Cauchy Problem and Green’s Functions for First Order Differential Operators and Algebraic Quantization

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Abstract. Existence and uniqueness of advanced and retarded fundamental solutions (Green’s functions) and of global solutions to the Cauchy problem is proved for a general class of first order linear differential operators on vector bundles over globally hyperbolic Lorentzian manifolds. This is a core ingredient to CAR-/CCR-algebraic constructions of quantum field theories on curved spacetimes, particularly for higher spin field equations.

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I. MOTIVATION AND PHYSICAL GOAL

In the setting of general relativity, various types of physical fields (and, in a suitable sense, also quantized fields) can be described as solutions $\Phi$ to a differential equation

$$P\Phi = 0, \quad \Phi \in \Gamma(E).$$

Here, $E$ is a real or complex vector bundle over the Lorentzian spacetime manifold $M$ with metric $g$, and $P$ is a linear differential operator while $\Gamma(E)$ denotes the space of smooth sections in $E$. For the scalar case where $P$ is the Klein-Gordon operator and $E$ is the complex line bundle on $M$ as well as for the spin $\frac{1}{2}$ case where $P$ is the Dirac operator and $E$ the bundle of Dirac spinors on $M$, Dimock (1980, 1982) proposed algebraic quantum field theory constructions based on certain $C^*$-algebras (CAR-/CCR-algebras) of local observables, derived from the field operators (for an outline of such a construction cf. sec. [V]). It was shown that these quantum field theories are in fact locally covariant, cf. Brunetti et al. (2003), Sanders (2010). Core ingredients to both the constructions and the general covariance result are the well-posedness of the Cauchy problem and the existence of unique advanced and retarded fundamental solutions (Green’s functions) for the differential operator $P$, which are both, in the end, consequences of global hyperbolicity of $(M, g)$.

It is of course desirable to have such a quantum field theory for every type of quantum fields (particularly for fields of higher spin) or at least to have established for which fields such a construction may be possible. This is why in this work, for a quite general class of first order differential operators on globally hyperbolic spacetimes we present results on existence of advanced and retarded Green’s functions and well-posedness of the Cauchy problem, which are crucial requirements for the CAR-/CCR-algebraic quantum field theory construction as indicated above.

While many of the classic references (like Dimock 1980, 1982) still had to refer back to the famous unpublished lecture notes by Leray 1953 when dealing with these questions for the scalar and spin $\frac{1}{2}$ cases, meanwhile we have a powerful theory of second order normally hyperbolic differential operators on globally hyperbolic spacetimes at hands, presented in modern mathematical language by Bär, Ginoux, Pfaffel (2007). We shall make use of this work to derive our results on the first order case.

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The starting point of our work was an investigation of the generalized Dirac equations for higher spin fields as proposed by Buchdahl (1982), for which so far there existed only local results on the well-posedness of the Cauchy problem (cf. Wünsch, 1985). It turned out that for wide parts of our considerations (which may be found in full detail in Mühlhoff, 2007), committing to a special case of differential operator was not necessary. The present paper presents the first and general part of our work, while in a forthcoming publication we are planning to address applications to Buchdahl’s and other higher spin field equations.

II. INTRODUCTION AND NOTATION

Throughout this article, let \((M, g)\) be an \(n\)-dimensional smooth, time oriented, connected, globally hyperbolic Lorentzian manifold of signature \((+ - - \ldots)\) and let \(\mathcal{E}\) be a smooth, finite rank vector bundle on \(M\). We denote the space of smooth sections of \(\mathcal{E}\) by \(\Gamma(\mathcal{E})\) and the space of compactly supported smooth sections by \(\Gamma_0(\mathcal{E})\).

A second order linear differential operator \(L: \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})\) is called \textit{normally hyperbolic}, if its principal symbol \(\sigma_L\) “is given by the metric”:

\[
\forall x \in M \forall \xi \in T^*_x M: \sigma_L(\xi) = g(\xi, \xi) \text{Id}_{\mathcal{E}_x},
\]

where \(g\) also denotes the inverse metric of \(g\) on \(T^*M\). (For a definition of the principal symbol \(\sigma_L\) cf. e.g. Bär et al. (2007).) Equivalently this means that \(L\) can locally be written as

\[
L \Phi = g^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \Phi + \text{lower order derivatives of } \Phi, \quad \Phi \in \Gamma(\mathcal{E}).
\]

For normally hyperbolic (hence, second order) operators, the global Cauchy problem and existence of advanced and retarded Green’s functions were extensively treated in Bär et al. (2007). As this article is about first order operators, we shall now define a property which will play a role in our theory analogous to normal hyperbility for the second order case:

\textbf{Definition 1.} \textit{We call a first order linear differential operator }\(P: \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})\) \textit{prenormally hyperbolic}, if there is another first order linear differential operator \(Q: \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})\), such that \(PQ\) is normally hyperbolic. We call such a pair of \(P\) and \(Q\) a \textit{complementary pair of prenormally hyperbolic first order operators}.

As a first useful property of prenormally hyperbolic differential operators, we shall prove:

\textbf{Lemma 1.} \textit{Let }\(P: \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})\) \textit{be a prenormally hyperbolic first order differential operator. Denote by }\(\sigma_P\) \textit{its principal symbol. Then for every }\(x \in M\) \textit{and for every }\(\xi \in T^*_x M\) \textit{with }\(g_x(\xi, \xi) \neq 0\), \textit{the endomorphism }\(\sigma_P(\xi): \mathcal{E}_x \rightarrow \mathcal{E}_x\) \textit{is invertible.}

\textit{Proof.} Let \(Q: \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})\) be another prenormally hyperbolic first order differential operator such that \(Q, P\) form a complementary pair. Denote by \(\sigma_Q\) the principal symbol of \(Q\). For the principal symbol \(\sigma_{QP}\) of \(QP\) we find at \(x\):

\[
\sigma_{QP}(\xi) = \sigma_Q(\xi) \circ \sigma_P(\xi) = g(\xi, \xi) \text{Id}_{\mathcal{E}_x}.
\]

Hence, \(\sigma_P(\xi)\) is an automorphism if \(g(\xi, \xi) \neq 0\). \(\square\)

It turns out that in definition 1 instead of \(PQ\) being normally hyperbolic we could equivalently demand that \(QP\) be normally hyperbolic:

\textbf{Lemma 2.} \textit{Let }\(P, Q: \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})\) \textit{be first order linear differential operators. Then if }\(PQ\) \textit{is normally hyperbolic, }\(QP\) \textit{is normally hyperbolic, too.}

\textit{Proof.} \(PQ\) being normally hyperbolic means at every point \(x \in M\):

\[
\forall \xi \in T^*_x M: \sigma_P(\xi) \circ \sigma_Q(\xi) = g_x(\xi, \xi) \text{Id}_{\mathcal{E}_x}.
\]

First, let the co-vector \(\xi\) be non-null. Then by lemma 1 \(\sigma_P(\xi)\) and \(\sigma_Q(\xi)\) are invertible and thus elements of \(\text{GL}(\mathcal{E}_x)\). As in groups right inverses are also left inverses, we can deduce that

\[
\sigma_Q(\xi) \circ \sigma_P(\xi) = g_x(\xi, \xi) \text{Id}_{\mathcal{E}_x}.
\]
Finally, as both sides of this equation depend continuously on $\xi$, it also holds for null co-vectors. Hence we conclude that $QP$ is normally hyperbolic. \hfill $\square$

**Example 1.** Let $M$ have a spin structure, let $\mathcal{D}M$ be the spinor bundle on $M$, let $D: \Gamma(\mathcal{D}M) \to \Gamma(\mathcal{D}M)$ be the Dirac operator and let $A: \Gamma(\mathcal{D}M) \to \Gamma(\mathcal{D}M)$ be any operator of order $0$ (i.e. a linear map on the fiber).

Then $D + A$ is prenormally hyperbolic since using the Lichnerowicz formula we find:

$$(D + A)(D - A) = \Box + \frac{\text{scal}}{4} - [D, A] - A^2,$$

where $\text{scal}$ denotes the scalar curvature of the metric $g$ on $M$ and $\Box$ denotes the Laplacian with respect to the covariant derivative on $\mathcal{D}M$ induced by the Levi-Civita covariant derivative on $TM$ (d’Almbergtian). $\Box$ is the only second order term and it is normally hyperbolic. Hence, $(D + A)(D - A)$ is normally hyperbolic.

### III. Existence of Green’s Functions

As Dimock (1982) did for the Dirac special case, we shall now establish existence and uniqueness of advanced and retarded Green’s operators (in Dimock’s terminology: fundamental solutions) for our general class of prenormally hyperbolic operators. We do this by generalizing Dimock’s proof, facilitating results from Bär et al. (2007).

**Definition 2.** Let $P: \Gamma(E) \to \Gamma(E)$ be a first order linear differential operator on sections of $E$. Linear maps $G_{\pm}: \Gamma_0(E) \to \Gamma(E)$ are called **advanced** ($-\$ resp. **retarded** ($+$) Green’s functions for $P$, if

(i) $P \circ G_{\pm} = \text{Id}_{\Gamma_0(E)}$,

(ii) $G_{\pm} \circ P = \text{Id}_{\Gamma_0(E)}$,

(iii) $\forall \varphi \in \Gamma_0(E): \text{supp}(G_{\pm}\varphi) \subseteq J_{\pm}(\text{supp}(\varphi)).$

Here, for $A \subseteq M$, $J_+(A)$ resp. $J_-(A)$ denotes the **causal future** resp. **past** of the set $A$, i.e. the set of points in $M$, which are either points in $A$ or which can be reached from a point in $A$ by a future resp. past directed, piecewise $C^1$ curve.

**Theorem 1.** Let $P: \Gamma(E) \to \Gamma(E)$ be a prenormally hyperbolic first order linear differential operator on sections of $E$. Then there exist unique advanced and retarded Green’s functions $S_{\pm}: \Gamma_0(E) \to \Gamma(E)$ for $P$.

**Proof.** Choose $Q: \Gamma(E) \to \Gamma(E)$ such that $P, Q$ form a complementary pair of prenormally hyperbolic first order operators (i.e. $PQ$ and $QP$ are normally hyperbolic). Let $E^*$ denote the dual bundle of $E$ and let $P^*, Q^*: \Gamma(E^*) \to \Gamma(E^*)$ denote the formal adjoint of $P$ resp. $Q$.

As $QP$ and $PQ$ are normally hyperbolic, so are $(QP)^* = P^*Q^*$ and $(PQ)^* = Q^*P^*$. This enables us to apply Bär et al. (2007), corollary 3.4.3, which says that there exist unique advanced and retarded Green’s functions $G_{\pm}: \Gamma_0(E) \to \Gamma(E)$ for $PQ$ and $G_{\pm}': \Gamma_0(E^*) \to \Gamma(E^*)$ for $P^*Q^*$.

Define $S_{\pm} := QG_{\pm}$ and $S_{\pm}' := Q^*G_{\pm}'$. We will show that $S_{\pm}$ are advanced/retarded Green’s functions for $P$ and that they are unique. As a byproduct we will find that $S_{\pm}'$ are advanced/retarded Green’s functions for $P^*$. We have to check all three conditions from definition 2.

(i) $PS_{\pm}|_{\Gamma_0(E)} = PQG_{\pm}|_{\Gamma_0(E)} = \text{Id}_{\Gamma_0(E)}$ and $P^*S_{\pm}'|_{\Gamma_0(E^*)} = P^*Q^*G_{\pm}'|_{\Gamma_0(E^*)} = \text{Id}_{\Gamma_0(E^*)}$ as $G_{\pm}$ resp. $G_{\pm}'$ are Green’s functions for $PQ$ resp. $P^*Q^*$.

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1 For general terminology, cf. e.g. Berline et al. (1992) or Fewster and Verch (2002).

2 $P^*$ is the formal adjoint of $P$, if $\forall \varphi \in \Gamma_0(E) \forall \psi \in \Gamma_0(E^*): \int_M \langle P\varphi, \psi \rangle = \int_M \langle P^*\psi, \varphi \rangle$, where integration is with respect to the volume density induced by the Lorentz metric on $M$.

3 The formal adjoint of a normally hyperbolic operator is again normally hyperbolic (Mühlhoff, 2007, 7.2.4).
Here, we denote by $E|J$. Moreover, we set $J$.

Theorem 2. Let $\Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$ be a prenormally hyperbolic first order linear differential operator on the vector bundle $\mathcal{E}$ on $(M, g)$ and let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface. Then the Cauchy problem

\[
(P) \quad \begin{cases} P \Phi = 0, & \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_{\Sigma} = \Phi_0 \end{cases}
\]

has a unique solution for every initial datum $\Phi_0 \in \Gamma_0(\mathcal{E}|\Sigma)$. Moreover, this solution satisfies $\text{supp} \Phi \subseteq J(\text{supp} \Phi_0)$.

Here, we denote by $\mathcal{E}|_{\Sigma}$ the restriction of the bundle $\mathcal{E}$ on $M$ to the submanifold $\Sigma \subseteq M$. Moreover, we set $J(K) := J_+(K) \cup J_-(K)$.

Inspired by Dimock (1982), the idea of the proof is to reduce our first order problem to a Cauchy problem for a second order normally hyperbolic operator, and then we make use of the theory presented in Bär et al. (2007). Let $P, Q$ be a complementary pair of prenormally hyperbolic first order operators on $\mathcal{E}$ and let $n$ be the future directed unit normal vector field along $\Sigma$. We define the following auxiliary Cauchy problems:

\[
(QP) \quad \begin{cases} QP \Phi = 0, & \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_{\Sigma} = \Phi_0 \\ (\nabla_n \Phi)|_{\Sigma} = \Psi_0 \end{cases} \quad (Q\bar{P}) \quad \begin{cases} QP \Phi = 0, & \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_{\Sigma} = \Phi_0 \\ (Q\Phi)|_{\Sigma} = 0 \end{cases}
\]

for given $\Phi_0, \Psi_0 \in \Gamma_0(\mathcal{E}|\Sigma)$.

\[
(PQ) \quad \begin{cases} PQ \Phi = 0, & \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_{\Sigma} = \Phi_0 \\ (\nabla_n \Phi)|_{\Sigma} = \Psi_0 \end{cases} \quad (\bar{P}Q) \quad \begin{cases} PQ \Phi = 0, & \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_{\Sigma} = \Phi_0 \\ (Q\Phi)|_{\Sigma} = 0 \end{cases}
\]

for given $\Phi_0, \Psi_0 \in \Gamma_0(\mathcal{E}|\Sigma)$.

If $K, K' \subseteq M$ are two compact subsets of a globally hyperbolic $(M, g)$, then $J_+(K) \cap J_-(K')$ is compact \cite{Bär et al. 2007, A5.4).
Lemma 3.

(a) If $\Phi \in \Gamma(\mathcal{E})$ solves $\widetilde{Q}\Phi$ for initial datum $\Phi_0$, then $\Phi$ solves $(QP)$ for initial data $\Phi_0$ and $\Psi_0 := \nabla_a \Phi|_\Sigma$. The analogous result holds for $(PQ)$ and $(PQ)$.

(b) For given $\Phi_0 \in \Gamma_0(\mathcal{E}|\Sigma)$ there always exists a unique solution $\Phi \in \Gamma(\mathcal{E})$ to $(\widetilde{Q}\Phi)$. It satisfies $\text{supp} \Phi \subseteq J(\text{supp} \Phi_0)$. The analogous statement holds for $(PQ)$.

(c) $\Phi \in \Gamma(\mathcal{E})$ solves $(P)$ for initial datum $\Phi_0$ if and only if $\Phi$ solves $(\widetilde{Q}\Phi)$ for initial datum $\Phi_0$.

Proof. As a preparatory consideration, assume we are given $\Phi \in \Gamma(\mathcal{E})$ and $\Phi_0 \in \Gamma_0(\mathcal{E}|\Sigma)$ such that $\Phi|_\Sigma = \Phi_0$. Moreover, fix a point $x \in \Sigma$ and a local pseudo-orthogonal frame $(e_1, \ldots, e_n)$ on a neighborhood $U$, $x \in U \subseteq M$, such that $e_1|_{U \cap \Sigma} = n$ $(e_2, \ldots, e_n$ are then tangential to $\Sigma$ on $\Sigma \cap U$). Then we have the following equivalence of statements along $\Sigma \cap U$:

$$0 = (P\Phi)|_\Sigma = \sum_{i=1}^n \sigma_P(e_i^*) \nabla_{e_i} \Phi|_\Sigma + R_{P,U} \Phi|_\Sigma$$

$$\Leftrightarrow \sigma_P(n^*) \nabla_a \Phi|_\Sigma = -[\sigma_P(n^*)]^{-1} \left( \sum_{i=2}^n \sigma_P(e_i^*) \nabla_{e_i} \Phi_0 + R_{P,U} \Phi_0 \right)$$

Here, $\sigma_P \in \Gamma(T^* M \otimes \text{End}(\mathcal{E}))$ denotes the principal symbol of $P$, $R_{P,U} : \Gamma(\mathcal{E}|U) \to \Gamma(\mathcal{E}|U)$ is a linear operator of order 0, and for $v \in T_x M$, $\nu^- \in T_x^* M$ denotes the co-vector associated with $v$ by the metric $(\nu^* = g(v, \cdot))$, this is usual index-shifting. We also used invertibility of $\sigma_P$ in normal direction, which follows from lemma 1. We now prove the three statements:

(a) Let a solution $\Phi$ to $(\widetilde{Q}\Phi)$ for initial datum $\Phi_0$ be given. We only have to verify that $\Psi_0 := \nabla_a \Phi|_\Sigma$ has indeed compact support; but this follows immediately from (1).

(b) Existence and support property: Let $\Phi_0$ be given. Inspired by (1), we set using local pseudo-orthogonal frames $(n, e_2, \ldots, e_n)$ along $\Sigma$:

$$\Psi_0 := -[\sigma_P(n^*)]^{-1} \left( \sum_{i=2}^n \sigma_P(e_i^*) \nabla_{e_i} \Phi_0 + R_{P,U} \Phi_0 \right)$$

(so $\Psi_0 \in \Gamma_0(\mathcal{E}|\Sigma)$). Notice that the right hand side does not depend on the choice of local frame and hence $\Psi_0$ is well defined globally along $\Sigma$. Its support is a subset of $\text{supp}(\Phi_0)$ and thus compact. Using [Bär et al., 2007], thm. 3.2.11, there is a solution $\Phi$ to $(QP)$ for initial datum $(\Phi_0, \Psi_0)$, which moreover satisfies the support property. Finally, the preparatory consideration at the beginning of this proof shows that our choice of $\Psi_0$ is such that the solution $\Phi$ satisfies $(P\Phi)|_\Sigma = 0$; thus, $\Phi$ is a solution to $(\widetilde{Q}\Phi)$ for initial datum $\Phi_0$.

Uniqueness: If $\Phi'$ is another solution to $(\widetilde{Q}\Phi)$ then by (a) and (1), it solves $(QP)$ for $\Phi_0$ and $\Psi_0 = \Psi_0$. Thus, $\Phi' = \Phi$ by uniqueness of solutions to $(QP)$ (cf. [Bär et al., 2007], thm. 3.2.11).

(c) Fix $\Phi_0 \in \Gamma_0(\mathcal{E}|\Sigma)$. If $\Phi \in \Gamma(\mathcal{E})$ solves $(P)$ it is trivial that it also solves $(\widetilde{Q}\Phi)$. To prove the opposite direction, let $\Phi \in \Gamma(\mathcal{E})$ solve $(\widetilde{Q}\Phi)$, i.e. $QP\Phi = 0$, $\Phi|_\Sigma = \Phi_0$, $(P\Phi)|_\Sigma = 0$. By multiplying the first equation by $P$ from the left and by restricting the first equation to $\Sigma$ we easily obtain $PQ\Phi = 0$, $(P\Phi)|_\Sigma = 0$, $(QP\Phi)|_\Sigma = 0$, which means that $P\Phi$ solves $(PQ)$ for initial datum $\equiv 0$. According to (5), solutions to $(PQ)$ are unique, hence, $P\Phi = 0$ and $\Phi$ solves $(P)$. 

Proof of theorem 3. Now this follows immediately from 3.6 using 3.4.

Finally, as a corollary we may write down the following compatibility result for compactly supported Cauchy data on two different Cauchy hypersurfaces $\Sigma$ and $\Sigma'$:
Corollary 1 (compatibility of Cauchy data). Let $P : \Gamma(E) \to \Gamma(E)$ be as in theorem 3 and let $\Sigma, \Sigma' \subseteq M$ be two smooth spacelike Cauchy hypersurfaces. Then the spaces of compactly supported Cauchy data, $\Gamma_0(E|\Sigma)$ and $\Gamma_0(E|\Sigma')$ are in a one-one relation determined by having the same solution to the Cauchy problem: The map

$$\Gamma_0(E|\Sigma) \to \Gamma_0(E|\Sigma')$$

$$\Phi_0 \mapsto \Phi|_{\Sigma'},$$

for $\Phi$ the unique solution such that $\Phi|_{\Sigma} = \Phi_0$,

is an isomorphism of $\mathbb{C}$-vector spaces.

Proof. This follows immediately by uniqueness of solutions to the Cauchy problem. To establish that $\Phi|_{\Sigma'}$ is compact, make use of the well known fact on globally hyperbolic spacetimes that for compact $K \subseteq M$, $J(K) \cap \Sigma'$ is compact.

V. APPLICATION IN CAR-ALGEBRAIC QUANTIZATION

To illustrate an important application of our results, we shall briefly outline a basic quantization procedure using CAR-algebras for fermionic quantum fields described by a partial differential equation $P \Phi = 0$ for prenormally hyperbolic $P$ acting on sections of a vector bundle $E$ on $M$.

Let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface in $M$ with future directed unit normal vector field $n$ along $\Sigma$. We define the space of localized Cauchy data on $\Sigma$,

$$\mathcal{H}_\Sigma := \{ \Phi_0 \in \Gamma_0(E|\Sigma) \},$$

and the space of solutions with localized Cauchy data,

$$\mathcal{H} := \{ \Phi \in \Gamma(E) \mid P \Phi = 0 \text{ and } \Phi|_{\Sigma} \in \mathcal{H}_\Sigma \}.$$

Notice that due to the compatibility of compactly supported Cauchy data (corollary 1), $\mathcal{H}$ is independent of the choice of $\Sigma$.

Now, assume there is a physically distinguished Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ so that $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ forms a pre-Hilbert space. Then the system could be quantized by forming the CAR-algebra $\text{CAR}(\mathcal{H}, \langle \cdot, \cdot \rangle)$ (CAR = canonical anti-commutation relations, for definition of the unique CAR-algebra generated by the elements of a pre-Hilbert space cf. Bratteli and Robinson [1996]).

To perform such a construction, it may often be easier to work on a Cauchy hypersurface. For every smooth spacelike Cauchy hypersurface $\Sigma$, the canonical $\mathbb{C}$-vector space isomorphism

$$\mathcal{H} \to \mathcal{H}_\Sigma, \quad \Phi \mapsto \Phi|_{\Sigma},$$

(with inverse given by assigning to $\Phi_0$ the solution of the Cauchy problem with initial datum $\Phi_0$, cf. theorem 2) can be used to pull back a Hermitian inner product $\beta_\Sigma$ on $\mathcal{H}_\Sigma$ to a Hermitian inner product $\beta$ on $\mathcal{H}$, so that an isometric isomorphism $(\mathcal{H}, \beta) \cong (\mathcal{H}_\Sigma, \beta_\Sigma)$ would be obtained. Of course, such a construction will only be satisfactory if $\beta$ is independent of the choice of Cauchy hypersurface $\Sigma$.

Dimock [1982] presents such a construction for the spin $\frac{1}{2}$ Dirac equation

$$(\gamma_a \nabla^a + m) \Phi = 0, \quad \Phi \in \Gamma(DM),$$

where $DM$ denotes the bundle of Dirac spinors on $M$. Here, the principal symbol $\gamma_a \in T^*M \otimes \text{End}(DM)$ of the Dirac operator $D = \gamma_a \nabla^a$ is given by the Dirac matrices $\gamma_a$. The Hermitian product $\beta_\Sigma$ on $\mathcal{H}_\Sigma$ is given by

$$\beta_\Sigma(\Psi, \Phi) := \int_{\Sigma} n_a j^a(\Psi, \Phi) \, d\mu_\Sigma,$$
where $j_a(\Psi, \Phi) = \langle \Psi^+, \gamma_a(\Phi) \rangle$ is the Dirac current (depending sesquilinearly on two Dirac spinor fields $\Psi, \Phi \in \Gamma(DM)$), $\Phi^+ \in \Gamma(D^*M)$ is the Dirac adjoint co-spinor field of $\Phi$ (which depends complex anti-linearly on $\Phi$), $\langle \cdot, \cdot \rangle : \Gamma(D^*M) \times \Gamma(DM) \to C^\infty(M; \mathbb{C})$ denotes the canonical fiberwise pairing of co-spinor fields and spinor fields and $d\mu_\Sigma$ is the metric-induced volume density on $\Sigma$.

As can be shown, $\beta_\Sigma$ is indeed a Hermitian scalar product on $H_\Sigma$ (i.e. it is positive definite), and it does not depend on the choice of Cauchy hypersurface (which is basically a consequence of Stokes’ theorem), in the sense that if for a second smooth spacelike Cauchy hypersurface $\Sigma'$ we pull back $\beta_\Sigma'$ from $H_{\Sigma'}$ to $H_\Sigma$, the resulting Hermitian scalar product $\beta$ on $H_\Sigma$ equals the one obtained by pulling back $\beta_\Sigma$ from $H_{\Sigma'}$. In fact, we thus have a whole chain of isometric isomorphisms of pre-Hilbert spaces,

$$ (H, \beta) \cong (H_\Sigma, \beta_\Sigma) \cong (H_{\Sigma'}, \beta_{\Sigma'}) \cong \ldots, $$

resulting in a chain of canonically isomorphic CAR-algebras

$$ \text{CAR}(H, \beta) \cong \text{CAR}(H_\Sigma, \beta_\Sigma) \cong \text{CAR}(H_{\Sigma'}, \beta_{\Sigma'}) \cong \ldots. $$

Imitating this basic construction for fields of higher spin is a challenging task. Generally, one would try to construct the Hermitian inner product $\beta_\Sigma$ for a Cauchy hypersurface $\Sigma$ within a setting where the field equation is given by a Lagrangian density and by making use of a suitable concept of complex conjugation of fields. E.g. for the higher spin field equations proposed by Buchdahl (1982) and reformulated by Wünsch (1985), such a Lagrangian formulation was presented by Illge (1993). However, as was shown in Mühlhoff (2007), the resulting product $\beta_\Sigma$ fails to be positive definite (and thus it is not an Hermitian inner product) as soon as spin is $> 1$.

This state of affairs motivates future research, illuminating in which cases of higher spin prenormally hyperbolic partial differential equations a quantum field theory based on a canonical choice of CAR- (or CCR-) algebra can be constructed.

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