Character Formulae for Queer Lie Superalgebras and Canonical Bases of Types $A/C$

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Abstract: For the BGG category of $q(n)$-modules of half-integer weights, a Kazhdan–Lusztig conjecture à la Brundan is formulated in terms of categorical canonical basis of the $n$th tensor power of the natural representation of the quantum group of type $C$. For the BGG category of $q(n)$-modules of congruent non-integral weights, a Kazhdan–Lusztig conjecture is formulated in terms of canonical basis of a mixed tensor of the natural representation and its dual of the quantum group of type $A$. We also establish a character formula for the finite-dimensional irreducible $q(n)$-modules of half-integer weights in terms of type $C$ canonical basis of the corresponding $q$-wedge space.

Contents

1. Introduction .................................. 1091
2. The Tensor and Wedge Modules of Quantum Groups of Type $C$ ........ 1094
3. Canonical Basis on the Tensor Module and Wedge Module .............. 1097
4. Comparison of Canonical Bases of Types $A$ and $C$ ...................... 1101
5. Kazhdan–Lusztig Theory for Queer Lie superalgebra $q(n)$ .............. 1105
Appendix A. Category $\mathcal{O}_{n,\mathbb{Z}}$ and Canonical Bases ................ 1114
References ..................................... 1118

1. Introduction

1.1. The celebrated Kazhdan–Lusztig theory for semisimple Lie algebras is formulated in terms of canonical bases of the Hecke algebras associated to the corresponding Weyl
groups. For basic Lie superalgebras (which can be viewed as super counterparts of semisimple Lie algebras) as well as for the queer Lie superalgebras \( q(n) \) (see, e.g., [CW2]), triangular decomposition is available, which allows one to define the BGG category \( \mathcal{O} \). However, the linkage in \( \mathcal{O} \) is no longer controlled by the Weyl groups, and thus the conventional formulation of Kazhdan–Lusztig theory does not apply to Lie superalgebras.

In [CLW] Lam and two of the authors proved Brundan’s Kazhdan–Lusztig conjecture for the BGG category \( \mathcal{O} \) of integer-weight modules for the Lie superalgebra \( \mathfrak{gl}(m|n) \) (also see [BLW]); Brundan’s conjecture [Br1] is formulated in terms of the type A canonical basis of a tensor product module [Lu2]. It is shown in [CMW] that the irreducible character problem for the BGG category of \( \mathfrak{gl}(m|n) \)-modules of arbitrary weights can be reduced to the setting of integer weights by using various category equivalences induced by twisting, odd reflection, and parabolic induction functors.

A Kazhdan–Lusztig theory for the Lie superalgebras \( \mathfrak{osp}(2m+1|2n) \) has recently been formulated and established in [BW] via new canonical bases arising from quantum symmetric pairs. This approach has been adapted further to establish a Kazhdan–Lusztig theory for \( \mathfrak{osp}(2m|2n) \) [B]. In these papers, two variants of the category \( \mathcal{O} \) for every \( \mathfrak{osp} \)-type superalgebra were considered, one with integer weights and another with half-integer weights, and their corresponding Kazhdan–Lusztig theories utilize canonical bases from somewhat different quantum algebras.

1.2. An analogous (still open) conjecture for the characters of irreducible \( q(n) \)-modules of integer weights in the category \( \mathcal{O} \) was formulated in [Br2] in terms of a type B canonical basis [Lu1,Lu3,Ka]; see Remark A.3 for an update and a modification. Brundan established a character formula for the finite-dimensional irreducible modules of integer weights in terms of type B dual canonical basis of a \( q \)-deformed wedge space. However, there is no reason to restrict oneself to the integer-weight modules only. Characters for the finite-dimensional irreducible \( q(n) \)-modules of half-integer weights have recently been obtained in [CK], and they were shown to be closely related to those of the finite-dimensional irreducible characters of integer weights given in [Br2]. We remark here that an algorithm for all finite-dimensional irreducible characters of \( q(n) \)-modules was developed earlier in [PS].

We are interested in understanding the characters of irreducible modules in the whole BGG category of \( q(n) \)-modules, but not just those with integer-weight highest weights. A reduction similar to (but more complicated than) [CMW] is established for a queer Lie superalgebra by Chen in [Ch]. By Chen’s result, which confirmed our expectation, the problem of computing the characters of the irreducible modules of arbitrary highest weights in the category \( \mathcal{O} \) for a queer Lie superalgebra is reduced to the problem of computing them in the following three categories: (i) a BGG category \( \mathcal{O}_{n,Z} \) of the \( q(n) \)-modules of integer weights (see the main conjecture of [Br2, (4.56)]), (ii) a BGG category \( \mathcal{O}_{n,\mathbb{Z}/2\mathbb{Z}} \) (where \( \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2 \mathbb{Z} \)) of the \( q(n) \)-modules of half-integer weights, (iii) a BGG category \( \mathcal{O}_{n,s} \) of the \( q(n) \)-modules of “congruent \( \pm s \)-weights”, for \( s \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z} \) and \( l \in \{0, 1, \ldots, n\} \) (see Sect. 5.7 for a precise definition).

1.3. A main goal of this paper is to formulate a Kazhdan–Lusztig conjecture for the categories \( \mathcal{O}_{n,Z} \) (Conjecture 5.12) and \( \mathcal{O}_{n,s} \) (Conjecture 5.14). We also prove results and formulate various conjectures on connections among the canonical bases of types \( A/B/C \), which seem to indicate new connections between representations of \( q(n) \) and