On Flattenability of Graphs

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Flattenability

A graph $G$ is $d$-flattenable if for every linkage (bar-lengths in a given norm) that has a realization in some dimension also has a realization in $R^d$.\(^1\)

Clearly $d$-flattenable is a minor closed property. So it has a forbidden minor characterization.

Under the $l_2$ norm, the only forbidden minor for 2-flattenability is $K_4$.

Warm Up Question

What are the forbidden minors for 2-flattenability under $l_1$?

\(^1\)M. Belk and R. Connelly, “Realizability of graphs,” *Discrete Comput. Geom.*, vol. 37, no. 2, pp. 125–137, Feb. 2007.
$l_1$ 2-flattenability

$K_4$ is 2-flattenable under $l_1^2$

For 5 vertex graphs, we have shown the following:

- $K_5$ is not 2-flattenable (Known)
- Banana is not 2-flattenable
- 4-Wheel is unknown (OPEN)
- All others are 2-flattenable

If 4-wheel is not 2-flattenable, it is the only forbidden minor

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$^2$H. Witsenhausen, “Minimum dimension embedding of finite metric spaces,” *Journal of Combinatorial Theory, Series A*, vol. 42, no. 2, pp. 184–199, 1986.
This was shown in Sitharam-Gao\textsuperscript{3} for the $l_2$ norm:

**Theorem**

*For any $l_p$ norm, a graph $G$ is $d$-flattenable iff the set of attainable edge-length vectors for $G$ in $d$-dimensions is convex* \textsuperscript{4}.

Useful in many science or engineering applications.

Proof makes extensive use of the cone of pairwise distance vectors.

\textsuperscript{3} M. Sitharam and H. Gao, “Characterizing graphs with convex and connected cayley configuration spaces,” *Discrete & Computational Geometry*, vol. 43, no. 3, pp. 594–625, 2010.

\textsuperscript{4} Also called $d$-dimensional Cayley configuration space on $G$. 
Some Definitions

The cone \(5\) of all pairwise \(l^p\)-distance vectors on \(n\)-point configurations: \(\Phi_{n,l^p}\)

The \(d\)-dimensional stratum: pairwise distance vectors of \(d\)-dimensional point configurations: \(\Phi_{n,l^p}^d\)

The projection or shadow of this cone on an edge set \(G\): \(\Phi_{G,l^p}\)

Theorem (Restatement)

\(G\) is \(d\)-flattenable iff \(\Phi_{G,l^p}^d\) is convex

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\(^5\)K. Ball, “Isometric embedding in \(l^p\)-spaces,” European Journal of Combinatorics, vol. 11, no. 4, pp. 305–311, 1990
Connections to Rigidity

Using the structure of $\Phi_{n,l_p}$, we get a connection among:

- Flattenability
- Dimension of certain projections/strata of $\Phi_{n,l_p}$
- Rigidity and Independence

For norms other than $l_2$, we can use the formulation of the rigidity matroid of Kitson\textsuperscript{6}.

**Theorem**

*For general $l_p$ norms, there exists a generic $d$-flattenable framework of $G$ if and only if $G$ is independent in the $d$-dimensional generic rigidity matroid.*

\textsuperscript{6}D. Kitson, *Finite and infinitesimal rigidity with polyhedral norms*, 2014. eprint: arXiv:1401.1336.
Conjecture

$G$ is $d$-independent iff the projection of every face of $\Phi_{n,l_p}^d$ has dimension $|E|$.

Proof of another theorem raised another problem:

Question

1. Is $d$-flattening a continuous map over linkages
2. Is there a continuous path from high dimensional realization to $d$-dimensional realization for a $d$-flattenable linkage?
Can studying the Cayley configuration space of a certain class of graphs (partial 2-trees) lead to an extension of the Walker conjecture to partial 2-trees?

We may be able to better understand the entire structure of $\Phi_{n,l_2}^2$ by building it up from these partial 2-trees.
Our Paper: M. Sitharam and J. Willoughby, “On flattenability of graphs,” in *Post-proceedings of ADG*, ser. LNAI, Springer, 2014
M. Belk and R. Connelly, “Realizability of graphs,” *Discrete Comput. Geom.*, vol. 37, no. 2, pp. 125–137, Feb. 2007.

H. Witsenhausen, “Minimum dimension embedding of finite metric spaces,” *Journal of Combinatorial Theory, Series A*, vol. 42, no. 2, pp. 184–199, 1986.

M. Sitharam and H. Gao, “Characterizing graphs with convex and connected cayley configuration spaces,” *Discrete & Computational Geometry*, vol. 43, no. 3, pp. 594–625, 2010.

K. Ball, “Isometric embedding in lp-spaces,” *European Journal of Combinatorics*, vol. 11, no. 4, pp. 305–311, 1990.

D. Kitson, *Finite and infinitesimal rigidity with polyhedral norms*, 2014. eprint: arXiv:1401.1336.

M. Sitharam and J. Willoughby, “On flattenability of graphs,” in *Post-proceedings of ADG*, ser. LNAI, Springer, 2014.