ABSTRACT: In this work we derive for the first time the complete gravitational quartic-in-spin interaction of generic compact binaries at the next-to-leading order in the post-Newtonian (PN) expansion. The derivation builds on the effective field theory for gravitating spinning objects, and its recent extensions, in which new type of worldline couplings should be considered, and the effective action should be extended to quadratic order in the curvature. The latter entails a new Wilson coefficient that appears in this sector. This work pushes the precision with spins at the fifth post-Newtonian (5PN) order for maximally-spinning compact objects, and at the same time informs us of the gravitational Compton scattering with higher spins.
1 Introduction

In this work we pursue two interwoven objectives, each of which has a different nature. The first objective falls under the timely phenomenological efforts to improve the theoretical precision used in modeling gravitational waveforms, that are currently successfully being used to measure gravitational waves (GWs) from mergers of compact binaries as of 2016 [1]. The post-Newtonian (PN) approximation of classical Gravity [2] via the Effective-One-Body approach [3] provides the crucial input for the generation of GW templates, and accordingly significant progress has been carried out recently in pushing the state-of-the-art with the current frontier at the 4.5PN [4–6] and the 5PN [7–12] accuracy in the orbital dynamics. The complete state-of-the-art of the generic orbital dynamics of a compact binary to date is captured in table 1. In order to complete a certain PN accuracy, the sectors that are shown in table 1 should be addressed across its diagonals, which stands for two orthogonal types of efforts: The challenge of going along the horizontal axis of the table is that of computational multi-loop technology, whereas the challenge of going along the vertical axis of the table involves improving the conceptual understanding of spin in gravity and higher-spin theory.

Which brings us to the second objective of the present work, which is the theoretical effort to understand what happens in gravitational interactions when higher spins are involved. Within classical gravity the first work to provide the LO quadratic-in-spin interaction, including the spin-induced quadrupole was carried out decades ago [16]. Within
Table 1. The complete state-of-the-art of PN orbital dynamics of generic compact binaries. The PN corrections enter at the order \( n + l + \text{Parity}(l)/2 \), with the parity 0 or 1 for even or odd \( l \), respectively. The sectors with the entries in boldface have been addressed in [4, 5, 10, 13] and in the present work via the EFT of gravitating spinning objects [14]. The entries in the table indicate the loop computational scale within this framework as explained e.g. in [4]. The sectors up to the current complete state of the art at 4PN order except the top right one are available in the public EFTofPNG code at https://github.com/miche-levi/pncbc-eftofpng [15].

| \( l \) | \( n \) | (N\(^0\))LO | N\(^1\)LO | N\(^2\)LO | N\(^3\)LO | N\(^4\)LO |
|-------|-------|-------------|-----------|-----------|-----------|-----------|
| \( S^0 \) | 0 | 1 | 3 | 0 | 25 |
| \( S^1 \) | 1 | 2 | 7 | 32 | 174 |
| \( S^2 \) | 2 | 2 | 18 | 52 |
| \( S^3 \) | 4 | 24 |
| \( S^4 \) | 3 | 5 |

Figure 1. The gravitational Compton scattering with two massive spin particles and two gravitons.
of fixing the graviton Compton amplitude with higher spins, and thus these two inquiries may inform each other.

This work directly builds on the EFT for gravitating spinning objects introduced in [14], its implementation on [28] at LO, and the recent extension to the NLO cubic-in-spin interaction in [4], and the NNNLO quadratic-in-spin interaction in [10], to get an extension of the quartic-in-spin interaction to the NLO. This enters at the 5PN order for maximally-spinning compact objects, beyond the current complete PN state of the art in general, and with spins in particular [29]. This work extends the body of work of various studies to push PN precision with higher spin in [4, 10, 13, 14, 28, 30–37], where we derive the complete sector, including all interactions with all possible spin multipoles up to and including the hexadecapole. This sector is also the next complicated one after [4] in the intriguing gray area of table 1, and thus beyond pushing the state of the art in PN theory, we will be alert to the conceptual new effects that show up in it.

The paper is organized as follows. In section 2 we review briefly the formulation of the EFT of spinning gravitating objects from [14], and outline the ingredients relevant for this sector. Notably, in section 2.1 we extend the theory, in order to extract new contributions to the current sector. In section 3 we present the full perturbative expansion with the elementary worldline couplings of the EFT, where the linear momentum is taken as independent of the spin, and in section 4 we discuss new contributions, due to composite worldline couplings that emerge, first seen in the NLO cubic-in-spin sector [4]. In section 5 we discuss and evaluate new contributions arising from new worldline couplings that are quadratic in the curvature. The final result for the effective action of this sector is provided in section 6, and we conclude the work in section 7.

2 The EFT of gravitating spinning objects

The evaluation of this sector, that contains spins at quartic order with leading gravitational nonlinearities, builds on the EFT of gravitating spinning objects formulated in [14], the LO sectors in [28], the recent complete sector in [4] at the 4.5PN order, and the work in [10] at 5PN order, as well as the implementation from LO to the state of the art at the 4PN order in its implementation from LO up to the state of the art at the 4PN order in [13, 14, 38]. We follow all the conventions and gauge choices in the abovementioned papers, and we also use the beneficial Kaluza-Klein (KK) decomposition of the field to scalar, vector and tensor components, as in all high-order PN computations in the EFT approach [39–41].

We start from a two-particle effective action [41], in which each of the objects is captured by the one-particle effective action of a spinning particle, that was derived in [14]. This effective action contains a pure gravitational part, and two copies of a point-particle part that captures the coupling of gravity to the worldline degrees of freedom. The Feynman rules for the propagators and self-interacting vertices derived from the purely gravitational action are found in [42], and [38]. The action of a spinning particle including its spin-induced non-minimal coupling, and in which the gauge freedom of the rotational
DOFs was introduced into the action has the following form [14]:

$$S_{pp}(\sigma) = \int d\sigma \left[ -m\sqrt{u^2} - \frac{1}{2} \hat{\mathcal{B}}_{\mu\nu} \hat{\mathcal{Y}}^{\mu\nu} - \frac{\hat{S}_{\mu\nu} p_{\nu}}{p^2} D_{\mu} - L_{	ext{SI}} \right],$$  \hspace{1cm} (2.1)

with the four-velocity $u_\mu$, the conjugate linear momentum $p_\mu$ and the rotational DOFs, denoted with a hat e.g. $\hat{S}_{\mu\nu}$. The spin-induced part, labeled by “SI”, will contribute to this sector its three leading terms [14]:

$$L_{\text{SI}} = -\frac{C_{E} S^2}{2m} \frac{E_{\mu\nu}}{\sqrt{u^2}} S^\mu S^\nu - \frac{C_{B S}^3}{6m^2} D_{\lambda} \frac{B_{\mu\nu}}{\sqrt{u^2}} S^\mu S^\nu S^\lambda + \frac{C_{E S^4}}{24m^4} D_{\lambda} D_{\mu} \frac{E_{\mu\nu}}{\sqrt{u^2}} S^\mu S^\nu S^\lambda S^\kappa,$$  \hspace{1cm} (2.2)

where here we use the electric and magnetic components of the curvature tensor and a classical version of the Pauli-Lubanski pseudovector, $S^\mu$, as defined first in [28] via:

$$S_\mu = \frac{1}{2} \epsilon_{\alpha\beta\gamma\mu} S^{\alpha\beta} D_\gamma.$$  \hspace{1cm} (2.3)

Note that this is with a reverse sign with respect to the definition detailed in [14], which was implemented up to the quadratic-in-spin order, where the sign choice does not make a difference.

We remind the extra term in eq. (2.1), which is essentially the Thomas precession, that was derived from the introduction of gauge freedom to the rotational DOFs in [14] (and later recovered in e.g. [24] as “Hilbert space matching”). This term is relevant to all orders in spin, including all finite size spin effects, though it does not encapsulate any UV physics, but rather accounts for the extended measure of a relativistic gravitating spinning object.

Since we compute here the complete NLO quartic-in-spin sector our graphs will contain all spin-induced multipoles up to the hexadecapole, in addition to the mass and spin- and spin-induced quadrupole and octupole. Therefore we need the Feynman rules of worldline-graviton coupling to NLO for all of these multipoles, including further new rules for the hexadecapole couplings. The Feynman rules for the mass, the spin and the spin-induced quadrupole are found e.g. in [43], [14], and [38], and [13]. The spin-induced octupole couplings are found in [4, 28]. In this work the Feynman rule of the scalar component of the KK fields, which appeared at LO in [28] should be extended to the next PN order and is given as follows:

$$\begin{align*}
\int dt & \quad -\frac{C_{E S^4}}{24m^3} \left[ S^i S^j S^k S^l \left( \partial_i \partial_j \partial_k \partial_l \phi + \frac{3v^2}{2} \right) - 2v^l v^m \partial_m \partial_j \partial_k \partial_l \phi \right] + S^2 S^j S^k \left( v^m v^l \partial_m \partial_j \partial_k \partial_l \phi + 2v^l \partial_i \partial_j \partial_k \partial_l \phi + \partial_j \partial_k \partial^2 \partial_l \phi \right),
\end{align*}$$  \hspace{1cm} (2.4)

where the crossed black box represents the spin-induced hexadecapole.
We also need new rules for the one-graviton coupling of the KK vector field, and for the two-graviton coupling, with the KK scalar field given as follows:

\[
\mathcal{L} = \int dt \frac{C_{ES}}{24m^3} S^i S^j S^k S^l \left[ \nu^m \partial_t \partial_j \partial_k \partial_l A_m - \nu^m \partial_m \partial_j \partial_k \partial_l A_i - \partial_j \partial_k \partial_l \partial_t A_i \right], \quad (2.5)
\]

\[
\mathcal{L} = \int dt \left[ \frac{C_{ES}}{24m^3} \left( 16 \partial_t \partial_j \partial_k \partial_l \phi + 10 \partial_t \partial_j \partial_k \partial_l \phi + 5 \partial_t \partial_j \partial_k \partial_l \phi \right) 
- S^2 S^j S^k \left( 7 \partial_j \partial_k \partial_n \phi \partial_n \phi + 4 \partial_j \partial_k \partial_n \phi \partial_n \phi \right) \right]. \quad (2.6)
\]

In these rules the spin is fixed to the canonical gauge and all indices are Euclidean. Note the dominant role that the KK scalar field plays in the coupling to the even-parity hexadecapole, similar to the couplings to the even-parity mass monopole, and spin-induced quadrupole.

For this sector we also need to extend the non-minimal coupling part of the spinning particle action, and add higher dimensional operators beyond the linear-in-Riemann ones, which were derived in [14], but we also need to address the new subtle feature that became relevant as of the NLO cubic-in-spin sector (the simplest corner of the gray area in table 1). For the latter we need to take into account corrections to the linear momentum that are linear in Riemann and higher-order in the spin as in [4], which was explicitly noted already in [14]. These give rise to what we will refer as “composite” worldline couplings, which are discussed in section 4, after the “elementary” worldline couplings, in which the linear momentum is still independent of the spin (the white area of table 1), are used in section 3.

### 2.1 Extending the EFT of a spinning particle

As noted at the 5PN order the effective action should be extended [10], or more specifically, the non-minimal coupling part of the effective action of a spinning particle that is given in eq. (2.2). This extension should include operators that are higher-order in the curvature components, and describe the tidal deformations of the compact object. This extension should only include operators that are quadratic in the curvature, since here only corrections up to order \( G^2 \) are considered. Following the symmetries and the logic detailed in [14], and the recent extension in [10], we find that the new terms to quartic order in spin are of the form:

\[
L_{NMC(R^2)} = C_{E2} \frac{E_{\alpha \beta} E_{\alpha \beta}}{\sqrt{u^2}} + C_{B2} \frac{B_{\alpha \beta} B_{\alpha \beta}}{\sqrt{u^2}} + \ldots
+ C_{E2S2} S^\mu S^\nu E_{\mu \alpha} E_{\nu \alpha} \frac{1}{\sqrt{u^2}} + C_{B2S2} B_{\mu \alpha} B_{\nu \alpha} \frac{1}{\sqrt{u^2}}
+ C_{E2S4} S^\mu S^\nu S^\sigma S^\rho E_{\mu \nu \rho} E_{\sigma \rho} \frac{1}{\sqrt{u^2}} + C_{B2S4} B_{\mu \nu \rho} B_{\sigma \rho} \frac{1}{\sqrt{u^2}}
\]
\[ + C_{\mathcal{E}B^3 S^\mu} \frac{D_\mu E_{\alpha \beta} B_{\alpha \beta}}{\sqrt{u^2}} + C_{\mathcal{E}B^4 S^\mu S^\nu S^\kappa} \frac{D_\mu E_{\alpha \beta} B_{\alpha \beta}}{\sqrt{u^2}} + C_{\mathcal{E}B^4 S^\mu S^\nu S^\kappa} E_{\nu \alpha \beta} D_{\nu} B_{\alpha \beta} \]

\[ + C_{(\mathcal{E})^2 S^2 S^\mu S^\nu} \frac{D_\mu E_{\alpha \beta} E_{\alpha \beta}}{\sqrt{u^2}} + C_{(\mathcal{B})^2 S^2 S^\mu S^\nu} \frac{D_\mu B_{\alpha \beta} D_{\nu} B_{\alpha \beta}}{\sqrt{u^2}} + C_{(\mathcal{B})^2 S^2 S^\mu S^\nu S^\kappa} \frac{D_\mu B_{\nu \alpha \beta} D_{\nu} B_{\alpha \beta}}{\sqrt{u^2}}, \]

which in contrast with eq. (2.2) – are currently defined to absorb all numerical and mass factors. Notice the new tidal Wilson coefficients involved with these new operators.

As in [10] in the first line of eq. (2.7) the leading mass-induced quadrupolar tidal deformations, which are known to enter at the 5PN order are written [41], suppressing higher-order mass-induced tidal operators, that were provided in [44]. We have then written the adiabatic tidal operators with spin up to quartic order. From power-counting considerations, which were detailed in [14, 41]), it can be deduced that the second and third lines of eq. (2.7) contribute as of the 5PN order, whereas the fourth and fifth lines, and the sixth and seventh lines, of eq. (2.7), contribute as of the 6.5PN and 7PN orders, respectively. Therefore at the 5PN order considered in this work there are two new operators which are quartic in the spin and quadratic in the curvature of the form:

\[ L_{S^4(\mathcal{R}^2)} = \frac{C_{\mathcal{E}^2 S^4}}{24m^3} S^\mu S^\nu S^\rho E_{\mu \nu} E_{\rho \eta} \frac{E_{\eta \rho} E_{\mu \nu}}{\sqrt{u^2}} + \frac{C_{\mathcal{B}^2 S^4}}{24m^3} S^\mu S^\nu S^\rho B_{\mu \nu} B_{\rho \eta} \frac{B_{\eta \rho} B_{\mu \nu}}{\sqrt{u^2}}, \]

where the numerical and mass factors have now been adjusted in order to render the Wilson coefficients dimensionless.

The consequent new Feynman rule that contributes from eq. (2.8) is then a two-graviton coupling of two KK scalars given by

\[ \int dt \frac{C_{\mathcal{E}^2 S^4}}{24m^3} \left[ S_i S_j S_k S_l \phi_{i j} \phi_{k l} - 2 S^2 S_i S_j \phi_{i j} \phi_{k k} + S^4 \phi_{i i} \phi_{j j} \right], \]

that arises from the quadratic electric operator, and is represented here by the black “star of Lakshmi”. As to the quadratic magnetic operator in eq. (2.8), it contributes a two-graviton coupling of two KK vectors, which will become relevant only at the 6PN order.

### 3 Elementary worldline couplings

Let us first evaluate the Feynman graphs from the diagrammatic expansion of the effective action, that contain the elementary worldline couplings, i.e. those obtained under the leading approximation of the linear momentum that is independent of spin. With spins all of the three relevant topologies up to the $G^2$ order are realized even with the useful KK decomposition [14, 41, 43, 45], and we have here a total of 23 = 10 + 8 + 5 graphs, distributed among
the topologies of one- and two-graviton exchanges and cubic self-interaction, as shown in figures 2-4 (drawn with Jaxodraw [46, 47] based on [48]), respectively. As noted in table 1 only 5 of the graphs require a one-loop evaluation, though as we consider the nonlinear graphs in the sector, the options to assemble a quartic-in-spin interaction multiply.

At the linear level there are only three types of interaction, similar to the LO in [28], namely either a quadrupole-quadrupole, an octupole-dipole or a hexadecapole-monopole interaction. These graphs are easily constructed following the nice analogies among these types of interaction according to the parity of the multipole moments [4, 28]. Yet, more types of interactions emerge once we proceed to nonlinear level, in which there are interactions with various multipoles on two different points of the worldline, as of the NLO spin-squared sector [4, 13, 14], which add up to interactions that are quartic in the spin, such as a spin and a spin-induced quadrupole, or two spin quadrupoles on the same worldline.

Note that all the graphs in this sector are to be included together with their mirror images, in which the worldline labels are 1 ↔ 2 exchanged, and that more details on the generation and the evaluation of the Feynman graphs can be found in [41] and references therein.

3.1 One-graviton exchange

There are 10 graphs of one-graviton exchange in this sector, as seen in figure 2, in which most already contain time derivatives to be evaluated. As in previous works of one of the authors, all of the higher order time derivative terms that appear in the evaluations of the graphs are kept, to be be treated rigorously later via the redefinitions of variables procedure, that was explained in [49]). Notice that there are 3 graphs that appeared at the LO, to which we add the quadratic time insertions on the propagators at graphs 2(a2),(b2),(c2), and a new hexadecapole coupling of the KK vector field at graph 2(c3).

The graphs in figure 2 have the following values:

\[
\text{Fig. 2(a1)} = \frac{3C_1(ES^2)C_2(ES^2)}{8} \frac{G}{m_1 m_2 r^5} \left[ 70 (\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_2 \cdot \vec{n})^2 \left( 1 + \frac{3v_1^2}{2} + \frac{3v_2^2}{2} \right) 
- 10 (\vec{S}_1 \cdot \vec{n})^2 S_2^2 \left( 1 + \frac{3v_1^2}{2} + \frac{5v_2^2}{2} - 7 (\vec{v}_2 \cdot \vec{n})^2 \right) 
- 40 (\vec{S}_1 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{n}) \left( \vec{S}_1 \cdot \vec{v}_1 \right) \left( \vec{S}_2 \cdot \vec{v}_2 \right) \left( \vec{v}_2 \cdot \vec{n} \right) 
+ 10 (\vec{S}_1 \cdot \vec{n}) \left( \vec{S}_1 \cdot \vec{v}_1 \right) S_2^2 \left( \vec{v}_1 \cdot \vec{n} \right) - 40 (\vec{S}_2 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) (\vec{v}_2 \cdot \vec{n}) \right] 
- 20 (\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_2 \cdot \vec{v}_2) (\vec{v}_2 \cdot \vec{n}) + 20 (\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_1 \cdot \vec{v}_2) (\vec{v}_1 \cdot \vec{n}) 
- 10 (\vec{S}_1 \cdot \vec{v}_1)^2 (\vec{S}_1 \cdot \vec{v}_2)^2 (\vec{v}_1 \cdot \vec{n}) + 4 (\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_1 \cdot \vec{v}_2) (\vec{S}_2 \cdot \vec{v}_1) \left( \vec{S}_2 \cdot \vec{n} \right) - 4 (\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_2 \cdot \vec{v}_1) (\vec{S}_2 \cdot \vec{v}_2) \left( \vec{S}_2 \cdot \vec{n} \right)
\]
Figure 2. The Feynman graphs of one-graviton exchange, that comprise the NLO quartic-in-spin interaction at the 5PN order for maximally rotating compact objects. At the linear level we only have three types of interactions, similar to the LO in [28], of either a quadrupole-quadrupole, octupole-dipole or a hexadecapole-monopole type. The graphs are easily constructed following the nice analogies pointed out in [28] among the interactions according to the parity of the multipole moments. Notice that we have here the three graphs that appeared at the LO with the quadratic time insertions on the propagators at graphs (a2), (b2), (c2), and a new hexadecapole coupling with the KK vector field at graph (c3).

\[
- 10S_1^2 (\vec{S}_2 \cdot \vec{n})^2 \left(1 + \frac{5v_1^2}{2} + \frac{3v_2^2}{2} - 7(\vec{v}_1 \cdot \vec{n})^2\right) + 4S_1^2 (\vec{S}_2 \cdot \vec{v}_1)^2 - 2S_1^2 (\vec{S}_2 \cdot \vec{v}_2)^2 \\
- 40S_1^2 (\vec{S}_2 \cdot \vec{n}) (\vec{v}_1 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{v}_1) + 10S_1^2 (\vec{S}_2 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{v}_2) (\vec{v}_2 \cdot \vec{n}) \\
+ 2S_1^2 S_2^2 \left(1 + \frac{5v_1^2}{2} + \frac{5v_2^2}{2} - 5(\vec{v}_1 \cdot \vec{n})^2 - 5(\vec{v}_2 \cdot \vec{n})^2\right) \\
+ \frac{3C_{1(ES)^2}C_{2(ES)^2}G}{2m_1m_2r^4} \left[5(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_2 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{v}_2) \\
- 10(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_2 \cdot \vec{v}_2) (\vec{v}_2 \cdot \vec{n}) + 5(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_2 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{v}_2) \\
+ 5(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_2 \cdot \vec{v}_2) (\vec{S}_2 \cdot \vec{v}_2) + (\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_1) S_2^2 - 5(\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_2 \cdot \vec{n})^2 \\
- 2(\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) (\vec{S}_2 \cdot \vec{v}_2) + 4(\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) (\vec{S}_2 \cdot \vec{v}_2) \\
- 2(\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) (\vec{S}_2 \cdot \vec{v}_2) - 5(\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) (\vec{S}_2 \cdot \vec{n})^2 + (\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) S_2^2 \\
- 2(\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) (\vec{S}_2 \cdot \vec{n}) - S_1^2 (\vec{S}_2 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{n}) + 2S_1^2 (\vec{S}_2 \cdot \vec{n}) (\vec{v}_2 \cdot \vec{n}) \\
+ 2(\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_1 \cdot \vec{v}_2) (\vec{S}_2 \cdot \vec{n}) - 5(\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_2 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{n})^2 + (\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_2 \cdot \vec{n}) S_2^2 \\
+ 2(\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) (\vec{S}_2 \cdot \vec{n}) - S_1^2 (\vec{S}_2 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{v}_2) - S_1^2 (\vec{S}_2 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{v}_2)
\]
\[\begin{align*}
&+ 2(\dot{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{S}_2) + 10(\dot{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{n})^2 (\vec{v}_1 \cdot \vec{n}) \\
&- 4(\dot{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{n}) (\vec{S}_2 \cdot \vec{v}_1) - 2(\dot{S}_1 \cdot \vec{S}_1)S^2_2 (\vec{v}_1 \cdot \vec{n}) \\
&- \frac{C_{1(ES)^2}C_{2(ES)^2}G}{m_1 m_2 r^3} \left[ 3(\vec{S}_1 \cdot \vec{n})^2 S^2_2 + 3(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_2 \cdot \vec{S}_2) - S^2_1 S^2_2 - S^2_1 (\vec{S}_2 \cdot \vec{S}_2) \\
&+ 3S^2_2 (\vec{S}_2 \cdot \vec{n})^2 - S^2_2 S^2_2 - (\vec{S}_1 \cdot \vec{S}_1)S^2_2 + 3(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{n})^2 \right].
\end{align*}\]
\[ + 2(\vec{S}_1 \cdot \vec{v}_2)(\hat{\vec{S}}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) - 2(\vec{S}_2 \cdot \vec{v}_2)(\hat{\vec{S}}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{n}) \\
+ 2(\vec{S}_2 \cdot \vec{v}_2)(\hat{\vec{S}}_1 \cdot \vec{S}_2)(\hat{\vec{S}}_1 \cdot \vec{n}) - (\vec{S}_1 \cdot \vec{v}_2)(\hat{\vec{S}}_1 \cdot \vec{n}) S_2^2 \\
- 10(\vec{S}_1 \cdot \vec{S}_2)(\hat{\vec{S}}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) + 2(\vec{S}_1 \cdot \vec{S}_2)(\hat{\vec{S}}_1 \cdot \vec{S}_2)(\vec{v}_2 \cdot \vec{n}) \\
+ 2(\vec{S}_1 \cdot \vec{S}_2)(\hat{\vec{S}}_1 \cdot \vec{v}_2)(\vec{S}_2 \cdot \vec{n}) - 2(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_2)(\vec{v}_1 \cdot \vec{n}) \\
- 5S_1^2(\vec{S}_2 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n})(\vec{v}_1 \cdot \vec{n}) + 3S_2^2(\vec{S}_2 \cdot \vec{S}_2)(\vec{v}_1 \cdot \vec{n}) + S_1^2(\vec{S}_2 \cdot \vec{n})(\vec{S}_2 \cdot \vec{v}_1) \\
+ S_1^2(\vec{S}_2 \cdot \vec{n})(\vec{S}_2 \cdot \vec{v}_1) - 3(\vec{S}_1 \cdot \vec{S}_1) S_2^2(\vec{v}_2 \cdot \vec{n}) + 5(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n}) \]

\[ + \frac{C_{1(ES^2)} C_{2(ES^2)} G}{2m_1 m_2 r^3} \left[ 3(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) - 3(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) \\
- 3(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) + 15(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) \\
+ (\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_2) - 3(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) + 3(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) \\
- 3(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{S}_2) + (\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_2) - 3(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) \right], \]

(3.2)

\[ \text{Fig. 2(a3)} = \frac{3C_{1(ES^2)} C_{2(ES^2)} G}{m_1 m_2 r^3} \left[ 5(\vec{S}_1 \cdot \vec{n})^2 S_2^2(\vec{v}_1 \cdot \vec{v}_2) - 35(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_2 \cdot \vec{n})^2 \right. \]

\[ + 20(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) - S_1^4 S_2^2 + 5S_1^2(\vec{S}_2 \cdot \vec{n})^2 - 2(\vec{S}_1 \cdot \vec{S}_2)^2(\vec{v}_1 \cdot \vec{v}_2) \]

\[ + \frac{3C_{1(ES^2)} C_{2(ES^2)} G}{m_1 m_2 r^4} \left[ 10(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_2 \cdot \vec{S}_2)(\vec{v}_1 \cdot \vec{n}) - 5(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_2 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{n}) \\
- 5(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_2 \cdot \vec{n})(\vec{S}_2 \cdot \vec{v}_1) - (\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_2) S_2^2 - 4(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{S}_2) \\
+ 5(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_2 \cdot \vec{n})^2 + 2(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{v}_1) \\
+ 2(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{v}_1) - 2(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) - (\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_1 \cdot \vec{n}) S_2^2 \\
+ 5(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{v}_2) - 2(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) \\
+ S_1^2(\vec{S}_2 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{n}) - 2S_2^2(\vec{S}_2 \cdot \vec{S}_2)(\vec{v}_1 \cdot \vec{n}) + S_1^2(\vec{S}_2 \cdot \vec{n})(\vec{S}_2 \cdot \vec{v}_1) \\
+ 4(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{n})(\vec{S}_2 \cdot \vec{v}_2) - 10(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n}) \]

\[ + 2(\vec{S}_1 \cdot \vec{S}_1) S_2^2(\vec{v}_2 \cdot \vec{n}) \right] + \frac{C_{1(ES^2)} C_{2(ES^2)} G}{m_1 m_2 r^3} \left[ 3(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) \\
- 12(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) + 3(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) \\
+ 3(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) - 2(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_2) + 8(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{S}_2) \\
- 12(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_2 \cdot \vec{n})(\vec{S}_2 \cdot \vec{S}_2) - 2(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{S}_1) \\
+ 3(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) \right], \]

(3.3)
Fig. 2(b1) = \frac{C_{1(BS)}^3}{4} \frac{G}{m_1^7r^3} \left[ 70(\vec{S}_1 \cdot \vec{n})^3 (\vec{S}_2 \cdot \vec{n}) \left( 1 + \frac{3\nu_1^2}{2} + \frac{3\nu_2^2}{2} \right) 
- 30(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{S}_2) \left( 1 + \frac{13\nu_1^2}{6} + \frac{13\nu_2^2}{6} - \frac{14(\vec{n}_1 \cdot \vec{n})^2}{3} - \frac{14(\vec{n}_2 \cdot \vec{n})^2}{3} \right) 
+ 65S_1^2(\vec{S}_1 \cdot \vec{S}_2) \left( 1 + \frac{13\nu_1^2}{6} + \frac{13\nu_2^2}{6} - \frac{10(\vec{n}_1 \cdot \vec{n})^2}{3} - \frac{10(\vec{n}_2 \cdot \vec{n})^2}{3} \right) 
- 30S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{n}) \left( 1 + \frac{3\nu_1^2}{2} + \frac{3\nu_2^2}{2} \right) - 2(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{S}_2) 
- 140(\vec{S}_1 \cdot \vec{n})^3 (\vec{n}_2 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) - 105(\vec{S}_1 \cdot \vec{n})^3 (\vec{n}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) 
- 140(\vec{S}_1 \cdot \vec{n})^3 (\vec{n}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) + 85(\vec{S}_1 \cdot \vec{n})^2 (\vec{n}_2 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{n}) 
- 175(\vec{S}_1 \cdot \vec{n})^2 (\vec{n}_2 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{n}) + 65(\vec{S}_1 \cdot \vec{n})^2 (\vec{n}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) 
+ 50(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_1)^2 (\vec{v}_2 \cdot \vec{n}) + 40(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_2)^2 (\vec{v}_2 \cdot \vec{n}) 
- 30(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_1)(\vec{n}_2 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) - 40(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_2)(\vec{n}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) 
+ 60S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) + 45S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) 
+ 25S_2^2(\vec{n}_2 \cdot \vec{v}_1)(\vec{n}_2 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) - 13S_2^2(\vec{n}_2 \cdot \vec{v}_2)(\vec{n}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) 
+ 20S_2^2(\vec{n}_2 \cdot \vec{v}_1)(\vec{n}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) - 17S_2^2(\vec{n}_2 \cdot \vec{v}_1)(\vec{n}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) 
+ \frac{C_{1(BS)}^3}{2} \frac{G}{m_1^7r^3} \left[ 10(\vec{S}_1 \cdot \vec{n})^3 (\vec{n}_2 \cdot \vec{a}_1) 
- 10(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{a}_2)(\vec{S}_2 \cdot \vec{n}) - 10(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{a}_2)(\vec{v}_1 \cdot \vec{n}) 
+ 10(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{S}_2)(\vec{n}_2 \cdot \vec{n}) - 10(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{S}_2)(\vec{a}_1 \cdot \vec{n}) 
+ 10(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{S}_2)(\vec{n}_2 \cdot \vec{n}) - 10(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{S}_2)(\vec{a}_1 \cdot \vec{n}) 
+ 30(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_1) + 10(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_1)(\vec{n}_2 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) 
+ 5S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_1)(\vec{a}_1 \cdot \vec{n}) - 7S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{a}_1) 
- 20(\vec{S}_1 \cdot \vec{n})(\vec{n}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) + 4(\vec{S}_1 \cdot \vec{n})(\vec{n}_1 \cdot \vec{a}_1)(\vec{n}_2 \cdot \vec{S}_2) 
+ 4(\vec{S}_1 \cdot \vec{n})(\vec{n}_1 \cdot \vec{v}_1)(\vec{n}_2 \cdot \vec{S}_2) + 4(\vec{S}_1 \cdot \vec{n})(\vec{n}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{S}_2) 
- 14(\vec{S}_1 \cdot \vec{n})(\vec{n}_1 \cdot \vec{v}_1)(\vec{n}_2 \cdot \vec{v}_1) + 5S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{n})(\vec{v}_1 \cdot \vec{n}) 
+ 2S_1^2(\vec{n}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{n}) - S_1^2(\vec{n}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{n}) + 4(\vec{S}_1 \cdot \vec{n})(\vec{n}_1 \cdot \vec{v}_1)(\vec{n}_2 \cdot \vec{S}_2) 
- 2S_1^2(\vec{n}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{n}) = 2(\vec{S}_1 \cdot \vec{S}_1)(\vec{n}_1 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) + 4S_1^2(\vec{S}_1 \cdot \vec{S}_2)(\vec{a}_1 \cdot \vec{n}) 
- 2S_1^2(\vec{n}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{n}) - 2(\vec{S}_1 \cdot \vec{S}_1)(\vec{n}_1 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) - S_1^2(\vec{n}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{n}) 
+ S_1^2(\vec{n}_1 \cdot \vec{n})(\vec{v}_1 \cdot \vec{n}) + 2S_1^2(\vec{n}_1 \cdot \vec{v}_2)(\vec{n}_2 \cdot \vec{n}) - 7S_1^2(\vec{n}_1 \cdot \vec{n})(\vec{n}_2 \cdot \vec{v}_1) \right], \quad (3.4)

Fig. 2(b2) = \frac{C_{1(BS)}^3}{4} \frac{G}{m_1^7r^3} \left[ 35(\vec{S}_1 \cdot \vec{n})^3 (\vec{n}_2 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{n}) + 35(\vec{S}_1 \cdot \vec{n})^3 (\vec{n}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{n}) 
+ 35(\vec{S}_1 \cdot \vec{n})^3 (\vec{n}_2 \cdot \vec{n})(\vec{v}_1 \cdot \vec{v}_2) - 315(\vec{S}_1 \cdot \vec{n})^3 (\vec{n}_2 \cdot \vec{n})(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) 
+ 35(\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_1 \cdot \vec{S}_2)(\vec{n}_2 \cdot \vec{n})(\vec{v}_1 \cdot \vec{v}_2) + 105(\vec{S}_1 \cdot \vec{n})^2 (\vec{n}_2 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1) \right]
\[
\begin{align*}
&-35(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_m)G + 20(\vec{S}_1 \cdot \vec{S}_2)G + 2(\vec{S}_1 \cdot \vec{S}_2)G + 10(\vec{S}_1 \cdot \vec{S}_2)G + 5(\vec{S}_1 \cdot \vec{S}_2)G - 30(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_m)G,
\end{align*}
\]
\[ + 5(\vec{S}_1 \cdot \vec{S}_2)(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) - 5(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_2 \cdot \vec{n})(\vec{v}_1 \cdot \vec{n}) + 4(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_2 \cdot \vec{v}_1) \right\}

\text{(3.6)}

\text{Fig. 2(b4) =}

\[ \frac{C_{1(BS^3)}}{2} \frac{G}{m_1 r^5} \left[ 35(\vec{S}_1 \cdot \vec{n})^3(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{n}) - 70(\vec{S}_1 \cdot \vec{n})^3(\vec{S}_2 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{n}) \right. \]

\[ - 70(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{n}) + 35(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{n}) \]

\[ + 10(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{n}) + 35(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_1 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) \]

\[ - 15S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{n}) + 10(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_1 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) \]

\[ + 30S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) - 20(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_1 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{n}) \]

\[ - 5(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_1 \cdot \vec{S}_2)(\vec{v}_1 \cdot \vec{v}_2) - 20(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{v}_2) \]

\[ + 40(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_1 \cdot \vec{S}_2)(\vec{v}_1 \cdot \vec{v}_1) + 10S_1^2(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{n}) \]

\[ - 5S_1^2(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{n}) - 5S_1^2(\vec{S}_1 \cdot \vec{S}_2)(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) \]

\[ + S_1^2(\vec{S}_1 \cdot \vec{S}_2)(\vec{v}_1 \cdot \vec{v}_2) + 4S_1^2(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{v}_2) \]

\[ - 8S_1^2(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_2 \cdot \vec{v}_1) \bigg] + \frac{C_{1(BS^3)}}{2} \frac{G}{m_1 r^5} \left[ 5(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})(\vec{v}_1 \cdot \vec{n}) \right. \]

\[ + 8(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{v}_2) - 2(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_2 \cdot \vec{n}) \]

\[ - 10(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{n}) - 6(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1) \]

\[ + 10(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \vec{n}) - 6(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_2) \]

\[ + 20(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{n}) - 6(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_2 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1) \]

\[ + 10(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_2)(\vec{V}_2 \cdot \vec{n}) - 15(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{v}_2 \cdot \vec{n}) \]

\[ - 5S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) - S_1^2(\vec{S}_1 \cdot \vec{v}_2)(\vec{S}_2 \cdot \vec{n}) + 4S_1^2(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_2 \cdot \vec{v}_2) \bigg],

\text{(3.7)}

\text{Fig. 2(c1) =}

\[ \frac{C_{1(EST^4)}}{16} \frac{G m_2}{m_1 r^5} \left\{ 35(\vec{S}_1 \cdot \vec{n})^4 - 30S_1^2(\vec{S}_1 \cdot \vec{n})^2 + 3S_1^4 \right\} \bigg] (2 + 3v_1^2 + 3v_2^2) \]

\[ - 140(\vec{S}_1 \cdot \vec{n})^3(\vec{S}_1 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) + 60(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_1 \cdot \vec{v}_1)^2 - 12S_1^2(\vec{S}_1 \cdot \vec{v}_1)^2 \]

\[ + 60S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{n}) \right\} + \frac{C_{1(EST^4)}}{8} \frac{G m_2}{m_1 r^5} \left[ 5S_1^2(\vec{S}_1 \cdot \vec{n})^2(\vec{a}_1 \cdot \vec{n}) \right. \]

\[ - 2S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{a}_1) - S_1^4(\vec{a}_1 \cdot \vec{n}) \bigg] + \frac{C_{1(EST^4)}}{12} \frac{G m_2}{m_1 r^5} \left[ 3(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_1 \cdot \vec{n})^2 \right. \]

\[ + 3S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_1) - 2S_1^2(\vec{S}_1 \cdot \vec{S}_1) + 3S_1^2(\vec{S}_1 \cdot \vec{n})^2 + 3S_1^2(\vec{S}_1 \cdot \vec{S}_1)^2 \]

\[ + 12(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_1)(\vec{S}_1 \cdot \vec{n}) - 2S_1^2S_1^2 - 4(\vec{S}_1 \cdot \vec{S}_1)^2 \bigg],

\text{(3.8)}

\text{Fig. 2(c2) =}

\[ - \frac{C_{1(EST^4)}}{16} \frac{G m_2}{m_1 r^5} \left\{ 315(\vec{S}_1 \cdot \vec{n})^4(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) - 210S_1^2(\vec{S}_1 \cdot \vec{n})^2(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) \right. \]

\[ + 15S_1^4(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) - 140(\vec{S}_1 \cdot \vec{n})^3(\vec{S}_1 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{n}) + 60S_1^2(\vec{S}_1 \cdot \vec{v}_1)(\vec{S}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) \]
even at second order, whereas at the LO no higher-order time derivatives appeared yet. Notice that almost all these graphs contain higher order time derivatives terms, notably such as a spin dipole and a spin-induced quadrupole or two spin quadrupoles on the same worldline. The graph (a1) contains a new two-graviton coupling to the hexadecapole.

Fig. 2(c3) = \frac{C_{1(ES^4)} G m_2}{2 m_1^{3/2}} \left( \hat{v}_1 \cdot \hat{v}_2 \right) \left[ 30 S_1^2 (\hat{S}_1 \cdot \hat{n})^2 - 35 (\hat{S}_1 \cdot \hat{n})^4 - 3 S_1^4 \right]

\begin{align*}
&- 140 (\hat{S}_1 \cdot \hat{n})^3 (\hat{S}_1 \cdot \hat{v}_2)(\hat{v}_1 \cdot \hat{n}) + 60 S_1^2 (\hat{S}_1 \cdot \hat{n}) (\hat{S}_1 \cdot \hat{v}_2)(\hat{v}_1 \cdot \hat{n}) - 35 (\hat{S}_1 \cdot \hat{n})^4 (\hat{v}_1 \cdot \hat{v}_2) \\
&+ 60 (\hat{S}_1 \cdot \hat{n})^2 (\hat{S}_1 \cdot \hat{v}_1)(\hat{S}_1 \cdot \hat{v}_2) - 12 S_1^2 (\hat{S}_1 \cdot \hat{v}_1)(\hat{S}_1 \cdot \hat{v}_2) + 30 S_1^2 (\hat{S}_1 \cdot \hat{n})^2 (\hat{v}_1 \cdot \hat{v}_2) \\
&- 3 S_1^4 (\hat{v}_1 \cdot \hat{v}_2) \left[ \frac{C_{1(ES^4)} G m_2}{4 m_1^{3/2}} \left[ 5 (\hat{S}_1 \cdot \hat{v}_2)(\hat{S}_1 \cdot \hat{n})^3 + 15 (\hat{S}_1 \cdot \hat{S}_1)(\hat{S}_1 \cdot \hat{n})^2 (\hat{v}_2 \cdot \hat{n}) \\
&- 35 (\hat{S}_1 \cdot \hat{n})(\hat{S}_1 \cdot \hat{n})^3 (\hat{v}_2 \cdot \hat{n}) - 3 S_1^2 (\hat{S}_1 \cdot \hat{v}_2)(\hat{S}_1 \cdot \hat{n}) - 3 S_1^2 (\hat{S}_1 \cdot \hat{S}_1)(\hat{v}_2 \cdot \hat{n}) \\
&+ 15 S_1^4 (\hat{S}_1 \cdot \hat{n})(\hat{S}_1 \cdot \hat{n})(\hat{v}_2 \cdot \hat{n}) - 6 (\hat{S}_1 \cdot \hat{S}_1)(\hat{S}_1 \cdot \hat{v}_2)(\hat{S}_1 \cdot \hat{n}) \\
&+ 15 (\hat{S}_1 \cdot \hat{n})(\hat{S}_1 \cdot \hat{n})^2 (\hat{S}_1 \cdot \hat{v}_2) - 3 S_1^2 (\hat{S}_1 \cdot \hat{n})(\hat{S}_1 \cdot \hat{v}_2) \right], \quad (3.9)
\end{align*}

Fig. 3. The Feynman graphs of two-graviton exchange at the NLO quartic-in-spin interaction. These graphs contain all relevant interactions among the mass, spin and spin-induced multipoles up to hexadecapole, in particular at this nonlinear level there are also interactions with the various multipoles on two different points of the worldline, as of the NLO spin-squared sector \([4, 13, 14]\), such as a spin dipole and a spin-induced quadrupole or two spin quadrupoles on the same worldline. The graph (a1) contains a new two-graviton coupling to the hexadecapole.

\begin{align*}
&- \frac{C_{1(ES^4)} G m_2}{2 m_1^{3/2}} \left( \hat{v}_1 \cdot \hat{v}_2 \right) \left[ 30 S_1^2 (\hat{S}_1 \cdot \hat{n})^2 - 35 (\hat{S}_1 \cdot \hat{n})^4 - 3 S_1^4 \right] \\
&+ \frac{C_{1(ES^4)} G m_2}{2 m_1^{3/2}} \left[ 5 (\hat{S}_1 \cdot \hat{v}_2)(\hat{S}_1 \cdot \hat{n})^3 + 15 (\hat{S}_1 \cdot \hat{S}_1)(\hat{n} \cdot \hat{S}_1)^2 (\hat{S}_1 \cdot \hat{v}_2) \\
&- 3 S_1^4 (\hat{S}_1 \cdot \hat{v}_2)(\hat{S}_1 \cdot \hat{n}) - 6 (\hat{S}_1 \cdot \hat{S}_1)(\hat{S}_1 \cdot \hat{v}_2)(\hat{S}_1 \cdot \hat{n}) - 3 S_1^2 (\hat{S}_1 \cdot \hat{n})(\hat{S}_1 \cdot \hat{v}_2) \right]. \quad (3.10)
\end{align*}

Notice that almost all these graphs contain higher order time derivatives terms, notably even at second order, whereas at the LO no higher-order time derivatives appeared yet [28].

### 3.2 Two-graviton exchange

There are 8 graphs of two-graviton exchange in this sector, shown in figure 3, none of which contains time derivatives. The graph 3(a1) contains a new two-graviton coupling to the hexadecapole.
There are 5 graphs of cubic self-gravitational interaction, i.e. at one-loop level, at the NLO quartic-in-spin interaction. There are no vertices with time dependence here, similar to the NLO even-parity quadratic-in-spin sectors \cite{14}. These graphs contain all possible interactions among the mass, spin and spin-induced multipoles up to hexadecapole, similar to the nonlinear graphs of two-graviton exchange.

The graphs in figure 3 have the following values:

Fig. 3(a1) = $-\frac{3C_{1(ES_1^1)}G^2m_2^2}{8m_1^2r^6} \left[ 95(\vec{S}_1 \cdot \vec{n})^4 - 81S_1^2(\vec{S}_1 \cdot \vec{n})^2 + 8S_1^4 \right]$, (3.11)

Fig. 3(a2) = $-\frac{C_{1(ES_1^1)}G^2m_2}{8m_1^2r^6} \left[ 3S_1^4 - 30S_1^2(\vec{S}_1 \cdot \vec{n})^2 + 35(\vec{S}_1 \cdot \vec{n})^4 \right]$, (3.12)

Fig. 3(b1) = $-\frac{C_{1(BS_2^1)}G^2m_2}{3m_1^2r^6} \left[ 441(\vec{S}_1 \cdot \vec{n})^3(\vec{S}_2 \cdot \vec{n}) - 189S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) - 183(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})^2 + 35S_1^2(\vec{S}_1 \cdot \vec{S}_2) \right]$, (3.13)

Fig. 3(b2) = $-2\frac{C_{1(BS_2^1)}G^2}{m_1^2r^6} \left[ 3S_1^2(\vec{S}_1 \cdot \vec{S}_2) - 15S_1^2(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) - 15(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})^2 + 35(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) \right]$, (3.14)

Fig. 3(c1) = $-\frac{9C_{1(ES_1^2)}C_{2(ES_2^1)}G^2}{2m_1^2r^6} \left[ S_1^2 S_2 - 4S_1^2(\vec{S}_2 \cdot \vec{n}) - 4S_2^2(\vec{S}_1 \cdot \vec{n}) + (\vec{S}_1 \cdot \vec{S}_2) \right]$

$-12(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) + 24(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n})^2 \right]$, (3.15)

Fig. 3(c2) = $-\frac{C_{1(ES_2^1)}G^2m_2}{8m_1^2r^6} \left[ 3(\vec{S}_1 \cdot \vec{n})^2 - S_1^2 \right]$, (3.16)

Fig. 3(d1) = $-\frac{C_{1(ES_2^1)}G^2}{2m_1^2r^6} \left[ S_1^2 S_2 + (\vec{S}_1 \cdot \vec{S}_2)^2 + 3S_1^2(\vec{S}_2 \cdot \vec{n})^2 + 9(\vec{S}_1 \cdot \vec{n})^2(\vec{S}_2 \cdot \vec{n})^2 \right]$

$-6(\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_2 \cdot \vec{n}) \right]$, (3.17)

Fig. 3(d2) = $-2\frac{C_{1(ES_2^1)}G^2}{m_1^2r^6} \left[ 3(\vec{S}_1 \cdot \vec{n})^2 - S_1^2 \right] \left[ 3(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) - (\vec{S}_1 \cdot \vec{S}_2) \right]$, (3.18)

3.3 Cubic self-interaction

There are 5 graphs of cubic self-interaction in this sector, shown in figure 4, none of which contains time-dependent self-interaction, as in the NLO even-parity quadratic-in-spin sectors \cite{14}. These graphs contain all possible interactions among the mass, spin and spin-induced multipoles up to hexadecapole, similar to the nonlinear graphs of two-graviton exchange.
The graphs in figure 4 have the following values:

\[ \text{Fig. 4(a1)} = - \frac{4C_{1(BS^3)} G^2 m_1}{m_1 r^6} \left[ 9S_1^2 (S_1 \cdot \vec{n}) (S_2 \cdot \vec{n}) - 24 (S_1 \cdot \vec{n})^3 (S_2 \cdot \vec{n}) + 12 (S_1 \cdot \vec{n})^2 (S_1 \cdot S_2) - 2S_1^2 (S_1 \cdot S_2) \right], \]

\[ \text{Fig. 4(a2)} = -3C_{1(BS^3)} \frac{G^2 m_2}{m_1 r^6} \left[ 4S_1^2 (S_1 \cdot \vec{n}) (S_2 \cdot \vec{n}) - 10 (S_1 \cdot \vec{n})^3 (S_2 \cdot \vec{n}) + 5 (S_1 \cdot \vec{n})^2 (S_1 \cdot S_2) - S_1^2 (S_1 \cdot S_2) \right], \]

\[ \text{Fig. 4(a3)} = C_{1(BS^3)} \frac{G^2 m_2}{m_1 r^6} (S_1 \cdot \vec{n})^2 \left[ 3S_1^2 - 5 (S_1 \cdot \vec{n})^2 \right], \]

\[ \text{Fig. 4(b1)} = -C_{1(ES^2)} \frac{G^2}{m_1 r^6} \left[ 24 (S_1 \cdot \vec{n}) (S_2 \cdot \vec{n})^3 (S_2 \cdot \vec{n}) - 6S_1^2 (S_2 \cdot \vec{n})^2 + (S_1 \cdot S_2)^2 - S_1^2 S_2 \right.

\[ - 12 (S_1 \cdot \vec{n}) (S_1 \cdot S_2) (S_2 \cdot \vec{n}) \right], \]

\[ \text{Fig. 4(b2)} = -C_{1(ES^2)} \frac{G^2}{m_1 r^6} \left[ 6 (S_1 \cdot \vec{n})^3 (S_2 \cdot \vec{n}) - 3 (S_1 \cdot \vec{n})^2 (S_1 \cdot S_2) - S_1^2 (S_1 \cdot S_2) \right]. \]

\[ (3.19) \]

\[ (3.20) \]

\[ (3.21) \]

\[ (3.22) \]

\[ (3.23) \]

\section{4 Composite worldline couplings}

The formulation of the EFT of a spinning gravitating particle in [14] assumed an initial covariant gauge of the rotational DOFs in terms of the linear momentum as originally put forward by by Tulczyjew [50], and proven to be uniquely distinguished in [51, 52]. As detailed in section 4 of [4], and pointed out already in [14], this gives rise to composite worldline couplings in sectors of higher-spin as of the NLO cubic-in-spin as the linear momentum can no longer be considered independent of the spin:

\[ p_\mu = -\frac{\partial L}{\partial u^\mu} = m \frac{u_\mu}{\sqrt{u^2}} + \mathcal{O}(RS^2). \]

(4.1)

The correction to the linear momentum which was already required for the NLO cubic-in-spin sector is given by [4]:

\[ \Delta p_\kappa [S^2] \equiv p_\kappa [S^2] - \bar{p}_\kappa \simeq \frac{C_{ES^2}}{2m} S^\mu S^\nu \left( \frac{2}{u^3} R_{\mu\nu\kappa} u^\alpha - \frac{1}{u^3} R_{\mu\nu\beta} u^\alpha u^\beta u_\kappa \right), \]

(4.2)

where \( \bar{p}_\kappa \equiv \frac{m}{u} u_\kappa \) is the leading approximation to the linear momentum. At this order it is clear from eq. (4.8) of [14] that the spin-induced multipole has no effect on the linear momentum. At the current NLO quartic-in-spin sector, we also have to consider the next correction to the linear momentum:

\[ \Delta p_\kappa [S^3] \equiv p_\kappa [S^3] - p_\kappa [S^2] \simeq \frac{C_{BS^3}}{12m^2} S^\mu S^\nu S^\lambda \left[ \frac{1}{u^3} \left( D_\lambda \epsilon_{\alpha \beta \gamma \kappa} R^{\alpha \beta}_{\delta \gamma} u^\delta + D_\lambda \epsilon_{\alpha \beta \gamma \mu} R^{\alpha \beta}_{\nu \gamma} u^\nu u_\kappa \right) \right. \]

\[ \left. - \frac{1}{u^3} \epsilon_{\alpha \beta \gamma \kappa} R^{\alpha \beta}_{\delta \gamma} u^\delta u_\gamma u^\nu u_\kappa \right]. \]

(4.3)
These corrections should be implemented first in the minimal coupling part of the spinning particle, which is recast in the form [14]:

\[ L_S = -\frac{1}{2} \hat{S}_{ab} \dot{\hat{\Omega}}^{ab}_{\text{flat}} - \frac{1}{2} \hat{S}_{abc} \dot{u}^c u^a - \frac{\dot{\hat{S}}_{ab} p^b}{p^2} \frac{Dp^a}{D\lambda}, \]  

(4.4)

with lowercase Latin indices for the locally flat frame, and where the Ricci rotation coefficients, \( \omega_{\mu}^{ab} \), are used. The new couplings arise from substituting in the linear momentum in the canonical gauge into the linear-in-spin couplings, and into the extra term that appears last in eq. (4.4), which as noted in section 2 stands for the Thomas precession and was related in [14] to the gauge of the rotational DOFs.

It is important to stress that the issue here is not about going from a covariant gauge to a ‘non-covariant’ gauge, rather it is about going from the spin-independent approximation of the linear momentum to its spin-dependent completion.

From eq. (4.4) we obtain the following terms that yield new higher-order in spin couplings [4, 14]:

\[ L_S \to S_3, S_4 = \omega_{ij}^\mu u^\mu \frac{\hat{S}_{ik}^j p^i}{p (p + p^0)} - \omega_{ij}^\mu u^\mu \frac{\hat{S}_{ij}^j p^i}{p (p + p^0)} + \frac{\hat{S}_{ij}^j p^i p^j}{p (p + p^0)}. \]  

(4.5)

where all the indices are in the locally flat frame, and the canonical gauge is applied. The first two terms in eq. (4.5) yield new two-graviton couplings, and the last term yields new one-graviton couplings with higher-order time derivatives. Plugging in the corrections to the linear momentum from eqs. (4.2), (4.3) in eq. (4.5) to linear order yields new worldline-graviton couplings that are cubic and quartic in the spin, respectively.

As for the new couplings that are cubic in the spin, these can be found in [4]. The new couplings that are quartic in the spin have the following Feynman rules

\[ \begin{align*}
\begin{array}{c}
\hline
\hline
\end{array}
\end{align*}
\int dt \frac{C_{(BS^3)}}{6m^5} S_i S_j \phi, i j k \left( 2 \left( \hat{S}^2 a^k - \hat{S} \cdot \hat{a} S_k \right) + \hat{S} \cdot \hat{S} v^k - \hat{S} \cdot \hat{v} S_k \right),
\end{align*} \]  

(4.6)

for the one-graviton coupling, and for a two-graviton coupling

\[ \begin{align*}
\begin{array}{c}
\hline
\hline
\end{array}
\end{align*}
\int dt \frac{C_{(BS^3)}}{3m^5} S_i S_j S^k S^l \left[ \phi, i j k \phi, l - \delta_{k l} \phi, i j m \phi, m \right],
\end{align*} \]  

(4.7)

where a gray rectangle mounted on an oval blob represents this new type of “composite” quartic-in-spin worldline couplings. Further KK field couplings of this type enter beyond the 5PN order. The Wilson coefficients in these new rules for quartic-in-spin couplings indicate that these cannot be identified with the elementary hexadecapole couplings.

The composite worldline couplings from [4] and the new quartic-in-spin couplings above give rise to four additional graphs in this sector, as shown in figure 5, which are similar to those in figure 2 (b1), (c1) and in figure 3 (a1), (b1). The graphs in figure 5 have the following values:

Fig. 5(a1) = \[ \frac{3 C_{(ES^2)}}{2 m^2_1} G \left[ 2 S_1^2 (\hat{S}_2 \cdot \hat{a}_1) + (\hat{S}_1 \cdot \hat{S}_1) (\hat{S}_2 \cdot \hat{v}_1) - (\hat{S}_1 \cdot \hat{S}_2) (\hat{S}_1 \cdot \hat{v}_1) \right] (\hat{S}_1 \cdot \hat{n}) \]  

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5 Quadratic-in-curvature worldline couplings

In this sector yet a new type of composite coupling, which is quartic in the spin and quadratic in the curvature, should be considered: this should arise from plugging the correction in eq. (4.2) back into the leading non-minimal coupling $L_{ES^2}$ in eq. (2.2), such that $L_{S^2\rightarrow S^4}$, and the resulting coupling will be preceded by the coefficient $(C_{ES^2})^2$. However, this coupling turns out to show up beyond the 5PN order, and is thus not relevant for this sector.

We are left then with the elementary coupling of the hexadecapole to the quadratic electric operator from eq. (2.9) as the single contribution to this sector that is quadratic in Riemann. The Feynman rule in eq. (2.9) gives rise to a single two-graviton exchange.
Figure 6. The Feynman graph of two-graviton exchange at the NLO quartic-in-spin interaction, which originates from the new quadratic-in-curvature operator with the hexadecapole and with a new Wilson coefficient.

graph, shown in figure 6, the value of which is given by

\[ \text{Fig. } 6 = \frac{C_1(\text{ES}^4) m_1^2 G^2}{24 m_1^4 r^6} \left[ S_1^4 - 6 S_1^2 (S_1 \cdot \vec{n})^2 + 9 (S_1 \cdot \vec{n})^4 \right], \quad (5.1) \]

Notice that this introduces a new Wilson coefficient that appears first in this sector.

6 Next-to-leading gravitational quartic-in-spin action

Putting together the graph values from sections 3, 4, and 5, where the summation also takes into account the exchange of particle labels 1 ↔ 2, we get the final effective action for this sector:

\[ L_{S_1}^{\text{NLO}} = L_{S_1 S_2}^{\text{NLO}} + L_{S_1 S_2}^{\text{NLO}} + L_{S_1}^{\text{NLO}} + (1 \leftrightarrow 2), \quad (6.1) \]

where we have:

\[ L_{S_1 S_2}^{\text{NLO}} = \frac{1}{2} C_1(\text{ES}^2) C_2(\text{ES}^2) \frac{G}{m_1 m_2} \left( \frac{3 L_{(1)}}{8 r^5} + \frac{3 L_{(2)}}{4 r^4} + \frac{L_{(3)}}{2 r^3} \right) - \frac{9}{2} C_1(\text{ES}^2) C_2(\text{ES}^2) \frac{G^2}{m_1 r^6} L_{(4)} + \frac{1}{2} C_1(\text{ES}^2) \frac{G^2}{m_1 r^6} L_{(5)} \quad (6.2) \]

with the following pieces:

\[
\begin{align*}
L_{(1)} &= +10 \left[ \vec{S}_1 \cdot \vec{v}_1 \right] \left[ \vec{S}_2 \cdot (\vec{v}_1 - \vec{v}_2) \right] \left[ \vec{S}_2 \cdot \vec{n} \right] \left[ \vec{S}_2 \cdot (\vec{v}_1 - \vec{v}_2) \right] \left[ \vec{S}_1 \cdot \vec{n} \right] \\
&- 4 \left[ \vec{S}_1 \cdot (\vec{v}_1 - \vec{v}_2) \right] \left[ \vec{S}_2 \cdot (\vec{v}_1 - \vec{v}_2) \right] \left[ \vec{S}_1 \cdot \vec{S}_2 \right] \\
&- 5 (S_1 \cdot \vec{n}) (S_2 \cdot \vec{n}) \\
&+ 4 S_1^2 \left[ \vec{S}_2 \cdot \vec{v}_1 \right]^2 - 2 S_1^2 \left[ \vec{S}_2 \cdot (\vec{v}_1 + \vec{v}_2) \right] \left[ \vec{S}_2 \cdot \vec{v}_2 \right] \\
&+ 4 \left( \vec{S}_1 \cdot \vec{v}_2 \right)^2 S_2^2 - 2 \left[ \vec{S}_1 \cdot (\vec{v}_1 + \vec{v}_2) \right] \left( \vec{S}_1 \cdot \vec{v}_1 \right) S_2^2 \\
&+ S_1^2 S_2^2 \left[ 5 v_1^2 + 5 v_2^2 - 11 (v_1 \cdot v_2) - 10 (v_1 \cdot \vec{n})^2 - 10 (v_2 \cdot \vec{n})^2 + 15 (v_1 \cdot \vec{n}) (v_2 \cdot \vec{n}) \right] \\
&+ 10 S_1^2 \left[ \vec{S}_2 \cdot \vec{v}_2 \right] \left[ \vec{S}_2 \cdot \vec{n} \right] \left[ (v_1 + \vec{v}_2) \cdot \vec{n} \right] - 10 S_1^2 \left[ \vec{S}_2 \cdot \vec{v}_1 \right] \left[ \vec{S}_2 \cdot \vec{n} \right] \left[ 4 (v_1 \cdot \vec{n}) - (v_2 \cdot \vec{n}) \right] \\
&+ 10 \left( \vec{S}_1 \cdot \vec{v}_1 \right) \left( \vec{S}_1 \cdot \vec{n} \right) S_2^2 \left[ (v_1 + \vec{v}_2) \cdot \vec{n} \right] - 10 S_2^2 \left[ \vec{S}_2 \cdot \vec{v}_2 \right] \left( \vec{S}_1 \cdot \vec{n} \right) \left[ 4 (v_2 \cdot \vec{n}) - (v_1 \cdot \vec{n}) \right] \\
&- 5 (S_1 \cdot \vec{n})^2 S_2^2 \left[ 3 v_1^2 + 3 v_2^2 - 14 (v_2 \cdot \vec{n})^2 - 9 (v_1 \cdot \vec{n}) (v_2 \cdot \vec{n}) \right] \\
&- 5 S_1^2 \left( \vec{S}_2 \cdot \vec{n} \right)^2 \left[ 3 v_1^2 + 3 v_2^2 - 14 (v_1 \cdot \vec{n})^2 - 9 (v_1 \cdot \vec{n}) (v_2 \cdot \vec{n}) \right]
\end{align*}
\]
\[ L_{(2)} = -2(\mathbf{S}_1 \cdot \mathbf{S}_2) (\mathbf{S}_1 \cdot \mathbf{n}) [2(\mathbf{S}_2 \cdot \mathbf{a}_2) + \dot{\mathbf{S}}_2 \cdot (2\mathbf{v}_2 - 3\mathbf{v}_1) - 5(\dot{\mathbf{S}}_1 \cdot \mathbf{n})(\mathbf{v}_1 \cdot \mathbf{n})] \\
+ 2(\mathbf{S}_1 \cdot \mathbf{S}_2) (\mathbf{S}_2 \cdot \mathbf{n}) [2(\mathbf{S}_1 \cdot \mathbf{a}_1) + \dot{\mathbf{S}}_1 \cdot (2\mathbf{v}_1 - 3\mathbf{v}_2) - 5(\dot{\mathbf{S}}_1 \cdot \mathbf{n})(\mathbf{v}_2 \cdot \mathbf{n})] \\
- 2(\mathbf{S}_1 \cdot \mathbf{S}_2) [(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{v}_1 \cdot \mathbf{n}) - (\dot{\mathbf{S}}_1 \cdot \mathbf{S}_2)(\mathbf{v}_2 \cdot \mathbf{n}) + (\dot{\mathbf{S}}_1 \cdot \mathbf{v}_1)(\mathbf{S}_2 \cdot \mathbf{n}) - (\dot{\mathbf{S}}_2 \cdot \mathbf{v}_2)(\mathbf{S}_1 \cdot \mathbf{n})] \\
+ 2(\dot{\mathbf{S}}_1 \cdot \mathbf{S}_2)(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{v}_2) - 2(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_2 \cdot \mathbf{n})(\mathbf{S}_1 \cdot \mathbf{v}_1) \\
- 10(\dot{\mathbf{S}}_1 \cdot \mathbf{S}_2)(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n})(\mathbf{v}_2 \cdot \mathbf{n}) + 10(\dot{\mathbf{S}}_1 \cdot \mathbf{S}_2)(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n})(\mathbf{v}_1 \cdot \mathbf{n}) \\
- 2(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_1 \cdot \mathbf{n})[\mathbf{S}_2 \cdot (2\mathbf{v}_2 - 3\mathbf{v}_1)] + 2(\dot{\mathbf{S}}_1 \cdot \mathbf{S}_2)(\mathbf{S}_2 \cdot \mathbf{n})[\mathbf{S}_1 \cdot (2\mathbf{v}_1 - 3\mathbf{v}_2)] \\
- 2[S_1^2 - 5(S_1 \cdot \mathbf{n})^2][(\dot{\mathbf{S}}_2 \cdot \mathbf{a}_2)(\mathbf{S}_2 \cdot \mathbf{n}) + (\dot{\mathbf{S}}_2 \cdot \mathbf{v}_2)(\mathbf{S}_2 \cdot \mathbf{n}) + (\mathbf{S}_2 \cdot \mathbf{v}_2)(\dot{\mathbf{S}}_2 \cdot \mathbf{n})] \\
+ 2[S_2^2 - 5(S_2 \cdot \mathbf{n})^2][(\dot{\mathbf{S}}_1 \cdot \mathbf{a}_1)(\mathbf{S}_1 \cdot \mathbf{n}) + (\dot{\mathbf{S}}_1 \cdot \mathbf{v}_1)(\mathbf{S}_1 \cdot \mathbf{n}) + (\mathbf{S}_1 \cdot \mathbf{v}_1)(\dot{\mathbf{S}}_1 \cdot \mathbf{n})] \\
- 5[S_2^2 - 3(S_2 \cdot \mathbf{n})^2][(\mathbf{S}_1 \cdot \mathbf{v}_2)(\mathbf{S}_1 \cdot \mathbf{n}) + (\mathbf{S}_2 \cdot \mathbf{v}_1)(\dot{\mathbf{S}}_2 \cdot \mathbf{n})] \\
+ 5[S_1^2 - 3(S_1 \cdot \mathbf{n})^2][(\dot{\mathbf{S}}_2 \cdot \mathbf{v}_1)(\mathbf{S}_2 \cdot \mathbf{n}) + (\mathbf{S}_2 \cdot \mathbf{v}_1)(\dot{\mathbf{S}}_2 \cdot \mathbf{n})] \\
+ 5(\mathbf{S}_1 \cdot \mathbf{n})(\dot{\mathbf{S}}_1 \cdot \mathbf{n})[S_2^2 + 7(S_2 \cdot \mathbf{n})^2]\mathbf{(v}_2 \cdot \mathbf{n}) \\
- 5(\mathbf{S}_1 \cdot \mathbf{n})(\dot{\mathbf{S}}_2 \cdot \mathbf{n})[S_1^2 + 7(S_1 \cdot \mathbf{n})^2]\mathbf{(v}_1 \cdot \mathbf{n}) \\
+ 8(\mathbf{S}_1 \cdot \mathbf{v}_2)(\mathbf{S}_1 \cdot \mathbf{n})(\dot{\mathbf{S}}_2 \cdot \mathbf{S}_2) - 8(\mathbf{S}_2 \cdot \mathbf{v}_1)(\dot{\mathbf{S}}_2 \cdot \mathbf{S}_1)(\dot{\mathbf{S}}_1 \cdot \mathbf{S}_1) \\
- 2(\mathbf{S}_1 \cdot \mathbf{v}_1)(\mathbf{S}_1 \cdot \mathbf{n})[7(\dot{\mathbf{S}}_2 \cdot \mathbf{S}_2) - 5(\dot{\mathbf{S}}_2 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n})] \\
+ 2(\mathbf{S}_2 \cdot \mathbf{v}_2)(\mathbf{S}_2 \cdot \mathbf{n})[7(\dot{\mathbf{S}}_1 \cdot \mathbf{S}_1) - 5(\dot{\mathbf{S}}_1 \cdot \mathbf{n})(\mathbf{S}_1 \cdot \mathbf{n})] \\
+ 5(\mathbf{S}_1 \cdot \mathbf{S}_1)[S_2^2 - 7(S_2 \cdot \mathbf{n})^2]\mathbf{(v}_2 \cdot \mathbf{n}) - 4(\mathbf{S}_1 \cdot \mathbf{S}_1)[S_1^2 - 5(S_1 \cdot \mathbf{n})^2]\mathbf{(v}_1 \cdot \mathbf{n}) \\
- 5[S_1^2 - 7(S_1 \cdot \mathbf{n})^2]\mathbf{(S}_2 \cdot \mathbf{S}_2)(\mathbf{v}_1 \cdot \mathbf{n}) + 4[S_2^2 - 5(S_2 \cdot \mathbf{n})^2]\mathbf{(S}_2 \cdot \mathbf{S}_2)(\mathbf{v}_2 \cdot \mathbf{n}). \quad (6.4) \]

\[ L_{(3)} = +3(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n}) - 3(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_1 \cdot \mathbf{S}_2) + 3(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n})(\mathbf{S}_1 \cdot \mathbf{S}_2) \\
+ 13(\mathbf{S}_1 \cdot \mathbf{S}_1)(\mathbf{S}_2 \cdot \mathbf{S}_2) - 3(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_1 \cdot \mathbf{S}_2) + 15(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n})(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n}) \\
- 21(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{S}_2) - 21(\mathbf{S}_1 \cdot \mathbf{S}_1)(\mathbf{S}_2 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n}) \\
+ 3(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n}) + 3(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_1 \cdot \mathbf{n})(\mathbf{S}_2 \cdot \mathbf{n}) \\
+ 2[S_1^2 + (\mathbf{S}_1 \cdot \mathbf{S}_1)][S_2^2 - 3(S_2 \cdot \mathbf{n})^2] + 2[S_2^2 - 3(S_2 \cdot \mathbf{n})^2][S_2^2 + (\mathbf{S}_2 \cdot \mathbf{S}_2)], \quad (6.5) \]
\[ L_{(4)} = + S_1^2 S_2^2 - 4 S_1^2 (\vec S_2 \cdot \vec n) - 4 S_2^2 (\vec S_1 \cdot \vec n)^2 + (\vec S_1 \cdot \vec S_2)^2 \]
\[-12 (\vec S_1 \cdot \vec S_2) (\vec S_1 \cdot \vec n) (\vec S_2 \cdot \vec n) + 24 (\vec S_1 \cdot \vec n)^2 (\vec S_2 \cdot \vec n)^2, \] (6.6)

\[ L_{(5)} = + 30 (\vec S_1 \cdot \vec n) (\vec S_1 \cdot \vec S_2) (\vec S_2 \cdot \vec n) - 57 (\vec S_1 \cdot \vec n)^2 (\vec S_2 \cdot \vec n)^2 \]
\[-3 (\vec S_1 \cdot \vec S_2)^2 + 9 S_1^2 (\vec S_2 \cdot \vec n)^2 + S_1^2 S_2^2, \] (6.7)

and:

\[ L_{\text{NLO}}^{\text{S}_1 \text{S}_2} = C_{1(BS^3)} \frac{G}{m_1^4} \left( \frac{L_{[1]}}{4r^5} + \frac{L_{[2]}}{4r^4} + \frac{L_{[3]}}{12r^3} \right) \]
\[+ C_{1(BS^3)} \frac{G^2}{m_1^6 r^6} L_{[4]} + C_{1(BS^3)} \frac{m_2 G^2}{m_1^4 r^6} L_{[5]} + C_{1(ES^2)} \frac{G^2}{m_1^6 r^6} L_{[6]} \]
\[+ \frac{3}{2} C_{1(ES^2)} \frac{G}{m_1^4 r^4} L_{[7]} + 3 C_{1(ES^2)} \frac{m_2 G^2}{m_1^4 r^6} L_{[8]}, \] (6.8)

with the pieces:

\[ L_{[1]} = - 2 (\vec S_1 \cdot \vec S_2) (\vec S_1 \cdot \vec v_1)^2 + 2 (\vec S_1 \cdot \vec S_2) (\vec S_1 \cdot \vec v_1) (\vec S_1 \cdot \vec v_2) \]
\[-30 (\vec S_1 \cdot \vec S_2) (\vec S_1 \cdot \vec v_1) (\vec S_1 \cdot \vec n) [(\vec v_1 - \vec v_2) \cdot \vec n] \]
\[+ 10 (\vec S_1 \cdot \vec S_2) (\vec S_1 \cdot \vec v_2) (\vec S_1 \cdot \vec n) [(\vec v_1 - 4\vec v_2) \cdot \vec n] \]
\[+ (\vec S_1 \cdot \vec S_2) \left[ 13(v_1^2 + v_2^2) - 29(\vec v_1 \cdot \vec v_2) - 20(\vec v_1 \cdot \vec n)^2 - 20(\vec v_2 \cdot \vec n)^2 + 25(\vec v_1 \cdot \vec n)(\vec v_2 \cdot \vec n) \right] \]
\[-5 (\vec S_1 \cdot \vec n)^2 \left[ 13(v_1^2 + v_2^2) - 29(\vec v_1 \cdot \vec v_2) - 28(\vec v_1 \cdot \vec n)^2 - 28(\vec v_2 \cdot \vec n)^2 + 35(\vec v_1 \cdot \vec n)(\vec v_2 \cdot \vec n) \right] \]
\[-15S_1^2 - 7(\vec S_1 \cdot \vec n)^2 \left( \vec S_1 \cdot \vec v_1 \right) \left( \vec S_2 \cdot \vec v_1 \right) [(4\vec v_1 - 5\vec v_2) \cdot \vec n] \]
\[-15S_2^2 - 7(\vec S_1 \cdot \vec n)^2 \left( \vec S_2 \cdot \vec v_1 \right) \left( \vec S_2 \cdot \vec v_2 \right) [(\vec v_1 - \vec v_2) \cdot \vec n] \]
\[-50 (\vec S_1 \cdot \vec v_1)^2 (\vec S_1 \cdot \vec n) (\vec S_2 \cdot \vec n) + 40 (\vec S_1 \cdot \vec v_2)^2 (\vec S_1 \cdot \vec n) (\vec S_2 \cdot \vec n) \]
\[-90 (\vec S_1 \cdot \vec v_1) (\vec S_1 \cdot \vec v_2) (\vec S_1 \cdot \vec n) (\vec S_2 \cdot \vec n) \]
\[+ 25 [S_1^2 - 7(\vec S_1 \cdot \vec n)^2] (\vec S_1 \cdot \vec v_1) (\vec S_2 \cdot \vec n) [(\vec v_1 - \vec v_2) \cdot \vec n] \]
\[- [S_1^2 - 5(\vec S_1 \cdot \vec n)^2] (\vec S_1 \cdot \vec v_1) \left[ \vec S_2 \cdot (17\vec v_1 - 11\vec v_2) \right] \]
\[+ [S_1^2 - 5(\vec S_1 \cdot \vec n)^2] (\vec S_1 \cdot \vec v_2) \left[ \vec S_2 \cdot (19\vec v_1 - 13\vec v_2) \right] \]
\[-5 [S_1^2 - 7(\vec S_1 \cdot \vec n)^2] (\vec S_1 \cdot \vec v_2) (\vec S_2 \cdot \vec n) [(7\vec v_1 - 4\vec v_2) \cdot \vec n], \] (6.9)
\[ L_{[3]} = \left[ \mathbf{\hat{S}} \cdot \mathbf{\hat{v}} \right] \mathbf{\hat{S}} \cdot \mathbf{\hat{a}} + \left( \mathbf{\hat{S}} \cdot \mathbf{\hat{a}} \right) \left( \mathbf{\hat{S}} \cdot \mathbf{\hat{v}} \right) - \mathbf{\hat{S}} \cdot \mathbf{\hat{a}} \mathbf{\hat{S}} \cdot \mathbf{\hat{v}} - \mathbf{\hat{S}} \cdot \mathbf{\hat{a}} \mathbf{\hat{S}} \cdot \mathbf{\hat{v}} - \mathbf{\hat{S}} \cdot \mathbf{\hat{a}} \mathbf{\hat{S}} \cdot \mathbf{\hat{v}}, \quad (6.11) \]

\[ L_{[4]} = -3(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \left[ \mathbf{\hat{S}} \cdot \mathbf{\hat{S}} - 5(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \right] + 2(9\mathbf{\hat{S}} \cdot \mathbf{\hat{S}} - 20(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \right] (\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}), \quad (6.12) \]

\[ L_{[5]} = -9(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \left[ \mathbf{\hat{S}} \cdot \mathbf{\hat{S}} - 5(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \right] + 51(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \mathbf{\hat{S}} \cdot \mathbf{\hat{S}} - 115(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \mathbf{\hat{S}} \cdot \mathbf{\hat{S}}, \quad (6.13) \]

\[ L_{[6]} = -(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \left[ \mathbf{\hat{S}} \cdot \mathbf{\hat{S}} - 9(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \right] + 6(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \left[ \mathbf{\hat{S}} \cdot \mathbf{\hat{S}} - 4(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \right] (\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}), \quad (6.14) \]

\[ L_{[7]} = \left[ 2\mathbf{\hat{S}} \cdot \mathbf{\hat{S}} - 9(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \right] (\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) - 2(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \mathbf{\hat{S}} \cdot \mathbf{\hat{S}} - 5(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \mathbf{\hat{S}} \cdot \mathbf{\hat{S}} \right] (\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}), \quad (6.15) \]

\[ L_{[8]} = S_{[4]}^2 \left[ \mathbf{\hat{S}} \cdot \mathbf{\hat{S}} - 4(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \right] (\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) - (\mathbf{\hat{S}} \cdot \mathbf{\hat{S}})^2 \left[ 2(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) - 5(\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}) \right] (\mathbf{\hat{S}} \cdot \mathbf{\hat{S}}), \quad (6.16) \]

and finally:

\[ L_{NLO}^{S_1} = C_{1(ES^4)} \frac{G m_2}{m_1^2} \left( \frac{L_{[1]}^{(1)}}{16\pi^2} + \frac{L_{[2]}^{(2)}}{32\pi^4} + \frac{L_{[3]}^{(3)}}{12\pi^2} \right) - \frac{1}{8} C_{1(ES^4)} \frac{G^2 m_2}{r_0 m_1^2} L_{[4]} - \frac{3}{8} C_{1(ES^4)} \frac{G^2 m_2^2}{r_0 m_1^2} L_{[5]} \]
\begin{align*}
\frac{G^2 m_2}{r_6 m_1^2} L_{(6)} & \quad - \frac{1}{8} C_1^{(ES^2)} \frac{G^2 m_2}{r_6 m_1^2} L_{(7)} \\
+ \frac{1}{2} C_1^{(BS^3)} \frac{G m_2}{r_4 m_1^2} L_{(8)} & \quad + C_1^{(BS^3)} \frac{G^2 m_2}{r_6 m_1^2} L_{(9)} + \frac{C_1^{(ES^4)}}{24} \frac{G^2 m_2}{r_6 m_1^2} L_{(10)},
\end{align*}

with the pieces:

\begin{align*}
L_{(1)} &= + 12 (\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_1 \cdot \vec{v}_2) \left[ S^2_1 - 5 (\vec{S}_1 \cdot \vec{n})^2 \right] - 12 (\vec{S}_1 \cdot \vec{v}_1) (\vec{S}_1 \cdot \vec{n})^2 \\
& \quad + 20 (\vec{S}_1 \cdot \vec{n}) [3 S^2_1 - 7 (\vec{S}_1 \cdot \vec{n})^2] [(\vec{S}_1 \cdot \vec{v}_1) (\vec{v}_1 - \vec{v}_2) \cdot \vec{n} - (\vec{S}_1 \cdot \vec{v}_2) (\vec{v}_1 \cdot \vec{n})] \\
& \quad + [3 S^4_1 - 30 S^2_1 (\vec{S}_1 \cdot \vec{n})^2 + 35 (\vec{S}_1 \cdot \vec{n})^4] [3 v^2_1 + 3 v^2_2 - 7 (\vec{v}_1 \cdot \vec{v}_2)] \\
& \quad - 15 [S^4_1 - 14 S^2_1 (\vec{S}_1 \cdot \vec{n})^2 + 21 (\vec{S}_1 \cdot \vec{n})^4] (\vec{v}_1 \cdot \vec{n}) (\vec{v}_2 \cdot \vec{n}),
\end{align*}

\begin{align*}
L_{(2)} &= - 6 (\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) \left[ S^2_1 - 5 (\vec{S}_1 \cdot \vec{n})^2 \right] - 10 (\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{n}) [3 S^2_1 - 7 (\vec{S}_1 \cdot \vec{n})^2] (\vec{v}_2 \cdot \vec{n}) \\
& \quad - 12 (\vec{S}_1 \cdot \vec{S}_1) (\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{v}_2) + 6 (\vec{S}_1 \cdot \vec{S}_1) [S^2_1 - 5 (\vec{S}_1 \cdot \vec{n})^2] (\vec{v}_2 \cdot \vec{n}) \\
& \quad - 2 (\vec{S}_1 \cdot \vec{v}_2) (\vec{S}_1 \cdot \vec{n}) [3 S^2_1 - 5 (\vec{S}_1 \cdot \vec{n})^2] - S^2_1 [S^2_1 - 5 (\vec{S}_1 \cdot \vec{n})^2] (\vec{a}_1 \cdot \vec{n}) \\
& \quad - 2 (\vec{S}_1 \cdot \vec{a}_1) (\vec{S}_1 \cdot \vec{n}) S^2_1,
\end{align*}

\begin{align*}
L_{(3)} &= 3 (\vec{S}_1 \cdot \vec{n}) \vec{S}_1 \cdot \vec{n} S^2_1 - \left[ S^2_1 + (\vec{S}_1 \cdot \vec{S}_1) \right] [2 S^2_1 - 3 (\vec{S}_1 \cdot \vec{n})^2] \\
& \quad + 12 (\vec{S}_1 \cdot \vec{n}) (\vec{S}_1 \cdot \vec{S}_1) (\vec{S}_1 \cdot \vec{n}) + 3 (\vec{S}_1 \cdot \vec{n})^2 S^2_1 - 4 (\vec{S}_1 \cdot \vec{S}_1)^2,
\end{align*}

\begin{align*}
L_{(4)} &= 3 S^4_1 - 30 S^2_1 (\vec{S}_1 \cdot \vec{n})^2 + 35 (\vec{S}_1 \cdot \vec{n})^4,
\end{align*}

\begin{align*}
L_{(5)} &= 95 (\vec{S}_1 \cdot \vec{n})^4 - 81 S^2_1 (\vec{S}_1 \cdot \vec{n})^2 + 8 S^4_1,
\end{align*}

\begin{align*}
L_{(6)} &= (\vec{S}_1 \cdot \vec{n})^2 [3 S^2_1 - 5 (\vec{S}_1 \cdot \vec{n})^2],
\end{align*}

\begin{align*}
L_{(7)} &= [S^2_1 - 3 (\vec{S}_1 \cdot \vec{n})^2]^2,
\end{align*}

\begin{align*}
L_{(8)} &= \left[ 2 S^2_1 (\vec{a}_1 \cdot \vec{n}) - 2 (\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{a}_1) + (\vec{S}_1 \cdot \vec{S}_1)(\vec{v}_1 \cdot \vec{n}) - (\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{v}_1) \right] \\
& \quad \times \left[ S^2_1 - 5 (\vec{S}_1 \cdot \vec{n})^2 \right],
\end{align*}

\begin{align*}
L_{(9)} &= S^4_1 - 6 S^2_1 (\vec{S}_1 \cdot \vec{n})^2 + 5 (\vec{S}_1 \cdot \vec{n})^4,
\end{align*}

\begin{align*}
L_{(10)} &= S^4_1 - 6 S^2_1 (\vec{S}_1 \cdot \vec{n})^2 + 9 (\vec{S}_1 \cdot \vec{n})^4.
\end{align*}
The result has been ordered according to the Wilson coefficients, mass ratios, and the total number/order of higher-order time derivatives. The higher-order time derivatives of both the velocity and the spin are to be removed following the procedure that was shown in [49] via variable redefinitions. Then the lengthy result will reduce to an ordinary action, and will significantly shrink. But before we handle the higher-order time derivatives in this sector, we should take into account additional contributions to this sector from lower-order redefinitions of variables made at lower order sectors, as detailed in section 6 of [14]. In a forthcoming publication we will provide the reduced effective action along with other important quantities and observables from this sector.

7 Conclusions

In this work we derived for the first time the complete NLO gravitational quartic-in-spin interaction of generic compact binaries. The derivation built on the EFT for gravitating spinning objects in [14], and its recent extensions in [4, 10], in which new type of worldline couplings should be considered, and the effective action should be extended to quadratic order in the curvature. This sector enters at the 5PN order for maximally-spinning compact objects, and together with [10] and its upcoming completion, provides all finite size spin effects up to this PN order.

Following [4] a careful intricate analysis was required to recover new type of composite worldline couplings that emerge due the fact the linear momentum can no longer be considered independent of the spin at these nonlinear higher-spin sectors. These new worldline couplings that contribute here are of cubic and quartic order in the spin. It is interesting whether these new couplings can be thought of the classical manifestation of the total spin of a composite particle.

The analysis in this work shows clearly that the spin-dependent correction to the linear momentum will have to be taken into account at quadratic order as of the NLO quintic-in-spin level, that corresponds to the quantum spin $s = 5/2$, which may render such derivation impossibly complex, if not ill-defined. This connection between the classical and the quantum levels will clarified in a forthcoming publication.

At the loop computational scale, there was no new special difficulty in this sector, and in fact it demonstrated once again that even-in-spin sectors are easier to handle compared to odd-in-spin ones, a trend that is clear from inspecting table 1. Yet, a new conceptual feature in this sector was a first relevant operator that is quadratic in the curvature, and entails a new Wilson coefficient.

In order to find out what is the effect of the contributions from section 4, we will provide the reduced action, EOMs, Hamiltonian, and the consequent gauge-invariant observables in a forthcoming publication.

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