RIGID IDEALS BY DEFORMING QUADRATIC LETTERPLACE IDEALS

GUNNAR FLØYSTAD AND AMIN NEMATBAKHSH

Abstract. We compute the deformation space of quadratic letterplace ideals \( L(2, P) \) of finite posets \( P \) when its Hasse diagram is a rooted tree. These deformations are unobstructed. The deformed family has a polynomial ring as the base ring. The ideal \( J(2, P) \) defining the full family of deformations is a rigid ideal and we compute it explicitly. In simple example cases \( J(2, P) \) is the ideal of maximal minors of a generic matrix, the Pfaffians of a skew-symmetric matrix, and a ladder determinantal ideal.

1. Introduction

Monomial ideal theory has much developed into a branch of its own. But before that one studied polynomial ideals in general. Monomial ideals came about since they are specializations, typically initial ideals, of such ideals. One should then ask: Can monomial ideal theory give something back? Can one start with monomial ideals and derive interesting classes of polynomial ideals in general? Yes one can, and here we do this for a reasonably large class of monomial ideals. We get a full understanding of the polynomial ideals which specialize to the monomial ideals we start out from.

The ideals we work with. More precisely we consider quadratically generated letterplace ideals \( L(2, P) \) associated to a finite poset \( P \). These are precisely the edge ideals of Cohen-Macaulay bipartite graphs. Its generators are the monomials \( x_{1,p}x_{2,q} \) where \( p \leq q \) in the poset \( P \). That edge ideals of Cohen-Macaulay bipartite graphs have this form, is an astonishing discovery of J.Herzog and T.Hibi [13]. This class of ideals were generalized in [9] and further studied and generalized in [10] were they were called letterplace ideals, see Section 2.

Results. When the Hasse diagram of \( P \) has the form of a rooted tree we get a complete algebraic understanding of all ideals which are deformations of the quadratic letterplace ideals \( L(2, P) \). This is all the more unusual and suprising for the following reason: Monomial ideals are degenerations of polynomial ideals. Thus whenever a monomial ideal is on a Hilbert scheme, it tends to be a singular point on the Hilbert scheme. Its infinitesimal deformations are then obstructed and it is a very hard and messy task to compute the space of all deformations.

However for the letterplace ideals \( L(2, P) \) we consider, it turns out that every nice thing one could wish for, actually happens:

- The ideals \( L(2, P) \) are unobstructed, i.e. every infinitesimal deformations lifts. In particular whenever this ideal is on a Hilbert scheme, it is a smooth point.
- The full deformation space which a priori is defined only over a complete local ring, actually lifts to a deformation over a polynomial ring. This follows from our computation of the first cotangent cohomology of \( L(2, P) \), Corollary 6.11, and the explicit family we give in Section 3.

Date: May 25, 2016.
• The full family of deformations over this polynomial ring is defined by a rigid ideal \( J(2, P) \), Corollary 8.3. So deformations of \( L(2, P) \) come from a coordinate change in \( J(2, P) \).

• We explicitly compute the ideal \( J(2, P) \) by a simple recursive procedure, see Section 3 and Equation 3.1.

Deforming Borel-fixed ideals. As said monomial ideals are usually obstructed. Any ideal can be specialized to a Borel-fixed monomial ideal, which in characteristic zero is the same as a strongly stable ideal. One could then envision a path to classify ideals by deforming strongly stable ideals. Unfortunately this is rather hopeless since strongly stable ideals typically are very obstructed. (A notable exception to this is the lexsegment ideal. When the polynomial ring is given the standard grading the lexsegment ideal is a smooth point on the Hilbert scheme \([17]\), thus giving a distinguished component of the Hilbert scheme for every Hilbert function \( h : \mathbb{Z} \to \mathbb{N} \).)

Deforming Stanley-Reisner ideals. Deformation theory applied to Stanley-Reisner ideals has been developed by K.Altman and J.Christophersen. In \([2]\) they give the basic deformation theory for Stanley-Reisner rings. In \([3]\) and \([5]\) they consider triangulations of spheres, which deform to Calabi-Yau manifolds, and triangulations of tori, which deform to Abelian varieties. For classes of triangulations they compute the versal deformation space (base space) of the (infinitesimal) deformation functor. This space is typically not smooth, i.e. typically not a power series ring, but they give equations for the relations of this space, and give a detailed description of it. Here, for quadratic letterplace ideals, we find that the base space both is smooth and global and that we can give explicit equations for the whole family of deformations. Recently \([1]\) applied the theory developed by Christophersen and Altman to investigate when monomial ideals are rigid. For edge ideals they develop a number of results for when this holds. They also classify the (few) letterplace ideals which are rigid.

Rigid ideals. The notion of a rigid ideal occurs in (infinitesimal) deformation theory. Although it is a well-known notion, we have not been able to find many examples of rigid ideals in the literature. Classically determinantal ideals of generic matrices are known to be rigid, \([4]\). Recently \([6]\) shows that the coordinate rings of Grassmannians for the Plücker embedding are rigid ideals. As mentioned above \([1]\) also gives classes of rigid monomial ideals. With the present article we therefore believe we make a substantial contribution to the known classes of rigid ideals.

Multigraded Hilbert schemes. While rigidity is an infinitesimal notion, one obtains global families of deformations, the (multigraded) Hilbert schemes, when one endows the ambient polynomial ring with a grading by an abelian group \( A \) \([11]\). The \( A \)-graded infinitesimal deformations are a subset of all infinitesimal deformations. For global families there is then a situation close to rigidity. An ideal \( I \subseteq \mathbb{k}[X] \) may not be rigid, but there nevertheless is a rigid ideal \( J \) in a larger polynomial ring \( \mathbb{k}[Y] \) such that any deformation of \( I \) comes from a coordinate change in \( J \) and then restricting to \( \mathbb{k}[X] \). J.Kleppe \([15]\) shows that this is the case when \( I \) is a determinantal ideal associated to a matrix of linear forms, but where there may be dependencies between the linear entries. We show that the same phenomenon happens here when we consider the letterplace ideals \( I = L(2, P) \) and their \( A \)-graded deformations, Theorem 9.5 and the applications after it.

Organization of the paper. In Section 2 we recall letterplace ideals as defined in \([10]\). In this article we are concerned with the quadratic letterplace ideals \( L(2, P) \) where the Hasse
diagram of $P$ is a rooted tree. In Section 3 we give an explicit recursive procedure for computing the family $J(2, P)$ of deformations of $L(2, P)$. Section 4 contains examples of these deformed families for posets $P$ of cardinality 3 and 4, and also for two other simple classes of posets. In the first cases we get the ideal of two-minors of a $2 \times 5$-matrix and the ideal of Pfaffians of a skew-symmetric $5 \times 5$-matrix. Section 5 shows that the family of ideals $J(2, P)$ is very finely graded, by a free abelian group of cardinality $2|P|$. Section 6 investigates the deformation theory of the ideals $L(2, P)$ and we compute the non-trivial first order deformations of $L(2, P)$ for any finite poset $P$. These are given by the first cotangent cohomology group. This module turns out to have an extremely nice set of generators. For each generator there is a single monomial $x_{1,p}x_{2,p}$ mapping to another monomial while all other monomials map to zero. Section 7 shows flatness of $J(2, P)$ over the base polynomial ring. In Section 8 we show rigidity of $J(2, P)$ (an infinitesimal notion). Section 9 considers global families of deformations. We show that the letterplace ideals $L(2, P)$ are smooth points on the Hilbert schemes and that the general point on the Hilbert scheme comes from the ideal $J(2, P)$ by a coordinate change and then restricting. Sections 8 and 9 are developed in a general setting, and the results concerning $L(2, P)$ are just particular instances of general results. In the end we give Conjecture 9.8, that the results of this article holds for any finite poset $P$ and not just for posets whose Hasse diagram is a rooted tree.

Acknowledgement. We thank Jan Christophersen for useful discussions which significantly influenced the form of this paper.

2. Letterplace ideals of posets

Let $k$ be a field. If $R$ is a set, denote by $k[x_R]$ the polynomial ring $k[x_{i \in R}]$. For a natural number $n$ let the chain poset be $[n] = \{1 < 2 < \cdots < n\}$, so $[2] = \{1, 2\}$.

Given a finite poset $P$. We shall in this paper be concerned with the monomial ideal $L(2, P)$ in $k[x_{[2] \times P}]$ generated by quadratic monomials $x_{1,p}x_{2,p}$ where $p \leq q$. These ideals are by [13] precisely the edge ideals of Cohen-Macaulay bipartite graphs, see Section 9 of [14] for more on this. The ideals $L(2, P)$ are a special case of letterplace ideals $L(n, P)$ introduced in [9] and [10], generated by monomials

$$x_{1,p_1}x_{2,p_2}, \cdots, x_{n,p_n}$$

where $p_1 \leq \cdots \leq p_n$ are weakly increasing chains in $P$. By [9, Corollary 2.5], see also [10, Corollary 2.4], $L(n, P)$ is a Cohen-Macaulay ideal of codimension equal to the cardinality $|P|$. The multiplicity of $L(2, P)$ is the cardinality of the distributive lattice of poset ideals of $P$, see Section 2 of [8]. This is the same as the degree of the algebraic subscheme of the affine space $k^{2|P|}$ defined by $L(2, P)$.

For ease of notation we shall rather write a variable $x_{i,p}$ as $p_i$. Thus $L(2, P)$ is generated by quadratic monomials $p_1q_2$ where $p \leq q$.

For more on letterplace ideals $L(n, P)$ and their Alexander duals $L(P,n)$ and the omnipresence of these monomial ideals, see [10]. In [8] we compute the Betti tables of the letterplace ideals $L(n, P)$, in particular of $L(2, P)$.

3. The family of deformations

We here describe the main object of study in this article: the ideals $J(2, P)$ which are deformations of the letterplace ideals $L(2, P)$. The generators of $L(2, P)$ are monomials $p_1q_2$ where $p \leq q$. We shall deform each such generator, and the ideal $J(2, P)$ will be
the ideal generated by these deformations. We do this for the situation that the Hasse diagram of the poset $P$ is a rooted tree. Except for Section 5, this is our assumption throughout the paper.

The root of $P$ is at the bottom. If an element $b$ covers $a$ we say that $a$ is a parent of $b$ and we write $a \prec b$. Two elements $b$ and $c$ are called siblings if they have the same parent.

For each pair $q,p$ where the meet of $q$ and $p$ is the parent of $p$, we introduce a variable $u_{q,p}$. Let $b$ and $c$ be distinct siblings. We define

$$T_c(b) = - \sum_{q \geq c} q_2 u_{q,b}.$$ 

If $a$ is a parent of $b$ we let

$$T_a(b) = -a_2 u_{a,b}.$$ 

We also define

$$T(b) = T_b(b) = -T_a(b) - \sum_c T_c(b),$$

where we sum over siblings $c$ of $b$, distinct from $b$. If $\rho$ is the root of $P$ we define

$$T(\rho) = u_{\emptyset,\rho}.$$

The rationale for introducing these variables will become clear in Subsection 6.4 where we compute the cotangent cohomology of the ring $k[x_{[2]} \times P]/L(2,P)$, Corollary 6.11. Let $B$ be the polynomial ring in these variables

$$B = k[u_{\emptyset,\rho}, u_{q,p}]$$

ranging over all pairs $(q,p)$ such that the meet of $q$ and $p$ is the parent of $p$. This will be the base ring for our family of deformations. Let $B(2,P)$ be the ring $B \otimes_k k[x_{[2]} \times P]$. This is the ring where the ideal of the full deformation family lives.

For an element $p$ in the poset $P$ we define the depth of $p$, depth$(p)$ to be the length of the longest chain upwards, starting from $p$. Thus if $p$ is a maximal element, then depth$(p) = 0$.

We now define the following:

- If $a \prec b$ so $a$ is the parent of $b$, we shall define determinants $D(a)^b$ lying in $B(2,P)$, as well as determinants $D(a)^a$.
- If $a \leq b$ we shall define polynomials $S_a(b_2)$ lying in $B(2,P)$. Then $S_a$ extends uniquely to a linear map on linear combinations of elements $b_2$ where $b \geq a$. For short we shall often write $S_a(b)$ for $S_a(b_2)$.

We shall do this inductively on depth$(a)$.

Given these definitions, we also define the following:

- If $b$ and $c$ are distinct siblings let

$$S_c T_c(b) = S_c(T_c(b)).$$

This definition will appear by induction on depth$(c)$ as we define $S_c$.

We also define the following:

- $S_b T_b(b) = b_1$.
- If $a \prec b$ so $a$ is the parent of $b$, let $S_a T_a(b) = -u_{a,b}$. 

Note that these last two definitions are not compositions, i.e. $S_b T_b(b)$ is not $S_b(T_b(b))$. Rather we think of these definitions as symbolic expressions.

Now let us start the inductive definitions. If $a$ is maximal, that is depth$(a) = 0$ we define $D_a(a) = 1$ and $S_a(a) = 1$.

Otherwise let $b^1, \ldots, b^m$ be the children of $a$. For uniformity we denote $b^0 = a$. We form the $m \times (m + 1)$ matrix $M(a) = [S_b T_b(b)]$ where the column index $i = 0, \ldots, m$ and the row index $j = 1, \ldots, m$.

Let $M(a)^{b^i}$ be the matrix obtained by deleting column $i$. Define the signed determinant

$$D(a)^i = D(a)^{b^i} = (-1)^i |M(a)^{b^i}|.$$ 

Note that in order to define this, we need to have defined $S_p$ for all $p$ strictly bigger than $a$.

For $a \leq b$ define $R(a, b) = 1$ if $a = b$ and if $a < b$ define

$$R(a, b) = \prod_{a \leq p < q \leq b} D(p)^q$$

where the product is over all covering relations $p < q$ between $a$ and $b$.

Now define

$$S_a(b_2) = R(a, b)D(b).$$

As said before this definition extends in a natural way to linear combinations of variables $b_2$ such that $b \geq a$. Also we shall often for short write $S_a(b)$ for $S_a(b_2)$. This completes the inductive definitions. Now we give the ideal defining the full family of deformations of $L(2, P)$.

**Definition 3.1.** Let $J(2, P)$ be the ideal in $B(2, P)$ generated by

$$p_1 q_2 - T(p) S_p(q)$$

for all $p \leq q$.

The following shows the various entities defined, in the simplest situation. It is useful in the next section where we give examples of the ideals $J(2, P)$.

**Lemma 3.2.** Suppose $a \in P$ has a single child $b$. Then:

a. $S_a(a) = b_1$,  
b. $D(a)^a = b_1$,  
c. $D(a)^b = u_{a, b}$,  
d. $T(b) = a_2 u_{a, b}$.

**Proof.** In this case the matrix $M(a) = [−u_{a, b} \quad b_1]$. □
We shall in the Section 7 show that the ring $B(2,P)/J(2,P)$ is a deformation of the letterplace ring $k[2,P]/L(2,P)$, flat over the base ring $B$.

4. Examples

We consider here four examples of posets and give the deformed equations explicitly. We also identify the variety they define.

4.1. Determinantal variety. Let $P$ be the totally ordered poset $[n] = \{1 < 2 < \cdots < n\}$. For simplicity we assume $n = 4$ and write $P = \{a < b < c < d\}$.

The deformations of monomials $p_1p_2$ for $p \in P$ are

$$p_1p_2 - T(p)S_p(p).$$

Since in this case $p$ has one or none child, and also no sibling, we apply Lemma 3.2 and these deformations are:

$$a_1a_2 - u_{\emptyset,a}b_1$$
$$b_1b_2 - a_2u_{a,b}c_1$$
$$c_1c_2 - b_2u_{b,c}d_1$$
$$d_1d_2 - c_2u_{c,d}.$$

Furthermore we have the deformed polynomials:

$$a_1b_2 - u_{\emptyset,a}u_{a,b}c_1$$
$$b_1c_2 - a_2u_{a,b}u_{b,c}d_1$$
$$c_1d_2 - b_2u_{b,c}u_{c,d}.$$

These binomials are the 2-minors of the following $2 \times 5$ matrix

$$\begin{bmatrix}
a_1 & b_1 & u_{a,b}c_1 & u_{a,b}d_1 & u_{a,b}u_{b,c}u_{c,d} \\
u_{\emptyset,a} & a_2 & b_2 & c_2 & d_2
\end{bmatrix},$$

after we localize by inverting $u_{a,b}$ and $u_{b,c}$. This matrix can also be written as:

$$\begin{bmatrix}
a_1 & S_a(a) & S_a(b) & S_a(c) & S_a(d) \\
u_{\emptyset,a} & a_2 & b_2 & c_2 & d_2
\end{bmatrix}.$$

4.2. The Pfaffians of $5 \times 5$ skew-symmetric matrices. Consider the star poset $P$

We will show the deformations when the root has two or three children. First suppose we have two children so the poset is
The matrix $M(a)$ is:

$$
\begin{bmatrix}
S_a T_a(b) & S_b T_b(b) & S_c T_c(b) \\
S_a T_a(c) & S_b T_b(c) & S_c T_c(c)
\end{bmatrix} = 
\begin{bmatrix}
-u_{a,b} & b_1 & -u_{c,b} \\
-u_{a,c} & -u_{b,c} & c_1
\end{bmatrix}.
$$

There are five generating monomials in $L(2,P)$ and the deformed polynomials are:

- $b_1b_2 + T(b) \cdot 1$
- $c_1c_2 + T(c) \cdot 1$
- $a_1b_2 + T(a) S_a(b)$
- $a_1c_2 + T(a) S_a(c)$
- $a_1a_2 + T(a) S_a(a)$

which are:

- $b_1b_2 - a_2u_{a,b} - c_2u_{c,b}$
- $c_1c_2 - a_2u_{a,c} - b_2u_{b,c}$
- $a_1b_2 - u_{\emptyset,a}u_{a,b}u_{c,b} - u_{\emptyset,a}u_{a,b}c_1$
- $a_1c_2 - u_{\emptyset,a}u_{a,b}u_{b,c} - u_{\emptyset,a}u_{a,c}b_1$
- $a_1a_2 - u_{\emptyset,a}b_1c_1 + u_{\emptyset,a}u_{c,b}u_{b,c}$

These are (after diving by $u_{\emptyset,a}$, the Pfaffians of the following skew-symmetric matrix:

$$
\begin{bmatrix}
0 & u_{b,c} & c_2 & b_1 & a_1 \\
-u_{b,c} & 0 & u_{a,c} & -1 & u_{\emptyset,a}a_1 & c_1 \\
-c_2 & -u_{a,c} & 0 & u_{a,b} & b_2 \\
-b_1 & -u_{\emptyset,a}a_1 & -u_{a,b} & 0 & u_{c,b} \\
-a_2 & -c_1 & -b_2 & -u_{c,b} & 0
\end{bmatrix}.
$$

Setting $u_{\emptyset,a} = 1$, they are also the Plücker relations defining the Grassmann variety $G(2,5)$ embedded in projective space $\mathbb{P}^9$.

Now let us consider the case of the star poset $P$ with three children:

```
          b  c  d
          |
          |
          a
```

The matrix $M(a)$ is:

$$
\begin{bmatrix}
-u_{a,b} & b_1 & -u_{c,b} & -u_{d,b} \\
-u_{a,c} & -u_{b,c} & c_1 & -u_{d,c} \\
-u_{a,d} & -u_{b,d} & -u_{c,d} & d_1
\end{bmatrix},
$$

and the $D(a)^x$ are the signed maximal minors of this matrix. There are seven generating monomials in $L(2,P)$. Their deformations are the following:

- $b_1b_2 - a_2u_{a,b} - c_2u_{c,b} - d_2u_{d,b}$
- $c_1c_2 - a_2u_{a,c} - b_2u_{b,c} - d_2u_{d,c}$
- $d_1d_2 - a_2u_{a,d} - b_2u_{b,d} - c_2u_{c,d}$
- $a_1b_2 - u_{\emptyset,a}D(a)^b$
- $a_1c_2 - u_{\emptyset,a}D(a)^c$

(1)
Question 4.1. Is there a natural description of the variety defined by these equations?

4.3. Ladder determinantal varieties. Consider now the poset $P$:

\[
\begin{array}{c}
\begin{array}{ccc}
b^r & & c^s \\
& b^{r-1} & \\
& & c^1 \\
\end{array}
\end{array}
\]

where $r, s \geq 2$. We let $b_1^{i+1} = 1 = c_1^{j+1}$, and for short write $b = b^1$ and $c = c^1$. The matrix $M(a)$ is:

\[
\begin{bmatrix}
-u_{a,b} & b_1 & S_cT_c(b) \\
-u_{a,c} & S_bT_b(c) & c_1
\end{bmatrix}.
\]

The monomials in $L(2, P)$ deform to the following polynomials generating $J(2, P)$:

\begin{align*}
(2) \quad & b_i^j b_2^j - b_2^{i-1}(\prod_{k=1}^{j} u_{b^{k-1}, b^k})b_1^{j+1} \\
& \quad \quad 2 \leq i \leq j \leq r \\
(3) \quad & b_1^j b_2^j - (a_2 u_{a,b^1} + \sum_{j=1}^{s} c_2^j u_{c^j,b^1})(\prod_{k=2}^{j} u_{b^{k-1}, b^k})b_1^{j+1} \\
& \quad \quad 1 \leq j \leq r \\
(4) \quad & a_1 b_2^j - u_{\emptyset,a}D(a)^b(\prod_{k=2}^{j} u_{b^{k-1}, b^k})b_1^{j+1} \\
& \quad \quad 1 \leq j \leq r \\
(5) \quad & a_1 c_2^j - u_{\emptyset,a}D(a)^c(\prod_{k=2}^{j} u_{c^j,b^j})c_1^{j+1} \\
& \quad \quad 1 \leq j \leq s \\
(6) \quad & a_1 a_2 - u_{\emptyset,a}D(a)^a
\end{align*}
Write \( T = a_2 u_{a,b} u_{a,c} + u_{a,b} T_b(c) + u_{a,c} T_c(b) \). We claim that the ideal \( J(2, P) \) they generate is precisely the ideal of two-minors of the ladder:

\[
\begin{array}{cccc}
S_e(c^2) & c_2^2 & 0 & 0 \\
S_e(c^1) & c_1^2 & 0 & 0 \\
D(a)^b & T & b_1^2 & b_2^2 \\
u_{\emptyset,a}^{-1} a_1 & D(a)^c & S_0(b^1) & S_0(b^2) \\
\end{array}
\]

if we localize by inverting \( u_{\emptyset,a}, u_{a,b} \) and \( u_{a,c} \). Let us call the part of the above ladder starting from the column with \( b_2^1 \), the left leg. The minors of the left leg are precisely the equations (2). The minors formed by taking the first column (only the two lowest entries) and a column of the left leg, gives the equations (4). Now by multiplying the column of \( b_2^1 \) with \( u_{a,b} u_{b,c} \) and subtracting the sum of all these scaled columns from the column with \( T \), we obtain a column

\[
\begin{bmatrix}
u_{a,c} T_c(b) + a_2 u_{a,b} u_{a,c} \\
u_{a,c} & b_1
\end{bmatrix}
\]

Taking the determinant of this column and the columns of the left leg, we obtain the equations (3) after inverting \( u_{a,c} \).

Lastly we want to obtain the equation (5). Take the determinant of the lower left 2 \times 2 matrix in the ladder. This is (after multiplying with \( u_{\emptyset,a} \)):

\[
a_1 T - u_{\emptyset,a} D(a)^b D(a)^c = u_{a,b} u_{a,c} a_1 a_2 + a_1 u_{a,b} T_b(c) + a_1 u_{a,c} T_c(b) + u_{a,b} u_{a,c} S_b T_b(c) S_c T_c(b) + u_{a,b} u_{a,c} S_b T_b(c) S_c T_c(b)
\]

By subtracting the following polynomial, obtained as a linear combination of the \( r \) first equations in (4):

\[
u_{a,b} [a_1 T_b(c) - u_{\emptyset,a} D(a)^b S_b T_b(c)]
\]

and the polynomial, obtained as a linear combination of the \( s \) second equations in (4):

\[
u_{a,c} [a_1 T_c(b) - u_{\emptyset,a} D(a)^c S_c T_c(b)]
\]

we obtain the last equation (5).

4.4. Variety of complexes. For definition of variety of complexes refer to [7]. Let \( P \) be the poset with Hasse diagram:

![Hasse diagram](image)

where \( s \geq 2 \). We write \( c = c^1 \) and define \( c_1^{s+1} = 1 \). We have

\[
M(a) = \begin{bmatrix}
u_{a,b} & b_1 & \sum_{j=1}^{s} u_{\emptyset,b} c_1^{j+1} \prod_{k=2}^{j} u_{c^k-1,c^k} \\
u_{a,c} & -u_{\emptyset,c}
\end{bmatrix}
\]
The deformed relations of $L(2, P)$ are the following:

\[(6)\quad c_1^i c_2^j - c_2^{-1} \left( \prod_{k=i}^{j} u_{k-1,c,k} \right) c_1^{j+1} \quad 2 \leq i \leq j \leq s\]

\[(7)\quad c_1^i c_2^j - (a_2 u_{a,c} + b_2 u_{b,c}) \prod_{k=2}^{j} u_{k-1,c,k} c_1^{j+1} \quad 1 \leq j \leq s\]

\[(8)\quad b_1 b_2 - a_2 u_{a,b} - \sum_{j=1}^{s} c_2^j u_{c,j,b}\]

\[(9)\quad a_1 c_2^j - u_{\emptyset,a} D(a)^c \left( \prod_{k=2}^{j} u_{k-1,c,k} \right) c_1^{j+1} \quad 1 \leq j \leq s\]

\[(10)\quad a_1 b_2 - u_{\emptyset,a} D(a)^b\]

\[(11)\quad a_1 a_2 - u_{\emptyset,a} D(a)^a\]

The 2-minors of the matrix

\[X = \begin{bmatrix}
  a_1 & c_1^1 u_{\emptyset,a} & c_1^2 u_{\emptyset,a} u_{a,c^1} & \cdots & c_1^{s+1} u_{\emptyset,a} u_{a,c^1} \prod_{k=2}^{s} u_{k-1,c,k} \\
  b_1 + u_{a,b} u_{a,c}^{-1} u_{b,c} & a_2 + b_2 u_{a,c}^{-1} u_{b,c} & c_2^1 & \cdots & c_2^s
\end{bmatrix}\]

except the one associated to the first two columns give us the relations (6), (7) and (9). Now define the matrix $Y$ as

\[\begin{bmatrix}
  b_2 & -u_{a,b} & -u_{c^1,b} & \cdots & -u_{c^s,b}
\end{bmatrix}\]

The two entries of the matrix $XY$ give the relations (8) and (10). If we multiply equation (10) with $u_{a,c}^{-1} u_{b,c}$ and subtract it from the minor of the first two columns of $X$, we get equation (11).

In the appendix we give a larger poset $P$ and the generators of $J(2, P)$. As we see these polynomials grow quickly in size.

5. The fine positive grading on the ideal $J(2, P)$

In this section we show that the ideal $J(2, P)$ is very finely graded. In fact it is graded by a free abelian group on $2|P|$ free generators. We show that this grading is positive in the sense of [11]. This enables us to state a simple criterion for flatness of homogeneous ideals, which we will apply in Section 7 to conclude that the quotient ring $B(2, P)/J(2, P)$ is flat over its base ring $B = \mathbb{k}[u_{\emptyset,p}, u_{q,p}]$.

5.1. The grading on $J(2, P)$. Let $\mathbb{Z}([2] \times P)$ be the free abelian group of order $2|P|$ generated by the $p_1$'s and $p_2$'s for $p \in P$. The ideal $J(2, P)$ lives in the polynomial ring

\[B(2, P) = \mathbb{k}[x_{[2] \times P}] \otimes_{\mathbb{k}} \mathbb{k}[u_{\emptyset,p}, u_{q,p}]\]

The pairs $(q, p)$ are all pairs such that the meet of $q$ and $p$ is the parent of $p$. They, together with $u_{\emptyset,p}$, correspond to the minimal generators of the first cotangent module $T^1(\mathbb{k}[x_{[2] \times P}]/L(2, P))$ by Corollary 6.11. For an element $p \in P$ let $b_1, \ldots, b_m$ be its children. Denote by $\hat{p}$ the degree $p_2 - b_1^1 - b_1^2 - \cdots - b_1^m$ in $\mathbb{Z}([2] \times P)$.

Now define a grading on the $B(2, P)$ by letting the variable $x_{i,p}$ (which we write as $p_i$) have degree (with some abuse of notation) $p_i$ in the abelian group $\mathbb{Z}([2] \times P)$. Also define the degree of $u_{q,p}$ to be $p_1 - q_2 + \hat{p}$. 
Proposition 5.1. The ideal \( J(2, P) \) is homogeneous for this \( \mathbb{Z}([2] \times P) \)-grading. Moreover:

1. \( T(p) \) is homogeneous of degree \( p_1 + \hat{p} \).
2. If \( p \leq q \) then \( S_p(q) \) is homogeneous of degree \( q_2 - \hat{p} \).
3. When \( p \) and \( q \) are siblings, \( S_q T_q(p) \) is homogeneous of degree \( p_1 + \hat{p} - \hat{q} \).
4. Let \( q \) be a child of \( p \). The determinant \( D(p)^q \) is homogeneous of degree \( \hat{q} - \hat{p} \). The determinant \( D(p)^p \) is homogeneous of degree \( p_2 - \hat{p} \).

Proof. (1) easily follows from the definitions. Note that if \( \rho \) is the root then \( \deg(T(\rho)) = \deg(u_{\rho, \rho}) = \rho_1 + \hat{\rho} \).

We prove (2) and (4) simultaneously since they are both dependent on each other. (3) follows from (2) by definition. Let \( m \) be the cardinality of \( P \) and \( \eta : P \rightarrow [m] \) a linear extension.

We prove (2) and (4) by descending induction on \( \eta(p) \). Suppose \( \eta(p) = m \), the maximal possible. Then \( D(p)^p = S_p(p) = 1 \) has degree \( p_2 - \hat{p} = 0 \).

Now we show that if for any two elements \( p \leq q \) such that \( \eta(p) = k + 1 \) we have

\[
\deg D(p)^p = p_2 - \hat{p}, \quad \deg D(p)^q = \hat{q} - \hat{p}, \quad \deg S_p(q) = q_2 - \hat{p}
\]

then the above statement is also true for any two elements \( p \leq q \) with \( \eta(p) = k \).

So suppose \( \eta(p) = k \) and let \( b^1, \ldots, b^m \) be children of \( p \). These have all \( \eta \)-value \( \geq k + 1 \). By assumption we have

\[
\deg(D(p)^p) = \deg \left( \sum_{\sigma \in S_m} \prod_{i=1}^m S_{b_i} T_{b_i} (b^{\sigma(i)}) \right) = m \sum_{i=1}^m b_i^p + m \sum_{i=1}^m b_i^q - \sum_{i=1}^m \hat{b}_i = p_2 - \hat{p}.
\]

where \( S_m \) is symmetric group on \( m \) letters.

Similarly when \( q \) is a child of \( p \), we have \( \deg(D(p)^q) = \hat{q} - \hat{p} \). For \( p \leq q \) in \( P \) let \( p = x^0, \ldots, x^m = q \) be the maximal chain between \( p \) and \( q \) in \( P \). Then

\[
\deg(S_p(q)) = \deg \left( D(p)^x D(x^1)^x \cdots D(x^{m-1})^y D(q)^y \right) = (x^1 - \hat{p}) + (x^2 - \hat{x}^1) + \cdots + (\hat{q} - x^m) + (q_2 - \hat{q}) = q_2 - \hat{p},
\]

This completes the proof of (2)-(4). The main assertion now follows from (1) and (2). \( \square \)

5.2. Positive gradings by an abelian group. Let \( Y \) be a finite-dimensional vector space graded by an abelian group \( A \). This gives an \( A \)-graded polynomial ring \( \mathbb{k}[Y] \). This grading is positive if the only elements in \( \mathbb{k}[Y] \) of degree 0 are the constants in \( \mathbb{k} \). By [16, Prop. 8.6], this is equivalent to each graded piece \( \mathbb{k}[Y]_a, a \in A \) being a finite-dimensional vector space over \( \mathbb{k} \). The following is another characterization.

Lemma 5.2. The grading by \( A \) on \( \mathbb{k}[Y] \) is positive, iff there is a homomorphism \( A \rightarrow \mathbb{Z} \) such that the \( \mathbb{Z} \)-degree of any \( A \)-homogeneous element of \( Y \) is positive, i.e. is in \( \mathbb{Z}_{>0} \).

Proof. The if direction is clear. The only if direction follows by [16, Cor. 8.7] by applying this to the torsion free part of \( A \), which must be nonzero. (Note that the definition of positive grading in [16, Chap.8] requires \( A \) to be torsion-free. We follow the convention of [11] which does not have this requirement.) \( \square \)
Proposition 5.3. The grading on $B(2, P)$ given in Proposition 5.1 is positive.

Proof. Define a map from $\mathbb{Z}([2] \times P)$ to $\mathbb{Z}$ by letting every $d(p_2) = 1$ and define $d(p_1)$ inductively on depth($p$) such that the $d(p)$ is positive and larger than the sum $\sum_b d(b)$ where we sum over the children $b$ of $p$. Then it is seen that all variables $p_i$ have positive $d$-values, as well as all variables $u_{q,p}$. □

Let $J \subseteq \mathbb{k}[Y]$ be an ideal which is homogeneous for a positive $A$-grading on $Y$. Let $U \subseteq Y$ be a homogeneous subspace for this grading, and $I \subseteq \mathbb{k}[Y/U]$ be the ideal such that

$$\mathbb{k}[Y/U]/I = (\mathbb{k}[Y]/J) \otimes_{\mathbb{k}[Y]} \mathbb{k}[Y/U].$$

Let $f_1, \ldots, f_k$ be generators of $J$ and let $\overline{f}_1, \ldots, \overline{f}_k$ denote their images in $I$, which will generate $I$. The following is the criterion we use, in Theorem 7.6, to show that $J(2, P)$ is a flat deformation of $L(2, P)$.

Proposition 5.4. Suppose the $A$-grading on $\mathbb{k}[Y]$ is positive. If every relation of $f_1, \ldots, f_k$ lifts to a relation for $\overline{f}_1, \ldots, \overline{f}_k$, then $\mathbb{k}[Y]/J$ is flat over $\mathbb{k}[U]$.

Proof. We apply the local criterion of flatness [18, Thm.A.5] and the criterion of lifting of relations [18, Thm.A.10], to the algebra homomorphism $\mathbb{k}[Y]/J \to \mathbb{k}[Y/U]/I$. The situation in [18, Thm.A.5] is a local homomorphism of local noetherian rings. Our situation is a graded homomorphism of positively graded rings. This situation works just as well for the arguments. □

6. First Order deformations and the cotangent cohomology

This section presents the basic general deformation theory that we need. General references for deformation theory are [18, Chap.3], [12, Chap.1.3] or [19, Sec.3]. We compute explicitly the first cotangent cohomology $T^1(\mathbb{k}[x_2] \times P/L(2, P))$ for any quadratic letterplace ideal of a poset $P$. As we shall see the elements of this module are remarkably simple in form.

6.1. The deformation functor. We consider a $\mathbb{k}$-algebra $R$ and an ideal $I$ in $R$. Let $B$ be another $\mathbb{k}$-algebra with a distinguished $\mathbb{k}$-point $b \in \text{Spec } B$ corresponding to a morphism $B \to \mathbb{k}$. A deformation over $B$ of the ideal $I \subseteq R$, is an ideal $J \subseteq R \otimes_{\mathbb{k}} B$ such that:

- $(R \otimes_{\mathbb{k}} B)/J$ is flat over $B$,
- The natural map $R \otimes_{\mathbb{k}} B \to R$ induces a map $(R \otimes_{\mathbb{k}} B)/J \to R/I$ which becomes an isomorphism

$$(R \otimes_{\mathbb{k}} B)/J \otimes_{B} \mathbb{k} \xrightarrow{\cong} R/I.$$

Let $\text{Set}$ be the category of sets, and $\mathbb{k} - \text{Art}$ be the category of local artinian $\mathbb{k}$-algebras with residue field $\mathbb{k}$.

The functor of infinitesimal deformations of $I$

$$\text{Def}_I : \mathbb{k} - \text{Art} \to \text{Set},$$

is given by letting $\text{Def}_I(A)$ be the set of such deformations $J$ over the ring $A$. 

6.2. The tangent space. Let $k[[t]] = \mathbb{k}[t]/(t^2)$. The deformations over this ring are called first order deformations. The tangent space of the deformation functor is $\text{Def}_I(k[[t]])$ (see below). This space identifies as $\text{Hom}_{\text{Set}}(I, R/I)$. Let $\mathcal{K} \to \mathcal{K}$ be the category of finite-dimensional vector spaces. For a vector space $V$ denote by $V^*$ its dual space. There is a functor

$$\mathcal{K} \to \mathcal{K}$$

sending $V$ to $k[V^*]/(V^*)^2$. By abuse of notation we get a restricted functor $\text{Def}_f$ from $\mathcal{K}$ where $\text{Def}_f(V) = \text{Def}_I(k[V^*]/(V^*)^2)$. We also have a functor

$$\text{Hom}_k(-, \text{Hom}_R(I, R/I)) : \mathcal{K} \to \text{Set},$$

sending $V$ to the set of linear maps $V \to \text{Hom}_R(I, R/I)$. The following is standard, see [18] Proposition 3.2.1 and Definition 3.2.3.

**Proposition 6.1.** There is a natural isomorphism of functors between the two functors

$$\text{Def}_f, \text{Hom}_k(-, \text{Hom}_R(I, R/I)) : \mathcal{K} \to \text{Set}.$$

The upshot is that the tangent space $\text{Def}_f(\mathbb{k}[e])$ of the functor $\text{Def}_f$ on $\mathcal{K} \to \text{Set}$ identifies as $\text{Hom}_R(I, R/I)$. We describe the isomorphism of functors in more detail.

1. Choose a basis $v_1, \ldots, v_r$ for $V$. We obtain a dual basis $v^*_1, \ldots, v^*_r$ for $V^*$. A linear map $\phi : V \to \text{Hom}_R(I, R/I)$ gives an ideal

$$J = \{f + \sum_i v_i^* \phi(v_i)(f) \mid f \in I\}.$$

This is the flat deformation corresponding to $\phi$.

Alternatively $\phi$ gives an element of $V^* \otimes_k \text{Hom}_R(I, R/I)$ and so a map

$$\overline{\phi} : I \to I \otimes_k V^* \otimes_k \text{Hom}_R(I, R/I) \to V^* \otimes_k R/I.$$

Then $J$ is the ideal

$$J = \{f + \overline{\phi}(f) \mid f \in I\}.$$

2. Conversely given a deformation $J \subseteq R \otimes_k k[V^*]/(V^*)^2$ of $I \subseteq R$, flat over $k[V^*]/(V^*)^2$. Note first that the inclusion $k \to k[V^*]/(V^*)^2$ induces $R \to R \otimes_k k[V^*]/(V^*)^2$, and so the ideal $I$ embeds as a linear subspace of the right ring. There is a short exact sequence

$$0 \to R \otimes_k V^* \to R \otimes_k k[V^*]/(V^*)^2 \to R \to 0.$$

Tensoring this with $- \otimes_{R \otimes_k} R$ we obtain, recall that $J$ is flat, a short exact sequence

$$0 \to V^* \otimes_{k[V^*]/(V^*)^2} J \to J \to (V^* \cdot J) = I \to 0.$$

The term $V^* \otimes_{k[V^*]/(V^*)^2} J$ identifies as $V^* \cdot J = V^* \otimes_k I$. An $f \in I$ lifts by $\pi$ to an $\tilde{f} = f + p$ where $p \in R \otimes_k V^*$. Any two liftings differ by an element of $V^* \otimes_k I$, and so we get a well-defined $R$-module map $I \to V^* \otimes_k (R/I)$ sending

$$f \mapsto p \in V^* \otimes_k (R/I).$$

But such an $R$-module map corresponds to a linear map $V \to \text{Hom}_R(I, R/I)$.

Now consider the situation of a subset $T$ of $\text{Hom}_k(V^*, R) = V \otimes_k R$. It gives a map $V^* \to R \otimes_k T^*$. This gives a map of algebras:

$$\tau : R \otimes_k k[V^*]/(V^*)^2 \to R \otimes_k R \otimes_k k[T^*]/(T^*)^2 \to R \otimes_k k[T^*]/(T^*)^2,$$

where in the last map we have used the multiplication on $R$. 
Lemma 6.2. Let $J \subseteq R \otimes k k[V^*]/(V^*)^2$ be a flat deformation of $I \subseteq R$ over $k[V^*]/(V^*)^2$. Then the image $J' = \tau(J)$ is a flat deformation of $I$ over $k[V^*]/(V^*)^2$. If the deformation $J$ corresponds to the linear map $V \to \text{Hom}_R(I, R/I)$, then the deformation $J'$ corresponds to the linear map $T \to V \otimes_k R \to \text{Hom}_R(I, R/I)$, where to define the latter map we have used the $R$-module structure on $\text{Hom}_R(I, R/I)$.

Proof. Let the $v_j$'s form a basis for $V$ and the $t_i$'s form a basis for $T$. If the map $T \to V \otimes_k R$ is given by

$$t_i \mapsto \sum_j v_j \otimes_k r_{ij},$$

then the map $V^* \to T^* \otimes_k R$ is given by

$$v^*_j \mapsto \sum_i t^*_i \otimes_k r_{ij}.$$ 

Let the deformation $J$ be given by

$$f + \sum_j v^*_j f_j, \quad f \in I.$$ 

The linear map $V^* \to T^* \otimes_k R$ gives the deformation

$$f + \sum_j \sum_i (t^*_i \otimes_k r_{ij}) f_j, \quad f \in I$$

and the ideal $J'$ consists of

$$f + \sum_j \sum_i t^*_i r_{ij} f_j, \quad f \in I$$

But this is the deformation corresponding to the linear map that sends each $t_i$ to the $R$-module map

$$\sum_j r_{ij} v_j : I \to R/I$$

$$f \mapsto \sum_j r_{ij} f_j,$$

and so is a flat deformation. \qed

6.3. **Deformations over polynomial rings.** Now assume the ring $R$ is a polynomial ring $k[X]$ where $X$ is a finite-dimensional vector space.

A first order deformation of $I$ over $k[\epsilon]$ is **trivial** if it is the image of

$$I \otimes_k k[\epsilon] \subseteq k[X] \otimes_k k[\epsilon]$$

by an (infinitesimal) coordinate change in $k[X] \otimes_k k[\epsilon]$, meaning a linear map

$$X \to k[X] \otimes_k k[\epsilon]$$

lifting the canonical inclusion $X \to k[X]$. This linear map induces the infinitesimal coordinate change, an algebra homomorphism

$$k[X] \otimes_k k[\epsilon] \to k[X] \otimes_k k[\epsilon].$$
Example 6.3. Let $k[X] = k[x, y]$ and $I = (xy)$. The map
\[ x \mapsto x + \epsilon y^3, \quad y \mapsto y + \epsilon xy \]
is an infinitesimal coordinate change. The ideal $I$ is mapped to the ideal $(xy + \epsilon x^2y + \epsilon y^4)$ by this coordinate change.

Let $\text{Der}_K(k[X])$ be the derivations of $k[X]$. If the $x_i$ form a basis for $X$, this is a free $k[X]$-module generated by the derivatives $\partial/\partial x_i$ for $i = 1, \ldots, n$. There is a map
\[ (12) \quad \delta^* : \text{Der}_K(k[X]) \to \text{Hom}_{k[X]}(I, k[X]/I) \]
which sends $\partial$ to the homomorphism sending $f_i$ to $\partial f_i + I$. The image of the homomorphism $\delta^*$ consists of all the trivial infinitesimal deformations of $k[X]/I$.

There is a diagonal map $X \to X \oplus X$ inducing
\[ \tau : k[X] \to k[X \oplus X] = k[X] \otimes_k k[X] \to k[X] \otimes_k k[X]/(X)^2. \]

Example 6.4. Let $X$ have basis $x_1, x_2$ and let the other copy of $X$ have basis $t_1, t_2$. The image by $\tau$ of the ideal $(x_1^2 x_2)$ in $k[x_1, x_2]$ is then the ideal generated by
\[ (x_1 + t_1)^2(x_2 + t_2) = x_1^2 x_2 + t_1 2x_1 x_2 + t_2 x_1^2. \]

The following characterizes the images of the partial derivatives in a coordinate free way.

Lemma 6.5. Let $I \subseteq k[X]$ be an ideal. Then $\bar{I} = \tau(I)$ is a flat deformation of $I$ over $k[X]/(X^2)$. The corresponding linear map $X^* \to \text{Hom}(I, k[X]/I)$ sends a dual basis element $x_i^*$ to the derivation $\partial/\partial x_i$.

Proof. By tensoring with $k[X]$ we get a flat (trivial) deformation $I \otimes_k k[X]$ in $k[X] \otimes_k k[X]$ over $k[X]$. Denote the generators of the algebra $k[X] \otimes_k k[X] = k[X \oplus X]$ by $x_i \otimes 0$ and $0 \oplus t_i$. We now perform the coordinate change $x_i \mapsto x_i \oplus t_i$ and $t_i \mapsto t_i$. Note that this coordinate change is a $k[X]$-module map for the inclusion $x_i \mapsto 0 \oplus t_i$. The image of $I \otimes_k k[X]$ by this coordinate change is denoted $\bar{I}$. Since
\[ (k[X]/I) \otimes_k k[X] \to k[X \oplus X]/\bar{I} \]
is a $k[X]$-module isomorphism, the right side will be flat over $k[X]$. The result now follows by the base change $k[X] \to k[X]/(X)^2$. \qed

The cokernel $\text{Coker} \delta^*$ of the map $\delta^*$ in (12) is called the first cotangent module of $k[X]/I$ and is denoted by $T^1(k[X]/I)$. Its elements are in one-to-one correspondence with the non-trivial infinitesimal deformations of $k[X]/I$.

Definition 6.6. The ideal $I$ is a rigid ideal if $T^1(k[X]/I)$ vanishes or equivalently the map $\delta^*$ is surjective.

This means that every deformation of $I$ comes from a coordinate change. We show in Section 8 that $J(2, P) \subseteq B(2, P)$ is a rigid ideal.
6.4. The first cotangent cohomology for letterplace ideals. We consider a finite poset \( P \). Let \( S \) be the polynomial ring \( k[x_{1,p}] \). Recall that we denote the variable \( x_{1,p} \) as \( p \). The letterplace ideal \( L = L(2, P) \) is generated by all monomials \( p_1 q_2 \) where \( p \leq q \). We shall compute the first cotangent cohomology group \( T^1(S/L) \), or rather its minimal generating set as an \( S \)-module. The general theory for doing this for Stanley-Reisner rings is developed in [2]. The results for \( T^1(k[x_{2,|x|}, P/L(2, P)] \) are however so simple that we do it from first principles, without recalling technicalities in loc.cit.

If \( J \) and \( U \) are subsets of \( P \) we say that \( U \) is an upper bound set for \( J \) if for every \( p \in J \) there is a \( q \in U \) with \( p \leq q \). Similarly if \( F \) and \( D \) are subset of \( P \), we say \( D \) is a lower bound set for \( F \) if for every \( p \in F \) there is a \( r \in D \) with \( r \leq p \). For \( p \in P \) let \( J(\leq p) \) (resp. \( J(\leq p) \)) be the order ideal of \( P \) consisting of all \( r \in P \) with \( r < p \) (resp. \( r \leq p \)), and let \( F(> p) \) (resp. \( F(\geq p) \)) be the order filter consisting of all \( q \in P \) with \( q > p \) (resp. \( q \geq p \)).

The first cotangent cohomology \( T^1(S/L) \) is the quotient of \( \text{Hom}_S(L, S/L) \) by the derivations \( \partial/\partial x_{1,p} \) as we range over the variables of \( S \). The following describes the minimal generators of \( T^1(S/L) \). It is remarkable in that they come from maps in \( \text{Hom}_S(L, S/L) \) where only one monomial is mapped to something nonzero. Also all monomials \( p_1 q_2 \in I \) where \( p < q \) are always mapped to zero.

**Theorem 6.7.** Given \( p \in P \). Let \( U \) be a an inclusion minimal subset of \( P - F(\geq p) \) which is an upper bound set for \( J(\leq p) \), and let \( D \) be an inclusion minimal subset of \( P - J(\leq p) \) which is a lower bound set for \( F(> p) \). Suppose also that whenever \( r \in D \) and \( s \in U \), we do not have \( r \leq s \). Then the map

\[
\phi : L \to S/L
\]

sending

\[
p_1 p_2 \mapsto \prod_{r \in D} r_1 \prod_{s \in U} s_2
\]

and all other minimal monomial generators in \( L \) to zero, is a nonzero map of \( S \)-modules.

Moreover as we vary over \( p \in P \) and sets \( D \) and \( U \) with this property, the maps \( \phi \) form a minimal generating set of the cotangent cohomology \( T^1(S/L) \).

Note that \( D \) and \( U \) above will both be antichains. We denote the map above as

\[
\phi = (p_1 p_2 \mapsto \prod_{r \in D} r_1 \prod_{s \in U} s_2).
\]

Before proving the above theorem we develop some lemmata.

**Lemma 6.8.** Let \( \phi \in \text{Hom}_S(L, S/L) \). Modulo \( \text{Im} \delta^* \) the homomorphism \( \phi \) is equal to a homomorphism \( \phi' \) with the property that for any \( p \leq q \in P \), \( \phi'(p_1 q_2) \) can be written as a linear combination of monomials \( m \) not divisible by \( p_1 \) or \( q_2 \).

**Proof.** Suppose \( \phi(p_1 q_2) \) contains a nonzero term of the form \( c p_1 m \) for some monomial \( m \) and constant \( c \). We show that modulo \( \text{Im} \delta^* \) we can eliminate this term. More precisely we show that for any generator of \( L \) of form \( r_1 q_2 \), for some \( r \leq q \), \( \phi(r_1 q_2) \) contains the term \( c r_1 m \). Therefore by subtracting \( c m \delta_{q_2} \) from \( \phi \) we can eliminate the terms \( c r_1 m \) without adding any extra terms to \( \phi(r_1 q_2) \).

Suppose \( r_1 q_2 \) is a generator of \( L \) that contains \( q_2 \). The syzygy \( r_1 (p_1 q_2) - p_1 (r_1 q_2) \) induces the relation \( r_1 \phi(p_1 q_2) - p_1 \phi(r_1 q_2) \). The term \( c r_1 p_1 m \) either belongs to \( L \) or for some \( m' \) in \( \phi(r_1 q_2) \) it cancels out by the term \( c p_1 m' \). If \( r_1 p_1 m \) is in \( L \) then \( r_1 m \in L \). If it cancels out then \( m' = c r_1 m \). Therefore in either case \( \phi(r_1 q_2) \) contains \( c m \delta_{q_2} (r_1 q_2) \). Note that if
for some generator $g$ of $L$, $q_2 \nmid g$ then $m_2 \partial \nabla_\varphi(q) = 0$. The proof for the case where $\varphi(p_1q_2)$ contains a term of the form $mq_2$ is similar.

For any homomorphism $\varphi \in \text{Hom}_S(L,S/L)$, the following lemma shows which monomials can appear in the image of generators of $L$.

**Lemma 6.9.** Let $\varphi \in \text{Hom}_S(L,S/L)$. For $p \leq q$ let $p_1q_2$ be a generator of $L$. We may write $\varphi(p_1q_2)$ as a linear combination of monomials $m$ that are not in $L$.

1. If $p \neq q$ then $\text{gcd}(p_1q_2,m) \neq 1$.
2. If $p = q$ then either $\text{gcd}(p_1p_2,m) \neq 1$ or, for all $r < p$ there is $t \geq r$ such that $t_2|m$ and for all $r' > p$, there is $s \leq r'$ such that $s_1|m$.

**Proof.** 1. Suppose $p \neq q$ and $\text{gcd}(p_1q_2,m) = 1$. We have a relation $p_2\varphi(p_1q_2) - q_2\varphi(p_1p_2)$ in $S/L$ since $p_2(p_1q_2) - q_2(p_1p_2)$ is a syzygy of $L$. If $p_2m$ is canceled with some monomial in $q_2\varphi(p_1p_2)$ then $q_2m$, against our initial assumption above.

Hence $p_2m \in L$, and this implies that there is some $s \leq p$ such that $s_1|m$. In a similar manner the syzygy $q_1(p_1q_2) - q_1(q_1q_2)$ shows that there is some $t \geq q$ such that $t_2|m$. So $s_1t_2|m$ for $s \leq t$ which is a contradiction. The upshot is that we must have $\text{gcd}(p_1q_2,m)$ must be nontrivial.

2. Suppose $p = q$ and $\text{gcd}(p_1p_2,m) = 1$. Let $x(p_1p_2) - y(p_1q_2)$ be a relation involving the generator $p_1p_2$, so $y$ contains at least one of $p_1$ or $p_2$ (and $x$ does not). If $xm$ is going to be canceled by some monomial in $y\varphi(p_1q_2)$ then $p_1p_2$ and $m$ cannot be relatively prime. Therefore if $\text{gcd}(p_1p_2,m) = 1$ then for any such syzygy $xm$ belongs to $L$. Note that if $x(p_1q_2) - y(p_1q_2)$ is not a linear syzygy then $xm$ belongs to $L$, so we only need to consider linear syzygies. For any $r < p$ the linear syzygy $r_1(p_1p_2) - p_1(r_1p_2)$ implies $r_1m \in L$. Hence for some $t \geq r$, $t_2$ divides $m$. Similarly for any $r' > p$, $m$ should be divided by a variable $s_1$ for some $s \leq r'$.

**Proof of Theorem 6.7.** Let $\varphi \in \text{Hom}(L,S/L)$ be a homomorphism. By Lemmata 6.8 and 6.9 we can decompose $\varphi$ as $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1$ belongs to the image of $\delta^*$ and for any $p \in P$, all the monomials in $\varphi_2(p_1p_2)$ are relatively prime to $p_1p_2$ and also for any $p < q$, $\varphi_2(p_1q_2) = 0$. Now let $m$ be a nonzero monomial in $\varphi(p_1p_2)$, let $\psi$ be a map that sends $p_1p_2$ to $m$ and any other generator of $L$ to zero. Since $m$ is relatively prime to $p_1p_2$ the second part of proof of 6.9 shows that this map satisfies all the relations of $L$. Hence it is a well-defined homomorphism. By Lemma 6.9 (2), there exists some $U$ and $D$ as in 6.7 such that $\Pi_{r \in D} \Pi_{s \in U} \delta_s|m$. Note that if $U$ contains an element $s$ in $F(\geq p)$ then $D$ contains an element $r$ such that $r \leq s$ and $m$ belongs to $L$ which is a contradiction. Hence $U \subseteq P - F(\geq p)$. Similarly $D \subseteq P - J(\leq p)$. Therefore the homomorphisms in 6.7 generate $T^1(S/L)$.

**Example 6.10.** Let $P$ be a poset with following Hasse diagram.

```
   e
  / \
 c   d
 /    \
 a    b
```
For the point $a$, $J(< a) = \emptyset$ so the empty set is the only inclusion minimal subset of $P - F(\geq a)$ which is an upper bound for $J(< a)$. We also have $F(> a) = \{c, d, e\}$ and the inclusion minimal subsets of $P - J(\leq a)$ that are lower bounds for $F(> a)$ are $\{c, d\}$ and $\{b\}$. Therefore for $a$, we have two first order deformations corresponding to maps

$$a_1a_2 \mapsto c_1d_1, \quad a_1a_2 \mapsto b_1.$$ 

Consider the point $c$. $J(< c) = \{a, b\}$ and the minimal upper bounds in $P - F(\geq c)$ are $\{a, b\}$ and $\{d\}$. We also have $F(> c) = \{e\}$ and the minimal lower bounds in $P - J(\leq c)$ are $\{d\}$ and $\{e\}$. Note that since $d_1d_2$ is in $L(2, P)$ we only have 3 maps corresponding to the point $c$ of $P$.

$$c_1c_2 \mapsto a_2b_2d_1, \quad c_1c_2 \mapsto a_2b_2e_1, \quad c_1c_2 \mapsto d_2e_1.$$ 

Finally, one can show that the first cotangent cohomology module is minimally generated by the following 11 maps.

$$a_1a_2 \mapsto c_1d_1, \quad a_1a_2 \mapsto b_1,$$

$$b_1b_2 \mapsto c_1d_1, \quad b_1b_2 \mapsto a_1,$$

$$c_1c_2 \mapsto a_2b_2d_1, \quad c_1c_2 \mapsto a_2b_2e_1, \quad c_1c_2 \mapsto d_2e_1,$$

$$d_1d_2 \mapsto a_2b_2c_1, \quad c_1c_2 \mapsto a_2b_2e_1, \quad c_1c_2 \mapsto c_2e_1,$$

$$e_1e_2 \mapsto c_2d_2.$$ 

**Corollary 6.11.** Suppose the Hasse diagram of $P$ is a rooted tree. Let $p$ in $P$ and $b_1, \ldots, b_n$ its children.

- Let $q$ be such that the meet of $q$ and $p$ is the parent of $p$. The map sending $p_1p_1 \mapsto q_2 \prod_{i=1}^n b_i^i$ and all other monomials to zero, is in $T^1(S/L)$.
- Let $\rho$ be the root of $P$. The map sending $p_1p_2 \mapsto \prod_{i=1}^n b_i^i$, and all other monomials to zero, is in $T^1(S/L)$.

As $q$ and $p$ vary, these maps generate $T^1(S/L)$.

**7. Flatness of deformation family**

We show that the ring $B(2, P)/J(2, P)$ is a flat deformation of $k(2, P)/L(2, P)$ over the base ring $B = k[u_{0, p}, u_{q, p}]$. We do this in Theorem 7.6, but before that we develop some auxiliary results.

**Proposition 7.1.** Given element $p, b, c \in P$ with $p \leq b$ and $p \leq c$. Then

$$S_p(b)c_2 - b_2S_p(c)$$

is in $J(2, P)$.

We shall prove this by induction on depth$(p)$. For the below lemmata we assume the above proposition holds for a given $p$, and the lemmata are consquences of this.

**Lemma 7.2.** Assume Proposition 7.1 holds for a given $p$. Let $q$ be sibling of $p$ (possibly equal) and $b \geq p$. Then

$$S_pT_p(q)b_2 - T_p(q)S_p(b)$$

is in $J(2, P)$. 


Proof. If $p = q$, then (13) is

$$p_1 b_2 - T(p)S_p(b)$$

and so is in $J(2, P)$ by definition. So assume $p \neq q$. Let $T_p(q) = -\sum_{c \geq p} c_2 u_{c,q}$. Since

$$S_p(c)b_2 - c_2 S_p(b)$$

is in $J(2, P)$ we immediately get the lemma. \qed

Lemma 7.3. Assume Proposition 7.1 holds for a given $p$. Let $q, r$ and $p$ be siblings (some possibly equal). Then

$$S_p T_p(q) T_p(r) - T_p(q) S_p T_p(r)$$

is in $J(2, P)$.

Proof. If all three are equal this clearly holds. Assume then that $p$ is distinct from either $q$ or $r$, say distinct from $r$. Then $T_p(r) = -\sum_{b \geq p} b_2 u_{b,r}$. The statement then follows by Lemma (7.2) above. \qed

Let $b^1, \ldots, b^m$ be the children of $a = b^0$. Let $b^i, b^j$ be elements of $\{b^0, \ldots, b^m\}$ and $b^k$ element of $\{b^1, \ldots, b^m\}$. Let $M(a)_{b^k}^{b^i b^j}$ be the submatrix of $M(a)$ obtained by deleting columns $i$ and $j$ and row $k$, and define the signed determinant

$$D(a)_{b^k}^{b^i b^j} = \begin{cases} (-1)^{i+j+k} |M(a)_{b^k}^{b^i b^j}| & i < j \\ (-1)^{i+j+k-1} |M(a)_{b^k}^{b^i b^j}| & i > j \end{cases}$$

More generally for a sequence of indices $i : i_0, \ldots, i_\ell$ from $0, 1, \ldots, m$ let $|i|$ be its length $\ell + 1$ and $\sigma(i)$ the sign of the permutation that puts them in strictly increasing order. Similarly for a sequence $k : k_1, \ldots, k_\ell$ from $1, 2, \ldots, m$, and let

$$D(a)_{b^k}^{b^i} = (-1)^{|i|+|k|+\sigma(i)+\sigma(k)} |M(a)_{b^k}^{b^i}|,$$

where the last matrix is obtained by deleting the columns from $i$ and the rows from $k$. It may be verified that we may expand $D(a)_{b^k}^{b^i}$ along a new row $r$ as

$$D(a)_{b^k}^{b^i} = \sum_c S_c T_c(r) D(a)_{b^k b^i c}^{b^i}$$

and similarly when expanding along a column.

Suppose now Lemma 7.1 is proven for all $p$ with depth($p$) $\leq n$.

Lemma 7.4. Let depth($a$) $\leq n + 1$, and let $b, c, d$ be distinct children of $a$.

1. $\sum_{x=b^1} b^m D(a)_{x}^{b^c} T_d(x)$ is in $J(2, P)$.
2. $\sum_{x=b^1} b^m D(a)_{x}^{b^c} T_a(x)$ is in $J(2, P)$.
3. $\sum_{x=b^1} b^m D(a)_{x}^{b^c} T_c(x)$ is in $J(2, P)$.

Proof. 1. We expand the determinant $D_x^{b^c}(a)$ by column $d$. We then get

$$D_x^{b^c}(a) T_d(x) = \sum_{y=b^1, y \neq x} b^m D_y^{b^c d}(a) S_d T_d(y) T_d(x).$$

We now sum these over $x$. Thus for each pair $(x, y)$ both $xy$ and $yx$ occur as indices of the determinant. But these determinants will then be negatives of each other. We obtain

$$\sum_{x=b^1} b^m D(a)_{x}^{b^c} T_d(x) = \sum_{x<y} b^m D(a)_{xy}^{b^c d} [S_d T_d(y) T_d(x) - T_d(y) S_d T_d(x)].$$
By Lemma 7.3 the last bracket is in the ideal $J(2, P)$.

2. We do as above but expand along the column $a$. The sum in the statement is:

$$\sum_{x=b_1^m} D(a)_x^{bc} T_a(x) = \sum_{x<y} D(a)_x^{ab} [S_a T_a(y) T_a(x) - T_a(y) S_a T_a(x)].$$

But the expression in the last bracket is $u_{a,y} a_2 u_{a,x} - u_{a,y} a_2 u_{a,x}$ which is zero.

3. We now expand by column $c$. Again in the same way we get that the sum in the statement is:

$$\sum_{x=b_1^m} D(a)_x^{ab} T_a(x) = \sum_{x<y} D(a)_x^{abc} [S_a T_a(y) T_a(x) - T_a(y) S_a T_a(x)].$$

By Lemma 7.3, again the last bracket is in the ideal $J(2, P)$. \hfill $\Box$

We are now in a position to prove Proposition 7.1.

**Proof of Proposition 7.1.** We shall split into two cases. The first is when $p < c$ and $p < b$ strictly, and the second is when $p = b$ and $p < c$.

So consider the first case. Let $q$ and $r$ be children of $p$ such that

$$p < q \leq b, \quad p < r \leq c,$$

so the depths of $q$ and $r$ are less than the depth of $p$. Then

$$S_p(b)c_2 - b_2 S_p(c) = D(p)^q S_q(b)c_2 - b_2 D(p)^r S_r(c).$$

Now we expand $D(p)^q$ by its column $r$, and we expand $D(p)^r$ by its column $q$. Then the above is:

$$S_q(b) \sum_{x=b_1^m} D(p)_x^{qr} S_r(x)c_2 - S_r(c) \sum_{x=b_1^m} D(p)_x^{rq} S_q T_q(x)b_2$$

Now we do a little trick by subtracting and adding the same terms, to make this:

$$S_q(b) \sum_{x=b_1^m} D(p)_x^{qr} (S_r(x)c_2 - T_r(x)S_r(c)) + S_r(c) \sum_{x=b_1^m} D(p)_x^{qr} (S_q T_q(x)b_2 - T_q(x)S_q(b))$$

$$+ S_q(b) S_r(c) \sum_{x=b_1^m} D(p)_x^{qr} (T_r(x) + T_q(x)).$$

By Lemma 7.2 the first two summands are in $J(2, P)$. For the lower sum note that

$$T_r(x) + T_q(x) = -T_p(x) - \sum_{b' \neq q, r} T_{b'}(x).$$

By Lemma 7.4 this terms is also in the ideal $J(2, P)$.

Now assume $p = b$. The change to the above is that we first get

$$S_p(b)c_2 - b_2 S_p(c) = D(p)^q \cdot 1 \cdot c_2 - b_2 D(p)^r S_r(c),$$

so $S_q(b)$ is replaced by 1. The sum (15) is replaced by

$$\sum_{x=b_1^m} D(p)_x^{qr} (S_r(x)c_2 - T_r(x)S_r(c)) + S_r(c) \sum_{x=b_1^m} D(p)_x^{qr} (S_p T_p(x)p_2 - T_p(x) \cdot 1)$$
Let Proposition 7.5. second term is:

\[ u_{p,x}p_2 - p_2u_{p,x} \]

and so the second term vanishes. Thus this expression is also in \( J(2, P) \).

**Proposition 7.5.** Let \( a \leq b \). Then

\[ a_1T(b) - T(a)R(a, b)b_1 \]

is in the ideal \( J(2, P) \).

**Proof.** Suppose \( p \) is the parent of \( b \) and let \( b = b^1, b^2, \ldots, b^m \) be the children of \( p \). Then

\[ a_1T(b) - T(a)R(a, b)b_1 = a_1p_2u_{p,b} - \sum_{x=b^2}^{b^m} a_1T_x(b) - b_1T(a)R(a, p)D(p)^b \]

Expanding \( D(p)^b \) by row \( b \), it is

\[ D(p)^b = -u_{p,b}D(b)^{bp} + \sum_{x=b^2}^{b^m} S_xT_x(b)D(p)^{bx}. \]

We insert this in the above expression and it becomes:

\[ a_1p_2u_{p,b} - \sum_{x=b^2}^{b^m} a_1T_x(b) - b_1T(a)R(a, p)[u_{p,b}D(p)^{bp} + \sum_{x=b^2}^{b^m} S_xT_x(b)D(p)^{bx}] \]

Since

\[ D(p)^p = b_1D(p)^{bp} + \sum_{x=b^2}^{b^m} S_xT_x(b)D(p)^{bx}, \]

the above equals

\[ a_1p_2u_{p,b} - \sum_{x=b^2}^{b^m} a_1T_x(b) - T(a)R(a, p)u_{p,b}D(p)^p + \sum_{x=b^2}^{b^m} T(a)R(a, p)u_{p,b}S_xT_x(b)D(p)^{bx} \]

\[ - \sum_{x=b^1}^{b^m} b_1T(a)R(a, p)S_xT_x(b)D(p)^{bx}. \]

Note that

\[ u_{p,b}[a_1p_2 - T(a)R(a, p)D(p)^p] \]

is in the ideal \( J(2, P) \). Adding this to the above, the above will modulo \( J(2, P) \) be:

\[ - \sum_{x=b^2}^{b^m} a_1T_x(b) + \sum_{x=b^2}^{b^m} T(a)R(a, p)S_xT_x(b)[u_{p,b}D(p)^{bx}] - \sum_{x=b^2}^{b^m} b_1T(a)R(a, p)S_xT_x(b)D(p)^{bx} \]

Expanding \( D(p)^x \) by row \( b \) it is:

\[ D(p)^x = -u_{p,b}D(p)^{xp} + \sum_{y=b^2}^{b^m} S_yT_y(b)D(p)^{xy} + b_1D(p)^{xb}. \]
Using this we replace $u_{p,b}D_b^{px}$ in (17) and get
\begin{equation}
- \sum_{x=b^2}^{b^m} a_1 T_x(b) + \sum_{x=b^2}^{b^m} T(a) R(a, p) S_x T_x(b) [D(p)^x - b_1 D(p)^{zb} - \sum_{y=b^2}^{b^m} S_y T_y(b) D(p)^{xy}] \\
- \sum_{x=b^1}^{b^m} b_1 T(a) R(a, p) S_x T_x(b) D(p)^{bx}
\end{equation}
Here
\[a_1 T_x(b) - T(a) R(a, p) S_x T_x(b) D(p)^x = a_1 T_x(b) - T(a) S_a(T_x(b))\]
is in the ideal $J(2, P)$. Furthermore the terms with $xy$ superscripts cancels against the terms with $yx$ superscripts. Thus (18) is in the ideal $J(2, P)$. □

**Theorem 7.6.** The ring $B(2, P)/J(2, P)$ is a flat deformation of of $\mathbb{k}[x_{[2]}]/L(2, P)$ over the ring $B = \mathbb{k}[u_{q,p}, u_{q,p}]$.

**Proof.** By Proposition 5.4 it is enough to show that all relations between the generators of $L(2, P)$ lift to relations between the corresponding generators of $J(2, P)$.

The relations of $L(2, P)$ are the following.

1. $c_2 \cdot a_1 b_2 - b_2 \cdot a_1 c_2$ where $a \leq b$ and $a \leq c$.
2. $b_1 \cdot a_1 c_2 - a_1 \cdot b_1 c_2$ where $a \leq b \leq c$.

The monomial $a_1 b_2$ deforms to $a_1 b_2 - T(a)S_a(b_2)$, and similarly for $a_1 c_2$. Let $m_B$ be the maximal ideal of $B = \mathbb{k}[u_{q,p}, u_{q,p}]$ generated by the $u$’s. For relations of type 1, it will be enough to show that
\begin{equation}
c_2(a_1 b_2 - T(a) S_a(b_2)) - b_2(a_1 c_2 - T(a) S_a(c_2)) \in J(2, P) \cap (m_B \cdot B(2, P)).
\end{equation}
But (19) is
\[T(a)[S_a(c) b_2 - c_2 S_a(b)].\]
By Proposition 7.1 the second factor is in $J(2, P)$ and since $T(a)$ is in $m_B$ we are done.

Consider now relations of type 2. It is enough to show that
\begin{equation}
b_1(a_1 c_2 - T(a) S_a(c_2)) - a_1(b_1 c_2 - T(b) S_b(c_2)) \in J(2, P) \cap (m_B \cdot B(2, P)).
\end{equation}
But (20) is
\[S_b(c_2)[a_1 T(b) - T(a) R(a, b) b_1].\]
By Proposition 7.5 below, the second factor is in $J(2, P)$. It is also in $m_B$ due to the $T$’s in the second factor, and so we are done. □

Recall the positive $\mathbb{Z}([2] \times P)$-grading on $B(2, P)$ from Section 5.

**Corollary 7.7.** Let $(L(2, P))$ be the ideal generated by $L(2, P)$ in $B(2, P)$. The $\mathbb{Z}([2] \times P)$-graded rings $B(2, P)/J(2, P)$ and $B(2, P)/(L(2, P))$ have the same Hilbert function $h : \mathbb{Z}([2] \times P) \to \mathbb{N}$.

**Proof.** We consider the situation in greater generality. We have an $A$-graded polynomial ring $S$, and a homogeneous ideal $I \subseteq S$. Let $B = \mathbb{k}[u_{k,k}]$ be an $A$-graded polynomial ring and $I \subseteq S \otimes \mathbb{k} B$ a flat deformation of $I$ over $B$. Suppose the $A$-grading on $(S \otimes \mathbb{k} B)/J$ is positive. Then we show that $(S \otimes \mathbb{k} B)/J$ and $(S \otimes \mathbb{k} B)/(I)$ have the same Hilbert function.
Let \( u_1 \) be a variable in \( B \), and \( B' = B / u_1 \cdot B \). Note that \( u_1 \) is \( A \)-homogeneous. Consider \( 0 \to B \overset{u_1}{\to} B' \to 0 \). Tensoring this exact sequence with the rings above and taking into account their flatness over \( B \), we have exact sequences

\[
0 \to (S \otimes_k B) / J \overset{u_1}{\to} (S \otimes_k B') / J \to (S \otimes_k B') / (I) \to 0
\]

By induction on the number of \( u \)-variables, we may assume that \((S \otimes_k B') / J'\) and \((S \otimes_k B') / (I)\) have the same Hilbert function. Then the same must be true for \((S \otimes_k B) / J\) and \((S \otimes_k B) / (I)\).

\[ \square \]

8. Rigidity of the deformation family

We show that the ideal \( J(2, P) \subseteq B(2, P) \) is a rigid ideal, meaning that every deformation of \( J(2, P) \) comes from a change of coordinates.

Let \( Y \) be a finite-dimensional vector space and \( U \subseteq Y \) a subspace. We also denote \( Y / U = X \). Fixing a splitting the reader may think of \( Y \) as \( X \oplus U \). We shall use such a direct decomposition in the arguments but will not need it for our statements. The space \( X \) may usually be thought of as the space generated by the variables \( p_1 \) and \( p_2 \) as \( p \) ranges over \( P \), and \( U \) as the space of variables \( u_{q, p} \) and \( u_{q, p} \) from Section 3. Our basic situation is an ideal \( J \subseteq \mathbb{k}[Y] \) such that \( \mathbb{k}[Y] / J \) is flat over \( \mathbb{k}[U] \). It is then a flat deformation of the ideal \( I = J \otimes_{\mathbb{k}[Y]} \mathbb{k}[Y/U] \) in \( \mathbb{k}[Y/U] \). The situation we have in mind is when \( J = J(2, P) \) and \( I = L(2, P) \).

Now we take a base change \( \mathbb{k}[U] \to \mathbb{k}[U]/(U)^2 \) and get an ideal

\[ J \otimes_{\mathbb{k}[U]} \mathbb{k}[U]/(U)^2 \subseteq \mathbb{k}[Y]/(U)^2 \]

which is a flat deformation of \( I \subseteq \mathbb{k}[Y] \) over \( \mathbb{k}[U]/(U)^2 \). Hence by Proposition 6.1 it induces a map

\[ U^* \xrightarrow{\alpha} \text{Hom}_{\mathbb{k}[Y/U]}(I, \mathbb{k}[Y/U]/I) \]

Recall the first cotangent cohomology group \( T^1(\mathbb{k}[Y/U]/I) \). It parametrizes the first order non-trivial deformations of \( I \). The following says that if the deformation of \( I \) over \( \mathbb{k}[U]/(U)^2 \) encompasses all non-trivial deformations, then all first order deformations of \( I \) come from infinitesimal coordinate changes of \( J \subseteq \mathbb{k}[Y] \).

**Lemma 8.1.** Suppose the composition

\[ U^* \xrightarrow{\alpha} \text{Hom}_{\mathbb{k}[Y/U]}(I, \mathbb{k}[Y/U]/I) \xrightarrow{\pi} T^1(\mathbb{k}[Y/U]/I) \]

maps \( U^* \) to a generating set for \( T^1 \).

Then the image of the composition (the first map is the map from Lemma 6.5)

\[ Y^* \xrightarrow{\beta} \text{Hom}_{\mathbb{k}[Y]}(J, \mathbb{k}[Y]/J) \xrightarrow{\eta} \text{Hom}_{\mathbb{k}[Y/U]}(I, \mathbb{k}[Y/U]/I) \]

is a generating set for \( \text{Hom}_{\mathbb{k}[Y/U]}(I, \mathbb{k}[Y/U]/I) \).

**Proof.** Let \( X = Y / U \) and fix a splitting \( Y = X \oplus U \). Let the \( u_i \)'s form a basis for \( U \). Consider elements in the ideal \( J \) written as

\[ f = f_0 + \sum_i u_i f_i + \text{terms involving degree two monomial in the } u_i, \]

where \( f_0 \in I \) and the \( f_i \) are in \( \mathbb{k}[X] \). Then the image of \( u_i^* \in U^* \subseteq Y^* \) by \( \beta \) is the map sending

\[ f \mapsto f_i + \text{ terms of degrees } \geq 1 \text{ in the } u_i. \]
The image of $u_i^*$ by $\alpha$ is the map sending
\[ f_0 \mapsto f_i. \]

Hence we see that restricted to $U^* \subseteq Y^*$ there is a commutative diagram
\[
\begin{array}{ccc}
U^* & \xrightarrow{=} & U^* \\
\downarrow{\beta} & & \downarrow{\alpha} \\
\text{Hom}_{k[Y]}(J, k[Y]/J) & \xrightarrow{\hat{\beta}} & \text{Hom}_{k[Y/U]}(I, k[Y/U]/I)
\end{array}
\]
(We do not get a commutative diagram if we replace $U^*$ by $Y^*$ at the upper left.) Consider the map
\[
\k[X] \otimes_k (X^* \oplus U^*) = \k[X] \otimes_k Y^* \xrightarrow{\hat{\beta}} \text{Hom}_{k[X]}(I, k[X]/I)
\]
coming from the composing the maps in (21). We have established that the map on the second factor identifies with $\alpha$. The cokernel of
\[
\k[X] \otimes_k X^* \rightarrow \text{Hom}_{k[X]}(I, k[X]/I).
\]
is $T^1(k[X]/I)$. Hence we get the lower row in the diagram below, and a commutative diagram
\[
\begin{array}{ccc}
\k[X] \otimes_k X^* & \longrightarrow & \k[X] \otimes_k (X^* \oplus U^*) \\
\downarrow{\beta} & & \downarrow{r_{\alpha}} \\
\text{Im}(\k[X] \otimes_k X^*) & \longrightarrow & \text{Hom}_{k[X]}(I, k[X]/I) \longrightarrow T^1
\end{array}
\]
Since the left and right maps are surjective, the middle one is also by the snake lemma.

\[\square\]

**Theorem 8.2.** Suppose $k[Y]$ has a positive grading by an abelian group and the subspace $U \subseteq Y$ and the ideal $J$ are homogeneous for this grading. If the composition
\[ U^* \rightarrow \text{Hom}_{k[Y/U]}(I, k[Y/U]/I) \rightarrow T^1(k[Y/U]/I) \]
is surjective, then $J \subseteq k[Y]$ is a rigid ideal.

**Proof.** By the lemma above we know that the composition
\[ Y^* \otimes k[Y] \xrightarrow{\hat{\beta}} \text{Hom}_{k[Y]}(J, k[Y]/J) \xrightarrow{q} \text{Hom}_{k[Y/U]}(I, k[Y/U]/I) \]
is surjective. We shall show by induction on $U$ that whenever the composition $q \circ \hat{\beta}$ is surjective, then $\hat{\beta}$ is surjective. This will prove that $J$ is rigid.

Let $V$ be a subspace of $U$ of codimension one, homogeneous for the grading. Let $\overline{\mathcal{J}}$ be $J \otimes_{k[Y]} k[Y/V]$. By base extension $k[Y/V]/\overline{\mathcal{J}}$ is flat over $k[Y/V]$. We have a factorization
\[ Y^* \otimes k[Y] \xrightarrow{\hat{\beta}} \text{Hom}_{k[Y]}(J, k[Y]/J) \xrightarrow{q_1} \text{Hom}_{k[Y/V]}(\overline{\mathcal{J}}, k[Y/V]/\overline{\mathcal{J}}) \xrightarrow{r} \text{Hom}_{k[Y/U]}(I, k[Y/U]/I) \]
By induction, it is sufficient to prove that $q_1 \circ \hat{\beta}$ is surjective. Let us rename $Y/V$ as $\overline{Y}$. Note that $U/V = (u)$ generated by one element. We must then show that if the composition
\[ Y^* \otimes k[Y] \xrightarrow{\hat{\beta}_1} \text{Hom}_{k[Y]}(\overline{\mathcal{J}}, k[\overline{Y}]/\overline{\mathcal{J}}) \xrightarrow{r} \text{Hom}_{k[Y/(u)]}(I, k[Y/(u)]/I) \]
is surjective, then the map $\hat{\beta}_1$ is surjective. Let $\phi$ in $\text{Hom}_{k[Y]}(\overline{\mathcal{J}}, k[\overline{Y}]/\overline{\mathcal{J}})$ map to zero in $\text{Hom}_{k[Y/(u)]}(I, k[\overline{Y}/(u)]/I)$. We first show that $\phi = u \psi$ for some $\psi \in \text{Hom}_{k[Y]}(\overline{\mathcal{J}}, k[\overline{Y}]/\overline{\mathcal{J}})$. Let $\{f_i\}$ be a generating set for $\overline{\mathcal{J}}$ and suppose $f_i \mapsto u g_i$ by $\phi$. Let $\sum_i r_i f_i$ be a relation
between the $f_i$. Then $u \sum r_i g_i$ is zero in $k[Y]/J$. But due to flatness of $k[Y]/J$ over $k[u]$, the element $u$ is a nonzero divisor. Hence $\sum r_i g_i$ is zero in $J$ and so $f_i \mapsto g_i$ gives an element $\psi \in \text{Hom}_{k[Y]}(J, k[Y]/J)$. So there is an injection
\[
\text{Hom}_{k[Y]}(J, k[Y]/J) \otimes_{k[Y]} k[Y]/(u) \hookrightarrow \text{Hom}_{k[Y]_u}(I, k[Y]/(u))/I).
\]
But this must also be a surjection since the composition $r \circ \hat{\beta}_1$ is surjective. Hence the above is an isomorphism, and so
\[
\text{Hom}_{k[Y]}(J, k[Y]/J) \otimes_{k[Y]} k \cong \text{Hom}_{k[Y]_u}(I, k[Y]/(u))/I) \otimes_{k[Y]_u} k.
\]
We apply Nakayama’s lemma applied to the finitely generated module $\text{Hom}_{k[Y]}(J, k[Y]/J)$ over the positively graded ring $k[Y]$. Since the image of $Y^*$ generates $\text{Hom}_{k[Y]_u}(I, k[Y]/(u))/I)$ its image must then also generate $\text{Hom}_{k[Y]}(J, k[Y]/J)$.

\[\text{Corollary 8.3.} \quad \text{The ideal } J(2, P) \subseteq B(2, P) \text{ is a rigid ideal.}\]

9. THE MULTIGRADED HILBERT SCHEMES

In the previous section we established that the ideals $J(2, P)$ were rigid. This is a property of (infinitesimal) deformation theory, concerned with first order deformations. This section is concerned with the global deformation family. The moduli spaces of families of quotient rings of a polynomial ring, are the Hilbert schemes. We shall establish that the family of quotient rings $B(2, P)/J(2, P)$ by coordinate changes maps dominantly onto any component of the Hilbert scheme containing $L(2, P)$. In order to have a Hilbert scheme, we need a grading on the ring $B(2, P)$. We shall follow the most general such approach, that of Haiman and Sturmfels considering multigraded Hilbert schemes [11]. Continuing the setting of Section 8, we assume that $Y$ is graded by an abelian group $A$. Then $A$ gives a grading on the polynomial ring $k[Y]$. We assume that $U$ is a homogeneous subspace of $Y$, and that $J$ is a homogeneous ideal for this $A$-grading. The grading on $k[Y]/U)/I$ is admissible if for each degree $a$, the graded piece $(k[Y]/U)/a$ is a finite-dimensional vector space.

9.1. The generic coordinate change. Choose a finite subspace $T \subseteq \text{Hom}_{k}(Y, k[Y]/U))_0$. We get a map
\[Y \to k[Y/U] \otimes_k T^* \to k[Y/U] \otimes_k k[T^*],\]
giving a morphism of algebras
\[k[Y] \to k[Y/U] \otimes_k k[T^*].\]
Each $t \in T$ corresponds to a point in $\text{Spec } k[T^*]$ or a map $k[T^*] \to k$. This induces from (22) a map $k[Y] \to k[Y/U]$. This is the natural map coming from $t \in \text{Hom}_{k}(Y, k[Y]/U))_0$. The map (22) induces the map
\[\tau : k[Y] \to k[Y/U] \otimes_k k[T^*],\]
where in the last map we have used the multiplication on $k[Y/U]$. Let $J = \tau(J)$. Note that the fiber over $0 \in \text{Spec } k[T^*]$ is $I \subseteq k[Y/U]$. The ideal $J \subseteq k[Y/U] \otimes_k k[T^*]$ is the ideal we get from $J$ by performing a generic coordinate change of $J$ and then restricting to $k[Y/U]$.
Example 9.1. Let $Y$ be $\langle x_1, x_2, u \rangle$ where $x_1$ and $x_2$ have degree 1 and $u$ has degree 2. The subspace $U$ is generated by $u$. Let $T = \text{Hom}(\langle x_1, x_2, u \rangle, \mathbb{k}[x_1, x_2, u])_0$. The space $T$ is eight-dimensional. A basis of $T$ are the maps

$$
t_{11} : x_1 \mapsto x_1, \quad x_2 \mapsto 0, \quad u \mapsto 0
$$

$$
t_{12} : x_1 \mapsto x_2, \quad x_2 \mapsto 0, \quad u \mapsto 0
$$

$$
\ldots
$$

$$
t_{u,22} : x_1 \mapsto 0, \quad x_2 \mapsto 0, \quad u \mapsto x_2^2
$$

The basis of $T$ is (the meaning of the maps should be clear from the above)

$$
t_{11}, t_{12}, t_{21}, t_{22}, t_{u,u}, t_{u,11}, t_{u,12}, t_{u,22}.
$$

and their dual elements give a basis for $T^*$. Denote the other copy of $Y$ by $\langle y_1, y_2, v \rangle$.

We consider the homogeneous polynomial $x_1^2 x_2 + u x_1$ in $\mathbb{k}[Y]$. Applying the map $\tau$ this becomes

$$
x_1^2 x_2 + u x_1 \mapsto (x_1 + y_1)^2(x_2 + y_2) + (u + v)(x_1 + y_1)
$$

$$
\alpha \circ \beta \circ \gamma : (x_1 + t_{11}^* x_1 + t_{12}^* x_2)^2(x_2 + t_{21}^* x_1 + t_{22}^* x_2)
$$

$$
+ (t_{u,u}^* x_1 + t_{u,11}^* x_1 x_2 + t_{u,22}^* x_2)(x_1 + t_{11}^* x_1 + t_{12}^* x_2).
$$

This is the form we get by performing a generic coordinate change on the polynomial $x_1^2 x_2 + u x_1$ and then taking the image in the quotient ring $\mathbb{k}[Y/U] = \mathbb{k}[x_1, x_2]$.

Proposition 9.2. Suppose the $A$-grading on $\mathbb{k}[Y/U]/I$ is admissible. There is a localization $\mathbb{k}[T^*_f]$ with $f(0) \neq 0$ such that $J_f \subseteq \mathbb{k}[Y/U] \otimes \mathbb{k}[T^*_f]$ is flat over $\mathbb{k}[T^*_f]$.

Proof. Let $X = Y/U$ and fix a splitting $Y/U \rightarrow Y$ such that $Y = X \oplus U$. The elements in $T$ have degree 0. Note that for each degree $a \in A$ then $(\mathbb{k}[X, T^*/J])_a$ is a finitely generated $\mathbb{k}[T^*_f]$-module.

Let $\mathfrak{m}$ be the maximal ideal in $\mathbb{k}[T^*]$ corresponding to the origin. We obtain a localized ideal $((J^*)_\mathfrak{m} \subseteq \mathbb{k}[X, T^*]_\mathfrak{m}$. We first show that $((\mathbb{k}[X, T^*/J])_\mathfrak{m}$ is flat over the local ring $\mathbb{k}[T^*_\mathfrak{m}]$. By the local criterion of flatness, Theorem A.5 in [18, App.A], it will be enough to show that $\mathbb{k}[X, T^*/((T^*_p + J)/T^*_p]$ is flat over $\mathbb{k}[T^*/((T^*_p$) for each natural number $p$. (Strictly speaking this theorem applies to the situation of a local homomorphism of local noetherian rings, but the argument works just as well when the codomain ring has positive grading.)

Let the $f_k$ form a generating set for $J$. We write $f_k'$ for their images in $J/(U)^p \cdot \mathbb{k}[X, U]/(U)^p$. Let $f_k$ denote the image in $I \subseteq \mathbb{k}[X]$, and $f_k$ the images of $f_k$ in $((T^*_p + J)/(T^*_p)^p$. Due to flatness of the deformation over $\mathbb{k}[U]/(U)^p$, all relations $0 = \sum_k \gamma_k \bar{f}_k$ in $\mathbb{k}[X]$ lift to a relation $0 = \sum_k \gamma_k f_k'$ in $\mathbb{k}[Y]/(U)^p$. Substituting

$$
x_j \mapsto x_j + \sum_{\deg(m) = \deg(x_j)} t_{j,m}^* m, \quad u_k \mapsto \sum_{\deg(u_k) = \deg(n)} t_{k,n}^* n,
$$

we get a relation

$$
0 = \sum_k \gamma_k \bar{f}_k \text{ in } \mathbb{k}[X, T^*/(T^*_p]
$$

Hence all relations for $I$ lift to relations for $((T^*_p + J)/(T^*_p)^p$. By Corollary A.11 in [18, App.A] the quotient ring $\mathbb{k}[X, T^*/((T^*_p + J]$ is flat over $\mathbb{k}[T^*/(T^*_p]$. Hence $J_m \subseteq \mathbb{k}[X, T^*_m]$ is a deformation of $I$, flat over $\mathbb{k}[T^*_m]$. 

GUNNAR FLOYSTAD AND AMIN NEMATBAKHSH
In the following we use: Let $M$ be a finitely generated module over an integral domain $R$, and $p \in \text{Spec } R$. i) If the localization $M_p$ is a free $R_p$ module of rank $r$, then there is an open subset $\text{Spec } R_f \subseteq \text{Spec } R$ containing $p$ such that $M_f$ is free of rank $r$ on $R_f$. ii) If for some $p \in \text{Spec } R$, the fiber $M_{k(p)}$ is generated by $r'$ elements, there is an open subset $\text{Spec } R_g \subseteq \text{Spec } R$ containing $p$ such that $M_g$ is generated on $R_g$ by $r'$ elements. Hence $r' \geq r$ since $\text{Spec } R_g$ and $\text{Spec } R_f$ intersect nonempty. (They both contain the zero ideal).

We also use the following [11, Prop.3.2]:

Let $k[X]$ be an $A$-graded polynomial ring, and $h : A \to \mathbb{N}$ a Hilbert function. There is a finite subset of degrees $D \subseteq A$ such that the following holds. Let $I \subseteq k[X]$ be a monomial ideal which is i) generated in degrees $D$ and ii) its Hilbert function $h_I$ has $h_I(a) = h(a)$ for $a \in D$. Then $h_I(a) = h(a)$ for all $a \in A$.

Let $h : A \to \mathbb{N}$ be the Hilbert function of $k[X]/I$, and let $D$ be a finite set of degrees given by the above proposition. Every $((k[X,T^*/J])_a)_m$ is a free $T^*_m$-module of finite rank $h(a)$. We may find an elements $f_a \in k[T^*_a]$ with $f(0) \neq 0$ such that $((k[X,T^*/J])_a)_m$ is a free module of rank $h(a)$ for $a \in D$. Let $f = \prod_{a \in D} f_a$.

Suppose now there is some $a_0 \in A$ such that $((k[X,T^*/J])_{a_0})_f$ is not locally free. The free $k[T^*_m]$-module $((k[X,T^*/J])_{a_0})$ has rank $h(a_0)$. There must then be $t \in \text{Spec } k[T^*_f]$ such that the fiber $(k[X,T^*/J])_{k(t)}$ has dimension $> h(a_0)$ in degree $a_0$. We may write $(k[X,T^*/J])_{k(t)} = k(t)[X]/J'$ where $J'$ is the image of $J_{k(t)}$ in $k(t)[X]$. Fix a monomial order on $k(t)[X]$. Let $M$ be ideal generated by the initial monomials of the ideal of $J' \subseteq k(t)[X]$ in degrees $D$. Then $h_M(a) = h(a)$ for $a \in D$, but

$$h_M(a_0) = \dim_k((k(t)[X]/J')_{a_0} > h(a_0).$$

This contradicts [11, Prop.3.2] given above. Hence $((k[X,T^*/J])_f)_a$ is a locally free $k[T^*_f]$ module of rank $h(a)$ for every degree $a \in A$.

To the flat family $\tilde{J} \subseteq k[Y/U] \otimes_k k[T^*_f]$ we apply the base change $k[T^*_f] \to k[T^*/(T^*)^2]$. 

**Corollary 9.3.** Let $J' \subseteq k[Y/U] \otimes_k k[T^*/(T^*)^2]$ be the image of $J$ by this base change. The ideal $J'$ is a flat deformation of $J$ over $k[T^*/(T^*)^2]$.

Note that by Lemma 6.2 the map corresponding to this deformation:

$$T \to \text{Hom}_{k[Y/U]}(I, k[Y/U]/I)$$

is the map obtained from the composition

$$Y^* \to \text{Hom}_{k[Y]}(J, k[Y]/J) \to \text{Hom}_{k[Y/U]}(I, k[Y/U]/I)$$

by using the $k[Y/U]$-module structure on the right module.

9.2. The multigraded Hilbert scheme. We recall the multigraded Hilbert scheme as introduced by Haiman and Sturmfels in [11]. As before $A$ is an abelian group and $X$ a finite dimensional $A$-graded vectorspace over $k$, so the polynomial ring $k[X]$ becomes $A$-graded. The Hilbert schemes come about by introducing a Hilbert function $h : A \to \mathbb{N}$. There is then a Hilbert scheme $H^h_{k[X]}$ parametrizing all ideals $I \subseteq k[X]$ such that $\dim_k(k[X]/I)_a = h(a)$ for $a \in A$. More precisely, let $k - \text{Alg}$ be the category of $k$-algebras. Then $H^h_{k[X]}$ is a $k$-scheme which represents the point functor

$$\hat{H}^h_{k[X]} : k - \text{Alg} \to \text{Set}.$$
where \( \hat{H}^h_{k[X]}(R) \) is the set of all ideals \( J \subseteq R[X] \) such that \( (R[X]/J)_{\alpha} \) is a locally free \( R \)-module of rank \( h(\alpha) \). In particular an ideal \( I \subseteq k[X] \) with Hilbert function \( h \) corresponds to a \( k \)-point \( \text{Spec} k \xrightarrow{p} H^h_{k[X]} \) of the Hilbert scheme.

We may restrict \( \hat{H}^h_{k[X]} \) to the category \( k – \text{Art} \) of local artinian \( k \)-algebras with residue field \( k \). If \( A \to k \) is the augmentation map, it induces

\[
\hat{H}^h_{k[X]}(A) \to \hat{H}^h_{k[X]}(k).
\]

(23) Let \( \hat{H}^h_{k[X]}(A) \) be the inverse image of the element \( (I \subseteq k[X]) \) in the right set of (23). Thus \( \hat{H}^h_{k[X]} \) is a subfunctor of \( \text{Def}_I \), giving the \( A \)-graded deformations of \( I \). Again we may restrict \( \hat{H}^h_{k[X]} \) to \( k – \text{vect} \) and we get the following specialization of Proposition 6.1.

**Proposition 9.4.** There is a natural isomorphism of functors between

\[
\hat{H}^h_{k[X]} : \text{Hom}_k(-, \text{Hom}_{k[X]}(I, k[X]/I)_0) : k – \text{vect} \to \text{Set}.
\]

Since \( \hat{H}^h_{k[X]} \) on \( k – \text{Alg} \) is represented by the Hilbert scheme \( H^h_{k[X]} \), the tangent space of this scheme at \( I \) is \( \text{Hom}_{k[X]}(I, k[X]/I)_0 \).

We now assume that \( U \subseteq Y \) is graded by the abelian group \( A \), and that \( I \) and \( J \) are \( A \)-graded ideals in \( k[Y/U] \) resp. \( k[Y] \). We assume the \( A \)-grading is admissible on \( k[Y/U]/I \) and so we have a Hilbert function:

\[
h : A \to \mathbb{N}, \quad h(\alpha) = \dim_k(k[Y/U]/I)_{\alpha}.
\]

There is a map

\[
\text{Hom}_k(Y, k[Y/U])_0 \to \text{Hom}_k(Y, k[Y/U]/I)_0.
\]

The right Hom-space is a finite dimensional vector space due to the grading being admissible.

**Theorem 9.5.** Suppose the \( A \)-grading on \( k[Y/U]/I \) is admissible and that the composition

\[
U^* \to \text{Hom}_k(Y/U)(I, k[Y/U]/I) \to T^1(k[Y/U]/I)
\]

maps \( U^* \) to a generating set of \( T^1 \). Let \( T \) be a finite dimensional subspace of \( \text{Hom}_k(Y, k[Y/U])_0 \) which maps surjectively to \( \text{Hom}_k(Y, k[Y/U]/I)_0 \).

Then the induced morphism from Theorem 9.2

\[
\text{Spec} k[T^*]_{f} \to H^h_{k[Y/U]}
\]

is surjective on tangent spaces at the origin \( 0 \in \text{Spec} k[T^*]_{f} \).

**Proof.** Let \( m \subseteq k[T^*]_{f} \) be the maximal ideal corresponding to the origin in \( k[T^*]_{f} \). The morphism

\[
\text{Spec} k[T^*]_{f}/m^2 \to H^h_{k[Y/U]}
\]

is then

\[
\text{Spec} k[T^*]/(T^*)^2 \to H^h_{k[Y/U]}.
\]

By Proposition 9.4 this corresponds to the map of tangent spaces

\[
T \to \text{Hom}_k(Y/U)(I, k[Y/U]/I)_0.
\]

(24) By the comment after Corollary 9.3 and Lemma 6.2 this map is obtained from the composition

\[
Y^* \to \text{Hom}_k(Y)(J, k[Y]/J) \to \text{Hom}_k(Y/U)(I, k[Y/U]/I)
\]
as
\[ T \to \text{Hom}(Y, k[Y/U]/I)_0 = (Y^* \otimes k[Y/U]/I)_0 \to \text{Hom}_{k[Y/U]}(I, k[Y/U]/I)_0. \]

Since the right map above is surjective by Lemma 8.1 the map (24) on tangent spaces, is surjective.

\[ \square \]

**Corollary 9.6.** The ideal \( I \) is a smooth point on the Hilbert scheme \( H^h_{k[Y/U]} \). The morphism \( \text{Spec} k[T^*] \to H^h_{k[Y/U]} \) is dominant on the component of the Hilbert scheme containing \( I \). So there is an open subset of the Hilbert scheme \( H^h_{k[Y/U]} \) such that the ideals in this open subset are obtained from \( J \subseteq k[Y] \) by a coordinate change, and then restricting to \( k[Y/U] \).

Let \( Y \subseteq \hat{Y} \) be an inclusion of finite-dimensional \( A \)-graded vector spaces. Consider the ideal \( (J) \subseteq k[\hat{Y}] \) generated by \( J \). Note that this identifies as \( J \cong (J) \subseteq k[\hat{Y}] \). Similarly we get an ideal \( (I) \subseteq k[\hat{Y}/U] \). Let \( \hat{h} \) denote the Hilbert function of the quotient ring, if \( h \) is the Hilbert function of the quotient ring of \( I \subseteq k[Y/U] \). The cotangent cohomology
\[ T^1((k[\hat{Y}/U]/(I)) = T^1(k[Y/U]/I) \otimes_{k[Y]} k[\hat{Y}]. \]

The theorem and corollary above still applies to this situation.

**Applications.** Let \( Z([2] \times P) \to A \) be a homomorphism of abelian groups. We take \( Y \) to be the space generated by the linear forms of \( B(2, P) = k[x_{[2] \times P}] \otimes_k [u_{q,p}, u_{q,p}] \), so \( Y \) is generated by the \( x \) and \( u \)-variables in this ring, and \( U \) the linear forms in \( k[u_{q,p}, u_{q,p}] \).

1. If \( A \) gives an admissible grading on \( k[X]/L(2,P) \), then for an open subset of the Hilbert scheme \( H^h_{k[X]} \), the ideals in this open subset come from a change of coordinates in \( J(2,P) \) and then restricting the ideal to \( k[X] \).

2. Let \( Y \subseteq \hat{Y} \) be such that \( \hat{Y}/Y \cong U \) as \( A \)-graded spaces, or equivalently \( \hat{Y}/U \cong Y \).

3. Often the \( A \)-grading on \( k[\hat{Y}/U]/(L(2,P)) \) is not admissible, but there is a space \( Y \subseteq Y^+ \subseteq \hat{Y} \) such that \( k[Y^+/U]/(L(2,P)) \) is admissible. For instance take \( A = Z \) and let the \( p \) map to positive values in \( Z \). Some of the \( u \)-variables typically map to negative values. Then \( k[Y] \) is infinite-dimensional in degree 0. Let \( \hat{Y} = Y \oplus V \) where \( V \) is a copy of \( U \) and let \( Y^+ \subseteq U \) and \( U^- \subseteq U \) be the variables of non-negative and negative degrees, and similarly for \( V \). Let
\[ J'(2,P) = J(2,P) \otimes_{k[Y]} k[Y/U^-] \]
so we are setting the \( u \)-variables of negative degrees equal to zero. Let \( Y^+ = Y \oplus V^+ \). Then there is an open subset of the Hilbert scheme of \( (L(2,P)) \subseteq k[Y^+/U] \) where the ideals in this open subset are obtained from \( (J(2,P)) \subseteq k[Y^+] \) by coordinate changes and then restricting to \( k[Y^+/U] \). Since \( Y^+/U \) only has elements of positive degree, all elements
of negative degree in $k[Y^+]$ map to zero. Also $(J(2, P))$ is generated by $J(2, P) \subseteq k[Y]$. Since $Y/U^- \cong Y^+/U$, this is then really simply a coordinate change of $J'(2, P)$.

**Example 9.7.** Let $P$ be the star poset with unique minimal element $a$ and 3 maximal elements $b, c, d$. Consider the standard $\mathbb{Z}$-grading on $k[x_{[2] \times P}]$. Since $\deg(u_{q,a}) = a_1 + a_2 - b_1 - c_1 - d_1$ with respect the standard grading on $k[x_{[2] \times P}]$ we have $\deg(u_{q,a}) = -1$. For any other $u_{q,p}$ we have $\deg(u_{q,p}) = 1$. The space $U^-$ is generated by $u_{q,a}$, the grading is positive (and so admissible) on $k[Y/V] = k[x_{[2] \times P}, u_{a,b}, u_{a,c}, u_{b,c}, u_{e,b}]$. Let $J' \subseteq k[Y/U^-]$ be the ideal obtained from $J$ by setting $u_{q,a} = 0$. This is the ideal generated by the forms (1) after setting $u_{q,a} = 0$. Note that $Y^+ = Y \oplus V^+$, and $Y^+/U \cong Y/U^-$ as $\mathbb{A}$-graded spaces. Then there is an open subset of the Hilbert scheme component of $(L(2, P)) \subseteq k[Y^+/U]$ such that the ideals in this open subset come from a coordinate change of $J(2, P) \subseteq k[Y]$, where $Y \cong Y'/U$ and $u_{q,a}$ will always be sent to a scalar.

**Conclusion** The letterplace ideal $I = L(2, P)$ is usually not rigid, but we see that something nearly as good holds when the Hasse diagram of the poset $P$ has tree structure. There is a “lifting” to a rigid ideal $J = J(2, P)$, and for an open set of the Hilbert scheme component of $L(2, P)$, all the ideals come from a coordinate change of $J(2, P)$.

We have also done computations investigating simple cases when $P$ does not have tree structure, f.ex. the four element diamond poset. It seems everything we show in this article also goes through. We therefore make the following conjecture.

**Conjecture 9.8.** For any finite poset $P$, the letterplace ideal $L(2, P) \subseteq k[x_{[2] \times P}]$ deforms to a rigid ideal $J(2, P)$ in a ring $k[x_{[2] \times P}] \otimes_k k[U]$, with the quotient ring by the ideal flat over a polynomial base ring $k[U]$. The ring $k[x_{[2] \times P}] \otimes_k k[U]$ is naturally positively graded by $\mathbb{Z}[[2 \times P]$ with $J(2, P)$ homogeneous for this grading.

10. **Appendix**

**Example 10.1.** Let $P$ be a poset with following Hasse diagram.

```
    f  g
   / \ /
 b c d
```

We start from the outer branch and compute the deformed relations.

Since $f$ is a maximal point we have

$$f_1f_2 - T(f) = f_1f_2 - u_{e,f}e_2 - u_{g,f}g_2$$
and similarly for $g$ we get the relation $g_1g_2 - u_{e,g}e_2 - u_{f,g}f_2$. By definition
\[
M(e) = \begin{bmatrix}
-u_{e,f} & f_1 & -u_{g,f} \\
-f_1 & g_1 \\
-u_{f,g} & -u_{g,f}
\end{bmatrix}
\]
and
\[
e_1e_2 - T(e)D(e)^c = e_1e_2 - u_{d,e}d_2(f_1g_1 - u_{g,f}u_{f,g}).
\]
Similarly we have $e_1f_2 - T(e)D(e)^f = e_1f_2 - g_1d_2u_{e,d}u_{d,e} - d_2u_{g,f}u_{e,g}u_{d,e}$. For $d_1f_2$ we have
\[
d_1f_2 - T(d)D(d)^eD(e)^f = d_1f_2 - (u_{a,d}a_2 + u_{b,d}b_2 + u_{c,d}c_2)u_{d,e}(g_1u_{e,f} + u_{g,f}u_{e,g})
\]
For the deformed relation of $a_1f_2$ we need the matrix $M(a)$ for the branch point $a$. The matrix $M(a)$ is given by
\[
\begin{bmatrix}
-u_{a,b} & b_1 & -u_{e,b} - (u_{d,b}e_1 + u_{e,b}u_{d,e}D(e)^c + u_{f,b}u_{d,e}D(e)^c + u_{f,b}u_{d,e}D(e)^f + u_{g,b}u_{d,e}D(e)^g) \\
-u_{a,c} & -u_{c,b} & c_1 - (u_{d,c}e_1 + u_{e,c}u_{d,e}D(e)^c + u_{f,c}u_{d,e}D(e)^c + u_{f,c}u_{d,e}D(e)^f + u_{g,c}u_{d,e}D(e)^g) \\
-u_{a,d} & -u_{b,d} & -u_{c,d}
\end{bmatrix}
\]
d_1

Now we have
\[
a_1f_2 - T(a)S_a(f) = a_1f_2 - u_{g,a}D(a)^dD(d)^eD(e)^fD(f)^f = a_1f_2 - u_{g,a}D(a)^dD(d)^e(u_{e,g}u_{g,f} + u_{e,f}g_1)
\]

Analogously, we can compute the remaining deformed relations and get the following flat family.

(1) $a_1a_2 - b_1c_1d_1u_{a,b} + b_1c_1u_{a,d}u_{b,c} + b_1f_1g_1u_{a,c}u_{d,c} + b_1f_1u_{g,a}u_{c,d}u_{d,e} + u_{g,b}u_{g,c}u_{f,d} - b_1u_{a,b}u_{a,c}u_{d,c}u_{d,e}u_{e,f} + b_1u_{g,b}u_{g,c}u_{f,d}u_{e,f} + b_1u_{a,b}u_{a,c}u_{d,c}u_{d,e}u_{e,f} + b_1u_{g,b}u_{g,c}u_{f,d}u_{e,f} + b_1u_{a,b}u_{a,c}u_{d,c}u_{d,e}u_{e,f}
\]
(2) $b_1b_2 - c_1d_1u_{a,b} - c_1u_{a,b}u_{b,c} - c_1u_{a,b}u_{b,c}u_{c,d} - c_1u_{a,b}u_{b,c}u_{c,d}u_{d,e} + c_1u_{a,b}u_{b,c}u_{c,d}u_{d,e} + c_1u_{a,b}u_{b,c}u_{c,d}u_{d,e} + c_1u_{a,b}u_{b,c}u_{c,d}u_{d,e} + c_1u_{a,b}u_{b,c}u_{c,d}u_{d,e}
\]
(3) $b_1b_2 - a_2u_{a,b} - a_2u_{a,b} - d_2u_{b,c} - d_2u_{b,c} - b_2u_{a,b}
\]
(4) $a_1c_2 - b_1d_1u_{a,c}u_{b,d} - b_1u_{a,c}u_{b,d}u_{a,c}u_{b,d}u_{c,d} - b_1u_{a,c}u_{b,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e} + b_1u_{a,c}u_{b,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e}u_{e,f} + b_1u_{a,c}u_{b,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e}u_{e,f} + b_1u_{a,c}u_{b,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e}u_{e,f} + b_1u_{a,c}u_{b,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e}u_{e,f}
\]
(5) $c_1c_2 - a_2u_{a,c} - b_2u_{a,c} - d_2u_{c,e} - c_2u_{c,e} - g_2u_{a,c}
\]
(6) $a_1d_2 - b_1c_1u_{a,d}u_{b,c} - c_1d_1u_{a,b}u_{b,c} - c_1u_{a,b}u_{b,c}u_{c,d} - c_1u_{a,b}u_{b,c}u_{c,d}u_{d,e} - c_1u_{a,b}u_{b,c}u_{c,d}u_{d,e}u_{e,f} - c_1u_{a,b}u_{b,c}u_{c,d}u_{d,e}u_{e,f} - c_1u_{a,b}u_{b,c}u_{c,d}u_{d,e}u_{e,f}
\]
(7) $d_1d_2 - c_2u_{a,d} - c_2u_{a,d} - c_1e_2u_{d,c}
\]
(8) $a_1c_2 - b_1c_1u_{a,b}u_{c,d} - b_1c_1u_{a,b}u_{c,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e} + b_1u_{a,b}u_{c,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e}u_{e,f} - c_1f_1g_1u_{a,b}u_{c,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e}u_{e,f} - c_1f_1g_1u_{a,b}u_{c,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e}u_{e,f} - c_1f_1g_1u_{a,b}u_{c,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e}u_{e,f}
\]
(9) $d_1c_2 - f_1g_1u_{a,b}u_{c,d} - f_1g_1u_{a,b}u_{c,d}u_{a,c}u_{b,d}u_{c,d}u_{d,e} + a_2u_{a,d}u_{d,e}u_{f,g} + b_2u_{a,d}u_{d,e}u_{f,g} + c_2u_{a,d}u_{d,e}u_{f,g}
\]
(10) $e_1e_2 - f_1g_1u_{a,d}u_{d,e} + d_2u_{d,e}u_{f,g}. $
\[ 11. \] Jurgen Herzog and Takayuki Hibi, Deformation theory of algebraic combinatorics and the monomial ideals.

\[ 12. \] Robin Hartshorne, Determinantal rings (Lecture Notes in Mathematics, 1327).

\[ 13. \] GUNNAR FLØYSTAD AND AMIN NEMATBAKHSH

\[ \text{Vanishing cotangent cohomology for plane monomial ideals.} \]

Deforming Stanley–Reisner schemes, Monomial ideals, and determinantal rings. Mathematische Annalen 338(4), 2007.

\[ \text{Manuscripta Mathematica 115, 2007.} \]

\[ \text{References} \]

1. Klaus Altmann, Mina Bigdeli, Jurgen Herzog, and Dancheng Lu, Algebraically rigid simplicial complexes and graphs, arXiv preprint arXiv:1503.08080 (2015).

2. Klaus Altmann and Jan Arthur Christophersen, Cotangent cohomology of Stanley-Reisner rings, Manuscripta Mathematica 115 (2004), no. 3, 361–378.

3. , Deforming Stanley–Reisner schemes, Mathematische Annalen 348 (2010), no. 3, 513–537.

4. W Bruns and U Vetter, Determinantal Rings (Lecture Notes in Mathematics, 1327), 1988.

5. Jan Arthur Christophersen, Deformations of equivariant Stanley–Reisner abelian surfaces, Advances in Mathematics 227 (2011), no. 2, 801–829.

6. Jan Arthur Christophersen and Nathan Owen Ilten, Vanishing cotangent cohomology for \( p \backslash \text{"aucker algebras}, \) arXiv preprint arXiv:1409.3432 (2014).

7. Corrado De Concini and Elisabetta Strickland, On the variety of complexes, Advances in Mathematics 41 (1981), no. 1, 57 – 77.

8. Alessio D’Ali, Gunnar Fløystad, and Amin Nematbakhsh, Resolutions of letterplace ideals of posets, preprint, arXiv (2016).

9. Viviana Ene, Jurgen Herzog, and Fatemeh Mohammadi, Monomial ideals and toric rings of Hibi type arising from a finite poset, European Journal of Combinatorics 32 (2011), no. 3, 404–421.

10. Gunnar Fløystad, Bjorn Moller Greve, and Jurgen Herzog, Letterplace and co-letterplace ideals of posets, preprint, arXiv:1501.04523 (2015).

11. Mark Haiman and B Sturmfels, Multigraded Hilbert schemes, Journal of Algebraic Geometry 13 (2004), no. 4, 2, 10, 12, 25, 27.

12. Robin Hartshorne, Deformation theory, vol. 257, Springer Science & Business Media, 2009.

13. Jurgen Herzog and Takayuki Hibi, Distributive lattices, bipartite graphs and alexander duality, Journal of Algebraic Combinatorics 22 (2005), no. 3, 289–302.

14. Monomial ideals, Springer, 2011.

15. Jan O Kleppe, Deformations of modules of maximal grade and the Hilbert scheme at determinantal schemes, Journal of Algebra 407 (2014), 246–274.

16. Ezra Miller and Bernd Sturmfels, Combinatorial commutative algebra, vol. 227, Springer Science & Business Media, 2005.

17. Alyson Reeves and Mike Stillman, Smoothness of the lexicographic point, Journal of Algebraic Geometry 6 (1997), no. 2, 235–246.

18. Edoardo Sernesi, Deformations of algebraic schemes, vol. 334, Springer Science & Business Media, 2007.

19. Jan Stevens, Deformations of singularities, Lecture Notes in Mathematics, vol. 1811, Springer-Verlag, Berlin, 2003.

Universitet i Bergen, Matematisk Institutt, Postboks 7803, 5020 Bergen, NORWAY
E-mail address: gunnar@mi.uib.no

Institute for Research in Fundamental Sciences (IPM), Tehran, IRAN
E-mail address: nematbakhsh@ipm.ir