Existence results of fractional delta–nabla difference equations via mixed boundary conditions

Jiraporn Reunsumrit and Thanin Sitthiwirattham

Abstract

In this article, we purpose existence results for a fractional delta–nabla difference equations with mixed boundary conditions by using Banach contraction principle and Schauder’s fixed point theorem. Our problem contains a nonlinear function involving fractional delta and nabla differences. Moreover, our problem contains different orders in four fractional delta differences, four fractional nabla differences, one fractional delta sum, and one fractional nabla sum. Finally, we present some illustrative examples.

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1 Introduction

Simultaneously with the development of the theory and application of differential calculus, difference calculus has also received more intense attention. In this article, we study the evolution of fractional difference calculus. Recently, fractional difference calculus became an attractive field to researchers since it can be used in ecology, biology, and other applied sciences [1–4].

In general, difference calculus is divided into two types, namely delta and nabla difference calculus. The fractional delta and nabla difference calculus has been studied in many research works such as [5–25] and [26–37], respectively. However, there are a few papers studying delta–nabla calculus, such as the delta–nabla calculus of variations [38–40], systems of delta–nabla fractional difference inclusions [41], and the discrete delta–nabla fractional boundary value problems with p-Laplacian [42].

The results mentioned above are the motivation for this research. In this paper, we study the existence of solutions of a fractional delta–nabla difference equation with mixed fractional delta–nabla difference–sum boundary conditions given by

\[ \Delta^\alpha u(t) = F[t^\alpha - 1, u(t^\alpha + 1), \Delta^\theta u(t^\alpha - \theta + 1), \nabla^\gamma u(t^\alpha + 1)]. \]
\[ \Delta^{\beta-k} u(\alpha - \beta - 1) = \eta_k \nabla^{\omega-k} u(\alpha - 1 - k), \quad k = 0, 1, \tag{1.1} \]
\[ \Delta^{-\beta} u(T + \alpha + \beta) = \lambda \nabla^{\omega} u(T \alpha), \]

where \( t \in \mathbb{N}_{0,T} := \{0, 1, \ldots, T\}; \alpha \in (2, 3]; \theta, \gamma, \beta, \omega \in (1, 2]; T \in \mathbb{N}; \eta_0, \eta_1, \lambda \) are given constants; and \( F \in C(\mathbb{N}_{\alpha-3,T+\alpha} \times \mathbb{R}^3, \mathbb{R}) \).

In Sect. 2, we provide some basic knowledge about delta and nabla difference calculus and investigate results for a linear variant of the boundary value problem (1.1). In Sect. 3, we present the existence results of (1.1) by using Banach contraction principle and Schauder’s theorem. Then, we give some examples to illustrate our results.

## 2 Preliminaries

This section is divided into two parts. The first contains the notations, definitions, and lemmas which are used in the main results. In the second part, we provide a lemma presenting a linear variant of problem (1.1).

The forward jump operator is defined by \( \sigma(t) := t + 1 \), and the backward jump operator is defined by \( \rho(t) := t - 1 \).

For \( t, \alpha \in \mathbb{R} \), the generalized falling function is defined by

\[ t^{\alpha} := \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \alpha)}, \]

where \( t + 1 - \alpha \) is not a pole of the Gamma function. If \( t + 1 - \alpha \) is a pole and \( t + 1 \) is not a pole, then \( t^{\alpha} = 0 \).

The generalized rising function is defined by

\[ t^\alpha := \frac{\Gamma(t + \alpha)}{\Gamma(t)}, \]

where \( t \) and \( t + \alpha \) are not poles of the Gamma function. If \( t \) is a pole and \( t + \alpha \) is not a pole, then \( t^\alpha = 0 \).

**Definition 2.1** ([10]) For \( \alpha > 0 \) and \( f \) defined on \( \mathbb{N}_a := \{a, a+1, \ldots\} \), the \( \alpha \)-order fractional delta sum of \( f \) is defined by

\[ \Delta_a^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t - \sigma(s))^{\alpha-1} f(s), \quad t \in \mathbb{N}_{a+\alpha}, \]

and the \( \alpha \)-order Riemann–Liouville fractional delta difference of \( f \) is defined by

\[ \Delta_a^{\alpha} f(t) := \Delta^N \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t - \sigma(s))^{\alpha-1} f(s), \quad t \in \mathbb{N}_{a+N-\alpha}, \]

where \( N \in \mathbb{N} \) is such that \( 0 \leq N - 1 < \alpha < N \).

For convenience, the notations \( \Delta^{-\alpha} f(t) \) and \( \Delta^{\alpha} f(t) \) are used instead of \( \Delta_a^{-\alpha} f(t) \) and \( \Delta_a^{\alpha} f(t) \), respectively.
Definition 2.2 ([29]) For $\alpha > 0$ and $f$ defined on $\mathbb{N}_a$, the $\alpha$-order fractional nabla sum of $f$ is defined by

$$\nabla^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t - \rho(s))^{-\alpha-1} f(s), \quad t \in \mathbb{N}_a,$$

and the $\alpha$-order Riemann–Liouville fractional nabla difference of $f$ is defined by

$$\nabla^\alpha f(t) := \nabla^N \nabla^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t} (t - \rho(s))^{-\alpha-1} f(s), \quad t \in \mathbb{N}_{a,N},$$

where $N \in \mathbb{N}$ is such that $0 \leq N-1 < \alpha < N$.

Lemma 2.1 ([11]) Let $0 \leq N - 1 < \alpha \leq N$, $N \in \mathbb{N}$ and $y : \mathbb{N}_a \rightarrow \mathbb{R}$. Then,

$$\Delta^{-\alpha} \Delta^\alpha y(t) = y(t) + C_1 (t-a)^{\alpha-1} + C_2 (t-a)^{\alpha-2} + \cdots + C_N (t-a)^{\alpha-N},$$

for some $C_i \in \mathbb{R}$, with $1 \leq i \leq N$.

Lemma 2.2 ([29]) Let $0 \leq N - 1 < \alpha \leq N$, $N \in \mathbb{N}$ and $y : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$. Then,

$$\nabla^{-\alpha} \nabla^\alpha y(t) = \begin{cases} y(t), & \alpha \notin \mathbb{N}, \\ y(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k)} \nabla^k f(a), & \alpha = N, \end{cases}$$

for all $t \in \mathbb{N}_{a+1}$.

The solution of a linear variant of the boundary value problem (1.1) is given in the following lemma.

Lemma 2.3 Let $\Lambda \neq 0$, $\alpha \in (2,3]$; $\beta, \omega \in (1,2]$; $T \in \mathbb{N}$; $\eta_0, \eta_1, \lambda$ be given constants; and $h \in C(\mathbb{N}_{a-3,T+a}, \mathbb{R})$. Then,

$$\Delta^{-\alpha} u(t) = h(t + \alpha - 1), \quad t \in \mathbb{N}_{0,T}, \quad (2.1)$$
$$\Delta^{-\omega} u(\alpha - \beta - 1) = \eta_k \nabla^{\omega-k} u(\alpha - 1 - k), \quad k = 0,1, \quad (2.2)$$
$$\Delta^{-\beta} u(T + \alpha + \beta) = \lambda \nabla^{-\omega} u(T + \alpha) \quad (2.3)$$

has the unique solution given by

$$u(t) = \frac{\Phi[h]}{\Lambda} \left[ A_1 t^{\alpha-1} + A_2 t^{\alpha-2} + A_3 t^{\alpha-3} \right]$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t - \sigma(s))^{\alpha-1} h(s + \alpha - 1), \quad (2.4)$$
where the functional $\Phi[h]$ and the constants $\Lambda, A_1, A_2, A_3$ are defined as

$$\Phi[h] = \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha}^{t+\alpha} \left[ \lambda \left( \frac{(T + \alpha - \rho(s))^{\beta-1}}{\Gamma(\omega)} - \frac{(T + \alpha + \beta - \sigma(s))^{\beta-1}}{\Gamma(\beta)} \right) \right]$$

$$\times (s - \sigma(r))^{\alpha-1}h(r + \alpha - 1),$$

(2.5)

$$\Lambda = \left[ \Gamma(\alpha - 1)(\eta_0\omega - \beta) + \Gamma(\alpha)(\eta_0 - 1) \right] \Gamma(\alpha - 2)((1 - \beta) - \eta_1(1 - \omega))$$

$$+ \Gamma(\alpha - 1)(1 - \eta_1) - \frac{\Gamma(\alpha - 2)}{2} (\eta_0(1 - \omega)\omega - (1 - \beta)\beta)$$

$$+ \Gamma(\alpha - 1)(\eta_0\omega - \beta) + \Gamma(\alpha)(1 - \eta_0) \right] \Gamma(\alpha - 1)(1 - \eta_1),$$

(2.6)

$$A_1 = \left[ \Gamma(\alpha - 1)(\eta_0\omega - \beta) + \Gamma(\alpha)(\eta_0 - 1) \right] \Gamma(\alpha - 2)((1 - \beta) - \eta_1(1 - \omega))$$

$$+ \Gamma(\alpha - 1)(1 - \eta_1) - \frac{\Gamma(\alpha - 2)}{2} (\eta_0(1 - \omega)\omega - (1 - \beta)\beta)$$

$$+ \Gamma(\alpha - 1)(\eta_0\omega - \beta) + \Gamma(\alpha)(1 - \eta_0) \right] \Gamma(\alpha - 1)(1 - \eta_1),$$

(2.7)

$$A_2 = \Gamma(\alpha)(\eta_0 - 1) \Gamma(\alpha - 2)((1 - \beta) - \eta_1(1 - \omega)) + \Gamma(\alpha - 1)(1 - \eta_1),$$

(2.8)

$$A_3 = \Gamma(\alpha)(\alpha - 1)(\eta_0 - 1)(1 - \eta_1).$$

(2.9)

Proof: By using the fractional delta sum of order $\alpha$ in (2.1), we have

$$u(t) = C_1t_{\alpha-1} + C_2t_{\alpha-2} + C_3t_{\alpha-3} + \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha}^{t+\alpha} (t - \sigma(s))^{\alpha-1}h(s + \alpha - 1),$$

(2.10)

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$. Then taking the fractional delta difference of order $\beta - k$ in (2.10) where $k = 0, 1$, we get

$$\Delta^{\beta-k}u(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha}^{t+\beta-k} (t - \sigma(s))^{\alpha-1}h(s + \alpha - 1),$$

(2.11)
Taking the fractional nabla difference of order $\omega - k$ of (2.10) where $k = 0, 1$, we obtain

$$
\nabla^{\omega-k}u(t) = \frac{1}{\Gamma(k-\omega)} \sum_{s=\alpha-3}^{t} (t - \rho(s))^{\omega+1-k-1} \left[ C_1 s^{\alpha-1} + C_2 s^{\alpha-2} + C_3 s^{\alpha-3} \right]
$$

$$
+ \frac{1}{\Gamma((k-\omega)\Gamma(\alpha))} \sum_{s=\alpha-3}^{t} (t - \rho(s))^{\omega+1-k-1} (s - \sigma (r))^{\alpha-1} h(r + \alpha - 1),
$$

(2.12)

for $t \in \mathbb{N}_{\alpha-k-1, T+\alpha}$.

We now substitute $t = \alpha - \beta - 1$ into (2.11) and $t = \alpha - 1 - k$ into (2.12), then apply condition (2.2). So, we have

$$
\frac{1}{\Gamma(k-\omega)} \sum_{s=\alpha-3}^{\alpha-1} (\alpha - \beta - 1 - \sigma(s))^{\omega+1-k-1} \left[ C_1 s^{\alpha-1} + C_2 s^{\alpha-2} + C_3 s^{\alpha-3} \right]
$$

$$
= \frac{\eta_k}{\Gamma(k-\omega)} \sum_{s=\alpha-3}^{\alpha-1} (\alpha - k - 1 - \rho(s))^{\omega+1-k-1} \left[ C_1 s^{\alpha-1} + C_2 s^{\alpha-2} + C_3 s^{\alpha-3} \right],
$$

(2.13)

With $k = 0$ and $k = 1$ in (2.13), we obtain the equations

\begin{align*}
\text{(E1)} & \quad C_1 \Gamma(\alpha)(1 - \eta_0) + C_2 \left\{ \Gamma(\alpha - 1)(\eta_0 \omega - \beta) + \Gamma(\alpha)(\eta_0 - 1) \right\} + C_3 \left[ \frac{\Gamma'(\alpha - 2)}{2} \right] \\
& \quad \times \left( \eta_0(1 - \omega)\omega - (1 - \beta)\beta \right) + \Gamma'(\alpha - 1)(\eta_0 \omega - \beta) + \Gamma'(\alpha)(1 - \eta_0) = 0, \\
\text{(E2)} & \quad C_2 \Gamma'(\alpha - 1)(1 - \eta_1) + C_3 \left[ \Gamma'(\alpha - 2)((1 - \beta) - \eta_1 (1 - \omega)) + \Gamma'(\alpha - 1)(1 - \eta_1) \right] = 0.
\end{align*}

After taking the fractional delta sum of order $\beta$ for (2.10), we get

$$
\Delta^{-\beta}u(t) = \frac{1}{\Gamma(\beta)} \sum_{s=\alpha-3}^{t-\beta} (t - \rho(s))^{\beta-1} \left[ C_1 s^{\alpha-1} + C_2 s^{\alpha-2} + C_3 s^{\alpha-3} \right]
$$

$$
+ \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \sum_{s=\alpha-3}^{t-\beta} (t - \rho(s))^{\beta-1} (s - \sigma (r))^{\alpha-1} h(r + \alpha - 1),
$$

(2.14)

for $t \in \mathbb{N}_{\alpha-\beta-3, T+\alpha+\beta}$.

Using the fractional nabla sum of order $\omega$ for (2.10), we obtain

$$
\nabla^{-\omega}u(t) = \frac{1}{\Gamma(\omega)} \sum_{s=\alpha-3}^{t} (t - \rho(s))^{\omega-1} \left[ C_1 s^{\alpha-1} + C_2 s^{\alpha-2} + C_3 s^{\alpha-3} \right]
$$

$$
+ \frac{1}{\Gamma(\omega)\Gamma(\alpha)} \sum_{s=\alpha-3}^{t} (t - \rho(s))^{\omega-1} (s - \sigma (r))^{\alpha-1} h(r + \alpha - 1),
$$

(2.15)

for $t \in \mathbb{N}_{\alpha-3, T+\alpha}$.
We now substitute $t = T + \alpha + \beta$ into (2.14) and $t = T + \alpha$ into (2.15), then apply condition (2.3). So, we have

\[
(E_3) \quad C_i \sum_{\sigma=a}^{T+\alpha} \left[ \frac{(T + \alpha + \beta - \sigma(s))^{\beta-1}}{\Gamma(\beta)} - \frac{\lambda (T + \alpha - \rho(s))^{\alpha-1}}{\Gamma(\alpha)} \right]^{\frac{1}{\beta}}
\]

\[
+ C_2 \sum_{\sigma=a}^{T+\alpha} \left[ \frac{(T + \alpha + \beta - \sigma(s))^{\beta-1}}{\Gamma(\beta)} - \frac{\lambda (T + \alpha - \rho(s))^{\alpha-1}}{\Gamma(\alpha)} \right]^{\frac{1}{\beta}}
\]

\[
+ C_3 \sum_{\sigma=a}^{T+\alpha} \left[ \frac{(T + \alpha + \beta - \sigma(s))^{\beta-1}}{\Gamma(\beta)} - \frac{\lambda (T + \alpha - \rho(s))^{\alpha-1}}{\Gamma(\alpha)} \right]^{\frac{1}{\beta}}
\]

\[
= \frac{1}{\Gamma(a)} \sum_{\sigma=a}^{T+\alpha} \sum_{r=0}^{T+\alpha} \left[ \frac{\lambda (T + \alpha - \rho(s))^{\alpha-1}}{\Gamma(\beta)} - \frac{(T + \alpha + \beta - \sigma(s))^{\beta-1}}{\Gamma(\beta)} \right]
\]

\[
\times (s - \sigma(r))^{\frac{\beta-1}{\beta}} h(r + \alpha - 1).
\]

Finding the solution of equations $(E_1)-(E_3)$, we have

\[
C_i = \frac{\Phi[h]}{A} A_i, \quad i = 1, 2, 3,
\]

where $\Phi[h]$, $A$, $A_1$, $A_2$, and $A_3$ are defined by (2.5)–(2.9), respectively. Substituting the constants $C_1$ through $C_3$ into (2.10), we get the unique solution as (2.4).

3 Main results

In this section, we show existence results of problem (1.1). Let $C = C(\mathbb{N}_{\alpha-3,T+\alpha};\mathbb{R})$ be the Banach space of functions $u$ with the norm defined by

\[
\|u\|_C = \max \{ \|u\|, \|\Delta^\theta u\|, \|\nabla^\gamma u\| \},
\]

where $\|u\| = \max_{t \in \mathbb{N}_{\alpha-3,T+\alpha}} |u(t)|$, $\|\Delta^\theta u\| = \max_{t \in \mathbb{N}_{\alpha-3,T+\alpha}} |\Delta^\theta u(t - \theta + 2)|$ and $\|\nabla^\gamma u\| = \max_{t \in \mathbb{N}_{\alpha-3,T+\alpha}} |\nabla^\gamma u(t + 2)|$. We define the operator $F: C \to C$ by

\[
(Fu)(t) = \frac{\Phi[F(u)]}{A} \left[ A_1 t^{\alpha-1} + A_2 t^{\alpha-2} + A_3 t^{\alpha-3} \right] + \frac{1}{\Gamma(a)} \sum_{s=0}^{T+\alpha} (t - \sigma(s))^{\alpha-1}
\]

\[
\times \left( s + \alpha - 1, u(s + \alpha - 1), \Delta^\theta u(s + \alpha - \theta + 1), \nabla^\gamma u(s + \alpha + 1) \right),
\]

where $A \neq 0$, $A_1$, $A_2$, and $A_3$ are given in Lemma 2.3 and the functional $\Phi[F(u)]$ is given by

\[
\Phi[F(u)] = \frac{1}{\Gamma(a)} \sum_{\sigma=a}^{T+\alpha} \sum_{r=0}^{T+\alpha} \left[ \frac{\lambda (T + \alpha - \rho(s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(T + \alpha + \beta - \sigma(s))^{\beta-1}}{\Gamma(\beta)} \right] (s - \sigma(r))^{\frac{\beta-1}{\beta}}
\]

\[
\times \left( r + \alpha - 1, u(r + \alpha - 1), \Delta^\theta u(r + \alpha - \theta + 1), \nabla^\gamma u(r + \alpha + 1) \right).
\]

The boundary value problem (1.1) has solutions if and only if operator $F$ has fixed points.
**Theorem 3.1** Let $F : \mathbb{N}_{\alpha-3,T+\alpha} \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function and suppose that the following conditions hold:

(H1) There exist constants $L_1, L_2, L_3 > 0$ such that for each $t \in \mathbb{N}_{\alpha-3,T+\alpha}$ and $u, v, \iota \in \mathbb{R},$ $i = 1, 2, 3,$

$$|F(t, u_1, u_2, u_3) - F(t, v_1, v_2, v_3)| \leq L_1|u_1 - v_1| + L_2|u_2 - v_2| + L_3|u_3 - v_3|,$$

(H2) $[L_1 + L_2 + L_3]\max(\Omega_1, \Omega_2, \Omega_3) < 1,$

then problem (1.1) has a unique solution on $\mathbb{N}_{\alpha-3,T+\alpha},$ where

$$\Theta = \frac{(T + \alpha)\mu}{\Gamma(\alpha + 1)} \left[ (T + \omega)\mu - \frac{(T + \beta)^\mu}{\Gamma(\alpha + 1)} \right],$$

(3.3)

$$\Omega_1 = \left| A \right| \frac{\Theta}{(T + \alpha + 2)|\alpha|} \left[ |A_1|(T + \alpha + 2)|\alpha| + |A_2|(T + \alpha + 2)|\alpha| \right] + \frac{(T + \alpha)\mu}{\Gamma(\alpha + 1)},$$

(3.4)

$$\Omega_2 = \left| A \right| \frac{\Theta}{(T + \alpha + 2)|\alpha|} \left[ |A_1|(T + \alpha + 2)|\alpha| + |A_2|(T + \alpha + 2)|\alpha| \right] + \frac{(T + \alpha + 2)\mu}{\Gamma(\alpha + 1)},$$

(3.5)

$$\Omega_3 = \left| A \right| \frac{\Theta}{(T + \alpha + 2)|\alpha|} \left[ |A_1|(T + \alpha + 2)|\alpha| + |A_2|(T + \alpha + 2)|\alpha| \right] + \frac{(T + \alpha + 2)\mu}{\Gamma(\alpha + 1)},$$

(3.6)

**Proof** Letting $u, v \in C,$ for each $t \in \mathbb{N}_{\alpha-3,T+\alpha},$ we have

$$\left| \Phi[F(u)] - \Phi[F(v)] \right| \\
\leq \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T_{m+1}} \sum_{\sigma=0}^{T_{m+1}} \left[ \left( T + \alpha - \rho(s) \right)^{\mu - 1} \frac{\Gamma(\omega)}{\Gamma(\omega)} - \frac{(T + \alpha + \beta - \sigma(s))^\mu}{\Gamma(\beta)} \right] (s - \sigma(r))^{\mu - 1} \\
\times \left[ L_1 |u(r + \alpha - 1) - v(r + \alpha - 1)| + L_2 |\Delta^\alpha u(r + \alpha - \theta + 1) - \Delta^\alpha v(r + \alpha - \theta + 1)| \\
+ L_3 |\nabla^\alpha u(r + \alpha - 1) - \nabla^\alpha v(r + \alpha - 1)| \right] \\
\leq \left[ L_1 \|u - v\| + L_2 \|\Delta^\alpha u - \Delta^\alpha v\| + L_3 \|\nabla^\alpha u - \nabla^\alpha v\| \right] \frac{(T + \alpha)\mu}{\Gamma(\alpha + 1)} \\
\times \left[ \sum_{s=0}^{T_{m+1}} \left( T + \alpha - \rho(s) \right)^{\mu - 1} \frac{\Gamma(\omega)}{\Gamma(\omega)} - \sum_{s=0}^{T_{m+1}} \left( T + \alpha + \beta - \sigma(s) \right)^{\mu - 1} \frac{\Gamma(\beta)}{\Gamma(\beta)} \right] \\
= \|u - v\|_C \left[ L_1 + L_2 + L_3 \right] \Theta$$

(3.7)

and

$$\left| (F)(t) - (F)(t) \right| \\
\leq \left| \Phi[F(u)] - \Phi[F(v)] \right| \left[ |A_1|(T + \alpha)^{\mu - 1} + |A_2|(T + \alpha)^{\mu - 2} + |A_3|(T + \alpha)^{\mu - 3} \right] \\
+ \sum_{s=0}^{T_{m+1}} \left( T + \alpha - \rho(s) \right)^{\mu - 1} \frac{\Gamma(\omega)}{\Gamma(\omega)} \left[ L_1 |u(s + \alpha - 1) - v(s + \alpha - 1)| + L_2 |\Delta^\alpha u(s + \alpha - \theta + 1) \right]$$

(3.7)
We find that

\[ (\Delta^\theta F u)(t - \theta + 2) = \frac{\Phi[F(u)]}{\Gamma(\theta)} \sum_{s=\alpha-3}^{t+2} (t + 2 - \sigma(s))^{-\alpha-1} \left[ |A_1|s^{-\alpha-1} + |A_2|s^{-\alpha-2} + |A_3|s^{-\alpha-3} \right] \]

\[ + \frac{1}{\Gamma(\gamma)\Gamma(\alpha)} \sum_{s=\alpha-3}^{t+2} \sum_{r=0}^{s-\alpha} \frac{(t + 2 - \rho(s))^{-\gamma-1}(s - \sigma(r))^{\alpha-1}}{r!} \]

\[ \times F[r + \alpha - 1, u(r + \alpha - 1), \Delta^\theta u(r + \alpha - 1), \nabla^\gamma u(r + \alpha + 1)] \]  \hspace{5cm} (3.9)

and

\[ (\nabla^\gamma F u)(t + 2) = \frac{\Phi[F(u)]}{\Gamma(\gamma)} \sum_{s=\alpha-3}^{t+2} (t + 2 - \sigma(s))^{-\gamma-1} \left[ |A_1|s^{-\gamma-1} + |A_2|s^{-\gamma-2} + |A_3|s^{-\gamma-3} \right] \]

\[ + \frac{1}{\Gamma(\gamma)\Gamma(\alpha)} \sum_{s=\alpha-3}^{t+2} \sum_{r=0}^{s-\alpha} \frac{(t + 2 - \rho(s))^{-\gamma-1}(s - \sigma(r))^{\alpha-1}}{r!} \]

\[ \times F[r + \alpha - 1, u(r + \alpha - 1), \Delta^\theta u(r + \alpha - 1), \nabla^\gamma u(r + \alpha + 1)] \]  \hspace{5cm} (3.10)

Since

\[ \left| (\Delta^\theta F u)(t - \theta + 2) - (\Delta^\theta F v)(t - \theta + 2) \right| \leq \| u - v \|_{C}[L_1 + L_2 + L_3]\Omega_1, \]  \hspace{5cm} (3.11)

\[ \left| (\nabla^\gamma F u)(t + 2) - (\nabla^\gamma F v)(t + 2) \right| \leq \| u - v \|_{C}[L_1 + L_2 + L_3]\Omega_2, \]  \hspace{5cm} (3.12)

we get

\[ \| (F u) - (F v) \|_{C} \leq [L_1 + L_2 + L_3] \max\{\Omega_1, \Omega_2, \Omega_3\} \| u - v \|_{C}. \]  \hspace{5cm} (3.13)

By (H2), we get \( \| (F u)(t) - (F v)(t) \|_{C} < \| u - v \|_{C} \).

Hence, \( F \) is a contraction. By the Banach contraction principle, we conclude that \( F \) has a unique fixed point which is a unique solution of the problem (1.1) for \( t \in \mathbb{N}_{u-3, T+\alpha}. \) \( \square \)

We next show that our problem (1.1) has at least one solution as follows.

**Lemma 3.1** (Arzelà–Ascoli theorem, [43]) A set of functions in \( C[a, b] \) with the sup-norm is relatively compact if and only if it is uniformly bounded and equicontinuous on \( [a, b] \).
Lemma 3.2 ([43]) If a set is closed and relatively compact, then it is compact.

Lemma 3.3 (Schauder’s fixed point theorem, [44]) Let \( (D, d) \) be a complete metric space, \( U \) a closed convex subset of \( D \), and \( T : D \to D \) a map such that the set \( Tu : u \in U \) is relatively compact in \( D \). Then, the operator \( T \) has at least one fixed point \( u^* \in U : Tu^* = u^* \).

Theorem 3.2 Suppose that \( (H_1) \) and \( (H_2) \) hold. Then problem (1.1) has at least one solution on \( \mathbb{N}_{d-3, T, \alpha} \).

Proof Step I. We verify that \( F \) maps bounded sets into bounded sets in \( B_R \), where we consider \( B_R = \{ u \in C : \| u \|_C \leq R \} \).

Let \( \max_{u \in [\Theta, T, \omega]} |F(t, 0, 0, 0)| = M \) and choose a constant

\[
R \geq \frac{M \max(\Omega_1, \Omega_2, \Omega_3)}{1 - [L_1 + L_2 + L_3] \max(\Omega_1, \Omega_2, \Omega_3)}.
\]

(3.14)

For each \( u \in B_R \), we obtain

\[
|\Phi[F(\omega)]| \leq \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T \omega} \sum_{r=0}^{s \omega} \left[ \frac{\lambda (T + \alpha - \rho(s))^{\omega-1}}{\Gamma(\omega)} - \frac{(T + \alpha + \beta - \sigma(s))^{\beta-1}}{\Gamma(\beta)} \right] (s - \sigma(r))^{\alpha-1} \times \left[ |F[r + \alpha - 1, u(r + \alpha - 1), \Delta^0 u(r + \alpha - \theta + 1), \nabla^\gamma u(r + \alpha + 1)] - F(r + \alpha - 1, 0, 0, 0)| + |F(r + \alpha - 1, 0, 0, 0)| \right] \leq \left[ (L_1 + L_2 + L_3) \| u \|_C + M \right] \Theta \]  

(3.15)

and

\[
|\xi[u](\omega)| \leq \frac{\Phi[F(\omega)]}{A} \left[ |A_1|(T + \alpha)^{\omega-1} + |A_2|(T + \alpha)^{\omega-2} + |A_3|(T + \alpha)^{\omega-3} \right] \sum_{s=0}^{T} \left( T + \alpha - \sigma(s) \frac{\omega}{\Gamma(\alpha)} \right) \left[ |F[s + \alpha - 1, u(s + \alpha - 1), \Delta^0 u(s + \alpha - \theta + 1), \nabla^\gamma u(s + \alpha + 1)] - F(s + \alpha - 1, 0, 0, 0)| + |F(s + \alpha - 1, 0, 0, 0)| \right] \leq \left[ (L_1 + L_2 + L_3) \| u \|_C + M \right] \left\{ \frac{\Theta}{|A|} \left[ |A_1|(T + \alpha)^{\omega-1} + |A_2|(T + \alpha)^{\omega-2} + |A_3|(T + \alpha)^{\omega-3} \right] + \frac{\Gamma(T + \alpha + 1)}{\Gamma(\alpha + 1) \Gamma(T + 1)} \right\} \leq \left[ (L_1 + L_2 + L_3) \| u \|_C + M \right] \Omega_1.
\]

(3.16)
Since

\[
\left| (\Delta^\delta F u)(t - \theta + 2) \right| \leq \left[ (L_1 + L_2 + L_3) \| u \|_C + M \right] \Omega_2, \\
\left| (\nabla^\gamma F u)(t + 2) \right| \leq \left[ (L_1 + L_2 + L_3) \| u \|_C + M \right] \Omega_3,
\]

this implies that

\[
\| (Fu)(t) \|_C \leq \left[ (L_1 + L_2 + L_3) \| u \|_C + M \right] \max\{\Omega_1, \Omega_2, \Omega_3\} \\
\leq R.
\]

(3.19)

We find that \( \| Fu \|_C \leq R \). Hence, \( F \) is uniformly bounded.

**Step II.** Since \( F \) is a continuous function, the operator \( F \) is continuous on \( B_R \).

**Step III.** We show that \( F \) is equicontinuous on \( B_R \). For any \( \epsilon > 0 \), there exists a positive constant \( \rho^* = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\} \) such that for \( t_1, t_2 \in \mathbb{N}_{n-3, T+\alpha} \),

\[
|t_2^\alpha - t_1^\alpha| < \frac{\epsilon |A|}{4|A|}, \quad \text{if } |t_2 - t_1| < \delta_1, i = 1, 2, 3,
\]

\[
|t_2^\alpha - t_1^\alpha| < \frac{\epsilon \Gamma(\alpha + 1)}{4\|F\|}, \quad \text{if } |t_2 - t_1| < \delta_2,
\]

\[
(1 - \alpha - \theta + 5)^\frac{\alpha}{\alpha - 1} - (1 - \alpha - \theta + 4)^\frac{\alpha}{\alpha - 1}\n
\]

\[
< \left( \epsilon \left| \Gamma(1 - \theta) \right| \right) \left( \left| \frac{\frac{\theta}{\Gamma}}{|A|} \right| \left| (T + \alpha + 2)^{\frac{\alpha - 1}{\alpha - 1}} + |A_2|(T + \alpha + 2)^{\frac{\alpha - 2}{\alpha - 1}} + |A_3|(T + \alpha + 2)^{\frac{\alpha - 2}{\alpha - 1}} \right| + \frac{(T + \alpha + 2)^{\frac{\alpha - 1}{\alpha - 1}}}{\Gamma(\alpha + 1)} \right)^{-1},
\]

if \( |t_2 - t_1| < \delta_4 \),

\[
|t_2^\alpha - t_1^\alpha| < \frac{\epsilon \Gamma(\alpha + 1)}{4\|F\|}, \quad \text{if } |t_2 - t_1| < \delta_3,
\]

\[
(1 - \alpha - \theta + 5)^\frac{\alpha}{\alpha - 1} - (1 - \alpha - \theta + 4)^\frac{\alpha}{\alpha - 1}\n
\]

\[
< \left( \epsilon \left| \Gamma(1 - \gamma) \right| \right) \left( \left| \frac{\frac{\theta}{\Gamma}}{|A|} \right| \left| (T + \alpha + 2)^{\frac{\alpha - 1}{\alpha - 1}} + |A_2|(T + \alpha + 2)^{\frac{\alpha - 2}{\alpha - 1}} + |A_3|(T + \alpha + 2)^{\frac{\alpha - 2}{\alpha - 1}} \right| + \frac{(T + \alpha + 2)^{\frac{\alpha - 1}{\alpha - 1}}}{\Gamma(\alpha + 1)} \right)^{-1},
\]

if \( |t_2 - t_1| < \delta_5 \).

Then, for \( |t_2 - t_1| < \rho^* \), we have

\[
\left| (Fu)(t_2) - (Fu)(t_1) \right| 
\]

\[
\leq \left| \Phi[F(u)] \right| \left[ |A_1|\left| \frac{\alpha - 1}{\alpha - 1} - \frac{\alpha - 1}{\alpha - 1} \right| + |A_2|\left| \frac{\alpha - 1}{\alpha - 1} - \frac{\alpha - 2}{\alpha - 1} \right| + |A_3| \right] 
\]

\[
\times \left| t_2^\alpha - t_1^\alpha \right| + \sum_{s=0}^{t_2} \left( t_2 - s \right)^{\frac{\alpha - 1}{\alpha - 1}} F[s + \alpha + 1, u(s + \alpha + 1)] - \sum_{s=0}^{t_1} \left( t_1 - s \right)^{\frac{\alpha - 1}{\alpha - 1}} F[s + \alpha - 1, \nabla^\gamma u(s + \alpha + 1)]
\]
\[ u(s + \alpha - 1), \Delta^\theta u(s + \alpha - \theta + 1), \nabla^\gamma u(s + \alpha + 1) \]

\[
< \frac{A_1}{1} \left( \Theta \|F\| \left| t_2^{\alpha - 1} - t_1^{\alpha - 1} \right| + \frac{A_2}{1} \left( \Theta \|F\| \left| t_2^{\alpha - 2} - t_1^{\alpha - 2} \right| + \frac{A_3}{1} \right) \right) \left( \Theta \|F\| \left| t_2^{\alpha - 3} - t_1^{\alpha - 3} \right| \right) \\
+ \frac{\|F\|}{\Gamma(\alpha + 1)} \left| t_2^{\alpha} - t_1^{\alpha} \right|
\]

\[
\epsilon \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\] (3.20)

Similarly, we have

\[
\left| (\Delta^\theta F u)(t_2 - \theta + 2) - (\Delta^\theta F u)(t_1 - \theta + 2) \right| < \epsilon,
\] (3.21)

\[
\left| (\nabla^\gamma F u)(t_2 + 2) - (\nabla^\gamma F u)(t_1 + 2) \right| < \epsilon.
\] (3.22)

So,

\[
\| (Fu)(t_2) - (Fu)(t_1) \|_C < \epsilon.
\] (3.23)

Hence, the set \( F(B_R) \) is equicontinuous. Combining the results of Steps I to III with the Arzelá–Ascoli theorem, we get that \( F : C \rightarrow C \) is completely continuous. By using Schauder fixed point theorem, we can conclude that boundary value problem (1.1) has at least one solution. \( \square \)

4 Some examples

In this section, we provide a mixed boundary value problem for fractional delta–nabla difference equations and apply our results from the previous section as follows:

\[
\Delta^{\frac{5}{4}} u(t) = F \left[ t + \alpha - 1, u(t + \alpha - 1), \Delta^\theta u(t + \alpha - \theta + 1), \nabla^\gamma u(t + \alpha + 1) \right],
\]

\[
\Delta^{\frac{5}{4} - k} u \left( \frac{1}{4} \right) = \left( \frac{k + 1}{3} \right) \nabla^{\frac{5}{4} - k} u \left( \frac{3}{2} - k \right), \quad k = 0, 1,
\] (4.1)

\[
\Delta^{\frac{5}{4}} u \left( \frac{35}{4} \right) = 2 \nabla^{\frac{5}{4}} u \left( \frac{15}{2} \right).
\]

Here \( \alpha = \frac{5}{4}, \theta = \frac{4}{5}, \gamma = \frac{8}{7}, \beta = \frac{5}{8}, \omega = \frac{7}{6}, \eta_0 = \frac{1}{2}, \eta_1 = \frac{2}{3}, \) and \( \lambda = 2, T = 5. \) We find that

\[
|A| = 3.711, \quad \Theta = 529.938, \quad \Omega_1 = 5151.475, \quad \Omega_2 = 87.307,
\]

\[
\Omega_3 = 53.338.
\]

(i) Let

\[
F \left[ t, u(t), \Delta^\theta u(t - \theta + 2), \nabla^\gamma u(t + 2) \right] = e^{-\cos^2 t} \cdot \frac{|u(t)| + 2|\Delta^{\frac{5}{4}} u(t + \frac{5}{3})| + 3|\nabla^{\frac{5}{4}} u(t + 2)|}{(t + 100)^2}.
\]
Since \((H_1)\) holds for each \(t \in \mathbb{N}_{1/2,15}\), we obtain
\[
\left| F[t, u(t), \Delta^\theta u(t-\theta + 2), \nabla^\gamma u(t + 2)] - F[t, v(t), \Delta^\theta v(t-\theta + 2), \nabla^\gamma v(t + 2)] \right|
\leq \frac{4}{46,225} \| u - v \| + \frac{8}{46,225} \left\| \Delta^\theta u - \Delta^\theta v \right\| + \frac{12}{46,225} \left\| \nabla^\gamma u - \nabla^\gamma v \right\|,
\]
so \(L_1 = \frac{4}{46,225}, L_2 = \frac{8}{46,225}, L_3 = \frac{12}{46,225}\).

Finally, we can show that \((H_2)\) holds with
\[
[L_1 + L_2 + L_3] \max \{\Omega_1, \Omega_2, \Omega_3\} = 0.275 < 1.
\]
Hence, by Theorem 3.1, Problem 4.1 has a unique solution on \(\mathbb{N}_{1/2,15}\). In addition, by Theorem 3.2, Problem 4.1 has at least one solution on \(\mathbb{N}_{1/2,15}\).

(ii) Let
\[
F[t, u(t), \Delta^\theta u(t-\theta + 2), \nabla^\gamma u(t + 2)] = \frac{e^{-\sin^2 t}}{2t + 10} \frac{3|u(t)| + 5|\Delta^{1/2} u(t + 2)| + 2|\nabla^{1/2} u(t + 2)|}{[1 + |u(t)|]}.
\]
Since \((H_1)\) holds for each \(t \in \mathbb{N}_{1/2,15}\), we obtain
\[
\left| F[t, u(t), \Delta^\theta u(t-\theta + 2), \nabla^\gamma u(t + 2)] - F[t, v(t), \Delta^\theta v(t-\theta + 2), \nabla^\gamma v(t + 2)] \right|
\leq \frac{3}{115} \| u - v \| + \frac{5}{115} \left\| \Delta^\theta u - \Delta^\theta v \right\| + \frac{2}{115} \left\| \nabla^\gamma u - \nabla^\gamma v \right\|,
\]
so \(L_1 = \frac{3}{115}, L_2 = \frac{5}{115}, L_3 = \frac{2}{115}\).

Finally, we show that \((H2)\) not holds with
\[
[L_1 + L_2 + L_3] \max \{\Omega_1, \Omega_2, \Omega_3\} = 41.473 > 1.
\]
Therefore, Problem 4.1 is inconsistent with Theorem 3.1 and 3.2, which makes it impossible to conclude the existence results for this problem.

5 Conclusions
We consider a fractional delta–nabla difference equation with fractional delta–nabla sum-difference boundary value conditions. In our studies, we employ the Banach contraction principle to investigate the conditions for the existence and uniqueness of solution for our problem. In addition, the conditions for at least one solution is obtained by using the Schauder’s fixed point theorem.

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Author details
1 Department of Mathematics, Faculty of Applied Science, King Mongkut’s University of Technology North Bangkok, Bangkok, Thailand. 2 Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, Thailand.

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References
1. Goodrich, C.S., Peterson, A.C.: Discrete Fractional Calculus. Springer, New York (2015)
2. Wu, G.C., Baleanu, D.: Discrete fractional logistic map and its chaos. Nonlinear Dyn. 75, 283–287 (2014)
3. Wu, G.C., Baleanu, D.: Chaos synchronization of the discrete fractional logistic map. Signal Process. 102, 96–99 (2014)
4. Wu, G.C., Baleanu, D., Xie, H.P., Chen, F.L.: Chaos synchronization of fractional chaotic maps based on stability results. Physica A 460, 374–383 (2016)
5. Agarwal, R.P., Baleanu, D., Rezapour, S., Salehi, S.: The existence of solutions for some fractional finite difference equations via sum boundary conditions. Adv. Differ. Equ. 2014, 282 (2014)
6. Goodrich, C.S.: On discrete sequential fractional boundary value problems. J. Math. Anal. Appl. 385, 111–124 (2012)
7. Goodrich, C.S.: On a discrete fractional three-point boundary value problem. J. Differ. Equ. Appl. 18, 397–415 (2012)
8. Lv, W.: Existence of solutions for discrete fractional boundary value problems with p-Laplacian operator. Adv. Differ. Equ. 2012, 163 (2012)
9. Ferreira, R.: Existence and uniqueness of solution to some discrete fractional boundary value problems of order less than one. J. Differ. Equ. Appl. 19, 712–718 (2013)
10. Abdeljawad, T.: On Riemann and Caputo fractional differences. Comput. Math. Appl. 62(3), 1602–1611 (2011)
11. Atici, F.M., Elie, P.W.: Two-point boundary value problems for finite fractional difference equations. J. Differ. Equ. Appl. 17, 445–456 (2011)
12. Atici, F.M., Elie, P.W.: A transform method in discrete fractional calculus. Int. J. Differ. Equ. 2(2), 165–176 (2007)
13. Atici, F.M., Elie, P.W.: Initial value problems in discrete fractional calculus. Proc. Am. Math. Soc. 137(3), 981–989 (2009)
14. Sitthiwirattham, T., Tariboon, J., Ntouyas, S.K.: Existence results for fractional difference equations with three-point fractional sum boundary conditions. Discrete Dyn. Nat. Soc. 2013, Article ID 104276 (2013)
15. Sitthiwirattham, T., Tariboon, J., Ntouyas, S.K.: Boundary value problems for fractional difference equations with three-point fractional sum boundary conditions. Adv. Differ. Equ. 2013, 296 (2013)
16. Sitthiwirattham, T.: Existence and uniqueness of solutions of sequential nonlinear fractional difference equations with three-point fractional sum boundary conditions. Math. Methods Appl. Sci. 38, 2809–2815 (2015)
17. Sitthiwirattham, T.: Boundary value problem for p-Laplacian Caputo fractional difference equations with fractional sum boundary conditions. Math. Methods Appl. Sci. 39(8), 1522–1534 (2016)
18. Reunsumrit, J., Sitthiwirattham, T.: Positive solutions of three-point fractional sum boundary value problem for Caputo fractional difference equations via an argument with a shift. Positivity 20(4), 861–876 (2016)
19. Reunsumrit, J., Sitthiwirattham, T.: On positive solutions to fractional sum boundary value problems for nonlinear fractional difference equations. Math. Methods Appl. Sci. 39(10), 2737–2751 (2016)
20. Kaewwisetkul, B., Sitthiwirattham, T.: On nonlocal fractional sum-difference boundary value problems for Caputo fractional functional difference equations with delay. Adv. Differ. Equ. 2017, Article ID 219 (2017)
21. Chasreechai, S., Sitthiwirattham, T.: Existence results of initial value problems for hybrid fractional sum-difference equations. Discrete Dyn. Nat. Soc. 2018, Article ID 5268528 (2018)
22. Chasreechai, S., Sitthiwirattham, T.: On separate fractional sum-difference boundary value problems with n-point fractional sum-difference boundary conditions via arbitrary different fractional orders. Mathematics 2019(7), Article ID 471 (2019)
23. Kunawuttipreecahan, E., Promsakon, C., Sitthiwirattham, T.: Nonlocal fractional sum boundary value problem for a coupled system of fractional sum-difference equations. Dyn. Syst. Appl. 28(1), 73–92 (2019)
24. Promsakon, C., Chasreechai, S., Sitthiwirattham, T.: Positive solution to a coupled system of singular fractional difference equations with fractional sum boundary value conditions. Adv. Differ. Equ. 2019, Article ID 218 (2017)
25. Soontharanon, J., Chasreechai, S., Sitthiwirattham, T.: On a coupled system of fractional difference equations with nonlocal fractional sum boundary value conditions on the discrete half-line. Mathematics 2019(7), Article ID 256 (2019)
26. Setniker, A.: Sequential differences in nabla fractional calculus. PhD Thesis, University of Nebraska, Lincoln, NE, USA (2019)
27. Anastassiou, G.A.: Nabla discrete calculus and nabla inequalities. Math. Comput. Model. 51, 562–571 (2010)
28. Anastassiou, G.A.: Foundations of nabla fractional calculus on time scales and inequalities. Comput. Math. Appl. 59, 3750–3762 (2010)
29. Abdeljawad, T., Atici, F.M.: On the definitions of nabla fractional operators. Abstr. Appl. Anal. 2012, Article ID 406757 (2012)
30. Abdeljawad, T.: On delta and nabla Caputo fractional differences and dual identities. Discrete Dyn. Nat. Soc. 2013, Article ID 406910 (2013)
31. Abdeljawad, T., Abdall, B.: Monotonicity results for delta and nabla Caputo and Riemann fractional differences via dual identities. Filomat 31(12), 3671–3683 (2017)
32. Ahrendt, K., Castle, L., Holm, M., Yochman, K.: Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula. Commun. Appl. Anal. 16(3), 317–347 (2012)
33. Atici, F.M., Eloe, P.W.: Discrete fractional calculus with the nabla operator. Electron. J. Qual. Theory Differ. Equ. 3, 1 (2009)
34. Atici, F.M., Eloe, P.W.: Linear systems of fractional nabla difference equations. Rocky Mt. J. Math. 41, 353–370 (2011)
35. Baoguoa, J., Erbe, L., Peterson, A.: Convexity for nabla and delta fractional differences. J. Differ. Equ. Appl. 21(4), 360–373 (2015)
36. Baoguoa, J., Erbe, L., Peterson, A.: Two monotonicity results for nabla and delta fractional differences. Arch. Math. 104, 589–597 (2015)
37. Martins, N., Torres, D.F.M.: Calculus of variations on time scales with nabla derivatives. Nonlinear Anal. 71(12), 763–773 (2008)
38. Malinowska, A.B., Torres, D.F.M.: The delta–nabla calculus of variations. Fasc. Math. 44(44), 75–83 (2009)
39. Dryl, M., Torres, D.F.M.: The delta–nabla calculus of variations for composition functionals on time scales. Int. J. Difference Equ. 8(1), 27–47 (2003)
40. Dryl, M., Torres, D.F.M.: A general delta–nabla calculus of variations on time scales with application to economics. Int. J. Dyn. Syst. Differ. Equ. 5(1), 42–71 (2014)
41. Ghorbanian, V., Rezapour, S.: A two-dimensional system of delta–nabla fractional difference inclusions. Novi Sad J. Math. 47(1), 143–163 (2017)
42. Liu, H., Jin, Y., Hou, C.: Existence of positive solutions for discrete delta–nabla fractional boundary value problems with $p$-Laplacian. Bound. Value Probl. 2017, 60 (2017)
43. Griffel, D.H.: Applied Functional Analysis. Ellis Horwood, Chichester (1981)
44. Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cone. Academic Press, Orlando (1988)