Phase Locking, Devil’s Staircases, Farey Trees, and Arnold Tongues in Driven Vortex Lattices with Periodic Pinning

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Abstract

Using numerical simulations, we observe phase locking, Arnold tongues, and Devil’s staircases for vortex lattices driven at varying angles with respect to an underlying superconducting periodic pinning array. This rich structure should be observable in transport measurements. The transverse $V(I)$ curves have a Devil’s staircase structure, with plateaus occurring near the driving angles along symmetry directions of the pinning array. Each of the plateaus corresponds to a different dynamical phase with a distinctive vortex structure and flow pattern.

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Introduction. — Numerous nonlinear driven systems in physics, astronomy, and engineering exhibit striking responses with complex phase-locking plateaus characterized by Devil’s staircases, Arnold tongues, and Farey trees [1–4]. Here, we present the first evidence that these structures can be observed in bulk superconductors.

Driven vortex lattices (VLs) interacting with either random or periodic disorder have attracted growing interest due to the rich variety of nonequilibrium dynamic phases which are observed in these systems. These phases include the elastic and plastic flow of vortices which can be related to VL order and transport properties [5–9]. Periodic pinning arrays interacting with VLs are now attracting increasing attention as recent experiments with patterns of holes [10] and magnetic dots [11] have produced interesting commensurability effects and enhanced pinning. These systems are an excellent realization of an elastic lattice interacting with a periodic substrate that is found in a wide variety of condensed matter systems including charge-density-waves, Josephson-junction arrays, and Frenkel-Kontorova-type models of friction (see, e.g., [12]). An interesting aspect of periodic pinning arrays that has not been addressed so far is how the symmetry properties of the array affect the transport properties as the VL is driven at different angles.

We find that as a slowly increasing transverse force is applied to a VL already moving in the longitudinal direction, the VL undergoes a remarkable series of locking transitions that significantly affect both the VL ordering and transport properties. These locking phases occur when the direction of the vortex motion locks with a symmetry direction of the pinning array. As the VL passes through these phases, the transverse velocity component as a function of increasing transverse drive shows a series of plateaus which form a Devil’s staircase structure [1–3]. At the boundaries of certain locked phases the VL undergoes a transition to a plastic flow phase in which defects are generated in the VL. In the locked phases the VL undergoes elastic flow in static 1D channels and the overall VL has a variety of orderings, including triangular and square.

Simulation. — We consider a 2D slice of $N_v$ 3D rigid vortices interacting with a square array of $N_p$ parabolic wells, with lattice constant $a$, and periodic boundary conditions. We
integrate the equations of vortex motion \( f_i = f_{i,\text{vv}}^i + f_{i,\text{vp}}^i + f_d = \eta v_i \). The total force \( f_i \) on vortex \( i \) includes interactions with other vortices \( f_{i,\text{vv}}^i \), pinning \( f_{i,\text{vp}}^i \) by parabolic wells, and an applied driving force \( f_d = f_x \hat{x} \). The vortex-vortex interaction between vortex \( i \) and the other \( N_v \) vortices is \( f_{i,\text{vv}}^i = \sum_{j=1}^{N_v} f_0 K_1(\|r_i - r_j\|/\lambda) \hat{r}_{ij} \), where \( K_1(r/\lambda) \) is a modified Bessel function, \( \lambda \) is the penetration depth, \( f_0 = \Phi_0^2 / 8\pi^2\lambda^3 \), \( \hat{r}_{ij} = (r_i - r_j) / \|r_i - r_j\| \), and we set \( \eta = 1 \). All lengths, fields, and forces are given in units of \( \lambda \), \( \Phi_0 / \lambda^2 \), and \( f_0 \), respectively. For most of the results presented here the number of vortices is close to the number of pinning sites, \( N_v = 1.062 N_p \). We have conducted a series of simulations with different pinning parameters so that accurate phase diagrams of the dynamic phases can be obtained. In order to investigate finite size effects we have examined system sizes varying between \( 36\lambda \times 36\lambda \) and \( 108\lambda \times 108\lambda \), with \( N_v \) between \( N_v = 550 \) and \( N_v = 4955 \).

Voltage–Current Response.— First, the VL ground state at zero applied driving force is found by simulated annealing (i.e., by cooling the VL from high-\( T \)). After a low energy ground state is found, a slowly increasing driving force, \( f_x \), is applied along the horizontal symmetry axis of the square pinning. We find that increasing \( f_x \) in increments of 0.001\( f_0 \) every 400 MD steps, from \( f_x = 0 \) to \( f_x = 3.0 f_0 \), is slow enough that the vortex dynamics does not depend on the rate of increase of \( f_x \). Once \( f_x \) is brought to 3.0\( f_0 \) it is held constant while a force, which we label \( f_y \), is applied in the transverse or \( y \)-direction. We increase \( f_y \) from 0 to 3.25\( f_0 \), also in increments of 0.001\( f_0 \) every 400 MD steps. The total driving force has a net magnitude of \( f_d = (f_x^2 + f_y^2)^{1/2} \) at an angle \( \theta = \tan^{-1}(f_y/f_x) \) with respect to the \( x \)-direction. We compute the average velocity of the moving vortices in both the longitudinal \( V_x = (1/N_v) \sum_{i=1}^{N_v} v_i \cdot \hat{x} \) and the transverse \( V_y = (1/N_v) \sum_{i=1}^{N_v} v_i \cdot \hat{y} \) direction, as \( f_y \) is increased. Velocity versus driving plots correspond to experimentally measurable voltage-current \( V(I) \) curves.

In Fig. 1(a) we present a typical plot of \( V_x \) and \( V_y \). For \( f_y \lesssim 0.4 f_0 \), \( V_y = 0 \) indicating that the VL is pinned in the \( y \)-direction even though the VL is moving in the \( x \)-direction. Depinning in the transverse direction occurs at \( f_y = 0.4 f_0 \), as indicated by the sharp jump up in \( V_y \). We label this critical transverse depinning force \( f_y^c \). A jump up in \( V_x \) is also observed
at $f_y^c$. As $f_y$ is linearly increased, $V_y$ does not grow linearly but instead in a remarkable series of jumps and plateaus of varying sizes \[3\]. Along the plateaus $V_y$ is constant or increasing very slowly, indicating that the vortex motion is locked in a certain direction for a finite range of increasing $f_y$. The small jumps and dips in $V_x$ correspond to the onset of plateaus in $V_y$. The plateaus in $V_y$ occur when the ratio of $f_y$ to $f_x$ is near a rational value: $f_y/f_x = p/q$, where $p$ and $q$ are integers. In Fig. 1(a) the largest plateaus occur at $p/q = 0, 1/3, 1/2, 2/3$ and 1. Fig. 1(b) shows a blow-up of a region in Fig. 1(a) for values of $f_y = 0.6f_0$ to $2.1f_0$, where additional plateaus at $p/q = 1/5, 1/4, 2/5, 3/7$, and $3/5$ are highlighted. For larger system sizes we find exactly the same behavior in $V_y$ and $V_x$ as observed in Fig. 1, indicating that it is independent of the system size.

**Vortex Dynamics and the Origin of the Plateaus.**—To understand why the plateaus occur as well as the VL dynamics in the plateau and non-plateau regions, in Fig. 2(a-d) we plot the vortex trajectories for rational ratios of $f_y/f_x = 0, 1/2, 1$, and the irrational ratio $f_y/f_x = 2\pi/11 = 0.571...$ In Fig. 2(a), where $f_y < f_y^c$, the vortex motion traverses pin sites periodically and it is only along the $x$-direction—with the vortex flow restricted in 1D paths along the pinning rows. This periodic 1D motion persists up to $f_y = f_y^c$, at which point the vortices also begin to flow in the $y$-direction. In Fig. 2(b), for $p/q = 1/2$ where a large plateau in $V_y$ is observed in Fig. 1, the vortices again exhibit periodic motion and flow in 1D channels along the pinning sites—and along a symmetry axis of the pinning array at an angle $\theta = \tan^{-1}(1/2)$ from the $x$-axis. A similar periodic 1D motion is seen in (d) for $f_y/f_x = 1$, with the VL motion at $45^\circ$ from the $x$-axis. In Fig. 2(c), at the irrational $f_y/f_x$ ratio, the vortex trajectories are different than those observed in Fig. 2(a,b,d). Here the quasiperiodic vortex trajectories drift over time, eventually covering the sample (i.e., ergodic-like motion). In general, the plateau regions (with rational $f_y/f_x$ in $V_y(I)$) correspond to periodic 1D vortex trajectories, while the non-plateau regions produce quasiperiodic trajectories.[133]

To understand how the vortex motion locks into certain driving angles, we first consider the case $f_y/f_x = 0$. Here the vortices move along the pinning rows in 1D paths, with each vortex traversing a distance $a - 2r_p$ between pinning sites, as seen in Fig. 2(a). An
application of a transverse force $f_y$ causes the moving vortices to drift a small distance in the $y$–direction. Once the vortices interact with the pinning sites, they feel a force that moves them towards the center of the pinning site which keeps them locked along the $x$-direction. When $f_y$ is large enough, $f_y > f_x \tan(r_p/a)$, the vortices are able to break off from moving only along the $x$-direction and start moving in the $y$-direction as well.

As $f_y$ is increased beyond $f_y^c$, the net driving force vector will be at an angle with the horizontal. Due to the symmetry of the square pinning array, along the angles where $\theta = \tan^{-1}(p/q)$, the vortices encounter pinning sites periodically spaced a distance $a_\theta$ apart. This distance is related to the pinning lattice constant $a$ by $a_\theta = a(p^2 + q^2)^{1/2}$. Along these commensurate angles, the vortex motion will be periodic and locked in 1D channels in a similar manner as the $f_y/f_x = 0$ case. The force needed to depin the vortices from the commensurate angles will vary since $a_\theta$ varies. For values where $a_\theta$ is small, the vortices will move only a small distance between pinning sites, so a higher depinning force is needed. For large $a_\theta$ the vortices will move a much longer distance before encountering the pinning sites, so a much smaller depinning force is needed. This is in agreement with Fig. 1 where the largest plateaus (due to enhanced pinning) occur for values of $p/q$ that produce the lowest distance between pinning sites, that is the smallest $a_\theta$ (i.e., $p/q = 0/1, 1/1$, and $1/2$).

The onset of certain plateaus coincide with a variety of structural transitions in the VL. We quantify this angle-dependent evolving topological order by using the Voronoi (or Wigner-Seitz) construction to obtain the fraction of vortices with coordination numbers six, $P_6$, and four, $P_4$. In Fig. 3(a,b) we show the evolution of $P_6$ and $P_4$ as $f_y$ is increased, for the same system as in Fig. 1. For $f_y < f_y^c$, $P_6 \approx 0.68$, indicating a mostly triangular VL. At $f_y = f_y^c$, a dip in $P_6$, along with direct observation of the VL flow, show that the VL disorders due to plastic deformations. Right after the initial dip in $P_6$ the VL suddenly regains considerable triangular ordering, as indicated by $P_6 \approx 0.95$. Small dips in $P_6$ can be seen near the $1/4, 1/3$, and $2/3$ locking regions. At the $1/2$ locking region the VL is considerably disordered, as indicated by the sharp drop in $P_6$. This is consistent with Fig. 3(c), where both the vortex positions and Voronoi polygons are shown for a $12\lambda \times 12\lambda$
region in the 1/2 locking region. At the 1/1 locking region $P_6$ drops almost to zero while $P_4$ increases to about 0.9, indicating a structural phase transition from a triangular to a square VL. Here, the $f_y = f_x$ symmetric drive is what produces a moving square VL. The less symmetric drives ($2f_y = f_x$ and $3f_y = f_x$), produce more distorted squares. For the special case when $f_y = 0$ and for the $B$ used in Fig. 3, correlations between nearby VL rows are strong, and near 2/3 of the VL has triangular order (which diminishes for weaker $B$’s).

In Fig. 3(d,e) the vortex positions and Voronoi polygons are shown for (d) right before the transition to the 1/1 locking region and (e) in the 1/1 locking region showing the triangular and square ordering of the VL respectively. Right at the boundaries of the 1/1 phase, the VL is strongly disordered and has a similar structure to Fig. 3(c).

Phase Diagrams with Arnold Tongues.— We have derived five phase diagrams which indicate the evolution of the plateau regions versus the following parameters: $f_p$, $n_p$, $r_p$, commensurability, and disorder. These five phase diagrams are all very similar, and thus here we present only one: Fig. 4(a). This is obtained by conducting a series of simulations in which the maximum pinning force $f_p$ is varied between $0.25 \leq f_p/f_0 \leq 2.75$. The phase diagram shows 18 clearly defined (shaded) Arnold tongues or plateaus. As $f_p$ is decreased the widths of the tongues also show a corresponding decrease. For $f_p/f_0 > 2.5$ several locking phases are lost (i.e., 1/6, 4/7, 5/6) due to overlapping by other locking regions. For $f_p/f_0 < 1.0$ only the strongest plateau regions can be resolved within the accuracy of our calculations.

The phase diagram in Fig. 4(a) has the same structure as Arnold tongues found in phase locking systems where the widths of the tongues, or locking regions, increase as the nonlinear coupling increases. Here, the coupling is between the vortices and the pinning array, and is increased with increasing $f_p$, $r_p$, $n_p$, density of vortices (i.e., the commensurability $B/B_\phi$), and pin-location order. In Fig. 4(b) we present the width of the 0/1 locking region for varying pinning density in units of the pinning lattice constant $a$. As $a$ decreases the width of the locked region increases. This can be understood by considering that as $a$ decreases the vortices in the locked region will move a smaller distance between
pinning sites; thus a higher transverse force is needed to break the vortices away from the locked region. The widths of the other locked regions show the same behavior as the 0/1 region for increasing $a$. 

We have also examined the effects of pin disorder on the width of the locking regions by conducting a series of simulations in which the pinning sites are randomly displaced up to an amount $\delta r$ away from the perfectly square pinning lattice. We consider the case where $\delta r = a/2$ to be a good approximation to a random pinning array. In Fig. 4(c), we examine how the width of the 0/1 locking region, $f_y^c$, decreases as $\delta r$ is increased. It is of interest to compare our results for large disorder with Ref [6](a) in which a nonzero transverse critical force $f_y^c$ was predicted. Recent $T = 0$ MD simulations have observed extremely small transverse barriers [6](b). We find that for large disorder, $\delta r = a/2$, a true transverse barrier (i.e., $V_y = 0$) is not observed. Also, for a triangular array of pins, the plateaus occur for $\theta = \tan^{-1}(\sqrt{3}p/(2q + 1))$.

**Summary.** — In conclusion, we have found that as an increasing transverse force is applied to a strongly driven VL interacting with a periodic pinning array, the VL undergoes a remarkable series of locking transitions in which both the VL order and flow patterns change. As the VL passes through these transitions, $V_y$ exhibits a striking series of plateaus forming a Devil’s staircase structure. The width variations of these plateaus with different pinning form Arnold tongues which can be indexed via a Farey tree construction. These locking effects occur whenever the VL is driven along a symmetry angle of the pinning array. For a square pinning array, the locking phases occur when driving in the longitudinal direction is a rational ratio, $f_y/f_x = p/q$. These predictions can be tested experimentally and we hope that this work will motivate several novel experiments. Moreover, other candidate systems where these predictions may be accessible include: driven Wigner crystals interacting with a periodic array of donors, driven colloids interacting with optical-trap arrays, spin- and charge-density waves, Josephson-junction arrays, and solid friction experiments.

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[3] The overall structure of the transverse voltage $V_y(I)$ is that of a Devil’s staircase, in which plateaus appear at rational ratios of $f_y/f_x = p/q$ with the largest plateaus occurring when $p/q$ has the smallest denominator, in agreement with Fig. 1. The hierarchy of plateau sizes follows the Farey tree construction, which orders all rationals in $[0, 1]$ with increasing denominators $q$ according to the rule that the largest plateau between $p/q$ and $p'/q'$ is $(p+p')/(q+q')$, and orders all mode-locking steps with $w = p/q$ according to their decreasing widths.

[4] Note that Arnold tongues are typically studied in the context of nonlinear circle maps with two degrees of freedom. Here we have very many degrees of freedom which are interacting with each other, and also with numerous sites in the substrate. Moreover our dynamics is continuous, not a discrete map. Thus, our VL system is far more complex than the standard discrete 2D maps used to describe phase-locked structures.

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FIG. 1. (a) Average longitudinal $V_x$ (upper curve) and transverse $V_y$ velocities versus the transverse driving force $f_y$ for a $36\lambda \times 36\lambda$ sample with a square pinning array, density of field lines $B$ satisfying $B/B_\phi = 1.062$, matching field $B_\phi = 0.4 \Phi_0/\lambda^2$, density of pinning sites $n_p = 0.4/\lambda^2$, $f_p = 2.5f_0$, $a = 1.57\lambda$, and $r_p = 0.3\lambda$. $f_x$ is fixed at $f_x = 3.0f_0$. Plateaus are seen in $V_y$ near values where $f_y/f_x = p/q$, where $p$ and $q$ are integers. The largest plateaus (at $0/1$, $1/3$, $1/2$, $2/3$, and $1/1$) are clearly seen. (b) shows a blow up of $V_y$ from (a), for $f_y = 0.6f_0$ to $2.1f_0$, where additional plateaus at $1/5$, $1/4$, $2/5$, $3/7$, and $3/5$ can be seen more clearly. The overall structure in $V_y$ is that of a Devil’s staircase [1-3].

FIG. 2. The vortex trajectories for a subset of the system in Fig. 1 at the plateau regions (a) $f_y/f_x = 0$, (b) $f_y/f_x = 1/2$, and (d) $f_y/f_x = 1/1$, and at the non-plateau region (c) $f_y/f_x = 2\pi/11 = 0.571...$ At the plateau regions, the vortices move in 1D channels, periodically along the pinning rows; while at the non-plateau regions the vortices exhibit quasiperiodic trajectories.
FIG. 3. The fraction of (a) six-fold $P_6$ and (b) four-fold $P_4$ coordinated vortices versus transverse driving force $f_y$, for the same system as in Fig. 1. Large drops in $P_6$ can be seen at $f_y^c$, as well as at the 1/2 and 1/1 locking regions. Smaller dips in $P_6$ can be seen at the 1/4, 1/3, and 2/3 plateaus regions. At the 1/1 transition $P_4$ rises to $\approx 0.9$ indicating a transition to a square VL. In (c,d,e) both the vortex positions and Voronoi polygons for a subset of the VL can be seen for: (c) the 1/2 locking region, where a disordered VL is observed; (d) right before the 1/1 plateau, with a triangular VL; and (e) at the 1/1 plateau, with a square VL.

FIG. 4. (a) Phase diagram, for the system in Fig. 1, showing Arnold tongues (shaded); i.e., the widths of the locking regions versus $f_p$. As $f_p$ decreases, the tongues (locking regions with periodic trajectories) shrink. In (b,c) the width of the first locking region or $f_y^c$ is shown versus pinning lattice constant $a$ (b) and disorder $\delta r/2a$ (c).
FIG. 1.

Transverse Force $f_y / f_0$

Transverse Velocity $V_y$

Velocities: $V_y, V_x$

$f_x = 3.0 f_0$

(a) Transverse Force $f_y / f_0$

(b) Transverse Velocity $V_y$
FIG. 2.
FIG. 3.
FIG. 4.