ABSTRACT. We define a weak notion of universality in symbolic dynamics and, by generalizing a proof of Mike Hochman, we prove that this yields necessary conditions on the forbidden patterns defining a universal subshift: These forbidden patterns are necessarily in a Medvedev degree greater or equal than the degree of the set of subshifts for which it is universal.

We also show that this necessary condition is optimal by giving constructions of universal subshifts in the same Medvedev degree as the subshifts they simulate and prove that this universality can be achieved by the sofic projective subdynamics of such a subshift as soon as the necessary conditions are verified.

This could be summarized as: There are obstructions for the existence of universal subshifts due to the theory of computability and they are the only ones.

An important problem in topological dynamics is to understand when one system can “simulate” another one: For example, two systems will be essentially the same if they are conjugate, a system will be simpler than another one if the former is a factor of the latter, etc. In symbolic dynamics, most of the one-dimensional cases have been settled by algebraic invariants: An irreducible SFT factors onto another SFT of strictly smaller entropy if and only if the necessary conditions on the periodic points are met [4], and recent results have settled the sofic case [16]. On the other hand, even if we know some sufficient conditions [15], the situation tends to be much more complex in multi-dimensional symbolic dynamics [5].

In theoretical computer science, a fundamental result is the existence of a universal Turing machine [29]: This particular machine enables constructing a good part of the theory of computability [24] and will be used implicitly in this article. While the one-dimensional SFTs are quite simple and well understood [18], multi-dimensional SFTs can exhibit computational properties [2, 23]. It then becomes natural to ask whether there exist multi-dimensional SFTs that can “simulate” every other SFT, that is, as in the Turing machine case, a universal SFT. Unfortunately, Mike Hochman proved that there cannot exist such an SFT [11].

Usually, the notion of simulation between subshifts is the factoring relation, or sofic projective subdynamics [22] if we want more flexibility. In this article, we extend this notion of simulation to any computable function that preserves the subshift structure: Computability between subshift is modelled by recursive operators and preserving the structure is what we define as subshift-preserving. A subshift will then simulate another one when there exists a computable function (a recursive operator) mapping the former onto a subsystem of the latter. Recursive operators allow to define a pre-order between sets of functions (a
configuration of a subshift may be seen as a function) known as the Medvedev reduction and its equivalence classes are the Medvedev degrees [19] (or degrees of difficulty [24]). The idea behind Medvedev degrees is that a set will be simpler than another one if from any element of the latter we can compute an element of the former. For example, if we consider factor maps, a subshift will be simpler than another one if the latter factors onto a subsystem of the former. Based on these relations we define a notion of universality in symbolic dynamics (Definition 1.3): A subshift is universal for a set of subshifts if it can simulate, in that way, a subsystem of every subshift in the class. Being honest, this definition is purely academic and is not what one would consider a natural definition of universality. However, this notion includes most of what one would call universality: A universal subshift could be a subshift that factors onto all the subshifts of a given set, that admits all these subshifts as sofic projective subdynamics or even that admits a rescaling factoring on a rescaling of every subshift in the set such as in [3]. All these notions fall under our definition of universality.

Even with this weak notion of universality, we are able to generalize M. Hochman’s proof [11] to prove that the Medvedev degree of the class of simulated subshift cannot be greater than the difficulty of defining the universal subshift (Theorems 2.2 and 2.3). We prove that these conditions are optimal by constructing universal subshifts of the smallest possible Medvedev degree: If we allow any computable function as a simulation then we can always construct a one-dimensional universal subshift (Theorem 3.3). If we restrict the simulation to sofic projective subdynamics then the simulation cannot increase the dimension and we are able to prove that universality can be achieved by sofic projective subdynamics as soon as the necessary conditions are met (Theorem 3.5). If we only allow factor maps then entropy has to be bounded since it cannot increase with factor maps but then we do not know if, additionally, bounding the entropy is sufficient for the existence of such a universal subshift.

1. Definitions, notations and codings

1.1. Elements of symbolic dynamics. Let $\Sigma$ be a finite set, called the alphabet. For an integer $d$, $\Sigma^{\mathbb{Z}^d}$ is the set of configurations of dimension $d$ over $\Sigma$, i.e., the functions from $\mathbb{Z}^d$ to $\Sigma$, called the fullshift of dimension $d$ over $\Sigma$. If $x \in \Sigma^{\mathbb{Z}^d}$ is a configuration, for $i \in \mathbb{Z}^d$, we denote by $x_i$ the value of $x$ at $i$. We endow $\Sigma$ with the discrete topology and $\Sigma^{\mathbb{Z}^d}$ with its product which turns it into a compact metrizable space. A pattern $P$ is a mapping from a finite square subset $D_P = [-n;n]^d$ of $\mathbb{Z}^d$ to $\Sigma$. A pattern $P$ is said to be supported over $D$ if $D_P \subseteq D$. For $D \subseteq \mathbb{Z}^d$, $x_D$ denotes the restriction of $x : \mathbb{Z}^d \to \Sigma$ to $x_D : D \to \Sigma$. The shift of vector $i$, for $i \in \mathbb{Z}^d$, is defined by: $\sigma_i(x)_j = x_{i+j}$, for every $j \in \mathbb{Z}^d$. A subset $X$ of $\Sigma^{\mathbb{Z}^d}$ is said to be shift-invariant if for all $i \in \mathbb{Z}^d$ and all $x \in X$, $\sigma_i(x)$ belongs to $X$. A subshift is a closed and shift-invariant subset of $\Sigma^{\mathbb{Z}^d}$.

A configuration $c \in \Sigma^{\mathbb{Z}^d}$ is said to contain a pattern $P$ if there exists $i \in \mathbb{Z}^d$ such that $\sigma_i(c)|_{D_P} = P$. Given a set of patterns $\mathcal{F}$, $X_{\mathcal{F}}$ is the subshift defined by forbidding $\mathcal{F}$: The set of configurations that contain no pattern of $\mathcal{F}$. Such $X_{\mathcal{F}}$’s are closed and shift-invariant; for any closed and shift-invariant subset of $\Sigma^{\mathbb{Z}^d}$ there exists such an $\mathcal{F}$ defining it [10]. For a subshift $X$, if there exists a recursively enumerable $\mathcal{F}$ such that $X = X_{\mathcal{F}}$ then $X$ is an effective subshift: if $\mathcal{F}$ is finite then $X$ is a subshift of finite type (SFT in short).

An onto function $\varphi : X \to Y$ between two subshifts is called a factor map if it is continuous and shift commuting (i.e., $\varphi \circ \sigma_i = \sigma_i \circ \varphi$ for all $i \in \mathbb{Z}^d$). By Hedlund’s theorem [10], factor maps correspond to sliding block codes: For $\varphi : X \to Y$ where $X \subseteq \Sigma^{\mathbb{Z}^d}$ and $Y \subseteq \Gamma^{\mathbb{Z}^d}$ there exists a finite subset $D$ of $\mathbb{Z}^d$ and a function $\Phi : \Sigma^D \to \Gamma$. 


such that for any \( x \in X \), \( \varphi(x)_i = \Phi(\sigma_i(x)_{|D}) \); conversely, a sliding block code is a factor map between its domain and its image.

A subgroup \( G \) of \( \mathbb{Z}^d \) is isomorphic to some \( \mathbb{Z}^k \) for \( k \leq d \). Denote this isomorphism \( h : \mathbb{Z}^k \to G \). Considering a subshift \( X \) over \( \Sigma^\mathbb{Z} \), we can define its projective subdynamics over \( G \) and see it as a subshift of \( \Sigma^\mathbb{Z} \) by \( H \) defined by \( H(x)_i = x_{h(i)} \). Given, in addition, a factor map \( \varphi \) with domain \( X \), we can define the \( (h, \varphi) \) sofic projective subdynamics of \( X \) by \( X^{(h, \varphi)} = H \circ \varphi(X) \) [12, 22]. Remark that \( X^{(h, \varphi)} \) can have greater entropy than \( X \) even if \( G \) is a subgroup of finite index of \( \mathbb{Z}^d \), and in this case the operation is very similar to the bulking used for simulations between cellular automata [6, 7, 3]. The case where \( G \) is a subgroup of finite index of \( \mathbb{Z}^d \), meaning the projective subdynamics do not decrease the dimension, is maybe the most natural notion of sofic projective subdynamics in the context of universality since it avoids the entropy constraints while remaining close to a factor map.

When \( G \) is a subgroup of finite index of \( \mathbb{Z}^d \) we say that sofic projective subdynamics over \( G \) are finite index sofic projective subdynamics.

1.2. **Mass problems and subshifts.** In this paper, we want to study the Medvedev degrees [19, 24] of subshifts and the links that can be obtained between universality and the relations among those degrees. A mass problem is a collection of total functions from \( \mathbb{N} \) to \( \mathbb{N} \), i.e., a subset of \( \mathbb{N}^\mathbb{N} \). We can assume, without loss of generality, that subshifts are defined on an alphabet that is a subset of \( \mathbb{N} \). The collection of all configurations of any dimension can then be effectively coded and decoded as elements of \( \mathbb{N}^\mathbb{N} \): For an integer \( d \), let \( B_d \) be a (fixed) computable bijection from \( \mathbb{N} \) onto \( \mathbb{Z}^d \); \( c \in \Sigma^{\mathbb{Z}^d} \) is coded by \( w^c \in \mathbb{N}^\mathbb{N} \) such that:

\[
\begin{align*}
    w_0^c &= |\Sigma|, \quad w_1^c = d, \quad w_{n+2}^c = c_{B_d(n)}.
\end{align*}
\]

The decoding of \( w \in \mathbb{N}^\mathbb{N} \) is done with the same idea:

Let \( \Sigma = \{0, \ldots, w_0 - 1\} \), \( d = w_1 \), and then decode \( w \in \mathbb{N}^\mathbb{N} \) as \( c^w \in \Sigma^{\mathbb{Z}^d} \) such that:

\[
    c_i^w = w_{B_d^{-1}(i)+2} \mod |\Sigma|.
\]

Denote by \( \Theta \) the function that codes \( c \) as \( w^c \in \mathbb{N}^\mathbb{N} \) and \( \Theta^{-1} \) the function that decodes \( w \in \mathbb{N}^\mathbb{N} \) as \( c^w \). For any \( \Sigma \) and any \( d, \Theta \) is a bijection between \( \Sigma^{\mathbb{Z}^d} \) and \( \mathbb{N}^\mathbb{N} \) and we denote by \( \Theta^{-1} \) its inverse. As such, any subshift can (effectively) be seen as a mass problem via the \( \Theta \) function.

Since coding and decoding between \( \Sigma^{\mathbb{Z}^d} \) and \( \mathbb{N}^\mathbb{N} \) should really be seen as a technical detail, we say that a mass problem \( \mathcal{C} \) is a subshift, SFT or effective subshift if \( \Theta^{-1}(\mathcal{C}) \) is a subset of some \( \Sigma^{\mathbb{Z}^d} \) that is, respectively, a subshift, SFT or effective subshift.

1.3. **Recursive operators.** Turing machines [29] are often taken as the standard model of computation. We mainly use the, more abstract, recursive operators defined below. The reader who is not familiar with the concept of Turing machines may understand it as an algorithm in any programming language, or refer to standard textbooks on the subject such as [24, 21, 25]. While (classical) Turing machines deal with finite words on a finite alphabet, with mass problems we have to deal with infinite words over a countable alphabet (elements of \( \mathbb{N}^\mathbb{N} \)), hence we define the analogous of Turing machines in this context:

**Definition 1.1.** A partial function \( \Psi : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N} \) is said to be a recursive operator if there exists an oracle Turing machine \( M_\Psi \) such that for any \( f \in \mathbb{N}^\mathbb{N} \) where \( \Psi \) is defined, for any \( n \in \mathbb{N} \), \( M_\Psi \) given as oracle \( f \) and input \( n \) eventually halts with output \( \Psi(f)_n \in \mathbb{N} \).

\( M_\Psi \) is said to compute \( \Psi \).

Recursive operators are continuous for the natural topology over \( \mathbb{N}^\mathbb{N} \): The Turing machine \( M_\Psi \) computing \( \Psi \), given any input \( n \), halts after a finite number of steps and thus needs only a finite number of values from \( f \); for any integer \( n \), there exists \( k_n \) such that the output of the machine is independent of the values of \( f_i \), for \( i \geq k_n \). Therefore, \( \Psi \) is a continuous function.
Since Turing machines are recursively enumerable, (partial) recursive operators are recursively enumerable too; in the remaining of this article, we fix such a recursive enumeration and denote by $\Psi_n$ the $n^{th}$ recursive operator in this fixed enumeration.

A factor map $\varphi : \Sigma^Z \rightarrow \Pi^Z$ is a computable function since it is a sliding block code [10]; See, e.g., [27, Theorem 1]. $\psi = \Theta \circ \varphi \circ \Theta^{-1} : \Theta(\Sigma^Z) \rightarrow \Theta(\Pi^Z)$ can, therefore, be seen as a partial recursive operator. By the same type of arguments, the sofic projective subdynamics operation $(X \rightarrow X^{(k,\varphi)})$ is a partial recursive operator.

**Definition 1.2.** For a subshift $X$, a recursive operator $\Psi$ is said to be $X$-subshift-preserving if $\Psi$ is defined on the coding by $\Theta$ of $X$ and its image is a subshift.

For example, factor maps, sofic projective subdynamics or the $\leq$ simulation from [3] give subshift-preserving recursive operators when transformed as such. Since recursive operators are functions from $N^N$ to $N^N$ and we will often have to apply them to the coding of subshifts, when there is no ambiguity, for a subshift $X$, we will denote $\Psi_n(X)$ for $\Theta^{-1} \circ \Psi_n \circ \Theta(X)$.

### 1.4. Medvedev degrees

A mass problem $A$ is said to be **Medvedev reducible** to the mass problem $B$ if there exists a recursive operator $\Psi_n$ such that $\Psi_n$ is defined over $B$ and $\Psi_n(B) \subseteq A$. We denote this by $A \leq_n B$. We write $A \leq B$ if there exists $n$ such that $A \leq_n B$. Intuitively, $A \leq B$ means that there exists an automatic procedure such that when given any solution of a problem $B$ finds a solution to the problem $A$. The relation $\leq$ is a pre-order. When quotiented by its induced equivalence relation ($\leq \land \geq$), the set of mass problems becomes the **Medvedev lattice** [19] (also known as the **degrees of difficulty** [24]).

The empty mass problem is in the highest degree of difficulty, denoted by $1$: a mass problem containing a computable function is always in the minimal degree, denoted by $0$, and any mass problem in $0$ contains a computable function. Simpson showed that the sublattice of the $\Pi^0_1$ subsets of $\{0,1\}^N$ is exactly the Medvedev lattice of $2$-dimensional SFTs [27]. Remark that if a subshift $X$ factors onto $Y$ then $Y \leq X$ since factor maps can be seen as recursive operators. The Medvedev degree is therefore a conjugacy invariant.

Medvedev degrees ordered by the relation $\leq$ form a lattice: There is a sup, denoted by $\lor$ and representing the product of the two sets, and an inf denoted by $\land$ and representing the disjoint union of the two sets; these are defined formally by:

\[
A \lor B = \{x \in N^N | \exists a \in A, \exists b \in B, \forall n \in N, x(2n) = a(n), x(2n+1) = b(n)\}
\]

\[
i \lor A = \{x \in N^N | x(0) = i, \exists a \in A, \forall n \in N, x(n+1) = a(n)\}
\]

\[
A \land B = (0 \lor A) \lor (1 \lor B)
\]

Strictly speaking, $(A \lor B) \lor C$ is different from $A \lor (B \lor C)$, but they are in the same Medvedev degree (the same applies for $\land$). Therefore, we can use the $\lor$ and $\land$ operations associatively on Medvedev degrees.

### 1.5. Codings

We want to compare the Medvedev degrees of sets of subshifts and those of the forbidden patterns defining them; therefore we also need to define how to code these sets as mass problems.

We can associate to any subset $N$ of $N$ a canonical mass problem $M_{set}(N) \subseteq N^N$ defined by: $M_{set}(N) = \{x \in N^N | n \in N \iff \exists x_1 \in N, x_1 = n\}$. $N$ is recursively enumerable if and only if $M_{set}(N)$ contains a computable point, thus if and only if $M_{set}(N)$ is in the minimal degree of difficulty $0$. Intuitively, $M_{set}(A) \leq M_{set}(B)$ means that there exists a Turing machine that computes an enumeration of $A$ when given an enumeration of $B$.

Given an alphabet $\Sigma$ and a dimension $d$, one can fix a recursive bijection between $N$ and the set of finite patterns of dimension $d$ over the alphabet $\Sigma$. Therefore we can represent
these forbidden patterns as a $w \in \mathbb{N}^\mathbb{N}$: $w_0 = |\Sigma|$, $w_1 = d$ and for $n \geq 2$, $w_n$ is the code of a forbidden pattern. Effective subshifts can thus be expressed in terms of recursive operators: To an integer $n$ associate the effective subshift $X_n$ whose set of forbidden patterns is coded in that way by $w = \Psi_n(0^{n^\infty}) \in \mathbb{N}^\mathbb{N}$.

A set of effective subshifts $C$ is therefore coded by $C$, a subset of $\mathbb{N}$, and has its associated mass problem $\mathcal{M}_{\text{set}}(C)$. Again, there exists a recursively enumerable set $K \subseteq \mathbb{N}$ such that $C = \{X_n | n \in K\}$ if and only if $\mathcal{M}_{\text{set}}(C)$ is in 0. Since any subshift can be written as the intersection of SFTs, we can also code an arbitrary subshift in the same way: A that $C$ in that way by $w$.

**Proof.** Let $\mathcal{R}$ be a reduction class $\mathcal{R} = \{\Psi_n | n \in \mathcal{R}\}$. Remark that, e.g., the set of all factor maps forms a recursively enumerable reduction class since they correspond to sliding block codes [10] and such codes can be recursively enumerated. The same applies to the set of all sofic projective subdynamics. We can now define what we call universality:

**Definition 1.3.** Let $\mathcal{R}$ be a reduction class, $\mathcal{C}$ and $\mathcal{G}$ sets of effective subshifts and $\mathcal{B}$ a subset of $\mathbb{N}^\mathbb{N}$, we say that a subshift $X$ is $(\mathcal{R}, \mathcal{G}, \mathcal{B})$—universal for $\mathcal{C}$ when:

$$i \in \mathcal{C} \iff \{i \in \mathcal{G} \text{ and } \exists n \in \mathcal{R}, (i, b) \in \mathcal{B}, \Psi_n(X)_{[-b;b]^d} \text{ does not contain any of the first } b \text{ forbidden patterns in } X_i\}$$

Similarly, we say that $X$ is $(\mathcal{R}, \mathcal{G}, \infty)$—universal for $\mathcal{C}$ when the following holds:

$$i \in \mathcal{C} \iff \{i \in \mathcal{G} \text{ and } \exists n \in \mathcal{R}, \Psi_n(X) \subseteq X_i\}$$

The set $\mathcal{B}$ shall be understood as the precision we require from the simulation: The biggest $b$ is, closer the image of $X$ to elements of $X_i$ will be. If $b$ could be chosen infinite then the image of $X$ would not contain any $X_i$-forbidden pattern. This is why, in the second definition, we replaced $\mathcal{B}$ by $\infty$ to emphasize that we require infinite precision, i.e., complete inclusion. In some cases, with sufficiently big $b$’s, we can force the inclusion directly: If $\mathcal{R}$ is the set of all factor maps, $\mathcal{G}$ the set of all SFTs and for any $b_i$ such that $(i, b_i)$ is in $\mathcal{B}$, $b_i$ must be big enough so that any forbidden pattern defining $X_i$ is supported on $[-b_i; b_i]^d$. This can actually be proved for any subshift-preserving recursive operator:

**Lemma 1.4.** Given a set of SFTs $\mathcal{G}$, there exists $\mathcal{B} \subseteq 2^{\mathbb{N}}$ such that $\mathcal{M}_{\text{set}}(\mathcal{B}) \leq \mathcal{M}_{\text{set}}(\mathcal{G})$ and for any subshift $X$ and any reduction class $\mathcal{R}$ of $X$-subshift-preserving recursive operators, for any set of SFTs $\mathcal{C}$, $X$ is $(\mathcal{R}, \mathcal{G}, \mathcal{B})$—universal for $\mathcal{C}$ if and only if $X$ is $(\mathcal{R}, \mathcal{G}, \infty)$—universal for $\mathcal{C}$.

**Proof.** We are given an enumeration of $i \in \mathcal{G}$ and want to compute an enumeration of the set $\mathcal{B}$. Let $b_i$ be a big enough integer such that any forbidden pattern of $X_i$ is supported on $[-b_i; b_i]^d$, where $X_i$ is an SFT of dimension $d$. When $i$ is given, this $b_i$ can be easily computed, so we define: $\mathcal{B} = \{(i, b_i) | i \in \mathcal{G}\}$ and the relation $\mathcal{M}_{\text{set}}(\mathcal{B}) \leq \mathcal{M}_{\text{set}}(\mathcal{G})$ holds.

Suppose $X$ is $(\mathcal{R}, \mathcal{G}, \mathcal{B})$—universal for $\mathcal{C}$. Let $(i, b_i) \in \mathcal{B}$ and $y \in \Psi_n(X)$. We supposed $\Psi_n$ to be $X$-subshift-preserving: For any $v \in \mathbb{Z}^d$, there exists $x_v \in X$ such that $\Psi_n(x_v) = \sigma_v(y)$. Since all the forbidden patterns of $X_i$ are supported on $[-b_i; b_i]^d$, $y$ does not contain
any pattern forbidden in \( X_i \). Since \( y \) is arbitrary, we conclude that \( \Psi_n(X) \subseteq X_i \), thus \( X \) is \((R, G, \infty)\)-universal for \( C \). The other direction is trivial.

This definition of universality is weak: \( \Psi_n(X) \subseteq X_i \) implies that any subshift \( X_{i'} \in G \) such that \( X_i \subseteq X_{i'} \) also verifies \( \Psi_n(X) \subseteq X_{i'} \). The artifact introduced by considering universality relative to a set of subshifts \( G \) is what allows to avoid trivial necessary conditions on the set \( C \) for the existence of an universal subshift. The artifact introduced by the set \( B \) in the definition of \((R, G, B)\)-universal is there to make proofs simpler: in fact, we are interested in subshift-preserving reduction classes and with Lemma 1.4 being \((R, G, B)\)-universal is equivalent to \((R, G, \infty)\)-universal for a carefully chosen \( B \).

A stronger notion of universality would be to ask \( X \) to simulate every point of \( X_i \) by asking \( \Psi_n(X) = X_i \) in the definition of universality, instead of only some of them (what we get with \( \Psi_n(X) \subseteq X_i \)). We will see in Section 2 that even this weak notion of universality imposes some conditions on the universal subshifts. In Section 3 we will show how to construct universal subshifts for this stronger notion of universality \( (\Psi_n(X) = X_i) \) as soon as the necessary conditions developed in Section 2 are met. Therefore, considering this weaker definition of universality only makes our results stronger.

2. Consequences of the Existence of Universal Subshifts

M. Hochman proved that there cannot exist any universal effective subshift (in any dimension) for the set of non-empty SFTs of dimension at least 2 by proving that this would give an algorithm enumerating all the non-empty 2-dimensional SFTs [11]. This would contradict Berger’s theorem [2] stating that it is undecidable, given a finite set of patterns \( \mathcal{F} \), to know if \( X_\mathcal{F} \) is empty in dimension \( d \geq 2 \) because the set of empty SFTs is recursively enumerable [31]. Hochman’s definition of universality is, in our context, considering the reduction class of sofic projective subdynamics. With a very similar proof, we generalize his result to our notion of universality and give some easy consequences that were not obvious from the original result.

As we are dealing with effective continuous functions, we can expect to obtain classical topological results in an effective way; the one that we will use later is that a continuous function over a compact set is uniformly continuous, and that “uniform continuity can be computed”:

Lemma 2.1 (Straightforward extension of [21, Chapter II.3],[30, 20]). There exists a Turing machine with oracle that when given as oracle an element of \( M_{\text{subshift}}(X) \), where \( X \) is a subshift of \( \Sigma^{Z^d} \), and as input \( n \) and \( r \) computes, when \( \Psi_n \) is defined on \( X \), \( l = \mathcal{U}(n, r, X) \) such that for any \( x, y \in X \), we have:

\[
x[[-l:l]^d] = y[[-l:l]^d] \Rightarrow \Psi_n(x)[0;r] = \Psi_n(y)[0;r]
\]

In simpler words: if two configurations on a finite alphabet match on a “sufficiently big” domain around the origin then their images have a long common prefix; and this “sufficiently big” can be computed. Note that this proof is classical, as it follows trivially from the proof of classical results, but we have not been able to find a reference to use for our case. The same proof applies to compact subsets of \( \mathbb{N}^N \) but since we have the formalism for dealing with subshifts, we only prove it in this case.

Proof. Since \( \Psi_n \) is continuous over a compact set \( X \), it is uniformly continuous: \( \forall r \in \mathbb{N}, \exists l_r \in \mathbb{N}, \forall x, y \in X, x[[-l_r,l_r]^d] = y[[-l_r,l_r]^d] \Rightarrow \Psi_n(x)[0;r] = \Psi_n(y)[0;r] \). This \( l_r \), or any larger integer, is the \( \mathcal{U}(n, r, X) \) we are looking for.
For an integer $i$ growing from 0 to $\infty$, for any pattern $P$ of $\Sigma^{2d}$ defined on $[-i; i]^d$, check that $P$ does not contain any of the first $i$ forbidden patterns of $X$ given by the oracle $M_{\text{subshift}}(X)$. If it does not contain such a pattern then we simulate $M_{\Psi_n}$ given as oracle $P$ on every input in $\{0, \ldots, r\}$. In this simulation, we distinguish two cases:

1. If for at least one $P$ and one input, before halting, the machine asks for the value of its oracle $P$ outside of the domain of $P ([−i; i]^d)$ then continue with the next $i$.
2. Otherwise define $U(n, r, X) = i$.

By construction, $U(n, r, X)$ matches the conclusions of our lemma since the machine never reads anything outside of $[−U(n, r, X); U(n, r, X)]^d$ and thus values can be changed outside of this domain without altering the image on its first $r$ values. However, we have to prove that case 2 always happens for a given $n, r$ and $X$ so that $U$ is well defined.

Suppose case 2 does not happen: For any integer $i$, there exists a pattern $P_i$ with domain $[−i; i]^d$ that does not contain any of the first $i$ forbidden patterns of $X$, and an input $p_i \in \{0, \ldots, r\}$ such that $M_{\Psi_n}$ asks for a value of its oracle outside of $[−i; i]^d$ with input $p_i$.

Since the input $p_i$ can take only finitely many different values, this implies: There exists an input $p$ such that for infinitely many $i$'s, there exists a $P_i$ defined over $[−i; i]^d$ such that $M_{\Psi_n}$ asks for a value of its oracle outside of the domain of $P_i$ with input $p$. Let $v_1 \in \mathbb{N}$ be the first value asked by the Turing machine. Since we still have infinitely many patterns $P_i$, infinitely many of them have the same value at $v_1$, thus, without loss of generality we assume that all of the $P_i$'s have the same value at $v_1$. Since the Turing machine is deterministic and has a fixed input, the second value it will ask is the same for every $P_i$: $v_2$. Again, w.l.o.g., we may assume that all the $P_i$'s have the same value at $v_2$. By iterating this process we obtain a configuration $c \in \Sigma^{2d}$ such that the Turing machine asks for the value $v_1$, then $v_2$, etc. when given $c$ as oracle. Since this machine asks for infinitely many values of its oracle, it never halts. However, by its construction, $c$ does not contain any forbidden pattern for $X$, hence $c \in X$ and $\Psi_n$ is not defined on all $X$.

Note that in the proof of Lemma 2.1, if $\Psi_n$ is not defined on all $X$ then the algorithm may not halt, so that $U(n, r, X)$ is only defined for $n$’s such that $\Psi_n$ is defined on $X$. We now have all the tools to generalize M. Hochman’s proof [11]:

**Theorem 2.2.** If a subshift $X$ is $(R, G, B)$–universal for a set of effective subshifts $C$ then:

$$M_{\text{set}}(C) \leq M_{\text{subshift}}(X) \vee M_{\text{set}}(R) \vee M_{\text{set}}(G) \vee M_{\text{set}}(B)$$

**Proof.** We are given as oracle $x \in M_{\text{subshift}}(X) \vee M_{\text{set}}(R) \vee M_{\text{set}}(G) \vee M_{\text{set}}(B) \subseteq \mathbb{N}$; by decoding $x$, we may assume that we are given $\Sigma$ and $d$ such that $X$ is a subshift of $\Sigma^{2d}$ defined by a family of forbidden patterns $(F_i)_{i \in \mathbb{N}}$, that we can enumerate given $x$.

To prove our theorem, we need to prove that we can enumerate all the subshifts of $C$ given such an oracle, which will give us an element of $M_{\text{set}}(C)$. The following algorithm is an adaptation to our context of M. Hochman’s one [11, Section 5]:

- Enumerate all the integers $i \in G$ and decode $k_i, d_i$ and $(F(i))_{n \in \mathbb{N}}$ which are, respectively, the size of the alphabet, the dimension and the forbidden patterns defining $X_i$.
- Enumerate all the integers $n$ in $R$ and all the integers $b$ such that $(i, b) \in B$.
- By hypothesis, $\Psi_n$ is defined on $X$ since it is $X$-subshift-preserving: Compute $l = U(n, b, X)$ from Lemma 2.1.
  - For every $j > l$, compute all the patterns defined on $[−j; j]^d$ that do not contain any of the $F_1, \ldots, F_j$ and denote them by $P_1, \ldots, P_h$.
  - If all the patterns $\Psi_{n}(P_1)[−b;b]^d, \ldots, \Psi_{n}(P_h)[−b;b]^d$ do not contain any of the first $b$ forbidden patterns in $X_i$ then claim that $X_i \in C$: Enumerate it.
By definition, this algorithm enumerates only elements of \( C \). It remains to prove that we actually enumerate the whole set \( C \). Suppose it is not the case: If there exists \( n \in \mathcal{R}, i \in C \), \((i, b) \in B \) such that \( \Psi_n(X)_{[b, b]} \neq X_{[b, b]} \) does not contain any of the \( b \) forbidden patterns in \( X_i \) and the algorithm does not enumerate \( i \): For arbitrary large \( j \), there is a pattern \( P_k \) not containing any of the \( F_1, \ldots, F_j \) defined on \([-j;j]^d \) such that \( \Psi_n(P_k)_{[b, b]} \) contains one of the \( b \) forbidden patterns in \( X_i \). By compactness, there would exist \( x \in X \) such that \( \Psi_n(x)_{[b, b]} \) contains one of the first \( b \) forbidden patterns in \( X_i \).

Combining it with Lemma 1.4, we get a result for \((\mathcal{R}, \mathcal{G}, \infty)\)–universal subshifts:

**Theorem 2.3.** For any set of SFTs \( \mathcal{G} \), any subshift \( X \) and any reduction class of \( X \)-subshift-preserving recursive operators \( \mathcal{R} \), if \( X \) is \((\mathcal{R}, \mathcal{G}, \infty)\)-universal for \( C \) then:

\[
\mathcal{M}_{\text{set}}(C) \leq \mathcal{M}_{\text{subshift}}(X) \lor \mathcal{M}_{\text{set}}(\mathcal{R}) \lor \mathcal{M}_{\text{set}}(\mathcal{G})
\]

**Proof.** Find \( B \) from Lemma 1.4; \( \mathcal{M}_{\text{set}}(B) \leq \mathcal{M}_{\text{set}}(\mathcal{G}) \) and if \( X \) is \((\mathcal{R}, \mathcal{G}, B)\)-universal for \( C \) then \( X \) is \((\mathcal{R}, \mathcal{G}, \infty)\)-universal for \( C \). From Theorem 2.2: \( \mathcal{M}_{\text{set}}(C) \leq \mathcal{M}_{\text{subshift}}(X) \lor \mathcal{M}_{\text{set}}(\mathcal{R}) \lor \mathcal{M}_{\text{set}}(\mathcal{G}) \) and if \( \mathcal{M}_{\text{set}}(B) \leq \mathcal{M}_{\text{set}}(\mathcal{G}) \) we get our result. \( \square \)

In our formalism, M. Hochman considers \( \mathcal{R} \) to be a recursively enumerable set (the set of all sofic projective subdynamics maps) and \( \mathcal{G} \) to be the set of all (possibly empty) SFTs, which is also recursively enumerable, the theorem can thus be reformulated as \( \mathcal{M}_{\text{set}}(C) \leq \mathcal{M}_{\text{subshift}}(X) \) with these conditions and obtain Hochman’s result [11]. By considering \( \mathcal{R} \) to be the set of recursive operators corresponding to the \( \preceq \) relation from [3] we can also recover [3, Theorem 1].

Note that the condition on \( \mathcal{R} \) is important: Consider an SFT \( X \) that is maximal in the lattice of Medvedev degrees of 2–dimensional SFTs which we know exists by Simpson’s work [27]. We also know that for any non-empty SFT \( S \) there exists a recursive operator \( \Psi \) such that \( \Psi(X) = S \) [26, Lemma 3.16, point 1]. If we take \( \mathcal{R} \) to be the reduction class of all such \( \Psi \)’s (they all are \( X \)-subshift-preserving by definition), \( \mathcal{G} \) to be the set of all SFTs, we get \( \mathcal{M}_{\text{subshift}}(X) \lor \mathcal{M}_{\text{set}}(\mathcal{G}) \) \( \in 0 \) but the set \( C \) of non-empty SFTs is not in 0 so that \( \mathcal{M}_{\text{set}}(\mathcal{R}) \notin 0 \) by Theorem 2.3.

**Corollary 1.** Given any (non-empty) SFT \( S \), there exists no effective subshift that factors (nor projects by sofic projective subdynamics) onto all the non-empty SFTs of entropy equal to the entropy of \( S \) in dimension greater than one.

**Proof.** By [13, Corollary 3.3], let \( M \) be a Turing machine computing a right approximation of the entropy of \( S \): \( M \) enumerates a sequence of rational numbers \( (h_n)_{n \in \mathbb{N}} \) such that \( \lim_{n \to \infty} h_n = h(S) \) and \( \forall n \in \mathbb{N}, h_n \geq h_{n+1} \geq h(S) \).

Let \( M_n \) be the Turing machine executing \( M \) one step out of two and the \( n \)th Turing machine the other step. \( M_n \) halts whenever one of the two machines halts, that is, \( M_n \) halts if and only if the \( n \)th Turing machine halts. With this Turing machine \( M_n \), construct an SFT \( S_n \) as in the proof of [13, Theorem 1.1], with a slight modification: If \( M_n \) halts then we force a state that is incompatible with every other state so that \( S_n \) is empty. Thus, by [13, Theorem 1.1], \( S_n \) is empty if and only if \( M_n \) halts and \( h(S_n) = h(S) \) otherwise.

Let \( \mathcal{G} \) be the mass problem associated to the set \( \{\mathcal{S}_n\}_{n \in \mathbb{N}} \). Let \( \mathcal{R} \) be the recursively enumerable reduction class of all factor maps. If we suppose that there exists an effective subshift factoring on all the SFTs of entropy equal to the entropy of \( S \), by Theorem 2.2 we get that the set \( \{\mathcal{S}_n|\mathcal{S}_n \neq \emptyset\} \) is recursively enumerable since \( \mathcal{M}_{\text{subshift}}(X), \mathcal{M}_{\text{set}}(\mathcal{G}) \) and \( \mathcal{M}_{\text{set}}(\mathcal{R}) \) all belong to the minimal degree of difficulty 0. This would mean that the set of Turing machines that do not halt is recursively enumerable, which is known not to be. \( \square \)

**Corollary 2.** No effective subshift can factor onto all the zero-entropy SFTs.
Proof. Apply Corollary 1 with $S$ being the SFT consisting of only one uniform configuration (this SFT has zero-entropy). □

We also note that an extended version of [9] (personal communication) should prove that minimal non-empty SFTs are not recursively enumerable. Together with Theorem 2.2, this would answer the question of M. Hochman whether or not there exists universal effective subshifts for the set of non-empty minimal SFTs [11]: Such universal subshifts would not exist.

3. Construction of Universal Subshifts

After proving necessary conditions for the existence of universal subshifts, we construct universal subshifts that achieve optimal conditions in the sense of Theorem 2.2.

3.1. Toeplitz sequences as foundations. The main ingredient of all the following results is summarized by Theorem 3.1: There exists an effective way to code independently an infinity of configurations into a single one and the decoding is effective too.

Theorem 3.1. There exists a finite alphabet $\Sigma$ and an effective 1−dimensional subshift $X \subseteq \Sigma^\mathbb{Z}$ such that each configuration of $X$ contains $\omega$ independent layers, in the following sense:

1. There exists a recursively enumerable set of factor maps $(\varphi_n : \Sigma^\mathbb{Z} \to (\{0, 1\}^n)^\mathbb{Z})_{n \in \mathbb{N}}$ and a computable sequence of integers $(m_n)_{n \in \mathbb{N}}$ defining finite index sofic projective subdynamics maps $L_n : \Sigma^\mathbb{Z} \to (\{0, 1\}^n)^\mathbb{Z}$ such that $L_n(c)_i = \varphi_n(c)_{m_n,i}$.

2. Each $L_n$ is a surjection from $X$ onto $(\{0, 1\}^n)^\mathbb{Z}$

3. There exists an algorithm such that when given as input a (possibly infinite) list of forbidden patterns $F$, defining a 1−dimensional subshift $X_F$, the algorithm computes an integer $n_i$ and enumerates an infinite list of forbidden patterns $A(F)$ defining a 1−dimensional subshift $X_A(F)$ such that:

   $L_{n_i}(X \cap X_A(F)) = X_F$

   and, for $j \neq n_i$, if $X_A(F) \neq \emptyset$:

   $L_j(X \cap X_A(F)) = L_j(X)$

The intuition behind Theorem 3.1 is as follows: $X$ forces each of its configurations to contain an infinity of independent layers; we use $m_n$ cells in $X$ to code $n$ bits in the $n^{th}$ layer. $L_n$ is the decoding function: It reads the $n^{th}$ layer, gets the $n$ bits value and decodes it to get a configuration over $(\{0, 1\}^n)^\mathbb{Z}$. Point 3 states that we can forbid coded patterns in any layer independently of the other layers.

Proof. Our proof is in four parts: First we describe how to code $\omega$ independent layers and then we show how to code required information for our theorem, how to decode it and finally why the layers are independent.

Layers: Since each configuration contains an infinite number of cells, we use half of our cells to code the first layer, then half of what is remaining for the second layer, etc. In fact, the base construction is a Toeplitz sequence constructed as in [28, 14] in which each step of the construction is used to obtain a new independent layer.

Fix an integer $k$ greater than 2. A meta-cell is a group of $k$ cells (the shaded cells in Figure 1) bordered by left brackets (■) on their left and right brackets (■) on their right. To the right of a left bracket (■) there are exactly $k$ coding cells (■ and then a right bracket (■). To the
right of a right bracket (■) there are exactly \( k + 2 \) blank cells (□) and then a left bracket (■). For simplicity we use \( k + 2 \) here because it is the size of a meta-cell. This gives us a set of forbidden words for coding the first layer as depicted in Figure 1.

![Figure 1. First layer (with \( k = 4 \)).](image)

Now we will fill the blank cells (□) in order to code the second layer. First, remark that if we group the cells by arrays of \( k + 2 \) cells we see that half the cells are occupied by the first layer and the other half is left free as depicted on Figure 2.

![Figure 2. Occupied and free cells after the first layer (here one cell represents a block of \( k + 2 \) real cells).](image)

Now, we see the discrete line as divided in blocks of size \( 2(k + 2) \), each of these blocks having \( k + 2 \) blank cells. For the second layer, we use meta-cells consisting of \( k + 2 \) identical real cells. Those meta-cells will fill the cells left blank by the first layer. After that, this is basically the same construction as for the first layer. A \{left bracket, right bracket, coding cell, blank cell\} for the second layer consists of \( k + 2 \) "real" \{left brackets, right brackets, coding cells, blank cells\}. The same rules as for the cells of the first layer apply to these meta-cells. The global picture of a configuration with the first and second layers is depicted in Figure 3.

![Figure 3.](image)

Note that adding the second layer does not affect the first one if we modify a bit our rules: A left bracket of the first layer is the only one with a coding cell at its right and a right bracket has a coding cell on its left; hence we can still force a given number of blank and coding cells around the first layer brackets by considering the cells it expects to be blank as blank.

The second layer, as the first one, occupies blocks of \( k + 2 \) meta-cells and leaves \( k + 2 \) meta-blank cells between them. By dividing again the line in blocks of \( 2(k + 2) \) meta-cells, each block contains \( k + 2 \) meta-blank cell; that is, each block consists of \( (2(k + 2))^2 \) real cells and contains \( (k + 2)^2 \) real blank cells. The third layer is the same as the second one except we use meta-meta-cells of meta-cells, and so on for each layer.

We described a procedure to produce the forbidden patterns defining a subshift, hence the obtained subshift is effective. As a conclusion of this part of the proof, we have:

- \( \omega \) layers in each configuration of this effective subshift;
- cells belonging to a given layer can be recognized locally: The window needed depends only on the layer of cells we are looking at;
- each meta-coding cell of the \( n^{th} \) layer is an array of \( k(k + 2)^{n-1} \) real coding cells.
**Coding:** The real coding cells (the ■ cells in Figure 1), in fact, are two: ■₀ and ■₁ and thus code one bit of information. Remark that a meta-coding cell from the nᵗʰ layer contains \( k(k+2)^{n-1} \) real coding cells. The leftmost cell of meta-coding cells in the nᵗʰ layer are separated by \( 2^n(k+2)^n \) cells, we therefore define \( m_n = 2^n(k+2)^n \) in point 1.

**Decoding:** In this paragraph, we define the factor maps \( \varphi_n \) for proving our point 1. We define \( \varphi_n \) as a block map; \( \varphi_n \) has a radius of \( m_n/2 \). \( \varphi_n \), when applied to a configuration of our subshift, at every cell, “sees” at least one entire meta-coding cell from the nᵗʰ layer: choose, e.g., the leftmost one. A meta-coding cell from the nᵗʰ layer codes \( k(k+2)^{n-1} \) bits of information. We want \( \varphi_n \) to take its values in \( \{0, 1\}^n \) in point 1; since \( k > 2, 
\]

\( k(k+2)^{n-1} > n \), and there exists a surjection from \( \{0, 1\}^{k(k+2)^{n-1}} \) onto \( \{0, 1\}^n \), which will be the value of \( \varphi_n \) at this position. Note that we lose a lot of information with this factor map but it does not matter since \( n \) can be taken as large as we want.

The construction of \( \varphi_n \) and \( m_n \) is effective, which proves point 1. Since, with these \( \varphi_n \) and \( m_n \), \( L_n \) is a surjection from \( X \) onto \( \{0, 1\}^n \), this proves point 2.

**Independence:** Given an enumeration \( F \) of patterns defining a 1−dimensional subshift, defined on an alphabet of size smaller than \( 2^n \), for a given integer \( n \), it is clear that this alphabet can be sent into \( \{0, 1\}^n \) and decoded back. \( n \) or any greater integer, is thus the \( n_i \) we need for point 3. Given an enumeration of \( F \), we can effectively transform these patterns so that they correspond to their codings on the nᵗʰ layer; these transformed patterns are \( A(F) \). Since \( L_n \) depends only on the value of the configuration on its nᵗʰ layer, this completes the proof of point 3 and of the whole theorem.

We can reformulate Point 1 as follows: \( m_n \mathbb{Z} \) is a subgroup of \( \mathbb{Z} \) and define \( h_n(i) = m_ni : \mathbb{Z} \to m_n\mathbb{Z}. \) \( L_n(X) \) is simply \( X^{(h_n, \varphi_n)} \), the \( (h_n, \varphi_n) \) finite index sofic projective subdynamics of \( X \). Also, from the definition of the \( L_n \)'s, these define a recursively enumerable reduction class \( R = \{r_n| n \in \mathbb{N}\} \subseteq \mathbb{N} \) of sofic projective subdynamics maps.

### 3.2. Universality with subshift-preserving recursive operators

We start by an almost straightforward application of Theorem 3.1 to construct a one-dimensional universal subshift with sofic projective subdynamics as soon as the necessary conditions on the dimension and the computability conditions imposed by Theorem 2.2 are fulfilled.

**Theorem 3.2.** For any set of non-empty 1−dimensional effective subshifts \( C \), there exists a 1−dimensional subshift \( X \) such that \( \mathcal{M}_{\text{subshift}}(X) \leq \mathcal{M}_{\text{set}}(C) \) and there exists a recursively
enumerate reduction class \( \mathcal{R} \) such that \( C = \{ \Psi_n(X) | n \in \mathcal{R} \} \) and all the \( \Psi_n \)'s are (finite index) sofic projective subdynamics maps.

By Theorem 3.1, we know how to code an infinite number of configurations in any configuration of an effective subshift and how to keep the coding and decoding effective. The idea of this proof is that every element of \( X \) must in some sense contain an element of any subshift of \( C \), that is where the \( \omega \) layers will play their key role.

**Proof.** First, let us fix some notations: \( C \) codes a set of non-empty effective subshifts \( C: C = \{ c_i | i \in \mathbb{N} \} \) \( (C = \{ X_c | c \in C \}) \), \( X_c \) is a subshift of \( \Sigma \) defined by the (recursively enumerable) set of forbidden patterns \( F_i \). Since, by definition, \( \Sigma_i \) and \( F_i \) can be recursively enumerated from \( c_i \), Theorem 3.1 is almost everything we need. The sofic projective subdynamics and the recursively enumerable reduction class \( \mathcal{R} \) are given by point 1 in Theorem 3.1.

We enumerate the forbidden patterns defining the subshift \( X \), knowing the \( c_i \)'s as follows: Enumerate the patterns defining the effective subshift \( X \) from Theorem 3.1. Assume that we have coded up to \( c_{i-1} \) and filled up to the \( (n-1)^{th} \) layer of our subshift. We are trying to code \( c_i \) into the \( n^{th} \) layer. If \( n \) bits are not enough to code \( \Sigma_i \) then code \( c_{i-1} \) in the \( n^{th} \) layer and continue with the next layer. This does not work for \( c_1 \) but this is only a detail: We may use a special code telling that this layer codes nothing and still continue with the next layer, or, alternatively, choose the initial length \( k \) of the first layer in Theorem 3.1 to be big enough to contain it.

Now assume now that \( n \) bits are enough to code \( \Sigma_i \). Enumerate the patterns of \( F_i \) and transform them as in point 3 of Theorem 3.1 to get \( A_i(F_i) \). All these forbidden patterns are enumerated in parallel and define a subshift \( X \), which is non-empty since we assumed \( C \) to contain only non-empty subshifts. It is clear that \( \mathcal{M}_{\text{subshift}}(X) \leq \mathcal{M}_{\text{set}}(C) \) and point 3 of Theorem 3.1 ensures us that \( C = \{ X_c | c \in C \} = \{ L_n(X) | n \in \mathbb{N} \} \).

We now construct universal subshifts where we do not impose any restriction on the reduction class, besides being subshift-preserving recursive operators. This construction allows us to get a general result about universality, proving that in the context of subshift-preserving recursive operators, dimension does not matter: As long as the necessary conditions from Theorem 2.2 are fulfilled, there exists a 1-dimensional universal subshift! This shall be understood as a converse to Theorem 2.2.

**Theorem 3.3.** There exists a recursively enumerable reduction class \( \mathcal{R} \) such that for any set of non-empty effective subshifts \( C \), there exists a 1-dimensional subshift \( X \) such that \( \mathcal{R} \) is \( X \)-subshift-preserving, \( \mathcal{M}_{\text{subshift}}(X) \leq \mathcal{M}_{\text{set}}(C) \) and \( C = \{ \Psi_n(X) | n \in \mathcal{R} \} \).

**Proof.** There exists a recursive injection of \( \bigcup_{d \in \mathbb{N}, \Sigma \text{ finite}} \Sigma^{Z^d} \) into \( \{0, 1 \}^N \). With the construction of Theorem 3.1, there exists an effective 1-dimensional subshift \( K \) where \( \{0, 1 \}^N \) can be recursively injected into \( K \); given \( x \in \{0, 1 \}^N \), code \( x_i \) as a uniform configuration in the \( i^{th} \) layer of the construction. Therefore, there exists a recursive operator \( \Psi \) such that for any effective subshift \( K \) of \( \Sigma^{Z^d} \) (where \( \Sigma \) and \( d \) are not fixed a priori), we can construct an effective 1-dimensional subshift \( K \) such that \( \Psi(K_S) = S \). \( \Psi \) is therefore \( K \)-subshift-preserving. There exists an algorithm transforming the forbidden patterns defining \( S \) into those defining \( K \). Denote by \( \varphi \) the computable function from \( \mathbb{N} \) to \( \mathbb{N} \) such that for all \( n \), \( \varphi(n) = K_{X_S} \). Applying Theorem 3.2 to \( \varphi(C) \) allows us to conclude since we can get back to \( C \) via \( \Psi \) which is \( K \)-subshift-preserving for all \( K \).

If we consider a recursively enumerable set of non-empty effective subshifts, Theorem 3.3 gives an effective universal subshift of dimension 1. We know that effective
subshifts of dimension 1 can be “implemented” by 2—dimensional SFTs [1, 8] in the sense that the 1—dimensional effective subshift is the sofic projective subdynamics of the 2—dimensional SFT. We can therefore obtain the following corollary to Theorem 3.3:

**Corollary 3.** There exists a recursively enumerable reduction class \( R \) such that for any recursively enumerable set of non-empty effective subshifts \( C \) there exists a 2—dimensional SFT \( S \) such that \( R \) is \( S \)-subshift-preserving and \( C = \{ \Psi_n(S) \mid n \in R \} \).

For example, one may take \( C \) to be the set of non-empty 2—dimensional SFTs with a periodic point, of non-empty 1—dimensional SFTs, of SFTs of dimension prime with 42 with a uniform configuration, limit sets of cellular automata, etc.

3.3. **Universality with sofic projective subdynamics.** In Corollary 3, one may want to replace “subshift-preserving recursively enumerable reduction class” by simply “sofic projective subdynamics”: This restriction is usually imposed in order not to hide the “simulation power” of the subshift in the reduction itself. This is, of course, not possible if we allow any dimension since the dimension of the sofic projective subdynamics of a subshift \( X \) is never greater than the dimension of \( X \). Even if we only consider subshifts of dimension 2, the finite index sofic projective subdynamics of an SFT would be a sofic subshift and there exist non-sofic effective subshifts, so that Corollary 3 cannot be true if we do not impose the set \( C \) of simulated subshifts to be sofic. If, moreover, we impose \( C \) to be a set of SFTs (or sofic shifts) then it is possible and has already been done by Lafitte and Weiss [17, Theorem 10]: In our formalism, this theorem states precisely that given any recursively enumerable set of non-empty two-dimensional SFTs, there exists a two-dimensional SFT that admits all these SFTs as finite index sofic projective subdynamics. Because of Theorem 2.2, this result is optimal since there is no hope to be universal for a set of subshifts that is not recursively enumerable and the sofic projective subdynamics of a non-empty subshift is necessarily non-empty.

In this section, we give a converse to Theorem 2.2 with sofic projective subdynamics and, in some sense, extend the aforementioned result of Lafitte and Weiss to any set of non-empty effective subshifts: As soon as the set of simulated effective subshifts is of bounded dimension \( d \), and the computability conditions of Theorem 2.2 are fulfilled, there exists a universal subshift of dimension \( d \) for this set of effective subshifts.

**Theorem 3.4.** For any set of non-empty \( d \)—dimensional effective subshifts \( C \) there exists a \( d \)—dimensional subshift \( X \) such that any subshift of \( C \) is realized as the finite index sofic projective subdynamics of \( X \) and \( M_{\text{subshift}}(X) \leq M_{\text{set}}(C) \).

**Proof.** We extend the construction from Theorem 3.1 to any dimension by extending the layered construction and the way we code the effective subshifts.

**Extending the layered construction:** Put the layered construction from Theorem 3.1 onto one axis of \( \mathbb{Z}^d \). On all the other axes, impose that the cells have a constant type: A \( \square \) cell must be next to a \( \square \) cell, etc. For example, in dimension 2, we would obtain configurations like the one depicted in Figure 4.

**Extending the coding and decoding of the subshifts:** The subgroup of \( \mathbb{Z}^d \) we consider is \( m_n \mathbb{Z} \times \mathbb{Z}^{d-1} \) where \( m_n \) is from Point 1 in Theorem 3.1 and the sofic projective subdynamics consists in applying \( L_n \) independently on every row of the configuration. The forbidden patterns in the universal subshift are now those that are mapped by this transformation to a forbidden pattern in \( X_i \in C \) when trying to code \( X_i \) on the \( n^{th} \) layer. With these adaptations, the same proof as for Theorem 3.2 works.

We are now able to construct universal subshifts for sets of subshifts that all have the same dimension; since we can easily go down in dimension, we state the result here:
Theorem 3.5. For any set $C$ of non-empty effective subshifts of bounded dimension $d$, there exists $X$, a subshift of dimension $d$, such that any subshift of $C$ is realized as the projective subdynamics of $X$ and $\mathcal{M}_{\text{subshift}}(X) \leq \mathcal{M}_{\text{set}}(C)$.

Proof. The coding bits in the construction of Theorem 3.1 take their values in $\{0, 1\}$. Since the dimension is bounded and known we can make them take their values in $\{0, 1\} \times \{1, \ldots, d\}$; the second part will be called the coded dimension. Coding bits that belong to the same layer must have the same coded dimension. If the coded dimension on a given layer is $k$, then the coding bits must be equal in this layer on the $d - k + 1$ last axes of $\mathbb{Z}^d$. That way, we encode a $k$-dimensional subshift in a $d$-dimensional subshift in the first $k$ dimensions of $\mathbb{Z}^d$ and can obtain the wanted projective subdynamics by taking a subgroup of $\mathbb{Z}^d$ that “forgets” about the useless dimensions: $m_n \mathbb{Z} \times \mathbb{Z}^{k-1} \times \{0\}^{d-k}$. $\square$

4. Conclusions

In this paper we proved three main results: First, the Medvedev degree of the forbidden patterns defining a subshift cannot be lower than the degree of the subshifts it simulates (Theorems 2.2 and 2.3). Second, that this necessary condition is optimal by giving a general construction for obtaining a universal subshift given a set of effective subshifts to simulate (Theorem 3.3 is the strongest result in this sense). Finally, by extending our construction of universal subshifts, we showed that the simulation used can be restricted to sofic projective subdynamics by obtaining the equivalent of Theorem 3.3 with Theorem 3.5 (the most general result of section 3.3). Theorem 3.5 is again optimal as the conditions on the dimension are imposed by the nature of sofic projective subdynamics and the conditions on Medvedev degree are imposed by Theorem 2.2.

One may want to continue the weakening of the reduction class by replacing “sofic projective subdynamics” with “factor” in Theorem 3.4. Such a general statement cannot be true because the entropy of a factor of a subshift is never greater that the entropy of the
subshift itself. Also, it seems difficult to use our construction from Theorem 3.1 since the ratio of coding cells divided by the distance between them tends to zero as we consider increasing layers. Nevertheless, we can ask if bounding the entropy is again sufficient for the existence of a universal subshift:

**Problem 1.** Given a set of non-empty $d$-dimensional effective subshifts $C$ of entropy bounded by $h$, does there exist a $d$-dimensional subshift $X$ such that any subshift of $C$ is a factor of $X$ and $M_{\text{subshift}}(X) \leq M_{\text{set}}(C)$? Such that $X$ factors onto a subsystem of every subshift of $C$? Can $X$ be of entropy $h$?

**REFERENCES**

1. Nathalie Aubrun and Mathieu Sablik, *Simulation of effective subshifts by two-dimensional subshifts of finite type*, to appear in Acta Applicandae Mathematicae (2012).
2. Robert Berger, *The undecidability of the domino problem*, Ph.D. thesis, Harvard University, July 1964.
3. L. Boyer and G. Theyssier, *On Factor Universality in Symbolic Spaces*, Mathematical Foundations of Computer Science 2010 - Lecture Notes in Computer Science 6281 (2010), 209–220.
4. Mike Boyle, *Lower entropy factors of sofic systems*, Ergodic theory and dynamical systems 4 (1984), 541–557.
5. Mike Boyle, Ronnie Pavlov, and Michael Schraudner, *Multidimensional sofic shifts without separation and their factors*, Transactions of the American Mathematical Society 362 (2010), no. 9, 4617–4653.
6. M. Delorme, J. Mazoyer, N. Ollinger, and G. Theyssier, *Bulking I: An abstract theory of bulking*, Theoretical Computer Science 412 (2011), no. 30, 3866 – 3880.
7. Marianne Delorme, Jacques Mazoyer, Nicolas Ollinger, and Guillaume Theyssier, *Bulking II: Classifications of cellular automata*, Theoretical Computer Science 412 (2011), no. 30, 3881–3905.
8. Bruno Durand, Andrei Romashchenko, and Alexander Shen, *Effective Closed Subshifts in 1D Can Be Implemented in 2D*, Fields of Logic and Computation, Lecture Notes in Computer Science, no. 6300, Springer, 2010, pp. 208–226.
9. Pierre Guillou and Charalampos Zinoviadis, *Hierarchy and expansiveness*, Accepted contribution at Computability in Europe (CiE) (2013).
10. Gustav Arnold Hedlund, *Endomorphisms and automorphisms of the shift dynamical systems*, Mathematical Systems Theory 3 (1969), no. 4, 320–375.
11. Michael Hochman, *A note on universality in multidimensional symbolic dynamics*, Discrete and Continuous Dynamical Systems S 2 (2009), no. 2, 301–314.
12. ________, *On the dynamics and recursive properties of multidimensional symbolic systems*, Inventiones Mathematica 176 (2009), no. 1, 131–167.
13. Michael Hochman and Tom Meyerovitch, *A Characterization of the Entropies of Multidimensional Shifts of Finite Type*, Annals of Mathematics 171 (2010), no. 3, 2011–2038.
14. Konrad Jacobs and Michael Keane, *0-1-sequences of toeplitz type*, Probability Theory and Related Fields 13 (1969), 123–131, 10.1007/BF00537017.
15. Aimee Johnson and Kathleen Madden, *Factoring higher-dimensional shifts of finite type onto the full shift*, Ergodic Theory and Dynamical Systems 25 (2005), no. 3, 811–822.
16. Wolfgang Krieger, *On images of sofic systems*, arXiv math.DS:1101.1750 (2011).
17. Grégory Lafitte and Michael Weiss, *Tilings: Simulation and universality*, Mathematical Structures in Computer Science 20 (2010), no. 5, 813–850.
18. Douglas A. Lind and Brian Marcus, *Tilings: Simulation and universality*, Mathematical Structures in Computer Science 20 (2010), no. 5, 813–850.
19. Yu. T. Medvedev, *Degrees of difficulty of the mass problem*, Doklady Akademii Nauk SSSR 104 (1955), 501–504.
20. A. Nerode, *General topology and partial recursive functionals*, Talks Cornell Summ. Inst. Symb. Log. (1957), 247–251.
21. Piergiorgio Odifreddi, *Classical recursion theory*, ”Studies in Logic and the Foundations of Mathematics”, vol. 125, Elsevier, 1989.
22. Ronnie Pavlov and Michael Schraudner, *Classification of sofic projective subdynamics of multidimensional shifts of finite type*, Submitted (2012).
23. R. M. Robinson, *Undecidability and nonperiodicity for tilings of the plane*, Inventiones Mathematicae 12 (1971), 177–209.
24. Hartley Rogers, *Theory of recursive functions and effective computability*, The MIT Press, 1967.
25. S. Barry Cooper, *Computability Theory*, Chapman & Hall/CRC, 2004.
26. Stephen G Simpson, $\Pi^0_1$ Sets and Models of WKL₀, Reverse Mathematics 2001 (2000), 352–378.
27. ———, Medvedev degrees of 2-dimensional subshifts of finite type, Ergodic Theory and Dynamical Systems FirstView (2012), 1–10.
28. Otto Toeplitz, *Ein beispiel zur theorie der fastperiodischen funktionen*, Mathematische Annalen 98 (1928), 281–295, 10.1007/BF01451594.
29. Alan Turing, *On computable numbers, with an application to the ‘Entscheidungsproblem’*, Proceedings of the London Mathematical Society 42 (1937), 230–265.
30. V.A. Uspenskii, *On enumeration operators*, Dokl. Acad. Nauk 103 (1955), 773–776.
31. Hao Wang, *Notes on a class of tiling problems*, Fundamenta Mathematicae 82 (1975), 295–305.

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