Communication over a Channel that Wears Out

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Abstract—This work investigates the limits of communication over a noisy channel that wears out, in the sense of signal-dependent catastrophic failure. In particular, we consider a channel that starts as a memoryless binary-input channel and when the number of transmitted ones causes a sufficient amount of damage, the channel ceases to convey signals. We restrict attention to constant composition codes. Since infinite blocklength codes will always wear out the channel for any finite threshold of failure and therefore convey no information, we make use of finite blocklength codes to determine the maximum expected transmission volume at a given level of average error probability. We show that this maximization problem has a recursive form and can be solved by dynamic programming. A discussion of damage state feedback in channels that wear out is also provided. Numerical results show that a sequence of block codes is preferred to a single block code for streaming sources.

I. INTRODUCTION

In reliability theory, there are two basic modes of catastrophic failure: independent damage and cumulative damage [1]. With an independent damage process, a shock is either large enough to cause failure or it has no effect on the state of the system. With a cumulative damage process, however, each shock degrades the state of the system in an additive manner such that once the cumulative effect of all shocks exceeds a threshold, the system fails. Translating these notions to communication channels, failure can either be signal-independent or signal-dependent. A typical channel with signal-dependent failure is in visible light communication under on-off signaling where light sources may burn out with ‘on’ signals [2].

Here we consider optimizing communication over noisy channels that may wear out, i.e. suffer from signal-dependent failure. As depicted in Tab. I this is a novel setting that is distinct from channels that die [3] since failure time is dependent on the realized signaling scheme, and from meteor-burst channels [4], [5] and channels that heat up [6], [7] since the channel noise level does not change with time. The model is also distinct from Gallager’s “panic button” [8, p. 103] or “child’s toy” [9, p. 26] channel, since there is not a special input symbol that causes channel failure.

For example, consider a channel with finite input alphabet $\mathcal{X} = \{0, 1\}$ and finite output alphabet $\mathcal{Y} = \{0, 1, \gamma\}$. It has an alive state $\sigma = a$ when it acts like a binary symmetric channel (BSC) with crossover probability $0 < \varepsilon < 1$, i.e. the transmission matrix is

$$p(y|x, \sigma = a) = p_a(y|x) = \begin{bmatrix} 1 - \varepsilon & \varepsilon & 0 \\ \varepsilon & 1 - \varepsilon & 0 \end{bmatrix}, \quad (1)$$

and a dead state $\sigma = d$ when it erases the input, i.e. the transmission matrix is

$$p(y|x, \sigma = d) = p_d(y|x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2)$$

The channel starts in state $\sigma = a$ and then transitions to $\sigma = d$ at some random time $T$, where it remains for all time thereafter. That is, the channel is in state $a$ for times $i = 1, 2, \ldots, T$ and in state $d$ for times $i = T + 1, T + 2, \ldots$.

Since it is inevitable for the channel to fail at a finite time for any non-trivial signaling scheme, the Shannon capacity of the channel is zero. Rather than invoking infinite blocklength asymptotics, we must construct schemes that convey information via finite blocklength code(s) before the channel wears out. Thus results in the finite blocklength regime [10]–[12] and their refinements [13], [14] can be built upon to determine limits on expected transmission volume at a given average error probability.

Similar to channels that die [3], we find that a sequence of finite blocklength codes may perform better than a single code. Finite blocklength analysis, however, cannot be directly applied since there is a restriction on transmitting too many 1 symbols so that the channel stays alive. A principle of finite blocklength code design is therefore maximizing transmission volume while having a minimal number of 1s. To facilitate this, we restrict attention to constant composition codes [15]: the probability of successively transmitting a sequence of finite-length constant composition codes is studied and the approximation of the fundamental communication limit of using constant composition codes in [16] is applied.

To maximize the expected transmission volume, all possible sequences of finite-length constant composition codes with different types have to be tested exhaustively. Here we propose

| Classes of Channel Models |
|----------------------------|
| signal-to-noise | signal-independent | signal-dependent |
| Failure time     | meteor-burst channels | channels that heat up |
|                 | channels that die     | channels that wear out |

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a recursive formulation to maximize the expected transmission volume in an efficient manner. The corresponding dynamic program and its graphical representation are provided. In considering the possibility of damage count feedback being available at the transmitter, we find that this does not change the probability of successively transmitting a sequence of finite-length constant composition codes over the channel. Some numerical results are also provided to give insights into code design.

The rest of this paper is organized as follows. Section II defines the wearing out process. The maximum expected transmission volume of using constant composition codes is discussed in Section III. The dynamic programming formulation and the discussion of feedback are given as well. Section IV provides some numerical results, and Section V concludes this paper by suggesting some possible future directions.

II. CHANNEL FAILURE MODEL

Consider a channel with binary input alphabet \( \mathcal{X} = \{0, 1\} \) and finite output alphabet \( \mathcal{Y} = \{0, 1, \ldots, |\mathcal{Y}| - 2, ?\} \), and alive/dead states as indicated above. There is a probability \( \gamma \) of the channel getting damaged when a 1 is transmitted through the channel. The channel starts at state \( \sigma = a \) and transitions to state \( \sigma = d \) when the extent of damage exceeds a certain threshold \( S \), where \( S \) could be deterministic or random. For simplicity, deterministic \( S \) is assumed throughout this paper. The damage while transmitting a 1 can be simulated by the independent Bernoulli random variable \( D_k \), which takes the value 1 with probability \( \gamma \) and 0 with probability \( 1 - \gamma \). Thus, the channel that wears out can be defined as a sextuple \((X, p_a, p_d, \gamma, S, Y)\).

The communication system over the channel that wears out \((X, p_a, p_d, \gamma, S, Y)\) is defined as follows.

- An information stream is designed to be transmitted \((W(1), W(2), \ldots, W(m))\). Each \( W(i) \) is chosen from the set \( Y^{m(i)} \). The sequence of codewords \( (c(i))_i \) with \( m(i) \) symbols in total, is then transmitted through the channel that wears out. Let such a sequence of \( m \) codebooks be an \((M(i), n(i), \gamma, S, Y)\) - code.
- The received sequence \( r \in Y^{\sum_{i=1}^{n(i)} m(i)} \) is decoded into \((\hat{W}(1), \ldots, \hat{W}(m))\) by the decoders \( g(i) \), where \( g(i)(r) = \hat{W}(i) \) for \( r = r^{(i)} \) and \( 1 \leq i \leq m \). If all \( n(i) \) channel outputs for decoder \( g(i) \) are not ?, then \( \hat{W}(i) \in W(i) \); otherwise, \( \hat{W}(i) = e \).
- The decoding error probability for a codebook \((M(i), n(i))\) is defined as
  \[
P_e = \max_{i \in \{1, \ldots, m\}} P_e(i).
\]

III. THE MAXIMAL LOG-VOLUME OF USING CONSTANT COMPOSITION CODES

In this section, we restrict attention to finite blocklength constant composition codes, denoted as \( C_{ccc} \), in which all codewords from the same codebook have the same number of
1s. Given a $C_{ccc}$ with length $n$, the Hamming weight of each codeword can be denoted as $wt(P) \triangleq nP(1)$, where $P$ is a type from $\mathcal{P}_n(\mathcal{X})$, the set of all types of length $n$. Define an $(n, M, \eta)_P$-code to be an $\eta$-achievable constant composition code with type $P$, blocklength $n$, number of messages $M$, and average error probability no larger than $\eta$. Hence, when the constant composition code corresponding to $P$ is transmitted, the damage count for such a code is

$$U(P) \triangleq \sum_{k=1}^{\text{wt}(P)} D_k.$$  \hfill (11)

Suppose an $(n(i), M(i), \eta)_{P(i)}$-code, denoted as $(C(i)_{ccc})_{i=1}^m$, is conveyed through the channel. The individual codes need not be the same, and so the full concatenation is much like a constant subblock composition code [17]. The probability of the channel $(\mathcal{X}, p_n, p_d, \gamma, S, \mathcal{Y})$ staying alive after conveying the first $j$ codes $(C(i)_{ccc})_{i=1}^m$ in (9) can be further derived as

$$\Pr \left[ \left( C(i)_{ccc} \right)_{i=1}^j \text{ alive} \right] = B \left( S \sum_{i=1}^j \text{wt} \left( P(i) \right), \gamma \right). \tag{12}$$

Similar to the result in [3], the expected log-volume for transmitting $(C(i)_{ccc})_{i=1}^m$ with a maximum average error probability $\eta$ can be derived as

$$\sum_{j=1}^m \Pr \left[ \left( C(i)_{ccc} \right)_{i=1}^j \text{ alive} \right] \log M_{P(i)} \left( n(i), \eta \right) =$$

$$\sum_{j=1}^m B \left( S \sum_{i=1}^j \text{wt} \left( P(i) \right), \gamma \right) \log M_{P(i)}^* \left( n(i), \eta \right) \tag{13}$$

where

$$M_{P}^*(n, \eta) = \max \{ M \mid \exists \text{ an } (n, M, \eta)_P \text{-code for the alive channel } p_n \} \tag{14}$$

is the maximum transmission volume of the constant composition code over the binary-input DMC when channel is alive.

To maximize the expected log-volume (13) given a total length $N = \sum_{i=1}^m n(i)$, the $(C(i)_{ccc})_{i=1}^m$ for $1 \leq m \leq N$ to maximize (13) needs to be found. Let $Z_{ccc}(N, H, \eta)$ be the set of all possible $(n(i), M(i), \eta)_{P(i)}$-codes for all $m \in \{1, \ldots, N\}$, which has total length $N$ and total Hamming weight $H$, i.e.,

$$Z_{ccc}(N, H, \eta) \triangleq \left\{ \left( C(i)_{ccc} \right)_{i=1}^m \right\}_{n(i)}$$

$C(i)$ is an $(n(i), M(i), \eta)_{P(i)}$-code, $0 < m < N$, $\sum_{i=1}^m n(i) = N$ and $\sum_{i=1}^m \text{wt} \left( P(i) \right) = H$. \hfill (15)

Then the maximum expected log-volume with the given $N$ and $H$ of transmitting the constant composition codes is denoted as follows:

$$V_{ccc}^*(N, H, \eta) =$$

$$\max_{(C(i)_{ccc})_{i=1}^m \in Z_{ccc}(N, H, \eta)} \left\{ \sum_{j=1}^m B \left( S \sum_{i=1}^j \text{wt} \left( P(i) \right), \gamma \right) \log M_{P(i)}^* \left( n(i), \eta \right) \right\}. \tag{16}$$

and the maximum expected log-volume with $N$ is

$$V_{ccc}^*(N, \eta) = \max_{0 < H < N} V_{ccc}^*(N, H, \eta). \tag{17}$$

It should be noted that $m$ does not need to be specified explicitly in the maximization in (16), since $Z_{ccc}(N, H, \eta)$ in (15) consists of all collections of codes $(C(i)_{ccc})_{i=1}^m$ for each $1 \leq m \leq N - 1$.

### A. Dynamic Programming Formulation

To solve the maximization problem in equation (16), dynamic programming formalism is adopted. A recursive form of (16) can be formulated as:

$$V_{ccc}^*(N, H, \eta) = \max_{1 \leq n \leq N} \left\{ V_{ccc}^*(N - n, H - h, \eta) + B(S, H, \gamma) \log M_{P}^*(n, \eta) \right\}. \tag{18}$$

where $P(1) = h/n$.

A graphical representation of the recursive form (18) is illustrated in Fig. 1. In this trellis diagram, the metric of the branch from node $(x_1, y_1)$ to node $(x_2, y_2)$ is

$$B(S, y_2, \gamma) \log M_{P}^*(x_2 - x_1, \eta). \tag{19}$$

where $P(1) = \frac{y_2 - y_1}{x_2 - x_1}$.

The path with the maximum accumulated branch metric from node $(0, 0)$ to node $(N, H)$ is the solution for $V_{ccc}^*(N, H, \eta)$. Thus the optimization problem in (16) can be reduced to finding the longest path in Fig. 1.
B. Damage Count Feedback Does Not Improve Performance

The expected log-volume in (13) contains two parts: the probability of successfully conveying $C^{(i)}$ and the volume of $C^{(i)}$. How the volume of $C^{(i)}$ over DMCs is affected by full feedback has been studied in [18], [19, Ch. 20], which is beyond the scope of this paper. Instead, the following structural question regarding damage count feedback (rather than full feedback) is raised: Will the probability of successfully conveying $C^{(i)}$ be different due to feedback of the damage count?

Suppose $C^{(1)}$ has been sent through the channel and instant feedback tells the transmitter the damage count. The probability of causing $d$ damages from $\text{wt}(P^{(1)})$ transmitted 1s is

$$\Pr \left[ U(P^{(1)}) = d \right].$$

(20)

Since the transmitter knows that $d$ damage events have been incurred, the channel is still capable of handling $S - d$ damages. The probability of successfully transmitting $C^{(2)}$ given $d$ damages without wearing out the channel is

$$\Pr \left[ U(P^{(2)}) \leq S - d \right].$$

(21)

Combing (20) and (21), the overall probability of successfully transmitting $C^{(2)}$ is

$$\sum_{d=0}^{S} \Pr \left[ U(P^{(1)}) = d \right] \times \Pr \left[ U(P^{(2)}) \leq S - d \right]$$

(22)

$$= \Pr \left[ U(P^{(1)}) + U(P^{(2)}) \leq S \right]$$

(23)

$$= B \left( S, \text{wt}(P^{(1)}) + \text{wt}(P^{(2))}, \gamma \right).$$

(24)

The above result can be extended to have the following probability of successfully conveying $C^{(i)}$

$$B \left( S, \sum_{j=1}^{i} \text{wt}(P^{(j)}), \gamma \right),$$

(25)

which coincides with (13). Hence the probability of successively transmitting $C^{(i)}$ remains the same with or without damage state feedback.

This implies that the achievable transmission volume cannot be increased by providing damage state feedback.

IV. NUMERICAL RESULTS

This section presents some numerical results. Here, the BSC channel with crossover probability $\varepsilon$, denoted by BSC($\varepsilon$), is taken when the channel is alive, i.e., the channel $(X, p_x, p_0, p_1, S, Y)$ with $Y = \{0, 1, \ldots\}$, $p_a(1|0) = p_a(0|1) = \varepsilon$, and $p_a(0|0) = p_a(1|1) = 1 - \varepsilon$. To evaluate $M_p(n, \eta)$ for BSC($\varepsilon$), the approximation of (16) Eq. (21)) (ignoring the $o(1)$ term) is used, i.e.,

$$\log M_p(n, \eta) \approx n I(P; p_a) + \sqrt{n \rho(P; p_a)Q^{-1}(\eta)} + \frac{1}{2} \log n + A_\eta(P; p_a) + \Delta_{ccc}(P; p_a),$$

(26)

where $P$ is the type of input, $I(P; p_a)$ is the mutual information, $\rho(P; p_a)$ is the channel dispersion, $Q^{-1}(\eta)$ is the inverse Q-function, and $A_\eta(P; p_a) + \Delta_{ccc}(P; p_a)$ is the constant part of the approximation in (16). The channel is damaged with a probability $\gamma = 0.1$ when a 1 is transmitted, and worn out after the amount of damage exceeds $S = 10$. The maximum expected rates ($V^*_{ccc}(N, \eta)/N$) up to $N = 1000$ of using constant composition codes for BSC(0.01) and BSC(0.1) are depicted in Fig. 2 in which the average transmission error was assumed to be lower than $\eta = 0.001$.

Notice that the curves corresponding to BSC(0.01) and BSC(0.1) in Fig. 2 reach the highest rates 0.74 and 0.29 at $N = 114$ and $N = 153$, respectively. The expected rate keeps decreasing after passing through the highest point since the channel is less likely to survive for transmitting more bits. Both curves also indicate that more codes are favorable to be transmitted when the channel is worse.

For both curves in Fig. 2 the number of transmitted codes and the accumulated Hamming weights (H) are given in Fig. 3(a) and Fig. 3(b), respectively. Fig. 3(a) shows that, as $N$ is increasing, it is better to send a sequence of codes instead of a single code. It can be observed that the accumulated weight in Fig. 3(b) has a sudden rise while the more number of codes is favorable. This means that, by increasing the number of codes, more 1s can be conveyed through the channel at some certain $N$.

To further investigate how the length of each transmitted block changes as $N$ changes, the lengths of transmitted blocks of curves BSC(0.01) and BSC(0.1) are depicted in Fig. 4(a) and Fig. 4(b), respectively. In both subfigures, the length of each transmitted block is plotted in a different color. For example, there are five codes transmitted when $N = 700$ in Fig. 4(a), and the corresponding lengths ($n^{(1)}, n^{(2)}, \ldots, n^{(5)}$) are (412, 127, 74, 50, 37).

V. CONCLUSION AND FUTURE WORK

Further increasing the connections between reliability theory and information theory, we have proposed a channel that wears out and found the maximum expected transmission
the weight of each codeword, the fundamental limits of input-constrained transmission can help us. The extension of current results in input-constrained transmission [20], [21] to have constraint on the exact weight spectrum can be used to evaluate the expected transmission volume without the restriction of using constant composition codes.

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