ON DISTRIBUTIONAL ADJUGATE AND DERIVATIVE OF THE INVERSE

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Abstract. Let $\Omega \subset \mathbb{R}^3$ be a domain and let $f \in BV_{\text{loc}}(\Omega, \mathbb{R}^3)$ be a homeomorphism such that its distributional adjugate $\text{Adj} \, Df$ is a finite Radon measure. Very recently in [14] it was shown that its inverse has bounded variation $f^{-1} \in BV_{\text{loc}}$. In the present paper we show that the components of $\text{Adj} \, Df$ are equal to components of $Df^{-1}(f(U))$ as measures and that the absolutely continuous part of the distributional adjugate $\text{Adj} \, Df$ equals to the pointwise adjugate $\text{adj} \, Df(x)$ a.e. We also show the equivalence of two approaches to the definition of the distributional adjugate.

1. Introduction

Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and let $f: \Omega \rightarrow f(\Omega) \subset \mathbb{R}^n$ be a homeomorphism. In this paper we study the weak differentiability of the inverse of a Sobolev or $BV$-homeomorphism. This problem is of particular importance as Sobolev and $BV$ spaces are commonly used as initial spaces for existence problems in PDE’s and the calculus of variations. For instance, elasticity is a typical field where both invertibility problems and Sobolev (or $BV$) regularity issues are relevant (see e.g. [2], [3] and [20]).

The problem of the weak regularity of the inverse has attracted a big attention in the past decade. It started with the result of [15], [17] and [3] where it was shown that for homeomorphisms we have

$$\begin{align*}
(f \in BV_{\text{loc}}(\Omega, \mathbb{R}^2) \Rightarrow f^{-1} \in BV_{\text{loc}}(f(\Omega), \mathbb{R}^2))
\text{and (} f \in W_{\text{loc}}^{n-1}(\Omega, \mathbb{R}^n) \Rightarrow f^{-1} \in BV_{\text{loc}}(f(\Omega), \mathbb{R}^n)).
\end{align*}$$

Moreover, it was shown there that these results are sharp in the scale of Sobolev spaces and moreover under additional assumption one can prove that even $f^{-1} \in W^{1,1}$.

By results of [7] and [11] we know that for $f \in W^{1,n-1}$ we have not only $f^{-1} \in BV$ but also the total variation of the inverse satisfies

$$\begin{align*}
|Df^{-1}|(f(\Omega)) = \int_{\Omega} |\text{adj} \, Df(x)| \, dx
\end{align*}$$

the where $\text{adj} \, A$ denotes the adjugate matrix to $A$, i.e. the matrix of $(n-1) \times (n-1)$ subdeterminants arranged in such a way that

$$A \text{adj} \, A = I \det A.$$

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This indicates that the adjugate of $Df$ could be significant also for the problem of existence of $Df^{-1}$. One could think that the integrability of $|Df|^{n-1}$ in (1.1) is good only to guarantee the integrability of $\text{adj } Df$. However, for $n \geq 3$ it is possible to construct a $W^{1,1}$ homeomorphism with $\text{adj } Df \in L^1$ such that $f^{-1} \notin BV$ (see [13]).

The problem of characterization of the $BV$-regularity of the inverse demands distributional approach to the adjugate of the gradient matrix. We use the symbols $\text{Adj } Df$, $\text{ADJ } Df$ and $\text{ADJ } Df$ for various versions of this concept. The definitions of the distributional adjugate are presented and compared in Section 4. In fact, we show that they are equivalent within the class of measures.

The distributional approach has been successfully used in [14] to find a necessary and sufficient condition for the regularity of the inverse for 3-dimensional $BV$ homeomorphisms:

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^3$ be a domain and $f \in BV(\Omega, \mathbb{R}^3)$ be a homeomorphism. Then

$$
\text{ADJ } Df \in \mathcal{M}(\Omega, \mathbb{R}^{3\times 3}) \iff f^{-1} \in BV(f(\Omega), \mathbb{R}^3).
$$

Here $\mathcal{M}$ stands for the class of all finite (possibly signed or vector-valued) Radon measures and $BV(\Omega, \mathbb{R}^3)$ is the homogeneous $BV$ space, namely, the class of all $BV_{\text{loc}}$ mappings $f$ on $\Omega$ such that the total variation of $Df$ is finite, whereas the global integrability of $f$ is not required. Note that the integrability of $f$ is an issue only if $\Omega$ or $f(\Omega)$ is unbounded.

The classical inverse mapping theorem states that the formula

$$
\nabla f^{-1}(f(x)) J_f(x) = \text{adj } \nabla f(x)
$$

holds for $f$ if $f$ is a regular $C^1$ mapping. Our main goal is to show that a similar identity holds also under the assumptions of previous theorem. In the planar case, the corresponding formula has been proved by D’Onofrio, Malý, Sbordone and Schiattarella [8].

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^3$ be a domain and $f \in BV(\Omega, \mathbb{R}^3)$ be a homeomorphism such that $\text{Adj } Df \in \mathcal{M}(\Omega, \mathbb{R}^{3\times 3})$. Then $f^{-1} \in BV(f(\Omega), \mathbb{R}^3)$ and

$$(\text{Adj}_{ij} Df)(U) = (D_j(f^{-1})_i)(f(U)) \text{ for all open sets } U \subset \Omega \text{ and } i, j \in \{1, 2, 3\}.$$ 

As a corollary of our results we show that in some cases it is possible to verify the somewhat technical assumption $\text{Adj } Df \in \mathcal{M}$ easily using coordinate functions $f = (f_1, f_2, f_3)$.

**Corollary 1.3.** Let $\Omega \subset \mathbb{R}^3$ be a domain and $f \in BV(\Omega, \mathbb{R}^3)$ be a continuous mapping. Assume that

(a) $f_i \in W^{1,p_i}(\Omega)$ where $p_1, p_2, p_3 \in [1, \infty]$ and $\frac{1}{p_i} + \frac{1}{p_j} \leq 1$ for each distinct $i, j$ (with the convention $\frac{1}{\infty} = 0$),

or that

(b) at least two coordinates of $f$ are in $C^1(\Omega)$.

Then $\text{Adj } Df \in \mathcal{M}(\Omega, \mathbb{R}^{3\times 3})$. Therefore, $f^{-1} \in BV(f(\Omega), \mathbb{R}^3)$ if $f$ is a homeomorphism.

It is known that the absolutely continuous part of the distributional Jacobian equals to the pointwise Jacobian a.e. (see De Lellis [6] Lemma 4.7) and Müller [19] for nice enough $f$. Similar statement holds also for the distributional adjugate.
Theorem 1.4. Let $\Omega \subset \mathbb{R}^3$ be a domain and $f \in BV(\Omega, \mathbb{R}^3)$ be a continuous mapping such that $\text{Adj} Df \in \mathcal{M}(\Omega, \mathbb{R}^{3 \times 3})$. Then the absolutely continuous part of $\text{Adj} Df$ (with respect to Lebesgue measure) equals to the pointwise adjugate $\text{adj} Df(x)$ for a.e. $x \in \Omega$.

2. Preliminaries

For a domain $\Omega \subset \mathbb{R}^n$ we denote by $\mathcal{D}(\Omega)$ those smooth functions $\varphi$ whose support is compactly contained in $\Omega$, i.e. supp $\varphi \subset \subset \Omega$.

Given a distribution $T$ on an open set $\Omega$, the action of $T$ on a test function $\varphi \in \mathcal{D}(\Omega)$ is denoted by $\langle T, \varphi \rangle$. This can be extended to more general test functions according to the quality of $T$, for example, to $T$-integrable test functions if $T$ is a measure.

The total variation of an $\mathbb{R}^n$-valued Radon measure $\mu$ is the measure $|\mu|$ such that

$$\langle |\mu|, \psi \rangle := \sup \left\{ \int_{\mathbb{R}^n} \varphi \cdot d\mu : \varphi \in C_0(A; \mathbb{R}^n), \ |\varphi| \leq \psi \right\}, \quad \psi \in C^+_0(\mathbb{R}^n).$$

Given a vector $v \in \mathbb{R}^2$, we use the notation $*v$ for the Hodge star of $v$, i.e. the rotation of $v$ by $\pi/2$ to the left, so that

$$u \cdot (*v) = -\det(u, v), \quad v \in \mathbb{R}^2.$$

Given two vectors $u, v \in \mathbb{R}^3$ we denote by $u \times v$ their cross product, defined by the property

$$w \cdot (u \times v) = \det(w, u, v), \quad w \in \mathbb{R}^3.$$

2.1. Slicing of BV function. Let $f : \Omega \to \mathbb{R}^m$ be a BV function and $\varphi \in \mathcal{D}(\Omega)$. For simplicity we assume that $\Omega = (0, 1)^3$. Then

$$\langle D_1 f, \varphi \rangle = \int_{(0,1)^2} \langle D_1 f(\cdot, x_2, x_3), \varphi(\cdot, x_2, x_3) \rangle \, dx_2 \, dx_3$$

and

$$\langle |D_1 f|, \varphi \rangle = \int_{(0,1)^2} \langle |D_1 f(\cdot, x_2, x_3)|, \varphi(\cdot, x_2, x_3) \rangle \, dx_2 \, dx_3,$$

see e.g. [1] Theorem 3.103. Integrating with respect to $x_2$ we obtain

$$\langle D_1 f, \varphi \rangle = \int_{(0,1)} \langle D_1 f(\cdot, x_3), \varphi(\cdot, x_3) \rangle \, dx_3.$$ 

and

$$\langle |D_1 f|, \varphi \rangle = \int_{(0,1)} \langle |D_1 f(\cdot, x_3)|, \varphi(\cdot, x_3) \rangle \, dx_3.$$ 

Similarly we express $D_2 f$ by integration over $x_3$ (but not $D_3 f$). By approximation we observe that these identities can be extended to test functions $\varphi \in C_0(\Omega)$. 

2.2. **Topological degree.** For a bounded open set \( \Omega \subset \mathbb{R}^n \) and a given smooth map \( f : \overline{\Omega} \to \mathbb{R}^n \) we define the *topological degree* as
\[
\deg(f, \Omega, y_0) = \sum_{x \in \Omega \cap f^{-1}(y_0)} \text{sgn}(J_f(x))
\]
for a point \( y_0 \in \mathbb{R}^n \setminus f(\partial \Omega) \) if \( J_f(x) \neq 0 \) for each \( x \in f^{-1}(y_0) \). This definition can be extended to arbitrary continuous mappings and each point \( y_0 \notin f(\partial \Omega) \), see e.g. [10, Section 1.2] or [16, Chapter 3.2]. For our purposes the following property of the topological degree is crucial; see [10, Definition 1.18].

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( f : \overline{\Omega} \to \mathbb{R}^n \) be a continuous function. Then for any point \( y_0 \in \mathbb{R}^n \setminus f(\partial \Omega) \) and any continuous mapping \( g : \overline{\Omega} \to \mathbb{R}^n \) satisfying
\[
|f - g| \leq \text{dist}(y_0, f(\partial \Omega)) \quad \text{on } \partial \Omega
\]
we have \( \deg(f, \Omega, y_0) = \deg(g, \Omega, y_0) \).

Moreover, we need to use also degree composition formula see [21, Proposition IV.6.1].

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Let \( h : \overline{\Omega} \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \) be continuous function. Assume that \( y \notin g(h(\partial \Omega)) \) and let \( \Delta_i \) be the bounded connected components of \( \mathbb{R}^n \setminus h(\partial \Omega) \). Then
\[
\deg(g \circ h, \Omega, y) = \sum_i \deg(g, \Delta_i, y) \deg(h, \Omega, \Delta_i).
\]

2.3. **Hausdorff measure.** Given \( k \geq 0 \) we define
\[
\mathcal{H}^k(A) = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(A).
\]
where
\[
\mathcal{H}^k_\delta(A) = \inf \left\{ \alpha_k \sum_i \left( \frac{1}{2} \text{diam } A_i \right)^k : A \subset \bigcup_i A_i, \text{ diam } A_i \leq \delta \right\}, \quad 0 < \delta \leq \infty
\]
and
\[
\alpha_k = \frac{\pi^{k/2}}{\Gamma(1 + \frac{k}{2})},
\]
See e.g. [9].

2.4. **Degree formula.** Let \( h : \Omega \to \mathbb{R}^n \) be a \( C^1 \) smooth mapping. Then the change of variables formula
\[
(2.1) \quad \int_G v(h(x)) J_h(x) \, dx = \int_{\mathbb{R}^n} v(y) \deg(h, G, y) \, dy
\]
holds for each open set \( G \subset \subset \Omega \) and each measurable \( v : h(\Omega) \to [0, \infty) \).
2.5. Disintegration. Let $Q = (0, 1)^n$, $\mu \in \mathcal{M}(Q)$ and $\nu$ be a nonnegative finite Radon measure on $(0, 1)$. We denote the $k$-dimensional Lebesgue measure by $\lambda_k$. We still abbreviate “$\lambda_k$-a.e.” as “a.e.”. For Lebesgue decomposition of measures we refer to [1, Theorem 1.28].

A system $(\mu_t)_{t \in (0, 1)}$, where $\mu_t$ are signed Radon measures on $(0, 1)^{n-1}$, is called a disintegration of $\mu$ with respect to $\nu$ if

$$
(2.2) \quad \mu(A) = \int_0^1 \mu_t(A) \, d\nu(t), \quad A \subset Q \text{ Borel.}
$$

Note that this is equivalent to the validity of

$$
(2.3) \quad \int_Q \varphi(y, t) \, d\mu(x, t) = \int_0^1 \left( \int_{(0,1)^{n-1}} \varphi(y, t) \, d\mu_t(y) \right) \, d\nu(t)
$$

for each bounded Borel measurable disintegration $\varphi$ of $\lambda$ with respect to $\nu$.

**Theorem 2.3.** Let $\mu \in \mathcal{M}(Q)$. Then there exists a disintegration $(\mu_t)_{t \in (0, 1)}$ of $\mu$ with respect to $\nu$ as in (2.4)

$$
(2.4) \quad \nu : E \mapsto |\mu|((0, 1)^{n-1} \times E), \quad E \subset (0, 1) \text{ Borel.}
$$

Moreover, if $(\mu_t)$ and $(\sigma_t)$ are disintegrations of $\mu$ with respect to $\nu$, then $\mu_t = \sigma_t$ for $\nu$-a.e. $t \in (0, 1)$.

**Proof.** See e.g. [1, Theorem 2.28]. \(\square\)

**Corollary 2.4.** Let $\mu \in \mathcal{M}(Q)$. Let $(\mu_t)$ and $(\sigma_t)$ are disintegrations of $\mu$ with respect to the Lebesgue measure $\lambda_1$ on $(0, 1)$. Then $\mu_t = \sigma_t$ for a.e. $t \in (0, 1)$.

**Proof.** Let $\nu$ be as in (2.4), $\rho$ be the absolutely continuous part of $\mu_1$ with respect to $\nu$ and $a$ be the Radon-Nikodym derivative of $\rho$ with respect to $\nu$. Then there is a Borel set $E \subset (0, 1)$ such that $\nu(E) = 0$ and $\rho = \lambda_1$ on $(0, 1) \setminus E$. Then $(a(t)\mu_t)$ and $(a(t)\sigma_t)$ are disintegrations of $\mu$ with respect to $\nu$. By the uniqueness part of Theorem 2.3 we have $\mu_t = \sigma_t$ $\nu$-a.e. in $(0, 1)$, and by the absolute continuity, $\mu_t = \sigma_t$ $\rho$-a.e. in $(0, 1)$, which means $\mu_t = \sigma_t$ a.e. in $(0, 1) \setminus E$.

For each cube $M \subset (0, 1)^{n-1}$ and Borel set $E' \subset E$ we have

$$
\left| \int_{E'} \mu_t(M) \, dt \right| = |\mu(M \times E')| \leq \nu(E') = 0.
$$

It follows that $\mu_t(M) = 0$ for a.e. $t \in E$. The same argument shows that $\sigma_t(M) = 0$ for a.e. $t \in E$. We find a joint set $Z \subset E$ of $\lambda_1$-measure 0 such that $\mu_t(M) = 0 = \sigma_t(M)$ for $t \in E \setminus Z$ and each cube $M$ from a dense family of cubes in $[0, 1]^{n-1}$. It follows that $\mu_t = \sigma_t$ a.e. also in $E$. \(\square\)

**Remark 2.5.** Let $\mu \in \mathcal{M}(Q)$, $\nu$ be as in (2.4), $(\mu_t)$ be a disintegration of $\mu$ with respect to $\nu$ and $(|\mu|_t)$ be a disintegration of $|\mu|$ with respect to $\nu$. Then $|\mu|_t = |\mu_t|$ for $\nu$-a.e. $t \in (0, 1)$. Indeed, consider

$$
\sigma_t(M) = \int_M \vartheta(y, t) \, d|\mu|_t, \quad M \subset (0, 1)^{n-1} \text{ Borel},
$$

where $\vartheta = \frac{d\mu}{d|\mu|}$. Then the claim follows from the uniqueness part of Theorem 2.3.

Similar observation holds for the positive and negative parts of $\mu$.

If follows that $|\mu_t|((0, 1)^{n-1}) = 1$ for a.e. $t \in (0, 1)$. 

Lemma 2.6. Let \( Q = (0,1)^n \) and \( \mu \in \mathcal{M}(Q) \). Let \((\mu_t)_t\) be a disintegration of \( \mu \) with respect to \( \lambda_1 \). Let \( \mu_a \) be the absolutely continuous part of \( \mu \) with respect to \( \lambda_n \) and \((\mu_t)_a\) denote the absolutely continuous parts of \( \mu_t \), \( t \in (0,1) \), with respect to \( \lambda_{n-1} \). Then \((\mu_t)_a\) is a disintegration of \( \mu_a \) with respect to \( \lambda_1 \).

Proof. Let \( \mu_s \) be the singular part of \( \mu \) and \( g \) be a Borel-measurable representative of the Radon-Nikodym derivative of \( \mu_a \) with respect to \( \lambda_n \). Then there is a Borel set \( E \subset Q \) of measure zero such that

\[
\mu_s(A) = \mu(E \cap A), \quad A \subset Q \text{ Borel.}
\]

By the Fubini theorem, the set

\[
E_t = \{ y \in (0,1)^{n-1} : (y,t) \in E \}
\]

has \((n-1)\)-dimensional measure zero for almost every \( t \in (0,1) \). Set

\[
\tilde{E} = \{ (y,t) \in E : \lambda_{n-1}(E_t) = 0 \}.
\]

Then \( \tilde{E} \) can be used in place of \( E \) in (2.5). Set

\[
\sigma_t = (\sigma_t)_a + (\sigma_t)_s,
\]

where for each Borel set \( M \subset (0,1)^{n-1} \) we define

\[
(\sigma_t)_s(M) = \mu_t(M \cap \tilde{E}_t),
\]

\[
(\sigma_t)_a(M) = \int_M g(y,t) \, dy.
\]

Then for each \( t \in (0,1) \), \((\sigma_t)_a\) is absolutely continuous with respect to \( \lambda_{n-1} \) and \((\sigma_t)_s\) is singular with respect to \( \lambda_{n-1} \). It is easily seen that \((\sigma_t)_t\) is a disintegration of \( \mu \) with respect to \( \lambda_1 \) and thus by Corollary 2.4, \( \sigma_t = \mu_t \) for a.e. \( t \in (0,1) \). It follows that \((\mu_t)_a\) is a disintegration of \( \mu_a \) with respect to the Lebesgue measure on \( (0,1) \).

\[ \square \]

3. DISTRIBUTIONAL JACOBIAN

Let \( G \subset \mathbb{R}^2 \) be open and \( h \in \dot{BV}(G, \mathbb{R}^2) \) be continuous. The distributional Jacobian of \( h \) is the distribution

\[
\langle \text{Det} \, Dh, \varphi \rangle := - \langle Dh_2, *h_1 \nabla \varphi \rangle = \langle D_1h_2, h_1D_2\varphi \rangle - \langle D_2h_2, h_1D_1\varphi \rangle, \quad \varphi \in \mathcal{D}(G).
\]

We use also the symbol \( \mathcal{J}_h \) for \( \text{Det} \, Dh \). It is not difficult to show (see e.g. [16, Proposition 2.10]) that for smooth enough mappings \( h \in C^2 \) we have

\[
\mathcal{J}_h(\varphi) = \int_G \varphi(x) J_h(x) \, dx
\]

integrating by parts once and using interchangeability of second order derivatives. By approximation this extends to any \( h \in W^{1,2}(G, \mathbb{R}^2) \).
3.1. Two-dimensional degree and the Distributional Jacobian.

Lemma 3.1. Let $W \subset \mathbb{R}^2$ be a bounded open set and $g \in C(\overline{W}, \mathbb{R}^2) \cap BV(W, \mathbb{R}^2)$. Let $\eta \in \mathcal{D}(\mathbb{R}^2)$ have support in $\mathbb{R}^2 \setminus g(\partial W)$. Let $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^1$ mapping such that

$$\text{div } \Phi = \eta$$

and $\varphi \in \mathcal{D}(W)$ be such that $\{ \varphi \neq 1 \} \cap \{ \eta \circ g \neq 0 \} = \emptyset$. Then

$$\langle Dg_1, *(\Phi_2 \circ g) \nabla \varphi \rangle - \langle Dg_2, *(\Phi_1 \circ g) \nabla \varphi \rangle = \int_{\mathbb{R}^2} \eta(y) \deg(g, W, y).$$

Proof. If $g$ is smooth, we have by analogy of (3.2), (3.1) and (2.1)

$$\langle Dg_1, *(\Phi_2 \circ g) \nabla \varphi \rangle - \langle Dg_2, *(\Phi_1 \circ g) \nabla \varphi \rangle$$

$$= - \int_W (\Phi_2 \circ g) \det(\nabla g_1, \nabla \varphi) \, dx + \int_W (\Phi_1 \circ g) \det(\nabla g_2, \nabla \varphi) \, dx$$

$$= \int_W (D_2 \Phi_2 \circ g) \det(\nabla g_1, \nabla g_2) \varphi \, dx - \int_W (D_1 \Phi_1 \circ g) \det(\nabla g_2, \nabla g_1) \varphi \, dx$$

$$= \int_W \text{div } (\Phi(g(x))) J_g(x) \varphi(x) \, dx = \int_W \eta(g(x)) J_g(x) \, dx$$

$$= \int_{\mathbb{R}^2} \eta(y) \deg(g, W, y) \, dy.$$  

(3.3)

In the general case we approximate $g$ by standard mollifications $g^{(j)}$. The passage to the limit on the left of (3.3) is easy, as $\Phi \circ g^{(j)} \to \Phi \circ g$ uniformly and $Dg^{(j)} \to Dg$ weak$^*$ in measures. The passage on the right follows from the fact that $g^{(j)} \to g$ uniformly and $\eta$ has compact support in $\mathbb{R}^2 \setminus g(\partial W)$ (see Lemma 2.1). \hfill $\square$

Corollary 3.2. Let $W \subset \mathbb{R}^2$ be a bounded open set and $g \in C(\overline{W}, \mathbb{R}^2) \cap BV(W, \mathbb{R}^2)$. Let $\eta \in \mathcal{D}(\mathbb{R}^2)$ have support in $\mathbb{R}^2 \setminus g(\partial W)$. Let $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^1$ mapping such that

$$\text{div } \Phi = \eta$$

and $\varphi \in \mathcal{D}(W)$ be such that $\{ \varphi \neq 1 \} \cap \{ \eta \circ g \neq 0 \} = \emptyset$. Then

$$\left| \langle Dg_1, *(\Phi_2 \circ g) \nabla \varphi \rangle - \langle Dg_2, *(\Phi_1 \circ g) \nabla \varphi \rangle \right| \leq \int_{\mathbb{R}^2} |\deg(g, W, y)| \, dy.$$  

Lemma 3.3. Let $Q = Q(\bar{x}, r)$ be a square in $\mathbb{R}^2$ and $0 < \rho_j < r$, $\rho_j \not> r$. Let $g \in BV(Q, \mathbb{R}^2)$ be a continuous $BV$ mapping. Let $\eta_j \in \mathcal{D}(\mathbb{R}^2)$ have support in $\mathbb{R}^2 \setminus g(Q \setminus Q(\bar{x}, \rho_j))$. Let $\Phi^{(j)}: \mathbb{R}^2 \to \mathbb{R}^2$ be $C^1$ mappings such that

$$\text{div } \Phi^{(j)} = \eta_j$$

and $\varphi_j \in \mathcal{D}(Q)$ be such that $\varphi_j = 1$ on $Q(\bar{x}, \rho_j)$. Suppose that $J_g \in \mathcal{M}(Q)$ and

$$\Phi^{(j)}_1(y) \to y_1, \quad \Phi^{(j)}_2(y) \to 0 \text{ uniformly on } g(Q),$$

(3.4)

$$|\nabla \varphi_j| \leq \frac{C}{r - \rho_j}$$

(3.5)
and

\[
\limsup_{j \to \infty} \frac{|Dg|(Q \setminus Q(\bar{x}, \rho_j))}{r - \rho_j} < \infty.
\]

Then

\[
\lim_{j \to \infty} \left( \langle Dg_1, * (\Phi^{(j)}_2 \circ g) \nabla \varphi_j \rangle - \langle Dg_2, * (\Phi^{(j)}_1 \circ g) \nabla \varphi_j \rangle \right) = J_g(Q).
\]

**Proof.** Taking into account that (see (3.1))

\[-\langle Dg_2, * g_1 \nabla \varphi_j \rangle = \langle J_g, \varphi_j \rangle,
\]

we have

\[
\left| \left( \langle Dg_1, * (\Phi^{(j)}_2 \circ g) \nabla \varphi_j \rangle - \langle Dg_2, * (\Phi^{(j)}_1 \circ g) \nabla \varphi_j \rangle \right) - J_g(Q) \right|
\]

\[
\leq \left| \langle Dg_1, * (\Phi^{(j)}_2 \circ g) \nabla \varphi_j \rangle - \langle Dg_2, * (\Phi^{(j)}_1 \circ g - g_1) \nabla \varphi_j \rangle \right|
\]

\[+ \left| \langle J_g, \varphi_j \rangle - J_g(Q) \right| \to 0.
\]

The second term is easy, for the first on we use (3.4)–(3.6).

\[
\square
\]

**Lemma 3.4.** Let

\[
L(x) = -\frac{1}{2\pi} \log |x|, \quad K(x) = -\frac{1}{2\pi} \frac{x}{|x|^2}, \quad x \in \mathbb{R}^2 \setminus \{0\}.
\]

Then

\[
\text{div}(K \ast \psi) = \psi, \quad \psi \in \mathcal{D}(\mathbb{R}^n) \text{ supported in } B(0, R)
\]

and

\[
|K \ast \psi(x)| \leq CR^{1/2}\|\psi\|_{L^3(B(0,R))}, \quad x \in \mathbb{R}^2.
\]

**Proof.** Let \(u = L \ast \psi\) be the Newtonian (alias logarithmic) potential of \(\psi\). Then

\[
\text{div } K \ast \psi = -\Delta u = \psi.
\]

The estimate (3.8) follows from the H"older inequality as

\[
\|K\|_{L^{3/2}(B(x,R))} \leq CR^{1/2}.
\]

\[
\square
\]

**Theorem 3.5.** Let \(Q = Q(\bar{x}, r)\) be a square in \(\mathbb{R}^2\). Let \(g \in BV(Q, \mathbb{R}^2) \cap C(\overline{Q}, \mathbb{R}^2)\).

Suppose that

\[
|g(\partial Q)| = 0
\]

and

\[
s := \sup_{0 < \rho < r} \frac{|Dg|(Q \setminus Q(\bar{x}, \rho))}{r - \rho} < \infty.
\]

Then

\[
\int_{\mathbb{R}^2} \deg(g, Q, y) = J_g(Q).
\]
Proof. Let $B(0, R)$ be a ball containing $g(Q)$. Let $\eta$ be a smooth function with support in $B(0, R) \setminus g(\partial Q)$ such that $|\eta| \leq 1$. Set $\Phi = K \ast \eta$, where $K$ is as in (3.7). Then $\text{div} \Phi = \eta$ by Lemma 3.4. We find $\rho < r$ such that 
\[ \{ \eta \circ g \neq 0 \} \subset Q(\bar{x}, \rho) \]
and a test functions $\varphi \in \mathcal{D}(Q)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on $Q(\bar{x}, \rho)$ and 
\[ |\nabla \varphi| \leq \frac{C}{r - \rho}. \]

By Lemma 3.1 we have 
\[ \int_{\mathbb{R}^2} \eta(y) \deg(g, Q, y) \, dy = \langle Dg_1, *(\Phi_2 \circ g) \nabla \varphi \rangle - \langle Dg_2, *(\Phi_1 \circ g) \nabla \varphi \rangle, \]
and thus from (3.10) we infer that 
\[ \left| \int_{\mathbb{R}^2} \eta(y) \deg(g, Q, y) \, dy \right| \leq C \sup_{y \in \mathbb{R}^2} |\Phi(y)| \sup_{0 < \rho < r} \frac{|Dg|(Q \setminus Q(\bar{x}, \rho))}{r - \rho} \leq CsR^{1/2}. \]

Since this holds for all functions $\eta$ with the above listed properties, we deduce that 
\[ (3.11) \quad \int_{\mathbb{R}^2} |\deg(g, Q, y)| \, dy < \infty. \]

Now, let $\eta_0$ be a smooth function with compact support such that $0 \leq \eta_0 \leq 1$ and $\eta_0 = 1$ on a neighborhood of $g(\overline{Q})$. Consider a sequence $\eta_j$ of smooth functions such that $\eta_j = 0$ on a neighborhood of $g(\partial Q)$, $j = 1, 2, \ldots$, $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_0$ and $\eta_j \to \eta_0$ a.e. Let $K$ be as in (3.7). Set 
\[ \tilde{\Phi}^{(j)} = K \ast \eta_j, \]
\[ \Phi_1^{(j)}(y) = \tilde{\Phi}_1^{(j)}(y) - \tilde{\Phi}_1^{(0)}(y) + y_1, \]
\[ \Phi_2^{(j)}(y) = \tilde{\Phi}_2^{(j)}(y) - \tilde{\Phi}_2^{(0)}(y), \quad j = 1, 2, \ldots. \]

From Lemma 3.4 we obtain that 
\[ \text{div} \Phi^j = \eta_j \quad \text{in} \quad g(\overline{Q}) \]
(as it is $\eta_j - \eta_0 + 1$ in $\mathbb{R}^2$). Further, $\Phi_1^{(j)}(z) \to z_1$ and $\Phi_2^{(j)}(z) \to 0$ uniformly on $g(\overline{Q})$ as $\eta_j \to 0$ in $L^3(B(0, R))$. Next, we find $\rho_j \nearrow r$ such that 
\[ \{ \eta_j \circ g \neq 0 \} \subset Q(\bar{x}, \rho_j) \]
and test functions $\varphi_j \in \mathcal{D}(Q)$ such that $0 \leq \varphi_j \leq 1$, $\varphi_j = 1$ on $Q(\bar{x}, \rho_j)$ and 
\[ |\nabla \varphi_j| \leq \frac{C}{r - \rho_j}. \]

By Lemma 3.1 we have 
\[ \langle Dg_1, *(\Phi_2^{(j)} \circ g) \nabla \varphi_j \rangle - \langle Dg_2, *(\Phi_1^{(j)} \circ g) \nabla \varphi_j \rangle = \int_{\mathbb{R}^2} \eta_j(y) \deg(g, Q, y) \, dy \]
and passing to the limit as $j \to \infty$ we obtain 
\[ \mathcal{J}_g(Q) = \int_{\mathbb{R}^2} \deg(g, Q, y) \, dy. \]
Indeed, the passage to the limit on the left follows from Lemma 3.3 and the passage to the limit on the right is justified by (3.11).

\[\square\]

Remark 3.6. Since for continuous \( g \in \text{BV}(\Omega, \mathbb{R}^2) \), “almost every” square \( Q \subset \Omega \) satisfies (3.11), we have obtained an alternative proof of [14, Theorem 4.1].

4. On Various Definitions of Distributional Adjugate

Throughout this section, we use the symbol \( i' \) for the action of the cyclic permutation on \( i \), namely \( 1' = 2, 2' = 3, 3' = 1, i'' = (i')' \). Also, we use the maps

\[
\begin{align*}
\kappa_1^i(y) &= (t, y_1, y_2), \\
\kappa_2^i(y) &= (y_2, t, y_1), \\
\kappa_3^i(y) &= (y_1, y_2, t), \\
y & \in \mathbb{R}^2, \\
t & \in \mathbb{R}.
\end{align*}
\]

The following notion of the distributional adjugate has been introduced in [14].

Definition 4.1. Let \( f = (f_1, f_2, f_3) : \Omega \rightarrow \mathbb{R}^3 \) be a continuous \( \text{BV} \) mapping. The distributional adjugate of the first kind of \( f \) is defined as

\[
\langle \text{ADJ}_{ij} Df, \varphi \rangle = \int_{-\infty}^{\infty} \langle \text{Det}(D(f_{i'} \circ \kappa_1^i), D(f_{i''} \circ \kappa_1^i)), \varphi \circ \kappa_1^i \rangle \, dt, \quad \varphi \in \mathcal{D}(\Omega).
\]

Here the duality between \( \text{Det}(Df \circ \kappa_1^i) \) and \( \varphi \circ \kappa_1^i \) is considered on \( (\kappa_1^i)^{-1}(\Omega) = \{ x \in \Omega : x_i = t \} \).

We use the symbol \( \text{ADJ} Df \) for \( \text{ADJ}_{ij} Df \) if we know that the distributional Jacobians \( \text{Det}(D(f_{i'} \circ \kappa_1^i), D(f_{i''} \circ \kappa_1^i)) \) are signed Radon measures for a.e. \( t \) and all \( i, j \).

Following directly the idea of integration by parts (see [3.1]) we consider another approach to the distributional adjugate.

Definition 4.2. Let \( f = (f_1, f_2, f_3) : \Omega \rightarrow \mathbb{R}^3 \) be a continuous \( \text{BV} \) mapping. The distributional adjugate of the second kind of \( f \) is defined as

\[
\langle \text{Adj}_{ij} Df, \varphi \rangle = \langle D_{i'} f_{i''} D_{i'} \varphi, f_{i'} D_{i'} \varphi \rangle - \langle D_{i''} f_{i''}, f_{i'} D_{i'} \varphi \rangle, \quad \varphi \in \mathcal{D}(\Omega).
\]

Proposition 4.3. Let \( f \in \text{BV}(\Omega, \mathbb{R}^3) \) be a continuous mapping, \( i, j \in \{1, 2, 3\} \) Then

\[
\text{ADJ}_{ij} Df = \text{Adj}_{ij} Df.
\]

If \( \text{ADJ}_{ij} Df \in \mathcal{M}(\Omega) \), then for almost every \( t \in \mathbb{R} \) it holds that the distribution \( \delta_i := \text{Det}(D(f_{i'} \circ \kappa_1^i), D(f_{i''} \circ \kappa_1^i)) \) is a signed Radon measure on \( \Omega_i := (\kappa_1^i)^{-1}(\Omega) \) and the function

\[
t \mapsto |\delta_i|(\Omega_i)
\]

is Lebesgue integrable.

Therefore, \( \text{Adj} Df = \text{ADJ} Df = \text{ADJ} Df \) if \( \text{ADJ} Df \in \mathcal{M}(\Omega) \).

Proof. We prove the result only for \( i = j = 3 \) as all the other cases are similar. Without loss of generality we will also assume that \( \Omega = (0,1)^3 \). Let \( \varphi \in C_0^\infty(\Omega) \). Using this \( \varphi \) as a test function, for almost every \( t \in (0,1) \) we obtain

\[
\begin{align*}
\langle \text{Det}(D(f_1 \circ \kappa_3^i), D(f_2 \circ \kappa_3^i)), \varphi \circ \kappa_3^i \rangle \\
= \langle D_1 f_2(\cdot, \cdot, t), f_1(\cdot, \cdot, t)D_2 \varphi(\cdot, \cdot, t) \rangle - \langle D_2 f_2(\cdot, \cdot, t), f_1(\cdot, \cdot, t)D_1 \varphi(\cdot, \cdot, t) \rangle.
\end{align*}
\]

Integrating with respect to $t$ of zero Lebesgue measure such that $\nu$ is nontrivial measure there is an index $k$ (4.3)

$$\nu\text{ will first show that } C \text{ on } (0, 1), \text{exists a disintegration } (\mu_t)\text{ such that }$$

$$\mu_t \text{ has } 1\text{-dimensional measure zero.}$$

This proves (4.2). Now, assume that $\mu := \text{Adj}_{33} f \in M(\Omega)$. By Theorem 2.3 there exists a disintegration $(\mu_t)_{t\in (0,1)}$ of $\mu$ with respect to $\nu$, where $\nu$ is as in (2.4). We will first show that $\nu$ is absolutely continuous with respect to the Lebesgue measure on $(0,1)$.

Assume that $\nu$ is not absolutely continuous. Then there exists a set $E' \subset (0,1)$ of zero Lebesgue measure such that $\nu(E') > 0$. We choose a test function $\psi \in C_0^\infty((0,1)^2)$ such that

$$\int_{(0,1)^2} \psi d\mu_t > 1$$

for every $t \in E$ where $E$ is a compact subset of $E'$ with $\nu(E) > 0$. This can be done as follows. Let $\{\psi_k\}_{k\in \mathbb{N}}$ be a dense sequence in $C_0^1((0,1)^2)$. Given any $t$ such that $\mu_t$ is nontrivial measure there is an index $k$ such that

$$\int_{(0,1)^2} \psi_k d\mu_t > 1. \quad (4.4)$$

By countable additivity of measures there has to be at least one $k$ such that (4.4) holds for every $t \in E$, where $E \subset E'$ and $\nu(E) > 0$. Without loss of generality we may assume that $E$ is compact and, of course, $E$ has 1-dimensional measure zero.

Now, take a sequence $\theta_k$ of smooth functions on $(0,1)$ with compact support such that $0 \leq \theta_k \leq 1$, $\theta_k = 1$ on $E$ and $\theta_k \searrow 0$ on $(0,1) \setminus E$. Plugging $\theta_k(t)\psi(x_1, x_2)$ into (2.2) and (4.3) we obtain

$$\int_0^1 \theta_k(t) \left( \int_{(0,1)^2} \psi(y) d\mu_t(y) \right) dt \quad (4.5)$$

where $f(\cdot, t)$ is the function $y \mapsto f(y_1, y_2, t)$. The integrand on the right is estimated by

$$C(|D_1 f(\cdot, t)| + |D_2 f(\cdot, t)|),$$

which is integrable, see Subsection 2.1. Since the limit is zero a.e., the limit on the right hand part of (4.5) is zero by the Lebesgue dominated convergence theorem. Similarly we proceed on the left, as

$$t \mapsto \left| \int_{(0,1)^2} \psi d\mu_t \right|$$
is integrable with respect to $\nu$, however, here the limit of integrals is
\[ \int_E \left( \int_{(0,1)^2} \psi \, d\mu_t \right) \, d\nu(t) \geq \nu(E). \]
This contradiction shows that $\nu$ is absolutely continuous with respect to the Lebesgue measure. Let $a$ be the density $d\nu/dt$. Consider a dense sequence $\{\psi_k\}_{k \in \mathbb{N}}$ in $C^1_0((0,1)^2)$. Analogously to (4.5), for any $k \in \mathbb{N}$ we have
\[ \int_0^1 \theta(t) \langle \text{Det } D(f_1(\cdot,t), f_2(\cdot,t)), \psi_k \rangle = \int_0^1 a(t) \theta(t) \langle \nu_t, \psi_k \rangle, \quad \theta \in C_0((0,1)). \]
Hence there exists a Lebesgue null set $N_k \subset (0,1)$ such that
\[ \langle \text{Det } D(f_1(\cdot,t), f_2(\cdot,t)), \psi_k \rangle = a(t) \langle \nu_t, \psi_k \rangle, \quad t \in (0,1) \setminus N_k. \]
It follows that for each $t \in (0,1) \setminus \bigcup_k N_k$ we have
\[ \text{Det } D(f_1(\cdot,t), f_2(\cdot,t)) = a(t) \mu_t. \]
We conclude that the distributions $\text{Det } D(f_1(\cdot,t), f_2(\cdot,t))$ are signed Radon measures. Since by Remark 2.5 and (2.4)
\[ \int_0^1 a(t) \, d|t| \nu((0,1)) = |t|(\Omega), \]
the function $t \mapsto \text{Det } D(f_1(\cdot,t), f_2(\cdot,t))$ is integrable. 

5. FROM GRADIENT TO DEGREE

5.1. The meaning of integration over the graph. The graph mapping $x \mapsto (x, f(x))$ is denoted by $\Gamma$.

Let $i, j, k \in \{1, 2, 3\}$. We define the measure $dx_i \, dy_j \, dy_k$ on the graph $\Gamma(\Omega)$ of $f$ as
\[ \int_{\Gamma(U)} dx_i \, dy_j \, dy_k = \int_{\mathbb{R}^3} \deg \left( (x_i, f_j, f_k); U, (x_i, y_j, y_k) \right) dx_i \, dy_j \, dy_k \]
if $U$ is an open set such that $|(x_i, f_j, f_k)(\partial U)| = 0$. Here $(x_i, f_j, f_k)$ denotes a continuous mapping $g : \Omega \to \mathbb{R}^3$ defined as $g(x) = (x_i, f_j(x), f_k(x))$. Similarly we define
\[ \int_{\Gamma(U)} dx_i \, dx_j \, dy_k = \int_{\mathbb{R}^3} \deg \left( (x_i, x_j, f_k); U, (x_i, x_j, y_k) \right) dx_i \, dx_j \, dy_k \]
for such good open sets. Note that this measure does not depend on the choice whether we consider $(x_i, y_j, y_k)$ and $(x_i, x_j, y_k)$, respectively, as a function of $x$ or as a function of $y$. This follows from the composition formula for the degree (see the proof of Lemma 6.3 for details). The existence and uniqueness of the measure with the desired properties follows from Theorem 6.10.

Our aim is to prove the following theorem

**Theorem 5.1.** Let $U \subset \Omega$ be an open set and $i, j \in \{1, 2, 3\}$. Then
\[ (\text{Adj}_{ij} DF)(U) = (D_j(f^{-1})_i)(f(U)). \]
We postpone the proof to Section 6. One of useful ideas is to use an intermediate term consisting in integration over the graph of \( f \). For example, for \( i = j = 3 \) this sounds as
\[
(\text{Adj}_{33} D f)(U) = \int_{\Gamma(U)} dy_1 dy_2 dx_3 = (D_3(f^{-1}))_3(f(U)).
\]

5.2. From gradient to degree. Throughout this subsection we suppose that \( u \in BV(\Omega) \) is continuous, in applications this will be the third coordinate of a BV homeomorphism.

We consider the projection of the graph mapping to two horizontal and one vertical coordinates. We define
\[
h(x) = (x_1, x_2, u(x)).
\]
Our aim is to prove that
\[
D_3 u(U) = \int_{\mathbb{R}^2} \deg(h, U, z) \, dz
\]
provided that \( |h(\partial U)| = 0 \).

**Lemma 5.2.** Let \( U \subset \subset \Omega \) be an open set. Let \( \eta \in D(\mathbb{R}^3) \) and supp \( \eta \cap h(\partial U) = \emptyset \). Then
\[
\int_{\mathbb{R}^3} \eta(z) \deg(h, U, z) \, dz = \langle D_3 u, \eta \circ h \rangle.
\]

**Proof.** Assume first that \( h \) is smooth. Then (taking into account that \( h(x) = (x_1, x_2, u(x)) \)) the degree formula (2.1) yields
\[
\int_{\mathbb{R}^3} \eta(z) \deg(h, U, z) \, dz = \int_U \eta(h(x)) \, J_h(x) \, dx = \int_U \eta(h(x)) \frac{\partial u(x)}{\partial x_3} \, dx.
\]
Passing to the limit with convolution approximation we obtain the required formula. \( \square \)

**Lemma 5.3.** Let \( U \) be as above and \( |h(\partial U)| = 0 \). Then the function \( \deg(h, U, \cdot) \) is integrable.

**Proof.** Let \( \eta \) be a \( C^\infty \) function on \( \mathbb{R}^3 \) with supp \( \eta \subset h(\overline{U}) \setminus h(\partial U) \) and \( |\eta| \leq 1 \). By Lemma 5.2
\[
\left| \int_{\mathbb{R}^3} \eta(z) \deg(h, U, z) \, dz \right| = \left| \langle D_3 u, \eta \circ h \rangle \right| \leq |Du|(U).
\]
Passing to the supremum over admissible \( \eta \) we obtain
\[
\int_{h(\overline{U}) \setminus h(\partial U)} |\deg(h, U, z)| \, dz \leq |Du|(U).
\]
Since \( \deg(h, U, \cdot) = 0 \) on \( \mathbb{R}^3 \setminus h(\overline{U}) \) and \( |h(\partial U)| = 0 \), the integrability of \( \deg(h, U, \cdot) = 0 \) is verified. \( \square \)
**Theorem 5.4.** Let $U \subset \subset \Omega$ be an open set and $|h(\partial U)| = 0$. Then
\[
\int_{\mathbb{R}^3} \text{deg}((x_1, x_2, u), U, z) \, dz = D_3 u(U).
\]

**Proof.** Let $\eta_j \in \mathcal{D}(\mathbb{R}^3)$ be smooth functions satisfying $\text{supp} \, \eta_j \cap h(\partial U) = \emptyset$ and $\eta_j \not\to 1$ on $h(U) \setminus h(\partial U)$. By Lemma 5.2,
\[
\int_{\mathbb{R}^3} \text{deg}((x_1, x_2, u), U, z) \, dz = \lim_{j \to \infty} \int_{\mathbb{R}^3} \eta_j(z) \, \text{deg}(h, U, z) \, dz = \lim_{j \to \infty} \langle D_3 u, \eta_j \circ h \rangle = D_3 u(U).
\]
The passage to the limit is justified as $\text{deg}(h, U, \cdot)$ is integrable by Lemma 5.3 and $D_3 u$ is a finite measure. \qed

6. From adjugate to degree

Throughout this section we consider a continuous mapping $f \in \dot{BV}(\Omega, \mathbb{R}^3)$ with a continuous BV inverse.

Recall that the distributional adjugate $\text{Adj} Df_{i,j}$ was defined in Definition 4.2. Its definition can be also expressed as
\[
\langle \text{Adj}_{ij} Df, \varphi \rangle = \langle Df_{j''}, f_j' \nabla \varphi \times e_i \rangle, \quad \varphi \in \mathcal{D}(\Omega).
\]
We study the projection of the graph mapping to one horizontal and two vertical coordinates. We demonstrate the proof on one choice of coordinates, namely $i = j = 3$. Then
\[
\langle \text{Adj}_{33} Df, \varphi \rangle = \langle Df_2, f_1 \nabla \varphi \times e_3 \rangle, \quad \varphi \in \mathcal{D}(\Omega).
\]
We define
\[
g(x) = (f_1(x), f_2(x), x_3).
\]
We are going to prove that there is a sufficiently rich collection of open sets $U \subset \subset \Omega$ in $\Omega$ such that for each such $U$ we have
\[
(6.1) \quad \text{Adj}_{33} Df(U) = \int_{\mathbb{R}^3} \text{deg}(g, U, z) \, dz.
\]

**Lemma 6.1.** Let $K \subset \mathbb{R}^3$ be a compact set and $u : K \to \mathbb{R}$ be a continuous function. If $\mathcal{H}^2(K) < \infty$, then $\mathcal{H}^3(\Gamma_u(K)) = 0$, where $\Gamma_u$ is the mapping $x \mapsto (x, u(x))$.

**Proof.** Choose $\varepsilon > 0$ and find $\delta \in (0, \varepsilon)$ such that
\[
x, x' \in K, \quad |x - x'| < \delta \implies |u(x') - u(x)| < \varepsilon.
\]
Let $(A_j)_j$ be a covering of $K$ by sets of diameter $< \delta$ and choose $x_j \in A_j$. A simple partition argument shows that
\[
\mathcal{H}^3_{\infty}(\Gamma_u(A_j)) \leq \mathcal{H}^3_{\infty}(A_j \times (u(x_j) - \varepsilon, u(x_j) + \varepsilon)) \leq C\varepsilon (\text{diam } A_j)^2.
\]
Summing over $j$ we obtain
\[
\mathcal{H}^3_{\infty}(\Gamma_u(K)) \leq C\varepsilon \sum_j (\text{diam } A_j)^2
\]
and passing to the infimum over all coverings we conclude
\[
\mathcal{H}^3_{\infty}(\Gamma_u(K)) \leq C\varepsilon \mathcal{H}^2_{\delta}(K).
\]
\qed
Lemma 6.2. Let \( Q \subset \subset \Omega \) be a cube such that \( |g(\partial Q)| = 0 \). Let \( \eta \in \mathcal{D}(\mathbb{R}^3) \) and \( \text{supp} \eta \cap g(\partial Q) = \emptyset \). Let \( \Phi : \mathbb{R}^3 \to \mathbb{R}^3 \) be a smooth function such that \( \Phi_3 = 0 \) and
\[
(6.2) \quad D_1\Phi_1 + D_2\Phi_2 = \eta.
\]
Let \( \varphi \in \mathcal{D}(Q) \) be a test function such that \( \varphi = 1 \) on \( \{ \eta \neq 0 \} \). Then
\[
\int_{\mathbb{R}^3} \eta(z) \deg(g, Q, z) \, dz = \langle Df_2, (\Phi_1 \circ g) \nabla \varphi \times e_3 \rangle - \langle Df_1, (\Phi_2 \circ g) \nabla \varphi \times e_3 \rangle.
\]

Proof. In this proof we don’t need invertibility. Thus we may use an approximation argument and assume first that \( g \) is smooth. Then a direct computation together with interchangeability of second derivatives gives
\[
\text{div}(\Phi_1 \circ g \nabla g_2 \times \nabla g_3 + \Phi_2 \circ g \nabla g_3 \times \nabla g_1 + \Phi_3 \circ g \nabla g_1 \times \nabla g_2) = ((\text{div} \Phi) \circ g) \, J_g
\]
so that (taking into account that \( \text{div} \Phi = \eta \), \( \Phi_3 = 0 \) and \( \nabla g_3 = e_3 \)), the degree formula (2.1) yields
\[
\int_{\mathbb{R}^3} \eta(z) \deg(g, Q, z) \, dz = \int_Q \eta(g(x)) \, J_g(x) \, dx = \int_Q \varphi(x) \, J_g(x) \, dx
\]
\[
= \int_Q \varphi(x) \, \text{div}(\Phi_1 \circ g \nabla g_2 \times \nabla g_3 + \Phi_2 \circ g \nabla g_3 \times \nabla g_1) \, dx
\]
\[
= \int_Q \Phi_1 \circ g \nabla g_2 \cdot \nabla \varphi \times e_3 \, dx - \int_Q \Phi_2 \circ g \nabla g_1 \cdot \nabla \varphi \times e_3 \, dx.
\]
Passing to the limit with convolution approximations of true \( g \) we obtain the required formula. \( \square \)

Lemma 6.3. Let \( Q \subset \subset \Omega \) be a cube such that \( |g(\partial Q)| = 0 \). Then the function
\[
\deg(g, Q, \cdot)
\]
is integrable.

Proof. It is easy to see that mappings
\[
g(x) = (f_1(x), f_2(x), x_3) \quad \text{and} \quad h(y) = (y_1, y_2, (f^{-1})_3(y))
\]
satisfy \( g = h \circ f \) and that the degree of a homeomorphism \( f \) is 1. By the degree composition formula Lemma 2.2, we thus have \( \deg(g, Q, \cdot) = \deg(h, f(Q), \cdot) \). Now, the conclusion follows from Lemma 5.3. \( \square \)

Definition 6.4. Let \( \bar{x}_i \in \mathbb{R}, i \in \{1, 2, 3\} \). We say that \( H = H_{i, \bar{x}_i} := \{ x : x_i = \bar{x}_i \} \) is a good plane if the following properties hold:
\[
(6.3) \quad |Df|(H \cap \Omega) = 0, \quad |\text{Adj} \, Df|(H \cap \Omega) = 0.
\]
\[
(6.4) \quad |g(H \cap \Omega)| = 0.
\]
\[
(6.5) \quad \limsup_{r \to 0} \frac{|Df|(\Omega \cap \{ x : |x_i - \bar{x}_i| < r \})}{r} < \infty.
\]
If \( i = 1 \),
\[
(6.6) \quad \limsup_{r \to 0} \frac{|Df(\cdot, \cdot, x_3)|(\Omega \cap ((\bar{x}_1 - r, \bar{x}_1 + r) \times \mathbb{R} \times \{ x_3 \}))}{r} < \infty
\]
for a.e. \( x_3 \in \mathbb{R} \).
Lemma 6.5.

If \( i = 2 \),

\[
(6.7) \quad \limsup_{r \to 0} \frac{|Df(\cdot, \cdot, x_3)|}{r} < \infty
\]

for a.e. \( x_3 \in \mathbb{R} \).

We say that a cube \( Q(\bar{x}, r) \subset \mathbb{R}^3 \) is a good cube if all its faces are subsets of good planes.

It is obvious that almost all \( \bar{x}_i \) satisfy (6.3) and (6.5). The validity of (6.3) and (6.6) – (6.7) for almost all \( \bar{x}_i \) will be verified in Lemma 6.5 and Lemma 6.6.

Now, consider \( \bar{z} \in \mathbb{R}^3 \) such that for each \( i = 1, 2, 3 \) and each dyadic rational \( q \), the plane \( \{ x : x_i = \bar{z}_i + q \} \) is good. We see that almost each \( \bar{z} \in \mathbb{R}^3 \) has this property. It follows that we can consider arbitrarily fine regular translated-dyadic partitions of \( \mathbb{R}^3 \) consisting of good cubes. For simplicity (and without loss of generality), we assume that the origin of coordinates has the property described above and thus all dyadic cubes \( \{(2^{-k}z_1, 2^{-k}(z_1 + 1)) \times (2^{-k}z_2, 2^{-k}(z_2 + 1)) \times (2^{-k}z_3, 2^{-k}(z_3 + 1))\} \), \( z \in \mathbb{Z}^3 \), are good.

Lemma 6.5. Almost every \( \bar{x}_i \in \mathbb{R} \) satisfies (6.4).

Proof. By [14, Theorem 3.1], for almost every \( \bar{x}_i \in \mathbb{R} \) we have \( \mathcal{H}^2(f(H)) < \infty \), where \( H = \{ x : x_i = \bar{x}_i \} \). Pick such \( \bar{x}_i \). Let \( K \subset H \) be a compact set. Then \( \mathcal{H}^2(f(K)) < \infty \) and by Lemma 6.1 for \( u = (f^{-1})_3 \) we have

\[
\mathcal{H}^3(\{(f_1(x), f_2(x), f_3(x), x_3) : x \in K\}) = 0.
\]

Hence

\[
0 = \mathcal{H}^3(\{(f_1(x), f_2(x), x_3) : x \in K\}) = |g(K)|.
\]

Lemma 6.6. Almost every \( \bar{x}_1 \in \mathbb{R} \) satisfies (6.6) and almost every \( \bar{x}_2 \in \mathbb{R} \) satisfies (6.7).

Proof. It is enough to consider (6.6). We may assume that \( \Omega \) is the cube \((0, 1)^3\). We consider the function

\[
\psi(x_1, x_3) = |Df(\cdot, \cdot, x_3)|((0, 1) \times (0, 1) \times \{x_3\}).
\]

It can be rewritten as

\[
\psi(\bar{x}_1, x_3) = \sup_{j \in \mathbb{N}} \int_{(0, x_3) \times (0, 1)} f(x_1, x_2, x_3) \text{div} \varphi_j(x_1, x_2) \ dx_1 \ dx_2,
\]

where \( \{\varphi_j\} \) is a dense sequence in the collection of all \( \varphi \in \mathcal{D}((0, 1)^2, \mathbb{R}) \) with \( \sup_{(0,1)^2} |\varphi| \leq 1 \). Therefore \( \psi \) is measurable. Since \( \psi \) is increasing in \( x_1 \), we can express the upper partial derivative of \( \psi \) at \((x_1, x_3)\) with respect to \( x_1 \) as

\[
\overline{D}_1 \psi(x_1, x_3) = \inf_{m \in \mathbb{N}} \sup_{q \in \mathbb{Q} \cap (-\frac{1}{m}, \frac{1}{m}) \setminus \{0\}} \frac{\psi(x_1 + q, x_3) - \psi(x_1, x_3)}{q},
\]

where \( \mathbb{Q} \) is the set of all rationals, similarly for the lower partial derivative. It follows that the set where the partial derivative of \( \psi \) at \((x_1, x_3)\) with respect to \( x_1 \) exists is measurable. Taking into account again that \( \psi \) is increasing in \( x_1 \), we infer that there exists a set \( N \subset \mathbb{R}^2 \) of measure zero such that the partial derivative \( \frac{\partial \psi}{\partial x_1} \) exists outside
Let $\bar{x} \in \Omega$ and $0 < r < r_0 = \text{dist}(\bar{x}, \partial \Omega)$. Let $Q = Q(\bar{x}, r)$ be a good cube. Let $\eta_0$ be a smooth function with compact support such that $0 \leq \eta_0 \leq 1$ and $\eta_0 = 1$ on $g(\Omega)$. As in the proof of Theorem 3.3 consider a sequence $\eta_j$ of smooth functions such that $\eta_j = 0$ on a neighborhood of $g(\partial Q)$, $j = 1, 2, \ldots$, $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_0$ and $\eta_j \to \eta_0$ a.e. Let $K$ be as in (3.7). Set

$$
\Phi^{(j)}(z) = (K \ast \eta(\cdot, \cdot, z_3))(z_1, z_2),
$$

$$
\Phi_1^{(j)}(z) = \Phi^{(j)}(z) - \Phi_1^{(0)}(z) + z_1,
$$

$$
\Phi_2^{(j)}(z) = \Phi_2^{(j)}(z) - \Phi_2^{(0)}(z), \quad j = 0, 1, 2, \ldots
$$

Then

$$
D_1\Phi_1^{(j)} + D_2\Phi_2^{(j)} = \eta_j \quad \text{on } g(\Omega).
$$

Further, for almost every $x_3 \in \mathbb{R}$, $\eta_j \to \eta_0$ in $L^3(\mathbb{R}^2 \times \{x_3\})$. From Lemma 3.4 we obtain that $\Phi_1^{(j)}(z) \to z_1$ and $\Phi_2^{(j)}(z) \to 0$ uniformly on $g(\Omega) \cap (\mathbb{R}^2 \times \{x_3\})$. Next, we find $\rho_j \nearrow r$ such that

$$
\eta_j = 0 \text{ on } g(Q \setminus Q(\bar{x}, \rho_j))
$$

and test functions $\varphi_j \in \mathcal{D}(Q)$ such that $0 \leq \varphi_j \leq 1$, $\varphi_j = 1$ on $Q(\bar{x}, \rho_j)$ and

$$
|\nabla \varphi_j| \leq \frac{C}{r - \rho_j}.
$$

Lemma 6.7. Let $\bar{x} \in \Omega$ and $0 < r < r_0 = \text{dist}(\bar{x}, \partial \Omega)$. Let $Q = Q(\bar{x}, r)$ be a good cube. Then

$$
\lim_{j \to \infty} \left( \langle Df_2, (\Phi_1^{(j)} \circ g) \nabla \varphi_j \times e_3 \rangle - \langle Df_1, (\Phi_2^{(j)} \circ g) \nabla \varphi_j \times e_3 \rangle \right) = \text{ADJ}_{33} Df(Q).
$$

Proof. By Lemma 3.3, Fubini theorem, (6.6) and (6.7) (recall that $Q$ is a good cube)

$$
\langle Df_2, (\Phi_1^{(j)} \circ g) \nabla \varphi_j \times e_3 \rangle - \langle Df_1, (\Phi_2^{(j)} \circ g) \nabla \varphi_j \times e_3 \rangle
$$

$$
= \int_{x_3 - r}^{x_3 + r} \left( \langle Df_2(\cdot, \cdot, x_3), (\Phi_1^{(j)} \circ g) \nabla \varphi_j(\cdot, \cdot, x_3) \times e_3 \rangle - \langle Df_1(\cdot, \cdot, x_3), (\Phi_2^{(j)} \circ g) \nabla \varphi_j(\cdot, \cdot, x_3) \times e_3 \rangle \right) dx_3
$$

$$
= \int_{x_3 - r}^{x_3 + r} \left( \langle Df_2(\cdot, \cdot, x_3), * (\Phi_1^{(j)} \circ g) \nabla (\varphi_j(\cdot, \cdot, x_3)) \rangle - \langle Df_1(\cdot, \cdot, x_3), * (\Phi_2^{(j)} \circ g) \nabla (\varphi_j(\cdot, \cdot, x_3)) \rangle \right) dx_3
$$

$$
= \int_{x_3 - r}^{x_3 + r} \left( \langle Df_1(\cdot, \cdot, x_3), Df_2(\cdot, \cdot, x_3)(Q) \rangle dx_3 = \text{ADJ}_{33} Df(Q).
$$

To justify the passage to limit in (6.8) under the integral sign we need the pointwise convergence a.e., which is verified by Lemma 3.3 and a convergent majorant. By
Corollary 3.2 for almost each $x_3 \in (\bar{x}_3 - r, \bar{x}_3 + r)$ we have

$$
\left| (D(f_2(\cdot, \cdot, x_3)), \ast (\Phi^{(j)}_1 \circ g) \nabla (\varphi_j(\cdot, \cdot, x_3))) - (D(f_1(\cdot, \cdot, x_3)), \ast (\Phi^{(j)}_2 \circ g) \nabla (\varphi_j(\cdot, \cdot, x_3))) \right|
\leq C \int_{\mathbb{R}^2} |\text{deg}((f_1(\cdot, \cdot, x_3), f_2(\cdot, \cdot, x_3)), g, Q, z, (\bar{x}_1, \bar{x}_2), r, (y_1, y_2))| dy_1 dy_2
\leq C \int_{\mathbb{R}^2} N((f_1(\cdot, \cdot, x_3), f_2(\cdot, \cdot, x_3)), g, U, z, (\bar{x}_1, \bar{x}_2), r, (y_1, y_2)) dy_1 dy_2
\leq C |\mathcal{H}^2(f(Q_2((\bar{x}_1, \bar{x}_2), r) \times \{x_3\}))|
$$

where the estimate of degree by multiplicity is from [14, Lemma 6.1]. For the last inequality see [18, Theorem 7.7]. From [14, Theorem 3.1] we deduce that the function $x_3 \mapsto \mathcal{H}^2(f(Q_2((\bar{x}_1, \bar{x}_2), r) \times \{x_3\}))$ is integrable over $(\bar{x}_3 - r, \bar{x}_3 + r)$. \qed

**Theorem 6.8.** Let $Q \subset \subset \Omega$ be a good cube. Then

$$
\int_{\mathbb{R}^3} \text{deg}(g, Q, z) dz = \text{Adj}_{33} Df(Q).
$$

**Proof.** By Lemma 6.7 and Lemma 6.2

$$
\text{Adj}_{33} Df(Q) = \lim_{j \to \infty} \left( (Df_j, (\Phi^{(j)}_1 \circ g) \nabla \varphi_j \times e_3) - (Df_1, (\Phi^{(j)}_2 \circ g) \nabla \varphi_j \times e_3) \right)
$$

$$
= \lim_{j \to \infty} \int_{\mathbb{R}^3} \eta_j(z) \text{deg}(g, Q, z) dz = \int_{\mathbb{R}^3} \text{deg}((f_1, f_2, x_3), Q, z) dz.
$$

The passage to the limit in the last equality is justified as $\text{deg}(g, Q, \cdot)$ is integrable by Lemma 6.3. The equality $\text{Adj}_{33} Df(Q) = \text{ADJ}_{33} Df(Q)$ follows from Proposition 4.3. \qed

**Definition 6.9.** We say that a set $F$ is a closed dyadic figure if it is a finite union of closed dyadic cubes. An interior of a closed dyadic figure is called an open dyadic figure.

In the following theorem we justify the definition of measures used for integration $dy_i$, $dy_j$, $dx_k$ or $dx_i$, $dx_j$, $dy_k$.

**Theorem 6.10.** There exist a unique signed Radon measure $\mu$ on $\Gamma(\Omega)$ such that for each open set $U \subset \subset \Omega$ such that $|g(\partial U)| = 0$ we have

$$
\mu(\Gamma(U)) = \int_{\mathbb{R}^3} \text{deg}(g, U, z) dz = \int_{\mathbb{R}^3} \text{deg}(g \circ f^{-1}, f(U), z) dz.
$$

**Proof.** Such a measure exists by Theorem 5.4 applied to $y \mapsto (y_1, y_2, (f^{-1})_3)$ taking into account the degree composition formula Lemma 2.2 which justifies the second equality in (6.9). The value $\mu(\Gamma(U))$ is uniquely determined if $U \subset \subset \Omega$ is open with $|g(\partial U)| = 0$. However, under the convention proposed at the end of Definition 6.4 each dyadic cube $Q \subset \subset \Omega$ is good. If $U$ is an open dyadic figure, then $|g(\partial U)| = 0$ holds as well. Thus, the value $\mu(\Gamma(U))$ is uniquely determined for each open set
$U \subset \Omega$ as $U$ can be written as an union of an increasing sequence of dyadic figures. This is enough to verify uniqueness on Borel sets.

Proof of Theorem 5.7 For symmetry reasons we can demonstrate the proof for $i = j = 3$. Let $Q \subset \subset \Omega$ be a good cube. We apply Theorem 6.8 to $f$ and Theorem 5.4 to the mapping $y \mapsto (y_1, y_2, (f^{-1})_3(y))$. By (5.1) we thus obtain

$$D_2(f^{-1})_3(f(Q)) = \int_{\Gamma(Q)} dy_1 dy_2 dx_3 = \text{Adj}_{33} Df(Q).$$

The general case follows by means of exhaustion of a general open set $U$ by open dyadic figures as in the proof of Theorem 6.10.

7. Corollaries of the main result

Proof of Corollary 1.3 To prove the corollary it suffices to show that $\text{Adj}_{i,j} Df \in \mathcal{M}(\Omega)$ for each $i, j \in \{1, 2, 3\}$. We demonstrate this on $i = j = 3$.

(a) We claim that

$$\langle \text{Adj}_{33} Df, \varphi \rangle = \int \varphi \det(D_j f_i)_{i,j=1,2} dx, \quad \varphi \in \mathcal{D}(\Omega).$$

This fact demonstrates that $\text{Adj}_{33} Df$ is a measure, as $\det(D_j f_i)_{i,j=1,2}$ is an measure (it is even an $L^1$ function). To prove (7.1), we first assume that $f_1$ and $f_2$ are smooth, then it is just integration by parts. Next step is to assume that $f_2$ is smooth. By mollification we obtain a sequence $\{f_2^{(k)}\}_k$ such that $Df_2^{(k)}$ converge weak* to $Df_2$, so it is easy to observe that (7.1) holds in this case as well. Finally, we use the preceding step and mollify $f_1$ to obtain a sequence $\{f_1^{(k)}\}_k$ such that $Df_1^{(k)}$ converge to $Df_1$, strongly if $p_1 < \infty$ or weak* if $p_1 = \infty$. In any case we can conclude that (7.1) holds for the limit function.

(b) The statement is trivial if $f_1$ and $f_2$ are smooth. Assume e.g. that $f_2$ and $f_3$ are smooth. We proceed as in the first two steps of (a) replacing (7.1) by

$$\langle \text{Adj}_{33} Df, \varphi \rangle = \langle Df_1, Df_2 \varphi \times e_3 \rangle.$$

7.1. Absolutely continuous part of $\text{Adj} Df(x)$. Let $B$ be the unit disc in $\mathbb{R}^2$ and $u, v$ are continuous $BV$ functions on $B$. We express $u$ and $v$ in polar coordinates, writing

$$\bar{u}(\rho, t) = u(\rho \cos t, \rho \sin t), \quad \bar{v}(\rho, t) = v(\rho \cos t, \rho \sin t).$$

Then

$$\int_{\partial B(0, \rho)} u \, dv$$

is the Riemann-Stieltjes integral

$$\int_0^{2\pi} \bar{u}(\rho, \cdot) \, d\bar{v}(\rho, \cdot).$$
Lemma 7.1. Let \( h = (u,v): B(0,1) \to \mathbb{R}^2 \) be a continuous \( BV \) mapping. Suppose that \( J_h \in \mathcal{M}(B(0,1)) \). Then for a.e. \( \rho \in (0,1) \) we have

\[
J_h(B(0,\rho)) = \int_{\partial B(0,\rho)} u \, dv.
\]

Proof. Let \( \eta \) be a smooth function on \([0,1]\) such that \( \eta'(0^+) = \eta'(1-) = 0, \eta(0) = 1 \) and \( \eta' < 0 \) on \((0,1)\). Let \( \varphi(x) = \eta(|x|) \) and \( \mu = J_h \). We have

\[
\int_0^1 |\eta'(r)| \mu(B(0,r)) \, dr = \int_0^1 \mu(\{\varphi > t\}) \, dt
= \int_{B(0,1)} \left( \int_0^{\eta(|x|)} dt \right) d\mu(x) = \int_{B(0,1)} \varphi \, d\mu = -\langle Dv, *u \nabla \varphi \rangle
= \int_0^1 |\eta'(r)| \int_{\partial B(\rho)} u \, dv,
\]

where the last equality is obtained by slicing in the polar coordinates, see \([1, \text{Theorem 3.107}]\). Varying \( \eta \) we obtain (7.2).

\( \square \)

Lemma 7.2. Let \( u_j, v_j \) be continuous functions on \( \partial B(0,\rho), j = 1,2,\ldots, \infty \). Let \( u_j, v_j \) converge to \( u_\infty, v_\infty \) strongly in \( BV(\partial B(0,\rho)) \). Then

\[
\int_{\partial B(0,\rho)} u_j \, dv_j \to \int_{\partial B(0,\rho)} u_\infty \, dv_\infty.
\]

Proof. It is an immediate consequence of the fact that strong convergence of continuous functions in the \( BV \) norm implies the uniform convergence (if the dimension is one).

\( \square \)

Theorem 7.3. Let \( U \subset \mathbb{R}^2 \) be an open set and \( h \in BV(U) \) be continuous. Suppose that \( J_h \in \mathcal{M}(U) \). Then the absolutely continuous part of \( J_h \) is \( J_h \) for a.e. \( x \in \Omega \).

Proof. Write \( h \) is coordinates as \( h = (u,v) \). Let \( \mu = J_h \) and \( \theta \) be the density of the absolutely continuous part of \( \mu \). Recall that the approximative derivative \( \nabla h \) is the density of the absolutely continuous part of \( Dh \) and \( J_h = \det \nabla h \). Further, \( D_s h \) is the singular part of \( Dh \) and \( \mu_s \) is the singular part of \( \mu \).

Let \( x_0 \) be a point satisfying the following properties:

\[
(7.3) \quad x_0 \text{ is a Lebesgue point for } \nabla h \text{ and } \theta,
\]

\[
(7.4) \quad \lim_{r \to 0} \frac{|\mu_s|(B(x_0,r)) + |D_s h|(B(x_0,r))}{|B(x_0,r)|} = 0
\]

Then almost every point \( x_0 \in U \) has the desired properties, (see \([18, \text{Theorems 2.12 and 2.17}]\)). For simplicity assume that \( x_0 = h(x_0) = 0 \). Choose a sequence \( r_j \searrow 0 \) such that \( B(0,r_j) \subset \subset \Omega \) and denote

\[
h_j(y) = (u_j(y), v_j(y)) = \frac{1}{r_j} h(r_j y), \quad y \in B(0,1),
\]

\[
h_\infty(y) = (u_\infty(y), v_\infty(y)) = \nabla h(0) y.
\]
Now, consider a radius $\rho \in (0, 1)$ with the following properties:

\begin{equation}
\int_{\partial B(0, \rho r_j)} u \, dv = \mu(B(0, \rho r_j)), \tag{7.5}
\end{equation}

\begin{equation}
u_j \to \nu_\infty \text{ and } v_j \to v_\infty \text{ strongly in } BV(\partial B(0, \rho)), \tag{7.6}
\end{equation}

The proof of existence of such a radius is postponed for a while. We have by (7.4) and Lemma 7.2

\[
\theta(0) = \lim_{j \to \infty} \int_{B(0, \rho r_j)} \theta(y) \, dy = \lim_{j \to \infty} \frac{\mu(B(0, \rho r_j))}{|B(0, \rho r_j)|} = \frac{1}{|B(0, \rho)|} \int_{\partial B(0, \rho)} u_j \, dv_j = \int_{B(0, \rho)} \nu_\infty \, dv_\infty = \int_{B(0, \rho)} J_{h_\infty}(y) \, dy = J_h(0).
\]

Now, by Lemma 7.1 almost every $\rho \in (0, 1)$ satisfies (7.5). To show (7.6) we show first that $h_j$ converges to $h_\infty$ strongly in $BV(B)$. As the $L^1$-convergence follows from the definition of approximate differentiability it suffices to consider the convergence of the derivative.

By [1, Remark 3.18] we have for every Borel set $A \subset B$

\begin{equation}
Dh_j(A) = \frac{1}{r_j^2} \left( \int_{r_j A} \nabla h(x) \, dx + D_s h(r_j A) \right). \tag{7.7}
\end{equation}

We now establish the strong convergence of the derivative. Using (7.7) we estimate

\[
|Dh_j - Dh_\infty|(B) = \sup_{\{\varphi \in C_0(B) : |\varphi| \leq 1\}} \int_B \varphi (Dh_j - Dh_\infty) \leq \sup_{\varphi} \frac{1}{r_j} \int_{r_j B} |\nabla h(x) - \nabla h(0)| \, dx + \frac{|D_s h|(r_j B)}{r_j^2} \to 0
\]

Here the convergence on the last step follows from (7.3) and (7.4). Finally the strong convergence on almost every $\rho \in (0, 1)$ follows from this and [1, Theorem 3.103] applied to polar coordinates.

\begin{proof}
Proof of Theorem 1.4. This follows from Theorem 7.3 using Lemma 2.6.
\end{proof}

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