Compton Scattering on Black Body Photons

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Abstract

We examine Compton scattering of electrons on black body photons in the case where the electrons are highly relativistic, but the center of mass energy is small in comparison with the electron mass. We derive the partial lifetime of electrons in the LEP accelerator due to this form of scattering in the vacuum beam pipe and compare it with previous results.
I. INTRODUCTION

Vacuum beam pipes of modern particle accelerators closely approach the ideal limit of a pipe completely devoid of gas molecules. However, even an ideal vacuum beam pipe in a laboratory at room temperature is filled with photons having an energy distribution given by Planck’s law. Some time ago, Telnov\(^1\) noted that the scattering of electrons on these black body photons could be a significant mechanism for the depletion of the beam. This scattering of the electrons in the Large Electron Positron collider at CERN (LEP) on the black body radiation has been detected\(^2-4\). There is a long history of theoretical investigations on the scattering of high-energy electrons on black body photons, centering around the role this plays as an energy loss mechanism for cosmic rays, which is summarized by Blumenthal and Gould\(^5\). More recently, Domenico\(^6\) and Burkhardt\(^7\) have considered this effect for the LEP experiments and the consequent limit on the beam lifetime by using numerical Monte Carlo methods. In view of the intrinsic interest of the problem of high-energy electron scattering on black body photons, we believe that it is worthwhile to present here a simplified calculation of the effect. We compute the total cross section analytically. The cross section as a function of the energy loss — which is the important quantity for the beam lifetime — is also done analytically except for a final straightforward numerical integration. Our calculations use relativistic invariant methods, and are thus of some pedagogical interest.

Analytic computations can be performed because the problem involves two small dimensionless parameters. On the one hand, the electron of mass \(m\) has a very large laboratory energy \(E\) and it is ultrarelativistic, as characterized by the parameter \(m^2/E^2\). (We use natural units in which the velocity of light \(c = 1\), Planck’s constant \(\hbar = 1\), and Boltzmann’s constant \(k = 1\), so that temperature is measured in energy units.) At LEP, \(m^2/E^2 \approx 10^{-10}\). We shall neglect terms of order \(m^2/E^2\). On the other hand, the temperature \(T\) of the black body radiation is very small in comparison with the electron mass \(m\). Thus, although the electron is ultrarelativistic, the energy in the center of mass of the electron-photon system is still small in comparison with the electron mass. The head-on collision of a photon of
energy $T$ with an relativistic electron of energy $E$ produces, with the neglect of the electron mass, the squared center-of-mass energy $4ET$. We shall use the dimensionless parameter (which gives an average value)

$$s = \frac{2ET}{m^2}. \quad (1.1)$$

At LEP, $s \approx 10^{-2}$. Thus it is a good approximation to use the non-relativistic limit in the center of mass, with the relativistic Compton cross section replaced by its constant, non-relativistic Thomson limit. To assess this approximation, we shall also compute the first corrections in $s$.

In the next section we use simple relativistic techniques to compute the total cross section for the scattering of an ultrarelativistic electron on the Planck distribution of black body photons. The third section describes the more detailed calculation needed for the cross section in which the electron loses an energy greater than $\Delta E$. If the energy loss $\Delta E$ in an electron-photon collision is too large, the electron’s motion falls outside of the acceptance parameters of the machine. At LEP this happens when $\Delta E/E$ is greater than about 1%. As we shall see, this means that even if the beam were in a perfect vacuum, it would decay with a half life of about two days. The vacuum in the LEP accelerator is so good that the beam scattering of the black body photons is, in fact, the primary mechanism for beam loss when the machine is run with a single beam. Scattering on the residual gas in the beam pipe gives a considerably longer half life of about six days. When the machine is run in the usual mode with two beams for $e^+ e^-$ collision experiments, beam-beam collisions reduce the beam half life to about 14 hours.

II. TOTAL SCATTERING RATE

In the general case of an electron scattering off some photon distribution, the scattering rate $\Gamma$ in the electron’s rest frame may be computed using the formula

$$\Gamma = \int \frac{(d^3k)}{(2\pi)^3} \tilde{f}(k)\sigma(k), \quad (2.1)$$
where \( f(\vec{k}) \) is the photon phase-space density [with the normalization defined such that \( \vec{j}^0 \) in Eq. (2.7) is the photon number density] as a function of the photon momentum and \( \sigma(\vec{k}) \) is the scattering cross section, which is similarly a function of the photon momentum. Here all quantities are evaluated in the electron’s rest frame as indicated by the over bar. This scattering rate may be viewed as a time derivative

\[
\bar{\Gamma} = \frac{dn}{d\tau}, \quad (2.2)
\]

where \( \tau \) is the time in the electron’s rest frame. Since numbers are Lorentz invariant and \( \tau \) may be defined to be the invariant proper time of the electron, the rate \( dn/d\tau \) is, in fact, a Lorentz invariant. Thus, if the integral on the right-hand side of the rate formula (2.1) is written in a Lorentz invariant manner, we can immediately evaluate the rate in the laboratory frame. In the lab frame, the electron moves with four momenta

\[
p^\mu = m \frac{dz^\mu}{d\tau}, \quad (2.3)
\]

where \( z^\mu(\tau) \) is the world line of this particle, its space-time position as a function of proper time, and \( m \) is the electron mass. This gives the familiar time-dilation formula

\[
\frac{dt}{d\tau} = \frac{dz^0}{d\tau} = \frac{p^0}{m} = \frac{E}{m}, \quad (2.4)
\]

where \( E \) is the electron’s total energy. Thus the scattering rate \( \Gamma \) in the lab frame may be easily evaluated using

\[
\Gamma = \frac{dn}{dt} = m \frac{dn}{E d\tau}. \quad (2.5)
\]

In the non-relativistic limit, \( \sigma \) may be replaced with the Thomson cross section, \( \sigma_T = \frac{8\pi r_0^2}{3} \), where \( r_0 = e^2/4\pi m \) is the classical electron radius. Since this is independent of \( \vec{k} \), the scattering rate may be rewritten as

\[
\bar{\Gamma}_0 = \sigma_T \vec{j}^0, \quad (2.6)
\]

where
\[ j^\mu = \int \frac{(d^3 \bar{k}) \bar{k}^\mu}{(2\pi)^3 k^0} \bar{f}(\bar{k}) \]  

(2.7)

is the photon number flux four vector. Since \( \bar{p}^\mu/m = (1, 0) \) in the electron’s rest frame, we may write this leading approximation, denoted with a 0 subscript, as

\[ \left( \frac{dn}{d\tau} \right)_0 = -\sigma_T \bar{p}^\mu \frac{\bar{p}_\mu}{m}, \]  

(2.8)

with the minus sign arising from our Lorentz metric convention in which the metric has the signature \((-,+,+,+\)). The result (2.8) is now in an invariant form which holds in any frame. With a thermal photon distribution in the lab frame,

\[ f(k) = \frac{2}{e^{\omega/T} - 1}, \]  

(2.9)

where \( \omega = k^0 \) is the photon energy, the photon number distribution is isotropic, and so only the number density component \( j^0 \) is nonvanishing. Thus, in the lab frame,

\[ \Gamma_0 = \frac{m}{E} \sigma_T j^0 \bar{p}^0 \frac{p^0}{m} = \sigma_T j^0. \]  

(2.10)

The lab photon number density obtained from integrating (2.7) with the distribution (2.9) is the familiar result

\[ j^0 = \frac{2\zeta(3)}{\pi^2} T^3, \]  

(2.11)

in which \( \zeta(3) = 1.202 \ldots \) is the Riemann zeta function. Thus, \( \Gamma_0 \) is given by

\[ \Gamma_0 = \frac{2\zeta(3)}{\pi^2} T^3 \sigma_T. \]  

(2.12)

The first order relativistic correction to this result is obtained with the use of the corrected cross section

\[ \sigma = \sigma_T \left( 1 + \frac{2p_k}{m^2} \right). \]  

(2.13)

Note that the product \( kp = k^\mu p_\mu \) of the two four-momenta is negative with our metric. Because \( \sigma \) is no longer independent of \( k \), the corresponding form of Eq. (2.10) is
\[
\Gamma_1 = \frac{m}{E} \sigma_T \int \frac{(d^3k)}{(2\pi)^3} \frac{-kp}{k^0 m} f(k) \left( 1 + \frac{2pk}{m^2} \right) \\
= -\frac{1}{E} \sigma_T \left( j^\mu p_\mu + \frac{2}{m^2} T^{\mu\nu} p_\mu p_\nu \right),
\]

where

\[
T^{\mu\nu} = \int \frac{(d^3k)}{(2\pi)^3} \frac{k^\mu k^\nu}{k^0} f(k)
\]
is the stress-energy tensor of the photons. Due to the isotropy of the thermal photons in the lab frame, \(T^{\mu\nu}\) has no off diagonal components, and it is also traceless because the photon is massless, \(k^\mu k_\mu = 0\). Therefore, in the lab frame,

\[
T^{\mu\nu} p_\mu p_\nu = \left( E^2 + \frac{1}{3} |p|^2 \right) T^{00} = \frac{4}{3} E^2 \left( 1 - \frac{m^2}{4E^2} \right) T^{00}.
\]

The \(m^2/E^2\) term is very small, and it may be neglected. Integrating over the photon distribution in Eq. (2.15) gives the well-known black body energy density

\[
T^{00} = \frac{6\zeta(4)}{\pi^2} T^4,
\]

where \(\zeta(4) = \pi^4/90 = 1.082\ldots\). This yields the corrected scattering rate

\[
\Gamma_1 = \frac{2\zeta(3)}{\pi^2} T^3 \sigma_T \left[ 1 - 4s \frac{\zeta(4)}{\zeta(3)} \right].
\]

For the temperature in the LEP beam pipe we take \(T = 291 \text{ K} = 0.0251 \text{ eV}\), which is about room temperature. This gives the leading rate \(\Gamma_0 = 9.98 \times 10^{-6} \text{ s}^{-1}\) corresponding to the mean life \(\tau_0 = 1/\Gamma_0 = 28 \text{ hr}\). A typical LEP beam energy \(E = 46.1 \text{ GeV}\) is just above half the \(Z^0\) mass — within the width, but on the high side of resonance curve. Together with the previous value of the temperature, this gives \(s = 0.00886\), and the first-order corrected rate \(\Gamma_1 = 9.66 \times 10^{-6} \text{ s}^{-1}\), which is about 3% smaller than the leading rate. This gives a mean life \(\tau_1 = 1/\Gamma_1 = 29 \text{ hr}\).

**III. RATE WITH ENERGY LOSS**

The calculation of the scattering rate in which the electron loses an energy greater than \(\Delta E\) is facilitated by going back to the basic formula that expresses the rate in terms of
Lorentz invariant phase space integrals, an energy-momentum conserving $\delta$ function, and the square of the scattering amplitude $|T|^2$. The total electron scattering rate as observed in the lab frame reads

$$\Gamma = \frac{1}{2E} \int \frac{(d^3k)}{(2\pi)^3} \frac{1}{2\omega} f(k) \int \frac{(d^3k')}{(2\pi)^3} \frac{1}{2\omega'} \int \frac{(d^3p')}{(2\pi)^3} \frac{1}{2E'} (2\pi)^4 \delta^{(4)}(k' + p' - k - p) |T|^2,$$

(3.1)

where $p$ and $p'$ are the initial and final electron four momenta, $k$ and $k'$ the initial and final photon four momenta, with $E = p^0$, $E' = p'^0$, $\omega = k^0$, $\omega' = k'^0$ the time components of these four vectors. Except for the initial factor of $1/2E$ which is the lab energy of the initial electron and which converts the invariant proper time into the lab time, the right-hand side of this expression is a Lorentz invariant. The problem proves to be greatly simplified if the integrals are evaluated in the rest frame of the electron, because Compton scattering of a photon on an electron at rest has a very simple nonrelativistic limit. This complicates the initial photon distribution, but, if we introduce a four vector $\beta^\mu$, whose time component in the lab frame is one over the temperature of the photon distribution and whose spatial components are zero in the lab frame, the distribution in an arbitrary frame still has the simple form

$$f(k) = \frac{2}{e^{-\beta^\mu k_\mu} - 1}.$$

(3.2)

From the definition of $\beta^\mu$, $-\beta^\mu \beta_\mu = 1/T^2$ and $-\beta^\mu p_\mu = E/T$, because multiplication by $\beta^\mu$ selects out the time component in the lab frame. In the electron rest frame, $\beta^\mu$ therefore takes on the value

$$\beta^\mu = \left( \frac{E}{Tm}, -\frac{p}{Tm} \right),$$

(3.3)

where $E$ and $p$ are taken in the lab frame. We have not yet taken into account the lower bound on the electron energy loss in the lab frame. Because multiplication by $\beta^\mu$ selects the time component in the lab frame, this limit may be instituted by the inclusion of an “energy loss” step function in the integrand of Eq. (3.1),

$$\theta \left( -\beta^\mu (p_\mu - p'_\mu) - \frac{\Delta E}{T} \right),$$

(3.4)
where the $1/T$ in the second term has been included to compensate for the factor of $1/T$ which the first term has picked up by being multiplied by $\beta^\mu$. Using the identity
\[
\int \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} = \int \frac{d^4p'}{(2\pi)^4} \frac{\theta(E'-m)}{2\pi} \delta(p'^2 + m^2),
\]
the final electron four momentum $p'$ may be integrated over to leave
\[
\Gamma(\Delta E) = \frac{1}{2E} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} f(k) \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2\omega'} \theta(\omega - \omega') 2\pi \delta(-2m\omega + 2m\omega' - 2kk')
\theta\left(-\beta k' - \left(\frac{\Delta E}{T} - \beta k\right)\right)|T|^2.
\]
(3.6)

We do the $k'$ integral in spherical coordinates and take the $z$ axis to be parallel to $k$, with $\theta$ the angle between these two vectors. The angle $\theta$ is the photon scattering angle in the electron rest frame, and
\[
- kk' = \omega\omega'(1 - \cos \theta).
\]
(3.7)
The $\delta$ function can now be solved for $\omega'$ to yield
\[
\delta(-2m\omega + 2m\omega' - 2kk') = \frac{1}{2(m + \omega(1 - \cos \theta))} \delta\left(\omega' - \frac{m\omega}{m + \omega(1 - \cos \theta)}\right),
\]
(3.8)
which requires that $\omega' < \omega$ and thus makes the $\theta(\omega - \omega')$ step function redundant. The scattered photon energy $\omega'$ given by the $\delta$ function is, of course, just the Compton energy.

To deal with the “energy loss” step function, we note that the Lorentz transformation from the lab frame to the initial electron rest frame turns the lab frame isotropic black body photon distribution into a very narrow pencil in the electron rest frame in which we are now working. Thus the initial photon distribution is sharply peaked about the average value
\[
\langle k\mu/\omega \rangle = \beta^\mu / \beta^0.
\]
(3.9)

Hence we can approximate
\[
- \beta k' \approx \beta^0 \omega'\langle -kk' \rangle = \beta^0 \omega'(1 - \cos \theta).
\]
(3.10)

To verify this and assess the order of accuracy, we define the average more precisely by
\[ \langle X \rangle = \frac{\int \left( \frac{d^3k}{(2\pi)^3} \right) \frac{1}{2\omega} f(-k\beta) X}{\int \left( \frac{d^3k}{(2\pi)^3} \right) \frac{1}{2\omega} f(-k\beta)} , \]  

(3.11)

Then, by virtue of the relativistic invariance of this definition,

\[ \langle k^\mu k^\nu \rangle = \left( \beta^\mu \beta^\nu - \frac{1}{4} \beta^2 g^{\mu\nu} \right) A(\beta^2) , \]  

(3.12)

since \( \beta^\lambda \) is the only four vector available and \( k^\mu k_\mu = 0 \). Remembering that \( \omega = k^0 \), this presents the squared fluctuation about the average as

\[ \left\langle \frac{(k^\mu - \beta^\mu \frac{\omega}{\beta^0}) (k^\nu - \beta^\nu \frac{\omega}{\beta^0})}{\langle \omega^2 \rangle} \right\rangle = B \left( g^{\mu\nu} - \frac{\beta^\nu}{\beta^0} g^{\mu0} - \frac{\beta^\mu}{\beta^0} g^{\nu0} + \frac{\beta^\mu \beta^\nu}{(\beta^0)^2} \right) , \]  

(3.13)

where

\[ B = \frac{-\beta^2}{4(\beta^0)^2 + \beta^2} \approx \frac{m^2}{4E^2} . \]  

(3.14)

Thus the deviations away from our approximation may be neglected because they involve the very small quantity \( m^2/E^2 \). Using this approximation for \( -k'\beta \) simplifies the “energy loss” step function to

\[ \theta \left( \frac{\omega'}{\beta^0 T(1 - \cos \theta)} - \frac{\Delta E}{\beta^0 E(1 - \cos \theta)} \right) = \theta \left( \frac{\omega'}{E' (1 - \cos \theta)} - \frac{\Delta E}{E'E(1 - \cos \theta)} \right) , \]  

(3.15)

where the \( -\beta k \) in the original step function has been neglected because it is much less than \( \Delta E/T \). Inserting the value of \( \omega' \) given by the energy-conserving \( \delta \) function \[ \text{into the step function gives} \]

\[ \theta \left( \frac{m \omega}{m + \omega(1 - \cos \theta)} - \frac{\Delta E}{E (1 - \cos \theta)} \right) = \theta \left( \frac{\omega - \Delta E}{E' (1 - \cos \theta)} \right) , \]  

(3.16)

where on the right hand side we have solved for \( \omega \) and defined

\[ E' = E - \Delta E , \]  

(3.17)

which is the maximum final electron energy in the lab frame.

We perform the \( k \) integral in spherical coordinates, with the polar angle \( \chi \) taken to be the angle between \( k \) and \( \beta \), so that
\[ -k\beta = \omega (\beta^0 - |\beta| \cos \chi). \] (3.18)

We rewrite the angular integral for \( k \) in terms of an integration over \( k\beta \) by noting the limits

\[ -k\beta < \omega (\beta^0 + |\beta|) \simeq 2\omega \beta^0 = \frac{2\omega E}{mT}, \] (3.19)

and

\[ -k\beta > \omega (\beta^0 - |\beta|) = \frac{\omega \beta^2}{\beta^0 + |\beta|} \simeq \frac{\omega m}{2TE}. \] (3.20)

Thus, with the neglect of order \( m^2/E^2 \) corrections, and remembering that \( |\beta| \simeq E/mT \),

\[ \int_{-1}^{1} d\cos \chi = \frac{mT}{\omega E} \int_{\omega /m/2TE}^{2\omega E/mT} d(-k\beta). \] (3.21)

We shall do the \(-k\beta\) integral last, due to its dependence on the initial photon distribution. In order to interchange the order of the \( \omega \) and \(-k\beta\) integrations, we note that the lower limit on \(-k\beta\),

\[ -k\beta > \frac{\omega m}{2TE}, \] (3.22)

gives the upper bound on \( \omega \),

\[ \omega < \frac{2TE(-k\beta)}{m} = s(-k\beta)m. \] (3.23)

The upper limit on \(-k\beta\),

\[ -k\beta < \frac{2\omega E}{mT}, \] (3.24)

gives the lower bound on \( \omega \),

\[ \omega > \frac{(-k\beta)mT}{2E} = s(-k\beta)m \frac{m^2}{4E^2}. \] (3.25)

In view of the extreme smallness of \( m^2/E^2 \), we may replace this lower limit by \( \omega = 0 \). Hence, switching the order of integration gives

\[ \int_{0}^{\infty} d\omega \int_{\omega m/2TE}^{2\omega E/mT} d(-k\beta) = \int_{0}^{\infty} d(-k\beta) \int_{0}^{s(-k\beta)m} d\omega. \] (3.26)
We perform this reversal of integrals, do the two trivial azimuthal integrals, and do the $\omega'$ integral using the $\delta$ function to obtain

$$
\Gamma(\Delta E) = \frac{m^2T}{16E^2(2\pi)^3} \int_0^\infty d(-k\beta)f(-k\beta) \int_{-1}^1 d\cos \theta \int_0^{s(-k\beta)m} d\omega \frac{\omega}{[m + \omega(1 - \cos \theta)]^2} |T|^2 \theta \left(\omega - \frac{\Delta E}{E'} \frac{m}{1 - \cos \theta}\right).
$$

(3.27)

To work out the integrals which appear here, it is convenient to first introduce the appropriate, dimensionless variables,

$$
x = -k\beta, \quad z = 1 - \cos \theta, \quad \nu = \frac{\omega}{s(-k\beta)m},
$$

(3.28)

and define

$$
u = \frac{\Delta E}{2sE'}.
$$

(3.29)

With this new notation, we have

$$
\Gamma(\Delta E) = \frac{T^3}{4m^2(2\pi)^3} \int_u^\infty dx x^2 f(x) \int_0^2 dz \int_0^{\nu} d\nu \frac{\nu}{(1 + s\nu xz)^2} |T|^2 \theta \left(\nu - \frac{2u}{xz}\right).
$$

(3.30)

The final step function provides the lower limit $\nu = 2u/xz$. This lower limit must not exceed the upper limit $\nu = 1$. Hence we must have condition $z > 2u/x$ on the $z$ integration. But again, this must not exceed the upper limit $z = 2$. Thus $x > u$, and imposing all these limits gives

$$
\Gamma(\Delta E) = \frac{T^3}{4m^2(2\pi)^3} \int_u^\infty dx x^2 f(x) \int_{2u/x}^{2u/xz} dz \int_{2u/xz}^{1} d\nu \frac{\nu}{(1 + s\nu xz)^2} |T|^2.
$$

(3.31)

This will be evaluated in the nonrelativistic limit, keeping first order corrections in $s$. The exact squared amplitude differs from its nonrelativistic limit

$$
|T|^2 = 2e^4(1 + \cos^2 \theta) = 2e^4(2 - 2z + z^2)
$$

(3.32)

by corrections of order $\omega\omega'/m^2$. These corrections involve $s^2$ and are thus negligible. To first order in $s$,

$$
\frac{1}{(1 + s\nu xz)^2} \simeq 1 - 2s\nu xz.
$$

(3.33)
The $\zeta$ and $\nu$ integrations are now straightforward. We express the result as

$$\Gamma(\Delta E) = \Gamma_0 \left[ F_0(u) - 4s \frac{\zeta(4)}{\zeta(3)} F_1(u) \right],$$

where $\Gamma_0$ is the approximate total scattering rate from Eq. (2.12). The straightforward integrations give

$$F_0(u) = \frac{1}{2\zeta(3)} \int_u^\infty \frac{dx}{e^x - 1} \left( x^2 - 3ux + 3u^2 \ln\left( \frac{x}{u} \right) + \frac{2u^3}{x} \right),$$

and

$$F_1(u) = \frac{1}{6\zeta(4)} \int_u^\infty \frac{dx}{e^x - 1} \left( x^3 - \frac{9}{2} u^2x - u^3 + 6u^3 \ln\left( \frac{x}{u} \right) + \frac{9u^4}{2x} \right).$$

It can be seen that in the $\Delta E \to 0$ limit, $F_0(0) = F_1(0) = 1$, so $\Gamma(\Delta E)$ reduces to the result (2.18) for $\Gamma_1$.

At this stage, one must resort to numerical calculations to evaluate the integrals. However, analytic calculations of the energy weighted moments of the distribution can still be performed. The simplest of these is the average energy loss observed in the lab frame, $\langle E - E' \rangle$. Because $\Gamma(\Delta E)$ describes the rate due to all scattering events where $E - E' > \Delta E$, this average value may be computed by

$$\langle E - E' \rangle = -\int_0^\infty d\Delta E \Delta E \frac{d}{d\Delta E} \left( \frac{\Gamma(\Delta E)}{\Gamma_1} \right),$$

where $\Gamma_1$ is the total scattering rate including the first correction in $s$ given by Eq. (2.18). Changing variables to $u$ and integrating by parts gives

$$\langle E - E' \rangle = \int_0^\infty du \frac{\Gamma(\Delta E)}{\Gamma_1} \frac{d\Delta E}{du},$$

with, in view of Eq. (3.29),

$$\frac{d\Delta E}{du} = \frac{2sE}{(1 + 2su)^2}.$$

Inserting the expressions for $\Gamma(\Delta E)$ and $\Gamma_1$ into the integral and expanding in powers of $s$ gives.
\[
\langle E - E' \rangle = 2sE \int_0^\infty du \left\{ F_0(u) - 4s \left[ uF_0(u) - \frac{\zeta(4)}{\zeta(3)} F_0(u) + \frac{\zeta(4)}{\zeta(3)} F_1(u) \right] + O(s^2) \right\} \tag{3.40}
\]

Inserting the expressions for the \(F\)'s from Eq. (3.35) and Eq. (3.36) and interchanging the order of the \(x\) and \(u\) integrals, the integrals may be evaluated analytically, and we find that

\[
\langle E - E' \rangle = 2sE \frac{\zeta(4)}{\zeta(3)} \left\{ 1 + s \left[ 4 \frac{\zeta(4)}{\zeta(3)} - \frac{63}{5} \frac{\zeta(5)}{\zeta(4)} \right] + O(s^2) \right\} = 1.80 \ sE \left[ 1 - 8.5 \ s \right]. \tag{3.41}
\]

To check that no mistakes have been made in our calculation of \(F_0(u)\) and \(F_1(u)\) given in Eq. (3.35) and Eq. (3.36), we have independently evaluated the average energy loss \(\langle E - E' \rangle\) starting from Eq. (3.6) and only making the small \(s\) approximation towards the end of the calculation. We find the same result with this different method.

The integrals in the definitions (3.35) and (3.36) of the functions \(F_0(u)\) and \(F_1(u)\) have been calculated numerically, and the results are displayed in Fig. 1 and Fig. 2. As a check on this numerical result, we have used it to evaluate the integrals in Eq. (3.40) numerically, and the results agree with the analytic expression (3.41) to within 0.2 percent.

We may compare our calculations with the those of Domenico who employed a Monte Carlo method. He used the values \(E = 46.1\ \text{GeV}\) and \(T = 291\ \text{K}\) (which we have previously employed) that give \(s = 0.0089\). He also took \(\Delta E = 0.012E\) which places \(u = 0.69\). Numerical integration gives \(F_0(0.69) = 0.44\) and \(F_1(0.69) = 0.83\), and from these values we calculate a mean beam lifetime of 64 hours to zeroth order in the nonrelativistic limit, and of 68 hours when the first order relativistic corrections are included. This is to be compared with Domenico's value of 90 hours for the same input parameters. We do not understand the reason for this discrepancy.

We may also compare our results with those of H. Burkhardt who also used a Monte Carlo method. Burkhardt uses parameters that are slightly different than those used by Domenico, namely \(E = 45.6\ \text{GeV}\) and \(T = 295\ \text{K}\). These parameters yield the same \(s = 0.0089\). However, since the overall rate scales as \(T^3\), which is 4% larger with Burkhardt's
FIG. 1. The dimensionless $F_0(u)$ defined in Eq. (3.35) as a function of the dimensionless variable $u$ defined in Eq. (3.29).

temperature, our value of the beam lifetime for $\Delta E = 0.012E$ with his parameters is reduced from 68 to 65 hours. Burkhardt finds 83 hours. (To compare with Domenico, we note that modifying his result of 90 hours by the 4% change in $T^3$ produces 86 hours.) Burkhardt and Kleiss also state that the average fractional energy loss $\langle E - E' \rangle / E$ is 1.1% for this value of $s$, but our result (3.41) gives the larger value 1.5% corresponding to our shorter beam lifetime. Again, we we can only state that we do not understand the reason for these discrepancies.

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FIG. 2. The dimensionless $F_1(u)$ defined in Eq. (3.36) as a function of $u$.

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10. The result for $F_0(u)$ given in Eq. (3.35) can be obtained from an integration of Eq. (2.42) of Blumenthal and Gould, but the result for $F_1(u)$ in Eq. (3.36) appears to be new.