Analytical solution of Brill waves

Piotr Koc

Independent researcher,
ul. Królowej Jadwigi 29/12, 30-209 Kraków, Poland
(Dated: August 12, 2021)

A class of analytical solutions of axially symmetric vacuum initial data for a self-gravitating system has been found. The active region of the constructed gravitational wave is a thin torus around which the solution is conformally flat. For higher values of gravitational wave amplitude the resulting hypersurface contains apparent horizons.

INTRODUCTION

In 1959 Dieter Brill in his dissertation [1] considered the problem of axially and time-symmetric, vacuum initial data for the Einstein equations. Although this is the simplest non-trivial case of vacuum constraints of general relativity, no analytical solutions are known so far. Over the last half-century an enormous effort has been made to study numerical solutions of initial data for axially symmetric gravitational waves. Due to the lack of analytical solutions, sophisticated numerical methods were developed [2–10]. Some properties of this system have also been deducted by advanced indirect methods [11–13]. The motivation to search for this solution comes from the fact that until now we have not known any non-trivial analytical vacuum initial data in general relativity.

In this paper I construct a class of explicit analytical solutions of Brill waves. I assume that the initial hypersurface of vacuum self-gravitating system in the moment of time symmetry is axially symmetric, asymptotically flat and regular.

PURE GRAVITATIONAL RADIATION INITIAL VALUE PROBLEM

Momentarily static vacuum constraint equations of general relativity reduces to:

\[(3) \quad R = 0, \]

where \((3) R\) denotes 3-dimensional scalar curvature of the initial hypersurface. The axially symmetric line element in cylindrical coordinates may be written as follows [1]:

\[ds^2 = \Phi^4(\rho, z) \left(e^{2q(\rho,z)}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right).\]  

(2)

For the above metric the Hamiltonian constraint [1] takes the form of coupled Schrödinger-like and Poisson equations:

\[\Delta_3 \Phi + V\Phi = 0,\]  

(3)

\[\Delta_2 q = 4V.\]  

(4)

The same function \(V\) acts respectively as a potential and source in the corresponding above equations. \(\Delta_3\) denotes 3-dimensional flat Laplace operator while \(\Delta_2\) is 2-dimensional flat laplacian on \(\rho z\) plane. Regularity of metric on the axis and asymptotic flatness implies the boundary conditions for \(\Phi\) and \(q\):

\[q(0, z) = 0, \quad \frac{\partial q}{\partial \rho}(0, z) = 0, \quad \frac{\partial q}{\partial z}(\rho, 0) = 0,\]

(5)

\[q = \mathcal{O}(r^{-2}) \text{ as } r \to \infty,\]

\[\frac{\partial \Phi}{\partial \rho}(0, z) = 0, \quad \frac{\partial \Phi}{\partial z}(\rho, 0) = 0,\]

(6)

\[\Phi \to 1 + \frac{m}{2r} \text{ as } r \to \infty,\]

where \(r = \sqrt{\rho^2 + z^2}\) and \(m\) is ADM mass of the system. Moreover the determinant of the metric must have no zeros. Therefore solutions for \(\Phi\) have to be everywhere positive \(\Phi > 0\).

The most common approach to this system of equations is to choose the function \(q(\rho, z)\) that meets the boundary conditions [5] and then solve for \(\Phi(\rho, z)\). Such a procedure is effective only through a numerical approach and has been extensively applied in many previous studies [2–8, 10]. Brill [1], Wheeler [11], Holz et al [12], Beig and Murchadha [13] proved many interesting properties of the described system but they have not found any \(q(\rho, z)\) for which (3) could be solved analytically.

In this research I propose an alternative approach. We first select the appropriate function \(V\) and then solve equations (3) and (4) for \(q\) and \(\Phi\). The main problem is how to choose \(V\) so that the resulting \(q\) would meet the boundary conditions [5].

The plan of this work is as follows. We will change variables of equations (3) and (4) to toroidal coordinates. Next, I will propose an appropriate function \(V\) that will generate \(q\) satisfying the boundary conditions [5]. Subsequently, the solution of equations (4) and (3) for \(q\) and \(\Phi\) will be constructed. Lastly in the resulting analytical solution I will numerically analyze the existence and properties of apparent horizons.
Let’s transform equations (3) and (4) from the cylindrical to the toroidal coordinates:

\[ \rho = a \sinh \nu \cosh \nu - \cos u, \quad z = a \sin u \cosh \nu - \cos u \]

(7)

Parameter \( a \) is a major radius of the torus. Azimuth angle \( \varphi \) is the same in both coordinate systems. Now equations (3) and (4) take the following form:

\[
\left( \cosh \nu - \cos u \right) \left( \frac{a^2}{2} \sinh \nu \frac{\partial}{\partial u} \left( \frac{\sinh \nu}{\cosh \nu - \cos u} \frac{\partial}{\partial u} \right) + \frac{\partial}{\partial \nu} \left( \frac{\sinh \nu}{\cosh \nu - \cos u} \frac{\partial}{\partial \nu} \right) \right) \Phi + V \Phi = 0,
\]

(8)

\[
\frac{(\cosh \nu - \cos u)^2}{a^2} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial \nu^2} \right) q = 4V.
\]

(9)

Respectively also transform the boundary conditions (5) and (6).

Let’s choose the function \( V \) such that it vanishes outside the thin torus. So assume that the minor radius of the torus \( R_2 \) is negligibly small compared to the large one: \( R_2 \ll a \). The interior of this torus will be an active region of a constructed gravitational wave. Near the inner circle of the torus \( e^\nu \gg 1 \). So it would be more convenient to specify a new variable \( \tau \), which measures the distance from the inner circle of the torus:

\[
\tau = 2ae^{-\nu}.
\]

(10)

Additionally, let’s assume that \( V \) forms a double-well potential inside toroidal active region:

\[
V(\tau, u) = \begin{cases} 
  h_1 & \text{for } \tau \leq R_1 \\
  h_2 & \text{for } R_1 < \tau \leq R_2 \\
  0 & \text{for } \tau > R_2.
\end{cases}
\]

(11)

Potential \( V \) is depicted schematically in Fig. 1. In the next section, analyzing solution for \( q \), I will choose the constants \( h_1 \) and \( h_2 \) such that also function \( q \) would vanish outside active region \( \tau > R_2 \).

**SOLUTION FOR FUNCTION \( q \)**

Wheeler [11] interpreted function \( q(\rho, z) \) as ”distribution of gravitational wave amplitude”. In constructed solution it will be concentrated only on a thin toroidal active region. Inside our thin torus \( e^\nu \gg 1 \). In such a limit equation (9) simplifies to:

\[
\frac{e^{2\nu}}{4a^2} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial \nu^2} \right) q = 4V.
\]

(12)

Changing variable to \( \tau = 2ae^{-\nu} \) we get

\[
\frac{\partial^2 q}{\partial \tau^2} + \frac{2}{\tau} \frac{\partial q}{\partial \tau} + \frac{1}{\tau^2} \frac{\partial^2 q}{\partial u^2} = 4V.
\]

(13)

This equation is solved by:

\[
q(\tau, u) = \begin{cases} 
  c_1 \ln(\tau) + c_2 + h_1 \tau^2 & \text{for } \tau \leq R_1 \\
  d_1 \ln(\tau) + d_2 + h_2 \tau^2 & \text{for } \tau \in (R_1, R_2) \\
  0 & \text{for } \tau > R_2.
\end{cases}
\]

(14)

We have four conditions for the continuity of the function \( q \) and its derivative at \( R_1 \) and \( R_2 \) and additionally

**FIG. 1.** Function \( V \) in toroidal coordinates \( \nu, u \). Dotted circles are surfaces of constant \( u \) and solid ones are toruses of constant \( \nu \). Potential \( V \) vanishes outside of toroidal active region \( \tau > R_2 \). Inside active region \( V \) forms double-well potential with values \( V_1 \) and \( V_2 \).
the requirement of regularity at $\tau = 0$. This way, we get all constants of integration and an additional requirement that must be met by potential $V$:
\begin{align*}
c_1 &= 0, \quad c_2 = 2h_2R_2^2 \ln \left( \frac{R_2}{R_1} \right), \\
d_1 &= -2h_2R_2^2, \quad d_2 = h_2R_2^2(2\ln(R_2) - 1),
\end{align*}
(15)

Changing variable to $\tau$ the torus that must be met by potential all constants of integration and an additional requirement the requirement of regularity at $\tau$ second kind.

\begin{align*}
h_1 &= h_2 \left( 1 - \frac{R_2^2}{R_1^2} \right), \quad (16)
\end{align*}

\[ q(\nu, u) = 2h_2 \cdot \begin{cases} R_2^2 \ln \left( \frac{R_2}{R_1} \right) + 2 \left( 1 - \frac{R_2^2}{R_1^2} \right) a^2 e^{-2\nu} & \text{for } 2ae^{-\nu} \leq R_1 \\
R_2^2 \ln \left( \frac{R_2}{2a\sqrt{e}} \right) + \nu + 2a^2 e^{-2\nu} & \text{for } R_1 < 2ae^{-\nu} \leq R_2 \\
0 & \text{for } 2ae^{-\nu} > R_2.
\end{cases} \quad (17)
\]

\[ q = 0 \rightarrow q(v, u) = 0 \]

\[ q = \frac{1}{2} \left( \frac{\partial^2}{\partial v^2} + \frac{1}{v} \frac{\partial}{\partial v} \right) \Phi + V\Phi = 0. \quad (18)\]

Changing variable to $\tau = 2ae^{-\nu}$ we get
\[ \frac{\partial^2 \Phi}{\partial \tau^2} + \frac{1}{\tau} \frac{\partial \Phi}{\partial \tau} + \frac{1}{\tau^2} \frac{\partial^2 \Phi}{\partial \nu^2} + V\Phi = 0. \quad (19)\]

For our double-well potential (11) above equation reduces to Helmholtz equation and may be solved by:
\[ \Phi(\tau, u) = \begin{cases} \tilde{c}_1 J_0(\sqrt{h_2} \tau) + \tilde{c}_2 Y_0(\sqrt{h_2} \tau) & \text{for } \tau \leq R_1 \\
\tilde{d}_1 J_0(\sqrt{h_2} \tau) + \tilde{d}_2 Y_0(\sqrt{h_2} \tau) & \text{for } \tau \in (R_1, R_2) \\
\end{cases} \quad (20)\]

where $J_0$ and $Y_0$ are Bessel functions of the first and second kind.

\[ e^{2\nu} \frac{1}{4a^2} \left( \frac{\partial^2}{\partial v^2} + \frac{1}{v} \frac{\partial}{\partial v} \right) \Phi + V\Phi = 0. \quad (21)\]

Outside active region $V = 0$ so equation (21) reduces to Laplace equation that in toroidal coordinates is separable by standard methods (14). Taking into account the boundary conditions (15), and local polar symmetry of the active region, the solution has to take the form:
\[ \Phi(\nu, u) = 1 + \frac{m}{2a\sqrt{2}} \sqrt{\cosh \nu - \cos u} \ P_{-\frac{1}{2}}(\cosh \nu) \quad (21)\]

$P_{-\frac{1}{2}}$ is Legendre function with half-integer index that is also known as toroidal harmonic. This particular harmonic $P_{-\frac{1}{2}}$ is related to the elliptic integral of the first kind: $P_{-\frac{1}{2}}(x) = \frac{1}{2} \pi K \left( \frac{1-x^2}{1+x^2} \right)$. Same as before we have to glue function $\Phi$ smoothly on $R_1$ and $R_2$. Near the active region where $e^{\nu} \gg 1$ equation (21) simplifies to:
\[ \Phi(\nu, u) = 1 + \frac{m}{2\pi a} \ln \left( \frac{e^{\nu}}{2} \right) = 1 + \frac{m}{2\pi a} \ln \frac{a}{\tau} \quad (22)\]

The regularity of the function $\Phi$ in equation (21) at $\tau = 0$ implies that $\tilde{c}_2 = 0$. Using the above formula and (20) we may impose continuity conditions on the function $\Phi$ and its derivative at $R_1$ and $R_2$. Taking into account also condition (16), we get the following set of equations:
\[ \frac{1}{2} \left( \frac{\partial^2}{\partial v^2} + \frac{1}{v} \frac{\partial}{\partial v} \right) \Phi + V\Phi = 0. \quad (21)\]
Explicit solution (for \( \bar{c}_1, \bar{d}_1, \bar{d}_2 \) and \( m \)) of this system of equations leads to obvious but extremely lengthy formulas. After returning to the variable \( \nu \) finally \( \Phi \) takes the following form:

\[
\Phi(\nu, u) = \begin{cases} 
\bar{c}_1 J_0 \left( 2 \sqrt{h_2} \left( 1 - \frac{R_2^2}{\nu^2} \right) ae^{-\nu} \right) & \text{for } 2ae^{-\nu} \leq R_1 \\
\bar{d}_1 J_0(2\sqrt{h_2} ae^{-\nu}) + \bar{d}_2 Y_0(2\sqrt{h_2} ae^{-\nu}) & \text{for } R_1 < 2ae^{-\nu} \leq R_2 \\
1 + \frac{m}{2a\sqrt{2}} \sqrt{\cosh \nu - \cos u} P_{-\frac{1}{2}}(\cosh \nu) & \text{for } 2ae^{-\nu} > R_2
\end{cases}
\] (24)

Since \( h_1 \) and \( h_2 \) are constrained by the equation (16), they have opposite signs. Therefore in equations (24) some of the Bessel functions have a purely imaginary argument. Nevertheless all the resulting functions all real because Bessel functions of purely imaginary argument reduce to modified Bessel functions \([15]\) that are real.

**SOME PROPERTIES OF THE SOLUTION**

Lastly, let’s also transform the line element (2) to the toroidal coordinates:

\[
ds^2 = a^2 \Phi^4(\nu, u) \frac{e^{2q(\nu, u)}(du^2 + dv^2) + \sinh^2 \nu dv^2}{(\cosh \nu - \cos u)^2}.
\] (25)

Function \( q(\nu, u) \) is specified by equation (17). Conformal factor \( \Phi(\nu, u) \) is given by equation (24), where constants \( \bar{c}_1, \bar{d}_1, \bar{d}_2 \) and \( m \) are determined by (23). So the final solution depends on four parameters: \( a, R_1, R_2 \) and potential well depth \( h_2 \) which determines the strength of the field in the active thin toroidal region.

To illustrate the properties of the metric, let’s assume that: \( R_1 = 1, R_2 = 2, a = 100 \) and \( h_2 \) is the only free parameter determining the strength of the gravitational field. Relationship between ADM mass \( m \) and \( h_2 \) for these set of parameters is shown in Fig. 2.

\[
\nu'' + \nu'^3 \left( \frac{4 \Phi_u}{\Phi} - \frac{2 \sin u}{\cosh \nu - \cos u} \right) - \nu'^2 \left( \frac{4 \Phi_u}{\Phi} - \frac{2 \sinh \nu}{\cosh \nu - \cos u} \right) + \coth \nu \left( \frac{4 \Phi_u}{\Phi} - \frac{2 \sinh \nu}{\cosh \nu - \cos u} + \coth \nu \right) = 0
\] (26)

Now using the solution (24) we may solve the above equation numerically. A several examples of external apparent horizons are depicted in Fig. 3. The mass of the system increases with \( h_2 \) and the corresponding apparent horizons are further from the center of the system and become more spherical, which is in accordance with our physical intuition. For non-vacuum systems analogical study of trapped surfaces in toroidal geometries has also been conducted recently by Karkowski et al. [16].
SUMMARY

An analytical solution of Brill waves has been found. Currently, this is the only one known (nontrivial) solution of vacuum initial data in general relativity. In this construction, gravitational wave amplitude $q$ is concentrated in the thin toroidal region. Coefficients of the metric are given in terms of elementary functions and the elliptic integral. Numerical analysis of the obtained analytical solution shows the existence of apparent horizons for some higher values of gravitational wave amplitude $q$. Further studies might include the analysis of the evolution of these initial data.

ACKNOWLEDGEMENTS

The author would like to thank Anna Klecha, Wojciech Grygiel and Tadeusz Palasz for discussions and reading the manuscript.

[1] D. R. Brill, On the positive definite mass of the Bondi-Weber-Wheeler time-symmetric gravitational waves, Ann. Phys. N.Y. 7, 466 (1959).
[2] K. Eppley, Evolution of time-symmetric gravitational waves: Initial data and apparent horizons, Phys. Rev. D 16, 1609–14 (1977).
[3] A. P. Gentle, Simplicial Brill wave initial data, Class. Quantum Grav. 16, 1987–2003 (1999).
[4] J. Karkowski, P. Koc, and Z. Świerczyński, Penrose inequality for gravitational waves., Class. Quantum Grav. 11.6, 1535–1538 (1994).
[5] O. Korobkin, B. Aksoylu, M. Holst, E. Pazos, and M. Tiggia, Solving the Einstein constraint equations on multi-block triangulations using finite element methods, Classical and Quantum Gravity 26, 145007 (2009).
[6] H. P. de Oliveira and E. L. Rodrigues, Brill wave initial data: Using the Galerkin-collocation method, Phys. Rev. D 86, 064007 (2012).
[7] E. Sörr, On critical collapse of gravitational waves, Classical and Quantum Gravity 28, 025011 (2011).
[8] D. Hilditch, A. Weyhausen, and B. Bruegmann, Evolution of centered Brill waves with a pseudospectral method, Phys. Rev. D 96, 104051 (2017).
[9] D. Hilditch, A. Weyhausen, and B. Bruegmann, Pseudospectral method for gravitational wave collapse, Physical Review D 93, 063006 (2016).
[10] D. Garfinkle and G. C. Duncan, Numerical evolution of Brill waves, Phys. Rev. D 63, 044011 (2001).
[11] J. Wheeler, Geometrodynamics and the issue of the final state, in Relativity, Groups and Topology, edited by C. Witt and B. DeWitt (Gordon and Breach, New York, 1964) p. 317–522.
[12] D. E. Holz, W. A. Miller, M. Wakano, and J. A. Wheeler, Coalescence of primal gravity waves to make cosmological mass without matter, in Directions in General Relativity: Proc. 1993 Int. Symp. (Maryland); Papers in Honour of Dieter Brill, edited by B. L. Hu and T. Jacobson (Cambridge University Press, Cambridge, 1993) p. 339–359.
[13] R. Beig and N. O. Murchadha, Trapped surfaces due to concentration of gravitational radiation, Phys. Rev. Lett. 66, 2421 (1991).
[14] S. Loh, On toroidal functions, Canadian Journal of Physics 37, 619 (1959).
[15] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, ninth dozen printing, tenth gpo printing ed. (Dover, New York, 1964) pp. 374–377.
[16] J. Karkowski, P. Mach, E. Malec, N. O. Murchadha, and N. Xie, Toroidal trapped surfaces and isoperimetric inequalities, Phys. Rev. D 95, 064037 (2017).