The Bottom Mass Prediction in Supersymmetric Grand Unification: Uncertainties and Constraints

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Abstract

Grand unified theories often predict unification of Yukawa couplings (e.g., $h_b = h_\tau$), and thus certain relations among fermion masses. The latter can distinguish these from models that predict only coupling constant unification. The implications of Yukawa couplings of the heavy-family in the supersymmetric extension of the standard model (when embedded in a GUT) are discussed. In particular, uncertainties associated with $m_t$ and $m_b$, threshold corrections at the low-scale, and threshold and nonrenormalizable-operator corrections associated with a grand-unified sector at the high-scale are parametrized and estimated. The implication of these and of the correlation between $m_t$ and the prediction for $\alpha_s$ are discussed. Constraints on the tan $\beta$ range in such models and an upper bound on the $t$-quark pole mass are given and are shown to be affected by the $\alpha_s - m_t$ correlation. Constraints on the low-scale thresholds are found to be weakened by uncertainties associated with the high-scale.

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I. INTRODUCTION

Recent LEP \[1\] and other precision electroweak data is known \[2\] to be consistent with coupling constant unification within the minimal supersymmetric standard model (MSSM) \[3\], in which the standard model (SM) matter is minimally extended, i.e., the Higgs sector contains one pair of Higgs doublets and there is a grand desert (up to small perturbations) between the weak (low) and unification (high) scales. Recently, it was further shown \[4\] that corrections associated with the $t$-quark and Higgs scalar thresholds, sparticle spectrum (for example, see Ref. \[5\]), Yukawa couplings, a possible embedding of the MSSM in a grand unified theory (GUT) \[3\], and nonrenormalizable effects \[4\], as well as constraints \[8,9\] from proton decay non-observation \[10\], introduce theoretical uncertainties but do not alter the successful unification; e.g., the prediction of $\alpha_s(M_Z) \approx 0.125 \pm 0.010$ \[4\] agrees well with the observed value. Such uncertainties depend on seven different effective parameters in addition to the $t$-quark mass and Yukawa coupling. (The $\pm 0.010$ is a sum (in quadrature) of the different theoretical uncertainties estimated using reasonable ranges for the various parameters.) This theoretical uncertainty is sufficiently large that few meaningful constraints can be derived from the $\alpha_s(M_Z)$ prediction by itself. Similar conclusions were reached by Barbieri, Hall and Sarid \[11\].

If, indeed, coupling constant unification is a hint for a supersymmetric (SUSY) GUT, then a next step is to study the predicted relationships among fermion masses in such theories \[12\], in a way that consistently incorporates the different theoretical uncertainties listed above. (The nature of the theoretical corrections, and in particular the presence of adjoint representations, also distinguishes such models from many string-inspired ones.) Let us assume in the following (in addition to the MSSM) that we have (i) Coupling constant unification, and (ii) Third-family two-Yukawa unification. That is, at the unification point $M_G$ (the point above which all the GUT gauge group supermultiplets are complete) we have $h_b(M_G) = h_\tau(M_G)$, as is the case \[12\] in a minimal $SU(5)$ unification, which we will assume below for definiteness, and in similar unification schemes\[4\]. $h_\alpha$ is the MSSM Yukawa coupling of a fermion of type $\alpha$ and $M_G \approx 10^{16} - 10^{17}$ GeV.

Assumption (ii) can be incorporated into more ambitious attempts \[13\] to explain the origin of all fermion masses. Such models, which assume extended high-scale structures ("textures"), were shown recently \[14,20\] to have successful predictions as well as possible implications for neutrino masses. However, limiting our analysis to assumptions (i) and (ii), we neglect hereafter the Yukawa couplings of the first two families (where empirically $m_\nu \approx 10 m_e$, rather than $m_\nu \approx m_e$; the latter would be implied by extending assumption (ii) to the first two families and their negligibly small Yukawa couplings) and also flavor mixings. The usual argument goes that some perturbation modifies the couplings or masses of the first two families\[6\] without significantly altering (ii). We do not elaborate on any

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1 This holds in models such as $SU(5)$, $SO(10)$ and $E_6$ for the Yukawa coupling of Higgs fields in the fundamental $(5,10,27)$ representations.

2 In the texture models mentioned above, such a mechanism is realized by introducing large Higgs representations (e.g., $45$ of $SU(5)$ or $126$ and $210$ of $SO(10)$) and (in most cases) a set of flavor
such mechanism. A special case of (ii) is a third-family three-Yukawa unification, i.e., \( h_t(M_G) = h_b(M_G) = h_\tau(M_G) \), which is the situation in some \( SO(10) \) models involving a single complex Higgs 10-plet. We will consider such a possibility as well.

Let us stress that we do not take (ii) to be independent of (i). The coupling constant unification assumption by itself is not enough to significantly constrain the MSSM parameter space. Here, we examine whether more can be said when imposing (ii) as an additional assumption that can possibly distinguish GUT models from some GUT-like string-inspired models (where (ii) is not expected to hold in general). Assumption (ii) was considered recently by several groups. Some \[22\], either (a) carried out a one-loop analysis, (b) assumed a low \( \alpha_s(M_Z) \) (e.g., \( \alpha_s(M_Z) \sim 0.11 \), which is lower than the value expected from coupling constant unification) as an input, (c) ignored the correlation between \( m_t \) and the predicted value of \( \alpha_s(M_Z) \), and/or (d) allowed the running \( b \)-quark mass, \( m_b \), to be as high as 5 GeV (which, as we discuss below, is a more appropriate upper bound on the pole mass). More recent results of two-loop analyses \[3,18,23\] imply a very constrained parameter space, i.e., only a small allowed area in the \( m_t^{\text{pole}} - \tan \beta \) plane, where \( m_t^{\text{pole}} \) is the \( t \)-quark pole mass and \( \tan \beta \) is the ratio of the two Higgs doublet expectation values, \( \nu_{h_{up}}/\nu_{h_{down}} \). Therefore, one would hope that linking (i) with (ii) (and considering uncertainties associated with \( m_t \) and \( m_b \)) will result in some useful constraints on the MSSM parameters, assuming a minimal \( SU(5) \)-type unification (for example, see Ref. \[3\]).

Below, we carry out a careful analysis under the above assumptions and consider various theoretical uncertainties in the calculation. We find that requiring (i) and predicting \( \alpha_s(M_Z) \) as a function of \( m_t^{\text{pole}} \) and of \( \tan \beta \) \[4\] in the range of \( \sim 0.12 - 0.13 \) (see Figure 1), constrains the \( \tan \beta \) range allowed by (ii) more severely than suggested by previous analyses. On the other hand, various theoretical uncertainties can relax the constraints. We also obtain \( \sim 215 \) GeV for the upper bound on \( m_t^{\text{pole}} \) (where \( \alpha_s - m_t \) correlations were taken into account). Some information about the low-scale mass parameters can be extracted. However, corrections associated with the high-scale contribute significantly to the theoretical uncertainties and weaken any constraints. The only spectrum parameter that is strongly constrained is \( \tan \beta \). In agreement with other authors, we find low- (\( \sim 0.6 - 3 \)) and high- (\( \sim 40 - 58 \)) \( \tan \beta \) allowed regions (branches). The former saturates the \( h_t \) infra-red fixed-point \[24\] line (the divergence line). The \( \alpha_s - m_t \) correlation modifies the fixed-point value for \( h_t \) and diminishes the dependence of the allowed \( \tan \beta \) range on \( m_t^{\text{pole}} \lesssim 215 \) GeV. Theoretical uncertainties (and in particular, those associated with the high-scale) determine the width of each branch and, thus, the separation between the two branches.

The various data (and in particular, the \( b \)-quark mass) and the procedure are reviewed in section \[14\]. The constraints on the \( m_t^{\text{pole}} - \tan \beta \) plane and the role of the strong coupling are presented and discussed in section \[11\]. The different correction terms are described and evaluated in greater detail in section \[15\]. We summarize our conclusions in section \[16\]. Throughout this work, we keep the philosophy (and where relevant, the notation) we introduced previously \[4\].

...symmetries. For a different possibility involving nonrenormalizable operators, see Ref. \[21\].
II. INPUT DATA AND PROCEDURE

At the $Z$-pole,

$$M_Z = 91.187 \pm 0.007 \ \text{GeV},$$  \hspace{1cm} (1)

and using the modified minimal subtraction scheme ($\overline{\text{MS}}$) \cite{25} the weak angle and couplings\cite{4,26,27} are \cite{126,27}

$$s^2(M_Z) = 0.2324 - 1.03 \times 10^{-7} (m_t^{\text{pole}} \text{(GeV)}^2 - (138)^2) \pm 0.0003, \hspace{1cm} (2)$$

$$\frac{1}{\alpha(M_Z)} = 127.9 \pm 0.1, \hspace{1cm} (3)$$

$$\alpha_s(M_Z) = 0.12 \pm 0.01, \hspace{1cm} (4)$$

where we displayed explicitly the quadratic dependence of $s^2(M_Z)$ on $m_t^{\text{pole}}$, which is decoupled from the 0.0003 uncertainty \cite{4}. The $U(1)_Y$, $SU(2)_L$, and $SU(3)_c$ couplings are given by

$$\frac{1}{\alpha_1(M_Z)} = \frac{3}{5} \frac{1 - s^2(M_Z)}{\alpha(M_Z)} \quad \text{and} \quad \frac{1}{\alpha_2(M_Z)} = \frac{s^2(M_Z)}{\alpha(M_Z)},$$

and

$$\frac{1}{\alpha_3(M_Z)} = \frac{1}{\alpha_s(M_Z)} \quad \text{respectively.}$$

For the fermion masses, from electroweak precision data\cite{4} we have for the $t$-quark \cite{4}

$$m_t^{\text{pole}} = 138^{+20}_{-25} \pm 5 \ \text{GeV} \hspace{1cm} (5)$$

for a Higgs mass in the range 50 – 150 GeV, which is appropriate for the MSSM. The pole mass is related to the $\overline{\text{MS}}$ running mass, $m_t$, to leading order in $\alpha_s$ by $m_t = (1 - \frac{4}{3} \alpha_s) m_t^{\text{pole}}$. The $\tau$-lepton ($\overline{\text{MS}}$ running) mass \cite{31} is given at the $Z$-pole by

$$m_{\tau}(M_Z) = 1.7486 \pm 0.0006 \ \text{GeV},$$

which corresponds to \cite{3} and $m_{\tau}^{\text{pole}} = 1.7771 \pm 0.0005 \ \text{GeV} \ [32]$.\cite{31}

The situation regarding the $b$-quark mass is more complicated. There are ambiguities in the extraction of the $\overline{\text{MS}}$ running mass $m_b$. Gasser and Leutwyler \cite{33} point out that there

\footnote{A predicted $\alpha_s(M_Z)$ slightly above 0.13, as it is for a heavy enough $t$-quark (see Figure 1), does not contradict \cite{4}; the $\alpha_s(M_Z)$ prediction still has a fairly large theoretical uncertainty of $\sim \pm 0.008$.}

\footnote{Slightly more recent data yields \cite{28} $m_t^{\text{pole}} = 134^{+23}_{-28} \pm 5 \ \text{GeV}$ (for $m_{h,0} \sim 60 - 150 \ \text{GeV}$ and including two-loop $\alpha \alpha_s m_t^2$ corrections) and $s^2(M_Z) = 0.2326 \pm 0.0006$ ($m_t$ free). For our present purposes the difference with \cite{2} and \cite{3} is negligible. The new data will be incorporated in future analyses \cite{29}.}

\footnote{The next-to-leading correction \cite{30,31} is $\sim 2\%$ (depending on $\alpha_s$). The leading correction given here is $\sim 5\%$. (See also the discussion of $m_b/m_t^{\text{pole}}$ below.) We neglect the former (together with other subleading $m_t$ and $m_t^{\text{pole}}$ effects) while keeping the latter.}

\[4\]
is no universal prescription for the relevant scale where $\alpha_s$ is to be evaluated, which suggests that the extraction of $m_b$ is to be carried out case by case, or alternatively, for a range of $\alpha_s$. (We will adopt the latter.) Gasser and Leutwyler identify (to leading order in $\alpha_s$) the running mass $m_b(m_b)$ with the euclidean mass parameter. This point was emphasized by Narison, who offers an alternative definition of $m_b(m_b^{\text{pole}})$ \[34\]. The different definitions introduce a scale ambiguity. Another theoretical difficulty may arise from the role of nonperturbative effects in the interpretation of potential models \[33\]. The next-to-leading correction to the ratio of the $\overline{MS}$ running mass to the pole mass was given more recently by Gray et al. \[30\], i.e.,

\[
m_b = m_b^{\text{pole}} \left( 1 - \frac{4 \alpha_s}{3 \pi} - 12.4 \left( \frac{\alpha_s}{\pi} \right)^2 \right). \tag{7}
\]

The above comments call for some caution, especially given our aim of exploring the strongly constrained $m_t^{\text{pole}} - \tan \beta$ plane. Let us then adopt a conservative attitude, i.e.,

\[
m_b(5 \text{ GeV}) \leq 4.45 \text{ GeV}, \tag{8}
\]

which corresponds, for example, to $m_b^{\text{pole}} \leq 5 \text{ GeV}$, $\alpha_s \geq 0.17$, and using \[7\]. The next-to-leading correction term in \[7\] reduces $m_b$, and $m_b^{\text{pole}} \approx 4.5 \text{ GeV}$, $\alpha_s \approx 0.2$ implies $m_b < 4 \text{ GeV}$. For example, $m_b^{\text{pole}} \approx 4.5 \text{ GeV}$, $\alpha_s \approx 0.25$ gives $m_b \approx 4.0 \text{ GeV}$ when neglecting the next-to-leading term, and $m_b \approx 3.7 \text{ GeV}$ when using \[7\]. The $m_b$ prediction, on the other hand, lies (in general) above 4 GeV. Given the above, we do not specify a lower bound equivalent to \[5\]. Also, requiring $m_b(4.45 \text{ GeV}) \leq 4.45 \text{ GeV}$ (which will correspond to $m_b(m_b) = 4.25 \pm 0.20 \text{ GeV}$ \[33\], where we have doubled the uncertainty) is somewhat more constraining (e.g., $m_b(4.45 \text{ GeV}) - m_b(5 \text{ GeV}) \approx 0.05 - 0.15 \text{ GeV}$ depending on $\alpha_s$).

We use $\alpha(M_Z)$, $s^2(M_Z)$, and the $\tau$-lepton and $t$-quark Yukawa couplings,

\[
h_\tau(M_Z) = \frac{m_\tau(M_Z)}{174 \text{ GeV} \times \cos \beta}, \tag{9}
\]

and

\[
h_t(M_Z) = \left( 1 - \frac{4 \alpha_s(M_Z)}{3 \pi} \right) \frac{m_t^{\text{pole}}}{174 \text{ GeV} \times \sin \beta}, \tag{10}
\]

to predict $\alpha_s(M_Z)$ and $h_b(M_Z)$, for a definite point in the $m_t^{\text{pole}} - \tan \beta$ plane. One should note that $h_t$ depends on $m_t^{\text{pole}}$ also via the $\alpha_s(M_Z)$ correction in \[10\] (and via the $\alpha_3$ contribution to the running — see below). As we pointed out, $s^2(M_Z)$ depends quadratically on $m_t^{\text{pole}}$. Therefore, we neglect all subleading logarithmic dependencies on $m_t^{\text{pole}}$ (for a discussion, see Ref. \[4\]), including small corrections to \[10\]. We further neglect the error bars in \[1\] - \[3\] and in \[5\]. Also, $\alpha_i$ are all converted to the $\overline{DR}$ scheme, using the proper

\[6\]

The constituent mass parameter in these models is identified with the pole mass.

\[7\]
i.e., case (b) in the notation of Ref. \[4\].
step functions. Using two-loop renormalization group equations (RGE’s) iteratively, we are able to predict $\alpha_s(M_Z)$ and $h_b(M_Z)$ as functions of $m_t^{pole}$ and $\tan \beta$. We take $100 < m_t^{pole} < 200 \text{ GeV}$ as a reasonable conservative range, and constrain $\tan \beta$ only by requiring the Yukawa couplings to stay perturbative, i.e., $h_\alpha(\mu) < 3$ where $M_Z < \mu < M_G$ and $\alpha = t, b, \tau$. (This range can also be justified by requiring two-loop contributions to the RGE’s to be less than a quarter of the one-loop ones.) We then run down using three-loop QCD and two-loop QED RGE’s to find $m_b^0(5 \text{ GeV})$, where the $m_b^0$ prediction is that of $m_b$, but without (theoretical) corrections to the RG calculation.

In any realization of the MSSM, there are small perturbations (order of magnitude of two-loop terms) to the grand desert and unification assumptions, as described above. Thus, in general, $m_b = \rho^{-1} m_b^0$ where $\rho^{-1} \neq 1$ is a correction parameter which incorporates the uncertainties in the running from $M_Z$ up to $M_G$. Let us stress that in our formulation one does not change the MSSM $\beta$-functions to those of the SM at $m_t$ or at some other effective scale. Rather, leading $m_t^{pole}$ effects are accounted for in (2), and all other such effects determine $\rho^{-1}$. A point in the $m_t^{pole} - \tan \beta$ is excluded if either $h_t > 3$ (tan $\beta \lesssim 1 - 2$ and/or $m_t^{pole} \gtrsim 215 \text{ GeV}$), $h_b \sim h_\tau > 3$ (tan $\beta \gtrsim 58$), or

$$m_b(5 \text{ GeV}) = \rho^{-1} m_b^0(5 \text{ GeV}) > 4.45 \text{ GeV}. \quad (11)$$

Incorporating uncertainties associated with $m_t^{pole}$ and Yukawa couplings (in addition to the $\overline{DR}$ conversion step functions) in the numerical procedure we have to further consider uncertainties associated with the sparticle and Higgs thresholds, high-scale thresholds, and Planck-scale nonrenormalizable operators. For simplicity, we will assume that we have one heavy ($M_H \gg M_Z$) Higgs doublet that decouples with the sparticles, and another light ($m_h \sim M_Z$) SM-like doublet that is responsible for all fermion masses. We are able to obtain an (approximate) analytic expression for $\rho^{-1}$ by expanding one-loop expressions around their unperturbed values. This will be carried out in section [V], where we study the different contributions to $\rho^{-1}$ in GUT models, and estimate $\rho^{-1}$ in the minimal $SU(5)$ model. High-scale corrections to the coupling constant unification (and not the details of the sparticle spectrum) constitute the larger uncertainty. We take

$$\rho^{-1} = 1.00 \pm 0.15, \quad (12)$$

which is a conservative estimate derived for reasonable ranges of the various correction parameters. Using (12), the exclusion condition (11) reads

$$m_b(5 \text{ GeV}) \geq 0.85 m_b^0(5 \text{ GeV}) > 4.45 \text{ GeV}. \quad (13)$$

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8In the notation of ref. [1], $\Delta_i^{\text{conversion}}, \Delta_i^{\text{top}},$ and $\Delta_i^{\text{Yukawa}}$, are all directly incorporated in the calculation.

9In such a case, $SU(2)$ breaking effects are, in general, negligible above $m_t$. 

6
III. THE $m^\text{pole}_t - \tan \beta$ PLANE

Given the above, we find that assumptions (i) and (ii) allow a low-tan $\beta$ branch and a high-tan $\beta$ branch. The allowed parameter space is shown in Figure 2, where the narrow strip corresponding to three-Yukawa unification is also indicated. The low-tan $\beta$ branch is shown in greater detail in Figure 3, where the lines corresponding to $\rho^{-1} = 1$ and $h_t(M_G) = 2$ are displayed for comparison. The former is, in fact, the $h_t$ infra-red fixed-point [24] line, which is the $h_t$-divergence line. ($h_t(M_Z) > h_t^{\text{fixed}} \Rightarrow h_t(\mu) \gg 1$ for $\mu < M_G$.) This point was also discussed recently in Ref. [18,23]. $\rho^{-1} \neq 1$ only slightly extends the allowed low-tan $\beta$ range. It is also interesting to note that constraints from proton decay via dimension-five operators would exclude the high-tan $\beta$ branch for $\rho^{-1} \equiv 1$ (i.e., $\tan \beta \lesssim 4.7$ [8]). However, once correction terms are included, $M_G$ can grow significantly [14,11] and no useful constraints on $\tan \beta$ can be derived from proton decay non-observation [9]. For comparison, we show in Figure 4 the equivalent parameter space with (13) replaced by $0.85m_b^0(4.45 \text{ GeV}) > 4.45 \text{ GeV}$. The allowed $\tan \beta$ range is reduced by $\sim 0.03 - 0.10$ for the low-tan $\beta$ branch and by $3 - 4$ for the high-tan $\beta$ one. Replacing (13) with $0.85m_b^0(\sim 5.1 \text{ GeV}) \gtrsim 4.6 \text{ GeV}$ would have a similar but opposite effect (i.e., slightly decreasing the separation between the two branches). A smaller (larger) uncertainty in (12) will have an effect similar to the former (latter). The $\rho^{-1}$-range estimate (and the $m_b$ upper bound) determine the width of each branch, and thus the excluded intermediate $\tan \beta$ range. Perturbative consistency (i.e., the divergence lines discussed above) excludes the very small and the very large $\tan \beta$ ranges and determines the upper bound on $m^\text{pole}_t$, $m^\text{pole}_t \lesssim 215 \pm 10 \text{ GeV}$, where the $\pm 10 \text{ GeV}$ uncertainty is due to $\rho_{\text{top}}$.

The $h_t$-divergence line eventually becomes approximately parallel to the $\tan \beta$-axis (and determines the upper bound $m^\text{pole}_t \lesssim 215 \text{ GeV}$). Some intermediate values of $\tan \beta$ are thus allowed for $m^\text{pole}_t > 200 \text{ GeV}$. (The $h_b$ and $h_t$ divergence lines meet near the $h_b = h_t$ line.) Otherwise, Yukawa unification at the grand unification scale is ruled out [34] if $2.7 \lesssim \tan \beta \lesssim 40$. Furthermore, the low-tan $\beta$ branch, where $h_t \sim 1$ and which many would consider a more natural choice, saturates the fixed-point line and has to be adjusted to a few parts in a hundred (a few parts in a thousand, if $\rho^{-1} = 1$) for a given $m^\text{pole}_t$ (see Figure 3). The large-tan $\beta$ branch is more spread and implies, in general, a much lower $h_t$ and $h_t < h_b$. ($h_t$ can still be large for a large enough $m^\text{pole}_t$, and $h_t > h_b$ above the three-Yukawa unification strip.) While we find no constraints on $m^\text{pole}_t \lesssim 215 \text{ GeV}$ from two-Yukawa unification, three-Yukawa unification is ruled out unless $169 \lesssim m^\text{pole}_t \lesssim 196 \text{ GeV}$. (A slightly larger range, i.e., $m^\text{pole}_t \gtrsim 160 \text{ GeV}$, is allowed when one includes corrections to the $h_t/h_b$ ratio, which induce a $\sim 5\%$ theoretical uncertainty. We comment more on this point in section [V].) One expects mutual implications [29] between the above observations and radiative-breaking of $SU(2) \otimes U(1)$, an attractive feature of the MSSM that prefers $h_t > h_b$ [35].

To demonstrate the effect of calculating $m_b$ using the predicted $\alpha_s(M_Z)$ rather than a fixed input value, i.e., of associating the Yukawa coupling unification with the rather high

\footnote{For a large $\tan \beta$ ($\nu_{h_{\text{down}}} \ll \nu_{h_{\text{top}}}$, $\nu$) some caution may be required regarding the scale at which the Higgs potential is minimized and $\tan \beta$ is defined, as was pointed out by Bando et al. [22] and by Chankowski [37].}
values of $\alpha_s(M_Z)$ predicted by $\alpha_1 - \alpha_2$ unification, we compare Figure 2 with Figures 5 – 7. There, $\alpha_s(M_Z)$ is fixed ($\alpha_s(M_Z) = 0.11, 0.12, 0.13$, in Figure 5, 6, 7, respectively), and thus assumption (i) is relaxed; i.e., for $\alpha_s(M_Z) = 0.11 (0.12, 0.13)$ there is a $\sim 7\%$ ($\sim 3\%$) split between $\alpha_3(M_G)$ and the $\alpha_G$ defined by $\alpha_1$ and $\alpha_2$. Let us stress that the different corrections are not treated on equal footing in this case, because some are included in $\rho^{-1}$ while others (like NRO’s) are absorbed in the fixed value of $\alpha_s(M_Z)$. Furthermore, the appropriateness of this decomposition depends on which type of uncertainties shift the predicted $\alpha_s(M_Z)$. (We elaborate more on this point in section IV). Nevertheless, the comparison illustrates that a low $\alpha_s(M_Z)$ is preferred by $m_b$. The allowed parameter space for $\alpha_s(M_Z) = 0.11$ (Figure 5) is much larger than that for $\alpha_s(M_Z) = 0.13$ (Figure 7). For a lower value of $\alpha_s$, the radiative corrections that reduce $h_b$ are diminished, and thus a given $h_\tau(M_G) = h_b(M_G)$ implies a lower $h_b(M_Z)$. However, the low value $\alpha_s(M_Z) = 0.11$ requires large corrections to the coupling constant unification.

The above discussion also explains the slight differences between our results and those of previous analyses. Requiring (i) and using (2) for $s^2(M_Z)$ imply that $\alpha_s(M_Z)$ grows with $m_{t\text{pole}}$, e.g., $\alpha_s(M_Z) \sim 0.12$ for $m_{t\text{pole}} \sim 100$ GeV, and $\alpha_s(M_Z) \sim 0.13$ for $m_{t\text{pole}} \sim 180$ GeV (see Figure 1). Indeed, Figure 2 roughly coincides with Figure 6 for the former and with Figure 7 for the latter, $h_t(M_Z)$ in $\langle Z \rangle \sim m_{s\text{pole}} Z$, but is diminished by (the $m_{t\text{pole}}$ dependent) $\alpha_s(M_Z)$. These all affect the balance between positive and negative contributions to the Yukawa coupling RGE’s (i.e., the fixed-points), and thus modify the $h_b(M_Z)$ prediction and increase the upper bound on $m_{t\text{pole}}$.

### IV. THE CORRECTION TERMS

We now turn to a detailed discussion of the correction parameter, $\rho^{-1}$. The coupling constant two-loop RGE’s are solvable analytically, and it is convenient to write [39]

$$\frac{1}{\alpha_i(M_Z)} = \frac{1}{\alpha_G} + b_it + \theta_i + H_i - \Delta_i \text{ for } i = 1, 2, 3, \quad (14)$$

where $t = \frac{1}{2\pi} \ln \frac{M_G}{M_Z} \approx 5.3$ is the relevant scale parameter, and $\alpha_G \approx \frac{1}{24}$ is the coupling constant at the unification point, $M_G$. $b_i = 6.6, 1, -3$, for $i = 1, 2, 3$, respectively, are the one-loop $\beta$-function coefficients; $\theta_i \approx 0.7, 1.1, 0.6$, for $i = 1, 2, 3$, are the two-loop corrections; $H_i$ are negligible Yukawa coupling two-loop contributions; and the functions $\Delta_i$ incorporate all other corrections to the calculation of order of magnitude consistent with $\theta_i$. In our scheme, $\alpha_1$ and $\alpha_2$ are inputs. By taking linear combinations we obtain three predictions, i.e.,

$$\alpha_s(M_Z) = \alpha_s^0(M_Z)[\alpha_1, \alpha_2, \theta_i] + (\alpha_s^0(M_Z))^2 \Delta_s[\Delta_1, \Delta_2, \Delta_3], \quad (15a)$$

$$\frac{1}{\alpha_G} = \frac{1}{\alpha_G^0}[\alpha_1, \alpha_2, \theta_1, \theta_2] + \Delta_{\alpha_G}[\Delta_1, \Delta_2], \quad (15b)$$

$$t = t^0[\alpha_1, \alpha_2, \theta_1, \theta_2] + \Delta_t[\Delta_1, \Delta_2], \quad (15c)$$
where we explicitly separated the two-loop predictions with no corrections ($\Delta_i = 0$) from the contribution of the correction functions, $\Delta_i$. The expressions for $\Delta_{\alpha_s}$, $\Delta_{\alpha_G}$ and $\Delta_t$ are given in Appendix A.

The integration of the two-loop RGE’s for the Yukawa couplings is rather complicated and has to be done numerically. To estimate the theoretical correction terms it is useful to display the (one-loop) RGE’s, i.e.,

$$\frac{dy_\alpha}{y_\alpha} = \left[ \sum_{i=1,2,3} b_{\alpha;i} \alpha_i + \sum_{\beta=t,b,\tau} b_{\alpha;\beta} y_\beta + \ldots \right] dt', \quad \text{(16)}$$

where $y_\alpha = \frac{\beta^2}{4\pi}$ for $\alpha = t, b, \tau$; $t' = \frac{1}{2\pi} \ln \frac{\mu'}{M}$, and we have omitted higher-order terms. $b_{\beta;i} = -\frac{7}{15}, -3, -\frac{16}{3}$ for $i = 1, 2, 3$, respectively; and $b_{\beta;\beta} = 1, 6, 1$, for $\beta = t, b, \tau$. ($b_{t;i} = -\frac{9}{5}, -3, 0; b_{b;i} = 0, 3, 4; b_{t;i} = -\frac{13}{15}, -3, -\frac{16}{3};$ and $b_{b;\beta} = 6, 1, 0$.) The balance between the negative $b_{\alpha;i} \alpha_i$ and the positive $b_{\alpha;\beta} y_\beta$ terms determines the infra-red fixed point in the Yukawa coupling renormalization flow. From (16) we obtain

$$h_b(M_G) = h_b(M_Z) \times \prod_{i=1}^{3} \left( \frac{\alpha_{i,OL}(M_G)}{\alpha_{i,OL}(M_Z)} \right)^{\frac{b_{b;i}}{2\pi}} \times F_b \times \Theta_b \times \rho_b, \quad \text{(17)}$$

and similarly for $h_t(M_G)$. The $\alpha_{i,OL}$ are the one-loop ($OL$) couplings (i.e., $\theta_i = H_i = 0$ in (14)). Substituting instead two-loop ($TL$) (or input) expressions one has to compensate by properly modifying the two-loop correction, $\Theta_b$. $F_b$ is the correction due to the non-negligible Yukawa contribution at one-loop, i.e., $\int b_{\alpha;\beta} y_\beta dt'$. $\rho_b$ incorporates the theoretical uncertainties in the RG calculation.

From (17) and the equivalent expression for $h_t$ (and assuming $h_b(M_G) = h_t(M_G)$) we have

$$m_b(M_Z) = m_t(M_Z) \times \left( \frac{\alpha_{3,OL}(M_Z)}{\alpha_{G,OL}(M_Z)} \right)^{\frac{5}{9}} \left( \frac{\alpha_{3,OL}(M_Z)}{\alpha_{G,OL}(M_Z)} \right)^{\frac{10}{9}} \times F^{-1} \times \Theta^{-1} \times \rho^{-1}, \quad \text{(18)}$$

where $F = \frac{F_b}{F_t}$, $\Theta = \frac{\Theta_b}{\Theta_t}$, and $\rho = \frac{\rho_b}{\rho_t}$. Setting $\Theta = \rho = 1$, substituting the one-loop expressions for $\alpha_i$ and $\alpha_{G}$, and assuming negligible Yukawa couplings (i.e., $F \approx 1$) gives an exact well-known one-loop expression. $F^{-1}$ can be estimated analytically for $h_t \gg h_b, h_t$ [10], i.e.,

$$F^{-1} \approx (1 + 11 h_t^2(M_G))^{-\frac{1}{12}}, \quad \text{(19)}$$

which gives $F^{-1} \sim 0.68$ for $h_t(M_G) \sim 3$. In general, however, a numerical analysis is required to fully incorporate ($F, \Theta \neq 1$). The $m_b^0(M_Z)$ that we calculate is given by (18) with $\rho^{-1} = 1$ and numerical values for $F$ and $\Theta$.

Before we turn to a rather technical derivation of the correction parameter $\rho^{-1}$, let us discuss a simple toy model and point out the ways in which it gets complicated. If the ideal desert and unification assumptions hold, then (neglecting two-loop terms)

$$\frac{1}{\alpha_1(M_Z)} = \frac{1}{\alpha_{G}^0} + b_1 t^0, \quad \text{(20a)}$$
\[
\frac{1}{\alpha_2(M_Z)} = \frac{1}{\alpha^0_G} + b_2 t^0, \tag{20b}
\]

\[
\frac{1}{\alpha_s^0(M_Z)} = \frac{1}{\alpha^0_G} + b_3 t^0. \tag{20c}
\]

We use (20a) and (20b) to define \( \alpha^0_G \) and \( t^0 \) in terms of the (input) \( \alpha_{1,2}(M_Z) \). We now turn on the \( \Delta_1 \) and \( \Delta_2 \) correction functions and assume that no other corrections contribute to \( \rho^{-1} \). The coupling constants are now given by

\[
\frac{1}{\alpha_1(M_Z)} = \frac{1}{\alpha^0_G} + b_1 t^0 + \Delta_{\alpha_G} + b_1 \Delta_t - \Delta_1, \tag{21a}
\]

\[
\frac{1}{\alpha_2(M_Z)} = \frac{1}{\alpha^0_G} + b_2 t^0 + \Delta_{\alpha_G} + b_2 \Delta_t - \Delta_2, \tag{21b}
\]

\[
\frac{1}{\alpha_s(M_Z)} = \frac{1}{\alpha^0_s(M_Z)} - \Delta_{\alpha_s} = \frac{1}{\alpha^0_G} + b_3 t^0 + \Delta_{\alpha_G} + b_3 \Delta_t. \tag{21c}
\]

\( \Delta_{\alpha_G} \) and \( \Delta_t \) are determined by the condition \( \Delta_{\alpha_G} + b_i \Delta_t - \Delta_i = 0 \) for \( i = 1, 2 \), while (in the present approximation) \( \Delta_{\alpha_s} \) is due entirely to the change in \( \alpha_G \) and \( t \), i.e., \(-\Delta_{\alpha_s} = \Delta_{\alpha_G} + b_3 \Delta_t \). Also,

\[
\frac{1}{\alpha_3(M_G)} = \frac{1}{\alpha^0_G} + \Delta_{\alpha_G}, \tag{22}
\]

and the \( \alpha_3 \) term in (18) now reads

\[
\left( \frac{\alpha_s^0(M_Z)}{\alpha^0_G} \right)^{\frac{8}{9}} \times \left( 1 + \frac{8}{9} [\alpha_s^0(M_Z) \Delta_{\alpha_s} + \alpha^0_G \Delta_{\alpha_G}] \right). \tag{23}
\]

We thus obtain (in the toy model)

\[
\rho^{-1} = e^{\frac{8}{9} [(\alpha_s(M_Z) - \alpha_G) \Delta_{\alpha_s} - b_3 \alpha_G \Delta t]} \tag{24}
\]

In the more general case \( \Delta_3 \neq 0 \) and \(-\Delta_{\alpha_s} = \Delta_{\alpha_G} + b_3 \Delta_t - \Delta_3 \), where \( \Delta_3 = \Delta^{SUSY}_3 + \Delta^{heavy}_3 + \Delta^{NRO}_3 \) (for the low-scale threshold, high-scale threshold, and NRO contributions, respectively). NRO's \( (\Delta^{NRO}_3) \) modify only the \( \alpha_3 \) value and not any RGE coefficients (see Ref. [4] and below) and can be easily incorporated in our toy model, i.e., (24) is still correct if \(-\Delta_{\alpha_s} = \Delta_{\alpha_G} + b_3 \Delta_t - \Delta^{NRO}_3 \). The high-scale thresholds are more complicated because they not only affect \( \alpha_s(M_Z) \) but also change the \( \beta \)-function coefficient \( b_3 \) and the coefficient of the RGE for \( y_b \) (the \( b_{1,2,3} \)) at the various thresholds. The expression for \( \rho^{-1} \) will be derived below. Ignoring for now the threshold changes in \( b_{1,2,3} \) the \( \Delta^{heavy}_3 \) contribution to \( \rho^{-1} \)

\[
\sim e^{\frac{8}{9} [(\alpha_s(M_Z) - \alpha_G) \Delta_{\alpha_s} - b_3 \alpha_G \Delta t]} \tag{23}, \]

i.e., only the shift in \( \alpha_s(M_Z) \), which affects the entire \( t' \) range in (13), is relevant. The effect of the change in \( b_3 \) above the threshold is of second order because it only affects a small region of the \( t' \) integral. Similarly, the leading contribution of \( \Delta^{SUSY}_3 \) is
\( \rho^{-1} \sim e^{\frac{\pi}{2}[\alpha_s(M_Z) - \alpha_3(\sim TeV)] \Delta^{SUSY}_3 t} \), which is a second order in small quantities (because it only affects a small region of the \( t' \) integral) and is therefore negligible.

Hence, the corrections to gauge couplings lead to

\[
\rho^{-1} = e^{\frac{\pi}{2}[\alpha_s(M_Z) - \alpha_3(\sim TeV)] \Delta^{SUSY}_3 t + \Delta b \alpha_3 \Delta t},
\]

(25)

where \( -\Delta'_{\alpha_s} = -\Delta_{\alpha_s} + \Delta^{SUSY}_3 = \Delta_{\alpha_G} + b_3 \Delta t - \Delta^{NRO}_3 - \Delta^{heavy}_3 \) includes all the shifts in \( \alpha_s(M_Z) \) except those induced by \( \Delta^{SUSY}_3 \). The additional corrections associated with the changes in \( b_{\alpha_3} \) at thresholds will be discussed below.

A different complication is due to the non-negligible role of the Yukawa couplings. \( F \) is modified when thresholds are decoupled. In particular, once the heavy Higgs doublet is decoupled the Yukawa operators and their evolution are modified. (Recall that we assume that we have one heavy \( (M_H \gg M_Z) \) Higgs doublet that decouples with the sparticles, and another light \( (m_h \sim M_Z) \) SM-like doublet that is responsible for all fermion masses.) Also, \( h_3(M_G) > 1 \) near either the \( h_t \) (low-tan \( \beta \)) or \( h_b \) (large-tan \( \beta \)) fixed points, and the most significant high-scale effect of correcting \( t^0 \rightarrow t^0 + \Delta t \) is due to the large Yukawa couplings and not to the \( \alpha_G \Delta t \) term. We will therefore treat high-scale \( \Delta t \) effects \( (\rho^{-1}_t) \) separately from \( \Delta_{\alpha_s} \) and \( \Delta_{\alpha_G} \) effects. \( \Delta_{\alpha_s} \) will include \( \Delta_3 \) contributions which will be partially cancelled by decoupling thresholds from both the \( \alpha_3 \) and \( y_b \) RGE’s. Thus, \( \Delta_{\alpha_s} \) and \( \Delta_{\alpha_G} \) effects will be described by \( \rho^{-1}_{\alpha_1} \), which we derive first. (Using the input value of \( \alpha_1, \rho^{-1}_{\alpha_1} \sim 1 \) – see below.) We will then consider corrections to \( F (\rho^{-1}_F) \). Lastly, we will derive \( \rho^{-1}_F \) and rewrite \( \rho^{-1} \) in a way that reflects the correlations among \( \rho^{-1}_{\alpha_3}, \rho^{-1}_F \) and \( \rho^{-1}_t \). We will also comment on the role of the high-scale corrections, the case of using \( \alpha_s(M_Z) \) as an input, and on corrections to the \( h_t/h_b \) ratio.

Allowing a complicated threshold structure near \( M_Z \) (and/or near \( M_G \)) gives a modified one-loop expression for \( m_b \),

\[
m_b(M_Z) = m_t(M_Z) \times \prod_{i=1}^3 \prod_{k=0}^{n-1} \left( \frac{\alpha_i(\mu^k)}{\alpha_i(\mu^{k+1})} \right)^{\frac{1}{\mu^k - \mu^{k+1}}} \times F^{-1} \times (1 + \Delta F),
\]

(26)

where \( k \) runs over the various thresholds, i.e., \( \mu^0 = M_Z \) and \( \mu^n = M_G \). \( b^k \) is the one-loop coefficient of the respective RGE between \( \mu^k \) and \( \mu^{k+1} \), and \( \Delta F \) represents the threshold corrections to \( F \). By expanding (26) around (18) (in a similar way to (23)) and using the results of Ref. [4] we can obtain an approximate expression for \( \rho^{-1} \). This yields a better insight into the role of the different correction parameters than purely numerical estimates.

The important effects of the coupling constant uncertainties are in the \( \alpha_3 \) terms. \( \alpha_2 \) (in our approximation[4]) drops out from (26) and the residual uncertainties from \( \alpha_1 \) are small when the input value is used. Recall that our strategy is to use the experimental values

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11 Once sparticles are decoupled the degeneracy among various operators is lifted, e.g., the gaugino - sfermion - fermion coupling is different from the respective gauge coupling and the higgsino - sfermion - fermion Yukawa coupling is different from the Higgs-boson - fermion - fermion one (see, for example, Chankowski [37]). In [29] we ignored this effect, which is negligible for sparticles and the Higgs-doublet below the TeV scale.
of $\alpha_1(M_Z)$ and $\alpha_2(M_Z)$ to predict $\alpha_3$. The dominant corrections to the $m_b$ prediction are the uncertainties in $\alpha_G$ and $t$ due to $\Delta_1$ and $\Delta_2$, and the explicit uncertainties in $\Delta_3$ (as was illustrated by our toy model). The latter can be divided into low-scale ($\Delta_3^{SU3}$) and to high-scale ($\Delta_3^{\text{heavy}} + \Delta_3^{\text{NRO}}$) contributions. The low-scale uncertainties have only a small effect on $m_b$ because they only affect a small $t'$ range in $\mu$ (see the toy model). High-scale corrections affect the entire $t'$ range. They modify both $\alpha_s(M_Z)$ (high-scale contributions to $\Delta_3$ constitute a part of $\Delta_{\alpha_3}$) and either the $\beta_3$ function near $M_G$ ($\Delta_3^{\text{heavy}}$) or the $\alpha_3(M_G)$ value ($\Delta_3^{\text{NRO}}$). All (high- and low-scale threshold) corrections to $\beta_3$ affect the $\alpha_3$ terms in (20).

We denote the heavy X and Y vector; color-triplet; and the adjoint color-octet, $SU(2)$-triplet (and singlet) superfield thresholds by $M_V$, $M_5$, and $M_{24}$, respectively. Some of the high-scale thresholds are strongly constrained by proton decay, i.e., in the minimal $SU(5)$ model (which we assume) $M_5 \sim M_G$ and perturbative consistency constrains $M_G \lesssim 3M_V$ [8,9]. $M_{24} \ll M_G$ is possible, and $\Delta_t$ in this scenario can be $\sim +0.5$ and the constraints on $M_5$ are relaxed (i.e., $M_5 \gtrsim 0.1M_G$) [1]. Also, proton decay constraints can be removed by a simple modification of the model [1].

The sparticles and the Higgs doublet decouple from the $\alpha_i$ RGE at an effective scale, $M_i$, defined in Ref. [1] (see also Carena et al. [23]), i.e.,

$$\sum \frac{b_\zeta}{(2\pi)} \ln \frac{M_\zeta}{M_Z} = \frac{b_i^{\text{MSSM}} - b_i^S}{(2\pi)} \ln \frac{M_i}{M_Z} \quad \text{for } i = 1, 2, 3. \quad (27)$$

The summation is over all relevant thresholds, i.e., sparticles and the heavy Higgs doublet, and $b_i^\zeta$ is the $\zeta$-particle contribution to the respective $\beta$-function. $M_i$ can be split by a factor of a few. In general, $M_1$ grows most significantly with the scalar mass; $M_3$ with the gaugino mass; and $M_1$ and $M_2$ grow the same with the higgsino mass; and $M_2 \ll M_1$ and/or $M_2 \ll M_3$. $M_1$, $M_2$ and $M_3$ all appear in $\Delta_{\alpha_s}$, $\Delta_{\alpha_G}$ and $\Delta_t$. On the other hand, once either the gluinos or the squarks are decoupled, all squark - gluino loops are eliminated and $b_{v,3} = b_{v,3}^{SM}$ [17], and two other scales of relevance are (in the approximation of degenerate squark masses) $M_3 = \min (M_{\text{gluino}}, M_{\text{squark}})$ and $M_3 = \max (M_{\text{gluino}}, M_{\text{squark}})$. One has $M_3^2 = M_5 M_3$.

We consider high-scale thresholds and NRO’s ($\sim [\alpha_s(M_Z) - \alpha_G]\Delta$), low-scale thresholds ($\sim [\alpha_s(M_Z) - \alpha_3(\sim \text{TeV})]\Delta$), and corrections to the coupling constant unification predictions for $\alpha_s(M_Z)$ and $\alpha_G$. We will discuss corrections to $F$ and to $t$ below. A more detailed treatment of low-scale effects will be needed if either some of the spectrum parameters are better known or if one assumes particle thresholds above the TeV scale. We will take $\mu^1 = M_3$, $\mu^2 = M_3$, $\mu^3 = M_{24}$ and $\mu^4 = M_5$. The couplings and coefficients (to be substituted in (20)) read

$$\alpha_3(\mu^0) = \alpha_s^0(M_Z) + (\alpha_s^0(M_Z))^2 \Delta_{\alpha_s}. \quad (28a)$$

\footnote{The generalization to $M_5 < M_{24}$ is straightforward. The $M_V < M_G$ case is much more difficult to describe. The heavy X and Y supervectors couple to the $SU_3 \times SU_2 \times U_1$ Yukawa operators in a complicated way. However, $M_V \gtrsim \frac{1}{3} M_G$ and the effects cannot be large.}
\[(\alpha_3(\mu^1))^{-1} = (\alpha_s^0(M_Z))^{-1} - b_3^2 t_3 - \Delta_{\alpha_s}, \quad (28b)\]

\[(\alpha_3(\mu^2))^{-1} = (\alpha_s^0(M_Z))^{-1} - b_3^0 t_3 - b_3^1 \delta t_3 - \Delta_{\alpha_s}, \quad (28c)\]

\[(\alpha_3(\mu^3))^{-1} = (\alpha_3(\mu^2))^{-1} - b_3^2 \ln \frac{M_{24}}{M_3} = (\alpha_3(\mu^4))^{-1} \quad (28d)\]

\[(\alpha_3(\mu^4))^{-1} = (\alpha_0^G)^{-1} - b_3^0 t_5 + \Delta_{\alpha_G} - \Delta_3^{NRO}, \quad (28e)\]

\[(\alpha_3(\mu^5))^{-1} = (\alpha_s^0)^{-1} = (\alpha_0^G)^{-1} + \Delta_{\alpha_G} - \Delta_3^{NRO}, \quad (28f)\]

\[b_3^0 = b_3^{SM} = -7, \quad b_3^1 = -5, \quad b_3^2 = b_3^{MSSM} = -3, \quad b_3^3 = 0, \quad b_3^4 = 1, \quad (28g)\]

\[b_{b_3}^0 = b_{b_3}^1 = b_{b_3}^{SM} = -8, \quad b_{b_3}^2 = b_{b_3}^3 = b_{b_3}^4 = b_{b_3}^{MSSM} = -\frac{16}{3}, \quad (28h)\]

\[t_3 = \frac{1}{2\pi} \ln \frac{M_3}{M_2}, \quad \delta t_3 = \frac{1}{2\pi} \ln \frac{M_3}{M_3}, \quad t_5 = \frac{1}{2\pi} \ln \frac{M_5}{M_3}. \quad (\text{We replaced } \alpha_{s,G}^\text{OL} \text{ by the two-loop } \alpha_{s,G}^0 \text{ which introduces a negligible inconsistency.}) \quad \text{In the } [M_{24}, M_5] \text{ interval we cannot use (26) because } b_3^2 = 0, \text{ and instead we have } b_{b_3}^0 \alpha_3(\mu^4) f_{\mu^4} d\ln \mu^4, \text{ which contributes } -\frac{4}{3} \alpha_3 \ln \frac{M_{24}}{M_3} - \ln \frac{M_5}{M_3} = \alpha_G [-\frac{8}{9} \Delta_3^{24} + \frac{4}{3} t_5] \text{ to } \ln \rho^{-1}.\]

We obtain (for the \(\alpha_3\) terms in (26))

\[\left(\frac{\alpha_3^\text{OL}(M_Z)}{\alpha_G^\text{OL}}\right)^{\frac{3}{5}} \times \rho_3^{-1}, \quad (29)\]

where

\[\rho_3^{-1} \equiv e^{[\frac{3}{5} \alpha_0^G \Delta_{\alpha_G} + \frac{3}{5} \alpha_0^G \Delta_{\alpha_G} - \frac{4}{3} \alpha_0^G \delta t_3 - \frac{4}{3} \alpha_0^G \delta t_3 - \frac{4}{3} \alpha_0^G \delta t_3 - \frac{4}{3} \alpha_0^G \delta t_3 - \frac{4}{3} \alpha_0^G \Delta_{\alpha_G}^{24} - \frac{4}{3} \alpha_0^G \Delta_{\alpha_G}^{NRO}]} \quad (30)\]

\(\alpha_1\) (and \(\alpha_2\)) uncertainties feed into \(\Delta_{\alpha_s}, \Delta_{\alpha_G}\) and \(\Delta_t\) (we discuss the latter below). There are also \(\rho_3^{-1}\) corrections from thresholds and NRO’s analogous to (30) from the \((\alpha_1(M_Z)/\alpha_1(M_G))^{10/3}\) factor. However, these are negligible (\(\lesssim 1\%\) or \(\rho_3^{-1} \sim 1\)) when the experimental input value for \(\alpha_1(M_Z)\) is used. We take in (26) \(\rho_3^{-1} = 1\). The \(\alpha_1\) term in (6) does, however, lead to a small contribution to the \(\rho_t^{-1}\) term.

\(\Delta_{\alpha_s}, \Delta_{\alpha_G}, \Delta_{\alpha_G}^{NRO}\) and \(\Delta_{\alpha_G}^{24}\) are defined in Ref. [1] and are given in Appendix A for completeness. They involve the low- and high-scale mass parameters introduced above, as well as the NRO effective strength, \(\eta\). To leading order \(\eta\) is the only NRO free parameter and it incorporates the degrees of freedom associated with the strength, sign, scale, and normalization of the dimension-five operators \(-\frac{1}{2} \frac{n}{M_{\text{planck}}} \text{Tr}(F_{\mu\nu} \Phi F^{\mu\nu})\), where \(F_{\mu\nu}\) is the field strength tensor and \(\Phi\) is the adjoint scalar field. The range \(-10 \leq \eta \leq 10\) suggested in Ref. [1] is constrained only by perturbative consistency of the analysis.
Threshold corrections also affect the one-loop contribution from the Yukawa sector, i.e., $F \to F(1 + \Delta_F)$, and it is convenient to define

$$\rho_F^{-1} = 1 + \Delta_F.$$  \hspace{1cm} (31)

$F^{-1}$ is a correction term, but it can be as large as a $\sim 30\%$ correction (which, in fact, is responsible for the successful $m_h$ prediction in the MSSM), and, as we shall show, $\Delta_F \approx 2\% - 4\%$. $M_{24} \ll M_G$ will not contribute since the adjoint superfield couples (to one-loop) to the Yukawa operators via its coupling to the Higgs doublets, which drops out from the ratio. However, new and large Yukawa couplings will (radiatively) increase $h_a(\mu)$ and thus affect the infra-red fixed points and the perturbative limit; i.e., they affect $F_{\alpha}$ rather than the ratio $F$. (Such an effect may shift the $h_t$ and $h_b$ divergence lines in Figures 2–7 inwards towards each other.) New Yukawa operators (that do contribute to the ratio) are also generated if $M_5 < M_G$ (see, for example, Hisano et al. [9]). The exact magnitude of such effects will be determined by the details of the high-scale Lagrangian.

There are, however, low-scale corrections to $F^{-1}$. We naively change the Yukawa coupling RGE’s below the heavy Higgs doublet threshold ($t_H = \frac{1}{2\pi} \ln \frac{M_G}{M_Z}$) to those which are appropriate given the SM fermion spectrum with one SM-like Higgs doublet (for example, see Giveon et al. [22]). We will also neglect (near $M_Z$) $h_b \cos \beta$, $h_\tau \cos \beta \sim 0$. We obtain

$$\rho_F^{-1} = e^{\frac{1}{2}t_H + \frac{3}{4}t_H \sin^2 \beta}t_H,$$ \hspace{1cm} (32)

where here $y_t$ is taken at $M_Z$ (or more correctly, between $M_Z$ and $M_H$), and $t_H < 0.38$. $\rho_F^{-1}$ increases $m_b$ by slightly diminishing the effect of $F^{-1}$ in (30). Note that the $F$ behavior distinguishes the MSSM, where only $h_b$ gets corrected (to one-loop) by $h_t$, from the SM where all fermions couple to only one Higgs doublet, and both $h_b$ and $h_\tau$ get corrected. For $h_t < 1.1$ (as is reasonable at the low-scale) $\rho_F^{-1} \lesssim 1.04$, which is a naive overestimate. Including a $h_b(M_Z) - h_\tau(M_Z)$ (\(\approx 0.4\)) contribution can increase $\rho_F^{-1}$ by less than $\sim 2\%$ (the upper bound is for a large tan $\beta$). In most parts of the plane the correction is moderate, i.e., $\rho_F^{-1} \lesssim 1.02$ if either $\sin \beta \sim 0$ or $h_t \ll 1$. Let us stress that this is a somewhat naive description which gets complicated in many ways. For example, a light $t$-squark and a light chargino will still couple to the SM-like effective Yukawa operators. Such effects will have to be accounted for if and when the spectrum is better known and a refined analysis is required.

Lastly, $t$ (which is determined by $\alpha_1 - \alpha_2$ unification) can be corrected by either corrections to the coupling constant unification (see Eq. (A3)) or by a split between the coupling constant and Yukawa coupling unification points. In the latter case, from our definition of $M_G$, $\Delta_t < 0$ (and it is reasonable to take $\Delta_t \gg -1$). (Effects (e.g., NRO’s) that may split $h_b(M_G)$ and $h_\tau(M_G)$ can be also expressed in terms of the split between the unification points, but then $\Delta_t$ has no fixed sign.) Taking the approximation that $\Delta_t \ll t$ so that $\alpha_i(t) \approx \alpha_i(t + \Delta_t)$, $h_t(t) \approx h_t(t + \Delta_t)$ we find

$$\rho_t^{-1} = e^{(-\frac{1}{2}t_H + 2\alpha_0^0)\Delta_t},$$ \hspace{1cm} (33)

Their effect can be estimated by observation of the $SU(5)$ invariant operators, i.e., $F_b/F_\tau \to 1$ above $M_5$, (19) is slightly modified for $M_5 < M_G$, and the divergence lines move slightly outwards.
where \( y_t \) in (33) is taken at \( M_G \), i.e., \( y_t(M_G) = \frac{h_t^2(M_Z)}{2\pi} \), and \( h_b \approx h_t \) dropped out. For small values of \( h_t(M_G) \), a longer running time reduces \( h_b(M_G) \) (and thus, increases the predicted \( h_b(M_Z) \)) and vice versa. The situation reverses for \( h_t(M_G) \gtrsim \sqrt{16\pi\alpha_G} \sim \sqrt{2} \).

Collecting our results, we have

\[
\rho^{-1} = \rho^{-1}_a \times \rho^{-1}_b \times \rho^{-1}_c \equiv \rho^{-1}_z \times \rho^{-1}_f,
\]

where

\[
\rho^{-1}_z = \left( \frac{M_1}{M_Z} \right)^{25C_1 + 15C_8} \times \left( \frac{M_2}{M_Z} \right)^{-25C_1 + 25C_8} \times \left( \frac{M_3}{M_z} \right)^{C_1 + C_3 - C_4} \times \left( \frac{M_3}{M_Z} \right)^{C_1 + C_4} \times \left( \frac{M_H}{M_Z} \right)^{C_7}.
\]

and

\[
\rho^{-1}_f = \left( \frac{M_V}{M_G} \right)^{-25C_1 - 25C_8} \times \left( \frac{M_9}{M_G} \right)^{-25C_1 - 25C_8} \times \left( \frac{M_5}{M_Z} \right)^{C_1 + C_2 + C_6 + 12C_8} \times \left( 1 + 0.29C_1 + 0.24C_2 + C_8 \right) \eta,
\]

represent the low-scale and high-scale corrections, respectively. The coefficients \( C_i \) are defined and estimated in Table II. Note that \( C_1 + C_3 - C_4 = 0 \), i.e., \( M_3 \) drops out. This is because \( M_3 \) is associated with the change in \( \alpha_s(M_Z) \) due the threshold, which is a second order effect (see the discussion above). The \( M_3 \) dependence, on the other hand, is due to the change of the \( b_{\eta_3} \) coefficient and is of first order. We used \( M_3^2 = \overline{m}_3 \overline{M}_3 \) and added \( \Delta_{\alpha_G} \) and \( \Delta_{\eta_3} \) terms.

It is instructive to rewrite (using Table II)

\[
\rho^{-1}_z \approx \left( \frac{M_1}{M_Z} \right)^{0.02} \left( \frac{M_2}{M_Z} \right)^{-0.08} \left( \frac{M_3}{M_Z} \right)^{0.025} \left( \frac{M_H}{M_Z} \right)^{0.01}.
\]

(Different values of \( C_7 \) and \( C_8 \) were averaged.) If the spectrum were all degenerate at \( M_{SUSY} \), then \( \rho^{-1}_z \sim \left( \frac{M_{SUSY}}{M_Z} \right)^{-0.025} \gtrsim 0.94 \). We can invert the logic and use (37) to define an effective scale that gives \( \rho^{-1}_z \) correctly. For example, in Ref. [4] we defined an effective scale parameter, \( \Lambda_{SUSY} \),

\[
25 \ln \frac{M_1}{M_Z} - 100 \ln \frac{M_2}{M_Z} + 56 \ln \frac{M_3}{M_Z} = -19 \ln \frac{\Lambda_{SUSY}}{M_Z}.
\]

(\( \Lambda_{SUSY} \) here is \( M_{SUSY} \) of Ref. [4], and we have changed notation in order to avoid confusion with other definitions of \( M_{SUSY} \).) \( \Lambda_{SUSY} \) gives correctly the corrections to the \( \alpha_s(M_Z) \) (or \( s^2(M_Z) \)) prediction, but does not contain any information on the spectrum – it can be as low as a few GeV for sparticle \( \gg M_Z \). (See also Carena et al. [23].) Here, we can similarly define

\[
\rho^{-1}_z = \left( \frac{B_{SUSY}}{M_Z} \right)^{-25C_1 + 25C_8} \times \left( \frac{B_{SUSY}}{M_Z} \right)^{-0.025} \sim \left( \frac{B_{SUSY}}{M_Z} \right)^{-0.025}.
\]


The slightly negative exponent implies in many cases (for non-degenerate spectra) $B_{SUSY} \lesssim M_Z$ (i.e., $M_2 < M_1, M_3, M_H$ usually implies $\rho^{-1}_z \gtrsim 1$). $B_{SUSY} \geq M_Z$ for a strongly degenerate spectrum. For the spectra of Ref. [3] we find $\rho^{-1}_z \sim 1$ ($A_{SUSY} \approx 32, 21$ GeV, $B_{SUSY} \approx M_Z$). Taking the limits of heavy gluinos and of a degenerate spectrum we find $0.94 \lesssim \rho^{-1}_z \lesssim 1.06$. Away from the limits $\rho^{-1}_z \to 1$.

Similarly to (37) we can rewrite

$$\rho^{-1}_G \approx \left( \frac{M_V}{M_G} \right)^{-0.030} \left( \frac{M_{24}}{M_G} \right)^{-0.004} \left( \frac{M_5}{M_G} \right)^{0.015} (1 + 0.007\eta).$$

A scenario in which $M_{24} \ll M_G; M_5 \sim (0.1 - 0.5)M_G; M_V = M_G$; and $\eta \approx -10$; would give $\rho^{-1}_G \approx 0.9$. This scenario is also consistent with limits from proton decay [8,9]. Furthermore, NRO’s contribute only negligibly to $\Delta\alpha_S$ and to $\Delta t$ (unless one allows NRO effects to be very large [7,12]). $M_{24} \ll M_G$ on the other hand can increase $t$ significantly, i.e., $M_G \lesssim 5 \times 10^{17}$ GeV (which is the reason that we can have $M_5 < M_G$). A large negative $\eta$ maintains an acceptable value of $\alpha_s(M_Z)$ in such a scenario. Lifting proton decay constraints (e.g., see Ref. [11]), we can have $M_5 \ll M_G$ and $\rho^{-1}_G \approx 0.8 - 0.9$. Taking these limits and that of a degenerate spectrum and a large positive $\eta$ we obtain $0.8 \lesssim \rho^{-1}_G \lesssim 1.1$.

The high-scale corrections to the coupling constant unification emerge as the leading contribution to $\rho^{-1} \neq 1$. We would like to stress that $\eta$ is not just a new ad-hoc parameter. Given the precision to which we know the low-scale observables, one cannot ignore the likely possibility of unknown physics at the high-scale where the (supergravity-induced) MSSM breaks down, and which is parameterized in terms of NRO’s (whose form is defined in SU(5) models). Furthermore, similar corrections may arise in supergravity from non-minimal (and non-universal) gauge kinetic functions (see, for example, Ref. [14]). Unfortunately, this, in turn, introduces some ambiguity in RG calculations (via high-scale boundary conditions). It should also be noted that adding large representations [13–16,20], e.g., 126 of SO(10), does not introduce (for nearly degenerate heavy components) large threshold corrections to $\alpha_s(M_Z)$ and $t$. This is because the decoupled heavy components constitute a nearly complete representation (which acts equally on all the $b_i$’s). Thus, the threshold corrections in the minimal model give a good estimate of $\rho^{-1}_G$ (in models with a GUT sector, which are the relevant ones for Yukawa unification). A model independent treatment of high-scale threshold effects on coupling constant unification was given in Ref. [14]. The heavy Yukawa sectors of different models may affect the infra-red fixed-points differently.

An arbitrary splitting of the two unification points induces a $\sim 5\%$ uncertainty. By combining all the contributions in quadrature (as a guideline only) we obtain

$$0.80 \lesssim \rho^{-1} \lesssim 1.15.$$  

$\rho^{-1} = 1 \pm 0.1$ is thus a reasonable range, and $\rho^{-1} = 1 \pm 0.15$ (that we adopted) is somewhat more extreme, but well within the allowed range. We would like to stress that all ranges extracted here are a guideline only. This range, which is controlled by high-scale corrections, is still valid when the sparticle spectrum is explicitly calculated (and, e.g., decoupled numerically).

As we pointed out above, corrections that either change the prediction for $\alpha_s(M_Z)$ or the positive contribution to (16) from Yukawa terms, affect the infra-red fixed points and can
sightly shift the corresponding divergence lines in Figures 2 – 7. They may also affect the upper bound on $m_t^{pole}$, and thus they induce a $\sim -10$ GeV uncertainty to the upper bound value. However, the corrected range remains $\gtrsim 200$ GeV (see also Barger et al. [18]). In particular, it is much higher than the the upper bound suggested by precision data (see Eq. (1)).

We would also like to point out that if $\alpha_1(M_Z)$, $\alpha_2(M_Z)$ and $\alpha_3(M_Z)$ are all used as inputs, then one arbitrarily adjusts $\Delta_\alpha$ so that $\alpha_0^0 + (\alpha_0^0)^2 \Delta_\alpha$ is fixed at some desired value. The coupling constants do not unify unless one consistently corrects $\alpha_G$ and $t$ as well. However, such a procedure is a reasonable approximation if $\Delta_\text{SUSY}$ is small (or is known and corrected for). In that case one can minimize the residual uncertainty by calculating $\alpha_3(M_G)$ from the input value of $\alpha_s(M_Z)$ and from $M_G$ ($M_G$ is calculated from $\alpha_1 - \alpha_2$ unification). Then only $\rho_t^{-1}$, $\rho_f^{-1}$ and $b_{b,3}$ terms contribute to $\rho^{-1}$ (i.e., one can obtain their contribution by setting $C_1 = C_2 = 0$ in (35) and (36)), and the residual uncertainty is small. Some caution is, however, needed. The coupling constant unification constraints are not integral in such a procedure (e.g., compare Figures 2 and 5). In particular, the correlation between $\alpha_s(M_Z)$ and $m_t$ is not manifest. A large $m_t$ value implies larger values of $\alpha_s(M_Z)$ or, alternatively, very large corrections to coupling constant unification. Also, only $\Delta_1$, $\Delta_2$, $\Delta_3^{NRO}$ and $\Delta_3^{\text{heavy}}$ can induce first order corrections to $m_b$, and thus can be used to fix $\alpha_s(M_Z)$. (NRO’s renormalize and split $\alpha_i(M_G)$, and thus honestly modify the boundary conditions. The simplest way to adjust the $\alpha_s(M_Z)$ prediction to a given input value is by adjusting $\eta$ – i.e., $\eta \sim -10$ corrects the $\alpha_s(M_Z)$ prediction to $\alpha_1(M_Z) \sim 0.11$.) As was illustrated by our toy model, $\Delta_3^{\text{SUSY}}$ contributes to $\Delta_{\alpha}$, but does not affect $m_t$ to first order in small terms. Thus, unless one knows and corrects for the $\Delta_3^{\text{SUSY}}$ contribution to the input $\alpha_s(M_Z)$ one introduces a significant theoretical uncertainty. Lastly, the experimental uncertainty in $\alpha_s(M_Z)$ is large, and arbitrarily varying $\alpha_s(M_Z)$ in that range is not very instructive. Nevertheless, it is useful in demonstrating the role of $\alpha_s(M_Z)$ in predicting $m_b$, as we saw in section III.

Finally, the three-Yukawa unification strip (see Figure 2) has uncertainties in both the $\tan \beta$ and the $m_t^{pole}$ ranges, coming from corrections to the $h_b/h_{\tau}$ and $h_t/h_b$ ratios, respectively. To one-loop $(h_t/h_b) \sim (\alpha_L^{\alpha_L}/\alpha_G^{\alpha_G})^{-\frac{1}{2}} \times (F_i/F_i)$ and any uncertainties in the $\alpha_1$ term are negligible. However, variation of $-0.5 < \Delta_t < 0.5$ generates $\sim \pm 2\%$ ($\sim \pm 8\%$) correction if $h_t \sim h_b \sim h_{\tau} \sim 1$ ($\sim 2$), i.e., $(\rho_b/\rho_{top})_{t} \sim e^{\frac{3}{2} \gamma_s(M_G) \Delta_t}$. Additional uncertainty of $0.95 < (\rho_b/\rho_{top})_{F} \sim e^{(\gamma_t(M_Z) - \gamma_t(M_Z)_{\text{pole}})} \sim 1$ (we assume $\tan \beta \gg 1$) is associated with the decoupling of the heavy Higgs doublet. We estimate a $\sim \pm 5 - 10\%$ uncertainty in the $m_t^{pole}$ range that corresponds to three-Yukawa unification.

V. CONCLUSIONS

Grand unified theories typically predict $h_b = h_{\tau}$ at $M_G$, and contain non-fundamental Higgs representations. These distinguish such models from some other realizations of the MSSM, e.g., string-inspired GUT-like models. Above, we explicitly embedded the MSSM in a minimal $SU(5)$ model, and concluded that such a model is constrained to a small area of the parameter space. We showed that corrections to a two-loop calculation of the bottom mass (when assuming grand unification) are manifested in various ways. Parametrizing
those corrections, we were able to relate them to the correction parameters identified in Ref. 4, and to study their magnitude and behavior in some detail. The theoretical uncertainty in the bottom mass prediction is typically $\lesssim 15\%$. We thus took (given the ambiguities in the extraction of $m_b$ from experiment) $0.85m_b^5(5\text{ GeV}) < 4.45\text{ GeV}$ as a (conservative) constraint. Requiring this, as well as requiring perturbative Yukawa couplings up to $M_G$ and identifying the coupling constant and the (third-family) Yukawa coupling unification points, we found that the range $2.7 \lesssim \tan \beta \lesssim 40$ is excluded (as well as $m_t^{pole} \gtrsim 215\text{ GeV}$), and that, in agreement with other authors, the allowed area in the $m_t^{pole} - \tan \beta$ plane is described by low- and high-$\tan \beta$ branches (where the former saturates the $h_t$ fixed-point line). The separation between the two branches is determined by the correction factor. Requiring all three (third-family) Yukawa couplings to meet constraints $160\text{ GeV} \lesssim m_t^{pole}$ and requires a large $\tan \beta$. We demonstrated that the allowed parameter space grows for lower (input) values of $\alpha_s(M_Z)$, but that the MSSM prefers higher values. We further argued that the $s^2(M_Z)$ quadratic dependence on $m_t^{pole}$ cannot be ignored as it correlates the $\alpha_s(M_Z)$ prediction with $m_t^{pole}$, and thus affects the $m_t^{pole}$ dependence of the $m_b$ prediction, as well as the the upper bound on $m_t^{pole}$ and the range of $m_t^{pole}$ for which intermediate values of $\tan \beta$ are allowed. Finally, we expect the above observations and radiative breaking of $SU(2) \otimes U(1)$ to have mutual implications, and suggest that the above constraint is still valid in a calculation in which the sparticle spectrum, and therefore $\rho^{-1}_Z$, is calculated explicitly. (The larger uncertainty in the calculation comes from the unification-scale physics rather than from the details of the sparticle spectrum.) Our hope is that a careful study of various correction terms will eventually result in reliable constraints on the MSSM parameter space, and in a way that can distinguish different realizations of the MSSM. Here we have showed (in agreement with others) that by measuring $\tan \beta$ one can exclude simple (and some extended) GUT structures at the high-scale.

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APPENDIX A: THE CORRECTION FUNCTIONS

For completeness, we give the correction functions to the coupling constant unification (in the minimal $SU(5)$ MSSM). For more details, see Ref. 4. Corrections that depend on $m_t^{pole}$ or the conversion to the $\overline{DR}$ scheme are included in the numerical procedure, and are not quoted below. All the parameters are defined above (see section V).

\[
28\pi \Delta_{\alpha_s} = -12 \ln \frac{M_V}{M_G} - 6 \ln \frac{M_{24}}{M_G} + 18 \ln \frac{M_5}{M_G} + 25 \ln \frac{M_1}{M_Z} - 100 \ln \frac{M_2}{M_Z} + 56 \ln \frac{M_3}{M_Z} + 8.00\eta.
\]

\[
-336\pi \Delta_{\alpha_G} = +888 \ln \frac{M_V}{M_G} - 396 \ln \frac{M_{24}}{M_G} + 12 \ln \frac{M_5}{M_G}
\]

\[
(A1)
\]
\[ +75 \ln \frac{M_1}{M_Z} - 825 \ln \frac{M_2}{M_Z} + 50.0 \eta. \] (A2)

\[ \frac{336 \pi}{5} \Delta_t = -24 \ln \frac{M_V}{M_G} - 12 \ln \frac{M_{24}}{M_G} + \frac{12}{5} \ln \frac{M_3}{M_G} \]
\[ +15 \ln \frac{M_1}{M_Z} - 25 \ln \frac{M_2}{M_Z} + 1.0 \eta. \] (A3)

\[ \Delta_3^{24} = \frac{3}{2\pi} \ln \frac{M_{24}}{M_G}. \] (A4)

\[ \Delta_3^{NRO} \approx 0.03 \eta. \] (A5)
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FIG. 1. The predicted strong coupling at the $Z$-pole, $\alpha_s(M_Z)$, for different values of the $t$-quark pole mass, $m_t^{pole}$, and of the two Higgs doublet expectation value ratio, $\tan\beta$. $h_b(M_G) = h_\tau(M_G)$ is assumed. $m_t^{pole}$ (in GeV) is indicated on the right-hand-side above the relevant line.

FIG. 2. The $m_t^{pole} - \tan\beta$ plane is divided into five different regions. Two areas (low- and high-$\tan\beta$ branches) are consistent with perturbative two-Yukawa unification ($h_b(M_G) = h_\tau(M_G)$) and with $0.85 m_b^0(5 \text{ GeV}) < 4.45 \text{ GeV}$. Between the two branches the $b$-quark mass is too high. For a too low (high) $\tan\beta$, $h_t/h_b$ diverges. The strip where all three (third-family) Yukawa couplings unify intersects the allowed high-$\tan\beta$ branch and is indicated as well (dash-dot line). Corrections to the $h_t/h_b$ ratio induce a $\sim \pm 5\%$ (vertical) uncertainty in the $m_t^{pole}$ range that corresponds to each of the points in the three-Yukawa unification strip. $\alpha_s(M_Z)$, $\alpha_G$, and the unification scale used in the calculation are the ones predicted by the MSSM coupling constant unification, and are sensitive to the $t$-quark pole mass, $m_t^{pole}$ (see Figure 1). The $m_t^{pole}$ range suggested by the electroweak data is indicated (dashed lines) for comparison. $m_t^{pole}$ is in GeV.

FIG. 3. The low-$\tan\beta$ branch of Figure 2 is shown in greater detail. The lines corresponding to $\rho^{-1} = 1$ (thick) and $h_t(M_G) = 2$ (dashed) are indicated. To the left of the allowed branch one obtains $h_t(M_G) > 3$.

FIG. 4. The same as Figure 2, except the constraint is replaced with the more restrictive one, $0.85 m_b^0(4.45 \text{ GeV}) < 4.45 \text{ GeV}$. The allowed $\tan\beta$ range is reduced by $\sim 0.03 - 0.10$ for the low-$\tan\beta$ branch (the effect is hardly seen in the figure) and by $\sim 3 - 4$ for the high-$\tan\beta$ branch (where the corresponding range for $0.85 m_b^0(5 \text{ GeV}) < 4.45 \text{ GeV}$ is indicated – dashed line – for comparison).

FIG. 5. The area in the $m_t^{pole} - \tan\beta$ plane which is consistent with perturbative two-Yukawa unification and with $0.85 m_b^0(5 \text{ GeV}) < 4.45 \text{ GeV}$ assuming $\alpha_s(M_Z) = 0.11$. The unification scale and $\alpha_G$ used in the calculation are those predicted by $\alpha_1 - \alpha_2$ unification. We chose $\rho^{-1} = 0.85$ for comparison with Figures 2 – 3. $m_t^{pole}$ is in GeV.

FIG. 6. The same as Figure 5, except $\alpha_s(M_Z) = 0.12$.

FIG. 7. The same as Figure 5, except $\alpha_s(M_Z) = 0.13$. 
TABLE I. The coefficients $C_i$ are defined and estimated using $s^2(M_Z) = 0.2324$, $\alpha_0^G = 0.125$, and $\alpha_0^0 = 0.040$.

| definition                           | estimate | comments |
|--------------------------------------|----------|----------|
| $C_1$                                | $\frac{8}{\pi} \alpha_0^G(M_Z)$ | +0.035  |
| $C_2$                                | $-\frac{8}{\pi} \alpha_0^G$     | -0.011  |
| $C_3$                                | $-\frac{10}{\pi} \alpha_0^G(M_Z)$ | -0.044 |
| $C_4$                                | $-\frac{2}{\pi} \alpha_0^0(M_G)$ | -0.009 |
| $C_5$                                | $-\frac{4}{\pi} \alpha_0^G$     | -0.017  | $M_{24} < M_5$ |
| $C_6$                                | $-\frac{4}{\pi} \alpha_0^G$     | -0.006  | $M_{24} < M_5$ |
| $C_7$                                | $\frac{2+3\sin\beta y_t(M_Z)}{\pi}$ | +0.010 | $h_t \sim 0.8, \beta \sim \frac{\pi}{2}$ |
|                                       |          | +0.008  | $h_t \sim 1.1, \beta \sim 0$ |
|                                       |          | +0.019  | $h_t \sim 1.1, \beta \sim \frac{\pi}{2}$ |
| $C_8$                                | $\frac{5}{6\pi} \frac{-y_t(M_G)+4\alpha_0^0}{\pi}$ | +0.0002 | $h_t \sim 1$ |
|                                       |          | -0.0013 | $h_t \sim 3$ |