PROPERTY (u) IN $JH \tilde{\otimes}_c JH$

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Abstract. It is shown that the tensor product $JH \tilde{\otimes}_c JH$ fails Pelczyński’s property (u). The proof uses a result of Kwapień and Pelczyński on the main triangle projection in matrix spaces.

The Banach space $JH$ constructed by Hagler [1] has a number of interesting properties. For instance, it is known that $JH$ contains no isomorph of $\ell^1$, and has property (S): every normalized weakly null sequence has a subsequence equivalent to the $c_0$-basis. This easily implies that $JH$ is $c_0$-saturated, i.e., every infinite dimensional closed subspace contains an isomorph of $c_0$. In answer to a question raised originally in [1], Knaust and Odell [2] showed that every Banach space which has property (S) also has Pelczyński’s property (u). In [4], the author showed that the Banach space $JH \tilde{\otimes}_c JH$ is $c_0$-saturated. It is thus natural to ask whether $JH \tilde{\otimes}_c JH$ has also the related properties (S) and/or (u). In this note, we show that $JH \tilde{\otimes}_c JH$ fails property (u) (and hence property (S) as well). Our proof makes use of a result, due to Kwapień and Pelczyński, that the main triangle projection is unbounded in certain matrix spaces.

We use standard Banach space notation as may be found in [5]. Recall that a series $\sum x_n$ in a Banach space $E$ is called weakly unconditionally Cauchy (wuC) if there is a constant $K < \infty$ such that $\|\sum_{n=1}^k \epsilon_n x_n\| \leq K$ for all choices of signs $\epsilon_n = \pm 1$ and all $k \in \mathbb{N}$. A Banach space $E$ has property (u) if whenever $(x_n)$ is a weakly Cauchy sequence in $E$, there is a wuC series $\sum y_k$ in $E$ such that $x_n - \sum_{k=1}^n y_k \to 0$ weakly as $n \to \infty$. If $E$ and $F$ are Banach spaces, and $L(E', F)$ is the space of all bounded linear operators from $E'$ into $F$ endowed with the operator norm, then the tensor product $E \tilde{\otimes}_c F$ is the closed subspace of $L(E', F)$ generated by the weak*-weakly continuous operators of finite rank. In particular, for any $x \in E$, and $y \in F$, one obtains an element $x \otimes y \in E \tilde{\otimes}_c F$ defined by $(x \otimes y)x' = \langle x, x'\rangle y$ for all $x' \in E'$.

Let us also recall the definition of the space $JH$, as well as fix some terms and notation. Let $T = \bigcup_{n=0}^\infty \{0, 1\}^n$ be the dyadic tree. The elements of $T$ are called nodes. If $\phi$ is a node of the form $(\epsilon_i)_{i=1}^n$, we

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\end{itemize}
say that $\phi$ has length $n$ and write $|\phi| = n$. The length of the empty node is defined to be 0. For $\phi, \psi \in T$ with $\phi = (\epsilon_i)_{i=1}^n$ and $\psi = (\delta_i)_{i=1}^m$, we say that $\phi \leq \psi$ if $n \leq m$ and $\epsilon_i = \delta_i$ for $1 \leq i \leq n$. The empty node is $\leq \phi$ for all $\phi \in T$. Two nodes $\phi$ and $\psi$ are incomparable if neither $\phi \leq \psi$ nor $\psi \leq \phi$ hold. If $\phi \leq \psi$, we say that $\psi$ is a descendant of $\phi$, and we set

$$S(\phi, \psi) = \{ \xi : \phi \leq \xi \leq \psi \}.$$ 

A set of the form $S(\phi, \psi)$ is called a segment, or more specifically, an $m$-$n$ segment provided $|\phi| = m$, and $|\psi| = n$. A branch is a maximal totally ordered subset of $T$. The set of all branches is denoted by $\Gamma$. A branch $\gamma$ (respectively, a segment $S$) is said to pass through a node $\phi$ if $\phi \in \gamma$ (respectively, $\phi \in S$). If $x : T \to \mathbb{R}$ is a finitely supported function and $S$ is a segment, we define (with slight abuse of notation) $Sx = \sum_{\phi \in S} x(\phi)$. In case $S = \{\phi\}$ is a singleton, we write simply $\phi x$ for $Sx$. Similarly, if $\gamma \in \Gamma$, we define $\gamma(x) = \sum_{\phi \in \gamma} x(\phi)$. A set of segments $\{S_1, \ldots, S_r\}$ is admissible if they are pairwise disjoint, and there are $m, n \in \mathbb{N} \cup \{0\}$ such that each $S_i$ is an $m$-$n$ segment. The James Hagler space $JH$ is defined as the completion of the set of all finitely supported functions $x : T \to \mathbb{R}$ under the norm:

$$\|x\| = \sup \left\{ \sum_{i=1}^r |S_i x| : S_1, \ldots, S_r \text{ is an admissible set of segments} \right\}.$$ 

Clearly, all $S$ and $\gamma$ extend to norm 1 functionals on $JH$. It is known that the set $T$ of all node functionals, and the set $\Gamma$ of all branch functionals together span a dense subspace of $JH'$ (cf. p. 301 of [1]). Finally, if $x : T \to \mathbb{R}$ is finitely supported, and $n \geq 0$, let $P_n x : T \to \mathbb{R}$ be defined by

$$(P_n x)(\phi) = \begin{cases} x(\phi) & \text{if } |\phi| \geq n \\ 0 & \text{otherwise.} \end{cases}$$ 

Obviously, $P_n$ extends uniquely to a norm 1 projection on $JH$, which we denote again by $P_n$. The proof of the following lemma is left to the reader. We thank the referee for the succinct formulation.

**Lemma 1.** For any $n \in \mathbb{N}$, construct a sequence $(\pi(1), \pi(2), \ldots, \pi(n))$ by writing the odd integers in the set $\{1, \ldots, n\}$ in increasing order, followed by the even integers in decreasing order. Then

$$(-1)^{\min(\pi(i), \pi(j)) + 1} = 1 \iff i + j \leq n + 1.$$ 

For any $n \in \mathbb{N}$ and $n \times n$ real matrix $M = [M(i, j)]_{i,j=1}^n$, let $E(M)$ be the matrix $[(-1)^{\min(i,j)+1}M(i, j)]$. Denote by $\sigma(M)$ the norm of $M$. 


considered as a linear map from $\ell^\infty(n)$ into $\ell^1(n)$, i.e.,

$$\sigma(M) = \sup \left\{ \sum_{i,j=1}^{n} a_i b_j M(i,j) : \sup_{1 \leq i,j \leq n} \{|a_i|, |b_j|\} \leq 1 \right\}.$$ 

**Lemma 2.** There is a constant $C > 0$ such that for every $n \in \mathbb{N}$, there is an $n \times n$ real matrix $M_n$ such that $\sigma(M_n) = 1$ and $\sigma(E(M_n)) \geq C \log n$.

**Proof.** It follows easily from [3, Proposition 1.2] that there are a constant $C > 0$ and real $n \times n$ matrices $N_n = [N_n(i,j)]$ for every $n$ such that $\sigma(N_n) = 1$, and $\sigma([\epsilon(i,j)N_n(i,j)]) \geq C \log n$, where

$$\epsilon(i,j) = \begin{cases} 1 & \text{if } i + j \leq n + 1, \\ -1 & \text{otherwise.} \end{cases}$$

Let $\pi$ be the permutation in Lemma 1. Define $M_n(i,j) = N_n(\pi^{-1}(i), \pi^{-1}(j))$, $1 \leq i, j \leq n$, and let $M_n = [M_n(i,j)]$. Clearly $\sigma(M_n) = \sigma(N_n) = 1$ for all $n$. Also,

$$\sigma(E(M_n)) = \sigma\left( [(-1)^{\min(\pi(i),\pi(j))+1} M_n(\pi(i), \pi(j))] \right) = \sigma\left( [\epsilon(i,j)N_n(i,j)] \right) \geq C \log n,$$

as required. \[\square\]

Let $\psi_n$ denote the node $0$, and $\psi_n = (1 \ldots 1 0)$ for $n \geq 2$. For convenience, define $s_0 = 0$ and $s_k = \sum_{i=1}^{k} i$ for $k \geq 1$. Now choose a strictly increasing sequence $(n_k)$ in $\mathbb{N}$, and a sequence of pairwise distinct nodes $(\phi_i)$ such that $\phi_i$ is a descendant of $\psi_{n_k}$ and belongs to $\gamma_i$. Finally, let $R_k = [R_k(i,j)]_{i,j=s_k-1}^{s_k}$ be $k \times k$ real matrices such that $\sum_{k=1}^{n} \sigma(R_k) < \infty$. Then define a sequence of elements in $JH \hat{\otimes} JH$ as follows:

$$U_l = \sum_{k=1}^{l} \sum_{i,j=s_{k-1}+1}^{s_k} R_k(i,j) e_{\phi(i,l)} \otimes e_{\phi(j,l)}$$

for $l \in \mathbb{N}$. Here, $e_{\phi} \in JH$ is the characteristic function of the singleton set $\{\phi\}$. Since the sequence $(e_{\phi(i,l)})_{i=s_{k-1}+1}^{s_k}$ is isometrically equivalent to the $\ell^1(k)$-basis whenever $k \leq l$,

$$\left\| \sum_{i,j=s_{k-1}+1}^{s_k} R_k(i,j) e_{\phi(i,l)} \otimes e_{\phi(j,l)} \right\| = \sigma(R_k),$$

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Lemma 3. The sequence \((U_t)\) is a weakly Cauchy sequence in \(JH \hat{\otimes}_e JH\).

Proof. It is well known that a bounded sequence \((W_n)\) in a tensor product \(E \hat{\otimes}_e F\) is weakly Cauchy if and only if \((W_n x')\) is weakly Cauchy in \(F\) for all \(x' \in E'\). Since \((U_t)\) is a bounded sequence, and \([T \cup \Gamma] = JH'\), it suffices to show that \((U_t x')\) is weakly Cauchy in \(JH\) for every \(x' \in T \cup \Gamma\). Now for all \(\phi \in T\), we clearly have \(U_t \phi = 0\) for all large enough \(t\). Next, consider any \(\gamma \in \Gamma\). If \(\gamma\) does not pass through any \(\psi_k\), then it cannot pass through any \(\phi(i, k)\) either. So \(U_t \gamma = 0\) for all \(t\). Otherwise, due to the pairwise incomparability of \((\psi_k)\), there is a unique \(k_0\) such that \(\psi_{k_0} \in \gamma\). If \(\gamma\) is distinct from \(\gamma_i\) for all \(s_{k_0-1} < i \leq s_{k_0}\), then again \(U_t \gamma = 0\) for all sufficiently large \(t\). Now suppose \(\gamma = \gamma_{i_0}\), where \(s_{k_0-1} < i_0 \leq s_{k_0}\). Then, for \(l \geq k_0\),

\[
U_t \gamma = \sum_{j=s_{k_0-1}+1}^{s_{k_0}} R_{k_0}(i_0, j)e_{\phi(j, l)}.
\]

Since each sequence \((e_{\phi(j, l)})_{l=k_0}^{\infty}\) is weakly Cauchy in \(JH\), so is \((U_t \gamma)\). □

Now if \(JH \hat{\otimes}_e JH\) has property \((u)\), then it is easy to observe that there must be a block sequence of convex combinations \((V_r)\) of \((U_t)\) such that \(\sum(V_r - V_{r+1})\) is a \(wuC\) series. Write \(V_r = \sum_{l=1}^{l_r} a_t U_t\) (convex combination), where \((l_r)\) is a strictly increasing sequence in \(\mathbb{N}\). Fix \(r \in \mathbb{N}\). For \(s_{r-1} < i \leq s_r\), let \(\xi_i\) be a branch such that \(\phi(i, l_r+i-s_{r-1})\) is the node of maximal length which it shares with \(\gamma_i\). Then if \(s_{r-1} < i, j \leq s_r\), and \(r \leq l\),

\[
\langle e_{\phi(i, l)}, \xi_j \rangle = 1 \iff i = j \text{ and } l \leq l_r+j-s_{r-1}.
\]

Hence, if \(r \leq l\),

\[
\langle U_t \xi_i, \xi_j \rangle = \begin{cases} R_r(i, j) & \text{if } l \leq \min(l_r+i-s_{r-1}, l_r+j-s_{r-1}), \\ 0 & \text{otherwise}. \end{cases}
\]

Thus, if \(s_{r-1} < i, j, k \leq s_r\),

\[
\langle V_{r+k-s_{r-1}} \xi_i, \xi_j \rangle = \begin{cases} R_r(i, j) & \text{if } k \leq \min(i, j), \\ 0 & \text{otherwise}. \end{cases}
\]

It follows that

\[
\left\langle \sum_{k=s_{r-1}+1}^{s_r} (-1)^{k+1-s_{r-1}} (V_{r+k-s_{r-1}} - V_{r+k+1-s_{r-1}}) \xi_i, \xi_j \right\rangle
\]

\[
= (-1)^{\min(i-s_{r-1}, j-s_{r-1})+1} R_r(i, j).\]
Notice that $k > s_{r-1}$ implies $l_{r+k-s_{r-1}} \geq l_{r+1} \geq r$, hence
\[
\langle V_{r+k-s_{r-1}}P'_{n_i}\xi_i, P'_{n_j}\xi_j \rangle = \langle V_{r+k-s_{r-1}}\xi_i, \xi_j \rangle.
\]
Also, $(P'_{n_i}\xi_i)_{i=s_{r-1}+1}^{s_r}$ is isometrically equivalent to the $\ell^\infty(r)$-basis. Therefore,
\[
\left\| \sum_{k=s_{r-1}+1}^{s_r} (-1)^{k+1-s_{r-1}} (V_{r+k-s_{r-1}} - V_{r+k+1-s_{r-1}}) \right\| 
\geq \sigma \left( \left[ (-1)^{\min(i-s_{r-1},j-s_{r-1})+1} R_r(i,j) \right] \right) = \sigma(E(R_r)).
\]
But since $\sum (V_r - V_{r+1})$ is a $\text{wuC}$ series, there is a constant $K < \infty$ (which may depend on the sequence $(R_k)$) such that
\[
\left\| \sum_{k=s_{r-1}+1}^{s_r} (-1)^{k+1-s_{r-1}} (V_{r+k-s_{r-1}} - V_{r+k+1-s_{r-1}}) \right\| \leq K
\]
for any $r$. Consequently, $\sup_r \sigma(E(R_r)) \leq K$.

Now choose a strictly increasing sequence $(r_m)$ such that $\lim_m 2^{-m} \log r_m = \infty$. Then let
\[
R_k = \begin{cases} 
\frac{M_{r_m}}{2^m} & \text{if } k = r_m \text{ for some } m, \\
0 & \text{otherwise,}
\end{cases}
\]
where $M_{r_m}$ is the matrix given by Lemma 4. Then $\sum_k \sigma(R_k) = \sum_m 2^{-m} \sigma(M_{r_m}) = 1$. So the preceding argument yields a finite constant $K$ such that
\[
K \geq \sup_m \frac{\sigma(E(M_{r_m}))}{2^m} \geq C \sup_m \frac{\log r_m}{2^m},
\]
contrary to the choice of $(r_m)$. We have thus proved the following result.

**Theorem 4.** The Banach space $JH \tilde{\otimes}_\epsilon JH$ fails property (u).

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