Equivalence of the Dirichlet and Neumann problems for the Laplace operator in planar doubly-connected regions

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Abstract

Motivated by recent results regarding the equivalence of the Dirichlet and Neumann problems for the Laplace operator in the case of simply connected regions, the present paper takes a step further and provides a similar equivalence between the above mentioned problems in the case of planar doubly-connected regions. The equivalence means that solving any of these problems leads by an explicit formula to a solution of the other problem. In addition, sufficient conditions guaranteeing the uniform H"older continuity of the higher order partial derivatives of the solutions to the Neumann problem are provided as a consequence of this equivalence.

Keywords — Dirichlet problem, Neumann problem, Laplace operator, harmonic functions, analytic functions, conformal transformations.

Mathematics Subject Classification (2010) - 31B05, 31B10, 35J05, 42B37, 30D05, 30C20.

1 Introduction

The Dirichlet and Neumann problems are fundamental in the theory of differential equations, while still capturing the interest of the mathematical community. Some representations of the solutions of the Dirichlet and Neumann problems which can be found in the literature are: by single/double layer potential and spherical harmonics (see for instance [9, Chapters 2, 3]) and by probabilistic methods (see [10] for the Dirichlet problem and [11] for the Neumann problem).

Recently the connection between these problems was investigated and it was shown that in the case of the Laplace operator (and other differential operators satisfying certain homogeneity conditions) there is an equivalence between these problems, in the sense that solving one of them leads by an explicit formula to a solution of the other problem (for details see [5], [6]). Moreover it was shown that this equivalence leads to a new probabilistic representation of
the solution of the Neumann problem, for the case of the (unit) ball (see \cite{7}). The domains taken into consideration in these papers were simply connected. In the present paper the author takes a step further and shows that a similar equivalence between the Dirichlet and the Neumann problems holds in the case of planar doubly-connected regions. Incidentally it is shown that this intimate connection can be used for proving the uniform Hölder continuity of the higher order partial derivatives of the solutions of the Neumann problem. This aspect is in concordance with a well-known theorem in potential theory which is attributed to O. D. Kellogg, where the smooth extension of the higher order partial derivatives of the solution of the Dirichlet problem, under some appropriate smoothness assumption, was investigated (see \cite{8}). The contributions of this paper can be summarized as follows:

- First, an equivalence between the solutions of the Dirichlet and Neumann problems for the Laplace operator in annular regions $A_{r_1,r_2}$ satisfying $0 < r_1 < r_2 < \infty$, formulated in polar coordinates, is provided (see Theorem 1).

- Second, sufficient conditions for the uniform Hölder continuity of the higher order partial derivatives of the solutions of the Neumann problem (9) are provided as a consequence of Theorem 1 (see Theorem 2).

- A consistent definition of the Neumann problem in the special case of the punctured unit disk is given in Definition 4 using a polar coordinates representation (see also Remark 3).

- When the boundary data is symmetric, a simplified formula for a solution of the Neumann problem (9) in terms of the solution of the Dirichlet problem (8) is presented (see Theorem 3).

- An equivalence between the solutions of the Neumann and Dirichlet problems (3) and (2), respectively, as well as sufficient conditions for the uniform Hölder continuity of the higher order partial derivatives of the solutions of the Neumann problem (3) are provided in Theorem 4, in the case of annular regions whose radii satisfy the above mentioned condition.

- As a consequence, two main results of \cite{5} (namely Theorems 1 and 3) are recaptured in the case of $\mathbb{R}^2$ as particular instances in Corollary 1. In addition Corollary 1 also provides sufficient conditions for the uniform Hölder continuity of the higher order partial derivatives of the solutions of the Neumann problem (7) for the case of the (unit) disk.

- For the general case of planar, doubly-connected regions, Theorem 5 gives an equivalence of the solutions of the Dirichlet and the Neumann problems (2) and (3), respectively, while Theorem 6 provides sufficient conditions for the uniform Hölder continuity of the higher order partial derivatives
of the solutions of the Neumann problem \(3\), as a consequence of this equivalence.

The structure of the paper is the following. In Section 2 the notation is established and some preparations are made. In Section 3 the author presents all the results of the paper which were announced above. Section 4 draws some final conclusions and some potential research directions.

## 2 Preliminaries

### 2.1 Notations

Denote by \(U = \{z \in \mathbb{C} : |z| < 1\}\) the unit disk, by \(\hat{U}\) the punctured unit disk, by \(C_r\) the circle of radius \(r\) (centered in origin), and the annulus with radii \(0 \leq r_1 < r_2\) by \(A_{r_1,r_2} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}\), respectively. In addition, for any region \(D\), \(C^1(D)\) will stand for the set of all functions \(h \in C^1(D)\) for which the gradient \(\nabla h\) can be continuously extended to \(\overline{D}\), and if \(D\) is in addition smooth and bounded, \(\sigma\) will be denoting the length measure on its boundary. If \(E\) is a subset of \(\mathbb{C}\) or \(\mathbb{R}\), then the real or complex valued function \(f\) belongs to \(C^m,\alpha(E)\), for some non-negative integer \(m\) and some \(\alpha \in (0,1]\), if the \(m\) order partial derivatives of \(f\) exist and are locally \(\alpha\) Hölder continuous on \(E\) (in the case \(f\) is complex-valued, the derivative should be understood in the sense of Complex Analysis). Also if \(w\) is any function defined on some set \(X\), then we define \(\|w\| = \sup_{x \in X} |w(x)|\). Throughout the paper the author will switch between the complex and the \(\mathbb{R}^2\) notations, depending on the context to discriminate between them. For example if \(u\) is a harmonic function defined on some region containing the point \((\cos \theta, \sin \theta)\) then \(u(e^{i\theta})\) is a shorthand representing the value of \(u\) in \((\cos \theta, \sin \theta)\). Next if \(u\) is differentiable at \(z = (r \cos \theta, r \sin \theta)\), \(r > 0\), then the directional derivative of \(u\) in the direction of the unit vector \((\cos \theta, \sin \theta)\) evaluated in \(z\) will be denoted \(\frac{\partial u}{\partial a}(z)\). If \(h_1\) and \(h_2\) are any two functions then \(h_1 \sim h_2\), \(h_1 \lesssim h_2\) provided that there is some constant \(C\) such that \(h_1 = Ch_2\), and \(h_1 \leq Ch_2\), respectively. In the last case any constant \(C\) with that property will be referred to as a proportionality constant. Last but not least if \(z\) is any complex number we will let \(x\) denote its real part, \(x = \Re(z)\), and \(y\) denote its imaginary part, \(y = \Im(z)\) (unless otherwise specified).

### 2.2 Definitions and preliminary aspects

Throughout the paper \(D \subset \mathbb{C}\) will denote a bounded, doubly-connected region of the complex plane.

**Definition 1.** We say that \(D \in C^{m,\alpha}\) for some non-negative integer \(m\) and some real \(\alpha \in (0,1]\) if for any point \(a \in \partial D\) there exists an open interval \(I\) and a real-valued function \(\beta : I \rightarrow (c,d)\) satisfying \(\beta \in C^{m,\alpha}(I)\), such that the set \(U := I \times (c,d)\) is a neighborhood of \(a\) and, eventually up to a rotation

\[
D \cap U = \{ (x,t) : x \in I, t < \beta(x) \}.
\]  

(1)
Notice that $I$, $c$, $d$, as well as the function $\beta$ may depend on $a$.

**Definition 2.** Let $f$ be a real-valued function defined on the boundary of $D$ and assume $D \in C^{m, \alpha}$. Then $f \in C^{m, \alpha}(\partial D)$ if for any point $a \in \partial D$ and any local parametrization $\beta$ as above the $m$ order derivative of the function $f \circ \beta$ is locally $\alpha$ Hölder continuous on $I$.

If $D$ is also smooth, consider the corresponding Dirichlet and Neumann problems for the Laplace operator

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } D, \\
u u &= g \quad \text{on } \partial D,
\end{align*}
\]

and

\[
\begin{align*}
\Delta U &= 0 \quad \text{in } D, \\
u U \nu^{-1} &= f \quad \text{on } \partial D,
\end{align*}
\]

respectively, where $\nu$ is the unitary outward normal to the boundary of $D$. In the particular case when $D = A_{r_1, r_2}$, $r_1 > 0$, we have

\[
\nu(z) = \begin{cases} \frac{z}{r_2}, & \text{if } |z| = r_2, \\
-\frac{z}{r_1}, & \text{if } |z| = r_1. \end{cases}
\]

By a solution of the Dirichlet/Neumann problems above, it is understood a function $u \in C^2(D) \cap C^0(\partial D)$, respectively $U \in C^2(D) \cap C^1(\partial D)$, which satisfies (2), respectively (3).

**Remark 1.** Assume $D$ is the annulus $A_{r_1, r_2}$, $r_1 > 0$. Using the maximum principle for harmonic functions (see, e.g., [1, Theorem 2.2.4]), it can be seen that for continuous boundary data $g$ the Dirichlet problem (2) has a unique solution. Also if $f$ is a continuous function satisfying $\int_{\partial D} f d\sigma = 0$ then it can be argued that the (classical) Neumann problem always has a solution, which is unique up to additive constants.

The existence of solutions of the Dirichlet and the Neumann problems in the case of the punctured disk $A_{0, r_2}$ requires special attention. As shown by Zaremba’s example, for continuous boundary data $g$ and $r_1 = 0$, the Dirichlet problem (2) has a solution iff $g(0) = \frac{1}{2\pi r_2} \int_0^{2\pi} g(r_2 e^{i\theta}) d\theta$. Also, for continuous boundary data $f$ and $r_1 = 0$, the boundary condition at the origin of the Neumann problem (3) should be ignored (the exterior normal to $\partial A_{0, r_2}$ at the origin cannot be properly defined), and a solution of (3) satisfying the boundary condition just on $\partial A_{0, r_2} \setminus \{0\}$ exists only if $\int_0^{2\pi} f(r_2 e^{i\theta}) d\theta = 0$.

When $D = A_{r_1, r_2}$, due to the radial symmetry of the region, it is natural to consider polar coordinates $(r, \theta)$, defined by $r = |z|$ and $\theta = \text{Arg}(z) := \{\text{arg}(z) + 2k\pi : k \in \mathbb{Z}\}$ for $z \in A_{r_1, r_2}$. Here $\text{arg}(z)$ denotes the principal argument of $z \neq 0$, and this notation will be in force for the rest of the paper.

The link between the cartesian and polar coordinates formulation of the Dirichlet and Neumann problems (2) – (3), when $D = A_{r_1, r_2}$, is given by the following proposition.
Proposition 1. If \( w \in C^2(A_{r_1,r_2}) \) satisfies \( \Delta w = 0 \) in \( A_{r_1,r_2} \), then the function \( \hat{w} : (r_1, r_2) \times \mathbb{R} \to \mathbb{R} \) defined by \( \hat{w}(r, \theta) = w(re^{i\theta}) \) is \( 2\pi \)-periodic in the second variable, has continuous second order partial derivatives and satisfies
\[
\hat{w}_{rr} + \frac{1}{r} \hat{w}_r + \frac{1}{r^2} \hat{w}_{\theta\theta} = 0 \quad \text{in} \quad (r_1, r_2) \times \mathbb{R}.
\]

Conversely, if the function \( \hat{w} : (r_1, r_2) \times \mathbb{R} \to \mathbb{R} \) is \( 2\pi \)-periodic in the second variable, has continuous second order partial derivatives and satisfies \( \hat{w}(re^{i\theta}) = \hat{w}(r, \theta) \quad \text{for} \quad (r, \theta) \in [r_1, r_2] \times \mathbb{R} \), then the function \( w : A_{r_1,r_2} \to \mathbb{R} \) defined by \( w(z) = \hat{w}(|z|, \arg(z)) \) belongs to \( C^2(A_{r_1,r_2}) \) and satisfies \( \Delta w = 0 \) in \( A_{r_1,r_2} \).

Moreover, \( w \) has a continuous extension to \( \overline{A_{r_1,r_2}} \) if and only if \( \hat{w} \) has a continuous extension to \( [r_1, r_2] \times \mathbb{R} \) as well, and in this case
\[
w(re^{i\theta}) = \hat{w}(r, \theta) \quad \text{for} \quad (r, \theta) \in [r_1, r_2] \times \mathbb{R}.
\]

Also \( w \) has (outer) normal derivative at a point \( re^{i\theta} \in \partial A_{r_1,r_2} \) if and only if \( \hat{w} \) has partial derivative with respect to the first variable at the point \( (r, \theta) \in \{r_1, r_2\} \times \mathbb{R} \), and in this case
\[
\frac{\partial w}{\partial \nu}(re^{i\theta}) = \begin{cases} 
\hat{w}_r(r, \theta), & \text{if } r = r_2, \\
-\hat{w}_r(r, \theta), & \text{if } r = r_1.
\end{cases}
\]

Finally \( w \in C^1(\overline{A_{r_1,r_2}}) \) if and only if \( \hat{w} \in C^1([r_1, r_2] \times \mathbb{R}) \).

Proof. The direct implication is immediate. For the converse, by using the \( 2\pi \)-periodicity of \( \hat{w} \) in the second variable and the fact that it has continuous second order partial derivatives, direct computations show that \( w \in C^2(A_{r_1,r_2}) \). Also, it is not difficult to check that
\[
\Delta w(z) = \hat{w}_{rr}(|z|, \arg(z)) + \frac{1}{|z|} \hat{w}_r(|z|, \arg(z)) + \frac{1}{|z|^2} \hat{w}_{\theta\theta}(|z|, \arg(z)) = 0, \quad z \in A_{r_1,r_2},
\]
where the last equality follows by using hypothesis \( [5] \).

The fact that \( w \) has a continuous extension to the boundary of the domain if and only if \( \hat{w} \) has is immediate.

Next notice that for any \( \theta \in \mathbb{R} \) the corresponding directional derivatives are given by:
\[
\begin{align*}
\frac{\partial w}{\partial a_0}(re^{i\theta}) &= \lim_{t \to 0} \frac{w(r(|\cos \theta, \sin \theta|)+t(|\cos \theta, \sin \theta|))-w(r(|\cos \theta, \sin \theta|))}{t} = \hat{w}_r(r, \theta), \\
\frac{\partial w}{\partial a_1}(re^{i\theta}) &= \lim_{t \to 0} \frac{w(r_2(|\cos \theta, \sin \theta|)+t(|\cos \theta, \sin \theta|))-w(r_2(|\cos \theta, \sin \theta|))}{t} = \hat{w}_r(r, \theta), \\
\frac{\partial w}{\partial a_2}(re^{i\theta}) &= -\lim_{t \to 0} \frac{w(r_1(|\cos \theta, \sin \theta|)+t(|\cos \theta, \sin \theta|))-w(r_1(|\cos \theta, \sin \theta|))}{t} = -\hat{w}_r(r_1, \theta),
\end{align*}
\]
where the first and second equalities above hold for any \( r \in (r_1, r_2) \).

For the last claim it can also be checked by direct computations that the following equalities hold:
\[
\begin{align*}
w_x(re^{i\theta}) &= -\frac{r \sin \theta}{r} \hat{w}_q(r, \theta) + \frac{r \cos \theta}{r} \hat{w}_r(r, \theta) \quad \text{if } \cos \theta < 0, \ \sin \theta \neq 0, \\
w_y(re^{i\theta}) &= \frac{r \cos \theta}{r} \hat{w}_q(r, \theta) + \frac{r \sin \theta}{r} \hat{w}_r(r, \theta) \quad \text{if } \cos \theta < 0, \ \sin \theta \neq 0, \\
w_x(re^{-i\pi}) &= -\hat{w}_r(-r, -\pi), \\
w_y(re^{-i\pi}) &= -\frac{r \sin \theta}{r} \hat{w}_q(-r, -\pi).
\end{align*}
\]
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Combining equations (7) with the fact that arg(·) is harmonic in \( C \setminus \{ z : \Re(z) < 0, \, \Im(z) = 0 \} \) (and hence arg(·) \( \in C^2 (C \setminus \{ z : \Re(z) < 0, \, \Im(z) = 0 \}) \)) and \( \hat{w}_r, \hat{w}_\theta \) are 2\( \pi \)-periodic in the second variable, the conclusion follows. This ends the proof.

The above result shows that in the case of annuli one can reformulate the Dirichlet and the Neumann problems (2) – (3) in polar coordinates as follows: find \( u = u(r, \theta) \in C^2 ((r_1, r_2) \times \mathbb{R}) \cap C^0 ([r_1, r_2] \times \mathbb{R}) \) which is 2\( \pi \)-periodic in the second variable and satisfies

\[
\begin{align*}
  u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0 & \text{in } (r_1, r_2) \times \mathbb{R}, \\
  u &= \varphi & \text{on } \{r_1, r_2\} \times \mathbb{R},
\end{align*}
\]  

respectively find \( U = U(r, \theta) \in C^2 ((r_1, r_2) \times \mathbb{R}) \cap C^4 ([r_1, r_2] \times \mathbb{R}) \) which is 2\( \pi \)-periodic in the second variable and satisfies

\[
\begin{align*}
  U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} &= 0 & \text{in } (r_1, r_2) \times \mathbb{R}, \\
  U_r &= \phi & \text{on } \{r_1, r_2\} \times \mathbb{R}.
\end{align*}
\]  

and the boundary data \( \varphi, \phi : \{r_1, r_2\} \times \mathbb{R} \) is related to the boundary data \( f, g : \partial A_{r_1, r_2} \to \mathbb{R} \) in (2) – (3) by

\[
\varphi(r, \theta) = g(re^{i\theta}) \quad \text{and} \quad \phi(r, \theta) = \begin{cases} f(re^{i\theta}) & \text{if } r = r_2, \\ -f(re^{i\theta}) & \text{if } r = r_1. \end{cases}
\]

Notice that, in particular, the functions \( \varphi, \phi \) are 2\( \pi \)-periodic in the second variable.

**Remark 2.** The compatibility condition \( \int_{\partial A_{r_1, r_2}} f \, d\sigma = 0 \) for the existence of a solution of the Neumann problem (3) in cartesian coordinates becomes, in polar coordinates, the following:

\[
\int_{0}^{2\pi} r_1 \phi(r_1, \theta) \, d\theta = \int_{0}^{2\pi} r_2 \phi(r_2, \theta) \, d\theta.
\]  

### 3 Main results

This section is divided into two parts: Subsection 3.1 is devoted to the study of the equivalence between the solutions of the Dirichlet and Neumann problems in the case of annular regions, while Subsection 3.2 is devoted to the study of the equivalence of these two problems for generally doubly connected regions.

#### 3.1 Annular regions

At this point we are prepared to state and prove the main result of this section.
Theorem 1. Let \( 0 < r_1 < r_2 < \infty \) and assume \( \phi : \{r_1, r_2\} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous, \( 2\pi \)-periodic in the second variable, and satisfies the compatibility condition \( \int_0^{2\pi} r_1 \phi(r_1, \theta) \, d\theta = \int_0^{2\pi} r_2 \phi(r_2, \theta) \, d\theta \). If \( U \) is the solution of the Neumann problem \( (9) \) with boundary data \( \phi \), satisfying \( U(\sqrt{r_1r_2}, 0) = 0 \), then for any \( (r, \theta) \in [r_1, r_2] \times \mathbb{R} \)

\[
U(r, \theta) = \frac{1}{\sqrt{r_1r_2}} \frac{1}{r} \int_{\frac{1}{r}}^{1} \frac{u(r\rho, \theta)}{\rho} \, d\rho + \sqrt{r_1r_2} \int_0^\theta \left( \mathcal{C} - \int_0^t u_r(\sqrt{r_1r_2}, \tau) \, d\tau \right) \, dt,
\]

(11)

where \( u \) is the solution of the Dirichlet problem \( (8) \) with boundary values \( \varphi(r, \theta) = r\varphi(r, \theta) \) on \( \{r_1, r_2\} \times \mathbb{R} \) and

\[
\mathcal{C} = \frac{\sqrt{r_1r_2}}{2\pi} \int_0^t \int_0^\tau u_r(\sqrt{r_1r_2}, \tau) \, d\tau \, dt.
\]

(12)

Conversely if \( \varphi : \{r_1, r_2\} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous, \( 2\pi \)-periodic in the second variable, satisfies \( \int_0^{2\pi} \varphi(r_1, \theta) \, d\theta = \int_0^{2\pi} \varphi(r_2, \theta) \, d\theta \), and if \( U \) is a solution of the Neumann problem \( (9) \) with \( \phi(r, \theta) = \frac{\varphi(r, \theta)}{r} \) for \( (r, \theta) \in \{r_1, r_2\} \times \mathbb{R} \), then

\[
u(r, \theta) = rU_r(r, \theta), \quad (r, \theta) \in [r_1, r_2] \times \mathbb{R},
\]

(13)

is the solution of the Dirichlet problem \( (8) \).

Proof. Denote by \( \mathcal{U} \) the right-hand side of (11). Let us first consider that \( r_2 = \frac{1}{a} = a > 1 \), in which case the problem reduces to showing that the function

\[
\mathcal{U}(r, \theta) = \frac{1}{\sqrt{r_1r_2}} \frac{1}{r} \int_{\frac{1}{a}}^{1} \frac{u(r\rho, \theta)}{\rho} \, d\rho + \sqrt{r_1r_2} \int_0^\theta \left( \mathcal{C} - \int_0^t u_r(1, \tau) \, d\tau \right) \, dt,
\]

(14)

satisfying \( \mathcal{U}(1, 0) = 0 \) is the desired solution of the Neumann problem \( (9) \) on \( A_{\frac{1}{a},a} \) with boundary data

\[
\phi(r, \theta) = \begin{cases} f(re^{i\theta}) & \text{if } r = a, \; \theta \in \mathbb{R}, \\
-f(re^{i\theta}) & \text{if } r = \frac{1}{a}, \; \theta \in \mathbb{R}, \end{cases}
\]

where \( u \) is the solution of the Dirichlet problem \( (8) \) with boundary values \( \varphi(r, \theta) = r\varphi(r, \theta) \) on \( \left\{ \frac{1}{a}, a \right\} \times \mathbb{R} \). To this end we will first show that

\[
\int_0^{2\pi} u_r(1, \tau) \, d\tau = 0.
\]

(15)
To this end compute the proof of (15).

Indeed using the definition of \(U\) we get

\[
\frac{\partial}{\partial r} \int_0^1 \frac{u(r, \theta)}{p} \, dp = \frac{\partial}{\partial r} \int \frac{u(r, \theta)}{p} \, dp = \frac{\partial}{\partial r} \int \frac{u(r, \theta)}{r} \, d\theta, \quad (r, \theta) \in \left(\frac{1}{a}, a\right) \times \mathbb{R}.
\]

Consequently, the partial derivative of \(U\) with respect to the first variable can be continuously extended to \(\left[\frac{1}{a}, a\right] \times \mathbb{R}\) and thus for any \(\theta \in \mathbb{R}\) one obtains

\[
U_r(a, \theta) = \frac{u(a, \theta)}{a} = \frac{\varphi(a, \theta)}{a} = \phi(a, \theta)
\]

and also

\[
U_r(\frac{1}{a}, \theta) = \frac{u(\frac{1}{a}, \theta)}{\frac{1}{a}} = \frac{\varphi(\frac{1}{a}, \theta)}{\frac{1}{a}} = \phi(\frac{1}{a}, \theta).
\]

Define \(W : A_{\frac{1}{a}; a} \to \mathbb{R}\), \(W(z) := u(|z|, \arg(z))\). Since \(u\) is the solution of the Dirichlet problem (8) it follows by Proposition 1 that \(W\) is harmonic in \(A_{\frac{1}{a}; a}\) and using a continuity argument \(W(re^{i\theta}) = \varphi(r, \theta), \quad \forall (r, \theta) \in \left(\frac{1}{a}, a\right)\). Then there exist real constants \(\alpha, \beta \in \mathbb{R}\) such that

\[
2\pi \int_0^1 W(re^{i\theta}) \, d\theta = \alpha \log r + \beta, \quad \forall r \in \left[\frac{1}{a}, a\right] \quad (\text{see [2, Chapter 4, Theorem 20]}).
\]

But then

\[
-\alpha \log a + \beta = \int_{C_{\frac{1}{a}}} W \left(\frac{1}{a} e^{i\theta}\right) \, d\theta = \int_{C_a} W \left(\frac{1}{a} e^{i\theta}\right) \, d\theta = \frac{2\pi}{a} u \left(\frac{1}{a}, \theta\right) \, d\theta = \frac{2\pi}{a} \int_0^\frac{1}{a} u \left(\frac{1}{a}, \theta\right) \, d\theta = \frac{2\pi}{a} \int_0^\frac{1}{a} \phi \left(\frac{1}{a}, \theta\right) \, d\theta = \int_{C_a} a\phi(a, \theta) \, d\theta = \int_{C_a} W(a e^{i\theta}) = \alpha \log a + \beta
\]

which implies \(\alpha = 0\).

To sum up

\[
\int W \left(\frac{1}{a} e^{i\theta}\right) \, d\theta = \int_0^\frac{1}{a} u(r, \theta) \, d\theta = 0.
\]

Since \(1 \in \left(\frac{1}{a}, a\right)\), an application of the Dominant Convergence theorem together with the above identity concludes the proof of (15).

The next step is to show that whenever \((r, \theta) \in \left(\frac{1}{a}, a\right) \times \mathbb{R}\)

\[
U(r, \theta + 2\pi) = U(r, \theta).
\]

To this end compute

\[
U(r, \theta + 2\pi) = \int_0^\frac{1}{a} \frac{u(r, \theta, \theta + 2\pi)}{p} \, dp - \int_0^\frac{1}{a} \frac{u(r, \theta, 2\pi)}{p} \, dp = \frac{\theta + 2\pi}{\theta} \int_0^\frac{1}{a} u_r(1, \theta) \, d\theta + C(\theta + 2\pi) - \frac{\theta + 2\pi}{\theta} \int_0^t u_r(1, \theta) \, d\theta + 2\pi C.
\]

As \(u_r(\theta) = u(r, \theta + 2\pi), u_r(r_0, \theta + 2\pi) = \lim_{r \to r_0} \frac{u(r, \theta + 2\pi) - u(r_0, \theta + 2\pi)}{r - r_0} = \lim_{r \to r_0} \frac{u(r, \theta) - u(r_0, \theta)}{r - r_0} = u_r(r_0, \theta)\) \(\forall (r_0, \theta)\). Thus, \(u_r(1, \cdot)\) is \(2\pi\)-periodic and so

\[
\int_0^\frac{1}{a} u_r(1, \theta) \, d\theta = 2\pi C.
\]

Consequently it follows that the function \(t \to \int_0^t u_r(1, \tau) \, d\tau\) is \(2\pi\)-periodic, showing in turn that

\[
\int_0^{\theta + 2\pi} \int_0^t u_r(1, \tau) \, d\tau \, dt = \int_0^{\theta + 2\pi} \int_0^t u_r(1, \tau) \, d\tau \, dt = 2\pi C.
\]
which proves relation (16).

Proceeding further we need to show that \( U \) satisfies (5) for any pair \((r, \theta)\) in \((\frac{1}{a}, a) \times \mathbb{R}\). But this follows easily using the Leibniz-Newton formula in the definition (11) of \( U \), which gives \( U_r(r, \theta) = \frac{u(1, \theta)}{r} + \int \frac{1}{r^2} u_r(r \rho, \theta) d \rho, \quad U_{rr}(r, \theta) = -\frac{u(1, \theta)}{r^2} + \int \frac{1}{r^3} u_{rr}(r \rho, \theta) d \rho, \)

whenever \((r, \theta) \in (\frac{1}{a}, a) \times \mathbb{R}\). Adding them up one obtains \( U_{rr}(r, \theta) + \frac{1}{r} U_r(r, \theta) + \frac{1}{r^2} u_{\theta \theta}(r \rho, \theta) \) \( d \rho \), where the quantity on the right-hand side is identically 0 since \( u \) verifies relation (5). Let us now show that the derivative of \( U \) with respect to the first argument exists, is finite, and equals \( \phi(r, \theta) \) at all points \((r, \theta) \in \{r_1, r_2\} \times \mathbb{R}\). Indeed \( \lim_{r \to a} \frac{U(r, \theta) - U(a, \theta)}{r - a} = \phi(a, \theta) \), and likewise \( \lim_{r \to a} \frac{U(r, \theta) - U(\frac{1}{a}, \theta)}{r - \frac{1}{a}} = \phi(\frac{1}{a}, \theta) \). It only remains to be proved that the partial derivatives of \( U \) extend continuously to \([r_1, r_2] \times \mathbb{R}\). To see that this is indeed the case define \( \tilde{U}(r e^{i \theta}) = U(r, \theta), \quad (r, \theta) \in (r_1, r_2) \times \mathbb{R} \), and also \( f(re^{i \theta}) = \begin{cases} \phi(r, \theta) & \text{if} \quad r = r_2; \\ -\phi(r, \theta) & \text{if} \quad r = r_1. \end{cases} \) Thus, using Proposition 1, it can be easily seen that \( \tilde{U} \) is harmonic on \( A_{r_1, r_2} \) and that the directional derivative of \( \tilde{U} \) along any ray is \( f \). Let \( V \) be any solution of the Neumann problem (5) on \( A_{r_1, r_2} \) having boundary data \( f \). It will be proved that \( W := \tilde{U} - V \) is constant on \( A_{r_1, r_2} \). Indeed let \( r_{1,n} \) and \( r_{2,n} \) be two sequences with positive terms such that \( r_{1,n} \) is decreasing, \( r_{1,n} \to r_1 \), and \( r_{2,n} \) is increasing, \( r_{2,n} \to r_2 \), respectively and denote \( A_n = A_{r_{1,n}, r_{2,n}} \). According to Green’s first identity applied to \( W \) on \( A_n \) it follows that \( \int_{A_n} (W \Delta W + \|\nabla W\|^2) \, dm = \int_{\partial A_n} \frac{\partial W}{\partial \nu} \, d \sigma \) and since \( \Delta W = 0 \) on \( A_n \)

\[
\int_{A_n} \|\nabla W\|^2 \, dm = \int_{\partial A_n} \frac{\partial W}{\partial \nu} \, d \sigma, \tag{17}
\]

where \( m \) is the Lebesgue measure. Since the sets \( A_n \) increase to \( A_{r_1, r_2} \), the sequence of non-negative real-valued functions \( 1_{A_n} \|\nabla W\|^2 \) increases to the function \( 1_{A_{r_1, r_2}} \|\nabla W\|^2 \) (where \( 1_E \) is the indicator function of the set \( E \subset \mathbb{C} \)) and hence an application of the Monotone Convergence theorem to the left-hand side of (17) gives

\[
\int_{A_{r_1, r_2}} \|\nabla W\|^2 \, dm = \lim_{n \to \infty} \int_{A_n} \|\nabla W\|^2 \, dm = \lim_{n \to \infty} \int_{\partial A_n} \frac{\partial W}{\partial \nu} \, d \sigma. \tag{18}
\]

On the other hand one can notice that on \( \partial A_n \) the normal derivative of \( W \) is given by \( \frac{\partial W}{\partial \nu} = \frac{\partial W}{\partial \nu} - \langle \nabla W, \nu \rangle \). By definition \( \nabla V \) extends continuously
to \( A_{r_1, r_2} \). Also, in view of Proposition 1 \( \frac{\partial \hat{u}}{\partial r}(re^{i\theta}) = \begin{cases} \hat{U}_r(re^{i\theta}) & \text{if } r = r_{n;2} \\ -\hat{U}_r(re^{i\theta}) & \text{if } r = r_{n;1} \end{cases} \)

and since it has already shown that \( \hat{U}_r(r, \theta) = \frac{u(r, \theta)}{r} \) it follows that \( \frac{\partial \hat{u}}{\partial r} \) extends continuously to \( A_{r_1, r_2} \) as well and one can conclude that \( \frac{\partial \hat{u}}{\partial r} \) is bounded on \( A_{r_1, r_2} \). Consequently using the Dominant Convergence theorem in (15) it follows that \( \nabla W = 0 \) in \( A_{r_1, r_2} \) and hence \( \hat{U} \in C^1(A_{r_1, r_2}) \). Invoking again Proposition 1 shows that \( \hat{U} \in C^1([r_1, r_2] \times \mathbb{R}) \), as desired. This completes the proof of the first part in the case \( r_2 = a > 1 > \frac{1}{a} = r_1 \).

For the general case \( 0 < r_1 < r_2 \) define \( \lambda = \frac{1}{\sqrt{r_1 r_2}} \), \( a = \sqrt{\frac{r_1}{r_2}} \) and let \( \hat{u} \) be the solution of the Dirichlet problem (6) on \( A_{\frac{1}{a}, a} \) with boundary data \( \hat{\phi}(r, \theta) = \varphi(\frac{r}{a}, \theta) = \frac{1}{a} \phi(\frac{r}{a}, \theta) \), \( (r, \theta) \in \{ \frac{1}{a}, a \} \times \mathbb{R} \). By the previous part the function \( \hat{U}(r, \theta) := \int_0^r \hat{u}(r, \theta) d\rho + \int_0^\theta \left( C - \int_0^1 \hat{u}_r(1, \tau)d\tau \right) dt \) is the solution of the Neumann problem (9) with boundary data \( \hat{\phi}(r, \theta) = \frac{\hat{\phi}(r, \theta)}{r} \) on \( \{ \frac{1}{a}, a \} \times \mathbb{R} \), satisfying \( \hat{U}(1, 0) = 0 \). Consequently defining \( U(R, \theta) = \hat{U}(\lambda R, \theta) \), \( (R, \theta) \in (r_1, r_2) \times \mathbb{R} \), it follows that \( \frac{\partial}{\partial R} U(R, \theta) = \lambda \hat{U}_r(R, \theta) \) from where \( \frac{\partial U}{\partial R}(r_2, \theta) = \lambda \hat{U}_r(a, \theta) = \lambda \hat{\phi}(a, \theta) = \hat{\phi}(r_2, \theta) \) and also \( \frac{\partial U}{\partial \theta}(r_1, \theta) = \lambda \hat{U}_r(\frac{1}{a}, \theta) = \lambda \hat{\phi}(\frac{1}{a}, \theta) = \hat{\phi}(r_1, \theta) \). In addition notice that equation (11) is fulfilled for \( U \) on \( (r_1, r_2) \times \mathbb{R} \), and since \( U(\sqrt{r_1 r_2}, 0) = \hat{U}(1, 0) = 0 \) one can conclude that \( U = \hat{U} \).

The proof of the second part is immediate and follows directly from equation (11) by taking the derivative with respect to \( r \).

If an additional assumption on the smoothness of \( \phi \) is added, the result in Theorem 1 can be strengthened. The main idea is that on \( A_{r_1, r_2} \) the solution \( w \) of the Dirichlet problem (6) with boundary data \( g \in C^{m,\alpha}(\partial A_{r_1, r_2}) \), where \( m \geq 2 \) is integer and \( \alpha \in (0, 1] \), has the remarkable property that its \( m \) order partial derivatives are locally \( \alpha \) Hölder continuous in a sufficiently small neighborhood of each point \( \alpha \in \partial A_{r_1, r_2} \). This result is often referred to as Kellogg’s theorem (for further details see [8]). But we can link \( w \) with \( U \) given by (11) and thus obtain important results on the continuous extensions of the higher order partial derivatives of \( U \) to the closure of the domain where it is defined. Before stating and proving explicitly these results, we need to introduce a lemma which will be of crucial importance in the subsequent proofs.

**Lemma 1.** Let again \( 0 < r_1 < r_2 < \infty \) and assume \( \varphi : \{ r_1, r_2 \} \times \mathbb{R} \to \mathbb{R} \) is \( 2\pi \)-periodic in the second variable, satisfies the condition \( \int_0^{2\pi} \varphi(r_1, \theta) d\theta = 0 \)

and in addition suppose there exists \( \alpha \in (0, 1] \) such that \( \varphi(r, \cdot) \)

belongs to \( C^{m,\alpha}(\mathbb{R}) \) for some positive integer \( m \geq 2 \), whenever \( r \in \{ r_1, r_2 \} \). If
the function $g : \partial \mathcal{A}_{r_1, r_2} \to \mathbb{R}$ satisfies $g(re^{i\theta}) = \varphi(r, \theta)$ $\forall (r, \theta) \in \{r_1, r_2\} \times \mathbb{R}$ and $w$ is the solution of the Dirichlet problem \(\mathcal{D}_{r_1, r_2}\) with boundary data $g$ then $w$, together with all its partial derivatives up to order $m$, are uniformly Hölder continuous with exponent $\alpha$ on $\overline{\mathcal{A}_{r_1, r_2}}$.

**Proof.** For simplicity the lemma will only be proved for the case $m = 2$, since the case of a positive integer $m \geq 3$ follows in the same way, using backward induction. To begin with, choose any $z_1, z_2 \in \mathcal{A}_{r_1, r_2}$. By Kellogg’s theorem the second order partial derivatives of $w$ are locally $\alpha$ Hölder continuous in some neighborhood of any point $a \in \partial \mathcal{A}_{r_1, r_2}$. Since $r_2 < \infty$ the closure of $\mathcal{A}_{r_1, r_2}$ is a compact subset of the complex-plane and consequently the second-order partial derivatives of $w$ turn out to be uniformly Hölder continuous with exponent $\alpha$. Indeed the second-order partial derivatives of $w$ are locally Lipschitz continuous in some neighborhood of any point contained in $\mathcal{A}_{r_1, r_2}$, and if we choose the neighborhood small enough it follows that they are also locally $\alpha$ Hölder continuous ($|z_2 - z_1| \leq \|z_2 - z_1\|^\alpha$ when $z_1$ and $z_2$ are sufficiently close). Next assume $w_{xx}$ is not uniformly Hölder continuous with exponent $\alpha$. If so, there exist two sequences $z_n$ and $\xi_n$ in $\mathcal{A}_{r_1, r_2}$ such that $\frac{|w_{xx}(z_n) - w_{xx}(\xi_n)|}{|z_n - \xi_n|^\alpha} \to \infty$. Since $w_{xx}$ is in particular continuous on $\overline{\mathcal{A}_{r_1, r_2}}$ it is also bounded there and hence there are subsequences $z_{n_k}$ and $\xi_{n_k}$ converging to some $z$ in the closure of $\mathcal{A}_{r_1, r_2}$. But this contradicts the fact that $w_{xx}$ is locally $\alpha$ Hölder continuous at $z$. The exact same reasoning can also be applied to $w_{yy}$ and $w_{xy}$, respectively. It is also easy to prove that $w, w_x$ and $w_y$ are uniformly $\alpha$ Hölder continuous. This can be seen using an integral representation in terms of the higher-order partial derivatives in a sufficiently small convex neighborhood of each point $z \in \overline{\mathcal{A}_{r_1, r_2}}$, together with the compactness of the closed annulus.

\[\square\]

**Theorem 2.** Let $0 < r_1 < r_2 < \infty$ and assume $\phi : \{r_1, r_2\} \times \mathbb{R} \to \mathbb{R}$ is $2\pi$-periodic in the second variable, satisfies $2\pi \int_0^{2\pi} r_1 \phi(r_1, \theta) \, d\theta = 2\pi \int_0^{2\pi} r_2 \phi(r_2, \theta) \, d\theta$, and in addition suppose there exists $\alpha \in (0, 1]$ such that $\phi(r, \cdot)$ belongs to $C^{m, \alpha}(\mathbb{R})$ for some positive integer $m \geq 2$, whenever $r \in \{r_1, r_2\}$. If $U$ is any solution of the Neumann problem \(\mathcal{N}_{r_1, r_2}\) on $[r_1, r_2] \times \mathbb{R}$ with boundary data $\phi$, then $U$ is uniformly Hölder continuous with exponent $\alpha$, and likewise are all its partial derivatives up to order $m + 1$.

**Proof.** Define $g : \partial \mathcal{A}_{r_1, r_2} \to \mathbb{R}, g(re^{i\theta}) = r\phi(r, \theta), \theta \in \mathbb{R}$, and let $w$ be the solution of the Dirichlet problem \(\mathcal{D}_{r_1, r_2}\) with boundary data $g$. According to Lemma the harmonic function $w$ together with all its partial derivatives up to order $m$ are uniformly Hölder continuous with exponent $\alpha$ on $\overline{\mathcal{A}_{r_1, r_2}}$. The theorem will be proved for the case $m = 2$, as the case of a general positive integer greater than or equal to two follows exactly in the same manner, using induction. With the same notations as those used in Theorem notice that Proposition together with the uniqueness of the solution to Dirichlet problem implies that $u$ is just the representation in polar coordinates of $w$; more precisely we have
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\[ u(r, \theta) = w(re^{i\theta}) \quad \forall r \in [r_1, r_2], \ \forall \theta \in \mathbb{R}. \]
Thus for all pairs \((r, \theta)\) in \((r_1, r_2) \times \mathbb{R}\) we obtain the following relations

\[
\begin{align*}
\frac{\partial u}{\partial \theta}(r, \theta) &= w_x(z) \cos \theta + w_y(z) \sin \theta, \\
\frac{\partial u}{\partial r}(r, \theta) &= -rw_x(z) \sin \theta + rw_y(z) \cos \theta, \\
\frac{\partial^2 u}{\partial r^2}(r, \theta) &= w_{xx}(z) \sin \theta + r^2 w_{yy}(z) \sin^2 \theta, \\
\frac{\partial^2 u}{\partial r \partial \theta}(r, \theta) &= w_y(z) \cos \theta - w_x(z) \sin \theta + r w_{xy}(z) \cos 2\theta + \frac{r^2}{2} w_{yy}(z) - w_{xx}(z) \sin 2\theta, \\
\frac{\partial^2 u}{\partial \theta^2}(r, \theta) &= -r^2 w_{yy}(z) \sin \theta + r^2 \left( w_{yy}(z) \cos^2 \theta + w_{xx}(z) \sin^2 \theta \right) - rw_y(z) \sin \theta - r w_x(z) \cos \theta, \\
\end{align*}
\]

where \(z = x + iy = r \cos \theta + ir \sin \theta\). According to Lemma 1, \(w, w_x, w_y, w_{xx}, w_{yx}, w_{yy}\) can be continuously extended to \(A_{r_1, r_2}\), and so \(u\) and all its partial derivatives up to order two can be continuously extended to \([r_1, r_2] \times \mathbb{R}\), due to the above relations. Further choose any \(\alpha_1, \alpha_2 \in [r_1, r_2]\) and any \(\theta_1, \theta_2 \in \mathbb{R}\) and denote \(z_1 = \alpha_1 e^{i\theta_1}, \ z_2 = \alpha_2 e^{i\theta_2}\). Then one has \(|u(\alpha_2, \theta_2) - u(\alpha_1, \theta_1)| = |w(z_2) - w(z_1)|\). Using again Lemma 1, there is a positive constant dubbed \(H\) such that the Hölder constants corresponding to \(w\) and all its partial derivatives up to order two, respectively, are upper bounded by it. Consequently this implies in particular that \(|w(z_2) - w(z_1)| \leq H_{w}|z_2 - z_1|^\alpha\). On the other hand notice that the geometry of the annulus \(A_{r_1, r_2}\) reveals that \(z_1\) and \(z_2\) must satisfy \(|z_2 - z_1| \leq |\alpha_2 - \alpha_1| + \max(\alpha_1, \alpha_2)||\theta_2 - \theta_1|| \leq |\alpha_2 - \alpha_1| + r_2|\theta_2 - \theta_1||\) (see Figure 1). Hence

\[
|z_2 - z_1| \leq (r_2 + 1)\sqrt{|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2}. \tag{20}
\]

To sum up we have just proved that

\[
|u(\alpha_2, \theta_2) - u(\alpha_1, \theta_1)| \leq H_{w}(r_2 + 1)^\alpha \left(|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2\right)^{\frac{\alpha}{2}}. \tag{21}
\]

As for the first order partial derivatives of \(u\), using the first two relations in (19), we notice that \(|\partial u(\alpha_2, \theta_2) - \partial u(\alpha_1, \theta_1)| \leq \left|w_x(z_2) \cos \theta_2 - w_x(z_1) \cos \theta_1 + w_y(z_2) \sin \theta_2 - w_y(z_1) \sin \theta_1\right| \leq \left|w_x(z_2) - w_x(z_1)\right| + \left|\cos \theta_2 - \cos \theta_1\right| \left|w_x(z_1)\right| + \left|w_y(z_2) - w_y(z_1)\right| + \left|\sin \theta_2 - \sin \theta_1\right| \left|w_y(z_1)\right| \leq 2H_{w}|z_2 - z_1| + (|w_x| + |w_y|)||\theta_2 - \theta_1||.\] If the euclidean distance between the pairs of points \((\alpha_1, \theta_1), (\alpha_2, \theta_2)\) is less than one then \(\sqrt{|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2} \leq (|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2)^{\frac{1}{2}}\), and using (20)

\[
|\partial u(\alpha_2, \theta_2) - \partial u(\alpha_1, \theta_1)| \leq \left(|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2\right)^{\frac{\alpha}{2}}, \tag{22}
\]
where a proportionality constant is given by

\[
2H_{w}(r_2 + 1)^\alpha + |w_x| + |w_y|. \]

In a similar way \(|\partial u(\alpha_2, \theta_2) - \partial u(\alpha_1, \theta_1)| \leq \left|\alpha_2 w_x(z_2) \sin \theta_2 - \alpha_1 w_x(z_1) \sin \theta_1 + \alpha_2 w_y(z_2) \cos \theta_2 - \alpha_1 w_y(z_1) \cos \theta_1\right| \leq \left|\alpha_2 w_x(z_2) - \alpha_1 w_x(z_1)\right| + \left|\alpha_2 w_y(z_2) - \alpha_1 w_y(z_1)\right| + \left|\alpha_1 \cos \theta_2 - \cos \theta_1\right| \left|w_y(z_1)\right| \leq 2r_2 H_{w}(r_2 + 1)^\alpha \left(|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2\right)^{\frac{\alpha}{2}} + (r_2 + 1)\left(|w_x| + |w_y|\right)\sqrt{|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2},\]
and if \((\alpha_1, \theta_1)\) and \((\alpha_2, \theta_2)\) are close enough in the euclidean distance (that is if the euclidean distance is less than one) then

\[
|u_\theta(\alpha_2, \theta_2) - u_\theta(\alpha_1, \theta_1)| \lesssim \left(|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2\right)^{\frac{1}{2}},
\]

where a proportionality constant is

\[
2r_2 H_w (r_2 + 1)^\alpha + (r_2 + 1) (\|w_x\| + \|w_y\|).
\]

Proceeding further the last three equations in (19) will be used in order to derive similar conclusions on the second-order partial derivatives of \(u\). To this end notice first that

\[
|u_{rr}(\alpha_2, \theta_2) - u_{rr}(\alpha_1, \theta_1)| \leq |w_{xx}(z_2) - w_{xx}(z_1)| + 2|\cos \theta_2 - \cos \theta_1||w_{xx}(z_1)| + |w_{yx}(z_2) - w_{yx}(z_1)| + |\sin 2\theta_2 - \sin 2\theta_1||w_{yx}(z_1)| + |w_{yy}(z_2) - w_{yy}(z_1)| + 2|\theta_2 - \theta_1|,
\]

and if the pairs \((\alpha_2, \theta_2)\), \((\alpha_1, \theta_1)\) are again assumed to be close in the euclidean distance then

\[
|u_{rr}(\alpha_2, \theta_2) - u_{rr}(\alpha_1, \theta_1)| \lesssim \left(|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2\right)^{\frac{1}{2}},
\]

where a proportionality constant is given by

\[
3H_w (r_2 + 1) + 2 (\|w_{xx}\| + \|w_{yx}\| + \|w_{yy}\|).
\]

Also \(|u_{r\theta}(\alpha_2, \theta_2) - u_{r\theta}(\alpha_1, \theta_1)| \leq |w_y(z_2) \cos \theta_2 - w_y(z_1) \cos \theta_1| + |w_x(z_2) \sin \theta_2 - w_x(z_1) \sin \theta_1| + |\alpha w_{yx}(z_2) \sin 2\theta_2 - \alpha w_{yx}(z_1) \sin 2\theta_1| + |\alpha w_{yy}(z_2) - w_{xx}(z_2)| \cos 2\theta_1 + 2|\alpha w_{yy}(z_1) - w_{xx}(z_1)| \sin 2\theta_1 + |\alpha w_{yy}(z_1) - w_{xx}(z_1)| \sin 2\theta_1|.
\]

Again if \((\alpha_1, \theta_1)\) and \((\alpha_2, \theta_2)\) are sufficiently close in the euclidean distance then

\[
|u_{r\theta}(\alpha_2, \theta_2) - u_{r\theta}(\alpha_1, \theta_1)| \lesssim \left(|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2\right)^{\frac{1}{2}},
\]

with a proportionality constant equal to

\[
\|w_x\| + \|w_y\| + (r_2 + 0.5) (\|w_{xx}\| + \|w_{yy}\|) + (2r_2 + 1) \|w_{xy}\| + 2H_w (r_2 + 1)^{\alpha + 1}.
\]

Finally we compute

\[
|u_{\theta\theta}(\alpha_2, \theta_2) - u_{\theta\theta}(\alpha_1, \theta_1)| \leq |\alpha_2^2 w_{xy}(z_2) \sin 2\theta_2 - \alpha_1^2 w_{xy}(z_1) \sin 2\theta_1| + |\alpha_2^2 (w_{yy}(z_2) \cos 2\theta_2 + w_{xx}(z_2) \sin 2\theta_2) - \alpha_1^2 (w_{yy}(z_1) \cos 2\theta_1 + w_{xx}(z_1) \sin 2\theta_1)| + |\alpha_2 w_y(z_2) \sin \theta_2 - \alpha_1 w_y(z_1) \sin \theta_1| + |\alpha_2 w_x(z_2) \cos \theta_2 - \alpha_1 w_x(z_1) \cos \theta_1|,
\]

and if we assume once more that the pairs \((\alpha_1, \theta_1)\) and \((\alpha_2, \theta_2)\) are close in the euclidean distance then

\[
|u_{\theta\theta}(\alpha_2, \theta_2) - u_{\theta\theta}(\alpha_1, \theta_1)| \lesssim \left(|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2\right)^{\frac{1}{2}},
\]

where a proportionality constant is

\[
(r_2 + 1) \|w_x\| + \|w_y\| + 2r_2 (\|w_{xx}\| + \|w_{xy}\| + \|w_{yy}\|) + 3H_w r_2 (r_2 + 1)^{\alpha + 1}.
\]

To sum up equations (21)-(26) show that \(u\) together with all its partial derivatives up to the second order are locally \(\alpha\) Hölder continuous on \([r_1, r_2] \times \mathbb{R}\) with uniformly bounded constants. Hence \(u\) and its partial derivatives up to
order two are uniformly Hölder continuous with exponent \( \alpha \) on any compact subset of \( [r_1, r_2] \times \mathbb{R} \) and so on \( [r_1, r_2] \times [0, 3\pi] \) in particular. To show that this latter fact suffices to conclude that \( u \) together with its partial derivatives up to order two are uniformly Hölder continuous with exponent \( \alpha \) on \( [r_1, r_2] \times \mathbb{R} \) choose any \( \alpha_1, \alpha_2 \in [r_1, r_2] \) and any \( \theta_1, \theta_2 \in \mathbb{R} \) and let \( k_1, k_2 \) be two integers such that defining \( \theta_1' = \theta_1 - 2k_1\pi, \theta_2' = \theta_2 - 2k_2\pi, \theta_1'' \) and \( \theta_2'' \) satisfy \( |\theta_2'' - \theta_1''| \in [0, 2\pi] \). There are two possible cases.

i. \( |\theta_2' - \theta_1'| \leq \pi \), in which case we claim that \( |\theta_2 - \theta_1| \geq |\theta_2'' - \theta_1''| \). Indeed if \( k_2 > k_1 \) then \( |\theta_2 - \theta_1| = |\theta_2' - \theta_1'| + 2(k_2 - k_1)\pi = 2(k_2 - k_1)\pi - (\theta_2' - \theta_1') \geq 2(k_2 - k_1)\pi - |\theta_2' - \theta_1'| \geq 2\pi \), \( |\theta_2 - \theta_1| \geq 2\pi - |\theta_2'' - \theta_1''| \geq |\theta_2'' - \theta_1''| \). If \( k_1 > k_2 \) switch the indexes 1 and 2, and if \( k_1 = k_2 \) the inequality is trivial.

ii. \( |\theta_2' - \theta_1'| > \pi \). Assume first that \( \theta_2' > \theta_1' \), in which case we find that \( 3\pi > \theta_2' + 2\pi =: \theta_2'' = \theta_2'' = \theta_2'' = \theta_2'' = \theta_2'' \). Also \( |\theta_2'' - \theta_1''| = |\theta_2'' - \theta_2'' - \theta_2'' - \theta_2''| = 2\pi - |\theta_2'' - \theta_1''| < \pi \). But setting \( k_1 := k_1 - 1 \) and \( k_2 := k_2 \) it follows that \( \theta_2 = \theta_2'' + 2k_1\pi \) and \( \theta_1 = \theta_1'' + 2k_1\pi \), respectively. Since \( |\theta_2'' - \theta_1''| < \pi \) the previous point shows \( |\theta_2 - \theta_1| \geq |\theta_2'' - \theta_1''| \). If \( \theta_2' > \theta_1' \) then \( 3\pi > \theta_2' + 2\pi =: \theta_2'' = \theta_2'' = \theta_2'' = \theta_2'' \) and also \( |\theta_2'' - \theta_1''| = 2\pi - |\theta_2'' - \theta_1''| < \pi \). Proceeding similarly one obtains \( |\theta_2 - \theta_1| \geq |\theta_2'' - \theta_1''| \). We conclude thus that in the case when \( |\theta_2' - \theta_1'| > \pi \) there also exist two integers, dubbed \( k_1' \) and \( k_2' \), such that denoting \( \theta_2'' = \theta_2 - 2k_2\pi \) and \( \theta_1'' = \theta_1 - 2k_1\pi \), respectively, it follows that \( \theta_1'' \), \( \theta_2'' \) \( \in [\pi, 3\pi] \) and in addition \( |\theta_2 - \theta_1| \geq |\theta_2'' - \theta_1''| \).

Hence for any pairs \( (\alpha_1, \theta_1), (\alpha_2, \theta_2) \in [r_1, r_2] \times \mathbb{R} \) one can always find pairs \( (\alpha_1, \theta_1') \) and \( (\alpha_2, \theta_2') \), respectively, such that \( \theta_1', \theta_2' \in [0, 3\pi] \) and in addition \( \sqrt{|\alpha_2 - \alpha_1|^2 + |\theta_2' - \theta_1'|^2} \leq \sqrt{|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2} \). Consequently this shows, using the \( 2\pi \)-periodicity in the second argument of \( u \) and of all its partial derivatives up to order two, that there is a positive constant, call it \( H_u \), such that

\[
|u(\alpha_2, \theta_2) - u(\alpha_1, \theta_1)| = |u(\alpha_2, \theta_1') - u(\alpha_1, \theta_1')| \leq H_u \left( |\alpha_2 - \alpha_1|^2 + |\theta_2' - \theta_1'|^2 \right)^{\frac{\alpha}{2}},
\]

and the same holds true for \( u_r, u_{\theta}, u_{rr}, u_{\theta r}, u_{\theta \theta} \), respectively, on \( [r_1, r_2] \times \mathbb{R} \).

Finally the uniform Hölder continuity of \( u \) and of its partial derivatives will be used in order to draw conclusions about the uniform Hölder continuity of \( U \) and of its partial derivatives up to order three. In this respect relations \((11)\) and \((13)\) will be a key element. More precisely choose any \( (\alpha_1, \theta_1), (\alpha_2, \theta_2) \in [r_1, r_2] \times \mathbb{R} \) and notice that

\[
|U(\alpha_2, \theta_2) - U(\alpha_1, \theta_1)| \leq \left| \int_{\sqrt{r_1^2 r_2}}^{\sqrt{r_1^2 r_2}} \int_{\theta_1}^{\theta_2} \left( C - \int_{0}^{t} u_r(\sqrt{r_1 r_2}, \tau) d\tau \right) d\rho \right| + \sqrt{r_1^2 r_2} \int_{\theta_1}^{\theta_2} \left( -\int_{0}^{t} u_r(\sqrt{r_1 r_2}, \tau) d\tau \right) d\tau.
\]

According to the proof of Theorem 1 the real-valued function \( h(t) := C - \int_{0}^{t} u_r(\sqrt{r_1 r_2}, \tau) d\tau \) is \( 2\pi \)-periodic and so one can readily see that \( \sup_{t \in [0, 2\pi]} |h(t)| < \infty \). Consequently \( |U(\alpha_2, \theta_2) - U(\alpha_1, \theta_1)| \leq \)
Taking the derivative with respect to the second argument in (11) and using the assumption that \((\alpha_1, \theta_1)\) and \((\alpha_2, \theta_2)\) are close enough in the euclidean distance we observe that

\[
|U(\alpha_2, \theta_2) - U(\alpha_1, \theta_1)| \lesssim (|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2)^{\frac{2}{\sigma}},
\]

where a proportionality constant is given by

\[
\frac{||u||}{r_1} + \frac{r_2 - r_1}{r_1} + \sqrt{r_1 r_2}||h||.
\]

Using relation (13) \(|U_\rho(\alpha_2, \theta_2) - U_\rho(\alpha_1, \theta_1)| \leq \frac{1}{r_1^2}||\alpha_1 u(\alpha_2, \theta_2) - \alpha_2 u(\alpha_1, \theta_1)|| \leq \frac{r_2^2}{r_1^2}||u|||\alpha_2 - \alpha_1|\) and assuming \((\alpha_1, \theta_1), (\alpha_2, \theta_2)\) are sufficiently close then

\[
|U_\rho(\alpha_2, \theta_2) - U_\rho(\alpha_1, \theta_1)| \lesssim (|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2)^{\frac{1}{2}},
\]

where a proportionality constant is

\[
\frac{r_2 + ||u||}{r_1^2}.
\]

Taking the derivative with respect to the second argument in (11) and using the \(2\pi\)-periodicity in the second argument for \(U_\theta\) gives \(|U_\theta(\alpha_2, \theta_2) - U_\theta(\alpha_1, \theta_1)|| \leq \frac{\alpha_1 u_\theta|\alpha_2|}{2} + \int R_1 R_2 \frac{\theta_1 \delta_2}{\theta_1 \delta_2} ||u_\theta|| d\theta + \int \frac{\alpha_1 u_\theta|\alpha_2 - |\alpha_1|\theta_1|\theta_2 - \theta_1||d\theta + \int U_\theta(\sqrt{r_1 r_2}, t)}{2} dt \leq |\alpha_2 - \alpha_1| |u_\theta|| + \frac{r_2 - r_1}{2} H_\rho |\theta_2 - \theta_1|^2 + \frac{r_2 - r_1}{2} |u_\theta||,\) and if \((\alpha_1, \theta_1), (\alpha_2, \theta_2)\) are sufficiently close then

\[
|U_\theta(\alpha_2, \theta_2) - U_\theta(\alpha_1, \theta_1)| \lesssim (|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2)^{\frac{1}{2}},
\]

where a proportionality constant is found to be

\[
\frac{||\alpha_\theta||}{r_1} + \frac{r_2 - r_1}{r_1} H_\theta + r_2 ||u_\theta||.
\]

For the partial derivatives of order two of \(U\) notice that, whenever \((r, \theta) \in (r_1, r_2) \times \mathbb{R}\), they are given by \(U_{rr} = \frac{u_r(r, \theta)}{r}, U_{r\theta}(r, \theta) = \frac{u_{\theta}(r, \theta)}{r}\), and using relations (5) and (13) \(U_{\theta\theta}(r, \theta) = -\frac{1}{r} U_{rr}(r, \theta) - \frac{r}{r} U_{r\theta}(r, \theta)\). These relations show that \(U_{rr}, U_{r\theta}\) and \(U_{\theta\theta}\) can be continuously extended to \([r_1, r_2] \times \mathbb{R}\) and the same notation will be kept for their continuous extensions. Choose now any pairs \((\alpha_1, \theta_1), (\alpha_2, \theta_2) \in [r_1, r_2] \times \mathbb{R}\) and compute

\[
|U_{rr}(\alpha_2, \theta_2) - U_{rr}(\alpha_1, \theta_1)| \leq \left| \frac{u_r(\alpha_2, \theta_2)}{\alpha_2} - \frac{u_r(\alpha_1, \theta_1)}{\alpha_1} \right| + \left| \frac{u_{\theta}(\alpha_2, \theta_2)}{\alpha_2} - \frac{u_{\theta}(\alpha_1, \theta_1)}{\alpha_1} \right| \leq
\]
Equivalence in planar doubly-connected regions

$\frac{r_2}{r_1^2} |u_r(\alpha_2, \theta_2) - u_r(\alpha_1, \theta_1)| + \frac{1}{r_1} |\alpha_2 - \alpha_1||u_r(\alpha_1, \theta_1)| + \frac{r_2^2}{r_1^2} |u(\alpha_2, \theta_2) - u(\alpha_1, \theta_1)| + \frac{r_1 + r_2}{r_1^2} |u(\alpha_1, \theta_1)||\alpha_2 - \alpha_1|$. Again if $(\alpha_1, \theta_1), (\alpha_2, \theta_2)$ are close in the euclidean distance then

$$|U_{rr}(\alpha_2, \theta_2) - U_{rr}(\alpha_1, \theta_1)| \lesssim (|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2)^{\frac{2}{2}}$$

where a proportionality constant is given by

$$\frac{r_2^2 + \|u_r\| + (r_1 + r_2)||u|}{r_1^2}.$$ 

Also $|U_{r\theta}(\alpha_2, \theta_2) - U_{r\theta}(\alpha_1, \theta_1)| \leq \frac{r_2}{r_1} |u_r(\alpha_2, \theta_2) - u_r(\alpha_1, \theta_1)| + \frac{1}{r_1} |u_{\theta}(\alpha_1, \theta_1)||\alpha_2 - \alpha_1|$, and under the same closeness assumption on the pairs $(\alpha_1, \theta_1), (\alpha_2, \theta_2)$ one has

$$|U_{r\theta}(\alpha_2, \theta_2) - U_{r\theta}(\alpha_1, \theta_1)| \lesssim (|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2)^{\frac{2}{2}}$$

where a proportionality constant can be easily found to be

$$\frac{r_2 + \|u_{\theta}\|}{r_1^2}.$$ 

Finally $|U_{\theta\theta}(\alpha_2, \theta_2) - U_{\theta\theta}(\alpha_1, \theta_1)| \leq r_2 |u_r(\alpha_2, \theta_2) - u_r(\alpha_1, \theta_1)| + |u_{\theta}(\alpha_1, \theta_1)|$. 

$|\alpha_2 - \alpha_1|$, and if $(\alpha_1, \theta_1), (\alpha_2, \theta_2)$ are close enough in the euclidean distance, then

$$|U_{\theta\theta}(\alpha_2, \theta_2) - U_{\theta\theta}(\alpha_1, \theta_1)| \lesssim (|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2)^{\frac{2}{2}}$$

where a proportionality constant is

$$r_2 H_u + \|u_r\|.$$ 

Proceeding further $U_{r\theta r}(r, \theta) = \frac{\partial}{\partial r} (\frac{r u_r(r, \theta) - u_r(r, \theta)}{r^2}) = \frac{r^2 u_{rr}(r, \theta) - 2r u_r(r, \theta) + 2u_r(r, \theta)}{r^2} + u_{rr}(r, \theta)$, $U_{r\theta\theta}(r, \theta) = U_{\theta\theta\theta}(r, \theta) = U_{r\theta r}(r, \theta) = -u_r(r, \theta) - r u_{rr}(r, \theta)$, $U_{\theta\theta r}(r, \theta) = U_{r\theta\theta}(r, \theta) = \frac{\partial}{\partial \theta} (\frac{r u_r(r, \theta) - u_r(r, \theta)}{r^2}) = \frac{r u_{r\theta}(r, \theta) - u_{\theta}(r, \theta)}{r^2}$, and $U_{\theta\theta\theta}(r, \theta) = r u_{r\theta}(r, \theta)$. But then a similar reasoning as above, using the triangle inequality, shows that all the third order partial derivatives of $U$ can be continuously extended to $[r_1, r_2] \times \mathbb{R}$ and satisfy

$$|U_{abc}(\alpha_2, \theta_2) - U_{abc}(\alpha_1, \theta_1)| \lesssim (|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2)^{\frac{2}{2}}, \quad a, b, c \in \{r, \theta\},$$

(33)

with proportionality constants depending only on $r_1$, $r_2$, $H_u$, $\|u_r\|$, $\|u_{r\theta}\|$, $\|u_{r\theta\theta}\|$, for any $(\alpha_1, \theta_1), (\alpha_2, \theta_2) \in [r_1, r_2] \times \mathbb{R}$ which are close enough in the euclidean distance.

In conclusion it has been shown so far that $U$ together with its partial derivatives up to order three are locally $\alpha$ Hölder continuous on $[r_1, r_2] \times \mathbb{R}$, and using a compactness argument we can argue that they are uniformly Hölder continuous with exponent $\alpha$ on $[r_1, r_2] \times [0, 3\pi]$. By considering the same argument as the one used earlier in the proof for $u$ and its partial derivatives up to order
two, we can see that $U$ and its partial derivatives up to order three are actually uniformly Hölder continuous with exponent $\alpha$ on $[r_1, r_2] \times \mathbb{R}$. This concludes the whole proof.

Figure 1: Some geometric properties of the annulus.

Next some remarks will be provided. Notice first that for $r_1 \searrow 0$ and $r_2 = 1$, the region $A_{r_1;r_2}$ becomes the punctured unit disk $A_{0;1} = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = \mathbb{U}$. If $w : \mathbb{U} \to \mathbb{R}$ is a harmonic function having a finite limit at the origin (an isolated boundary point of the domain), then it is known that $w$ can be extended by continuity at the origin, and the resulting function is harmonic in $\mathbb{U}$. If $w$ has a continuous extension to $\overline{\mathbb{U}}$, with boundary values $w(0) = \varphi(0, \cdot)$ (a constant function of $\theta \in \mathbb{R}$) and $w(e^{i\theta}) = \varphi(1, \theta)$, $\theta \in \mathbb{R}$, then the condition

$$
\frac{2\pi}{0} \int \varphi(0, \theta) \, d\theta = \frac{2\pi}{0} \int \varphi(1, \theta) \, d\theta
$$

in Theorem 1 is a necessary condition for the solvability of the Dirichlet problem in $\mathbb{U}$ with continuous boundary data $\varphi$;

$$
w(0) = \varphi(0, \cdot) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(0, \theta) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \varphi(1, \theta) \, d\theta.
$$

Subtracting a constant if necessary (i.e. considering $w - w(0)$ instead), without loss of generality it can be assumed that $w(0) = 0$, or equivalently $0 = \ldots$
The continuous extensions of \( \partial \) finally gives \( \partial u \) difficult to show, using Poisson’s formula as well as the Dominant Convergence

Also \( \partial r \) \( \phi \) continuous boundary data punctured disk \( \dot{\mathbb{D}} \). Equivalence in planar doubly-connected regions where \( u \) second order partial derivatives, and satisfies equation (5) in (0, 0) by Proposition 1 that \( \hat{U} \) ally coincides with the representation in polar coordinates of \( U \), and in addition it has finite partial derivative with respect to the first variable at any point \((1, \theta_0)\), \( \theta_0 \in \mathbb{R} \). Moreover \( \frac{\partial \tilde{U}_0}{\partial r}(0, \theta) = \lim_{r \to 0} \frac{U_{0}(r e^{i\theta}) - U_{0}(0, \theta)}{r} = \lim_{r \to 0} \frac{U_{0}(r \theta)}{r} = \lim_{r \to 0} \frac{U_{0}(r \theta)}{r} \).

Similarly if \( \phi(r, \cdot) \in C^0(\mathbb{R}) \), \( r \in \{0, 1\} \), satisfies \( 0 = \frac{1}{2\pi} \int_0^{2\pi} \phi(1, \theta) \, d\theta \), \( \phi(0, \theta) = \frac{1}{\pi} \int_0^{2\pi} \cos(t - \theta) \phi(1, t) \, dt \), \( \theta \in \mathbb{R} \) and if \( U_0 \) is the solution of the Neumann problem [1] on \( U \) which vanishes for \( z = 0 \) and has boundary data \( \Phi_0(z) := \phi(1, \arg(z)) \), then the Neumann problem [1] has a solution which actually coincides with the representation in polar coordinates of \( U_0 \). Indeed, by applying [1] Theorem 1 (or Corollary 1 below) it follows that \( \tilde{U}_0(z) = \int_0^{\rho(r,z,2)} \rho \, d\rho \), where \( u_0 \) is the solution of the Dirichlet problem in \( U \) with boundary data \( \varphi_0 := \Phi_0 \). But then defining \( \hat{U}_0(r, \theta) = U_0(r e^{i\theta}) \), \( (r, \theta) \in [0, 1] \times \mathbb{R} \), it follows by Proposition 1 that \( \hat{U}_0 \) is \( 2\pi \)-periodic in the second variable, has continuous second order partial derivatives, and satisfies equation (5) in (0, 1) \( \times \mathbb{R} \), and in addition it has finite partial derivative with respect to the first variable at any point \((1, \theta_0)\), \( \theta_0 \in \mathbb{R} \). Moreover \( \frac{\partial \tilde{U}_0}{\partial r}(0, \theta) = \lim_{r \to 0} \frac{U_{0}(r \theta) - U_{0}(0, \theta)}{r} = \lim_{r \to 0} \frac{U_{0}(r \theta)}{r} = \lim_{r \to 0} \frac{U_{0}(r \theta)}{r} \).

Also \( \frac{\partial \tilde{U}_0}{\partial \theta}(1, \theta) = \frac{\partial \tilde{U}_0}{\partial \theta}(\theta, \cdot) = \Phi_0(\theta, \cdot) = \phi(1, \theta) \), whenever \( \theta \in \mathbb{R} \). It is not difficult to show, using Poisson’s formula as well as the Dominant Convergence theorem (see also [1] Theorem 2.27), that \( \frac{\partial u_0}{\partial x}(0) = \frac{1}{\pi} \int_0^{2\pi} \Phi_0(e^{i\theta}) \cos t \, dt = \frac{1}{\pi} \int_0^{2\pi} \phi(1, t) \cos t \, dt \) and \( \frac{\partial u_0}{\partial y}(0) = \frac{1}{\pi} \int_0^{2\pi} \Phi_0(e^{i\theta}) \sin t \, dt = \frac{1}{\pi} \int_0^{2\pi} \phi(1, t) \sin t \, dt \), which finally gives \( \frac{\partial \tilde{U}_0}{\partial \theta}(0, \theta) = \frac{1}{\pi} \int_0^{2\pi} \cos(t - \theta) \phi(1, t) \, dt \). To sum up it can be concluded that

\[
\frac{\partial \tilde{U}_0}{\partial r}(r, \theta) = \begin{cases} \frac{1}{\pi} \int_0^{2\pi} \cos(t - \theta) \phi(1, t) \, dt, & \text{if } r = 0, \\ \phi(1, \theta), & \text{if } r = 1. \end{cases}
\]

The continuous extensions of \( \frac{\partial \tilde{U}_0}{\partial r} \) and \( \frac{\partial \tilde{U}_0}{\partial \theta} \) to \([0, 1] \times \mathbb{R}\) are easily justified by Corollary 1 (below) and Proposition 1.

With this preamble the following two definitions will be introduced, with
the convention that for both of them \( D = A_{0,1} \).

**Definition 3.** If \( \varphi : \mathbb{R} \to \mathbb{R} \) is continuous, 2\( \pi \)-periodic, and satisfies \( \int_0^{2\pi} \varphi(\theta) \, d\theta = 0 \), then the Dirichlet problem in polar coordinates for \( D \) consists in finding \( u = u(r, \theta) \in C^2((0,1) \times \mathbb{R}) \cap C^0([0,1] \times \mathbb{R}) \) which is 2\( \pi \)-periodic in the second variable and satisfies

\[
\begin{align*}
&u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{in} \ (0,1) \times \mathbb{R}, \\
&u(1,\cdot) = \varphi(\cdot), \\
&u(0,\cdot) = 0.
\end{align*}
\]

**Definition 4.** If \( \phi : \{0;1\} \times \mathbb{R} \to \mathbb{R} \) is continuous, 2\( \pi \)-periodic in the second argument, and satisfies \( \int_0^{2\pi} \phi(1,\theta) \, d\theta = 0 \) as well as \( \phi(0,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \cos(t - \theta) \phi(1,t) \, dt \), then the Neumann problem in polar coordinates for \( D \) consists in finding \( U \in C^2((0,1) \times \mathbb{R}) \cap C^1([0,1] \times \mathbb{R}) \) which is 2\( \pi \)-periodic in the second variable, and satisfies

\[
\begin{align*}
&U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0 \quad \text{in} \ (0,1) \times \mathbb{R}, \\
&U_r = \phi \quad \text{in} \ \{0;1\} \times \mathbb{R}, \\
&U(0,\cdot) = 0.
\end{align*}
\]

**Remark 3.** As we have already remarked, Definition 3 is nothing but the polar coordinates version of the Dirichlet problem (2) for \( D = \mathbb{U} \) and boundary data \( g(z) = \varphi(\arg(z)) \) on \( \partial\mathbb{U} \). Definition 4 instead, comes with a novelty which allows one to formulate the Neumann problem in a consistent way, for the punctured disk as well. This fact is in contrast with the classical Neumann problem where the (outward) normal derivative at \( \{0\} \) can not be defined. In addition it reveals that if \( \hat{U} \) is the solution of the Neumann problem (35) on \( A_{0,1} \), then \( \hat{U} \) is just the representation in polar coordinates of the solution \( U \) to the Neumann problem (3) on \( \mathbb{U} \), with boundary data \( f(z) = \phi(1,\arg(z)) \) and \( U(0) = 0 \).

In the particular case when the boundary data is symmetric, the result in Theorem 3 has the following simplified form.

**Theorem 3.** Let \( 0 < r_1 < r_2 < \infty \) and assume \( \phi : \{r_1,r_2\} \times \mathbb{R} \to \mathbb{R} \) is continuous, 2\( \pi \)-periodic in the second argument, verifies the Dirichlet conditions as a function of \( \theta \), and satisfies \( r_1 \phi(\rho,1) = r_2 \phi(\rho,\theta) \) for \( \theta \in \mathbb{R} \). If \( U \) is the solution of the Neumann problem (3) with boundary data \( \phi \), satisfying \( U(\sqrt{r_1^2 - r_2^2},0) = 0 \), then

\[
U(r,\theta) = \frac{1}{\sqrt{r_1^2 - r_2^2}} \int_{r_2}^{r_1} \frac{u(\rho,\theta)}{\rho} \, d\rho, \quad (r,\theta) \in [r_1,r_2] \times \mathbb{R},
\]

where \( u \) is the solution of the Dirichlet problem (3) with boundary values \( \varphi(r,\theta) = r \phi(r,\theta) \) on \( \{r_1,r_2\} \times \mathbb{R} \). Conversely, if \( \varphi : \{r_1,r_2\} \times \mathbb{R} \to \mathbb{R} \) is continuous, 2\( \pi \)-periodic in the second variable, and satisfies \( \int_0^{2\pi} \varphi(r_1,\theta) \, d\theta = \int_0^{2\pi} \varphi(\rho,\theta) \, d\theta \), and
if \( U \) is a solution of the Neumann problem \( \square \) with boundary data \( \phi(r, \theta) = \frac{\varphi(r, \theta)}{r}, r \in \{r_1, r_2\} \), then

\[
u(r, \theta) = rU_r(r, \theta), \quad (r, \theta) \in [r_1, r_2] \times \mathbb{R},
\]

is the solution of the Dirichlet problem \( \square \).

**Proof.** For the first part it will be shown that under the additional hypothesis
\[
\alpha \phi(r_1, \theta) = \beta \phi(r_2, \theta), \quad \theta \in \mathbb{R},
\]
one has \( u(r, \theta) = u(\alpha r_1, \beta \theta) \) \( \forall \) \( (r, \theta) \in [r_1, r_2] \times \mathbb{R} \) from where it follows by derivation with respect to the first argument that
\[
u(r, \theta) = \frac{\beta}{\alpha} u_r(\alpha r_1, \beta \theta)
\]
and taking \( r = \sqrt{r_1 r_2} \) it follows that \( u_r(\sqrt{r_1 r_2}, \theta) = -u_r(\sqrt{r_1 r_2}, \theta) \) which in turn implies \( u_r(\sqrt{r_1 r_2}, \theta) = 0 \) \( \forall \) \( \theta \in \mathbb{R} \), and so \( U \) will have the desired expression. Notice that it is enough to prove the result for the special case \( r_2 = a \), \( r_1 = 1/a \), since the general case follows from this one by means of scalarization, in the same way it was done in the proof of Theorem \( \square \). Hence it can be assumed without loss of generality that \( r_2 = a \), \( r_1 = 1/a \), \( a > 1 \). Writing again the Fourier expansions for \( \varphi(r_1, \cdot) = \varphi(r_1, \cdot) \) it is obtained
\[
\varphi(r_2, \theta) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos k \theta + b_k \sin k \theta \right) = \varphi(r_1, \theta) \quad \forall \theta \in \mathbb{R}.
\]
But then the solution of the Dirichlet problem \( \square \) on \( \left( \frac{1}{a}, a \right) \times \mathbb{R} \) with boundary data \( \varphi \) is given by

\[
u(r, \theta) = A + B \log r + \sum_{k=1}^{\infty} \left[ (C_k r^k + D_k r^{-k}) \cos k \theta + (E_k r^k + G_k r^{-k}) \sin k \theta \right],
\]

with
\[
\begin{align*}
a_k &\left( C_k a^k + D_k a^{-k} \right) = a_k, \\
C_k a^k + D_k a^{-k} &= a_k, \\
E_k a^k + G_k a^{-k} &= b_k,
\end{align*}
\]

\( A = a_0, \ B = 0, \ C_k = D_k = \frac{a_k}{a^k + a^{-k}}, \ E_k = G_k = \frac{b_k}{a^k + a^{-k}}, k \in \mathbb{N}^*, \ \text{and hence}
\]
\[
u(r, \theta) = a_0 + \sum_{k=1}^{\infty} \left[ \frac{\cos k \theta + b_k \sin k \theta}{a^k + a^{-k}} \right], \quad (r, \theta) \in \left[ \frac{1}{a}, a \right] \times \mathbb{R}, \ \text{which}
\]
finally gives \( u(r, \theta) = u(\frac{1}{a}, \theta) \) for any pair \( (r, \theta) \in \left[ \frac{1}{a}, a \right] \times \mathbb{R} \).

The second part is just the second part of Theorem \( \square \). This concludes the proof. \( \square \)

Combining Proposition \( \square \), Theorem \( \square \), and Theorem \( \square \), an important result in cartesian coordinates is obtained. Before presenting and proving it, the following lemma must be provided.

**Lemma 2.** There exists some positive constant \( L > 0 \) such that if \( z_1 = \alpha_1 e^{i \theta_1} \) and \( z_2 = \alpha_2 e^{i \theta_2} \) are any two points in \( A_{r_1, r_2}, r_1 > 0, \) and \( |\theta_2 - \theta_1| \leq \pi \), then

\[
|\theta_2 - \theta_1| \leq L|z_2 - z_1|.
\]

(37)
Equivalence in planar doubly-connected regions

Proof. Denote \( z'_1 = r_1 e^{i\theta_1} \) and \( z'_2 = r_2 e^{i\theta_2} \) and notice that \( |z'_2 - z'_1| \leq |z_2 - z_1| \) (see Figure 1). Hence it suffices to show that \( |\theta_2 - \theta_1| \leq L|z'_1 - z'_2| \). Using the Cosine Rule we can readily notice that \( |z'_2 - z'_1| = |z'_1|^2 + |z'_2|^2 - 2|z'_1||z'_2| \cos (\theta_2 - \theta_1) = 2r_1^2 (1 - \cos (\theta_2 - \theta_1)) = 4r_1^2 \sin^2 \left( \frac{\theta_2 - \theta_1}{2} \right) \). Since we have assumed \( |\theta_2 - \theta_1| \leq \pi \), it follows that

\[
\sin \left( \frac{|\theta_2 - \theta_1|}{2} \right) = \frac{|z'_2 - z'_1|}{2r_1}. \tag{38}
\]

The equality \( \lim_{t \to 0} \frac{\sin t}{t} = 1 \) implies the existence of some sufficiently small \( \epsilon_0 > 0 \) such that \( \sin t \geq (1 - \epsilon_0) t \geq \frac{t}{2} \) as soon as \( t \in [0, \epsilon_0] \). On the other hand if \( t \in [\epsilon_0, \frac{\pi}{2}] \) then \( \sin t \geq \epsilon_0 \) and thus \( 2\sin \epsilon_0 t \leq \sin \epsilon_0 \leq \sin t \). Consequently define \( L_0 = \min \left( \frac{2\sin \epsilon_0}{\pi}, \frac{1}{2} \right) \), which shows that \( \sin \left( \frac{|\theta_2 - \theta_1|}{2} \right) \geq L_0 \frac{|\theta_2 - \theta_1|}{2} \) from where, using relation (38), one obtains \( L_0 \frac{|\theta_2 - \theta_1|}{2} \leq \sin \left( \frac{|\theta_2 - \theta_1|}{2} \right) = \frac{|z'_2 - z'_1|}{2r_1} \).

Finally, putting

\[
L = \frac{1}{L_0 r_1} = \frac{1}{r_1 \min \left( \frac{2\sin \epsilon_0}{\pi}, \frac{1}{2} \right)} \tag{39}
\]

the lemma is proved. \( \square \)

**Theorem 4.** Let \( f : \partial A_{r_1, r_2} \to \mathbb{R} \) be a continuous function satisfying \( \int_{\partial A_{r_1, r_2}} f \, d\sigma = 0 \). If \( U \) is the solution of the Neumann problem (3) with boundary data \( f \), satisfying \( U(\sqrt{r_1 r_2}) = 0 \), then for any point \( re^{i\theta} \in \partial A_{r_1, r_2} \)

\[
U(re^{i\theta}) = \int_{\sqrt{r_1^2 + r_2^2}}^{1} \frac{u(\rho e^{i\theta})}{\rho} \, d\rho + \sqrt{r_1 r_2} \int_{0}^{\theta} \left( \mathcal{C} - \int_{0}^{t} \frac{\partial u}{\partial \alpha_r}(\sqrt{r_1 r_2} e^{ir\tau}) \, d\tau \right) \, dt, \tag{40}
\]

where \( u \) is the solution of the Dirichlet problem (2) with boundary values \( g(z) = \begin{cases} r_2 f(z) & \text{if } |z| = r_2, \\ -r_1 f(z) & \text{if } |z| = r_1, \end{cases} \) and where the constant \( \mathcal{C} \) is given by

\[
\frac{\sqrt{r_1 r_2}}{2\pi} \int_{0}^{2\pi} \int_{0}^{t} \frac{\partial u}{\partial \alpha_r}(\sqrt{r_1 r_2} e^{ir\tau}) \, d\tau \, dt. \tag{41}
\]

If in addition \( f \in C^{m, \alpha}(\partial A_{r_1, r_2}) \) for some positive integer \( m \geq 2 \) and some \( \alpha \in (0, 1] \), then \( U \) given in (40) together with all its partial derivatives up to order \( m + 1 \) can be continuously extended to \( \mathcal{A}_{r_1, r_2} \) and their extensions are uniformly Hölder continuous with exponent \( \alpha \). Conversely if \( g : \partial A_{r_1, r_2} \to \mathbb{R} \) is a continuous function satisfying \( \int_{0}^{2\pi} g(r_2 e^{i\theta}) \, d\theta = \int_{0}^{2\pi} g(r_1 e^{i\theta}) \, d\theta \), and if \( U \) is a solution of the Neumann problem (3) with boundary data \( f(z) = \)
Proof. The only thing left to be proved is that if \( f \in C^{m,\alpha} (\partial A_{r_1 r_2}) \) for some positive integer \( m \geq 2 \) and some \( \alpha \in (0, 1] \), then \( U \) given in (40) together with all its partial derivatives up to order \( m + 1 \) can be continuously extended to \( \overline{A_{r_1 r_2}} \) and their extensions are uniformly Hölder continuous with exponent \( \alpha \).

As explained earlier in the proof of Theorem 2, this proof will be done for the case \( m = 2 \) as the case of a positive integer \( m \geq 3 \) follows exactly in the same way, using induction. To begin with assume first that \( r_2 \leq 1/2 \) and let \( \hat{U} \) be the solution of the Neumann problem (9) on \([r_1, r_2] \times \mathbb{R}\) with boundary data \( \phi(r, \theta) = \begin{cases} f(re^{i\theta}) & \text{if } r = r_2, \\ -f(re^{i\theta}) & \text{if } r = r_1, \end{cases} \)

satisfying \( \hat{U}(\sqrt{r_1 r_2}, 0) = 0 \). Then, since \( f \in C^{m,\alpha}(\partial A_{r_1 r_2}) \), it follows by Theorem 2 that \( \hat{U} \) together with all its partial derivatives up to order \( m + 1 \) can be continuously extended to \([r_1, r_2] \times \mathbb{R}\) and their extensions are uniformly Hölder continuous with exponent \( \alpha \) there. We will transfer this property to \( U \) in (40) and its partial derivatives up to order \( m + 1 \) as follows. Choose any \( z_1, z_2 \in \overline{A_{r_1 r_2}} \) and let \( \alpha_1, \alpha_2, \in [r_1, r_2] \) and \( \theta_1, \theta_2 \in \mathbb{R} \) be such that \( z_1 = \alpha_1 e^{i\theta_1}, z_2 = \alpha_2 e^{i\theta_2}, |\theta_2 - \theta_1| \leq \pi \). Notice then that \( U \) and \( \hat{U} \) given by (40) are related through the equation

\[
\hat{U}(r, \theta) = U(re^{i\theta}), \quad (r, \theta) \in (r_1, r_2) \times \mathbb{R}. \tag{43}
\]

Next \( |U(z_2) - U(z_1)| = |\hat{U}(\alpha_2, \theta_2) - \hat{U}(\alpha_1, \theta_1)| \leq H_U \left( |\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2 \right)^{\frac{\alpha}{2}} \leq H_U \left( |z_2 - z_1|^2 + L^2 |z_2 - z_1|^2 \right)^{\frac{\alpha}{2}} \), for some positive constant \( H_U \), where the first inequality is due to Theorem 2 whereas the second one is due to Lemma 2. Hence

\[
|U(z_2) - U(z_1)| \leq H_U (1 + L^2)^{\frac{\alpha}{2}} |z_2 - z_1|^\alpha, \tag{44}
\]

which proves that \( U \) is indeed uniformly Hölder continuous with exponent \( \alpha \) on \( \overline{A_{r_1 r_2}} \). Proceeding further take the derivatives with respect to \( r \) and \( \theta \) in (43) and letting \( z = (r \cos \theta, r \sin \theta) \) we obtain

\[
\begin{cases}
U_x(z) = \hat{U}_r(r, \theta) \cos \theta - \frac{1}{\alpha_1} \hat{U}_\theta(r, \theta) \sin \theta, \\
U_y(z) = \hat{U}_r(r, \theta) \sin \theta + \frac{1}{\alpha_1} \hat{U}_\theta(r, \theta) \cos \theta.
\end{cases} \tag{45}
\]

To prove the locally \( \alpha \) Hölder continuity property on \( \overline{A_{r_1 r_2}} \) for the partial derivatives of \( U \) up to order three, assume in addition that \( z_1 \) and \( z_2 \) are close enough. Consequently compute \( |U_x(z_2) - U_x(z_1)| \leq |\hat{U}_r(\alpha_2, \theta_2) \cos \theta_2 - \hat{U}_r(\alpha_1, \theta_1) \cos \theta_1| + \frac{1}{\alpha_2} |\hat{U}_\theta(\alpha_2, \theta_2) \sin \theta_2 - \hat{U}_\theta(\alpha_1, \theta_1) \sin \theta_1| \leq |\hat{U}_r(\alpha_2, \theta_2) - \hat{U}_r(\alpha_1, \theta_1)| + |\hat{U}_\theta(\alpha_2, \theta_2) - \hat{U}_\theta(\alpha_1, \theta_1)|. \)
| cos \theta_2 - \cos \theta_1 | + \frac{1}{\alpha_2} | \hat{U}_\theta(\alpha_2, \theta_2) - \hat{U}_\theta(\alpha_1, \theta_1) | + | \hat{U}_\theta(\alpha_1, \theta_1) | \sin \frac{\theta_2}{\alpha_2} - \sin \frac{\theta_1}{\alpha_1} |.

Focusing on the last inequality the first term is upper-bounded by $H_U (1 + L^2)^\frac{3}{2} | z_2 - z_1 |^\alpha$, the second term is upper-bounded by $L \| \hat{U}_r \| | z_2 - z_1 |$, the third term is upper-bounded by $(1 + L^2)^\frac{3}{2} \frac{H_U}{r_1} | z_2 - z_1 |^\alpha$, and lastly the fourth term is upper-bounded by $\frac{1 + L r_2}{r_1^2} \| \hat{U}_\theta \| | z_2 - z_1 |$. Since $r_2$ was assumed to be less than 0.5 it follows that any $z_1, z_2 \in \overline{A_{r_1, r_2}}$ satisfy $| z_2 - z_1 | \leq | z_2 - z_1 |^\alpha$, and thus

$$| U_x(z_2) - U_x(z_1) | \lesssim | z_2 - z_1 |^\alpha,$$

where a proportionality constant is readily found to be

$$H_U (1 + L^2)^\frac{3}{2} \left( 1 + \frac{1}{r_1} \right) + L \| \hat{U}_r \| + \frac{1 + L r_2}{r_1^2} \| \hat{U}_\theta \|.$$

In a similar way compute $| U_y(z_2) - U_y(z_1) | \leq | \hat{U}_r(\alpha_2, \theta_2) \sin \theta_2 - \hat{U}_r(\alpha_1, \theta_1) \sin \theta_1 | + | \frac{1}{\alpha_2} \hat{U}_\theta(\alpha_2, \theta_2) \cos \theta_2 - \frac{1}{\alpha_1} \hat{U}_\theta(\alpha_1, \theta_1) \cos \theta_1 | \leq | \hat{U}_r(\alpha_2, \theta_2) - \hat{U}_r(\alpha_1, \theta_1) | + | \hat{U}_r(\alpha_1, \theta_1) |$, and notice that all four terms have the same upper bounds as above. Hence one can conclude that

$$| U_y(z_2) - U_y(z_1) | \lesssim | z_2 - z_1 |^\alpha,$$

where a proportionality constant is thus

$$H_U (1 + L^2)^\frac{3}{2} \left( 1 + \frac{1}{r_1} \right) + L \| \hat{U}_r \| + \frac{1 + L r_2}{r_1^2} \| \hat{U}_\theta \|.$$

To show the locally $\alpha$ Hölder continuity property on $\overline{A_{r_1, r_2}}$ for the second order partial derivatives of $U$ plug $z = (r \cos \theta, r \sin \theta)$ in (45) and take again derivatives with respect to both $r$ and $\theta$ to obtain after some elementary algebraic manipulations

$$U_{xx}(z) = \hat{U}_{rr}(r, \theta) \cos^2 \theta - \hat{U}_{r\theta}(r, \theta) \sin \frac{2 \theta}{r} + \hat{U}_{\theta\theta}(r, \theta) \frac{\sin^2 \theta}{r^2} + \hat{U}_\theta(r, \theta) \left( \cos (1 + \sin \theta) \right) + \hat{U}_r(r, \theta) \frac{\sin \theta}{r},$$

$$U_{xy}(z) = \hat{U}_{rr}(r, \theta) \frac{\sin 2 \theta}{2} + \hat{U}_{r\theta}(r, \theta) \frac{\cos 2 \theta}{r} - \hat{U}_{\theta\theta}(r, \theta) \frac{\sin 2 \theta}{2 r^2} - \hat{U}_\theta(r, \theta) \left( \frac{\sin \theta}{r} + \frac{\cos^2 \theta}{r^2} \right) - \hat{U}_r(r, \theta) \frac{\sin 2 \theta}{2 r},$$

$$U_{yy}(z) = \hat{U}_{rr}(r, \theta) \sin^2 \theta + \hat{U}_{r\theta}(r, \theta) \frac{\sin 2 \theta}{r} + \hat{U}_{\theta\theta}(r, \theta) \frac{\cos^2 \theta}{r^2} - \hat{U}_\theta(r, \theta) \sin \frac{2 \theta}{r} - \hat{U}_r(r, \theta) \frac{\cos \theta}{r}.$$
where a proportionality constant is found to be

\[ \frac{\sin^2 \theta_2}{\alpha_2} - \frac{\sin^2 \theta_1}{\alpha_1} \]

\[ \|\dot{\mathbf{U}}_{\mathbf{R}}(\alpha_1, \theta_1)\| + \frac{\sin^2 \theta_2}{\alpha_2} - \frac{\sin^2 \theta_1}{\alpha_1} \left\| \dot{\mathbf{U}}_{\mathbf{R}}(\alpha_1, \theta_1) \right\| + \frac{2}{\sqrt{2}} \left\| \dot{\mathbf{U}}_{\mathbf{R}}(\alpha_2, \theta_2) - \dot{\mathbf{U}}_{\mathbf{R}}(\alpha_1, \theta_1) \right\| + \frac{\sin^2 \theta_2}{\alpha_2} - \frac{\sin^2 \theta_1}{\alpha_1} \left\| \dot{\mathbf{U}}_{\mathbf{R}}(\alpha_2, \theta_2) - \dot{\mathbf{U}}_{\mathbf{R}}(\alpha_1, \theta_1) \right\|.

Let \( \mathcal{H} > 0 \) be such that \( \mathcal{H}(|\alpha_2 - \alpha_1|^2 + |\theta_2 - \theta_1|^2) \geq 2 \) upper-bounds \( |\dot{U}_{\mathbf{R}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{R}}(\alpha_1, \theta_1)|, |\dot{U}_{\mathbf{O}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{O}}(\alpha_1, \theta_1)|, |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)|, |\dot{U}_{\mathbf{0}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{0}}(\alpha_1, \theta_1)|, |\dot{U}_{\mathbf{0}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{0}}(\alpha_1, \theta_1)|, \) as well as \( |\dot{U}_{\mathbf{abc}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{abc}}(\alpha_1, \theta_1)|, \mathbf{abc} \in \{\mathbf{r}, \mathbf{th}, \mathbf{r} \mathbf{th} \}, \) for any \((\alpha_1, \theta_1), (\alpha_2, \theta_2) \in [r_1, r_2] \times \mathbb{R} \) (see Theorem 2). Consequently using simple algebraic manipulations it is easy to check that

\[ |U_{\mathbf{x} \mathbf{y}}(z_2) - U_{\mathbf{x} \mathbf{y}}(z_1)| \lesssim |z_2 - z_1|^\alpha, \quad (49) \]

where a proportionality constant is

\[ 2L\|\dot{U}_{\mathbf{R}}\| + \frac{2Lr_2 + 1}{r_1^2} \|\dot{U}_{\mathbf{r}}\| + \frac{2Lr_2 + 1}{r_1^2} \|\dot{U}_{\mathbf{O}}\| + \frac{3Lr_2 + 4r_2}{r_1^2} \|\dot{U}_{\mathbf{th}}\| + \frac{2Lr_2 + 1}{r_1^2} \|\dot{U}_{\mathbf{th}}\| + \mathcal{H}(L^2 + 1)^{\frac{3}{2}} \left( 1 + \frac{2}{r_1^2} + \frac{3}{r_1} \right). \]

Similarly \( |U_{\mathbf{xy}}(z_2) - U_{\mathbf{xy}}(z_1)| \leq \frac{1}{2} |\dot{U}_{\mathbf{R}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{R}}(\alpha_1, \theta_1)| + |\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| + |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)| + |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)| + |\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| + \frac{3}{2} |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)| + \frac{1}{2} \left[ |\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| + |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)| + |\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| \right] + \frac{3}{2} |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)|, \)

and it follows that

\[ |U_{\mathbf{xy}}(z_2) - U_{\mathbf{xy}}(z_1)| \lesssim |z_2 - z_1|^\alpha, \quad (50) \]

where a proportionality constant is found to be

\[ L\|\dot{U}_{\mathbf{R}}\| + \frac{2Lr_2 + 1}{r_1^2} \|\dot{U}_{\mathbf{r}}\| + \frac{Lr_2 + 1}{r_1^2} \|\dot{U}_{\mathbf{O}}\| + \frac{2Lr_2 + 1}{r_1^2} \|\dot{U}_{\mathbf{th}}\| + \frac{2Lr_2 + 1}{r_1^2} \|\dot{U}_{\mathbf{th}}\| + \mathcal{H}(L^2 + 1)^{\frac{3}{2}} \left( 1 + \frac{2}{r_1^2} + \frac{3}{r_1} \right). \]

Finally \( |U_{\mathbf{yy}}(z_2) - U_{\mathbf{yy}}(z_1)| \leq |\dot{U}_{\mathbf{R}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{R}}(\alpha_1, \theta_1)| + 2|\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| + \frac{1}{2} |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)| + \frac{1}{2} |\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| + \frac{1}{2} |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)| + \frac{1}{2} \left[ |\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| + |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)| + \frac{1}{2} |\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| \right] + \frac{1}{2} \left[ |\dot{U}_{\mathbf{th}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{th}}(\alpha_1, \theta_1)| + \frac{1}{2} |\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| \right] + \frac{1}{2} \left[ |\dot{U}_{\mathbf{r}}(\alpha_2, \theta_2) - \dot{U}_{\mathbf{r}}(\alpha_1, \theta_1)| \right], \)

and one obtains

\[ |U_{\mathbf{yy}}(z_2) - U_{\mathbf{yy}}(z_1)| \lesssim |z_2 - z_1|^\alpha, \quad (51) \]
where a proportionality constant can be taken as

\[
\frac{2Lr_2 + 1}{r^4_1} \|\dot{U}_{r\theta}\| + \frac{2Lr_2^2 + r_2}{r^4_1} \|\dot{U}_{\theta\theta}\| + \|\dot{U}_r\| \frac{2Lr_2 + 1}{r^4_1} + \frac{Lr_2^2(r_1^2 + 1) + r_2}{r^4_1} \|\dot{U}_\theta\|
\]

\[+ 2L \|\ddot{U}_{rr}\| + 3\left(\frac{2r_1^2}{r^2_1} + 4r_1 + 3(L_1^2 + 1)^\alpha\right).\]

To show the locally \(\alpha\) Hölder continuity property on \(\mathcal{A}_{r_1;r_2}\) for the third order partial derivatives of \(U\) plug \(z = (r \cos \theta, r \sin \theta)\) in the first and last equations of (18) and take again derivatives with respect to both \(r\) and \(\theta\) to obtain after some algebraic manipulations

\[
U_{xxx}(z) = \dot{U}_{rrr}(r, \theta) \cos^3 \theta - \frac{3}{2} \dot{U}_{\theta rr}(r, \theta) \frac{\sin 2\theta \cos \theta}{r} + \frac{\sin^2 \theta (1 + 2 \cos \theta)}{r^2},
\]

\[
\dot{U}_{\theta \theta \theta}(r, \theta) - \dot{U}_{\theta \theta \theta}(r, \theta) \frac{\sin^3 \theta}{r^3} + \dot{U}_{\theta r}(r, \theta) \frac{3 \cos^2 \theta \sin \theta + \cos^2 \theta + 2 \cos 2 \theta \sin \theta}{r^2} + \dot{U}_{rr}(r, \theta) \frac{3 \sin^2 \theta \cos \theta}{r} + \dot{U}_{\theta \theta}(r, \theta) \frac{\sin^2 \theta (1 + \sin \theta + 3 \cos \theta)}{r^3} - \frac{3 \sin^2 \theta \cos \theta}{r^2},
\]

\[
\dot{U}_{\theta}(r, \theta) - \dot{U}_{\theta}(r, \theta) \frac{2 \cos^2 \theta (1 + \sin \theta) - \sin^2 \theta \sin \theta \cos 2 \theta}{r^3},
\]

\[
U_{xyy}(z) = \ddot{U}_{\theta \theta \theta}(r, \theta) \frac{\sin^2 \theta \cos \theta}{r^3} + \ddot{U}_{\theta r}(r, \theta) \frac{2 \sin 2 \theta \sin \theta + \cos \theta (\sin \theta - 2 \cos 2 \theta)}{r^2} + \ddot{U}_{\theta \theta r}(r, \theta) \frac{\cos^3 \theta - \sin 2 \theta \sin \theta}{r^3} + \ddot{U}_{\theta \theta}(r, \theta) \frac{\cos \theta (2 \cos 2 \theta - 2 \sin \theta - \sin^2 \theta)}{r^3} + \ddot{U}_{rr}(r, \theta) \frac{\sin \theta (\sin^2 \theta - 2 \cos^2 \theta)}{r^3} + \ddot{U}_{\theta \theta \theta}(r, \theta) \frac{\sin^3 \theta - \sin 2 \theta \cos \theta}{r^2} + \ddot{U}_{rr}(r, \theta),
\]

\[
\sin \theta (2 \cos^2 \theta - \sin^3 \theta),
\]

(52)

\[
U_{yyx}(z) = -\dot{U}_{\theta r}(r, \theta) \frac{\sin \theta \cos \theta + \sin \theta \cos^2 \theta (r^2 + 1) + 2 \cos 2 \theta \sin \theta + \cos^2 \theta \sin \theta}{r^2} + \dot{U}_{\theta}(r, \theta) \frac{\sin 2 \theta \cos \theta + \cos 2 \theta \sin \theta (r^2 + 1)}{r^3} + \dot{U}_{\theta r}(r, \theta) \frac{\sin 2 \theta \cos \theta - \sin^3 \theta}{r} + \dot{U}_{\theta \theta r}(r, \theta) \frac{\cos^3 \theta - \sin 2 \theta \sin \theta}{r^2} - \dot{U}_{\theta \theta \theta}(r, \theta) \frac{\cos^2 \theta \sin \theta}{r^3} + \sin^2 \theta \cos \theta.
\]

\[
\dot{U}_{rrr}(r, \theta) + \dot{U}_{rr}(r, \theta) \frac{\cos^3 \theta - \sin 2 \theta \sin \theta}{r} + \frac{\sin^2 \theta \cos \theta (r^2 + 3)}{r^3} - 2 \cos^3 \theta.
\]

\[
\dot{U}_{\theta \theta}(r, \theta) + \dot{U}_{r}(r, \theta) \frac{\sin 2 \theta \sin \theta - \cos^3 \theta}{r^2},
\]
\[ U_{yuy}(z) = \hat{U}_{\theta r}(r, \theta) \frac{\cos^3 \theta + 2 \cos 2 \theta \cos \theta - \sin^2 \theta - \cos \theta \sin^2 \theta (r^2 + 1)}{r^2} + \sin^3 \theta. \]
\[ \hat{U}_{rrr}(r, \theta) + \hat{U}_\theta(r, \theta) \frac{\sin 2 \theta \sin \theta - \cos 2 \theta \cos \theta (r^2 + 1)}{r^3} + \frac{3 \cos^2 \theta \sin \theta}{r}. \]
\[ \hat{U}_{\theta r}(r, \theta) - \hat{U}_\theta(r, \theta) \frac{\cos^2 \theta \sin \theta (r^2 + 5)}{r^3} + \hat{U}_{\theta \theta r}(r, \theta) \frac{3 \sin^2 \theta \cos \theta}{r} + \frac{\cos^3 \theta}{r^3}. \]
\[ \hat{U}_{\theta \theta \theta}(r, \theta) + \hat{U}_{\theta \theta r}(r, \theta) \frac{3 \cos^2 \theta \sin \theta}{r^2} + \hat{U}_{r}(r, \theta) \frac{3 \cos^2 \theta \sin \theta}{r^2}. \]

Using the above relations and the same ideas as for the first and second order partial derivatives of \( U \), it can be checked that the third order partial derivatives of \( U \) are locally \( \alpha \) Hölder continuous on \( A_{r_1, r_2} \) as well.

To sum up, it has been proved that the partial derivatives of \( U \) up to order three are locally \( \alpha \) Hölder continuous on \( A_{r_1, r_2} \), and a compactness argument shows that this is enough to argue that they are in fact uniform Hölder continuous with exponent \( \alpha \). In conclusion, the theorem is now proved for the case when \( r_2 \leq 0.5 \). If \( r_2 > 0.5 \) performing a scaling of the annulus by \( (2r_2)^{-1} \) and applying the previous conclusions to the function \( U_s(w) := U(2r_2w) \) defined on the scaled annulus, it is immediately seen that \( U \) together with all its partial derivatives up to order three are uniformly Hölder continuous with exponent \( \alpha \) on \( A_{r_1, r_2} \). The proof is now completed. \( \square \)

**Remark 4.** The constant \( C \) which appears in both \([12]\) and \([41]\) has an interesting interpretation. Cutting the annulus \( A_{r_1, r_2} \) along the negative real axis for example, that is defining \( A_{r_1, r_2}^- = \{ z \in A_{r_1, r_2} \mid z \not\in \mathbb{R}_- \} \), the solution \( u \) of the Dirichlet problem \([2]\) having boundary data \( g(z) = \begin{cases} r_2 f(z) & \text{if } |z| = r_2, \\ -r_1 f(z) & \text{if } |z| = r_1, \end{cases} \) an harmonic-conjugate function \( v_0 \) on \( A_{r_1, r_2}^- \) satisfying \( v_0(\sqrt{r_1 r_2}) = 0 \). Then it is claimed that

\[ C = \frac{1}{2\pi} \int_{-\pi}^{\pi} v_0(\sqrt{r_1 r_2} e^{i\theta}) \, d\theta. \]  

(53)

Indeed assume first \( r_1 = 1/a < a = r_2 \) and denoting \( \hat{u}(r, \theta) = u(re^{i\theta}) \), \( (r, \theta) \in \left[ \frac{1}{a}, a \right] \times \mathbb{R} \), we have \( C = \frac{1}{2\pi} \int_{0}^{t} \int_{0}^{2\pi} \frac{\partial u}{\partial e^{-i\tau}}(e^{i\tau}) \, d\tau dt = \frac{1}{2\pi} \int_{0}^{t} \int_{0}^{2\pi} \hat{u}_r(1, \tau) \, d\tau dt \). According to the proof of Theorem \([1]\) the application \( t \to \int \hat{u}_r(1, \tau) d\tau \) is 2\pi-periodic and thus \( C = \frac{1}{2\pi} \int_{0}^{t} \int_{0}^{2\pi} \hat{u}_r(1, \tau) \, d\tau dt + \frac{1}{2\pi} \int_{0}^{t} \int_{2\pi}^{0} \hat{u}_r(1, \tau) \, d\tau dt = -\frac{1}{2\pi} \int_{0}^{t} \int_{0}^{2\pi} d^* u + \frac{1}{2\pi} \int_{0}^{t} \int_{0}^{2\pi} \hat{u}_r(1, \tau) \, d\tau dt + \frac{1}{2\pi} \int_{0}^{t} \int_{2\pi}^{0} \hat{u}_r(1, \tau) \, d\tau dt = -\frac{1}{2\pi} \int_{0}^{t} \int_{0}^{2\pi} d^* u + \frac{1}{2\pi} \int_{0}^{t} \int_{0}^{2\pi} \hat{u}_r(1, \tau) \, d\tau dt + \frac{1}{2\pi} \int_{0}^{t} \int_{2\pi}^{0} \hat{u}_r(1, \tau) \, d\tau dt = -\frac{1}{2\pi} \int_{0}^{t} \int_{0}^{2\pi} d^* u + \frac{1}{2\pi} \int_{0}^{t} \int_{0}^{2\pi} \hat{u}_r(1, \tau) \, d\tau dt + \frac{1}{2\pi} \int_{0}^{t} \int_{2\pi}^{0} \hat{u}_r(1, \tau) \, d\tau dt \), where \( \gamma_i^- : [t, 0] \to \mathbb{C} \), \( \gamma_i^+(\tau) = e^{i\tau}, \ t < 0 \), \( \gamma_i^+ : [0, t] \to \mathbb{C} \), \( \gamma_i^+(\tau) = e^{i\tau}, \ t \geq 0 \), and where \( d^* u \) denotes the conjugate differential of \( u \) (see \([2\) Chapter 4.6.1]). But \( -\int_{\gamma_i^-} d^* u = v_0(e^{i\tau}) - v_0(1) \) and also \( \int_{\gamma_i^+} d^* u = v_0(e^{it}) - v_0(1) \),
and since \( v_0(1) = 0 \) the proof is complete for the case when \( r_1 = 1/a < a = r_2 \). The general case \( 0 < r_1 < r_2 < \infty \) follows by the previous one by performing a scaling with \( \lambda = \frac{1}{\sqrt{r_1 r_2}} \) and defining \( a = \sqrt{\frac{r_1}{r_2}} \).

The next corollary shows that Theorem 4 (or equivalently Theorem 1) is a generalization of the main result in [5] (actually its first part is exactly Theorem 1 in [5] when the unit ball has dimension 2). This will show, in particular, that the theory presented so far is a more powerful tool in \( \mathbb{R}^2 \) which embeds the main result in [5] as a particular case. Moreover, if additional assumptions on the smoothness of \( f \) are provided, the result in [5] can be strengthened to uniform Hölder continuity of \( U \) and its partial derivatives.

**Corollary 1.** Assume \( f : \partial \mathcal{U} \to \mathbb{R} \) is continuous and satisfies \( \int_0^{2\pi} f \, d\theta = 0 \).

If \( U \) is the solution of the Neumann problem (3) on \( \mathcal{U} \) with boundary data \( f \), satisfying \( U(0) = 0 \), then

\[
U(z) = \int_0^1 \frac{u(\rho z)}{\rho} \, d\rho, \quad z \in \mathcal{U},
\]

(54)

where \( u \) is the solution of the Dirichlet problem (2) on \( \mathcal{U} \) with boundary data \( g = f \). If \( f \in C^m,\alpha(\partial \mathcal{U}) \) for some positive integer \( m \geq 2 \) and some \( \alpha \in (0, 1] \), then \( U \) and all its partial derivatives up to order \( m+1 \) are uniformly Hölder continuous with exponent \( \alpha \) on \( \mathcal{U} \). Conversely if \( g : \partial \mathcal{U} \to \mathbb{R} \) is continuous, satisfies \( \int_0^{2\pi} g \, d\theta = 0 \), and if \( U \) is a solution of the Neumann problem (3) on \( \mathcal{U} \) with boundary data \( f = g \), then the solution \( u \) of the Dirichlet problem (2) on \( \mathcal{U} \) with boundary data \( g \) is given by

\[
u(r e^{i\theta}) = r \frac{\partial U}{\partial a_\theta}(r e^{i\theta}), \quad r e^{i\theta} \in \mathcal{U}.
\]

(55)

**Proof.** Define \( r_n = r_1(n) = \frac{1}{n\pi} \), \( A_n = A_{r_1,1}, \, n \in \mathbb{N} \setminus \{0, 1\} \). On \( A_n \) let \( u_n(\cdot) \) be the solution of the Dirichlet problem (2) with boundary data \( g_n = u_{|\partial A_n} \). By the uniqueness of the solution of the Dirichlet problem it follows that \( u_n = u \) on \( A_n \).

Next define \( U_n(re^{i\theta}) = \int_0^{2\pi} \frac{u_n(re^{i\theta})}{\rho} \, d\rho + \frac{1}{n} \int_0^t \tilde{c}_n - \int_0^t \frac{\partial u_n}{\partial a_\theta}(z^{\tau}) \, d\tau \) \( dt \), \( r e^{i\theta} \in \mathcal{A}_n \),

where \( \tilde{c}_n = \frac{1}{2\pi n} \int_0^{2\pi} \frac{\partial u_n}{\partial a_\theta}(z^{\tau}) \, d\tau \). It follows by Theorem 4 that \( U_n \) is harmonic in \( \mathcal{A}_n \), has normal derivative \( f_n(z) = \begin{cases} f(z) & \text{if } |z| = 1, \\ -n^2 u(z) & \text{if } |z| = \frac{1}{n^2}, \end{cases} \) and satisfies \( U_n(\frac{1}{n}) = 0 \). Further let \( K \subset A_{0,1} \) be any compact set. Hence \( \exists N \in \mathbb{N} \setminus \{0, 1\} \) (possibly depending on \( K \)) such that \( \forall n \geq N \Rightarrow K \subset A_n \). So choose any \( n \geq N \) and any \( p \in \mathbb{N}^* \), and consequently \( |U_{n+p}(re^{i\theta}) - U_n(re^{i\theta})| \leq \int_0^{2\pi} \left| \frac{u(re^{i\theta})}{\rho} \right| \, d\rho + \int_0^{2\pi} \frac{\partial u}{\partial \theta}(z^{\tau}) \, d\tau \).
\[
\left| \frac{1}{n+p} \int_0^\theta \left( C_n + \frac{t}{\theta} \frac{\partial u}{\partial \theta} \left( \frac{\rho}{n+p} e^{i\theta} \right) \right) dt - \frac{1}{n} \int_0^\theta \left( C_n + \frac{t}{\theta} \frac{\partial u}{\partial \theta} \left( \frac{\rho}{n} e^{i\theta} \right) \right) dt \right|. 
\]

To evaluate the first term, notice first that since \( u \in C^0(\overline{U}) \cap C^1(U) \) we have \( \lim_{\rho \to 0} \frac{u(\rho e^{i\theta})}{\rho} = u_x(0) r \cos \theta + u_y(0) r \sin \theta = r \frac{\partial u}{\partial \theta}(0), \ \forall \ \rho \in U.\) Hence, \( \exists M_1 > 0 \) such that \( \left| \frac{u(\rho e^{i\theta})}{\rho} \right| \leq M_1, \ \forall \ \rho \in [0,1], \ \forall \ \rho e^{i\theta} \in U.\) On the other hand, since \( 0 \not\in K, \) there is a \( \delta > 0 \) (which may depend on \( K \) as well) such that \( d(0,K) = \delta.\) The last two observations in turn imply \( \int_{\{ \rho \in (n+p) \}} \left| \frac{u(\rho e^{i\theta})}{\rho} \right| d\rho \leq \frac{M_1}{\delta n(n+p)}, \ \forall \ \rho \in \mathbb{N}^*, \ \forall \ n \geq N, \ \forall \ \rho \in \mathbb{N}^+.\) Next since \( u \in C^1(U) \) it can be concluded that \( \nabla u \) is bounded on, say, \( |z| \leq 2/3 \) which in turn shows that one can choose \( M_2 > 0 \) for which \( |\nabla u(\frac{\rho}{n})| \leq M_2, \ \forall \ \tau \in \mathbb{R}. \) So \( |E_n| \leq \frac{\pi M_2}{n}, \ \forall n \in \mathbb{N}^* \setminus \{1\}.\)

Finally, we obtain \( \frac{pM_1}{\delta n} + \theta \pi M_2 \left( \frac{1}{n} - \frac{1}{(n+p)^2} \right) + \theta^2 \frac{M_2}{n} \geq \int_{\{ \rho \in (n+p) \}} \left| \frac{u(\rho e^{i\theta})}{\rho} \right| d\rho + \int_{\{ \rho \in (n+p) \}} \left| \frac{1}{n+p} \int_0^\theta \left( \frac{\rho}{n} \frac{\partial u}{\partial \theta} \left( \frac{\rho}{n+p} e^{i\theta} \right) \right) dt - \frac{1}{n} \int_0^\theta \left( \frac{\rho}{n} \frac{\partial u}{\partial \theta} \left( \frac{\rho}{n} e^{i\theta} \right) \right) dt \right|, \)

where the last term is greater than \( |U_{n+p}(\rho e^{i\theta}) - U_n(\rho e^{i\theta})|\). Since we can consider without loss of generality \( \theta \in (-\pi, \pi], \) it follows that the sequence of harmonic functions \( \{U_n\}_{n \geq 2} \) is uniformly Cauchy on \( K, \) and hence on any compact subset of \( A_{0,1}. \) Setting \( \mathcal{U}(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \ z \in U, \) it is easy to see that

\[
\lim_{n \to \infty} U_n(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho = \mathcal{U}(z) \text{ on } A_{0,1}. \]

Hence \( \mathcal{U} \) is harmonic in \( A_{0,1}. \) In addition, using the Dominant Convergence theorem, it follows that \( \lim_{z \to 0} \mathcal{U}(z) = 0. \)

This shows that \( \mathcal{U} \) can be (uniquely) extended to a harmonic function in the whole unit disk, which shall also be denoted for brevity \( \mathcal{U}. \) It is not difficult to check that \( \mathcal{U} \) can actually be extended by continuity to \( \overline{U}. \) Finally \( \frac{\partial u}{\partial \rho}(e^{i\theta}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\{ \rho \in (n+p) \}} \left( \frac{u(\rho e^{i\theta}) - u(\epsilon e^{i\theta})}{\epsilon} \right) \rho d\rho = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\{ \rho \in (n+p) \}} \frac{u(\rho e^{i\theta})}{\rho} d\rho = u(e^{i\theta}) = g(e^{i\theta}) = f(e^{i\theta}), \ \forall \ \theta \in \mathbb{R}.\)

The continuous extension of \( \nabla \mathcal{U} \) to \( U \) follows by exactly the same arguments as those invoked in the proof of Theorem 4 that is choosing any solution \( V \) of the Neumann problem \( \{3\} \) on \( U \) with boundary data \( f, \) and approximating the unit disk by an increasing sequence of disks of radii \( r_n, \ r_n \nearrow 1, \) the function \( \mathcal{U} - V \) turns out to be constant on \( U. \) In conclusion we have proved so far that \( \mathcal{U} = U, \) and so the solution of the Neumann problem \( \{3\} \) on \( U \) with boundary data \( f \) has the desired expression provided by relation \( \{5\}. \) To complete the proof of the first part, it only remains to show that if \( f \in C^{m,\alpha}(\partial U) \) for some positive integer \( m \geq 2 \) and some \( \alpha \in (0,1], \) then \( U \) and all its partial derivatives up to order \( m + 1 \) are uniformly Hölder continuous with exponent \( \alpha \) on \( U. \) To this
and denote by $U_1$ and $U_2$ the restrictions of $U$ to $\frac{1}{2}U$ and $\overline{A_{\frac{1}{4},1}}$, respectively. Further let $f_2 : \partial A_{\frac{1}{4},1} \rightarrow \mathbb{R}$, $f_2(\rho e^{i\theta}) = \begin{cases} f(\rho e^{i\theta}), & \text{if } r = 1, \
-U(z_r(\rho e^{i\theta}) \cos \theta - U(z_\nu(\rho e^{i\theta}) \sin \theta), & \text{if } r = \frac{1}{4}, \end{cases}$ and observe that $f_2 \in C^{m,\alpha}(\partial A_{\frac{1}{4},1})$; in addition $f_2$ satisfies the compatibility condition $\int_{\partial A_{\frac{1}{4},1}} f_2 d\sigma = 0$. According to Theorem 1, then, all the partial derivatives of $U_2$ up to order $m + 1$ can be continuously extended to $\overline{A_{\frac{1}{4},1}}$ and their extensions are uniformly Hölder continuous with exponent $\alpha$ there. Also, all the partial derivatives of $U_1$ are locally Lipschitz continuous on $\frac{1}{2}U$ (due to the harmonicity of $U$), and consequently they are locally $\alpha$ Hölder continuous there. Appealing to the definitions of $U_1$ and $U_2$ it follows that $U$ together with all its partial derivatives up to order $m + 1$ are locally $\alpha$ Hölder continuous on $\overline{U}$, and using again a compactness argument concludes the first part of the proof.

For the second part denote $\check{U}(r, \theta) = U(\rho e^{i\theta})$, $\check{u}(r, \theta) = u(\rho e^{i\theta})$, $\check{e}^i \theta \in \overline{U}$, where one can choose $U(0) = 0$. Using the first part $\check{U}(r, \theta) = \int_0^r \frac{\check{u}(\rho, \theta)}{\rho} d\rho = \int_0^r \frac{\check{u}(\rho, \theta)}{\rho} d\rho, \ \rho e^{i\theta} \in \overline{U}$. Taking the derivative with respect to the first argument one obtains $\check{U}_r(r, \theta) = \frac{\check{u}(r, \theta)}{r}$ or equivalently $\check{u}(r, \theta) = r \check{U}_r(r, \theta)$, for any $r \in (0; 1)$. Since $\check{U}_r(r, \theta) = \frac{\partial U_r}{\partial \nu}(\rho e^{i\theta}) \forall \theta \in \mathbb{R}$ the conclusion follows.

### 3.2 General smooth, bounded, doubly-connected regions

Using the conformal invariance of harmonic functions and Theorem 1 an important general result is obtained. Before stating it, some preparations are needed. First let $D \subset \mathbb{C}$ be some smooth, doubly connected region whose boundary consists of two bounded Jordan curves which are the images of $\Gamma_i$, $i \in \{1, 2\}$. It will be assumed that $\Gamma_1$ corresponds to the inner contour. Following the approach in [2] Chapter 6] let $\omega_1$ be the harmonic measure of $\{\Gamma_1\}$ with respect to the region $D$, and define $\alpha_1 = \int_{\Gamma_1} \frac{d\omega}{\partial \nu} d\nu$. Consequently define $\omega = \lambda_1 \omega_1$, where $\lambda_1 = \frac{2\pi}{\alpha_1}$, and letting $w = \xi + i\eta$ be the variable on $D$ we also define $p = \frac{\partial \omega}{\partial \xi} - i \frac{\partial \omega}{\partial \eta}$, $q = \int p$ (where the integral is considered over any rectifiable curve having an extremity in $w_0$) and finally $G = e^q$.

$w_0$ is an arbitrary point in $D$ which is assumed to be fixed. Notice that $q$ is not single-valued, in general. However we will see in the lemma below that $G$ is actually a single-valued analytic function in $D$.

**Lemma 3.** Assume $D \in C^{2,\alpha}$ for some $\alpha \in (0, 1)$. Then $G$ defined above has the following properties.
i. $G$ is well defined on $\overline{D}$;

ii. $G(\overline{D}) = A_{1;\epsilon_1^\lambda}$ and the mapping is one-to-one. In addition $G(\Gamma_2) = C_1$ and $G(\Gamma_1) = C_{e\lambda_1}$, respectively;

iii. $G$ is a conformal representation of $D$ on $A_{1;\epsilon_1^\lambda}$;

iv. If $F = G^{-1}$ then the limit $\lim_{z \to z^*} \frac{F(z^*) - F(z)}{z^* - z}$ exists at all points $z^* \in A_{1;\epsilon_1^\lambda}$, and $F'$ can be extended by continuity to $A_{1;\epsilon_1^\lambda}$.

v. The limit $\lim_{z \to z^*} \frac{F(z^*) - F(z)}{z^* - z} = F''(z^*)$ exists at all points $z^* \in A_{1;\epsilon_1^\lambda}$, and $F''$ can be extended by continuity to $A_{1;\epsilon_1^\lambda}$.

Proof. For the proof of i. – iii. see [2], Chapter 6, Theorem 10. For point iv. notice first that the assumption $\partial D \in C^{2;\alpha}$ implies (using Kellogg’s theorem) that $\nabla \omega$ can be continuously extended to $\overline{D}$. Consequently $G$ extends continuously to $\overline{D}$. Using this aspect, the compactness of $\overline{D}$, as well as points i. and ii. it is easy to notice that $F$ can be continuously extended to $A_{1;\epsilon_1^\lambda}$.

The next step is to evaluate the limit $\lim_{w \to w^*} \frac{G(w^*) - G(w)}{w^* - w}$ when $w^* \in \partial D$ and $w \in D$. To this end it is helpful to notice that one may assume without loss of generality that the points $w_0, w, w^*$ belong to a single rectifiable curve as $w$ approaches $w^*$. With this observation in mind it is quite easy to see that $\lim_{w \to w^*} \frac{G(w^*) - G(w)}{w^* - w} = p(w^*)G'(w^*), \ \forall w^* \in \partial D$. Hence one can continuously extend the derivative of $G$ to $\partial D$ by setting $G'(w) = p(w)G(w)$ if $w \in \partial D$. Then $\lim_{z \to z^*} \frac{F(z^*) - F(z)}{z^* - z} = \lim_{w \to F(z^*)} \frac{1}{G'(w)} \frac{G'(w) - G'(w^*)}{w^* - w} = G''(F(z^*))$, $z^* \in \partial A_{1;\epsilon_1^\lambda}$. In order to conclude, it only remains to prove that $G'_{\partial D}$ does not vanish at any point. Suppose by contradiction that there is a point $w^* \in \partial D$ such that $G'(w^*) = 0$, and one may assume without loss of generality that this point belongs to the exterior contour (for the case when $w^*$ belongs to the inner contour, the reasoning is similar, with the only difference that the conformal mapping $T$ defined right below will be considered from the exterior of $\Omega_t$ to the interior of the unit disk). Define $\Omega_t$ to be the region bounded by the image of $\Gamma_1$ and let $\Omega$ be the region bounded by the image of $\Gamma_2$. Since $\Omega \neq \mathbb{C}$ is a simply connected region, by Riemann Mapping theorem there exists a (unique) conformal transformation $T$ of $\mathbb{U}$ onto $\Omega$ such that $T(0) = w_1$, $T'(0) > 0$ for some $w_1 \in \Omega_t$ (see Figure 2). Letting $V_t = T^{-1}(\Omega_t)$ it follows by the Reflection Principle that the map $J : \mathbb{U} \setminus V_t \to A_{1;\epsilon_1^\lambda}$, $J = GoT$ can be analytically extended to $(\mathbb{U} \setminus V_t) \cup B(\lambda, 2\epsilon)$ at any point $\lambda \in \partial \mathbb{U}$ (where $\epsilon$ may depend on $\lambda$) and in addition when restricting $J$ to $B(\lambda, 2\epsilon)$, the disk of radius $2\epsilon$ centered in $\lambda$, the only points in $B(\lambda, 2\epsilon)$ which are mapped on $\partial A_{1;\epsilon_1^\lambda}$ are those which lie on $\partial \mathbb{U}$ and the correspondence is one-to-one. But defining $\lambda^* = T^{-1}(w^*)$, the assumption $G'(w^*) = 0$ implies $J'((\lambda^*)) = 0$ (this is true since $T'$ can be continuously extended to $\partial \mathbb{U}$ according to [3] Theorem 3.6] and is thus bounded on $\mathbb{U}$). Restricting $J$ to $B(\lambda^*, 2\epsilon)$ an
application of the Argument Principle shows that for any \( z \in J(B(\lambda^*, \epsilon)) \) the equation \( z - J(\lambda) = 0 \) has either at least two solutions in \( B(\lambda^*, \epsilon) \), or \( \lambda \) is a solution of order at least two in which case \( J'(\lambda) = 0 \). But if \( z \) also lies on \( \partial A_{1, \epsilon^2} \) then the earlier discussion shows that there is a unique \( \lambda \in B(\lambda^*, \epsilon) \) for which \( J(\lambda) = z \) and furthermore \( \lambda \) also belongs to \( \partial U \). Consequently it must be the case that \( J'(\lambda) = 0 \), and since this is true for any \( \lambda \in \partial U \cap B(\lambda^*, \epsilon) \) it follows that \( J \) must be a constant function. This is obviously a contradiction and so \( G(w^*) \neq 0 \). Finally putting \( F'(z) = \frac{1}{G'(F(z))} \) whenever \( z \in A_{1, \epsilon^2} \) and noticing that \( G'(w) = p(w)G(w) \forall w \in D \) the proof of iii. is complete.

For the last point fix any arbitrary \( z^* \in \partial A_{1, \epsilon^2} \) and denoting \( w = F(z) \), \( w^* = F(z^*) \),

\[
\lim_{z \to z^*, \; z \in A_{1, \epsilon^2}} \frac{F'(z) - F'(z^*)}{z - z^*} = \lim_{w \to w^*, \; w \in \overline{D}} \left[ \frac{1}{G'(w)} \frac{G'(w^*) - G'(w)}{w^* - w} \right] \frac{1}{G'(w^*)}
\]

\[
= -\frac{1}{G'(w^*)} \lim_{w \to w^*, \; w \in \overline{D}} \left[ \frac{G'(w^*) - G'(w)}{w^* - w} \frac{1}{G'(w)} \right] \frac{1}{G'(w^*)} \frac{1}{G'(w)}.
\]

Also we notice that

\[
\lim_{w \to w^*, \; w \in \overline{D}} \frac{G'(w^*) - G'(w)}{w^* - w} = \lim_{w \to w^*, \; w \in \overline{D}} \left[ p(w^*) \frac{G(w^*) - G(w)}{w^* - w} + G(w) \frac{p(w^*) - p(w)}{w^* - w} \right].
\]

Let \( B_{w^*} \) be a simply connected, relatively open (with respect to \( \overline{D} \)) neighborhood of \( w^* \), and let \( w_0^* \) be some point in \( B_{w^*} \) which will be chosen later on. It is easy to see that the function \( \int_{w_0^*}^w p'(\lambda) d\lambda + p(w_0^*) \) is well-defined and coincides with \( p \), on \( B_{w^*} \setminus \partial D \). In addition, since \( p' \) extends continuously to \( \partial D \) (use the Cauchy-Riemann equations as well as Kellogg’s theorem for \( \omega_1 \)), it follows that \( \int_{w_0^*}^w p'(\lambda) d\lambda + p(w_0^*) \) can actually be extended by continuity to \( B_{w^*} \). It is claimed that one can always choose the point \( w_0^* \neq w \) in such way that the line segments with edges \((w_0^*, w), (w, w^*)\) are in \( B_{w^*} \) and furthermore the ratio \[ \left| \frac{w_0^* - w}{w^* - w} \right| \] stays
bounded as \( w \to w^* \). Indeed since \( D \in C^{m,\alpha} \), \( m \geq 2 \), letting \( t^* = \Re(w^*) \) there is a real-valued function \( h \) defined on some open interval \( I \) containing \( t^* \) such that \( h \in C^{m,\alpha}(I) \) and, eventually after performing an appropriate rotation, the boundary of \( D \) around \( w^* \) coincides with the graph of \( h \). Assume without loss of generality that \( B_{w^*} \cap D \) is below the graph of \( h \). If \( h''(t^*) < 0 \) then \( h \) is locally (strictly) concave around \( t^* \) and so one can choose \( w_0^* \) to be the middle of the segment with edges \( w \) and \( w^* \). If \( h''(t^*) > 0 \) then \( h \) is locally strictly convex around \( t^* \) and one can choose \( w_0^* \) to be the projection of \( w \) on the tangent to the graph of \( h \) in \( t^* \). Last but not least if \( h''(t^*) = 0 \) then \( h \) is convex at the left-hand side of \( t^* \) and concave at the right-hand side of \( t^* \), or vice-versa, and we choose \( w_0^* \) as above according to whether it lies on the convex or the concave side of the graph (see Figure 3), with the amendment that whenever the line segment with edges \( (w, w^*) \) is included in \( D \) the point \( w_0^* \) can be chosen the middle of the segment. Having proved the claim we can go back and thus obtain

\[
\lim_{w \to w^*, \ w \in D} \frac{p(w^*) - p(w)}{w^* - w} = \lim_{w \to w^*, \ w \in D} \frac{(p(w^*) - p(w^*_1)) + (p(w^*_1) - p(w))}{w^* - w} = \lim_{w \to w^*, \ w \in D} \left[ \frac{p(w^*) - p(w^*_1)}{w^* - w} + \frac{p(w^*_1) - p(w)}{w^*_1 - w} \right] \frac{w^*_1 - w}{w^* - w},
\]

where in the last expression, using the integral representation for \( p \) as well as the Dominant Convergence theorem, the first term approaches \( \frac{\partial^2 \omega}{\partial z \partial \bar{z}}(w^*) - i \frac{\partial^2 \omega}{\partial z \partial \bar{\eta}}(w^*) \) and the second term approaches 0. In conclusion

\[
\lim_{w \to w^*, \ w \in D} \frac{p(w^*) - p(w)}{w^* - w} = \frac{\partial^2 \omega}{\partial z \partial \bar{z}}(w^*) - i \frac{\partial^2 \omega}{\partial z \partial \bar{\eta}}(w^*) =: p'(w^*). \]

Returning, we observe that

\[
\lim_{z \to z^*, \ z \in \bar{A}_{1,\epsilon^\lambda_1}} \frac{F(z) - F(z^*)}{z - z^*} = \lim_{w \to w^*, \ w \in D} \left[ \frac{G'(w^*) - G'(w)}{w^* - w} \frac{1}{w - w^*} \right] = \left[ p^2(F(z^*)) + p'(F(z^*)) \right] \cdot \frac{z^*(-F'(z^*))^3}{z^*} =: F''(z^*). \]

This expression obviously holds for all points \( z^* \in \bar{A}_{1,\epsilon^\lambda_1} \) as well, and it is thus seen that \( F'' \) can be continuously extended to \( \bar{A}_{1,\epsilon^\lambda_1} \). This ends the proof of the lemma.

\[\square\]

**Remark 5.** The lemma above can be generalized by exactly the same arguments to the case when \( D \in C^{m,\alpha} \) for some positive integer \( m \geq 2 \) and some \( \alpha \in (0,1) \), in which case the higher derivatives of \( F \) up to order \( m \) can be defined for the boundary points of \( A_{1,\epsilon^\lambda_1} \) as well, and in addition they extend continuously to \( \bar{A}_{1,\epsilon^\lambda_1} \).

**Theorem 5.** Let \( D \in C^{m+1,\alpha} \) for some positive integer \( m \geq 2 \) and some \( \alpha \in (0,1) \), and in addition assume \( \Phi \in C^{m,\alpha}(\partial D) \) satisfies the compatibility condition \( \int_D \Phi \ d\sigma = 0 \). If \( U \) is a solution of the Neumann problem \( \partial D \) with boundary data \( \Phi \) then \( U \) and all its partial derivatives up to order \( m + 1 \) are uniformly Hölder continuous with exponent \( \alpha \) on \( D \).

Before proceeding with the proof of the theorem notice that due to the condition \( D \in C^{m+1,\alpha} \), \( m \geq 2 \), Kellogg’s theorem guarantees that \( \nabla \omega \) can be continuously extended to \( \bar{D} \), in which case its continuous extension is also
denoted by $\nabla \omega$. Furthermore, as seen in the proof of Lemma 3, $G' \neq 0$ on $\overline{D}$ and since $G' = pG$, it follows that $\nabla \omega \neq 0$ on $\overline{D}$.

**Proof.** Set $r_1 = 1$, $r_2 = e^{\lambda_1}$ and let $F : A_{r_1:r_2} \to D$ be the conformal map given in Lemma 3. Without loss of generality assume $U(F(\sqrt{r_1r_2})) = 0$. Since $A_{r_1:r_2} \subset C^\infty$ it can be also assumed, without restricting the generality, that $\Gamma_1$, $\Gamma_2 \subset C^{2,\alpha}$. Defining $\tilde{f} : \partial A_{r_1:r_2} \to \mathbb{R}$, $\tilde{f} = (\Phi \circ F) |F'|$, we claim that $f \in C^{m,\alpha}(\partial A_{r_1:r_2})$. In an attempt to keep the proof as clear as possible the author will prove the claim only for the case $m = 2$; the case of a general $m \geq 2$ follows by induction in the same spirit as for this simpler case. To begin with notice that point iv. of Lemma 3 shows that $F$ is locally Lipschitz continuous on $A_{r_1:r_2}$ (use an integral representation of $F$ in terms of $F'$ in some convex neighborhood of any point of $A_{r_1:r_2}$). Secondly, point v. of Lemma 3 reveals that $F'$ is locally Lipschitz continuous on $A_{r_1:r_2}$ (use again an integral representation of $F'$ in terms of $F''$ in some convex neighborhood of any point of $A_{r_1:r_2}$) and hence locally $\alpha$ Hölder continuous there. Using the compactness of $A_{r_1:r_2}$ we can argue that $F''$ is in fact uniformly Hölder continuous with exponent $\alpha$ on $A_{r_1:r_2}$. For $F''$ choose any two points $z_1$, $z_2 \in A_{r_1:r_2}$ and notice that $|F''(z_2) - F''(z_1)| = |z_2 (p'(F(z_2)) + p^2(F(z_2))) (F'(z_2))^3 - z_1 (p'(F(z_1)) + p^2(F(z_1))) (F'(z_1))^3| \leq (|p' \circ F| + |p^2 \circ F|) (F'(z_2))^3 |z_2 - z_1| + r_2 |F'(z_2) - F'(z_1)| (|F''|)^2 + |p' \circ F| + |p^2 \circ F|) + r_2 (|F'(z_2)| - |p'(F(z_1))| + 2|p(F(z_2)) - p(F(z_1))|).$ Recall that $p = \omega - i \omega_\eta$ and by Kellogg’s theorem combined with the compactness of $\overline{D}$ the $m = 2$ order partial derivatives of $\omega$ are uniformly Hölder continuous with exponent $\alpha$ on $\overline{D}$. Thus using the last inequality and the locally Lipschitz continuity of $F$ on $\overline{D}$ it follows that $F''$ is locally $\alpha$ Hölder continuous on $A_{r_1:r_2}$ and hence uniformly Hölder continuous with exponent $\alpha$ there. Finally $F'''(z) = -(F'(z))^2[p'(F(z)) + p^2(F(z))F'(z) + z(p''(F(z)) + 2p(F(z)))(F'(z))^2 - 3z(p'(F(z)) + p^2(F(z)))F''(z)]$.

![Figure 3: Smooth boundary property.](image-url)
and using the locally Lipschitz property for \( F \), the uniform Hölder continuity with exponent \( \alpha \) of \( F' \) and \( F'' \) as well as the uniform Hölder continuity with exponent \( \alpha \) of the partial derivatives of \( \omega \) up to order \( m = 2 \), it can be deduced after several applications of the triangle inequality that \( F'' \) is locally \( \alpha \) Hölder continuous on \( A_{r_1,r_2} \) and so uniform Hölder continuous with exponent \( \alpha \) there.

To complete the proof of the claim notice that since \( 0 \notin \overline{A_{r_1,r_2}} \) it follows that \( | \cdot | \in C^\infty(A_{r_1,r_2}) \) and so we have
\[
\frac{d^2}{dt^2} f(\Gamma_i(t)) = \frac{d^2}{dt^2} \Phi(F(\Gamma_i(t))) \cdot |F'(\Gamma_i(t))| + 2 \frac{d}{dt} \Phi(F(\Gamma_i(t))) \cdot \frac{d}{dt} |F'(\Gamma_i(t))| + \Phi(F(\Gamma_i(t))) \cdot \frac{d^2}{dt^2} |F'(\Gamma_i(t))| \].
But \( \frac{d^2}{dt^2} F(\Gamma_i(t)) = F''(\Gamma_i(t))(\Gamma_i')^2 + F'(\Gamma_i(t))\Gamma_i''(t) \) which is readily seen to be locally \( \alpha \) Hölder continuous. Also \( \frac{d}{dt} |F'(\Gamma_i(t))| = |F'(\Gamma_i(t))| \Re (F''(\Gamma_i(t))\Gamma_i'(t)G(\Gamma_i(t))) \) and thus
\[
\frac{d^2}{dt^2} |F'(\Gamma_i(t))| = \Re \left( \frac{(F''(\Gamma_i(t))\Gamma_i'(t))^2 + F'(\Gamma_i(t))\Gamma_i''(t)(F''(\Gamma_i(t))\Gamma_i'(t) - (F''(\Gamma_i(t))\Gamma_i'(t))^2)}{|F'(\Gamma_i(t))|^2} \right)
\].

When putting everything together, the proof of the claim follows easily. Having proved that \( f \) belongs to \( C^{m,\alpha}(\partial A_{r_1,r_2}) \) let \( V \) be the solution of the Neumann problem \( 3 \) on \( A_{r_1,r_2} \) with boundary data \( f \), satisfying \( V(\sqrt{r_1r_2}^2) = 0 \) (using direct computations together with the assumption \( \int \Phi \, d\sigma = 0 \) it is not difficult to see that \( f \) satisfies the compatibility condition \( \int_{\partial D} f \, ds = 0 \)). Then, according to Theorem \( 4 \) the gradient of \( V \) can be continuously extended to the closure of \( A_{r_1,r_2} \) (as before, its continuous extension will also be denoted \( \nabla V \)). Now set \( W = V \circ G \) which shows that \( W \) is harmonic in \( D \) and furthermore \( W(F(\sqrt{r_1r_2}^2)) = 0 \). Also taking the partial derivatives of \( W \) with respect to \( \xi \) and \( \eta \) it follows that for any \( w \in D \)

\[
\begin{align*}
\frac{\partial W}{\partial \xi}(w) &= \Re \left( \frac{\nabla V(G(w))}{F'(G(w))} \right), \\
\frac{\partial W}{\partial \eta}(w) &= -\Im \left( \frac{\nabla V(G(w))}{F'(G(w))} \right).
\end{align*}
\]

Hence \( \nabla W = \sum_{r=0}^{\infty} \frac{\partial W}{\partial \nu^r}G \) on \( D \) which proves, together with point iv. of Lemma \( 3 \) that \( \nabla W \) extends continuously to \( \overline{D} \). Consequently it follows by the Mean Value theorem that
\[
\frac{\partial W}{\partial \nu}(w) = \langle \nabla W(w); \nu(w) \rangle, \quad w \in \partial D,
\]
where \( \nu \) stands for the (outward) normal derivative at \( \partial D \) and it is given by
\[
\nu(F(z)) = \begin{cases} 
\frac{zF'(z)}{|z|F'(z)} = \frac{\nabla \omega(F(z))}{|\nabla \omega(F(z))|} & \text{if } |z| = r_2, \\
-\frac{zF'(z)}{|z|F'(z)} = -\frac{\nabla \omega(F(z))}{|\nabla \omega(F(z))|} & \text{if } |z| = r_1.
\end{cases}
\]
Returning \( \frac{\partial W}{\partial \nu}(w) = \langle \nabla W(w); \nu(w) \rangle = \frac{|zG|}{|\nabla G(w)||G(w)|} = \Phi, \quad \forall \ w \in \partial D \). To sum up \( W \) is harmonic in \( D \), has boundary data \( \Phi \), satisfies \( W(F(\sqrt{r_1r_2}^2)) = 0 \), and in addition \( \nabla W \in C^1(D) \). So \( W - U = \text{constant} \). But then fixing any arbitrary points \( w_1, w_2 \) in \( D \) and letting \( z_1 = G(w_1), \ z_2 = G(w_2) \) one obtains
\[
|U(w_2) - U(w_1)| = |W(w_2) - W(w_1)| = |V(z_2) - V(z_1)| \leq C_0|z_2 - z_1|^\alpha = C_0|G(w_2) - G(w_1)|^\alpha
\]
for some positive constant \( C_0 \). Using the fact that \( G' \)
can be continuously extended to $\overline{D}$, one concludes that $G$ is locally Lipschitz continuous on $\overline{D}$ and hence $|U(w_2) - U(w_1)| \leq C_0|G(w_2) - G(w_1)|^\alpha \lesssim |w_2 - w_1|^\alpha$, thus proving that $U$ is locally $\alpha$ Hölder continuous on $\overline{D}$ and hence uniformly Hölder continuous with exponent $\alpha$ by means of a compactness argument. Also $U_\xi(w) = W_\xi(w) = \Re(\nabla V(G(w))G'(w))$ and so using Theorem 3.5.13, we notice that $U_\xi(w_2) - U_\xi(w_1) = |W_\xi(w_2) - W_\xi(w_1)| = |\Re(\nabla V(G(w_2))G'(w_2)) - \Re(\nabla V(G(w_1))G'(w_1))| \leq |\nabla V(G(w_2))G'(w_2) - \nabla V(G(w_1))G'(w_1)| \leq |G(w_2) - G(w_1)|\alpha \|G\|\|G'\|\|G''\| |w_2 - w_1| + |w_2 - w_1|L_0\|\nabla V \circ G\|\|G\|\|G''\|$, where $C_1$, $H_1$, $L_0$ are positive constants. But if $w_1$ and $w_2$ are close enough then one can replace $|w_2 - w_1|$ in the last term above by $|w_2 - w_1|^\alpha$ and thus conclude that $U_\xi$ is locally $\alpha$ Hölder continuous on $\overline{D}$ and so uniformly Hölder continuous with exponent $\alpha$ there since $D$ is bounded. Similarly $U_\eta(w) = W_\eta(w) = -\Im(\nabla V(G(w))G'(w))$ and thus $|U_\eta(w_2) - U_\eta(w_1)| = |W_\eta(w_2) - W_\eta(w_1)| = |\Im(\nabla V(G(w_2))G'(w_2)) - \Im(\nabla V(G(w_1))G'(w_1))| \leq |\nabla V(G(w_2))G'(w_2) - \nabla V(G(w_1))G'(w_1)| \leq |G(w_2) - G(w_1)|\alpha \|G\|\|G'\|\|G''\| |w_2 - w_1| + |w_2 - w_1|L_0\|\nabla V \circ G\|\|G\|\|G''\|$, and so $U_\eta$ also turns out to be uniformly Hölder continuous with exponent $\alpha$ on $D$. For the second order partial derivatives notice that $V$ harmonic implies that $\nabla V$ is an analytic function, when considering $\nabla V$ as a complex number, and so $U_{\xi\xi}(w) = \Re\left(\nabla V(G(w))G''(w) + \nabla V'(G(w))(G'(w))^2\right)$. But then it is obtained that $|U_{\xi\xi}(w_2) - U_{\xi\xi}(w_1)| = |W_{\xi\xi}(w_2) - W_{\xi\xi}(w_1)| \leq \Re\left|\nabla V(G(w_2))G''(w_2) - \nabla V(G(w_1))G''(w_1)\right| + \Re\left|\nabla V'(G(w_2))(G'(w_2))^2 - \nabla V'(G(w_1))(G'(w_1))^2\right| \leq \Re\left|\nabla V(G(w_2))G''(w_2) - \nabla V(G(w_1))G''(w_1)\right| + \Re\left|\nabla V'(G(w_2))(G'(w_2))^2 - \nabla V'(G(w_1))(G'(w_1))^2\right| + |\nabla V'(G(w_2))G'(w_2) - \nabla V'(G(w_1))G'(w_1)| \leq |G''(w_2) - G''(w_1)|\|G''\|\|G'(w_2)\|^2 + |G'(w_2) - G'(w_1)|\|G'(w_2)\|^2 + 2\|G''\|^2\|\nabla V'\|^2\|G'(w_2) - G'(w_1)\|$. Using the uniform Lipschitz continuity of $G$ and $G'$, the uniform Hölder continuity with exponent $\alpha$ of $V_\xi, V_\eta, V_{\xi\xi}, V_{\xi\eta}, p, p'$, as well as the compactness of $D$ it follows that $U_{\xi\xi}$ is uniformly Hölder continuous with exponent $\alpha$. Proceeding further we notice that $U_{\xi\eta}(w) = -\Im\left(\nabla V(G(w))G''(w) + \nabla V'(G(w))(G'(w))^2\right)$, and also $U_{\eta\eta}(w) = -\Re\left(\nabla V(G(w))G''(w) + \nabla V'(G(w))(G'(w))^2\right)$; so one gets the same upper-bounds for $|U_{\eta\eta}(w_2) - U_{\eta\eta}(w_1)|$ and $|U_{\xi\xi}(w_2) - U_{\xi\xi}(w_1)|$, respectively. In conclusion $U_{\xi\eta}$ and $U_{\eta\eta}$ are uniformly Hölder continuous with exponent $\alpha$ on $D$ as well. For the third order partial derivatives of $U$ notice that applying the Cauchy-Riemann equations for the expressions of $U_{\xi\xi\xi}$ and $U_{\eta\eta\eta}$ we obtain that $U_{\xi\xi\xi}(w) = \Re(F(w)), U_{\eta\eta\eta}(w) = U_{\xi\xi\xi}(w) = U_{\eta\eta\eta}(w) = -3\Im(F(w)), U_{\xi\eta\eta}(w) = U_{\eta\eta\xi}(w) = -\Re(F(w))$, and finally $U_{\eta\eta\eta}(w) = 3\Im(F(w))$, where $F(w) = \nabla V'(G(w))G'(w)G''(w) + \nabla V(G(w))G''(w) + \nabla V'(G(w))(G'(w))^2 + 2\nabla V'(G(w))G'(w)G''(w)$ is an analytic function on the other hand $G''(w) = p(w)G'(w), G''(w) = (p'(w) + p(w))G'(w), G'''(w) = (p'(w) + 3p(w)p'(w) + p''(w))G'(w)$, and $p''(w) = \omega_{\xi\xi\xi}(w) = -i\omega_{\xi\eta\eta}(w), p''(w) = \omega_{\eta\eta\eta}(w) - i\omega_{\xi\xi\xi}(w)$, where the latter two are uniformly Hölder continuous with exponent $\alpha$ on $D$. 



due to Kellogg’s theorem. Using these relations and proceeding similarly as we did for the second order partial derivatives, it follows that the third order partial derivatives of \( U \) are locally \( \alpha \) Hölder continuous on \( \overline{D} \), and hence uniformly Hölder continuous with exponent \( \alpha \) due to the compactness of \( \overline{D} \). This concludes the proof for the case when \( m = 2 \). For an arbitrary positive integer \( m \geq 2 \) one can proceed by induction using the same arguments as for the case \( m = 2 \). The proof is now completed.

The next theorem shows the equivalence of the solutions of Dirichlet and Neumann problems for the Laplace operator in the case of bounded, planar, doubly-connected regions.

**Theorem 6.** Let \( F : A_1 r_2 \to D \) be the conformal map given in Lemma \[3\] where \( r_2 = e^{\lambda_1} \), and assume \( D \in \mathcal{C}^{2, \alpha} \) for some \( \alpha \in (0, 1) \). If \( \Phi \in \mathcal{C}^0(\partial D) \) satisfies the compatibility condition \( \int_{\partial D} \Phi \ d\sigma = 0 \) and if \( U \) is the solution of the Neumann problem \[3\] on \( D \) with boundary data \( \Phi \), satisfying \( U(F(\sqrt{r_2})) = 0 \), then for any point \( w \in D \)

\[
U(w) = \int_{\frac{\pi}{2\arg(G(w))}}^{\pi} \frac{u(F(\rho G(w)))}{\rho} d\rho + \sqrt{r_2} \int_0^t \left[ \hat{c} - \Re \left( \int_0^t \nabla u(F(\sqrt{r_2} e^{i\tau})) F'(\sqrt{r_2} e^{i\tau}) e^{i\tau} d\tau \right) \right] dt,
\]

where \( u \) is the solution of the Dirichlet problem \[2\] on \( D \) with boundary values

\[
\varphi(w) = \begin{cases} \frac{\Phi(w)}{|\nabla \omega(w)|} & \text{if } |G(w)| = r_2, \\ -\frac{\Phi(w)}{|\nabla \omega(w)|} & \text{if } |G(w)| = 1, \end{cases}
\]

and where the constant \( \hat{c} \) is given by

\[
\frac{\sqrt{r_2}}{2\pi} \int_0^{2\pi} \Re \left( \int_0^t \nabla u(F(\sqrt{r_2} e^{i\tau})) F'(\sqrt{r_2} e^{i\tau}) e^{i\tau} d\tau \right) dt.
\]

Conversely if \( \varphi \in \mathcal{C}^0(\partial D) \) satisfies \( \int_0^{2\pi} \varphi(F(r_2 e^{i\theta})) \ d\theta = \int_0^{2\pi} \varphi(F(e^{i\theta})) \ d\theta \) and if \( U \) is the solution of the Neumann problem \[3\] on \( D \) with boundary data

\[
\Phi(w) = \begin{cases} |\nabla \omega(w)| \varphi(w), & \text{if } |G(w)| = r_2, \\ -|\nabla \omega(w)| \varphi(w), & \text{if } |G(w)| = 1, \end{cases}
\]

then the solution \( u \) of the Dirichlet problem \[3\] on \( D \) with boundary values \( \varphi \) is

\[
u(w) = \frac{\langle \nabla U(w); \nabla \omega(w) \rangle}{|\nabla \omega(w)|^2}, \ w \in \overline{D}.
\]
Proof. For brevity the following notations will be adopted $V = U \circ F$, $\nu : \partial D \to \mathbb{R}$ is the outward normal derivative at $D$, $n : \partial A_{1;r_2} \to \mathbb{R}$ is the outward normal derivative at $A_{1;r_2}$. We have $\Phi(w) = \langle \nabla U(w), \nu(w) \rangle$ on $\partial D$ and notice that

$$
\nu(F(z)) = \begin{cases} \frac{zF'(z)}{r_2|F'(z)|} = \frac{\nabla \omega(F(z))}{|\nabla \omega(F(z))|} & \text{if } |z| = r_2, \\ \frac{zF'(z)}{r_1|F'(z)|} = \frac{\nabla \omega(F(z))}{|\nabla \omega(F(z))|} & \text{if } |z| = 1, \end{cases}
$$

from where it is obtained that

$$\Phi(w) = \Re(\nabla U(w) \nu(w)), \ \forall w \in \partial D.$$ 

Also $\nabla V(z) = \frac{\partial}{\partial \bar{w}}(U(F(z))) + i \frac{\partial}{\partial w}(U(F(z)))$ and letting $w = F(z)$ compute successively

$$\frac{\partial}{\partial x}(U(F(z))) = U\xi(w)\xi_x(z) + U\eta(w)\eta_x(z) = \Re(\nabla U(F(z))F'(z)),$$

$$\frac{\partial}{\partial y}(U(F(z))) = U\xi(w)\xi_y(z) + U\eta(w)\eta_y(z)$$

C-R equations $-U\xi(w)\eta_x(z) + U\eta(w)\xi_x(z) = -\Im(\nabla U(F(z))F'(z))$.

In conclusion $\nabla V(z) = \Re(\nabla U(F(z))F'(z)) - i \Im(\nabla U(F(z))F'(z))$, $z \in A_{1;r_2}$, from where it follows by means of a continuity argument that

$$\nabla V(z) = \nabla U(F(z))F'(z), \ z \in A_{1;r_2},$$

yielding $\frac{\partial V}{\partial n}(z) \big|_{z=r_2} = \Re(\nabla U(F(z))F'(z)\frac{\bar{z}}{r_2}) = \Re(\nabla U(F(z))F'(z))$.

$\cdot \nu(F(z))) = |F'(z)|\Phi(F(z))$, and similarly one obtains $\frac{\partial V}{\partial n}(z) \big|_{z=1} = \Re(-z\nabla U(F(z))F'(z)) = |F'(z)|\Re(\nabla U(F(z))\nu(F(z))) = |F'(z)|\Phi(F(z))$. To sum up

$$\frac{\partial V}{\partial n}(z) = \Phi(F(z))|F'(z)| = \Phi(F(z))|\nabla \omega(F(z))|^{-1}, \ \forall z \in \partial A_{1;r_2}, \ (61)$$

where the second equality is due to the relation $G' = pG$. But then defining $\Phi_V : \partial A_{1;r_2} \to \mathbb{R}$, $\Phi_V = \frac{\partial V}{\partial n}$, since $V$ is harmonic in $A_{1;r_2}$ and $\nabla V$ can be extended by continuity to $A_{1;r_2}$, it follows that $V$ is a solution of the Neumann problem on $A_{1;r_2}$ with boundary data $\Phi_V$. In addition $V(\sqrt{r_2}) = U(F(\sqrt{r_2})) = 0$.

So applying Theorem 3 for $\Phi_V$

$$V(z) = \int_0^1 \frac{v(\rho z)}{\rho} \ d\rho + i \int_0^{\arg(z)} \left( \mathcal{C} - \int_0^t \frac{\partial v}{\partial \alpha c}(\sqrt{r_2}e^{i\tau}) \ d\tau \right) \ d\tau , \ (62)$$

where $\mathcal{C} = \sqrt{r_2} \int_0^2 \int_0^t \frac{\partial v}{\partial \alpha c}(\sqrt{r_2}e^{i\tau}) \ d\tau dt$, and where $v$ is the solution of the Dirichlet problem on $A_{1;r_2}$ with boundary data $\varphi_V(z) = \begin{cases} r_2 \Phi_V(z) & \text{if } |z| = r_2, \\ -\Phi_V(z) & \text{if } |z| = 1. \end{cases}$
Consequently if \( u = v \circ G \) then \( u \) is harmonic in \( D \), extends continuously to \( D \), and has continuous boundary data \( \varphi V \circ G \), which coincides with \( \varphi \) given in the statement of the theorem. In addition denoting \( \mathcal{G} = \{ F(C, \sqrt{z}) \} \) (the image through \( F \) of \( C, \sqrt{z} \) ) one obtains the following two relations

\[
\begin{align*}
  v_x(w) &= (\nabla u(w) + F'(G(w)) = \Re(\nabla u(w) F'(G(w))) \\
  v_y(w) &= (\nabla u(w), -\eta \xi(z)) = -\Im(\nabla u(w) F'(G(w))) \quad \forall w \in \mathcal{G}.
\end{align*}
\]

To this end letting \( z = \sqrt{r_2} e^{i\tau} \) it follows that \( \frac{\partial v}{\partial n_e}(\sqrt{r_2} e^{i\tau}) = (\nabla v(z), \frac{z}{\sqrt{r_2}}) = \frac{1}{\sqrt{r_2}} \Re(\nabla v(z) z) = \Re(\nabla u(w) F'(G(w)) e^{i\tau}), \quad \forall \tau \in \mathbb{R}, \ w = F(\sqrt{r_2} e^{i\tau}). \)

Combining this with the expression of \( V \) given in [62] it follows that \( U = V \circ G \) has the desired expression [67].

For the second part observe first that the assumption \( \int_0^{2\pi} \varphi(F(r_2 e^{i\theta})) \ d\theta = 0 \) implies \( \int_0^{2\pi} \frac{\partial}{\partial \varphi} \Phi \ d\sigma = 0 \). Next putting \( G(w) = z = re^{i\theta} \) and using the first part of the theorem one obtains

\[
\frac{\partial}{\partial \varphi} U(F(re^{i\theta})) = \frac{u(F(re^{i\theta}))}{r}.
\]

Consequently compute \( \frac{\partial}{\partial \varphi} U(F(re^{i\theta})) = U_{\xi}(w) (\frac{\partial}{\partial \varphi} \xi(re^{i\theta})) + U_{\eta}(w) (\frac{\partial}{\partial \varphi} \eta(re^{i\theta})) \).

But \( \frac{\partial}{\partial \varphi} \xi(re^{i\theta}) = \frac{1}{r} \Re \left( \frac{1}{p(w)} \frac{\omega_{\xi}(w)}{\omega_1(w) + \omega_2(w)} \right) \) and similarly \( \frac{\partial}{\partial \varphi} \eta(re^{i\theta}) = \frac{1}{r} \Im \left( \frac{1}{p(w)} \right) \).

\[
\begin{align*}
  \frac{1}{r} U_{\eta}(w) \cdot \frac{\omega_1(w)}{\omega_2(w) + \omega_4(w)} \quad \text{which concludes the whole proof.}
\end{align*}
\]

Though this section is devoted to the case of doubly-connected regions, it will be ended with a result concerning the bounded simply-connected regions in the plane. More precisely Theorem 5 in [5] will be completed with a result concerning the smooth extension of the higher order partial derivatives of a solution to the Neumann problem in the case of a smooth, bounded, simply-connected region \( D \subset \mathbb{C}, \ D \neq \mathbb{C}. \)

**Theorem 7.** Let \( D \) be a smooth, bounded, simply-connected region of the complex plane and let \( f : U \rightarrow D \) be the conformal transformation of \( U \) onto \( D \) with \( f(0) = w_0 \) and \( f'(0) > 0 \); define \( g = f^{-1} \) and assume there is some positive integer \( m \geq 2 \) and some \( \alpha \in (0, 1) \) such that \( D \in C^{m+1, \alpha} \). If \( \Phi \in C^{m, \alpha}(\partial D) \) satisfies the compatibility condition \( \int \Phi \ d\sigma = 0 \) and if \( U \) is the solution of the Neumann problem [3] on \( U \) with boundary data \( f \), satisfying \( U(w_0) = 0 \), then

\[
U(w) = \int_0^1 \frac{u(f(pg(w))))}{\rho} \ d\rho, \ w \in \partial D,
\]

(63)
where \( u \) is the solution of the Dirichlet problem \(^2\) on \( U \) with boundary values \( \varphi = \frac{\partial f}{\partial \nu} \). Moreover \( U \) and all its partial derivatives up to order \( m + 1 \) are uniformly Hölder continuous with exponent \( \alpha \) on \( D \). Conversely if \( \varphi \in C^0(\partial D) \) satisfies \( \int_0^{2\pi} \varphi \, d\theta = 0 \) and if \( U \) is a solution of the Neumann problem \(^3\) on \( U \) with boundary data \( \Phi = \varphi|g'| \), then the solution \( u \) of the Dirichlet problem \(^2\) on \( U \) with boundary data \( g \) is given by

\[
u(w) = g(w) \frac{|g'(w)|}{g'(w)}, \quad w \in \partial D
\]

Proof. For the first part, in light of \(^3\) Theorem 5, it only remains to prove that \( U \) together with all its partial derivatives up to order \( m + 1 \) are uniformly Hölder continuous with exponent \( \alpha \) provided that \( \Phi \in C^{m,\alpha}(D) \), \( D \in C^{m+1,\alpha} \).

Again, to keep the derivations simple, the theorem will be proved only for the case \( m = 2 \), as the general case \( m \geq 2 \) follows similarly using induction. To see that our claim for the case \( m = 2 \) is indeed true notice that if \( D \in C^{3,\alpha} \) then in view of \(^3\) Chapter 3 \( f^{(3)} \) is uniformly Hölder continuous with exponent \( \alpha \) and moreover the continuous extension of \( f' \) to \( \overline{U} \) does not vanish. Consequently \( g' = \frac{f'}{|f'} \) extends continuously to \( D \) and \( g \) turns out to be uniformly Lipschitz continuous. Also it is easy to see that \( f \in C^{3,\alpha} \) and \( \inf_{z \in \overline{U}} |f'(z)| > 0 \) imply that \( g' \) as well as \( g'' \) and \( g''' \) are uniformly Hölder continuous with exponent \( \alpha \). On the other hand if \( \Gamma \) is any \( C^{2,\alpha} \) parameterization of the unit circle then, assuming without loss of generality that \( 0 \notin \partial D \), \( |f' \circ \Gamma| \) is readily seen to belong to \( C^{2,\alpha}(\mathbb{R}) \) and thus the function \( (\Phi \circ f)|f'| \in C^{2,\alpha}(\partial U) \). Defining now \( V = U \circ f \) it follows that \( V \) is harmonic in \( U \) and \( \nabla V(z) = \nabla U(f(z))f'(z) \). Thus \( \nabla V \) extends continuously to \( \overline{U} \), which in turn shows that the normal derivative of \( V \) is \( \frac{\partial V}{\partial n}(z) = \Re \left( z \nabla V(z) \right) = \Re \left( z \nabla U(f(z))f'(z) \right) = |f'(z)| \Re \left( \nabla U(f(z))\nu(f(z)) \right) = \Phi(f(z)) |f'(z)| \forall z \in \partial U \), where

\[
u(w) = g(w) \frac{|g'(w)|}{g'(w)}, \quad w \in \partial D
\]

is the unitary outward pointing normal to \( \partial D \) at \( w \) (see for instance \(^3\)). In conclusion \( V \) is the solution of the Neumann problem \(^3\) on \( U \) with boundary data \( (\Phi \circ f)|f'| \) and since the latter was proved to belong to \( C^{2,\alpha}(\partial U) \) Corollary \(^1\) guarantees that \( V \) and all its partial derivatives up to order \( m = 3 \) are uniformly Hölder continuous with exponent \( \alpha \). Finally \( \nabla U = (\nabla V \circ g) g' \) and since \( \nabla U \) and \( \nabla V \) are analytic functions in \( D \) and \( U \), respectively, taking the derivatives and using the Cauchy-Riemann equations as well as the relations...
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\[ \nabla U' = U_{\xi\xi} - iU_{\xi\eta}, \quad \nabla V' = V_{xx} - iV_{xy}, \]

one obtains

\[
\begin{align*}
U_\xi(w) &= \Re \left( \nabla V(g(w))g'(w) \right), \\
U_\eta(w) &= -\Im \left( \nabla V(g(w))g'(w) \right), \\
U_{\xi\xi}(w) &= \Re \left( \nabla V'(g(w))(g'(w))^2 + \nabla V(g(w))g''(w) \right), \\
U_{\xi\eta}(w) &= -\Im \left( \nabla V'(g(w))(g'(w))^2 + \nabla V(g(w))g''(w) \right), \\
U_{\eta\eta}(w) &= -U_{\xi\xi}(w) = -\Re \left( \nabla V'(g(w))(g'(w))^2 + \nabla V(g(w))g''(w) \right), \\
U_{\xi\xi\eta}(w) &= U_{\xi\eta\xi}(w) = U_{\eta\xi\xi}(w) = -\Im(\mathcal{F}), \\
U_{\xi\eta\eta}(w) &= U_{\eta\xi\eta}(w) = U_{\eta\eta\xi}(w) = -\Re(\mathcal{F}),
\end{align*}
\]

with \( \mathcal{F}(w) = \nabla' V''(g(w))(g'(w))^2 + 2\nabla' V(g(w))g'(w)g''(w) + \nabla V(g(w))g''(w) \) analytic. The theorem now follows using the properties of \( \nabla V \) and \( g \) discussed above, as well as using the triangle inequality in the above relations in a similar way it was done in the proof of Theorem 5.

\[ \square \]

4 Conclusions

The paper provides an equivalence between the solutions of the Neumann and the Dirichlet problems for planar, smooth, bounded, doubly-connected regions. This equivalence is expressed by the fact that solving any of these two problems leads by an analytic formula to an explicit solution of the other problem. As an application of this intimate connection the theory developed in this paper shows that under additional smoothness assumptions on the boundary of the region as well as on the boundary data, the higher order partial derivatives of the solutions of the Neumann problem are uniformly Hölder conditions. These assumptions are similar to those appearing in Kellogg's theorem, where the problem of continuous extensions of the higher order partial derivatives for the solution of the Dirichlet problem was investigated. This fact comes to enhance the connection between the Dirichlet and the Neumann problems thus showing that for some types of regions of the complex plane these two problems should be studied simultaneously in a unified approach. In the end, a closer examination of the results reveals that the dependency between the solutions of the Dirichlet and the Neumann problems is more complex in the case of doubly-connected regions, and a natural question that arises is how does this connection look like for regions of connectivity greater than or equal to three and how could this be extended to regions of \( \mathbb{R}^d \) for \( d \geq 3 \)? A positive answer might help us dive deeper into this intimate connection and obtain new interesting results regarding these fundamental problems.
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5 APPENDIX

This section is entirely devoted to the connection between the solution of the Dirichlet problem and a particular solution of the Neumann problem, in the case of elliptic regions. Although the elliptic regions are obviously planar, smooth, bounded, simply-connected regions for which the connection between the two problems has been explicitly provided (see [5, Theorem 5] or [6, Theorem 5]), the conformal mapping on which this connection relies on is somewhat cumbersome, thus making the representation of the Neumann problem in terms of the solution of the Dirichlet problem somehow redundant for a direct application. This issue is fixed in the paper at hand by considering another approach for obtaining the desired connection. This approach is based on the Joukowsky transform.

Let \( J : \mathbb{C}^* \to \mathbb{C}, \ J(w) = \frac{1}{2} \left( w + \frac{1}{w} \right) \) be the Joukowsky transform, and define \( J_+ = J \big|_{U^c \cup \{ e^{i\theta} \mid \theta \in (-\pi, 0) \}}, \ T_+ = J_+^{-1}, \ J_- = J \big|_{\partial U \cup \{ e^{i\theta} \mid \theta \in (-\pi, 0) \}}, \ T_- = J_-^{-1}. \)

Throughout this section, the argument of a complex number will be defined as taking values in \((-\pi, \pi]\) and if \( z \) is any complex number then its square root will be defined as

\[
\sqrt{z} = \begin{cases} 
|z| e^{i \frac{\arg(z)}{2}} & \text{if } z \neq 0, \\
0 & \text{if } z = 0.
\end{cases}
\]

If \( \rho > 1 \) let \( E_\rho \) be the interior of the ellipse with equation

\[
\frac{x^2}{\rho^2 + \rho - 1} + \frac{y^2}{(\rho - \rho^{-1})^2} = 1
\]

and for \( \theta \in (-\pi, \pi] \) the hyperbola described by

\[
\frac{x^2}{\cos^2 \theta} - \frac{y^2}{\sin^2 \theta} = 1
\]

will be denoted \( H_\theta \). Finally we let \( \mathcal{H} = \{ H_\theta \mid \theta \in (-\pi, \pi] \} \). It is not difficult to notice that \( \mathcal{H} \) is actually the set of all hyperbolas orthogonal to the family of confocal ellipses having foci \( \{( -1, 0), (1, 0) \} \). This aspect will play an essential role in the proof of Theorem 8 below.

**Lemma 4.** \( T_+ \) has the following properties.

1. It is well defined on the whole \( \mathbb{C}; \)

2. It is analytic in \( \mathbb{C} \setminus [-1, 1] \) with nonzero derivative;

3. For any point \( \xi \in [-1, 1] \) and any sequence \( \{ z_n \}_{n=1}^\infty \) satisfying
   a. there exist \( H_\theta \in \mathcal{H} \) for which \( z_n \in H_\theta \cap \{ \Im(z) > 0 \} \) \( \forall \ n \in \mathbb{N}^*; \)
   b. \( z_n \to \xi; \)

   one has \( \lim_{n \to \infty} T_+(z_n) = T_+(\xi); \)

4. \( T_+(z) = z + \sqrt{z^2 - 1}, \ z \in \mathbb{C}. \)

**Proof.** 1. It is not difficult to see that \( J_+ \left( \mathbb{C} \setminus \{ e^{i\theta} : \theta \in (-\pi, 0) \} \right) = \mathbb{C} \) and that \( J_+ \) is invertible there (see for instance [2, Chapter 4.2]). This property was actually used in the definition of \( T_+. \)
2. Indeed choose any point \( z \in \mathbb{C} \setminus [-1, 1] \) and let \( w = T_+(z) \). As the set 
\( T_+(\mathbb{C} \setminus [-1, 1]) = \{ \xi \in \mathbb{C} : |\xi| > 1 \} \) is open and \( J'_+(w) = J'(w) \neq 0 \) it
follows that there is an open neighborhood \( U_w \) of \( w \) in \( \{ \xi \in \mathbb{C} : |\xi| > 1 \} \) such that
\( J'_+(u) \neq 0 \) \( \forall u \in U_w \). Define \( V_z = J_+(U_w) \). Then \( V_z \) is an open
subset of \( \mathbb{C} \) and we have \( T'_+(l) = \frac{1}{J'_+(T+(l))} \in \mathbb{C}^* \forall l \in V_z \).

3. Since \( z_n \in \mathcal{H}_\theta \cap \{ 3(z) > 0 \} \) it follows that \( z_n = \frac{1}{2} \left( \rho_n + \frac{1}{\rho_n} \right) \cos \theta + \frac{i}{2} \left( \rho_n - \frac{1}{\rho_n} \right) \sin \theta \) for some \( \rho_n > 1 \) and some \( \theta \in [0, \pi] \). On the other hand
since \( z_n \to \xi \in [-1, 1] \) it follows that \( \rho_n \to 1 \) and thus \( \xi = \cos \theta \). One
can also deduce that \( T_+(z_n) = \rho_ne^{i\theta} \) and thus \( \lim_{n \to \infty} T_+(z_n) = e^{i\theta} \). But
\( J_+(e^{i\theta}) = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) = \cos \theta = \xi \) and so \( T_+(z_n) \to e^{i\theta} = T_+(\xi) \).

4. See [2, Chapter 3].

**Theorem 8.** Fix some \( \rho > 1 \) and assume that \( f \in C^0(\partial E_\rho) \) satisfies
the compatibility condition \( \int f \, d\sigma = 0 \). If \( U \) is the solution of the Neumann problem
\( \mathcal{D} \) on \( E_\rho \) having boundary data \( f \) and satisfying \( U(1) = 0 \), then letting
\( R(z)e^{i\theta(z)} := T_+(z) \), \( \Theta(z) \in (-\pi, \pi] \), one has

\[
U(z) = \int_0^1 \frac{u(tT_+(z))}{t} \, dt - \Theta(z) \int_0^1 \int_0^\infty \frac{\partial u}{\partial a_x}(e^{i\tau}) \, d\tau \, dt, \quad z \in E_\rho,
\]

(65)

where \( u \) is the solution of the Dirichlet problem on \( \{ \frac{1}{\rho} < |w| < \rho \} \) with boundary values

\[
\varphi_- = \frac{f \circ J}{\rho |T'_- \circ J|} \quad \text{on} \quad C_{\frac{1}{\rho}},
\]

\[
\varphi_+ = \frac{f \circ J}{|T'_+ \circ J|} \quad \text{on} \quad C_\rho.
\]

**Proof.** Let \( U : E_\rho \to \mathbb{R} \) be as in the statement of the theorem. Define \( V = U \circ J_{\rho^{-1}} \) \( \mathbf{A}_{\rho^{-1}} \) and notice that \( V \) thus obtained is harmonic on \( \mathbf{A}_{\rho^{-1}} \) and also

\[
\frac{\partial V}{\partial \rho}(w) = V_\xi(w) \frac{\partial w}{\rho} + V_\eta(w) \frac{\partial \eta}{\rho}, \quad \forall w \in C_\rho,
\]

where \( \omega = \xi + i\eta \) and \( \mathbf{V} \) is the unitary outward pointing normal to \( \partial \mathbf{A}_{\rho^{-1}} \). Defining

\[
\omega(w) = V_\xi(w) - iV_\eta(w) = \nabla V(w), \quad \{ \frac{1}{\rho} < |w| < \rho \},
\]

one obtains alternatively \( \frac{\partial V}{\partial \rho}(w) = \mathbb{R} \left( \omega(w) \frac{\omega}{\rho} \right), \quad \forall w \in C_\rho \). Using the Cauchy-
Riemann equations together with the harmonicity of \( V \) it follows that \( \omega \) is
analytic on \( \{ \frac{1}{\rho} < |w| < \rho \} \). Furthermore \( \frac{\partial V}{\partial \rho}(w) = \mathbb{R} \left( \omega(w) \frac{\omega}{\rho} \right), \quad w \in C_\rho \). On the
other hand let $G$ be an analytic function such that $U = \Re(G)$ on $E_\rho$ (which is always possible since $U$ is harmonic and $E_\rho$ is a simply connected region). But then setting $F = G \circ J_{\frac{1}{2\rho-1},\rho}$ one obtains $V = \Re(F)$. Consequently it follows that $F' = \omega$ on $\{ \frac{1}{\rho} < |w| < \rho \}$, which gives $F(w) = F(w_0) + \int_{w_0}^w \omega(o) \, do$, or equivalently

$$G(J(w)) = G(J(w_0)) + \int_{w_0}^w \omega(o) \, do,$$

(66)

where $w_0 = w_0(w)$ is to be specified later on. Notice now that any $o \in \{ 1 < |w| < \rho \}$ is of the form $T_+(\lambda)$ for some $\lambda \in E_\rho \setminus [-1, 1]$, and hence according to point (2) of Lemma 4 $o'(\lambda) = T'_+(\lambda) = \frac{2\rho^2(\lambda)}{\sigma(\lambda)-1}$ and consequently

$$G(z) = G(z_0) + \int_{z_0}^z \omega(T_+(\lambda)) T'_+(\lambda) d\lambda = G(z_0) + 2 \int_{z_0}^z \omega(T_+(\lambda)) \frac{T_{2+}^2(\lambda)}{T_{2+}^2(\lambda) - 1} d\lambda,$$

where $z_0 = J(w_0)$. Most of the remaining part of the proof will be divided into several steps.

Step 1. Denote $w = T_+(z)$ for any $w \in A_{1,\rho}$ and consequently define the curve

$$\gamma_w^\epsilon(t) := \begin{cases} \frac{wt}{t}, & t \in \left[ \frac{1+\epsilon}{|w|}, 1 \right]; \\ (1+\epsilon) \exp \left( i \arg(w) \frac{t-\epsilon}{|w|-\epsilon} \right), & t \in \left[ \epsilon, \frac{1+\epsilon}{|w|} \right]; \\ 1+t, & t \in \left[ \frac{\epsilon}{2}, \epsilon \right), \end{cases}$$

from where it follows that

$$\dot{\gamma}_w^\epsilon(t) = \begin{cases} \frac{w}{t}, & t \in \left( \frac{1+\epsilon}{|w|}, 1 \right); \\ i \frac{\arg(w)}{|w|-\epsilon} (1+\epsilon) \exp \left( i \arg(w) \frac{t-\epsilon}{|w|-\epsilon} \right), & t \in \left( \epsilon, \frac{1+\epsilon}{|w|} \right); \\ 1, & t \in \left( \frac{\epsilon}{2}, \epsilon \right) \end{cases}$$

for any $w \in \{ 1 < |w| \leq \rho \}$ and any $\epsilon > 0$ small enough. Now define $\lambda_\epsilon^w(t) = J_+(\gamma_\epsilon^w(t))$ which gives $\dot{\lambda}_\epsilon^w(t) = \frac{\gamma_\epsilon^w(t)}{T_{2+}(\lambda)^2(t)}$. Also $\lim_{\epsilon \to 0} G(J_+(1+\frac{\epsilon}{2})) = \lim_{\epsilon \to 0} G(J_+(1+\frac{\epsilon}{2})) = G(0)$. To this end setting $z_0 = z_0(\epsilon) = J_+(1+\frac{\epsilon}{2})$ in (66) one obtains

$$G(z) = G(J_+(1+\frac{\epsilon}{2})) + \int_{1+\frac{\epsilon}{2}}^{1+\epsilon} \frac{\omega(tw)}{t} \frac{dt}{tw} + \int_{\frac{\epsilon}{2}}^{\epsilon} \omega(1+t) \frac{dt}{tw}$$

$$+ i \frac{\arg(w)}{|w|-\epsilon} \int_{\epsilon}^{\frac{1+\epsilon}{2}} \omega(\gamma_\epsilon^w(t)) \gamma_\epsilon^w(t) \, dt, \quad \epsilon > 0.$$
Step 2. Link the first integral term in (67) to the solution of some Dirichlet
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Combining now (68) and (69) one obtains

\[ U(z) = \frac{1}{2\pi i} \int \frac{\Re [\{\mathbf{T}_+(z)\} \mathbf{w}(t\mathbf{T}_+(z))]}{t^2} \, dt - \frac{1}{2\pi i} \int \frac{\Im [\mathbf{w}(\gamma^w(t)) \gamma^w(t)]}{t} \, dt, \]

where the normalization \( U(1) = 0 \) was used.

Setting \( \epsilon \to 0 \) it follows by the use of the Dominant Convergence theorem
that \( G(z) = G(1) + \frac{1}{2\pi i} \int \frac{\mathbf{T}_+(z)\mathbf{w}(t\mathbf{T}_+(z))}{t} \, dt + iR(z)\Theta(z) \int \mathbf{w}(\gamma^w(t)) \gamma^w(t) \, dt, \)
where \( \gamma^w = \lim_{\epsilon \to 0} \gamma^w_\epsilon \). Taking the real part in the equation above it follows that

\[ \omega (\Lambda^w_{\epsilon}(t)) \mathbf{T}_+^\epsilon (\Lambda^w_{\epsilon}(t)) \dot{\Lambda}^w_{\epsilon}(t) = \omega (\mathbf{T}_+ (\Lambda^w_{\epsilon}(t))) \mathbf{w} = \omega (tw) \mathbf{w}, \] (68)

and also

\[ \omega (\Lambda^w_{\epsilon}(t)) \mathbf{T}_+^\epsilon (\Lambda^w_{\epsilon}(t)) \dot{\Lambda}^w_{\epsilon}(t) = \nabla U(\Lambda^w_{\epsilon}(t)) \dot{\Lambda}^w_{\epsilon}(t), \] (69)

both (68) and (69) being true for any \( \epsilon > 0 \), \( w \in \{1 < |w| < \rho\} \), \( t \in \left[ \frac{1}{|w|} + \epsilon, 1 \right] \), where in the derivation of (69) the following important ob-

\[ \nabla U(z) = \omega (\mathbf{T}_+(z)) \cdot \mathbf{T}_+(z), \quad \forall z \in \mathbf{E}_p \setminus [-1, 1]. \] (70)

Combining now (68) and (69) one obtains

\[ \Re [\omega (tw) \mathbf{w}] = \langle \nabla U(\Lambda^w_{\epsilon}(t)); \dot{\Lambda}^w_{\epsilon}(t) \rangle, \]

whenever \( \epsilon > 0 \), \( w \in \{1 < |w| \leq \rho\} \), \( t \in \left[ \frac{1}{|w|} + \epsilon, 1 \right] \) and choosing any

\[ z^* \in \partial \mathbf{E}_p \]

\[ \Re [\omega (\mathbf{T}_+(z^*)) \mathbf{T}_+(z^*)] = \langle \nabla U(z^*); \Lambda^w (1) \rangle, \] (71)
where \( \dot{\Lambda}^z_+ (1) \) is thus the outward normal derivative in \( z^* \) at \( \partial E_\rho \). Consequently compute \( |\dot{\Lambda}^z_+ (1)| = \frac{|T^z_+ (z^*)|}{|T^z_+ (z^*)|} = \frac{\rho}{|T^z_+ (z^*)|} \) which shows, using relation \((71)\), that

\[
\Re \left[ \omega (T_+ (z^*)) \right] T_+ (z^*) = \langle \nabla U (z^*) ; \frac{\dot{\Lambda}^z_+ (1)}{|\dot{\Lambda}^z_+ (1)|} \rangle = f(z^*) \frac{\rho}{|T^z_+ (z^*)|},
\]

or equivalently

\[
\Re \left[ \omega \left( w^*_+ \right) w^*_+ \right] = \frac{\rho f (J_+ (w^*_+))}{|T^z_+ (J_+ (w^*_+))|} = \rho f (J_+ (w^*_+)) |J_+ (w^*_+)|, \quad w^*_+ \in C_\rho,
\]

where \( w^*_+ = T_+ (z^*) \). On the other hand \( U (z) = V (T_- (z)) \forall z \in E_\rho \) which gives (exactly as it was done for the previous case) \( \nabla U (z) = \omega (T_- (z)) T'_- (z) \), \( \forall z \in E_\rho \setminus [-1, 1] \). In the same way define \( \Gamma^w_\epsilon, \rho : [1, \frac{1}{\rho}] \to \{ \frac{1}{\rho} \leq |w| < 1 \}, \quad \Gamma^w_\epsilon, \rho (t) = tw \) for any \( w \in \{ \frac{1}{\rho} \leq |w| < 1 \} \) and any \( \epsilon > 0 \) small enough. Thus \( \Gamma^w_\epsilon, \rho (t) = w \) and by defining \( \Lambda^z_\epsilon, \rho (t) := J_- (\Gamma^w_\epsilon, \rho (t)) \) it follows that the image of \( \Lambda^z_\epsilon, \rho \) is also a branch of some hyperbola in \( H \) orthogonal to \( \partial E_\rho \) which approaches some \( z^*_0 = z^*_0 (\epsilon) \in [-1, 1] \) as \( \epsilon \to 0 \). Also notice that \( \dot{\Lambda}^z_\epsilon, \rho (t) = J'_- (\Gamma^w_\epsilon, \rho (t)) w \) and hence

\[
\frac{\partial}{\partial t} (\dot{\Lambda}^z_\epsilon, \rho (t)) = \frac{T^z_- (z^*_0 (\epsilon))}{T^z_- (\dot{\Lambda}^z_\epsilon, \rho (t))}.
\]

But then one obtains, similarly as for \((68)\) and \((69)\)

\[
\omega \left( T'_- (\Lambda^z_\epsilon, \rho (t)) \right) T'_- (\Lambda^z_\epsilon, \rho (t)) \dot{\Lambda}^z_\epsilon, \rho (t) = \omega \left( T_- (\Lambda^z_\epsilon, \rho (t)) \right) w = \omega (tw) w,
\]

and also

\[
\omega \left( T'_- (\Lambda^z_\epsilon, \rho (t)) \right) T'_- (\Lambda^z_\epsilon, \rho (t)) \dot{\Lambda}^z_\epsilon, \rho (t) = \nabla U (\Lambda^z_\epsilon, \rho (t)) \dot{\Lambda}^z_\epsilon, \rho (t),
\]

both \((73)\) and \((74)\) being true for any \( \epsilon > 0 \), \( w \in \{ \frac{1}{\rho} \leq |w| < 1 \} \) and any \( t \in \left[ \frac{1}{|w| + \epsilon}, 1 \right] \), respectively. Combining \((73)\) and \((74)\) it follows that

\[
\Re \left[ \omega (tw) w \right] = \langle \nabla U (\Lambda^z_\epsilon, \rho (t)) ; \dot{\Lambda}^z_\epsilon, \rho (t) \rangle,
\]

\( \epsilon > 0 \), \( w \in \{ \frac{1}{\rho} \leq |w| < 1 \} \), \( t \in \left[ \frac{1}{|w| + \epsilon}, 1 \right] \) and choosing any \( z^* \in \partial E_\rho \)

\[
\Re \left[ \omega (T_- (z^*)) T_- (z^*) \right] = \langle \nabla U (z^*) ; \dot{\Lambda}^z_+ (1) \rangle = f(z^*) \frac{\rho}{|T^z_- (z^*)|}, \quad z^* \in \partial E_\rho,
\]

where \( \dot{\Lambda}^z_+ (1) \) is thus the outward normal derivative in \( z^* \) at \( \partial E_\rho \). Consequently compute \( |\dot{\Lambda}^z_+ (1)| = \frac{|T^z_- (z^*)|}{|T^z_- (z^*)|} = \frac{1}{\rho |T^z_- (z^*)|} \) which shows, using relation \((75)\), that

\[
\Re \left[ \omega (T_- (z^*)) T_- (z^*) \right] = \langle \nabla U (z^*) ; \frac{|\dot{\Lambda}^z_+ (1)|}{|\dot{\Lambda}^z_+ (1)|} \rangle = \frac{f(z^*)}{\rho |T^z_- (z^*)|}, \quad z^* \in \partial E_\rho.
\]
or equivalently

$$\Re \left[ \omega \left( w^* \right) w^* \right] = \frac{f \left( J_\rho (w^*) \right)}{\rho |T_\rho (J_\rho (w^*))|} = \frac{f \left( J_\rho (w^*) \right) |J_\rho' (w^*)|}{\rho}, \quad w^* \in C_{\frac{1}{\rho}},$$

where \( w^* = T_\rho (z^*) \). Finally, using equation (67), equations (72) and (76) together with the analyticity of \( w \omega (w) \) on \( A_{\frac{1}{\rho}, \rho} \), the continuity (and hence boundedness) of \( u(w) := \Re \{ w \omega (w) \} \) on \( A_{\frac{1}{\rho}, \rho} \), and the uniqueness of the solution to the Dirichlet problem, it follows that \( u \) is the solution of the Dirichlet problem (2) on \( \Omega \), where \( \Omega \) is the solution to the Dirichlet problem, it follows that \( \| \omega \| \) is the solution of the Dirichlet problem on \( \{ \frac{1}{\rho} < |w| < \rho \} \) with boundary values

$$\varphi_- : = \frac{f \circ J_\rho}{|T_\rho' \circ J_\rho|} \quad \text{on } C_{\frac{1}{\rho}},$$

$$\varphi_+ : = \frac{\rho f \circ J_\rho}{|T_\rho' \circ J_\rho|} \quad \text{on } C_\rho.$$

Step 3. Link the second integral in (77) to \( u \). To do so notice that since \( w \omega (w) \) is an analytic function on \( A_{\frac{1}{\rho}, \rho} \) which extends continuously to \( \overline{A}_{\frac{1}{\rho}, \rho} \) it follows that \( \Im \{ w \omega (w) \} \) is a harmonic function on \( A_{\frac{1}{\rho}, \rho} \) which extends continuously to \( \overline{A}_{\frac{1}{\rho}, \rho} \). Letting \( w \omega (w) =: u(w) + iv(w) \) it follows that \( v \) is a harmonic function in \( A_{\frac{1}{\rho}, \rho} \) which extends continuously to \( \overline{A}_{\frac{1}{\rho}, \rho} \) and the idea is to determine \( v \) from \( u \). Using (70)

$$\omega (T_\rho (z)) = \frac{\nabla U(z)}{T_\rho'(z)}, \quad z \in E_\rho \setminus [-1, 1].$$

Define the sequence \( z_n = 1 + \frac{1}{n} \) for any \( n \) large enough so that \( z_n \in E_\rho \). Then \( T_\rho (z_n) \in A_{1, \rho} \cap \mathbb{R}_+ \) and by point (3.) of Lemma 4, it follows that \( T_\rho (z_n) \to T_\rho (1) = 1 \). In addition \( T_\rho' (z_n) \) is well defined by point (2.) of the same lemma and using point (4.) of the same result it follows that \( T_\rho' (z_n) = 1 + \frac{z_n}{\sqrt{z_n^2 - 1}} \) which shows that \( |T_\rho' (z_n)| \geq \frac{|z_n|}{|\sqrt{z_n^2 - 1}|} - 1 > \frac{1}{|\sqrt{z_n^2 - 1}|} - 1 \) for \( n \) large. Since \( |\sqrt{z_n^2 - 1}| \to 0 \) it follows that \( |T_\rho' (z_n)| \geq \frac{1}{2|\sqrt{z_n^2 - 1}|} \to \infty \). But \( U \in C^1(E_\rho) \) and since \( (1, 0) \in E_\rho \), \( E_\rho \) open, it follows that there is some neighborhood of \( (1, 0) \) contained in \( E_\rho \) on which
\(|\nabla U| \leq M\), for some \(M > 0\). Consequently, it follows that \(\exists N \in \mathbb{N}^*\) such that \(\forall n \geq N\) one has \(|\omega(T_+(z_n))| = \frac{|\nabla U(z_n)|}{|T_+(z_n)|} \leq \frac{M}{\pi} \to 0\). To sum up \(\omega(w_n) \to 0\) as \(w_n \to 1\), \(w_n \in A_1, \rho \cap \mathbb{R}_+\). Since \(\omega\) is continuous on \(A_1, \rho \supset A_1, \rho \Rightarrow \omega(1) = \lim_{w_n \to 1, w_n \in A_1, \rho \cap \mathbb{R}_+} \omega(w_n) = 0\). This gives \(v(1) = 0\). Since \(u\) and \(v\) are conjugate-harmonic functions and \(v(1) = 0\), it follows that one can precisely determine \(v\) solely from \(u\). Indeed using the Cauchy-Riemann equations it follows that \(v(a, b) = \int_{\gamma} dv = \int_{\gamma} \frac{\partial u}{\partial a} (e^{i\theta}) \, d\theta\) whenever \(a + ib \in \gamma\), where \(\gamma\) is considered to be the curve \(e^{it}\) for \(t \in [0, \arg(a + ib)]\). Hence

\[
\int_0^{1/2} \Im [\omega(w(t)) w(t)] \, dt = \int_0^{1/2} \int_0^{1/2} \frac{\partial u}{\partial a} (e^{i\tau}) \, d\tau dt, \tag{78}
\]

so combining equation (77) with equation (78) and using a change of variable, Theorem 8 is proved for any \(z \in \mathbb{E}_\rho \setminus [-1, 1]\).

If \(z \in [-1, 1]\) choose any sequence \(\{z_n\}_{n=1}^\infty\) as in point (3) of Lemma 4 such that \(z_n \to z\). Then using the same point of Lemma 4 a continuity argument for \(U\), the Dominant Convergence theorem, as well as the boundedness of \(\frac{\partial u}{\partial a}\) on \(C_1\), the proof is completed.

**Remark 6.** The proof of Theorem 8 provides in addition an interesting interpretation of the second term in the right-hand side of equality (65). Indeed

\[
\Theta(z) \int_0^{1/2} \int_0^{1/2} \frac{\partial u}{\partial a} (e^{i\tau}) \, d\tau dt = -U(z_0^\Theta(z)), \quad z \in \mathbb{E}_\rho \setminus [-1, 1], \tag{79}
\]

where \(z_0^\Theta(z)\) is the intersection of the (unique) hyperbola \(H_\theta\) containing \(z\) with the line segment \([-1, 1]\).