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Optimal control and bifurcation diagram for a model nonlinear fractional SIRC

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Abstract In this article, the optimal control for nonlinear SIRC model is studied in fractional order using the Caputo fractional derivative. Graph signal flow is given of the model and simulated by Simulink/Matlab which helps in discussing the topological structure of the model. Dynamics of the system versus certain parameters are studied via bifurcation diagrams, Lyapunov exponents and Poincare maps. The existence of a uniformly stable solution is proved after control. The obtained results display the activeness and suitability of the Mittag Generalized-Leffler function method (MGLFM). The approximate solution of the fractional order SIRC model using MGLFM is explained by giving the figures of solutions before and after control. Also, we plot the 3D relationships with different alpha (fractional order) which display the originality and suitability of the results.

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1. Preparatory

Studiying the notion of disease discovery and diffusion mathematically was done for the first time in [2,46]. The discovery of diseases and epidemics using mathematical models have got a great attention of many researchers [1–9,18–29,33,35,40–44,49–53].

Mathematical models are considered as a great tool in studying the sawing and controlling of diseases epidemic. Recognition the propagation manner of diseases infectious in society, zones, and nations is very helpful in reducing the propagation of these diseases [18–22].

Several researchers have interested in applying the new notions of derivatives to solve some of the ordinary cases [13]. In [40], via Atangana–Baleanu fractional derivative, the the polluted lakes system’s dynamics were investigated. Also,
the fractal-fractional model was solved utilizing a novel proposed technique. In addition, they proved the existence and uniqueness of their results. In [41], via Atangana–Baaleanu fractional derivative, the authors investigated the dynamics of the competition between rural and commercial banks. The authors of [42] described the mathematical modeling and dynamical behavior of a novel coronavirus (2019-nCoV) in utilizing the fractional derivatives. The authors in [43] presented the dynamics of a fractional SIR model with a generalized incidence rate using two differential derivatives, which are the dynamics of the fractal-fractional model was solved utilizing a novel procedure to present the fractional SIRC system. The authors in [22] found the two equilibrium points and study the system dynamics.

Here, the fractional-order model of the SIRC [18,21] is studied that has the following form:

\[
\begin{align*}
D^q S(t) & = \mu(1 - S) - \beta SI + \gamma C, \\
D^q I(t) & = \beta SI + \sigma CI - (\mu + \theta)I, \\
D^q R(t) & = (1 - \sigma)\beta CI + \theta I - (\mu + \delta)R, \\
D^q C(t) & = \delta R - \beta CI - (\mu + \gamma)C,
\end{align*}
\]

(1-1)

with given initial condition:

\[
S(0) = S_1, I(0) = I_1, R(0) = R_1, C(0) = C_1,
\]

(1-2)

where \( D^q \) is derivative Caputo fractional 0 < \( q \leq 1 \) and following are the used symbols:

| The variables | The concept |
|---------------|-------------|
| \( S(t) \)   | proportions of susceptible |
| \( I(t) \)   | infectious |
| \( R(t) \)   | recovered, |
| \( C(t) \)   | cross-immune, |
| \( N(t) \)   | total population of constant size, |
| \( \theta \)  | average of headway from infective to recovered per year, |
| \( \mu \)    | mortality rate, |
| \( \sigma \)  | average of headway from recovered to susceptible per year, |
| \( \gamma \)  | average of headway from recovered to cross-immune per year, |
| \( \delta \)  | average of strain-averaged into the infective |

0 ≤ \( \sigma \) ≤ 1.

The model (1) with initial conditions must be split up of each other and confirm the law \( N(t) = S(t) + I(t) + R(t) + C(t) \) where \( N \) is the size of the overall population.

The paper is built in 7 sections. We discuss the fractional optimal control for SIRC modeling and we prove the existence of a uniformly stable solution after control in Section 2. In Section 3, a graph of signal flow is proposed for the model which helps in understanding the structure of the system from graph theory point of view. In Section 4, we give the numerical implementation to show the efficiency of using FMGLM. Section 5, we display a simulation of this model using FMGLM before and after control and construct its Simulink simulation scheme. Section 6 represents the dynamics of the system versus certain parameters via bifurcation diagrams, Lyapunov exponents and Poincare maps. We give the conclusions, in Section 7.

2. Fractional optimal control (FOCP) for SIRC model

Here, the fractional optimal control for SIRC system is to be discussed.
\[ D^\alpha S(t) = \mu(1 - S) - \beta SI + v_2C, \]
\[ D^\alpha I(t) = \beta SI + \sigma CI - (\mu + v_1)I, \]
\[ D^\alpha R(t) = (1 - \sigma)\beta CI + v_1I - (\mu + \delta)R, \]
\[ D^\alpha C(t) = \delta R - \beta CI - (\mu + v_2)C. \]  

(2-1)

where \(T_f\) is the final time, \(v_1(\cdot)\) is the average of headway from infective to recovered per year and \(v_2(\cdot)\) average of headway from recovered to susceptible per year. \(v_1(\cdot)\) and \(v_2(\cdot)\) are the control functions. The \(L^\infty\) norm \(\|x\|_\infty = \max |x|\) is a function space. This definition of the \(L^\infty\) norm is equivalent to taking the limit as \(p \to \infty\) of the \(L^p\) norm.

The objective functional is defined as follows (quadratics are the control variables)

\[ J(v_1, v_2) = \int_0^{T_f} \left[ A_1 I(t) + A_2 v_1^2(t) + A_3 v_2^2(t) \right] dt, \]

(2-2)

where \(A_1, A_2\) and \(A_3\) represent the measure of infectious, an average of headway from infective to recovered per year and an average of headway from recovered to susceptible per year respectively.

Find the optimal controls \(v_1(\cdot)\) and \(v_2(\cdot)\) is the main task in FOCPs, in order to minimize the selected fitness function:

\[ J(v_1, v_2) = \int_0^{T_f} \left[ \phi(S, I, R, C, v_1, v_2, t) \right] dt, \]

(2-3)

where \(\phi(S, I, R, C, v_1, v_2, t) = \left[ A_1 I(t) + A_2 v_1^2(t) + A_3 v_2^2(t) \right], \)

subjected to the constraint

\[ D^\alpha S = \xi_1, \quad D^\alpha I = \xi_2, \quad D^\alpha R = \xi_3, \quad D^\alpha C = \xi_4, \]

(2-4)

where \(\xi_i = \xi(S, I, R, C, v_1, v_2, t), \quad i = 1, 2, 3, 4.\)

The following initial conditions are satisfied:

\[ S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0, \quad C(0) = C_0, \]

(2-5)

To design the FOCP, taking into account the new fitness function as \([26-29]:\)

\[ J = \int_0^{T_f} \left[ H(S, I, R, C, v_1, v_2, t) + \sum_{i=1}^{4} \lambda_i \xi_i(S, I, R, C, v_1, v_2, t) \right] dt, \]

(2-6)

where \(i = 1, 2, 3, 4.\)

The Hamiltonian for the objective (cost) functional (2-6) and the control fractional order SIRC model (2-1) is given as follows:

\[ H(S, I, R, C, v_1, v_2, t) = \phi(S, I, R, C, v_1, v_2, t) \]

\[ + \sum_{i=1}^{4} \lambda_i \xi_i(S, I, R, C, v_1, v_2, t), \]

(2-7)

then

\[ H = A_1 I + A_2 v_1^2 + A_3 v_2^2 + \lambda_4[\mu - \mu S - \beta SI + v_2C] \]

\[ + \lambda_3[\beta SI + \sigma CI - (\mu + v_1)I] \]

\[ + \lambda_2[(1 - \sigma)\beta CI + v_1I - (\mu + \delta)R] \]

\[ + \lambda_1[\delta R - \beta CI - (\mu + v_2)C]. \]

(2-8)

From (2-6) and (2-8), the necessary and sufficient conditions of FOPC can be derived (see \([26-29, 36-39]:\) as follows:

\[ D^\alpha \lambda_1 = \frac{\partial H}{\partial S}, \quad D^\alpha \lambda_2 = \frac{\partial H}{\partial I}, \quad D^\alpha \lambda_3 = \frac{\partial H}{\partial R}, \quad D^\alpha \lambda_4 = \frac{\partial H}{\partial C}, \]

(2-9)

\[ \frac{\partial H}{\partial v_k} = 0, \quad k = 1, 2 \Rightarrow \frac{\partial H}{\partial v_1} = 0, \quad \frac{\partial H}{\partial v_2} = 0, \]

(2-10)

\[ D^\alpha S = \frac{\partial H}{\partial \lambda_1}, \quad D^\alpha I = \frac{\partial H}{\partial \lambda_2}, \quad D^\alpha R = \frac{\partial H}{\partial \lambda_3}, \quad D^\alpha C = \frac{\partial H}{\partial \lambda_4}, \]

(2-11)

additionally,

\[ \lambda_i \quad (T_f) = 0, \]

(2-12)

where \(\lambda_i, \quad i = 1, 2, 3, 4\) are the Lagrange multipliers.

Theorem 1. If \(v_1\) and \(v_2\) are optimal controls with corresponding state \(S, I, R, C\) then there exist adjoint variables \(\lambda_i^* \quad i = 1, 2, 3, 4\) satisfies the following:

(i) Co-state equations (adjoint equations)

Applying the conditions in the text theorem and applying Eq. (2.9) see, \([26-29]:\) we get the following four equations that can be written as follows:-

\[ D^\alpha \lambda_1^* = \lambda_2^*[\sigma \beta I] + \lambda_3^*[-\mu - \beta], \]

(2-13)

\[ D^\alpha \lambda_2^* = A_4 + \lambda_1^*[-\beta S] + \lambda_2^*[\beta S + \sigma C] - (\mu + v_1)I \]

\[ + \lambda_3^*[(1 - \sigma)\beta C + v_1I] + \lambda_4^*[-\beta C], \]

(2-14)

\[ D^\alpha \lambda_3^* = \lambda_1^*[v_2] + \lambda_2^*[\sigma \beta I] + \lambda_3^*[(1 - \sigma)\beta I] \]

\[ + \lambda_4^*[-\beta R - (\mu + v_2)], \]

(2-15)

\[ D^\alpha \lambda_4^* \quad (T_f) = 0, \quad i = 1, 2, 3, 4. \]

(2-16)

(ii) Transversality conditions:

\[ (2-17) \]

\[ \lambda_i \quad (T_f) = 0, \quad i = 1, 2, 3, 4. \]

(iii) Optimality conditions

\[ H(S', I', R', C', v_1', v_2', \lambda_i) \]

\[ = \min_{v_1', v_2', \lambda_i} H(S', I', R', C', v_1', v_2', \lambda_i), \]

(2-18)

moreover, by applying Eq. (2.10), the control functions \(v_1', v_2'\) are given as follows:

\[ \frac{\partial H}{\partial v_1} = 0 \Rightarrow v_1 = \frac{\Gamma[\lambda_2^* - \lambda_1^*]}{2A_2}, \]

(2-19)
\begin{align}
\frac{\partial H}{\partial v_2} = 0 & \Rightarrow v_2 = \frac{C_1 v_1^2 - C_2 v_1}{2 A_2}, \quad (2-20) \\
v_i' = \min\left\{1, \max\left\{0, \frac{I v_1 - C_2 v_1}{2 A_2}\right\}\right\}, \quad (2-21) \\
v_2' = \min\left\{1, \max\left\{0, \frac{C_1 v_1^2 - C_2 v_1}{2 A_2}\right\}\right\}. \quad (2-22)
\end{align}

**Proof.** The co-state system Eqs. (2-13)-(2-16) are found from Eq. (2-11) where the Hamiltonian $H'$ is given by

\begin{equation}
H' = A_1 \Gamma + A_2 v_1^2 + A_3 v_2^2 + \frac{1}{2} D^T S' + \frac{1}{2} D^T T + \frac{1}{2} S^T R' + \frac{1}{2} D^T C'. \quad (2-23)
\end{equation}

Further, the condition in Eq. (2-12) also satisfied, and the optimal control written in Eqs. (2-21)-(2-22) can be derived from Eq. (2-10).

Putting $v_i', i = 1, 2$ in (2-1), the following state system can be found as:

\begin{align}
D^T S' (t) &= \mu (1 - S') - \beta S' \Gamma + v_2 C', \\
D^T T (t) &= \beta S' \Gamma + \sigma \beta C' \Gamma - (\mu + v_1) \Gamma, \\
D^T R' (t) &= (1 - \sigma) \beta C' \Gamma + v_i I - (\mu + \delta) R', \\
D^T C' (t) &= \delta R' - \beta C' \Gamma - (\mu + v_2) C'. \quad (2-24)
\end{align}

For more details about fractional optimal control see the Ref. [26–29,36–39,49–51,54,56,58].

2.1. Existence of uniformly stable solution after control:

The existence of the control system (2-13)-(2-16) is to be proved here, the same can be found in [5,7,52] as follows:

Let

\begin{align}
f_1 (x_1, x_2, x_3, x_4) &= \lambda_1 [\mu - \beta \Gamma] + \lambda_2 [\beta \Gamma], \\
f_2 (x_1, x_2, x_3, x_4) &= \lambda_3 [\lambda_2 [-\beta S'] + \lambda_4 [-\beta C'] - (\mu + v_1)] \\
&+ \lambda_3 [1 - \sigma] \beta C' + v_1 + \lambda_3 (-\beta C'), \\
f_3 (x_1, x_2, x_3, x_4) &= \lambda_3 [-\mu - \delta] + \lambda_3 [\delta], \\
f_4 (x_1, x_2, x_3, x_4) &= \lambda_3 [v_2] + \lambda_3 [\sigma \beta \Gamma] + \lambda_3 [1 - \sigma] \beta \Gamma \\
&+ \lambda_3 [-\beta \Gamma - (\mu + v_2)].
\end{align}

Let $\Omega = \{x_i \in R^4: 0 = x_i, r = 1, 2, 3, 4, t \in [0, T]\}.

We have, at $\Omega$:

\begin{align}
\frac{\partial f_1}{\partial x_1} &= -\mu - \beta \Gamma, & \frac{\partial f_1}{\partial x_2} &= \beta \Gamma, & \frac{\partial f_1}{\partial x_3} &= 0, & \frac{\partial f_1}{\partial x_4} &= 0, \\
\frac{\partial f_2}{\partial x_1} &= -\beta S', & \frac{\partial f_2}{\partial x_2} &= \beta S' + \sigma \beta C' - (\mu + v_1), & \frac{\partial f_2}{\partial x_3} &= 1 - \sigma, & \frac{\partial f_2}{\partial x_4} &= -\beta C', \\
\frac{\partial f_3}{\partial x_1} &= 0, & \frac{\partial f_3}{\partial x_2} &= 0, & \frac{\partial f_3}{\partial x_3} &= -\mu - \delta, & \frac{\partial f_3}{\partial x_4} &= \delta, \\
\frac{\partial f_4}{\partial x_1} &= v_2, & \frac{\partial f_4}{\partial x_2} &= \sigma \beta \Gamma, & \frac{\partial f_4}{\partial x_3} &= (1 - \sigma) \beta \Gamma, & \frac{\partial f_4}{\partial x_4} &= -\beta \Gamma - (\mu + v_2).
\end{align}

3. Graph of signal flow

The signal flow between the system’s states is proposed as shown in Fig. 1, the signal flow graph is indicated by $\overrightarrow{G}$ which characterized by a couple of two sets: the vertices set $v(\overrightarrow{G})$ and the edges set $E(\overrightarrow{G})$, in which every vertex corresponds to a state variable and named by its symbol in (1-1). So, The cardinality of vertices set is $|v(\overrightarrow{G})| = 6$. In $\overrightarrow{G}$ (x1, x2) $E(\overrightarrow{G})$ if the state corresponding to vertex x1 affects directly the state corresponding to vertex x2 in model (1–1). The graph of signal flow $\overrightarrow{G}$ can be represented by its adjacency matrix $A(\overrightarrow{G})$.

Then

\begin{align}
A(\overrightarrow{G}) &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\end{align}

The eigenvalues of $A(\overrightarrow{G})$ are calculated as:

![Fig. 1](image-url)

The proposed SIRC model’s graph of signal flow.
Using Eqs. (4-1) and (4-2) in the model (1-1), as follows:

\[
\sum_{n=0}^{\infty} a^{n+1} \frac{e_1^{n+1}}{r^{n+1}} = \mu - \mu \left( \sum_{n=0}^{\infty} a^n \frac{e_1^n}{r^{n+1}} \right) - \beta \left( \sum_{n=0}^{\infty} f_1^{n+1} \right) + \gamma \left( \sum_{n=0}^{\infty} b^n \frac{e_1^n}{r^{n+1}} \right),
\]

\[
\sum_{n=0}^{\infty} b^{n+1} \frac{e_1^{n+1}}{r^{n+1}} = \beta \left( \sum_{n=0}^{\infty} f_1^{n+1} \right) + \delta \beta \left( \sum_{n=0}^{\infty} f_1^n \right) + \delta \left( \sum_{n=0}^{\infty} d^n \frac{e_1^n}{r^{n+1}} \right) - \left( \mu + \beta \right) \left( \sum_{n=0}^{\infty} b^n \frac{e_1^n}{r^{n+1}} \right).
\]

\[
\sum_{n=0}^{\infty} d^{n+1} \frac{e_1^{n+1}}{r^{n+1}} = \left( 1 - \sigma \right) \beta \left( \sum_{n=0}^{\infty} f_1^n \right) + \theta \left( \sum_{n=0}^{\infty} b^n \frac{e_1^n}{r^{n+1}} \right) - \left( \mu + \gamma \right) \left( \sum_{n=0}^{\infty} d^n \frac{e_1^n}{r^{n+1}} \right).
\]

where \( f'_1 = \sum_{n=0}^{\infty} a_n \frac{d^2 e_1^n}{r^{n+2}} \), \( f'_1 = \sum_{n=0}^{\infty} a_n \frac{d^2 e_1^n}{r^{n+2}} \).

After compilation

\[
\sum_{n=0}^{\infty} \left\{ \frac{e_1^{n+1}}{r^{n+2}} + \frac{e_1^n}{r^{n+1}} + \beta f'_1 + \frac{b^n}{r^{n+1}} \right\} = 0,
\]

\[
\sum_{n=0}^{\infty} \left\{ \frac{e_1^{n+1}}{r^{n+2}} + \beta f'_1 - \sigma \beta f'_1 + \frac{\left( \mu + \beta \right) b^n}{r^{n+1}} \right\} = 0,
\]

\[
\sum_{n=0}^{\infty} \left\{ \frac{e_1^{n+1}}{r^{n+2}} + \left( 1 - \sigma \right) \beta f'_1 - \frac{\beta f'_1}{r^{n+1}} + \frac{\left( \mu + \delta \right) d^n}{r^{n+1}} \right\} = 0,
\]

\[
\sum_{n=0}^{\infty} \left\{ \frac{e_1^{n+1}}{r^{n+2}} - \frac{\delta \beta f'_1}{r^{n+1}} + \frac{\left( \mu + \delta \right) d^n}{r^{n+1}} \right\} = 0.
\]

At \( n = 0 \) in Eq. (4-3), we find:

\[
a^1 = \mu - \mu a^0 - \beta f'_1 + \gamma b^0,
\]

\[
b^1 = \beta f'_1 - \sigma \beta f'_1 + \left( \mu + \beta \right) b^0,
\]

\[
d^1 = \left( 1 - \sigma \right) \beta f'_1 + \theta b^0 - \left( \mu + \delta \right) d^0,
\]

\[
e^1 = \delta d^0 - \beta f'_1 - \left( \mu + \gamma \right) e^0.
\]

Then system (4-4) become

\[
\sum_{n=1}^{\infty} \left\{ \frac{e_1^{n+1}}{r^{n+2}} + \frac{e_1^n}{r^{n+1}} + \beta f'_1 + \frac{b^n}{r^{n+1}} \right\} = 0,
\]

\[
\sum_{n=1}^{\infty} \left\{ \frac{e_1^{n+1}}{r^{n+2}} + \beta f'_1 + \frac{\left( \mu + \beta \right) b^n}{r^{n+1}} \right\} = 0,
\]

\[
\sum_{n=1}^{\infty} \left\{ \frac{e_1^{n+1}}{r^{n+2}} + \left( 1 - \sigma \right) \beta f'_1 + \frac{\beta f'_1}{r^{n+1}} + \frac{\left( \mu + \delta \right) d^n}{r^{n+1}} \right\} = 0,
\]

\[
\sum_{n=1}^{\infty} \left\{ \frac{e_1^{n+1}}{r^{n+2}} - \frac{\delta \beta f'_1}{r^{n+1}} + \frac{\left( \mu + \delta \right) d^n}{r^{n+1}} \right\} = 0.
\]

In Eq. (4-5), \( e_1^n \) is not to equal zero so the coefficient that equal zero and we get the recurrence relationship from which we calculate the constants \( a^0 \), \( c^0 \), \( e^0 \), \( q^n \), \( g^n \), \( n = 1, 2, 3, ..., \infty \).

\[
a^{n+1} = -\mu a^n + \beta f'_1 \Gamma (n+1) + \gamma b^n,
\]

\[
b^{n+1} = \beta f'_1 \Gamma (n+1) + \sigma \beta f'_1 \Gamma (n+1) + b^n,
\]

\[
d^{n+1} = \left( 1 - \sigma \right) \beta f'_1 \Gamma (n+1) + \theta \Gamma (n+1) + \delta \Gamma (n+1),
\]

\[
e^{n+1} = \delta d^n - \beta f'_1 \Gamma (n+1) + \gamma e^n.
\]

At \( n = 1 \)
\[ a^2 = \mu a^1 - \beta f^1 \Gamma(a + 1) + \gamma b^1, \]
\[ b^2 = \beta f^1 \Gamma(a + 1) - \beta^2 r^1 \Gamma(a + 1) - (\mu + \theta) b^1, \]
\[ c^2 = (1 - \sigma) \beta f^1 \Gamma(a + 1) + \theta b^1 - (\mu + \delta) d^1, \]
\[ e^2 = \delta d^1 - \beta f^1 \Gamma(a + 1) - (\mu + \gamma) e^1. \]

At the same way, we find \( a^3, b^3, d^3 \) and \( e^3 \)...

Setting (4-6) and (4-7) in (4-2), we have the settling in the infinite series:

\[ S(t) = d^0 + \frac{d^1}{\Gamma(a + 1)} + \frac{d^2}{\Gamma(2a + 1)} + \frac{d^3}{\Gamma(3a + 1)} + \cdots, \]
\[ I(t) = b^0 + \frac{b^1}{\Gamma(a + 1)} + \frac{b^2}{\Gamma(2a + 1)} + \frac{b^3}{\Gamma(3a + 1)} + \cdots, \]
\[ R(t) = d^0 + \frac{d^1}{\Gamma(a + 1)} + \frac{d^2}{\Gamma(2a + 1)} + \frac{d^3}{\Gamma(3a + 1)} + \cdots, \]
\[ C(t) = e^0 + \frac{e^1}{\Gamma(a + 1)} + \frac{e^2}{\Gamma(2a + 1)} + \frac{e^3}{\Gamma(3a + 1)} + \cdots. \]

5. System simulation of fractional-order SIRC model

5.1. Simulation before control

Here, in Fig. 2, the fractional SIRC model’s solution is shown before control at fraction derivative order, \( x = 1 \), \( S(0) = 0.8 \), \( I(0) = 0.1 \), \( R(0) = 0.05 \), and \( C(0) = 0.05 \). Figs. 3–6, show the dynamics of \( S(t), I(t), R(t) \) and \( C(t) \) of the approximate solution of fractional order SIRC model before control with different \( \alpha \). In Fig. 7, the originality and proper of the results are clear in view of the figure using a 3D plot. In Fig. 8, we display a simulation of this model using Simulink. The diagram of Simulink is very important because it shows the dependency on the model states on each other.
Fig. 4  The dynamics of I(t) before control with different $\alpha$.

Fig. 5  The dynamics of R(t) before control with different $\alpha$. 
Fig. 6  The dynamics of $C(t)$ before control with different $\alpha$.

Fig. 7  The various variables’ 3D plots before control.
Fig. 8 Simulation of system by Simulink/MATLAB.
5.2. Simulation after control

Here, in Figs. 9–12, we show the behavior of approximate solution of $S(t)$, $I(t)$, $R(t)$ and $C(t)$ after control at $\alpha = 1$. Figs. 13–16 show the solution of $S(t)$, $I(t)$, $R(t)$ and $C(t)$ after control at $\alpha = 0$. Figs. 17–20 show the solution of $S(t)$, $I(t)$, $R(t)$ and $C(t)$ after control at $\alpha = 0.9$. Figs. 21–23 show the solution of $S(t)$, $I(t)$, $R(t)$ and $C(t)$ after control at $\alpha = 0.8$. From Figs. 21–23, the originality and proper of the results are clear in view of the 3D plot with $\alpha = 1$, 0.9 and 0.8 respectively after control.
Fig. 11  The dynamics of $R(t)$ after control.

Fig. 12  The dynamics of $C(t)$ after control.

Fig. 13  The dynamics of $S(t)$ after control.
Fig. 14  The dynamics of I(t) after control.

Fig. 15  The dynamics of R(t) after control.

Fig. 16  The dynamics of C(t) after control.
Optimal control and bifurcation diagram

**Fig. 17** The dynamics of $S(t)$ after control.

**Fig. 18** The dynamics of $I(t)$ after control.

**Fig. 19** The dynamics of $R(t)$ after control.
Fig. 20  The dynamics of $C(t)$ after control.

Fig. 21  The various variables by the 3D plot after control.

Fig. 22  The various variables' 3D plots after control.
6. Dynamics of the system versus certain parameters

Figs. 24 and 25 represent the bifurcation diagram of $S(t)$ versus the parameter beta. Figs. 26 and 27 represent Lyapunov exponents in two time periods. Figs. 28 and 36 represent the Poincaré map of the system for three different values of alfa (alfa is the parameter $\theta$). All of which ensure the system stability (see Figs. 26–36).

7. Conclusions

In this essay, we study the fractional optimal control for nonlinear SIRC modeling using the Caputo fractional derivative. The model is presented by the graph of signal flow and simulated using Simulink/Matlab. FMGLM is to find the approximate solutions of the model. We discuss the existence of uniformly stable solution after control. We have presented the behavior of approximate solution by giving the figures
Fig. 26  Lyapunov exponents of the studied model for short time.

Fig. 27  Lyapunov exponents of this model for long time.

Fig. 28  Poincare map of this model.
Fig. 29  Poincare map of this model.

Fig. 30  Poincare map of this model.

Fig. 31  Poincare map of this model.
Fig. 32  Poincare map of this model.

Fig. 33  Poincare map of this model.

Fig. 34  Poincare map of this model.
before and after control. Dynamics of the system versus certain parameters are studied via bifurcation diagram, Lyapunov exponents and Poincare maps. The results are consistent with that obtained in [21].

8. Data availability

The information about this research is ready from the authors upon request.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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**References**

[1] A.H. Mokdad, J.S. Marks, D.F. Stroup, J.L. Gerberding, Actual causes of death in the United States, JAMA 291 (2004) 1238–1245.

[2] O.D. Makinde, Adomian decomposition approach to a SIR epidemic model with constant vaccination strategy, Appl. Math. Comput. 184 (2017) 842–848.

[3] F. Haq, K. Shah, G. Rahman, M. Shazad, Numerical solution of fractional order smoking model via laplace Adomian decomposition method, Alexandr. Eng. J. 57 (2) (2018) 1061–1069.

[4] A.M.A. El-Sayed, S.M. Salman, On a discretization process of fractional order Riccati’s differential equation, J. Fract. Calc. Appl. 4 (2013) 251–259.

[5] A.A. Elsadany, A.E. Matouk, Dynamical behaviors of fractional-order Lotka-Volterra predator-prey model and its discretization, Appl. Math. Comput. 49 (2015) 269–283.
M. El-Shahed, J.J. Nieto, A.M. Ahmed, I.M.E. Abdelstar, Fractional-order model for biocontrol of the lesser date moth in palm trees and its discretization, Adv. Diff. Equat. 2017 (2017) 1–16.

H.A.A. El-Saka, The fractional order SIS epidemic model with variable population size, J. Egypt. Math. Soc. 22 (2014) 50–54.

Y.A. Amer, A.M.S. Mahdy, E.S.M. Youssef, Solving systems of fractional differential equations using Sumudu transform method, Asian Res. J. Math. 7 (2) (2017) 1–15.

S.Z. Rida, Y.Gh. Gouda, H.M. Ali, M.M. Farag, Approximate solution for SIR epidemic model by generalized Mittag-Lefﬂer function method, International J. of Advanced Research in Science, Eng. Technol. 5 (4) (2018) 5525–15519.

A.A.M. Arafa, S.Z. Rida, H.M. Ali, Generalized Mittag-Lefﬂer function method for solving Lorenz system, Int. J. Innovat. Appl. Stud. 3 (2013) 105–111.

A.A.M. Arafa, S.Z. Rida, A.A. Mohamadein, H.M. Ali, Solving nonlinear fractional differential equation by generalized Mittag-Lefﬂer function method, Commun. Theor. Phys. J. 59 (2013) 661–663.

S.Z. Rida, A.A.M. Arafa, New method for solving linear fractional differential equations, Int. J. Diff. Eq. 2011 (2011) 1–8.

I. Podlubny, Fractional Differential Equations, Academic Press, London, 1999.

S.Z. Rida, A.S. AbdEl-Rady, A.A.M. Arafa, H.R. AbdEl-Rahim, A Donian decomposition Sumudu transform method for solving fractional nonlinear equations, Math. Sci. Lett. 5 (2016) 39–48.

M.M. Khader, N.H. Sweilam, A.M.S. Mahdy, Two computational algorithms for the numerical solution for system of fractional, Arab J. Math. Sci. 21 (1) (2015) 39–52.

H. Schmid, A. Huber, Analysis of switched-capacitor circuits using driving-point signal-ﬂowgraphs, AnalogIntegr. Circ. Sig. Process 96 (2018) 495–507.

A.M.S. Mahdy, M. Higazy, Numerical different methods for solving the nonlinear Biochemical Reaction model, Int. J. Appl. Comput. Math. 5 (6) (2019) 1–17.

M.M. Khader, N.H. Sweilam, A.M.S. Mahdy, N.K. Abdel, Moniem, Numerical simulation for the fractional SIRC model and Inﬂuenza A, Appl. Math. Inf. Sci. 8 (3) (2014) 1–8.

R. Casagrandi, L. Bolzoni, S.A. Levin, V. Andreassen, The SIRC model and inﬂuenza A, Math. Biosci. 200 (2006) 152–169.

W. Chinviriyasit, Numerical modeling of the transmission dynamics of inﬂuenza, in: The First International Symposium on Optimization and Systems Biology, Beijing, China, 2007, pp. 52–59.

M. El-Shahed, A. Alsaeedi, The fractional SIRC model and inﬂuenza A, Math. Probl. Eng. 2011 (2011) 1–9.

L. Jodar, R.J. Villanueva, A.J. Arenas, G.C. Gonzalez, Nonstandard numerical methods for a mathematical model for inﬂuenza disease, Math. Comput. Simul. 79 (2008) 622–633.

K.A. Gepreel, A.M.S. Mahdy, M.S. Mohamed, A. Al-Amiri, Reduced differential transform method for solving nonlinear Biomathematics models, Comput. Mater. Contin. 61 (3) (2019) 979–994.

J. Singh, D. Kumar, M. Al Qurashi, D. Baleanu, A new fractional model for giving up smoking dynamics, Adv. Diff. Eq. 2017 (2017) 1–16.

M.I.A. Othman, A.M.S. Mahdy, Differential transformation method and variation iteration method for Cauchy reaction-diffusion problems, J. Math. Comput. Sci. 1 (2) (2010) 61–75.

N.H. Sweilam, S.M. Al-Mekhlafi, Optimal control for a nonlinear Mathematical model of tumor under immune suppression a numerical approach, Optim. Control Appl. Meth. 39 (2018) 1581–1596.

N.H. Sweilam, S.M. Al-Mekhlafi, D. Baleanu, Optimal control for a fractional tuberculosis infection model including the impact of diabetes and resistant strains, J. Adv. Res. 17 (2019) 125–137.

N.H. Sweilam, O.M. Saad, D.G. Mohamed, Fractional optimal control in transmission dynamics of West Nile model with state and control time delay a numerical approach, Adv. Diff. Eq. 2019 (210) (2019) 1–25.

N.H. Sweilam, O.M. Saad, D.G. Mohamed, Numerical treatments of the transmission dynamics of West Nile virus and its optimal control, Electronic J. Math. Anal. Appl. 7 (2) (2019) 9–38.

A.M.S. Mahdy, Numerical studies for solving fractional integro-differential equations, J. Ocean Eng. Sci. 3 (2) (2018) 127–132.

A.Y. Amer, A.M.S. Mahdy, E.S.M. Youssef, Solving fractional integro- differential equations by using Sumudu transform method and Hermite Spectral Collocation Method, Comput. Mater. Contin. 54 (2) (2018) 161–180.

M. Higazy, Suborthogonal double covers of the complete bipartite graphs by all Bipartite Subgraphs with five edges over ﬁnite groups, Far East J. Appl. Math. 91 (1) (2015) 63–80.

A.M.S. Mahdy, N.H. Sweilam, M. Higazy, Approximate solutions for solving nonlinear fractional order smoking model, Alexand. Eng. J. 59 (2) (2020) 739–752.

M. Higazy, Orthogonal double covers of circulant graphs by corona product of certain inﬁnite graph classes, Ind. J. Pure Appl. Math. (accepted October 4, 2019).

E.E. Mahmoud, M. Higazy, T.M. Al-Harthi, A new nine-dimensional chaotic Lorenz system with quaternion variables: complicated dynamics, electronic circuit design, anti-anticipating synchronization, and chaotic masking communication application, Mathematics 7 (10) (2019) 1–26.

O.P. Agrawal, On a general formulation for the numerical solution of optimal control problems, Int. J. Control 28 (1–4) (2004) 323–337.

O.P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, J. Math. Anal. Appl. 272 (2002) 368–379.

O.P. Agrawal, A formulation and numerical scheme for fractional optimal control problems, IFAC Proc. 39 (11) (2006) 68–72.

O.P. Agrawal, O. Defertri, D. Baleanu, Fractional optimal control problems with several state and control variables, J. Vib. Control 16 (13) (2010) 1967–1976.

M.M. El-Dessoky, A.M. Khan, Modeling and analysis of the polluted lakes system with various fractional approaches, Chaos, Solit. Fract. 134 (2020) 1–14, https://doi.org/10.1016/j.chaos.2020.109720.

W. Wang, M.A. Khan, Analysis and numerical simulation of fractional model of bank data with fractal-fractional Atangana-Baleanu derivative, J. Comput. Appl. Math. (2019) 1–23, https://doi.org/10.1016/j.cam.2019.112646.

M.A. Khan, A. Atangana, Modeling the dynamics of novel coronavirus (2019-nCov) with fractional derivative, Alexand. Eng. J. (2020) 1–11, https://doi.org/10.1016/j.aej.2020.02.033.

M.A. Khan, M. Ismail, S. Ullah, M. Farhan, Fractional order SIR model with generalized incidence rate, AIMSMath. 5 (3) (2020) 1856–1880.

A. Atangana, M.A. Khan, Fatmawati, Modeling and analysis of competition model of bank data with fractal-fractional Caputo-Fabrizio operator, Alexand. Eng. J. (2020) 1–14, https://doi.org/10.1016/j.aej.2019.12.032.

P. Palese, J.F. Young, Variation of inﬂuenza A, B, and C viruses, Science 215 (4539) (1982) 1468–1474.

R. Anderson, R. May, Infectious Disease of Humans, Dynamics and Control, Oxford University Press, Oxford, UK, 1995.

R.G. Webster, W.J. Bean, O.T. Gorman, T.M. Chambers, Y. Kawaoka, Evolution and ecology of inﬂuenza A viruses, Microbiol. Rev. 56 (1) (1992) 152–179.
[48] W.O. Kermack, A.G. McKendrick, Contributions to the mathematical theory of epidemics, Proc Roy. Soc. London, A 115 (1927) 700–721.

[49] E.F.D. Goufo, C.B. Tabi, On the chaotic pole of attraction for Hindmarsh–Rose neuron dynamics with external current input, Chaos 29 (2019) 1–9, https://doi.org/10.1063/1.5083180.

[50] E.F.D. Goufo, A. Kubeka, Approximation result for non-autonomous and non-local rock fracture models, Japan J. Indust. Appl. Math. 35 (2018) 217–233, https://doi.org/10.1007/s13160-017-0287-3.

[51] E.F.D. Goufo, Evolution equations with a parameter and application to transport-convection differential equations, Turkish J. Math. 41 (2017) 636–654.

[52] A.M.A. El-Sayed, On the existance and stbility of positive solution for a nonlinear fractional-order differential equation and some applications, Alexand. Eng. J. 1 (2010) 1–10.

[53] M.M. Khader, Mohammed M. Babatin, Numerical treatment for solving fractional SIRC model and influenza A, Comput. Appl. Math. 33 (2014) 543–556, https://doi.org/10.1007/s40314-013-0079-6.

[54] K.A. Gepreel, M. Higazy, A.M.S. Mahdy, Optimal control, signal flow graph, and system electronic circuit realization for nonlinear Anopheles Mosquito model, Int. J. Mod. Phys. C (2020) (accepted May, 16, 2020).

[55] A.M.S. Mahdy, Kh. Lotfy, M.H. Ahmed, A. El-Bary, Electromagnetic Hall Current Effect and Fractional Heat order for Microtemperature Photo-Excited Semiconductor Medium with Laser Pulses, Res. Phys. (2020), https://doi.org/10.1016/j.rinp.2020.103161.

[56] A.M.S. Mahdy, A.A.H. Mtawa, Numerical study for the fractional optimal control problem using Sumudu transform method and Picard method, Mitteilung. Klosterneub. 66 (2) (2016) 41–59.

[57] A.M.S. Mahdy, A.S. Mohamed, A.A.H. Mtawa, Implementation of the homotopy perturbation Sumudu transform method for solving Klein-Gordon equation, Appl. Math. 6 (3) (2015) 617–628.

[58] N.H. Sweilam, S.M. Al–Mekhlafi, O.O. Albalawi, Optimal control for a fractional order malaria transmission dynamics mathematical model, Alexand. Eng. J. 59 (2020) 1677–1692, https://doi.org/10.1016/j.aej.2020.04.020.