tangency type $(\alpha, \beta)$ with $D$. Let

$$\Lambda^{\Omega}_{\alpha, \beta, \delta}(Y, D, E) := c_{rN-c}(E^{[N]}) \cap [Y^{[N]}(\alpha, \beta, \delta)]$$

in the Chow group of $Y^{[N]}$.

**Lemma 5.1.** $\Lambda^{\Omega}_{\alpha, \beta, \delta}(Y, D, E)$ is a zero cycle.

**Proof.** Since $\text{codim}(\alpha, \beta) = (n-1)|\alpha| + \tau(\alpha) + \tau(\beta)$,

$$rN - c = r(N(\alpha) + N(\beta) + N(\delta)) - (n-1)|\alpha| - \tau(\alpha) - \tau(\beta) - \tau(\delta)$$

$$= (rN(\alpha) - (n-1)|\alpha| - \tau(\alpha)) + (rN(\beta) - \tau(\beta)) + (rN(\delta) - \tau(\delta))$$
Next we compute the dimension of $Y_{\Omega}(\alpha, \beta, \delta)$. If $f$ is a representative of a single ICIS, since $\delta$ is $k(\delta)$-determined,

$$\mathcal{O}_{\mathbb{C}^n,0}/((f_1, f_2, \ldots, f_r) + m^{k(\delta)+1}) \cong \mathcal{O}_{\mathbb{C}^n,0}/((g_1, g_2, \ldots, g_r) + m^{k(\delta)+1})$$

holds if and only if $g = (g_1, \ldots, g_r)$ also defines $\delta$. These two subschemes are equal for general choices of $g_i$ in $(f_1, f_2, \ldots, f_r) + m^{k(\delta)+1} / m^{k(\delta)+1}$ and the dimension for all choices of $g_1, \ldots, g_r$ is

$$r \cdot \left( \dim (f_1, f_2, \ldots, f_r) + m^{k(\delta)+1} / m^{k(\delta)+1} \right).$$

Let $H$ be the direct sum of $r$ copies of degree at most $k(\delta)$ polynomials in $x_1, \ldots, x_n$. A deformation of $f$ over $H$ is given by $F(x,h) = f(x) + h(x)$, since $\delta$ is $k(\delta)$-determined, this deformation of $f$ is versal and the dimension of $H$ equals $\tau(\delta)$ plus the dimension of $k(\delta)$-jets $g$ which also define $\delta$. As a result, the dimension of choosing an subscheme $\eta$ corresponding to $\delta$ is

$$\dim H - \tau(\delta) - r \cdot \left( \dim (f_1, f_2, \ldots, f_{n-1}) + m^{k(\delta)+1} / m^{k(\delta)+1} \right)$$

$$= r \dim \mathcal{O}_{\mathbb{C}^n,0}/((f_1, f_2, \ldots, f_r) + m^{k(\delta)+1}) - \tau(\delta) = rN(\delta) - \tau(\delta).$$

Therefore the dimension of choosing $\coprod_{k=1}^{[\delta]} \eta_{|\alpha|+|\beta|+k}$ is $rN(\delta) - \tau(\delta)$.

For $\beta$, the only difference is that the subschemes are supported on $D$. If we change $n$ to $n-1$ above, the same argument holds and the dimension of choosing $\coprod_{k=1}^{[\beta]} \eta_{|\alpha|+k}$ is $rN(\beta) - \tau(\beta)$. For $\alpha$, there are total $(n-1)|\alpha|$ less dimensions due to $\eta_1, \ldots, \eta_{|\alpha|}$ are supported at assigned points on $D$. So we can conclude the dimension of $Y_{\Omega}(\alpha, \beta, \delta)$ is

$$(rN(\alpha) - (n-1)|\alpha| - \tau(\alpha)) + (rN(\beta) - \tau(\beta)) + (rN(\delta) - \tau(\delta)) = rN - c.$$

\begin{lemma}
The degree of $\Lambda_{\alpha,\beta,\delta}(Y,D,E)$ is independent of the collection of points $\Omega$.
\end{lemma}

\begin{proof}
Since $\Omega$ is a collection of $|\alpha|$ distinct points in $D$, $\Omega$ can be parametrized by $D_0^{\times |\alpha|}$, the open set of $D^{\times |\alpha|}$ consisting of distinct points. Let $\mathcal{Y}(\alpha, \beta, \delta)$ be the family of $Y^{[N]} \times D_0^{\times |\alpha|}$ whose fiber over $\Omega$ is $Y_{\Omega}(\alpha, \beta, \delta)$. This family $\mathcal{Y}(\alpha, \beta, \delta)$ is flat over $D_0^{\times |\alpha|}$ and its intersection with the pullback of $c_{rN-c}(E^{[N]})$ has constant degree $\Lambda_{\alpha,\beta,\delta}(Y,D,E)$ in every fiber. Therefore the degree of $\Lambda_{\alpha,\beta,\delta}(Y,D,E)$ is independent of $\Omega$.
\end{proof}

\begin{proposition}
If $\alpha$, $\beta$ are two collections of ICIS of dimension $n-r-1$, $\Omega$ is a set of $|\alpha|$ distinct points in $D$ and $\delta$ is a collection of ICIS of dimension $n-r$. For all $(N(\alpha) + N(\beta) + N(\delta) + n-r+1)$-very ample vector bundle $E$ of rank $r$ on a smooth variety $Y$ of dimension $n$, if $\mathbb{P}(V) \subset \mathbb{P}(H^0(E))$ is a general linear subspace of dimension $\text{codim}(\alpha, \beta) + \tau(\delta)$, then there are precisely $d_{\alpha,\beta,\delta}(Y,D,E)$ subvarieties defined by zero sets of sections in $V$ which have non-induced singularity type $\delta$ and tangency type $(\alpha, \beta)$ with $D$.
\end{proposition}
Proof. Recall we write $N = N(\alpha) + N(\beta) + N(\delta)$, $c = \text{codim}(\alpha, \beta) + \tau(\delta)$.

Let $\{s_0, s_1, \ldots, s_c\} \subset H^0(E)$ be a coordinate system of $V$, then $\{q_s p^* s_i\}_{i=0}^c$ are global sections of $E^{[N]}$. By the ampleness assumption, $H^0(E) \to H^0(E \otimes \mathcal{O}_X)$ is surjective for every closed subscheme $Z$ in $Y^{[N]}$, therefore $E^{[N]}$ are generated by sections from $H^0(E)$. By applying [9, Example 14.3.2] and a dimensional count, the locus $W$ where $\{q_s p^* s_i\}_{i=0}^c$ are linearly dependent is Poincaré dual to $c_{rN-c}(E^{[N]})$ for general $V$ and

$$[W \cap Y(\alpha, \beta, \delta)] = c_{rN-c}(E^{[N]}) \cap [Y(\alpha, \beta, \delta)] = \Lambda_{\alpha, \beta, \delta}(Y, D, E).$$

Applying [9, Example 14.3.2] again to $Y(\alpha, \beta, \delta) \setminus Y^0(\alpha, \beta, \delta)$ with a dimensional count implies that $W \cap Y(\alpha, \beta, \delta)$ is only supported on $Y^0(\alpha, \beta, \delta)$ for general $V$. If $Z \in Y^{[N]}$, the kernel of $H^0(E) \to H^0(E|_Z)$ is a $rN$-codimensional subspace of $H^0(E)$ so it defines a morphism $\phi_N : Y^{[N]} \to \text{Grass}(h^0(E) - rN, H^0(E))$. $W \cap Y(\alpha, \beta, \delta)$ is the preimage of the Schubert cycle $\{U \in \text{Grass}(h^0(E) - rN, H^0(E)) | U \cap V = 0 \}$ under $\phi_N$. By [15], $W \cap Y(\alpha, \beta, \delta)$ is smooth for general $V$.

Every point in $W \cap Y(\alpha, \beta, \delta)$ corresponds to a section $s \in \text{span}\{s_0, \ldots, s_c\}$ which vanishes at a point in $Y^0(\alpha, \beta, \delta)$. We need to show the vanishing locus of such $s$ must be a subvariety with non-induced singularity types $\delta$ and tangent type $(\alpha, \beta)$ with $D$. Denote the zero locus of $s$ by $X$ and let $X$ contain the zero-dimensional subscheme $\eta = \bigcap_{i=1}^{[\alpha]+[\beta]+[\delta]} \eta_i$ in $Y^0(\alpha, \beta, \delta)$. The argument is divided into three steps.

**Step 1.** For general $V$, we show all such subvarieties $X \in \Lambda_{\alpha, \beta, \delta}(Y, D, E)$ must have dimension $n - r$ and every singular point of $X$ must lie in $\text{supp} \eta$. We will prove it by contradiction.

Suppose $X$ has a singular point disjoint from the support of $\eta$ or the dimension of $X$ is greater than $n - r$. then $X$ must contain a zero-dimensional subscheme in $Y'(\alpha, \beta, \delta) := \bigcap_{i=1}^{[\alpha]+[\beta]+[\delta]} \eta_i \cup Z$ where

$$\prod_{i=1}^{[\alpha]+[\beta]+[\delta]} \eta_i \in Y^0(\alpha, \beta, \delta) \text{ and } Z \cong \text{Spec } \mathbb{C}[x_1, \ldots, x_{n-r+1}]/(x_1, \ldots, x_{n-r+1})^2.$$

$Y'(\alpha, \beta, \delta)$ is a subset of $Y^{[N+n-r+2]}$. Since $V$ is general and $E$ is $(N + n - r + 1)$-very ample, the number of such subvarieties can be computed by

$$c_{r(N+n-r+2)-c}(E^{[N+n-r+2]}) \cap [Y'(\alpha, \beta, \delta)].$$

But the cycle vanishes because

$$\dim Y'(\alpha, \beta, \delta) = \dim Y(\alpha, \beta, \delta) + n + (r-1)(n-r+1)$$

$$= rN - c + n + (r-1)(n-r+1) < r(N + n - r + 2) - c.$$ 

So such $X$ does not exist.

**Step 2.** We prove that subvarieties in $\Lambda_{\alpha, \beta, \delta}(Y, D, E)$ must have non-induced singularity type $\delta$. Let $X$, $s$ and $\eta$ be same as above. By Step 1 we know $X$ has exactly $|\delta|$ non-induced singular points on the support of $\bigcap_{i=1}^{[\delta]} \eta_{[\alpha]+[\beta]+i}$. If the non-induced singularity type of $X$ is not $\delta$, without loss of generality we can assume the singularity type at the point $y_1$ (which is defined to be the support of $\eta_{[\alpha]+[\beta]+1}$) is not $\delta_1$. 

Fix an isomorphism between \((Y, y_1)\) with \((\mathbb{C}^n, 0)\). Let \(g_1 = \cdots = g_r = 0\) be the local equations of \(X \subset Y\) at \(y_1\) and \(\eta_{|\alpha|+|\beta|+1}\) be \(\text{Spec} \mathcal{O}_{\mathbb{C}^n, 0}/(f_1, \ldots, f_r, m_{\mathbb{C}^n, 0}^{k(\delta_1)+1})\). Because \(\eta_{|\alpha|+|\beta|+1}\) is a subscheme of \(X\), the ideal \((f_1, \ldots, f_r, m_{\mathbb{C}^n, 0}^{k(\delta_1)+1}) \subset \mathcal{O}_{\mathbb{C}^n, 0}\) contains but can not equal \(\langle g_1, \ldots, g_r, m_{\mathbb{C}^n, 0}^{k(\delta_1)+1}\rangle\) otherwise by the finite determinacy theorem the singularity type of \(X\) at \(y_1\) is \(\delta_1\). Then there must be an ideal \(J\) satisfying

\[
\langle g_1, \ldots, g_r, m_{\mathbb{C}^n, 0}^{k(\delta_1)+1}\rangle \subset J \subset \langle f_1, \ldots, f_r, m_{\mathbb{C}^n, 0}^{k(\delta_1)+1}\rangle
\]

and \(\dim \mathbb{C}(\mathcal{O}_{\mathbb{C}^n, 0}/J) = \text{length}(\eta_{|\alpha|+|\beta|+1}) + 1\). If we fix \(\eta_{|\alpha|+|\beta|+1}\), every ideal \(J\) satisfying \(m_{\mathbb{C}^n, 0}^{k(\delta_1)+1} \subset J \subset \langle f_1, \ldots, f_r, m_{\mathbb{C}^n, 0}^{k(\delta_1)+1}\rangle\) and \(\text{length}(\mathcal{O}_{\mathbb{C}^n, 0}/J) = \text{length}(\eta_{|\alpha|+|\beta|+1}) + 1\) corresponds to a codimension 1 subspace in the vector space spanned by the images of \(f_1, \ldots, f_r\) in \(\mathcal{O}_{\mathbb{C}^n, 0}/m_{\mathbb{C}^n, 0}^{k(\delta_1)+1}\). Therefore the dimension of all choices of \(J\) is no greater than \(r - 1\).

If we allow \(\eta_{|\alpha|+|\beta|+1}\) to vary, the dimension of such \(J\) is no greater than \(r - 1\) plus the dimension of all \(\eta_{|\alpha|+|\beta|+1}\). Since \(X\) contains \(\text{Spec} \mathcal{O}_{Y, y_1}/J\) and \(\eta \setminus \eta_{|\alpha|+|\beta|+1}\), a similar argument to step 1 imply such \(X\) must contribute to intersection of \(c_{r(N+1)-c}(E^{[N+1]})\) and the closure of all possible union of \(\text{Spec} \mathcal{O}_{Y, y_1}/J\) and \(\bigcup_{i \in \{1, 2, \ldots, |\alpha|+|\beta|+\delta\}} \{\{|\alpha|+|\beta|+1\} \eta_i\}\) in \(Y^{[N+1]}\). For dimension reason this cycle is empty, therefore for general \(V\), every \(X\) in \(\Lambda_{\alpha, \beta, \delta}(Y, D, E)\) must have singularity type \(\delta\) at the support of \(\bigcup_{i \in \{1, 2, \ldots, |\alpha|+|\beta|+\delta\}} \{\{|\alpha|+|\beta|+1\} \eta_i\}\).

All \(\eta\) with \(\eta_{|\alpha|+|\beta|+1}\) supported on \(D\) form a closed subvariety of \(Y(\alpha, \beta, \delta)\). For dimension reason the intersection of this subvariety and \(c_{rN-c}(E^{[N]})\) is empty. Therefore by symmetry for general \(V\), \(X\) has non-induced singularity type precisely \(\delta\).

**Step 3.** We prove for general \(V\), subvarieties in \(\Lambda_{\alpha, \beta, \delta}(Y, D, E)\) must have tangent type \((\alpha, \beta)\) with \(D\).

Since \(X\) contains \(\bigcup_{i \in \{1, 2, \ldots, |\alpha|+|\beta|+\delta\}} \eta_k \subset D\), the definition of \(\eta_k\) ensures that \(X\) has tangency type “at least” \((\alpha, \beta)\) with \(D\) at the support of \(\bigcup_{i \in \{1, 2, \ldots, |\alpha|+|\beta|+\delta\}} \eta_i\). If \(X\) is tangent to \(D\) at another point, then \(D \cap X\) is not smooth there and the tangent space at this point is at least \((n - r)\)-dimensional. Therefore, \(X \cap D\) contains a subscheme \(Z \cong \text{Spec} \mathbb{C}[x_1, \ldots, x_{n-r}]/(x_1, \ldots, x_{n-r})^2\) which is disjoint from \(\bigcup_{i \in \{1, 2, \ldots, |\alpha|+|\beta|+\delta\}} \eta_k\). All such \(Z\) form a subset of \(D^{[n-r+1]}\) of dimension \(n - 1 + (r - 1)(n - r)\) where \(n - 1\) comes from the choice of point on \(D\) and \((r - 1)(n - r)\) from the choice of \(x_1, \ldots, x_{n-r}\) in \(x_1, \ldots, x_{n-1}\). Using a similar argument in Step 2 and the fact that \(E\) is \((N+n-r)\)-very ample, those \(X\) containing \(Z \cup \eta\) in \(V\) can be computed by the intersection of \(c_{r(N+n-r+1)-c}(E^{[N+n-r+1]})\) and a \((rN - c + n - 1 + (r - 1)(n - r))\)-dimensional subvariety in \(Y^{[N+n-r+1]}\). The dimension of this cycle is negative so such \(X\) does not exist. Therefore we can conclude \(X\) is tangent to \(D\) at exactly \(|\alpha|+|\beta|\) points.

If the tangency type of \(X\) and \(D\) is still not \((\alpha, \beta)\), it must due to a different ICIS at one of the tangency points on \(D \cap X\). Suppose this happens at an unassigned point, which can be assumed to be \(q = \text{supp} \eta_{|\alpha|+1}\) without loss of generality. The argument in Step 2 can be applied here with \(Y\) replaced by \(D\). It shows \(X\) must contain the union of \(\text{Spec} \mathcal{O}_{D, q}/J\) with \(\bigcup_{i \in \{1, 2, \ldots, |\alpha|+|\beta|+\delta\}} \{\{|\alpha|+1\} \eta_i\}\) for some \(J\) such that the length of \(\text{Spec} \mathcal{O}_{D, q}/J\) is \(N(\beta_1) + 1\). Since \(E\) is \(N\)-very ample, such \(X\) must contribute to intersection of \(c_{r(N_1)-c}(E^{[N+1]})\) and the closure of all possible \(\text{Spec} \mathcal{O}_{D, q}/J\) union
\[ \prod_{i \in \{1, 2, \ldots, |\alpha|+|\beta|+|\delta|\} \setminus \{|\alpha|+1\}} \eta_i \text{ in } Y^{[N+1]}. \] All possible \( \text{Spec} \mathcal{O}_{D,q}/J \) form a subset of dimension no greater than \( r-1 \) plus the dimension of all \( \eta_{|\alpha|+1} \). Thus for dimension reason this cycle is empty and tangency type of \( X \) with \( D \) at unassigned points must be \( \beta \).

If the tangency type of \( D \) and \( X \) is not \( \alpha_k \) at an assigned point \( p_k \), we can modify the argument above by considering \( q = p_k \). Then all possible \( \text{Spec} \mathcal{O}_{D,p_k}/J \) still form a subset of dimension no greater than \( r-1 \) plus the dimension of all \( \eta_k \). Thus for dimension reason it can not happen and the tangency type of \( X \) with \( D \) must be \( (\alpha, \beta) \).

Currently there is no method to evaluate \( d_{\alpha, \beta, \delta}(Y, D, E) \) directly. In the next subsection we will derive a degeneration formula for \( d_{\alpha, \beta, \delta} \) under double point relations.

5.2. Relative Hilbert schemes. To set up, we recall some useful facts about the relative Hilbert schemes constructed by Li and Wu [21]. Let \( U \) be a smooth irreducible curve and \( \infty \in U \) be a specialized point. Consider a flat projective family of schemes \( \pi : Y \rightarrow U \) which satisfies

1. \( Y \) is smooth and \( \pi \) is smooth away from the fiber \( \pi^{-1}(\infty) \);
2. \( \pi^{-1}(\infty) =: Y_1 \cup Y_2 \) is a union of two irreducible smooth components \( Y_1 \) and \( Y_2 \) which intersect transversally along a smooth divisor \( B \).

In [21], Li and Wu constructed a family of Hilbert schemes of \( n \) points \( \pi^{[n]} : Y^{[n]} \rightarrow U \), whose smooth fiber over \( t \neq \infty \) is \( Y^{[n]}_t \), the Hilbert scheme of \( n \) points on the smooth fiber \( Y_t \) of \( \pi \). To compactify this moduli space, one can replace \( Y \) by a new space \( Y' \) so that \( Y \) and \( Y'^{[n]} \) have same smooth fibers over \( t \neq \infty \), but over \( \infty \) the fiber of \( Y^{[n]} \) is a semistable model

\[ Y^{[n]}_\infty = Y_1 \cup \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_{n-1} \cup Y_2, \]

where \( \Delta_i \cong \mathbb{P}_B(O_B \oplus N_{Y_1/B}) \). The fiber of \( Y^{[n]} \) over \( \infty \) consists of length \( n \) zero-dimensional subschemes supported on the smooth locus of \( Y^{[n]}_\infty \). Since \( Y^{[n]}_\infty \) is a chain, any zero-dimensional subscheme \( Z \) on \( Y^{[n]}_\infty \) can be decomposed into \( Z_1 \cup Z_2 \) where \( Z_1 \) is supported on \( Y_1 \cup \Delta_1 \cup \ldots \cup \Delta_i \) and \( Z_2 \) is supported on \( \Delta_{i+1} \cup \ldots \cup \Delta_{n-1} \cup Y_2 \) for some \( i \). These \( Z_1 \) and \( Z_2 \) belongs to the relative Hilbert scheme \( (Y_t/B)^{[k]} \) and the decomposition gives

\[ Y^{[n]}_\infty = \bigcup_{k=0}^n (Y_1/B)^{[k]} \times (Y_2/B)^{[n-k]}. \]

Li and Wu proved that the moduli stack \( Y^{[n]} \) is a separated and proper Deligne-Mumford stack of finite type over \( U \).

For a vector bundle \( E \) on \( Y \), let \( E_i \) denote the restriction of \( E \) to \( Y_i \) for \( i = 0, 1, 2 \) and call \( E_3 \) the pullback of the restriction of \( E \) to \( B \) via the morphism \( Y_3 \rightarrow B \). The tautological bundle \( E^{[n]} \) on \( Y^{[n]} \) exists. A straightforward generalization of [27] Lemma 3.7 shows the restriction of \( E^{[n]} \) on smooth fibers \( Y^{[n]}_t \) is the tautological bundle \( (E|_{Y_1})^{[n]} \) and the restriction on \( (Y_1/B)^{[k]} \times (Y_2/B)^{[n-k]} \) is the direct sum of tautological bundles on relative Hilbert schemes \( E_1^{[k]} \oplus E_2^{[n-k]} \).

Let \( D \) be a smooth effective divisor on \( Y \) which intersects \( Y_1 \) and \( Y_2 \) transversally along smooth divisors. Denote the intersection \( D \cap Y_i \) by \( D_i \) for \( i = 0, 1, 2 \) and call \( D_3 \) the inverse image of \( D \cap B \) via the morphism \( Y_3 \rightarrow B \). Because \( D \) intersects \( B \) transversally

\[ 3 \text{More precisely, there is a natural way to extend bundles to the expanded relative pairs of } Y_1 \cup B Y_2. \]

See Section 2.3 of [21].
and \( \mathcal{Y}[n] \) is constructed by blowups, \( \mathcal{D} \) can be naturally lifted to a divisor \( \mathcal{D}[n] \) in \( \mathcal{Y}[n] \) and its restriction \( \mathcal{D}[n]_\infty \) on \( \mathcal{Y}[n]_\infty \) are union of smooth divisors on \( Y_1, Y_2 \) and all \( \Delta_i \).

Recall in Section 4.3 the subsets \( Y_i^0(\alpha, \beta, \delta) \) and \( Y_{\Omega}(\alpha, \beta, \delta) \) of \( Y^{|N(\alpha)+N(\beta)+N(\delta)|} \) are defined. If the following assumption are satisfies, these subsets can be extended to the family \( \mathcal{Y} \to U \).

**Assumption 5.4.** Assume there are sections \( \{\sigma_i\}_{1 \leq i \leq |\alpha|} \) of the family \( \mathcal{D} \to U \) whose images are smooth curves in \( \mathcal{D} \) and disjoint from \( B \). In addition, assume these sections are disjoint over an open subset of \( U \) which contains 0 and \( \infty \). By replacing \( U \) with this open subset, we can assume the sections are disjoint everywhere.

Call \( \Omega_0 = \{\sigma_i(0)\}_{1 \leq i \leq |\alpha|}, \Omega_1 = \{\sigma_i(\infty)\}_{1 \leq i \leq |\alpha|} \cap D_1, \Omega_2 = \{\sigma_i(\infty)\}_{1 \leq i \leq |\alpha|} \cap D_2. \) Let \( \alpha_0 = \alpha \) and \( \alpha = \alpha_1 \cup \alpha_2 \) where \( \alpha_j \) be the collections of ICIS in \( \alpha \) corresponding to \( \Omega_j \) for \( j = 1, 2 \). On the relative Hilbert scheme \( (Y_j/B)^{|N(\alpha_j)+N(\beta_j)+N(\delta_j)|} \), the subset \( (Y_j/B)^{\Omega_j}_0(\alpha_j, \beta_j, \delta_j) \) can be defined in a similarly way to \( Y_i^0(\alpha, \beta, \delta) \) except now all subschemes are in the relative Hilbert scheme of \( Y_j/B \) and \( \bigcup_{i=1}^{n} \alpha_j \cap \beta_j \eta_i \) are supported on \( \mathcal{D}[n]_\infty \) for some \( n \). Let \( (Y_j/B)_{\Omega_j}(\alpha_j, \beta_j, \delta_j) \) be the closure of \((Y_j/B)^{\Omega_j}_0(\alpha_j, \beta_j, \delta_j)\) with reduced induced structure and \( d_{\alpha_j, \beta_j, \delta_j}(Y_j/B, D_j, E_j) \) be the degree of the zero cycle

\[
c_{r(N(\alpha_j)+N(\beta_j)+N(\delta_j))} \cdot \text{codim}(\alpha_j, \beta_j, \delta_j)(E_j^{[N(\alpha_j)+N(\beta_j)+N(\delta_j)]} \cap [(Y_j/B)_{\Omega_j}(\alpha_j, \beta_j, \delta_j)]).
\]

The proof of Lemma 5.2 can be applied to show \( d_{\alpha_j, \beta_j, \delta_j}(Y_j/B, D_j, E_j) \) is independent of \( \Omega_j \).

**Lemma 5.5.** If \( \mathcal{Y} \to U \) satisfies Assumption 5.4. For every \( \alpha, \beta, \delta \), there is a flat 1-cycle \( \mathcal{Y}(\alpha, \beta, \delta) \subset \mathcal{Y}^{[N(\alpha)+N(\beta)+N(\delta)]} \) such that

\[
\mathcal{Y}(\alpha, \beta, \delta) \cap (Y_0)^{[N(\alpha)+N(\beta)+N(\delta)]} = (Y_0)^{\Omega_0}(\alpha, \beta, \delta)
\]

\[
\mathcal{Y}(\alpha, \beta, \delta) \cap \left((Y_1/B)^{[N(\alpha)+N(\beta)+N(\delta)-m]} \times (Y_2/B)^{[N(\alpha)+N(\beta)+N(\delta)-m]}\right) \cup (Y_1/B)^{\Omega_1}(\alpha_1, \beta_1, \delta_1) \times (Y_2/B)^{\Omega_2}(\alpha_2, \beta_2, \delta_2),
\]

where the union is over all \( \beta_j, \delta_j \) satisfying \( N(\alpha_1) + N(\beta_1) + N(\delta_1) = m, \beta = \beta_1 \cup \beta_2 \) and \( \delta = \delta_1 \cup \delta_2 \).

**Proof.** For simplicity we write \( N = N(\alpha)+N(\beta)+N(\delta) \) and \( \Omega_t = \{\sigma_t(t)\} \). Let \( \mathcal{Y}^0(\alpha, \beta, \delta) \) be the union of all \( (Y_t)^{\Omega_t}_0(\alpha, \beta, \delta) \) for all smooth fibers \( Y_t \) over \( t \in U \). Define \( \mathcal{Y}(\alpha, \beta, \delta) \) to be the closure of \( \mathcal{Y}^0(\alpha, \beta, \delta) \) in \( \mathcal{Y}^{[N]} \), then by definition \( \mathcal{Y}(\alpha, \beta, \delta) \cap Y_0^{[N]} = (Y_0)^{\Omega_0}(\alpha, \beta, \delta) \).

Since the fibers of \( \pi \) are smooth away from \( B \), for every point \( p \in \mathcal{Y} \) one can choose an analytic neighborhood \( V_p \) such that \( \pi : V_p \to U \) is a trivial fibration over its image in \( U \). Since \( \mathcal{D} \) and the image of sections \( \sigma_t \) are submanifolds of \( \mathcal{Y} \), we can further assume \( \mathcal{D} \) is the zero locus of a coordinate function of \( V_p \) if \( p \in D \) and if \( p \) is in the image of a section \( \sigma_t \) then the image locally gives the coordinate along the direction of \( U \).

Assume \( z \) is a point in \( \mathcal{Y}(\alpha, \beta, \delta) \) which lies over \( \infty \). Write \( z = \cup z_k \) so that each \( z_k \) is only supported at one point \( p_k \). Then every \( p_k \) must belong to the smooth locus of a component \( \Delta \) of a semistable model \( \mathcal{Y}[n]_\infty = Y_1 \cup \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_{n-1} \cup Y_2 \). Since \( z \) is in the closure of \( \mathcal{Y}^0(\alpha, \beta, \delta) \), there is a sequence of points \( \{z_j\} \) in \( \mathcal{Y}^0(\alpha, \beta, \delta) \) approaching \( z \). By shrinking \( V_{p_k} \) if necessary and taking large enough \( j \), we can assume \( \{V_{p_k}\} \) are pairwise disjoint and all \( z_j \) are contained in \( \cup_k V_{p_k} \). As a result, the sequence \( \{z_j\} \) can
be decomposed into disjoint sequences $z_{jk} \to z_k$ in $V_{pk}$. Since $V_{pk} \subset \mathcal{Y} \to U$ is a trivial fibration over its image, there is another trivial projection in the complementary direction called $q_k : V_{pk} \to \Delta$ which projects $V_{pk}$ along the direction of $U$ onto a small open neighborhood of $p_k$ on $\Delta$.

Because $z_{jk}$ are subschemes of fibers, the trivial projection $q_k$ does not change their isomorphism types. In addition, our choice of coordinate functions assures that $q_k$ sends points on $D$ to $D$ and $\sigma_i(t)$ to $\sigma_i(\infty)$. Therefore $\{\cup_k q_k(z_{jk})\}$ is a sequence in $(Y_1/B)^0_{\Omega_1}(\alpha_1, \beta_1, \delta_1) \times (Y_2/B)^0_{\Omega_2}(\alpha_2, \beta_2, \delta_2)$ for some $\beta_j, \delta_j$ and $\Omega_j$ and its limit $z$ is in the closure $(Y_1/B)_{\Omega_1}(\alpha_1, \beta_1, \delta_1) \times (Y_2/B)_{\Omega_2}(\alpha_2, \beta_2, \delta_2)$.

On the other hand, $(Y_1/B)^0_{\Omega_1}(\alpha_1, \beta_1, \delta_1) \times (Y_2/B)^0_{\Omega_2}(\alpha_2, \beta_2, \delta_2)$ is a subset of $\mathcal{Y}(\alpha, \beta, \delta)$ because the trivialized neighborhood allows subschemes to be moved away from the fiber of $\infty$ along fibers of $q_k$. So for every point in $(Y_1/B)^0_{\Omega_1}(\alpha_1, \beta_1, \delta_1) \times (Y_2/B)^0_{\Omega_2}(\alpha_2, \beta_2, \delta_2)$, we can create a sequence in $\mathcal{Y}^0(\alpha, \beta, \delta)$ approaching it. Therefore $\mathcal{Y}(\alpha, \beta, \delta)$, defined as the closure of $\mathcal{Y}^0(\alpha, \beta, \delta)$, contains $(Y_1/B)^0_{\Omega_1}(\alpha_1, \beta_1, \delta_1) \times (Y_2/B)^0_{\Omega_2}(\alpha_2, \beta_2, \delta_2)$ because the latter is the closure of $(Y_1/B)^0_{\Omega_1}(\alpha_1, \beta_1, \delta_1) \times (Y_2/B)^0_{\Omega_2}(\alpha_2, \beta_2, \delta_2)$.

The open part $\mathcal{Y}^{0}(\alpha, \beta, \delta)$ is irreducible and it dominates the curve $U$. Since the scheme structure of $\mathcal{Y}(\alpha, \beta, \delta)$ is the induced reduced structure, $\mathcal{Y}(\alpha, \beta, \delta)$ is flat over $U$. \hfill \Box

Suppose $(\pi : \mathcal{Y} \to \mathbb{P}^1, \mathcal{D}, \mathcal{E})$ gives a double point relation in the algebraic cobordism ring of varieties and bundles constructed in Section 3. By definition, the fiber of $\pi$ over 0 is smooth but other fibers may not be smooth.

**Corollary 5.6.** If $(\pi : \mathcal{Y} \to \mathbb{P}^1, \mathcal{D}, \mathcal{E})$ gives a double point relation

\[
[Y_0, D_0, E_0] - [Y_1, D_1, E_1] - [Y_2, D_2, E_2] + [Y_3, D_3, E_3]
\]

and there is an open set $U$ of $\mathbb{P}^1$ containing 0 and $\infty$ such that $\mathcal{Y} \times_{\mathbb{P}^1} U \to U$ satisfies Assumption 5.4. Then

\[
d_{\alpha, \beta, \gamma}(Y_0, D_0, E_0) = \sum d_{\alpha_1, \beta_1, \gamma_1}(Y_1/B, D_1, E_1)d_{\alpha_2, \beta_2, \gamma_2}(Y_2/B, D_2, E_2)
\]

where the sum is over all $\beta_j$ and $\delta_j$ satisfying $\beta = \beta_1 \cup \beta_2$ and $\delta = \delta_1 \cup \delta_2$.

**Proof.** By Lemma 5.1,

\[
c_{r,N-c}(E^{[N]} \cap [\mathcal{Y}(\alpha, \beta, \delta)])
\]

is a flat 1-cycle in $\mathcal{Y}^{[N]}$. The degree of its restriction on the fiber over 0 is $d_{\alpha, \beta, \gamma}(Y_0, D_0, E_0)$ and its restriction on the fiber over $\infty$ is the right side of the desired equation by Lemma 5.5. This establishes the formula. \hfill \Box

If $\alpha$ is the empty set $\emptyset$, it is easy to see Lemma 5.5 still holds without assuming Assumption 5.4. Therefore we have

**Corollary 5.7.** If $(\pi : \mathcal{Y} \to \mathbb{P}^1, \mathcal{D}, \mathcal{E})$ gives a double point relation

\[
[Y_0, D_0, E_0] - [Y_1, D_1, E_1] - [Y_2, D_2, E_2] + [Y_3, D_3, E_3].
\]

Then

\[
d_{\emptyset, \beta, \gamma}(Y_0, D_0, E_0) = \sum d_{\emptyset, \beta_1, \gamma_1}(Y_1/B, D_1, E_1)d_{\emptyset, \beta_2, \gamma_2}(Y_2/B, D_2, E_2)
\]

where the sum is over all $\beta_j$ and $\delta_j$ satisfying $\beta = \beta_1 \cup \beta_2$ and $\delta = \delta_1 \cup \delta_2$.\hfill \Box
6. Universal polynomials and generating series

In this section, we will combine the degeneration formula (Corollary 5.7) and the algebraic cobordism group of bundles and divisors on varieties \( \nu_{n,1,r} \) in Section 3 to prove the existence of universal polynomials of subvarieties with given singular type and tangency type \((\emptyset, \beta)\). Moreover, the generating series of these polynomials is multiplicative, which can be used to determine their leading terms.

The reason we only consider the case \( \alpha \) is empty is because not all double point degenerations satisfy Assumption 5.4. If tangency conditions at assigned points are proposed, the generating series of the numbers of such subvarieties is not simply a multiplicative invariant of \( \nu_{n,1,r}(k) \). The case involving general \( \alpha \) will be discussed in another upcoming article [26].

For each ICIS \( \delta \), let \( z_\delta \) be a formal variable indexed by \( \delta \). If \( \delta = \{ \delta_i \} \) is a collection of ICIS then write \( z_\delta = \prod z_{\delta_i} \). It is easy to see \( z_\delta_1 \cdot z_\delta_2 \) is equal to \( z_\delta \) if and only if \( \delta \) is the union of \( \delta_1 \) and \( \delta_2 \) and the multiplication is commutative. For collections of ICIS \((\alpha, \beta)\), variables and their products \( x_\alpha \) and \( y_\beta \) are defined similarly.

**Definition 6.1.** For a rank \( r \) vector bundle \( E \) and a smooth divisor \( D \) on a smooth \( n \)-dimensional variety \( Y \), define the generating series

\[
T_\alpha(Y, D, E) = \sum_{\beta, \delta} d_{\alpha, \beta, \gamma}(Y, D, E)x_\alpha y_\beta z_\delta.
\]

If \( B \) is a smooth divisor of \( Y \) which intersects \( D \) transversally, let

\[
T_\alpha(Y/B, D, E) = \sum_{\beta, \delta} d_{\alpha, \beta, \gamma}(Y/B, D, E)x_\alpha y_\beta z_\delta.
\]

**Remark.** By definition if \( \alpha \), \( \beta \) and \( \delta \) are all empty, \( d_{\alpha, \beta, \gamma}(Y, D, E) = 1 \) because 1 is the degree of \( \mathbb{P}(H^0(E)) \) in itself.

**Proposition 6.1.** (Degeneration formula) Suppose \((\pi : \mathcal{Y} \rightarrow \mathbb{P}^1, D, E)\) give a double point relation

\[
[Y_0, D_0, E_0] - [Y_1, D_1, E_1] - [Y_2, D_2, E_2] + [Y_3, D_3, E_3]
\]

in the algebraic cobordism group of divisors and bundles constructed in Section 3. Then

\[
T_\emptyset(Y_0, D_0, E_0) = \frac{T_\emptyset(Y_1, D_1, E_1)T_\emptyset(Y_2, D_2, E_2)}{T_\emptyset(Y_3, D_3, E_3)}.
\]

In other words, \( T_\emptyset \) induces a homomorphism from the algebraic cobordism group \( \nu_{n,1,r} \) to \( \mathbb{C}[[y_\beta, z_\delta]]^\times \).

**Proof.** Recall \( B = Y_1 \cap Y_2 \). By Lemma 5.7 we have

\[
d_{\emptyset, \beta, \gamma}(Y_0, D_0, E_0) = \sum d_{\emptyset, \beta_1, \gamma_1}(Y_1/B, D_1, E_1)d_{\emptyset, \beta_2, \gamma_2}(Y_2/B, D_2, E_2)
\]

where the sum is over all \( \beta_j \) and \( \delta_j \) satisfying \( \beta = \beta_1 \cup \beta_2 \) and \( \delta = \delta_1 \cup \delta_2 \). This is equivalent to

\[
T_\emptyset(Y_0, D_0, E_0) = T_\emptyset(Y_1/B, D_1, E_1)T_\emptyset(Y_2/B, D_2, E_2).
\]
To derive a relation of generating series without relative terms, we apply this equality to four families: \( \mathcal{Y} \), the blowup of \( Y_1 \times \mathbb{P}^1 \) along \( B \times \{ \infty \} \), the blowup of \( Y_2 \times \mathbb{P}^1 \) along \( B \times \{ \infty \} \), and the blowup of \( Y_3 \times \mathbb{P}^1 \) along \( B \times \{ \infty \} \). The results are

\[
T_\emptyset(Y_0, D_0, E_0) = T_\emptyset(Y_1/B, D_1, E_1)T_\emptyset(Y_2/B, D_2, E_2),
\]

\[
T_\emptyset(Y_1, D_1, E_1) = T_\emptyset(Y_1/B, D_1, E_1)T_\emptyset(Y_3/B+, D_3, E_3),
\]

\[
T_\emptyset(Y_2, D_2, E_2) = T_\emptyset(Y_2/B, D_2, E_2)T_\emptyset(Y_3/B-, D_3, E_3),
\]

\[
T_\emptyset(Y_3, D_3, E_3) = T_\emptyset(Y_3/B+, D_3, E_3)T_\emptyset(Y_3/B-, D_3, E_3).
\]

Here \( B^+ \) and \( B^- \) are the section at \( \infty \) and \( 0 \) in the \( \mathbb{P}^1 \)-bundle \( Y_3 \to B \). Multiply the first and fourth equations and divide by the second and third equations prove the proposition. \( \square \)

Proposition 6.1 implies the torsion elements in \( \nu_{n,1,r}(k) \) has generate series 1 and \( T_\emptyset(Y, D, E) \) can be determined by a basis of \( \nu_{n,1,r} \otimes \mathbb{Q} \), which corresponds to a basis of Chern numbers of \( Y, D, \) and \( E \).

**Theorem 6.2.** If \( \{ \Theta_1, \Theta_2, \ldots, \Theta_m \} \) forms a basis of the finite-dimensional \( \mathbb{Q} \)-vector space of graded degree \( n \) polynomials in the Chern classes of

\[
\{c_i(T_Y)\}^{n}_{i=0}, c_1(\mathcal{O}(D)), \{c_i(E)\}^{r}_{i=0}.
\]

Then there exist power series \( A_1, A_2, \ldots, A_m \) in \( \mathbb{Q}[[y, z]] \) such that

\[
T_\emptyset(Y, D, E) = \prod_{k=1}^{m} A_k^{\Theta_k(c_i(T_Y), c_1(D), c_i(E))}.
\]

It follows that \( T_\emptyset \) (and therefore its coefficients) is an invariant of the algebraic cobordism group \( \nu_{n,1,r} \).

**Proof.** Proposition 6.1 shows \( T_\emptyset \) defines a multiplicative homomorphism from \( \nu_{n,1,r} \) to \( \mathbb{Q}[[y, z]] \). Recall \( \nu_{n,1,r} \otimes \mathbb{Q} \) is isomorphic to the finite-dimensional \( \mathbb{Q} \)-vector space of graded degree \( n \) polynomials in the Chern classes of \( \{c_i(T_Y)\}^{n}_{i=0}, c_1(\mathcal{O}(D)), \{c_i(E)\}^{r}_{i=0} \) (Corollary 6.2) by sending \( [Y, D, E] \) to its Chern numbers. Therefore \( T_\emptyset \) factors through this vector space and has the desired product form. \( \square \)

The following expression reduces the computation of universal polynomials to finding linear polynomials.

**Corollary 6.3.** The generating series \( T_\emptyset(Y, D, E) \) has an exponential description

\[
T_\emptyset(Y, D, E) = \exp \left( \sum_{\beta, \delta} \frac{a_{\beta, \delta}y^\beta z^\delta}{\#\text{Aut}(\beta) \cdot \#\text{Aut}(\delta)} \right)
\]

so that every \( a_{\beta, \delta} \) is a linear polynomial in Chern numbers of \( Y, D \) and \( E \) and \( a_{\emptyset, \emptyset} = 0 \).

**Proof.** Take natural log of two sides of Equation 6.2:

\[
\ln(T_\emptyset(Y, D, E)) = \sum_{k=1}^{m} \Theta_k(c_i(T_Y), c_1(D), c_i(E)) \cdot \ln A_k.
\]

\( d_{\emptyset, \emptyset, \emptyset}(Y, D, E) = 1 \) for all possible \( Y, D, \) and \( E \) and their Chern numbers form a full rank lattice in the vector space of all Chern numbers. It implies the leading coefficient
every $A_i$ is 1. Therefore every $\ln A_i$ is a series of $y_\beta$ and $z_\delta$ without constant term by the Taylor series of $\ln(1 + x)$. The sum of $\Theta_k$ multiplied by the coefficient of $y_\beta z_\delta$ in $\ln A_k$ gives $\frac{a_{\beta, \delta}}{\#Aut(\beta) \#Aut(\delta)}$.

Remark. By [14], for nodal curves on surfaces with at most eight nodes and no tangency condition, the coefficients of $a_{\beta, \delta}$ are all integers. This might still be true in general, but we do not know how to prove it.

The existence of universal polynomials can be easily deduced now.

*proof of Theorem 5.3.* Take a binomial expansion of Equation 6.2 and use Proposition 5.3. □

**Example 6.1.** The leading terms of the universal polynomials can be computed as follows. Consider curves in the linear system of a line bundle $L$ on a surface $S$. From [14], the universal polynomial of curves with a node ($A_1$-singularity) is

$$T_{0,0,A_1} = 3L^2 + 2LK_S + c_2(S).$$

The universal polynomial of curves with an ordinary triple point is

$$T_{0,0,D_4} = 15L^2 + 20LK_S + 5c_1(S)^2 + 5c_2(S).$$

By De Jonquières’s Formula or using discriminant, if $S$ contains a line $D \cong \mathbb{P}^1$, then the number of curves in $|L|$ tangent to $D$ is $2\deg(L|_D) - 2$. Then by Corollary 6.3, the number of curve in $|L|$ which has $a$ nodes and $b$ ordinary triple points and tangent to $D$ at $c$ points has leading term

$$(3L^2 + 2LK_S + c_2(S))^a(15L^2 + 20LK_S + 5c_1(S)^2 + 5c_2(S))^{b}(2\deg(L|_D) - 2)^c/\text{albhl}.$$

**Example 6.2.** There are seven Chern numbers of a line bundle $L$ and a divisor $D$ on a surface $S$:

$$L^2, LK, c_1(S)^2, c_2(S), D^2, \deg(L|_D), DK_S$$

In this case, Theorem 6.2 takes the form

$$T_S(S, D, L) = A_1L^2 A_2LK_S A_3^{c_1(S)^2} A_4^{c_2(S)} A_5^{D^2} A_6^{\deg(L|_D)} A_7^{DK_S},$$

where $A_i \in \mathbb{Q}[y_\beta, z_\delta]$ for $1 \leq i \leq 7$. By plugging in $y_\beta = 0$ in $A_1$, $A_2$, $A_3$ and $A_4$, we obtain the $A_i$ in [20] Theorem 3.2] which encode universal formulas of singular curves of any type. If we further let all variables zero except the one for node, the results are the $A_i$ in [27] Theorem 1.3].

On the other hand, if we let $z_\delta = 0$ for all $\delta$ in $T_S(S, D, L)$, the result is the generating series of numbers of smooth curves in $|L|$ tangent to the divisor $D$. These numbers only depend on $D$ and $L|_D$, not on $S$. Therefore the coefficients of $z_\delta = 0$ in $A_i$ is 1 for $1 \leq i \leq 4$ and the coefficients of $z_\delta = 0$ in $A_5$, $A_6$ and $A_7$ are totally determined by De Jonquières’s Formula.

7. Formulas

Although we have expressed the universal polynomials as intersection numbers on Hilbert schemes, computing them from this point of view is not easy because the cycle $Y(\alpha, \beta, \delta)$ is usually very singular. Therefore the existing results come from wide range of methods and usually only work in special situations. In our framework, those
results contribute to coefficients of generating series $T_\alpha(Y,D,E)$ or equivalently to universal polynomials. As long as there are enough special cases to generate the algebraic cobordism group, all coefficients can be computed.

For general K3 surfaces and primitive line bundles, the generating series of nodal curves can be expressed in modular forms and quasi-modular forms [27, Theorem 1.1]. So it is natural to ask if other generating series (e.g., on CY3 or log canonical manifold) will be in modular form and quasi-modular forms. Even if not, then is it possible to find a closed form of generating series? So far we can not answer these questions. But it will be very helpful if more computations can be made.

Below we list existing results on the enumeration of singular varieties and/or with tangency conditions. In Example 6.1, we compute leading terms of universal polynomials from the case of one singular point or one tangency point. The same method works in general by Corollary [5,3] so the formulas below do not only apply to their original setting but also contribute to infinitely many universal polynomials.

7.1. Nodal curves on surfaces.
   - The universal polynomial of $r$-nodal curves on surfaces can be determined by combining any two of [5, 6] and [30], or by [16]. The explicit formulas for $r \leq 6$ and $r \leq 8$ were found by Vainsencher and Kleiman-Piene ([28, 14]). Kool, Shende and Thomas [16] provided an algorithm for any $r$.
   - Caporaso-Harris’ recursive formula determines the numbers of plane nodal curves with arbitrary tangency conditions with a given line without any ampleness assumption.
   - Vakil [30] proves a similar recursive formula for nodal curves on Hirzebruch surfaces. $T_{\alpha,\beta,r,A_1}$ can be determined from Caporaso-Harris’ and Vakil’s formulas.
   - Block [4] used tropical geometry to prove the Caporaso-Harris invariants are polynomials when degree of the curves are relatively large. He computed $T_{\alpha,\beta,r,A_1}$ for $(\mathbb{P}^2,\mathcal{O}(d))$ when $1 \leq r \leq 6$ and any $\alpha, \beta$.

7.2. Curves with arbitrary singularity on surfaces.
   - In the same paper [14] Kleiman and Piene also computed the universal polynomials for many singularities of low codimensions such as $\delta = (D_4), (D_4, A_1), (D_4, A_1, A_1), (D_4, A_1, A_1, A_1), (D_6), (D_6, A_1)$ and $(E_7)$.
   - When there is only one singular point, Kerner [13] found an algorithm to enumerate the number of plane curves with one fixed topological type singularity, provided that the normal form is known.
   - If there is no singularity condition, then it is equivalent to finding sections of line bundles on smooth curves which vanish at points of given multiplicities. The answer is given by De Jonqui`ere’s Formula [1, p.359].
   - Recently Basu and Mukherjee [3, 2] used topological methods to compute the number of plane curves with one or two singularities when the first singularity is $ADE$-singularity or ordinary fourfold points and the additional one is a node.

7.3. General Dimension.
   - Vainsencher [29] computed the number hypersurfaces with less or equal to six double points.
In a series of paper ([11] [12] and a lot more on line), M. Kazarian showed Legendre characteristic classes should give universal polynomials for singular hypersurfaces, and they are independent of the dimension! The explicit formulas of Legendre characteristic classes when the codimension is less or equal to six are listed in [11]. They agree with Kleiman and Piene’s results on surfaces. The transversality conditions and the domain of universality are quite subtle.

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