Fusion of twisted representations

Matthias R. Gaberdiel† ‡
Department of Applied Mathematics and Theoretical Physics
University of Cambridge, Silver Street
Cambridge, CB3 9EW, U. K.

July 1996

Abstract

The comultiplication formula for fusion products of untwisted representations of the chiral algebra is generalised to include arbitrary twisted representations. We show that the formulae define a tensor product with suitable properties, and determine the analogue of Zhu’s algebra for arbitrary twisted representations. As an example we study the fusion of representations of the Ramond sector of the $N = 1$ and $N = 2$ superconformal algebra. In the latter case, certain subtleties arise which we describe in detail.

---

* e-mail: M.R.Gaberdiel@damtp.cam.ac.uk
† Address from 1 Sep 1996: Department of Physics, Harvard University, Cambridge MA, 02138, USA.
1 Introduction

Some years ago, Richard Borcherds proposed that fusion in (chiral) conformal field theory should be regarded as the ring-like tensor product of representations of a quantum ring, a generalisation of rings and vertex algebras, for which the holomorphic fields of a chiral theory form the natural example. From this point of view, conformal field theory is just the representation theory of this quantum ring, and fusion corresponds to taking the canonical ring-like tensor product of modules. A somewhat similar definition has now also been given by Huang and Lepowsky [13].

In [7], this approach was formulated for untwisted representations in terms of the chiral algebra, the mode expansion of the holomorphic fields. Using standard arguments of conformal field theory, two different actions (comultiplications) of the chiral algebra on the vector space tensor product of representations were derived. The fusion product was then defined as the (ring-like) quotient of the original tensor product by all relations which enforce the equality of the two different actions.

The two different actions were given explicitly, and as a consequence it was possible to prove various properties of the resulting fusion product. It was also checked for a number of example [7, 8] that the resulting fusion rules reproduce the known restrictions.

It has become apparent that this approach to fusion does not only give a satisfactory description from a conceptual point of view, but that it also allows one to prove various structural properties of fusion [21]. In addition, it provides a rather powerful method for the calculation of the fusion rules. This seems, in particular, to be the case in the situation, where the correlation functions contain logarithms [14], whence the fusion product is not completely reducible [21, 6, 18].

The above analysis was only performed for the situation in which all representations are untwisted. For these representations, the moding of the chiral algebra is the same as in the vacuum representation, and the interpretation of the fusion product as a tensor product is rather suggestive. In this paper we show that this interpretation is not limited to this simple situation, and that the general case of twisted representations can be treated similarly. We believe that this is important from a conceptual point of view, but, as indicated before, it should also allow us to calculate fusion rules fairly efficiently. As an example, and in order to check our formulae, we derive the fusion rules of the \( N = 1 \) superconformal algebra in this way.

Part of the original motivation was to understand the \( N = 2 \) fusion rules for the Ramond and Neveu-Schwarz sectors, and consequently, quite a substantial part of the paper deals with this problem. One of the central themes is to describe the way in which the fusion rules of the \( N = 2 \) algebra respect the automorphism symmetry from the spectral flow [22]. A number of subtleties arise in this context, and we explain them in detail. Some of them seem to have been overlooked so far.

The paper is organised as follows. In the next section we introduce the relevant notation, and derive the generalised comultiplication formula. We also indicate how some of its properties can be shown. In section 3, we use the explicit formulae to find the analogue of Zhu’s algebra [23] for the twisted case, thereby generalising further the analysis of [16] and [8]. We also
explain how to define the special subspace \[\mathbb{P}\] in the more general situation. In the reminder of the paper we calculate explicitly the fusion rules for the Ramond and Neveu-Schwarz sector of the \(N = 1\) (section 4) and \(N = 2\) (section 5) superconformal algebra.

2 The general comultiplication formula

2.1 Definitions and Notation

Let \(S^i(w), i = 1, \ldots, n\) denote (a suitable subset of) the holomorphic quasiprimary fields of a conformal field theory, where \(S^i\) has conformal weight \(h_i \in \mathbb{Z}/2\). We define the vacuum sector \(H_0\) of the corresponding chiral conformal field theory as a certain completion of the vector space spanned by products of the form

\[
S^{j_1}(w_1) \cdots S^{j_n}(w_n) \Omega, \tag{2.1.1}
\]

where \(\Omega\) is the vacuum vector, and \(w_i \neq w_j\) for \(i \neq j\) (see \[1\] and \[2\] for more details). We can associate to a state \(\Psi \in H_0\) a field \(V(\Psi, z)\) which then satisfies \(V(\Psi, 0)\Omega = \Psi\).

The correlation functions of the holomorphic fields are meromorphic (single-valued) functions. This implies, in particular, that the holomorphic fields satisfy an operator product expansion

\[
S^1(w)S^2(z) = \sum_{l \in \mathbb{Z}+h_1} (w - z)^{-h_1} V(S^1, S^2(0)\Omega, z), \tag{2.1.2}
\]

where \(|w - z|\) is sufficiently small. If the spectrum of the scaling operator \(L_0\) is positive (as we shall assume), then the sum in (2.1.2) is bounded by \(l \geq -h_2\).

Because of (2.1.2) it is possible to expand the holomorphic fields in terms of modes as (cf. \[3\])

\[
S(w) = \sum_{l \in \mathbb{Z}+h} w^{-h} S_{-l}. \tag{2.1.3}
\]

The singular part of the operator product expansion gives rise to commutation relations for the modes; these generate an infinite dimensional algebra (typically a \(W\)-algebra) which is called the chiral algebra \(\mathcal{A}\) of the conformal field theory.

The other sectors of the chiral theory can be interpreted as representations \(\mathcal{H}\) of the vacuum sector \(H_0\). A representation is defined by a set of amplitudes, involving an arbitrary number of holomorphic and two special fields, which we may imagine to correspond to a state in a representation \(\mathcal{H}\), and one in its dual \(\mathcal{H}^*\). The crucial property which has to be satisfied by these amplitudes is that they respect the relations coming from the vacuum amplitudes, and in particular (2.1.2) (see again \[1\] and \[2\] for more details). Furthermore, the amplitudes have to possess suitable analytic properties, and the short distance behaviour of the holomorphic field \(S^i(w)\) with each of the two special fields has to be of the form

\[
S^i(w)V(\psi, z) = \sum_{l \geq h_\psi} (w - z)^{-h_i} V(\psi^i, z), \tag{2.1.4}
\]

where \(\psi^i\) defines another set of allowed amplitudes, \(l \in \mathbb{Z} + h_i + \alpha^i\), and \(h_\psi\) is some finite number which is independent of \(S^i\). We can use this to define an action of the modes of \(S^i\)
on states in $H$, by defining (in analogy with (2.1.2)) $S_{-l} \psi := \psi_l$. As the correlation functions satisfy the conditions coming from (2.1.2), this action defines a representation of the chiral algebra. Furthermore, the null-relations of the vacuum sector are respected, and the resulting representation is of positive energy, as the sum is bounded from below.

There are two different types of representations, usually referred to as untwisted and twisted. In the untwisted case, the sum in (2.1.4) runs over $l \in \mathbb{Z} + h_i$ (for all $S^i$), and the correlation functions are meromorphic functions of the arguments of the holomorphic fields. (In particular, the modes of the holomorphic fields have then the same moding as in the vacuum sector.) In the twisted case, there exists at least one holomorphic field $S_j$ which is not single-valued, and for which the sum runs over $l \in \mathbb{Z} + h_j + \alpha_j$, where $\alpha_j \not\in \mathbb{Z}$. The monodromy of the field $S_j$ around states in $H$ is characterised by $\alpha_j$ (mod $\mathbb{Z}$). We shall call $\alpha_j$ in the following the twist of the representation (with respect to the field $S_j$).

An important aspect of the chiral theory is to understand the correlation functions involving more than two non-holomorphic fields. Because of the analyticity of the amplitudes it is sufficient to analyse the situation where there are three such fields, as the general case can be reduced inductively to this case. These three-point functions are largely determined by the constraints which come from the compatibility with the chiral algebra. Indeed, in [7] the point of view was put forward, that one should think of the three-point functions as the decomposition of a ring-like tensor product of two of the representations into the conjugate of the third. This was developed further in [8], where the general formula for the action of the chiral algebra on the tensor product was derived for the case that both representations are untwisted. In this paper, we want to find the generalisation of this formula to the case where the representations in question are not necessarily untwisted. We shall see that the same interpretation can be given to these products. This demonstrates that this approach is rather general.

### 2.2 Derivation of the Comultiplication Formula

Let $\psi_1$ and $\psi_2$ be two vectors in representations $H_1$ and $H_2$, respectively, and let the twist of the representation $H_i$ (with respect to the field $S$) be described by $\alpha_i$. The product of the two fields will then define a representation whose twist (with respect to $S$) is given by $\alpha = \alpha_1 + \alpha_2$.

We want to derive a formula for the action of modes of $S$ (in a representation with twist $\alpha$) on the product of the two states. This means that we want to calculate the contour integral

$$\oint_C dw \, w^{l + h_1 + \alpha - 1} S(w) V(\psi_1, z_1) V(\psi_2, z_2) \Omega, \tag{2.2.5}$$

where $C$ is a contour encircling the two insertion points. This will give a formula for the comultiplication $\Delta_{z_1, z_2}(S_1)(\psi_1 \otimes \psi_2)$.

Unfortunately, (2.2.5) only converges in correlation functions with vectors of finite energy (at infinity), and thus it is not possible to evaluate the integral independently of the state at infinity. To circumvent this problem we consider, following Friedan [9], a slightly modified integral

$$\Delta_{z_1, z_2}(\tilde{S}_1)(\psi_1 \otimes \psi_2) = \oint_C dw \, w^{l + h_1 - \alpha_1 (w - z_1)^{-\alpha_1} (w - z_2)^{-\alpha_2}} S(w) V(\psi_1, z_1) V(\psi_2, z_2) \Omega \tag{2.2.6}$$
for which this problem does not arise. It is possible to express $S_l$ in terms of $\tilde{S}_m$ by expanding the function $(w - z_1)^{-\alpha_1}(w - z_2)^{-\alpha_2}$ for large $w$,

$$\tilde{S}_l = \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \left( \frac{-\alpha_1}{p_1} \right) \left( \frac{-\alpha_2}{p_2} \right) (-z_1)^{p_1} (-z_2)^{p_2} S_{l-(p_1+p_2)}, \quad (2.2.7)$$

and similarly to write $S_l$ in terms of $\tilde{S}_m$. It is therefore equivalent to give an action of the chiral algebra in terms of the modes $\tilde{S}_m$ or in terms of the modes $S_l$.

The important property of the relation between $S_l$ and $\tilde{S}_m$ is that it is of the form

$$\tilde{S}_l = S_l + \sum_{m<l} a_m S_m \quad S_l = \tilde{S}_l + \sum_{m<l} b_m \tilde{S}_m, \quad (2.2.8)$$

which will be important in later calculations. This depends on the fact that $\alpha = \alpha_1 + \alpha_2$, and we shall always choose $\alpha$ in this way.

To find an explicit formula for $\Delta_{z_1,z_2}(\tilde{S}_l)$, let us (as in [7]) introduce the function

$$\mathcal{F}(w; z_1, z_2) := (w - z_1)^{-\alpha_1}(w - z_2)^{-\alpha_2} \langle \phi, S(w)V(\psi_1, z_1)V(\psi_2, z_2) \rangle, \quad (2.2.9)$$

where $\phi$ is any vector in the dense subspace of finite energy vectors of the chiral conformal field theory. (It is clear that $\mathcal{F}$ can only be different from zero, if $\phi$ is in a representation with twist $\alpha$.) By construction, this function (as a function of $w$) has no branch cuts, and its poles are given by

$$\sum_{l \leq \alpha_1 + h - 1} (w - z_1)^{l-h-\alpha_1}(w - z_2)^{-\alpha_2} \langle (S_{-l} \otimes \mathbb{1}) \rangle + \varepsilon_1 \sum_{m \leq \alpha_2 + h - 1} (w - z_1)^{-\alpha_1}(w - z_2)^{m-h-\alpha_2} \langle (\mathbb{1} \otimes S_{-m}) \rangle, \quad (2.2.10)$$

where we use the short-hand notation

$$\langle (S_{-l} \otimes \mathbb{1}) \rangle = \langle \phi, V(S_{-l}\psi_1, z_1)V(\psi_2, z_2) \rangle, \quad (2.2.11)$$

and likewise for $\langle (\mathbb{1} \otimes S_{-m}) \rangle$. Here $\varepsilon_1$ is a complex number which is defined by the identity

$$S(w)V(\psi_1, z_1) = \varepsilon_1 V(\psi_1, z_1)S(w), \quad (2.2.12)$$

where $|w| > |z_1|$, and the right-hand-side is to be understood as the (clockwise) analytic continuation from $|w| < |z_1|$ (cf. [13]). In particular, we have

$$\varepsilon_1 = e^{\pm \pi i \alpha_1}. \quad (2.2.13)$$

As in [7], we can then define the regular part of $\mathcal{F}$ by subtracting the poles (2.2.10) from an expansion of $\mathcal{F}$, using the short distance expansion of $S$ with $V(\psi_2, z_2)$

$$\mathcal{F}(w; z_1, z_2) = \varepsilon_1 \sum_{m \in \mathbb{Z} + h + \alpha_2} (w - z_1)^{-\alpha_1}(w - z_2)^{m-h-\alpha_2} \langle (\mathbb{1} \otimes S_{-m}) \rangle. \quad (2.2.14)$$

We obtain
\[ F_{\text{reg}}(w; z_1, z_2) = - \sum_{l \leq \alpha_1 + h - 1} (w - z_1)^{l-h-\alpha_1} (w - z_2)^{-\alpha_2} \langle (S_l \otimes \mathbb{1}) \rangle + \varepsilon_1 \sum_{m \geq \alpha_2 + h} (w - z_1)^{-\alpha_1} (w - z_2)^{m-h-\alpha_2} \langle (\mathbb{1} \otimes S_m) \rangle. \]

It is clear that the regular part \( F_{\text{reg}} \) of \( F \) only contributes to \((2.2.6)\) for \( n + h + \alpha \geq 0 \). For \( n + h + \alpha \geq 1 \), therefore only the singular part \((2.2.10)\) of \( F \) contributes to the comultiplication formula \((2.2.6)\), and we easily find (for \( n + h + \alpha \geq 1 \))

\[
\Delta_{z_1, z_2}(\tilde{S}_n) = \sum_{l=1}^{\infty} \sum_{q=0}^{\infty} \left( n + h + \alpha - 1 \right) \left( -\alpha_2 \right) \left( h + \alpha_1 + l - 1 - q \right) (z_1 - z_2)^{-\alpha_2-h-\alpha_1-l+1+q} (S_l \otimes \mathbb{1}) + \varepsilon_1 \sum_{m=1}^{\infty} \sum_{q=0}^{\infty} \left( n + h + \alpha - 1 \right) \left( h + \alpha_2 + m - 1 - q \right) (z_2 - z_1)^{-\alpha_1-h-\alpha_2-m+1+q} (\mathbb{1} \otimes S_m).\tag{2.2.15}
\]

For \( n + h + \alpha \geq 0 \), both the regular and the singular part contribute, and the explicit formula therefore depends on the expansion chosen in \((2.2.14)\). As in \([7]\), there will be two different formulae, whose action on states has to agree in all correlation functions. Using the expansion with \( V(\psi_2, z_2) \) (as in \((2.2.14)\)), we get (for \( n - h - \alpha \geq 0 \))

\[
\Delta_{z_1, z_2}(\tilde{S}_{-n}) = \sum_{l=1}^{\infty} \sum_{q=0}^{\infty} \left( n + l - \alpha + \alpha_1 - 1 - q \right) \left( -1 \right)^{h+\alpha_1+l-1-q} z_1^{l-n+\alpha-\alpha_1+q} \left( -\alpha_2 \right) \left( h + \alpha_1 + 1 - q \right) (z_1 - z_2)^{-\alpha_2-q} (S_l \otimes \mathbb{1}) + \varepsilon_1 \sum_{m=1}^{\infty} \sum_{q=0}^{\infty} \left( h + \alpha_2 + m - 1 + q \right) \left( -\alpha_1 \right) \left( n + m + \alpha_2 - \alpha + q \right) (z_2 - z_1)^{-m-n-q} (\mathbb{1} \otimes S_m) + \varepsilon_1 \sum_{m=\alpha_2+h}^{\infty} \sum_{q=0}^{n-h-\alpha} \left( m - h - \alpha_2 \right) \left( -\alpha_1 \right) \left( n - h - \alpha - q \right) (z_2)^{-m-h-\alpha_2-q} + \varepsilon_1 \sum_{m=\alpha_2+h}^{\infty} \sum_{q=0}^{n-h-\alpha} \left( m - h - \alpha_2 \right) \left( -\alpha_1 \right) \left( n - h - \alpha - q \right) (z_2)^{-m-h-\alpha_2-q} (\mathbb{1} \otimes S_{-m}).\tag{2.2.16}
\]

Here we have chosen to expand the functions as power series of \( z_2 \) about 0, so that the limit \( z_2 \to 0 \) is well-defined. It is then easy to see that they reduce to the expressions given in \([3]\) for \( \alpha_1 = \alpha_2 = \alpha = 0 \).
As mentioned before, there exists a second formula which can be obtained using the expansion of $S$ with $V(\psi_1, z_1)$ in (2.2.14). Denoting, as in [8] the relevant formula by $\Delta_{z_1,z_2}(\tilde{S}_{-n})$, it is given by (2.2.16) upon exchanging $z_1 \leftrightarrow z_2$, $\alpha_1 \leftrightarrow \alpha_2$, and the two factors in the tensor product. The fusion product is then defined as the quotient of the vector space tensor product by the relations which come from the equality of $\Delta_{z_1,z_2}(\tilde{S}_{-n})$ and $\Delta_{z_1,z_2}(\tilde{S}_{-n})$.

These explicit formulae are useful for actual calculations as we shall demonstrate in the following sections. For more structural considerations, it is sometimes better to use a formula, where the various residues have not yet been evaluated

$$
\Delta_{z_1,z_2}(\tilde{S}_{-n}) = \sum_l \text{Res}_{w=z_1} \left( (w - z_2)^{-\alpha_2} w^{n+h+\alpha-1} (w - z_1)^{(l+h+\alpha_1)} \right) \left( S_l \otimes \mathbb{1} \right)
$$

$$
= \sum_m \text{Res}_{w=z_2} \left( (w - z_1)^{-\alpha_1} w^{n+h+\alpha-1} (w - z_2)^{(l+h+\alpha_2)} \right) \left( \mathbb{1} \otimes S_m \right). \quad (2.2.17)
$$

Here the residue means

$$
\text{Res}_{w=z_1} = \oint_{C_1} dw, \quad \text{Res}_{w=z_2} = \oint_{C_2} dw, \quad (2.2.18)
$$

where the contour $C_1$ encircles $z_1$, but not $z_2$ or 0, and the contour $C_2$ encircles $z_2$ and 0, but not $z_1$.

### 2.3 Some Properties

As in [4, 5], we expect that the comultiplication formula will have a natural transformation property under translation. This will indeed turn out to be true, but the situation here is slightly more complicated, as the $\tilde{S}_l$ (with respect to which we have formulated our formulae) also depend on the two insertion points (as is obvious from (2.2.7)). To be more explicit about this dependence let us denote for the moment the modes in (2.2.7) by $\tilde{S}_{l,z_1,z_2}$.

Then it is easy to see that in the sector described by the twist $\alpha$ we have the transformation law

$$
e^{nL-1} \tilde{S}_{l,z_1,z_2} e^{-uL-1} = \left\{ \begin{array}{ll}
\sum_{m=1}^{l} \left( \begin{array}{c}
l + h + \alpha - 1 \\
m + h + \alpha - 1
\end{array} \right) (-u)^{l-m} \tilde{S}_{m,z_1+z_2} & \text{if } l \geq 1 - h - \alpha \\
\sum_{m=-l}^{\infty} \left( \begin{array}{c}
m - h - \alpha \\
-m - h - \alpha
\end{array} \right) u^{m+l} \tilde{S}_{-m,z_1+z_2} & \text{if } l \leq -h - \alpha.
\end{array} \right. \quad (2.3.19)
$$

This is respected by the comultiplication, i.e. we have

$$
\Delta_{z_1+u,z_2+u} \left( e^{uL-1} \tilde{S}_{l,z_1,z_2} e^{-uL-1} \right) = \Delta_{z_1,z_2} \left( \tilde{S}_{l,z_1,z_2} \right), \quad (2.3.20)
$$

and likewise for the other comultiplication. This implies in particular that the formulae can at most depend on the difference of the two insertion points.

Next, we want to discuss the coassociativity of the fusion product. In order to analyse this it is advantageous to use the form of the comultiplication formula in which the various residues

\footnote{The formula given in [8] has the wrong $\varepsilon$ factors. This is corrected in [4].}
have not yet been explicitly evaluated (2.2.17). We also have to be careful to convert the \( S_i \)
modes into the appropriate \( \tilde{S}_m \) modes. Keeping this in mind it is then not hard to see that
\[
(\Delta_{z_2-w,z_1-w} \otimes \mathbb{I}) \circ \Delta_{w,z} = (\mathbb{I} \otimes \Delta_{z_1-w,z-w}) \circ \Delta_{z_2,w},
\]
and similarly for \( \tilde{\Delta} \) (see [7]). This establishes then that the fusion product is coassociative up
to equivalence.

Finally, we should mention that by the same arguments as in [7, 8], the comultiplication
formulae must satisfy
\[
[\Delta_{z_1,z_2}(a), \Delta_{z_1,z_2}(b)] = \Delta_{z_1,z_2}([a,b])
\]
for all \( a, b \in A \). However, as the formulae are formulated in terms of \( \tilde{S}_m \) (which do not satisfy
simple commutation relations), this is rather difficult to check explicitly.

### 3 Zhu’s algebra

#### 3.1 Derivation of Zhu’s algebra

It was pointed out in [4] that it is possible to rederive Zhu’s algebra [25] using an approach
based on the comultiplication formulae. In this section we shall follow this idea to find the
generalisation of Zhu’s algebra for the general twisted case. Zhu’s algebra has been generalised
to the fermionic (untwisted) case in [16], and to the twisted bosonic case in [3].

The main idea of the derivation is to consider the quotient of the fusion product of a representation \( \mathcal{H} \) (which
we assume without loss of generality to be inserted at \( z_2 = 0 \)) with the vacuum representation
at \( z_1 = z \). Then we have \( \alpha_1 = 0, \ z_2 = 0, \) and writing \( z = z_1, \alpha = \alpha_2 \) the formulae simplify considerably
\[
\Delta_{z,0}(\tilde{S}_n) = \sum_{l=1-h}^{\infty} \begin{pmatrix} n + h - 1 \\ l + h - 1 \end{pmatrix} z^{n-l} (S_l \otimes \mathbb{I}) + \varepsilon_1 (\mathbb{I} \otimes S_n),
\]
where \( n \geq 1 - h - \alpha \), and
\[
\Delta_{z,0}(\tilde{S}_{-n}) = \sum_{l=1-h}^{\infty} \begin{pmatrix} n + l - 1 \\ l + h - 1 \end{pmatrix} (-1)^{l+h-1} z^{-(n+l)} (S_l \otimes \mathbb{I}) + \varepsilon_1 (\mathbb{I} \otimes S_{-n}),
\]
where \( n \geq h + \alpha \). We can assume (by using the ambiguity in defining \( \alpha \)) that \( h + \alpha \geq 2 \).

The idea of the derivation is that we consider the quotient of the fusion product of \( \mathcal{H} \) with
\( \mathcal{H}_0 \) by all states of the form
\[
\Delta_{x,0}(\mathcal{A}_-) (\mathcal{H}_0 \otimes \mathcal{H})_f,
\]
where \( \mathcal{A}_- \) is the algebra generated by all negative modes. (In the conventional approach to
fusion in terms of 3-point functions, all such states vanish if there is a highest weight vector
at infinity.) Using (3.1.2), it is clear that we can identify this quotient space with a certain
subspace of
\[
(\mathcal{H}_0 \otimes \mathcal{H})_f / \Delta_{z,0}(\mathcal{A}_-) (\mathcal{H}_0 \otimes \mathcal{H})_f \subset (\mathcal{H}_0 \otimes \mathcal{H}^{(0)})_f,
\]
where \( \mathcal{H}^{(0)} \) is the highest weight space of the representation \( \mathcal{H} \). The idea is now to analyse this quotient space for the universal highest weight representation \( \mathcal{H} = \mathcal{H}_{\text{univ}} \) corresponding to the twist \( \alpha \), i.e. to use no property of \( \psi \in \mathcal{H}_{\text{univ}}^{(0)} \), other than that it is a highest weight state. We can then identify this quotient space with a certain quotient of the vacuum representation \( \mathcal{H}_0 \), thus defining \( A(\mathcal{H}_0) \),

\[
\left( A(\mathcal{H}_0) \otimes \mathcal{H}_{\text{univ}}^{(0)} \right) = (\mathcal{H}_0 \otimes \mathcal{H}_{\text{univ}}) / \Delta_{z,0}(A_-) (\mathcal{H}_0 \otimes \mathcal{H}_{\text{univ}}) .
\]

(3.1.5)

We should mention that \( A(\mathcal{H}_0) \) will depend on the set of twists \( \alpha \) for each of the holomorphic fields \( S \).

The crucial ingredient we shall be using is the observation that, because of (2.2.7) and (2.3.20),

\[
\tilde{\Delta}_{0,-z}(\tilde{S}_{-m}) (\mathcal{H}_0 \otimes \mathcal{H})
\]

is for \( m \geq h + \alpha \) in the space by which we quotient (3.1.3). In more detail we have

\[
\tilde{\Delta}_{0,-z}(\tilde{S}_{-m}) = \sum_{l=-m+\alpha}^{\infty} \left( \frac{-\alpha}{l + m - \alpha} \right) z^{-(m+l)} (S_l \otimes \mathbb{1}) + \varepsilon_1 \sum_{k=1-h-\alpha}^{\infty} \left( \frac{m + k - 1}{m - h - \alpha} \right) (-1)^{k+h+\alpha-1} z^{-(m+k)} (\mathbb{1} \otimes S_k) ,
\]

(3.1.6)

where \( m \geq h + \alpha \).

The structure of the space \( A(\mathcal{H}_0) \) depends on whether \( \alpha + h \in \mathbb{Z} \) or not. In the first case, the field \( S \) has a zero mode for the representations with twist \( \alpha \), and we have to find a formula for

\[
(\mathbb{1} \otimes S_0)(\mathcal{H}_0 \otimes \psi) \mod \Delta_{z,0}(A_-)(\mathcal{H}_0 \otimes \mathcal{H}_{\text{univ}}) ,
\]

(3.1.7)

in terms of modes acting on the left-hand factor in the tensor product. To do this, we use the explicit formula (3.1.6) with \( m = h + \alpha \), and then (3.1.2) for \( n = 1 - h - \alpha, \ldots, -1 \) to rewrite terms of the form \( (\mathbb{1} \otimes S_n) \). After a short calculation we obtain

\[
\varepsilon_1 (\mathbb{1} \otimes S_0) \cong \sum_{m=0}^{h-1} \left( \begin{array}{c} h - 1 \\ m \end{array} \right) z^{h-m} (S_{m-h} \otimes \mathbb{1}) .
\]

(3.1.8)

This is independent of \( \alpha \) (as it should be), and reproduces precisely the formula of \( \mathbb{1} \). Using this expression, the action of the zero mode on the fusion product then becomes

\[
\Delta_{z,0}(S_0) \cong \Delta_{z,0}(\tilde{S}_0) = \sum_{m=0}^{\infty} z^{h-m} \left( \begin{array}{c} h \\ m \end{array} \right) (S_{m-h} \otimes \mathbb{1}) ,
\]

(3.1.9)

which generalises Zhu’s product formula \([25]\). Here we have used that \( S_0 \) differs from \( \tilde{S}_0 \) only by negative modes which follows from (2.2.7).

Next, in order to obtain the relations which characterise the quotient space \( A(\mathcal{H}_0) \), we repeat the above calculation for \( m = h + \alpha + p \), where \( p \geq 1 \), and also use (3.1.8) to replace the term
of the form \((\mathbb{1} \otimes S_0)\). We then find that, modulo terms in the quotient, we have

\[
0 \cong \sum_{m=-h-p}^{-h} \left( \frac{-\alpha}{m+h+p} \right) z^{-(h+\alpha+p+m)} (S_m \otimes \mathbb{1})
+ \left( \frac{h+\alpha+p-1}{p} \right) (-1)^{p+1} z^{-\alpha+p} (S_{-h} \otimes \mathbb{1})
+ \sum_{m=1-h}^{\infty} \left[ C_{m,p} \left( \frac{h-1}{m+h} \right) \left( \frac{h+\alpha+p-1}{p} \right) (-1)^{p+1} \right] z^{-(h+\alpha+p+m)} (S_m \otimes \mathbb{1}),
\]

where \(C_{m,p}\) is given as above.

\[
C_{m,p} = \left( \frac{-\alpha}{m+h+p} \right) + \sum_{l=1-h-\alpha}^{M_{h,\alpha}} \left( \frac{h+\alpha+p+l-1}{p} \right) (-1)^p \left( \frac{l+h-1}{m+h-1} \right),
\]

and \(M_{h,\alpha} < 0\) is the largest number of the form \(r - h - \alpha\), where \(r \in \mathbb{Z}\). (In the present case, \(M_{h,\alpha} = -1\)) In general, this formula is rather complicated. However, for \(p = 1\), using the same identities as in \([4]\), the expression simplifies to

\[
0 \cong z^{-\alpha} \sum_{m=0}^{\infty} \left( \frac{h}{m} \right) z^{-m} (S_{m-1-h} \otimes \mathbb{1}),
\]

which gives conditions independent of \(\alpha\). This generalises the formula of Zhu for \(O(H_0)\) \([25]\).

If \(\alpha + h \not\in \mathbb{Z}\), then the field \(S\) has no zero mode (for this twist). In this case there is no analogue of the formula (3.1.8), but we can similarly do the calculation which lead to (3.1.10) for \(p \geq 0\). Then the result is

\[
0 \cong \sum_{m=-h-p}^{-h} \left( \frac{-\alpha}{m+h+p} \right) z^{-(h+\alpha+p+m)} (S_m \otimes \mathbb{1})
+ \sum_{m=1-h}^{\infty} C_{m,p} z^{-(h+\alpha+p+m)} (S_m \otimes \mathbb{1}),
\]

where \(C_{m,p}\) is given as above. For \(p = 0\), the formula simplifies to

\[
0 \cong \sum_{m=0}^{\infty} \left( \frac{M_{h,\alpha} + h}{m} \right) z^{-(\alpha+m)} (S_{m-h} \otimes \mathbb{1}).
\]

This generalises the formula of \([3]\) (which was only derived for integer \(h\)). We note that \(M_{h,\alpha} + h\) is independent of the ambiguity in choosing \(\alpha\), i.e. invariant under \(\alpha \mapsto \alpha + r\), where \(r \in \mathbb{Z}\).

### 3.2 Special Subspaces

It was observed by Nahm \([20]\) that for untwisted representations

\[
(\mathcal{H}_1 \otimes \mathcal{H}_2)_{\Delta_{z,0}} (\mathcal{A}_-) (\mathcal{H}_1 \otimes \mathcal{H}_2)_{\Delta} \subset (\mathcal{H}_1^0 \otimes \mathcal{H}_2^0),
\]

where

\[
(\mathcal{H}_1 \otimes \mathcal{H}_2)_{\Delta_{z,0}} (\mathcal{A}_-) (\mathcal{H}_1 \otimes \mathcal{H}_2)_{\Delta} \subset \left( \mathcal{H}_1^0 \otimes \mathcal{H}_2^0 \right)
\]
where $H_s^1$ is the special subspace of $H_1$, and $H_2^{(0)}$ is the space of highest weight vectors in $H_2$. The special subspace can be defined as the quotient space

$$H_s^1 = H_1 / \mathcal{A}_- H_1,$$  

where $\mathcal{A}_-$ is the algebra generated by the modes which do not annihilate the vacuum, i.e. $\mathcal{A}_-$ is generated by $S_{-n}, n \geq h$. Representations whose special subspace is finite dimensional are called quasi rational. It follows from (3.2.15) that the fusion product of a highest weight representation with a quasi-rational representation contains only finitely many highest weight representations.

It follows from (3.1.1, 3.1.2, 3.1.6) that (3.2.15) also holds if $H_1$ is an untwisted representation, and $H_2$ is any twisted highest weight representation. It also makes sense to extend the above definition (3.2.14) and the definition of quasi-rationality to cover the twisted case as well. In particular, we can choose in (2.2.16) $\alpha_i$ such that $1 \geq h + \alpha > 0$ (so that $n = h + \alpha$ is the first negative mode) and $1 \geq h + \alpha_2 > 0$ (so that for $z_2 = 0$ and $n = h + \alpha$, $1 \otimes S_{-(h+\alpha_2)}$ is the only negative mode on the right-hand-side). Then $1 - h - \alpha_1 > -h$, and it follows by the same argument as in Nahm [20] that the fusion product of any quasi-rational representation with any highest weight representation contains only finitely many subrepresentations, irrespective of whether they are twisted or untwisted.

4 Fusion in the $N = 1$ algebra

In this (and the following) chapter we want to use the explicit formulae to (re)derive the fusion rules for the Neveu-Schwarz (NS) and Ramond (R) sector of the $N = 1$ and $N = 2$ superconformal algebra. We shall only consider the case of quasi-rational representations at generic $c$. The fusion rules for the “minimal” models (at specific values of the central charge) follow from these calculations if the relevant representations are identified.

In this section we analyse the simpler case of the $N = 1$ algebra. This algebra is generated by the Virasoro algebra $\{L_n\}, n \in \mathbb{Z}$, and the modes of the superfield $G$ with $h = \frac{3}{2}$, subject to the relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12} cn (n^2 - 1) \delta_{n,-m}$$

$$[L_n, G_r] = \left( \frac{1}{2} n - r \right) G_{n+r}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{3} c \left( r^2 - \frac{1}{4} \right) \delta_{r,-s}. \tag{4.1}$$

The untwisted (NS) sector has $r \in \mathbb{Z} + \frac{1}{2}$, whereas the twisted (R) sector has $r \in \mathbb{Z}$. Let us parametrise as in [24]

$$c(t) = \frac{15}{2} - \frac{3}{t} - 3t. \tag{4.2}$$

The representations we are concerned with are highest weight representations of the chiral algebra, i.e. representations generated by the action of the algebra from a state which is
annihilated by the action of the positive modes. In the NS sector, there is only one zero mode ($L_0$), and the representation is uniquely determined by the eigenvalue of the highest weight vector with respect to $L_0$. We usually denote this highest weight state by $|h⟩$, where $h$ is the eigenvalue of $L_0$. If $h$ is of the form $h_{p,q}(t) = \frac{(1 - pq)}{4} + (q^2 - 1)\frac{t}{8} + \frac{(p^2 - 1)}{(8t)}$, (4.4)

where $p, q \in \mathbb{Z}$ and $p + q \in 2\mathbb{Z}$, then the representation generated from the corresponding state has a singular vector at level $pq/2$. The case $p = q = 1$ corresponds to the vacuum representation $h = 0$; the first non-trivial case is $(p, q) = (1, 3)$ with singular vector

$$\begin{pmatrix} G_\frac{3}{2} - \frac{1}{t}L_{-1}G_{-\frac{3}{2}} \end{pmatrix} |h_{1,3}(t)⟩ \quad (4.5)$$

and $(p, q) = (3, 1)$ with singular vector

$$\begin{pmatrix} G_{-\frac{3}{2}} - tL_{-1}G_{-\frac{3}{2}} \end{pmatrix} |h_{3,1}(t)⟩. \quad (4.6)$$

To determine the special subspace of the representations corresponding to $(1, 3)$ and $(3, 1)$ we use the singular vector equations to eliminate $L_{-1}G_{-\frac{3}{2}}|h⟩$, and the $G_{-1}^2$ descendant of the singular vector equation to eliminate $L_{-1}L_{-1}|h⟩$. It then follows that the dimension is in both cases three.

We shall also consider below the NS representation $(2, 2)$ which has a singular vector at level two

$$\begin{pmatrix} L_{-1}^2 - \frac{4}{3}h_{(2,2)}(t)L_{-2} - G_{-\frac{3}{2}}^2G_{-\frac{3}{2}} \end{pmatrix} |h_{2,2}(t)⟩,$$

and whose special subspace is two dimensional.

In the Ramond sector, there are two zero modes, and the highest weight state is uniquely characterised by

$$G_0|λ⟩ = λ|λ⟩, \quad (4.7)$$

as the commutation relations then imply $L_0|λ⟩ = (λ^2 + c/24)|λ⟩$. If $λ$ is of the form $λ_{p,q}(t) = \frac{p - qt}{2\sqrt{2t}}$, (4.8)

where $p, q \in \mathbb{Z}$ and $q - p$ odd, then the corresponding representation has a singular vector at level $pq/2$. The first most simple cases are $(p, q) = (1, 2)$ with singular vector

$$\begin{pmatrix} L_{-1} + \sqrt{\frac{t}{2}}G_{-1} \end{pmatrix} |λ_{1,2}(t)⟩ = 0 \quad (4.9)$$

and $(p, q) = (2, 1)$ with singular vector

$$\begin{pmatrix} L_{-1} - \frac{1}{\sqrt{2t}}G_{-1} \end{pmatrix} |λ_{2,1}(t)⟩ = 0. \quad (4.10)$$
We should note that the representation corresponding to \( \lambda \) and \(-\lambda\) define the same vertex operator in correlation functions: it follows from (2.1.4) that the short distance expansion of \( G \) with the highest weight state has a branch cut. If we move \( G \) once around the highest weight state, all coefficients change sign, and thus, in particular, the eigenvalue \( \lambda \) itself. This point was overlooked in \[24\].

To determine the special subspace for these two representations, we use the singular vector equation to eliminate \( G - 1 |\lambda\rangle \), and the \( L - 1 \) and \( G - 1 \) descendants of the singular vector equation to eliminate \( L - 1 G - 1 |\lambda\rangle \) and \( L - 1 L - 1 |\lambda\rangle \). This then demonstrates that the dimension of the special subspace is two for both cases.

To illustrate how we can calculate the fusion products, let us consider the situation, where we have a NS field at \( z_1 = z \), and a R field at \( z_2 = 0 \). Choosing \( \alpha_1 = 0, \alpha_2 = \alpha = 1/2 \), the comultiplication formulae simplify to

\[
\Delta_{z,0}(\tilde{G}_n) = \sum_{l=-1/2}^{\infty} \left( \frac{n + 1/2}{l + 1/2} \right) z^{n-l}(G_l \otimes 1) + \varepsilon_1(1 \otimes G_n),
\]

(4.11)

where \( n \geq -1 \), and

\[
\Delta_{z,0}(\tilde{G}_{-n}) = \sum_{l=-1/2}^{\infty} \left( \frac{n + l - 1}{l + 1/2} \right) z^{-(n+l)}(-1)^{l+1/2}(G_l \otimes 1) + \varepsilon_1(1 \otimes G_{-n})
\]

(4.12)

for \( n \geq 2 \).

Let us now consider the case, where the representation at \( z \) is any NS highest weight representation, and the R representation at 0 is \((1, 2)\) or \((2, 1)\). The singular vector equation implies that

\[
0 = \langle 1 \otimes L_{-1} \rangle + \alpha \langle 1 \otimes G_{-1} \rangle = -z^{-1} \kappa \langle 1 \otimes 1 \rangle + \alpha \langle 1 \otimes G_{-1} \rangle,
\]

(4.13)

where \( \alpha = \sqrt{t/2} \) for \((p, q) = (1, 2)\) and \( \alpha = -\sqrt{1/2t} \) for \((p, q) = (2, 1)\). Here the modes act on the highest weight states of the corresponding representations. In the second line we have used the standard relation (see e.g. \[7\]) to replace the \( L_{-1} \) mode by inserting \( \Delta(L_0) \); \( \kappa \) is then

\[
\kappa = h_3 - h_1 - h_2,
\]

(4.14)

where \( h_3 \) is the conformal weight of the state at infinity, \( h_1 \) the conformal weight of the state at \( z \), and \( h_2 = h_{(1,2)} \) or \( h_2 = h_{(2,1)} \). We can also use the equation coming from the \( G_{-1} \) descendant of the singular vector equation to give

\[
0 = \langle 1 \otimes G_{-1} L_{-1} \rangle + \alpha \langle 1 \otimes G_{-1}^2 \rangle = -\frac{1}{2} \langle 1 \otimes G_{-2} \rangle + \langle 1 \otimes L_{-1} G_{-1} \rangle + \alpha \langle 1 \otimes L_{-2} \rangle
\]

\[
= -z^{-1}(\kappa - 1) \langle 1 \otimes G_{-1} \rangle - \frac{1}{2} \langle 1 \otimes G_{-2} \rangle + \alpha \langle 1 \otimes L_{-2} \rangle.
\]

(4.15)
Using the comultiplication of $L_{-2}$, it is easy to see that
\[
\langle 1 \otimes L_{-2} \rangle = (h_1 - \kappa)z^{-2}\langle 1 \otimes 1 \rangle .
\] (4.16)

Furthermore, from the comultiplication of $G_{-2}$ and $G_{-1}$ we learn that
\[
0 = z^{-\frac{3}{2}}\langle G_{-\frac{1}{2}} \otimes 1 \rangle + \varepsilon_1\langle 1 \otimes G_{-2} \rangle ,
\] (4.17)
and this implies
\[
\langle 1 \otimes G_{-2} \rangle = z^{-\frac{1}{2}}\langle 1 \otimes G_{-1} \rangle .
\] (4.18)

Taking (4.16), (4.18) and (4.15) together, we find then that
\[
0 = -z^{-1}\left((\kappa - 1) + \frac{1}{2}\right)\langle 1 \otimes G_{-1} \rangle + \alpha(h_1 - \kappa)z^{-2} .
\] (4.19)

Putting this together with (4.13), we obtain
\[
z^{-2}\left[\alpha^2(h_1 - \kappa) - \kappa(\kappa - \frac{1}{2})\right] = 0 .
\] (4.20)

Thus a necessary condition for the fusion to be allowed is that the bracket $[\ldots]$ vanishes. This gives rise to the relations.

\[
(p, q)_{NS} \otimes (1, 2)_R = (p, q + 1)_R \oplus (p, q - 1)_R
\]
\[
(p, q)_{NS} \otimes (2, 1)_R = (p + 1, q)_R \oplus (p - 1, q)_R .
\] (4.21)

Using similar techniques we can derive the restrictions for the fusion with another Ramond field. The result is identical to (4.21), if we replace the suffices $NS$ on the left-hand side and $R$ on the right-hand-side by $R$ and $NS$, respectively. We can also determine the restrictions coming from the fusion of the $(1, 3)$, $(3, 1)$ and $(2, 2)$ field in the NS sector. We obtain (after similar calculations)
\[
(1, 3)_{NS} \otimes (p, q)_U = (p, q)_U \oplus (p, q + 2)_U \oplus (p, q - 2)_U
\]
\[
(3, 1)_{NS} \otimes (p, q)_U = (p, q)_U \oplus (p + 2, q)_U \oplus (p - 2, q)_U
\]
\[
(2, 2)_{NS} \otimes (p, q)_U = (p + 1, q + 1)_U \oplus (p - 1, q - 1)_U ,
\] (4.22)

where $U = NS$ or $U = R$. Taking all of these results together and using the associativity and commutativity of the fusion product, we can give a compact formula for all different fusion products
\[
(p_1, q_1) \otimes (p_2, q_2) = \bigoplus_{p=|p_1-p_2|+1}^{|p_1+p_2-1|} \bigoplus_{q=|q_1-q_2|+1}^{|q_1+q_2-1|} (p, q) ,
\] (4.23)

where $p$ and $q$ are integers, and the sum is over every other integer. The representation denoted by $(p, q)$ is in the NS sector if $p + q \in 2\mathbb{Z}$, and in the R sector otherwise. This reproduces the results of [23, 24].

\footnote{As always, we can only derive necessary conditions for the fusion rules in this way. To derive sufficient conditions, a much more detailed analysis is necessary. We also assume that all fusion products are completely reducible — this is justified for generic $c$.}
5 Fusion in the $N = 2$ algebra

In this section we want to analyse the fusion of quasirational representations of the $N = 2$ superconformal algebra for generic $c$. Let us recall that this algebra is generated by the Virasoro algebra $\{L_n\}$, the modes of an $U(1)$-current with $h = 1 \{T_n\}$ and the modes of two superfields of conformal dimension $h = \frac{3}{2}$, $\{G_r^\pm\}$, subject to the relations

\[
\begin{align*}
[L_m, G_{a}^\pm] &= \left(\frac{1}{2} m - \alpha\right) G_{a+m}^\pm \\
[L_m, T_n] &= -n T_{m+n} \\
[T_m, T_n] &= \tilde{c} m \delta_{m,-n} \\
[T_m, G_{a}^\pm] &= \pm G_{a+m}^\pm \\
\{G_{a}^+, G_{\beta}^-\} &= 0 \\
\{G_{a}^+, G_{\beta}^+\} &= 2 L_{\alpha+\beta} + (\alpha - \beta) T_{\alpha+\beta} + \tilde{c} \left(\alpha^2 - \frac{1}{4}\right) \delta_{\alpha,-\beta},
\end{align*}
\]

(5.1)

where $\tilde{c} = c/3$. In the NS and R sector, the modes of $L_m$ and $T_n$ are both integral, whereas the modes of $G_r^\pm$ are half-integral in the NS sector, and integral in the R sector. There also exists the so-called twisted sector, where the modes of $T_n$ and $G_r^1$ are half-integral, those of $G_r^2$ and $L_n$ are integral, and where $G^\pm = G^1 \pm iG^2$ \([1]\). Here we shall only discuss the NS and the R sector.

It was observed in \([22]\) that there exists a family of automorphisms $\alpha_\eta : \mathcal{A} \to \mathcal{A}$ which map the chiral algebra to itself. They are explicitly given as

\[
\begin{align*}
\alpha_\eta(G_r^\pm) &= G_{r+\eta}^\pm \\
\alpha_\eta(L_n) &= L_n - \eta T_n + \frac{1}{2} \eta^2 \tilde{c} \delta_{n,0} \\
\alpha_\eta(T_n) &= T_n - \eta \tilde{c} \delta_{n,0}.
\end{align*}
\]

(5.2)

For $\eta \in \mathbb{Z}$, this gives an automorphism of each of the sectors of the algebra, whereas for $\eta = \mathbb{Z} + \frac{1}{2}$, it relates the NS and the R sector to each other. We shall demonstrate in this section that the fusion rules respect the automorphism symmetry (see (5.25) below). The fusion rules for the $(NS \otimes NS)$ and the $(R \otimes R)$ case are in principle known \([19]\) (although, strictly speaking, only the unitary minimal case is discussed there), but in order to see that the automorphism symmetry is respected generically, certain subtleties have to be taken into account which were not discussed in \([13]\). We shall also derive the fusion rules for the $(NS \otimes R)$ case.

Following \([1]\), we want to parametrise the central charge

\[
\tilde{c} = 1 - \frac{2}{m}.
\]

(5.3)

We shall only discuss the situation, where $m$ is generic, i.e. not a positive integer $m \geq 2$. Then the theory is not unitary, and thus, as has been shown in \([3]\), not rational. The unitary
fusion rules can be obtained from the ones discussed in this paper by identifying certain
representations (see for example [19]).
In the NS sector, there are two zero modes, and a highest weight representation is uniquely
determined by giving the eigenvalue \( h \) and \( q \) of the highest weight vector with respect to the
action of \( L_0 \) and \( T_0 \), respectively. We parametrise
\[
q_{NS}(j,k) = \frac{j-k}{m},
\]
\[
h_{NS}(j,k) = \frac{jk-\frac{1}{4}}{m}.
\]
For \( 0 < j, j \in \mathbb{Z} + \frac{1}{2} \) the corresponding highest weight representation has a fermionic singular
vector at level \( j \), and likewise for \( k \). For \( n := m - (j+k) \in \mathbb{Z}, n > 0 \), the highest weight
representation has a bosonic singular vector at level \( n \) [1].
We observe that for \( jk = 0 \), the two highest weight states are not related
by the action of \( G^\pm_0 \), as the coefficients in (5.5) vanish. In this case, we denote by \((j,k)\pm\)
the two different representations generated from the highest weight states corresponding to
\( q_{R}(j,k) \pm \frac{1}{2} \).
Before we start analysing the fusion rules, let us describe how the automorphism (5.2) acts
on the various representations. Let us first consider the NS representation \((j,k)\), and consider
the representation defined by
\[
\hat{a}|(j,k)\rangle := \alpha_{\eta}(a)|(j,k)\rangle,
\]
(5.7)
where $| (j, k) \rangle$ denotes an arbitrary state in the highest weight representation corresponding to $(j, k)$, and $a$ is an element in the chiral algebra. We denote this representation by

\[ (\hat{j}, \hat{k}) = \alpha_\eta(j, k) . \]

Let us first consider the case $\eta = 1$. Then evaluating (5.7) on the highest weight state of the representation $(\hat{j}, \hat{k})$, we find that

\[ \hat{G}_+^+ | h(j, k) \rangle = G_+^+ | h(j, k) \rangle, \]
\[ \hat{G}_-^- | h(j, k) \rangle = G_-^- | h(j, k) \rangle = 0, \]

and thus that (generically) the highest weight state of the representation $(\hat{j}, \hat{k})$ is $G_+^+ | h(j, k) \rangle$, rather than $| h(j, k) \rangle$ itself. We can then evaluate $\hat{L}_0$ and $\hat{T}_0$ on this state, and find that

\[ \alpha_1(j, k) = (j + 1, k - 1). \]

In the above discussion something special happens for $k = 1/2$, as then

\[ G_-^+ | h \left( j, \frac{1}{2} \right) \rangle = 0. \]

Thus the state $| h(j, k) \rangle$ is the highest weight state itself, and we find

\[ \alpha_1 \left( j, \frac{1}{2} \right) = \left( \frac{1}{2}, m - j - 1 \right). \]

The same phenomena occurs for $\eta = -1$,

\[ \alpha_{-1}(j, k) = \begin{cases} (j - 1, k + 1) & \text{if } j \neq \frac{1}{2}, \\ (m - k - 1, \frac{1}{2}) & \text{if } j = \frac{1}{2}, \end{cases} \]

In the Ramond sector, similar considerations lead to

\[ \alpha_1(j, k)_+ = (j + 1, k - 1)_-, \]

and

\[ \alpha_1(j, k)_- = \begin{cases} (j + 1, k - 1)_- & \text{if } jk \neq 0, \\ (1, m - j - 1) & \text{if } jk = 0, \end{cases} \]

and similarly,

\[ \alpha_{-1}(j, k)_- = (j - 1, k + 1)_+, \]

and

\[ \alpha_{-1}(j, k)_+ = \begin{cases} (j - 1, k + 1)_+ & \text{if } jk \neq 0, \\ (m - k - 1, 1) & \text{if } jk = 0. \end{cases} \]

As a consistency check, we note that

\[ \alpha_{-1} \alpha_1(j, 0)_- = \alpha_{-1}(1, m - j - 1) = (0, m - j)_+ = (j, 0)_-. \]
Finally, for $\eta = \pm \frac{1}{2}$, we have maps from the NS sector in the R sector,

$$\alpha_{\frac{1}{2}}(j, k) = \left( j + \frac{1}{2}, k - \frac{1}{2} \right)_-, \quad \alpha_{-\frac{1}{2}}(j, k) = \left( j - \frac{1}{2}, k + \frac{1}{2} \right)_+,$$

and, conversely, maps from the R sector into the NS sector

$$\alpha_{\frac{1}{2}}(j, k)_+ = \left( j + \frac{1}{2}, k - \frac{1}{2} \right), \quad \alpha_{\frac{1}{2}}(j, k)_- = \left\{ \begin{array}{ll}
(j + \frac{1}{2}, k - \frac{1}{2}) & \text{if } jk \neq 0, \\
(\frac{1}{2}, m - j + k - \frac{1}{2}) & \text{if } jk = 0,
\end{array} \right. \quad (5.17)$$

$$\alpha_{-\frac{1}{2}}(j, k)_- = \left( j - \frac{1}{2}, k + \frac{1}{2} \right), \quad \alpha_{-\frac{1}{2}}(j, k)_+ = \left\{ \begin{array}{ll}
(j - \frac{1}{2}, k + \frac{1}{2}) & \text{if } jk \neq 0, \\
(m - k + j - \frac{1}{2}, \frac{1}{2}) & \text{if } jk = 0.
\end{array} \right. \quad (5.18)$$

Let us now turn to analysing the fusion rules. First of all, from the analysis of the $u(1)$ subalgebra, it follows that the eigenvalue of the $T_0$-operator is additive under fusion. However, it is not clear which states in the tensor product give a contribution with a highest weight space at infinity, and therefore the quantum numbers corresponding to the highest weight states do not always add up. In fact, in general there are three different classes labelled by

$$j_3 - k_3 = (j_1 - k_1) + (j_2 - k_2) + \delta m,$$

where $\delta$ is either 0, whence the fusion is called even, or $\delta = \pm 1$, whence the fusion is called odd \cite{18, 19}.

The (even) fusion rules of a generic NS representation with the $(\frac{3}{2}, \frac{1}{2})$ and the $(\frac{1}{2}, \frac{3}{2})$ representation have been determined before using the comultiplication formula \cite{8}. The result is

$$\left( j, k \right) \otimes \left( \frac{3}{2}, \frac{1}{2} \right) = \left( j + 1, k \right) \oplus \left( j, k - 1 \right), \quad (5.19)$$

$$\left( j, k \right) \otimes \left( \frac{1}{2}, \frac{3}{2} \right) = \left( j + 1, k \right) \oplus \left( j - 1, k \right).$$

For generic $(j, k)$, this gives already the whole fusion rule, as the odd fusion rule is forbidden. (This can be seen by combining the restrictions coming from the two independent singular vectors. We shall describe this in detail in appendix A.) The only values for which an odd fusion is allowed is $k = \frac{1}{2}$ in the first line, and $j = \frac{1}{2}$ in the second. For these values of $j$ (or $k$), the representation $(j, k)$ has a singular vector at level $\frac{1}{2}$, and this will rule out one of the two representations in the fusion product $(5.19)$. (As a matter of fact, the representation which would contain a negative value for $j_3$ or $k_3$ is absent.) Taking this together, the fusion rules are $(5.19)$, if $k \neq \frac{1}{2}$, or $j \neq \frac{1}{2}$, respectively, and

$$\left( j, \frac{1}{2} \right) \otimes \left( \frac{3}{2}, \frac{1}{2} \right) = \left( j + 1, \frac{1}{2} \right) \oplus \left( \frac{1}{2}, m - j \right), \quad (5.20)$$

$$\left( \frac{1}{2}, k \right) \otimes \left( \frac{1}{2}, \frac{3}{2} \right) = \left( \frac{1}{2}, k + 1 \right) \oplus \left( m - k, \frac{1}{2} \right).$$
Furthermore, if both \( j = k = \frac{1}{2} \), then only the first term on the right hand side survives. (This is consistent with \((\frac{1}{2}, \frac{1}{2})\) being the vacuum representation.) Using the commutativity and associativity of the fusion product, these results are already sufficient to derive restrictions for the general fusion rules. We find

\[
(j_1, k_1) \otimes (j_2, k_2) = \bigoplus_{j = \max(j_2 - k_1, j_1 - k_2) + \frac{1}{2}}^{j_1 + j_2 - \frac{1}{2}} \left[ (j, j - (j_1 + j_2) + (k_1 + k_2)) \right], \quad (5.21)
\]

where \( j_i \) and \( k_m \) are positive half-integers, and

\[
[(j, k)] = \begin{cases} 
(j, k) & \text{if } j, k > 0, \\
(-k, m - j) & \text{if } j > 0, k < 0, \\
(m - k, -j) & \text{if } k > 0, j < 0.
\end{cases} \quad (5.22)
\]

Similarly, we can determine the fusion rules involving the field \((\frac{1}{2}, m - \frac{5}{2})\)

\[
(j, k) \otimes \left( \frac{1}{2}, m - \frac{5}{2} \right) = \begin{cases}
(j + 2, k - 1) \oplus (j + 1, k - 2) & \text{for } k \neq \frac{1}{2}, \frac{3}{2}, \\
(j + 2, k - 1) \oplus (2 - k, m - j - 1) & \text{for } k = \frac{3}{2}, \\
(1 - k, m - j - 2) \oplus (2 - k, m - j - 1) & \text{for } k = \frac{1}{2},
\end{cases} \quad (5.23)
\]

and

\[
(j, k) \otimes \left( m - \frac{5}{2}, \frac{1}{2} \right) = \begin{cases}
(j - 1, k + 2) \oplus (j - 2, k + 1) & \text{for } j \neq \frac{1}{2}, \frac{3}{2}, \\
(j - 1, k + 2) \oplus (m - k - 1, 2 - j) & \text{for } j = \frac{3}{2}, \\
(m - k - 2, 1 - j) \oplus (m - k - 1, 2 - j) & \text{for } j = \frac{1}{2},
\end{cases} \quad (5.24)
\]

Again, further cancellations arise if \( j = \frac{1}{2} \) in (5.23) and \( k = \frac{1}{2} \) in (5.24).

It is not difficult to see that (5.20) and (5.22) are covariant under the automorphism (5.2), i.e.

\[
\alpha_{\eta_1} (j_1, k_1) \otimes \alpha_{\eta_2} (j_2, k_2) = \alpha_{\eta_1 + \eta_2} ((j_1, k_1) \otimes (j_2, k_2)). \quad (5.25)
\]

Indeed, for example applying the automorphisms with \( \eta_1 = 0 \) and \( \eta_2 = -1 \) to the first line of (5.20), we get for \( j, k \neq \frac{1}{2} \),

\[
(j, k) \otimes \alpha_{-1} \left( \frac{3}{2}, \frac{1}{2} \right) = (j, k) \otimes \left( \frac{1}{2}, \frac{3}{2} \right) = (j, k + 1) \oplus (j - 1, k) = \alpha_{-1} ((j + 1, k) \oplus (j, k - 1)) = \alpha_{-1} \left( (j, k) \otimes \left( \frac{3}{2}, \frac{1}{2} \right) \right), \quad (5.26)
\]

and for \( j = \frac{1}{2}, k \neq \frac{1}{2} \).
\[
\left(\frac{1}{2}, k\right) \otimes \alpha_{-1} \left(\frac{3}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, k\right) \otimes \left(\frac{1}{2}, \frac{3}{2}\right) \\
= \left(\frac{1}{2}, k + 1\right) \oplus \left(m - k, \frac{1}{2}\right) \\
= \alpha_{-1} \left(\left(\frac{3}{2}, k\right) \oplus \left(\frac{1}{2}, k - 1\right)\right) \\
= \alpha_{-1} \left(\left(\frac{1}{2}, k\right) \otimes \left(\frac{3}{2}, \frac{1}{2}\right)\right). \quad (5.27)
\]

The other cases are similar. As these fusion rules already determine the general case, this implies that (5.25) holds in general. We can therefore deduce the general fusion rules of fields \((j, k)\), where \(j\) and \(k\) are either positive half-integers or of the form \(m - s\), where \(s\) is a positive half-integer, from (5.21) using the automorphism (5.25). For example we have for \(k > j\)

\[
(j, m - k) = \alpha_{j + \frac{1}{2}} \left(k - j - \frac{1}{2}, \frac{1}{2}\right),
\]

and so

\[
(j_1, k_1) \otimes (j_2, m - k_2) = \bigoplus_{j = \max(k_2 - k_1, j_1 + j_2) + \frac{1}{2}}^{j_1 + k_2 - \frac{1}{2}} \left[(j, j - (j_1 + j_2) + (k_1 - k_2))\right]. \quad (5.28)
\]

The other cases are analogous. All of this holds for generic \(m\), \(i.e.\) generic \(c\). For integer \(m\), representations of the different types can be identified, and the fusion rules are restricted by the intersection of the various separate rules.

We also determined the fusion rules of the \(NS \otimes R\) sector, and the \(R \otimes R\) sector; the explicit results of our calculations can be found in appendix B. The main feature of these results is that all fusion rules are covariant under the automorphism, where now also \(\eta \in \frac{1}{2} \mathbb{Z}\) is admissible. As they are sufficient to derive the general restrictions, it follows that the automorphism is respected in general. The most general fusion rules are therefore already described by (5.21), provided we use the automorphism (5.25) as described before.

**Appendix**

**A \(N = 2\) calculations**

In this appendix we shall give some details for the derivation of the simplest interesting case where a cancellation appears, the fusion of a NS field with \((\frac{3}{2}, \frac{1}{2})\).

The derivation of the even fusion rules was explained in [8]. For the odd case, because of the null-vector

\[
\mathcal{N}_1^i = G^+_{-\frac{1}{2}} \phi_3^{\frac{1}{2}} = 0,
\]

We should mention that the notation here differs from that in [3] by \(j \leftrightarrow k\).
only one of the two possible odd fusion rules is non-trivial, the one where \( q_3 = q_1 + q_2 - 1 \). Furthermore, the representation \( (\frac{3}{2}, \frac{1}{2}) \) has a second null vector,

\[
\mathcal{N}_2 = \mathcal{O}_2 \phi_{\frac{3}{2}, \frac{1}{2}} = \left( -\frac{1}{m} G_{-\frac{3}{2}}^- + L_{-1} G_{-\frac{1}{2}}^- + T_{-1} G_{-\frac{1}{2}}^- \right) \phi_{\frac{3}{2}, \frac{1}{2}} = 0 ,
\]

and this implies the following restriction

\[
0 = \langle (\mathbb{1} \otimes \mathcal{O}_2) \rangle = z^{-1} \langle (\mathbb{1} \otimes G_{-\frac{3}{2}}^-) \rangle \left[ -\frac{1}{m} - \left( \kappa - \frac{1}{2} \right) - q_1 \right],
\]

where \( \kappa \) is as before (4.14), and \( q_1 \) denotes the \( T_0 \) eigenvalue of the highest weight state at \( z = 0 \). (To derive this equation, we have, as usual, considered the action of the comultiplication \( \Delta_{z,0} \) of \( G_{-\frac{3}{2}}^-, G_{-\frac{1}{2}}^-, L_{-1} \) and \( T_{-1} \) on the product, which vanishes in correlation functions because of the highest weight property of the state at infinity.) If the (odd) fusion is allowed, the correlation function does not vanish, and we must have that the bracket is zero. This implies that

\[
\kappa = \frac{1}{2} - \frac{1}{m} - q_1 . \tag{A.2}
\]

Next we consider the condition coming from the \( G_{-\frac{3}{2}}^- G_{-\frac{1}{2}}^- \) descendant of \( \mathcal{N}_2 \), which simplifies to

\[
\mathcal{N}_3 = \left( -\left( \frac{2}{m} + 2 \right) \left( G_{-\frac{3}{2}}^- + L_2 G_{-\frac{1}{2}}^- \right) - \left( \frac{1}{m} - 1 \right) T_{-2} G_{-\frac{1}{2}}^- \right) + 2L_{-1}^2 G_{-\frac{3}{2}}^- + 2T_{-1} L_{-1} G_{-\frac{1}{2}}^- + 2L_{-1} G_{-\frac{1}{2}}^- \right) \phi_{\frac{3}{2}, \frac{1}{2}} = 0 . \tag{A.3}
\]

Using the same techniques as before, this implies the equation

\[
0 = z^{-2} \langle (\mathbb{1} \otimes G_{-\frac{3}{2}}^-) \rangle \left[ \left( \frac{2}{m} + 2 \right) \left( 1 + h_1 - \left( \kappa - \frac{1}{2} \right) \right) - \left( \frac{1}{m} - 1 \right) q_1 \right. \\
-2 \left( \kappa - \frac{3}{2} \right) \left( \kappa - \frac{1}{2} \right) - 2q_1 \left( \kappa - \frac{1}{2} \right) - 2 \left( \kappa - \frac{3}{2} \right) \] . \tag{A.4}
\]

We can now evaluate (A.4), plugging in the value for \( \kappa \) from (A.2), and the equation then becomes

\[
0 = \frac{(1 + 2j)(1 - 2k)(1 + m)}{2m^2} , \tag{A.5}
\]

where \( (j, k) \) labels the \( L_0 \) and \( T_0 \) eigenvalues of the highest weight state at \( z \) as in (5.6). We conclude that the odd fusion is only allowed for \( k = \frac{1}{2} \), as we can restrict \( j \) and \( k \) to be positive half-integers or of the form \( m - p \), where \( p \) is a half-integer. (As we mentioned before, we are considering here the generic case, where \( m \) is not an integer.) For \( k = \frac{1}{2} \), we can evaluate (A.2), and this gives the fusion, as described in (5.20).
To see that only one of the even fusions is allowed for \( k = \frac{1}{2} \), we observe that in this case, the field at \( z \) has the null-vector \((A.1)\), and we thus have
\[
0 = \langle \Delta_{z,0}(G^+_{1/2}) (\phi_{j,1/2} \otimes G^-_{1/2} \phi_{k,1/2}) \rangle \\
= 2 \langle (\phi_{j,1/2} \otimes L_{-1} \phi_{k,1/2}) \rangle \\
= -2kz^{-1} \langle (\phi_{j,1/2} \otimes \phi_{k,1/2}) \rangle .
\]
This implies for the even fusion rules
\[
\begin{align*}
  j_3 - k_3 & = j + \frac{1}{2} \\
  j_3 k_3 & = \frac{1}{2} j + \frac{1}{2},
\end{align*}
\]
which has the unique solution \( j_3 = j + 1, k_3 = \frac{1}{2} \).

**B** Some \( N = 2 \) fusion rules

The results for the \((R \otimes NS)\) fusion are
\[
(j,k)_- \otimes \left( \frac{3}{2}, \frac{1}{2} \right) = \begin{cases} 
  (j+1,k) \oplus (j,k-1) & \text{if } jk \neq 0, \\
  (j+1,k)_- \oplus (1,m+k-j) & \text{if } jk = 0,
\end{cases} \quad (B.1)
\]
\[
(j,k)_- \otimes \left( \frac{1}{2}, m - \frac{5}{2} \right) = \begin{cases} 
  (j+2,k-1) \oplus (j+1,k-2) & \text{if } k \neq 0, 1, j \neq 0, \\
  (j+2,k-1)_- \oplus (1,m+k-j-2) & \text{if } k = 1, j \neq 0, \\
  (1,m+k-j-2) \oplus (2,m+k-j-1) & \text{if } jk = 0,
\end{cases} \quad (B.2)
\]
\[
(1,1) \otimes (j,k) = \begin{cases} 
  (j+\frac{1}{2},k+\frac{1}{2}) \oplus (j-\frac{1}{2},k-\frac{1}{2})_- & \text{if } j \neq \frac{1}{2}, \\
  (j+\frac{1}{2},k+\frac{1}{2}) \oplus (m-k+\frac{1}{2},0)_- & \text{if } j = \frac{1}{2},
\end{cases} \quad (B.3)
\]
and
\[
(2,0)_- \otimes (j,k) = \begin{cases} 
  (j+\frac{3}{2},k-\frac{1}{2}) \oplus (j+\frac{1}{2},k-\frac{3}{2})_- & \text{if } k \neq \frac{1}{2}, \\
  (j+\frac{3}{2},k-\frac{1}{2})_- \oplus (1,m+k-j-1) & \text{if } k = \frac{1}{2},
\end{cases} \quad (B.4)
\]
where for the \( R \) representation \((j,k)\), for which one of the labels might be zero, a suffix ± has been included in order to indicate which of the two representations is referred to. The fusion rules of the fields \((\frac{1}{2}, \frac{3}{2})\) and \((m - \frac{5}{2}, \frac{1}{2})\) can be obtained from \((B.1)\) and \((B.2)\), using the obvious symmetry.

We also determined the following \((R \otimes R)\) fusion rules
\[
(1,1) \otimes (j,k)_+ = \begin{cases} 
  (j+\frac{1}{2},k+\frac{1}{2}) \oplus (j-\frac{1}{2},k-\frac{1}{2}) & \text{if } jk \neq 0, \\
  (j+\frac{1}{2},k+\frac{1}{2}) \oplus (m-k+j+\frac{1}{2},\frac{1}{2}) & \text{if } jk = 0,
\end{cases} \quad (B.5)
\]
and

\[(2, 0)_- \otimes (j, k)_- = \begin{cases} 
(j + \frac{3}{2}, k - \frac{1}{2}) \oplus (j + \frac{1}{2}, k - \frac{3}{2}) & \text{if } k \neq 1, 0, j \neq 0, \\
(j + \frac{3}{2}, k - \frac{1}{2}) \oplus \left( \frac{1}{2}, m + k - j - \frac{3}{2} \right) & \text{if } k = 1, j \neq 0, \\
\left( \frac{3}{2}, m + k - j - \frac{1}{2} \right) \oplus \left( \frac{1}{2}, m + k - j - \frac{3}{2} \right) & \text{if } jk = 0.
\end{cases} \]

\[(B.6)\]

**Acknowledgements**

I would like to thank Adrian Kent for suggesting this problem and for useful discussions. I acknowledge useful conversations with Matthias Dörرزapf, Wolfgang Eholzer, Peter Goddard and Gérard Watts.

This work was supported by a Research Fellowship of Jesus College, Cambridge, and partly by PPARC and EPSRC, grant GR/J73322.

**References**

[1] W. Boucher, D. Friedan, A. Kent: *Determinant formulae and unitarity for the \(N = 2\) superconformal algebras in two dimensions or exact results on string compactification*, Phys. Lett. **172 B**, 316-322 (1986)

[2] M. Dörرزapf: *Analytic expressions for singular vectors of the \(N = 2\) superconformal algebra*, preprint DAMTP-94-53, [hep-th/9601056](https://arxiv.org/abs/hep-th/9601056), to appear in Commun. Math. Phys.

[3] C. Dong, H. Li, G. Mason: *Twisted representations of vertex operator algebras*, preprint [q-alg/9509005](https://arxiv.org/abs/q-alg/9509005)

[4] W. Eholzer, M.R. Gaberdiel: *Unitarity of rational \(N = 2\) superconformal theories*, DAMTP-96-06, [hep-th/9601163](https://arxiv.org/abs/hep-th/9601163)

[5] H. Eichenherr: *Minimal operator algebras in superconformal quantum field theory*, Phys. Lett. **151 B**, 26-31 (1985)

[6] D. Friedan, Z. Qiu, S. Shenker: *Superconformal invariance in two dimensions and the tricritical Ising model*, Phys. Lett. **151 B**, 37-43 (1985)

[7] M.R. Gaberdiel: *Fusion in conformal field theory as the tensor product of the symmetry algebra*, Int. Journ. Mod. Phys. **A 9**, 4619-4636 (1994)

[8] M.R. Gaberdiel: *Fusion rules of chiral algebras*, Nucl. Phys. **B 417**, 130-150 (1994)

[9] M.R. Gaberdiel, H.G. Kausch: *Indecomposable fusion products*, DAMTP-96-36, [hep-th/9604026](https://arxiv.org/abs/hep-th/9604026)

[10] M.R. Gaberdiel, H.G. Kausch: *A rational logarithmic conformal field theory*, DAMTP-96-54, [hep-th/9606050](https://arxiv.org/abs/hep-th/9606050)

[11] M.R. Gaberdiel: *Chiral conformal field theory*, Ph.D. thesis, Cambridge University, 1995.
[12] M.R. Gaberdiel, P. Goddard: in preparation.

[13] P. Goddard: *Meromorphic conformal field theory*, in: V.G. Kac (ed.) Infinite dimensional Lie Algebras and Lie Groups. Proceedings of CIRM-Luminy Conference 1988, p. 556. Singapore, New Jersey, Hong Kong: World Scientific 1989

[14] V. Gurarie: *Logarithmic operators in conformal field theory*, Nucl. Phys. B **410**, 535-549 (1993)

[15] Y.-Z. Huang and J. Lepowsky, *A theory of tensor products for module categories for a vertex operator algebra I, II*, preprints (1993), hep-th/9309076, hep-th/9309159, to appear in Selecta Mathematica

[16] V.G. Kac, W. Wang: *Vertex operator superalgebras and their representations*, Contemp. Math. **175**, 161-191 (1994), hep-th/9312065

[17] E. Kiritsis: *Structure of N = 2 superconformally invariant unitary “minimal” theories: operator algebra and correlation functions*, Phys. Rev. D**36**, 3048-3065 (1987)

[18] G. Mussardo, G. Sotkov, M. Stanishkov: *Fusion rules, four-point functions and discrete symmetries of N = 2 superconformal models*, Phys. Lett. B**218**, 191-199 (1989)

[19] G. Mussardo, G. Sotkov, M. Stanishkov: *N = 2 superconformal minimal models*, Int. Journ. Mod. Phys. A **4**, 1135-1206 (1989)

[20] W. Nahm: *Quasirational fusion products*, Int. J. Mod. Phys. B **8**, 3693-3702 (1994)

[21] S. Odake: *c = 3d conformal algebra with extended supersymmetry*, Mod. Phys. Lett. A **5**, 561-580 (1990)

[22] A. Schwimmer, N. Seiberg: *Comments on the N = 2, 3, 4 superconformal algebras in two dimensions*, Phys. Lett. B**184**, 191-196 (1987)

[23] G.M. Sotkov, M.S. Stanishkov: *N = 1 superconformal operator product expansions and superfield fusion rules*, Phys. Lett. B**177**, 361-376 (1986)

[24] G.M.T. Watts: *Null vectors of the superconformal algebra: The Ramond sector*, Nucl. Phys. B **407**, 213-236 (1993)

[25] Y. Zhu: *Vertex operator algebras, elliptic functions, and modular forms*, PhD thesis, Yale University (1990)