Energy flow in a dispersive qubit read-out

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Abstract. We analyze a superconducting charge qubit that is dispersively coupled to an electric resonator. The system is connected to a transmission line that allows a reflection measurement. In this paper we derive the equations of motion of the system by using the quantum network theory. We assume that the measurement signal is so strong that the resonator behaves classically. The time evolution of the qubit is calculated with the Bloch equations. We have simulated the system in the adiabatic eigenbasis of the qubit to bring out the effects of the changing band curvatures under strong driving. We use circuit theory to calculate the energy flow in different parts of the circuit.

1. Introduction
Circuits built of Josephson junctions are macroscopic quantum objects that behave as scalable artificial atoms [1]. One of the most intriguing applications of these atoms is the use in quantum computation. It has been shown in many experiments [2, 3, 4] that superconducting atoms can be used as qubits that are the basic building blocks of a quantum computer. Josephson qubits are strongly coupled to the environment which on one hand allows a good connection with other qubits but on the other hand causes relaxation and dephasing due to the fluctuations [5]. By coupling the qubit dispersively to an electric $LC$ resonator one can reduce these undesired effects and do a fast quantum limited readout [6, 7]. This kind of qubit-oscillator is reminiscent of a real atom in a one-mode cavity and thus the system can be studied in terms of cavity quantum electrodynamics (cQED) [8].

In this paper we study theoretically a charge qubit in a dispersive L-SET read-out scheme [9]. In our setup a transmission line is coupled capacitively to the qubit-resonator system which is used for a reflection measurement. We derive the Hamiltonian of the circuit by using the quantum network theory [10, 11] and form the classical equations of motion with the Ehrenfest theorem. The time evolution of the qubit’s density matrix is calculated with the Bloch equations. We also study the flow of energy in the circuit by calculating numerically the power consumption in the resonator and in the qubit.

2. Circuit Analysis
In Fig. 1 we present the schematics of a circuit where a charge qubit, formed by a single Cooper pair transistor (SCPT), is coupled to a resonator and to a transmission line for a read-out. We model the transmission line and the dissipation in the resonator with infinite chains of harmonic oscillators. By choosing the node fluxes as the canonical coordinates one can derive the Hamiltonian for the whole circuit [11, 12, 13]

$$H = H_q + H_{osc} + H_{coll} + H_{int} + H_T + H_R$$
coordinates and corresponding momenta that are used in the derivation of the Hamiltonian. The resistor are modelled as chains of harmonic oscillators. The transmission line but the effect is neglected here by assuming that the measurement signal is so strong that the transmission line and resonator degrees of freedom, \( \Phi \) and \( q \), are canonically conjugated variables. In this paper we take a semiclassical approach and defined \( \Phi \) and \( q \), respectively. The island of the SCPT is also coupled to the transmission line and in the resonator (\( \Phi \), \( q \), and \( \Theta \), \( \Theta_1 \), \( \Theta_2 \)). This leads to an intuitive set of differential equations describing the behaviour of the apparatus.

\[
\begin{align*}
H_q &= \frac{(Q - Q_g)^2}{2C_S} - (E_{J_1} + E_{J_2}) \frac{\pi \Phi}{2\Phi_Q} \cos \frac{\pi \Phi}{\Phi_Q} \cos \frac{2\pi \Theta}{\Phi_Q} \sin \frac{\pi \Phi}{\Phi_Q} \sin \frac{2\pi \Theta}{\Phi_Q} \\
H_{osc} &= \frac{q_0^2}{2C} + \frac{(\Phi - \Phi_h)^2}{2L}, \quad H_{eovt} = \frac{q_0^2}{2C}, \quad H_{int} = \frac{q_0^2}{2C} \\
H_T &= \frac{1}{\Delta x L_T} \sum_{i=0}^{N_T} \left[ \frac{q_i}{2C} + \frac{(\Phi_{i+1} - \Phi_i)^2}{2L} \right], \quad H_R = \frac{1}{\Delta y} \sum_{i=0}^{N_T} \left[ \frac{p_i^2}{2C_R} + \frac{(\Phi_i - \Phi_{i-1})^2}{2L_R} \right],
\end{align*}
\]

where \( \Phi_Q = h/2e \) is the flux quantum and \( C_S = C_1 + C_2 + C_g \). We have assumed that the system is capacitively symmetric, i.e. \( 2C = C_3 = C_4 \) and \( C_1 = C_2 \). The first two terms in the Hamiltonian describe the SCPT coupled with an electric resonator via the flux degree of freedom \( \Phi \). One can use a constant magnetic flux \( \Phi_h \) through the inductance loop \( L \) for biasing purposes. The third term is the capacitive energy at the end of the transmission line. It is coupled to the SCPT-resonator with \( H_{int} \). The last two terms are responsible for the dissipation in the transmission line and in the resonator. The capacitances \( C_R, C_T \) and the inductances \( L_R, L_T \) per unit length determine the resistances \( R = \sqrt{L_R/C_R} \) and \( Z_0 = \sqrt{L_T/C_T} \) for the resonator and transmission line, respectively. The island of the SCPT is also coupled to the transmission line but the effect is neglected here by assuming \( C_g \ll C \). We have also set the coupling capacitance \( C_e \ll C \) and defined \( \Delta x = l_T/N_T \) and \( \Delta y = l_R/N_R \), where \( l_T \) and \( l_R \) are the lengths of the transmission line and the resonator, respectively. The charge \( p \) is a constant of motion according to Lagrange’s equation and can be set \( p \equiv Q_g \).

We quantize the Hamiltonian by assuming the commutation relation \( [\Phi, Q] = i\hbar \) between all canonically conjugated variables \( \Phi_i \) and \( Q_i \). In this paper we take a semiclassical approach and assume that the measurement signal is so strong that the transmission line and resonator degrees of freedom behave classically. We calculate the expectation values of the conjugated operators with the Ehrenfest theorem. In the end, we go to the limit of infinite number of oscillators in the transmission line and in the resonator \( (N_T, N_R \to \infty) \). This leads to an intuitive set of differential equations describing the behaviour of the apparatus:

\[
\begin{align*}
\dot{\Phi} &= -\frac{\partial H_q}{\partial q} = \frac{q}{C} + \frac{q_0}{2C}, \\
\dot{\Phi}_h &= -\frac{\partial H_q}{\partial \Phi_h} = -\frac{\Phi - \Phi_h}{L} - \frac{\Phi}{R} \\
\dot{\Phi} &= \frac{\partial H}{\partial \Phi} = \frac{q}{C} + \frac{q_0}{2C} \\
\dot{\Phi}_h &= \frac{\partial H_q}{\partial \Phi_h} = \frac{q}{C} + \frac{q_0}{2C},
\end{align*}
\]
\[ \dot{\psi}_0 = \frac{1}{Z_0} (\dot{\phi}^{in}_0 - \dot{\phi}^{out}_0) \]
\[ \dot{\phi}_1 = \dot{\phi}^{in}_1 + \dot{\phi}^{out}_1 = \frac{g_0}{C_e} + \frac{q + g_0}{2C_c}. \]

Quantum mechanics comes into the play in the expectation value of the current through the SCPT which can be calculated with the master equation for the density matrix.

Part of the measurement signal \( \dot{\phi}^{in}_m = V_{in} \sin \omega_{rf} t \) is reflected at the end of the transmission line due to the mismatch of the transmission line and the system impedances. Experimentally one can monitor the reflection coefficient \( \Gamma = V_{out}/V_{in} \) where \( V_{out} \) is the Fourier component of the reflected signal at the measurement frequency.

**3. Adiabatic qubit basis**

We are interested in SCPT at the charge qubit limit. The energy levels of the SCPT can be controlled with the gate charge \( Q_g \) and external flux \( \Phi_b \). In the vicinity of the avoided crossing point \( (Q_g = \varepsilon, \Phi_b = 0.5\Phi_Q) \) we can approximate the SCPT with the qubit Hamiltonian

\[ H_q = \frac{1}{2} \left[ E_d \sigma_z - E_J \cos \frac{\pi}{\Phi_Q} \sigma_x + dE_{j0} \sin \frac{\pi}{\Phi_Q} \sigma_y \right], \tag{3} \]

were \( E_d = 4E_c(1 - Q_g/e) \) is the electric energy of the qubit and \( E_c = e^2/2C_c \). We denote the total Josephson energy by \( E_{j0} = E_{j1} + E_{j2} \) and the asymmetry parameter by \( d = (E_{j1} - E_{j2})/(E_{j1} + E_{j2}) \).

We make a transformation into the adiabatic eigenbasis of the qubit since the driving of the resonator induced by the measurement signal can be strong. This way we can take into account the effects of the changing band curvatures. We use a unitary transformation

\[ U^\dagger(t) = \begin{pmatrix} \cos(\eta/2) e^{-i\gamma/2} & -\sin(\eta/2) e^{-i\gamma/2} \\ \sin(\eta/2) e^{i\gamma/2} & \cos(\eta/2) e^{i\gamma/2} \end{pmatrix}, \tag{4} \]

where \( \tan \eta = -E_J/E_d, E_J = 2E_{j0} \sqrt{\cos^2(\pi/\Phi_Q) + d^2 \sin^2(\pi/\Phi_Q)} \) and \( \tan \gamma = -d \tan(\pi/\Phi_Q) \). The resulting rotated Hamiltonian \( \tilde{H} = UHU^\dagger + i\hbar \tilde{U}U^\dagger \) can be written in the pseudo-spin form \( \tilde{H} = -\hbar \tilde{\sigma} \cdot \mathbf{B} \) where the magnetic field is defined as

\[ B_x = -\gamma \sin \eta \]
\[ B_y = \eta \]
\[ B_z = -\Omega_0 + \gamma \cos \eta \tag{5} \]

The qubit level splitting \( \hbar \Omega_0 = \sqrt{E_d^2 + E_J^2} \) has a minimum at \( \Phi = \Phi_Q/2 \) for every \( Q_g \). The time evolution of the of the qubit’s density matrix is solved with Bloch equations. The current through the qubit can be calculated from \( I_Q = \langle \frac{\partial H}{\partial \sigma_x} \rangle = \text{Tr} (\rho \frac{\partial H}{\partial \sigma_x}) \).

**4. Energy flow**

We have kept track of the energy economics of the cQED system by studying the division of the transmitted power \( P_T = (1 - |\Gamma|^2)P_{in} \) between the resonator and the qubit. Here \( P_{in} = V_{in}^2/2Z_0 \). The total power dissipated in the resonator is \( P_R = V_{rms}^2/R \) where \( V_{rms} \) is the root mean square value of the voltage of the resonator. The qubit radiates energy into the environment in the steady state at power

\[ P_Q = \frac{\hbar \Omega_0}{T_{rf}} \int_0^{T_{rf}} \frac{S_z - S_{\sigma_0}}{T_1}, \tag{6} \]
where $T_{rf} = 2\pi/\omega_{rf}$, $S_z$ is the equilibrium value of z-component of the pseudo-spin $S_z = \hbar \langle \sigma_z \rangle / 2$ and $T_1$ is the relaxation time of the qubit. One expects $P_T - P_R - P_Q = 0$ in the steady state.

We have visualized the energy flow in the system at charge degeneracy $Q_g = e$ in Fig. 2. When the driving is weak one expects the qubit to stay in the ground state and the transmitted power is dissipated in the resonator (blue area). At larger drives, the qubit can make Landau-Zener (LZ) transitions to the excited state by absorbing oscillator quanta [14, 15]. Interference of the LZ-transitions can occur at bias points corresponding to $k = 10, 11, 12, 13$ oscillator quanta can be seen. We have used $E_f = 3.2$ K, $E_c = 0.8$ K, $d = 0.04$, $C = 196.2$ pF, $L = 169$ pH, $C_z = 7$ pF, $R = 2800$ Ω, $P_m = -129$ dBm, $f_{rf} = 872$ MHz and $Z_0 = 50$ Ω. We have also used the relaxation time $T_1 = 40$ ns and dephasing time $T_2 = 2$ ns for the qubit.

![Figure 2. Energy flow in cQED system as a function of flux bias at charge degeneracy $Q_g = e$. The thick black line is the total transmitted power. The blue area describes the power dissipated in the resonator. The red area in between denotes the power loss in the qubit. Interferences of LZ-transitions that take place at bias points corresponding to $k = 10, 11, 12, 13$ oscillator quanta can be seen. We have used $E_f = 3.2$ K, $E_c = 0.8$ K, $d = 0.04$, $C = 196.2$ pF, $L = 169$ pH, $C_z = 7$ pF, $R = 2800$ Ω, $P_m = -129$ dBm, $f_{rf} = 872$ MHz and $Z_0 = 50$ Ω. We have also used the relaxation time $T_1 = 40$ ns and dephasing time $T_2 = 2$ ns for the qubit.](image)

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