TOTAL POSITIVITY AND CONJUGACY CLASSES

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Abstract. In this paper we study the interaction between the totally positive monoid $G_{\geq 0}$ attached to a connected reductive group $G$ with a pinning and the conjugacy classes in $G$. In particular we study how a conjugacy class meets the various cells of $G_{\geq 0}$. We also state a conjectural Jordan decomposition for $G_{\geq 0}$ and prove it in some special cases.

INTRODUCTION

Let $G$ be a connected reductive algebraic group over $\mathbb{C}$. Let $G_{>0} \subset G_{\geq 0}$ be the totally positive submonoids of $G$ associated to a pinning of $G$ in [8]. (In the case where $G = GL_n(\mathbb{C})$, the definition of $G_{>0}, G_{\geq 0}$ goes back to Schoenberg and to Gantmacher-Krein in the 1930’s.) In this paper we are mainly interested in the interaction between these monoids and the conjugacy classes of $G$. Here are some examples of such interaction in the earlier literature.

In [4], Gantmacher and Krein showed that when $G = GL_n(\mathbb{C})$, any matrix in $G_{>0}$ has $n$ distinct eigenvalues which are all in $\mathbb{R}_{>0}$. This result was generalized in [8] to a general $G$ by showing that

(a) any $g \in G_{>0}$ is regular semisimple and contained in a $\mathbb{R}$-split maximal torus.

In [2], it is shown that when $G = GL_n(\mathbb{C})$, any matrix in $G_{\geq 0}$ has all eigenvalues in $\mathbb{R}_{\geq 0}$. This result was generalized in [10] to a general $G$ by showing that

(b) any $g \in G_{\geq 0}$ acts on any finite dimensional representation of $G$ with all eigenvalues in $\mathbb{R}_{>0}$.

In [8] it is shown that $G_{>0}$ contains “relatively few” unipotent elements in the sense that

(c) if $C$ is a unipotent class in $G$ then $C \cap G_{>0}$ has dimension less than or equal to half the dimension of the set of real points of $C$ (and it can be empty).

Let $G_{\text{reg}}$ be the open subset of $G$ consisting of regular elements in the sense of Steinberg. Let $G_{\text{reg,ss}}$ be the open subset of $G$ consisting of regular semisimple elements.

In [8] a partition of $G_{\geq 0}$ into cells indexed by two Weyl group elements was defined. In this paper we show that each cell of $G_{\geq 0}$ contains regular elements (and even regular semisimple elements). We describe explicitly the cells of $G_{\geq 0}$ which are entirely contained in $G_{\text{reg}}$ and also the cells which are entirely contained in $G_{\text{reg,ss}}$.

We show that if $G$ is almost simple, then the set of regular unipotent elements in $G_{\geq 0}$ is contained in the union of the unipotent radicals of the two Borel subgroups which are part of the pinning.

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If \( g \in G \), we have a Jordan decomposition \( g = g_s g_u = g_u g_s \) with \( g_u \) unipotent and \( g_s \) semisimple. Let \( H \) be the centralizer of \( g_s \) in \( G \) (by [10], \( H \) is a connected reductive group). If \( g \in G_{>0} \) then \( g_u, g_s \) are not necessarily contained in \( G_{>0} \).

As a substitute, in this paper we state a conjecture which says that \( H \) has something close to a pinning with respect to which we can define \( H_{>0} \) and we have \( g_u \in H_{>0}; g_s \in H_{>0} \). This conjecture is proved in some cases in this paper.

A consequence of this conjecture and of (c) is that, if \( C' \) is a non-semisimple conjugacy class in \( G \), defined over \( \mathbb{R} \), then \( C' \cap G_{>0} \) has dimension less than the dimension of the set of real points of \( C' \).

1. PINNINGS AND TOTAL POSITIVITY

1.1. Totally nonnegative monoid. We assume that \( G \) is defined and split over \( \mathbb{R} \). We fix a pinning \( P = (B^+, B^-, T, x_i, y_i; i \in I) \). Let \( W = N_G(T)/T \) be the Weyl group of \( G \) and \( \{ s_i \}_{i \in I} \) be the set of simple reflection in \( W \). Let \( G_{>0} \) be the totally nonnegative submonoid of \( G \) introduced in [8]. For any \( w \in W \) and \( w = s_{i_1} s_{i_2} \cdots s_{i_n} \) be a reduced expression of \( w \). We set

\[
U_{w,>0}^+ = \left\{ x_{i_1}(a_1)x_{i_2}(a_2)\cdots x_{i_n}(a_n); a_1, \ldots, a_n > 0 \right\};
\]
\[
U_{w,>0}^- = \left\{ y_{i_1}(a_1)y_{i_2}(a_2)\cdots y_{i_n}(a_n); a_1, \ldots, a_n > 0 \right\}.
\]

For any \( w_1, w_2 \in W \), we set \( G_{w_1, w_2, >0} = U_{w_1,>0}^+ T_{>0} U_{w_2,>0}^- \). By [8, §2.11], \( G_{w_1, w_2, >0} = U_{w_2,>0}^- T_{>0} U_{w_1,>0}^+ \) and

\[
G_{>0} = \bigcup_{w_1, w_2 \in W} G_{w_1, w_2, >0}.
\]

For any \( i \in I \), let \( \alpha_i \) be the corresponding simple root. Set \( \hat{s}_i = x_i(1)y_i(-1)x_i(1) \). For any \( w \in W \), it is known that for any reduced expression \( w = s_{i_1} s_{i_2} \cdots \) of \( w \), the element \( \hat{s}_{i_1} \hat{s}_{i_2} \cdots \) is independent of the choice of the reduced expression. We denote this element by \( \hat{w} \). For any element \( w \in W \), let \( \text{supp}(w) \) be the set of simple reflections occurring in some (or equivalently, any) reduced expression of \( w \).

For any \( J \subset I \), let \( P_J^+ \supset B^+ \) be the standard parabolic subgroup of \( G \), \( P_J^- \supset B^- \) be the opposite parabolic subgroup and \( \bar{L}_J = P_J^+ \cap P_J^- \) be the standard Levi subgroup. For any parabolic subgroup \( P \), let \( U_P \) be its unipotent radical. Then \( P_J^+ = L_J U_{P_J^+} \). Let \( \Phi \) be the root system of \( G \) and \( \Phi_J \subset \Phi \) be the subsystem attached to \( \bar{L}_J \). The Weyl group \( W_J \) of \( \bar{L}_J \) is the subgroup of \( W \) generated by \( s_j \) for \( j \in J \). We denote by \( w_J \) the longest element in \( W_J \). We simply write \( U_{w_J,>0}^\pm \) for \( U_{w_J,>0}^\pm \) and write \( G_{>0} \) for \( G_{w_J, w_J, >0} \). The set \( G_{>0} \) is the totally positive part of \( G \).

For any \( J \subset I \), let \( W_J \) (resp. \( J' \)) be the set of minimal length elements in their cosets in \( W/W_J \) (resp. \( W_J \backslash W \)). For any \( J, J' \subset I \), we simply write \( J' W_{J''} \) for \( J' W \cap J'' W \).

1.2. Weak pinningns. We define an equivalence relation on the set of pinnings of \( G \). Two pinnings \( P = (B^+, B^-, T, x_i, y_i; \ i \in I) \), \( P' = (B'^+, B'^-, T', x'_i, y'_i; \ i \in I') \) are said to be equivalent if \( T = T', I = I' \) and if there exists \( c = (c_i)_{i \in I} \in \mathbb{R}_{>0}^I \) such that for any almost simple “factor” \( G_1 \) of the derived subgroup of \( G \) we have

- either \( B^+ \cap G_1 = B'^+ \cap G_1, B^- \cap G_1 = B'^- \cap G_1, x'_i(a) = x_i(c_i a), y'_i(a) = y_i(c_i^{-1} a) \) for all \( i \in I, a \in \mathbb{C} \);
or \( B^+ \cap G_1 = B^- \cap G_1, B^- \cap G_1 = B'^+ \cap G_1 \), \( x'_i(a) = y_i(c, a), y'_i(a) = x_i(c^{-1}a) \) for all \( i \in I, a \in \mathbb{C} \).

This is an equivalence relation on the set of pinnings of \( G \). An equivalence class of pinnings is called a weak pinning of \( G \). We denote by \( \mathcal{P} \) the weak pinning containing a pinning \( \mathcal{P} \).

The monoids \( G_{ \geq 0}, G_{ > 0 } \) attached in [8] to a pinning of \( G \) depend only on the equivalence class of that pinning hence can be viewed as being associated to a weak pinning of \( G \). To show that the monoid \( G_{ \geq 0 } \) determines a unique weak pinning of \( G \), we first prove the following lemma.

**Lemma 1.1.** Suppose that \( G \) is almost simple. Let \( G_{ \text{reg,uni}} \) be the set of regular unipotent elements in \( G \). Then

\[
G_{ \text{reg,uni}} \cap G_{ \geq 0 } = \bigcup_{w \in W, \supp(w) = I} U_{w, > 0}^+ \sqcup \sqcup_{w \in W, \supp(w) = I} U_{w, > 0}^-.
\]

In particular, \( G_{ \text{reg,uni}} \cap G_{ \geq 0 } \) has two connected components and they are dense in the unipotent radical of two opposite Borel subgroups \( B^+ \) and \( B^- \).

**Proof.** Let \( G_{ \text{uni}} \) be the set of unipotent elements in \( G \). By [8, Theorem 6.6],

\[
G_{ \text{uni}} \cap G_{ \geq 0 } = \bigcup_{w_1, w_2 \in W, \supp(w_1) \cap \supp(w_2) = \emptyset} U_{w_1, > 0}^+ U_{w_2, > 0}^-.
\]

Suppose that \( G \) is almost simple. Let \( w_1, w_2 \in W \) and \( g \in U_{w_1, > 0}^+ U_{w_2, > 0}^- \). Let \( J_i = \supp(w_i) \) for \( i = 1, 2 \). We assume furthermore that \( J_1 \cap J_2 = \emptyset \). Then \( w_1 g \hat{w}_1^{-1} J_2 \subseteq U^+ \). If \( J_2 \neq \emptyset \) and \( J_2 \neq I \), then there exists \( i \in I - J_2 \) such that in the Dynkin diagram of \( G \), the vertex corresponds \( i \) is connected to some vertex in \( J_2 \). In this case, \( w_1 g \hat{w}_1^{-1} \in U_1^+ \). Hence \( g \) is not regular.

Similarly, if \( J_1 \neq \emptyset \) and \( J_1 \neq I \), then \( g \) is not regular. It is obvious that if \( J_1 = J_2 = \emptyset \), then \( g = 1 \) and hence is not regular. Finally if \( (J_1, J_2) = (I, \emptyset) \) or \( (\emptyset, I) \), then \( g \) is regular.

The “in particular” part follows from that the fact that \( \sqcup_{w \in W, \supp(w) = I} U_{w, > 0}^+ \) is connected and dense in \( U^+ \).

1.3. **Uniqueness of weak pinnings.** Let \( G_{ \geq 0 } \) be the totally nonnegative part of \( G \) associated to pinnings \( \mathcal{P} \) and \( \mathcal{P}' = (B'^+, B'^-, T', x', y'; i \in I') \). We show that \( \mathcal{P} \) and \( \mathcal{P}' \) are equivalent.

Let \( G_{ \text{der}} \) be the derived subgroup of \( G \). Let \( G_k \) for \( 1 \leq k \leq l \) be the almost simple factors of \( G_{ \text{der}} \).

For any \( k \), \( G_k \cap G_{ \geq 0 } \) is a totally nonnegative part of \( G_k \) associated to the pinnings \( (B'^+ \cap G_k, B^- \cap G_k, T \cap G_k, \ldots), (B'^+ \cap G_k, B^- \cap G_k, T' \cap G_k, \ldots) \) of \( G_k \).

By Lemma 1.1, \( G_k \cap G_{ \geq 0 } \) has two connected components and the closure of these connected components are the unipotent radicals of two opposite Borel subgroups

\[
\{B'^+ \cap G_k, B^- \cap G_k\} = \{B'^+ \cap G_k, B'^- \cap G_k\}.
\]

In particular,

\[
T \cap G_k = (B'^+ \cap G_k) \cap (B^- \cap G_k) = (B'^+ \cap G_k) \cap (B'^+ \cap G_k) = T' \cap G_k.
\]

Hence \( T = (T \cap G_1) \cdots (T \cap G_l) Z(G) = (T' \cap G_1) \cdots (T' \cap G_l) Z(G) = T' \).

Recall that \( \Phi \) is the set of roots of \( G \) with respect to \( T \). Then \( \Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_l \), where \( \Phi_k \) is the set of roots of \( G_k \) with respect to \( T \cap G_k \). Let \( \Sigma \) be the set of positive roots determined by \( B^+ \) and \( \Sigma' \) be the set of positive roots determined by \( B'^+ \).
Since \( \{B^+ \cap G_k, B^- \cap G_k\} \equiv \{B'^+ \cap G_k, B'^- \cap G_k\} \), we have \( \Sigma' \cap \Phi_k = \Sigma \cap \Phi_k \) or \( -\Sigma \cap \Phi_k \). In particular, we may identify \( I \) with \( I' \). If \( B'^+ \cap G_k = B^+ \cap G_k \), then for any \( i \in I \) with \( \alpha_i \in \Phi_K \), we have \( x_i'(\mathbb{R}_{>0}) = x_i(\mathbb{R}_{>0}) \) and thus there exists \( c_i > 0 \) such that \( x_i'(a) = x_i(c_i a) \) for all \( a \in \mathbb{C} \). Similarly, if \( B'^- \cap G_k = B^- \cap G_k \), then for any \( i \in I \) with \( \alpha_i \in \Phi_K \), we have \( x_i'(\mathbb{R}_{>0}) = y_i(\mathbb{R}_{>0}) \) and thus there exists \( c_i > 0 \) such that \( x_i'(a) = y_i(c_i a) \) for all \( a \in \mathbb{C} \).

Thus \( \mathcal{P}' \) and \( \mathcal{P} \) are equivalent.

1.4. Weak bases. Let \( V \) be a \( \mathbb{C} \)-vector space. A half line in \( V \) is a subset of \( V \) of the form \( \{av; a \in \mathbb{R}_{\geq 0}\} \) for some \( v \in V - \{0\} \). Assume now that \( \dim(V) < \infty \). A weak basis of \( V \) is a collection of half lines in \( V \) such that if we choose a nonzero vector on each of these half lines, the resulting nonzero vectors form a basis of \( V \). Note that any basis of \( V \) gives rise to a weak basis of \( V \), which consists of the half lines containing the various elements in that basis.

Let \( \beta \) be a weak basis of \( V \). Let \( V_{\beta, \geq 0} \) be the set of \( \mathbb{R}_{\geq 0} \)-linear combination of vectors in the half lines in \( \beta \). This subset is closed under addition and under scalar multiplication by \( \mathbb{R}_{\geq 0} \). Let \( V_{\beta, > 0} \) be the set of \( \mathbb{R}_{>0} \)-linear combination of nonzero vectors in the half lines in \( \beta \). This subset is closed under addition and under scalar multiplication by \( \mathbb{R}_{>0} \). If \( \beta, \beta' \) are two weak bases of \( V \), we have

\[
V_{\beta, \geq 0} = V_{\beta', \geq 0} \Rightarrow V_{\beta, > 0} = V_{\beta', > 0} \Rightarrow \beta = \beta'.
\]

We now assume that \( G \) is simply laced.

We fix a weak pinning \( \mathcal{P} \) of \( G \). Let \( X_{\mathcal{P}} \) be the set of Borel subgroups \( B \) of \( G \) such that \( B = B^+ \) for some pinning \( \mathcal{P} = (B^+, B^-, \ldots) \) in \( \mathcal{P} \).

Let \( \text{Irr}(G) \) be the collection of irreducible finite dimensional rational representations of \( G \). Let \( V \in \text{Irr}(G) \). For any Borel subgroup \( B \) of \( G \) we denote by \( V_B \) the unique \( B \)-invariant \( \mathbb{C} \)-line in \( V \).

For \( B \in X_{\mathcal{P}} \) let \( e_B \) be the set of half lines in \( V_B \). For any pinning \( \mathcal{P} = (B^+, B^-, \ldots) \) in \( \mathcal{P} \) and any \( v \in V_{B^+} - \{0\} \) we denote by \( B_{V, \mathcal{P}, v} \) the canonical basis of \( V \) containing \( v \) defined in terms of \( \mathcal{P} \) in [6].

If \( \mathcal{P}' = (B'^+, B'^-, \ldots) \) is another pinning in \( \mathcal{P} \) and if \( v' \in V_{B'^+} - \{0\} \) is defined by the condition that \( v' \in B_{V, \mathcal{P}', v} \) then, using [7, Proposition 21.1.2], we see that the canonical basis \( B_{V, \mathcal{P}, v} \) of \( V \) gives rise to the same weak basis of \( V \) as \( B_{V, \mathcal{P}', v} \); moreover, the half line in \( V_{B'^+} \) containing \( v' \) depends only on the half line containing \( v \). Thus the sets \( e_B \) for various \( B \in X_{\mathcal{P}} \) can be identified with a single set \( e_{V, \mathcal{P}} \). We also see that \( \mathcal{P} \) together with a choice of an element \( \epsilon \in e_{V, \mathcal{P}} \) give rise to a weak basis \( \beta_{V, \mathcal{P}, \epsilon} \) of \( V \).

Hence the subsets \( V_{\beta_{V, \mathcal{P}, \epsilon, \geq 0}}, V_{\beta_{V, \mathcal{P}, \epsilon, > 0}} \) are defined.

According to [8, §3.2], if \( g \in \mathcal{G}_{\mathcal{P}, > 0} \) then \( g : V \to V \) maps \( V_{\beta_{V, \mathcal{P}, \epsilon, \geq 0}} \) into \( V_{\beta_{V, \mathcal{P}, \epsilon, > 0}} \). Conversely, according to [3, Theorem 1.11], if \( g \in \mathcal{G} \) is such that for any \( V \in \text{Irr}(G) \), \( g : V \to V \) maps \( V_{\beta_{V, \mathcal{P}, \epsilon, \geq 0}} \) into \( V_{\beta_{V, \mathcal{P}, \epsilon, > 0}} \) (for some \( \epsilon \)), then \( g \in \mathcal{G}_{\geq 0} \).

2. Regular elements

2.1. The main results. Recall that \( G^{\text{reg}} \) is the set of regular elements in \( G \) and \( G^{\text{reg, ss}} \) is the set of regular semisimple elements in \( G \). Now we state the main results of this section.

Theorem 2.1. Let \( w_1, w_2 \in W \). Then
(1) $G_{w_1, w_2, > 0} \cap G_{\text{reg, ss}} \neq \emptyset$.
(2) $G_{w_1, w_2, > 0} \subset G_{\text{reg, ss}}$ if and only if $\text{supp}(w_1) = \text{supp}(w_2) = I$.
(3) Suppose that $G$ is almost simple. Then $G_{w_1, w_2, > 0} \subset G_{\text{reg}}$ if and only if $\text{supp}(w_1) = I$ or $\text{supp}(w_2) = I$.

2.2. Some semigroups. Let $*: W \times W \to W$ be the monoid product (see [10, §2.11]). Then for any $w, w' \in W$, we have $\text{supp}(w * w') = \text{supp}(w) \cup \text{supp}(w')$. By [10, §2.11(d)], we have $G_{w_1, w_2, > 0}G_{w_3, w_4, > 0} = G_{w_1 * w_2, w_3 * w_4, > 0}$. Now as a consequence of Theorem 2.1, we have

(a) the union of the totally nonnegative cells consisting of regular semisimple group is an open sub semigroup of $G_{\geq 0}$;
(b) the union of the totally nonnegative cells consisting of regular elements is an open sub semigroup of $G_{\geq 0}$.

2.3. Oscillatory elements. We call an element $g \in G_{\geq 0}$ oscillatory if $g^m \in G_{> 0}$ for some $m \in \mathbb{N}$. By [8, Proposition 2.19], the subset of oscillatory elements is $G_{\geq 0}^O$. By [8, Corollary 5.7], any oscillatory element is regular and semisimple. This is the “if” part of Theorem 2.1 (2).

We have the following result, which generalizes [8, Corollary 8.10] from totally positive elements to oscillatory elements.

**Lemma 2.2.** Let $g$ be an oscillatory element. Then there exists a unique element $u \in U_{\geq 0}$ such that $u^{-1}gu \in B^+$. Moreover, $g = uu'tu^{-1}$ for some $u' \in U_{> 0}^+$ and $t \in T$ with $\alpha_i(t) > 1$ for all $i \in I$.

**Proof.** Let $B = G/B^+$ be the flag variety of $G$. We denote the action of $G$ on $B$ by $g \cdot B := gBg^{-1}$. Let $B_{> 0} = U_{> 0}^+.B^+ \subset B$ be the totally positive flag manifold and $B_{\geq 0} = B_{> 0} \cup B_{\leq 0}$ be the totally nonnegative flag manifold. By [8, Corollary 8.11], there exists $B \in B_{\geq 0}$ with $g \in B$. We have $g^m \in G_{> 0}$ for $m \in \mathbb{N}$. By [8, Lemma 8.2], there exists a unique $B' \in B_{> 0}$ with $g^n \in B'$. Since $g \in B$, we have $g^n \in B$. Thus $B = B'$. By [8, Lemma 8.5], $B' \subset B_{> 0}$. Hence $B = uB^+u^{-1}$. So $u^{-1}gu \in B^+$. The uniqueness of $u$ follows from the uniqueness of $B$.

Let $u^{-1}gu = u't$ with $u' \in U^+$ and $t \in T$. Then we have $uu't = gu \in G_{> 0}$. Therefore we have $u' \in U_{> 0}$ and $t \in T_{> 0}$. Moreover, $g^m = u(u't)^m u^{-1} = uu'_m t^m u^{-1}$ for some $u'_m \in U_{> 0}$. By [8, Corollary 8.10], $\alpha_i(t^m) > 1$ for all $i \in I$. Hence $\alpha_i(t) > 1$ for all $i \in I$. \hfill $\square$

2.4. Projection maps. For $J \subset I$, let $\pi_J^+ : P_J^+ \to L_J$ be the projection map. Then the restriction of $\pi_J^+$ to $T_{> 0}$ is the identity map. Moreover, $\pi_J^+(y_i(a)) = y_i(a)$ and $\pi_J^+(x_i(a)) = x_i(a)$ for $i \in J$. We also have

$$
\pi_J^+(x_i(a)) = \begin{cases} x_i(a), & \text{if } i \in J; \\ 1, & \text{if } i \notin J \end{cases}
$$

and

$$
\pi_J^-(y_i(a)) = \begin{cases} y_i(a), & \text{if } i \in J; \\ 1, & \text{if } i \notin J \end{cases}
$$

In particular, $\pi_J^+(P_J^+ \cap G_{\geq 0}) = L_{J, \geq 0}$. We define a morphism of monoids $\pi_J : W \to W_J$ by $s_i \mapsto s_i$ if $i \in J$ and $s_i \mapsto 1$ if $i \notin J$. Although not needed in this paper, it is worth pointing out that the map $W \to W_J$ can be regarded as the map $\pi_J^+ : U^+(K) \to (U^+ \cap L_J)(K)$ in the case where $K$ is the semifield of one element.
2.5. Proof of Theorem 2.1 (1) and (2). Let \( J_i = \text{supp}(w_i) \) for \( i = 1, 2 \). Let \( u_1 \in U^+_{w_1,0} \), \( u_2 \in U^-_{w_2,0} \).

Set \( K = J_1 \cap J_2 \). Let \( v_i = \pi_K(w_i) \). By our assumption, \( \text{supp}(v_1) = \text{supp}(v_2) = K \). Let \( u_1' = \pi_K^+(u_1) \) and \( u_2' = \pi_K^-(u_2) \). Then \( u_1' \in U^+_{w_1,0} \) and \( u_2' \in U^-_{w_2,0} \).

By Lemma 2.2, there exists \( u'' \in U^+_{w_1,0} \) and \( t_1 \in T_{>0} \) such that \( (u'')^{-1}u_1'u_2'' \in (U^+ \cap L_K)t_1 \) and \( \alpha(t_1) > 1 \) for all \( i \in K \). Since \( u_1u_2 \in (U_{K}^+ \cap L_{J_1})u_1'u_2(U_{K}^+ \cap L_{J_2}) \), we have

\[
(u'')^{-1}u_1u_2u'' \in (U_{K}^+ \cap L_{J_1})(u'')^{-1}u_1'u_2''(U_{K}^+ \cap L_{J_2}) \\
\subseteq (U_{K}^+ \cap L_{J_1})(U^+ \cap L_K)t(U_{K}^+ \cap L_{J_2}) \\
= (U^+ \cap L_{J_1})t_1(U_{K}^+ \cap L_{J_2}).
\]

Set \( g = w_{J_1}w_K \). We have \( g(U_{K}^+ \cap L_{J_i})g^{-1} \in U^+ \cap L_{J_i} \). Note that \( w_{J_1}w_K \in W^{J_1} \).

Then \( g(U^+ \cap L_{J_i})g^{-1} \in U^+ \). So \( g(u'')^{-1}u_1'u_2''g^{-1} \in U^+(gt_1g^{-1}) \).

Let \( T' = \{ t \in T_{>0}; \alpha(t) = 1 \} \) for all \( i \in K \). Then \( g(u'')^{-1}u_1'u_2'tu''g^{-1} \in U^+(gt_1tg^{-1}) \) for any \( t \in T' \).

Note that there exists \( t' \in T' \) such that \( \alpha(t_1t) > 1 \) for all \( i \in K \). Moreover, \( \alpha(t_1t) = \alpha_1(t_1) > 1 \) for \( i \in K \). Thus \( t_1t \) is a regular semisimple element in \( T \). So any element in \( U^+(gt_1tg^{-1}) \) is conjugate by an element in \( U^+ \) to \( gt_1tg^{-1} \), which is regular semisimple. In particular, \( u_1u_2t \) is regular semisimple. This proves part (1) of Theorem 2.1.

Now we come to the part (2) of Theorem 2.1. The “if” part is proved in [8]. Now we prove the “only if” part.

Without loss of generality, we assume that \( J_2 \neq I \). Let \( j \notin J_2 \). There exists \( t \in T' \) such that \( \alpha_2(gt_1tg^{-1}) = 1 \). The element \( u_1u_2t \in G_{w_1,w_2,0} \) is conjugate to an element in \( U^+(gt_1tg^{-1}) \). Note that the semisimple part of any element in \( U^+(gt_1tg^{-1}) \) is conjugate by \( U^+ \) to \( gt_1tg^{-1} \). In particular, if \( u_1u_2t \) is semisimple, then it is conjugate to \( gt_1tg^{-1} \). Since \( \alpha_2(gt_1tg^{-1}) = 1 \), \( gt_1tg^{-1} \) is not regular. Thus \( u_1u_2t \notin G_{\text{reg},\text{ss}} \).

\[ \square \]

2.6. A technical lemma on the root system. Before proving the “only if” part of Theorem 2.1(3), we establish a lemma on the root system.

**Lemma 2.3.** Suppose that \( G \) is almost simple and \( J, J' \) are proper subsets of \( I \). Then there exists \( j \notin J' \) such that \( w_{J'}(\alpha_j) \notin \Phi_J \).

**Proof.** If \( J' \subset J \), then we may take \( j \) to be any vertex not in \( J \).

Now assume that \( J' \notin J \). Set \( K = J \cap J' \). Then \( J' - K \neq \emptyset \). We consider \( J \) as a sub-diagram of the Coxeter diagram of \( I \). Let \( C \) be a connected component of \( J' \) that contains some vertex \( j_1 \) in \( J' - K \). Since \( J' \subset I \), there exists \( j \notin J' \) such that \( j \) is connected to a vertex \( j' \) in \( C \).

Let \( \omega_j \) be the fundamental weight of \( j_1 \) with respect to \( \Phi \) and \( \omega_{J',j_1} \) be the fundamental weight of \( j_1 \) with respect to \( \Phi_{J'} \). Then \( \omega_j \in \sum_{i \in I} \mathbb{Q}_{>0} \alpha_i \) and \( \omega_{J',j_1} \in \sum_{i \in C} \mathbb{Q}_{>0} \alpha_i \). Note that \( w_{J'}(\omega_{J',j_1}) = -\omega_{J',j_2} = \omega_{J',j_1} - (\omega_{J',j_1} + \omega_{J',j_2}) \) for some \( j_2 \in C \). We have \( \omega_{J',j_2} \in \sum_{i \in C} \mathbb{Q}_{>0} \alpha_i \) and hence \( \omega_{J',j_2} + \omega_{J',j_1} \in \sum_{i \in C} \mathbb{Q}_{>0} \alpha_i \).

Since the restriction of \( \omega_j \) to the root system \( \Phi_{J'} \) equals to \( \omega_{J',j_1} \), we have

\[ w_{J'}(\omega_j) = \omega_j - (\omega_{J',j_1} + \omega_{J',j_2}) \in \omega_j - \sum_{i \in C} \mathbb{Q}_{>0} \alpha_i. \]
Now \( \langle w_J, (\alpha_j), \omega_j \rangle = \langle \alpha_j, w_J(\omega_j) \rangle \in \langle \alpha_j, \omega_j - \sum_{i \in C} Q_0 \alpha_i \rangle = -\langle \alpha_j, \sum_{i \in C} Q_0 \alpha_i \rangle \).
Since \( \langle \alpha_j, \alpha_j \rangle \leq 0 \) for any \( i \in C \) and \( \langle \alpha_j, \alpha_j \rangle < 0 \), we have \( \langle w_J, (\alpha_j), \omega_j \rangle > 0 \).
Since \( j \notin J \), we have \( w_J(\alpha_j) \notin \Phi_J \). \( \square \)

2.7. Proof of the “only if” part of Theorem 2.1 (3). Let \( J_i = \text{supp}(w_i) \) for \( i = 1, 2 \). Set \( K = J_1 \cap J_2 \). Let \( T' = \{ t \in T_{>0}; \alpha_i(t) = 1 \text{ for all } i \in K \} \). Let \( u_1 \in U_{w_1,>0}, u_2 \in U_{w_2,>0} \). We show that there exists \( t \in T' \) such that \( u_1 u_2 t \) is not regular. Our strategy is similar to §2.5 but the choice of \( t \) is more involved.

By §2.5, there exists \( u'' \in U_{w_K,>0} \) and \( t_1 \in T_{>0} \) such that
\[
(u'')^{-1}u_1u_2t'' \in (U^+ \cap L_{J_1})t_1(U_{P_K^-} \cap L_{J_2}).
\]
By Lemma 2.3, there exists \( j \notin J_2 \) such that \( w_K w_{J_2}(\alpha_j) \notin \Phi_{J_1} \). Set \( g = w_{J_2}w_K \).
We have \( g(U_{P_K^-} \cap L_{J_2})g^{-1} \in U^+ \cap L_{J_2} \subset U_{P^-_{I_{-}(j)}} \). Note that \( w_{J_2}w_K \in W_{J_i} \), we have \( g(U^+ \cap L_{J_2})g^{-1} \in U_{P^-_{I_{-}(j)}} \). Hence
\[
g(U^+ \cap L_{J_2})t_1(U_{P_K^-} \cap L_{J_2})g^{-1} \subset U_{P^-_{I_{-}(j)}} g t_1 g^{-1}.
\]
Let \( t \in T' \) with \( \alpha_j(gt_1g^{-1}) = 1 \).
Now
\[
g(u'')^{-1}u_1u_2tu''g^{-1} \in g(U^+ \cap L_{J_2})t_1(U_{P_K^-} \cap L_{J_2})g^{-1} \subset U_{P^-_{I_{-}(j)}} g t_1 g^{-1}.
\]

The conjugation by \( T \) preserves \( U_{P_{I_{-}(j)}^+}(gt_1g^{-1}) \). Since \( \alpha_j(gt_1g^{-1}) = 1 \), the conjugation by the root subgroups \( U_{\pm \alpha_j} \) preserves \( U_{P_{I_{-}(j)}^+}(gt_1g^{-1}) \). Therefore, the \( P_{I_{-}(j)}^+ \)-conjugacy class \( O_1 \) of \( g(u'')^{-1}u_1u_2tu''g^{-1} \) is contained in \( U_{P_{I_{-}(j)}^+}(gt_1g^{-1}) \).
In particular, \( \dim O_1 \leq \dim U_{P_{I_{-}(j)}^+} = \dim P_{I_{-}(j)}^+ - \text{rank } G - 2 \).

Let \( O \) be the conjugacy class of \( uu't \) in \( G \). It is also the conjugacy class of \( g(u'')^{-1}u_1u_2tu''g^{-1} \) in \( G \). We have
\[
\dim O \leq \dim O_1 + \dim G - \dim P_{I_{-}(j)}^+ \leq \dim G - \text{rank } G - 2.
\]
Therefore \( u_1u_2t \) is not regular. \( \square \)

2.8. Regularity for \( GL_n \). Now we study the “if” part of Theorem 2.1 (3). The key part is to show that any element in \( B_{>0}^2 = T_{>0}U_{>0}^+ \) is regular. In this subsection, we prove this claim via linear algebra.

Let \( G = GL_n \) and \( g = (a_{ij})_{i,j \in [1,n]} \) in \( B_{>0}^+ \). In particular, \( a_{ij} = 0 \) if \( i < j \), \( a_{ij} > 0 \) if \( i \geq j \) and various minors are \( > 0 \). For any \( J_1, J_2 \subset \{1,2,\ldots,n\} \), let \( \Delta_{J_1, J_2} \) be the \((J_1, J_2)\)-minor of \( g \), i.e., the determinant of the submatrix \( (a_{ij})_{i \in J_1, j \in J_2} \). To show that \( g \) is regular, it suffices to show that any eigenspace of \( g : \mathbb{R}^n \to \mathbb{R}^n \) is one-dimensional.

Lemma 2.4. Let \( 1 \leq k \leq n \). Set
\[
V_k = \{(A_1,\ldots,A_n) \in \mathbb{R}^n; \sum_j a_{ij}A_j = a_{kk}A_i \text{ for all } i \in [1,n]\}.
\]

Then \( \dim V_k = 1 \).

Proof. We argue by induction on \( n \).
If \( n = 1 \), the result is obvious.
Assume now that \( n \geq 2 \). Let \( c = a_{kk} \) and \( I_c = \{ i \in [1, n] ; a_{ii} = c \} \). Let \( (A_1, ..., A_n) \in V_k \). If \( A_n = 0 \), then \( (A_1, ..., A_{n-1}) \) satisfy a condition similar to that defining \( V_k \). In this case, the statement follow from the induction hypothesis.

If \( n \notin I_c \), then \((a_{nn} - c)A_n = 0\) and \( A_n = 0 \).

We assume that \( n \in I_c \). If \( n - 1 \in I_c \), then \( a_{n-1,n}A_n = 0 \). Since \( a_{n-1,n} \neq 0 \), we have \( A_n = 0 \) and the statement follows again from the induction hypothesis.

Now we assume that \( n \in I_c, n - 1 \notin I_c \). If \( n - 2 \in I_c \), then we have the linear system

\[
\begin{cases}
    a_{n-2,n-1}A_n - a_{n-2,n}A_n = 0 \\
    (a_{n-1,n-1} - c)A_n + a_{n-1,n}A_n = 0
\end{cases}
\]

The determinant of this linear system is

\[
a_{n-2,n-1}a_{n-1,n} - (a_{n-1,n-1} - c)a_{n-2,n} = \Delta_{\{n-2,n-1\},\{n-1,n\}} + ca_{n-2,n} > 0.
\]

It follows that \( A_{n-1} = A_n = 0 \). In particular \( A_n = 0 \) and we are done.

We then assume that \( n \in I_c, n - 1 \notin I_c, n - 2 \notin I_c \). If \( n - 3 \in I_c \), we have the linear system

\[
\begin{cases}
    a_{n-3,n-2}A_n - a_{n-3,n-1}A_n - a_{n-3,n}A_n = 0 \\
    (a_{n-2,n-2} - c)A_n + a_{n-2,n-1}A_n + a_{n-2,n}A_n = 0 \\
    (a_{n-1,n-1} - c)A_n + a_{n-1,n}A_n = 0
\end{cases}
\]

The determinant of this system of linear equations is

\[
a_{n-3,n-2}a_{n-2,n-1}a_{n-1,n} + (a_{n-2,n-2} - c)(a_{n-1,n-1} - c)a_{n-3,n} - a_{n-3,n-1}(a_{n-2,n-2} - c)a_{n-1,n} - a_{n-3,n-2}(a_{n-1,n-1} - c)a_{n-2,n} = \Delta_{\{n-3,n-2,n-1\},\{n-2,n-1,n\}} + \Delta_{\{n-3,n-2\},\{n-2,n\}} + c^2a_{n-3,n}.
\]

Again the determinant is > 0. It follows that \( A_{n-2} = A_{n-1} = A_n = 0 \). In particular \( A_n = 0 \) and we are done.

Thus we can assume \( n \in I_c, n - 1 \notin I_c, n - 2 \notin I_c, n - 3 \notin I_c \). Continuing in this way we see that we can assume \( n \in I_c \) and \( n - 1, n - 2, ..., 1 \) are not in \( I_c \).

Then we can choose \( A_n \) at random. From \((a_{n-1,n-1} - c)A_n - a_{n-1,n}A_n = 0\) where \( a_{n-1,n-1} - c \neq 0 \) we determine uniquely \( A_{n-1} \). From \((a_{n-2,n-2} - c)A_n + a_{n-2,n-1}A_n + a_{n-2,n}A_n = 0\) where \( a_{n-2,n-2} - c \neq 0 \) we determine uniquely \( A_{n-2} \). Continuing in this way we see that \( A_{n-1}, A_{n-2}, ..., A_1 \) are uniquely determined by \( A_n \). Hence \( \dim V_k = 1 \). \( \square \)

2.9. Conjugating the torus part. To handle the general case, we use a more Lie-theoretic approach.

For any \( t \in T_{>0} \), let \( I(t) = \{ i \in I; \alpha_i(t) = 1 \} \). For any \( t \in T_{>0} \) and \( J \subset I \), let \( \overline{I}_J \) be the unique element \( t' \) in the \( W_J \)-orbit of \( t \) with \( \alpha_i(t') \geq 1 \) for all \( i \in J \). Let \( w_t,J \) be the unique element in \( W_{I(\overline{I}_J)} \backslash W_J/W_{I(t)} \) such that \( \overline{I}_J = \overline{w_t,J} \).

Lemma 2.5. Let \( w \in W \) and \( J = \text{supp}(w) \). Then for any \( t \in T_{>0} \) and \( u \in U_{w,>0}^+ \), there exists \( u' \in U_{w,>0}^{-1} \) such that \( (u')^{-1}tuu' \in \overline{I}_J U_{w,>0}^+ \).

Proof. We prove that

(a) Let \( w \in W, u \in U_{w,>0}^+ \) and \( t \in T_{>0} \). Let \( i \in J \). If \( \alpha_i(t) < 1 \), then there exists \( b > 0 \) such that \( y_i(-b)tu_i(b) \in s_i(t)U_{w,>0}^+ \).
Let \( c = \alpha_i(t) < 1 \). We have \( \pi_{ji}^+(u) = x_i(a) \) for some \( a > 0 \). Set \( b = \frac{1}{ca} > 0 \). By [8, §2.11], \( y_i(-b)ty_y(i)b \in y_i(-b)y_i(b)t'U' \) for some \( b' > 0, t' \in T_{>0} \) and \( U' \in U_{w,>0} \). In particular, \( y_i(-b)ty_y(i)b \in P^+_{ji} \). Then

\[
\pi_{ji}^+(y_i(-b)ty_y(i)b) = y_i(-b)tx_i(a)y_i(b) = ty_i(-\frac{1 - c}{a})x_i(a)y_i(\frac{1 - c}{ca})
\]

\[
= tx_i(\frac{a}{c})\alpha_i(\frac{1}{c}).
\]

Note that \( ta_i(\frac{1}{c}) = s_i(t) \). (a) is proved.

Now the statement follows from (a) by induction on \( \ell(w_1,\text{supp}(w)) \). \( \square \)

2.10. Proof of the “if” part of Theorem 2.1 (3). Let \( g \in G_{w_1,w_2,>0} \). We assume that \( \text{supp}(w_1) = I \). The case where \( \text{supp}(w_2) = I \) is proved in a similar way.

Let \( J = \text{supp}(w_2) \). By definition, \( \pi_j^+(g) \in G_{\pi_j(w_1),w_2,>0} \subset L_J \).

Since \( \text{supp}(\pi_j(w_1)) = \text{supp}(w_2) = J \), we have \( \pi_j^+(g) \) is an oscillatory element of \( L_{J,>0} \). Thus by Lemma 2.2, there exists \( u \in U_{w,J,>0} \) such that \( u^{-1}\pi_j^+(g)u \in B^+ \cap L_J \). Then \( u^{-1}gu \in B^+ \). Similar to the proof of Lemma 2.2, we also have that \( u^{-1}gu \in G_{w_1,1,>0} \).

By Lemma 2.5, \( u^{-1}gu \) is conjugate to an element in \( tU_{w_1,>0}^+ \) for some \( t \in T \) with \( \alpha_i(t) \geq 1 \) for all \( i \). In particular, \( Z_G(t) \) is a standard Levi subgroup \( L_{J'} \) of \( G \) for some \( J' \subset I \). Note that \( tU_{w_1,>0}^+ = tU_{\pi_j(w_1),>0}^+u_{J'} \). Since \( Z_G(t) = L_{J'} \), any element in \( tU_{w_1,>0}^+ \) is conjugate to an element in \( tU_{\pi_j(w_1),>0}^+ \).

For any \( u' \in U_{\pi_j(w_1),>0}^+ \), we have \( tu' = u't \). This is the Jordan decomposition of the element \( tu' \). Since \( \text{supp}(\pi_j(w_1)) = J' \), we have that \( u' \) is a regular unipotent element in \( L_{J'} \). Thus \( tu' \) is a regular element in \( G \). So \( g \) is also a regular element in \( G \). \( \square \)

2.11. Regular semisimple conjugacy classes in \( G_{>0} \). Set

\[
T_{>1} = \{ t \in T_{>0}; \alpha_i(t) > 1 \text{ for all } i \in I \}.
\]

Then any element in \( T_{>1} \) is regular and semisimple. By [8, Theorem 5.6 & Corollary 8.10], any element \( G_{>0} \) is conjugate to an element in \( T_{>1} \).

Now we prove that the converse is also true. This provides a stronger statement than Theorem 2.1 (1) for the cell \( G_{>0} \). We also conjecture that the similar statement holds for any cell in \( G_{>0} \).

Theorem 2.6. Let \( t \in T_{>1} \) and \( C \) be the regular semisimple conjugacy class of \( t \) in \( G \). Then \( C \cap G_{>0} \neq \emptyset \).

Proof. It suffices to consider the case where that \( G \) is semisimple. Let \( \mathfrak{g} \) be the Lie algebra of \( G \). For \( i \in I \), let \( e_i \in \mathfrak{g} \) (resp. \( f_i \in \mathfrak{g} \)) be the differential of \( x_i \) (resp. \( y_i \)). Let \( \mathfrak{h} \) be the Cartan subalgebra of \( \mathfrak{g} \) corresponding to \( T \). Let \( \mathfrak{h}_R \) be the set of real points of \( \mathfrak{h} \) and let \( \mathfrak{h}_+ \) be the fundamental open Weyl chamber in \( \mathfrak{h}_R \).

Let \( \exp : \mathfrak{g} \to G \) be the exponential map. By our assumption on \( C \), we can find \( y \in \mathfrak{h}_+ \) such that \( \exp(y) = t \).

Let \( \mathfrak{h}/W \) be the set of orbits of the standard Weyl group action on \( \mathfrak{h} \). For \( x \in \mathfrak{g} \), the semisimple part of \( x \) is \( G \)-conjugate to an element \( x' \in \mathfrak{h} \) whose Weyl group
orbit depends only on \(x\); this defines a map \(\mathcal{I} : \mathfrak{g} \to \mathfrak{h}/W\). Following [5], we define the set of “Jacobi elements”

\[
\mathfrak{g}_{>0} = \{ \sum a_i f_i + h + \sum b_i e_i; a_i \in \mathbb{R}_{>0}, b_i \in \mathbb{R}_{>0}, h \in \mathfrak{h}_{\mathbb{R}} \}.
\]

By [5, (3.5.26)], we have \(\mathcal{I}(\mathfrak{g}_{>0}) = \mathcal{I}(\mathfrak{h}_{\mathbb{R}})\). Hence any element in \(\mathfrak{g}_{>0}\) is regular semisimple and contained in an \(\mathbb{R}\)-split Cartan subalgebra and we can find \(x \in \mathfrak{g}_{>0}\) such that the semisimple part of \(x\) is \(G\)-conjugate to \(y\); since \(x\) is semisimple we see that \(x\) is \(G\)-conjugate to \(y\). Since \(\exp(y) \in C\), we see that \(\exp(x) \in C\).

It remains to show that \(\exp\) maps \(\mathfrak{g}_{>0}\) into \(G_{>0}\). This property is stated in [8, 5.8(c)] where a property close to it is proved (see [8, 5.9(c)]). But as mentioned in [8, p.550, footnote], a similar proof yields [8, 5.8(c)]. \(\square\)

3. JORDAN DECOMPOSITION

3.1. Jordan decomposition. Let \(g \in G_{>0}\) and \(g = g_s g_u = g_u g_s\) be the Jordan decomposition of \(g\) in \(G\). According to [10] the centralizer \(H = Z_G(g_s)\) of \(g_s\) is the Levi subgroup of a parabolic subgroup of \(G\); in particular it is connected.

Note that \(g_s\) lies in the identity component of \(H\). Since \(g_s\) is a central element of \(H\), we have that \(g_s\) is automatically contained in \(H_{P_{>0}}\) associated to any weak pinning \(P_H\) of \(H\). We conjecture that there exists a weak pinning \(P_H\) of \(H\) so that the associated totally nonnegative part \(H_{P_{>0}}\) of \(H\) contains the unipotent part \(g_u\) of \(g\) (and hence contains \(g\) itself). Moreover, the desired weak pinning \(P_H\) of \(H\) has a representation-theoretic interpretation.

To do this, we use the positivity property of the canonical basis for the simply-laced groups. For arbitrary group, we may apply the “folding method” of [8] once the desired result is proved for the simply-laced groups.

By [10, §9], if \(g \in G_{>0}\), then \(g_s\) is conjugate by an element in \(G(\mathbb{R})\) to an element in \(T_{>0}\). Let \(\lambda\) be a dominant weight of \(G\) and \(V_\lambda\) be the corresponding irreducible finite dimensional representation of \(G\). We have \(V_\lambda = \oplus_{c \in \mathbb{R}_{>0}} V_{\lambda_c}(g, c)\), where \(V_{\lambda_c}(g, c)\) is the generalized eigenspace of \(g : V_\lambda \to V_\lambda\) (see [10, §9]). Now \(H = Z_G(g_s)\) acts naturally on each \(V_{\lambda_c}(g_s, c)\). Let \(c_M = \max\{c \in \mathbb{R}_{>0}; V_\lambda(c) \neq 0\}\). We show that

(a) For any \(g \in G_{>0}\) and any dominant weight \(\lambda\) of \(G\), \(V_{\lambda_c}(g, c_M)\) is an irreducible representation of \(H\) with highest weight \(\lambda\).

We call an element \(t \in T_{>0}\) a standard element if \(\alpha_i(t) \geq 1\) for all \(i \in I\). If \(g_s\) is a standard element, then \(H\) is a standard Levi subgroup \(L_J\) for some \(J \subset I\). By definition, \(V_{\lambda_c}(g, c_M)\) is spanned by the weight vectors with \(\mu\) such that \(\mu(g_s) = \lambda(g_s)\). Note that the condition \(\mu(g_s) = \lambda(g_s)\) holds if and only if \(\lambda - \mu \in \sum_{i \in J} \mathbb{Z} \alpha_i\). In this case, \(V_{\lambda_c}(g, c_M)\) is the \(L_J\)-subrepresentation of \(V_\lambda\) generated by the highest weight vectors of \(V_{\lambda}c\). In particular, \(V_{\lambda_c}(g, c_M)\) is an irreducible finite dimensional representation of \(H\) with the highest weight \(\lambda\).

Note that any element in \(T_{>0}\) is conjugate to a standard element. Thus for any \(g \in G_{>0}\), there exists \(h \in G(\mathbb{R})\) such that \(h g_s h^{-1} \in T_{>0}\) and \(\alpha_i(h g_s h^{-1}) \geq 1\) for all \(i \in I\). Set \(g' = h g_s h^{-1}\). Then \(g'_i = h g_s h^{-1}\), \(Z_G(g'_i) = h H h^{-1}\) and \(V_{\lambda_c}(g', c_M) = h \cdot V_{\lambda_c}(g, c_M)\). Hence (a) is proved.

Now we state the conjectural totally nonnegative Jordan decomposition.
Conjecture 3.1. Assume that $G$ is simply laced. Let $g \in G_{\geq 0}$ and $H = Z_G(g_s)$. Then there exists a weak pinning $P_H$ of $H$ such that

1. $g_u \in H_{P_H; \geq 0}$;
2. $V(g, c_M)_{P_H; \geq 0} = V(g, c_M) \cap V_{\geq 0}$ for any irreducible representation $V$ of $G$.

By §1.4 and §1.2, the weak pinning on $H$, if it exists, is uniquely determined by $V(g, c_M) \cap V_{\geq 0}$. Moreover, as the totally nonnegative part of $H$ is determined by a representation-theoretic method, it only depends on the semisimple part $g_s$, but not on the element $g$ itself. In particular, Conjecture 3.1 claims that the totally nonnegative part $H_{\geq 0}$ is “canonical”.

3.2. Non simply-laced groups. We follow [8, §1.6]. Let $\hat{G}$ be the simply connected (algebraic) covering of the derived group of $G$ and $\pi : \hat{G} \to G$ be the covering homomorphism. The pinning on $G$ induces a pinning on $\hat{G}$. Then there exists a semisimple, simply connected, simply laced group $\hat{G}$, a pinning on $\hat{G}$ and an automorphism $\sigma : \hat{G} \to \hat{G}$ compatible with the pinning such that there is an isomorphism $\hat{G}^\sigma = \hat{G}$ so that the associated pinning on $G^\sigma$ (defined in [8, §1.5]) is compatible with the pinning on $\hat{G}$. In this case, we have $G_{\geq 0} = (Z_G)_{\geq 0} \pi(G^\sigma_{\geq 0})$.

Let $g \in G_{\geq 0}$. Then there exists $z \in (Z_G)_{\geq 0}$ and $\hat{g} \in G^\sigma_{\geq 0}$ such that $g = z\pi(\hat{g})$. Then $g_s = z\pi((\hat{g})_s)$ and $g_u = \pi((\hat{g})_u)$. Set $H = Z_G(g_s)$ and $\hat{H} = Z_G((\hat{g})_s)$. Then $H = Z_G \pi(\hat{H}^\sigma)$.

We assume that Conjecture 3.1 holds for $\hat{G}$. Since $\hat{H} = s(\hat{H})$ and the totally nonnegative part $\hat{H}_{\geq 0}$ is “canonical”, we have in particular that $H_{\geq 0}$ is $\sigma$-stable. Set $H_{\geq 0} = (Z_G)_{\geq 0} \pi(H^\sigma_{\geq 0})$. By Conjecture 3.1, $g_u \in \hat{H}^\sigma_{\geq 0}$. Thus $g_u = \pi(g_u) \in \pi(H^\sigma_{\geq 0}) \subset H_{\geq 0}$.

As a summary, we have

(a) Suppose that Conjecture 3.1 holds for $\hat{G}$. Then for any Levi subgroup $H$ of a parabolic subgroup of $G$, there is a “canonical” totally nonnegative part $H_{\geq 0}$ such that for any $g \in G_{\geq 0}$ with $Z_G(g_s) = H$, we have $g_u \in H_{\geq 0}$.

3.3. Some evidence of Conjecture 3.1. In this subsection, we verify Conjecture 3.1 in some special cases. We start with a lemma on $V_{\geq 0}$.

Lemma 3.2. Let $V$ be an irreducible finite dimensional representation of $G$ and $\beta$ be the canonical basis of $V$. Let $J \subset I$ and $V_J \subset V$ be the $L_J$-submodule spanned by the highest weight vector of $V$. Let $V_{\geq 0} = \sum_{b \in \beta} \mathbb{R}_{\geq 0} b$ and $(V_J)_{\geq 0} = V_J \cap V_{\geq 0}$. Then for any $w \in W^J$ and $u \in U_{w_{\geq 0}}$, we have

$$u \cdot (V_J)_{\geq 0} = (u \cdot V_J) \cap V_{\geq 0}.$$ 

Proof. By [8, §3.2], $u \cdot V_{\geq 0} \subset V_{\geq 0}$. Thus $u \cdot (V_J)_{\geq 0} \subset (u \cdot V_J) \cap V_{\geq 0}$. Now we prove the other direction. By definition, $(V_J)_{\geq 0} = \sum_{b \in \beta \cap V_J} \mathbb{R}_{\geq 0} b$. We recall the following result.

(a) Let $i \in I$ and $b \in \beta$ such that $x_i(a) \cdot b = b$ for all $a \in \mathbb{R}$. Then $s_i \cdot b \in \beta$.

This is a special case of the braid group action on the canonical basis [9]. See also [1, Lemma 2.8].

Let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression. Since $w \in W^J$, for any $j$ we have $(s_{i_{j+1}} \cdots s_{i_n})^{-1} a_{ij} \notin \Phi_J$. By definition, $x_{i_j}(a)(s_{i_{j+1}} \cdots s_{i_n}) \cdot b = (s_{i_{j+1}} \cdots s_{i_n}) \cdot b$ for
any $1 \leq j \leq n$, $a \in \mathbb{R}$ and $b \in V_J$. Thus by (a), for any $b \in \beta \cap V_J$, we have $\hat{s}_{i_n} \cdot b \in \beta, \hat{s}_{i_{n-1}} \hat{s}_{i_n} \cdot b \in \beta, \ldots, \hat{w} \cdot b \in \beta$.

By definition,
\[ u \cdot b \in \mathbb{R}_0 \hat{w} \cdot b + \sum_{b' \in \beta, \hat{w}(b') \in \mathbb{R}_0 \hat{w}} \mathbb{R}_0 \mathbb{R}_0 b'. \tag{b} \]

Since $w \in W_J$, $w(\Phi_J) \cap \sum_{l=1}^n Ns_{i_1} \cdots s_{i_{l-1}} \alpha_{i_l} = \{0\}$. In other words, the only element in $\{\hat{w} \cdot b; b' \in \beta \cap V_J\}$ that occurs in the right hand of (b) with positive coefficient is $\hat{w} \cdot b$.

Let $\sum_{b \in \beta \cap V_J} c_b b \in V_J$ with $u \cdot \sum_{b \in \beta \cap V_J} c_b b \in V_{>0}$. Note that
\[ u \cdot \sum_{b \in \beta \cap V_J} c_b b = \sum_{b \in \beta \cap V_J} c_b u \cdot b + \sum_{b \in \beta \cap V_J} c_b \mathbb{R}_0 \hat{w} \cdot b + \sum_{b' \in \beta \cap \hat{w} \cdot b} \mathbb{R}_0 \mathbb{R}_0 b'. \]

Then $c_b \geq 0$ for all $b \in \beta \cap V_J$. Thus $\sum_{b \in \beta \cap V_J} c_b b \in (V_J)_{>0}$. The Lemma is proved.

\begin{proposition}
Let $w_1, w_2 \in W$ with $\text{supp}(w_2) \subset \text{supp}(w_1)$. Then for any $g \in G_{w_1,w_2,>0}$ or $g \in G_{w_2,w_1,>0}$, conjecture 3.1 holds.
\end{proposition}

\begin{proof}
We prove the case where $g \in G_{w_1,w_2,>0}$. The case $g \in G_{w_2,w_1,>0}$ is proved in the same way.

Let $J = \text{supp}(w_1)$. By Lemma 2.2, there exists $u_1 \in U_{w_2,J,>0}^{-}$ such that $u_1^{-1} g u_1 \in tU_{w_1,>0}^{+}$ for some $t \in T_{>0}$ with $\alpha_i(t) < 1$ for all $i \in J_2$. By Lemma 2.5, there exists $u_2 \in U_{w_1,J_1,>0}^{-}$ such that $u_2^{-1} g u_1 u_2 \in \tilde{t}_J t_J U_{w_1,>0}^{+}$. By definition, $w_{t,J_1} \in W_{t,J_1} \cap W_{J_2}$. Let $\ell(w_{t,J_1} w_{t_1,J_1}) = \ell(w_J) + \ell(w_{t,J_1})$. Set $u_{-} = u_1 u_2 \in U_{w_2,J_1,>0}^{-}$. It is also easy to see that there exists $u_{+} \in L_{J_1} \cap U_{P_{w_{t,J_1}}}^{+}$ such that

- the semisimple part $u_{+}^{-1} u_{-}^{-1} g u_{-} u_{+}$ of $u_{+}^{-1} u_{-}^{-1} g u_{-} u_{+}$ is $\tilde{t}_J$;
- the unipotent part $u_{+}^{-1} u_{-}^{-1} g u_{-} u_{+}$ of $u_{+}^{-1} u_{-}^{-1} g u_{-} u_{+}$ is in $U_{>0} \cap L_{I(t)}$.

Let $\tilde{t} = \tilde{t}_J$ and $J = I(\tilde{t})$. Let $w$ be the unique element in $J_{>0} W_{J}^{+}$ with $\tilde{t} = \hat{w}^{-1} I_{t_1} \hat{w}$. Hence the semisimple part of $w_{>0}^{-1} u_{-}^{-1} u_{-}^{-1} g u_{-} u_{+}$ is $\tilde{t}$. Let $H = Z_G(g_a)$. Since $Z_G(\tilde{t}) = L_J$, we have $H = (u_{-} u_{+} \hat{w}) L_{J} (u_{-} u_{+} \hat{w})^{-1}$. The pinning $P_H$ of $H$ is defined to be the conjugate of the standard pinning of $L_J$ by the element $u_{-} u_{+} \hat{w}$.

In particular, $H_{P_{H},>0} = (u_{-} u_{+} \hat{w}) L_{J,>0} (u_{-} u_{+} \hat{w})^{-1}$.

Since $\tilde{t} = \hat{w}^{-1} I_{t_1} \hat{w}$, we have $\Phi_H(\tilde{t}_J) = \Phi_H(\tilde{t}) \cap \Phi_J$. Since $w \in J_{>0} W_{J}^{+}$, we have $w_{-1}(I(\tilde{t}_J)) \subset J$. Therefore $w_{-1}(U_{>0} \cap L_{I(\tilde{t}_J)} \hat{w}) \subset U_{>0} \cap L_{J}$. Thus $g_u \in (u_{-} u_{+} \hat{w}) (U_{>0} \cap L_{I(\tilde{t}_J)} \hat{w}) (u_{-} u_{+} \hat{w})^{-1} \subset (u_{-} u_{+} \hat{w}) L_{J,>0} (u_{-} u_{+} \hat{w})^{-1} = H_{P_{H},>0}$.

Part (1) of Conjecture 3.1 is proved.

Let $V$ be an irreducible finite dimensional representation of $G$ and $v$ be a highest weight vector of $V$. Let $V_J \subset V$ be the $L_J$-submodule generated by $v$. Then $V(\bar{t}, c_M) = V_J$ and $V(g, c_M) = u_{-} u_{+} \hat{w} \cdot V_J$. The nonnegative part $V(g, c_M)_{P_{H},>0}$ of $V(g, c_M)$ determined by the pinning $P_H$ of $H$ equals $u_{-} u_{+} \hat{w} \cdot V_{J,>0}$. Since $\Phi_H(\tilde{t}_J) = W(\Phi_J) \cap \Phi_J$ and $w \in J_{>0} W_{J}^{+}$, we have $w_{-1} u_{-} u_{+} \hat{w} \in U_{P_J}^{+}$. Thus $u_{-} u_{+} \hat{w} \cdot V_J = u_{-} \hat{w} u \cdot V_J = u_{-} \hat{w} \cdot V_J$ for some $u \in U_{P_J}^{+}$.

\[ V(g, c_M)_{P_{H},>0} = u_{-} \hat{w} \cdot V_{J,>0}. \]
By the proof of Lemma 3.2, $\mathbf{w} \cdot V_{J, > 0} \subset V_{\geq 0}$. Since $u_- \in U_{> 0}$, we have $u_- \cdot V_{\geq 0} \subset V_{\geq 0}$. Thus $V(g, c_M)_{P_M, \geq 0} \subset V_{\geq 0}$.

On the other hand, let $\beta$ be the canonical basis of $V$. Suppose that $\sum_{b \in \beta \cap V_J} c_b b \in V_J$ with $u_- \mathbf{w} \cdot \sum_{b \in \beta \cap V_J} c_b b \in V_{\geq 0}$. By (b) in the proof of Lemma 3.2, we have

$$u_\mathbf{w} \cdot \sum_{b \in \beta \cap V_J} c_b b \in V_J \subseteq \sum_{b \in \beta \cap V_J} c_b \mathbb{R}_{> 0} \mathbf{w} \cdot b + \sum_{b \in \beta \cap V_J} \mathbb{R}_{> 0} b'. $$

Thus $c_b \geq 0$ for all $b \in \beta \cap V_J$ and $\sum_{b \in \beta \cap V_J} c_b b \in (V_J)_{\geq 0}$.

Therefore $V(g, c_M)_{P_M, \geq 0} = u_\mathbf{w} \cdot V_{J, \geq 0} = u_\mathbf{w} \cdot V_J \cap V_{\geq 0} = V(g, c_M) \cap V_{\geq 0}$. Part (2) of Conjecture 3.1 is proved.

### 3.4. Verification of conjecture 3.1 for $GL_3$.

In this subsection, we prove conjecture 3.1 for $G = GL_3$. Let $W = S_3$ be the Weyl group of $G$ and $\{s_1, s_2\}$ be the set of simple reflections of $W$. By Proposition 3.3, if $g \in G_{\geq 0}$ but $g \notin G_{s_1, s_2, > 0} \cup G_{s_2, s_1, > 0}$, then the conjecture 3.1 holds for $g$. Now we prove the conjecture 3.1 for $g \in G_{s_1, s_2, > 0}$. The elements in $G_{s_1, s_2, > 0}$ are handled in the same way.

The element $g$ can be written as $g = t u_1 u_2$ for some $u_1 \in U_{s_1, > 0}$, $u_2 \in U_{s_2, > 0}$ and $t \in T_{> 0}$.

If $\alpha_1(t) \neq 1$ and $\alpha_2(t) \neq 1$, then $g$ is regular semisimple and the conjecture 3.1 is obvious for $g$.

If $\alpha_1(t) = \alpha_2(t) = 1$, then $g_s = t$ and $H = G$. In this case, the weak pinning on $H$ is the equivalence class of the pinning on $G$ we fixed in the beginning. The conjecture 3.1 holds for $g$.

It remains to consider the case where $\alpha_1(t) = 1$ and $\alpha_2(t) \neq 1$. (The case where $\alpha_1(t) \neq 1$ and $\alpha_2(t) = 1$ is proved in the same way.) In this case, there exists $a \in \mathbb{R}$ such that $y_2(-a) g y_2(a) = t u_1$. We have $Z_G(t) = L_{\{1\}}$ and $H = y_2(a) L_{\{1\}} y_2(-a)$. The pinning on $G$ induces the pinning $P_{\{1\}} = (T, B^+ \cap L_{\{1\}}, B^- \cap L_{\{1\}}, x_1, y_1)$ on the Levi subgroup $L_{\{1\}}$. The pinning $P_H$ on $H$ is obtained from $P_{\{1\}}$ by conjugating $y_2(a)$. Moreover, $V(g_s, c_M) = y_2(a) \cdot V(t, c_M)$ and $V(g_s, c_M)_{P_{M}, \geq 0} = y_2(a) \cdot V(t, c_M)_{\geq 0}$.

If $\alpha_2(t) < 1$, then $V(t, c_M)$ is the $L_{\{1\}}$-submodule of $V$ generated by the lowest weight vector of $V$ and $V(g_s, C_M) = V(t, c_M)$. In this case, $V(g_s, c_M)_{P_M, \geq 0} = V(t, c_M)_{\geq 0} = V(g_s, C_M) \cap V_{\geq 0}$.

If $\alpha_2(t) > 1$, then $V(t, c_M)$ is the $L_{\{1\}}$-submodule of $V$ generated by the highest weight vector of $V$. Moreover, we have $a > 0$ since $u_2 \in U_{s_2, > 0}$. By Lemma 3.2, $V(g_s, c_M)_{P_M, \geq 0} = y_2(a) \cdot V(t, c_M)_{\geq 0} = V(g_s, C_M) \cap V_{\geq 0}$.

This finishes the verification of the conjecture 3.1 for $G = GL_3$.

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