On the stability of flat-band modes in a rhombic nonlinear optical waveguide array

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Abstract

A quasi-one-dimensional rhombic array of waveguides is considered. In the nonlinear case the system of equations describing coupled waves in the waveguides has the particular solutions that represent the superposition of flat-band modes. The stability of these solutions is considered. It was found that in one case the flat-band solution is unstable until the intensity threshold is attained. In another case the stability of the flat-band solution breaks down when the intensity per waveguide exceeds the threshold value.

Keywords: nonlinear waveguides, couplers, waveguide array, instability

1. Introduction

Optical simulations of different phenomena of condensed matter physics [1–4], quantum physics [5, 6], and cosmology [7–10] are objects of recent investigation. In 2D electron systems and magnetic systems [11–14] it was found that the presence of a third atom in the unit cell of the lattice leads to the emergence of a flat band between conventional energy zones. The number of lattices characterized by a flat band was investigated in [15], where the connection between the existence of flat-band states and the appearance of Fano resonances for wave propagation was related. Similar optical lattices can be realized by means of waveguides as nodes of the lattice [16–18]. Some kinds of optical lattices having a photonic spectrum with a flat band have been discussed in [19–21]. If the electric field in the optical lattice is made up of flat-band modes, then this field demonstrates diffractionless propagation along the waveguide array. Parity–time symmetry in a flat-band system was also considered in [22, 23].

Recently a quasi-one-dimensional array of waveguides in three linear chains was considered [19]. The central chain, designated as an A-type chain, is placed between two chains of waveguides, which are designated as B- and C-type chains. These chains of waveguides are shifted relative to the A-chain at one half of the period of the lattice. The interaction between the waveguides is due to tunnel coupling. Furthermore, coupling takes place only between A and B waveguides and between A and C waveguides. This waveguide system is akin to a double-zigzag array or to an array of rhombuses (see figure 1). This waveguide array, named a quasi-one-dimensional rhombic array, has been studied in linear optical waveguides [15, 19, 24, 25].

The purpose of this paper is to study the stability of the electromagnetic field distribution in a quasi-one-dimensional rhombic array of nonlinear waveguides. Nonlinearity is described by the susceptibility of the third order. In section 3 two nonlinear analogies of the superposition of flat-band modes are found. Discrete diffraction for these electric field distributions over the waveguides is absent. The stability of these solutions is investigated by the use of linear stability analysis in section 4. Taking into account that the field distribution over the waveguide is the superposition of the band mode, we can make inferences about the stability band mode from analysis of the stability properties of the field distribution. It is worth noting that one of the electromagnetic field distributions is unstable at weak intensity per waveguide but stable when the intensity exceeds some threshold value. Conversely, the second electric field distribution is stable when the intensity is lower than the threshold value.

2. Model and basic equations

It is assumed that all waveguides are manufactured from a dielectric characterized by Kerr nonlinearity. The system of...
The expression in brackets can be interpreted as the discrete continuity equation. Let us consider the following two constraints:

(a) \( A_n = 0, \ B_n = -C_n \),

(b) \( A_n = (-1)^n A, \ B_n = (-1)^n B, \ C_n = (-1)^n C \).

For all these constraints equation (4) holds. In the case of (a), the current density \( f_n \) is equal to zero at any \( B_n (C_n = -B_n) \).

\[ A_n = c_1 (B_n + B_{n-1}) + c_2 (C_n + C_{n-1}) + \mu_1 |A_n|^2 A_n, \]
\[ B_n = c_1 (A_{n+1} + A_n) + \mu_2 |B_n|^2 B_n, \]
\[ C_n = c_1 (A_{n+1} + A_n) + \mu_3 |C_n|^2 C_n. \]

Here \( A_n, B_n \) and \( C_n \) are the dimensionless slowly varying amplitudes of the electric fields propagating in the waveguides of the RNOWA. The sub-indices are the number of unit cells (see figure 1). It is assumed that the phase matching condition is satisfied. The coefficients \( c_1, c_2, c_3 \) specify the coupling between waveguides from different chains. The parameters \( \mu_1, \mu_2 \) and \( \mu_3 \) represent the self-interaction effect in the waveguides. If \( c_1 = c_2 = 1 \) and \( \mu_3 = \mu_2 = 0 \), system-of-equations (1) is reduced to the system of linear equations considered in [24-26]. The symmetric rhombic array where \( c_1 = c_2 = 1 \) and \( \mu_3 = \mu_2 \) will be considered. This contraction allows us to reduce system-of-equations (1) to the following equations:

\[ \frac{\partial A_n}{\partial \zeta} = (B_n + B_{n-1}) + (C_n + C_{n-1}) + \mu_1 |A_n|^2 A_n, \]
\[ \frac{\partial B_n}{\partial \zeta} = (A_{n+1} + A_n) + \mu_2 |B_n|^2 B_n, \]
\[ \frac{\partial C_n}{\partial \zeta} = (A_{n+1} + A_n) + \mu_3 |C_n|^2 C_n. \]

If the intensity per unit cell is defined by the expression \( W_n = |A_n|^2 + |B_n|^2 + |C_n|^2 \), the equation on the right-hand side of (2) can be obtained from system-of-equations (2). The local value \( f_n \) is introduced, where \( D_n = B_n + C_n \). So, the equation for \( W_n \) can be rewritten now as

\[ \frac{\partial W_n}{\partial \zeta} + (f_{n+1} - f_n) = 0. \]

The expression in brackets can be interpreted as the discrete divergence of the current density \( f_n \) in 1D space. Equation (4) is the discrete continuity equation.

In the linear case the distribution of (a) corresponds to the superposition of the flat-band modes of [19, 24, 25]. In the RNOWA the electromagnetic field distributions under constraints (a) and (b) can be considered as a nonlinear version of the superposition of the flat-band modes.

3. The flat-band solutions

Given constraint (a), \( A_n = 0, \ B_n = -C_n \), system-of-equations (2) can be represented as

\[ \frac{\partial}{\partial \zeta} A_n = 0, \ \frac{\partial}{\partial \zeta} B_n = \mu_2 |B_n|^2 B_n, \]
\[ \frac{\partial}{\partial \zeta} C_n = \mu_2 |C_n|^2 C_n. \]

Defining the real variables \( a_n, b_n, c_n, \varphi_n, \varphi_b, \varphi_c \) from the formulas \( A_n = a_n \exp(i\varphi_n), \ B_n = b_n \exp(i\varphi_b), \ C_n = c_n \exp(i\varphi_c) \) and deriving the following real equations:

\[ \frac{\partial a_n}{\partial \zeta} = 0, \ \frac{\partial b_n}{\partial \zeta} = 0, \ \frac{\partial c_n}{\partial \zeta} = 0, \]
\[ \frac{\partial \varphi_n}{\partial \zeta} = -\mu_1 a_n^2, \ \frac{\partial \varphi_b}{\partial \zeta} = -\mu_2 b_n^2, \ \frac{\partial \varphi_c}{\partial \zeta} = -\mu_3 c_n^2. \]

As the case \( A_n = 0 \) is considered, the phase \( \varphi_n = \varphi_b = \varphi_c \) is indeterminate. Amplitudes \( b_n \) and \( c_n \) are constant, with \( b_{n0} \) and \( c_{n0} = -b_{n0} \).

Given this result, the solutions of these equations can be written as

\[ \varphi_n = \varphi_{n0} = -\varphi_{n0}^2 \zeta. \]

The initial phases are constants of integration and can be chosen to be zero. Thus the solution of system-of-equations (5) reads

\[ \tilde{A}_n = 0, \ \tilde{B}_n = b_{n0} e^{-i\varphi_{n0}^2 \zeta}, \ \tilde{C}_n = -b_{n0} e^{-i\varphi_{n0}^2 \zeta}. \]
The electric field distribution (6) characterized by diffractionless propagation along the waveguides will be referred to as the flat-band solution of system-of-equations (2). In what follows, the homogeneous electric field distribution,

\[ A_n = 0, \quad B_n = b_0 e^{-i\omega k_n z}, \quad C_n = -b_0 e^{-i\omega k_n z}, \]

where for all \( n \), will be discussed.

Now we consider the solutions of system-of-equations (2), taking account of constraint (b): \( A_n = (-1)^nA, \quad B_n = (-1)^nB, \quad C_n = (-1)^nC \). In this case, system-of-equations (2) reduces to

\[
\begin{align*}
\frac{i}{\partial \zeta}A &= \mu_1|A|^2 A, \quad \frac{i}{\partial \zeta}B = \mu_2|B|^2 B, \\
\frac{i}{\partial \zeta}C &= \mu_3|C|^2 C.
\end{align*}
\]

The solutions of these equations can be found above. The electric field distribution reads

\[ A_n = (-1)^n a_0 e^{-i\omega k_n z}, \quad B_n = (-1)^n b_0 e^{-i\omega k_n z}, \quad C_n = (-1)^n c_0 e^{-i\omega k_n z}. \]

The current density \( J_n \) in this case is

\[ J_n = 2\omega [b_0 \sin(\mu_1 a_0^2 - \mu_2 b_0^2)\zeta + c_0 \sin(\mu_1 a_0^2 - \mu_2 c_0^2)\zeta]. \]

Diffraction in the waveguide array will be absent if \( a_0 = 0 \). However, if \( a_0 \neq 0 \), then the conditions \( \mu_1 a_0^2 = \mu_2 b_0^2 \) and \( \mu_1 a_0^2 = \mu_2 c_0^2 \) will be fulfilled simultaneously. They can be obtained by taking the condition \( c_0^2 = b_0^2 \). Thus the diffractionless propagation of electromagnetic waves along \( \zeta \) can be realized if the electric field distribution is

\[ A_n = (-1)^n a_0 e^{-i\omega k_n z}, \quad B_n = (-1)^n b_0 e^{-i\omega k_n z}, \quad C_n = (-1)^n c_0 e^{-i\omega k_n z}, \]

where \( \varphi_1 = \mu_1 a_0^2 \) and \( \varphi_0 = b_0^2 \).

### 4. Stability of the flat-band solutions

#### 4.1. Solution corresponding to constraint (a)

Let us consider the homogeneous distribution of the electric field amplitudes over the waveguide array, \( b_{00} = b_0 \). The stability of solution (7) can be analyzed by introducing small perturbations into the electric field amplitudes:

\[ A_n = p_n e^{-i\omega k_n z}, \quad B_n = \tilde{B}_n + b_n = (b_0 + q_n) e^{-i\omega k_n z}, \quad C_n = \tilde{C}_n + c_n = (-b_0 + r_n) e^{-i\omega k_n z}, \]

where \( p_n, q_n \) and \( r_n \) are the small perturbations of the fields in the \( n \)th unit cell of the RNOWA.

The linearized system of equations for these perturbations takes the form

\[
\begin{align*}
\frac{i}{\partial \zeta}p_n &= -\varphi p_n + (q_n + q_{n-1}) + (r_n + r_{n-1}), \\
\frac{i}{\partial \zeta}q_n &= (p_n + p_{n-1}) + \varphi (q_n + q_{n}^*), \\
\frac{i}{\partial \zeta}r_n &= (p_n + p_{n-1}) + \varphi (r_n + r_{n}^*).
\end{align*}
\]

Here \( \varphi = \mu_2 b_0^2 \).

Let there be \( N = 2M + 1 \) waveguides in the RNOWA. The fields in the \( n \)th unit cell are presented as a Fourier series:

\[
\begin{align*}
p_n &= \sum_{s=-M}^{s=M} (p_s e^{2\pi i s/M} + \tilde{p}_s e^{2\pi i s/M}), \\
q_n &= \sum_{s=-M}^{s=M} (q_s e^{2\pi i s/M} + \tilde{q}_s e^{2\pi i s/M}), \\
r_n &= \sum_{s=-M}^{s=M} (r_s e^{2\pi i s/M} + \tilde{r}_s e^{2\pi i s/M}).
\end{align*}
\]

Substitution of (12) into equation (11) considering the orthogonality of the harmonic functions results in the following equations for the modes:

\[
\begin{align*}
\frac{i}{\partial \zeta} p_s &= -\varphi p_s + \kappa(s) q_s + r_s, \\
\frac{i}{\partial \zeta} q_s &= -\varphi q_s + \kappa(s) q_0^* + \tilde{r}_s, \\
\frac{i}{\partial \zeta} r_s &= \kappa(s) q_s + \varphi (q_0 + q_0^*), \\
\frac{i}{\partial \zeta} \tilde{p}_s &= \kappa(s) p_s + \varphi (\tilde{q}_s + \tilde{q}_s^*), \\
\frac{i}{\partial \zeta} \tilde{q}_s &= \kappa(s) \tilde{p}_s + \varphi (\tilde{r}_s + \tilde{r}_s^*), \\
\frac{i}{\partial \zeta} \tilde{r}_s &= \kappa(s) \tilde{p}_s + \varphi (\tilde{r}_s + \tilde{r}_s^*).
\end{align*}
\]

Here

\[ \kappa(s) = 2\cos(\pi s/M) e^{\pi i s/M}. \]

In the following the mode mark \( s \) can be omitted, as long as it is not necessary. If the new functions \( w = \kappa p, \quad \tilde{w} = \kappa \tilde{p} \) are introduced, this system of equations will take the form

\[
\begin{align*}
\frac{i}{\partial \zeta} w &= -\varphi w + |\kappa|^2 (q + r), \\
\frac{i}{\partial \zeta} \tilde{w} &= -\varphi \tilde{w} + |\kappa|^2 (\tilde{q} + \tilde{r}), \\
\frac{i}{\partial \zeta} q &= w + \varphi (q + q^*), \\
\frac{i}{\partial \zeta} \tilde{q} &= \tilde{w} + \varphi (\tilde{q} + \tilde{q}^*), \\
\frac{i}{\partial \zeta} r &= w + \varphi (r + r^*), \\
\frac{i}{\partial \zeta} \tilde{r} &= \tilde{w} + \varphi (\tilde{r} + \tilde{r}^*).
\end{align*}
\]

In these equations one can find a closed system of three
equations:
\[-\frac{\partial^2 q}{\partial \xi^2} = -\varrho \tilde{w}^* + |\kappa|^2 (q + r),\]
\[-\frac{\partial^2 r}{\partial \xi^2} = -\varrho \tilde{w}^* + |\kappa|^2 (q + r),\]
\[-\frac{\partial^2 \tilde{w}^*}{\partial \xi^2} = (2|\kappa|^2 + \varrho^2) \tilde{w}^* + \varrho |\kappa|^2 (q + r).\]

If the variables \( u = q + r, \tilde{u} = q - r, \tilde{w}^* = v \) are used, the system of equations can be written as
\[-\frac{\partial^2 v}{\partial \xi^2} + (2|\kappa|^2 + \varrho^2)v + \varrho |\kappa|^2 u = 0,\]
\[-\frac{\partial^2 u}{\partial \xi^2} - 2\varrho v + 2|\kappa|^2 u = 0,\]
\[-\frac{\partial^2 \tilde{u}}{\partial \xi^2} = 0.\] (15)

Thus the variable \( \tilde{u} = q - r \) varies as \( \tilde{u} = \tilde{u}_0 + \tilde{u}_1 \xi \). Hence, the small perturbations vary proportionally with the distance \( \xi \).

If the initial variables \( q, r, \) and \( v = \tilde{w}^* \) are considered, the corresponding characteristic equation takes the form
\[\lambda^2 + (2|\kappa|^2 + \varrho^2) \varrho |\kappa|^2 - \varrho \lambda^2 + 2|\kappa|^2 |\kappa|^2 - \varrho |\kappa|^2 = 0.\] (16)

The roots of equation (16) \( \lambda^2 = 0 \) are evidence for the linear increase of the small perturbations. Another root can be found from the reduced characteristic equation
\[(\lambda^2 + 2|\kappa|^2 + \varrho^2)(\lambda^2 + 2|\kappa|^2) + 2\varrho^2|\kappa|^2 = 0.\] (17)

Changing to \( \lambda^2 = 2|\kappa|^2 \xi \) results in the following equation:

\[(1 + \xi)(1 + \xi + \mu) + \mu = 0,\]

where \( \mu = \varrho^2/(2|\kappa|^2) \). It follows that the roots of this equation read
\[\xi_{1,2} = -\left(1 + \frac{\mu}{2}\right) \pm \sqrt{D_2},\]

where \( D_2 = \mu^2/4 - \mu \). Thus, the roots of equation (17) are given by the expressions
\[\lambda_{1,2}^4 = \pm \sqrt{2|\kappa|}\left[-\left(1 + \frac{\mu}{2}\right) \pm \sqrt{D_2}\right].\] (18)

Instability takes place if \( \text{Re}(\lambda_{1,2}^4) > 0 \) or \( \text{Re}(\lambda_{1,2}^4) > 0 \).

In a linear case, where \( \mu = 0, \lambda_{1,2}^4 = \pm 2|\kappa| \). This means that there is no exponential increase in the perturbations. However, any small perturbation leads to the spreading of electromagnetic waves in the transversal direction. This is due to \( \lambda^2 = 0 \). Thus discrete diffraction takes place in a linear 1D rhombic waveguide array. However, the amplitudes of the electromagnetic waves remain limited [27]. In this case the solutions are considered stable but not asymptotically stable.

The roots of equation (17) can be written as

\[\lambda_{2, \pm}^4 = \pm \sqrt{2|\kappa|}\sqrt{\xi_1}, \quad \lambda_{2, \pm}^4 = \pm \sqrt{2|\kappa|}\sqrt{\xi_2}.\]

If \( 0 < \mu < 4 \), the discriminant \( D_2 \) is negative, \( D_2 = -\mu(4 - \mu)/4 \); hence
\[\xi_{1,2} = -\left(1 + \frac{\mu}{2}\right) \pm \sqrt{D_1}.\]

By extracting the square root from \( \xi_{1,2} \), one can obtain the expressions for the roots of equation (17):

\[\lambda_{1,2}^4 = \Omega(\cosh \phi^0 + i \sinh \phi^0), \quad \lambda_{2, \pm}^4 = \Omega(\cosh \phi^0 + i \sinh \phi^0),\] (19)

where
\[\Omega = \sqrt{2|\kappa|}\left(1 + \frac{\mu}{2}\right)^{1/2}, \quad \text{sinc} 2\phi^0 = \pm \sqrt{D_1}/\left(1 + \mu/2\right).\]

As \( \text{Re}(\lambda_{1,2}^4) > 0 \) and \( \text{Re}(\lambda_{2, \pm}^4) < 0 \), the flat-band solution under consideration is unstable in the region \( 0 < \mu < 4 \). If \( \mu > 4 \), the discriminant \( D_2 = (\mu - 4)/4 \) is positive. Hence \( \xi_{1,2} \) is a real value, and
\[\xi_1 = -\left(1 + \frac{\mu}{2}\right) + \sqrt{D_2}, \quad \xi_2 = -\left(1 + \frac{\mu}{2}\right) - \sqrt{D_2}.\]

For \( \xi_1 \) the expression
\[\xi_1 = -\left(1 + \frac{\mu}{2}\right) + \frac{\mu}{2} \sqrt{1 - \frac{4}{\mu^2}} = -1 \mu \left(1 - \frac{4}{\mu^2}\right),\]

can be found. It is negative at \( \mu > 4 \). From the definition of \( \xi_2 \) it follows that \( \xi_2 < 0 \). Thus, \( \text{Re}(\lambda_{1,2}^4) = 0 \) and \( \text{Re}(\lambda_{2, \pm}^4) = 0 \). Hence, the flat-band solution is stable in the region \( \mu > 4 \).

So, the flat-band solution (7) is unstable (i.e., there is no exponential increase of the amplitudes) if the intensity of radiation in the waveguide is lower than some threshold intensity. This solution will be stable if the intensity is higher than the threshold value. Figure 2 shows a contour map for the increment of the instability of the 5th mode. The increment is defined as \( \Gamma_i = \text{Re}(\lambda_{1,2}^4) = 2\sqrt{2}|\cos \Lambda|\text{Re}(\xi_{1,2}^4)/\Lambda \), where \( \Lambda = \pi S/M \) and the variable \( Y = \mu_2 b_0^2/\sqrt{2} \) is the normalized intensity. The area above the curve \( Y = 2\cos \Lambda \) corresponds to the stability area for the electric field distribution (7).

Using the definition of the relevant parameters,
\[\varrho = \mu_2 b_0^2, \quad |\kappa(s)| = 2|\cos(\pi s/M)|,\]

one can write the stability condition in the form
\[\mu_2 b_0^2 \geq 2\sqrt{2}|\cos(\pi s/M)| = 2\sqrt{2}|\cos(\Lambda)|.\] (20)

If the normalized intensity per mode \( b_0 \) is greater than the critical value \( (b_0^2 = 2\sqrt{2}\mu_2^{-1}\cos(\pi s/M)|) \), the perturbations
The stability of solution can be analyzed by introducing small perturbations into the electric field amplitudes:

\[ A_n = [(−1)^n a_0 + p_n] e^{−i\kappa}, \]
\[ B_n = [(−1)^n b_0 + q_n] e^{−i\kappa}, \]
\[ C_n = [(−1)^n c_0 + r_n] e^{−i\kappa}, \]  

where \( p_n, q_n \) and \( r_n \) are the small perturbations of the amplitudes for the homogeneous distributions \( A_0, B_0, \) and \( C_0 \). Substitution of \( (21) \) into system-of-equations \( (2) \) results in a linearized system of equations for the small perturbations:

\[
\begin{align*}
\frac{i\partial p_n}{\partial \zeta} &= (q_n + q_{n−1}) + (r_n + r_{n−1}) + \varphi_1(p_n + p_n^*), \\
\frac{i\partial q_n}{\partial \zeta} &= (p_n + p_{n+1}) + \varphi_1(q_n + q_n^*), \\
\frac{i\partial r_n}{\partial \zeta} &= (p_n + p_{n+1}) + \varphi_1(r_n + r_n^*). 
\end{align*}
\]  

(22)

Figure 2. Contour map of the increment of instability for the homogeneous distribution \( \theta = 0 \).

will not increase exponentially. It should be mentioned that the critical value \( b_0 \) is dependent on the mode mark \( s \). Hence only part of the modes having marks, which belong to the interval

\[ \frac{\pi}{2} > \frac{s\pi}{M} \geq \arccos \frac{\mu_2 b_0^2}{2\sqrt{2}}, \]

will be stable. However, all modes of the flat band will be stable if the condition \( \mu_2 b_0^2 \geq 1 \) holds.

Inferences about the instability of the homogeneous distribution \( \theta = 0 \) were tested by the numerical solution of system-of-equations \( (2) \) with the following initial conditions:

\[ A_n(0) = p_n, \quad B_n(0) = b_0 + q_n, \quad C_n(0) = −b_0 + r_n, \]

where the small perturbations of solution \( \theta = 0 \) are chosen as \( p_0 = q_0 = r_0 = 0.05 \). Amplitude \( b_0 \) was chosen in the range 0.7 to 1, 7. If the perturbations are zero, the numerical solution of \( (2) \) results in the analytical solution \( \theta = 0 \). At \( p_0 = q_0 = r_0 = 0.05 \) the numerical solution gives rise to a non-monotonic increase of these perturbations. This demonstrates that the diffractionless propagation of electromagnetic waves in the RNOWA is terminated by the small perturbations.

4.2. Solution corresponding to constraint (b)

The stability of solution \( \theta = 0 \) can be analyzed by introducing small perturbations into the electric field amplitudes:

\[ A_n = [(−1)^n a_0 + p_n] e^{−i\kappa}, \]
\[ B_n = [(−1)^n b_0 + q_n] e^{−i\kappa}, \]
\[ C_n = [(−1)^n c_0 + r_n] e^{−i\kappa}, \]  

(21)

where \( p_n, q_n \) and \( r_n \) are the small perturbations of the amplitudes for the homogeneous distributions \( \theta = 0 \). Substitution of \( (21) \) into system-of-equations \( (2) \) results in a linearized system of equations for the small perturbations:

\[
\begin{align*}
\frac{i\partial p_n}{\partial \zeta} &= (q_n + q_{n−1}) + (r_n + r_{n−1}) + \varphi_1(p_n + p_n^*), \\
\frac{i\partial q_n}{\partial \zeta} &= (p_n + p_{n+1}) + \varphi_1(q_n + q_n^*), \\
\frac{i\partial r_n}{\partial \zeta} &= (p_n + p_{n+1}) + \varphi_1(r_n + r_n^*). 
\end{align*}
\]  

(22)

Substitution of \( (12) \) into equation \( (22) \) considering the orthogonality of the harmonic functions results in the following equations for the amplitudes of modes \( p_s, q_s, \) and \( r_s \):

\[
\begin{align*}
\frac{i\partial p_s}{\partial \zeta} &= \kappa(s)(q_s + r_s) + \varphi_1(p_s + p_s^*), \\
\frac{i\partial q_s}{\partial \zeta} &= \kappa(s)(p_s + q_s^*) + \varphi_1(p_s^* + p_s), \\
\frac{i\partial r_s}{\partial \zeta} &= \kappa(s)(p_s^* + q_s) + \varphi_1(r_s + r_s^*). 
\end{align*}
\]

The coupling constant \( \kappa(s) \) is defined by \( (13) \). As discussed above, this system of equations can be reduced to the following system of equations:

\[
\begin{align*}
\frac{\partial^2 w}{\partial \zeta^2} + 2|\kappa|^2 w + 2 \varphi_1|\kappa|^2 (q + r) &= 0, \\
\frac{\partial^2 q}{\partial \zeta^2} + 2 \varphi_1 w + |\kappa|^2 (q + r) &= 0, \\
\frac{\partial^2 r}{\partial \zeta^2} + 2 \varphi_1 w + |\kappa|^2 (q + r) &= 0. 
\end{align*}
\]  

(23)

where \( w = \kappa(s)p_s, \kappa = \kappa(s), q = q_s, \) and \( r = r_s. \)

The corresponding characteristic equation takes the form

\[ \lambda^2 [\lambda_+^2 + 2|\kappa|^2] - 8 \varphi_1^2 |\kappa|^2] = 0. \]  

(24)

The roots of this equation \( \lambda^2 = 0 \) are related to the linear dependence of the small perturbations on the distance \( \zeta \). The exponential increase or decrease of the perturbations is related to the roots of the following equation:

\[ (\lambda_+^2 + 2|\kappa|^2) - 8 \varphi_1^2 |\kappa|^2] = 0. \]

These roots are given by

\[ \lambda_+^2 = -2|\kappa|^2 \pm 2\sqrt{2} \varphi_1|\kappa|. \]

As \( \lambda_+^2 < 0 \), these roots \( (\pm|\lambda_+|) \) describe the periodical behavior of the perturbations. If \( |\kappa| > \sqrt{2} \varphi_1 \), then \( \lambda_+^2 > 0 \). In this case, solution \( (10) \) is stable.

However, if \( |\kappa| < \sqrt{2} \varphi_1 \), then \( \Re \lambda_+^2 = 0 \), which indicates the instability of the flat-band solution. The threshold amplitude \( a_{00}(s) \) of the instability of the \( s \)th mode is defined...
by the expression
\[ \mu_1 a_0^2(s) = \sqrt{2} |\cos(\pi s/M)|. \] (25)

The intervals of the wave numbers of the \( st \)th mode \( \Lambda \) can be written as
\[ \Lambda_{\pm} \leq \Lambda \leq \pi/2, \quad -\pi/2 \leq \Lambda \leq -\Lambda_{\pm}, \]
where \( \Lambda = \pi s/M \) and \( \pm \Lambda_{\pm} \) is defined by the expression \( \mu_1 a_0^2 = \sqrt{2} |\cos(\Lambda)| \).

The increment of the instability of the \( st \)th mode \( G_s \), also called the modulation instability gain, is determined by the expression \( G_s = \text{Re} \Lambda_s^+ \). That can be read as
\[ G_s^2 = 4 \sqrt{2} \cos\Lambda(\mu_1 a_0^2 - \sqrt{2} \cos\Lambda), \] (26)
under the condition \( \mu_1 a_0^2 \geq \sqrt{2} |\cos(\Lambda)| \). The contour lines of the increment \( G_s^2/8 \) are shown in figure 3. The abscissa corresponds with the wave numbers of the \( st \)th mode \( \Lambda \). The ordinate represents the normalized intensity \( Y = \mu_1 a_0^2 / \sqrt{2} \).

The contour line mark is equal to the value of \( G_s^2/8 \). The area below the line marked by 0 corresponds to the area of stability for the electric field distribution (10). If the condition \( \mu_1 a_0^2 > \sqrt{2} \) holds, then all modes will be unstable; hence it follows that the electric field distribution under consideration will be unstable.

From equation (26) it follows that the amplitudes of the modes having the numbers of \( \Lambda_m \), where
\[ \cos\Lambda_m = \frac{1}{2\sqrt{2}} \mu_1 a_0^2, \]
increase exponentially with the distance \( \zeta \). In this case the increment is equal to \( G_{\text{max}} = \mu_1 a_0^2 / 2 \).

5. Conclusion

The RNOWA is considered in this paper. The array of linear waveguides has been investigated in [19, 24–26]. It was shown that all (normal) modes of this waveguide array are separated into three groups or bands in the 1D space of the wave vectors. Two bands are populated by modes describing discrete diffraction in the waveguide array. The third band contains modes that describe wave propagation without diffraction. This band was named the flat band.

In the RNOWA a flat-band analog exists. There are solutions of the system of equations of the RNOWA that describe diffractionless wave propagation. However, in both the linear and nonlinear cases the flat-band solution is weak and unstable. Small perturbations increase directly with the first power of distance along the waveguide. In the case of nonlinear waveguides small perturbations grow exponentially. But with intensity increasing, part of the modes begin to be stable. All modes become stable if the intensity of radiation in the waveguide is greater than the threshold value. These flat-band solutions are stable but not asymptotically stable.

As pointed above, there are two kinds of electric field distributions: (a) \( A_n = 0, B_n = B_0 = -C_0 = -C_0 \) and (b) \( A_n = (-1)^n A, B_n = (-1)^n B, C_n = (-1)^{-n+1} C \), for which the power flux between unit cells of the RNOWA is equal to zero. The case of (a) can be correlated with discrete self-focusing. Weak intensity in waveguides can tunnel to neighboring waveguides. This is analogous to diffraction. Diffraction is arrested if the intensity exceeds the critical value (20).

As for the second case, there is discrete analogy of the modulation instability for a cubic nonlinear bulk material. There is a threshold intensity for each \( st \)th mode (25), and the threshold intensity for all bands is \( a_0^2 = \sqrt{2} / \mu_1 \). If the intensity of radiation in the waveguide of the RNOWA is greater than this threshold, perturbations increase exponentially with distance.

Recently [28] a binary Bose–Einstein condensate loaded into an optically imprinted ribbon with the structure of a rhombic chain was considered. The model describing this system includes interactions between its components, both nonlinear and linear ones, with the latter mediated by spin–orbit coupling. Stable compactons (i.e., compact localized states) and discrete solitons for nonlinear spinor waves were found. Within the linear limit and when spin–orbit coupling is terminated, the model equation (8) from [28] takes the form of equation (2). The compacton equation (13) from [28] will take the following form:

\[ \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \text{const} \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} (\delta_n 0 + \delta_{n+1,0}). \]

Here only non-zero components are presented. It is worth noting that the compactons also persist and remain available in the exact analytical form of equation (13) in the nonlinear case. Furthermore, they are stable in narrow areas adjacent to the flat band. I believe that the stability of the compactons is due to the spin–orbit coupling, and in the simple case considered here the compactons will be unstable. This hypothesis clearly warrants further study.
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