Relation-algebraic and Tool-supported Control of Condorcet Voting

Rudolf Berghammer and Henning Schnoor
Institut für Informatik, Christian-Albrechts-Universität Kiel
Olshausenstraße 40, 24098 Kiel, Germany

Abstract. We present a relation-algebraic model of Condorcet voting and, based on it, relation-algebraic solutions of the constructive control problem via the removal of voters. We consider two winning conditions, viz. to be a Condorcet winner and to be in the (Gilles resp. upward) uncovered set. For the first condition the control problem is known to be NP-hard; for the second condition the NP-hardness of the control problem is shown in the paper. All relation-algebraic specifications we will develop in the paper immediately can be translated into the programming language of the BDD-based computer system RelView. Our approach is very flexible and especially appropriate for prototyping and experimentation, and as such very instructive for educational purposes. It can easily be applied to other voting rules and control problems.

1 Introduction

Elections have been studied by scientists from different disciplines for more than a thousand years. In addition to the obvious moral and political issues, elections also give rise to several computational questions, which are studied in the field of Computational Social Choice. The most prominent of these questions is the requirement of an algorithm that efficiently computes the winner(s) of an election. Surprisingly, such algorithms do not exist for all natural election systems, see [12] for an example. However, elections also give rise to computational problems which ideally should be hard to solve:

– The manipulation problem (see [1]) asks to determine a way for a group of voters to vote that serves their interest best, even though the vote might not represent their true preferences. Unfortunately, classical results show that every reasonable voting system gives voters incentives to vote strategically in this way (Gibbard-Satterthwaite theorem, cf. [11,17]).
– The control problem (see, e.g., [2]) asks for determining a way for the coordinator of an election to set up the election in a way that serves his or her personal interest. In order to achieve this, the coordinator might remove or add alternatives or voters from the election or partition the election.

Following the above-mentioned paper [1], numerous papers have studied the complexity of manipulation and control problems for elections (see, e.g., [7,13]).
For many election systems, it can be shown that the studied control or manipulation problem is NP-hard, and thus the election system is deemed to be ‘secure’ against this attempt to influence the outcome of the election. However, it has long been observed that efficient algorithms that work for many cases can still exist for NP-hard problems, the very successful history of SAT solvers being an impressive example. In the context of Computational Social Choice, [6] demonstrates a fast and very simple algorithm that works correctly on ‘most’ inputs (according to a suitably chosen probability distribution) and is allowed to compute an incorrect result on the remaining inputs.

In this paper, we study an alternative approach to show that NP-hard election problems may be solvable in practice. We apply the Computer Algebra system RELVIEW (see [3, 20]), which uses mathematical tools from relation algebra in the sense of [18, 19], to implement algorithms for the control problem of an election. Our implementations are provably correct for all instances; hence, as the problems we study are NP-hard, our algorithms do not run in polynomial time in general. Instead, we rely on RELVIEW’s optimization to exploit the simple structure of most practical instances of the problems we study, which allows for an algorithmic treatment.

Concretely, we study the following problem: Given an election consisting of a set of alternatives (also sometimes called candidates), voters along with information on how they will vote, and a preferred alternative \( a^* \), determine a minimum set \( Y \) of voters such that removing all voters in \( Y \) makes \( a^* \) win the election. The election system we study is the Condorcet voting system with the uncovered set winning condition (in case there is no Condorcet winner). To the best of our knowledge, this is the first paper where a relation-algebraic approach is used to solve problems related to elections that directly take the individual votes into account. An advantage of our approach is that it is very general and allows to treat related problems for different election systems with only small modifications. In particular, we could also treat elections in a generalized setting, where voters’ preferences are not linear orders (such a setting is studied in [10]). A further advantage is that the correctness proofs for our algorithms are formalized in such a way that, in principle, their automatic verification is possible. Our results and the performance of our algorithms demonstrate that Computer Algebra tools can be used successfully to solve NP-hard problems, where the data structures used in the Computer Algebra package automatically allow to exploit the ‘easyness’ that may be present in practical instances. In our case, RELVIEW uses BDDs to efficiently represent relations that are exponential in the input size. Thus, relation-algebraic algorithms can be obtained without specific knowledge about the problem domain.

2 Relation-algebraic Preliminaries

Given sets \( X \) and \( Y \), we write \( R : X \leftrightarrow Y \) if \( R \) is a (binary) relation with source \( X \) and target \( Y \), i.e., a subset of \( X \times Y \). If the sets of \( R \)’s type \( X \leftrightarrow Y \) are finite, then we may consider \( R \) as a Boolean matrix. Since such an interpretation is...
well suited for many purposes and also used by RELVIEW as the main possibility to visualize relations, in this paper we frequently use matrix terminology and notation. Especially, we speak about the entries, rows and columns of a relation/matrix and write \( R \) instead of \((x, y) \in R \) or \( x R y \). We assume the reader to be familiar with the basic operations on relations, viz. \( R^\top \) (transposition), \( \overline{R} \) (complement), \( R \cup S \) (union), \( R \cap S \) (intersection) and \( R; S \) (composition), the predicates \( R \subseteq S \) (inclusion) and \( R = S \) (equality), and the special relations \( \emptyset \) (empty relation), \( \mathbb{L} \) (universal relation) and \( I \) (identity relation). In case of \( \emptyset, \mathbb{L} \) and \( I \) we overload the symbols, i.e., avoid the binding of types to them.

For \( R : X \leftrightarrow Y \) and \( S : X \leftrightarrow Z \), by \( \text{syq}(R, S) = \overline{R^\top \overline{S}} \cap \overline{R^\top S} \) their symmetric quotient \( \text{syq}(R, S) : Y \leftrightarrow Z \) is defined. In the present paper we will only use its point-wise description, saying that for all \( y \in Y \) and \( z \in Z \) it holds \( \text{syq}(R, S)_{y,z} \) iff for all \( x \in X \) the relationships \( R_{x,y} \) and \( S_{x,z} \) are equivalent.

In relation algebra vectors are a well-known means to model subsets of a given set \( X \). Vectors are relations \( r : X \leftrightarrow \mathbb{1} \) (we prefer in this context lower case letters) with a specific singleton set \( \mathbb{1} = \{ \bot \} \) as target. They can be considered as Boolean column vectors. To be consonant with the usual notation, we omit always the second subscript, i.e., write \( r_x \) instead of \( r_{x,\bot} \). Then \( r \) describes the subset \( Y \) of \( X \) if for all \( x \in X \) it holds \( r_x \) iff \( x \in Y \). A point \( p : X \leftrightarrow \mathbb{1} \) is a vector with precisely one 1-entry. Consequently, it describes a singleton subset \( \{x\} \) of \( X \) and we then say that it describes the element \( x \) of \( X \). If \( r : X \leftrightarrow \mathbb{1} \) is a vector and \( Y \) the subset of \( X \) it describes, then \( \text{inj}(r) : Y \leftrightarrow X \) denotes the embedding relation of \( Y \) into \( X \). In Boolean matrix terminology this means that \( \text{inj}(r) \) is obtained from \( 1 : X \leftrightarrow X \) by deleting all rows which do not correspond to an element of \( Y \) and point-wisely this means that for all \( y \in Y \) and \( x \in X \) it holds \( \text{inj}(r)_{y,x} \) iff \( y = x \).

In conjunction with power sets \( 2^X \) we will use membership relations \( \text{M} : X \leftrightarrow 2^X \) and size comparison relations \( S : 2^X \leftrightarrow 2^X \). Point-wisely they are defined for all \( x \in X \) and \( Y, Z \in 2^X \) as follows: \( \text{M}_{x,Y} \) iff \( x \in Y \) and \( \text{S}_{Y,Z} \) iff \( |Y| \leq |Z| \). A combination of \( \text{M} \) with embedding relations allows a column-wise enumeration of an arbitrary subset \( \mathcal{S} \) of \( 2^X \). Namely, if the vector \( r : 2^X \leftrightarrow \mathbb{1} \) describes \( \mathcal{S} \) in the sense defined above and we define \( S = \text{M}; \text{inj}(r)^\top \), then we get \( X \leftrightarrow \mathcal{S} \) as type of \( S \) and that for all \( x \in X \) and \( Y \in \mathcal{S} \) it holds \( S_{x,Y} \) iff \( x \in Y \). In the Boolean matrix model this means that the sets of \( \mathcal{S} \) are precisely described by the columns of \( S \), if the columns are considered as vectors of type \( X \leftrightarrow \mathbb{1} \).

To model direct products \( X \times Y \) of sets \( X \) and \( Y \) relation-algebraically, the projection relations \( \pi : X \times Y \leftrightarrow X \) and \( \rho : X \times Y \leftrightarrow Y \) are the convenient means. They are the relational variants of the well-known projection functions and, hence, fulfill for all \( u \in X \times Y \), \( x \in X \) and \( y \in Y \) the following equivalences: \( \pi_{u,x} \) iff \( u_1 = x \) and \( \rho_{u,y} \) iff \( u_2 = y \). Here \( u_1 \) denotes the first component of \( u \) and \( u_2 \) the second component. As a general assumption, in the remainder of the paper we always assume a pair \( u \) to be of the form \( u = (u_1, u_2) \). Then \( \hat{u} \) denotes the transposed pair \((u_2, u_1)\). The projection relations enable us to specify the well-known pairing operation of functional programming relation-algebraically. The pairing of \( R : Z \leftrightarrow X \) and \( S : Z \leftrightarrow Y \) is defined as \([R, S] = R; \pi^\top \cap S; \rho^\top :\)
Z ↔ X × Y. where π and ρ are as above. Point-wisely this definition says that
\([R,S]_{z,u}\) iff \(R_{z,u_1}\) and \(S_{z,u_2}\), for all \(z \in Z\) and \(u \in X \times Y\). Based on π and ρ we are also able to establish a bijective correspondence between the relations of type
\(X \leftrightarrow Y\) and the vectors of type
\(X \times Y \leftrightarrow 1\). The transformation of \(R : X \leftrightarrow Y\) into
its corresponding vector \(\text{vec}(R) : X \times Y \leftrightarrow 1\) is given by
\(\text{vec}(R) = (\pi; R\cap\rho); L\) and the step back from \(r : X \times Y \leftrightarrow 1\) to its corresponding relation \(\text{rel}(r) : X \leftrightarrow Y\) by
\(\text{rel}(r) = \pi^T; (\rho \cap r; L)\). Point-wisely this means that for all \(u \in X \times Y\) the following equivalences are true:
\(\text{vec}(R)_{u_1,u_2}\) iff \(R_{u_1,u_2}\) and \(\text{rel}(r)_{u_1,u_2}\) iff \(r_{u_1,u_2}\).

3 A Relation-algebraic Model of Condorcet Voting

Usually, an election consists of a non-empty and finite set \(N\) of voters (agents), normally \(N = \{1, \ldots, n\}\), a non-empty and finite set \(A\) of alternatives (candidates), the individual preferences (choices, wishes) of the voters and a voting rule that aggregates the winners from the individual preferences. A well-known voting rule is the Condorcet voting rule. Here it is usually assumed that each voter ranks the alternatives from top to bottom, i.e., the individual preferences of the voters \(i \in N\) are expressed via linear strict orders \(>_i : A \leftrightarrow A\). From them the dominance relation \(C : A \leftrightarrow A\) is computed that specifies the collective preferences. An instance of a Condorcet election consists of the sets \(N, A\), and the relations \(>_i\) for all \(i \in N\). In the following we consider the approach that \(C_{a,b}\) if the number of voters \(i\) with \(a >_i b\) is (strictly) greater than the number of voters \(i\) with \(b >_i a\). In this case we also say that \(a\) beats \(b\) with \(p\) points, where \(p\) is the (positive) difference between these numbers. It is known that \(C\) may contain cycles and that an alternative that dominates all other ones – a so-called Condorcet winner – does not necessarily exist. To get around this problem, in the literature so-called choice sets have been introduced which take over the role of the best alternative and specify the winners (see e.g., \([14]\) for more details). In this paper, we will study the choice set Uncovered Set.

For a relation-algebraic treatment of Condorcet voting, we first model its input, i.e., the individual preferences of the voters, accordingly.

**Definition 3.1** The relation \(P : N \leftrightarrow A^2\) models the instance \((N, A, (<_i)_{i \in N})\) of a Condorcet election if \(P_{i,u}\) is equivalent to \(u_1 >_i u_2\), for all \(i \in N\) and \(u \in A^2\).

In the following RELVIEW picture an input relation \(P\) is shown. The labels of the rows and columns indicate that the voters are the natural numbers from 1 to 13 and the alternatives are the eight letters from \(a\) to \(h\).
It is troublesome to identify from this picture the individual preferences. But if we select the single rows, transpose them to obtain vectors of type $A^2 \leftrightarrow 1$ and apply the function $\text{rel}$ of Section 2 to the latter, then RELView depicts the individual preferences as Boolean matrices. For the rows 1, 4, 7 and 10 we get, in the same order, the following Boolean matrices for $>_1, >_4, >_7$ and $>_11$:

Now, the preferences of the single voters are easy to see. Voters 1 to 3 rank their alternatives from top to bottom as $a,c,e,g,b,d,f,h$, voters 4 to 6 as $a,b,c,d,e,f,g,h$, voters 7 to 9 as $b,a,d,c,f,e,h,g$ and the remaining voters 10 to 13 as $h,g,f,e,a,b,c,d$. The procedure also shows how to construct, in general, the input $P : N \leftrightarrow A^2$ from strict orders $>_i ; A \leftrightarrow A$ by inverting it. We have to number the voters from 1 to $n$, then to transform each relation $>_i$ into $\text{vec}(>_i)^T : 1 \leftrightarrow A^2$, i.e., the transpose of its corresponding vector, and finally to combine the transposed vectors row by row into a Boolean matrix. The latter means that we have to form the relation-algebraic sum $\text{vec}(>_1)^T + \cdots + \text{vec}(>_n)^T$.

We won’t go into details with regard to sums of relations and refer to [19], where a relation-algebraic specification via injection relations is given. Instead, we demonstrate how to get from the individual preferences relation $P$ the collective preferences, i.e., the dominance relation $C$. In what follows, we assume the projection relations $\pi, \rho : A^2 \leftrightarrow A$ of the direct product $A^2$ to be at hand as well as the membership relation $M : N \leftrightarrow 2^N$ and the size comparison relation $S : 2^N \leftrightarrow 2^N$. Each of these relations is available in RELView via a pre-defined function and their BDD-implementations are rather small. See [15,16] for details.

**Theorem 3.1** Suppose that $P : N \leftrightarrow A^2$ models an instance of Condorcet voting. If we specify relations $E, F : A^2 \leftrightarrow 2^N$ and $C : A \leftrightarrow A$ by

$$E = \text{syq}(P, M) \quad F = \text{syq}(P; [\rho, \pi], M) \quad C = \text{rel}((E \cap F; (S \cap S^T )); L),$$

then $C_{u_1, u_2}$ is equivalent to $|\{i \in N \mid P_{i,u_1}\}| > |\{i \in N \mid P_{i,u_2}\}|$, for all $u \in A^2$.

**Proof.** For the given $u \in A^2$ we prove in a preparatory step for all $Y \in 2^N$ that

$$E_{u,Y} \iff \text{syq}(P, M)_{u,Y} \iff \forall i \in N : P_{i,u} \leftrightarrow M_{i,Y} \iff \forall i \in N : P_{i,u} \leftrightarrow i \in Y \iff \{i \in N \mid P_{i,u}\} = Y.$$  

Using that the exchange relation $[\rho, \pi] : A^2 \leftrightarrow A^2$ relates the pair $u$ precisely with its transposition $\hat{u} = (u_2, u_1)$, in a rather similar way we can prove that for

---

1 A still more appropriate method is to compute for each relation $>_i$, its Hasse diagram in the sense of [18] and to draw the latter in RELView as directed graphs.
all $Z \in 2^N$ the following property holds:

$$F_{u,Z} \iff \{i \in N \mid P_{i,u}\} = Z$$

By means of these two auxiliary results, we now conclude the proof as follows:

$$C_{u_1,u_2} \iff \text{rel}((E \cap F; (S \cap \overline{S}))_{u_1,u_2})$$

$$\iff (E \cap F; (S \cap \overline{S}))_{u_1,u_2}$$

The specifications of Theorem 3.1 can be executed by means of RELView after a straightforward translation into its programming language. In case of the above input relation $P$ the tool computed the following dominance relation $C$. From the first row of $C$ we see that alternative $a$ is the Condorcet winner since it dominates all other alternatives.

![Dominance relation](image)

This relation is not only asymmetric (i.e., satisfies $C \cap C^T = \emptyset$) but also complete (i.e., satisfies $I \subseteq C \cup C^T$). Altogether, $C$ is a tournament relation and this property implies the uniqueness of a Condorcet winner in the case that one exists. How to compute, in general, from the dominance relation $C$ the choice sets using relation-algebraic means is demonstrated in [4].

## 4 Control of Condorcet Voting by Deleting Voters

We only consider the constructive variant of the control problem for Condorcet voting, where control is done by deleting voters. Usually, the task is formulated as a minimization problem: Given a specific alternative $a^*$, determine a minimum set of voters $Y$ such that the removal of $Y$ from the set $N$ of all voters makes $a^*$ to a winner. To allow for an easier relation-algebraic representation, we consider the dual maximization problem, i.e., we ask for a maximum set of voters $X$ such that $a^*$ wins subject to the condition that only voters from $X$ are allowed to vote. It is obvious that from $X$ then a desired $Y$ is obtained via $Y = N \setminus X$.

We start with the assumption that ‘to win’ means ‘to be a Condorcet winner’. As shown in [2], Condorcet voting is computationally resistant to our control type.

\[^2\] The destructive variant of our control problem asks for a minimum set of voters the removal of which prevents win of $a^*$. 
in case of this specification of winners. I.e., it is NP-hard to decide, for \( a^* \in A \) and \( k \in \mathbb{N} \) as inputs, whether it is possible to find \( k \) voters whose removal makes \( a^* \) to a Condorcet winner.

As a first step towards a solution of the maximization-problem, we relativize the dominance relation \( C \) by additionally considering the sets of voters \( X \) which only are allowed to vote. Concretely this means that we specify a relation \( R \) that relates \( X \in 2^N \) with \( a, b \in A \) iff \(|\{i \in X \mid a >_i b\}| > |\{i \in X \mid b >_i a\}|\).

Since we work with binary relations, we have to combine two of the three objects \( X, a \) and \( b \) to a pair. We do this with \( a \) and \( b \), i.e., relate \( X \) with \( u \) under the assumption that \( u_1 \) equals \( a \) and \( u_2 \) equals \( b \). Then the following theorem shows how the relativized dominance relation \( R : 2^N \leftrightarrow A^2 \) can be specified relation-algebraically. Again we assume the relations \( \pi, \rho : A^2 \leftrightarrow A \), \( M : N \leftrightarrow 2^N \) and \( S : 2^N \leftrightarrow 2^N \) to be at hand.

**Theorem 4.1** Suppose again that \( P : N \leftrightarrow A^2 \) models an instance of Condorcet voting. If we specify relations \( E, F : 2^N \times A^2 \leftrightarrow 2^N \) and \( R : 2^N \leftrightarrow A^2 \) by

\[
E = \text{syq}(\text{[M, P]}, M) \quad F = \text{syq}(\text{[M, P], [\rho, \pi]}), M) \quad R = \text{rel}(E \cap F; (S \cap \overline{S}^T)); L),
\]

then \( R_{X,u} \) is equivalent to \(|\{i \in X \mid P_{i,u}\}| > |\{i \in X \mid P_{i,\hat{a}}\}|\), for all \( X \in 2^N \) and \( u \in A^2 \).

**Proof.** Assume arbitrary objects \( X \in 2^N \) and \( u \in A^2 \) to be given. Then, we have for all \( Y \in 2^N \) the following equivalence:

\[
E_{(X,u),Y} \iff \text{syq}(\text{[M, P]}, M)_{(X,u),Y} \\
\iff \forall i \in N : [M, P]_{i,(X,u)} \leftrightarrow M_{i,Y} \\
\iff \forall i \in N : M_{i,X} \land P_{i,u} \leftrightarrow M_{i,Y} \\
\iff \forall i \in N : i \in X \land P_{i,u} \leftrightarrow i \in Y \\
\iff \{i \in X \mid P_{i,u}\} = Y
\]

In a similar way we can show for all \( Z \in 2^N \) the following fact, using the property of the exchange relation \([\rho, \pi] : A^2 \leftrightarrow A^2 \) mentioned in the proof of Theorem 3.1

\[
F_{(X,u),Z} \iff \{i \in X \mid P_{i,\hat{a}}\} = Z.
\]

Now, the following calculation shows the claim:

\[
R_{X,u} \iff \text{rel}(E \cap F; (S \cap \overline{S}^T)); L)_{X,u} \\
\iff ((E \cap F; (S \cap \overline{S}^T)); L)_{(X,u)} \\
\iff \exists Y \in 2^N : E_{(X,u),Y} \land (F; (S \cap \overline{S}^T))_{(X,u),Y} \land L_Y \\
\iff \exists Y \in 2^N : E_{(X,u),Y} \land \exists Z \in 2^N : F_{(X,u),Z} \land S_{Z,Y} \land \neg S_Y; Z \\
\iff \exists Y, Z \in 2^N : \{i \in X \mid P_{i,u}\} = Y \land \{i \in X \mid P_{i,\hat{a}}\} = Z \land |Z| < |Y| \\
\iff \{i \in X \mid P_{i,u}\} > \{i \in X \mid P_{i,\hat{a}}\} \quad \square
\]
In the second step, we now take the relativized dominance relation $R$ of Theorem 4.1 and specify with its help a vector $\text{cand} : 2^N \leftrightarrow 1$ that describes the subset of the members of which are the sets $X$ which are candidates for the solution of our control problem. The latter property means that $a^*$ is a Condorcet winner, provided that only voters from $X$ are allowed to vote. From the vector $\text{cand}$ we then finally compute the vector description $\text{sol} : 2^N \leftrightarrow 1$ of the maximum candidate sets, which are the solutions we are looking for. The next theorem shows how to get $\text{cand}$ and $\text{sol}$ from $R$ and $a^*$.

**Theorem 4.2** Suppose that $R : 2^N \leftrightarrow A^2$ is the relation specified in Theorem 4.1 and that the specific alternative $a^* \in A$ is described by the point $p : A \leftrightarrow 1$.

If we specify vectors $\text{cand}$, $\text{sol} : 2^N \leftrightarrow 1$ by

$$\text{cand} = \overline{R ; (\pi; p \cap \overline{p; p})} \quad \text{sol} = \text{cand} \cap \overline{\bigtriangledown ; \text{cand}},$$

then the set $\{X \in 2^N \mid \forall b \in A \setminus \{a^*\} : |\{i \in X \mid P_{i,(a^*,b)}\}| > |\{i \in X \mid P_{i,(b,a^*)}\}| \}$ is described by $\text{cand}$ and the set of its maximum sets by $\text{sol}$.

**Proof.** Since $p$ describes $a^*$, for all $u \in A^2$ we have $(\pi;p)_u$ iff $u_1 = a^*$ and $\overline{p; p}_u$ iff $u_2 \neq a^*$. We now assume an arbitrary set $X \in 2^N$ and calculate as follows, where in the fifth step Theorem 4.1 is applied:

$$\text{cand}_X \iff \overline{R ; (\pi; p \cap \overline{p; p})}_X$$
$$\iff \neg\exists u \in A^2 : \overline{R}_{X,u} \land (\pi; p)_u \land \overline{p; p}_u$$
$$\iff \neg\exists u \in A^2 : \overline{R}_{X,u} \land u_1 = a^* \land u_2 \neq a^*$$
$$\iff \forall u \in A^2 : u_1 = a^* \land u_2 \neq a^* \rightarrow R_{X,u}$$
$$\iff \forall u \in A^2 : u_1 = a^* \land u_2 \neq a^* \rightarrow |\{i \in X \mid P_{i,u}\}| > |\{i \in X \mid P_{i,\overline{a}}\}|$$
$$\iff \forall b \in A : b \neq a^* \rightarrow |\{i \in X \mid P_{i,(a^*,b)}\}| > |\{i \in X \mid P_{i,(b,a^*)}\}|$$
$$\iff \forall b \in A \setminus \{a^*\} : |\{i \in X \mid P_{i,(a^*,b)}\}| > |\{i \in X \mid P_{i,(b,a^*)}\}|$$

Hence, the first claim follows from the definition of the set a vector describes. To prove the second claim, we take again an arbitrary set $X \in 2^N$. Then, we get:

$$\text{sol}_X \iff (\text{cand} \cap \overline{\bigtriangledown ; \text{cand}})_X$$
$$\iff \text{cand}_X \land \overline{\bigtriangledown ; \text{cand}}_X$$
$$\iff \text{cand}_X \land \neg\exists Y \in 2^N : \overline{\bigtriangledown}_{Y,X} \land \text{cand}_Y$$
$$\iff \text{cand}_X \land \forall Y \in 2^N : \text{cand}_Y \rightarrow \overline{\bigtriangledown}_{Y,X}$$
$$\iff \text{cand}_X \land \forall Y \in 2^N : \text{cand}_Y \rightarrow |Y| \leq |X|$$

This equivalence implies that $\text{sol}$ describes the set of maximum sets of voters $X$ for which $\text{cand}_X$ holds, that is, for which $a^*$ wins subject to the condition that only voters from $X$ are allowed to vote. □

Using RELVIEW we have solved our control problem with Condorcet winners as winning alternatives for the above input relation $P$ and each of the eight
alternatives. The tool showed that only the alternatives $a$, $b$ and $h$ can be made to Condorcet winners by deleting voters. Some of the results for these alternatives are presented in the following six RELView pictures:

The vector on position 1 says that $a$ is a Condorcet winner if all voters are allowed to vote and the corresponding dominance relation on position 2 is the original dominance relation $C$. To make $b$ a Condorcet winner at least eight voters must be deleted. Altogether there are 45 possibilities for this. The vector on position 3 shows one of them, where the voters from 1 to 6 and the voters 10 and 11 are deleted. On position 4 the resulting dominance relation is depicted. To get $h$ as Condorcet winner requires a removal of at least six voters. According to RELView there are 85 possibilities for this. One of them and the resulting dominance relation are depicted at positions 5 and 6.

Since Condorcet winners do not always exist, choice sets have been introduced as a general concept that always allows to define the winners of Condorcet voting. In the remainder of this section we treat a well-known example, the uncovered set. This choice set is usually defined via an induced transitive subrelation of the dominance relation $C$, called covering relation. In the literature different such relations are discussed. We concentrate on a relation $G : A \leftrightarrow A$ that in [8] is called Gilles covering and in [5] upward covering. Its usual point-wise definition says that $G_{a,b}$ iff $C_{a,b}$ and for all $c \in A$ from $C_{c,a}$ it follows $C_{c,b}$, for all $a,b \in A$.

This relation-algebraically can be specified as equation $G = C \cap C^T$. The (Gilles or upward) uncovered set is the set of all $a \in A$ such that there exists no $b \in A \setminus \{a\}$ with $G_{b,a}$. It is non-empty because $G$ is a strict-order and $A$ is finite. To the best of our knowledge, the computational complexity of control problems for Condorcet elections with winning conditions different from being a Condorcet winner has not been studied in the literature. We obtain the first result in this direction by proving that the problem to control Condorcet elections with upward covering by deleting voters is NP-hard (see Section 5). To solve our control problem for this specification of winners we use the same idea as in the relativization of the relation $C$ to the relation $R$ by additionally considering the set of voters $X$ which are allowed to vote. The next theorem shows how to obtain the relativized covering relation $U$ from the relativized dominance relation $R$.

**Theorem 4.3** Suppose again that $R : 2^N \leftrightarrow A^2$ is the relation specified in Theorem 4.1. If we specify relations $E : A^2 \times A^2 \leftrightarrow A^2$ and $U : 2^N \times A^2 \leftrightarrow 2^N$ by

$$E = [\pi; \rho^T, \rho; \rho^T]^T \cap \text{vec}(\pi; \pi^T); L \quad U = R \cap [\overline{R}; \overline{R}; \overline{E}];$$

then for all $X \in 2^N$ and $u \in A^2$ we have

$$U_{X,u} \iff R_{X,u} \land \forall c \in A : R_{X,(c,u_1)} \rightarrow R_{X,(c,u_2)}.$$
Proof. Let $X \in 2^N$ and $u \in A^2$ be given. In a first step we show for all $v, w \in A^2$ the following property, where we use the equivalence of $(\pi; \rho^T)_{u,v}$ and $u_1 = v_2$, of $(\rho; \rho^T)_{u,v}$ and $u_2 = w_2$, and of $(\pi; \pi^T)_{v,w}$ and $v_1 = w_1$:

$$E_{(v,w),u} \iff ([\pi; \rho^T, \rho; \rho^T]_u \cap \text{vec}(\pi; \pi^T); L)(v,w),u$$

$$\iff [\pi; \rho^T, \rho; \rho^T]_{u,v} \cap \text{vec}(\pi; \pi^T); L)(v,w),u$$

$$\iff (\pi; \rho^T)_{u,v} \cap (\rho; \rho^T)_{u,w} \cap \text{vec}(\pi; \pi^T)_{v,w}$$

$$\iff u_1 = v_2 \land w_2 = w_2 \land (\pi; \pi^T)_{v,w}$$

$$\iff u_1 = v_2 \land u_2 = w_2 \land v_1 = w_1$$

We now can calculate as follows to conclude the proof:

$$U_{X,u} \iff (R \cap [R, \overline{R}]; E)_{X,u}$$

$$\iff R_{X,u} \land [R, \overline{R}]; E_{X,u}$$

$$\iff R_{X,u} \land \exists v, w \in A^2 : [R, \overline{R}]_{X,(v,w)} \land E_{(v,w),u}$$

$$\iff R_{X,u} \land \exists v, w \in A^2 : R_{X,v} \land \overline{R}_{X,w} \land u_1 = v_2 \land u_2 = w_2 \land v_1 = w_1$$

$$\iff R_{X,u} \land \exists c \in A : R_{X,(c,u_1)} \land \overline{R}_{X,(c,u_2)}$$

$$\iff R_{X,u} \land \forall c \in A : R_{X,(c,u_1)} \rightarrow R_{X,(c,u_2)} \quad \square$$

After this result we are able to solve our control problem also for the uncovered set as set of winners. We use again a vector $\text{cand}$ for the description of the candidate sets and a vector $\text{sol}$ for the description of the solutions.

Theorem 4.4 Suppose that $U : 2^N \leftrightarrow A^2$ is the relation specified in Theorem 4.3 and that the specific alternative $a^* \in A$ is described by the point $p : A \leftrightarrow 1$. If we specify vectors $\text{cand}, \text{sol} : 2^N \leftrightarrow 1$ by

$$\text{cand} = U_1(\vec{\pi}, \vec{\rho} \cap \vec{p}; p)$$

$$\text{sol} = \text{cand} \cap \text{vec}(\overline{S^T}; \text{cand})$$

then the set \{ $X \in 2^N | \exists b \in A \setminus \{a^*\} : U_{X,(b,a^*)}$ \} is described by $\text{cand}$ and the set of its maximum sets by $\text{sol}$.

Proof. Because $a^*$ is described by $p$, for all $u \in A^2$ we have $\pi_u = \rho_u$ if $u_1 \neq a^*$ and $\rho_u$ if $u_2 = a^*$. Now, for all $X \in 2^N$ we can calculate as follows to show the first claim (for the second claim cf. the proof of Theorem 4.2).

$$\text{cand}_X \iff U_1(\vec{\pi}, \vec{\rho} \cap \vec{p}; p)_X$$

$$\iff \exists u \in A^2 : U_{X,u} \land \pi_u \land (\rho; p)_u$$

$$\iff \exists u \in A^2 : U_{X,u} \land u_1 \neq a^* \land u_2 = a^*$$

$$\iff \exists b \in A : U_{X,(b,a^*)} \land b \neq a^*$$

$$\iff \exists b \in A \setminus \{a^*\} : U_{X,(b,a^*)} \quad \square$$

As already mentioned, the uncovered set is always non-empty. The degenerate case is that no voter is allowed to vote. Then the resulting dominance relation
as well as the induced covering relation are empty and, thus, the uncovered set equals \( N \). \textsc{RelView} showed that in our running example this situation occurs if \( c \) or \( d \) shall win. We already know that \( a \) wins without a removal of voters. By reason of the tool at least five voters must be deleted to ensure win for \( e,f,g \) or \( h \) and the corresponding numbers of possibilities are 11, 111, 15 and 126. And, finally, \( b \) becomes winning if at least seven voters are not allowed to vote. To reach the goal there exist 120 possibilities. We end this section with the following three \textsc{RelView} pictures that concern alternative \( e \):

The vector shows that \( e \) is in the uncovered set if the voters 1, 2, 4, 5 and 6 are deleted, the relation in the middle is the dominance relation resulting from this, and the relation on the right is the induced covering relation. The empty columns show that, besides \( e \), the removal also make \( a,f \) and \( h \) uncovered.

5 Control Remains Hard if Uncovered Alternatives Win

As already mentioned, in [2] it is shown that for Condorcet voting constructive control by deleting voters is NP-hard if Condorcet winners are defined as winners. In this section we prove that this result remains true if instead of Condorcet winners the uncovered alternatives are taken. To this end we first introduce the following problem that we will be used in our reduction.

\textbf{Definition 5.1} The problem \( X4C \) (exact cover by 4-sets) is the following:

\textit{Input:} Sets \( S_1, \ldots, S_k \in 2^{\{1, \ldots, n\}} \) such that for all \( i \in \{1, \ldots, k\} \) it holds \( |S_i| = 4 \) and \( |\{i \mid j \in S_i\}| = 3 \) for all \( j \in \{1, \ldots, n\} \).

\textit{Question:} Is there some set \( I \in 2^{\{1, \ldots, k\}} \) such that \( \bigcup_{i \in I} S_i = \{1, \ldots, n\} \) and \( S_i \cap S_j = \emptyset \) for all \( i,j \in I \) with \( i \neq j \)?

Note that if an \( I \) as required exists, then \( |I| = \frac{1}{4} n \), since each \( S_i \) has cardinality 4 and the union must have cardinality \( n \). On the other hand, if an \( I \) with \( \bigcup_{i \in I} S_i = \{1, \ldots, n\} \) exists and \( |I| = \frac{1}{4} n \), then by a simple counting argument, \( S_i \cap S_j = \emptyset \) for all \( i,j \in I \) with \( i \neq j \). Also, the value \( k \) in the problem instance must necessarily be equal to \( \frac{3}{4} n \), since each \( S_i \) has 4 elements and each \( j \in \{1, \ldots, n\} \) appears in exactly 3 of the sets \( S_i \). In particular, it follows that \( n \) is a multiple of 4 in every instance for \( X4C \). The following result is mentioned without proof in [9], we give the complete proof:

\textbf{Lemma 5.1} The problem \( X4C \) is NP-hard.
Proof. We reduce from a special version of the 1-in-3-satisfiability problem, called 1-in-3-Sat′ and introduced in [9]. An instance of 1-in-3-Sat′ is a formula of the form \( \varphi = \bigwedge_{i=1}^{n} 1\text{-in-3}(x_i^1, x_i^2, x_i^3) \), where 1-in-3\((x, y, z)\) is a clause which is true iff exactly one of the variables \(x\), \(y\), and \(z\) is true. Additionally, \( \varphi \) has the following properties: In each clause the 3 appearing variables are distinct, and each variable appears in exactly 4 clauses. Note that this implies that the number of distinct variables in \( \varphi \) is \( \frac{3}{4}n \).

An instance \( \varphi \) of the problem 1-in-3-Sat′ can be transferred into an instance of X4C as follows:

- Each of the \( n \) clauses in \( \varphi \) becomes a set element containing the set element corresponding to the clause is define instance consisting of the sets \( S_1, \ldots, S_n \).
- Each variable \( x_i \) becomes a set \( S_i \) containing the clauses in which \( x_i \) appears.

First assume that \( \varphi \) is satisfiable. Then there is an assignment \( I \) with \( I \models \varphi \). Since \( I \) satisfies exactly one variable in each clause, we know that \( n \) variable occurrences are satisfied by \( I \). Since each variable, in particular each of the satisfied variables, appears in 4 clauses, we get that \( \frac{1}{4}n \) many variables are satisfied by \( I \). We can naturally interpret \( I \) as the set of indices \( i \) with \( I \models x_i \), and claim that \( I \) satisfies the conditions of X4C. As mentioned above, since \( |I| = \frac{1}{4}n \), it suffices to show that \( \bigcup_{i \in I} = \{1, \ldots, n\} \). This follows from the construction: Since \( I \) (seen as a truth assignment to the variables) satisfies each clause, we know that for each clause, there is a variable satisfied by \( I \). For the X4C instance, this implies that for each element \( i \in \{1, \ldots, n\} \), there is an index \( j \in I \) with \( i \in S_j \).

For the converse, assume that there is an index set \( I \) satisfying the conditions of X4C. We can interpret \( I \) as a truth assignment for the variables in \( \varphi \) in the obvious way: A variable is set to 1 iff its corresponding set is in the selection \( I \). We show that \( I \), seen as a truth assignment, satisfies the formula \( \varphi \). Hence let \( 1\text{-in-3}(x_1^i, x_2^i, x_3^i) \) be a clause in \( \varphi \). Since \( I \) is a set cover, we know that for this clause, an element containing the set element corresponding to the clause is selected in \( I \). Hence \( I \) satisfies at least one of the variables \( x_1^i, x_2^i, \) and \( x_3^i \). Since \( I \) is an exact cover, we also know that each set element appears only in one of the selected sets, hence only one of the variables is true, and we are done.  

We can now show the main theorem of this section.

Theorem 5.1 For Condorcet voting the constructive control problem by deleting voters is NP-hard if the uncovered alternatives are specified as the winners.

Proof. We reduce from X4C, which is NP-hard due to Lemma [11]. So, let an X4C-instance consisting of the sets \( S_1, \ldots, S_{\frac{2}{3}n} \) be given. Without loss of generality we assume \( n \geq 16 \). To do so, we construct an election \( E \) as follows. First we define \( t = \frac{1}{4}n - 2 \). Define \( s_1, \ldots, s_n \) and \( b_1, \ldots, b_n \). Finally, we introduce the following four groups of individual preferences, where \( S_{\neq i} = \{ s_j \mid j \neq i \} \), \( B_{\neq i} = \{ b_j \mid j \neq i \} \), \( B_{\notin S_i} = \{ b_j \mid j \notin S_i \} \) and \( B_{\in S_i} = \{ b_j \mid j \in S_i \} \).
1. For each $i \in \{1, \ldots, n\}$ we use $t$ linear strict orders of the form $S_\neq i > s_i > b_i > B_{\neq i} > a^\ast$.
2. For each $i \in \{1, \ldots, n\}$ we use $t$ linear strict orders of the form $B_{\neq i} > a^\ast > s_i > b_i > S_{\neq i}$.
3. For each set $S_i$ we use a linear strict order of the form $B_{\in S_i} > a^\ast > S > B_{\in S_i}$.
4. We use a linear strict order of the form $a^\ast > S > B$.

The notation of preferences (linear strict orders) using sets means that the order of the alternatives inside the sets is irrelevant. For instance, $a^\ast > S > B$ means that in the linear strict order $a^\ast$ is the greatest element, then the alternatives $s_1, \ldots, s_n$ follow in any order and, finally, the alternatives $b_1, \ldots, b_n$ follow, again in any order. Note that, by definition of $X4C$, we get $|B_{\in S_i}| = n - 4$ and $|B_{\in S_j}| = 4$. Now, the question in our constructed instance of the control problem whether the specific alternative $a^\ast$ can be made uncovered by deleting at most $\frac{1}{4}n$ linear strict orders (i.e., voters).

We first study the relationship between each of the relevant alternatives in the constructed election before any deletion of voters is performed. Note that if the point difference between two alternatives is at least $\frac{1}{4}n + 1$, then deleting at most $\frac{1}{4}n$ linear strict orders cannot change which of these alternatives dominates the other.

**Each $b_i$ beats $a^\ast$ with at least $\frac{1}{4}n + 1$ points.** To see this, we consider all preferences introduced in the election. For each $j \neq i$, the $2t$ linear strict orders of the first two groups place $b_i$ ahead of $a^\ast$. From the linear strict orders introduced for $i$, one puts $b_i$ ahead of $a^\ast$ and the other puts $a^\ast$ ahead of $b_i$. We now consider the linear strict orders introduced for the sets $S_j$: There are 3 sets $S_j$ in which $i$ appears (these place $a^\ast$ ahead of $b_i$) and $i$ does not appear in the remaining $\frac{3}{4}n - 3$ many (these place $b_i$ ahead of $a^\ast$).

Finally, $a^\ast > S > B$ put $a^\ast$ ahead of $b_i$. Hence the lead of $b_i$ over $a^\ast$ is

$$\frac{(n - 1) \cdot 2 \cdot t + \frac{3}{4}n - 6 - 1}{2(n - 1)t + \frac{3}{4}n} = 2(n - 1) + \frac{3}{4}n - 7$$

which is at least $\frac{1}{4}n + 1$, since we assumed $n \geq 16$.

**Alternative $a^\ast$ beats each $s_i$ with at least $\frac{1}{4}n + 1$ points.** Note that half of the linear strict orders introduced in the first two groups place $s_i$ ahead of $a^\ast$ and the other half put $a^\ast$ ahead of $s_i$. Hence $a^\ast$ and $s_i$ tie in the sub-election consisting of these linear strict orders. In the $\frac{3}{4}n$ linear strict orders introduced for the sets $S_i$, however, $a^\ast$ is always placed ahead of $s_i$. Finally, $a^\ast$ is ahead of $s_i$ in $a^\ast > S > B$. As a consequence $a^\ast$ beats each $s_i$ with $\frac{3}{4}n + 1$ many points, which is at least $\frac{1}{4}n + 1$.

**If $i \neq j$, then $b_i$ beats $s_j$ with at least $\frac{1}{4}n + 1$ points.** To see that this is true, note that the linear strict orders introduced in the first two groups are neutral between $b_i$ and $s_j$, as half of them have $b_i$ ahead of $s_j$ and the other half have $s_j$ ahead of $b_i$ (recall that $i \neq j$). Now consider the linear strict orders introduced for the sets $S_i$. There are 3 such orders which place $s_j$ ahead of $b_i$ (the ones corresponding to sets $S_i$ with $i \in S_i$), and the remaining $\frac{3}{4}n - 3$ many place $b_i$ ahead of $s_j$ (these are the ones corresponding
to sets $S_i$ with $i \not\in S_j$). In $a^* > S > B$, $s_i$ is voted ahead of $b_i$. Hence $b_i$ beats $s_j$ by $\frac{3}{4}n - 7$ points, which is at least $\frac{1}{4}n + 1$, since again $n \geq 16$.

**Alternative $b_i$ beats $s_i$ with exactly $\frac{1}{4}n - 3$ points.** This holds due to the following: The linear strict orders introduced in the first two groups for $j \neq i$ are neutral with respect to the relationship between $s_i$ and $b_i$ (half of them put $s_i$ ahead of $b_i$, the other half put $b_i$ ahead of $s_i$). The $2t$ many linear strict orders introduced for $i$ in the first two groups all put $s_i$ ahead of $b_i$.

Now we consider the linear strict orders introduced for the sets $S_j$. If $i \in S_j$, then $s_i$ is ahead of $b_i$ here, this happens 3 times. In the remaining $\frac{3}{4}n - 3$ linear strict orders introduced for the sets $S_j$, we have that $i \notin S_j$ and hence in these linear strict orders, $b_i$ is ahead of $s_i$. In $a^* > S > B$ the alternative $s_i$ is voted ahead of $b_i$. Together we have that $b_i$ beats $s_i$ with

$$-2t - 3 + \frac{3}{4}n - 3 - 1 = \frac{3}{4}n - 2t - 7$$

votes. Since $t = \frac{1}{4}n - 2$, it follows that $\frac{1}{4}n - 2t - 7 = \frac{1}{4}n - 3$ as required.

In particular, it follows that by deleting at most $\frac{1}{4}n$ voters, the only relevant relationships that can be influenced are those between $b_i$ and $s_i$ (we will see that the relationships between $b_i$ and $b_j$ or $s_i$ and $s_j$ for $i \neq j$ are not relevant).

We now show that the reduction is correct: The instance of X4C is positive iff $a^*$ can be made a winner of the election using the Condorcet criterion with uncovered set by deleting at most $\frac{1}{4}n$ linear strict orders.

First, assume that the instance is positive, and let $I$ be a corresponding index set. We delete the $\frac{1}{4}n$ linear strict orders corresponding to the elements in $I$ and denote the resulting election with $E'$. Then $a^*$ indeed is uncovered in $E'$. To show this, it suffices to prove that none of the $b_i$ covers $a^*$, since $a^*$ wins against all of the $s_i$ (since $a^*$ leads against $s_i$ with at least $\frac{1}{4}n + 1$ linear strict orders, this remains true also after deleting at most $\frac{1}{4}n$ linear strict orders). Hence, assume that some $b_i$ covers $a^*$ in $E'$. It suffices to prove that $s_i$ dominates $b_i$ in $E'$, then, since $a^*$ dominates $s_i$ in $E'$, it follows that $b_i$ does not cover $a^*$. Note that in the original election $E$ the alternative $b_i$ beats $s_i$ with $\frac{1}{4}n - 3$ points. Deleting the $\frac{1}{4}n$ linear strict orders corresponding to $I$ has the following effect:

a) For the deleted linear strict orders corresponding to sets $S_j$ with $i \notin S_j$, the alternative $s_i$ gains a point against $b_i$. Since $i$ appears in exactly one of the chosen and $\frac{1}{4}n$ linear strict orders are deleted, this means that $s_i$ gains $\frac{1}{4}n - 1$ points against $b_i$ from these linear strict orders.

b) For the single deleted linear strict order corresponding to a set $S_j$ with $i \in S_j$, the alternative $s_i$ loses a point against $b_i$.

Hence altogether, $s_i$ gains $\frac{1}{4}n - 2$ points against $b_i$ and, thus, now beats $b_i$ with a single point. Therefore, as claimed, $b_i$ does not cover $a^*$.

For the converse direction, assume that it is possible to make $a^*$ a winner of the election by deleting at most $\frac{1}{4}n$ linear strict orders. Again, let $E'$ be the election resulting from $E$ by the deletions. Since the relationship between the
$b_i$’s and $a^*$ cannot be changed by deleting at most $\frac{1}{4}n$ linear strict orders and $b_i$ wins against $a^*$ in the original election $E$, all $b_i$ also win against $a^*$ in $E'$. Since $a^*$ is a winner in $E'$, it follows that for each $b_i$ there must be an alternative dominating $b_i$ who does not dominate $a^*$. Since for $i \neq j$, we know that $b_i$ wins against $s_j$ in the election $E'$, it follows that for all relevant $i$ the alternative $s_i$ wins against $b_i$ in $E'$. Since $b_i$ wins against $s_i$ with $\frac{1}{4}n - 3$ points, it follows that each $s_i$ must gain at least $\frac{1}{4}n - 2$ points against $b_i$ by the removal of linear strict orders. Hence $n(\frac{1}{4}n - 2) = \frac{1}{4}n^2 - 2n$ points need to be gained collectively by all $s_i$ against their corresponding $b_i$. Obviously, only deleting linear strict orders introduced for the sets $S_j$ helps to let $s_i$ gain points against $b_i$. Deleting one of these linear strict orders gains $n - 8$ points (since it hurts for the 4 values of $i$ with $i \in S_j$, and helps the remaining $n - 4$ ones). Hence, by deleting $\frac{1}{4}n$ linear strict orders we can gain at most $\frac{1}{4}n \cdot (n - 8) = \frac{1}{4}n^2 - 2n$ points. Since this is the total number of points that need to be gained, we know that exactly $\frac{1}{4}n$ linear strict orders are deleted to obtain the election $E'$, and each of these linear strict orders is one introduced for a set $S_j$. Now assume that there is some $i$ such that two linear strict orders corresponding to sets $S_{j_1}$ and $S_{j_2}$ are deleted, where $i \in S_{j_1}$ and $i \in S_{j_2}$ and $j_1 \neq j_2$. Then $s_i$ gains a point against $b_i$ in at most $\frac{1}{4}n - 2$ of the deletions and loses in at least 2 of them. Hence, $s_i$ gains at most $\frac{1}{4}n - 4$ points against $b_i$, and this implies that $s_i$ loses against $b_i$ in $E'$, which is a contradiction. Therefore, it follows that each $i$ is contained in at most one of the $S_j$ whose corresponding linear strict order is deleted. Due to cardinality reasons ($\frac{1}{4}n$ linear strict orders corresponding to sets of 4 elements each are deleted), it follows that each $i$ appears in exactly one set. As a consequence, we have obtained a set cover as required. $\square$

6 Conclusion

In this paper, we have demonstrated that the relation-algebraic approach can be used to solve NP-hard problems from Social Choice Theory. In particular, this shows how Computer Algebra tools can be used to obtain practical algorithms for hard problems without relying on domain knowledge for optimizations. Our results support the point of view that proving NP-hardness is not sufficient in order to conclude that a voting system is “safe” from attempts to influence the outcome of an election. In addition to the execution of algorithms, RELView also provides us with visualizations of both the input and output of the algorithms and some further features that support scientific experiments, like step-wise execution, test of properties and generation of random relations. All this makes the approach especially appropriate for prototyping and experimentation, and as such very instructive scientific research as well as for university education.

An interesting open question is whether similar problems from the Social Choice literature, as for example the manipulation problem mentioned in the introduction, can also be solved with RELView or other Computer Algebra tools.
References

1. J.J. Bartholdi, J. Orlin: Single transferable vote resists strategic voting. Soc. Choice and Welf. 8, 341-354 (1991).
2. J.J. Bartholdi, C.A. Tovey, M.A. Trick: How hard is it to control an election? Math. and Comput. Modelling 16, 27-40 (1992).
3. R. Berghammer, F. Neumann: RELVIEW – An OBDD-based Computer Algebra system for relations. In: V.G. Gansha, E.W. Mayr, E. Vorozhtsov (eds.): Computer Algebra in Scientific Computing, LNCS 3718, Springer, 40-51 (2005).
4. R. Berghammer, A. Rusinowska, H. de Swart: Computing tournament solutions using relation algebra and RELVIEW. Europ. J. of Operat. Res. 226, 636-645 (2013).
5. F. Brandt, F. Fischer: Computing the minimal covering set. Mathematical Social Sciences 56, 254-268 (2008).
6. V. Conitzer, T. Sandholm: Nonexistence of voting rules that are usually hard to manipulate. In: Proc. 21. AAAI, AAAI Press, 627–634 (2006).
7. V. Conitzer, T. Sandholm, J. Lang: When are elections with few candidates hard to manipulate? J. ACM 54, Article 14 (2007).
8. J. Duggan: Uncovered sets. Wallis Institute of Political Economy, Working Paper Nr. 63, University of Rochester (2011).
9. P. Faliszewski, E. Hemaspaandra, H. Schnoor: Copeland voting: Ties matter. In: Proceedings of AAMAS-08, 983-990 (2008).
10. P. Faliszewski, E. Hemaspaandra, L. Hemaspaandra, J. Rothe: Llull and Copeland voting computationally resist bribery and constructive control. J. Artif. Intell. Res. 35, 275-341 (2009).
11. A. Gibbard: Manipulation of voting schemes: A general result. Econometrica, 41, 587-601 (1973).
12. E. Hemaspaandra, L. Hemaspaandra, J. Rothe: Exact analysis of dodgson elections: Lewis carroll’s 1876 voting system is complete for parallel access to np. J. ACM 44, 806-825 (1997).
13. E. Hemaspaandra, L. Hemaspaandra, J. Rothe: Anyone but him: The complexity of precluding an alternative. Artif. Intell. 171, 255-285 (2007).
14. J.-F. Laslier: Tournament solutions and majority voting. Springer (1997).
15. B. Leoniuk: ROBDD-based implementation of relational algebra with applications (in German). Dissertation, Universität Kiel (2001).
16. U. Milanese: On the implementation of a ROBDD-based tool for the manipulation and visualization of relations (in German). Dissertation, Universität Kiel (2003).
17. M.A. Satterthwaite: Strategy-proofness and arrow’s conditions: Existence and correspondence theorems for voting procedures and social welfare functions. J. Econom. Theory 10, 187-217 (1975).
18. G. Schmidt, T. Ströhlein: Relations and graphs, Discrete mathematics for computer scientists, Springer (1993).
19. G. Schmidt: Relational mathematics. Cambridge University Press (2010).
20. http://www.informatik.uni-kiel.de/~progsys/relview/