Homology Vanishing Theorems for Pinched Submanifolds

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Abstract

We investigate the geometry and topology of submanifolds under a sharp pinching condition involving extrinsic invariants like the mean curvature and the length of the second fundamental form. Homology vanishing results are given that strengthen and sharpen previous ones. In addition, an integral bound is provided for the Bochner operator of compact Euclidean submanifolds in terms of the Betti numbers.

Keywords

Bochner operator · Betti numbers · Homology groups · Pinching · Mean curvature · Length of the second fundamental form

Mathematics Subject Classification 53C40 · 53C42

1 Introduction

A fundamental problem in differential geometry is to investigate the relationship between geometry and topology of Riemannian manifolds. The same question can be raised from the point of view of submanifold geometry. Indeed, it has been an active field of research to study the effect of pinching conditions, involving intrinsic and extrinsic curvature invariants, on the geometry or the topology of submanifolds. An important result in this direction was given by Simons [33] for minimal submanifolds of spheres. Since then, plenty of geometric and topological rigidity results have been obtained, under various pinching conditions, see, for instance, [1, 2, 7, 15, 16, 18, 20, 29, 30, 35, 36, 41]. In particular, Lawson and Simons [24] showed that certain bounds on the second fundamental form for submanifolds of spheres force homology groups to vanish. In their approach, the second variation of area was exploited to rule

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out stable minimal currents in certain dimensions, and since one can minimize area in
a homology class, this trivializes integral homology.

The aim of the present paper is to study the geometry and topology of submanifolds
under a sharp pinching condition involving the mean curvature and the length of
the second fundamental form. Throughout the paper, $S$ denotes the squared length
of the second fundamental form $\alpha_f$ of an isometric immersion $f$, while the mean
curvature is defined as the length $H = \|\mathcal{H}\|$ of the mean curvature vector field given
by $\mathcal{H} = (\text{tr} \alpha_f)/n$, where tr means taking the trace. The choice of the pinching
condition is inspired by the standard immersion of a torus $T^p_n(r) = \mathbb{S}^p(r) \times \mathbb{S}^{n-p}(\sqrt{1-r^2})$,
into the unit sphere $\mathbb{S}^{n+1}$, where $\mathbb{S}^p(r)$ denotes the $p$-dimensional sphere of radius
$r < 1$. The principal curvatures are $\sqrt{1-r^2}/r$ and $-r/\sqrt{1-r^2}$ of multiplicity $p$ and
$n-p$, respectively. A direct computation gives that $S = a(n, p, H, t) = nc + n^3 t^2 - n|n-2p|t\sqrt{n^2 t^2 + 4p(n-p)c}$, $t, c \in \mathbb{R}$.

This example served as the motivation to study isometric immersions $f : M^n \rightarrow \tilde{M}^m$ between Riemannian manifolds that satisfy the pinching condition

$$S \leq a(n, p, H, c),$$

pointwise, where $p$ is an integer with $1 \leq p \leq n/2$, under the mild assumption that the
curvature operator of the ambient manifold $\tilde{M}^m$ is bounded from below by a constant
$c$. In particular, this includes the case of submanifolds of spaces of constant sectional
curvature. Submanifolds that satisfy condition (1) have already been investigated by
several authors under geometric assumptions (see for instance [1, 6, 27, 29, 30, 39,
40, 43, 44]). In addition, the topology of submanifolds that satisfy strict inequality in
(1) was studied in [18, 25, 32, 37, 42] without any geometric assumptions.

In the present paper, we investigate the geometry and topology of submanifolds that
satisfy the pinching condition (1). We show that this pinching condition either forces
homology to vanish in a range of intermediate dimensions, or completely determines
the pinched submanifold. Our results are sharp and strengthen previous ones (see
[2, 9, 15, 17, 25, 27, 29, 30, 32, 37]). A key ingredient in our approach is the Bochner
technique that is based on the Bochner–Weitzenböck formula. This states that the
Laplacian of every $p$-form $\omega \in \Omega^p(M^n)$ on a manifold $M^n$ is given by

$$\Delta \omega = \nabla^* \nabla \omega + \mathcal{B}^{[p]} \omega,$$

where $\nabla^* \nabla$ is the rough Laplacian and

$$\mathcal{B}^{[p]} : \Omega^p(M^n) \rightarrow \Omega^p(M^n),$$
is a certain symmetric endomorphism of the bundle of $p$-forms, called the Bochner operator. The Bochner–Weitzenböck formula implies that every harmonic $p$-form vanishes on compact manifolds, provided that $\mathcal{B}[p]$ is positive definite. Since every de Rham cohomology class is represented by a harmonic form, it follows that the $p$-th de Rham cohomology group $H^p(M^n; \mathbb{R})$ vanishes.

The difficulty when dealing with the Bochner operator $\mathcal{B}[p]$ arises from the fact that it is unmanageable for $p > 1$. Our approach is inspired by the interesting paper due to Savo [29]. The key point is to establish a sharp inequality for the Bochner operator in terms of the second fundamental form of any immersed submanifold in a Riemannian manifold whose curvature operator is bounded from below by a constant. Moreover, we determine the structure of the second fundamental form at points where equality holds. All the above mentioned, combined with Bochner technique and Morse theory, enable us to provide homology vanishing results, bounds for the Betti numbers, or determine the submanifolds that satisfy the pinching condition (1) regardless codimension. Examples show that the pinching condition is sharp.

2 The Results

This section is devoted to the statements of our results. Throughout the paper, all submanifolds under consideration are assumed to be oriented. Our first global result is stated as follows.

**Theorem 1** Let $M^n, n \geq 3$, be a compact $n$-dimensional Riemannian manifold isometrically immersed in a Riemannian manifold whose curvature operator is bounded from below by a constant $c$. Suppose that inequality (1) is satisfied for an integer $1 \leq p \leq n/2$ and $H^2 + c \geq 0$. The following assertions hold:

(i) The Betti numbers of $M^n$ satisfy $b_i(M^n) \leq \binom{n}{i}$ for each $p \leq i \leq n - p$.

(ii) If strict inequality holds in (1) at a point, then $b_i(M^n) = 0$ for all $p \leq i \leq n - p$. If in addition $p = 1$, then $M^n$ is a real homology sphere, i.e., $H^i(M^n; \mathbb{R}) = 0$ for each $1 \leq i \leq n - 1$.

(iii) If $b_p(M^n) > 0$, then equality holds in (1) everywhere on $M^n$ and every harmonic $p$-form is parallel. In particular, $M^n$ supports a nontrivial parallel $p$-form.

The above extends previous results due to Cui and Sun [9] and Savo [29, Th. 8].

Our second main result is about (not necessarily compact) submanifolds carrying a nontrivial parallel $p$-form and can be formulated as follows (see [17] for related results).

**Theorem 2** Let $M^n, n \geq 3$, be a Riemannian manifold isometrically immersed in a Riemannian manifold whose curvature operator is bounded from below by a constant $c$ such that $H^2 + c \geq 0$. If $M^n$ supports a nontrivial harmonic $p$-form of constant length (in particular, a parallel $p$-form), then $S \geq a(n, p, H, c)$.

Since Kähler manifolds support nontrivial parallel forms in all even degrees, we obtain the following:
Corollary 3 If a Kähler manifold $M^{2m}$ is isometrically immersed in a Riemannian manifold whose curvature operator is bounded from below by a constant $c$ such that $H^2 + c \geq 0$, then $S \geq a(2m, 2[m/2], H, c)$.

In the above results, the assumption that the curvature operator of the ambient manifold is bounded from below can be weakened by merely assuming that the induced Bochner operator is bounded from below.

Now we focus on submanifolds that satisfy the pinching condition (1) in complete simply connected space forms $Q^n_m$ of constant sectional curvature $c$, that is, the Euclidean space $\mathbb{R}^m$, the sphere $S^m$ or the hyperbolic space $\mathbb{H}^m$, according to whether $c = 0$, $c = 1$, or $c = -1$, respectively. The next result deals with minimal submanifolds in spheres and extends the results in [2, 25].

Theorem 4 Let $f : M^n \to S^m$, $n \geq 3$, be a minimal isometric immersion of a compact Riemannian manifold $M^n$. If $S \leq n$ on $M^n$, then one of the following assertions holds:

(i) $M^n$ is a real homology sphere, with finite fundamental group, that admits a Riemannian metric of positive Ricci curvature.
(ii) $f(M^n)$ is isometric to the Clifford torus $T_p^n(\sqrt{p/n})$ for an integer $1 \leq p \leq n/2$.
(iii) The immersion $f$ is the standard embedding $\psi : \mathbb{C}P^2_{4/3} \to S^7$ of the complex projective plane of constant holomorphic curvature $4/3$.

In particular, for minimal submanifolds of dimension three and four, we have the following result that improves the bound in [30, Th. 3]:

Corollary 5 Let $f : M^n \to S^m$ be a minimal isometric immersion of a compact Riemannian manifold $M^n$ with $S \leq n$ on $M^n$.

(i) If $n = 3$, then either $f(M^3)$ is isometric to the Clifford torus $T_1^3(\sqrt{1/3})$, or $M^3$ is diffeomorphic to a spherical space form.
(ii) If $n = 4$, then $f(M^4)$ is isometric to the Clifford torus $T_p^4(\sqrt{p/n})$, $p = 1, 2$, the immersion $f$ is the standard embedding $\psi : \mathbb{C}P^2_{4/3} \to S^7$, or the universal cover of $M^4$ is homeomorphic to $S^4$.

We recall that the Cartan minimal hypersurface in $S^4$ satisfies $S = 6$, while the minimal submanifold in $S^5$, constructed in [11] as a circle bundle over a flat minimal torus, has $S = 8$. In addition, the minimal immersion of the $n$-dimensional complex projective $\mathbb{C}P^2_{2n/(n+1)}$ space of constant holomorphic curvature $2n/(n+1)$, $n > 2$, into $S^{n(n+2)}$ (see [38]) satisfies $S = 2n(n-1)$. These examples show that Theorem 4 and Corollary 5 are optimal.

Our results for not necessarily minimal submanifolds of spheres can be stated as follows:

Theorem 6 Let $f : M^n \to S^m$, $n \geq 3$, be an isometric immersion of a compact Riemannian manifold $M^n$. Suppose that inequality (1) is satisfied for an integer $1 \leq p \leq n/2$. Then either $b_i(M^n) = 0$ for every $p \leq i \leq n-p$, or the following assertions hold:
(i) If $b_p(M^n) > 0$, then $M^n$ has the homotopy type of a CW-complex with cells only in dimensions $0, p, n - p$ or $n$. In particular, the homology groups of $M^n$ satisfy $H_i(M^n; G) = 0$ for each $i \neq 0, p, n - p, n$, where $G$ is any coefficient group.

(ii) If $b_q(M^n) > 0$ for an integer $p < q \leq n/2$, then $f$ is minimal and either $f(M^n)$ is isometric to the Clifford torus $\mathbb{T}_q^n(\sqrt{q/n})$, or $f$ is the standard embedding as in Theorem 4(iii).

If in addition $H > 0$ on $M^n$, then the following facts hold:

(iii) If $b_p(M^n) > 0$ and $p < n/2$, then $f(M^n)$ is isometric to a torus $\mathbb{T}_p^n(r)$ with $r > \sqrt{p/n}$.

(iv) If $p < (n - 1)/2$, then either $H_i(M^n; \mathbb{Z}) = 0$ for all $p \leq i < n - p$, or $f(M^n)$ is as in (iii) above.

The above result extends the sphere theorem proved in [32] and [29, Th. 9]. The standard embedding $\psi : \mathbb{CP}^2_{4/3} \to \mathbb{S}^7$ justifies the necessity of the assumptions in parts (iii) and (iv).

**Theorem 7** Let $f : M^n \to \mathbb{S}^m, n \geq 3$, be an isometric immersion of a compact Riemannian manifold $M^n$. If inequality (1) holds for $p = 1$, then one of the following assertions holds:

(i) $M^n$ is a real homology sphere, with finite fundamental group, that admits a Riemannian metric of positive Ricci curvature. Furthermore, $M^n$ is diffeomorphic to a spherical space form if $n = 3$ and its universal cover is homeomorphic to $\mathbb{S}^4$ if $n = 4$.

(ii) $M^n$ has the homotopy type of a CW-complex with cells only in dimensions $0, 1, n - 1, n$.

(iii) $f(M^n)$ is isometric to the Clifford torus $\mathbb{T}_q^n(\sqrt{q/n})$ for an integer $1 < q \leq n/2$, or $f$ is the standard embedding as in Theorem 4(iii).

If in addition $H > 0$ on $M^n$, then either $M^n$ is homeomorphic to $\mathbb{S}^n$, or $f(M^n)$ is isometric to a torus $\mathbb{T}_q^n(r)$ with $r > 1/\sqrt{n}$.

The pinching condition (1) is sharp in both Theorems 6 and 7. In fact, let $g$ be the Cartan minimal hypersurface in $\mathbb{S}^4$, the minimal submanifold in $\mathbb{S}^5$ constructed in [11], or the minimal immersion of the $n$-dimensional complex projective $\mathbb{CP}^2_{2n/(n+1)}$ space of constant holomorphic curvature $2n/(n+1), n > 2$, into $\mathbb{S}^{2n(n+2)}$. For each $0 < \rho < 1$, we consider the isometric immersion $f_\rho = j \circ G_\rho \circ g$, where $G_\rho$ is the homothety of center at the origin and ratio $\rho$, and $j$ is the standard inclusion of the sphere of radius $\rho$ as a hypersurface in the unit sphere. It is easy to see that for $\rho$ sufficiently close to 1 the nonminimal immersion $f_\rho$ satisfies the strict reverse inequality in (1).

The following corollary improves a result due to Lawson and Simons [24, Cor. 2].

**Corollary 8** Let $f : M^n \to \mathbb{S}^m, n \geq 3$, be an isometric immersion of a compact Riemannian manifold $M^n$. If $S \leq 2\sqrt{n - 1}$ on $M^n$, then either $f(M^n)$ is isometric to a torus $\mathbb{T}_q^n(r)$ with appropriate $r > 1/\sqrt{n}$, or $M^n$ is a real homology sphere of positive Ricci curvature. Furthermore, $M^n$ is diffeomorphic to a spherical space form if $n = 3$ and its universal cover is homeomorphic to $\mathbb{S}^4$ if $n = 4$. 
For Euclidean and spherical submanifolds that are contained in an open hemisphere, we prove the following result:

**Theorem 9** Let \( f : M^n \rightarrow Q^m_c, n \geq 3 \), be an isometric immersion of a compact Riemannian manifold \( M^n \) and let \( c = 0, 1 \). If \( c = 1 \), suppose further that \( f(M^n) \) is contained in an open hemisphere. If inequality (1) holds for an integer \( 1 \leq p \leq n/2 \), then the following assertions hold:

(i) Either \( b_i(M^n) = 0 \) for each \( p \leq i \leq n - p \), or \( p = n/2 \) and \( M^n \) has the homotopy type of a CW-complex with cells only in dimensions 0, \( n/2 \) or \( n \).

(ii) If \( p = 1 \), then \( M^n \) is a real homology sphere, with finite fundamental group, that admits a Riemannian metric of positive Ricci curvature. Furthermore, \( M^n \) is diffeomorphic to a spherical space form if \( n = 3 \) and its universal cover is homeomorphic to \( S^4 \) if \( n = 4 \).

(iii) If \( H > 0 \) on \( M^n \) and \( p < (n - 1)/2 \) when \( c = 0 \), then \( H_i(M^n; \mathbb{Z}) = 0 \) for each \( p \leq i < n - p \).

The assumption in the above result that the submanifold is contained in an open hemisphere is necessary, as shown by the Clifford tori in Theorem 4(ii). It is worth noticing that the even dimensional torus \( \mathbb{T}^n_{n/2}(1/\sqrt{2}) \), viewed as a submanifold of the Euclidean space, satisfies equality in the pinching condition (1). The above result extends Theorem 1.6 in [9] and Theorem 1.2 in [27]. A stronger result was proved in [18] for the case \( p = 1 \) under the additional assumption that the mean curvature is positive everywhere.

The following improves the bound given in [15] for submanifolds of a hyperbolic space.

**Theorem 10** Let \( M^n, n \geq 3 \), be a compact Riemannian manifold isometrically immersed in a hyperbolic space \( H^m \) with mean curvature \( H \geq 1 \). If inequality (1) holds for an integer \( 1 \leq p < n/2 \), then \( b_i(M^n) = 0 \) for every \( p \leq i \leq n - p \).

Next, we provide an integral bound for the Bochner operator in terms of the mean curvature, the length of the traceless part \( \Phi \) of the second fundamental form and the Betti numbers \( b_i(M^n; \mathbb{F}) \) over an arbitrary coefficient field \( \mathbb{F} \).

**Theorem 11** Given integers \( n \geq 4, 1 \leq p < n/2 \) and \( k \geq 1 \), there exists a positive constant \( c(n, p, k) \) such that if \( M^n \) is a compact Riemannian manifold isometrically immersed in \( \mathbb{R}^{n+k} \), then the lowest eigenvalue \( \varrho_p \) of its Bochner operator \( \mathcal{B}^{[p]} \) satisfies

\[
\int_{M^n} |\varrho_p - F_{n,p}(H, \|\Phi\|)|^{n/2} \, dM \geq c(n, p, k) \sum_{p < i < n - p} b_i(M^n; \mathbb{F}),
\]

for any coefficient field \( \mathbb{F} \), where \( F_{n,p} \) is the function given by

\[
F_{n,p}(x, y) = \frac{p(n-p)}{n} \left( nx^2 - \frac{n(n-2p)}{\sqrt{n(p-n)}} xy - y^2 \right).
\]
Moreover, if
\[
\int_{M^n} \left| \varrho_p - F_{n,p}(H, \|\Phi\|) \right|^{n/2} dM < c(n, p, k),
\]
then $M^n$ has the homotopy type of a CW-complex with no cells of dimension $p < i < n - p$. In particular, if $p = 1$, then the fundamental group $\pi_1(M^n)$ is a free group on $b_1(M^n)$ generators, and $M^n$ is homeomorphic to $\mathbb{S}^n$ if $\pi_1(M^n)$ is finite.

### 3 Algebraic Preliminaries and Auxiliary Results

This section is devoted to some algebraic preliminaries inspired by the ideas developed by Savo [29]. Let $V$ be a real $n$-dimensional vector space, $n \geq 3$, equipped with a positive definite inner product denoted by $\langle \cdot, \cdot \rangle$. We denote by $\text{End}(V)$ the set of all self-adjoint endomorphisms of $V$.

For every integer $1 \leq p \leq n$, we consider the set $I_p$ of $p$-multi-indices
\[
I_p = \{ \{i_1, \ldots, i_p\} : 1 \leq i_1 < \cdots < i_p \leq n \}.
\]
Let $A \in \text{End}(V)$ be an arbitrary endomorphism of $V$ with eigenvalues $k_1, \ldots, k_n$. For each $a = \{i_1, \ldots, i_p\} \in I_p$, the associated $p$-algebraic curvature $K_a$ of $A$ is the number given by
\[
K_a = k_{i_1} + \cdots + k_{i_p}.
\]

Let $\Lambda^p V^*$ be the $\binom{n}{p}$-dimensional real vector space, defined as the $p$-th exterior power of the dual vector space $V^* = \text{Hom}(V, \mathbb{R})$ of $V$. To each $A \in \text{End}(V)$, we associate the endomorphism $T_A^{[p]} \in \text{End}(\Lambda^p V^*)$ defined by
\[
T_A^{[p]} = (\text{tr} A) A^{[p]} - A^{[p]} \circ A^{[p]},
\]
where $A^{[p]} \in \text{End}(\Lambda^p V^*)$ is given by
\[
A^{[p]} \omega(v_1, \ldots, v_p) = \sum_{i=1}^{p} \omega(v_1, \ldots, Av_i, \ldots, v_p),
\]
with $\omega \in \Lambda^p V^*$ and $v_1, \ldots, v_p \in V$. The endomorphism $T_A^{[p]}$ is self-adjoint with respect to the natural inner product $\langle \cdot, \cdot \rangle$ in $\Lambda^p V^*$. It follows from [29, Lem. 2] that the $\binom{n}{p}$ eigenvalues $\lambda_a(T_A^{[p]})$ of $T_A^{[p]}$ are given by
\[
\lambda_a(T_A^{[p]}) = K_a K_{\star a}, \quad a = \{i_1, \ldots, i_p\} \in I_p,
\]
where $\star a \in I_p$ is defined by $\star a = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\}$.
The lowest eigenvalue of \( T_A[p] \) is given by

\[
\min_{a \in I_p} \lambda_a(T_A[p]) = \min_{\omega \in \Lambda^p V^*, \|\omega\|=1} \langle T_A[p] \omega, \omega \rangle.
\]

We denote by \( \hat{A} = A - H_A \text{id}_V \) the traceless part of every \( A \in \text{End}(V) \), where \( H_A = (1/n) \text{tr}A \) and \( \text{id}_V : V \rightarrow V \) being the identity map on \( V \).

The inequality in the following lemma was essentially proved in [29, Lem. 14].

**Lemma 12** For every \( A \in \text{End}(V) \) with \( \text{tr}A \geq 0 \) and each integer \( 1 \leq p \leq n/2 \), the lowest eigenvalue of \( T_A[p] \in \text{End}(\Lambda^p V^*) \) satisfies

\[
\min_{a \in I_p} \lambda_a(T_A[p]) \geq -\frac{1}{4} (\text{tr}A)^2 - \frac{1}{4} \left( \frac{n-2p}{n} \text{tr}A + \sqrt{4p(n-p)/n} \|\hat{A}\| \right)^2.
\]

If equality holds, then \( A \) has at most two distinct eigenvalues \( \lambda \) and \( \mu \) with multiplicities \( p \) and \( n-p \), respectively. If in addition \( \mu = 0 \) and \( p < n/2 \), then \( A = 0 \).

**Proof** Let \( k_1, \ldots, k_n \) be the eigenvalues of \( A \). We distinguish two cases.

**Case 1.** Suppose that \( \text{tr}A = 0 \) and let \( a = \{i_1, \ldots, i_p\} \in I_p \). It follows from (3) that the eigenvalue \( \lambda_a(T_A[p]) \) of \( T_A[p] \) satisfies

\[
\lambda_a(T_A[p]) = -\left( \sum_{i \in a} k_i \right)^2 \geq -p \sum_{i \in a} k_i^2,
\]

\[
\lambda_a(T_A[p]) = -\left( \sum_{i \in a} k_i \right)^2 \geq -(n-p) \sum_{i \in a} k_i^2.
\]

Summing up, we obtain

\[
\lambda_a(T_A[p]) \geq -\frac{p(n-p)}{n} \|A\|^2
\]

which proves the desired inequality in this case. If equality holds in (4), then all the above inequalities become equalities, and this implies that \( A \) has at most two distinct eigenvalues

\[
\lambda = k_{i_1} = \cdots = k_{i_p} \quad \text{and} \quad \mu = k_{i_{p+1}} = \cdots = k_{i_n},
\]

with multiplicities \( p \) and \( n-p \), respectively, where \( \star a = \{i_{p+1}, \ldots, i_n\} \).

**Case 2.** Suppose that \( \text{tr}A > 0 \). The eigenvalues of the traceless part \( \hat{A} \) of \( A \) are \( k_1 - (\text{tr}A)/n, \ldots, k_n - (\text{tr}A)/n \). It follows from (3) that, for each \( a = \{i_1, \ldots, i_p\} \in I_p \), the eigenvalue \( \lambda_a(T_A[p]) \) of \( T_A[p] \) is given by
\[
\lambda_a(T_A^{[p]}) = (K_a - \frac{p}{n} \text{tr} A) (\frac{n-p}{n} K_{*a} - \frac{n-p}{n} \text{tr} A)
\]
\[
= \lambda_a(T_A^{[p]}) + \frac{p(n-p)}{n^2} (\text{tr} A)^2 - \frac{n-p}{n} K_a \text{tr} A - \frac{p}{n} K_{*a} \text{tr} A.
\]

Using that \( K_a + K_{*a} = \text{tr} A \), we obtain
\[
\lambda_a(T_A^{[p]}) = \lambda_a(T_A^{[p]}) - \frac{p^2}{n^2} (\text{tr} A)^2 + \frac{2p-n}{n} K_a \text{tr} A,
\]
\[
\lambda_a(T_A^{[p]}) = \lambda_a(T_A^{[p]}) - \frac{(n-p)^2}{n^2} (\text{tr} A)^2 + \frac{n-2p}{n} (K_{*a} - K_a) \text{tr} A.
\]

Summing up, we have
\[
2\lambda_a(T_A^{[p]}) = 2\lambda_a(T_A^{[p]}) - \frac{(n-p)^2 + p^2}{n^2} (\text{tr} A)^2 + \frac{n-2p}{n} (K_{*a} - K_a) \text{tr} A.
\]

Since \( \hat{A} \) is trace free, inequality (5) applies and yields
\[
\lambda_a(T_A^{[p]}) \geq -\frac{p(n-p)}{n} \| \hat{A} \|^2.
\]

Combining the above, we obtain
\[
2\lambda_a(T_A^{[p]}) - \frac{(n-p)^2 + p^2}{n^2} (\text{tr} A)^2 + \frac{n-2p}{n} (\text{tr} A)^2 + \frac{2p-n}{n} (K_{*a} - K_a) \text{tr} A.
\]

Using (3), it follows that
\[
|K_{*a} - K_a| = ((\text{tr} A)^2 - 4\lambda_a(T_A^{[p]}) )^{1/2},
\]
and consequently
\[
2\lambda_a(T_A^{[p]}) - \frac{(n-p)^2 + p^2}{n^2} (\text{tr} A)^2 + \frac{2p-n}{n} (K_{*a} - K_a) \text{tr} A.
\]

Setting \( \rho = (\text{tr} A)^2 - 4\lambda_a(T_A^{[p]}) )^{1/2} \), the above is written as
\[
n^2 \rho^2 - 2n(n-2p) \text{tr} A \rho + (n-2p)^2 (\text{tr} A)^2 - 4np(n-p) \| \hat{A} \|^2 \leq 0.
\]

Thus
\[
\rho \leq \frac{1}{n} (n-2p) \text{tr} A + \frac{\sqrt{4p(n-p)/n} \| \hat{A} \|}{n},
\]
or equivalently

\[
\lambda_a(T_A^{[p]}) \geq \frac{1}{4}(\text{tr}A)^2 - \frac{1}{4}\left(\frac{n-2p}{n}\right)\text{tr}A + \sqrt{4p(n-p)/n} \|A\| \geq \frac{1}{4}\left(\frac{n-2p}{n}\right)\text{tr}A + \sqrt{4p(n-p)/n} \|A\|^2.
\]

Inequality (4) follows immediately from the above. If equality holds in (4), then all the above inequalities become equalities, and this implies that \( A \) has at most two distinct eigenvalues

\[
\lambda = k_{i_1} = \cdots = k_{i_p} \quad \text{and} \quad \mu = k_{i_{p+1}} = \cdots = k_{i_n},
\]

with multiplicities \( p \) and \( n-p \), respectively, where \( \star a = \{i_{p+1}, \ldots, i_n\} \).

Next we suppose that equality holds in (4) and

\[
\lambda_a(T_A^{[p]}) = \min_{b \in I_p} \lambda_b(T_A^{[p]}),
\]

for some \( a = \{i_1, \ldots, i_p\} \in I_p \). In both Cases 1 and 2, the endomorphism \( A \) has at most two distinct eigenvalues

\[
\lambda = k_{i_1} = \cdots = k_{i_p} \quad \text{and} \quad \mu = k_{i_{p+1}} = \cdots = k_{i_n},
\]

with multiplicities \( p \) and \( n-p \), respectively, where \( \star a = \{i_{p+1}, \ldots, i_n\} \). A direct computation shows that the eigenvalues of \( T_A^{[p]} \) are given by

\[
\lambda_b(T_A^{[p]}) = ((p - n_b)\lambda + n_b\mu)(n_b\lambda + (n-p-n_b)\mu),
\]

for each \( b = \{j_1, \ldots, j_p\} \in I_p \) with \( 1 \leq j_1 < \cdots < j_p \leq n \). Here \( 0 \leq n_b \leq p \) is the number of common elements of \( b \) and \( a = \{i_1, \ldots, i_p\} \).

In addition, we suppose that \( \mu = 0 \) and \( p < n/2 \). From the above, it follows that

\[
\min_{b \in I_p} \lambda_b(T_A^{[p]}) = 0.
\]

Computing the right-hand side in (4) in terms of the eigenvalues of \( A \), and since equality holds in (4), we obtain

\[
\min_{b \in I_p} \lambda_b(T_A^{[p]}) = \frac{2p^2}{n^2}(n-p)(2p-n)\lambda^2.
\]

It is now obvious that \( A = 0 \). \( \Box \)

Let \( W \) be a \( k \)-dimensional real vector space equipped with a positive definite inner product which, by abuse of notation, is again denoted by \( \langle \cdot, \cdot \rangle \). We denote by \( \text{Hom}(V \times V, W) \) the space of all bilinear forms with values in \( W \) and by \( \text{Sym}(V \times V, W) \) the subspace that consists of all symmetric bilinear forms. The space \( \text{Sym}(V \times V, W) \)
can be viewed as a complete metric space with respect to the usual Euclidean norm \(\| \cdot \|\). For each \(\beta \in \text{Sym}(V \times V, W)\), we define the map

\[
\beta^\sharp : W \to \text{End}(V), \quad \xi \mapsto \beta^\sharp(\xi),
\]

such that

\[
\langle \beta^\sharp(\xi)v_1, v_2 \rangle = \langle \beta(v_1, v_2), \xi \rangle \quad \text{for all } v_1, v_2 \in V.
\]

For every integer \(1 \leq p \leq n\), we define the map

\[
B[p] : \text{Sym}(V \times V, W) \to \text{End}(\Lambda^p V^*),
\]

given by

\[
B[p](\beta) = \sum_{i=1}^{k} T^{[p]}_{\beta^\sharp(\xi_i)},
\]

where \(\xi_1, \ldots, \xi_k\) is an arbitrary orthonormal basis of \(W\). Observe that \(B[p](\beta)\) is a self-adjoint endomorphism of \(\Lambda^p V^*\). Its lowest eigenvalue is denoted by \(\varrho_p(\beta)\).

For convenience, we set \(H_\beta = (1/n) \text{tr}\beta\) and denote by \(\hat{\beta} = \beta - (\cdot, \cdot)H_\beta\) the traceless part of each \(\beta \in \text{Sym}(V \times V, W)\). The image of \(\beta\) is the subspace \(\text{Im} \beta \subset W\) given by

\[
\text{Im} \beta = \text{span} \{\beta(v_1, v_2) : v_1, v_2 \in V\}.
\]

The following will be an important tool in our proofs.

**Proposition 13** For every \(\beta \in \text{Sym}(V \times V, W)\) and each integer \(1 \leq p \leq n/2\), the lowest eigenvalue \(\varrho_p(\beta)\) of \(B[p](\beta) \in \text{End}(\Lambda^p V^*)\) satisfies

\[
\varrho_p(\beta) \geq \frac{p(n-p)}{n} \left(n\|H_\beta\|^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}} \|H_\beta\| \|\hat{\beta}\| - \|\hat{\beta}\|^2\right). \quad (6)
\]

Moreover, if equality holds in (6) for some \(\beta\), then the following hold:

(i) For every unit vector \(u \in W\), \(\beta^\sharp(u)\) has at most two distinct eigenvalues of multiplicities \(p\) and \(n-p\). In addition, if \(p < n/2\) and the eigenvalue with multiplicity \(n - p\) vanishes, then \(\beta^\sharp(u) = 0\).

(ii) If \(H_\beta \neq 0\) and \(p < n/2\), then \(\text{Im} \beta = \text{span} \{H_\beta\}\).

**Proof** Let \(\beta \in \text{Sym}(V \times V, W)\). If \(\dim W = 1\), then the proof follows directly from Lemma 12. We suppose that \(\dim W > 1\) and distinguish two cases.

**Case 1.** Suppose that \(H_\beta \neq 0\) and consider an orthonormal basis \(\xi_1, \ldots, \xi_k\) of \(W\) such that \(\xi_1 = H_\beta/\|H_\beta\|\). Then we have \(\text{tr}\beta^\sharp(\xi_i) = 0\) for each \(2 \leq i \leq k\). From Lemma 12, it follows that the eigenvalues of \(T^{[p]}_{\beta^\sharp(\xi_i)}\), \(1 \leq i \leq k\), satisfy
\begin{equation}
\min_{a \in I_p} \lambda_a(T^{[p]}_{\beta^\sharp(\xi_1)}) \geq \frac{n^2}{4} \|H_\beta\|^2 - \frac{1}{4} ((n-2p)\|H_\beta\| + \sqrt{4p(n-p)/n} \|\beta^\sharp(\xi_1)\|)^2
\end{equation}

(7)

and

\begin{equation}
\min_{a \in I_p} \lambda_a(T^{[p]}_{\beta^\sharp(\xi_i)}) \geq -\frac{p(n-p)}{n} \|\beta^\sharp(\xi_i)\|^2 \text{ for each } 2 \leq i \leq k.
\end{equation}

(8)

Note that

\[\|\beta^\sharp(\xi_1)\|^2 = \|\beta^\sharp(\xi_1)\|^2 - n\|H_\beta\|^2 \text{ and } \beta^\sharp(\xi_i) = \beta^\sharp(\xi_i) \text{ for all } 2 \leq i \leq k.\]

Summing up (7) and (8) and since

\[\varrho_p(\beta) = \min_{\omega \in \Lambda_p V^*} \langle B^{[p]}(\beta)\omega, \omega \rangle = \min_{a \in I_p} \lambda_a(B^{[p]}(\beta)),\]

we obtain

\[\varrho_p(\beta) \geq p(n-p)\|H_\beta\|^2 - \frac{p(n-p)}{n} (\|\beta\|^2 - n\|H_\beta\|^2)
- (n-2p)\|H_\beta\|\sqrt{p(n-p)/n}\sqrt{\|\beta^\sharp(\xi_1)\|^2 - n\|H_\beta\|^2}
\geq p(n-p)\|H_\beta\|^2 - \frac{p(n-p)}{n} (\|\beta\|^2 - n\|H_\beta\|^2)
- (n-2p)\|H_\beta\|\sqrt{p(n-p)/n}\sqrt{\|\beta\|^2 - n\|H_\beta\|^2},\]

and this is inequality (6).

Now, we suppose that equality holds in (6) for some \(\beta \in \text{Sym}(V \times V, W)\). Since all inequalities above become equalities, we obtain \(\beta^\sharp(\xi_i) = 0\) for each \(2 \leq i \leq k\) provided that \(p < n/2\). Thus \(\text{Im} \beta = \text{span} \{H_\beta\}\) if \(p < n/2\), and the rest of the proof for this case follows from Lemma 12.

Case 2. Suppose that \(\beta \in \text{Sym}(V \times V, W)\) is traceless. Let \(u \in W\) be an arbitrary unit vector. We choose an orthonormal base \(\xi_1, \ldots, \xi_k\) of \(W\) such that \(\xi_1 = u\). Since \(\text{tr}\beta^\sharp(\xi_i) = 0\) for every \(1 \leq i \leq k\), from Lemma 12, it follows that

\[\min_{a \in I_p} \lambda_a(T^{[p]}_{\beta^\sharp(\xi_i)}) \geq -\frac{p(n-p)}{n} \|\beta^\sharp(\xi_i)\|^2 \text{ for each } 1 \leq i \leq k.
\]

Summing over \(i\), we obtain

\[\varrho_p(\beta) = \min_{a \in I_p} \lambda_a(B^{[p]}(\beta)) \geq -\frac{p(n-p)}{n} \sum_{i=1}^{k} \|\beta^\sharp(\xi_i)\|^2 = -\frac{p(n-p)}{n} \|\beta\|^2,
\]

and this proves inequality (6).
If equality holds in (6) for some $\beta \in \text{Sym}(V \times V, W)$, then the above inequalities become equalities and the proof follows from Lemma 12. \hfill \Box

Next we assume that the dimension of $V$ is $n \geq 4$. For each integer $1 \leq p < n/2$ and for every $\beta \in \text{Sym}(V \times V, W)$, we denote by $\Lambda_p(\beta)$ the subset of the unit $(k-1)$-sphere $S^{k-1}$ in $W$ given by

$$
\Lambda_p(\beta) = \left\{ u \in S^{k-1} : p < \text{Index } \beta^\sharp(u) < n - p \right\}.
$$

The inequality that follows is crucial for the proof of Theorem 11.

**Proposition 14** Given integers $n \geq 4$, $k \geq 1$ and $1 \leq p < n/2$, there exists a constant $\varepsilon(n, k, p) > 0$ such that the following inequality holds

$$
\varrho_p(\beta) - \frac{p(n-p)}{n} \left( n\|H_\beta\|^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}} \|H_\beta\| \|\hat{\beta}\| - \|\hat{\beta}\|^2 \right)
\geq \varepsilon(n, k, p) \left( \int_{\Lambda_p(\beta)} |\det \beta^\sharp(u)| dS_u \right)^{2/n},
$$

for all $\beta \in \text{Sym}(V \times V, W)$, where $dS_u$ stands for the volume element of the unit sphere in $W$.

**Proof** We consider the functions $\phi_p, \psi_p : \text{Sym}(V \times V, W) \to \mathbb{R}$ defined by

$$
\phi_p(\beta) = \varrho_p(\beta) - \frac{p(n-p)}{n} \left( n\|H_\beta\|^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}} \|H_\beta\| \|\hat{\beta}\| - \|\hat{\beta}\|^2 \right),
$$

and

$$
\psi_p(\beta) = \int_{\Lambda_p(\beta)} |\det \beta^\sharp(u)| dS_u.
$$

From Proposition 13, we know that the function $\phi_p$ is nonnegative. To prove the desired inequality, it is sufficient to show that $\phi_p$ attains a positive minimum on the level set

$$
\Sigma_{n,k,p} = \left\{ \beta \in \text{Sym}(V \times V, W) : \psi_p(\beta) = 1 \right\}.
$$

Let $\{\beta_m\}$ be a sequence in $\Sigma_{n,k,p}$ such that

$$
\lim_{m \to \infty} \phi_p(\beta_m) = \inf \phi_p(\Sigma_{n,k,p}) \geq 0.
$$

**Claim I:** The sequence $\{\beta_m\}$ is bounded. Suppose to the contrary that there exists a subsequence of $\{\beta_m\}$, which by abuse of notation is again denoted by $\{\beta_m\}$, such that $\lim_{m \to \infty} \|\beta_m\| = \infty$. Since $\beta_m \neq 0$ for all $m \in \mathbb{N}$, we may write $\beta_m = \|\beta_m\|\hat{\beta}_m$. \hfill \Box
where \( \|\hat{\beta}_m\| = 1 \), and assume that \( \{\hat{\beta}_m\} \) converges to some \( \hat{\beta} \in \text{Sym}(V \times V, W) \) with \( \|\hat{\beta}\| = 1 \). It can be easily checked that \( \phi_p \) is homogeneous of degree 2, that is
\[
\phi_p(t\beta) = t^2 \phi_p(\beta) \quad \text{for all } t > 0.
\]

Therefore, we obtain \( \phi_p(\hat{\beta}_m) = \phi_p(\beta_m)/\|\beta_m\|^2 \). Thus \( \lim_{m \to \infty} \phi_p(\hat{\beta}_m) = 0 \) and consequently \( \phi_p(\hat{\beta}) = 0 \), which means that \( \hat{\beta} \) satisfies equality in inequality (6) in Proposition 13. To reach a contradiction, we distinguish two cases.

**Case 1.** Suppose that \( H_{\hat{\beta}} \neq 0 \). Proposition 13 implies that \( \hat{\beta}(\cdot, \cdot) = \langle \hat{\beta}^\vee(\hat{\xi}), \cdot, \cdot \rangle \hat{\xi} \) and \( \hat{\beta}^\vee(\hat{\xi}) \) has at most two distinct eigenvalues \( \hat{\lambda} \) and \( \hat{\mu} \) with multiplicities \( p \) and \( n - p \), respectively, where \( \hat{\xi} = H_{\beta}/\|H_{\hat{\beta}}\| \). Clearly \( \hat{\mu} \neq 0 \), since otherwise \( \hat{\beta} = 0 \), and this is a contradiction.

Since \( \beta_m \in \Sigma_{n,k,p} \), there exists an open subset \( \hat{U}_m \subset S^{k-1} \subset W \) such that \( \hat{U}_m \subset \Lambda_{\hat{\beta}}(\hat{\beta}_m) \) and \( \det \beta_m^\vee(u) \neq 0 \) for all \( u \in \hat{U}_m \) and \( m \in \mathbb{N} \). Let \( \{\hat{u}_m\} \) be any convergent sequence such that \( \hat{u}_m \in \hat{U}_m \) for all \( m \in \mathbb{N} \) and set \( \hat{u} = \lim_{m \to \infty} \hat{u}_m \). From \( \lim_{m \to \infty} \beta_m^\vee(\hat{u}_m) = \hat{\beta}^\vee(\hat{u}) \) and since \( \hat{u}_m \in \hat{U}_m \), it follows that \( \text{Index } \hat{\beta}^\vee(\hat{u}) < n - p \). Note that
\[
\hat{\beta}^\vee(\hat{u}) = \langle \hat{u}, \hat{\xi} \rangle \hat{\beta}^\vee(\hat{\xi}).
\]

We claim that \( \langle \hat{u}, \hat{\xi} \rangle = 0 \). Suppose to the contrary that \( \langle \hat{u}, \hat{\xi} \rangle > 0 \). If \( \hat{\mu} > 0 \) (respectively, \( \hat{\mu} < 0 \)), then \( \text{Index } \hat{\beta}^\vee(\hat{u}) \leq p \) (respectively, \( \text{Index } \hat{\beta}^\vee(\hat{u}) \geq n - p \)). Hence we have \( \text{Index } \hat{\beta}_m^\vee(\hat{u}_m) \leq p \) (respectively, \( \text{Index } \hat{\beta}_m^\vee(\hat{u}_m) \geq n - p \)) for \( m \) large enough, and that is a contradiction since \( \hat{u}_m \in \Lambda_{\hat{\beta}}(\hat{\beta}_m) \). Similarly, we reach a contradiction if \( \langle \hat{u}, \hat{\xi} \rangle < 0 \). Thus we conclude that \( \langle \hat{u}, \hat{\xi} \rangle = 0 \).

In fact, we proved that \( \langle \lim_{m \to \infty} \hat{u}_m, \hat{\xi} \rangle = 0 \) for any convergent sequence \( \{\hat{u}_m\} \) such that \( \hat{u}_m \in \hat{U}_m \) for all \( m \in \mathbb{N} \). We may choose convergent sequences \( \{\hat{u}^{(1)}_m\}, \ldots, \{\hat{u}^{(k)}_m\} \) in \( \hat{U}_m \) such that \( \hat{u}^{(1)}_m, \ldots, \hat{u}^{(k)}_m \) span \( W \) for all \( m \in \mathbb{N} \). We have \( \lim_{m \to \infty} \hat{u}^{(a)}_m = \hat{u} \) for all \( a \in \{1, \ldots, k\} \). Moreover, we may write \( \hat{\beta}_m = \sum_{a=1}^k \gamma_m^{(a)} \hat{u}^{(a)}_m \), with \( \gamma_m^{(a)} \in \text{Sym}(V \times V, \mathbb{R}) \) being bounded sequences. Letting \( m \to \infty \) in the following
\[
\langle \hat{\beta}_m(\cdot, \cdot), \hat{\xi} \rangle = \sum_{a=1}^k \gamma_m^{(a)}(\cdot, \cdot)\langle \hat{u}^{(a)}_m, \hat{\xi} \rangle,
\]
we obtain \( \langle \hat{\beta}(\cdot, \cdot), \hat{\xi} \rangle = 0 \). This yields \( \hat{\beta} = 0 \), which is a contradiction.

**Case 2.** Suppose that \( \hat{\beta} \) is traceless. Given that \( \hat{\beta} \) satisfies equality in inequality (6), Proposition 13 yields that for all \( \hat{u} \in S^{k-1} \subset W \), the endomorphism \( \hat{\beta}^\vee(\hat{u}) \) has at most two distinct eigenvalues \( \hat{\lambda}(\hat{u}) \) and \( \hat{\mu}(\hat{u}) \) with multiplicities \( p \) and \( n - p \), respectively.

Since \( \beta_m \in \Sigma_{n,k,p} \), it follows that there exists an open subset \( \hat{U}_m \subset S^{k-1} \subset W \) such that \( \hat{U}_m \subset \Lambda_{\hat{\beta}}(\hat{\beta}_m) \) and \( \det \beta_m^\vee(u) \neq 0 \) for all \( u \in \hat{U}_m \) and \( m \in \mathbb{N} \). Let \( \{\hat{u}_m\} \) be any convergent sequence such that \( \hat{u}_m \in \hat{U}_m \) for all \( m \in \mathbb{N} \) and set \( \hat{u} = \lim_{m \to \infty} \hat{u}_m \). Since \( \lim_{m \to \infty} \beta_m^\vee(\hat{u}_m) = \hat{\beta}^\vee(\hat{u}) \) and \( \hat{u}_m \in \hat{U}_m \), it follows that \( \text{Index } \hat{\beta}^\vee(\hat{u}) < n - p \). We claim that \( \hat{\mu}(\hat{u}) = 0 \). Observe that
p\lambda(\hat{u}) + (n - p)\hat{\mu}(\hat{u}) = 0.

If \(\hat{\mu}(\hat{u}) > 0\) (respectively, \(\hat{\mu}(\hat{u}) < 0\)), then Index \(\hat{\beta}^{\nu}(\hat{u}) \leq p\) (respectively, Index \(\hat{\beta}^{\nu}(\hat{u}) \geq n - p\)). Thus Index \(\hat{\beta}^{\nu}_m(\hat{u}_m) \leq p\) (respectively, Index \(\hat{\beta}^{\nu}_m(\hat{u}_m) \geq n - p\)) for \(m\) large enough, and this contradicts the fact that \(\hat{u}_m \in \Lambda_p(\hat{\beta}_m)\). Hence \(\hat{\mu}(\hat{u}) = 0\) and Proposition 13 yields \(\hat{\beta}^{\nu}(\hat{u}) = 0\).

In fact, we proved that

\[
\lim_{m \to \infty} \hat{\beta}^{\nu}_m(\hat{u}_m) = 0,
\]

for any convergent sequence \(\{\hat{u}_m\}\) such that \(\hat{u}_m \in \hat{U}_m\) for all \(m \in \mathbb{N}\). We may choose convergent sequences \(\{\hat{u}_m^{(1)}\}, \ldots, \{\hat{u}_m^{(k)}\}\) in \(\hat{U}_m\) such that \(\hat{u}_m^{(1)}\), \ldots, \(\hat{u}_m^{(k)}\) span \(W\) for all \(m \in \mathbb{N}\).

Using the Gram–Schmidt process, we obtain sequences \(\{\xi_m^{(1)}\}, \ldots, \{\xi_m^{(k)}\}\) such that

\[
\xi_m^{(i)} \in \text{span}\{\hat{u}_m^{(1)}, \ldots, \hat{u}_m^{(i)}\} \quad \text{for each } 1 \leq i \leq k.
\]

Thus \(\xi_m^{(a)} = \sum_{\ell=1}^a x_m^{(\ell)} \hat{u}_m^{(\ell)}\), where \(\{x_m^{(\ell)}\}\) are convergent sequences for each \(1 \leq \ell \leq k\), and consequently we have

\[
\hat{\beta}_m(\cdot, \cdot) = \sum_{a=1}^k \langle \hat{\beta}^{\nu}_m(\xi_m^{(a)}), \cdot, \xi_m^{(a)} \rangle, \quad (10)
\]

and \(\hat{\beta}_m(\xi_m^{(a)}) = \sum_{\ell=1}^a x_m^{(\ell)} \hat{\beta}^{\nu}_m(\hat{u}_m^{(\ell)})\). Using (9), we obtain

\[
\lim_{m \to \infty} \hat{\beta}^{\nu}_m(\xi_m^{(a)}) = \lim_{m \to \infty} \sum_{\ell=1}^a x_m^{(\ell)} \hat{\beta}^{\nu}_m(\hat{u}_m^{(\ell)}) = 0,
\]

for all \(1 \leq a \leq k\). From (10) and the above, it follows that \(\lim_{m \to \infty} \hat{\beta}_m = 0\). Thus \(\hat{\beta} = 0\), which is a contradiction, and this completes the proof of Claim I.

Thus we may assume that \(\lim_{m \to \infty} \beta_m = \beta_\infty \in \text{Sym}(V \times V, W)\). We first argue that \(\beta_\infty \neq 0\). Indeed, if \(\beta_\infty = 0\), then \(\beta_m^{\nu}(u) = 0\) for every \(u \in \mathbb{S}^{k-1}\). Since \(\beta_m \in \Sigma_{n,k,p}\), there exists \(\xi_m \in \Lambda_p(\beta_m)\) such that

\[
|\det \beta_m^{\nu}(\xi_m)| \, \text{Vol}(\Omega(\beta_m)) = 1 \quad \text{for all } m \in \mathbb{N}.
\]

We may assume that the sequence \(\{\xi_m\}\) converges to some \(\xi \in \mathbb{S}^{k-1}\). Hence we have \(\lim_{m \to \infty} \beta_m^{\nu}(\xi_m) = \beta_\infty^{\nu}(\xi) = 0\), which contradicts (11).

Claim II: The bilinear form \(\beta_\infty\) is not a zero of \(\phi_p\). Arguing by contradiction, suppose that \(\phi_p(\beta_\infty) = 0\), that is \(\beta\) satisfies equality in inequality (6). To reach a contradiction and according to Proposition 13, we distinguish two cases.
Case 3. Suppose that $H_{β_∞} \neq 0$. Hence $β_∞ = \langle β_{∞}^x(ξ), · \rangle ξ$, where $ξ = H_{β_∞}/∥H_{β_∞}\parallel$ and $β_{∞}^x(ξ)$ has at most two distinct eigenvalues $λ$ and $μ$ with multiplicities $p$ and $n − p$, respectively. Clearly $μ = 0$, since otherwise $β_∞ = 0$, which is a contradiction.

Given that $β_m ∈ Σ_{n,k,p}$, there exists an open subset $U_m ∈ S^{k−1} ⊂ W$ such that $U_m ⊂ Λ_p(β_m)$ and $det β_m^x(u) \neq 0$ for each $u ∈ U_m$ and $m ∈ N$. Let $\{u_m\}$ be any convergent sequence such that $u_m ∈ U_m$ for all $m ∈ N$ and set $u = lim_{m→∞} u_m$. Since $lim_{m→∞} β_m^x(u_m) = β_{∞}^x(u)$ and $u_m ∈ U_m$, it follows that Index $β_{∞}^x(u) < n − p$. Observe that

$$β_{∞}^x(u) = (u, ξ)β_{∞}^x(ξ).$$

We claim that $⟨u, ξ⟩ = 0$. Suppose to the contrary that $⟨u, ξ⟩ > 0$. If $μ > 0$ (respectively, $μ < 0$), then Index $β_{∞}^x(u) ≤ p$ (respectively, Index $β_{∞}^x(u) ≥ n − p$) and consequently Index $β_m^x(u_m) ≤ p$ (respectively, Index $β_m^x(u_m) ≥ n − p$) for $m$ large enough, and that is a contradiction since $u_m ∈ Λ_p(β_m)$. A contradiction is reached in a similar manner if $⟨u, ξ⟩ < 0$. Thus we conclude that $⟨u, ξ⟩ = 0$.

In fact, we proved that $(lim_{m→∞} u_m, ξ) = 0$ for any convergent sequence $\{u_m\}$ such that $u_m ∈ U_m$ for all $m ∈ N$. We may choose convergent sequences $\{u_m^{(1)}\}, \ldots, \{u_m^{(k)}\}$ in $U_m$ such that $u_m^{(1)}$, $\ldots$, $u_m^{(k)}$ span $W$ for all $m ∈ N$. We have $(lim_{m→∞} u_m^{(a)}, ξ) = 0$ for all $a ∈ \{1, \ldots, k\}$. Moreover, we may write $β_m = \sum_{a=1}^{k} γ_m^{(a)} u_m^{(a)}$, with $γ_m^{(a)} ∈ Sym(V × V, R)$ being bounded. Letting $m → ∞$ in the equality

$$⟨β_m(·, ·), ξ⟩ = \sum_{a=1}^{k} γ_m^{(a)}(·, ·)⟨u_m^{(a)}, ξ⟩,$$

it follows that $⟨β_{∞}(·, ·), ξ⟩ = 0$. Thus $β_{∞} = 0$, which is a contradiction.

Case 4. Suppose that $β_{∞}$ is trace free. Given that $β_{∞}$ satisfies equality in inequality (6), Proposition 13 yields that for all $u ∈ S^{k−1} ⊂ W$, the endomorphism $β_{∞}^x(u)$ has at most two distinct eigenvalues $λ(u)$ and $μ(u)$ with multiplicities $p$ and $n − p$, respectively.

Since $β_m ∈ Σ_{n,k,p}$, there exists an open subset $U_m ∈ S^{k−1} ⊂ W$ such that $U_m ⊂ Λ_p(β_m)$ and $det β_m^x(u) \neq 0$ for all $u ∈ U_m$ and $m ∈ N$. Let $\{u_m\}$ be any convergent sequence such that $u_m ∈ U_m$ for all $m ∈ N$ with $u = lim_{m→∞} u_m$. Since $lim_{m→∞} β_m^x(u_m) = β_{∞}^x(u)$ and $u_m ∈ U_m$, it follows that Index $β_{∞}^x(u) < n − p$.

We claim that $μ(u) = 0$. Observe that

$$pλ(u) + (n − p)μ(u) = 0.$$

If $μ(u) > 0$ (respectively, $μ(u) < 0$), then Index $β_{∞}^x(u) ≤ p$ (respectively, Index $β_{∞}^x(u) ≥ n − p$). Hence Index $β_m^x(u_m) ≤ p$ (respectively, Index $β_m^x(u_m) ≥ n − p$) for $m$ large enough, and that is a contradiction since $u_m ∈ Λ_p(β_m)$. Thus we have $μ(u) = 0$, which implies that $β_{∞}^x(u) = 0$. 

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In fact, we proved that
\[ \lim_{m \to \infty} \beta_m^\sharp (u_m) = 0, \] (12)
for every convergent sequence \( \{u_m\} \) such that \( u_m \in U_m \) for all \( m \in \mathbb{N} \). We may choose convergent sequences \( \{u_m^{(1)}\}, \ldots, \{u_m^{(k)}\} \) in \( U_m \) such that \( u_m^{(1)}, \ldots, u_m^{(k)} \) span \( W \) for all \( m \in \mathbb{N} \).

Using the Gram–Schmidt process, we obtain sequences \( \{\xi_m^{(1)}\}, \ldots, \{\xi_m^{(k)}\} \) such that
\[ \xi_m^{(i)} \in \text{span}\{u_m^{(1)}, \ldots, u_m^{(i)}\} \text{ for each } 1 \leq i \leq k. \]
Thus \( \xi_m^{(a)} = \sum_{\ell=1}^{a} x_m^{(\ell)} u_m^{(\ell)} \), where \( \{x_m^{(\ell)}\} \) are convergent sequences for every \( 1 \leq \ell \leq k \). Clearly we have
\[ \beta_m (\cdot, \cdot) = \sum_{a=1}^{k} \langle \beta_m^{\sharp} (\xi_m^{(a)}) \cdot, \cdot \rangle \xi_m^{(a)}, \] (13)
and therefore \( \beta_m^{\sharp} (\xi_m^{(a)}) = \sum_{\ell=1}^{a} x_m^{(\ell)} \beta_m^{\sharp} (u_m^{(\ell)}) \). Using (12), it follows that
\[ \lim_{m \to \infty} \beta_m^{\sharp} (\xi_m^{(a)}) = \lim_{m \to \infty} \sum_{\ell=1}^{a} x_m^{(\ell)} \beta_m^{\sharp} (u_m^{(\ell)}) = 0, \]
for all \( 1 \leq a \leq k \). From (13) and the above it follows that \( \beta_\infty = 0 \), which is a contradiction. This completes the proof of Claim II.

Thus the function \( \phi_p \) attains a positive minimum \( \epsilon(n, k, p) = \phi_p(\beta_\infty) \) on \( \Sigma_{n,k,p} \), which depends only on \( n, k \), and \( p \). Let \( \beta \) be an arbitrary bilinear form in \( \text{Sym}(V \times V, W) \). Suppose that \( \psi_p(\beta) \neq 0 \) and set \( \gamma = \beta/ (\psi_p(\beta))^{1/n} \). Since \( \gamma \in \Sigma_{n,k,p} \), we have \( \phi_p(\gamma) \geq \epsilon(n, k, p) \) and the desired inequality follows from the homogeneity of the function \( \phi_p \).

\( \square \)

4 Submanifolds Satisfying the Pinching Condition

Let \( M^n \) be an oriented Riemannian manifold of dimension \( n \). For each integer \( 0 \leq p \leq n \), the Hodge–Laplace operator acting on differential \( p \)-forms is defined by
\[ \Delta = d \delta + \delta d : \Omega^p (M^n) \to \Omega^p (M^n), \]
where \( d \) and \( \delta \) are the differential and the co-differential operators, respectively. When \( p = 0 \), the Hodge–Laplace operator is just the Laplace–Beltrami operator acting on 0-forms, i.e., scalar functions.

We recall the following well-known result (cf. [29, Prop. 3]).

**Proposition 15** Let \( M^n \) be a compact, oriented Riemannian manifold and let \( 1 \leq p \leq n/2 \). The following assertions hold:
(i) If \( B[p] \geq 0 \), then \( b_p(M^n) \leq \binom{n}{p} \).

(ii) If \( B[p] \geq 0 \) and the strict inequality holds at some point, then \( H^p(M^n; \mathbb{R}) = H^{n-p}(M^n; \mathbb{R}) = 0 \).

(iii) If \( B[p] \geq 0 \) and \( H^p(M^n; \mathbb{R}) \neq 0 \), then every harmonic \( p \)-form is parallel. In particular, \( M^n \) supports a nontrivial parallel \( p \)-form.

**Proof** Let \( \omega \in \Omega^p(M^n) \). From the Bochner–Weitzenböck formula (2), we obtain

\[ \langle \Delta \omega, \omega \rangle = \| \nabla \omega \|^2 + \langle B[p] \omega, \omega \rangle + \frac{1}{2} \| \Delta \omega \|^2. \quad (14) \]

If \( \omega \) belongs to the space \( \mathcal{H}^p(M^n) \) of harmonic \( p \)-forms, then using the assumption, we have

\[ 0 \leq \int_{M^n} \langle B[p] \omega, \omega \rangle dM = -\int_{M^n} \| \nabla \omega \|^2 dM. \]

Thus \( \omega \) is parallel and therefore

\[ b_p(M^n) = \dim H^p(M^n; \mathbb{R}) = \dim \mathcal{H}^p(M^n) \leq \dim \Lambda^p(T^*_x M^n) = \binom{n}{p}. \]

This proves (i). For the proofs of (ii) and (iii) see [29, Prop. 3].

Let \( f : M^n \to \tilde{M}^{n+k} \) be an isometric immersion whose second fundamental form \( \alpha_f \) is viewed as a section of the vector bundle \( \text{Hom}(TM \times TM, N_f M) \), where \( N_f M \) is the normal bundle. For each unit normal vector field \( \xi \in \Gamma(N_f M) \), the associated shape operator \( A_\xi \) is given by

\[ \langle A_\xi X, Y \rangle = \langle \alpha_f(X, Y), \xi \rangle, \quad X, Y \in TM. \]

Savo [29, Th. 1] proved that the Bochner operator \( B[p] \) splits as

\[ B[p] = B[p]_{\text{res}} + B[p]_{\text{ext}}, \quad (15) \]

where \( B[p]_{\text{res}} \) depends on the geometry of the ambient manifold \( \tilde{M}^{n+k} \) restricted to the submanifold \( f(M^n) \) and satisfies

\[ B[p]_{\text{res}} \geq p(n - p)c, \quad (16) \]

where \( c \) is a lower bound for the curvature operator of \( \tilde{M}^{n+k} \). In particular, if \( \tilde{M}^{n+k} \) has constant sectional curvature \( c \), then

\[ B[p]_{\text{res}} = p(n - p)c. \quad (17) \]
The second operator $B_{\text{ext}}^{[p]}$ is given explicitly in terms of the second fundamental form of the immersion $f$. For each unit vector field $\xi \in \Gamma(N_f M)$, we define the endomorphism

$$T_{A_\xi}^{[p]} = (\text{tr} A_\xi) A_\xi^{[p]} - A_\xi^{[p]} \circ A_\xi^{[p]},$$

where

$$A_\xi^{[p]} : \Omega^p(M^n) \to \Omega^p(M^n),$$

is the self-adjoint extension of the shape operator $A_\xi$ in the direction $\xi$ defined by

$$A_\xi^{[p]} \omega(X_1, \ldots, X_p) = \sum_{i=1}^{p} \omega(X_1, \ldots, A_\xi X_i, \ldots, X_p), \quad X_1, \ldots, X_p \in TM.$$

Then $B_{\text{ext}}^{[p]}$ is given by (see [29])

$$B_{\text{ext}}^{[p]} = \sum_{i=1}^{k} T_{A_{\xi_i}}^{[p]},$$

where $\xi_1, \ldots, \xi_k$ is an orthonormal frame of the normal bundle of $f$.

The next proposition provides a sharp estimate for the Bochner operator in terms of the second fundamental form in arbitrary codimension and gives its structure at points where equality holds. Inequality (18) was proved by Savo [29, Prop. 15] for hypersurfaces. Note that, since $B^{[1]}$ is nothing but the Ricci tensor, inequality (18) for $p = 1$ reduces to the inequality due to Leung [26]. We recall that the traceless part of the second fundamental form is given by $\Phi = \alpha_f - \langle \cdot, \cdot \rangle \mathcal{H}$.

**Proposition 16** Let $f : M^n \to \tilde{M}^m$, $n \geq 3$, be an isometric immersion. If the curvature operator of $\tilde{M}^m$ is bounded from below by a constant $c$, then the Bochner operator of $M^n$, for any $1 \leq p \leq n/2$, satisfies pointwise the inequality

$$\min_{\omega \in \Omega^p(M^n)} \langle B^{[p]} \omega, \omega \rangle \geq \frac{p(n-p)}{n} (n (H^2 + c) - \frac{n(n-2p)}{\sqrt{np(n-p)}} \|H\|^2 - \|\Phi\|^2).$$

(18)

If equality holds in (18) at a point $x \in M^n$, then the following hold:

(i) The shape operator $A_\xi(x)$ has at most two distinct eigenvalues with multiplicities $p$ and $n-p$ for every unit vector $\xi \in N_f M(x)$. If in addition $p < n/2$ and the eigenvalue of multiplicity $n-p$ vanishes, then $A_\xi(x) = 0$.

(ii) If $H(x) \neq 0$ and $p < n/2$, then $\text{Im} \alpha(x) = \text{span} \{\mathcal{H}(x)\}$.

**Proof** It is a direct consequence of Proposition 13, (15) and (16).
Proof of Theorem 1. We observe that
\[ a(n, p, H, c) = (r(n, p, H, c))^2 + nH^2, \]  \hspace{0.5cm} (19)
where \( r(n, p, H, c) \) is the largest root of the quadratic polynomial
\[ P(t; p, H, c) = t^2 + \frac{n(n - 2p)}{\sqrt{np(n - p)}} Ht - n(H^2 + c). \]  \hspace{0.5cm} (20)
Using that \( \|\Phi\|^2 = S - nH^2 \), our pinching assumption \( S \leq a(n, p, H, c) \) turns out to be equivalent to \( \|\Phi\| \leq r(n, i, H, c) \). Since
\[ 0 \leq r(n, 1, H, c) \leq \cdots \leq r(n, \lfloor n/2 \rfloor, H, c), \]
we have \( \|\Phi\| \leq r(n, i, H, c) \), or equivalently
\[ P(\|\Phi\|; i, H, c) \leq 0 \]  for each \( p \leq i \leq n/2 \).

Proposition 16 implies that \( B[i] \geq 0 \) for all \( p \leq i \leq n/2 \). Now, part (i) of the theorem follows from Proposition 15(i) and the Poincaré duality, while part (ii) follows from Propositions 15(ii) and 16. Part (iii) follows from part (ii) and Proposition 15(iii). \( \square \)

Proof of Theorem 2. Let \( \omega \in \mathcal{H}^p(M^n) \) be a harmonic form of constant length. Equation (14) yields
\[ \langle B[p] \omega, \omega \rangle = -\|\nabla \omega\|^2 \leq 0. \]
From Proposition 16, it follows that
\[ n(H^2 + c) - \frac{n(n - 2p)}{\sqrt{np(n - p)}} H\|\Phi\| - \|\Phi\|^2 \leq 0. \]
Hence we have \( \|\Phi\| \geq r(n, p, H, c) \), where \( r(n, p, H, c) \) is the largest root of the polynomial (20). Since \( \|\Phi\|^2 = S - nH^2 \), using (19) we obtain \( S \geq a(n, p, H, c) \). \( \square \)

We recall that an isometric immersion \( f: M^n \to \mathbb{Q}_c^{n+k} \), into a space form of constant sectional curvature \( c \), is said to reduce codimension to \( m < k \) if there exists a totally geodesic submanifold \( \mathbb{Q}_c^{n+m} \) of \( \mathbb{Q}_c^{n+k} \) such that \( f(M) \subset \mathbb{Q}_c^{n+m} \). We need the following proposition (a related result was given in [8] for hypersurfaces in spheres).

Proposition 17 Let \( f: M^n \to \mathbb{Q}_c^{n+k}, n \geq 3, \) be an isometric immersion into a nonflat space form that satisfies (1) for an integer \( 1 \leq p < n/2 \) and \( H^2 + c \geq 0 \). Suppose that \( M^n \) supports a nontrivial harmonic \( p \)-form (in particular, a parallel \( p \)-form) with constant length. If \( H > 0 \) on \( M^n \), then the codimension of \( f \) is reduced to one. Moreover, the hypersurface \( f: M^n \to \mathbb{Q}_c^{n+1} \) has at most two distinct principal curvatures \( \lambda \) and \( \mu \) of multiplicities \( p \) and \( n - p \), respectively, such that \( \lambda \mu + c = 0 \).
Proof Let $\omega \in H^p(M^n)$ be a nontrivial harmonic $p$-form of constant length. Equation (14) yields $\langle B[p] \omega, \omega \rangle \leq 0$. Theorem 2 implies that $S = a(n, p, H, c)$ on $M^n$, which is equivalent to

$$n(H^2 + c) - \frac{n(n - 2p)}{\sqrt{np(n - p)}} H \|\Phi\| - \|\Phi\|^2 = 0.$$  

Thus equality holds in (18) at each point. According to Proposition 16(ii), the second fundamental form of $f$ satisfies

$$\alpha_f(X, Y) = \langle A_\xi X, Y \rangle \xi \quad \text{for all } X, Y \in TM,$$

where $\xi = H/H$ and the shape operator $A_\xi$ has at most two distinct eigenvalues $\lambda$ and $\mu$ of multiplicities $p$ and $n - p$, respectively. The Gauss equation is written as

$$R(X, Y)Z = c (\langle Y, Z \rangle X - \langle X, Z \rangle Y) + (A_\xi Y, Z)A_\xi X - (A_\xi X, Z)A_\xi Y,$$

where the curvature tensor $R$ is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}, \quad X, Y \in \mathcal{X}(M^n).$$

Then we find that

$$R(X, Y)\omega = c X^\# \wedge i_Y \omega - c Y^\# \wedge i_X \omega + (A_\xi X)^\# \wedge i_{A_\xi Y} \omega - (A_\xi Y)^\# \wedge i_{A_\xi X} \omega.$$  

(21)

Here $i_X \omega$ is the interior multiplication of the form $\omega$ by the vector field $X$ and $X^\#$ is the 1-form dual to $X$. Let $X_1, \ldots, X_n$ be an arbitrary local orthonormal frame with dual frame $\theta_1, \ldots, \theta_n$ and set

$$\Psi_{ij} = \theta_i \wedge i_{\hat{X}_j} \omega - \theta_j \wedge i_{X_i} \omega, \quad 1 \leq i \leq p < j \leq n.$$  

We claim that $\Psi_{ij} \neq 0$. In fact, since $\omega$ is parallel and nontrivial, we may assume after reordering that $\omega(X_1, \ldots, X_p) \neq 0$. Thus we have

$$\Psi_{ij}(X_1, \ldots, \hat{X}_i, \ldots, X_p, X_j) = \pm \omega(X_1, \ldots, X_p) \neq 0.$$  

Now we choose the local orthonormal frame $X_1, \ldots, X_n$ such that

$$A_\xi X_i = \lambda X_i, \quad 1 \leq i \leq p \quad \text{and} \quad A_\xi X_j = \mu X_j, \quad p + 1 \leq j \leq n.$$  

Since $\omega$ is parallel, we have $R(X_i, X_j)\omega = 0$ and Eq. (21) yields $(\lambda \mu + c)\Psi_{ij} = 0$, or equivalently $\lambda \mu + c = 0$. For each $\eta \in \Gamma(N_f M)$ perpendicular to $\xi$, the Codazzi equation yields

$$A_{\nabla_{\hat{X}_i} \eta} X_j = A_{\nabla_{X_i} \eta} X_j, \quad 1 \leq i \leq p < j \leq n,$$
or equivalently
\[
\mu \langle \nabla_{\hat{X}_i} \eta, \xi \rangle X_j = \lambda \langle \nabla_{\hat{X}_j} \eta, \xi \rangle X_i,
\]
for all \(1 \leq i \leq p < j \leq n\).

This implies that
\[
\langle \nabla_{\hat{X}_i} \eta, \xi \rangle = \langle \nabla_{\hat{X}_j} \eta, \xi \rangle = 0,
\]
for all \(1 \leq i \leq p\) and \(p + 1 \leq j \leq n\). Therefore, the first normal bundle \(N^1_f M = \text{Im} \alpha_f\) is a parallel subbundle of the normal bundle of constant rank one. Thus, the codimension of \(f\) is reduced to one (see for instance [10, Prop. 2.1]). \(\square\)

**Proof of Theorem 4.** Suppose that \(M^n\) is not a real homology sphere and let \(p\) be an integer such that \(b_p(M^n) > 0\) with \(1 \leq p \leq n/2\). Since \(S \leq n = a(n, p, 0, 1)\), it follows from Theorem 1(iii) that \(S = n\) on \(M^n\) and that every harmonic form \(\omega \in H^p(M^n)\) is parallel. Let \(\omega\) be a nonvanishing parallel form. Equation (14) yields \(\langle \mathcal{B}^p \omega, \omega \rangle = 0\). Hence equality holds in inequality (18) at every point. Proposition 16(i) implies that the second fundamental form in every normal direction has at most two distinct principal curvatures of multiplicities \(p\) and \(n - p\).

If \(m = n + 1\), then it follows from [7] that \(f(M^n)\) is isometric to the Clifford torus \(\mathbb{T}_n^{\sqrt{p/n}}\). If \(m > n + 1\), then it follows from [22, Th. 1.5] that \(f\) is a standard embedding of a projective plane over the complex \(\mathbb{C}\), quaternions \(\mathbb{H}\), or Cayley numbers \(\mathbb{O}\). In other words, \(f\) is one of the standard minimal isometric embeddings
\[
\psi_1 : \mathbb{CP}^2 \to \mathbb{S}^7, \quad \psi_2 : \mathbb{HP}^2 \to \mathbb{S}^{13}, \quad \psi_3 : \mathbb{OP}^2 \to \mathbb{S}^{25}.
\]

The holomorphic curvature of \(\mathbb{CP}^2\) is \(4/3\), the \(\mathbb{H}\)-sectional curvature of \(\mathbb{HP}^2\) is \(4/3\), and the \(\mathbb{O}\)-sectional curvature of the Cayley plane is \(4/3\) (see [16, p. 780] or [21]). It can be easily checked that \(\psi_1 : \mathbb{CP}^2 \to \mathbb{S}^7\) is the only one of the above satisfying \(S = n\).

Now suppose that \(M^n\) is a real homology sphere. It remains to prove that \(M^n\) admits a metric of positive Ricci curvature. Using our assumptions, it follows from inequality (18) for \(p = 1\) that the Ricci curvature of \(M^n\) is nonnegative everywhere.

We claim that there is a point where the Ricci curvature is positive for every tangent direction. Suppose to the contrary that at each point, there exists a tangent direction where the Ricci curvature vanishes. This implies that equality holds in inequality (18) for \(p = 1\) at every point. Proposition 16(i) implies that the second fundamental form in every normal direction has at most two distinct principal curvatures, one with multiplicity \(n - 1\). Arguing as before, it follows from [7] and [22, Th. 1.5] that either \(f(M^n)\) is isometric to the Clifford torus \(\mathbb{T}_n^{\sqrt{1/n}}\), or \(f\) is the standard embedding \(\psi : \mathbb{CP}^2_{4/3} \to \mathbb{S}^7\) of the complex projective plane of constant holomorphic curvature \(4/3\). Hence either \(M^n\) is diffeomorphic to \(\mathbb{S}^1 \times \mathbb{S}^{n-1}\) (cf. [13, Th. 1]), or diffeomorphic to \(\mathbb{CP}^2\). This clearly contradicts the fact that \(M^n\) is a real homology sphere.

Thus there exists a point where the Ricci curvature is positive for every tangent direction. It follows from Aubin [3] that \(M^n\) carries a metric with positive Ricci curvature. The Bonnet–Myers theorem implies that the fundamental group of \(M^n\) is finite. \(\square\)
Proof of Corollary 5. Suppose that \( n = 3 \) and \( f(M^3) \) is not isometric to the Clifford torus \( \mathbb{T}^3(\sqrt{1/3}) \). Theorem 4 implies that \( M^3 \) carries a metric with positive Ricci curvature. It follows from [19, Th. 1.1] that \( M^3 \) is diffeomorphic to a spherical space form.

Now assume that \( n = 4 \) and the minimal submanifold is neither a Clifford torus, nor the standard embedding \( \psi \) of the complex projective plane. Theorem 4 implies that \( M^4 \) is a real homology sphere and its universal cover \( \tilde{M}^4 \) is compact. We claim that \( \tilde{M}^4 \) is a real homology sphere. If otherwise, we consider the minimal isometric immersion given by \( \tilde{f} = f \circ \pi \), where \( \pi: \tilde{M}^4 \to M^4 \) is the covering map. It is obvious that the squared length of its second fundamental form satisfies \( \tilde{S} \leq 4 \). Then Theorem 4 would imply that either \( f(M^4) \) has to be isometric to a minimal Clifford torus, or \( \tilde{f} \) is the standard isometric embedding of the complex projective plane. This is clearly a contradiction.

From the Poincaré duality and the universal coefficient theorem (see [34, Cor. 4, p. 244]), it follows that the torsion subgroups of \( H_i(\tilde{M}^4; \mathbb{Z}) \) and \( H_{3-i}(\tilde{M}^4; \mathbb{Z}) \) are isomorphic for \( i = 1, 2, 3 \). Since \( \tilde{M}^4 \) is both simply connected and a real homology sphere, we have that it is also a homology sphere over the integers. Therefore \( \tilde{M}^4 \) is a homotopy sphere. By the proof of the Poincaré conjecture for \( n = 4 \) due to Freedman [14], the universal cover \( M^4 \) is homeomorphic to \( S^4 \). \( \square \)

Proof of Theorem 6. Suppose that \( b_p(M^n) > 0 \) and let \( \omega \in \mathcal{H}_p(M^n) \) be a non-vanishing harmonic form. Theorem 1 (iii) implies that the form \( \omega \) is parallel and \( S = a(n, p, H, 1) \) on \( M^n \), or equivalently

\[
n(H^2 + 1) - \frac{n(n - 2p)}{\sqrt{n}p(n - p)} H \| \Phi \| - \| \Phi \|^2 = 0.
\]

Then Proposition 16 gives \( \langle B[p] \omega, \omega \rangle \geq 0 \). On the other hand, since \( \omega \) is parallel, it follows from Eq. (14) that \( \langle B[p] \omega, \omega \rangle = 0 \). Hence equality holds in inequality (18) at every point, and Proposition 16(i) implies that the second fundamental form of \( f \) has at most two distinct principal curvatures of multiplicities \( p \) and \( n - p \) in each normal direction.

Let \( v \in \mathbb{R}^{m+1} \) be a vector such that the height function \( h: M^n \to \mathbb{R} \) defined by \( h = \langle g, v \rangle \) is a Morse function, where \( g \) is the isometric immersion \( g = j \circ f \), and \( j: S^m \to \mathbb{R}^{m+1} \) is the inclusion. The Hessian of \( h \) is given by

\[
\text{Hess } h(X, Y) = \langle \alpha_g(X, Y), v \rangle, \quad X, Y \in TM.
\]

Obviously, the second fundamental form of \( g \) has at most two distinct principal curvatures of multiplicities \( p \) and \( n - p \) in every normal direction. Hence the index of each nondegenerate critical point of \( h \) is 0, \( p \), \( n - p \) or \( n \). That the manifold \( M^n \) has the homotopy type of a CW-complex with cells only in dimensions 0, \( p \), \( n - p \) or \( n \) follows from standard Morse theory (cf. [28, Th. 3.5] or [5, Th. 4.10]), and this proves part (i) of the theorem.
(ii) Our pinching assumption \( S \leq a(n, p, H, 1) \) turns out to be equivalent to

\[
\|\Phi\| \leq r(n, p, H, 1) \leq r(n, q, H, 1).
\]

Since \( b_q(M^n) > 0 \), Theorem 1 (iii) implies that equality holds everywhere in the above inequalities, and consequently

\[
n(H^2 + 1) - \frac{n(n - 2p)}{\sqrt{np(n - p)}} H \|\Phi\| - \|\Phi\|^2 = 0,
\]

and

\[
n(H^2 + 1) - \frac{n(n - 2q)}{\sqrt{nq(n - q)}} H \|\Phi\| - \|\Phi\|^2 = 0.
\]

It follows directly from the above that the submanifold is minimal and \( S = n \). Hence Theorem 4 implies that either \( f(M^n) \) is isometric to the Clifford torus \( T^n_q(\sqrt{q/n}) \), or \( f \) is the standard embedding of the complex projective plane.

Hereafter, we suppose that \( H > 0 \) on \( M^n \).

(iii) Since \( b_p(M^n) > 0 \), Theorem 1 implies that there exists a nontrivial parallel \( p \)-form. Using our assumption on the mean curvature, it follows from Proposition 17 that the codimension of \( f \) reduces to one, and the proof of part (iii) follows from [29, Th. 9].

(iv) Under our assumption, we obtain

\[
S = a(n, p, H, 1) < a(n, p + 1, H, 1),
\]

on \( M^n \). From [37, Th. 1], it follows that the homology groups satisfy \( H_i(M^n; \mathbb{Z}) = 0 \) for every \( p < i < n - p \). Now suppose that \( f(M^n) \) is not isometric to a torus as in the statement of part (iii) of the theorem. Then we have \( b_p(M^n) = 0 \). From the universal coefficient theorem (see [34, Cor. 4, p. 244]) and using \( H_{n-p-1}(M^n; \mathbb{Z}) = 0 \), we conclude that \( H^{n-p}(M^n; \mathbb{Z}) \) has no torsion and neither does \( H_p(M^n; \mathbb{Z}) \) by Poincaré duality. Since \( b_p(M^n) = 0 \), we obtain \( H_p(M^n; \mathbb{Z}) = 0 \).

**Proof of Theorem 7.** At first, we observe that \( M^n \) has nonnegative Ricci curvature. Indeed, our assumption \( S \leq a(n, 1, H, 1) \) is equivalent to \( \|\Phi\| \leq r(n, 1, H, 1) \), where \( r(n, 1, H, 1) \) is the largest root of the polynomial (20). Hence, the right-hand side of inequality (18) for \( p = 1 \) is nonnegative, and consequently, \( M^n \) has nonnegative Ricci curvature.

If \( b_1(M^n) > 0 \), then it follows from Theorem 6(i) that \( M^n \) has the homotopy type of a CW-complex with cells only in dimensions 0, 1, \( n - 1, n \). If \( b_q(M^n) > 0 \) for an integer \( 1 < q \leq n/2 \), then Theorem 6(ii) yields that \( f(M^n) \) is isometric to the Clifford torus \( T^n_q(\sqrt{q/n}) \) for an integer \( 1 < q \leq n/2 \), or \( f \) is the standard embedding as in Theorem 4(iii).

Suppose now that none of the above holds. Hence, according to Theorem 6, \( M^n \) is a real homology sphere. We claim that there is a point where all Ricci curvatures

\[Springer\]
are positive. If otherwise, equality holds in (18) for $p = 1$ everywhere, and consequently, the second fundamental form has at most two distinct principal curvatures of multiplicities 1 and $n - 1$ in every normal direction. Then arguing as in the proof of Theorem 6 using Morse theory, we conclude that $M^n$ has the homotopy type of a CW-complex with cells only in dimensions 0, 1, $n - 1$, $n$. This is a contradiction since this case has been excluded.

Hence there is a point where all Ricci curvatures are positive. It follows from Aubin [3] that $M^n$ carries a metric of positive Ricci curvature. By Bonnet–Myers theorem, the fundamental group of $M^n$ is finite and its universal covering $\tilde{M}^n$ is compact. We claim that $\tilde{M}^n$ is a real homology sphere. In fact, the isometric immersion $\tilde{f} = f \circ \pi$ clearly satisfies our pinching condition for $p = 1$, where $\pi : \tilde{M}^n \to M^n$ is the covering map. At first, we observe that $b_1(\tilde{M}^n) = 0$. If otherwise, Theorem 6(i) would imply that $\tilde{M}^n$ has the homotopy type of a CW-complex with cells only in dimensions 0, 1, $n - 1$, $n$, and this is clearly a contradiction. If $b_q(\tilde{M}^n) > 0$ for an integer $1 < q \leq n/2$, then Theorem 6(ii) yields that $f(\tilde{M}^n)$ is isometric to the Clifford torus $\mathbb{T}^{q^2}((\sqrt{q}/n)$, or $f$ is the standard embedding as in Theorem 4(iii). This is again a contradiction. Thus $\tilde{M}^n$ is a real homology sphere. The assertion for $n = 3, 4$ follows as in the proof of Corollary 5. This completes the proof of parts (i)-(iii).

Now suppose that $H > 0$ on $M^n$. Assume that there exists a point where the Ricci curvature is positive. Then according to Aubin [3], the manifold $M^n$ admits a metric of positive Ricci curvature. The Bonnet–Myers theorem implies that the fundamental group of $M^n$ is finite. Since we have

$$S \leq a(n, 1, H, 1) < a(n, 2, H, 1),$$

on $M^n$, it follows from [37, Cor. 1] that $M^n$ is homeomorphic to $\mathbb{S}^n$.

Now suppose that $M^n$ is not homeomorphic to $\mathbb{S}^n$. Thus at every point, there exists a tangent direction where the Ricci curvature vanishes. This implies that the left-hand side of inequality (18) is nonpositive, being the right-hand side nonnegative. Hence, equality holds in (18) at every point. Proposition 16(ii) implies that the second fundamental form of $f$ satisfies

$$\alpha_f(X, Y) = \langle A_\xi X, Y \rangle \xi$$

for all $X, Y \in TM$, where $\xi = \mathcal{H}/H$ and the shape operator $A_\xi$ has at most two distinct eigenvalues $\lambda$ and $\mu$ of multiplicities 1 and $n - 1$, respectively. We choose a local orthonormal frame $X_1, \ldots, X_n$ such that

$$A_\xi X_1 = \lambda X_1 \quad \text{and} \quad A_\xi X_i = \mu X_i, \ 2 \leq i \leq n.$$

For each $\eta \in \Gamma(N_f M)$ perpendicular to $\xi$, the Codazzi equation yields

$$\mu(\nabla_{X_1}^\perp \eta, \xi) X_i = \lambda(\nabla_{X_i}^\perp \eta, \xi) X_1, \ 2 \leq i \leq n,$$
and
\[ \mu \langle \nabla_{X_i} \eta, \xi \rangle X_j = \mu \langle \nabla_{X_j} \eta, \xi \rangle X_i, \quad 2 \leq i \neq j \leq n. \]

We claim that \( \mu \) cannot vanish. Indeed, if \( \mu \) vanishes at a point \( x \in M^n \), then Proposition 16(i) gives that \( x \) is a totally geodesic point and this contradicts our assumption on the mean curvature. Hence, we have \( \langle \nabla_{X_i} \eta, \xi \rangle = 0 \) for all \( 1 \leq i \leq n \), and therefore, the first normal bundle is a parallel subbundle of the normal bundle of constant rank one. Thus \( f \) reduces codimension to one and can be regarded as a hypersurface in a totally geodesic \( S^{n+1} \) of \( S^m \).

Recall that the Ricci curvature is nonnegative and at every point, there exists a tangent direction where the Ricci curvature vanishes. Using that the hypersurface \( f \) has a principal curvature of multiplicity \( n-1 \), it follows from the Gauss equation that \( M^n \) has nonnegative sectional curvature. Given that \( M^n \) is not homeomorphic to \( S^n \) and oriented, the main result in [13] yields that \( M^n \) is diffeomorphic to \( S^1 \times S^{n-1} \). It follows from Theorem 6 (iii) that \( f(M^n) \) is isometric to a torus \( T^n_1(r) \) with \( r > 1/\sqrt{n} \).

\[ \square \]

**Proof of Corollary 8.** Observe that
\[ S \leq 2\sqrt{n-1} \leq a(n, 1, H, 1). \tag{22} \]

At first, we claim that if \( M^n \) is not a homology sphere, then \( f(M^n) \) is isometric to a torus \( T^n_1(r) \) for appropriate \( r \). Suppose that \( M^n \) is not a homology sphere. We claim that \( b_1(M^n) > 0 \). Assume to the contrary that \( b_1(M^n) = 0 \), and let \( q \) be an integer such that \( b_q(M^n) > 0 \) with \( 1 < q \leq n/2 \). Theorem 6(ii) implies that \( f \) is minimal with \( S = n \) and this is a contradiction. Hence \( b_1(M^n) > 0 \) and Theorem 1(ii) shows that equality holds in (22) pointwise. In particular, we have \( H > 0 \) on \( M^n \). The claim now follows from Theorem 6(iii).

Now suppose that \( f(M^n) \) is not isometric to any torus \( T^n_1(r) \). Hence, \( M^n \) is a homology sphere. Using (19), it is obvious that (22) is written equivalently as
\[ \| \Phi \|^2 \leq 2\sqrt{n-1} - nH^2 \leq r^2(n, 1, H, 1), \]
where \( r(n, 1, H, 1) \) is the positive root of the polynomial \( P \) given by (20). Hence, we have
\[ P(\| \Phi \|, 1, H, 1) \geq P \left( 2\sqrt{n-1} - nH^2, 1, H, 1 \right) < 0, \]
at points where \( 2\sqrt{n-1} > nH^2 \). It follows from inequality (18) for \( p = 1 \) that the Ricci curvature of \( M^n \) at these points satisfies
\[ \text{Ric} \geq -\frac{n-1}{n} P \left( 2\sqrt{n-1} - nH^2, 1, H, 1 \right) > 0. \]

\[ \square \]
On the other hand, the points where $2\sqrt{n-1} = nH^2$ are totally umbilical points and consequently the Ricci curvature is

$$\text{Ric} = (n-1)(H^2 + 1).$$

It is clear from the above that the Ricci curvature of $M^n$ is bounded from below by a positive constant. The rest of the proof is a consequence of Corollary 7. \hfill \Box

**Proof of Theorem 9.** Since $M^n$ is compact and $f(M^n)$ is contained in an open hemisphere in case where $c = 1$, there exists a point $x_0 \in M^n$ and a unit normal vector $\eta \in N_f M(x_0)$ such that the shape operator $A_\eta(x_0)$ is positive definite.

At first, we observe that $b_q(M^n) = 0$ for every integer $p < q \leq n/2$. Indeed, note that

$$S \leq a(n, p, H, c) \leq a(n, q, H, c),$$
on $M^n$ and $S(x_0) < a(n, q, H(x_0), c)$. Then Theorem 1(ii) implies that $b_q(M^n) = 0$ for every $p < q \leq n/2$.

Now we claim that if $b_p(M^n) > 0$, then $p = n/2$ and $M^n$ has the homotopy type of a CW-complex with cells only in dimensions $0, n/2$, or $n$. Indeed, if $b_p(M^n) > 0$, then it follows from Theorem 1(iii) that $S = a(n, p, H, c)$ on $M^n$ and that every harmonic form $\omega \in \mathcal{H}^p(M^n)$ is parallel. Thus $M^n$ supports a nontrivial parallel $p$-form $\omega$. We distinguish two cases.

**Case c = 1.** Suppose to the contrary that $p < n/2$. We argue on an open neighborhood of $U$ of $x_0$ where the mean curvature is positive. Proposition 17 implies that $f$ is a hypersurface in a totally geodesic sphere $\mathbb{S}^{n+1}$ of $\mathbb{S}^m$ with two distinct principal curvatures $\lambda$ and $\mu$ of multiplicities $p$ and $n - p$, respectively, that satisfy $\lambda \mu + 1 = 0$. Since the shape operator $A_\eta(x_0)$ is positive definite, obviously $A_\xi(x_0)$ has to be either positive, or negative definite. This contradicts the fact that the principal curvatures satisfy $\lambda \mu + 1 = 0$. Thus $p = n/2$. That $M^n$ has the homotopy type of a CW-complex with cells only in dimensions $0, n/2$, or $n$ follows from Theorem 6(i).

**Case c = 0.** Suppose to the contrary that $p < n/2$. Equation (14) yields $\langle B^{[p]} \omega, \omega \rangle = 0$. Hence, equality holds in inequality (18) at every point. Let $U$ be an open neighborhood of $x_0$ where the mean curvature is positive. Hereafter, we argue on $U$. Proposition 16(ii) yields that

$$\alpha_f(X, Y) = \langle A_\xi X, Y \rangle_\xi \text{ for all } X, Y \in TU,$$

where $\xi = \mathcal{H}/H$ and the shape operator $A_\xi$ has at most two distinct eigenvalues $\lambda$ and $\mu$ with multiplicities $p$ and $n - p$, respectively. The Gauss equation is written as

$$R(X, Y)Z = \langle A_\xi Y, Z \rangle A_\xi X - \langle A_\xi X, Z \rangle A_\xi Y,$$

and consequently

$$R(X, Y)\omega = (A_\xi X)^\# \wedge i_{A_\xi Y} \omega - (A_\xi Y)^\# \wedge i_{A_\xi X}\omega.$$
We choose a local orthonormal frame $X_1, \ldots, X_n$ with dual frame $\theta_1, \ldots, \theta_n$ such that

$$A_\xi X_i = \lambda X_i, \; 1 \leq i \leq p \quad \text{and} \quad A_\xi X_j = \mu X_j, \; p + 1 \leq j \leq n.$$ 

Since $\omega$ is parallel, we have $R(X_i, X_j)\omega = 0$. From the above, it follows that

$$\lambda \mu \Psi_{ij} = 0 \quad \text{for} \quad 1 \leq i \leq p, \; p + 1 \leq j \leq n,$$

where

$$\Psi_{ij} = \theta_i \wedge i X_j \omega - \theta_j \wedge i X_i \omega.$$ 

Arguing as in the proof of Proposition 17, we have $\Psi_{ij} \neq 0$ for all $1 \leq i \leq p < j \leq n$. Therefore, $\lambda \mu = 0$. It is clear that the shape operator $A_\xi(x_0)$ has to be either positive, or negative definite, since $A_\eta(x_0)$ is positive definite. This contradicts the fact that the principal curvatures satisfy $\lambda \mu = 0$. Thus $p = n/2$. Since equality holds in inequality (18) at every point, Proposition 16 implies that the second fundamental form of $f$ has at most two distinct principal curvatures both of multiplicity $p$ in each normal direction. That $M^n$ has the homotopy type of a CW-complex with cells only in dimensions $0, n/2, \text{or} n$, follows by using Morse theory, as in the proof of Theorem 6(i).

Hence, we conclude that either $b_i(M^n) = 0$ for all $p \leq i \leq n - p$, or $p = n/2$ and $M^n$ has the homotopy type of a CW-complex with cells only in dimensions $0, n/2, \text{or} n$, and this completes the proof of part (i).

Next we assume that $p = 1$. From the above, we have that $M^n$ is a real homology sphere. Our pinching condition implies that the right-hand side in inequality (18) is nonnegative. Thus the Ricci curvature of $M^n$ is nonnegative. We claim that there is a point where all Ricci curvatures are positive. Arguing by contradiction, we suppose that at every point, there exists a tangent direction where the Ricci curvature vanishes. Hence, equality holds in (18) at every point. Using that the mean curvature is positive at the point $x_0 \in M^n$, it follows from Proposition 16(ii) that

$$\alpha_f(X, Y) = \langle A_\xi X, Y \rangle \xi \quad \text{for all} \quad X, Y \in T_{x_0}M,$$

where $\xi \in N_f(M)(x_0)$ is a unit normal vector and the shape operator $A_\xi$ has an eigenvalue with multiplicity $n - 1$. Given that the shape operator $A_\eta(x_0)$ is positive definite, it is obvious that $A_\xi(x_0)$ is either positive or negative definite. In either case, it follows from the Gauss equation that the Ricci curvature at $x_0$ is positive for each tangent direction, and this is a contradiction.

Since the Ricci curvature is nonnegative and there exists a point where all Ricci curvatures are positive, it follows from Aubin [3] that $M^n$ admits a metric with positive Ricci curvature. Then the Bonnet–Myers theorem implies that the fundamental group of $M^n$ is finite. In analogy with the proof of Corollary 5, we can prove that $M^n$ is diffeomorphic to a spherical space if $n = 3$, and its universal cover is homeomorphic to $S^4$ if $n = 4$. 

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Suppose now that \( H > 0 \) on \( M^n \), \( p < (n - 1)/2 \) and \( c = 0 \). Then, we have

\[
S \leq a(n, p, H, 0) < a(n, p + 1, H, 0).
\]

From [37, Th. 1], it follows that \( H_i(M^n; \mathbb{Z}) = 0 \) for each \( p < i < n - p \). Since \( H_{n-p-1}(M^n; \mathbb{Z}) = 0 \), the universal coefficient theorem for cohomology (see [34, Cor. 4, p. 244]) implies that \( H^{n-p}(M^n; \mathbb{Z}) \) has no torsion and neither does \( H_p(M^n; \mathbb{Z}) \) by Poincaré duality. Since \( b_p(M^n) = 0 \), we obtain \( H_p(M^n; \mathbb{Z}) = 0 \).

\section*{5 Integral Bound for the Bochner Operator}

For the proof of Theorem 11, we need to recall some well-known facts on the total curvature and how Morse theory provides restrictions on the Betti numbers. Let \( f \) be a Morse function for all \( u \) with \( \mu_i(u) \) the number of critical points of \( h_u \) of index \( i \) for each \( u \in S^{n+k-1} \setminus E \) and set \( \mu_i(u) = 0 \) for every \( u \in E \). Following Kuiper [23], we define the total curvature of index \( i \) of \( f \) by

\[ S \leq a(n, p, H, -1) \leq a(n, i, H, -1). \]

Repeating the above argument, we obtain \( b_i(M^n) = 0 \) for every \( p < i \leq n/2 \).
\[ \tau_i(f) = \frac{1}{\text{Vol}(S^{n+k-1})} \int_{S^{n+k-1}} \mu_i(u) dS, \]

where \( dS \) denotes the volume element of the sphere \( S^{n+k-1} \). From the weak Morse inequalities (cf. [28]), we have

\[ \mu_i(u) \geq b_i(M^n; \mathbb{F}) \text{ for all } u \in S^{n+k-1} \setminus E. \]

By integrating over \( S^{n+k-1} \), we obtain

\[ \tau_i(f) \geq b_i(M^n; \mathbb{F}). \quad (23) \]

There is a natural volume element \( d\Sigma \) on the unit normal bundle \( UN_f \). In fact, if \( dV \) is a \((k - 1)\)-form on \( UN_f \) such that its restriction to a fiber of the unit normal bundle at \((x, \eta)\) is the volume element of the unit \((k - 1)\)-sphere of the normal space of \( f \) at \( x \), then \( d\Sigma = dM \wedge dV \), where \( dM \) is the volume element of \( M^n \). Shiohama and Xu [31, p. 381] proved that

\[ \int_{U^i N_f} |\det A_\eta| d\Sigma = \int_{S^{n+k-1}} \mu_i(u) dS, \quad (24) \]

where \( U^i N_f \) is the subset of the unit normal bundle of \( f \) defined by

\[ U^i N_f = \{(x, \eta) \in UN_f : \text{Index } A_\eta = i \}, \quad 0 \leq i \leq n. \]

**Proof of Theorem 11.** Let \( f : M^n \to \mathbb{R}^{n+k} \) be an isometric immersion with second fundamental form \( \alpha_f \) and shape operator \( A_\eta \) with respect to \( \eta \), where \((x, \eta) \in UN_f \). Using (15), (17) and Proposition 14, we have

\[ |\varrho_x - F_{n,p}(H, \|\Phi\|)|^{n/2} \geq (\varepsilon(n, k, p))^{n/2} \int_{A_\eta(\alpha_f(x))} |\det A_\eta| dV_\eta, \]

for every point \( x \in M^n \). Integrating over \( M^n \) and using (24), we obtain

\[ \int_{M^n} |\varrho_x - F_{n,p}(H, \|\Phi\|)|^{n/2} dM \geq (\varepsilon(n, k, p))^{n/2} \text{Vol}(S^{n+k-1}) \sum_{i=p+1}^{n-p-1} \tau_i(f). \]

Using (23), it follows that

\[ \int_{M^n} |\varrho_x - F_{n,p}(H, \|\Phi\|)|^{n/2} dM \geq c(n, k, p) \sum_{i=p+1}^{n-p-1} \tau_i(f) \geq c(n, k, p) \sum_{i=p+1}^{n-p-1} b_i(M; \mathbb{F}), \quad (25) \]
where \( c(n, k, p) = (\epsilon(n, k, p))^{n/2} \text{Vol}(\mathbb{S}^{n+k-1}) \).

Now, suppose that
\[
\int_{M^n} \left| \varrho_p - F_{n,p}(H, \|\Phi\|) \right|^{n/2} dM < c(n, p, k).
\]

It follows from (25) that
\[
\sum_{i=p+1}^{n-p-1} \tau_i(f) < 1.
\]

Thus, there exists \( u \in \mathbb{S}^{n+k-1} \) such that the height function \( h_u \) is a Morse function whose number of critical points of index \( i \) satisfies \( \mu_i(u) = 0 \) for every \( p + 1 \leq i \leq n - p - 1 \). By Morse theory, the manifold \( M^n \) has the homotopy type of a CW-complex with no cells of dimension \( p + 1 \leq i \leq n - p - 1 \).

If \( p = 1 \), there are no 2-cells and thus by the cellular approximation theorem the inclusion of the 1-skeleton \( X^{(1)} \hookrightarrow M^n \) induces isomorphism between the fundamental groups. Therefore, \( \pi_1(M^n) \) is a free group on \( b_1(M^n; \mathbb{Z}) \) elements and \( H_1(M^n; \mathbb{Z}) \) is a free Abelian group on \( b_1(M^n; \mathbb{Z}) \) generators. If \( \pi_1(M^n) \) is finite, then \( \pi_1(M^n) = 0 \) and hence \( H_1(M^n; \mathbb{Z}) = 0 \). From Poincaré duality and the universal coefficient theorem, it follows that \( H_{n-1}(M^n; \mathbb{Z}) = 0 \). Thus, \( M^n \) is a simply connected homology sphere and hence a homotopy sphere. By the generalized Poincaré conjecture (Smale \( n \geq 5 \), Freedman \( n = 4 \)) \( M^n \) is homeomorphic to \( \mathbb{S}^n \).

\[\Box\]

6 Concluding Remarks

(i) Since homology with real coefficients encodes less topological information that homology with integers ones, it should be interesting to know whether our homology vanishing results hold for homology with integers coefficients.

(ii) It is worth noticing that the sharp estimate of the Bochner operator in terms of the second fundamental form of a submanifold (see Proposition 16) can be used to obtain homology vanishing results for submanifolds in different context. For instance, one can strengthen the results in [4], concerning minimal submanifolds in balls with free boundary, by dropping the assumption therein on the flatness of the normal bundle.

(iii) DeTurck and Ziller [12] obtained minimal isometric embeddings of spherical space forms into spheres. None of these minimal submanifolds satisfies the condition \( S \leq n \). In view of Theorem 4, this observation raises the question whether any minimal isometric immersion of spherical space forms into spheres is totally geodesic provided that \( S \leq n \).
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