Thermostatistics of deformed bosons and fermions

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Abstract  Based on the $q$-deformed oscillator algebra, we study the behavior of the mean occupation number and its analogies with intermediate statistics and we obtain an expression in terms of an infinite continued fraction, thus clarifying successive approximations. In this framework, we study the thermostatistics of $q$-deformed bosons and fermions and show that thermodynamics can be built on the formalism of $q$-calculus. The entire structure of thermodynamics is preserved if ordinary derivatives are replaced by the use of an appropriate Jackson derivative (JD) and $q$-integral. Moreover, we derive the most important thermodynamic functions and we study the $q$-boson and $q$-fermion ideal gas in the thermodynamic limit. 

Keywords  Deformed quantum thermodynamics · $q$-calculus

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1 Introduction

The founding principle of the quantum theory of many body-systems and quantum statistical mechanics is based on the spin-statistics theorem. The contrasting nature of spin one-half particles, fermions, and spin integer, bosons, affects sensibly the quantum statistical behavior of many particle systems [12].

The power of statistical mechanics lies not only in the derivation of the general laws of thermodynamics from microscopic theory but also in determining the
meaning of all the thermodynamic functions in terms of microscopic interparticle interaction. In the recent literature, a great deal of effort has been devoted to studying the possible violation of standard statistical behavior derived from the indistinguishability of identical fermions and bosons. The investigation of statistical distribution interpolating between Fermi-Dirac (FD) and Bose-Einstein (BE) distribution functions has led to many studies of intermediate statistics, where creation and annihilation operators obey commutation relation interpolating between bosons and fermions [3,4,5,6,7,8,9]. Experimental high precision methods have been developed to verify small deviations from the Pauli principle [10,11,12,13,14]. Implications of such deviations are reported in atomic, nuclear, high energy physics, condensed matter, astrophysics and cosmology [15,16,17,18,19].

In the above context, describing complex systems, quantum algebra and quantum groups have been the subject of intense research in several physical fields. The quantum group $SU_q(2)$ is well-known to play an important role in classical and quantum integrable systems and arose in the work on Yang-Baxter equations [20,21]. The main aspect of quantum theory can be described by the quantum Heisenberg uncertainty relation and hence a consequence of $q$-deformation can best be understood in terms of the modified uncertainty relation, as revealed in the seminal works of Biedenharn [22] and Macfarlane [23]. The uncertainty relation, expressed as the commutation relation between the coordinate and momentum, has the Planck constant, namely $\hbar$, on the right hand side in the standard quantum mechanics, is multiplied by a function, $F(q) = \cosh((2n+1)(\ln q)/2)/\cosh((\ln q)/2)$ (where $n$ represents the number of quanta of oscillators) in the case of the $q$-deformed quantum theory. In other words, an important consequence of $q$-deformation can be stated as a modification of the Planck constant itself. Furthermore, it was early clear that the $q$-calculus, originally introduced by Heine [24] and by Jackson [25] in the study of the basic hypergeometric series [26], plays a central role in the representation of the quantum groups with a deep physical meaning [27,28,29,30]. A basic hypergeometric function is a generalization of the ordinary hypergeometric function, signified by the addition of an extra parameter $q$, so that in the limit when $q \rightarrow 1$ this tends to the normal hypergeometric function. These functions have a range of representation, from the simple basic series to a generalized basic hypergeometric function. Various ordinary functions have thus been generalized to introduce the basic functions with applications in many fields of science. Furthermore, it is remarkable to observe that the $q$-calculus, based on the so-called Jackson Derivative (JD) operator and its inverse operator $q$-integral, is indeed well suited for describing fractal and multifractal systems. As soon as the system exhibits a discrete-scale invariance, the natural tool is provided by Jackson $q$-derivative and $q$-integral, which constitute the natural generalization of the regular derivative and integral for discretely self-similar systems [31,32].

As a consequence, we expect $q$-calculus to play a central role in $q$-deformed thermostatistics. It is well-known that the theory of harmonic oscillators in standard thermodynamics can be modified by means of the $q$-deformed algebra of the oscillators. This modifies the thermodynamic distribution function and all the thermodynamic functions such as the entropy, partition function, equation of state etc. The $q$-deformation affects the energy spectrum in a fundamental manner and in this manner the deformation may be thought of as describing complex ensembles or real gases, contrary to the standard thermodynamics as the theory describ-
ing ideal gases. The extent of the deformation i.e., the value of \( q \) in reality, is of course to be determined by confronting the predictions of the theory against experiment and this has been accomplished so far by means of numerical calculations. The physical meaning of \( q \)-deformation can be stated in terms of the JD i.e., the \( q \)-deformation corresponds to difference equations as opposed to continuous derivatives and continuous differential equations. Several investigators have studied the equilibrium statistical mechanics of noninteracting \( q \)-deformed bosons and fermions and many of the details of the analysis are well known. It is remarkable that the main premise consists of demonstrating that the \( q \)-deformed thermostatistics can be consistently formulated by replacing the standard ordinary derivatives of thermostatistics by JD. Of course care must be exercised in observing that many of the standard rules of calculus such as the usual Leibniz chain rule are ruled out and in this manner one formulates what may be termed as the \( q \)-calculus. Accordingly, one can formulate the entropy, mean occupation number, bose condensation etc. and thus the entire \( q \)-thermostatistics [33,34,35]. The main goal of this paper is to extend our previous investigations employing consistently \( q \)-deformed integral (inverse operator of the JD) in the formulation of the quantum thermodynamics and to study the behavior of the mean occupation number considering its development in terms of infinite continued fractions and outlining its analogies with intermediate statistics.

2 Deformed algebra and \( q \)-calculus

We shall review the principal relations of \( q \)-oscillators defined by the \( q \)-Heisenberg algebra of creation and annihilation operators introduced by Biedenharn and McFarlane [22,23], derivable through a map from SU\(_q\)(2). Furthermore, we will review the main features of the strictly connected \( q \)-calculus useful in the present investigation.

The symmetric \( q \)-oscillator algebra is defined, in terms of the creation and annihilation operators \( c, c^\dagger \) and the \( q \)-number operator \( N \), by [36,37,38,39]

\[
[c, c]_\kappa = [c^\dagger, c^\dagger]_\kappa = 0, \quad cc^\dagger - \kappa c^\dagger c = q^{-N},
\]

\[
[N, c^\dagger] = c^\dagger, \quad [N, c] = -c,
\]

where the deformation parameter \( q \) is real and \([x, y]_\kappa = xy - \kappa yx\), where, as before, \( \kappa = 1 \) for \( q \)-bosons with commutators and \( \kappa = -1 \) for \( q \)-fermions with anticommutators.

Furthermore, the operators obey the relations

\[
c^\dagger c = [N], \quad cc^\dagger = [1 + \kappa N],
\]

where the \( q \)-basic number is defined as

\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}}.
\]

In the limit \( q \to 1 \), the basic number \([x]\) reduces to the ordinary number \( x \) and all the above relations reduce to the standard boson and fermion relations.
The transformation from Fock observables to the configuration space may be accomplished by the replacement (Bargmann holomorphic representation) \[ a^\dagger \rightarrow x, \quad a \rightarrow \mathcal{D}_x, \] where \( \mathcal{D}_x \) is the JD \[ \mathcal{D}_x = \frac{D_x - (D_x)^{-1}}{(q - q^{-1})x}, \] and \( D_x = q^x \partial_x \), is the dilatation operator. Its action on an arbitrary real function \( f(x) \) is defined by \[ \mathcal{D}_x f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}. \]

In contrast to the usual derivative, which measures the rate of change of the function in terms of an incremental translation of its argument, the JD measures its rate of change with respect to a dilatation of its argument by a factor of \( q \). The JD plays an important role in the general \( q \)-Taylor expansion as discussed in Ref. \[40\] and occurs naturally in \( q \)-deformed structures; we will see that it plays a crucial role in the \( q \)-generalization of the thermodynamics relations.

The JD satisfies some simple properties which will be useful in the following. For instance, its action on a monomial \( f(x) = x^n \), where \( n \geq 0 \), is given by \[ \mathcal{D}_x (ax^n) = a[n]x^{n-1}, \] where \( a \) is a real constant.

Moreover, it is easy to verify the following basic-version of the Leibnitz rule
\[ \mathcal{D}_x \left( f(x) g(x) \right) = \mathcal{D}_x f(x) g(q^{-1}x) + f(qx) \mathcal{D}_x g(x), \]
\[ = \mathcal{D}_x f(x) g(qx) + f(q^{-1}x) \mathcal{D}_x g(x). \] (10)

In addition the following property holds
\[ \mathcal{D}_{ax} f(x) = \frac{1}{a} \mathcal{D}_x f(x). \] (11)

In order to formulate a self-consistent \( q \)-deformed theory, the standard integral must be generalized to the basic-integral defined, for \( 0 < q < 1 \) in the interval \([0, a]\), as \[ \int_0^a f(x) dq x = a(q^{-1} - q) \sum_{n=0}^{\infty} q^{2n+1} f(q^{2n+1} a), \] while in the interval \([0, \infty)\)
\[ \int_0^\infty f(x) dq x = (q^{-1} - q) \sum_{n=-\infty}^{\infty} q^{2n+1} f(q^{2n+1}). \] (13)
The indefinite $q$-integral is defined as
\[
\int f(x) \, dq x = (q^{-1} - q) \sum_{n=0}^{\infty} q^{2n+1} x f(q^{2n+1} x) + \text{constant}. \tag{14}
\]

One can easily see that the $q$-integral approaches the Riemann integral as $q \to 1$ and also that $q$-differentiation and $q$-integration are inverse of each other, thus \[ D_x \int_0^x f(t) \, dq t = f(x) - f(0), \tag{15} \]
where the second identity occurs when the function $f(x)$ is $q$-regular at zero, i.e. \[ \lim_{n \to \infty} f(x q^n) = f(0). \tag{16} \]

By using the deformed Leibnitz rule of Eq.(10), analogous formulas for integration by parts may easily be deduced as
\[
\int_0^a f(qx) D_x g(x) \, dq x = f(x) g(x)|_{x=a} - \int_0^a D_x f(x) g(q^{-1} x) \, dq x, \tag{17}
\]
\[
\int_0^a f(q^{-1} x) D_x g(x) \, dq x = f(x) g(x)|_{x=0} - \int_0^a D_x f(x) g(qx) \, dq x. \tag{18}
\]

From the above relations, as also pointed out in Ref. [31,43], it appears evident that JD and $q$-calculus provides a custom made formalism in which to express scaling relations. When $x$ is taken as the distance from a critical point, JD thus quantifies the discrete self-similarity of the function $f(x)$ in the vicinity of the critical point and can be identified with the generator of fractal and multifractal sets with discrete dilatation symmetries.

### 3 Thermal average and mean occupation number

Let us start from the following Hamiltonian of non-interacting $q$-deformed oscillators (fermions or bosons)
\[
H = \sum_i (E_i - \mu) N_i, \tag{19}
\]
where $\mu$ is the chemical potential and $E_i$ is the kinetic energy in the state $i$ with the number operator $N_i$. It is important to recognize that the latter Hamiltonian, in spite of the appearance, does include deformation (since the number operator is deformed by means of Eq.(3)), as will become evident from the form of the average occupation number.

Thermal average of an observable can be computed by following the usual prescription of quantum mechanics, as follows
\[
\langle \mathcal{O} \rangle = Tr (\rho \mathcal{O}), \tag{20}
\]
where $\rho$ is the density operator and $\mathcal{Z}$ is the grand canonical partition function defined as
\[
\rho = \frac{e^{-\beta H}}{\mathcal{Z}}, \quad \mathcal{Z} = Tr (e^{-\beta H}), \tag{21}
\]
and $\beta = 1/T$ (hence forward we shall set Boltzmann constant to unity). We observe that the structure of the density matrix $\rho = e^{-\beta H}$ and the thermal average are undeformed. As a consequence, the structure of the partition function is also unchanged. This is a very common assumption in the literature of deformed thermodynamics associate to the quantum $q$-deformed algebra \cite{37,38,39,44,45}. We emphasize that this is not a trivial assumption because its validity implicitly amounts to an unmodified structure of the Boltzmann-Gibbs entropy, $S = \log W$, where $W$ stands for the number of states of the system corresponding to the set of occupation numbers $\{n_i\}$. Obviously the number $W$ is modified in the $q$-deformed case \cite{33}. It may be pointed out that in the subject literature, statistical generalizations are present, such as the so-called nonextensive thermostatistics or superstatistics with a completely different origin \cite{46,47,48,49,50,51,52}. We like to stress that in this paper we are dealing with a many body statistical theory of particles that obey an intermediate behavior between fermions and bosons, corresponding to a deformed algebra of the creation and annihilation operators. The deformation contained into the quantum algebra is reflected into the quantum statistical behavior. The other way round is realized in other statistical generalizations present in literature where the deformation starts from generalized assumptions in statistical mechanics altering the classical (or quantum) statistical behavior of complex systems. For example, in the case of the nonextensive deformed Tsallis statistics, the structure of the entropy is deformed via the logarithm function and a deformed (classical) algebra related to a generalized exponential and logarithm functions emerges in a natural way \cite{53,54}.

The above assumptions allow us to calculate the average occupation number $n_i$ defined by the relation $[n_i] = Tr \left( e^{-\beta H c_i^\dagger c_i} / Z \right)$. Repeated application of the algebra of $c,c^\dagger$ along with the use of the cyclic property of the trace leads to the result

$$n_i = \frac{1}{q - q^{-1}} \log \left( \frac{e^{\eta_i} - \overline{\kappa}^{-1}}{e^{\eta_i} - \kappa} \right), \quad (22)$$

where we have set $\eta_i = \beta (E_i - \mu)$ and $\overline{\kappa} = \kappa q^\kappa$ (with the property $\kappa^{-1} = \kappa$).

Now we need to further express the occupation number in a useful form. From Eq.\,(22), we arrive at the result in the form of a power series

$$n = \frac{1}{y} + \left( \frac{1}{3y^3} + \frac{\kappa}{2y^2} \right) \varepsilon^2 + \left( \frac{1}{3y^3} + \frac{\kappa}{2y^2} \right) \varepsilon^3 + \left( \frac{1}{3y^3} + \frac{\kappa}{2y^2} + \frac{2}{3y^3} + \frac{\kappa}{2y^2} \right) \varepsilon^4 + \cdots, \quad (23)$$

where $y = e^\eta - \kappa$ and we have taken $\overline{\kappa} = \kappa (1 - \varepsilon)$, $\varepsilon \ll 1$. At this point, if we use the approximation by retaining only the leading term, we then arrive at the form

$$n \approx \frac{1}{e^\eta - \kappa}. \quad (24)$$

More generally, for $\varepsilon$ not very small, we have the series form in Eq.\,(23) describing the various powers of the deformation parameter $\varepsilon$. 
The detailed thermodynamic properties stemming from this form, such as the equation of state, virial expansion etc. for the $q$-bosons and $q$-fermions have been studied in Ref. [55]. It is also observed that the equality of the specific heats of boson-type and fermion-type intermediate statistics particles [56] also prevails as shown in Ref. [55], if we utilize the approximate forms for the occupation numbers. This is a very interesting result and may be true more generally for the exact forms of the occupation numbers formulated in the present work.

4 The occupation number as an infinite continued fraction

It is possible to obtain an expression for the mean occupation number in terms of the infinite Continued Fraction (CF). Let us begin with the series expression which may be expressed conveniently in the form

\[ n = \frac{\alpha_1}{y} + \frac{\alpha_2}{y^2} + \frac{\alpha_3}{y^3} + \cdots, \]  

where, as before, we have set $y = e^{\eta} - \kappa$ (for simplicity, we have dropped the particle index $i$) and the parameters $\alpha_1, \alpha_2$ etc. are determined from the previous sections, specifically Eq.(23), such as

\[ \alpha_1 = 1, \quad \alpha_2 = \frac{\kappa}{2}(\epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots), \]

\[ \alpha_3 = \frac{1}{3}(\epsilon^2 + \epsilon^3 + 2\epsilon^4 + \cdots), \]  

etc. by combining terms containing various powers of $\epsilon$ in Eq.(23). There is a standard method by which this infinite series can be put in the form of CF. The method of determining the CF form of a function given by an infinite series is well-known in the literature [57,58].

We shall briefly summarize the procedure here. The general continued fraction of order $r$ is of the form

\[ C_r = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}}, \]

where the constants $b_0, b_1, \cdots, a_0, a_1, \cdots$ can be determined by a straightforward procedure. The various convergents are $C_0, C_1, \cdots$ corresponding to $r = 0, 1, 2, \cdots, \infty$. Accordingly we have

\[ C_0 = b_0 = A_0/B_0; \]

\[ C_1 = b_0 + a_1/b_1 = b_0 b_1 + a_1 = A_1/B_1; \]

\[ C_2 = b_0 + a_1/(b_1 + a_2/b_2) = b_0 + a_1 b_2/(b_1 b_2 + a_2) = A_2/B_2, \]
etc. The parameters \( A_n, B_n \) satisfy the two-term recurrence relations \[58\]:

\[
\begin{align*}
A_n &= b_n A_{n-1} + a_n A_{n-2}; \quad A_{-1} = 1; \\
B_n &= b_n B_{n-1} + a_n B_{n-2}; \quad B_{-1} = 0.
\end{align*}
\]  

By solving the recurrence relations, the general CF can be determined. We may quote two examples of this procedure. The standard sine series may be expressed in the form of a CF as:

\[
\sin x = \frac{x}{1 + \frac{x^2}{2 \cdot 3 - \frac{x^2}{4 \cdot 5 - x^2 + \cdots}}}. \tag{30}
\]

Furthermore, we can also deal with the inverse problem, i.e., given the standard series form of the cosine function,

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \tag{31}
\]

we can employ the above procedure and obtain the CF form for the cosine function as:

\[
\cos x = \frac{1}{1 + \frac{x^2}{2 \cdot 1 - \frac{x^2}{4 \cdot 3 - x^2 + \cdots}}}. \tag{32}
\]

Employing this procedure for our present problem, after some algebra, the final result can be expressed by accordingly obtaining the various convergents (approximants):

\[
\begin{align*}
n_1 &= \frac{\alpha_1}{y}, \\
n_2 &= -\frac{\alpha_2 y}{\alpha_1 y + \alpha_2}, \\
n_3 &= -\frac{\alpha_3 y}{\alpha_2 y + \alpha_3},
\end{align*}
\]  

etc.

In the literature on CF, the convergents, which may be obtained in a straightforward manner after some algebra, play an important role.

The general form of the CF is given by the form \( C_r \) as in Eq.\(\text{(27)}\) and the procedure can be extended to many convergents. The meaning of the convergent or the approximant is evident.

Now the question which might arise is: what is the advantage of CF? Other than the elegant mathematical form, we remark that there is a distinct advantage. The Pade approximant is a well-known application. Moreover there is a theorem \[57\], involving the convergents \( n_1, n_2, n_3 \cdots \) which may be stated as:

\[
n_1 < n_3 < \cdots < n \quad \text{and} \quad n_2 > n_4, \cdots > n. \tag{36}
\]
This immediately provides a clarifying definition of successive approximations i.e., the above inequality tells us how to obtain successive approximations of the quantity \( n \). Indeed, the above tells us immediately that the exact form of \( n \) lies between \( n_1 \) and \( n_2 \), hence its importance. We can thus establish that the exact \( n \) is bigger than the first convergent \( n_1 = \alpha_1/y \) but smaller than \( n_2 \) obtained above.

5 Thermodynamics of \( q \)-deformed bosons and fermions

In Ref. \[33\], we have shown that the entire structure of thermodynamics is preserved if the ordinary derivatives are replaced by the use of an appropriate JD

\[
\frac{\partial}{\partial z} \Rightarrow \mathcal{D}_z(q).
\]

Consequently, posing the fugacity \( z = e^{\beta \mu} \), the number of particles in the \( q \)-deformed theory can be derived from the relation

\[
N = z \mathcal{D}_z(q) \ln \mathcal{Z} \equiv \sum_i n_i,
\]

where \( n_i \) is the mean occupation number expressed in Eq.(22).

The usual Leibniz chain rule is ruled out for the JD and therefore derivatives encountered in thermodynamics must be modified as follows. First we observe that the JD applies only with respect to the variable in the exponential form such as \( z = e^{\beta \mu} \) or \( y_i = e^{-\beta \epsilon_i} \). Therefore for the \( q \)-deformed case, any thermodynamic derivative of functions which depend on \( z \) or \( y_i \) must be transformed to derivatives in one of these variables by using the ordinary chain rule and then evaluating the JD with respect to the exponential variable. For instance, in the case of the internal energy in the \( q \)-deformed case, we can write this prescription explicitly as

\[
U = -\frac{\partial}{\partial \beta} \ln \mathcal{Z} \bigg|_z = \kappa \sum_i \frac{\partial y_i}{\partial \beta} \mathcal{D}_z(q) \ln(1 - \kappa z y_i).
\]

In this case we obtain the correct form of the internal energy

\[
U = \sum_i \epsilon_i n_i,
\]

where \( n_i \) is the mean occupation number expressed in Eq.(22).

In the thermodynamic limit, for a large volume \( V \) and a large number of particles, the sum over states can be replaced by the integral. However, as previously discussed, in a \( q \)-deformed theory the standard integral should be consistently generalized to the \( q \)-integral, inverse operator of the JD. In this manner, we extend our previous formulation by employing the \( q \)-integral operator and, following the above prescriptions, we have

\[
\sum_i f(u_i) \implies I_q = g_\kappa \frac{V}{(2\pi)^3} \int f[u(k)] d_4 k_x d_4 k_y d_4 k_z,
\]

where \( g_\kappa \) is the spin degeneracy factor, \( u(k) = \beta \hbar^2 k^2/2m \) and satisfies the constraint: \( k^2 = k_x^2 + k_y^2 + k_z^2 \) [59]. By taking into account the rules related to changing
the variable of \( q \)-integration \(^{[42]} \), we have verified that for \( 0.6 < q < 1.4 \) the above integration can be well approximately expressed as

\[
I_q \approx g \kappa \frac{V}{\sqrt{\pi}} \frac{2}{q + q^{-1}} \int_0^\infty f(u) dQ_u,
\]

where \( Q = q^2 \) (a change of variable \( u = \beta h^2 k^2 / 2m \) also involves a corresponding change of base) and \( \lambda = h / (2\pi m T)^{1/2} \) is the thermal wavelength. Therefore, in the thermodynamic limit, Eq.\(^{[38]} \) and Eq.\(^{[39]} \), respectively, becomes

\[
N_\kappa(T, z) = g \kappa \frac{V}{\lambda^3} h_{3/2}^\kappa(z, q),
\]

\[
U_\kappa(T, z) = g \kappa \frac{3}{2} \beta \lambda^3 h_{5/2}^\kappa(z, q),
\]

where we have defined the \( q \)-deformed \( h_{n}^\kappa(z, q) \) as

\[
h_{n}^\kappa(z, q) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{u^{n-1}}{q - q^{-1}} \ln \left( \frac{z^{-1} e^u - \kappa q^{-1}}{z^{-1} e^u - \kappa q} \right) dQ_u.
\]

It must be stressed that the above equation is quite different from the definition of the generalized function introduced in Eq.\(^{[21]} \) in our earlier work \(^{[34]} \). This is an important notion in our present work. It should also be noted that, to the best of our knowledge, this is the first time that \( q \)-integrals are numerically employed in thermostatistics calculations. In the limit \( q \to 1 \), the deformed \( h_{n}^\kappa(z, q) \) functions reduce to the standard \( h_{n}^\kappa(z) \) for bosons and fermions.

As in the undeformed boson case, we need to set the range of the \( q \)-boson fugacity \( z_B \) which will correspond to non-negative occupation number. In the case of \( q \)-bosons we see that the condition is \( z_B < 1/q \) for \( q > 1 \) and \( z_B < 1 \) for \( q < 1 \). Moreover, it should be pointed out that we also have to require the existence of the JD of the mean occupation number which is encountered in the calculation of thermodynamic quantities such as the specific heat and this changes the upper bound of the fugacity \( z_B \). In the following, we thus will require the condition \( z_B < z_q \), where we have defined

\[
z_q = \begin{cases} 
q^{-2} & \text{if } q > 1; \\
q^2 & \text{if } q < 1.
\end{cases}
\]

6 Specific heat of boson and fermion systems

We are now able to calculate the specific heat of the \( q \)-boson and \( q \)-fermion gas, starting from the thermodynamic definition

\[
C_v = \left. \frac{\partial U}{\partial T} \right|_{V,N}.
\]

Carrying out the JD prescription, described earlier, Eq.\(^{[47]} \) in the \( q \)-deformed theory can be written as

\[
C_v = -\beta^2 \sum_i \epsilon_i \frac{1}{q - q^{-1}} \frac{\partial \gamma_i}{\partial \beta} \ln \left( \frac{1 - \kappa q^{-1} \gamma_i}{1 - \kappa q \gamma_i} \right),
\]
where $\gamma = ze^{-\beta \epsilon}$ and
\[
\frac{\partial \gamma}{\partial \beta} = \left( \frac{1}{z} \frac{\partial z}{\partial \beta} - \epsilon_i \right) \gamma.
\] (49)

For this purpose we first need, therefore, the derivative of the fugacity with respect to $T$ (or $\beta$), keeping $V$ and $N$ constant. Accordingly, we observe that the following identity holds (since the number of particles is kept constant)
\[
\frac{\partial}{\partial \beta} \sum_i \ln \left( \frac{1}{1 - \kappa q - \kappa \gamma_i} \right) = 0.
\] (50)

In accordance with the JD recipe about the thermodynamical relations, the above equation can be written as
\[
\sum_i \frac{\partial \gamma_i}{\partial \beta} \phi^{(q)}_\beta \ln \left( \frac{1}{1 - \kappa q - \kappa \gamma_i} \right) = 0.
\] (51)

Evaluating in the thermodynamical limit ($V \to \infty$) and by using the definition in Eq.(45), we obtain
\[
1 \frac{\partial z}{\partial \beta} \bigg|_{V,N} = \frac{3}{2} \beta \frac{\phi^{(q)}_\beta h_{x/2}^{(q)}(z,q)}{\phi^{(q)}_\beta h_{x/2}^{(q)}(z,q)}.
\] (52)

By using the above relation in Eq.(48), we obtain the specific heat for a system of bosons and fermions at fixed $T$ and $N$
\[
\frac{C_v \lambda^3}{V} = g_k \left\{ \frac{15}{4} \frac{\phi^{(q)}_\beta h_{x/2}^{(q)}(z,q)}{\phi^{(q)}_\beta h_{x/2}^{(q)}(z,q)} \right. \right. \left. \left. - \frac{9}{4} \frac{\phi^{(q)}_\beta h_{x/2}^{(q)}(z,q))^2}{\phi^{(q)}_\beta h_{x/2}^{(q)}(z,q)} \right\}.
\] (53)

In Fig. 1 and 2 we display the specific heat $C_v \lambda^3/(g_k V)$ as a function of the fugacity for bosons (above the critical point of boson condensation) and fermions, respectively (note that the range of meaningful fugacities $z_B$, for boson gas, is limited by the condition (46)) for different values of $q$. We remember that we are employing the symmetric $q \leftrightarrow q^{-1}$ deformed quantum algebra, therefore the displayed graphs are identical for the transformation $q \leftrightarrow q^{-1}$ (more explicitly: the long dashed curves in Fig. 1 and Fig. 2 stand for $q = 0.8$ and $q = 1/0.8$, the short dashed curves stand for $q = 0.7$ and $q = 1/0.7$). Finally, let us observe that the modification of the specific heat increasing with the value of the deformation parameter $q$ becomes very remarkable in the fermion case.

7 Conclusion

In this paper, we have investigated the structure of symmetric $q \leftrightarrow q^{-1}$ deformed quantum thermostatistics by working, consistently in the framework of the $q$-calculus, with the use of the JD and the $q$-integration. We have shown that the entire structure of thermodynamics is preserved if the ordinary derivatives and integrals are replaced by the JD and $q$-integral, respectively. This prescription indeed gives us a recipe to obtain the fundamental thermodynamic functions such
Fig. 1 The specific heat $C_v \frac{\lambda^3}{g_B V}$ for bosons as a function of fugacity $z_B$ for different values of $q$. Same values are obtained for the transformation $q \leftrightarrow q^{-1}$.

Fig. 2 The specific heat $C_v \frac{\lambda^3}{g_F V}$ for a fermion gas as a function of fugacity $z_F$ for different values of $q$. Same values are obtained for the transformation $q \leftrightarrow q^{-1}$.

the mean occupation number, the specific heat etc. We may point out that this is a relevant premise of our original approach since, to the best of our knowledge, this is the first time $q$-derivatives and $q$-integrals are consistently employed in thermo-dynamical investigations.

Our formulation of the mean occupation number and other thermodynamic parameters in terms of the infinite continued fraction is a new feature, not known in the literature. Its importance stems from the possibility of approximations i.e., the validity of approximations in the theory. The behavior different from the undeformed quantum theory can be dealt with in the statistical behavior of a complex
system, intrinsically contained in the $q$-deformation, whose underlying dynamics is spanned in many body interactions and other long time memory effects. This aspect has been outlined in many papers in the recent literature.

The different behavior from the undeformed quantum theory can be dealt with in the statistical behavior of a complex systems, intrinsically contained in $q$-deformation, whose underlying dynamics is spanned in many-body interactions and/or long-time memory effects. This aspect has been outlined in several papers. For example in Ref. [60] it has been shown that $q$-deformation plays a significant role in understanding higher-order effects in many-body nuclear interactions. Moreover, the strong effects on the deformation, that we have found especially in the $q$-fermion specific heat, could be connected to an intrinsic presence of complex many-body effective interactions on $q$-deformation theory. In this context, it appears relevant to observe that nonanalytic temperature behavior of the specific heat of Fermi liquid can be explained within two dimensional interactions beyond the weak-coupling limit [61].

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