Gonihedric String Equation II

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Abstract

Arguing that the equation for the gonihedric string should have a generalized Dirac form, we found a new equation which corresponds to a symmetric solution of the Majorana commutation relations and has non-Jacobian form. The corresponding generalized gamma-matrices are anticommuting and guarantee unitarity at all orders of $v/c$. The previous solution was in a Jacobian form and admits unitarity at zero order. Explicit formulas for the mass spectrum lead to nonzero string tension $M_j^2 \geq M^2(j+1)^2$. The equation does not admit tachyonic solutions, but still has unwanted ghost solutions. We discuss also new dual transformation of the Dirac equation and of the proposed generalizations.

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There is some experimental and theoretical evidence for the existence of a string theory in four dimensions which may describe strong interactions and represent the solution of QCD [1].

One of the possible candidates for that purpose is the gonihedric string which has been defined as a model of random surfaces with an action which is proportional to the linear size of the surface [2]

\[ A(M) = m \sum_{\langle ij \rangle} \lambda_{ij} \cdot \Theta(\alpha_{ij}), \quad \Theta(\alpha) = |\pi - \alpha|, \quad (1) \]

where \( \lambda_{ij} \) is the length of the edge \( \langle ij \rangle \) of the triangulated surface \( M \) and \( \alpha_{ij} \) is the dihedral angle between two neighbouring triangles of \( M \) sharing a common edge \( \langle ij \rangle \).

The model has a number of properties which make it very close to the Feynman path integral for a point-like relativistic particle. In the limit when the surface degenerates into a single world line the action becomes proportional to the length of the path and the classical equation of motion for the gonihedric string is reduced to the classical equation of motion for a free relativistic particle. At the classical level the string tension is equal to zero and, as it was demonstrated in [2], quantum fluctuations generate the nonzero string tension

\[ \sigma_{\text{quantum}} = \frac{d}{a^2} \left( 1 - \ln \frac{d}{\beta} \right), \quad (2) \]

where \( d \) is the dimension of the spacetime, \( \beta \) is the coupling constant, \( a \) is the scaling parameter and \( \varsigma = (d - 2)/d \) in (1).

It is natural therefore to ask what type of equation may describe this string theory in the continuum limit. The aim of the article [3] was to suggest a possible answer to this question. The analysis of the transfer matrix shows [3] that the desired equation should describe propagation of fermionic degrees of freedom distributed over the space contour. When this contour shrinks to a point, the equation should describe propagation of a free Dirac fermion. Thus each particle in this theory should be viewed as a state of a complex fermionic system and the system should have a point-particle limit when there is no excitation of the internal motion. In the given case this restriction should be understood as a principle according to which the infinite sequence of particles should contain the spin one-half fermion and the equation should has the Dirac form [3].

\[ \text{1} \]

The angular factor \( \Theta \) and the index \( \varsigma \) define the rigidity of the random surfaces [4], and the convergence of the partition function. As it was proved in [6], the partition function \( Z_T(\beta) \) for the given triangulation \( T \) is convergent when the parameter \( \varsigma \) is in the interval \( 0 < \varsigma \leq 1 \). In [6] it has been proved that the full partition function \( Z(\beta) = \sum_T Z_T(\beta) \) (the sum is over all triangulations \( \{T\} \)) is divergent for \( \frac{d-2}{d} < \varsigma \leq 1 \), where \( d \) is the dimension of the spacetime.

In [6] it was demonstrated that for \( 0 < \varsigma \leq \frac{d-2}{2d} \) the \( Z(\beta) \) is convergent and the scaling limit should be taken exactly at the point \( \varsigma = \frac{d-2}{2d} \) so that the string tension (2) is generated. In addition to the formulation of the theory in the continuum space the system allows an equivalent representation on Euclidean lattices where a surface is associated with a collection of plaquettes [5] and it has been proved that the entropy has exponential behaviour and not factorial. The Monte Carlo simulations [4] demonstrate that the gonihedric system undergoes the second order phase transition and the string tension is generated by quantum fluctuations, as it was expected theoretically [4].
\[ \{ i \Gamma_\mu \partial^\mu - M \} \Psi = 0. \]  
(3)

The invariance of this equation under Lorentz transformations \( x'_\mu = \Lambda^\mu_\nu x_\nu, \) \( \Psi'(x') = \Theta(\Lambda) \Psi(x) \) leads to the following equation for the gamma matrices [7, 10]

\[ \Gamma_\nu = \Lambda^\nu_\mu \Theta \Gamma_\mu \Theta^{-1}. \]  
(4)

If we use the infinitesimal form of Lorentz transformations \( \Lambda^\mu_\nu = \eta^\mu_\nu + \varepsilon^\mu_\nu, \) \( \Theta = 1 + \frac{1}{2} \varepsilon^\mu_\nu I^{\mu\nu} \) it follows that gamma matrices should satisfy the Majorana commutation relation [7]

\[ [\Gamma_\mu, I_{\lambda\rho}] = \eta^\mu_\lambda \Gamma^\rho_\rho - \eta^\mu_\rho \Gamma^\rho_\lambda \]  
(5)

where \( I^{\mu\nu} \) are the generators of the Lorentz algebra. These equations allow to find the \( \Gamma_\mu \) matrices when the representation of the \( I^{\mu\nu} \) is given[7]. The original Majorana solution for \( \Gamma_\mu \) matrices is infinite-dimensional (see equation (14) in [7]) and the mass spectrum of the theory is equal to

\[ M^j = \frac{M}{j + 1/2}, \]  
(6)

where \( j = 1/2, 3/2, 5/2, \ldots \) in the fermion case and \( j = 0, 1, 2, \ldots \) in the boson case. The main problems of Majorana theory are the decreasing mass spectrum [8], absence of antiparticles and troublesome tachyonic solutions - the problems common to high spin theories [8].

Nevertheless in [8] the Majorana theory has been interpreted as a natural way to incorporate additional degrees of freedom into the relativistic Dirac equation. Unlike Majorana the authors consider the infinite sequence of high-dimensional representations of the Lorentz group with nonzero Casimir operators \((\vec{a} \cdot \vec{b})\) and \((\vec{a}^2 - \vec{b}^2)\). These representations \((j_0, \lambda)\) and their adjoint \((j_0, -\lambda)\) are enumerated by the index \( r = j_0 + 1/2, \) where \( r = 1, \ldots, N \) and \( j_0 = 1/2, 3/2, \ldots \) is the lower spin in the representation \((j_0, \lambda)\), thus \( j = j_0, j_0 + 1, \ldots \). We took the free complex parameter \( \lambda \) in the real interval \(-3/2 \leq \lambda \leq 3/2\) in order to have real matrix elements for the Lorentz boosts operator \( \vec{b} \) (see (10) and (11)). These representations are infinite-dimensional except of the case \( j_0 = 1/2, \lambda = \pm 3/2\). At the same time their dual representations were also used [8]. The dual transformation \((j_0; \lambda) \rightarrow (\lambda; j_0)\), defined in [8], leads to a subsequent restriction on a free parameter \( \lambda \) and requires \( \lambda = 1/2 \) so that the dual representations become finite-dimensional \((1/2, \pm(1/2+r)). \) The corresponding \( \Sigma^\text{dual} \) equation is not in contradiction with no-go theorem of [8], because dual representations are finite-dimensional.

In the present article we found a new \( \Sigma \Delta \) equation which corresponds to a symmetric solution of the Majorana commutation relations and has non-Jacobian form.

\(^2\)Ettore Majorana suggested this extension of the Dirac equation in 1932 [2] by constructing an infinite-dimensional extension of the gamma matrices. An alternative way to incorporate the internal motion into the Dirac equation was suggested by Pierre Ramond in 1971 [2]. In his extension of the Dirac equation the internal motion is incorporated through the construction of operator-valued gamma matrices. In both cases one can see effectively an extensions of Dirac gamma matrices into the infinite-dimensional case. For our purposes we shall follow Majorana’s approach to incorporate the internal motion in the form of an infinite-dimensional wave equation.
It is based on the same dual representations \((1/2, \pm (1/2 + r))\) of the Lorentz algebra and is a natural extension of the previous \(\Sigma_2^{\text{dual}}\)-equation of \(\mathbb{R}\). The corresponding gamma-matrices are anticommuting

\[
\{\Gamma_\mu, \Gamma_\nu\} = 2 \, g_{\mu\nu} \Gamma_0^2,
\]

and guarantee unitarity at all orders of \(v/c\). The \(\Sigma_2^{\text{dual}}\)-equation admits unitarity at zero order.

For the completeness we shall review the logical and analytical steps which lead to \(\Sigma_2^{\text{dual}}\)-equation \(\mathbb{R}\) and then will derive the new equation which has anticommuting gamma-matrices. In terms of \(SO(3)\) generators \(\vec{a}\) and Lorentz boosts \(\vec{b}\) \(\left( a_x = iI^{23}, \, a_y = iI^{31}, \, a_z = iI^{12}, \, b_x = iI^{10}, \, b_y = iI^{20}, \, b_z = iI^{30} \right)\) the algebra of the \(SO(3, 1)\) generators can be rewritten as \(\mathbb{R}\) (we use Majorana’s notations)

\[
[a_x, a_y] = ia_z \quad [a_x, b_y] = ib_z \quad [b_x, b_y] = -ia_z.
\]

The irreducible representations \(R^{(j)}\) of the \(SO(3)\) subalgebra \(\mathbb{R}\) are

\[
\begin{align*}
\langle j, m | a_z | j, m \rangle &= m \\
\langle j, m | a_+ | j, m-1 \rangle &= \sqrt{(j + m)(j - m + 1)} \\
\langle j, m | a_- | j, m+1 \rangle &= \sqrt{(j + m + 1)(j - m)},
\end{align*}
\]

where \(m = -j, ..., +j\), the dimension of \(R^{(j)}\) is \(2j + 1\) and \(j = 0, 1/2, 1, 3/2, ...\)

The representation \(\Theta = (j_0; \lambda)\) of the Lorentz algebra can be parameterized as \(\mathbb{R}\)

\[
\begin{align*}
\langle j, m | b_z | j, m \rangle &= \lambda_j \cdot m \\
\langle j - 1, m | b_z | j, m \rangle &= \varsigma_j \cdot \sqrt{(j^2 - m^2)} \\
\langle j, m | b_z | j - 1, m \rangle &= \varsigma_j \cdot \sqrt{(j^2 - m^2)}
\end{align*}
\]

plus similar formulas for the \(b_x\) and \(b_y\) generators. The amplitudes \(\lambda_j\) describe diagonal transitions inside the \(SO(3)\) multiplet \(R^{(j)}\), while \(\varsigma_j\) describe nondiagonal transitions between \(SO(3)\) multiplets which form the representation \(\Theta\) of \(SO(3, 1)\). Thus \(\Theta(j_0, \lambda) = \oplus \sum_{j = j_0}^{\infty} R^{(j)}\), where \(j_0\) defines the lower spin in the representation and \(\lambda\) is a free complex parameter. The amplitudes \(\lambda_j\) and \(\varsigma_j\) can be found from the commutation relations \(\mathbb{R}\)

\[
\lambda_j = \frac{i \, j_0 \, \lambda}{j(j + 1)}, \quad \varsigma_j^2 = \frac{(j^2 - j_0^2) \, (j^2 - \lambda^2)}{j^2 \, (4j^2 - 1)},
\]

where \(\lambda\) appears as an essential dynamical parameter which cannot be determined solely from the kinematics of the Lorentz group \(\mathbb{R}\). The adjoint representation is defined as \(\hat{\Theta} = (j_0; -\lambda)\). We shall consider the case \(\Theta_r = (r - 1/2, \lambda)\) and \(-3/2 \leq \lambda \leq r + 1/2\) for all \(r \geq 1\). The representation is infinite-dimensional if \(\lambda = j_0 + r, \, r = 1, 2, 3, ...,\) as it is easy to see from \(\mathbb{R}\) and \(\mathbb{R}\) \(j_0\) and \(\mathbb{R}\) \(j_0\). The representations used in the Dirac equation are \((1/2, -3/2)\) and \((1/2, 3/2)\) and in the Majorana equation they are \((0, 1/2)\) in the boson case and \((1/2, 0)\) in the fermion case. The infinite-dimensional Majorana representation \((1/2, 0)\) contains \(j = 1/2, 3/2, ...\) multiplets of the \(SO(3)\) while \((0, 1/2)\) contains \(j = 0, 1, 2, ...\) multiplets.
\( \lambda \leq 3/2 \) to have \( \varsigma_j \) real for all values of \( r = 1, 2, \ldots \). The Casimir operators \((\vec{a} \cdot \vec{b})\) and \((\vec{a}^2 - \vec{b}^2)\) for the representation \( \Theta_r \) are equal correspondingly to \( < j, m | \vec{a} \cdot \vec{b} | j, m > = i \lambda (r - 1/2), \quad < j, m | (\vec{a}^2 - \vec{b}^2) | j, m > = (r - 1/2)^2 + \lambda^2 - 1 \). As it is easy to see from these formulas the Casimir operator \((\vec{a} \cdot \vec{b})\) is nonzero only if \( \lambda \neq 0 \).

The Majorana commutation relation \((\vec{a})\) together with the last equations allow to find \( \Gamma_\mu \) matrices when a representation \( \Theta \) of the Lorentz algebra \( J_{\mu \nu} \) is given \((\vec{b})\). Because \( \Gamma_0 \) commutes with spatial rotations \( \Theta = (\Theta_N, \cdots, \Theta_1, \cdots, \Theta_N) \) with \( j_0 = 1/2, \cdots, N - 1/2 \). Thus \( \gamma_{rr'} \) is \( 2N \times 2N \) matrix which should satisfy the equation for \( \Gamma_0 \) which follows from \((\vec{a})\) \((\vec{b})\)

\[
\Gamma_0 b_z^2 - 2 b_z \Gamma_0 b_z + b_z^2 \Gamma_0 = - \Gamma_0. \tag{13}
\]

In \((\vec{a})\) the authors were searching the solution of the above equation in the form of Jacoby matrices

\[
\gamma_j = \begin{pmatrix}
0 & \gamma_j^{N-1} & \gamma_j^{N-2} \\
\gamma_j^{N-1} & 0 & \gamma_j^{N-2} \\
\gamma_j^{N-2} & \gamma_j^{N-2} & 0
\end{pmatrix}, \quad \Psi_j = \begin{pmatrix}
\psi_j^N \\
\vdots \\
\psi_j^N
\end{pmatrix}. \tag{14}
\]

It should be understood that \( \vec{a}^{rr'} = \delta^{rr'} \cdot \vec{a} \quad \vec{b}^{rr'} = \delta^{rr'} \cdot \vec{b} \) and the corresponding matrices \( \vec{\gamma}^{rr'} \) and \( \vec{\psi}^{rr'} \) are defined by \((\vec{a})\) and \((\vec{b})\). In the present work we found a new solution which has additional nonvanishing antidiagonal elements \( \gamma_j^{rr} \).

The equation \((\vec{a})\) together with the matrix elements \((\vec{b})\) \((\vec{a})\) and \((\vec{b})\) completely define the problem. The solutions of the equations \((\vec{a})\) are defined up to a set of constant factors which are independent from \( j \). Indeed, because Jacoby matrices \((\vec{a})\) have a very specific form, the original equation \((\vec{a})\) factorize into separate equations for every element \( \gamma_j^{rr+1} \) of the Jacoby matrix and one can check that the solution has the form \((\vec{a})\)

\[
\gamma_j^{rr+1} = \text{Const} \sqrt{\left(1 - \frac{r^2}{N^2}\right)} \cdot \sqrt{\left(\frac{j^2 + j}{4r^2 - 1} - \frac{1}{4}\right)} \tag{15}
\]

and has therefore \((4N - 2) j\)-independent free constant. This freedom allows to impose necessary physical constraints on a solution requiring: i) correct behaviour of the spectrum, ii) Hermitian property of the system, iii) reality and the positivity of the current density matrix \( \rho = \Omega \Gamma_0 \). For that one should study the spectral properties of the matrices of infinite size with matrix elements \( \gamma_j^{rr+1} \) and \( \gamma_j^{rr} \) which have complicated ”root” dependence. The first inspection of the solution \((\vec{a})\) simply shows that every element \( \gamma_j^{rr+1} \) grows like \( \approx j \) and in general all eigenvalues \( \epsilon_j \) will also grow with \( j \). Therefore the mass spectrum \( M_j = M/\epsilon_j \) will have Majorana-like
behaviour \((6)\) \(M_j \approx M/j\). To avoid this general behaviour of the spectrum one should carefully inspect eigenvalues of the matrix \(\Gamma_0\) for small values of \(N\) and then for arbitrary \(N\). The parameter \(N\) plays the role of a natural regularization. The solutions \(B - H - \Sigma - \Sigma_1 - \Sigma_2\)-solutions which appear (see \(8\) and below) have exceptional behaviour: half of the eigenvalues of the spectrum are increasing and the other half are decreasing. One can achieve this exceptional behaviour of the solution by tuning the free constants in the general solution \((15)\). However these solutions have not been accepted \([3]\) because half of the eigenvalues produce a mass spectrum which has an accumulation point at zero mass. This phenomena can be understood on the example of the Dirac equation. For that let us define the dual representation \(\Theta = (j_0; \lambda) \rightarrow (\lambda; j_0) = \Theta^\text{dual}\). From formulas \((11)\) and \((10)\) it is easy to see that representations \((j_0, \lambda)\) and \((-j_0, -\lambda)\) should be considered as identical. Therefore the dual transformation of the adjoint representation \(\hat{\Theta} = (j_0; -\lambda)\) which is defined as \((-\lambda; j_0)\) is identical with \((\lambda; -j_0)\), thus \(\hat{\Theta} = (j_0; -\lambda) \leftrightarrow (\lambda; -j_0) = \Theta^\text{dual}\). For the dual representations \(\Theta\) and \(\Theta^\text{dual}\) the matrix elements of Lorentz generators \(I_{\mu\nu}\) are precisely the same, the only difference between them is that the lower spin is equal to \(j_0\) for the representation \(\Theta\) and is equal to \(\lambda\) for its dual \(\Theta^\text{dual}\) (see formulas \((11)\) and \((10)\)). Therefore any solution \(\Gamma_\mu\) of the Majorana commutation relations \((3)\) for \(\Theta\) can be translated into the corresponding solution \(\Gamma^\text{dual}_\mu\) for \(\Theta^\text{dual}\) by exchanging \(j_0\) for \(\lambda\) \([3]\). This symmetry transformation imposes constraints on the free parameter \(\lambda\), so that it should be integer or half-integer.

The dual transformation of the Dirac representations \((1/2, -3/2)\) and \((1/2, 3/2)\) would be infinite-dimensional \((3/2, -1/2)\) and \((3/2, 1/2)\) with \(j = 3/2, 5/2, \ldots\) and the corresponding solution \(\Gamma^\text{dual}_0\) has the form \(\gamma^1_j = \gamma_{j+1}^1 = j + 1/2\) with the following mass spectrum

\[
M_j^{\text{Dirac dual}} = \frac{M}{j + 1/2}; \quad j = 3/2, 5/2, \ldots
\]

This Majorana-like mass spectrum is dual to the physical spectrum of the Dirac equation

\[
M_j^{\text{Dirac}} = M, \quad j = 1/2.
\]

The dual equation is simply unphysical, but we have to admit that the whole decreasing mass spectrum of the dual equation corresponds or is dual to a physical Dirac fermion. From this point of view we have to ask about physical properties of the equations which are dual to "unphysical" ones \(B - H - \Sigma - \Sigma_1 - \Sigma_2\). The dual transformation completely improves the decreasing mass spectrum of these equations \([3]\) as it take place in \((10)\) and \((17)\). Indeed the last \(\Sigma^\text{dual}_2\)-equation has the spectrum of particle and antiparticles of increasing half-integer spin lying on quasilinear trajectories. The \(\Sigma^\text{dual}_2\)-equation admits unitarity only at zero order of \(v/c\).

Before going on to review \(B - H - \Sigma - \Sigma_1 - \Sigma_2\)-solutions and present the new \(\Sigma\Delta\) equation let us introduce the invariant scalar product \(\langle \Theta \Psi_1 | \Theta \Psi_2 \rangle = \langle \Psi_1 | \Psi_2 \rangle\), where \(\Theta = 1 + \frac{1}{2} \epsilon_{\mu\nu} I^{\mu\nu}\) and the matrix \(\Omega\) is defined as \(\Psi_1 | \Psi_2 \rangle = \Psi_1^+ \Omega \Psi_2 = \Psi_1^* r^j m \Omega_{jm}^{jr'} \Psi_2^{r'} j'_m\) with the properties

\[
\Omega a_k = a_k \Omega \quad \Omega b_k = b_k^* \Omega \quad \Omega = \Omega^+.
\]
From the first relation it follows that $\Omega = \omega_j^{\mu\nu} \cdot \delta_{j\nu} \cdot \delta_{mm'}$ and from the last two equations, for our choice of the representation $\Theta$ and for a real $\lambda$ in the interval $-3/2 \leq \lambda \leq 3/2$, that $\omega_j^{\mu\nu} = \omega_j^{\nu\mu} = 1 \quad \omega_j^2 = 1$, thus $\Omega$ is an antidiagonal matrix. The conserved current density is equal to $J_\mu = \bar{\Psi} \Gamma_\mu \Psi$, $\partial^\mu J_\mu = 0$. The current density $J_0$ should be real and positive definite, which is equivalent to the requirement that

$$\Gamma_\mu^+ \Omega = \Omega \Gamma_\mu,$$

and to the positivity of the eigenvalues of the matrix $\rho = \Omega \Gamma_0$.

The basic solution (B-solution) of the equation (13) for the $\Gamma_0$ has the form (14), (15) with all set of constant factors equal to $\text{Const} = i [\delta]$.

$$\gamma_r^{j+1} r = \gamma_r^j r^{+1} = \gamma_r^{j+1} \hat{r} = \gamma_r^{j+1} = i \sqrt{(1 - \frac{r^2}{N^2}) \left( \frac{j^2 + j}{4r^2 - 1} - \frac{1}{4} \right)} \quad j \geq r + 1/2 \quad (20)$$

and $\gamma_{j+1}^1 = \gamma_1^{j+1} = j + 1/2$, where $r = 1, \ldots, N - 1$. These matrices grow in size with $j$ until $j = N - 1/2$, for greater $j$ the size of the matrix $\gamma_j$ remains the same and is equal to $2N \times 2N$. The number of states with angular momentum $j$ grows as $j + 1/2$ and this takes place up to spin $j = N - 1/2$. For higher spins $j \geq N - 1/2$ the number of states remains constant and is equal to $N$. The positive eigenvalues $\epsilon_j$ can now be found [3]

$$\begin{align*}
1 & \quad j = 1/2 \\
1 - 1/N & \quad 1 + 1/N \quad \text{if } j = 3/2 \\
\cdots & \\
1 + (j - 5/2)/N, & 1 + (j - 1/2)/N \quad j \geq N - 1/2
\end{align*} \quad (21)$$

The last formulas show that the coefficient of proportionality behind $j$ drops N times compared with the one in the Majorana solution $\epsilon_j = j + 1/2$ in (6) and now many eigenvalues are less than unity and the corresponding masses $M_j = M/\epsilon_j$ are bigger than the ground state mass $M$. This actually means that by increasing the number of representations in $\Theta = (\Theta_\mu, \ldots, \Theta_1, \Theta_1, \ldots, \Theta_N)$ one can slow down the growth of the eigenvalues. To have the mass spectrum bounded from below one should have spectrum with all eigenvalues $\epsilon_j$ less than unity. In the limit $N \to \infty$ the B-solution (20) is being reduced to the form

$$\gamma_r^{j+1} r = \gamma_r^j r^{+1} = \gamma_r^{j+1} \hat{r} = \gamma_r^{j+1} = i \sqrt{(\frac{j^2 + j}{4r^2 - 1} - \frac{1}{4})} \quad j \geq r + 1/2 \quad (22)$$

and $\gamma_j^{j+1} = \gamma_1^{j+1} = j + 1/2$, where $r = 1, 2, \ldots$. As it is easy to see from the previous formulas, all eigenvalues $\epsilon_j$ tend to unity when the number of representations $N \to \infty$. The characteristic equation which is satisfied by the gamma matrix in this limit is

$$(\gamma_j^2 - 1)^{j+1/2} = 0 \quad j = 1/2, 3/2, 5/2, \cdots \quad (23)$$

with all eigenvalues $\epsilon_j = \pm 1$. Therefore all states have equal masses $M_j = 1$ and the spectrum is bounded from below, but the Hamiltonian is not Hermitian ($\Gamma_0^+ \neq \Gamma_0$).
The matrix \( \Omega \Gamma_0 \) has the characteristic equation \( (\omega_j \gamma_j - 1)^{2j+1} = 0 \) with all eigenvalues equal to \( \rho_j = +1 \). Thus the matrix \( \Omega \Gamma_0 \) is positive definite and all its eigenvalues are equal to one, but the relations \( \Omega \Gamma_0 \neq \Gamma_0^+ \Omega \), \( \Gamma_0^+ \neq \Gamma_0 \) do not hold. What is crucial here is that we can improve the B-solution without disturbing its determinant which is equal to one (23) \((Det \Gamma_0 = 1)\). The last property of the determinant is necessary to keep in order that the spectrum will be symmetrically distributed around unity.

The Hermitian solution \((H\text{-solution})\) of (23) for \( \Gamma_0 \) can be found as a phase modification of the basic B-solution \((20)\)

\[
\gamma_j^{r+1} = -\gamma_j^r \gamma_j^{r+1} = -\gamma_j^{r+1} \gamma_j^r = \gamma_j^r \gamma_j^{r+1} = i \sqrt{ \frac{j^2 + j - 1}{4r^2 - 1} } \quad j \geq r + 1/2.
\]

These matrices are Hermitian \( \Gamma_0^+ = \Gamma_0 \), but the characteristic equations are more complicated now. These polynomials \( p(\varepsilon) \) have reflective symmetry and are even \( p_j(\varepsilon) = e^{2j+1} p_j(1/\varepsilon), \ p_j(-\varepsilon) = p_j(\varepsilon) \) therefore if \( \varepsilon_j \) is a solution then \( 1/\varepsilon_j, -\varepsilon_j \) and \(-1/\varepsilon_j\) are also solutions \(4\). The eigenvalues \( \varepsilon_j \) can be found \(4\)

\[
\begin{align*}
&\sqrt{2} - 1 & \sqrt{2} + 1 & \\
&j = 1/2 & j = 3/2 & \\
&\ldots & \ldots & \ldots
\end{align*}
\]

The changes in the phases of the matrix elements of (24) result in a different behavior of eigenvalues. The half of the eigenvalues (decreasing eigenvalues), produce quasilinear trajectories with nonzero string tension and the other half (increasing eigenvalues) affect the low spin states on trajectories, so that smallest mass on a given trajectory tends to zero (see \(4\)). The matrix \( \Omega \Gamma_0 \) has again the characteristic equation \( (\omega_j \gamma_j - 1)^{2j+1} = 0 \) and all eigenvalues are equal to one. Thus again the matrix \( \Omega \Gamma_0 \) is positive definite because all eigenvalues are equal to one, but the important relation \( \Omega \Gamma_0 = \Gamma_0^+ \Omega \) does not hold. The solution of (23) for \( \Gamma_0 \) with both properties \( \Gamma_0^+ = \Gamma_0 \) and \( \Omega \Gamma_0 = \Gamma_0^+ \Omega \) can be found by using the basic solutions \(20\) rewritten with arbitrary phases of the matrix elements and then by requiring that \( \Gamma_0 \) should be Hermitian \( \Gamma_0^+ = \Gamma_0 \) and should satisfy the relation \( \Omega \Gamma_0 = \Gamma_0^+ \Omega \). This solution, \( \Sigma \text{-solution} \), is symmetric and has the form \(\]

\[
\gamma_j^{r+1} = \gamma_j^r \gamma_j^{r+1} = \gamma_j^{r+1} \gamma_j^r = \gamma_j^r \gamma_j^{r+1} = \sqrt{ \frac{j^2 + j - 1}{4r^2 - 1} } \quad j \geq r + 1/2.
\]

In this case the Hermitian matrix \( \Gamma_0^+ = \Gamma_0 \) has the desired property \( \Gamma_0^+ \Omega = \Omega \Gamma_0 \). This means that the current density is equal to \( \rho = \Omega \Gamma_0 \). In addition, all of the gamma matrices now have this property \(19\) \( \Gamma^+_k \Omega = \Omega \Gamma_k \) \( k = x, y, z \) which follows from the equation \( \Gamma_k = i[b_k \Gamma_0] \) and equation \(18\) \( \Omega b_k = b_k \Omega \).

\footnote{The determinant and the trace are equal to \( Det \gamma_j = \pm 1 \), \( Tr \gamma_j^2 = 2j + 1 \), thus \( \varepsilon_1^2 \gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1 \), \( \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = j + 1/2 \).}

\footnote{Computing the traces and determinants of these matrices one can get the following general relation for the eigenvalues \( \varepsilon_1^2 \gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1 \), \( \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = j(2j + 1) \).}
The characteristic equations and the spectrum (25) are the same for the Hermitian H-solution and symmetric Σ-solution, but the corresponding characteristic equations for the matrices $p_j$ are different and the eigenvalues of the density matrix are not positive definite any more

$$
\begin{align*}
1 & \quad 1 & j = 1/2 \\
1 - \sqrt{2} & \quad \sqrt{2} + 1 & j = 3/2 \\
\end{align*}
$$

Both states with $j = 1/2$ have positive norms, the $j = 3/2$ level has two positive and two negative norm states, the $j = 5/2$ has four positive and two negative norm states, and so on. The positive norm physical states are lying on the quasilinear trajectories of different slope and the negative norm ghost states are also lying on the quasilinear trajectories [3]. Thus the equation has the increasing mass spectrum, but the smallest mass on a given trajectory still tends to zero and in addition there are many ghost states (see also bellow).

In the case when some of the transition amplitudes in (26) are set to zero

$$
\begin{align*}
\gamma^1_{\bar{1}} &= \gamma^2_{\bar{3}} = \gamma^4_{\bar{5}} = \ldots = 0 \\
\gamma^1_{\bar{1}} &= \gamma^2_{\bar{3}} = \gamma^4_{\bar{5}} = \ldots = 0
\end{align*}
$$

and all other elements of the $\Gamma_0$ matrix remain the same as in (26) we have a new $\Sigma_1$-solution with the important property that $\Gamma_0^2$ is a diagonal matrix and that the antihermitean part of $\Gamma_k$ anticommutes with $\Gamma_0$. Thus in this case we recover the nondiagonal part of the Dirac commutation relations for gamma matrices $\{\Gamma_0, \tilde{\Gamma}_k\} = 0 \quad k = x, y, z$. For the solution (28) one can explicitly compute the mass spectrum and the slope of the trajectories [4]

$$
M_n^2 = \frac{2M^2}{n} \frac{j^2 - (2n - 1)j + n(n - 1)}{j - (n - 1)/2} \quad n = 1, 2, \ldots
$$

where $j = n + 1/2, n + 5/2, \ldots$. The string tension $\sigma_n = 1/2\pi\alpha'_n$ varies from one trajectory to another and is equal to

$$
2\pi\sigma_n = \frac{1}{\alpha'_n} = \frac{2M^2}{n} \quad n = 1, 2, \ldots
$$

Thus we have the string equation which has trajectories with different string tension and that trajectories with large $n$ are almost "free" because the string tension tends to zero. The smallest mass on a given trajectory $n$ has spin $j = n+1/2$ and decreases as $M_n^2(j = n+1/2) = \frac{3M^2}{n(n+3)}$. The other solution, $\Sigma_2$-solution, which shares the above properties of $\Sigma_1$-solution is (26) with

$$
\begin{align*}
\gamma^j_{\bar{j}} &= \gamma^3_{\bar{4}} = \ldots = 0 \\
\gamma^j_{\bar{j}} &= \gamma^3_{\bar{4}} = \ldots = 0
\end{align*}
$$

The difference between these last two solutions is that in the first case the lower spin is $j = 3/2$ and in the second case it is $j = 1/2$. The unwanted property of all these solutions $\Sigma$, $\Sigma_1$ and $\Sigma_2$ is that the smallest mass $M_n^2(min)$ tends to zero and the
spectrum is not bounded from below. We have to remark also that both equations, \(\Sigma_1\) and \(\Sigma_2\), which correspond to (28) and to (31) have natural constraints \([3]\).

The unwanted property of the \(\Sigma\)-solutions, that is the decreasing of the smallest mass on a given trajectory, can be solved by dual transformation of the system \([4]\). This exact symmetry transforms two solutions of the Majorana commutation relations one into another. Indeed, interchanging \(j_0\) and \(\lambda\) in the representation \(\Theta = (j_0, \lambda)\) does not affect the matrix elements of the Lorentz generators \(J_{\mu\nu}\), therefore a solution \(\gamma_j\) for \(\Theta = (j_0, \lambda)\) can be translated into a solution \(\gamma_j^{\text{dual}}\) for the dual representation \(\Theta^{\text{dual}}\) by exchanging \(j_0\) for \(\lambda\) and letting spin \(j\) to run in a different interval \(j = \lambda, \lambda + 1, \ldots [4]\). The dual transformation does not change the actual \(j\) dependence of \(\gamma_j\), but what is important here is that despite the fact that the dual solution \(\gamma_j^{\text{dual}}\) is almost identical with \(\gamma_j\) (we mean the \(j\) dependence), the spectrum essentially changes for the low spin states and does not affect the high spin states. This is because the number of states with spin \(j\), which was equal to \(j + 1/2\) before the dual transformation becomes infinite now. Therefore the spin contents of the dual equation is different and the equation has different spectrum. The dual transformation does not affect the higher spin states and thus does not change the slope of the trajectories \([3]\), and has the commulative effect on lower spin states keeping them bounded from below.

Indeed under the dual transformation \(\Theta = (j_0; \lambda) \rightarrow (\lambda; j_0) = \Theta^{\text{dual}}\) the representation \(\Theta = (\Theta_N, \ldots, \Theta_1, \Theta_0, \ldots, \Theta_N)\) is transformed into its dual \(\Theta^{\text{dual}} = \ldots (\lambda; -5/2) (\lambda; -3/2) (\lambda; -1/2) (\lambda; 1/2) (\lambda; 3/2) (\lambda; 5/2)\ldots\) and we are lead to take \(\lambda\) to be half-integer and to \(\lambda = 1/2\) in order to have the Dirac representation incorporated in \(\Theta\) \([4]\). The solution which is dual to \(\Sigma_2\) \((20)\) and \((31)\) is equal to \([3]\)

\[
\gamma_j^{r+1} = \gamma_j^r + 1 = \gamma_j^{r+1} i = \gamma_j^r i^{1/2} = \sqrt{\left(\frac{1}{4} - \frac{j^2 + j}{4r^2 - 1}\right)} \quad r \geq j + 3/2 \quad (32)
\]

where \(j = 1/2, 3/2, 5/2, \ldots\), \(r = 2, 4, 6, \ldots\) and the rest of the elements are equal to zero

\[
\gamma^{1 \ 1} = \gamma^{1 \ 2} = \gamma^{3 \ 4} = \ldots = 0 \quad \gamma^{1 \ 1} = \gamma^{1 \ 2} = \gamma^{3 \ 4} = \ldots = 0 \quad (33)
\]

The Lorentz boost operators \(\vec{b}\) are antihermitian in this case \(b_k^+ = -b_k\), because the amplitudes \(\varsigma_i\) are pure imaginary and therefore the \(\Gamma_k\) matrices are also antihermitian \(\Gamma_k^+ = -\Gamma_k\). The matrix \(\Omega\) changes and is now equal to the parity operator \(P\), the relation \(\Omega \Gamma_\mu = \Gamma_\mu \Omega\) remains valid. The diagonal part of \(\Gamma_k\) anticommutes with \(\Gamma_0\) as it was before \(\{\Gamma_0, \Gamma_k\} = 0\) \(k = x, y, z\). The mass spectrum is equal to

\[
M_n^2 = \frac{2M^2}{n} \frac{(j + n)(j + n + 1)}{j + (n + 1)/2} \quad (34)
\]

where \(n = 1, 2, 3, \ldots\) and enumerates the trajectories. The lowest spin on a given trajectory is either \(1/2\) or \(3/2\) depending on \(n\): if \(n\) is odd then \(j_{\min} = 1/2\), if \(n\) is even \(j_{\min} = 3/2\). This is an essential new property of the dual equation because now we have an infinite number of states with a given spin \(j\) instead of \(j + 1/2\). The string

\[\text{The obvious consequence of the dual transformation is that the free parameter } \lambda \text{ should be integer or half-integer.}\]

\[\text{These representations do not coincide with the ones in Ramond equation [9].}\]
The mass spectrum is highly degenerated and is given by the formula
\[ M_n^2(j = 1/2) = \frac{4M^2(2n+1)(2n+3)}{n+2} \to (4M)^2 \]
and the spectrum is bounded from below by positive mass.

The last \( \Sigma^2 \) dual-equation has the property that only the diagonal matrix elements
of the anticommutator \( \{ \Gamma_0, \Gamma_z \} \) are equal to zero
\[ < j, m, r | \{ \Gamma_0, \Gamma_z \} | r, j, m > = \gamma_j^{rr+1} m (\lambda_j^{r+1} - \lambda_j^r) \gamma_j^{r+1} + im(\lambda_j^r - \lambda_j^{r+1})\gamma_j^{rr+1} = 0, \tag{35} \]
and that nondiagonal elements are not equal to zero
\[ < j - 1, m, r | \{ \Gamma_0, \Gamma_z \} | r, j, m > = i\sqrt{j^2 - m^2} \gamma_j^r \left[ (\gamma_j^{r+1})^2 - (\gamma_j^{r-1})^2 \right]. \tag{36} \]

Let us search the solution of the Majorana commutation relation \( (13) \) in the same \( \Sigma^2 \) dual-Jacoby form \( (44) \) but with an additional nonvanishing antidiagonal matrix elements \( \gamma_j^r \). The solution has the form
\[ \gamma_j^r = \gamma_j^r - \gamma_j^{r+1} r+1 - \gamma_j^{r+1} r+1 = \frac{j + 1/2}{\sqrt{4r^2 - 1}} \tag{37} \]
where \( j = 1/2, 3/2, 5/2, \ldots \), \( r = 2, 4, 6, \ldots \) and \( r \geq j + 3/2 \) and one can check directly that \( \Gamma_0 \) is the solution of \( (13) \). These additional matrix elements in \( \Gamma_0 \) will not change the diagonal matrix elements of the anticommutator \( (53) \) because
\[ < j, m, r | \{ \Gamma_0, \Gamma_z \} | r, j, m > = \gamma_j^{rr} m (\lambda_j^r - \lambda_j^{r+1}) \gamma_j^{r+1} + im(\lambda_j^r - \lambda_j^{r+1})\gamma_j^{rr+1} = 0, \tag{38} \]
and will cancel nondiagonal matrix elements of \( (36) \)
\[ < j - 1, m, r | \{ \Gamma_0, \Gamma_z \} | r, j, m > = i\sqrt{j^2 - m^2} \gamma_j^r \left[ (\gamma_j^{r+1})^2 - (\gamma_j^{r-1})^2 \right] = 0. \tag{39} \]

One can check this fact also using the relation \( \{ \Gamma_0, \Gamma_z \} = i [b_z, \Gamma_z^2] \) which follows from \( (9) \). Using the relations \( (3) \) \( \Gamma_y = -i [\Gamma_z, \Gamma_x] \) and \( [\Gamma_0, \Gamma_x] = 0 \) one can see that \( \{ \Gamma_0, \Gamma_y \} = 0 \) and in the same way, that \( \{ \Gamma_0, \Gamma_x \} = 0 \). Finally using the relation \( (8) \) \( \Gamma_k = -i [\Gamma_0, b_k] \) one can prove by direct calculation that \( \{ \Gamma_k, \Gamma_l \} = 0 \) for \( k \neq l \) and then using \( (13) \) and the fact that \( [b_k, \Gamma_x^2] = 0 \) to prove that \( \Gamma_k^2 = -\Gamma_x^2 \) therefore
\[ \{ \Gamma_\mu, \Gamma_\nu \} = 2g_{\mu\nu} \Gamma_0^2, \tag{40} \]
where \( \Gamma_0^2 \) is a diagonal matrix. Now the theory is Hermitian in all orders of \( v/c \).

The mass spectrum is highly degenerated and is given by the formula
\[ M_j^2 = M^2 \frac{4r^2 - 1}{4r^2} \quad r \geq j + 3/2. \tag{41} \]

New mass terms \( (\vec{a} \cdot \vec{b}) \Gamma_5 \) and \( (\vec{a}^2 - \vec{b}^2) \) can be added into the string equation \( (8) \) in order to increase the string tension
\[ \{ i \Gamma_\mu \partial^\mu - M (\vec{a} \cdot \vec{b}) \Gamma_5 - gM (\vec{a}^2 - \vec{b}^2) \} \Psi = 0 \tag{42} \]
where \( (\vec{a} \cdot \vec{b}) \) and \( (\vec{a}^2 - \vec{b}^2) \) are the Casimir operators of the Lorentz algebra. The commutation relations which define the \( \Gamma_5 \) matrix are \( \Gamma_5 \ a_k = a_k \Gamma_5, \quad \Gamma_5 \ b_k = \ldots \)
that $P a \Gamma_5$, $\Gamma_5^2 = 1$, thus $\Gamma_5 = \Gamma_{5j}^{rr'} \delta_{jj'} \delta_{mm'}$ and $\Gamma_{5j}^{rr} = -\Gamma_{5j}^{jr} = (-1)^{r+1}$. One can check that $\Gamma_5 \Gamma_0 = -\Gamma_0 \Gamma_5$, $\Gamma_5 \Gamma_k = -\Gamma_k \Gamma_5$, $\Gamma_5 P = -P \Gamma_5$, $\Gamma_5 \Omega = -\Omega \Gamma_5$, where the parity operator $P$ defined as $P a_k = a_k P$, $P b_k = -b_k P$, $P^2 = 1$.

We have again $P = P_j^{rr'} \delta_{jj'} \delta_{mm'}$ and that $P_j^{rr} = P_j^{rr'} = (-1)^{[j]}$. One can check that $P \Gamma_0 = \Gamma_0 P$, $P \Gamma_k = -\Gamma_k P$, $P \Omega = \Omega P$. Thus the additional new mass matrix $(\vec{a} \cdot \vec{b}) \Gamma_5$ is diagonal and is equal to $<j,m,r | (\vec{a} \cdot \vec{b}) \Gamma_5 | r,j,m> = <j,m,r | (\vec{a} \cdot \vec{b}) \Gamma_5 | \hat{r},j,m>$, $= i \frac{1}{2} (-1)^{r+1}(r-1/2)$.

Including the $\Gamma_5$ mass term one can see that mass spectrum grows as $j^2$ and all trajectories acquire a nonzero slope

$$M_j^2 = \frac{M^2}{4} \frac{4r^2 - 1}{r^2} (r - 1/2)^2 \quad r \geq j + 3/2$$

where $r = 2, 4, 6, \ldots, j = 1/2, 3/2, 5/2, \ldots$, thus $M_j^2 \geq M^2 (j + 1)^2$. If we “turn on” the pure Casimir mass term $gM (\vec{a}^2 - \vec{b}^2)$ the spectrum will grow as $j^4$

$$M_j^2 = (gM)^2 \frac{4r^2 - 1}{r^2} (r - 1/2)^4 \quad r \geq j + 3/2,$$

thus $M_j^2 \geq (2gM)^2 (j + 1)^4$. In general case the formula is

$$M_j^2 = \frac{M^2}{4} \frac{4r^2 - 1}{r^2} (r - 1/2)^2 (1 + 2g(r - 1/2))^2 \quad r \geq j + 3/2.$$

Thus the spectrum of the theory consists of particles and antiparticles of increasing half-integer spin lying on quasilinear trajectories of different slope. It is difficult to say at the moment what is the physical reason for this nonperturbative behaviour.

The equation is explicitly Lorentz invariant, but has unwanted ghost solutions. The tachyonic solutions which appear in Majorana equation (see (20) in [4]) do not show up here. This is because the nondiagonal transition amplitudes of the form $<..j.. | \Gamma_k | ..j \pm 1..>$ are small here and the diagonal amplitudes $<..j.. | \Gamma_k | ..j..>$ are large. The problem of ghost states is more subtle here and we shall analyze the natural constraints appearing in the system to ensure that they decouple from the physical space of states.

We will present the derivation of the above equation from the gonihe dric string, which was formulated as a model of random surfaces, in a separate place. The equation has its own value independent of the motivation advocated in this article.

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