ON ISOTROPIC LAGRANGIAN SUBMANIFOLDS IN THE HOMOGENEOUS NEARLY KÄHLER $\mathbb{S}^3 \times \mathbb{S}^3$

ZEJUN HU AND YINSHAN ZHANG

Abstract. In this paper, we show that isotropic Lagrangian submanifolds in a 6-dimensional strict nearly Kähler manifold are totally geodesic. Moreover, under some weaker conditions, a complete classification of the $J$-isotropic Lagrangian submanifolds in the homogeneous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ is also obtained. Here, a Lagrangian submanifold is called $J$-isotropic, if there exists a function $\lambda$, such that $g(\langle \nabla h \rangle(v, v, v), Jv) = \lambda$ holds for all unit tangent vector $v$.

1. Introduction

Nearly Kähler (abbrev. NK) manifolds are a special class of almost Hermitian manifolds with almost complex structure $J$ satisfying that $\nabla J$ is skew-symmetric. In 1970s, A. Gray made systemically studies on the geometry of NK manifolds (cf. [13, 14]). Towards the important problem of classifying the NK manifolds, P. A. Nagy made significant contributions on the decomposition of complete, simply connected, strict NK manifolds, and in his works [21, 22] it was shown that 6-dimensional NK manifolds play an distinguished role for the study of generic NK manifolds. In [2, 3], J. B. Butruille further proved that homogeneous 6-dimensional NK manifolds must be the NK $\mathbb{S}^6$, $\mathbb{S}^3 \times \mathbb{S}^3$, the complex projective space $\mathbb{C}P^3$ and the flag manifold $SU(3)/U(1) \times U(1)$. On the other hand, we would mention the recent remarkable development that L. Foscolo and M. Haskins [12] have constructed inhomogeneous NK structures on both manifolds of $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$.

Amongst the geometry of submanifolds of the NK manifolds, most researches concentrate on the homogeneous NK $\mathbb{S}^6$, and there are rich literatures (cf. [4, 5, 6, 11, 18, 26] and the references therein). We also noticed that recently a broader study of submanifolds in NK manifolds was investigated in [24] by Schäfer and Smoczyk.

In this paper we mainly restrict to study submanifolds of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$. Recall that a submanifold of an almost Hermitian manifold is called almost complex, if the almost complex structure $J$ preserves the tangent space. The study of almost complex surfaces of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$ was initiated in [11], where all almost complex surfaces of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$.

This project was supported by NSFC (Grant No. 11371330).

2010 Mathematics Subject Classification. 53B35, 53C30, 53C42, 53D12.

Key words and phrases. Nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, Lagrangian submanifold, isotropic submanifold, $J$-parallel, totally geodesic.
surfaces with parallel second fundamental form were classified. In [7, 8, 16], we can find further developments on the study of the almost complex surfaces in NK $S^3 \times S^3$.

Recall also that a submanifold $M$ of an almost Hermitian manifold $N$ is called Lagrangian, if the almost complex structure $J$ interchanges the tangent and the normal spaces and thus the dimension of $M$ is half the dimension of $N$. After the fruitful study on the Lagrangian submanifolds of the NK $S^6$ during the past several decades, in recent years the study of the Lagrangian submanifolds of the homogeneous NK $S^3 \times S^3$ attracts much attention. It is well known that both factors, $S^3 \times \{pt\}$ and $\{pt\} \times S^3$, and the diagonal $\{(x, x) | x \in S^3\}$ are examples of totally geodesic Lagrangian submanifolds in the NK $S^3 \times S^3$, in addition to these simplest ones, A. Moroianu and U. Semmelmann [20] constructed several interesting and new examples of Lagrangian submanifolds of the NK $S^3 \times S^3$. In fact, as usual denoting $\mathbb{H}$ the quaternion space with $i, j, k$ its imaginary units and regarding the unit 3-sphere $S^3$ as the set of all the unit quaternions in $\mathbb{H}$, it was shown in [20] that the graphs of the product of some simple functions on $S^3$ yield Lagrangian submanifolds of the NK $S^3 \times S^3$ with significantly properties. Furthermore, very recent investigations reveal that actually all the totally geodesic Lagrangian submanifolds can be produced in such natural way (see [28]), and so do the Lagrangian submanifolds with nonzero constant sectional curvature (see [9]). In this respect, we would introduce the following interesting result:

**Theorem 1.1** ([28]). A Lagrangian submanifold of any 6-dimensional strict NK manifold is of parallel second fundamental form if and only if it is totally geodesic.

In fact, as the main result of [28], after finishing the proof of Theorem 1.1 a complete classification of all totally geodesic Lagrangian submanifolds of the homogeneous NK $S^3 \times S^3$ was explicitly demonstrated, which consists of six non-congruent examples.

Still focusing on the study of Lagrangian submanifolds of the NK manifolds, in this paper we first study its subclass of the so-called isotropic ones. According to B. O’Neill [23], a submanifold of a Riemannian manifold is called isotropic if and only if for any tangent vector $v$ at a point $p$, we have the relation that

$$g(h(v, v), h(v, v)) = \mu^2(p)(g(v, v))^2,$$

where $h$ denotes the second fundamental form of the immersion and $\mu$ is a non-negative function on the submanifold. Similarly as for Lagrangian submanifolds of Kähler manifolds, studied by Montiel and Urbano [19], in present setting the following result holds, which gives a remarkable counterpart of Theorem 1.1.

**Theorem 1.2.** A Lagrangian submanifold of any 6-dimensional strict NK manifold is isotropic if and only if it is totally geodesic.

We remark that Theorem 1.2 together with Theorem 1.1 implies that for Lagrangian submanifolds of any 6-dimensional strict NK manifold the isotropic condition is equivalent to the condition that the second fundamental form is parallel. As generally a Lagrangian
submanifold of arbitrary 6-dimensional strict NK manifold is not obviously curvature-invariant (see [19] for the notion and (2.10) for the assertion), Theorem 1.2 gives a meaningful generalization of Proposition 1 in [19] which states that if a Lagrangian submanifold in a Kähler manifold is curvature-invariant, then the constant isotropic condition implies that the second fundamental form is parallel.

Next to isotropic submanifolds which correspond to satisfying the restriction (1.1), for Lagrangian submanifolds of a NK manifold, one can further consider the subclass of Lagrangian submanifolds which satisfy the condition that for any tangent vector \( v \) at a point \( p \), we have the \textit{special isotropic} relation

\[
g((\nabla h)(v, v, v), Jv) = \lambda(p)(g(v, v))^2
\]

for a function \( \lambda \) on the submanifold.

For simplicity, a Lagrangian submanifold satisfying (1.2) will be called as \( J \)-isotropic. In particular, following the terminology of [10] (where the authors only considered the NK \( S^6 \)), we will call a \( J \)-isotropic Lagrangian submanifold with vanishing \( \lambda \) being \( J \)-parallel.

Recall that Djorić and Vrancken [10] proved that for Lagrangian manifolds of the homogeneous NK \( S^6 \), the notions of \( J \)-isotropic and \( J \)-parallel are in fact equivalent (Theorem B in [10]), whereas the \( J \)-parallel condition gives more examples than that of parallel second fundamental from. In fact, by the classification Theorem A in [10], besides the totally geodesic 3-sphere, there are two other examples of \( J \)-parallel Lagrangian submanifolds in the homogeneous NK \( S^6 \).

The main purpose of the present paper is to extend the results of [10] by replacing the ambient homogeneous NK \( S^6 \) with the homogeneous NK \( S^3 \times S^3 \). Although the NK \( S^6 \) is a space form whereas the NK \( S^3 \times S^3 \) is much more complicated with the particular properties that it is even neither locally symmetric nor Chern flat (cf. [16]), it turns out that, once overcoming the difficulty brought by the complicated expression of its Riemannian curvature tensor, we can still succeed in achieving a complete classification of the \( J \)-isotropic Lagrangian submanifolds in the homogeneous NK \( S^3 \times S^3 \). To speak accurately, we will generalize the results of [10] by showing that the case for NK \( S^3 \times S^3 \) is totally similar as for NK \( S^6 \), i.e., for Lagrangian submanifolds of the homogeneous NK \( S^3 \times S^3 \), the two conditions of \( J \)-isotropic and \( J \)-parallel are equivalent. As the main result, a classification theorem can be obtained as follows:

\textbf{Main Theorem.} Let \( M \) be a Lagrangian submanifold in the homogeneous NK \( S^3 \times S^3 \). If \( M \) is \( J \)-isotropic in the sense of (1.2), then the isotropic function \( \lambda \) vanishes, and \( M \) is locally given by one of the following immersions:

\begin{align*}
(1) & \quad f_1 : S^3 \to S^3 \times S^3 \text{ defined by } u \mapsto (1, u). \\
(2) & \quad f_2 : S^3 \to S^3 \times S^3 \text{ defined by } u \mapsto (u, 1). \\
(3) & \quad f_3 : S^3 \to S^3 \times S^3 \text{ defined by } u \mapsto (u, u). \\
(4) & \quad f_4 : S^3 \to S^3 \times S^3 \text{ defined by } u \mapsto (u, ui).
\end{align*}
(5) $f_5: \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$ defined by $u \mapsto (u^{1}, u{-1})$.
(6) $f_6: \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$ defined by $u \mapsto (u^{-1}, u^{1})$.
(7) $f_7: \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$ defined by $u \mapsto (u^{1}, u^{1})$.
(8) $f_8: \mathbb{R}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$ defined by $(u, v, w) \mapsto (p(u, w), q(u, v))$, where $p$ and $q$ are constant mean curvature tori in $\mathbb{S}^3$, given respectively by

$$p(u, w) = \left( \cos(\sqrt{2}u) \cos(\sqrt{2}w), \cos(\sqrt{2}u) \sin(\sqrt{2}u), \sin(\sqrt{2}u) \cos(\sqrt{2}w), \sin(\sqrt{2}u) \sin(\sqrt{2}w) \right),$$

$$q(u, v) = \frac{1}{\sqrt{2}} \left( \cos(\sqrt{2}v) \sin(\sqrt{2}u) + \cos(\sqrt{2}u) \sin(\sqrt{2)v} \sin(\sqrt{2}u) + \cos(\sqrt{2}u), \sin(\sqrt{2}u) \sin(\sqrt{2}u) - \cos(\sqrt{2}u) \right).$$

**Remark 1.1.** The immersions $\{f_i\}_{i=1}^5$ are totally geodesic, and the immersions $f_1, f_2, f_3, f_7, f_8$ are of constant sectional curvature. It should be pointed out that due to the restriction of the almost product $P$ of the NK $\mathbb{S}^3 \times \mathbb{S}^3$, the two pairs of $f_1$ and $f_2$, as well as $f_5$ and $f_6$, are not congruent (cf. [27]). Therefore, combining the main results of [11] and [27], the above Main Theorem shows that the $J$-isotropic condition (1.2) is a very nice concept which gives a unified characterization of all the totally geodesic Lagrangian submanifolds together with those of constant sectional curvature.

**Remark 1.2.** Related to our result, it is worth mentioning that isotropic Lagrangian submanifolds of the complex space forms have been completely classified (cf. [17, 19, 25]). Also, Wang, Li and Vrancken [27] have considered Lagrangian submanifolds of the complex space forms with an *isotropic cubic form*, and a complete classification is obtained if the submanifolds are of dimension 3.

The paper is organized as follows. In section 2, we review relevant materials for NK manifolds and their Lagrangian submanifolds, particularly concern is for the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$. In section 3, a proof of Theorem 7 is given. In section 4, restricting to the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$, we first derive an equivalent statement of the $J$-isotropic condition, then using Ricci identity we prove Proposition 4.2 which becomes crucial for our purpose. In section 5, we discuss the examples $f_7$ and $f_8$ as described in the Main Theorem and show that both of them are $J$-parallel. Finally in section 6, we complete the proof of the Main Theorem.

**Acknowledgements.** We are greatly indebted to Prof. Luc Vrancken for his very helpful suggestions as well as his valuable comments during the time when we were working on this project.

## 2. Preliminaries

### 2.1. The homogeneous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

In this subsection, we recall from [1] the homogeneous NK structure on $\mathbb{S}^3 \times \mathbb{S}^3$. By the natural identification $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong ...
holds for any $X$ and it is anti-commutative with $J$ metric on $S$, allows to express $\tilde{(2.6)}$

$T_0S^3 \oplus T_qS^3$, we can write a tangent vector at $(p,q)$ as $Z(p,q) = (U_{(p,q)}, V_{(p,q)})$, or $Z = (U, V)$ for simplicity. The well known almost complex structure $J$ on $S^3 \times S^3$ is given by

\[ JZ(p,q) = \frac{1}{\sqrt{3}}(2pq^{-1}V - U, -2qp^{-1}U + V). \]

Define the following Hermitian metric $g$

\[ g(Z, Z') = \frac{1}{3}(\langle Z, Z' \rangle + \langle JZ, JZ' \rangle) = \frac{1}{3}(U, U') + \langle V, V' \rangle - \frac{2}{3}(p^{-1}U, q^{-1}V') + (p^{-1}U', q^{-1}V), \]

where $Z = (U, V), Z' = (U', V')$ are tangent vectors and $\langle \cdot, \cdot \rangle$ is the standard product metric on $S^3 \times S^3$. Then $(S^3 \times S^3, g, J)$ is a homogeneous NK manifold, i.e., $(\tilde{\nabla}Z)JZ = 0$ holds for any $X \in T_{(p,q)}(S^3 \times S^3)$, where $\tilde{\nabla}$ is the Levi-Civita connection of $g$.

As usual we denote the tensor field $\tilde{\nabla}J$ by $G$. Straightforward computations show that $G$ has the following properties (cf. [1]):

\[ G(X, Y) + G(Y, X) = 0, \quad (2.3) \]

\[ G(X, JY) + JG(Y, X) = 0, \quad (2.4) \]

\[ g(G(X, Y), Z) + g(G(X, Z), Y) = 0, \quad (2.5) \]

\[ (\tilde{\nabla}_X G)(Y, Z) = \frac{1}{3}(g(Y, JZ)X + g(X, Z)JY - g(X, Y)JZ). \]

On $S^3 \times S^3$, an almost product structure $P$ can be defined by (cf. [1])

\[ PZ = (pq^{-1}V, qp^{-1}U), \quad \forall Z = (U, V) \in T_{(p,q)}(S^3 \times S^3). \]

It is easily seen that the operator $P$ is compatible and symmetric with respect to $g$, and it is anti-commutative with $J$. We particularly mention that $P$ plays an important role in the study of submanifolds of the homogeneous NK $S^3 \times S^3$. The following formula allows to express $\tilde{\nabla}P$ in terms of $P$ and $G$ (cf. [1]):

\[ JG(X, PY) + JPG(X, Y) = 2(\tilde{\nabla}_X P)Y. \]

By definition, the above useful relation immediately yields

\[ -G(X, PY) + PG(X, Y) = 2(\tilde{\nabla}_X (PJ))Y. \]

The curvature tensor $\tilde{R}$ of the homogeneous NK $S^3 \times S^3$ is given by (cf. [1]):

\[ \tilde{R}(X, Y)Z = \frac{1}{12}(g(Y, Z)X - g(X, Z)Y) \]

\[ + \frac{1}{12}(g(JY, Z)X - g(JX, Z)JY - 2g(JX, Y)JZ) \]

\[ + \frac{1}{3}(g(PY, Z)PX - g(PX, Z)PY) \]

\[ + g(JPY, Z)JPX - g(JPX, Z)JPY). \]
2.2. Lagrangian submanifolds of the NK $\mathbb{S}^3 \times \mathbb{S}^3$. Let $M$ be a Lagrangian submanifold of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$, and therefore $JTM = T^\perp M$. By a result of Schäfer and Smoczyk in [24] (see also [15]), $M$ is orientable and minimal, and that $G(X, Y)$ is a normal vector for any $X, Y \in TM$. We denote by $\nabla$ and $\nabla^\perp$ respectively the induced connection and normal connection on $M$. Then the formulas of Gauss and Weingarten can be expressed as

\begin{equation}
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in TM,
\end{equation}

\begin{equation}
\nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi, \quad \forall X \in TM, \xi \in T^\perp M,
\end{equation}

where $h$ is the second fundamental form, and it is related to the shape operator $A_\xi$ by $g(h(X, Y), \xi) = g(A_\xi X, Y)$. From (2.11) and the property of $G$, we derive

\begin{equation}
\nabla_X^\perp JY = G(X, Y) + J\nabla_X Y, \quad A_\xi Y = -Jh(X, Y).
\end{equation}

Clearly, the second formula in (2.12) shows that $g(h(X, Y), JZ)$ is totally symmetric.

Denote the curvature tensor of $\nabla$ and $\nabla^\perp$ by $R$ and $R^\perp$, respectively. Then the equations of Gauss, Codazzi and Ricci are given by

\begin{equation}
R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),
\end{equation}

\begin{equation}
g((\nabla h)(X, Y, Z) - (\nabla h)(Y, X, Z), \xi) = g(\tilde{R}(X, Y)Z, \xi),
\end{equation}

\begin{equation}
g(R^\perp(X, Y)\xi, \eta) = g(\tilde{R}(X, Y)\xi, \eta) + g([A_\xi A_\eta]X, Y),
\end{equation}

where $(\nabla h)(X, Y, Z) = \nabla^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

Applying the equations of Gauss and Ricci, and using (2.10) as well as (2.12), we find

\begin{equation}
g(R(X, Y)Z, W) = g(R^\perp(X, Y)JZ, JW) + \frac{1}{2}(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)).
\end{equation}

Related to $\nabla h$, we also have the following useful formula (cf. Lemma 2.3 in [28]):

\begin{equation}
g((\nabla h)(X, Y, Z), JW) - g((\nabla h)(X, Y, W), JZ) = g(h(X, Y), G(W, Z)).
\end{equation}

Define the second covariant derivative $\nabla^2 h$ by

\begin{equation}
(\nabla^2 h)(X, Y, Z, W) = \nabla^\perp \left((\nabla h)(Y, Z, W) - (\nabla h)(\nabla_X Y, Z, W) - (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y, Z, \nabla_X W)\right).
\end{equation}

Then $\nabla^2 h$ satisfies the Ricci identity

\begin{equation}
(\nabla^2 h)(X, Y, Z, W) - (\nabla^2 h)(Y, X, Z, W) = R^\perp(X, Y)h(Z, W) - h(R(X, Y)Z, W) - h(Z, R(X, Y)W).
\end{equation}

Since $M$ is Lagrangian, the pull-back of $T(\mathbb{S}^3 \times \mathbb{S}^3)$ to $M$ splits into $TM \oplus JTM$. We can consider on a dense open set of $M$, and choose an orthonormal frame field $\{e_1, e_2, e_3\}$ such that (cf. [9], p.8)

\begin{equation}
P e_1 = \lambda_1 e_1 + \mu_1 J e_1, \quad P e_2 = \lambda_2 e_2 + \mu_2 J e_2, \quad P e_3 = \lambda_3 e_3 + \mu_3 J e_3,
\end{equation}
where $\lambda_i = \cos 2\theta_i$, $\mu_i = \sin 2\theta_i$, $i = 1, 2, 3$. The functions $\theta_1, \theta_2$ and $\theta_3$ are called the angles of $M$. Taking into account of the properties of $G$, we can further assume

$$\sqrt{3} J G(e_i, e_j) = \sum_k \epsilon^{k}_{ij} e_k,$$

where $\epsilon^{k}_{ij}$ is the Levi-Civita symbol, i.e.,

$$\epsilon^{k}_{ij} := \begin{cases} 1, & \text{if } (ijk) \text{ is an even permutation of } (123), \\ -1, & \text{if } (ijk) \text{ is an odd permutation of } (123), \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $\omega^k_{ij}$ the coefficient of induced connection, determined by

$$\nabla_{E_i} E_j = \sum_k \omega^k_{ij} E_k, \quad \omega^k_{ij} = -\omega^j_{ik}.$$

In [9], applying the properties of $P$, it was proved that the angle functions $\{\theta_1, \theta_2, \theta_3\}$ satisfy the following important relations:

**Lemma 2.1** (Lemma 3.8 of [9]). Let $M$ be Lagrangian submanifold of the homogeneous $\text{NK } S^3 \times S^3$. With respect to the above chosen frame field $\{e_1, e_2, e_3\}$, we have

1. $\theta_1 + \theta_2 + \theta_3$ is a multiple of $\pi$,
2. $e_1(\theta_j) = -h^{j}_{jj}$,
3. $h^k_{ij} \cos(\theta_j - \theta_k) = \left(\frac{\sqrt{3}}{2} \epsilon^{k}_{ij} - \omega^k_{ij}\right) \sin(\theta_j - \theta_k), \quad \forall j \neq k$,

where $h^k_{ij} = g(h(e_i, e_j), Je_k)$.

It is worth noting that the angle functions $\{\theta_i\}_{i=1}^3$ play an important role in the study of Lagrangian submanifolds of $S^3 \times S^3$. In fact, by carefully analyzing the angle functions, a complete classification of the Lagrangian manifolds in the homogeneous $\text{NK } S^3 \times S^3$ with constant sectional curvature was obtained in [9]. Also in [9], the authors established the following interesting characterization of totally geodesic Lagrangian submanifolds in terms of the angle functions.

**Lemma 2.2** (Lemma 3.8 of [9]). If two of the angles $\{\theta_1, \theta_2, \theta_3\}$ are equal modulo $\pi$, then the Lagrangian submanifold is totally geodesic.

We remark that the totally geodesic Lagrangian submanifolds in the homogeneous $\text{NK } S^3 \times S^3$ have been classified in [28]. Actually such a Lagrangian submanifold is locally given by one of the immersions $\{f_i\}_{i=1}^6$ described in the Main Theorem.

3. **Proof of Theorem 1.2**

To give a proof of Theorem 1.2 we are sufficient to consider the "only if" part.

Let $M$ be an isotropic Lagrangian submanifold of a 6-dimensional strict NK manifold. Suppose on the contrary that the assertion is not true, then we have a point $p \in M$ such that it is not totally geodesic. Consider the function $F$ defined on the unit sphere
$U_p M$ in $T_p M$ by $F(v) = g(h(v, v), Jv)$. Noting that the cubic form $g(h(\cdot, \cdot), J\cdot)$ is totally symmetric (cf. Proposition 3.2 of [24]), we can choose an orthonormal basis $\{e_1, e_2, e_3\}$ of $T_p M$ as in Lemma 1 of [19], such that
\begin{equation}
 h(e_j, e_j) = \mu_j e_j, \quad j = 1, 2, 3,
\end{equation}
where $\mu_1$ is the maximum of the function $F$ on $U_p M$.

As $M$ satisfies the condition
\begin{equation}
 g(h(v, v), h(v, v)) = \mu_2 (g(v, v))^2, \quad \forall \ v \in TM,
\end{equation}
we derive from (3.1) and (3.2) that $\mu_1 = \mu \geq 0$. From the relation (3.2) we also have (see (3.2) in [19]):
\begin{equation}
 (4.1)
 \mu^2 - g(h(x, x), h(y, y)) - 2g((h(x, y), h(x, y)) = 0,
\end{equation}
for all orthonormal vectors $x, y$. Now in (4.1) taking $x = e_1, y = e_j$ for $j = 2$ and $j = 3$, respectively, then using (4.1) we derive $\mu^2 - \mu \mu_j - 2\mu_j^2 = 0$. Solving this equation for $\mu_j$, we get $\mu_j = -\mu$, or $\mu_j = \frac{1}{2}\mu$. From Theorem A of [24] we know that $M$ is minimal, thus we have
\begin{equation}
 0 = g(h(e_1, e_1) + h(e_2, e_2) + h(e_3, e_3), Je_1) = \mu + \mu_2 + \mu_3,
\end{equation}
which, together with the previous relations, yields $\mu = 0$. Hence, by (3.2), we easily derive $h(u, v) = 0$ for all tangent vector $u$ and $v$, i.e., $M$ is totally geodesic at $p$, we get the desired contradiction.

4. Implications of the $J$-isotropic condition

Now we assume that $M$ is a $J$-isotropic Lagrangian submanifold in the homogeneous NK $S^3 \times S^3$, such that
\begin{equation}
 (4.1)
 g((\nabla h)(v, v), Jv) = \lambda (g(v, v))^2, \quad \forall \ v \in TM,
\end{equation}
where $\lambda$ is a function on $M$.

First of all, for later use we present an equivalent condition of the $J$-isotropic property (4.1). For notational simplicity, in sequel we will use the symbol $\otimes$ to denote cyclic sum.

**Proposition 4.1.** A Lagrangian submanifold $M$ of the homogeneous NK $S^3 \times S^3$ is $J$-isotropic if and only if the following equation holds:
\begin{align*}
 12g((\nabla h)(Y, Z, W), Jv) + 3 \otimes_{YZW} g(h(Y, Z), G(W, V)) \\
 2 \otimes_{ZWV} [g(PY, Z)g(PJW, V) - g(PYJ, Z)g(PW, V)] \\
 - 4\lambda \otimes_{ZWV} g(Y, Z)g(W, V) = 0,
\end{align*}
where $Y, Z, W, V$ are any vector fields tangent to $M$. 

□
Proof. It is easily seen that the “if” part is trivial.

Now, we consider the “only if” part. Taking $v = a_1 Y + a_2 Z + a_3 W + a_4 V$ for arbitrary real numbers $\{a_1, a_2, a_3, a_4\}$ in (4.1), and comparing the coefficient of the term $a_1 a_2 a_3 a_4$, we obtain

$$
\begin{align*}
\mathcal{S}_{YZW} g((\nabla h)(Y, Z, W), JV) + \mathcal{S}_{YWV} g((\nabla h)(Y, Z, V), JW) \\
+ \mathcal{S}_{YWV} g((\nabla h)(Y, W, V), JZ) + \mathcal{S}_{ZWV} g((\nabla h)(Z, W, V), JY) \\
- 4\lambda \mathcal{S}_{ZWV} g(Y, Z)g(W, V) = 0.
\end{align*}
$$

(4.3)

Applying the symmetry of $h$, and the Codazzi equation (2.14), such as

$$
g((\nabla h)(Z, Y, W), JV) = g((\nabla h)(Y, Z, W), JV) + \tilde{R}(Z, Y, W, JV),
$$

(4.4)

to (4.3), we can get

$$
0 = 3g((\nabla h)(Y, Z, W), JV) + \tilde{R}(Z, Y, W, JV) + \tilde{R}(W, Y, Z, JV) \\
+ 3g((\nabla h)(Y, Z, V), JW) + \tilde{R}(Z, Y, V, JW) + \tilde{R}(V, Y, Z, JW) \\
+ 3g((\nabla h)(Y, W, V), JZ) + \tilde{R}(W, Y, V, JZ) + \tilde{R}(V, Y, W, JZ) \\
+ 3g((\nabla h)(Z, W, V), JY) + \tilde{R}(W, Z, V, JY) + \tilde{R}(V, Z, W, JY) \\
- 4\lambda \mathcal{S}_{ZWV} g(Y, Z)g(W, V).
$$

(4.5)

By the use of (2.17), we can reduce (4.5) to be

$$
0 = 9g((\nabla h)(Y, Z, W), JV) + 3g((\nabla h)(Z, Y, W), JV) \\
+ 3 \mathcal{S}_{ZWV} g(h(Y, Z), G(W, V)) \\
+ \tilde{R}(Z, Y, W, JV) + \tilde{R}(W, Y, Z, JV) + \tilde{R}(Z, Y, V, JW) \\
+ \tilde{R}(V, Y, Z, JW) + \tilde{R}(W, Y, V, JZ) + \tilde{R}(V, Y, W, JZ) \\
+ \tilde{R}(W, Z, V, JY) + \tilde{R}(V, Z, W, JY) - 4\lambda \mathcal{S}_{ZWV} g(Y, Z)g(W, V).
$$

(4.6)

From (4.6), and applying the Codazzi equation (4.4) once more, we further obtain

$$
12g((\nabla h)(Y, Z, W), JV) + 3 \mathcal{S}_{ZWV} g(h(Y, Z), G(W, V)) \\
+ 4\tilde{R}(Z, Y, W, JV) + \tilde{R}(W, Y, Z, JV) + \tilde{R}(Z, Y, V, JW) \\
+ \tilde{R}(V, Y, Z, JW) + \tilde{R}(W, Y, V, JZ) + \tilde{R}(V, Y, W, JZ) \\
+ \tilde{R}(W, Z, V, JY) + \tilde{R}(V, Z, W, JY) - 4\lambda \mathcal{S}_{ZWV} g(Y, Z)g(W, V) = 0.
$$

(4.7)

From (4.7) and (2.10), we obtain the expression (4.2) immediately. □
Next, to achieve further implications of the \( J \)-isotropic condition \( (4.1) \), we differentiate the equation \((4.2)\). Then, using \((2.6)\), we can get

\[
\begin{align*}
&\text{Proposition 4.2.} \\
&g((\nabla h)(X,Y,Z),JW) + g((\nabla h)(Y,Z,W),G(X,V)) \\
&\quad + 12\left[ g((\nabla h)(X,Y,Z),JW) + g((\nabla h)(Y,Z,W),G(X,V)) \right] \\
&\quad + 3g((\nabla h)(X,Y,Z),G(W,V)) + g(h(Y,Z),JW)g(X,V) \\
&\quad - g(h(Y,Z),JW)g(X,W) + 2\left( g(PY,h(X,Z))g(PJW,V) \\
&\quad + g(\nabla X P)Y + Ph(X,Y),Z)g(PJW,V) + g(PY,Z)g((\nabla X P)W + PJh(X,W),V) - g(PJY,h(X,Z))g(PW,V) \\
&\quad - g((\nabla X (PJ))Y + PJh(X,Y),Z)g(PW,V) - g(PJY,Z)g(PW,h(X,V)) \\
&\quad - g(PJY,Z)g((\nabla X P)W + Ph(X,W),V) - 2X(\lambda)g(Y,Z)g(W,V) \right] = 0,
\end{align*}
\]

(4.8)

where, besides the basic formulas \((2.11)\) and \((2.12)\), we have used \((4.1)\) for the expressions of such terms \( g((\nabla h)(\nabla X Y,Z),JW),\ldots,g((\nabla h)(Y,Z,W),J\nabla X V) \).

From the equation \((4.8)\), we have the following crucial proposition.

**Proposition 4.2.** If \( M \) is \( J \)-isotropic in the homogeneous \( NK \mathbb{S}^3 \times \mathbb{S}^3 \), then we have

\[
\begin{align*}
&12g(R^l (X,Y)h(Z,W) - h(R(X,Y)Z,W) - h(Z,R(X,Y)W),JW) \\
&\quad + 9g((\nabla h)(Y,Z,W),G(X,V)) - g((\nabla h)(X,Y,Z),G(Y,V)) \\
&\quad + 3g(h(Y,Z),JW)g(X,V) - 3g(h(X,Z),JW)g(Y,V) \\
&\quad + 3\left[ g(h(X,Z),JW)g(Y,W) - g(h(Y,Z),JW)g(X,W) \\
&\quad + g(PY,Z)g(PX,G(W,V)) - g(PX,Z)g(PY,G(W,V)) \\
&\quad + g(JPY,Z)g(JPX,G(W,V)) - g(JPX,Z)g(JPY,G(W,V)) \right] \\
&\quad + 2\left( g(X,Y,Z,W) = 0,
\end{align*}
\]

(4.9)

where \( I(X,Y,Z,W,V) \) is defined by

\[
I(X,Y,Z,W,V) := g(PY,h(X,Z))g(PJW,V) \\
+ g((\nabla X P)Y + Ph(X,Y),Z)g(PJW,V) + g(PY,Z)g(PJW,h(X,V)) \\
+ g(PY,Z)g((\nabla X (PJ))W + PJh(X,W),V) - g(PX,h(Y,Z))g(PJW,V) \\
- g((\nabla Y P)X + Ph(X,Y),Z)g(PJW,V) - g(PX,Z)g(PJW,h(Y,V)) \\
- g(PX,Z)g((\nabla Y (PJ))W + PJh(Y,W),V) - g(PJY,h(X,Z))g(PW,V) \\
- g((\nabla Y (PJ))Y + PJh(X,Y),Z)g(PW,V) - g(PJY,Z)g(PW,h(X,V)) \\
- g(PJY,Z)g((\nabla Y P)W + Ph(X,W),V) + g(PJX,h(Y,Z))g(PW,V) \\
+ g((\nabla Y (PJ))X + PJh(X,Y),Z)g(PW,V) + g(PJX,Z)g(PW,h(Y,V)) \\
+ g(PJX,Z)g((\nabla Y P)W + Ph(Y,W),V) - 2X(\lambda)g(Y,Z)g(W,V) \\
+ 2Y(\lambda)g(X,Z)g(W,V).
\]

(4.10)
Proof. This is a direct consequence of (4.8) and the following Ricci identity
\[
g((\nabla^2 h)(X, Y, Z, W), JV) - g((\nabla^2 h)(Y, X, Z, W), JV)
= g(R^1(X, Y)h(Z, W) - h(R(X, Y)Z, W) - h(Z, R(X, Y)W), JV).
\]

In fact, the totally symmetry of \(g(h(\cdot, \cdot), J\cdot)\) implies that
\[
\begin{align*}
&\begin{align*}
&3g[h(Y, Z), JW]g(X, V) - g(h(Y, Z), JV)g(X, W)
&= g(h(Z, W), JV)g(X, Y) - g(h(W, Y), JV)g(X, Z),
\end{align*}
\end{align*}
\]
from which we obtain the term
\[
3g[h(Y, Z), JW]g(X, V) - 3g[h(Y, Z), JW]g(Y, V)
+ \begin{align*}
\begin{array}{c}
\begin{aligned}
g(h(X, Z), JV)g(Y, W) - g(h(Y, Z), JV)g(X, W)\end{aligned}
\end{array}
\end{align*}
\]
in (4.9) immediately.

Next, the Codazzi equation implies that
\[
g((\nabla h)(X, Y, Z), G(W, V)) - g((\nabla h)(Y, X, Z), G(W, V)) = g(\tilde{R}(X, Y)Z, G(W, V)),
\]
by which, and using (2.10), we get
\[
\begin{align*}
&\begin{align*}
&3g[h(Y, Z), JW]g(X, V) - g(h(Y, Z), JV)g(X, W)
&= g(h(X, Z), JV)g(Y, W) - g(h(Y, Z), JV)g(X, W)
\end{align*}
\end{align*}
\]
in (4.9) immediately.

Thus the term
\[
\begin{align*}
&\begin{align*}
&3g[h(Y, Z), JW]g(X, V) - g(h(Y, Z), JV)g(X, W)
&= g(h(X, Z), JV)g(Y, W) - g(h(Y, Z), JV)g(X, W)
\end{align*}
\end{align*}
\]
in (4.9) is derived.

The remaining terms in (4.9) can be easily obtained by direct calculations. □

5. Examples of \(J\)-isotropic and non-totally geodesic

As have been shown in [9], the two Lagrangian immersions \(f_7\) and \(f_8\) into \(S^3 \times S^3\) (see Main Theorem for their definitions) are of constant sectional curvature \(3/16\) and zero, respectively. Moreover, both of them are not totally geodesic.

In this section, we further show that these two immersions are the simplest examples next to the totally geodesic. Precisely, we have the following fact:

Proposition 5.1. The immersions \(f_7\) and \(f_8\) are both \(J\)-isotropic with \(\lambda = 0\), i.e., they are in fact \(J\)-parallel.
Proof. According to the calculations in [9], with respect to the frame field \(\{e_1, e_2, e_3\}\) that is assumed satisfying (2.19), we have

\[
(5.1) \quad h_{12}^3 = \frac{1}{4}, \quad h_{ij}^k = 0 \text{ for other } i, j, k; \quad \omega_{ij}^k = \frac{\sqrt{3}}{4} e_{ij}^k,
\]

for the immersion \(f_7\); while for the immersion \(f_8\) we have

\[
(5.2) \quad h_{12}^3 = -\frac{1}{2}, \quad h_{ij}^k = 0 \text{ for other } i, j, k; \quad \omega_{ij}^k = 0.
\]

Moreover, the angles of \(f_7\) and \(f_8\) are both given by

\[
(5.3) \quad (2\theta_1, 2\theta_2, 2\theta_3) = (0, 2\pi/3, 4\pi/3).
\]

As \(h_{ij}^k\) and \(\omega_{ij}^k\) are constant for both immersions, we calculate that

\[
g((\nabla_{e_i} h)(e_j, e_k), J e_l) = g(\nabla_{e_i}^j h(e_j, e_k), J e_l) - g(h(\nabla_{e_i} e_j, e_k), J e_l) - g(h(e_j, \nabla_{e_i} e_k), J e_l) = \sum_m [h_{ij}^m g(\nabla_{e_i}^j, J e_m, J e_l) - \omega_{ij}^m h_{mk}^l - \omega_{ik}^m h_{mj}^l].
\]

Noting that

\[
\nabla_{e_i}^j J e_m = G(e_i, e_m) + J \nabla_{e_i} e_m, \quad \sqrt{3} J G(e_i, e_j) = \sum_k \varepsilon_{ij}^k e_k,
\]

(5.4) implies that

\[
(5.5) \quad g((\nabla_{e_i} h)(e_j, e_k), J e_l) = \sum_m [h_{ij}^m (\frac{1}{\sqrt{3}} e_{im}^l + \omega_{im}^l) - \omega_{ij}^m h_{mk}^l - \omega_{ik}^m h_{mj}^l].
\]

To complete the proof, we next show that (4.2) holds for \(f_7\) and \(f_8\) with \(\lambda = 0\).

In fact, using (5.6), we see that (4.2) becomes equivalent to the following:

\[
12 \sum_m [h_{jk}^m (\frac{1}{\sqrt{3}} e_{mi}^l + \omega_{im}^l) - \omega_{ij}^m h_{mk}^l - \omega_{ik}^m h_{mj}^l]
+ 3[g(h(e_i, e_j), G(e_k, e_l)) + g(h(e_i, e_k), G(e_j, e_l)) + g(h(e_j, e_k), G(e_i, e_l))]
+ 2[g(P e_i, e_j) g(P J e_k, e_l) - g(P J e_i, e_j) g(P e_k, e_l)]
+ g(P e_i, e_k) g(P J e_i, e_j) - g(P J e_i, e_k) g(P e_l, e_j)
+ g(P e_i, e_l) g(P J e_j, e_k) - g(P J e_i, e_l) g(P e_j, e_k)]
- 4\lambda [g(e_i, e_j) g(e_k, e_l) + g(e_i, e_k) g(e_i, e_j) + g(e_i, e_l) g(e_j, e_k)] = 0, \quad \forall i, j, k, l.
\]

Noticing that at least two indices of \(\{i, j, k, l\}\) are the same, by the facts (5.1)-(5.3), we easily see that (5.6) holds for arbitrary \(\lambda\) when \(\{i, j, k, l\} = \{1, 2, 3\}\), or three elements of \(\{i, j, k, l\}\) are equal.

Therefore, to show that \(f_7\) and \(f_8\) are \(J\)-parallel, it is sufficient to prove that (5.6) holds for \(\lambda = 0\) in the three cases: \(i = j \neq k = l\), \(i = k \neq j = l\) or \(i = l \neq j = k\).

As the calculations for the above three cases are straightforward and totally similar, in below we will only taking for example treat the case \(i = j \neq k = l\). In this case, the
equation (5.6) reduces to

\begin{equation}
\sum_m [2\sqrt{3}h_{k \alpha}^m \omega_{\alpha i}^k + 12h_{k \alpha}^m \omega_{\alpha i}^k + \sqrt{3}h_{k \alpha}^m \epsilon_{\alpha i}^k + \sin 2(\theta_k - \theta_i)] - 2\lambda = 0, \quad \forall i \neq k.
\end{equation}

Using the facts (5.1) and (5.2), we further see that (5.7) becomes equivalent to

\begin{equation}
2\lambda = \sqrt{\frac{\epsilon_{\alpha i}^k}{2}} + \sin 2(\theta_k - \theta_i), \quad m \neq i, k.
\end{equation}

From the fact (5.3), now it is trivial to check that (5.8) does hold for \( \lambda = 0 \).

This completes the proof of Proposition 5.1. \qed

6. PROOF OF THE MAIN THEOREM

For simplicity consideration, let us denote \( R_{ijkl} = g(R(e_i, e_j)e_k, e_l) \).

First, we take \( X = e_2, Y = Z = W = e_1 \) and \( V = e_3 \) in (4.9) to obtain

\begin{equation}
0 = 12g(R^2(e_2, e_1)h(e_1, e_1) - 2h(R(e_2, e_1)e_1, e_1), J_3) + 2g(h(e_2, e_1), J_3)
\end{equation}

\begin{equation}
+ 9g((\nabla h)(e_1, e_1, e_1), G(e_2, e_3)) - 9g((\nabla h)(e_2, e_1, e_1), G(e_1, e_3))
\end{equation}

\begin{equation}
+ 2[g(P e_1, e_1)g(P e_2, G(e_1, e_3)) + g(J P e_1, e_1)g(J P e_2, G(e_1, e_3))] + 2I,
\end{equation}

where \( I = I(e_2, e_1, e_1, e_1, e_3) \) is given by

\begin{equation}
I = g(P e_1, h(e_2, e_3))g(P J e_1, e_1) + g((\nabla e_2) P)e_1 + P h(e_2, e_1), e_3)g(P J e_1, e_1)
+ g(P e_1, e_1)g(P J e_1, h(e_2, e_3)) + g(P J e_3, h(e_2, e_1)) + g(P e_1, e_1).
\end{equation}

\begin{equation}
[g((\nabla e_2)(P J))e_1 + P J h(e_2, e_1), e_3] + g((\nabla e_2)(P J))e_3 + P J h(e_2, e_3), e_1]
- g(P e_2, h(e_1, e_3))g(P J e_1, e_1) - g((\nabla e_2)P)e_2 + P h(e_1, e_2), e_3)g(P J e_1, e_1)
\end{equation}

\begin{equation}
- g(P J e_1, h(e_2, e_3))g(P e_1, e_1) - g((\nabla e_2)P)e_1 + P J h(e_2, e_1), e_3)g(P e_1, e_1)
- g(P J e_1, e_1)g(P e_1, h(e_2, e_3)) + g(P e_3, h(e_2, e_1))
- g(P J e_1, e_3)[g((\nabla e_2)P)e_1 + P h(e_2, e_1), e_3] + g((\nabla e_2)P)e_3 + P h(e_2, e_3), e_1]
\end{equation}

\begin{equation}
+ g(P J e_2, h(e_1, e_3))g(P e_1, e_1) + g((\nabla e_1)(P J))e_2 + P J h(e_1, e_2), e_3)g(P e_1, e_1).
\end{equation}

By using (2.19) and Proposition 4.1, we can calculate that

\begin{equation}
- 9g((\nabla h)(e_2, e_1, e_1), G(e_1, e_3))
= \frac{3}{\sqrt{6}}g(h(e_1, e_2), G(e_1, e_2))
\end{equation}

\begin{equation}
+ \frac{\sqrt{3}}{2}[g(P e_2, e_2)g(P J e_1, e_1) - g(P J e_2, e_2)g(P e_1, e_1)]
\end{equation}

\begin{equation}
= - \frac{3}{2}h_{12}^3 + \frac{\sqrt{3}}{2}(\lambda_2^1 - \lambda_1^2).
\end{equation}

Putting (6.3) into (6.4), and using (2.19) as well as (2.19), we derive

\begin{equation}
12[R_{2113}(h_{11}^3 - h_{13}^1) + R_{2123}h_{12}^3 - 2R_{2112}h_{12}^3]
\end{equation}

\begin{equation}
+ \frac{\sqrt{3}}{6}(\lambda_1^2 - \lambda_2^1) + \frac{1}{2}h_{12}^3 + 2I = 3\sqrt{3}\lambda.
\end{equation}
Next, we take $X = e_2$, $Y = Z = V = e_1$ and $W = e_3$ in (6.4) to derive

$$0 = 12g(R^1(e_2, e_1)h(e_1, e_3) - h(R(e_2, e_1)e_1, e_3) - h(e_1, R(e_2, e_1)e_3), Je_1)$$

$$+ 9g((\nabla h)(e_1, e_1, e_3), G(e_2, e_1)) - 3g(h(e_2, e_1), Je_3) + g(h(e_2, e_3), Je_1)$$

$$+ g(Pe_1, e_1)g(Pe_2, G(e_3, e_1)) + g(JPe_1, e_1)g(JPe_2, G(e_3, e_1)) + 2I',$$

where, by definition, we can check that $I' = I(e_2, e_1, e_1, e_3, e_1) = I(e_2, e_1, e_1, e_1, e_3) = I$ as defined by (6.2).

By using (2.19) and Proposition 4.1, we can calculate that

$$9g((\nabla h)(e_1, e_1, e_3), G(e_2, e_1)) = -g_2 g(h(e_1, e_3), G(e_1, e_3))$$

$$- \sqrt{g_2} [g(Pe_1, e_1)g(PJe_1, e_1) - g(PJe_1, e_1)g(Pe_3, e_3)]$$

$$= -\frac{3}{2}h_{12} - \frac{\sqrt{g_2}}{2}(\lambda_1\mu_3 - \lambda_3\mu_1).$$

Putting (6.6) into (6.5), and using (2.16) and (2.19), we also derive

$$12[R_{1213}(h_{11} - 2h_{33}) + R_{2131}(h_{11} - 2h_{11})]$$

$$- \sqrt{g_2}(\lambda_1\mu_3 - \lambda_2\mu_1) = \sqrt{g_2}(\lambda_1\mu_3 - \lambda_2\mu_1) + \frac{1}{2}h_{12} + 2I = 0.$$

Now from (6.4) and (6.7), we easily obtain the following relation:

$$\lambda = \frac{1}{6}(\lambda_1\mu_3 - \lambda_2\mu_1 + \lambda_3\mu_3 - \lambda_3\mu_1).$$

Similarly, by changing the roles of $e_1, e_2, e_3$ played in the above discussions in circular order, we also have the following relations:

$$\lambda = \frac{1}{6}(\lambda_2\mu_3 - \lambda_3\mu_2 + \lambda_2\mu_1 - \lambda_1\mu_2),$$

$$\lambda = \frac{1}{6}(\lambda_3\mu_1 - \lambda_1\mu_3 + \lambda_3\mu_2 - \lambda_2\mu_3).$$

Now, taking the summation of (6.8), (6.9) and (6.10) immediately yields $\lambda = 0$, which verifies the first assertion of the Main Theorem.

To prove the remaining part of the Main Theorem, we need a more careful calculation for the expression of $I$ defined by (6.2). For that purpose, we make use of (2.19) to simplify $I$ as follows:

$$I = \mu_1^2h_{12}^3 + [g(\nabla e_2 P)e_1, e_3] + \mu_3h_{12}^3\mu_1 - \lambda_1(\lambda_1 + \lambda_3)h_{12}^3$$

$$+ \lambda_1[\nabla (\nabla e_2 (P)J)e_1, e_3] - \lambda_3h_{12}^3 + g(\nabla e_2 (P)J)e_3, e_1) - \lambda_1h_{12}^3]$$

$$- \mu_2\mu_1h_{12}^3 - [g(\nabla e_3 P)e_2, e_3] + \mu_3h_{12}^3\mu_1 + \lambda_2h_{12}^3$$

$$- [g(\nabla e_3 (P)J)e_1, e_3) - \lambda_3h_{12}^3\lambda_1 - \mu_1(\mu_1 + \mu_3)h_{12}^3$$

$$- \mu_1[\nabla (\nabla e_3 P)e_1, e_3) + \mu_3h_{12}^3 + g(\nabla e_3 (P)J)e_3, e_1) + \mu_1h_{12}^3]$$

$$- \lambda_2\lambda_1h_{12}^3 + g((\nabla e_1 (P)J)e_2, e_3) - \lambda_3h_{12}^3\lambda_1.$$
Noting that \( \lambda_i^2 + \mu_i^2 = 1, \ i = 1, 2, 3 \), the above expression reduces to that

\[
I = - \left[ 1 + 2(\lambda_1 \lambda_3 + \mu_1 \mu_3) + (\lambda_1 \lambda_2 + \mu_1 \mu_2) \right] h_{12}^3
\]

(6.12)

- \mu_1 \left[ g((\tilde{\nabla}_e)P)e_2, e_3 \right] + g((\tilde{\nabla}_e P)e_3, e_1) + \lambda_1 \left[ g((\tilde{\nabla}_e (PJ))e_2, e_3 \right] + g((\tilde{\nabla}_e (PJ))e_3, e_1) \right].

Moreover, by use of the formulas (2.8) and (2.9), we have the calculations:

\[
g((\tilde{\nabla}_e)P)e_2, e_3) = \frac{1}{2\sqrt{3}}(\lambda_2 - \lambda_3), \quad g((\tilde{\nabla}_e P)e_3, e_1) = \frac{1}{2\sqrt{3}}(\lambda_3 - \lambda_1),
\]

(6.13)

\[
g((\tilde{\nabla}_e (PJ))e_2, e_3) = \frac{1}{2\sqrt{3}}(\mu_2 - \mu_3), \quad g((\tilde{\nabla}_e (PJ))e_3, e_1) = \frac{1}{2\sqrt{3}}(\mu_3 - \mu_1).
\]

Substituting (6.13) into (6.12) yields

\[
I = - \left[ 1 + 2(\lambda_1 \lambda_3 + \mu_1 \mu_3) + (\lambda_1 \lambda_2 + \mu_1 \mu_2) \right] h_{12}^3
\]

(6.14)

- \frac{1}{2\sqrt{3}} h_1^3(\lambda_2 - \lambda_1) + \frac{1}{2\sqrt{3}} \lambda_1(\mu_2 - \mu_1)
\]

= - \left[ 1 + 2(\lambda_1 \lambda_3 + \mu_1 \mu_3) + (\lambda_1 \lambda_2 + \mu_1 \mu_2) \right] h_{12}^3 + \frac{1}{2\sqrt{3}}(\lambda_1 \mu_2 - \lambda_2 \mu_1).

Putting (6.14) into (6.4), with the fact that \( \lambda = 0 \), we eventually obtain

\[
12[R_{2113}(h_{11}^1 - 2h_{13}^1) + R_{2123}h_{11}^1 - 2R_{2113}h_{12}^1]
\]

- \frac{1}{2\sqrt{3}}(\lambda_1 \mu_2 - \lambda_2 \mu_1) + \frac{1}{2} h_{12}^3 = 0.

Completion of the proof of the Main Theorem.

If \( M \) is a totally geodesic Lagrangian submanifold of the homogeneous \( NK \mathbb{S}^3 \times \mathbb{S}^3 \), then it is trivially \( J \)-isotropic, in that case, according to the classification theorem of \( [28] \), \( M \) should be given by one of immersions \( \{ f_i \}_{i=1}^6 \).

Next, we assume that \( M \) is \( J \)-isotropic but not totally geodesic. We are sufficient to prove that \( M \) is given by the immersion \( f_7 \) or \( f_8 \).

As we have already proved that \( \lambda = 0 \), the relations \( \lambda = \cos 2\theta_i \) and \( \mu = \sin 2\theta_i \) enable us to rewrite the equations (6.8), (6.9) and (6.10) as below:

\[
\cos(\theta_3 - \theta_2) \sin(\theta_3 + \theta_2 - 2\theta_1) = 0,
\]

(6.16)

\[
\cos(\theta_1 - \theta_3) \sin(\theta_1 + \theta_3 - 2\theta_2) = 0,
\]

\[
\cos(\theta_2 - \theta_1) \sin(\theta_2 + \theta_1 - 2\theta_3) = 0.
\]

From (6.16), we are sufficient to consider the following three cases.

**Case-I.** For any distinct \( i, j, k \), it hold the relations

\[
\sin(\theta_i + \theta_j - 2\theta_k) = 0, \ \cos(\theta_i - \theta_j) \neq 0.
\]

In this case, we have \( \theta_i + \theta_j - 2\theta_k \equiv 0 \mod \pi \), for any distinct \( i, j, k \). Noting that by Lemma (2.4) \( \theta_i + \theta_2 + \theta_3 \equiv 0 \mod \pi \). It follows that all the angles \( \theta_i \equiv 0 \mod \pi/3 \).

As \( M \) is not totally geodesic, then by Lemma (2.2) all the angles are different modulo \( \pi \).
So after rearranging the order of the angle functions if necessary, only the possibility as 
\((\theta_1, \theta_2, \theta_3) = (0, \pi/3, 2\pi/3)\) can occur, and therefore we have 
\[ \lambda_1 = 1, \; \lambda_2 = \lambda_3 = -1/2, \; \mu_1 = 0, \; \mu_2 = -\mu_3 = \sqrt{3}/2. \]

Since all the angles functions are constant, from Lemma 2.1 we see that \(h_{jj}^1 = 0\) for all \(i, j\). Substituting these relations into (6.15), we get 
\[ 0 = -24R_{1221}h_{12}^3 + 2h_{12}^3 + \sqrt{3}(\lambda_1\mu_2 - \lambda_2\mu_1) \]
(6.17) 
\[ -2h_{12}^3[1 + 2(\lambda_1\lambda_3 + \mu_1\mu_3) + (\lambda_1\lambda_2 + \mu_1\mu_2)] \]
\[ = -24R_{1221}h_{12}^3 + \frac{3}{2}h_{12}^3 + \frac{3}{4}. \]

Using the following Gauss equation 
(6.18) 
\[ R_{2112} = \frac{5}{12} + 2(\lambda_1\lambda_2 + \mu_1\mu_2) - (h_{12}^3)^2 = \frac{1}{4} - (h_{12}^3)^2, \]
we can rewrite (6.17) as 
(6.19) 
\[ 32(h_{12}^3)^3 - 6h_{12}^3 + 1 = 0. \]

The above equation for \(h_{12}^3\) has exactly only two different solutions, i.e. \(h_{12}^3 = 1/4\) or \(h_{12}^3 = -1/2\). Then the Gauss equations give that 
(6.20) 
\[ R_{i\bar{j}j} = \frac{5}{12} + 2(\lambda_i\lambda_j + \mu_i\mu_j) - (h_{12}^3)^2 = \frac{1}{4} - (h_{12}^3)^2, \; \forall i \neq j. \]

If \(h_{12}^3 = 1/4\), then \(M\) has constant sectional curvature 3/16. By Theorem 5.3 of [9], 
\(M\) is locally given by the immersion \(f_7\).

If \(h_{12}^3 = -1/2\), then \(M\) is flat. By Theorem 5.4 of [9], \(M\) is locally given by the 
immersion \(f_8\).

**Case-II.** In \(\{\cos(\theta_2 - \theta_1), \cos(\theta_3 - \theta_2), \cos(\theta_1 - \theta_3)\}\) exactly one is zero, say 
\[ \cos(\theta_2 - \theta_1) = 0, \; \cos(\theta_3 - \theta_2) \neq 0, \; \cos(\theta_1 - \theta_3) \neq 0, \]
and thus \(\sin(\theta_3 + \theta_2 - 2\theta_1) = \sin(\theta_1 + \theta_3 - 2\theta_2) = 0\).

In this case, we have 
(6.21) 
\[ \theta_1 - \theta_2 \equiv \frac{\pi}{2} \; \text{mod} \; \pi, \]
\[ \theta_3 + \theta_2 - 2\theta_1 \equiv 0 \; \text{mod} \; \pi, \]
\[ \theta_1 + \theta_3 - 2\theta_2 \equiv 0 \; \text{mod} \; \pi. \]

As by Lemma 2.1 we also have \(\theta_1 + \theta_2 + \theta_3 \equiv 0 \; \text{mod} \; \pi\), which is clearly a contradiction to (6.21). Hence **Case-II** does not occur.

**Case-III.** At least two elements of \(\{\cos(\theta_2 - \theta_1), \cos(\theta_3 - \theta_2), \cos(\theta_1 - \theta_3)\}\) are zero, 
say \(\cos(\theta_2 - \theta_1) = \cos(\theta_1 - \theta_3) = 0\).

In this case, it holds that 
\[ \theta_2 - \theta_1 \equiv \frac{\pi}{2} \; \text{mod} \; \pi, \]
\[ \theta_1 - \theta_3 \equiv \frac{\pi}{2} \; \text{mod} \; \pi. \]
This yields that $\theta_2 - \theta_3 \equiv 0 \mod \pi$, and then by Lemma 2.2 we know that $M$ is totally geodesic. This is a contradiction to our assumption. Hence Case-III does not occur either.

In conclusion, we have completed the proof of the Main Theorem. □

References

[1] J. Bolton, F. Dillen, B. Dioos and L. Vrancken, Almost complex surfaces in the nearly Kähler $S^3 \times S^3$, Tôhoku Math. J. (2) 67 (2015), 1–17.
[2] J. B. Butruille, Classification des variétés approximativement kähleriennes homogènes, Ann. Glob. Anal. Geom. 27 (2005), 201–225.
[3] J. B. Butruille, Homogeneous nearly Kähler manifolds, in: Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. Theor. Phys., 16, Eur. Math. Soc., Zürich, 2010. pp. 399–423.
[4] J. Bolton, L. Vrancken and L. M. Woodward, On almost complex curves in the nearly Kähler 6-sphere, Quart. J. Math. Oxford. Ser. 45 (1994), 407–427.
[5] F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken, On totally real 3-dimensional submanifolds of the nearly Kaehler 6-sphere, Proc. Amer. Math. Soc. 99 (1987), 741–749.
[6] F. Dillen, L. Verstraelen and L. Vrancken, Classification of totally real 3-dimensional submanifolds of $S^6(1)$ with $K \geq 1/16$, J. Math. Soc. Japan. 42 (1990), 565–584.
[7] B. Dioos, H. Li, H. Ma and L. Vrancken, Flat almost complex surfaces in $S^3 \times S^3$. Preprint, 2014.
[8] B. Dioos, J. Van der Veken and L. Vrancken, Sequences of harmonic maps in the 3-sphere, Math. Nachr. 288 (2015), 2001–2015.
[9] B. Dioos, L. Vrancken and X. Wang, Lagrangian submanifolds in the nearly Kähler $S^3 \times S^3$, arXiv:1604.05060v1.
[10] M. Djorić and L. Vrancken, On J-parallel totally real three-dimensional submanifolds of $S^6(1)$, J. Geom. Phys. 60 (2010), 175–181.
[11] N. Ejiri, Totally real submanifolds in a 6-sphere, Proc. Amer. Math. Soc. 83 (1981), 759–763.
[12] L. Foscolo and M. Haskins, New $G_2$-holonomy cones and exotic nearly Kähler structures on $S^6$ and $S^1 \times S^5$, arXiv:1501.07838v2.
[13] A. Gray, Nearly Kähler manifolds, J. Diff. Geom. 4 (1970), 283-309.
[14] A. Gray, The structure of nearly Kähler manifolds, Math. Ann. 223 (1976), 233-248.
[15] J. Gutowski, S. Ivanov and G. Papadopoulos, Deformations of generalized calibrations and compact non-Kähler manifolds with vanishing first Chern class, Asian J. Math. 7 (2003), 39–79.
[16] Z. Hu and Y. Zhang, Rigidity of the almost complex surfaces in the nearly Kähler $S^3 \times S^3$, J. Geom. Phys. 100 (2016), 80–91.
[17] H. Li and X. Wang, Isotropic Lagrangian submanifolds in complex Euclidean space and complex hyperbolic space, Results Math. 56 (2009), 387–403.
[18] J. D. Lotay, Ruled Lagrangian submanifolds of the 6-sphere, Trans. Amer. Math. Soc. 366 (2011), 2305–2339.
[19] S. Montiel and F. Urbano, Isotropic totally real submanifolds, Math. Z. 199 (1988), 55–60.
[20] A. Moroianu and U. Semmelmann, Generalized Killing spinors and Lagrangian graphs, Diff. Geom. Appl. 37 (2014), 141–151.
[21] P. A. Nagy, Nearly Kähler geometry and Riemannian foliations, Asian J. Math. 6 (2002), 481-504.
[22] P. A. Nagy, On nearly-Kähler geometry, Ann. Glob. Anal. Geom. 22 (2002), 167–178.
[23] B. O’Neill, Isotropic and Kähler immersions, Canad. J. Math. 17 (1965), 905–915.
[24] L. Schäfer and K. Smoczyk, *Decomposition and minimality of Lagrangian submanifolds in nearly Kähler manifolds*, Ann. Glob. Anal. Geom. **37** (2010), 221–240.

[25] L. Vrancken, *Some remarks on isotropic submanifolds*, Publ. Inst. Math. (N.S.) **51(65)** (1992), 94–100.

[26] L. Vrancken, *Special Lagrangian submanifolds of the nearly Kähler 6-sphere*, Glasg. Math. J. **45** (2003), 415–426.

[27] X. Wang, H. Li and L. Vrancken, *Lagrangian submanifolds in 3-dimensional complex space forms with isotropic cubic tensor*, Bull. Belg. Math. Soc. Simon Stevin **18** (2011), 431–451.

[28] Y. Zhang, B. Dioos, Z. Hu, L. Vrancken and X. Wang, *Lagrangian submanifolds in the 6-dimensional nearly Kähler manifolds with parallel second fundamental form*, J. Geom. Phys. **108** (2016), 21–37.

**School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, People’s Republic of China.**

E-mails: huzj@zzu.edu.cn; zhangysookk@163.com

**Present address of Yinshan Zhang:**

**School of Sciences, Henan University of Engineering, Zhengzhou, 451191, People’s Republic of China.**