MAKING MULTIGRAPHS SIMPLE BY A SEQUENCE OF DOUBLE EDGE SWAPS

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Abstract. We show that any loopy multigraph with a graphical degree sequence can be transformed into a simple graph by a finite sequence of double edge swaps with each swap involving at least one loop or multiple edge. Our result answers a question of Janson motivated by random graph theory, and it adds to the rich literature on reachability of double edge swaps with applications in Markov chain Monte Carlo sampling from the uniform distribution of graphs with prescribed degrees.

1. Introduction

We will consider different classes of undirected graphs, the most general being loopy multigraphs where both multiple edges and multiple loops are allowed. Specifically, we are interested in graphs where each vertex has a prescribed degree, the degree of a vertex being the number of stubs (half-edges) attached to it (so the contribution from a loop is two). The list of the degrees of all vertices, sorted in weakly decreasing order, is called the degree sequence of the graph, and a weakly decreasing sequence is said to be graphical if it is the degree sequence of some simple graph (no loops and no multiple edges). The most popular basic graph operation that preserves the degree sequence is the replacement of any two edges \((v_1, v_2)\) and \((v_3, v_4)\) by \((v_2, v_3)\) and \((v_4, v_1)\). This is called a double edge swap and was first introduced by Petersen [12]. It has been reinvented several times and has many alternative names in the literature [1]: degree-preserving rewiring, checkerboard swap, tetrad or alternating rectangle.

The main motivation for our work comes from the theory of random graphs. There is a simple direct method of generating a uniformly random stub-labelled (where the stubs have identity) loopy multigraph with prescribed degrees: Attach the prescribed number of stubs to each vertex, then choose a random matching of all stubs. This is called the configuration model and was introduced by Bollobás 1980 [2]. The simplicity of the method makes it very useful for theoretical analyses of random graphs, but in many applications one wants to study simple graphs rather than multigraphs. There are several possible solutions to this issue. Sometimes it is

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possible to simply condition the random loopy multigraph from the configuration model on the event that it is a simple graph. This yields a uniform distribution of simple graphs with the given degree sequence. Recently, Janson [9] proposed another method, the switched configuration model, where the random loopy multigraph is transformed into a simple graph by a sequence of random double edge swaps. Each swap is required to have the property that at least one of the two swapped edges is a loop or a multiple edge. The resulting distribution on simple graphs is not exactly uniform, but for a certain class of degree sequences Janson showed that it is asymptotically uniform in the sense that the total variation distance to the uniform distribution tends to zero when the number of vertices goes to infinity. Motivated by his construction, he posed the following question to us in person:

**Question 1.** Can any loopy multigraph with a graphical degree sequence be transformed into a simple graph by a finite sequence of double edge swaps involving at least one loop or multiple edge?

In this paper, we answer the question affirmatively. In fact, we show a stronger statement that Janson conjectured in [9, Remark 3.4], namely that it is always possible to reach a simple graph even if an evil person chooses which loop or multiple edge should be involved in each double edge swap.

Our result adheres to a rich literature of reachability of double edge swaps, a topic that has an important application in the context of Markov chain Monte Carlo sampling; see Fosdick et al. [1] for a comprehensive discussion. In the simplest case, we want to sample from the uniform distribution of all graphs (of some class) with prescribed degrees. Basically, one starts with any graph with the given degrees and performs random double edge swaps for a while; the stationary distribution is uniform. (Exactly how the random double edge swaps should be chosen depends on the class of graphs and the type of labelling of the graph, see [1].) To show uniformity, one has to verify that the Markov chain satisfies three conditions:

(i) that the transition matrix of the chain is doubly stochastic,
(ii) that the chain is irreducible,
(iii) and that the chain is aperiodic.

The irreducibility condition means that for any pair of graphs $G$ and $G'$ with the same degree sequence there is a sequence of double edge swaps that transforms $G$ to $G'$. If this is true or not depends on the particular class of graphs we are interested in. It is true for simple graphs [4, 8, 6], connected simple graphs [13], 2-connected simple graphs [14], loop-free multigraphs [8], simple-loopy multigraphs (multiple edges and simple loops) [10] and loopy multigraphs [5], but not for simple-loopy simple graphs (simple edges and simple loops) [11] and loopy simple graphs (where multiple loops are allowed but no other multiple edges) [10].

Note how our result differs from that of Eggleton and Holton [5]. While they show that any loopy multigraph can be transformed into any other loopy multigraph with the same degree sequence by a sequence of double
edge swaps, we show that this can be accomplished with admissible swaps only, where a swap is admissible if it involves at least one loop or multiple edge. In the situation where Janson posed Question \[1\] this condition is natural since the goal is to reach a simple graph. In applications, when using the switched configuration model to sample from an approximately uniform distribution of simple graphs with a given degree sequence, one wants to obtain a simple graph by as few double edge swaps as possible, so swapping away “bad” edges is essential for the efficiency of this method.

The paper is organized as follows. First, in Section 2 we fix the notation and recall the Erdős-Gallai theorem. In Section 3 we present our results and in Sections 4 and 5 we prove them. Finally, in Section 6 we discuss some open questions.

2. Notation and prerequisites

The terminology on multigraphs is not standardized, so let us start by defining it. Figure 1 shows some examples.

A loop is an edge connecting a vertex to itself. A loopy multigraph is an undirected graph where loops are allowed and where there might be multiple edges between the same pair of vertices and multiple loops at the same vertex.

A loop-free multigraph is a loopy multigraph without loops.

An edge is said to be simple if it has multiplicity one and is not a loop, and a graph is simple if all its edges are simple.

The degree of a vertex is the number of half-edges adjacent to it (so each loop contributes with two to the degree). The list of the degrees of all vertices, sorted in weakly decreasing order, is called the degree sequence of the graph.

We will denote an edge between \(v_1\) and \(v_2\) with curly braces \(\{v_1, v_2\}\) and sometimes, to stress the difference between an edge and an unordered pair of vertices, we will talk about an edge of type \(\{v_1, v_2\}\).

Two edges are said to be incident if they share at least one vertex.

**Definition 1.** Suppose there are two edges of types \(\{v_1, v_2\}\) and \(\{v_3, v_4\}\). Then we define the double edge swap \((v_1, v_2)(v_3, v_4)\) as the operation of removing two such edges and adding two edges of type \(\{v_2, v_3\}\) and \(\{v_4, v_1\}\).
The swap is admissible if the edges \{v_1, v_2\} and \{v_3, v_4\} are not incident and not both of them are simple (before the swap).

See Figure 2 for an illustration. Clearly, a double edge swap (admissible or not) leaves the degree sequence unchanged. Note also that an admissible double edge swap never introduces a new loop since the edges are not incident.

A weakly decreasing sequence is said to be graphical if it is the degree sequence of some simple graph. The following theorem characterizes those sequences.

**Theorem 1** (Erdős-Gallai [7]). A sequence of nonnegative integers \(d_1 \geq d_2 \geq \cdots \geq d_n\) is graphical if and only if \(d_1 + d_2 + \cdots + d_n\) is even and

\[
\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)
\]

holds for each \(1 \leq k \leq n\).

We will need this theorem later on, but only the “only if” part, and since its proof is a simple double counting argument we include it here for completeness: Consider a simple graph with vertices \(v_1, v_2, \ldots, v_n\) with degrees \(d_1 \geq d_2 \geq \cdots \geq d_n\), respectively. The left-hand side of (1) gives the number of edge-vertex adjacencies among \(v_1, v_2, \ldots, v_k\). The edge of each such adjacency must have either one or two endpoints among \(v_1, v_2, \ldots, v_k\); the \(k(k-1)\) term on the right-hand side gives the maximum possible number
Figure 3. A loop-free multigraph made simple by two admissible double edge swaps. Note that in the original graph there is no admissible double edge swap that does not create a new multiple edge.

3. Results

Our main result is the following.

**Theorem 2.** Any loop-free multigraph whose degree sequence is graphical can be transformed into a simple graph by a finite sequence of admissible double edge swaps.

Figure 4 shows an example.

Let us state a simple consequence of Theorem 2.

**Theorem 3.** Any loopy multigraph whose degree sequence is graphical can be transformed into a simple graph by a finite sequence of admissible double edge swaps.

**Proof.** Consider a loopy multigraph whose degree sequence is graphical. If there is a loop at some vertex \( v \), since the degree sequence is graphical there must be at least one edge \( \{v_1, v_2\} \) not incident to \( v \), and the double edge swap \( (v, v)(v_1, v_2) \) is admissible and reduces the number of loops. After removing all loops this way, the resulting loop-free graph can be transformed into a simple graph by Theorem 2. \(\square\)

As mentioned in the introduction, Janson [9, Remark 3.4] conjectured a stronger version of Question 1, perhaps best phrased in terms of a combinatorial game.

The **loopy multigraph game** is played by the Angel and the Devil as follows. The starting position is a loopy multigraph \( G \) with a graphical degree sequence. In each move, the Devil chooses any loop or multiple edge \( e \) and then the Angel chooses any edge \( e' \) and performs a double edge swap on \( e \) and \( e' \). The Angel wins if she reaches a simple graph, and the Devil wins if the game goes on forever.
Conjecture 1 (Janson 2018). In the loopy multigraph game, the Angel has a winning strategy for any starting position.

We will prove this conjecture by showing that the Angel has a winning strategy even if we change the rules of the game to make it much harder for her. Let $H$ be a loop-free multigraph on a vertex set $V$. In the loopy multigraph game with target $H$, starting from a loopy multigraph $G$ on $V$ such that every vertex has the same degree in $G$ as in $H$, in each move the Devil chooses any edge $e$ in $G$ such that $G$ has more edges than $H$ of the same type as $e$ (that is, with the same endpoints), and then the Angel chooses any edge $e'$ in $G$ not incident to $e$ and performs a double edge swap on $e$ and $e'$ in $G$. The Angel wins if she reaches $H$, and the Devil wins if the game goes on forever or if the Angel cannot make a move.

Theorem 4. In the loopy multigraph game with target $H$, the Angel has a winning strategy for any starting position.

It is not hard to see that Conjecture 1 follows from Theorem 4: Given a loopy multigraph $G$ with a graphical degree sequence, there is a simple (and therefore loop-free) graph $H$ with the same degree sequence. In the loopy multigraph game, a valid move for the Devil is to choose any edge which is a loop or whose multiplicity is larger than one. Any such move is also valid in the loopy multigraph game with target $H$, since the multiplicity of the same edge in $H$ is at most one. Thus, the Angel’s winning strategy in the target game is a winning strategy in the loopy multigraph game as well.

In the next section we will prove Theorem 4 from which Conjecture 1 follows by the reasoning above. Clearly, Theorem 2 follows from Theorem 4 since we can choose the target $H$ to be any simple graph with the same degree sequence as the original graph. However, in Section 5 we will present an alternative proof of Theorem 2 that is more elementary in the sense that it does not rely on a choice of a simple graph as a target but depends only on the "only if" part of the Erdős-Gallai theorem.

4. Proof of Theorem 4

Proof. Let $G$ be the current position in the game. For each unordered pair of vertices $\{u, v\}$, let $m_G(\{u, v\})$ and $m_H(\{u, v\})$ be the multiplicities of $\{u, v\}$ in $G$ and $H$, respectively. If $m_G(\{u, v\}) > m_H(\{u, v\})$, choose $m_G(\{u, v\}) - m_H(\{u, v\})$ of the edges of type $\{u, v\}$ in $G$ and call them exceeding. Analogously, if $m_H(\{u, v\}) > m_G(\{u, v\})$, choose $m_H(\{u, v\}) - m_G(\{u, v\})$ of the edges of type $\{u, v\}$ in $H$ and call them exceeding. All loops in $G$ are said to be exceeding as well. Define the distance to $H$ to be the total number of exceeding edges.

Suppose the devil chooses an edge $e$. We may assume that $e$ is exceeding since there are more edges in $G$ than in $H$ of the same type as $e$.

Colour some of the exceeding edges in $G$ blue and some of the exceeding edges in $H$ red such that
I e is blue,  
II for each vertex, equally many blue as red edges are incident to it,  
III among all colourings with the above properties, we choose one where  
the number of coloured edges is minimal.

Clearly, there is at least one such colouring since colouring all exceeding  
edges in \( G \) blue and all exceeding edges in \( H \) red satisfies conditions [II] and [III].

Let \( u_1 \) and \( v_1 \) be the endpoints of \( e \). Since \( e \) is blue, by property [II] there  
must be a red edge from \( v_1 \) to some vertex \( v_2 \) and a blue edge from \( v_2 \) to  
some vertex \( v_3 \). Analogously, there must be a red edge from \( u_1 \) to some  
vertex \( u_2 \) and a blue edge from \( u_2 \) to some vertex \( u_3 \).

If \( u_1 = v_3 \) and \( v_1 = u_3 \), the red-blue alternating circuit \( v_1, v_2, u_1, u_2, v_1 \)  
can be uncoloured without violating conditions [II] and [III] which is impossible  
by the minimality condition [III]. Thus, at least one of the inequalities \( u_1 \neq v_3 \)  
and \( v_1 \neq u_3 \) holds. For symmetry reasons, we may assume that \( u_1 \neq v_3 \).  
Since no two vertices can have both a blue and red edge between them,  
we have \( u_1 \neq v_2 \) and \( v_1 \neq v_3 \), and since \( H \) is loop-free, we have \( v_1 \neq v_2 \).  
These four inequalities show that the edges \( e = (u_1, v_1) \) and \( (v_2, v_3) \) are not  
incident, so the Angel can perform the double edge swap \( (u_1, v_1)(v_2, v_3) \).  
This removes a blue exceeding edge of type \( \{u_1, v_1\} \), a blue exceeding edge  
of type \( \{v_2, v_3\} \) and at least one red exceeding edge of type \( \{v_1, v_2\} \) while  
introducing at most one new blue exceeding edge of type \( \{v_3, u_1\} \), so the  
distance to \( H \) decreases. (Note that this holds also for the case where \( u_1 = v_1 \)  
and \( v_2 = v_3 \).)

Thus, the Angel can always decrease the distance to \( H \). Eventually the  
distance will be zero and \( H \) is reached. \( \square \)

5. Alternative proof of Theorem 2

Note that Theorem 2 follows from Theorem 4 since we can choose the  
target \( H \) to be any simple graph with the same degree sequence as the  
original graph. In this section we present an alternative proof of Theorem 2  
which does not rely on such a choice. In fact, the only way it exploits the  
graphicality of the degree sequence is via the inequalities guaranteed by the  
easy “only if” direction of the Erdős-Gallai theorem. With this in mind,  
the harder “if” direction of the Erdős-Gallai theorem follows easily from  
Theorem 3 since any sequence of nonnegative integers with an even sum  
clearly is the degree sequence of some loopy multigraph.

5.1. Ordering vertices, edges and graphs. Fix a vertex set \( V \) with  
prescribed degrees given by a function \( d : V \to \mathbb{N} \). Choose a total order on  
\( V \) with the property that \( d(u) < d(v) \) implies \( u < v \). This induces a total  
order on unordered pairs of vertices defined by, for \( u_1 > u_2 \) and \( v_1 > v_2 \),  
letting \( \{u_1, u_2\} \leq \{v_1, v_2\} \) if either \( u_1 < v_1 \) or \( u_1 = v_1 \) and \( u_2 \leq v_2 \). This in  
turn induces a (strict) partial order on loop-free multigraphs with vertex set  
\( V \) and the prescribed degrees \( \deg v = d(v) \) \( \forall v \in V \), by letting \( G < H \) if \( H \)
is not simple and its maximal non-simple edge is larger than all non-simple edges in $G$, or the maximal non-simple edges of $G$ and $H$ are equal but its multiplicity is strictly larger in $H$ than in $G$.

**Proposition 1.** Any non-simple loop-free multigraph whose degree sequence is graphical can be transformed into a smaller graph by a finite sequence of admissible double edge swaps.

Before we prove the proposition we present a technical device that can sometimes reduce the multiplicity of an edge without adding too many new multiple edges.

5.2. **A swapping lemma.**

**Lemma 1.** Let $m \geq 2$ be an integer and let $v_1, v_2, \ldots, v_{2m}$ be a sequence of vertices in a loop-free multigraph. Suppose the following holds.

(a) $v_1$ is distinct from all of $v_2, v_3, \ldots, v_{2m}$.

(b) $v_i \neq v_{i+1}$ for $i = 1, 2, \ldots, 2m - 1$, and the corresponding unordered pairs $\{v_i, v_{i+1}\}$ are all distinct from each other and from $\{v_1, v_{2m}\}$.

(c) There are edges of type $\{v_{2j}, v_{2j+1}\}$ for $j = 1, 2, \ldots, m - 1$.

(d) There are multiple edges of type $\{v_1, v_{2m}\}$.

Then there is a sequence of admissible double edge swaps that reduces the multiplicity of $\{v_1, v_{2m}\}$ by one without adding any new non-simple edge except possibly those edges of types $\{v_{2j-1}, v_{2j}\}$, $j = 1, 2, \ldots, m$ that were already present.

**Proof.** Since the unordered pairs $\{v_{2m-2}, v_{2m-1}\}$ and $\{v_{2m-1}, v_{2m}\}$ are distinct, the vertices $v_{2m-2}, v_{2m-1}$ and $v_{2m}$ are all distinct, and we can perform the admissible swap $(v_1, v_{2m})/(v_{2m-1}, v_{2m-2})$, see Figure 4. That reduces the multiplicity of $\{v_1, v_{2m}\}$ by one and introduces only two new edges, of type $\{v_1, v_{2m-2}\}$ and $\{v_{2m-1}, v_{2m}\}$. We are done unless $\{v_1, v_{2m-2}\}$ is now neither simple nor equal to any $\{v_{2j-1}, v_{2j}\}$, $j = 1, 2, \ldots, m$. In that case, $v_{2m-2} \neq v_2$ and $m \geq 3$, so the sequence $v_1, v_2, \ldots, v_{2(m-1)}$ satisfies properties (a) to (d). Then, by induction, there is a sequence of admissible double edge swaps that reduces the multiplicity of $\{v_1, v_{2(m-1)}\}$ to its original value without adding any new non-simple edge except possibly those edges of type $\{v_{2j-1}, v_{2j}\}$, $j = 1, \ldots, m - 1$ that were already present. \hfill $\Box$

5.3. **Proof of Proposition 1**. Let $G$ be a non-simple loop-free multigraph that cannot be transformed to a smaller graph by a finite number of admissible double edge swaps. To prove the proposition, we must show that the degree sequence of $G$ is not graphical.

To this end we will need a bunch of lemmas, and they are all implicitly referring to $G$ and to the following notation and terminology.

Let $\{u_1, u_2\}$ with $u_1 > u_2$ be the maximal non-simple edge in $G$. The vertices other than $u_1$ and $u_2$ will be called *ordinary* vertices. For $i = 1, 2$, let $V_i$ and $V_i'$ be the sets of ordinary vertices that have an edge to $u_i$ and
that do not have an edge to $u_i$, respectively. An ordinary vertex is called small if it is smaller than $u_1$ and large if it is larger than $u_1$.

**Lemma 2.** There is no edge between a vertex $v_1$ in $V_1$ and a vertex $v_2$ that is small or belongs to $V_2$.

*Proof.* If there was such an edge $\{v_1, v_2\}$ the admissible double edge swap $(u_1, u_2)(v_2, v_1)$ would reduce the multiplicity of $\{u_1, u_2\}$ without creating any new non-simple edge, except possibly for $\{u_2, v_2\}$ if $v_2$ is small, and then $\{u_2, v_2\}$ is smaller than $\{u_1, u_2\}$. This contradicts the assumptions on $G$. □

**Lemma 3.** All large vertices belong to $V_1 \cap V_2$, and every large vertex is adjacent to some vertex in $V_1$ and to some vertex in $V_2$.

*Proof.* Let $v$ be any large vertex. By the maximality of the non-simple edge $\{u_1, u_2\}$ all edges from $v$ are simple, so the degree of $v$ equals the number of vertices adjacent to $v$. Since $\deg v \geq \deg u_1$ and $u_1$ has multiple edges to $u_2$, $v$ is adjacent to some vertex $v_1$ in $V_1$. Lemma 2 now yields that $v$ belongs to $V_2$. Analogously, since $\deg v \geq \deg u_2$ and $u_2$ has multiple edges to $u_1$, $v$ is adjacent to some vertex $v_2$ in $V_2$, and, by Lemma 2, $v$ belongs to $V_1$. □

**Lemma 4.** Any ordinary vertex adjacent to a small vertex must be adjacent to all large vertices, except for itself (if it is large).

*Proof.* Suppose an ordinary vertex $v_1$ is adjacent to a small vertex $v_2$ but not to some large vertex $v_3 \neq v_1$. By Lemma 3, $v_3$ is adjacent to some vertex $v_4$ in $V_1$. (Note that $v_4$ might be identical to $v_2$.) Applying Lemma 1 to the sequence $u_1, v_4, v_3, v_1, v_2, u_2$ shows that there is a sequence of admissible double edge swaps that reduces the multiplicity of $\{u_1, u_2\}$ without adding any new non-simple edge except possibly those edges among $\{u_1, v_4\}$, $\{v_3, v_1\}$ and $\{v_2, u_2\}$ that were already present. By construction, $\{u_1, v_4\}$ and $\{v_3, v_1\}$ were not present, so the only possible new non-simple edge is $\{v_2, u_2\}$, which is smaller than $\{u_1, u_2\}$ since $v_2$ is small. This contradicts the assumptions on $G$, and we conclude that the first sentence in the lemma holds. □

**Lemma 5.** All vertices in $L \cup \{u_1, u_2\}$ are adjacent.
Proof. By Lemma 3, $u_1$ and $u_2$ are adjacent to all large vertices. Again by Lemma 8 any large vertex is adjacent to some small vertex, since all vertices in $V_1$ are small. Now it follows from Lemma 4 that any large vertex is adjacent to all other large vertices. Finally, $u_1$ and $u_2$ are adjacent by definition. □

Lemma 6. A small vertex not adjacent to any small vertex must be smaller than $u_2$.

Proof. Suppose there is a small vertex $v > u_2$ not adjacent to any small vertex. By the maximality of the non-simple edge $\{u_1, u_2\}$, all edges from $v$ to any vertex greater than or equal to $u_1$ are simple. Thus, the degree of $v$ is at most $\ell + 1 + m$, where $\ell$ is the number of large vertices and $m$ is the multiplicity of the edge $\{v, u_2\}$ (possibly zero). By Lemma 8 $u_2$ is adjacent to all large vertices, so its degree is at least $\ell + m + 2$. This shows that $\deg u_2 > \deg v$, which contradicts the assumption that $v > u_2$. □

Lemma 7. If there is an edge between small vertices somewhere in the graph, then every small vertex in $V_2$ is adjacent to some small vertex.

Proof. Suppose there are small adjacent vertices $v_1$ and $v_2$ and a small vertex $v$ in $V_2$ not adjacent to any small vertex. By Lemma 2, $v_2$ is adjacent to $u_1$. Applying Lemma 1 on the sequence $u_1, v, u_2, v_1, v_2, u_2$ shows that there is a sequence of admissible double edge swaps that reduces the multiplicity of $\{u_1, u_2\}$ and adds no new non-simple edge except possibly $\{u_1, v\}$, $\{u_2, v_1\}$ and $\{v_2, u_2\}$. But all these edges are smaller than $\{u_1, u_2\}$, the first one since $v < u_2$ by Lemma 6. This contradicts the assumptions on $G$. □

Let $L$ be the set of large vertices and let $\ell$ be the number of them.

Lemma 8. If there is an edge between small vertices somewhere in the graph, then every small vertex $v$ has at least $\min(\ell + 1, \deg v)$ edges to vertices in $L \cup \{u_1\}$.

Proof. Suppose there is an edge between small vertices somewhere in the graph, and consider a small vertex $v$.

First suppose $v$ is adjacent to a small vertex. Then by Lemma 3 it is adjacent to all large vertices, and by Lemma 2 it belongs to $V_1$, so it is adjacent to all $\ell + 1$ vertices in $L \cup \{u_1\}$.

Now, suppose instead that $v$ is not adjacent to any small vertex. Then, by Lemma 7 it does not belong to $V_2$, and clearly the degree of $v$ equals the number of edges from $v$ to $L \cup \{u_1\}$. □

Lemma 9. The degree sequence of $G$ is not graphical.

Proof. We treat two cases separately.

Case 1: No two small vertices are adjacent.

By Lemma 6 all small vertices are smaller than $u_2$. Also, by Lemma 8 all vertices in $L \cup \{u_1, u_2\}$ are adjacent, and $\{u_1, u_2\}$ is non-simple. It follows
that
\[ \sum_{v \geq u_2} \deg v \geq (\ell + 2)(\ell + 1) + 2 + m, \]
where \( m \) is the number of edges (counted with multiplicity) between a vertex in \( L \cup \{u_1, u_2\} \) and a small vertex. Since no two small vertices are adjacent, we have \( \sum_{v < u_2} \deg v = m \). Plugging this into the inequality above yields
\[ \sum_{v \geq u_2} \deg v \geq (\ell + 2)(\ell + 1) + 2 + \sum_{v < u_2} \deg v. \]
Note that, since all small vertices are smaller than \( u_2 \), there are exactly \( \ell + 2 \) vertices larger than or equal to \( u_2 \), namely \( u_1, u_2 \) and all the large vertices. Letting \( d_i \) be the degree of the \( i \)-th largest vertex, the above inequality can be written
\[ \sum_{i=1}^{\ell+2} d_i \geq (\ell + 2)(\ell + 1) + 2 + \sum_{i=\ell+3}^{n} d_i, \]
where \( n \) is the number of vertices. Now it follows from the Erdős-Gallai theorem (with \( k = \ell + 2 \)) that the degree sequence of \( G \) is not graphical.

**Case 2:** There are at least two adjacent small vertices.

As before, by Lemma 5 all vertices in \( L \cup \{u_1, u_2\} \) are adjacent, and \( \{u_1, u_2\} \) is non-simple. It follows that the number of edges (counted with multiplicity) between a vertex in \( L \cup \{u_1\} \) and a vertex in \( L \cup \{u_1, u_2\} \) is at least \( (\ell + 1)^2 + 1 \), so
\[ \sum_{v \geq u_1} \deg v \geq (\ell + 1)^2 + 1 + m', \]
where \( m' \) is the number of edges between a vertex in \( L \cup \{u_1\} \) and a small vertex. By Lemma 8,
\[ m' \geq \sum_{\text{small } v} \min(\ell + 1, \deg v), \]
and thus
\[ \sum_{v < u_1} \min(\ell + 1, \deg v) \leq \ell + 1 + m'. \]
Combining this with (2) gives
\[ \sum_{v \geq u_1} \deg v \geq (\ell + 1)^2 \ell + 1 + \sum_{v < u_1} \min(\ell + 1, \deg v), \]
and it follows from the Erdős-Gallai theorem (with \( k = \ell + 1 \)) that the degree sequence of \( G \) is not graphical. \( \square \)

Proposition 1 follows from Lemma 9, and Theorem 2 then follows from the proposition.
6. Open questions

In this paper, we have focused on the existence of a sequence of admissible double edge swaps that makes a graph simple. For further research, it seems natural to ask about the length of such a sequence.

**Question 2.** What is the minimum number of admissible double edge swaps needed to transform a given loopy multigraph $G$ with a graphical degree sequence into a simple graph?

As we saw in Figure 3, it is not always possible to decrease the number of multiple edges by a double edge swap. On the other hand, some double edge swaps might decrease the number of multiple edges by two.

A related question is whether the word “admissible” in Question 2 matters. Are there situations where the fastest road to simplicity require double edge swaps with only simple edges?

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