Degeneration of spectral sequences and complex Lagrangian submanifolds

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Abstract

There is a local-to-global Ext spectral sequence $E_2^{p,q} = H^p(L, \Omega^q_L) \Rightarrow \text{Ext}^{p+q}(i_*\mathcal{O}_L, i_*\mathcal{O}_L)$ for a smooth Lagrangian subvariety in a hyperkähler variety. We prove its degeneration on $E_2$, and various generalisations thereof.

Introduction. Let $i : Z \hookrightarrow X$ be a locally complete intersection over an algebraically closed field $k$ of characteristic 0 with normal bundle $\mathcal{N}_{Z/X}$. For any $\mathcal{L} \in \text{Pic}(Z)$, there’s a local-to-global Ext spectral sequence:

$$E_2^{p,q} = H^p(Z, \wedge^q \mathcal{N}_{Z/X}) \Rightarrow \text{Ext}^{p+q}(i_*\mathcal{L}, i_*\mathcal{L}).$$

Notice that the left-hand side is independent of $\mathcal{L}$, but the differentials are not. We are interested in the case where $i : L \hookrightarrow X$ is a smooth Lagrangian in a hyperkähler variety over $k$. In this case we have $\mathcal{N}_{L/X} \cong \Omega_L$, the second page therefore becomes $E_2^{p,q} = H^p(L, \Omega^q_L)$. Our main results go as follows:

**Theorem 0.1.** Let $X/k$ be a (projective) hyperkähler variety, let $i : L \hookrightarrow X$ be a smooth Lagrangian, denote the Kähler form on $L$ by $\omega \in H^1(L, \Omega^1_L)$, and suppose $\mathcal{L}$ is a line bundle on $L$. Then the local-to-global Ext spectral sequence

$$E_2^{p,q} = H^p(L, \Omega^q_L) \Rightarrow \text{Ext}^{p+q}(i_*\mathcal{L}, i_*\mathcal{L})$$

degenerates (multiplicatively) on the second page if and only if $d_2(\omega) = 0$.

**Theorem 0.2.** Let $X/k$ be a (projective) hyperkähler variety, $i : L \hookrightarrow X$ a smooth Lagrangian, and let $\mathcal{L}$ be a line bundle on $L$, extending to the first infinitesimal neighbourhood of $L$ in $X$, such as $\mathcal{O}_L$. Then the local-to-global Ext spectral sequence

$$E_2^{p,q} = H^p(L, \Omega^q_L) \Rightarrow \text{Ext}^{p+q}(i_*\mathcal{L}, i_*\mathcal{L})$$

degenerates on the second page. Hence $H^*(L/k) = \oplus_{p,q} H^p(L, \Omega^q_L) = \text{Ext}(i_*\mathcal{L}, i_*\mathcal{L})$, as graded algebras.

**Theorem 0.3.** Let $i : L \hookrightarrow X$ be as above, and let $\mathcal{K}$ be any (existing) rational power of the canonical bundle of $L$. Then the local-to-global Ext spectral sequence

$$E_2^{p,q} = H^p(L, \Omega^q_L) \Rightarrow \text{Ext}^{p+q}(i_*\mathcal{K}, i_*\mathcal{K})$$

degenerates (multiplicatively) on the second page.

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(1) (See §2, Definition 2.1)
Method of proof. The condition $d_2(\omega) = 0$ is equivalent to commutativity of the differentials with the Lefschetz operator. The proof of Theorem 0.1, given in §2, is an extension of some ideas of Deligne in [5, 6, 7], which use Lefschetz-like operators to prove degeneration of spectral sequences. These are based on the following simple observation that if $d : H^*(X/k) \to H^*(X/k)$ is a linear map of degree $-1$, which commutes with the Lefschetz operator, then $d = 0$. As noted above, in our setting a local calculation shows that $E_2^{p,q} = H^p(L, \Omega^q_L)$, and thinking of $E_2^{p,q} \subset H^{p+q}(L/k)$, the differentials $d_r$ of our spectral sequence are of (total) degree $+1$ and, even if they commute with Lefschetz, Deligne’s observation doesn’t apply immediately. In the case of commutativity with the Lefschetz operator, we are, however, able to apply Deligne’s idea to a part of the differential, and the outcome is that $d_r$ preserves primitive cohomology and vanishes on middle primitive cohomology. Then a downward induction argument allows us to conclude that $d_r = 0$.

To prove that $d_2(\omega) = 0$ under the hypotheses of Theorem 0.2 and Theorem 0.3, we need a second technical ingredient - it involves explicitly identifying some of the differentials on the second page. These remarks apply in the general case of a locally complete intersection $i : Z \hookrightarrow X$ and occupy most of §1. Here we describe the simplest and also most important case. There is an obstruction class $\alpha_L \in \text{Ext}^2(L, \Omega_L)$ to extending $L$ from $Z$ to $2Z$. It is a morphism

$$\mathcal{L} \to \mathcal{L} \otimes \mathcal{N}_{Z/X}[2].$$

Taking the adjoint, we get a morphism

$$\mathcal{N}_{Z/X} \to \mathcal{O}_Z[2],$$

which induces our differential

$$d_2^{p-1} : H^p(Z, \mathcal{N}_{Z/X}) \to H^{p+2}(Z, \mathcal{O}_Z).$$

In particular, we see this vanishes if $\mathcal{L}$ lifts to the first infinitesimal neighbourhood.

Generality. We note that over $\mathbb{C}$ the results obtained here hold in much broader generality. It is enough to require $X$ holomorphic symplectic (not necessarily proper!) and $L$ compact Kähler.

Context. The results obtained here are motivated by and seem to mirror the degeneration of $H^\ast(L/\mathbb{C}) \Rightarrow HF^\ast(L, L)$ of Solomon and Verbitsky in [15]. In [3] Joyce et al. construct a perverse sheaf $\mathcal{P}$ on the intersection of two oriented Lagrangians in a homolorphic symplectic variety following Joyce’s notion of an algebraic d-critical locus. This gives a first categorification of the intersection numbers of Lagrangians. It follows by Corollary 3.11, and a calculation of Grosse-Brauckmann in [10], that we have:

**Proposition 0.4.** $\dim(\text{Ext}^i(i_\ast K^1_L/2, j_\ast K^1_M/2)) = \dim(H^{i-n}(\mathcal{P}_{L,M})).$

In [13] Kapustin and Rozansky study the RW model. They suggest a 2-category associated to a holomorphic symplectic variety whose simplest objects are Lagrangians. In the special case of (a deformation of) the cotangent bundle, the endomorphism category of the zero section in the conjectured 2-category is a monoidal deformation of the 2-periodic derived category of the

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\(\text{(2)}\) Defined by $\mathcal{L}_Z^2$
underlying complex manifold. Then the Ext groups could be realised as Hochschild homology of the 2-periodic derived category, which might hint at the collapse of the Ext spectral sequence.

We mention the paper of D’Agnolo and Schapira\(^{(3)}\) [4] where simple holonomic deformation-quantisation modules \(\mathcal{D}_L\) on smooth orientable\(^{(4)}\) Lagrangians \(L\) in holomorphic symplectic varieties are constructed. It is a deep result of \(^{(14)}\) that there is a deformation of \(\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}(i_*K^{1/2}_L, i_*K^{1/2}_L)\) to the de Rham complex of \(L\) (pushed forward to \(X\)) which easily implies degeneration of the spectral sequence in the \(K^{1/2}_L\) case (see Proposition 3.1 and Remark 3.12).

**Plan of paper.** §1 contains general results on the homological algebra of locally complete intersections. In §2 we prove our main results, while §3 is devoted to various generalisations and applications: we give the DQ modules proof hinted at in the previous paragraph, describe a relative version of Theorem 0.1 as well as a generalisation to pairs of Lagrangians, which also has a DQ modules proof (see Remark 3.12). Finally, we mention that our results imply a modified version of a conjecture of Behrend and Fantechi in [2], which they describe as a generalisation of Hodge theory.

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## 1 General results

**Notation.** We shall be working throughout over an algebraically closed field \(k\) of characteristic 0. By \(X\) we denote, in general, a scheme, \(\text{Coh}(X)\) is the abelian category of coherent sheaves on \(X\) and its bounded derived category is \(\mathcal{D}^b(X)\). Sometimes we shall need the derived categories \(\mathcal{D}^\pm(X)\) of bounded above or below complexes. These are triangulated categories, so come with a shift functor \(\mathcal{F} \mapsto \mathcal{F}[1]\) and exact triangles \(\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \mathcal{F}_1[1]\). All functors we consider will be implicitly derived, except for \(\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}, \text{Hom}\) and \(\Gamma\). For a complex \(\mathcal{F}\), we let \(\mathcal{H}^i(\mathcal{F})\) be its \(i\)th cohomology sheaf, so \(\mathcal{H}^i(\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}(\mathcal{F}, \mathcal{G})) = \mathcal{E}\mathcal{X}\mathcal{T}^i(\mathcal{F}, \mathcal{G}), \mathcal{H}^i(\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}(\mathcal{F}, \mathcal{G})) = \mathcal{E}\mathcal{X}\mathcal{T}^i(\mathcal{F}, \mathcal{G}), \) in particular, we get the useful identity \(\mathcal{E}\mathcal{X}\mathcal{T}^i(\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, \mathcal{G}[i])\); similarly \(\mathcal{H}^{-i}(\mathcal{F} \otimes \mathcal{G}) = \mathcal{F} \otimes \mathcal{G}[i]\) and so forth. We say that \(\mathcal{F}\) is a perfect complex, if it is locally isomorphic, in the derived category, to a finite complex of locally free sheaves of finite rank. The category of perfect complexes is triangulated and denoted by \(\text{Perf}(X)\).

Similarly, we let \(\text{FCoh}(X)\) be the abelian category of finitely filtered objects of \(\text{Coh}(X)\). We denote \(\mathcal{D}^\pm F(X)\) the filtered derived categories of Deligne which are localisations with respect to the class of filtered quasi-isomorphisms. Naturally we have filtered derived functors and for a left exact \(T: \text{FCoh}(X) \to \text{Vect}\) and any \((K, F) \in \mathcal{D}^\pm F(X)\), we get a spectral sequence

\[
E_{1}^{p,q} = R^{p+q}T(\text{Gr}^F_p(K)) \Rightarrow R^{p+q}T(K).
\]  

\(^{(3)}\) (see also \([14]\))

\(^{(4)}\) i.e. the canonical bundle admits a square root
This construction is a functor from the filtered derived category to the category of cohomological spectral sequences of vector spaces.

**Homological algebra.** We begin by collecting some general results on multiplication in spectral sequences and sheaves on subvarieties.

**Definition 1.1.** Let $E_p^{i,j} \Rightarrow H^{i+j}$, $E_q^{i,j} \Rightarrow H^{i+p+q}$, $E_r^{i,j} \Rightarrow H^{i+p+q}$ be spectral sequences. A pairing of $E_p^{i,j} \Rightarrow H^{i+p}$ and $E_q^{i,j} \Rightarrow H^{i+p+q}$ to $E_r^{i,j} \Rightarrow H^{p+q}$ is a family of maps

$$
\cup_r : E_r^{i,j} \otimes E_r^{s,t} \rightarrow E_r^{p+q+r+s,t}
$$

such that $\cup_r$ are compatible with the differentials, $\cup_{r+1}$ is induced by $\cup_r$, colim $\cup_r = \cup_\infty$ and $\cup_\infty = \text{gr}(\cup)$. 

**Definition 1.2.** Given a Grothendieck spectral sequence $E_r^{p,q}(S) = R^pTR^qG(S) \Rightarrow H^{p+q}(S)$ such that the functors $R^p$, $RG$ and $H$ have cup products, we shall say it has cup products if, given a pairing $S_1 \otimes S_2 \rightarrow S$, we have a natural pairing of the corresponding spectral sequences and, in addition, the products on $E_2$ and $H$ are the ones induced by the functors.

**Proposition 1.3.** Let $\mathcal{F}$ be a coherent sheaf on $X$. Then the local-to-global Ext spectral sequence

$$
E_2^{p,q} = H^p(X, Ext^q(\mathcal{F}, \mathcal{F})) \Rightarrow Ext^{p+q}(\mathcal{F}, \mathcal{F})
$$

has cup products.

**Remark 1.4.** More generally, there are also pairings of the local-to-global Ext in the case of different sheaves arising from the composition in local $R \mathcal{H}om$.

The local-to-global Ext spectral sequence arises, classically, from the natural isomorphism of derived functors $R \mathcal{H}om \cong R \Gamma \circ R \mathcal{H}om$. We shall give an alternative way of constructing it - via filtered complexes. This has the advantage of giving a geometric interpretation of the differentials as obstructions to formality of certain objects. For $(\mathcal{F}, d) \in D(X)$, define the canonical filtrations:

$$
\tau^{\leq p} \mathcal{F} = \begin{cases} 
\mathcal{F}, & \text{for } i < p \\
\ker(d^p), & \text{for } i = p \\
\tau^{\geq p} \mathcal{F}, & \text{for } i > p
\end{cases}
$$

The inclusion $\tau^{\leq p} \mathcal{F} \hookrightarrow \mathcal{F}$ induces isomorphisms on $\mathcal{H}^i$ for $i \leq p$ and the canonical projection $\mathcal{F} \rightarrow \tau^{\geq p} \mathcal{F}$ induces isomorphisms on $\mathcal{H}^i$ for $i \geq p$. There are canonical exact triangles:

$$
\tau^{\leq p-1} \mathcal{F} \rightarrow \tau^{\leq p} \mathcal{F} \rightarrow \mathcal{H}^p(\mathcal{F})[-p] \rightarrow \tau^{\leq p-1} \mathcal{F}[1],
$$

$$
\mathcal{H}^{p-1}(\mathcal{F})[-p + 1] \rightarrow \tau^{\geq p-1} \mathcal{F} \rightarrow \tau^{\geq p} \mathcal{F} \rightarrow \mathcal{H}^{p-1}(\mathcal{F})[-p + 2].
$$

Let's set $\tau^{[p-1]} \mathcal{F} = \tau^{\geq p-1} \tau^{\leq p} \mathcal{F}$, so that there's an exact triangle:

$$
\mathcal{H}^{p-1}(\mathcal{F})[-p + 1] \rightarrow \tau^{[p-1]} \mathcal{F} \rightarrow \mathcal{H}^p(\mathcal{F})[-p] \rightarrow \mathcal{H}^{p-1}(\mathcal{F})[-p + 2],
$$

and the elements $\delta_{-p}(\mathcal{F}) \in Ext^2(\mathcal{H}^p(\mathcal{F}), \mathcal{H}^{p-1}(\mathcal{F}))$ realise universally the second differential $d_2$ of the spectral sequence of any (filtered) derived functor applied to $\mathcal{F}$.
Remark 1.5. In particular, if \( i : Z \to X \) is a locally complete intersection, \( \mathcal{L} \in \text{Pic}(Z) \), the spectral sequence

\[
E_2^{p,q} = H^p(X, \mathcal{E}xt^q(i_! \mathcal{L}, i_* \mathcal{L})) \Rightarrow \mathcal{E}xt^{p+q}(i_* \mathcal{L}, i_* \mathcal{L})
\]

can be realised from \( i^! i_* \mathcal{L} \) with its canonical filtration. Indeed, Grothendieck-Verdier duality gives a filtered quasi-isomorphism between canonically filtered complexes

\[
R\mathcal{H}om(i_* \mathcal{L}, i_* \mathcal{L}) \simeq i_* R\mathcal{H}om(\mathcal{L}, i^! i_* \mathcal{L})
\]

and applying filtered derived global sections functor, we get a spectral sequence using \( (1) \) which, by Deligne’s décalage theorem (see [6]), is the local-to-global Ext spectral sequence after renumbering \( E_{p,q} \to E_{p+q-n, q} \).

Let \( i : Z \to X \) be a locally complete intersection. The normal bundle of \( Z \) in \( X \) is denoted by \( \mathcal{N}_{Z/X} \).

**Proposition 1.6.** Suppose \( c = \text{codim}(Z, X) \), and let \( \mathcal{F} \) be a coherent sheaf on \( Z \). Then

\[
\mathcal{H}^{-i}(i^! i_* \mathcal{F}) \cong \mathcal{F} \otimes i^* \mathcal{N}_{Z/X}^\vee,
\]

for \( 0 \leq i \leq c \).

**Proposition 1.7.** With the above notation, let \( \mathcal{F} \) and \( \mathcal{G} \) be coherent sheaves on \( Z \) and assume \( \mathcal{F} \) locally free, then we have \( \mathcal{E}xt^i(i_* \mathcal{F}, i_* \mathcal{G}) \cong i_*(\wedge^i \mathcal{N}_{Z/X} \otimes \mathcal{F}^\vee \otimes \mathcal{G}) \), where \( 0 \leq i \leq c \). Furthermore, under these isomorphisms, the Yoneda product coincides with the usual cup product. More precisely, let \( \mathcal{F}, \mathcal{G} \) be locally free sheaves, \( \mathcal{H} \) any coherent sheaf, then the Yoneda multiplication

\[
\mathcal{E}xt^i(i_* \mathcal{F}, i_* \mathcal{H}) \otimes \mathcal{E}xt^j(i_* \mathcal{G}, i_* \mathcal{H}) \to \mathcal{E}xt^{i+j}(i_* \mathcal{F}, i_* \mathcal{H})
\]

corresponds under the above isomorphisms to

\[
i_*(\wedge^i \mathcal{N}_{Z/X} \otimes \mathcal{F}^\vee \otimes \mathcal{H}) \otimes i_*(\wedge^j \mathcal{N}_{Z/X} \otimes \mathcal{G}^\vee \otimes \mathcal{H}) \to i_* (\wedge^{i+j} \mathcal{N}_{Z/X} \otimes \mathcal{G}^\vee \otimes \mathcal{H}),
\]

given by exterior product and the natural map \( \mathcal{G} \otimes \mathcal{G}^\vee \to \mathcal{O}_Z \).

For any coherent \( \mathcal{F} \) on \( Z \), we have a canonical exact triangle

\[
\mathcal{F} \otimes \mathcal{N}_{Z/X}^\vee[1] \to \tau_{\geq -1} i^* i_* \mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{N}_{Z/X}^\vee[2].
\]

**Definition 1.8.** The extension class of the above triangle \( \alpha_{\mathcal{F}} \in \text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}_{Z/X}^\vee) \) is called the deformation-obstruction class of \( \mathcal{F} \).

**Remark 1.9.** In the special case of \( \mathcal{N}_{Z/X}^\vee \) the class \( \alpha_{\mathcal{N}_{Z/X}} \) is sometimes called the HKR class because of its close relationship to Hochschild-Kostant-Rosenberg-like theorems.

Given our locally complete intersection \( i : Z \to X \), we have the conormal exact sequence

\[
0 \to \mathcal{N}_{Z/X}^\vee \to i^* \Omega_X \to \Omega_Z \to 0
\]

whose class is called the Kodaira-Spencer class \( \text{KS} \in \text{Ext}^1(\Omega_Z, \mathcal{N}_{Z/X}^\vee) \).

We would like to extend the definitions of the Atiyah class and the obstruction class \( \alpha_{\mathcal{F}} \) to the derived category. Consider the diagonal embedding \( \Delta : X \to X \times X \). Let \( \Delta(X)^{(2)} \) be the second
infinitesimal neighbourhood of \( \Delta(X) \) in \( X \times X \), i.e. it is defined by the square of the ideal sheaf of the diagonal. The canonical exact sequence of a closed embedding becomes

\[
0 \to \Delta_* \Omega_X \to \mathcal{O}_{\Delta(X)} \to \Delta_* \mathcal{O}_X \to 0
\]

Its extension class is called the universal Atiyah class \( \text{At} \in \text{Ext}^1(\Delta_* \mathcal{O}_X, \Delta_* \Omega_X) \). Now we use Fourier-Mukai functors to evaluate these universal classes at particular objects in the derived category: the Atiyah class of an object \( \mathcal{F} \in \text{D}^b(X) \) is then \( \text{At}(\mathcal{F}) = \Phi_{\text{At}}(\mathcal{F}) \), where

\[
\Phi_{\text{At}}(\mathcal{F}) : \Phi_{\Delta_* \mathcal{O}_X}(\mathcal{F}) \to \Phi_{\Delta_* \Omega_X}(\mathcal{F}).
\]

To define the universal obstruction class, consider the closed embedding \( \tilde{i} = \text{id} \times i : Z \times Z \hookrightarrow Z \times X \), so that the complex \( \tilde{i}^* \tilde{i}^* \Delta_* \mathcal{O}_Z \) gives an \( \text{Ext}^2 \) class \( \alpha_{\Delta_* \mathcal{O}_Z} : \Delta_* \mathcal{O}_Z \to \Delta_* \mathcal{N}_{Z/X}^\vee[2] \).

As above we can evaluate it at any object in the derived category \( \text{D}^b(X) \).

**Proposition 1.10.** ([1], [11]) Suppose \( i : Z \hookrightarrow X \) is a locally complete intersection, and consider \( \mathcal{F} \in \text{D}^b(X) \). The class \( \alpha_{\mathcal{F}} \) is the product of \( \text{At}(\mathcal{F}) \) and \( \text{KS} \), i.e. \( \alpha_{\mathcal{F}} = (\text{id}_\mathcal{F} \otimes \text{KS}) \circ \text{At}(\mathcal{F}) \).

The next theorem gives a geometric interpretation of the obstruction class defined above (and justifies the terminology!).

**Theorem 1.11.** (Huybrechts-Thomas [11], Grivaux [9]) Let \( j : X \hookrightarrow X^{(1)} \) be a first order thickening of a Noetherian separated scheme. Suppose \( \mathcal{F} \in \text{D}^b(X) \) is a perfect complex. Then \( \mathcal{F} \) extends to a perfect complex \( \mathcal{F}^{(1)} \) on \( X^{(1)} \) iff \( \alpha_{\mathcal{F}} = 0 \).

**Theorem 1.12.** (Arinkin-Căldăraru [1]) Let \( i : Z \hookrightarrow X \) be a locally complete intersection. Suppose \( \mathcal{F} \) is locally free, \( \mathcal{F}_0 \) - quasi-coherent on \( Z \). Consider a morphism \( m : \mathcal{F} \otimes \mathcal{N}_{Z/X} \to \mathcal{F}_0 \). Then the obstruction to existence of an exact sequence

\[
0 \to i_* \mathcal{F}_0 \to \mathcal{G} \to i_* \mathcal{F} \to 0,
\]

such that the ideal sheaf of \( Z \) acts on \( \mathcal{G} \) via \( m \) is \( m \circ \alpha_{\mathcal{F}} \).

We are ready explain the relationship between these obstruction classes and the differentials on the second page of the spectral sequence. Thinking of \( \alpha_{\mathcal{F}} \) as an extension class, it is clear that \( \delta_0(i^* i_* \mathcal{Z}) = \alpha_{\mathcal{F}} \) as \( i^* i_* \mathcal{Z} \) is concentrated in non-positive degrees. For the other Ext classes, we have to work harder. Let us factor \( i : Z \hookrightarrow X \) as follows:

\[
Z \overset{i}{\hookrightarrow} Z^{(1)} \overset{\text{incl}}{\longrightarrow} X,
\]

where as usual \( Z^{(1)} \) is the first infinitesimal neighbourhood of \( Z \) in \( X \). In the case of \( \mathcal{O}_Z \), one can calculate \( \delta_1(i^* i_* \mathcal{O}_Z) \) in terms of the deformation-obstruction classes:

**Proposition 1.13.** (Arinkin-Căldăraru) Let \( i : Z \hookrightarrow X \) be a locally complete intersection. Then we have

\[
\delta_1(i^* i_* \mathcal{O}_Z) = \alpha_{\mathcal{N}_{Z/X}^\vee}.\]
Proof. Consider the exact sequence
\[ 0 \to j_*N_{Z/X}^\vee \to O_{Z} \to j_*O_{Z} \to 0 \] (2)
apply $j^*$ to get an exact triangle
\[ j^*j_*N_{Z/X}^\vee \to O_{Z} \to j^*j_*O_{Z} \to j^*j_*N_{Z/X}^\vee[1], \]
hence an isomorphism
\[ \tau^{<0}j_*O_{Z} \cong j^*j_*N_{Z/X}^\vee[1] \] (3)
in $D^b(X)$. This implies that
\[ \delta_1(j^*j_*O_{Z}) = \delta_1(j^*j_*N_{Z/X}^\vee[1]) = \delta_0(j^*j_*N_{Z/X}^\vee) = \alpha_{N_{Z/X}^\vee}. \]
We let $\text{Alt} : \mathcal{F}^\otimes p \to \bigwedge^p \mathcal{F}$ be the natural quotient. Consider an exact triangle
\[ \text{Sym}^2 N_{Z/X}^\vee[2] \to \tau^{\geq-2}j_*O_{Z} \to \mathcal{C} \to \text{Sym}^2 N_{Z/X}^\vee[3], \]
where the first map is the inclusion
\[ \text{Sym}^2 N_{Z/X}^\vee \to N_{Z/X}^\vee \otimes N_{Z/X}^\vee \]
followed by the canonical morphisms
\[ N_{Z/X}^\vee \otimes N_{Z/X}^\vee \to \mathcal{H}^{-2}(j^*j_*O_{Z}) \text{ and } \mathcal{H}^{-2}(j^*j_*O_{Z}) \to \tau^{\geq-2}j_*O_{Z}. \]
By construction we have $\delta_1(\mathcal{C}) = \text{Alt} \circ \delta_1(j^*j_*O_{Z})$, and since
\[ \tau^{\geq-2}i_*i^*O_{Z} \to \tau^{\geq-2}j^*j_*O_{Z} \to \mathcal{C} \]
is an isomorphism in $D^b(X)$, we conclude
\[ \delta_1(i^*i_*O_{Z}) = \delta_1(\mathcal{C}) = \text{Alt} \circ \delta_1(j^*j_*O_{Z}) = \text{Alt} \circ \alpha_{N_{Z/X}^\vee}. \]

Theorem 1.12, and the exact sequence
\[ 0 \to i_*\text{Sym}^2 N_{Z/X}^\vee = \mathcal{I}_Z^2/\mathcal{I}_Z^3 \to \mathcal{I}_Z/\mathcal{I}_Z^3 \to i_*N_{Z/X}^\vee \to 0 \]
imply that
\[ \text{Alt} \circ \alpha_{N_{Z/X}^\vee} = \alpha_{N_{Z/X}^\vee}, \]
since $\text{Sym} \circ \alpha_{N_{Z/X}^\vee} = 0$, that is, the obstruction class of the conormal bundle is skew-symmetric and we are done.

Remark 1.14. It would be nice to have similar geometric interpretation for the higher $\delta$'s in the general case.

Remark 1.15. In general, if $\mathcal{F} \in \text{Perf}(Z)$ lifts to the first infinitesimal neighbourhood, we have $\delta_1(i^*i_*\mathcal{F}) = \alpha_{\mathcal{F} \otimes N_{Z/X}^\vee}$. We note that Arinkin and Căldăraru in [1] show that $i^*i_*\mathcal{F}$ is formal iff $\alpha_{\mathcal{F}}$ and $\alpha_{\mathcal{F} \otimes N_{Z/X}^\vee}$ vanish. In the case of structure sheaf, formality is understood in the (stronger) sense of differential graded algebras.
Lemma 1.16. Let $i : Z \hookrightarrow X$ be a locally complete intersection, $c = \text{codim}(Z, X)$, and consider $\mathcal{L} \in \text{Pic}(Z)$. Define

$$\delta_q(\mathcal{L}) := \delta_{-q}(i^* i_* \mathcal{L}) \otimes \text{id}_{\mathcal{L}^\vee} \otimes \text{id}_{\det \mathcal{N}_Z/X}.$$ 

The differential $d_2$ of

$$E^{p,q}_2 = H^p(X, \mathcal{E}xt^q(i_* \mathcal{L}, i_* \mathcal{L})) \Rightarrow \mathcal{E}xt^{p+q}(i_* \mathcal{L}, i_* \mathcal{L})$$

can be described as

$$d^{p,q}_2 = R^p \Gamma \delta_q(\mathcal{L}) : H^p(Z, \wedge^q \mathcal{N}_{Z/X}) \rightarrow H^{p+2}(Z, \wedge^{q-1} \mathcal{N}_{Z/X}).$$

Proof. As already explained Grothendieck-Verdier duality gives an isomorphism in $D^b \mathcal{F}(X)$ between canonically filtered complexes

$$R\mathcal{H}om(i_* \mathcal{L}, i_* \mathcal{L}) \simeq i_* R\mathcal{H}om(\mathcal{L}, i^! i_* \mathcal{L}).$$

By definition (see Remark 1.5),

$$d^{p,q}_2 = R^p \Gamma \delta_q(\mathcal{L}) \in R^{p+q} \mathcal{H}om(i_* \mathcal{L}, i_* \mathcal{L}),$$

hence we are done since there is a canonical isomorphism in $D^b \mathcal{F}(X)$

$$i_* R\mathcal{H}om(\mathcal{L}, i^! i_* \mathcal{L}) \simeq i_* (\mathcal{L}^\vee \otimes i^! i_* \mathcal{L}).$$

\[ \square \]

Lemma 1.17. Let $i : Z \hookrightarrow X$ be a locally complete intersection and consider any $\mathcal{F} \in \text{Perf}(Z)$. Then $i_* \alpha_{\mathcal{F}} = 0$. Furthermore, the morphism

$$\text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}_{Z/X}^\vee) \rightarrow \text{Ext}^2(i_* \mathcal{F}, i_* (\mathcal{F} \otimes \mathcal{N}_{Z/X}^\vee))$$

is injective iff $\alpha_{\mathcal{F}} = 0$.

Remark 1.18. We observe that there are two obvious cases $\mathcal{L} = \mathcal{O}_Z$ and $\mathcal{L} = \det \mathcal{N}_{Z/X}$ in which the differential $d^{p,c}_2 : H^p(Z, \wedge^c \mathcal{N}_{Z/X}) \rightarrow H^{p+2}(Z, \wedge^{c-1} \mathcal{N}_{Z/X})$ vanishes. For the structure sheaf, it’s enough to note that $\delta_0(i^* i_* \mathcal{O}_Z) = 0$, while for $\det \mathcal{N}_{Z/X}$, we use Lemma 1.17.

Traces. We briefly review traces, mainly to fix notation. Suppose $\mathcal{F} \in \text{Perf}(X)$ and let $\mathcal{V}$ be a vector bundle on $X$. Then we define the sheaf trace

$$\text{Tr} : R\mathcal{H}om(\mathcal{F}, \mathcal{F} \otimes \mathcal{V}) \rightarrow \mathcal{V}$$

as the composition of the natural isomorphism $R\mathcal{H}om(\mathcal{F}, \mathcal{F} \otimes \mathcal{V}) \simeq \mathcal{F}^\vee \otimes \mathcal{F} \otimes \mathcal{V}$ and the natural map $\mathcal{F}^\vee \otimes \mathcal{F} \rightarrow \mathcal{O}_X$. Applying $R\Gamma$, and taking cohomology, we get the cohomological trace map of Illusie (see [12]):

$$\text{Tr} : \text{Ext}^k(\mathcal{F}, \mathcal{F} \otimes \mathcal{V}) \rightarrow H^k(X, \mathcal{V}).$$

If $\mathcal{F}, \mathcal{G} \in \text{Perf}(X)$, we also have a partial trace:

$$\text{Tr}_\mathcal{G} : \text{Ext}^k(\mathcal{F} \otimes \mathcal{G}, \mathcal{F} \otimes \mathcal{G} \otimes \mathcal{V}) \rightarrow \text{Ext}^k(\mathcal{G} \otimes \mathcal{V}).$$
is additive in the following sense: let \( \mathcal{F} \to \mathcal{F}_2 \to \mathcal{F}_3 \to \mathcal{F}_1[1] \) be an exact triangle, and suppose \( \alpha_i \in \text{Ext}^k(\mathcal{F}_i, \mathcal{F}_i \otimes \mathcal{V}) \) induce a morphism of exact triangles (notice \( \mathcal{V} \) is flat so tensoring with \( \mathcal{V} \) is exact), then \( \text{Tr} \alpha_1 - \text{Tr} \alpha_2 + \text{Tr} \alpha_3 = 0 \). It is also multiplicative: if \( \alpha \in \text{Ext}^k(\mathcal{F}, \mathcal{F} \otimes \mathcal{V}) \) and \( \beta \in \text{Ext}^j(\mathcal{V}, \mathcal{E}) \), for a vector bundle \( \mathcal{E} \), then \( \text{Tr}(\text{id} \otimes \beta \circ \alpha) = \beta \circ \text{Tr}(\alpha) \) in \( H^{i+j}(X, \mathcal{E}) \). Applying this to the case of an obstruction class \( \alpha_V \), we get
\[
\text{Tr} \alpha_V = \text{Tr}(\text{id} \otimes \text{KS} \circ \text{At} V) = \text{KS} \circ \text{Tr}(\text{At} V) = \alpha_{\det V}.
\] (4)

2 Applications

The absolute case. We begin with the definition of a hyperkähler variety in the algebraic setting. Afterwards, we consider the compatibility of the local-to-global spectral sequence with Serre duality and give a brief reminder on Lefschetz structures, before going into our main results.

Definition 2.1. A smooth, proper, connected scheme \( X/k \) is a hyperkähler variety (over \( k \)) if \( \dim(H^0(X, \Omega^2_X)) = 1 \), generated by a non-degenerate form \( \sigma \) (a symplectic form), \( \pi_{\text{et}}(X) = 1 \), and \( K_X \cong O_X \).

A subvariety \( L \) of \( X \) is called Lagrangian if \( \sigma \big|_L = 0 \) and \( 2\dim L = \dim X \). If \( i : L \hookrightarrow X \) is a smooth Lagrangian we have \( T_X \cong \Omega^1_X \) via the symplectic form, hence \( i^* T_X \cong i^* \Omega^1_X \). There is a commutative diagram:
\[
\begin{array}{cccccc}
0 & \to & \mathcal{T}_L & \to & i^* \mathcal{T}_X & \to & \mathcal{N}_{Z/X} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{N}_{Z/X}^\vee & \to & i^* \Omega^1_X & \to & \Omega^1_L & \to & 0
\end{array}
\]

which shows we have isomorphisms \( \Omega^q_L \cong \wedge^q \mathcal{N}_{Z/X} \). Hence the second page in the Lagrangian case is \( E^2_{p,q} = H^p(L, \Omega^q_L) \) and, using the results of previous sections, we have differentials
\[
\begin{align*}
R^p \delta_1(L) &= d^p_1 : H^p(L, \Omega^1_L) \to H^{p+2}(L, \mathcal{O}_L), \\
R^p \delta_2(L) &= d^p_2 : H^p(L, \Omega^2_L) \to H^{p+2}(L, \Omega^1_L).
\end{align*}
\]

Serre duality asserts that the pairing
\[
H^p(L, \Omega^q_L) \otimes H^{n-p}(L, \Omega^{n-q}_L) \to H^n(L, K_L)
\]
is perfect, where we set \( n = \dim L = \text{codim}(L, X) \) and \( K_L \) denotes the canonical bundle of \( L \). Since \( d_2 \) kills \( H^p(L, K_L) \), and the higher differentials kill the appropriate subquotients, and \( d_2 \) is multiplicative, we see that the diagram
\[
\begin{array}{cccc}
H^p(L, \Omega^q_L) \otimes H^{n-p}(L, \Omega^{n-q}_L) & \xrightarrow{\text{id} \otimes d_2} & H^{p+2}(L, \Omega^q_L) \otimes H^{n-p-2}(L, \Omega^{n-q+1}_L) & \xleftarrow{d_2 \otimes \text{id}} \\
\end{array}
\]
is commutative.
commutes up to sign, hence the following diagram, and its variants for the higher differentials, commutes up to sign, i.e. the differentials are compatible with Serre duality:

\[
\begin{array}{ccc}
H^p(L, \Omega^q_L) & \xrightarrow{d_2} & H^{p+2}(L, \Omega^{q-1}_L) \\
\downarrow & & \downarrow \\
H^{n-p}(L, \Omega^{n-q}_L) & \xrightarrow{d_2'} & H^{n-p-2}(L, \Omega^{n-q+1}_L).
\end{array}
\]

**Definition 2.2.** A graded Lefschetz structure, in an abelian category \(\mathcal{A}\), is a pair \((H, L)\), where \(H \in \mathcal{A}\) is a graded object and \(L\) is a nilpotent endomorphism of \(H\), such that \(L(H^i) \subset H^{i+2}\) and \(L^i : H^{-i} \xrightarrow{\sim} H^i\).

**Example 2.3.** Let \(X\) be a smooth projective variety of dimension \(n\) with an ample line bundle \(\mathcal{L}\). Then, if \(\omega\) is the first Chern class of \(\mathcal{L}\) and \(L_\omega := \omega \cup : H^{n-\bullet}(X/k) \to H^{n-\bullet}(X/k)\), the pair \((H^{n-\bullet}(X/k), L_\omega)\) is a graded Lefschetz structure (in fact, it is a \((-\)Hodge-Lefschetz structure - an example of a mixed Hodge structure).

**Remark 2.4.** We shall abuse terminology, calling the first Chern class of an ample line bundle a Kähler form.

**Theorem 2.5.** Let \(X/k\) be a (projective) hyperkähler variety, let \(i : L \to X\) be a smooth Lagrangian, denote the Kähler form on \(L\) by \(\omega \in H^1(\Omega^1_L)\), and suppose \(\mathcal{L}\) is a line bundle on \(L\). Then the local-to-global Ext spectral sequence

\[E^2_{p,q} = H^p(L, \Omega^q_L) \Rightarrow \text{Ext}^{p,q}(\mathcal{i}_*\mathcal{L}, \mathcal{i}_*\mathcal{L})\]

degenerates (multiplicatively) on the second page if and only if \(d_2(\omega) = 0\).

**Proof.** The only if part is trivial. Suppose \(d_2(\omega) = 0\). Then for all \(r \geq 2\), \(d_r(\omega) = 0\) and we get

\[d_r(L_\omega \alpha) = d_r(\omega) \cup \alpha + \omega \cup d_r(\alpha) = L_\omega d_r(\alpha),\]

where \(L_\omega\) is the Lefschetz operator associated to \(\omega\), i.e. the differentials commute with the Lefschetz operator. Let \(\text{codim}(L, X) = n\), and consider the diagram:

\[
\begin{array}{ccc}
H^p(L, \Omega^q_L) & \xrightarrow{d_2} & H^{p+2}(L, \Omega^{q-1}_L) \\
\downarrow \text{L}_\omega^{-p-q+1} & & \downarrow \text{L}_\omega^{-p-q+1} \\
H^{n-q+1}(L, \Omega_L^{n-p+1}) & \xrightarrow{d_2} & H^{n-q+3}(L, \Omega_L^{n-p}).
\end{array}
\]

Restricting to the primitive cohomology \(H^p_0(L, \Omega^q_L)\), we see that

\[d_2(H^0_0(L, \Omega^q_L)) \subset \ker(L_\omega^{-p-q+1}|_{H^{p+2}(L, \Omega_L^{n-1})}) = H^{p+2}_0(L, \Omega_L^{n-1}) \oplus L_\omega H^{p+1}_0(L, \Omega_L^{n-2}).\]

So we can write \(d_2 = d_2^0 + L_\omega d_2'\). Deligne’s argument is essentially the diagram:

\[
\begin{array}{ccc}
H^0_0(L, \Omega^q_L) & \xrightarrow{d_2'} & H^{p+1}_0(L, \Omega_L^{n-2}) \\
\downarrow \text{L}_\omega^{-p+q+1} & & \downarrow \text{L}_\omega^{-p+q+1} \\
H^{n-q+1}(L, \Omega_L^{n-p+1}) & \xrightarrow{d_2'} & H^{n-q+2}(L, \Omega_L^{n-p}).
\end{array}
\]
The left vertical arrow is 0, while the right one is injective, hence \( d'_2 = 0 \). This means that \( d_2 \) preserves primitive cohomology and \( d_2^{p,q} \) vanishes for all \( p + q = n \). Let us assume that \( d_2 \) vanishes for \( p + q = k + 1 \), base case being \( p + q = n \). Suppose \( \alpha \in H^p_0(L, \Omega^q_L) \) with \( p + q = k \). Then for any \( \beta \in H^{q-1}_0(L, \Omega^{p+2}_L) \), we have \( L^{\omega^{p-q-1}(\alpha \cup \beta)} = 0 \), so

\[
0 = d_2(L^{\omega^{p-q-1}(\alpha \cup \beta)}) = L^{\omega^{p-q-1}(d_2(\alpha) \cup \beta + (-1)^{\text{deg} \alpha} \alpha \cup \beta)} = L^{\omega^{p-q-1}(d_2(\alpha) \cup \beta)}.
\]

We note that the pairing

\[
H^{p+2}_0(L, \Omega^{q-1}_L) \otimes H^{q-1}_0(L, \Omega^{p+2}_L) \to k : \gamma \otimes \beta \mapsto \int_L \omega^{p-q-1} \cup \gamma \cup \beta
\]

is non-degenerate. Therefore \( d_2(\alpha) = 0 \) and by induction we conclude that \( d_2 = 0 \). Assuming by induction \( d_r = 0 \), we run the same procedure to \( d_{r+1} \) to complete the induction, hence concluding that for all \( r \geq 2 \), \( d_r = 0 \), and the spectral sequence collapses on the second page. \( \square \)

**Theorem 2.6.** Let \( X/k \) be a (projective) hyperkähler variety, \( i : L \to X \) a smooth Lagrangian, and let \( \mathcal{L} \) be a line bundle on \( L \), extending to the first infinitesimal neighbourhood of \( L \) in \( X \), such as \( \mathcal{O}_L \). Then the local-to-global Ext spectral sequence

\[
E^2_p = H^p(L, \Omega^q_L) \Rightarrow \text{Ext}^{p+q}(i_* \mathcal{L}, i_* \mathcal{L})
\]

degenerates on the second page. Hence \( H^*(L/k) = \oplus_p \text{Ext}^p(L, \Omega^q_L) = \text{Ext}(i_* \mathcal{L}, i_* \mathcal{L}) \), as graded algebras.

**Proof.** It would suffice to show that \( d_2(\omega) = 0 \). By Serre duality, \( d_2^{p-1} = 0 \) if \( d_2^n = 0 \). We have already seen that \( d_2^{p,n} = R^p \Gamma(\alpha_{\mathcal{L}} \otimes \text{id}_{\mathcal{L}}^\vee \otimes \text{id}_{K_L}) \). The class \( \alpha_{\mathcal{L}} \) is the obstruction to extending \( \mathcal{L} \) to a line bundle on the first infinitesimal neighbourhood, so \( \omega \) vanishes iff \( \mathcal{L} \) extends and hence \( d_2(\omega) = 0 \). \( \square \)

**Theorem 2.7.** Let \( i : L \to X \) be as above, and let \( \mathcal{K} \) be any (existing) rational power of the canonical bundle of \( L \). Then the local-to-global Ext spectral sequence

\[
E^2_p = H^p(L, \Omega^q_L) \Rightarrow \text{Ext}^{p+q}(i_* \mathcal{K}, i_* \mathcal{K})
\]

degenerates (multiplicatively) on the second page.

**Proof.** Let \( \mathcal{K} = K_L^{s/t} \) and note that since \( \mathcal{K} \) is a line bundle

\[
\alpha_{\mathcal{K}} = KS \cup c_1(\mathcal{K}) \in H^2(L, \mathcal{H}_L),
\]

hence \( \alpha_{\mathcal{K}} = (s/t)\alpha_{K_L} \) in \( H^2(L, \mathcal{H}_L) \), and then \( \hat{\delta}_n(\mathcal{K}) = (s/t)\hat{\delta}_n(K_L) \), so the proof below shows we may suppose that \( \mathcal{K} = K_L \) is the canonical bundle of \( L \). We would like to show that \( d_2(\omega) = 0 \) and apply the theorem. Notice, as in the proof of *Theorem 2.6*, Serre duality implies that it suffices to show that \( d_2^{p,n} = 0 \). However, in the present case, we have

\[
d_2^{p,n} = R^p \Gamma(\alpha_{K_L} \otimes \text{id}_{K_L^\vee} \otimes \text{id}_{K_L}) = R^p \Gamma(\alpha_{K_L}) = 0,
\]

where the last equality follows from *Remark 1.18.* \( \square \)
3 Variants

Proof by DQ modules. We give an alternative way of proving a version of Theorem 2.7 over C in the case \( \mathcal{X} = K_L^{1/2} \) which works without assuming the Lagrangian Kähler.

**Proposition 3.1.** Let \( X/C \) be holomorphic symplectic, and consider a smooth compact Lagrangian \( i : L \to X \) whose canonical bundle admits a square root. Then the local-to-global Ext spectral sequence

\[
E_{p,q}^2 = H^p(L, \Omega^q_L) \Rightarrow \text{Ext}^{p+q}(i_*K_L^{1/2}, i_*K_L^{1/2})
\]

degenerates on the second page.

**Proof.** This proof was envisaged by Thomas, and Petit helped us make the initial sketch rigorous. It will be enough to show that \( \dim(\text{Ext}^i(i_*K_L^{1/2}, i_*K_L^{1/2})) \geq \dim(H^i(L/C)) \).

Let \( \mathcal{A} \) be the quantisation of \( X \) as in [14]. We fix a square root \( K_L^{1/2} \) of the canonical bundle. It quantises (see [4], [14]), so we get an \( \mathcal{A} \)-module \( D_L^0 \). Let \( \mathcal{A}^\text{loc} \) be the localisation \( \mathbb{C}((h)) \otimes_{\mathbb{C}[h]} \mathcal{A} \). Then \( D_L^0 \) localises to a simple holonomic deformation-quantisation module \( D_L \) over \( \mathcal{A}^\text{loc} \).

It’s well-known that \( R\text{Hom}(D_L, D_L) \cong \mathbb{C}((h)) \), so we get \( \text{Ext}^i_{\mathcal{A}^\text{loc}}(D_L, D_L) = H^i(L, \mathbb{C}((h))) \), hence the universal coefficients theorem implies that \( \dim(\text{Ext}^i_{\mathcal{A}^\text{loc}}(D_L, D_L)) = \dim(H^i(L/C)) \). By [14, Corollary 3.2.3], which requires \( L \) compact, we have that

\[
\text{RHom}_{\mathcal{A}}(D_L^0, D_L^0) \in \text{Perf}(\text{Spec}(\mathbb{C}[h])),
\]

hence we can apply the semi-continuity theorem on \( \mathbb{C}[h] \) to get

\[
\dim(H^i(\mathbb{C} \otimes_{\mathbb{C}[h]} \text{RHom}_{\mathcal{A}}(D_L^0, D_L^0))) \geq \dim(H^i(\mathbb{C}((h)) \otimes_{\mathbb{C}[h]} \text{RHom}_{\mathcal{A}}(D_L^0, D_L^0)))).
\]

The projection formula implies \( \mathbb{C} \otimes_{\mathbb{C}[h]} \text{RHom}_{\mathcal{A}}(D_L^0, D_L^0) \simeq \text{RHom}(i_*K_L^{1/2}, i_*K_L^{1/2}) \) and

\[
\mathbb{C}((h)) \otimes_{\mathbb{C}[h]} \text{RHom}_{\mathcal{A}}(D_L^0, D_L^0) \simeq \text{RHom}_{\mathcal{A}^\text{loc}}(D_L, D_L),
\]

thus \( \dim(\text{Ext}^i(i_*K_L^{1/2}, i_*K_L^{1/2})) \geq \dim(\text{Ext}^i_{\mathcal{A}^\text{loc}}(D_L, D_L)) = \dim(H^i(L/C)) \). \( \square \)

**Formality.** It would be interesting to know if \( \text{RHom}(i_*L, i_*L) \) is a formal differential graded algebra, provided that the associated spectral sequence degenerates, as in the mirror case of Solomon and Verbitsky this holds. In their paper, the differential graded algebra is essentially the differential graded algebra of differential forms (the product is twisted but this doesn’t affect the conclusion) and they are able to apply a celebrated formality result of Deligne-Griffiths-Morgan-Sullivan. From the degeneration of the spectral sequence, we get a morphism \( \text{RHom}(i_*L, i_*L) \to \text{Ext}(i_*L, i_*L) \) that is quasi-isomorphism of complexes but it needn’t be a morphism of differential graded algebras. In fact, Kontsevich’s formality in the case of the diagonal shows that we would need to twist by a square root of the inverse Todd class at the very least.
The relative case. We briefly outline a relative version of the results obtained in the previous paragraph.

It is evident that Definition 2.1 generalises to any base scheme $S$ of characteristic 0, so we have a notion of hyperkähler schemes $X/S$. Consider the following situation:

$$
\begin{array}{c}
L \\
\downarrow^p \\
S,
\end{array}
\quad
\xrightarrow{i}
\begin{array}{c}
X
\end{array}
$$

where $L$ is a smooth Lagrangian in $X$, and set $p' = p \circ i$. Just as in the absolute case, we obtain $\mathcal{M}_{L/X} \cong \Omega^1_{L/S}$. If $X$ is projective, we have a relatively ample line bundle on $X$, inducing a relative Kähler form $\omega \in H^1(L, \Omega^1_{L/S})$. We get Lefschetz operators

$$
R^p_p : R^p_p \mathcal{O}_{L/S} \to R^p_p \Omega^{p+1}_{L/S}[1],
$$

satisfying the hard Lefschetz theorem. Furthermore, $R^p_p \mathcal{O}_{L/S}$ are formal (as complexes), and their cohomology sheaves are locally free, hence Serre duality holds. Therefore, we obtain the following generalisation of the absolute case considered in the previous paragraph:

**Theorem 3.2.** Let $X/S$ be a projective hyperkähler variety, where $S$ is of characteristic 0 and consider a smooth Lagrangian $L \hookrightarrow X$.

$$
\begin{array}{c}
L \\
\downarrow^p \\
S,
\end{array}
\quad
\xrightarrow{i}
\begin{array}{c}
X
\end{array}
$$

Let $\omega : \mathcal{O}_L \to \Omega_{L/S}[1]$ be the relative Kähler form induced by a relatively ample line bundle and suppose $\mathcal{L} \in \text{Pic}(L)$. Then the spectral sequence

$$
E_2^{p,q} = R^p_p i_* \Omega^q_{L/S} \Rightarrow R^p_p i_* R^q \mathcal{H}om(i_* \mathcal{L}, i_* \mathcal{L})
$$

degenerates (on $E_2$) if and only if $d_2 \circ R^p_p \omega = 0$, acting on $R^p_p \mathcal{O}_L$.

Coisotropic subvarieties. Another possible generalisation, suggested to us by Sabin Cautis, involves replacing Lagrangian by coisotropic. So let $i : L \hookrightarrow X$ be a coisotropic subvariety, then it has a characteristic foliation $p : L \to B$, which we assume smooth projective, where $B$ is the space of leaves. Then $\mathcal{M}_{L/X} \cong \Omega_{L/B}$ and we get a spectral sequence

$$
E_2^{p,q} = R^p_p \Omega^q_{L/B} \Rightarrow R^{p+q} p_*(\mathcal{L}^\vee \otimes i^! i_* \mathcal{L}).
$$

One may speculate to what extent it degenerates. Our approach doesn’t immediately generalise to this setting, and the reason is that $\mathcal{L}^\vee \otimes i^! i_* \mathcal{L}$ doesn’t seem to carry any (natural) algebra structure, so while its cohomology sheaves have cup products, it’s not clear if the differentials of the spectral sequence will be compatible with these cup products.

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Pairs of Lagrangians. So far, we have been focusing on the self-intersection of a single Lagrangian \( i: L \hookrightarrow X \) in a hyperkähler variety. Now, we shall consider (smooth) intersections \( Z \) of two Lagrangians \( i: L \hookrightarrow X \) and \( j: M \hookrightarrow X \) in a (projective) hyperkähler variety \( X \). We shall be suppressing pushforwards under the inclusion \( Z \hookrightarrow X \). We also don’t need hyperkähler and Lagrangian conditions for the calculations in the two propositions below.

**Proposition 3.3.** Let \( \mathcal{F}, \mathcal{G} \) be coherent sheaves on \( L \) and \( M \), respectively. Assume \( \mathcal{F} \) locally free. Then we have

\[
\mathcal{E}xt^p(i_* \mathcal{F}, j_* \mathcal{G}) \cong \mathcal{E}xt^p(i_* \mathcal{O}_L, j_* \mathcal{O}_M) \otimes \mathcal{F}^\vee|_Z \otimes \mathcal{G}|_Z
\]

**Proposition 3.4.** Assuming \( L, M \) and \( Z \) smooth, we have

\[
\mathcal{E}xt^p(i_* \mathcal{O}_L, j_* \mathcal{O}_M) \cong \wedge^{p-c} \mathcal{N} \otimes \det \mathcal{N}_Z/M,
\]

where \( c = \text{rk} \mathcal{N}_Z/M, \mathcal{N} := \mathcal{T}_X|_Z / (\mathcal{T}_L|_Z + \mathcal{T}_M|_Z) \) is the excess normal bundle.

In the Lagrangian case, which we assume from now on, we have an exact sequence:

\[
0 \to \mathcal{T}_Z \to \mathcal{T}_L|_Z \oplus \mathcal{T}_M|_Z \to \mathcal{T}_X|_Z \to \Omega_Z \to 0,
\]

hence

\[
\mathcal{E}xt^p(i_* \mathcal{O}_L, j_* \mathcal{O}_M) \cong \wedge^{p-c} \Omega_Z \otimes \det \mathcal{N}_Z/M.
\]

**Lemma 3.5.** Let \( L, M \) be Lagrangians with a smooth intersection \( Z \), there is an isomorphism

\[
K_Z \otimes K_Z \cong K_L|_Z \otimes K_M|_Z.
\]

Combine this lemma with the adjunction formula to obtain \( \det \mathcal{N}_Z/L \otimes \det \mathcal{N}_Z/M \cong \mathcal{O}_Z \).

**Corollary 3.6.** Assuming there exist square roots \( K^{1/2}_L \) and \( K^{1/2}_M \), define the orientation bundle

\[
\mathcal{K}_Z := \left( K^{1/2}_L|_Z \otimes K^{1/2}_M|_Z \right)^\vee \otimes K_Z
\]

and set \( n = \text{codim}(L, X) \), so \( c = n - \text{dim} Z \). Then

\[
\mathcal{E}xt^p(i_* K^{1/2}_L, j_* K^{1/2}_M) \cong \wedge^{p-c} \Omega_Z \otimes \mathcal{K}_Z.
\]

**Remark 3.7.** Notice that the line bundle \( \mathcal{K}_Z \) is torsion, in fact of order 2. Hence the monodromy representation associated to the local system \( \mathcal{K}_Z \), arising from \( \mathcal{K}_Z \), is unitary.

**Remark 3.8.** More generally, for line bundles \( \mathcal{L} \) and \( \mathcal{M} \), we have an orientation bundle

\[
\mathcal{L}_Z := \det \mathcal{N}_Z/L \otimes \mathcal{L}|_Z \otimes \mathcal{M}|_Z,
\]

whose flatness is the obstruction to applying our Hodge theoretic arguments.

There is a natural map

\[
R \mathcal{H}om(i_* \mathcal{L}, j_* \mathcal{M}) \otimes R \mathcal{H}om(i_* \mathcal{L}, i_* \mathcal{L}) \to R \mathcal{H}om(i_* \mathcal{L}, j_* \mathcal{M}),
\]

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Theorem (5)
sequence

One direction is trivial. Let us assume
Proof.

Indeed, if \( \omega \) is a Kähler form on \( Z \), and similarly for \( \omega_M \), then the lemma implies \( L_{\omega_L} = L_\omega = L_{\omega_M} \) when acting on \( H^p(Z, \Omega_X^Z \otimes L_{\omega_L}) \), where \( \omega \) is the Kähler form on \( Z \) and \( L_{(-)} \) stands for "cup with \((-)\)". We are ready to prove:

**Theorem 3.10.** Let \( X/k \) be a (projective) hyperkähler variety, let \( i : L \hookrightarrow X, j : M \hookrightarrow X \) be smooth Lagrangians with \( L \cap M = Z \) smooth, and suppose \( L, M \) are line bundles on \( L \) and \( M \), respectively, such that the line bundle \( L_{\omega_L} = \det(N_{L/M} \otimes \mathcal{L}^{\vee} \otimes \mathcal{M})_Z \), with corresponding local system \( L_{\omega_L} \), is 2-torsion. Then the spectral sequence

\[
E^{p,q}_2 = H^p(X, \mathcal{E}xt^q(i_* \mathcal{L}, j_* \mathcal{M})) \Rightarrow Ext^{p+q}(i_* \mathcal{L}, j_* \mathcal{M})
\]

(5)

degenerates on the second page if and only if \( d_2 \) commutes with the Lefschetz operator \( L_{\omega_L} \).

**Proof.** One direction is trivial. Let us assume \( d_2 \) commutes with \( L_{\omega_L} \) - this makes sense after identifying \( L_{\omega_L} = L_{\omega_L} \) acting on \( H^p(Z, \Omega_X^Z \otimes L_{\omega_L}) \). The idea is similar to what we did in the proof of **Theorem 2.5**, the only difference being that we do not have a natural pairing of the spectral sequence (5) with itself. So, instead, we are going to pair it with

\[
E^{p,q}_2 = H^p(X, \mathcal{E}xt^q(j_* \mathcal{M}, i_* \mathcal{L})) \Rightarrow Ext^{p+q}(j_* \mathcal{M}, i_* \mathcal{L}),
\]

(6)
whose differentials are denoted $d^*$. Since $\mathcal{L}_{or}$ is 2-torsion, (5) and (6) have isomorphic second pages, but the differentials are, a priori, not the same. However, $d^*$ still commutes with $L_{or}$, since it does not act on $H^1(L, \Omega^1_L)$. The same reasoning as in Theorem 2.5 implies that both $d_{2p,q}^*$ and $d_{2p,q}^*$ preserve primitive cohomology and vanish for $p + q = n$, where $n = \text{codim}(L, X)$. By Serre duality and Lefschetz, the standard pairing

$$H^0_0(Z, \Omega^q_Z \otimes \mathcal{L}_{or}) \otimes H^0_0(Z, \Omega^p_Z \otimes \mathcal{L}_{or}) \to k : \alpha \otimes \beta \mapsto \int_Z \omega^{d-p-q} \cup \alpha \cup \beta$$

is non-degenerate, and hence the commutative diagram

$$\begin{align*}
H^p(Z, \Omega^p_Z \otimes \mathcal{L}_{or}) & \otimes H^q(Z, \Omega^q_Z \otimes \mathcal{L}_{or}) \longrightarrow H^{p+q}(L, \Omega^p_L) \\
\downarrow & \\
H^{p+q}(Z, \Omega^{p+q}_Z) \longrightarrow H^{p+q}(Z, \Omega^{p+q}_Z)
\end{align*}$$

shows that the pairing

$$H^0_0(Z, \Omega^q_Z \otimes \mathcal{L}_{or}) \otimes H^0_0(Z, \Omega^p_Z \otimes \mathcal{L}_{or}) \to H^{p+q}(L, \Omega^p_L) \xrightarrow{L_{or}} H^d(L, \Omega^d_L) \to k$$

is non-degenerate, where the last map is restriction to $H^d(Z, \Omega^d_Z) \cong k$, $\dim Z = d$. Hence the induction argument on $p + q$ goes through - more precisely, setting $c = \text{codim}(Z, L)$, we have an intertwined downward induction with base case $p + q = n$:

$$
\begin{align*}
d_{2p,q}^* = 0 \implies d_{2q-c-2p+c+1}^* &= 0 \\
d_{2p,q}^* = 0 \implies d_{c-2p+c+1}^* &= 0,
\end{align*}
$$

so the spectral sequences (5) and (6) degenerate on $E_2$. \hfill \Box

Using Remark 3.7 and Theorem 2.7, we get at once the following corollary:

**Corollary 3.11.** Let $X/k$ be a (projective) hyperkähler variety, let $i : L \hookrightarrow X$, $j : M \hookrightarrow X$ be smooth Lagrangians with a smooth intersection $Z$. Then the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(i_*K^{1/2}_L, j_*K^{1/2}_M)) \Rightarrow \text{Ext}^{p+q}(i_*K^{1/2}_L, j_*K^{1/2}_M) \quad (7)$$

degenerates on the second page.

**Remark 3.12.** Similarly to Proposition 3.1, we can prove Corollary 3.11 using DQ modules. The proof is almost identical, using that the complex $\mathcal{K}om(\mathcal{P}_L, \mathcal{P}_M)$ is a perverse sheaf, isomorphic to $\mathcal{P}_{L,M}$ over $\mathcal{C}((h))$, and the latter is in fact a flat local system for $X = L \cap M$ smooth. This proof shows the result is true more generally for any holomorphic symplectic $X$, and not necessarily Kähler Lagrangians. We do not require $X$ be proper, but the Lagrangians are assumed proper.

**Behrend and Fantechi.** In [2] Behrend and Fantechi construct a virtual de Rham complex $(\mathcal{E}, d)$ on Lagrangian intersections. Their construction is missing some orientation data, i.e. one should replace $\mathcal{E}^p := \mathcal{E}xt^p(i_*, \mathcal{O}_L, j_*, \mathcal{O}_M)$ by $\mathcal{E}xt^p(i_*K^{1/2}_L, j_*K^{1/2}_M)$. We prove the accordingly modified version of [2, Conjecture 5.8]:

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Proposition 3.13. Let $X/C$ be holomorphic symplectic, and consider smooth proper Lagrangians $L, M$. Assuming $Z = L \cap M$ smooth, we have

$$H^p(\mathcal{E}, d) \simeq \text{Ext}^p(i_*K_1^{1/2}, j_*K_M^{1/2}).$$

Remark 3.14. Notice [2, Conjecture 5.8] requires $Z$ satisfy some analogue of the Kähler condition, but we demand it only be smooth proper.

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