On the Identity Problem and the Group Problem for subsemigroups of unipotent matrix groups

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Abstract

Let $G$ be a finite set of matrices in a unipotent matrix group $G$ over $\mathbb{Q}$, where $G$ has nilpotency class at most ten. We exhibit a polynomial time algorithm that computes the subset of $G$ which generates the group of units of the semigroup $\langle G \rangle$ generated by $G$. In particular, this result shows that the Identity Problem and the Group Problem are decidable in polynomial time for finitely generated subsemigroups of the groups $\text{UT}(11, \mathbb{Q})^n$. Another important implication of our result is the decidability of the Identity Problem and the Group Problem within finitely generated nilpotent groups of class at most ten. Our main idea is to analyze the logarithm of the matrices appearing in $\langle G \rangle$. This allows us to employ Lie algebra methods to study this semigroup. In particular, we prove several new properties of the Baker-Campbell-Hausdorff formula, which help us characterize the convex cone spanned by the elements in $\log(\langle G \rangle)$.

Furthermore, we formulate a sufficient condition in order for our results to generalize to unipotent matrix groups of class $d > 10$. For every such $d$, we exhibit an effective procedure that verifies this sufficient condition in case it is true.

1 Introduction

1.1 Algorithmic problems in matrix semigroups. The computational theory of matrix groups and semigroups is one of the oldest and most well-developed parts of computational algebra. The area has, in particular, proven influential in the development of randomized algorithms [1] and interactive proof systems [5]. Among the most prominent algorithmic problems for matrix semigroups are the Membership Problem, the Identity Problem and the Group Problem. For these three decision problems, we work in a fixed matrix group or semigroup $G$. The input is a finite set of matrices $\mathcal{G} = \{A_1, \ldots, A_K\} \subseteq G$, plus a distinguished matrix $A$ for the Membership Problem. Denote by $\langle \mathcal{G} \rangle$ the semigroup generated by $\mathcal{G}$ (that is, the smallest subset of $G$ containing $\mathcal{G}$ which is closed under matrix multiplication).

Definition 1.1. Given as input the set of matrices $\mathcal{G} = \{A_1, \ldots, A_K\}$ and $A$.

(i) The Membership Problem is to decide whether the matrix $A$ is in $\langle \mathcal{G} \rangle$.

(ii) The Identity Problem is to decide whether the identity matrix $I$ is in $\langle \mathcal{G} \rangle$.

(iii) The Group Problem is to decide whether the semigroup $\langle \mathcal{G} \rangle$ is a group.

For general matrices, the Membership Problem is undecidable by a classical result of Markov [31]. It is one of the earliest known results in the algorithmic theory of matrix semigroups (indeed, it is one of the oldest undecidability results tout court). Most of these algorithmic problems remain undecidable in low dimension. For example, the Membership Problem for $3 \times 3$ integer matrices was
shown to be undecidable by Paterson [34], using an embedding of the Post correspondence problem. In dimension four, the Membership Problem, the Identity Problem and the Group Problem are all undecidable for matrices in \( \text{GL}(4, \mathbb{Z}) \) [7, 32]. The undecidability results stem from the fact that \( \text{GL}(4, \mathbb{Z}) \) contains as a subgroup the direct product of two free groups. On the other hand, the Membership Problem in \( \text{GL}(2, \mathbb{Z}) \) was shown to be decidable by Choffrut and Karhumaki [12] using automata theory, while the Identity Problem in \( \text{SL}(2, \mathbb{Z}) \) was shown to be NP-complete by Bell, Hirvensalo, and Potapov [6]. It remains an intricate open problem whether any of these three problems is decidable in \( \text{GL}(3, \mathbb{Z}) \) or \( \text{GL}(2, \mathbb{Q}) \).

1.2 Unipotent matrix groups and related work. Computation on matrix groups becomes easier in the presence of structural restrictions such as commutativity and nilpotence. In [2], Babai et al. gave a PTIME algorithm for the Group Membership Problem for finitely generated commutative subgroups of \( \text{GL}(n, \mathbb{Q}) \). (The Group Membership Problem is to decide whether a matrix \( A \) lies in the group generated by a given set of invertible matrices \( \{A_1, \ldots, A_k\} \)). Babai also showed the decidability of the (semigroup) Membership Problem for commutating matrices [2], generalizing earlier work of Cai, Lipton, and Zalcstein [9]. In this paper we work in the more general setting of nilpotent groups.

Definition 1.2. Given a group \( G \) and a subgroup \( H \) of \( G \), define the commutator \( [G, H] \) to be the group generated by the elements in \( \{ghg^{-1}h^{-1} \mid g \in G, h \in H\} \). The lower central series of a group \( G \) is the inductively defined descending sequence of subgroups

\[
G = G_1 \geq G_2 \geq G_3 \geq \cdots,
\]

in which \( G_k = [G, G_{k-1}] \). A group \( G \) is called nilpotent if its lower central series terminates with \( G_{d+1} = \{1\} \) for some \( d \). In this case, the smallest such \( d \) is called the nilpotency class of \( G \).

In particular, commutative groups are nilpotent of class one. It is a well known fact that subgroups of nilpotent groups are nilpotent. The most prominent examples of nilpotent groups are the groups \( \text{UT}(n, \mathbb{Q}) \) of unitriangular \( n \times n \) matrices.

Definition 1.3. Denote by \( \text{UT}(n, \mathbb{Q}) \) the group of \( n \times n \) upper triangular rational matrices with ones along the diagonal:

\[
\text{UT}(n, \mathbb{Q}) := \left\{ \begin{pmatrix}
1 & * & \cdots & * \\
0 & 1 & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & *
\end{pmatrix}, \text{ where } * \text{ are elements of } \mathbb{Q} \right\}
\]

A group \( G \) is called a unipotent matrix group\(^1\) over \( \mathbb{Q} \) if it is a subgroup of \( \text{UT}(n, \mathbb{Q}) \).

The group \( \text{UT}(n, \mathbb{Q}) \) is of nilpotency class \( n - 1 \) [21, Example 16.1.2]. A strong motivation for studying this group is that every finitely generated torsion-free nilpotent group can be embedded as a subgroup of \( \text{UT}(n, \mathbb{Q}) \) for some \( n \) [21]. In fact, every finitely generated nilpotent group is isomorphic to a subgroup of a direct product of \( \text{UT}(n, \mathbb{Q}) \) with a finite group [4]. For this reason, we can now focus our study on unipotent matrix groups over \( \mathbb{Q} \).

\(^1\)In the literature, a unipotent matrix group is sometimes defined as a matrix group conjugate to a subgroup of \( \text{UT}(n, \mathbb{Q}) \). For simplicity, in this paper we suppose that it is simply a subgroup of \( \text{UT}(n, \mathbb{Q}) \), since the answers to all three algorithmic problems in Definition 1.1 do not change under group isomorphism.
While the Group Membership Problem for $\text{UT}(n, \mathbb{Q})$ is well-known to be decidable [26], corresponding algorithmic problems for semigroups are much more difficult. In [23], Ko et al. showed the PTIME decidability of the Identity Problem in $\text{UT}(3, \mathbb{Q})$. They also extended their decidability result to the Heisenberg groups $\text{H}_{2n+1}$. Later, utilising the special structure of the first term in the Baker-Campbell-Hausdorff formula, Colcombet et al. proved the decidability of the Membership Problem in $\text{UT}(3, \mathbb{Q})$ and $\text{H}_{2n+1}$ by encoding it into a Parikh automaton [13]. Recently, Dong [15] showed the PTIME decidability of the Identity Problem in $\text{UT}(4, \mathbb{Z})$, a result easily generalizable to $\text{UT}(4, \mathbb{Q})$. His main idea is to introduce arguments from convex geometry as well as to use computational algebraic geometry for technical results. It was left as an open problem in [15] whether the Identity Problem in $\text{UT}(n, \mathbb{Q})$ is decidable for $n \geq 5$. On the undecidability side, Lefaucheux [27] extended a result of König et al. [25] to show that the Membership Problem in $\text{UT}(3, \mathbb{Q})^k$ is undecidable for sufficiently large $k$. Their main technique is to embed Hilbert’s tenth problem into the Membership Problem in $\text{UT}(3, \mathbb{Q})^k$.

1.3 Invertible subsets and the group of units. The main focus of this paper is on the Identity Problem and the Group Problem for unipotent matrix groups of relatively small nilpotency class. It turns out that both problems are special cases of the more difficult problem of computing invertible subsets.

**Definition 1.4.** Let $G$ be a matrix group. Given a finite set of elements $\mathcal{G} = \{A_1, \ldots, A_K\} \subseteq G$, the invertible subset of $\mathcal{G}$ is the set of matrices in $\mathcal{G}$ who inverse lies in $\langle \mathcal{G} \rangle$.

For a semigroup $S$, the group of units of $S$ is the set of elements who have an inverse in $S$. This set forms a group when it is not empty [20]. The algorithmic problem of computing the group of units of a semigroup is interesting in its own right, and is important for understanding the structure of the semigroup. The following proposition shows that the computation of invertible subsets subsumes the Identity Problem and the Group Problem, as well as gives a generator set for the group of units. See Appendix A for a straightforward proof.

**Proposition 1.5.** Given a finite set of matrices $\mathcal{G} = \{A_1, \ldots, A_K\}$ in a matrix group $G$. Denote by $\mathcal{G}_{\text{inv}}$ the invertible subset of $\mathcal{G}$.

(i) The Identity Problem for $\mathcal{G}$ has a positive answer if and only if $\mathcal{G}_{\text{inv}}$ is non-empty.

(ii) If $\mathcal{G}_{\text{inv}}$ is non-empty, then it generates the group of units of $\langle \mathcal{G} \rangle$ as a semigroup.

(iii) The Group Problem for $\mathcal{G}$ has a positive answer if and only if $\mathcal{G}_{\text{inv}} = \mathcal{G}$.

1.4 Main results and applications. The main result of this paper is the following theorem.

**Theorem 1.6.** Let $G$ be a unipotent matrix group over $\mathbb{Q}$ with nilpotency class at most ten. Given any finite set $\mathcal{G} \subseteq G$, the invertible subset of $\mathcal{G}$ is computable in polynomial time.

Here, the input size of $\mathcal{G}$ is defined as the total bit length of all the entries in the matrices of $\mathcal{G}$. The proof of Theorem 1.6 will be given in Section 3 and 4. Together with Proposition 1.5, Theorem 1.6 implies that the Identity Problem and the Group Problem are decidable in PTIME for unipotent matrix groups of nilpotency class at most ten. Since for any $k$, the direct product $\text{UT}(11, \mathbb{Q})^k$ is unipotent and has nilpotency class ten [21], an immediate application of Theorem 1.6 is the following.

**Corollary 1.7.** Let $k$ be an integer. The Identity Problem and the Group Problem are decidable in polynomial time in the group $G = \text{UT}(11, \mathbb{Q})^k$. 

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Our result also shows a fundamental difference between the Identity Problem and the Membership Problem. In particular, Corollary 1.7 subsumes the PTIME decidability of the Identity Problem in $\text{UT}(3, \mathbb{Q})^k$ for all $k$, by the obvious embedding of $\text{UT}(3, \mathbb{Q})$ as a subgroup of $\text{UT}(11, \mathbb{Q})$. However, as mentioned earlier, for sufficiently large $k$ the Membership Problem in $\text{UT}(3, \mathbb{Q})^k$ is undecidable.

Using classical embedding theorems, the following corollary extends Theorem 1.6 to arbitrary finitely generated nilpotent groups. However, the complexity will depend on specific group embeddings, which we do not analyse. A proof of Corollary 1.8 subject to Theorem 1.6 is given in Appendix A.

**Corollary 1.8.** Let $G$ be a finitely generated nilpotent group of class at most ten, given by a finite presentation or a consistent polycyclic presentation (see [19, Chapter 8]). Then the Identity Problem and the Group Problem are decidable within $G$.

### 1.5 Main techniques and outline

The main technical tools used in this paper include convex geometry, Lie algebra and combinatorics. The highlight of our approach is using Lie algebra to study semigroup algorithmic problems, which to the best of our knowledge is a new method in this domain. Our inspiration is drawn from [13], where a special case of the Baker-Campbell-Hausdorff (BCH) formula is used. The most significant contribution of our paper includes proving several intricate properties of the $k$-th term of the BCH formula, from which our main result follows. All but one of these properties are proven for every term of the BCH formula, whereas the remaining one needs to be verified term by term using assistance from computer algebra software. This is due to the complexity of the BCH formula. The huge computational power needed to verify this particular property is the reason why our result stops at nilpotency class ten.

Our paper also exhibits several novel convex geometry arguments, whose roots can be traced back to [15]. In that paper, the author used the fact that the commutator subgroup $U_1$ of $\text{UT}(4, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^3$, and analysed all possible shapes of $\langle G \rangle \cap U_1$. However, this method seems to be limited to situations when the commutator subgroup is abelian and has fixed dimension. We observe that convex geometry fits particularly well within our Lie algebra context, since Lie algebras have a natural vector space structure. By combining convex geometry arguments with Lie algebra, we are able to bypass the complicated case analysis performed in [15] and significantly improve the result from $\text{UT}(4, \mathbb{Z})$ to arbitrary unipotent matrix groups of class ten.

The organization of this paper is as follows. In Section 2, we state the preliminaries on word combinatorics, convex geometry and Lie algebra. In Section 3, we exhibit the algorithm (Algorithm 1) claimed by Theorem 1.6. We prove its PTIME complexity and its the correctness subject to a core technical theorem (Theorem 3.1). Section 4 is dedicated to the proof of Theorem 3.1: this is the most technical part of our paper, where Lie algebra techniques and computer assistance are heavily employed. Finally, in Section 5, we state a conjecture with a variable $d$. Subject to this conjecture, Theorem 1.6 still holds for nilpotency class $d$. For each $d$, we exhibit a procedure that can verify this conjecture in case it is true.

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2 The Identity Problem for an arbitrary group $G$ is defined similarly to the case where $G$ is a matrix group. Given a finite subset $\mathcal{G} \subseteq G$, it asks whether the semigroup $\langle \mathcal{G} \rangle$ generated by $\mathcal{G}$ contains the neutral element of $G$.

3 Nilpotent groups of large classes have intrinsically complicated structures. As a comparison, there exists a plethora of work dedicated to another related problem: classifying nilpotent Lie algebras of small dimensions. These efforts generally stop at dimension seven due to their high complexity (see, e.g. [17]).
2 Preliminaries

2.1 Words, convex geometry and linear programming. Given a finite set of matrix \( G = \{A_1, \ldots, A_K\} \), one can consider \( G \) as a finite alphabet. Let \( G^\ast \) denote the set of words over \( G \), and let \( G^+ \) denote the set of non-empty words over \( G \). Given a word \( w \) over the alphabet \( G \), by multiplying consecutively the matrices appearing in \( w \), we can evaluate \( w \) as a matrix in \( \langle G \rangle \). We denote this matrix by \( \pi(w) \). For the empty word \( \varepsilon \), define \( \pi(\varepsilon) = I \).

We now define some concepts necessary for analysing words with linear algebra.

Definition 2.1 (Parikh Image). Given a finite alphabet \( G = \{A_1, \ldots, A_K\} \), the Parikh Image of a word \( w = B_1 \cdots B_m \) in \( G^\ast \) is the vector \( \ell = (\ell_1, \ldots, \ell_K) \in \mathbb{Z}_{\geq 0}^K \) defined by \( \ell_i = \text{card}(\{j \mid B_j = A_i\}) \) (that is, \( \ell_i \) is the number of times \( A_i \) appears in \( w \)). The Parikh Image of \( w \) over the alphabet \( G \) is denoted by \( \text{PI}_G(w) \).

Definition 2.2 (Cones). Let \( V \) be a \( \mathbb{Q} \)-linear space. A subset \( C \subseteq V \) is called a \( \mathbb{Q}_{\geq 0} \)-cone if \( a \in C \implies a\mathbb{Q}_{\geq 0} \subseteq C \), and \( a, b \in C \implies a + b \in C \). Given a set of vectors \( S \subseteq V \), denote by \( \langle S \rangle_{\mathbb{Q}_{\geq 0}} \) the \( \mathbb{Q}_{\geq 0} \)-cone generated by \( S \), that is, the smallest \( \mathbb{Q}_{\geq 0} \)-cone of \( V \) containing \( S \). These definitions can be naturally extended to \( \mathbb{R} \)-linear spaces and \( \mathbb{R}_{\geq 0} \)-cones.

Definition 2.3 (Support). A subset \( \Lambda \subseteq \mathbb{Z}_{\geq 0}^K \) is called a \( \mathbb{Z}_{\geq 0} \)-cone if \( a, b \in \Lambda \implies a + b \in \Lambda \), and \( 0 \in \Lambda \). The support of a vector \( \ell = (\ell_1, \ldots, \ell_K) \in \mathbb{Z}_{\geq 0}^K \) is defined as the set of indices where the entry of \( \ell \) is non-zero:

\[
\text{supp}(\ell) := \{ i \in \{1, \ldots, K\} \mid \ell_i > 0 \}.
\]

The support of a \( \mathbb{Z}_{\geq 0} \)-cone \( \Lambda \) is defined as the union of supports of all vectors in \( \Lambda \):

\[
\text{supp}(\Lambda) := \bigcup_{\ell \in \Lambda} \text{supp}(\ell) = \{ i \mid \exists (\ell_1, \ldots, \ell_K) \in \Lambda, \ell_i > 0 \}.
\]

Let \( V \) be a \( \mathbb{Q} \)-linear subspace of \( \mathbb{Q}^K \), represented as the solution set of linear homogeneous equations. Then \( \mathbb{Z}_{\geq 0}^K \cap V \) is a \( \mathbb{Z}_{\geq 0} \)-cone. In this paper, we will need to compute the support of \( \mathbb{Z}_{\geq 0} \)-cones of the form \( \Lambda = \mathbb{Z}_{\geq 0}^K \cap V \) (namely, in Algorithm 1).

Lemma 2.4. Given \( V \) represented as the solution set of linear homogeneous equations, one can compute the support of \( \Lambda = \mathbb{Z}_{\geq 0}^K \cap V \) in polynomial time.

Proof. For \( i = 1, \ldots, K \), we can check whether \( i \in \text{supp}(\Lambda) \) in the following way. By definition, \( i \in \text{supp}(\Lambda) \) if and only if the system

\[
(\ell_1, \ldots, \ell_K) \in V, \ell_1 \geq 0, \ldots, \ell_i > 0, \ldots, \ell_K \geq 0
\]

has an integer solution \( (\ell_1, \ldots, \ell_K) \in \mathbb{Z}^K \). By the homogeneity of the system (1), it has an integer solution if and only if it has a rational solution. The existence of a rational solution to system (1) can be decided by linear programming in polynomial time. Therefore, the support of \( \Lambda \) can be computed in polynomial time by checking whether \( i \in \text{supp}(\Lambda) \) for every \( i = 1, \ldots, K \).

2.2 Lie algebra and the Baker-Campbell-Hausdorff formula.

Definition 2.5 (Lie algebra \( \mathfrak{u}(n) \)). The Lie algebra \( \mathfrak{u}(n) \) is defined as the \( \mathbb{Q} \)-linear space of \( n \times n \) upper triangular rational matrices with zeros on the diagonal. There exist maps

\[
\log : \text{UT}(n, \mathbb{Q}) \to \mathfrak{u}(n), \quad A \mapsto \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} (A - I)^k
\]
and
\[ \exp : u(n) \to UT(n, \mathbb{Q}), \quad X \mapsto \sum_{k=0}^{n} \frac{1}{k!} X^k \]
which are inverse of one another. In particular, \( \log I = 0 \) and \( \exp(0) = I \).

The Lie algebra \( u(n) \) is equipped with a \textit{Lie bracket} \( [\cdot, \cdot] : u(n) \times u(n) \to u(n) \) given by \( [X, Y] = XY - YX \). The Lie bracket is bilinear, anticommutative (meaning \( [X, Y] = -[Y, X] \)), and it additionally satisfies the \textit{Jacobi Identity}:
\[ [X, [Y, Z]] + [Y, [X, Z]] + [Z, [X, Y]] = 0 \]
for all \( X, Y, Z \in u(n) \).

**Notation 2.6** (Logarithm of sets and long Lie brackets). Given a set of matrices \( G \subseteq UT(n, \mathbb{Q}) \), define the set
\[ \log G := \{ \log A \mid A \in G \} \]
of logarithms of matrices in \( G \). It is a subset of \( u(n) \). If \( G \) is a subgroup of \( UT(n, \mathbb{Q}) \), then by slight abuse of notation, \( \log G \) is similarly defined by considering \( G \) as a set \( \log G := \{ \log A \mid A \in G \} \).

Given a set of elements \( H \subseteq u(n) \) and an integer \( k \geq 2 \), define the set
\[ [H]_k := \{ \ldots [[[X_1, X_2], X_3], \ldots, X_k] \mid X_1, X_2, \ldots, X_k \in H \} \]
That is, \([H]_k \) is the set of all “left bracketing” of length \( k \) of elements in \( H \).

It is a standard result that, using the bilinearity, anticommutativity and the Jacobi identity, any \( k \)-iteration of Lie brackets of elements in \( H \) can be written as a linear combination of elements in \([H]_k \). For example, for \( k = 4 \), one can write
\[ [[[X_1, X_2], [X_3, X_4]], X_5] = -([[X_2, [X_3, X_4]], X_1] - [[[X_3, X_4], X_1], X_2] \quad \text{(Jacobi identity)} \]
\[ = [[[X_3, X_4], X_2], X_1] - [[[X_3, X_4], X_1], X_2] \quad \text{(Anticommutativity)} \]

The following lemma relates the nilpotency class of a unipotent matrix group \( G \) over \( \mathbb{Q} \) to the integer \( d \) such that the set \( \log G |_{d+1} \) vanishes. This result has deep roots in what is called the Mal'cev correspondence between unipotent Lie groups and nilpotent Lie algebras. A proof is given in Appendix A.

**Lemma 2.7.** Let \( G \) be a unipotent matrix group over \( \mathbb{Q} \). If the nilpotency class of \( G \) is \( d \), then \( \log G |_{d+1} = \{0\} \).

We now introduce the Baker-Campbell-Hausdorff formula, one of the main tools of this paper.

**Theorem 2.8** (Baker-Campbell-Hausdorff Formula [3, 10, 18]). Let \( G \) be a unipotent matrix group over \( \mathbb{Q} \), whose nilpotency class is at most \( d \). Let \( B_1, \ldots, B_m \) be elements of \( G \). We have
\[ \log(B_1 \cdots B_m) = \sum_{i=1}^{m} \log B_i + \sum_{k=2}^{d} H_k(\log B_1, \ldots, \log B_m), \tag{2} \]
where the terms \( H_k(\log B_1, \ldots, \log B_m), k = 2, 3, \ldots, \) can be expressed as \( \mathbb{Q} \)-linear combinations of elements in \( \{[[\log B_1, \ldots, \log B_m]]]_k \).
In theory, one can compute the expressions $H_k$ effectively using recursion (see, for example [11]). An explicit expression for the term $H_k$ has been discovered by Dynkin (see [29, Proposition 3.4 and Proposition 4.2]). However, as $k$ grows, these expressions quickly become very complicated. For example, here are the explicit expressions of the first two terms.

\[
H_2(C_1, \ldots, C_m) = \frac{1}{2} \sum_{i<j} [C_i, C_j]
\]

\[
H_3(C_1, \ldots, C_m) = \sum_{i<j<k} \left( \frac{1}{3} [C_i, [C_j, C_k]] + \frac{1}{6} [[C_i, C_k], C_j] \right) + \frac{1}{12} \sum_{i<j} ([C_i, [C_i, C_j]] + [[C_i, C_j], C_j])
\]

### 3 Polynomial time algorithm for Theorem 1.6

In this section, we exhibit the algorithm that proves the main result of this paper (Theorem 1.6). In order to describe our algorithm, we need to introduce the following notation. Let $H$ be a finite set of elements in the Lie algebra $u(n)$, denote

\[
\mathcal{L}_{\geq k}(H) := \left( \bigcup_{i \geq k} [H]_i \right)_Q.
\]

That is, $\mathcal{L}_{\geq k}(H)$ is the linear space spanned by the set of all “left bracketing” of length at least $k$ of elements in $H$. By Lemma 2.7, if a unipotent matrix group $G$ has nilpotency class $d$, then for any $\mathcal{H} \subseteq \log G$, we have $\mathcal{L}_{\geq k}(\mathcal{H}) = ([\mathcal{H}]_k)_Q + ([\mathcal{H}]_{k+1})_Q + \cdots + ([\mathcal{H}]_d)_Q$, and $\mathcal{L}_{\geq d+1}(\mathcal{H}) = \{0\}$. We have thus a descending series of linear spaces $\mathcal{L}_{\geq 1}(\mathcal{H}) \supseteq \mathcal{L}_{\geq 2}(\mathcal{H}) \supseteq \cdots \supseteq \mathcal{L}_{\geq d+1}(\mathcal{H}) = \{0\}$ such that $[\mathcal{L}_{\geq i}(\mathcal{H}), \mathcal{L}_{\geq j}(\mathcal{H})] \subseteq \mathcal{L}_{\geq i+j}(\mathcal{H})$.

Now let $G \leq \text{UT}(n, Q)$ be a unipotent matrix group of nilpotency class at most ten, and fix $\mathcal{G} = \{A_1, \ldots, A_K\}$ to be a finite alphabet of elements in $G$. For any vector $\ell = (\ell_1, \ldots, \ell_K) \in \mathbb{Z}_{\geq 0}^K$, define

\[
\log \mathcal{G}_{\text{supp}(\ell)} := \{\log A_i \mid A_i \in \mathcal{G}, i \in \text{supp}(\ell)\}
\]

as the set of logarithm of matrices in $\mathcal{G}$ whose index appears in $\text{supp}(\ell)$.

We now give an overview of the algorithm that computes the invertible subset of $\mathcal{G}$. The key ingredient of our algorithm is the following Theorem 3.1, which provides a criterion for the existence of a non-empty word $w \in \mathcal{G}^+$ satisfying $\log \pi(w) = 0$ (a condition equivalent to $\pi(w) = I$ as noted in Definition 2.5).

**Theorem 3.1.** Let $\mathcal{G} = \{A_1, \ldots, A_K\}$ be a finite set of matrices in $\text{UT}(n, Q)$ that satisfies $|\log \mathcal{G}|_{11} = \{0\}$. Given a non-zero vector $\ell = (\ell_1, \ldots, \ell_K) \in \mathbb{Z}_{\geq 0}^K$,

(i) If there exists a word $w \in \mathcal{G}^+$ with $\text{PI}_\mathcal{G}(w) = \ell$ and $\log \pi(w) = 0$, then

\[
\sum_{i=1}^K \ell_i \log A_i \in \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}).
\]

(ii) If $\ell$ satisfies (3), then there exists a non-empty word $w \in \mathcal{G}^+$, with $\text{PI}_\mathcal{G}(w) \in \mathbb{Z}_{\geq 0} \cdot \ell$, such that $\log \pi(w) = 0$.

Part (i) of Theorem 3.1 is relatively easy to prove:

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\[\text{In mathematical terms, the linear spaces } L_{\geq 1}(H) \supseteq L_{\geq 2}(H) \supseteq \cdots \supseteq L_{\geq d+1}(H) = \{0\} \text{ give the Lie algebra } L_{\geq 1}(H) \text{ the structure of a filtered Lie algebra (for an exact definition, see [24]).}\]
Therefore, we have
\[ \sum_{i \in S} \ell_i \in \mathbb{Q} \]

Therefore, \( \sum_{i=1}^{K} \ell_i \log A_i = \sum_{k=1}^{n} H_k \log B_1, \ldots, \log B_m = 0. \)

The higher order terms \( H_k, k \geq n \) vanish because \( \log |\mathcal{G}|_n = \{0\} \) (a consequence of \( \mathcal{G} \subseteq \mathbb{U}(n, \mathbb{Q}) \)). Therefore, \( \sum_{i=1}^{K} \ell_i \log A_i = -\sum_{k=2}^{n-1} H_k (\log B_1, \ldots, \log B_m). \)

Since the Parikh Image of the word \( B_1 \cdots B_m \) is \( \ell \), the matrices \( B_i \) all lie in the subset \( \{ A_i \mid i \in \text{supp}(\ell) \} \) of \( \mathcal{G} \). Therefore, \( \log B_i \in \log \mathcal{G}_{\text{supp}(\ell)} \) for all \( i \). By Theorem 2.8, for all \( k \geq 2 \) we have \( -H_k (\log B_1, \ldots, \log B_m) \in \langle \{ \log B_i \mid i = 1, \ldots, m \} \rangle \mathbb{Q} \subseteq \mathcal{L}_{\geq 2} (\log \mathcal{G}_{\text{supp}(\ell)}) \subseteq \mathcal{L}_{\geq 2} (\log \mathcal{G}_{\text{supp}(\ell)}). \)

Therefore, we have \( \sum_{i=1}^{K} \ell_i \log A_i = -\sum_{k=2}^{n-1} H_k (\log B_1, \ldots, \log B_m) \in \mathcal{L}_{\geq 2} (\log \mathcal{G}_{\text{supp}(\ell)}). \)

\[ \square \]

Part (ii) of Theorem 3.1 is highly non-trivial and will be the main focus of Section 4. One can see that the above proof did not use the condition \( |\log \mathcal{G}|_1 = \{0\} \), hence part (i) would still hold without it. However, the condition \( |\log \mathcal{G}|_1 = \{0\} \) will be needed in the proof of part (ii).

Note that finding solutions of Equation (3) only relies on linear algebra. Assuming Theorem 3.1, we can devise the following Algorithm 1 that computes the invertible subset of any finite set \( \mathcal{G} \subseteq G \).

**Algorithm 1:** Computing the invertible subset of \( \mathcal{G} \)

**Input:** A finite set of elements \( \mathcal{G} = \{ A_1, \ldots, A_K \} \) in \( G \).

**Output:** The invertible subset \( \mathcal{G}_{\text{inv}} \) of \( \mathcal{G} \).

**Step 1** **Initialization.** Set \( S := \{1, \ldots, K\} \).

**Step 2** Repeat the following

(a) Represent the \( \mathbb{Q} \)-linear subspace of \( \mathbb{Q}^K \):

\[ V := \left\{ (\ell_1, \ldots, \ell_K) \in \mathbb{Q}^K \mid \sum_{i=1}^{K} \ell_i \log A_i \in \mathcal{L}_{\geq 2} (\{ \log A_i \mid i \in S \}) \right\} \]

as the solution set of homogeneous linear equations.

(b) Define \( \Lambda := \mathbb{Z}_S^K \cap V \) and compute \( \text{supp}(\Lambda) \) using Lemma 2.4.

(c) If \( \text{supp}(\Lambda) = S \), terminate the algorithm and return \( \mathcal{G}_{\text{inv}} = \{ A_i \mid i \in S \} \). Otherwise let \( S := \text{supp}(\Lambda) \) and continue.

**Proof of Theorem 1.6 and proof of correctness of Algorithm 1 (assuming Theorem 3.1).** After each iteration of Step 2, the set \( S = \text{supp}(\Lambda) \) strictly decreases. Therefore, the algorithm terminates after at most \( K \) iterations of Step 2.

Since \( G \) has nilpotency class at most ten, by Lemma 2.7, its subset \( \mathcal{G} \) satisfies \( |\log \mathcal{G}|_1 = \{0\} \). For the correctness of the algorithm, we start by showing that, when the algorithm terminates, every element of \( \{ A_i \mid i \in S \} \) has an inverse in the semigroup \( \langle \mathcal{G} \rangle \). When the algorithm terminates at Step 2(c), we have \( \text{supp}(\Lambda) = S \). By the additivity of \( \Lambda \) (that is, \( a, b \in \Lambda \implies a + b \in \Lambda \)), there exists a vector \( \ell = (\ell_1, \ldots, \ell_K) \in \Lambda \) such that \( \text{supp}(\ell) = \text{supp}(\Lambda) = S \). This vector then satisfies

\[ \sum_{i=1}^{K} \ell_i \log A_i \in \mathcal{L}_{\geq 2} (\{ \log A_i \mid i \in \text{supp}(\ell) \}) \]
by the definition of $V$. By Theorem 3.1(ii), this shows that there exists a non-empty word $w$, with $\text{PI}_G(w) \in \mathbb{Z}_{>0} \cdot \ell$ such that $\log \pi(w) = 0$ (that is, $\pi(w) = I$). For any $i \in S$, since $\text{supp}(\ell) = S$, the letter $A_i$ appears in the word $w$. Write $w = w_1 A_i w_2$; then since $\pi(w_1 A_i w_2) = I$, we have $\pi(w_1) A_i \pi(w_2) = I$. That is, either $A_i = I$ in which case $A_i^{-1} = I \in \langle G \rangle$, or $A_i \neq I$ in which case $A_i^{-1} = \pi(w_2) \pi(w_1) \in \langle G \rangle \cup \{I\}$, so $A_i^{-1} \in \langle G \rangle$.

We then show that for every matrix $A_i$ invertible in $\langle G \rangle$, $i$ is in the set $S$ at the termination of the algorithm. Suppose $A_i^{-1}$ is equal to $\pi(w)$, where $w$ is a non-empty word $w \in G^+$. Then the product of the word $w' = w A_i$ is equal to the identity, that is, $\log \pi(w') = 0$. By Theorem 3.1(i), the Parikh Image $\ell = \text{PI}_G(w')$ satisfies $\sum_{i=1}^{K} \ell_i \log A_i \in \mathcal{L}_{\geq 2}(\{\log A_i \mid i \in \text{supp}(\ell)\})$.

We show that $\text{supp}(\ell) \subseteq S$ is an invariant of the algorithm. At initialization, we obviously have $\text{supp}(\ell) \subseteq S$. Before each iteration of Step 2(b), suppose we have $\text{supp}(\ell) \subseteq S$. Then after the iteration of Step 2, $\text{supp}(\ell)$ is an invariant of the algorithm. At initialization, we obviously have $\text{supp}(\ell) \subseteq S$. Before each iteration of Step 2(b), suppose we have $\text{supp}(\ell) \subseteq S$. Then after the iteration of Step 2, $\text{supp}(\ell)$ is an invariant of the algorithm. Therefore, the overall complexity of Algorithm 1 is polynomial with respect to the input $G$. 

\section{Proof of Theorem 3.1(ii)}

\subsection{Overview of three technical propositions.}

The proof of Theorem 3.1(ii) relies on the following three technical propositions. For $k \in \mathbb{Z}_{>0}$, denote by $S_k$ the permutation group of the set $\{1, \ldots, k\}$. Theorem 2.8 showed that the term $H_k[\log B_1, \ldots, \log B_m]$ can be written as a linear combination of $k$-iterated Lie brackets $[\ldots [\log B_{i_1}, \log B_{i_2}], \log B_{i_3}], \ldots, \log B_{i_k}]$. Our first proposition shows that a converse of it is true: that for any $k \geq 2$, the $k$-iterated Lie bracket $[\ldots [\log B_1, \log B_2], \log B_3], \ldots, \log B_k]$ can be written as a linear combination of expressions in $H_k$.

\begin{proposition}
For every $k \geq 2$, there exists a function $\mu_k : S_k \to \mathbb{Z}$, such that for any set of elements $C_1, \ldots, C_m$ in the Lie algebra $u(n)$ we have
\begin{equation}
[\ldots [[C_1, C_2], C_3], \ldots, C_k] = \sum_{\sigma \in S_k} \mu_k(\sigma) H_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}, C_{k+1}, \ldots, C_m)
\end{equation}
\end{proposition}

Let $G = \{A_1, \ldots, A_K\}$ be a finite alphabet of elements in $G$. For a word $B \in G^*$, by abuse of notation we will write $\log B$ for $\log \pi(B)$. For any vector $\ell = (\ell_1, \ldots, \ell_K) \in \mathbb{Z}_{\geq 0}^K$, define inductively the following $\mathbb{Q}$-cones $\mathcal{R}_k(\ell)$ for $k = 10, 9, \ldots, 2$:

\begin{equation}
\mathcal{R}_{10}(\ell) := \left\{ H_{10}(\log B_1, \ldots, \log B_m) \mid m \geq 1, B_i \in G^*, \sum_{i=1}^m \text{PI}_G(B_i) \in \{\ell, 2\ell\} \right\}_{\mathbb{Q}_{\geq 0}},
\end{equation}

(5)
\[ \mathcal{R}_k(\ell) := \mathcal{R}_{k+1}(\ell) + \left\{ H_k(\log B_1, \ldots, \log B_m) \mid m \geq 1, B_i \in \mathcal{G}^*, \sum_{i=1}^{m} \text{PL}_{\mathcal{G}}(B_i) \in \{ \ell, 2\ell \} \right\}_{Q \geq 0} \quad (6) \]

That is, \( \mathcal{R}_k(\ell) \) is the \( \mathbb{Q} \)-cone generated by the elements \( H_j(\log B_1, \ldots, \log B_m), j \geq k \), where \( B_1, \ldots, B_m \) are words in \( \mathcal{G}^* \), and the Parikh Images of \( B_i \) sum up to \( \ell \) or \( 2\ell \). Recall the definition of
\[
\log \mathcal{G}_{\text{supp}(\ell)} := \{ \log A_i \mid i \in \text{supp}(\ell) \}
\]
as the set of logarithm of matrices in \( \mathcal{G} \) whose index appears in \( \text{supp}(\ell) \).

Our second technical proposition characterizes the cones \( \mathcal{R}_k(\log \mathcal{G}_{\text{supp}(\ell)}) \) up to the quotient of a certain vector space.

**Proposition 4.2.** Let \( \mathcal{G} = \{ A_1, \ldots, A_K \} \) be a finite set of matrices in \( \text{UT}(n, \mathbb{Q}) \) that satisfies \( \log \mathcal{G}_{11} = \{ 0 \} \). Given a non-zero vector \( \ell = (\ell_1, \ldots, \ell_K) \in \mathbb{Z}_{\geq 0}^K \) satisfying \( \ell_i \geq 10 \) for all \( i \in \text{supp}(\ell) \). Consider the quotient vector space \( \mathbb{Q}(n)/\mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) \). For any cone (or linear space) \( \mathcal{C} \subseteq \mathbb{Q}(n) \), denote by \( \overline{\mathcal{C}} \) the cone (or linear space) of \( \mathbb{Q}(n)/\mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) \) consisting of the equivalence classes of elements in \( \mathcal{C} \). Then we have the following equalities:

(i) The cone \( \overline{\mathcal{R}_{10}(\log \mathcal{G}_{\text{supp}(\ell)})} \) is equal to the linear space \( \mathcal{L}_{\geq 10}(\log \mathcal{G}_{\text{supp}(\ell)}) \).

(ii) If \( \sum_{i=1}^{K} \ell_i \log A_i \in \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) \), then for \( k = 9, 8, \ldots, 2 \), the cone \( \overline{\mathcal{R}_k(\ell)} \) is equal to the linear space \( \mathcal{L}_{\geq k}(\log \mathcal{G}_{\text{supp}(\ell)}) \).

Our third technical proposition concerns convex geometry.

**Proposition 4.3.** Let \( V \) be a finite dimensional \( \mathbb{Q} \)-linear space. Let \( d \) be a positive integer, \( \mathcal{I} \) be a finite index set, and \( a_{i_1}, \ldots, a_{i_d}; i \in \mathcal{I} \) be vectors in \( V \).

For any \( t \in \mathbb{Z}_{>0} \), define the vectors
\[
P_i(t) := t^1 a_{i_1} + \cdots + t^d a_{i_d}, \quad \text{for } i \in \mathcal{I}.
\]

Suppose the following two conditions hold:

(i) The \( \mathbb{Q}_{\geq 0} \)-cone \( \mathcal{C}_d := \langle a_{i} \mid i \in \mathcal{I} \rangle_{\mathbb{Q}_{\geq 0}} \) is a linear space.

(ii) For \( k = d - 1, d - 2, \ldots, 1 \), the inductively defined \( \mathbb{Q}_{\geq 0} \)-cone \( \mathcal{C}_k := \langle a_{i_k} \mid i \in \mathcal{I} \rangle_{\mathbb{Q}_{\geq 0}} + \mathcal{C}_{k+1} \) are linear spaces.

Then the \( \mathbb{Q}_{\geq 0} \)-cone \( \langle P_i(t) \mid i \in \mathcal{I}, t \in \mathbb{Z}_{>0} \rangle_{\mathbb{Q}_{\geq 0}} \) is equal to \( \mathcal{C}_1 \).

Proposition 4.2 is the only one among the three technical propositions that is limited by the nilpotency class. This constitutes the main obstacle to generalizing our main result (Theorem 1.6) to higher nilpotency classes (cf. Footnote 3).

The general idea of proving Theorem 3.1(ii) is as follows: consider matrices of the form \( A_j = B_1^t \cdots B_m^t; j = 1, 2, \ldots \), where \( t \geq 1 \) and \( B_1, \ldots, B_m \in \mathcal{G}^* \) range over words satisfying \( \sum_{i=1}^{m} \text{PL}_{\mathcal{G}}(B_i) \in \{ \ell, 2\ell \} \). We want to show that the identity matrix lies in the semigroup generated by these matrices \( A_j \). To this end, we characterize the \( \mathbb{Q} \)-cone generated by \( \log(B_1^t \cdots B_m^t) \) up to the quotient defined in Proposition 4.2, by applying Proposition 4.3 to the vectors

\[
P(t) = \log(B_1^t \cdots B_m^t) = t \sum_{i=1}^{m} \log B_i^t + \sum_{k=2}^{10} t^k H_k(\log B_1, \ldots, \log B_m). \quad (7)
\]

Equation (7) can be obtained by applying the BCH formula to the matrices \( B_1^t, \ldots, B_m^t \); note that \( \log B_i^t = t \log B_i \) and \( \log[B_1^t, \ldots, B_m^t] = t^k[\log C_1, C_2, \ldots, C_k] \). The terms \( H_k, k \geq 11 \) vanish because of the condition \( \log \mathcal{G}_{11} = \{ 0 \} \). We will use Proposition 4.2(i) and (ii) respectively to
guarantee that the conditions (i) and (ii) of Proposition 4.3 are satisfied. Condition (3) will imply that \( \log A' \in \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(f)}) \). We then set a new alphabet \( \mathcal{G}' \) consisting of a finite selection of these matrices \( A' \). Repeating the above process on the alphabet \( \mathcal{G}' \) will yield matrices \( A''_k, k = 1, 2, \ldots, \) with \( \log A''_k \in \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(f)}) \). Repeating this again will yield matrices \( A'''_k, k = 1, 2, \ldots, \) with \( \log A'''_k \in \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(f)}) \). Eventually, we will obtain a matrix \( A'''' \) with \( \log A'''' \in \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(f)}) \). The word representing this matrix will be non-empty and thus satisfies the requirements in Theorem 3.1(ii).

This concludes the overview. The rest of this section will be dedicated to proving Proposition 4.1 - 4.3, as well as formalizing the above idea to give the full proof for Theorem 3.1(ii).

### 4.2 Proof of Proposition 4.1

For a permutation \( \sigma \in S_k \), define \( d(\sigma) \) to be the number of descents in \( \sigma \), that is, the number of \( i \in \{1, \ldots, k-1\} \) such that \( \sigma(i) > \sigma(i+1) \). In order to prove Proposition 4.1, we need an explicit expression for the terms \( H_k \). This expression is provided by Dynkin\(^5\):

#### Lemma 4.4 (Dynkin formula [16], [29, Proposition 3.4 and Proposition 4.2]).

We have

\[
H_k(C_1, \ldots, C_m) = \sum_{i_1 + \cdots + i_m = k} \frac{1}{i_1! \cdots i_m!} \varphi_k(C_1, \ldots, C_m), \quad (8)
\]

where the indices \( i_1, \ldots, i_m \) are non-negative integers, and

\[
\varphi_k(X_1, \ldots, X_k) = \sum_{\sigma \in S_k} \frac{(-1)^{d(\sigma)}}{k^2(\sigma)} \left[ [X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}], \ldots, X_{\sigma(k)} \right]. \quad (9)
\]

Define recursively the following maps \( \mu_k : S_k \to \mathbb{Z}, k = 2, 3, \ldots \):

For \( k = 2 \), let \( \mu_2(\text{id}) = 1, \mu_2((12)) = -1 \), where \( \text{id} \) is the constant permutation and \( (12) \) is the permutation that swaps 1 and 2. For \( k \geq 3 \), denote by \( (j_1 j_2 \cdots j_m) \) the cyclic permutation that sends \( j_i \) to \( j_{i+1} \), \( i = 1, \ldots, m-1 \), and sends \( j_m \) to \( j_1 \). Suppose \( \mu_{k-1} \) already defined, we then define

\[
\mu_k(\sigma) :=
\begin{cases}
\mu_{k-1}(\sigma) & k = \sigma(k) \\
-\mu_{k-1}(\sigma \circ (12 \cdots k)) & k = \sigma(1) \\
0 & k = \sigma(i), i = 2, \ldots, k-1.
\end{cases} \quad (10)
\]

In the first two cases, the permutation \( \sigma \) and \( \sigma \circ (12 \cdots k) \) fix \( k \), so they can be considered as elements in \( S_{k-1} \), hence \( \mu_{k-1}(\sigma) \) is well defined. For example, \( \mu_3(\sigma) = 1 \) when \( \sigma = \text{id} \) or \( (13) \); \( \mu_3(\sigma) = -1 \) when \( \sigma = (12) \) or \( (132) \); and \( \mu_3(\sigma) = 0 \) otherwise. We will show that, for this \( \mu_k \), the Equation (4) in Proposition 4.1 is satisfied.

#### Proposition 4.1

For every \( k \geq 2 \), there exists a function \( \mu_k : S_k \to \mathbb{Z} \), such that for any set of elements \( C_1, \ldots, C_m \) in the Lie algebra \( \mathfrak{u}(n) \) we have

\[
[\ldots [C_1, C_2], C_3], \ldots, C_k] = \sum_{\sigma \in S_k} \mu_k(\sigma) H_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}, C_{k+1}, \ldots, C_m) \quad (4)
\]

**Proof.** Take \( \mu_k \) to be the function defined recursively in (10). For every \( j \geq 2 \), there is a natural embedding \( f_j : S_j \to S_{j+1} \), defined by \( f_j(\sigma)(i) = \sigma(i), i = 1, \ldots, j, f_j(\sigma)(j + 1) = j + 1 \). It is easy to verify that under this natural embedding, \( \mu_j \) and \( \mu_{j+1} \) are identified, that is, \( \mu_j = \mu_{j+1} \circ f_j \). Therefore, we can denote by \( \mu \) the map \( \cup_{k \geq 2} S_k \to \mathbb{Z} \) as \( \mu(\sigma) = \mu_k(\sigma) \), where \( \sigma \in S_k \). We prove Equation (4) in three steps.

\(^5\)Dynkin originally only proved the bivariate case of Lemma 4.4. It was later been generalized to the multivariate case without much difficulty.
(Step 1.) First, we simplify the right hand side of Equation (4) by showing

\[ \sum_{\sigma \in S_k} \mu(\sigma) H_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}, C_{k+1}, \ldots, C_m) = \sum_{\sigma \in S_k} \mu(\sigma) \varphi_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}), \]  

(11)

where \( \varphi_k \) is defined in Lemma 4.4.

Thanks to Lemma 4.4, \( H_k(C_1, \ldots, C_m) \) can be written as

\[ H_k(C_1, \ldots, C_m) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq m} \varphi_k(C_{j_1}, \ldots, C_{j_k}) + \sum_{l=2}^{k-1} \sum_{1 \leq j_1 < j_2 < \cdots < j_l \leq m} H_{kl}(C_{j_1}, \ldots, C_{j_l}), \]  

(12)

where \( H_{kl}(C_{j_1}, \ldots, C_{j_l}) \) is some linear combination of elements in \([\{C_{j_1}, \ldots, C_{j_l}\}]_k\). By abuse of notation, for \( \sigma \in S_k \) and \( x > k \), we define \( \sigma(x) = \sigma^{-1}(x) = x \). For any \( l = 2, \ldots, k-1 \), we have

\[ \sum_{\sigma \in S_k} \sum_{1 \leq j_1 < j_2 < \cdots < j_l \leq m} \mu(\sigma) H_{kl}(C_{\sigma(j_1)}, \ldots, C_{\sigma(j_l)}) \]

\[ = \sum_{\sigma \in S_k} \sum_{t_1, t_2, \ldots, t_l \in \{1, \ldots, m\} \text{ pairwise distinct}} H_{kl}(C_{t_1}, \ldots, C_{t_l}) \sum_{\sigma \in S_k} \sum_{\sigma^{-1}(t_1) < \cdots < \sigma^{-1}(t_l)} \mu(\sigma). \]  

(13)

We claim that, for any pairwise distinct \( t_1, t_2, \ldots, t_l \in \{1, \ldots, m\} \), \( l < k \), we have

\[ \sum_{\sigma \in S_k} \sum_{\sigma^{-1}(t_1) < \cdots < \sigma^{-1}(t_l)} \mu(\sigma) = 0. \]  

(14)

We show (14) by induction on \( k \). When \( k = 2 \), by the definition of \( \mu \), (14) holds. Suppose (14) holds for \( k-1 \). Denote by \( c \) the cyclic permutation \( (12\cdots k) \), then by the recursive definition of \( \mu \),

\[ \sum_{\sigma \in S_k} \mu(\sigma) = \sum_{\sigma \in S_{k-1}} \mu(\sigma) - \sum_{\sigma \in S_{k-1}} \mu(\sigma). \]  

(15)

Without loss of generality, suppose the sum on the left hand side is not empty. That is, there exists at least one permutation \( \sigma \in S_k \) such that \( \sigma^{-1}(t_1) < \cdots < \sigma^{-1}(t_l) \). Since \( \sigma^{-1} \in S_k \) does not permute any \( t_j \) with \( t_j > k \), the elements of \( \{t_1, \ldots, t_l\} \) which are larger than \( k \) must appear after the elements which are smaller or equal to \( k \), and must appear in increasing order. In other words, there exists some \( s \geq 1 \), such that \( t_i \leq k \) for all \( i < s \), and \( k < t_s < \cdots < t_l \). (\( s \) could be \( l+1 \), in which case \( t_i \leq k \) for all \( i = 1, \ldots, l \).) Since \( \sigma \in S_k \) does not change the value of \( t_s, \cdots, t_l \), one can discard them without changing the sum. Hence, we suppose without loss of generality \( t_1, \ldots, t_l \in \{1, \ldots, k\} \).

(a) If \( t_i = k \) for some \( i = 2, \ldots, l-1 \). Then no permutation \( \sigma \in S_{k-1} \) can satisfy \( \sigma^{-1}(t_1) < \sigma^{-1}(t_l) = k < \sigma^{-1}(t_i) \) or \( c \circ \sigma^{-1}(t_1) < c \circ \sigma^{-1}(t_l) = 1 < c \circ \sigma^{-1}(t_i) \). Hence, both sums on the right hand side of Equation (15) are empty. The claim (14) follows.

(b) If \( t_k = k \). Then no permutation \( \sigma \in S_{k-1} \) can satisfy \( \sigma^{-1}(t_1) < \sigma^{-1}(t_k) = k < \sigma^{-1}(t_l) \), so the first sum on the right hand side of Equation (15) is empty. As for the second sum, because \( c \circ \sigma^{-1}(t_1) = c(k) = 1 \), we have \( c \circ \sigma^{-1}(t_1) < \cdots < c \circ \sigma^{-1}(t_l) \) if and only if
\[ \sigma^{-1}(t_2) < \cdots < \sigma^{-1}(t_l). \] Hence, using the induction hypothesis on \( t_2, \ldots, t_l \in \{1, \ldots, k\} \) yields
\[
\sum_{\sigma \in S_{k-1}} \mu(\sigma) = \sum_{\sigma \in S_{k-1}} \mu(\sigma) = 0.
\]

Therefore both sums on the right hand side of Equation (15) equal zero. The claim (14) follows.

(c) If \( t_l = k \). Similar to the previous case, the second sum on the right hand side of Equation (15) is empty. As for the first sum, because \( \sigma^{-1}(t_l) = k \), we have \( \sigma^{-1}(t_1) < \cdots < \sigma^{-1}(t_l) \) if and only if \( \sigma^{-1}(t_1) < \cdots < \sigma^{-1}(t_{l-1}) \). Hence, using the induction hypothesis on \( t_1, \ldots, t_{l-1} \in \{1, \ldots, k\} \) shows the sum is zero. The claim (14) follows.

(d) If \( t_i \neq k \) for all \( i = 1, \ldots, l \). Then \( \sigma^{-1}(t_1) < \cdots < \sigma^{-1}(t_l) \) if and only if \( c \circ \sigma^{-1}(t_1) < \cdots < c \circ \sigma^{-1}(t_l) \). Hence, the two sums on the right hand side of Equation (15) are the same. The claim (14) follows.

Using the claim (14) on Equation (13) yields
\[
\sum_{\sigma \in S_k} \mu(\sigma) H_{kl}(C_{\sigma(j_1)}, \ldots, C_{\sigma(j_l)}) = 0, \tag{16}
\]
and this combined with Equation (12) yields
\[
\sum_{\sigma \in S_k} \mu(\sigma) H_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}, C_{k+1}, \ldots, C_m)
= \sum_{\sigma \in S_k} \mu(\sigma) H_k(C_{\sigma(1)}, \ldots, C_{\sigma(m)}) \quad \text{(define } \sigma(s) = s \text{ for } \sigma \in S_k \text{ and } s > k) \nonumber
\]
\[
= \sum_{\sigma \in S_k} \mu(\sigma) \sum_{1 \leq j_1 < j_2 < \cdots < j_l \leq m} \varphi_k(C_{\sigma(j_1)}, \ldots, C_{\sigma(j_l)}) \quad \text{(by (12) and (16))} \nonumber
\]
\[
= \sum_{t_1, t_2, \ldots, t_l \in \{1, \ldots, m\} \ \text{pairwise distinct}} \varphi_k(C_{t_1}, \ldots, C_{t_l}) \sum_{\sigma \in S_k} \mu(\sigma) \sum_{\sigma^{-1}(t_1) < \cdots < \sigma^{-1}(t_l)} \nonumber
\]
\[
= \sum_{l=0}^{k} \sum_{t_1, t_2, \ldots, t_l \in \{1, \ldots, k\} \ \text{pairwise distinct}} \varphi_k(C_{t_1}, \ldots, C_{t_l}) \sum_{\sigma \in S_k} \mu(\sigma) \sum_{\sigma^{-1}(t_1) < \cdots < \sigma^{-1}(t_l)} \quad \text{(17)}
\]

Because Equation (14) holds for \( l < k \), that is, the sum \( \sum_{\sigma \in S_k} \mu(\sigma) \) vanishes whenever \( l < k \), the above expression (17) is equal to
\[
\sum_{t_1, t_2, \ldots, t_k \in \{1, \ldots, k\} \ \text{pairwise distinct}} \varphi_k(C_{t_1}, \ldots, C_{t_k}) \sum_{\sigma \in S_k} \mu(\sigma) = \sum_{\sigma \in S_k} \mu(\sigma) \varphi_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}). \nonumber
\]

We have hence shown Equation (11):
\[
\sum_{\sigma \in S_k} \mu(\sigma) H_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}, C_{k+1}, \ldots, C_m) = \sum_{\sigma \in S_k} \mu(\sigma) \varphi_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}),
\]

(Step 2.) The second step is to show
\[
\sum_{\sigma \in S_k} \mu(\sigma) \varphi_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}) = \sum_{T \in S_k} \frac{\mu(T)}{k} \left[ \ldots \left[ [C_{T(1)}, C_{T(2)}], C_{T(3)}], \ldots, C_{T(k)} \right] \right]. \tag{18}
\]
Using the exact expression for $\varphi_k$ in Lemma 4.4, we have
\begin{equation}
\sum_{\sigma \in S_k} \mu(\sigma) \varphi_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)})
= \sum_{\sigma \in S_k} \sum_{\tau \in S_k} \frac{(-1)^{\delta(\tau)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(\tau)} \right)} \left[ \cdots \left[ \frac{C_{\sigma \tau(1)}, C_{\sigma \tau(2)}, \ldots, C_{\sigma \tau(k)}}{C_{\sigma(1)}, C_{\sigma(2)}, \ldots, C_{\sigma(k)}} \right] \right]
= \sum_{T \in S_k} \left[ \cdots \left[ \frac{C_{T(1)}, C_{T(2)}, \ldots, C_{T(k)}}{C_{T(1)}, C_{T(2)}, \ldots, C_{T(k)}} \right] \right] \sum_{\sigma \in S_k} \frac{(-1)^{\delta(\sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(\sigma^{-1} o T)} \right)} \tag{19}
\end{equation}

We will compute the value of $\sum_{\sigma \in S_k} \frac{(-1)^{\delta(\sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(\sigma^{-1} o T)} \right)}$ depending on the permutation $T$. We show by induction on $k$ that
\begin{equation}
\sum_{\sigma \in S_k} \frac{(-1)^{\delta(\sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(\sigma^{-1} o T)} \right)} = \frac{\mu(T)}{k}. \tag{20}
\end{equation}

When $k = 2$, by direct computation, $\sum_{\sigma \in S_k} \frac{(-1)^{\delta(\sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(\sigma^{-1} o T)} \right)}$ is equal to $\frac{1}{2}$ if $T = \text{id}$ and to $-\frac{1}{2}$ if $T = (12)$. This matches the values of $\frac{\mu(T)}{k}$. If $k \geq 3$, suppose (20) proven for $k - 1$. Again denote by $c$ the cyclic permutation $(12 \cdots k)$, by the recursive definition of $\mu$ we have
\begin{equation}
\sum_{\sigma \in S_k} \frac{(-1)^{\delta(\sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(\sigma^{-1} o T)} \right)} = \sum_{\sigma \in S_{k-1}} \frac{(-1)^{\delta(\sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(\sigma^{-1} o T)} \right)} - \sum_{\sigma \in S_{k-1}} \frac{(-1)^{\delta(c o \sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(c o \sigma^{-1} o T)} \right)}. \tag{21}
\end{equation}

(a) If $T(i) = k$ for some $i = 2, \ldots, k - 1$. We claim that $d(\sigma^{-1} o T) = d(c o \sigma^{-1} o T)$ for all $\sigma \in S_{k-1}$. In fact, for $\sigma \in S_{k-1}$, we have $\sigma^{-1} o T(i) = k$ and $c o \sigma^{-1} o T(i) = 1$. Therefore $\sigma^{-1} o T(i) > \sigma^{-1} o T(i+1)$, $\sigma^{-1} o T(i) > \sigma^{-1} o T(i-1)$, whereas $c o \sigma^{-1} o T(i) < c o \sigma^{-1} o T(i+1)$, $c o \sigma^{-1} o T(i) < c o \sigma^{-1} o T(i-1)$. And for $j \neq i - 1$, we have $\sigma^{-1} o T(j) > \sigma^{-1} o T(j+1)$ and only if $c o \sigma^{-1} o T(j) > c o \sigma^{-1} o T(j+1)$. This shows $d(\sigma^{-1} o T) = d(c o \sigma^{-1} o T)$. Hence, the two sums on the right hand side of (21) are equal, and $\sum_{\sigma \in S_k} \frac{(-1)^{\delta(\sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(\sigma^{-1} o T)} \right)} = 0 = \frac{\mu(T)}{k}$. 

(b) If $T(1) = k$. Similar to the above discussion, we can show that $d(\sigma^{-1} o T) = d(c o \sigma^{-1} o T) + 1$. Hence the right hand side of (21) is equal to
\begin{align*}
&= -\sum_{\sigma \in S_{k-1}} \frac{(-1)^{\delta(\sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(\sigma^{-1} o T)+1} \right)} + \frac{(-1)^{\delta(c o \sigma^{-1} o T)} \mu(\sigma)}{k^2 \left( \frac{k-1}{d(c o \sigma^{-1} o T)} \right)} \\
&= -\sum_{\sigma \in S_{k-1}} \frac{(-1)^{\delta(c o \sigma^{-1} o T)} \mu(\sigma)}{k(k-1) \left( \frac{k-2}{d(c o \sigma^{-1} o T)} \right)} \\
&= -\frac{(k-1)}{k} \sum_{\sigma \in S_{k-1}} \frac{(-1)^{\delta(c o \sigma^{-1} o T)} \mu(\sigma)}{(k-1)^2 \left( \frac{k-2}{d(c o \sigma^{-1} o T)} \right)}
\end{align*}

We claim that $d(c o \sigma^{-1} o T) = d(\sigma^{-1} o T o c)$. This is because $\sigma^{-1} o T o c(k-1) < \sigma^{-1} o T o c(k) = k$, $1 = c o \sigma^{-1} o T(1) < c o \sigma^{-1} o T(2)$, and $c o \sigma^{-1} o T(i+1) > c o \sigma^{-1} o T(i)$.
if and only if \( \sigma^{-1} \circ T \circ c(i) > \sigma^{-1} \circ T \circ c(i - 1) \), for \( i = 2, 3, \ldots, k - 1 \). Hence,

\[
\frac{-(k-1)}{k} \sum_{\sigma \in S_{k-1}} \frac{(-1)^{d(c \sigma^{-1} \circ T)} \mu(\sigma)}{(k-1)^2 (d_{c (c \sigma^{-1} \circ T)})} = \frac{-(k-1)}{k} \sum_{\sigma' \in S_{k-1}} \frac{(-1)^{d(c \sigma^{-1} \circ T \circ c)} \mu(\sigma)}{(k-1)^2 (d_{c \sigma^{-1} \circ T \circ c})} = \frac{-(k-1)}{k} \frac{\mu(T \circ c)}{k-1} = \frac{(k-1) \mu(T)}{k} \text{ (by induction hypothesis)}.
\]

(c) If \( T(k) = k \). Similar to the above discussion, we can show that \( d(c \circ \sigma^{-1} \circ T) = d(\sigma^{-1} \circ T) + 1 \). And hence the right hand side of (21) is equal to

\[
\frac{(k-1)}{k} \sum_{\sigma \in S_{k-1}} \frac{(-1)^{d(\sigma^{-1} \circ T)} \mu(\sigma)}{(k-1)^2 (d(\sigma^{-1} \circ T))} = \frac{(k-1) \mu(T)}{k-1} = \frac{\mu(T)}{k}.
\]

by the induction hypothesis, where \( T \) can be considered as an element in \( S_{k-1} \) since it stabilizes \( k \).

We have thus shown the claim (20). Putting this into Equation (19) shows Equation (18):

\[
\sum_{\sigma \in S_k} \mu(\sigma) \varphi_k(C_{\sigma(1)}, \ldots, C_{\sigma(k)}) = \sum_{T \in S_k} \frac{\mu(T)}{k} \cdot [[[C_{T(1)}, C_{T(2)}], C_{T(3)}], \ldots, C_{T(k)}].
\]

(Step 3.) The third and last step is to show

\[
\sum_{T \in S_k} \mu(T) \cdot [[[C_{k+1}, C_{T(1)}], C_{T(2)}], \ldots, C_{T(k)}] = k \cdot [[[C_{1}, C_{2}], C_{3}], \ldots, C_{k}]. \tag{22}
\]

First, using induction on \( k \), we will show that

\[
\sum_{T \in S_k} \mu(T) \cdot [[[C_{k+1}, C_{T(1)}], C_{T(2)}], \ldots, C_{T(k)}] = -[[[C_{1}, C_{2}], C_{3}], \ldots, C_{k+1}]. \tag{23}
\]

The case where \( k = 2 \) is immediate. Suppose Equation (23) hold for \( k - 1 \), then

\[
\sum_{T \in S_k} \mu(T) \cdot [[[C_{k+1}, C_{T(1)}], C_{T(2)}], \ldots, C_{T(k)}] = \sum_{T \in S_{k-1}} \mu(T) \cdot [[[C_{k+1}, C_{T(1)}], C_{T(2)}], \ldots, C_{T(k-1)}], C_k] - \sum_{T \in S_{k-1}} \mu(T) \cdot [[[C_{k+1}, C_k], C_{T(1)}], \ldots, C_{T(k-1)}]. \tag{24}
\]

A direct way of proving Equation (22) is to use the Dynkin-Specht-Wever theorem [16], which states that if a non-commutative polynomial \( f \in \mathbb{Q}(C_1, \ldots, C_k) \) is Lie, then one can replace all monomials \( C_1 C_2 \cdots C_k \) by \( [\ldots [C_{i_1}, C_{i_2}], \ldots, C_{i_k}] / k \) without changing its value. Writing the right hand side of (22) as an element in \( \mathbb{Q}(C_1, \ldots, C_k) \) gives \( k \sum_{\sigma \in S_k} \mu(\sigma) C_{\sigma(1)} C_{\sigma(2)} \cdots C_{\sigma(k)} \) (we can check this using the definition of \( \mu \), which is equal to the left hand side by replacing the monomials \( C_{\sigma(1)} C_{\sigma(2)} \cdots C_{\sigma(k)} \) by the Lie brackets \( [[[C_{\sigma(1)}, C_{\sigma(2)}], C_{\sigma(3)}], \ldots, C_{\sigma(k)}] / k \). Nevertheless, here we will give a self-contained proof without using the Dynkin-Specht-Wever theorem.
By the induction hypothesis, the first sum on the right hand side is equal to
\[-[[...[[C_1, C_2, C_3], ..., C_{k-1}], C_{k+1}], C_k],\]
and the second sum on the right hand side is equal to
\[-[[...[[C_1, C_2, C_3], ..., C_{k-1}], C_{k+1}, C_k]].\]

Using the Jacobi identity and the anticommutativity of Lie brackets, we have
\[-[[...[[C_1, C_2, C_3], ..., C_{k-1}], C_{k+1}, C_k] + [[...[[C_1, C_2, C_3], ..., C_{k-1}], C_{k+1}, C_k]]

Hence, Equation (24) yields
\[
\sum_{T \in S_k} \mu(T)[...[[C_{k+1}], C_{T(1)}], C_{T(2)}, ..., C_{T(k)}] = -[[...[[C_1, C_2, C_3], ..., C_k], C_{k+1}],
\]
concluding the proof by induction for Equation (23).

Next, we will again use induction on \(k\) to prove Equation (22):
\[
\sum_{T \in S_k} \mu(T)[...[[C_{T(1)}, C_{T(2)}], C_{T(3)}, ..., C_{T(k)}] = k[...[[C_1, C_2, C_3], ..., C_k].
\]

The case of \(k = 2\) results from direct computation. Suppose (22) hold for \(k - 1\), then
\[
\sum_{T \in S_k} \mu(T)[...[[C_{T(1)}, C_{T(2)}], C_{T(3)}, ..., C_{T(k)}] = \sum_{T \in S_{k-1}} \mu(T)[...[[C_{k}, C_{T(1)}], C_{T(2)}, ..., C_{T(k-1)}] - \sum_{T \in S_{k-1}} \mu(T)[...[[C_{k}, C_{T(1)}], C_{T(2)}, ..., C_{T(k-1)}] - \sum_{T \in S_{k-1}} \mu(T)[...[[C_{k}, C_{T(1)}], C_{T(2)}, ..., C_{T(k-1)}] = (k - 1)[...[[C_1, C_2, C_3], ..., C_k]
\]

We have thus shown Equation (22).

Combining the Equations (11), (18) and (22) obtained in the three steps gives us
\[
\sum_{\sigma \in S_k} \mu(\sigma)H_k(C_{\sigma(1)}, ..., C_{\sigma(k)}, C_{k+1}, ..., C_m) = [...[[C_1, C_2, C_3], ..., C_k].
\]

\[\square\]

4.3 Proof of Proposition 4.2. In this subsection we prove Proposition 4.2. Again, the key is understanding the structure of the expressions for \(H_k\). First, we need the following lemma, which shows that for even \(k\), the expression \(H_k(C_1, ..., C_m)\) is “antisymmetric”. 

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Lemma 4.5. When $k$ is even, we have
\[ H_k(C_1, \ldots, C_m) = -H_k(C_m, \ldots, C_1) \]

Proof. Define a new variable $t$. Replacing $B_i$ by $\exp(tC_i)$ in the Baker-Campbell-Hausdorff Formula (2), we have
\[
\log(\exp(tC_1) \cdots \exp(tC_m)) = t \sum_{i=1}^m C_i + t^k \sum_{k=2}^{d-1} H_k(C_1, \ldots, C_m)
\]  
(25)

because the term $H_k$ is homogeneous of degree $k$. Now, replace $B_i$ by $\exp(-tC_{m+1-i})$, $i = 1, \ldots, m$, in the Baker-Campbell-Hausdorff Formula (2), we have
\[
\log(\exp(-tC_m) \cdots \exp(-tC_1)) = -t \sum_{i=1}^m C_i + (-t)^k \sum_{k=2}^{d-1} H_k(C_m, \ldots, C_1).
\]  
(26)

Since $\log(\exp(tC_1) \cdots \exp(tC_m)) = -\log(\exp(-tC_m) \cdots \exp(-tC_1))$, comparing the coefficients of $t^k$ in (25) and (26) yields
\[ H_k(C_1, \ldots, C_m) = -H_k(C_m, \ldots, C_1) \]
for even $k$.

Next, we need the following lemmas regarding the odd terms $H_3, H_5, H_7$ and $H_9$.

Lemma 4.6. Let $G$ be a unipotent matrix group over $\mathbb{Q}$, and $\mathcal{H} \subseteq G$ be a finite set of matrices. Given matrices $B_1, \ldots, B_m$ in $G$ such that $\log B_i \in \mathcal{L}_{\geq 1}(\log \mathcal{H})$, $i = 1, \ldots, m$, and $\sum_{i=1}^m \log B_i \in \mathcal{L}_{\geq 2}(\log \mathcal{H})$, then
\[ \sum_{\sigma \in \mathcal{S}_m} H_3(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(m)}) \in \mathcal{L}_{\geq 4}(\log \mathcal{H}). \]

Proof. Denote $C_i := \log B_i$, $i = 1, \ldots, m$, we will show the following identity
\[
\sum_{\sigma \in \mathcal{S}_m} H_3(C_{\sigma(1)}, \ldots, C_{\sigma(m)}) = \frac{m!}{12} \sum_{i=1}^m \left[ C_i, \left[ C_i, \sum_{j=1}^m C_j \right] \right].
\]  
(27)

Write
\[ H_3(C_{\sigma(1)}, \ldots, C_{\sigma(m)}) = \sum_{i<j<k} H_{33}(C_{\sigma(i)}, C_{\sigma(j)}, C_{\sigma(k)}) + \sum_{i<j} H_{32}(C_{\sigma(i)}, C_{\sigma(j)}) \]
where
\[ H_{33}(X, Y, Z) = \frac{1}{3}[X, [Y, Z]] + \frac{1}{6}[[X, Z], Y], \]
\[ H_{32}(X, Y) = \frac{1}{12}([X, [X, Y]] + [[X, Y], Y]). \]

Using the Jacobi identity, we have
\[ H_{33}(C_i, C_j, C_k) + H_{33}(C_j, C_k, C_i) + H_{33}(C_k, C_i, C_j) \]
\[ = \frac{1}{3} \left( [[C_i, [C_j, C_k]] + [C_j, [C_k, C_i]] + [C_k, [C_i, C_j]]] \right) + \frac{1}{6} \left( [[[C_i, C_j], C_k] + [[C_j, C_k], C_i] + [[C_k, C_i], C_j]] \right) \]
\[ = 0 \]

\[ \blacksquare \]
for any $i, j, k$. Similarly,

$$H_{33}(C_k, C_j, C_i) + H_{33}(C_j, C_i, C_k) + H_{33}(C_i, C_k, C_j) = 0.$$ 

Hence,

$$\sum_{\sigma \in S_m} \sum_{i<j<k} H_{33}(C_{\sigma(i)}, C_{\sigma(j)}, C_{\sigma(k)})$$

$$= \frac{m!}{6} \sum_{i<j<k} (H_{33}(C_i, C_j, C_k) + H_{33}(C_j, C_k, C_i) + H_{33}(C_k, C_i, C_j))$$

$$+ \frac{m!}{6} \sum_{i<j<k} (H_{33}(C_k, C_j, C_i) + H_{33}(C_j, C_i, C_k) + H_{33}(C_i, C_k, C_j))$$

$$= 0.$$

While

$$\sum_{\sigma \in S_m} \sum_{i<j} H_{3,2}(C_{\sigma(i)}, C_{\sigma(j)})$$

$$= \frac{m!}{2} \sum_{i \neq j} H_{3,2}(C_i, C_j)$$

$$= \frac{m!}{2} \sum_{i \neq j} \left( \frac{1}{12} [C_i, [C_i, C_j]] + \frac{1}{12} [[C_i, C_j], C_j] \right)$$

$$= \frac{m!}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \left( \frac{1}{12} [C_i, [C_i, C_j]] + \frac{1}{12} [[C_i, C_j], C_j] \right)$$

$$= \frac{m!}{12} \sum_{i=1}^{m} \left[ C_i, \left[ C_i, \sum_{j=1}^{m} C_j \right] \right] + \frac{m!}{2} \sum_{j=1}^{m} \frac{1}{12} \left[ \sum_{i=1}^{m} C_i, C_j \right]$$

$$= \frac{m!}{12} \sum_{i=1}^{m} \left[ C_i, \left[ C_i, \sum_{j=1}^{m} C_j \right] \right].$$

Adding up the two above expressions yields Equation (27). Since $\log B_i \in \mathcal{L}_{\geq 1}(\log \mathcal{H})$ for all $i$ and $\sum_{i=1}^{m} \log B_i \in \mathcal{L}_{\geq 2}(\log \mathcal{H})$, Equation (27) yields

$$\sum_{\sigma \in S_m} H_3(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(m)})$$

$$= \frac{m!}{12} \sum_{i=1}^{m} \left[ \log B_i, \left[ \log B_i, \sum_{j=1}^{m} \log B_j \right] \right]$$

$$\in \frac{m!}{12} \sum_{i=1}^{m} [\log B_i, [\log B_i, \mathcal{L}_{\geq 2}(\log \mathcal{H})]]$$

$$\in \mathcal{L}_{\geq 4}(\log \mathcal{H})$$
The following Lemmas 4.7, 4.9 and 4.10 regarding $H_5, H_7, H_9$ are found using computer assistance from the software SageMath [35]. In what follows, we give a sketch of their proof. Details of the full proof along with the algorithm used for computer assistance is given in Appendix B. Links to the code can be found in the respective proofs.

**Lemma 4.7.** Let $G$ be a unipotent matrix group over $\mathbb{Q}$, and $\mathcal{H} \subseteq G$ be a finite set of matrices. There exists a permutation $(j_1, j_2, \ldots, j_{12})$ of the tuple $(1, 1, 2, 2, \ldots, 6, 6)$, such that for any given set of matrices $B_1, \ldots, B_6$ in $G$ with $\log B_i \in \mathcal{L}_{\geq 1}(\log \mathcal{H})$ and $\sum_{i=1}^{6} \log B_i \in \mathcal{L}_{\geq 2}(\log \mathcal{H})$, we have

$$\sum_{\sigma \in S_6} H_5(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(6)}) + \sum_{\sigma \in S_6} H_5(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_{12})}) \in \mathcal{L}_{\geq 6}(\log \mathcal{H}) + \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{H})). \tag{28}$$

Namely, we can take $(j_1, j_2, \ldots, j_{12}) = (1, 2, 3, 4, 4, 5, 6, 6, 1, 2, 3)$.

**Sketch of proof of Lemma 4.7.** For $x, y \in \mathfrak{u}(n)$, denote $x \sim y$ if $x - y \in \mathcal{L}_{\geq 6}(\log \mathcal{H}) + \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{H}))$. The claim (28) can be written as

$$\sum_{\sigma \in S_6} H_5(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(6)}) + \sum_{\sigma \in S_6} H_5(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_{12})}) \sim 0.$$

By the Dynkin formula (Lemma 4.4), the two expressions

$$\sum_{\sigma \in S_6} H_5(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(6)})$$

and

$$\sum_{\sigma \in S_6} H_5(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_{12})})$$

can be expressed as a sum in the form of

$$\sum_{j=(j_1, \ldots, j_5) \in \{1, \ldots, 6\}^5} \alpha_j \sum_{\sigma \in S_6} \varphi_5(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_5)}), \tag{29}$$

where $\alpha_j$ are rational numbers.

Since $\sum_{i=1}^{6} \log B_i \in \mathcal{L}_{\geq 2}(\log \mathcal{H})$, for any tuple $j = (j_1, \ldots, j_5) \in \{1, \ldots, 6\}^5$, the expression

$$\sum_{\sigma \in S_6} \varphi_5(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_{12})})$$

is equivalent (under $\sim$) to a rational multiple of

$$\sum_{i \neq j} [\log B_i, \log B_j], [\log B_i, \log B_j], [\log B_i, \log B_i]. \quad (\text{See Appendix B for detailed justification.})$$

In particular, using computer assistance, we can compute these rational multiples and show

$$\sum_{\sigma \in S_6} H_5(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(6)}) \sim \sum_{i \neq j} [\log B_i, \log B_j], [\log B_i, \log B_j], [\log B_i, \log B_i], [\log B_i, \log B_i]$$

and

$$\sum_{\sigma \in S_6} H_5(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_{12})}) \sim -\sum_{i \neq j} [\log B_i, \log B_j], [\log B_i, \log B_j], [\log B_i, \log B_i], [\log B_i, \log B_i].$$

This yields

$$\sum_{\sigma \in S_6} H_5(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(6)}) + \sum_{\sigma \in S_6} H_5(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_{12})}) \sim 0.$$

The code used for computer assistance can be found at [https://doi.org/10.6084/m9.figshare.20124146.v1](https://doi.org/10.6084/m9.figshare.20124146.v1).

**Remark 4.8.** The added expression of $\mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{H}))$ on the right hand side of Equation (28) is crucial for its correctness. In fact, we can consider Equation (28) in the quotient Lie algebra $L := \mathcal{L}_{\geq 1}(\log \mathcal{H})/\mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{H}))$. $L$ is *metabelian*, meaning $[[L, L], [L, L]] = 0$. (Free) metabelian Lie algebras have significantly fewer dimensions compared to (free) Lie algebras having the same number
of generators. Moreover, (free) metabelian Lie algebras admit a relatively simple basis (sometimes called the Gröbner-Shirshov basis) \[8\], making it computationally viable to find identities such as Equation (28). In this paper, we are using a heavily modified version of this basis to compute Equation (28), as well as Equations (30) and (31) in the following lemmas. See Appendix B for details.

**Lemma 4.9.** Let \( G \) be a unipotent matrix group over \( \mathbb{Q} \), and \( \mathcal{H} \subseteq G \) be a finite set of matrices. There exist positive rational numbers \( \alpha_1, \alpha_2 \), as well as, for \( s = 1, 2 \), permutations \((j_{s, 1}, j_{s, 2}, \ldots, j_{s, 16})\) of the tuple \((1, 1, 2, 2, \ldots, 8, 8)\), such that for any given set of matrices \( B_1, \ldots, B_8 \) in \( G \) with \( \log B_i \in \mathcal{L}_{\geq 1}(\log \mathcal{H}) \) and \( \sum_{i=1}^{8} \log B_i \in \mathcal{L}_{\geq 2}(\log \mathcal{H}) \), we have

\[
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(8)}) + \sum_{s=1}^{2} \alpha_s \sum_{\sigma \in S_8} H_7(\log B_{\sigma(j_{s, 1})}, \ldots, \log B_{\sigma(j_{s, 16})}) \\
\in \mathcal{L}_{\geq 8}(\log \mathcal{H}) + \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{H})). \tag{30}
\]

Namely, we can take \( \alpha_1 = \frac{1}{15}, \alpha_2 = \frac{8}{15} \), and

\[
(j_{1, 1}, j_{1, 2}, \ldots, j_{1, 16}) = (1, 2, 3, 4, 5, 5, 6, 6, 7, 7, 8, 8, 1, 2, 3, 4), \\
(j_{2, 1}, j_{2, 2}, \ldots, j_{2, 16}) = (1, 2, 3, 4, 5, 4, 6, 7, 1, 2, 8, 3, 5, 6, 7, 8).
\]

**Sketch of proof of Lemma 4.9.** Similar to Lemma 4.7, define the equivalence relation

\[ x \sim y \iff x - y \in \mathcal{L}_{\geq 8}(\log \mathcal{H}) + \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{H})). \]

By the Dynkin formula (Lemma 4.4), the two expressions

\[
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(8)}) \quad \text{and} \quad \sum_{\sigma \in S_8} H_7(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_{16})})
\]

can be expressed as a sum in the form of

\[
\sum_{j = (j_1, \ldots, j_7) \in \{1, \ldots, 8\}^7} \alpha_j \sum_{\sigma \in S_8} \varphi_7(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_7)}),
\]

where \( \alpha_j \) are rational numbers.

Denote \( C_i := \log B_i, i = 1, \ldots, m \). Since \( \sum_{i=1}^{8} C_i \in \mathcal{L}_{\geq 2}(\log \mathcal{H}) \), for any tuple \( j = (j_1, \ldots, j_7) \in \{1, \ldots, 8\}^7 \), the expression \( \sum_{\sigma \in S_8} \varphi_7(C_{\sigma(j_1)}, \ldots, C_{\sigma(j_7)}) \) is equivalent to a linear combination (with rational coefficient) of

\[
\sum_{i \neq j} [[[[[[C_i, C_j], C_j], C_i], C_i], C_i], C_i], \\
\sum_{i \neq j} [[[[[[C_i, C_j], C_j], C_j], C_i], C_i], C_i],
\]

and

\[
\sum_{i, j, k \text{ distinct}} [[[[[[C_i, C_j], C_j], C_k], C_k], C_i], C_i].
\]

(See Appendix B for detailed justification.) In fact, using computer assistance, we show that

\[
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(8)}) \sim \frac{34}{15} \sum_{i \neq j} [[[[[[C_i, C_j], C_j], C_i], C_i], C_i], C_i]
\]

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\[-\frac{34}{45} \sum_{i \neq j} [[[C_i, C_j], C_j, C_j, i, i], i, i] + \frac{68}{15} \sum_{i, j, k \text{ distinct}} [[[C_i, C_j], C_j, k, k], k, k], i, i], i, i],\]

\[
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(j_1,1)}, \ldots, \log B_{\sigma(j_1,16)}) \sim \frac{34}{15} \sum_{i \neq j} [[[C_i, C_j], C_j, i, i], i, i] + \frac{238}{45} \sum_{i \neq j} [[[C_i, C_j], C_j, i, i], i, i] - \frac{68}{5} \sum_{i, j, k \text{ distinct}} [[[C_i, C_j], C_j, k, k], k, k], i, i], i, i],
\]

and

\[
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(j_2,1)}, \ldots, \log B_{\sigma(j_2,16)}) \sim -\frac{68}{15} \sum_{i \neq j} [[[C_i, C_j], C_j, i, i], i, i] + \frac{34}{45} \sum_{i \neq j} [[[C_i, C_j], C_j, i, i], i, i] - \frac{34}{5} \sum_{i, j, k \text{ distinct}} [[[C_i, C_j], C_j, k, k], k, k], i, i], i, i].
\]

This yields

\[
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(8)}) + \sum_{s=1}^{2} \alpha_s \sum_{\sigma \in S_8} H_7(\log B_{\sigma(j_s,1)}, \ldots, \log B_{\sigma(j_s,16)}) \sim 0,
\]

where \(\alpha_1 = \frac{1}{15}, \alpha_2 = \frac{8}{15}\). The code used for computer assistance can be found at https://doi.org/10.6084/m9.figshare.20124113.v1.

\[\square\]

**Lemma 4.10.** Let \(G\) be a unipotent matrix group over \(\mathbb{Q}\), and \(\mathcal{H} \subseteq G\) be a finite set of matrices. There exist positive rational numbers \(\alpha_1, \ldots, \alpha_6\), as well as, for \(s = 1, \ldots, 6\), permutations \((j_{s,1}; j_{s,2}; \ldots, j_{s,20})\) of the tuple \((1, 1, 2, 2, \ldots, 10, 10)\), such that for any given set of matrices \(B_1, \ldots, B_{10}\) in \(G\) with \(\log B_i \in \mathcal{L}_{\geq 1}(\log \mathcal{H})\) and \(\sum_{i=1}^{10} \log B_i \in \mathcal{L}_{\geq 2}(\log \mathcal{H})\), we have

\[
\sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(10)}) + \sum_{s=1}^{6} \alpha_s \sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(j_{s,1}), \ldots, \log B_{\sigma(j_{s,1})}}, \log B_{\sigma(j_{s,20})})
\]

\[
\in \mathcal{L}_{\geq 10}(\log \mathcal{H}) + \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{H})). \tag{31}
\]

Namely, we can take \(\alpha_1 = \frac{41506633}{13702601}, \alpha_2 = \frac{557040}{13702601}, \alpha_3 = \frac{205175}{3915040}, \alpha_4 = \frac{1307207}{13702601}, \alpha_5 = \frac{80275225}{27405322}, \alpha_6 = \frac{1957623}{4105194}\), and

\[
(j_{1,1}; j_{1,2}; \ldots, j_{1,20}) = (5, 4, 7, 10, 2, 8, 3, 8, 1, 9, 7, 6, 5, 6, 2, 3, 9, 10, 1, 4),
\]

\[
(j_{2,1}; j_{2,2}; \ldots, j_{2,20}) = (8, 3, 5, 7, 10, 6, 8, 2, 1, 10, 2, 4, 9, 1, 5, 9, 3, 6, 7, 4),
\]

\[
(j_{3,1}; j_{3,2}; \ldots, j_{3,20}) = (7, 10, 2, 6, 4, 9, 6, 4, 1, 5, 3, 5, 1, 9, 3, 7, 10, 2, 8, 8),
\]

\[
(j_{4,1}; j_{4,2}; \ldots, j_{4,20}) = (10, 2, 2, 6, 7, 1, 9, 3, 9, 4, 8, 7, 8, 5, 5, 1, 4, 10, 6, 3),
\]

\[
(j_{5,1}; j_{5,2}; \ldots, j_{5,20}) = (3, 5, 10, 1, 4, 8, 6, 9, 3, 2, 7, 6, 1, 10, 9, 7, 2, 4, 5, 8),
\]

\[
(j_{6,1}; j_{6,2}; \ldots, j_{6,20}) = (4, 7, 2, 10, 2, 1, 3, 5, 8, 1, 6, 9, 10, 7, 6, 8, 3, 5, 9, 4).
\]
Sketch of proof of Lemma 4.10. Similar to Lemma 4.7, and Lemma 4.9, denote \( C_i = \log B_i, i = 1, \ldots, m \). Since \( \sum_{i=1}^{10} C_i \in \mathfrak{S}_{\geq 2}(\log \mathcal{H}) \), for any tuple \( j = (j_1, \ldots, j_9) \in \{1, \ldots, 10\}^9 \), the expression \( \sum_{\sigma \in S_{10}} \varphi_9(C_{\sigma(j_1)}, \ldots, C_{\sigma(j_9)}) \) is equivalent to a linear combination (with rational coefficient) of

\[
\begin{align*}
&\sum_{i \neq j} [[[[[C_i, C_j], C_i], C_i], C_i], C_i], \\
&\sum_{i \neq j} [[[[[C_i, C_j], C_i], C_i], C_i], C_i], \\
&\sum_{i \neq j} [[[[[C_i, C_j], C_i], C_i], C_i], C_i], \\
&\sum_{i,j,k \text{ distinct}} [[[C_i, C_j], C_j], C_i], C_i], \\
&\sum_{i,j,k \text{ distinct}} [[[C_i, C_j], C_j], C_i], C_i], \\
&\sum_{i,j,k \text{ distinct}} [[[C_i, C_j], C_j], C_j], C_i], C_i], \\
\end{align*}
\]

and

\[
\sum_{i,j,k \text{ distinct}} [[[C_i, C_j], C_j], C_j], C_i].
\]

Similar to the previous lemmas, the rest of the prove can be done by computer assistance. The code can be found at [https://doi.org/10.6084/m9.figshare.2012979.v1](https://doi.org/10.6084/m9.figshare.2012979.v1).

Using these lemmas, we can finally prove Proposition 4.2.

**Proposition 4.2.** Let \( \mathcal{G} = \{A_1, \ldots, A_K\} \) be a finite set of matrices in \( \text{UT}(n, \mathbb{Q}) \) that satisfies \( [\log \mathcal{G}]_{11} = \{0\} \). Given a non-zero vector \( \ell = (\ell_1, \ldots, \ell_K) \in \mathbb{Z}_{\geq 0}^K \) satisfying \( \ell_i \geq 10 \) for all \( i \in \supp(\ell) \). Consider the quotient vector space \( u(n)/\mathfrak{S}_{\geq 2}(\Sigma_{\geq 2}(\log \mathcal{G}_{\supp(\ell)})) \). For any cone (or linear space) \( \mathcal{C} \subseteq u(n) \), denote by \( \overline{\mathcal{C}} \) the cone (or linear space) of \( u(n)/\mathfrak{S}_{\geq 2}(\Sigma_{\geq 2}(\log \mathcal{G}_{\supp(\ell)})) \) consisting of the equivalence classes of elements in \( \mathcal{C} \). Then we have the following equalities:

(i) The cone \( \overline{\mathcal{R}}_{10}(\log \mathcal{G}_{\supp(\ell)}) \) is equal to the linear space \( \Sigma_{\geq 10}(\log \mathcal{G}_{\supp(\ell)}) \).

(ii) If \( \sum_{i=1}^{K} \ell_i \log A_i \in \Sigma_{\geq 2}(\log \mathcal{G}_{\supp(\ell)}) \), then for \( k = 9, 8, \ldots, 2 \), the cone \( \overline{\mathcal{R}}_k(\ell) \) is equal to the linear space \( \Sigma_{\geq k}(\log \mathcal{G}_{\supp(\ell)}) \).

**Proof of Proposition 4.2.** (i) By Lemma 4.5, we have

\[-H_{10}(\log B_1, \ldots, \log B_m) = H_{10}(\log B_m, \ldots, \log B_1).\]

Therefore, the cone

\[\mathcal{R}_{10}(\ell) = \left\{ H_{10}(\log B_1, \ldots, \log B_m) \left| m \geq 1, B_i \in \mathcal{G}^*, \sum_{i=1}^{m} \prod_{i} B_i \in \{\ell, 2\ell\} \right. \right\} \]

is a linear space, because the inverse of any its element is still in the cone. We now prove that \( \mathcal{R}_{10}(\ell) \) is equal to \( \mathfrak{S}_{\geq 10}(\log \mathcal{G}_{\supp(\ell)}) \).

On one hand, for any \( i_1, i_2, \ldots, i_{10} \in \supp(\ell) \), take a tuple of words \( (B_{i_1}', \ldots, B_{i_{10}}') \) with \( B_{i_1}' = A_{i_1}, B_{i_2}' = A_{i_2}, \ldots, B_{i_{10}}' = A_{i_{10}}, B_{i_{11}}' \in \mathcal{G}^* \), such that \( \sum_{i=1}^{11} \prod_{i} B_i = \ell \). Such a tuple can always be
found because \( \ell \) satisfies \( \ell_i \geq 10, i \in \text{supp}(\ell) \). We have, by Proposition 4.1,
\[
[\ldots[[\log A_{i_1}, \log A_{i_2}], \log A_{i_3}], \ldots, \log A_{i_{10}}]
= [\ldots[[\log B'_{i_1}, \log B'_{i_2}], \log B'_{i_3}], \ldots, \log B'_{i_{10}}]
= \sum_{\sigma \in S_{10}} \mu(\sigma) H_k \left( \log B'_{\sigma(1)}, \log B'_{\sigma(2)}, \ldots, \log B'_{\sigma(10)}, \log B'_{11} \right)
\in \left\langle H_k(\log B_1, \ldots, \log B_{11}) \mid B_i \in G^*, \sum_{i=1}^{11} \text{Pl}_G(B_i) = \ell \right\rangle_Q
\subseteq R_{10}(\ell) \quad \text{(because } R_{10}(\ell) \text{ is a linear space).}
\]

Therefore, \([\log G_{\text{supp}(\ell)}]_{10} \subseteq R_{10}(\ell)\). And since \(L_{\geq 11}(\log G_{\text{supp}(\ell)}) = \{0\}\), we have
\[
L_{\geq 10}(\log G_{\text{supp}(\ell)}) = \langle [\log G_{\text{supp}(\ell)}]_{10} \rangle_Q \subseteq R_{10}(\ell). \tag{32}
\]

On the other hand, take any tuple \((B_1, \ldots, B_m) \in (G^*)^m, \sum_{i=1}^m \text{Pl}_G(B_i) = \ell \) or \(2\ell\). For \(i = 1, \ldots, m\), since \(B_i \in G^*\) and \(\sum_{i=1}^m \text{Pl}_G(B_i) = \ell \) or \(2\ell\), the word \(B_i\) can only use letters in \(G\) whose index is in \(\text{supp}(\ell)\). Therefore, by the Baker-Campbell-Hausdorff formula, \(\log B_i \in L_{\geq 1}(\log G_{\text{supp}(\ell)})\) for all \(i\). Hence, the expression \(H_k(\log B_1, \log B_2, \ldots, \log B_m)\) can be written as a linear combination of elements in \([L_{\geq 1}(\log G_{\text{supp}(\ell)})]_{10}\). That is,
\[
R_{10}(\ell) \subseteq \left\langle \left[ L_{\geq 1}(\log G_{\text{supp}(\ell)}) \right]_{10} \right\rangle_Q \subseteq L_{\geq 10}(\log G_{\text{supp}(\ell)}). \tag{33}
\]

Combining (32) and (33) and taking the equivalence class, we have the desired equality.

(ii) We show that the claim in (ii) is true for \(k = 10, 9, \ldots, 2\) using induction with reverse order on \(k\). For \(k = 10\), \(R_k(\ell) = L_{\geq k}(\log G_{\text{supp}(\ell)})\) has been proved in (i). Now for some \(9 \geq k \geq 2\), suppose \(\sum_{i=1}^k \ell_i \log A_i \in L_{\geq 2}(\log G_{\text{supp}(\ell)})\) as well as \(R_{k+1}(\ell) = L_{\geq k+1}(\log G_{\text{supp}(\ell)})\) by induction hypothesis. We will show that \(R_k(\ell) = L_{\geq k}(\log G_{\text{supp}(\ell)})\). Consider the two following cases:

1. If \(k\) is even. Similar to the proof for \(k = 10\), by Lemma 4.5, \(H_k(\log B_1, \ldots, \log B_m) = -H_k(\log B_m, \ldots, \log B_1)\), so the cone
\[
S_k(\ell) := \left\langle H_k(\log B_1, \ldots, \log B_m) \mid m \geq 1, B_i \in G^*, \sum_{i=1}^m \text{Pl}_G(B_i) \in \{\ell, 2\ell\} \right\rangle_Q \geq 0
\]
is a linear space. We now prove that \(R_k(\ell) = S_k(\ell) + R_{k+1}(\ell)\) equal to \(L_{\geq k}(\log G_{\text{supp}(\ell)})\).

On one hand, for any \(i_1, i_2, \ldots, i_k \in \text{supp}(\ell)\), take a tuple of words \((B'_{i_1}, \ldots, B'_{i_k+1})\) with \(B'_{i_1} = A_{i_1}, B'_{i_2} = A_{i_2}, \ldots, B'_{i_k} = A_{i_k}, B'_{i_k+1} \in G^*\), such that \(\sum_{i=1}^{k+1} \text{Pl}_G(B'_i) = \ell\). Such a tuple can always be found because \(\ell\) satisfies \(\ell_i \geq 10 \geq k, i \in \text{supp}(\ell)\). Similar to the proof of (i), by Proposition 4.1,
\[
[\ldots[[\log A_{i_1}, \log A_{i_2}], \log A_{i_3}], \ldots, \log A_{i_k}]
= \sum_{\sigma \in S_k} \mu(\sigma) H_k \left( \log B'_{\sigma(1)}, \log B'_{\sigma(2)}, \ldots, \log B'_{\sigma(k+1)} \right)
\in S_k(\ell) \quad \text{(because } S_k(\ell) \text{ is a linear space)}
\subseteq R_k(\ell)
\]

2. If \(k\) is odd. Similar to the proof for even \(k\), but use a different cone construction. 

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Therefore, \(|\log G_{\text{supp}(\ell)}|_k \subseteq R_k(\ell)|. And since \(L_{2 k+1}(\log G_{\text{supp}(\ell)}) \subseteq R_{2 k+1}(\ell) \subseteq R_k(\ell)| by the induction hypothesis, we have
\[
\Pi_{2 k}(\log G_{\text{supp}(\ell)}) = \langle |\log G_{\text{supp}(\ell)}|_k \rangle_{Q} + \Pi_{2 k+1}(\log G_{\text{supp}(\ell)}) \subseteq R_k(\ell). \tag{34}
\]
On the other hand, take any tuple \((B_1, \ldots, B_m) \in (G^*)^m, \sum_{i=1}^{m} \Pi_G(B_i) = \ell\) or \(2\ell\). Same as in the proof for (i), the expression \(H_k(\log B_1, \log B_2, \ldots, \log B_m)\) can be written as a linear combination of elements in \([\Pi_{\geq 1}(\log G_{\text{supp}(\ell)})]_k\). That is,
\[
R_k(\ell) \subseteq \left\langle \left[\Pi_{\geq 1}(\log G_{\text{supp}(\ell)})\right]_k \right\rangle_Q \subseteq \Pi_{\geq k}(\log G_{\text{supp}(\ell)}). \tag{35}
\]
Combining (34) and (35) we have the desired equality.

2. If \(k\) is odd. First, we show that under the condition \(\sum_{i=1}^{K} \ell_i \log A_i \in \Pi_{\geq 2}(\log G_{\text{supp}(\ell)})\), for any \(i_1, i_2, \ldots, i_k \in \text{supp}(\ell),\) we have
\[
[\ldots [\log A_{i_1}, \log A_{i_2}], \log A_{i_3}], \ldots, \log A_{i_k}] \in R_k(\ell) + \Pi_{\geq 2}(\log G_{\text{supp}(\ell)})\).
\]
Take a tuple of words \((B'_1, \ldots, B'_{k+1})\) with \(B'_1 = A_{i_1}, B'_2 = A_{i_2}, \ldots, B'_{k} = A_{i_k}, B'_{k+1} \in G^*\), such that \(\sum_{i=1}^{k+1} \Pi_G(B'_i) = \ell\). Such a tuple can always be found because \(\ell\) satisfies \(\ell_i \geq 10 \geq k, \ i \in \text{supp}(\ell)\). For this tuple, the BCH formula gives us
\[
\sum_{i=1}^{k+1} \log B'_i \in \sum_{i=1}^{K} \ell_i \log A_i + \Pi_{\geq 2}(\log G_{\text{supp}(\ell)}) \subseteq \Pi_{\geq 2}(\log G_{\text{supp}(\ell)}). \]
Hence, for any \(\sigma \in S_k\), according to whether \(k = 3, 5, 7\) or \(9\), Lemma 4.6, 4.7, 4.9 or 4.10 (with \(\mathcal{H} = \log G_{\text{supp}(\ell)}\)) shows that
\[
-H_k(\log B'_1, \log B'_2, \ldots, \log B'_k, \log B'_{k+1})
\]
\[
\in \left\langle H_k(\log B_1, \ldots, \log B_{k+1}) \bigg| B_i \in G^*, \sum_{i=1}^{k+1} \Pi_G(B_i) = \ell \right\rangle_{Q_{\geq 0}}
\]
\[
+ \left\langle H_k(\log B_1, \ldots, \log B_{2 k+2}) \bigg| B_i \in G^*, \sum_{i=1}^{2 k+2} \Pi_G(B_i) = 2\ell \right\rangle_{Q_{\geq 0}}
\]
\[
+ \Pi_{\geq k+1}(\log G_{\text{supp}(\ell)}) + \Pi_{\geq 2}(\log G_{\text{supp}(\ell)})
\]
\[
\subseteq R_k(\ell) + \Pi_{\geq k+1}(\log G_{\text{supp}(\ell)}) + \Pi_{\geq 2}(\log G_{\text{supp}(\ell)})
\]
\[
= R_k(\ell) + \Pi_{\geq 2}(\log G_{\text{supp}(\ell)}). \tag{36}
\]
The last equality come from \(\Pi_{\geq k+1}(\log G_{\text{supp}(\ell)}) = \Pi_{k+1}(\ell) \subseteq R_k(\ell)\) by the induction hypothesis.
Hence, by Proposition 4.1,
\[
[\ldots [\log A_{i_1}, \log A_{i_2}], \log A_{i_3}], \ldots, \log A_{i_k}] = [\ldots [\log B'_1, \log B'_2], \log B'_3], \ldots, \log B'_k]
\]
\[
= \sum_{\sigma \in S_k} \mu(\sigma) H_k(\log B'_1, \log B'_2, \ldots, \log B'_k, \log B'_{i_k+1})
\]
\[
\in R_k(\ell) + \Pi_{\geq 2}(\log G_{\text{supp}(\ell)}). \quad \text{(by Equation (36) and } \mu(\sigma) \in \mathbb{Q})} \]
Therefore, \(|\log \mathcal{G}_{\supp(\ell)}|k| \subseteq \mathcal{R}_k(\ell) + \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{G}_{\supp(\ell)}))\), that is, \(\log \mathcal{G}_{\supp(\ell)}|k| \subseteq \mathcal{R}_k(\ell)\).

And since \(\mathcal{L}_{\geq k+1}(\log \mathcal{G}_{\supp(\ell)}) \subseteq \mathcal{R}_k(\ell) \subseteq \mathcal{R}_k(\ell)\) by the induction hypothesis, we have

\[
\mathcal{L}_{\geq k}(\log \mathcal{G}_{\supp(\ell)}) = \langle \log \mathcal{G}_{\supp(\ell)}|k| \rangle_{\mathcal{Q}} + \mathcal{L}_{\geq k+1}(\log \mathcal{G}_{\supp(\ell)}) \subseteq \mathcal{R}_k(\ell).
\] (37)

Next, take any tuple \((B_1, \ldots, B_m) \in (\mathcal{G}^*)^m\), \(\sum_{i=1}^m \text{PI}_G(B_i) = \ell \) or \(2\ell\). Same as in the proof for (i), the expression \(H_k(\log B_1, \log B_2, \ldots, \log B_m)\) can be written as a linear combination of elements in \(\mathcal{L}_{\geq 1}(\log \mathcal{G}_{\supp(\ell)})\). That is,

\[
\mathcal{R}_k(\ell) \subseteq \left\langle \left[\mathcal{L}_{\geq 1}(\log \mathcal{G}_{\supp(\ell)})\right]_k \right\rangle_{\mathcal{Q}} + \mathcal{R}_{k+1}(\ell)
\subseteq \mathcal{L}_{\geq k}(\log \mathcal{G}_{\supp(\ell)}) + \mathcal{R}_{k+1}(\ell) = \mathcal{L}_{\geq k+1}(\log \mathcal{G}_{\supp(\ell)}).
\] (38)

Combining (37) and (38) we have the desired equality. This concludes the induction and thus the whole proof.

\[\square\]

4.4 Proof of Proposition 4.3. In this subsection, we give a proof of Proposition 4.3.

Proposition 4.3. Let \(V\) be a finite dimensional \(\mathbb{Q}\)-linear space. Let \(d\) be a positive integer, \(I\) be a finite index set, and \(a_{i_1}, \ldots, a_{i_d}, i \in I\) be vectors in \(V\).

For any \(t \in \mathbb{Z}_{\geq 0}\), define the vectors

\[P_i(t) := ta_{i_1} + \cdots + td_{a_{i_d}}, \quad \text{for } i \in I.\]

Suppose the following two conditions hold:

(i) The \(\mathbb{Q}_{\geq 0}\)-cone \(C_d := \langle a_{i_d} \mid i \in I \rangle_{\mathbb{Q}_{\geq 0}}\) is a linear space.
(ii) For \(k = d - 1, d - 2, \ldots, 1\), the inductively defined \(\mathbb{Q}_{\geq 0}\)-cone \(C_k := \langle a_{i_k} \mid i \in I \rangle_{\mathbb{Q}_{\geq 0}} + C_{k+1}\) are linear spaces.

Then the \(\mathbb{Q}_{\geq 0}\)-cone \(\langle P_i(t) \mid i \in I, t \in \mathbb{Z}_{\geq 0} \rangle_{\mathbb{Q}_{\geq 0}}\) is equal to \(C_1\).

Proof. For convenience, define \(C_{d+1} := \{0\}\). We will prove that, for all \(k = 2, \ldots, d + 1\), the cone \(\langle P_i(t) \mid i \in I, t \in \mathbb{Z}_{\geq 0} \rangle_{\mathbb{Q}_{\geq 0}} + C_k\) is equal to \(C_1\). Notice that the claim in the proposition is the case where \(k = d + 1\). We use induction on \(k\).

For \(k = 2\), since \(a_{i_k} \in C_2\) for \(k \geq 2\), we have \(P_i(t) + C_2 = ta_{i_1} + C_2\), so

\[\langle P_i(t) \mid i \in I, t \in \mathbb{Z}_{\geq 0} \rangle_{\mathbb{Q}_{\geq 0}} + C_2 = \langle ta_{i_1} \mid i \in I, t \in \mathbb{Z}_{\geq 0} \rangle_{\mathbb{Q}_{\geq 0}} + C_2 = C_1.\] (39)

For the induction step, suppose now that the cone \(\langle P_i(t) \mid i \in I, t \in \mathbb{Z}_{\geq 0} \rangle_{\mathbb{Q}_{\geq 0}} + C_k\) is equal to \(C_1\), we want to prove that \(\langle P_i(t) \mid i \in I, t \in \mathbb{Z}_{\geq 0} \rangle_{\mathbb{Q}_{\geq 0}} + C_{k+1}\) is equal to \(C_1\).

By the induction hypothesis, there exist indices \(i_1, \ldots, i_m \in I\), and positive integers \(t_1, \ldots, t_m \in \mathbb{Z}_{\geq 0}\), such that

\[\langle P_{i_j}(t_j) \mid j = 1, \ldots, m \rangle_{Q_{\geq 0}} + C_k = C_1.\]

Condition (ii) of the proposition shows that there exist indices \(i'_{1}, \ldots, i'_{m'} \in I\) such that

\[\langle a_{k_{j'}} \mid j = 1, \ldots, m' \rangle_{Q_{\geq 0}} + C_{k+1} = C_k.\]

Hence

\[\langle P_{i_j}(t_j) \mid j = 1, \ldots, m \rangle_{Q_{\geq 0}} + \langle a_{k_{j'}} \mid j = 1, \ldots, m' \rangle_{Q_{\geq 0}} + C_{k+1} = C_1.\] (39)
We show that there exists \( t \in \mathbb{Z}_{>0} \) such that
\[
\langle P_j(t_j) \mid j = 1, \ldots, m \rangle_{Q_{\geq 0}} + \langle P_{j'}(t) \mid j = 1, \ldots, m' \rangle_{Q_{\geq 0}} + C_{k+1} = C_1.
\]

Suppose the contrary, that for every \( t \in \mathbb{Z}_{>0} \),
\[
\langle P_j(t_j) \mid j = 1, \ldots, m \rangle_{Q_{\geq 0}} + \langle P_{j'}(t) \mid j = 1, \ldots, m' \rangle_{Q_{\geq 0}} + C_{k+1} \subsetneq C_1.
\]

For any \( \mathbb{Q}_{\geq 0} \)-cone \( C \), define the normal cone of \( C \) as the set of vectors \( v \in V \) such that \( v \cdot c^\top \leq 0 \) for all \( c \in C \). For every \( t \), take a normalized vector \( v_t \in C_1 \) (meaning the norm of \( v_t \) is 1) in the normal cone of \( \langle P_j(t_j) \mid j = 1, \ldots, m \rangle_{Q_{\geq 0}} + \langle P_{j'}(t) \mid j = 1, \ldots, m' \rangle_{Q_{\geq 0}} + C_{k+1} \). That is,
\[
P_j(t_j) \cdot v_t^\top \leq 0 \text{ for all } j, \quad P_{j'}(t) \cdot v_t^\top \leq 0 \text{ for all } j, \quad v_t \bot C_{k+1}.
\]

Such a vector must exist because \( \langle P_j(t_j) \mid j = 1, \ldots, m \rangle_{Q_{\geq 0}} + \langle P_{j'}(t) \mid j = 1, \ldots, m' \rangle_{Q_{\geq 0}} + C_{k+1} \) is a strict sub-cone of the linear space \( C_1 \). The \( \mathbb{R} \)-linear space \( V_R = V \otimes \mathbb{Q} \mathbb{R} \) is finite dimensional and hence compact. Embed \( V \) into \( V_R \) canonically, then the sequence \( \{v_t\}_{t \in \mathbb{Z}_{>0}} \) has a limit point in \( V_R \). Denote by \( v_{lim} \) this limit point. As all the vectors \( v_t \) are in \( C_1 \), \( v_{lim} \) must be in \( C_1 \otimes \mathbb{Q} \mathbb{R} \). Since the inner product of \( V \) canonically extends to the inner product of \( V_R \), taking the limit of \( (40) \), we have
\[
P_j(t_j) \cdot v_{lim}^\top \leq 0 \text{ for all } j, \quad v_{lim} \bot C_{k+1}, \quad (41)
\]

and
\[
a_{k,j'} \cdot v_{lim}^\top = \lim_{t \to \infty} \left( \frac{P_{j'}(t)}{t^k} - t a_{k+1,j'} - \cdots - t^{d-j} a_{d,j'} \right) \cdot v_{lim}^\top
\]
\[
= \lim_{t \to \infty} \frac{P_{j'}(t)}{t^k} \cdot v_{lim}^\top \leq 0, \quad j = 1, \ldots, m'. \quad (42)
\]

The second equality is due to \( a_{k+1,j'}, \ldots, a_{d,j'} \in C_{k+1} \bot v_{lim} \). Hence, \( (41) \) and \( (42) \) show that \( v_{lim} \) is in the normal cone of the \( \mathbb{R}_{\geq 0} \)-cone generated by \( \langle P_j(t_j) \mid j = 1, \ldots, m \rangle_{Q_{\geq 0}} + \langle a_{k,j'} \mid j = 1, \ldots, m' \rangle_{Q_{\geq 0}} + C_{k+1} \) \( \equiv \) \( C_1 \). Since \( v_{lim} \) is non-zero (it has norm 1) and is in \( C_1 \otimes \mathbb{Q} \mathbb{R} \), this yields a contradiction. We have thus shown that there exists \( t \in \mathbb{Z}_{>0} \) such that
\[
\langle P_j(t_j) \mid j = 1, \ldots, m \rangle_{Q_{\geq 0}} + \langle P_{j'}(t) \mid j = 1, \ldots, m' \rangle_{Q_{\geq 0}} + C_{k+1} = C_1.
\]

Since \( P_i(t) \in C_1, i \in I, t \in \mathbb{Z}_{>0} \), this means
\[
\langle P_i(t) \mid i \in I, t \in \mathbb{Z}_{>0} \rangle_{Q_{\geq 0}} + C_{k+1} = C_1,
\]

concluding the induction.

Finally, take \( k = d + 1 \). This yields \( \langle P_i(t) \mid i \in I, t \in \mathbb{Z}_{>0} \rangle_{Q_{\geq 0}} = C_1 \).

\[\square\]

4.5 Full proof of Theorem 3.1. In this subsection, with Propositions 4.1 - 4.3 at our disposal, we will show the proof of Theorem 3.1. We need the following lemma.

Lemma 4.11. Let \( H \) be a finite subset of the Lie algebra \( \mathfrak{u}(n) \). Let \( W, V \) be linear subspaces of \( \mathcal{L}_{\geq 1}(H) \) such that \( W + \mathcal{L}_{\geq 2}(V) = V \), then \( \mathcal{L}_{\geq 2}(W) = \mathcal{L}_{\geq 2}(V) \).

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Theorem 3.1. Since \( W + \mathcal{L}_{\geq 2}(V) = V \), we have \( W \subseteq V \), and thus \( \mathcal{L}_{\geq 2}(W) \subseteq \mathcal{L}_{\geq 2}(V) \). Therefore, it suffices to prove the opposite inclusion \( \mathcal{L}_{\geq 2}(W) \supseteq \mathcal{L}_{\geq 2}(V) \).

We use induction on \( k \) to show that

\[
[V]_k \subseteq [W]_k + \mathcal{L}_{\geq k+1}(V). \tag{43}
\]

For \( k = 1 \) this immediately results from the equation \( W + \mathcal{L}_{\geq 2}(V) = V \). Suppose Equation (43) hold for \( k - 1 \). Then, take any elements \( x \in V, y \in [V]_{k-1} \), by the induction hypothesis and by \( W + \mathcal{L}_{\geq 2}(V) = V \), there exist \( x' \in W, y' \in [W]_{k-1} \), such that \( x - x' \in \mathcal{L}_{\geq 2}(V), y - y' \in \mathcal{L}_{\geq k}(V) \). Then,

\[
[y, x] = [y', x'] + [y - y', x] + [y - y', x - x']
\subseteq [W]_k + [\mathcal{L}_{\geq k}(V), V] + [\mathcal{L}_{\geq k}(V), \mathcal{L}_{\geq 2}(V)] \subseteq [W]_k + \mathcal{L}_{\geq k+1}(V).
\]

Taking the linear span for all \( x \in V, y \in [V]_{k-1} \) shows \( [V]_k \subseteq [W]_k + \mathcal{L}_{\geq k+1}(V) \), concluding the induction.

Now, for any \( l = 2, \ldots, d \), take the sum of Equation (43) for \( k = l, \ldots, d \), we have

\[
\mathcal{L}_{\geq l}(V) = \sum_{k \geq l} [V]_k \subseteq \sum_{k \geq l} [W]_k + \sum_{k \geq l} \mathcal{L}_{\geq k+1}(V) = \mathcal{L}_{\geq l}(W) + \mathcal{L}_{\geq l+1}(V).
\]

Therefore,

\[
\mathcal{L}_{\geq 2}(V) \\
\subseteq \mathcal{L}_{\geq 2}(W) + \mathcal{L}_{\geq 3}(V) \\
\subseteq \mathcal{L}_{\geq 2}(W) + \mathcal{L}_{\geq 3}(W) + \mathcal{L}_{\geq 4}(V) \\
\vdots \\
\subseteq \mathcal{L}_{\geq 2}(W) + \mathcal{L}_{\geq 3}(W) + \cdots + \mathcal{L}_{\geq d}(W) \\
= \mathcal{L}_{\geq 2}(W).
\]

This shows the inclusion \( \mathcal{L}_{\geq 2}(W) \supseteq \mathcal{L}_{\geq 2}(V) \). \( \Box \)

We now prove Theorem 3.1. Although part (i) has already been proven when the theorem is first stated, we will restate it for the sake of completeness.

**Theorem 3.1.** Let \( \mathcal{G} = \{A_1, \ldots, A_K\} \) be a finite set of matrices in \( \text{UT}(n, \mathbb{Q}) \) that satisfies \( \log \mathcal{G}_{11} = \{0\} \). Given a non-zero vector \( \ell = (\ell_1, \ldots, \ell_K) \in \mathbb{Z}_{\geq 0}^K \).

(i) If there exists a word \( w \in \mathcal{G}^+ \) with \( \text{PI}_\mathcal{G}(w) = \ell \) and \( \log \pi(w) = 0 \), then

\[
\sum_{i=1}^{K} \ell_i \log A_i \in \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\supp(\ell)}). \tag{3}
\]

(ii) If \( \ell \) satisfies (3), then there exists a non-empty word \( w \in \mathcal{G}^+ \), with \( \text{PI}_\mathcal{G}(w) \in \mathbb{Z}_{\geq 0} \cdot \ell \), such that \( \log \pi(w) = 0 \).

**Proof.** (i) Let \( w \) be a word with \( \text{PI}_\mathcal{G}(w) = \ell \). Write \( w = B_1 B_2 \cdots B_m \) \( B_i \in \mathcal{G}, i = 1, \ldots, m \). Regrouping by letters, we have \( \sum_{i=1}^{K} \ell_i \log A_i = \sum_{i=1}^{m} \log B_i \).
If \( \log \pi(w) = 0 \), then by the Baker-Campbell-Hausdorff formula, we have

\[
\sum_{i=1}^{m} \log B_i + \sum_{k=2}^{n-1} H_k (\log B_1, \ldots, \log B_m) = \log(B_1 B_2 \cdots B_m) = 0.
\]

The higher order terms \( H_k, k > n \) vanish because \( [\log \mathcal{G}]_n = \{0\} \) (a consequence of \( \mathcal{G} \subseteq \mathcal{U}(n, \mathbb{Q}) \)). Therefore, \( \sum_{i=1}^{K} \ell_i \log A_i = -\sum_{k=2}^{n-1} H_k (\log B_1, \ldots, \log B_m) \).

Since the Parikh Image of the word \( B_1 \cdots B_m \) is \( \ell \), the matrices \( B_i \) all lie in the subset \( \{ A_i \mid i \in \text{supp}(\ell) \} \) of \( \mathcal{G} \). Therefore, \( \log B_i \in \log \mathcal{G}_{\text{supp}(\ell)} \) for all \( i \). By Theorem 2.8, for all \( k \geq 2 \) we have

\[
-H_k (\log B_1, \ldots, \log B_m) \in \langle \{ [\log B_i \mid i = 1, \ldots, m] \} \rangle \subseteq \mathcal{L}_{2k}(\log \mathcal{G}_{\text{supp}(\ell)}) \subseteq \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}).
\]

Therefore, we have \( \sum_{i=1}^{K} \ell_i \log A_i = -\sum_{k=2}^{n-1} H_k (\log B_1, \ldots, \log B_m) \in \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) \).

(ii) Suppose condition (3) hold for the vector \( \ell \). Our proof for (ii) proceeds in four steps. As the first step, we want to fabricate some matrices \( A_1', \ldots, A_K' \in \langle \mathcal{G} \rangle \), such that

\[
\langle \log A_i' \mid i = 1, \ldots, K' \rangle_{Q_{20}} + \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) = \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) + \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}).
\] (44)

The candidates for the matrices \( A_1', \ldots, A_K' \) are of the form \( B_1' \cdots B_m' \), where \( m \geq 1 \), \( t \in \mathbb{Z}_{>0} \), \( B_i \in \mathcal{G}^* \), \( i = 1, \ldots, m \) and \( \sum_{i=1}^{m} \Pi_{\mathcal{G}}(B_i) = \ell \) or \( 2\ell \). The general strategy is to invoke Proposition 4.3 while using Proposition 4.2 to guarantee that the conditions (i) and (ii) of Proposition 4.3 are satisfied.

As the second step, we work in the new alphabet \( \mathcal{G}' = \{ A_1', \ldots, A_K' \} \) of matrices found in the previous step. We want to fabricate some matrices \( A_1'', \ldots, A_K'' \in \langle \mathcal{G}' \rangle \), such that

\[
\langle \log A_i'' \mid i = 1, \ldots, K'' \rangle_{Q_{20}} + \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) = \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) + \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}).
\] (45)

The candidates for the matrices \( A_1'', \ldots, A_K'' \) are of the form \( B_1'' \cdots B_m'' \), where \( B_i \in (\mathcal{G}')^* \), \( i = 1, \ldots, m \). The idea is to again invoke Proposition 4.3 and to use Proposition 4.2 for the new alphabet \( \mathcal{G}' \) and a suitable vector \( \ell'' \).

As the third step, we work in the new alphabet \( \mathcal{G}'' = \{ A_1'', \ldots, A_K'' \} \) of matrices found in the previous step. We want to fabricate some matrices \( A_1''', \ldots, A_K''' \in \langle \mathcal{G}'' \rangle \), such that

\[
\langle \log A_i''' \mid i = 1, \ldots, K'''' \rangle_{Q_{20}} = \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}).
\] (46)

(Note that \( \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) = \{0\} \). The candidates for the matrices \( A_1'', \ldots, A_K'''' \) are of the form \( B_1'''' \cdots B_m'''' \), where \( B_i \in (\mathcal{G}'')^* \), \( i = 1, \ldots, m \). The idea is to again invoke Proposition 4.3 and to use Proposition 4.2 for the new alphabet \( \mathcal{G}'''' \) and a suitable vector \( \ell''' \).

As the fourth and last step, we work in the new alphabet \( \mathcal{G}''' = \{ A_1''', \ldots, A_K''' \} \) of matrices found in the previous step. We then observe that the matrices \( A_1'', \ldots, A_K''' \) commute with each other, because \( \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)})) = \{0\} \). Hence, it is very easy to search for the desired word \( w \in (\mathcal{G}''')^* \) with \( \log \pi(w) = 0 \).

We now give the detailed account of each step.

Step 1: **Find matrices** \( A_1', \ldots, A_K', \in \langle \mathcal{G} \rangle \) **satisfying condition** (44). Since the right hand side of Equation (3) is a linear space, we can replace \( \ell \) by \( 10\ell \), and thus suppose \( \ell \) satisfy \( \ell_i \geq 10, i \in \text{supp}(\ell) \). Since \( \mathcal{Z}_{>0} \cdot 10\ell \subseteq \mathcal{Z}_{>0} \cdot \ell \), the resulting word \( w \) will still satisfy \( \Pi_{\mathcal{G}}(w) \in \mathcal{Z}_{>0} \cdot \ell \). Since \( \ell \) satisfies \( \sum_{i=1}^{K} \ell_i \log A_i \in \mathcal{L}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) \), we are able to use Proposition 4.2 for the vector \( \ell \).
Our aim is to apply Proposition 4.3 in the quotient space \( V = u(n)/\Sigma_{\geq 2}(\mathcal{G}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) \)), for the index set

\[
\mathcal{I} = \left\{ (B_1, \ldots, B_m) \mid m \geq 1, B_i \in G^*, \sum_{i=1}^{m} \text{PI}_G(B_i) \in \{\ell, 2\ell\} \right\}
\]

that is, the set of tuples of words whose concatenation has Parikh Image \( \ell \) or \( 2\ell \). For any element \( x \in u(n) \), denote \( x := x + L \geq 2 \quad (L \geq 2 (\log G_{\text{supp}(\ell)})) \) its equivalence class in

\[
V := u(n)/\Sigma_{\geq 2}(\mathcal{G}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)})).
\]

For any tuple \( v = (B_1, \ldots, B_m) \in \mathcal{I} \), consider the vectors in \( V \):

\[
a_{1v} := \sum_{i=1}^{m} \log B_i,
\]

\[
a_{kv} := H_k(\log B_1, \ldots, \log B_m), \quad k = 2, \ldots, 10,
\]

and

\[
P_v(t) := \log(B_1^t \cdots B_m^t) = t a_{1v} + \sum_{k=2}^{10} t^k a_{kv},
\]

due to the Baker-Campbell-Hausdorff formula for \( B_1^t, \ldots, B_m^t \) (Equation (7)). We now apply Proposition 4.3 to these vectors: we need to verify that the cones \( C_k, k = 10, \ldots, 1 \) as defined in Proposition 4.3 are indeed linear spaces. Proposition 4.2 shows that

\[
C_{10} = \langle a_{10v} \mid v \in \mathcal{I} \rangle_{Q \geq 0} = \mathcal{R}_{10}(\ell)
\]

is a linear space, and

\[
C_k = \langle a_{kv} \mid v \in \mathcal{I} \rangle_{Q \geq 0} + C_{k+1} = \mathcal{R}_k(\ell), \quad k = 9, \ldots, 2,
\]

are linear subspaces of \( u(n)/\Sigma_{\geq 2}(\mathcal{G}_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}) \)). Furthermore, by the condition

\[
\sum_{i=1}^{K} \ell_i \log A_i \in \Sigma_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)})
\]

we have

\[
a_{1v} \in \left\{ \sum_{i=1}^{K} \ell_i \log A_i, 2 \cdot \sum_{i=1}^{K} \ell_i \log A_i \right\} \subseteq \Sigma_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)})
\]

for all \( v \in \mathcal{I} \). Hence,

\[
C_1 = \langle a_{1v} \mid v \in \mathcal{I} \rangle_{Q \geq 0} + C_2 = \Sigma_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)})
\]

is also a linear space. The conditions (i) and (ii) in Proposition 4.3 are thus satisfied. We can thus apply Proposition 4.3, which yields

\[
\langle P_v(t) \mid v \in \mathcal{I}, t \in \mathbb{Z}_{\geq 0} \rangle_{Q \geq 0} = C_1 = \Sigma_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)})
\]

In other words,

\[
\left\langle \log(B_1^t \cdots B_m^t) \right\rangle_{Q \geq 0} = \Sigma_{\geq 2}(\log \mathcal{G}_{\text{supp}(\ell)}).
\]
Step 2: Find matrices $A'_1, \ldots, A'_{K'} \in \langle G' \rangle$ satisfying condition (45). Since the right hand side of Equation (44) is a linear space, we have $-\log A'_j \in \langle \log A'_i | i = 1, \ldots, K' \rangle_{Q\geq 0} + \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon'))$ for $j = 1, \ldots, K'$. Hence, there exists a non-zero vector $\ell' = (\ell'_1, \ldots, \ell'_{K'})$ in $\mathbb{Z}_{\geq 0}^{K'}$, satisfying $\supp(\ell') = \{1, \ldots, K'\}$, $\ell_i \geq 10$ for all $i$, and

$$\sum_{i=1}^{K'} \ell'_i \log A'_i \in \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')).$$  

(47)

Define $\log G'_\supp(\epsilon') := \{ \log A'_i | i \in \supp(\ell') \} = \log G'$, because $\supp(\ell') = \{1, \ldots, K'\}$. First, we claim that

$$\mathcal{L}_{Q\geq 2}(\mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon''))) = \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')).$$

(48)

Indeed, Equation (44) shows that

$$\langle \log G'_\supp(\epsilon') \rangle_Q + \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon'))
= \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')) + \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')) = \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')).$$

Applying Lemma 4.11 with $W = \langle \log G'_\supp(\epsilon') \rangle_Q, V = \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon'))$ to the above equation yields the equality (48).

Consequently, we have $\sum_{i=1}^{K'} \ell'_i \log A'_i \in \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon'))$ by (47). Apply Proposition 4.2 for the alphabet $G'$ and the vector $\ell'$, then we have that, in the quotient space $u(n)/\mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon'))$, the equalities $\mathcal{R}_k(\ell') = \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')), k = 10, \ldots, 2$, hold.

Then, applying Proposition 4.3 in the quotient linear space $V = u(n)/\mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon'))$ as in the previous step, we have

$$\left\langle \log(B_1^t \cdots B_m^t) | t \in \mathbb{Z}_{\geq 0}, m \geq 1, B_i \in \langle G' \rangle^*, \sum_{i=1}^m \Pi_{G'}(B_i) \in \{ \ell', 2\ell' \} \right\rangle_{Q\geq 0}
= \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')).$$

Hence, there exist $K'' > 0$ tuples of words $(B'_1, \ldots, B'_{l_1}), \ldots, (B'_{K''1}, \ldots, B'_{K''m})$ in $\langle G' \rangle^*$ with $\sum_{i=1}^m \Pi_{G'}(B_{ji}) = \ell'$ or $2\ell'$ for all $j$, as well as positive integers $t'_1, \ldots, t'_{K''} \in \mathbb{Z}_{\geq 0}$, such that

$$\langle \log(B'_{i1}^t \cdots B'_{im}^t) | i = 1, \ldots, K'' \rangle_{Q\geq 0} + \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon'))
= \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')) + \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')) = \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')).$$

(49)

Substituting with $\mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon')) = \mathcal{L}_{Q\geq 2}(\log G'_\supp(\epsilon'))$, Equation (49) can be rewritten as
Step 3: Find matrices 

\[ A''_1, \ldots, A''_{K''}, i = 1, \ldots, K'' \] 

satisfying condition (46). Similar to the previous step, one can find a vector \( \ell'' = (\ell''_1, \ldots, \ell''_{K''}) \in \mathbb{Z}_{\geq 0}^{K''} \), satisfying \( \supp(\ell'') = \{1, \ldots, K''\} \), \( \ell_j \geq 10, i = 1, \ldots, K'' \), and

\[
\sum_{i=1}^{K''} \ell''_i \log A''_i \in \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})).
\]

Define \( \mathfrak{G}'_{\supp(\ell'')} := \{ \log A''_i | i \in \supp(\ell'') \} = \log \mathfrak{G}''. \) As in the previous step, we have

\[
\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')}) = \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})).
\]

Combining it with \( \mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')}) = \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})) \), we have

\[
\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')}) = \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})).
\]

Apply Proposition 4.2 for the alphabet \( \mathfrak{G}'' \) and the vector \( \ell'' \), then we have that, in the quotient space \( u(n)/\mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})) \), the equalities \( R_k(\ell'') = \mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')}), k = 10, \ldots, 2 \), hold.

Then, applying Proposition 4.3 in the quotient linear space \( V = u(n)/\mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})) \) as in the previous steps, we have

\[
\left\langle \log(B_1 \cdots B_m) \mid t \in \mathbb{Z}_{\geq 0}, m \geq 1, B_t \in (\mathfrak{G}'')^*, \sum_{i=1}^{m} \prod_{i}^{(\mathfrak{G}'')} (B_i) \in \{\ell'', 2\ell''\} \right\rangle_{\mathbb{Q}_{\geq 0}} = \mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')}).
\]

Hence, there exist \( K'' > 0 \) tuples of words \( (B''_{11}, \ldots, B''_{1m}), \ldots, (B''_{K''1}, \ldots, B''_{K''m}) \) in \( (\mathfrak{G}'')^* \) with \( \sum_{i=1}^{m} \prod_{i}^{(\mathfrak{G}'')} (B_{ij}) = \ell'' \) or \( 2\ell'' \) for all \( j \), as well as positive integers \( t''_1, \ldots, t''_{K''} \in \mathbb{Z}_{\geq 0} \), such that

\[
\langle \log(B''_{i1} \cdots B''_{im}) \mid i = 1, \ldots, K'' \rangle_{\mathbb{Q}_{\geq 0}} + \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})) = \mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')} + \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')}))).
\]

Since \( \mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')}), \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})), \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})) \), we have

\[
\mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})) \subseteq \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')})) = \{0\}.
\]

Thus, Equation (50) can be rewritten as

\[
\langle \log(B''_{i1} \cdots B''_{im}) \mid i = 1, \ldots, K'' \rangle_{\mathbb{Q}_{\geq 0}} = \mathfrak{L}_2(\mathfrak{L}_2(\mathfrak{G}'_{\supp(\ell'')}))
\]

Hence, the matrices \( A''_1 = B''_{i1} \cdots B''_{im}, i = 1, \ldots, K'' \) satisfy the Equation (46). Define the new alphabet \( \mathfrak{G}'' = \{A''_1, \ldots, A''_{K''}\} \).
Step 4: **Find a word** \( w \in (G'') \) with \( \log \pi(w) = 0 \). Since the right hand side of Equation (46) is a linear space, we have \(- \log A''_i \in (\log A''_i \mid i = 1, \ldots, K'')_{j=1}^{\infty} \) for \( j = 1, \ldots, K'' \). Hence, there exists a non-zero vector \( \ell'' = (\ell''_1, \ldots, \ell''_{K''}) \in \mathbb{Z}_{\geq 0}^{K''} \), satisfying
\[
\sum_{i=1}^{K''} \ell''_i \log A''_i = 0.
\]
Since \( \log G'' \in \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log \mathcal{G}_{\supp(\ell)})) \subseteq \mathcal{L}_{\geq 8}(\log \mathcal{G}_{\supp(\ell)}) \), we have that \( \mathcal{L}_{\geq 2}(\log G'') \subseteq \mathcal{L}_{\geq 16}(\log \mathcal{G}_{\supp(\ell)}) = \{0\} \). Hence, by the Baker-Campbell-Hausdorff formula,
\[
\log(A''_{11}^{m''} \cdots A''_{K''}^{m''}) = \sum_{i=1}^{K''} \ell''_i \log A''_i = 0,
\]
because the terms \( H_k, k \geq 2 \) are in \( \mathcal{L}_{\geq 2}(\log G'') \), which vanishes. Therefore, we have found the word \( w = A''_{11}^{m''} \cdots A''_{K''}^{m''} \in (G'') \) satisfying \( \log \pi(w) = 0 \). By replacing \( A''_i \) with their corresponding words \( B''_{i1}^{m''}, \ldots, B''_{i,im}^{m''} \) in \( (G'') \), then replacing \( A''_i \) with corresponding words in \( (G) \), \( (G') \), then replacing \( A''_i \) with corresponding words in \( (G') \), we see that \( w \) considered as a word in \( G \) has Parikh Image in \( \mathbb{Z}_{\geq 0} \cdot \ell \), because the words \( B''_{i1}^{m''}, \ldots, B''_{i,im}^{m''} \) corresponding to \( A''_i \) all have Parikh Image in \( \mathbb{Z}_{\geq 0} \cdot \ell \).

\[\square\]

5 Conjecture for higher nilpotency class

In the previous sections, we have shown that the invertible subset of any finite set \( G \subseteq G \) is computable in polynomial time, where \( G \) is a unipotent matrix group of nilpotency class at most ten. The only obstacle for generalizing this result to higher nilpotency class is to find similar identities like those in Lemma 4.7-4.10 for the terms \( H_k, k \geq 11 \). If these identities exist, then they can be found and proved with the same computer aided procedure as the one used in Lemma 4.7-4.10 (given in Appendix B). This section aims to discuss this idea.

Given \( k \geq 11 \), we propose the following conjecture, which is a generalization of Lemma 4.7-4.10:

**Conjecture 5.1.** Let \( G \) be a unipotent matrix group over \( \mathbb{Q} \), and \( H \subseteq G \) be a finite set of matrices. There exist a non-negative integer \( r \), positive rational numbers \( \alpha_1, \ldots, \alpha_r \), as well as, for \( s = 1, \ldots, r \), words \( j_s = j_{s,1}j_{s,2} \cdots j_{s,m_s} \) in the alphabet \( I = \{1, 2, \ldots, k+1\} \), such that \( \Pi_I(j_s) \in \mathbb{Z}_{\geq 0} \cdot (1, 1, \ldots, 1) \), and
\[
\sum_{\sigma \in S_{k+1}} H_k(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(k+1)}) + \sum_{s=1}^r \alpha_s \sum_{\sigma \in S_{k+1}} H_k(\log B_{\sigma(j_{s,1})}, \ldots, \log B_{\sigma(j_{s,m_s})})
\in \mathcal{L}_{\geq k+1}(\log H) + \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(\log H)) \tag{51}
\]
for all matrices \( B_1, \ldots, B_{k+1} \) in \( G \) satisfying \( \log B_i \in \mathcal{L}_{\geq 1}(\log H) \), and \( \sum_{i=1}^{k+1} \log B_i \in \mathcal{L}_{\geq 2}(\log H) \).

For even \( k \), taking \( r = 0 \), Conjecture 5.1 is correct by the antisymmetry of \( H_k \) (see Lemma 4.5): in fact, we can deduce
Lemma 4.7, 4.9, 4.10, where the words \( j \) and \( k \) for \( j \) satisfy Conjecture 5.1. Namely, starting with \( q \) (can write each expression \( \hat{\sigma} \) as a linear combination of expressions \( -\sum_{\alpha} \sigma \) positive rational numbers and the set \( \overline{\mathcal{P}}(G) \) whether it is in the quotient space \( u \).

For any \( k \), using Algorithm 2 in Appendix B, we can search for words \( j \) that potentially verify Conjecture 5.1. Namely, starting with \( q = 2 \), take all the words \( j \) satisfying \( \overline{\mathcal{P}}(j) = (l, l, \ldots, l) \), \( 2 \leq l \leq q \). Under the equivalence relation \( \sim \) (defined in the proof of Lemma 4.7), we can write each expression

\[
h_k(j) := -\sum_{\sigma \in S_{k+1}} H_k(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(k+1)})
\]

as a linear combination of expressions \( \bar{M}(P, q) \) (see Appendix B) using Algorithm 2. Then, writing

\[-\sum_{\sigma \in S_{k+1}} H_k(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(k+1)}) \] also as a linear combination of \( \bar{M}(P, q) \), we can verify whether it is in the \( \mathbb{Q}_{\geq 0} \)-cone generated by the elements \( h_k(j) \). If this is the case, then there exist positive rational numbers \( \alpha_s \) satisfying Equation (51). If this is not the case, we can increase \( q \) and repeat the above procedure.

If there exists a relation of the form (51), then the above procedure terminates for some \( q \) and returns this relation. Otherwise it does not terminate. In practice, it is more computationally viable to not take all the words \( j \) satisfying \( \overline{\mathcal{P}}(j) = (l, l, \ldots, l) \), but only a small amount of them chosen randomly.

Due to the restraint on computational power, we have only verified Conjecture 5.1 for all \( k \leq 10 \), this is the reason why the main result of this paper stops at nilpotency class ten. However, if we can verify Conjecture 5.1 for larger \( k \) (it suffices to verify for odd \( k \)), then we can extend the result of this paper to higher nilpotency class. This is formalized by the following theorem.

**Theorem 5.2.** Let \( G \) be a unipotent matrix group over \( \mathbb{Q} \) whose nilpotency class is at most \( d \). If Conjecture 5.1 holds for all \( k \leq d \), then Algorithm 1 correctly computes the invertible subset of any finite set \( \mathcal{G} \subseteq G \) in polynomial time.

**Proof.** For any \( \ell \in \mathbb{Z}_{\geq 0}^K \), similar to Equation (5), define recursively the cones

\[
\mathcal{R}_0(\ell) := \left\{ H_d(\log B_1, \ldots, \log B_m) \left| m \geq 1, B_i \in \mathcal{G}^*, \sum_{i=1}^m \overline{\mathcal{P}}(B_i) \in \mathbb{Z}_{>0} \cdot \ell \right. \right\}_{\mathbb{Q}_{>0}}
\]

\[
\mathcal{R}_k(\ell) := \mathcal{R}_{k+1}(\ell) + \left\{ H_k(\log B_1, \ldots, \log B_m) \left| m \geq 1, B_i \in \mathcal{G}^*, \sum_{i=1}^m \overline{\mathcal{P}}(B_i) \in \mathbb{Z}_{>0} \cdot \ell \right. \right\}_{\mathbb{Q}_{>0}}
\]

\[
k = d - 1, d - 2, \ldots, 3, 2,
\]

and the set

\[
\log \mathcal{G}_{\text{supp}(\ell)} := \{ \log A_i \mid A_i \in \mathcal{G}, i \in \text{supp}(\ell) \}.
\]

Suppose \( \ell \) satisfies \( \ell_i \geq d, i \in \text{supp}(\ell) \). Following the pattern in the proof of Proposition 4.2, we can show that the following holds in the quotient space \( u(n) / \mathcal{L}_{\geq 2}(\mathcal{L}_{\geq 2}(J(\ell))) \):
(i) The cone \( \mathcal{R}_d(\ell) \) is equal to the linear space \( L_{\geq d}(J(\ell)) \).

(ii) If \( \sum_{i=1}^{K} \ell_i \log A_i \in L_{\geq 2}(J(\ell)) \), then for \( k = d - 1, d - 2, \ldots, 2 \), the cone \( \mathcal{R}_k(\ell) \) is equal to the linear space \( L_{\geq k}(J(\ell)) \).

Here, the cone \( \mathcal{C} \) means the equivalence class \( \mathcal{C} + L_{\geq 2}(L_{\geq 2}(J(\ell))) \). These two claims can be shown following exactly the proof of Proposition 4.2, replacing the usage of Lemma 4.6-4.10 by Conjecture 5.1 for \( k = 3, 5, \ldots, d - 1 \) (or \( d \)).

Then, using the same arguments as Theorem 3.1, we can show the following generalization of Theorem 3.1:

(i) If there exists a word \( w \in G^+ \) with \( \text{PI}_G(w) = \ell \) and \( \log w = 0 \), then

\[
\sum_{i=1}^{K} \ell_i \log A_i \in L_{\geq 2}(J(\ell)).
\]

(ii) If \( \ell \) satisfies (52), then there exists a non-empty word \( w \in G^+ \), with \( \text{PI}_G(w) \in \mathbb{Z}_{>0} \cdot \ell \), such that \( \log w = 0 \).

From here, the proof of correctness of Algorithm 1 and its complexity analysis is identical to the proof of Theorem 1.6, replacing the property \([\log G]_{11} = \{0\}\) by \([\log G]_{d+1} = \{0\}\).

6 Conclusion and outlook

In this paper we have proven that the invertible subset is computable in PTIME for any finite set in a unipotent matrix group of nilpotency class at most ten. A natural question is whether this result can be extended to arbitrary dimensions. This can either be done by proving Conjecture 5.1 for higher \( k \) or by finding another way to attack this problem.

Another natural follow-up problem is whether the Lie algebra approach introduced in this paper can be used to deal with more general matrix groups. Naturally, the closest candidate to applying this method would be the group \( T(n, \mathbb{Q}) \) of upper triangular rational matrices (with not necessarily ones on the diagonal), and the group \( T(n, \mathbb{K}) \) of upper triangular matrices with entries from a number field \( \mathbb{K} \). Although the BCH formula in this case would have infinite terms, it would still be plausible to give detailed analysis, taking into account its rate of convergence. However, it is not clear whether our method can still apply for non-solvable groups, since the logarithm map may be well-defined only on a small neighbourhood of the identity matrix.

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[21] M. I. Kargapolov and J. I. Merzljakov. *Fundamentals of the Theory of Groups*, volume 62. Springer, 1979.
A Semigroup and group theory results

In this section of the appendix we give the proofs of several semigroup and group theory results omitted in the main paper.

**Proposition 1.5.** Given a finite set of matrices $G = \{A_1, \ldots, A_K\}$ in a matrix group $G$. Denote by $G_{inv}$ the invertible subset of $G$.

(i) The Identity Problem for $G$ has a positive answer if and only if $G_{inv}$ is non-empty.
(ii) If \( G_{inv} \) is non-empty, then it generates the group of units of \( \langle G \rangle \) as a semigroup. 
(iii) The Group Problem for \( G \) has a positive answer if and only if \( G_{inv} = G \).

Proof. (i) If the Identity Problem has a positive answer, let \( w \in G^+ \) be a non-empty word such that \( \pi(w) = I \). Write \( w = A_iw' \), \((w' \) could be the empty word), then \( A_i^{-1} = \pi(w') \). If \( A_i = I \) then obviously \( A_i^{-1} = A_i \in \langle G \rangle \). If \( A_i \neq I \) then \( \pi(w') \neq I \) so \( w' \) is not the empty word and \( \pi(w') \in \langle G \rangle \). Therefore \( A_i^{-1} \in \langle G \rangle \). Conversely, if \( A_i \in G_{inv} \), then either \( A_i = I \) in which case \( I = A_i \in \langle G \rangle \), or \( A_i^{-1} = \pi(w') \) for some non-empty word \( w' \), so \( I = \pi(A_iw') \in \langle G \rangle \).

(ii) Since every element in \( G_{inv} \) is invertible in \( \langle G \rangle \), the semigroup \( \langle G_{inv} \rangle \) it generates also only contains invertible elements. Hence, it suffices to show that no element in \( \langle G \rangle \setminus \langle G_{inv} \rangle \) is invertible. Suppose on the contrary that there exists a word \( w \in G^* \) such that \( \pi(w) \in \langle G \rangle \setminus \langle G_{inv} \rangle \) and \( \pi(w)^{-1} \in \langle G \rangle \). Since \( \pi(w) \notin \langle G_{inv} \rangle \), \( w \) must contain a letter \( A \in G \setminus G_{inv} \). But because \( \pi(w)^{-1} \in \langle G \rangle \), there exists a word \( v \in G^* \) such that \( \pi(v) = \pi(w)^{-1} \). Thus, \( \pi(wv) = I \). But \( wv \) is a word containing the letter \( A \). Writing \( wv = w_1Aw_2 \), we have \( A^{-1} = \pi(w_2)\pi(w_1) \in \langle G \rangle \), a contradiction to \( A \notin G_{inv} \).

(iii) Since a semigroup is a group if and only if it is its own group of unit, (iii) is a direct consequence of (ii). \( \square \)

Corollary 1.8. Let \( G \) be a finitely generated nilpotent group of class at most ten, given by a finite presentation or a consistent polycyclic presentation (see [19, Chapter 8]). Then the Identity Problem and the Group Problem are decidable within \( G \).

Proof. A consistent polycyclic presentation of \( G \) can be computed from a finite presentation of \( G \) [33], so we can suppose that a consistent polycyclic presentation of \( G \) is given. Let \( G \) be a finitely generated nilpotent group of class at most ten. The set of torsion elements in \( G \) forms a normal subgroup \( T \) of \( G \). A set of generators of \( T \) along with a presentation can be effectively computed by [30, Theorem 8]. Then, by [19, Lemma 8.38], a consistent polycyclic presentation for the torsion-free nilpotent group \( G/T \) can be computed. Note that \( G/T \) is still nilpotent and its nilpotency class does not exceed that of \( G \). An embedding of \( G/T \) as a subgroup of \( UT(n, \mathbb{Q}) \) for some \( n \) can then be effectively computed ([28], [14]). Using this embedding, the Identity Problem and the Group Problem can be decided in the quotient group \( G/T \) by Theorem 1.6 and Proposition 1.5(i), (iii).

By [4, Theorem 2.1], \( G \) can be embedded (injectively) as a subgroup of a direct product \( A \times G/T \), where \( A \) is a finite group. Let \( \phi: G \rightarrow A \times G/T \) denote this embedding, and let \( p: A \times G/T \rightarrow G/T \) be the natural projection.

By the injectivity of \( \phi \), the Identity Problem has a positive answer for \( G \subseteq G \) if and only if it has a positive answer for \( \phi(G) \subseteq A \times G/T \). We claim that the Identity Problem has a positive answer for \( \phi(G) \subseteq A \times G/T \) if and only if it has a positive answer for \( p(\phi(G)) \subseteq G/T \). Indeed, for any group \( H \), denote \( e_H \) its natural element. If \( e_{A \times G/T} \in \langle \phi(G) \rangle \), then obviously \( e_{G/T} \in \langle p(\phi(G)) \rangle \) because \( p \) is a semigroup homomorphism. If \( e_{G/T} \in \langle p(\phi(G)) \rangle \), then there exists \( a \in A \) such that \( \langle a, e_{G/T} \rangle \in \langle \phi(G) \rangle \). Because \( A \) is finite, there exists a positive integer \( k \), such that \( a^k = e_A \) for all \( a \in A \). Then \( e_{A \times G/T} = (e_A, e_{G/T}) = (a^k, e_{G/T}^{k}) \in \langle \phi(G) \rangle \). This proves the claim. Therefore, the Identity Problem for \( G \subseteq G \) is equivalent to the Identity Problem for \( p(\phi(G)) \subseteq G/T \), which is decidable by the first part of the proof.

The injectivity of \( \phi \) also shows that the Group Problem has a positive answer for \( G \subseteq G \) if and only if it has a positive answer for \( \phi(G) \subseteq A \times G/T \). We claim that the Group Problem has a positive answer for \( \phi(G) \subseteq A \times G/T \) if and only if it has a positive answer for \( p(\phi(G)) \subseteq G/T \). Indeed, suppose \( \langle \phi(G) \rangle \) is a group, then there exists a non-empty word \( w \in \phi(G)^+ \) where every letter of \( \phi(G) \) appears at least once, and whose product is equal to the neutral element \( e_{A \times G/T} \).
This is because every letter $B \in \phi(G)$ has an inverse in $\langle \phi(G) \rangle$, hence multiplying $B$ with the word representing its inverse yields a word $w_B$ whose product is the neutral element, and where the letter $B$ appears. Concatenating the words $w_B$ for all $B \in \phi(G)$ yields the word $w$. Next, projecting each letter in $w$ with $p$ yields a non-empty word $p(w) \in p(\phi(G))^+$ where every letter of $p(\phi(G))$ appears at least once, and whose product is equal to the neutral element $e_{G/T}$. Thus every element in $p(\phi(G))$ is invertible in $(p(\phi(G)))$. This shows that if $\langle \phi(G) \rangle$ is a group then $\langle p(\phi(G)) \rangle$ is a group. For the opposite implication, suppose $\langle p(\phi(G)) \rangle$ is a group, then there exists a non-empty word $w \in \phi(G)^+$ where every letter of $\phi(G)$ appears at least once, and whose product is equal to some element $(a,e_{G/T}) \in A \times e_{G/T}$. Because $A$ is finite, there exists a positive integer $k$, such that $a^k = e_A$. Hence the product of the word $w^k \in \phi(G)^+$ is equal to $(a^k,e_{G/T}^k) = e_{A \times G/T}$. As every letter of $\phi(G)$ appears in $w^k$ at least once, every element in $\phi(G)$ is invertible in $\langle \phi(G) \rangle$. Thus $\langle \phi(G) \rangle$ is a group. Therefore, the Group Problem for $\mathcal{G} \subseteq G$ is equivalent to the Group Problem for $p(\phi(G)) \subseteq G/T$, which is decidable by the first part of the proof. \hfill \Box

Lemma 2.7. Let $G$ be a unipotent matrix group over $\mathbb{Q}$. If the nilpotency class of $G$ is $d$, then $[\log G]_{d+1} = \{0\}$.

Proof. For an element $g \in G$ and a rational number $q \in \mathbb{Q}$, define $g^q := \exp(q \log g)$. A group $G \leq UT(n, \mathbb{Q})$ is called $\mathbb{Q}$-powered if for every element $g \in G$ and $q \in \mathbb{Q}$, we have $g^q \in G$. A unipotent matrix group over $\mathbb{Q}$ is torsion-free, because $A^n = I \iff \log A = 0 \iff A = I$. Therefore, by [22, Theorem 9.20(a)], $G$ can be embedded in a $\mathbb{Q}$-powered group $\tilde{G}$ of the same nilpotency class $d$.\footnote{One can take the group $\tilde{G}$ to be Mal’cev completion of $G$.} By [22, Theorem 10.3(d)], $\log \tilde{G}$ is a Lie algebra over $\mathbb{Q}$, and $\log \tilde{G}$ is of nilpotency class $d$ (meaning $[\log \tilde{G}]_{d+1} = \{0\}$). Therefore, $[\log G]_{d+1} \subseteq [\log \tilde{G}]_{d+1} = \{0\}$. \hfill \Box

\section*{B \ Computer-aided proof of Lemma 4.7-4.10}

In this section of the appendix we give the detailed account for the proof of Lemma 4.7-4.10 using computer assistance with the software SageMath.

Fix an integer $k$. Let $\mathcal{H}$ be a subset of the unipotent matrix group $G$. For $x,y \in u(n)$, denote $x \sim_2 [\log_2(\mathcal{H})] y$ if $x - y \in \mathcal{L}_{\geq 2}([\log_2(\mathcal{H})])$. Denote $x \sim_2 [\log_2(\mathcal{H})] y$ if $x - y \in \mathcal{L}_{\geq k+1}([\log_2(\mathcal{H})])$. Obviously, $\mathcal{L}_{\geq 2}([\log_2(\mathcal{H})])$ and $\mathcal{L}_{\geq k+1}([\log_2(\mathcal{H})])$ are equivalence relations and denote by $\sim$ the transitive closure of these two relations.

The following lemma shows the effect of the relation $\mathcal{L}_{\geq 2}([\log_2(\mathcal{H})])$. In fact, the quotient Lie algebra $L := \mathcal{L}_{\geq 1}([\log(\mathcal{H})])/\mathcal{L}_{\geq 2}([\log_2(\mathcal{H})])$ is metabelian, meaning $[[L,L],[L,L]] = 0$. This property allows us to permute elements in iterated Lie brackets:

Lemma B.1. For $C_1, \ldots, C_k \in \mathcal{L}_{\geq 1}([\log_2(\mathcal{H})])$ and $i = 3, \ldots, k - 1$, we have

$$[\ldots [[\ldots [C_1,C_2],\ldots,C_i],C_{i+1}],\ldots,C_k] \sim_2 [\log_2(\mathcal{H})] [\ldots [[\ldots [C_1,C_2],\ldots,C_i],C_{i+1}],\ldots,C_k].$$

Proof. For $i = 3, \ldots, k - 1$, by the Jacobi Identity,

$$[\ldots [[\ldots [C_1,C_2],\ldots,C_i],C_{i+1}],\ldots,C_k] - [\ldots [[\ldots [C_1,C_2],\ldots,C_{i+1}],C_i],\ldots,C_k]$$

$$\in [\ldots \mathcal{L}_{\geq 2}([\log_2(\mathcal{H})]),\ldots,C_k].$$
\[ \subseteq \ldots [L_{\geq 2}(L_{\geq 2}(\log H)), C_{i+2}], \ldots, C_k]. \tag{53} \]

We then show that
\[ X \in L_{\geq 2}(L_{\geq 2}(\log H)), Y \in L_{\geq 1}(\log H) \implies [X,Y] \in L_{\geq 2}(L_{\geq 2}(\log H)). \tag{54} \]

Since \( X \) is in \( L_{\geq 2}(L_{\geq 2}(\log H)) \), it can be written as a linear combination of elements of the form \([\ldots [X_1, X_2], \ldots, X_s] \) where \( s \geq 2, X_i \in L_{\geq 2}(\log H) \). Therefore it suffices to show the implication (54) for the case \( X = [\ldots [X_1, X_2], \ldots, X_s] \) where \( X_i \in L_{\geq 2}(\log H) \). In this case, \( X' := [\ldots [X_1, X_2], \ldots, X_{s-1}] \in L_{\geq 2(s-1)}(\log H) \subseteq L_{\geq 2}(\log H) \), so \( X = [X', X_s] \) with \( X', X_s \in L_{\geq 2}(\log H) \).

Then by the Jacobi Identity,
\[ [X,Y] = [[X', X_s], Y] = -[[X_s, Y], X'] - [[Y, X'], X_s], \]

where
\[ [[X_s, Y], X'] \in [[L_{\geq 2}(\log H), L_{\geq 1}(\log H)], L_{\geq 2}(\log H)] \subseteq [L_{\geq 2}(\log H), L_{\geq 2}(\log H)] \subseteq L_{\geq 2}(L_{\geq 2}(\log H)) \]

and
\[ [[Y, X'], X_s] \in [[L_{\geq 1}(\log H), L_{\geq 2}(\log H)], L_{\geq 2}(\log H)] \subseteq [L_{\geq 2}(\log H), L_{\geq 2}(\log H)] \subseteq L_{\geq 2}(L_{\geq 2}(\log H)). \]

Therefore \([X,Y] \in L_{\geq 2}(L_{\geq 2}(\log H))\), showing the implication (54).

Using this to \( Y = C_{i+2}, C_{i+3}, \ldots, C_k \) in Equation (53) shows
\[ \ldots [L_{\geq 2}(L_{\geq 2}(\log H)), C_{i+2}], \ldots, C_k] \subseteq \ldots [L_{\geq 2}(L_{\geq 2}(\log H)), C_{i+3}], \ldots, C_k] \]
\[ \quad \vdots \]
\[ \subseteq [L_{\geq 2}(L_{\geq 2}(\log H)), C_k] \subseteq L_{\geq 2}(L_{\geq 2}(\log H)) \]

Hence Equation (53) yields
\[ \ldots [[\ldots C_1, C_2], \ldots, C_i], C_{i+1}], \ldots, C_k \] \( \sim \) \[ \ldots [[\ldots C_1, C_2], \ldots, C_i+1], C_{i+2}, \ldots, C_k]. \]

Fix an integer \( k \). Define an integer partition \( P \) (of \( k \)) to be a series of numbers \( (a_1, \ldots, a_s) \) such that \( a_1 \geq a_2 \geq \cdots \geq a_s \geq 1 \) and \( k = a_1 + \cdots + a_s \). Define \( \max(P) = a_1, \min(P) = a_s \) and \( \text{set}(P) = \{ k \mid \exists a_i = k \} \). Define a set partition \( S \) (of \( \{1, \ldots, k\} \)) to be a set of non-empty disjoint sets \( S = \{A_1, \ldots, A_s\} \) such that \( A_1 \cup \cdots \cup A_s = \{1, \ldots, k\} \). For any \( k \)-tuple \( \mathbf{j} = (j_1, \ldots, j_k) \in \{1, \ldots, k+1\}^k \), define the associated set partition of \( \mathbf{j} \) the set partition consisting of sets of indices of its distinct elements
\[ \text{SP}(\mathbf{j}) := \left\{ A_i := \{l \mid j_l = i\} \mid i = 1, \ldots, k + 1, A_i \neq \emptyset \right\}. \]
For example, if \( k = 6, \ j = (4, 2, 7, 2, 2, 4) \), then \( \text{SP}(j) = \{\{1, 6\}, \{2, 4, 5\}, \{3\}\} \).

We now fix elements \( C_1, \ldots, C_k \in \mathcal{L}_{\geq 1}(\log \mathcal{H}) \). For a given tuple \( j = (j_1, \ldots, j_k) \in \{1, \ldots, k+1\}^k \), define the symmetric sums

\[
\Phi(j) := \sum_{\sigma \in S_{k+1}} \varphi_k(C_{\sigma(j_1)}, C_{\sigma(j_2)}, \ldots, C_{\sigma(j_k)}),
\]

\[
M(j) := \sum_{\sigma \in S_{k+1}} [\ldots [C_{\sigma(j_1)}, C_{\sigma(j_2)}], \ldots, C_{\sigma(j_k)}].
\]

By their symmetry, \( \Phi(j) \) and \( M(j) \) only depend on their associated set partition \( \text{SP}(j) \). Hence we can denote

\[
\Phi(S) := \Phi(j), \quad M(S) := M(j), \quad \text{where} \ \text{SP}(j) = S.
\]

Define the associated integer partition \( \text{IP}(S) \) of a set partition \( S \) to be the series of set cardinalities in \( S \) in decreasing order. For example, if \( k = 6, \ S = \{\{1, 6\}, \{2, 4, 5\}, \{3\}\} \), then \( \text{IP}(S) = (3, 2, 1) \). Define a partition-integer pair to be a pair \( (P, c) \), where \( P \) is an integer partition and \( c \) is a number in \( \text{set}(P) \). For a partition-integer pair \( (P, c) \), define the following symmetric sum.

\[
\hat{M}(P, c) := M(S),
\]

where \( S \) is a set partition such that \( \text{IP}(S) = P \), and \( 1 \in A \in S \) with \( \text{card}(A) = \max(P) \) and \( 2 \in A' \in S \) with \( \text{card}(A') = c \). Note that this definition depends on the choice of the set partition \( S \). However, we will show that under the equivalence relation \( \mathcal{L}_{\geq 2}^{\leq 2}(\log \mathcal{H}) \), different choices on \( S \) result in the same equivalence class. For example, a possible definition of \( \hat{M}((3, 2, 1), 1) \) can be

\[
\hat{M}((3, 2, 1), 1) := M(\{\{1, 3, 4\}, \{2\}, \{5, 6\}\}) = \sum_{\sigma \in S_7} [\ldots [C_{\sigma(2)}, C_{\sigma(7)}], C_{\sigma(2)}], C_{\sigma(4)}], C_{\sigma(4)}].
\]

We now justify the claim that under the equivalence relation \( \mathcal{L}_{\geq 2}^{\leq 2}(\log \mathcal{H}) \), different choices on \( S \) result in the same equivalence class. Let \( j \) be a tuple whose associated set partition is \( S \). By Lemma B.1, any exchange of order among the elements \( j_3, j_4, \ldots, j_k \) will not change the equivalence class of \( \ldots [C_{\sigma(j_1)}, C_{\sigma(j_2)}], \ldots, C_{\sigma(j_k)}] \), so it will not change the equivalence class of \( M(j) \). This means that the equivalent class of \( M(S) \) does not change when we permute the numbers \( 3, 4, \ldots, k \). For example, \( M(\{\{1, 3, 4\}, \{2\}, \{5, 6\}\}) \sim M(\{\{1, 3, 5\}, \{2\}, \{4, 6\}\}) \).

Hence, the equivalent class of \( M(S) \) only depends on the integer partition \( \text{IP}(j) \) as well as the cardinality of the sets where \( 1 \) and \( 2 \) belong. This is uniquely determined by the partition-cardinality pair \( (P, c) \).

The next lemma shows the effect of the relation \( \sim \) for symmetric sums.

**Lemma B.2.** Let \( \mathcal{H} \) be a subset of \( G \). Suppose \( C_1, \ldots, C_k+1 \in \mathcal{L}_{\geq 1}(\log \mathcal{H}) \) and \( \sum_{i=1}^{k+1} C_i \in \mathcal{L}_{\geq 2}(\log \mathcal{H}) \), then for any set partition \( S \), the symmetric sum \( M(S) \) is equivalent (under \( \sim \)) to a linear combination of \( \hat{M}(P, c) \), where \( (P, c) \) are partition-integer pairs satisfying \( c \neq \max(P) \) and \( \min(P) \geq 2 \).

In order words, there exist integers \( \alpha_{(P, c)} \), where \( (P, c) \) ranges over all partition-integer pairs with \( c \neq \max(P) \) and \( \min(P) \geq 2 \), such that

\[
M(S) \sim \sum_{(P, c)} \alpha_{(P, c)} \hat{M}(P, c).
\]
Proof. For two set partitions \( S_1 \) and \( S_2 \), \( S_2 \) is called a coarsening of \( S_1 \) if for every \( A \in S_1 \), there exists \( A' \in S_2 \) such that \( A \subseteq A' \). For example, \( \{\{1,3,4\},\{2,5,6\}\} \) is a coarsening of \( \{\{1,3,4\},\{2\},\{5,6\}\} \). In particular, any set partition is a coarsening of itself. Denote \( S_2 \succ S_1 \) if \( S_2 \) is a coarsening of \( S_1 \).

The first step of our proof is to show that \( M(S) \) is equivalent to a linear combination of expressions \( M(S') \) where \( \min(S') \geq 2 \). Using \( \sum_{i=1}^{k+1} C_i \in \mathfrak{L}_{\geq 2}(\log H) \), we now show that, if a set partition \( S \) satisfies \( \min(S) = 1 \), then

\[
\sum_{S' \succ S} M(S') \gtrapprox_{k+1}(\log H) 0. \tag{55}
\]

As an example, if \( k = 6 \), \( S = \{\{1,3,4\},\{2\},\{5,6\}\} \), then there are five coarsenings of \( S \), which are \( S, \{\{1,3,4\},\{2,5,6\}\}, \{\{1,3,4,2\},\{5,6\}\}, \{\{1,3,4,5,6\},\{2\}\} \) and \( \{\{1,3,4,2,5,6\}\} \). Correspondingly,

\[
M(S) + M(\{\{1,3,4\},\{2,5,6\}\}) + M(\{\{1,3,4,2\},\{5,6\}\}) + M(\{\{1,3,4,5,6\},\{2\}\}) \\
+ M(\{\{1,3,4,2,5,6\}\}) \\
= \sum_{\sigma \in \mathfrak{S}_7} [[[C_{\sigma(1)},C_{\sigma(2)},C_{\sigma(1)},C_{\sigma(3)},C_{\sigma(3)}]]] \\
+ \sum_{\sigma \in \mathfrak{S}_7} [[[C_{\sigma(1)},C_{\sigma(2)},C_{\sigma(1)},C_{\sigma(2)},C_{\sigma(2)}]]] \\
+ \sum_{\sigma \in \mathfrak{S}_7} [[[C_{\sigma(1)},C_{\sigma(1)},C_{\sigma(1)},C_{\sigma(2)},C_{\sigma(2)}]]] \\
+ \sum_{\sigma \in \mathfrak{S}_7} [[[C_{\sigma(1)},C_{\sigma(2)},C_{\sigma(1)},C_{\sigma(1)},C_{\sigma(1)}]]] \\
+ \sum_{\sigma \in \mathfrak{S}_7} [[[C_{\sigma(1)},C_{\sigma(1)},C_{\sigma(1)},C_{\sigma(1)},C_{\sigma(1)}]]] \\
= \sum_{i,j,k \text{ distinct}} [[[C_i,C_j,C_i,C_i,C_k,C_k]]] + \sum_{i \neq j = k} [[[C_i,C_j,C_i,C_i,C_k,C_k]]] \\
+ \sum_{i = j \neq k} [[[C_i,C_j,C_i,C_i,C_k,C_k]]] + \sum_{i = k \neq j} [[[C_i,C_j,C_i,C_i,C_k,C_k]]] \\
+ \sum_{i = j = k} [[[C_i,C_j,C_i,C_i,C_k,C_k]]] \\
= \sum_{i=1}^{7} \sum_{j=1}^{7} \sum_{k=1}^{7} [[[C_i,C_j,C_i,C_i,C_k,C_k]]] \\
= \sum_{i=1}^{7} \sum_{k=1}^{7} [[[C_i,\sum_{j=1}^{7} C_j,C_i,C_k,C_k]]] \\
\in \sum_{i=1}^{7} \sum_{k=1}^{7} [[[C_i,\mathfrak{L}_{\geq 2}(\log H),C_i,C_k,C_k,C_k]]] \\
\subseteq \mathfrak{L}_{\geq 7}(\log H).
\]

So \( \sum_{S' \succ S} M(S') \gtrapprox_{k+1}(\log H) 0 \) for this particular example.
For the general case, write \( S = \{A_1, \ldots, A_s\} \) with \( \text{card}(A_1) = 1 \), then

\[
\sum_{S' \supseteq S} M(S')
= \sum_{S' \supseteq S} \sum_{j \in \{1, \ldots, k+1\}^k} \left[ \ldots [C_{j_1}, C_{j_2}], \ldots, C_{j_k}] \right]
= \sum_{j_i = j'_i \text{ if } i, i' \text{ are in the same set of } S} \left[ \ldots [C_{j_1}, C_{j_2}], \ldots, C_{j_k}] \right]
\]

\[
= \sum_{i_1=1}^{k+1} \cdots \sum_{i_k=1}^{k+1} \left[ \ldots [C_{i_f(1)}, C_{i_f(2)}], \ldots, C_{i_f(k)}] \right] \quad \text{where } f(r) \text{ is defined by } r \in A_{f(r)}. 
\]

\[
= \sum_{i_2=1}^{k+1} \cdots \sum_{i_k=1}^{k+1} \left[ \ldots [C_{i_f(1)}, C_{i_f(2)}], \ldots, \sum_{i_1=1}^{k+1} C_{i_1}], \ldots, C_{i_f(k)}] \right]
\]

\[
\subseteq \mathcal{L}_{\geq k+1}(\log \mathcal{H}).
\]

Hence \( \sum_{S' \supseteq S} M(S') \sim 0 \). This shows that if \( \text{min}(S) = 1 \), then under the equivalence \( \mathcal{L}_{\geq k+1}(\log \mathcal{H}) \), we can replace \( M(S) \) by \( -\sum_{S' \supseteq S, S' \not\supseteq S} M(S') \). Repeat this procedure for \( M(j) = M(\text{SP}(j)) \) for sufficiently many times, we can thus rewrite \( M(\text{SP}(j)) \) as a linear combination of expressions \( M(S') \) where \( \text{min}(S') \geq 2 \).

The second step of our proof is to show that any expression \( M(S') \) where \( \text{min}(S') \geq 2 \) is equivalent to a linear combination of the expressions \( \hat{M}(P, c) \), where \( (P, c) \) are partition-integer pairs satisfying \( \text{min}(P) \geq 2 \). Write \( S' = \{A_1, \ldots, A_s\} \) with \( \text{card}(A_1) = \text{max}(S') \). By Lemma B.1, the equivalent class of \( M(S') \) does not change when we permute the numbers \( 3, 4, \ldots, k \). We can therefore suppose \( 3 \in A_1 \). Take any tuple \( j = (j_1, \ldots, j_k) \in \{1, \ldots, k+1\}^k \) with \( \text{SP}(j) = S' \). By the Jacobi Identity,

\[
[\ldots [C_{\sigma(j_1)}, C_{\sigma(j_2)}], C_{\sigma(j_3)}], \ldots, C_{\sigma(j_k)}] =
[\ldots [C_{\sigma(j_1)}, C_{\sigma(j_2)}], C_{\sigma(j_1)}], \ldots, C_{\sigma(j_k)}] - [\ldots [C_{\sigma(j_1)}, C_{\sigma(j_1)}], C_{\sigma(j_2)}], \ldots, C_{\sigma(j_k)}] 
\]

(56)

Summing up for \( \sigma \in S_{k+1} \), the expression \( \sum_{\sigma \in S_{k+1}} [\ldots [C_{\sigma(j_1)}, C_{\sigma(j_2)}], C_{\sigma(j_1)}], \ldots, C_{\sigma(j_k)}] \) is equivalent to \( \hat{M}(\text{IP}(S'), c) \), with \( c = \text{card}(A_j) \) where \( j_2 \in A_i \). Similarly, the expression

\[
\sum_{\sigma \in S_{k+1}} [\ldots [C_{\sigma(j_1)}, C_{\sigma(j_1)}], C_{\sigma(j_2)}], \ldots, C_{\sigma(j_k)}] 
\]

is equivalent to \( \hat{M}(\text{IP}(S'), c') \), with \( c' = \text{card}(A_{j_1}) \) where \( j_1 \in A_{j_1} \).

We claim that if \( c = \text{max}(S') \), then \( \hat{M}(\text{IP}(S'), c) \sim 0 \). This is because, writing \( \hat{M}(\text{IP}(S'), c) = [\ldots [C_{\sigma(j_1)}, C_{\sigma(j_2)}], C_{\sigma(j_1)}], \ldots, C_{\sigma(j_k)}] \), if \( j_2 \in A_i \) with \( \text{card}(A_i) = \text{max}(S') \), then swapping 2 and 3 in the set partition \( \text{SP}(j) \) does not change its associated integer partition. Therefore, we have

\[
\hat{M}(\text{IP}(S'), \text{max}(S')) \sim M(\text{SP}(j)) = [\ldots [C_{\sigma(j_1)}, C_{\sigma(j_2)}], C_{\sigma(j_1)}], \ldots, C_{\sigma(j_k)}] \sim
\]
- \[\ldots [C_{\sigma(j_2)}, C_{\sigma(j_3)}], C_{\sigma(j_1)}], \ldots, C_{\sigma(j_k)}] \sim -\widehat{M}(\text{IP}(S'), \max(S')),

so \(\widehat{M}(\text{IP}(S'), \max(S')) \sim 0\).

We can thus conclude by summing up Equation (56) for \(\sigma \in S_{k+1}\) that

\[M(S') = \sum_{\sigma \in S_{k+1}} \ldots [C_{\sigma(j_2)}, C_{\sigma(j_3)}], C_{\sigma(j_1)}], \ldots, C_{\sigma(j_k)}\]

is equivalent to a linear combination of \(\widehat{M}(\text{IP}(S'), c)\), where \(c \neq \max(S')\).

Combining the two steps of the proof, we have shown that \(M(S)\) is equivalent (under \(\sim\)) to a linear combination of \(\widehat{M}(P, c)\), where \((P, c)\) are partition-integer pairs satisfying \(c \neq \max(P)\) and \(\min(P) \geq 2\). Furthermore, this linear combination can be supposed to have integer coefficients. \(\square\)

For any \(k\), all such partition-integer pairs can be effectively listed. For example, when \(k = 5\), there is only one pair \(((3, 2), 2)\). When \(k = 7\), there are three pairs \(((5, 2), 2), ((4, 3), 3), ((3, 2), 2)\). When \(k = 9\), there are six pairs \(((7, 2), 2), ((6, 3), 3), ((5, 4), 4), ((5, 2, 2), 2), ((4, 3, 2), 3), ((4, 3, 2), 3)\).

Suppose \(C_1, \ldots, C_{k+1} \in \mathcal{L}_{\geq 1}(\log H)\) and \(\sum_{i=1}^{k+1} C_i \in \mathcal{L}_{\geq 2}(\log H)\). Lemma B.1 and B.2 give an effective procedure of writing \(M(j)\) as a linear combination of \(\widehat{M}(P, c)\), under the equivalence \(\sim\). Here, \((P, c)\) ranges over all partition-integer pairs with \(c \neq \max(P)\) and \(\min(P) \geq 2\). In fact, by the Dynkin formula (Lemma 4.4), for any tuple \(j' \in \{1, \ldots, k+1\}^k\), the expression \(\Phi(j') = \sum_{\sigma \in S_{k+1}} \varphi_k(C_{\sigma(j'_1)}, \ldots, C_{\sigma(j'_k)})\) is equivalent to a linear combination of \(M(j'')\), \(j'' \in \{1, \ldots, k+1\}^k\), and is thus equivalent to a linear combination \(\sum_{(P, c)} \beta_{(P, c)} \widehat{M}(P, c)\), where \(\beta_{(P, c)} \in \mathbb{Q}\), and any expression \(\sum_{\sigma \in S_{k+1}} H_k(C_{\sigma(j_1)}, \ldots, C_{\sigma(j_k)})\) can be rewritten into a linear combination of \(\Phi(j')\), where \(j'\) are subsequences (with possible repetition) of \(j\).

In sum, any expression \(\sum_{\sigma \in S_{k+1}} H_k(C_{\sigma(j_1)}, \ldots, C_{\sigma(j_k)})\) is equivalent to a linear combination of \(\widehat{M}(P, c)\), where \((P, c)\) ranges over all partition-integer pairs with \(c \neq \max(P)\) and \(\min(P) \geq 2\). Furthermore, this linear combination can be effectively computed. The effective procedure is summarized by the following Algorithm 2. Note that for the algorithm we fix the integer \(k\), so all
set partitions in the algorithm refer to set partitions of $k$.

**Algorithm 2:** For a fixed $k$, computing coefficients $\gamma_{(P,c)}$ such that
$$\sum_{\sigma \in S_{k+1}} H_k(C_{\sigma(j_1)}, \ldots, C_{\sigma(j_k)}) \sim \sum_{(P,c)} \gamma_{(P,c)} \hat{M}(P,c)$$

**Input:** An integer $k$. A tuple $j = (j_1, \ldots, j_m) \in \{1, \ldots, k+1\}^m$.

**Output:** Rational numbers $\gamma_{(P,c)}$, where $(P,c)$ ranges over all partition-integer pairs with $c \neq \max(P)$ and $\min(P) \geq 2$.

**Step 3:** Compute rational numbers $a_S$ such that
$$\sum_{\sigma \in S_{k+1}} H_k(C_{\sigma(j_1)}, \ldots, C_{\sigma(j_m)}) = \sum_{\text{set partition } S} a_S \Phi(S) \quad (57)$$

in the following way:

(a) Initialize with $a_S := 0$ for all set partitions $S$.

(b) For every tuple $(i_1, \ldots, i_m) \in \mathbb{Z}_{\geq 0}^m$ such that $i_1 + \cdots + i_m = k$, compute the sequence
$$t := (\underbrace{j_1, \ldots, j_1}_{i_1}, \underbrace{j_2, \ldots, j_2}_{i_2}, \ldots, \underbrace{j_m, \ldots, j_m}_{i_m})$$

and update $a_{SP(i)} := a_{SP(i)} + \frac{1}{i_1! \cdots i_m!}$.

**Step 2:** Compute rational numbers $b_S$ such that
$$\sum_{\text{set partition } S} a_S \Phi(S) = \sum_{\text{set partition } S} b_S M(S) \quad (58)$$

in the following way:

(a) Initialize with $b_S := 0$ for all set partitions $S$.

(b) For every set partition $S$ and every permutation $\sigma \in S_k$, compute the set partition
$$S^{\sigma} := \left\{ \{\sigma(j) \mid j \in A\} \mid A \in S \right\}$$

and update $b_{S^{\sigma}} := b_{S^{\sigma}} + a_S \cdot \frac{(-1)^{d(\sigma)}}{k^2 d(\sigma)}$ (where $d(\cdot)$ denotes the number of descents).

**Step 3:** Compute rational numbers $g_S$ such that
$$\sum_{\text{set partition } S} b_S M(S) = \sum_{\text{set partition } S} g_S M(S) \quad (59)$$

in the following way:

(a) Initialize with $g_S := b_S$ for all set partitions $S$.

(b) Order all set partitions $S$ into $S_1, S_2, \ldots, S_p$, such that if $S_j \supseteq S_i$ then $j \geq i$.

(c) For $i = 1, 2, \ldots, p$:

- If $\min(IP(S_i)) = 1$, then update $g_{S_i} := 0$ and $g_{S_j} := g_{S_j} - g_{S_i}$ for all $S_j \supseteq S_i$.

**Step 4:** Compute all partition-integer pairs $(P,c)$ with $c \neq \max(P)$ and $\min(P) \geq 2$.

(To be continued in the next page)

**Algorithm 2:** (continued)

Step 5 Compute rational numbers $\gamma_{(P,c)}$ such that
We can now give computer assisted proofs of Lemma 4.7 - 4.10 based on Algorithm 2.

**Proof of Lemma 4.7.** (The code used can be found at [https://doi.org/10.6084/m9.figshare.20124146.v1](https://doi.org/10.6084/m9.figshare.20124146.v1))

Set $k = 5$. Using Algorithm 2 on the tuples $(1, 2, 3, 4, 5, 6)$ and $(1, 2, 3, 4, 5, 6, 1, 2, 3)$, we get

$$
\sum_{\sigma \in S_6} H_5(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(6)}) \sim \hat{M}((3, 2), 2),
$$

$$
\sum_{\sigma \in S_6} H_5(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_{12})}) \sim -\hat{M}((3, 2), 2).
$$

Therefore,

$$
\sum_{\sigma \in S_6} H_5(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(6)}) + \sum_{\sigma \in S_6} H_5(\log B_{\sigma(j_1)}, \ldots, \log B_{\sigma(j_{12})}) \sim 0.
$$

**Proof of Lemma 4.9.** (The code used can be found at [https://doi.org/10.6084/m9.figshare.20124113.v1](https://doi.org/10.6084/m9.figshare.20124113.v1))

Set $k = 7$. Using Algorithm 2 on the tuples $(1, \ldots, 8)$ and

$$
j_1 = (j_{1,1}, j_{1,2}, \ldots, j_{1,16}) = (1, 2, 3, 4, 5, 5, 6, 7, 7, 8, 8, 1, 2, 3, 4),
j_2 = (j_{2,1}, j_{2,2}, \ldots, j_{2,16}) = (1, 2, 3, 4, 5, 4, 6, 7, 1, 2, 8, 3, 5, 6, 7, 8).
$$

We get

$$
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(8)}) \sim \frac{34}{15} \hat{M}((5, 2), 2) - \frac{34}{45} \hat{M}((4, 3), 3) + \frac{68}{15} \hat{M}((3, 2, 2), 2),
$$

$$
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(j_{1,1})}, \ldots, \log B_{\sigma(j_{1,16})}) \sim \frac{34}{15} \hat{M}((5, 2), 2) + \frac{238}{45} \hat{M}((4, 3), 3) - \frac{68}{5} \hat{M}((3, 2, 2), 2),
$$

$$
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(j_{2,1})}, \ldots, \log B_{\sigma(j_{2,16})}) \sim -\frac{68}{15} \hat{M}((5, 2), 2) + \frac{34}{45} \hat{M}((4, 3), 3) - \frac{34}{5} \hat{M}((3, 2, 2), 2).
$$

Therefore,

$$
\sum_{\sigma \in S_8} H_7(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(8)}) + \sum_{s=1}^{2} \alpha_s \sum_{\sigma \in S_8} H_7(\log B_{\sigma(j_{s,1})}, \ldots, \log B_{\sigma(j_{s,16})}) \sim 0
$$

with $\alpha_1 = \frac{1}{15}, \alpha_2 = \frac{8}{15}$. 

**Proof of Lemma 4.10.** (The code used can be found at [https://doi.org/10.6084/m9.figshare.20122979.v1](https://doi.org/10.6084/m9.figshare.20122979.v1))

Set $k = 9$. Using Algorithm 2 on the tuples $(1, \ldots, 10)$ and

$$
(j_{1,1}, j_{1,2}, \ldots, j_{1,20}) = (5, 4, 7, 10, 2, 8, 3, 8, 1, 9, 7, 6, 5, 6, 2, 3, 9, 10, 1, 4),
(j_{2,1}, j_{2,2}, \ldots, j_{2,20}) = (8, 3, 5, 7, 10, 6, 8, 2, 1, 10, 2, 4, 9, 1, 5, 9, 3, 6, 7, 4),
(j_{3,1}, j_{3,2}, \ldots, j_{3,20}) = (7, 10, 2, 6, 4, 9, 6, 4, 1, 5, 3, 5, 1, 9, 3, 7, 10, 2, 8, 8),
$$
\((j_{4,1}, j_{4,2}, \ldots, j_{4,20}) = (10, 2, 2, 6, 7, 1, 9, 3, 9, 4, 8, 7, 8, 5, 5, 1, 4, 10, 6, 3),\)
\((j_{5,1}, j_{5,2}, \ldots, j_{5,20}) = (3, 5, 10, 1, 4, 8, 6, 9, 3, 2, 7, 6, 1, 10, 9, 7, 2, 4, 5, 8),\)
\((j_{6,1}, j_{6,2}, \ldots, j_{6,20}) = (4, 7, 2, 10, 2, 1, 3, 5, 8, 1, 6, 9, 10, 7, 6, 8, 3, 5, 9, 4).\)

We get
\[
\sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(10)}) \sim \frac{347}{105} \hat{M}((7, 2), 2) + \frac{347}{315} \hat{M}((6, 3), 3)
\]
\[
+ \frac{347}{105} \hat{M}((5, 4), 4) + \frac{1388}{105} \hat{M}((5, 2, 2), 2) - \frac{347}{21} \hat{M}((4, 3, 2), 3) + \frac{4511}{315} \hat{M}((4, 3, 2), 2)
\]
\[
\sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(j_{1,1})}, \ldots, \log B_{\sigma(j_{1,20})}) \sim -\frac{4511}{315} \hat{M}((5, 4), 4) + 0 \cdot \hat{M}((5, 2, 2), 2) + \frac{3817}{63} \hat{M}((4, 3, 2), 3) + \frac{1735}{63} \hat{M}((4, 3, 2), 2)
\]
\[
\sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(j_{2,1})}, \ldots, \log B_{\sigma(j_{2,20})}) \sim \frac{347}{45} \hat{M}((7, 2), 2) + \frac{18391}{945} \hat{M}((6, 3), 3)
\]
\[
+ \frac{347}{14} \hat{M}((5, 4), 4) - \frac{1388}{315} \hat{M}((5, 2, 2), 2) + \frac{9022}{63} \hat{M}((4, 3, 2), 3) - \frac{694}{63} \hat{M}((4, 3, 2), 2)
\]
\[
\sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(j_{3,1})}, \ldots, \log B_{\sigma(j_{3,20})}) \sim \frac{16309}{42} \hat{M}((7, 2), 2) + \frac{85709}{630} \hat{M}((6, 3), 3)
\]
\[
+ \frac{241859}{1260} \hat{M}((5, 4), 4) + \frac{30883}{126} \hat{M}((5, 2, 2), 2) - \frac{8675}{63} \hat{M}((4, 3, 2), 3) + \frac{94037}{630} \hat{M}((4, 3, 2), 2)
\]
\[
\sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(j_{4,1})}, \ldots, \log B_{\sigma(j_{4,20})}) \sim \frac{20473}{210} \hat{M}((7, 2), 2) - \frac{314729}{1890} \hat{M}((6, 3), 3)
\]
\[
+ \frac{4511}{140} \hat{M}((5, 4), 4) + \frac{137759}{630} \hat{M}((5, 2, 2), 2) - \frac{23249}{315} \hat{M}((4, 3, 2), 3) + \frac{33659}{210} \hat{M}((4, 3, 2), 2)
\]
\[
\sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(j_{5,1})}, \ldots, \log B_{\sigma(j_{5,20})}) \sim \frac{347}{210} \hat{M}((7, 2), 2) + \frac{35741}{1890} \hat{M}((6, 3), 3)
\]
\[
- \frac{18391}{1260} \hat{M}((5, 4), 4) + \frac{1041}{70} \hat{M}((5, 2, 2), 2) - \frac{347}{63} \hat{M}((4, 3, 2), 3) + \frac{1735}{126} \hat{M}((4, 3, 2), 2)
\]
\[
\sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(j_{6,1})}, \ldots, \log B_{\sigma(j_{6,20})}) \sim -\frac{1388}{105} \hat{M}((7, 2), 2) - \frac{56561}{945} \hat{M}((6, 3), 3)
\]
\[
+ \frac{4511}{126} \hat{M}((5, 4), 4) - \frac{3123}{70} \hat{M}((5, 2, 2), 2) - \frac{28454}{315} \hat{M}((4, 3, 2), 3) - \frac{51703}{630} \hat{M}((4, 3, 2), 2)
\]

Therefore,
\[
\sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(1)}, \ldots, \log B_{\sigma(10)}) + \sum_{s=1}^{6} \alpha_s \sum_{\sigma \in S_{10}} H_9(\log B_{\sigma(j_{s,1})}, \ldots, \log B_{\sigma(j_{s,20})}) \sim 0
\]

with \(\alpha_1 = \frac{44566633}{13702661}, \alpha_2 = \frac{557040}{13702661}, \alpha_3 = \frac{205175}{3915046}, \alpha_4 = \frac{1307207}{13702661}, \alpha_5 = \frac{86275275}{27406322}, \alpha_6 = \frac{4105194}{1957523}.\)