ON THE EXTINCTION PROFILE OF SOLUTIONS TO
FAST-DIFFUSION

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Abstract. We study the extinction behavior of solutions to the fast diffusion
equation \( u_t = \Delta u^m \) on \( \mathbb{R}^N \times (0, T) \), in the range of exponents \( m \in (0, \frac{N-2}{N}) \),
\( N > 2 \). We show that if the initial data \( u_0 \) is trapped in between two Barenblatt solutions vanishing at time \( T \), then the vanishing behaviour of \( u \) at \( T \) is
given by a Barenblatt solution. We also give an example showing that for such
a behavior the bound from above by a Barenblatt solution \( B \) (vanishing at \( T \))
is crucial: we construct a class of solutions \( u \) with initial data \( u_0 = B (1+o(1)) \),
near \( |x| >> 1 \), which live longer than \( B \) and change behaviour at \( T \). The be-
havior of such solutions is governed by \( B(\cdot, t) \) up to \( T \), while for \( t > T \) the
solutions become integrable and exhibit a different vanishing profile. For the
Yamabe flow \( (m = \frac{N-2}{N+2}) \) the above means that these solutions \( u \) develop a
singularity at time \( T \), when the Barenblatt solution disappears, and at \( t > T \)
they immediately smoothen up and exhibit the vanishing profile of a sphere.

In the appendix we show how we remove the assumption on the bound on
\( u_0 \) by a Barenblatt from below.

1. Introduction

We consider the Cauchy problem for the fast diffusion equation

\[
\begin{aligned}
 \frac{\partial u}{\partial t} &= \Delta u^m & \text{in } \mathbb{R}^N \times (0, T) \\
u(x, 0) &= u_0(x) & x \in \mathbb{R}^N
\end{aligned}
\]

in the range of exponents \( 0 < m < (\frac{N-2}{N}) \), in dimensions \( N \geq 3 \). The initial
data \( u_0 \) is assumed to be non-negative and locally integrable.

Equation (1.1) arises as a model of various diffusion processes. It is found in
plasma physics and in particular as the Okuda-Dawson low, when \( m = 1/2 \). Also,
King studies (1.1) in a model of diffusion of impurities in silicon [5].

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When \( m = (N - 2)/(N + 2) \) equation (1.1) is equivalent to the evolution of the conformally flat metric \( g_{ij} = u^{4/N+2} \, dx_i dx_j \) by the Yamabe flow

\[
\frac{\partial g}{\partial t} = -R \, g
\]

where \( R \) denotes the scalar curvature with respect to the metric \( g \). The equivalence easily follows from the observation that the conformal metric \( g_{ij} = u^{4/N+2} \, dx_i dx_j \) has scalar curvature

\[
R = -\frac{\Delta u^{N+2}}{u}.
\]

For an introduction to the Yamabe flow see in [11].

Our goal in this paper is to study the vanishing behavior of solutions to the Cauchy problem (1.1), under the assumption that the initial data \( u_0 \) satisfies the growth condition

\[ (1.2) \quad u_0(x) = \left( \frac{C}{|x|^2} \right)^{\frac{1}{1-m}} (1 + o(1)), \quad \text{as} \quad |x| \to \infty. \]

The self-similar Barenblatt solutions of equation (1.1) given explicitly by

\[ (1.3) \quad B_k(x, t) = \left( \frac{C^* (T - t)}{k(T - t)^{\beta} + |x|^2} \right)^{\frac{1}{1-m}} \]

with

\[ (1.4) \quad \beta = \frac{N}{N - 2 - Nm} \quad \text{and} \quad \gamma = -\frac{\beta}{N} \]

satisfy the growth condition (1.2). The constant \( C^* \) depends only on \( m \) and \( N \) and is given explicitly by

\[ (1.5) \quad C^* = \frac{2m (N - 2 - m N)}{1 - m}. \]

We will assume in the first part of this paper that the initial condition \( u_0 \) is trapped in between two Barenblatt solutions, i.e.,

\[ (1.6) \quad \left( \frac{C^* T}{k_1 + |x|^2} \right)^{\frac{1}{1-m}} \leq u_0(x) \leq \left( \frac{C^* T}{k_2 + |x|^2} \right)^{\frac{1}{1-m}} \]

for some constants \( k_1 > k_2 > 0 \). As a direct consequence of the maximum principle we then have

\[ (1.7) \quad B_{k_1}(x, t) \leq u(x, t) \leq B_{k_2}(x, t) \quad \text{for} \quad 0 < t < T. \]

In particular, \( u \) vanishes at time \( T \). We will show in the first part of this paper that the vanishing profile of \( u \) is given by a Barenblatt solution.
Consider the rescaled function
\begin{equation}
\tilde{u}(x, \tau) = (T - t)^{-\beta} u(x(T - t)^{\gamma}, t), \quad \tau = -\log(T - t)
\end{equation}
with \(\beta\) and \(\gamma\) given by \((1.4)\). It follows by direct computation, that \(u\) satisfies the equation
\begin{equation}
\tilde{u}_{\tau} = \Delta \tilde{u}^m + |\gamma| \text{div}(x \cdot \tilde{u})
\end{equation}
and due to condition \((1.6)\), the inequality
\begin{equation}
\left( \frac{C^*}{|x|^2 + k_1} \right)^{1-m} \leq \tilde{u}(x, \tau) \leq \left( \frac{C^*}{|x|^2 + k_2} \right)^{1-m}
\end{equation}
holds, for \((x, \tau) \in \mathbb{R}^N \times [-\log T, \infty)\). We denote by
\begin{equation}
\tilde{B}_k(x) = \left( \frac{C^*}{k + |x|^2} \right)^{1-m}
\end{equation}
the rescaled Barenblatt solution.

Our convergence results are described in the following two Theorems. The first result is concerned with the range of exponents \(\frac{N-4}{N-2} < m < \frac{N-2}{N}\), for which the difference \(u - B_k\), with \(B_k\) a Barenblatt solution, is integrable, namely we have \(\int_{\mathbb{R}^N} (u - B_k)(x, t) \, dx < \infty\). Notice that this range of exponents includes the Yamabe flow, \(m = (N - 2)/(N + 2)\), when \(N < 6\).

**Theorem 1.1.** Let \(u\) solve the equation \((1.1)\) for \(\frac{N-4}{N-2} < m < \frac{N-2}{N}\), with initial value \(u_0\) satisfying \((1.6)\), for some constants \(k_1, k_2\). Then, the rescaled function \(\tilde{u}\) given by \((1.8)\) converges, as \(\tau \to \infty\), uniformly on \(\mathbb{R}^N\), and also in \(L^1(\mathbb{R}^N)\), to the rescaled Barenblatt solution \(\tilde{B}_{k_0}\) given by \((1.11)\), for some \(k_0 > 0\). The constant \(k_0\) is uniquely determined by the equality
\[ \int_{\mathbb{R}^N} u_0 \, dx = \int_{\mathbb{R}^N} B_{k_0} \, dx. \]

The second result deals with the range of exponents \(0 < m \leq \frac{N-4}{N-2}\), for which the difference \(u - B_k\), with \(B_k\) a Barenblatt solution, is non-integrable, namely \(\int_{\mathbb{R}^N} (u - B_k)(x, t) \, dx = \infty\).

**Theorem 1.2.** Let \(u\) solve the equation \((1.1)\), for \(0 < m \leq \frac{N-4}{N-2}\), \(N > 4\), with initial value \(u_0\) satisfying \((1.6)\), for some constants \(k_1, k_2\). Assume, in addition that
\begin{equation}
u_0 = B_{k_0} + f
\end{equation}
for some $k_0 > 0$, where $B_{k_0}$ is a Barenblatt solution and $f$ is in $L^1(\mathbb{R}^N)$. Then, the rescaled function $\tilde{u}$ given by (1.8) converges, as $\tau \to \infty$, uniformly on $\mathbb{R}^N$, to the Barenblatt solution $\tilde{B}_{k_0}$.

Remark 1.3. The condition $0 < m \leq \frac{N-4}{N-2}$ in Theorem 1.2 implies that for any two Barenblatt solutions $B_k$ and $B_{k'}$, $B_k - B_{k'} \in L^1(\mathbb{R}^N)$ if and only if $k = k'$.

One may ask whether condition (1.2) is necessary for Theorems 1.1 and 1.2 to hold true. We will show in section 5 that this is indeed the case. We will present an example of a class of initial conditions $u_0$ which satisfy the growth condition

$$u_0(x) = \left( \frac{C^* T}{|x|^2} \right) \left( 1 + o(1) \right), \quad \text{as } |x| \to \infty$$

with $C^*$ given by (1.5), for which the solution $u$ of (1.1) with initial data $u_0$ satisfies the following Theorem.

**Theorem 1.4.** There exists a class of solutions $u$ of the Cauchy problem (1.1) with initial data $u_0$ satisfying (1.13) and with the following properties:

(i) The vanishing time $T^*$ of $u$ satisfies $T^* > T$.

(ii) The solution $u$ satisfies as $|x| \to \infty$, the growth conditions

$$u(x,t) \geq \left( \frac{C^* (T - t)}{1 + |x|^2} \right)^{\frac{1}{1-m}}, \quad \text{on } 0 < t < T$$

and

$$u(x,t) \leq C(t) \frac{1}{|x|^{N-2}}, \quad \text{on } T < t < T^*.$$ 

In particular, $u$ becomes integrable on $t > T$.

(iii) The vanishing behavior of $u$ is given by one of the self-similar solutions $\Theta(x,t)$ (see section 5 for the explanation of $\theta(x,t)$).

The vanishing behavior of the solution $u$ in this case is described in the results of Galaktionov and Peletier [2], and del Pino and Sáez [11] (in the case $m = \frac{N-2}{N+2}$), also formally shown by King [7].

The case of a special interest is $m = \frac{N-2}{N+2}$, when the equation for $u$ is equivalent to the Yamabe flow of a corresponding conformally flat metric. The previous Theorem gives the following corollary in the Yamabe case.

**Corollary 1.5.** If $u$ is a solution of the Cauchy problem (1.1) with initial data satisfying (1.13). Then the vanishing time of $u$ is $T^* > T$, (ii) in Theorem 1.4.
holds and
\[(T^* - t)^{-\frac{m}{2}} u(x, t) \to \left( \frac{C_1}{C_2 + |x - \bar{x}|^2} \right)^{\frac{N-4}{2}},\]
as \(t \to T^*,\) where \(\bar{x} \in \mathbb{R}^N\) and \(C_1, C_2 > 0.\)

Geometrically speaking, Yamabe flow starting at \(u_0\) (described above) develops a singularity at \(T < T^*\) at which the Barenblatt solution (cylinder) pinches off and immediately at \(t > T\) the solution becomes integrable. Due to the results of Del Pino and Saez it smoothens up at \(t > T\) and exhibits the behaviour of a compact sphere as \(t \to T^*\).

The next section will be devoted on preliminary estimates for solutions \(u\) of (1.1) with initial data in \(L^1_{loc}.\) The proof of Theorem 1.1 will be given in section 3. It will follow from the strong \(L^1\)-contraction principle, Lemma 4.1, which holds for the difference of the rescaled solutions \(\hat{u} - \hat{B}_k,\) for any Barenblatt solution \(B_k.\) This method, based upon the ideas of Osher and Ralston [9], was previously used by S-Y Hsu in [3]. Since the difference \(\hat{u} - \hat{B}_k \notin L^1(\mathbb{R}^N)\) in the range of exponents \(0 < m < \frac{N-4}{N-2},\) for the proof of Theorem 1.2 we will need to weight the \(L^1\)-norms with an appropriate power \(\tilde{B}_k^a.\) The proof in this other case is more involved and will be given in section 4. The last section will be devoted to the construction of the examples described in Theorem 1.4.

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2. Preliminary Estimates

Our goal in this section is to establish the \(L^1\)-contracting for solutions of (1.1) and (1.9) and some other preliminary results which can also be of independent interest. We begin by showing the following integrability lemma for the difference of any two solutions of (1.1).

**Lemma 2.1.** Assume that \(u, v\) are two solutions of (1.1) on \(\mathbb{R}^N \times (0, T).\) If \(f = u_0 - v_0 \in L^1(\mathbb{R}^N)\) and \(f\) is compactly supported, then \(u(\cdot, t) - v(\cdot, t) \in L^1(\mathbb{R}^N)\) for all \(t \in [0, T).\)
Proof. Our proof is based on the well known technique of Herrero and Pierre [4].

We introduce the potential function

\[ w(x, t) = \int_0^t |(u^m - v^m)(x, s)| \, ds \]  

which satisfies the inequality

\[ \Delta w \geq -|f|, \quad \text{on } \mathbb{R}^N. \]

Indeed, by Kato’s inequality [6], we have

\[ \Delta |u^m - v^m| \geq \text{sign}(u - v) \Delta (u^m - v^m) \]

so from equation (1.1), we obtain

\[ \frac{\partial}{\partial t} |u - v| \leq \Delta |u^m - v^m|. \]

Integrating the previous inequality in time, and using that \(|f| = |u_0 - v_0|\), we obtain (2.2).

Let

\[ Z(x) = \frac{1}{N(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2}} |f(y)| \, dy \]

denote the Newtonian potential of \(|f|\), so that from (2.2) we have

\[ \Delta (w - Z) \geq 0. \]

Also, since \(|f|\) is integrable and compactly supported, there exists a constant \(C < \infty\) for which

\[ Z(x) \leq \frac{C}{|x|^{N-2}}, \quad \forall x >> 1, \]

where the range of \(x\) for which the estimate holds depends on the support of \(f\).

The mean value property implies that

\[ w(x) \leq Z(x) + \frac{1}{\omega_N \rho^N} \int_{B_\rho(x)} (w(y) - Z(y)) \, dy \]

\[ \leq Z(x) + \frac{1}{\omega_N \rho^N} \int_{B_\rho(x)} w(y) \, dy \]

for all \(x \in \mathbb{R}^N, \rho > 0.\)

We next claim that

\[ \lim_{\rho \to \infty} \frac{1}{\rho^N} \int_{B_\rho(x)} w(y) \, dy = 0. \]
Indeed, by Lemma 3.1 in [4] we have
\[
\int_{B_r(x)} |u - v|(y, t) \, dy \leq C \left( \| u_0 - v_0 \|_{L^1(\mathbb{R}^N)} + \rho^N \left( \frac{t}{\rho^2} \right)^{1/m} \right)
\]
which yields
\[
\int_{B_r(x)} w(y, t) \, dx = \int_0^t \int_{B_r(x)} |u - v|^m(y, s) \, dy \, ds \\
\leq C \int_0^t \rho^{N(1-m)} \left( \int_{B_r(x)} |u - v| \, dy \right)^m \, ds \\
\leq C \rho^{N(1-m)} \int_0^t \left( \| f \|_{L^1(\mathbb{R}^N)} + \rho^N \left( \frac{s}{\rho^2} \right)^{1/m} \right)^m \, ds \\
\leq C(T, \| f \|) \rho^{N-2m/(1-m)}. \tag{2.6}
\]
Combining (2.4), (2.5), (2.6) and letting \( \rho \to \infty \), we conclude the estimate
\[
(2.7) \quad w(x) \leq Z(x) \leq \frac{C}{|x|^{N-2}}.
\]
We will use (2.7) along the lines of the proof of Theorem 2.3 in [4] to bound \( \|(u - v)(\cdot, t)\|_{L^1(\mathbb{R}^N)} \). For \( 0 \leq \eta_R \leq 1 \), let \( \eta_R \in C_0^\infty(\mathbb{R}^N) \) be a test function such that \( \eta_R = 1 \) for \( |x| \leq R \) and \( \eta_R = 0 \) for \( |x| \geq 2R \). Then \( |\Delta \eta_R| \leq C/R^2 \) and \( |\nabla \eta_R| \leq C/R \). Using equation (1.1), estimate (2.7) and the integrability of \( f \), we get
\[
\int |u - v|(\cdot, t) \eta_R \, dx \leq \int_{\mathbb{R}^N} |f| \, dx + \int_0^t \int_{B_{2R} \setminus B_R} |u^m - v^m| \Delta \eta_R \, dx \, ds \\
\leq \| f \|_{L^1(\mathbb{R}^N)} + \frac{C}{R^2} \int_{B_{2R} \setminus B_R} w(x, t) \, dx \\
\leq C(\| f \|_{L^1(\mathbb{R}^N)}) + \bar{C}
\]
where \( \bar{C} \) can be taken to be independent of \( R \) because of (2.7). Letting \( R \to \infty \) in the previous estimate gives that
\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^N} |u - v|(x, t) \, dx \leq C(\| f \|_{L^1(\mathbb{R}^N)}) + \bar{C} < \infty
\]
finishing the proof of the lemma.

As a consequence of the previous Lemma, we will now establish the following \( L^1 \) contraction principle for the solutions to (1.1) that are bounded from below by a Barenblatt solution \( B \).
Corollary 2.2. Let $u, v$ be two solutions of (1.1) with initial values $u_0, v_0$ respectively and that $f = u_0 - v_0 \in L^1(\mathbb{R}^N)$. Assume in addition that $u, v \geq B$, for some Barenblatt solution $B$ given by (1.3). Then,

\[ \int_{\mathbb{R}^N} |u(\cdot, t) - v(\cdot, t)| \, dx \leq \int_{\mathbb{R}^N} |u_0 - v_0| \, dx, \quad \forall t \in [0, T). \]

Proof. Let $\eta_R$ be a cut off function as in the proof of Lemma 2.1, with the support contained in $B_{2R}$. Then, as before, we have

\[ \frac{\partial}{\partial t} |u(x, t) - v(x, t)| \leq \Delta |u^m - v^m|. \]

If we multiply the above inequality by $\eta_R$ and integrate over $\mathbb{R}^N$, since $|\Delta \eta_R| \leq C R^2$, we get

\[ \frac{d}{dt} \int_{\mathbb{R}^N} (u-v)(x,t) \eta_R \, dx \leq \frac{C}{R^2} \int_{B_{2R}\setminus B_R} a(x,t) |(u-v)(x,t)| \]

with

\[ a(x,t) = \int_0^1 \frac{d\theta}{(\theta u + (1-\theta)v)^{1-m}} \leq CR^2, \quad \text{in } B_{2R} \]

(since $(\theta u + (1-\theta)v)^{1-m} \geq B^{1-m} = C_1(|x|^2 + C_2)$). Hence, fixing $t \in [0, T)$, we obtain the estimate

\[ \frac{d}{dt} \int_{\mathbb{R}^N} |u-v|(x,t) \eta_R \, dx \leq C \int_{B_{2R}\setminus B_R} |u-v|(x,t) \, dx. \]

Assume that $u_0, v_0$ are compactly supported, so that $f = u_0 - v_0$ is compactly supported as well. Then, the right hand side of the above inequality converges to zero as $R \to \infty$, due to Lemma 2.1. This gives

\[ \frac{d}{dt} \int_{\mathbb{R}^N} |u-v|(x,t) \, dx \leq 0 \]

which implies (6.2).

To remove the assumption that $u_0, v_0$ are compactly supported, we use a standard approximation argument. For any $k > 1$, we set $u_k^0 = u_0 \chi_{B_k(0)}$, $v_k^0 = v_0 \chi_{B_k(0)}$. Let $u^k, v^k$ be the solutions of (1.1) on $\mathbb{R}^N \times (0, \infty)$ with initial values $u_k^0, v_k^0$ respectively. By standard arguments $u^k \to u$ and $v^k \to v$ uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$. Also, by the previous argument

\[ \int_{\mathbb{R}^N} |u^k(\cdot, t) - v^k(\cdot, t)| \, dx \leq \int_{\mathbb{R}^N} |u_k^0 - v_k^0| \, dx \leq \int_{\mathbb{R}^N} |u_0 - v_0| \, dx \]

for all $t > 0$ and for all $k$. Letting $k \to \infty$ we readily obtain (6.2). \qed

As an immediate corollary of Lemma 2.1 we have the following result concerning the rescaled solutions $\tilde{u}, \tilde{v}$. 
Corollary 2.3. Let \( u \) and \( v \) be as above. If \( u_0 - v_0 \in L^1(\mathbb{R}^N) \), then for every \( \tau > -\log T \) there is \( C(\tau) \) such that

\[
\int_{\mathbb{R}^N} |\tilde{u}(x, \tau) - \tilde{v}(x, \tau)| \, dx \leq C(\tau).
\]

Let \( u, v \) be two solutions of (1.1) satisfying (1.6). We observe that the difference \( q = \tilde{u} - \tilde{v} \) satisfies the equation

\[
q_\tau = \Delta(a(x, \tau) q) + |\gamma| \text{div}(x \cdot q(x, \tau))
\]

on \( \mathbb{R}^N \times [-\log T, \infty) \), with

\[
a(x, \tau) = \int_0^1 \frac{m}{(\theta \tilde{u} + (1 - \theta) \tilde{v})^{1-m}} \, d\theta.
\]

Since both \( \tilde{u} \) and \( \tilde{v} \) satisfy (1.10), for some constants \( k_1, k_2 \), it is clear that \( a(x, \tau) \) is smooth and satisfies the growth estimate

\[
m (k_2 + |x|^2) \leq a(x, \tau) \leq m (k_1 + |x|^2).
\]

Hence, (2.9) is uniformly parabolic on any compact subset of \( \mathbb{R}^N \times (0, \infty) \).

Let \( F(x, t) \) be a solution of

\[
F_1 = \Delta(a_1(x, t) F)
\]

with

\[
a_1(x, t) = \int_0^1 \frac{m}{(\theta u(x, t) + (1 - \theta) v(x, t))^{1-m}} \, d\theta.
\]

A direct computation shows that \( \tilde{F}(x, \tau) = F(x, (T - t)^\gamma, t) \), \( \tau = -\log(T - t) \), is a solution of equation

\[
\tilde{F}_\tau = \Delta(a(x, \tau) \tilde{F}) + |\gamma| \text{div}(\tilde{F} \cdot x)
\]

where \( a(x, \tau) \) is given by (2.10). Similarly as before we have the following result.

Corollary 2.4. Let \( F(x, t) \) be a solution of (2.12). If \( F(x, 0) \in L^1(\mathbb{R}^N) \), then \( F(x, t) \in L^1(\mathbb{R}^N) \). Moreover, \( \tilde{F}(x, \tau) \in L^1(\mathbb{R}^N) \) and for every \( \tau > -\log T \) there is a \( C(\tau) \) such that

\[
\|F(\cdot, \tau)\|_{L^1(\mathbb{R}^N)} \leq C(\tau).
\]
This section is devoted to the proof of Theorem 1.1 which deals with solutions of equation (1.1) in the range of exponents $\frac{N-4}{N-2} < m < \frac{N-2}{N}$. In this case the difference of two solutions $u, v$ satisfying (1.7) is integrable. We begin this section with the following strong contraction principle, which constitutes the main step in the proof of Theorem 1.1. Its proof as well as the rest of the argument is very similar to the proof of Theorem 2 in [5]. To facilitate future references we will sketch the proof of the strong contraction principle.

**Lemma 3.1.** Let $u, v$ be two solutions of (1.1), for $m \in (0, \frac{N-2}{N})$, with initial values $u_0, v_0$, satisfying (1.6). If

$$\min\{\|\tilde{u}_0 - \tilde{v}_0\|_{L^\infty}, \|\tilde{v}_0 - \tilde{u}_0\|_{L^\infty}\} > 0,$$

then

$$\|\tilde{u}(\cdot, \tau)\|_{L^1(\mathbb{R}^N)} < \|\tilde{u}_0\|_{L^1(\mathbb{R}^N)}, \quad \tau \geq -\log T.$$

**Proof.** Notice that by the comparison principle, $\tilde{u}(x, \tau), \tilde{v}(x, \tau)$ satisfy (1.10). The proof is almost the same as that of Lemma 2.1 by S-Y Hsu in [5], using the results in [9] and [12].

Set $q = \tilde{u} - \tilde{v}$ and observe, as above, that $q$ satisfies equation (2.9). Fix $R > 0$. By the standard parabolic theory, there exist solutions $q^+_R, q^-_R$ of (2.9) in $Q_R = B_R \times (-\log T, \infty)$, with initial values $q(\cdot, -\log T)_+, q(\cdot, -\log T)_-$ and boundary values $q_+, q_-$ on $\partial B_R \times (0, \infty)$, respectively. Notice that $q^+_R - q^-_R$ is a solution of (2.9) in $Q_R$, with initial value $q(\cdot, -\log T)$ and boundary values $q_+ - q_-$. By the maximum principle, $q = q^+_R - q^-_R$ on $Q_R$. Similarly there are solutions $\bar{q}^+_R$ and $\bar{q}^-_R$ of (2.9) in $Q_R$ with initial values $q^+_R, q^-_R$, and zero lateral boundary value. By the maximum principle, $0 \leq q^+_R \leq q^+_R$ and $0 \leq \bar{q}^+_R \leq \bar{q}^-_R$. Furthermore let $\bar{q}_R$ be the solution of (2.9) with initial value and lateral boundary value $\bar{B}_{k_2} - \bar{B}_{k_1}$. By the maximum principle, we have

$$0 \leq q^+_R, q^-_R \leq \bar{q}_R.$$

Let $\eta \in C_c^\infty(\mathbb{R}^2)$ be a cut-off function such that $\eta(x) = 1$ on $|x| \leq 1/2$, $\eta(x) = 0$ for all $|x| \geq 1$ and $0 \leq \eta \leq 1$. Denote by $\eta_R = \eta(x/R)$. The same computation as
in the proof of Lemma 2.1 in [1] gives

\[
\int_{\mathbb{R}^N} |q(x, \tau)| \eta_R \, dx - \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}| \eta_R \, dx = \int_{\mathbb{R}^N} a(x, \tau') q_+^R(x, \tau') \Delta \eta_R - |\gamma| q_+^R(x, \tau') x \cdot \nabla \eta_R \, dx \, d\tau' \\
+ \int_{\mathbb{R}^N} a(x, \tau') q_-^R(x, \tau') \Delta \eta_R - |\gamma| q_-^R(x, \tau') x \cdot \nabla \eta_R \, dx \, d\tau'
\]

\[= -2 \int_{\mathbb{R}^N} \min\{q_+^R(x, \tau'), q_-^R(x, \tau')\} \eta_R \, dx.
\]

The families of solutions \( q_+^R(x, \tau) \) and \( q_-^R(x, \tau) \) are monotone increasing in \( R \) and uniformly bounded above, which implies that

\[
\bar{q}_1 = \lim_{R \to \infty} q_+^R
\]

and

\[
\bar{q}_2 = \lim_{R \to \infty} q_-^R
\]

exist and are both solutions of (2.14) on \( \mathbb{R}^N \times (\log T, \infty) \). This implies

\[
\int_{\mathbb{R}^N} |q(x, \tau)| \eta_R \, dx - \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}| \eta_R \, dx \leq \frac{C}{R^2} \int_{-\log T}^{\log T} \int_{R/2 \leq|x| \leq R} a(x, \tau') q^R(x, \tau') \, dxd\tau' \\
+ C \int_{-\log T}^{\log T} \int_{R/2 \leq|x| \leq R} \q_+^R(x, \tau') \, dxd\tau' \leq 2 \int_{B_{R_0}} \min\{q_+^R(x, \tau), q_-^R(x, \tau)\} \eta_R \, dx.
\]

By the same computation as in [1], after letting \( R \to \infty \), we get

\[
\int_{\mathbb{R}^N} |q(x, \tau)| \, dx - \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}| \, dx \leq -2 \int_{B_{R_0}} \min\{\bar{q}_1(x, \tau), \bar{q}_2(x, \tau)\} \, dx
\]

for all \( R_0 > 0 \) and \( \tau > -\log T \). Since \( \bar{q}_1 \geq q_+^{2R_0} \) and \( \bar{q}_2 \geq q_-^{2R_0} \), we obtain

\[
\int_{\mathbb{R}^N} |q(x, \tau)| \, dx - \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}| \, dx \leq -2 \int_{B_{R_0}} \min\{q_+^{2R_0}(x, \tau), q_-^{2R_0}(x, \tau)\} \, dx.
\]

Since \( q_+^{2R_0} \) and \( q_-^{2R_0} \) are the solutions of (2.13) in \( Q_{2R_0} \) with zero boundary value and initial values \( q_+(\cdot, -\log T), q_-(\cdot, -\log T) \), respectively, by the Green’s function
representation for solutions, for any $\tau > -\log T$, there exists a constant $c(\tau)$ such that

$$\min_{x \in B_{2R_0}} q_x^{2R_0} \geq c(\tau) > 0 \quad \text{and} \quad \min_{x \in B_{2R_0}} q_x^{2R_0} \geq c(\tau) > 0.$$ 

That finishes the proof of Lemma 4.1. \hfill \Box

The rest of the proof relies on the following result of Osher and Ralston (9).

**Lemma 3.2** (Lemma 1 in [9]). Suppose that $\tilde{u}(\cdot, \tau_i) \to \bar{u}$ in $L^1(\mathbb{R}^N)$, as $i \to \infty$, for some sequence $\tau_i \to \infty$. Let $\tilde{B}_k$ be any stationary solution of (1.9). If $\tilde{v}$ is the solution of (1.9) in $\mathbb{R}^N \times [0, \infty)$ with initial value $\tilde{v}(x, 0) = \bar{u}(x)$, then

$$\|\tilde{v}(\cdot, \tau) - \tilde{B}_k\|_{L^1(\mathbb{R}^N)} = \|\bar{u} - \tilde{B}_k\|_{L^1(\mathbb{R}^N)}, \quad \forall \tau > 0, \ k > 0.$$ 

**Proof of Theorem 1.1** We claim that in this integrable case there is a unique $k_0$ so that

$$\int_{\mathbb{R}^N} (u_0 - B_{k_0}) \, dx = 0.$$ 

To prove that, let

$$f(k) = \int_{\mathbb{R}^N} (u_0 - B_k) \, dx$$

and observe that $f(k)$ is a continuous, monotone increasing function with $f(k_1) \geq 0$ and $f(k_2) \leq 0$ due to (1.9). Therefore, by the intermediate value theorem there exists a unique $k_0$ such that $f(k_0) = 0$. The rest of the proof is the same as in [5], based on the strong contraction Lemma 3.1.

4. **The non-integrable case**

This section will be devoted to the proof of Theorem 1.2 which is concerned with the range of exponents $0 < m \leq \frac{N-4}{N-2}$, $N > 4$. Assume that $u$ is a solution of (1.1) which satisfies the bound (1.7). Throughout this section $\tilde{u}$ will denote the rescaled solution defined by (1.8), and $\tilde{B}_k$ the rescaled Barenblatt solution given by (1.11).

Since a difference of any two solutions $u$, $v$ of equations (1.1) is not always integrable in the range of exponents $0 < m \leq \frac{N-4}{N-2}$, $N > 4$, we need to depart in this section from the techniques used in [5] in which we heavily used the integrability of a difference of any two Barenblatt solutions.

We define the weighted $L^1$-space with weight $\tilde{B}^\alpha := (C^*/(k_2 + |x|^2))^{\alpha/(1-m)}$, as

$$L^1(\tilde{B}^\alpha, \mathbb{R}^N) := \{ f | \int_{\mathbb{R}^N} f(x) \tilde{B}^\alpha(x) \, dx < \infty \}.$$
Lemma 4.1. Let \( u, v \) be two solutions of (1.9), for \( 0 < m < \frac{N - 2}{N} \), with initial values \( u_0, v_0 \) satisfying (1.10) and \( u_0 - v_0 \in L^1(\bar{B}^\alpha, \mathbb{R}^N) \), with \( \alpha = (N - 2)(1 - m)/2 - 1 \) (we will explain later such a choice of \( \alpha \)). Let \( \bar{B} := \bar{B}_{k_2} \). If

\[
\max_{\mathbb{R}^N} |\bar{u}_0 - \bar{v}_0| \neq 0
\]

then,

\[
\| (\bar{u} - \bar{v})(\cdot, \tau) \bar{B}^\alpha \eta_R \|_{L^1} < \| (\bar{u}_0 - \bar{v}_0) \bar{B}^\alpha \eta_R \|_{L^1}, \quad \tau \geq -\log T.
\]

Proof. Set \( q = \bar{u} - \bar{v} \). After rescaling, (2.3) becomes

\[
|q|_\tau \leq \Delta (a |q|) + |\nabla (x \cdot |q|)|,
\]

where \( a(x, \tau) \) is as in (2.10). Let \( \eta \in C_0^\infty(\mathbb{R}^2) \) be a cut off function as before. Denote by \( \eta_R = \eta(x/R) \) so that \( |\nabla \eta_R| \leq C/R, |\Delta \eta_R| \leq C/R^2 \). Then, the above equation and integration by parts yield to

\[
\int_{\mathbb{R}^N} |q(x, \tau)| \eta_R \bar{B}^\alpha (x) \, dx - \int_{\mathbb{R}^N} |\bar{u}_0 - \bar{v}_0|_R \eta_R \bar{B}^\alpha (x) \, dx =
\]

\[
= \int_{-\log T}^{\tau} \int_{\mathbb{R}^N} \left( a(x, \tau') |q|(x, \tau') \{ \bar{B}^\alpha(x) \Delta \eta_R + \Delta \bar{B}^\alpha(x) \eta_R + 2 \nabla \eta_R \nabla \bar{B}^\alpha(x) \} \right.
\]

\[
- |\gamma| |q|(x, \tau') x \cdot \{ \bar{B}^\alpha(x) \nabla \eta_R + \nabla \bar{B}^\alpha(x) \eta_R \} \, dx \, d\tau'.
\]

Moreover,

\[
\int_{\mathbb{R}^N} |q(x, \tau)| \eta_R \bar{B}^\alpha (x) \, dx - \int_{\mathbb{R}^N} |\bar{u}_0 - \bar{v}_0|_R \eta_R \bar{B}^\alpha (x) \, dx
\]

\[
\leq \frac{C}{R^2} \int_{-\log T}^{\tau} \int_{\mathbb{R}^2 \leq |x| \leq R} a(x, \tau') |q|(x, \tau') \bar{B}^\alpha (x) \, dx \, d\tau'
\]

\[
+ \frac{C}{R} \int_{-\log T}^{\tau} \int_{\mathbb{R}^2 \leq |x| \leq R} a(x, \tau') |q|(x, \tau') |\nabla \bar{B}^\alpha (x)| \, dx \, d\tau'
\]

\[
+ C \int_{-\log T}^{\tau} \int_{\mathbb{R}^2 \leq |x| \leq R} |q|(x, \tau') \bar{B}^\alpha (x) \, dx \, d\tau'
\]

\[
+ \int_{-\log T}^{\tau} \int_{\mathbb{R}^N} \left\{ a(x, \tau') \Delta \bar{B}^\alpha - |\gamma| x \cdot \nabla \bar{B}^\alpha \right\} |q|(x, \tau') \eta_R \, dx \, d\tau'
\]

\[
= I_1(R) + I_2(R) + I_3(R) + I_4(R) \quad \forall \ R \geq R_0 > 0.
\]

We fix

\[
\alpha = \frac{(N - 2)(1 - m)}{2} - 1.
\]

Claim i. If \( \max_{\mathbb{R}^N} |\bar{u}(x, \tau) - \bar{v}(x, \tau)| \neq 0 \), there exists a constant \( C(\tau) > 0 \), such that \( I_4(R) < -C(\tau) \).
Again, by direct computation (using also that $|\cdot|$ and the claim i. in the proof of Claim iii). Estimate (4.4) now follows from the computation in (4.2).

Proof of Claim i. We recall that

$$m(|x|^2 + k_2) \leq a(x, \tau) \leq \frac{m(|x|^2 + k_1)}{C^*}$$

with $C^* = \frac{2m(N-2-mN)}{1-m}$. A direct computation shows that

$$\Delta B^\alpha = \frac{2\alpha [2\alpha - (1-m)(N-2)] |x|^2 - k_2 (1-m) N}{(1-m)^2 (1+|x|^2)^2} B^\alpha \leq 0$$

provided that $\alpha \leq (1-m)(N-2)/2$. Hence

$$a(x, \tau) \Delta B^\alpha(x) - |\gamma| x \cdot \nabla B^\alpha(x) \leq \frac{m(|x|^2 + k_2)}{C^*} \Delta B^\alpha(x) - |\gamma| x \cdot \nabla B^\alpha(x).$$

Again, by direct computation (using also that $|\gamma| = \frac{1}{N-2-m}$ and that $C^* = \frac{2m(N-2-mN)}{1-m}$) we find

$$\frac{m(|x|^2 + k_2)}{C^*} \Delta B^\alpha(x) - |\gamma| x \cdot \nabla B^\alpha(x)$$

$$= -\frac{k_2 (N-4-m(N-2)) N}{2(N(1-m)+2) (k_2 + |x|^2)} B^\alpha = -\frac{\theta(m, n, k_2)}{(k_2 + |x|^2)^{\frac{4}{2} - \frac{N}{m}}} < 0$$

for $m < \frac{N-4}{N-2}$ and $\alpha = \frac{(N-2) (1-m)}{2} - 1$. From this the claim easily follows.

We will now compare the terms $I_i$ in (5.1) in order to get a strong contraction principle with the weight $B^\alpha$.

Claim ii. There is a uniform constant $C$ (independent of $R$) such that

$$(4.4) \quad \int_{R^N} |q(x, \tau)| \eta_R \tilde{B}^\alpha(x) \, dx \leq \int_{R^N} |\tilde{u}_0 - \tilde{v}_0| \eta_R \tilde{B}^\alpha(x) \, dx + C.$$

In particular, if $\tilde{u}_0 - \tilde{v}_0 \in L^1(\tilde{B}^\alpha, R^N)$, then $\tilde{u}(x, \tau) - \tilde{v}(x, \tau) \in L^1(\tilde{B}^\alpha, R^N)$.

Proof of Claim ii. The proof of Claim ii is similar to the proof of Lemma 2.1 once we know

$$I_i(R) \leq C \int_{-\log T}^T \int_{R/2 \leq |x| \leq R} |q(x, \tau)| \tilde{B}^\alpha(x) \, dx \, d\tau$$

$$\leq C \int_{-\log T}^T \int_{R/2 \leq |x| \leq R} \frac{dx}{|x|^{\frac{2+1}{1+m}}} \leq C(\tau), \quad i \in \{1, 2, 3\},$$

for our choice of $\alpha$, where $C$ is independent of $R$ (for $I_2(R)$ see also the arguments in the proof of Claim iii). Estimate (4.4) now follows from the computation in (4.2) and the claim i.

Claim iii. We have, $\lim_{R \to \infty} I_i(R) = 0, i = 1, 2, 3.$
Proof of Claim iii. By the Claim ii we have \(|q(\cdot, \tau') \in L^1(\tilde{B}^\alpha, \mathbb{R}^N)\), for \(\tau' \in [-\log T, \tau]\). Hence, by choosing \(R\) sufficiently big, the integral
\[
\int_{-\log T}^{\tau} \int_{R/2 \leq |x| \leq R} |q(x, \tau') \tilde{B}^\alpha(x) dxd\tau'
\]
can be made arbitrarily small. This readily implies that \(\lim_{R \to \infty} I_3(R) = 0\). In addition, since \(a(x, \tau) \leq C|x|^2\), this also implies that \(\lim_{R \to \infty} I_1(R) = 0\). Finally, observing that \(|\nabla \tilde{B}^\alpha| \approx B^\alpha/R\) on \(R/2 \leq |x| \leq R\), we conclude that \(\lim_{R \to \infty} I_2(R) = 0\), finishing the proof of the claim.

Combining the above two claims and (4.2), concludes the proof of Lemma 4.1. □

Since we have Lemma 4.1, the following result holds due to Osher and Ralston if we replace the usual \(L^1\) norm by the weighted \(L^1(\tilde{B}^\alpha, \mathbb{R}^N)\) norm. This replacement will leave the proof of the following Lemma unchanged.

Lemma 4.2 (Lemma 1 in [9] by Osher and Ralston). Let \(R \geq R_0\). Suppose \(\|\tilde{u}(\cdot, \tau_i) - \tilde{v}_0\|_{L^1(\tilde{B}^\alpha, \mathbb{R}^N)} \to 0\) as \(i \to \infty\), and let \(\tilde{B}_k\) be any stationary solution of (1.9). If \(\tilde{v}\) is a solution of (1.9) in \(\mathbb{R}^N \times [0, \infty)\) with initial value \(\tilde{v}(x, 0) = \tilde{v}_0(x)\), then
\[
\|\tilde{v}(\cdot, \tau) - \tilde{B}_k\|_{L^1(\tilde{B}^\alpha, \mathbb{R}^N)} = \|\tilde{v}_0 - \tilde{B}_k\|_{L^1(\tilde{B}^\alpha, \mathbb{R}^N)},
\]
for all \(\tau > 0\) and all \(k > 0\).

Remark on the proof of Lemma 4.2: Define \(T(t)\tilde{u}_0 = \tilde{u}(t)\), where \(\tilde{u}(t)\) is a solution of (1.9) starting at \(\tilde{u}_0\). The proof of Lemma 4.2 uses only that \(T(t)\) is a semi-group on an \(L^1(\tilde{B}^\alpha, \mathbb{R}^N)\)-closed subset of \(L^\infty\), satisfying the contraction principle (6.2) and fixing \(\tilde{B}_k\).

The following simple convergence result will be used in the proof of Theorem 1.2.

Lemma 4.3. Let \(u_0\) satisfy (1.6) for some constants \(k_1, k_2\). Take any \(\tau_i \to \infty\) and let \(\tilde{u}_i(\cdot, \tau) = \tilde{u}(\cdot, \tau_i + \tau)\). Then, passing to a subsequence, \(\tilde{u}_i\) converges, as \(i \to \infty\), uniformly on compact subsets of \(\mathbb{R}^N \times (-\infty, \infty)\) to \(\tilde{v}(x, \tau)\), an eternal solution of (1.9), satisfying (1.10).

Proof. Since \(\tilde{u}\) satisfies (1.10), equation (1.9) is uniformly parabolic on \(B_{2R} \times [-\log T/2 - \tau, \infty)\), for any \(R > 0\). By standard parabolic estimates, the sequence \(\{\tilde{u}_i\}\) is equicontinuous on compact subsets of \(\mathbb{R}^N \times (-\infty, \infty)\). Hence, by Arzela-Ascoli theorem and the diagonalization argument any sequence \(\{\tilde{u}_i\}\) will
have a convergent subsequence, converging uniformly on compact subsets to \( \tilde{v} \), an eternal solution of (1.9) satisfying (1.10). \( \quad \square \)

Recall that we denote \( \tilde{B}_{k_0} \) simply by \( \tilde{B} \).

**Claim 4.4.** The sequence \( \tilde{u}_i(x, \tau) \) converges to \( \tilde{v}(x, \tau) \) in \( L(\tilde{B}^\alpha, \mathbb{R}^N) \) norm.

**Proof.** By Lemma 4.1 and Lemma 3.1 we have

\[
\int_{\mathbb{R}^N} |\tilde{u}_i(x, \tau) - \tilde{B}(x)| \, dx \leq C, \quad \text{and}
\]

\[
\int_{\mathbb{R}^N} |\tilde{u}_i(x, \tau) - \tilde{B}(x)| \cdot |\tilde{B}^\alpha(x) \, dx \leq C,
\]

where \( C \) is a constant independent of \( i \) and \( \tau \) and therefore,

\[
\int_{\mathbb{R}^N} |\tilde{u}_i(x, \tau) - \tilde{v}(x, \tau)| \cdot |\tilde{B}^\alpha(x) \, dx \leq
\]

\[
\leq \int_{B_R} |\tilde{u}_i(x, \tau) - \tilde{v}(x, \tau)| \cdot |\tilde{B}^\alpha(x) \, dx +
\]

\[
+ \left( \int_{|x| \geq R} |\tilde{u}_i(x, \tau) - \tilde{B}(x)| \cdot |\tilde{B}^\alpha(x) \, dx + \int_{|x| \geq R} |\tilde{v}(x, \tau) - \tilde{B}(x)| \cdot |\tilde{B}^\alpha(x) \, dx \right)
\]

\[
\leq \int_{B_R} |\tilde{u}_i(x, \tau) - \tilde{v}(x, \tau)| \cdot |\tilde{B}^\alpha(x) \, dx +
\]

\[
+ \frac{C}{R^{2\alpha/(1-m)}} \left( \int_{|x| \geq R} |\tilde{u}_i(x, \tau) - \tilde{B}(x)| \, dx + \int_{|x| \geq R} |\tilde{v}(x, \tau) - \tilde{B}(x)| \cdot |\tilde{B}^\alpha(x) \, dx \right) \leq
\]

\[
\leq \int_{B_R} |\tilde{u}_i(x, \tau) - \tilde{v}(x, \tau)| \cdot |\tilde{B}^\alpha(x) \, dx + \frac{C}{R^{2\alpha/(1-m)}}.
\]

If we let \( i \to \infty \) in the previous estimate, since \( \tilde{u}_i(x, \tau) \to \tilde{v}(x, \tau) \) uniformly on compact sets, we get

\[
\lim_{i \to \infty} \int_{\mathbb{R}^N} |\tilde{u}_i(x, \tau) - \tilde{v}(x, \tau)| \cdot |\tilde{B}^\alpha(x) \, dx \leq \frac{C}{R^{2\alpha/(1-m)}},
\]

which holds for every \( R > 0 \) and therefore \( ||\tilde{u}_i(\cdot, \tau) - \tilde{v}(\cdot, \tau)||_{L^1(\tilde{B}^\alpha, \mathbb{R}^N)} \to 0 \) as \( i \to \infty \). \( \quad \square \)

**Proof of Theorem 1.2**

**Proof.** For a sequence \( \tau_i \to \infty \), let \( \tau_{i_k} \) a subsequence for which \( \tilde{u}(\cdot, \tau_i) \to \tilde{v}_0 \), as \( i_k \to \infty \), uniformly on compact sets of \( \mathbb{R}^N \), as shown in Lemma 1.3. We will show that \( \tilde{v}_0 = \tilde{B} \), as stated in the Theorem. By the previous claim and Lemma 4.2 we have that

\[
(4.5) \quad ||\tilde{v}(\cdot, \tau) - \tilde{B}(\cdot)||_{L^1(\tilde{B}^\alpha, \mathbb{R}^N)} = ||\tilde{v}(\cdot, \tau_0) - \tilde{B}(\cdot)||_{L^1(\tilde{B}^\alpha, \mathbb{R}^N)},
\]
for all $\tau > -\log T$. On the other hand, if $\max|v(x,0) - \tilde{B}| > 0$ we have the strong contraction principle \ref{1.5}, which contradicts \ref{4.2}. Therefore, $\tilde{v}(x,0) = \tilde{B}(x)$. □

5. Solutions that live longer

In the previous sections we established the vanishing profile of solutions $u$ of equation \ref{1.1} with initial data $u_0$ trapped in between two Barenblatt solutions with the same vanishing time $T$, i.e. when \ref{1.2} holds true. We showed that if $u_0$ satisfies \ref{1.6}, then $u$ vanishes at time $T$ and the rescaled solution $\bar{u}(x,\tau) = (T-t)^{-\beta}u(x(T-t)^{\gamma},t)$, with $\tau = 1/(T-t)$, converges, as $\tau \to \infty$, to a rescaled Barenblatt solution $\bar{B}$.

In this section we will show the condition \ref{1.6} is necessary for Theorems 1.1 and 1.2 to hold true. We will prove Theorem 1.4 which presents an example of a class of initial conditions $u_0$ which satisfy the growth condition

\begin{equation}
(5.1) \quad u_0(x) = \left(\frac{C^* T}{|x|^2}\right)^{\frac{m}{2(1-m)}} (1 + o(1)), \quad \text{as } |x| \to \infty
\end{equation}

with $C^*$ given by \ref{1.5}, for which the solution $u$ of \ref{1.1} with initial data $u_0$ vanishes at time $T^* > T$. In addition, we will show that the solution $u$ remains strictly positive for $t < T^*$ and it satisfies, as $|x| \to \infty$, the growth conditions $u(x,t) \approx C(t) |x|^{-\frac{2}{1-m}}$, with $C(t) > 0$ on $0 < t < T$ and $u(x,t) = O(|x|^{-\frac{2}{1-m}})$, on $T < t < T^*$. In particular, $u$ becomes integrable on $t > T$ and its vanishing behaviour is given by a class of self-similar solutions $\theta(x,t)$.

It is well known that the Barenblatt solutions given by \ref{1.3} are not the only self-similar solutions of equation \ref{1.1}. It was shown in \cite{ref10} that \ref{1.1} possesses self-similar solutions of the form

$$\Theta(r, t) = (T-t)^{\alpha} f(\eta), \quad \eta = \frac{r}{(T-t)^{\gamma}}, \quad \alpha = \frac{1-2\theta}{1-m},$$

where the function $f$ is a solution of an elliptic non-linear eigenvalue problem with eigenvalue $\theta$ satisfying the bound

$$-\frac{m}{(1-m)N-2} < \theta < \frac{1}{2},$$

$f'(0) = 0$ and $f(\eta) = O(\eta^{-(N-2)/m})$ as $\eta \to \infty$. The solution $f$ was shown to be unique apart from a scaling due to the invariance of $f$ under the transformation $f(\eta; \lambda) = \lambda^{\frac{2}{1-m}} f(\eta/\lambda; 1)$ and satisfies

\begin{equation}
(5.2) \quad f(\eta; 1) \approx \eta^{-\frac{2}{1-m}}, \quad \text{as } \eta \to \infty.
\end{equation}
It was shown in [3] that for any radially symmetric solution of (1.1) with initial data satisfying the growth condition
\[ u_0(r) = O(r^{-\frac{N-2}{2}}) \] as \( r \to \infty \), then the vanishing behavior of \( u \) is described by the self-similar solutions \( \Theta \), i.e. there exists \( \lambda > 0 \) such that the rescaled solution satisfies
\[ (T-t)^{-\alpha} u(\eta (T-t)^{\theta}, t) \to f(\eta; \lambda) \]
uniformly in \( \eta \geq 0 \).

In the proof of Theorem 1.4 we will use the following lemma, which is also of independent interest.

**Lemma 5.1.** Assume that \( v \) is a solution of (1.1) on \( \mathbb{R}^N \times (0, T) \). Assume that \( v_0 \leq f \) with \( f \in L^1(\mathbb{R}^N) \) and radially symmetric. Then, at time \( t > 0 \) the solution \( v \) satisfies the bound
\[ v(x, t) \leq \frac{C}{|x|^{N-2}}, \quad |x| > 1. \]

**Proof.** We introduce the potential function
\[ w(x, t) = \int_0^t v^m(x, s) \, ds \]
which satisfies the inequality
\[ \Delta w \geq -f, \quad \text{on } \mathbb{R}^N. \]

Let
\[ Z(x) = \frac{1}{N(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} f(y) \, dy \]
denote the Newtonian potential of \( f \), so that we have
\[ \Delta(w - Z) \geq 0. \]

Since \( f \) is integrable and radially symmetric, there exists a constant \( C < \infty \) for which
\[ Z(x) \leq \frac{C}{|x|^{N-2}}, \quad \forall x \in \mathbb{R}^N. \]

Indeed, this follows from the observation that for a radially symmetric \( f \) the Newtonian potential of \( f \) is also given by
\[ Z(r) = \int_r^\infty \frac{1}{\rho^{N-1}} \int_{|y| \leq \rho} f(y) \, dy \, d\rho. \]

Similarly as in the proof of Lemma 2.1 we get
\[ w(x) = \int_0^t v^m(x, s) \, ds \leq Z(x) \leq \frac{C}{|x|^{N-2}}. \]
We will now use this bound together with the Aronson-Bénilan inequality
\[(5.6)\quad v_t \leq \frac{1}{(1 - m) t} v\]
to conclude the desired bound on \(v\). Indeed, we first integrate (5.6) in time to obtain the inequality
\[
\frac{v(x, t_2)}{v(x, t_1)} \leq \left( \frac{t_2}{t_1} \right)^{\frac{1}{1 - m}}, \quad \text{if } 0 < t_1 < t_2.
\]
Hence
\[
w(x) \geq \int_{t/2}^{t} v^m(x, s) \, ds \geq v^m(x, t) \int_{t/2}^{t} \left( \frac{s}{t} \right)^{\frac{m}{1 - m}} = c(t) v^m(x, t)
\]
which combined with (5.5) implies the bound
\[
v(x, t) \leq \frac{C(t)}{|x|^{\frac{N}{2} - m}}
\]
as desired. \qed

We now proceed with the proof of the Theorem.

Proof of Theorem 1.4. We begin with the following observation: If \(w\) is a solution of (1.1) which vanishes at time \(T\), then for any \(a > 0\), the solution of (1.1) given by \(W(x, t) = w(ax, a^2 t)\) has vanishing time \(T' = T/a^2\). Hence, we can make \(T'\) arbitrarily large by choosing \(a\) sufficiently small.

Let \(f \geq 0\), be any radially symmetric integrable function such that \(f(x) = o(|x|^{-\frac{N}{2} - m})\), as \(|x| \to \infty\). In particular, we can take \(f(x)\) to satisfy \(f(x) = O(|x|^{-\frac{N + 2}{2}}), \, |x| \to \infty\). Choose \(a\) sufficiently large so that the vanishing time \(T'\) of the solution \(w\) of (1.1) with initial data \(w_0(x) = f(ax)\) satisfies \(T' > T\), with \(T\) as in (5.1). Set
\[
u_0 = \left( \frac{C^* T}{|x|^2 + 1} \right)^{\frac{1}{1 - m}} + w_0
\]
and let \(B(x, t) = \left( \frac{C^* (T-t)}{|x|^2 + 1} \right)^{\frac{1}{1 - m}}\) the Barenblatt solution with \(B(x, 0) = \left( \frac{C^* T}{|x|^2 + 1} \right)^{\frac{1}{1 - m}}\). Denoting by \(u\) the solution of (1.1) with initial data \(u_0\) it is clear application of the comparison principle that \(u \geq w\) so that the vanishing time \(T^*\) of \(u\) satisfies \(T^* \geq T' > T\). This proves (i). Also, since \(f \geq 0, \, u \geq B\), so that (1.14) is satisfied as well.

Since \(u_0 - B(\cdot, 0) = f \in L^1(\mathbb{R}^N)\), by Corollary 2.2
\[
\|u(\cdot, t) - B(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|f(\cdot)\|_{L^1(\mathbb{R}^N)}.
\]
Since $B(\cdot, t) \equiv 0$ for $t \geq T$, this implies

$$\|u(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)} \quad \text{for } t \in [T, T^*].$$

Combining the estimate (5.7) and Lemma 5.1, if we take $v(x, 0) = u(x, T)$, yields to (1.15).

The statement (iii) of our Theorem now immediately follows by the result of Galaktionov and Peletier in [3].

The proof of Corollary 1.5 now easily follows.

Proof of Corollary 1.5. It is known that if $m = \frac{N-2}{N+2}$ then $\alpha = \frac{N+2}{4}$ and $\theta = 0$ in the definition of self-similar solutions $\Theta(x, t)$. In this case function $f$ is given explicitly by

$$f(\eta, \lambda) = \left(\frac{K_N \lambda}{\lambda^2 + \eta^2}\right)^{N+2}.$$

The proof of Theorem 1.4 implies that for $t > T$, we have the bound

$$u(x, t) \leq \frac{C(t)}{|x|^{N+2}}.$$

By the result of Del Pino and Saez in [1], if the vanishing time of $u$ is $T^*$ with $T^* > T$, then there is $\lambda > 0$ so that

$$(T^* - t)^{-\frac{1}{1-m}} u(x, t) \to f(|x|, \lambda) \quad \text{as } t \to T^*.$$

We will end the paper with the following remark regarding the examples of solutions $u$ constructed in Theorem 1.4. We know that these solutions $u$ in live up to $T^* > T$, and satisfy $u(x, t) \geq B(x, t)$ for $t < T$ and $u(x, t) \leq C(t) |x|^{-\frac{N-2}{1-m}}$ for $t > T$.

Remark 5.2. In the examples of solutions $u$ constructed in Theorem 1.4, the Barenblatt $B(x, t)$ dies off exactly at time $T$, in the sense that there is no sequence $|x_i| \to \infty$ and $A > 0$ so that

$$u(x_i, T) \geq \frac{A}{|x_i|^{1-m}}.$$

Proof. Let us argue by contradiction. Assume there is a sequence $|x_i| \to \infty$ so that (5.8) holds for some $A > 0$. Take $t > T/2$ so that $2^m(C^*(T - t))^{1-m} < A/2$ and let $\delta = (T - t)/2$. Notice that by our choice of $f$, $0 \leq u(x, 0) - B(x, 0) \in L^1(\mathbb{R}^N)$. By
the comparison principle, we have $u(x, t) \geq B(x, t)$ for $t > 0$, and by Lemma 2.1 $u(x, t) - B(x, t) \in L^1(\mathbb{R}^N)$. Let

$$w(x, t) = \int_t^{t+\delta} (u^m - B^m) \, ds.$$ 

As before, $\Delta w \geq -(u - B)$ and since $||(u - B)(\cdot, t)||_{L^1(\mathbb{R}^N)} \leq ||f||_{L^1(\mathbb{R}^N)}$, using Newtonian potentials we obtain that

$$w(x, t) \leq \frac{C}{|x|^{N-2}}$$

for a uniform constant $C$. This yields the existence of $s \in (t, t+\delta)$ such that

$$u^m(x, s) \leq B^m(x, s) + \frac{C}{\delta |x|^{N-2}} \leq \left( \frac{C^* (T-s)}{|x|^2} \right)^{\frac{m}{1-m}} + \frac{C}{\delta |x|^{N-2}}.$$ 

(5.9)

For each $x_i$ choose $s_i \in [t, t+\delta]$ so that (5.9) is satisfied. The Aronson-Bénilan inequality gives

$$u(x_i, T) \leq \frac{T}{s_i} u(x_i, s_i) \leq 2 u(x_i, s_i).$$

Combining (5.8) and (5.9) yields

$$\frac{A}{|x_i|^{\frac{2m}{1-m}}} \leq u^m(x_i, T) \leq \frac{2^m (C^* (T-s_i))^{\frac{m}{1-m}}}{|x_i|^{\frac{2m}{1-m}}} + \frac{C}{\delta |x_i|^{N-2}}$$

and therefore

$$\frac{A}{2 |x_i|^{\frac{2m}{1-m}}} \leq \frac{C}{\delta |x_i|^{N-2}}$$

which can not be fulfilled for $|x_i| >> 1$. This finishes the proof of our claim. □

6. Appendix

In this appendix we improve Theorem 1.2. The goal is to remove the assumption $u_0(x) \geq B_{k_1}(x, 0)$, where $B_{k_1}(x, t) = (C^* (T-t)/(k_1 + |x|^2))^{\frac{1}{1-m}}$ is a Barenblatt solution with the same vanishing time as $u$. This was assumed in (1.6). The bound from above is necessary, as was proven in the previous section. Denoting by $B(x, t) = (C^* (T-t)/(k + |x|^2))^{\frac{1}{1-m}}$, for some $k > 0$, we will show the following result.

**Theorem 6.1.** If $0 < m \leq \frac{N-4}{N-2}$ for $N > 4$ and

$$|u_0(x) - B(x, 0)| \leq f(|x|) \in L^1(\mathbb{R}^N) \quad \text{and} \quad u_0(x) \leq B(x, 0)$$

(6.1)
for a positive, radial function \( g \), then \( u \) vanishes at the same time as \( B(x,t) \) and the rescaled solution \( \tilde{u}(x,\tau) \) converges, as \( \tau \to \infty \), uniformly \( \mathbb{R}^N \), to the rescaled Barenblatt \( \tilde{B} \).

To simplify the notation we will assume that \( B(x,t) = (C^*(T-t)/(1+|x|^2))^{1/(1-m)} \).

The proof of the theorem will be based on a sequence of observations.

We begin by noticing that condition (6.1) implies the \( L^1 \)-contraction principle
\[
\int_{\mathbb{R}^N} |u(\cdot,t) - B(\cdot,t)| \, dx \leq \int_{\mathbb{R}^N} |u_0 - B(\cdot,0)| \, dx, \quad \forall t \in [0,T).
\]
The proof is the same as the proof of Corollary 2.2, which goes through under the weaker assumption that only \( v_0 \geq B \) (and not necessarily \( u_0 \geq B \)). Furthermore, this implies that \( u \) and \( B \) have the same vanishing time.

In order to be able to take the limit of the rescaled solution, we need to establish the necessary a’priori estimates. It turns out that it is possible to do so, just by using the fact that the difference \( u(\cdot,0) - B(\cdot,0) \in L^1(\mathbb{R}^N) \) and that \( B \) and \( u \) have the same vanishing time. Introducing as in the previous sections the rescaling
\[
\tilde{u}(x,\tau) = (T-t)^{-\beta} u(x(T-t)^{\gamma}, \tau), \quad \tau = -\ln(T-t)
\]
(with \( \beta, \gamma \) given by (1.4)), which satisfies the rescaled equation
\[
(\tilde{u})_\tau = \Delta \tilde{u}^m + |\gamma| \text{div}(x \cdot \tilde{u}), \quad \text{on} \ \mathbb{R}^N \times (-\log T, \infty)
\]
we have:

**Proposition 6.2.** If \(|u_0(x) - B(x,0)| \leq f(|x|) \in L^1(\mathbb{R}^N)\), then there are positive constants \( C_1, C_2, r_0, \tau_0 \) such that
\[
\frac{C_1}{(r^2 + 1)^{1/(1-m)}} \leq \tilde{u}(r,\tau) \leq \frac{C_2}{(r^2 + 1)^{1/(1-m)}} \quad \text{for} \ r \geq r_0
\]
for all \( \tau \in [\tau_0, \infty) \).

**Proof.** We will first prove (6.5) under the assumption that \( u_0 \) is radially symmetric. At the end of the proof we will remove this assumption.

As we have observed in the proof of Lemma 2.1, absolute value \(|u - B|\) satisfies the differential inequality
\[
\frac{d}{dt} |u - B| \leq \Delta |u^m - B^m|.
\]

Hence, for any fixed numbers \( T/2 < s < t < T \), the function
\[
w(x) = \int_s^t |u^m - B^m|(x,l) \, dl
\]
satisfies
\[ \Delta w \geq -|u - B|(s). \]
Let \( Z \) be such that \( \Delta Z = -|u - B|(s) \) and therefore
\[ \Delta(w - Z) \geq 0 \]
which, by the mean value property, as in the proof of Lemma 2.1, yields to the inequality
\[ w(x) \leq Z(x, s), \quad r = |x|. \]
Since, \( u - B \) is radially symmetric, the function potential \( Z \) is given by
\[ Z(x, s) = c_n \int_0^\infty \frac{1}{\rho^{N-1}} \int_{|y| \leq \rho} |u - B|(s) \, dy \, d\rho, \quad r = |x| \]
for an appropriate constant \( c_n \). We conclude that
\[ w(x) \leq \frac{\|(u - B)(s)\|_{L^1}}{r^{N-2}}, \quad r \geq 1 \]
which combined with (6.2) and our assumption (6.1) implies the bounds
\[ \int_s^t B^m(x, l) \, dl - C \frac{\|f\|_{L^1}}{r^{N-2}} \leq \int_s^t u^m(x, l) \, dl \leq \int_s^t B^m(x, l) \, dl + C \frac{\|f\|_{L^1}}{r^{N-2}}. \]
Next, fix \( t \in [3T/4, T) \) and choose \( s \in [T/2, T) \) such that \( T - t = t - s \), so that
\[ 2^{-\frac{m}{1-m}} (t - s) B^m(x, s) \leq \int_s^t B^m(x, l) \, dl \leq 2^{-\frac{m}{1-m}} (t - s) B^m(x, t). \]
Combining the two last inequalities gives
\[ 2^{-\frac{m}{1-m}} B^m(x, s) - C \frac{\|f\|_{L^1}}{r^{N-2}} \leq \frac{1}{t - s} \int_s^t u^m(x, l) \, dl \leq 2^{-\frac{m}{1-m}} B^m(x, t) + C \frac{\|f\|_{L^1}}{r^{N-2}}. \]
From the above we conclude a pointwise bound from above and below for \( u^m(x, t) \) with the aid of the Aronson-Benilan inequality, \( (\log u)_t \leq 1/((1 - m) t) \), which after integration implies the bound
\[ \left( \frac{s}{t} \right)^{\frac{m}{1-m}} u^m(x, t) \leq \frac{1}{t - s} \int_s^t u^m(x, l) \, dl \leq \left( \frac{t}{s} \right)^{\frac{m}{1-m}} u^m(x, s). \]
Due to our choice for \( s \) (for a given \( t \)) we have \( t/s \leq 2 \). Hence, combining the last two inequalities gives
\[ u^m(r, t) \leq C_1 B^m(r, t) + \frac{C}{(T - t) r^{N-2}} \]
which holds for all \( t \in [3T/4, T) \), since \( t \) is arbitrary, and also
\[ u^m(r, s) \geq C_2 B^m(r, s) - \frac{C}{(T - s) r^{N-2}} \]
The previous Proposition yields the existence of Proof of Theorem 6.1. \[ \square \]

Since, by the comparison principle, \((1.1)\) with initial data \(u_0\) which also holds for all \(s \in [3T/4, T]\) since \(t\) is arbitrary and \(T-t = t-s\). Rescaling inequalities (6.6) and (6.7), and using (1.4) we conclude

\[
\frac{C_1}{(r^2 + 1)^{m/(1-m)}} - \frac{C}{r^{N-2}} \leq u^m(r, \tau) \leq \frac{C_2}{(r^2 + 1)^{m/(1-m)}} + \frac{C}{r^{N-2}}
\]

for some uniform constants \(C_1, C_2, C\) and \(\tau \geq \tau_0 := -\log(T/4)\). This readily implies (6.3) for \(r \geq r_0\) (independent of \(\tau\)) if \(N-2 > \frac{2m}{1-m}\). The last is equivalent to \(m < \frac{N-2}{N}\) and is implied by our assumption \(m \leq (N-4)/(N-2)\).

In the case where \(u(x, 0)\) is nonradial and \(B(x, 0) - u(x, 0)\) is bounded from above by a radial function in \(L^1(\mathbb{R}^N)\), define

\[
\underline{u}_0(r) := \inf_{|x|=r} u(x, 0) \quad \text{and} \quad \overline{u}_0(r) := \sup_{|x|=r} u(x, 0).
\]

By our assumption, \(B(x, 0) - f(r) \leq u(x, 0) \leq B(x, 0) + f(r)\), where \(r = |x|\) and \(f \in L^1(\mathbb{R}^N)\). Since \(B(x, 0)\) is a radial function itself, we have \(B(x, 0) - \underline{u}_0(r) \in L^1(\mathbb{R}^N)\) and \(B(x, 0) - \overline{u}_0(r) \in L^1(\mathbb{R}^N)\). Let \(\underline{u}(x, t)\) and \(\overline{u}(x, t)\) be the solutions to (1.1) with initial data \(\underline{u}_0(r)\) and \(\overline{u}_0(r)\), respectively. Then, the radial result implies the bounds

\[
\frac{C_1}{(r^2 + 1)^{m/(1-m)}} - \frac{C}{r^{N-2}} \leq \underline{u}^m(r, \tau) \leq \frac{C_2}{(r^2 + 1)^{m/(1-m)}} + \frac{C}{r^{N-2}}
\]

and

\[
\frac{C_1}{(r^2 + 1)^{m/(1-m)}} - \frac{C}{r^{N-2}} \leq \overline{u}^m(r, \tau) \leq \frac{C_2}{(r^2 + 1)^{m/(1-m)}} + \frac{C}{r^{N-2}}.
\]

Since, by the comparison principle, \(\underline{u}(r, \tau) \leq u(x, \tau) \leq \overline{u}(r, \tau)\), the above inequalities imply (6.3) in the nonradial case. \(\square\)

**Proof of Theorem 6.1** The previous Proposition yields the existence of \(r_0 > 0\) and \(\tau_0 < \infty\) so that

\[
\frac{C_1}{(r^2 + 1)^{1/(1-m)}} \leq \underline{u}(r, \tau) \leq \frac{C_2}{(r^2 + 1)^{1/(1-m)}}
\]

for \(\tau \in [\tau_0, \infty)\) and \(r \geq r_0\). Let \(Q_{r_0} = B_{r_0} \times [\tau_0, \infty)\). Hence, there exists a constant \(c_0 = c(r_0, \tau_0) > 0\) such that

\[
\underline{u}(r, \tau) \geq \frac{C_0}{(r^2 + 1)^{1/(1-m)}}, \quad \text{on } \partial_p Q_{r_0}.
\]

By the maximum principle

\[
\inf_{Q_{r_0}} \underline{u} \geq \frac{C_0}{(r^2 + 1)^{1/(1-m)}},
\]

which, combined with the lower bound in (6.3) implies that

\[
\underline{u}(x, \tau) \geq \frac{C_1}{(r^2 + 1)^{1/(1-m)}}, \quad \text{on } \mathbb{R}^N \times [\tau_0, \infty)
\]
for a constant $C_1$ that depends on $r_0$.

By our assumption we have $\tilde{u}(x, \tau) \leq \tilde{B}(x)$ on $\mathbb{R}^N \times [-\log T, \infty)$. Hence, there are uniform constants $C_1$ and $C_2$ such that

$$
(6.8) \quad \frac{C_1}{(r^2 + 1)^{1/(1-m)}} \leq \tilde{u}(x, \tau) \leq \frac{C_2}{(r^2 + 1)^{1/(1-m)}}, \quad \text{on } \mathbb{R}^N \times [\tau_0, \infty).
$$

We conclude that the difference $\tilde{u} - \tilde{B}$ satisfies the equation

$$
(\tilde{u} - \tilde{B})_\tau = \Delta (\tilde{a}(\tilde{u} - \tilde{B})) + |\gamma| \text{div}(x \cdot (\tilde{u} - \tilde{B})),
$$

with $\tilde{a}(x, \tau) = \int_0^1 \frac{d\theta}{(\theta \tilde{u} + (1-\theta)\tilde{B})^{1-m}}$ satisfying the bounds

$$
(6.9) \quad \tilde{C}_1 (r^2 + 1) \leq \tilde{a}(x, \tau) \leq \tilde{C}_2 (r^2 + 1)
$$

on $R^N \times [\tau_0, \infty)$. The rest of the proof of Theorem 6.1 is the same as that of Theorem 1.2. \hfill \Box

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