Global small solution and optimal decay rate for the Korteweg system in Besov spaces

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August 17, 2018

Abstract: In this paper we consider the Cauchy problem to the Korteweg system with the general pressure in dimension $d \geq 2$, and establish the global well-posedness of strong solution for the small initial data in $L^p$ type critical Besov spaces by using the Friedrich method and compactness arguments. Furthermore, we also obtain the optimal decay rate for the Korteweg system in $L^2$ type Besov spaces.

Keywords: Korteweg system; global small solution; Besov spaces; optimal decay rate

MSC (2010): 35Q35; 76N10

1 Introduction

This paper focuses on the Cauchy problem to the following isothermal model of capillary fluids derived by Dunn and Serrin [6], which can be used as a phase transition model

\begin{equation}
\begin{cases}
\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0, & x \in \mathbb{R}^d, t > 0, \\
\partial_t (\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \text{div}(\mu(\rho) \mathbb{D} \mathbf{u}) - \nabla (\lambda(\rho) \text{div} \mathbf{u}) + \nabla P(\rho) = \text{div} \mathbf{K}, & x \in \mathbb{R}^d, t > 0, \quad (1.1) \\
(\mathbf{u}, \rho)|_{t=0} = (\mathbf{u}_0, \rho_0), & x \in \mathbb{R}^d, t > 0,
\end{cases}
\end{equation}

here $\mathbf{u}(t, x)$ denotes the velocity field and $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density, respectively. $\mu(\rho)$ and $\lambda(\rho)$ are the shear and bulk viscosity coefficients of the flow which fulfill the standard strong parabolicity assumption:

$$
\mu > 0 \quad \text{and} \quad 2\mu + \lambda > 0.
$$

In the physical case the viscosity coefficients satisfy $2\mu + d\lambda > 0$ which is a special case of the previous assumption. The strain tensor $\mathbb{D} \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^t \mathbf{u})$ is the symmetric part of the velocity gradient. The pressure $P$ is suitably smooth increasing function of the density $\rho$. The Kortewe...
tensor \( \text{div} K \) allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid-vapor, which can be written as follows:

\[
\text{div} K = \nabla \left( \rho \kappa (\rho) \Delta \rho + \frac{\kappa (\rho) + \rho \kappa' (\rho)}{2} |\nabla \rho|^2 \right) - \text{div} (\kappa (\rho) \nabla \rho \otimes \nabla \rho),
\]

where the regular function \( \kappa \) denotes the capillary coefficient. It is worth mentioning that a typical pressure term \( P(\rho) \) is assumed to obey the following polytropic law,

\[
P(\rho) = A \rho^\gamma,
\]

where \( A \) is the entropy constant and \( \gamma \geq 1 \) is called the adiabatic index.

When \( \text{div} K \equiv 0 \), the system (1.1) reduces to the compressible Navier-Stokes equations (CNS). There have been huge literatures on the study of CNS by many physicists and mathematicians due to its physical importance, complexity, rich phenomena and mathematical challenges (see [5, 13, 14, 7, 8, 9] and the references therein). To explore the scaling invariance property, Danchin first introduce in his series papers the “Critical Besov Spaces” which were inspired by those efforts on the incompressible Navier-Stokes. By critical, we mean that the solution space that we shall consider has the same scaling invariance by time and space dilations as (1.1) itself. Precisely speaking, we can check that if \((\rho, u)\) solves (1.1), so does \((\rho_\ell, u_\ell)\) where:

\[(\rho_\ell, u_\ell)(t, x) = (\rho(\ell^2 t, \ell x), \ell u(\ell^2 t, \ell x)),
\]

provided that the pressure term \( P \) has been changed into \( \ell^2 P \). This suggests us to choose initial data \((\rho_0, u_0)\) in “critical spaces” whose norm is invariant for all \( \ell > 0 \) by the transformation \((\rho_0, u_0) = (\rho(\ell), \ell u(\ell))\). Now, we briefly review some results concerned with the multi-dimensional compressible Korteweg system in the framework of critical Besov spaces which are more relatively with our problem. Danchin and Desjardins in [11] have been the first to obtain the existence of global strong solution with small initial data when \((\rho_0 - \bar{\rho}, u_0) \in \dot{B}^{\frac{d}{2}}_{2,1} \times \dot{B}^{\frac{d}{2} - 1}_{2,1} \). Recently, Haspot consider the cases when the viscosity coefficients \( \mu(\rho), \lambda(\rho) \) and the pressure \( P(\rho) \) depends linearly on the density for the system (1.1)-(1.2) with \( \kappa(\rho) = \frac{\gamma}{\rho} \), and obtain the global solution with the suitable small initial data in the \( L^2 \) framework. Subsequently, Haspot continue to investigate the Cauchy problem of the system (1.1)-(1.2) with \( \mu(\rho) = \mu \rho, \lambda(\rho) = 0 \) and the pressure \( P(\rho) = \rho \), and establish the global solution under the setting of slightly subcritical \( L^p \) type initial data, where the specific choice of the pressure is crucial since it provides a gain of integrability on the effective velocity \( v \). However, for the general pressure which fulfills (1.3), the existence of global strong solutions with large initial data is still an open problem even in dimension \( d = 2 \). Motivated by this work, our goal in the present paper is devoted to study the global well-posedness of strong solution for the system (1.1)-(1.2) with the small initial data in the \( L^p \) framework. Concerning the existence of global weak solutions, we refer to [2, 18, 19] and the references therein.

So far there are lots of mathematical results on the large time behavior of the solution of (CNS) for the initial data \((\rho_0, u_0)\) which is a small perturbation of an equilibrium state \((\bar{\rho}, 0)\). In this research direction, the first effort is due to Matsumura and Nishida [22, 23], they showed the global in time existence of solution to (CNS) in \( \mathbb{R}^3 \) assumed that the initial small perturbation \((\rho_0 - \bar{\rho}, u_0)\) in \( H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \) and obtained the following decay estimates

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To the best of our knowledge, the approach relies heavily on the analysis of the linearization of the system and the decay estimates of the semi-group which is the reason why almost all the convergence rate of solution as \( t \to \infty \) are restricted to the regime which is near the equilibrium state. We also mention those works by [20, 21, 26] and the references therein. Okita [24] showed the global in time existence of solution to (CNS) for \( d \geq 3 \) when the initial perturbation \((\rho_0 - \bar{\rho}, \mathbf{u}_0)\) is sufficiently small in \(((B_{2,1}^\frac{d}{2} \cap \dot{B}_1^0) \times (B_{2,1}^{\frac{d-1}{2}} \cap \dot{B}_1^{0}))\). Furthermore, Okita apply the decay estimates for the low frequency part of \( E(t) \) which is generated by the linearized operator at the constant state \((\bar{\rho}, 0)\) and invoke the energy method for the high frequency part, and obtain the optimal \( L^2 \) decay rate for strong solutions in critical Besov spaces, namely,

\[
|(|\rho - \bar{\rho}, \mathbf{u}|(t))|_{B_{2,1}^\frac{d}{2}+1} \lesssim (1 + t)^{-\frac{d}{2}} \quad \text{for} \quad t \geq 0.
\]

Recently, Danchin established the optimal time-decay estimate of solution to (CNS) in the general critical Besov framework (for more details, see [15]). All the decay results we mentioned above are obtained for the system (CNS) which is corresponds to (1.1) with \( \text{div} \mathbb{K} = 0 \). Inspired by Okita’s work, our second purpose is to consider the longtime behavior of the solution of (CNS) with the Korteweg tensor \( \text{div} \mathbb{K} \) in the present paper.

1.1 Reformulation of the System

The main difficulties in the study of the compressible fluid flows when dealing with the vacuum is that the momentum equation loses its parabolic regularizing effect. That is why in the present paper we suppose that the initial data \( \rho_0 \) is a small perturbation of an equilibrium state \( \bar{\rho} = 1 \) (just for convenience). Following the specific choice on the capillary coefficient \( \kappa(\rho) = \frac{\rho}{\bar{\rho}} \) with \( \kappa = \mu^2 \) in [16], then from (1.2), we have \( \text{div} \mathbb{K} = \mu^2 \text{div}(\rho \nabla \nabla \ln \rho) \). Denoting \( a = \rho - 1 \) and introducing the effective velocity \( \mathbf{v} = \mathbf{u} + \mu \nabla \ln(1 + a) \), hence, as long as \( \rho \) does not vanish, we can reformulate the system (1.1)-(1.2) equivalently as follows

\[
\begin{aligned}
\partial_t a - \mu \Delta a + \text{div} \mathbf{v} &= f(a, \mathbf{v}) := -\text{div}(\mathbf{a} \cdot \nabla \mathbf{v}), \\
\partial_t \mathbf{v} - \mu \Delta \mathbf{v} + \nabla a &= g(a, \mathbf{v}) := 2\mu \nabla \ln(1 + a) \cdot \nabla \mathbf{v} - I(a) \nabla a - \mathbf{v} \cdot \nabla \mathbf{v}, \\
(a, \mathbf{v})|_{t=0} &= (a_0, \mathbf{v}_0),
\end{aligned}
\tag{1.4}
\]

where, we denote \( I(a) := \frac{P'(1+a)}{1+a} - 1 \) and assume that \( P'(1) = 1 \) without loss of generality. Here, denoting \( G(a) := \int_0^a \frac{P'(1+t)}{1+t} - 1 dt \), we also have \( \nabla G(a) = I(a) \nabla a \).

1.2 Main Results

We denote

\[
\mathcal{E} = \left\{ (a, \mathbf{v}) \in \left( L^1(\mathbb{R}^+; \mathcal{B}_{2,p}^{\frac{d+1}{p}+1}) \cap L^\infty(\mathbb{R}^+; \mathcal{B}_{2,p}^{\frac{d-1}{p}+1}) \right) \times \left( L^1(\mathbb{R}^+; \mathcal{B}_{2,p}^{\frac{d}{p}+1}) \cap L^\infty(\mathbb{R}^+; \mathcal{B}_{2,p}^{\frac{d-1}{p}+1}) \right)^d \right\}
\]

Our first result of this paper as follows.
Theorem 1.1 Let $d \geq 2, 2 \leq p < d$ and $p \leq \min \{4, \frac{2d}{d-2}\}$. Assume that $(\nabla a_0, v_0) \in \dot{B}_p^{\frac{d}{p} - 1}$ and $(a_0^\ell, v_0^\ell) \in \dot{B}_2^{\frac{d}{2} - 1}$. There exists a constant $\varepsilon$ such that if
\[
\|(a_0, v_0)\|_{\dot{B}_2^{\frac{d}{2} - 1}}^{\ell} + \|\nabla a_0, v_0\|_{\dot{B}_p^{\frac{d}{p} - 1}}^h \leq \varepsilon,
\]
then (1.4) has a unique global-in-time solution $(a, v) \in \mathcal{E}$. Moreover, there holds for all $T \geq 0$,
\[
X_{p,T}(a, v) \leq C_0 X_{p,0},
\]
where
\[
X_{p,0} = \|(a_0, v_0)\|_{\dot{B}_2^{\frac{d}{2} - 1}}^{\ell} + \|\nabla a_0, v_0\|_{\dot{B}_p^{\frac{d}{p} - 1}}^h
\]
and
\[
X_{p,t}(a, v) = \|(a, v)\|_{L_p^p(\dot{B}_2^{\frac{d}{2} - 1}) \cap L^1(\dot{B}_p^{\frac{d}{p} + 1})}^\ell + \|\nabla a, v\|_{L_p^p(\dot{B}_p^{\frac{d}{p} - 1}) \cap L^1(\dot{B}_p^{\frac{d}{p} + 1})}^h.
\]

Remark 1.1 When $p = 2$ in Theorem 1.1, we then have $a_0 \in \dot{B}_2^{\frac{d}{2} - 1} \cap \dot{B}_2^{\frac{d}{2} - 1}$ and $v_0 \in \dot{B}_2^{\frac{d}{2} - 1}$. Let $d \geq 3$. Assume that $a_0 \in B_{2,1}^{\frac{d}{2}}$ and $v_0 \in B_{2,1}^{\frac{d}{2} - 1}$ satisfying $\|a_0\|_{\dot{B}_2^{\frac{d}{2} - 1}} + \|v_0\|_{\dot{B}_2^{\frac{d}{2} - 1}} \leq \varepsilon$ for small enough $\varepsilon > 0$, we also have $a \in C([0, T]; B_{2,1}^{\frac{d}{2}})$, $v \in C([0, T]; B_{2,1}^{\frac{d}{2} - 1})$ for all $T \geq 0$.

We now state our second result of this paper which gives the optimal $L^2$ decay rate for strong solutions.

Theorem 1.2 Let $d \geq 3$. Assume that $a_0 \in B_{2,1}^{\frac{d}{2}} \cap \dot{B}_{1,\infty}^{0}$ and $v_0 \in B_{2,1}^{\frac{d}{2} - 1} \cap \dot{B}_{1,\infty}^{0}$. There exists a constant $\varepsilon$ such that if
\[
M_0 := \|a_0\|_{\dot{B}_2^{\frac{d}{2} - 1} \cap \dot{B}_{1,\infty}^0} + \|v_0\|_{\dot{B}_2^{\frac{d}{2} - 1} \cap \dot{B}_{1,\infty}^0} \leq \varepsilon,
\]
then (1.4) has a unique global-in-time solution $(a, v)$. Furthermore, we have for all $t \geq 0$,
\[
\|a(t)\|_{\dot{B}_2^{\frac{d}{2}}} + \|v(t)\|_{\dot{B}_2^{\frac{d}{2} - 1}} \leq \widetilde{C}_0(1 + t)^{-\frac{d}{4}}.
\]

Remark 1.2 By $\dot{B}_2^{\frac{d}{2} - 1} \subset L^2$, the convergence rate of Theorem 1.2 is optimal.

The rest of this paper is structured as follows. In Section 2 we present some notions and basic tools. In Section 3 we establish the a priori estimates for the linearized equation of the system (1.1) which will be crucial in the proof of Theorem 1.1. In Section 4 we prove the Theorem 1.1 by utilizing the Friedrich method and compactness arguments. In Section 5 we complete the proof of Theorem 1.2 by spectral analysis to the linearized system.

2 Preliminaries

In this section, we introduce some notations and conventions, and recall some standard theories of Besov space which will be used throughout this paper.
2.1 Notations

In the following, we denote by $(\cdot | \cdot)$ the $L^2$ scalar product and use the convention that $C$, with or without subscripts, to denote strictly positive constants whose values are insignificant and may change from line to line. $A \lesssim B$ means that there is a uniform positive constant $c$ independent of $A$ and $B$ such that $A \leq cB$. For $X$ a Banach space and $T > 0$, we denote by $L^p_T(X)$ with $p \in [1, \infty]$ stands for the set of measurable functions on $[0, T]$ with values in $X$, such that $t \mapsto \|f(t)\|_X \in L^p([0, T])$.

2.2 Littlewood-Paley theory and Besov spaces

The following material involving the theories of Littlewood-Paley is standard, we refer the readers to Bahouri, Chemin and Danchin [1].

Let $C$ denote the annulus $\{ \xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3 \}$ and $B$ denote the ball $\{ \xi \in \mathbb{R}^d : |\xi| \leq 4/3 \}$. There exist two radial functions $\chi \in C_\infty^c(B(0, 4/3))$ and $\varphi \in C_\infty^c(C)$ both taking values in $[0, 1]$ such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^d \setminus \{0\} \quad \text{and} \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^d.$$

For every $f \in \mathcal{S}'(\mathbb{R}^d)$, the homogeneous (or nonhomogeneous) dyadic blocks $\hat{\Delta}_j$ (or $\Delta_j$) and homogeneous (or nonhomogeneous) low-frequency cut-off operator $\hat{S}_j$ (or $S_j$) are defined as follows

$$\forall j \in \mathbb{Z}, \quad \hat{\Delta}_j f = \varphi(2^{-j}D)f \quad \text{and} \quad \hat{S}_j f = \chi(2^{-j}D)f = \sum_{q \leq j-1} \hat{\Delta}_q f;$$

$$\Delta_j f = \begin{cases} 
0, & j \leq -2; \\
\chi(D)f, & j = -1; \\
\varphi(2^{-j}D)f, & j \geq 0;
\end{cases}$$

and

$$S_j f = \sum_{q=-1}^{j-1} \Delta_q f.$$

Unfortunately, for the homogeneous case, the Littlewood-Paley decomposition is invalid. We need to a new space to modify it, namely,

$$\mathcal{S}'_h \triangleq \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \lim_{j \to -\infty} \|\chi(2^{-j}D)f\|_{L^\infty} = 0 \right\}.$$

Then we have the formal Littlewood-Paley decomposition in the homogeneous case

$$f = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j f, \quad \forall f \in \mathcal{S}'_h.$$

With a suitable choice of $\varphi$, one can easily verify that the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\hat{\Delta}_j \hat{\Delta}_k f \equiv 0 \quad \text{if} \quad |j - k| \geq 2 \quad \text{and} \quad \hat{\Delta}_j (\hat{S}_{k-1} f \hat{\Delta}_k f) \equiv 0 \quad \text{if} \quad |j - k| \geq 5. \quad (2.1)$$
Next we recall Bony’s decomposition from [1]:
\[ uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v), \]
with
\[ \dot{T}_u v = \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u, v) = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}_j v, \quad \dot{\Delta}_j v = \sum_{|j' - j| \leq 1} \Delta_{j'} v. \]

The operators \( \Delta_j \) and \( \dot{\Delta}_j \) help us recall the definition of the inhomogenous Besov spaces, the homogenous Besov spaces and hybrid-Besov spaces (see [1])

**Definition 2.1** Let \( s \in \mathbb{R}, T > 0 \) and \( 1 \leq p, r, q \leq \infty \). The homogeneous Besov space \( B_{p, r}^s \) is the set of tempered distribution \( f \in S'_h \) satisfying
\[ \| f \|_{B_{p, r}^s} \triangleq \left\| (2^{js} \| \dot{\Delta}_j f \|_{L^p})_j \right\|_{\ell^r} < \infty. \]
The time-space homogeneous Besov spaces is the set of tempered distribution \( f \) satisfying
\[ \lim_{j \to -\infty} \| \dot{S}_j f \|_{L^q_{\infty}(L^\infty)} = 0 \]
and
\[ \| f \|_{\dot{B}_{p, r}^s} \triangleq \left\| (2^{js} \| \dot{\Delta}_j f \|_{L^p})_j \right\|_{\ell^r} < +\infty. \]
The nonhomogeneous Besov space \( B_{p, r}^s \) is the set of tempered distribution \( f \) satisfying
\[ \| f \|_{B_{p, r}^s} \triangleq \left\| (2^{js} \| \Delta_j f \|_{L^p})_j \right\|_{\ell^r} < \infty. \]

**Remark 2.1**

- Restricting the above norms to the low or high frequencies parts of distributions will be fundamental in our approach. Fix some integer \( j_0 \), we denote
\[ \| f \|_{\dot{B}_{p, 1}^s} = \sum_{j \leq j_0} 2^{js} \| \dot{\Delta}_j f \|_{L^p} \quad \text{and} \quad \| f \|_{\dot{B}_{p, \infty}^s} = \sum_{j \geq j_0 + 1} 2^{js} \| \dot{\Delta}_j f \|_{L^p}; \]
\[ \| f \|_{\dot{L}^q_{\infty}(\dot{B}_{p, 1}^s)} = \sum_{j \leq j_0} 2^{js} \| \dot{\Delta}_j f \|_{\dot{L}^q_{\infty}(L^\infty)} \quad \text{and} \quad \| f \|_{\dot{L}^q_{\infty}(\dot{B}_{p, \infty}^s)} = \sum_{j \geq j_0 + 1} 2^{js} \| \dot{\Delta}_j f \|_{\dot{L}^q_{\infty}(L^\infty)}. \]

- By Minkowski’s inequality, it is easy to find that
\[ \| f \|_{\dot{L}^q_{\infty}(\dot{B}_{p, r}^s)} \leq \| f \|_{L^q_{\infty}(\dot{B}_{p, r}^s)} \quad \text{if} \quad q \leq r \quad \text{and} \quad \| f \|_{\dot{L}^q_{\infty}(\dot{B}_{p, r}^s)} \geq \| f \|_{L^q_{\infty}(\dot{B}_{p, r}^s)} \quad \text{if} \quad q \geq r. \]

Now we introduce the hybrid-Besov space we will work with in this paper. Let \( j_0 > 0 \) be as in Lemma 3.1.

**Definition 2.2** Let \( s, \sigma \in \mathbb{R} \) and \( 1 \leq p \leq \infty \). The hybrid-Besov space \( \dot{B}_{2, p}^{s, \sigma} \) is the set of tempered distribution \( f \in S'_h \) satisfying
\[ \| f \|_{\dot{B}_{2, p}^{s, \sigma}} \triangleq \sum_{j \leq j_0} 2^{js} \| \dot{\Delta}_j f \|_{L^2} + \sum_{j \geq j_0 + 1} 2^{j\sigma} \| \dot{\Delta}_j f \|_{L^p} < \infty. \]
The time-space hybrid-Besov space \( L^q_T(\dot{B}_{2, p}^{s, \sigma}) \) is the set of tempered distribution \( f \) satisfying
\[ \lim_{j \to -\infty} \| \dot{S}_j f \|_{L^q_{\infty}(L^\infty)} = 0, \]
and
\[ \| f \|_{L^q_T(\dot{B}_{2, p}^{s, \sigma})} \triangleq \sum_{j \leq j_0} 2^{js} \| \dot{\Delta}_j f \|_{L^q_T(L^2)} + \sum_{j \geq j_0 + 1} 2^{j\sigma} \| \dot{\Delta}_j f \|_{L^q_T(L^p)} < \infty. \]
2.3 Basic Properties

The following Bernstein lemma will be stated as follows (see [1]):

**Lemma 2.1** ([1]) Let $1 \leq p \leq q \leq \infty$ and $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^d$. Assume that $f \in L^p$, then for any $\alpha \in \mathbb{N}^d$, there is a constant $C$ independent of $f$, $j$ such that

$$
\text{Supp } \hat{f} \subset \lambda \mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p},
$$

$$
\text{Supp } \hat{f} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|f\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} \leq C^{k+1} \lambda^k \|f\|_{L^p}.
$$

As a result of Bernstein’s inequalities, we have the following Besov embedding theorem.

**Lemma 2.2** ([1]) Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then we have for $s \in \mathbb{R}$

$$
\dot{B}^{s}_{p_1,r_1} \hookrightarrow \dot{B}^{s-d(\frac{1}{p_1} - \frac{1}{r_2})}_{p_2,r_2}.
$$

Next, we give the important product acts on homogenous Besov spaces and composition estimate which will be also often used implicity throughout the paper.

**Lemma 2.3** ([5]) Let $d \geq 2, 1 \leq p \leq \infty$ and $s \leq d/p, t \leq d/p$ with $s + t > d \max\{0, \frac{2}{p} - 1\}$. Then we have for $(f, g) \in \dot{B}^{s}_{p,1}(\mathbb{R}^d) \times \dot{B}^{s}_{p,1}(\mathbb{R}^d)$

$$
\|fg\|_{\dot{B}^{s+t-d/p}_{p,1}} \leq C \|f\|_{\dot{B}^{s}_{p,1}} \|g\|_{\dot{B}^{t}_{p,1}}.
$$

**Lemma 2.4** ([1]) Let $s > 0, p \in [1, \infty]$ and $f, g \in \dot{B}^{s}_{p,1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

1. Assume that $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^d)$ with $F(0) = 0$, then there exists a function $C$ depending only on $s, p, d$ and $F$ such that

$$
\|F(f)\|_{\dot{B}^{s}_{p,1}} \leq C(\|f\|_{L^\infty}) \|f\|_{\dot{B}^{s}_{p,1}}.
$$

2. Assume that $H \in W^{[s]+3,\infty}_{loc}(\mathbb{R}^d)$ with $H'(0) = 0$ and $g \in \dot{B}^{s}_{p,1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then there exists a function $C$ depending only on $s, p, d$ and $H$ such that

$$
\|H(f) - H(g)\|_{\dot{B}^{s}_{p,1}} \leq C(\|f\|_{L^\infty}, \|g\|_{L^\infty}) \|f - g\|_{\dot{B}^{s}_{p,1} \cap L^\infty}(\|f\|_{\dot{B}^{s}_{p,1} \cap L^\infty} + \|g\|_{\dot{B}^{s}_{p,1} \cap L^\infty}).
$$

**Remark 2.2** Let $s > 0, p \in [1, \infty]$ and $f, g \in \dot{B}^{s}_{p,1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $\|f\|_{L^\infty} \leq \frac{1}{2}, \|g\|_{L^\infty} \leq \frac{1}{2}$. Let $\delta(x)$ be a smooth function with value in $[0, 1]$, supported in the ball $B(0, \frac{3}{5})$ and equal to 1 on $B(0, \frac{3}{7})$. If $E(x) = \ln(1 + x)\delta(x)$, then there exists a function $C$ depending only on $s, p, d$ and $H$ such that

$$
\|E(f)\|_{\dot{B}^{s}_{p,1}} \leq C(\|f\|_{L^\infty}) \|f\|_{\dot{B}^{s}_{p,1}},
$$

$$
\|E(f) - E(g)\|_{\dot{B}^{s}_{p,1}} \leq C(\|f\|_{L^\infty}, \|g\|_{L^\infty}) \|f - g\|_{\dot{B}^{s}_{p,1} \cap L^\infty}(\|f\|_{\dot{B}^{s}_{p,1} \cap L^\infty} + \|g\|_{\dot{B}^{s}_{p,1} \cap L^\infty}).
$$

Finally, we end this section with the following interpolation inequality.

**Lemma 2.5** ([1]) For $(p, r_1, r_2, r) \in [1, \infty]^4$, $s_1 \neq s_2$ and $\theta \in (0, 1)$, the following inequality holds

$$
\|u\|_{\dot{B}^{s_1+(1-\theta)s_2}_{p,r_1}} \leq C \|u\|_{\dot{B}^{s_1}_{p,r_1}} \|u\|_{\dot{B}^{s_2}_{p,r_2}}^{1-\theta} \|u\|_{\dot{B}^{s_2}_{p,r_2}}^{\theta}.
$$
2.4 Some useful lemmas

Lemma 2.6 ([24]) (i) Let $a, b > 0$ satisfying $\max\{a, b\} > 1$. Then
\[
\int_0^t (1 + s)^{-a}(1 + t - s)^{-b}ds < C(1 + t)^{-\min\{a, b\}}, \quad t > 0.
\]
(ii) Let $a, b > 0$ and $f \in L^1(0, \infty)$. Then
\[
\int_0^t (1 + s)^{-a}(1 + t - s)^{-b}f ds < C(1 + t)^{-\min\{a, b\}} \int_0^t |f| ds, \quad t > 0.
\]

Lemma 2.7 ([12]) If $\text{Supp } F f \subset \{ \xi \in \mathbb{R}^d : R_1 \lambda \leq |\xi| \leq R_2 \lambda \}$, then there exists $c$ depending only on $d$, $R_1$ and $R_2$ such that for all $1 < p < \infty$,
\[
c\lambda^2 \left( \frac{p-1}{p} \right) \int_{\mathbb{R}^d} |f|^p dx \leq (p-1) \int_{\mathbb{R}^d} |\nabla f|^2 |f|^{p-2} dx = - \int_{\mathbb{R}^d} \Delta f |f|^{p-2} dx.
\]

Lemma 2.8 Let $d \geq 2, 2 \leq p < 2d$ and $p \leq \min\{4, \frac{2d}{d-2}\}$. Then for $f, g \in \dot{B}^{d/p}_{p,1}(\mathbb{R}^d) \cap \dot{B}^{d/p-1}_{p,1}(\mathbb{R}^d)$, we have
\[
\|fg\|_{\dot{B}^{d/2-1}_{2,1}} \leq C(\|f\|_{\dot{B}^{d/p}_{p,1}} \|g\|_{\dot{B}^{d/p-1}_{p,1}} + \|f\|_{\dot{B}^{d/p}_{p,1}} \|g\|_{\dot{B}^{d/p-1}_{p,1}}).
\]

Proof In order to prove our claim, by Bony’s decompose, we write $fg$ as follows
\[
fg = \dot{T}_f g + \dot{T}_g f + \dot{R}(f, g).
\]

Let $\frac{1}{q} = \frac{1}{2} - \frac{1}{p}$. Since $p \leq \min\{4, \frac{2d}{d-2}\}$, we have $2 \leq p \leq 4 \leq q$ and $q \geq d$.

For the term $\dot{T}_f g$, we deduce that
\[
||\dot{T}_f g||_{\dot{B}^{d/2-1}_{2,1}} \leq \sum_{j \in \mathbb{Z}} \sum_{j-k \leq 4} 2^{j(d/2-1)} ||\dot{\Delta}_j (\dot{S}_{k-1} f \dot{\Delta}_k g)||_{L^2}
\]
\[
\leq \sum_{j \in \mathbb{Z}} \sum_{j-k \leq 4} 2^{j(d/2-1)} ||\dot{S}_{k-1} f||_{L^q} ||\dot{\Delta}_k g||_{L^p} \quad (\text{by Hölder’s inequality})
\]
\[
\leq \sum_{j \in \mathbb{Z}} \sum_{j-k \leq 4} 2^{j(d/2-1)} \sum_{l \leq k-2} ||\dot{\Delta}_l f||_{L^q} ||\dot{\Delta}_k g||_{L^p}
\]
\[
\leq \sum_{j \in \mathbb{Z}} \sum_{j-k \leq 4} 2^{j(d/2-1)} \sum_{l \leq k-2} 2^{(d/p+1-k/2)(d/2-1)} ||\dot{\Delta}_l f||_{L^p} ||\dot{\Delta}_k g||_{L^p} \quad (\text{by Bernstein inequality})
\]
\[
\leq \|f\|_{\dot{B}^{d/p}_{p,1}} \sum_{j \in \mathbb{Z}} \sum_{j-k \leq 4} 2^{(j-k)(d/2-1)} 2^{k/2} ||\dot{\Delta}_k g||_{L^p}
\]
\[
\leq \|f\|_{\dot{B}^{d/p}_{p,1}} \|g\|_{\dot{B}^{d/p-1}_{p,1}},
\]
where we have used that $p \leq 4$ in the fourth step and $p \leq 2d/(d-2)$ in the fifth step.

Similarly, for the term $\dot{T}_g f$, we have
\[
||\dot{T}_g f||_{\dot{B}^{d/2-1}_{2,1}} \lesssim \|g\|_{\dot{B}^{d/p-1}_{p,1}} \|f\|_{\dot{B}^{d/p}_{p,1}}.
\]
By (2.1)-(2.2), we see that the Fourier transform of $\hat{\Delta}_k u \hat{\Delta}_k v$ is supported in $2^k B(0,8)$, which implies

$$\hat{\Delta}_j (\hat{\Delta}_k u \hat{\Delta}_k v) = 0 \quad \text{for} \quad j \geq k + 4.$$ 

For the last term $\dot{R}(f, g)$, we get

$$\|\dot{R}(f, g)\|_{B^{d/p-1}_{2,1}} \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{j(\frac{d}{2} - 1)} \|\hat{\Delta}_j (\hat{\Delta}_k f \hat{\Delta}_k g)\|_{L^2} \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{j(\frac{d}{2} - 1)} \|\hat{\Delta}_k f\|_{L^q} \|\hat{\Delta}_k g\|_{L^p} \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{j(\frac{d}{2} - 1)} 2^{(\frac{d}{2} - 1)k} \|\hat{\Delta}_k f\|_{L^p} \|g\|_{B^{d/p}_{p,1}} \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{(\frac{d}{2} - 1)(j-k)} 2^{(\frac{d}{2} - 1)k} \|\hat{\Delta}_k f\|_{L^p} \|g\|_{B^{d/p}_{p,1}} \lesssim \|f\|_{B^{d/p}_{d/p-1}} \|g\|_{B^{d/p}_{p,1}}.$$

This completes the proof of this lemma.

## 3 A priori estimates for the linearized equation

In this section, we consider the following linearized equation of the system which paly an important role in the proof of our theorem:

$$\begin{cases}
\partial_t a - \mu \Delta a + \text{div} \, \mathbf{v} = f, \\
\partial_t \mathbf{v} - \mu \Delta \mathbf{v} + \nabla a = g, \\
(a, \mathbf{v})|_{t=0} = (a_0, \mathbf{v}_0).
\end{cases} \tag{3.1}$$

We have the following lemma.

**Lemma 3.1** Let $(a, \mathbf{v})$ be the smooth solution of system (3.1). Then there holds for all $t \in [0, T]$,

$$\|(a, \mathbf{v})\|_{L^\infty_t (B^{\frac{d}{2}+1}_{2,1}) \cap L^h_t (B^{\frac{d}{2}+1}_{p,1})} + \|((\nabla a, \mathbf{v}))\|_{L^\infty_t (B^{\frac{d}{2}+1}_{p,1}) \cap L^h_t (B^{\frac{d}{2}+1}_{p,1})} \lesssim \|(a_0, \mathbf{v}_0)\|_{L^\infty_t (B^{\frac{d}{2}+1}_{2,1})} + \|((\nabla a_0, \mathbf{v}_0))\|_{L^\infty_t (B^{\frac{d}{2}+1}_{p,1})} + \|((f, g))\|_{L^\infty_t (B^{\frac{d}{2}+1}_{p,1})} + \|((\nabla f, \mathbf{g}))\|_{L^\infty_t (B^{\frac{d}{2}+1}_{p,1})}.$$ 

**Proof** Applying the operator $\hat{\Delta}_j$ to Equations (3.1)$_1$ and (3.1)$_2$, respectively, yields that

$$\begin{cases}
\partial_t a_j - \mu \Delta a_j + \text{div} \, \mathbf{v}_j = f_j, \\
\partial_t \mathbf{v}_j - \mu \Delta \mathbf{v}_j + \nabla a_j = g_j,
\end{cases} \tag{3.2}$$

here and in the sequel, we always denote $\phi_j = \hat{\Delta}_j \phi$. 


Taking the $L^2$ inner product of Equations (3.2)$_1$ and (3.2)$_2$ with $a_j$ and $v_j$, respectively, then integrating by parts, we get

\[ \begin{cases} \frac{1}{2} \frac{d}{dt} ||a_j||^2_{L^2} + \mu ||\nabla a_j||^2_{L^2} + (\text{div} v_j | a_j) = (f_j | a_j), \\ \frac{1}{2} \frac{d}{dt} ||v_j||^2_{L^2} + \mu ||\nabla v_j||^2_{L^2} + (\nabla a_j | v_j) = (g_j | v_j). \end{cases} \tag{3.3} \]

Noticing the fact that $(\text{div} v_j | a_j) = -(\nabla a_j | v_j)$ and using Lemma 2.1, then we get from (3.3) that

\[ \frac{d}{dt} ||(a_j, v_j)||^2_{L^2} + c_0 \mu 2^{2j} ||(a_j, v_j)||^2_{L^2} \leq C ||(f_j, g_j)||_{L^2} ||(a_j, v_j)||_{L^2}, \]

which leads to

\[ \frac{d}{dt} ||(a_j, v_j)||_{L^2} + c_0 \mu 2^{2j} ||(a_j, v_j)||_{L^2} \leq C ||(f_j, g_j)||_{L^2}. \tag{3.4} \]

Multiplying both sides of (3.4) by $2^j (\frac{j}{d} - 1)$ and summing up over $j \leq j_0$, we infer that

\[ ||(a, v)||^{\ell}_{L^p(B_{x_1}^j \cap \mathbb{R}^d_{0})} \leq C ||(a_0, v_0)||^{\ell}_{L^p(B_{x_1}^j \cap \mathbb{R}^d_{0})} + C ||(f, g)||^{\ell}_{L^p(B_{x_1}^j \cap \mathbb{R}^d_{0})}. \tag{3.5} \]

Now, we need to bound the high frequencies of the solution in $L^p$ framework. First, multiplying each component of Equation (3.2)$_2$ by $v_j^i | v_j^i|^{p-2}$ and integrating over $\mathbb{R}^d$ gives for $i=1, \ldots, d$,

\[ \frac{1}{p} \frac{d}{dt} ||v_j||^p_{L^p} + \int \mu \Delta v_j^i \cdot v_j^i | v_j^i|^{p-2} dx + \int \partial_t a_j v_j^i | v_j^i|^{p-2} dx = \int g_j^i v_j^i | v_j^i|^{p-2} dx. \]

After summation on $i=1, \ldots, d$, we conclude from Lemma 2.1 and Lemma 2.7 that

\[ \frac{1}{p} \frac{d}{dt} ||v_j||^p_{L^p} + c_0 \mu 2^{2j} ||v_j||^p_{L^p} \leq C (||\nabla a_j||_{L^p} + ||g_j||_{L^p}) ||v_j||^{p-1}_{L^p}, \]

which leads to

\[ \frac{d}{dt} ||v_j||_{L^p} + c_0 \mu 2^{2j} ||v_j||_{L^p} \leq C ||g_j||_{L^p} + C ||\nabla a_j||_{L^p}. \tag{3.6} \]

Applying the operator $\partial_t$ to the first equation of (3.2), we get

\[ \partial_t \partial_t a_j - \mu \Delta \partial_t a_j + \partial_t \text{div} v_j = \partial_t f_j. \tag{3.7} \]

Then, multiplying (3.7) by $|\partial_t a_j|^{p-2} \partial_t a_j$ and integrating over $\mathbb{R}^d$ gives for $i=1, \ldots, d$

\[ \frac{1}{p} \frac{d}{dt} ||\partial_t a_j||^p_{L^p} + \mu \int \partial_t \partial_t a_j \partial_t a_j |\partial_t a_j|^{p-2} dx + \int \partial_t \text{div} v \partial_t a_j |\partial_t a_j|^{p-2} dx = \int \partial_t f_j \partial_t a_j |\partial_t a_j|^{p-2} dx. \]

After summation on $i=1, \ldots, d$, we infer from Lemma 2.1 and Lemma 2.7 that

\[ \frac{1}{p} \frac{d}{dt} ||\nabla a_j||^p_{L^p} + c_0 \mu 2^{2j} ||\nabla a_j||^p_{L^p} \leq C (||\nabla f_j||_{L^p} + 2^{2j} ||v_j||_{L^p}) ||\nabla a_j||^{p-1}_{L^p}. \]

which implies

\[ \frac{d}{dt} ||\nabla a_j||_{L^p} + c_0 \mu 2^{2j} ||\nabla a_j||_{L^p} \leq C ||\nabla f_j||_{L^p} + C 2^{2j} ||v_j||_{L^p}. \tag{3.8} \]
Hence, adding the inequality \((3.8) \times \gamma\) to the inequality \((3.6)\), we find that for \(j \geq j_0 + 1\),
\[
\frac{d}{dt}(\gamma \|
abla a_j\|_{L^p} + \|v_j\|_{L^p}) + c_p \mu 2^{2j} \|v_j\|_{L^p} + c_p \mu \gamma 2^{2j} \|
abla a_j\|_{L^p} \\
\leq C \gamma 2^{2j} \|v_j\|_{L^p} + C 2^{2j} \|\nabla a_j\|_{L^p} + C \|\gamma \nabla f_j, g_j\|_{L^p}.
\]
Choosing \(\gamma\) suitably small and \(j_0\) sufficiently large and absorbing the first two terms of RHS of \((3.9)\), we discover that
\[
\frac{d}{dt}(\gamma \|
abla a_j\|_{L^p} + \|v_j\|_{L^p}) + 2^{2j} c_p \mu (\|v_j\|_{L^p} + \gamma \|
abla a_j\|_{L^p}) \leq C \|\nabla f_j, g_j\|_{L^p}.
\]
(3.10)
Hence, multiplying \((3.10)\) by \(2^{j(\frac{d}{2} - 1)}\) and summing up over \(j \geq j_0 + 1\) yields that
\[
\|\nabla (a, v)\|_{h, L^\infty(B_{p,1}^\frac{d}{2} - 1)} \leq C \|\nabla (a_0, v_0)\|_{h, L^\infty(B_{p,1}^\frac{d}{2} - 1)} + C \|\nabla f, g\|_{h, L^1(B_{p,1}^\frac{d}{2} - 1)}.
\]
(3.11)
Therefore, combining \((3.5)\) and \((3.11)\), we complete the proof of lemma 3.1.

4 Proof of the main theorem

Now, we will divide the proof of Theorem 1.1 into several steps. The method of the proof is a very classical one.

**Step 1: Building of the approximation sequence.**
Let \(L_n^p\) be the set of \(L^2\) function spectrally supported in the Balls \(2^n B\) and let \(\Omega_n\) be the set of functions \((a, v)\) of \((L_n^2)^{1+d}\) such that \(\|a(x)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2^n}\). Let us consider the following approximate system
\[
\begin{align*}
\partial_t a^n - \mu \Delta a^n + \text{div} v^n &= -S_n \left(\text{div}(a^n v^n)\right) := f^n, \\
\partial_t v^n - \mu \Delta v^n + \nabla a^n &= S_n(2\mu\nabla(1 + a^n))\nabla v^n - I(a^n)\nabla a^n - v^n \cdot \nabla v^n := g^n, \\
(a^n, v^n)|_{t=0} &= S_n(a_0, v_0).
\end{align*}
\]
(4.1)
It is easy to show that system \((4.1)\) has a unique solution \((a^n, v^n)_{n \in \mathbb{N}}\) in the space \(C^1([0, T_n^*); \Omega_n)\).

**Step 2: Uniform estimates.**
Applying Lemma 3.1 to system \((4.1)\), we have for any \(T \in [0, T_n^*)\)
\[
X_{p,T}(a^n, v^n) \lesssim X_{p,0} \|f^n, g^n\|_{L^1(B_{p,1}^\frac{d}{2} + 1)} + \|\nabla f^n, g^n\|_{L^1(B_{p,1}^\frac{d}{2} - 1)}.
\]
(4.2)
According to the definition for the Besov spaces, it is easy to show that
\[
\|a^n, v^n\|_{L^\infty(B_{p,1}^\frac{d}{2} - 1) \cap L^1(B_{p,1}^\frac{d}{2} + 1)} + \|a^n\|_{L^\infty(B_{p,1}^\frac{d}{2} - 1) \cap L^1(B_{p,1}^\frac{d}{2} + 1)} \lesssim X_{p,T}(a^n, v^n).
\]
In order to bound the high frequency of \(f^n\) and \(g^n\), we see from Lemmas 2.3-2.4, Remark 2.2 and Lemma 2.8 that
\[
|\text{div}(a^n v^n)|_{L^1(B_{p,1}^\frac{d}{2} + 1)} + |\text{div}(a^n v^n)|_{L^1(B_{p,1}^\frac{d}{2} - 1)} \\
\lesssim |v^n|_{L^\infty(B_{p,1}^\frac{d}{2} - 1)} |a^n|_{L^1(B_{p,1}^\frac{d}{2} + 1)} + |v^n|_{L^\infty(B_{p,1}^\frac{d}{2} - 1)} |a^n|_{L^1(B_{p,1}^\frac{d}{2} + 1)} + |a^n|_{L^\infty(B_{p,1}^\frac{d}{2} - 1)} |v^n|_{L^1(B_{p,1}^\frac{d}{2} + 1)} \\
+ |v^n|_{L^1(B_{p,1}^\frac{d}{2} + 1)} |a^n|_{L^1(B_{p,1}^\frac{d}{2} + 1)} + |v^n|_{L^1(B_{p,1}^\frac{d}{2} + 1)} |a^n|_{L^1(B_{p,1}^\frac{d}{2} + 1)} \\
\lesssim \left(X_{p,T}(a^n, v^n)\right)^2.
\]
and
\[ ||\nabla (\ln (1 + a^n))\nabla v^n||_{L^1_T(B^{d-1}_{2,1})} + ||I(a^n)\nabla a^n||_{L^1_T(B^{d-1}_{2,1})} + ||v^n \cdot \nabla v^n||_{L^1_T(B^{d-1}_{2,1})} \]
\[ \lesssim ||a^n||_{L^2_T(B^{d+1}_{p,1})} \ ||v^n||_{L^2_T(B^{d}_{p,1})} + ||a^n||_{L^\infty_T(B^{d}_{p,1})} \ ||v^n||_{L^\infty_T(B^{d+1}_{p,1})} + ||v^n||_{L^\infty_T(B^{d}_{p,1})} \ ||a^n||_{L^1_T(B^{d+1}_{p,1})} \]
\[ + ||(a^n, v^n)||^2_{L^2_T(B^{d}_{p,1})} + ||v^n||_{L^\infty_T(B^{d}_{p,1})} \ ||v^n||_{L^1_T(B^{d+1}_{p,1})} \]
\[ \lesssim (X_{p,T}(a^n, v^n))^2. \]

Hence, adding up the above inequalities into (4.2), we get
\[ X_{p,T}(a^n, v^n) \lesssim X_{p,0} + (X_{p,T}(a^n, v^n))^2. \]

Since \( X_{p,0} \) is small enough and \( X_{p,T}(a^n, v^n) \) depends continuously on the time variable, a standard bootstrap argument will ensure that \( T_n^* = +\infty \). Moreover, there holds for \( T \in [0, \infty) \),
\[ X_{p,T}(a^n, v^n) \leq C_0 X_{p,0}, \quad ||a^n||_{L^\infty_T(L^\infty)} < \frac{1}{2}. \]

This implies \( X_{p,T}(a^n, v^n) \) is bounded independent of \( n \) for all \( T \in [0, \infty) \).

**Step 3: Existence of the solution.** A classical compactness method as in [3] show that we can find a global solution \((a, v) \in \mathcal{E}\) satisfying system (1.4) with the initial data \((a_0, v_0)\).

**Step 4: Uniqueness of the solution.** Assume that \((a^1, v^1)\) and \((a^2, v^2)\) are two solutions of the system (1.4) with the same initial data \((a_0, v_0)\). Setting \( \delta a = a^1 - a^2 \) and \( \delta v = v^1 - v^2 \), we find that \((\delta a, \delta v)\) satisfies
\[
\begin{align*}
\partial_t \delta a - \mu \Delta \delta a + \text{div} \delta v &= f(a^1, v^1) - f(a^2, v^2), \\
\partial_t \delta v - \mu \Delta \delta v + \nabla \delta a &= g(a^1, v^1) - g(a^2, v^2), \\
(\delta a, \delta v)|_{t=0} &= (0, 0). 
\end{align*}
\]
(4.3)

Applying the Lemma 3.1 to system (4.3), we have for any \( T \in [0, \infty) \),
\[ X_{p,T}(\delta a, \delta v) \lesssim ||(f(a^1, v^1) - f(a^2, v^2), g(a^1, v^1) - g(a^2, v^2))||^\ell_{L^1_T(B^{d-1}_{2,1})} \]
\[ + ||(\nabla [f(a^1, v^1) - f(a^2, v^2)], g(a^1, v^1) - g(a^2, v^2))||^h_{L^1_T(B^{d-1}_{2,1})} \]
\[ \lesssim ||f(a^1, v^1) - f(a^2, v^2)||^\ell_{L^1_T(B^{d-1}_{2,1})} + ||f(a^1, v^1) - f(a^2, v^2)||^h_{L^1_T(B^{d}_{p,1})} \]
\[ + ||g(a^1, v^1) - g(a^2, v^2)||^h_{L^1_T(B^{d-1}_{2,1})}. \]

Note that
\[ f(a^1, v^1) - f(a^2, v^2) = -\text{div}(\delta a v^1) - \text{div}(a^2 \delta v), \]
we infer from Lemma 2.3 and Lemma 2.8 that

\[
\|\text{div}(\delta v^1) + \text{div}(a^2 \delta v)\|_{L^h(B_{z,1}^{g-1})} + \|\text{div}(\delta a^1) + \text{div}(a^2 \delta v)\|_{L^h(B_{p,1}^{g-1})} \\
\lesssim \|v^1\|_{L^p(B_{p,1}^{g-1})} \|\delta a\|_{L^2(B_{p,1}^{g-1})} + \|v^1\|_{L^2(B_{p,1}^{g-1})} \|\delta a\|_{L^2(B_{p,1}^{g-1})} \\
+ \|\delta v\|_{L^2(B_{p,1}^{g-1})} \|\delta a\|_{L^2(B_{p,1}^{g-1})} + \|\delta v\|_{L^2(B_{p,1}^{g-1})} \|\delta a\|_{L^2(B_{p,1}^{g-1})} \\
+ \|\delta v\|_{L^2(B_{p,1}^{g-1})} \|a^2\|_{L^2(B_{p,1}^{g-1})} + \|\delta v\|_{L^2(B_{p,1}^{g-1})} \|a^2\|_{L^2(B_{p,1}^{g-1})} \\
\lesssim \varepsilon X_{p,T}(\delta a, \delta v).
\]

By direct calculation, we also have

\[
g(a^1, v^1) - g(a^2, v^2) = \underbrace{2\mu \nabla \ln(1 + a^1) - \ln(1 + a^2) \nabla v^1}_I + \underbrace{2\mu \nabla \ln(1 + a^2) \nabla v}_J + \underbrace{-I(a^2) \nabla \delta a - [I(a^1) - I(a^2)] \nabla a^1}_K - \underbrace{(\delta v \cdot \nabla v^2 + v^1 \cdot \nabla \delta v)}_L.
\]

From Lemmas 2.3-2.4, Remark 2.2 and Lemma 2.8, we obtain

\[
\|I_1\|_{L^1(B_{z,1}^{g-1})} \lesssim \|\ln(1 + a^1) - \ln(1 + a^2)\|_{L^\infty(B_{p,1}^{g-1})} \|v^1\|_{L^p(B_{p,1}^{g-1})} \\
+ \|\ln(1 + a^1) - \ln(1 + a^2)\|_{L^2(B_{p,1}^{g-1})} \|v^1\|_{L^2(B_{p,1}^{g-1})} \\
\lesssim \|v^1\|_{L^2(B_{p,1}^{g-1})} \|\delta a\|_{L^2(B_{p,1}^{g-1})} + \|v^1\|_{L^2(B_{p,1}^{g-1})} \|\delta a\|_{L^2(B_{p,1}^{g-1})} \\
\lesssim \varepsilon X_{p,T}(\delta a, \delta v),
\]

and

\[
\|I_2\|_{L^1(B_{z,1}^{g-1})} \lesssim \|a^2\|_{L^\infty(B_{p,1}^{g-1})} \|\delta v\|_{L^2(B_{p,1}^{g-1})} + \|a^2\|_{L^2(B_{p,1}^{g-1})} \|\delta v\|_{L^2(B_{p,1}^{g-1})} \lesssim \varepsilon X_{p,T}(\delta a, \delta v).
\]

To estimate the composition term $I_3$, it suffices to note that for any sufficiently smooth function $H$, we have

\[
H(y) - H(x) = \left( H'(0) + \int_0^1 [H'(x + \tau(y - x)) - H'(0)] d\tau \right)(y - x).
\]

Therefore, using $G'(0) = 0$ and $I_3 = -\nabla[G(a^1) - G(a^2)]$, we obtain

\[
\|I_3\|_{L^1(B_{z,1}^{g-1})} \lesssim \|\nabla \left( \int_0^1 G'(a^1 + \tau(a^2 - a^1))d\tau \cdot \delta a \right)\|_{L^h(B_{z,1}^{g-1})} \\
\lesssim \|\delta a\|_{L^2(B_{p,1}^{g-1})} \left( \|a^1\|_{L^2(B_{p,1}^{g-1})} + \|a^2\|_{L^2(B_{p,1}^{g-1})} \right) \\
+ \|\delta a\|_{L^2(B_{p,1}^{g-1})} \left( \|a^1\|_{L^\infty(B_{p,1}^{g-1})} + \|a^2\|_{L^\infty(B_{p,1}^{g-1})} \right) \\
+ \|\delta a\|_{L^2(B_{p,1}^{g-1})} \left( \|a^1\|_{L^2(B_{p,1}^{g-1})} + \|a^2\|_{L^2(B_{p,1}^{g-1})} \right) \\
\lesssim \varepsilon X_{p,T}(\delta a, \delta v).
\]
For the last term $I_4$, according to Lemma 2.3 and Lemma 2.8, we get
\[
\|I_4\|_{L^1_t(B^\frac{d}{2} \ast \ast_{2,1})} \lesssim \|
abla \mathbf{v} \cdot \nabla \mathbf{v}^2\|_{L^1_t(B^\frac{d}{2} \ast \ast_{2,1})} + \|\mathbf{v}^1 \cdot \nabla \mathbf{v}\|_{L^1_t(B^\frac{d}{2} \ast \ast_{2,1})}
\lesssim \|
abla \mathbf{v}\|_{L^\infty_t(B^\frac{d}{2} \ast \ast_{p,1})} \|\nabla \mathbf{v}\|_{L^1_t(B^\frac{d}{2} \ast \ast_{p,1})} + \|\mathbf{v}^1\|_{L^1_t(B^\frac{d}{2} \ast \ast_{p,1})} \|\mathbf{v}^2\|_{L^1_t(B^\frac{d}{2} \ast \ast_{p,1})}
\lesssim \varepsilon X_{p,t}(\delta a, \delta \mathbf{v}).
\]

Combining the above estimates (4.4)-(4.9) and Lemma 3.1, we can deduce
\[
X_{p,t}(\delta a, \delta \mathbf{v}) \lesssim \varepsilon X_{p,t}(\delta a, \delta \mathbf{v}),
\]
which implies that for all $t \in [0, +\infty)$
\[
\mathbf{v}^1(t) = \mathbf{v}^2(t) \quad \text{and} \quad a^1(t) = a^2(t).
\]

Therefore, we complete the Proof of Theorem 1.1.

**Proof of Theorem 1.2.** Let $d \geq 3$ and $a_0 \in B^\frac{d}{2} \ast \ast_{2,1}$ and $\mathbf{v}_0 \in B^\frac{d}{2} \ast \ast_{2,1}$. By the Theorem 1.1 and the construction of system (1.2), we easily conclude that $(a, \mathbf{v}) \in L^\infty_t(L^2)$ for any $T \geq 0$ under the assumption of Theorem 1.1. Let $(E(t))_{t \geq 0}$ be the semi-group associated with left-hand side of (1.2), we get after spectral localization
\[
U(t) = E(t)U_0 + \int_0^t E(t-\tau)F(U(\tau))d\tau, \quad U(t) = (a(t), \mathbf{v}(t)) \quad \text{and} \quad F = (f, \mathbf{g}).
\]

To simplify the notation, we set
\[
M_e(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{d}{2}} \sum_{-1 \leq j \leq j_0} (\|\Delta_j \mathbf{v}\|_{L^2} + \|\Delta_j a\|_{L^2}),
\]
\[
M_h(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{d}{2}} \sum_{j \geq j_0 + 1} 2^{j(d+1)}(\|\Delta_j \mathbf{v}\|_{L^2} + 2^j \|\Delta_j a\|_{L^2}),
\]
which implies
\[
\|a(\tau)\|_{B^\frac{d}{2} \ast \ast_{2,1}} + \|\mathbf{v}(\tau)\|_{B^\frac{d}{2} \ast \ast_{2,1}} \leq (1 + \tau)^{-\frac{d}{4}} M(t), \quad \text{for} \quad \tau \in [0, t].
\]

where $M(t) = M_e(t) + M_h(t)$.

To obtain the result of Theorem 1.2, we state the estimate of the semi-group operator.

**Lemma 4.1** Let $U_0 = (a_0, \mathbf{v}_0)$. Then the operator $E(t)$ satisfies the estimates
\[
\sum_{j \leq j_0} \|E(t)\Delta_j U_0\|_{L^2} \leq C(1 + t)^{-\frac{d}{4}} \|U_0\|_{B^0_{2,1}},
\]
\[
\sum_{j \geq j_0 + 1} 2^{j(d+1)} \|E(t)\Delta_j (\nabla a_0, \mathbf{v}_0)\|_{L^2} \leq C(1 + t)^{-\frac{d}{4}} \|\nabla a_0, \mathbf{v}_0\|_{B^\frac{d}{2} \ast \ast_{2,1}}.
\]
Proof  By Lemma 3.1, it is easy to see that
\[ ||F(E(t)\hat{\Delta}_jU_0)(\xi)||_{L^2} \leq Ce^{-c_0 \mu t^{2j}} ||\hat{\Delta}_jU_0||_{L^2}, \quad \text{for} \quad j \leq j_0, \]
and
\[ ||F(E(t)\hat{\Delta}_j(\nabla a_0, v_0))(\xi)||_{L^2} \leq Ce^{-c_0 \mu t^{2j}} ||\hat{\Delta}_j(\nabla a_0, v_0)||_{L^2}, \quad \text{for} \quad j \geq j_0 + 1. \]
Due to the fact that: for any \( \sigma > 0 \) there exists a constant \( C_\sigma \) so that
\[ \sup_{t \geq 0} \sum_{k \in \mathbb{Z}} t^\sigma 2^{k\sigma} e^{-c_0 \sigma t} \leq C_\sigma. \]
Direct calculation shows that
\[ t^{\frac{d}{2}} \sum_{j \leq j_0} ||E(t)\hat{\Delta}_jU_0||_{L^2} \leq C t^{\frac{d}{4}} \sum_{j \leq j_0} 2^{j\frac{d}{2}} e^{-\mu t^{2j}} ||\hat{\Delta}_jU_0||_{L^1} \]
\[ \leq C||U_0||_{B^0_{\frac{d}{4},\infty}} \sum_{j \leq j_0} (\sqrt{t}2^j)^{\frac{d}{4}} e^{-\mu(\sqrt{t}2^j)^2} \leq C||U_0||_{B^0_{\frac{d}{4},\infty}} \]
and
\[ \sum_{j \leq j_0} ||E(t)\hat{\Delta}_jU_0||_{L^2} \leq C \sum_{j \leq j_0} 2^{j\frac{d}{2}} e^{-\mu t^{2j}} ||\hat{\Delta}_jU_0||_{L^1} \]
\[ \leq C||U_0||_{B^0_{\frac{d}{4},\infty}} \sum_{j \leq j_0} 2^{j\frac{d}{2}} \leq C||U_0||_{B^0_{\frac{d}{4},\infty}}. \]
This implies the first estimate of our result. To obtain the second estimate of this lemma, we find that
\[ t^{\frac{d}{4}} \sum_{j \geq j_0 + 1} 2^{j(\frac{d}{2} - 1)} ||E(t)\hat{\Delta}_j(\nabla a_0, v_0)||_{L^2} \]
\[ \leq C t^{\frac{d}{4}} \sum_{j \geq j_0 + 1} 2^{j\frac{d}{2}} e^{-\mu t^{2j}} ||\hat{\Delta}_j(\nabla a_0, v_0)||_{L^2} \]
\[ \leq C||(\nabla a_0, v_0)||_{B^2_{\frac{d}{4},\infty}} \sum_{j \geq j_0 + 1} (\sqrt{t}2^j)^{\frac{d}{4}} e^{-\mu(\sqrt{t}2^j)^2} \]
\[ \leq C||((\nabla a_0, v_0)||_{B^2_{\frac{d}{4},\infty}} \leq C||((\nabla a_0, v_0)||_{B^4_{\frac{d}{4},1}}, \]
and
\[ \sum_{j \geq j_0 + 1} 2^{j(\frac{d}{4} - 1)} ||E(t)\hat{\Delta}_j(\nabla a_0, v_0)||_{L^2} \]
\[ \leq C \sum_{j \geq j_0 + 1} 2^{j(\frac{d}{4} - 1)} e^{-\mu t^{2j}} ||\hat{\Delta}_j(\nabla a_0, v_0)||_{L^2} \]
\[ \leq C \sum_{j \geq j_0 + 1} 2^{j(\frac{d}{4} - 1)} ||\hat{\Delta}_j(\nabla a_0, v_0)||_{L^2} \]
\[ \leq C||((\nabla a_0, v_0)||_{B^4_{\frac{d}{4},1}}. \]
This completes the proof of this lemma.

Now, we will give the details proof of Theorem 1.2. According to Lemma 2.6, Lemma 4.1 and the following estimate

\[
\|\text{div}(av)\|_{L^1} + \|\nabla(\ln(1+a))\nabla v\|_{L^1} + \|I(a)\nabla a\|_{L^1} + \|v \cdot \nabla v\|_{L^1} \\
\lesssim \|v\|_{L^2}\|\nabla a\|_{L^2} + \|a\|_{L^2}\|\text{div}v\|_{L^2} + \|\nabla v\|_{L^2}\|a\|_{H^1} + \|a\|_{B_{2,1}^{\frac{d}{2}}} + \|v\|_{L^2}\|\nabla v\|_{L^2} \\
\lesssim (\|a\|_{B_{2,1}^{\frac{d}{2}}} + \|v\|_{B_{2,1}^{\frac{d}{2}+1}})(\|v\|_{B_{2,1}^{\frac{d}{2}+1}} + \|a\|_{B_{2,1}^{\frac{d}{2}}} + \|v\|_{B_{2,1}^{\frac{d}{2}+1}}),
\]

we have for \(\tau \in [0, t]\),

\[
\int_0^\tau \sum_{j \leq j_0} \|\Delta_j E(\tau - s)F(U(s))\|_{L^2} ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{d}{2}} \|F(U(s))\|_{B_{1,\infty}} ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{d}{2}} \|F(U(s))\|_{L^1} ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{d}{2}} (1 + s)^{-\frac{d}{2}} M(\tau)(1 + s)^{-\frac{d}{2}} M(\tau) + \|v(s)\|_{B_{2,1}^{\frac{d}{2}+1}} ds \\
\leq C(1 + \tau)^{-\frac{d}{2}} M(\tau) X_2(\tau) + C(1 + \tau)^{-\frac{d}{2}} M^2(\tau),
\]

where

\[X_2(t) = \|\langle a, v \rangle\|_{L^2(B_{2,1}^{\frac{d}{2}+1})} + \|a\|_{L^2(B_{2,1}^{\frac{d}{2}+1})} + \|v\|_{L^2(B_{2,1}^{\frac{d}{2}+1})} + \|\nabla a\|_{L^2(B_{2,1}^{\frac{d}{2}+1})} + \|\nabla v\|_{L^2(B_{2,1}^{\frac{d}{2}+1})} + \|\Delta a\|_{L^2(B_{2,1}^{\frac{d}{2}+1})} + \|\Delta v\|_{L^2(B_{2,1}^{\frac{d}{2}+1})}.
\]

Due to the fact that

\[
\|\langle \nabla f, g \rangle\|_{B_{2,1}^{\frac{d}{2}+1}} \\
\leq C(\|\text{div}(av)\|_{B_{2,1}^{\frac{d}{2}}} + \|\nabla(\ln(1+a))\nabla v\|_{B_{2,1}^{\frac{d}{2}}} + \|I(a)\nabla a\|_{B_{2,1}^{\frac{d}{2}}} + \|v \cdot \nabla v\|_{B_{2,1}^{\frac{d}{2}}}) \\
\leq C(\|a\|_{B_{2,1}^{\frac{d}{2}}} \|v\|_{B_{2,1}^{\frac{d}{2}+1}} + \|v\|_{B_{2,1}^{\frac{d}{2}}} \|v\|_{B_{2,1}^{\frac{d}{2}+1}} + \|a\|_{B_{2,1}^{\frac{d}{2}}} \|a\|_{B_{2,1}^{\frac{d}{2}+1}} + \|a\|_{B_{2,1}^{\frac{d}{2}}} \|v\|_{B_{2,1}^{\frac{d}{2}+1}}) \\
\leq C \left( (\|a\|_{B_{2,1}^{\frac{d}{2}}} + \|v\|_{B_{2,1}^{\frac{d}{2}+1}})(\|a\|_{B_{2,1}^{\frac{d}{2}+1}} + \|v\|_{B_{2,1}^{\frac{d}{2}+1}}) + \|a\|_{B_{2,1}^{\frac{d}{2}}} \|a\|_{B_{2,1}^{\frac{d}{2}+1}} \|v\|_{B_{2,1}^{\frac{d}{2}+1}} \|v\|_{B_{2,1}^{\frac{d}{2}+1}} \right) \\
\leq C \left( (\|a\|_{B_{2,1}^{\frac{d}{2}}} + \|v\|_{B_{2,1}^{\frac{d}{2}+1}})(\|a\|_{B_{2,1}^{\frac{d}{2}+1}} + \|v\|_{B_{2,1}^{\frac{d}{2}+1}}) + \|a\|_{B_{2,1}^{\frac{d}{2}}} \|a\|_{B_{2,1}^{\frac{d}{2}+1}} \|v\|_{B_{2,1}^{\frac{d}{2}+1}} \|v\|_{B_{2,1}^{\frac{d}{2}+1}} \right),
\]

we get from Lemma 2.6 that for \(\tau \in [0, t]\),

\[
\int_0^\tau \|E(\tau - s)F(U(s))\|_{B_{2,1}^{\frac{d}{2}} \times B_{2,1}^{\frac{d}{2}+1}} ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{d}{2}} \|\langle \nabla f, g \rangle\|_{B_{2,1}^{\frac{d}{2}}} ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{d}{2}} (1 + s)^{-\frac{d}{2}} M(t)(\|\langle a, v \rangle\|_{B_{2,1}^{\frac{d}{2}+1}} + \|a\|_{B_{2,1}^{\frac{d}{2}+2}}) ds \\
\leq C(1 + \tau)^{-\frac{d}{2}} M(t) X_2(t).
\]
By Lemma 4.1, we have
\[
M(t) \lesssim M_0 + \sup_{0 \leq \tau \leq t} (1 + \tau)^4 \int_0^\tau \sum_{j \leq j_0} |\hat{\Delta}_j E(\tau - s) F(U(s))|_{L^2} ds
\]
\[
+ \sup_{0 \leq \tau \leq t} (1 + \tau)^4 \int_0^\tau |E(\tau - s) F(U(s))|_{B_{2,1}^{4} \cap B_{2,-1}^{4}}^h ds
\]
(4.12)
Noticing that \( X_2(t) \lesssim M_0 \) and adding up (4.10)-(4.11) into (4.12), we conclude that
\[
M(t) \lesssim M_0 + M(t) M_0 + M^2(t).
\]
If \( M_0 \) is sufficiently small enough, the standard continuous method show that for all \( t \geq 0 \),
\[
M(t) \leq \tilde{C}_0 M_0,
\]
for some positive constant \( \tilde{C} \). Thus, we complete the proof of Theorem 1.2.

Acknowledgements. This work was partially supported by NSFC (No.11801090).

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