Entanglement Localization and Optimal Measurement

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The entanglement can be localized between two noncomplementary parts of a many-body system by performing measurements on the rest of the system. This localized entanglement (LE) depends on the chosen basis set of measurement (BSM). We derive here a generic optimality condition for the LE, which, besides being helpful in studying tripartite systems in pure states, can also be of use in studying mixed states of general bipartite systems. We further discuss a canonical way of localizing entanglement, where the BSM is not chosen arbitrarily, but is fully determined by the properties of the system. The LE obtained in this way, we call the localized entanglement by canonical measurement (LECM), is not only operationally meaningful and easy to calculate in practice (without needing any demanding optimization procedure), it provides a nice way to define the entanglement length in many-body systems. For spin-1/2 systems, the LECM is shown to be optimal in some important cases. At the end, some numerical results are presented for \( j_1 - j_2 \) spin model to demonstrate how the LECM behaves.

Besides its importance in interpreting and understanding quantum mechanics, the entanglement has gained immense interest in recent times as it has the potential to play a significant role in modern technology. In addition, it has become an important tool to study quantum many-body systems [1]. Some of the very useful measures used here for studying quantum non-local nature of a system are pairwise entanglement [2,3], local entropy [4], localizable entanglement [5] and negativity [6]. Here we study entanglement localization which is important for two reasons - from a practical point of view, it can be a useful method of producing entangled pairs (especially from three-body systems) and secondly, we get an alternative theoretical way for studying quantum many-body systems. Here it may be stressed that, almost all the measures which try to quantify mutual quantum behavior between two disjoint parts of a many-body system are either difficult to calculate (often needing optimization procedure) or they do not have any operational meaning. The measure we present here (LECM) is not only easy to calculate, it also has some operational meaning.

Let \( S_1 \) and \( S_2 \) be any two noncomplementary parts of a total system \( U \). The rest of the system is called the environment (E), which generally consists of many sites (Fig. 1a). A measurement on \( E \) by some basis set would result in \( S (= S_1 + S_2) \) assuming different pure states with appropriate probabilities. Unlike in the case of localizable entanglement [5] where only local measurements on the individual sites of \( E \) are allowed, we allow all possible measurements (including the joint measurements on the sites) in our localization process. It may be noted here that, all the measurements in this work are considered to be non-selective projective-type. In the next part, we derive a simple but generic optimality condition for the LE, which will help us find optimal (more generally, stationary) solutions and check whether a given solution is optimal. Studying a general bipartite system in mixed state is notoriously difficult. There can be innumerable ways of decomposing a mixed state, where each decomposition corresponds to an average entanglement (entropy). The maximum and the minimum possible values of the average entanglement are termed as entanglement of assistance (EoA) [7] and entanglement of formation (EoF) [8] respectively. The optimality condition derived here may be of use in finding them. In this regard, a brief discussion is given after arriving at the condition.

The optimality condition.- When expressed in the product basis states of \( E \) and \( S \) (Fig. 1a), the given wave function (that we study) becomes, \( |\Psi\rangle = \sum_{i,j=1}^{D_E,D_S} C_{i,j}|\xi_i\rangle^E|\phi_j\rangle^S \). Here \( |\xi_i\rangle^E \) and \( |\phi_j\rangle^S \) are some orthonormal basis vectors of the state space of \( E \) (\( S \)) with dimensionality \( D_E \) (\( D_S \)). The state can also be written as

\[
|\Psi\rangle = \sum_{i=1}^D \sqrt{p_i}|\xi_i\rangle^E|\xi_i\rangle^S,
\]

with \( p_i = \sum_{j'=1}^{D_S} C_{i,j'}C_{i,j'}^\ast \) and \( |\xi_i\rangle^S = \sum_{j=1}^{D_S} C_{i,j}|\phi_j\rangle^S \). Here the summation runs over nonzero \( p_i \)'s, numbering \( D (\leq D_E) \). In general, states \( |\xi_i\rangle^S \) are not orthonormal.
The operational interpretation of the later expression of the state $|\Psi\rangle$ is that, if we perform measurement on $E$ by the basis set $\{\xi^E\}$, the state will collapse and we will get $S$ in different pure states $|\xi_i^S\rangle$’s with corresponding probabilities $p_i$’s.

If $S_i$ be the entropy of $|\xi_i^S\rangle$, then the average entropy (entanglement) localized between $S_1$ and $S_2$ would be,

$$\bar{S}\{\xi^E\} = \sum_{i=1}^{D} p_i S_i. \quad (2)$$

As both $p_i$’s and $S_i$’s depend on the choice of the BSM, the average entropy (or LE) $S$ will also depend on the choice of the BSM. We need to derive a condition for the choice of BSM ($\{\xi^E\}$) which optimizes $S$.

We first note that, any general basis set can be obtained from an initial basis set $\{\xi^E\}$ by application of a series of elementary transformations (ETs). Here an ET is a small-angle orthonormal transformation between any two initial basis states keeping others unchanged. We now derive first order change in $S$ due to an ET. If $|\xi_i^E\rangle$ and $|\xi_j^E\rangle$ be any two initial basis states, then the two new basis states obtained by an ET would be,

$$|\xi_i^E\rangle = |\xi_i^E\rangle + \epsilon |\xi_j^E\rangle \quad \text{and} \quad |\xi_j^E\rangle = |\xi_j^E\rangle - \epsilon |\xi_i^E\rangle. \quad (3)$$

Here $\epsilon$ is the small angle (a parameter) whose higher order terms can be neglected. Due to change in these basis states, corresponding probabilities and states of the $S$ would also change (see eq. 4). We need to relate these new probabilities and states with the old ones.

At this stage it is advantageous to express all the probabilities as the diagonal elements of a density operator (matrix), which is, in our case, the reduced density matrix (RDM) of $E$ ($\rho^E$). The elements of the RDM are given by $\rho_{ij}^E = \sum_{j=1}^{D} C_{ij}^* C_{ij}$. Using this RDM, probability corresponding to a state $|\xi_i^E\rangle$ would be $p = \langle \xi_i | \rho^E | \xi_i \rangle$. This allows us to write the new probabilities as (using eq. 3),

$$p_i’ = p_i + \epsilon k_{ii} \quad \text{and} \quad p_j’ = p_j - \epsilon k_{jj}, \quad (4)$$

with $k_{ij} = k_{ji} = \langle \xi_i | \rho^E | \xi_j \rangle + \langle \xi_j | \rho^E | \xi_i \rangle$. Let us first consider the case when none of the $p_i$ and $p_j$ is zero. Now if $|\xi_i^S\rangle$ and $|\xi_j^S\rangle$ be the new states of the $S$, then, in the new scenario, the state $|\Psi\rangle$ can be rewritten as,

$$|\Psi\rangle = \sqrt{p_i} |\xi_i^E\rangle |\xi_i^S\rangle + \sqrt{p_j} |\xi_j^E\rangle |\xi_j^S\rangle + \cdots \quad (5)$$

Here we focus only on $i$-th and $j$-th states, as other states are unchanged. Now using eqs. 3 and 4 in the above expression and then comparing the terms associated with the initial basis states $|\xi_i^E\rangle$ and $|\xi_j^E\rangle$ from the two different expressions of $|\Psi\rangle$ (in eqs. 1 and 4), we get the following solutions for the new states of $S$:

$$|\xi_i^S\rangle = |\xi_i^S\rangle + \epsilon (a_{ij} |\xi_j^S\rangle + b_{ij} |\xi_j^S\rangle) \quad (6)$$

$$|\xi_j^S\rangle = |\xi_j^S\rangle - \epsilon (a_{ij} |\xi_j^S\rangle + b_{ij} |\xi_i^S\rangle) \quad (7)$$

Here $a_{ij} = -\frac{1}{2} k_{ij} p_i^{-1}$ and $b_{ij} = p_i^{1/2} p_j^{-1/2}$ in eq. 6. Interchanging the indices $i$ and $j$ we get similar terms in eq. 7.

Now let $Q = \{Q_{lm}\}$ and $R = \{R_{lm}\}$ be the matrices representing respectively the states $|\xi_i^S\rangle$ and $|\xi_j^S\rangle$ in some product basis states of the parts $S_1$ and $S_2$. In terms of these matrices, the RDMs for $S_1$ would be $\rho^S_i(|\xi_i^S\rangle = Q Q^\dagger$ and $\rho^S_i(|\xi_j^S\rangle = R R^\dagger$ while $S$ is respectively in $|\xi_i^S\rangle$ and $|\xi_j^S\rangle$. Similarly, the RDMs for $S_1$ corresponding to the new states, given in eqs. 6 and 7 would be,

$$\rho^S_i(|\xi_i^S\rangle = \rho^S_i(|\xi_j^S\rangle + \epsilon (2 a_{ij} p_i^S(|\xi_j^S\rangle + 2 b_{ij} \Delta_{ij}) \quad (8)$$

$$\rho^S_i(|\xi_j^S\rangle = \rho^S_i(|\xi_i^S\rangle - \epsilon (2 a_{ij} p_i^S(|\xi_i^S\rangle + 2 b_{ij} \Delta_{ij}) \quad (9)$$

Here $\Delta_{ij} = \frac{1}{2} (QR^\dagger + RQ^\dagger)$, a Hermitian matrix. Let us denote here the changes in the RDMs in eqs. 8 and 9 as $\epsilon p_i^S(|\xi_j^S\rangle$ and $-\epsilon p_i^S(|\xi_i^S\rangle$ respectively. It is worth mentioning that, as trace (Tr) of any RDM is 1, we have

$$\text{Tr} \rho_i^S(|\xi_j^S\rangle = \text{Tr} \rho_i^S(|\xi_i^S\rangle = 0. \quad (10)$$

We now use the relation $\rho^S_i(|\xi_i^S\rangle \log_2 \rho^S_i(|\xi_i^S\rangle = \rho^S_i(|\xi_i^S\rangle \log_2 \rho^S_i(|\xi_j^S\rangle + \epsilon p_i^S(|\xi_j^S\rangle + \epsilon p_i^S(|\xi_i^S\rangle \log_2 \rho^S_i(|\xi_i^S\rangle \quad (9)$

for obtaining entropy corresponding to the new state $|\xi_i^S\rangle$. This operator relation is not ill-defined due to last term as both $\rho^S_i(|\xi_i^S\rangle$ and $\rho_i^S(|\xi_j^S\rangle$ go to zero simultaneously (this can be understood by singular value decomposition of the matrix Q). Now tracing over both sides of this relation and a similar relation for the $j$-th state, we respectively get the following entropies for the new states of $S$,

$$S' = S_i - \epsilon \sum_{j=1}^{D} p_j S_{ij} = S + \epsilon S_i, \quad (11)$$

Here we used eq. 10 to get these relations. Let us now denote the changes in entropies in eqs. 11 and 12 as $-\epsilon S_{ij}^1$ and $\epsilon S_{ji}^1$, respectively.

Now using these new entropies along with the new probabilities in eqn. 4 we get the new average entropy:

$$S' = \sum_{i=1}^{D} p_i S_{ij} = S + \epsilon S_1, \quad (13)$$

where, $S_1 = k_{ij} S_i - p_i S_{ij} - k_{ji} S_j + p_j S_{ji}$.

Before we set the optimality condition, we now check the cases when both are or one of $p_i$ and $p_j$ is zero. When $p_i = p_j = 0$, then $k_{ij} = k_{ji} = 0$. Therefore, $p_i' = p_j' = 0$ (see eq. 4). Which implies that $S_1$ is zero. On the other hand, when $p_i \neq 0$ and $p_j = 0$, we gave again $k_{ij} = k_{ji} = 0$. From eq. 4 we have $p_i' = p_i$ and $p_j' = 0$. Now it is clear from eq. 4 (with second term being zero) that, this type of ETs are not allowed (within the first order calculation).

So, the desired optimality condition is $\bar{S}_1 = 0$ or,

$$k_{ij} S_i - p_i S_{ij} = k_{ji} S_j - p_j S_{ji}, \quad (14)$$
for all \(i\) and \(j\) for which corresponding probabilities are nonzero. The second order change in \(S\) due to different ETs can also be derived but they cannot confirm actual character of an optimum \(\Omega\).

For the derivation of the above optimality condition, we have assumed the existence of a fixed environment \((E)\). Therefore, the condition will be helpful when we have a definite tripartite system and we want to localize entanglement between two parts in an optimal way by performing measurement on the third part. Now the question is whether it also can be of any use for calculating EoA and EoF of a mixed state. We note that, for a bipartite system in a mixed state, it is always possible to construct a pure state by augmenting the bipartite system with an ancilla in such a way that the RDM of the system becomes the given mixed state. By performing all possible measurements on all possible ancillas, we get all possible decompositions of the mixed state. By a theorem of Hughston-Jozsa-Wootters \([11]\), we can relate each decomposition to a \(M_{r×k}\) matrix with \(k\) orthonormal column vectors. Here the \(r\) and \(k\) are the number of terms in the decomposition and the rank of the mixed state respectively. This implies that, for a particular ancilla, we can express the optimality condition in terms of a matrix \(M_{r×k}\). Now as in principle number of terms in a decomposition can be anything, it may appear that the optimality condition obtained for a fixed ancilla would be of no use for the said purpose. Fortunately, at least in the case of EoF, it is seen that, consideration of a few number of terms in the decomposition is enough for extremization \([12]\). Hence we hope that, optimality condition given here may also be useful in calculating EoA and EoF of a mixed state.

**The LECM.-** Our canonical way of localizing entanglement between \(S_1\) and \(S_2\) is to take eigenstates of \(\rho^E\) (the RDM of \(E\)) as the BSM and perform measurement on \(E\). We will find now the expression for the entanglement localized in this way (which we call LECM).

Expression of the LECM can easily be obtained from Schmidt decomposition (SD) of the state under study. The SD of the state \(|\Psi\rangle\) into the product states of \(E\) and \(S\) is given by

\[
|\Psi\rangle = \sum_i \sqrt{\lambda_i} |\xi_i\rangle^S |\xi_i\rangle^E.
\]  

(15)

Here, \(|\xi_i\rangle^S\) (\(|\xi_i\rangle^E\)) is the \(i\)-th eigenstate of the RDM of \(S\) (\(E\)) corresponding to the eigenvalue \(\lambda_i\). The index \(i\) runs from 1 to \(D_s\) (Schmidt number). An operational interpretation of eq. (15) is as follows: measurement on \(E\) by the basis set \(|\xi_i\rangle^E\) would result in \(S\) assuming state \(|\xi_i\rangle^S\) with probability \(\lambda_i\). Now if \(S_i\) is the entropy of \(|\xi_i\rangle^S\) then, we can write directly from eq. (2)

\[
\bar{S} = \sum_{i=1}^{D_s} \lambda_i S_i.
\]  

(16)

This is the desired expression for the LECM.

This localization procedure may appear to face some problems when \(\rho^S\) has degenerate eigenstates. Most of the time this difficulty can easily be resolved by the use of conserved quantities and the symmetries of the system \([10]\).

We now discuss the important cases when the LECM can be shown to be optimal. We note that when measurement is performed by eigenstates of RDM \((\rho^E)\), \(k_{ij} = k_{ji} = 0\) and the eq. (14) reduces to

\[
\text{Tr} \Delta_{ij} \log_2 \rho^{S_i}(\xi_i) = \text{Tr} \Delta_{ji} \log_2 \rho^{S_i}(\xi_i).
\]  

(17)

For spin-1/2 systems, with both the parts \(S_1\) and \(S_2\) taken to be single-sites, use of conserved quantity (here \(Z\)-component of total spin) and parity or \(C_2\) symmetry (for finite systems; for translationally invariant systems this will be automatically satisfied) leads the four eigenstates of \(\rho^S\) into the following form: \(|↑↑\rangle\), \(|↑↓\rangle\), \(|↓↑\rangle\), \(|↓↓\rangle\) with \(↑\) (down) being up (down) spin. Any two of the \(2 \times 2\) matrices representing these states are seen to satisfy the condition given in eq. (17).

Since entropies of first and fourth states are zero, and second and third states (a singlet and a triplet) are one, the LECM in this case simply becomes, \(\bar{S} = \lambda_s + \lambda_t\), with \(\lambda_s (\lambda_t)\) is the eigenvalue of \(\rho^S\) corresponding to singlet (triplet). Note, after measurement, two sites will be in the statistical mixture of all the four eigenstates of \(\rho^S\) and if \(\lambda_s = \lambda_t\), then we will not be able to extract any useful entangled pair from the ensemble.

**Numerical Result.-** We study the LECM \((\bar{S})\) between two sites of a frustrated antiferromagnetic Heisenberg chain \((j_1 - j_2 \text{ spin-1/2 model})\). Two sites are placed symmetrically as in fig. 1b; this arrangement makes \(\bar{S}\) to be an optimal.

Its behavior against the distance between the sites \((R)\) for different values of \(j_2\) is shown in fig. 2.

We see that, in the Néel phase \((j_2 < 0.5)\), \(\bar{S}\) falls with increasing \(R\) and reaches a constant value at large \(R\). With increase in \(R\), all the four eigenvalues of the \(\rho^S\) become equal (hence \(\lambda_s = \lambda_t\)), which results in the LECM assuming a value of 0.5 (which we call residual value or \(S_\text{r}\)). In fact, this particular value of the LECM is obtained if we take two sites (symmetrically) one each from two totally separate chains (unentangled) and perform canonical measurement on remaining parts of the chains. Physically this implies that, in case of a single chain (where sites are connected), when \(R\) is large, two sites become unentangled, i.e., we can not localize ‘extractable’ or useful entanglement between them by a canonical measurement. Keeping this in mind, we therefore, can quantify actual extractable entanglement in our localization process as \(\Delta S = S - S_\text{r}\). Any positive value of the quantity \(\Delta S\) will give us the actual ‘gain’ in our localization process. In case of our single chain, the quantity \(\Delta S\) falls with increasing \(R\), which can be understood qualitatively by the Valence Bond (VB) theory \([14, 15]\).
A groundstate can be expressed by linear combination of many VB basis states where basis states with nearest neighbor bonds (a bond represents an entangled pair) contribute more towards the groundstate compared to the ones with distant neighbor bonds. This says why for large $R$ two sites become decoherent or unentangled. This fact naturally leads us to the notion of entanglement length ($\xi_E$), which is the typical length scale upto which it is possible to localize useful or extractable entanglement between two sites. If fall in $\Delta \mathcal{S}$ with increasing $R$ is assumed to be exponential in nature, we can define $\xi_E$ as $\xi_E = 3$, $\xi_E^{-1} = – \ln \Delta \mathcal{S}$ for large $R$. Since we have a finite system (24 sites), we can use two values of $R$ and corresponding values of $\Delta \mathcal{S}$ to estimate the value of $\xi_E$ in the following way: $\xi_E^{-1} = \ln \Delta \mathcal{S} \rightarrow -\ln \Delta \mathcal{S}$. We take $R = 7$ and 11 for this purpose (this particular values are chosen to avoid odd-even effect of a finite chain [13]), and calculated values of $\xi_E$ as a function of $j_2$ can be seen from fig. 3.

The fall in the value of $\xi_E$ with increasing value of $j_2$ is not unexpected, as the contribution of VB basis states with long bonds decreases with the increasing value of $j_2$. This is supported by the fact that at the Majumdar-Ghosh (MG) point ($j_2 = 0.5$) groundstate has only nearest neighbor bonding.

The large oscillations in the value of $\mathcal{S}$ for $j_2 > 0.5$ (fig. 2) can be understood by the fact that the phase of the system in this range is spiral in nature. The degree of entanglement between two sites depends on the relative phase factor between the sites. This is why for some distances the value of $\mathcal{S}$ is very low.

**Conclusion.** In this letter, we have derived a simple but generic optimality condition, which would be helpful in finding optimal values of the entanglement localized between two disjoint parts of a many-body system by doing measurement on the remaining part of the system. Besides, we also discussed how it can be useful in studying mixed states of a general bipartite system. We further discussed a canonical way of localizing entanglement which in some important and not-too-restricted cases (shown for spin-$1/2$ systems) gives optimal value of the localized entanglement. The entanglement localized by canonical measurement (or LECM) is operationally meaningful and easy to calculate in practice. Unlike other measurement-based quantifications of entanglement, the LECM does not require any demanding optimization procedure. Another important advantage of this LECM is that, since it does not depend on arbitrary choice of BSM, it provides a general framework for comparative study of different types of quantum many-body systems. We studied a $j_1 - j_2$ spin model to demonstrate the behavior of LECM. In this context, we also discussed extractable or useful part of LECM and defined an entanglement length scale upto which one can localize extractable entanglement between two sites. It may be stressed here that, all the concepts in this letter are quite general, virtually applicable to any kind of quantum many-body systems.

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