Exact Fourier expansion in cylindrical coordinates for the three-dimensional Helmholtz Green function

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Abstract
A new method is presented for Fourier decomposition of the Helmholtz Green function in cylindrical coordinates, which is equivalent to obtaining the solution of the Helmholtz equation for a general ring source. The Fourier coefficients of the Helmholtz Green function are split into their half advanced+half retarded and half advanced−half retarded components. Closed form solutions are given for these components in terms of a Horn function and a Kampé de Fériet function, respectively. The systems of partial differential equations associated with these two-dimensional hypergeometric functions are used to construct a fourth-order ordinary differential equation which both components satisfy. A second fourth-order ordinary differential equation for the general Fourier coefficient is derived from an integral representation of the coefficient, and both differential equations are shown to be equivalent. Series solutions for the various Fourier coefficients are also given, mostly in terms of Legendre functions and Bessel/Hankel functions. These are derived from the closed form hypergeometric solutions or an integral representation, or both. Numerical calculations comparing different methods of calculating the Fourier coefficients are presented.

1 Introduction and overview
The inhomogeneous Helmholtz wave equation is

$$ (\nabla^2 + \beta^2) \Phi (\beta, \mathbf{r}) = \rho (\mathbf{r}) $$

(1)
and this has the well known free-space retarded Green function [1, p. 284]

\[ G_H (\beta, \mathbf{r} - \mathbf{r}') = -\frac{\exp (i\beta |\mathbf{r} - \mathbf{r}'|)}{4\pi |\mathbf{r} - \mathbf{r}'|} \]  

(2)

where \( \mathbf{r} \) is a field point, \( \mathbf{r}' \) is a source point and \( \beta \) is the wave number, here considered to be a general complex number. The free-space Green function (2) is restricted to values of \( \beta \) such that \( |G_H (\beta, \mathbf{r} - \mathbf{r}')| \to 0 \) as \( |\mathbf{r} - \mathbf{r}'| \to \infty \). For general dispersive waves with \( \beta = \alpha + i\sigma \) where \( \alpha \) and \( \sigma \) are real, then \( \sigma \geq 0 \) is a condition for this to hold. In the limit as \( \beta \to 0 \) these equations reduce to the Poisson equation and its corresponding Green function \( G_P (\mathbf{r} - \mathbf{r}') \). The general retarded solution \( \Phi (\beta, \mathbf{r}) \) of the Helmholtz equation at a field point \( \mathbf{r} \) for a general source density \( \rho (\mathbf{r}) \), subject to the boundary condition that \( \Phi (\beta, \mathbf{r}) \to 0 \) as \( |\mathbf{r}| \to \infty \), is given in terms of the Green function as

\[ \Phi (\beta, \mathbf{r}) = \iiint G_H (\beta, \mathbf{r} - \mathbf{r}') \rho (\mathbf{r}') \, d^3\mathbf{r}' \]  

(3)

where the volume integral is to be taken over all regions of space where the source density \( \rho (\mathbf{r}) \) is non zero.

Many problems of practical interest have some element of axial symmetry and are best treated in cylindrical coordinates \((r, \phi, z)\), the Cartesian components \((x, y, z)\) of \( \mathbf{r} \) being related to the cylindrical components by \( (x, y, z) = (r \cos \phi, r \sin \phi, z) \). It follows immediately from this relation that the distance between a source point and a field point is given by

\[ |\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 + (z - z')^2 - 2rr' \cos (\phi - \phi')} \]  

(4)

The solution for \( \Phi (\beta, \mathbf{r}) \) when \( \rho (\mathbf{r}) \) is a general circular ring source is of particular interest, with applications such as circular loop antennas [2], [3], [4], the acoustics of rotating machinery [5] and acoustic and electromagnetic scattering [6]. For the simpler Poisson equation most of the analytical solutions found in the literature for cylindrical geometry are either ring source solutions or can be easily constructed from them by integration or summation. Examples are gravitating rings and disks, ring vortices and vortex disks, and circular current loops and solenoids.

The source density \( \rho_c (\mathbf{r}, R, z) \) for a thin circular ring of radius \( R \) located in the plane \( z = Z \) is of the form

\[ \rho_c (\mathbf{r}, R, Z) = f (\phi) \delta (r - R) \delta (z - Z) \]  

(5)

where \( f (\phi) \) is the angular distribution of the source strength around the ring. This can be most conveniently described by a Fourier series of the form

\[ f (\phi) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos (m\phi) + b_m \sin (m\phi)) \]  

(6)
where the Fourier coefficients \( a_m \) and \( b_m \) are given by [7, p. 1066]

\[
a_m = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \cos(m\phi) \, d\phi
\]

(7)

\[
b_m = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \sin(m\phi) \, d\phi.
\]

(8)

From equation (4), the Green function (3) is even in the variable \( \psi \equiv \phi' - \phi \), where \( \phi \) is the angular coordinate of \( r \) and \( \phi' \) is the angular coordinate of \( r' \). It is convenient to exploit this symmetry when substituting equations (5) and (6) into (3). From the identity

\[
f(\phi + \psi) \equiv f(\phi')
\]

we obtain

\[
f(\phi') = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos(m\phi) + b_m \sin(m\phi)) \cos(m\psi)
\]

\[
+ \sum_{m=1}^{\infty} (-a_m \sin(m\phi) + b_m \cos(m\phi)) \sin(m\psi)
\]

(9)

and on substituting equations (5) and (9) into (3) and performing the volume integration, the odd terms proportional to \( \sin(m\psi) \) in equation (9) do not contribute to the solution \( \Phi(\beta, r) \) as \( G_H(\beta, r - r') \) is even in \( \psi \). The remaining integrals from the even terms can be calculated over the reduced interval from 0 to \( \pi \). This gives the solution \( \Phi_c(\beta, r, R, Z) \) of the Helmholtz equation for a circular ring source with general \( f(\phi) \) in the form

\[
\Phi_c(\beta, r, R, Z) = -\frac{a_0}{2} G_H^0(\beta, r, R, z - Z)
\]

\[
- \sum_{m=1}^{\infty} (a_m \cos(m\phi) + b_m \sin(m\phi)) G_H^m(\beta, r, R, z - Z)
\]

(10)

where

\[
G_H^m(\beta, r, R, z - Z) = \frac{1}{\pi} \int_0^{\pi} \exp \left( i\beta \sqrt{r^2 + R^2 + (z - Z)^2 - 2rR \cos \psi} \right) \cos(m\psi) \, d\psi
\]

(11)

and where the explicit dependence of the solution on the constant ring parameters \( R \) and \( Z \) has been introduced in these definitions. Introducing the Neumann factor \( \epsilon_m \) such that \( \epsilon_m = 1 \) for \( m = 0 \) and \( \epsilon_m = 2 \) for \( m > 0 \), and defining \( b_0 = 0 \) allows (10) to be expressed more concisely as

\[
\Phi_c(\beta, r, R, Z) = -\frac{1}{2} \sum_{m=0}^{\infty} (a_m \cos(m\phi) + b_m \sin(m\phi)) \epsilon_m G_H^m(\beta, r, R, z - Z).
\]

(12)
Apart from a constant factor, the terms \( \epsilon_m G^m_H(\beta, r, R, z - Z) \) in (12) are also the coefficients in the Fourier expansion of the Green function (2) itself, when the source point is given by \( r' = (R, \phi', Z) \). From equations (6), (7) and (8) this is given by

\[
G_H(\beta, r - r') = -\frac{1}{4\pi} \sum_{m=0}^{\infty} \epsilon_m G^m_H(\beta, r, R, z - Z) \cos(m(\phi - \phi')).
\] (13)

Thus the solution \( \Phi_c(\beta, r, R, Z) \) of the Helmholtz equation for a general ring source can be constructed directly from the coefficients \( G^m_H(\beta, r, R, z) \) in the Fourier expansion of the Green function (3). This provides in large measure the motivation to analytically construct the Fourier series for the Helmholtz Green function.

For the Poisson equation with \( \beta = 0 \) the corresponding Fourier expansion of the Green function has already been given in closed form as [8]:

\[
G_P(r - r') = -\frac{1}{4\pi^2 \sqrt{rr'}} \sum_{m=0}^{\infty} \epsilon_m Q_{m-1/2}(\omega) \cos(m(\phi - \phi')).
\] (14)

where

\[
\omega = \frac{r^2 + R^2 + (z - Z)^2}{2rr'}
\] (15)

is a toroidal variable such that \( \omega \geq 1 \) and the \( Q_{m-1/2}(\omega) \) are the Legendre functions of the second kind and half-integral degree, which are also toroidal harmonics. The Fourier expansion given by equations (14) and (15) can be obtained immediately by writing the Green function (2) for \( \beta = 0 \) in the form

\[
G_P(r - r') = -\frac{1}{4\pi \sqrt{rr'} \sqrt{2\omega - 2\cos(\phi - \phi')}}
\] (16)

where \( \omega \) is given by (15), and noting that the function \( Q_{m-1/2}(\omega) \) has the simple integral representation [7, eqn 8.713]

\[
Q_{m-1/2}(\omega) = \int_0^\pi \frac{\cos(m\psi) d\psi}{\sqrt{2\omega - 2\cos\psi}}.
\] (17)

An alternative derivation of (14) employs the Lipschitz integral [7, eqn 6.611]

\[
\int_0^\infty J_0(sa) \exp(-s|b|) ds = \frac{1}{\sqrt{a^2 + b^2}}
\] (18)

and Neumann’s addition theorem [9, eqn 11.2]

\[
J_0\left(s\sqrt{r^2 + R^2 - 2Rr\cos\psi}\right) = \sum_{m=0}^{\infty} \epsilon_m \cos(m\psi) J_m(sr) J_m(sR)
\] (19)
to obtain the well known eigenfunction expansion

$$G_P (r - r') = -\frac{1}{4\pi} \times \sum_{m=0}^{\infty} \epsilon_m \cos \left( m \left( \phi - \phi' \right) \right) \int_0^{\infty} J_m (sr) J_m (sR) \exp \left( -s \mid z - Z \mid \right) ds. \quad (20)$$

This reduces to (14) on employing the integral [7, eqn 6.612 3],[9, eqn 13.22]:

$$\int_0^{\infty} J_m (sr) J_m (sR) \exp \left( -s \mid Z - z \mid \right) ds = \frac{1}{\pi \sqrt{RR}} Q_{m-1/2} (\omega) . \quad (21)$$

The generalization of (20) for the Helmholtz case is also well known [10, p. 888]

$$G_H (\beta, r - r') = -\frac{i}{4\pi} \times \sum_{m=0}^{\infty} \epsilon_m \cos \left( m \left( \phi - \phi' \right) \right) \int_0^{\infty} \exp \left( i \mid Z - z \mid \sqrt{\beta^2 - s^2} \right) J_m (sr) J_m (sR) \frac{sds}{\sqrt{\beta^2 - s^2}}. \quad (22)$$

This can be similarly obtained from Neumann’s theorem by employing the integral [7, eqn 6.616 2]

$$\int_1^{\infty} \exp \left( -ax \right) J_0 \left( b \sqrt{x^2 - 1} \right) dx = \frac{\exp \left( -\sqrt{a^2 + b^2} \right)}{\sqrt{a^2 + b^2}} \quad (23)$$

instead of the Lipschitz integral. Equation (22) gives the Fourier coefficients of the Helmholtz Green function in the form

$$G_H^m (\beta, r, R, z - Z) = i \int_0^{\infty} \exp \left( i \mid Z - z \mid \sqrt{\beta^2 - s^2} \right) J_m (sr) J_m (sR) \frac{sds}{\sqrt{\beta^2 - s^2}}. \quad (24)$$

This reduces to (20) in the limit as $\beta \to 0$ but unfortunately the integral in (24) is not given in standard tables for $\beta \neq 0$. Numerical evaluation of this integral requires care, as the integrand is oscillatory and singular in an infinite range of integration, though the integrand tends exponentially to zero as $s \to \infty$. Equation (11) is a convenient alternative numerical evaluation of the Fourier coefficients, provided $m$ is not too large.

The integrals (11) and (24) contain the additional parameter $\beta$ which is not contained in (17) and (21). As a consequence of this, the closed form generalization of (14) for the Helmholtz case involves two-multidimensional Gaussian hypergeometric series, and the main purpose of this article is to present these
solutions and various related results. The core idea leading to the solution is the expansion of the exponential in (2) as the absolutely convergent power series

\[ G_H(\beta, r - r') = -\frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{(i\beta)^n |r - r'|^{n-1}}{n!} \]  

(25)

where

\[ |r - r'|^{n-1} = \left( r^2 + R^2 + (z - Z)^2 - 2rR \cos \psi \right)^{(n-1)/2}. \]  

(26)

Hence

\[ G_H^m(\beta, r, R, z - Z) = \sum_{n=0}^{\infty} \frac{(i\beta)^n}{n!} I_{m,n}(r, R, z - Z) \]  

(27)

where

\[ I_{m,n}(r, R, z - Z) = \frac{1}{\pi} \int_0^\pi \left( r^2 + R^2 + (z - Z)^2 - 2rR \cos \psi \right)^{(n-1)/2} \cos (m\psi) \; d\psi. \]  

(28)

The integral (28) can be evaluated as a series by binomial expansion and this gives a double series for the Fourier coefficient \( G_H^m(\beta, r, R, z - Z) \). The expansion of (28) gives an infinite number of terms for \( n \) even and a finite number of terms for \( n \) odd. These two cases are best treated separately and it is therefore convenient to split the summation over \( n \) in (25) into odd and even terms. This is equivalent to splitting the Green function (2) such that

\[ G_H(\beta, r - r') = \Lambda_+(\beta, r - r') + \Lambda_-(\beta, r - r') \]  

(29)

where

\[ \Lambda_+(\beta, r - r') = -\frac{1}{8\pi} \left( \frac{\exp(i\beta |r - r'|)}{|r - r'|} + \frac{\exp(-i\beta |r - r'|)}{|r - r'|} \right) \]  

(30)

is the half advanced+half retarded Green function and

\[ \Lambda_-(\beta, r - r') = -\frac{1}{8\pi} \left( \frac{\exp(i\beta |r - r'|)}{|r - r'|} - \frac{\exp(-i\beta |r - r'|)}{|r - r'|} \right) \]  

(31)

is the half advanced−half retarded Green function. The corresponding Fourier coefficients are split in the same manner such that

\[ \Lambda^m_+(\beta, r, R, z - Z) = \frac{1}{2} \left( G_H^m(\beta, r, R, z - Z) + G_H^m(-\beta, r, R, z - Z) \right) \]  

(32)

\[ \Lambda^m_-(\beta, r, R, z - Z) = \frac{1}{2} \left( G_H^m(\beta, r, R, z - Z) + G_H^m(-\beta, r, R, z - Z) \right). \]  

(33)

For real \( \beta \), splitting the Green function in this way is equivalent to dividing it into its real and imaginary parts, but this is not the case for general complex \( \beta \).
It is shown in Section 2 that the Fourier coefficients in (32) and (33) are given respectively by

\[ \bar{\Lambda}_m^{+} (rR, \gamma, k) = \frac{\Gamma \left( m + \frac{1}{2} \right)}{m! \sqrt{\pi rR}} \left( \frac{k}{2} \right)^{2m+1} H_3 \left( m + 1/2, m + 1/2, 2m + 1, k^2, \frac{\gamma^2}{4} \right) \] (34)

and

\[ \bar{\Lambda}_m^{-} (rR, \gamma, k) = \frac{i}{\sqrt{rR}} \frac{\left( \gamma k^2 \right)^{2m+1}}{(2m+1)!} \times F_{0;1;0}^{6;1;0} \left[ \begin{array}{c} - : m + 1/2; - : \gamma^2 k^2; - : 4 \gamma^2 \end{array} \right] (35) \]

where

\[ k = \sqrt{\frac{4rR}{(r + R)^2 + (z - Z)^2}} \] (36)

\[ \gamma = \beta \sqrt{(r + R)^2 + (z - Z)^2} \] (37)

and

\[ \bar{\Lambda}_m^{+} (rR, \gamma, k) \equiv \Lambda_m^{+} (\beta, r, R, z - Z). \] (38)

The variable \( k \) is the usual modulus contained in elliptic integral solutions of elementary ring problems and is related to the toroidal variable \( \omega \) by

\[ \omega = \frac{2 - k^2}{k^2}. \] (39)

The function \( H_3 \) in equation (34) is one of the standard Horn functions [11, eqn 5.7.1 31] and is equivalent to the double hypergeometric series

\[ H_3 \left( m + 1/2, m + 1/2, 2m + 1, k^2, \frac{\gamma^2}{4} \right) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(m + 1/2)_{n-p} (m + 1/2)_n}{(2m + 1)_n n! p!} (k^2)^n \left( \frac{\gamma^2}{4} \right)^p. \] (40)

The Kampé de Fériet function [12, p. 27] in (35) is equivalent to the double hypergeometric series

\[ F_{0;1;0}^{6;1;0} \left[ \begin{array}{c} - : m + 1/2; - : \gamma^2 k^2; - : 4 \gamma^2 \end{array} \right] (35) \]

\[ \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(m + 1/2)_n}{(2m + 1)_n n! p!} (k^2)^n \left( \frac{\gamma^2}{4} \right)^p. \] (41)
The integral (28) can also be evaluated using an integral representation for the associated Legendre function of the first kind, and it is shown in Appendix A that this gives the series expansion:

\[
\hat{G}^m_{H}(rR, \lambda, \omega) = \frac{(-1)^m}{(\omega^2 - 1)^{1/4}} \sqrt{2rR} \times \sum_{n=0}^{\infty} \frac{(i\lambda (\omega^2 - 1)^{1/4})^n}{n!} \frac{\Gamma ((n+1)/2)}{\Gamma (m + (n+1)/2)} P^m_{(n-1)/2} \left( \frac{\omega}{\sqrt{\omega^2 - 1}} \right) \]  

(42)

where

\[
\lambda = \beta \sqrt{2rR} \]  

(43)

and

\[
\hat{G}^m_H (rR, \lambda, \omega) \equiv G^m_H (\beta, r, R, z - Z). \]  

(44)

The Legendre function in equation (42) reduces to an associated Legendre polynomial for odd \(n\). The series in (42) can be split into even and odd terms such that

\[
\hat{G}^m_{H}(rR, \lambda, \omega) = \frac{1}{2} \left( \hat{\Lambda}^m_{+} (rR, \lambda, k) + \hat{\Lambda}^m_{-} (rR, \lambda, \omega) \right) \]  

(45)

where

\[
\hat{\Lambda}^m_{\pm} (rR, \lambda, \omega) \equiv \Lambda^m_{\pm} (\beta, r, R, z - Z) \]  

(46)

and it is shown in Appendix A that the even and odd series can be expressed respectively as:

\[
\hat{\Lambda}^m_{+} (rR, \lambda, \omega) = \frac{(-1)^m}{\sqrt{rR}} \sum_{p=0}^{\infty} \left( \frac{\lambda^2 \sqrt{\omega^2 - 1}}{4} \right)^p \frac{Q^{p+m}_{m-1/2} (\omega)}{p! \Gamma (p + m + 1/2) \Gamma (p + m + 1)} \]  

\[
\hat{\Lambda}^m_{-} (rR, \lambda, \omega) = \frac{(-1)^m}{\sqrt{rR}} \sum_{p=0}^{\infty} \left( \frac{\lambda^2 \sqrt{\omega^2 - 1}}{4} \right)^{p+m+1/2} \frac{Q^{p+m+1/2}_{m-1/2} (\omega)}{p! \Gamma (p + m + 3/2) \Gamma (p + 2m + 1)} \]  

(47)

(48)

In equation (48), the Legendre function is purely imaginary for real \(\lambda\). In the static limit as \(\lambda \to 0\) then \(\Lambda^m_{-} (rR, 0, \omega) = 0\) and from the gamma function identity [7, eqn 8.334 2]

\[
\Gamma (1/2 - m) \Gamma (1/2 + m) = (-1)^m \pi \quad [m \in \mathbb{N}_0] \]  

then equation (47) reduces to

\[
\hat{\Lambda}^m_{+} (rR, 0, \omega) = \frac{Q^{m-1/2}_{m-1/2} (\omega)}{\pi \sqrt{rR}} \quad [m \in \mathbb{N}_0] \]  

(50)

as it must do for consistency with (14).

The solutions in terms of two-dimensional hypergeometric functions defined by Equations (34)-(38) and (40)-(41) can be summed over either index to give
the solutions as series of special functions. It is shown in Section 3 that summation over the index \( n \) in equation (40) gives equation (47), exactly as given by the integral representation. However, summation over the index \( n \) in equation (41) gives instead the series solution

\[
\bar{\Lambda}_m^i (rR, \gamma, k) = \frac{i \sqrt{\pi}}{m! \sqrt{rR}} \left( \frac{\gamma k^2}{4} \right)^{2m+1} \times \sum_{p=0}^{\infty} \frac{1}{\Gamma(p + m + 3/2)p!} \left( -\frac{\gamma^2}{4} \right)^p \binom{1}{p} \binom{1}{2m+1} \left( m + 1/2; n + m + 3/2, 2m + 1; \frac{\gamma^2 k^2}{4} \right).
\]

(51)

A hypergeometric identity to reduce the hypergeometric function in equation (51) to other well-known special functions does not seem to be available in standard tabulations. It might nevertheless be conjectured that (51) could somehow be reducible to equation (48), but this is not in fact the case. It is easily verified numerically that although equations (48) and (51) both converge rapidly to the same limit, the individual terms do not match. Hence, equation (51) is a distinct series from equation (48). It is also shown in Section 3 that summation over the index \( p \) in equations (40) and (41) gives the Bessel function series:

\[
\bar{\Lambda}_m^i (rR, \gamma, k) = \frac{-1}{2 \sqrt{rR}} \sum_{n=0}^{\infty} \frac{\Gamma(n + m + 1/2)}{\Gamma(n + 2m + 1)n!} \left( \frac{\gamma k^2}{2} \right)^{n+m+1/2} \sqrt{\pi x} H_3 \left( \alpha, \alpha, 2 \alpha, x, y \right)
\]

and

\[
\bar{\Lambda}_m^i (rR, \gamma, k) = \frac{i}{2 \sqrt{rR}} \sum_{n=0}^{\infty} \frac{\Gamma(n + m + 1/2)}{\Gamma(n + 2m + 1)n!} \left( \frac{\gamma k^2}{2} \right)^{n+m+1/2} \sqrt{\pi x} J_n \left( \alpha, \alpha, 2 \alpha, x, y \right).
\]

(52)

(53)

and these two series can be conveniently combined to give a series of Hankel functions of the first kind:

\[
\bar{G}_m^i (rR, \gamma, k) = \frac{i}{2 \sqrt{rR}} \sum_{n=0}^{\infty} \frac{\Gamma(n + m + 1/2)}{\Gamma(n + 2m + 1)n!} \left( \frac{\gamma k^2}{2} \right)^{n+m+1/2} H_{n+m+1/2}^{(1)} (\gamma).
\]

(54)

From the solutions (34) and (35) it can be seen that dimensionless Fourier coefficients defined by

\[
g_m^i (k^2, \gamma^2/4) \equiv \sqrt{rR} \bar{\Lambda}_m^i (rR, \gamma, k)
\]

depend only on the two dimensionless variables \( x \equiv k^2 \) and \( y \equiv \gamma^2/4 \). The functions \( g_m^i (x, y) \) are given explicitly by equations (34) and (35) as:

\[
g_m^+ (x, y) = \frac{\Gamma(m + 1/2)}{2^{2m+1} m! \sqrt{\pi}} x^{\alpha} H_3 \left( \alpha, \alpha, 2 \alpha, x, y \right)
\]

(56)

and

\[
g_m^- (x, y) = \frac{i}{(2m+1)!} (xy)^{\alpha} F_0:1:0^{\alpha: \alpha; -; \alpha; -; xy, -y}
\]

(57)
where
\[ \alpha = m + 1/2. \] (58)

Two-dimensional hypergeometric series such as (56) and (57) are associated with pairs of partial differential equations [11, section 5.9] and these can be used to construct ordinary differential equations for \( g^m_\pm (x, y) \) with \( y \) fixed and \( x \) as the independent variable. It is shown in Section 4 that for constant \( y \) the coefficients \( g^m_\pm (x, y) \) both satisfy the same fourth-order ordinary differential equation in \( x \):

\[
(1 - x) x^4 \frac{d^4 g^m_\pm}{dx^4} + (6 - 9x) x^3 \frac{d^3 g^m_\pm}{dx^3} + (\alpha (1 - \alpha) + 6 - 18x - xy + 2y) x^2 \frac{d^2 g^m_\pm}{dx^2} + (2 \alpha (1 - \alpha) - 2x (3 - y)) x \frac{dg^m_\pm}{dx} + (y^2 + 2 \alpha (1 - \alpha) y - 3 \alpha (\alpha + 1) (\alpha + 2)) g^m_\pm = 0. \] (59)

In Section 5 an integral representation is derived for

\[ \hat{y}_m (\lambda, \omega) \equiv \pi \sqrt{2r} \hat{G}^m_{\lambda R} (rR, \lambda, \omega) \] (60)

and this is used to derive a fourth-order ordinary differential equation for \( \hat{y}_m (\lambda, \omega) \) in terms of \( \omega \):

\[
(1 - \omega^2) \frac{d^4 \hat{y}_m}{d\omega^4} - 6 \omega \frac{d^3 \hat{y}_m}{d\omega^3} + (m^2 - \frac{\lambda^2 \omega}{2} - \frac{25}{4}) \frac{d^2 \hat{y}_m}{d\omega^2} - \lambda^2 \frac{d\hat{y}_m}{d\omega} - \left( \frac{\lambda^2}{4} \right)^2 \hat{y}_m = 0. \] (61)

In the static limit as \( \lambda \to 0 \), equation (61) reduces to:

\[
\frac{d^2}{d\omega^2} \left[ (1 - \omega^2) \frac{d^2 \hat{y}_m}{d\omega^2} - 2\omega \frac{d\hat{y}_m}{d\omega} + \left( m^2 - \frac{1}{4} \right) \hat{y}_m \right] = 0 \] (62)

where

\[
(1 - \omega^2) \frac{d^2 \hat{y}_m}{d\omega^2} - 2\omega \frac{d\hat{y}_m}{d\omega} + \left( m^2 - \frac{1}{4} \right) \hat{y}_m = 0 \] (63)

is Legendre’s equation of degree \( m - 1/2 \) [7, eqn 8.820]. It is also shown in Section 5 that the differential equations (59) and (61), obtained by quite different routes, are equivalent. The special functions used in the analysis are given in Table 1.

Recurrence relations for the Fourier coefficients for the Helmholtz equation were investigated by Matviyenko [6], but the closed form solutions and differential equations presented here appear to be new. Werner [3] presented an expansion of the Fourier coefficient as a series of spherical Hankel functions, superficially similar to equation (64), but the two expansions are distinct. The two-dimensional hypergeometric series approach applied here to obtain the Fourier expansion for the Helmholtz Green function has recently been applied to obtain the Fourier expansion in terms of the amplitude \( \phi \) for the Legendre incomplete elliptic integral of the third kind [13].

The numerical performance of the various expressions for the Fourier coefficients was investigated using Mathematica [14] and this is examined in Appendix C.
2 Solution in terms of two-dimensional hypergeometric series

The power series expansion (27) for the Fourier coefficient \( G_{m}^{m} (\beta, r, R, z - Z) \) can be expressed in the form

\[
\bar{G}_{m}^{m} (rR, \gamma, k) = \frac{(-1)^{m} k}{\pi \sqrt{r R}} \sum_{n=0}^{\infty} \frac{(i \gamma)^{n}}{n!} \int_{0}^{\pi/2} \cos (2m \theta) \left( 1 - k^2 \sin^2 \theta \right)^{(n-1)/2} d\theta
\]  

where

\[
G_{m}^{m} (\beta, r, R, z - Z) = \bar{G}_{m}^{m} (rR, \gamma (\beta, r, R, z - Z), k (r, R, z - Z))
\]

and \( k \) and \( \gamma \) are defined by (36) and (37). The term \( (1 - k^2 \sin^2 \theta)^{(n-1)/2} \) in (64) can be expanded binomially to give \( G_{m}^{m} (rR, \gamma, k) \) as a double series containing integrals of the form

\[
\bar{I}_{m,p} = \int_{0}^{\pi/2} \sin^{2p} \theta \cos (2m \theta) d\theta.
\]

This integral is given by Gradshteyn and Ryzhik [7, eqns 3.631 8,12] in a form which can be recast as

\[
\bar{I}_{m,p} = \frac{(-1)^{m} \pi}{22p+1(2p+1)} \frac{1}{B(p+m+1,p-m+1)} \text{ for } p \geq m
\]

\[
\bar{I}_{m,p} = 0 \text{ for } p < m
\]

Table 1: Special Functions Used
and expressing the beta function in (67) in terms of gamma functions and employing the duplication theorem [7, eqn 8.335 1]

\[ \Gamma (2x) = \frac{\Gamma (x) \Gamma (x + 1/2)}{\sqrt{\pi}} \]

(69)
gives after some reduction the alternative form

\[ I_{m,p} = \left( -1 \right)^m \sqrt{\pi} \frac{\Gamma (p + 1/2) p!}{\Gamma (p + m + 1) \Gamma (p - m + 1)} \quad \text{for } p \geq m. \]

(70)
The binomial expansion of (63) gives an infinite series for \( n \) zero or even and a finite sum for \( n \) odd, and these two cases must be treated separately. It is therefore convenient to split the series for \( \bar{G}^m \) such that

\[ \bar{G}^m (rR, \gamma, k) = \bar{\Lambda}^m_+ (rR, \gamma, k) + \bar{\Lambda}^m_- (rR, \gamma, k) \]

(71)

and on employing equation (69) the divided series are given by

\[ \bar{\Lambda}^m_+ (rR, \gamma, k) = \frac{(-1)^m}{\sqrt{\pi} rR} \sum_{n=0}^{\infty} \frac{(-\gamma^2/4)^n}{\Gamma (n + 1/2) n!} \frac{\pi/2}{\int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{n-1/2} d\theta} \]

\[ \bar{\Lambda}^m_- (rR, \gamma, k) = \frac{i(-1)^m k \gamma}{2\sqrt{\pi} rR} \sum_{n=0}^{\infty} \frac{(-\gamma^2/4)^n}{\Gamma (n + 3/2) n!} \frac{\pi/2}{\int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^n d\theta}. \]

(72)

Binomial expansion of the integrals in equations (72) and (73) gives respectively

\[ \bar{\Lambda}^m_+ (rR, \gamma, k) = \frac{(-1)^m}{\sqrt{\pi} rR} \sum_{n=0}^{\infty} \sum_{s=m}^{\infty} \frac{(-\gamma^2/4)^n}{\Gamma (n + 1/2) n!} \frac{\Gamma (s - n + 1/2) (k^2)^s}{\Gamma (1/2 - n) s!} \bar{I}_{m,s} \]

\[ \bar{\Lambda}^m_- (rR, \gamma, k) = \frac{i(-1)^m k \gamma}{2\sqrt{\pi} rR} \sum_{n=0}^{\infty} \sum_{s=m}^{\infty} \frac{(-\gamma^2/4)^n}{\Gamma (n + 3/2) n!} \frac{(-k^2)^s}{(n - s)!} \bar{I}_{m,s} \]

(74)

(75)

and employing the explicit formula (60) for \( I_{m,s} \) in (74) and (75) gives respectively

\[ \bar{\Lambda}^m_+ (rR, \gamma, k) = \frac{k}{2\pi \sqrt{rR}} \times \sum_{n=0}^{\infty} \sum_{s=m}^{\infty} \frac{\Gamma (s - n + 1/2) \Gamma (s + 1/2) (\gamma^2/4)^n (k^2)^s}{\Gamma (s + m + 1) \Gamma (s - m + 1) n!} \]

\[ \bar{\Lambda}^m_- (rR, \gamma, k) = \frac{ik \gamma}{4\sqrt{rR}} \times \sum_{n=m}^{\infty} \sum_{s=m}^{n} \frac{\Gamma (s + 1/2) (-\gamma^2/4)^n (-k)^s}{(n - s)! \Gamma (n + 3/2) \Gamma (s + m + 1) \Gamma (s - m + 1)} \]

(76)

(77)

where the gamma identity (49) has been used to simplify equation (76).
2.1 The series for $\tilde{\Lambda}_m^\pm (rR, \gamma, k)$

The substitution $s = p + m$ in equation (76) yields after some reduction the double series:

$$\tilde{\Lambda}_m^+ (rR, \gamma, k) = \frac{\Gamma (m + 1/2)}{m! \sqrt{\pi rR}} \left( \frac{k}{2} \right)^{2m+1} \times$$

$$\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(m + 1/2)_p - n (m + 1/2)_p}{(2m + 1)_p} (k^2)^p \left( \frac{\gamma^2}{4} \right)^n n! p! \quad (78)$$

where in (78)

$$(a)_p = \frac{\Gamma (a + p)}{\Gamma (a)} \quad (79)$$

is the Pochhammer symbol. The double hypergeometric function in (78) can be identified as one of the confluent Horn functions [11, eqn 5.7.31] and hence

$$\tilde{\Lambda}_m^+ (rR, \gamma, k) = \frac{\Gamma (m + 1/2)}{m! \sqrt{\pi rR}} \left( \frac{k}{2} \right)^{2m+1} \times$$

$$H_3 \left( m + 1/2, m + 1/2, 2m + 1, k^2, \frac{\gamma^2}{4} \right). \quad (80)$$

The convergence condition given in [11, eqn 5.7.31] for the double series in equation (78) is $k^2 < 1$, which always holds. The order of summation in equation (78) can be reversed, but the order of the arguments $k^2$ and $\gamma^2/4$ in equation (80) cannot be exchanged.

2.2 The series for $\tilde{\Lambda}_m^- (rR, \gamma, k)$

Equation (77) can be converted to a doubly infinite series by reversing the order of summation, which gives

$$\tilde{\Lambda}_m^- (rR, \gamma, k) = \frac{i k \gamma}{4 \sqrt{rR}} \times$$

$$\sum_{s=m}^{\infty} \sum_{n=s}^{\infty} \frac{\Gamma (s + 1/2) (-\gamma^2/4)^n (-k^2)^s}{(n - s)! \Gamma (n + 3/2) \Gamma (s + m + 1) \Gamma (s - m + 1)} \quad (81)$$

The substitution $n = s + p$ in (81) gives

$$\tilde{\Lambda}_m^- (rR, \gamma, k) = \frac{i k \gamma}{4 \sqrt{rR}} \times$$

$$\sum_{s=m}^{\infty} \sum_{p=0}^{\infty} \frac{\Gamma (s + 1/2)}{p! \Gamma (p + s + 3/2) \Gamma (s + m + 1) \Gamma (s - m + 1)} \left( \frac{\gamma^2 k^2}{4} \right)^s \left( - \frac{\gamma^2}{4} \right)^p \quad (82)$$

and the further substitution $s = n + m$ in (82) gives

$$\tilde{\Lambda}_m^- (rR, \gamma, k) = \frac{i}{2 \sqrt{rR}} \times \left( \frac{\gamma^2 k^2}{4} \right)^{m+1/2} \times$$
\[
\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{\Gamma(n+m+1/2)}{\Gamma(p+n+m+3/2)} \frac{1}{\Gamma(n+2m+1)} \frac{\gamma^{2k^2}}{4} \left( -\frac{\gamma^2}{4} \right)^p n! p! \left( -\frac{\gamma^2}{4} \right)^p (83)
\]

Expressing equation (83) in terms of Pochhammer symbols gives after some reduction the double hypergeometric series

\[
\tilde{\Lambda}_m^n (rR, \gamma, k) = \frac{i}{\sqrt{rR} (2m+1)!} \left( \frac{\gamma k}{2} \right)^{2m+1} \times \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(m+1/2)_n}{(m+3/2)_n p!} \left( \frac{\gamma^{2k^2}}{4} \right)^{n} \left( -\frac{\gamma^2}{4} \right)^p (84)
\]

and this can be expressed as a Kampé de Fériet function as defined by Srivastava and Karlsson [12, p. 27]:

\[
F_{p+q}^{p+q} : \left[ \begin{array}{cccc}
\{a_p\} & \{b_q\} & \{c_r\} & \{d_t\} \\
\{\alpha_i\} & \{\beta_m\} & \{\gamma_n\} & \{\delta_o\} \\
\end{array} \right] ; x, y = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \prod_{u=1}^{p} (a_u)_{j+i} \prod_{u=1}^{q} (b_u) \prod_{u=1}^{r} (c_u) \prod_{u=1}^{m} (d_u) \prod_{u=1}^{n} (\alpha_i) \prod_{u=1}^{\beta_m} \prod_{u=1}^{\gamma_n} \frac{x^i y^j}{i! j!} (85)
\]

In the definition (85), \(\{a_p\} \equiv a_1, \ldots, a_p\) and \(\{b_q\}\) and so on, are the lists of the arguments of the Pochhammer symbols of the various types which appear in the products on the right-hand side of the equation. If a list has no members, it is represented by a hyphen. Comparing (84) with (85) gives

\[
\tilde{\Lambda}_m^n (rR, \gamma, k) = \frac{i}{\sqrt{rR} (2m+1)!} \left( \frac{\gamma^{2k^2}}{4} \right)^{m+1/2} \left( -\frac{\gamma^2}{4} \right)^p (86)
\]

3 Consequences of the hypergeometric formulas

In the static limit as \(\beta \to 0\) then equation (34) reduces to

\[
\Lambda_+^m (rR, 0, k) = \frac{\Gamma(m+1/2)}{m! \sqrt{\pi r R}} \left( \frac{k}{2} \right)^{m+1/2} 2F_1 (m+1/2, m+1/2; 2m+1; k^2) (87)
\]

and from (87) and the standard hypergeometric identity [15, eqn 7.3.171]:

\[
2F_1 (a, b; 2b; z) = \frac{2^b \Gamma(b+1/2)}{\sqrt{\pi} \Gamma(2b)} z^{-b} (1-z)^{(b-a)/2} \exp (i \pi (a-b)) Q_{b-1}^{b-a} \left( \frac{2}{z} - 1 \right) (88)
\]

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this reduces to
\[ \Lambda^m_+ (rR, 0, k) = \frac{1}{\pi \sqrt{rR}} Q_{m-1/2} \left( \frac{2 - k^2}{k^2} \right) \] (89)
in agreement with equations (44) and (37).

### 3.1 Fourier coefficients as series of special functions

The double series given by equation (34) can be summed with respect to either the index \( n \) or the index \( p \) in the definition (40). Summing with respect to \( n \) in (40) gives a series of Bessel functions of the second kind:
\[ \Lambda^m_+ (rR, \gamma, k) = \frac{-1}{2\sqrt{rR}} \left( \frac{k^2 \gamma}{2} \right)^{m+1/2} \sum_{p=0}^{\infty} \frac{\Gamma (m + 1/2 + p)}{\Gamma (2m + 1 + p) p!} \left( \frac{k^2 \gamma}{2} \right)^p Y_{m+\frac{1}{2}+p} (\gamma) \] (90)
where the gamma function identity (49) and the Bessel function identity
\[ Y_{\nu} (\gamma) = \frac{1}{\sin \nu \pi} [\cos (\nu \pi) - J_{-\nu} (\gamma)] \] (91)
have been employed to obtain equation (90). Summing instead over the index \( p \) in (40) gives the alternative series
\[ \Lambda^m_+ (rR, \gamma, k) = \frac{1}{m! \sqrt{rR}} \left( \frac{k}{2} \right)^{2m+1} \times \sum_{n=0}^{\infty} \frac{\Gamma (m - n + 1/2) \left( \gamma^2/4 \right)^n}{n!} \, _2F_1 \left( m + 1/2 - n, m + 1/2; 2m + 1; k^2 \right) \] (92)
This can be reduced using (88) and (49) to give:
\[ \Lambda^m_+ (rR, \gamma, k) = \left( -1 \right)^m \frac{\sqrt{2 \lambda}}{k} \times \sum_{n=0}^{\infty} \left( \frac{\gamma^2 \sqrt{1 - k^2}}{4} \right)^n \frac{Q_{m-1/2} \left( (2 - k^2) / k^2 \right)}{n! \Gamma (n - m + 1/2) \Gamma (n + m + 1/2)} . \] (93)
The dimensionless variables \( \gamma \) and \( \lambda \) defined by equations (37) and (43) respectively are related by
\[ \gamma = \frac{\sqrt{2 \lambda}}{k} \] (94)
and substituting this equation and equation (39) in equation (93) gives immediately equation (47).

Summing with respect to \( p \) in equation (83) gives the Bessel series
\[ \Lambda^m_+ (rR, \gamma, k) = \frac{i}{2\sqrt{rR}} \sum_{p=0}^{\infty} \frac{\Gamma (p + m + 1/2)}{\Gamma (p + 2m + 1) p!} \left( \frac{\gamma^2}{2} \right)^{p + m + 1/2} J_{m+\frac{1}{2}+p} (\gamma) \] (95)
The corresponding summation over the index \( n \) gives
\[
\Lambda_m^m (rR, \gamma, k) = \frac{i \sqrt{\pi}}{m! \sqrt{rR}} \left( \frac{\gamma k^2}{4} \right)^{m+1/2} \times \\
\sum_{n=0}^{\infty} \frac{1}{\Gamma(n+m+3/2) n!} \left( \frac{-\gamma^2}{4} \right)^n \left( m + 1/2; n + m + 3/2, 2m + 1; \frac{\gamma^2 k^2}{4} \right).
\]

There seems to be no hypergeometric transformation listed in standard tables suitable for directly reducing the hypergeometric function in this equation.

Equations (90) and (95) can be conveniently combined to give a series of Hankel functions of the first kind:
\[
\tilde{G}_m^m (rR, \gamma, k) = \frac{i}{2 \sqrt{rR}} \sum_{p=0}^{\infty} \frac{\Gamma(p+m+1/2)}{\Gamma(p+2m+1)} \left( \frac{\gamma k^2}{2} \right)^{p+m+1/2} H_{m+\frac{1}{2}+p}^{(1)} (\gamma),
\]
where \( H_{m+\frac{1}{2}+p}^{(1)} (\gamma) \equiv J_{m+\frac{1}{2}+p} (\gamma) + iY_{m+\frac{1}{2}+p} (\gamma) \).

### 4 Differential equations from the two-dimensional hypergeometric solutions

Erdélyi et. al. [11, section 5.9] tabulate the partial differential equations satisfied by all the functions in Horn’s list. They employ the notation reproduced below for the various partial derivatives
\[
p = \frac{\partial z}{\partial x}; \quad q = \frac{\partial z}{\partial y}; \quad r = \frac{\partial^2 z}{\partial x^2}; \quad s = \frac{\partial^2 z}{\partial x \partial y}; \quad t = \frac{\partial^2 z}{\partial y^2}
\]
where \( z \) is any function on the list. Each function in the list satisfies two partial differential equations, and unfortunately those given in [11, 5.9 34] for \( z (\alpha, \beta, \delta, x, y) \equiv H_3 (\alpha, \beta, \delta, x, y) \) contain typographical errors. The correct equations can be shown by the methods given in [11, section 5.7] to be:
\[
x (1-x) p + xys + [\delta - (\alpha + \beta + 1) x] p + \beta yq - \alpha \beta z = 0 \quad (98)
\]
\[
yt - xs + (1-\alpha) q + z = 0 \quad (99)
\]
and for the particular case considered here we have \( \beta = \alpha \) and \( \delta = 2\alpha \) so that (98) reduces to
\[
x (1-x) p + xys + [2\alpha - (2\alpha + 1) x] p + \alpha yq - \alpha^2 z = 0. \quad (100)
\]
Similar equations can also be derived for the Kampé de Fériet function defined by equation (41). For the definition
\[
\tilde{z} (\alpha, u, v) \equiv F_{1;0}^{0;1;0} \left[ \begin{array}{c} - \alpha; \alpha+1; -u, v \end{array} \right]
\]
(101)
and the notation
\[ \bar{p} = \frac{\partial \bar{z}}{\partial u}; \quad \bar{q} = \frac{\partial \bar{z}}{\partial v}; \quad \bar{r} = \frac{\partial^2 \bar{z}}{\partial u^2}; \quad \bar{s} = \frac{\partial^2 \bar{z}}{\partial u \partial v}; \quad \bar{t} = \frac{\partial^2 \bar{z}}{\partial v^2} \]
then the equations corresponding to (99) and (100) can be shown to be
\[ \ddot{z} = vt + u\bar{s} + (\alpha + 1)\bar{q} \quad (102) \]
\[ \ddot{z} = 2\alpha\bar{p} + u\bar{r} + \bar{q} + vt \quad (103) \]

4.1 Fourth-order differential equation from the Horn Function \( H_3(\alpha, \beta, \delta, x, y) \)

Writing
\[ z(\alpha, x, y) = H_3(\alpha, \alpha, 2\alpha, x, y) \quad (104) \]
then (56) can be expressed as
\[ z(\alpha, x, y) = \frac{2^{2m+1}m!\sqrt{\pi}}{\Gamma(m+1/2)}x^{-\alpha}g^m_+(x, y). \quad (105) \]
The ordinary differential equation for \( g^m_+(x, y) \) in terms of \( x \equiv k^2 \) can be derived by first obtaining the corresponding differential equation for \( z(\alpha, x, y) \) from (99) and (100) and then substituting (105) into this equation. Although straightforward in principle, this procedure is rather intricate in practice, and only the essential elements of the derivation are given below.

4.1.1 Differential equation for \( z(\alpha, x, y) \)

It is convenient to define a differential operator \( D \) such that:
\[ D \equiv x \frac{d}{dx} \quad (106) \]
which has by definition the properties:
\[ Dx = x \quad (107) \]
\[ Dy = -y \quad (108) \]
\[ D(xy) = 0 \quad (109) \]
\[ Dz = xp - yq \quad (110) \]
\[ Dp = xr - ys \quad (111) \]
\[ Dq = xs - yt. \quad (112) \]
Combining equations (99) and (112) gives the equation
\[ Dq = z + (1 - \alpha)q \quad (113) \]
and applying the operators \( \partial/\partial x \) and \( \partial/\partial y \) to this equation gives respectively:

\[
Ds = p - \alpha s \tag{114}
\]
\[
Dt = q + (2 - \alpha)t. \tag{115}
\]

Eliminating the variable \( r \) gives a system of four coupled equations:

\[
Dz + yq = xp \tag{116}
\]
\[
(1 - x)xDp = -yxs + [(2\alpha + 1)x - 2\alpha]xp - \alpha xyq + \alpha^2xz \tag{117}
\]
\[
Dq = z + (1 - \alpha)q \tag{118}
\]
\[
xDs = xp - \alpha xs. \tag{119}
\]

Eliminating \( p \) gives a system of 3 equations:

\[
D^2z - xD^2z + (2\alpha - 1)Dz - 2\alpha xDz + yz - xyz - \alpha^2xz = -yxs + (1 - \alpha)yq \tag{120}
\]
\[
Dq = z + (1 - \alpha)q \tag{121}
\]
\[
xDs = Dz + yq - \alpha xs. \tag{122}
\]

Eliminating \( s \) gives the two equations:

\[
D^3z - xD^3z - (3\alpha + 1)xDz + (3\alpha - 1)D^2z - \alpha (3\alpha + 2)xDz + 2yDz + \alpha (2\alpha - 1)Dz - xyDz - \alpha^2(\alpha + 1)xz + 2(\alpha - 1)yz - \alpha xyz = -y^2q \tag{123}
\]
\[
Dq = z + (1 - \alpha)q. \tag{124}
\]

Eliminating \( q \) finally gives the fourth-order equation:

\[
(1 - x) D^4z + [4\alpha - (4\alpha + 3)x]D^3z
+ [5\alpha^2 + \alpha - 1 - (6\alpha^2 + 9\alpha + 2)x - xy + 2y]D^2z
+ \left[ \alpha (2\alpha - 1) (\alpha + 1) - \alpha (4\alpha^2 + 9\alpha + 4)x + 2(2\alpha - 1)y - (2\alpha + 1)xy \right] Dz
+ \left[ 2\alpha (\alpha - 1) y - \alpha^2 (\alpha + 1) (\alpha + 2)x - \alpha (\alpha + 1) xy + y^2 \right] z = 0. \tag{125}
\]

This equation can be converted to standard differential form using the identity:

\[
Dz = x d\frac{dz}{dx} \tag{126}
\]

which gives:

\[
(1 - x) x^3 \frac{d^4z}{dx^4} + [2(2\alpha + 3) - (4\alpha + 9)x] x^3 \frac{d^3z}{dx^3}
+ [(\alpha + 2)(5\alpha + 3) - 3(\alpha + 2)(2\alpha + 3)x - xy + 2y] x^2 \frac{d^2z}{dx^2}
\]
\[ + [2\alpha (\alpha + 1) (\alpha + 2) - (\alpha + 1) (\alpha + 2) (4\alpha + 3) x + 4\alpha y - 2 (\alpha + 1) xy] x \frac{dz}{dx} \]
\[ + [2\alpha (\alpha - 1) y - \alpha^2 (\alpha + 1) (\alpha + 2) x - \alpha (\alpha + 1) xy + y^2] z = 0. \] (127)

From equation (105), the Fourier coefficient \( g^m_+ (x, y) \) is related to \( z(\alpha, x, y) \) by
\[ z = C x^{-\alpha} g^m_+ (x, y) \] (128)
where the constant \( C \) is given by equation (105). Differentiating equation (128) gives the relations
\[ x \frac{dz}{dx} = C x^{-\alpha} \alpha g^m_+ + \alpha \frac{dg^m_+}{dx} \] (129)
\[ x^2 \frac{d^2 z}{dx^2} = C x^{-\alpha} (\alpha + 1) g^m_+ - 2 \alpha x \frac{dg^m_+}{dx} + x^2 \frac{d^2 g^m_+}{dx^2} \] (130)
\[ x^3 \frac{d^3 z}{dx^3} = C x^{-\alpha} x \]
\[ [\alpha (\alpha + 1) (\alpha + 2) g^m_+ + 3 \alpha (\alpha + 1) x \frac{dg^m_+}{dx} - 3 \alpha x^2 \frac{d^2 g^m_+}{dx^2} + x^3 \frac{d^3 g^m_+}{dx^3}] \] (131)
\[ x^4 \frac{d^4 z}{dx^4} = C x^{-\alpha} \]
\[ [\alpha (\alpha + 1) (\alpha + 2) (\alpha + 3) g^m_+ - 4 \alpha (\alpha + 1) (\alpha + 2) x \frac{dg^m_+}{dx} + \]
\[ 6 \alpha (\alpha + 1) x^2 \frac{d^2 g^m_+}{dx^2} - 4 \alpha x^3 \frac{d^3 g^m_+}{dx^3} + x^4 \frac{d^4 g^m_+}{dx^4}] \]. (132)

and substituting these relations into equation (128) gives after much reduction the fourth-order differential equation:
\[ (1 - x) x^4 \frac{d^4 g^m_+}{dx^4} + (6 - 9x) x^3 \frac{d^3 g^m_+}{dx^3} + [6 - \alpha (\alpha - 1) - 18x + y (2 - x)] x^2 \frac{d^2 g^m_+}{dx^2} + \]
\[ - 2 [\alpha (\alpha - 1) + x (y + 3)] x \frac{dg^m_+}{dx} + y^2 g^m_+ = 0. \] (133)

### 4.2 Fourth-order differential equation from the Kampé de Fériet function

Equation (57) can be written in the form
\[ g^m_+ (x, y) = \frac{i}{(2m + 1)!} u^\alpha \bar{z} (\alpha, u, v) \] (134)
where
\[ u = xy \] (135)
and
\[ v = -y. \] (136)
The equation for \( g_m^{\pm} (x, y) \) can be established in the same manner as for equation (133), but having already derived (133), it is enough to establish that \( g_m^{\pm} (x, y) \) and \( g_m^0 (x, y) \) both obey the same differential equation. From equations (135) and (136) then

\[
\frac{\partial}{\partial u} = \frac{1}{y} \frac{\partial}{\partial x} \quad (137)
\]

and with the definition

\[
\bar{w} (\alpha, x, y) = z (\alpha, xy, -y) \quad (139)
\]

and the notation

\[
\bar{p} = \frac{\partial \bar{w}}{\partial x}, \quad \bar{q} = \frac{\partial \bar{w}}{\partial y}, \quad \bar{r} = \frac{\partial^2 \bar{w}}{\partial x^2}, \quad \bar{s} = \frac{\partial^2 \bar{w}}{\partial x \partial y}, \quad \bar{t} = \frac{\partial^2 \bar{w}}{\partial y^2}
\]

then we have

\[
\bar{p} = \frac{\hat{p}}{y} \quad (140)
\]

\[
\bar{q} = \frac{x}{y} \hat{p} - \hat{q} \quad (141)
\]

\[
\bar{r} = \frac{\hat{r}}{y^2} \quad (142)
\]

\[
\bar{s} = \frac{x}{y^2 \hat{r}} + \frac{\hat{p}}{y^2} - \frac{\hat{s}}{y} \quad (143)
\]

\[
\bar{t} = \hat{t} + 2 \frac{x}{y^2} \hat{p} + \frac{x^2}{y^2} \hat{r} - 2 \frac{x}{y} \hat{s}. \quad (144)
\]

Employing equations (139)-(144) in the Kampé de Fériet differential equations (102) and (103) gives

\[
y \bar{w} = -y^2 \hat{t} + xy \hat{s} + x \hat{p} - (\alpha + 1) y \hat{q} \quad (145)
\]

\[
y \bar{w} = (2\alpha - x) \hat{p} + x (1 - x) \hat{r} - y \hat{q} - y^2 \hat{t} + 2 xy \hat{s}. \quad (146)
\]

To compare these equations with the partial differential equations from the Horn function, we note that

\[
g^+_m (\alpha, x, y) = C x^\alpha z (\alpha, x, y) \quad (147)
\]

whereas

\[
g^0_m (\alpha, x, y) = E x^\alpha y^\alpha \bar{w} (\alpha, x, y) \quad (148)
\]

where \( C \) and \( E \) are constants. Defining

\[
w (\alpha, x, y) = y^\alpha \bar{w} (\alpha, x, y) \quad (149)
\]
then $g_m^n (x, y)$ and $g^n_m (x, y)$ will satisfy the same ordinary differential equation if $z (\alpha, x, y)$ and $w (\alpha, x, y)$ satisfy the same pair of partial differential equations. With the notation

$$\hat{p} = \frac{\partial w}{\partial x}; \hat{q} = \frac{\partial w}{\partial y}; \hat{r} = \frac{\partial^2 w}{\partial x^2}; \hat{s} = \frac{\partial^2 w}{\partial x \partial y}; \hat{t} = \frac{\partial^2 w}{\partial y^2},$$

then

$$\hat{w} (\alpha, x, y) = y^{-\alpha} w (\alpha, x, y) \quad (150)$$

$$\hat{p} = y^{-\alpha} \hat{p} \quad (151)$$

$$\hat{q} = y^{-\alpha} \left( \hat{q} - \frac{\alpha}{y} w \right) \quad (152)$$

$$\hat{r} = y^{-\alpha} \hat{r} \quad (153)$$

$$\hat{s} = y^{-\alpha} \left( \hat{s} - \frac{\alpha}{y} \hat{p} \right) \quad (154)$$

$$\hat{t} = y^{-\alpha} \left( \hat{t} + \frac{\alpha (\alpha + 1)}{y^2} w - 2 \frac{\alpha}{y} \hat{q} \right) \quad (155)$$

and substituting these relations in equations (145) and (146) gives

$$y \hat{t} - x \hat{s} + (1 - \alpha) \hat{q} + w = 0. \quad (156)$$

$$yw = [2\alpha - (2\alpha + 1) x] \hat{p} + x (1 - x) \hat{r} - y^2 \hat{t} - \alpha^2 w + (2\alpha - 1) y \hat{q} + 2xy \hat{s}. \quad (157)$$

Eliminating $yw$ from (157) using equation (156) gives

$$x (1 - x) \hat{r} + xy \hat{s} + [2\alpha - (2\alpha + 1) x] \hat{p} + \alpha y \hat{q} - \alpha^2 w = 0. \quad (158)$$

Equations (156) and (158) obtained from the Kampé de Fériet function are identical to equations (99) and (100) obtained from the Horn function, so $g_m^n (x, y)$ also satisfies the differential equation (133).

5 Fourth-order differential equations from an integral representation

The integral representation (11) for the Fourier coefficient can be written in the form:

$$\hat{G}_m^N (rR, \lambda, \omega) = \frac{\hat{y}_m (\lambda, \omega)}{\pi \sqrt{2rR}} \quad (159)$$

where

$$\hat{y}_m (\lambda, \omega) = \int_0^\pi \frac{\exp (i\lambda\sqrt{\omega - \cos \psi})}{\sqrt{\omega - \cos \psi}} \cos (m\psi) \, d\psi. \quad (160)$$
The function \( y_m(\lambda, \omega) \) satisfies the partial differential equation

\[
\frac{\partial^2 y_m(\lambda, \omega)}{\partial \omega \partial \lambda} = -\frac{\lambda}{2} y_m(\lambda, \omega) \tag{161}
\]

and this has the elementary separated solution:

\[
Y_m(s, \lambda, \omega) = C_m(s) \exp \left[ \pm i \left( s\omega + \frac{\lambda^2}{4s} \right) \right] \tag{162}
\]

where \( s \) is the separation constant. The solution \( y_m(\lambda, \omega) \) can be constructed as a superposition of the allowable (i.e. finite at infinity) elementary solutions given by (162). This gives \( y_m(\lambda, \omega) \) in the form:

\[
y_m(\lambda, \omega) = \int_0^\infty C_m(s) \exp \left[ \pm i \left( s\omega + \frac{\lambda^2}{4s} \right) \right] ds \tag{163}
\]

where the \( \pm \) sign is chosen so that the integral converges. As \( \omega \) is real and positive, this depends only on the imaginary part of \( \lambda \equiv \alpha + i\sigma \), the appropriate sign being the same as that of \( \sigma \), which will be assumed positive here. Setting \( \lambda = 0 \) in (160) and (163) gives

\[
\int_0^\infty C_m(s) \exp (is\omega) ds = \sqrt{2} Q_{m-1/2}(\omega) \tag{164}
\]

and \( C_m(s) \) can be determined from the integral [7, eqn 6.621 1]:

\[
\int_0^\infty \exp (-s\delta) J_\nu(s) s^{\nu-1} ds =
\]

\[
\frac{\Gamma(\nu + \mu)}{2^{\nu+\mu} \Gamma(\nu + 1)} \ _2F_1 \left( \frac{\nu + \mu}{2}, \frac{\nu + \mu + 1}{2}; \nu + 1; -\frac{1}{\delta^2} \right) . \tag{165}
\]

Setting \( \delta = -i\omega \), \( \nu = m \) and \( \mu = 1/2 \) in equation (165) gives

\[
\int_0^\infty \exp (i\omega s) J_m(s) s^{-1/2} ds =
\]

\[
\frac{\Gamma(m + 1/2)}{(-i)^{m+1/2} 2^m \omega^{m+1/2} \Gamma(m + 1)} \ _2F_1 \left( \frac{m + 1/2}{2}, \frac{m + 3/2}{2}; m + 1; \frac{1}{\omega^2} \right) . \tag{166}
\]

and the expression for \( Q_{m-1/2}(\omega) \) in terms of the Gauss hypergeometric function is [16, eqn 8.1.3]

\[
Q_{m-1/2}(\omega) = \frac{1}{2^m} \sqrt{\frac{\pi}{\omega}} \frac{\Gamma(m + 1/2)}{\omega^{m+1/2} \Gamma(m + 1)} \ _2F_1 \left( \frac{m + 1/2}{2}, \frac{m + 3/2}{2}; m + 1; \frac{1}{\omega^2} \right) . \tag{167}
\]
From equations (164), (166) and (167) it follows that

$$C_m(s) = \sqrt{\pi} (-i)^{m+1/2} J_m(s) s^{-1/2}$$

(168)

and hence

$$y_m(\lambda, \omega) = \sqrt{\pi} (-i)^{m+1/2} \int_0^\infty \exp \left[ i \left( \omega s + \frac{\lambda^2}{4s} \right) \right] J_m(s) s^{-1/2} ds.$$  

(169)

For the special case of evanescent waves such that \( \lambda = i\sigma \) with \( \sigma > 0 \) then with a suitable transformation in the complex plane, this equation can be expressed in the form

$$y_m(i\sigma, \omega) = \sqrt{\pi} \int_0^\infty \exp \left[ - \left( \omega s + \frac{\sigma^2}{4s} \right) \right] I_m(s) s^{-1/2} ds.$$  

(170)

The details of this transformation are given in Appendix B. Equation (170) is straightforward to evaluate numerically as the integrand is not oscillatory and decays exponentially to zero as \( s \to \infty \) provided \( \omega > 1 \), which from equation (15) is always the case. This follows immediately from the leading term in the asymptotic approximation as \( s \to \infty \) of \( I_m(s) \), which is [7, eqn 8.451 5]:

$$I_m(s) \sim \frac{\exp(s)}{\sqrt{2\pi s}}.$$  

(171)

5.1 Fourth-order differential equation in terms of \( \omega \)

The integral representation (169) allows the ordinary differential equations in terms of \( \omega \) or \( \lambda \) satisfied by \( y_m(\lambda, \omega) \) to be constructed in a straightforward manner. It is convenient to define a new variable \( \chi \) such that

$$\chi = \frac{\lambda^2}{4}$$

(172)

and also a new dependent variable \( \hat{y}_m(\chi, \omega) \) such that

$$\hat{y}_m(\chi, \omega) \equiv y_m(\lambda, \omega).$$

(173)

Then \( \hat{y}_m(\chi, \omega) \) is given by

$$\hat{y}_m(\chi, \omega) = \int_0^\infty J_m(s) f(s, \omega, \chi) ds$$

(174)

where \( \chi \) is to be regarded as a constant embedded parameter in the ODE satisfied by \( \hat{y}_m(\chi, \omega), \) and where \( f(s, \omega, \chi) \) is given by

$$f(s, \omega, \chi) \equiv \sqrt{\pi} (-i)^{m+1/2} s^{1/2} \exp \left[ i \left( \omega s + \frac{\chi}{s} \right) \right].$$

(175)
The various derivatives of \( \hat{y}(\omega, \chi) \) are then given by
\[
\frac{d^n \hat{y}_m}{d\omega^n} \equiv (i)^n \int_0^\infty s^n J_m(s) f(s, \omega, \chi) ds. \tag{176}
\]

The Bessel function \( J_m(s) \) satisfies the differential equation \[7, eqn 8.401\]
\[
\frac{1}{s} \frac{d}{ds} \left( s \frac{d J_m(s)}{ds} \right) = \left( \frac{m^2}{s^2} - 1 \right) J_m(s) \tag{177}
\]
and therefore:
\[
\int_0^\infty \left( \frac{m^2}{s^2} - 1 \right) J_m(s) f(s, \omega, \chi) ds = \int_0^\infty \frac{1}{s} \frac{d}{ds} \left( s \frac{d J_m(s)}{ds} \right) f(s, \omega, \chi) ds. \tag{178}
\]

Differentiating (178) twice with respect to \( \omega \) and utilizing equations (174)-(176) gives:
\[
\int_0^\infty (m^2 - s^2) J_m(s) f(s, \omega, \chi) ds = \int_0^\infty s \frac{d}{ds} \left( s \frac{d J_m(s)}{ds} \right) f(s, \omega, \chi) ds
\]
\[
\frac{d^2 \hat{y}_m}{d\omega^2} + m^2 \hat{y}_m = \int_0^\infty s \frac{d}{ds} \left( s \frac{d J_m(s)}{ds} \right) f(s, \omega, \chi) ds. \tag{179}
\]

Integrating this twice by parts yields
\[
\frac{d^2 \hat{y}_m}{d\omega^2} + m^2 \hat{y}_m = \int_0^\infty J_m(s) \frac{d}{ds} \left( s \frac{d}{ds} [s f(s, \omega, \chi)] \right) ds. \tag{180}
\]

Since
\[
\frac{d}{ds} \left( s \frac{d}{ds} [s f(s, \omega, \chi)] \right) = \left( \frac{1}{4} + 2i \omega s - \omega^2 s^2 + 2 \omega \chi - \chi^2 s^2 \right) f(s, \omega, \chi) \tag{181}
\]
then
\[
\frac{d^2 \hat{y}_m}{d\omega^2} + m^2 \hat{y}_m = \int_0^\infty J_m(s) \left( \frac{1}{4} + 2i \omega s - \omega^2 s^2 + 2 \omega \chi - \chi^2 s^2 \right) f(s, \omega, \chi) ds. \tag{182}
\]

For the Poisson case with \( \chi = 0 \), employing (176) in (182) gives Legendre’s equation (60) of degree \( m - 1/2 \), as must be the case for consistency with equation (14). For the Helmholtz case, (182) must be differentiated twice with respect to \( \omega \) before employing (176). This yields the fourth-order linear ODE
\[
(1 - \omega^2) \frac{d^4 \hat{y}_m}{d\omega^4} - 6\omega \frac{d^3 \hat{y}_m}{d\omega^3} + (m^2 - 2 \omega \chi - \frac{25}{4} d^2 \hat{y}_m d\omega^2 - 4 \chi \frac{d \hat{y}_m}{d\omega} - \chi^2 \hat{y}_m = 0 \tag{183}
\]
which becomes equation (58) on substituting \( \chi \equiv \lambda^2/4 \).
5.2 Equivalence of the differential equations

Equation (183) can be converted to an equation in terms of $x \equiv k^2$ by making the substitutions:

\[ \chi = \frac{xy}{2} \quad (184) \]
\[ \omega = \frac{2 - x}{x} \quad (185) \]
\[ 1 - \omega^2 = \frac{4(x - 1)}{x^2} \quad (186) \]
\[ m^2 - 2\chi \omega - \frac{25}{4} = \alpha^2 - \alpha - 6 + (x - 2) y \quad (187) \]
\[ \frac{dy}{d\omega} = -\frac{x^2 \hat{y}}{2 \ dx} \quad (188) \]
\[ \frac{d^2 y}{d\omega^2} = \frac{x^4 \hat{y}^2}{4 \ dx^2} + \frac{x^3 \hat{y}}{2 \ dx} \quad (189) \]
\[ \frac{d^3 \hat{y}}{d\omega^3} = \frac{x^5 \hat{y}^2}{8 \ dx^3} - \frac{3x^5 \hat{y}}{4 \ dx^2} + \frac{3x^4 \hat{y}}{4 \ dx} \quad (190) \]
\[ \frac{d^4 y}{d\omega^4} = \frac{x^8 \hat{y}^2}{16 \ dx^4} + \frac{3x^7 \hat{y}^2}{4 \ dx^3} + \frac{9x^6 \hat{y}^2}{4 \ dx^2} + \frac{3x^5 \hat{y}}{2 \ dx} \quad (191) \]

On collecting terms and simplifying this yields

\[ (1 - x) x^4 \frac{\hat{y}}{dx^4} + (6 - 9x) x^3 \frac{\hat{y}}{dx^3} + \]
\[ [\alpha - \alpha^2 + 6 - 18x + (2 - x) y] x^2 \frac{\hat{y}}{dx^2} + \]
\[ -2 [\alpha (\alpha - 1) + x (3 + y)] x \frac{\hat{y}}{dx} + y^2 \hat{y} = 0. \quad (192) \]

Inspection of equations (133) and (192), obtained by totally different methods, shows that they are identical.

6 Comments and conclusions

The Fourier coefficients for the Helmholtz Green function have been split into their half advanced+half retarded and half advanced−half retarded components, and these components have been given in closed form in terms of two-dimensional hypergeometric functions. These solutions generalize the well-known solutions of Poisson’s equation for ring sources, and reduce to them in the static limit when the wave number $\beta = 0$. The two-dimensional hypergeometric functions can be considered as double series, with the order of summation arbitrary. The two summation choices give different series of special functions for each of the Fourier components, and all of these series have been numerically verified, as have the closed form solutions themselves. One series is
given in terms of Hankel functions, and only a few terms are need far from the ring source for accurate results. A second series in terms of associated Legendre functions only requires a few terms in the neighborhood of the ring to give accurate results.

The systems of partial differential equations associated with each of the two generalized hypergeometric functions have been used to derive a fourth-order ordinary differential equation in terms of $x \equiv k^2$ for the Fourier coefficients. A completely different approach involving integral representations of the Fourier coefficients has been presented in tandem, which derives many of the same results, as well as some new ones. Both approaches give exactly the same fourth-order differential equation for the general Fourier coefficient, despite the algebra being rather intricate in both cases. Another fourth order ordinary differential equation in terms of the wave number parameter $\lambda$ can also be derived by the methods presented here.

A Series from an integral representation

The Fourier coefficient $G_n^m (\beta, r, R, z - Z) \equiv \hat{G}_n^m (rR, \lambda, \omega)$ given by equations (27) and (28) can be expressed in the form

$$\hat{G}_n^m (rR, \lambda, \omega) = \frac{1}{\pi \sqrt{2rR}} \sum_{n=0}^{\infty} \int_0^{\pi} \frac{(i\lambda)^n}{n!} (\omega - \cos \psi)^{(n-1)/2} \cos (m\psi) d\psi$$

(193)

where $\omega$ and $\lambda$ are defined by equations (15) and (43) respectively. Evaluation of (193) requires the integral

$$\hat{I}_{m,n} (\omega) = \int_0^{\pi} \cos (m\psi) (\omega - \cos \psi)^{(n-1)/2} d\psi$$

(194)

which can be evaluated for $m \in \mathbb{N}_0$ using the integral representation [7, eqn 8.711 2]:

$$P^m_\nu (\xi) = \frac{\Gamma (\nu + m + 1)}{\pi \Gamma (\nu + 1)} \int_0^{\pi} \cos (m\theta) \left( \xi + \sqrt{\xi^2 - 1} \cos \theta \right)^\nu d\theta$$

(195)

which is equivalent to

$$P^m_\nu (\xi) = \frac{(-1)^m \Gamma (\nu + m + 1) (\xi^2 - 1)^{\nu/2}}{\pi \Gamma (\nu + 1)} \int_0^{\pi} \cos (m\psi) \left( \frac{\xi}{\sqrt{\xi^2 - 1}} - \cos \psi \right)^\nu d\psi.$$  

(196)

The substitutions

$$\omega = \frac{\xi}{\sqrt{\xi^2 - 1}}$$

(197)
\[ \nu = \frac{n - 1}{2} \quad (198) \]

then give
\[ \hat{I}_{m,n}(\omega) = \frac{(-1)^m \pi \Gamma \left(\frac{n+1}{2}\right) (\omega^2 - 1)^{(n-1)/4}}{\Gamma (m + (n+1)/2)} P_{(n-1)/2}^m \left( \frac{\omega}{\sqrt{\omega^2 - 1}} \right) \quad (199) \]
and
\[ \hat{G}^m_{H}(r_R, \lambda, \omega) = \frac{(-1)^m}{(\omega^2 - 1)^{1/4} \sqrt{2 r_R}} \times \sum_{n=0}^{\infty} \frac{(i \lambda (\omega^2 - 1)^{1/4})^n}{n!} \frac{\Gamma \left(\frac{n+1}{2}\right)}{\Gamma (m + (n+1)/2)} P_{(n-1)/2}^m \left( \frac{\omega}{\sqrt{\omega^2 - 1}} \right) \cdot (200) \]

Splitting the series \[ (200) \] into even and odd terms gives after some reduction
\[ \hat{\Lambda}^m_+(r_R, \lambda, \omega) = \frac{(-1)^m \sqrt{\pi} \lambda}{(\omega^2 - 1)^{1/4} \sqrt{2 r_R}} \times \sum_{p=0}^{\infty} \frac{\left(-\lambda^2 \sqrt{\omega^2 - 1}\right)^p}{4^p} \frac{1}{p! \Gamma (p+m+1/2)} P_{p-1/2}^m \left( \frac{\omega}{\sqrt{\omega^2 - 1}} \right) \quad (201) \]
\[ \hat{\Lambda}^m_-(r_R, \lambda, \omega) = \frac{(-1)^m \sqrt{\pi} \lambda}{2 \sqrt{2 r_R}} \times \sum_{s=m}^{\infty} \frac{\left(-\lambda^2 \sqrt{\omega^2 - 1}\right)^s}{4^s} \frac{1}{\Gamma (s+3/2) \Gamma (m+s+1)} P_{s}^m \left( \frac{\omega}{\sqrt{\omega^2 - 1}} \right) \cdot (202) \]

where the factorial formulas
\[ (2p)! = \frac{2^{2p} \Gamma (p+1/2) p!}{\sqrt{\pi}} \quad (203) \]
\[ (2s + 1)! = \frac{2^{2s+1} \Gamma (s+3/2) s!}{\sqrt{\pi}} \quad (204) \]

have been used to obtain \[ (201) \] and \[ (202) \]. The index in \[ (202) \] runs from \( s = m \) rather than from \( s = 0 \) as the associated Legendre polynomial \( P^m_s (\xi) \) is zero for \( s < m \). The substitution \( s = p + m \) in \[ (202) \] gives the alternative form
\[ \hat{\Lambda}^m_-(r_R, \lambda, \omega) = \frac{\sqrt{\pi} \lambda}{2 \sqrt{2 r_R}} \left( \frac{\lambda^2 \sqrt{\omega^2 - 1}}{4} \right)^m \times \sum_{p=0}^{\infty} \frac{\left(-\lambda^2 \sqrt{\omega^2 - 1}\right)^p}{4^p} \frac{1}{\Gamma (p+m+3/2) \Gamma (p+2m+1)} P_{p+m}^m \left( \frac{\omega}{\sqrt{\omega^2 - 1}} \right) \cdot (205) \]
The indices in equations (201) and (205) can be switched to negative values using the relations [16, eqns 8.2.5, 8.2.1]:

\[ P^m_\nu (\xi) = \frac{\Gamma (\nu + m + 1)}{\Gamma (\nu - m + 1)} P^{-m}_\nu (\xi) \quad [m \in \mathbb{N}_0] \quad (206) \]

\[ P^m_{\nu-1} (\xi) = P^m_\nu (\xi) \quad (207) \]

which gives

\[ \hat{\Lambda}^m_+ (rR, \lambda, \omega) = \frac{(-1)^m \sqrt{\pi}}{(\omega^2 - 1)^{1/4} \sqrt{2rR}} \times \sum_{p=0}^{\infty} \left( \frac{-\lambda^2 \sqrt{\omega^2 - 1}}{4} \right)^p \frac{1}{p! \Gamma (p - m + 1/2)} P^{-m}_{-p-1/2} \left( \frac{\omega}{\sqrt{\omega^2 - 1}} \right)^m \quad (208) \]

\[ \hat{\Lambda}^m_- (rR, \lambda, \omega) = \frac{\sqrt{\pi} i \lambda}{2 \sqrt{2rR}} \left( \frac{\lambda^2 \sqrt{\omega^2 - 1}}{4} \right)^m \times \sum_{p=0}^{\infty} \left( \frac{-\lambda^2 \sqrt{\omega^2 - 1}}{4} \right)^p \frac{1}{\Gamma (p + m + 3/2) \Gamma (p + 1)} P^{-m}_{-p-1} \left( \frac{\omega}{\sqrt{\omega^2 - 1}} \right). \quad (209) \]

The kind of Legendre functions in equations (208) and (209) can be switched using the Whipple relation [16, eqn 8.2.7],[17]:

\[ P^{-\alpha-1/2}_{-\mu-1/2} \left( \frac{\omega}{\sqrt{\omega^2 - 1}} \right) = \frac{(\omega^2 - 1)^{1/4} \exp (-i\mu\pi)}{(\pi/2)^{1/2} \Gamma (\alpha + \mu + 1)} Q^\mu_\alpha (\omega) \quad (210) \]

which gives after some reduction

\[ \hat{\Lambda}^m_+ (rR, \lambda, \omega) = \frac{(-1)^m \sqrt{\pi}}{\sqrt{2rR}} \times \sum_{p=0}^{\infty} \left( \frac{\lambda^2 \sqrt{\omega^2 - 1}}{4} \right)^p \frac{Q^p_\mu (\omega)}{p! \Gamma (p - m + 1/2) \Gamma (m + p + 1/2)} \quad (211) \]

\[ \hat{\Lambda}^m_- (rR, \lambda, \omega) = \frac{(-1)^m \sqrt{\pi}}{\sqrt{2rR}} \times \sum_{p=0}^{\infty} \left( \frac{\lambda^2 \sqrt{\omega^2 - 1}}{4} \right)^{p+m+1/2} \frac{Q^{p+m+1/2}_\mu (\omega)}{p! \Gamma (p + m + 3/2) \Gamma (p + 2m + 1)}. \quad (212) \]
B Transformation of an integral in the complex plane

The integral for \( y_m(i\sigma, \omega) \) in equation (169) for \( \lambda = i\sigma \) can be considered to be the contribution along the real axis of the contour integral

\[
T_m(\sigma, \omega) = \sqrt{\pi} (-i)^{m+1/2} \oint_C \exp \left[ i \left( \omega s - \frac{\sigma^2}{4s} \right) \right] J_m(s) s^{-1/2} ds
\]  \hspace{1cm} (213)

where \( C \) is the closed contour shown in the figure below, in the limits as \( \epsilon \to 0 \) and \( R \to \infty \).

Figure 1: Contour \( C \) in the complex \( s \)–plane where \( s = x + iy \). The smaller quarter circle is of radius \( \epsilon \to 0 \) and the larger quarter circle is of radius \( R \to \infty \). The integrand of the contour integral has an isolated essential singularity at the origin of the \( s \)–plane and is analytic on and within the contour \( C \).

The corresponding contribution to the contour integral along the imaginary axis is given by

\[
U_m(\omega, \sigma) = -\sqrt{\pi} (-i)^{m+1/2} \int_0^\infty \exp \left[ - \left( \omega y + \frac{\sigma^2}{4y} \right) \right] J_m(iy) y^{-1/2} (i)^{1/2} dy
\]  \hspace{1cm} (214)

and this can be stated in terms of the modified Bessel function of the first kind.
using the identity [7, eqn 8.406 3]

\[ J_m (iy) = (i)^m I_m (y) \] (215)

which gives immediately

\[ U_m (\sigma, \omega) = -\sqrt{\pi} \int_0^\infty \exp \left[ - \left( \omega y + \frac{\sigma^2}{4y} \right) \right] I_m (y) y^{-1/2} dy. \] (216)

The integrand of (213) is analytic everywhere within the contour \( C \) and therefore from Cauchy’s theorem, the integral (213) is zero. Therefore if the contributions to the contour integral along the two quarter circles vanish in the limits as \( \epsilon \to 0 \) and \( R \to \infty \), then

\[ y_m (i\sigma, \omega) + U_m (\sigma, \omega) = 0 \] (217)

and equation (170) has been proven. Along the smaller quarter circle we set \( s = \epsilon \exp (i\theta) \) and the contribution to the contour integral becomes

\[
V_m (\sigma, \omega) = -\sqrt{\pi} (-i)^{m+1/2} \int_0^{\pi/2} \exp \left[ i \left( \omega \epsilon \exp (i\theta) - \frac{\sigma^2}{4\epsilon} \exp (-i\theta) \right) \right] \times J_m (\epsilon \exp (i\theta)) i \epsilon^{1/2} d\theta
\] (218)

and this clearly vanishes as \( \epsilon \to 0 \). Along the larger quarter circle we set \( s = R \exp (i\theta) \) and the contribution to the integral is given by

\[
W_m (\sigma, \omega) = \sqrt{\pi} (-i)^{m+1/2} \int_0^{\pi/2} \exp \left[ i \left( \omega R \exp (i\theta) - \frac{\sigma^2}{4R} \exp (-i\theta) \right) \right] \times J_m (R \exp (i\theta)) i R^{1/2} \exp (i\theta/2) d\theta.
\] (219)

The leading term in the asymptotic approximation of \( J_m (s) \) is given by [7, eqn 8.451 1]

\[ J_m (s) = \sqrt{\frac{2}{\pi s}} \cos \left( s - \frac{\pi m}{2} - \frac{\pi}{4} \right) \] (220)

and therefore

\[
W_m (\sigma, \omega) = \sqrt{2} (-i)^{m+1/2} \int_0^{\pi/2} \exp \left[ i \left( \omega R \exp (i\theta) - \frac{\sigma^2}{4R} \exp (-i\theta) \right) \right] \times \frac{(i)^{1/2}}{2} \left[ \exp \left( iR \exp (i\theta) - \frac{i \pi m}{2} - \frac{i \pi}{4} \right) + \exp \left( -iR \exp (i\theta) + \frac{i \pi m}{2} + \frac{i \pi}{4} \right) \right] d\theta.
\] (221)

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which can be expressed as

\[
\frac{(-i)^m}{\sqrt{2}} \frac{\pi/2}{\int_0^\infty} \left[ \exp \left( -i \frac{\sigma^2}{4R} \exp (-i\theta) \right) \right] \times \\
\left( \exp \left[ i \left( (\omega + 1) R \exp (i\theta) - \frac{i\pi m}{2} - \frac{i\pi}{4} \right) \right] + \exp \left[ i \left( (\omega - 1) R \exp (i\theta) + \frac{i\pi m}{2} + \frac{i\pi}{4} \right) \right] \right) d\theta.
\]

Inspection of (222) shows that the integrand vanishes exponentially in the limit as \( R \to \infty \) provided \( \omega > 1 \). This condition always holds and is also the condition for the integral in (170) to converge.

C Numerical results

The series solutions for the Fourier coefficients given by equation (54) and equations (47)- (48) were evaluated using Mathematica and the numerical performance was explored for various geometric parameters and wave numbers. For comparison, the two integrals (11) and (22) for the Fourier coefficients were also evaluated numerically for the same parameters. All four methods give identical results at locations which are neither too far away nor too close to the ring source. The numerical integration (11) performs very well at all distances from the ring, whereas the numerical integration (22) fails when either very close to the ring or too far away. No cases were identified where equation (22) was superior. The Hankel function series (54) requires fewer and fewer terms for convergence as the distance from the loop increases, and conversely performance decreases as the ring is approached. The associated Legendre function series (47) and (48) have precisely the opposite performance, with great accuracy close to the ring and failure at large distances from the ring. The two series (47) and (48) are well suited to calculations close to the ring as the associated Legendre functions themselves each contain the ring singularity as \( \omega \to 1 \). By contrast, the Hankel functions (54) are not singular at the ring and hence an increasing number of terms are required to model the singularity as the ring is approached. In all cases, there are always at least one numerical integration and one series solution which can be used to cross check each other.

Sample numerical results are given in Table 2 for moderate distances from the ring source, and shows the number of terms required by each series to match the numerical integrations exactly. Table 3 shows the performance of the Hankel series with increasing distance from the ring. The number of Hankel terms decreases to very few at large distances from the ring. Table 4 shows the performance of the two associated Legendre series (47) and (48) as the ring is approached. The real part of \( G^m_H(\beta, r, R, z - Z) \) diverges logarithmically as \( z \to Z \), whereas the imaginary part tends to a finite limit. It can be seen immediately from Table 4 that only 8 terms in each Legendre series is sufficient to calculate \( G^m_H(\beta, r, R, z - Z) \) for the range \( \{0 < z \leq 1\} \).
Table 2: Numerical solution for the Fourier coefficient $G_H^m (\beta, r, R, z - Z)$. $R = 1$, $Z = 0$ and the other parameters are given in the table. $N_1$ is the number of terms in the Hankel function series (54) needed to give the accuracy given. $N_2$ is the number of terms in the associated Legendre series (47) and (48) to provide the accuracy given. Of the two associated Legendre series, (47) requires more terms than (48) for the stated accuracy.

| $m$ | $\beta$ | $r$ | $z$ | $N_1$ | $N_2$ | $G_H^m (\beta, r, R, z - Z)$ |
|-----|---------|-----|-----|-------|-------|-------------------------------|
| 0   | 2       | 1/2 | 1/2 | 94    | 10    | $-0.4332208795 + 0.6063507453 i$ |
| 0   | 2       | 1/2 | 3/2 | 24    | 12    | $-0.4324244083 - 0.2593676946 i$ |
| 0   | 2       | 1/2 | 5   | 9     | 24    | $-0.1320141290 - 0.1413052175 i$ |
| 0   | 2       | 1/2 | 10  | 8     | 37    | $0.02892833221 + 0.0948283693 i$ |
| 0   | 2       | 1/2 | 20  | 6     | 63    | $-0.0355277935 + 0.0350269752 i$ |
| 3   | 5       | 3/2 | 1/2 | 224   | 24    | $-0.2152817201 - 0.2085849956 i$ |
| 3   | 5       | 3/2 | 1   | 97    | 25    | $0.1382226177 - 0.2010980843 i$ |
| 3   | 5       | 3/2 | 5   | 17    | 47    | $-0.009794158906 - 0.0005460392 i$ |
| 3   | 5       | 3/2 | 10  | 13    | 80    | $-0.0004846328044 + 0.000634053250 i$ |

Table 3: Series solution for the Fourier coefficient $G_H^m (\beta, r, R, z - Z)$. $m = 1$, $\beta = 6$, $R = 1$, $Z = 0$ and $r = 3/2$. $z$ is given in the table. $N_1$ is the number of terms in the Hankel function series (54) needed for the accuracy given.

| $z$   | $N_1$ | $G_H^m (\beta, r, R, z - Z)$ |
|-------|-------|-------------------------------|
| 1/2   | 227   | $0.0785417676 - 0.2281496125 i$ |
| 1     | 99    | $0.1318397799 + 0.0959755332 i$ |
| 5     | 20    | $0.04717552085 - 0.0981998477 i$ |
| 50    | 7     | $-0.001762722093 - 0.0003136682 i$ |
| 100   | 6     | $-0.0000246835014 + 0.0004487208692 i$ |
| 200   | 5     | $-4.36384289 \times 10^{-6} - 0.0001237773355 i$ |
| 1000  | 4     | $-1.884281670 \times 10^{-6} - 4.086433833 \times 10^{-6} i$ |
| 5000  | 3     | $-1.446923432 \times 10^{-7} + 1.070704963 \times 10^{-7} i$ |
| 10000 | 2     | $4.307310056 \times 10^{-8} + 1.302718185 \times 10^{-8} i$ |
| $10^{-5}$ | 1 | $-2.303180610 \times 10^{-14} + 3.865922798 \times 10^{-14} i$ |
Table 4: Series solution for the Fourier coefficient $G^m_H(\beta, r, R, z - Z)$. $m = 2$, $\beta = 1$, $R = 1$, $Z = 0$ and $r = 1$. $z$ is given in the table, and as $z \to 0$, the ring is approached. $N_2$ is the number of terms in the associated Legendre function series (47) and (48) needed for the accuracy given.

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