A survey on the theory of multiple Dirichlet series with arithmetical coefficients as numerators

Kohji Matsumoto

Graduate School of Mathematics, Nagoya University, Furocho, Chikusa-ku, Nagoya 464-8602, Japan
(e-mail: kohjimat@math.nagoya-u.ac.jp)

Dedicated to Professor Jonas Kubilius on the occasion of his 100th anniversary

Received January 12, 2021; revised March 18, 2021

Abstract. We survey some recent developments in the analytic theory of multiple Dirichlet series with arithmetical coefficients as numerators.

MSC: 11M32, 11M26, 11M41

Keywords: multiple Dirichlet series, arithmetical coefficients, meromorphic continuation, natural boundary

1 A personal recollection

I first met Professor Jonas Kubilius at Kyoto in July 1986 when the 5th USSR–Japan Symposium on Probability was held there. Professor Kubilius was one of the members of the Soviet team, and on this occasion, some Japanese number theorists organized a small satellite meeting on probabilistic number theory with him. At that time, I was a post-doctoral researcher, just after getting my degree of Dr. Sci. from Rikkyo University in March of the same year. On the meeting, I gave a talk on the contents of my thesis, concerning the value-distribution of the Riemann zeta-function \( \zeta(s) \). After my talk, Professor Kubilius came close to me and said:

“Do you know the name of Antanas Laurinčikas?”
“No.”

“He is a Lithuanian mathematician, and he wrote a lot of papers in which you are surely interested.”

Further he mentioned that many papers of Laurinčikas can be found in Lietuvos Matematikos Rinkinys. I thanked Professor Kubilius, and when I went back to Rikkyo University, I immediately visited the library. Unfortunately, in the library, there were only the Russian original version of the journal back numbers, which I could not read, so at that time, I gave up reading the papers of Laurinčikas. (Several years later, I noticed the existence of the English translation of Liet. Mat. Rink.)

In the next year, I got a job as lecturer at Iwate University, and in 1995, I moved to Nagoya University, and then, in 1996, I first visited Lithuania to attend the 2nd Palanga Conference on Analytic and Probabilistic Number Theory. I met again Professor Kubilius, and found many new Lithuanian friends. Since then, I became a regular member of the Palanga Conferences. In September 2011, I attended the 5th Palanga Conference; it
was just one and a half month before the death of Professor Kubilius. When I arrived at the university villa in Palanga, at the entrance of the villa, I met Professor Kubilius. When he noticed me, he said “Ça va?”
I replied “oui, ça va” and said greetings in English. It was my last conversation with him.

2 Multiple Dirichlet series with or without coefficients

One of the most favorite topics investigated by Professor Kubilius is the theory of arithmetical functions. Therefore the author feels it a natural choice here to report some recent developments in the theory of multiple Dirichlet series with arithmetical coefficients as numerators.

We begin, however, with the definition of multiple zeta-functions without coefficients:

\[ \zeta_r(s_1, \ldots, s_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{m_1^{-s_1}(m_1 + m_2)^{-s_2} \cdots (m_1 + \cdots + m_r)^{-s_r}}{m_1^{s_1}(m_1 + m_2)^{s_2} \cdots (m_1 + \cdots + m_r)^{s_r}}, \quad (2.1) \]

where \( s_1, \ldots, s_r \in \mathbb{C} \). This multiple series is sometimes called the Euler–Zagier \( r \)-fold zeta-function. This series is absolutely convergent when

\[ \Re(s_{r-k+1} + \cdots + s_r) > k \quad (1 \leq k \leq r) \]

and can be continued meromorphically to the whole space \( \mathbb{C}^r \). Now lots of analytic, algebraic, and arithmetic properties of this series (2.1) and its special values are known.

It is natural to consider some generalization of (2.1), which has some coefficients as numerators. We introduce the following two types of generalizations:

Type III: \( \Phi_{rI}(s_1, \ldots, s_r; a_1, \ldots, a_r) \)

\[ \Phi_{rI}(s_1, \ldots, s_r; a_1, \ldots, a_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)a_2(m_2)\cdots a_r(m_r)}{m_1^{s_1}(m_1 + m_2)^{s_2} \cdots (m_1 + \cdots + m_r)^{s_r}}, \]

and

Type *: \( \Phi_{r*}(s_1, \ldots, s_r; a_1, \ldots, a_r) \)

\[ \Phi_{r*}(s_1, \ldots, s_r; a_1, \ldots, a_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)a_2(m_1 + m_2)\cdots a_r(m_1 + \cdots + m_r)}{m_1^{s_1}(m_1 + m_2)^{s_2} \cdots (m_1 + \cdots + m_r)^{s_r}}. \]

Here the notation III and * is according to the philosophy of Arakawa and Kaneko [2]. The aim of the present paper is to survey recent results related with these two types of multiple series.

Note that the present survey is by no means complete. In this paper, we mainly discuss the analytic point of view, so many important results on special values are not mentioned. Moreover, we do not discuss several multiple series with arithmetical numerators, similar to our series. For example, there are a lot of references on multiple Dirichlet series with some kind of twisted factors as numerators. In the present paper, however, we only discuss the case of Dirichlet characters later, without mentioning the details of other papers, for example, Cassou-Noguès [8, 9], de Crisenoy [12], de Crisenoy and Essouabri [13], Essouabri and the author [17], and so on.

More generally, de la Bretèche [14] studied multiple series of the form

\[ \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{f(m_1, \ldots, m_r)}{m_1^{s_1}m_2^{s_2} \cdots m_r^{s_r}}, \]
whereas several mathematicians treated

\[ \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{f(m_1, \ldots, m_r)}{P(m_1, \ldots, m_r)^s}, \]

where \( P \) is a polynomial, or more generally

\[ \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{f(m_1, \ldots, m_r)}{P_1(m_1, \ldots, m_r)^s_1 \cdots P_n(m_1, \ldots, m_r)^s_n}, \]

where \( P_1, \ldots, P_n \) are polynomials (see, e.g., Lichtin’s series of papers such as [28, 29], Peter [32], Essouabri [16], and so on). The contents of those researches are also not treated in the present paper.

The series introduced and studied by Goldfeld, Bump, Friedberg, Hoffstein et al. (see, e.g., [7]) are also called “multiple Dirichlet series” but are different from the series discussed in the present paper.

3 The case of type \( \ast \)

In this section, we consider the case of type \( \ast \).

The simplest situation is where \( a_j \) (\( 1 \leq j \leq r \)) are periodic functions. In this case the sum \( \Phi_r^* (s_1, \ldots, s_r; a_1, \ldots, a_r) \) can be easily written as a linear combination of multiple series with numerator = 1.

The case \( a_j = \chi_j \) (Dirichlet characters) was studied by Akiyama and Ishikawa [1]. They wrote \( \Phi_r^* (s_1, \ldots, s_r; \chi_1, \ldots, \chi_r) \) as a linear combination of multiple zeta-functions of the form

\[ (m_1 + \alpha_1)^{-s_1} (m_1 + m_2 + \alpha_2)^{-s_2} \cdots (m_1 + \cdots + m_r + \alpha_r)^{-s_r} \]  

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)

(3.1)
**Theorem 1.** (See [18].) If \( a_i \) is at most of polynomial order, satisfies the above recurrence condition, and if 1 is not an eigenvalue of any of the matrices \( T_1, \ldots, T_r \), then

\[
\sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)a_2(m_1+m_2) \cdots a_r(m_1+\cdots+m_r)}{m_1^{s_1}(m_1+m_2)^{s_2} \cdots (m_1+\cdots+m_r)^{s_r}}
\]

can be holomorphically continued to the whole space \( \mathbb{C}^r \).

As a typical example, a double series with Fibonacci numbers \( F_n \) on the numerator is discussed. In fact, it was shown that the double series

\[
\phi(s) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(i\alpha^{-1})^{2m+n}F_{2m+n} + (i\alpha^{-1})^{m+2n}F_{m+2n}}{m^s(m+n)^{s}}
\]

\( (\alpha = (1 + \sqrt{5})/2) \) can be continued meromorphically to \( \mathbb{C} \). Moreover, the evaluation of a special value

\[
\phi(0) = \frac{1}{18} (6 - \sqrt{5} + (2 - 3\sqrt{5})i)
\]

and the sum formula

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha^{-m}F_m + \alpha^{-n}F_n - \alpha^{-m-n}F_{m+n}}{m(m+n)^2} = \sum_{m=1}^{\infty} \frac{\alpha^{-m}F_m}{m^3}
\]

are given.

An important auxiliary tool in [18] is a kind of vectorial zeta-functions. Special values of those vectorial zeta-functions satisfy “vectorial sum formulas”. Such formulas were proved in [18] in the double and triple cases. The formula in the general case was proposed as a conjecture in [18] and proved by Yamamoto [35].

### 4 Application of the Mellin–Barnes integral formula

Hereafter we discuss the case of type III. Let \( \varphi_k(s) = \sum_{m=1}^{\infty} a_k(m)m^{-s} \) \((1 \leq k \leq r)\), and we sometimes write \( \Phi_{r}^{III}(s_1, \ldots, s_r; \varphi_1, \ldots, \varphi_r) \) instead of \( \Phi_{r}^{III}(s_1, \ldots, s_r; a_1, \ldots, a_r) \).

The advantage of type III is that this type of series is more suitable to analytic study. Indeed, write

\[
\Phi_{r}^{III}(s_1, \ldots, s_r; \varphi_1, \ldots, \varphi_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)a_2(m_2) \cdots a_r(m_r)}{m_1^{s_1}(m_1+m_2)^{s_2} \cdots (m_1+\cdots+m_{r-1})^{s_{r-1}}} \times (m_1+\cdots+m_{r-1})^{-s_r} \left( 1 + \frac{m_r}{m_1+\cdots+m_{r-1}} \right)^{-s_r},
\]

and to the last factor apply the classical Mellin–Barnes integral formula

\[
(1 + \lambda)^{-s} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \lambda^z \, dz,
\]
where \( s, \lambda \in \mathbb{C}, \Re s > 0, \lambda \neq 0, |\arg \lambda| < \pi, \) and \(-\Re s < c < 0\). We get

\[
\left(1 + \frac{m_r}{m_1 + \cdots + m_{r-1}} \right)^{-s_r} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \left( \frac{m_r}{m_1 + \cdots + m_{r-1}} \right)^z dz,
\]

and hence

\[
\Phi^{\text{III}}_r(s_1, \ldots, s_r; \varphi_1, \ldots, \varphi_r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \prod_{m_r = 1}^{\infty} \prod_{m_{r-1} = 1}^{\infty} \frac{a_1(m_1)a_2(m_2) \cdots a_r(m_r)}{m_1^{s_1}(m_1 + m_2)^{s_2} \cdots (m_1 + \cdots + m_{r-1})^{s_{r-1} + s_r + z}} \sum_{m_r = 1}^{\infty} a_r(m_r)m_r^z dz
\]

\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \Phi^{\text{III}}_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z; \varphi_1, \ldots, \varphi_{r-1}) \varphi_r(-z) dz.
\]

(Here the important point is that the sum with respect to \( m_r \) can be separated.)

Using this expression, we can reduce the study of \( \Phi_r \) to that of \( \Phi_{r-1} \) and \( \varphi_r \), and to that of \( \Phi_{r-2} \) and \( \varphi_{r-1} \), and finally we obtain the following:

**Theorem 2.** (See [31].) Assume that \( \varphi_k(s) (1 \leq k \leq r) \) are convergent absolutely for \( s \) with sufficiently large real part, continued meromorphically to the whole complex plane, with finitely many poles, and are of polynomial order. Then \( \Phi^{\text{III}}_r(s_1, \ldots, s_r; \varphi_1, \ldots, \varphi_r) \) can be continued meromorphically to the whole \( \mathbb{C}^r \), and the location of possible singularities can be given explicitly. In particular, if all \( \varphi_k(s) (1 \leq k \leq r) \) are entire, then \( \Phi^{\text{III}}_r(s_1, \ldots, s_r; \varphi_1, \ldots, \varphi_r) \) is also entire.

In particular,

\[
\Phi^{\text{III}}_2(s_1, s_2; 1, \varphi_2) = \sum_{m_1 = 1}^{\infty} \sum_{m_2 = 1}^{\infty} \frac{a_2(m_2)}{m_1^{s_1}(m_1 + m_2)^{s_2}}
\]

can be continued. This double series satisfies a certain “functional equation”, which is written in terms of confluent hypergeometric functions (see [10, 11]):

**Theorem 3.** (See [10].) We have

\[
\Phi^{\text{III}}_2(s_1, s_2; 1, \varphi_2) = \frac{\Gamma(1 - s_1)\Gamma(s_1 + s_2 - 1)}{\Gamma(s_2)} \varphi_2(s_1 + s_2 - 1)
\]

\[
+ \Gamma(1 - s_1) \left\{ F_{2,+}(1 - s_2, 1 - s_1) + F_{2,-}(1 - s_2, 1 - s_1) \right\},
\]

where

\[
F_{2, \pm}(s_1, s_2) = \sum_{l \geq 1} A_{s_1 + s_2 - 1}(l) \Psi(s_2, s_1 + s_2; \pm 2\pi il),
\]

\[
A_c(l) = \sum_{n \mid l} n^c a_2(n), \text{ and } \Psi \text{ is the confluent hypergeometric function defined by}
\]

\[
\psi(a, b; x) = \frac{1}{\Gamma(a)} \int_0^{e^{x\infty}} e^{-xy} y^{a-1}(y + 1)^{b-a-1} dy.
\]
Furthermore, when $\varphi_2$ is an automorphic $L$-function, then by using the modular relation a different type of functional equation can also be proved.

Why we may call Theorem 3 a “functional equation”? There are mainly two reasons. First, it can be compared with the functional equation of the Hurwitz zeta-function $\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s}$ ($0 < \alpha \leq 1$):

$$\zeta(1 - s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ e^{\pi i s/2} \phi(s, -\alpha) + e^{-\pi i s/2} \phi(s, \alpha) \right\}$$

with $\phi(s, \alpha) = \sum_{n=1}^{\infty} e^{2\pi i n\alpha} n^{-s}$.

Second, when $a_2(n) \equiv 1$, a symmetric form of the functional equation (i.e., a relation connecting $\Phi_{X^2}(s_1, s_2)$ with $\Phi_{X^2}(1 - s_2, 1 - s_1)$) can be deduced from Theorem 3 on the hyperplanes $s_1 + s_2 = 2k + 1$ ($k \in \mathbb{Z} \setminus \{0\}$).

This second point was already observed by Komori, Tsumura, and the author [26]. We note that a generalization of the results in [26] to the case of double $L$-functions twisted by Dirichlet characters was discussed in [27].

In Theorem 2 the assumption that each $\varphi_k$ has only finitely many poles is important. The analytic behavior of $\Phi_{X^2}$ may become different when some of $\varphi_k$ has infinitely many poles. In the next section, we discuss such cases.

## 5 The case where some $\varphi_k$ has infinitely many poles

Typical examples of Dirichlet series with arithmetical coefficients that have infinitely many poles are

$$\sum_{m=1}^{\infty} A(m) m^{-s} = - \frac{\zeta'(s)}{\zeta(s)}, \quad \sum_{m=1}^{\infty} \mu(m) m^{-s} = \frac{1}{\zeta(s)},$$

where $A(m)$ is the von Mangoldt function, and $\mu(m)$ is the Möbius function. Both series have infinitely many poles at the zeros of $\zeta(s)$.

We now consider the behavior of $\Phi_{X^2}$, where some of associated $\varphi_k$ has infinitely many poles (as in the above examples). Let

$$N(\Phi_{X^2}) = \#\{k \mid 1 \leq k \leq r, \varphi_k \text{ has infinitely many poles}\}.$$  

In a recent paper, Nawashiro, Tsumura, and the author [30] studied several examples satisfying $N(\Phi_{X^2}) = 1$. For example:

**Theorem 4.** (See [30].) The double series

$$\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{A(m_2)}{m_1^{s_1}(m_1 + m_2)^{s_2}}$$

can be continued meromorphically to the whole $\mathbb{C}^2$, and the location of possible singularities can be described explicitly.

Next, take $\varphi_1(s) = \sum_{m=1}^{\infty} a_1(m)m^{-s}$, which has only finitely many poles, and consider the convolution of $a_1$ and $\mu$:

$$\tilde{a}(m) = \sum_{d|m} a_1\left(\frac{m}{d}\right) \mu(d).$$
In other words,
\[
\frac{\varphi_1(s)}{\zeta(s)} = \sum_{m=1}^{\infty} \tilde{a}(m)m^{-s}.
\]

**ASSUMPTION.** All nontrivial zeros of \( \zeta(s) \) are simple, and
\[
\frac{1}{\zeta'(\rho_n)} = O\left(|\rho_n|^B\right) \quad (\rho_n \text{ is the } n\text{th zero, } B > 0).
\]

(This is a quantitative version of the well-known “simplicity conjecture” for the zeros of \( \zeta(s) \), consistent with the Gonek–Hejhal conjecture [19, 22].)

**Theorem 5.** (See [30]). Under the above assumption, the double series
\[
\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{\tilde{a}(m_2)}{m_1^s (m_1 + m_2)^{s_2}}
\]
can be continued meromorphically to the whole \( \mathbb{C}^2 \), and the location of possible singularities can be described explicitly.

The paper [30] only considers the double zeta case, but a generalization of [30] to the general multiple case was treated by Rei Kawashima [25]. She also discussed the case where \( \Lambda \) is replaced by the Liouville function \( \lambda \).

The above two theorems show that when \( N(\Phi^{III}_1) = 1 \), the analytic behavior of \( \Phi^{III}_1 \) is not so different from the case where \( N(\Phi^{III}_2) = 0 \). However, if \( N(\Phi^{III}_2) = 2 \) or, more generally, if \( N(\Phi^{III}_r) \geq 2 \), then the analytic behavior of the multiple series is totally different.

Consider
\[
F_2(s) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\Lambda(k)\Lambda(l)}{(k + l)^s},
\]
which is equal to \( \Phi_2(0, s; -\zeta'/\zeta, -\zeta'/\zeta) \), so both \( \varphi_1 \) and \( \varphi_2 \) have infinitely many poles. We can rewrite
\[
F_2(s) = \sum_{m=1}^{\infty} \frac{G_2(m)}{m^s}, \quad G_2(m) = \sum_{k+l=m} \Lambda(k)\Lambda(l).
\]

Note that this \( G_2(m) \) is the counting function of the classical Goldbach problem:
\[
G_2(m) = \sum_{\substack{r_1 \geq 1, r_2 \geq 1 \\ p_1^r + p_2^r = m}} \log p_1 \log p_2 = \sum_{p_1 + p_2 = m} \log p_1 \log p_2 + \text{(error)},
\]
where \( p_1, p_2 \) denote primes. Using the Mellin–Barnes formula, we have
\[
F_2(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s+z)\Gamma(-z)\zeta'(s+z)}{\Gamma(s)\zeta'(s)\zeta(-z)} \frac{\zeta'(s+z)}{\zeta(s+z)} \frac{\zeta'(s)}{\zeta(s)} \frac{\zeta'(z)}{\zeta(z)} \, dz.
\]

Shifting the path of integration suitably, we find that there are poles of \( F_2(s) \) at \( s = 2, s = \rho + 1 \), and \( s = \rho + \rho' \) (where \( \rho \) and \( \rho' \) denote the nontrivial zeros of \( \zeta(s) \)).
Now assume the RH (Riemann hypothesis). Then \( \Re(\rho + \rho') = 1 \), and we can show that \( \rho + \rho' \) are dense on the line \( \Re s = 1 \). Therefore \( \Re s = 1 \) seems a kind of barrier if we want to continue \( F_2(s) \) meromorphically.

It is believed that \( \gamma = \Im \rho > 0 \) are linearly independent over \( \mathbb{Q} \) (the linear independence conjecture, LIC).

**Theorem 6.** If RH and LIC are true, then \( \Re s = 1 \) is the natural boundary of \( F_2(s) \).

This was first proved by Egami and the author [15] under the RH and a stronger quantitative version of LIC, and then in the above form by Bhowmik and Schlage-Puchta [6]. Moreover, in [6], it is also shown that

\[
F_2(s) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} A(k_1) \cdots A(k_r) (k_1 + \cdots + k_r)^s \quad (r \geq 3)
\]

has the natural boundary \( \Re s = r - 1 \) if RH is true and \( \Re s = 1 \) is the natural boundary of \( F_2(s) \).

A generalization to the case with congruence conditions has been also studied. Let

\[
G_2(m; q, a, b) := \sum_{k+l=m, \ k \equiv a, \ l \equiv b \ (\text{mod} \ q)} A(k) A(l),
\]

where \( a, b, q \) are positive integers with \( (ab, q) = 1 \), and define the associated Dirichlet series by

\[
F_2(s; q, a, b) := \sum_{m=1}^{\infty} \frac{G_2(m; q, a, b)}{m^s}.
\]

The behavior of \( F_2(s; q, a, b) \) was first treated by Rüppel [33], and then further studied by Suzuki [34].

Let

\[
S(x; q, a, b) := \sum_{m \leq x} G_2(m; q, a, b).
\]

We can reduce the study of \( S(x; q, a, b) \), via its associated Dirichlet series \( F_2(s; q, a, b) \), to that of the behavior (especially the distribution of zeros) of Dirichlet \( L \)-functions. Thereby we can establish the connection between Goldbach counting functions and the zeros of \( L \)-functions (especially the generalized Riemann hypothesis, GRH). The existence of this connection was first suggested by Granville [20, 21] in the Riemann zeta case and then fully developed in Bhowmik et al. [3, 4, 5]. We conclude this paper with the statement of some theorems proved in those papers.

Let \( B_{\chi} = \sup \{ \Re \rho_{\chi} \} \) and \( B_q = \sup \{ B_{\chi} | \chi \ (\text{mod} \ q) \} \).

**Theorem 7.** (See [4].) For any \( \delta > 0 \), we have

\[
S(x; q, a, b) = \frac{x^2}{2\varphi(q)^2} + O(x^{1+B_q^*}),
\]

where \( B_q^* = \min \{ B_q, 1 - \eta \} \) with

\[
\eta = \frac{c(\delta)}{\max\{q^\delta, (\log x)^{2/3}(\log\log x)^{1/3}\}}
\]

(where \( c(\delta) \) is a small positive constant).

**Remark 1.** In particular, if we assume GRH, then \( B_q^* = B_q = 1/2 \), and

\[
S(x; q, a, b) = \frac{x^2}{2\varphi(q)^2} + O(x^{3/2}).
\]
Now recall the well-known conjecture (DZC): Any two distinct Dirichlet $L$-functions (mod $q$) do not have a common nontrivial zero (except for a possible multiple zero at $s = 1/2$).

**Theorem 8.** (See [4, 5].) Assume that the DZC is true and $\chi(a) + \chi(b) \neq 0$ for all $\chi$ (mod $q$). If the asymptotic formula

$$S(x; q, a, b) = \frac{x^2}{\varphi(q)^2} + O\left(x^{1+\varepsilon} \right) \quad \left( \frac{1}{2} \leq d < 1 \right)$$

holds for any $\varepsilon > 0$, then either $B_q \leq d$ or $B_q = 1$. Moreover, we can remove the possibility of $B_q = 1$ when $a = b$.

**Remark 2.** In particular, if $a = b$ and the above formula holds with the error $O(x^{3/2+\varepsilon})$ (i.e., $d = 1/2$), then $B_q = 1/2$, that is, the GRH holds.

**References**

1. S. Akiyama and H. Ishikawa, On analytic continuation of multiple $L$-functions and related zeta-functions, in C. Jia and K. Matsumoto (Eds.), *Analytic Number Theory*, Dev. Math., Vol. 6, Kluwer, Dordrecht, 2002, pp. 1–16.

2. T. Arakawa and M. Kaneko, On multiple $L$-values, *J. Math. Soc. Japan*, 56:967–991, 2004.

3. G. Bhowmik and K. Halupczok, Asymptotics of Goldbach representations, in H. Mishou, T. Nakamura, M. Suzuki, and Y. Umegaki (Eds.), *Various Aspects of Multiple Zeta Functions – in Honor of Professor Kohji Matsumoto’s 60th Birthday*, Adv. Stud. Pure Math., Vol. 84, Mathematical Society of Japan, Tokyo, 2020, pp. 1–21.

4. G. Bhowmik, K. Halupczok, K. Matsumoto, and Y. Suzuki, Goldbach representations in arithmetic progressions and zeros of Dirichlet $L$-functions, *Mathematika*, 65:57–97, 2019.

5. G. Bhowmik and I.Z. Ruzsa, Average Goldbach and the quasi-Riemann hypothesis, *Anal. Math.*, 44:51–56, 2018.

6. G. Bhowmik and J.-C. Schlage-Puchta, Meromorphic continuation of the Goldbach generating function, *Funct. Approximatio, Comment. Math.*, 45(1):43–53, 2011.

7. B. Brubaker, D. Bump, and S. Friedberg, *Weyl Group Multiple Dirichlet Series*, Ann. Math. Stud., Vol. 175, Princeton Univ. Press, Princeton, NJ, 2011.

8. P. Cassou-Noguès, Applications arithmétiques de l’étude des valeurs aux entiers négatifs des séries de Dirichlet associées à un polynôme, *Ann. Inst. Fourier*, 31(4):1–35, 1981.

9. P. Cassou-Noguès, Valeurs aux entiers négatifs des séries de Dirichlet associées à un polynôme. I, *J. Number Theory*, 14(1):32–64, 1982.

10. Y. Choie and K. Matsumoto, Functional equations for double series of Euler type with coefficients, *Adv. Math.*, 292:529–557, 2016.

11. Y. Choie and K. Matsumoto, Functional equations for double series of Euler–Hurwitz–Barnes type with coefficients, in K. Ihara (Ed.), *Various Aspects of Multiple Zeta Values*, RIMS Kôkyûroku Bessatsu B68, RIMS, Kyoto University, Kyoto, 2017, pp. 91–109.

12. M. de Crisenoy, Values at $T$-tuples of negative integers of twisted multivariable zeta series associated to polynomials of several variables, *Compos. Math.*, 142:1373–1402, 2006.

13. M. de Crisenoy and D. Essouabri, Relations between values at $T$-tuples of negative integers of twisted multivariable zeta series associated to polynomials of several variables, *J. Math. Soc. Japan*, 60:1–16, 2008.

14. R. de la Brèteche, Estimation de sommes multiples de fonctions arithmétiques, *Compos. Math.*, 128:261–298, 2001.

15. S. Egami and K. Matsumoto, Convolutions of the von Mangoldt function and related Dirichlet series, in S. Kanemitsu and J. Liu (Eds.), *Number Theory: Sailing on the Sea of Number Theory*, Ser. Number Theory Appl., Vol. 2, World Scientific, Singapore, 2007, pp. 1–23.
16. D. Essouabri, Height zeta functions on generalized projective toric varieties, Contemp. Math., 566:65–98, 2012.
17. D. Essouabri and K. Matsumoto, Values at non-positive integers of partially twisted multiple zeta-functions. I, Comment. Math. Univ. St. Pauli, 67(1):83–100, 2019.
18. D. Essouabri, K. Matsumoto, and H. Tsumura, Multiple zeta-functions associated with linear recurrence sequences and the vectorial sum formula, Can. J. Math., 63:241–276, 2011.
19. S.M. Gonek, On negative moments of the Riemann zeta-function, Mathematika, 36:71–88, 1989.
20. A. Granville, Refinements of Goldbach’s conjecture, and the generalized Riemann hypothesis, Funct. Approximatio, Comment. Math., 37:159–173, 2007.
21. A. Granville, Corrigendum to “Refinements of Goldbach’s conjecture, and the generalized Riemann hypothesis”, Funct. Approximatio, Comment. Math., 38:235–237, 2008.
22. D. Hejhal, On the distribution of $\log |\zeta(\frac{1}{2} + it)|$, in K.E. Aubert, E. Bombieri, and D. Goldfeld (Eds.), Number Theory, Trace Formula and Discrete Groups. Symposium in Honor of Atle Selberg Oslo, Norway, July 14–21, Ser. Number Theory Appl., Vol. 2, Academic Press, Boston, MA, 1989, pp. 343–370.
23. H. Ishikawa, On analytic properties of a multiple $L$-function, in S. Saitoh, N. Hayashi, and M. Yamamoto (Eds.), Analytic Extension Formulas and Their Applications, Int. Soc. Appl. Anal. Comput., Vol. 9, Kluwer, Dordrecht, 2001, pp. 105–122.
24. H. Ishikawa, A multiple character sum and a multiple $L$-function, Arch. Math., 79:439–448, 2002.
25. R. Kawashima, On Various Properties of Multiple Dirichlet Series Associated with Arithmetical Functions, Master thesis, Tokyo Metropolitan University, Tokyo, Japan, 2019 (in Japanese).
26. Y. Komori, K. Matsumoto, and H. Tsumura, Functional equations and functional relations for the Euler double zeta-function and its generalization of Eisenstein type, Publ. Math. Debrecen, 77(1–2):15–31, 2010.
27. Y. Komori, K. Matsumoto, and H. Tsumura, Functional equations for double $L$-functions and values at non-positive integers, Int. J. Number Theory, 7:1441–1461, 2011.
28. B. Lichtin, Generalized Dirichlet series and $b$-functions, Compos. Math., 65:81–120, 1988.
29. B. Lichtin, The asymptotics of a lattice point problem determined by a hypoelliptic polynomial, in M. Kashiwara, T. Monteiro Fernandes, and P. Schapira (Eds.), $D$-Modules and Microlocal Geometry. Proceedings of the International Conference Held at the University of Lisbon, Portugal, October 29–November 2, 1990, Walter de Gruyter, Berlin, 1993, pp. 75–106.
30. K. Matsumoto, A. Nawashiro, and H. Tsumura, Double Dirichlet series associated with arithmetic functions, Kodai Math. J., to appear, arXiv:1801.04444v3.
31. K. Matsumoto and Y. Tanigawa, The analytic continuation and the order estimate of multiple Dirichlet series, J. Théor. Nombres Bordeaux, 15(1):267–274, 2003.
32. M. Peter, Dirichlet series associated with polynomials, Acta Arith., 84:245–278, 1998.
33. F. Rüppel, Convolutions of the von Mangoldt function over residue classes, Šiauliai Math. Semin., 7(15):135–156, 2012.
34. Y. Suzuki, A mean value of the representation function for the sum of two primes in arithmetic progressions, Int. J. Number Theory, 13:977–990, 2017.
35. S. Yamamoto, A sum formula of multiple $L$-values, Int. J. Number Theory, 11:127–137, 2015.