Assouad-like dimensions of random Moran measures

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Abstract. In this paper, we determine the almost sure values of the Φ-dimensions of random measures supported on random Moran sets that satisfy a uniform separation condition. The Φ-dimensions are intermediate Assouad-like dimensions, the (quasi-)Assouad dimensions and θ-Assouad spectrum being special cases. Their values depend on the size of Φ, with one size coinciding with the Assouad dimension and the other coinciding with the quasi-Assouad dimension. We give many applications, including to equicontractive self-similar measures and 1-variable random Moran measures such as Cantor-like measures with probabilities that are uniformly distributed. We can also deduce the Φ-dimensions of the underlying random sets.

1. Introduction

Many notions of dimension have been developed to quantify the size of sets and measures, in part, to better understand the geometry of the set and the nature of the measure. Perhaps the most well known are Hausdorff and box dimensions, global notions of size. More recently, there has been much interest in understanding the local complexity of sets and measures and for this various dimensions have been introduced including the (upper and lower) Assouad dimensions, which quantify the ‘thickest’ and ‘thinnest’ parts of sets and measures (see [1, 6, 20, 21]), the less extreme quasi-Assouad dimensions ([2, 16, 17, 22]), the θ-Assouad spectrum ([9]), and the most general, intermediate Φ-dimensions ([9, 11, 15]). The Φ-dimensions range between the box and Assouad dimensions. They are localized, like the Assouad dimensions, but vary in the depth of the scales considered, thus they provide very refined information. In this paper, we continue the study of the upper and lower Φ-dimensions of measures, with a focus on random measures supported on random Moran sets.

There is a long history of the study of dimensional properties of random sets, with early important early papers including [3, 14], for example. More recently, Assouad-like dimensions of random sets have been investigated in papers such as [7, 8, 12, 24, 25].

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By a random Moran measure, we will mean a probability measure supported on a random Moran set in $\mathbb{R}^D$ that can be thought of as arising from a random homogenous model of uncountably many equicontractive iterated function systems satisfying a suitable uniform strong separation condition. The number of children at each level, the ratios of child to parent diameters and the probability weights assigned to each child that specify the measure, are all to be iid random variables. We refer the reader to Section 2.2 for the technical details.

The values of the $\Phi$-dimensions of these random measures depend on how the dimension function $\Phi$ compares with the function $\Psi(x) = \log |\log x|/|\log x|$ near 0, as was also seen to be the case in [12] where the $\Phi$-dimensions of random rearrangements of Cantor-like sets were considered. Under mild technical assumptions on the probability and ratio distributions, when $\Phi(x) \gg \Psi(x)$ (such as for the quasi-Assouad dimensions), then almost surely the value of all the upper $\Phi$-dimensions of the random measure is $E(\log m)/E(\log r)$ where $m$ is the minimum probability and $r$ is the child/parent diameter ratio. For the lower $\Phi$-dimension we simply replace the minimum probability $m$ by the maximum $M$. When $\Phi(x) \ll \Psi(x)$ (such as for the Assouad dimensions) and the essential infimum of either the ratios or the probabilities is bounded away from 0, then the upper (and lower) $\Phi$-dimensions again coincide, being the almost sure extreme behaviour of $\log m/\log r$ (resp., $\log M/\log r$). These results are formally stated and proven in Sections 3 and 4.

One special case to which our theorems apply is when the set is deterministic and the probabilities are chosen uniformly distributed. For example, take the classical middle-third Cantor set. If the two probabilities, $p, 1-p$, are equal, all the dimensions (of either the set or the measure) equal $\log 2/\log 3$. In contrast, if the probabilities are chosen with $p$ uniformly distributed over $(0,1)$, then for $\Phi \gg \Psi$, the upper $\Phi$-dimension of the random measure is almost surely $(1 + \log 2)/\log 3$ which, perhaps surprisingly, do not average to $\log 2/\log 3$. For $\Phi \ll \Psi$ the upper and lower $\Phi$-dimensions are almost surely $\infty$ and 0 respectively. More complicated formulas hold if the Cantor set has $T$ children, with $T > 2$, and the probabilities are chosen uniformly over the simplex $\{(p_i)_{i=1}^T : \sum_{i=1}^T p_i = 1, p_i \geq 0\}$.

Another special case to which our theorems apply is a random 1-variable (homogeneous) model with finitely many equicontractive iterated function systems, satisfying the strong separation condition. In [24] it was shown that the upper quasi-Assouad dimensions for these random sets coincide almost surely with their Hausdorff dimensions. If we take the measure with uniform probabilities, the $\Phi$-dimensions of the measure equal those of the random set, thus the Hausdorff dimension coincides almost surely with the $\Phi$-dimensions for all $\Phi \gg \Psi$ (including both the upper and lower quasi-Assouad dimensions). A different formula applies for $\Phi \ll \Psi$, (including the Assouad dimension).

These examples are discussed in Section 5 along with others. Section 2 contains the definitions of the $\Phi$-dimensions, as well as their basic properties, and details the random setup.

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1We will write $f \gg g$ if there is a function $A$ and $\delta > 0$ such that $f(x) \geq A(x)g(x)$ for all $0 < x < \delta$ and $A(x) \to \infty$ as $x \to 0^+$. 
2. Preliminaries

2.1. Dimensions of sets and measures. Given a bounded metric space $X$, we denote the open ball centred at $x \in X$ and radius $R$ by $B(x, R)$. By a measure, we will always mean a Borel probability measure on the metric space $X$.

**Definition 1.** By a dimension function, we mean a map $\Phi : (0, 1) \to \mathbb{R}^+$ such that $x^{1+\Phi(x)}$ decreases to 0 as $x$ decreases to 0.

Interesting examples include the constant functions $\Phi(x) = \delta \geq 0$, $\Phi(x) = 1/|\log x|$ and $\Phi(x) = \log|\log x|/|\log x|$.

**Definition 2.** Let $\Phi$ be a dimension function and let $\mu$ be a measure on $X$. The upper and lower $\Phi$-dimensions of $\mu$ are given by

$$\overline{\dim}_\Phi \mu = \inf \left\{ d : (\exists C_1, C_2 > 0)(\forall 0 < r < R^{1+\Phi(R)} \leq R \leq C_1) \right.$$

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_2 \frac{R^d}{r^d} \ \forall x \in \text{supp} \mu$$

$$\left. \right\}$$

and

$$\underline{\dim}_\Phi \mu = \sup \left\{ d : (\exists C_1, C_2 > 0)(\forall 0 < r < R^{1+\Phi(R)} \leq R \leq C_1) \right.$$ 

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq C_2 \frac{R^d}{r^d} \ \forall x \in \text{supp} \mu$$

$$\left. \right\}$$

**Remark 1.** (i) The upper and lower Assouad dimensions of $\mu$ (also known as the upper and lower regularity dimensions) studied by Käenmäki et al in [19, 20] and Fraser and Howroyd in [6], and denoted $\dim_A \mu$ and $\dim_L \mu$ respectively, are the upper and lower $\Phi$-dimensions with $\Phi$ the constant function 0. It is well known that a measure $\mu$ is doubling if and only if $\dim_A \mu < \infty$ and uniformly perfect if and only if $\dim_L \mu > 0$.

(ii) If we let $\Phi_\theta = 1/\theta - 1$, then $\overline{\dim}_{\Phi_\theta} \mu$ and $\underline{\dim}_{\Phi_\theta} \mu$ are (basically) the upper and lower $\theta$-Assouad spectrum introduced in [9]. The upper and lower quasi-Assouad dimensions of $\mu$ developed in [16, 17] are given by

$$\dim_{qA} \mu = \lim_{\theta \to 1} \overline{\dim}_{\Phi_\theta} \mu, \ \dim_{qL} \mu = \lim_{\theta \to 1} \underline{\dim}_{\Phi_\theta} \mu.$$

As noted in [15], there are always dimension functions which give rise to the quasi-Assouad dimensions, but these need to be tailored to the particular measure.

(iii) The upper Minkowski dimension, $\overline{\dim}_M \mu$, and the Frostman dimension, $\dim_F \mu$, coincide with the upper and lower $\Phi$-dimensions respectively for $\Phi \to \infty$; see [4] and Proposition [7].

The upper and lower $\Phi$-dimensions of a measure were introduced in [15] to provide more refined information about the local behaviour of a measure than that given by the upper and lower Assouad dimensions. They were motivated, in part, by related definitions for dimensions of sets. To recall these, we use the notation $N_r(Y)$ to mean the least number of balls of radius $r$ that cover $Y \subseteq X$.

**Definition 3.** The upper and lower $\Phi$-dimensions of $E \subseteq X$ are given by

$$\overline{\dim}_\Phi E = \inf \left\{ \alpha : (\exists C_1, C_2 > 0)(\forall 0 < r \leq R^{1+\Phi(R)} \leq R < C_1) \right.$$ 

$$N_r(B(z, R) \cap E) \leq C_2 \frac{R^\alpha}{r^\alpha} \ \forall z \in E$$

$$\left. \right\}$$

and

$$\underline{\dim}_\Phi E = \sup \left\{ \alpha : (\exists C_1, C_2 > 0)(\forall 0 < r \leq R^{1+\Phi(R)} \leq R < C_1) \right.$$ 

$$N_r(B(z, R) \cap E) \geq C_2 \frac{R^\alpha}{r^\alpha} \ \forall z \in E$$

$$\left. \right\}.$$
The $\Phi$-dimensions were first thoroughly studied in [11], expanding upon the earlier work of [9]. The (quasi-) upper and lower Assouad dimensions and the upper and lower $\theta$-Assouad spectrum are again special cases, arising in the same manner as for measures. It is known that

$$\dim_L E \leq \dim_\Phi E \leq \dim_B E \leq \dim_G E \leq \dim_\Phi E \leq \dim_A E$$

and for closed sets $\dim_L E \leq \dim_H E$. For further background and proofs, we refer the reader to the references mentioned in the introduction, as well as Fraser’s monograph, [5].

Obviously, if $\Phi(x) \leq \Psi(x)$ for all $x > 0$, then for any measure $\mu$ we have

$$\overline{\dim}_\Phi \mu \leq \overline{\dim}_\Psi \mu$$

A similar statement holds for the $\Phi$-dimension of sets. The next Proposition summarizes other relationships between these dimensions. For the proofs of these facts and many other properties of the $\Phi$-dimensions of measures, we refer the reader to [15] and the references cited there.

**Proposition 1.** Let $\Phi$ be a dimension function and $\mu$ be a measure.

(i) Then $\dim_\Phi \mu \geq \dim_\Phi \text{supp} \mu \geq \dim_H \text{supp} \mu$ and

$$\dim_L \mu \leq \dim_\Phi \mu \leq \dim_F \mu \leq \dim_M \mu \leq \overline{\dim}_\Phi \mu \leq \dim_A \mu.$$  \hspace{1cm} (2.1)

If $\mu$ is doubling, then $\dim_\Phi \mu \leq \dim_H \text{supp} \mu$.

(ii) If $\Phi(x) \to 0$ as $x \to 0$, then $\dim_\Phi \mu \leq \dim_\Phi \mu$ and $\dim_A \mu \leq \overline{\dim}_\Phi \mu$.

(iii) If there exists $x_0 > 0$ such that $\Phi(x) \leq C/|\log x|$ for $0 < x \leq x_0$, then $\overline{\dim}_\Phi \mu = \dim_A \mu$ and $\dim_\Phi \mu = \dim_L \mu$.

(iv) If $\Theta = \limsup_{x \to 0} \Phi(x)^{-1}$, then

$$\overline{\dim}_\Phi \mu \geq \dim_F \mu - \Theta(\overline{\dim}_M \mu - \dim_F \mu),$$

$$\overline{\dim}_\Phi \mu \leq \overline{\dim}_M \mu + \Theta(\overline{\dim}_M \mu - \dim_F \mu).$$

**2.2. The random set-up.** Let $I \subseteq \mathbb{R}^D$. We denote by $\text{diam}(I)$ the diameter of $I$. Given $r > 0$, we say the subset $J \subseteq I$ is an $r$-similarity of $I$ if there is a similarity $S_\tau$ such that $J = S_\tau(I)$ and $\text{diam}(J) = r \cdot \text{diam}(I)$. Thus $S_\tau$ has contraction factor $r$.) We say the collection of $r$-similarities, $J_1, \ldots, J_t$, is $\tau$-separated if $d(J_i, J_j) \geq \tau \cdot r \cdot \text{diam}(I)$ for all $i \neq j$. When such a collection of $t$ sets exists, we say $I$ has the $(t, r, \tau)$-separation property.

Of course, if $I$ contains a non-empty open set, and $\tau > 0$ is given, then $I$ will have the $(t, r, \tau)$-separation property for all $t \in \mathbb{N}$ and $r \leq r_t$, for some suitably small $r_t > 0$. For example, if $I_0 = [0, 1] \subseteq \mathbb{R}$, then $r_t = 1/(t + \tau(t - 1))$ will work. This condition can be viewed as a uniform strong separation condition.

From here on, $I_0$ will denote a (fixed) compact subset of $\mathbb{R}^D$ with diameter 1 and non-empty interior. We fix $0 < \tau < 1$ and for each $t \in \mathbb{N}$ we choose $r_t \in (0, 1/2]$ so that $I_0$ has $(t, r, \tau)$-separation property for all $r \leq r_t$. For each $t = 2, 3, \ldots$ we define the probability simplices:

$$S_t = \{(x_1, \ldots, x_t) \in \mathbb{R}^t : x_i > 0, \sum_{i=1}^t x_i = 1\},$$

$$\Omega_t = (0, r_t] \times S_t \subseteq (0, 1) \times S_t.$$
and
\[ \Omega_0 = \bigcup_{t \geq 2} \Omega_t. \]

Let \( \pi \) be a Borel probability measure on \( \Omega_0 \) and let \( \mathbb{P} \) be the product measure on the infinite product \( \Omega = \Omega_0^\mathbb{N} \) induced by \( \pi \).

Continuing, we define random variables \( T \) and \( r \), and a random vector \( p \) on \( \Omega_0 \). To do this, for \( \omega \in \Omega_0 \), we have \( \omega \in \Omega_t \) for some \( t = 2, 3, \ldots \) with \( \omega = (r, p^{(1)}, p^{(2)}, \ldots, p^{(t)}) \). Using this, we define
\[ T(\omega) = t, \quad r(\omega) = r, \quad p(\omega) = (p^{(1)}, p^{(2)}, \ldots, p^{(t)}) \in S_t. \]

A common way to define the measure \( \pi \) on \( \Omega_0 \) is as a two-step process where one first chooses the integer \( t \) randomly according to some distribution and then independently choose \( r \) uniformly from \((0, r_1] \) and \( p \) uniformly from \( S_t \).

With this framework, \( \omega \in \Omega \) drawn according to \( \mathbb{P} \) represents an independent and identically distributed random sample \((T_n, r_n, p_n)\) from \( \pi \) on \( \Omega_0 \).

Using this iid sample, we can now construct a random Moran fractal. Beginning with the compact set \( I_0 \) and \( \omega \in \Omega \), we select a collection of \( T_1(\omega) \) subsets that are \( r_1(\omega) \)-similarities of \( I_0 \), \( \{I_0^{(j)}\}_{j=1}^{T_1(\omega)} \), that are \( \tau \)-separated. This is possible as \( r_1(\omega) \leq r_T \), so \( I_0 \) has the \((T_1(\omega), r_1(\omega), \tau)\)-separation property. We call the sets \( I_0^{(j)}, j = 1, \ldots, T_1(\omega) \), the Moran sets of step (or level) 1. The Moran sets of step 1 have diameter \( r_1 \) and the distance between any two is at least \( \tau r_1 \).

Being similar to \( I_0 \), the sets \( I_0^{(j)} \) also have the \((t, r, \tau)\)-separation property for all \( r \leq r_1 \), so we may repeat this process. Assume inductively that we have chosen the \( \prod_{j=1}^n T_j(\omega) \) Moran sets of step \( n \). From each such set, \( I_n \), we select \( T_{n+1}(\omega) \) subsets that are \( r_{n+1}(\omega) \)-similarities of \( I_n \) and are \( \tau \)-separated. These sets all have diameter \( r_1 \cdots r_{n+1} \), are separated by a distance of at least \( \tau r_1 \cdots r_{n+1} \) and are known as the Moran sets of step \( n+1 \). The Moran sets of step \( n+1 \) that are subsets of a given Moran set \( I_n \) of step \( n \) are known as the children of the parent set \( I_n \). We will use the notation \( I_N(x) \) for the unique Moran set of step \( N \) containing \( x \in E(\omega) \). Of course, \( I_{N-1}(x) \) is its parent.

If we let \( \mathcal{M}_n(\omega) \) be the union of the step \( n \) Moran sets, then the random Moran set \( E(\omega) \) is the compact set
\[ E(\omega) = \bigcap_{n=1}^{\infty} \mathcal{M}_n(\omega). \]

We define a random Moran measure \( \mu = \mu_\omega \) inductively by the rule that if the \( T_{n+1} \) children of the step \( n \) Moran set \( I \) are labelled \( I^{(1)}, \ldots, I^{(T_{n+1})} \), then \( \mu(I^{(j)}) = \mu(I)p^{(j)}_{n+1} \) where \( \mu(I_0) = 1 \). The support of \( \mu \) is the random Moran set \( E(\omega) \).

It is not difficult to see that the \( \Phi \)-dimensions of these random measures is a tail event and hence our interest is in almost sure results.

**Example 1.** The classical middle-third Cantor set is a simple example of a Moran set where \( I_0 = [0, 1] \), \( \tau = 1/3 \), \( T_n(\omega) = 2 \) and \( r_n(\omega) = 1/3 \) for all \( \omega \) and \( n \). The uniform Cantor measure is a special case of this construction with probabilities \( (1/2, 1/2) \). More generally, any self-similar set/measure arising from an iterated function system (IFS) \( \{S_j, p_j\}_{j=1}^M \) of equicontinuous similarities \( S_j \) acting
on \( I_0 \) and satisfying the strong separation condition, (meaning, the sets \( S_j(I_0) \) are disjoint) and associated probabilities \( p_j \), is a random Moran set/measure in our sense.

Example 2. Another natural class of examples of random Moran sets are the finite random 1-variable (homogeneous) models which arise from a finite family of iterated function systems, \( \{F_i\} \), where each IFS \( F_i \) consists of equicontractive similarities acting on (the common set) \( I_0 \) and satisfying the strong separation condition, where at each step in the construction we randomly choose one IFS to apply at that level. See \[24\].

More generally, our construction can be thought of as a random 1-variable IFS construction where we have an uncountable family of IFSs with associated probabilities.

Other examples are given in Section \[5\].

2.3. Preliminary Results. Throughout the paper, we will assume \( L \) is the smallest integer such that

\[ 2^{-L} \leq \tau/2. \]

First, we will verify that we can replace balls by suitable Moran sets for the calculation of the \( \Phi \)-dimensions. The \( \tau \)-separation condition is required for this.

Lemma 1. Fix \( \omega \in \Omega, \ R > 0 \) and \( x \in E(\omega) \). Choose \( N = N(\omega) \) such that

\[ r_1(\omega) \cdots r_{N+1}(\omega) \leq R < r_1(\omega) \cdots r_N(\omega). \]

Then

\[ I_{N+1}(x) \cap E(\omega) \subseteq B(x, R) \cap E(\omega) \subseteq I_{N-L}(x) \]

and hence

\[ \mu(I_{N+1}(x)) \leq \mu(B(x, R)) \leq \mu(I_{N-L}(x)). \]

Proof. Since \( R \) is at least the diameter of any Moran set of step \( N+1 \), \( I_{N+1}(x) \subseteq B(x, R) \).

Now consider all the step \( N \) Moran sets that intersect \( B(x, R) \). If two of these sets, say \( I^{(1)}, I^{(2)} \), are subsets of different Moran sets of level \( N-k \), then the distance between these two sets, call it \( \delta \), is at least the minimum distance between any two level \( N-k \) Moran sets and hence is at least \( \tau r_1 \cdots r_{N-k} \). But \( \tau \geq 2^{1-L} \)

and \( r_j \leq 1/2 \), hence

\[ \delta \geq 2 \cdot 2^{-L} r_1 \cdots r_{N-k} \geq 2 r_1 \cdots r_{N-k} r_{N-k+1} \cdots r_{N-k+L}. \]

As \( I^{(1)} \) and \( I^{(2)} \) both intersect \( B(x, R) \), we must have \( \delta \leq 2R < 2r_1 \cdots r_N \) and therefore \( k < L \). Thus all the step \( N \) Moran sets that intersect \( B(x, R) \) are subsets of the same Moran set of level \( N-L \), namely \( I_{N-L}(x) \).

Next, we introduce the positive-valued random variables

\[ M_n = \max_j p_n^{(j)}, \ m_n = \min_j p_n^{(j)}, \]

\[ X_n = -\log M_n, \ Y_n = -\log m_n, \ \text{and} \ Z_n = -\log r_n. \]

Of course, the collections \((M_n)_n, (m_n)_n, (X_n)_n, (Y_n)_n \) and \((Z_n)_n \) are all independent and identically distributed.
Since \( \sum_{j=1}^{T_n} p_n^{(j)} = 1 \) for each \( n \), \( M_n \geq 1/T_n \), so if the number of children is bounded, then \( E(e^{\lambda X_1}) < \infty \) for any \( \lambda \). More generally, \( E(e^{\lambda W}) < \infty \) for \( W \) any of \( X_1, Y_1 \) or \( Z_1 \) if there exists \( \delta > 0 \) such that \( W_1(\omega) > \delta \) a.s.

**Basic Assumption:** Throughout this paper we will assume that there exists some \( A > 0 \) such that \( E(e^{\lambda X}) < \infty \) for all \( |\lambda| \leq A \).

Note that this happens if and only if \( E(r_1^{-A}) < \infty \), as is true, for example, when the ratios \( r_n \) are uniformly distributed. This is an important assumption because we will make heavy use of the following probabilistic result, sometimes known as the Chernoff technique.

**Theorem 1.** (Thm. 2.6) Suppose \((F_n)\) are iid rv’s and for some \( A > 0 \), \( E(e^{\lambda F_n}) < \infty \) for all \( |\lambda| \leq A \). Then for all \( a > 0 \), there exists \( b > 0 \) such that

\[
P \left( \left| \sum_{j=1}^{k} F_j - kE(F_1) \right| \geq ak \right) \leq \exp(-bk)
\]

for all \( k \in \mathbb{N} \).

**Remark 2.**
(i) If \( W \) is a non-negative random variable with \( P(W \leq x) \leq Cx^\theta \) for small \( x \) and \( \theta > 0 \), then \( E(e^{-\lambda \log W}) = E(W^{-\lambda}) < \infty \) for small \( |\lambda| \). In particular this holds if \( W \) has a probability density function \( f(x) \) with \( f(x) \leq Cx^q \) and \( q > -1 \).

(ii) Since \( \{ p \in S_t : \min_i p_i \leq z \} \subseteq (\partial S_t)_{2z} \) (the \( 2z \)-dilation of the boundary of \( S_t \)), part (i) means it is enough that \( P((\partial S_t)_{2z}) \leq Cz^\theta \) for some \( C, \theta > 0 \) and small \( z \) to have \( E(e^{\lambda Y}) < \infty \) for small \( |\lambda| \). Thus if the probability distribution on \( S_t \) has bounded density function, this will be true.

(iii) Since \( \max_i p_i \geq 1/t \) for \( p \in S_t \), we know that \( M_n \) is close to zero only when \( T_n \), the number of children, is very large. Very roughly, \( P(M_n \leq \lambda) \leq P(T_n \geq 1/\lambda) \). If \( P(T = t) \leq Ct^{-\theta} \) with \( \theta > 1 \) and the distribution on \( S_t \) is uniform, then \( E(e^{\lambda X}) < \infty \) for small \( |\lambda| \). Thus this can happen even if \( E(T) = \infty \) (take any \( \theta \in (1, 2) \)).

**Notation 1.** Given a dimension function \( \Phi \) and random Moran set \( E(\omega) \), we define the associated depth function \( \phi = \phi_\omega : \mathbb{N} \to \mathbb{N} \) by the rule that \( \phi_\omega(n) \) is the minimal positive integer \( k \) such that

\[
r_1(\omega) \cdots r_{n+k}(\omega) \leq (r_1 \cdots r_n)^{1+\Phi(r_1 \cdots r_n)}.
\]

Consequently,

\[
(2.2) \quad r_1 \cdots r_{n+\phi(n)} \leq (r_1 \cdots r_n)^{1+\Phi(r_1 \cdots r_n)} \quad \text{and} \quad r_1 \cdots r_{n+\phi(n)-1} > (r_1 \cdots r_n)^{1+\Phi(r_1 \cdots r_n)}.
\]

This notion was introduced in [11] to study the formulas for the \( \Phi \)-dimensions of (deterministic) Cantor sets. There it was shown that if \( C \) is the central Cantor set with intervals of length \( r_1 \cdots r_n \) at step \( n \) and \( \inf_n r_n > 0 \), then

\[
\overline{\dim}_\Phi C = \limsup_n \left( \sup_{k \geq \phi(n)} \frac{n \log 2}{\log r_n + \cdots r_{n+k}} \right), \quad \underline{\dim}_\Phi C = \liminf_n \left( \inf_{k \geq \phi(n)} \frac{n \log 2}{\log r_n + \cdots r_{n+k}} \right).
\]
In this situation, there is a simple relationship between \( \phi \) and \( \Phi \): with \( C = \inf r_n/\sup r_n \) we have \( \phi(n) - 1 \leq Cn\Phi(r_1 \cdots r_n) \leq \phi(n) \).

For the random problem, it will be helpful to have information about the size of \( \phi_\omega(n) \) that is independent of \( \omega \) (in an almost sure sense). As we will see in the next result, the answer depends on how \( \Phi(x) \) compares with the function \( \log |\log x|/|\log x| \).

**Notation 2.** Given functions \( G, H : (0,1) \to \mathbb{R}^+ \), we define

\[
\zeta_N^{(G)} = \frac{G(2^{-N}) \log(N \log 2)}{2\mathbb{E}(Z_1)}
\]

and

\[
\chi_N^{(H)} = \frac{H(2^{-N}) \log(2N\mathbb{E}(Z_1))}{\log 2}.
\]

Apply Theorem 1 to choose a constant \( B \) so that for all \( k \in \mathbb{N} \),

\[
P \left( \sum_{j=1}^{k} Z_j - k\mathbb{E}(Z_1) \geq k\mathbb{E}(Z_1) \right) \leq \exp(-Bk).
\]

**Lemma 2.** (i) Suppose \( \Phi(x) \geq G(x) \log |\log x| / |\log x| \) where \( G \) is non-decreasing as \( x \) decreases to 0 and \( G(x) \geq 4\mathbb{E}(Z_1)/B \) for all \( x \in (0, 1) \). Then

\[P(\omega : \phi_\omega(N) < \zeta_N^{(G)} \text{ i.o.}) = 0.\]

(ii) If \( \Phi(x) \leq H(x) \log |\log x| / |\log x| \) where \( H \) is non-increasing as \( x \) decreases to 0, then \( P(\omega : \phi_\omega(N) > \chi_N^{(H)} \text{ i.o.}) = 0.\)

**Proof.** (i) If \( \phi_\omega(N) < \zeta_N^{(G)} = \zeta_N \), then

\[r_1r_2 \cdots r_{N+\zeta_N} \leq r_1r_2 \cdots r_{N+\phi_\omega(N)} \leq (r_1 \cdots r_N)^{1+\Phi(r_1 \cdots r_N)},\]

so

\[r_{N+1}r_{N+2} \cdots r_{N+\zeta_N} \leq (r_1 \cdots r_N)^{\Phi(r_1 \cdots r_N)}\]

and therefore

\[\sum_{j=N+1}^{N+\zeta_N} \log r_j \geq \Phi(r_1 \cdots r_N) |\log r_1 \cdots r_N| \geq G(r_1 \cdots r_N) \log |\log r_1 \cdots r_N| .\]

But \( r_1 \cdots r_N \leq 2^{-N} \), so by monotonicity, \( G(r_1 \cdots r_N) \geq G(2^{-N}) \) and \( |\log r_1 \cdots r_N| \geq N \log 2 \). Hence if \( \phi_\omega(N) < \zeta_N \), then

\[\sum_{j=N+1}^{N+\zeta_N} Z_j = \sum_{j=N+1}^{N+\zeta_N} \log r_j \geq G(2^{-N}) \log(N \log 2) = 2\zeta_N \mathbb{E}(Z_1),\]

so the choice of \( B \) gives

\[P(\omega : \phi_\omega(N) < \zeta_N) = P \left( \sum_{j=1}^{\zeta_N} Z_j \geq 2\zeta_N \mathbb{E}(Z_1) \right) \leq P \left( \sum_{j=1}^{\zeta_N} Z_j - \zeta_N \mathbb{E}(Z_1) \geq \zeta_N \mathbb{E}(Z_1) \right) \leq \exp(-B\zeta_N).\]
Since \( G(2^{-N}) \geq 4E(Z_1)/B \), there is a constant \( c \) such that

\[
\exp(−Bζ_N) = (N \log 2)^{-BG(2^{-N})/2E(Z_1)} \leq cN^{-2}.
\]

Thus

\[
\sum_N P(\omega : \phi_\omega(N) < \zeta_N) \leq c \sum N^{-2} < \infty
\]

and the Borel Cantelli lemma implies that \( P(\omega : \phi_\omega(N) < \zeta_N^{(G)}) \) i.o. = 0.

(ii) The arguments are similar, but easier, for (ii). If \( \phi_\omega(N) > \chi_N^{(H)} \), then

\[
r_1 \cdots r_{N+X} \geq r_1 \cdots r_{N+\phi(N)-1} \geq (r_1 \cdots r_N)^{1+\Phi(r_1 \cdots r_N)}.
\]

Hence

\[
\chi_N^{(H)} \log 2 \leq -\log r_{N+1} \cdots r_{N+X} \leq -\Phi(r_1 \cdots r_N) \log(r_1 \cdots r_N)
\]

\[
= H(r_1 \cdots r_N) \log |\log r_1 \cdots r_N| \leq H(2^{-N}) \log |\log r_1 \cdots r_N|.
\]

Putting in the formula for \( \chi_N^{(H)} \) and taking exponentials gives

\[
2NE(Z_1) \leq |\log r_1 \cdots r_N| = \sum_{i=1}^N -\log r_i.
\]

Applying Theorem \( \square \) again, we obtain

\[
P(\phi_\omega(N) > \chi_N^{(H)}) \leq P \left( \sum_{i=1}^N Z_i \geq 2NE(Z_1) \right) \leq \exp(-BN)
\]

and the Borel Cantelli lemma gives the result. \( \square \)

### 3. Dimension Results for Large \( \Phi \)

**Terminology:** We call a dimension function \( \Phi \) **large** if it satisfies \( \Phi(x) \gg \log |\log x|/|\log x| \).

The constant functions \( \Phi = \delta > 0 \) or any dimension function \( \Phi \to \infty \) are examples of large dimension functions.

**Theorem 2.** There is a set \( \Gamma \), of full measure in \( \Omega \), with the following properties:

(i) If there exists \( A > 0 \) such that \( E(e^{\lambda X_1}) < \infty \) for all \( |\lambda| \leq A \), then \( \dim_\Phi \mu_\omega = E(X_1)/E(Z_1) \) for all large dimension functions \( \Phi \) and for all \( \omega \in \Gamma \).

(ii) If there exists \( A > 0 \) such that \( E(e^{\lambda Y_1}) < \infty \) for all \( |\lambda| \leq A \), then \( \dim_\Phi \mu_\omega = E(Y_1)/E(Z_1) \) for all large dimension functions \( \Phi \) and for all \( \omega \in \Gamma \).

Since positive constant functions are large dimension functions, the following corollary is immediate.

**Corollary 1.** If there exists \( A > 0 \) such that \( E(e^{\lambda X_1}), E(e^{\lambda Y_1}) < \infty \) for all \( |\lambda| \leq A \), then \( \dim_{qL} \mu_\omega = \dim_F \mu_\omega = E(X_1)/E(Z_1) \) and \( \dim_{qA} \mu_\omega = \dim_M \mu_\omega = E(Y_1)/E(Z_1) \) almost surely.

We will abbreviate the statement ‘there exists \( A > 0 \) such that \( E(e^{\lambda W}) < \infty \) for all \( |\lambda| \leq A \)’ by \( E(e^{\lambda W}) < \infty \) for small \( |\lambda| \) and set

\[
\underline{d} = E(X_1)/E(Z_1), \quad \bar{d} = E(Y_1)/E(Z_1).
\]

We begin with two lemmas. We recall that \( \zeta_N^{(i)} \) was defined in \( 2.3 \).
Lemma 3. (i) Assume $\mathbb{E}(e^{\lambda X_1}) < \infty$ for small $|\lambda|$. For each $\varepsilon > 0$, there is a constant $K_\varepsilon \geq 1$ such that

$$F_N(\varepsilon) = \{ \omega : \exists k \geq \zeta_N^{(K_\varepsilon)} - 1 \text{ with } \prod_{i=N+2}^{N+k-L-1} M_i^{-1} < \prod_{i=N+1}^{N+k} r^{-d(1-\varepsilon)} \},$$

then $\mathbb{P}(F_N(\varepsilon) \text{ i.o.}) = 0$.

(ii) Assume $\mathbb{E}(e^{\lambda Y_1}) < \infty$ for small $|\lambda|$. For each $\varepsilon > 0$, there is a constant $K_\varepsilon \geq 1$ such that if

$$G_N(\varepsilon) = \{ \omega : \exists k \geq \zeta_N^{(K_\varepsilon)} - 1 \text{ with } \prod_{i=N-L+1}^{N+k} m_i^{-1} > \prod_{i=N+2}^{N+k} r^{-\lambda(1+\varepsilon)} \},$$

then $\mathbb{P}(G_N(\varepsilon) \text{ i.o.}) = 0$.

Remark 3. We remark we can take the same constant $K_\varepsilon$ in both parts; the choice of constant will be clear from the proof. We also note that once $N$ is suitably large, we will have $\zeta_N^{(K_\varepsilon)} > \max(3, L+3)$, whence the products above are well defined.

Proof. To simplify notation, we will let $\zeta_N^{(K_\varepsilon)} = \zeta_\varepsilon^N$.

(i) Upon taking logarithms, we have

$$F_N(\varepsilon) = \{ \omega : \exists k \geq \zeta_\varepsilon^N - 1 \text{ with } \sum_{j=N+2}^{N+k-L-1} X_j < d(1-\varepsilon) \sum_{j=N+1}^{N+k} Z_j \}.$$

The definition of $d$ ensures that

$$d(1-\varepsilon) \sum_{j=N+1}^{N+k} Z_j = d(1-\varepsilon) \left[ \sum_{j=N+1}^{N+k} Z_j - k\mathbb{E}(Z_1) \right] + (1-\varepsilon)k\mathbb{E}(X_1).$$

Pick $N_0$ such that if $k \geq \zeta_\varepsilon^{N_0} - 1$, then $(1-\varepsilon)k \leq (1-\varepsilon/2)(k-L-2)$ and $(k-L-2)\varepsilon/4 \geq k\varepsilon/8$. For all $N \geq N_0$, the set $F_N(\varepsilon)$ is contained in

$$\bigcup_{k \geq \zeta_\varepsilon^{N_0} - 1} \left\{ \sum_{j=N+2}^{N+k-L-1} X_j < d(1-\varepsilon) \left[ \sum_{j=N+1}^{N+k} Z_j - k\mathbb{E}(Z_1) \right] + (1-\varepsilon/2)(k-L-2)\mathbb{E}(X_1) \right\} \subseteq \bigcup_{k \geq \zeta_\varepsilon^{N_0} - 1} \left\{ \left[ \sum_{j=N+2}^{N+k-L-1} X_j - (k-L-2)\mathbb{E}(X_1) \right] + d(1-\varepsilon) \left[ k\mathbb{E}(Z_1) - \sum_{j=N+1}^{N+k} Z_j \right] < -\frac{(k-L-2)\varepsilon}{4} \mathbb{E}(X_1) - \frac{k\varepsilon}{4} \mathbb{E}(X_1) \right\}.$$

Hence $F_N(\varepsilon) \subseteq \bigcup_{k \geq \zeta_\varepsilon^{N_0} - 1} (A_k \cup B_k)$ where

$$A_k = \left\{ \omega : \left| \sum_{i=2}^{k-L-1} X_i - (k-L-2)\mathbb{E}(X_1) \right| > \frac{(k-L-2)\varepsilon}{4} \mathbb{E}(X_1) \right\}$$

and

$$B_k = \left\{ \omega : \left| k\mathbb{E}(Z_1) - \sum_{i=1}^{k} Z_i \right| > \frac{k\varepsilon}{8} \mathbb{E}(X_1) \right\} \subseteq \left\{ \omega : \left| k\mathbb{E}(Z_1) - \sum_{i=1}^{k} Z_i \right| > \frac{k\varepsilon}{8} \mathbb{E}(X_1) \right\}.$$
From Theorem 1 there are constants \(a_\varepsilon, b_\varepsilon > 0\) such that \(\mathbb{P}(A_k) \leq e^{-a_\varepsilon(k-L-2)}\) and \(\mathbb{P}(B_k) \leq e^{-b_\varepsilon k}\) for all \(k\). Thus there are constants \(C_\varepsilon, c_\varepsilon > 0\) such that for large enough \(N\),
\[
\mathbb{P}(F_N(\varepsilon)) \leq \sum_{k \geq \zeta N} \mathbb{P}(A_k) + \mathbb{P}(B_k) \leq C_\varepsilon e^{-c_\varepsilon \zeta N}.
\]
Choose \(K_\varepsilon\) sufficiently large to ensure that \(\exp(-c_\varepsilon \zeta N) \leq N^{-2}\) for all \(N\). With this choice, \(\sum_{N=1}^{\infty} \mathbb{P}(F_N(\varepsilon)) < \infty\) and hence the Borel Cantelli lemma gives the desired result.

(ii) This follows in a very similar fashion. The details are left for the reader. \(\square\)

**Lemma 4.** (i) Assume \(\mathbb{E}(e^{\lambda X_1}) < \infty\) for small \(|\lambda|\). For each \(\varepsilon > 0\), there is a constant \(K_\varepsilon' \geq 1\) such that \(\mathbb{P}(F_N'(\varepsilon) \text{ i.o.}) = 0\) where
\[
F_N'(\varepsilon) = \{\omega : \exists K \geq \zeta_N^{(K_\varepsilon')} \text{ with } \prod_{i=N+1}^{N+k} M_i^{-1} > \prod_{i=N+1}^{N+k} r_i^{-d(1+\varepsilon)}\}.
\]
(ii) Assume \(\mathbb{E}(e^{\lambda Y_1}) < \infty\) for small \(|\lambda|\). For each \(\varepsilon > 0\), there is a constant \(K_\varepsilon' \geq 1\) such that \(\mathbb{P}(G_N'(\varepsilon) \text{ i.o.}) = 0\) where
\[
G_N'(\varepsilon) = \{\omega : \exists K \geq \zeta_N^{(K_\varepsilon')} \text{ with } \prod_{i=N+1}^{N+k} m_i^{-1} < \prod_{i=N+1}^{N+k} r_i^{-d(1+\varepsilon)}\}.
\]

**Proof.** Again we will only prove (i) as (ii) is similar and to simplify notation we will write \(\zeta_N^{(K_\varepsilon')} = \zeta_N^\varepsilon\).

It is straightforward to see that
\[
F_N'(\varepsilon) = \bigcup_{k \geq \zeta_N} \left\{ \sum_{j=N+1}^{N+k} X_j > d(1+\varepsilon) \sum_{j=N+1}^{N+k} Z_j \right\}
\]
\[
\subseteq \bigcup_{k \geq \zeta_N} \left\{ \left[ \sum_{j=N+1}^{N+k} X_j - \mathbb{E}(X_1)k \right] + d(1+\varepsilon)\mathbb{E}(Z_1) - \sum_{j=N+1}^{N+k} Z_j \right] \geq \varepsilon k \mathbb{E}(X_1) \right\}
\]
\[
\subseteq \bigcup_{k \geq \zeta_N} \left\{ \left| \sum_{j=1}^{k} X_j - \mathbb{E}(X_1) \right| \geq \varepsilon k \mathbb{E}(X_1) \right\} \bigcup \left\{ \mathbb{E}(Z_1) - \sum_{j=1}^{k} Z_j \right\} \geq \varepsilon k \mathbb{E}(Z_1) \frac{\varepsilon}{2(\varepsilon + 1)} \right\}.
\]

From Petrov’s theorem, \(\mathbb{P}(F_N'(\varepsilon)) \leq C_\varepsilon e^{-c_\varepsilon \zeta N}\) for suitable constants \(C_\varepsilon, c_\varepsilon > 0\). Again, make the choice of \(K_\varepsilon'\) to ensure \(\sum_{N} \mathbb{P}(F_N'(\varepsilon)) < \infty\).

**Proof.** [of Theorem 2] We begin with some initial observations and notation that will be relevant to both the lower and upper \(\Phi\)-dimensions.

Let \(\varepsilon > 0\) and take \(K_\varepsilon = \max(K_\varepsilon, 4\mathbb{E}(Z_1)/B)\) where \(K_\varepsilon\) is the constant from Lemma 3 and \(B\) arises from the probability theorem as outlined in (2.6). Let
\[
\Phi^{(K_\varepsilon)}(x) = K_\varepsilon \log |\log x| / |\log x|
\]
and take \(\zeta_N^{(K_\varepsilon)}\) as defined in (2.4). To simplify notation, we will write \(\Phi^\varepsilon, \phi^\varepsilon\) and \(\zeta_N^\varepsilon\).
According to Lemma 2, there is a set, $\Gamma_\varepsilon$, of full measure, with the property that for every $\omega \in \Gamma_\varepsilon$, there is some integer $N_\omega$ such that for all $N \geq N_\omega$ we have $\phi^c(\varepsilon)(N) \geq \zeta^c(N)$. Let $\Gamma_j(j)$ be the subset of $\Gamma_\varepsilon$ with $N_\omega = j \in \mathbb{N}$.

Since $\zeta^c \to \infty$, we can choose $J_0$ such that if $N \geq J_0$, then $\zeta^c \geq L + 4$. Take $\omega \in \Gamma_j(j)$ and put $J_j = \max(j, J_0)$.

Given $0 < r < R < r_1 \cdots r_j$, choose $N = N(\omega)$ and $n = n(\omega) > N$ such that

$$r_1 \cdots r_{n+1} \leq R < r_1 \cdots r_N$$
$$r_1 \cdots r_n \leq r < r_1 \cdots r_{n-1}.$$

Note that $N \geq J_j$. Furthermore, we have the bounds

$$\frac{(r_{n+1} \cdots r_{n-1})^{-1}}{r} \leq \frac{(r_{n+1} \cdots r_n)^{-1}}{r}.$$

If $r \leq r_1 \cdots r_{n+1}^\varepsilon(N+1)$, then since the function $x^{1+\Phi}(x)$ is increasing, (2.2) tells us that

$$r \leq (r_1 \cdots r_{n+1})^{1+\Phi}(r_1 \cdots r_{n+1}) \leq R^{1+\Phi}(R).$$

On the other hand, if $r \leq R^{1+\Phi}(R)$, then by (2.3),

$$r \leq (r_1 \cdots r_n)^{1+\Phi}(r_1 \cdots r_n) \leq r_1 \cdots r_{n+\Phi}(N+1)$$

and hence $n \geq N + \phi^c(N) - 1 \geq N + \zeta^c - 1$, so that $n - L - 1 \geq N + 2$.

From Lemma 4 we know

$$\frac{\mu(I_{n+1}(x))}{\mu(I_{n-1}(x))} \leq \frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \frac{\mu(I_N(x))}{\mu(I_n(x))}.$$

Since

$$\frac{\mu(I_j(x))}{\mu(I_{j+s}(x))} = \frac{\mu(I_j(x))}{\mu(I_j(x))p_{j+1}^{(j+1)} \cdots p_{j+s}^{(j+s)}}$$

for a suitable choice of probabilities $p_{\ell}^{(i)}$, $\ell = j + 1, \ldots, j + s$, we have

$$(M_{n+2} \cdots M_{n+1})^{-1} \leq \frac{\mu(B(x, R))}{\mu(B(x, r))} \leq (m_{n+1} \cdots m_n)^{-1}.$$

**Computation of the almost sure lower $\Phi$-dimension:** Assume $\mathbb{E}(e^{\lambda X_1}) < \infty$ for small $|\lambda|$.

Since $\phi^c(\varepsilon)(N) \geq \zeta^c$ for $N \geq J_j$, with the notation $F_N(\varepsilon)$ from Lemma 3(i) we have

$$\omega \in \Gamma_\varepsilon(j) : \exists k \geq \phi^c(\varepsilon)(N) - 1 \text{ with } \prod_{i=N+2}^{N+k-1} M_i^{-1} < \prod_{i=N+1}^{N+k} r_i^{-\lambda(1-\varepsilon)} \subseteq F_N(\varepsilon).$$

By Lemma 4 $\mathbb{P}(F_N(\varepsilon) \text{ i.o}) = 0$, so $(F_N(\varepsilon) \text{ i.o}) = \Lambda_\varepsilon$ where $\Lambda_\varepsilon$ has full measure. Thus the set $\Delta_\varepsilon(j) = \Lambda_\varepsilon \cap \Gamma_\varepsilon(j)$ has full measure in $\Gamma_\varepsilon(j)$. Moreover, it has the property that for each $\omega \in \Delta_\varepsilon(j)$, there is an integer $N_{\varepsilon,j}(\omega) \geq J_j$ so that if $N \geq N_{\varepsilon,j}(\omega)$, then for every $k \geq \phi^c(\varepsilon)(N) - 1$ we have

$$(M_{N+2} \cdots M_{N+k-L-1})^{-1} \geq (r_{N+1} \cdots r_{N+k})^{-\lambda(1-\varepsilon)}.$$

For $\omega \in \Delta_\varepsilon(j)$, set $\rho(\varepsilon, \omega) = r_1 \cdots r_{N_{\varepsilon,j}(\omega)}$. If $R \leq \rho(\varepsilon, \omega)$ satisfies (3.1), then $N \geq N_{\varepsilon,j}(\omega)$, hence there can be no choice of $k \geq \phi^c(\varepsilon)(N) - 1$ with

$$(M_{N+2} \cdots M_{N+k-L-1})^{-1} < (r_{N+1} \cdots r_{N+k})^{-\lambda(1-\varepsilon)}.$$
We deduce that for all \( r \leq R^{1+\Phi(R)} \leq \rho(\epsilon, \omega) \) and for all \( x \in E(\omega) \),
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq (M_{N+2} \cdots M_{N-L-1})^{-1} \geq (r_{N+1} \cdots r_n)^{-d(1+\epsilon)} \geq \left( \frac{R}{r} \right)^{d(1+\epsilon)}.
\]

Let \( \Delta_\epsilon = \bigcup_{j=1}^{\infty} \Delta_\epsilon(j) \). This is a set of measure one and we have just shown that 
\[
\dim_{\Phi} \mu_\omega \geq d(1-\epsilon) \text{ for all } \omega \in \Delta_\epsilon.
\]
Take a sequence \( \epsilon_i \to 0 \) and let \( \Delta = \bigcap_{i=1}^{\infty} \Delta_{\epsilon_i} \).
Of course, \( \mathbb{P}(\Delta) = 1 \) and \( \dim_{\Phi} \mu_\omega \geq d(1-\epsilon_i) \) for all \( \omega \in \Delta \).

Now suppose \( \Phi \) is any large dimension function. Given any \( \epsilon_i \), we have \( \Phi(x) \geq \Phi_0(\epsilon_i)(x) \) for small enough \( x \) and therefore \( \dim_{\Phi} \mu_\omega \geq \dim_{\Phi_0} \mu_\omega \) for all \( \omega \). We conclude that \( \dim_{\Phi} \mu_\omega \geq d(1-\epsilon_i) \) for all \( \omega \in \Delta \) and for all \( i \), and consequently, \( \dim_{\Phi} \mu_\omega \geq d \) on \( \Delta \).

For the opposite inequality, fix \( \epsilon > 0 \) and choose \( K'_\epsilon \) and \( F_\epsilon'(\epsilon) \) as in Lemma 4(i) so that \( \{F'_\epsilon(\epsilon) \ i.o.\}' = \Delta'_\epsilon \) is a set of measure one. That means, for all \( \omega \in \Delta'_\epsilon \) there are arbitrarily large \( N \) with the property that for all large enough \( k \),
\[
(M_{N+1} \cdots M_{N+k})^{-1} \leq (r_{N+1} \cdots r_{N+k})^{-d(1+\epsilon)}.
\]

Let \( \Phi \) be any large dimension function. For \( \omega \in \Delta'_\epsilon \), take
\[
R_N = \tau r_1(\omega) \cdots r_N(\omega)/2.
\]
As \( r/2 \geq 2^{-L} \geq r_{N+1} \cdots r_{N+L} \), we have \( R_N \geq r_1 \cdots r_{N+L} \) and thus if \( k \geq \phi_\omega(N+L) \),
then
\[
\theta_{N,k} := r_1 \cdots r_{N+L+k} \leq R_N^{1+\Phi(R_N)}.
\]
Fix some Moran set of level \( N \) and for each \( k \geq L + \phi(N+L) \) consider a descendant Moran set of step \( N + k \) where at each step from \( N + 1 \) to \( N + k \) we select the child that is assigned the maximum probability. Choose \( x = x_{N,k} \in E(\omega) \) in that descendant, so \( B(x, \theta_{N,k}) \supseteq I_{N+k}(x) \). Since the distance from \( x \) to any Moran set of level \( N \), other than \( I_N(x) \), is at least \( 2R_N \), we have \( B(x, R_N) \cap E(\omega) \subseteq I_N(x) \), and therefore
\[
\frac{\mu(B(x, R_N))}{\mu(B(x, \theta_{N,k}))} \leq \frac{\mu(I_N(x))}{\mu(I_{N+k}(x))} = (M_{N+1} \cdots M_{N+k})^{-1}.
\]
Hence there are choices of \( k \geq \phi_\omega(N+L) \) such that
\[
\frac{\mu(B(x, R_N))}{\mu(B(x, \theta_{N,k}))} \leq (r_{N+1} \cdots r_{N+k})^{-d(1+\epsilon)} = C_\epsilon \left( \frac{R_N}{\theta_{N,k}} \right)^{d(1+\epsilon)}
\]
(with constant \( C_\epsilon = (2/\tau)^{d(1+\epsilon)} \). It follows that \( \dim_{\Phi} \mu_\omega \leq (1+\epsilon)d \) for every \( \omega \in \Delta'_\epsilon \). Again, take a sequence \( \epsilon_i \to 0 \) and let \( \Delta' = \bigcap_{i=1}^{\infty} \Delta'_{\epsilon_i} \), a set of full measure.

Combining these facts, we see that \( \dim_{\Phi} \mu_\omega \leq d \) for all large dimension functions \( \Phi \) and for every \( \omega \in \Delta \cap \Delta' \), a set of measure one.

**Computation of the almost sure upper \( \Phi \)-dimension:** This computation is very similar to that of the lower \( \Phi \)-dimension. First, use the upper bound in [3.4],
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \left( m_{N-L+1} \cdots m_n \right)^{-1},
\]

together with Lemma 3(ii) to prove that for each \( \varepsilon > 0 \) there is a set of full measure, \( \Delta_\varepsilon \), such that \( \dim_{\Phi^{(K_\varepsilon)}} \mu_\omega \leq d(1 + \varepsilon) \) for a suitable choice of constant \( K_\varepsilon \). But any large dimension function \( \Phi \) dominates the function \( \Phi^{(K_\varepsilon)} \) for small enough \( x \), hence \( \dim_{\Phi} \mu_\omega \leq d(1 + \varepsilon) \) on \( \Delta_\varepsilon \), as well. Then use Lemma 3(ii), along with \( R_N = r_1 \cdots r_N \), \( g_{N,k} = r_1 \cdots r_{N+k}/2 \) and \( x_{N,k} \in E(\omega) \) belonging to a Moran descendent of level \( N + k \), where at each step \( N + 1 \) to \( N + k \) we choose a Moran set where the minimal probability is assigned, to deduce that \( \dim_{\Phi} \mu_\omega \geq d(1 - \varepsilon) \) on some set \( \Delta_\varepsilon \) of full measure (independent of the choice of \( \Phi \)). To complete the proof, take a sequence \( \varepsilon_i \to 0 \) as before.

Of course, we can take as \( \Gamma \) the intersection of the two sets of full measure where the upper and lower \( \Phi \) dimensions are the appropriate values. \( \square \)

**Remark 4.** Note that the proof actually shows that if \( \varepsilon > 0 \), then (in the notation of the proof) \( \dim_{\Phi^{(K_\varepsilon)}} \mu_\omega \geq d(1 - \varepsilon) \) and \( \dim_{\Phi^{(K_\varepsilon)}} \mu_\omega \leq d(1 + \varepsilon) \) a.s.

### 4. Dimension Results for Small \( \Phi \)

**Terminology:** We call a dimension function \( \Phi \) small if it satisfies \( \Phi(x) \ll \log |\log x|/|\log x| \).

The constant function \( \Phi = 0 \) (giving the Assouad dimension) is an example of a small \( \Phi \).

**Notation 3.** Put

\[
\alpha = \inf \{ s : \log m_1/\log r_1 \leq s \ a.s. \} = \essinf \log m_1/\log r_1,
\]

and

\[
\beta = \sup \{ t : \log M_1/\log r_1 \geq t \ a.s. \} = \essinf \log M_1/\log r_1.
\]

Obviously \( \alpha \geq \beta \). Here are some other easy facts.

**Lemma 5.** (i) If \( \esssup M_1 = 1 \), or \( \essinf r_1 = 0 \) but \( \essinf M_1 \neq 0 \), then \( \beta = 0 \). If \( \essinf M_1 = 0 \), but \( \essinf r_1 \neq 0 \), then \( \alpha = \infty \).

(ii) If \( \{ r_1, m_1 \} \) are independent random variables and \( \essinf m_1 = 0 \), then \( \alpha = \infty \).

**Proof.** Part (i) is obvious.

(ii) Since \( r_1(\omega) > 0 \) for all \( \omega \), there must be some \( \varepsilon > 0 \) such that \( \mathbb{P}(r_1 \geq \varepsilon) > 0 \).

The independence of \( r_1 \) and \( m_1 \) ensures that for any \( q \in \mathbb{R}^+ \),

\[
\begin{align*}
\mathbb{P}(- \log m_1 \geq q \log r_1) &= \mathbb{P}(- \log m_1 \geq -q \log \varepsilon) \mathbb{P}(- \log r_1 \leq -\log \varepsilon) \\
&= \mathbb{P}(- \log m_1 \geq -q \log \varepsilon) \mathbb{P}(r_1 \geq \varepsilon) \\
&= \mathbb{P}(m_1 \leq \varepsilon^q) \mathbb{P}(r_1 \geq \varepsilon).
\end{align*}
\]

But \( \mathbb{P}(m_1 \leq \varepsilon^q) > 0 \) as \( \essinf m_1 = 0 \) and therefore \( \alpha = \infty \). \( \square \)

**Theorem 3.** There is a set \( \Gamma \), of full measure in \( \Omega \), with the following properties:

(i) \( \dim_{\Phi} \mu_\omega \leq \beta \) and \( \dim_{\Phi} \mu_\omega \geq \alpha \) for all small dimension functions \( \Phi \) and all \( \omega \in \Gamma \).

(ii) If \( \essinf M_j > 0 \) or \( \essinf r_j > 0 \), then \( \dim_{\Phi} \mu_\omega \geq \beta \) for all dimension functions \( \Phi \) and all \( \omega \in \Gamma \).

(iii) If \( \essinf m_j > 0 \) or \( \essinf r_j > 0 \), then \( \dim_{\Phi} \mu_\omega \leq \alpha \) for all dimension functions \( \Phi \) and all \( \omega \in \Gamma \).
Here are some immediate consequences.

**Corollary 2.** If \( \text{ess inf} r_j > 0 \), then \( \dim_{\Phi} \mu_\omega = \dim_L \mu_\omega = \beta \) and \( \overline{\dim}_{\Phi} \mu_\omega = \dim_A \mu_\omega \). For all small \( \Phi \) and almost every \( \omega \). If the number of children is bounded, then \( \dim_{\Phi} \mu_\omega = \dim_L \mu_\omega = \beta \) for all small \( \Phi \) and a.e. \( \omega \).

**Proof.** [of Theorem 3] (i) We will assume \( \alpha < \infty \); the proof for \( \alpha = \infty \) is similar and left for the reader. The definition of \( \alpha \) ensures that given \( \varepsilon > 0 \), we can choose \( q(\varepsilon) \leq \exp(-3) \) with

\[
0 < q(\varepsilon) \leq P(-\log m_1 \geq (\alpha - \varepsilon)(-\log r_1)).
\]

Let \( \phi^\varepsilon(x) = \frac{\log 2}{2|\log q(\varepsilon)|} \log |x| \) and \( \chi^\varepsilon_N = \frac{\log(2NE(Z_1))}{2|\log q(\varepsilon)|} \)

and obtain the set, \( \Gamma_\varepsilon \), of full measure from Lemma 2, with the property that \( \phi^\varepsilon(N) \leq \chi^\varepsilon_N \) eventually for all \( \omega \in \Gamma_\varepsilon \). Set \( J_N(\varepsilon) = \lceil \chi^\varepsilon_N \rceil \) (meaning, the next integer) and let

\[
G^\varepsilon_N = \left\{ (m_{N+1} \cdots m_{N+J_N(\varepsilon)})^{-1} \geq (r_{N+1} \cdots r_{N+J_N(\varepsilon)})^{-(\alpha - \varepsilon)} \right\}.
\]

As the random variables \( \{m_i\} \) are independent and identically distributed, as are the random variables \( \{r_i\} \),

\[
P(G^\varepsilon_N) = P \left( \sum_{i=1}^{J_N(\varepsilon)} -\log m_i \geq (\alpha - \varepsilon) \sum_{i=1}^{J_N(\varepsilon)} -\log r_i \right)
\]

\[
\geq P \left( \sum_{i=1}^{J_N(\varepsilon)} -\log m_i \geq (\alpha - \varepsilon)(-\log r_1) \right.
\]

\[
\left. \quad \text{for each } i = 1, \ldots, J_N(\varepsilon) \right) = \prod_{i=1}^{J_N(\varepsilon)} P(-\log m_i \geq (\alpha - \varepsilon)(-\log r_1)) \geq q(\varepsilon)^{-J_N(\varepsilon)}.
\]

It is easy to check that \( q(\varepsilon)^{J_N(\varepsilon)} \geq 1/N \) for large \( N \), thus if we set \( N_k = k \log k \), then

\[
\sum_k P(G^\varepsilon_N) \geq \sum_k q(\varepsilon)^{-J_N(\varepsilon)} \geq \sum_k \frac{1}{k \log k} = \infty.
\]

The fact that \( |\log q(\varepsilon)| \geq 3 \) ensures that \( N_{k+1} - N_k \geq (k+1) - J_N(\varepsilon) \) for large \( k \), hence the events \( G^\varepsilon_N \) are independent and the Borel Cantelli lemma implies \( G^\varepsilon_N \) occurs infinitely often with probability one, say on \( \Gamma_\varepsilon \).

For \( \omega \in \Gamma_\varepsilon \cap \Gamma_\varepsilon^c = \Delta_\varepsilon \), choose \( N_\omega \) such that for \( N \geq N_\omega \), \( \phi^\varepsilon(N) \leq \chi^\varepsilon_N \). Put \( R_N = r_1(\omega) \cdots r_N(\omega) \) and \( \varrho N = r_1 \cdots r_{N+J_N(\varepsilon)} / 2 \). As \( J_N(\varepsilon) \geq \phi^\varepsilon(\varepsilon) \),

\[
\varrho N \leq R_{N+J_N(\varepsilon)}^{1+\Phi^\varepsilon(R_N)}.
\]

Take any Moran set \( I_N \) of step \( N \) and consider the descendant at step \( N+J_N(\varepsilon) \) where the minimal probability was chosen each time. Let \( x_N \in E(\omega) \) belong to that descendant. The separation property and size of \( \varrho N \) ensures that \( B(x_N, \varrho N) \cap E(\omega) \subseteq I_{N+J_N(\varepsilon)}(x) \), while \( B(x_N, R_N) \supseteq I_N(x) \). Hence

\[
\frac{\mu(B(x_N, R_N))}{\mu(B(x_N, \varrho N))} \geq \frac{\mu(I_N(x))}{\mu(I_{N+J_N(\varepsilon)}(x))} = (m_{N+1} \cdots m_{N+J_N(\varepsilon)})^{-1}.
\]

For infinitely many \( N \in \{N_k\} \),

\[
(m_{N+1} \cdots m_{N+J_N(\varepsilon)})^{-1} \geq (r_{N+1} \cdots r_{N+J_N(\varepsilon)})^{-(\alpha - \varepsilon)} = \left( \frac{\tau}{2} \right)^{\alpha - \varepsilon} \left( \frac{R}{r} \right)^{\alpha - \varepsilon},
\]
thus for suitable arbitrarily small \( R, r \leq R^{1+\Phi(R)} \) and \( x \in E(\omega) \), we have

\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq C_\varepsilon \left( \frac{R}{r} \right)^{\alpha - \varepsilon}
\]

for a constant \( C_\varepsilon > 0 \). That implies \( \dim_{\Phi} \mu_\omega \geq \alpha - \varepsilon \) for all \( \omega \in \Delta_\varepsilon \).

If \( \Phi \) is any small dimension function, then \( \Phi(x) \leq \Phi^\varepsilon(x) \) for small \( x \) and consequently \( \dim_{\Phi} \mu_\omega \geq \dim_{\Phi^\varepsilon} \mu_\omega \geq \alpha - \varepsilon \) for all \( \omega \in \Delta_\varepsilon \). Taking a sequence \( \varepsilon_i \to 0 \), we conclude that \( \dim_{\Phi} \mu_\omega \geq \alpha \) for all \( \omega \in \Delta = \bigcap \Delta_\varepsilon_i \), a set of full measure.

The arguments are similar for the lower dimension, but in this case we begin with

\[
0 < q(\varepsilon) \leq \mathbb{P}( -\log M_1 \geq (\beta + \varepsilon)(-\log r_1) )
\]

and define \( \Phi^\varepsilon \) and \( \chi_\varepsilon \) accordingly. Put

\[
G_{N_\varepsilon} = \left\{ \omega : (M_{N+1} \cdots M_{N+L+J_{N+L}(\varepsilon)})^{-1} \leq (r_{N+1} \cdots r_{N+L+J_{N+L}(\varepsilon)})^{-(\beta + \varepsilon)} \right\}.
\]

For large \( k \), \( N_{k+1} \geq N_k + L + J_{N_k+L}(\varepsilon) \), so analogous reasoning to the above with the Borel Cantelli lemma implies \( G_{N_k} \) occurs infinitely often with probability one.

Put \( R_N = r r_{N+1} \cdots r_N / 2 \), \( x_N = x_{r_{N+1} \cdots r_N+L} \). Take a Moran set of step \( N \) and consider the descendent at step \( N + L + J_{N+L}(\varepsilon) \) where we choose the maximum probability each time. For \( x_N \in E(\omega) \) belonging to this descendent and for suitable arbitrarily large \( N \in \{N_k\} \), we have

\[
\frac{\mu(B(x_N, R_N))}{\mu(B(x_N, \varnothing_N))} \leq \frac{\mu(I_N(x_N))}{\mu(I_N+L+J_{N+L}(\varepsilon))(x)} = (M_{N+1} \cdots M_{N+L+J_{N+L}(\varepsilon)})^{-1}
\]

\[
\leq (r_{N+1} \cdots r_{N+L+J_{N+L}(\varepsilon)})^{-(\beta + \varepsilon)} \leq C_\varepsilon \left( \frac{R}{r} \right)^{\beta + \varepsilon}.
\]

Thus \( \dim_{\Phi} \mu \leq \beta + \varepsilon \) on a set of full measure, \( \Delta' \). As above, we deduce that for any small dimension function \( \Phi \), \( \dim_{\Phi} \mu \leq \beta \) on the set of full measure \( \Delta' = \bigcap \Delta'_\varepsilon \).

(ii) Fix \( \omega \) and assume \( r \leq R^{1+\Phi(R)} \leq R \) as in (3.1), so \( R/r \leq (r_{N+1} \cdots r_n)^{-1} \). For any \( x \in E(\omega) \), Lemma 1 implies

\[
(4.1) \quad \frac{\mu(B(x, R))}{\mu(B(x, r))} \geq (M_{N+2} \cdots M_{n-L-1})^{-1} \text{ if } n > N + L + 2.
\]

Regardless of \( n \), \( \mu(B(x, R))/\mu(B(x, r)) \geq 1 \).

If \( \text{ess inf } M_j = \delta > 0 \), then in either case

\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq C_\delta (M \cdots M_n)^{-1}
\]

for some constant \( C_\delta > 0 \). If, instead, \( \text{ess inf } r_j = \delta > 0 \) and \( n \leq N + L + 2 \), then \( R/r \leq \delta^{-(L+2)} \), while if \( n > N + L + 2 \), then \( R/r \leq C_\delta (r_{N+2} \cdots r_{n-L-1})^{-1} \).

It follows that for any \( \omega \) in the set of full measure, \( \Omega_\varepsilon \), where \( -\log M_i / -\log r_1 \geq \beta - \varepsilon \) for all \( i \), there is a constant \( c > 0 \) such that

\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq c \left( \frac{R}{r} \right)^{\beta - \varepsilon}.
\]

We conclude that for any \( \Phi \), \( \dim_{\Phi} \mu \geq \beta \) on \( \bigcap \Omega_\varepsilon \), where \( \varepsilon_i \to 0 \).
(iii) This is similar to (ii) (but easier) since if $R$ and $r \leq R^{1+\Phi(R)}$ satisfy (3.1), then for all $n > N$,
\[
\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq \frac{\mu(I_{N-L}(x))}{\mu(I_{n}(x))} \leq (m_{N-L+1} \cdots m_{n})^{-1}.
\]

\[\square\]

Remark 5. In Subsection 5.4, we see that it is possible to have $\dim \Phi \mu > \alpha$ and $\dim \Phi \mu < \beta$ on sets of positive measure.

Remark 6. It would be interesting to know formulas for the $\Phi$-dimensions when $\Phi(x) \sim \log |\log x|/|\log x|$.

5. Applications

In this section we will consider some classes of examples of random Moran sets and measures to which our results apply.

5.1. Fixed probabilities.

Example 3. One obvious special case of this set up is the deterministic model where $T_j(\omega) = T$, $r_j(\omega) = r \leq r_T$ and $(p^{(1)}_j(\omega), \ldots, p^{(T)}_j(\omega)) = (p^{(1)}, \ldots, p^{(T)})$ for all $j$ and all $\omega$. Formally, this arises by choosing the probability measure to be a single point mass measure. We clearly have $\mathbb{E}(e^{\lambda X_j}), \mathbb{E}(e^{\lambda Y_j}), \mathbb{E}(e^{\lambda Z_j}) < \infty$ for all $\lambda$ and $\mathbb{E}(X_1) = -\log(\max p^{(j)}_1)$, $\mathbb{E}(Y_1) = -\log(\min p^{(j)}_1)$, and $\mathbb{E}(Z_1) = -\log r_1$.

An example of this situation is to take an iterated function system on $\mathbb{R}^D$, satisfying the strong separation condition, with $T$ similarities all having contraction factor $r > 0$, and probabilities $p^{(j)}$. The strong separation condition ensures the $(T, r, \tau)$-separation condition is automatically satisfied for a suitable choice of $\tau > 0$. Thus the self-similar measure has
\[\dim \Phi \mu = \frac{\log(\max p^{(j)}_1)}{\log r} \text{ and } \dim \Phi \mu = \frac{\log(\min p^{(j)}_1)}{\log r}\]
for all dimension functions $\Phi$. This class of examples was also worked out in [15].

But we do not require the rigid structure of a self-similar set. For instance, we need not apply the same similarities to different parents and we need not have the same distances between children of different parents.

Example 4. Another special case is when we have a fixed set of probabilities, so $\mathbb{E}(e^{\lambda X_1}), \mathbb{E}(e^{\lambda Y_1})$ are finite for all $\lambda$, but the ratios are chosen independently and identically distributed from some (non-trivial) set. Provided $\mathbb{E}(e^{\lambda Z_1}) < \infty$ for small $|\lambda|$, then the $\Phi$-dimensions follow from the theorems.

If we choose the probabilities to be equal ($1/T$ if there are $T$ children), then there are constants $c, C > 0$ such that for any $r \leq R, x \in E(\omega)$ we have
\[cN_r(B(x,R)) \leq \frac{\mu_\omega(B(x,R))}{\mu_\omega(B(x,r))} \leq CN_r(B(x,R)).\]

It follows that the $\Phi$-dimensions of the random measure $\mu_\omega$ and the random set $E(\omega)$ coincide.
For example, if we begin with \( I_0 = [0, 1] \subseteq \mathbb{R} \), take \( T = 2 \) and assume \( r_j \) is chosen uniformly over \((0, 1/2)\), then \( \mathbb{E}(e^{\lambda Z_1}) < \infty \) for \( |\lambda| < 1 \) and \( \mathbb{E}(Z_1) = 1 + \log 2 \). Thus for large \( \Phi \),

\[
\dim_\Phi E(\omega) = \overline{\dim}_\Phi E(\omega) = \frac{\log 2}{1 + \log 2} \text{ a.s.}
\]

Since \( \text{ess inf } r_j = 0 \) and \( \text{ess sup } r_j = 1/2 \), we have \( \beta = 0 \) and \( \alpha = 1 \), thus for small \( \Phi \)

\[
\overline{\dim}_\Phi E(\omega) = 1 \text{ and } \underline{\dim}_\Phi E(\omega) = 0 \text{ a.s.}
\]

5.2. Variable probabilities. Another interesting class is when the ratios are constant, \( r_j = r \) for all \( j \), but the probabilities, \( \{p_j^{(1)}, \ldots, p_j^{(T)}\} \), are iid random variables chosen from a non-trivial probability space. In this case, we obviously have \( \mathbb{E}(e^{\lambda X_1}) < \infty \) for all \( \lambda \). Further, the separation condition forces \( T_j \leq T \) for some \( T < \infty \), so \( M_j \geq 1/T \) and \( \mathbb{E}(e^{\lambda Y_1}) < \infty \). Thus we deduce the following results from the two theorems.

**Corollary 3.** (i) There is a set, \( \Gamma \), of measure one, such that if \( \Phi(x) \) is any large dimension function, then

\[
\dim_\Phi \mu(\omega) = \frac{\mathbb{E}(X_1)}{\log r} = \frac{\mathbb{E}(\| \log \max_k p_1^{(k)} \|)}{\log r} \text{ for all } \omega \in \Gamma.
\]

Further, if there exists \( A > 0 \) such that \( \mathbb{E}(e^{\lambda Y_1}) < \infty \) for all \( |\lambda| \leq A \), then

\[
\overline{\dim}_\Phi \mu(\omega) = \frac{\mathbb{E}(Y_1)}{\log r} = \frac{\mathbb{E}(\| \log \min_k p_1^{(k)} \|)}{\log r} \text{ for all } \omega \in \Gamma.
\]

(ii) There is a set, \( \Gamma \), of measure one, such that if \( \Phi(x) \) is any small dimension function, then

\[
\underline{\dim}_\Phi \mu(\omega) = \text{ess inf} \| \log \max_k p_1^{(k)} \| / \log r \text{ and } \underline{\dim}_\Phi \mu(\omega) = \text{ess sup} \| \log \min_k p_1^{(k)} \| / \log r \forall \omega \in \Gamma.
\]

**Example 5.** When \( T_j = T \) and the probabilities \( \{p_j^{(1)}, \ldots, p_j^{(T)}\} \) are uniformly distributed over the simplex \( \mathcal{S}_T \) it follows that for small \( \Phi \), almost surely \( \underline{\dim}_{\Phi,M} = 0 \) and \( \overline{\dim}_{\Phi,M} = \infty \).

When \( T = 2 \) and the probabilities are uniformly distributed, it is easy to compute \( \mathbb{E}(\| \log \max p_1^{(k)} \|) \) and \( \mathbb{E}(\| \log \min p_1^{(k)} \|) \) since \( \min(p, 1-p) \) and \( \max(p, 1-p) \) are uniformly distributed over \((0, 1/2]\) and \([1/2, 1)\) respectively. However, for \( T > 2 \) this is a more challenging combinatorial problem answered in the next Proposition. The proof is given in the Appendix.

**Proposition 2.** Suppose the probabilities \( \{p_j^{(1)}, \ldots, p_j^{(T)}\} \) are uniformly distributed over the simplex \( \mathcal{S}_T \). Then \( \mathbb{E}(e^{\lambda X_1}), \mathbb{E}(e^{\lambda Y_1}) < \infty \) for \( |\lambda| < 1 \) and

\[
\mathbb{E}(X_1) = \sum_{j=1}^{T} (-1)^{j+1} \binom{T}{j} \log j + \sum_{j=1}^{T-1} \frac{1}{j}, \quad \mathbb{E}(Y_1) = \log T + \sum_{j=1}^{T-1} \frac{1}{j}.
\]

Coupled with Corollary 8 this gives many further examples. Here are two.
Example 6. (1) Suppose $E(\omega)$ is the classical middle-third Cantor set. Let $\mu$ be the random measure where the probabilities $(p_j, 1 - p_j)$ are independent and uniformly distributed random variables. If $\Phi(x)$ is large, then almost surely

$$\dim_{\Phi, \mu} = \frac{1 + \log 2}{\log 3}, \dim_{\Phi, \mu} = \frac{1 - \log 2}{\log 3}.$$ 

(2) Suppose $E(\omega)$ is a Cantor-like set, but with three children and all $r_j = r < 1/3$. Let $\mu$ be the random measure where the probabilities $(p_j(1), p_j(2), p_j(3))$ are independent and uniformly distributed rv’s. If $\Phi(x)$ is large, then almost surely

$$\dim_{\Phi, \mu} = \frac{3/2 + \log 3}{|\log r|}, \dim_{\Phi, \mu} = \frac{3/2 - 3 \log 2 + \log 3}{|\log r|}.$$ 

5.3. Random 1-variable model. Another important class of examples are the random 1-variable Moran sets and measures described in Example 6. Hence our theorems give formulas for the almost sure upper and lower $\Phi$-dimensions for these measures.

If we choose the probabilities associated with each of the IFS to be equal, then the dimension of the random measure and set $E(\omega)$ coincide. In [24], Troscheit showed that $\dim_{qA} E = \dim_{H} E$ almost surely. In fact, our results show that the upper and lower $\Phi$-dimensions for all large $\Phi$ coincide almost surely, so even $\dim_{qA} E = \dim_{qL} E$ a.s.

If, for example, the finitely many IFS are chosen with equal likelihood, then the dimension formulas are very simple. Assume there are a total of $N$ families of iterated function systems, where the $j$’th family consists of $K_j$ similarities, having (common) contraction factor $r_j$ and probabilities $1/K_j$. Then there is a set, $\Gamma$, of full measure, such that for all $\omega \in \Gamma$,

$$\dim_{\Phi} E(\omega) = \dim_{\Phi} E(\omega) = \frac{\sum_{i=1}^{N} \log K_i}{\sum_{i=1}^{N} \log r_i} \text{ for all large } \Phi,$$

and

$$\dim_{\Phi} E(\omega) = \max \frac{\log K_i}{\log r_i} \dim_{\Phi} E(\omega) = \min \frac{\log K_i}{\log r_i} \text{ for all small } \Phi.$$ 

5.4. Examples of strict inequality in Theorem 3. We conclude with examples that show we need not have $\dim_{\Phi, \mu} = \alpha$ or $\dim_{\Phi, \mu} = \beta$ a.s. (in the notation of Theorem 3).

For these examples, we will work in $\mathbb{R}$ with the initial compact set $I_0 = [0, 1]$. Thus all the Moran sets of each step will be intervals. We will also assume a stronger separation condition, namely that the gaps adjacent to the Moran intervals of step $n$ which are assigned measure equal to that of the parent interval times $m_n$ (resp., $M_n$) have length at least $\tau r_1 \cdots r_{n-1}$.

Proposition 3. In addition to assuming $I_0 = [0, 1] \subseteq \mathbb{R}$ and the stronger separation condition described above, we will assume that for each $n$, $\{m_{n+1}, r_{n+1}\}$ is independent of $\{m_i, r_l : l = 1, \ldots, n\}$. Suppose $\Phi$ is a small dimension function.

If there are constants $\theta, c > 0$ such that for all small $z > 0$, $\mathbb{P}(m_1 \leq z) \geq cz^\theta$, (resp., $\mathbb{P}(r_1 \leq z) \geq cz^\theta$), then $\dim_{\Phi} = \infty$ (resp., $\dim_{\Phi} = 0$) almost surely.

Proof. Being small, we can assume $\Phi \leq H(x) \log |\log x|/|\log x|$ where $H$ decreases as $x \downarrow 0$. Let $J_N = \left\lfloor \frac{\chi_N^{(H)}}{\log x} \right\rfloor$. For any $\omega$ in the set of full measure where
$\phi_\omega(N) \leq \chi_N^{(H)}$ eventually, put $R_N = r_1 \cdots r_N$ and $\varrho_N = \tau r_1 \cdots r_{N+J_N}$. For large enough $N$, $\varrho_N \leq R_N^{1+\Phi(R_N)}$. Take any Moran interval $I_N$ of step $N$ and consider the descendant at step $N + J_N + 1$ where the minimal probability was chosen each time. Let $x_N \in E(\omega)$ belong to that descendant. The stronger separation condition ensures that for each $x_N \in E(\omega)$, $\frac{\mu(B(x_N, R_N))}{\mu(B(x_N, \varrho_N))} \geq \frac{\mu(I_N(x_N))}{\mu(I_{N+1+J_N}(x_N))} = (m_{N+1} \cdots m_{N+J_N+1})^{-1}$.

For $\gamma \in \mathbb{R}^+$, let

$$F_N = \{ \omega : (m_{N+1} \cdots m_{N+J_N+1})^{-1} \geq (r_{N+1} \cdots r_{N+J_N})^{-\gamma} \}$$

$$= \left\{ m_{N+J_N+1}^{-1} \geq \prod_{i=1}^{J_N} (m_{N+i} r_{N+i}^{-\gamma}) \right\}.$$ 

Since $m_1 r_1^{-\gamma}$ is real valued, there must be some $K$ such that $0 < \mathbb{P}(m_1 r_1^{-\gamma} \leq K) =: \delta$.

Now,

$$\mathbb{P}(F_N) \geq \mathbb{P} \left( m_{N+J_N+1}^{-1} \geq K^{J_N} \text{ and } \prod_{i=1}^{J_N} (m_{N+i} r_{N+i}^{-\gamma}) \leq K^{J_N} \right)$$

$$\geq \mathbb{P} \left( m_{N+J_N+1}^{-1} \geq K^{J_N} \text{ and } (m_{N+i} r_{N+i}^{-\gamma}) \leq K \text{ each } i = 1, ..., J_N \right),$$

so by independence and the hypothesis on the distribution of $m_1$, we have

$$\mathbb{P}(F_N) \geq \mathbb{P}(m_{N+J_N+1}^{-1} \geq K^{J_N}) \prod_{i=1}^{J_N} \mathbb{P}(m_{N+i} r_{N+i}^{-\gamma} \leq K) \geq c(K^{-\theta} \delta)^{J_N}.$$ 

Arguing with the Borel Cantelli lemma, as in the proof of Theorem 4, we deduce that for each $\gamma$ and a.a. $\omega$, $\dim_{\phi} \mu \geq \gamma$ and hence $\dim_{\phi} \mu = \infty$ a.s.

The arguments to see $\dim_{\phi} \mu = 0$ are similar, but this time begin with $R_N = \tau r_1 \cdots r_{N-1}$ and $\varrho_N = r_1 \cdots r_{N+L+J_N+L}$, which is dominated by $R_N^{1+\Phi(R_N)}$ eventually. Take any interval $I_N$ of level $N$, consider the descendant at level $N + J_N$ where the maximal probability was chosen each time and let $x_N \in E(\omega)$ belong to that interval. The stronger separation condition ensures that in this case,

$$\frac{\mu(B(x_N, R_N))}{\mu(B(x_N, \varrho_N))} \leq \frac{\mu(I_N(x_N))}{\mu(I_{N+L+J_N+L}(x_N))} = (M_{N+1} \cdots M_{N+L+J_N+L})^{-1}.$$ 

For $\varepsilon > 0$, choose $K > 0$ such that $0 < \mathbb{P}(M_1 r_1^{-\varepsilon} > 1/K) =: \delta$ and set

$$G_N = \{ \omega : (M_{N+1} \cdots M_{N+L+J_N+L})^{-1} \leq (r_N \cdots r_{N+L+J_N+L})^{-\varepsilon} \}$$

$$= \left\{ r_N^{-\varepsilon} \leq \prod_{i=1}^{L+J_N+L} (M_{N+i} r_{N+i}^{-\varepsilon}) \right\}.$$ 

Then

$$\mathbb{P}(G_N) \geq \mathbb{P}(r_N^{-\varepsilon} \leq K^{-(L+J_N+L)}) \delta^{L+J_N+L} \geq c(K^{-\theta/\varepsilon} \delta)^{L+J_N+L}.$$ 

Again, we apply the Borel Cantelli lemma and deduce that for any $\varepsilon > 0$ and a.a. $\omega$, $\dim_{\phi} \mu \leq \varepsilon$. \square
Remark 7. Notice that the proof shows that if multiple children of a given parent are assigned the minimum (or maximum) probability, it is enough that the gaps adjacent to one of these children has the suitable size.

Example 7. An example with $\overline{\dim} \mu > \alpha$: Take $T = 2$ and choose $p_j$ independently and uniformly distributed over $(0, 1)$. Set $r_j = m_j/2$ and construct the associated random Cantor set so that the $2^n$ Moran intervals at step $n$ have length $r_1 \cdots r_n$. As $m_j \leq 1/2$, all the gaps at step $n$ have length at least $r_1 \cdots r_{n-1}$. Since $\mathbb{P}(m_1 \geq z) = 1 - 2z$, an appeal to Proposition 3 shows that $\overline{\dim} \mu = \infty$ a.s. But, $\log m_1/\log r_1 = \log m_1/(\log m_1 - \log 2)$ and $\inf \log m_1 = -\infty$ a.s., thus $\alpha = 1$.

More generally, it can be seen in the proof of Proposition 3 that if there are $T$ children at each step and the probabilities are chosen uniformly distributed over $S_T$, then $\mathbb{P}(m_1 \geq z) = (1 - T z)^{T-1}$. Thus $\mathbb{P}(m_1 \leq z) = 1 - (1 - T z)^{T-1} \geq cz$ for suitable $c > 0$.

Example 8. An example with $\overline{\dim} \mu > \alpha$ and $\underline{\dim} \mu < \beta$: Consider the special case where we take $r_n(\omega) = 1/(4t)$ and $p_n(\omega) = (1/t, \ldots, 1/t)$ for all $n$, whenever $\omega \in \Omega_t$. Formally, we can do this by (for example) defining the discrete probability measure $\pi$ on $\Omega_0$ by $\pi = c \sum_{t=2}^{\infty} t^{-2} \delta_{x_t}$, where $x_t = (1/4t, (1/t, \ldots, 1/t)) \in \Omega_t$ and $c = (\sum_{t=2}^{\infty} t^{-2})^{-1}$. Thus $\mathbb{P}(\Omega_t) = ct^2$ and $\mathbb{E}(e^{-\log r_1}) < \infty$.

Define the Moran set by beginning with $I_0 = [0, 1]$ and applying the rule that at step $n$, the $T_n$ children of each parent interval are placed starting at the left endpoint of the parent, with gaps between them of length $r_1 \cdots r_n$, except for the final child of each parent, which will be placed at the right end of the parent. This construction ensures that the stronger separation condition of Proposition 3 as noted in Remark 7 is satisfied with the right-most child. Obviously,

$\mathbb{P}(m_1 \leq 1/t) = \mathbb{P}(r_1 \leq 1/(4t)) = \mathbb{P}(M_1 \leq 1/t) = \mathbb{P}(\omega : T(\omega) \geq t) \geq ct^{-2},$

so Proposition 3 gives $\overline{\dim} \mu = \infty$ and $\underline{\dim} \mu = 0$ almost surely. But $\alpha = \beta = 1$. Of course, $M_n = m_n = 4r_n = 1/t$ on $\Omega_t$ and $\mathbb{P}(\Omega_t) > 0$ for all $t = 2, 3, \ldots$, so the essential infimum of each of $m_1, M_1$ and $r_1$ equals 0.

For the special cases of the Assouad dimensions more can be said.

Corollary 4. Assume we begin with $I_0 = [0, 1]$ and the stronger separation condition as in Proposition 3

(i) If either $\essinf m_1 = 0$ or $\essinf r_1 = 0$, then $\underline{\dim} \mu_\omega = \infty$ a.s.

(ii) If either $\essinf M_1 = 0$ or $\essinf r_1 = 0$, then $\overline{\dim} \mu_\omega = 0$ a.s.

Proof. (i) Take $R_N = r_1 \cdots r_N$ and $\varrho_N = \tau r_1 \cdots r_N$. As $\tau < 1$, $\varrho_N < R_N$ and with a suitable choice of $x_N$ as in Proposition 3

$\frac{\mu_\omega(B(x_N, R_N))}{\mu_\omega(B(x_N, \varrho_N))} \geq m_{N+1}^{-1}(\omega),$

while $R_N/\varrho_N = \tau^{-1}$. If $\essinf m_1 = 0$, then it follows from the Borel Cantelli lemma that for any $\varepsilon > 0$, $\mathbb{P}(m_n \leq \varepsilon \ i.o.) = 1$. Thus for a.e. $\omega$ and any $\gamma \in \mathbb{R}^+$, we have $m_{N+1}^{-1} \geq (R_N/\varrho_N)\gamma$ for large $N$. That implies $\underline{\dim} \mu_\omega = \infty$ a.s.

If $\essinf r_1 = 0$, but $\essinf m_1 \neq 0$, then $\alpha = \infty$ and hence Theorem 3 (with $\Phi = 0$) implies $\overline{\dim} \mu_\omega = \infty$ a.s.

(ii) Since $M_n \geq 1/T_n \geq r_n$, if $\essinf M_1 = 0$, then also $\essinf r_1 = 0$, thus we may assume the latter. The argument is similar to (i). Take $\omega$ from
\{ \inf r_1 = 0 \}. For infinitely many \( N \), \( r_N(\omega) < \tau \) and for such \( N \), put \( R_N = \tau r_1 \cdots r_{N-1} \) and \( q_N = r_1 \cdots r_N < R_N \). For suitable \( x_N \), the gap assumption gives
\[ B(x_N, R_N) \cap E(\omega) = B(x_N, q_N) \cap E(\omega). \]
Thus \( \mu(B(x_N, R_N))/\mu(B(x_N, q_N)) = 1 \), while \( R_N/q_N = \tau r_N^{-1} \to \infty \). It follows that \( \dim_L \mu_\omega = 0 \) a.s.

6. Appendix: Proof of Proposition 2

PROOF. [of Proposition 2] As \( \sum_{j=1}^{T} p_j^{(j)} = 1 \), we have \( \min p_1^{(k)} \leq 1/T \), so for every \( z \in (0, 1/T) \),
\[
\{(p_1^{(1)}, \ldots, p_1^{(T)}) : \min p_1^{(k)} \geq z \} = (z, \ldots, z) + (1 - Tz)S_T.
\]
Thus \( \mathbb{P}(\min p_1^{(k)} \geq z) = (1 - Tz)^{-1} \) and the probability density function for \( m_1 = \min p_1^{(k)} \) is the function \( f(x) = T(T-1)(1-Tx)^{T-2} \) for \( 0 \leq x \leq 1/T \).

Using the binomial theorem, it follows that
\[
E(Y_1) = E(-\log m_1) = \int_0^{1/T} T(T-1)(1-Tx)^{T-2}(-\log x)dx
\]
\[
= -T(T-1) \sum_{k=0}^{T-2} (-1)^k T^k \binom{T-2}{k} \int_0^{1/T} x^k \log x
\]
\[
= -T(T-1) \sum_{k=0}^{T-2} (-1)^k T^k \binom{T-2}{k} \frac{T^{-k-1}}{k+1} \left(-\log T - \frac{1}{k+1}\right).
\]
Now we apply special cases of Melzak’s formula (c.f. [13] vol. 5: 1.3, 1.56]):
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+1} = \frac{1}{n+1}
\]
and
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(k+1)^2} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{k+1}.
\]
This gives
\[
E(Y_1) = \log T + (T-1) \sum_{k=0}^{T-2} \binom{T-2}{k} \frac{(-1)^k}{(k+1)^2} = \log T + \sum_{k=1}^{T-1} \frac{1}{k^2}.
\]
Similarly, for \(|\lambda| < 1\),
\[
E(e^{\lambda Y_1}) = T(T-1) \int_0^{1/T} x^{-\lambda}(1-Tx)^{T-2}dx
\]
\[
= T(T-1) \sum_{k=0}^{T-2} (-1)^k T^k \binom{T-2}{k} \int_0^{1/T} x^{k-\lambda}dx
\]
\[
= (T-1)^T \sum_{k=0}^{T-2} \frac{(-1)^k}{k-\lambda+1} = T^T \prod_{k=1}^{T-1} \frac{k}{k-\lambda}.
\]
Here the last equality is another consequence of Melzak’s formula ([13] vol. 5: 1.3]):
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+1 - \lambda} = \frac{n!}{(1-\lambda)(n+\lambda-1) \cdots (2-\lambda)} \text{ for } \lambda \neq 1, 2, \ldots, n+1.
\]
Thus, $E(e^{\lambda Y_1})$ is finite if $|\lambda| < 1$.

We have $E(e^{\lambda X_1}) < \infty$ since $\max p_1^{(k)} \geq 1/T$. Lastly, we compute $E(X_1)$. It was proven in [18] that

$$P(M_1 \leq z) = \mathbb{P}\left(\max_{k=1}^T p_1^{(k)} \leq z\right) = 1 + \sum_{k=1}^T (-1)^k \binom{T}{k} (1 - k z)^{T-1} := F(z)$$

where $x_+ = \max\{x, 0\}$. Note that $(1 - k z)_+ = 0$ if $z \geq 1/k$.

Since $M_1 \geq 1/T$ with equality only on a set of measure zero, integration by parts and the binomial theorem give

$$E(X_1) = \mathbb{E}(\log M_1) = \int_{1/T}^1 -F'(x) \log(x) \,dx = \int_{1/T}^1 \frac{F(x)}{x} \,dx$$

$$= \int_{1/T}^1 \frac{dx}{x} + \sum_{k=1}^T (-1)^k \binom{T}{k} \int_{1/T}^1 \frac{(1 - k x)^{T-1}}{x} \,dx$$

$$= \log T + \sum_{k=1}^T (-1)^k \binom{T}{k} \sum_{j=0}^{T-1} (-1)^j \binom{T-1}{j} \int_{1/T}^1 x^{j-1} \,dx.$$

Evaluating the integrals gives

$$E(X_1) = \log T + \sum_{k=1}^T (-1)^k \binom{T}{k} (\log T - k)$$

$$+ \sum_{k=1}^T (-1)^k \binom{T}{k} \sum_{j=1}^{T-1} \binom{T-1}{j} (-1)^j \frac{(k^{-j} - T^{-j})}{j}$$

$$= \log T + \sum_{k=1}^T (-1)^k \binom{T}{k} (\log T - k) + A + B$$

where

$$A = \left( \sum_{k=1}^T (-1)^k \binom{T}{k} \right) \left( \sum_{j=1}^{T-1} \binom{T-1}{j} \frac{(-1)^j}{j} \right)$$

and

$$B = -\sum_{k=1}^T (-1)^k \binom{T}{k} \sum_{j=1}^{T-1} \binom{T-1}{j} (-1)^j \frac{k^T}{j}.$$

Another application of the binomial theorem shows that

$$1 + \sum_{k=1}^T (-1)^k \binom{T}{k} = 0.$$

Together with the combinatorial identity (c.f. [13], vol. 5: 1.4)]

$$\sum_{j=1}^{T-1} \binom{T-1}{j} \frac{(-1)^j}{j} = -\sum_{n=1}^{T-1} \frac{1}{n},$$
this proves \( A = \sum_{n=1}^{T-1} 1/n \). Euler’s finite difference formula (c.f. \([13]\) vol. 4: 10.1) implies
\[
\sum_{k=1}^{T} (-1)^k \binom{T}{k} k^j = 0 \text{ for } T \geq 2, 1 \leq j \leq T - 1.
\]
Changing the order of the summation and applying this formula shows that
\[
B = \sum_{j=1}^{T-1} \binom{T-j}{T-j} \left( \sum_{k=1}^{T} (-1)^k \binom{T}{k} k^j \right) = 0.
\]
Using (6.1), we conclude that
\[
E(X_1) = \sum_{k=1}^{T} (-1)^{k+1} \binom{T}{k} \log k + \sum_{k=1}^{T-1} \frac{1}{k}.
\]

\[\square\]

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