Overcritical $\mathcal{PT}$-symmetric square well potential in the Dirac equation

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Abstract

We study scattering properties of a $\mathcal{PT}$-symmetric square well potential with real depth larger than the threshold of particle-antiparticle pair production as the time component of a vector potential in the Dirac equation. Spontaneous pair production inside the well becomes tiny beyond the strength at which discrete bound states with real energies disappear, consistently with a spontaneous breakdown of $\mathcal{PT}$ symmetry.

1 Introduction

The study of $\mathcal{PT}$-symmetric potentials originated in the seminal papers[1],[2] by Bender and coworkers, dedicated to the analysis of spectra of non-Hermitian Hamiltonians of anharmonic oscillators, which turn out to be entirely real on condition that exact $\mathcal{PT}$ symmetry holds, i.e., the Hamiltonian, $H$, commutes with the $\mathcal{PT}$ operator and all eigenfunctions of $H$ are also eigenstates of $\mathcal{PT}$. In case of exact $\mathcal{PT}$ symmetry, it is possible to formulate an equivalent quantum mechanical description by defining a new metric operator in the representation space, a concept already established[3] before the study of the properties of $\mathcal{PT}$-symmetric potentials and generalized later to the definition of pseudo-Hermitian quantum mechanics[4]. If the condition that the eigenfunctions of the $\mathcal{PT}$ invariant Hamiltonian are also eigenfunctions of $\mathcal{PT}$ is relaxed, the $\mathcal{PT}$ symmetry is spontaneously broken and complex eigenvalues appear in the spectrum of $H$. Since then many authors have examined bound state problems with $\mathcal{PT}$-symmetric potentials in non relativistic quantum mechanics, while relatively few studies are dedicated to relativistic models, or even to $\mathcal{PT}$-symmetric quantum field theories[5, 6]. The present work intends to be a contribution to the study of the $\mathcal{PT}$ invariant Dirac equation, with emphasis on scattering aspects, within the framework of standard relativistic quantum mechanics. $\mathcal{PT}$-symmetric potentials will be treated phenomenologically as effective potentials, without attempting to formulate an equivalent Hermitian theory [6].
Solvable models in the Dirac equation are most easily constructed in (1+1) space-time dimensions and the related $\mathcal{PT}$-symmetric potentials may have different behaviour under Lorentz transformations, and are classified as vectors, pseudovectors, scalars, or pseudoscalars. In the cases of scalar and pseudoscalar potentials, affecting particles and antiparticles in the same way, the relevant hidden symmetry of the Dirac equation can be classified as pseudosupersymmetry\cite{7}. The analysis of vector potentials has been focused mainly on bound states, like in the generalized Hultén potential of Ref.\cite{8}, or in the logarithmic derivative of a suitable position-dependent effective mass of Ref.\cite{9}, while scattering aspects have not been examined in detail. Among the latter, an interesting peculiarity of strong potentials, with $|V| > 2m$, the threshold for production of a particle-antiparticle pair, with $m$ the particle mass, is the possibility that bound states merge with the negative-energy continuum of scattering states, thus appearing as transmission resonances at negative energies inside the potential well, which becomes overcritical with respect to spontaneous creation of particle-antiparticle pairs.

Such overcritical vector potentials are well studied in the Hermitian case in (1+1) dimensions, from the simplest example of the square well\cite{10}, to the cusp potential\cite{11},\cite{12}, or the Woods-Saxon potential\cite{13}. In the present work, we study scattering aspects of overcritical $\mathcal{PT}$-symmetric potentials, in particular how overcriticality is interrelated with spontaneous breakdown of $\mathcal{PT}$ symmetry. It is worthwhile to remark from the very beginning that our framework is standard quantum mechanics with complex potentials, phenomenologically treated as effective potentials. For the sake of simplicity, the analysis is carried out in detail for the $\mathcal{PT}$-symmetric square well, taken as the time component of a vector potential.

In this short note, we shall not enter into discussion of construction of positive definite norms via a linear operator, $\mathcal{C}$, commuting with the $\mathcal{PT}$ operator and the Hamiltonian, $H$, introduced by Bender and collaborators in nonrelativistic quantum mechanics\cite{14},\cite{15},\cite{16} and later extended to quantum field theory, in particular $\mathcal{PT}$-symmetric quantum electrodynamics, with imaginary electric charge and axial vector potential\cite{17}.

We already know from previous work on bound states in the Schrödinger equation that the $\mathcal{PT}$ symmetry of the square well is spontaneously broken\cite{18},\cite{19} and that the perturbative derivation of the $\mathcal{C}$ operator below the critical value of the imaginary part of the potential is by no means trivial\cite{16}. As far as scattering states are concerned, however, we have pointed out in our previous work on nonrelativistic scattering\cite{20}, that, even in the case of spontaneous breakdown of $\mathcal{PT}$ symmetry, transmission and reflection coefficients for progressive waves, travelling from left to right on the real axis, and for regressive waves, travelling from right to left, have definite non trivial relations, which are not valid for a general complex potential. These relations are connected with the fact that the imaginary part of a $\mathcal{PT}$-symmetric potential is an odd function of the space coordinate $x$, so that its integral over the $x$ axis vanishes.

The present work is thus intended as a phenomenological investigation of relativistic features, such as overcriticality, in presence of a spontaneously broken
$\mathcal{PT}$ symmetry, which would hardly emerge for different reasons, either in the case of a general complex potential, or in the case of a potential with exact asymptotic $\mathcal{PT}$ symmetry [20].

2 Formalism

In the present work we assume a $\mathcal{PT}$-symmetric square well potential

$$V(x) = \begin{cases} 
0, & x < -b \ (I) \\
q(V_0 - iV_1), & -b \leq x < 0 \ (II) \\
q(V_0 + iV_1), & 0 < x \leq +b \ (III) \\
0, & x > +b \ (IV)
\end{cases}$$

(1)

where the real and imaginary depths, $V_0$ and $V_1$, respectively, and the half-width, $b$, are positive numbers, while the elementary charge is assumed to be $q = -1$ for particles, as the time component of a vector potential in the Dirac equation in $(1+1)$ dimensions

$$i\frac{\partial}{\partial t}\Psi(x,t) = H_D(q)\Psi(x,t).$$

(2)

Here, the Dirac Hamiltonian, $H_D$, reads

$$H_D = V(x) - i\alpha_x \frac{\partial}{\partial x} + \beta m.$$  

(3)

Formula (3) is written in natural units, $\hbar = c = 1$, which will be used throughout the present work, and the metric is $g_{00} = -g_{11} = +1$. The $2 \times 2$ Hermitian matrices $\alpha_x$ and $\beta$ anticommute and are traceless with square unity; it is thus possible to identify them with two of the Pauli matrices: the choice we make corresponds to the standard Dirac representation[21]

$$\alpha_x = \sigma_x, \quad \beta = \sigma_z,$$

(4)

particularly suited to the study of the nonrelativistic limit of Eq. (2).

As is evident from the formulae given above, the solution, $\Psi$, to the Dirac equation in $(1+1)$ dimensions can be written as a spinor with two components. The parity operator, $\mathcal{P}$, and the time reversal operator, $\mathcal{T}$, are to be defined in a consistent way. In the adopted Dirac representation, we obtain [22]

$$\mathcal{P} = e^{i\theta_{\mathcal{P}}} P_0 \sigma_z,$$

(5)

where $\theta_{\mathcal{P}}$ is an arbitrary constant, and $P_0$ changes $x$ into $-x$. The Pauli matrix $\sigma_z$ ensures that the upper and lower components of $\Psi$ have opposite parities. With formula (5) as definition of the parity operator, it is immediate to check that $\Psi_{\mathcal{P}}(x,t) \equiv \mathcal{P}\Psi(x,t)$ is a solution to the Dirac equation (2) with potential $\mathcal{P}V(x)\mathcal{P}^{-1} = V(-x)$.

For the time reversal operator, $\mathcal{T}$, we consistently adopt the following form

$$\mathcal{T} = e^{i\theta_{\mathcal{T}}} \sigma_z \mathcal{K},$$

(6)
where \( \theta_T \) is a constant and \( K \) performs complex conjugation. \( \Psi_T (x,t) \equiv T \Psi (x,t) \) satisfies the equation

\[
- \frac{i}{\partial t} \Psi_T (x,t) = \left( V^* (x) - i \sigma_x \frac{\partial}{\partial x} + m \sigma_z \right) \Psi_T (x,t) .
\]  

(7)

For simplicity’s sake, we may assume \( \theta_T = - \theta_P \), so that

\[
\mathcal{P} T = P_0 K ,
\]

(8)

since \( \sigma_2 ^ z \) is the identity matrix. Definition (8) is consistent with the one commonly adopted in nonrelativistic quantum mechanics (see, e.g., section 4.1 of Ref.[20]).

Moreover, it is easy to show that the operator

\[
\mathcal{C}' = e^{i \theta_C} \sigma_y ,
\]

(9)

with \( \theta_C \) a real number, meets the condition

\[
\mathcal{C}' H_D (q) \mathcal{C}'^{-1} = - H_D (-q) .
\]

(10)

Therefore, \( \Psi_C (x,t) \equiv \mathcal{C}' \Psi (x,t) \) fulfills a modified Dirac equation

\[
- \frac{i}{\partial t} \Psi_C (x,t) = H_D (-q) \Psi_C (x,t) .
\]

(11)

Note that \( \mathcal{C}' \mathcal{P} T \) meets condition (10), too, provided that the Dirac Hamiltonian, \( H_D \), commutes with \( \mathcal{P} T \), i.e. \( V(x) = V^*(-x) \).

It is worthwhile to point out that the above definitions are different from those commonly adopted in textbooks[10]. In particular, the transformed wave function \( \Psi_{C' \mathcal{P} T} (x,t) \equiv \mathcal{C}' \mathcal{P} T \Psi (x,t) \) satisfies the Dirac equation for "antiparticles"

\[
i \frac{\partial}{\partial t} \Psi_{C' \mathcal{P} T} (x,t) = H_D (-q) \Psi_{C' \mathcal{P} T} (x,t) .
\]

(12)

In each of the four regions of the \( x \) axis defined by formula (1), we search for particular solutions, \( \Phi(x,t) = \Phi_0 (x) e^{-iEt} \), whose spatial part, \( \Phi_0 (x) \), can be written in the compact form

\[
\Phi_0 (x) = u_\pm (k) \cdot e^{\pm ikx} = \left( \begin{array}{c}
u_+^n (k) \\ \nu_-^n (k) \end{array} \right) \cdot e^{\pm ikx} .
\]

(13)

Direct replacement of formula (13) in Eqs. (2-3) yields for momentum \( k \)

\[
k^2 (x) = (E - V(x))^2 - m^2
\]

(14)

and for the ratio, \( \lambda \), of lower and upper components

\[
u_\pm = \pm \frac{k(x)}{E - V(x) + m} u_\pm \equiv \pm \lambda(x) u_\pm ,
\]

(15)
where the upper components, \( u_+^x \), turn out to be arbitrary non-zero constants, set to 1 for convenience. Adopting the matrix notation of Ref.\[21\], the general stationary solution, \( \Psi_J(x) \), to the Dirac equation in the \( J \)-th region of the \( x \) axis \( (J = I, \ldots, IV) \) can be written in the form

\[
\Psi_J(x) = \Omega_J(x) \begin{pmatrix} A_J \\ B_J \end{pmatrix},
\]

(16)

where \( A_J \) and \( B_J \) are constant and

\[
\Omega_J(x) \equiv \begin{pmatrix} 1 & 1 \\ \lambda_J & -\lambda_J \end{pmatrix} \cdot \begin{pmatrix} e^{ik_Jx} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_J & -\lambda_J \end{pmatrix} \cdot e^{ik_Jx},
\]

(17)

is the matrix whose columns are the two linearly independent solutions \( \Phi_J(x) \) in region \( J \), apart from a normalization factor, which does not affect the derivation of transmission and reflection coefficients. Here, \( \lambda_J \equiv k_J/(E - V_J + m) \) and \( k_J = \sqrt{(E - V_J)^2 - m^2} \). It is worthwhile to notice that \( \lambda_I = \lambda_{IV} \equiv \lambda = k/(E + m) \), with \( k = \sqrt{E^2 - m^2} > 0 \), for scattering states, since \( V_I = V_{IV} = 0 \), while \( \lambda_{II} = \lambda_{III}^* = \Lambda \) and \( k_{II} = k_{III}^* = K \), since \( V_{II} = V_{III}^* \).

By imposing continuity of the general solution, \( \Psi \), at the boundary, \( x_J^b \), between regions \( J \) and \( J + 1 \)

\[
\Omega_{J+1}(x_J^b) \cdot \begin{pmatrix} A_{J+1} \\ B_{J+1} \end{pmatrix} = \Omega_J(x_J^b) \cdot \begin{pmatrix} A_J \\ B_J \end{pmatrix},
\]

it is easy to express the coefficients of the general solution in region \( IV \) \( (x \to +\infty) \) as linear functions of those in region \( I \) \( (x \to -\infty) \)

\[
\begin{pmatrix} A_{IV} \\ B_{IV} \end{pmatrix} = M_D \cdot \begin{pmatrix} A_I \\ B_I \end{pmatrix},
\]

(18)

where

\[
M_D = \Omega_{IV}^{-1}(+b) \cdot \Omega_{III}^{-1}(+b) \cdot \Omega_{II}^{-1}(0) \cdot \Omega_{II}(0) \cdot \Omega_{III}^{-1}(-b) \cdot \Omega_I(-b)
\]

(19)

is the Dirac matching matrix\[21\]. Since \( \det \Omega_J = -2\lambda_J \) and \( \det \Omega_J^{-1} = 1 / \det \Omega_J \) are independent of \( x \), it is immediate to check that

\[
\det M_D = \det \Omega_I(-b) / \det \Omega_{IV}(+b) = \lambda_I / \lambda_{IV} = 1.
\]

(20)

It is worthwhile to mention that, in the matching method applied to the \( \mathcal{PT} \)-symmetric square well in the one-dimensional Schrödinger equation in a previous work of ours\[20\], use was made of a matching matrix, \( M \), which expresses the coefficients of the general solution in region \( I \) in terms of those in region \( IV \); therefore, the \( M \) matrix of Ref.\[20\] is the nonrelativistic limit of \( (M_D)^{-1} \), easily derivable from formula (19).
Elementary quantum mechanics immediately yields transmission and reflection coefficients in terms of $M^D$ matrix elements. For a plane wave travelling from left to right ($L \rightarrow R$), we must have $B_{IV} = 0$, so that, from formulae (18-19)

$$T_{L \rightarrow R} = \frac{A_{IV}}{A_I} = \frac{\det M^D_{22}}{M^D_{22}} = \frac{1}{M^D_{22}},$$

(21)

and

$$R_{L \rightarrow R} = \frac{B_I}{A_I} = - \frac{M^D_{12}}{M^D_{22}}.$$

(22)

For a plane wave travelling from right to left ($R \rightarrow L$), $A_I = 0$, so that

$$T_{R \rightarrow L} = \frac{B_I}{B_{IV}} = \frac{1}{M^D_{22}},$$

(23)

and

$$R_{R \rightarrow L} = \frac{A_{IV}}{B_{IV}} = \frac{M^D_{12}}{M^D_{22}}.$$

(24)

Thus, for a $\mathcal{PT}$-symmetric square well, $T_{L \rightarrow R} = T_{R \rightarrow L}$, while $R_{L \rightarrow R} \neq R_{R \rightarrow L}$. The nonrelativistic case [20] suggests the equality of the two transmission coefficients as a consequence of the intertwining relation $TH = H^\dagger T$, satisfied by the Hamiltonian [3] with any local $\mathcal{PT}$-symmetric potential. The intertwining relation would be broken by a non-local potential [20] [23]; in that case, we would have $T_{L \rightarrow R} \neq T_{R \rightarrow L}$, too.

Transmission and reflection coefficients are entries of the scattering matrix, $S$ [21], [20]

$$S = \begin{pmatrix} T_{L \rightarrow R} & R_{R \rightarrow L} \\ R_{L \rightarrow R} & T_{R \rightarrow L} \end{pmatrix}.$$

(25)

As a consequence of formulae [20], [21], [24], $S$ is not unitary. As already explained in Ref. [20], our approach uses $\mathcal{PT}$-symmetric potentials as effective potentials, and our purpose is different from that of Ref. [24], which considers them as fundamental and thus searches for a different Hilbert-space metric that permits conservation of probability. As shown in the same reference, however, one should face, in this latter approach to scattering by localized potentials, conceptual problems connected with the non-locality of the metric that ensures unitarity of the $S$ matrix.

From the definition of transmission and reflection coefficients in terms of $M^D$ matrix elements, it is easy to check that the determinant meets the condition $|\det S| = 1$.

Formulae (17) allow us to obtain compact expressions of the $M^D$ matrix elements and, consequently, of the transmission and reflection coefficients (21) [24] in terms of real $k$ and $\lambda$ defined in region $I$ and complex $K$ and $\Lambda$ defined in region $II$. For the latter quantities it is convenient to use the parameterization

$$K = \sqrt{\alpha_+ \alpha_-} e^{i(\varphi_+ + \varphi_-)/2}, \quad \Lambda = \sqrt{\alpha_+ / \alpha_-} e^{i(\varphi_+ - \varphi_-)/2},$$

(26)
with
\[ \alpha_\pm = \sqrt{(E - qV_0 \pm m)^2 + q^2V_1^2}, \quad \varphi_\pm = \arctan \left( \frac{qV_1}{E - qV_0 \pm m} \right). \] (27)

After some algebra, we obtain

\[ M_{11}^D = e^{-2i\kappa b} \left\{ \frac{\text{Re} \Lambda}{|\Lambda|} - \text{cosh} (2b \text{ Im} K) + \frac{\text{Re} \Lambda}{|\Lambda|} \text{ cos} (2b \text{ Re} K) \right. \]
\[ + i \left[ \text{Im} \Lambda \text{ sinh} (2b \text{ Im} K) \left( \frac{\Lambda^2 - |\Lambda|^2}{2|\Lambda|^2} \right) + \text{Re} \Lambda \text{ sinh} (2b \text{ Re} K) \left( \frac{\Lambda^2 + |\Lambda|^2}{2|\Lambda|^2} \right) \right] \} \quad (28) \]

The \( M_{22}^D \) matrix element turns out to be the complex conjugate of \( M_{11}^D \)

\[ M_{22}^D = (M_{11}^D)^*, \quad (29) \]

and the off-diagonal elements read

\[ M_{12}^D = i \left\{ \frac{\text{Re} \Lambda \text{ Im} \Lambda}{|\Lambda|^2} \left[ \text{ cos} (2b \text{ Re} K) - \text{ cosh} (2b \text{ Im} K) \right] \right. \]
\[ - \text{ Re} \Lambda \text{ sinh} (2b \text{ Re} K) \left( \frac{\Lambda^2 - |\Lambda|^2}{2|\Lambda|^2} \right) - \text{ Im} \Lambda \text{ sinh} (2b \text{ Im} K) \left( \frac{\Lambda^2 + |\Lambda|^2}{2|\Lambda|^2} \right) \} \quad (30) \]

and

\[ M_{21}^D = i \left\{ \frac{\text{Re} \Lambda \text{ Im} \Lambda}{|\Lambda|^2} \left[ \text{ cos} (2b \text{ Re} K) - \text{ cosh} (2b \text{ Im} K) \right] \right. \]
\[ + \text{ Re} \Lambda \text{ sinh} (2b \text{ Re} K) \left( \frac{\Lambda^2 - |\Lambda|^2}{2|\Lambda|^2} \right) + \text{ Im} \Lambda \text{ sinh} (2b \text{ Im} K) \left( \frac{\Lambda^2 + |\Lambda|^2}{2|\Lambda|^2} \right) \}. \quad (31) \]

In the \( V_1 \to 0 \) limit, corresponding to a real square well, the diagonal matrix elements \( 28, 29 \) reduce to the corresponding ones of Ref.\[21\], the off-diagonal elements \( 30, 31 \) differ from those of Ref.\[21\] by phase factors due to the different choice of the origin of the \( x \) axis (left edge of the well in Ref.\[21\], centre of the well in the present work): more precisely, \( M_{22}^D = e^{2i\kappa b}M_{12}^D \[21\], \( M_{21}^D = e^{-2i\kappa b}M_{22}^D \[21\], as expected\[20\]. The square moduli of reflection coefficients are obviously not affected by these phase differences.

It is also of some interest to compute the nonrelativistic limits of the \( M^D \) matrix elements, in order to compare them with the corresponding expressions for the \( \mathcal{PT} \)-symmetric square well in the Schrödinger equation obtained in Ref.\[20\]. To this aim, we need the limits of the basic quantities \( k_j^2 \) and \( \lambda_j^2 \) for \( E \to m + \epsilon \), where \( \epsilon (< m) \) is the kinetic energy. In that limit, \( k_j^2 = k_{jV}^2 = k^2 \to 2m \epsilon, \]
\( \lambda_j^2 = \lambda_{jV}^2 = \lambda^2 \to \epsilon/(2m), \quad k_{jH}^2 = (k_{jH}^2)^* = K^2 \to 2m(\epsilon - qV_0 + iqV_1), \]
\( \lambda_{jH}^2 = (\lambda_{jH}^2)^* = (\epsilon - qV_0 + iqV_1)/(2m). \) Using units \( 2m = 1 \), as in Ref.\[20\], \( k_j^2 \) and \( \lambda_j^2 \) coincide. In the same units, one easily verifies, after some algebra, that \( M^D \to M^{-1} \) of Ref.\[20\], as expected.

The formulation gives above is suited to the description of scattering states, with \( E < -m \) or \( E > +m \). The energies of discrete bound states, in the \(-m < E < +m \) range, appear as poles of the transmission coefficient, \( T_{L \to R} = T_{R \to L} \), or, equivalently, as zeros of the \( M_{22}^D \) matrix element. For bound states, \( k \) and \( \lambda \) in the asymptotic regions become imaginary, \( k = ik' = \sqrt{m^2 - E^2} \) and
\( \lambda = i \lambda' \equiv i k'/(m + E) \). The equation satisfied by real bound-state energies thus reads

\[
e^{-2b k'(E)} \left\{ \left( \frac{\text{Im} \Lambda(E)}{|\text{Im} \Lambda(E)|} \right)^2 \cosh (2b |K(E)|) + \left( \frac{\text{Re} \Lambda(E)}{|\text{Re} \Lambda(E)|} \right)^2 \cos (2b |K(E)|) \right. \\
+ \left. \text{Im} \Lambda(E) \sinh (2b |K(E)|) \left( \frac{\Lambda^2(E) + |\Lambda(E)|^2}{2K(E)|\Lambda(E)|} \right) \right. \\
+ \left. \text{Re} \Lambda(E) \sin (2b |K(E)|) \left( \frac{\Lambda^2(E) - |\Lambda(E)|^2}{2K(E)|\Lambda(E)|} \right) \right\} = 0
\]

(32)

In the \( V_1 \to 0 \) limit, corresponding to the real square well, formula (32) reduces to the well-known textbook expression\[10\].

The scattering eigenfunctions of the \( \mathcal{PT} \)-invariant Hamiltonian \([3]\) are not eigenstates of \( \mathcal{PT} \): this corresponds to a spontaneous breakdown of \( \mathcal{PT} \) symmetry. A different scenario would be obtained if the \( \mathcal{PT} \)-symmetric square well potential were the space component of a vector potential: in that case, if one assumes minimal coupling, \( p_x \rightarrow p_x + qV_x \), the time-independent Dirac equation reads, with our choice of Dirac matrices

\[
E \Psi(x) = H'_D \Psi(x) = \left\{ -i \frac{\partial}{\partial x} + qV_x(x) \right\} \sigma_x + \sigma_z m \right\} \Psi(x),
\]

(33)

By assuming for \( qV_x(x) \) a square well potential of type \([1]\), and repeating the calculations of the \( M \) matrix as in the previous case, one would obtain the following relation between the coefficients of the asymptotic solutions in region \( I \,(x \to -\infty) \) and \( IV \,(x \to +\infty) \)

\[
\begin{pmatrix}
A_{IV} \\
B_{IV}
\end{pmatrix} = e^{-2iqV_0b} \begin{pmatrix}
A_I \\
B_I
\end{pmatrix}.
\]

(34)

For a wave travelling from left to right \( (L \to R) \), with the boundary conditions \( A_I = 1, B_{IV} = 0, \) formula (34) yields \( T_{L\to R} = A_{IV} = e^{-2iqV_0b} \), \( R_{L\to R} = B_I = 0 \), while, for a wave travelling from right to left \( (R \to L) \), with \( B_{IV} = 1, A_I = 0 \), one gets \( T_{R\to L} = B_I = e^{2iqV_0b}, \) \( R_{R\to L} = A_{IV} = 0 \). The square well potential thus becomes reflectionless and conserves the probability flux. A more general \( \mathcal{PT} \)-symmetric local potential of finite range, \( V(x) = V^*(-x), \) \((-b \leq x + b \leq +b)\), would maintain this property, since, in that case, the argument \( 2qV_0b \) of the exponential in formula (34) would be replaced with the real integral \( q \int_{-b}^{+b} V_R(x) dx \), where \( V_R(x) \) is the real part of \( V \), owing to the fact that the imaginary part is an odd function of \( x \) and does not contribute to it.

It is an easy matter to check that the asymptotic wave functions are eigenstates of \( \mathcal{PT} \), in keeping with the proof given in Ref.\[20\] that an exact asymptotic \( \mathcal{PT} \) symmetry necessarily implies that the potential is reflectionless and conserves unitarity. It is worthwhile to note that in this case \( T_{R\to L} \neq T_{L\to R} \), as a consequence of the fact that \( TH^D_D \neq H^D_D T \). The connection between exact asymptotic \( \mathcal{PT} \) symmetry and potential reflectionlessness was proved in Ref.\[20\] for finite range potentials, but it might be more general, since it was already pointed out in Ref.\[25\] for infinite range potentials of the class \( V(x) = -x^{2K+2} \,(K = 1, 2, 3, ...) \).
3 Results and Comments

In order to explore scattering properties of an overcritical $\mathcal{PT}$-symmetric square well, we have performed several calculations of transmission and reflection coefficients as functions of an increasing imaginary depth, $V_1$, while keeping real depth, $V_0 > 2m$, and half-width, $b$, fixed. In our calculations, $V_0$ and $V_1$ are expressed in unit of particle mass, $m$, and $b$ in Compton wavelengths $\lambda_C = 1/m$. As an example of our results, Figure 1 gives $|T_L \rightarrow R|^2$ ($= |T_R \rightarrow L|^2 \equiv |T|^2$),

$|R_L \rightarrow R|^2$ and $|R_R \rightarrow L|^2$ versus energy, $E$, with $m = 1$, $V_0 = 3$, $b = 5$ and $V_1 = 0$ (top panels), $V_1 = 0.25$ (intermediate panels) and $V_1 = 0.5$ (bottom panels).

As a general comment, the signature of spontaneous pair production inside the well is represented by the transmission resonances at negative energies in the $-2 \leq E/m \leq -1$ range, adjacent to the bound-state region ($-1 < E/m < +1$), where the transmission coefficient may have poles on the real axis, corresponding to bound states. The half-plane $E/m > +1$ corresponds to positive-energy
scattering, the half-plane $E/m < -4 (= -1 - V_0/m)$ to negative-energy scattering.

The top panels refer to a real well ($V_1 = 0$). In this case, the two reflection coefficients are equal: $|R_{L\rightarrow R}|^2 = |R_{R\rightarrow L}|^2 \equiv |R|^2$ and unitarity holds: $|T|^2 + |R|^2 = 1$. The intermediate panels refer to a $\mathcal{PT}$-symmetric well with $V_1 = 0.25m$: spontaneous pair creation is still sizable, the two reflection coefficients differ by more than an order of magnitude at their maxima and unitarity is broken. The bottom panels refer to a $\mathcal{PT}$-symmetric well with $V_1 = 0.5m$: with increasing $V_1$, spontaneous pair creation is drastically reduced, albeit still present; the two reflection coefficients differ by order of magnitudes and $|R_{L\rightarrow R}|^2$ is sharply peaked at positive energies (particle reflection), $|R_{R\rightarrow L}|^2$ at negative energies (antiparticle reflection). The potential is neither absorptive ($|T_{i\rightarrow j}|^2 + |R_{i\rightarrow j}|^2 < 1$ at all incident energies), nor generative ($|T_{i\rightarrow j}|^2 + |R_{i\rightarrow j}|^2 > 1$ at all energies), but shows an intermediate behaviour.

As for bound states, the real well considered in our example has four bound states at negative energies and four at positive energies, which still persist when the imaginary part is weak, such as $V_1 = 0.25m$, considered in the intermediate panels of Fig.1. Real bound states begin to disappear at a critical value $V_{1\text{crit}} \simeq 0.272m$. At $V_1 = 0.5m$ bound states with real energies do not exist any more. The behaviour of the real bound state spectrum with increasing $V_1$ appears to be consistent with a spontaneous breakdown of $\mathcal{PT}$ symmetry, which deserves, in any case, a more detailed formal treatment in relativistic quantum mechanics, including a discussion of spectral degeneracies.

The present work should be considered as a phenomenological exploration of properties of $\mathcal{PT}$-symmetric local vector potentials in standard relativistic quantum mechanics; as such, it might be easily extended to non-local potentials \cite{26}. Further developments could involve either the study of the $\mathcal{C}$ operator \cite{16} when $\mathcal{PT}$ symmetry holds, or the search for a more general symmetry than $\mathcal{PT}$ for a Dirac equation that is not $\mathcal{PT}$-symmetric, as discussed, for instance, in Ref.\cite{27}, for the Schrödinger equation.

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