On massive spin-$3/2$ in the Fradkin–Vasiliev formalism

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Abstract

One of the possible approaches to the construction of massive higher spin interactions is to use their gauge invariant description based on the introduction of the appropriate set of Stueckelberg fields. Recently, the general properties of such approach were investigated by Boulanger et al (2018 J. High Energy Phys. JHEP07(2018)021). The main findings of this work can be formulated in two statements. At first, there always exist enough field redefinitions to bring the vertex into abelian form where there are some corrections to the gauge transformations but the gauge algebra is undeformed. At second, with the further (as a rule higher derivative) field redefinitions one can bring the vertex into trivially gauge invariant form expressed in terms of the gauge invariant objects of the free theory. Our aim in this work is to show (using a simple example) how these general properties are realised in the so-called Fradkin–Vasiliev formalism and to see the effects (if any) that the presence of massless field, and hence of some unbroken gauge symmetries, can produce. As such example we take the gravitational interaction for massive spin-$3/2$ field so we complete the investigation started by Buchbinder et al (2014 Eur. Phys. J. C 74 3153) relaxing all restrictions on the number of derivatives and allowed field redefinitions. We show that in spite of the presence of massless spin-2 field, the first statement is still valid, while there exist two abelian vertices which are not equivalent on-shell to the trivially gauge invariant ones. Moreover, it is one of this abelian vertices that reproduce the minimal interaction for massive spin-$3/2$.

Keywords: massive higher spins, Fradkin–Vasiliev formalism, gravitational interactions, cubic vertices
1. Introduction

Gauge invariance plays a crucial role in the construction of the consistent interaction vertices for massless higher spin $s \geq 1$ fields. At the same time, massive fields in their minimal covariant formulation do not have any gauge symmetry so that one has to look for other approaches. But the gauge invariant description for massive fields, where gauge invariance is achieved with the introduction of the appropriate set of Stueckelberg fields, also exists both for the metric-like [3, 4] as well as frame-like [5–7] formalism. Thus we can try to follow the same line as for the massless fields though the procedure appears to be quite different in many aspects. Recently, the general properties of such approach were investigated [1] using metric-like formalism with some concrete examples as illustrations. Let us briefly formulate the main finding of this work in the language we adopt in this paper. First statement (or first part of the theorem if one likes) is that there always exist enough field redefinitions to bring the vertex into the abelian form. (We call the vertex to be trivially gauge invariant if it does not change initial gauge transformations; abelian if it changes the gauge transformations but not the algebra; non-abelian if it changes both the gauge transformations and the algebra.) The second statement (part) is that with the further (as a rule higher derivative) field redefinitions one can bring the vertex into trivially gauge invariant form expressed in terms of gauge invariant objects of the free theory. Let us stress here that all these three forms (non-abelian, abelian and trivially gauge invariant) are equivalent on-shell.

Our aim in this work is twofold. First of all, using the most simple (after Yang–Mills theory) example to show how these general properties are realised in the so-called Fradkin–Vasiliev approach [8–10] based on the frame-like formalism. The second point is to see the effects (if any) that the presence of massless field, and hence some unbroken gauge symmetries, can produce. As such example we take the gravitational interaction for massive spin-3/2 field. This case has been already considered in [2] with the implicit restrictions on the number of derivatives and allowed field redefinitions. (Note that there were other applications of such formalism where explicitly or implicitly restrictions on the number of derivatives and allowed field redefinitions were imposed [11–13].) Here we complete this investigation relaxing any such constraints.

Let us recall the main points of the Fradkin–Vasiliev formalism and compare its application to massless and massive cases. Massless fields in the frame-like formalism are described by a set of one-forms $\Phi$ (physical, auxiliary and extra ones) each one playing the role of the gauge transformations

$$\delta \Phi \sim D\xi + \cdots$$

For each field a gauge invariant two-form (curvature) can be constructed

$$\mathcal{R} \sim D\Phi + \cdots$$

On-shell most of these curvatures vanish so that all auxiliary and extra fields are equivalent to the higher derivatives of the physical one. At last, in $AdS$ space with the non-zero cosmological constant the free Lagrangian can be written in an explicitly gauge invariant form

$$\mathcal{L}_0 \sim \sum \mathcal{R}\mathcal{R}.$$
To non-linearly deform the free theory one considers the most general quadratic deformation for all the curvatures

\[ R \Rightarrow \hat{R} = R + \Delta R, \quad \Delta R \sim \Phi\Phi. \]

Naturally, these deformed curvatures ceased to be gauge invariant and the main requirement on this step is that they must transform covariantly under all gauge transformations, i.e.

\[ \delta \hat{R} \sim R\xi. \]

This allows one directly read out the corrections to the gauge transformations

\[ \delta_1 \Phi \sim \Phi \xi \]

and severely restricts all parameters of the deformations. Let us stress that having only one-forms we cannot introduce any field redefinitions without spoiling the very nature of the frame-like formalism (e.g. mixing the world and local indices using the inverse frame). Thus the procedure appears to be straightforward and unambiguous (and especially simple in \( d = 4 \), see e.g. [14, 15]). At the second step one replaces the initial curvatures in the free Lagrangian with the deformed ones and requires the Lagrangian to be invariant. As has been shown by Vasiliev [10], to construct the full set of cubic vertices one also has to add all possible abelian terms of the form \( R/R\Phi \) to this deformed Lagrangian.

Now let us turn to the massive case. Frame-like gauge invariant description for the massive fields [5–7] also requires a set of one-forms \( \Phi \) as well as a similar set (there is a one-to-one correspondence) of zero-forms \( W \) playing the role of Stueckelberg fields:

\[ \delta \Phi \sim D\xi + \cdots, \quad \delta W \sim m\xi. \]

Each of them has its own gauge invariant object (two-form or one-form which we collectively call curvatures):

\[ \mathcal{R} \sim D\Phi + \cdots, \quad \mathcal{C} \sim DW - m\Phi + \cdots \]

As in the massless case the free Lagrangian can be written in the explicitly gauge invariant form

\[ \mathcal{L}_0 \sim \sum [\mathcal{R} R + \mathcal{R} C + \mathcal{C} C]. \]

Note however that in the massive case this is possible not only in \( AdS \) space but also in Minkowski space and even in \( dS \) space inside the unitary allowed region. The allowed range of the parameters \((m^2, \Lambda)\) appeared already at the construction of the initial gauge invariant Lagrangians in terms of the physical and auxiliary fields [5, 7]. The fact that for any allowed values the Lagrangian can be rewritten in terms of the gauge invariant curvatures has been shown by direct calculations in [7].

Now let us turn to the interaction. The first step is again the most general quadratic deformation for all curvatures:

\[ \Delta \mathcal{R} \sim \Phi\Phi + \Phi W + W W, \]
\[ \Delta \mathcal{C} \sim \Phi W + W W, \]

with the corresponding corrections to the gauge transformations:

\[ \delta_1 \Phi \sim \Phi \xi + W\xi, \quad \delta_1 W \sim W\xi, \]
and with the requirement that the deformed curvatures transform covariantly. Due to the presence of zero-forms one can construct a lot of possible field redefinitions

\[ \Phi \Rightarrow \Phi + \Phi W + W \Phi, \quad W \Rightarrow W + W W, \]

so that the procedure drastically differs form that in the massless case.

The second step is the same as before: one has to replace the initial curvatures in the free Lagrangian with the deformed ones

\[ L_0 \Rightarrow \hat{L} \sim \sum [\hat{R} \hat{R} + \hat{R} \hat{C} + \hat{C} \hat{C}], \]

add all possible abelian terms and require the total Lagrangian to be invariant.

The paper is organized as follows. In section 2 we provide all necessary information on the description of massless spin-2 and massive spin-3/2 in AdS_4 background. In sections 3 and 4 we consider the most general quadratic deformations for spin-3/2 and spin-2 curvatures correspondingly. In both cases we show that there exist enough field redefinitions to reduce the vertex to the abelian form so that the first part of the theorem in [1] works even with the presence of massless spin-2. In section 5 we consider the most general abelian vertex and show that two of them are independent in a sense that they are not on-shell equivalent to the trivially gauge invariant ones contrary to the purely massive case. Moreover, it is one of these vertices that gives the minimal gravitational interactions for massive spin-3/2. At last, in section 6 using the field redefinitions we bring the vertex back to its non-abelian incarnation.

2. Kinematics

In this section we briefly recall all necessary information on the description of massless spin-2 and massive spin-3/2 fields living in AdS_4 background.

2.1. Massless spin 2

We work in the frame-like multispinor formalism (we use the same notation and conventions as in our previous works [7, 14, 15]), where the massless spin-2 is described by the dynamical frame one-form \( h^{\alpha \dot{\alpha}} \) and dynamical Lorentz connection one-forms \( \omega^{\alpha (2)}, \omega^{\dot{\alpha} (2)} \). The Lagrangian (four-form in our formalism) for the free field in AdS_4 looks like:

\[
\frac{1}{i} L_0 = \omega^{\alpha \beta} E_{\beta}^{\gamma} \omega_{\alpha \gamma} + \omega^{\dot{\alpha} \dot{\beta}} E_{\dot{\beta}}^{\alpha} dh_{\alpha \dot{\alpha}} + 2 \lambda^2 h^{\alpha \dot{\alpha}} E_{\alpha}^{\beta} h_{\beta \dot{\alpha}} + \text{h.c.}
\]

Here \( e^{\alpha \dot{\alpha}} \) is the background frame and \( D \) is the Lorentz covariant derivative of AdS_4 defined so that

\[
De^{\alpha \dot{\alpha}} = 0, \quad DD \zeta^{\alpha} = -2 \lambda^2 E^{\alpha} \zeta^{\beta}
\]

and the two-forms \( E^{(2)}, E^{(2)} \), three-form \( E^{\alpha \dot{\alpha}} \) and four-form \( E \) are defined as follows:

\[
\begin{align*}
& e^{\alpha \dot{\alpha}} e^{\beta \dot{\beta}} = e^{\alpha \beta} E^{\alpha \beta} + e^{\dot{\alpha} \dot{\beta}} E^{\dot{\alpha} \dot{\beta}}, \\
& E^{\alpha \dot{\alpha}} e^{\beta \dot{\beta}} = e^{\alpha \beta} E^{\beta \dot{\alpha}} + e^{\dot{\alpha} \dot{\beta}} E^{\dot{\alpha} \beta}, \\
& E^{\alpha \dot{\alpha}} e^{\beta \dot{\beta}} = e^{\alpha \beta} E^{\beta \dot{\alpha}} + e^{\dot{\alpha} \dot{\beta}} E^{\dot{\alpha} \beta}.
\end{align*}
\]
This Lagrangian is invariant under the following gauge transformations:

\[
\delta \omega^{\alpha(2)} = D \eta^{\alpha(2)} + \lambda^{2} e^{\alpha}_{\beta} \epsilon^{\beta \gamma}, \\
\delta \omega^{\dot{\alpha}(2)} = D \eta^{\dot{\alpha}(2)} + \lambda^{2} e_{\beta}^{\dot{\alpha}} \epsilon^{\beta \dot{\gamma}}, \\
\delta h^{\alpha \dot{\alpha}} = D \xi^{\alpha \dot{\alpha}} + e^{\alpha}_{\beta} \eta^{\beta \dot{\gamma}} + e^{\alpha}_{\dot{\beta}} \eta^{\dot{\beta} \beta}. 
\] (4)

There are two gauge invariant two-forms (curvature and torsion, but in what follows all such gauge invariant two- and one-forms we collectively call curvatures):

\[
R^{\alpha(2)} = D \omega^{\alpha(2)} + \lambda^{2} e^{\alpha}_{\alpha} h^{\alpha \dot{\alpha}}, \\
R^{\dot{\alpha}(2)} = D \omega^{\dot{\alpha}(2)} + \lambda^{2} e^{\dot{\alpha}}_{\beta} h^{\beta \dot{\gamma}}, \\
T^{\alpha \dot{\alpha}} = D h^{\alpha \dot{\alpha}} + e^{\alpha}_{\beta} \omega^{\beta \dot{\gamma}} + e^{\alpha}_{\dot{\beta}} \omega^{\dot{\beta} \beta}. 
\] (5)

They satisfy the following differential identities:

\[
DR^{\alpha(2)} = -\lambda^{2} e^{\alpha}_{\alpha} T^{\alpha \dot{\alpha}}, \\
DR^{\dot{\alpha}(2)} = -\lambda^{2} e^{\dot{\alpha}}_{\beta} T^{\beta \dot{\gamma}}, \\
DT^{\alpha \dot{\alpha}} = -e^{\dot{\alpha}}_{\beta} R^{\alpha \beta} - e^{\alpha}_{\beta} R^{\beta \dot{\alpha}}. 
\] (6)

On-shell, i.e. on zero torsion \( T^{\alpha \dot{\alpha}} \approx 0 \), we obtain:

\[
DR^{\alpha(2)} \approx 0, \quad DR^{\dot{\alpha}(2)} \approx 0, \quad e^{\dot{\alpha}}_{\beta} R^{\alpha \beta} + e^{\alpha}_{\beta} R^{\beta \dot{\alpha}} \approx 0. 
\] (7)

Using these curvatures the free Lagrangian can be rewritten in the explicitly gauge invariant form:

\[
\mathcal{L}_0 = \frac{i}{2\lambda^2} [R_{\alpha(2)} R^{\alpha(2)} - R_{\dot{\alpha}(2)} R^{\dot{\alpha}(2)}]. 
\] (8)

### 2.2. Massive spin 3/2

The frame-like gauge invariant description uses the one-forms \( \Phi^{\alpha}, \Phi^{\dot{\alpha}} \) and zero-forms \( \phi^{\alpha}, \phi^{\dot{\alpha}} \).

Free Lagrangian in \( AdS_4 \):

\[
\mathcal{L}_0 = -\Phi^{\alpha} e^{\alpha}_{\beta} D \Phi^{\beta} - \phi^{\alpha} E^{\alpha}_{\beta} D \phi^{\beta} \\
- M \Phi^{\alpha} E^{\alpha}_{\dot{\beta}} + c_0 \Phi^{\alpha} E^{\alpha}_{\beta} \phi^{\beta} + M E \phi^{\alpha} \phi^{\beta} + \text{h.c.}, 
\] (9)

where

\[
M^2 = m^2 + \lambda^2, \quad c_0^2 = 6m^2. 
\] (10)

This Lagrangian is invariant under the following gauge transformations:

\[
\delta \Phi^{\alpha} = D \xi^{\alpha} + M e^{\alpha}_{\dot{\alpha}} \xi^{\dot{\alpha}}, \quad \delta \Phi^{\dot{\alpha}} = c_0 \zeta^{\dot{\alpha}}. 
\] (11)

Corresponding gauge invariant curvatures (two-form and one-form) can be constructed:
\[ F^\alpha = D \Phi^\alpha + M e^\alpha_\beta \Phi^\beta - \frac{c_0}{3} E^\alpha_\beta \Phi^\beta, \]

\[ C^\alpha = D \Phi^\alpha - c_0 \Phi^\alpha + M e^\alpha_\beta \Phi^\beta. \]

They also satisfy their differential identities:

\[ DF^\alpha = -M e^\alpha_\beta F^\beta + \frac{c_0}{3} E^\alpha_\beta C^\beta, \]

\[ DC^\alpha = -c_0 F^\alpha - M e^\alpha_\beta C^\beta. \]

It is important to what follows that on-shell these curvatures can be expressed in terms of the zero-forms [7]:

\[ F^\alpha \approx E^\alpha_\beta Y^{\alpha \beta(2)}, \quad C^\alpha \approx e^\alpha_\beta \dot{Y}^{\alpha \beta}. \]

At last, the free Lagrangian can be rewritten as:

\[ L_0 = c_1 F^\alpha F^\alpha + c_2 F^\alpha e^\alpha_\beta C^\beta + c_3 C^\alpha E^\alpha_\beta C^\beta + h.c., \]

where

\[ c_1 = -\frac{c_1}{3}, \quad 2M c_1 - c_0 c_2 = \frac{1}{2}. \]

The remaining ambiguity is related with an identity (here we use the fact that the Lagrangians are defined up to a total derivative):

\[ 0 = D[F^\alpha C^\alpha] = DF^\alpha C^\alpha + F^\alpha DC^\alpha \]

\[ = -c_0 F^\alpha F^\alpha - 2M F^\alpha e^\alpha_\beta C^\beta + \frac{c_0}{3} C^\alpha E^\alpha_\beta C^\beta. \]

Thus the Lagrangian is defined up to a shift

\[ L_0 \Rightarrow L_0 + \kappa_0 \left[ -c_0 F^\alpha F^\alpha - 2M F^\alpha e^\alpha_\beta C^\beta + \frac{c_0}{3} C^\alpha E^\alpha_\beta C^\beta \right], \]

which corresponds to

\[ c_1 \Rightarrow c_1 - \kappa_0 c_0, \quad c_2 \Rightarrow c_2 - 2M \kappa_0, \quad c_3 \Rightarrow c_3 + \frac{\kappa_0 c_0}{3}. \]

It is easy to see that the equations on the parameters \( c_1, c_2, c_3 \) are invariant under this shift.

### 3. Deformations for spin-3/2

Let us consider quadratic deformations for the spin-3/2 curvatures:

\[ F^\alpha \Rightarrow \tilde{F}^\alpha = F^\alpha + \Delta F^\alpha, \quad C^\alpha \Rightarrow \tilde{C}^\alpha = C^\alpha + \Delta C^\alpha. \]

The most general ansatz appears to be:

\[ \Delta F^\alpha = a_1 \omega^\alpha_\beta \Phi^\beta + a_2 \omega^\alpha_\beta \dot{e}^\beta + a_3 \omega^\beta_\alpha \dot{\Phi}^\alpha + a_4 h^\alpha_\beta \dot{\Phi}^\beta + a_5 \dot{h}^\alpha_\beta \dot{\Phi}^\beta, \]

\[ \Delta C^\alpha = a_6 \omega^\alpha_\beta \dot{\Phi}^\beta + a_7 \dot{h}^\alpha_\beta \dot{\Phi}^\beta. \]
Naturally, these deformed curvatures cease to be gauge invariant and the main requirement of the formalism is that they must transform covariantly under all gauge transformations. Let consider them in turn.

**Supertransformations.** The appropriate corrections can be directly read from the structure of deformations:

\[ \delta_{1} \Phi^\alpha = a_{1} \omega_{\beta} \phi^{\beta} + a_{4} h_{\alpha}^{\alpha} \phi^{\alpha}. \tag{22} \]

Now we calculate the variations for the deformed curvatures and require:

\[ \delta_{1} \hat{F}^{\alpha} = a_{1} R_{\beta}^{\alpha} \phi^{\beta} + a_{4} T_{\alpha}^{\alpha} \phi^{\alpha}, \quad \delta_{1} \hat{C}^{\alpha} = 0, \tag{23} \]

this gives us a number of equations on the free parameters:

\[ a_{2} = a_{3}, \quad a_{5} = a_{6}, \quad Ma_{4} + c_{0}a_{5} = a_{1} \lambda^{2}, \]
\[ Ma_{1} + c_{0}a_{2} = a_{4}, \quad a_{7} = a_{1}, \quad a_{8} = a_{4}. \]

**Lorentz transformations.** In this case the corrections have the form:

\[ \delta_{3} \Phi^\alpha = -a_{1} \eta_{\beta} \phi^{\beta}, \]
\[ \delta_{3} \Phi^{\dot{\alpha}} = a_{2} e_{\alpha}^{\dot{\alpha}} \phi^{\beta}, \]
\[ \delta \phi^{\alpha} = -a_{3} \phi^{\alpha}. \tag{24} \]

For the deformed curvatures to be covariant we must apply:

\[ a_{7} = a_{1}, \quad -a_{8} + c_{0}a_{2} = -Ma_{7}. \tag{25} \]

**Pseudo-translations.** At last, here the corrections look like:

\[ \delta_{8} \Phi^\alpha = -a_{4} \xi_{\alpha}^{\alpha} \phi^{\beta} - a_{5} \xi_{\alpha}^{\alpha} e_{\alpha}^{\dot{\alpha}} \phi^{\dot{\alpha}} + a_{6} e_{\alpha}^{\dot{\alpha}} \phi^{\beta} + a_{7} e_{\alpha}^{\dot{\alpha}} \phi^{\dot{\alpha}}, \]
\[ \delta \phi^{\alpha} = -a_{8} \phi^{\alpha}. \tag{26} \]

and we obtain the last set of equations.

\[ a_{8} = a_{4}, \quad c_{0}a_{5} - \lambda^{2}a_{7} = -Ma_{8}, \quad \lambda^{2}a_{1} - c_{0}a_{2} - Ma_{8} = 0. \tag{27} \]

All the equations obtained appear to be consistent and their general solution has two free parameters. It is not an accident because due to the presence of zero-forms \( \phi^{\alpha}, \phi^{\dot{\alpha}} \) there exists a pair of possible field redefinitions, namely:

\[ \Phi^\alpha \Rightarrow \Phi^{\alpha} + \kappa_{1} \omega_{\beta}^{\alpha} \phi^{\beta} + \kappa_{2} h_{\alpha}^{\alpha} \phi^{\alpha}. \tag{28} \]

It is straightforward to calculate their effect on the curvatures:

\[ \Delta F^{\alpha} = \kappa_{1} D \omega_{\beta}^{\alpha} \phi^{\beta} - \kappa_{1} \omega_{\beta}^{\alpha} D \phi^{\beta} + M \kappa_{1} e_{\beta}^{\alpha} \phi^{\beta} \]
\[ = -\kappa_{1} R_{\beta}^{\alpha} \phi^{\beta} - \kappa_{1} \omega_{\beta}^{\alpha} \phi^{\beta} \]
\[ = -\kappa_{1} c_{0} \omega_{\beta}^{\alpha} \phi^{\beta} + M \kappa_{1} (\omega_{\beta}^{\alpha} e_{\alpha}^{\dot{\alpha}} + e_{\alpha}^{\dot{\alpha}} \omega_{\beta}^{\alpha}) \phi^{\beta} \]
\[ = -\kappa_{1} \lambda^{2} (h_{\alpha}^{\alpha} e_{\alpha}^{\dot{\alpha}} + e_{\alpha}^{\dot{\alpha}} h_{\alpha}^{\alpha}) \phi^{\beta}. \tag{29} \]
\[ \Delta C^\alpha = -\kappa_1 c_0 \omega_\beta^{\alpha} \phi^\beta. \]  

(30)

\[ \Delta F^\alpha = \kappa_2 D^\alpha \phi^\beta - \kappa_3 h^\alpha_{\beta \phi^\beta} + M \kappa_2 e^\alpha_{\beta \phi^\beta} D^\alpha. \]

(31)

\[ \Delta C^\alpha = -\kappa_2 c_0 h^\alpha_{\alpha \phi^\beta}. \]

(32)

Thus these redefinitions shift the deformation parameters:

\[ a_{1,7} \Rightarrow a_{1,7} - \kappa_1 c_0, \]
\[ a_{2,3} \Rightarrow a_{2,3} + M \kappa_1 - \kappa_2, \]
\[ a_{4,8} \Rightarrow a_{4,8} - \kappa_2 c_0, \]
\[ a_{5,6} \Rightarrow a_{5,6} - \kappa_1 \lambda^2 + M \kappa_2, \]

(33)

and also generate a number of abelian corrections:

\[ \Delta F^\alpha = -\kappa_1 R^\alpha_{\beta \phi^\beta} - \kappa_1 \omega_\beta^{\alpha} \phi^\beta + \kappa_2 T^\alpha_{\alpha \phi^\beta} D^\alpha. \]

(34)

and corresponding corrections to the gauge transformations:

\[ \delta_1 \Phi = \kappa_1 \eta^\alpha_{\beta \phi^\beta} - \kappa_2 \xi^\alpha_{\alpha \phi^\beta}. \]

(35)

All the equations on the parameters \( a_{1-8} \) given above are invariant under these shifts and it serves as a non-trivial independent check for our calculations. Moreover, using these redefinitions all parameters \( a_{1-8} \) can be set to zero leaving us with the abelian deformations only in agreement with the general analysis in [1].

### 4. Deformations for spin-2

Now let us turn to the quadratic deformations for massless spin-2. The most general ansatz looks like:

\[ \frac{1}{l} \Delta R^{(2)} = b_1 \Phi^\alpha \Phi^\alpha + b_2 e^\alpha_{\beta \phi^\beta} + b_3 e^\alpha_{\alpha \phi^\beta} \phi^\beta, \]

(36)

\[ \frac{1}{l} \Delta T^{\mu \nu} = b_4 \Phi^\mu \Phi^\nu + b_7 (e^\alpha_{\beta \phi^\beta} + e^\alpha_{\alpha \phi^\beta}) \phi^\beta, \]

(37)
Note that one more possible term \( e^\alpha(\Phi^0 \phi_\beta + \Phi^\beta \phi^\alpha) \) is equivalent to the combination of the terms with parameters \( b_{7,8} \). Let us consider the variations for the deformed curvatures and require that they transform covariantly.

### \( \zeta^\alpha \) - transformations

Here the corrections to supertransformations have the form:

\[
\delta_1 \omega^{(2)} = 2b_1 \Phi^\alpha \zeta^\alpha + b_2 \epsilon^\alpha_\beta \zeta^\alpha \phi^\beta, \\
\delta_1 h^{\alpha \beta} = b_6 \Phi^\alpha \zeta^\alpha - b_7 \epsilon^\alpha_\beta \phi^\beta \zeta^\alpha - b_8 \epsilon^\alpha_\beta \phi^\alpha \zeta^\beta. 
\]

(38)

and the covariance of the deformed curvatures gives a number of equations on the parameters:

\[
\begin{align*}
&c_0 b_3 + \lambda^2 b_6 = 2M b_1 - c_0 b_2, \\
&-2M b_3 - c_0 b_4 + 2\lambda b_8 = 0, \\
&c_0 b_4 - 2\lambda^2 b_7 = -\frac{2b_1 c_0}{3} + 2M b_2, \\
&2b_1 - M b_6 + c_0 b_7 + c_0 b_8 = 0, \\
&2b_1 - M b_6 + c_0 b_7 + c_0 b_8 = 0, \\
&-M b_7 - M b_8 - c_0 b_9 + b_2 - 3b_3 = 0, \\
&-M b_7 - M b_8 + c_0 b_9 - 3b_2 + b_3 = -\frac{c_0 b_6}{3}. 
\end{align*}
\]

\( \zeta^\alpha \) - transformations.

In this case we have:

\[
\begin{align*}
&\delta_1 \omega^{(2)} = -b_3 \epsilon^\alpha_\beta \phi^\beta \zeta^\alpha, \\
&\delta_1 h^{\alpha \beta} = b_6 \Phi^\alpha \zeta^\alpha - b_7 \epsilon^\alpha_\beta \phi^\beta \zeta^\alpha - b_8 \epsilon^\alpha_\beta \phi^\beta \zeta^\beta. 
\end{align*}
\]

(40)

and obtain a couple of additional equations:

\[
\begin{align*}
-2M b_1 + c_0 b_2 + \lambda^2 b_6 &= -c_0 b_3, \\
2M b_2 + 2c_0 b_3 - 2\lambda^2 b_7 + 2\lambda^2 b_8 &= 2M b_3. 
\end{align*}
\]

(41)

As in the previous case, these equations appear to be consistent and their general solution has four arbitrary parameters. Here this is also connected with the existence of four possible field redefinitions:

\[
\begin{align*}
\omega^{(2)} &\Rightarrow \omega^{(2)} + i\rho_1 \Psi^\alpha \phi^\alpha + i\rho_2 \epsilon^\alpha_\beta \phi^\beta \\
\omega^{(2)} &\Rightarrow \omega^{(2)} + i\rho_1 \Psi^\alpha \phi^\alpha + i\rho_2 \epsilon^\alpha_\beta \phi^\beta \\
h^{\alpha \beta} &\Rightarrow h^{\alpha \beta} + i\rho_3 (\Phi^\alpha \phi^\alpha + \Phi^\beta \phi^\beta) + i\rho_4 \epsilon^\alpha_\beta (\phi^\alpha \phi^\beta + \phi^\beta \phi^\alpha). 
\end{align*}
\]

(42)

Their effects on the curvatures:

\[
\begin{align*}
\frac{1}{i\rho_1} \Delta R^{(2)} &= D \Phi^\alpha \phi^\alpha - \Phi^\alpha D \phi^\alpha \\
&= F^\alpha \phi^\alpha - \Phi^\alpha C^\alpha - c_0 \Phi^\alpha \Phi^\alpha \\
&- M \epsilon^\alpha_\beta [\Phi^\alpha \phi^\alpha + \Phi^\beta \phi^\beta] + \frac{c_0}{9} E^{(2)} \phi^\alpha \phi^\beta. 
\end{align*}
\]

(43)
\[
\frac{1}{i\rho_1} \Delta R^{\alpha \beta} = e^{\alpha \beta} (\Phi^\alpha \Phi^\beta + \Phi^\beta \Phi^\alpha) + \rho_1 e^{\alpha \beta} (\Phi^\alpha \Phi^\beta + \Phi^\beta \Phi^\alpha),
\]
(44)

\[
\frac{1}{i\rho_2} \Delta T^{\alpha \beta} = -e^{\alpha \beta} D\Phi^\alpha \Phi^\beta - e^{\alpha \beta} \Phi^\alpha C^\alpha - c_0 e^{\alpha \beta} (\Phi^\alpha \Phi^\beta - \Phi^\beta \Phi^\alpha)
\]
(45)

\[
\frac{1}{i\rho_2} \Delta T^{\alpha \beta} = -4E^{\alpha \beta} \Phi^\alpha \Phi^\beta + 4E^{\alpha \beta} \Phi^\alpha \Phi^\beta,
\]
(46)

\[
\frac{1}{i\rho_3} \Delta R^{\alpha \beta} = \lambda^2 e^{\alpha \beta} (\Phi^\alpha \Phi^\beta + \Phi^\beta \Phi^\alpha),
\]
(47)

\[
\frac{1}{i\rho_3} \Delta T^{\alpha \beta} = D\Phi^\alpha \Phi^\beta - D\Phi^\alpha \Phi^\beta - \Phi^\alpha D\Phi^\beta
\]
(48)

\[
\frac{1}{i\rho_4} \Delta R^{\alpha \beta} = 4\lambda^2 E^{\alpha \beta} (\Phi^\alpha \Phi^\beta + \Phi^\beta \Phi^\alpha),
\]
(49)

\[
\frac{1}{i\rho_4} \Delta T^{\alpha \beta} = -2e^{\alpha \beta} (D\Phi^\beta \Phi^\alpha + D\Phi^\alpha \Phi^\beta)
\]
(50)

Thus they also produce the shifts of the deformation parameters:

\[
\begin{align*}
    b_1 &\Rightarrow b_1 - \rho_1 c_0, \\
    b_2 &\Rightarrow b_2 - \rho_1 M - \rho_2 c_0 + \rho_3 \lambda^2, \\
    b_3 &\Rightarrow b_3 - \rho_1 M + \rho_2 c_0 + \rho_3 \lambda^2, \\
    b_4 &\Rightarrow b_4 + \frac{\rho_1 c_0}{3} - 2\rho_2 M + 4\rho_4 \lambda^2, \\
    b_5 &\Rightarrow b_5 + 2\rho_2 M + 4\rho_4 \lambda^2, \\
    b_6 &\Rightarrow b_6 - 2\rho_4 c_0, \\
    b_7 &\Rightarrow b_7 + \rho_1 - \rho_3 M + 2\rho_4 c_0, \\
    b_8 &\Rightarrow b_8 + \rho_1 - \rho_3 M - 2\rho_4 c_0, \\
    b_9 &\Rightarrow b_9 - 4\rho_2 + \frac{\rho_1 c_0}{3}
\end{align*}
\]
(51)

and generate a number of abelian deformations:

\[
\Delta R^{\alpha \beta} = i\rho_1 (F^{\alpha \beta} - \Phi^\alpha C^\beta) - i\rho_2 e^{\alpha \beta} (C^\alpha \Phi^\beta + \Phi^\beta C^\alpha),
\]
\[
\Delta T^{\alpha} = i\rho_3 (F^{\alpha} - \Phi^\alpha C^\alpha) - i\rho_4 e^{\alpha} C^\alpha \Phi + \text{h.c.},
\]
(52)
as well as the corresponding corrections to the gauge transformations:

\[ \delta_1 \omega^{\alpha(2)} = i \rho_1 \xi^{\alpha} C^\alpha, \quad \delta_1 h^{\alpha \dot{\alpha}} = i \rho_3 \zeta^{\alpha} C^\dot{\alpha} + \text{h.c.} \] (53)

We have explicitly checked that all equations on the parameters \( b_{1-9} \) are invariant under these shifts. Moreover, using these redefinitions one can set all deformation parameters \( b_{1-9} \) to zero, once again in agreement with the general analysis in [1].

5. Abelian vertices

As we have seen in the previous two subsections, using all field redefinitions we can reformulate the cubic interactions as a combination of the abelian and/or trivially gauge invariant terms. Note first of all, that there exists just one trivially gauge invariant vertex that does not vanish on-shell:

\[ L_\ell \sim R_\alpha \beta C^\alpha C^\beta + \text{h.c.} \] (54)

Formally, it contains terms with up to four derivatives, but up to the total derivative it is equivalent to the particular combination of the abelian ones, as can be seen from

\[
L_\ell = [D \omega_{\alpha \beta} - \lambda^2 (e_\alpha^{\dot{\alpha}} h_{\beta \dot{\alpha}} + e_\beta^{\dot{\alpha}} h_{\alpha \dot{\alpha}})] C^\alpha C^\beta + \text{h.c.} \\
= 2 c_{0} F_\alpha C_{\beta} e^{\alpha \beta} + 2 \lambda^2 C_\alpha C_\beta e^{\alpha \beta} + 2 \lambda^3 C_\alpha C_\beta e^{\alpha \beta} h^{\beta \dot{\alpha}} + \text{h.c.} \] (55)

Recall that we call abelian vertices those constructed out of two gauge invariant curvatures and one explicit field. Thus in our case we have two types of abelian vertices, namely those with the massive spin\(^3/2\) components \( \Phi^\alpha \) or \( \phi^\alpha \) and those with \( h^{\alpha \dot{\alpha}} \) or \( \omega^{\alpha(2)} \). As for the first ones, it is easy to show that any gauge invariant combinations of them is on-shell equivalent to the trivially gauge invariant one given above, once again in agreement with [1]. But for the vertices with the massless spin-2 the situation appears to be different. The most general ansatz for such vertices is:

\[ L_a = d_1 F_\alpha C_{\beta} e^{\alpha \beta} + d_2 C_\alpha C_\beta e^{\alpha \beta} + d_3 F_\alpha C_\beta h^{\alpha \dot{\alpha}} + d_4 C_\alpha C_\beta e^{\alpha \beta} h^{\beta \dot{\alpha}} + \text{h.c.} \] (56)

If we require that these terms be gauge invariant, then we find that all \( \eta^{\alpha(2)}\)-variations vanish on-shell (14), while for the \( \epsilon^{\alpha \beta} \)-transformations we obtain

\[ \delta_\epsilon L_a \approx [\lambda^2 d_1 - M d_3 - c_0 d_4] F_\alpha C_{\beta} e^{\alpha \beta} e^{\beta \dot{\alpha}}. \] (57)

Thus we have only one equation on four parameters:

\[ \lambda^2 d_1 - M d_3 - c_0 d_4 = 0 \] (58)

and this leaves us with the three possible solutions. One of them as it was shown above is equivalent to the trivially gauge invariant one, but two other ones are independent and have to be considered separately. The most simple way to determine which combination reproduces the minimal vertex is to consider the so-called unitary gauge \( \phi^\alpha = 0 \). The reason is that all field redefinitions are necessarily contain zero-forms \( \phi^\alpha \) and do not change this part of the vertex. Using explicit expressions for the curvatures we obtain:

\[
L_a = c_0 d_1 (D \Phi_\alpha - M e_\alpha^{\dot{\alpha}} \Phi_\dot{\alpha}) \Phi_\alpha e^{\alpha \beta} + c_0^2 d_2 F_\alpha \Phi_\beta e^{\alpha \beta} h^{\beta \dot{\alpha}} + c_0^2 d_3 F_\alpha \Phi_\beta e^{\alpha \beta} h^{\beta \dot{\alpha}} + c_0^2 d_4 F_\alpha \Phi_\beta e^{\alpha \beta} h^{\beta \dot{\alpha}}. \] (59)
It appears that the minimal vertex having no more than one derivative corresponds to the choice:
\[ d_1 = 0, \quad d_3 = c_0 d_2, \quad d_4 = -M d_2. \] (60)

As a result we obtain
\[ \frac{1}{c_0^2} \mathcal{L}_a = d_2 [\Phi_\alpha \Phi_\beta \epsilon_\gamma \omega_\alpha^\beta - D \Phi_\alpha \Phi_\beta h^\alpha_\beta - 2M \Phi_\alpha \Phi_\beta \epsilon_\gamma^\alpha \beta h^\beta_\gamma + \text{h.c.}] \] (61)

and this indeed agrees with the minimal substitution rule for the corresponding part of the free Lagrangian. We have explicitly checked that all the terms with two derivatives (beyond the unitary gauge) vanish on-shell.

6. Comeback

Thus we managed to reproduce the minimal gravitational vertex for massive spin-$3/2$ as a purely abelian one. Abelian vertices are easy to construct and simple to deal with, but such a form drastically differs from what we used to working with the massless fields where the most important part (and in $d = 4$ very often the only one) is non-abelian one. So it seems instructive to bring the result in the form close to that of massless field, in particular to see can the massive theory be interpreted as the spontaneously broken massless one as the presence of the Stueckelberg fields suggests.

As the first step we use the field redefinition to restore the non-abelian part of the curvature deformations. We chose
\[ \Delta F^\alpha = a_1 [\omega_\alpha^\beta \Phi^\beta + \kappa_0 \omega_\alpha^\beta \epsilon_\gamma^\beta \phi^\alpha + \kappa_0 \epsilon_\alpha^\alpha \omega_\beta^\gamma \phi^\beta] + \lambda a_1 [h^\alpha_\alpha \Phi_\alpha h^\alpha_\beta \phi^\beta] + \lambda a_1 [h^\alpha_\alpha \phi^\alpha], \] (62)
\[ \Delta C^\alpha = a_1 \omega_\alpha^\beta \phi^\beta + \lambda a_1 h^\alpha_\alpha \phi^\alpha, \] (63)

for the spin-$3/2$ curvatures, where
\[ \kappa_0 = \frac{M - \lambda}{c_0}, \] (64)

while for the spin-2 ones:
\[ \Delta R^{(2)} = b_1 [\Phi^\alpha \Phi_\alpha - 2\kappa_0 \epsilon_\alpha^\alpha \Phi_\alpha \phi^\alpha - 2\kappa_0^2 E^{(2)} \Phi_\alpha \phi^\alpha], \] (65)
\[ \Delta T^{(2)} = b_6 \left[ \Phi^\alpha \Phi_\alpha - \kappa_0 (\epsilon_\alpha^\alpha \Phi_\alpha \phi^\alpha + \epsilon_\alpha^\gamma \Phi_\alpha \phi^\gamma) - \kappa_0^2 (E^{(2)} \phi^\alpha \phi^\alpha - E^{(2)} \phi^\alpha \phi^\alpha) \right], \] (66)

where
\[ 2b_1 = \lambda b_6 = -2c_1 a_1 \lambda^2. \] (67)

The reason for these particular choice will become clear in a moment.

As our second step let us introduce new variable:
\[ \bar{\phi}^\alpha = \Phi^\alpha + \kappa_0 \epsilon_\alpha^\alpha \phi^\alpha, \] (68)
so that it transforms exactly as the massless spin-3/2 field:

$$\delta \tilde{\Phi}^\alpha = D_{\zeta}^\alpha + \lambda e_\dot{\alpha}^\alpha \zeta^\dot{\alpha}. \quad (69)$$

In turn, for the gauge invariant curvatures we obtain:

$$\tilde{F}_\alpha = F_\alpha - \kappa_0 e_\dot{\alpha}^\alpha C^\dot{\alpha} = D_{\tilde{\Phi}}^\alpha + \lambda e_\dot{\alpha}^\alpha \tilde{\Phi}^\dot{\alpha},$$

$$C^\alpha = D_{\Phi}^\alpha - c_0 \Phi^\alpha + \lambda e_\dot{\alpha}^\alpha \Phi^\dot{\alpha}. \quad (70)$$

Now we at last fix our choice for the form of the free Lagrangian written in terms of curvatures. We choose

$$c_2 = -2c_1 \kappa_0 \Rightarrow c_1 = \frac{1}{4\lambda}, \quad (71)$$

then the free Lagrangian takes the form:

$$L_0 = \frac{1}{4\lambda} \tilde{F}_\alpha \tilde{F}^\alpha - \frac{1}{6(M + \lambda)} C_\alpha E^\alpha_\beta C^\beta + h.c., \quad (72)$$

so that the first term coincides with the usual Lagrangian for massless spin-3/2 field in AdS_4, while the second one contains mass terms dressed with the Stueckelberg fields.

In these new variables the quadratic deformations take the form

$$\Delta \tilde{F}_\alpha = a_1 \omega_\alpha^\beta \tilde{\Phi}^\beta + \lambda h_\alpha^\beta \tilde{\Phi}^\beta,$$

$$\Delta C_\alpha = a_1 \omega_\alpha^\beta \Phi^\beta + \lambda h_\alpha^\beta \Phi^\beta, \quad (73)$$

$$\Delta R^{(2)} = -\frac{a_1}{4} \tilde{\Phi}^\alpha \tilde{\Phi}^\alpha,$$

$$\Delta T^{(3)} = -\frac{a_1}{2} \tilde{\Phi}^\alpha \tilde{\Phi}^\alpha \quad (74)$$

while the interacting Lagrangian looks like

$$L = \frac{i}{2\lambda^2} \tilde{R}^{(2)} \hat{R}^{(2)} + \frac{1}{4\lambda} \tilde{F}_\alpha \tilde{F}^\alpha$$

$$- \frac{1}{6(M + \lambda)} \hat{C}_\alpha E^\alpha_\beta \hat{C}^\beta + \frac{a_1}{12(M + \lambda)} C_\alpha C_\beta e_\dot{\alpha}^\alpha h^\dot{\beta} + h.c., \quad (75)$$

where the first two terms coincide with corresponding results for the massless supergravity. Note that we still have a combination of the non-abelian deformation and one abelian vertex.

### 7. Conclusion

In this work we applied the so-called Fradkin–Vasiliev formalism to the system of massless spin-2 and massive spin-3/2 fields. Initially, this formalism was developed for the construction of interactions for massless higher spins, but using the frame-like gauge invariant description for massive ones, it can be straightforwardly extended to any system of massless and/or massive fields. In massless case, the interaction vertices appears as a result of most general quadratic deformation for all gauge invariant curvatures in combination with all possible abelian (or Chern–Simons like) terms. But in the massive case [1] due to the presence of Stueckelberg fields there always exist enough field redefinition to bring the vertex into purely abelian form. We have shown here that this statement remains valid in the presence of massless spin-2 (with
its unbroken gauge symmetries) as well. At the same time, our investigation of the abelian vertices show that there exist two such vertices which are not equivalent on-shell to any trivially gauge invariant ones contrary to the case where only massive fields are present. Moreover, it is one of these vertices that reproduces standard minimal gravitational cubic vertex for massive spin-$3/2$.

Recall, that the results on the cubic interactions are universal in a sense that for any three spins they do not depend on the presence or absence of any other fields in the system. The dependence appears when one tries to proceed beyond the cubic level. In this, without introduction of any new fields one necessarily comes to the model with non-linearly realized supersymmetry of Volkov–Akulov type. Another possibility—to make goldstino a member of vector or chiral supermultiplet (in the most general case the goldstino is a linear combination of the spinors from the vector and chiral supermultiplets). This leads to the models with linearly realized spontaneously broken supersymmetry. Note that these two possibilities are the same as in the spontaneous symmetry breaking in the Yang–Mills theory, see e.g. discussion in [16]. We leave to the future work the extension of our investigations to other higher spins fields, first of all to massive (bi)gravity and supergravity along the lines of [13, 17].

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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