Operators of the \( q \)–oscillator

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Abstract. We scrutinize the possibility of extending the result of [19] to the case of \( q \)-deformed oscillator for \( q \) real; for this we exploit the whole range of the deformation parameter as much as possible. We split the case into two depending on whether a solution of the commutation relation is bounded or not. Our \textit{leitmotif} is subnormality.

The deformation parameter \( q \) is reshaped and this is what makes our approach effective. The newly arrived parameter, the operator \( C \), has two remarkable properties: it separates in the commutation relation the annihilation and creation operators from the deformation as well as it \( q \)-commutes with those two. This is why introducing the operator \( C \) seems to be far-reaching.

\( q \)-deformations of the quantum harmonic oscillator (the abbreviation the \textit{q–oscillator} stands here for it) has been arresting attention of many\footnote{\textit{q–oscillator} stands here for it} resulting among other things in \textit{quantum groups}. Besides realizing the ever lasting temptation to generalize matters, it brings forth new attractive findings. This paper exhibits the spatial side of the story.

The \( q \)-oscillator algebra, which is the \textit{milieu} of our considerations, is that generated by three objects \( a_+, a_- \) and 1 (the latter being a unit in the algebra) satisfying the commutation relations

\[
a_-a_+ - qa_+a_- = 1; \tag{1}
\]

it goes back to the seventies with [1] as a specimen. The other versions which appear in the literature are equivalent to that and this is described completely in [8] where a list of further references can be found.

Looking for \( \ast \)-representations of (1) usually means assuming that \( a_- = a_+ \ast \), with the asterisk denoting the Hilbert space adjoint. Thus what we start with is a \textit{given} Hilbert space and the commutation relation

\[
S^*S - qSS^* = I, \quad (O_{q,\text{op}})
\]

in it. Of course, \( q \) must be perforce real then; this is what assume in the paper.

An easy-going consequence is

\textbf{Sample Theorem.} If \( S \) is a weighted shift with respect to the basis \( \{e_n\}_{n=0}^\infty \) and

\[
S^*Sf - qSS^*f = f, \quad f \in \text{lin}\{e_n\}_{n=0}^\infty,
\]

then \( S e_n = \sqrt{1 + q + \cdots + q^n} e_{n+1} \), \( n \geq 0 \).

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\item \footnote{\textit{q–oscillator} stands here for it} \textit{q–oscillators} are vastly disseminated in Mathematical Physics and we would like to acknowledge here with pleasure [8] for bringing them closer to Mathematics
\end{itemize}
‘If \( S \) is a weighted shift’ – this is usually tacitly assumed when dealing with the relation 
\((\Omega_{qqp}), \text{like in } [5] \). It is sometimes made a bit more explicit in stating that a vacuum vector 
(or a ground state, depending on denomination in Mathematical Physics an author belongs to) of \( S \) exists. The point here (as it was in \([19]\) for \( q = 1 \)) is to discuss the case. It 
turns out that, like in \([19]\), subnormality plays an important role in the matter (and this, 
the case \( q = 1 \) at least, is parallel to Rellich-Dixmier \([12,7]\) characterization of solutions 
to the CCR). Luckily, the above coincides with our belief that subnormality is the missing 
counterpart of complex variable in the quantization scheme.

Preliminary essentials

A short guide to subnormality. Recall that a densely defined operator \( A \) is said to 
be hyponormal if \( \mathcal{D}(A) \subset \mathcal{D}(A^*) \) and \( \|A^* f\| \leq \|Af\|, \ f \in \mathcal{D}(A) \). A hyponormal operator 
\( N \) is said to be formally normal if \( \|N f\| = \|N^* f\|, \ f \in \mathcal{D}(N) \). Specifying more, a formally 
normal operator \( N \) is called normal if \( \mathcal{D}(N) = \mathcal{D}(N^*) \). Finally, a densely defined operator \( S \) is called (formally) subnormal if there is a Hilbert space \( \mathcal{K} \) containing \( \mathcal{H} \) isometrically 
and a (formally) normal operator \( N \) in \( \mathcal{K} \) such that \( S \subset N \).

The following diagram relates these notions.

\[
\begin{array}{ccc}
\text{normal} & \Rightarrow & \text{formally normal} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\text{hyponormal} & \Rightarrow & \text{formally subnormal} \\
\end{array}
\]

Though the definitions of formal normality and normality look much alike, with a little 
difference concerning the domains involved, the operators they define may behave in a 
totally incomparable manner. However, needless to say, these two notions do not differ at 
all in the case of bounded operators.

If \( A \) and \( B \) are densely defined operators in \( \mathcal{H} \) and \( \mathcal{K} \) resp such that \( \mathcal{H} \subset \mathcal{K} \) and \( A \subset B \) then

\[
\mathcal{D}(A) \subset \mathcal{D}(B) \cap \mathcal{H}, \quad \mathcal{D}(B^*) \cap \mathcal{H} \subset PD(B^*) \subset \mathcal{D}(A^*)
\]

where \( P \) stands for the orthogonal projection of \( \mathcal{K} \) onto \( \mathcal{H} \); moreover,

\[
A^* Px = PB^* x, \quad x \in \mathcal{D}(B^*).
\]

If \( B \) closable, then so is \( A \) and both \( A^* \) as well as \( B^* \) are densely defined. The extension 
\( B \) of \( A \) is said to be tight if \( \mathcal{D}(\bar{A}) = \mathcal{D}(\bar{B}) \cap \mathcal{H} \) and \( *-tight \) if \( \mathcal{D}(B^*) \cap \mathcal{H} = \mathcal{D}(A^*) \). If 
\( \mathcal{D}(B) \subset \mathcal{D}(B^*) \) (and this happens for formally normal operators as we already know), the 
two chains in (2) glue together as

\[
\mathcal{D}(A) \subset \mathcal{D}(B) \cap \mathcal{H} \subset \mathcal{D}(B^*) \cap \mathcal{H} \subset PD(B^*) \subset \mathcal{D}(A^*). \quad (4)
\]

As we have already said a densely defined operator having a normal extension is just sub-
normal. However, normal extensions may not be uniquely determined in unbounded case 
as their minimality becomes a rather fragile matter, see \([17]\); even though the inclusions 
(4) hold for any of them. Moreover, even if all of them turn into equalities none of the 
normal extensions may be minimal of cyclic type (this is what ensures uniqueness); this

\(^2\) Description of domains of weighted shifts and their adjoint can be found in \([15]\).
will become effective when we pass to the case of \( q > 1 \). So far we have got an obvious fact.

**Proposition 1.** A subnormal operator \( S \) has a normal extension which is both tight and \( \ast \)-tight if and only if

\[
D(S) = D(S^\ast). \tag{5}
\]

If this happens then any normal extension is both tight and \( \ast \)-tight.

Because equality (5) is undoubtedly decisive for a solution of the commutation relation of (any of) the oscillators to be a weighted shift, subnormality is properly settled into this context.

**q-notions.** For \( x \) an integer and \( q \) real, \([x]_q \overset{\text{def}}{=} (1 - q^x)(1 - q^{-1})^{-1} \) if \( q \neq 1 \) and \([x]_1 \overset{\text{def}}{=} x\).

If \( x \) is a non-negative integer, \([x]_q = 1 + q + \cdots + q^{x-1} \) and this is usually referred to as a basic or \( q \)-number. A little step further, the \( q \)-factorial is like the conventional, \([0]_q! = 1 \) and \([n]_q! = [0]_q \cdot [1]_q \cdot \cdots \cdot [n-1]_q \cdot [n]_q \) and so is the \( q \)-binomial \([m]_{aq}! \overset{\text{def}}{=} \frac{[m]_q!}{[m-aq]_q! \cdot [aq]_q!} \). Thus, if \(-1 \leq q < 0 \) and \( x \in \mathbb{N} \) the basic number \([x]_q \) is non-negative.

For arbitrary complex numbers \( a \) and \( q \) one can always define \((a; q)_k \) as follows

\[
(a; q)_k \overset{\text{def}}{=} (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{k-1}), \quad k = 1, 2, 3, \ldots
\]

Then for \( n > 0 \) one has \([n]_q! = (q, q)_n (1 - q)^n \). Moreover, there are (at least) two possible definitions of \( q \)-exponential functions

\[
e_q(z) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{(q; q)_k} z^k, \quad z \in \omega_q,
\]

\[
E_q(z) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{q^k}{(q; q)_k} z^k, \quad z \in \omega_q^{-1}, \quad q \neq 0,
\]

where

\[
\omega_q = \begin{cases} 
\{z; \ |z| < 1 \} & \text{if } |q| < 1, \\
\mathbb{C} & \text{otherwise.}
\end{cases}
\]

These two functions are related via

\[
e_q(z) = E_{q^{-1}}(-z), \quad z \in \omega_q, \quad q \neq 0.
\]

**The q oscillator**

**Spatial interpretation of** \((O_{q,op})\). The relation \((O_{q,op})\) has nothing but a symbolic meaning unless someone says something more about it; this is because some of the solutions may be unbounded. By reason of this we distinguish two, extreme in a sense, ways of looking at the relation \((O_{q,op})\):

The first meaning of \((O_{q,op})\) is

\[
S \text{ closable, } D \text{ is dense in } \mathcal{H} \text{ and } D \subset D(S^\ast \tilde{S}) \cap D(\tilde{S} S^\ast), \quad S^\ast S f - q S S^\ast f = f, \quad f \in D. \tag{O_{q,d}}
\]

The other is

\[
\langle S f, S g \rangle - q \langle S^\ast f, S^\ast g \rangle = \langle f, g \rangle, \quad f, g \in D(S) \cap D(S^\ast \tilde{S}) \tag{O_{q,wm}}
\]

and, because this is equivalent to

\[
\|S f\|^2 - q \|S^\ast f\|^2 = \|f\|^2, \quad f \in D(S) \cap D(S^\ast)
\]
it implies for $S$ to be closable, $(O_{q,w})$ in turn is equivalent to
\[
\langle S f, S g \rangle - q \langle S^* f, S^* g \rangle = \langle f, g \rangle, \quad f \in \mathcal{D}(S) \cap \mathcal{D}(S^*).
\]

The occurring interdependence, which follows, let us play variation on the theme of $(O_{q,w})$.

1° $(O_{q,w})$ with $D$ being a core of $S$ $\implies$ $(O_{q,w})$ and $\mathcal{D}(\bar{S}) \subset \mathcal{D}(S^*)$.

Indeed, for $f \in \mathcal{D}(\bar{S})$ there is a sequence $(f_n)_n \subset D$ such that $f_n \to f$ and $S f_n \to \bar{S} f$. Because $S^*$ is closed we get from $(O_{q,D})$ that $S^* f_n \to S^* f$ and consequently $f \in \mathcal{D}(S^*)$ as well as $(O_{q,w})$.

2° $(O_{q,D})$ with $D$ being a core of $S^*$ $\implies$ $(O_{q,w})$ and $\mathcal{D}(S^*) \subset \mathcal{D}(S)$.

This uses the same argument as that for 1°.

3° $(O_{q,w})$ $\implies$ $(O_{q,D})$ with $D = \mathcal{D}(S^* S) \cap \mathcal{D}(S^* S^*)$.

This is because $\mathcal{D}(S^* S) \cap \mathcal{D}(S^* S^*) \subset \mathcal{D}(S) \cap \mathcal{D}(S^*)$.

4° $(O_{q,w})$ and $\mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*)$ a core of $S$ and $S^*$ $\implies$ $\mathcal{D}(S^* S^*) = \mathcal{D}(\bar{S} S^*)$.

Take $f \in \mathcal{D}(S^* S)$. This means $f \in \mathcal{D}(\bar{S})$ and $\bar{S} f \in \mathcal{D}(S^*)$. Because of this, picking $(f_n)_n \in \mathcal{D}(S) \cap \mathcal{D}(S^*)$, we get from $(O_{q,w})$ in limit
\[
\langle S^* \bar{S} f, g \rangle - q \langle S^* f, S^* g \rangle = \langle f, g \rangle
\]
for $g \in \mathcal{D}(S^*)$ and, because $g \in \mathcal{D}(S) \cap \mathcal{D}(S^*)$ is a core of $S^*$, we get (6) to hold for $g \in \mathcal{D}(S^*)$. Finally, $S^* f \in \mathcal{D}(\bar{S})$. The reverse inequality needs the same kind of argument.

The above results in

5° $(O_{q,w})$ and $\mathcal{D}(\bar{S}) = \mathcal{D}(S^*)$ $\implies$ $S$ satisfies $(O_{q,D})$ on $D = \mathcal{D}(S^* S) = \mathcal{D}(\bar{S} S^*)$.

Remark 2. Notice that when $q \neq -1$ and $S$ satisfying $(O_{q,D})$ with $D = \mathcal{D}(S^* \bar{S}) = \mathcal{D}(\bar{S} S^*)$ for $\mathcal{D}$ to be a core of $S^*$ is necessary and sufficient $R(S^* S)$ to be dense in $\mathcal{H}$.

The following is a kind of general observation and settles hyponormality (or bound-
edness) in the context of $(O_{q,D})$.

Proposition 3. (a) For $0 \leq q < 1$ and for $S$ satisfying $(O_{q,D})$, $S|_D$ is hyponormal if and only if $S$ is bounded and $\|S\| \leq (1 - q)^{-1/2}$. (b) For $q < 0$ and for $S$ satisfying $(O_{q,D})$, $S^*|_D$ is hyponormal if and only if $S$ is bounded and $\|S\| \leq (1 - q)^{-1/2}$.

Proof. Write $(O_{q,D})$ as
\[
(1 - q)\|S f\|^2 = q (\|S^* f\|^2 - \|S f\|^2) + \|f\|^2, \quad f \in D.
\]
and look at this. □

The selfcommutator. Assuming $D \subset \mathcal{D}(S S^*) \cap \mathcal{D}(S^* S)$ we introduce the following operator
\[
C \overset{\text{def}}{=} I + (q - 1) S S^*, \quad \mathcal{D}(C) \overset{\text{def}}{=} \mathcal{D}.
\]
This operator turns out to be an important invention in the matter. In particular there are two immediate consequences of this definition. The first says if $S$ satisfies $(O_{q,D})$ with $D$ invariant for both $S$ and $S^*$ then $D$ is invariant for $C$ as well and
\[
CS f = q SC f, \quad q CS^* f = S^* Cf, \quad f \in D.
\]
The other is that $(O_{q,D})$ takes now the form
\[
S^* S f - SS^* f = Cf, \quad f \in D,
\]
which means that $C$ is just the selfcommutator of $S$ on $D$.

We would like to know the instances when $C$ is a positive operator.
Proposition 4. (a) For $q \geq 1$, $C > 0$ always. (b) For $q < 1$, $C \geq 0$ if and only if $S$ is bounded and $\|S\| \leq (1 - q)^{-1/2}$. (c) For $S$ satisfying $(\mathcal{O}_{q,D})$, $C > 0$ if and only if $S$ is hyponormal.

Proof. While (a) is apparently trivial (b) comes out immediately from
\[ \langle Cf, f \rangle = \|f\|^2 + (q - 1)\|S^*f\|^2, \quad f \in \mathcal{D}. \]
For (c) write (using $(\mathcal{O}_{q,D})$) with $f \in \mathcal{D}$
\[ \langle Cf, f \rangle = \|f\|^2 + (q - 1)\|S^*f\|^2 = \|f\|^2 + q\|S^*f\|^2 - \|S^*f\|^2 = \|Sf\|^2 - \|S^*f\|^2. \]

Example 5. On the other hand, with any unitary $U$ the operator
\[ S \equiv (1 - q)^{-1/2}U \tag{10} \]
satisfies $(\mathcal{O}_{q,D})$ if $q < 1$. The operator $S$ is apparently bounded and normal. Consequently (the Spectral Theorem) it may have a bunch of nontrivial reducing subspaces (even not necessarily one dimensional) or may be irreducible and this observation ought to be dedicated to all those who start too fast generating algebras from formal commutation relations.

Proposition 6. For $q < 1$ the only formally normal operators satisfying $(\mathcal{O}_{q,D})$ are those of the form $(10)$. For $q \geq 1$ there is no formally normal solution of $(\mathcal{O}_{q,D})$.

Proof. Straightforward.

Example 7. An ad hoc illustration can be given as follows. Take a separable Hilbert space with a basis $(e_n)_{n=-\infty}^{\infty}$ and look for a bilateral (or rather two-sided) weighted shift $T$ defined as $Te_n = \tau_n e_{n+1}$, $n \in \mathbb{Z}$. Then, because $T^* e_n = \tau_{n-1} e_{n-1}$, $n \in \mathbb{Z}$, for any $\alpha \in \mathbb{C}$ and $N \in \mathbb{Z}$ we get $|\tau_n|^2 = \alpha q^{n+N} + (1 - q^{n+N})(1 - q)^{-1} = q^{\alpha+n+N} + [n + N]_q$ for all $n$ if $q \neq 1$ and $|\tau_n|^2 = \alpha + n$ if $q = 1$; this is for all $n \in \mathbb{Z}$. The only possibility for the right hand sides to be non–negative (and in fact positive) footnote We avoid weights which are not defined as $\alpha \geq (1 - q)^{-1}$ for $0 < q < 1$ and $\alpha = (1 - q)^{-1}$ for $q < 0$; the latter corresponds to Example 10 Thus the only bilateral weighted shifts satisfying $(\mathcal{O}_{q,D})$, with $\mathcal{D} = \text{lin}[e_n; \ n \in \mathbb{Z}]$, are those $Te_n = \tau_n e_{n+1}$, $n \in \mathbb{Z}$ which have the weights
\[ \tau_n = \begin{cases} \sqrt{(1-q)^{-1}}, & \text{if } q < 0 \\ \sqrt{\alpha q^{n+N} + [n + N]_q}, & \alpha > (1 - q)^{-1}, \ N \in \mathbb{Z}, \\ \text{none}, & 0 \leq q < 1 \\ 1 \leq q \end{cases} \]
However, $T$ violates hyponormality (pick up $f = e_0$ as a sample) if $0 < q < 1$. Also $C$ defined by (7) is neither positive nor negative $(C e_0, e_0) = a > 0$ while $(C e_{-1}, e_{-1}) < 0)$. Let us mention that $T$ is $q^{-1}$–hyponormal in the sense of (13). Anyway, $T$ is apparently unbounded if $q > 0$. The case of $q \leq 0$ is precisely that of Example 10.

Example 8. Repeating the way of reasoning of Example 7 we get that the only unilateral weighted shifts satisfying $(\mathcal{O}_{q,D})$ are those $T$, defined as $Te_n = \tau_n e_{n+1}$ for $n \in \mathbb{N}$, which have the weights
\[ \tau_n = \sqrt{[n + 1]_q}, \quad -1 \leq q. \]
This is so because the virtual, in this case, $'\tau_{-1}'$ is 0 ($T^* e_0 = 0$). If $-1 \leq q < 0$ they are bounded and not hyponormal, if $0 \leq q < 1$ they are again bounded and hyponormal and if $1 \leq q$ the are unbounded and hyponormal; the two latter are even subnormal (cf. Theorem 19 and 21 resp.).
Remark 9. According to Lemma 2.3 of [10] for \( 0 < q < 1 \) the only cases which may happen are the orthogonal sums of the operators considered in Examples [7] [8] and given by formula [10]. For \( q > 1 \), due to the same Lemma, the orthogonal sum of that from Example [8] can be taken into account.

An auxiliary lemma of [14]. We state here a result, [14] Lemma 2.4, which authorizes the examples above. We adapt the notation of [14] to ours as well as improve a bit the syntax of the conclusion therein.

**Lemma 10.** Let \( 0 < p < 1 \) and \( \varepsilon \in \{-1, +1\} \). Assume \( T \) is a closed densely defined operator in \( \mathcal{H} \). Then

\[
T^*Tf - p^2TT^*f = \varepsilon(1 - p^2)f, \quad f \in \mathcal{D}(T^*T) = \mathcal{D}(TT^*)
\]

if and only if \( T \) is unitarily equivalent to an orthogonal sum of operators of the following type:

- **in the case of \( \varepsilon = 1 \)**

  \[
  (I) \quad T_I: \quad f_n \to (1 - p^{2(n+1)})^{1/2}f_{n+1} \text{ in } \mathcal{H} = \bigoplus_{n=0}^{+\infty} \mathcal{H}_n \text{ with each } \mathcal{H}_n \overset{\text{def}}{=} \mathcal{H}_0;
  \]

  \[
  (II) \quad T_{II}: \quad f_n \to (1 + q^{2(n+1)}A^2)^{1/2}f_{n+1} \text{ in } \mathcal{H} = \bigoplus_{n=-\infty}^{+\infty} \mathcal{H}_n \text{ with each } \mathcal{H}_n \overset{\text{def}}{=} \mathcal{H}_0 \text{ and } A \text{ being a selfadjoint operator in } \mathcal{H}_0 \text{ with } \text{sp}(A) \subset [p, 1] \text{ and either } p \text{ or } 1 \text{ not being an eigenvalue of } A;
  \]

  \[
  (III) \quad T_{III} \text{ a unitary operator;}
  \]

- **in the case of \( \varepsilon = -1 \)**

  \[
  (IV) \quad T_{IV}: \quad f_n \to (p^{2n} - 1)^{1/2}f_{n-1} \text{ in } \mathcal{H} = \bigoplus_{n=0}^{+\infty} \mathcal{H}_n \text{ with each } \mathcal{H}_n \overset{\text{def}}{=} \mathcal{H}_0 \text{ and always } f_{-1} \overset{\text{def}}{=} 0.
  \]

A couple of remarks seem to be absolutely imperative.

**Remark 11.** The conclusion of Lemma [10] is a bit too condensed. Let us provide with some hints to reading it. First of all the way of understanding the meaning of \( f_n \)'s appearing in (I), (II) and (IV) should be as follows: take \( f \in \mathcal{H}_0 \) and define \( f_n \) as a (one sided or two sides, depending on circumstances) sequence having all the coordinates zero except that of number \( n \) which is equal to \( f \). Then, with a definition

\[
\mathcal{D}(E) \overset{\text{def}}{=} \lim_{n \to \infty} f_n, \quad f \in E \subset \mathcal{H}_0, \quad n \in \mathbb{Z} \text{ or } n \in \mathbb{N} \text{ depending on the case},
\]

one has to guess that \( \mathcal{D}(T_1) = \mathcal{D}(T_{IV}) = \mathcal{D}(\mathcal{H}_0) \) and \( \mathcal{D}(T_{II}) = \mathcal{D}(\mathcal{D}(A)) \). Passing to closures in (I), (II) and (IV) we check that \( T_1 \) as well as \( T_{IV} \) are everywhere defined bounded operators (use \( 0 < p < 1 \) while \( T_{II} \) is always unbounded (though satisfying \( \mathcal{D}(T_{II}^*T_{II}) = \mathcal{D}(T_{II}T_{II}^*) \)).

**Remark 12.** To relate [11] to \((O_{q,S})\) set \( \varepsilon = 1 \), \( p = \sqrt{q} \) and \( T = \sqrt{1 - p^{-2}S} \) when \( 0 < q < 1 \), and \( \varepsilon = -1, p^{-1} = \sqrt{q} \) and \( T = p^{-1} \sqrt{p^{-2} - 1}S^* \) when \( q > 1 \).

**Positive definiteness from \((O_{q,S})\).** The following formalism will be needed.

**Proposition 13.** If \( S \) satisfies \((O_{q,S})\) with \( \mathcal{D} \) being invariant for both \( S \) and \( S^* \), then

\[
S^{-i}S^jf = \sum_{k=0}^{\infty} |k_q|! \left[ \begin{array}{c} j \cr k \end{array} \right] S^{-j-k}S^jS^{*(j-k)}f, \quad f \in \mathcal{D}, \quad i, j = 0, 1, \ldots,
\]

\[\text{In this matter we have implications } 4^0 \text{ and } 5^0 \text{ on p. [11]}\]
If, moreover, \( C \geq 0 \) then
\[
\sum_{i,j=0}^{p} \langle S^i f_j, S^j f_i \rangle = \sum_{k=0}^{\infty} [k]_q! \left| \sum_{i=0}^{p} \left[ \frac{i}{k} \right] \right|^2 \left| S^{k/2} S^{(i-k)} f_i \right|^2, \quad f_0, \ldots, f_p \in \mathcal{D}. \tag{13}
\]

All this under convention \( S^l = (S^*)^l = 0 \) for \( l < 0 \) and \( \left[ \frac{i}{j} \right] = 0 \) for \( j > i \).

**Proof.** Formula (12) is in [6 formula (35)]. Formula (13) is an immediate consequence of (12). \( \Box \)

As a direct consequence of Fact A and (13) we get

**Corollary 14.** Suppose \( S \) satisfies \((O,q,\mathcal{D})\) with \( \mathcal{D} \) being invariant for \( S \) and \( S^* \) as well as \( \mathcal{D} \) is a core of \( S \). If \( C \geq 0 \), then
\[
\sum_{i,j=0}^{p} \langle S^i f_j, S^j f_i \rangle \geq 0, \quad f_0, \ldots, f_p \in \mathcal{D}. \tag{PD}
\]

**A useful Lemma.**

**Lemma 15.** Let \( q > 0 \). Consider following conditions:

(a) \( S \) satisfies \((O,q,\mathcal{D})\) and \( \mathcal{D}(\tilde{S}) = \mathcal{D}(S^*) \);

(b) \( \mathcal{N}(S^*) \neq \{0\} \) and for \( n = 0, 1, \ldots \),
\[
f \in \mathcal{N}(S^*) \implies \tilde{S}^n f \in \mathcal{D}(\tilde{S}), \quad \tilde{S}^{(n-1)} f \in \mathcal{D}(S^*) \lor S^* \tilde{S}^{n-1} f = (n-1)\tilde{S}^{n-2} f; \tag{14}
\]

(c) there is \( f \neq 0 \) such that \( \tilde{S}^n f \in \mathcal{D}(\tilde{S}), n = 0, 1, \ldots \) and \( \tilde{S}^n f \perp \tilde{S}^m f \) for \( m \neq n \).

Then (a) \( \implies \) (b) \( \implies \) (c).

**Proof.** (a) \( \implies \) (b). The polar decomposition for \( S^* \) is \( S^* = V[S^*] \) where \( V \) is a partial isometry with the initial space \( \mathcal{R}(S^*) \) and the final space \( \mathcal{R}(S S^*) \). Suppose \( \mathcal{N}(S^*) = \{0\} \). Then, because \( \mathcal{N}(V) = R([S^*]) = \mathcal{N}(S^*) \), \( V \) is unitary. Since \( \tilde{S} = [S^*]V^* \), from \( S^n \) we get \( V[S^*]^2 V^* = q[S^*]^2 + I \). Consequently, for the spectra we have \( sp(S^*) \subset q sp([S^*]) + 1 \subset [0, +\infty) \) which is an absurd. Thus \( \mathcal{N}(S^*) \neq \{0\} \).

We show (14) by induction. Of course, \( \mathcal{N}(S^*) \subset \mathcal{D}(\tilde{S}) = \mathcal{D}(S^*) \), which establishes (14) for \( n = 0 \). Suppose \( \mathcal{N}(S^*) \subset \mathcal{D}(\tilde{S}^n) \) and \( S^* \tilde{S}^{n-1} f = (n-1)\tilde{S}^{n-2} f \). Then, for \( g \in \mathcal{D}(\tilde{S}) = \mathcal{D}(S^*) \),
\[
\langle S^* \tilde{S}^{n-1} f, S^* g \rangle = (n-1)\langle \tilde{S}^{n-2} f, \tilde{S} g \rangle. \tag{15}
\]
Because already \( \tilde{S}^{(n-2)} f \in \mathcal{D}(\tilde{S}) = \mathcal{D}(S^*) \), we have
\[
\|\langle S^* \tilde{S}^{n-1} f, S^* g \rangle \leq C\|g\|. \tag{16}
\]
Because \( \tilde{S}^{(n-1)} f \in \mathcal{D}(\tilde{S}) = \mathcal{D}(S^*) \), we can use \((O,q,\mathcal{D})\) so as to get
\[
\langle S^n f, \tilde{S} g \rangle = \langle \tilde{S} S^{(n-1)} f, \tilde{S} g \rangle = \langle S^* S^{(n-1)} f, S^* g \rangle. \tag{14}
\]
This, by (16), implies \( \tilde{S}^n f \in \mathcal{D}(S^*) \) and, consequently, by (15), gives us \( S^* \tilde{S}^n f = n\tilde{S}^{n-1} f \) which completes the induction argument. Now a straightforward application of (14) gives \( \tilde{S}^n(\mathcal{N}(S^*)) \subset \mathcal{D}(\tilde{S}) \cap \mathcal{D}(S^*) \) for \( n = 0, 1, \ldots \).

(b) \( \implies \) (c). Take any \( f \in \mathcal{N}(S^*) \) and using (14) and (12) write
\[
\langle S^m f, S^n f \rangle = \sum_{k=0}^{\min(m,n)} [k]_q! \left[ \frac{m}{k} \right] \left[ \frac{n}{k} \right] \left[ k \right]_q \langle S^{(m-k)} C^k S^{(n-k)} f, f \rangle = 0, \quad m > n.
\]
\( \Box \)
A matrix formation. Suppose \( q > 0 \) and \( S \) is a weighted shift with respect to \( (e_k)_{k=0}^\infty \) with the weights \( \sqrt{[1 + 1/y]_{k=0}^\infty} \). With
\[
S_0 \defeq S, \quad S_n \defeq q^{n/2} S, \quad D_n \defeq \sqrt{n} \diag(q^{k/2})_{k=0}^\infty, \quad n = 1, 2 \ldots
\] (17)
the matrix
\[
\begin{pmatrix}
S_0 & D_1 & 0 & 0 \\
0 & S_1 & D_2 & 0 \\
0 & 0 & S_2 & D_3 \\
& & & \ddots
\end{pmatrix}
\] (18)
defines an operator \( N \) in \( \bigoplus_{n=0}^\infty \mathcal{H}_n \) with domain comprised of all those \( \bigoplus_{n=0}^\infty f_n \) for which \( f_n = 0 \) but a finite number of \( n \)'s. This matrix, for the familiar creation operator was set out in \([21]\).

First we need to determine \( \mathcal{D}(N^*) \) and relate it to \( \mathcal{D}(N) \). If \( 0 < q < 1 \) then each \( D_n \) is bounded. In that case Remark 9 in \([20]\) gives us
\[
\mathcal{D}(N^*) = \bigoplus_{n=0}^\infty \mathcal{D}(S_n^*).
\] (19)
If \( q > 1 \) then each \( S_nD_n^{-1} \) is bounded. According to Proposition 4.5 in \([11]\) and Corollary 8 in \([20]\) we can deduce \([19]\) as well. In either case, what we get is the adjoint of \( N \) can be taken as a matrix of adjoints (which is rather an exceptional case). Because the same argument concerning the adjoint of a matrix operator applies now to \( N^* \) we can assert that the closure operation for the operator \( N \) goes entrywise as well. Now, due to the fact that the apparent norm equality for \( N \) and \( N^* \) holds on \( \mathcal{D}(N) \), we get essential normality of \( N \). Consequently,
\[
S \text{ is subnormal and } \bar{N} \text{ is its tight and } \ast\text{-tight normal extension.} \quad (20)
\]

Subnormality in the \( q \)-oscillator

The case of \( S \) bounded. The next says a little bit more about boundedness of solutions of \((O_{q,2})\).

**Proposition 16.** Suppose \( S \) is bounded and satisfies \((O_{q,2})\). (a) If \( q < 0 \) then \( \|S\| \geq (1 - q)^{-1/2} \). (b) If \( 0 \leq q < 1 \) then \( \|S\| \leq (1 - q)^{-1/2} \). (c) If \( q \geq 1 \) then no such an \( S \) exists.

**Proof.** For (a) look at \( \|Sf\|^2 = |f|^2 + q|S^* f|^2 \geq |f|^2 + q\|S\|^2\|f\|^2 \), for (b) do at \( \|Sf\|^2 = |f|^2 + q|S^* f|^2 \leq |f|^2 + q\|S\|^2\|f\|^2 \). For (c) write \( \|Sf\|^2 = |f|^2 + q\|S^* f\|^2 \geq q\|S\|^2\|f\|^2 \) which gives \( 1 \geq q \). The case of \( q = 1 \) is excluded by the well known result of Winter.

**The case of \( q < 0 \).** Here we get at once

**Corollary 17.** For \( q < 0 \) the only bounded operator \( S \) with norm \( \|S\| = (1 - q)^{-1/2} \) satisfying \((O_{q,2})\) is that given by \([10]\).

**Proof.** By Proposition16(a) and Proposition5(b) \( S^* |_{\Omega} \) is hyponormal. On the other hand, by Proposition5(b) and (c) \( S |_{\Omega} \) is hyponormal too. Proposition6 makes the conclusion.
Pauli matrices, which are neither hyponormal nor cohyponormal\(^4\), provide an example of operators satisfying \((O_{-1,\text{op}})\) with norm \(1 > 2^{-1/2} = (1 - q)^{-1/2}\). Are there bounded operators satisfying \((O_{q,\text{op}})\) with norm not to be equal \((1 - q)^{-1/2}\) for arbitrary \(q < 0\), different from \(-1\) say?

The case of \(0 \leq q < 1\). We list two results which hold in this case

**Proposition 18.** Suppose \(S\) satisfies \((O_{q,D})\) with \(D\) dense in \(\mathcal{H}\). If \(0 \leq q < 1\), then the following facts are equivalent

(i) \(S\) is bounded and \(\|S\| \leq (1 - q)^{-1/2}\);

(ii) \(S\) is bounded;

(iii) \(S\) is subnormal;

(iv) \(S\) is hyponormal.

**Proof.** Because of conclusion (a) of Proposition\(^4\) the only remaining implication to argue for is (ii) \(\Rightarrow\) (iii). But, in virtue of (13), this comes out from the Halmos-Bram characterization\(^4\) of subnormality of bounded operators. \(\square\)

**Theorem 19.** If \(0 \leq q < 1\), then the following facts are equivalent

(i) there is an orthonormal basis \((e_n)_{n=0}^{\infty}\) in \(\mathcal{H}\) such that \(Se_n = \sqrt{n+1}q^n e_{n+1}\), \(n = 0, 1, \ldots\);

(ii) \(S\) is irreducible\(^5\) satisfies \((O_{q,D})\) with some \(D\) dense in \(\mathcal{H}\), is bounded and \(\|S\| = (1 - q)^{-1/2}\);

(iii) \(S\) is irreducible, satisfies \((O_{q,D})\) with some \(D\) dense in \(\mathcal{H}\), is bounded and \(\|S\| \leq (1 - q)^{-1/2}\);

(iv) \(S\) is irreducible, satisfies \((O_{q,D})\) with some \(D\) dense in \(\mathcal{H}\) and is bounded;

(v) \(S\) is irreducible, satisfies with some \(D\) dense in \(\mathcal{H}(O_{q,D})\) and is subnormal;

(vi) \(S\) is irreducible, satisfies \((O_{q,D})\) with some \(D\) dense in \(\mathcal{H}\) and is hyponormal.

**Proof.** Proposition\(^18\) establishes the equivalence of (ii) up to (vi). Because \(\sup \{\sqrt{n+1}q^n; n \geq 0\} = (1 - q)^{-1}\) and for \(S\) as being a weighted shift \(\|S\| = \sup \{\sqrt{n+1}q^n; n \geq 0\}\), we get (i) \(\Rightarrow\) (ii).

Assume (iv). Because \(\mathcal{D}(S) = \mathcal{D}(S^*)\), condition (c) of Lemma\(^15\) let calculate the weights of \(S\) while starting with \(e_0 \in \mathcal{N}(N^*)\). Because \(S\) is irreducible the sequence \((e_n)_{n=0}^{\infty}\) is complete. This establishes (i). \(\square\)

**Remark 20.** From Theorem\(^19\) and Example\(^8\) we get that there are two, of different nature, solutions of \((O_{q,D})\). Is there any other at all?

The case of \(q > 1\). No bounded solution exits at all, cf. Proposition\(^16\) part (c).

Let us memorize what is known already in the bounded case by the following tableau.

\(^4\) An operator \(A\) is said to be cohyponormal if \(A^*\) is hyponormal; for unbounded \(A\) this may not be the same as \(A^* \mid_{\mathcal{D}(A)}\) to be hyponormal.

\(^5\) Let us recall relevant definitions: a subspace \(D \subset \mathcal{D}(A)\) is invariant for \(A\) if \(AD \subset D; A|_D\) stands for the restriction of \(A\) to \(D\). On the other hand, a closed subspace \(L\) is invariant for \(A\) if \(A(L \cap \mathcal{D}(A)) \subset \mathcal{D}(A); \) then the restriction \(A|_L \equiv \mathcal{A}|_L / \mathcal{D}(A)\). A step further, a closed subspace \(L\) reduces an operator \(A\) if both \(L\) and \(L^\perp\) are invariant for \(A\) as well as \(PD(A) \subset \mathcal{D}(A)\), where \(P\) is the orthogonal projection of \(\mathcal{H}\) onto \(L\); all this is the same as to require \(PA \subset AP\). Then the restriction \(A|_L\) is called a part of \(A\) in \(L\). \(A\) is irreducible if it has no nontrivial reducing subspace. Comparing to the more familiar case of bounded operators some nuances become requisite here. Therefore, if \(L\) reduces \(A\), then \(A|_L = A|_L\) and \((A|_L)^* = A^*|_L\).
The case of $S$ unbounded.

The case of $q < 0$. There is no hope to look for subnormal solutions of $(O_{q,\text{op}})$ among weighted shifts, neither one- nor two-sided.

The only one-sided weighted shifts satisfying $(O_{q,\text{op}})$ are for $-1 < q < 0$ and they are given as in (i) of Theorem[19]. They are apparently not hyponormal (their weights are not increasing).

The only two-sided weighted shifts which satisfy $(O_{q,\text{op}})$ are those of Example[7]. They are normal bilateral weighted shifts. So if there are subnormal operators satisfying $(O_{q,\text{op}})$ they must not be weighted shifts or bounded operators of norm less or equal $(1 - q)^{-1/2}$, cf. Corollary[17].

The case of $0 \leq q < 1$. Lemma[10] does not leave any hope subnormal solutions different than those in Theorem[19] but they must necessarily be bounded.

The case of $q \geq 1$. This is the right case for unbounded solutions to exist.

**Theorem 21.** For a densely defined closable operator $S$ in a complex Hilbert space $\mathcal{H}$ consider the following conditions

(i) $\mathcal{H}$ is separable and there is an orthonormal basis in it of the form $\{e_n\}_{n=0}^\infty$ contained in $\mathcal{D}(\tilde{S})$ and such that

$$\tilde{S} e_n = \sqrt{[n+1]_q} e_{n+1}, \quad n = 0, 1, \ldots; \quad (21)$$

(ii) $S$ is irreducible, satisfies $(O_{q,\mathcal{D}})$ with some $\mathcal{D}$ being invariant for $S$ and $S^*$ and being a core of $S$, and $S$ is a subnormal operator having a tight and $*$-tight normal extension;

(iii) $S$ is irreducible, satisfies $(O_{q,\mathcal{D}})$ with some $\mathcal{D}$ being a core of both $S$ and $S^*$;

(iv) $S$ is irreducible, satisfies $(O_{q,\mathcal{D}})$ and $\mathcal{D}(\tilde{S}) = \mathcal{D}(S^*)$;

(v) $S$ is irreducible, satisfies $(O_{q,\mathcal{D}})$ with $\mathcal{D}(\tilde{S}) \cap \mathcal{D}(S^*)$ being dense in $\mathcal{H}$, $\mathcal{N}(S^*) \neq \{0\}$ and $\tilde{S}^n(\mathcal{N}(S^*)) \subset \mathcal{D}(\tilde{S}) \cap \mathcal{D}(S^*)$ for $n = 0, 1, \ldots$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

Proof. The implication (i) $\Rightarrow$ (ii) comes out from (20). Proposition[1] leads us from (ii) to (iii), from there using Lemma[15] comes it up to (v). Now, like in the proof of Theorem[19] calculating the weights rounds up the chain of implications.
Now we visualize this section findings in the following tableau.

|       | $q < 0$ | $0 \leq q < 1$ | $1 \leq q$ |
|-------|---------|----------------|-------------|
| normal | general | NONE           | Prop. 6     |
|       |         | unilat. shift  | Prop. 3(b)  |
| subnormal | bilat. shift | NONE           | NONE        |
|       |         | Prop. 3(a)     | Prop. 4(a)  |
| others |         |                | MAY         |
| hyponormal | bilat. shift | NONE           | Prop. 4(b)  |
| others |         |                |             |

**The $q$ oscillator: models in RKHS**

*A general look at.* A reproducing kernel Hilbert space $\mathcal{H}$ and its kernel $K$ which suits our considerations is of the form

$$K(z, w) \overset{\text{def}}{=} \sum_{n=0}^{+\infty} c_n z^n w^n, \quad z, w \in D, \quad D = \mathbb{C} \text{ or } D = \{z; |z| < R \leq 1\}. \quad (22)$$

Notice $(\sqrt[n]{c_n Z^n})_{n=0}^{+\infty}$ is an orthonormal basis of $\mathcal{H}$.

The following fact comes out, as a byproduct, from some general results on subnormality in [16]; we give here an *ad hoc* argument. Let us make a shorthand notation

$$\mathcal{H} \subset L^2(\mathbb{C}, \mu) \text{ isometrically.} \quad (23)$$

**Proposition 22.** There is a measure $\mu$ such that (23) holds if and only if there is a Stieltjes moment sequence $(a_n)_{n=0}^{+\infty}$ such that

$$a_{2n} = c_n^{-1}, \quad n = 0, 1, \ldots \quad (24)$$

If this happens than a measure $\mu$ can be chosen to be rotationally invariant\(^{6}\) that is such that $\mu(e^{i\theta} \sigma) = \mu(\sigma)$ for all $\theta$’s and $\sigma$’s.

Proof. Suppose (23) to hold. Because $(\sqrt[n]{c_n Z^n})_{n=0}^{+\infty}$ is an orthonormal sequence in $L^2(\mathbb{C}, \mu)$, we have

$$c_n^{-1} = \int_{\mathbb{C}} |z|^{2n} \mu(dz), \quad n = 0, 1, \ldots$$

Let $m_\mu$ be the measure on $[0, +\infty)$ transported from $\mu$ via the mapping $\mathbb{C} \ni z \rightarrow |z| \in [0, +\infty)$. Then

$$a_n \overset{\text{def}}{=} \int_0^{+\infty} r^n m_\mu(dr) = \int_{\mathbb{C}} |z|^n \mu(dz), \quad n = 0, 1, \ldots \quad (25)$$

satisfies (24) as well as the sequence $(a_n)_{n=0}^{+\infty}$ is a Stieltjes moment sequence.

\(^6\) Or radial as some authors say.
If \((a_n)_{n=0}^{\infty}\) is any Stieltjes moment sequence with a representing measure \(m\) and satisfying \((24)\) then the rotationally invariant measure
\[
\mu(\sigma) = (2\pi)^{-1} \int_0^{+\infty} \int_0^{2\pi} \chi_\sigma(r e^{i\theta}) m(\theta) \, d\theta,
\]
\(\sigma\) Borel subset of \(\mathbb{C}\)
\[(26)\]
makes the imbedding \((23)\) happen. \(\Box\)

**Theorem 23.** Under the circumstances of Proposition \(22\) there exists a not rotationally invariant measure \(\mu\) such that \((23)\) holds if and only if there is a sequence \((a_n)_{n=0}^{\infty}\) satisfying \((24)\) which is not Stieltjes determinate.

**Proof.** Suppose \((26)\) with \(\mu\) not rotationally invariant and define \((a_n)_{n=0}^{\infty}\) as in \((25)\). Thus there is and \(s \in \mathbb{R}\) such that \(\mu(\tau) \neq \mu(e^{i\theta} \tau)\) for some subset \(\tau\) of \(\mathbb{C}\); make \(\tau\) maximal closed with respect to this property. Let \(\nu\) be a measure on \(\mathbb{C}\) transported from \(\mu\) via the rotation \(z \to e^{-i\theta} z\) and let \(m_\nu\) be the the measure on \([0, +\infty)\) constructed from \(\nu\) in the way \(m_\mu\) was from \(\mu\), cf. \((25)\). Because, what is a matter of straightforward calculation, \(m_\mu\) and \(m_\nu\) differ on \([|z|: z \in \tau]\), we get indeterminacy of \((a_n)_{n=0}^{\infty}\) at once.

The other way around, if \(m_1\) and \(m_2\) are two different measures on \([0, +\infty)\) representing the Stieltjes moment sequence \((a_n)_{n=0}^{\infty}\) satisfying \((24)\), then the measure \(\mu\) on \(\mathbb{C}\) defined by
\[
\mu(\sigma) = (2\pi)^{-1} s \int_0^{+\infty} \int_0^{2\pi} \chi_\sigma(r e^{i\theta}) m_1(\theta) + (1 - s) \int_0^{+\infty} \int_0^{2\pi} \chi_\sigma(r e^{i\theta}) (sm_2(\theta)) \, d\theta,
\]
\(\sigma\) Borel subset of \(\mathbb{C}\), \(0 < s < 1\), \(0 < a < 2\pi\)
is not rotationally invariant while still \((23)\) is maintained. \(\Box\)

**Résumé.** Define two linear operators \(M\) and \(D_q\) acting on functions
\[
(Mf)(z) \overset{\text{def}}{=} zf(z), \quad (D_q f)(z) \overset{\text{def}}{=} \left\{ \begin{array}{ll} f(z) & \text{if } q \neq 1 \\ f'(z) & \text{if } q = 1. \end{array} \right.
\]
\[(27)\]
It turns out that for \(a_+ = M\) and \(a_- = D_q\) the commutation relation \((1)\) is always satisfied. What Bargmann did in \([3]\) was to find, for \(q = 1\), a Hilbert space of entire functions such that \(M\) and \(D_1\) are formally adjoint. This for arbitrary \(q > 0\) leads to the reproducing kernel Hilbert space \(\mathcal{H}_q\) of analytic functions with the kernel
\[
K(z, w) \overset{\text{def}}{=} e_q((1 - q)z \bar{w}), \quad z, w \in |1 - q|^{-1/2} \omega_q
\]
where
\[
\omega_q = \begin{cases} \{z: |z| < 1\} & \text{if } 0 < q < 1 \\ \mathbb{C} & \text{if } q > 1 \end{cases}
\]
Under these circumstances we always have
\[
\langle Z^n, Z^m \rangle_{\mathcal{H}_q} = \delta_{m,n} [m]_q!
\]
and the operator \(S = M\) act as a weighted shift with the weights \(\sqrt{|n + 1|_q}\) as in Sample Theorem on p.\[1\]

Our keynote, subnormality of \(M\) now means precisely \((23)\) with some \(\mu\) is retained. Here we have three qualitatively different situations:
(a) for \(0 < q < 1\) the multiplication operator \(M\) is bounded and subnormal, this implies uniqueness of \(\mu\);
(b) for \(q = 1\) the multiplication operator is unbounded and subnormal, it has a normal extension of cyclic type in the sense of \([17]\) and consequently \(\mu\) is uniquely determined as well;
(c) for \( q > 1 \) the multiplication operator is unbounded and subnormal, it has no normal extension of cyclic type in the sense of [17] though it does plenty of those of spectral type in the sense of [17], which are not unitary equivalent [1].

That is, there is no unitary map between the \( L^2 \) spaces in question, which is the identity on \( H_q \).

The author's afterword. The fundamentals of this paper have been presented on several occasions for the last couple of years, recently at the Będlewo 9th Workshop Non-commutative Harmonic Analysis with Applications to Probability. It was Marek Bożejko’s contagious enthusiasm what catalysed converting at long last my distracted notes into a cohesive exposition.

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