ABSTRACT. We consider a wide class of the two-particle Schrödinger operators $H_\mu(k) = H_0(k) + \mu V$, $\mu > 0$, with a fixed two-particle quasi-momentum $k$ in the $d$-dimensional torus $\mathbb{T}^d$, associated to the Bose-Hubbard hamiltonian $H_\mu$ of a system of two identical quantum-mechanical particles (bosons) on the $d$-dimensional hypercubic lattice $\mathbb{Z}^d$ interacting via short-range pair potentials. We study the existence of eigenvalues of $H_\mu(k)$ below the threshold of the essential spectrum depending on the interaction energy $\mu > 0$ and the quasi-momentum $k \in \mathbb{T}^d$ of particles. We prove that the threshold (bottom of the essential spectrum), as a singular point (a threshold resonance or a threshold eigenvalue), creates eigenvalues below the essential spectrum under perturbations of both the coupling constant $\mu > 0$ and the quasi-momentum $k$ of the particles. Moreover, we show that if the threshold is a regular point, then it does not create any eigenvalues under small perturbations of the coupling constant $\mu > 0$ and the quasi-momentum $k$.

2010 Mathematics Subject Classification. Primary: 81Q10, Secondary: 47A10, 47B39

Keywords and phrases: discrete Schrödinger operator, two-particle system, hamiltonian, conditionally negative, dispersion relation, resonance, eigenvalue.

1. INTRODUCTION

The main goal of the present paper is to give a thorough mathematical treatment of the spectral properties for the two-particle lattice Schrödinger operators $H_\mu(k) = H_0(k) + \mu V$, $\mu > 0$, where $k \in \mathbb{T}^d$ being the two-particle quasi-momentum in the $d$-dimensional torus, associated to the Bose-Hubbard hamiltonian $H_\mu$ of a system of two identical quantum-mechanical particles (bosons) on the $d$-dimensional hypercubic lattice $\mathbb{Z}^d$ interacting via short-range pair attractive potentials, with an emphasis on new threshold phenomena that are not present in the continuous case (see, e.g., [6], [8], [13], [16], [19]–[24], [25] for relevant discussions).

Throughout physics, stable composite objects are usually formed by way of attractive forces, which allows the constituents to lower their energy by binding together. Repulsive forces separate particles in the free space. However, in a structured environment such as a periodic potential and in the absence of dissipation, stable composite objects can exist even for repulsive interactions that arise from the lattice band structure [35].

The Bose-Hubbard model, which is used to describe the attractive and repulsive pairs is the theoretical basis for applications. The work [35] exemplifies the important correspondence between the Bose-Hubbard model [9, 17] and atoms in optical lattices, and helps to pave the way for many more interesting developments and applications [34]. In particular, the dynamics of ultracold atoms loaded in the lower band is well described by the Bose-Hubbard hamiltonian.

In the continuum space (continuum), due to rotational invariance, the multi-particle hamiltonian separates into a free hamiltonian for the center of mass and a hamiltonian for the relative motion. Bound states are eigenstates of the latter hamiltonian.
The fundamental difference between the continuum and the lattice multi-particle hamiltonian is that in lattice, the free hamiltonian is not rotationally invariant. Therefore, in contrast to the continuum, in the lattice there is excess mass phenomenon, i.e., the effective mass of the bound state of an \(N\)-particle system is, in general, strictly greater than the sum of the effective masses of the constituent quasi–particles. This has been discussed, e.g., in [24], [25].

For the Schrödinger operators with short-range potentials on \(\mathbb{R}^3\) and \(\mathbb{Z}^3\) and on perturbations of Schrödinger operators with periodic potentials one observes the emission of negative bound states from the continuous spectrum at the so-called critical potential strength (see, e.g., [2], [3], [6], [14], [18], [19], [20], [21], [27], [36], [37]). This phenomenon is closely related to the existence of generalized eigenfunctions, which are, vanishing at infinity, solutions of the Schrödinger equation with zero energy, but are not square integrable. These solutions are usually called zero-energy resonance functions and, in this case, the Hamiltonian is called a critical one and the Schrödinger operator is said to have a zero-energy resonance (virtual level).

The appearance of the negative eigenvalues for the critical (non-negative) two–particle Schrödinger operators on \(\mathbb{R}^3\) under infinitesimally negative perturbations, i.e., the presence of the threshold resonances, leads to the existence of infinitely many bound states (Efimov’s effect) for the corresponding three-particle system (see, e.g., [4], [11], [26], [31], [32], [33], [36]).

The threshold of the essential spectrum for the Schrödinger operator is either regular or singular point (threshold resonance or threshold eigenvalue). In the continuum, whether the threshold is a regular or a singular point for the two-particle Schrödinger operator depends only on the interaction \(\mu V\). In the lattice case it depends not only on the interaction \(\mu V\) of particles, but also to the quasi-momentum \(k\) of the particle pair.

In the current paper, we study the emission mechanisms of eigenvalues at the thresholds of the essential spectrum for the operator \(\hat{H}_\mu(k)\), depending on the interaction \(\mu \hat{V}\) and the quasi-momentum \(k \in \mathbb{T}^d\), for all dimensions \(d \geq 3\). We prove the existence of bound states in the following two cases:

\[(\text{low } d)\] In the case \(d = 1, 2\), for any nonzero potential \(\mu \hat{V}\) and each quasi-momentum \(k \in \mathbb{T}^d\) of the particle pair (Theorem 4.1 and 4.2).

\[(\text{high } d)\] In the case \(d \geq 3\), for large potentials \(\mu \hat{V}\) and for each \(k \in \mathbb{G}\), where \(\mathbb{G} \subset \mathbb{T}^d\) is a region that includes the point \(0 \in \mathbb{T}^d\) (Theorem 4.6 and Remark 4.7).

We establish the existence of eigenvalues \(E_\mu(k)\) of \(\hat{H}_\mu(k)\) for each non-zero \(k \in \mathbb{T}^d\) provided that the bottom of the essential spectrum of \(\hat{H}_\mu(0)\) is its threshold resonance or threshold eigenvalue, by applying a generalization of the well known Birman-Schwinger principle (see, e.g., [12], [30]). The method we use gives implicit forms of the eigenfunctions (bound states) by means of eigenfunctions of the generalized Birman-Schwinger operator.

We find a two-sided assessments for the eigenvalue \(E(k)\) of \(\hat{H}_1(k)\) depending on the quasi-momentum \(k \in \mathbb{T}^d\) by means of \(E_{\text{min}}(k)\), the threshold of the essential spectrum of \(\hat{H}_1(k)\) and the eigenvalue \(E(0)\) of \(\hat{H}_1(0)\) (Corollary 4.5). This result shows that for any nonzero \(k \in \mathbb{T}^d\) the eigenvalue \(E(k)\) is strictly greater than \(E(0)\) and lies below the threshold \(E_{\text{min}}(k)\).

We prove, for \(d \geq 3\), that the threshold \(E_{\text{min}}(k)\), as a singular point, i.e., as a threshold resonance or a threshold eigenvalue, creates eigenvalues below the essential spectrum under small perturbations of both the effective mass (by changing the quasi-momentum of the particles) and the coupling constant \(\mu > 0\). However, if the threshold \(E_{\text{min}}(k)\) is a regular point, then no eigenvalues are created under such perturbations.

Moreover, we show that for each \(k_0 \in \mathbb{G}\), there exists a neighborhood \(G(k_0) \subset \mathbb{G}\) and a manifold \(M=(k_0) \subset \mathbb{G}\) of codimension one, such that for each \(k \in M=(k_0)\) the threshold \(E_{\text{min}}(k)\)
is a singular point of \( \hat{H}_\mu(k) \), but it is a regular point if \( k \in \mathcal{G} \setminus \mathcal{M}_\infty(k_0) \). Furthermore, if the threshold \( \mathcal{E}_{\min}(k_0) \) of \( \hat{H}_\mu(k_0) \): (i) is a regular point, then for each \( k \) lying in a neighborhood of \( k_0 \in \mathcal{G} \), the number of eigenvalues of \( \hat{H}_\mu(k) \) lying below the threshold \( \mathcal{E}_{\min}(k) \) remains unchanged (Theorem 4.9). (ii) is a singular point, then there is an open set \( \mathcal{M}_\infty(k_0) \subset \mathcal{G} \) with \( k_0 \in \mathcal{M}_\infty(k_0) \) such that for all \( k \in \mathcal{M}_\infty(k_0) \) the operator \( \hat{H}_1(k) \) has an eigenvalue below \( \mathcal{E}_{\min}(k) \) (Theorem 4.12).

We observe that for \( k_0 = 0 \in \mathbb{T}^d \), the case (i) above yields Efimov’s effect for the three-particle lattice Schrödinger operators \( \mathbf{H}(K) \), \( K \in \mathbb{T}^d \), associated to the Bose-Hubbard Hamiltonian of a system of three particles on \( \mathbb{Z}^3 \) interacting via short-range pair potentials: the operator \( \mathbf{H}(0) \) has infinitely many eigenvalues below the bottom of the three particle essential spectrum, whereas for all non-zero \( K \in \mathbb{T}^d \) close to \( K = 0 \), the operator \( \mathbf{H}(K) \) has finitely many eigenvalues (see, e.g., [11, 5, 7, 20]).

It is also shown that for \( d \geq 3 \), the total number of both the eigenvalues (counting multiplicities) below the threshold of essential spectrum and the multiplicity of the singular point \( \mathcal{E}_{\min}(k) \) is a non-decreasing function of the quasi-momentum \( k \in \mathbb{T}^d \) and \( \mu > 0 \) (Corollary 4.15 and 4.17).

We note that unlike the case of Schrödinger operators in \( \mathbb{R}^d \), the lattice Schrödinger operators may have eigenvalues above the threshold of the essential spectrum, when the sign of the potential is changed. The repulsive case can be investigated exactly in the same way as the attractive one treated here.

The paper is organized as follows: In Section 2 we introduce the lattice two-particle Hamiltonians, decompose them into the von Neumann direct integrals and introduce the Schrödinger operators

\[
\text{(2.1) } (\hat{H}_0 \hat{f})(x_1, x_2) = \sum_{s_1, s_2 \in \mathbb{Z}^d} [\hat{\epsilon}(x_1 - s_1) + \hat{\epsilon}(x_2 - s_2)] \hat{f}(s_1, s_2), \quad \hat{f} \in \ell^2, s(\mathbb{Z}^d \otimes \mathbb{Z}^d),
\]

where \( \hat{\epsilon} \in \ell^1(\mathbb{Z}^d) \) is a real valued even function.

The interaction operator \( \hat{\nabla} \) of two bosons, in the position space representation, is the multiplication operator by the non-positive function \( \hat{v} \in \ell^1(\mathbb{Z}^d; \mathbb{R}_0^-) \), i.e.,

\[
\text{(2.2) } (\hat{\nabla} \hat{f})(x_1, x_2) = \hat{v}(x_1 - x_2) \hat{f}(x_1, x_2), \quad \hat{f} \in \ell^2(\mathbb{Z}^d \otimes \mathbb{Z}^d).
\]

The total Hamiltonian \( \hat{H}_\mu \) of a system of two identical particles (bosons) with the pair non-positive interaction \( \hat{v} \in \ell^1(\mathbb{Z}^d; \mathbb{R}_0^-) \), in the position space representation, is associated with the bounded self-adjoint operator on \( \ell^2, s(\mathbb{Z}^d \otimes \mathbb{Z}^d) \):

\[
\text{(2.3) } \hat{H}_\mu = \hat{H}_0 + \mu \hat{\nabla}, \quad \mu > 0.
\]
2.2. The two-particle hamiltonian – momentum space representation. Let \( T^d = (\mathbb{R}/2\pi\mathbb{Z})^d \equiv [-\pi,\pi)^d \) be the \( d \)-dimensional torus, the Pontryagin dual group of \( \mathbb{Z}^d \), equipped with the (normalized) Haar measure
\[
\eta(dp) = \frac{dp}{(2\pi)^d}.
\]
Let \( L^2(T^d, \eta) \) be the Hilbert space of square-integrable functions on \( T^d \) and \( \mathcal{F} : L^2(\mathbb{Z}^d) \to L^2(T^d, \eta) \) be the Fourier transform
\[
(2.4) \quad f(p) := (\mathcal{F}\hat{f})(p) = \sum_{x\in\mathbb{Z}^d} e^{i(p,x)} \hat{f}(x)
\]
and \( \mathcal{F}^* \) is its inverse
\[
(2.5) \quad [\mathcal{F}^* f](x) := \int_{T^d} e^{-i(p,x)} f(p) \eta(dp).
\]

The free hamiltonian \( \mathbb{H}_0 = (\mathcal{F} \otimes \mathcal{F}) \hat{\mathbb{H}}_0(\mathcal{F}^* \otimes \mathcal{F}^*) \) of a system of two identical particles (bosons), in the momentum space representation, is the multiplication operator by the function \( \varepsilon(k_1) + \varepsilon(k_2) \) on the Hilbert space \( L^{2,\varepsilon}(T^d \otimes T^d, \eta \otimes \eta) \) of symmetric functions on the cartesian product \( T^d \otimes T^d \) of the torus \( T^d \):
\[
(2.6) \quad (\mathbb{H}_0 f)(k_1, k_2) = (\varepsilon(k_1) + \varepsilon(k_2)) f(k_1, k_2),
\]
where the continuous function (dispersion relation) \( \varepsilon \) is given by
\[
(2.7) \quad \varepsilon(p) = [\mathcal{F} \varepsilon](p) = \sum_{x\in\mathbb{Z}^d} e^{i(p,x)} \hat{\varepsilon}(x).
\]

The interaction operator \( \mathcal{V} = (\mathcal{F} \otimes \mathcal{F}) \hat{\mathcal{V}}(\mathcal{F}^* \otimes \mathcal{F}^*) \) is the integral operator of convolution type acting in \( L^{2,\varepsilon}(T^d \otimes T^d, \eta \otimes \eta) \) as
\[
(2.8) \quad (\mathcal{V} f)(k_1, k_2) = \int_{T^d} v(k_1 - q) f(q, k_1 + k_2 - q) \eta(dq),
\]
where the kernel function \( v(\cdot) \) is given by
\[
(2.9) \quad v(p) = [\mathcal{F} \hat{v}](p) = \sum_{x\in\mathbb{Z}^d} e^{i(p,x)} \hat{v}(x).
\]

The total two-particle hamiltonian \( \mathbb{H}_\mu \) of a system of two identical quantum-mechanical particles (bosons) interacting via a short range attractive potentials \( \hat{v} \), in the momentum space representation, is the bounded self–adjoint operator acting in \( L^{2,\varepsilon}(T^d \otimes T^d, \eta \otimes \eta) \) as
\[
(2.10) \quad \mathbb{H}_\mu = \mathbb{H}_0 + \mu \mathcal{V}.
\]

2.3. Decomposition of the two–particle hamiltonians into the von Neumann direct integrals. Let \( k = k_1 + k_2 \in T^d \) be the quasi–momentum of a pair of particles. For any fixed \( k \in T^d \), we define the set \( Q_k \subset T^d \otimes T^d \) as
\[
Q_k = \{(q, k - q) : q \in T^d\}.
\]
We further define the map
\[
\pi : T^d \otimes T^d \to T^d, \quad \pi((k_1, k_2)) = k_1.
\]
Denote by $\pi_k = \pi|_{Q_k}$, $k \in \mathbb{T}^d$ the restriction of $\pi$ on $Q_k \subset \mathbb{T}^d \otimes \mathbb{T}^d$. At this point it is useful to remark that $Q_k$ is a $d$-dimensional manifold isomorphic to $\mathbb{T}^d$. The map $\pi_k$ is bijective from $Q_k \subset \mathbb{T}^d \otimes \mathbb{T}^d$ onto $\mathbb{T}^d$ with the inverse

$$\pi_k^{-1}(q) = (q, k - q).$$

Consequently, the Hilbert space $L^{2,\varepsilon}(\mathbb{T}^d \otimes \mathbb{T}^d, \eta \otimes \eta)$ can be decomposed into the von Neumann direct integral as

$$(2.11) \quad L^{2,\varepsilon}(\mathbb{T}^d \otimes \mathbb{T}^d, \eta \otimes \eta) \simeq \bigoplus_{k \in \mathbb{T}^d} L^{2,\varepsilon}(\mathbb{T}^d, \eta)(dk),$$

where $L^{2,\varepsilon}(\mathbb{T}^d, \eta)$ is the Hilbert space of square-integrable even functions on $\mathbb{T}^d$. The total hamiltonian $H_\mu$ of a system of two particles, in the position space representation, obviously commutes with the representation of the discrete group $\mathbb{Z}^d$ by shift operators on the lattice, and hence $H_\mu$ can be decomposed into the integral (see, e.g., [6])

$$(2.12) \quad H_\mu(k) = \int_{\mathbb{T}^d} H_\mu(k)(\eta)(dk).$$

In the physical literature, the parameter $k \in \mathbb{T}^d$ is called two-particle quasi-momentum and the corresponding operator $H_\mu(k)$ is called Schrödinger operator with fixed quasi-momentum $k$.

2.4. Schrödinger operators for particle pairs with fixed quasi-momentum. For any $\mu > 0$ and $k \in \mathbb{T}^d$, the Schrödinger operator $H_\mu(k)$ from the decomposition (2.12), in the momentum space representation, is bounded self-adjoint operator acting in $L^{2,\varepsilon}(\mathbb{T}^d, \eta)$ as

$$(2.13) \quad H_\mu(k) = H_0(k) + \mu V.$$

Here the non-perturbed operator $H_0(k), k \in \mathbb{T}^d$ is the multiplication operator by the function $\mathcal{E}_k$ (quasi-momentum-dependent pair dispersion relation) acting in $L^{2,\varepsilon}(\mathbb{T}^d, \eta)$ as

$$(2.14) \quad (H_0(k)f)(p) = \mathcal{E}_k(p)f(p),$$

where

$$(2.15) \quad \mathcal{E}_k(p) = \varepsilon \left( \frac{k}{2} + p \right) + \varepsilon \left( \frac{k}{2} - p \right)$$

and $\varepsilon(\cdot)$ is defined in (2.7).

Note that we identified the torus $\mathbb{T}^d$ with $[-\pi, \pi)^d \subset \mathbb{R}^d$ so that the operation change of variables is well-defined on $\mathbb{T}^d$.

The perturbation operator $V$ in (2.13) is defined as

$$(2.16) \quad (Vf)(p) = \int_{\mathbb{T}^d} v(p - q)f(q)\eta(dq), \quad f \in L^{2,\varepsilon}(\mathbb{T}^d, \eta).$$

In the position space representation, the Schrödinger operator $\hat{H}_\mu(k)$ with a fixed quasi-momentum $k \in \mathbb{T}^d$ acts in the Hilbert space $\ell^{2,\varepsilon}(\mathbb{Z}^d)$ of all square-summable even functions on $\mathbb{Z}^d$ as

$$(2.17) \quad \hat{H}_\mu(k) = \hat{H}_0(k) + \mu \hat{V}, \quad \mu > 0,$$
Here, \( \hat{E}_k(x) := \int_{\mathbb{T}^d} e^{-i(p,x)} E_k(p) \eta(dp) \).

and \( \hat{V} \) in (2.17) is defined as

\[
(\hat{V} \hat{f})(x) = \hat{\nu}(x) \hat{f}(x), \quad \hat{f} \in \ell^{2,c}(\mathbb{Z}^d).
\]

3. Spectral properties of \( \hat{H}_k(k), k \in \mathbb{T}^d \)

Since the perturbation operator \( \hat{V} \) is compact, according to Weyl’s theorem [29] Theorem XIII.14, the essential spectrum \( \sigma_{\text{ess}}(\hat{H}_k(k)) \) of the operator \( \hat{H}_k(k), k \in \mathbb{T}^d \) coincides with the spectrum \( \sigma(\hat{H}_0(k)) \) of the non-perturbed operator \( \hat{H}_0(k) \). Explicitly, one has

\[
\sigma_{\text{ess}}(\hat{H}_k(k)) = [\varepsilon_{\min}(k), \varepsilon_{\max}(k)],
\]

with

\[
\varepsilon_{\min}(k) := \min_{q \in \mathbb{T}^d} E_k(q), \quad \varepsilon_{\max}(k) := \max_{q \in \mathbb{T}^d} E_k(q).
\]

From the non-positivity of \( \hat{V} \) and the min-max principle, it follows that all isolated eigenvalues of finite multiplicity lie below \( \varepsilon_{\min}(k) \), the bottom of the essential spectrum \( \sigma_{\text{ess}}(\hat{H}_k(k)) \).

3.1. Birman-Schwinger principle. Let \( d \geq 1, k \in \mathbb{T}^d \), and \( E_k(\cdot) \) be the quasi-momentum-dependent pair dispersion relation and \( \hat{\nu} \in \ell^2(\mathbb{Z}^d; \mathcal{B}_0) \). For any \( z < E_{\min}(k) \), we define the Birman-Schwinger operator \( \mathcal{B}_k(k, z) \), which is a non-negative compact operator acting in \( \ell^{2,c}(\mathbb{Z}^d) \) as

\[
\mathcal{B}_k(k, z) := \mu|\hat{V}|^{1/2} \hat{R}_0(k, z) |\hat{V}|^{1/2}.
\]

Here, \( \hat{R}_0(k, z), z \in \mathbb{C} \setminus [E_{\min}(k), E_{\max}(k)] \) is the resolvent of the operator \( \hat{H}_0(k) \) and \( |\hat{V}|^{1/2} \) is the (unique) positive square root of the (positive) operator \( |\hat{V}| \):

\[
(|\hat{V}|^{1/2} \hat{\psi})(x) = |\hat{\nu}(x)|^{1/2} \hat{\psi}(x), \quad \hat{\psi} \in \ell^{2,c}(\mathbb{Z}^d).
\]

The kernel function \( \mathcal{B}_k(k, z; \cdot, \cdot), k \in \mathbb{T}^d, z < E_{\min}(k) \), associated to the Birman-Schwinger operator \( \mathcal{B}_k(k, z) \) is given by

\[
\mathcal{B}_k(k, z; x, y) = \mu |\hat{\nu}(x)|^{1/2} \hat{R}_0(k, z; x - y) |\hat{\nu}(y)|^{1/2}, \quad x, y \in \mathbb{Z}^d,
\]

where

\[
\hat{R}_0(k, z; x) := \int_{\mathbb{T}^d} \frac{e^{i(q,x)}}{E_k(q) - z} \eta(dq), \quad x \in \mathbb{Z}^d.
\]

Now we recall, for the convenience of the reader, the well known Birman-Schwinger principle (see, e.g., [12], p.180 and [30], p. 89.), associated to the Schrödinger operator \( \hat{H}_k(k) \), the proof of which can be found in [8].

**Proposition 3.1. (Birman-Schwinger principle)** Let \( k \in \mathbb{T}^d \) and \( z < E_{\min}(k) \).

i) If \( \hat{f} \in \ell^{2,c}(\mathbb{Z}^d) \) solves \( \hat{H}_k(k) \hat{f} = z \hat{f} \), then \( \hat{\psi} := \mu |\hat{V}|^{1/2} \hat{\nu} \in \ell^{2,c}(\mathbb{Z}^d) \) solves \( \hat{\psi} = \mathcal{B}_k(k, z) \hat{\psi} \).
Hypothesis 3.3. has a unique minimum at continuous conditionally negative definite function and hence

Remark 3.5. The following subclass of the one-particle systems is of certain interest (see, e.g., \[10\]). It is introduced by the above Hypotheses 3.3 on 3.2.

A generalization of the Birman-Schwinger principle. In what follows we assume the following Hypotheses [3.3] on \(\varepsilon(\cdot)\) and \(\hat{\varepsilon}(\cdot)\).

Hypothesis 3.3. (i) The function \(\varepsilon(\cdot) \in C^{(3)}(\mathbb{T}^d)\), \(d \geq 1\) is a real–valued even Morse function and has a unique minimum at \(0 \in \mathbb{T}^d\).

(ii) The function \(\hat{\varepsilon}(\cdot) \in \ell^2(\mathbb{Z}^d; \mathbb{R}_-^d)\), \(d \geq 1\) is a real–valued even Morse function and hence

Lemma 3.2. Let \(\hat{\varepsilon} \in \ell^1(\mathbb{Z}^d; \mathbb{R}_-^d)\) be a non-positive function. Then for any \(x, y \in \mathbb{Z}^d\), the function \(B_{\mu}(k, x, y)\) is analytic in \(z \in \mathbb{C} \setminus [e_{\min}(k), e_{\max}(k)]\).

Proof. The equality (3.3) and the analyticity of the function \(R_{\mu}(k, z; L)\) in \(z \in \mathbb{C} \setminus [e_{\min}(k), e_{\max}(k)]\) yield the proof of Lemma 3.2. \(\square\)

3.2. A generalization of the Birman-Schwinger principle. In what follows we assume the following Hypotheses [3.3] on \(\varepsilon(\cdot)\) and \(\hat{\varepsilon}(\cdot)\).

Hypothesis 3.3. (i) The function \(\varepsilon(\cdot) \in C^{(3)}(\mathbb{T}^d)\), \(d \geq 1\) is a real–valued even Morse function and has a unique minimum at \(0 \in \mathbb{T}^d\).

(ii) The function \(\hat{\varepsilon}(\cdot) \in \ell^1(\mathbb{Z}^d; \mathbb{R}_-^d)\), \(d \geq 1\) is a real–valued even Morse function and hence

Lemma 3.4. (i) The minimum \(e_{\min} = \min_{p \in \mathbb{T}^d} \varepsilon(p)\) is attained at the point \(p = 0 \in \mathbb{T}^d\).

(ii) There is a maximal open neighborhood \(G \subset \mathbb{T}^d\) containing the point \(0 \in \mathbb{T}^d\) such that, for any \(k \in G\), the pair dispersion relation \(E_k(\cdot) \in C^{(3)}(\mathbb{T}^d)\) is a Morse function and for each \(k \in G\), the minimum \(\hat{e}_{\min}(k) = \min_{q \in \mathbb{T}^d} \hat{e}_k(q)\) is attained at the point \(p(k) = 0 \in \mathbb{T}^d\).

The proof of Lemma 3.4 can be found in [10, Lemma 3].

Remark 3.5. The following subclass of the one-particle systems is of certain interest (see, e.g., [10]). It is introduced by the additional requirement that the dispersion relation \(\varepsilon(p)\) is a real-valued continuous conditionally negative definite function and hence

(i) \(\varepsilon\) is an even function,

(ii) \(\varepsilon(p)\) has a minimum at \(p = 0\).

Recall (see, e.g., [6, 29]) that a complex-valued bounded function \(\varepsilon : \mathbb{T}^d \rightarrow \mathbb{C}\) is called conditionally negative definite if \(\varepsilon(p) = \varepsilon(-p)\) and

\[
\sum_{i,j=1}^{n} \varepsilon(p_i - p_j)z_i\bar{z}_j \leq 0
\]

for any \(n \in \mathbb{N}\), \(p_1, p_2, \ldots, p_n \in \mathbb{T}^d\) and for all \(z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n\) satisfying \(\sum_{i=1}^{n} z_i = 0\).
Definition 3.6. Assume \( d \geq 3 \) and Hypothesis \( 3.3 \). Let
\[
\hat{\mathcal{R}}_0(k, \mathcal{E}_{\min}(k); x) := \int_{\mathbb{T}^d} \frac{e^{i(q \cdot x)}}{\mathcal{E}_k(q) - \mathcal{E}_{\min}(k)} \eta(dq), \; x \in \mathbb{Z}^d.
\]
For \( k \in \mathbb{G} \subset \mathbb{T}^d \), we define the generalized Birman-Schwinger operator \( \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k)) \) by means of the kernel function
\[
\mathbb{B}_\mu(k, \mathcal{E}_{\min}(k); x, y) := \mu |\hat{\nu}(x)|^2 \hat{\mathcal{R}}_0(k, \mathcal{E}_{\min}(k); x - y)|\hat{\nu}(y)|^2, \; x, y \in \mathbb{Z}^d
\]
as follows
\[
(\mathbb{B}_\mu(k, \mathcal{E}_{\min}(k)) \hat{f})(x) = \sum_{y \in \mathbb{Z}^d} \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k); x, y) \hat{f}(y), \; \hat{f} \in \ell^2, \mathcal{E}(\mathbb{Z}^d).
\]

Now we formulate some properties of the generalized Birman-Schwinger operator.

Lemma 3.7. Let \( d \geq 3 \) and assume Hypothesis \( 3.3 \).

(i) For any \( x \in \mathbb{Z}^d \), the kernel function \( \hat{\mathcal{R}}_0(\cdot, \mathcal{E}_{\min}(\cdot); x) \) is continuous in \( k \in \mathbb{G} \).

(ii) For any \( x, y \in \mathbb{Z}^d \), the kernel function \( \mathbb{B}_\mu(\cdot, \mathcal{E}_{\min}(\cdot); x, y) \) is continuous in \( k \in \mathbb{G} \).

(iii) For any \( k \in \mathbb{G} \), the operator \( \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k)) \), acting in \( \ell^2, \mathcal{E}(\mathbb{Z}^d) \), belongs to the Hilbert-Schmidt class and the map \( k \in \mathbb{G} \mapsto \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k)) \) is continuous.

Remark 3.8. Lemma 3.7 yields that for each \( d \geq 3 \) the kernel function \( \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k); x, y) \) defines a generalized Birman-Schwinger operator \( \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k)) \) on \( \ell^2, \mathcal{E}(\mathbb{Z}^d) \), which is non-negative, compact and hence self-adjoint Hilbert-Schmidt operator.

Let \( \ell_0(\mathbb{Z}^d) \) be the Banach space of all functions defined on \( \mathbb{Z}^d \) and vanishing at infinity.

3.3. Regular or singular point. Since for each \( k \in \mathbb{G} \) the operator \( \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k)) \) is compact (self-adjoint), only one of the following two cases may happen

(i) The number 1 is an eigenvalue for \( \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k)) \).

(ii) The number 1 is not an eigenvalue for \( \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k)) \).

Definition 3.9. Let \( d \geq 3 \). For each \( k \in \mathbb{G} \) the threshold \( z = \mathcal{E}_{\min}(k) \) of the essential spectrum \( \sigma_{\text{ess}}(\hat{H}_\mu(k)) \) is called a singular point of multiplicity \( n \) (resp. regular point) of the operator \( \hat{H}_\mu(k) \), if the number 1 is an eigenvalue of multiplicity \( n \) (resp. not an eigenvalue) for the operator \( \mathbb{B}_\mu(k, \mathcal{E}_{\min}(k)) \).

Remark 3.10. If the threshold \( \mathcal{E}_{\min}(k) \) is a regular point of the essential spectrum of \( \hat{H}_\mu(k) \), then Theorem 3.7 yields that the equation \( \hat{H}_\mu(k) \hat{f} = \mathcal{E}_{\min}(k) \hat{f}, \hat{f} \in \ell^2, \mathcal{E}(\mathbb{Z}^d) \) has only the trivial solution and the number of eigenvalues of the operator \( \hat{H}_\mu(k) \) below the threshold \( \mathcal{E}_{\min}(k) \) remains unchanged under small perturbations of \( k \in \mathbb{G} \) and \( \mu \hat{V} \) (see Theorem 4.7).

Remark 3.11. Notice that the threshold singular point is either a threshold resonance or a threshold eigenvalue of \( \hat{H}_\mu(k) \) and our definition of the threshold resonance is the direct analogue of those, which have been introduced in the continuous and lattice cases (see, e.g., \([6], [31]\) and references therein).

In the following theorem the Birman–Schwinger principle is extended to \( z = \mathcal{E}_{\min}(k) \).

Theorem 3.12. Assume Hypothesis \( 3.3 \). Then the following statements hold:
(i) Let \( d \geq 3 \). If \( \hat{f} \in \ell_0(\mathbb{Z}^d) \) solves \( \hat{H}_\mu(k) \hat{f} = \varepsilon_{\text{min}}(k) \hat{f} \), then \( \hat{\psi} := |\hat{V}|^{1/2} \hat{f} \in \ell^{2,\varepsilon}(\mathbb{Z}^d) \) solves \( \hat{\psi} = B_\mu(k, \varepsilon_{\text{min}}(k)) \hat{\psi} \).

(ii) Let \( d = 3, 4 \). If \( \hat{\psi} \in \ell^{2,\varepsilon}(\mathbb{Z}^d) \) solves \( \hat{\psi} = B_\mu(k, \varepsilon_{\text{min}}(k)) \hat{\psi} \), then \( \hat{f} := \hat{R}_0(k, \varepsilon_{\text{min}}(k)) |\hat{V}|^{1/2} \hat{\psi} \in \ell_0(\mathbb{Z}^d) \) solves \( \hat{H}_\mu(k) \hat{f} = \varepsilon_{\text{min}}(k) \hat{f} \).

(iii) Let \( d \geq 5 \). If \( \hat{\psi} \in \ell^{2,\varepsilon}(\mathbb{Z}^d) \) solves \( \hat{\psi} = B_\mu(k, \varepsilon_{\text{min}}(k)) \hat{\psi} \), then \( \hat{f} := \hat{R}_0(k, \varepsilon_{\text{min}}(k)) |\hat{V}|^{1/2} \hat{\psi} \in \ell^{2,\varepsilon}(\mathbb{Z}^d) \) solves \( \hat{H}_\mu(k) \hat{f} = \varepsilon_{\text{min}}(k) \hat{f} \).

(iv) Let \( d \geq 3 \). The threshold \( \varepsilon_{\text{min}}(k) \) is a singular point (a resonance or an eigenvalue) of the essential spectrum of \( \hat{H}_\mu(k) \) with multiplicity \( m \) if and only if the number \( 1 \) is an eigenvalue of \( B_\mu(k, \varepsilon_{\text{min}}(k)) \) with multiplicity \( m \).

(v) Let \( d \geq 3 \). Counting multiplicities, the number of eigenvalues of \( \hat{H}_\mu(k) \) less than \( \varepsilon_{\text{min}}(k) \) equals to the number of eigenvalues of \( B_\mu(k, \varepsilon_{\text{min}}(k)) \) greater than \( 1 \), i.e.,

\[
\mathcal{N}_-(\varepsilon_{\text{min}}(k), \hat{H}_\mu(k)) = \mathcal{N}_+(1, B_\mu(k, \varepsilon_{\text{min}}(k))).
\]

**Remark 3.13.** Let the threshold \( z = \varepsilon_{\text{min}}(k), k \in \mathbb{G} \) be a singular point of the essential spectrum of \( \hat{H}_\mu(k) \), i.e., the equation \( B_\mu(k, \varepsilon_{\text{min}}(k)) \hat{\psi} = \hat{\psi} \) has a non-trivial solution \( \hat{\psi} \in \ell^{2,\varepsilon}(\mathbb{Z}^d) \) and wherein the function

\[
\hat{f}(x) = \sum_{y \in \mathbb{Z}^d} \int_{\mathbb{Z}^d} \frac{e^{i(p \cdot x - y)} \eta(dp)}{\varepsilon_k(p) - \varepsilon_{\text{min}}(k)} |\hat{\psi}(y)|^2 \hat{\psi}(y)
\]

is a non-trivial solution of the Schrödinger equation \( \hat{H}_\mu(k) \hat{f} = \varepsilon_{\text{min}}(k) \hat{f} \).

(i) If \( d = 3 \) or \( 4 \), then the solution \( \hat{f} \) of the equation \( \hat{H}_\mu(k) \hat{f} = \varepsilon_{\text{min}}(k) \hat{f} \) in (3.11) belongs to \( \ell_0(\mathbb{Z}^d) \setminus \ell^{2,\varepsilon}(\mathbb{Z}^d) \).

(ii) If \( d \geq 5 \), then the solution \( \hat{f} \) in (3.11) belongs to \( \ell^{2,\varepsilon}(\mathbb{Z}^d) \) and hence the singular point \( \varepsilon_{\text{min}}(k) \) is an eigenvalue of the operator \( \hat{H}_\mu(k) \).

**Definition 3.14.** In the case (i) of Remark 3.13 the singular point \( \varepsilon_{\text{min}}(k), k \in \mathbb{G} \) is called a threshold resonance (virtual level) for \( \hat{H}_\mu(k) \).

### 3.4. Example. The discrete Laplacian.

The dispersion relation \( \varepsilon \) associated to the discrete Laplacian \( \Delta_{\mathbb{Z}^d} \) is given as

\[
\varepsilon(p) = \sum_{j=1}^{d} [1 - \cos p_j], \quad p = (p_1, ..., p_d) \in \mathbb{T}^d
\]

and hence it is a Morse function satisfying the Hypothesis 3.3. The corresponding two-particle dispersion relation \( \varepsilon_k(\cdot) \) is of the form

\[
\varepsilon_k(p) = 2 \sum_{j=1}^{d} [1 - \cos \frac{k_j}{2} \cos p_j], \quad p = (p_1, ..., p_d) \in \mathbb{T}^d.
\]

The function \( \varepsilon_k(\cdot) \) can be degenerate only for \( k \in \Pi_n \), where

\[
\Pi_n = \{ k \in \mathbb{T}^d : \text{ n coordinates of k is equal to } \pi \}, \quad 1 \leq n \leq d.
\]

The set \( \Pi_d \) consists of exactly one point \((\pi, ..., \pi)\) and the set \( \Pi_n \subset \mathbb{T}^d, 1 \leq n \leq d \) is a surface (manifold) of co-dimension \( d - n \) and in this case the operator \( \hat{H}_\mu(k) \) defined in (2.17) becomes the Schrödinger operator on \( \mathbb{Z}^{d-n} \). Note that the set \( \mathbb{G} = \mathbb{T}^d \setminus \cup_{n=1}^{d} \Pi_n \) is an open (maximal) set in \( \mathbb{T}^d \) satisfying the Hypothesis 3.3 and the function \( \varepsilon_k(p) \) is a Morse function for any \( k \in \mathbb{G} \).
Remark 3.15. If the dispersion relation $\varepsilon(\cdot)$ is associated to the discrete Laplacian $\Delta_{Z^d}$, then for the Schrödinger operator $\hat{H}_\mu(k)$, given in Eq. (2.17), the region $\mathcal{G}$ in (ii) of Lemma 3.4 can be defined precisely as $\mathcal{G} = \mathbb{T}^d \setminus \{\cup_{i=1}^d \Pi_n\}$.

4. STATEMENT OF THE MAIN RESULTS

The first result is the existence of the eigenvalues for each nonzero potential $\mu \hat{v}$ and quasi-momentum $k \in \mathbb{T}^d$ for $d = 1, 2$.

Theorem 4.1. Let $d = 1, 2$. Assume the Hypothesis 3.3 and $\hat{v} \neq 0$. Then for any $k \in \mathbb{T}^d$ and $\mu > 0$ the operator $\hat{H}_\mu(k)$ has an eigenvalue $z_\mu(k)$ below the threshold $\varepsilon_{\min}(k)$ of the essential spectrum $\sigma_{\text{ess}}(\hat{H}_\mu(k))$.

The next results are on dependence of the eigenvalues of quasi-momentum $k \in \mathbb{T}^d$ and the conservation of the number of eigenvalues, which lies below the essential spectrum of the operator $\hat{H}_1(0) = \hat{H}_0(0) + \hat{V}$ for all non-zero $k$.

Theorem 4.2. Let $d \geq 1$. Assume that $\varepsilon$ be conditionally negative definite function on $\mathbb{T}^d$ and the inequality $\hat{H}_1(0) = \hat{H}_0(0) + \hat{V} \geq z_0 I$ holds for $z_0 < \varepsilon_{\min}(0)$. Then for all non-zero $k \in \mathbb{T}^d$, the strict inequality $\hat{H}_1(k) = \hat{H}_0(k) + \hat{V} > z_0 I$ holds.

Corollary 4.3. Let $d \geq 1$. Let $\varepsilon$ be conditionally negative definite function on $\mathbb{T}^d$ and for any $k \in \mathbb{T}^d$ the numbers $z_1(k) \leq \ldots \leq z_m(k)$ be eigenvalues of the operator $\hat{H}_\mu(k)$ (counting multiplicities) lying below $\varepsilon_{\min}(k)$. Then for any nonzero $k \in \mathbb{T}^d$, the inequalities

$$z_j(0) < z_j(k), j = 1, ..., m$$

hold.

Theorem 4.4. Let $d \geq 1$. Let $\varepsilon$ be conditionally negative definite function on $\mathbb{T}^d$ and the numbers $z_1(0) \leq \ldots \leq z_m(0)$ be $m$ eigenvalues of the operator $\hat{H}_1(0)$ (counting multiplicities) lying below $\varepsilon_{\min}(0)$. Then there exists $z_0$, $z_0 < \varepsilon_{\min}(0)$ such that for any nonzero $k \in \mathbb{T}^d$, the operator $\hat{H}_1(k)$ has at least $m$ eigenvalues $z_1(k) \leq \ldots \leq z_m(k)$ (counting multiplicities) satisfying the inequalities

$$z_j(k) < z_0 + [\varepsilon_{\min}(k) - \varepsilon_{\min}(0)], j = 1, ..., m.$$

Theorems 4.2 and 4.4 yield the following corollary.

Corollary 4.5. Assume the assumptions of Theorem 4.4. Then for any nonzero $k \in \mathbb{T}^d$

$$z_j(0) < z_j(k) < z_j(0) + [\varepsilon_{\min}(k) - \varepsilon_{\min}(0)], j = 1, ..., m.$$
Remark 4.7. Let $d \geq 3$. Under the assumptions of Theorem 4.6 the integral at the left-hand side of the inequality (4.1) is continuous function in $k \in \mathbb{G}$ (see Lemma 3.7). Hence, for any fixed $k_0 \in \mathbb{G}$ there exists a neighborhood $G(k_0) \subseteq \mathbb{G}$ of $k_0$ such that for all $k \in G(k_0)$ the inequality

$$\mu \max_{x \in \mathbb{T}^d} |\hat{v}(x)| \int_{\mathbb{T}^d} \eta(dq) \frac{\eta(dq)}{E_k(q) - E_{\min}(k)} > 1$$

holds. Therefore, the operator $\hat{H}_\mu(k)$ has an eigenvalue $z_\mu(k)$ below the threshold $E_{\min}(k)$ for all $k \in G(k_0)$.

The following theorem states that for smooth dispersion relations and absolutely convergent potentials on $\mathbb{Z}^d$, the Schrödinger operators $\hat{H}_\mu(k)$ have only finitely many eigenvalues below the essential spectrum.

Theorem 4.8. Let $d \geq 3$. Assume the Hypothesis 3.3. Then for each $\mu > 0$ and $\hat{v} \neq 0$, the number of eigenvalues of the operator $\hat{H}_\mu(k)$, $k \in \mathbb{G}$ lying below the threshold $E_{\min}(k)$ is finite.

The following results describe the sets of the coupling constants $\mu > 0$ and the quasi-momenta $k \in \mathbb{G}$, for which $E_{\min}(k)$ is a regular or a singular point of the essential spectrum of $\hat{H}_\mu(k)$.

Theorem 4.9. Assume $d \geq 3$ and Hypothesis 3.3. Let for some $\mu_0 > 0$ and $k_0 \in \mathbb{G}$ the threshold $E_{\min}(k_0)$ is a regular point of the essential spectrum of the operator $\hat{H}_\mu_0(k_0) = \hat{H}_0(k_0) + \mu_0 \hat{V}$. Then there exist neighborhoods $U(\mu_0) \subseteq \mathbb{R}$ and $G(k_0) \subseteq \mathbb{G}$ of $\mu_0$ and $k_0 \in \mathbb{G}$, respectively, such that for all $\mu \in U(\mu_0)$ and $k \in G(k_0)$, the number of eigenvalues of operator $\hat{H}_\mu(k)$ below the threshold $E_{\min}(k)$ remains unchanged.

Corollary 4.10. The set $U \subset \mathbb{R}_+$ of all $\mu \in \mathbb{R}_+$ resp. $G \subset \mathbb{G}$ of all $k \in \mathbb{G}$, for which $E_{\min}(k)$, the threshold of the essential spectrum is regular point of the operator $\hat{H}_\mu(k)$ is an open set in $\mathbb{R}_+$ resp. in $\mathbb{G}$, i.e., there exist intervals $U_\alpha \subset \mathbb{R}_+$ resp. connected components $G_\alpha \subset \mathbb{G}, \alpha = 1, 2, ..., \text{such that} U = \bigcup_\alpha U_\alpha$ resp. $G = \bigcup_\alpha G_\alpha$. Consequently, the sets $\mathbb{R}_+ \setminus U$ and $\mathbb{G} \setminus G$ consist of the points $\mu \in \mathbb{R}_+$ and quasi-momenta $k \in \mathbb{G}$, respectively, for which $E_{\min}(k)$, the threshold of the essential spectrum is a singular point of the operator $\hat{H}_\mu(k)$.

4.1 The threshold is a singular point. Let $d \geq 3$. Now, we study the emission of eigenvalues of the operator $\hat{H}_1(k), k \in \mathbb{G}$ from $E_{\min}(k)$, the bottom of the essential spectrum $\sigma_{\text{ess}}(\hat{H}_1(k))$, when the threshold is a singular point of the essential spectrum $\sigma_{\text{ess}}(\hat{H}_1(k))$. Recall that for any $\psi \in \ell^2.c(\mathbb{Z}^d)$, the Fourier transform $\mathcal{F} \circ |\hat{V}|^{1/2} \hat{\psi}$ of the function $|\hat{V}|^{1/2} \hat{\psi}$ is continuous in $p \in \mathbb{T}^d$.

For each $k_0 \in \mathbb{G}$, $s \in \mathbb{Z}^d$ and $\hat{\psi} \in \ell^2.c(\mathbb{Z}^d)$ under Hypothesis 3.3, the function

$$E_s(k, k_0; \psi) = \int_{\mathbb{T}^d} \frac{[1 - \cos(p, s)] \mathcal{F} \circ |\hat{V}|^{1/2} \hat{\psi}(p)^2 \eta(dp)}{(E_k(p) - E_{\min}(k))(E_{k_0}(p) - E_{\min}(k_0))} \geq 0$$

is continuous in $k \in \mathbb{G}$ and can be estimated by a constant not depending on $s \in \mathbb{Z}^d$.

Remark 4.11. (i) Let for some $k_0 \in \mathbb{G}$, $E_{\min}(k_0)$, the threshold of the essential spectrum of $\hat{H}_1(k_0)$ be a singular point of multiplicity $n \geq 1$, i.e., the number 1 is an eigenvalue for $B_1(k_0, \mathbb{E}_{\min}(k_0))$ of multiplicity $n \geq 1$. Let $\mathcal{H}_n$ be an $n$-dimensional subspace of the operator $B_1(k_0, \mathbb{E}_{\min}(k_0))$, associated to the eigenvalue 1. Then for any non-zero $\psi \in \mathcal{H}_n$, the inequality $E_s(k, k_0; \psi) > 0$, $s \in \mathbb{Z}^d$ holds.
(ii) Let the threshold $\mathcal{E}_{\min}(k_0), k_0 \in \mathbb{G}$ be a regular point of $\hat{H}_\mu(k_0)$, i.e., the equation $B_\mu(k_0, \mathcal{E}_{\min}(k_0)) \hat{\psi} = \hat{\psi}$ has only the trivial solution $\hat{\psi} = 0 \in \ell^2, \mathbb{R}^d$. Then for all $s \in \mathbb{Z}^d$, the equality $\mathcal{E}_{s}(k, k_0; \hat{\psi}) = 0$ holds.

We define the function $L(k, k_0; \hat{\psi})$ on $\mathbb{G} \subset \mathbb{T}^d$ as

$$L(k, k_0; \hat{\psi}) = 2 \sum_{s \in \mathbb{Z}^d, s \neq 0} \hat{\epsilon}(s) [\cos(k/2, s) - \cos(k_0/2, s)] \cdot \mathcal{E}_{s}(k, k_0; \hat{\psi}).$$

Since $\hat{\epsilon} \in \ell^1(\mathbb{Z}^d)$, the series in the right hand side of Eq. (4.3) is absolutely convergent and defines a continuous function in $k \in \mathbb{G}$.

For any fixed $k_0 \in \mathbb{G}$ and non-zero $\hat{\psi} \in \ell^2, \mathbb{R}^d$ we define the following sets

$$M_>(k_0; \hat{\psi}) := \{ k \in \mathbb{G} : L(k, k_0; \hat{\psi}) > 0 \} \subset \mathbb{G},$$

$$M_=(k_0; \hat{\psi}) := \{ k \in \mathbb{G} : L(k, k_0; \hat{\psi}) = 0 \} \subset \mathbb{G},$$

$$M_<(k_0; \hat{\psi}) := \{ k \in \mathbb{G} : L(k, k_0; \hat{\psi}) < 0 \} \subset \mathbb{G}.$$

Repeating the argument of the proof of Lemma[3.7] one can show that the map $k \in \mathbb{G} \mapsto L(k, k_0; \hat{\psi})$ is continuous. The sets $M_>(k_0; \hat{\psi})$ and $M_<(k_0; \hat{\psi})$ are open subsets of $\mathbb{G} \subset \mathbb{T}^d$ and

$$\mathbb{G} = M_>(k_0; \hat{\psi}) \cup M_=(k_0; \hat{\psi}) \cup M_<(k_0; \hat{\psi}).$$

For the $n$ be dimensional subspace $\mathcal{H}_n$, associated to the eigenvalue 1 of the operator $B_\mu(k_0, \mathcal{E}_{\min}(k_0)), k_0 \in \mathbb{G}$, we define the following subsets of $\mathbb{G}$

$$M_>(k_0; \mathcal{H}_n) = \bigcup_{\hat{\psi} \in \mathcal{H}_n, ||\hat{\psi}|| = 1} M_>(k_0; \hat{\psi}),$$

$$M_=(k_0; \mathcal{H}_n) = \bigcup_{\hat{\psi} \in \mathcal{H}_n, ||\hat{\psi}|| = 1} M_=(k_0; \hat{\psi}),$$

$$M_>(k_0; \mathcal{H}_n) = \bigcap_{\hat{\psi} \in \mathcal{H}_n, ||\hat{\psi}|| = 1} M_>(k_0; \hat{\psi}),$$

$$M_=(k_0; \mathcal{H}_n) = \bigcap_{\hat{\psi} \in \mathcal{H}_n, ||\hat{\psi}|| = 1} M_=(k_0; \hat{\psi}),$$

$$M_<(k_0; \ell^2, \mathbb{R}^d) = \bigcap_{\hat{\psi} \in \ell^2, \mathbb{R}^d, ||\hat{\psi}|| = 1} M_<(k_0; \hat{\psi}).$$

The following results precisely describes the emission of eigenvalues at the threshold $\mathcal{E}_{\min}(k)$ depending on the quasi-momentum $k \in \mathbb{G}$ and the potential $\hat{V}$.

**Theorem 4.12.** Let $d \geq 3$. Assume Hypothesis[3.4] and that for $k_0 \in \mathbb{G}$ the threshold $\mathcal{E}_{\min}(k_0)$ is a singular point (of multiplicity $n = 1, 2, …$) for the operator $\hat{H}_1(k_0) = \hat{H}_0(k_0) + \hat{V}$ satisfying the inequality $\hat{H}_1(k_0) \geq \mathcal{E}_{\min}(k_0)I$. Then:

(i) For any $k \in M_>(k_0; \mathcal{H}_n)$, the operator $\hat{H}_1(k)$ has an eigenvalue below the threshold $\mathcal{E}_{\min}(k)$.

(ii) For any $k \in M_=(k_0; \mathcal{H}_n)$, the threshold $\mathcal{E}_{\min}(k)$ is a singular point of the essential spectrum of $\hat{H}_\mu(k)$.

(iii) For any $k \in M_>(k_0; \mathcal{H}_n)$, the operator $\hat{H}_1(k)$ has at least $n \geq 1$ eigenvalues $z_1(k), ..., z_n(k)$ (counting multiplicities) below the threshold $\mathcal{E}_{\min}(k)$. 


(iv) For any \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \), the threshold \( \varepsilon_{\min}(k) \) is a singular point for the essential spectrum of \( \hat{H}_1(k) \) (with multiplicity \( n \)).

(v) For any \( k \in \mathcal{M}_\varepsilon(k_0; \ell^2, \varepsilon(\mathbb{Z}^d)) \), the operator \( \hat{H}_1(k) \) has no eigenvalues below the threshold \( \varepsilon_{\min}(k) \).

**Corollary 4.13.** Let the assumptions of Theorem 4.12 are fulfilled. Then:

(i) For any \( \mu > 1 \) and \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \), the operator \( \hat{H}_\mu(k) = \hat{H}_0(k) + \mu \hat{V} \) has at least one eigenvalue below the threshold \( \varepsilon_{\min}(k) \).

(ii) For any \( \mu > 1 \) and \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \cup \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \), the operator \( \hat{H}_\mu(k) = \hat{H}_0(k) + \mu \hat{V} \) has at least \( n \) eigenvalues (counting multiplicities) below the threshold \( \varepsilon_{\min}(k) \).

**Remark 4.14.** Theorem 4.12 yields that for any \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \), there is an open ball \( B(k) \) with the center \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \) such that the sets \( B(k) \cap \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \) and \( B(k) \cap \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \) are non-empty. Thus, for any \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \), we define the subset \( \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \) such that for any \( \mu > 1 \) and \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \), the operator \( \hat{H}_\mu(k) = \hat{H}_0(k) + \mu \hat{V} \) has at least \( n \) eigenvalues (counting multiplicities) below the threshold \( \varepsilon_{\min}(k) \) and it has no eigenvalues for \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \).

**Corollary 4.15.** Let the assumptions of Theorem 4.12 are fulfilled. Then for each \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \), there exists \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \) such that the number of eigenvalues of \( \hat{H}_1(k) \) below \( \varepsilon_{\min}(k) \) is greater than the number of eigenvalues (counting multiplicities) of \( \hat{H}_1(k) \) below \( \varepsilon_{\min}(k) \).

**Remark 4.16.** Let the operator \( \hat{H}_1(k_0), k_0 \in \mathcal{G} \) has \( m \geq 1 \) eigenvalues (counting multiplicities) lying below the threshold \( \varepsilon_{\min}(k_0) \). Then by the Theorem 4.12 the operator \( \mathcal{B}_1(k_0, \varepsilon_{\min}(k_0)) \) has \( m \) eigenvalues (counting multiplicities) \( \lambda_1 \geq \ldots \geq \lambda_m > 1 \). Let \( \mathcal{H}_m^{\varepsilon_0} \) be an \( m \)-dimensional subspace, spanned to the eigenfunctions of the operator \( \mathcal{B}_1(k_0, \varepsilon_{\min}(k_0)) \), \( k_0 \in \mathcal{G} \) associated to \( \lambda_1, \ldots, \lambda_m \). Then for any non-zero \( \psi \in \mathcal{H}_m^{\varepsilon_0} \), the inequality \( C_s(k, k_0; \psi) > 0 \), \( s \in \mathbb{Z}^d \) holds, where \( C_s(k, k_0; \psi) \) is defined in 4.12.

For the \( m \)-dimensional subspace \( \mathcal{H}_m^{\varepsilon_0} \), associated to the eigenvalues lying below the threshold \( \varepsilon_{\min}(k_0) \) of \( \hat{H}_1(k_0) \), we define the subset \( \mathcal{M}_\varepsilon(k_0; \mathcal{H}_m^{\varepsilon_0}) \subset \mathcal{G} \) as

\[
\mathcal{M}_\varepsilon(k_0; \mathcal{H}_m^{\varepsilon_0}) = \bigcap_{\psi \in \mathcal{H}_m^{\varepsilon_0}, \|\psi\| = 1} \mathcal{M}_\varepsilon(k_0; \hat{\psi}).
\]

Theorems 4.4 and 4.12 yield that the total number of both the multiplicity of singular point \( \varepsilon_{\min}(k) \) and the eigenvalues (counting multiplicities) below the threshold \( \varepsilon_{\min}(k) \), \( k \in \mathcal{G} \) of \( \hat{H}_\mu(k) = \hat{H}_0(k) + \mu \hat{V} \) is a nonincreasing function in \( k \in \mathcal{G} \) and \( \mu > 0 \) in the sense of the following Corollary.

**Corollary 4.17.** Let \( d \geq 3 \). Assume Hypothesis 3.3 and that \( \varepsilon_{\min}(k_0), k_0 \in \mathcal{G} \) is a singular point of multiplicity \( n \) and the operator \( \hat{H}_\mu(k_0) \) has only \( m \) eigenvalues (counting multiplicities) below the threshold \( \varepsilon_{\min}(k_0) \).

(i) If \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \cap \mathcal{M}_\varepsilon(k_0; \mathcal{H}_m^{\varepsilon_0}) \), then for all \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \) the operator \( \hat{H}_\mu(k) \) has at least \( n + m \) eigenvalues (counting multiplicities) lying below \( \varepsilon_{\min}(k) \).

(ii) For all \( \mu > 1 \) and \( k \in \mathcal{M}_\varepsilon(k_0; \mathcal{H}_n) \cap \mathcal{M}_\varepsilon(k_0; \mathcal{H}_m^{\varepsilon_0}) \), the operator \( \hat{H}_\mu(k) \) has at least \( n + m \) eigenvalues (counting multiplicities) lying below \( \varepsilon_{\min}(k) \).

5. The Proof of the Results

The proof of Lemma 3.7 (i) Since \( \varepsilon(\cdot) \) satisfies Hypothesis 3.3 for any \( k \in \mathcal{G} \) the point \( 0 \in \mathbb{T}^d, d \geq 3 \) is a non-degenerate minimum of the function \( \varepsilon_k(\cdot) \).
According to the parametrical Morse lemma (see, e.g., [15], p. 113) on smooth functions, one concludes that for any \( k \in \mathbb{G} \) there exists \( C^1(W_\gamma(0)) \)-diffeomorphisms \( \phi(k, \cdot) : W_\gamma(0) \to \mathbb{U}(0) \) of the ball \( W_\gamma(0) \subset \mathbb{R}^d \) to a neighborhood \( \mathbb{U}(0) \subset \mathbb{T}^d \) of the point \( 0 \in \mathbb{T}^d \), so that the function \( \mathcal{E}_k(\phi(k, q)) \) can be represented as

\[
\mathcal{E}_k(\phi(k, q)) = \mathcal{E}_{\min}(k) + q^2 = \mathcal{E}_{\min}(k) + q_1^2 + \ldots + q_d^2, \quad \mathcal{E}_{\min}(k) \in C^1(\mathbb{G}).
\]

The Jacobian \( J(k, \phi(k, q)) \) of the mapping \( \phi \) is continuous in \( (k, q) \in \mathbb{G} \times W_\gamma(0) \) and \( J(k, 0) > 0 \) for any \( k \in \mathbb{G} \).

Therefore, for any fixed \( k \in \mathbb{G} \) and \( x \in \mathbb{Z}^d \), the kernel function (of resolvent) \( \mathcal{R}_0(k, \mathcal{E}_{\min}(k); x) \) can be written as the sum

\[
\mathcal{R}_0(k, \mathcal{E}_{\min}(k); x) = \mathcal{R}^{(1)}_0(k, \mathcal{E}_{\min}(k); x) + \mathcal{R}^{(2)}_0(k, \mathcal{E}_{\min}(k); x),
\]

where

\[
\begin{align*}
\mathcal{R}^{(1)}_0(k, \mathcal{E}_{\min}(k); x) &= \int_{\mathbb{U}(0)} \frac{e^{i(p,x)} \eta(dp)}{\mathcal{E}_k(p) - \mathcal{E}_{\min}(k)}, \\
\mathcal{R}^{(2)}_0(k, \mathcal{E}_{\min}(k); x) &= \int_{\mathbb{T}^d \setminus \mathbb{U}(0)} \frac{e^{i(p,x)} \eta(dp)}{\mathcal{E}_k(p) - \mathcal{E}_{\min}(k)}.
\end{align*}
\]

By making a change of variables \( p = \phi(k, q) \), we obtain

\[
\mathcal{R}^{(1)}_0(k, \mathcal{E}_{\min}(k); x) = \int_{W_\gamma(0)} \frac{e^{i(\phi(k, q), x)}}{q_1^2 + \ldots + q_d^2} J(k, \phi(k, q)) \eta(dq).
\]

In the spheroidal coordinates, by denoting \( q = r\omega \), it can be rewritten as

\[
(5.1) \quad \mathcal{R}^{(1)}_0(k, \mathcal{E}_{\min}(k); x) = \int_0^\gamma r^{d-3} dr \int_{\Omega_{d-1}} e^{i(\phi(k, r\omega), x)} J(k, \phi(k, r\omega)) d\omega,
\]

where \( S^{d-1} \) is the unit sphere in \( \mathbb{R}^d \) and \( d\omega \) is its element.

Observe that the function

\[
\mathcal{R}^{(2)}_0(k, \mathcal{E}_{\min}(k); x) = \int_{\mathbb{T}^d \setminus \mathbb{U}(0)} \frac{e^{i(p,x)} \eta(dp)}{\mathcal{E}_k(p) - \mathcal{E}_{\min}(k)}
\]

is continuously differentiable in \( k \in \mathbb{G} \). Since \( J(k, \phi(k, q)) \) is continuous in \( (k, q) \in \mathbb{G} \times W_\gamma(0) \) the functions \( \mathcal{R}^{(1)}_0(k, \mathcal{E}_{\min}(k); x) \) and \( \mathcal{R}_0(k, \mathcal{E}_{\min}(k); x) = \mathcal{R}^{(1)}_0(k, \mathcal{E}_{\min}(k); x) + \mathcal{R}^{(2)}_0(k, \mathcal{E}_{\min}(k); x) \) are continuous in \( k \in \mathbb{G} \).

(ii) For any fixed \( x, y \in \mathbb{Z}^d \), the continuity in \( k \in \mathbb{G} \) of the kernel function \( \mathcal{B}_U(k, \mathcal{E}_{\min}(k); x, y) \) defined in (3.8) can be proven by the same way as (i) of Lemma 3.7.
(iii) For any $k \in \mathbb{G}$, the Hilbert-Schmidt norm $||\mathcal{B}_\mu(k, E_{\min}(k))||_2$ of the operator $\mathcal{B}_\mu(k, E_{\min}(k))$ can be estimated as

$$||\mathcal{B}_\mu(k, E_{\min}(k))||_2^2 = \sum_{x, y \in \mathbb{Z}^d} |\mathcal{B}_\mu(k, E_{\min}(k); x, y)|^2$$

$$\leq \mu^2 \sum_{x, y \in \mathbb{Z}^d} |\hat{\psi}(x)| \left| \int_{\mathbb{T}^d} e^{i(p \cdot x - y)} \eta(dp) \right|^2 |\hat{\psi}(y)|$$

$$\leq \mu^2 \left[ \int_{\mathbb{Z}^d} \frac{\eta(dp)}{E_k(p) - E_{\min}(k)} \right]^2 ||\hat{V}||^2_{L^1(\mathbb{Z}^d)},$$

i.e., the operator $\mathcal{B}_\mu(k, E_{\min}(k))$ belongs to $\Sigma_2$. Since the operator norm $||\mathcal{B}_\mu(k, E_{\min}(k))||$ satisfies the inequality

$$||\mathcal{B}_\mu(k, E_{\min}(k))||^2 \leq ||\mathcal{B}_\mu(k, E_{\min}(k))||^2_2$$

the operator $\mathcal{B}_\mu(k, E_{\min}(k))$ is continuous in $k \in \mathbb{G}$. □

**The proof of Theorem 3.12** We only prove the items (i), (ii) and (iii), since the case (iv) can be proven as in Lemma 2.2 of ref. [8] and the case (v) similarly to Theorem 6 of ref. [22].

Let $d \geq 3$ and $\hat{\psi} \in \ell^1(\mathbb{Z}^d; \mathbb{R}_{-})$.

(i) Note that $\hat{f} \in \ell_0(\mathbb{Z}^d)$ and $\hat{\psi} \in \ell^1(\mathbb{Z}^d; \mathbb{R}_{-})$ yield $\hat{\psi} = |\hat{V}|^{1/2} \hat{f} \in \ell^{2,\eta}(\mathbb{Z}^d)$. Now let $\hat{f} \in \ell_0(\mathbb{Z}^d)$ be a solution of $(\hat{R}_0(k) - E_{\min}(k)I)\hat{f} = 0$, i.e., the equality

$$\hat{R}_0(k) - E_{\min}(k)I \hat{f} = -\mu \hat{V} \hat{f}$$

holds. Then the conditions for $\hat{\psi}$ of Hypothesis 3.3 yields that $\hat{V} \hat{f} \in \ell^1(\mathbb{Z}^d)$. Using the definition of $\hat{R}_0(k, E_{\min}(k))$ and the properties of the Fourier transform, it can be shown that

$$[\hat{R}_0(k, E_{\min}(k))][\hat{R}_0(k) - E_{\min}(k)I] = I,$$

i.e., the operator $\hat{R}_0(k, E_{\min}(k))$ is the resolvent of $\hat{R}_0(k)$ at the threshold $z = E_{\min}(k)$.

The equality (5.2) yields the relation

$$\hat{f} = \hat{R}_0(k, E_{\min}(k)) \left( \hat{R}_0(k) - E_{\min}(k)I \right) \hat{f} = \mu \hat{R}_0(k, E_{\min}(k))|\hat{V}|^{1/2} \hat{f}$$

and

$$|\hat{V}|^{1/2} \hat{f} = \mu [|\hat{V}|^{1/2} \hat{R}_0(k, E_{\min}(k))]|\hat{V}|^{1/2} \hat{f},$$

i.e., $\hat{\psi} = |\hat{V}|^{1/2} \hat{f}$ is a solution of $\mathcal{B}_\mu(k, E_{\min}(k))\hat{\psi} = \hat{\psi}$.

(ii) Let $d = 3, 4$. Since $\hat{\psi} \in \ell^1(\mathbb{Z}^d; \mathbb{R}_{-})$ is a non-positive function, for any $\hat{\psi} \in \ell^{2,\eta}(\mathbb{Z}^d)$, the Cauchy–Schwarz inequality leads to the relation $|\hat{V}|^{1/2} \hat{\psi} \in \ell^1(\mathbb{Z}^d)$.

Since the function $\frac{1}{E_k(p) - E_{\min}(k)}$ is integrable on $\mathbb{T}^d$, the Riemann–Lebesgue lemma yields

$$\hat{R}_0(k, E_{\min}(k); x) := \int_{\mathbb{T}^d} e^{i(p \cdot x)} \eta(dp) - E_k(p) - E_{\min}(k) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty.$$

The inclusion $|\hat{V}|^{1/2} \hat{\psi} \in \ell^1(\mathbb{Z}^d)$ and (5.3) lead to

$$\hat{f}(x) = \sum_{y \in \mathbb{Z}^d} \hat{R}_0(k, E_{\min}(k); y - x)|\hat{\psi}(y)|^{1/2} \hat{\psi}(y) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty,$$
i.e., \( \hat{f} \in L_0(\mathbb{Z}^d) \).

Now let \( \hat{\psi} \in \ell^{2,c}(\mathbb{Z}^d) \) be a solution of the equation

\[
\mu|\hat{V}| \hat{\mu} \hat{\Delta}_0(k, \epsilon_{\min}(k))|\hat{V}| \hat{\mu} \hat{\Delta}_0(k, \epsilon_{\min}(k)) \hat{\psi} = \hat{\psi}.
\]

By denoting

\[
\hat{f} = \hat{\Delta}_0(k, \epsilon_{\min}(k))|\hat{V}| \hat{\mu} \hat{\psi}
\]

we have that \( \hat{f} \in L_0(\mathbb{Z}^d) \). Equation (5.5) yields that

\[
\mu|\hat{V}| \hat{f} = \hat{\psi} \quad \text{and} \quad \mu|\hat{V}| \hat{f} = |\hat{V}| \hat{\mu} \hat{\psi}.
\]

The equality (5.6) and the second equality in (5.7) imply that

\[
\mu|\hat{V}| \hat{f} = \hat{\psi}
\]

holds and hence the Plancherel theorem leads to

\[
\hat{R}_0(k, \epsilon_{\min}(k); x) := \int_{\mathbb{T}^d} e^{i(p,x)} \eta(dp) / \epsilon_k(p) - \epsilon_{\min}(k) \in \ell^{2,c}(\mathbb{Z}^d).
\]

Since for any \( \hat{\psi} \in \ell^{2,c}(\mathbb{Z}^d) \) the relation \( |\hat{V}| \hat{\psi} \in \ell^1(\mathbb{Z}^d) \) holds, the Young convolution inequality [28 IX.4] and the relation (5.9) yield that

\[
\hat{f} = \hat{R}_0(k, \epsilon_{\min}(k))|\hat{V}| \hat{\psi} \in \ell^{2,c}(\mathbb{Z}^d).
\]

The equality \( \hat{H}_\mu(k) \hat{f} = \epsilon_{\min}(k) \hat{f} \), \( \hat{f} \in \ell^{2,c}(\mathbb{Z}^d) \) can be proven as the case above. □

**Proof of Theorem 4.1**

Let \( d = 1, 2 \) and \( v \leq 0 \) is not identically zero. Then the number \( \lambda = v(s), s \in \mathbb{Z}^d \) is an eigenvalue of the operator \( \hat{V} \) and the kronecker delta function \( \hat{\psi}_s(x) = \delta_{sx} \in \ell^{2,c}(\mathbb{Z}^d) \) of the point \( s \in \mathbb{Z}^d \) is the associated eigenfunction. Then for any \( z < \epsilon_{\min}(k) \)

\[
(\mathbb{B}_\lambda(k, z) \hat{\psi}_s, \hat{\psi}_s)
= \mu(|\hat{V}| \hat{\Delta}_0(k, z)|\hat{V}| \hat{\psi}_s, \hat{\psi}_s)
= \mu(|\hat{R}_0(k, z)|\hat{V}| \hat{\psi}_s, |\hat{V}| \hat{\psi}_s)
= \mu \int_{\mathbb{T}^d} \frac{|\hat{F} \circ |\hat{V}| |^{1/2} \hat{\psi}_s|^2 \eta(dp)}{\epsilon_k(p) - z}
= \mu \int_{\mathbb{T}^d} \frac{\sum_{x \in \mathbb{Z}^d} e^{i(p,x)} |\hat{\psi}(x)| |\hat{V}| \hat{\psi}_s(x)|^2 \eta(dp)}{\epsilon_k(p) - z}
= \mu |\hat{\psi}(s)| \int_{\mathbb{T}^d} \frac{\eta(dp)}{\epsilon_k(p) - z}.
\]
Proposition 3.1 yields the existence of \( m \) the eigenvectors, associated to the eigenvalues \( z \) of the operator \( B \) which can be proven analogously to [22, Theorem 6].

Therefore, for some \( z < \varepsilon_{\min}(k) \), the inequality \( (B_\mu(k, z) \hat{\psi}_z, \hat{\psi}_z) > 1 \) holds, i.e., the self-adjoint compact operator \( B_\mu(k, z) \) has an eigenvalue in the interval \((1, +\infty)\). Then, Proposition 3.1 yields that the operator \( \tilde{H}_\mu(k) \) has an eigenvalue in the interval \(( -\infty, \varepsilon_{\min}(k) )\).

The proof of Theorem 4.2 can be proven analogously to Theorem 1 of [22].

Proof of Theorem 4.2. Let \( d = 1, 2 \) and operator \( \tilde{H}_1(0) \), \( 0 \in T^d \) has \( m \geq 1 \) eigenvalues \( z_1(0) \leq ... \leq z_m(0) < \varepsilon_{\min}(0) \) (counting multiplicities). Then for any \( z_0 \in (z_m(0), \varepsilon_{\min}(0)) \), the Proposition 3.1 yields the existence of \( m \)-dimensional linear subspace \( \mathcal{H}_m \subset \ell_2, e \left( \mathbb{Z}^d \right) \) spanned to the eigenvectors, associated to the eigenvalues \( \lambda_1(0) \geq ... \geq \lambda_m(0) > 1 \) (counting multiplicities), of the operator \( B_1(0, z_0) \), such that for any non-zero \( \psi \in \mathcal{H}_m \) the relations \( |\mathcal{F} \circ \hat{V}^{1/2} \hat{\psi}| \neq 0 \) and

\[
(\mathbb{B}_1(0, z_0) \hat{\psi}, \hat{\psi}) = (\hat{V}^{1/2} \hat{R}_0(0, z_0) \hat{V}^{1/2} \hat{\psi}, \hat{\psi}) = (\hat{R}_0(0, z_0) |\hat{V}^{1/2} \hat{\psi}|, |\hat{V}^{1/2} \hat{\psi}|)
\]

hold.

Since the function \( \varepsilon \) is conditionally negative definite, the minimum \( \varepsilon_{\min}(k) = \min_{p \in T^d} \varepsilon_k(p) = 2 \varepsilon(\hat{\psi}), k \in T^d \) of the function \( \varepsilon_k(p) \) is attained at the point \( p(k) = 0 \). Hence, for any non-zero \( k \in T^d \) and \( z_0 \in (z_m(0), \varepsilon_{\min}(0)) \), the inequality

\[
\varepsilon_0(p) - z_0 > \varepsilon_k(p) - [z_0 + \varepsilon_{\min}(k) - \varepsilon_{\min}(0)], \ p \in T^d
\]

holds, which can be proven similarly to [6] Lemma 5. Then for any non-zero \( k \in T^d \) and \( \hat{\psi} \in \mathcal{H}_m \) we have

\[
\int_{T^d} \frac{|(\mathcal{F} \circ \hat{V}^{1/2} \hat{\psi})(p)|^2 \eta(dp)}{\varepsilon_0(p) - z_0} < \int_{T^d} \frac{|(\mathcal{F} \circ \hat{V}^{1/2} \hat{\psi})(p)|^2}{\varepsilon_k(p) - [z_0 + \varepsilon_{\min}(k) - \varepsilon_{\min}(0)]} \eta(dp).
\]

Therefore, for any non-zero \( k \in T^d \) and \( \hat{\psi} \in \mathcal{H}_m \), the relation

\[
(\mathbb{B}_\mu(k, z_0 + [\varepsilon_{\min}(k) - \varepsilon_{\min}(0)] \hat{\psi}, \hat{\psi}) > (\hat{\psi}, \hat{\psi})
\]

holds, which implies that the compact operator \( \mathbb{B}_\mu(k, z_0 + [\varepsilon_{\min}(k) - \varepsilon_{\min}(0)]) \) has \( m \) eigenvalues (counting multiplicity) greater than 1. Proposition 3.1 gives that the operator \( \tilde{H}_\mu(k) \) has \( m \) eigenvalues \( z_1(k) \leq ... \leq z_m(k) \) (counting multiplicity), which satisfy the relations

\[
z_j(k) < z_0 + [\varepsilon_{\min}(k) - \varepsilon_{\min}(0)], \ j = 1, ..., m.
\]

This completes the proof of Theorem 4.2. □

Theorem 4.6 can be proven by the same way as Theorem 4.1.

Proof of the Theorem 4.8 follows from the equality

\[
N_-(\mathbb{B}_\mu(k), \tilde{H}_\mu(k)) = N_+(1, \mathbb{B}_\mu(k, \varepsilon_{\min}(k))),
\]

which can be proven analogously to [22] Theorem 6.
Proof of Theorem 4.9 If $\varepsilon_{\min}(k_0)$ is a regular point, then the number 1 is not an eigenvalue of the operator $B_\mu(k_0, \varepsilon_{\min}(k_0))$ and hence there exists a bounded operator

$$(I - B_\mu(k_0, \varepsilon_{\min}(k_0)))^{-1}.\$$

Since, the operator $B_\mu(k, \varepsilon_{\min}(k))$ is continuous in $k \in \mathbb{G}$ (see (iii) of Lemma 3.7), there exists a neighborhood $G(k_0) \subset \mathbb{G}$ of the point $k_0 \in \mathbb{G}$, such that the operator $(I - B_\mu(k, \varepsilon_{\min}(k)))^{-1}$ exists and continuous in $k \in G(k_0)$. So, for any $k \in G(k_0)$, the number of non-zero solutions of $B_\mu(k, \varepsilon_{\min}(k))\psi = \hat{\psi}$ in $L^2(\mathbb{Z}^d)$ remains unchanged.

The existence of a neighborhood of $\mu_0 > 0$ can be proven similarly. □

Proof of Theorem 4.12 We prove the cases (iii) and (v) of Theorem 4.12 since the cases (i), (ii) and (iv) can be proven in the same way as (iii).

(iii) Under the assumptions of Theorem 4.12 the bottom $\varepsilon_{\min}(k_0)$ is a singular point of $\hat{H}_1(k_0), k_0 \in \mathbb{G}$ of multiplicity $n$. Definition 3.9 of the singular point and Theorem 3.12 yield the existence a $n$-dimensional subspace $\mathcal{H}_n \subset L^2(\mathbb{Z}^d)$, associated to eigenvalue 1 and for any non-zero $\hat{\psi} \in \mathcal{H}_n$ the relations

$$(B_1(k_0, \varepsilon_{\min}(k_0))\hat{\psi}, \hat{\psi}) = (\hat{R}_0(k_0, \varepsilon_{\min}(k_0))|\hat{V}|^{1/2}\hat{\psi}, |\hat{V}|^{1/2}\hat{\psi}) = \int_{\mathbb{T}^d} |\mathcal{T} \circ |\hat{V}|^{1/2}\hat{\psi}|^2\eta(dp)\frac{\varepsilon_{\min}(k_0) - \varepsilon_k(p)}{[\varepsilon_k(p) - \varepsilon_{\min}(k_0)]^2}\psi, \varepsilon_{\min}(k_0),\psi,$$

hold and therefore $|\mathcal{T} \circ |\hat{V}|^{1/2}\hat{\psi}| \neq 0$. Consequently, for any $k \in M_>(k_0; \mathcal{H}_n)$ and non-zero $\hat{\psi} \in \mathcal{H}_n$

$$(B_1(k, \varepsilon_{\min}(k))\hat{\psi}, \hat{\psi}) - (B_1(k_0, \varepsilon_{\min}(k_0))\hat{\psi}, \hat{\psi}) = \int_{\mathbb{T}^d} \frac{[\varepsilon_k(p) - \varepsilon_{\min}(k)]\varepsilon_{\min}(k_0) - \varepsilon_k(p)}{[\varepsilon_k(p) - \varepsilon_{\min}(k_0)]^2}\psi, \varepsilon_{\min}(k_0),\psi)$$

Applying the Fourier series expansion for $\varepsilon(p)$

$$\varepsilon_k(p) - \varepsilon_{\min}(k) = \varepsilon(p)\frac{k}{2} + \varepsilon(p)\frac{k}{2} - 2\varepsilon(p)\frac{k}{2} = 2\sum_{s \in \mathbb{Z}^d} \hat{\varepsilon}(s) \cos(\frac{k}{2}, s) \cdot \cos(p, s) - 1,$$

gives the representation

$$(B_1(k, \varepsilon_{\min}(k)) - B_1(k_0, \varepsilon_{\min}(k_0))\hat{\psi}, \hat{\psi}) = 2\sum_{s \in \mathbb{Z}^d, s \neq 0} \hat{\varepsilon}(s) [\cos(\frac{k}{2}, s) - \cos(\frac{k_0}{2}, s)] \mathcal{E}_s(k_0, \psi),$$

where

$$\mathcal{E}_s(k_0, \psi, k) = \int_{\mathbb{T}^d} \frac{[1 - \cos(p, s)]|\mathcal{T} \circ |\hat{V}|^{1/2}\hat{\psi}|^2\eta(dp)}{[\varepsilon_k(p) - \varepsilon_{\min}(k)]^2\varepsilon_k(p) - \varepsilon_{\min}(k_0)^2} > 0, s \in \mathbb{Z}^d \setminus \{0\}.$$
(v) For any $k \in \mathcal{M}_c(k_0; \ell^2(\mathbb{Z}^d))$ and non-zero $\hat{\psi} \in \ell^2(\mathbb{Z}^d)$, we have

\[
\| B_1(k, E_{\min}(k)) \| = \sup_{\hat{\psi} \in \ell^2(\mathbb{Z}^d), \| \hat{\psi} \|=1} (B_1(k, E_{\min}(k)) \hat{\psi}, \hat{\psi})
\]

\[
= \sup_{\hat{\psi} \in \ell^2(\mathbb{Z}^d), \| \hat{\psi} \|=1} (|\hat{\psi}|^2 \hat{R}_0(k, E_{\min}(k)) |\hat{\psi}|^2, \hat{\psi})
\]

\[
= \sup_{\hat{\psi} \in \ell^2(\mathbb{Z}^d), \| \hat{\psi} \|=1} (\hat{R}_0(k, E_{\min}(k)) |\hat{\psi}|^2, |\hat{\psi}|^2)
\]

\[
\leq \sup_{\hat{\psi} \in \ell^2(\mathbb{Z}^d), \| \hat{\psi} \|=1} \left( \int_{\mathbb{R}^d} |\mathcal{F} \circ |\hat{\psi}|^{1/2} \eta|^2 |dp\right) \frac{E_k(p) - E_{\min}(k)}{E_{k_0}(p) - E_{\min}(k_0)}
\]

\[
= (B_1(k_0, E_{\min}(k_0)) \hat{\psi}, \hat{\psi})
\]

\[
= \| B_1(k_0, E_{\min}(k_0)) \| = 1.
\]

Therefore, the non-negative compact operator $B_1(k, E_{\min}(k)), k \in \mathcal{M}_c(k_0; \ell^2(\mathbb{Z}^d))$ may have only non-negative eigenvalues smaller than 1. Consequently, the Theorem 3.12 yields that the operator $\hat{H}_1(k), k \in \mathcal{M}_c(k_0; \ell^2(\mathbb{Z}^d))$ has no eigenvalues below $E_{\min}(k)$, the bottom of the essential spectrum of $\hat{H}_1(k)$.

Acknowledgments S.N. Lakaev acknowledges to the Institute of Applied Mathematics of the University Mainz for its kind hospitality during his stay in the summer 2011 and also I.A. Ikromov and Sh. Yu. Kholmatov for useful discussions and remarks. This research was supported by the Foundation for Basic Research of the Republic of Uzbekistan (Grant No.OT-F4-66).

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\[ \text{E-mail address: slakaev@mail.ru} \]

[1] SAMARKAND STATE UNIVERSITY, SAMARKAND (UZBEKISTAN)

[2] INSTITUT FÜR ANALYSE UND ALGEBRA CARL-FRIEDRICH-GAUSS-FAKULTÄT TECHNISCHE UNIVERSITÄT BRAUNSCHWEIG (GERMANY)

[3] DEPARTAMENTO DE FÍSICA MATEMÁTICA DO INSTITUTO DE FÍSICA, UNIVERSIDADE DE SAO PAULO, 05314-970 SAO PAULO, SP, BRAZIL

\[ \text{E-mail address: v.bach@tu-bs.de} \]

\[ \text{E-mail address: wpedra@if.usp.br} \]