On the spectrum of the operator which is a composition of integration and substitution

by

Ignat Domanov (Donetsk)

Abstract. Let \( \phi : [0, 1] \rightarrow [0, 1] \) be a nondecreasing continuous function such that \( \phi(x) > x \) for all \( x \in (0, 1) \). Let the operator \( V_\phi : f(x) \rightarrow \int_0^x f(t)dt \) be defined on \( L_2[0, 1] \). We prove that \( V_\phi \) has a finite number of non-zero eigenvalues if and only if \( \phi(0) > 0 \) and \( \phi(1-\varepsilon) = 1 \) for some \( 0 < \varepsilon < 1 \). Also, we show that the spectral trace of the operator \( V_\phi \) always equals 1.

1. Introduction.

It is well known that the Volterra integration operator \( V : f(x) \rightarrow \int_0^x f(t)dt \) defined on \( L_2[0, 1] \) is quasinilpotent, that is, \( \sigma(V) = \{0\} \). Let \( \phi \in C[0, 1] \) such that \( \phi(0) = 0 \). It was pointed out in [9] and [10] that an operator \( V_\phi \) defined by

\[
V_\phi : f(x) \rightarrow \int_0^x f(t)dt
\]

is quasinilpotent on \( C[0, 1] \) whenever \( \phi(x) \leq x \) for all \( x \in [0, 1] \).

Let \( \phi : [0, 1] \rightarrow [0, 1] \) be a measurable function and let \( V_\phi : L_p[0, 1] \rightarrow L_p[0, 1] \) (\( 1 \leq p < \infty \)) be defined by (1.1). It was proved in [12] and [13] that \( V_\phi \) is quasinilpotent on \( L_p[0, 1] \) if and only if \( \phi(x) \leq x \) for almost all \( x \in [0, 1] \). It was noted in [13] and proved in [14] that the spectral radius of \( V_\alpha \) (defined on \( L_p[0, 1] \) or \( C[0, 1] \)) is \( 1 - \alpha \) (\( 0 < \alpha < 1 \)). The detailed investigation of the spectrum of the operator \( V_\alpha \) was done in [1], where it was shown that the point spectrum \( \sigma_p(V_\alpha) \) of \( V_\alpha \) is simple and \( \sigma_p(V_\alpha) = \{(1-\alpha)\alpha^{n-1}\} \) for all \( \varepsilon > 0 \).

The aim of this paper is to prove the following theorem.

**Theorem 1.1.** Let \( \phi : [0, 1] \rightarrow [0, 1] \) be a nondecreasing continuous function such that \( \phi(x) > x \) for all \( x \in (0, 1) \), and \( V_\phi \) be defined on \( L_2[0, 1] \) by (1.1). Set also \( \sigma_p(V_\phi) \setminus \{0\} = \{\lambda_n\}_{n=1}^\infty \) (\( 1 \leq \omega \leq \infty \)). Then:

1. \( \omega < \infty \) if and only if \( \phi(0) > 0 \) and \( \phi(1-\varepsilon) = 1 \) for some \( 0 < \varepsilon < 1 \);
2. \( \lim_{\varepsilon \to 0} \sum_{|\lambda_n| > \varepsilon} \lambda_n = 1 \);
3. \( \sum_{n=1}^\infty |\lambda_n|^{1+\varepsilon} < \infty \) for all \( \varepsilon > 0 \).

---

\[0\text{2000 Mathematics Subject Classification: 34l20, 45C05, 47A10, 47A75.}

Key words and phrases: eigenvalue, integral operator, Fredholm determinant.

This research was partially supported by NAS of Ukraine, Grant # 0105U006289.
The order of the material is as follows.

In section 2 we recall some classical results in the theory of trace-class operators, in the theory of Fredholm determinants and in the theory of entire functions. In section 3 we calculate the Fredholm determinant \( D_{V_\phi}(\lambda) \) of the operator \( V_\phi \). In section 4 we estimate the order of growth of \( D_{V_\phi}(\lambda) \) and prove Theorem 1.1. It turns out that the matrix trace of the operator \( V_\phi \) is not defined, but the spectral trace of \( V_\phi \) does not depend on \( \phi \) and always equals 1. This contrasts with the fact that \( \sigma_p(V_x) = \{\emptyset\} \). We find also the spectral(= matrix) traces of the \( V_2^2 \) and \( V_3^3 \). In section 5 we assume that \( \phi : [0,1] \to [0,1] \) is a strictly increasing continuous function such that \( \text{card}\{x : \phi(x) = x\} < \infty \) and describe the spectrum of \( V_\phi \). Then we consider \( V_\phi \) defined on the space \( L_p[0,1] \).

2. Preliminaries. Here we recall some facts about trace class operators, Fredholm determinants and entire functions.

2.1. Let \( K \) be a compact operator defined on an infinite dimensional Hilbert space \( \mathcal{H} \). Let \( s_n(K) (n \geq 1) \) be the eigenvalues of \( KK^* \). The operator \( K \) is said to be of class \( S_p \) if \( \sum_{n=1}^{\infty} s_n(K)^p < \infty \). The trace \( \text{tr}K \) of an operator \( K \in S_1 \) is defined as its matrix trace: \( \text{tr}K = \sum_{n=1}^{\infty} (Ke_n, e_n) \), where \( \{e_n\}_{n=1}^{\infty} \) is some orthonormal basis. It is known that \( \text{tr}K \) does not depend on the choice of \( \{e_n\}_{n=1}^{\infty} \) and the series \( \sum_{n=1}^{\infty} (Ke_n, e_n) \) converges absolutely.

The celebrated theorem of Lidskii (see [4]) says that the matrix trace of an operator \( K \in S_1 \) is equal to its spectral trace, which is defined as the sum of eigenvalues of \( K \) (counted with the algebraic multiplicity):

\[
\text{tr}K = \sum_{n=1}^{\infty} (Ke_n, e_n) = \sum_{n=1}^{\omega} \lambda_n, \quad \omega \leq \infty. \tag{2.2}
\]

Let \( K \) be an integral operator: \( (Kf)(x) = \int_{0}^{1} k(x,t)f(t)dt \) on \( L_2[0,1] \). It is well known (see [4]) that if \( k(x,t) \) is a continuous function on \([0,1] \times [0,1] \), then \( K \in S_1 \) and \( \text{tr}K \) is given by the integral of its diagonal:

\[
\text{tr}K = \int_{0}^{1} k(t,t)dt. \tag{2.3}
\]

2.2. Now let \( k(x,t) \) be a bounded function on \([0,1] \times [0,1] \). By definition, put

\[
D_K(\lambda) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n \lambda^n, \tag{2.4}
\]
where $A_0 := 1$ and

$$A_n := \int_0^1 \ldots \int_0^1 K(t_1, \ldots, t_n) dt_1 \ldots dt_n,$$

$$K(t_1, \ldots, t_n) := \det \begin{pmatrix} k(t_1, t_1) & \ldots & k(t_1, t_n) \\ \vdots & \ddots & \vdots \\ k(t_n, t_1) & \ldots & k(t_n, t_n) \end{pmatrix}$$ (2.5)

for $n \geq 1$. The function $D_K(\lambda)$ is called the Fredholm determinant of $K$.

Recall (see [6, 8, 11]) that:

1. $A_n = n! \int_0^1 \ldots \int_0^1 K(t_1, \ldots, t_n) dt_n \ldots dt_1$, $n \geq 1$; (2.6)

2. $D_K(\lambda)$ is an entire function of $\lambda$ of the order $\rho \leq 2$;

3. $D_K(\mu^*) = 0$ if and only if $\lambda^* := 1/\mu^* \in \sigma_p(K)$ and the multiplicity of $\mu^*$ as a root of the Fredholm determinant of $K$ is equal to the algebraic multiplicity of the eigenvalue $\lambda^*$.

### 2.3. From Hadamard’s theorem

From Hadamard’s theorem (Th 1, p.26, [7]) and Lindelöf’s theorem (Th 3, p.33, [7]), we get the following

**Theorem 2.2.** Let $f(z)$ be an entire function of order $\rho_f \leq 1$ and type $\sigma_f < \infty$. Let also $\{a_n\}_{n=1}^\omega$ ($\omega \leq \infty$) be all roots of $f(z)$ and $f(0) = 1$. Then

(i) if $\rho_f = 1$, $\sigma_f = 0$ and $\sum_{n=1}^\omega |a_n| < \infty$, then $\omega = \infty$, $f(z) = \prod_{n=1}^\infty (1 - \frac{z}{a_n})$ and $\sum_{n=1}^\infty \frac{1}{a_n} = -f'(0)$;

(ii) if $\rho_f < 1$, then $f(z) = \prod_{n=1}^\omega (1 - \frac{z}{a_n})$ and $\sum_{n=1}^\omega \frac{1}{a_n} = -f'(0)$;

(iii) if $\rho_f = 0$, then $\sum_{n=1}^\omega \frac{1}{|a_n|^r} < \infty$ for each $\epsilon > 0$;

(iv) if $\rho_f = 1$, $\sigma_f = 0$ and $\sum_{n=1}^\infty \frac{1}{|a_n|} = \infty$,

then $f(z) = e^{az} \prod_{n=1}^\omega \left(1 - \frac{z}{a_n}\right)^{z/a_n}$ and $\limsup_{r \to \infty} |a + \sum_{|a_n| < r} \frac{1}{a_n}| = 0$.

In particular, $\limsup_{r \to \infty} \left(\sum_{|a_n| < r} \frac{1}{a_n}\right) = -a = -f'(0)$.

(v) $\sum_{n=1}^\omega \frac{1}{|a_n|^{1+r}} < \infty$ for each $\epsilon > 0$.

### 3. The Fredholm determinant of the operator $V_\phi$.

We begin with an auxiliary lemma.
Lemma 3.3. Let \( A = (a_{ij})_{i,j=1}^n \) be an \( n \times n \) matrix all of whose elements are 0 or 1 and \( a_{ij} = 1 \) for \( 1 \leq j \leq i \leq n \). Then

\[
\det A = \prod_{i=2}^n (1 - a_{i-1i}) = \begin{cases} 
1, & a_{i-1i} = 0 \text{ for } 2 \leq i \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]

Proof. The proof is trivial. \( \square \)

Theorem 3.4. Let \( \phi : [0, 1] \to [0, 1] \) be a nondecreasing continuous function such that \( \phi(x) > x \) for all \( x \in (0, 1) \). Let also \( V_\phi \) be defined on \( L_2[0, 1] \) by (1.1). Then

\[
D V_\phi(\lambda) = 1 + \sum_{n=1}^\infty (-1)^n \lambda^n \int_0^1 \int_{\phi(t_1)}^1 \cdots \int_{\phi(t_{n-1})}^1 dt_n \cdots dt_1. \quad (3.7)
\]

Proof. It is clear that \((V_\phi f)(x) = \int_0^1 k(x, t)f(t)dt =: (Kf)(x)\), where

\[
k(x, t) = \chi(\phi(x) - t) = \begin{cases} 
1, & \phi(x) \geq t; \\
0, & \phi(x) < t.
\end{cases}
\]

Assume that \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1 \). Then \( k(t_i, t_j) = 1 \) for \( 1 \leq j \leq i \leq n \) and the matrix \((k(t_i, t_j))_{i,j=1}^n\) satisfies the assumptions of Lemma 3.3. Hence, \( K(t_1, \ldots, t_n) = \prod_{i=2}^n (1 - k(t_{i-1}, t_i)) \). Further, using (2.4), (2.5), and (2.6) we get

\[
A_n = n! \int_0^1 \int_{t_1}^1 \int_{t_2}^1 \cdots \int_{t_{n-1}}^1 \prod_{i=2}^n (1 - k(t_{i-1}, t_i)) dt_n \cdots dt_1 = n! \int_{\Omega_n} 1 dt_n \cdots dt_1,
\]

where

\[
\Omega_n := \{(t_1, \ldots, t_n) : 0 \leq t_1 \leq \cdots \leq t_n \leq 1, k(t_1, t_2) = \cdots = k(t_{n-1}, t_n) = 0\} = \{(t_1, \ldots, t_n) : 0 \leq t_1 \leq \phi(t_1) \leq t_2 \leq \phi(t_2) \leq \cdots \leq \phi(t_{n-1}) \leq t_n \leq 1\}.
\]

That is

\[
A_n = n! \int_0^1 \int_{\phi(t_1)}^1 \cdots \int_{\phi(t_{n-1})}^1 dt_n \cdots dt_1, \quad n \geq 1.
\]

This completes the proof. \( \square \)

4. The spectrum of the operator \( V_\phi \).

The following Proposition immediately follows from Theorem 3.4.

Proposition 4.5. Let \( \phi : [0, 1] \to [0, 1] \) be a nondecreasing continuous function such that \( \phi(x) > x \) for all \( x \in (0, 1) \). Then \( \sigma_p(V_\phi) \cap \mathbb{R}_- = \{0\} \).
Lemma 4.6. Suppose \( \phi : [0, 1] \rightarrow [0, 1] \) is a nondecreasing continuous function and \( \phi(x) > x \) for \( x \in (0, 1) \); then the following conditions are equivalent:

(i) \( \phi(0) > 0 \) and \( \phi(1 - \varepsilon) = 1 \) for some \( 0 < \varepsilon < 1 \);
(ii) there exists a unique \( N = N(\phi) \in \{2, 3, \ldots\} \) such that \( \phi^N(x) := \phi(\phi(\ldots \phi(x))) = 1 \) for all \( x \in [0, 1] \) and \( \phi^{N-1}(x_0) \neq 1 \) for some \( x_0 \in [0, 1] \).

Proof. The proof is left to the reader. \( \square \)

Theorem 4.7. Let \( \phi : [0, 1] \rightarrow [0, 1] \) be a nondecreasing continuous function such that \( \phi(x) > x \) for all \( x \in (0, 1) \). Suppose also that \( \phi(0) > 0 \), \( \phi(1 - \varepsilon) = 1 \) for some \( 0 < \varepsilon < 1 \), and \( N = N(\phi) \) is determined by Lemma 4.6 (ii). Then

(1) \( \sigma_p(V_\phi) \setminus \{0\} \) is a finite set. Moreover, \( \sigma_p(V_\phi) = \{0\} \cup (\lambda_1, \ldots, \lambda_N) \), where \( \lambda_n \neq 0 \);
(2) \( \sum_{n=1}^{N} \lambda_n = 1 \).

Proof. It is easily shown that \( 0 \in \sigma_p(V_\phi) \). Using Theorem 3.4, we get \( D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^{\infty} A_n\lambda^n \), where \( A_n = (-1)^n \int_{0}^{1} \int_{0}^{1} \ldots \int_{0}^{1} dt_n \ldots dt_1 \). It is easily shown that \( \phi^{n-1}(t_1) \leq t_n \leq 1 \). Since \( \phi^n(x) = 1 \) for \( n \geq N \), it follows that \( A_n = 0 \) for \( n \geq N + 1 \). Therefore \( D_{V_\phi}(\lambda) \) is a polynomial of degree \( N \) and (1) is proved. Further note that \( D_{V_\phi}(\lambda) = \prod_{n=1}^{N} (1 - \frac{1}{a_n}) \). Thus \( \sum_{n=1}^{N} \lambda_n = \sum_{n=1}^{N} \frac{1}{a_n} = -A_1 = 1 \). \( \square \)

Let \( \alpha_i(x), \beta_i(x) \in C[0, 1] \) (1 \( \leq i \leq n \)). By definition, put

\[
\left\{ \frac{\alpha_1 \ldots \alpha_n}{\beta_1 \ldots \beta_n} \right\} := \int_{\beta_1(x)}^{\alpha_1(x)} \int_{\beta_2(x)}^{\alpha_2(x)} \int_{\beta_3(x)}^{\alpha_3(x)} \ldots \int_{\beta_n(x)}^{\alpha_n(x)} dx_n \ldots dx_1.
\]

So \( \left\{ \frac{\alpha_1 \ldots \alpha_n}{\beta_1 \ldots \beta_n} \right\} \) is a function of \( x \). It is clear that

\[
\left\{ \frac{\alpha_1 \ldots \alpha_i \ldots \alpha_n}{\beta_1 \ldots \beta_i \ldots \beta_n} \right\} + \left\{ \frac{\alpha_1 \ldots \alpha_i \gamma_i \ldots \alpha_n}{\beta_1 \ldots \beta_i \ldots \beta_n} \right\} = \left\{ \frac{\alpha_1 \ldots \alpha_i \ldots \alpha_{i-1} \beta_i + \gamma_i \alpha_{i+1} \ldots \alpha_n}{\beta_1 \ldots \beta_i \ldots \beta_n} \right\}.
\]

The following lemmas are needed.
Lemma 4.8. Let \(0 < \varepsilon_1 < \varepsilon_2 < 1\) and

\[
\psi(x) = \begin{cases} 
\psi_1(x), & x \in [0, \varepsilon_1]; \\
\psi_2(x), & x \in [\varepsilon_1, \varepsilon_2]; \\
\psi_3(x), & x \in [\varepsilon_2, 1];
\end{cases}
\]

be a strictly increasing continuous function such that \(\psi(\varepsilon_1) = \varepsilon_1\) and \(\psi(\varepsilon_2) = \varepsilon_2\). Let also \(a_0 = b_0 = c_0 = 1\) and \(a_k, b_k, c_k, d_k\) \((k = 1, 2 \ldots)\) be \(k\)-multiple integrals defined by

\[
a_k := \{\varepsilon_1, \psi_1, 1\}, \quad b_k := \{\varepsilon_2, \psi_2, \psi_2\}, \quad c_k := \{1, \psi_3, \psi_3\}, \quad d_k := \{0, \psi, \psi\}.
\]

Then

\[
d_n = \sum_{k=0}^{n} c_k \sum_{l=0}^{n-k} b_l a_{n-k-l}, \quad n = 1, 2, \ldots.
\]

Proof. Using (4.8), we get

\[
d_n = \left\{\varepsilon_1 + \frac{\varepsilon_1}{\varepsilon_2} + \frac{1}{\varepsilon_2} \psi \cdot \psi \psi \right\} = \left\{\varepsilon_1, \psi_1, 1\right\} \cup \left\{0, 0, \ldots, 0\right\}
\]

\[
+ \left\{\frac{\varepsilon_1}{\varepsilon_2}, \frac{\psi_1}{\varepsilon_2} \psi \psi \psi \psi \right\} + \left\{\frac{1}{\varepsilon_1}, \frac{\varepsilon_2}{\varepsilon_2} \psi_3 \psi \psi \right\}
\]

\[
=: K_n + L_n + M_n.
\]

By definition \(K_n = a_n\). Further, again using (4.8), we get

\[
L_n = \left\{\varepsilon_2, \frac{\varepsilon_1}{\varepsilon_2} \psi_1, \psi_1, 1\right\} + \left\{\frac{\varepsilon_2}{\varepsilon_2} \psi_2, \frac{\varepsilon_1}{\varepsilon_2} \psi_2, \psi \right\} + \left\{\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_1} \psi_3, \psi \right\} + \left\{1, \psi_3, \psi_3\right\}
\]

\[
= b_1 a_{n-1} + b_2 a_{n-2} + b_3 a_{n-3} + \cdots = \sum_{k=1}^{n} b_k a_{n-k},
\]

\[
M_n = \left\{\frac{1}{\varepsilon_1}, \psi_1, 1\right\} + \left\{\frac{1}{\varepsilon_2}, \frac{\psi_2}{\varepsilon_2} \psi, \psi \right\} + \left\{\frac{1}{\varepsilon_2}, \psi_3, \psi \right\}
\]

\[
= c_1 a_{n-1} + c_1 L_n + \left\{\frac{1}{\varepsilon_2}, \varepsilon_1, \frac{\varepsilon_2}{\varepsilon_2} \psi_3, \psi_3, \psi \right\}
\]

\[
= c_1 a_{n-1} + c_1 L_n + c_2 a_{n-2} + c_2 L_n + \cdots
\]

\[
= \cdots = \sum_{k=1}^{n} c_k a_{n-k} + \sum_{k=1}^{n-1} c_k L_{n-k} = \sum_{k=1}^{n} c_k a_{n-k} + \sum_{k=1}^{n} c_k \sum_{l=1}^{n-k} b_l a_{n-k-l}.
\]
Finally, we obtain
\[d_n = K_n + L_n + M_n = c_0 a_n + \sum_{k=1}^n b_k a_{n-k} + \sum_{k=1}^n c_k a_{n-k} + \sum_{k=1}^n c_k \sum_{l=1}^{n-k} b_l a_{n-k-l}\]
\[= \sum_{k=0}^n c_k \sum_{l=0}^{n-k} b_l a_{n-k-l}.\]

\[\square\]

**Lemma 4.9.** Let \(0 < \varepsilon \leq 1/4, \beta > 1,\) and
\[
\psi_{\varepsilon, \beta}(x) = \begin{cases} x, & x \in [0, \varepsilon]; \\
\varepsilon + (1 - 2\varepsilon)^{1-\beta}(x - \varepsilon)^\beta, & x \in [\varepsilon, 1 - \varepsilon]; \\
x, & x \in [1 - \varepsilon, 1]; \end{cases}
\]
Then
\[d_n = \begin{cases} 1, & \psi_{\varepsilon, \beta} \\
0, & \psi_{\varepsilon, \beta} \end{cases} \quad (4.9)\]
\[+ \sum_{l=2}^n \frac{(1 - 2\varepsilon)^l (2\varepsilon)^{n-l}}{(n-l)! (1 + \beta) \ldots (1 + \beta + \ldots + \beta^{l-1})}, \quad n = 1, 2, \ldots;\]
\[< \text{const}(\varepsilon, \beta) \frac{(4\varepsilon)^n}{n!}, \quad n = 1, 2, \ldots,\]
where \(\text{const}(\varepsilon, \beta)\) does not depend on \(n\).

**Proof.** Substituting \(\psi_{\varepsilon, \beta}\) for \(\psi(x)\) in Lemma 4.8, we get (4.9). Indeed, it is easily proved that \(a_l = c_l = \frac{\varepsilon^l}{l!} (l = 0, 1, \ldots n)\). By definition, put \(\tilde{b}_1(x) := (1 - 2\varepsilon)^{1-\beta}(x - \varepsilon)^\beta, \psi_2(x) := \varepsilon + \tilde{b}_1(x),\) and
\[\tilde{\psi}_l(x) := \begin{cases} \psi_2, & l = 2, 3, \ldots; \\
\varepsilon, & l = 1. \end{cases}\]
Then \(\tilde{b}_{l+1}(x) = \int_{\varepsilon}^{\psi_2(x)} \tilde{b}_l(t) \, dt\). It can easily be checked (by induction on \(l\)) that
\[\tilde{b}_l(x) = \frac{(1 - 2\varepsilon)^{l-\beta - \ldots - \beta'}(x - \varepsilon)^\beta + \beta^2 + \ldots + \beta'}{(1 + \beta) \ldots (1 + \beta + \ldots + \beta^{l-1})}, \quad l = 2, 3, \ldots.
\]
Since \(b_l = \tilde{b}_l(1 - \varepsilon),\) we see that
\[b_0 = 1, \quad b_1 = 1 - 2\varepsilon, \quad b_l = \frac{(1 - 2\varepsilon)^l}{(1 + \beta) \ldots (1 + \beta + \ldots + \beta^{l-1})}, \quad l = 2, 3, \ldots (4.10)\]
Using Lemma 4.8, we get
\[
d_n = \sum_{k=0}^{n} c_k \sum_{l=0}^{n-k} b_l a_{n-k-l} = \sum_{l=0}^{n} b_l \sum_{k=0}^{n-l} c_k a_{n-k-l} = \sum_{l=0}^{n} b_l \sum_{k=0}^{n-l} \frac{\varepsilon^k}{k! (n-k-l)!} \sum_{l=0}^{n-l} \frac{\varepsilon^{n-l}}{(n-l)!} l!(n-l-k)! = \sum_{l=0}^{n} \frac{b_l (2\varepsilon)^{n-l}}{(n-l)!} n = 1, 2, \ldots.
\]
(4.11)

Substituting (4.10) for \(b_l\) in (4.11) we get (4.9).

(2) Taking into account the inequality of arithmetic and geometric means, we obtain
\[
(1 + \beta) \ldots (1 + \beta + \ldots \beta^{l-1}) \geq 2\beta^{1/2} \beta^{2/2} \ldots \beta^{(l-1)/2} = \beta^{(l-1)/2}.
\]
(4.12)

Hence,
\[
b_l \leq \frac{(1-2\varepsilon)^l}{l!} \left( \frac{1}{\beta^{1/4}} \right)^{l^2-l} < \frac{(1-2\varepsilon)^l}{l!}.
\]

Let \(N\) be a number such that \(\left( \frac{1}{\beta^{1/4}} \right)^{l^2-l} < \frac{(2\varepsilon)^l}{l!}\) for \(l > N\) (for example, \(N = [4\log_{\beta} (\frac{1}{2\varepsilon} - 1)] + 2\)). Then \(b_l < \frac{(2\varepsilon)^l}{l!}\) for \(l > N\). Using (4.11), we get for \(n > N\)
\[
d_n = \sum_{l=0}^{N} b_l \frac{(2\varepsilon)^{n-l}}{(n-l)!} + \sum_{l=N+1}^{n} b_l \frac{(2\varepsilon)^{n-l}}{(n-l)!} \\
\leq \frac{(2\varepsilon)^n}{n!} \sum_{l=0}^{N} \frac{n!}{l!(n-l)!} \left( \frac{1-2\varepsilon}{2\varepsilon} \right)^l + \frac{(2\varepsilon)^n}{n!} \sum_{l=N+1}^{n} \frac{n!}{l!(n-l)!} \\
\leq \frac{(2\varepsilon)^n}{n!} \left( \frac{1-2\varepsilon}{2\varepsilon} \right)^N \sum_{l=0}^{N} \frac{n!}{l!(n-l)!} + \frac{(2\varepsilon)^n}{n!} \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \\
\leq \frac{(4\varepsilon)^n}{n!} \left( \left( \frac{1-2\varepsilon}{2\varepsilon} \right)^N + 1 \right).
\]

This completes the proof.

\[\square\]

Lemma 4.10. Let \(\beta > 1\) and
\[
\psi_\beta(x) := \begin{cases} 2\beta^{-1}x^\beta, & x \in [0, 1/2]; \\ 2\beta^{-1}(x - 1/2)^\beta + 1/2, & x \in [1/2, 1]; \end{cases} =: \begin{cases} \psi_1(x), & x \in [0, 1/2]; \\ \psi_2(x), & x \in [1/2, 1]; \end{cases}
\]

Let also \(a_0 = b_0 = 1\) and \(a_k, b_k, d_k \ (k = 1, 2 \ldots)\) be \(k\)-multiple integrals defined by
\[
a_k := \{1/2, \psi_1, \psi_1, 0, 0, \ldots, 0\}, \quad b_k := \{1/2, 1/2, \ldots, 1/2\}, \quad d_k := \{1\psi_\beta, \psi_\beta, 0, 0, \ldots, 0\}.
\]

8
Then

\[(1) \quad d_n = \sum_{l=0}^{n} b_l a_{n-l}, \quad n = 1, 2, \ldots; \quad (4.13)\]

\[(2) \quad d_n < \frac{\beta^{-n^2/2+n}}{n!}, \quad n = 1, 2, \ldots.\]

**Proof.** Substituting \(1/2\) for \(\varepsilon_1\) and 1 for \(\varepsilon_2\) in Lemma 4.8, we get (4.13). Further, it is not hard to prove that

\[a_1 = b_1 = 1/2 \quad \text{and} \quad a_l = b_l = 2^{-l} ((\beta + 1) \cdots (\beta^{-1} + \cdots + 1))^{-1} \quad \text{for} \quad l \geq 2.\]

Now, by (4.12),

\[a_l \leq 2^{-l} \beta^{-l/4} \frac{n!}{l!(n-l)!} \frac{\psi_{\varepsilon,\beta}^{-1}(x)}{n!} \quad \text{and} \quad d_n \leq \sum_{l=0}^{n} \frac{2^{-l} \beta^{-l/4} \frac{n!}{l!(n-l)!} \frac{\psi_{\varepsilon,\beta}^{-1}(x)}}{n!} \beta^{-n^2/2+n} < \frac{\beta^{-n^2/2+n}}{n!}.\]

\[\square\]

**Proposition 4.11.** Let \(\phi : [0, 1] \rightarrow [0, 1]\) be a nondecreasing continuous function.

1. If \(\phi(x) > x\) for \(x \in (0, 1)\) then the order of \(D_{V_{\phi}}(\lambda)\) does not exceed 1, and if it equals 1, \(D_{V_{\phi}}(\lambda)\) is of minimal type;
2. if for some \(0 < a < b < 1\)

\[\phi(x) \geq f_{a,b}(x) := \begin{cases} \frac{b}{a} x, & x \in [0, a], \\ \frac{1-a}{b-a} x + \frac{b-a}{b-a} x, & x \in [a, b], \end{cases}\]

for \(x \in [0, 1]\), then the order of \(D_{V_{\phi}}(\lambda)\) equals 0.

**Proof.** (1) Taking into account Theorem 3.4 we obtain \(D_{V_{\phi}}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n A_n \lambda^n\), where \(A_n = \left\{ \begin{array}{l} 1, 1, \cdots, 1 \\ 0, \phi, \cdots, \phi \end{array} \right\}\). Since \(\phi(x) > x\) for each \(0 < \varepsilon < 1/4\), it follows that there exists \(\beta > 1\) such that \(\phi(x) \geq \psi_{\varepsilon,\beta}^{-1}(x)\). Using Lemma 4.9 we get

\[A_n = d_n = \left\{ \begin{array}{l} 1, 1, \cdots, 1 \\ 0, \phi, \cdots, \phi \end{array} \right\} < \left\{ \begin{array}{l} 1, 1, \cdots, 1 \\ 0, \psi_{\varepsilon,\beta}^{-1}, \cdots, \psi_{\varepsilon,\beta}^{-1} \end{array} \right\} = \left\{ \begin{array}{l} 1, \psi_{\varepsilon,\beta}, \cdots, \psi_{\varepsilon,\beta} \\ 0, 0, \cdots, 0 \end{array} \right\} < \text{const}(\varepsilon, \beta) \frac{(4\varepsilon)^n}{n!}.\]

Therefore the order of growth of \(D_{V_{\phi}}(\lambda)\) does not exceed 1. Assume that the order of growth of \(D_{V_{\phi}}(\lambda)\) is equal to 1. Then the type of \(D_{V_{\phi}}(\lambda)\) does not exceed \(4\varepsilon\) for each \(\varepsilon < 1/4\). Thus \(D_{V_{\phi}}(\lambda)\) is of minimal type.
Since \( \phi(x) \geq f_{a,b}(x) \) for some \( 0 < a < b < 1 \), it follows that there exists \( \beta > 1 \) such that \( \phi(x) \geq \psi^{-1}_\beta(x) \). Using Lemma \[4.10\] we get

\[
A_n = d_n = \begin{cases} 1 & 1 \\ 0 & \phi \\ \vdots & \vdots \end{cases} < \begin{cases} 1 & 1 \\ 0 & \psi^{-1}_\beta \\ \vdots & \vdots \end{cases} \begin{cases} 1 \\ \psi_\beta \\ \vdots \end{cases} = \begin{cases} 1 \\ \psi_\beta \\ \vdots \end{cases} = \frac{\beta^{n^{2/2+n}}}{n!}.
\]

Therefore the order of growth of \( D_{V_\phi}(\lambda) \) equals 0.

\[\square\]

**Theorem 4.12.** Let \( \phi : [0, 1] \rightarrow [0, 1] \) be a nondecreasing continuous function such that \( \phi(x) > x \) for all \( x \in (0, 1) \). Suppose that either \( \phi(0) = 0 \) or \( \phi(1 - \varepsilon) \neq 1 \) for all \( 0 < \varepsilon < 1 \). Then

1. \( \sigma_p(V_\phi) \setminus \{0\} := (\lambda_1, \ldots, \lambda_n, \ldots) \) — is an infinite set;
2. \( \lim_{\varepsilon \rightarrow 0} \sum_{|\lambda_n| > \varepsilon} \lambda_n = 1 \); 
3. \( \sum_{n=1}^\omega |\lambda_n|^{1+\varepsilon} < \infty \) for all \( \varepsilon > 0 \).

**Proof.** Using Theorem \[3.4\], we get \( D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^\infty (-1)^n A_n \lambda^n \), where \( A_n = \begin{cases} 1 & 1 \\ 0 & \phi \\ \vdots & \vdots \end{cases} \). It is easy to see that if either \( \phi(0) = 0 \) or \( \phi(1 - \varepsilon) \neq 1 \) for all \( 0 < \varepsilon < 1 \), then \( A_n > 0 \) for \( n \geq 0 \). Therefore \( D_{V_\phi}(\lambda) \) is not a polynomial in \( \lambda \). Now we apply Proposition \[4.11\] (1). Suppose that the order of \( D_{V_\phi}(\lambda) \) is less than 1; then using Theorem \[2.2\] (ii), we get

\[
D_{V_\phi}(\lambda) = \prod_{n=1}^\omega (1 - \frac{\lambda}{a_n}).
\]

Since \( D_{V_\phi}(\lambda) \) is not a polynomial, it follows that \( \omega = \infty \) and \( \sum_{n=1}^\infty \lambda_n = \sum_{n=1}^\infty \frac{1}{a_n} = -A_1/A_0 = 1 \). Now suppose that the order of \( D_{V_\phi}(\lambda) \) is equal to 1; then \( D_{V_\phi}(\lambda) \) is of minimal type. Thus the spectrum of \( V_\phi \) is an infinite set. Now, the application of Theorem \[2.2\] (i), (iv) yields (2).

(3) follows from Theorem \[2.2\].

\[\square\]

Now we are ready to prove the main result of the paper

**Proof of Theorem \[4.11\]**

(1) follows from Theorem \[4.7\] (1) and Theorem \[4.12\] (1).

(2)-(3) follow from Theorem \[4.7\] (2) and Theorem \[4.12\] (2)-(3).

\[\square\]

**Theorem 4.13.** Let \( \phi : [0, 1] \rightarrow [0, 1] \) be a nondecreasing continuous function and for some \( 0 < a < b < 1 \)

\[
\phi(x) \geq \begin{cases} \frac{b}{a} x, & x \in [0, a], \\ \frac{1-b}{1-a} x + \frac{b-a}{1-a}, & x \in [a, b], \\ \end{cases}
\]

for all \( x \in [0, 1] \). Let also either \( \phi(0) = 0 \) or \( \phi(1 - \varepsilon) \neq 1 \) for all \( 0 < \varepsilon < 1 \). Then
(1) \( \sigma_p(V_\phi) \setminus \{0\} := (\lambda_1, \ldots, \lambda_n, \ldots) \) is an infinite set;
(2) \( \sum_{n=1}^{\infty} \lambda_n = 1; \)
(3) \( \sum_{n=1}^{\infty} |\lambda_n|^\varepsilon < \infty \) for all \( \varepsilon > 0. \)

Proof. (1) follows from Theorem 4.12 (1). By Proposition 4.11 (2), the order of \( D_{V_\phi}(\lambda) \) equals 0. Thus (2) and (3) are implied by (ii) and (iii) of Theorem 2.2.

Remark 4.14. (i) Suppose \( \phi(x) \) is a strictly increasing function and \( \phi(x) > x \) for all \( x \in (0, 1). \) Let also \( \phi(x) \in C^1[0, 1] \) and \( (\phi'(x))^{-1/2} \in L_\infty[0, 1]. \) We claim that \( V_\phi \not\in S_1. \) Indeed, let \( c := \left( \int (\phi'(s))^{1/2} ds \right)^{-1} \) and let \( W_\phi \) and \( T_\phi \)
be linear operators defined on \( L_2[0, 1] \) by

\[
(W_\phi f)(x) = \int_0^x (\phi'(t))^{1/2} f(t) dt, \quad (T_\phi f)(x) = f(c \int_0^x (\phi'(s))^{1/2} ds).
\]

It can easily be checked (see \([2]-[3]\)) that \( T_\phi \) and \( T_\phi^{-1} \) are bounded operators and \( cV_\phi = T_\phi^{-1} W_\phi T_\phi. \) Hence, (see \([3]\)) \( s_n(W_\phi) \geq \|T_\phi^{-1}\|^{-1} \|T_\phi\|^{-1} \|T_\phi^{-1}\|^{-1} c^{-2/(2n-1)\pi}. \) Further,

\[
(V_\phi V_\phi^* f)(x) = \int_0^1 \int_0^1 f(s) ds dt = \int_0^x \phi'(t) \int_0^1 f(s) ds dt = (W_\phi W_\phi^* f)(x).
\]

Thus \( s_n(V_\phi) = s_n(W_\phi) \geq \|T_\phi^{-1}\|^{-1} \|T_\phi\|^{-1} \|T_\phi^{-1}\|^{-1} c^{-2/(2n-1)\pi}. \) Hence, \( V_\phi \not\in S_1. \)

(ii) Since \( V_\phi \not\in S_1, \) it follows that the matrix trace of an operator \( V_\phi \) is not defined. Hence we cannot use \([2]-[3]\) to prove Theorem 4.13 (2). Nevertheless, \([2]-[3]\) hold for \( K = V_\phi \) and the orthonormal basis \( \{e_n\}_{n=1}^{\infty} \) defined by: \( e_1 := 1, e_{2n} := e^{2\pi inx} \) and \( e_{2n+1} := e^{-2\pi inx} (n = 1, 2, \ldots). \) Indeed, since \( \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi-x}{2} \) for \( x \in (0, 2\pi), \) it follows that

\[
\sum_{n=0}^{\infty} (V_\phi e_n, e_n) = \int_0^{1} \phi(x) dx
\]

\[
+ \sum_{n=1}^{\infty} \left( \int_0^{1} \frac{(e^{2\pi inx} - 1)e^{-2\pi inx}}{2\pi in} dx + \int_0^{1} \frac{(e^{-2\pi inx} - 1)e^{2\pi inx}}{-2\pi in} dx \right)
\]

\[
= \int_0^{1} \phi(x) dx + \sum_{n=1}^{\infty} \int_0^{1} \frac{\sin(2\pi n(\phi(x) - x))}{\pi n} dx
\]

\[
= \int_0^{1} \phi(x) dx + \int_0^{1} \frac{1}{\pi} (\pi - 2\pi(\phi(x) - x)) \frac{dx}{2} = 1.
\]
Further, \( \int_0^1 \chi(\phi(x) - x)dx = 1 \). Thus formulas (2.2)–(2.3) hold. This contrasts with the fact that \( \sum_{n=0}^\infty (V_\phi e_n, e_n) = \infty \).

(iii) Theorem 4.1.4 states that the spectral trace of an operator \( V_\phi \) always equals 1. This also contrasts with the fact that an operator \( V_x \) is quasinilpotent.

To estimate the spectral radius \( r(V_\phi) \) of the operator \( V_\phi \) we recall (see [14]) some results on integral operators with nonnegative kernels. Let \( (Kf)(x) = \int_0^1 k(x, t)f(t)dt \) and \( k(x, t) \geq 0 \) for \( (x, t) \in [0, 1] \times [0, 1] \). If there exist \( \alpha > 0 \) and a nonnegative function \( f \) such that \( (Kf)(x) \geq \alpha f(x) \) for \( x \in [0, 1] \), then \( r(K) \in \sigma_p(K) \) and \( r(K) > \alpha \).

**Proposition 4.15.** Let \( \phi : [0, 1] \to [0, 1] \) be a strictly increasing continuous function such that \( \phi(x) \geq x \) for all \( x \in [0, 1] \). Set also \( \sigma_p(V_\phi) = \{ \lambda_n \}_{n=1}^\infty \) \( (\omega \leq \infty) \). Then

1. \( r(V_\phi) \geq \max_{x \in [0, 1]} (\phi(x) - x) \), \( r(V_\phi) \in \sigma_p(V_\phi) \).

Let also \( \phi(0) = 0 \). Then \( \omega = \infty \) and

2. \( \sum_{n=1}^\infty \lambda_n^2 = 2 \int_0^1 \phi(t)dt - 1 \); 
3. \( \sum_{n=1}^\infty \lambda_n^3 = 1 - 3 \int_0^1 \phi(t)\phi^{-1}(t)dt \).

**Proof.** (1) Let \( f_a(x) = 1 - \chi(a - x) \), \( a \in (0, 1) \) then

\[
(V_\phi f_a)(x) = \begin{cases} 0, & x \in [0, \phi^{-1}(a)], \\ \phi(x) - a, & x \in [\phi^{-1}(a), 1]. \end{cases}
\]

and (1) is proved.

(2), (3) It is easy to check that \( \phi^{-1}(x) \) is well defined and

\[
(V_\phi^2 f)(x) = \int_0^1 \chi(\phi^2(x) - t)(\phi(x) - \phi^{-1}(t))f(t)dt =: \int_0^1 k_2(x, t)f(t)dt.
\]

\[
(V_\phi^3 f)(x) = \int_0^1 \chi(\phi^3(x) - t)\int_{\phi^{-2}(t)}^{\phi(x)} (\phi(s) - \phi^{-1}(t))dsf(t)dt =: \int_0^1 k_3(x, t)f(t)dt.
\]

Further, \( k_2(x, t) \) and \( k_3(x, t) \) are continuous functions on \([0, 1] \times [0, 1]\). Hence, \( V_\phi^2 \in S_1 \) and \( V_\phi^3 \in S_1 \). Now if we recall (2.3), we get

\[
\sum_{n=1}^\infty \lambda_n^3 = \int_0^1 k_2(t, t)dt = \int_0^1 (\phi(t) - \phi^{-1}(t))dt = 2 \int_0^1 \phi(t)dt - 1,
\]

12
\[
\sum_{n=1}^{\infty} \lambda_n^3 = \int_0^1 k_3(t,t)dt = \int_0^1 \frac{1}{\phi(t)} (\phi(s) - \phi^{-1}(t))ds
\]

\[
= \int_0^1 (\phi(t)\phi^2(t) - 2\phi^{-1}(t)\phi(t) + \phi^{-1}(t)\phi^{-2}(t)) dt = 1 - 3 \int_0^1 \phi(t)\phi^{-1}(t)dt.
\]

\[\square\]

**Example 4.16.** Let \( \phi(x) = x^\alpha \) (\( 0 < \alpha < 1 \)). It can be proved by direct calculations that

\[
D_{V_{x^\alpha}}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \int_0^{\alpha} \cdots \int_0^{\alpha} dt_1 \cdots dt_n
\]

\[
= 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \left( \frac{\alpha^{n-1}}{(1-\alpha)^n} \right) = \prod_{n=1}^{\infty} \left( 1 - \frac{\lambda}{(1-\alpha)\alpha^{n-1}} \right).
\]

Hence, \( \sigma_p(V_{x^\alpha}) = \{(1-\alpha)\alpha^{n-1} \}_{n=1}^{\infty} \) and each eigenvalue of \( V_{x^\alpha} \) is of algebraic multiplicity one. Further, \( \sum_{n=1}^{\infty} (1-\alpha)\alpha^{n-1} = 1 \) and \( \sum_{n=1}^{\infty} ((1-\alpha)\alpha^{n-1})^\varepsilon = \frac{1-\alpha}{1-\alpha^\varepsilon} < \infty \) for each \( \varepsilon > 0 \).

5. Some generalizations.

5.1. The following Lemma can be proved by direct calculations.

**Lemma 5.17.** Let \( A \) be a compact operator defined on a Hilbert space \( \mathcal{H} \). Let also \( \mathcal{H} = \bigoplus_{i=1}^{k} \mathcal{H}_i \) and \( A_i := P_iA : \mathcal{H}_i \to \mathcal{H}_i \), where \( P_i \) be an orthoprojection in \( \mathcal{H} \) onto \( \mathcal{H}_i \). Suppose that \( \left\{ \bigoplus_{j=1}^{i} \mathcal{H}_j \right\}_{i=1}^{k} \) is invariant for \( A \); then \( 1/\lambda \) is an eigenvalue of \( A \) of the algebraic multiplicity \( m \geq 1 \) if and only if \( 1/\lambda \) is an eigenvalue of \( A_i \) of the algebraic multiplicity \( m_i \geq 0 \) and \( \sum_{i=1}^{k} m_i = m \).

**Proof.** The proof is omitted. \( \square \)

**Theorem 5.18.** Let \( \phi : [0, 1] \to [0, 1] \) be a strictly increasing continuous function. Let also \( \left\{ x : \phi(x) = x, \ x \in (0, 1) \right\} = \{a_i\}_{i=1}^{k-1}, \) where \( 0 < a_1 < \cdots < a_{k-1} < 1 \ (k \geq 2). \) By definition, put \( a_0 := 0, \ a_k := 1, \) and

\[
\phi_i(x) := (\phi(x_i - a_{i-1}) + a_i - a_{i-1})/(a_i - a_{i-1}), \quad 1 \leq i \leq k.
\]

\[
D_{V_{\phi_i}}(\lambda) := \begin{cases} 
1 + \sum_{n=0}^{\infty} (-\lambda)^n \left\{ \frac{1}{\phi_i}, \cdots, \frac{1}{\phi_i} \right\}, & \phi_i(x) > x \text{ for } x \in (0, 1); \\
1, & \phi_i(x) < x \text{ for } x \in (0, 1).
\end{cases}
\]

Then
(1) $1/\lambda \in \sigma_p(V_\phi)$ if and only if $\prod_{i=1}^{k} D_{V_{\phi_i}}((a_i-a_{i-1})\lambda) = 0$;

(2) the algebraic multiplicity of the eigenvalue $1/\lambda$ is equal to the multiplicity of $\lambda$ as a root of the entire function $\prod_{i=1}^{k} D_{V_{\phi_i}}((a_i-a_{i-1})\lambda)$.

Proof. By definition, put $\mathfrak{H} := L_2[0,1]$, $\mathfrak{H}_i := L_2[a_{i-1}, a_i]$ and

$$P_i : f(x) \to \begin{cases} f(x), & x \in [a_{i-1}, a_i]; \\ 0, & x \not\in [a_{i-1}, a_i]; \end{cases}, \quad \mathfrak{P}_i : \mathfrak{H}_i \to \mathfrak{H}_i,$$

$$A := V_\phi, \quad A_i := P_i A |_{\mathfrak{H}_i},$$

$$T_i : \mathfrak{H}_i \to \mathfrak{H}_i.$$

It follows easily that $\bigoplus_{j=1}^{i} \mathfrak{H}_j (= L_2[0, a_i])$ is invariant for $A$ and

$$A_i : \begin{cases} f(x), & x \in [a_{i-1}, a_i]; \\ 0, & x \not\in [a_{i-1}, a_i]; \end{cases} \to \begin{cases} \phi(x), & x \in [a_{i-1}, a_i]; \\ \int_{a_{i-1}}^{a_i} f(t)dt, & x \not\in [a_{i-1}, a_i]; \end{cases},$$

$$T_i^{-1} : f(x) \to \begin{cases} f(\frac{x-a_{i-1}}{a_i-a_{i-1}}), & x \in [a_{i-1}, a_i]; \\ 0, & x \not\in [a_{i-1}, a_i]; \end{cases}, \quad \mathfrak{T}_i : \mathfrak{H}_i \to \mathfrak{H}_i,$$

$$T_i A_i T_i^{-1} = (a_i-a_{i-1})V_{\phi_i}.$$

The application of Theorem 3.4 yields

$$1/\lambda \in \sigma_p(A_i) \iff 1/\lambda \in \sigma_p((a_i-a_{i-1})V_{\phi_i}) \iff D_{V_{\phi_i}}((a_i-a_{i-1})\lambda) = 0.$$  

The applying of Lemma 5.17 completes the proof. $\square$

**Corollary 5.19.** Suppose $\phi(x)$ satisfies the conditions of Theorem 5.18 and $\operatorname{mes}\{x : \phi(x) \geq x, \ x \in (0,1]\} > 0$. Set also $\sigma_p(V_\phi) \setminus \{0\} = \{\lambda_n\}_{n=1}^{\omega}$ ($1 \leq \omega \leq \infty$). Then

(1) $\omega < \infty$ if and only if $\phi(0) > 0$, $\phi(1-\varepsilon) = 1$ for some $0 < \varepsilon < 1$ and $\phi(x) > x$ for all $x \in (0,1)$;

(2) $\lim_{\varepsilon \to 0} \sum_{|\lambda_n| > \varepsilon} \lambda_n = \operatorname{mes}\{x : \phi(x) \geq x, \ x \in (0,1]\}.$

**Proof.** (1) follows from Theorems 4.7, 4.12, 5.18

(2) By definition, put

$$\Omega := \{i : \phi(x) \geq x \text{ for } x \in [a_{i-1}, a_i]\} = \{i : \phi_i(x) \geq x \text{ for } x \in [0,1]\};$$

$$\sigma_p(V_{\phi_i}) := \{\lambda_m\}_{n=1}^{\omega}, \quad 1 \leq \omega \leq \infty, \quad i \in \Omega.$$

14
By Theorem 5.18

\[ \{\lambda_n\}^\omega_{n=1} = \sigma_p(V_\phi) = \bigcup_{i \in \Omega} \sigma_p((a_i - a_{i-1})V_{\phi_i}) = \bigcup_{i \in \Omega} (a_i - a_{i-1}) \{\lambda_{in}\}^\omega_{n=1}. \]

By Theorem 4.12

\[ \lim_{\varepsilon \to 0} \sum_{|\lambda_{in}| > \varepsilon} \lambda_{in} = 1. \]

Thus

\[ \lim_{\varepsilon \to 0} \sum_{|\lambda_{in}| > \varepsilon} \lambda_{n} = \sum_{i \in \Omega} (a_i - a_{i-1}) \lim_{\varepsilon \to 0} \sum_{|\lambda_{in}| > \varepsilon} \lambda_{in} \]

\[ = \sum_{i \in \Omega} (a_i - a_{i-1}) = \text{mes}\{x : \phi(x) \geq x, \ x \in [0, 1]\}. \]

\[ \square \]

**Remark 5.20.** It is interesting to note that the case of nonincreasing function \( \phi \) can be more multifarious. In particular, if \( \phi(x) \) is a strictly decreasing continuous function such that \( \phi(0) = 1, \phi(1) = 0 \) and \( \phi(\phi(x)) = x \) then \( V_\phi \) is a selfadjoint operator in \( L^2[0, 1] \). For example, \( \sigma_p(V_{1-x}) = \{ \frac{2(-1)^n}{(2n+1)\pi} \}_{n=1}^\infty \) and \( \sum_{n=1}^\infty \frac{2(-1)^n}{(2n+1)\pi} = \frac{2}{\pi} \frac{\pi}{4} = \frac{1}{2} = \text{mes}\{x : 1 - x \geq x\} \).

5.2. In this subsection we consider an operator \( V_\phi \) defined on \( L_p[0, 1] \) (1 \(\leq p < \infty\)). Let \( A_i \) be a bounded operator defined on Banach space \( X_i \) (i = 1, 2). Recall that \( A_1 \) is said to be quasisimilar to \( A_2 \) if there exist deformations \( K : X_1 \to X_2 \) and \( L : X_2 \to X_1 \) (i.e. \( \overline{\text{Ker}}(K) = X_2 \), \( \text{Ker}K = \{0\} \), \( \overline{\text{Ker}}(L) = X_1 \), \( \text{Ker}L = \{0\} \)) such that \( A_1L = LA_2 \) and \( KA_1 = A_2K \). It is clear that \( \sigma_p(A_1) = \sigma_p(A_2) \).

**Proposition 5.21.** Let \( \phi : [0, 1] \to [0, 1] \) be a strictly increasing continuous function such that \( \phi(0) = 0 \) and \( \phi(1) = 1 \). Let \( A_1 \) denote an operator \( V_\phi \) defined on \( L_p[0, 1] \) (1 \(\leq p < \infty\)) and let \( A_2 \) denote an operator \( V_\phi \) defined on \( L_2[0, 1] \). Then \( A_1 \) is quasisimilar to \( A_2 \), and hence \( \sigma_p(A_1) = \sigma_p(A_2) \).

**Proof.** By definition, put \( K := V_\phi : L_p[0, 1] \to L_2[0, 1] \), \( L := V_\phi : L_2[0, 1] \to L_p[0, 1] \). It is clear that \( K \) and \( L \) are deformations and \( A_1L = LA_2, KA_1 = A_2K \). \( \square \)

5.3. Now we consider the operator \( (V_{\phi,q,w,f})(x) := q(x) \int_0^\phi(x) f(t)w(t)dt \) defined on \( L_2[0, 1] \).

The proof of the following theorem is similar to the proof of Theorem 3.4.
Theorem 5.22. Let $\phi : [0, 1] \rightarrow [0, 1]$ be a nondecreasing continuous function such that $\phi(x) > x$ for all $x \in (0, 1)$. Let also $q(x), w(x) \in L_2[0, 1]$. Then

$$D_{V_{\phi,q,w}}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \int_0^1 \int_0^1 \ldots \int_0^1 q(t_1)w(t_1) \ldots q(t_n)w(t_n)dt_n \ldots dt_1.$$

Corollary 5.23. Let the conditions of Theorem 5.22 hold and $q(x)w(x) > 0$ for a.a. $x \in [0, 1]$. Then $\sigma_p(V_{\phi,q,w}) \setminus \{0\}$ is a finite set if and only if $\phi(0) > 0$ and $\phi(1 - \varepsilon) = 1$ for some $0 < \varepsilon < 1$.

Acknowledgments. The author wishes to thank Professor J. Zemánek for setting the problem.

References

[1] I.Yu. Domanov, On the spectrum and eigenfunctions of the operator $(Vf)(x) = x^\alpha \int_0^x f(t)dt$, in: Perspectives in Operator Theory, Banach Center Publications 75 (2007), 137-142.

[2] I.Yu. Domanov, On cyclic subspaces and unicellularity of the operator $(Vf)(x) = q(x) \int_0^x w(t)f(t)dt$. (Russian) Ukr. Mat. Visn. 1 (2004), no. 2, 172–213, 283; translation in Ukr. Math. Bull. 1 (2004), no. 2, 177–219.

[3] I.Yu. Domanov, Spectral analysis of powers of the operator $(Vf)(x) = q(x) \int_0^x w(t)f(t)dt$, (Russian) Mat. Zametki 73 (2003), no. 3, 444–449; translation in Math. Notes 73 (2003), no. 3-4, 408–413

[4] I.C.Gohberg, M.G. Kreĭn, Introduction to the Theory of Linear Non-selfadjoint Operators. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18 American Mathematical Society, Providence, R.I. 1969 xv+378 pp.

[5] I.C.Gohberg, M.G. Kreĭn, Theory and applications of Volterra operators in Hilbert space, Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 24 American Mathematical Society, Providence, R.I. 1970 x+430 pp.

[6] E. Goursat, A course in mathematical analysis. Vol. III, Part Two: Integral equations. Calculus of variations, Translated by Howard G. Bergmann Dover Publications, Inc., New York 1964 xi+389 pp.
[7] B.Ya. Levin, *Lectures on Entire Functions*, In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko. Translated from the Russian manuscript by Tkachenko. Translations of Mathematical Monographs, 150. American Mathematical Society, Providence, RI, 1996. xvi+248 pp.

[8] W.V. Lovitt, *Linear Integral Equations*, Moscow, 1957 (in Russian).

[9] Yu.I. Lyubich, *Linear functional analysis*, (Russian) Current problems in mathematics. Fundamental directions, Vol. 19 (Russian), 5–305, 316, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988.

[10] Yu.I. Lyubich, *Composition of integration and substitution*, in: Linear and Complex Analysis Problem Book, Lect. Notes in Math. 1043, Springer, Berlin, 1974, 249-250.

[11] F. G. Tricomi, *Integral Equations*, Dover, New York, 1957.

[12] Yu Sun Tong, *Quasinilpotent integral operators*, (Chinese) Acta Math. Sinica 32 (1989), no. 6, 727–735.

[13] R. Whitley, *The spectrum of a Volterra composition operator*, Integral Equation and Operator Theory, 10(1987), no.1, 146-149.

[14] P. P. Zabreiko, A. I. Koshelev, M. A. Krasnosel’skii, S. G. Mikhlin, L. S. Rakovshchik, and V. Ya. Stetsenko, *Integral operators*, Nauka, Moscow, 1968 (in Russian).

[15] M. Zima, *A certain fixed point theorem and its applications to integral-functional equations*, Bull. Austral. Math. Soc., 46(1992), no. 2, 179-186.

Institute of Applied Mathematics and Mechanics
Ukrainian National Academy of Sciences
R. Luxemburg Str. 74
83114 Donetsk
Ukraine

Mathematical Institute of the Academy of Sciences of the Czech Republic
Zitna 25
CZ - 115 67 Praha 1
Czech Republic
e-mail: domanov@yahoo.com