Even-odd entanglement in boson and spin systems

R. Rossignoli, N. Canosa, J.M. Matera
Departamento de Física-IFLP, Universidad Nacional de La Plata, C.C. 67, La Plata (1900), Argentina
(Dated: January 20, 2013)

We examine the entanglement entropy of the even half of a translationally invariant finite chain or lattice in its ground state. This entropy measures the entanglement between the even and odd halves (each forming a “comb” of $n/2$ sites) and can be expected to be extensive for short range couplings away from criticality. We first consider bosonic systems with quadratic couplings, where analytic expressions for arbitrary dimensions can be provided. The bosonic treatment is then applied to finite spin chains and arrays by means of the random phase approximation. Results for first neighbor anisotropic XY couplings indicate that while at strong magnetic fields this entropy is strictly extensive, at weak fields important deviations arise, stemming from parity-breaking effects and the presence of a factorizing field (in which vicinity it becomes size-independent and identical to the entropy of a contiguous half). Exact numerical results for small spin $s$ chains are shown to be in agreement with the bosonic RPA prediction.

PACS numbers: 03.67.Mn, 03.65.Ud, 75.10.Jm

I. INTRODUCTION

The entanglement properties of many-body systems are of great interest for both quantum information theory [1] and condensed matter physics [2,3]. Their knowledge enables, on the one hand, to assess the potential of a given many-body system for quantum information processing tasks such as quantum teleportation [5] and quantum computation [1,6,7]. On the other hand, it provides a deep understanding of quantum correlations and their relation with criticality [2–4,8,9]. In non-critical systems with short range couplings, i.e., local couplings in boson or spin lattices, ground state entanglement is believed to satisfy a general area law by which the entropy of the reduced state of a given region, which measures its entanglement with the rest of the system, scales as the area of its boundary as the system size increases [4,10]. This behavior is quite different from that of standard thermodynamic entropy which scales as the volume. In one dimensional systems this statement has been quite generally and rigorously proved [2,11] and simply means that the entropy of a contiguous section saturates, i.e., approaches a size independent constant, as the size increases. Violation of this scaling is therefore an indication of criticality [8,9,12]. The exact expression of the entropy of a contiguous block in a one-dimensional XY spin $1/2$ chain in the thermodynamic limit has been obtained [13,14] and confirms the previous behavior.

The conventional area law holds for contiguous subsystems. For non-contiguous regions it actually implies that the entropy is proportional to the number of couplings broken by the partition. For instance, for comb-like regions like the subset of all even sites in a chain, the entropy should scale as the total number $n$ of sites for first neighbor or short range couplings. This was in fact verified in [11] for the harmonic cyclic chain, where the corresponding logarithmic negativity was calculated, and also verified numerically in [10] for some spin arrays and a 1-d half-filled Hubbard model, where the even entanglement entropy was computed. An exact treatment of general comb entropies for a large one-dimensional critical XX spin $1/2$ chain with first neighbor couplings was given in [17], showing that they are indeed proportional to the size $L$ plus a logarithmic correction.

The aim of this work is to analyze in detail the entanglement entropy of all even sites in finite boson and spin arrays, both in one dimension as well as in general $d$-dimensions. Such bipartition can be normally expected to be the maximally entangled bipartition at least for uniform nearest neighbor couplings, as it will there break all coupling links. We first analyze the bosonic case with general quadratic couplings, where a fully analytic treatment of this entropy is shown to be feasible and allows to derive simple general expressions in the weak coupling limit. Comparison with single site and block entropies is also made. The bosonic treatment is then applied to finite spin $s$ arrays with anisotropic ferromagnetic-type XY couplings in a uniform transverse field through the RPA approach [18]. This allows to predict in a simple way the main properties of the total even entropy in these systems. Comparison with exact numerical results indicate that the RPA prediction, while qualitatively correct, is also quite accurate outside the critical region already for low spin $s \gtrsim 2$, representing the high spin limit. Results corroborate that for strong fields, the total even entropy in these systems is extensive, i.e., directly proportional to the total number $n$ of sites. However, for low fields $B < B_c$, this entropy has an additive constant, which arises in the RPA from parity restoration [18]. Moreover, in the immediate vicinity of the factorizing field $B_s < B_c$ [19,22], extensivity is fully lost and the total even entropy reduces to this constant, which is the same as that for the block entropy and is exactly evaluated. The exact bosonic treatment is described in sec. II whereas its application to spin systems is discussed in sec. III. Conclusions are finally drawn in IV.
II. ENTANGLEMENT ENTROPY IN BOSONIC SYSTEMS

We start by considering a system of $n$ bosonic modes defined by boson creation operators $b^\dagger_i \ (b_i, b_i^\dagger = \delta_{ij})$, interacting through a general quadratic coupling. The Hamiltonian can be written as

$$H = \sum_{ij} \left( \lambda_i \delta_{ij} - \Delta_{ij}^+ \right) (b_i^\dagger b_j + \frac{1}{2} \delta_{ij}) - \frac{1}{2} \left( \Delta_{ij}^- b_i^\dagger b_j + \Delta_{ij}^+ b_i b_j \right)$$

$$= \frac{1}{2} Z^\dagger H Z = Z = \left( \begin{array}{c} b \\ b^\dagger \end{array} \right), \quad H = \left( \begin{array}{cc} \Lambda - \Delta^+ & -\Delta^- \\ -\Delta^- & \Lambda - \Delta^+ \end{array} \right),$$

where $Z = (b^\dagger, b)$, $\Lambda_{ij} = \lambda_i \delta_{ij}$ and the $2n \times 2n$ matrix $H$ is Hermitian. The system is assumed stable, such that the matrix $H$ is positive definite. We may then also write

$$H = \sum_k \omega_k (b_k^\dagger b_k + \frac{1}{2}), \quad (2)$$

where $\omega_k$ are the symplectic eigenvalues of $H$, i.e., the positive eigenvalues of the matrix $M^\dagger H M$, with $M = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$, which come in pairs of opposite sign and are all real non-zero when $H$ is positive definite \[23\], and $b_k^\dagger$ are the normal boson operators determined by the diagonalizing Bogoliubov transformation \[23\], $Z = WZ'$ satisfying $W^\dagger MW = M$ and $(W^\dagger H W)_{kk'} = \omega_k \delta_{kk'}$. The ground state is the vacuum $|0\rangle$ of the operators $b_k^\dagger$ and is nondegenerate.

Ground state entanglement properties can be evaluated through the general Gaussian state formalism \[11, 24, 25\], which we here recast in terms of the contraction matrix \[18, 23\] and the corresponding sub-matrix $W_k$ (Eq. (4)) with $i, j \in A$ and can be written as \[18\]

$$\rho_A = \text{exp}[-\frac{1}{2} Z^\dagger A \tilde{H}_A Z_A] / \text{Tr} \text{exp}[-\frac{1}{2} Z^\dagger A \tilde{H}_A Z_A], \quad (5)$$

where $\tilde{H}_A = M_A \ln[I + M_A D_A^{-1}]$. Eq. (3) represents a thermal-like state of suitable $n_A$ independent modes determined by the effective Hamiltonian $\tilde{H}_A$. The entanglement entropy of the $(A, \bar{A})$ partition, $S(\rho_A) = S(\rho_{\bar{A}})$, is then determined by the symplectic eigenvalues $f_k^\pm$ of $D_A$ (i.e., the positive eigenvalues of the matrix $D_A M_A$),

which has eigenvalues $f_k^\pm$ and $-1 - f_k^\pm$), and given by

$$S(\rho_A) = -\text{Tr} \rho_A \ln \rho_A = \sum_{k=1}^{n_A} h(f_k^\pm), \quad (6)$$

$$h(f) = -f \ln f + (1 + f) \ln(1 + f). \quad (7)$$

For instance, the entanglement of a single mode $i$ with the rest of the system is just

$$S(\rho_i) = h(f_i), \quad f_i = \sqrt{(F_{ii}^+ + \frac{1}{2})^2 - |F_{ii}^-|^2 - \frac{1}{2}}, \quad (8)$$

where $f_i$, the symplectic eigenvalue of the single mode contraction matrix $D_i$, represents the deviation from minimum uncertainty of the mode: $(F_{ii}^+ + \frac{1}{2})^2 - |F_{ii}^-|^2 = |q_i|^2 + |p_i|^2 - \left| \text{Re}((q_i p_i)') \right|^2 \geq 0$ for $q_i = \frac{b_i + b_i^\dagger}{\sqrt{2}}$, $p_i = \frac{b_i - b_i^\dagger}{\sqrt{2}}$.

A. Finite translationally invariant systems

Let us now associate each bosonic mode with a given site in a cyclic chain and consider a translationally invariant system of $n$ sites, such that $\lambda_i = \lambda$ and $\Delta_{ij} = \Delta^+(i-j)$, with $\Delta^+(l) = \Delta^+(n-l)$. We first consider for simplicity the one-dimensional case. Through a discrete Fourier transform $b_i^\dagger = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{i2\pi ki/n} b_k^\dagger$, we can diagonalize $H$ analytically and obtain an explicit expression for the contractions $F_{ij}^\pm$. We will assume $\Delta^+(l) = \Delta^+(n-l)$ $\forall l$, in which case the energies $\omega_k$ in \[23\] adopt the simple form \[18\]

$$\omega_k = \sqrt{(\lambda - \Delta_k^+)^2 - (\Delta_k^-)^2}, \quad (9)$$

where $\Delta_k^\pm$ are the Fourier transforms of the couplings:

$$\Delta_k^\pm = \sum_{l=0}^{n-1} e^{2i\pi kl/n} \Delta^\pm(l). \quad (10)$$

The contractions $F_{ij}^\pm$ depend just on the separation $l \equiv |i-j|$ and are given by

$$F_{ij}^\pm = \frac{1}{n} \sum_{k=0}^{n-1} e^{i2\pi kl/n} f_k^\pm, \quad (11)$$

$$f_k^+ = \langle b_k^\dagger b_k \rangle_{0^l} = \lambda - \Delta_k^+ - \frac{1}{2}, \quad f_k^- = \langle b_k b_{-k}^\dagger \rangle_{0^l} = \frac{\Delta_k^-}{2\omega_k} \quad (12)$$

The symplectic eigenvalues of the full contraction matrix \[41\] are of course $f_k = \sqrt{(\frac{1}{2} + f_k^+)^2 - (f_k^-)^2 - \frac{1}{2} = 0 \forall k}$. In the weak coupling limit $|\Delta_k^\pm| \ll \lambda \forall k$, $f_k^\pm$ become small and up to lowest non-zero order we obtain

$$f_k^\pm \approx \frac{\Delta_k^\pm}{2\lambda}, \quad f_k^\pm \approx \frac{(\Delta_k^\pm)^2}{4\lambda^2} \approx (f_k^\pm)^2, \quad (13)$$

which leads to

$$F_{ij}^- \approx \frac{\Delta^-(l)}{2\lambda}, \quad F_{ij}^+ \approx \frac{\sum_{l'} \Delta^-(l') \Delta^-(l-l')} {4\lambda^2}. \quad (14)$$
At this order just sites linked by $\Delta^+(l)$ or its convolution are correlated. The eigenvalues $f_k^\pm$ of subsystem contraction matrices will depend up to lowest non-zero order on $F_i^+$ and $(F_i^-)^2$, being then $O(\Delta^2/\lambda^2)$ for $\Delta^-(l) \propto \Delta_-$.

We can then use in (9) the approximation
\[ h(f) \approx -f(\ln f - 1) + O(f^2), \tag{15} \]
such that $S(\rho_A) = O(\Delta^2/\lambda^2 \ln \Delta^2/\lambda^2)$.

On the other hand, it is seen from Eq. (9) that the present system is stable provided $\lambda \geq \Delta_0^+ + |\Delta_0^-| \forall k$. For attractive couplings $\Delta^+(l) \geq 0 \forall l$, with all $\Delta^-(l)$ of the same sign, the strongest condition is obtained for $k = 0$, so that stability occurs for
\[ \lambda > \lambda_c = \Delta_0^+ + |\Delta_0^-| = \sum_l \Delta^+(l) + |\Delta^-(l)|. \tag{16} \]

For $\lambda \to \lambda_c$, $\omega_k \to 0$ (while all other $\omega_k$ remain finite in a finite system), implying a divergence of $f_0^\pm$ (Eq. (12)):
\[ |f_0^-| \approx \sqrt{\frac{|\Delta_0^-|}{8(\lambda - \lambda_c)}}, \quad f_0^+ \approx |f_0^-| - 1/2, \tag{17} \]
plus terms $O(\lambda^2/\Delta^2)$. This entails in turn a divergence $f_0^A \propto (\lambda/\lambda_c - 1)^{-1/4}$ of the largest eigenvalue of a subsystem contraction matrix $D_A$, with $S(\rho_A) \approx \ln f_0^A + 1 \approx -\frac{1}{4} \ln (\lambda/\lambda_c - 1)$ plus constant terms.

For example, the single site entropy (8) becomes
\[ S(\rho_i) = h(f), \quad f = \sqrt{\frac{1}{2} + F_0^+ - (F_0^-)^2} - \frac{1}{2}, \tag{18} \]
with $F_0^\pm = \frac{1}{n} \sum_k f_k^\pm$ (Eq. (11)). For weak coupling,
\[ f \approx F_0^+ - (F_0^-)^2 \approx \frac{\sum_{l\neq 0}(\Delta^-(l))^2}{4\lambda^2}, \tag{19} \]
which involves just the couplings $\Delta^-(l)$ connecting the site with the rest of the system. On the other hand, for $\lambda \to \lambda_c$, $f \propto \sqrt{\frac{1}{n} \ln (\lambda/\lambda_c - 1)^{-1/4}}$.

**B. Even-odd entanglement entropy**

We now evaluate the entropy of the reduced state of all even sites, $S(\rho_E) = S(\rho_0)$, which measures their entanglement with the complementary set of odd sites (Fig. 1 left). We will assume $n$ even, such that the even subsystem, defined by $(-1)^i = +1$, is again translationally invariant. The ensuing contraction matrix $D_E$ can be obtained by removing contractions between even and odd sites in the full matrix (4) and extracting then the even part. This leads to elements
\[ \tilde{F}_{ij}^\pm = \frac{1}{2} F_{ij}^\pm (1 + e^{i\pi(j-i)}), \tag{20} \]
whose Fourier transforms are, using Eq. (11),
\[ \tilde{f}_k^\pm = \frac{1}{2} (f_k^+ + f_{k+n/2}^+). \tag{21} \]

The final symplectic eigenvalues of $D_E$ then become
\[ \tilde{f}_k = \sqrt{\frac{1}{4} + (\tilde{f}_k^+)^2 - (\tilde{f}_k^-)^2} - \frac{1}{2}, \tag{22} \]
for $k = 0, \ldots, n/2 - 1$. We then obtain
\[ S(\rho_E) = \sum_{k=0}^{n/2-1} h(\tilde{f}_k) = \frac{1}{2} \sum_{k=0}^{n-1} h(\tilde{f}_k). \tag{23} \]

Whenever $\tilde{f}_k$ can be approximated by a smooth function $\tilde{f}(k)$ of $k \equiv k/n$, we may replace (23) by the integral
\[ S(\rho_E) \approx \frac{n}{2} \int_0^1 h(\tilde{f}(k))d\tilde{k}. \tag{24} \]

In these cases, we may then expect $S(\rho_E)$ extensive, i.e., proportional to the number $n/2$ of even sites. Let us remark, however, that this is not always the case: In a completely and uniformly connected system like the Lipkin model, the contraction matrix will have a single non-zero symplectic eigenvalue $f_{nA}$ for any subsystem, including the whole even set, and $S(\rho_E) = h(n/2)$ is no longer proportional to $n$. A similar lack of extensivity holds in a finite system in the vicinity of the instability ($\lambda \to \lambda_c$, see below).

For weak coupling, Eqs. (13), (14) and (21) lead to
\[ \tilde{f}_k \approx \frac{\Delta_k - \Delta_{k+n/2}}{16\lambda^2} = \frac{\sum_{l\neq 0} e^{i2\pi kl/n} \Delta^-(l)^2}{4\lambda^2}, \tag{25} \]
which involves again just the couplings $\Delta^-(l)$ connecting the even and odd subsystems. On the other hand, for $\lambda \to \lambda_c$ (Eq. (10)), $\tilde{f}_0 \approx \frac{1}{2} \sqrt{\frac{1}{4} + 2f_{n/2}^+ - 2f_{n/2}}$ diverges as $(\lambda/\lambda_c - 1)^{-1/4}$ whereas all other $\tilde{f}_k$ remain finite, and extensivity is lost.

**C. First neighbor coupling**

Let us now examine in detail the first neighbor case $\Delta^\pm(l) = \frac{1}{2} \Delta^\pm(\delta_{01} + \delta_{l,-1})$, where Eq. (10) becomes
\[ \Delta_k^\pm = \Delta^\pm \cos(2\pi k/n). \tag{26} \]
The exact $S(\rho_E)$ can be obtained from Eqs. (21)–(23). In the weak coupling limit, Eqs. (19) and (25) lead to
\[
f \approx \frac{(\Delta^-)^2}{8\lambda^2},
\]
\[
\tilde{f}_k \approx \frac{(\Delta^-)^2}{4\lambda^2} \approx 2f \cos^2(2\pi k/n).
\]
Using Eqs. (15)–(21), the single site and the total even entropies can then be expressed just in terms of $f$:
\[
S(\rho_i) \approx -f(\ln f - 1),
\]
\[
S(\rho_E) \approx -nf \int_0^1 \cos^2(2\pi k)(\ln[2f \cos^2(2\pi k)] - 1) \, dk
\approx -\frac{n}{2}f(\ln f - 2).
\]
Hence, in this limit $S(\rho_E)$ is extensive, becoming $n/2$ times the single site entropy (29) minus a $O(nf)$ correction accounting for the interaction between even sites:
\[
S(\rho_E) \approx \frac{n}{2}S(\rho_i) - \frac{n}{2}f(1 - \ln 2).
\]
The last term represents the even mutual entropy $\frac{n}{2}(S(\rho_i) - S(\rho_E))$, which is always a positive quantity and becomes here also extensive in this limit.

In contrast, the block entropy $S(\rho_L)$, where $\rho_L$ denotes a contiguous block of $L < n$ spins, rapidly saturates as $L$ increases [11]. In the weak coupling limit, it is verified that the ensuing contraction matrix $D_L$ possesses, up to lowest non-zero order, just two positive non-zero symplectic eigenvalues $f_k^\pm \approx \frac{1}{2}f$ for any $L \geq 2$, such that
\[
S(\rho_L) \approx -f(\ln f/2 - 1) \approx S(\rho_i) + f\ln 2,
\]
for $2 \leq L \leq n - 2$, i.e., it saturates already for $L = 2$. Hence, in this limit,
\[
S(\rho_E) \approx \frac{n}{2}S(\rho_L) - \frac{n}{2}f.
\]

Assuming $\Delta^+ > 0$ (if $\Delta^- < 0$ we can change its sign by a local change $b_i \rightarrow -b_i$ at odd sites) the present system is stable for $\lambda > \lambda_c = \Delta_+ + |\Delta_-|$ (Eq. (16)). For $\lambda \rightarrow \lambda_c$, $\omega_0 \rightarrow 0$ and all previous entropies diverge. In particular, Eq. (22) leads to
\[
\tilde{f}_0 \approx \frac{1}{2}j\sqrt{\frac{|\Delta^-|\lambda_c}{2|\Delta^+|\lambda_c + |\Delta^-|} - 1},
\]
being then verified that $S(\rho_E) \approx -\frac{1}{2}\ln(\lambda/\lambda_c - 1)$ plus a constant term up to leading order. Hence, in this limit $S(\rho_E)/S(\rho_i) \rightarrow 1$.

As illustration, the left panels in Fig. 2 depict the single site, block and even-odd entanglement entropies for a ring of $n = 36$ sites with $\Delta^- = \Delta^+/3$, where $\lambda_c = 4\Delta^+/3$.

D. Even-Odd entropy in d-dimensions

The whole previous treatment can be directly extended to a translationally invariant cyclic array in $d$ dimensions. We should just replace $l, k, n$ by vectors $l = (l_1, \ldots, l_d)$, $k = (k_1, \ldots, k_d)$ and $n = (n_1, \ldots, n_d)$, with $l_i, k_i = 0, \ldots, n_i - 1$. We will assume couplings satisfying $\Delta_{i,j}^x = \Delta^x(i - j)$, with $\Delta^x(-l) = \Delta^x(n - l) = \Delta^x(l)$.

The same previous expressions (9)–(12) then hold, with
\[
\Delta_k^\pm = \sum_l e^{i2\pi k \cdot l} \Delta^x(l),
\]
\[
F_k^\pm = \frac{1}{n} \sum_k e^{-i2\pi k \cdot l} \tilde{f}_k^\pm,
\]
where $\tilde{k} = (k_1/n_1, \ldots, k_d/n_d)$ and $n = \prod_{i=1}^d n_i$ is the total number of sites. Eqs. (13)–(14) remain unchanged with $i, k, l \rightarrow i, k, l$.

The subsystem of all even sites, like that formed by the blue sites in Fig. 1 right, is defined by $(-1)^{i_1 + \cdots + i_d} = +1$.

Its contraction matrix will then be the even block of
\[
\tilde{F}_{ij}^\pm = \frac{1}{2}F_{ij}^\pm (1 + e^{i\pi(i - j) \cdot 1})
\]
where $1 = (1, 1, \ldots)$. Assuming $n_i$ even $\forall i$, its Fourier transform is then given again by
\[
\tilde{f}_k^\pm = \frac{1}{2}[f_k^+ + f_{k+n/2}^-],
\]
where $k_i + n_i/2 \rightarrow k_i - n_i/2$ if $k_i \geq n_i/2$. The symplectic eigenvalues of $D_E$ are then given again by Eq. (22) with $k \rightarrow k$, and the even-odd entropy reads
\[
S(\rho_E) = \frac{1}{2} \sum_k h(\tilde{f}_k) \approx \frac{n}{2} \int h(\tilde{f}(\tilde{k})) d^d \tilde{k},
\]
where $k_i = 0, \ldots, n_i - 1$ in the sum and the integral is restricted to the unit cube $0 \leq \tilde{k}_i \leq 1$ and valid if $\tilde{f}_k$ is a smooth function $\tilde{f}(\tilde{k})$ of $\tilde{k}$.

In the case of first neighbor couplings
\[
\Delta^x(l) = \frac{1}{2} \sum_{i=1}^d \Delta^x_i (\delta_{l_i \epsilon_i} + \delta_{l_i \epsilon_i}),
\]
where $\epsilon_i = (0, \ldots, 1, \ldots, 0)$, Eq. (34) leads to
\[
\Delta_k^\pm = \sum_{i=1}^d \Delta_k^x \cos(2\pi k_i/n_i).
\]
with $\Delta_{k+n/2}^\pm = -\Delta_k^\pm$. In the weak coupling limit we then obtain
\[
f \approx \frac{|\Delta^-|^2}{8\lambda^2}, \quad |\Delta^-|^2 = \sum_{i=1}^d (\Delta_i^-)^2,
\]
\[
\tilde{f}_k \approx u(\tilde{k}) f, \quad u(\tilde{k}) = 2\sum_i \frac{\Delta_i^-}{|\Delta^-|} \cos \frac{2\pi k_i}{n_i}.
\]
Hence, the single site entropy is again $S(\rho_i) \approx -f(\ln f - 1)$ while Eq. (38) yields

$$S(\rho_E) \approx -\frac{n}{2} f(\ln f - 1 + \alpha)$$

$$S(\rho_L) \approx \frac{n}{2} S(\rho_i) - \frac{n}{2} f \alpha,$$

where $\alpha$ is a geometric entropy factor:

$$\alpha = \int u(\tilde{k}) \ln u(\tilde{k}) d^d \tilde{k},$$

$$(u(\tilde{k}) \geq 0, \int u(\tilde{k}) d^d \tilde{k} = 1).$$

In the isotropic case $\Delta_i = \Delta^+ \forall i$, we have $\alpha = \alpha_d$, with $\alpha_1 = 1 - \ln 2 \approx 0.307$ (Eq. 31), $\alpha_2 = 2\alpha_1 = 0.614$ and $\alpha_3 = 0.636$, approaching $\approx \ln 2$ for large $d$.

At fixed $\lambda$, and for $\Delta^\pm = \Delta^\pm$, $f = (\Delta^-)^2 d/(8\lambda^2)$ and hence both $S(\rho_i)$ and $S(\rho_E)$ increase as $d$ increases, reflecting the larger number of links. However, and assuming again $\Delta^+ \geq 0$, $\lambda_c = d(\Delta^+ + |\Delta^-|)$ also increases, entailing that at fixed $\lambda/\lambda_c$, $f$ (and so $S(\rho_i)$ and $S(\rho_E)$) decreases:

$$f \approx \frac{[\Delta^-/(\Delta^+ + |\Delta^-|)]^2}{8d(\lambda/\lambda_c)^2}.$$  \hspace{1cm} (45)

For example, the right panels in Fig. 2 depict $S(\rho_E)$ and $S(\rho_L)$ in an isotropic square lattice of $6 \times 6$ sites, with the same previous ratio $\Delta^-/\Delta^+ = 1/3$. At fixed $\lambda/\lambda_c$, their values are verified to be roughly half that of the similar one-dimensional case (Eq. (45)). Their ratio is also slightly smaller due to the increase in the parameter $\alpha$ in Eq. 32. On the other hand, for $\lambda \rightarrow \lambda_c$ there is again a single vanishing energy $\omega_0$, so that all entropies behave as $-\frac{1}{4}\ln(\lambda/\lambda_c - 1)$ up to leading order, with all ratios approaching 1.

We also depict there the entropy $S(\rho_L)$ of a contiguous half-size block $(n_x \times n_y/2 = 6 \times 3$ sites), which is now proportional to its boundary $2n_x$. For $\lambda \gg \lambda_c$, it is verified that the number of non-zero positive eigenvalues of the corresponding contraction matrix $D_L$ is just the number of couplings “broken” by the partition $(2n_x)$, being all approximately equal to $f/4$ up to leading non-zero
order. We then obtain
\[ S(\rho_L) \approx -\frac{n_z}{2} f(\ln f/4 - 1) \approx \frac{n_z}{2} [S(\rho_i) + 2f\ln 2], \tag{46} \]
whence \( S(\rho_L)/S(\rho_i) \propto n_z/2 \) in this limit, as verified in the right panels of Fig. 2.

III. APPLICATION TO SPIN SYSTEMS

The previous bosonic formalism can be directly applied to interacting spin \( s \) systems in an external magnetic field through the RPA approximation \[18\]. Denoting with \( s_{iz}/h \) at site \( i \), we will consider a cyclic translationally invariant finite array which can be described by an XY Hamiltonian of the form
\[
H = B \sum_i s_{iz} - \frac{1}{2s} \sum_{i \neq j} \left( J_{ij}^x s_{iz} s_{jx} + J_{ij}^y s_{iz} s_{jy} \right) \tag{47a}
\]
\[= B \sum_i s_{iz} - \frac{1}{2s} \sum_{i \neq j} \Delta^z s_{iz} s_{jz} + \frac{1}{2} \Delta^x (s_{iz} s_{jz} + \text{H.c.}) \tag{47b} \]
where \( s_{iz} = s_{jz} \pm is_{iz} \), \( J_{ij}^z = J_{ij}(i-j) \) and
\[
\Delta^\pm = \frac{1}{2}(J_{ij}^x \pm J_{ij}^y) = \Delta^\pm (i-j). \tag{48} \]
We note that \( x, y, z \) may in principle also denote local intrinsic axes at each site, in which case the field is assumed to be directed along the local \( z \) axis. The \( s^{-1} \) scaling of the couplings ensures a spin-independent mean field and effective RPA boson Hamiltonian (see below).

Normal RPA. For sufficiently strong field \( B \), the lowest mean field state (i.e., the separable state with lowest energy) is the aligned state \( |0\rangle = |0_1 \rangle \otimes \cdots \otimes |0_n \rangle \), where \( |0_i \rangle \) denotes the local state with maximum spin along the \( -z \) axis \( (s_{iz}|0_i\rangle = -s|0_i\rangle) \). In such a case, RPA implies the approximate bosonization \[18\]
\[
s_{iz} \rightarrow \sqrt{2s} b_i, \quad s_{iz} \rightarrow \sqrt{2s} b_i, \quad s_{iz} \rightarrow b_i^\dagger b_i - \frac{1}{2}, \tag{49} \]
which is similar to the Holstein-Primakoff bosonization \[23, 26\] and leads to the quadratic boson Hamiltonian \[11\] with the parameters \[18\] and \( \lambda = B \). We may then directly apply all previous expressions.

The bosonic RPA scheme becomes exact for strong fields \( |B| \gg B_c \) for any size \( n \), spin \( s \), geometry or interaction range, since for weak coupling it corresponds to the exact first order perturbative expansion of the ground state wave function \[18\]. As a check, in the case of the spin 1/2 one-dimensional chain with first neighbor XY coupling, an analytic expression of the block entropy in the limit \( n \rightarrow \infty \) has been obtained in \[13\]. For \( \lambda = B > \Delta^+ \), it is given in present notation by \[13\]
\[
S(\rho_A) \approx \frac{1}{2} \left[ \ln \frac{1}{\pi \alpha} + (\alpha^2 - \alpha'^2) \frac{2I(\alpha)I(\alpha')}{\alpha^2} \right], \tag{50} \]
where \( \alpha = \Delta / \sqrt{\lambda^2 + \Delta^2 - \Delta^+} \), \( \alpha' = \sqrt{1 - \alpha^2} \) and \( I(\alpha) = \int_0^1 dx / \sqrt{(1-x^2)(1-\alpha^2x^2)} \) is the elliptic integral of the first kind. An expansion of \(50\) for \( \lambda \gg \Delta^\pm \) leads exactly to present Eq. \[13\], with \( f \) given by \[27\]. We can then expect the asymptotic expressions \(40\) and \(43\) for \( S(\rho_E) \) to be exact in this limit also in spin systems.

Parity breaking RPA. Considering now the anisotropic ferromagnetic-type case \( |J_y(l)| \leq J_x(l) \forall l \) in \( \{17\} \), the previous normal RPA scheme will hold, according to Eq. \[16\], for \( B \geq B_c = J_x^0 \sum J_x(l) \), i.e., when the corresponding boson system is stable.

For \( |B| < B_c \), the normal RPA becomes unstable \( (\omega_0 \text{ becomes imaginary}) \). The lowest mean field state corresponds here to degenerate states \( \{\pm \Theta\} \) fully aligned along an axis \( z' \) forming an angle \( \pm \theta \) with the \( z \) axis in the \( x, z \) plane: \( \{\Theta\} = \{\theta_1\} \otimes \cdots \otimes \{\theta_n\} \), with \( \{\theta_i\} = \exp[-i\theta s_{ix}(0_i)] \). We are assuming here an anisotropic XY coupling such that \( H \) commutes with the \( S_z \) parity \( P_z = e^{i\pi (\sum, s_{iz}+ns)} \), but not with an arbitrary rotation around the \( z \) axis (as in the \( XX \) case). Such states break then parity symmetry, satisfying \( P_z(\Theta) = -\Theta \). The angle \( \theta \) is to be determined from \[18\]
\[
\cos \theta = B/B_c, \quad B_c = \sum J_x(l) \tag{51} \]
For \( |B| < B_c \), the bosonization \[23\] is then to be applied in the RPA to the rotated spin operators \( s_{iz} = s_{iz} \cos \theta + s_{ix} \sin \theta \), \( s_{iz'} = s_{iz} \cos \theta - s_{ix} \sin \theta \) and \( s_{iy'} = s_{iy} \). This leads again to a stable Hamiltonian of the form \[11\] with \[18\]
\[
\lambda = B_c, \quad \Delta^\pm(l) = \frac{1}{2} [J^x(l) \cos^2 \theta \pm J^y(l) \tag{52} \]
For \( |B| < B_c \), we should also take into account the important effects from parity restoration for a proper RPA estimation of entanglement entropies \[18\]. The exact ground state in a finite array will have a definite parity \( P_z \) outside crossing points \[21\], implying that the actual RPA ground state should be taken as a definite parity superposition of the RPA spin states constructed around \( \{\pm \Theta\} \) \[18\]. This leads to reduced RPA spin densities of the form \( \rho_A \approx \frac{1}{2} \rho_A(\theta) + \rho_A(-\theta) \) if the complementary overlap \( O_A = (-\Theta_A | \Theta_A) \) can be neglected. If the subsystem overlap \( O_A = (-\Theta_A | \Theta_A) = \cos^2 nA \) is also negligible, such that \( \rho_A(\theta) \rho_A(-\theta) \approx 0 \), then \[18\]
\[
S(\rho_A) \approx S(\rho_A(\theta)) + \delta, \tag{53} \]
where \( \delta = \ln 2 \). The final effect is then the addition of a constant shift to the bosonic subsystem entropy for \( |B| < B_c \). This is applicable to both \( S(\rho_E) \) and \( S(\rho_L) \) if \( \theta \), \( n \) and the block size \( L \) are not too small.

For first neighbor couplings with anisotropy \( \chi = J_y/J_x \in (0, 1) \) (if \( \chi > 1 \) we just redefine the \( x, y \) axes) as well as for arbitrary range couplings with a common anisotropy \( \chi = J_y(l)/J_x(l) \in (0, 1) \), another fundamental feature for \( |B| < B_c \) is the existence of a transverse factorizing field \( B_s = B_c \sqrt{\chi} \) where the mean field states \( \{\pm \Theta\} \) become exact ground states \[19\]. As seen from \[26\], at this field \( \Delta^\pm(l) = 0 \forall l \), so that the RPA vacuum remains the same as the mean field vacuum \[18\] and
all contractions $F^\pm_{ij}$ vanish, implying $S(\rho_A(\theta)) = 0$. All RPA entropies at $B_s$ reduce then to the correction term $\delta$ arising from parity restoration [13].

This is essentially also the exact result at $B_s$: The transverse factorizing field corresponds to the last ground state parity transition as $B$ increases from 0 [21] and the ground state side-limits for $B \rightarrow B_s$ are actually the definite parity combinations of the mean field states $| \pm \Theta \rangle$ [21, 28]. These definite parity states have Schmidt number 2 for any bipartition, implying that the side-limits of the exact entropy of the reduced state of any subsystem at $B_s$ do not approach 0 but rather the values [28]

$$S(\rho^A_s(B_s)) = \sum_{\nu=\pm} q^\nu \ln q^\nu + \frac{(1 + \nu O_A)(1 + \nu O_A)}{2(1 + \nu O_A O_A)}, \tag{54}$$

where $+ (-)$ corresponds to positive (negative) parity, i.e. the right (left) side limit at $B_s$. Eq. [57] is valid for any size or spin. For small complementary overlap $\bar{O}_A$, $q^\nu \approx \frac{1}{2}(1 + \nu O_A)$ and both side limits coincide, while if $O_A$ is also small, $q^\nu \approx 1/2$ and Eq. [54] reduces to $\ln 2$. This is also in agreement with the exact limit of the block entropy of the large one dimensional $s = 1/2$ XY chain at $B_s$ [13].

Illustrative exact results for the even-odd entanglement entropy in a finite linear cyclic spin $s$ chain with first neighbor couplings are plotted in the top left panel of Fig. [3] for spins $s = 1/2$, 1 and 2, together with the bosonic RPA estimation. The exact definite parity ground state was employed in all cases. We also depict for comparison the entropy of a contiguous half (top right), and the ratio $S(\rho_E)/S(\rho_L)$ (bottom left). The anisotropy of the coupling is the same as in Fig. [2] ($\Delta / \Delta_+ = 1/3$). The RPA result (independent of $s$ for the scaling used in [17]) represents the large spin limit but is already quite close to the exact results for $s = 2$ except in the vicinity of $B_c$, where the exact entropies remain of course finite in a finite chain. The ratio $S(\rho_E)/S(\rho_L)$ is nonetheless quite accurately reproduced and shows the extensive character of $S(\rho_E)$ for $B > B_c$, in agreement with [33], where the entropies for all spin values rapidly approach the RPA result and become spin independent. For $|B| < B_c$ the shift $\delta$ in [33] ($\delta = +1$ in Figs [9, 14] since base 2 logarithm was employed) is essential for the agreement and explains the lack of direct extensivity in this region. The collapse of all entropies to the value $\delta$ at $B_s$ is also verified, and
FIG. 4. (Color online) Top: Exact entanglement entropy of all $n/2$ even sites in the ground state of a spin $1/2$ cyclic chain for different values of $n$. Couplings are the same as in Fig. 3. Bottom: The intensive entropy $S(\rho_E)/n_E$ ($n_E = n/2$). All curves coalesce for $B > B_c$. The inset depicts the intensive shifted entropy $(S(\rho_E) - \delta)/n_E$, where $\delta = 0$ for $B > B_s$ and $\delta = 1$ for $B < B_s$, which makes curves for $n \geq 8$ coalesce also for $B < B_s$. for $s = 1/2$ even the small discontinuity at $B_s$ predicted by Eq. (54) can be appreciated (together with the other parity transitions for $B < B_s$). The bottom right panel depicts $S(\rho_E)$ in a $4 \times 2$ square lattice with identical couplings in both directions and the same ratio $\Delta_-/\Delta_+$, where a similar behavior is obtained. Exact results for $s = 2$ are now even closer to the RPA prediction, indicating that the accuracy of the latter tends to improve, for stable mean fields, as the connectivity increases [29].

Exact results for a spin $1/2$ chain for different sizes are depicted in Fig. 4. Even though RPA is not accurate for such low spin with a first neighbor coupling [29], the exact results are again in qualitative agreement with its predictions away from the critical region: Direct extensivity $S(\rho_E) \propto n$ is verified for strong fields $B \gtrsim \Delta_+$ (bottom panel), whereas for $B < B_s$ it holds for the shifted entanglement entropy $S(\rho_E) - \delta$, as seen in the inset. Complete lack of extensivity takes place at the factorizing field $B_s$, where the discontinuity implied by (54) is appreciable for $n = 8$ and becomes quite noticeable for $n = 4$.

IV. CONCLUSIONS

We have shown that the total even-odd entanglement entropy displays a strict extensive behavior in both bosonic and spin chains or lattices for weak first neighbor couplings (i.e., strong fields in a spin chain), providing explicit asymptotic expressions for the general $d$ dimensional case. Extensivity of the associated mutual information is also implied by these expressions. Deviations from this behavior, however, were shown to arise for stronger couplings, i.e., proximity to the instability in the finite bosonic case or low fields in the spin case. In the latter, a constant shift is essential to understand the exact results for $|B| < B_c$, which has an evident meaning as a symmetry restoration effect in the RPA. Besides, full loss of extensivity occurs in the vicinity of the factorizing field. Present results confirm the validity of the RPA approach (with inclusion of symmetry-restoration effects) for obtaining a simple direct understanding of the main aspects of ground state entanglement in spin chains, at least in those regions where a well defined mean field minimum exists.

The authors acknowledge support from CIC (RR) and CONICET (NC,JMM) of Argentina.
[14] A.R. Its, B-Q. Jin, V. Korepin, J.Phys. A 38 2975 (2005).
[15] I. Peschel, J. Stat. Mech. P12005 (2004).
[16] Y. Chen, Z.D. Wang and F.C. Zhang, Phys. Rev. B 73, 224414 (2006).
[17] J.P. Keating, F. Mezzadri, M. Novaes, Phys. Rev. A 74 012311 (2006).
[18] J.M. Matera, R. Rossignoli, N. Canosa, Phys. Rev. A 82, 052332 (2010).
[19] J. Kurmann, H. Thomas, G. Müller, Physica A 112, 235 (1982).
[20] L. Amico et al, Phys. Rev. A 74, 022322 (2006); F. Baron et al, J. Phys. A 40 09845 (2007).
[21] R. Rossignoli, N. Canosa, J.M. Matera, Phys. Rev. A 77, 052322 (2008).
[22] S.M. Giampaolo, G. Adesso, F. Illuminati, Phys. Rev. Lett. 100, 197201 (2008); Phys. Rev. B 79, 224434 (2009).
[23] Peter Ring and Peter Schuck, The Nuclear Many-Body Problem (Springer-Verlag, NY, 1980).

[24] M. Cramer, J. Eisert, M.B. Plenio, J. Dreißig, Phys. Rev. A 73 012309 (2006).
[25] G. Adesso, A. Serafini, F. Illuminati, Phys. Rev. A 70, 022318 (2004); A. Serafini, G. Adesso, F. Illuminati, Phys. Rev. A 71, 032349 (2005); G. Adesso, F. Illuminati, Phys. Rev. A 78, 042310 (2008).
[26] T. Barthel, S. Dusuel, and J. Vidal, Phys. Rev. Lett. 97, 220402 (2006); S. Dusuel and J. Vidal, Phys. Rev. B 71, 224420 (2005); J Vidal, S. Dusuel, and T. Barthel, J. Stat. Mech. P01015 (2007).
[27] H. Wichterich, J. Vidal, and S. Bose Phys. Rev. A 81, 032311 (2010).
[28] R. Rossignoli, N. Canosa, and J.M. Matera, Phys. Rev. A 80, 062325 (2009).
[29] J.M. Matera, R. Rossignoli, N. Canosa, Phys. Rev. A 78, 042319 (2008).