Rational solutions of KZ equation, case $S_4$.

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Abstract

We consider the Knizhnik-Zamolodchikov system of linear differential equations. The coefficients of this system are generated by elements of the symmetric group $S_n$. We separately investigate the case $S_4$. In this case we solve the corresponding KZ-equation in the explicit form.

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1 Introduction

1. We consider the Knizhnik-Zamolodchikov differential system (see [3])
\[
\frac{dW}{dz} = \rho A(z)W, \quad z \in C,
\]  \hspace{1cm} (1.1)
where \( A(z) \) and \( W(z) \) are \( n \times n \) matrices. We suppose that \( A(z) \) has the form
\[
A(z) = \sum_{k=1}^{s} \frac{P_k}{z - z_k}, \quad s \geq 2,
\]  \hspace{1cm} (1.2)
where \( z_k \neq z_\ell \) if \( k \neq \ell \). The matrices \( P_k \) are connected with matrix representation of the symmetric group \( S_n \) and are defined by formulas (2.1)-(2.4). A. Chervov and D. Talalaev formulated the following interesting conjecture [2].

Conjecture 1.1. The Knizhnik-Zamolodchikov system (1.1), (1.2) has a rational fundamental matrix solution when parameter \( \rho \) is integer.

We have proved this conjecture [5] for the case when \( \rho = \pm 1 \). In the present paper we separately consider the case \( S_4 \) and solve the corresponding KZ-equation in the explicit form.

2. In a neighborhood of \( z_k \) the matrix function \( A(z) \) can be represented in the form
\[
A(z) = \frac{a_{-1}}{z - z_k} + a_0 + a_1(z - z_k) + \ldots
\]  \hspace{1cm} (1.3)
where \( a_k \) are \( n \times n \) matrices. We investigate the case when \( z_k \) is either a regular point of \( W(z) \) or a pole. Hence the following relation
\[
W(z) = \sum_{p \geq m} b_p(z - z_k)^p, \quad b_m \neq 0
\]  \hspace{1cm} (1.4)
is true. Here \( b_p \) are \( n \times n \) matrices. We note that \( m \) can be negative.

Proposition 1.1. (necessary condition, see [4]) If the solution of system (1.1) has form (1.4) then \( m \) is an eigenvalue of \( a_{-1} \).

We denote by \( M \) the greatest integer eigenvalue of the matrix \( \rho a_{-1} \). Using relations (1.3) and (1.4)) we obtain the assertion.

Proposition 1.2. (necessary and sufficient condition, see [4]) If the matrix system
\[
[(q + 1)I_n - a_{-1}]b_{q+1} = \sum_{j + \ell = q} \rho a_j b_\ell,
\]  \hspace{1cm} (1.5)
where $m \leq q + 1 \leq M$ has a solution $b_m, b_{m+1}, \ldots, b_M$ and $b_m \neq 0$ then system (1.1) has a solution of form (1.4).

We introduce the matrices

$$P_k^- = I + P_k, \quad P_k^+ = I - P_k.$$ (1.6)

2 Explicit solution

1. We consider the natural representation of the symmetric group $S_n$ (see [1]). By $(i; j)$ we denote the permutation which transposes $i$ and $j$ and preserves all the rest. The $n \times n$ matrix which corresponds to $(i; j)$ is denoted by

$$P(i, j) = [p_k,\ell(i, j)], \quad (i \neq j).$$ (2.1)

The elements $p_k,\ell(i, j)$ are equal to zero except

$$p_k,\ell(i, j) = 1, \quad (k = i, \ell = j); \quad p_k,\ell(i, j) = 1, \quad (k = j, \ell = i),$$

$$p_k,k(i, j) = 1, \quad (k \neq i, k \neq j).$$ (2.2)

Now we introduce the matrices

$$P_k = P(1, k + 1), \quad 1 \leq k \leq n - 1.$$ (2.4)

and the $1 \times (n - 1)$ vector $e = [1, 1, \ldots, 1]$, and $n \times n$ matrix

$$S = \left[ \begin{array}{cc} 2 - n & e \\ e^\tau & 0 \end{array} \right].$$ (2.5)

Using relations (1.38) and (2.1)-(2.4) we deduce that

$$T = (n - 2)I_n + S.$$ (2.6)

The eigenvalues of $T$ are defined by the equalities

$$\lambda_1 = n - 1, \quad \lambda_2 = n - 2, \quad \lambda_3 = -1.$$ (2.7)

Hence we have $m_T = -1$, $M_T = n - 1$. We shall consider the case when $\rho = -1$. In this case the matrix solution $W(z)$ of system (1.1) can be written in the form

$$W(z) = \sum_{k=1}^{s} \frac{L_k}{z - z_k} + Q(z),$$ (2.8)
where $L_k$ are $n \times n$ matrices, $Q(z)$ is a $n \times n$ matrix polynomial of the form

$$Q(z) = Q_{-1}z + Q_0.$$  \hfill (2.9)

By substituting formulas (2.8) and (2.9) in relation (1.1) we receive the equalities

$$\begin{align*}
(I - P_k)L_k &= 0, \quad 1 \leq k \leq s, \quad (2.10) \\
\sum_{j \neq k} P_j L_j + P_j L_k &+ P_k Q(z_k) = 0, \quad (2.11) \\
Q_{-1} &= -TQ_{-1}. \quad (2.12)
\end{align*}$$

2. Further we consider the case $S_4$.

The vectors $L_k$ can be chosen in the following form

$$L_1 = \text{col}[1, 1, -1, -1], \quad L_2 = \alpha \text{col}[1, -1, 1, -1], \quad L_3 = \beta \text{col}[1, -1, -1, 1].$$ \hfill (2.13)

We note, that $L_k$ which are defined by relation (2.13) satisfy conditions (2.10). The numbers $\alpha$ and $\beta$ are defined by relations

$$\alpha = -\frac{z_3 - z_2}{z_3 - z_1}, \quad \beta = \frac{z_3 - z_2}{z_2 - z_1}. \quad (2.14)$$

Now we introduce the matrix polynomial $Q(z)$ (see (2.9)), where

$$\begin{align*}
Q_{-1} &= -[(z_2 - z_1)(z_3 - z_1)]^{-1}\text{col}[3, -1, -1, -1], \quad (2.15) \\
Q_0 &= [(z_2 - z_1)(z_3 - z_1)]^{-1}(z_1 N_1 + z_2 N_2 + z_3 N_3). \quad (2.16)
\end{align*}$$

Here the vectors $N_k$ are defined by relations

$$N_1 = L_1 \quad N_2 = \text{col}[1, -1, 1, -1], \quad N_3 = \text{col}[1, -1, -1, 1].$$ \hfill (2.17)

By direct calculations we see that that matrices $L_k$, $(k = 1, 2, 3)$ and matrix polynomial $Q(z)$ which are defined by relations (2.9) and (2.13)-(2.17),satisfy the relations (2.10)-(2.12). Hence the vector function

$$Y_1(z) = \sum_{k=1}^{3} \frac{L_k}{z - z_k} + Q(z) \quad (2.18)$$
is the rational solution of system (1.1) when $\rho = -1$.

In order to construct the second rational solution of system (1.1) we introduce the vector
\[ M_k = \beta_k \text{col}[1, 1, 1, 1], \quad k = 1, 2, 3, \tag{2.19} \]
where the numbers $\beta_k$ are defined by the relations
\[ \beta_1 = z_2 - z_3, \quad \beta_2 = z_3 - z_1, \quad \beta_3 = z_1 - z_2. \tag{2.20} \]

By direct calculations we see that the numbers $\beta_k$ satisfy the equations
\[ \frac{\beta_1 + \beta_2}{z_1 - z_2} + \frac{\beta_1 + \beta_3}{z_1 - z_3} = 0, \tag{2.21} \]
\[ \frac{\beta_1 + \beta_2}{z_2 - z_1} + \frac{\beta_2 + \beta_3}{z_2 - z_3} = 0. \tag{2.22} \]

It follows from relations (2.19)-(2.22) that conditions (2.10)-(2.12) will be fulfilled if $L_k = M_k, \quad Q_{-1} = Q_0 = 0$. Hence the vector function
\[ Y_2(z) = \sum_{k=1}^{3} \frac{M_k}{z - z_k} \tag{2.23} \]
is the rational solution of system (1.1) when $\rho = -1$.

In order to construct the next solution $Y_3(z)$ of system (1.1) we introduce the vectors
\[ m_1 = \beta_1 \text{col}[0, 0, 1, a], \quad m_2 = \beta_2 \text{col}[0, b, 0, c], \quad m_3 = \beta_3 \text{col}[0, 1, a, 0], \tag{2.24} \]
where
\[ a = -\frac{\beta_1}{\beta_3}, \quad b = -\frac{\beta_3}{\beta_2}, \quad c = \frac{\beta_1^2}{\beta_2 \beta_3}. \tag{2.25} \]

We note, that the numbers $\beta_k$ as in the previous case are defined by relations (2.20). It follows from relations (2.24) and (2.25) that conditions (2.10)-(2.12) will be fulfilled if $L_k = m_k, \quad Q_{-1} = Q_0 = 0$. Hence the vector function
\[ Y_3(z) = \sum_{k=1}^{3} \frac{m_k}{z - z_k} \tag{2.26} \]
is the rational solution of system (1.1) when $\rho = -1$.

In order to construct the next solution $Y_4(z)$ of system (1.1) we introduce the vectors
\[ \ell_1 = \alpha_1 \text{col}[0, 0, 1, a], \quad \ell_2 = \alpha_2 \text{col}[0, b, 0, c], \quad \ell_3 = \alpha_3 \text{col}[0, d, e, 0]. \tag{2.27} \]
where
\[ a = -\frac{\alpha_1}{\alpha_3}, \quad b = -\frac{\alpha_3}{\alpha_2}d, \quad c = \frac{\alpha_1^2}{\alpha_2\alpha_3}, \]
(2.28)
\[ d = -\frac{\alpha_1^2}{\alpha_2\alpha_3^2}(\alpha_1\alpha_2 + \alpha_3^2), \quad e = 1 + \frac{\alpha_1}{\alpha_2}. \]
(2.29)

We note, that the numbers \( \alpha_k \) are defined by the relations
\[ \alpha_1 = (z_3 - z_2)^{-1}, \quad \alpha_2 = (z_1 - z_3)^{-1}, \quad \alpha_3 = (z_2 - z_1)^{-1}. \]
(2.30)

It follows from relations (2.27)-(2.30) that conditions (2.10)-(2.12) will be fulfilled if \( L_k = \ell_k, \quad Q_{-1} = Q_0 = 0. \) Hence the vector function
\[ Y_4(z) = \sum_{k=1}^{3} \frac{\ell_k}{z - z_k} \]
(2.31)
is the rational solution of system (1.1) when \( \rho = -1. \)

**Proposition 2.1.** Let us consider the case \( S_4 \) when \( \rho = -1. \) In this case the fundamental solution of system (1.1) has the form
\[ Y(z) = \sum_{k=1}^{4} c_k Y_k(z), \]
(2.32)
where \( c_k \) are arbitrary constants, the functions \( Y_k(z) \) are defined by relations (2.18), (2.23), (2.26) and (2.31).

**Remark 2.1.** The explicit solution for the case \( S_3 \) was constructed in the paper [5].

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