Compactness Results for $\mathcal{H}$–Holomorphic Maps

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Abstract

$\mathcal{H}$–holomorphic maps are a parameter version of $J$–holomorphic maps into contact manifolds. They have arisen in efforts to prove the existence of higher–genus holomorphic open book decompositions, the existence of finite energy foliations and the Weinstein conjecture [ACH05], as well as in folded holomorphic maps [vB07]. For all these applications it is essential to understand the compactness properties of the space of $\mathcal{H}$–holomorphic maps.

We prove that the space of $\mathcal{H}$–holomorphic maps with bounded periods into a manifold with stable Hamiltonian structure possesses a natural compactification. Limits of smooth maps are neck-nodal maps, i.e. their domains can be pictured as nodal domains where the node is replaced by a finite cylinder that converges to a twisted cylinder over a closed characteristic or a finite length characteristic flow line. We show by examples that compactness fails without the condition on the periods, and we give topological conditions that ensure compactness.

1 Introduction

The theory of $J$–holomorphic curves has become an indispensable tool for symplectic and contact geometry and topology. However in many potential applications for $J$–holomorphic curves the index for the curves of interest turns out to be negative, or have dimension too low. In particular, this happens when one tries to foliate a 4–manifold, or the symplectization of a contact 3–manifold, by $J$–holomorphic surfaces of genus $g \geq 1$. In this situation one considers embedded curves with trivial normal bundle, so the index is $2 - 2g$, when one would like the index to be 2, the dimension of the leaf space of the foliation. To remedy this Abbas Cieliebak and Hofer suggested using families of $J$–holomorphic curves parameterized by $H^1(\Sigma; \mathbb{R})$ [ACH05]. The same parameter space is also needed for index reasons in [vB07]. These family version of $J$–holomorphic maps are called $\mathcal{H}$–holomorphic maps.

The basic setup for $\mathcal{H}$–holomorphic maps goes as follows (see also [ACH05] and [vB07]). Let $Z$ be a closed oriented manifold of dimension $2n + 1$. 


Definition 1.1. \((Z, \alpha, \omega)\) is a stable Hamiltonian structure if \(\alpha \in \Omega^1(Z)\) and \(\omega \in \Omega^2(Z)\) satisfy
\[
\alpha \wedge \omega^n > 0, \quad d\omega = 0, \quad \ker(\omega) \subset \ker(d\alpha) \tag{1.1}
\]

The stable Hamiltonian structure induces as splitting \(TZ = L \oplus F\), where \(F = \ker(\alpha)\), \(L = \ker(d\alpha)\).

\(L\) is called the characteristic foliation and the section \(R\) of \(L\) defined by \(\alpha(R) = 1\) is called the characteristic vector field. \((F, \omega)\) is a symplectic vector bundle. In the case that \(d\alpha = \omega\), \(\alpha\) is called a contact form, \(F\) a contact structure and \(R\) the Reeb vector field.

Let \(J\) be the set of almost complex structures on \(F\) that are compatible with \(\omega\), i.e.
\[
J = \{ J \in \text{End}(F) | J^2 = -\text{Id}, \; \omega(Ju, Jv) = \omega(u, v), \; g_F(u, v) = \omega(Ju, v) \text{ is a Riemannian metric on } F \}\.
\]

Often we will refer to a stable Hamiltonian structure as including a choice of \(J \in J\). This gives rise to a metric \(g = \alpha \otimes \alpha + g_F\) on \(Z\).

Definition 1.2 (\(\mathcal{H}\)-Holomorphic Maps). Let \((\Sigma, j)\) be a punctured Riemann surface. A map \(v : \hat{\Sigma} \longrightarrow Z\) is called \(\mathcal{H}\)-holomorphic if
\[
\bar{\partial}_j^F v = 0, \quad \bar{\partial}_j^F = \frac{1}{2} (\pi_F dv + J \pi_F dv j) \tag{1.2}
\]
\[
d(v^* \alpha \circ j) = 0, \tag{1.3}
\]
\[
\int_{\partial B_p(\xi)} v^* \alpha \circ j = 0 \quad \forall \ p \in \Sigma \setminus \hat{\Sigma} \text{ and } \xi \text{ small enough.} \tag{1.4}
\]

This is a system of elliptic differential equations, a (first order) Cauchy-Riemann type equation in the \(F\) directions and a (second order) Poisson equation in the \(L\) direction. The last equation demands that the periods of \(v^* \alpha \circ j\) vanish at the punctures.

\(\mathcal{H}\)-holomorphic maps can be viewed as \(J\)-holomorphic maps with parameter space \(H^1(\Sigma; Z)\) in the following way. Let \(\mathcal{H} = \mathcal{H}(\Sigma, j)\) of harmonic 1–forms on \(\Sigma\), i.e.
\[
\mathcal{H} = \{ \nu \in \Omega^1(\Sigma) | d\nu = d(\nu \circ j) = 0 \}.
\]

For an \(\mathcal{H}\)-holomorphic map \(v : \Sigma \longrightarrow Z\) there is a unique \(\eta \in \mathcal{H}\), and a function \(a : \hat{\Sigma} \longrightarrow \mathbb{R}\) that is unique up to addition of a constant, so that
\[
v^* \alpha + da \circ j = \eta \in \mathcal{H}. \tag{1.5}
\]

Given an \(\mathcal{H}\)-holomorphic map \(v\) we will often make implicit use of the splitting of closed 1–forms given in Equation (1.5) without mention. The pair
\[
\tilde{v} = (a, v) : \hat{\Sigma} \longrightarrow \mathbb{R} \times Z
\]
is called the canonical lift of \( v \) to the symplectization. It is unique up to translation in the \( \mathbb{R} \)–factor. With this notation the map \( v \) is \( \mathcal{H} \)–holomorphic if and only if
\[
\bar{\partial} J \tilde{v} \in \mathcal{H}_{C}^{0,1},
\]
where \( J \) is the canonical \( \mathbb{R} \)–invariant almost complex structure on the symplectization and \( \mathcal{H}_{C}^{0,1} \) is the \((0, 1)\)–part of the complexification of the space of harmonic 1–forms \( \mathcal{H} \) on \( \Sigma \) viewed as taking values in the trivial complex subbundle \( \mathbb{C} = \tilde{v}^*(\mathbb{R} \oplus L) \subset \tilde{v}^*T(\mathbb{R} \times Z) \). In particular, every \( J \)–holomorphic map is also \( \mathcal{H} \)–holomorphic.

The important feature of the space \( \mathcal{H} \) is that it gives a complement of the coexact 1–forms in the space of coclosed 1–forms. Sometimes it is convenient to choose a different complement \( \tilde{\mathcal{H}} \) with some prescribed properties and consider lifts to the symplectization w.r.t. \( \tilde{\mathcal{H}} \), i.e. maps \( \tilde{v} = (a, v) \) with \( v^*\alpha + da \circ j \in \tilde{\mathcal{H}} \).

Locally all closed forms are exact, so \( \mathcal{H} \)–holomorphic maps inherit all local properties of \( J \)–holomorphic maps. By standard theory (see e.g. \[HWZ96\]) \( \mathcal{H} \)–holomorphic maps that satisfy certain energy assumptions limit to closed characteristics at the punctures and the maps extend to a continuous map from the radial compactification \( \hat{\Sigma} \) of \( \dot{\Sigma} \).

In order for \( \mathcal{H} \)–holomorphic maps to be useful for applications in symplectic and contact geometry, it is important for the moduli space of \( \mathcal{H} \)–holomorphic maps to possess a natural compactification. The non–compactness of this parameter space \( H^1(\Sigma;\mathbb{R}) \) is the source for the compactness issues of the space of maps.

![Figure 1: A map in the compactification of smooth \( \mathcal{H} \)–holomorphic maps. The boundary components of \( \hat{\Sigma} \) map to closed characteristics \( x_1, x_2 \text{ and } x_3 \), and the necks \( N_1, N_2 \text{ and } N_4 \) map to closed characteristics \( y_1, y_2 \text{ and } y_4 \). The null–homologous neck \( N_1 \) has vanishing twist ("bubbles connect") while \( N_2 \text{ and } N_4 \) have in general non–vanishing twist. The neck \( N_3 \) maps to a finite length characteristic flow line \( y_3 \).](image)

The idea of using a parameter space to change the index of an equation to fit applications
has a long history, and usually requires a delicate analysis of solutions. For example, Junho Lee [Lee04] considered families of $J$–holomorphic maps into Kähler manifolds with non–compact parameter space given by harmonic 1–forms on the target. While in that case the space of maps is in general not compact, he was able to show that in certain interesting cases the space of maps stay in a compact subset of the parameter space and thus are compact.

We show that the situation for $H$–holomorphic maps is quite similar. The space of maps is in general not compact (Theorem 2.9), and boundedness of the “periods” (Definition 2.7) is a necessary and sufficient condition on a sequence of smooth $H$–holomorphic maps to have a convergent subsequence (Theorem 2.8, see Figure 1). This condition is automatically satisfied in situations arising for important applications (Theorem 4.2). In a sequel to this we are applying these results to nicely embedded $H$–holomorphic maps and open book decompositions [vB09].

2 Main Results

In order to understand the precise compactness statement we briefly survey some related compactness results in the literature.

Bubble tree convergence for $J$-holomorphic maps was established in the early ’90s ([Gro85], [PW93], [Ye94]). These results build on the bubbling phenomenon of conformally invariant elliptic equations with uniform energy bounds first studied by Sachs and Uhlenbeck [SU81].

For harmonic maps, bubble tree convergence in a fixed homology class with fixed domain complex structure was proved by Parker [Par96]. There it was also observed that a similar result with varying complex structure on the domain does not hold due to loss of control over the “neck” regions.

Chen and Tian proved [CT99] that compactness for energy minimizing finite energy harmonic maps with domain complex structure converging in $\bar{M}_g$ can be achieved if one fixes the homotopy class of maps, rather than just homology. Then the neck maps converge to finite length geodesics.

$J$-holomorphic maps into contact manifolds do not have uniform (or even finite) $W^{1,2}$-energy bounds. To overcome this Hofer, Wysocki and Zehnder ([HWZ96]) tailored a suitable notion of energy that is a homological invariant and guarantees bubble tree convergence where nodes (and punctures) can “open up” to wrap closed characteristics. In [BEH+03] this has been extended to the case of manifolds with stable Hamiltonian structure as targets and allowing certain degenerations of the almost complex structure on the target.

In the case of $H$-holomorphic maps there are again no uniform $W^{1,2}$-energy bounds. Roughly speaking, the $H$-holomorphic map equation into a $(2n+1)$- dimensional manifold with stable Hamiltonian structure is a mixture of a $2n$-dimensional $J$-holomorphic map equation in the almost contact planes and a 1-dimensional harmonic map equation in the characteristic direction. This dual nature is reflected in the compactness statement as neck maps converge to “twisted cylinders” over closed characteristics or finite length characteristic flow lines.
In order to account for the possibility of necks converging to characteristic flow lines we make the following definition for the space of domains.

**Definition 2.1.** Fix a genus $l$ reference surface $\hat{\Sigma}$ with $m$ boundary components. Denote the surface obtained by collapsing each boundary component to a point by $\Sigma$. A neck-nodal domain of genus $l$ with $m$ boundary components and $k$ necks is given by a map

$$\pi : \hat{\Sigma} \rightarrow C$$

to a nodal curve $C$ with $k$ nodes and $m$ marked points such that

1. each boundary component is mapped to a marked point in $C$, called a puncture,
2. there are $k$ embedded loops $\gamma_i$ with pairwise disjoint tubular neighborhoods $\nu(\gamma_i)$ bounded away from each other and the boundary of $\hat{\Sigma}$ and not containing any of the marked points so that each neck domain

$$N_i = \nu(\gamma_i) = \left[ -\frac{1}{2}, \frac{1}{2} \right] \times S^1, \quad 1 \leq i \leq k$$

maps to a distinct node of $C$, and
3. $\pi$ is a diffeomorphism from the smooth part of $\hat{\Sigma}$

$$\Sigma_0 = \hat{\Sigma} \setminus (\partial \Sigma \cup \mathcal{N}) \quad \text{where} \quad \mathcal{N} = \bigcup_{i=1}^{m} N_i$$

onto the curve $C^0$ obtained from $C$ by removing the punctures and nodes.

Thus $\pi$ induces a complex structure on the punctured surface $\Sigma_0$. We denote the space of neck-nodal domains of genus $l$ with $m$ boundary components and $n$ marked points modulo diffeomorphisms of $\hat{\Sigma}$ preserving the boundary components by $\mathcal{M}_N^{l,m,n}$.

![Figure 2: Neck–nodal domain.](image-url)
to the space of decorated nodal surfaces defined in [BEH+03], and we endow $M_{l,m+n}^S$ with the same topology as $M_{l,m+n}^S$.

The neck domains $N_i$ don’t carry a well defined conformal structure. Intuitively they should be viewed as flat cylinders with infinitesimal circumference.

**Definition 2.2.** A neck map $v : N = [-\frac{1}{2}, \frac{1}{2}] \times S^1 \to Z$ is a map of the form

$$v(s,t) = x(S \cdot s + T \cdot t)$$

where $x : \mathbb{R} \to Z$ is a flow line of the characteristic vector field and $S, T \in \mathbb{R}$. $T$ is called the period of the neck and $S$ is called the twist of the neck.

Note that if $T \neq 0$ then $x$ is necessarily a $T$-periodic orbit.

Each neck region $N \subset \hat{\Sigma}$ defines an element $[N] \in H_1(\Sigma; \mathbb{Z})$, where $\Sigma$ is as always defined to be the surface obtained from $\hat{\Sigma}$ by collapsing the boundary components.

**Definition 2.3.** Let $\hat{\Sigma}$ be a neck-nodal domain with neck domains $N = \bigcup_{i=1}^k N_i$. Then the collection of neck maps $v : N \to Z$ has minimal twist if whenever $\sum_{i \in I} [N_i] = 0 \in H_1(\Sigma; \mathbb{Z})$ for some index set $I$, then there exists a collection of non-negative real numbers $l_i, i \in I$ with $\sum_{i \in I} l_i = 1$ so that

$$\sum_{i \in I} l_i S_i = 0$$

where $S_i$ is the twist of $v|_{N_i}$.

We need one more definition ensuring that maps from a singular domain can be lifted to the symplectization.

**Definition 2.4.** A neck region $N$ is called non–separating if both boundary components of $N$ are adjacent to the same connected component of the smooth part $\Sigma_0$ of $\hat{\Sigma}$.

An $\mathcal{H}$–holomorphic map $v : \Sigma_0 \to Z$ from the smooth part $\Sigma_0 = \hat{\Sigma} \setminus (N \cup \partial \hat{\Sigma})$ is called exact if it lifts to a map to the symplectization so that the $\mathbb{R}$–component of the lift extends continuously by a constant over the non–separating components of the necks.

We are now prepared for the definition of neck–nodal maps.

**Definition 2.5.** An $\mathcal{H}$–holomorphic map from a neck-nodal domain $(\hat{\Sigma}, j)$ with neck domains $N = \{N_i\}_{1 \leq i \leq k}$ into $Z$ is a continuous map from $\hat{\Sigma}$ into $Z$ that restricts to an exact $\mathcal{H}$–holomorphic map from the smooth part $\Sigma_0$ into $Z$, and a minimal twist neck map on the neck domains.

This definition allows for $\mathcal{H}$–holomorphic maps in the compactification with qualitatively different behavior from $J$–holomorphic maps. This is necessary as such maps occur in examples (see the end of Section 4). If a neck region $N$ is homologically trivial in $\Sigma$, then a
minimal twist neck map has vanishing twist on $N$. This means that $\mathcal{H}$–holomorphic maps exhibit “zero distance bubbling”, just like in the $J$–holomorphic and harmonic map case.

To illustrate the meaning of the minimal twist condition further let $g$ be the genus of $\Sigma$ and denote the genus of the normalization of corresponding nodal curve $C$ by $\tilde{g}$. Let $n$ be the number of neck regions and $r$ the number of independent relations of the neck regions in $H_1(\Sigma; \mathbb{Z})$. Then $g - \tilde{g} = n - r$. In this equation $g - \tilde{g}$ is half the number of harmonic 1–forms that are lost in the singular domain. Half of the lost harmonic 1–forms are fixed as the periods of necks, and the other half is encoded in the $n$ twist parameters of the necks satisfying the $r$ relations.

There are several notions of energy that are important for $\mathcal{H}$–holomorphic maps. With $\mathcal{A}$ the space of probability measures on the real line we make use of the following standard definition for $J$–holomorphic maps.

**Definition 2.6.** Let $\mathcal{A}$ be the space of smooth probability measures on the real line. The $\alpha$–energy $E_\alpha(v)$ and $\omega$–energy $E_\omega(v)$ are

$$E_\alpha(v) = \sup_{f \in \mathcal{A}} \int_\Sigma f \circ a \, da \wedge j \, da, \quad E_\omega(v) = \int_\Sigma v^* \omega. \quad (2.6)$$

The integrands are pointwise non–negative functions on $\tilde{\Sigma}$ and both energies are invariants of the relative homology class of the map $v$. By the definition of stable Hamiltonian structure Definition 1.1 there exists a constant $M > 0$ so that

$$\left| \int_S v^* da \right| \leq M \int_S v^* \omega$$

for any $\mathcal{H}$–holomorphic map $v$ from a Riemann surface, possibly with boundary, $S$. In particular, finite $\omega$–energy of an $\mathcal{H}$–holomorphic map $v : \tilde{\Sigma} \to Z$ implies that $|\int_S v^* da|$ is also finite for any subdomain $S \subset \tilde{\Sigma}$.

We will see that in order to prove compactness of a family of $\mathcal{H}$–holomorphic maps it is necessary and sufficient that the parameters stay in a compact subset of $H^1(\Sigma; \mathbb{R})$. Compactness is to be understood with respect to the topology induced by the period map (on a basis of $H_1(\Sigma; \mathbb{Z})$). Here it is essential that the harmonic 1–form $\eta$ in question is defined as the harmonic part of $v^* \alpha$, and not of $v^* \alpha \circ j$ if one wishes to consider sequences of complex structures converging to the boundary of $\mathcal{M}_{g,m}$.

We wish to find a useful criterion to check if the periods of the harmonic parts of $v_n^* \alpha$ of a sequence of $\mathcal{H}$–holomorphic maps remains bounded. To this end, we associate to a canonical family of curves along which we will evaluate the integrals of $v^* \alpha$. It turns out that one–cylinder Strebel differentials are a convenient tool for this. We quickly outline the relevant portions of the theory, for more details we refer the interested reader to [Str84].

If $\Sigma$ has genus 0, then $H^1(\Sigma; \mathbb{R})$ is trivial, so every $\mathcal{H}$–holomorphic map from a domain of genus 0 is automatically $J$–holomorphic. Since the compactness properties of $J$–holomorphic maps is already well understood we will restrict our attention to domains of genus $\geq 1$. 

A holomorphic quadratic differential is a tensor, locally in complex coordinates \( z \), given as \( \phi(z)dz^2 \), where \( \phi(z) \) is a holomorphic function. \( \phi \) defines a singular Euclidean metric \( |\phi(z)||dz|^2 \) on \( \Sigma \) with finitely many singular points corresponding to the zeros of \( \phi \). \( \phi \) determines a pair of transverse measured foliations \( F_v(\phi) \) and \( F_h(\phi) \) called the horizontal and vertical foliations given by the preimages of the real and imaginary axes under \( \phi \), respectively. Near a singular point of \( \phi \) of order \( k \), \( \phi \) is given in local coordinates \( z \) as \( z^k dz^2 \). The union of the leaves both beginning and ending at a critical point is called the critical graph \( \Gamma \).

Given a non–separating simple closed curve \( \gamma \) on \( (\Sigma, j) \), there exists a holomorphic quadratic differential \( \phi \), called the Strebel differential, so that its horizontal foliation has closed leaves in the free homotopy class \([\gamma]\). Denote the set of such Strebel differential associated with \([\gamma]\) and \( j \) by \( \Phi(\gamma, j) \).

The complement \( \Sigma \setminus \Gamma \) of the critical graph \( \Gamma \) is a metric (w.r.t. \( |\phi(z)||dz|^2 \)) cylinder \( R \subset \Sigma \). If \( \Sigma \) has genus 1, then there is no critical graph and we use one regular leaf for \( \Gamma \). For simplicity we normalize \( \phi \) so that \( R = (0, 1) \times S^1 \) has height 1. For details see [Str84, Theorem 21.1]. Let \( \{\sigma_s(\phi) : S^1 \to R\}_{s \in (0, 1)} \) be a parametrization the closed leaves of the horizontal foliation \( F_h(\phi) \).

Armed with this definition we are ready to define the periods of \( H \)–holomorphic maps.

**Definition 2.7.** Let \( \gamma \) be a non–separating simple closed curve in \( \Sigma \) and \( \mu \) a 1–form on \( \Sigma \). Then the period of \( \mu \) along \( \gamma \) are

\[
P_{[\gamma]}(\mu) = \sup_{\phi \in \Phi(\gamma, j)} \sup_{s \in (0, 1)} \left| \int_{\sigma_s(\phi)} \mu \right|.
\]

For an \( H \)–holomorphic map \( v : \hat{\Sigma} \to \mathbb{Z} \) define the period

\[
P_{[\gamma]}(v) = P_{[\gamma]}(v^*\alpha).
\]

We say that a family of maps \((v_n, j_n)\) has bounded periods if there exists a collection of simple closed curves \( \gamma_i \) that form a basis of \( H_1(\Sigma, \mathbb{Z}) \) so that the associated periods \( P_{[\gamma_i]}(v_n) \) are uniformly bounded.

The definition of the periods along a curve \( \gamma \) is somewhat abstract. Intuitively, we think of the periods as the periods of the non–closed form \( v^*\alpha \). The one–cylinder Strebel differentials allow us to define the periods in a way that is invariant under the gauge action by diffeomorphisms, and independent of the choice of conformal metric on the domain. It turns out that the periods are essentially given by the periods of the harmonic part of the co–closed form \( v^*\alpha \) (see Lemma [3.2]).

Bounded periods are a necessary condition for any meaningful compactness result, as the a sequence of maps with unbounded periods has unbounded diameter in the image. The following theorem shows that the converse is also true, i.e. that bounded periods lead to compactness.
Theorem 2.8. Let \((Z, \alpha, \omega, J)\) be a stable Hamiltonian structure so that all periodic orbits are Morse or Morse-Bott. The space of smooth \(H\)-holomorphic maps into \(Z\) with uniformly bounded \(\omega\) and \(\alpha\)-energies with uniformly bounded periods has compact closure in the space of neck-nodal \(H\)-holomorphic maps.

In Section 4 we give examples of topological conditions that guarantee that the periods of families of maps are uniformly bounded, leading to compact moduli spaces. Another such condition guaranteeing bounded periods is given in a follow-up paper [vB09] when considering nicely embedded maps.

The condition on the periods is not vacuous as the following result shows.

Theorem 2.9. Let \((\hat{T}, \hat{i})\) be the twice-punctured standard torus and let \(S^3\) be equipped with the standard contact form and complex structure. There exists a smooth family \(v_t\) of \(H\)-holomorphic maps parametrized by \(\mathbb{R}\) so that a sequence \(v_{t_n}, t_n \in \mathbb{R}\) has a convergent subsequence if and only if \(t_n\) has a convergent subsequence in \(\mathbb{R}\) as \(n \to \infty\). The width of the image becomes unbounded as \(t_n \to \infty\).

The existence of non-compact smooth families of maps stands in stark contrast to the case of \(J\)-holomorphic maps and destroys any hope for a general compactness theorem for \(H\)-holomorphic maps.

3 Bounded Periods and Compactness

In this section we prove Theorem 2.8. In the first part, we show that the requirement of bounded periods of the maps leads to bounded periods of the harmonic 1-forms, and that every sequence with bounded periods possesses a subsequence that converges on compact subsets of the complement of the necks.

In the following subsection we investigate the convergence of the maps on the necks, where the requirement of bounded periods will lead to neck maps with minimal twist.

Putting these results together we then proceed to prove Theorem 2.8.

First we explain the metrics we are using on the domains. Following the construction in Section 4 of [IP04] we choose a family of metrics on the space of the domains \((\hat{\Sigma}, j), j \in \overline{\mathcal{M}}_{g,n}\) of \(H\)-holomorphic maps that comes from a metric on the universal curve on the thick part of the domain and is given by the cylindrical metric on (a cofinite subset of) the thin part. More specifically we adopt the definition of the weight function \(\rho : (\hat{\Sigma}, j) \to \mathbb{R}\) and work with the metric \(g = \rho^{-2} \cdot \tilde{g}\), where \(\tilde{g}\) is the restriction of a Riemannian metric on the universal curve. In particular, near a puncture we have local coordinates \(C_0 = [0, \infty) \times S^1 \subset \hat{\Sigma}\) with metric \(g = ds^2 + dt^2, s \in [0, \infty)\) and \(t \in S^1\), and \(\rho^2(s, t) = 8e^{-2s}\), and on a neck cylinder \(C_R = [-R, R] \times S^1 \subset \hat{\Sigma}\) we have the same flat metric and \(\rho^2(s, t) = 8e^{-2R\cosh(2s)}\). Given a sequence of maps and conformal structures \((v_m, j_m)\) we may adjust the space \(\overline{\mathcal{M}}_{g,n}\) every time we rescale (a finite number of times) by adding marked points as needed, and we will adjust the metrics accordingly without making explicit mention of this.
We refer to this metric as the \textit{cylindrical metric} and will use it throughout for estimates. For the final statement of the compactness theorem we will however use a different metric, namely the non–conformal metric where the cylinders of the thin part are rescaled along the height of the cylinders to $[-1, 1] \times S^1$, where the scaling function depends on the asymptotic approach of the maps to a closed characteristic. This metric extends to a smooth metric on the space of neck–nodal domains, and the convergence results in the compactness statement are to be understood with respect to this metric.

\textbf{Lemma 3.1.} Let $(v, j)$ be an $\mathcal{H}$–holomorphic map and let $E = E_{da}(v) + |T|$ be the sum of the $\omega$ energy of $v$ and $T$ be the of the sum of the absolute values of the periods of $v$ at the punctures. Then, for any non–trivial free homotopy-separating simple closed curve $\gamma$, the periods $P[\gamma](da \circ j) \leq E$, where $-da \circ j$ is the co-exact part of $v^*\alpha$, i.e. $v^*\alpha + da \circ j \in \mathcal{H}$.

\textit{Proof.} Let $\sigma_s$ be a foliation with compact leaves given by a Strebel differential $\phi$ with ring domain $R$ of height 1 associated with $\gamma$, and let $\Gamma$ denote it’s singular leaf. So $\sigma_s = \{s\} \times S^1 \subset R = (0, 1) \times S^1$.

$$\int_0^1 \left( \int_{\sigma_s} da \circ j \right) ds = \int_R da \circ j \wedge ds + \int_R da \wedge dt = \int_{\partial R} a dt = \int_\Gamma a dt - \int_\Gamma a dt = 0.$$  

Here we interpret integrals on a leaf containing a puncture in the sense of Cauchy. Then for any $s, \tilde{s} \in (0, 1)$

$$\left| \int_{\sigma_s} da \circ j - \int_{\sigma_{\tilde{s}}} da \circ j \right| \leq E_{da}(v) + |T|$$

where $|T|$ is the sum of the absolute value of the periods, so

$$\left| \int_{\sigma_s} da \circ j \right| \leq \int_R da \circ j \wedge ds + E_{da}(v) + |T| \leq E_{da}(v) + |T|. \quad (3.7)$$

Since this is true for any Strebel differential $\phi$ and any leaf $\sigma_s$ we conclude that

$$P[\gamma](da \circ j) = \sup_{\phi \in \Phi([\gamma], j)} \sup_{s \in (0, 1)} \left| \int_{\sigma_s} da \circ j \right| \leq E.$$

\hfill $\square$

We immediately obtain the following results.

\textbf{Lemma 3.2.} Let $(v_n, j_n)$ be a sequence of $\mathcal{H}$–holomorphic maps with fixed asymptotics and uniformly bounded $\omega$–energy. Then $v_n$ has uniformly bounded periods if and only if the periods of $\eta_n = v_n^*\alpha + da_n \circ j_n \in \mathcal{H}$ are uniformly bounded.

\textbf{Lemma 3.3.} Let $(v_n, j_n)$ be a sequence of $\mathcal{H}$–holomorphic maps so that the periods of $\eta_n = v_n^*\alpha + da_n \circ j_n \in \mathcal{H}$ are uniformly bounded. Then, after finitely many rescalings, there is a subsequence so that $dv_n$ is uniformly bounded.
Proof. By Lemma A.3 we see that $\eta_n$ is uniformly bounded. The result now follows from the standard bubbling off analysis.

**Definition 3.4.** For an $H$–holomorphic map $v : C_R \rightarrow Z$ from a cylinder $C_R = [-R, R] \times S^1$ we define the twist

$$S = \int_{S^1} \left( \int_{[-R,R] \times \{t\}} v^* \alpha \right) dt$$

and the average twist

$$\bar{S} = \frac{S}{2R} = \frac{1}{2R} \int_{S^1} \left( \int_{[-R,R] \times \{t\}} v^* \alpha \right) dt.$$

The twist and the average twist only depend on $\eta$ and are independent of $da$, since

$$\int_{S^1} \left( \int_{[-R,R] \times \{t\}} da \circ j \right) dt = \int_{[-R,R] \times S^1} a_t \ dt \ ds = \int_{[-R,R] \times \{t\}} \left( \int_{S^1} da \right) ds = 0. \tag{3.8}$$

In particular, the twist of a neck–region is uniformly bounded in terms of the periods of $\eta$ by Lemma A.2 and the relative twist is bounded in terms of $\|\eta\|_{\infty}$.

**Theorem 3.5.** Let $v_n : (\hat{\Sigma}, j_n) \rightarrow Z$ be a sequence of $H$–holomorphic maps with $j_n \in \mathcal{M}_{g,n}$ and uniformly bounded $E_\alpha$–energy and $E_\omega$–energy and uniformly bounded periods.

Then there exists a constant $C > 0$ and a subsequence so that with $v_n^* \alpha = \eta_n - da_n \circ j_n$

$$\|\eta_n\|_{\infty} < C, \quad \|da_n\|_{\infty} < C, \quad \text{and} \quad \|\pi_F dv_n\|_{\infty} < C$$

and the twist of all neck–maps is uniformly bounded.

**Proof.** By Lemma 3.2 we see that the periods of $\eta_n$ are uniformly bounded. Then Lemma 3.3 shows that we can, after finitely many rescalings, choose a subsequence so that $dv_n$ is uniformly bounded. By Lemma A.3 we see that the sup norm of $\eta_n$ is also uniformly bounded, and thus $da_n$ and $\pi_F dv_n$ must also be uniformly bounded in the sup norm. The twists of the neck maps are uniformly bounded by Lemma A.2.

3.1 Long Cylinders

To prove the compactness statement we need to understand the behavior of long $H$–holomorphic cylinders with small $\omega$–energy and uniformly bounded derivative, center action, and twist. We reduce the argument to the $J$–holomorphic case discussed in [HWZ02]. The main difference between the $J$ and $H$–holomorphic settings is that $H$–holomorphic maps may have non–zero (uniformly bounded) twist, whereas $J$–holomorphic maps have vanishing twist.

To reduce the question about $H$–holomorphic cylinders to $J$–holomorphic cylinders, let $\phi_t : Z \rightarrow Z$ denote the time–$t$ characteristic flow. The bundle map $d\phi_t : TZ \rightarrow \phi_t TZ$
is an isomorphism preserving the splitting $TZ = L \oplus F$. Given an $\mathcal{H}$–holomorphic map $v : C_R \longrightarrow Z$ with average twist $\tilde{S}$ let $\tilde{v} : C_R \longrightarrow Z$ be given by

$$\tilde{v}(s,t) = \phi_{-\tilde{S}} v(s,t)$$

and define the 1–parameter family of almost complex structures

$$J_s \in \text{End}(F), \quad J_s(z) = (\phi^*_s J)(z) = d\phi_{-\tilde{S}}(\phi_{\tilde{S}}(z)) \circ J(\phi_{\tilde{S}}(z)) \circ d\phi_{\tilde{S}}(z).$$

Let $\tilde{J} = \tilde{J}(s,t,z) : C_R \times Z \longrightarrow \text{End}(F)$ be the domain dependent almost complex structure defined by $\tilde{J}(s,t,z) = J_s(z)$. Then $\tilde{v}$ is $\tilde{J}$–holomorphic, that is $\tilde{v}^* \alpha = v^* \alpha - \tilde{S} ds$ is coexact and

$$\partial^F \tilde{v}(s,t) = \frac{1}{2} \{ \pi_F d\tilde{v}(s,t) + J_s(\tilde{v}(s,t)) \circ \pi_F d\tilde{v}(s,t) \circ j \} = \frac{1}{2} d\phi_{-\tilde{S}}(v(s,t)) \{ \pi_F dv(s,t) + J(v(s,t)) \circ \pi_F dv(s,t) \circ j \} = 0.$$

Now suppose the twists $S_n$ of a family of $\mathcal{H}$–holomorphic maps are uniformly bounded by $|S_n| < C$. Then for each $z \in Z$, the family $J_s(z)$, of almost complex structures varies in the compact set

$$\mathcal{J}(z,C) = \{ d\phi_{-\sigma}(\phi_{\sigma}(z)) \circ J(\phi_{\sigma}(z)) \circ d\phi_{\sigma}(z) \mid \sigma \in [-C/2,C/2] \}.$$

independent of how large $R$ is.

We note that all the results in [HWZ02] remain valid when the fixed almost complex structure in [HWZ02] is replaced by a domain–dependent almost complex structure varying in a compact set.

Before we proceed we need the following definition.

**Definition 3.6.** The period spectrum of $(Z, \alpha, \omega)$ is

$$\mathcal{P} = \{0\} \cup \{ T > 0 \mid \exists \text{ closed characteristic} \ x \text{ of period } T \}.$$

For any $E > 0$ the period gap w.r.t $E$ is the largest number $\hbar = h(E)$ so that

$$|T - T'| < \hbar \ \forall T, T' \in \mathcal{P} \text{ with } T, T' < E. \quad (3.9)$$

**Lemma 3.7.** Let $E_0 > 0$ so that all closed characteristics of period $T \leq E_0$ are non–degenerate. Let $h$ be the period gap between closed characteristics of period $\leq E_0$ as in Equation (3.9). Let $1 > \delta > 0$ be smaller than the lowest eigenvalue of any asymptotic operator governing the transverse approach to any closed characteristic of period $T \leq E_0$.

Fix $\gamma$ satisfying $0 < \gamma < h \leq E_0$ and $N \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists a constant $h > 0$ so that the following holds.
For every $R > h$ and every $\mathcal{H}$–holomorphic cylinder

$$v : C_R = [-R, R] \times S^1 \to Z$$

satisfying $E_\omega(v) \leq \gamma$ and gradient and twist bounded by $C$ and center action $T = \int_{\{0\} \times S^1} \eta$ satisfies $T \leq E_0 - \gamma$ there exist a characteristic flow line $x$ so that

$$d(v(s, t), x(\bar{\delta} s + T t)) \leq \varepsilon e^{-\delta(R-h)} \cosh(\delta s), \quad \forall (s, t) \in C_{R-h}$$

$$|D^\nu(dv(s, t) - \bar{\delta} ds - T dt)| \leq \varepsilon e^{-\delta(R-h)} \cosh(\delta s), \quad \forall (s, t) \in C_{R-h}, \quad \forall \nu, \ |\nu| \leq N.$$

**Proof.** Consider the $\tilde{J} = \tilde{J}(\bar{\delta})$–holomorphic map

$$\tilde{v}(s, t) = \phi_{-\bar{\delta}} v(s, t)$$

and note that $E_\omega(\tilde{v}) = E_\omega(v)$ and $\tilde{v}$ lifts to a finite $\alpha$–energy $\tilde{J}$–holomorphic map into the symplectization. Moreover the $\alpha$ energy is a–priori bounded in terms of the center action and the $\omega$–energy.

To prove the theorem we need to show that there exists $h > 0$ so that

$$d(\tilde{v}(s, t), x(T t)) \leq \varepsilon e^{-\delta(R-h)} \cosh(\delta s), \quad \forall (s, t) \in C_{R-h}$$

$$|D^\nu(d\tilde{v}(s, t) - T dt)| \leq \varepsilon e^{-\delta(R-h)} \cosh(\delta s), \quad \forall (s, t) \in C_{R-h}, \quad \forall \nu, \ |\nu| \leq N.$$

But this follows directly from Theorems 1.2 and 1.3 of [HWZ02].

### 3.2 Proof of Theorem 2.8

Before we proof Theorem 2.8 we observe some relations among the neck lengths.

**Lemma 3.8.** Let $j_n \to j_0 \in \mathcal{M}_{g,n}$ be a sequence of complex structures on $\Sigma$ and $\eta_n$ a sequence of harmonic 1–forms with converging period integrals. Then the twists $S^n_i$ of $\eta_n$ on each neck $N_i$ converge to real numbers $S_i$, and there exists a subsequence so that whenever

$$\sum_{i \in I} [N_i] = 0 \in H_1(\Sigma; \mathbb{Z})$$

for some index set $I$ there exist non–negative real numbers $l_i$ with

$$\sum_{i \in I} l_i = 1$$

so that

$$\sum_{i \in I} l_i \cdot S_i = 0.$$

In particular, homologically trivial necks have vanishing twist.

**Proof.** Consider $\eta_n$ on the necks $N_i = [-R^n_i, R^n_i] \times S^1$. With $N^0_i = \{0\} \times S^1$ the center loops of each neck. The periods of $\eta_n \circ j$ on $N^0_i$ satisfy

$$\check{S}^n_i = \int_{N^0_i} \eta_n \circ j.$$

Let $I$ be an index set so that

$$\sum_{i \in I} [N_i] = 0 \in H_1(\Sigma; \mathbb{Z})$$

and define, for $j \in I$

$$l^n_j = \frac{R^n_j}{R^n_i} \in (0, 1], \quad R^n_i = \left(\sum_{i \in I} \frac{1}{R^n_i}\right).$$
The twist $S_i^n$ of $\eta_n$ on the neck = $N_i$ satisfy,

$$\sum_{i \in I} l_i^n \cdot S_i^n = 2R^n_I \sum_{j \in I} \tilde{S}_j^n = 0.$$ 

Set $S_i = \lim_{n \to -\infty} S_i^n$, which exists by assumption, and choose a subsequence so that $l_i = \lim_{n \to -\infty} l_i^n \in [0, 1]$ exists. For each $n$ we have

$$\sum_{i \in I} l_i^n = \sum_{i \in I} \frac{R^n_I}{R^n_J} = 1$$

so

$$\sum_{i \in I} l_i = 1, \quad \text{and} \quad \sum_{i \in I} l_i \cdot S_i = 0.$$ 

Since there are only finitely many index sets $I$ so that $\sum_{i \in I} |N_i| = 0$ there exists a subsequence so that the Lemma holds true. 

\[\square\]

**Proof of Theorem 2.8.** Let $v_n : \hat{\Sigma} \to Z$ be a sequence of smooth $\mathcal{H}$–holomorphic maps with bounded $\omega$ and $\alpha$–energies and periods bounded by $C$. We need to show that there exists a subsequence that converges to a neck–nodal $\mathcal{H}$–holomorphic map.

By Theorem 3.5 we may pass to a subsequence so that $j_n \to j_0$ and $|dv_n|, |\eta_n|$ and the relative twists of the neck maps are uniformly bounded. By elliptic regularity and Arzela–Ascoli we extract a convergent (in $C^\infty$) on the thick part of $(\Sigma, j_0)$.

By Lemma 3.8 we may extract a subsequence so that the twists of $v_n$ on the necks are bounded, convergent, and the twists are minimal.

By our assumption we have that the center action of each neck is bounded by some constant $E \leq E_0$, so by Lemma 3.7 we see that there exists constants $C, h, \delta > 0$ and a characteristic flow line $x$ so that $d(v_n(s, t) - x(s, t)) < Ce^{-\delta(R_n - h)} \cosh(\delta s)$ and $|dv_n(s, t) - (\tilde{S} ds + T dt) \otimes R| < Ce^{-\delta(R_n - h)} \cosh(\delta s)$ for $(s, t) \in C_R_{n-h} = [-R_n + h, R_n - h] \times S^1$. Since the gradient of $v_n$ is uniformly bounded on $C_{R_n}$ we may assume, by adjusting the constant $C$ in the above formulas, that $h = 0$.

Set $\mu = \delta/2$. We may split a neck region $C_{R_n} = [-R_n, R_n] \times S^1$ into regions

$$A_n^- = [-R_n, -R_n + \ln(R_n)] \times S^1$$
$$B_N = [-R_n + \ln(R_n), R_n - \ln(R_n)] \times S^1$$
$$A_n^+ = [R_n - \ln(R_n), R_n] \times S^1.$$ 

We similarly split up the cylinder $C_1 = [-1, 1] \times S^1$ into regions

$$\tilde{A}^+ = [-1, -\frac{1}{2}] \times S^1, \quad N = [-1/2, 1/2] \times S^1, \quad \tilde{A}^+ = [\frac{1}{2}, 1] \times S^1$$

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and define the piecewise diffeomorphism \( \phi_n : C_1 \rightarrow C_{R_n} \) via the diffeomorphisms
\[
\phi_n^- : \tilde{A}^- \rightarrow A_n^-, \quad \phi_n^-(r, t) = (-R_n - \mu^{-1} \ln[1 - 2(1 - R_n^{-\mu})(1 + r)], t)
\]
\[
\phi_n^N : N \rightarrow B_n, \quad \phi_n^N(r, t) = (2(R_n - \ln(R_n))r, t)
\]
\[
\phi_n^+ : \tilde{A}^+ \rightarrow A_n^+, \quad \phi_n^+(r, t) = (R_n + \mu^{-1} \ln[1 - 2(1 - R_n^{-\mu})(1 - r)], t).
\]

Define the map \( \tilde{v}_n = v_n \circ \phi_n : C_1 \rightarrow Z \) and consider the restrictions to the subdomains \( \tilde{v}_n^\pm = \tilde{v}_n|_{\tilde{A}^\pm} \) and \( \tilde{v}_n^N = \tilde{v}_n|_N \). Then, with \((r, t)\) coordinates on \( \tilde{A} \) and remembering that \( \delta/\mu = 2 \)
\[
|((d\tilde{v}_n^\pm(r, t) - T dt)(\partial_t))| \leq C e^{-\delta R_n} \cosh(\pm \delta R_n \pm 2 \ln[1 - 2(1 - R_n^{-\mu})(1 \mp r)]) \leq C[1 - 2(1 - R_n^{-\mu})(1 \mp r)]^2
\]
which is uniformly bounded by \( C \) and converges to zero as \( r \rightarrow \pm \frac{1}{2} \) and \( n \rightarrow \infty \). Similarly
\[
|((dv_n^\pm(r, t))(\partial_r))| = |dv_n \circ d\phi_n^\pm(r, t)(\partial_r)| \leq C[1 - 2(1 - R_n^{-\mu})(1 \mp r)]^2 \frac{2(1 - R_n^{-\mu})}{\mu 1 - 2(1 - R_n^{-\mu})(1 \mp r)} = \frac{2C}{\mu}[1 - 2(1 - R_n^{-\mu})(1 \mp r)]
\]
which is also uniformly bounded and converges to zero for \( r \rightarrow \pm \frac{1}{2} \) and \( n \rightarrow \infty \). So there exists a reparametrization and a subsequence so that \( v \) converges to an \( \mathcal{H} \)-holomorphic map from the smooth part \( \Sigma_0 \) of the neck–nodal domain \((\hat{\Sigma}, j_0)\). Note that since the average twist on each neck converges to zero uniformly we have that \( \int_{\{s\} \times S^1} v^* \alpha \circ j \rightarrow 0 \) uniformly, so in the limit condition 1.4 of Definition 1.2 is also satisfied along the necks which are now punctures for \( \Sigma_0 \).

Similarly, we compute
\[
|d\tilde{v}_n^N(r, t) - (T dt + S ds)(\partial_t)| \leq \frac{C}{R_n}
\]
\[
|d\tilde{v}_n^N(r, t) - (T dt + S ds)(\partial_r)| \leq C \frac{R_n}{2(R_n - \ln(R_n))} + |S - 2(R_n - \ln(R_n))\tilde{S}|
\]
\[
\leq C \left( 1 + \frac{\ln(R_n)}{R_n - \ln(R_n)} \right) + |S| \left| 1 - \frac{R_n - \ln(R_n)}{R_n} \right|
\]
\[
\leq \left( \frac{C}{2} + |S| \right) \frac{\ln(R_n)}{R_n}
\]
which converges to zero uniformly, so a subsequence of \( \tilde{v}_n^N \) converges uniformly to a neck map.

Using the diffeomorphisms \( \phi_n \) we reinterpret our sequence of maps as maps from a fixed reference surface by gluing the thick part to the cylinders \( C_1 \). After passing to a subsequence the resulting domains with their induced complex structure converge to a neck–nodal domain, and the resulting maps converge uniformly in \( C^0 \) to a minimal twist neck–nodal \( \mathcal{H} \)-holomorphic map \( v_0 \).
Standard arguments show that the canonical lifts to the symplectization also converge, so $v_0$ is exact.

In fact, it is not hard to extend these results to make the convergence piecewise smooth, so that the convergence is $C^\infty$ on the neck regions $N$ and the smooth part $\Sigma_0$ of the domain.

4 $S^1$–Invariant Stable Hamiltonian Manifolds

In this section we consider circle–invariant stable Hamiltonian manifolds of any (odd) dimension. We give topological conditions under which the periods of families of $\mathcal{H}$–holomorphic maps are always uniformly bounded, and thus obtain compact moduli spaces of maps. $\mathcal{H}$–holomorphic maps into circle–invariant manifolds are needed for applications to folded holomorphic maps \cite{vB07}. They also allow for the explicit construction of examples highlighting the features of the compactness theorem, which we give at the end of this section, as well as the counterexample to a general compactness theorem which is constructed in Section 5.

**Definition 4.1.** An stable Hamiltonian manifold $(Z,\alpha,\omega,J)$ is called $S^1$–invariant if the characteristic flow defines a free $S^1$–action that preserves $J$.

Any circle–invariant manifold is an $S^1$–bundle over a symplectic manifold $(V,\omega_V)$ with projection $\pi_V : Z \to V$ so that $\omega = \pi_V^*\omega_V$. The almost complex structure $J$ descends to an $\omega_V$–compatible almost complex structure $J_V$ on $V$, so $J = \pi_V^*J_V$. For simplicity we will assume that $d\alpha = C \cdot \omega$, where $C = \langle c_1(Z),[V]\rangle/vol(V)$, where $c_1(Z)$ is the first Chern class of the bundle and $vol(V)$ is the volume with respect to $\omega_V$. We can always arrange for $\omega$ to be of this form. For details see Section 1 in \cite{vB07}.

There is a natural action of $\text{Map}(\hat{\Sigma},S^1)$ on the space of smooth maps into an $S^1$–invariant stable Hamiltonian manifold $Z$ given by

$$\text{Map}(\hat{\Sigma},S^1) \times \text{Map}(\hat{\Sigma},Z) \to \text{Map}(\hat{\Sigma},Z) \quad (f,v) \mapsto f * v$$

where

$$(f * v)(z) = \phi_{f(z)}(v(z))$$

and $\phi_t$ is the time–$t$ flow of the characteristic vector field giving the circle action on $Z$. Since the circle action preserves $J$, the action of of $\text{Map}(\hat{\Sigma},S^1)$ leaves Equation \eqref{eq:1.2} invariant. Moreover, $(f * v)^*\alpha = df + v^*\alpha$, so Equation \eqref{eq:1.3} is invariant under the action by $f \in \text{Map}(\hat{\Sigma},S^1)$ if and only if $f$ is harmonic on $\Sigma$. Let $H \subset \text{Map}(\hat{\Sigma},S^1)$ denote the space of harmonic circle–valued functions on $\Sigma$. $H$ carries a circle action by adding a constant, and $H/S^1 = H^1(\Sigma;\mathbb{Z})$. This space is not compact and the counter example Theorem \ref{thm:2.9} builds on this. The intuition behind the topological conditions in Theorem \ref{thm:4.2} is that they allow us to conclude that the action of $H/S^1$ is free on the relative homotopy classes of $\mathcal{H}$–holomorphic maps.
Theorem 4.2. Let $u_n$ be a sequence of $\mathcal{H}$–holomorphic maps into an $S^1$–invariant almost contact manifold asymptotic to the same closed characteristics at the punctures and in the same relative homotopy class. Let $\pi: Z \longrightarrow V$ be the projection to the base of $Z$. Assume that one of the following holds:

(i) $\pi_2(V)$ is trivial.

(ii) $V = S^2$ and $\hat{\Sigma}$ is the once–punctured torus and the image of no $u_n$ intersects the limit cycle and they are homotopic through maps that do not intersect the limit cycle.

(iii) The bundle $Z$ is trivial.

Then $u_n$ has a subsequence converging to a neck–nodal map.

Proof. By Theorem 2.8 it suffices to show that the period integrals $P_\gamma(u_n)$ are uniformly bounded for any non–separating simple closed curve $\gamma$. Let $\phi_n$ be any sequence of Strebel differentials associated with the free homotopy class $[\gamma]$ and $(\Sigma, j_n)$, normalized to have sup norm 1 (in the cylindrical metric). Choose a subsequence so that $\phi_n$ converge with all derivatives to $\phi_0$. In particular the associated horizontal foliations converge. W.l.o.g. assume by possibly choosing another subsequence that $\gamma$ is a closed leaf of $\phi_1$ and that each foliation associated with $\phi_n$ has a closed leaf $\gamma_n$ that is close to $\gamma$ in the sense that

$$\left|\int_{\gamma_n} u_n^* \alpha - \int_{\gamma_n} u_n^* \alpha\right| < 1.$$ 

Thus it suffices to prove that

$$P_n = \int_{\gamma} u_n^* \alpha$$

is uniformly bounded. We prove this by contradiction. Assume this was not the case, so there exists a subsequence that $|P_n| > n$.

Let $\bar{u}_n = \pi \circ u_n : \Sigma \longrightarrow V$ denote the projection of the maps into $V$, which extend naturally over the punctures. These maps are $J_V$–holomorphic, and by the usual Gromov compactness we may choose a further subsequence so that $\bar{u}_n$ converge to a map $\bar{u}_0 : \Sigma \longrightarrow V$. For $N$ large enough and any $n, m \geq N$ there exists a unique vector field $\xi_{n,m} \in \bar{u}_n^* TV$ so that

$$\bar{u}_m(z) = \exp_{\bar{u}_n(z)}(\xi_{n,m}(z))$$

satisfying $\|\xi_{n,m}\|_\infty \longrightarrow 0$ and $\int_{\Sigma} \|\xi_{n,m}\|^2 \longrightarrow 0$ as $n, m \longrightarrow 0$.

Let $H_{n,m} : [0, 1] \longrightarrow Z$ be a relative homotopy between $u_n$ and $u_m$ and consider the “flux” given by the difference in period integrals

$$\mathcal{F}(v_m, v_n)(\gamma) = P_m - P_n = \int_{\gamma} u_m^* \alpha - u_n^* \alpha = \int_{[0,1] \times \gamma} H^* d\alpha = \int_{[0,1] \times \gamma} (\pi \circ H)^* c_\nu \omega_V$$

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and let \( G_{n,m} : [0, 1] \times \hat{\Sigma} \longrightarrow V \) be the homotopy, relative to the punctures, between \( \tilde{u}_n \) and \( \tilde{u}_m \) given by

\[
G_{n,m}(t, z) = \exp_{\tilde{u}_m}(t \xi_{m,n}(z)).
\]

Then the composition \( K = (\pi \circ H) \ast G \) is a homotopy between \( \tilde{u}_n \) and \( \tilde{u}_n \), and we choose another subsequence by dropping finitely many terms so that \( |\int_{\Sigma} G^* \omega_V| < 1 \), and thus

\[
|P_n| \leq |P_1| + |P_n - P_1| \leq |P_1| + 1 + \left| \int_{S^1 \times \gamma} K^* c_{v} \omega_V \right|. \tag{4.10}
\]

We finish the proof by showing that the integral in Equation (4.10) vanishes and thus the periods are uniformly bounded.

In the case (i) consider the fibration

\[
X = Map\left( (\hat{\Sigma}, \partial \hat{\Sigma}), (V, \{p_i\}) \right) \longrightarrow Map(\hat{\Sigma}, V) \longrightarrow Map(\partial \hat{\Sigma}, \{p_i\}).
\]

This gives rise to the long exact sequence in homotopy

\[
0 \longrightarrow \pi_1(X) \longrightarrow \pi_1(Map(\hat{\Sigma}, V)) = \pi_1(Map(\bigvee S^1, V)) = \bigoplus \pi_2(V) \longrightarrow 0.
\]

Thus \( \pi_1(X) = \bigoplus \pi_2(V) \) so in the case (i) the loop of maps \( K \) is contractible and the integral in Equation (4.10) vanishes.

In the case (ii) note that \( \omega_V = c \cdot PD[pt] \). For the once–punctured torus we may choose the Strebel cylinders so that the puncture is on the boundary of the cylinder. To compute the integral in Equation (4.10) we take the curve \( \gamma \) to be the central curve in the Strebel cylinder. For a family of degree \( d \) maps mapping the puncture \( p \) to the closed characteristic over \( \infty \subset S^2 \) and not intersecting the limit cycle so that \( u^{-1}(\infty) = p \) we have

\[
K^* \omega_V = c K^* PD[pt] = cPD[u^{-1}(\infty)] = c PD[S^1 \times d \cdot \{p\}]
\]

and thus

\[
\int_{S^1 \times \gamma} K^* c_{v} \omega_V = c c_v \#(S^1 \times \gamma_t, S^1 \times d[p]) = c c_v \#(\gamma, d[p]) = 0
\]

since the curve \( \gamma \) does not intersect \( p \) by construction.

In the case (iii) the integral vanishes since \( c_v = 0 \).

The obvious case absent from the theorem is when \( V = S^2 \) and we consider more general maps than case (ii). This will be addressed in Section 4 and it turns out that in general the space of maps is not compact in this case. The key to proving compactness of the space of maps was to show that the “flux” integrals are uniformly bounded. This can either be achieved by topological restrictions on the target as in cases (i) or (iii) or by assumptions on the space of domains as in (ii).
At this point it is convenient to give some explicit examples to illuminate aspects of the compactness theorem.

First we give an example of a sequence of maps converging to a neck–nodal map where the neck converges to a characteristic flow line.

**Example 4.3.** Consider $Z = S^1 \times S^2$, the trivial bundle with an $S^1$–invariant structure. Maps $v : \dot{\Sigma} \longrightarrow Z$ are $\mathcal{H}$–holomorphic if and only if the projection $\tilde{v} = \pi_{S^2}v$ is $J$–holomorphic and the projection $\theta = \pi_{S^1}v$ is harmonic. Let $(\tilde{v}_t, j_t) : T^2 \longrightarrow S^2$ be a family of $J$–holomorphic maps so that the domains converge to a once–pinched torus, pinched along a simple closed curve in the homology class $A \in H_1(T^2; \mathbb{Z})$ with a dual class $B$. Then $(v_t, j_t) : S^2 \longrightarrow Z$ $v_t = (\theta = 0, \tilde{v}_t)$ is $\mathcal{H}$–holomorphic (and in fact also $J$–holomorphic). Mark a point $p$ in $T^2$ away from the node.

Consider the family of $j_t$ harmonic circle–valued functions $\theta_t : T^2 \longrightarrow S^1$ with $\theta_t(p) = 0$ and periods 0 along $A$ and 1 along $B$. Set $u_t = \theta_t \ast v_t = (\theta_t, \tilde{v}_t)$. Then $u_t$ is $\mathcal{H}$–holomorphic and the image neck domain converges to a characteristic flow line of length 1 over the image of the node $q \in S^2$. More precisely, the neck map converges to a map $x : [-1/2, 1/2] \times S^1 \longrightarrow Z$, $x(s,t) = (s,q)$.

![Map with twist at node](image.png)

**Figure 3:** Map with twist at node.

Next we show that “breaking of trajectories” can happen in a compact subset in the symplectization.

**Example 4.4.** With the same notation as in the previous example, consider a family of $j_t$ harmonic circle–valued functions $\theta_t : T^2 \longrightarrow S^1$ with $\theta_t(p) = 0$ and periods 1 along $A$ and 0 along $B$. Set $u_t = \theta_t \ast v_t = (\theta_t, \tilde{v}_t)$. Then $u_t$ is $\mathcal{H}$–holomorphic and the neck domain converges to a cylinder over a closed characteristic over the image of the node $q \in S^2$. More precisely, the neck map converges to a map $x : [-1/2, 1/2] \times S^1 \longrightarrow Z$, $x(s,t) = x_q(t) = (t,q)$, where $x_q$ is the parametrized closed characteristic over $q$. The lift to the symplectization sits in a constant $\mathbb{R}$–slice.

To see how the minimality of the twist comes into the compactness statement, consider the following example.
Example 4.5. Let $\tilde{v} : T^2 \to S^2$ be a $J$–holomorphic map with two homologous neck regions $N_1$ and $N_2$ that is pinched along the two necks and lift this to a map $v : T^2 \to \mathbb{Z} = S^1 \times S^2$ via $v = (0, \tilde{v})$.

Orient the necks so that $[N_1] + [N_2] = 0 \in H_1(\Sigma; \mathbb{Z})$. The periods of $v$ are zero, and the twists $S_1$ and $S_2$ are zero. However, let $\chi$ be the function on $\Sigma$ that equals 1 on one component of the normalization of $\Sigma$ and 0 on the other. Then $v_t = (t\chi, \tilde{v})$ is also an $\mathcal{H}$–holomorphic map with twists $S_1^t = S_2^t$. So the periods still vanish since $S_1^t - S_2^t = 0$ for any $t \in \mathbb{R}$. This gives a non–compact family of maps with bounded periods, but $v_t$ does not have minimal twist for $t \neq 0$, so these are not minimal twist neck–nodal maps and cannot arise as limits of smooth $\mathcal{H}$–holomorphic maps unless $t = 0$.

We now consider an example where the lift to the symplectization develops an “infinite funnel”, i.e. a node is forming on one neck region (zero period), but the node is pushed off to infinity in the symplectization. These following two examples motivate why we have opted to focus our discussion on $\mathcal{H}$–holomorphic maps into the contact manifold instead of their lifts to the symplectization.

Example 4.6. Let $Z = S^3$ be the Hopf fibration over $S^2$ with the canonical contact from $\alpha$. Let $(v_t, j_t) : \mathbb{T}^2 \to \mathbb{Z}$ be a family of $J$–holomorphic maps from the once–punctured torus so that the lifts to the symplectization $\hat{v}_t = (a_t, v_t)$ converge to a building of height two where the torus is pinched along two homologous neck regions $N_1$ and $N_2$, oriented as the boundary of the component of the smooth part of the limit domain that does not contain the puncture. The necks converge to a cylinder over a closed characteristic of positive integer period $m_1$ and $m_2$ on $N_1$ and $N_2$.

Let $A = [N_1] = -[N_2] \in H_1(T^2; \mathbb{Z})$ and $B$ a dual class. Consider a family of $j_t$–harmonic circle–valued functions $f_t : T^2 \to S^1$ with periods $-m_1$ on $A$ and 0 on $B$, and let $u_t = f_t \ast v_t$ be the corresponding family of $\mathcal{H}$–holomorphic maps that lift to the symplectization as $\hat{u}_t = (a_t, u_t)$. Then $a_t$ is unchanged from before, so the limit is a level 2 curve. But the neck $N_1$ now has period 0 (and $N_2$ has period $m_1 + m_2 > 0$). So the neck $N_1$ converges to an “infinite funnel” in the symplectization.

This example can easily be modified to include non–vanishing twist at the funnel.
Example 4.7. With the same notation as in the previous example choose a function \( f_t : T^2 \to S^1 \) with periods \(-m_1\) on \( A \) and 1 on \( B \), and let \( u_t = f_t \ast v_t \) be the corresponding family of \( \mathcal{H} \)-holomorphic maps. Then the limit \( u_0 \) has non-zero twist \( S_1 = \frac{m_1}{m_1 + m_2} > 0 \) at the “infinite funnel”, and twist \( S_2 = \frac{m_2}{m_1 + m_2} \) at the neck \( N_2 \) converging to a closed characteristic.

5 Non-Compactness Results

In this section we prove Theorem 2.9. Let \( \pi : Z \to S^2 \) be a principle \( S^1 \)-bundle with connection 1-form \( \alpha \) and curvature form \( \omega = d\alpha > 0 \). Choose an \( S^1 \)-invariant almost complex structure \( J \), so \((Z, \alpha, \omega, J)\) is a circle-invariant stable Hamiltonian structure (see Definition 4.1) with bundle projection \( \pi : Z \to V = S^2 \) and first Chern class \( c_1(Z) \geq 0 \).

Consider the projection \( u = \pi_V v \) of such maps to the base \( S^2 \). These are degree \( d \) holomorphic maps with the punctures \( p_i \) mapping to \( \infty \in S^2 \). So \( u \) is a rational function with poles at \( p_i \) (with points repeated according to the multiplicity \( m_i \)). Up to \( C^* \) action on
$S^2$, degree $d$ rational functions with these prescribed poles are in 1–1 correspondence with divisors $D = \sum_{i=1}^{d} (p_i - q_i)$ of degree 0 on the domain with vanishing abelian sum

$$\mu(D) = \left( \sum_{i=1}^{d} \int_{q_i}^{p_i} \omega, \ldots, \sum_{i=1}^{d} \int_{q_i}^{p_i} \omega_g \right)$$

where $\omega_1, \ldots, \omega_g$ is a normalized basis of holomorphic 1–forms on $(\Sigma_g, j)$ (see e.g. [GH78] p 235).

Let $D_t = m_1 p_1(t) + \sum_{i=m_1+1}^{d} p_i - \sum_{i=1}^{d} q_i(t)$, $t \in \mathbb{R}/\mathbb{Z}$, be a smooth family of divisors on $\Sigma_g$ with $\mu(D_t) = 0$ so that the loop $\{p_1(t)\} \subset \Sigma$ is non–trivial in homology, i.e. there exists a loop $\gamma$ so that $\#(\gamma, \{p_1(t)\}) = 1$.

To give an explicit example, consider the flat torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with basis of holomorphic 1–forms $\omega_1 = dz = dx + idy$ and let $p_1(t) = (t, 0)$, $q_1(t) = (t + \frac{1}{2}, 0)$, $p_2 = (\frac{1}{2}, \frac{1}{2})$, $q_2 = (0, \frac{1}{2})$. Then, with $D_t = p_1(t) - q_1(t) + q_2 - q_2$,

$$\mu(D_t) = \int_{q_1(t)}^{p_1(t)} dz + \int_{q_2(t)}^{p_2(t)} dz = \int_{q_1(t)}^{p_1(t)} dx + \int_{q_2(t)}^{p_2(t)} dx + i \left( \int_{q_1(t)}^{p_1(t)} dy + \int_{q_2(t)}^{p_2(t)} dy \right)$$

$$= -\frac{1}{2} + \frac{1}{2} + 0i = 0$$

and $\{p_1(t)\}$ intersects the loop $\{y = 0\}$ once.

Now consider a corresponding loop of holomorphic maps $\tilde{u} : \mathbb{R}/\mathbb{Z} \times \Sigma_g \rightarrow V$, where $u_t = \tilde{u}(t, \cdot)$ is a rational function corresponding to $D_t$. We are interested in the flux

$$\mathcal{F}(u_0, u_1)[\gamma] = \int_{S^1 \times \gamma} u^* \omega = c \int_{S^1 \times \gamma} u^* PD[pt] = c \#(S^1 \times \gamma, |u^{-1}(\infty)|)$$

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Lift the family of maps \( \tilde{u} \) to a family of \( \mathcal{H} \)--holomorphic maps \( \tilde{v} : \mathbb{R} \times \hat{\Sigma}_g \rightarrow Z \) with the prescribed asymptotics at the punctures \( p_i \). Choose a sequence \( t_i \in \mathbb{R} \) and consider the sequence \( v_t = \tilde{v}(t, \cdot) \) of \( \mathcal{H} \)--holomorphic maps in the 1--parameter family \( \tilde{v} \). Then for the loop \( \gamma \) in \( \hat{\Sigma} \) the flux

\[
\mathcal{F}(v_1, v_t)(\gamma) = \int_{\gamma} (v^*_{t_1} - v^*_1) \alpha
\]

is of the order of \( m_1(t_i-t_1) \), more precisely it lies in the interval \( [m_1(t_i-t_1)-M, m_1(t_i-t_1)+M] \) where \( M = \sup_{t \in [0,1]} \mathcal{F}(v(0, \cdot), v(t, \cdot))(\gamma) \). Thus there exists a subsequence with bounded flux if and only if there exists a subsequence with bounded \( t_i \), i.e. if there exists a convergent subsequence of \( t_i \in \mathbb{R} \).

In the above example the harmonic 1--form becomes unbounded in the sup norm w.r.t. the cylindrical metric. But this is not the essential condition that keeps the space of \( \mathcal{H} \)--holomorphic maps in a fixed relative homotopy class from being compactified as the following example shows, where the harmonic 1--form \( |\eta| \) is bounded by \( |\alpha| \) in the cylindrical metric.

Consider maps from the once--punctured torus to \( S^3 \) with the standard structures and let \( v_n \) be a sequence of \( J \)--holomorphic maps so that the conformal structure is biholomorphic to

\[
S^1 \times n \cdot S^1 = \{(e^{2\pi i s}, e^{2\pi i t/n})\},
\]

so the complex structure \( j\partial_s = \partial_t \) degenerates, and assume that the sequence converges to a \( J \)--holomorphic map \( v_\infty \) from a bubble domain given by two spheres joined at two points, i.e. the complex structure pinched along two circles in the class of \( S^1 \times \{pt\} \). Further assume that the two nodes are non--trivial, i.e. the wrap a closed characteristic of period \( \tau \neq 0 \). In particular, \( v^* \alpha \approx \tau ds \) on the necks. The cylindrical metric is given by \( g_n = ds^2 + dt^2 \).

We construct new maps \( \hat{v}_n = f_n \ast v_n \), where \( f_n(s, t) = e^{2\pi i t} \) where \( (s, t) \in S^1 \times n \cdot S^1 \). Then \( df_n = dt \), so it is bounded in the cylindrical metric (and also bounded by \( da = v^* \alpha \) if \( \tau \) is large enough).

\section{Harmonic 1--Forms}

Here we establish some properties of harmonic 1--forms that we use.

\textbf{Lemma A.1.} \textit{Any harmonic 1--form} \( \eta \) \textit{on} \( C_R = [-R, R] \times S^1 \) \textit{satisfies}

\[
|\eta(s, t) - (\tilde{S} ds + T dt)| \leq \rho(s) \|\eta - (\tilde{S} ds + T dt)\|_{\partial C_{R, \infty}}.
\]

\textit{where} \( T \) \textit{is the period} \textit{and} \( \tilde{S} \) \textit{is the average twist.}

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Proof. Any harmonic 1–form \( \eta \) on \( C_R \) is of the form
\[
\eta(s,t) = (\tilde{S} \, ds + T \, dt) + f(s,t) \, ds + g(s,t) \, dt
\]
where \( f \) and \( g \) are harmonic function with vanishing average. Expanding \( f \) and \( g \) into Fourier series in the \( t \)–variable shows that \( f \) and \( g \) are sums of terms consisting of products of \( \sinh(ns) \) and \( \cosh(nt) \) with \( \sin(nt) \) and \( \cos(nt) \) plus a linear term in \( s \). Remembering that \( d\eta = d(\eta \circ j) = 0 \) we see that the term that is linear in \( s \) must in fact vanish. Since \( fds + gdt \) have vanishing average twist and center action, only terms with \( n > 0 \) appear in the sums. Now note that
\[
cosh(ns) \leq 2 \cosh(s) \cosh((n-1)s), \quad \text{and} \quad |\sinh(ns)| \leq 2 \cosh(s) |\sin((n-1)s)|
\]
and thus
\[
\frac{\cosh(ns)}{\cosh(nR)} \leq e^{-R} \cosh(s) \leq \rho(s), \quad \frac{\cosh(ns)}{\cosh(nR)} \leq e^{-R} \cosh(s) \leq \rho(s),
\]
remembering that \( \rho^2(s) = 8e^{-2R} \cosh(2s) \) and thus \( e^{-R} \cosh(s) \leq \rho(s) \).

Using the Fourier expansions of \( f \) we obtain
\[
|f(s,t)| \leq \rho(s) \|f(s,t)\|_{\partial C_R, \infty}
\]
with an analogous estimate holding for \( g \) which gives the desired result. \( \square \)

Lemma A.2. Fix a basis \( \{ \gamma_i \}_{i=1,...,2g} \) of \( H_1(\Sigma; \mathbb{Z}) \) and let \( C_R = [-R,R] \times S^1 \) with \( R > 4 \) be a neck cylinder in \( \Sigma \).

There exists a constant \( c > 0 \) so that for any \( \eta \) be a harmonic 1–form on \( \Sigma \) with periods on \( \{ \gamma_i \} \) bounded by \( C \) the twist \( S \) of \( \eta \) on \( C_R \) satisfies \( S_n < cC \).

Proof. Using a change of symplectic basis we may assume that \( \gamma_1 \) is in the class of \( C_R \) and \( \gamma = \gamma_{g+1} \) is its dual basis element, and that all periods with respect to this new basis are bounded by \( C/c2 \). Assume without loss of generality that \( \eta \) has vanishing periods on all of the these basis elements except \( \gamma \).

Let \( \beta : [-R, R] \rightarrow [0,1] \) be a bump function supported on the interior of the interval that equals 1 at all points with distance at least 1 from the boundary with integral \( \tilde{C} \). Note that \( 2(R_n - 2) \leq \tilde{C} \leq 2R_n \). Now consider the closed 1–form \( \nu(s,t) = \frac{C}{\tilde{C}} \beta(s) ds \) on \( C_R \). It has the same periods as \( \eta \), and \( L^2 \) norm
\[
\langle \nu, \nu \rangle \leq \frac{C^2}{\tilde{C}^2} 2R \leq C^2 \frac{R}{(R-2)^2} \leq 2 \frac{C^2}{R}.
\]
On the other hand \( \eta - \tilde{S} ds + \mu \), where \( \mu \) has average 0 on \( C_R \). Then
\[
\langle \eta, \eta \rangle = \langle \mu, \mu \rangle + \tilde{S}^2 2R \geq \frac{S^2}{2R}.
\]
Since \( \eta \) minimizes the \( L^2 \) norm among all closed forms with the same periods, we conclude that the twist \( S < cC \). \( \square \)
Lemma A.3. Let \( \{\gamma_i\} \) be a basis of \( H_1(\Sigma; \mathbb{Z}) \). There exists a constant \( C > 0 \), independent on the complex structure \( j \) on \( \Sigma \) so that for any harmonic 1–form \( \eta \) on \( (\Sigma, j) \)

\[
|\eta|_\infty \leq C |P(\eta)|
\]

where \( P(\eta) \) is the period map w.r.t. \( \{\gamma_i\} \) and the \( \|\cdot\|_\infty \) is taken with respect to the cylindrical metric.

Proof. Since both sides are linear under scaling, it suffices to prove the statement for harmonic 1–forms with \( P(\eta) = 1 \). If this was not true, there exists a sequence of surfaces \((\Sigma, j_n)\) and harmonic 1–forms \( \eta_n \) so that

\[
M_n = \|\eta_n\|_\infty > n\|P(\eta_n)\| = n.
\]

Consider the rescaled harmonic 1–forms \( \mu_n = \eta_n/M_n \) with periods less than \( \frac{1}{n} \). By elliptic regularity and Arzela Ascoli we can extract a subsequence that converges uniformly in \( C^\infty \) to a the thick part of \((\Sigma, j_n)\). Using Lemma A.1 we see that \( \eta_n \) then also converge uniformly in \( C^\infty \) on the necks, so \( \eta_n \) converge uniformly in \( C^\infty \) to a harmonic 1–form \( \eta \) with vanishing periods on the limit surface with cylindrical ends. The periods of \( \eta_n \) on the neck regions converge to 0 uniformly, and the twist converge to zero by Lemma A.2. By Lemma A.1 we then see that the limit \( \eta \) still has sup norm 1. The cylindrical ends are conformally equivalent to punctured disks, and \( \eta \) extends to a harmonic 1–form over the punctures to a 1–form on the normalization of the closed nodal surface with with vanishing periods, again using Lemma A.1. By the Hodge Theorem \( \eta \) vanishes identically on the normalization, contradicting that the sup norm of \( \eta \) is 1.

\[\square\]

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