Topological phonon polaritons in one-dimensional non-Hermitian nanoparticle chains

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Topological phonon polaritons (TPhPs) are highly protected and localized edge modes that are capable to achieve a strong confinement of electromagnetic waves and immune to impurities and disorder. Here we realize TPhPs by constructing one-dimensional dimerized silicon carbide nanoparticle chains, which mimic the topological property of the well-known Su-Schrieffer-Heeger (SSH) model. We analytically calculate the complex bandstructure of such chains by taking all near-field and far-field dipole-dipole interactions into account. Despite the non-Hermiticity and the breaking of the chiral symmetry, we find the band topology can be still characterized by the complex Zak phase, which is quantized and indicates a topological phase transition when the dimerization parameter \( \beta \) changes from less than 0.5 to larger than 0.5, like the conventional Hermitian SSH model. By calculating the eigenmodes of a finite chain as well as their inverse participation ratios, we find such a dimerized chain supports nontrivial topological eigenmodes localized over its edges, if the dimerization parameter \( \beta > 0.5 \), illustrating the validity of the bulk-boundary correspondence. These TPhPs can provide an efficient way for enhancing light-matter interaction in the mid-infrared.

I. INTRODUCTION

Phonon polaritons (PhPs) are bosonic quasi-particles originating from the strong coupling of phonons and photons [1, 2]. They are able to confine light into the deep subwavelength scale [3–7], and thus show a great potential in enhancing light-matter interactions in the infrared and terahertz (THz) range. As a consequence, phonon polaritons are very promising for nanophotonic applications like nanoscale thermal radiation heat transfer [8–10], enhanced infrared molecular nano-spectroscopy [11], enhanced infrared absorption [12] and radiative cooling [13], etc. Moreover, compared to plasmon polaritons that suffer a lot from the inherent optical losses of metals, phonon polaritons, which are usually excited in low-loss polar dielectrics, like silicon carbide (SiC) [1–3] as well as hexagonal boron nitride (hBN) [4–7], can have very high quality factors [3], even though their frequencies are generally much lower. On the other hand, the rise of fundamental building blocks of low-loss phonon-polariton based nanoscale circuits, following from their rapid developing counterpart of quantum plasmonics [20, 21].

Our interest for realizing TPhPs is also stimulated by recent works on topological plasmon polaritons [22–25]. The simplest nanostructure supporting topological plasmon polaritons is based on an optical analogy of the Su-Schrieffer-Heeger (SSH) model, by using one-dimensional (1D) periodic, dimerized plasmonic nanoparticle (NP) chains [22–25]. Hence, similarly in this paper, we aim to realize topologically protected phonon polaritons in 1D dimerized silicon carbide (SiC) NP chains also by mimicking the SSH model. Since a single SiC NP is capable to sustain localized phonon polaritons, a 1D periodic SiC NP chain can also support collective PhPs bound along the chain due to the collective excitation of these localized PhPs [8, 21]. If an appropriate dimerization parameter can be found to make the system fall into the topologically nontrivial phase, it is then possible to observe collective PhPs that spatially localize at the boundaries between topologically non-trivial and trivial phases, and spectrally reside at the gap of the optical bandstructure. These collective PhPs thus can be regarded as TPhPs.

However, as the Hamiltonian of conventional SSH model for electron transport in a diatomic chain is Hermitian, and only accounts for the nearest-neighbor electron hoppings [26], we are not able to directly draw conclusions about the topological properties of our system based on it. More specifically, this is because the long-range coupling and retardation effect are natural ingredients of electromagnetic (EM) interactions between the localized PhPs, and in this circumstance our system becomes non-Hermitian. The non-Hermiticity introduces difficulties in studying the topological properties of this system. Recently, there are extensive discussions on the topological properties of non-Hermitian Hamiltonians [27–47], since traditional topological physics is mainly developed on the basis of Hermitian Hamiltonians. In fact, there are still several crucial open questions...
not fully solved about the topological properties of non-Hermitian systems. For example, can appropriate topological invariants be defined in non-Hermitian systems to describe the band topology [28, 29, 33, 39, 42, 48]? And if the answer is positive, then is the bulk-boundary correspondence that allows us to predict the number of topological boundary (edge) states solely based on the bulk band topology still valid [38, 42, 48]? Since there are by now no standard periodic tables of topological insulators for non-Hermitian systems, like the Altland-Zirnbauer (AZ) classification for Hermitian systems [49], so far the analysis on topological properties in this field still remains case by case.

In a recent paper [47], we studied the topological optical states in 1D dimerized ultracold atomic chains, which are typically described by an effective non-Hermitian Hamiltonian if the photonic degrees of freedom in the reservoir (namely the quantized EM field) are integrated out. It was shown that the complex Zak phase is still quantized and becomes nontrivial when the dimerization parameter $\beta > 0.5$, despite the non-Hermiticity. We also verified the bulk-boundary correspondence for that system by analyzing the eigenstate distributions. In a similar way, in this paper, we also aim to demonstrate whether above conclusions are still valid for the present non-Hermitian system involving SiC NPs. For small enough SiC NPs, we treat them as electric dipoles. The complex bandstructure of such chains is analytically calculated by taking all near-field and far-field dipole interactions into account. We find that the band topology of this system can still be characterized by the complex Zak phase, which indicates a topological phase transition when the dimerization parameter changes from less than 0.5 to larger than 0.5. By analyzing the eigenmodes of a finite chain as well as their inverse participation ratios (IPRs), we demonstrate that the chain indeed supports nontrivial topological eigenmodes localized over the edges, as long as the dimerization parameter $\beta > 0.5$, implying the validity of the bulk-boundary correspondence. We further exhibit the excitation of TPhPs and show their enhancement to the photonic local density of states (LDOS). These TPhPs can offer an efficient interface for enhancing light-matter interaction in the mid-infrared.

II. MODEL

Consider a 1D dimerized chain composed of spherical $\alpha-$ (hexagonal) SiC NPs schematically shown in Fig.1a. The chain is well aligned along the $x$-axis, where the dimerization is introduced by using inequivalent spacings $d_1$ and $d_2$ for the two sublattices, denoted by $A$ and $B$. Here we define the dimerization parameter as $\beta = d_1/d$ where $d = d_1 + d_2$ is the overall lattice constant. SiC NPs support strong localized phonon polaritonic resonances in the infrared region around $11\mu m$ due to excitation of longitudinal optical phonons, and the permittivity function is modeled by a Lorentz model as [50]

$$\varepsilon_p(\omega) = \varepsilon_\infty \left(1 + \frac{\omega_p^2 - \omega_i^2}{\omega_p^2 - \omega^2 - i\omega\gamma}\right),$$

where $\omega$ is the angular frequency of the driving field in the unit of cm$^{-1}$, $\varepsilon_\infty = 6.7$ is the high-frequency limit of the permittivity, $\omega_i = 790$cm$^{-1}$ is the transverse optical phonon frequency, $\omega_p = 966$cm$^{-1}$ is the longitudinal optical phonon frequency, and $\gamma = 5$cm$^{-1}$ is the (non-radiative) damping coefficient [50]. Herein we set the radius of the spherical SiC NP as $a = 0.1\mu m$. In this situation, the single particle extinction efficiency $Q_{\text{ext}} = C_{\text{ext}}/(\pi a^2)$ is calculated using the Mie theory as shown in Fig.1b, where $C_{\text{ext}}$ is the extinction cross section. Actually, such a small NP can be well modeled by the electric dipole (ED) approximation. By considering electric dipole excitations only, the EM response of an individual SiC NP is described by the dipole polarizability with the radiative correction, which is given as [10, 51, 52]

$$\alpha(\omega) = \frac{4\pi a^3\alpha_0}{1 - 2i\alpha_0(ka)^3/3}$$

where

$$\alpha_0(\omega) = \frac{\varepsilon_p(\omega) - 1}{\varepsilon_p(\omega) + 2}.$$  

The extinction efficiency calculated under the ED approximation is also shown in Fig.1b, and a good agreement with the exact Mie theory is observed. Furthermore, when the distance between the centers of different spherical NPs is less than $3a$, the EM response of such a chain is described by the well-known set of coupled-dipole equations [10, 51, 52]:

$$p_j(\omega) = \alpha(\omega) \left[ E_{\text{inc}}(r_j) + \frac{\varepsilon_p^2}{c^2} \sum_{i=1,i\neq j}^{\infty} G_0(\omega, r_j, r_i) p_i(\omega) \right],$$

where $G_0(\omega, r_j, r_i)$ is the free space Green’s function.
where \( c \) is the speed of light in vacuum. \( E_{\text{inc}}(\mathbf{r}) \) is the external incident field and \( \mathbf{p}_j(\omega) \) is the excited electric dipole moment of the \( j \)-th NP. \( G_0(\omega, \mathbf{r}_j, \mathbf{r}_i) \) is the free-space dyadic Green’s function describing the propagation of field emitting from the \( i \)-th NP to \( j \)-th NP [51]. This model takes all types of near-field and far-field dipole-dipole interactions into account and is thus beyond the traditional nearest-neighbor approximation, which is implemented in the SSH model.

For 1D chains, there are two types of electromagnetic eigenmodes, including the transverse and longitudinal ones [53]. For the longitudinal eigenmodes, the dipole moments of the NPs are aligned to the \( x \)-axis. By applying the Bloch theorem in Eq. (4) for an infinitely periodic chain with zero incident field, we can analytically solve the longitudinal Bloch eigenmode with a momentum \( k_x \) along the \( x \)-axis. Such an eigenmode is

\[
\omega^2 \left( \sum_{n\neq 0} G_{0,xx}(nd) \exp(ik_xnd) \sum_{n=0} G_{0,xx}(nd \pm d_1) \exp(ik_xnd) \right) \left( \begin{array}{c} p_{A,k_x} \\ p_{B,k_x} \end{array} \right) = \alpha^{-1}(\omega) \left( \begin{array}{c} p_{A,k_x} \\ p_{B,k_x} \end{array} \right) .
\]

(7)

For convenience, this equation is equivalently expressed as

\[
\omega^2 \left( \sum_{n\neq 0} G_{0,xx}(nd) \exp(ik_xnd) \sum_{n=0} G_{0,xx}(nd \pm d_1) \exp(ik_xnd) \right) \left( \begin{array}{c} p_{A,k_x} \\ p_{B,k_x} \end{array} \right) = \alpha^{-1}(\omega) \left( \begin{array}{c} p_{A,k_x} \\ p_{B,k_x} \end{array} \right) .
\]

(8)

Hence we arrive at an eigenvalue problem whose solution corresponds to the dispersion relation (or bandstructure) of the longitudinal eigenmodes, and the matrix in the LHS can be regarded as the effective Hamiltonian \( H(k_x) \) in the reciprocal space. For PhPs with ultranarrow linewidths, the lattice sum matrix in the LHS of above equation can be regarded as frequency-independent [24]. Therefore, for a fixed \( k_x \), \( \alpha^{-1}(\omega) \) is the eigenvalue of that matrix. This fact allows us to straightforwardly calculate the eigenfrequency and thus the bandstructure. Using the polylogarithm (or Jonquières’ function) \( L_n(z) = \sum_{n=1}^{\infty} z^n/n^s \) sum series, the diagonal elements in the matrix are given by [47]

\[
a_{11}^L(k_x) = a_{22}^L(k_x) = -i \frac{L_{i1}(z^+)}{2\pi k^2 d^2} + L_{i2}(z^-) \frac{2\pi k^2 d^2}{2\pi k d^3},
\]

(9)

where \( z^+ = \exp(ik(k + k_x)d) \) and \( z^- = \exp(ik(k - k_x)d) \). Using the Lerch transcendent \( \Phi(z, s, a) = \sum_{n=0}^{\infty} z^n/(n + a)^s \), we can obtain the off-diagonal series sums as [47]

\[
a_{12}^L(k_x) = \left[ -i \frac{\Phi(z^+, 2, 1 - \beta)}{2\pi k^2 d^2} + \frac{\Phi(z^+, 3, 1 - \beta)}{2\pi k d^3} \right] \exp(i\beta d)
\]

\[
+ \left[ -i \frac{\Phi(z^-, 2, 1 - \beta)}{2\pi k^2 d^2} + \frac{\Phi(z^-, 3, 1 - \beta)}{2\pi k d^3} \right] \exp(-i\beta d),
\]

(10)

\[
a_{21}^L(k_x) = \left[ -i \frac{\Phi(z^+, 2, 1 - \beta)}{2\pi k^2 d^2} + \frac{\Phi(z^+, 3, 1 - \beta)}{2\pi k d^3} \right] \exp(-i\beta d)
\]

\[
+ \left[ -i \frac{\Phi(z^-, 2, 1 - \beta)}{2\pi k^2 d^2} + \frac{\Phi(z^-, 3, 1 - \beta)}{2\pi k d^3} \right] \exp(i\beta d),
\]

(11)

In the same way, for transverse eigenmodes whose dipole moments are perpendicular to the chain, by using the transverse (namely, \( yy \) or \( zz \)) component of the Green’s function as

\[
G_{0,yy}(x) = \left[ \frac{i}{k|x|} + \frac{1}{(k|x|)^2} \right] \exp(ik|x|),
\]

(12)

we have

\[
\omega^2 \left( \sum_{n\neq 0} G_{0,xx}(nd) \exp(ik_xnd) \sum_{n=0} G_{0,xx}(nd \pm d_1) \exp(ik_xnd) \right) \left( \begin{array}{c} p_{A,k_x} \\ p_{B,k_x} \end{array} \right) = \alpha^{-1}(\omega) \left( \begin{array}{c} p_{A,k_x} \\ p_{B,k_x} \end{array} \right) .
\]

(13)

The diagonal elements read

\[
a_{11}^T(k_x) = a_{22}^T(k_x) = \frac{L_{i1}(z^+) + L_{i1}(z^-)}{4\pi kd},
\]

\[
a_{12}^T(k_x) = a_{21}^T(k_x) = \frac{-i L_{i2}(z^+) - i L_{i2}(z^-)}{4\pi k^2 d^2} \frac{2\pi k^2 d^2}{2\pi k d^3}.
\]

(14)
The off-diagonal elements are given by

\[
a_{12}^T(k_x) = \left[ \frac{i}{4\pi k^2 d^2} \Phi(z^+, 2, \beta) - \frac{\Phi(z^+, 3, \beta)}{4\pi k^3 d^3} + \frac{\Phi(z^+, 1, \beta)}{4\pi k d} \right]
\times \exp(i k \beta d) + \left[ \frac{\Phi(z^+, 2, 1 - \beta)}{4\pi k^2 d^2} - \frac{\Phi(z^+, 3, 1 - \beta)}{4\pi k^3 d^3} \right] z^- \exp(-i k \beta d),
\]

and

\[
a_{21}^T(k_x) = \left[ \frac{i}{4\pi k^2 d^2} \Phi(z^+, 2, 1 - \beta) - \frac{\Phi(z^+, 3, 1 - \beta)}{4\pi k^3 d^3} \right]
\times \exp(-i k \beta d) + \left[ \frac{\Phi(z^+, 2, \beta)}{4\pi k^2 d^2} - \frac{\Phi(z^+, 1, \beta)}{4\pi k d} \right] \exp(i k \beta d).
\]

III. RESULTS AND DISCUSSION

A. Bandstructures

We first investigate the bandstructure of a dimerized chain with an overall lattice constant of \( d = 1 \mu m \). In Figs. 2a and 2b we show the real parts of the bulk bandstructures for different dimerization parameters \( \beta = 0.5, 0.6, 0.7 \) for both longitudinal and transverse eigenmodes, respectively. Note the bulk bandstructures for \( \beta = 0.3 \) and \( \beta = 0.4 \) are the same as those for \( \beta = 0.7 \) and \( \beta = 0.6 \) correspondingly. There are small discontinuities in the transverse band near the light-line \( k_x = \omega/c \). It is found that for \( \beta \neq 0.5 \), bandgaps in the real frequency space are opened in both cases and a larger \( |\beta - 0.5| \) (i.e., the deviation from a non-bipartite chain) gives rise to a wider bandgap. This behavior is consistent with the conventional SSH model [56]. In this paper, we mainly discuss the longitudinal eigenmodes because the transverse ones are more strongly coupled to free-space radiation (denoted by the light-line), and the bandgap of transverse eigenmodes is much narrower, which makes it difficult to observe topological eigenmodes experimentally. Note that the bandgap residing in these transverse eigenmodes in the real frequency space is even smaller than the linewidth, i.e., \( \gamma \), even for a highly dimerized chain \( \beta = 0.7 \). In the subsequent sections, we will omit the subscripts \( L \) that denotes the longitudinal eigenmodes for convenience.

B. The complex Zak phase

In the last several years, especially recent two years, there were fierce discussions on the topological properties of non-Hermitian systems [27–47]. Most of these discussions were devoted to the topologically nontrivial properties related to exceptional points (EPs) [57, 58], which are singularities in the energy spectra of non-Hermitian Hamiltonians where the eigenvalues and eigenwavefunctions coalesce [59]. On the other hand, for non-Hermitian systems without EPs, it was recently shown that the com-
plex Zak phase, which is defined by simultaneously using the left-eigenvectors and right-eigenvectors of the non-Hermitian Hamiltonian [19, 29, 36, 43, 60, 61], can be exploited to characterize the band topology of 1D systems. In our recent paper [47], we studied the non-Hermitian topological optical states in 1D dimerized ultracold atomic chains, and showed that as long as the bandstructure is separable in the complex plane, the complex Zak phase is always quantized and the bulk-boundary correspondence applies. In fact, Shen and Fu [39] presented a rigorous definition for the non-Hermitian bandstructures. In a non-Hermitian system, that a band with a band number of n is separable means that for any m ≠ n in the entire bandstructure, the energies (complex frequencies) also satisfy ̂ω_m,k ≠ ̂ω_n,k in the complex plane for all possible k [39]. Moreover, if their energies (complex frequencies) further fulfill ̂ω_m,k' ≠ ̂ω_n,k in the complex plane for all k and k', the band n is then regarded as isolated or gapped [39]. According to Eq.(17), since a_{12}(k_x)a_{21}(k_x) is always not exactly zero when β ≠ 0.5, the bulk bandstructures are always separable in the complex frequency plane (which can be seen in the examples exhibited below). Hence we anticipate that for the present non-Hermitian system the complex Zak phase is quantized and can describe the topological phase transition, where the transition point is the gap closing point, i.e., β = 0.5 [36, 47].

Another issue is necessary to be addressed before calculating the complex Zak phase is that the present non-Hermitian Hamiltonian exhibits a breaking of chiral symmetry (or sublattice symmetry) due to the existence of diagonal elements a_{11}(k_x) and a_{22}(k_x). According to the conventional AZ classification, this system belongs to the AI class and is topologically trivial [26, 62]. Nevertheless, since a_{11}(k_x) is always equal to a_{22}(k_x), the eigenvectors of the matrix are not affected by these diagonal terms at all [24, 47]. Actually, the normalized left and right eigenvectors for longitudinal eigenmodes are solved as follows:

\[ |p^L_{A,k_x}x⟩ = \frac{1}{\sqrt{2}} \left( \frac{a_{21}^*(k_x)}{a_{21}(k_x)} |p^R_{A,k_x}⟩ \right) \]  \hspace{1cm} (18)

\[ |p^R_{B,k_x}⟩ = \frac{1}{\sqrt{2}} \left( \frac{a_{21}^*(k_x)}{a_{21}(k_x)} |p^L_{B,k_x}⟩ \right) \]  \hspace{1cm} (19)

The left eigenvector is solved through the relation of \( H^\dagger(k_x)|p^L_{B,k_x}⟩ = E^\dagger_k |p^L_{A,k_x}⟩ \). Therefore, the eigenvectors of the present Hamiltonian is the same as those of its chirally-symmetric counterpart with zero diagonal elements. This chiral symmetry breaking can be viewed as trivial, as pointed out by Pocock et al [24]. Hence the complex Zak phase is still quantized in a similar way as in a chirally symmetric system. This complex Zak phase can be conveniently used to determine the topology of bulk bandstructure, as indicated by Lieu [36]. Moreover, this quantization does not require the inversion symmetry unlike the real Zak phase. The real Zak phase is defined solely based on right eigenvectors, and in that circumstance, the inversion symmetry is necessary to make it quantized [25, 36]. However, it is not appropriate to use the real Zak phase in a non-Hermitian system [36].

Based on the orthogonality of left and right eigenmodes (namely, biorthogonality) [19, 28, 30, 32, 34–37, 39, 61, 63–65] in 1D non-Hermitian systems, the complex Zak phase is expressed as

\[ θ_z = \int_{BZ} dk_x A(k_x) \]

\[ = i \int_{-\pi/d}^{\pi/d} \left[ p^R_{A,k_x} \frac{∂p^L_{A,k_x}}{∂k_x} + p^R_{B,k_x} \frac{∂p^L_{B,k_x}}{∂k_x} \right] dk_x \]

\[ = \frac{\arg(a_{21}(k_x)) - \arg(a_{12}(k_x))}{4} \pi/d \]

where \( A(k_x) \) is the Berry connection. According to Eq.(20), the complex Zak phase is actually a real quantity, and it is simply half the difference of the winding numbers of a_{21}(k_x) and a_{12}(k_x) encircling the origin multiplied by π. To calculate θ_z, the winding of a_{12}(k_x) and a_{21}(k_x) around the origin of the complex plane by sweeping k_x from 0 to 2π/d is illustrated in Fig.3, for the cases of β = 0.7, β = 0.3, β = 0.6 and β = 0.4 with the lattice constant d = 1µm. Note the directions of the encircling of a_{12}(k_x) and a_{21}(k_x) are always opposite because a_{12}(k_x) = a_{21}(−k_x). The winding numbers of a_{12}(k_x) and a_{21}(k_x) are +1 and -1 respectively when the dimerization parameter is β = 0.7 and β = 0.6, and are both zero when β = 0.3 and β = 0.4. As a consequence, the complex Zak phase for β = 0.7 and β = 0.6 is π and is 0 for β = 0.3 and β = 0.4. We further calculate a phase diagram of the complex Zak phase by varying the lattice constants and dimerization parameters, as plotted in Fig.4. It is observed that the gap closing point β = 0.5 indeed amounts to a topological transition from the topologically trivial phase to the topologically nontrivial phase as expected.

C. Bulk-boundary correspondence and midgap modes

In Hermitian systems, the principle of bulk-boundary correspondence indicates that the topological invariant (Zak phase in 1D) determines the existence of edge eigenmodes, and the total winding number \( W = θ_z/π \) is equivalent to the number of edge eigenmodes localized over the boundary of the systems [14, 15, 48, 66]. However, it was surprisingly found that the conventional bulk-boundary correspondence becomes invalid for some specific 1D non-Hermitian Hamiltonians [38, 40, 44]. This is because in those systems, the Bloch bandstructures under the periodic boundary condition are much different from the bandstructures calculated from the open boundary condition [40, 44]. Therefore, to appropriately investigate
where \( \omega \) and \( -1 \) are the wavenumber for an eigenmode is determined by \([24, 53]\). The topologically protected eigenmodes are exponentially localized over the boundary of the finite chain \([56]\). Therefore, to quantify the degree of eigenmode localization, we further calculate the inverse participation ratio (IPR) of an eigenmode from its eigenvector as \([47, 67]\).

\[
\text{IPR} = \frac{\sum_{n=1}^{N} |p_j|^4}{(\sum_{n=1}^{N} |p_j|^2)^2}.
\]

For an IPR approaches \(1/M\), where \(M\) is an integer, the corresponding eigenmode involves the excitation of \(M\) NPs \([67]\). Therefore, for a highly localized topological eigenmode, its IPR should be much larger compared to those of the bulk eigenmodes \([67]\).

The topological properties of the present non-Hermitian system, we must first directly compute the bandstructures of finite chains with open boundaries.

Here we investigate a dimerized chain with \(N = 100\) identical NPs. In fact, the calculation of the bandstructures of finite chains is rather straightforward by using Eq. (4) and setting the incident field to be zero \([24, 53]\). In this way, we obtain an eigenvalue equation in the form of \(\mathbf{G}|\mathbf{p}⟩ = α^{-1}(\omega)|\mathbf{p}⟩\). Here \(\mathbf{G}\) stands for the interaction matrix whose elements are derived from the Green’s function, and \(|\mathbf{p}⟩ = [p_1p_2…p_j…p_N]\) is the right eigenvector, which stands for the dipole moment distribution of an eigenmode, where \(p_j\) is the dipole moment of the \(j\)-th NP. Like the case of infinite chains, this equation also specifies a set of complex eigenfrequencies in the lower complex plane in the form of \(\omega = \omega - i\Gamma/2\). The physical significance of these complex eigenfrequencies are the same as the ones appearing in Section II. The corresponding wavenumber for an eigenmode is determined by \([24, 53]\)

\[
k_z = \frac{2\pi}{d}(N - 1)n + 1, \quad n = 1, 2, 3, \ldots, \quad N\text{,}
\]

plus the number of times of sign changes of \(\text{Re}(p_j)\) along the chain for that eigenmode \([24, 53]\).

FIG. 3. The winding of \(a_{12}(k_z)\) and \(a_{21}(k_z)\) over the origin in the complex plane. The lattice constant is \(d = 1\mu m\) with different dimerization parameters (a) \(\beta = 0.7\). (b) \(\beta = 0.3\). (c) \(\beta = 0.6\). (d) \(\beta = 0.4\).

FIG. 4. Phase diagram with respect to the complex Zak phase calculated from bulk bandstructures with different lattice constants and dimerization parameters.

FIG. 5. (a) Complex bandstructure of longitudinal eigenmodes of a dimerized chain with \(N = 100\) NPs under \(\beta = 0.7\) and \(d = 1\mu m\). Note there are two midgap eigenmodes. (b) Complex bandstructure of longitudinal eigenmodes of a dimerized chain with \(N = 100\) NPs under \(\beta = 0.3\) and \(d = 1\mu m\). (c) Dipole moment distribution of the midgap edge states. (d) Complex eigenfrequency spectrum of eigenmodes as a function of the lattice constant \(d\).

FIG. 6. (a) Longitudinal eigenmode distribution for a connected chain. Inset: Schematic of the connected chain. (e) Dipole moment distribution of the interface eigenmode at \((\omega/\gamma = 928.5277\text{cm}^{-1}, \Gamma = 5.0070\text{cm}^{-1})\) in (a), compared with those of bulk eigenmodes at \((\omega = 924.4771\text{cm}^{-1}, \Gamma = 5.0145\text{cm}^{-1})\) and \((\omega = 931.5735\text{cm}^{-1}, \Gamma = 5.0001\text{cm}^{-1})\).
In Fig.5a, we show the complex band structures of the longitudinal eigenmodes for $\beta = 0.7$ and $\beta = 0.3$ with a lattice constant of $d = 1\mu m$. In both cases we observe that bandgaps are opened in the complex frequency plane. The corresponding range of the bandgaps also agrees well with the Bloch band structure. The difference between the $\beta = 0.7$ and $\beta = 0.3$ cases is that two midgap eigenmodes with high IPRs emerge in the complex bandgap in the former case. The dipole moment distributions of the two midgap eigenmodes are shown in Fig.5c, which exhibit an exponential localization behavior from the boundary. Hence, by combining the non-trivial complex Zak phase of the $\beta = 0.7$ case, we can conclude that these midgap eigenmodes are topologically protected edge modes.

Along with the phase diagram presented in Fig.4, we discuss the effect of lattice constant by plotting the eigenfrequency spectrum as a function of the lattice constant, shown in Fig.5d. The dimerization parameter is set to be $\beta = 0.7$. The projections to real-frequency ($\omega$)-$k_x$ and imaginary-frequency ($\Gamma$)-$k_x$ planes are also given. It is obviously seen that the complex bandgaps are always open and eigenmodes with high IPRs are well situated in the bandgaps, despite that the bandgaps in the real-frequency plane are already closed at large lattice constants. Therefore taking the entire complex bandstructure into account in a non-Hermitian system is necessary and critical to correctly study its topological properties [24, 39, 47].

To further examine the bulk-boundary correspondence, we can check whether the topologically protected interface eigenmode can emerge at the boundary involving two topologically different media. Fig.6a presents the eigenmode distribution of a 1D connected chain, which consists of a topologically trivial chain with $\beta = 0.3$ in the left and a topologically nontrivial chain with $\beta = 0.7$ in the right. The distance between the two chains is set to be $d_1$ (more specifically, the distance between the centers of the rightmost NP in the left chain and the leftmost NP in the right chain.). Two midgap eigenmodes with high IPRs are also observed, in which one is the interface eigenmode while the other is the edge eigenmode localized at the right boundary of the right chain. Their IPRs are much larger than the midgap eigenmodes in the single chain case, both reaching unity. In Fig.6b we show the dipole moment distribution for the interface eigenmode, which is indeed highly (exponentially) localized over the interface, compared with those of two typical bulk eigenmodes in the upper band and lower band. Therefore, so far, we can conclude that the previously derived complex Zak phase is able to characterize these edge states, and the principle of bulk-boundary correspondence is still valid in our system.

**D. Field enhancement and LDOS**

We now briefly investigate the excitation and the electromagnetic field enhancement of these TPhPs. In Figs.7a and 7b, we show the electric field distribution near both edges of a dimerized SiC NP chain when a topological edge mode is excited using an evanescent plane wave with a wavenumber $k = \pm \pi/d$, that is the wavenumber of the midgap eigenmode, polarized along the $x$-axis, where the excitation frequency is $\omega = 928.5116cm^{-1}$, just lying in the bandgap. The SiC NP chain is composed of 100 identical NPs whose radius is $a = 0.1\mu m$, where the lattice constant is $d = 1\mu m$ and the dimerization parameter is $\beta = 0.7$. The first NP is centered at $x = 0\mu m$, and the last NP is located at $x = 50\mu m$. It is clearly observed in Figs.7a and 7b that the electric field is enormously enhanced in and near the NPs in the edges of the chain. In experiment, such an excitation method is available by using quantum dots [68] as well scanning near-field optical microscopy (SNOM) tips [5] whose radiation field contains large momentum components.

Furthermore, we calculate the optical local density of states (LDOS) near the edge of the chain. The LDOS can also be obtained under the framework of the coupled-dipole equations, while the incident field is replaced by that emitted from a point source [69, 70]:

\[
p_j(\omega) = \frac{\omega^2 \alpha(\omega)}{c^2} \left[ G_0(\omega, r_j, r_s)p_s + \sum_{i=1, i \neq j}^{N} G_0(\omega, r_j, r_i)p_i(\omega) \right], \tag{23}
\]

where $p_s$ is the dipole moment of the emitting point source whose position is $r_s$. After calculating the electromagnetic responses of all NPs based on Eq.(23), the total scattered field of the NP chain at an arbitrary position outside the NPs is computed as [69, 70]

\[
E_s(r) = \frac{\omega^2}{c^2} \sum_{i=1}^{N} G_0(\omega, r, r_i)p_i(\omega). \tag{24}
\]

From the scattered field, it is straightforward to obtain the full Green’s function with respect to the point source at $r_s$ as $G(\omega, r, r_s) = G_0(\omega, r, r_s) + S(\omega, r, r_s)$ [69, 70]. Here the elements in scattering field tensor $S(\omega, r, r_s)$ can be calculated through the relation $E_s(r) = S(\omega, r, r_s)p_s$ by aligning the dipole moment of the point source along different axes. Afterwards LDOS is obtained from the
full Green’s function as
\[ \rho(r_s, \omega) = \frac{2\omega}{\pi c^2} \text{Im} [\text{Tr} G(\omega, r_s, r_s)]. \] (25)

Note this total LDOS contains all electromagnetic eigenmodes including both longitudinal and transverse ones. Fig. 7c shows the total LDOS at \( x_s = -0.5 \mu m \) as a function of the angular frequency, where the point source is located along the negative \( x \)-axis and \( x_s \) is its \( x \)-coordinate. It is not surprising that the LDOS is greatly enhanced in the bandgap frequency for the topologically nontrivial chain (\( \beta = 0.7 \)), while in the topologically trivial chain (\( \beta = 0.3 \)), the LDOS is suppressed to be lower than the vacuum value (\( \rho_0 = \omega^2/(\pi^2 c^3) \)). Hence this enhancement in LDOS is attributed to the topologically protected midgap TPhPs. In Fig. 7d, we plot the distance dependence of the LDOS near the left-edge of the chain, where the excitation frequency is \( \omega = 928.5116 \text{cm}^{-1} \) and a frequency in the bulk band (\( \omega = 924.5 \text{cm}^{-1} \)). The LDOS at \( \omega = 928.5116 \text{cm}^{-1} \) is always substantially larger than that at \( \omega = 924.5 \text{cm}^{-1} \) all over the distance range (Note the logarithmic scale). As a result, we have verified that the existence of topological phonon polaritons gives rise to an appreciable enhancement to the electromagnetic field and LDOS. Moreover, this enhancement in return provides an indicator of TPhPs, and can aid us to experimentally locate and detect those modes.

IV. CONCLUSION

In conclusion, we achieve topologically protected phonon polaritons by constructing 1D dimerized silicon SiC NP chains, which mimic the celebrated SSH model. However, different from the conventional SSH model, we carry out this study beyond the nearest-neighbor approximation and taking all near-field and far-field dipole-dipole interactions into account. Despite the consequences of non-Hermiticity and the breaking of chiral symmetry brought by this treatment, we show that such dimerized chains can still support topological protected midgap eigenmodes, i.e., TPhPs. We reveal that in this non-Hermitian system, the band topology can be characterized by a quantized complex Zak phase, which indicates a topological phase transition point of \( \beta = 0.5 \), like its Hermitian counterpart. By analyzing the eigenmodes of a finite chain as well as their inverse participation ratios (IPRs), we find topologically protected midgap eigenmodes and unequivocally verify the principle of bulk-boundary correspondence. We also demonstrate the excitation of the topological phonon polaritons and show their enhancement to the photonic LDOS. These TPhPs offer an efficient tool for enhancing light-matter interaction in the mid-infrared.

FIG. 7. The distribution of the electric field \( |E| \) near the left (a) and right (b) edges of a dimerized SiC NP chain when a topological edge mode is excited using an evanescent plane wave with a wavevector \( k = \hat{x} \pi/d \) polarized along the \( x \)-axis, where the excitation frequency is \( \omega = 928.5116 \text{cm}^{-1} \). (c) Total LDOS at \( x_s = -0.5 \mu m \) as a function of the angular frequency, for both topologically nontrivial (\( \beta = 0.7 \)) and trivial (\( \beta = 0.3 \)) chains, where \( \rho_0 = \omega^2/(\pi^2 c^3) \) is the LDOS in the free space. (d) The distance dependence of the LDOS near the left-edge of the topologically nontrivial (\( \beta = 0.7 \)) chain (in logarithmic scale).

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