BRST invariant formulation of spontaneously broken
gauge theory in generalized differential geometry

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Abstract
Noncommutative geometry (NCG) on the discrete space successfully reproduces the Higgs mechanism of the spontaneously broken gauge theory, in which the Higgs boson field is regarded as a kind of gauge field on the discrete space. We could construct the generalized differential geometry (GDG) on the discrete space $M_4 \times Z_N$ which is very close to NCG in case of $M_4 \times Z_2$ and $M_4 \times Z_3$. GDG is a direct generalization of the differential geometry on the ordinary manifold into the discrete one. In this paper, we attempt to construct the BRST invariant formulation of spontaneously broken gauge theory based on GDG and obtain the BRST invariant Lagrangian with the t’Hooft-Feynman gauge fixing term.

1 Introduction
The standard model in particle physics has matched with all experimental data conducted so far. Only ingredient which remains undetermined is the Higgs boson field that causes the spontaneous breakdown of symmetry through its vacuum expectation value. The next targets of future accelerators such as LHC and Tevatron II are to detect the Higgs boson and supersymmetric partners of particles and to find the breakthrough out of the standard model. Thus, the great concerns about the Higgs boson field have been kept also from now.

Let us put emphasis on its similarity with the gauge boson as one of the characteristic features of the Higgs boson in the standard model. Regardless to say, the Higgs boson field is a boson field as the gauge field though it is scalar field. In addition, the Higgs field has the same type coupling with fermions as the gauge fields and it has trilinear and quartic self-couplings as well as weak gauge fields. From these similarity between the Higgs and gauge bosons, an idea that the Higgs boson may be a kind of gauge boson comes out. In fact, as models to realize this idea, there has been proposed the Kaluza-Klein model [1], and Noncommutative geometry (NCG) [2], [4] on the discrete space. Especially, NCG approach does not require any extra physical mode and realizes the unified picture of gauge and Higgs fields as the generalized connection on the discrete space $M_4 \times Z_2$.

Since the first formulation of NCG by Connes [2], many versions of NCG [3]-[5] has appeared and succeeded to reconstruct the spontaneously broken gauge theories. Morita and the present author [9] proposed the generalized differential geometry (GDG) on the discrete space $M_4 \times Z_2$ and reconstructed the Weinberg-Salam model. In this formulation on $M_4 \times Z_2$ the extra differential one-form $\chi$ is introduced in addition to the usual one-form $dx^\mu$ and so our formalism is the generalization of the ordinary differential geometry on the compact manifold. This formulation was generalized to GDG on the discrete space $M_4 \times Z_N$ [9], [10] by introducing the extra one-forms $\chi_k (k = 1, 2 \cdots N)$, which generalization enabled us to reconstruct the left-right symmetric gauge theory [11], SU(5) GUT [12] and SO(10) GUT [13] as spontaneously broken gauge theories on the discrete space $M_4 \times Z_N$.

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It is also very important to reconstruct the gauge fixing and ghost terms in NCG in order to ensure the quantization of gauge theory. Lee, Hwang and Ne'eman \cite{LHN} succeeded in incorporating these terms in NCG in the matrix derivative approach based on the super-connection formalism \cite{BS}. They obtained the BRS/anti-BRS transformation rules of the theory by applying the horizontality condition \cite{B} in the super-connection formalism and constructed BRST invariant Lagrangian including the gauge fixing and ghost terms. The present author subsequently applied their idea to the GDG formulation of gauge theory on $M_4 \times Z_2$ \cite{T1} and obtained the BRST invariant formulation. We apply in this article the similar method to more general formulation of GDG on $M_4 \times Z_N$ \cite{T2} based on our formulation of GDG. The last section is devoted to concluding remarks.

2 BRST transformation in generalized differential geometry

The generalized differential geometry (GDG) on $M_4 \times Z_N$ was formulated \cite{BL, BGL} to reconstruct the gauge invariant Lagrangian of the spontaneously broken gauge theories such as the standard model, the left-right symmetric gauge theory and SU(5) and SO(10) grand unified theories. In this section, we incorporate BRST transformation in GDG according to the super field formalism of Bonora and Tonin \cite{BT}. This formulation \cite{BGL} has already done in GDG on the discrete space $M_4 \times Z_2$. Here, we generalize it to $M_4 \times Z_N$ where $x_\mu$ and $n (n = 1, 2 \cdots N)$ are arguments in $M_4$ and $Z_N$, respectively.

Let us start with the equation of the generalized gauge field $A(x, n, \theta, \bar{\theta})$ written in one-form on the discrete space $M_4 \times Z_N$:

$$A(x, n, \theta, \bar{\theta}) = \sum_i a_i(x, n, \theta, \bar{\theta})da_i(x, n, \theta, \bar{\theta})$$

(2.1)

by adding the Grassmann numbers $\theta$ and $\bar{\theta}$ to $x_\mu$ and $n$ to produce the ghost and anti-ghost fields. The constituent $a_i(x, n, \theta, \bar{\theta})$ is the square-matrix-valued function with a subscript $i$ which is a variable of the extra internal space. Now, we simply regard $a_i(x, n, \theta, \bar{\theta})$ as the more fundamental field to construct gauge and Higgs fields because it has only mathematical meaning and never appears in final stage. The operator $d$ in Eq. (2.1) is the generalized exterior derivative defined as follows.

$$d = d + \sum_{k=1}^N d_\chi_k + d\theta + d\bar{\theta},$$

(2.2)

$$\begin{align*}
da_i(x, n, \theta, \bar{\theta}) &= \partial_\alpha a_i(x, n, \theta, \bar{\theta})dx^\alpha, \\
d_\chi_k a_i(x, n, \theta, \bar{\theta}) &= [-a_i(x, n, \theta, \bar{\theta})M_{nk} + M_{nk}a_i(x, k, \theta, \bar{\theta})]\chi_k, \\
d_\theta a_i(x, n, \theta, \bar{\theta}) &= \partial_\theta a_i(x, n, \theta, \bar{\theta})d\theta \\
d_{\bar{\theta}} a_i(x, n, \theta, \bar{\theta}) &= \partial_{\bar{\theta}} a_i(x, n, \theta, \bar{\theta})d\bar{\theta},
\end{align*}$$

(2.3)\sim (2.6)

where $dx^\alpha$ is ordinary one-form basis, taken to be dimensionless, in Minkowski space $M_4$, and $\chi_k$ is the one-form basis, assumed to be also dimensionless, in the discrete space $Z_N$. $d_\theta$ and $d_{\bar{\theta}}$ are also one-form base in super-space. We have introduced $x$-independent matrix $M_{nk}$ whose hermitian conjugation is given by $M_{nk}^\dagger = M_{kn}$. The matrix $M(y)$ turns out to determine the scale and pattern of the spontaneous breakdown of the gauge symmetry. Thus, the symmetry breaking mechanism is encoded in the $d_\chi = \sum_{k=1}^N d_\chi_k$ operation. In order to find the explicit forms of gauge, Higgs fields and ghost fields according to Eqs. (2.1) and (2.2)\sim (2.6), we need the following important algebraic rule of GDG.

$$\chi_k f(x, n, \theta, \bar{\theta}) = f(x, k, \theta, \bar{\theta})\chi_k,$$

(2.7)

where $f(x, n, \theta, \bar{\theta})$ is a field defined on the discrete space such as $a_i(x, n, \theta, \bar{\theta})$, gauge field, Higgs field, ghosts or fermion fields. It should be noticed that Eq. (2.7) never expresses the relation between the matrix elements of $f(x, n, \theta, \bar{\theta})$ and $f(x, k, \theta, \bar{\theta})$ but insures the product between the fields expressed in differential form on the discrete space. Equation (2.7) realizes the non-commutativity of our algebra in the
geometry on the discrete space $M_4 \times Z_N$. Inserting Eq.(2.2)~Eq.(2.6) into Eq.(2.1) and using Eq.(2.7), $A(x, n, \theta, \bar{\theta})$ is rewritten as

$$A(x, n, \theta, \bar{\theta}) = A_\mu(x, n, \theta, \bar{\theta})dx^\mu + \sum_{k=1}^{N} \Phi_{nk}(x, \theta, \bar{\theta})\chi_k + C(x, n, \theta, \bar{\theta})d\theta + \bar{C}(x, n, \theta, \bar{\theta})d\bar{\theta}, \quad (2.8)$$

where

$$A_\mu(x, n, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, n, \theta, \bar{\theta})\partial_\mu a_i(x, n, \theta, \bar{\theta}), \quad (2.9)$$

$$\Phi_{nk}(x, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, n, \theta, \bar{\theta})(-a_i(x, k, \theta, \bar{\theta})M_{nk} + M_{nk}a_i(x, k, \theta, \bar{\theta})), \quad (2.10)$$

$$C(x, n, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, n, \theta, \bar{\theta})\partial_\theta a_i(x, n, \theta, \bar{\theta}) \quad (2.11)$$

$$\bar{C}(x, n, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, n, \theta, \bar{\theta})\partial_{\bar{\theta}} a_i(x, n, \theta, \bar{\theta}). \quad (2.12)$$

$A_\mu(x, n, \theta, \bar{\theta})$, $\Phi_{nk}(x, \theta, \bar{\theta})$, $C(x, n, \theta, \bar{\theta})$ and $\bar{C}(x, n, \theta, \bar{\theta})$ are identified with the gauge field in the flavor symmetry, Higgs field, ghost and anti-ghost fields, respectively. In order to identify $A_\mu(x, n, \theta, \bar{\theta})$ with true gauge fields, the following conditions have to be imposed.

$$\sum_i a_i^\dagger(x, n, \theta, \bar{\theta})a_i(x, n, \theta, \bar{\theta}) = 1, \quad (2.14)$$

where $1$ in the right hand side is a unit matrix in the corresponding internal space. This equation is very important often used in later calculations and may suggest that the variable $i$ might be an argument in the internal space because the definition of gauge field in Eq.(2.9) is very similar to that in Berry phase $\phi$. If we define the operator $\partial_k$ as

$$\partial_k a_i(x, n) = -a_i(x, n)M_{nk} + M_{nk}a_i(x, k) \quad (2.15)$$

the Higgs field $\Phi_{nk}(x, \theta, \bar{\theta})$ is written as

$$\Phi_{nk}(x, \theta, \bar{\theta}) = \sum_i a_i^\dagger(x, n)\partial_k a_i(x, n), \quad (2.16)$$

which is the same form as the ordinary gauge field $A_\mu(x, n, \theta, \bar{\theta})$ in Eq.(2.9). For later convenience, we define the following one-form fields as

$$\dot{A}(x, n, \theta, \bar{\theta}) = A_\mu(x, n, \theta, \bar{\theta})dx^\mu, \quad (2.17)$$

$$\dot{\Phi}_{nk}(x, \theta, \bar{\theta}) = \Phi_{nk}(x, \theta, \bar{\theta})\chi_k, \quad (2.18)$$

$$\dot{C}(x, n, \theta, \bar{\theta}) = C(x, n, \theta, \bar{\theta})d\theta, \quad (2.19)$$

$$\dot{\bar{C}}(x, n, \theta, \bar{\theta}) = \bar{C}(x, n, \theta, \bar{\theta})d\bar{\theta}. \quad (2.20)$$

Before constructing the gauge covariant field strength, we address the gauge transformation of $a_i(x, y, \theta, \bar{\theta})$ which is defined as

$$a_i^g(x, n, \theta, \bar{\theta}) = a_i(x, n, \theta, \bar{\theta})g(x, n), \quad (2.21)$$

where $g(x, n)$ is the gauge function with respect to the corresponding flavor unitary group. Then, we can find from Eqs.(2.1) and (2.21) the gauge transformation of $A(x, n, \theta, \bar{\theta})$ to be

$$A^g(x, n, \theta, \bar{\theta}) = g^{-1}(x, n)A(x, n, \theta, \bar{\theta})g(x, n) + g^{-1}(x, n)dg(x, n), \quad (2.22)$$
where as in Eq.$(2.3)\sim$(2.4),
\[
 dg(x, n) = (d + \sum_{k=1}^{N} \partial_{x_k})g(x, n) = \partial_{\mu}g(x, n)dx^\mu + \sum_{k=1}^{N} \partial_k g(x, n)\chi_k
\]  
(2.23)

Using Eqs. (2.21) and (2.22), we can find the gauge transformations of gauge, Higgs, ghost and anti-ghost fields as
\[
 A^\mu_n(x, n, \theta, \bar{\theta}) = g^{-1}(x, n)A_\mu(x, n, \theta, \bar{\theta})g(x, n) + g^{-1}(x, n)\partial_\mu g(x, n),
\]
(2.24)
\[
 \Phi^\mu_{nk}(x, n, \theta, \bar{\theta}) = g^{-1}(x, n)\Phi_{nk}(x, \theta, \bar{\theta})g(x, n) + g^{-1}(x, n)\partial_{nk}g(x, n),
\]
(2.25)
\[
 C^\mu_n(x, n, \theta, \bar{\theta}) = g^{-1}(x, n)C(x, n, \theta, \bar{\theta})g(x, n),
\]
(2.26)
\[
 \bar{C}^\mu_n(x, n, \theta, \bar{\theta}) = g^{-1}(x, n)\bar{C}(x, n, \theta, \bar{\theta})g(x, n),
\]
(2.27)

Equation (2.25) is very similar to Eq. (2.24) that is the gauge transformation of the genuine gauge field $A_\mu(x, n, \theta, \bar{\theta})$ and so it strongly indicates that the Higgs field is a kind of gauge field on the discrete space $M_k \times Z_N$. From Eq. (2.23), Eq. (2.25) is rewritten as
\[
 \Phi^\mu_{nk}(x, \theta, \bar{\theta}) + M_{nk} = g^{-1}(x, n)(\Phi_{nk}(x, \theta, \bar{\theta}) + M_{nk})g(x, k),
\]
(2.28)

which makes obvious that
\[
 H_{nk}(x, \theta, \bar{\theta}) = \Phi_{nk}(x, \theta, \bar{\theta}) + M_{nk}
\]
(2.29)
is un-shifted Higgs field whereas $\Phi_{nk}(x, \theta, \bar{\theta})$ denotes shifted one with the vanishing vacuum expectation value. Equations (2.26) and (2.27) show that ghost and anti-ghost fields are transformed as the adjoint representation.

In addition to the algebraic rules in Eq. (2.3)$\sim$(2.6) we add one more important rule that
\[
 d_{x_k}(M_{nk}\chi_k) = (M_{nk}\chi_l) \wedge (M_{nk}\chi_k) = M_{nl}M_{lk}\chi_l \wedge \chi_k,
\]
(2.30)

and in addition, whenever the $d_{x_k}$ operation jumps over $M_{nl}\chi_l$, a minus sign is attached. For example,
\[
 d_{x_k}(M_{nl}\chi_l a(x, n)) = (d_{x_k}M_{nl}\chi_l) a(x, n) - M_{nl}\chi_l \wedge (d_{x_k}a(x, n)).
\]
(2.31)

which together with Eq. (2.4) yields the nilpotency of $d_{x} = \sum_{k=1}^{N} d_{x_k}$ and then the nilpotency of the generalized exterior derivative $d$ under the natural conditions that
\[
 dx^\mu \wedge \chi_k = -\chi_k \wedge dx^\mu, \quad dx^\mu \wedge \bar{\theta} = -\bar{\theta} \wedge dx^\mu, \quad dx^\mu \wedge \bar{\theta} = -\bar{\theta} \wedge dx^\mu,
\]
\[
 \chi_k \wedge d\bar{\theta} = -d\bar{\theta} \wedge \chi_k, \quad \chi_k \wedge d\theta = -d\theta \wedge \chi_k
\]
\[
 d\theta \wedge d\bar{\theta} = d\bar{\theta} \wedge d\theta, \quad \partial_{\theta}\partial_{\bar{\theta}} = -\partial_{\bar{\theta}}\partial_{\theta}.
\]
(2.32)

It should be noted that $\chi_l \wedge \chi_k$ is independent of $\chi_l \wedge \chi_k$ so that $\chi_k \wedge \chi_l \neq \chi_l \wedge \chi_k$. This independence is due to the noncommutative property of our generalized differential geometry. For the proof of nilpotency of $d_{x}$, see [11]. With these considerations we can construct the gauge covariant field strength:
\[
 F(x, n, \theta, \bar{\theta}) = dA(x, n, \theta, \bar{\theta}) + A(x, n, \theta, \bar{\theta}) \wedge A(x, n, \theta, \bar{\theta})
\]
(2.33)

From Eqs. (2.22) and (2.23) we can easily find the gauge transformation of $F(x, n, \theta, \bar{\theta})$ as
\[
 F^\gamma(x, n, \theta, \bar{\theta}) = g^{-1}(x, n)F(x, n, \theta, \bar{\theta})g(x, n),
\]
(2.34)

Here, according to Bonora and Tonin [14] we impose the horizontality condition [5] on $F(x, n, \theta, \bar{\theta})$ that
\[
 F(x, n, \theta, \bar{\theta})|_{\theta = \bar{\theta} = 0} = F(x, n),
\]
(2.35)
where $F(x, n)$ is the generalized field strength not accompanying one-form base $d\theta$ and $\bar{d}\theta$. Equation (2.35) yields the conditions that

$$d_\theta \hat{A}(x, n) + d\hat{C}(x, n) + \hat{A}(x, n) \wedge \hat{C}(x, n) + \hat{C}(x, n) \wedge \hat{A}(x, n) = 0,$$

$$d_\bar{\theta} \hat{A}(x, n) + d\hat{C}(x, n) + \hat{A}(x, n) \wedge \hat{C}(x, n) + \hat{C}(x, n) \wedge \hat{A}(x, n) = 0,$$

$$d_\theta \hat{\Phi}_{nk}(x) + d_{x,k} \hat{C}(x, n) + \hat{\Phi}_{nk}(x) \wedge \hat{C}(x, n) + \hat{C}(x, n) \wedge \hat{\Phi}_{nk}(x) = 0,$$

$$d_\theta \hat{\Phi}_{nk}(x) + d_{x,k} \hat{C}(x, n) + \hat{\Phi}_{nk}(x) \wedge \hat{C}(x, n) + \hat{C}(x, n) \wedge \hat{\Phi}_{nk}(x) = 0,$$

$$d_\theta \hat{C}(x, n) + \hat{\chi}(x, n) \wedge \hat{C}(x, n) = 0,$$

$$d_\theta \hat{C}(x, n) + \hat{\chi}(x, n) \wedge \hat{C}(x, n) = 0,$$

which determine the BRST/anti-BRST transformations of each field together with the definitions that

$$d_\theta \hat{C}(x, n) = -\hat{B}(x, n),$$

$$d_\theta \hat{C}(x, n) = -\hat{B}(x, n).$$

It should be noticed that nilpotencies of $d_\theta$ and $d_\bar{\theta}$ are consistent with Eqs. (2.36)~(2.43) and for example, $d_\theta \hat{C}(x, n) = \partial_\theta d\bar{\theta} \hat{C}(x, n) = -\partial_\theta (C(x, n)) d\bar{\theta}$ because $\theta$ and $C(x, n)$ are Grassmann numbers. Thus,

$$\partial_\theta \hat{C}(x, n) = B(x, n),$$

$$\partial_\theta \hat{C}(x, n) = B(x, n).$$

From Eqs. (2.36) and (2.37), the BRST/anti-BRST transformations of $A_{\mu}(x, n)$ follows as

$$\partial_\theta A_{\mu}(x, n) = D_\mu C(x, n) = \partial_\mu C(x, n) + [A(x, n), C(x, n)],$$

$$\partial_\bar{\theta} A_{\mu}(x, n) = D_\mu \hat{C}(x, n) = \partial_\mu \hat{C}(x, n) + [A(x, n), \hat{C}(x, n)].$$

From Eq. (2.18)~Eq. (2.20), the BRST/anti-BRST transformation of the Higgs field are rewritten as

$$\partial_\theta \Phi_{nk}(x) = \partial_k C(x, n) + \Phi_{nk}(x) C(x, k) - C(x, n) \Phi_{nk}(x),$$

$$\partial_\bar{\theta} \Phi_{nk}(x) = \partial_k \hat{C}(x, n) + \Phi_{nk}(x) \hat{C}(x, k) - \hat{C}(x, n) \Phi_{nk}(x),$$

which by use of $H_{nk}(x) = \Phi_{nk}(x) + M_{nk}$ lead to

$$\partial_\theta H_{nk}(x) = H_{nk}(x) C(x, k) - C(x, n) H_{nk}(x),$$

$$\partial_\bar{\theta} H_{nk}(x) = H_{nk}(x) \hat{C}(x, k) - \hat{C}(x, n) H_{nk}(x).$$

Equations (2.49) and (2.50) are the usual BRST/anti-BRST transformation of the Higgs field. From Eqs. (2.40) and (2.41), the BRST and anti-BRST transformations of ghost and anti-ghost fields are obtained, respectively.

$$\partial_\theta C(x, n) = -C(x, n) C(x, n),$$

$$\partial_\theta \hat{C}(x, n) = -\hat{C}(x, n) \hat{C}(x, n),$$

and from Eq. (2.42), the restriction about Nakanishi-Lautrup field follows as

$$B(x, n) + \bar{B}(x, n) = -C(x, n) \hat{C}(x, n) - \hat{C}(x, n) C(x, n).$$

With these equations, we can proceed to obtain the BRST invariant Lagrangian of gauge theory.

BRST invariant Yang-Mills-Higgs Lagrangian is obtained by

$$\mathcal{L}_{YMH} = -\frac{1}{g_n^2} \sum_{n=1}^{N} \text{Tr} < F(x, n), F(x, n) >$$

$$+ \frac{1}{g_n^2} \sum_{n=1}^{N} \partial_\theta \partial_\bar{\theta} \text{Tr} < A(x, n, \theta, \bar{\theta}), A(x, n, \theta, \bar{\theta}) > |_{\theta=\bar{\theta}=0}$$

$$+ \frac{1}{g_n^2} \sum_{n=1}^{N} \frac{1}{2} \text{Tr} < \hat{B}(x, n, \theta, \bar{\theta}), \hat{B}(x, n, \theta, \bar{\theta}) > |_{\theta=\bar{\theta}=0}. $$
with the vanishing values of other combinations. From Eqs.(2.56)∼(2.58), the second term in Eq.(2.54) gives the ghost term
\[ L \] of one-forms.

Calculation of the second term in Eq.(2.54) proceeds as
\[ Y \] and third terms of Eq.(2.54) give the ghost term
\[ L \] where the third term in the right hand side of Eq.(2.62) is the potential term of Higgs particle. The second
\[ F(x, n) = \frac{1}{2} F_{\mu\nu}(x, n) dx^\mu \wedge dx^\nu + \sum_{k \neq n} D_\mu \Phi_{nk}(x) dx^\mu \wedge \chi_k \]
\[ + \sum_{k \neq n} V_{nk}(x) \chi_k \wedge \chi_n \] (2.56)

where
\[ F_{\mu\nu}(x, n) = \partial_\mu A_\nu(x, n) - \partial_\nu A_\mu(x, n) + [A_\mu(x, n), A_\nu(x, n)], \]
\[ D_\mu \Phi_{nk}(x) = \partial_\mu \Phi_{nk}(x) + A_\mu(x, n)(M_{nk} + \Phi_{nk}(x)) \]
\[ - (\Phi_{nk}(x) + M_{nk}) A_\mu(x, k), \]
\[ V_{nk}(x) = (\Phi_{nk}(x) + M_{nk})(\Phi_{kn}(x) + M_{kn}) - Y_{nk}(x) \quad \text{for} \quad k \neq n, \] (2.59)

\[ Y_{nk}(x) \] in Eq.(2.59) is auxiliary field and expressed as
\[ Y_{nk}(x) = \sum_i a_i^\dagger(x, n) M_{nk} M_{kn} a_i(x, n), \] (2.60)

which may be independent or dependent of \( \Phi_{nk}(x) \) and/or may be a constant field.

In order to get the explicit expression of \( \mathcal{L}_{YMH} \) in Eq.(2.54) we have to determine the metric structure of one-forms.
\[ < dx^\mu, dx^\nu > = g^{\mu\nu}, \quad g^{\mu\nu} = \text{diag}(1, -1, -1, -1), \]
\[ < \chi_k, \chi_l > = - \delta_{kl}, \quad < d\theta, d\bar{\theta} > = < d\bar{\theta}, d\theta > = 1, \] (2.61)

with the vanishing values of other combinations. From Eqs.(2.56)∼(2.59), the first term of Eq.(2.54) that is denoted by \( \mathcal{L}_{YMH} \) is written as

\[ \mathcal{L}_{YMH} = - \text{Tr} \sum_{n=1}^N \frac{1}{2g^2_n} F_{\mu\nu}^\dagger(x, n) F^{\mu\nu}(x, n) \]
\[ + \text{Tr} \sum_{n,k=1}^N \frac{1}{g^2_n} (D_\mu \Phi_{nk}(x))^\dagger D^\mu \Phi_{nk}(x) \]
\[ - \text{Tr} \sum_{n,k=1}^N \frac{1}{2g_n} V_{nk}^\dagger(x) V_{nk}(x), \] (2.62)

where the third term in the right hand side of Eq.(2.62) is the potential term of Higgs particle. The second
third terms of Eq.(2.54) give the ghost term \( \mathcal{L}_{GH} \) and the gauge fixing term \( \mathcal{L}_{GF} \), respectively. The calculation of the second term in Eq.(2.54) proceeds as

\[ \partial_\theta \partial_{\bar{\theta}} \text{Tr} < A(x, n, \theta, \bar{\theta}), A(x, n, \theta, \bar{\theta}) > |_{\theta = \bar{\theta} = 0} = \]
\[ - \partial_\theta \partial_{\bar{\theta}} \text{Tr} \left[ A^\mu(x, n, \theta, \bar{\theta}) A_\mu(x, n, \theta, \bar{\theta}) + \sum_{k=1}^N \Phi_{kn}(x, \theta, \bar{\theta}) \Phi_{nk}(x, \theta, \bar{\theta}) \]
\[ + C(x, n, \theta, \bar{\theta}) C(x, n, \theta, \bar{\theta}) + \bar{C}(x, n, \theta, \bar{\theta}) \bar{C}(x, n, \theta, \bar{\theta}) \right] \bigg|_{\theta = \bar{\theta} = 0}, \] (2.63)
Let us first assign the fields on discrete space $M$. In this section we apply the previous results to the spontaneously broken SU(2) gauge model. We need $3$ Application to SU(2) Higgs-Kibble gauge model

are reconstructed according to the general framework in this section. Due to the nilpotency of BRST/anti-BRST transformation. Then, according to the BRST/anti-BRST transformations in Eqs. (2.45) $\sim$ (2.53), $L_{GH}$ is expressed as

$$L_{GH} = 2i \sum_{n=1}^{N} \frac{1}{g_n^2} \text{Tr} \partial_\mu \tilde{C}(x, n) \mathcal{D}^\mu C(x, n)$$

$$+ i \sum_{n=1}^{N} \frac{1}{g_n^2} \sum_{k \neq n} \text{Tr}(\partial_\mu \tilde{C}(x, k) \mathcal{D}_k C(x, n) + \partial_k \tilde{C}(x, n) \mathcal{D}_n C(x, k)),$$

where

$$\mathcal{D}^\mu C(x, n) = \partial^\mu C(x, n) + [A^\mu(x, n), C(x, n)],$$

$$\partial_k \tilde{C}(x, n) = -\tilde{C}(x, n) M_{nk} + M_{nk} \tilde{C}(x, k),$$

$$\mathcal{D}_k C(x, n) = \partial_k C(x, n) + \Phi_{nk}(x) C(x, k) = C(x, n) \Phi_{nk}(x)$$

$$= M_{nk}(x) C(x, k) - C(x, n) H_{nk}(x)$$

and $L_{GF}$ as

$$L_{GF} = \frac{\alpha}{2} \sum_{n=1}^{N} \frac{1}{g_n^2} \text{Tr} B(x, n)^2 - 2i \sum_{n=1}^{N} \frac{1}{g_n^2} \text{Tr} \partial_\mu B(x, n) A^\mu(x, n)$$

$$- i \sum_{n=1}^{N} \frac{1}{g_n^2} \sum_{k \neq n} \text{Tr}(\partial_\mu B(x, k) \Phi_{nk}(x) + \Phi_{kn}(x) \partial_k B(x, n)).$$

If we note the Hermitian conjugate conditions that

$$(\partial_\mu \tilde{C}(x, n))^\dagger = \partial_n \tilde{C}(x, k), \quad (\mathcal{D}_k C(x, n))^\dagger = \mathcal{D}_n C(x, k)$$

$$(\partial_\mu B(x, n))^\dagger = -\partial_n B(x, k)$$

because of $B(x, n)^\dagger = B(x, n)$, $C(x, n)^\dagger = -C(x, n)$ and $\tilde{C}(x, n)^\dagger = -\tilde{C}(x, n)$, we easily find the Hermiticity of Eqs. (2.66) and (2.70).

In next two sections, two special models, SU(2) Higgs-Kibble gauge model and the standard model are reconstructed according to the general framework in this section.

**3 Application to SU(2) Higgs-Kibble gauge model**

In this section we apply the previous results to the spontaneously broken SU(2) gauge model. We need the discrete space $M_4 \times Z_2(N = 2)$ to reproduce the Higgs mechanism in the SU(2) Higgs Kibble model. Let us first assign the fields on discrete space $M_4 \times Z_2$ to the physical fields. For gauge fields,

$$A_\mu(x, 1) = -\frac{i}{2} \sum_{i=1}^{3} \tau^i A^i_\mu(x),$$

$$A_\mu(x, 0) = 0.$$
where $A^i_{\mu}(x)$ denotes SU(2) adjoint gauge fields. $\tau^i(i = 1, 2, 3)$ are Pauli matrices. The Higgs field is assigned as

$$\Phi_{12}(x) = \Phi_{21}(x)^\dagger = \begin{pmatrix} \phi^0^* & \phi^+ \\ -\phi^- & \phi^0 \end{pmatrix},$$

$$M_{12} = M_{21}^\dagger = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix},$$ (3.2)

where $M_{12}$ must be chosen to give the correct symmetry breakdown. For ghost and anti-ghost fields which correspond with gauge fields in Eq.(3.1) we take

$$C(x, 1) = -\frac{i}{2} \sum_{i=1}^{3} \tau^i C^i(x),$$

$$C(x, 2) = 0$$ (3.3)

and

$$\bar{C}(x, 1) = -\frac{i}{2} \sum_{i=1}^{3} \tau^i \bar{C}^i(x),$$

$$\bar{C}(x, 2) = 0$$ (3.4)

Also for the Nakanishi-Lautrup field, we assign

$$B(x, 1) = \frac{1}{2} \sum_{i=1}^{3} \tau^i B^i(x),$$

$$B(x, 2) = 0$$ (3.5)

and

$$\bar{B}(x, 1) = \frac{1}{2} \sum_{i=1}^{3} \tau^i \bar{B}^i(x),$$

$$\bar{B}(x, 2) = 0$$ (3.6)

because $\partial_y \bar{C}^i = iB^i$ and $\partial_y \bar{C}^0 = iB^0$. We can take the gauge transformation functions as

$$g(x, 1) = g(x), \ g(x) \in \text{SU}(2),$$

$$g(x, 2) = 1.$$ (3.7)

The auxiliary field $Y(x, n)(n = 1, 2)$ becomes unit matrix because of the assignments of $M_{nk}$ in Eq.(3.2),

$$Y(x, 1) = \sum_i a_i^\dagger(x, 1) M_{12} M_{21} a_i(x, 1) = \sum_i a_i^\dagger(x, 1) a_i(x, 1) = 1^{24},$$

$$Y(x, 2) = \sum_i a_i^\dagger(x, 2) M_{21} M_{12} a_i(x, 2) = \sum_i a_i^\dagger(x, 2) a_i(x, 2) = 1^{24},$$ (3.8)

on account of Eq.(2.14).

With these considerations, we can obtain $L_{YMH}$, $L_{GH}$ and $L_{GF}$ in Eqs.(2.62), (2.66) and (2.70), respectively. After the definition of the Higgs doublet $h = \begin{pmatrix} \phi^+ \\ \phi^0 + \mu \end{pmatrix}$ and the following rescaling of fields

$$A^i_{\mu}(x) \rightarrow g A^i_{\mu}(x), \quad h(x) \rightarrow g_h h(x),$$ (3.9)
with \( g = g_1 \) and \( g_H = \sqrt{g_1^2g_2^2/(g_1^2 + g_2^2)} \), we find the standard Yang-Mills-Higgs Lagrangian for the Higgs Kibble gauge model.

\[
\mathcal{L}_{YMH} = -\frac{1}{4} \sum_{i=1}^{3} F^i_{\mu\nu}(x) \cdot F^{i\mu\nu}(x) + (D_\mu h(x))^\dagger (D^\mu h(x)) - \lambda (h^\dagger(x) h(x) - \mu^2)^2,
\]

(3.10)

where

\[
F^i_{\mu\nu}(x) = \partial_\mu A^i_\nu(x) - \partial_\nu A^i_\mu(x) + g \epsilon^{ijk} A^j_\mu(x) A^k_\nu(x),
\]

(3.11)

\[
D_\mu h(x) = \left[ \partial_\mu - i\frac{\alpha}{2} \sum_{i=1}^{3} \tau^i \cdot A^i_\mu(x) \right] h(x),
\]

(3.12)

with \( \lambda = g_H^2 \) and the rescaling of \( \mu \to \sqrt{g_H\mu} \). Equation (3.10) expresses Yang-Mills-Higgs Lagrangian of the Higgs-Kibble gauge theory with the symmetry SU(2) spontaneously broken to global SU(2).

Let us move to the ghost and gauge fixing terms expressed in Eqs.(2.66) and (2.70). For simplicity, hereafter we abbreviate the argument \( x \) in the respective fields. After the same rescaling of ghost and Nakanishi-Lautrup fields as in Eq.(3.9) we get the gauge fixing term \( \mathcal{L}_{GF} \) in Eq.(2.66) as

\[
\mathcal{L}_{GF} = \sum_{i=1}^{3} \alpha \left\{ \frac{1}{2} B^2_i + B_i (\partial^\mu A^i_\mu + m_W \phi^i) \right\},
\]

(3.13)

where \( m_W \) is the gauge boson mass and \( \phi^i (i = 1, 2, 3) \) is given by the following parametrization of \( h \).

\[
h = \frac{1}{\sqrt{2}} \left( \psi + v + i \sum_{i=1}^{3} \tau^i \phi^i \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right)
\]

(3.14)

with \( v = \sqrt{2}\mu \). The equations of motion eliminate the Nakanishi-Lautrup fields from Eq.(3.13), which yields

\[
\mathcal{L}_{GF} = -\sum_{i=1}^{3} \frac{1}{2\alpha} (\partial^\mu A^i_\mu + m_W \phi^i)^2.
\]

(3.15)

Eq.(3.15) enables us to obtain the gauge fixed Lagrangian with the ’t Hooft-Feynman gauge\([15]\) when \( \alpha = 1 \).

With the same notations as in Eq.(3.13) we get the explicit expression of ghost terms in Eq.(2.66) as follows:

\[
\mathcal{L}_{GH} = -i \sum_{i=1}^{3} (\partial^\mu \bar{C}^i D_\mu C^i - m_W^2 \bar{C}^i C^i) + \frac{g m_W}{2} \sum_{i=1}^{3} \left( \bar{C}^i C^i \psi + f^{ijk} \bar{C}^i \phi^j C^k \right),
\]

(3.16)

where \( D_\mu C^i = \partial_\mu C^i + g \epsilon^{ijk} A^j_\mu C^k \). The ghost fields become massive and the new interaction terms between ghosts and Higgs fields appear. This is natural because the Higgs field is a member of the generalized connection in GDG on the discrete space in the same way as the gauge field \( A_\mu \).

It should be noted that our definitions of the gauge fixing and ghost terms are equal to

\[
\mathcal{L}_{GH+GF} = -i\partial_\theta \left[ \sum_{i=1}^{3} \bar{C}^i \left( \partial^\mu A^i_\mu + m_W \phi^i + \frac{1}{2\alpha} B^i \right) \right]
\]

(3.17)

for the SU(2) Higgs-Kibble model. This type of prescription to determine the gauge fixing condition was proposed by Kugo and Uehara\([17]\).
4 Application to the standard model

The reconstruction of the standard model in GDG was completely performed in [16] by adopting the discrete space \( M_4 \times Z_2 \) on which the fermion fields are represented as vectors in 24 dimensional internal space including weak isospin, hypercharge, color and generation indices. Corresponding to this fermion representation, gauge fields, Higgs boson, ghost fields, Nakanishi-Lautrup fields are expressed in 24 \( \times \) 24 matrix forms as generators in 24-dimensional space. Here, we omit the color gauge field because it does not bring any significant difference from our results.

\[
A_\mu(x, 1) = -\frac{i}{2} \left( \sum_{i=1}^{3} \tau^i A^i_\mu(x) \otimes 1^4 + a B_\mu(x) \right) \otimes 1^3, \\
A_\mu(x, 2) = -\frac{i}{2} b B_\mu(x) \otimes 1^3, 
\]

(4.1)

where \( A^i_\mu(x) \) denotes SU(2) adjoint gauge field and \( B_\mu(x) \) is U(1) gauge field. \( a \) and \( b \) are the U(1) hypercharge matrices corresponding to the left and right handed fermions expressed as vectors in 24-dimensional space, respectively and are denoted as

\[
a = \text{diag} \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1 \right), \\
b = \text{diag} \left( \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 0, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, -2 \right). 
\]

(4.2)

The Higgs field is assigned in the same way as in Eq.(3.2) by making it a 24 \( \times \) 24 matrix. The symmetry breaking function \( M_{nk} \) is also given in the same way.

\[
\Phi_{12}(x) = \Phi_{21}(x)^\dagger = \left( \begin{array}{cc} \phi^0 & \phi^+ \\ \phi^- & \phi^0 \end{array} \right) \otimes 1^{12}, \\
M_{12} = M_{21}^\dagger = \left( \begin{array}{c} \mu \\ 0 \end{array} \begin{array}{c} 0 \\ \mu \end{array} \right) \otimes 1^{12}. 
\]

(4.3)

For ghost and anti-ghost fields which correspond with gauge fields in Eq.(4.1) we take

\[
C(x, 1) = -\frac{i}{2} \left( \sum_{i=1}^{3} \tau^i C^i(x) \otimes 1^4 + a C^0(x) \right) \otimes 1^3, \\
C(x, 2) = -\frac{i}{2} b C^0(x) \otimes 1^3, 
\]

(4.4)

and

\[
\bar{C}(x, 1) = -\frac{i}{2} \left( \sum_{i=1}^{3} \tau^i \bar{C}^i(x) \otimes 1^4 + a \bar{C}^0(x) \right) \otimes 1^3, \\
\bar{C}(x, 2) = -\frac{i}{2} b \bar{C}^0(x) \otimes 1^3, 
\]

(4.5)

Also for the Nakanishi-Lautrup field, we assign

\[
B(x, 1) = \frac{1}{2} \left( \sum_{i=1}^{3} \tau^i B^i(x) \otimes 1^4 + a B^0(x) \right) \otimes 1^3, \\
B(x, 2) = \frac{1}{2} b B^0(x) \otimes 1^3, 
\]

(4.6)
where

\[ L^Y \]

The auxiliary field \( \partial_0 \bar{C}^i = iB^i \) and \( \partial_0 \bar{C}^0 = iB^0 \). We can take the gauge transformation functions as

\[
g(x, 1) = g(x)e^{i \alpha(x)}, \quad g(x) \in SU(2), \quad e^{i \alpha(x)} \in U(1),
\]

\[
g(x, 2) = e^{i \beta(x)}, \quad e^{i \beta(x)} \in U(1).
\]

The auxiliary field \( Y(x, n)(n = 1, 2) \) becomes unit matrix because of the assignments of \( M_{nk} \) in Eq. (3.2). With these considerations, we can obtain \( L_{YMH} \), \( L_{GH} \) and \( L_{GF} \) in Eqs. (2.62), (2.66) and (2.70) respectively.

After the rescaling of fields

\[
A^i_\mu (x) \rightarrow gA^i_\mu (x), \quad B_\mu (x) \rightarrow g'B_\mu (x),
\]

\[
h(x) \rightarrow g_h h(x),
\]

with

\[
g^2 = \frac{g^2}{12},
\]

\[
g'^2 = \frac{2g_1^2g_2^2}{3g_2^2 \text{Tr}^2 + 3g_1^2 \text{Tr} b^2} = \frac{g_1^2 g_2^2}{16 g_1^2 + 4 g_2^2},
\]

\[
g_h^2 = \frac{g_1^2 g_2^2}{24(g_1^2 + g_2^2)}.
\]

we find the Yang-Mills-Higgs Lagrangian for the Standard model.

\[
L_{YMH} = -\frac{1}{4} \sum g_i^2 F_{\mu \nu}^i (x) \cdot F^{\mu \nu i} (x) - \frac{1}{4} B^2_{\mu \nu}
\]

\[
+ (D_\mu h(x))^\dagger (D^\mu h(x)) - \lambda (h^\dagger(x) h(x) - \mu^2)^2,
\]

where

\[
F_{\mu \nu}^i (x) = \partial_\mu A^i_\nu (x) - \partial_\nu A^i_\mu (x) + g e^{ij \kappa} A^j_\mu (x) A^\kappa_\nu (x),
\]

\[
B_{\mu \nu} (x) = \partial_\mu B_\nu (x) - \partial_\nu B_\mu (x),
\]

\[
D_\mu h(x) = \left[ \partial_\mu - \frac{i}{2} g \sum_{i=1}^3 \tau^i \cdot A^i_\mu (x) - \frac{i}{2} g' B^i_\mu (x) \right] h(x),
\]

with \( \lambda = g_0^2 \) and the rescaling of \( \mu \rightarrow \sqrt{g_0} \mu \). Except for color sector, Eq. (4.12) expresses Yang-Mills-Higgs Lagrangian of the standard model with the symmetry \( SU(2)_L \times U(1)_Y \) spontaneously broken to \( SU(1)_{em} \).

Let us express the ghost and gauge fixing terms in Eqs. (2.66) and (2.70) in the case of the standard model. For simplicity, hereafter we abbreviate the argument \( x \) in the respective fields. After the same rescaling of ghost and Nakanishi-Lautrup fields such as

\[
C^i (\bar{C}^i) \rightarrow gC^i (\bar{C}^i), \quad C^0 (\bar{C}^0) \rightarrow g' C^0 (\bar{C}^0),
\]

\[
B^i (\bar{B}^i) \rightarrow gB^i (\bar{B}^i), \quad B^0 (\bar{B}^0) \rightarrow g' B^0 (\bar{B}^0),
\]

\[
 B(x, 1) = \frac{1}{2} \left( \sum_{i=1}^3 \tau^i B^i (x) \otimes 1^4 + a \ B^0 (x) \right) \otimes 1^3,
\]

\[
 \bar{B}(x, 2) = \frac{1}{2} b \ \bar{B}^0 (x) \otimes 1^3,
\]

because \( \partial_0 \bar{C}^i = iB^i \) and \( \partial_0 \bar{C}^0 = iB^0 \).
we get the gauge fixing term $\mathcal{L}_{GF}$ in Eq. (2.66) as

$$\mathcal{L}_{GF} = \sum_{i=1}^{2} \left\{ \frac{\alpha}{2} B_i Z^2 + B_i \left( \partial^\mu A_i^\mu + m_w \phi^i \right) \right\} ,$$

$$+ \left\{ \frac{\alpha}{2} B_2 Z + B_2 \left( \partial^\mu Z^\mu + m_z \phi^3 \right) \right\}$$

$$+ \left\{ \frac{\alpha}{2} B_3 A + B_A \partial^\mu A^\mu \right\}$$

(4.18)

where $m_w$ and $m_z$ are the charged and weak neutral gauge boson masses, respectively, and $Z_\mu$, $A_\mu$, $B_2$ and $B_A$ are defined as

$$Z_\mu = \frac{g A_\mu^3 - g' B_0^\mu}{\sqrt{g^2 + g'^2}}, \quad A_\mu = \frac{g A_\mu^3 + g B_0^\mu}{\sqrt{g^2 + g'^2}} ,$$

$$B_2 = \frac{g B_3 - g' B_0^3}{\sqrt{g^2 + g'^2}}, \quad B_A = \frac{g' B_3 + g B_0^3}{\sqrt{g^2 + g'^2}}$$

(4.19)

The parametrization of $\phi^i(i = 1, 2, 3)$ is given in Eq. (3.14). By use of the equations of motion of the Nakanishi-Lautrup fields, Eq. (4.18) leads to

$$\mathcal{L}_{GF} = -\sum_{i=1}^{2} \frac{1}{2\alpha} \left( \partial^\mu A_i^\mu + m_w \phi^i \right)^2$$

$$- \frac{1}{2\alpha} \left( \partial^\mu Z^\mu + m_z \phi^3 \right)^2 - \frac{1}{2\alpha} \left( \partial^\mu A^\mu \right)^2 ,$$

(4.20)

which is the gauge fixed Lagrangian with the ’t Hooft-Feynman gauge \[15\] when $\alpha = 1$.

With the same notations of ghost and anti-ghost fields as

$$C_z = \frac{g C_3 - g' C_0}{\sqrt{g^2 + g'^2}}, \quad C_\lambda = \frac{g' C_3 + g C_0}{\sqrt{g^2 + g'^2}},$$

$$\bar{C}_z = \frac{g' C_3 - g C_0}{\sqrt{g^2 + g'^2}}, \quad \bar{C}_\lambda = \frac{g C_3 + g' C_0}{\sqrt{g^2 + g'^2}}$$

(4.21)

we obtain the explicit expression of ghost terms in Eq. (2.66) as follows:

$$\mathcal{L}_{GH} = -i \sum_{i=1}^{2} \partial^\mu \bar{C}_i D_\mu C^i - i \partial^\mu \bar{C}_z D_\mu C_z - i \partial^\mu \bar{C}_\lambda D_\mu C_\lambda$$

$$+ i \sum_{i=1}^{2} m_w^2 \bar{C}_i C^i + i m_z^2 \bar{C}_z C_z$$

$$+ \frac{i}{2} \left\{ m_w g \left( C_1 C_1^1 + \bar{C}^2 C^2 \right) + m_z \sqrt{g^2 + g'^2} \bar{C}_z C_z \right\} \psi$$

$$+ \frac{i}{2} \left\{ m_z g \bar{C}_z C^2 - m_w \frac{\left( g^2 - g'^2 \right) \sqrt{g^2 + g'^2}} \bar{C}_z C_z \right\} \phi^3$$

$$+ \frac{i}{2} \left\{ m_w \frac{g^2 - g'^2}{\sqrt{g^2 + g'^2}} \bar{C}_z C_z - m_z g \bar{C}_z C^3 \right\} \phi^3$$

$$+ \frac{i}{2} m_w g \left( \bar{C}^2 C^1 - \bar{C}_z C^3 \right) \phi^3$$

$$+ i m_w \frac{g g'}{\sqrt{g^2 + g'^2}} \left( C_1 \phi^2 - C^2 \phi^3 \right) C_\lambda ,$$

(4.22)
where covariant derivatives of ghost fields are calculated through Eqs. (4.19) and (4.21). Also in this case, the ghost fields become massive and the new interaction terms between ghosts and Higgs fields appear. It should be noted that $\mathcal{L}_{GH}$ coincides with that in Eq. (3.16) in the limit of $g' = 0$ except for U(1) ghost term.

It should be also noted that our definition of gauge fixing condition is connected with the prescription proposed by Kugo and Uehara [17] as

$$\mathcal{L}_{GH+GF} = -i\partial_\theta \left[ \sum_{i=1}^2 \bar{C}^i \left( \partial^\mu A_{\mu}^i + m_\omega \phi^i + \frac{1}{2} \alpha B^i \right) \right]$$

-$$i\partial_\theta \left[ \bar{C}_z \left( \partial^\mu Z_{\mu} + m_2 \phi^3 + \frac{1}{2} \alpha B_z \right) \right] - i\partial_\theta \left[ \bar{C}_A \left( \partial^\mu A_{\mu} + \frac{1}{2} \alpha B_A \right) \right]$$

(4.23)

for the standard model. It is interesting that our definition naturally leads to the more general gauge fixing condition.

5 Concluding remarks

The reconstructions of the spontaneously broken gauge theories based on the generalized differential geometry on the discrete space $M_4 \times \mathbb{Z}_N$ have consistently performed so far. Especially, the standard model is nicely reconstructed in GDG on $M_4 \times \mathbb{Z}_2$ [16] by introducing the 24-dimensional internal space where chiral fermions are represented as 24-dimensional vectors. It is well understood that the Higgs boson field is a connection on the discrete space $\mathbb{Z}_N$ and an unified picture of the ordinary gauge fields and Higgs boson field as the generalized connection is realized. This is a common feature of the NCG approach.

In this paper, BRST invariant formulation of the spontaneously broken gauge theory is presented in our scheme of a GDG on the discrete space $M_4 \times \mathbb{Z}_N$. According to the super space formulation by Bonora and Tonin [14], we introduce the Grassmann numbers $\theta$ and $\bar{\theta}$ as the arguments in super space in addition to $x_\mu$ in $M_4$ and $n$ in $\mathbb{Z}_N$. The horizontality condition [8] on the generalized field strength $\mathcal{F}(x, n, \theta, \bar{\theta})$ determines the BRST transformation of every field including the Higgs boson field. By use of the generalized gauge field $A(x, n, \theta, \bar{\theta})$ and the Nakanishi-Lautrup field $B(x, n, \theta, \bar{\theta})$, the gauge fixing and ghost terms are defined in Eq. (2.54) and written explicitly in Eqs. (2.66) and (2.70). Two applications to the SU(2) Higgs-Kibble model and the standard model show ghost fields to be massive and bring the new interaction terms between ghosts and Higgs fields. This is natural because the Higgs boson is a member of the generalized gauge field in GDG on the discrete space in the same way as ordinary gauge fields. Especially, our BRST formulation prefers the t’Hooft-Feynman gauge [13] as the gauge fixing condition, so in this case $\alpha = 1$ is required.

The Higgs mechanism necessary for the spontaneously broken gauge theory is well understood in the generalized differential geometry on the discrete space. In addition, the super space formalism by Bonora and Tonin [14] is nicely incorporated in this formulation to bring the BRST invariant Lagrangian. It is possible to discuss the anomaly of the spontaneously broken gauge theory in the present formulation as a future work. It is also an important purpose to incorporate the supersymmetry in the present formulation. If it would be possible, this approach would be more promising.

Acknowledgement

The author would like to express his sincere thanks to Professor H. Kase and Professor K. Morita for useful suggestion and invaluable discussions. He is grateful to all members at Department of Physics, Boston University for their warm hospitality.

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