Classification of $\mathbb{Z}_{p^k}$ orientation preserving actions on surfaces

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Dedicate to Askold Khovanskii on the occasion of his 60th birthday.

Abstract. In this article we classify actions of the groups $\mathbb{Z}_{p^k}$ ($p$ is a prime integer) on compact oriented surfaces.

1 Introduction

Finite abelian group actions on surfaces by autohomeomorphisms constitutes a classical subject, see the articles [N], [E], [J1], [J2], [Na1], [S], [Z], [CN]. In this article we present a direct and complete way to deal with the topological classification of $G$-actions, where $G = \mathbb{Z}_{p^k}^m = \mathbb{Z}_{p^k} \oplus \cdots \oplus \mathbb{Z}_{p^k}$, $p$ is a prime integer and $\mathbb{Z}_{p^k} = \mathbb{Z}/p^k\mathbb{Z}$.

Since $G$ is a finite group, given an action $(\tilde{S}, f)$ of $G$, it is possible to construct an analytic structure on $\tilde{S}$ such that $f(G)$ consists of automorphisms as Riemann surface (see [K]). Hence all the actions considered in this paper appear as automorphism group actions of complex algebraic curves. One of the motivations for our study is the description of the set of connected components in the moduli space $M$ of pairs $(C, G)$, where $C$ is a complex algebraic curve and $G \cong \mathbb{Z}_{p^k}^m$ is an automorphisms group of $C$. According to [Na2] the description of connected components of $M$ is reduced to the description of topological equivalence classes of pairs $(\tilde{S}, K)$, where $K \cong G$ is a group of autohomeomorphisms of $\tilde{S}$. Here we consider that $(\tilde{S}, K)$ and $(\tilde{S}', K')$ are topological equivalent if there exist a homeomorphism $\varphi : \tilde{S} \to \tilde{S}'$ such that

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1
In the context of $K' = \varphi \circ K \circ \varphi^{-1}$. Thus these topological equivalence classes are in one-to-one correspondence with weak equivalence classes for $G$-action on $\tilde{S}$ in sense of [E].

Our method for topological classification of $G$-action is a direct generalization of the method from [CN]. It is based on topological classification of representations $G$ in the group of all autohomeomorphisms of $\tilde{S}$. We prove that classes of strong equivalence, that appear here, one-to-one correspond to symplectic forms on $G = \mathbb{Z}^m_{p^k}$. Letter we prove, that the classes of weak equivalence one-to-one correspond to classes of algebraic equivalence of the symplectic forms on $G = \mathbb{Z}^m_{p^k}$. Using this algebraic description we find a full system of topological invariants, describing the classes of weak equivalence. This gives very simple and natural description for the classes of topological equivalence of the actions.

The classification of the actions of $\mathbb{Z}^m_{p^k}$ allow to obtain strong and weak classifications for actions by groups $\bigoplus_{p_i}^{p_k} \mathbb{Z}_{p_i}^{m_i}$, where $p_i$ are prime numbers and $p_i \neq p_j$ for $i \neq j$. Unfortunately, our methods do not work for $p_i = p_j$. This make very involved the problem of full classification of abelian actions.

Nevertheless, weak equivalence classes for fixed points free actions by $G = \mathbb{Z}^m_{p^2} \bigoplus \mathbb{Z}_p^m$ was found by Vinberg [V], using some another method. His method gives also a simple direct formula for the number of weakly equivalence classes for fixed points free actions by $\mathbb{Z}^m_{p^k}$.

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2 Algebraic preliminaries.

Let us consider the standard lattice $\mathbb{Z}^{2g} = \mathbb{Z} \oplus \mathbb{Z}^{2g} \oplus \mathbb{Z}$ with the standard basis $(e_i) = ((0, \ldots, 1^g, \ldots, 0))$. We define the skew symmetric bilinear form $(.,.) : \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \rightarrow \mathbb{Z}$, by $(e_i, e_j) = \delta_{i+j,2g+1}$ for $i < j$.

We consider also the group $\mathbb{Z}^{2g}_{p^k} = \mathbb{Z}_{p^k} \oplus \mathbb{Z}^{2g}_{p^k} \oplus \mathbb{Z}_{p^k}$ where $p$ is a prime and $\mathbb{Z}_{p^k} = \mathbb{Z}/p^k\mathbb{Z} = \{0, \overline{1}, \ldots, p^k-1\}$. Let $\varphi : \mathbb{Z}^{2g} \rightarrow \mathbb{Z}^{2g}_{p^k}$ be the natural projection defined by $\varphi(e_i) = \overline{e_i}$, where $\overline{e_i} = (0, \ldots, \overline{1}^{(i)}, \ldots, 0)$. Then we have a skew symmetric bilinear form $(.,.)_{p^k} : \mathbb{Z}^{2g}_{p^k} \times \mathbb{Z}^{2g}_{p^k} \rightarrow \mathbb{Z}_{p^k}$ defined by $(\overline{e_i}, \overline{e_j}) = (\varphi(e_i), \varphi(e_j))_{p^k} = (e_i, e_j) \mod p^k$. We shall denote by $\mathbb{Z}^{2g}_{p^k}$ the group $\mathbb{Z}^{2g}_{p^k}$ provided with the bilinear form $(.,.)_{p^k}$.

Let $Sp(2g, \mathbb{Z})$ and $Sp_{p^k}(2g, \mathbb{Z}_{p^k})$ be the subgroups of the automorphisms groups of $\mathbb{Z}^{2g}$ and $\mathbb{Z}^{2g}_{p^k}$ that preserve the bilinear forms $(.,.)$ and $(.,.)_{p^k}$ re-
spectively. The natural projection \( \varphi : \mathbb{Z}^{2g} \to \mathbb{Z}^{2g}_{p^k} \) induces a homomorphism \( \varphi_* : \text{Sp}(2g, \mathbb{Z}) \to \text{Sp}_{p^k}(2g, \mathbb{Z}_{p^k}) \) such that \( \varphi_*(f) \circ \varphi = \varphi \circ f \) for all \( f \in \text{Sp}(2g, \mathbb{Z}) \).

The following result is known:

**Lemma 1** (See Lemma 4.1, pg 177 of [E]) \( \varphi_* (\text{Sp}(2g, \mathbb{Z})) = \text{Sp}_{p^k}(2g, \mathbb{Z}_{p^k}) \).

We shall also need the following result, proving in [E]

**Theorem 2** Let \( G, G' \) be isomorphic to \( \mathbb{Z}^m_{p^k} \) subgroups of \( \hat{\mathbb{Z}}^{2g}_{p^k} \) and \( \psi : G \to G' \) be an isomorphism such that \( (\psi(a), \psi(b))_{p^k} = (a, b)_{p^k} \) for all \( a, b \in G \). Then there is an automorphism \( \tilde{\psi} \in \text{Sp}_{p^k}(2g, \mathbb{Z}_{p^k}) \), such that \( \psi \) is the restriction \( \tilde{\psi} \) to \( G \).

### 3 Strong classification of fixed point free actions

Let \( G \) be a finite group and \( \tilde{S} \) be a closed (compact without boundary) oriented surface. An (orientation preserving) action of the group \( G \) on \( \tilde{S} \) is a pair \( (\tilde{S}, f) \), where \( f \) is a monomorphism of \( G \) in the group of orientation preserving autohomeomorphisms of \( \tilde{S} \). Now we consider \( G = \mathbb{Z}^m_{p^k} \).

**Definition 3** (Strong equivalence). Two actions \( (\tilde{S}, f) \) and \( (\tilde{S}', f') \) are called strongly equivalent if there is a homeomorphism \( \tilde{\psi} : \tilde{S} \to \tilde{S}' \) sending the orientation of \( \tilde{S} \) to the orientation of \( \tilde{S}' \) and such that \( f'(h) = \tilde{\psi} \circ f(h) \circ \tilde{\psi}^{-1} \), for all \( h \in G \).

We denote by \( S = \tilde{S}/f(G) \) and by \( \varphi = \varphi(f) : \tilde{S} \to S \) the natural projection. In this and next section we shall consider the case when \( f(h) \) has no fixed points for any \( h \in G - \{\text{id}\} \), i.e. the action of \( (\tilde{S}, f) \) is fixed point free. Then the projection \( \varphi(f) : \tilde{S} \to S \) is an unbranched covering with deck transformation group \( f(G) \).

Let us consider \( \pi_1(S) \) as the group of deck transformations of the universal covering of \( S \). Then we have:

\[
\omega(\tilde{S}, f) : \pi_1(S) \to \pi_1(S)/\pi_1(\tilde{S}) = f(G)^{-1} \to G.
\]

The resulting epimorphism \( \omega(\tilde{S}, f) : \pi_1(S) \to G \) is the monodromy epimorphism of the covering \( \varphi(f) : \tilde{S} \to S \). The epimorphism \( \omega(\tilde{S}, f) : \pi_1(S) \to G \), induces the epimorphism \( \theta(\tilde{S}, f) : H_1(S, \mathbb{Z}_{p^k}) \to G \), since \( G \) is abelian.
Conversely, given an epimorphism \( \theta : H_1(S, \mathbb{Z}_{p^k}) \to G \), there is an action \((\widetilde{S}, f)\) such that \( \theta = \theta(\widetilde{S}, f) \). To obtain \( \widetilde{S} \) it is enough to consider the monodromy \( \omega : \pi_1(S) \to H_1(S, \mathbb{Z}_{p^k}) \) defined by \( \theta \) and then \( \widetilde{S} = U/\ker \omega \), where \( U \) is the universal covering of \( S \) and the action of \( G \) is given by \( G = \pi_1(S)/\ker \omega \).

**Definition 4** Let \( S \) and \( S' \) be two surfaces. Two epimorphisms \( \theta : H_1(S, \mathbb{Z}_{p^k}) \to G \) and \( \theta' : H_1(S', \mathbb{Z}_{p^k}) \to G \) are called strongly equivalent if there is an orientation preserving homeomorphism \( \psi : S \to S' \) inducing an isomorphism \( \psi_{p^k} : H_1(S, \mathbb{Z}_{p^k}) \to H_1(S', \mathbb{Z}_{p^k}) \) such that \( \theta = \theta' \circ \psi_{p^k} \).

Using covering space theory we have that:

**Theorem 5** Two actions \((\widetilde{S}, f)\) and \((\widetilde{S'}, f')\) are strongly equivalent if and only if the epimorphisms \( \theta(\widetilde{S}, f) \) and \( \theta(\widetilde{S'}, f') \) are strongly equivalent.

**Definition 6** Let \((\widetilde{S}, f)\) be an action of \( G \), \( S = \widetilde{S}/f(G) \), and \( \theta = \theta(\widetilde{S}, f) : H_1(S, \mathbb{Z}_{p^k}) \to G \) be the epimorphism defined by the action \((\widetilde{S}, f)\). Let us consider the spaces of homomorphisms \( G^* = \{ e : G \to \mathbb{Z}_{p^k} \} \) and \( H^1(S, \mathbb{Z}_{p^k}) = \{ e : H_1(S, \mathbb{Z}_{p^k}) \to \mathbb{Z}_{p^k} \} \). Then \( \theta \) generates a monomorphism \( \theta^* = \theta^*(\widetilde{S}, f) : G^* \to H^1(S, \mathbb{Z}_{p^k}) \). The intersection form \( (.,.)_{p^k} = (.,.)^S_{p^k} \) on \( H_1(S, \mathbb{Z}_{p^k}) \) induces an isomorphism \( i : H^1(S, \mathbb{Z}_{p^k}) \to H_1(S, \mathbb{Z}_{p^k}) \) defined by \( (a,) \to a \) and a form \( (.,.) : G^* \times G^* \to \mathbb{Z}_{p^k} \) such that \( (a, b)_{(\widetilde{S}, f)} = (i \circ \theta^*(a), i \circ \theta^*(b))_{p^k} \).

**Theorem 7** Two fixed point free actions \((\widetilde{S}, f)\) and \((\widetilde{S'}, f')\) of the group \( G \) are strongly equivalent if and only if \( \widetilde{S} \) and \( \widetilde{S'} \) have the same genus and \( (.,.)_{(\widetilde{S}, f)} = (.,.)_{(\widetilde{S'}, f')} \).

The proof is a direct adaptation to our case of the proof of Theorem 8 in [CN]. It is necessary only to use Lemma 1 and Theorem 2 in the above Section instead of Theorem 1 and Theorem 3 from [CN].

**Theorem 8** Let \( (.,.) : G \times G \to \mathbb{Z}_{p^k} \) be a skew symmetric bilinear form. Then there exists an action \((\widetilde{S}, f)\) such that \( (.,.) = (.,.)_{(\widetilde{S}, f)} \) and the genus of \( \widetilde{S}/f(G) \) is \( g \) if and only if there exists an monomorphism \( \phi : G \to \mathbb{Z}_{p^k}^{2g} \) such that \( (a, b) = (\phi(a), \phi(b))_{p^k} \).

**Proof.** Let \((\widetilde{S}, f)\) be an action such that \( (.,.) = (.,.)_{(\widetilde{S}, f)} \). Then it is obvious by the construction of \( (.,.)_{(\widetilde{S}, f)} \), that there exists the \( \phi : G \to \mathbb{Z}_{p^k}^{2g} \). To construct the action from the form and the numerical conditions, consider a basis \( \{ x_i, (i = 1, ..., m) \} \) of \( G \) and a surface \( S \) of genus \( g \). Then \( H_1(S, \mathbb{Z}_{p^k}) \) has a basis \( \{ \chi_i \} (i = 1, ..., m) \).
1,...,m), ν, (i = m + 1,...,2g)\}, such that (χ_1,χ_j) = (x_i,x_j). We define now the epimorphism \( \theta : H_1(S,\mathbb{Z}_p) \to G \), by \( \theta(\chi_i) = x_i \), if \( i \leq m \), \( \theta(\nu_i) = 0 \). Then the epimorphism \( \theta \) defines a homomorphism \( H_1(S,\mathbb{Z}) \to H_1(S,\mathbb{Z}_p) \to G \) and thus a regular covering \( \tilde{S} \to S \) with automorphism group \( G \). The action of \( G \) on \( \tilde{S} \) satisfies \( (\cdot,\cdot)_{(\tilde{S},f)} = (\cdot,\cdot) \).

If \( G = \bigoplus_{i=1}^{r} \mathbb{Z}_{p_i}^{m_i} \), where \( p_i \) are prime and \( p_i \neq p_j \) for \( i \neq j \), it is possible to generalize our study to this more general situation. We sketch some of the steps:

Let \( (\tilde{S},f) \) be an action of \( G = \bigoplus_{i=1}^{r} \mathbb{Z}_{p_i}^{m_i} \), where \( p_i \) are prime and \( p_i \neq p_j \) for \( i \neq j \). Put \( n = p_1^{k_1} \cdots p_r^{k_r} \). Put \( S = \tilde{S}/f(G) \), and \( \theta = \theta(\tilde{S},f) : H_1(S,\mathbb{Z}_n) \to G \) be the epimorphism defined by the action \( (\tilde{S},f) \). Let us consider the spaces of homomorphisms \( G^* = \{ e : G \to \mathbb{Z}_n \} \) and \( H^1(S,\mathbb{Z}_n) = \{ e : H_1(S,\mathbb{Z}_n) \to \mathbb{Z}_n \} \). Then \( \theta \) generates a monomorphism \( \theta^* = \theta^*(\tilde{S},f) : G^* \to H^1(S,\mathbb{Z}_n) \). For each \( i \in \{1,...,r\} \), the intersection form \( (\cdot,\cdot)_{(\tilde{S},f)}^i \) on \( H_1(S,\mathbb{Z}_n) \) induces a form \( (\cdot,\cdot)_{(\tilde{S},f)}^{(i)} : \mathbb{Z}_{p_i}^{m_i} \times \mathbb{Z}_{p_i}^{m_i} \to \mathbb{Z}_{p_i}^{m_i} \). The set of forms \( \{(\cdot,\cdot)_{(\tilde{S},f)}^{(i)}\} \) and the genus of the surface \( \tilde{S} \) gives now the complete set of invariants for the strong classification.

4 Weak classification of fixed point free actions

Definition 9 (Weak equivalence) Let \( (\tilde{S},f) \) and \( (\tilde{S}',f') \) be two actions of a group \( G \). We shall say that \( (\tilde{S},f) \) and \( (\tilde{S}',f') \) are weakly equivalent if there is a homeomorphism \( \tilde{\psi} : \tilde{S} \to \tilde{S}' \) and an automorphism \( \alpha \in \text{Aut}(G) \) such that \( f' \circ \alpha(h) = \tilde{\psi} \circ f(h) \circ \tilde{\psi}^{-1} \), \( h \in G \).

The map \( v \mapsto p^{k-1}v \) generate the natural epimorphism to the vector space \( \mu : \mathbb{Z}_{p}^{m} \to \mathbb{Z}_{p}^{m} \). For an action \( (\tilde{S},f) \) let us put \( q(\tilde{S},f) = (q_1(\tilde{S},f),...,q_k(\tilde{S},f)) \), where \( q_i(\tilde{S},f) \) is the dimension of the vector space \( \mu(\{ h \in G^* | (h,G^*)_{(\tilde{S},f)} \equiv 0 \mod p^i \}) \subset \mu(G^*) \approx \mathbb{Z}_{p}^{m} \). It is follow from [B, Ch IX,5,Theorem 1], that \( 0 \leq q_k \leq ... \leq q_1 \leq m \) and \( q_i \equiv m \mod 2 \).

The next Theorem solves the problem of weak classification of actions of \( \mathbb{Z}_{p}^{m} \) on surfaces:

Theorem 10 Let \( (\tilde{S},f) \) and \( (\tilde{S}',f') \) be two fixed point free actions of a group \( G = \mathbb{Z}_{p}^{m} \). Then the actions \( (\tilde{S},f) \) and \( (\tilde{S}',f') \) are weakly equivalent if and only if \( \tilde{S} \) and \( \tilde{S}' \) have the same genus and \( q(\tilde{S},f) = q(\tilde{S}',f') \).
Proof. Let \( S = \tilde{S}/f(G) \) and \( S' = \tilde{S}'/f'(G) \) have the same genus \( g \) and \( q(\tilde{S}, f) = q(\tilde{S}', f') \). Let \( \theta^*(\tilde{S}, f) \) and \( \theta^*(\tilde{S}', f') \) be the epimorphisms defined by the two actions. Consider the image \( \tilde{G} \) of \( G^* \) in \( H_1(S, \mathbb{Z}_{p^k}) \) by \( \theta^*(\tilde{S}, f) \) and the image \( \tilde{G}' \) of \( G^* \) in \( H_1(S', \mathbb{Z}_{p^k}) \) by \( \theta^*(\tilde{S}', f') \). Then there exists an isomorphism \( \psi : \tilde{G}' \to \tilde{G} \) such that \( (\psi(a), \psi(b))_{(\tilde{S}', f')} = (a, b)_{(\tilde{S}, f)} \). It follows from Theorem 2, Lemma 1 and [B, Ch IX, 5, Theorem 1], that there exists an isomorphism \( \tilde{\psi} : H^1(S', \mathbb{Z}) \to H^1(S, \mathbb{Z}) \) giving by restriction \( \psi \) and sending the intersection form of \( H^1(S', \mathbb{Z}) \) to the intersection form of \( H^1(S, \mathbb{Z}) \).

Let \( (\cdot, \cdot)_{(\tilde{S}, f)} \) and \( (\cdot, \cdot)_{(\tilde{S}', f')} \) be the skew symmetric bilinear forms induced by the two actions. Assume that \( (\cdot, \cdot)_{(\tilde{S}, f)} \) and \( (\cdot, \cdot)_{(\tilde{S}', f')} \) have the same signature then there exists an isomorphism \( \psi : \tilde{G}' \to \tilde{G} \) such that \( (\psi(a), \psi(b))_{(\tilde{S}', f')} = (a, b)_{(\tilde{S}, f)} \). By [MKS, page 178], there exists a homeomorphism \( \varphi : S \to S' \) inducing \( \tilde{\psi} \) on cohomology. Then by theorem 7, the actions \( (\tilde{S}, f) \) and \( (\tilde{S}, \varphi^{-1} \circ f' \circ \varphi) \) are strongly equivalent. The isomorphism \( \tilde{\psi} \), defines an automorphism of \( G \) giving the weak equivalence between \( (\tilde{S}, f) \) and \( (\tilde{S}', f') \). \( \blacksquare \)

Using a modification of the method from [CN, Theorem 9] we can construct a fixed point free actions \( (\tilde{S}, f) \) with \( q(\tilde{S}, f) = q \) for any set of integer numbers \( q = (q_1, \ldots, q_k) \), such that \( 0 \leq q_k \leq \ldots \leq q_1 \leq m \) and \( q_i \equiv m \mod 2 \). Moreover, an action \( (\tilde{S}, f) \) with these invariants exists if and only if the genus of the surface \( \tilde{S} \) is not less that \( p^k m (\frac{1}{2} (m + q_1) - 1) + 1 \). This gives a full description for the set of weakly equivalence classes.

As in the above section it is possible to extend the results of this section to the case \( G = \bigoplus_{j=1}^r \mathbb{Z}_{p_j}^{m_{j}} \), where \( p_i \) are prime and \( p_i \neq p_j \) for \( i \neq j \). In this case the invariant \( q(\tilde{S}, f) \) must be substituted to the set of invariants \( q_i^j(\tilde{S}, f) \), corresponding to the \( \mathbb{Z}_{p_j}^{m_{j}} \) actions.

5 Classification of actions with elements having fixed points.

Consider now an arbitrary effective action \( (\tilde{S}, f) \) of \( G = \mathbb{Z}_{p^k}^{m} \). Let \( B \subset S = \tilde{S}/f(G) \) be the set of all critical values of the natural projection \( \varphi : \tilde{S} \to S \). The construction from section 3 define now the epimorphism \( \theta(\tilde{S}, f) : H_1(S - B, \mathbb{Z}_{p^k}) \to G \).

The images by \( \theta \) of elements from \( H_1(S - B, \mathbb{Z}_{p^k}) \), surrounding points \( b \in B \) generate the subgroup \( G_{fix} \subset G \). This group contains all \( h \in G \), such that \( f(h) \) has fixed points, and it is generated by such \( h' \)s. Consider the function \( l = l(\tilde{S}, f) : G \to \mathbb{Z}_{\geq 0} = \{ n \in \mathbb{Z} | n \geq 0 \} \), where \( l(h) \) is the number of elements...
\( b \in B \), such that \( \theta(b) = h \). The functions with this properties will be called characteristic functions of \((\tilde{S}, f)\). It is obvious that \( \sum_{h \in G} l(h)h = 0 \). Two functions \( l, l' : G \to \mathbb{Z}_{\ge 0} \) we call equivalent if there exists a automorphism \( \alpha \in \text{Aut}(G) \) such that \( l'(h) = l(\alpha(h)) \).

Assume now \( G_{\text{free}} = G/G_{\text{fix}} \cong \mathbb{Z}_{p^k}^m \) and we consider the epimorphism \( \vartheta : H_1(S, \mathbb{Z}_{p^k}) \to G_{\text{free}} \), defined by \( H_1(S, \mathbb{Z}_{p^k}) \to H_1(S - B, \mathbb{Z}_{p^k})/G_{\text{fix}} \to G/G_{\text{fix}} = G_{\text{free}} \). In fact, the epimorphism \( \vartheta \) defined by fixed point-free action of \( G_{\text{free}} \) on \( \tilde{S}/G_{\text{fix}} \). Thus the construction from section 3 and 4 gives the skew symmetric bilinear form \((\cdot, \cdot)_{(\tilde{S}, f)} : G_{\text{free}}^* \times G_{\text{free}}^* \to \mathbb{Z}_{p^k} \) and the set of numbers \( q(\tilde{S}, f) \) for this form.

Using the same arguments that in Theorem 13,14,15 of [CN] we prove.

**Theorem 11** Two actions \((\tilde{S}, f)\) and \((\tilde{S}', f')\) of the group \( G = \mathbb{Z}_{p^k}^m \) with \( G_{\text{free}} \cong \mathbb{Z}_{p^k}^m \) are strongly equivalent if and only if \( \tilde{S} \) and \( \tilde{S}' \) have the same genus, \( l(\tilde{S}, f) = l(\tilde{S}', f') \) and \((\cdot, \cdot)_{(\tilde{S}, f)} = (\cdot, \cdot)_{(\tilde{S}', f')} \).

**Theorem 12** Two actions \((\tilde{S}, f)\) and \((\tilde{S}', f')\) of the group \( G = \mathbb{Z}_{p^k}^m \) with \( G_{\text{free}} \cong \mathbb{Z}_{p^k}^m \) are strongly equivalent if and only if \( \tilde{S} \) and \( \tilde{S}' \) have the same genus, \( l(\tilde{S}, f) = l(\tilde{S}', f') \) and \( q(\tilde{S}, f) = q(\tilde{S}', f') \).

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