Crumpled-to-tubule transition in anisotropic polymerized membranes: beyond $\epsilon$-expansion

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Anisotropic $D$-dimensional polymerized phantom membranes are investigated within a nonperturbative renormalization group (NPRG) framework. One focuses on the transition between a high-temperature, crumpled, phase and a low-temperature, tubular, phase where the membrane is flat along one direction and crumpled along the other ones. While the upper critical dimension $D_{uc}=5/2$ is close to $D=2$ the weak-coupling perturbative approach is qualitatively and quantitatively wrong. We show that our approach is free of the problems encountered within the perturbative framework and provides physically meaningful critical quantities.

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Polymerized phantom membranes display a remarkable behavior as the temperature is varied [1, 2]. While crumpled at high temperature due to fluctuations, they exhibit a low-temperature flat phase with long-range orientational order [3–5]. The existence of an ordered phase in $D=2$ despite Mermin-Wagner theorem originates in the existence of an anharmonic coupling between bending and elastic degrees of freedom or, equivalently, between out-of-plane capillary and in-plane phonon, modes [3]. This coupling induces a phonon-mediated long-range effective interaction between the out-of-plane fluctuations that stabilizes a low-temperature flat phase for two and even less – dimensional membranes [6–8]. Within a RG picture the coupling between the different modes is responsible for a stiffening of the bending rigidity constant at low-momentum $\kappa(q) \sim q^{-\eta}$ with $\eta > 0$ – called anomalous elasticity, that suppresses the destructive fluctuations in the transverse direction to the membrane [3, 6, 8]. Note that, although strongly reduced, the out-of-plane fluctuations are not completely suppressed and the membrane still display ripples [9–11].

This phenomenon of long-range order induced by coupling between bending and elastic degrees of freedom has provided the theoretical grounds for the existence and stability of polymerized membranes and, incidentally, of the recently discovered graphene [12], the first example of truly two-dimensional membrane. It has also underlined the importance of the nature of the membrane internal structure regarding the kind of order displayed at long distance. As a remarkable illustration of this fact is, in marked contrast with the case of polymerized membranes, the absence of long-range orientational order for membranes deprived of fixed connectivity, and thus of elasticity modulus, whose prototypical example are liquid membranes. Past studies have considered a wide variety of alterations of the conventional crystalline order of polymerized membranes: presence of topological defects, like dislocations or disclinations, vacancies, local variations of connectivity, random impurities, … giving rise to novel and rich critical behaviors including new universality classes (see [1] and [13] for reviews).

A fruitful modification of the translationally/rotationally invariant internal structure of the membrane is the inclusion of an in-plane anisotropy. In [14, 15] it has been shown that membranes with intrinsic one-directional anisotropy display, when the temperature is lowered, a tubular phase, flat in one direction and crumpled in the other ones. This phase lies in a temperature range intermediate between a high-temperature regime, corresponding to a fully crumpled phase and a low-temperature regime, where the membrane adopts a flat conformation in all directions, see Fig.1.

![FIG. 1: Flat, tubular and crumpled phases of anisotropic membranes as functions of the temperature.](image)

While the existence and stability of tubular phases have been numerically confirmed [16, 17], the crumpled-to-tubule transition remains numerically and experimentally unexplored. The main reason for this situation is the lack of accurate predictions for the critical quantities. This is a priori a rather surprising situation. Indeed, a striking consequence of the presence of anisotropy is that the upper critical dimension is lowered from $D_{uc}=4$, for isotropic membranes, to $D_{uc}=5/2$. From the proximity...
of \( D_{uc} \) with \( D = 2 \) one could expect the perturbative approach performed in the parameter \( \epsilon = 5/2 - D \) to be in a good position to evaluate the critical quantities in \( D = 2 \). However, as shown explicitly by Radzihovsky and Toner, due to the fractional nature of the upper critical dimension, the \( \epsilon \)-expansion is, on the contrary, "extremely unreliable" and even "qualitatively wrong" [15]. This is clear in view of the exponent \( \eta \) describing the correlation between the tangents of the membrane, which is found to be negative. A genuine negative value for this exponent would correspond to a downward renormalization - a decreasing - of the bending rigidity constant, in contradiction with what is expected from physical grounds [15].

Moreover alternative perturbative methods used to investigate the low-energy physics of membranes, e.g. the self consistent screening approximation (SCSA), that have successfully worked for isotropic membranes [18–20], appear here hard to implement given the complexity of the field theoretical formulation of the model.

In this article we investigate the crumpling-to-tubule phase transition of anisotropic membranes by means of a NPRG approach. This method [21] has been recently used to investigate the crumpling-to-flat phase transition and the flat phase of \( D \)-dimensional isotropic membranes embedded in a \( d \)-dimensional space [22] (see also [23]). One of its important features is that it does not rely on the proximity of an upper or lower critical dimension. This is the reason why it has allowed to confidently investigate the physics of membranes in the whole \((D, d)\) plane [22]. Such a method is appropriate in the present context where the \( \epsilon \)-expansion displays a pathological behavior.

The NPRG approach relies on the use of a running effective action [21] (see [24–28] for reviews), \( \Gamma_{k}[\mathbf{r}] \), a functional of \( \mathbf{r} = \mathbf{r}(\mathbf{x}) \) a \( D \)-dimensional external vector that describes the membrane in the embedding space, \( \mathbf{x} \) being a set of internal \( D \)-dimensional coordinates labeling a point of the membrane. The index \( k \) stands for a running scale that separates the high-momentum, with \( q > k \), from the low-momentum ones, with \( q < k \) and \( \Gamma_{k}[\mathbf{r}] \) represents a coarse grained free energy where only fluctuations with momenta \( q \geq k \) have been integrated out. The running of \( k \) towards the value \( k = 0 \) thus corresponds to gradually integrate more and more low-momentum fluctuations. The \( k \)-dependence, RG flow, of \( \Gamma_{k} \) is provided by an exact evolution equation [21]:

\[
\frac{\partial \Gamma_{k}}{\partial t} = \frac{1}{2} \text{Tr} \left\{ (\Gamma_{k}^{(2)} + R_{k})^{-1} i \frac{\partial R_{k}}{\partial t} \right\}
\]

where \( t = \ln k/\Lambda \), \( \Lambda \) being some microscopic, lattice, scale. The trace in (1) involves a \( D \)-dimensional momentum integral as well as a summation over vectorial indices. The function \( R_{k}(q) \) realizes the split between low- and high-momentum degrees of freedom. Several forms of \( R_{k}(q) \) will be considered in the following. Finally \( \Gamma_{k}^{(2)} \) is the second functional derivative of \( \Gamma_{k}[\mathbf{r}] \) with respect to \( \mathbf{r} \).

The effective action \( \Gamma_{k}[\mathbf{r}] \) relevant to study polymerized membranes must be invariant under the group of translations, which implies that it only depends on powers of the tangent vectors \( \partial_{\mu} \mathbf{r} \). For anisotropic membranes rotational invariance within the membrane is explicitly broken between one direction - named \( y \) - and the remaining \( D - 1 \) dimensions which are kept isotropic. The effective action thus writes:

\[
\Gamma_{k}[\mathbf{r}] = \int d^{D-1}x \; dy \left\{ \frac{Z_{y}}{2} (\partial_{\mu}^{2} \mathbf{r})^{2} + \tau_{\perp} (\partial_{\mu} \mathbf{r})^{2} \right\} + \frac{u_{y}}{2} (\partial_{\mu} \mathbf{r}, \partial_{\nu} \mathbf{r} - \zeta_{y}^{2})^{2}
\]

where \( Z_{y}, u_{y}, \tau_{\perp} \) and \( \zeta_{y} \) are running coupling constants: \( Z_{y} \) is a bending rigidity, \( u_{y} \) an elastic constant and \( \tau_{\perp} \) a tension term in the \( \perp \)-directions (indexed by \( \mu \) that runs from 1 to \( D - 1 \)). This last term, that plays the role of a temperature in the \( \perp \)-directions, is kept non-vanishing since these directions are not critical. Finally \( \zeta_{y} \), or rather the temperature along \( y \), \( t_{y} \equiv u_{y} \zeta_{y}^{2} \), parametrizes the approach to criticality in the \( y \)-direction. Note that, up to change of coupling constants, this action is that used in [14, 15]. Let us consider the mean-field treatment of this model for \( u_{y} > 0 \). When \( \zeta_{y} = 0 \), the minimum of \( \Gamma_{k} \) is given by a configuration where \( \partial_{y} \mathbf{r} \) vanishes which characterizes a crumpled, disordered, phase. When \( \zeta_{y} > 0 \), this minimum is given by a configuration \( \mathbf{r}(\mathbf{x}) = \zeta_{y} \mathbf{y} \mathbf{e}_{y} \) where \( \mathbf{e}_{y} \) is a unit vector that spans the one-dimensional submanifold along which long-range order occurs. Thus, when the temperature \( t_{y} \) is lowered, the system undergoes a phase transition between a crumpled phase at high temperature, with \( \zeta_{y} = 0 \), and a tubular phase at a low, intermediate, temperature, with \( \zeta_{y} \neq 0 \). When the temperature is further lowered, one recovers an isotropic flat phase along all the \( D \)-directions, see Fig.(1). A specificity of the model is that at the crumpling-to-tubule transition power counting leads to \( q_{\perp} \propto Q_{y}^{2} \), an anisotropic scaling which characterizes a Lifshitz-type behavior [29]. According to this scaling behavior the most relevant terms are those entering in Eq.(2). Note that since we treat nonperturbatively this action we are not supposed to base our computation on power-counting arguments. We do this nevertheless since i) the proximity of \( D = 2 \) with the upper critical dimension \( D_{uc} = 5/2 \) leads to guess that the terms present in Eq.(2) play the major role ii) we wish to compare our approach with the previous perturbative ones [14, 15]. The assumption i) must, of course, be checked, which can be partly done by evaluating the sensitivity of the results with respect to modification of the cut-off function \( R_{k}(q) \).

Let us now define the critical quantities. First, due to the anisotropy between the \( y \)-direction and the remain-
ing $D - 1$ $\perp$-directions one has the scale transformations: $x_\perp = k x'_\perp$ and $y = k^\eta y'$ that define the anisotropic scaling exponent $\eta$. Under a RG transformation the field $\vec{r}$ acquires an anomalous dimension $\eta$ such that $Z_y \sim k^{-\eta z}$, while $t_\perp \sim k^{-\eta_\perp}$ so the field $\vec{r}$ generically scales as: $\vec{r} = k^{(D-1-3\eta)/2} \vec{r}'$. One deduces the following relation between $z$, $\eta$ and $\eta_\perp$: $z = (2 - \eta_\perp)/(4 - \eta)$. Finally one defines the exponents $\eta_y$ and $\nu_\perp$ from the correlation length near criticality: $\xi_y \propto t_\perp^{-\nu_y}$ and $\xi_\perp \propto t_\perp^{-\nu_\perp}$ with $\nu_y = z \nu_\perp$ [15]. The flow equations for the couplings $Z_y$, $u_y$, $t_\perp$ and $\xi_y$ are obtained using their definitions in terms of functional derivatives of the effective action and applying the RG Eq. (1). One defines the dimensionless quantities: $\tilde{\xi}_y = k^{3-2D} Z_y \tilde{\xi}_y - t_\perp^2 \xi_y$ and $\tilde{\eta}_y = k^{2D - 5} \frac{\tilde{\xi}_y}{Z_y - \frac{1-D}{2} u_y}$ with $k = k^\perp$. In terms of these quantities the RG equations write, with $t = \ln k_y/\Lambda$:

\[
\begin{align*}
\partial_t t_\perp &= 0 \\
\partial_t \tilde{\xi}_y &= - (2D - 3 - \eta(D-3)/2) \xi_y^2 \\
&\quad - (d-1) \frac{\xi_y^2}{\tilde{t}_0} - 3 \frac{\xi_y^2}{\tilde{t}_{0,0}} \\
\partial_t \tilde{\eta}_y &= - (5 - 2D - \eta(5-D)/2) \eta_y + \\
&\quad (D-3) \frac{\tilde{\eta}_y^2}{\tilde{t}_{0,0}} - (d-1) \frac{\tilde{t}_{0,0}}{\tilde{t}_{0,0}} + \eta \frac{\tilde{t}_{0,0}}{\tilde{t}_{0,0}}
\end{align*}
\]

(3)

while the anomalous dimension function $\eta$ writes:

\[
\eta = \frac{1}{3} (D - 3) \tilde{\eta}_y \xi_y^2 - 108 \frac{\tilde{t}_{0,0}}{\tilde{t}_{0,0}} - 12(d-1) \frac{\tilde{t}_{0,0}}{\tilde{t}_{0,0}} - 540(D-5) \frac{\tilde{\eta}_y \xi_y \tilde{t}_{0,0}}{\tilde{t}_{0,0}} + (D-5)(D-7) \left( -288 \frac{\tilde{\eta}_y \xi_y}{\tilde{t}_{0,0}} + 9 \frac{\tilde{t}_{0,0}}{\tilde{t}_{0,0}} \right) + (d-1) \frac{\tilde{t}_{0,0}}{\tilde{t}_{0,0}} - 36 \frac{\tilde{\eta}_y \xi_y \tilde{t}_{0,0}}{\tilde{t}_{0,0}} \right). \tag{4}
\]

In Eqs. (3) and (4) $\tilde{t}_{0,0}^{\alpha,\beta}$, $\tilde{\eta}_{0,0}^{\alpha,\beta}$ and $\tilde{M}_{0,0}^{\alpha,\beta}$ are the de-dimensionlized "threshold functions" (see [25, 26]) $t_{0,0}^{\alpha,\beta}$, $\eta_{0,0}^{\alpha,\beta}$ and $M_{0,0}^{\alpha,\beta}$ that are given by:

\[
T_{0,0}^{\alpha,\beta} = K_D \frac{\partial}{\partial t} \int dq_y \tilde{q}_y^2 \frac{F[q_y]}{[P(q_y)]^a [P(q_y) + m^2 q_y^2]^b} \tag{5}
\]

where $K_D = (\pi/2)^{(D-1)/2} \Gamma[(3-D)/2]$, $P(q_y) = Z_y q_y + R_c(q_y)$, $m_y^2 = 4w_C q_y^2$ and where $\partial/\partial t$ only acts on $R_c$. In Eq.(5) the function $F[q_y]$ is given by 1, $dP/dq_y^2$ and $(dP/dq_y^2)^2$ for $l$, $N$ and $M$ respectively. These functions control the relative weight of the different modes: the single-phonon mode of mass $m_y$ and the $d-1$ capillary modes of zero mass, along the RG flow. Note, that these functions only depend on $q_y$. Indeed $\partial_\eta^l$ enters only quadratically in the action (2) so that one can exactly perform the integration on the $D - 1 \perp$-degrees of freedom in the $\beta$ functions.

Let us now discuss our results. First we find no renormalization of $t_\perp$, i.e. $\eta_\perp = 0$, for any value of $D$, in agreement with the all-order perturbative result [15]. Due to the one-loop structure of the RG equation (1) we recover the one-loop $\beta$ function for $\tilde{\xi}_y$ and $\tilde{\eta}_y = \eta \tilde{\xi}_y \tilde{t}_{0,0}$ found in [15] using our equations (3) expanded around $D = 5/2 - \epsilon$:

\[
\begin{align*}
\partial_t \tilde{\xi_y}^2 &= -2 \tilde{\xi}_{y}^2 + \frac{2}{3} (d + 2) K_{5/2} \\
\partial_t \tilde{\eta}_y &= -2 \epsilon \tilde{\eta}_y + (d + 8) K_{5/2} \tilde{\xi}_y^2
\end{align*}
\]

with, at this order, a vanishing anomalous dimension $\eta^\alpha$. Thanks again to the one-loop structure of (1), one gets the large-$d$ results from the set of equations (3)-(4), without almost any more computation. In this limit, assuming that $\tilde{\xi}_y \sim O(1/d)$ and $\tilde{\eta}_y^\alpha \sim O(d)$ one finds, in agreement with this assumption, and using, e.g., the cut-off function $R_k(q_y) = Z_y k^4 - q_y^4 \theta(k^2 - Q_0^2)$, a stable fixed point with coordinates:

\[
\begin{align*}
\tilde{\xi}_y^\alpha &= \frac{4 d}{3} (3 - D) K_{D} \\
\tilde{\eta}_y &= \frac{5}{4d} (5 - 2D) K_{D} \tilde{\xi}_y^\alpha
\end{align*}
\]

This fixed point exists for all values of $D$ between 5/2 and the lower critical dimension $D = 3/2$ (see [15]). The corresponding critical exponents are given by: $\nu = 1/(2D - 3) + O(1/d)$ and $\eta = O(1/d)$.

Going to finite values of $d$ one finds a stable fixed point for any value of $d$ in contrast to the isotropic case where it occurs only for $d$ greater that some critical value [5] $d_c(D)$ determined in [22]. One now specializes to the $d = 3$ case. One finds, for any dimension $D < 5/2$, a nontrivial anomalous dimension $\eta^\alpha$ from Eq.(4) at the fixed point. This exponent is found positive in any dimension between $D = 5/2$ and $D = 3/2$, in agreement with what is expected. The critical exponents $\eta^\alpha$ and $\nu_\perp$ in $D = 2$ are displayed in Figs.(2)-(3). They are represented as functions of a real number $\lambda$ that parametrizes three families of cut-off functions among those that we have used: $R_k^i(q_y) = \lambda R_k^i(q_y)$, $i = 1, 2, 3$ with $R_k^1(q_y) = Z_y k^4 \exp(-q_y^4/k^4)$, $R_k^2(q_y) = Z_y/\exp(q_y^4/k^4) - 1$ and $R_k^3(q_y) = Z_y(k^4 - q_y^4)\theta(k^2 - q_y^2)$. Variations of $\lambda$ allow to investigate the cut-off dependence of the critical quantities and to optimize each cut-off function inside its family [30], i.e. to (try to) find stationary values of the critical quantities. For each exponent one succeeds to find a single station point for closed values of $\lambda$, see Figs.(2)-(3). For $\nu_\perp$ one can extract from these considerations the value $\nu_\perp = 1.213(8)$ in agreement with
that found in [15]: $\nu_\perp \simeq 1.227$. For the anomalous dimension one finds $\eta^* \simeq 0.358(4)$ which differs considerably from the value found in [15]: $\eta \simeq -0.0015$. Finally one deduces from these results and the scaling relations: $z = 0.5490(6)$ and $\nu_y = 0.665(5)$ to be compared to $\nu_y \simeq 0.614$. Note that variations of the critical quantities appear to be very smooth with both variations $i)$ of the parameter $\lambda$ inside a family of cut-off function and $ii)$ of the family itself, see Figs.(2)-(3). This large insensitivity with respect to the cut-off functions constitutes a strong indication of trustability of our results, in agreement with the proximity of the upper critical dimension.

We have shown that the NPRG allows to overcome the main difficulty that plague the perturbative, weak coupling, expansion for anisotropic membranes. Our predictions can be easily tested through numerical investigations. This would validate the NPRG approach as an efficient alternative to the perturbative one, in particular in the perspective of probing the properties of membrane-like systems with various orders and geometries.

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FIG. 2: The exponent $\nu_\perp$ as function of the parameter $\lambda$. In solid line $\tilde{R}_1^\perp(q_y)$, in dashed line $\tilde{R}_2^\perp(q_y)$ and in dot-dashed line $\tilde{R}_3^\perp(q_y)$. The dot on each curve corresponds to a stationary value of the exponent.

FIG. 3: The exponent $\eta$ as function of the parameter $\lambda$. The conventions are the same as in Fig.(2).

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