Adaptive multiresolution for wavelet analysis

R Sturani\(^1,2\) and R Terenzi\(^3,4\)

\(^1\) Département de Physique Théorique, Université de Genève, Genève, Switzerland
\(^2\) INFN, Presidenza dell’INFN, Roma, Italy
\(^3\) INFN, Università di Roma “Tor Vergata”, Roma, Italy
\(^4\) INAF, Istituto Fisica Spazio Interplanetario, Roma, Italy

E-mail: riccardo.sturani@physics.unige.ch, roberto.terenzi@cern.ch

Abstract. We present a new wavelet packet decomposition method meant to gravitational wave detection. An issue in wavelet analysis is the choice of time-frequency resolution in order to best represent and reduce data, while in quest of a signal of unknown shape like a burst. In other wavelet methods currently employed, like the LIGO WaveBurst for instance, the analysis is performed at some trial resolutions. We propose a decomposition which automatically selects the best resolution at any frequency. The core of this best resolution selection criterion is the minimization of a function on the wavelet packet coefficients domain, named entropy, like the entropy function used in information theory. As a qualitative application we show how a multiresolution time-frequency scalogram looks like for a sample signal injected over Gaussian noise. As a quantitative application of the method, we use the wavelet packet decomposition to perform a non-linear filter of the data by rejecting the wavelet coefficients under a given threshold, analogously to what is done by the WaveBurst algorithm. Taking coincidences between the events obtained by WaveBurst on the original data and on the data filtered after the wavelet packet decomposition, the false alarm rate has been lowered with negligible effects on the efficiency.

1. Introduction

It is predicted that gravitational waves are emitted by a variety of cosmological and astrophysical sources and several detectors are now operating, or soon will, to directly observe a gravitational signal. Among all possible kind of signals, bursts are generically expected to be produced by compact astrophysical sources, like supernova explosions or compact object merging, as well as by high energy phenomena, like gamma rays bursts or more exotically by cosmic string cusps and kinks (see e.g. [1] and references therein). Not all of these processes are sufficiently well understood to predict a specific gravitational waveform. We use here “burst” generically, to designate a not better specified transient signals, with short duration, typically less than a second, and unknown shape. The identification of bursts, possibly drowned in the detector noise, is then an important goal in gravitational wave data analysis. Solid detection algorithms are required to generate trigger of events out of the time data stretch coming out of a detector.

Time-frequency decompositions for analysis of time series have been used since last decade in other research fields than physics, see e.g. [10] for an application on pattern recognition and [4] for an application to physiological time series. More recently they have been applied to the gravitational wave detector data analysis, see e.g. [2, 3]. In particular the WaveBurst algorithm [5, 6] has been employed in the the LIGO burst search, see also [7] for a wavelet method based on a different statistics than WaveBurst.
Here we focus our attention on the issue of selection of the time-frequency resolution. Since the signal one is looking for is unknown, it is a priori not clear what will be the resolution enabling to gather the power of the signal in the smallest possible number of wavelet coefficients, and consequently to better single it out from the noise. We then implemented a method to automatically recognize from the data themselves what are the best resolutions allowing to concentrate the power in the data in the least possible number of wavelet coefficients. That is obtained by considering an *entropy* function, defined so that it is maximum when the power is equally distributed among all the coefficients and minimum when all the power is concentrated in only one coefficient.

The plan of the paper is as follows. In Sec. 2 we explain how we applied the adaptive multiresolution method, by making use of wavelet packet decompositions, showing a case study of a signal injected over Gaussian noise and how it is represented in the wavelet decomposition with varying time-frequency resolution. In Sec. 3 we use this method as a filter, showing that in combination with the WaveBurst algorithm it can be used to lower the false alarm rate, leaving almost unaffected the detection efficiency of the WaveBurst trigger generator. We summarize our conclusion in Sec. 4.

2. The wavelet decomposition and the entropy criterion

We use wavelet packet bases to transform a set of data representing a time series into time-frequency domain. In such a time-frequency plane, orthonormal wavelet packet bases have the peculiarity of dividing the frequency axis in separate interval of various sizes (see e.g. [8]).

The output of a wavelet packet transform can be represented by a *binary wavelet packet tree* (or simply *binary tree* from here on) where each node, labeled $W_j^i$, is an orthogonal vector subspace at level $j$ and at layer $i$. Each of these levels will contain exactly the same information of the original time series, so that complete reconstruction of the original signal is possible from any level.

For instance, if the time resolution of the original series, made of $2^n$ samples, is $\Delta t_0$, information can be obtained from zero frequency up to a maximal frequency $f_{\text{max}} = (2\Delta t_0)^{-1}$ (we restrict here for simplicity to the case in which the number of sample is a power of two, but the method is completely general). Each level contains $2^n$ coefficients, arranged in $2^j$ layers, each of which has $2^{n-j}$ coefficients. At level $j$ the time resolution is $\Delta t_j = 2^j \Delta t_0$ and the frequency resolution is $\Delta f_j = 2^{-j} f_{\text{max}}$. The coefficients in the $i^{th}$ layer refer to the frequency bin characterized by $2^{-j} f_{\text{max}} < f < (i+1)2^{-j}$. Only a subset of these subspaces are needed to completely reconstruct the original time-signal: it can be shown (see again [8]) that the set of *leaves* of every *admissible tree* completely represents the original signal in the wavelets domain, where an admissible tree is a sub-tree of the original binary decomposition tree where every node has exactly 0 or 2 children nodes (see e.g. fig. 1, where the leaves of an admissible tree are marked by red circles).

Having such a redundancy, how to choose the set of $W_j^i$ to represent the signal? Here we present a way to choose the $W_j^i$ based on an *entropy* function $E^2(W_j^i)$, where $W_j^i$ is the set of the coefficients of the base $W_j^i$ of a node in the decomposition binary tree $T$, and defined as follows

$$E^2(W_j^i) = - \sum_k x_k^2 \log_2 \frac{x_k^2}{\|X\|^2},$$

where the $x_k \in W_j^i$ are the wavelet packet transform coefficients of a $W_j^i$ node and

$$\|X\|^2 = \sum_i x_i^2.$$
From the complete decomposition binary tree $T$ of a wavelet packet transform, we select the admissible tree $A_E \subset T$ whose leaves $W^j_i \in A$ have the minimum cost, i.e. where $E^2(W^j_i)$ is minimum for all the children nodes of $W^j_i$ in the original wavelet packet transform binary tree $T$ (entropy criterion [9], where a different notation is used).

From among all the admissible binary trees, $A_E$ is the one that represents the signal most efficiently. By “efficient” we mean that a signal can be represented by a small number of wavelet coefficients, that is, the basis for the decomposition is chosen such that the weight of the coefficients is concentrated on a small number of wavelet packet bases and a large number of coefficients are close to zero. In fact from eq.(1) we can see that if in a particular basis the decomposition produces all zero coefficients except one (i.e., the signal coincides with a wavelet packet transform wave form), then the entropy reaches its minimum value of zero. On the other hand, if in some basis the decomposition coefficients are all equally important, say $x_k = 1/N$ where $N$ is the length of the data, the entropy in this case is maximum, $\log_2 N$. Any other decomposition will fall in between these two extreme cases. In general, the smaller the entropy the fewer significant coefficients are needed to represent the signal.

As an illustration we show in figs. 2 and 3 two time-frequency decompositions, or scalograms. In fig. 3 we report an example of multiresolution scalogram, where we consider a time series with a sampling time of 0.2msec, and then performed a wavelet transform up to level 9 using our adaptive multiresolution algorithm which automatically detects the resolution which is better capable to “concentrate” the signal. As a comparison in fig. 2 we show the same data in a time-frequency analysis done at decomposition level 9.

![Figure 1. Schematic example of a binary tree for a $l = 3$ wavelet decomposition. The layers in each level are explicitly labeled for $l < 3$. The red circles show an example of a possible choice for wavelet packet basis. In this examples it is made by $W^0_0, W^2_1, W^3_2, W^1_3$. The central part of a 12.8sec-long periodogram has been shown.](image)

### 3. Test of the detection algorithm

Within the context of wavelet transforms, several detection procedures has been developed (see e.g. [5, 6] and [7]). Here we use the WaveBurst method as in [5] and we show a possible application of the wavelet packet analysis to improve the method. The WaveBurst algorithm is a trigger event generator, it identifies candidate events by looking for power excess in a wavelet decomposition and collecting clusters of higher coefficients, after performing the analysis at different time resolutions.

We suggest, as a possible application of our entropy multiresolution method, that a time series
Figure 2. Scalogram in the time-frequency plane. The diagram shows white noise data with an injection $h(t) = Ae^{-t^2/(2\Delta^2)} \sin(2\pi f)$, with $t_0 = 6.4$ sec, $\Delta = 0.06$ sec, $f = 930$ Hz and $A \simeq 0.76$. The wavelet transform is made on a Symlet basis with decomposition level equal to 8. Only the central half of the periodogram used in the analysis is shown. The absolute value of the wavelet coefficients is drawn.

Figure 3. Scalogram in the time-frequency plane as in fig. 2, but from entropy multiresolution method.

can be filtered by looking for excess power not in the wavelet decomposition with some given resolution (i.e. by choosing a basis belonging to the same decomposition level), but in the wavelet packet decomposition which makes use of the entropy method, see fig. 1.

After having zeroed the coefficients under a given threshold, one can reconstruct from
Figure 4. ROC curves for WaveBurst method applied on a time stretch of white data with sine Gaussian injections. The blue squares refer to the straight WaveBurst algorithm, the red circles to the coincidences between the events found by WaveBurst on the original data and on the data filtered after the adaptive multiresolution decomposition. On the ROC curves the percentage of coefficients which are zeroed varies.

the surviving wavelet packet coefficients a new, filtered, time series which can be fed to the WaveBurst algorithm. The original and the filtered data are both time series and the WaveBurst algorithm can be run on both of them to look for coincidence signals. We expect that if a signal is present in the original series, it should be stable under this analysis. So we can check the coincidences between the event found by WaveBurst on the original time series and on the one filtered as specified in the above lines.

The best wavelet packet base, or equivalently the choice of the binary tree that minimize the entropy, will depend on the strongest signal present in the stretch of data (or periodogram) under consideration. An independent choice of a binary tree can be made for each periodogram allowing an adaptive selection of the time-frequency resolution.

In particular our method could be usefully run on a stretch of data in which an event is known to be present (if e.g. WaveBurst has already detected a trigger in such data) to determine the best time-frequency resolution to represent that event.

To make a quantitative check we simulated one hour of white noise data, to which we added 71 injections separated by 50 seconds. The injections had the temporal profile

\[ h(t) = A \exp \left( -\frac{t^2}{2\Delta t^2} \right) \sin(2\pi ft) , \]

with \( f = 930 \text{Hz}, \Delta t = 0.06 \text{ sec and } A \approx 0.76. \)

We first considered how many injections were recovered (efficiency) and how many false alarms were found by applying the WaveBurst method to the original data to draw a Receiver
Operating Characteristic (ROC). We then considered the above mentioned coincidences between the events found on the original data and the events found on the data which had undergone our adaptive entropy multiresolution filter. The two ROC curve are shown in fig. 4 and they show how the coincidence method enables to lower substantially the false alarm rate without affecting the efficiency.

4. Conclusions

We presented a new method of wavelet packet decomposition to be used in gravitational wave detection. This entropy multiresolution method is aimed to better represent in the wavelets domain a signal embedded in noise in order to built a better filter in this domain. We presented also one of the possible applications of this method, showing how to use it in order to have an improved ROC curve by well tested methods, like WaveBurst. In following work we hope to present other applications of this entropy multiresolution method.

Acknowledgments

The authors wish to thank the ROG collaboration and A. Ortolan for useful discussions. R.T. wishes to thank G.V. Pallottino for his encouraging conversations during the study and development of the entropy-multiresolution method and R.S. wishes to thank G. Vedovato and M. Drago for their invaluable help and stimulating discussions.

5. References

[1] K. A. Thorne [for the LIGO Scientific Collaboration], to appear in the proceedings of 42nd Rencontres de Moriond: Gravitational Waves and Experimental Gravity, La Thuile, Valle d’Aosta, Italy, 11-18 Mar 2007. arXiv:0706.4301 [gr-qc].
[2] Mohanty S, 2000 Phys. Rev. D 61 122002.
[3] Sylvestre J, 2002 Phys. Rev. D 66 102004.
[4] Marrone A, Polosa A D, Scioscia G., Stramaglia S., Zenzola A. 1999 Phys. Rev. E 60, 1088.
[5] Klimenko S, Yakushin I, Rakhmanov M and Mitselmakher G 2004 Class.Quant.Grav. 21 S1685.
[6] Klimenko S and Mitselmakher G 2004 Class.Quant.Grav. 21 S1819.
[7] Camarda M and Ortolan A 2006 Phys. Rev. D 74 062001.
[8] Mallat S. "A wavelet tour in signal processing", Academic Press, 1998, pag. 319-326.
[9] Coifman, R. and Wickerhauser, M. V. (1992), IEEE Transactions on Information Theory, 38, No. 2, 713-718.
[10] J. Canny, IEEE Trans. Patt. Anal. and Mach. Intell., 36:961-1005, September 1986, Theory, 38, No. 2, 713-718.