Explicit Solutions to Fractional Stefan-like problems for Caputo and Riemann–Liouville Derivatives

SABRINA D. ROSCANI†, NAHUEL D. CARUSO‡, and DOMINGO A. TARZIA§

† CONICET - Depto. Matemática, FCE, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina
‡ Depto. Matemática, EFB, UNR, Pellegrini 250, Rosario, Argentina
§ CIFASIS - Centro Internacional Franco Argentino de Ciencias de la Información y de Sistemas, CONICET, Bv. 27 de Febrero 210 Bis, Rosario, S2000EZP Argentina

(sroscani@austral.edu.ar, ncaruso@fceia.unr.edu.ar, dtarzia@austral.edu.ar)

Abstract: Two fractional two-phase Stefan-like problems are considered by using Riemann-Liouville and Caputo derivatives of order $\alpha \in (0, 1)$ verifying that they coincide with the same classical Stefan problem at the limit case when $\alpha = 1$. For both problems, explicit solutions in terms of the Wright functions are presented. Even though the similarity of the two solutions, a proof that they are different is also given. The convergence when $\alpha \uparrow 1$ of the one and the other solutions to the same classical solution is given. Numerical examples for the dimensionless version of the problem are also presented and analyzed.

Keywords: Stefan-like problem; Caputo derivative; Riemann–Liouville derivative; Wright functions.

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1 Introduction

This paper deals with Stefan-like problems governed by fractional diffusion equations (FDE). A classical Stefan problem is a problem where a phase-change occurs, usually linked to melting (change from solid to liquid) or freezing (change from liquid to solid). In these problems the diffusion, considered as a heat flow, is expressed in terms of instantaneous local flow of temperature modeled by the Fourier Law. Therefore, the governing equations related to each phase are the well-known heat equations. There is also a latent heat-type condition at the interface connecting the velocity of the free boundary and the heat flux of the temperatures in both phases known as “Stefan condition”. A vast literature on Stefan problems is given in [1, 4, 5, 24, 25].

For example, the following is the mathematical formulation for a classical one-dimensional two-phase Stefan problem: Find the triple $\{u_1, u_2, s\}$ such that they have sufficiently regularity and they verify that:

(i) $\frac{\partial}{\partial t} u_2(x,t) = \lambda_2^2 \frac{\partial^2}{\partial x^2} u_2(x,t)$, $0 < x < s(t), 0 < t < T$,
(ii) $\frac{\partial}{\partial t} u_1(x,t) = \lambda_1^2 \frac{\partial^2}{\partial x^2} u_1(x,t)$, $x > s(t), 0 < t < T$,
(iii) $u_1(x,0) = U_i$, $0 \leq x$,
(iv) $u_2(0,t) = U_0$, $0 < t \leq T$,
(v) $u_1(s(t),t) = u_2(s(t),t) = U_m$, $0 < t \leq T$,
(vi) $\rho_1 k_1 \frac{d}{dt}s(t) = k_1 \frac{\partial}{\partial x} u_1(s(t),t) - k_2 \frac{\partial}{\partial x} u_2(s(t),t)$, $0 < t \leq T$,
(vii) $s(0) = 0$,

where $U_i < U_m < U_0$, $\lambda_j^2 = \frac{k_j}{\rho_j c_j}$, $j = 1$ (solid), $j = 2$ (liquid) and we have assumed that the thermo-physical properties are constant as well as the free boundary can be represented by an increasing

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1 This work was started at the beginning of 2018 when the second author was working also at Depto. Matemática, FCEIA, UNR, Pellegrini 250, Rosario, Argentina
where, the partial time derivative has been replaced by a fractional derivative in the sense of Caputo. Here, the partial time derivative has been replaced by a fractional derivative in the sense of Caputo.

Nevertheless, when discussing about FDE associated to fractional time derivatives, the reader may retract on the FDE for the Caputo derivative, that is

\[ \{0\}D^\alpha_t u(x,t) = \frac{\partial}{\partial t} \{0\}I^\alpha_t u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u(x,\tau) d\tau \]

for every \( u \in AC_{[0,T]} \{u \mid u(x,\cdot) \text{ is absolutely continuous on } [0,T] \text{ for every } x \in \mathbb{R}^+ \} \)

Nevertheless, when discussing about FDE associated to fractional time derivatives, the reader may retract on the FDE for the Caputo derivative, that is

\[ \{C\}D^\alpha_t u(x,t) = u_{xx}(x,t). \]

Here, the partial time derivative has been replaced by a fractional derivative in the sense of Caputo.

The Caputo derivative \( \{C\}D^\alpha_t \) is defined for every \( \alpha \in (0,1) \) as

\[ \{C\}D^\alpha_t u(x,t) = \left[ \{0\}D^\alpha_t (u_t) \right](x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u_t(x,\tau) d\tau \]

for every \( u \in AC_{[0,T]} \{u \mid u(x,\cdot) \text{ is absolutely continuous on } [0,T] \text{ for every } x \in \mathbb{R}^+ \} \).
for every $u \in AC_t[0,T]$.

As we said before, in this paper, problems like (1) governed by equations like (3) or (4) will be studied. The literature on fractional phase-change problems is rather scant. In [3] a fractional two-phase moving-boundary problem is approximated by a scale Brownian motion model for sub-diffusion. In [26] sharp and diffuse interface models of fractional Stefan problems are discussed. In [17] a formulation of a one-phase fractional phase-change problem is given arising a time dependence on the initial extreme of the fractional derivative. When the starting time considered in the fractional derivative of the governing equation is equal to 0, the mathematical point of view becomes interesting because they admit self-similar solutions in terms of the Wright functions (see [9, 10, 13, 18, 19]). It is worth noting that this kind of problems are not deduced as in [17, 27].

This paper is a continuation of a previous work [20], related to fractional one-phase change problems. In Section 2 some basic definitions and properties on fractional calculus are given. In Section 3, two fractional two-phase Stefan-like problems are considered, admitting both exact self-similar solutions. While the two governing equations are equivalent under certain assumptions for boundary-value-problems, when different “fractional Stefan conditions” are considered, the solutions obtained seem to be different. The uniqueness of the self-similar solution for one of the problems is obtained while it is an open problem for the other (see [19]). Finally, numerical examples and graphics of the solutions are presented by considering a dimensionless model in Section 4.

2 Basic definitions and properties

Proposition 1. [6] The following properties involving the fractional integrals and derivatives hold:

1. The fractional derivative of Riemann–Liouville is a left inverse operator of the fractional integral of Riemann–Liouville of the same order $\alpha \in \mathbb{R}^+$. If $f \in AC[a,b]$, then

$$RL_a D_\alpha^a I^\alpha f(t) = f(t) \quad \text{for every } t \in (a,b)$$

2. The fractional integral of Riemann–Liouville is not, in general, a left inverse operator of the fractional derivative of Riemann–Liouville.

In particular, if $0 < \alpha < 1$, then $a I^\alpha (RL_a D_\alpha^a f)(t) = f(t) - a I^{1-\alpha} f(a^+) \frac{\Gamma(\alpha)}{(t-a)^{1-\alpha}}$.

3. If there exist some $\phi \in L^1(a,b)$ such that $f = a I^\alpha \phi$, then

$$a I^\alpha RL_a D_\alpha^a f(t) = f(t) \quad \text{for every } t \in (a,b).$$

4. If $f \in AC[a,b]$, then

$$RL_a D_\alpha^a f(t) = \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + C_a D_\alpha^a f(t).$$

The fractional integral and derivatives of power functions can be easy calculated (see e.g. [14]). In fact, for every $t \geq a$ we have that

$$a I^\alpha ((t-a)^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \quad \text{for every } \beta > -1,$$

and that

$$RL_a D_\alpha^a ((t-a)^\beta) = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha} & \text{if } \beta \neq \alpha - 1, \\ 0 & \text{if } \beta = \alpha - 1. \end{cases}$$

In particular, if $\beta > 0$ $RL_a D_\alpha^a ((t-a)^\beta) = C_a D_\alpha^a ((t-a)^\beta)$ due to Proposition 1 item 4 and the Caputo derivative of $(t-a)^\beta$ is not defined for $-1 < \beta < 0$. 

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Proposition 2. The following limits hold:

1. If we set $a f^0 = 1d$ for the identity operator, then for every $f \in L^1(a, b)$,
   \[ \lim_{\alpha \searrow 0} a f^\alpha f(t) = a f^0 f(t) = f(t), \quad a.e. \]

2. For every $f \in AC[a, b]$, we have
   \[ \lim_{\alpha \searrow 0} C_a D^\alpha f(t) = f'(t) \quad \text{and} \quad \lim_{\alpha \searrow 0} C_a D^\alpha f(t) = f'(t) - f'(a^+) \quad \text{for all } t \in (a, b). \]

3. For every $f \in AC[a, b]$,
   \[ \lim_{\alpha \searrow 1} R^\alpha D^\alpha f(t) = f'(t) \quad \text{and} \quad \lim_{\alpha \searrow 1} R^\alpha D^\alpha f(t) = f'(t) \quad a.e. \]

Definition 1. For every $x \in \mathbb{R}$, the Wright function is defined as
   \[ W(x; \rho, \beta) = \sum_{k=0}^{\infty} \frac{x^k}{\rho^k (\rho k + \beta)}, \quad \rho > -1 \text{ and } \beta \in \mathbb{R}. \]  

An important particular case of the Wright function is the Mainardi function defined by
   \[ M_\rho(x) = W(-x, -\rho, 1 - \rho) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n! \rho^n (-\rho n + 1 - \rho)}, \quad 0 < \rho < 1. \]

Proposition 3. Let $\alpha > 0$, $\rho \in (0, 1)$ and $\beta \in \mathbb{R}$. Then the next assertions follows:

1. For every $x \in \mathbb{R}$ we have
   \[ \frac{\partial}{\partial x} W(x, \rho, \beta) = W(x, \rho, \rho + \beta). \]

2. For every $x > 0$ and $c > 0$,
   \[ a f^\alpha \left[ x^{\beta-1} W(-cx^{-\rho}, -\rho, \beta) \right] = x^{\beta + \alpha - 1} W(-cx^{-\rho}, -\rho, \beta + \alpha). \]

Proposition 4. For every $\beta \geq 0$, $\rho \in (0, 1)$:

1. The Wright function $W(-x, -\rho, \beta)$ is positive and strictly decreasing in $\mathbb{R}^+$. 

2. For every $x \geq 0$ the following equality holds
   \[ \rho x W(-x, -\rho, \beta - \rho) = W(-x, -\rho, \beta - 1) + (1 - \beta) W(-x, -\rho, \beta). \]

3. If, in addition $0 < \rho \leq \rho < \delta$, then for every $x > 0$ the following inequality holds
   \[ \Gamma(\delta) W(-x, -\rho, \delta) < \Gamma(\mu) W(-x, -\rho, \mu). \]

Proposition 5. For every $\beta \geq 0$ and $\rho \in (0, 1)$ the following limit holds
   \[ \lim_{x \to \infty} W(-x, -\rho, \beta) = 0. \]

Proposition 6. Let $x \in \mathbb{R}^+$. Then the following limits hold:

1. For $\alpha > 1$,
   \[ \lim_{\alpha \searrow 1} M_{\alpha/2}(2x) = \lim_{\alpha \searrow 1} W(-2x, -\frac{\alpha}{2}, 1 - \frac{\alpha}{2}) = M_{1/2}(2x) = \frac{e^{-x^2}}{\sqrt{\pi}}, \]

2. For $\alpha > 1$,
   \[ \lim_{\alpha \searrow 1} W(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2}) = \frac{e^{-x^2}}{\sqrt{\pi}}. \]


\[ \lim_{\alpha \nearrow 1} \left[ 1 - W \left(-2x, -\frac{\alpha}{2}, 1\right) \right] = \text{erf}(x), \]  
and \[ \lim_{\alpha \nearrow 1} \left[ W \left(-2x, -\frac{\alpha}{2}, 1\right) \right] = \text{erfc}(x), \]

where \( \text{erf}(\cdot) \) is the error function defined by \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} \, dz \) and \( \text{erfc}(\cdot) \) is the complementary error function defined by \( \text{erfc}(x) = 1 - \text{erf}(x) \). Moreover, the convergence is uniform over compact sets.

**Proposition 7.** The fractional initial-boundary-value problems (14) and (15) for the quarter plane are equivalent if there exists \( \beta > 0 \) and \( \delta > 0 \) such that \( \beta < \alpha < 1 \) and \( u_{xx}(x, \cdot) \) is an \( O(t^{-\beta}) \) in \((0, \delta)\):

(i) \( C \frac{\partial}{\partial x} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t), \quad 0 < x, 0 < t, \)

(ii) \( u(x, 0) = u_0(x), \quad 0 \leq x, \)

(iii) \( u(0, t) = g(t), \quad 0 < t, \)

(i) \( \frac{\partial}{\partial t} u(x, t) = \frac{RL}{\partial t} D_t^{1-\alpha} \left( \frac{\partial^2}{\partial x^2} u(x, t) \right), \quad 0 < x, 0 < t, \)

(ii) \( u(x, 0) = u_0(x), \quad 0 \leq x, \)

(iii) \( u(0, t) = g(t), \quad 0 < t, \)

Proof. Let \( u = u(x, t) \) be a function satisfying equation (14 - i). Applying \( RL D_t^{1-\alpha} \) to both sides and using Proposition 7 item 1 we get (15 - i).

Let now, for the inverse suppose that \( u \) satisfies equation (15 - i). Applying \( RL D_t^{1-\alpha} \) to both sides and using Proposition 7 item 2 yields that

\[ C \frac{\partial}{\partial x} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) - \lim_{\tau \searrow 0} \frac{D_0^{\alpha} \left( \frac{\partial^2}{\partial x^2} u(x, t) \right)}{\Gamma(1-\alpha)\tau^\alpha}, \quad 0 < x, 0 < t. \]  

Now, for every \( x \) fixed we have that \( u_{xx}(x, \cdot) \) is an \( O(t^{-\beta}) \) in \((0, \delta)\), then for \( t > 0 \) small it holds that

\[ -C \tau^{-\beta} \leq u_{xx}(x, \tau) \leq C \tau^{-\beta}, \quad 0 < \tau \leq t < \delta. \]  

Multiplying by \( \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \) in (17), integrating between 0 and \( t \) and applying formula (5) yields that

\[ -C \frac{\Gamma(1-\beta)\alpha^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \leq D_0^{\alpha} u_{xx}(x, t) \leq C \frac{\Gamma(1-\beta)\alpha^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}, \quad t < \delta. \]

Taking the limit when \( t \) tends to zero in (18) and being \( \beta < \alpha \) we conclude that equation (14 - i) holds as we wanted to see.

**Remark 1.** Equations (14 - i) and (15 - i) has been treated as equivalent in literature, as it can be seeing at [13], [14], [15], but the condition

\[ \lim_{\tau \searrow 0} D_0^{\alpha} \left( \frac{\partial^2}{\partial x^2} u(x, t) \right) = 0 \]

must be considered and should not be forget it.

**Remark 2.** It is easy to check that the following functions verifies equation (14 - i) and (15 - i) (we have taken \( \lambda = 1 \) without loss of generality)

\[ w_1(x, t) = x^2 + \frac{2}{\Gamma(\alpha+1)} t^\alpha. \]  
\[ w_2(x, t) = E_\alpha(t^\alpha) \exp \{-x\} \]
\( w_3(x, t) = W \left( \frac{x}{t^{\alpha/2}}, \frac{-\alpha}{2}, 1 \right) \). \hspace{1cm} (22)

The condition (19) trivially holds for function \( w_1 \) and \( w_2 \) and it is no difficult to check it for \( w_3 \) (by differentiating first and using Proposition 3 then).

## 3 The Fractional Stefan-like Problems

In this section, two fractional Stefan-like problems admitting both explicit self-similar solutions will be treated. Before that, some clarification about the used terminology is presented.

We refer to fractional Stefan problems when the governed equations in such problem are derived from physical assumptions, like considering memory fluxes.

For example, suppose that a process of melting of a semi–infinite slab (\( 0 \leq x < \infty \)) of some material is taking place, and the flux involved is a flux with memory. The melt temperature is \( U_m \), and a constant temperature \( U_0 > U_m \) is imposed on the fixed face \( x = 0 \). Let \( u_1 = u_1(x, t) \) and \( u_2 = u_2(x, t) \) be the temperatures at the solid and liquid phases respectively. Let \( J_1 = J_1(x, t) \) and \( J_2 = J_2(x, t) \) be the respective functions for the fluxes at position \( x \) and time \( t \) and let \( x = s(t) \) be the function representing the (unknown) position of the free boundary at time \( t \). Suppose further that:

(i) All the thermophysical parameters are constants.

(ii) The function \( s \) is an increasing function and consequently, an invertible function.

(iii) \( J_1 \) and \( J_2 \) are fluxes modeling the material with memory which verifies that “the weighted sum of the fluxes back in time at the current time, is proportional to the gradient of temperature”, that is, the following equations hold

\[
\nu_0 \alpha I_1^{\alpha - 1} J_1(x, t) = -k_1 \frac{\partial u_1}{\partial x}(x, t)
\]

(23)

and

\[
\nu_0 h(x) t^{1-\alpha} J_2(x, t) = -k_2 \frac{\partial u_2}{\partial x}(x, t)
\]

(24)

where the initial time in the fractional integral \( I_1^{\alpha - 1} \) is given by function \( h \) which gives us the time when the phase change occurs. That is,

\[
t = h(x) = s^{-1}(x) \quad \text{(i.e.} \quad x = s(t))
\]

The number \( \nu_0 \) is a parameter with physical dimension (see (70)) such that

\[
\lim_{\alpha \to 1} \nu_0 = 1,
\]

(25)

which has been added in order to preserve the consistency with respect to the units of measure in equations (23) and (24). Also, the parameter

\[
\mu_0 = \frac{1}{\nu_0}
\]

(26)

will be used in the following equations. More details about these parameters are given in Section 4.

Making an analogous reasoning for the two-phase free–boundary problem, than the one made in [17] for the one–phase free–boundary problem, the mathematical model for the problem described above is given by
satisfied self-similar solutions. These problems come from the assumption of consider the button limit said at the beginning of this section, this paper deals with Stefan-like problems admitting explicit

\[
\text{(27)}
\]

Note that self-similar solutions to problem (27) had not been yet founded, due to the difficulty

\[
\text{(27)}
\]

The Stefan-Like Problem for the Caputo derivative. The next problem was treated in
and can be obtained by replacing all the times derivatives in [1] by fractional derivatives in the Caputo sense of order \(\alpha \in (0, 1)\), i.e.

\[
\begin{align*}
(\text{i}) & \quad \frac{\partial}{\partial t} u_2(x, t) = \lambda_2^\alpha \mu_{\alpha} \frac{\partial}{\partial x} \left( RL_{h(x)} D_t^{1-\alpha} \left( \frac{\partial}{\partial x} u_2(x, t) \right) \right), & 0 < x < s(t), 0 < t < T, \\
(\text{ii}) & \quad \frac{\partial}{\partial t} u_1(x, t) = \lambda_1^\alpha \mu_{\alpha} \frac{\partial}{\partial x} \left( RL_{h(x)} D_t^{1-\alpha} \left( \frac{\partial}{\partial x} u_1(x, t) \right) \right), & x > s(t), 0 < t < T, \\
(\text{iii}) & \quad u_1(x, 0) = U_v, & 0 \leq x, \\
(\text{iv}) & \quad u_2(0, t) = U_0, & 0 < t \leq T, \\
(\text{v}) & \quad u_1(s(t), t) = u_2(s(t), t) = U_m, & 0 < t \leq T, \\
(\text{vi}) & \quad pl \frac{\partial}{\partial t} s(t) = k_1 \mu_{\alpha} \frac{\partial}{\partial x} \left( RL_{h(x)} D_t^{1-\alpha} \left( \frac{\partial}{\partial x} u_1(x, t) \right) \right) (s(t)^+, t) - k_2 \mu_{\alpha} \frac{\partial}{\partial x} u_2(x, t)(s(t)^-, t), & 0 < t \leq T, \\
(\text{vii}) & \quad s(0) = 0.
\end{align*}
\]

where \(U_v < U_m < U_0\) and \(\mu_{\alpha} = \frac{1}{\sqrt{\alpha}}\), (note that the parameter \(\mu_{\alpha}\) can be the same in equations
\[
\text{(27)–i and (27)–ii}, \text{and without loss of generality we will take from now on that } \mu_{2\alpha} = \mu_{\alpha}.\]

Note that self-similar solutions to problem (27) had not been yet founded, due to the difficulty imposed by the variable button limit in the fractional derivative for the liquid phase. As it was said at the beginning of this section, this paper deals with Stefan-like problems admitting explicit self-similar solutions. These problems come from the assumption of consider the button limit \(t_0 = 0\) in the fractional time derivatives in the Caputo or Riemann–Liouville sense.

The Stefan-Like Problem for the Caputo derivative. The next problem was treated in

\[
\begin{align*}
(\text{28})
\end{align*}
\]

where \(U_v < U_m < U_0\), \(\lambda_i, \alpha_i\), are positive parameters named as “subdiffusion coefficients” given by \(\lambda_i = \lambda_i \sqrt{\mu_{\alpha}}\) for \(i = 1, 2\), and \(k_i, \alpha_i\) are positive parameters named as “subdiffusion thermal conductivities” given by \(k_i = k_i \mu_{\alpha}, i = 1, 2\).

**Definition 2.** The triple \(\{u_1, u_2, s\}\) is a solution to problem (28) if the following conditions are satisfied

1. \(u_1\) is continuous in the region \(\mathcal{R}_T = \{(x, t) : 0 \leq x \leq s(t), 0 < t \leq T\}\) and at the point \((0, 0)\), \(u_1\) verifies that

\[
0 \leq \liminf_{(x,t) \to (0,0)} u_1(x, t) \leq \limsup_{(x,t) \to (0,0)} u_1(x, t) < +\infty.
\]

2. \(u_2\) is continuous in the region \(\{(x, t) : x > s(t), 0 < t \leq T\}\) and at the point \((0, 0)\), \(u_2\) verifies that

\[
0 \leq \liminf_{(x,t) \to (0,0)} u_2(x, t) \leq \limsup_{(x,t) \to (0,0)} u_2(x, t) < +\infty.
\]

3. \(u_1 \in C((0, \infty) \times (0, T)) \cap C_2^2((0, \infty) \times (0, T))\), such that \(u_1 \in AC_1[0, T]\)

4. \(u_2 \in C((0, \infty) \times (0, T)) \cap C_2^2((0, \infty) \times (0, T))\), such that \(u_2 \in AC_2[0, T]\).

5. \(s \in AC[0, T]\).

6. \(u_1, u_2\) and \(s\) satisfy (28).
Theorem 1. [9] A self-similar solution to problem (28) is given by

\[
\begin{align*}
&\left\{ \begin{array}{ll}
&u_2(x,t) = U_0 - \frac{U_0 - U_m}{1 - W(-2\xi_2, -\frac{\alpha}{2}, 1)} \left[ 1 - W\left(\frac{\alpha}{\alpha_2}e^{\alpha_2 t}, -\frac{\alpha}{2}, 1\right) \right] \\
&u_1(x,t) = U_i + \frac{U_m - U_i}{W(-2\xi_2, -\frac{\alpha}{2}, 1)} W\left(-\frac{x}{\alpha_2}e^{\alpha_2 t}, -\frac{\alpha}{2}, 1\right)
\end{array} \right.
\end{align*}
\]

(29)

where \(\xi_2\) is a solution to the equation

\[
\frac{k_\alpha_2 (U_0 - U_m) \Gamma(1 - \frac{\alpha}{2})}{\lambda_2} F_2(2\lambda x) - \frac{k_\alpha_1 (U_m - U_i) \Gamma(1 - \frac{\alpha}{2})}{\lambda_1} F_1(2x) = \Gamma\left(1 + \frac{\alpha}{2}\right) \lambda_\alpha_1 \rho^2 2x, \quad x > 0
\]

(30)

where \(\lambda = \frac{\lambda_1}{\lambda_2} = \frac{\lambda_1 \sqrt{\rho_\alpha}}{\lambda_2 \sqrt{\rho_\alpha}} = \frac{\alpha_1}{\alpha_2} > 0\), and \(F_1 : \mathbb{R}_0^+ \to \mathbb{R}\) and \(F_2 : \mathbb{R}_0^+ \to \mathbb{R}\) are the functions defined by

\[
F_1(x) = \frac{M_{\alpha/2}(x)}{W(-x, -\frac{\alpha}{2}, 1)} \quad \text{and} \quad F_2(x) = \frac{M_{\alpha/2}(x)}{1 - W(-x, -\frac{\alpha}{2}, 1)}.
\]

(31)

Note 1. The uniqueness of solution to equation (30) is still an open problem. However, the uniqueness of similarity solution will be achieved next for the Riemann–Liouville Stefan-like problem.

The Stefan-Like Problem for the Riemann–Liouville derivative. Consider now the following problem:

(i) \(\frac{\partial}{\partial t} w_2(x,t) = \lambda_2^2 \frac{\partial}{\partial x}\left(\hat{D}_t^{1-\alpha}\left(\frac{\partial}{\partial x} w_2(x,t)\right)\right)\), \(0 < x < r(t), 0 < t < T\),

(ii) \(\frac{\partial}{\partial x} w_1(x,t) = \lambda_2^2 \frac{\partial}{\partial x}\left(\hat{D}_t^{1-\alpha}\left(\frac{\partial}{\partial x} w_1(x,t)\right)\right)\), \(x > r(t), 0 < t < T\),

(iii) \(w_1(x,0) = U_i\), \(0 \leq x\),

(iv) \(w_2(0,t) = U_0\), \(0 < t \leq T\),

(v) \(w_1(r(t),t) = w_2(r(t),t) = U_m\), \(0 < t \leq T\),

(vi) \(\rho \frac{\partial}{\partial t} r(t) = k_\alpha_1 \hat{D}_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x,t)\big|_{(r(t))^{-}+}, -k_\alpha_2 \hat{D}_t^{1-\alpha} \frac{\partial}{\partial x} w_2(x,t)\big|_{(r(t))^{-}-}, 0 < t \leq T\),

(vii) \(r(0) = 0\).

where, as before, \(U_i < U_m < U_0\), \(\lambda_\alpha_i = \lambda_1 \sqrt{\rho_\alpha}\) for \(i = 1, 2\), and \(k_\alpha_i = k_i \rho_\alpha, i = 1, 2\).

Remark 3. The expression \(k_\alpha_1 \hat{D}_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x,t)\big|_{(r(t))^{-}+}\) is equivalent to

\[
\lim_{x \to r(t)^{+}} k_\alpha_1 \hat{D}_t^{1-\alpha} \left(\lim_{x \to r(t)^{-}} \frac{\partial}{\partial x} w_1(x,t)\right).
\]

(33)

which should not coincide with

\[
k_\alpha_1 \hat{D}_t^{1-\alpha} \left(\lim_{x \to r(t)^{-}} \frac{\partial}{\partial x} w_1(x,t)\right).
\]

(34)

Definition 3. The triple \(\{w_1, w_2, r\}\) is a solution of problem (32) if the following conditions are satisfied

1. \(w_1\) is continuous in the region \(R_T = \{(x,t) : 0 \leq x \leq s(t), 0 < t \leq T\}\) and at the point \((0,0)\), \(u_1\) verifies that

\[
0 \leq \liminf_{(x,t) \to (0,0)} w_1(x,t) \leq \limsup_{(x,t) \to (0,0)} w_1(x,t) < +\infty.
\]

2. \(w_2\) is continuous in the region \(\{(x,t) : x > r(t), 0 < t \leq T\}\) and at the point \((0,0)\), \(w_2\) verifies that

\[
0 \leq \liminf_{(x,t) \to (0,0)} w_2(x,t) \leq \limsup_{(x,t) \to (0,0)} w_2(x,t) < +\infty.
\]
3. $w_1 \in C((0, \infty) \times (0, T)) \cap C^2((0, \infty) \times (0, T))$, such that $w_{1x} \in AC((0, T))$.
4. $w_2 \in C((0, \infty) \times (0, T)) \cap C^2((0, \infty) \times (0, T))$, such that $w_{2x} \in AC(0, T)$.
5. $r \in C^1(0, T)$.
6. There exist $D_t^{1-\alpha} w_1(x, t)_{(s(t)+t)}$ and $D_t^{1-\alpha} w_1(x, t)_{(s(t)-t)}$ for all $t \in (0, T]$.
7. $w_1, w_2$ and $s$ satisfy (32).

**Theorem 2.** An explicit solution for the two-phase fractional Stefan-like problem (32) is given by

$$
\begin{align*}
w_2(x, t) &= U_0 - \frac{U_m - U_0}{1 - W(-2\eta_\lambda, -\frac{\alpha}{2}, 1)} \left[ 1 - W\left(-\frac{x}{\lambda_1^\alpha t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) \right] \\
w_1(x, t) &= U_1 + \frac{U_m - U_1}{W(-2\eta_\lambda, -\frac{\alpha}{2}, 1)} W\left(-\frac{x}{\lambda_1^\alpha t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) \\
(r(t) &= 2\eta_\lambda \lambda_1 t^{\alpha/2})
\end{align*}
$$

where $\eta_\lambda$ is the unique positive solution to the equation

$$
\frac{k_{02}(U_0 - U_m)}{\lambda_1 \lambda_2} G_2(2\lambda x) - \frac{k_{02}(U_0 - U_m)}{\lambda_2^2} G_1(2x) = \left( \rho + \frac{k_{02}(U_0 - U_m)}{\lambda_1^2} \right) 2x,
$$

where $\lambda = \frac{\lambda_1 \lambda_2 \sqrt{\rho}}{\lambda_2} = \frac{\lambda_1}{\lambda_2} > 0$, $U_1 < U_m < U_0$ and $G_1 : R^+_0 \to R$ and $G_2 : R^+_0 \to R$ are the functions defined by

$$
G_1(x) = \frac{W(-x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2})}{W(-x, -\frac{\alpha}{2}, 1)} \quad \text{and} \quad G_2(x) = \frac{2/\alpha W(-x, -\frac{\alpha}{2}, \frac{\alpha}{2})}{1 - W(-x, -\frac{\alpha}{2}, 1)}.
$$

**Proof.** Let the functions

$$
w_i : R^+_0 \times (0, T) \to R \quad \text{and} \quad w_i(x, t) = A_i + B_i \left[ 1 - W\left(-\frac{x}{\lambda_1^\alpha t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) \right]
$$

be the proposed solutions for $i = 1, 2$. Rewriting expression (8) for the variable $t$ and taking $c = \frac{\alpha}{\lambda_1}$ gives

$$
0 D_t^\alpha t^{\beta-1} W\left(-\frac{x}{\lambda_1^\alpha t^{\beta-1}}, -\rho, \beta \right) = t^{\beta+\alpha-1} W\left(-\frac{x}{\lambda_1^\alpha t^{\alpha/2}}, -\rho, \beta + \alpha \right).
$$

Then, by using (39) for $\beta = 1 - \frac{\alpha}{2}$ and Proposition 3, it is easy to check that $w_i$ verifies equations (32 – i) and (32 – ii) respectively for $i = 1, 2$.

From condition (32 – v) we deduce that $r(t)$ must be proportional to $t^{\alpha/2}$. Therefore we set

$$
r(t) = 2\eta_\lambda \lambda_1 t^{\alpha/2}, \quad t \geq 0
$$

where $\eta_\lambda$ is a constant to be determined and $\lambda_1$ was added for simplicity in the next calculations. Now, from conditions (32 – iii), (32 – iv) and (32 – v) it holds that

$$
A_1 = U_1 + \frac{U_m - U_1}{W(-2\eta_\lambda, -\frac{\alpha}{2}, 1)} \times \frac{U_m - U_1}{W(-2\eta_\lambda, -\frac{\alpha}{2}, 1)}, \quad B_1 = -\frac{U_m - U_1}{W(-2\eta_\lambda, -\frac{\alpha}{2}, 1)} = -\frac{U_m - U_1}{1 - W(-2\eta_\lambda, -\frac{\alpha}{2}, 1)}
$$

As before, by considering (39) for $\beta = 1 - \frac{\alpha}{2}$ and Proposition 3 it holds that

$$
R L D_t^{1-\alpha} w_i(x, t) = \frac{B_i \alpha / 2}{\lambda_1^\alpha \lambda_1 \lambda_1} W\left(-\frac{x}{\lambda_1^\alpha t^{\alpha/2}}, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2} \right) W\left(-\frac{x}{\lambda_1^\alpha t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right), \quad i = 1, 2.
$$
Then replacing (41) and (40) in equation (32) – viii), and evaluating the limits following (33) it yields that \( \eta_\alpha \) must verify the next equality

\[
\rho l 2\eta_\alpha \lambda_{\alpha_1} = -\frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}} W \left( -2\eta_\alpha, -\frac{\alpha}{2}, 1 + \frac{2}{\alpha} \right) - \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} 2\eta_\alpha - \frac{k_{\alpha_2}(U_0 - U_m) \ W \left( -2\alpha\eta_\alpha, -\frac{\alpha}{2}, 1 + \frac{2}{\alpha} \right) - k_{\alpha_2}(U_0 - U_m) 2\alpha \eta_\alpha W \left( -\alpha 2\eta_\alpha, -\frac{\alpha}{2}, 1 \right) }{\lambda_{\alpha_1} \lambda_{\alpha_2} (1 - W (-2\alpha\eta_\alpha, -\frac{\alpha}{2}, 1))} 2\alpha \eta_\alpha,
\]

which leads to conclude that \( \{w_1, w_2, r\} \) is a solution to (32) if and only if \( \eta_\alpha \) is a solution to the equation

\[
\frac{k_{\alpha_2}(U_0 - U_m) \ W (-\alpha 2x, -\frac{\alpha}{2}, 1 + \frac{2}{\alpha}) + 2\lambda x W (-\alpha 2x, -\frac{\alpha}{2}, 1) }{\lambda_{\alpha_1} \lambda_{\alpha_2} (1 - W (-2\alpha x, -\frac{\alpha}{2}, 1))} = \left( \rho l + \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \right) 2x, \quad x > 0.
\]

which, by using Proposition 4, leads to equation (46). The next step is to prove that Eq. (36) has unique solution. For that purpose we define function \( G \) in \( \mathbb{R}^+ \) as

\[
G(x) = \frac{k_{\alpha_2}(U_0 - U_m) G_2(2\lambda x) - k_{\alpha_1}(U_m - U_i) G_1(2x)}{\lambda_{\alpha_1} \lambda_{\alpha_2}} - \left( \rho l + \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \right) 2x.
\]

Note that \( G \) is continuous function such that

\[
G(0^+) = +\infty.
\]

From Proposition 3 for every \( x > 0 \) we have that

\[
0 < \frac{W (-2x, -\frac{\alpha}{2}, 1 + \frac{2}{\alpha})}{W (-2x, -\frac{\alpha}{2}, 1)} < \frac{1}{\Gamma \left( \frac{\alpha}{2} + 1 \right)},
\]

then \( G_1 \) is bounded. Also, from (45) it holds that

\[
-\frac{k_{\alpha_1}(U_i - U_m)}{\lambda_{\alpha_1}^2} \frac{1}{\Gamma \left( \frac{\alpha}{2} + 1 \right)} + \frac{k_{\alpha_2}(U_0 - U_m) G_2(2\lambda x) - \left( \rho l + \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \right) 2x < }{\lambda_{\alpha_1} \lambda_{\alpha_2}} G(x) < \frac{k_{\alpha_2}(U_0 - U_m) G_2(2\lambda x) - \left( \rho l + \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \right) 2x}{\lambda_{\alpha_1} \lambda_{\alpha_2}} 2x,
\]

and taking the limit when \( x \to \infty \) in (46) and using Proposition 5 we obtain that

\[
G(+\infty) = -\infty.
\]

Finally, consider the function \( K : \mathbb{R}^+ \to \mathbb{R} \) defined as

\[
K(x) = -\frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} [G_1(2x) + 2x] - \rho l 2x.
\]

Applying Proposition 3 item 1 and being \( \frac{(U_m - U_i)}{\lambda_{\alpha_1}^2} > 0 \) it results that \( K \) is a strictly decreasing function. By the other side, from Proposition 4 item 1 we have that \( G_2 \) is a strictly decreasing function. Then it can be concluded that \( G \) is a strictly decreasing function. Therefore Eq. (36) has a unique positive solution.

\[\square\]
Remark 4. The limits described in Remark 3 are different if we compute them for the functions $w_1$ and $r$. In fact, by using the computation made in the previous theorem, we get

\[
\lim_{x \to r(t)}^{RL} D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x,t) = \frac{B_1}{\lambda_{\alpha_1}} \left[ \frac{\alpha/2^\alpha/2 - 1}{W(-2\eta_\alpha, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2})} + \frac{\alpha}{2^\alpha/2 - 1} \right].
\]

and from Proposition 4-2 we have:

\[
W\left(-2\eta_\alpha, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right) + 2\eta_\alpha W\left(-2\eta_\alpha, -\frac{\alpha}{2}, 1\right) = \frac{2}{\alpha} W\left(-2\eta_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2}\right).
\]

Then

\[
\lim_{x \to r(t)}^{RL} D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x,t) = \frac{B_1}{\lambda_{\alpha_1}} \frac{\alpha/2^\alpha/2 - 1}{W(-2\eta_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2})}.
\]

whereas

\[
\frac{RL D_t^{1-\alpha}}{0} \left( \lim_{x \to r(t)} \frac{\partial}{\partial x} w_1(x,t) \right) = \frac{B_1}{\lambda_{\alpha_1}} \Gamma(1 - \frac{\alpha}{2}) \Gamma\left(\frac{\alpha}{2}\right), \quad \Gamma\left(\frac{\alpha}{2}\right).
\]

And we know that [51] and [52] are different due to Proposition 4-3.

Theorem 3. If $\lambda = 1$, the explicit solutions [53] to problem [52], and [49] to problem [28] are different.

Proof. Take $U_1 = -1$, $U_m = 0$ and $U_0 = 1$. Let $\{w_1, w_2, s\}$ be the solution to problem [28]. Then

\[
s(t) = 2\lambda_m, \xi(t) t \text{ where } \xi(t) \text{ is a positive solution to equation}
\]

\[
k_{\alpha_2} \frac{(1 - \frac{\alpha}{2})}{\lambda_{\alpha_1} \lambda_{\alpha_2}} M_{\alpha/2}(2\lambda x) W(-2\lambda x, -\frac{\alpha}{2}, 1) = \frac{k_{\alpha_2} (1 - \frac{\alpha}{2})}{\lambda_{\alpha_1}} M_{\alpha/2}(2x) W(-2x, -\frac{\alpha}{2}, 1) = \Gamma(1 + \frac{\alpha}{2}) \rho l 2x.
\]

By the other side, let $\{w_1, w_2, r\}$ be the solution to problem [52]. Then $\eta_\alpha$ is the positive solution to equation

\[
k_{\alpha_2} \frac{2/\alpha}{\lambda_{\alpha_1} \lambda_{\alpha_2}} W(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2}) W(-2x, -\frac{\alpha}{2}, 1) = \Gamma(1 + \frac{\alpha}{2}) \rho l 2x.
\]

or equivalently,

\[
k_{\alpha_2} \frac{(1 + \frac{\alpha}{2})}{\lambda_{\alpha_1} \lambda_{\alpha_2}} W(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2}) W(-2x, -\frac{\alpha}{2}, 1) = \Gamma(1 + \frac{\alpha}{2}) \rho l 2x.
\]

From Proposition 4-2 for every $x > 0$ we have that

\[
W\left(-2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right) + 2x W\left(-2x, -\frac{\alpha}{2}, 1\right) = \frac{2}{\alpha} W\left(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right).
\]

Then using the fact that the Gamma function verifies that $\Gamma(1 + \frac{\alpha}{2}) = \Gamma(\frac{\alpha}{2})$ and replacing [56] in [55] we deduce that $\eta_\alpha$ is the unique positive solution to the equation
\[ \frac{k_{\alpha_2}}{\lambda_{\alpha_1}} \frac{\Gamma \left( \frac{\alpha}{2} \right) W \left( -2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2} \right)}{1 - W \left( -2\lambda x, -\frac{\alpha}{2}, 1 \right)} - \frac{k_{\alpha_1}}{\lambda_{\alpha_1}} \frac{\Gamma \left( \frac{\alpha}{2} \right) W \left( -2x, -\frac{\alpha}{2}, \frac{\alpha}{2} \right)}{W \left( -2x, -\frac{\alpha}{2}, 1 \right)} = \Gamma \left( 1 + \frac{\alpha}{2} \right) \rho l 2x, x > 0. \] 

(57)

If we suppose then that \( \xi_\alpha = \eta_\alpha \), it result that there exist \( \xi_\alpha > 0 \) such that

\[ \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \frac{\Gamma \left( \frac{\alpha}{2} \right) W \left( -2\xi_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2} \right)}{W \left( -\xi_\alpha, -\frac{\alpha}{2}, 1 \right)} - \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \frac{\Gamma \left( 1 - \frac{\alpha}{2} \right) M_{\alpha/2} (2\xi_\alpha)}{1 - W \left( -\xi_\alpha, -\frac{\alpha}{2}, 1 \right)} = \frac{k_{\alpha_2}}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{\Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{\alpha}{2} \right) W \left( -\lambda \xi_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2} \right)}{1 - W \left( -\lambda \xi_\alpha, -\frac{\alpha}{2}, 1 \right)} - \frac{\Gamma \left( 1 - \frac{\alpha}{2} \right) M_{\alpha/2} (\lambda 2\xi_\alpha)}{1 - W \left( -\lambda \xi_\alpha, -\frac{\alpha}{2}, 1 \right)}. \]

(58)

By using the hypothesis that \( \lambda = 1 \), we conclude that

\[ \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \frac{\Gamma \left( \frac{\alpha}{2} \right) W \left( -\xi_\alpha, -\frac{\alpha}{2}, 1 \right)}{1 - W \left( -\xi_\alpha, -\frac{\alpha}{2}, 1 \right)} = \frac{k_{\alpha_2}}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{\Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{\alpha}{2} \right) W \left( -\xi_\alpha, -\frac{\alpha}{2}, 1 \right)}{1 - W \left( -\xi_\alpha, -\frac{\alpha}{2}, 1 \right)}. \]

(59)

which leads to

\[ W \left( -\xi_\alpha, -\frac{\alpha}{2}, 1 \right) = \frac{1}{1 + \frac{k_{\alpha_1} \lambda_{\alpha_2}}{k_{\alpha_1} \lambda_{\alpha_2}}}. \]

(60)

Replacing (60) in equation (53) yields that

\[ \rho l \lambda_{\alpha_1} 2\xi_\alpha = 0 \]

which leads to \( \xi_\alpha = 0 \), contradicting the fact that \( \xi_\alpha > 0 \).

\[ \square \]

**Note 2.** It is worth noting that an analogous proof for Theorem 3 but considering \( \lambda \neq 1 \) does not hold. In fact, if we define the function \( h_\alpha : \mathbb{R}^+ \rightarrow \mathbb{R} \) as

\[ h_\alpha(x) = \Gamma \left( \frac{\alpha}{2} \right) W \left( -x, -\frac{\alpha}{2}, \frac{\alpha}{2} \right) - \Gamma \left( 1 - \frac{\alpha}{2} \right) M_{\alpha/2} (x) \]

Figure 1: The function \( h_\alpha(x) = \Gamma \left( \frac{\alpha}{2} \right) W \left( -x, -\frac{\alpha}{2}, \frac{\alpha}{2} \right) - \Gamma \left( 1 - \frac{\alpha}{2} \right) M_{\alpha/2} (x) \) for different values of \( \alpha \),
then equality (58) can be expressed as
\[
\frac{k_{\alpha_2}}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{h_{\alpha}(\lambda 2 \zeta_{\alpha})}{1 - W(-\lambda 2 \zeta_{\alpha}, -\frac{a}{2}, 1)} = \frac{k_{\alpha_3}}{\lambda_{\alpha_3}} \frac{h_{\alpha}(2 \zeta_{\alpha})}{1 - W(-2 \zeta_{\alpha}, -\frac{a}{2}, 1)}.
\] (61)

If \(\lambda \neq 1\), it is not possible to cancel the expressions \(h_{\alpha}(\lambda 2 \zeta_{\alpha})\) and \(h_{\alpha}(2 \zeta_{\alpha})\) in equation (61). Moreover the graphics in Figure 2 lead us to suppose that there exists a positive solution to equation
\[
\frac{k_{\alpha_2}}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{h_{\alpha}(\lambda x)}{1 - W(-\lambda x, -\frac{a}{2}, 1)} = \frac{k_{\alpha_3}}{\lambda_{\alpha_3}} \frac{h_{\alpha}(x)}{1 - W(-x, -\frac{a}{2}, 1)}, \quad x > 0,
\] (62)
then, it is not possible to get a contradiction like (60).

![Figure 2: The left and right quotients of equation (62) for different values of \(\alpha\)](image)

However, if we take different values of \(\lambda\) (which are different to 1) and the parameters \(\xi_{\alpha}\) and \(\eta_{\alpha}\) are estimated numerically for different values of \(\alpha\), we show that they are different and converging both to the same value when \(\alpha \nearrow 1\). Numerical examples will be given in the next section.

**Theorem 4.** The explicit solution (35) to problem (32) converges, when \(\alpha \nearrow 1\), to the unique solution to the classical Stefan problem given by

\[
\begin{align*}
(i) & \quad \frac{\partial}{\partial t} u_2(x, t) = \lambda_1^2 \frac{\partial^2}{\partial x^2} u_2(x, t), & 0 < x < s(t), 0 < t < T, \\
(ii) & \quad \frac{\partial}{\partial t} u_1(x, t) = \lambda_1^2 \frac{\partial^2}{\partial x^2} u_1(x, t), & x > s(t), 0 < t < T, \\
(iii) & \quad u_1(x, 0) = U_i, & 0 \leq x, \\
(iv) & \quad u_2(0, t) = U_0, & 0 < t \leq T, \\
(v) & \quad u_1(s(t), t) = u_2(s(t), t) = U_m, & 0 < t \leq T, \\
(vi) & \quad \frac{\partial}{\partial t} s(t) = k_1 \frac{\partial}{\partial x} u_1(s(t), t) - k_2 \frac{\partial}{\partial x} u_2(s(t), t), & 0 < t \leq T, \\
(vii) & \quad s(0) = 0.
\end{align*}
\] (63)

**Proof.** The unique solution to problem (63) is the Neumann solution given in [28].

\[
\begin{align*}
z_2(x, t) &= U_0 - (U_0 - U_m) \frac{\text{erf} \left( \frac{x - s(t)}{\sqrt{\lambda_1}} \right)}{\text{erf} \left( \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}} \right)}, \\
z_1(x, t) &= U_i + (U_m - U_i) \frac{\text{erfc} \left( \frac{x - s(t)}{\sqrt{\lambda_1}} \right)}{\text{erfc} \left( \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}} \right)}, \\
w(t) &= 2 \eta_1 \lambda_1 \sqrt{t}
\end{align*}
\] (64)
where $\eta_1$ is the unique solution to the equation

$$\frac{k_2(U_0 - U_m)}{\lambda_1 \lambda_2} \frac{\exp \left\{-\lambda^2 x^2 \right\}}{\sqrt{\pi} \text{erf} (\lambda x)} - \frac{k_1(U_m - U_i)}{\lambda^2_i} \frac{\exp \left\{-x^2 \right\}}{\sqrt{\pi} \text{erf} (x)} = \rho lx, \quad x > 0. \quad (65)$$

Reasoning like in the previous theorem we can state that the solution to problem (32) is given by (35) where $\eta_2$ is the unique positive solution to the equation

$$\frac{k_{\alpha_2}(U_0 - U_m)}{\lambda_{\alpha_2} \alpha} \frac{W\left(-2\lambda x, -\frac{\alpha_2}{2}, \frac{\alpha}{2}\right)}{1 - W\left(-2\lambda x, -\frac{\alpha_2}{2}, 1\right)} - \frac{k_{\alpha_1}(U_m - U_i)}{\lambda^2_{\alpha_1} \alpha} \frac{W\left(-2x, -\frac{\alpha_2}{2}, \frac{\alpha}{2}\right)}{W\left(-2x, -\frac{\alpha_2}{2}, 1\right)} = \rho lx, \quad x > 0. \quad (66)$$

Clearly, if we take $\alpha = 1$ in equation (66) we recover equation (65). So, let the sequence $\{\eta_\alpha\}_\alpha$ be, where $\eta_\alpha$ is the unique positive solution to equation (66) for each $0 < \alpha < 1$. Defining the functions

$$f_\alpha(x) = \frac{k_{\alpha_2}(U_0 - U_m)}{\rho \lambda_{\alpha_2} \alpha} \frac{W\left(-2\lambda x, -\frac{\alpha_2}{2}, \frac{\alpha}{2}\right)}{1 - W\left(-2\lambda x, -\frac{\alpha_2}{2}, 1\right)} - \frac{k_{\alpha_1}(U_m - U_i)}{\rho \lambda^2_{\alpha_1} \alpha} \frac{W\left(-2x, -\frac{\alpha_2}{2}, \frac{\alpha}{2}\right)}{W\left(-2x, -\frac{\alpha_2}{2}, 1\right)}$$

for every $x \in \mathbb{R}^+$ and $0 < \alpha \leq 1$, it holds that $f_\alpha(\eta_\alpha) = \eta_\alpha$ for every $\alpha \in (0,1]$.

From [23] we know that $f_1$ is a strictly decreasing function in $\mathbb{R}^+$. Taking a close interval $[a,b] \subset \mathbb{R}^+$ such that $\eta_1 \in [a,b]$, using the uniform convergence over compact sets of all the positive functions given in Proposition [6] and proceeding like in [20] Theorem 2] we can state that

$$\lim_{\alpha \uparrow 1} \eta_\alpha = \eta_1. \quad (67)$$

Finally, by taking the limit when $\alpha \uparrow 1$ in solution (35) by applying Proposition [6] the thesis holds.

**Remark 5.** By using the same technique described before, we can improve the result given in [13] Theorem 3.3] by considering the functions $g_\alpha$ defined in $\mathbb{R}^+$ by

$$g_\alpha(x) = \frac{k_{\alpha_2}(U_0 - U_m)}{\rho \lambda_{\alpha_2} \alpha} \frac{\Gamma(1 - \alpha/2)}{\Gamma(1 + \alpha/2)} \frac{M_{\alpha/2}(-2\lambda x)}{1 - W\left(-2\lambda x, -\frac{\alpha_2}{2}, 1\right)} - \frac{k_{\alpha_1}(U_m - U_i)}{\rho \lambda^2_{\alpha_1} \alpha} \frac{M_{\alpha/2}(-2x)}{\Gamma(1 + \alpha/2) W\left(-2x, -\frac{\alpha_2}{2}, 1\right)}$$

and a sequence $\{\xi_\alpha\}_\alpha$ were $\xi_\alpha$ is a solution to equation $g_\alpha(x) = x, \quad x > 0$.

### 4 The dimensionless problems and numerical results

In the aim to give different graphics of the solutions obtained in Section 3, the problems (28) and (32) will be rewritten in their dimensionless form.

First, we give the following table exhibiting the usual heat conduction physical dimensions related to this work. Let us write $T$ for temperature, $t$ for time, $m$ for mass and $X$ for position.

| $u_1, u_2, w_1, w_2$ | temperatures |
| $k_1, k_2$ | thermal conductivities |
| $\rho$ | mass density |
| $c_1, c_2$ | specific heats |
| $\lambda_i^2 = \frac{k_i}{\rho c_i}, \quad i = 1, 2$ | diffusion coefficients |
| $l$ | latent heat per unit mass |

| \( T \) | \( m \) | \( X \) |
| \( \frac{t}{T} \) | \( \frac{m}{M} \) | \( \frac{X}{L} \) 

(68)
Proposition 8. For every \( \alpha \in (0, 1) \) it holds that

1. \( [0I^\alpha f] = [f]t^\alpha \) for every \( f \in L^1(0, T) \).

2. \( [R_0D^\alpha f] = \frac{[f]}{t^\alpha} \) for every \( f \in AC[0, T] \).

3. \( [\partial_t^\alpha D^\alpha f] = \frac{[f]}{t^\alpha} \) for every \( f \in AC[0, T] \).

Recall that the parameters \( \nu_\alpha \) and \( \mu_\alpha \) given in (25) where added to preserve the consistency with respect to the units of measure in equations (23) and (24). That is, being \( [J] = [ku_x] = \frac{m}{\nu_\alpha} \) and using Proposition 8 it holds that

\[
[0I_t^{1-\alpha} J(x, t)] = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{J(x, \tau)}{(t-\tau)^\alpha} d\tau = \frac{m}{t^{2+\alpha}}.
\]  

(69)

Then, replacing (69) in (23) one gets

\[
[\nu_\alpha] = \frac{\left[ \frac{\nu_\alpha}{\lambda^2} \right]}{\left[ b(x)I_t^{1-\alpha} J \right]} = \frac{1}{t^{1-\alpha}}.
\]

(70)

Therefore,

\[
[\mu_\alpha] = t^{1-\alpha}.
\]

(71)

Proposition 9. Let \( x_0 \) be a characteristic position and let \( U^* \) be a characteristic temperature. Then, if the following rescaling variable are considered

\[
y = \frac{x}{x_0}, \quad \tau = \frac{\lambda^2}{x_0^2} t \quad \text{and} \quad \tilde{w} = \frac{w}{U^*},
\]

(72)

it holds that

\[
0I_t^\alpha (w_x(x, t)) = \frac{U^* x_0}{\lambda_1^2} \left( \frac{\lambda^2}{x_0^2} \right)^{1-\alpha} 0I_t^\alpha (\tilde{w}_y(y, \tau)),
\]

(73)

\[
0I_t^\alpha (w_xx(x, t)) = \frac{U^*}{\lambda_1^2} \left( \frac{\lambda^2}{x_0^2} \right)^{1-\alpha} 0I_t^\alpha (\tilde{w}_{yy}(y, \tau))
\]

(74)

and

\[
R_0D^{1-\alpha}_t (w_xx(x, t)) = \frac{U^*}{x_0} \left( \frac{\lambda^2}{x_0^2} \right)^{1-\alpha} R_0D^{1-\alpha}_\tau (\tilde{w}_{yy}(y, \tau)).
\]

(75)

Proof. We prove here equation (73). By considering the rescaling (72), we have

\[
\tilde{w}(y, \tau) = \frac{w(x(y), t(\tau))}{U^*}.
\]

(76)

Then

\[
0I_t^\alpha (w_x(x, t)) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{w_x(x, z)}{(t-z)^{1-\alpha}} dz = \frac{U^*}{\Gamma(\alpha)} \int_0^t \frac{\tilde{w}_y(y, \tau(z))}{(t-\tau(z))^{1-\alpha}} dz
\]

\[
= \frac{U^*}{\Gamma(\alpha)} \int_0^{\frac{\lambda_1^2}{x_0^2} t} \frac{\tilde{w}_y(y, v)}{(\frac{\lambda^2}{x_0^2} t-v)^{1-\alpha}} \frac{x_0}{\lambda_1^2} dv = \frac{U^*}{x_0} \left( \frac{\lambda^2}{x_0^2} \right)^\alpha \int_0^{x_0} \tilde{w}_y(y, \tau(v)) dv.
\]

\[
0I_t^\alpha (w_xx(x, t)) = \frac{U^*}{\lambda_1^2} \left( \frac{\lambda^2}{x_0^2} \right)^{1-\alpha} 0I_t^\alpha (\tilde{w}_{yy}(y, \tau)).
\]

\[
R_0D^{1-\alpha}_t (w_xx(x, t)) = \frac{U^*}{x_0} \left( \frac{\lambda^2}{x_0^2} \right)^{1-\alpha} R_0D^{1-\alpha}_\tau (\tilde{w}_{yy}(y, \tau)).
\]

\[
\square
\]
Now, let us consider problems (28) and (32). By using Proposition 9 it is easy to state that the governing equation (32) is equivalent to the following equation

$$\frac{\partial}{\partial \tau} \tilde{w}_2(y, \tau) = \lambda^2 \mu_\alpha \left( \frac{\lambda_1^2}{x_0^2} \right) \frac{RL}{0} D^{1-\alpha}_\tau \tilde{w}_{2yy}(y, \tau).$$  \tag{77}$$

Note that \(\mu_\alpha = \left( \frac{x_0^2}{\lambda^2} \right)^{1-\alpha}\) is an admissible parameter because \([\mu_\alpha] = t^{1-\alpha}\) and that \(\lim_{\alpha \to 1} \mu_\alpha = 1\). Then, the parameter \(\mu_\alpha \left( \frac{\lambda_1^2}{x_0^2} \right)^{1-\alpha}\) in equation (77) can be omitted.

Analogously, transforming the governing equations, the Stefan conditions and the initial and boundary data in problems (28) and (32), and by taking \(U_m = 0\) and \(U^* = |U_i|\), it follows that the non-dimensional associated form are given by

\[
\begin{align*}
(i) \quad & \frac{\partial}{\partial \tau} \tilde{u}_2(y, \tau) = \lambda^2 \tilde{u}_{2yy}(y, \tau), & 0 < y < \tilde{s}(\tau), 0 < \tau < \tilde{T}, \\
(ii) \quad & \frac{\partial}{\partial \tau} \tilde{u}_1(y, \tau) = \tilde{u}_{2yy}(y, \tau), & y > \tilde{s}(\tau), 0 < \tau < \tilde{T}, \\
(iii) \quad & \tilde{u}_1(y, 0) = -1, & 0 \leq x, \\
(iv) \quad & \tilde{u}_2(0, \tau) = \frac{U_0}{|U_i|}, & 0 < \tau \leq \tilde{T}, \\
(v) \quad & \tilde{u}_1(\tilde{s}(\tau), \tau) = \tilde{u}_1(\tilde{s}(\tau), \tau) = 0, & 0 < \tau \leq \tilde{T}, \\
(vi) \quad & \frac{\partial}{\partial \tau} \tilde{s}(\tau) = \text{Ste} \left[ \tilde{u}_{1y}(\tilde{s}(\tau)^+ - \tilde{u}_{2y}(\tilde{s}(\tau)^-) \right], & 0 < \tau \leq \tilde{T}, \\
(vii) \quad & \tilde{s}(0) = 0.
\end{align*}
\]  \tag{78}

And

\[
\begin{align*}
(i) \quad & \tilde{w}_2(y, \tau) = \lambda^2 \frac{RL}{0} D^{1-\alpha}_\tau w_{2yy}(y, \tau), & 0 < y < \tilde{r}(\tau), 0 < \tilde{r} < \tilde{T}, \\
(ii) \quad & \tilde{w}_1(y, \tau) = \frac{RL}{0} D^{1-\alpha}_\tau w_{1yy}(y, \tau), & y > \tilde{r}(\tau), 0 < \tau < \tilde{T}, \\
(iii) \quad & \tilde{w}_1(y, 0) = -1, & 0 \leq y, \\
(iv) \quad & \tilde{w}_2(0, t) = \frac{U_0}{|U_i|}, & 0 < \tau \leq \tilde{T}, \\
(v) \quad & \tilde{w}_1(\tilde{r}(\tau), \tau) = \tilde{w}_2(\tilde{r}(\tau), \tau) = 0, & 0 < \tau \leq \tilde{T}, \\
(vi) \quad & \frac{\partial}{\partial \tau} \tilde{r}(\tau) = \text{Ste} \left[ \frac{RL}{0} D^{1-\alpha}_\tau w_{1y}(y, \tau) \bigg|_{\tilde{r}(\tau)^+}, \tau \right] \left. - \frac{k_2}{k_1} \frac{RL}{0} D^{1-\alpha}_\tau \tilde{w}_{2y}(y, \tau) \bigg|_{\tilde{r}(\tau)^-}, \tau \right] \right], & 0 < \tau \leq \tilde{T}, \\
(vii) \quad & \tilde{r}(0) = 0.
\end{align*}
\]  \tag{79}

where \(\lambda = \frac{k_2}{k_1}\) and \(\text{Ste} = \frac{|U_i|\eta_\alpha}{T}\) is the non-dimensional Stefan number.

In the following table there are different tests, i.e. sets of parameters for \(\lambda = \frac{k_2}{k_1}\), \(U = \frac{U_0}{|U_i|}\) and \(\text{Ste}\). For each test in Table 1 a corresponding graph of the comparison between the \(\xi_\alpha\) and \(\eta_\alpha\) is given in Figure 3.

| Test | \(\lambda\) | \(\frac{k_2}{k_1}\) | \(U\) | \(\frac{U_0}{|U_i|}\) | \(\text{Ste}\) |
|------|------------|-----------------|------|-----------------|--------|
| Test 1 | 0.5 | 0.5 | 1.0 | 0.5 |
| Test 2 | 2.0 | 2.0 | 1.0 | 0.5 |
| Test 3 | 0.5 | 0.5 | 1.0 | 1.2 |
| Test 4 | 2.0 | 2.0 | 1.0 | 1.2 |

Table 1: Different set of tests
Figure 3: $\xi_\alpha$ vs. $\eta_\alpha$ for different values of $\alpha$

At the end, we present in Figures 4 and 5 some color maps of temperature for tests 2 and 3, respectively. Three values of $\alpha$ are considered and as it is expected from Theorem 4, both solutions approach themselves when $\alpha \nearrow 1$. 
5 Conclusion

We have presented two different fractional two-phase Stefan-like problems for the Riemann-Liouville and Caputo derivatives of order $\alpha \in (0, 1)$ with the particularity that, if the parameter $\alpha = 1$ is replaced in both problems, we recover the same classical Stefan problem. In both cases, explicit solutions in terms of self-similar variables were given. It was interesting to see that, the role of the different “fractional Stefan conditions” associated to each problem was decisive to conclude that the solutions obtained where different. Also, as it was expected, we have proved that the two different solutions converge to the same triple of limits functions when $\alpha$ tends to 1, where this “limit solution” is the classical solution to the classical Stefan problem mentioned before.

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Figure 4: Caputos’s approach Solutions Vs. Riemann-Liouville’s aproach Solutions for Test

| α  | Caputo Stefan-Like Pb. | Riemann-Liouville Stefan-Like Pb. |
|----|------------------------|----------------------------------|
| 0.7| ![Graph](image1)       | ![Graph](image2)                 |
| 0.8| ![Graph](image3)       | ![Graph](image4)                 |
| 0.9| ![Graph](image5)       | ![Graph](image6)                 |
Figure 5: Caputos’s approach Solutions Vs. Riemann-Liouville’s aproach Solutions for Test

| $\alpha$ | Caputo Stefan-Like Pb. | Riemann-Liouville Stefan-Like Pb. |
|----------|------------------------|-----------------------------------|
| 0.7      | $\tilde{u}_\alpha(y,\tau)$ | $\tilde{w}_\alpha(y,\tau)$ |
|          | $\tilde{w}_\alpha(y,\tau)$ | $\tilde{\tilde{w}}_\alpha(y,\tau)$ |
| 0.8      | $\tilde{u}_\alpha(y,\tau)$ | $\tilde{w}_\alpha(y,\tau)$ |
|          | $\tilde{w}_\alpha(y,\tau)$ | $\tilde{\tilde{w}}_\alpha(y,\tau)$ |
| 0.9      | $\tilde{u}_\alpha(y,\tau)$ | $\tilde{w}_\alpha(y,\tau)$ |
|          | $\tilde{w}_\alpha(y,\tau)$ | $\tilde{\tilde{w}}_\alpha(y,\tau)$ |