Trajectory of a body in a resistant medium: an elementary derivation

Riccardo Borghi

Dipartimento di Elettronica Applicata, Università degli Studi ‘Roma tre’,
Via della vasca navale 84, I-00144 Rome, Italy
E-mail: borghi@uniroma3.it

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Abstract
A didactical exposition of the classical problem of the trajectory determination of a body, subject to the gravity in a resistant medium, is proposed. Our revisitation aims to show a derivation of the solution to the problem that is as simple as possible from a technical point of view, so that it can be understood by first-year undergraduates. A central role in our analysis is played by the so-called chain rule for derivatives, which is systematically used to remove the temporal variable from Newton’s law in order to derive the differential equation of the Cartesian representation of the trajectory, with a considerable reduction of the overall mathematical complexity. In particular, for a resistant medium exerting a force quadratic with respect to the velocity, our approach leads to the differential equation of the trajectory, which allows its Taylor series expansion to be derived in an easy way. A numerical comparison of the polynomial approximants obtained by truncating such series with the solution recently proposed through a homotopy analysis is also presented.

1. Introduction
The study of the motion of a body under the action of gravity is one of the first applications of Newton’s laws met by a student on a typical first-year undergraduate course in general physics. Since the topic is not particularly exciting to present, one is tempted to find unconventional ways to expose the subject in order to render it more appealing to the audience. The determination of the body trajectory, and of the range in particular, is a good candidate for this; if the body is in a vacuum the laws of motion assume, in the time domain, simple analytical expressions that make the trajectory determination an almost trivial task. On the other hand, when the body moves in air, where the resistance acting on it is customarily modelled as a force anti-parallel to its velocity and whose modulus is proportional to the first (for ‘slow’ objects) or to the
second (for ‘fast’ objects) power of the body speed, the mathematical complexity grows so rapidly that even now the study of the motion in a resistant medium remains an active research field, as witnessed by the recent literature [1–9].

The aim of this paper is to give a didactical presentation of the trajectory determination problem that is as simple as possible from a technical point of view, so that it can be grasped even by first-year undergraduates. We are aware of several nice didactical approaches to this problem that have appeared in the past [10–18]; nevertheless, we believe that it could be possible to further reduce the mathematical level of the presentation to avoid resorting to concepts such as the hodograph [19, 20], which will typically become available to students during the second year. Our presentation is organized as a sequence of steps of growing complexity: starting from the ideal, free-friction case, a resistance linear in velocity is considered, up to the most difficult (and still open) problem of the motion of a body in the presence of a quadratic resistant force. For all three cases we propose the same approach based on the use of Cartesian coordinates, which occur naturally in describing the shape of the body trajectory. Another aspect we are willing to emphasize is that the solution of several problems in mechanics often does not require knowledge of the complete solution of Newton’s equations in the time domain, but rather the overall mathematical complexity of an apparently nontrivial problem can be considerably reduced by resorting to some ‘tricks’ with which the student should become familiar as soon as possible. Among them, the use of the so-called chain rule for derivatives appears to be a technical instrument of invaluable help to allow nontrivial problems to be addressed at an elementary level of treatment. To help students to familiarize themselves with this technique, the classical parabolic trajectory in a vacuum is rederived without touching the temporal laws of motion, but rather by removing the temporal variable from Newton’s law, written in the ‘natural’ Cartesian reference frame, through the use of the chain rule. This is done to introduce in the most transparent way to the student the approach subsequently employed when a linear resistance force is added to the dynamical model and which allows the Cartesian equation of the trajectory to be exactly retrieved through only elementary quadratures. In the final part of the paper the same methodology is applied to the more realistic case of a quadratic resistance force, for which the determination of the analytical expression of the trajectory equation still remains an open problem. In this case, the proposed approach leads to the differential equation for the trajectory in Cartesian coordinates, which, within the small slope approximation, can be integrated in an elementary way to retrieve the classical solution provided by Lamb [11, 23]. Moreover, the differential equation turns out to be particularly suitable to be solved by a power series expansion, whose single terms can, in principle, be evaluated analytically up to arbitrarily high orders. Finally, the same differential equation is used to provide an elementary derivation of the hodograph of the motion, thus establishing a transition towards more advanced approaches.

2. Trajectory of a body in vacuum

We consider the classical parabolic motion of a point mass $m$ under the action of the sole gravity force, so that Newton’s second law reduces to

$$a = g,$$

where $a$ denotes the point acceleration and $g$ the gravity acceleration. In figure 1 the Cartesian reference frame $Oxy$ used to describe the motion of the point mass is depicted, with $v_0$ denoting

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2 Richard Feynman used to call the collection of such tricks the ‘box of tools’ of any physics student [21].
3 A celebrated example of the use of the chain rule in mechanics was provided by Arnold Sommerfeld in deriving the first Kepler’s law without resorting to energy conservation [22].
the initial velocity (with x- and y-components \( v_{0,x} \) and \( v_{0,y} \), respectively). With respect to this frame equation (1) splits into the pair of scalar differential equations

\[
\begin{align*}
\dot{v}_x &= 0, \\
\dot{v}_y &= -g,
\end{align*}
\tag{2}
\]

where \( v_x \) and \( v_y \) denote the x- and y-component of the velocity, respectively, and the dot denotes derivation with respect to the time \( t \). The trajectory of the motion is naturally described, in Cartesian coordinates, by the function \( y = y(x) \) and our aim is to extract the related differential equation without solving equation (2) in the time domain. In doing so, we have to formally remove from the equations of motion the temporal variable \( t \). This can be done in a very simple way by using the chain rule for derivatives.

To this end we start from the relation \( \dot{r} = \mathbf{v} \) written in \( Oxy \), i.e.

\[
\begin{align*}
\dot{x} &= v_x, \\
\dot{y} &= v_y,
\end{align*}
\tag{3}
\]

and, after dividing side by side the two equations, we obtain

\[
y'(x) = \frac{dy}{dx} = \frac{v_y}{v_x},
\tag{4}
\]

which coincides with the slope of the trajectory \( \eta = \tan \theta \), with \( \theta \) being the angle between \( \mathbf{v} \) and the x-axis, with the superscript denoting the derivative with respect to the spatial variable.

To find the differential equation for \( y(x) \) it is now sufficient to derive both sides of equation (4) with respect to \( x \) and to take the chain rule, i.e.

\[
\frac{d}{dx} = \frac{1}{v_x} \frac{d}{dv}.
\tag{5}
\]

into account. On substituting from equation (5) into equation (4) we have

\[
\frac{d^2 y}{dx^2} = \frac{1}{v_x} \frac{d}{dv} \left( \frac{v_y}{v_x} \right) = \frac{v_y v_x - \dot{v}_y v_x}{v_x^3},
\tag{6}
\]

which, on taking equation (2) into account, leads to

\[
\frac{d^2 y}{dx^2} = -\frac{g}{v_y^2} = -\frac{g}{v_{0,y}^2},
\tag{7}
\]

and where, in the last passage, use has been made of the fact that the x-component of the velocity is constant. Equation (7) can be solved via elementary quadratures; in particular, on imposing the ‘initial’ (i.e. at \( x = 0 \)) conditions

\[
\begin{align*}
y(0) &= 0, \\
y'(0) &= \tan \theta_0,
\end{align*}
\tag{8}
\]

the well-known equation for the parabolic trajectory of the projectile is obtained:

\[
y(x) = x \tan \theta_0 - \frac{g}{2v_{0,x}^2} x^2.
\tag{9}
\]
3. Trajectory in the presence of a linear resistance force

Consider now the motion of the point mass in the presence of a resistant force proportional to the point velocity, i.e. of the form \(-bv\), with \(b\) denoting a suitable parameter. In this case Newton’s law can be cast in the following form:

\[
a = g - \frac{1}{\tau} v,
\]

where the parameter \(\tau = m/b\) rules the temporal scale of the resulting dynamics. This, in particular, suggests the use of ‘natural’ and dimensionless normalized quantities: \(\tau\) will become the temporal unity, the product \(gt\) (the so-called limit speed) the speed unity, and \(gr^2\) the displacement unity. With such units the subsequent equations will be expressed in a dimensionless form by introducing the following scaled time, velocity, and displacement variables:

\[
\begin{align*}
T &= \frac{t}{\tau}, \\
(V_x, V_y) &= \frac{1}{gr} (v_x, v_y), \\
(X, Y) &= \frac{1}{gr^2} (x, y),
\end{align*}
\]

so that equation (10) leads to the dimensionless system

\[
\begin{align*}
\dot{V}_x &= -V_x, \\
\dot{V}_y &= -1 - V_y,
\end{align*}
\]

which, once inserted into the dimensionless counterpart of equation (6), i.e.

\[
\frac{d^2Y}{dX^2} = \frac{1}{V_x} \frac{d}{dT} \left( \frac{V_y}{V_x} \right) = \frac{V_y V_x - V_x V_y}{V_x^3},
\]

leads to the following differential equation for the trajectory:

\[
\frac{d^2Y}{dX^2} = \frac{(-1 - V_y)V_x + V_y V_x}{V_x^3} = -\frac{1}{V_x^2},
\]

formally identical to equation (7). However, differently from the previous case, the \(x\)-component of the velocity is no longer constant, and we have to find the dependence of \(V_x\) on the variable \(X\). This can be easily done on using the first row of equation (12) together with the chain rule in equation (5) to remove \(T\), thus obtaining

\[
\frac{dV_x}{dX} = -1,
\]

which gives

\[
V_x = C - X,
\]

where the constant \(C\) can be found by imposing the initial condition on \(V_x\), i.e. \(V_x(0) = V_{0,x}\), so that

\[
V_x(X) = V_{0,x} - X.
\]

On substituting from equation (17) into equation (14) we obtain

\[
\frac{d^2Y}{dX^2} = -\frac{1}{(V_{0,x} - X)^2},
\]

\(^4\) This is equivalent to formally set \(\tau = 1\) and \(g = 1\), in equation (10).
which can be solved by using elementary quadratures. In particular, on taking the initial conditions in equation (8) into account, after some algebra we have

\[ Y = X \tan \theta_0 + \frac{X}{V_{0,x}} \log \left( 1 - \frac{X}{V_{0,x}} \right), \]  

which, on using equation (11) in order to remove the normalization and to come back to physical quantities, becomes

\[ y(x) = x \tan \theta + g \tau^2 \left[ \frac{x}{v_{0,x} \tau} \log \left( 1 - \frac{x}{v_{0,x} \tau} \right) \right]. \]  

It should be noted that equation (20) is known from the literature, although its derivation is customarily carried out by first solving the time-domain dynamical equation (10) and only after removing the variable \( t \) among the functions \( x(t) \) and \( y(t) \). It is also worth showing to the student how, for small values of drag force, the shape of the trajectory tends to become identical to the ideal free-friction case of equation (9). To this end, it is sufficient to consider the limiting expression, for \( \tau \to \infty \), of the logarithmic function in equation (20) which, by using a second-order Taylor expansion, turns out to be

\[ \xi + \log(1 - \xi) \sim -\frac{\xi^2}{2}, \quad \xi \to 0. \]  

4. Trajectory with a quadratic-speed drag force

4.1. Derivation of the differential equation

When the resistant medium exerts on the body a force proportional to the squared modulus of the speed, Newton’s law takes on the following form:

\[ a = g - \frac{1}{\ell} v v, \]  

where \( \ell \) is a parameter whose physical dimensions are those of a length. Similarly, as done for the linear resistant medium, it is worth giving Newton’s law a dimensionless dress by introducing the following scaled variables:

\[ T = t \sqrt{\frac{g}{\ell}}, \]

\[ (V_x, V_y) = \frac{1}{\sqrt{g \ell}} (v_x, v_y), \]

\[ (X, Y) = \frac{1}{\ell} (x, y), \]  

so that equation (22) takes on the form [11]

\[ \begin{cases} \dot{V}_x = -VV_x, \\ \dot{V}_y = -1 - VV_y, \end{cases} \]  

5 See for instance [8, 24].
6 It could be worth including as evidence the fact that, as reported for instance in [24], Newton himself was aware that a linear resistance model was not the most suitable from a physical viewpoint, but rather the resistance offered by media without rigidity would have been expected to be proportional to the square of speed, as he wrote at the end of section 1 of book 2 of his Principia [25]: Cæterum resistentiam corporum esse in ratione velocitatis, Hypothesis est magis Mathematica quam Naturalis. Obtinet hæc ratio quamproxime ubi corpora in Mediis rigore aliquo preditis tardissime moventur. In Meditis autem que rigore omni vacant resistentiae corpora sunt in duplicata ratione velocitatis.
where the speed $V$, defined as
\[
V = \sqrt{V_x^2 + V_y^2},
\] (25)
acts as a coupling factor. On substituting from equation (24) into equation (13) we find again
\[
\frac{d^2Y}{dX^2} = \frac{-1 - VV_y V_x + VV_y V_x}{V_x^2} = -\frac{1}{V_x^2},
\] (26)
but now it is no longer possible to achieve an explicit $X$-dependence for $V_x$, a fact which prevents equation (26) from being solvable via simple quadratures. In fact, on applying the chain rule to the first row of equation (24) we have
\[
\frac{dV_x}{dT} = V_x \frac{dV_x}{dX} = -V_x \frac{\sqrt{V_x^2 + V_y^2}}{V_x},
\] (27)
so that
\[
\frac{dV_x}{dX} = -\frac{\sqrt{V_x^2 + V_y^2}}{V_x}.
\] (28)
Moreover, on deriving both sides of equation (26) with respect to $X$, on taking equation (28) into account, so that
\[
\frac{d^3Y}{dX^3} = \frac{2 \frac{dV_x}{dX}}{V_x^2} \frac{dV_x}{dX} = -\frac{2}{V_x^2} \sqrt{1 + \left(\frac{V_y}{V_x}\right)^2} = -\frac{2}{V_x^2} \sqrt{1 + \left(\frac{dY}{dX}\right)^2},
\] (29)
and on using again equation (26) we obtain
\[
\frac{d^3Y}{dX^3} \approx 2 \frac{d^2Y}{dX^2} \sqrt{1 + \left(\frac{dY}{dX}\right)^2},
\] (30)
which is the differential equation satisfied by the Cartesian representation of the body trajectory $Y = Y(X)$. It has to be solved together with the initial conditions
\[
Y(0) = 0, \\
Y'(0) = \tan \theta_0, \\
Y''(0) = -\frac{1}{V_{0.x}^2},
\] (31)
but unfortunately the solution of the Cauchy problem in equations (30) and (31) cannot be given a closed form; nevertheless, it is now easily achievable numerically due to the availability of several commercial computational platforms.

From a didactical point of view it could be worth showing how equation (30) leads in an elementary way to a well-known analytical approximation of $Y(X)$, called by Parker the ‘short-time approximation’ [11],\(^7\) which is valid in the limit of small velocity slopes. In fact, within such an approximation, it is sufficient to neglect the squared term into the square root of equation (30), in such a way that the differential equation reduces to
\[
\frac{d^3Y}{dX^3} \approx 2 \frac{d^2Y}{dX^2},
\] (32)
which can be solved simply by iterated elementary quadratures. In particular, on taking equation (31) into account, after simple algebra we obtain the approximated solution, say $Y_{st}$, as
\[
Y_{st} = \frac{1 - \exp(2X)}{4V_{0.x}^2} + X \left(\frac{1}{2V_{0.x}^2} + \tan \theta_0\right),
\] (33)
which coincides with the equation given at the end of section 3 in [11] and with equation (26) at p 297 of [23].

\(^7\) As the name suggests, the derivation of Parker was done in the time domain. A different derivation can also be found in the beautiful book by Lamb [23].
4.2. Power series expansion of the trajectory

It must be appreciated that the mathematical structure of equation (30) turns out to be particularly suitable to derive in a simple way the Taylor series expansion of the trajectory, namely

\[ Y(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k, \quad (34) \]

where \( a_k = Y^{(k)}(0) \) denotes the \( k \)-th-order spatial derivative of \( Y(X) \), evaluated at \( X = 0 \). In fact, starting from the initial conditions given in equation (31) we have for the coefficient \( a_3 \)

\[ a_3 = Y'''(0) = 2Y''(0)\sqrt{1 + [Y'(0)]^2} = -\frac{2}{V_{0,x}^2 \cos \theta_0}, \quad (35) \]

which differs from the corresponding coefficient in the power series expansion of \( Y_{st} \) by a factor \( \cos \theta_0 \) that, in the case of small slopes, can be replaced by the unity. As far as the coefficient \( a_4 \) is concerned, it is sufficient to derive both sides of equation (30) with respect to \( X \) to obtain

\[ Y^{(4)} = 2 Y''(1 + Y^2) + Y'Y''' \sqrt{1 + Y^2}, \quad (36) \]

which, after rearranging and simplifying, gives at once

\[ a_4 = -\frac{4}{V_{0,x}^4} \left( V_0^2 - \frac{\sin \theta_0}{2} \right), \quad (37) \]

whereas the short-time approximation would provide

\[ Y_{st}^{(4)}(0) = -\frac{4}{V_{0,x}^4}. \quad (38) \]

It is clear that, in principle, all terms of the power series expansion in equation (34) could be evaluated by iterating this approach; however, on increasing the truncation of the expansion the complexity of the analytical expressions of the coefficients \( a_k \) rapidly grows,\(^8\) although it is easy to implement their evaluation, up to arbitrarily high orders, by using any commercial symbolic computational platform.

To give an idea of the performance of the above polynomial approximation, we analyse a case already studied in the past, related to the motion of a volleyball in air [5]. Figure 2 shows the ball trajectory (within dimensionless spatial variables) for an initial velocity with components \((v_0, x, v_0, y) = (2, 5) \text{ m s}^{-1}\) in a medium characterized by \( \ell \approx 27.64 \text{ m} \). Such data refer to the results shown in figure 4 of [5]. The exact solution, obtained by solving numerically equation (30) through the standard command \texttt{NDSolve} of the symbolic language \textsc{Mathematica}, is represented by the circles, while the dashed and the solid curves are the polynomial approximations obtained by truncating the series in equation (34) up to \( k = 3 \) and \( k = 4 \), respectively. The dotted curve represents the trajectory corresponding to the ideal case in a vacuum. From the figure it is seen that the fourth-order polynomial displays an excellent agreement with the exact curve along the whole range of interest of \( x \). To make the comparison between our figure and figure 4 of [5] more complete, the trajectory corresponding to the initial condition \((v_{0,x}, v_{0,y}) = (6, 3/2) \text{ m s}^{-1}\) is also plotted. In this case the third- and the fourth-order approximants are practically undistinguishable.

Figure 3 shows the results corresponding to \( v_0 = 14 \text{ m s}^{-1} \) and to several values of the

\(^8\) As a further example we give the expression of the fifth-order coefficient:

\[ a_5 = \frac{-8 \cos \theta_0}{V_{0,x}^5} \left( V_0^4 - 2V_0^2 \sin \theta_0 + \frac{1}{4} \cos^2 \theta_0 \right), \]

which should be compared to its ‘short-time’ version, given by \(-8/V_{0,x}^4\).
4.3. An elementary derivation of the hodograph

In the present section we want to show how the third-order differential equation (30) leads in a simple way to the hodograph of the motion [19, 20]. To this end we first recast equation (30) in terms of the slope $\eta = Y'$ as follows:

$$\frac{d^2 \eta}{dX^2} = 2 \frac{d\eta}{dX} \sqrt{1 + \eta^2},$$

which, on letting

$$\frac{d\eta}{dX} = \eta',$$

becomes

$$\frac{d\eta'}{dX} = 2\eta' \sqrt{1 + \eta'^2}. \tag{41}$$

Although the differential equation cannot be solved in analytical terms, it is still possible to find an explicit functional expression for the quantity $\eta'$ whether or not it is thought of as a function of $\eta$. Again the trick is to use the chain rule, by letting

$$\frac{d}{dX} = \frac{d\eta}{dX} \frac{d}{d\eta} = \eta' \frac{d}{d\eta}. \tag{42}$$
so that equation (41) becomes
\[
\frac{d\eta'}{d\eta} = 2\sqrt{1 + \eta^2},
\]
and \(\eta'\) can be retrieved through a simple quadrature, which gives
\[
\eta' = \eta\sqrt{1 + \eta^2} + \arcsin\eta + C
= \eta\sqrt{1 + \eta^2} + \log(\eta + \sqrt{1 + \eta^2}) + C.
\]
The constant \(C\) has to be determined by imposing that at the start of the trajectory, when its slope is \(\eta_0 = \tan\theta_0\), the corresponding value of \(\eta'\), say \(\eta'_0\), is, according to equation (26),
\[
\eta'_0 = -\frac{1}{V_{0,x}} = -\frac{1}{V_0 \cos^2 \theta_0},
\] so that, after simple algebra, we obtain
\[
C = -\frac{1}{V_{0,x}} - f(\theta_0),
\]
where the function \(f(\cdot)\) is defined by
\[
f(\theta) = \frac{\sin\theta}{\cos^2\theta} + \log \frac{1 + \sin\theta}{\cos\theta}.
\]
On substituting from equations (46) and (47) into equation (44) the following differential equation for the function \(\eta(X)\) is then obtained:
\[
\frac{d\eta}{dX} = -\frac{1}{V_{0,x}^2} + \eta\sqrt{1 + \eta^2} + \log(\eta + \sqrt{1 + \eta^2}) - f(\theta_0),
\]
which, on recalling that \( \eta = \tan \theta \) and that the lhs does coincide with \( Y''(X) \), after algebra can be recast as follows:

\[
\frac{d^2 Y}{dX^2} = -\frac{1}{V_{0,x}^2} + f(\theta) - f(\theta_0), \tag{49}
\]

and, on taking equation (26) into account, after rearranging leads to the well-known expression of the hodograph [26]

\[
V = \frac{V_{0,x}}{\cos \theta \sqrt{1 + \frac{V_{0,x}^2}{V_0^2}[f(\theta) - f(\theta_0)]}}. \tag{50}
\]

5. Conclusions

An elementary exposition of the trajectory determination problem for a body launched in an arbitrary direction, and subject to the gravity in the presence of a resistant medium, has been proposed. The main task of such an exposition was to limit the mathematical complexity at most for it to be appreciable by first-year undergraduate students, especially in the case of a quadratic resistant medium. To this end, the use of time domain Newton’s laws of the motion was systematically avoided by removing, through the use of the derivative ‘chain rule’, the temporal variable, before solving the equations. In this way the differential equation for the representation of the trajectory has been derived in a Cartesian reference frame, which turns out to be naturally suited to represent the trajectory shape. In the study of the motion in a vacuum and in a resistant medium linear in the velocity, the proposed approach was able to retrieve, in an elementary way, the well-known exact expressions of the trajectory equation. For a body moving in a resistant medium that exerts a force quadratic with respect to the velocity, the proposed approach allowed the differential equation for the Cartesian representation of the trajectory to be derived in a simple way. From the same differential equation, simple polynomial approximants of the trajectory, having the form of truncated power series expansions, have been built up to arbitrary values of the truncation order and compared to the results recently obtained through a homotopy analysis-based approach. Finally, to make a transition toward more advanced mathematical treatments of the problem, our approach was also used to derive, again in an elementary way, the closed form of the motion hodograph, a well-known result from the literature.

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