ABCDP: Approximate Bayesian Computation Meets Differential Privacy

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Abstract

We develop a novel approximate Bayesian computation (ABC) framework, ABCDP, that obeys the notion of differential privacy (DP). Under our framework, simply performing ABC inference with a mild modification yields differentially private posterior samples. We theoretically analyze the interplay between the ABC similarity threshold $\epsilon_{abc}$ (for comparing the similarity between real and simulated data) and the resulting privacy level $\epsilon_{dp}$ of the posterior samples, in two types of frequently-used ABC algorithms. We apply ABCDP to simulated data as well as privacy-sensitive real data. The results suggest that tuning the similarity threshold $\epsilon_{abc}$ helps us obtain better privacy and accuracy trade-off.

1. Introduction

Approximate Bayesian computation (ABC) aims at identifying the posterior distribution over simulator parameters. The posterior distribution is of interest as it provides the mechanistic understanding of the stochastic procedure that directly generates data in many areas such as climate and weather, ecology, cosmology, and bioinformatics (Tavaré et al., 1997; Ratmann et al., 2007; Bazin et al., 2010; Schafer and Freeman, 2012). Under these complex models, directly evaluating the likelihood of data given the parameters is often intractable. ABC resorts to an approximation of the likelihood function using simulated data that are similar to the actual observations.

In the simplest form of ABC called rejection ABC, we proceed by sampling multiple model parameters from a prior distribution $\pi$: $\theta_1, \theta_2, \ldots \sim \pi$. For each $\theta_i$, a pseudo dataset $Y_i$ is generated from a simulator (the forward sampler associated with the intractable likelihood $p(y|\theta)$). The parameter $\theta_i$ for which the generated $Y_i$ are similar to the observed $Y^*$, as decided by $\rho(Y_i, Y^*) < \epsilon_{abc}$, are accepted. Here $\rho$ is a notion of distance, for instance, L2 distance between $Y_i$ and $Y^*$ in terms of a pre-chosen summary statistic. Whether the distance is small or large is determined by $\epsilon_{abc}$, a similarity threshold. The result is samples \{$\theta_i\}_{i=1}^M$ from a distribution, $\tilde{p}_c(\theta|Y^*) \propto \pi(\theta)\tilde{p}_c(Y^*|\theta)$, where $\tilde{p}_c(Y^*|\theta) = \int_{B_{\epsilon}(Y^*)} p(Y|\theta)dY$ and $B_{\epsilon}(Y^*) = \{y : \rho(Y, Y^*) < \epsilon_{abc}\}$. As the likelihood computation is approximate, so is the posterior distribution. Hence, this framework is named by “approximate Bayesian computation.”

Most ABC algorithms evaluate the data similarity in terms of summary statistics computed by an aggregation of individual datapoints (Joyce and Marjoram, 2008; Robert et al., 2011; Nunes and Balding, 2010; Aeschbacher et al., 2012; Drovandi et al., 2010; Aeschbacher et al., 2012; Fearnhead and Prangle, 2012). However, this seemingly innocuous step of similarity check could impose a privacy threat, as aggregated statistics could still reveal an individual’s participation to the dataset with the help of combining other publicly available datasets (see Homer et al. (2008); Johnson and Shmatikov (2013)). In addition, in some studies, the actual observations are privacy-sensitive in nature e.g., Genotype data for estimating Tuberculosis transmission parameters (Tanaka et al., 2006). Hence, there is in dire need of privatization in the step of similarity check in ABC algorithms.

In this light, we introduce an ABC framework that obeys the notion of differential privacy. The differential privacy definition provides a way to quantify the amount of information that the distance computed on the privacy-sensitive data contains on whether or not a single individual’s data is included (or modified) in the data (Dwork et al., 2006b). Differential privacy also provides rigorous privacy guarantees in the presence of arbitrary side information such as similar public data available.

A common form of applying DP to an algorithm is
by adding noise to outputs of the algorithm. In case of ABC, our theoretical results suggest that adding noise to the distance computed on the real observations and pseudo-data suffices the privacy guarantee of the resulting posterior samples. Besides, unlike other algorithms, ABC has a special parameter, the similarity threshold parameter, $\epsilon_{abc}$. What is the precise relationship between $\epsilon_{abc}$ and the final privacy level of the posterior samples? To date, the answer to this question is not well understood. Intuitively, one expects that the smaller the threshold $\epsilon_{abc}$, the closer the distance between the posterior samples to the true parameters that generated the real observations, implying the resulting posterior samples are less private. On the other hand, the larger the threshold $\epsilon_{abc}$, the farther the distance from the posterior samples to the true parameters, implying the posterior samples are more private. How could we formalize this intuition and formulate an ABC framework with a DP guarantee (ABCDP)? This is precisely a thesis of this work.

Furthermore, ABC inference requires a repeated use of sensitive data to compare the similarity for a new sample to the true parameters, implying the posterior samples are more private. How could we formalize this intuition and formulate an ABC framework with a DP guarantee (ABCDP)? This is precisely a thesis of this work.

Putting together, we conclude our introduction by summarizing our main contributions:

1. We provide two novel ABC frameworks: rejection-ABC and soft-thresholding-ABC. We take into account the intuition that the similarity threshold $\epsilon_{abc}$ could be used for introducing a privacy guarantee in the posterior samples. The resulting ABCDP frameworks can improve the trade-off between privacy and accuracy of the posterior samples, as the privacy loss under ABCDP is a function of the similarity threshold $\epsilon_{abc}$ as well as the level of injected noise for privacy.

2. We provide a tight privacy analysis using the Renyi differential privacy for our soft-thresholding ABCDP framework. This analysis results in a very small cumulative privacy loss after a repeated use of data, compared to the pure $\epsilon$ guarantee we have in our rejection-ABCDP framework.

3. We validate our theory in both frameworks in the experiments using simulated data as well as a privacy-sensitive real-world dataset.

2. Background

We start by describing some relevant background information.

2.1. Approximate Bayesian Computation

Given a set $Y^*$ containing observations, rejection ABC (Bazin et al., 2010) yields samples from an approximate posterior distribution by repeating the following three steps:

\begin{align}
\theta & \sim p(\theta) , \quad (1) \\
Y & \sim p(y|\theta), \quad (2) \\
p_{\epsilon_{abc}}(Y^*|\theta) & \sim p(Y^*|\theta)p(\theta), \quad (3)
\end{align}

where the pseudo dataset $Y$ is compared with the observations $Y^*$ via

\begin{align}
p_{\epsilon_{abc}}(Y^*|\theta) &= \int_{B_{\epsilon_{abc}}(Y^*)} p(Y|\theta) dY, \quad (4) \\
B_{\epsilon_{abc}}(Y^*) &= \{y|p(Y|Y^*) \leq \epsilon_{abc}\}, \quad (5)
\end{align}

where $\rho$ is a divergence measure between two datasets, and is often defined as a distance between the summary statistics of the two datasets i.e., $\rho(Y,Y^*) = D(S(Y),S(Y^*))$, with some distance measure $D$ on the statistics computed by $S$.

In soft-thresholding ABC (soft-ABC), on the other hand, parameter samples from the prior are weighted rather than accepted or rejected. An algorithm in this family is K2-ABC (Park et al., 2016) where the weight for each sampled parameter $\theta_j$ is given by $w_j = \frac{\exp(-\varphi^2(Y_j,Y^*)_{\epsilon_{abc}})}{\sum_{i=1}^N \exp(-\varphi^2(Y_i,Y^*)_{\epsilon_{abc}})}$, and $\kappa_q(Y,Y') = \exp\left(-\frac{\varphi^2(Y,Y')_{\epsilon_{abc}}}{\epsilon_{abc}}\right)$, for $q > 0$. Given a set of observations $Y^*$, K2-ABC yields weighted samples $\{(\theta_j, w_j)\}_{j=1}^M$. One can directly use these for estimating posterior expectations: given a function $f$, the expectation $\int_{\Theta} f(\theta)p(\theta|y^*)d\theta$ is estimated using

$$\hat{E}[f(\theta)] = \sum_{j=1}^M w_j f(\theta_j).$$

While any divergence metric can be used for $\rho$ in the soft-ABC framework, K2-ABC uses Maximum Mean Discrepancy (MMD, Gretton et al. (2012)), and we will also use MMD in some part of our experiments. MMD is described below.

Maximum Mean Discrepancy Assume that the data $Y \subset X$ and let $k : X \times X$ be a positive definite kernel. The MMD between two distributions $P, Q$ is defined as

\begin{align}
\text{MMD}(P,Q) := \\
(\mathbb{E}_{x,x' \sim P} k(x,x') + \mathbb{E}_{y,y' \sim Q} k(y,y') - 2\mathbb{E}_{x \sim P} \mathbb{E}_{y \sim Q} k(x,y))^\frac{1}{2}.
\end{align}
The Moore–Aronszajn theorem states that there is a unique Hilbert space $H$ on which $k$ defines an inner product. As a result, there exists a feature map $\phi: X \rightarrow H$ such that $k(x, y) = \langle \phi(x), \phi(y) \rangle_H$, where $\langle \cdot, \cdot \rangle_H$ denotes the inner product on $H$. The MMD in eq. 6 can be written as

$$\text{MMD}(P, Q) = ||E_{x \sim P}[\phi(x)] - E_{y \sim Q}[\phi(y)]||_H,$$

where $E_{x \sim P}[\phi(x)] \in H$ is known as the (kernel) mean embedding of $P$, and exists if $E_{x \sim P}\sqrt{k(x, x)} < \infty$ (Smola et al., 2007). The MMD can thus be interpreted as the distance between the mean embeddings of the two distributions. If $k$ is a characteristic kernel (Sriperumbudur et al., 2011), then $P \mapsto E_{x \sim P}[\phi(x)]$ is injective, and MMD forms a metric, implying that $\text{MMD}(P, Q) = 0$, if and only if $P = Q$.

Unlike a pre-chosen finite dimensional summary statistic typically used in ABC literature, MMD compares two distributions in terms of all the possible moments of the the random variables described by the two distributions. Hence, in theory, ABC frameworks using the MMD metric such as K2-ABC can avoid the problem of non-sufficiency of a chosen summary statistic that may incur in many ABC methods.

When $P, Q$ are observed through samples $X_n = \{x_i\}_{i=1}^n \sim P$ and $Y_n = \{y_i\}_{i=1}^n$ (as is the case in practice), eq. 6 can be estimated by simply replacing the population expectations with empirical averages (Gretton et al., 2012, eq. 3):

$$\widehat{\text{MMD}}(X_n, Y_n) = \left( \frac{1}{m} \sum_{i,j=1}^m k(x_i, x_j) + \frac{1}{n} \sum_{i,j=1}^n k(y_i, y_j) \right) - \frac{2}{mn} \sum_{i=1}^m \sum_{j=1}^n k(x_i, y_j)^\frac{3}{2}.$$

When applied in the ABC setting, one input set to MMD is $Y^*$ and the other is a pseudo dataset $Y \sim p(\cdot; \theta_i)$ generated from the simulator, for some $\theta_i \sim \pi(\theta)$. Note that the total computational cost of the estimator $\widehat{\text{MMD}}(X_n, Y_n)$ is $O(Tmn)$ for the $T$ number of posterior samples. To reduce the complexity, subquadratic time MMD estimators exist e.g., an unbiased linear-time estimator (Gretton et al., 2012, Section 6).

See Appendix Sec. C for MMD with random Fourier features (Rahimi and Recht, 2008) for another linear-time estimator.

### 2.2. Differential Privacy

An output from an algorithm that takes in sensitive data as input will naturally contain some information of the sensitive data $D$. The goal of differential privacy is to augment such an algorithm so that useful information about the population is retained, while sensitive information such as an individual’s participation in the dataset cannot be learned (Dwork and Roth, 2014). A common way to achieve these two seemingly paradoxical goals is by deliberately injecting a controlled-level of random noise to the to-be-released quantity. The modified procedure, known as a DP mechanism, now gives a stochastic output due to the injected noise. In the DP framework, higher level of noise provides stronger privacy guarantee at the expense of less accurate population-level information that can be derived from the released quantity. Less noise added to the output thus reveals more about an individual’s presence in the dataset.

More formally, given a mechanism $M$ and neighbouring datasets $D, D'$ differing by a single entry, the privacy loss of an outcome $o$ is defined by

$$L(o) = \log \frac{P(M(D) = o)}{P(M(D') = o)},$$

The mechanism $M$ is called $(\epsilon, \delta)$-DP if and only if

$$|L(o)| \leq \epsilon$$

for all possible outcomes $o$ and for all possible neighbouring datasets $D, D'$. A weaker version of the above notion is $(\epsilon, \delta)$-DP: $M$ is $(\epsilon, \delta)$-DP if $|L(o)| \leq \epsilon$, with probability at least $1 - \delta$. The definition states that a single individual’s participation in the data does not change the output probabilities by much; this limits the amount of information that the algorithm reveals about any one individual.

A differentially private algorithm is designed by adding noise to the algorithms’ outputs. Suppose a deterministic function $h : D \rightarrow \mathbb{R}^p$ computed on sensitive data $D$ outputs a $p$-dimensional vector quantity. In order to make $h$ private, we can add noise to the output of $h$, where the level of noise is calibrated to the global sensitivity (Dwork et al., 2006a), $\Delta_h$, defined by the maximum difference in terms of some norm $\|h(D) - h(D')\|$ for neighboring $D$ and $D'$ (i.e. differ by one data sample). For the Gaussian mechanism (Theorem 3.22 in Dwork and Roth (2014)), the perturbed output is given by

$$\tilde{h}(D) = h(D) + N(0, \sigma^2 I_p).$$

The perturbed function $\tilde{h}(D)$ is then $(\epsilon, \delta)$-DP, where $\sigma \geq \Delta_h \sqrt{2\log(1.25/\delta)}/\epsilon$, for $\epsilon \in (0, 1)$ and $\Delta_h$ uses L2 norm $\|h(D) - h(D')\|_2$. (See the proof of Theorem 3.22 in Dwork and Roth (2014) for more details).

There are two important properties of differential privacy that we will use in this work. The compossibility theorem (Dwork et al., 2006a) states that the strength of privacy guarantee degrades with repeated use of DP-algorithms. Formally, given an $(\epsilon_1, \delta_1)$-DP mechanism $M_1$ and an $(\epsilon_2, \delta_2)$-DP mechanism $M_2$, the mechanism $M(D) := (M_1(D), M_2(D))$ is...
\((\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)\)-DP. This composition is often-called linear composition, as the parameters are simply linearly summed up. Furthermore, the post-processing invariance property (Dwork et al., 2006a) tells us that the composition of any arbitrary data-independent mapping with an \((\epsilon, \delta)\)-DP algorithm is also \((\epsilon, \delta)\)-DP.

2.3. Rényi Differential Privacy

ABC algorithms are iterative in nature, and the simple linear composition would result in a high cumulative privacy loss. For a more refined calculation of cumulative privacy loss, we use the recently developed notion of privacy called Rényi Differential Privacy (RDP) (Mironov, 2017). As introduced below, RDP takes an expectation over the outcomes of the DP mechanism, rather than taking a single worst case as in pure DP (eq. 8). Hence, the upper bound in RDP is significantly smaller than the linear sum of worst cases in the pure DP case, resulting in a greatly reduced cumulative privacy loss for a repeated use of data.

**Definition 2.1** ((\(\alpha, \epsilon\))-RDP). A mechanism is called \(\epsilon\) Rényi differentially private with an order \(\alpha\) if for all neighbouring datasets \(D, D^\prime\),

\[
D_\alpha(M(D)||M(D^\prime)) \leq \epsilon,
\]

where \(D_\alpha\) denotes \(\alpha\)-Rényi Divergence.

\(D_\alpha(P||Q)\) is defined below.

**Definition 2.2** (\(\alpha\)-Rényi Divergence). For two probability distributions \(P, Q\) that have the same support, the \(\alpha\) Rényi divergence is

\[
D_\alpha(P||Q) = \frac{1}{\alpha - 1} \log \mathbb{E}_{x \sim Q(x)} \left( \frac{P(x)}{Q(x)} \right) ^ \alpha
\]

for \(\alpha \in (1, \infty)\).

For instance, a frequently used DP mechanism is Gaussian mechanism, which also satisfies the RDP notion as below:

**Corollary 2.1** (Gaussian mechanism). As \(D_\alpha(N(\tau, \sigma^2)||N(\mu, \sigma^2)) = \frac{\alpha ||\tau - \mu||^2}{2 \sigma^2}\), the Gaussian mechanism satisfies \((\alpha, \frac{\alpha ||\tau - \mu||^2}{2 \sigma^2})\)-RDP.

The repeated use of adaptive RDP mechanisms composes by the Theorem below:

**Theorem 2.2** (Composition of adaptive RDP mechanisms). Let \(f: D \mapsto \mathcal{R}_1\) be \((\alpha, \epsilon_1)\)-RDP. Let \(g: \mathcal{R}_1 \times D \mapsto \mathcal{R}_2\) be \((\alpha, \epsilon_2)\)-RDP. Then, the mechanism releasing \((X, Y)\), where \(X \sim f(D)\) and \(Y \sim g(D, X)\) satisfies \((\alpha, \epsilon_1 + \epsilon_2)\)-RDP.

Once we compute the cumulative privacy loss using the RDP composition, we convert the RDP notion to the original definition of DP by the following proposition.

**Proposition 1.** [From RDP to DP (Mironov, 2017)]

If \(M\) is a \((\alpha, \epsilon)\)-RDP mechanism, then it also satisfies \(\left(\epsilon + \frac{\log 1/\delta}{\alpha - 1}, \delta\right)\)-DP for any \(0 < \delta < 1\).

3. PROPOSAL: ABCDP

Here we present our ABCDP framework. We first describe the privacy setup we consider in this paper, then describe our algorithms.

3.1. Problem Setup

Suppose there is a data owner who owns valuable but sensitive data \(Y^*\) and is willing to contribute to the posterior inference. Suppose there is a modeler, whose aim is to identify the posterior distribution of the parameters of a simulator, which can well model the data.

We frame the ABCDP algorithm to be performed in the following two steps: (Step 1: non-private) the modeler draws a parameter sample \(\theta_t\) from a prior \(\pi(\theta)\) at \(t = 1, \ldots, T\). The modeler then runs a \(p(y|\theta)\) with \(\theta = \theta_t\) to obtain a pseudo dataset \(Y_t\). We assume that the the samples \(\{\theta_t\}\) from the prior and also the pseudo-data \(\{Y_t\}\) from the simulator given \(\{\theta_t\}\) are all public information (i.e., these are accessible by anyone including an adversary). (Step 2: private): the data owner takes \(\{\langle \theta_t, Y_t \rangle\}_{t=1}^T\) and outputs either a set of binary indicators determining whether each \(Y_t\) is similar to \(Y^*\) (i.e., accept \(\theta_t\) or not) in the case of rejection ABC or a vector of continuous weights representing the degree of similarity to \(Y^*\) in the case of soft ABC.

In either case, the data owner applies our proposed DP mechanism to the output (Section 3.2 for rejection ABC, and Section 3.3 for soft ABC) to guarantee the privacy on the sensitive data.

3.2. Proposed Mechanism for Rejection ABC

The first framework we introduce is a rejection ABC framework with a differentially private distance.

**Definition 3.1** (\(\mathcal{M}_{\text{rej}}\): DP mechanism for one-step rejection ABC). Given a dataset \(Y^*\), a pseudo-data sample \(Y_t\), noise variance \(\sigma_{\text{rej}}\), ABC rejection threshold \(\epsilon_{\text{abc}}\), and a distance \(p\), the mechanism \(\mathcal{M}_{\text{rej}}\) outputs a stochastic binary indicator given by

\[
\mathcal{M}_{\text{rej}}(Y^*, Y_t, \epsilon_{\text{abc}}, \sigma_{\text{rej}}) = I \left[ D(Y^*, Y_t) \leq \epsilon_{\text{abc}} \right] := \tau_t,
\]
Algorithm 1 Proposed rejection ABCDP algorithm

**Input:** Observations $Y^*$, posterior sample size $T$, ABC threshold $\epsilon_{abc}$, tolerable total privacy loss $\epsilon_{\text{total}}$, ABC distance $\rho$, and $\{(\theta_i,Y_i)\}_{i=1}^T$

**Output:** $\epsilon_{\text{total}}$-DP binary indicators $\{\tau_i\}_{i=1}^T$ for corresponding samples $\{\theta_i\}_{i=1}^T$ where $\tau_i \in \{0,1\}$

1. Per-step privacy loss $\epsilon = \epsilon_{\text{total}}/T$
2. Noise variance $\sigma_{\text{rej}} = h^{-1}(\epsilon)$. See Theorem 3.1.
3. for $t = 1, \ldots, T$
4. Draw noise $n_t \sim \mathcal{N}(0,\sigma_{\text{rej}}^2)$
5. $\tau_t = I[\rho(Y^*,Y_t) + n_t \leq \epsilon_{abc}]$
6. end for

where $D(Y^*,Y_t) = \rho(Y^*,Y_t) + n_t$, and $n_t$ is i.i.d. drawn from $\mathcal{N}(0,\sigma_{\text{rej}}^2)$.

Given a level of privacy loss $\epsilon$ that the data owner can tolerate, the noise variance $\sigma_{\text{rej}}^2$ can be chosen so as to guarantee that $M_{\text{rej}} = \epsilon$-DP. This is formally stated in Theorem 3.1.

**Theorem 3.1 ($M_{\text{rej}}$ is $\epsilon$-DP).** For any neighbouring datasets $X,X'$ of size $N$ and any dataset $Y$, assume that $\rho$ is such that $0 < \sup_{(x,x')\in X} \rho(x,x') \leq \Delta_p < \infty$. Assume that $\rho$ is bounded by $B_p$, i.e., $0 < \sup_{X,Y} \rho(x,x') \leq B_p < \infty$ where $X,Y$ are datasets of any size. Consider the mechanism $M_{\text{rej}}$ defined in Def. 3.1. Define

$$h(\sigma) = \min \left( 1, \frac{\Delta_p}{\sqrt{2\pi\sigma}} \right) \Phi \left( \frac{\min(\epsilon_{abc}-B_p,-\epsilon_{abc})}{\sigma} \right)^{-1},$$

where $\Phi$ is the CDF of the standard normal. Given an $\epsilon$, choose $\sigma_{\text{rej}} = h^{-1}(\epsilon)$ where $h^{-1}$ is the inverse of $h$. Then, the mechanism is $\epsilon$-DP where $\epsilon = h(\sigma_{\text{rej}})$.

A proof is given in Appendix Sec. A. Note that the function $h(\sigma)$ is strictly decreasing on $(0,\infty)$ and has a well-defined inverse $h^{-1}$. Thus, given a desired bound on privacy loss $\epsilon$, we can efficiently find the right noise variance $\sigma$ such that $\epsilon = h(\sigma)$ with binary search. In general, the upper bound $\Delta_p$ depends on the sample size. If $|Y^*| = N$ and $\rho(X,Y) = \text{MMD}(X,Y)$ with a bounded kernel, then $\Delta_p = O(1/N)$ as shown in Lemma 3.1.

**Lemma 3.1 ($\Delta_p = O(1/N)$ for MMD).** Assume that $Y^*$ and each pseudo dataset $Y_t$ are of the same cardinality $N$. Set $\rho(X,Y) = \text{MMD}(X,Y)$ with a kernel $k$ bounded by $B_k > 0$ i.e., $\sup_{x,y\in X} k(x,y) \leq B_k < \infty$. Then,

$$\sup_{(x,x'),Y} \rho(x,x') - \rho(x',y) \leq \Delta_p = \frac{2}{N} \sqrt{B_k}$$

and $\sup_{X,Y} \rho(x,y) \leq B_p = 2\sqrt{B_k}$.

Algorithm 2 Proposed soft ABCDP algorithm

**Input:** Observations $Y^*$, posterior sample size $T$, ABC threshold $\epsilon_{abc}$, tolerable total privacy loss $\epsilon_{\text{total}}$, ABC distance $\rho$ and $\{(\theta_i,Y_i)\}_{i=1}^T$

**Output:** $(\epsilon_{\text{total}},\delta_{\text{total}})$-DP weights $\{\omega_i\}_{i=1}^T$ for corresponding samples $\{\theta_i\}_{i=1}^T$

Compute $\sigma_{\text{soft}}$ by eq. 16 for $t = 1, \ldots, T$

$$\omega_t = \exp \left( -\Delta(Y^*,Y_t) \right)$$

where $\Delta(Y^*,Y_t)$ is in eq. 14

end for

$$\omega_t = \frac{\omega_t}{\sum_{j} \omega_j},$$

for all $t = 1, \cdots, T$

A proof is given in Appendix Sec. B. With the result in Lemma 3.1, the function $h$ in Theorem 3.1 is given by $h(\sigma) = \min \left( 1, \frac{2}{\sqrt{2\pi\sigma}} \right) \Phi \left( \frac{\min(\epsilon_{abc}-B_p,-\epsilon_{abc})}{\sigma} \right)^{-1}$. Note that $\lim_{N \to \infty} h(\sigma) = 0$ for any $\sigma > 0$. This means that as the size of the sensitive data $Y^*$ grows larger, we need smaller perturbation (i.e., smaller $\sigma_{\text{rej}}$) to the true output to guarantee the same privacy loss. For $\varrho = \text{MMD}$ using a Gaussian kernel, $k(x,y) = \exp \left( -\frac{|x-y|^2}{2\nu} \right)$ where $\nu > 0$ is the bandwidth of the kernel, $B_k = 1$ for any $\nu > 0$.

In practice, the mechanism $M_{\text{rej}}$ will be executed $T$ times where $T$ is the number of queries. To guarantee that the total privacy loss is no larger than $\epsilon_{\text{total}}$, it is sufficient to set the one-step privacy loss $\epsilon = \epsilon_{\text{total}}/T$, due to the linear composition. The full algorithm is shown in Algorithm 1.

### 3.3. Proposed Mechanism for Soft ABC

Now we present a DP mechanism, in which privacy analysis nicely identifies the relationship between the ABC similarity threshold and the privacy level. We propose a mechanism for releasing soft-ABC weight differentially privately. First, we define the mechanism.

**Definition 3.2 (The ABC weight mechanism for one-step soft ABC).** Given a dataset $Y^*$, a pseudo-data sample $Y_t$, ABC soft threshold $\epsilon_{\text{ABC}}$, privacy parameter $\sigma$ and features $\phi$, the Soft-ABC weight mechanism $M_{\text{SoftWeight}} : (Y^*,Y_t,\epsilon_{\text{ABC}},\sigma) \mapsto -\log \tilde{\omega}_t$ is defined by

$$M_{\text{SoftWeight}}(Y^*,Y_t,\epsilon_{\text{ABC}},\sigma_{\text{soft}}) = \hat{D}(Y^*,Y_t)$$

where the noisy distance is defined by

$$\hat{D}(Y^*,Y_t) = D_t + \epsilon_t,$$
\[ D_t := \frac{1}{e_{abc}} \rho(Y^*, Y_t), \] and the noise \( n_t \) is i.i.d. drawn from \( \mathcal{N}(0, \sigma^2_{soft}) \).

Note that when the computed noisy distance is negative, we project back to a small positive number so that the resulting weight (after exponentiation. See Algorithm 2) does not dominate other weights. This extra step does not alter the DP guarantee that we provide due to the post-processing invariance of DP.

Now we formally state that the proposed mechanism is RDP and also DP in the following theorem.

**Theorem 3.2.** The Soft-ABC weight mechanism defined in Def. 3.2 is \((\alpha, \epsilon)\)-RDP, where

\[
\epsilon = \frac{\alpha \Delta^2_p}{2 \sigma^2_{soft} \rho} \tag{15}
\]

and \( \Delta_p \) is the L2-sensitivity of \( \rho \).

A proof is given in Appendix Sec. D.

**Remark.** In Thm. 3.2, we can clearly see the interplay between the amount of noise \( \sigma_{abc} \) added for privacy and the soft threshold \( \epsilon_{abc} \). With a fixed amount of noise \( \sigma_{soft} \), the smaller the similarity threshold \( \epsilon_{abc} \), the larger the privacy loss \( \epsilon \). This is well aligned with our intuition that a smaller similarity threshold would encourage the resulting posterior sample to be closer to what the true data support, hence that posterior sample reveals more about the true data (less private, hence more privacy loss, i.e., higher \( \epsilon \)). On the other hand, with a fixed amount of noise \( \sigma_{soft} \), the larger the similarity threshold \( \epsilon_{abc} \), the smaller the privacy loss \( \epsilon \). In this case, the mean of perturbed distance gets smaller, resulting in many weights being larger and the similar to each other (i.e., from \( \omega_2 = \exp(-\hat{D}) \)), – at this point, the distance matters less – which forces taking more samples from the prior, i.e., the weights and weighted posterior samples become more private (i.e., becoming irrelevant to the data). With a fixed \( \epsilon_{abc} \), the larger the noise \( \sigma_{abc} \), the more private the weights, as the distance gets noisier, and hence the resulting \( \epsilon \) (privacy loss) becomes small.

In practice, to obtain an appropriate amount of noise to add to \( \rho \) in each sampling step, we convert the RDP level \((\epsilon\text{-RDP})\) to match the desired DP level \((\epsilon_{total}, \delta_{total})\)-DP due to Prop. 1

\[
\epsilon_{total} \geq \epsilon + \frac{\log(1/\delta_{total})}{\alpha - 1}. \tag{16}
\]

### 3.4. Privacy Analysis for Soft-ABCDP

Now we also formally state that the proposed algorithm is differentially private in the following theorem.

**Theorem 3.3.** Algorithm 2 is \((\alpha, \epsilon' (\alpha))\)-Renyi differentially private, where

\[
\epsilon'(\alpha) = \epsilon T = \frac{\alpha \Delta^2_T}{2 \sigma^2_{soft} \rho}.
\]

Algorithm 2 also satisfies \( \epsilon'(\alpha) + \frac{\log(1/\delta)}{\alpha - 1}, 0 \) for any \( 0 < \delta < 0 \).

A proof follows the composition theorem of RDP given in Thm. 2.2 and the proposition converting RDP to DP given in Prop. 1.

To find the smallest \( \sigma_{soft} \) and an optimal \( \alpha \) that satisfy eq. 16, we use the numerical tool\(^1\) published by Wang et al. (2019).

\[^1\)https://github.com/yuxiangw/autodp\].
A vector of mixing proportions is our model parameter, \( \pi \), where the ground truth is \( \theta^* = [0.25, 0.04, 0.33, 0.04, 0.34]^\top \) (see Fig. 1). The goal is to estimate \( \mathbb{E}[\theta|Y^*] \) where \( Y^* \) is generated with \( \theta^* \).

We first generated 5000 samples for \( Y^* \) drawn from eq. 17 with true parameters \( \theta^* \). Then we tested our two ABCDP frameworks with varying \( \epsilon_{abc} \) and \( \epsilon_{total} \). In these experiments, we set \( \rho = \text{MMD} \) with a Gaussian kernel. We set the bandwidth of the Gaussian kernel using the median heuristic computed on the simulated data (i.e., we did not use the real data for this, hence there is no privacy violation in this regard). For soft ABCDP, we drew 5000 pseudo-samples for \( Y_t \) at each time \( t \). We show the result of our soft ABCDP framework in Fig. 2.

\[ \pi(\theta) = \text{Dirichlet}(\theta; 1), \quad p(y|\theta) = \sum_{i=1}^{5} \theta_i \text{Uniform}(y; [i-1, i]). \] (17)

A vector of mixing proportions is our model parameter, \( \theta \), where the ground truth is \( \theta^* = [0.25, 0.04, 0.33, 0.04, 0.34]^\top \) (see Fig. 1). The goal is to estimate \( \mathbb{E}[\theta|Y^*] \) where \( Y^* \) is generated with \( \theta^* \).

We start by investigating the interplay between \( \epsilon_{abc} \) and \( \epsilon_{dp} = \epsilon \), in a simulated example where the ground truth parameters are known. Following (Park et al., 2016), we also consider a symmetric Dirichlet prior \( \pi \) and a likelihood \( p(y|\theta) \) given by a mixture of uniform distributions as

\[ \pi(\theta) = \text{Dirichlet}(\theta; 1), \quad p(y|\theta) = \sum_{i=1}^{5} \theta_i \text{Uniform}(y; [i-1, i]). \] (17)

A vector of mixing proportions is our model parameter, \( \theta \), where the ground truth is \( \theta^* = [0.25, 0.04, 0.33, 0.04, 0.34]^\top \) (see Fig. 1). The goal is to estimate \( \mathbb{E}[\theta|Y^*] \) where \( Y^* \) is generated with \( \theta^* \).

We first generated 5000 samples for \( Y^* \) drawn from eq. 17 with true parameters \( \theta^* \). Then we tested our two ABCDP frameworks with varying \( \epsilon_{abc} \) and \( \epsilon_{total} \). In these experiments, we set \( \rho = \text{MMD} \) with a Gaussian kernel. We set the bandwidth of the Gaussian kernel using the median heuristic computed on the simulated data (i.e., we did not use the real data for this, hence there is no privacy violation in this regard). For soft ABCDP, we drew 5000 pseudo-samples for \( Y_t \) at each time \( t \). We show the result of our soft ABCDP framework in Fig. 2.

\[ \pi(\theta) = \text{Dirichlet}(\theta; 1), \quad p(y|\theta) = \sum_{i=1}^{5} \theta_i \text{Uniform}(y; [i-1, i]). \] (17)

A vector of mixing proportions is our model parameter, \( \theta \), where the ground truth is \( \theta^* = [0.25, 0.04, 0.33, 0.04, 0.34]^\top \) (see Fig. 1). The goal is to estimate \( \mathbb{E}[\theta|Y^*] \) where \( Y^* \) is generated with \( \theta^* \).

For rejection ABCDP, we drew 1000 pseudo-samples for \( Y_t \) at each time. Since the bound in eq. 11 depends on the maximum of \( \rho \) denoted by \( B_\rho \), we pre-chose \( B_\rho \) to be either the top 5 or 20 percentile of the distances \( \rho(Y_t, Y_t') \) computed on simulated data. To ensure that actual maximum of \( \rho(Y_t, Y_t') \) values during the posterior sampling is indeed the pre-chosen values of \( B_\rho \), we took the minimum between \( \rho(Y_t, Y_t') \) and the pre-chosen \( B_\rho \). We show the result of rejection ABCDP in Fig. 3. Note that we do not intend to compare the results of private versions in Fig. 2 and in Fig. 3, as this rejection ABCDP provides a pure DP guarantee while soft ABCDP provides an approximate DP with a failure probability 1-\( \delta_{total} \) (where we set \( \delta_{total} = 10^{-4} \)).
could limit the sensitivity of $\rho(Y^*, Y_t)$ by taking the minimum between the evaluated distance $\rho(Y^*, Y_t)$ given $Y_t$ and a pre-chosen bound $B_\rho$ (this is precisely the same case as in the rejection ABC experiment in the earlier section). Once we ensure $\rho(Y^*, Y_t) \leq B_\rho$ for all $t$, we treat this maximum bound $B_\rho$ as a sensitivity and tune the noise accordingly. We show the prior and posterior samples obtained by soft ABC and soft ABCDP in Fig. 4.

As an error metric, we also computed the mean absolute distance between the posterior mean obtained by (Lintusaari et al., 2019) and ours in all three cases (2nd to 4th rows) shown in Fig. 4 assuming the posterior mean obtained by (Lintusaari et al., 2019) to be a ground truth. The errors averaged over 10 independent runs are 3.10 for non-private soft ABC with $\epsilon_{abc} = 20$ (2nd row in Fig. 4), 4.12 for soft ABCDP with $\epsilon_{abc} = 20$ and $\epsilon_{total} = 4$ and $\delta_{total} = 1e^{-4}$ (3rd row in Fig. 4), and 4.05 for soft ABCDP with $\epsilon_{abc} = 20$ and $\epsilon_{total} = 4$ and $\delta_{total} = 1e^{-4}$ (4th row in Fig. 4), respectively.

We presented two ABC algorithms that obey the notion of differential privacy. The resulting posterior samples are differentially private, either via releasing differentially private binary indicators (rejection ABC) or via releasing differentially private continuous weights (soft ABC). In our rejection ABCDP algorithm, we derived an upper bound for the privacy loss that guarantees $(\epsilon, \delta)$-DP, from which we can identify an appropriate amount of noise to privatize the posterior samples. In the second soft ABCDP algorithm, we used the RDP notion to get a tight bound on the privacy loss that guarantees $\epsilon$-DP, from which we can identify the same case as in the rejection ABC experiment in the earlier section). Once we ensure $\rho(Y^*, Y_t) \leq B_\rho$ for all $t$, we treat this maximum bound $B_\rho$ as a sensitivity and tune the noise accordingly. We show the prior and posterior samples obtained by soft ABC and soft ABCDP in Fig. 4.

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5. SUMMARY AND DISCUSSION

We presented two ABC algorithms that obey the notion of differential privacy. The resulting posterior samples are differentially private, either via releasing differentially private binary indicators (rejection ABC) or via releasing differentially private continuous weights (soft ABC). In our rejection ABCDP algorithm, we derived an upper bound for the privacy loss that guarantees $(\epsilon, \delta)$-DP, from which we can identify an appropriate amount of noise to privatize the posterior samples. In the second soft ABCDP algorithm, we used the RDP notion to get a tight bound on the privacy loss that guarantees $\epsilon$-DP, from which we can identify the same case as in the rejection ABC experiment in the earlier section). Once we ensure $\rho(Y^*, Y_t) \leq B_\rho$ for all $t$, we treat this maximum bound $B_\rho$ as a sensitivity and tune the noise accordingly. We show the prior and posterior samples obtained by soft ABC and soft ABCDP in Fig. 4.

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ABCDP: Approximate Bayesian Computation Meets Differential Privacy

Supplementary

A. PROOF OF THEOREM 3.1

We first recall Theorem 3.1 stated in the main text:

**Theorem 3.1** ($M_{rej}$ is $\epsilon$-DP). For any neighbouring datasets $X, X'$ of size $N$ and any dataset $Y$, assume that $\rho$ is such that $0 < \sup_{(X,X'),Y} |\rho(X,Y) - \rho(X',Y)| \leq \Delta_\rho < \infty$. Assume that $\rho$ is bounded by $B_\rho$ i.e., $0 < \sup_{X,Y} \rho(X,Y) \leq B_\rho < \infty$ where $X, Y$ are datasets of any size. Consider the mechanism $M_{rej}$ defined in Def. 3.1. Define

$$h(\sigma) = \min \left( 1, \frac{\Delta_\rho}{\sqrt{2\pi\sigma^2}} \right) \left[ \Phi \left( \frac{\min(\epsilon_{abc} - B_\rho, -\epsilon_{abc})}{\sigma} \right) \right]^{-1},$$ (11)

where $\Phi$ is the CDF of the standard normal. Given an $\epsilon$, choose $\sigma_{rej} = h^{-1}(\epsilon)$ where $h^{-1}$ is the inverse of $h$. Then, the mechanism is $\epsilon$-DP where $\epsilon = h(\sigma_{rej})$.

**Proof.** Recall that $M_{rej}(Y^*, Y_t, \epsilon_{abc}, \sigma) = I \left[ \tilde{D}(Y^*, Y_t) \leq \epsilon_{abc} \right] = \tau_t \in \{0, 1\}$ is the mechanism of the rejection ABC that outputs a binary value indicating whether or not $Y_t$ is close to $Y^*$, where

$$\tilde{D}(Y^*, Y_t) = \rho(Y^*, Y_t) + n_t,$$

$n_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$ and we write $\sigma$ for $\sigma_{rej}$. Here, $I[\cdot]$ denotes the Iverson bracket. For brevity, write $\tilde{D}_t := \tilde{D}(Y^*, Y_t)$ and $\rho(Y^*, Y_t) = \rho_t$. It follows that

$$\tilde{D}_t \sim \mathcal{N}(\rho_t, \sigma^2),$$

$$\epsilon_{abc} - \tilde{D}_t \sim \mathcal{N}(\epsilon_{abc} - \rho_t, \sigma^2),$$

$$\tau_t = I \left[ \epsilon_{abc} - \tilde{D}_t \geq 0 \right] \text{ implying that }$$

$$\tau_t \sim \text{Bernoulli}(p_t),$$

where $p_t = \Phi \left( \frac{\epsilon_{abc} - \rho_t}{\sigma} \right)$ (i.e., the probability that $\epsilon_{abc} - \tilde{D}_t \geq 0$) and $\Phi$ is the cumulative distribution function (CDF) of the standard normal distribution.

To show that $M_{rej}$ is $\epsilon$-DP, we need to show that $\sup_{(Y^*, Y')} (\epsilon_{abc}, \sigma) \max_{\epsilon \in \{0,1\}} \ln \frac{P(M_{rej}(Y^*, Y_t, \epsilon_{abc}, \sigma) = \epsilon)}{P(M_{rej}(Y^*, Y_t, \epsilon_{abc}, \sigma) = \epsilon)} \leq \epsilon$ where

$(Y^*, Y')$ denotes a pair of neighbouring datasets. Let $L^{(0)} := \ln \frac{P(M_{rej}(Y^*, Y_t, \epsilon_{abc}, \sigma) = \epsilon)}{P(M_{rej}(Y^*, Y_t, \epsilon_{abc}, \sigma) = \epsilon)}$, so $L^{(0)} = \ln \frac{p_t}{1 - p_t}$ and $L^{(1)} = \ln \frac{1 - p_t}{p_t}$. We have

$$\max(L^{(0)}, L^{(1)}) = \max \left( \ln \frac{1 - p_t}{p_t}, \ln \frac{p_t}{1 - p_t} \right)$$

$$= \max \left( - \ln \frac{1 - p_t}{p_t}, \ln \frac{1 - p_t}{1 - p_t} - \ln \frac{p_t}{p_t} \ln \frac{p_t}{1 - p_t} \right)$$

$$= \max \left( \ln \frac{1 - p_t}{p_t}, \ln \frac{1 - p_t}{1 - p_t} \ln \frac{p_t}{p_t} \right)$$

$$:= g(p_t, p_t').$$

It can be seen that $g$ is symmetric in its two arguments. Without loss of generality, assume that $p_t \leq p_t'$. If this is not true, simply swap the two variables to make this true. We have

$$0 \leq g(p_t, p_t') = \max \left( \ln \frac{1 - p_t}{1 - p_t'}, \ln \frac{p_t'}{p_t} \right).$$
Note that \( p_t, p'_t \in \left[ \Phi \left( \frac{\epsilon_{abc} - B_k}{\sigma} \right), \Phi \left( \frac{\epsilon_{abc}}{\sigma} \right) \right] := D \) and that \( \ln(x) \) is Lipschitz continuous on \( D \) with Lipschitz constant \( L_D \):

\[
L_D = \sup_{x \in D} \left| \frac{d \ln x}{dx} \right| = \sup_{x \in D} \left| \frac{1}{x} \right| = 1/\Phi \left( \frac{\epsilon_{abc} - B_k}{\sigma} \right).
\]

So,

\[
\ln \frac{p'_t}{p_t} = \ln p'_t - \ln p_t \leq \frac{1}{\Phi \left( \frac{\epsilon_{abc} - B_k}{\sigma} \right)} |p'_t - p_t|.
\] (18)

Consider \( |p'_t - p_t| \):

\[
|p'_t - p_t| = \left| \Phi \left( \frac{\epsilon_{abc} - p'_t}{\sigma} \right) - \Phi \left( \frac{\epsilon_{abc} - p_t}{\sigma} \right) \right|
\leq \min \left( 1, \frac{1}{\sqrt{2\pi}} \right) \left| \frac{\epsilon_{abc} - p'_t}{\sigma} - \frac{\epsilon_{abc} - p_t}{\sigma} \right|
\leq \min \left( 1, \frac{\Delta_p}{\sqrt{2\pi} \sigma^2} \right),
\] (19)

where at (a) we use the fact that \( x \mapsto \Phi(x) \) is \( \frac{1}{\sqrt{2\pi}} \)-Lipschitz i.e., for any \( x, x' \in \mathbb{R}, |\Phi(x) - \Phi(x')| \leq \frac{1}{\sqrt{2\pi}} |x - x'|. \)

Combining (18), and (19), we have

\[
\sup_{(Y, Y')} \ln \frac{p'_t}{p_t} \leq \min \left( 1, \frac{\Delta_p}{\sqrt{2\pi} \sigma^2} \right) \left[ \Phi \left( \frac{\epsilon_{abc} - B_k}{\sigma} \right) \right]^{-1}.
\]

Using the same proof structure to bound \( \sup_{(Y, Y')} \frac{1 - p_t}{1 - p'_t} \), we have

\[
\sup_{(Y, Y')} \ln \frac{1 - p_t}{1 - p'_t} \leq \min \left( 1, \frac{\Delta_p}{\sqrt{2\pi} \sigma^2} \right) \left[ \Phi \left( \frac{-\epsilon_{abc}}{\sigma} \right) \right]^{-1},
\]

implying that

\[
\sup_{(Y, Y')} \max \left( \ln \frac{1 - p_t}{1 - p'_t}, \frac{p'_t}{p_t} \right) \leq \min \left( 1, \frac{\Delta_p}{\sqrt{2\pi} \sigma^2} \right) \left[ 2 \frac{1}{\sigma} \min (\epsilon_{abc} - B_k, -\epsilon_{abc}) \right]^{-1}.
\]

We have the stated result by setting the upper bound to \( \epsilon \). \( \square \)

**B. Proof of Lemma 3.1**

Recall Lemma 3.1 stated in the main text:

**Lemma 3.1** (\( \Delta_p = O(1/N) \) for MMD). *Assume that \( Y^* \) and each pseudo dataset \( Y_t \) are of the same cardinality \( N \). Set \( \rho(X, Y) = \text{MMD}(X, Y) \) with a kernel \( k \) bounded by \( B_k > 0 \) i.e., \( \sup_{x,y \in X} k(x, y) \leq B_k < \infty \). Then,

\[
\sup_{(X, X'), Y} |\rho(X, Y) - \rho(X', Y)| \leq \Delta_p = \frac{2}{N} \sqrt{B_k}
\]

(12)

and \( \sup_{X, Y} \rho(X, Y) \leq B_p = 2\sqrt{B_k} \).
Proof. We will establish \( \Delta_\rho \) when \( \rho \) is MMD. Recall that \((X, X')\) is a pair of neighbouring datasets, and \(Y\) is an arbitrary dataset. Without loss of generality, assume that \(X = \{x_1, \ldots, x_N\}, X' = \{x_1', \ldots, x_N'\}\) such that \(x_i = x_i'\) for all \(i = 1, \ldots, N - 1\), and \(Y = \{y_1, \ldots, y_m\}\). We start with

\[
\sup_{(X, X') \neq (Y, Y)} |\rho(X, Y) - \rho(X', Y)|
\]

\[
= \sup_{(X, X') \neq (Y, Y)} |\tilde{\text{MMD}}(X, Y) - \tilde{\text{MMD}}(X', Y)|
\]

\[
= \sup_{(X, X') \neq (Y, Y)} \left\| \frac{1}{N} \sum_{i=1}^{N} \phi(x_i) - \frac{1}{m} \sum_{j=1}^{m} \phi(y_j) \right\|_{\mathcal{H}} - \left\| \frac{1}{N} \sum_{i=1}^{N} \phi(x_i') - \frac{1}{m} \sum_{j=1}^{m} \phi(y_j) \right\|_{\mathcal{H}}
\]

\[
\leq \sup_{(X, X') \neq (Y, Y)} \left( \frac{1}{N} \sum_{i=1}^{N} \phi(x_i) - \frac{1}{N} \sum_{i=1}^{N} \phi(x_i') \right)
\]

\[
= \sup_{(X, X') \neq (Y, Y)} \frac{1}{N} \sqrt{k(x_N, x_N) + k(x_N', x_N') - 2k(x_N, x_N')}
\]

\[
\leq \frac{2}{N} \sqrt{B_k},
\]

where at \((a)\) we use the reverse triangle inequality. To show that \(\sup_{X \neq Y} \rho(X, Y) = 2\sqrt{B_k}\), we note that

\[
\sup_{X, Y} \rho(X, Y)
\]

\[
\leq \sup_{X, Y} \left\| \frac{1}{N} \sum_{i=1}^{N} \phi(x_i) - \frac{1}{m} \sum_{j=1}^{m} \phi(y_j) \right\|_{\mathcal{H}}^2
\]

\[
= \sup_{X, Y} \frac{1}{N^2} \sum_{i, j=1}^{N} k(x_i, x_j) + \frac{1}{m^2} \sum_{i, j=1}^{m} k(y_i, y_j) - \frac{2}{mn} \sum_{i=1}^{N} \sum_{j=1}^{m} k(x_i, y_j)
\]

\[
= \sqrt{B_k + B_k + 2B_k} = 2\sqrt{B_k}.
\]

\[\square\]

C. MMD with Random Fourier Features

A fast linear MMD estimator can be achieved by considering an approximation to the kernel function \(k(x, y)\) with an inner product of finite dimensional feature vectors \(\phi(x) \cdot \phi(y)\) where \(\phi(x) \in \mathbb{R}^D\) and \(D\) is the number of features. Given the feature map \(\phi()\) such that, \(k(x, y) \approx \phi(x) \cdot \phi(y)\), MMD\(^2\) can be approximated as

\[
\tilde{\text{MMD}}^2_f(F_x, F_y) \approx \mathbb{E}_X \hat{\phi}(X) \mathbb{E}_{X'} \hat{\phi}(X') + \mathbb{E}_Y \hat{\phi}(Y) \mathbb{E}_{Y'} \hat{\phi}(Y')
\]

\[
- 2\mathbb{E}_X \hat{\phi}(X) \mathbb{E}_Y \hat{\phi}(Y) := \|\mathbb{E}_X \hat{\phi}(X) - \mathbb{E}_Y \hat{\phi}(Y)\|^2_2.
\]

A straightforward (biased) estimator is

\[
\tilde{\text{MMD}}^2_f(F_x, F_y) = \left\| \frac{1}{n_x} \sum_{i=1}^{n_x} \hat{\phi}(x^{(i)}) - \frac{1}{n_y} \sum_{i=1}^{n_y} \hat{\phi}(y^{(i)}) \right\|^2_2,
\]

which can be computed in \(O(D(n_x + n_y))\), i.e., linear in the sample size, leading to the overall cost of \(O(MD(n_x + n_y))\).

Given a kernel \(k\), there are a number of ways to obtain \(\hat{\phi}()\) such that \(k(x, y) \approx \hat{\phi}(x) \cdot \hat{\phi}(y)\). One approach which became popular in recent years is based on random Fourier features (Rahimi and Recht, 2008) which can
be applied to any translation invariant kernel. Assume that \( k \) is translation invariant i.e., \( k(x, y) = \tilde{k}(x - y) \) for some function \( \tilde{k} \). According to Bochner’s theorem (Rudin, 2013), \( \tilde{k} \) can be written as
\[
\tilde{k}(x - y) = \int e^{i\omega^\top(x - y)} d\Lambda(\omega) = \mathbb{E}_{\omega \sim \Lambda} \cos(\omega^\top(x - y)) = 2\mathbb{E}_{b \sim U[0, 2\pi]} \mathbb{E}_{\omega \sim \Lambda} \cos(\omega^\top x + b) \cos(\omega^\top y + b),
\]
where \( i = \sqrt{-1} \) and due to positive-definiteness of \( \tilde{k} \), its Fourier transform \( \Lambda \) is nonnegative and can be treated as a probability measure. By drawing random frequencies \( \{\omega_i\}_{i=1}^D \sim \Lambda \) and \( \{b_i\}_{i=1}^D \sim U[0, 2\pi] \), \( \tilde{k}(x - y) \) can be approximated with a Monte Carlo average. It follows that \( \hat{\phi}_j(x) = \sqrt{2/D} \cos(\omega_j^\top x + b_j) \) and \( \hat{\phi}(x) = (\hat{\phi}_1(x), \ldots, \hat{\phi}_D(x))^\top \). Note that a Gaussian kernel \( k \) corresponds to normal distribution \( \Lambda \).

## D. Proof of Soft ABCDP

Here is the proof of Thm. 3.2.

**Proof.**

\[
D_\alpha \left( \mathcal{M}(y^*, y', \epsilon_{abc}, \sigma_{soft}) \| \mathcal{M}(y'^*, y', \epsilon_{abc}, \sigma_{soft}) \right) = D_\alpha (N(D_t, \sigma_{soft}^2) \| N(D'_t, \sigma_{soft}^2)),
\]
\[
= \frac{\alpha}{2\sigma_{soft}^2 \epsilon_{abc}^2} (D_t - D'_t)^2, \text{ due to Corr. 2.1}
\]
\[
\leq \frac{\alpha}{2\sigma_{soft}^2 \epsilon_{abc}^2} \left( |\rho(Y_t, Y'^*) - \rho(Y_t, Y'^*)| \right)^2,
\]

if \( \rho = MMD \), then due to the reverse triangle inequality
\[
\leq \frac{\alpha \Delta_\rho^2}{2\sigma_{soft}^2 \epsilon_{abc}^2} \frac{2\alpha B_k}{N^2 \sigma_{soft}^2 \epsilon_{abc}^2},
\]

where \( (a) \) is due to
\[
\Delta_\phi = \max_{y^*, y'^*} \left| \frac{\sum_i \phi(y_i^*)}{N} - \frac{\sum_i \phi(y'_i^*)}{N} \right| \leq 2\sqrt{B_k} \frac{1}{N} \max_{y_i^*, y'_i^*} \left| \phi(y_i^*) - \phi(y'_i^*) \right| \leq 2\sqrt{B_k} \frac{1}{N}, \quad \text{(21)}
\]

with a L2-norm bounded feature vector \( \phi \) by \( \sqrt{B_k} \frac{|\phi(\cdot)|_2}{N} \leq \sqrt{B_k} \). Hence, following the definition of RDP, we conclude that the soft-weight mechanism is \((\alpha, \epsilon)\)-RDP, where \( \epsilon = \frac{2\alpha B_k}{N^2 \sigma_{soft}^2 \epsilon_{abc}} \). Using Prop. 1, we can compute the corresponding DP level. \( \square \)