ABSTRACT

We review some of the recent progress in the continuum formulation of two-dimensional string theory, i.e. two-dimensional quantum gravity coupled to $c = 1$ matter. Special attention is devoted to the discrete states and to the $w_\infty$ algebra they generate. To demonstrate the power of the infinite symmetry, we use the $w_\infty$ Ward identities to derive recursion relations among certain classes of correlation functions, which allow to calculate them exactly.

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Introduction

The field of two-dimensional quantum gravity has experienced remarkable progress in recent years. Application of matrix model techniques has led to exact solutions of various $c \leq 1$ conformal field theories coupled to gravity. $1,2,3,4,5$ Since string theory in the first-quantized formulation reduces to 2-d quantum gravity, these low-dimensional models should provide us with new insights into “stringy” phenomena. In particular, the $c = 1$ model corresponds to bosonic strings in two dimensions, the extra dimension originating from the world sheet conformal factor. $6$

We have learned that, after coupling to 2-d gravity, theories simplify considerably and become fully integrable. A good understanding of this exact integrability is still missing. With this goal in mind, it is important to develop the continuum path integral formulation of 2-d gravity, $7,8$ which so far has lagged behind the matrix models in its power. In these notes we will focus on the $c = 1$ model, i.e. quantum gravity coupled to one scalar field. $4,5$ This model is the richest among those exactly solved, and is also one of the simplest. Since there are several reviews documenting the recent progress in the $c = 1$ matrix models, $9,10$ we will discuss only the continuum formulation where some new insights have recently been obtained.

After reviewing the basics of the continuum path integral approach, we will proceed to the derivation of the $w_\infty$ symmetry structure. $11,12,13$ A similar $w_\infty$ symmetry has also appeared in the matrix model approach, $14$ but a precise connection between the two is still missing. We will review the construction of the ground ring operators, $11$ and of the $\omega$ currents. As an explicit application of the symmetries, we will use the Ward identities to calculate a large class of correlation functions. $15,16$

1. **Continuum description of 2D quantum gravity coupled to $c=1$ matter**

In this section we give a brief review of the continuum formulation of 2-dimensional string theory. We start by recalling the path integral approach to non-critical string theory in
$D = c + 1$ dimensions, and then apply it to the $c = 1$ case.

1.1 Non-critical string theory in $c$ dimensions

In the Polyakov approach, first-quantized strings propagating in $\mathbb{R}^c$ are described as a theory of $c$ free bosons coupled to 2-d quantum gravity. In other words, we begin with the path integral for 2-d quantum gravity coupled to $c$ scalar fields $X^i(\vec{\sigma})$.

$$Z = \frac{1}{\text{Vol} (\text{Diff})} \int \left[ Dg_{\mu\nu}(\vec{\sigma}) \right] \otimes \prod_{i=1}^{c} \left[ DX^i(\vec{\sigma}) \right] e^{-I(g_{\mu\nu},X^i)}$$

$$I(g_{\mu\nu},X^i) = \frac{1}{8\pi} \int d^2\sigma \sqrt{g} \left( g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i + \lambda \right). \quad (1.1)$$

Renewed interest in this problem was stimulated in part by the remarkable progress of the discretized (matrix model) approach, where $Z$ was calculated for $c \leq 1$. To study the problem in the continuum approach, it is convenient to choose the conformal gauge.

$$g_{\mu\nu}(\vec{\sigma}) = e^{-\gamma \phi(\vec{\sigma})} \hat{g}_{\mu\nu}(\vec{\sigma},\tau) \quad (1.2)$$

where $\hat{g}_{\mu\nu}(\vec{\sigma},\tau)$ is a family of reference metrics parametrized by the moduli space. Classically the Liouville field $\phi(\vec{\sigma})$ is non-dynamical, but in the quantum case there is a contribution from the path integral measure which gives rise to the kinetic term for it. After gauge fixing, the integration measure becomes

$$[Dg] \otimes [DX^i]_g = [D(\text{Diff})] \otimes [d\tau] \otimes [D\phi]_g \otimes [Db Dc]_g \otimes [DX^i]_g e^{-I(\hat{g},b,c)} \quad (1.3)$$

where $b$ and $c$ are the ghosts, and $I$ is the standard ghost action. $[d\tau]$ is the measure for integration over the moduli which will not interest us since eventually we will focus on the genus zero case.

Under a Weyl rescaling $g \rightarrow e^\psi g$, the matter and ghost actions are invariant, whereas the measures change according to

$$[DX]_e^\psi g \otimes [Db Dc]_e^\psi g = e^{\left(\frac{26-24}{8\pi}\right)S_L(\psi,g)} [DX]_g \otimes [Db Dc]_g \quad (1.4)$$

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\textsuperscript{a} This partition function can be seen as a sum over random surfaces (2D geometries) embedded in $c$ dimensions.

\textsuperscript{b} The interested reader is referred to the original literature\textsuperscript{17,8,18} for a more detailed derivation of the following results. For a review, see Ref. 19.
and the Liouville action is given by
\[
S_L(\psi, g) = \int d^2\sigma \sqrt{g} \left( \frac{1}{2} g^\mu{}^\nu \partial_\mu \psi \partial_\nu \psi + R \psi + \mu e^\psi \right). \tag{1.5}
\]
Since we have to integrate over \( \phi \), we would like to change from the measure defined with the field-dependent metric \( g \), to that defined with the fiducial metric \( \hat{g} \). While the transformation of the matter and ghost measures is simply given by Eq. (1.4), the treatment of \( [D\phi] \) is more subtle. According to a conjecture of David, Distler and Kawai, \(^8\) later confirmed in Ref. 18, the Jacobian for the change of measure from \( [D\phi]_g \) to \( [D\phi]_{\hat{g}} \) is given by the exponential of a renormalizable local action consistent with the general coordinate invariance of the underlying theory. Thus, the partition function on the sphere becomes
\[
Z = \int [D\phi]_{\hat{g}} \otimes [DbDc]_{\hat{g}} \otimes \prod_{i=1}^c [DX^i]_{\hat{g}} e^{-I'(\hat{g}, X, \phi) - I(b, c)} \tag{1.6}
\]
where the coefficients \( Q \) and \( \alpha \) are to be fixed by quantum Weyl invariance with respect to \( \hat{g} \), which is the remnant of the general covariance in the conformal gauge. We can now view this as a sigma model for ordinary string theory in flat \( D = c + 1 \) dimensions with a dilaton condensate \( \Phi = -Q\phi \), and a “tachyon” condensate \( T = \Delta e^{\alpha\phi} \).

To fix \( Q \) and \( \alpha \), one uses the requirement of cancellation of the conformal anomaly and the fact that a physical operator must have dimension \((1,1)\). First setting \( \Delta = 0 \), the chiral stress-energy tensor is
\[
T_{zz}^{(X,\phi)} = -\frac{1}{2}(\partial_z X^i)^2 - \frac{1}{2}(\partial_z \phi)^2 - \frac{1}{2}Q\partial_z^2 \phi
\]
\[
T_{zz}^{(b,c)} = -2b_{zz}\partial_z c^z + c^z \partial_z b_{zz} \tag{1.7}
\]
The Fourier components \( L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T_{zz}^{(X,\phi)} \) form the Virasoro algebra
\[
[L_n, L_m] = (n-m)L_{n+m} + \frac{c+1+3Q^2}{12} n(n^2-1) \delta_{n+m,0} \tag{1.8}
\]
To cancel the ghost contribution to the conformal anomaly one must set \( c+1+3Q^2-26 = 0 \), i.e.
\[
Q = \sqrt{\frac{25-c}{3}} \tag{1.9}
\]
Further, by requiring \( e^{\alpha\phi} \) to be a dimension \((1,1)\) perturbation, one finds \(^8\)
\[
\alpha = -\frac{1}{2\sqrt{3}} \left( \sqrt{25-c} - \sqrt{1-c} \right) \tag{1.10}
\]
For \( c > 1 \) we find the problem of complex \( \alpha \), which is indicative of the tachyonic nature of the string theory. Instead, we will set \( c \) to its critical value \( c = 1 \).
Applying the previous formalism, we obtain $Q = 2\sqrt{2}$, $\alpha = -\sqrt{2}$. The 2-d quantum gravity coupled to $c = 1$ matter can be viewed as a string theory in flat $D = c + 1 = 2$ dimensions with a dilaton condensate and a tachyon condensate (if $\Delta \neq 0$). This is the highest number of dimensions where bosonic strings are tachyon-free. We will see that the would-be tachyon of $D > 2$ theory is exactly massless for $D = 2$. The reader may be surprised by the lack of Poincaré invariance, since the two spacetime coordinates $(X, \phi)$ appear on a different footing. *c* Surprisingly, the theory instead possesses an infinite hidden symmetry. The main purpose of these notes is to make this infinite symmetry explicit.

Let us start by looking for the simplest physical states in this theory. We know that a physical state $|\psi\rangle$ must satisfy

$$L_n|\psi\rangle = \overline{L}_n|\psi\rangle = 0 \quad \text{for } n > 0$$
$$L_0|\psi\rangle = \overline{L}_0|\psi\rangle = 1 \cdot |\psi\rangle .$$  \hspace{1cm} (1.11)

The physical states are defined modulo pure gauge states, which are the Virasoro descendants. The simplest physical states are those with no oscillator excitations, characterizing motion of a string in its ground state,

$$|p, \epsilon\rangle = e^{ipX + \epsilon \phi}(0)|0\rangle .$$  \hspace{1cm} (1.12)

They satisfy $L_n|p, \epsilon\rangle = \overline{L}_n|p, \epsilon\rangle = 0$, $n > 0$, and

$$L_0|p, \epsilon\rangle = \overline{L}_0|p, \epsilon\rangle = \left[\frac{1}{2}p^2 - \frac{1}{2}\epsilon(\epsilon + 2\sqrt{2})\right]|p, \epsilon\rangle .$$  \hspace{1cm} (1.13)

Then the on-shell conditions Eqs. (1.11) imply $p^2 - \epsilon(\epsilon + 2\sqrt{2}) = 2$ or, defining $E = \epsilon + \sqrt{2}$,

$$p^2 - E^2 = 0 .$$  \hspace{1cm} (1.14)

This is a massless dispersion relation. In other words, the “tachyon” is massless in two dimensions, in accordance with the usual formula $m_T^2 = (2 - D)/12$. The solution of the

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* The lack of translation invariance was also found in the matrix model formulation of 2-d gravity coupled to $c = 1$ matter. 20
dispersion relation is \( \epsilon = -\sqrt{2} + \chi p \), where \( \chi = \pm 1 \) is the chirality. The associated tachyon vertex operators are

\[
T_\chi(p) = \int d^2 \sigma \sqrt{\hat{g}} e^{ipX + (\chi p - \sqrt{2})\phi} = \int d^2 \sigma \sqrt{\hat{g}} g_{st}(\phi)e^{ipX + \chi p\phi}. \tag{1.15}
\]

The factor \( e^{-\sqrt{2}\phi} = g_{st}(\phi) \) is the position-dependent string coupling constant. \( d \)

Setting \( p = 0 \) we find again \( \alpha = -\sqrt{2} \). The operator \( e^{-\sqrt{2}\phi} \) is believed to be exactly marginal.

We will be interested in the correlation functions of tachyons (on the sphere)

\[
\langle \prod_{i=1}^{N} T_{\chi_i}(p_i) \rangle. \tag{1.16}
\]

Since the theory is translationally invariant in \( X \), the correlator is non-vanishing only if \( \sum p_i = 0 \). On the other hand, \( \epsilon \) is not conserved. There is, however, a class of correlation functions where the integrand does not depend on the zero mode \( \phi_0 \) of \( \phi \) in the \( \Delta \to 0 \) limit. The \( \phi_0 \) dependence of the integrand in Eq. (1.16) is then given by

\[
\text{Exp} \left[ \sum_{i=1}^{N} \epsilon_i \phi_0 + \frac{1}{8\pi} \int d^2 \sigma \sqrt{\hat{g}} Q \hat{R} \phi_0 \right] = \text{Exp} \left[ \left( \sum_{i=1}^{N} \epsilon_i + 2\sqrt{2} \right) \phi_0 \right]. \tag{1.17}
\]

We will focus on the correlators which satisfy the sum rule

\[
\sum_{i=1}^{N} \epsilon_i = -2\sqrt{2} \tag{1.18}
\]

analogous to energy conservation. These correlators are sometimes called resonant because they are enhanced by the volume of \( \phi_0 \). \( e \)

Defining \( X = X_0 + \tilde{X} \) where \( X_0 \) is the zero mode of \( X \), and similarly for \( \phi \), we can explicitly perform the zero-mode integrals. Then the resonant correlators become \( 21,22 \)

\[
\langle \prod_{i=1}^{N} T_{\chi_i}(p_i) \rangle = \frac{1}{\sqrt{2}} | \log \Delta | \delta(\sum_{i=1}^{N} p_i) A(\chi_i, p_i) \tag{1.19}
\]

\( d \)

We may formally continue to Minkowski signature by sending \( \phi \to it \) and interpreting \( t \) as the time. In such a theory, \( T_\chi \) correspond to the ordinary plane waves. The unusual feature is that the string coupling \( g_{st}(t) = e^{-i\sqrt{2}t} \) is complex and oscillatory.

\( e \)

Note that in the \( \phi \to it \) continued theory, Eq. (1.18) is enforced by a delta-function constraint.
where
\[ A(\chi_i, p_i) = \int [D\tilde{X}][D\tilde{\phi}] \left( \prod_{i=1}^{N} \int d^2 \sigma \sqrt{\hat{g}} e^{ip_i \tilde{X} + \epsilon_i \tilde{\phi}} \right) e^{-S(\tilde{X}, \tilde{\phi})} \]
\[ S(\tilde{X}, \tilde{\phi}) = \frac{1}{8\pi} \int d^2 \sigma \sqrt{\hat{g}} \left( \partial_\mu \tilde{X} \partial^\mu \tilde{X} + \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \right) . \] (1.20)

The amplitudes \( A(\chi_i, p_i) \) are sometimes called the “Shifted Virasoro-Shapiro” amplitudes. Since they are formulated in terms of free fields \( \tilde{X}, \tilde{\phi} \), they can be written down explicitly and checked to be \( SL(2, C) \) invariant. After fixing the positions of three vertex operators, we find
\[ A_{\chi_1, \ldots, \chi_N}(p_1, \ldots, p_N) = \int \prod_{i=1}^{N-3} d^2 z_i \prod_{i<j} |z_i - z_j|^2 f_i \cdot f_j \] (1.21)
where \( f_i \cdot f_j = p_i p_j - \epsilon_i \epsilon_j \) (and \( \epsilon_i = -\sqrt{2} + \chi_i p_i \)).

These amplitudes exhibit a remarkable structure. They do not vanish only if there is exactly one particle with \( \chi = -1 \) (or with \( \chi = +1 \)). Then one finds
\[ A_{+...+}(p_1, \ldots, p_N, p_{N+1}) = \frac{\pi^{N-2}}{(N-2)!} \prod_{i=1}^{N} \frac{\Gamma(1 - \sqrt{2} p_i)}{\Gamma(\sqrt{2} p_i)} \] (1.22)
for amplitudes of type \((N, 1)\), i.e. for \( N \) particles with \( \chi = +1 \) and one particle with \( \chi = -1 \). The sum rules \( \sum_i p_i = 0 \) and \( \sum_i \epsilon_i = -2\sqrt{2} \) fix, in this case, the value of \( p_{N+1} \). Indeed, we have
\[ \sum_{i=1}^{N} p_i = -p_{N+1} = \frac{1}{\sqrt{2}} (N - 1) . \] (1.23)

The amplitudes of type \((1, N)\) are obtained from Eq. (1.22) by a parity flip.

1.3 The appearance of the “Discrete States”

The surprising feature of the amplitudes Eq. (1.22) is the presence of poles at \( p_i = n/\sqrt{2} \) for each \( \chi = 1 \) external leg. The infinite sequence of poles is related to the existence of operators other than the “tachyons”. Indeed, consider the four-point function of type \((3, 1)\), where \( p_4 = -\sqrt{2} \) is fixed by the sum rules. The poles result from the integration

\[ \text{\textsuperscript{f}} \text{ These states were first found in the matrix model formulation of 2-d quantum gravity.} \]
over the coordinates of $T_-(p_4)$ near one of the other vertex operators. Consider the region where $T_-(p_4)$ approaches $T_+(p_1)$. Let us examine the origin of the first pole which occurs at $p_1 = 1/\sqrt{2}$. The relevant operator product expansion (O.P.E.) is

$$p_1 = \frac{1}{\sqrt{2}} : \quad T_-(\sqrt{2}) \cdot T_+(\frac{1}{\sqrt{2}}) = e^{-i\sqrt{2}X(z, \bar{z})} \cdot e^{\frac{i}{\sqrt{2}}X - \frac{1}{\sqrt{2}}\phi}(0) \sim \frac{1}{|z|^2} e^{i\frac{i}{\sqrt{2}}X - \frac{1}{\sqrt{2}}\phi}(0) = \frac{1}{|z|^2} T_-(\frac{1}{\sqrt{2}}) . \quad (1.24)$$

After integrating over $z$, we find a divergence due to the appearance of the on-shell “tachyon” $T_-(\frac{1}{\sqrt{2}})$ in the intermediate channel. This only explains the origin of the first pole in $p_1$, however. Can an infinite sequence of Regge poles occur in a theory of a single scalar field? The answer is obviously negative. Indeed, repeating the calculation for $p_1 = \sqrt{2}$, we find that the operator appearing in the intermediate channel contains oscillator excitations,

$$p_1 = \sqrt{2} : \quad e^{-i\sqrt{2}X(z, \bar{z})} \cdot e^{i\sqrt{2}X}(0) \sim \frac{1}{|z|^2} \partial X \overline{\partial X} . \quad (1.25)$$

$\partial X \overline{\partial X}$ is a new operator called the “dilaton”, which can easily be checked to satisfy the Virasoro conditions. In fact, at each subsequent pole we find a new physical operator. The main feature of all these operators is that they are physical only at special values of the momenta and do not give rise to propagating states.

Let us look for such “discrete” states more systematically by constructing physical states (which satisfy Eq. (1.11)) using the oscillators of the $X$ and $\phi$ fields. Introducing the oscillators $\alpha_n$ and $\beta_n$ through

$$\partial z X = -i \sum_n \alpha_n z^{-n-1} , \quad \partial z \phi = -i \sum_n \beta_n z^{-n-1} \quad (1.26)$$

we define

$$\alpha^\mu_n = (i \beta_n, \alpha_n) , \quad f^\mu = (\epsilon, p) , \quad Q^\mu = (Q, 0) . \quad (1.27)$$

The Virasoro generators take the form

$$L_n = (f^\mu + \frac{n+1}{2}Q^\mu)\alpha_{\mu,n} + \sum_k : \alpha^\mu_{n+k} \alpha_{\mu,-k} : \quad n \neq 0$$

$$L_0 = f_\mu (f^\mu + Q^\mu) + \sum_k : \alpha^\mu_{-k} \alpha_{\mu,k} : \quad (1.28)$$

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where $[\alpha^\nu_m, \alpha^\mu_n] = n \delta_{n+m,0} \eta^{\mu\nu}$, $\eta_{\mu\nu} = \text{diag}(-1,1)$ and the indices are raised and lowered with the $\eta$ metric.

We have already seen that the “tachyons” are the physical states without any oscillator excitations. Consider now the states at level one, of the form

$$|\psi\rangle = \epsilon_{\mu\nu} \alpha^-_{-1} |f\rangle. \quad (1.29)$$

We simplify the discussion by considering only the chiral half of this state (relevant to the open string case) where

$$|\psi_L\rangle = \epsilon_\mu \alpha^-_{-1} |f\rangle. \quad (1.30)$$

We can later construct closed-string states by setting $\epsilon_{\mu\nu} = \epsilon_\mu \epsilon_\nu$. \textsuperscript{g}

The Virasoro conditions, Eq. (1.11), now give

$$(f_\mu + Q_\mu) e^\mu(f) = 0, \quad (f_\mu + Q_\mu) f_\mu = 0. \quad (1.31)$$

Notice also that the state $f_\mu \alpha^-_{-1} |f\rangle$ (with polarization $e_\mu = f^\mu$) is a pure gauge state, $L_{-1} |f\rangle$.

For a general $f_\mu$, the unique solution of the Virasoro conditions is $e_\mu \sim f_\mu$, which corresponds to the pure gauge state. This is in accord with the naïve light-cone argument that there are no physical oscillator states in two-dimensional string theory.

There are two exceptional momenta, however.

- $f^\mu = 0$. Here the gauge symmetry becomes trivial, and the polarization $e_\mu = (0,1)$ gives rise to a non-trivial physical state, whose vertex operator is $\partial X \overline{\partial} X$.
- $f^\mu = -Q^\mu$. In this case the constraints are trivially satisfied, and the polarization $e_\mu = (0,1)$ again gives a physical state. The corresponding operator is $\partial X \overline{\partial} X e^{-2\sqrt{2} \phi}$. Notice that the new vertex operators are two different Liouville “dressings” of the pure matter primary field $\partial X \overline{\partial} X$.

Continuing this analysis to higher levels, one can construct a family of physical operators \textsuperscript{12} which are matter ($X$) primary fields appropriately dressed by exponentials of the Liouville field $\phi$. These operators are exhibited in section §3.1. It was proven in Refs. 25, 26 that these are all the physical states not containing ghost excitations. However, there is also an infinite set of “ghostly” operators that turn out to play an important physical rôle in this theory. We will discuss them in the next section, after introducing the BRST formalism as a necessary tool.

\textsuperscript{8} Obviously it is not guaranteed that this procedure exhausts all the possible physical closed-string states. We will see an example of this in section §3.2.
2. Discrete states and the “Ground Ring”

First we change our conventions rescaling $\alpha'$ so as to eliminate the factors $\sqrt{2}$ from our formulas. Setting $\alpha' = 4$ (instead of $\alpha' = 2$) one has $Q = 2$, and the special states occur at momenta $p = n/2$. With these conventions $\langle X(z, \bar{z})X(w, \bar{w}) \rangle = -2 \log |z - w|^2$, and the tachyon operators are given by

$$T^\pm_k (z, \bar{z}) = e^{ikX + \epsilon \phi (z, \bar{z})} \quad \epsilon = -1 \pm k (2.1)$$

The sum rules for the resonant tachyon correlators become

$$\sum_{i=1}^{n} k_i = 0, \quad \sum_{i=1}^{n} \epsilon_i = -2. \quad (2.2)$$

We will keep these conventions for the rest of the paper.

2.1 “Discrete States” and the $c = 1$ primary fields

As we concluded in section §2.3, some of the discrete states of the two-dimensional string theory are simply primary fields of the $c = 1$ CFT dressed by exponentials of the Liouville field. To construct all such states, we review the structure of the $c = 1$ primary fields. First we consider the theory compactified at the self-dual radius $R = 2$, which is well known to have a chiral $SU(2)$ current algebra generated by

$$H^\pm (z) = \oint \frac{du}{2\pi i} : e^{\pm iX(u+z)} : , \quad H_3 (z) = \oint \frac{du}{4\pi} \partial X(u+z). \quad (2.3)$$

The allowed values of the chiral momenta are $p = n/2$, and the simplest primary fields are the $SU(2)$ highest weight states

$$\psi_{J,J} (z) = e^{iJX(z)} \quad (2.4)$$

\footnote{The presence of special discrete primary fields in the $c = 1$ CFT has been known for a long time.}
where $J = 0, 1/2, 1, 3/2, \ldots$. By repeatedly acting with the lowering operator $H_-$, one builds up a spin-$J$ $SU(2)$ multiplet of primary fields $\psi_{J,m}$, $m \in \{J, J-1, \cdots, -J\}$, with conformal dimension $\Delta = J^2$

$$\psi_{J,m}(z) = \left(\frac{(J+m)!}{(J-m)!(2J)!}\right)^{1/2} \left( \int \frac{du}{2\pi i} : e^{-iX(z+u)} : \right)^{J-m} \psi_{J,J}(z). \quad (2.5)$$

After performing the contractions and the contour integrals, $\psi_{J,m}$ become polynomials in the derivatives of $X$.

Upon coupling to gravity, we dress the $c = 1$ primaries to obtain the physical operators with dimension one

$$\Psi_{J,m}(z) = \left[[J+m]!(J-m)!(2J)!\right]^{1/2} \psi_{J,m}(z) : e^{\epsilon^\pm_J \phi(z)} : \quad (2.6)$$

where $\epsilon^\pm_J = -1 \pm J$, and the peculiar normalization is chosen to simplify the subsequent calculation of the operator algebra. It is easy to check that all the operators (2.6) satisfy the Virasoro conditions of the $c = 26$ $(X, \phi)$ CFT. The fact that they are not pure gauge is obvious, because the corresponding states have non-vanishing norms.

An important property of the chiral vertex operators $\Psi_{J,m}^\pm$ is that they form an interesting algebra under the O.P.E.\cite{11, 12}

$$\Psi_{J_1,m_1}^+(z) \Psi_{J_2,m_2}^+(0) = \cdots + \frac{2}{z} (J_1 m_2 - J_2 m_1) \Psi_{J_1+J_2-1,m_1+m_2}^+(0) + \cdots, \quad (2.7)$$

where we have shown the only physical operator appearing on the right-hand side. This is isomorphic to a wedge sub-algebra of $w_{\infty}$. Some of the consequences of this algebra will be explored in section §4.

To construct the physical vertex operators of closed-string theory, it is necessary to combine the $z$ and $\bar{z}$ sectors. Thus, at the self-dual radius we obtain a class of operators of the form $\Psi_{J,m}^\alpha(z) \cdot \overline{\Psi}_{J,m'}^{\bar{\alpha}}(\bar{z})$, where $\alpha$ (not summed over) can be $+$ or $-$. In fact, up to gauge equivalence, these are all the physical operators without ghost excitations.\cite{25, 26} At any radius of compactification we may construct combinations of the tachyon operators and of the “discrete primaries” appearing at the self-dual radius with their anti-holomorphic counterparts, that are permitted by the standard momentum restrictions. For example, in the uncompactified theory, in addition to the continuous spectrum of tachyons, we find the discrete states $\Psi_{J,m}^\alpha(z) \cdot \overline{\Psi}_{J,m}^{\bar{\alpha}}(\bar{z})$, with $|m| < J$. 

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2.2 BRST formalism and the “Ground Ring”

The structure of the discrete states’ sector in the two-dimensional string theory is richer than what we have seen up to now because there are additional states with ghost excitations and non-trivial ghost numbers. Such states can be studied only in the framework of the BRST quantization. There an important rôle is played by the nilpotent BRST charge

\[
Q_{BRST} = \frac{1}{2\pi i} \oint dz \, c(z) \left( T(X,\phi)(z) + \frac{1}{2} T(b,\bar{c})(z) \right).
\]  

(2.8)

Physical states are given by the cohomology classes of the BRST charge at ghost number one. In other words, the physical operators are the ghost number one vertex operators which commute with \(Q_{BRST}\), modulo commutators of \(Q_{BRST}\) with other operators. In the BRST formalism, the tachyon vertex operators are given by

\[
V_k^\pm (z) = c(z) e^{i k X + (-1 + \pm k) \phi}(z)
\]  

(2.9)

and the discrete vertex operators are

\[
Y_{J,m}^\pm (z) = c(z) \Psi_{J+1,m}^\pm (z).
\]  

(2.10)

Since these operators are non-trivial only at the discrete values of the momenta, it can be shown \(^{11}\) that each \(Y\) has a partner cohomology class of adjacent ghost number.

Indeed, in the complete analysis of the BRST cohomology \(^{25,26,28}\) it was found that the \(Y_{J,m}^+\) have “partners” at ghost number zero, whereas the \(Y_{J,m}^-\) have “partners” at ghost number two. The ghost number zero operators, usually denoted by \(O_{J,m}\), form the “Ground Ring” \(^{11}\)

\[
O_{J,m} O_{J',m'} = O_{J+J',m+m'}
\]  

(2.11)

which is to be interpreted as a fusion rule in BRST cohomology (i.e. up to BRST commutators). Their explicit form is given by

\[
\begin{align*}
O_{0,0} &= 1 \\
O_{\frac{1}{2}, \frac{1}{2}} &= (cb + \frac{i}{2} \partial X - \frac{1}{2} \partial \phi) e^{\frac{i}{2} (i X + \phi)} \\
O_{\frac{1}{2}, -\frac{1}{2}} &= (cb - \frac{1}{2} \partial X - \frac{1}{2} \partial \phi) e^{\frac{i}{2} (-i X + \phi)}
\end{align*}
\]  

(2.12)

\(^1\) For now, we are restricting ourselves to the chiral sector.
The latter two are the ring generators which determine the form of the entire ring via the fusion rules. One can think of the ring as a set of analogues of the identity operator, which occur at the discrete momenta. The presence of this ring, and other discrete states, endows the two-dimensional string theory with many special properties.

Witten has found a geometrical interpretation of the $w_\infty$ currents and of the ground ring. The first step is to note that the chiral ground ring can be interpreted as polynomial functions on a two dimensional plane with coordinates $x = O_{\frac{1}{2} \cdot \frac{1}{2}}, \ y = O_{\frac{1}{2} \cdot -\frac{1}{2}}$, i.e. $O_{J,m} \sim x^{J+m}y^{J-m}$. Next Witten studied the action of the chiral currents $\Psi_{J,m}^\pm$ on the chiral ground ring. He found that the $\Psi_{J,m}^+$ generate the area-preserving diffeomorphisms of the plane $x - y$ (a wedge sub-algebra of $w_\infty$). As an example, the first few $\Psi_{J,m}^+$ operators act on the ground ring as follows,

$$\begin{align*}
\Psi_{\frac{1}{2}, \frac{1}{2}}^+ & \sim \frac{\partial}{\partial y} \\
\Psi_{1,1}^+ & \sim x \frac{\partial}{\partial y} \\
\Psi_{1,0}^+ & \sim \frac{1}{2} \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) .
\end{align*}$$

(2.13)

Now we consider the complete set of cohomology classes of the chiral theory at the self-dual radius. The operators $O_{J,m}$ and $Y_{J,m}^+$ actually make up only half of the BRST cohomology (also known as the relative cohomology). This follows from the existence of the operator

$$a = [Q_{BRST}, \phi] = c\partial\phi + 2\partial c .$$

(2.14)

The operator $\phi$ is not a conformal field, but $a$ is. This implies that in the usual space of conformal fields, $a$ is not BRST trivial (but it is BRST invariant). Thus, by applying $a$ to an operator, one gets a new BRST invariant vertex operator. Following Ref. 13, one defines

$$aO_{J,m}(0) = \frac{1}{2\pi i} \oint dz \ a(z) \cdot O_{J,m}(0)$$

(2.15)

and similarly for $aY_{J,m}^+$. $aO_{J,m}$ is a new operator of ghost number one, whereas $aY_{J,m}^+$ has ghost number two. While $O_{J,m}$ and $Y_{J,m}^+$ are annihilated by $b_0 = \frac{1}{2\pi i} \oint dz \ b(z)$ (belong to

---

\textsuperscript{1} The algebra $w_\infty$ first appeared in the matrix model formulation of 2-d quantum gravity. \textsuperscript{14}
relative cohomology), \(a\mathcal{O}_{J,m}\) and \(aY^+_{J,m}\) are not. Thus we find the following cohomology classes labeled by the ghost number \(G\),

\[
G = 0 : \quad \mathcal{O}_{J,m} \\
G = 1 : \quad Y^+_{J,m}, \quad a\mathcal{O}_{J,m} \\
G = 2 : \quad aY^+_{J,m}.
\] (2.16)

These states are known to exhaust the absolute BRST cohomology for Liouville energy \(\epsilon > -1\). There are also their conjugate states with \(\epsilon < -1\) and ghost number \(3 - G\). Denoting the operator conjugate to \(Y^+_{J,m}\) by \(\mathcal{P}_{J,m}\), we find the complete list of cohomology classes with Liouville energy \(\epsilon < -1\),

\[
G = 1 : \quad Y^-_{J,m} \\
G = 2 : \quad \mathcal{P}_{J,m}, \quad aY^-_{J,m} \\
G = 3 : \quad a\mathcal{P}_{J,m}.
\] (2.17)

Consider now the two dimensional closed string theory. The closed string states are annihilated by the full BRST charge and satisfy the conditions \((L_0 - \bar{L}_0)|\psi\rangle = (b_0 - \bar{b}_0)|\psi\rangle = 0\).\(^k\) Witten and Zwiebach\(^{13}\) found that there exist states which satisfy the \(b_0 - \bar{b}_0\) condition but that are not annihilated separately by \(b_0\) and \(\bar{b}_0\). These states, which belong to the so-called semi-relative cohomology, cannot be written as products of the holomorphic and anti-holomorphic states. The complete list of \(\epsilon > -1\) semi-relative cohomologies is \(^{13}\)

\[
G = 0 : \quad \mathcal{O}_{J,m}\bar{\mathcal{O}}_{J,m'} \\
G = 1 : \quad Y^+_{J,m}\bar{\mathcal{O}}_{J,m'}, \quad \mathcal{O}_{J,m}Y^+_{J,m'}, \quad (a + \bar{a}) \cdot (\mathcal{O}_{J,m}\bar{\mathcal{O}}_{J,m'}) \\
G = 2 : \quad Y^+_{J,m}\bar{Y}^+_{J,m'}, \quad (a + \bar{a}) \cdot (Y^+_{J,m}\bar{\mathcal{O}}_{J,m'}), \quad (a + \bar{a}) \cdot (\mathcal{O}_{J,m}Y^+_{J,m'}) \\
G = 3 : \quad (a + \bar{a}) \cdot (Y^+_{J,m}\bar{Y}^+_{J,m'}).
\] (2.18)

The corresponding dual states with \(\epsilon < -1\) are \(^{13}\)

\[
G = 2 : \quad \bar{Y}^-_{J,m}\bar{Y}^-_{J,m'} \\
G = 3 : \quad \bar{Y}^-_{J,m}\bar{\mathcal{P}}_{J,m'}, \quad \mathcal{P}_{J,m}Y^-_{J,m'}, \quad (a + \bar{a}) \cdot (\bar{Y}^-_{J,m}\bar{Y}^-_{J,m'}) \\
G = 4 : \quad \mathcal{P}_{J,m}\bar{\mathcal{P}}_{J,m'}, \quad (a + \bar{a}) \cdot (\bar{Y}^-_{J,m}\bar{\mathcal{P}}_{J,m'}), \quad (a + \bar{a}) \cdot (\mathcal{P}_{J,m}\bar{Y}^-_{J,m'}) \\
G = 5 : \quad (a + \bar{a}) \cdot (\mathcal{P}_{J,m}\bar{\mathcal{P}}_{J,m'}).
\] (2.19)

\(^k\) See Ref. 13 for a detailed discussion of the \(b_0 - \bar{b}_0\) condition.
The dual states have ghost number $5 - G$ because of the necessary factor $b_0^{-}$ which we will discuss below. They can be interpreted as Batalin-Vilkovisky anti-field vertex operators.\textsuperscript{30}

The $G = 1$ vertex operators are of utmost importance because they give rise to the currents that generate the infinite symmetry. In the next section we will use these currents to calculate certain correlation functions through the Ward identities. We will make use only of those $G = 1$ operators that can be factorized into a product of chiral parts. The other operators, $(a + \overline{\alpha}) \cdot (O_{J,m} \overline{O}_{J,m'})$, which are not annihilated by $b_0$, do not appear to produce interesting symmetries.\textsuperscript{1}

3. Discrete states and Ward identities in 2-d closed string theory

In this section we will consider the uncompactified 2-d closed string theory, making extensive use of the symmetries generated by the $G = 1$ operators

$$Y_{J,m}^{\pm}(z) \overline{O}_{J,m}(\bar{z}) = c(z) W_{J,m}(z, \bar{z}),$$

(3.1)

where

$$W_{J,m}(z, \bar{z}) = \Psi_{J+1,m}^{+}(z) \overline{O}_{J,m}(\bar{z}).$$

(3.2)

We will write the bulk correlation functions of tachyons on a sphere,\textsuperscript{m}

$$\langle V_{k_1}^{\pm} \cdots V_{k_n}^{\pm} \rangle,$$

(3.3)

in two different ways.

In the path integral formalism,

$$\langle V_{k_1}^{\pm} \cdots V_{k_n}^{\pm} \rangle = \langle V_{k_1}^{\pm}(+\infty) V_{k_2}^{\pm}(1) \int T_{k_3}^{\pm} \cdots \int T_{k_{n-1}}^{\pm} V_{k_n}^{\pm}(0) \rangle$$

(3.4)

where $\int T_{k}^{\pm} \overset{\text{def}}{=} \int d^2 y T_{k}^{\pm}(y, \bar{y})$.

In the operator formalism,\textsuperscript{n}

$$\langle V_{k_1}^{\pm} \cdots V_{k_n}^{\pm} \rangle = \langle V_{k_1}^{\pm}(1) \Delta V_{k_2}^{\pm}(1) \Delta \cdots \Delta V_{k_{n-1}}^{\pm}(1) \mid V_{k_n}^{\pm} \rangle$$

+ permutations

(3.5)

\textsuperscript{1} Those operators that cannot be factorized into a product of chiral parts play a rather special rôle in the theory. Although they appear in the Ward identities, they seem to be unnecessary in the on-shell formulation of the theory because they correspond to auxiliary fields.\textsuperscript{31,32} Nevertheless, they are clearly important in the B-V formulation.

\textsuperscript{m} Correlation functions involving the discrete vertex operators were considered in Refs. 13, 33, 34, 35, 36, 37, 38, 39.

\textsuperscript{n} See for example Refs. 40, 41 and references therein.
where \( |V_{k_n}^\pm\rangle \overset{\text{def}}{=} \lim_{z \to 0} V_{k_n}^\pm(z) |0\rangle \), \( \langle V_{k_1}^\pm| \overset{\text{def}}{=} \lim_{z \to \infty} \langle 0| V_{k_1}^\pm(z) \). The propagator, including the ghost contribution, is

\[
\Delta = \frac{b_0^+ b_0^-}{L_0 + \bar{L}_0} \Pi_{L_0, \bar{L}_0}
\]

where \( b_0^\pm \overset{\text{def}}{=} b_0 \pm \bar{b}_0 \), and the Virasoro generators include the ghost pieces. \( \Pi_{L_0, \bar{L}_0} \) is the projector onto states that satisfy \( L_0 = \bar{L}_0 \).

In the following we will focus on the tachyon correlators of type \((N,1)\) and \((1,N)\), which are the only ones that do not vanish for generic momenta. As we have seen in §2.2, these correlators are given by the multiple integrals

\[
A_{N,1}(k_1, \ldots, k_N) = \langle V_{-k}(\infty) V_{k_2}^+(1) \ldots T_{k_N}^+ V_{k_N}(0) \rangle = \prod_l \int d^2 z_l \prod_i |z_i - 1|^{4s_{i1}} |z_i|^{4s_{i0}} \prod_{i<j} |z_i - z_j|^{4s_{ij}} = \frac{\pi^{N-2}}{(N-2)!} \prod_{r=1}^N \frac{\Gamma(1-2k_r)}{\Gamma(2k_r)}
\]

where \( l, i \) and \( j \) run from 2 to \( N-1 \), and \( s_{ij} = k_i k_j - \epsilon_i^+ \epsilon_j^+ = -1 + k_i + k_j \). The integrals of Eq. (3.7) are generalizations of the integrals calculated in Ref. 42 by contour deformation techniques. We will find that the integrals (3.7) can be calculated via simple recursion relations which follow from the \( w_\infty \) Ward identities. This is a mathematical result which does not seem to be well known. To accomplish this, we first need to review the systematic construction of conserved currents and charges from the discrete states. 13

3.1 BRST conserved currents and charges

In two dimensions a conserved current is a pair \((J_z, J_{\bar{z}})\) such that the one form

\[
\Omega^{(1)} = J_z dz - J_{\bar{z}} d\bar{z}
\]

is closed, \( d\Omega^{(1)} = 0 = \bar{\partial} J_z + \partial J_{\bar{z}} \). The associated conserved charge is \( A = \oint \Omega^{(1)} \). Actually, we may demand that a less stringent condition is fulfilled. In BRST quantization it is sufficient to require that

\[
d\Omega^{(1)} = \{ Q_{BRST}, \Omega^{(2)} \} \quad \text{or} \quad \bar{\partial} J_z + \partial J_{\bar{z}} = -\{ Q_{BRST}, \Omega^{(2)}_{z\bar{z}} \}
\]
where $\Omega^{(2)} = \Omega^{(2)}_{zz} dz \wedge d\bar{z}$. Then $A = \oint \Omega^{(1)}$ is conserved up to BRST trivial (pure gauge) operators.

The condition that $A$ be BRST invariant, $\{Q_{BRST}, A\} = 0$, implies that there exists a zero form $\Omega^{(0)}$ such that $d\Omega^{(0)} = \{Q_{BRST}, \Omega^{(1)}\}$. Moreover, $\{Q_{BRST}, \Omega^{(0)}\} = 0$ holds.

Thus, the existence of a conserved charge implies the descent equations

$$0 = \{Q_{BRST}, \Omega^{(0)}\}$$
$$d\Omega^{(0)} = \{Q_{BRST}, \Omega^{(1)}\}$$
$$d\Omega^{(1)} = \{Q_{BRST}, \Omega^{(2)}\}. \quad (3.10)$$

If we have found a $G = 1$ BRST invariant operator $\Omega^{(0)}$, these equations allow us to calculate the corresponding $G = 0$ current $\Omega^{(1)}$. Taking

$$\Omega_{J,m}^{(0)}(z, \bar{z}) = c(z)W_{J,m}(z, \bar{z}), \quad (3.11)$$

the associated charges are found to be

$$A_{J,m} = \oint \frac{dz}{2\pi i} W_{J,m}(z, \bar{z}) - \oint \frac{d\bar{z}}{2\pi i} c(z)\Psi_{J+1,m}(z) X_{J,m}(\bar{z}), \quad (3.12)$$

where $X_{J,m}(\bar{z}) = \bar{b}^{-1} \Omega_{J,m}(\bar{z})$. From Eqs. (2.7) and (2.11), the algebra of the charges is

$$[A_{J_1,m_1}, A_{J_2,m_2}] = 2((J_1 + 1)m_2 - (J_2 + 1)m_1) A_{J_1+J_2,m_1+m_2}. \quad (3.13)$$

Thus, the 2-d closed string has infinite symmetry isomorphic to the area-preserving diffeomorphism. Note that in the symmetry currents the usual chiral discrete states are sandwiched with the ground ring operators in such a way that the left and right momenta balance, producing a local quantum field. Thus, the presence of both types of operators is crucial to the demonstration of the infinite symmetry.

We may now consider the action of the charge $A_{J,m}$ on $n$ tachyons. Just from kinematical considerations, we conclude that the charge $A_{m+n-1,m}$ annihilates $l$ $T^+$ tachyons of generic momenta, for $l < n$. The first non-trivial action of this charge is on $n$ $T^+$ tachyons, producing only one $T^+$ tachyon:

$$A_{m+n-1,m} V^+_{k_1}(0) \int T^+_{k_2} \cdots \int T^+_{k_n} = F_{n,m}(k_1, \cdots, k_n) V^+_{k}(0) \quad (3.14)$$

where $k = \sum k_i + m$. A similar formula holds for $A_{-m+n-1,m}$ acting on $T^-$ tachyons.
The situation is more complex if one or more tachyons carry one of the discrete momenta. For example, let us consider the action of the charges $A_{m+n-1,m}$ on vertex operators $V_{s/2}^+$, with $s$ positive integers. For $n = 1$ the action is correctly determined by Eq. (3.14). However, for $n > 1$, the charge does not annihilate the vertex operator. Instead, it produces a discrete state in the “semi-relative” cohomology. \textsuperscript{13} We will not analyze in detail such action of the charges in this paper.

In the following we will need the explicit expression of $F_{n,m}(k_1, \ldots, k_n)$. To compute it, consider first the action of the charge $A_{\frac{1}{2},-\frac{1}{2}}$ on two $T^+$ tachyons of generic momenta:

$$A_{\frac{1}{2},-\frac{1}{2}} V_{k_1}^+ \int T_{k_2}^+ = \left[ \oint \frac{dz}{2\pi i} W_{\frac{1}{2},-\frac{1}{2}}(z, \bar{z}) - \oint \frac{d\bar{z}}{2\pi i} c(z) \Psi_{\frac{1}{2},-\frac{1}{2}}(z) \overline{X}_{\frac{1}{2},-\frac{1}{2}}(\bar{z}) \right] \cdot V_{k_1}^+(0,0) \int d^2 w T_{k_2}^+(w, \bar{w})$$

(3.15)

where the contour $\gamma$ encloses 0. To get a non-zero result from the action of the holomorphic part of the charge, it is necessary that

$$W_{\frac{1}{2},-\frac{1}{2}}(z, \bar{z}) V_{k_1}^+(0,0) \int d^2 w T_{k_2}^+(w, \bar{w}) = \frac{1}{z} F_{2, -\frac{1}{2}}(k_1, k_2) V_{k_1+k_2-\frac{1}{2}}^+(0,0) + \ldots$$

(3.16)

The only contributions to the residue come from the region where $z$ and $w$ approach each other and 0. The integral for $F_{2, -\frac{1}{2}}(k_1, k_2)$ was calculated in Ref. 15 using the results of Ref. 43

$$F_{2, -\frac{1}{2}}(k_1, k_2) = 2\pi(2k_1 + 2k_2 - 1) \frac{\Gamma(1 - 2k_1) \Gamma(1 - 2k_2) \Gamma(2k_1 + 2k_2 - 1)}{\Gamma(2k_1) \Gamma(2k_2) \Gamma(2 - 2k_1 - 2k_2)}.$$ 

(3.17)

Also, in Ref. 13 it was shown that the anti-holomorphic part of the charge gives a contribution proportional to $\bar{z}^s$ with $s > -1$. This is not singular enough to be relevant and, therefore, the anti-holomorphic part of the charge does not contribute in this case. One could anticipate this on general grounds because this part of the charge carries (left,right) ghost numbers equal to $(1, -1)$. The conservation of these ghost numbers forbids the physical tachyon from appearing in the O.P.E.

In an analogous way one can compute the action of the charge $A_{m,m}$ on one $T_k^+$ tachyon (see Ref. 15). Using these results and the $w_\infty$ algebra of charges, in Ref. 15 the action of the charge $A_{m+n-1,m}$ on $n$ $T^+$ tachyons was determined. The final result for Eq. (3.14) is

$$F_{n,m}(k_1, \ldots, k_n) = 2\pi^{n-1}(n!) k \frac{\Gamma(2k)}{\Gamma(1-2k)} \prod_{i=1}^{n} \frac{\Gamma(1-2k_i)}{\Gamma(2k_i)}$$

(3.18)
where \( k = m + \sum_{i=1}^{n} k_i \) and \( n \geq 1 \). A similar formula obviously holds for \( A_{-m+n-1,m} \) applied to \( n \) generic \( T^- \) vertex operators. Note that an explicit evaluation of this formula would require performing \( n-1 \) integrals, a very difficult task, which is avoided here thanks to the algebraic structure of the model.

### 3.2 \( w_\infty \) Ward identities

Perhaps, the simplest statement of the Ward identities \(^{11,46}\) is through the observation that the theory possesses states that are both pure gauge and identically zero. The vertex operator for such a state is

\[
\{Q_{BRST}, cW_{J,m}\} = 0 \tag{3.19}
\]

and formally carries the physical ghost number \( n_{gh} = 2 \). It vanishes identically because the zero-picture current \( cW_{J,m} \) is BRST invariant. A special feature of the theory under consideration is the presence of an infinite number of such BRST invariant zero-picture currents of ghost number 1. In general, linearized closed string gauge invariance assumes the form

\[
\delta \Psi = \{Q_{BRST}, \lambda\} \tag{3.20}
\]

where the ghost numbers of \( \Psi \) and \( \lambda \) are 2 and 1 respectively. Therefore, there is an infinite number of gauge parameters, \( \lambda = cW_{J,m} \), for which the linearized gauge invariance is trivial. In fact, as stated in Refs. 11, 46, the presence of such trivial gauge transformations guarantees that there are discrete states of ghost number 2 which cannot be gauged away. While at a general momentum there are enough gauge invariances to gauge away all the oscillator states, at the discrete momenta carried by \( cW_{J,m} \), some gauge invariances become trivial, Eq. (3.19), and there appear physical oscillator states that cannot be gauged away. Thus, in a theory with generally continuous momenta, the presence of symmetry charges seems intrinsically connected with the presence of discrete states, which are physical only at special discrete momenta. In higher-dimensional theories the only known cases of this phenomenon occur at zero momentum. Some familiar examples are the conservation of charge and the associated extra photon state at zero momentum, and the conservation of momentum and the extra graviton-dilaton states at zero momentum.

The Ward identities follow after inserting Eq. (3.19) into correlation functions. \(^{13,30}\) As usual, the BRST anti-commutator can be re-written as a sum over the boundaries

\(^{\circ}\) For another approach to the Ward identities see Refs. 44, 45.
of the moduli space of a sphere with \( n \) punctures where the sphere is pinched into two spheres with \( m \) and \( n - m \) punctures respectively. In the literature on string theory these boundaries of moduli space are sometimes referred to as the “canceled propagators”. The reason for this terminology is most apparent in the operator formalism. In the 26-dimensional string theory, the “canceled propagator argument” is the statement that such boundary terms on moduli space typically vanish (at least in the context of an appropriate analytic continuation), because the momentum that flows through the pinch is off-shell. In the 2-dimensional string theory the situation is very different. As emphasized in Ref. 15, there is an infinite set of special kinematical arrangements where the momentum that flows through the pinch is precisely on-shell. When this is the case, the “canceled propagator contribution” is non-vanishing and explicitly calculable. The Ward identity is nothing but the statement that the sum of all the canceled propagator contributions, which arise after the insertion of Eq. (3.19) into a correlation function, vanishes. We emphasize that one can search for the non-trivial contributions to the Ward identity simply on the basis of studying the kinematics: whenever the momentum that flows through the canceled propagator corresponds to an on-shell state, one expects, and usually finds, a non-vanishing contribution.

Thus the Ward identities for tachyon correlation functions assume the form

\[
\langle \{Q_{BRST}, cW_{J,m} \} V_{k_1}^\pm \cdots V_{k_n}^\pm \rangle = 0 .
\]

(3.21)

More explicitly, in the operator formalism we have

\[
0 = \langle V_{k_1}^\pm | \{Q_{BRST}, cW_{J,m} \} (1) \Delta V_{k_2}^\pm (1) \Delta \cdots \Delta V_{k_{n-1}}^\pm (1) | V_{k_n}^\pm \rangle
\]

+ permutations .

(3.22)

Having written the Ward identity in this form, we can now apply the “canceled propagator argument”. In the operator formalism this is simply obtained by explicitly commuting \( Q_{BRST} \) through each of the propagators \( \Delta \). First observe that

\[
[Q_{BRST}, \Delta] = \Pi_{L_0, \bar{L}_0} b_0^- = \sum_i | \Phi_i \rangle \langle \Phi_i | b_0^-
\]

(3.23)

where the sum is over a complete set of states that satisfy \((L_0 - \bar{L}_0) | \Phi_i \rangle = 0\). Thus, \( Q_{BRST} \) can literally cancel a propagator, replacing it with an insertion of \( b_0^- \). The conjugate states \(| \Phi_i \rangle \) and \(| \Phi^i \rangle \) satisfy \( \langle \Phi^j | \Phi_i \rangle = \delta_i^j \) (for continuous spectrum the Kroenecker symbol...
is replaced by the Dirac delta function). Each pair of conjugate states has their ghost numbers add up to 6, their momenta $k$ add up to 0, and their energies $\epsilon$ add up to $-2$. For instance, the state conjugate to $|V^\pm_k\rangle$ is

$$|V^-_k\rangle = c\partial c\bar{c}\partial\bar{c}T^\pm_{-k}(0)|0\rangle. \tag{3.24}$$

It is also convenient to define the states $|\tilde{V}^\pm_k\rangle \overset{\text{def}}{=} b_0^- |V^\pm_k\rangle$ which have ghost number 3 and are annihilated by $b_0^-$. In the Batalin-Vilkovisky quantization these are to be thought of as “anti-tachyons”.

Now, commuting $Q_{BRST}$ in Eq. (3.22) with the vertices and propagators until it annihilates against the vacua, one gets a sum of “canceled propagator contributions”. Each one has the form

$$\sum_i \langle V_{k_1}V_{k_2}\Delta\cdots\Delta V_{k_p} |\Phi_i\rangle \langle \tilde{\Phi}^i|V_{k_{p+1}}\Delta cW_{J,m}\Delta\cdots\Delta V_{k_{n-1}}|V_{k_n}\rangle, \tag{3.25}$$

where $|\tilde{\Phi}^i\rangle = b_0^- |\Phi^i\rangle$. By ghost number counting, $|\Phi_i\rangle$ must have the physical ghost number 2, and, therefore, $|\tilde{\Phi}^i\rangle$ carries ghost number 3. Also, the energy and momentum of $|\Phi_i\rangle$ are completely determined because the correlation functions obey the sum rules Eq. (2.2). It may happen that at this energy and momentum there is no physical state. Then the usual “canceled propagator argument” applies and we conclude that Eq. (3.25) vanishes. In this theory there is a class of cases, however, where there are physical states contributing to the sum over $i$. Here we will only analyze the situations where the energy and momentum that flow through the “canceled propagator” obey the tachyon dispersion relation. Then the sum over $i$ collapses to one non-vanishing term involving the tachyon states.

Note that Eq. (3.25) can be interpreted as the pinching of a sphere with $n$ punctures into two spheres with $p$ and $n-p$ punctures respectively. The amplitude corresponding to one of these spheres is perfectly conventional, with each vertex operator carrying ghost number 2. The other amplitude is unusual, since the current operator $cW_{J,m}$ carries ghost number 1, while $|\tilde{\Phi}^i\rangle$ carries ghost number 3. In the next section we will show that this amplitude can be interpreted as a matrix element of the charge operator $A_{J,m}$. For the correlation functions involving a current we introduce the notation

$$\langle \tilde{V}_{-k}V_{k_1}\cdots V_{k_n}cW_{J,m}\rangle = \sum_P \langle \tilde{V}_{-k} |V_{k_1}\Delta cW_{J,m}\Delta\cdots\Delta V_{k_{n-1}}|V_{k_n}\rangle. \tag{3.26}$$

The permutations $P$ involve all the vertex operators, except for $\tilde{V}$ which has to be kept as an out-state (in the leftmost position). Since $\tilde{V}$ and $cW$ are both anti-commuting, this
definition guarantees that all the permutations contribute with the same sign. In general, in dealing with correlators involving anti-commuting vertex operators, their ordering becomes important, and one may have to appeal to the operator formalism in order to construct their proper definition.

Taking into account the sum over all permutations in Eq. (3.22), the final form of the Ward identity is

\[ 0 = \sum_{I, \mathcal{L}} \langle V_{i_1} \ldots V_{i_I} \Phi \rangle \langle \tilde{\Phi} V_{i_1} \cdot \cdot \cdot V_{i_L} cW_{J,m} \rangle \]  

(3.27)

where the sum is over all possible partitions of the vertex operators \( V \) into two sets \( I \) and \( \mathcal{L} \) with \( I \) and \( L \) elements respectively.

Now, as we remarked earlier, there are cases when the kinematics restricts \( \Phi \) to be on the tachyon mass shell. Let us analyze the energy and momentum conservation laws for the correlator

\[ \langle \tilde{\Phi} V_{k_1}^+ \cdot \cdot \cdot V_{k_n}^+ cW_{J,m} \rangle . \]  

(3.28)

If \( \tilde{\Phi} \) carries momentum \( P \) and energy \( E \), we obtain

\[ m + P + \sum_{i=1}^{n} k_i = 0 \]

(3.29)

\[ J + E + \sum_{i=1}^{n} (-1 + k_i) = -2 . \]

For the number of particles \( n = J - m + 1 \), it follows that \( E = -1 + P \), which is the dispersion relation of a positive chirality tachyon. Remarkably, this applies for arbitrary momenta \( k_i \), which is a consequence of the two-dimensional kinematics! Therefore, whenever \( n = J - m + 1 \), Eq. (3.27) receives a contribution from the canceled propagator with \( \tilde{\Phi} = \tilde{V}_P^+ \), \( \Phi = V_{-P}^+ \). Similarly, if we flip all the chiralities, then the same situation arises for \( n = J + m + 1 \). It is not hard to see that these two cases are the only non-trivial amplitudes of type (3.28) involving tachyons of generic (not discrete) momenta.

3.3 Correlation functions from the Ward identities

We will now use the Ward identities of Eq. (3.27) to calculate the correlation functions of tachyons explicitly. As an intermediate step, we can use the \( w_\infty \) algebra to calculate the "unusual" correlators (3.26). In Ref. 16 they were reduced to the action of charges on
where the charge operator acts on all the tachyons to its right. From Eq. (3.14) it immediately follows that

$$\langle \tilde{V}^+ V^+ \cdots V^+ cW_{m+n-1,m} \rangle = \langle A_{N,m} V^+ \rangle \int T_{k_2}^+ \cdots \int T_{k_n}^+ \rangle$$

(3.30)

where $k = m + \sum_{i=1}^n k_i$, and $F_{n,m}$ is given by Eq. (3.18).

Now we can calculate all the correlation functions $A_{N,1}$, given by equation (3.7), from the Ward identity

$$\langle \{Q_{BRST}, cW_{-m,m}\} V^+ \cdots V^+ cW_{m+n-1,m} \rangle = 0$$

(3.32)

where $N \geq 3$, and the kinematics fixes $m = 1 - \frac{N}{2}$. Summing over all the non-vanishing canceled propagators, we find

$$0 = \sum_{i=1}^{N} \langle \tilde{V}^+ V^+ \cdots V^+ cW_{m+n-1,m} \rangle \langle V^+ V^+ V^- \rangle$$

(3.33)

where $p$ is the set of the first $N$ strictly positive integers except for $i$, and $p(j)$ is the $j^{th}$ element of the set. The momenta of the intermediate states, $r = \frac{1-N}{2}$ and $k = 1 - \frac{N}{2} + \sum_{j=1}^{N-1} k_{p(j)}$, have been fixed using the momentum and energy conservation laws.

Now we use Eq. (3.31), where $F_{n,m}$ is given by Eq. (3.18), to substitute the explicit expressions for the correlators involving the current. Thus, we obtain a linear relation expressing the tachyon $N+1$-point function in terms of the tachyon three-point function,

$$\left[(N-2)!\right]^2 (N-1) \cdot A_{N,1}(k_1, \ldots, k_N) = \sum_{i=1}^{N} F_{N-1,m}(k_{p(1)}, \ldots, k_{p(N-1)}) \cdot A_{2,1}(k, k_i).$$

(3.34)

Using the momentum conservation equation $2 \sum_{j=1}^{N} k_j = (N-1)$, one gets

$$F_{N-1,m}(k_{p(1)}, \ldots, k_{p(N-1)}) = \pi^{N-2}(N-1)! \frac{1}{(2k_j)}^{N} \prod_{l=1}^{N} \frac{\Gamma(1-2k_l)}{1(2k_l)}.$$  

(3.35)
The three-point function $A_{2,1}$ contains no integrations. Therefore, it is independent of the momenta and is normalized as $A_{2,1} = 1$. Now Eq. (3.34) gives

$$[(N - 2)!]^2(N - 1) \cdot A_{N,1}(k_1, \ldots, k_N) = \pi^{N-2}(N - 1)! \prod_{l=1}^{N} \frac{\Gamma(1-2k_l)}{\Gamma(2k_l)}$$

(3.36)

from which we recover the correct answer Eq. (3.7)

$$A_{N,1}(k_1, \ldots, k_N) = \frac{\pi^{N-2}}{(N - 2)!} \prod_{l=1}^{N} \frac{\Gamma(1-2k_l)}{\Gamma(2k_l)} .$$

(3.37)

4. Conclusions

The emergence of the $w_\infty$ symmetry structure in two-dimensional string theory is an interesting result, whose deep reason remains somewhat obscure. It may prove useful in elucidating the connection between the conventional path integral techniques and the matrix models, where the free fermions miraculously appear and also give rise to $w_\infty$ symmetry. Despite some recent progress, this connection is yet far from complete. In particular, we are missing a precise dictionary translating the matrix model objects into those of the continuum approach. If such a dictionary is found, it will undoubtedly provide us with new insight into closed string theory.

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References

1. V. Kazakov, Phys. Lett. 150B (1985) 282;
   F. David, Nucl. Phys. B257 (1985) 45;
   J. Ambjorn, B. Durhuus and J. Frolich, Nucl. Phys. B257 (1985) 433;
   V. Kazakov, I. Kostov and A. Migdal, Phys. Lett. 157B (1985) 295.
2. D.J. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127, 717, Nucl. Phys. B340 (1990) 333;
   M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635;
   E. Brezin and V. Kazakov, Phys. Lett. 326B (1990) 144.
3. M. Douglas, Phys. Lett. 238B (1990) 176.
4. D.J. Gross and N. Miljkovic, Phys. Lett. 238B (1990) 217;
   E. Brezin, V.A. Kazakov and A.B. Zamolodchikov, Nucl. Phys. B338 (1990) 673;
   P. Ginsparg and J. Zinn-Justin, Phys. Lett. B240 (1990) 333;
   G. Parisi, Phys. Lett. B238 (1990) 209, 213.
5. D.J. Gross and I.R. Klebanov, Nucl. Phys. B344 (1990) 375.
6. J. Polchinski, Nucl. Phys. B346 (1990) 253.
7. A.M. Polyakov, Phys. Lett. 103B (1981) 207.
8. F. David, Mod. Phys. Lett. A3 (1988) 1651;
   J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.
9. I. R. Klebanov, “String Theory in Two Dimensions”, PUPT-1271, in String Theory
   and Quantum Gravity ’91, World Scientific 1992.
10. V. Kazakov, in Random Surfaces and Quantum Gravity, O. Alvarez et al. eds.
11. E. Witten, Nucl. Phys. B373 (1992) 187.
12. I.R. Klebanov and A.M. Polyakov, Mod. Phys. Lett. A6 (1991) 3273.
13. E. Witten and B. Zwiebach, Nucl. Phys. B377 (1992) 55.
14. J. Avan and A. Jevicki, Phys. Lett. 266B (1991) 35, 272B (1991) 17, Mod. Phys.
    Lett. A7 (1992) 357; preprints BROWN-HET-847 and 869;
    D. Minic, J. Polchinski and Z. Yang, Nucl. Phys. B369 (1992) 324;
    G. Moore and N. Seiberg, Int. Jour. Mod. Phys. A7 (1992) 2601;
    S. Das, A. Dhar, G. Mandal and S. Wadia, preprints IASSNS-HEP-91/52, 91/89;
    Mod. Phys. Lett. A7 (1992) 71, 937, 2245.
15. I.R. Klebanov, Mod. Phys. Lett. A7 (1992) 723.
16. I.R. Klebanov and A. Pasquinucci, “Correlation functions from two-dimensional string
    Ward identities”, preprint PUPT-1313, hep-th/9204052, April 1992.
17. T.L. Curtright and C.B. Thorn, Phys. Rev. Lett. 48 (1982) 1309;
    E. Braaten, T.L. Curtright and C.B. Thorn, Phys. Lett. 118B (1982) 115, Ann.
    Phys. 147 (1983) 365;
    J.-L. Gervais and A. Neveu, Nucl. Phys. B199 (1982) 59, B209 (1982) 125, B224
(1983) 329, **B238** (1984) 125, 396, *Phys. Lett.* **151B** (1985) 271.

18. N. Mavromatos and J. Miramontes, *Mod. Phys. Lett.* **A4** (1989) 1849; E. D’Hoker and P.S. Kurzepa, *Mod. Phys. Lett.* **A5** (1990) 1411; E. D’Hoker, *Mod. Phys. Lett.* **A6** (1991) 745.

19. N. Seiberg, *Prog. Theor. Phys. Suppl.* **102** (1990) 319.

20. S. Das and A. Jevicki, *Mod. Phys. Lett.* **A5** (1990) 1639.

21. A.M. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 635.

22. D.J. Gross and I.R. Klebanov, *Nucl. Phys.* **B359** (1991) 3.

23. P. DiFrancesco and D. Kutasov, *Phys. Lett.* **261B** (1991) 385, *Nucl. Phys.* **B375** (1992) 119.

24. D.J. Gross, I.R. Klebanov and M.J. Newman, *Nucl. Phys.* **B350** (1991) 621; D.J. Gross and I.R. Klebanov, *Nucl. Phys.* **B352** (1991) 671; K. Demeterfi, A. Jevicki and J.P. Rodrigues, *Nucl. Phys.* **B362** (1991) 173, **B365** (1991) 499, *Mod. Phys. Lett.* **A6** (1991) 3199; U.H. Danielsson and D.J. Gross, *Nucl. Phys.* **B366** (1991) 3.

25. B. Lian and G. Zuckerman, *Phys. Lett.* **254B** (1991) 417, **266B** (1991) 21.

26. P. Bouwknegt, J. McCarthy and K. Pilch, *Comm. Math. Phys.* **145** (1992) 541, “Semi-infinite cohomology in conformal field theory and 2d gravity”, preprint USC-92/020, ADP-92-194/M12, CERN-TH.6646/92, [hep-th/9209034](https://arxiv.org/abs/hep-th/9209034), September 1992.

27. J. Goldstone, unpublished; V.G. Kac, in *Group Theoretical Methods in Physics*, Lecture Notes in Physics, vol. 94, Springer-Verlag, 1979.

28. S. Mukherji, S. Mukhi and A. Sen, *Phys. Lett.* **266** (1991) 337; K. Itoh and N. Ohta, *Nucl. Phys.* **B377** (1992) 113, “Spectrum of two-dimensional (Super) Gravity”, preprint OS-GE-22-91, [hep-th/9201034](https://arxiv.org/abs/hep-th/9201034), September 1991; C. Imbimbo, S. Mahapatra and S. Mukhi, *Nucl. Phys.* **B375** (1992) 399.

29. I. Bakas, *Phys. Lett.* **228B** (1989) 57; C. Pope, L. Romans and X. Shen, *Nucl. Phys.* **B339** (1990) 191; E. Bergshoeff, M.P. Blencowe and K.S. Stelle, *Comm. Math. Phys.* **128** (1990) 213.

30. E. Verlinde, *Nucl. Phys.* **B381** (1992) 141.

31. N. Sakai, Y. Taniii, “Physical Degrees of Freedom in 2-D String Theories”, preprint TIT/HEP-203, STUFP-92-129, July 1992.

32. S. Mahapatra, S. Mukherji and A. M. Sengupta, “Target space interpretation of new moduli in 2D string theory”, preprint TIFR/TH/92-30, [hep-th/9206111](https://arxiv.org/abs/hep-th/9206111), June 1992.

33. Y. Matsumura, N.Sakai and Y. Taniii, *Coupling of tachyons and discrete states in c = 1 2D gravity*, preprint TIT/HEP-186, STUFP-92-124, [hep-th/9201065](https://arxiv.org/abs/hep-th/9201065), “Interaction of tachyons and discrete states in c = 1 2D quantum gravity”, preprint TIT/HEP-187, [hep-th/9201066](https://arxiv.org/abs/hep-th/9201066), January 1992.

34. Y.-S. Wu and C.-J. Zhu, “The complete structure of the cohomology ring and associated symmetries in D=2 string theory”, preprint UU-HEP-92/6, June 1992.

35. I.Ya. Aref’eva and A.P. Zubarev, “Divergences of discrete states amplitudes and
effective lagrangians in 2D string theory”, preprint hep-th/9205020, May 1992.

36. N.Ohta and H. Suzuki, “Interaction of discrete states with non zero ghost number”, preprint OS-GE 25-92, hep-th/9205101, May 1992.

37. Vl.S. Dotsenko, Mod. Phys. Lett. A7 (1992) 2505.

38. D. Ghoshal, D.P. Jatkar and S. Mukhi, “Kleinian singularities and the ground ring of $c = 1$ string theory”, preprint TIFR/TH/92-34, June 1992.

39. M. Li, Nucl. Phys. B382 (1992) 242;
   J. Barbon, “Perturbing the ground ring of 2-d string theory”, preprint CERN-TH6379-92, January 1992.

40. M. Green, J. Schwarz and E. Witten, Superstring Theory, Vol. I and II, Cambridge University Press, Cambridge, 1987.

41. P. di Vecchia, R. Nakayama, J.L. Petersen and S. Sciuto, Nucl. Phys. B282 (1987) 103;
   P. di Vecchia, M. Frau, A. Lerda and S. Sciuto, Nucl. Phys. B298 (1988) 526.

42. Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B251 (1985) 291.

43. H. Kawai, D. Lewellen and S.H. Tye, Nucl. Phys. B269 (1986) 1.

44. D. Kutasov, E. Martinec and N. Seiberg, Phys. Lett. 276B (1992) 437.

45. S. Kachru, Mod. Phys. Lett. A7 (1992) 1419.

46. A.M. Polyakov, “Singular states in 2d quantum gravity”, preprint PUPT-1289, September 1991.

47. I. Batalin and G. Vilkovisky, Phys. Lett. 102B (1981) 27, 120B (1983) 166;
   see also M. Henneaux, “Lectures on the antifield-BRST formalism for gauge theories”, proceeding of the XX GIFT meeting, and M. Henneaux and C. Teitelboim, Quantization of gauge systems, to be published by Princeton University Press.