A FAMILY OF SIMPLE WEIGHT MODULES OVER THE VIRASORO ALGEBRA

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Abstract. Using simple modules over the derivation Lie algebra \( \mathbb{C}[t] \frac{d}{dt} \) of the associative polynomial algebra \( \mathbb{C}[t] \), we construct new weight Virasoro modules with all weight spaces infinite dimensional. We determine necessary and sufficient conditions for these new weight Virasoro modules to be simple, and determine necessary and sufficient conditions for two such weight Virasoro modules to be isomorphic. If such a weight Virasoro module is not simple, we obtain all its submodules. In particular, we completely determine the simplicity and the isomorphism classes of the weight modules defined in [CM] which are a small portion of the modules constructed in this paper.

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1. Introduction

We denote by \( \mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \) and \( \mathbb{C} \) the sets of all integers, nonnegative integers, positive integers, and complex numbers, respectively. For a Lie algebra \( L \) we denote by \( U(L) \) the universal enveloping algebra of \( L \).

The Virasoro algebra \( \mathfrak{V} \) is the universal central extension of the derivation algebra of the Laurent polynomial algebra \( \mathbb{C}[t, t^{-1}] \). More precisely, \( \mathfrak{V} \) is a Lie algebra over \( \mathbb{C} \) with the basis

\[
\{ t^{n+1} \frac{d}{dt}, z | n \in \mathbb{Z} \}
\]

and subject to the Lie bracket

\[
[t^{n+1} \frac{d}{dt}, t^{m+1} \frac{d}{dt}] = (m-n)t^{m+n+1} \frac{d}{dt} + \delta_{n,-m} \frac{n^3-n}{12} z,
\]

\[
[\mathfrak{V}, z] = 0.
\]

Denote \( d_n = t^{n+1} \frac{d}{dt} \). We will used both notations according to contexts.

The Virasoro algebra is one of the most important Lie algebras both in mathematics and in mathematical physics, see for example [KR] [IK] and references therein. Its theory has been widely used in many physics areas and other mathematical branches, for example, quantum physics.
The representation theory on the Virasoro algebra has been attracting a lot of attentions from mathematicians and physicists. There are two classical families of simple Harish-Chandra \( V \)-modules: highest weight modules (completely described in [FF]) and the so-called intermediate series modules. In [Mu] it is shown that these two families exhaust all simple weight Harish-Chandra modules. In [MZ1] it is even shown that the above modules exhaust all simple weight modules admitting a nonzero finite dimensional weight space.

Very naturally, the next important task is to study simple weight modules with infinite dimensional weight spaces. The first such examples were constructed by taking the tensor product of some highest weight modules and some intermediate series modules in [Zh] in 1997, and the necessary and sufficient conditions for such tensor product to be simple were recently obtained in [CGZ]. Conley and Martin gave another class of such examples with four parameters in [CM] in 2001 where some sufficient conditions were discussed for the modules to be simple. Then very recently, a big class of weight simple Virasoro modules were found in [LLZ]. We remark that the tensor products of intermediate series modules over the Virasoro algebra never gives irreducible modules [Zk]. In this paper, we construct a family of weight simple Virasoro modules which include all the modules defined in [CM, LLZ] as a small portion.

At the same time for the last decade, various other families of non-weight simple \( V \)-modules were studied in [OW1, LGZ, LZ, FJK, Ya, GLZ, OW2, MW, TZ]. These include various versions of Whittaker modules constructed using different tricks. In particular, all the above Whittaker modules and even more were described in a uniform way in [MZ2].

To introduce the contents of the present paper we need to define the following subalgebras of \( V \) where \( r \in \mathbb{Z}_+ \):

\[
\mathfrak{W} = \text{Der}(\mathbb{C}[t]) = \text{span}\{d_i | i \geq -1\},
\]

\[
\mathfrak{b} = \text{span}\{d_i | i \geq 0\},
\]

\[
\mathfrak{V}^{(r)} = \text{span}\{d_i | i \geq r\},
\]

\[
\mathfrak{a}_r = \mathfrak{b}/\mathfrak{V}^{(r+1)}.
\]

The Lie algebra \( \mathfrak{W} \) is usually called the Witt algebra of rank one. We denote by \( \mathcal{O}_\mathfrak{W} \) the category of all \( \mathfrak{W} \)-modules \( W \) satisfying

**Condition A:** For any \( w \in W \), there exists a nonnegative integer \( n \) depending on \( W \) such that \( d_i w = 0 \) for all \( i \geq n \).
Similarly we may define the categories $\mathcal{O}_b$, $\mathcal{O}_h$. It is clear that $\mathcal{O}_b$ consists of highest weight modules and the ones define in [M2].

The paper is organized as follows. In Sect.2, we determine all simple modules in $\mathcal{O}_b$ and all simple modules in $\mathcal{O}_{\mathfrak{g}^2}$. Actually, a simple module in $\mathcal{O}_b$ is a simple module over $\mathfrak{a}_r$ for some $r \in \mathbb{N}$, and all nontrivial simple modules in $\mathcal{O}_{\mathfrak{g}^2}$ are induced modules from a simple module over $\mathfrak{a}_r$. In Sect.3, for any $W \in \mathcal{O}_{\mathfrak{g}^2}$, $a, b \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$, we define our weight Virasoro modules $L(W, \lambda, a, b) = W \otimes \mathbb{C}[t, t^{-1}]$, and we prove that all the Virasoro modules $E_{\lambda}(b, \gamma, p)$ defined in [CM] are only very special cases of the modules $L(W, \lambda, a, b)$. We also establish a powerful technique for later use. In Sect.4 we determine necessary and sufficient conditions for $L(W, \lambda, a, b)$ to be simple. When it is not simple we determine all it submodules. In Sect.5 we determine necessary and sufficient conditions for two such simple Virasoro modules to be isomorphic. In Sect.6 we show that the simple Virasoro weight modules are new. The good presentation of the modules $L(W, \lambda, a, b)$ and the powerful technique in Proposition 5 enable us to establish the results in this paper.

2. Simple modules in $\mathcal{O}_{\mathfrak{g}^2}$

In this section, we determine all simple modules in $\mathcal{O}_{\mathfrak{g}^2}$.

Let $V$ be a module over a Lie algebra $L$. We say that the $L$-module $V$ is trivial if $LV = 0$. Denote by $\text{Soc}_L(V)$ the socle of the $L$-module $V$, i.e., $\text{Soc}_L(V)$ is the sum of the minimal nonzero submodules of $V$. For any $v \in V$, the annihilator of $v$ is defined as $\text{ann}_L(v) = \{g \in L | gv = 0\}$.

For any $\mathfrak{b}$-module $B$ and $0 \neq v \in B$, define $\text{ord}_\mathfrak{b}(v)$, the order of $v$, to be the minimal nonnegative integer $r$ with $d_{r+i}v = 0$ for all $i \geq 1$, or to be $\infty$ if such $r$ doesn’t exist. And $\text{ord}_\mathfrak{b}(B)$, the order of $B$, is defined to be the maximal order of all its elements or $\infty$ if it doesn’t exist.

**Lemma 1.** Suppose that $B \in \mathcal{O}_b$ is simple.

(a). $\text{ord}_\mathfrak{b}(B) = \text{ord}_\mathfrak{b}(v) < \infty$, for all nonzero $v \in B$.

(b). $\text{ord}_\mathfrak{b}(B) = 0$ if and only if $B$ is one dimensional.

(c). If $B$ is nontrivial, then the action of $d_r$ on $B$ is bijective, where $r = \text{ord}_\mathfrak{b}(B)$.

Consequently, $B$ is a simple $\mathfrak{a}_r$-module for some $r \in \mathbb{N}$.

**Proof.** For any nonzero $v, v' \in B$, since $B$ is simple, there exists some $u \in U(\mathfrak{b})$, such that $v' = uv$. It is straightforward to check that $d_iv' = d_iuv = 0$ for all $i > \text{ord}_\mathfrak{b}(v)$. So $\text{ord}_\mathfrak{b}(v) \geq \text{ord}_\mathfrak{b}(v')$. Similarly we have $\text{ord}_\mathfrak{b}(v') \geq \text{ord}_\mathfrak{b}(v)$. Thus $\text{ord}_\mathfrak{b}(v') = \text{ord}_\mathfrak{b}(v)$. So we have proved (a). Part (b) is trivial.

Now suppose that $B$ is nontrivial, and $r = \text{ord}_\mathfrak{b}(B)$. Consider the subspace $X = \{v \in B | d_r v = 0\}$ which is a proper subspace of $B$. Then $X$ and $d_r(B)$ are $\mathfrak{b}$-submodules of $B$. Since $B$ is simple, and $d_r B \neq 0$.
we deduce that $X = 0$ and $d_r(B) = B$, i.e., $d_r$ is bijective. Part (c) follows.

\[\text{Lemma 2. Let } B \in \mathcal{O}_b, W, W_1 \in \mathcal{O}_a \text{ be nontrivial simple modules.}\\
(1) \text{Ind}_b^a(B) \text{ is a simple module in } \mathcal{O}_a;\\
(2) \text{Soc}_b(W) \text{ is a simple } b\text{-module, and an essential } b\text{-submodule of } W, \text{ i.e. the intersection of all nonzero } b\text{-submodules of } V;\\
(3) W \cong \text{Ind}_b^a \text{Soc}_b(W), B = \text{Soc}_b(\text{Ind}_b^a B);\\
(4) W \cong W_1 \text{ if and only if } \text{Soc}_b(W) \cong \text{Soc}_b(W_1).\\
\text{Consequently, } W \text{ is the induced module from a simple } a_r\text{-module for some } r \in \mathbb{N}.\\
\text{Proof. (1). Let } M \text{ be a nonzero submodule of } \text{Ind}_b^a(B) = \mathbb{C}[d_{-1}] \otimes B.\\
\text{Choose } 0 \neq v \neq \sum_{i=0}^{s} d_{-1}^i \otimes v_i \in M \text{ with minimal } s, \text{ where } v_i \in B.\\
\text{Denote } r = \text{ord}_b(B). \text{ If } s > 0, \text{ then}\\
0 \neq d_{r+1} v \neq -s(r + 2)d_{-1}^s \otimes d_r v_s + \sum_{i=0}^{s-2} d_{-1}^i \otimes B \subset M,\\
\text{which contradicts the minimality of } s. \text{ So } s = 0, \text{ i.e., } v \in 1 \otimes B. \text{ Therefore } M = \text{Ind}_b^a(B), \text{ and } \text{Ind}_b^a(B) \text{ is simple.}\\
(2). \text{ Fix some } 0 \neq w \in W \text{ with minimal } \text{ord}_b w = r. \text{ Let } M = U(b)w. \text{ Then } \text{ord}_b M = r, \text{ and } W = \mathbb{C}[d_{-1}]M. \text{ For any } v = \sum_{i=0}^{s} d_{-1}^i \otimes w_i \in W \text{ with } w_s \neq 0 \text{ and } w_i \in M \text{ for } i = 0, \ldots, s, \text{ we have}\\
0 \neq d_{r+s} v = (-1)^s(r + s + 1)(r + s) \cdots (r + 2)d_r w_s \in M,\\
(2.1) d_{r+i} v = 0, \forall i > s.\\
\text{Thus } \text{ord}_b(v) = r + s. \text{ So } W \cong \mathbb{C}[d_{-1}] \otimes M \text{ and } W \cong \text{Ind}_b^a M. \text{ Since } W \text{ is a simple } a\text{-module, } M \text{ has to be simple as } b\text{-module, and it is essential from (2.1).}\\
\text{Part (3) is an obvious consequence of (1) and (2). Part (4) follows from (3).} \quad \square\\
\text{We remark that a classification for all simple modules over } a_1 \text{ was given in [B]}, \text{ while a classification for all simple modules over } a_2 \text{ was recently obtained in [MZ2]. The problem is open for all other } (r + 1)\text{-dimensional Lie algebras } a_r \text{ for } r > 2. \text{ However various simple modules over } a_r \text{ were given in [MZ2].}\\
\text{Example 1. Consider some } r \in \mathbb{N} \text{ and set}\\
\mu = (\mu_{r+1}, \mu_{r+2}, \ldots, \mu_{2r}) \in \mathbb{C}^r.\\
\text{Define the one dimensional } \mathfrak{g}^{(r)} \text{ module } \mathbb{C} \text{ with the action}\\
d_i 1 = 0, \forall i > 2r,\]
\[ d_k 1 = \mu_k, \forall k = r + 1, \ldots, 2r. \]

Then we have the induced module \( W_\mu = \text{Ind}_{W_{V^r}(\cdot)}^W \mathbb{C}, \) which is simple if and only if \( \mu_{2r} \neq 0 \) or \( \mu_{2r-1} \neq 0. \) See \([LGZ]\) or \([MZ2]\) for more details.

3. Constructing new Virasoro modules

In this section, we will introduce our new weight Virasoro modules to be studied in this paper. Then we provide our main technique for later use.

3.1. Constructing new Virasoro modules. In this subsection we will provide a method to construct new weight Virasoro modules from modules in \( \mathcal{O}_{2r}. \)

Let \( W \in \mathcal{O}_{2r}. \) Then \( W \) can be naturally regarded as a module over \( \mathbb{C}[[t]] \frac{d}{dt}. \) Regard \( \mathbb{C}[e^\pm t] \frac{d}{dt} \) as a subalgebra of \( \mathbb{C}[[t]] \frac{d}{dt}. \) We will use the expression

\[
e^{mt} = \sum_{k=0}^{\infty} \frac{(mt)^k}{k!} \in \mathbb{C}[[t]], \forall m \in \mathbb{Z}.
\]

In particular, we have

\[
[e^{mt} \frac{d}{dt}, e^{nt} \frac{d}{dt}] = (n-m)e^{(m+n)t} \frac{d}{dt},
\]

\[
[e^{mt} \frac{d}{dt}, \frac{d}{dt}] = -me^{mt} \frac{d}{dt},
\]

and

\[
(e^{mt} \frac{d}{dt}) \left( \frac{d}{dt} \right) = (e^{mt} \frac{d}{dt}) \left( \frac{d}{dt} - m \right), \forall m, n \in \mathbb{Z},
\]

in \( U(\mathfrak{W}) \). Now we can give the weight modules defined and studied in this paper.

Lemma 3. For any \( W \in \mathcal{O}_{2r}, a, b \in \mathbb{C} \) and \( \lambda \in \mathbb{C}^*, \) the vector space \( \mathcal{L}(W, \lambda, a, b) = W \otimes \mathbb{C}[t, t^{-1}] \) becomes a Virasoro module with the action

\[
z \cdot (w \otimes t^j) = 0,
\]

\[
d_k \cdot (w \otimes t^j) = ((\lambda^k e^{kt} - 1) \frac{d}{dt} + a + kb + j)w \otimes t^{k+j},
\]

for all \( k, j \in \mathbb{Z}, w \in W. \)

Proof. We only verify that \( [d_m, d_n] \cdot (v \otimes t^j) = (d_m d_n - d_n d_m) \cdot (v \otimes t^j) \) for all \( m, n, j \in \mathbb{Z} \) and \( v \in W, \) while other relations are obvious. We compute (for simplicity, we will denote \( \mu = a + j \))

\[
d_m d_n \cdot (v \otimes t^j)
\]

\[
= d_m \cdot ((\lambda^n e^{nt} - 1) \frac{d}{dt} + \mu + nb)v \otimes t^{j+n})
\]
Using the above formula, we deduce that

\[
\begin{align*}
&= (\lambda^m e^{mt} - 1) \frac{d}{dt} + \mu + mb + n)((\lambda^n e^{nt} - 1) \frac{d}{dt} + \mu + nb) v \otimes t^{m+n} \\
&= \lambda^{m+n} (e^{mt} \frac{d}{dt} (e^{nt} \frac{d}{dt}) - \lambda^m (\lambda^n \frac{d}{dt} - \mu - nb)) \\
&- \lambda^n (\frac{d}{dt} - \mu - mb) v \otimes t^{m+n+j} \\
&+ \left( \frac{d}{dt} - \mu - mb \right) \left( \frac{d}{dt} - \mu - nb \right) - n \left( \frac{d}{dt} - \mu \right) + n^2 b) v \otimes t^{m+n+j}.
\end{align*}
\]

Using the above formula, we deduce that

\[
(d_m d_n - d_n d_m) \cdot (v \otimes t^j)
= \lambda^{m+n} [e^{mt} \frac{d}{dt} e^{nt} \frac{d}{dt}] - (n - m) \left( \frac{d}{dt} - \mu \right) + (n^2 - m^2) b) v \otimes t^{m+n+j} \\
= (n - m) \left( \lambda^{m+n} e^{(m+n)j} \frac{d}{dt} - \frac{d}{dt} + a + j + (m + n) b \right) v \otimes t^{m+n+j} \\
= (n - m) d_{m+n} \cdot (v \otimes t^j) \\
= [d_m, d_n] \cdot (v \otimes t^j).
\]

From (3.2) we know that, if \( M \) is infinite dimensional, then the module \( \mathcal{L}(W, \lambda, a, b) \) is a weight Virasoro module with infinite dimensional weight spaces \( \mathcal{L}_{\alpha+n} = W \otimes t^n \) where

\[
\mathcal{L}_{\alpha+n} = \{ v \in \mathcal{L}(W, \lambda, a, b) \mid d_0 v = (\alpha + n) v \}.
\]

**Example 2.** When \( W = \mathbb{C} v \) is the trivial \( \mathcal{M} \)-module, the weight \( \mathcal{B} \)-module \( \mathcal{L}(W, \lambda, a, b) = v \otimes \mathbb{C}[t, t^{-1}] \) is determined by

\[
(3.3) \quad d_m v_n = (a + n + bm) v_{n+m},
\]

where \( v_n = v \otimes t^n \), which is exactly the module \( A_{a,b} \) of intermediate series (see [KR]).

### 3.2. Realizing Virasoro modules \( E_b(h, \gamma, p) \) defined in [CM].

Let \( W \) be the Verma \( \mathcal{M} \)-module with the highest weight vector \( w_0 \) of highest weight \( b' \in \mathbb{C} \), i.e., \( d_0 w_0 = b' w_0 \) and \( d_i w_0 = 0 \) for all \( i > 0 \). For any \( a, b \in \mathbb{C}, \lambda \in \mathbb{C}^* \), we have the weight \( \mathcal{B} \)-module \( \mathcal{L}(W, \lambda, a, b) = W \otimes \mathbb{C}[t, t^{-1}] \) with the action
Comparing this action with the one in Lemma 2.1 of [CM], we see that with \( \lambda \) algebra \( \mathcal{E} \) will be frequently used later. This technique is crucial to this paper.

To end this section with an important result of computations, which maps from \( \mathcal{Z} \) to end this section with an important result of computations, which maps from \( \mathcal{Z} \) to \( \mathbb{C}[t] \). In particular,

\[
d_n \cdot (d_{-1}^k w_0 \otimes t^i)
\]

\[
= \left( \lambda^n (n - d_{-1})^k (nb' + d_{-1}) w_0 \otimes t^{i+n} + T_{i+n}^{k+1} + (a + nb + i) T_{i+n}^k \right)
\]

\[
= \left( \lambda^n (nb' + d_{-1}) \sum_{j=0}^{k} \binom{k}{j} n^{k-j} (-d_{-1})^j \right) w_0 \otimes t^{i+n}
\]

\[
+ T_{i+n}^{k+1} + (a + nb + i) T_{i+n}^k
\]

\[
= \lambda^n nb' \sum_{j=0}^{k-1} \binom{k}{j} n^{k-j} T_{i+n}^j - \lambda^n \sum_{j=0}^{k-2} \binom{k}{j} n^{k-j} T_{i+n}^{j+1}
\]

\[
+ (1 - \lambda^n) T_{i+n}^{k+1} + (a + nb + i + \lambda^n nb' - \lambda^n nk) T_{i+n}^k.
\]

Comparing this action with the one in Lemma 2.1 of [CM], we see that \( \mathcal{L}(W, \lambda, a, b) \) is exactly the module \( E_h(a, -b', b + b') \) defined in [CM] with \( \lambda = e^h \) and \( \mu = a + i \). The main results in [CM] are discussions on some sufficient conditions for \( E_h(a, -b', b + b') \) to be simple.

### 3.3. Some properties of the \( \mathbb{C}[x] \)-module \( \mathcal{M}(\mathbb{Z}, \mathbb{C}) \).

We are going to end this section with an important result of computations, which will be frequently used later. This technique is crucial to this paper.

Let \( P \) be any vector space over \( \mathbb{C} \). Denote by \( \mathcal{M}(\mathbb{Z}, P) \) the set of all maps from \( \mathbb{Z} \) to \( P \), which naturally becomes a vector space over \( \mathbb{C} \).

It is easy to see that \( \mathcal{M}(\mathbb{Z}, P) \) becomes a module over the polynomial algebra \( \mathbb{C}[x] \) by the action

\[
\text{act-c}[x] \quad (x^i \cdot T)(m) = T(m + i), \forall m \in \mathbb{Z}, T \in \mathcal{M}(\mathbb{Z}, P).
\]

Now we consider the infinite dimensional vector space \( \mathcal{M}(\mathbb{Z}, \mathbb{C}) \). For any \( \lambda \in \mathbb{C}^* \) and \( k \in \mathbb{N} \), \( \lambda^m m^k \) is regarded as an element in \( \mathcal{M}(\mathbb{Z}, \mathbb{C}) \) by mapping \( m \in \mathbb{Z} \) to \( \lambda^m m^k \). In particular, \( x \cdot \lambda^m m^k = \lambda^{m+1}(m+1)^k \).
Lemma 4. Let $\lambda \in \mathbb{C}^*$, $k \in \mathbb{N}$, $p(x) \in \mathbb{C}[x]$ and $\lambda^m m^k \in \mathcal{M}(\mathbb{Z}, \mathbb{C})$.

(a) We have
\[
(x - \lambda)^k \cdot (\lambda^m m^k) = k! \lambda^{m+k},
\]
\[
(x - \lambda)^{k+1+i} \cdot (\lambda^m m^k) = 0, \quad \forall i \in \mathbb{Z}_+;
\]

(b) $p(x)(x - \lambda)^k \cdot (\lambda^m m^k) = k! \lambda^{m+k} p(\lambda)$.

(c) $p(x) \cdot (\lambda^m m^k) = 0 \in \mathcal{M}(\mathbb{Z}, \mathbb{C})$ if and only if $(x - \lambda)^{k+1} f(x)$.

Proof. (a). We compute
\[
(x - \lambda)^k \cdot (\lambda^m m^k) = (x - \lambda)^{k-1} (x - \lambda) \cdot (\lambda^m m^k)
\]
\[
= (x - \lambda)^{k-1} (\lambda (\lambda^m m^k) - \lambda \lambda^m m^k)
\]
\[
= (x - \lambda)^{k-1} (\lambda^{m+1} (m+1)^k - \lambda^{m+1} m^k)
\]
\[
= (x - \lambda)^{k-1} (\lambda^{m+1} (km^{k-1} + \text{lower terms of } m))
\]
\[
= ... = k! \lambda^{m+k}.
\]
The second formula follows after applying another $(x - \lambda)$.

Part (b) follows from the fact that $p(x) = q(x)(x - \lambda) + p(\lambda)$ for some $q(x) \in \mathbb{C}[x]$ by using (a).

(c). Note that $\text{ann}_{\mathbb{C}[x]}(\lambda^m m^k) = \{ g(x) \in \mathbb{C}[x] | p(x) \cdot (\lambda^m m^k) = 0 \}$ is an ideal of the principle ideal domain $\mathbb{C}[x]$. From (a) we have $(x - \lambda)^k \notin \text{ann}_{\mathbb{C}[x]}(\lambda^m m^k)$ and $(x - \lambda)^{k+1} \in \text{ann}_{\mathbb{C}[x]}(\lambda^m m^k)$. Thus $\text{ann}_{\mathbb{C}[x]}(\lambda^m m^k) = \mathbb{C}[x](x - \lambda)^{k+1}$. So we have proved (c). \qed

Now we are ready to provide our main tool for later use.

Proposition 5. Let $P$ be a vector space over $\mathbb{C}$, $P_1$ be a subspace of $P$. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathbb{C}^*$ are pairwise distinct, $v_{i,j} \in P$ and $f_{i,j}(t) \in \mathbb{C}[t]$ with $\text{deg}(f_{i,j}(t)) = j$ for $i = 1, 2, \ldots, s; j = 0, 1, 2, \ldots, k$.

If
\[
\sum_{i=1}^{s} \sum_{j=0}^{k} \lambda_i^m f_{i,j}(m)v_{i,j} \in P_1, \forall m \in \mathbb{Z},
\]
then $v_{i,j} \in P_1$ for all $i, j$.

Proof. Let $p(x) = ((x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_s))^{k+1}$, $q_j(x) = p(x)/(x - \lambda_j)^{k+1}$, and $p_j(x) = p(x)/(x - \lambda_j)$ for $j = 1, 2, \ldots, s$. Denote
\[
T_k(m) = \sum_{i=1}^{s} \sum_{j=0}^{k} \lambda_i^m f_{i,j}(m)v_{i,j} \in P_1, \forall m \in \mathbb{Z}.
\]
From (3.5) we have $T_k \in \mathcal{M}(\mathbb{Z}, P_1)$. By Lemma 4(b), we see that
\[
p_i(x) \cdot (\lambda_i^m f_{i,j}(m)v_{i,j}) = \delta_{i,j} \delta_{i,k} a_{i,k} q_i(\lambda_i) k! \lambda_i^{m+k} v_{i,k},
\]
where $a_{i,k}$ is the coefficient of $t^k$ in $f_{i,k}$. So
\[
p_i(x) \cdot T_k(m) = a_{i,k} q_i(\lambda_i) k! \lambda_i^{m+k} v_{i,k} \in P_1, \forall m \in \mathbb{Z}.
\]
Since \( q_i(\lambda_i) \neq 0 \), we see that \( v_{i,k} \in P_1 \) for all \( i = 1, 2, \ldots, s \), and

\[
T_{k-1}(m) = \sum_{i=1}^{s} \sum_{j=0}^{k-1} \lambda_i^m f_{i,j}(m)v_{i,j} \in P_1, \forall m \in \mathbb{Z}.
\]

In this manner we deduce that each \( v_{i,j} \in P_1 \). \( \square \)

**Remark 6.** In \([3,2]\), to satisfy the conditions for \( f_{i,j} \), many \( v_{i,j} \) may be zero. If we take \( P_1 = 0 \), the corresponding result in this proposition becomes that \( v_{i,j} = 0 \) for all \( i, j \).

### 4. Simplicity of Weight Virasoro Modules \( \mathcal{L}(W, \lambda, a, b) \)

For any \( W \in \mathcal{O}_{2\mathbb{N}} \), \( \lambda \in \mathbb{C}^* \), \( a, b \in \mathbb{C} \), we have defined the weight Virasoro module \( \mathcal{L}(W, \lambda, a, b) \) in Sect.3. In this section we will determine necessary and sufficient conditions for \( \mathcal{L}(W, \lambda, a, b) \) to be simple, and find all its submodules if it is not simple.

For any simple \( \mathfrak{b} \)-module \( B \) and any \( b \in \mathbb{C} \), we can have a new \( \mathfrak{b} \)-module structure on \( B \), denoted by \( B_{(b)} \), with the new action \( d_{0} \cdot v = (d_{0} + b)v \) and \( d_{i} \cdot v = d_{i}v \) for all \( v \in B \) and \( i > 0 \). The new \( \mathfrak{b} \)-module \( B_{(b)} \) is also simple.

Let \( W \in \mathcal{O}_{2\mathbb{N}} \) be simple. We know that \( \text{Soc}_{\mathfrak{b}} W \) is a simple \( \mathfrak{b} \)-module. Then \( \langle \text{Soc}_{\mathfrak{b}} W \rangle \otimes \mathbb{C}[t, t^{-1}] \) is a submodule of \( \mathcal{L}(W, 1, a, 0) \), which is exactly the Virasoro module \( \mathcal{N}(\langle \text{Soc}_{\mathfrak{b}} W \rangle, a) \) defined and studied in \([\text{LLZ}]\).

For \( n \in \mathbb{N} \), in \( W \) we define the subspace \( W^{(n)} = \sum_{i=0}^{n} d_{-1}^{i}(\langle \text{Soc}_{\mathfrak{b}} W \rangle) \). The following lemma solves the simplicity of \( \mathcal{L}(W, \lambda, a, b) \) for \( \lambda = 1 \).

**Lemma 7.** For \( a, b \in \mathbb{C} \) and any nontrivial simple \( W \in \mathcal{O}_{2\mathbb{N}} \), the Virasoro module \( \mathcal{L}(W, 1, a, b) \) has a filtration of submodules

\[
W^{(0)} \otimes \mathbb{C}[t, t^{-1}] \subset W^{(1)} \otimes \mathbb{C}[t, t^{-1}] \subset \cdots \subset W^{(n)} \otimes \mathbb{C}[t, t^{-1}] \subset \cdots,
\]

with \( (W^{(n)} \otimes \mathbb{C}[t, t^{-1}])/(W^{(n-1)} \otimes \mathbb{C}[t, t^{-1}]) \cong \mathcal{N}(\langle \text{Soc}_{\mathfrak{b}} W \rangle_{(b)}, a) \) for all \( n \in \mathbb{Z}^{+} \).

**Proof.** Since \( W \) is nontrivial, from Lemma 2 we know that \( W \cong \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} (\text{Soc}_{\mathfrak{b}} W) \). For any \( w \in \text{Soc}_{\mathfrak{b}} W, m, k \in \mathbb{Z}, n \in \mathbb{Z}^{+} \), we compute

\[
d_{m} \cdot ((d_{-1}^{m}w) \otimes t^{k}) = ((e^{mt} - 1) \frac{d}{dt} + bm + a + k)(d_{-1}^{m}w) \otimes t^{k+m}
= ((d_{-1} - m)^{n}(e^{mt} \frac{d}{dt}) + d_{-1}^{n}(-d_{-1} + bm + a + k))w \otimes t^{k+m}
≡ d_{-1}^{n}(m(d_{0} + b) + \sum_{j=2}^{\infty} \frac{m^{j}}{j!}d_{j-1} + a + k)w \otimes t^{k+m} \pmod{W^{(n-1)} \otimes \mathbb{C}[t, t^{-1}]}.
\]

We see that each \( W^{(n)} \otimes \mathbb{C}[t, t^{-1}] \) is a submodule of \( \mathcal{L}(W, 1, a, b) \) and \( (W^{(n)} \otimes \mathbb{C}[t, t^{-1}])/(W^{(n-1)} \otimes \mathbb{C}[t, t^{-1}]) \cong \mathcal{N}(\langle \text{Soc}_{\mathfrak{b}} W \rangle_{(b)}, a) \). \( \square \)
From Theorem 4 in [LLZ] we know that the above Virasoro module $\mathcal{N}((\text{Soc}_b W)_{(b)},a)$ is simple if $W$ is not a highest weight module.

The following lemma solves the simplicity of $\mathcal{L}(W,\lambda,a,b)$ for $b = 1$.

**Lemma 8.** For any nontrivial simple $W \in \mathcal{O}_\mathfrak{g}$, the subspace

$$\mathcal{L}'(W,\lambda,a,1) = \oplus_{n \in \mathbb{Z}} ((d_{-1} - a - n)W) \otimes t^n \subset \mathcal{L}(W,\lambda,a,1)$$

is a submodule which is isomorphic to $\mathcal{L}(W,\lambda,a,0)$ with the quotient $\mathcal{L}(W,\lambda,a,1)/\mathcal{L}'(W,\lambda,a,1) \cong \mathcal{N}((\text{Soc}_b W)_{(1)},a)$.

**Proof.** Since $W$ is nontrivial, from Lemma 2 we know that $W \cong \text{Ind}_{\mathfrak{g}'}^\mathfrak{g}(\text{Soc}_b W)$. Define the linear map $\tau : \mathcal{L}(W,\lambda,a,0) \to \mathcal{L}'(W,\lambda,a,1)$ by

$$\tau(w \otimes t^n) = ((d_{-1} - a - n)w) \otimes t^n, \forall w \in W, n \in \mathbb{Z}$$

which is clearly bijective. Now for any $m, n \in \mathbb{Z}$ and $w \in W$, we compute that

$$d_m \cdot ((d_{-1} - a - n)w) \otimes t^n)$$

$$= (\lambda^m e^m (d/dt) - d/dt + a + m + n)(d_{-1} - a - n)w \otimes t^{m+n}$$

$$= (\lambda^m e^m (d/dt) - d/dt + a + n)w \otimes t^{m+n} \in \mathcal{L}'(W,\lambda,a,1),$$

and

$$d_m \cdot \tau(w \otimes t^n) = (\lambda^m e^m (d/dt) - d/dt + a + n + m)(d/dt - a - n)w \otimes t^{m+n}$$

$$= (\lambda^m e^m (d/dt) - d/dt + a + n)w \otimes t^{m+n} \in \mathcal{L}'(W,\lambda,a,1),$$

we obtain that $d_m \cdot \tau(w \otimes t^n) = \tau(d_m \cdot (w \otimes t^n))$. So $\tau$ is a $\mathfrak{g}$-module isomorphism.

From

$$d_m \cdot (w \otimes \lambda^n t^n) = \lambda^n (\lambda^m e^m (d/dt) - d/dt + a + m + n)w \otimes t^{m+n}$$

$$\equiv \lambda^{m+n} e^m d/dt w \otimes t^{m+n} \mod \mathcal{L}'(W,\lambda,a,1)$$

we see that $\mathcal{L}(W,\lambda,a,1)/\mathcal{L}'(W,\lambda,a,1) \cong \mathcal{N}((\text{Soc}_b W)_{(1)},a)$. □
The following result shows that the Virasoro module $L(W, -1, a, b)$ is not simple if $W \in O_{m}$ is a nontrivial highest weight module with highest weight $b - 1$.

**Lemma 9.** Let $W \in O_{m}$ be a highest weight module with the highest weight vector $w_0$ of the highest weight $b' \neq 0$. Then

\[ L_0(a, b') = \text{span}\{d_i^1 \cdot (w_0 \otimes t^{2j})| i \in \mathbb{Z}_+, j \in \mathbb{Z}\}, \]

\[ L_1(a, b') = \text{span}\{d_i^1 \cdot (w_0 \otimes t^{2j+1})| i \in \mathbb{Z}_+, j \in \mathbb{Z}\} \]

are submodules of $L(W, -1, a, b' + 1)$, and

\[ L(W, -1, a, b' + 1) = L_0(a, b') \oplus L_1(a, b'). \]

**Proof.** By induction on $i \in \mathbb{Z}_+$ we can easily show that

\[ d_m d_i^1 \in \sum_{j=0}^i \mathbb{C}[d_i] d_{m+j}. \]

In $L(W, -1, a, b' + 1)$, for $k, j \in \mathbb{Z}$ we compute

\[ d_{2k} \cdot (w_0 \otimes t^{2j}) = (2kb' + a + 2j + 2k(b' + 1))w_0 \otimes t^{2j+2k}, \]

\[ d_{2k+1} \cdot (w_0 \otimes t^{2j}) = (-2d_{-1} + a + 2k + 2j + 1)w_0 \otimes t^{2k+2j+1} \]

\[ = d_1 \cdot (w_0 \otimes t^{2k+2j+1}) \in L_0(a, b'), \]

\[ d_{2k} \cdot (w_0 \otimes t^{2j+1}) = (2kb' + a + 2j + 1 + 2k(b' + 1))w_0 \otimes t^{2j+2k+1}, \]

\[ d_{2k+1} \cdot (w_0 \otimes t^{2j+1}) = (-2d_{-1} + a + 2k + 2j + 2)w_0 \otimes t^{2k+2j+2} \]

\[ = d_1 \cdot (w_0 \otimes t^{2k+2j+1}) \in L_1(a, b'). \]

Using the above formulas we deduce that $L_0(a, b'), L_1(a, b')$ are submodules of $L(W, -1, a, b' + 1)$.

Since $b' \neq 0$ we know that $W = \mathbb{C}[d_{-1}] w_0 = \mathbb{C}[d_{-1}] \otimes \mathbb{C} w_0$.

Denote $d_i^j \cdot (w_0 \otimes t^j) = f_{i,j} (d_{-1}) w_0 \otimes t^{i+j}$. Together with

\[ d_1 \cdot (f(d_{-1}) w_0 \otimes t^k) = (-2d_{-1} + a + k + b \]

\[ + \sum_{j=1}^{\infty} \frac{d_{j-1} f(d_{-1}) w_0 \otimes t^{k+j}}{j!}, \forall f(t) \in \mathbb{C}[t], m, k \in \mathbb{Z}, \]

we may deduce inductively that $f_{i,j}$ are polynomials of degree $i$. We see that $\{d_i^1 \cdot (w_0 \otimes t^j)| i \in \mathbb{Z}_+, j \in \mathbb{Z}\}$ is linearly independent. Consequently, $\{d_i^1 \cdot (w_0 \otimes t^j)| i \in \mathbb{Z}_+, j \in \mathbb{Z}\}$ is a basis for $L(W, -1, a, b' + 1)$. Therefore we have $L(W, -1, a, b' + 1) = L_0(a, b') \oplus L_1(a, b')$. Thus we have proved the lemma. \qed

**Lemma 10.** Let $\lambda, a, b \in \mathbb{C}$ with $\lambda \neq 0, 1$, and $W \in O_{m}$ be nontrivial and simple. Let $M$ be any submodule of $L(W, \lambda, a, b)$. If the subspace $\hat{M} = \{v \in W| v \otimes \mathbb{C}[t, t^{-1}] \subseteq M\}$ is nonzero, then $M = L(W, \lambda, a, b)$. 

Theorem 11. Let $\lambda \in \mathbb{C}^*, a, b \in \mathbb{C}$, and $W \in \mathcal{O}_{2b}$ be nontrivial simple. 

(a) If $W$ is a highest weight module with highest weight $b'$, then the Virasoro module $\mathcal{L}(W, \lambda, a, b)$ is simple if and only if $\lambda \neq 1, -1$ and $b \neq 1$; or $\lambda = -1$, $b = 1$ and $b \neq b' + 1$.

(b) If $W$ is not a highest weight module, then $\mathcal{L}(W, \lambda, a, b)$ is simple if and only if $\lambda \neq 1$ and $b \neq 1$.

(c) If $W$ is a highest weight module with highest weight $b' \neq 0$, then $\mathcal{L}_0(a, b')$ and $\mathcal{L}_1(a, b')$ defined in Lemma 9 are the only two non-trivial submodules of $\mathcal{L}(W, -1, a, b' + 1)$, which are simple.

Proof. In $\mathcal{L}(W, \lambda, a, b)$, for any $w \in W, l, j, m \in \mathbb{Z}$, we have 

$$d_{l-m}d_m \cdot (w \otimes t^l) = \left(\lambda^{l-m}e^{(l-m)t} \frac{d}{dt} - \frac{d}{dt} + a + (l - m)b + m + j\right)$$

$$= \left(\lambda^m e^{mt} \frac{d}{dt} - \frac{d}{dt} + a + mb + j\right)w \otimes t^{l+j}$$

$$= \left(\lambda^m e^{mt} \frac{d}{dt}\right) e^t w \otimes t^l + (-\frac{d}{dt} + a + (l - m)b + m + j)$$

$$(-\frac{d}{dt} + a + mb + j)w \otimes t^{l+j}$$

$$+ \lambda^{-m} \lambda^{l}(-\frac{d}{dt} + a + m(b-1) + j + l) e^{(l-m)t} \frac{d}{dt} w \otimes t^{l+j}$$

$$+ \lambda^m (-\frac{d}{dt} + a + lb + (1-b)m + j)(e^{mt} \frac{d}{dt}) w \otimes t^{l+j}.$$ 

Let $M$ be a nonzero submodule of $\mathcal{L}(W, \lambda, a, b)$. Take a nonzero

$$v = \left(\sum_{i=1}^s d_{l-1} w_i\right) \otimes t^s \in M$$

with $w_s \neq 0, w_i \in \text{Soc}_b(W)$, and $r = \text{ord}_b(\text{Soc}_b(W)) \geq 0$. From Lemma 11(c) we know that $d_r$ is bijective on the simple $b$ module $\text{Soc}_b(W)$.

Note that

$$(e^{mt} \frac{d}{dt})(d_{l} w_i) = (d_{l-1} - m)'(e^{mt} \frac{d}{dt}) w_i,$$
\[
(e^{(l-m)t}) \frac{d}{dt} (e^{mt} \frac{d}{dt}) (d_{-1} w_i) = (d_{-1} - l)^i (e^{(l-m)t}) \frac{d}{dt} (e^{mt} \frac{d}{dt}) w_i,
\]
and \(d_{r+i} w_i = 0\) for all \(k \in \mathbb{N}\). From (4.4), we may write
\[
d_{t-m} d_m \cdot (v) = \sum_{i=0}^{2r+2} m^i v^{(l,i_0)}_1 \otimes t^{l+i_0}
\]
\[
= \sum_{i=0}^{r+s+2} \lambda^{-m} m^i v^{(l,i_0)}_{\lambda,s} \otimes t^{l+i_0}
\]
\[
+ \sum_{i=0}^{r+s+2} \lambda m^i v^{(l,i_0)}_{\lambda,s} \otimes t^{l+i_0} \in M,
\]
where \(v^{(l,i_0)}_1, v^{(l,i_0)}_{\lambda,s}, v^{(l,i_0)}_{\lambda,s} \in W\) are independent of \(m\). In particular,
\[
v^{(l,i_0)}_{\lambda,r+s+2} = \frac{(1-b)(-1)^s}{(r+1)!} d_r w_s, \forall l \in \mathbb{Z},
\]
\[
v^{(l,i_0)}_{\lambda,r+s+2} = \frac{(-1)^{r+s}(1-b)\lambda^l}{(r+1)!} d_r w_s, \forall l \in \mathbb{Z}.
\]
And if \(r > 0\),
\[
v^{(l,i_0)}_{1,r+s+2} = \frac{(-1)^{r+1}\lambda^l}{((r+1)!)^2} (d_{-1} - l)^s d_r^2 w_s, \forall l \in \mathbb{Z}.
\]

Case 1. \(\lambda \neq 1, -1\) and \(b \neq 1\).

From (4.3), (4.4) and Proposition 5, we have
\[
d_r w_s \otimes t^{l+i_0} \in M, \forall l \in \mathbb{Z}.
\]
From Lemma 10, we have \(M = \mathcal{L}(W, \lambda, a, b)\). Thus \(\mathcal{L}(W, \lambda, a, b)\) is simple in this case.

Case 2. \(\lambda = -1\) and \(b \neq 1\).

From (4.4) and (4.5) we see that
\[
v^{(l,i_0)}_{1,r+s+2} + v^{(l,i_0)}_{\lambda,r+s+2} = \frac{(1-b)(-1)^s(1+(-1)^{l+r})}{(r+1)!} d_r w_s.
\]
Thus from (4.3) and Proposition 5, we have
\[
d_r w_s \otimes t^{l+s+2k+i_0} \in M, \forall k \in \mathbb{Z}.
\]
Now replacing \(v\) with \(d_r w_s \otimes t^{-r+i_0}\) if necessary, we may assume that
\[
0 \neq v = w \otimes t^{i_0+2k} \in M, \forall k \in \mathbb{Z}
\]
where \(w \in \text{Soc}_b(W)\). Now \(s = 0\) in (4.3)-(4.7).

Subcase 2.1. \(W\) is not a highest weight module.
Note that in this case we have $r > 0$. Then (4.6) becomes
\[ v_{l,i}^{(l,i_0)} = \frac{(-1)^{1+r+1}}{(r+1)!} d_2^r w. \]
So from Proposition 5 we have
\[ (d_2^r w) \otimes t^{l+i_0} \in M, \forall l \in \mathbb{Z}. \]
From Lemma 10 we have $M = \mathcal{L}(W, \lambda, a, b)$. Thus $\mathcal{L}(W, \lambda, a, b)$ is simple in this case.

**Subcase 2.2.** $W$ is a highest weight module with highest weight $b' \neq 0$, and $b = b' + 1$.

In this case, we have $r = s = 0$. From (4.9) and the proof of Lemma 9, we have either $L_0(a, b') \subset M$ if $i_0$ is even or $L_1(a, b') \subset M$ if $i_0$ is odd. Combining with Lemma 9 we see that $L_0(a, b')$ and $L_1(a, b')$ are the only two simple Virasoro submodules in $\mathcal{L}(W, \lambda, a, b)$. And we also see that $L_0(a, b')$ and $L_1(a, b')$ are not isomorphic.

**Subcase 2.3.** $W$ is a highest weight module with highest weight $b' \neq 0$, and $b \neq b' + 1$.

For this case, we have $r = s = 0$. In (4.9), $w$ is the highest weight vector of $W$. For any $m, l \in \mathbb{Z}$ we have
\[
(4.10) \quad d_{2m+1}(w \otimes t^{i_0-2m+2}) = 2m(b - b' - 1)w \otimes t^{i_0+2l+1} + (b - b' - 2d_{-1} + a + i_0 + 2l)w \otimes t^{i_0+2l+1} \in M.
\]
Since $b \neq b' + 1$, we have $w \otimes t^{i_0+2l+1} \in M$ for all $l \in \mathbb{Z}$. Thus $w \otimes \mathbb{C}[t, t^{-1}] \subset M$. From Lemma 10 we have $M = \mathcal{L}(W, \lambda, a, b)$. So $\mathcal{L}(W, \lambda, a, b)$ is simple in this case.

We know that $\mathcal{L}(W, \lambda, a, b)$ is not simple if $\lambda = 1$ (Lemma 7), or if $b = 1$ (Lemma 8). So we have completed the proof.

Note that, if $\mathbb{C}$ is the trivial $\mathfrak{W}$-module, then $A_{a,b} = \mathcal{L}(\mathbb{C}, 1, a, b)$ is the module of intermediate series. See [KR] and Example 2.

Applying Lemmas 7, 8, 9 and Theorem 11 to modules $E_h(b, \gamma, p)$ defined in [CM] (where we take the $\mathfrak{W}$-module $W$ as the Verma module of highest weight $-\gamma$), we can completely determine the structure of these modules.

**Corollary 12.** Let $h, b, \gamma, p \in \mathbb{C}$.

1. $E_h(b, \gamma, p)$ is simple if and only if $e^h \neq 1, -1$, and $\gamma \neq 0, 1 - p$; or $e^h = -1$, and $\gamma \neq 0, 1 - p, \frac{1-p}{2}$.
2. If $e^h = 1$ and $\gamma \neq 0$, then $E_h(b, \gamma, p)$ has a filtration
\[ 0 = M^{(0)} \subset M^{(1)} \subset \cdots \subset M^{(n)} \subset \cdots \]
with $M^{(i+1)}/M^{(i)} \cong A_{b,p}$ for all $i \in \mathbb{Z}$. 
(3) If \( e^h \neq 1 \) and \( \gamma \neq 0 \), then \( E_h(b, \gamma, 1 - \gamma) \) has a proper submodule 
\( M \cong E_h(b, \gamma, 1 - \gamma) / M \cong A_{b,1} \).

(4) If \( e^h = -1 \), and \( \gamma \neq 0 \), then \( E_h(b, \gamma, 1 - 2\gamma) \) has exactly two simple submodules \( E_\pm(b, \gamma, 1 - 2\gamma) \), and 
\[
E_h(b, \gamma, 1 - 2\gamma) = E_+(b, \gamma, 1 - 2\gamma) \oplus E_-(b, \gamma, 1 - 2\gamma)
\]
with \( E_+(b, \gamma, 1 - 2\gamma) \neq E_-(b, \gamma, 1 - 2\gamma) \).

(5) If \( e^h = -1 \), then \( E_h(b, 0, 1) \) has submodules \( M_2 \subset M_1 \) with 
\( M_1/M_2 \cong A_{b,0} \), \( E_h(b, 0, 1)/M_1 \cong A_{b,1} \), \( M_1 \cong E_h(b, 1, 0) \), \( M_2 \cong E_h(b, 1, -1) \oplus E_-(b, 1, -1) \).

(6) If \( e^h = -1 \), then \( E_h(b, 0, 0) \) has a submodule \( M \cong E_h(b, 1, -1) \) and \( E_h(b, 0, 0)/M \cong A_{b,0} \), and \( M \cong E_+(b, 1, -1) \oplus E_-(b, 1, -1) \).

(7) Let \( e^h \neq 1, -1 \) and \( p \neq 1 \), or \( e^h = -1 \) and \( p \neq 0, 1 \). Then 
\( E_h(b, 0, p) \) has an simple submodule \( M \) with \( M \cong E_h(b, 1, p - 1) \) and \( E_h(b, 0, p)/M \cong A_{b,p} \).

(8) If \( e^h \neq 1, -1 \), then \( E_h(b, 0, 1) \) has submodules \( M_2 \subset M_1 \) with 
\( M_1 \cong E_h(b, 1, 0) \), \( M_2 \cong E_h(b, 1, -1) \), \( M_1/M_2 = A_{b,0} \), and 
\( E_h(b, 0, 1)/M_1 \cong A_{b,1} \), where \( M_2 \) is simple.

**Proof.** From Sect.3.2, we know that if \( \gamma \neq 0 \), then \( E_h(b, \gamma, p) \cong \mathcal{L}(W, e^h, b, \gamma + p) \) where \( W \) is the simple highest weight \( \mathfrak{w} \)-module with highest weight \( -\gamma \).

Part (1) follows directly from Theorem 11(a), Part (2) follows directly from Lemma 8, Part (3) follows directly from Lemma 9, and Part (4) follows directly from Lemma 9.

Now let \( \gamma = 0 \), let \( W \) be the Verma \( \mathfrak{w} \)-module with highest weight 0, and let \( W' \) be the unique simple submodule of \( W \) with highest weight \( -1 \). (So \( \gamma = 1 \) for \( W' \)). We have \( E_h(b, 0, p) \cong \mathcal{L}(W, e^h, b, p) \). Note that in this case \( \mathcal{L}(W, e^h, b, p) \) has a submodule 
\[
M = \mathcal{L}(W', e^h, b, p) \cong E_h(b, 1, p - 1)
\]
with \( \mathcal{L}(W, e^h, b, p)/M \cong A_{b,p} \).

(5). We see that \( E_h(b, 0, 1) \cong \mathcal{L}(W, -1, b, 1) \). Then \( E_h(b, 0, 1) \) has submodule \( M_1 \cong E_h(b, 1, 0) \cong \mathcal{L}(W', -1, b, 1) \). Applying Lemmas 8 and 9 we see that \( M_1 \) has a submodule \( M_2 \cong E_h(b, 1, -1) \cong E_+(b, 1, -1) \oplus E_-(b, 1, -1) \). Also \( M_1/M_2 \cong A_{b,0} \).

(6). We see that \( E_h(b, 0, 0) \cong \mathcal{L}(W, -1, b, 0) \). Then \( E_h(b, 0, 0) \) has submodule \( M \cong E_h(b, 1, -1) \cong \mathcal{L}(W', -1, b, 0) \). Applying Lemma 9 we see that \( M \) has submodules \( M \cong E_h(b, 1, -1) \cong E_+(b, 1, -1) \oplus E_-(b, 1, -1) \). Also \( M_1/M_2 \cong A_{b,0} \).

Part (7) follows from Theorem 11(a), while Part (8) follows from Lemma 8.

**Example 3.** Let \( \mu = (\mu_1, \mu_2) \in \mathbb{C}^2 \), \( \lambda \in \mathbb{C}^* \). Let \( W_{\mu} \) be as in Example 1. Then \( W_{\mu} \) is simple if and only if \( \mu_1 \) or \( \mu_2 \) is nonzero. We can easily see that \( W_{\mu} = \mathbb{C}[d_{-1}] \otimes \mathbb{C}[d_0] \). For any \( \lambda, a, b \in \mathbb{C} \) with \( b \neq 1 \) and \( \lambda \notin
Lemma 13. Let \( \lambda \in \mathbb{C}^* \) and \( a, b, b_0 \in \mathbb{C} \). Let \( W, W_0 \in \mathcal{O}_{\mathfrak{g}} \) be the Verma modules with highest weight vectors \( w, w_0 \) of the highest weights \( b', b_0 \) respectively. Then the linear map

\[
\varphi : \mathcal{L}(W, \lambda, a, b_0 + 1) \rightarrow \mathcal{L}(W_0, \lambda^{-1}, a, b' + 1),
\]

\[
(d_{-1}^k w) \otimes t^l \mapsto \lambda^{-l}((a + l - d_{-1})^k w_0) \otimes t^l,
\]

is a \( \mathfrak{g} \)-module isomorphism.

**Proof.** We need only to verify that \( \varphi(d_m \cdot (d_{-1}^k w \otimes t^l)) = d_m \cdot \varphi(d_{-1}^k w \otimes t^l) \) for all \( m, l \in \mathbb{Z} \) and \( k \in \mathbb{Z}_+ \). We compute

\[
d_m \cdot (d_{-1}^k w \otimes t^l)
= d_m \cdot (\lambda^{-l}(a + l - d_{-1})^k w_0 \otimes t^l)
= \lambda^{-l}(\lambda^{-m} e^{mt} \frac{d}{dt} - d_{-1} + a + l + (1 + b')m)(a + l - d_{-1})^k w_0) \otimes t^{l+m}
= ((\lambda^{-l-m}(a + l + m - d_{-1})^k (d_{-1} + b'_0 m)
\quad + \lambda^{-l}(a + l + (1 + b')m - d_{-1})(a + l - d_{-1})^k w_0) \otimes t^{l+m},
\]
\[ \varphi(d_m \cdot (d_{-1}^k w \otimes t^l)) \]
\[ = \varphi((\lambda^m e^{m \frac{d}{dt}} - d_{-1} + a + l + m + b_0 m)d_{-1}^k w \otimes t^{l+m}) \]
\[ = \varphi((\lambda^m (d_{-1} - m)^k (d_{-1} + b' m) + (a + l + m - d_{-1} + b_0 m)d_{-1}^k w \otimes t^{l+m}) \]
\[ = \lambda^{-l-m}(\lambda^m (a + l - d_{-1})^k (a + l + m - d_{-1} + b' m)w_0 \otimes t^{l+m}) \]
\[ + \lambda^{-l-m}(d_{-1} + b_0 m)(a + l + m - d_{-1})^k w_0 \otimes t^{l+m} \]
\[ = ((\lambda^{-l-m}(a + l + m - d_{-1})^k (d_{-1} + b_0 m) \]
\[ + \lambda^{-l}(a + l + (1 + b') m - d_{-1})(a + l - d_{-1})^k w_0) \otimes t^{l+m}. \]

This completes the proof. \( \square \)

**Remark 14.** In [CM], it was proved that \( E_b(b, \gamma, p) \) and \( E_{-b}(b, 1 - \gamma - p, p) \) are equivalent. The lemma above shows that in fact we have \( E_b(b, \gamma, p) \cong E_{-b}(b, 1 - \gamma - p, p) \).

Now we are going to prove the main result in this section.

**Theorem 15.** Let \( \lambda, \lambda_0 \in \mathbb{C}, a, b, b', b_0', a_i, b_i \in \mathbb{C} \) for \( i = 0, 1, 2 \) with \( 0 \leq \text{Re} a, \text{Re} a_i < 1 \) and \( b'b_0' b_2 \neq 0 \). Let \( B \in \mathcal{O}_b \) and \( W, W_0 \in \mathcal{O}_{b_0} \) be nontrivial simple modules.

(a) Suppose that \( \mathcal{L}(W, \lambda, a, b) \) is simple. Then

\[ \mathcal{L}(W, \lambda, a, b) \cong \mathcal{L}(W_0, \lambda_0, a_0, b_0) \]

if and only if one of the following holds

(i). \( W \cong W_0, \lambda = \lambda_0, a = a_0, \) and \( b = b_0, \) or

(ii). \( W, W_0 \) are highest weight modules with highest weights \( b', b'_0 \)

respectively, and \( \lambda_0 = \lambda^{-1}, a = a_0, b = b'_0 + 1, b_0 = b' + 1. \)

(b) For \( i = 0, 1, \) we have \( \mathcal{L}_i(a_1, b_1) \cong \mathcal{L}_i(a_2, b_2) \) if and only if \( a_1 = a_2, \) and \( b_1 = b_2. \)

(c) The modules \( \mathcal{L}(W, \lambda, a, b), \mathcal{N}(B, a_0), \mathcal{L}_0(a_1, b_1), \mathcal{L}_1(a_2, b_2) \) are pairwise non-isomorphic.

**Proof.** (a). The sufficiency is trivial. Now suppose that \( \varphi : L(W, \lambda, a, b) \to L(W_0, \lambda_0, a_0, b_0) \) is a module isomorphism. It is clear that \( a = a_0. \)

Since \( \mathcal{L}(W, \lambda, a, b) \) is simple, \( \lambda, \lambda_0, b, b_0 \) are different from 1. Denote \( r = \text{ord}_0(W) \), \( r' = \text{ord}_0(W_0) \) and \( \varphi(t^n) = \varphi_n(t^n) \) \( \forall n \in \mathbb{Z} \). Then each \( \varphi_n : W \to W_0 \) is a vector space isomorphism.

Let \( M = \{ v \in W | \varphi_n(v) = \varphi_0(v) \text{ for all } n \in \mathbb{Z} \}. \) If \( M \neq 0 \), for any \( w \in M, t, m \in \mathbb{Z} \) we have

\[ \varphi_t((\lambda^m \sum_{j=0}^{\infty} \frac{m^j}{j!} d_{-1}^j - d_{-1} + a + l - m + b m) w) \otimes t^l \]
\[
\begin{align*}
\varphi((\lambda^m \sum_{j=0}^{\infty} \frac{m^j}{j!} d_{j-1} - d_{-1} + a + l - m + bm)w) \otimes t^i) \\
= \varphi(d_m \cdot (w \otimes t^{i-m})) \\
= d_m \cdot \varphi(w \otimes t^{i-m}) \\
= (\lambda^m \sum_{j=0}^{\infty} \frac{m^j}{j!} d_{j-1} - d_{-1} + a + l - m + b_0m)\varphi_0(w) \otimes t^i,
\end{align*}
\]
yielding that
\[
\lambda = \lambda_0, b = b_0, \varphi_l(d_jw) = d_j\varphi_0(w), \forall l \in \mathbb{Z}, j \geq -1, w \in M.
\]
Thus \( M \) is a \( \mathcal{W} \)-submodule of \( W \). Then \( M = W \) and \( \varphi_l = \varphi_0 \) is an \( \mathcal{W} \)-module isomorphism. So (i) holds in this case. Thus we only need to prove that \( M \neq 0 \) or (ii).

For any \( w \in \text{Soc}_l(W), l, m, n \in \mathbb{Z} \), similar to (4.2) we have

\[(4.11)\]
\[
\begin{align*}
(\lambda^l_0(e^{(l-m)t}) \frac{d}{dt})(e^{mt} \frac{d}{dt}) + (-\frac{d}{dt} + a + (l - m)b_0 + m + n) \\
- \lambda_0^{-m} \lambda^l_0(-\frac{d}{dt} + a + m(b_0 - 1) + n + l)(e^{(l-m)t}) \frac{d}{dt})\varphi_n(w) \otimes t^{l+n} \\
+ \lambda_0^{-m}(-\frac{d}{dt} + a + lb_0 + (1 - b_0)m + n)(e^{mt}) \varphi_n(w) \otimes t^{l+n} \\
= ((\lambda^l_0 e^{(l-m)t}) \frac{d}{dt} - \frac{d}{dt} + a + (l - m)b_0 + m + n) \\
(\lambda^m_0 e^{mt}) \frac{d}{dt} - \frac{d}{dt} + a + mb_0 + n)\varphi_n(w) \otimes t^{n+l} \\
= d_{l-m}d_m \cdot (\varphi_n(w) \otimes t^n) \\
= \varphi(d_{l-m}d_m \cdot (w \otimes t^{n+l})) \\
= \varphi_n + l((\lambda^{l-m} e^{(l-m)t}) \frac{d}{dt} - \frac{d}{dt} + a + (l - m)b + m + n) \\
(\lambda^m e^{mt}) \frac{d}{dt} - \frac{d}{dt} + a + mb + n)w) \otimes t^{n+l} \\
= \varphi_{l+n}((\lambda^l(e^{(l-m)t}) \frac{d}{dt})(e^{mt}) + (-\frac{d}{dt} + a + (l - m)b + m + n) \\
(-\frac{d}{dt} + a + mb + n)w) \otimes t^{l+n} \\
+ \lambda^{-m} \lambda^l \varphi_{l+n}((-\frac{d}{dt} + a + m(b - 1) + n + l)(e^{(l-m)t}) \frac{d}{dt})w) \otimes t^{l+n} \\
+ \lambda^m \varphi_{l+n}((-\frac{d}{dt} + a + lb + (1 - b)m + n)(e^{mt}) \frac{d}{dt})w) \otimes t^{l+n}.
\]
We can write $\varphi_n(w) = \sum_{i=1}^{s'} d_{i-1}^n w_i'$ with $w_i' \in \text{Soc}_b(W')$, and $w_i' \neq 0$.

Similarly as we have done in (4.3)-(4.6), we may write both sides of (5.1) as a linear combination of $\{m^i, \lambda^\pm i m^i, \lambda_0^{\pm 1} m^i\}$. By comparing with the highest degree of $m^i$ we see that $r = r'$. Further comparing with the highest degree of $\lambda^\pm m^i$, from the analogues of (4.3)-(4.6), we have $s' = 0$, and $\lambda = \lambda_0$ or $\lambda = \lambda_0^{-1}$. In particular, we have $\varphi(\text{Soc}_b(W) \otimes \mathbb{C}[t, t^{-1}]) \subset \text{Soc}_b(W_0) \otimes \mathbb{C}[t, t^{-1}]$, i.e., $\varphi_n(\text{Soc}_b(W)) = \text{Soc}_b(W_0)$ for any $n \in \mathbb{Z}$.

Now we can write (5.1) as

$$\sum_{i=0}^{2r+2} m^i v^{(l)}_{1,i} \otimes t^{l+n} + \sum_{i=0}^{2r+2} \lambda_0^{-m^i} m^i v^{(l)}_{\lambda_0,i} \otimes t^{l+n} + \sum_{i=0}^{2r+2} \lambda^m m^i v^{(l)}_{\lambda,i} \otimes t^{l+n}$$

$$= \sum_{i=0}^{r+2} m^i \varphi_{n+i}(w^{(l)}_{1,i}) \otimes t^{l+n} + \sum_{i=0}^{r+2} \lambda^{-m^i} \varphi_{n+i}(w^{(l)}_{\lambda_0,i}) \otimes t^{l+n}$$

$$+ \sum_{i=0}^{r+2} \lambda^m \varphi_{n+i}(w^{(l)}_{\lambda,i}) \otimes t^{l+n},$$

where $v^{(l)}_{1,i}, v^{(l)}_{\lambda_0,i}, v^{(l)}_{\lambda,i} \in W_0$, $w^{(l)}_{1,i}, w^{(l)}_{\lambda_0,i}, w^{(l)}_{\lambda,i} \in W$ are independent of $m$. In particular,

$$v^{(l)}_{\lambda_0,r+2} = \frac{(1 - b_0)}{(r+1)!} d_r \varphi_n(w), \forall l, n \in \mathbb{Z},$$

$$w^{(l)}_{\lambda,r+2} = \frac{(1 - b)}{(r+1)!} d_r w, \forall l, n \in \mathbb{Z},$$

$$v^{(l)}_{\lambda_0,r+2} = \frac{(-1)^r (1 - b_0) \lambda_0^l}{(r+1)!} d_r \varphi_n(w), \forall l, n \in \mathbb{Z},$$

$$w^{(l)}_{\lambda,r+2} = \frac{(-1)^r (1 - b) \lambda^l}{(r+1)!} d_r w, \forall l, n \in \mathbb{Z}.$$
From (5.3)-(5.6), we may deduce that \( \varphi \) for all \( l, n \in \mathbb{Z} \), \( w \in \text{Soc}_{t}(W) \). Thus \( M \neq 0 \).

Subcase 1.1. \( \lambda = \lambda_{0} \)

In this case, from (5.3) and (5.4) we have

\[
(1 - b_{0})d_{0}\varphi_{n}(w) = (1 - b)\varphi_{l+n}(d_{0}w),
\]

for all \( l, n \in \mathbb{Z} \), \( w \in \text{Soc}_{t}(W) \). Thus \( M \neq 0 \).

Subcase 1.2. \( \lambda = \lambda_{0}^{1} \)

In this case, by computing the coefficients of \( \lambda_{0}^{1}m^{2} \) in (5.2) and using (5.4) and (5.5), we have \( \lambda(1 - b)\varphi_{l+n}(d_{0}w) = (1 - b_{0})d_{0}\varphi_{n}(w) \) for all \( l \in \mathbb{Z} \), where \( w, \varphi_{n}(w) \) are the highest weight vectors in \( W \) and \( W_{0} \) respectively. We may assume that \( d_{0}w = b'w \) and \( d_{0}(\varphi_{n}(w)) = b'_{0}\varphi_{n}(w) \). Then \( b'b'_{0} \neq 0 \) because of the simplicity of the modules. We see that \( \lambda(1 - b)b'\varphi_{l+n}(w) = (1 - b_{0})b'_{0}\varphi_{n}(w) \). By taking \( l = 0 \) we deduce that \( (1 - b)b' = (1 - b_{0})b'_{0} \neq 0 \). Thus

\[
\varphi_{l}(w) = \lambda^{-l}\varphi_{0}(w), \quad \forall \ l \in \mathbb{Z}.
\]

We compute \( \varphi(d_{m} \cdot (w \otimes t^{l-m})) = d_{m} \cdot \varphi(w \otimes t^{l-m}) \):

\[
d_{m} \cdot \varphi(w \otimes t^{l-m}) = \lambda^{m-l}d_{m} \cdot (\varphi_{0}(w) \otimes t^{l-m})
\]

\[
= \lambda^{m-l}(\lambda^{-m}(d_{-1} + mb'_{0}) - d_{-1} + a + l - m + b_{0}m)\varphi_{0}(w) \otimes t^{l}
\]

\[
= (\lambda^{-l}(d_{-1} + mb'_{0}) + \lambda^{m-l}(-d_{-1} + a + l + (b_{0} - 1)m))\varphi_{0}(w) \otimes t^{l},
\]

\[
\varphi(d_{m} \cdot (w \otimes t^{l-m})) = \varphi((\lambda^{m}(d_{-1} + mb') - d_{-1} + a + l + (b_{0} - 1)m)w \otimes t^{l})
\]

\[
= (\lambda^{m} - 1)\varphi_{l}(d_{-1}w) \otimes t^{l} + (m\lambda^{m}b' + a + l + (b_{0} - 1)m)\lambda^{-l}\varphi_{0}(w) \otimes t^{l}.
\]

Comparing the coefficients of \( m \), \( m\lambda^{m} \) we obtain that

\[
(5.10) \quad b' - b_{0} + 1 = b'_{0} - b + 1 = 0.
\]

Thus (ii) holds in this case.

Case 2. \( \lambda = \lambda_{0} = -1 \).

Note that \( \varphi(\text{Soc}_{t}(W) \otimes \mathbb{C}[t, t^{-1}]) \subset \text{Soc}_{t}(W_{0}) \otimes \mathbb{C}[t, t^{-1}] \). If \( W \) is not a highest weight module, again by comparing the coefficient of \( m^{2l+2} \) in (5.2) and using (5.3) and (5.4), we have \( \varphi_{l+n}(d_{0}^{2}w) = d_{0}^{2}\psi_{n}(w) \), for all \( n, l \in \mathbb{Z} \), \( w \in \text{Soc}_{t}(W) \). Thus \( M \neq 0 \) in this case.

Now suppose that \( W \) is a highest weight module the highest weight vector \( w \). So \( r = r' = 0 \). Let where \( b'_{0} \) be highest weight of \( W_{0} \). Then \( b'b'_{0} \neq 0 \) because of the simplicity of the modules. From (5.3)-(5.6), we may deduce that \( \varphi_{2k}(w) = \varphi_{0}(w) \), \( \varphi_{2k+1}(w) = \varphi_{1}(w) \). We compute

\[
d_{2m+1} \cdot (\varphi_{i_{0}}(w) \otimes t^{i_{0} - 2m + 2l}) = \varphi_{i_{0}}(w) \otimes t^{i_{0} + 2l+1},
\]

\[
\varphi_{i_{0}}(w) \otimes t^{i_{0} - 2m + 2l}) = (-2d_{-1} + (2m + 1)(b_{0} - b'_{0}) + a + i_{0} - 2m + 2l)\varphi_{i_{0}}(w) \otimes t^{i_{0} + 2l+1},
\]
\[ \varphi(d_{2m+1} \cdot (w \otimes t^{i_0-2m+2})) \]
\[ = \varphi(-2d_{-1} + (2m+1)(b - b') + a + i_0 - 2m + 2l)w \otimes t^{i_0+2l+1}) \]
\[ = ((2m+1)(b-b')+a+i_0-2m+2l)\varphi_{m+1}(w) \otimes t^{i_0+2l+1} \]
\[ - 2\varphi(d_{-1}(w) \otimes t^{i_0+2l+1}). \]

Comparing the coefficient of \( m \), we obtain
\[ (b_0 - b'_0 - 1)\psi_{i_0}(w) = (b - b' - 1)\psi_{m+1}(w), \forall i_0 \in \mathbb{Z}, \]

Since \( \mathcal{L}(W, \lambda, a, b) \) is simple, \( b - b' - 1 \neq 0 \). By taking \( i_0 = 0 \) and 1 we obtain that \( b - b' - 1 = \pm(b_0 - b'_0 - 1) \). Thus we have either
\[ \varphi_i(w) = \varphi_0(w), \forall i \in \mathbb{Z} \]
\[ \varphi_i(w) = (-1)^i\varphi_0(w), \forall i \in \mathbb{Z}. \]

For the first case we have \( M \neq 0 \). Now we consider the second case, i.e., \( \varphi_i(w) = (-1)^i\varphi_0(w) \) for all \( i \in \mathbb{Z} \). By a same argument as in Subcase 1.2, we can prove that (ii) holds in this case.

(b). The sufficiency is obvious. We need only to consider the case \( \mathcal{L}_0(a_1, b_1) \cong \mathcal{L}_0(a_2, b_2) \). Now suppose \( \psi : \mathcal{L}_0(a_1, b_1) \to \mathcal{L}_0(a_2, b_2) \) is an isomorphism. Then it is clear that \( a_1 = a_2 \). Denote \( \psi(v \otimes t^n) = \psi_n(v) \otimes t^n \). Again we have (5.1). We may consider \( \lambda = \lambda_0 = -1, r = r' = 0 \) as in the argument in (a). From (5.3)-(5.6), we have \( \psi_{2k}(w) = \psi_{0}(w) \), where \( w, \psi_{0}(w) \) are the highest weight vectors. For any \( k \in \mathbb{Z} \) we compute \( \psi(d_{2k} \cdot (w \otimes t^0)) = d_{2k} \cdot \psi(w \otimes t^0); \)
\[ \psi(d_{2k} \cdot (w \otimes t^0)) = (2kb_1 + a_1 + 2k(b_1 + 1))\psi_{0}(w) \otimes t^{2k}, \]
\[ d_{2k} \cdot \psi((w \otimes t^0)) = (2kb_2 + a_1 + 2k(b_2 + 1))\psi_{0}(w) \otimes t^{2k}, \]

yielding that \( b_1 = b_2 \).

(c). Since \( \mathcal{L}(W, \lambda, a, b) \) is simple, then \( \lambda \neq 1 \) and \( b \neq 1 \). Let \( \varphi : \mathcal{L}(W, \lambda, a, b) \to \mathcal{N}(B, a_0) \) be a module isomorphism. We know that \( a = a_0 \). Since \( W \) is nontrivial, the action of \( \mathbb{C}[d_{-1}] \) on \( W \) is torsion-free. Let \( r = \mathrm{ord}_{d}(\mathrm{Soc}_{d}(W)) \) and \( r' = \mathrm{ord}_{d}(B) \). We also define \( \varphi_n(w \otimes t^n) = \varphi_{n}(w) \otimes t^n \) for any \( w \in \mathrm{Soc}_{d}(W) \) and any \( n \in \mathbb{Z} \). Then \( \varphi_n : W \to B \) is a vector space isomorphism. We can have a similar equation as in (5.1), while one side comes from \( d_{i-m}d_{m}(v(n)) \) in the proof of Lemma 3 in [LLZ]. From (5.3) and (5.4), we see that \( b = 1 \) is which impossible. So \( \mathcal{L}(W, \lambda, a, b) \) and \( \mathcal{N}(B, a_0) \) cannot be isomorphic.

To consider the isomorphisms between \( \mathcal{L}(W, \lambda, a, b) \) and \( \mathcal{L}_0(a_1, b_1) \) (or \( \mathcal{L}_1(a_1, b_1) \)), we let \( \varphi : \mathcal{L}(W, \lambda, a, b) \to \mathcal{L}(W', -1, a, b_1 + 1) \) be a nonzero one-to-one module homomorphism where \( W' \) is the highest weight \( \mathfrak{M} \)-module with highest weight \( b_1 \). Note that \( \mathcal{L}_0(a_1, b_1) \) and \( \mathcal{L}_1(a_1, b_1) \) are the only simple submodules of \( \mathcal{L}(W', -1, a, b_1 + 1) \). We also define \( \varphi_n(w \otimes t^n) = \varphi_{n}(w) \otimes t^n \) for any \( w \in \mathrm{Soc}_{d}(W) \) and any \( n \in \mathbb{Z} \). Then \( \varphi_n : W \to W' \) is a one-to-one vector space homomorphism. As
in the argument after (5.1) we deduce that $\lambda = -1$. By a similar argument to Case 2 in the proof for (a), we can see that $W$ is also a highest weight module. Continuing the argument as in Case 2 of the proof of (a) we see that $b = b' + 1$ which contradicts the simplicity of $\mathcal{L}(W, \lambda, a, b)$. So there is no such $\varphi$ exists. Thus $\mathcal{L}(W, \lambda, a, b)$ and $\mathcal{L}_0(a_1, b_1)$ (or $\mathcal{L}_1(a_1, b_1)$) cannot be isomorphic.

At last we consider isomorphisms between $\mathcal{N}(B, a_0)$ and $\mathcal{L}_0(a_1, b_1)$ (or $\mathcal{L}_1(a_1, b_1)$). Let $\varphi : \mathcal{N}(B, a_0) \to \mathcal{L}(W', -1, a, b_1 + 1)$ be a nonzero one-to-one module homomorphism where $W'$ is the highest weight $\mathfrak{m}$-module with highest weight $b_1$. We also define $\varphi_n(w \otimes t^n) = \varphi_n(w) \otimes t^n$ for any $w \in B$ and any $n \in \mathbb{Z}$. Then $\varphi_n : B \to W'$ is a one-to-one vector space homomorphism. We can have a similar equation as in (5.1). From (5.3) and (5.4), we deduce that $b_1 = 0$ which is impossible. So $\mathcal{N}(B, a_0)$ and $\mathcal{L}_0(a_1, b_1)$ (or $\mathcal{L}_1(a_1, b_1)$) cannot be isomorphic. \qed

6. Virasoro modules $\mathcal{L}(W, \lambda, a, b)$ are new

We need only to compare our simple Virasoro modules $\mathcal{L}(W, \lambda, a, b)$ and $\mathcal{L}_0(a, b)$ with the simple Virasoro modules obtained in [CGZ]. Let us first recall the modules in [CGZ].

Let $U := \mathbb{U}(\mathfrak{g})$ be the universal enveloping algebra of the Virasoro algebra $\mathfrak{g}$. For any $\hat{c}, \hat{h} \in \mathbb{C}$, let $I(\hat{c}, \hat{h})$ be the left ideal of $U$ generated by the set

$$\left\{ d_i \mid i > 0 \right\} \cup \left\{ d_0 - h \cdot 1, c - \hat{c} \cdot 1 \right\}.$$  

The Verma module with highest weight $(\hat{c}, \hat{h})$ for $\mathfrak{g}$ is defined as the quotient $\bar{V}(\hat{c}, \hat{h}) := U/I(\hat{c}, \hat{h})$. It is a highest weight module of $\mathfrak{g}$ and has a basis consisting of all vectors of the form

$$d_{-i_1}d_{-i_2}\cdots d_{-i_k}v_h; \quad k \in \mathbb{N} \cup \{0\}, i_j \in \mathbb{N}, i_1 \geq \cdots \geq i_2 \geq i_1 > 0.$$  

Then we have the simple highest weight module $V(\hat{c}, \hat{h}) = \bar{V}(\hat{c}, \hat{h})/J$ where $J$ is the maximal proper submodule of $\bar{V}(\hat{c}, \hat{h})$.

Theorem 16. Let $\lambda \in \mathbb{C}^*, a, b, \hat{c}, \hat{h}, a_1, b_1 \in \mathbb{C}$, and let $W \in \mathcal{O}_\mathfrak{g}$ be nontrivial simple. Then simple modules $\mathcal{L}(W, \lambda, a, b), \mathcal{L}_0(a, b), \mathcal{L}_1(a, b)$ are not isomorphic to any simple submodules of $V(\hat{c}, \hat{h}) \otimes A'_{a_1, b_1}$.

Proof. Let us introduce the operator in $U(\mathfrak{g})$:

$$X_{l,m} = d_{l-m-3}d_{m+3} - 3d_{l-m-2}d_{m+2} + 3d_{l-m-1}d_{m+1} - d_{l-m}d_m, \forall l, m \in \mathbb{Z}.$$  

Let $v_1$ be the highest weight vector of $V(\hat{c}, \hat{h})$. From [CGZ], there is a nonzero vector in any simple submodules of $V(\hat{c}, \hat{h}) \otimes A'_{a_1, b_1}$ of the form $v_1 \otimes v_2$ for some weight vector $v_2 \in A'_{a_1, b_1}$. From the proof of Theorem 7 in [LLZ], we know that

$$X_{l,m}(v_1 \otimes v_2) = v_1 \otimes \omega_{l,m}^{(3)}v_2 = 0, \forall m > 0, l > m + 3.$$
For any nonzero weight vector \( w \otimes t^k \in \mathcal{L}(W, \lambda, a, b) \), or \( \mathcal{L}_0(a, b) \), or \( \mathcal{L}_1(a, b) \), where \( w \in W, k \in \mathbb{Z} \), one can compute that

\[
X_{l,m}(w \otimes t^k) \in (\lambda - 1)^3(\lambda^{l-m-3} - \lambda^m)d_{-1}w \otimes t^{l+k} + (d_{-1}U(b) + U(b))w \otimes t^{l+k}.
\]

Recall that \( W \cong \text{Ind}_b^d(Soc_b W) = \mathbb{C}[d_{-1}] \otimes (Soc_b W) \). So \( X_{l,m}(w \otimes t^k) \) is nonzero for \( l > m + 3 \) with \( (\lambda - 1)^3(\lambda^{l-m-3} - \lambda^m) \neq 0 \). Thus the theorem follows.

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