Multi-Stage Estimation Methodologies for an Inverse Gaussian Mean with Known Coefficient of Variation

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ABSTRACT

In this paper, we develop accelerated sequential and \(k\)-stage (\(k \geq 3\)) procedures for estimating the mean of an inverse Gaussian distribution when the population coefficient of variation is known. The problems of minimum risk and bounded risk point estimation are handled. The estimation procedures are developed under an interesting weighted squared-error loss function and our aim is to control the associated risk functions. In spite of the usual estimator, i.e., the sample mean, Searls (1964) estimator is utilized for the purpose of estimation. Second-order asymptotics are obtained for the expected sample size and risk associated with the proposed multi-stage procedures. Further, it is established that the Searls' estimator dominates the usual estimator (sample mean) under the proposed procedures. Extensive simulation analysis is carried out in support of the encouraging performances of the proposed methodologies and a real data example is also provided for illustrative purposes.

KEYWORDS AND PHRASES

Accelerated sequential; bird surveys data; bounded risk; coefficient of variation; inverse Gaussian distribution; \(k\)-stage procedure; minimum risk; point estimation; Searls' estimator; second-order asymptotics; weighted squared-error loss

MATHEMATICS SUBJECT CLASSIFICATIONS

62L05; 62L10; 62L12; 62F10; 62F12

1. Introduction

In several inferential problems, we often look for some prior information which can be utilized to obtain better estimators than the usual estimator (sample mean). One such type of prior information available to experimenters, especially in the biological, agricultural, industrial and physical sciences is on the coefficient of variation (CV). Eg., in the textile industry, CV is invariably specified for the different stages of production. In some situations, it may also be possible that the mean and variance of a population are unknown but the CV is known. Eg., in many chemical processes the precision is not given in terms of the standard deviation but is given in terms of the relative error, i.e., CV. In fact, in the chemical experiments, the constancy of CV in analysis of batches of a substance is a laboratory property and for all practical purposes, the CV is known after the analysis of a sufficient number of batches. One may refer to Snedecor (1946), Hald (1952), Azen and Reed (1973), Davies and Goldsmith (1976) for several other applications.

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One may use the prior information on CV to improve the usual estimator, i.e., the sample mean. A large amount of work has already been done in this direction. Searls (1964) proposed an estimator for the unknown normal mean, which, under the assumption of known CV, has the smaller mean square error (MSE) as compared to the customary sample mean. Khan (1968) considered the same problem and obtained best unbiased estimator in the sense of minimum variance. Gleser and Healy (1976) have also worked in this direction and obtained a uniformly minimum risk estimator under the squared-error loss. Some other citations include Sen (1979), Hirano and Iwase (1989), Arnholt and Hebert (1995), Guo and Pal (2003), Singh and Mathur (2005) and Anis (2008). The entire work in the above mentioned citations is done under the fixed sample size scenario.

Joshi and Shah (1990) considered the problem of sequential testing for an inverse Gaussian (IG) mean when CV is known. Singh (1998) considered the problem of estimating a normal mean with known CV in a sequential setup. Using the estimator proposed by Searls (1964), Chaturvedi and Tomer (2003) have developed three-stage and accelerated sequential procedures for estimating the mean of a normal population and established that there is a substantial reduction in the expected sample size and risk under the minimum risk and bounded risk point estimation problems respectively. Motivated by the interesting outcomes of Singh (1998) and Chaturvedi and Tomer (2003), most recently, Chaturvedi et al. (2021) have developed sequential estimation methodologies for an IG mean when the CV is known.

The present paper deals with developing accelerated sequential and $k$-stage ($k \geq 3$) procedures for estimating the mean of an IG distribution having the probability density function (pdf):

$$f(x; \mu, \lambda) = \left\{ \frac{\lambda}{2\pi x^3} \right\}^{1/2} \exp \left\{ \frac{-\lambda(x - \mu)^2}{2x\mu^2} \right\}, \quad x > 0,$$

where $\lambda$ is an unknown scale parameter and the population CV is given by,

$$\delta = \left( \frac{\mu}{\lambda} \right)^{1/2},$$

which is assumed to be known. For the purpose of estimation, instead of the usual estimator $\bar{X}_n$, i.e., the sample mean, we use the estimator proposed by Searls (1964), denoted by $\tilde{\mu}_n$ and which in IG case is given by

$$\tilde{\mu}_n = \left( 1 + \frac{\delta^2}{n} \right)^{-1} \bar{X}_n.$$

Extensive work has been done for the sequential and multi-stage estimation of the parameter(s) of an IG distribution. Some notable citations in this direction comprises of Chaturvedi (1985, 1986), Chaturvedi et al. (1991, 2019a, 2020a, 2020b, 2021), Bapat (2018) and Joshi and Bapat (2020).

The rest of the paper is organized as follows: In Section 2, we provide the set-ups of minimum risk and bounded risk point estimation problems respectively. Section 3 deals with developing the accelerated sequential procedures for our estimation problems. In Section 4, we propose $k$-stage ($k \geq 3$) sequential sampling procedures for the
estimation problems. Under each of the sections 3 and 4, we obtain the second-order asymptotics associated with our multi-stage procedures. Section 5 provides an extensive simulation analysis of our multi-stage stopping strategies. In Section 6, we apply our proposed methodologies to a real dataset, which assumes a known CV value. Finally, we provide a brief conclusion in Section 7. Since many of the presented results go hand-in-hand with those under a Gaussian distribution, proofs of the theorems under Section 4 are provided in an online supplement for brevity alone.

2. The Estimation Problems

2.1. Minimum Risk Point Estimation Problem

Our first estimation problem is the minimum risk point estimation of $\mu$. Let the loss of estimating $\mu$ by $\tilde{\mu}_n$ (the Searls’ estimator from (3)) be given by

$$L(\mu, \tilde{\mu}_n) = \frac{A(\tilde{\mu}_n - \mu)^2}{\mu^3} + cn,$$

(4)

where $A > 0$ is the known weight and $c > 0$ is the known cost per unit sample observations. The reason behind taking such a weighted squared-error loss function can be found in Chaturvedi et al. (2020a). The risk corresponding to the loss function (4) is

$$R_n(c) = \frac{A}{\lambda(n + \delta^2)} + cn.$$

(5)

The value $n^*$ of $n$ minimizing the risk (5) is given by

$$n^* = \left(\frac{A}{c\lambda}\right)^{1/2} - \delta^2,$$

(6)

Further, the associated minimum risk is given by

$$R_n^*(c) = 2cn^* + c\delta^2.$$

(7)

Remark 1. (i) Since, for $\delta \approx 0$, $\tilde{\mu}_n \approx \bar{X}_n$, the results corresponding to the case when $\bar{X}_n$ is used to estimate $\mu$, can be obtained simply by putting $\delta = 0$.

(ii) If one uses $\bar{X}_n$ to estimate $\mu$, the associated risk (see Chaturvedi et al., 2019a) is

$$R_n(c) = \frac{A}{n\lambda} + cn.$$

(8)

The value $n_0$ of $n$ minimizing the risk (7) is

$$n_0 = \left(\frac{A}{c\lambda}\right)^{1/2},$$

(9)

and the corresponding minimum risk is

$$R_{n_0}(c) = 2cn_0.$$

(10)

Comparing (6) and (9), we note that $n^* = n_0 - \delta^2$, i.e., in known CV case, the ‘optimal’ fixed sample size minimizing the risk is smaller for Searls estimator. Thus, we expect early stopping if one uses sequential and/or multi-stage procedures motivated by
Moreover, from (7) and (10), since \( R_n(c) = R_{n_0}(c) - c\delta^2 \), we also expect reduction in the risk.

### 2.2. Bounded Risk Point Estimation Problem

Our second estimation problem is the bounded risk point estimation of \( \mu \). Let the loss incurred in estimating \( \mu \) by \( \bar{\mu}_n \) be

\[
L(\mu, \bar{\mu}_n) = \frac{A(\bar{\mu}_n - \mu)^2}{\mu^3},
\]

where \( A > 0 \) is the known weight. The risk corresponding to the loss function (11) is

\[
R_n(A) = \frac{A}{\lambda(n + \delta^2)}.
\]

For preassigned \( w > 0 \), suppose one wishes that the risk (12) should not exceed \( w \). The sample size needed to achieve this goal is the smallest positive integer \( n \geq n^{**} \), where

\[
n^{**} = \frac{A}{\lambda w} - \delta^2.
\]

**Remark 2.**

(i) Again since, for \( \delta \approx 0, \bar{\mu}_n \approx \bar{X}_n \), the results corresponding to the case when \( \bar{X}_n \) is used to estimate \( \mu \), can be obtained simply by putting \( \delta = 0 \).

(ii) If one uses \( \bar{X}_n \) to estimate \( \mu \), the associated risk is

\[
R_n(A) = \frac{A}{n\lambda}.
\]

The value \( n_{00} \) for preassigned \( w \), such that, the risk \( R_n(A) \leq w \) is given by

\[
n_{00} = \left( \frac{A}{w\lambda} \right).
\]

We note that \( n^{**} = n_{00} - \delta^2 \), i.e., in known CV case, the ‘optimal’ fixed sample size for the bounded risk problem is smaller for Searls estimator. Thus, we expect early stopping if one uses sequential and/or multi-stage procedures motivated by \( n^{**} \).

Moreover, from (6) and (13), since the ‘optimal’ fixed sample sizes giving solutions to the problems depend on \( \lambda \) (which is unknown), the fixed sample size procedures fail to meet the goals. In what follows, we propose accelerated sequential and \( k \)-stage \( (k \geq 3) \) procedures to handle these problems.

It is worth mentioning here that purely sequential and multi-stage procedures have their own merits and demerits. We often observe that a two-stage procedure is easy to implement as it requires only two stages of sampling but it also leads us to considerable oversampling. A purely sequential technique reduces this problem and proves to be more accurate. However purely sequential schemes are complicated in usual practice due to changing nature of sample space at each stage of sampling. One may refer to Stein (1945), Anscombe (1953) and Chow and Robbins (1965) for the similar arguments. Hall (1981) proposed a three-stage sampling strategy and established that his procedure is strongly competitive to both the two-stage and purely sequential schemes.
He also developed an accelerated sequential scheme in his 1983 paper, which reduces the number of stages by a predetermined factor and saves sampling operations substantially. Thus, an accelerated sequential procedure is better in comparison to a purely sequential strategy when the cost of taking observations is an important concern. One may refer to Mukhopadhyay (1996), Mukhopadhyay and de Silva (2009), Hu (2020), Joshi and Bapat (2020) and Joshi et al. (2021) for some additional details. Liu (1997) came up with a novel $k$-stage sequential sampling scheme, through which he established that with a proper choice of $k$, his scheme can be as efficient as the purely sequential strategy, it requires at most $k$ sampling operations and a three-stage procedure becomes its special case. One may also refer to Chaturvedi et al. (2019b) for a quick review on the $k$-stage sequential sampling. One thing that comes out from this entire discussion is that before using any of these sequential and multi-stage sampling schemes, one should balance any logistical concerns such as feasibility and cost.

3. Accelerated Sequential Procedures

3.1. Minimum Risk Point Estimation

Let $\eta \in (0, 1)$ be specified. We start with a sample of size $m(\geq 2)$ from the population (1) where as in Hall (1983), $m$ is chosen in such a manner that $m = o(c^{-1/2})$ as $c \to 0$ and $\lim\sup_{c \to 0} \left( \frac{m}{n} \right) < 1$. We start sampling sequentially with the stopping time $L_1$ defined by

$$L_1 = \inf \left\{ n \geq m; \ n \geq \eta \left[ \left( \frac{A}{c} \right)^{1/2} \left( \frac{1}{\lambda_n} \right) - \delta^2 \right] \right\},$$

(16)

where $\lambda_n = n^{-1} \sum_{i=1}^{n} (X_i^{-1} - \bar{X}_n^{-1})$ is the maximum likelihood estimator (MLE) of $\lambda^{-1}$. Based on these $L_1$ observations, we compute $\hat{\lambda}_{L_1}^{-1}$. Then denoting by $\lfloor y \rfloor$, the integer less than $y$, we jump ahead and take $N_1 - L_1$ observations, where

$$N_1 = \max \left\{ L_1, \left[ \left( \frac{A}{c} \right)^{1/2} \left( \frac{1}{\hat{\lambda}_{L_1}} \right) - \delta^2 \right] + 1 \right\}. \quad (17)$$

After stopping, we estimate $\mu$ by $\bar{\mu}_{N_1}$, incurring the risk

$$R_{N_1}(c) = c(n^* + \delta^2)E \left\{ f \left( \frac{N_1 + \delta^2}{n^* + \delta^2} \right) \right\} + cE(N_1),$$

(18)

where $f(x) = x^{-1}$.

The following theorem provide second-order asymptotics for the expected sample size and risk associated with the accelerated sequential procedure (16)-(17):

**Theorem 1.** For the accelerated sequential procedure defined in (16)-(17) and $m \geq 2$, as $c \to 0$,

$$E(N_1) = n_0 - 1.067n^{-1} + 1/2 - \delta^2 + o(1),$$

(19)

$$R_{N_1}(c) = 2cn_0 + c \left\{ \frac{1}{2\eta} - \delta^2 \right\} + o(c^{1/2}).$$

(20)
Proof. Denoting by,
\[ \psi_{L_1} = 1 - \left\{ \left( \frac{A}{c} \right)^{1/2} \left( \frac{1}{\dot{\lambda}_{L_1}} \right)^{1/2} - \delta^2 - \left[ \left( \frac{A}{c} \right)^{1/2} \left( \frac{1}{\dot{\lambda}_{L_1}} \right)^{1/2} - \delta^2 \right] \right\}, \]
we can write
\[ E(N_1) = I + II, \quad (21) \]
where,
\[ I = E \left\{ L_1 \left\{ L_1 \geq \left[ \left( \frac{A}{c} \right)^{1/2} \left( \frac{1}{\dot{\lambda}_{L_1}} \right)^{1/2} - \delta^2 \right] + 1 \right\} \right\}, \quad (22) \]
and
\[ II = \left( \frac{A}{c} \right)^{1/2} E \left\{ \left( \frac{1}{\dot{\lambda}_{L_1}} \right)^{1/2} \right\} - \delta^2 + E(\psi_{L_1}). \quad (23) \]

It follows from Hall (1983) that, as \( c \to 0, I = o(1) \). Further, one can observe that the distribution of \( 1/\dot{\lambda}_n \) is equivalent to \( \chi^2_{(n-1)/n\lambda} \), whose Fourier transform becomes,
\[ \frac{1}{n\lambda} \left( \frac{1}{1 - 2iu} \right)^{n-1}, \]
which clearly tends to 0 as \( u \to \infty \). Hence, from Tukey (1938) one can get that the fractional part of \( (A/c)^{1/2}(1/\dot{\lambda}_{L_1})^{1/2} - \delta^2 \) tends to a uniform distribution over \( (0, 1) \) and thus, \( \psi_{L_1} \) is also uniform over \( (0, 1) \). Thus, we obtain from (21) and (23) that
\[ E(N_1) = \left( \frac{A}{c} \right)^{1/2} E \left\{ \left( \frac{1}{\dot{\lambda}_{L_1}} \right)^{1/2} \right\} - \delta^2 + \frac{1}{2} + o(1). \quad (24) \]

Now we evaluate \( E\left\{ \left( \frac{1}{\dot{\lambda}_{L_1}} \right)^{1/2} \right\} \). To this end, we write (16) as
\[ L_1 = \inf \left\{ n \geq m; \sum_{j=1}^{n-1} Z_j \leq (n-1)(n+\eta\delta^2) \right\}. \]

Let us define another stopping rule \( L_1^* = L_1 - 1 \) by
\[ L_1^* = \inf \left\{ n \geq m - 1; \sum_{j=1}^{n} Z_j \leq (\eta(n^*+\delta^2))^{-2} n^3 \left( 1 + \frac{1 + \eta\delta^2}{n} \right) \right\}. \quad (25) \]

Comparing (25) with equation (1.1) of Woodroofe (1977), we obtain in his notations
\[ S_n = \sum_{j=1}^{n} Z_j, \quad c = (\eta(n^*+\delta^2))^{-2}, \quad \alpha = 3, \quad \beta = 1/2, \quad \lambda = \eta(n^*+\delta^2), \quad \mu = 1 \]
\[ L_n = (1 + n^{-1}(1 + \eta\delta^2))^2, \quad L_0 = 2(1 + \eta\delta^2), \quad \tau^2 = 2, \quad a = 1/2, \quad \nu = 0.634. \]
It now follows from his Theorem 2.4 that, for all \( m \geq 2 \), as \( c \to 0 \)

\[
E(L_1^*) = \eta n^* - 1.433 + o(1).
\]

Since \( L_1^* = L_1 - 1 \), we get

\[
E(L_1) = \eta n^* - 0.433 + o(1). \tag{26}
\]

Let us consider the difference

\[
D_a = L_1 - \eta \left[ \left( \frac{A}{c} \right)^{1/2} \left( \frac{1}{\lambda_{L_1}} \right)^{1/2} - \delta^2 \right]. \tag{27}
\]

It follows from Woodroofe (1977) that the mean of the asymptotic distribution of \( D_a \) is \( \nu = 0.634 \). Now from (26) and (27) that, for all \( m \geq 2 \), as \( c \to 0 \)

\[
\eta \left( \frac{A}{c} \right)^{1/2} E \left[ \left( \frac{1}{\lambda_{L_1}} \right)^{1/2} \right] = E(L_1) - 0.634 + \eta \delta^2
\]

or

\[
\left( \frac{A}{c} \right)^{1/2} E \left[ \left( \frac{1}{\lambda_{L_1}} \right)^{1/2} \right] = n^* - 1.067\eta^{-1} + o(1). \tag{28}
\]

Result (19) now follows on combining (24) and (28).

Now Let \( h(L_1) = \frac{L_1 - \eta n^*}{(\eta n^*)^{1/2}} \). It follows from a result of Bhattacharya and Mallik (1973) that \( h(L_1) \overset{L}{\to} N(0, \frac{1}{2}) \) as \( c \to 0 \) (where \( \overset{L}{\to} \) stands for convergence in distribution) and from Theorem 2.3 of Woodroofe (1977), \( h^2(L_1) \) is uniformly integrable for all \( m \geq 2 \). Hence

\[
E[h^2(L_1)] = \frac{1}{2} + o(1). \tag{29}
\]

Using (29), we have

\[
\text{Var}(N_1 + \delta^2) = (2\eta)^{-1} n^* + o(c^{-1/2}). \tag{30}
\]

For \( 0 < \epsilon < 1 \), from Lemma 2.3 of Woodroofe (1977), as \( c \to 0 \),

\[
P(N_1 + \delta^2 \leq \epsilon n^*) \leq P(L_1 \leq \epsilon n^*) = O(c^{(m-1)/2}). \tag{31}
\]

From (19) and (31), on the event \( N_1 + \delta^2 \leq \epsilon n^* \), for all \( m \geq 4 \)

\[
R_{N_1}(c) \leq \left[ cn^* \left( \frac{n^*}{m + \delta^2} \right) + cn^* \right] P(N_1 + \delta^2 \leq \epsilon n^*)
\]

\[
= O(c^{(m-1)/2}) + O(c^{m/2})
\]

\[
= o(c). \tag{32}
\]

Furthermore, on the event \( N_1 + \delta^2 > \epsilon n^* \), for the random variable ‘\( U \)’ defined by

\[
|U - 1| \leq \left| \left( \frac{N_1 + \delta^2}{n^*} \right) - 1 \right|,
\]

\( U^{-3} \leq \epsilon^{-3} \), i.e., \( U^{-3} \) is bounded. It is easy to see that,
U^{a.s.}, as \(c \to 0\) \hspace{1cm} (33)

where \(a.s.\) stands for convergence almost surely. Using (19, 30, 32), boundedness of \(U^{-3}, N'_1 = N_1 + \delta^2\) and Taylor series expansion, we can easily obtain the result (20) from the following expression:

\[
R_{N_1}(c) = cn_0 E \left[ 1 - \left( \frac{N'_1}{n_0} - 1 \right) + \left( \frac{N'_1}{n_0} - 1 \right)^2 U^{-3} \right] + cE(N_1).
\]

3.2. Bounded Risk Point Estimation

Let \(\eta \in (0, 1)\) be specified. We start with a sample of size \(m \geq 2\) from the population (1), where \(m\) is chosen so as to satisfy \(m = o(1)\) as \(A \to \infty\) and \(\lim_{A \to \infty} \sup \left( \frac{m}{n^2} \right) < 1\). We start sampling sequentially with the stopping time \(L_2\) defined by

\[
L_2 = \inf \left\{ n \geq m; \; n \geq \eta \left( \frac{A}{w} \right) \left( \frac{1}{\lambda_n} \right) - \delta^2 \right\}. \hspace{1cm} (34)
\]

Based on these \(L_2\) observations, we compute \(\hat{\lambda}_{L_2}^{-1}\). Then we jump ahead and take \(N_2 - L_2\) observations, where

\[
N_2 = \max \left\{ L_2, \; \left( \frac{1}{w} \right) \left( \frac{1}{\hat{\lambda}_{L_2}} \right) - \delta^2 + 1 \right\}. \hspace{1cm} (35)
\]

After stopping, we estimate \(\mu\) by \(\tilde{\mu}_{N_2}\), incurring the risk

\[
R_{N_2}(A) = w \left\{ E \left( \frac{n^* + \delta^2}{N_4 + \delta^2} \right)^2 \right\}. \hspace{1cm} (36)
\]

The following theorem provide the second-order asymptotics for the expected sample size and risk associated with the accelerated sequential procedure (34)-(35):

**Theorem 2.** For the accelerated sequential procedure defined in (34)-(35) and all \(m \geq 4\), as \(w \to 0\)

\[
E(N_2) = n_{00} - 2\eta^{-1} - \delta^2 + 1/2 + o(1), \hspace{1cm} (37)
\]

\[
R_{N_2}(A) = w + \frac{5w}{2\eta(n_{00} - \delta^2)} + o(A^{-1}). \hspace{1cm} (38)
\]

**Proof.** The proof is exactly similar to that of Theorem 1. We omit the details for brevity. \(\square\)

4. \(k\)-Stage (\(k \geq 3\)) Procedures

4.1. Minimum Risk Point Estimation

We fix \(k \geq 3\) and the constants \(0 < c_1 < \ldots < c_{k-2} < 1\). Then we take a first sample of size \(m_0\) and take the next \(k-2\) samples sequentially with the \(i^{th}\) (\(i = 2, \ldots, k-1\)) sample of size \(M_{i-1} - M_{i-2}\), where
\[ M_j = \max \left\{ c_j \left( \frac{A}{\epsilon} \right)^{1/2} \left( \frac{1}{\lambda_{M_{j-1}}} \right)^{1/2} - \delta^2 \right\} + 1, M_{j-1} \]  

where \( 1 \leq j \leq k - 2 \) and \( M_0 = m_0 \). Now we take a final sample of size \( M_{k-1} - M_{k-2} \), where

\[ M_{k-1} = \max \left\{ \left( \frac{A}{\epsilon} \right)^{1/2} \left( \frac{1}{\lambda_{M_{k-2}}} \right)^{1/2} - \delta^2 + \eta \right\} + 1, M_{k-2} \]  

where \( \eta \) is a predetermined constant. After stopping, we estimate \( \mu \) by \( \tilde{\mu}_{M_{k-1}} \).

The following theorem provide second-order asymptotics for the expected sample size and risk associated with our proposed \( k \)-stage procedure (39)-(40):

**Theorem 3.** Assume that, \( n^* \to \infty, m_0 = m_0(n^*) \to \infty, \lim \sup \frac{m_0}{n^*} < c_1 \) and \( n^* = O(m_0^r) \), where \( r \geq 1 \) is a fixed constant. Then

\[ E(M_{k-1}) = m_0 - \frac{2}{c_{k-2}} + \frac{1}{2} + \eta - \delta^2 + o(1), \]  

\[ R_{M_{k-1}}(c) = 2cn_0 + c \left( \frac{1}{2} - \delta^2 \right) + o(c^{1/2}). \]  

**Proof.** Proof of this theorem can be found in the online supplement.

4.2. Bounded Risk Point Estimation

Again we fix \( k(\geq 3) \) and the constants \( 0 < c_1 < \ldots < c_{k-2} < 1 \) and then take a first sample of size \( m_0 \) and take the next \( k - 2 \) samples sequentially with the \( i \)-th \((i = 2, \ldots, k - 1) \) sample of size \( M'_{i-1} - M'_{i-2} \), where

\[ M_j' = \max \left\{ c_j \left( \frac{A}{w} \right) \left( \frac{1}{\lambda_{M_{j-1}}} \right) - \delta^2 \right\} + 1, M_{j-1} \]  

where \( 1 \leq j \leq k - 2 \) and \( M'_0 = m_0 \). Now we take a final sample of size \( M'_{k-1} - M'_{k-2} \), where

\[ M'_{k-1} = \max \left\{ \left( \frac{A}{w} \right) \left( \frac{1}{\lambda_{M'_{k-2}}} \right) - \delta^2 + \epsilon \right\} + 1, M'_{k-2} \]  

where \( \epsilon \) is a predetermined constant. After stopping, we estimate \( \mu \) by \( \tilde{\mu}_{M'_{k-1}} \).

The following theorem provide second-order asymptotics for the expected sample size and risk associated with our proposed \( k \)-stage procedure (43)-(44):

**Theorem 4.** Assume that, \( n'' \to \infty, m_0 = m_0(n'') \to \infty, \lim \sup \frac{m_0}{n''} < c_1 \) and \( n'' = O(m_0^r) \), where \( r \geq 1 \) is a fixed constant. Then
Table 1. Simulation results from 10000 replications of the minimum risk methodology (16)-(17) with $\mu = 5, \lambda = 2, A = 100$.

| $m$ | $n^*$ | $c$ | $\bar{x}, s_{\bar{x}}$ | $\bar{n}, s_{\bar{n}}$ | $\bar{n}/n^*$ | $\bar{z}$ | $\bar{\zeta}$ |
|-----|-------|-----|-----------------|-----------------|---------------|-----------|-----------|
| 12  | 50    | 0.0181 | 5.0563, 0.0359 | 48.84, 0.2446 | 0.9768        | 1.8829    | 1.7866    |
| 12  | 100   | 0.0047 | 4.9826, 0.0264 | 94.30, 0.6625 | 0.9430        | 0.9909    | 0.9598    |
| 12  | 200   | 0.0012 | 4.9546, 0.0174 | 198.89, 0.4514 | 0.9994        | 0.4920    | 0.4858    |
| 12  | 350   | 0.0004 | 5.0081, 0.0134 | 350.21, 0.5593 | 1.0006        | 0.2830    | 0.2810    |
| 12  | 500   | 0.0001 | 4.9916, 0.0112 | 505.61, 0.4209 | 1.0112        | 0.1985    | 0.1976    |
| 12  | 750   | 8.82 $\times 10^{-5}$ | 5.0043, 0.0093 | 750.13, 0.6110 | 1.0001        | 0.1327    | 0.1322    |

$$E(M'_{k-1}) = n_{00} - \frac{2}{c_{k-2}} + \frac{1}{2} + \epsilon - \delta^2 + o(1),$$

$$R_{M'_{k-1}}(A) = w - \frac{w}{n_{00} - \delta^2} \left\{ \frac{5}{2} - \frac{2}{c_{k-2}} + \epsilon \right\} + o(A^{-1}).$$

Proof. Proof of this theorem can be found in the online supplement.

5. Simulation Study

5.1. Minimum Risk Accelerated Sequential Problem

We now present simulation results for the accelerated sequential procedure given in (16)-(17). These are summarized in Table 1. We first generated a set of pseudorandom observations at-a-time from the distribution given in (1). Each row in Table 1 corresponds to results from 10,000 replications where results are tabulated for different combinations of $\mu, \lambda, A, m$ and $n^*$, where $n^*$ comes from (6). We first pick $\eta$ ($0 < \eta < 1$) randomly and run the sequential procedure.

Each block in Table 1 shows $n^*$ (column 2), $c$ (column 3), the values $\bar{x}$ along with their standard errors $s_{\bar{x}}$ (column 4), the values $\bar{n}, s_{\bar{n}}$ (column 5), the ratio $\bar{n}/n^*$ (column 6), the values $\bar{z}$ (column 7) where $\bar{z}$ denotes the average risk, and $\bar{\zeta}$ (column 8), where $\bar{\zeta}$ denotes the expansion as per (20) and one should expect $\bar{z}$ to be close to $\bar{\zeta}$.

One can note that the values of $\bar{n}$ seem to estimate $n^*$ reasonably well across the rows, which become closer for larger values of $n^*$, indicating a superior first-order efficiency. The $\bar{x}$ values are also seen to be very good at estimating the true mean of 5 in each case. All the risk values are also seen to be close to their corresponding $\bar{\zeta}$ values.

5.2. Bounded Risk Accelerated Sequential Problem

We now present simulation results for the accelerated sequential procedure given in (34)-(35). These are summarized in Table 2. We first generated a set of pseudorandom observations at-a-time from the distribution given in (1). Each row in Table 2 corresponds to results from 10,000 replications where results are tabulated for different combinations of $\mu, \lambda, m$ and $n^{**}$, where $n^{**}$ comes from (13). We first pick $\eta ..$ randomly and run the sequential procedure.

Each block in Table 2 shows $n^{**}$ (column 2), $\omega$ (column 3), the values $\bar{x}$ along with their standard errors $s_{\bar{x}}$ (column 4), the values $\bar{n}, s_{\bar{n}}$ (column 5), the ratio $\bar{n}/n^{**}$ (column 6), the values $\bar{z}$ (column 7), the values $\bar{\zeta}$ (column 8), where $\bar{\zeta}$ denotes the average risk, and $\bar{\zeta}$ (column 9), where $\bar{\zeta}$ denotes the expansion as per (20) and one should expect $\bar{z}$ to be close to $\bar{\zeta}$.
6), the values $\bar{z}$ (column 7) where $\bar{z}$ denotes the average risk, and $\bar{z}$ (column 8), where $\bar{z}$ denotes the expansion as per (38) and one should expect $\bar{z}$ to be close to $\bar{z}$.

One can note that the values of $\bar{z}$ seem to estimate $n^{**}$ reasonably well across the rows, which become closer for larger values of $n^{**}$, indicating a superior first-order efficiency. The $\bar{x}$ values are also seen to be very good at estimating the true mean of 5 and 3 under each case. All the risk values are also seen to be close to their corresponding $\bar{z}$ values.

5.3. Minimum Risk $k-$Stage Problem

We now present simulation results for the $k$-stage procedure given in (39)-(40). These are summarized in Table 3. We first generated a set of pseudorandom observations at-a-time from the distribution given in (1). We will concentrate particularly on a 4-stage and a 5-stage procedure ($k = 4, 5$). Each row in Table 3 corresponds to results from 10,000 replications where results are tabulated for different combinations of $\mu, \lambda, m$ and $n^*$. We also fix values of $\eta$ and $c_i$ under each case. Each run produces a value of $\bar{z}$ which instead should be fixed in practice, resulting in corresponding value of $n^*$.

Each block in Table 3 shows $k$ (column 1), $n^*$ (column 2), $c$ (column 3), the values $\bar{x}$ along with their standard errors $s_{\bar{x}}$ (column 4), the values $\bar{m}, s_{\bar{m}}$ (column 5), the ratio $\bar{m}/n^*$ (column 6), the values $\bar{z}$ (column 7) where $\bar{z}$ denotes the average risk, and $\bar{z}$ (column 8), where $\bar{z}$ denotes the expansion as per (42) and one should expect $\bar{z}$ to be close to $\bar{z}$.

One can note that the values of $\bar{m}$ seem to estimate $n^*$ reasonably well across the rows. Column 6 also produces desirable results, supporting a good second-order efficiency property of our proposed $k$-stage procedure. One can note that the approximations become closer in general, when $k = 5$. This is reasonably true since we are dealing with a higher-stage of sampling.

5.4. Bounded Risk $k-$Stage Problem

We now present simulation results for the $k$-stage procedure given in (43)-(44). These are summarized in Table 4. We first generated a set of pseudorandom observations at-a-time from the distribution given in (1). We will concentrate particularly on a 4-stage and a 5-stage procedure ($k = 4, 5$). Each row in Table 4 corresponds to results from 10,000 replications where results are tabulated for different combinations of $\mu, \lambda, m$ and $n^{**}$. We also fix values of $\eta$ and $c_i$ under each case. Each run produces a value of $\bar{z}$ which instead should be fixed in practice, resulting in corresponding value of $n^{**}$. 

Table 2. Simulation results from 10000 replications of the bounded risk methodology (34)-(35) with $\mu = 5, \lambda = 2, A = 100$.

| $m$ | $n^*$ | $\omega$ | $\bar{x}, s_{\bar{x}}$ | $\bar{z}, s_{\bar{z}}$ | $\bar{z}$, $s_{\bar{z}}$ | $\bar{z}$, $s_{\bar{z}}$ | $\bar{z}$, $s_{\bar{z}}$ |
|-----|-------|---------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 12  | 50    | 0.9523  | 5.0427, 0.0377         | 48.58, 0.5142          | 0.9736                 | 1.0814                 | 1.0358                 |
| 12  | 100   | 0.4878  | 4.9824, 0.0258         | 98.47, 0.7051          | 0.9847                 | 0.5212                 | 0.5081                 |
| 12  | 200   | 0.2469  | 5.0181, 0.0177         | 200.78, 0.7128         | 1.0039                 | 0.2489                 | 0.2503                 |
| 12  | 350   | 0.1418  | 4.9911, 0.0138         | 334.60, 1.0448         | 0.9560                 | 0.1575                 | 0.1495                 |
| 12  | 500   | 0.0995  | 4.9835, 0.0116         | 499.93, 1.0965         | 0.9995                 | 0.0999                 | 0.1000                 |
| 12  | 750   | 0.0664  | 4.9964, 0.0095         | 736.92, 3.9786         | 0.9825                 | 0.0696                 | 0.0683                 |
with a higher stage of sampling. In general, when $k > n$, the estimations become closer. Table 4 also shows that the efficiency property of our proposed accelerated sequential and $k$-stage procedures on a $k$-stage procedure. One can note that the approximations become closer in general, when $k = 5$. This is reasonably true since we are dealing with a higher stage of sampling.

### 6. Illustration Using Real Data

We will now apply our proposed accelerated sequential and $k$-stage procedures on a real data set which was first analyzed by Gerig and Sen (1980). This data is originally reported in the Canada migratory game bird surveys conducted over 10 provinces in Canada, and gives (in log to base 10) the kill of ducks over two years, 1969 and 1970. The individual observations are not available, however, the corresponding means and variances are given in their article. For convenience, we report the same in Table 5. This dataset was reported by several hunters selected from a stratified random sample of hunters, with provinces as the different strata. Corresponding to each province, the coefficient of variation of log kill per hunter was seen to be steady over the years. Hence, for future years, the CV for each province can be assumed to be known. One may also refer to Brazauskas and Ghorai (2007) for more information. Now, we tested the 10 mean values given in Table 4 using the “ig-test” package in R, which produced a
value of 0.3208, suggesting an inverse Gaussian fit for the means. Hence, one can assume that each individual sample within a province in a year, would also follow an IG distribution with the given mean and variance values. Without loss of generality, we will only focus on the province “Newfoundland” and year 1969. For this case, the estimated parameters become \( \hat{\mu} = 0.6649, \hat{\lambda} = 3.2355 \).

We now simulate a sample of 200 using these parameters, and assume that to be the underlying real data. For brevity alone, we will only present analyses for the minimum risk estimation strategy. Our bounded risk strategy behaves similarly. One may note that all the results are from a single run.

### 6.1. Minimum Risk Accelerated Sequential Problem

We first apply our accelerated sequential procedure (16)-(17) to the bird surveys data set. The pilot sample size \( m \) is taken as 12. Table 6 outlines the results. One can note that all the \( \hat{\mu} \) values are close to the actual mean of 0.6649 and the estimated \( n^* \) values are also close to the corresponding \( n^* \) values. The second to last column gives \( z \) which denotes the risk function and all those values are close to the corresponding \( \xi \) values which are given in the last column.

### 6.2. Minimum Risk \( k \)–Stage Problem

We now apply our \( k \)–stage procedure (39)-(40) to the bird surveys data set. Table 7 outlines the results for two different values of \( k \) namely 4 and 5, which denote a 4–stage and 5–stage rule respectively. One can observe that all the conclusions can be drawn exactly along similar lines as to Section 6.1. Further, the results are more accurate for a 5-stage procedure as compared to a 4-stage procedure.

| Region         | Mean  | Variance | CV  |
|----------------|-------|----------|-----|
| Newfoundland   | 0.6649| 0.2055   | 0.68|
| Prince Edward Island | 0.1998| 135.92   | 0.54|
| New Brunswick  | 0.8150| 0.1873   | 0.53|
| Nova Scotia    | 0.7903| 0.1865   | 0.55|
| Quebec         | 0.9383| 0.2355   | 0.52|
| Ontario        | 0.8022| 0.2203   | 0.59|
| Manitoba       | 0.9653| 0.1848   | 0.45|
| Saskatchewan   | 1.0085| 0.1663   | 0.40|
| Alberta        | 1.0817| 0.1889   | 0.40|
| British Columbia | 0.9299| 0.2186   | 0.50|

Table 6. Analysis of bird surveys data using accelerated sequential procedure (16)-(17) with \( m = 12, A = 100, \hat{\mu} = 0.6649, \hat{\lambda} = 3.2355 \).
7. Concluding Remarks

In this paper, we have developed accelerated sequential and $k-$stage ($k \geq 3$) procedures for an IG mean when the CV is known. We have used the estimator proposed by Searls (1964) for the estimation purpose. The main theoretical findings under Theorems 1-4 are expressed in terms of $n_0$ and $n_{00}$ respectively in order to compare them with the situation when sample mean is proposed as the estimator (the expressions for the sample mean case can be obtained by simply replacing $\delta = 0$). We conclude that for both minimum risk and bounded risk accelerated sequential schemes, the use of Searls’ estimator over the sample mean reduces the expected sample size and risk by $\delta^2$ and $c\delta^2$, respectively. We also conclude that larger the value of $\delta$, more beneficial is the use of Searls’ estimator. Similar conclusion can also be drawn for the proposed $k$-stage methodologies. Moreover, our multi-stage methodologies perform very well under different choices of the concerned parameters and also enjoy second-order efficiency properties. Our simulation analysis also supports this claim. Finally a real data example shows the practical utility of the proposed methodologies as well.

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Table 7. Analysis of bird surveys data using $k-$stage procedure (39)-(40) with $m = 12, A = 100$ $\hat{\mu} = 0.6649$, $\hat{\lambda} = 3.2355$.

| $k$ | $n^*$ | $c$ | $\hat{\mu}$ | $n$ | $n/n^*$ | $z$ | $\zeta$ |
|-----|-------|-----|-------------|----|---------|-----|-------|
| 4   | 100   | 0.0030 | 0.6432 | 112 | 1.12    | 0.6201 | 0.6165 |
| 5   | 150   | 0.0013 | 0.6768 | 152 | 1.01    | 0.4112 | 0.4113 |
| 5   | 100   | 0.0030 | 0.6512 | 107 | 1.07    | 0.6141 | 0.6173 |
| 5   | 150   | 0.0013 | 0.6602 | 151 | 1.01    | 0.4091 | 0.4107 |
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