Plastic anisotropic constitutive equation based on stress-rate dependence related to non-associated flow rule for bifurcation analysis

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Abstract. In metal forming, progress in material models is required to construct a general and reliable fracture prediction framework because of the increased use of advanced materials and growing demand for higher prediction accuracy. In this study, a fracture prediction framework based on bifurcation theory is constructed. A novel material model based on the stress-rate dependence related to a non-associated flow rule is presented. This model is based on a non-associated flow rule with an arbitrary higher-order yield function and a plastic potential function for any anisotropic material. This formulation is combined with the stress-rate-dependent plastic constitutive equation, which is known as the Ito-Goya rule, to construct a generalized plastic constitutive model in which non-normality and non-associativity are reasonably included. Then, by adopting three-dimensional bifurcation theory, which is referred to the 3D theory, a new theoretical framework for fracture prediction based on the initiation of a shear band is constructed. Using virtual material data, a numerical simulation is carried out to produce a fracture limit diagram, which is used to investigate the characteristics of the proposed methodology.

1. Introduction

To conduct accurate metal-forming simulations, progress in material models is required because conventional models are not sufficiently accurate for the recent rapid development of advanced materials and there is growing demand for accurate simulation. In particular, fracture prediction has attracted attention as a means of avoiding forming failure when using advanced metals; however, it remains a fundamentally difficult problem. Therefore, we have been engaged in the development of a novel material model and fracture prediction framework to improve the accuracy and reliability of fracture prediction.

To construct a reliable fracture prediction model, bifurcation theory has been adopted in this research, and a novel material model for conducting bifurcation analysis was developed. Although the onset of localized bifurcation is not equivalent to the rupture of a material, this phenomenon is closely related to material fracture, particularly in sheet metals. Therefore, the development of an analytical methodology for fracture prediction based on bifurcation theory is worthwhile. The use of bifurcation theory as a fracture prediction method is advantageous in terms of generality and objectivity, but bifurcation analysis using conventional material models...
sometimes produces poor results. General bifurcation theory was established by Hill [1] for plastic materials, on the basis of which many studies were carried out. These were based on a plane-stress condition and a normality rule; however, under these assumptions, accurate prediction of the initiation of a shear band, which is considered to be a sign of fracture, is almost impossible. Even using S–R (Stören–Rice) theory [2], in which the stress-rate dependence is considered, there is still the restriction of the plane-stress condition. Thus, to conduct bifurcation analysis appropriately, a new framework that can deal with three-dimensional bifurcation modes and abrupt changes in the stress field should be created.

In this study, a material model based on the stress-rate dependence related to a non-associated flow rule is presented. This model is based on a non-associated flow rule with an arbitrary higher-order yield function and a plastic potential function for any anisotropic material [3][4]. This formulation is combined with the stress-rate-dependent plastic constitutive equation, which is known as the Ito–Goya plastic constitutive equation [5], to construct a generalized plastic constitutive model in which non-normality and non-associativity can be reasonably included. Then, by adopting three-dimensional bifurcation theory, which is known as the 3D theory, a new theoretical framework for fracture prediction based on the initiation of a shear band is constructed.

In this paper, the above-mentioned theoretical framework is described. Then, using virtual material data, a numerical simulation is carried out to produce a fracture limit diagram to demonstrate the effectiveness of the proposed methodology.

2. Proposed model and framework for fracture prediction

2.1. Material model

First, the material model proposed by the authors, which plays an essential role in this study, is described. This model is constructed to express deformation anisotropy and yield stress anisotropy by using a non-associated flow rule formulation with the same number of material constants as that in Hill’s 1948 model. In the following, we define the yield function, the plastic potential function, and the equivalent plastic strain increment.

In the proposed model, we defined the yield function \( f(\sigma) \) as being equal to the equivalent stress, namely,

\[
f(\sigma) = \sigma = 2^{m_y} \sqrt{\frac{3}{2(F + G + H)}} (s_{my} \cdot A \cdot s_{my}).
\]

Here, the matrix \( A \) has anisotropic parameters in its diagonal terms, and the pseudo vector \( s_{my} \) is a set of deviatomic stress components raised to the power of \( m_y \). This higher-order function preserves the form of Hill’s quadratic yield function, that is, it contains the same anisotropic parameters \( F, G, H, L, M, N \). This feature is important because it is possible to construct a higher-order yield function by changing the power \( m_y \) without increasing the number of undetermined variables.

In our non-associated flow-rule-based formulation, a function different from the yield function is adopted as the plastic potential function, which provides the direction of the plastic strain increment of the subsequent state of current stress. In this study, the previously introduced function \( f(\sigma) \) is used as the yield function, and another function \( g(\sigma) \), which takes the same form as \( f(\sigma) \) but has different anisotropic parameters \( F^*, G^*, H^*, L^*, M^*, N^* \), is adopted as the plastic potential function. In this expression, asterisks are used to distinguish \( f(\sigma) \) from \( g(\sigma) \). For example, the anisotropy matrix \( A \) is changed to \( A^* \), in which the original parameters \( F, G, H, L, M, N \) are also changed to \( F^*, G^*, H^*, L^*, M^*, N^* \), respectively. To express another order of the function, the power variable \( m_y \) is used instead of \( m_y \). Thus, the plastic
potential function of the proposed model takes the form

$$g(\sigma) = \sigma^* = \frac{3m_p}{2(F^* + G^* + H^*)} \left( s_{m_p} \cdot A^* \cdot s_{m_p} \right). \quad (2)$$

From the definition of plastic work, an explicit expression for the equivalent plastic strain increment is obtained as

$$d\varepsilon^p = \frac{m_p \sigma^* \varepsilon^p}{\sigma} \sqrt{\frac{2(F^* + G^* + H^*)}{3}} \left( D^s_{m_p} \cdot d\varepsilon^p \right)^T \cdot \left\{ A^* \cdot \left( D^s_{m_p} \cdot d\varepsilon^p \right) \right\}. \quad (3)$$

The main disadvantage of the non-associated flow rule models is the increase in the number of unknown variables. Usually, these variables can be obtained from experiments such as tensile tests; therefore, the use of the non-associated flow rule model could lead to an increased number of experiments and measurements. In addition, if a higher-order function is required, the burden increases, making this approach impractical. Thus, to benefit from the non-associated flow rule model, an increase in the number of unknown variables should be avoided, which is achieved using the proposed model.

The plastic anisotropy characteristics of materials are classified into two categories, namely, plastic flow stress anisotropy and plastic deformation anisotropy. The former and latter should be incorporated in the yield function and the plastic potential function, respectively. Consider cold-rolled metal sheets under plane stress. Under plane stress, the number of variables is halved. The anisotropic variables that must be determined are $F, G, H$, and $N$ and $F^*, G^*, H^*$, and $N^*$. The former set expresses the yield stress anisotropy and the latter set expresses the deformation anisotropy.

The parameters associated with stress anisotropy, $F, G, H$, are determined by the yield stresses, which are obtained from tensile tests in the rolling direction and transverse direction and an equibiaxial test. The parameters associated with deformation anisotropy, $F^*, G^*, H^*$, and $N^*$, are determined by the $r$-values in the rolling, diagonal, and transverse directions. The remaining parameter $N$ is determined by optimization using the tensile test data in the diagonal direction. Note that the diagonal yield stress, usually denoted as $\sigma_{45}$, is not used because the directions except for the rolling and transverse directions are not anisotropic principal axes and are difficult to separate from the shear component. The orders of the functions, $m_y$ and $m_p$, should be determined before these anisotropic parameters because they specify the function type and have a different physical meaning from the anisotropic parameters.

### 2.2. Ito–Goya plastic constitutive model

Local bifurcation abruptly changes the current strain rate direction. Since classical $J_2$ theory does not allow the rotation of the strain rate direction caused by the subsequent stress rate direction, it is not appropriate for use in bifurcation problems. Therefore, in this study, the Ito–Goya plastic constitutive equation [5] is applied because it can take the dependence of the strain rate direction into account. The Ito–Goya plastic constitutive equation is expressed as

$$d\varepsilon^p = \Lambda (n_F : l_p) |d\sigma^*| [K_CL + (1 - K_C)n_N], \quad (4)$$

where $n_N$ is unit tensor called the natural direction. This tensor indicates the direction of the deviatoric stress rate, which is identical to that of the plastic strain rate. The unit tensor $n_F$ is the direction of the gradient of the yield function and $l$ is the direction of the current deviatoric stress. In Eq. (4), the parameter $K_C$, which takes a value between 0 and 1, indicates the dependence of the direction of the strain rate on the stress rate. When $K_C$ is equal to 1, the Ito–Goya constitutive equation becomes $J_2$ flow theory.
2.3. 3D local bifurcation theory

On the basis of Hill’s general bifurcation theory, bifurcation occurs when the following condition is satisfied:

\[ I[\Delta v] = \int \Delta L : \Delta \dot{S} dV = 0, \]  

(5)

where \( \Delta v \) is the velocity field and \( L \) and \( \dot{S} \) are the velocity gradient tensor and the first Piola-Kirchhoff stress tensor rate, respectively. \( \dot{S} \) can be represented using the Cauchy stress tensor as

\[ \dot{S} = D : \dot{\varepsilon} + \omega \cdot \sigma - \sigma \cdot \omega - L \cdot \sigma = A : L, \]  

(6)

where \( \dot{\varepsilon}, \omega, \) and \( D \) are strain rate tensor, spin tensor, and tangent stiffness tensor, respectively. \( A \) is a fourth-rank tensor that relates the nominal stress rate and the velocity gradient tensor \( L \).

To characterize the bifurcation mode, the velocity gradient tensor is allowed to be discontinuous when the velocity gradient tensor crosses the bifurcation border \( \Gamma \). \( L \) can be expressed in terms of the normal vector \( n \) on the bifurcation border and the local deformation mode vector \( m \) that is normal to \( n \) as

\[ L = m \otimes n. \]  

(7)

The mode vector \( m \) can be composed of two different vectors in the \( \Gamma \) plane, namely, \( m_{sh} \) and \( m_{sv} \), which are vectors in the horizontal and vertical directions, respectively. These vectors are expressed using three angle parameters, \( \phi, \psi \) and \( \theta \), as shown in Fig. 1. The expressions for these vectors are as follows:

\[ n = (\sin \phi \cos \psi, \sin \phi \sin \psi, \cos \phi), \]  

(8)

\[ m_{sh} = (-\sin \phi, \cos \psi, 0), \]  

(9)

\[ m_{sv} = (\cos \phi \cos \psi, \cos \phi \sin \psi, -\sin \phi), \]  

(10)

\[ m = m_{sh} \cos \theta + m_{sv} \sin \theta. \]  

(11)

Substituting Eqs. (6) and (7) into Eq. (5), we have the following bifurcation criterion:

\[ I[m, n; \sigma] = h H[m, n; \alpha] - \sigma \Sigma[m, n; \alpha], \]  

(12)

where the first and second terms of this functional are expressed as

\[ H[m, n; \alpha] = m \cdot n \cdot \overline{D}(s) \cdot n \cdot m, \quad \text{where} \quad \overline{D}(s) \equiv D(s)/h, \]  

(13)

\[ \Sigma[m, n; \alpha] = \frac{1}{2} [m \cdot \alpha \cdot m - n \cdot \alpha \cdot n]. \]  

(14)

In an elasto-plastic material subjected to large strain, ignoring elastic deformation, the tangent stiffness tensor can be assumed to be proportional to the hardening rate \( h \).

Then, the current stress is expressed by

\[ \sigma = \sigma \alpha, \quad \sigma = \sqrt{\sigma \cdot \sigma}, \quad \alpha = \sigma / \sigma, \]  

(15)

where \( \sigma \) and \( \alpha \) are the norm of the current stress tensor and normalization tensor, which gives the stress ratio for each stress component, respectively.

On the basis of these relations, we have the following local bifurcation criterion.

\[ \left( \frac{\sigma}{h} \right)_{cr} = \min \left( \frac{H[m, n; \alpha]}{\Sigma[m, n; \alpha]} \right). \]  

(16)
The bifurcation criterion represented by Eq. (16) indicates that the local bifurcation, which is specified by the mode vectors $\mathbf{m}$ and $\mathbf{n}$ based on the current stress ratio tensor $\mathbf{\alpha}$, should be identified by the ratio $\sigma/h$, which is the ratio of the stress level to the work-hardening. Mechanically, the stress $\sigma$ indicates the intensity of fracture initiation, and the hardening coefficient $h$ indicates the resistance of the material against fracture. Therefore, the formability represented in the $\sigma/h$ plane is free from the strain-path dependence that is usually observed in a typical FLD (forming limit diagram) represented in the strain space. Thus, because it is mechanically reasonable to exhibit forming limits in the $\sigma/h$ plane, this new expression is called the SHFLD ($\sigma/h$ FLD).

The fracture limits in the SHFLD can show 3D local bifurcation limits. The actual fracture is considered to lie between the lower bound represented by the S–R limit and the upper bound represented by the 3D local bifurcation limit.

3. Bifurcation analysis

Using the theoretical framework described in the previous section, numerical analysis was conducted to investigate the characteristics of the proposed method. In this analysis, assuming a plane stress condition, an isotropic material was considered with the following constants: $F = G = H = F^* = G^* = H^* = 1$, $L = M = N = L^* = M^* = N^* = 3$, Young’s modulus = 210 GPa, Poisson’s ratio $\nu = 0.3$, $n = 0.2$, and $K = 5.0 \times 10^8$ in the $n$-power law for material hardening. The orders of the yield function and the plastic potential function, $m_y$ and $m_p$, respectively, were set to 1. The parameters used in this investigation were the $K_C$ value in the Ito–Goya model and the strain value $\varepsilon_h$, which determines the evaluation point of the work-hardening coefficient included in parameter $\lambda$ in Eq. (4). In the construction of the tangent stiffness tensor in Eq. (13), the component of the original tensor $\mathbf{D}$ was assumed to be linear in terms of the hardening coefficient $h$; however, the tensor $\mathbf{D}$ is not actually linear in $h$. In this study, instead of dealing with a nonlinear tensor $\mathbf{D}$, we assumed it to be a linear tensor as for the hardening $h$ for computational simplicity, and we investigated the effect of the hardening coefficient on the analysis.

The bifurcation analysis was conducted as follows. The minimum value of the functional on the right-hand side of Eq. (16) was searched for by changing the variables included in the fracture mode vectors $\mathbf{m}$ and $\mathbf{n}$. The simulated annealing algorithm was adopted in this

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**Figure 1.** Description of localized bifurcation in an oblique shear-band plane $\Gamma$ with two mode vectors. (a) Bifurcation plane direction vector $\mathbf{n}$ and (b) bifurcation mode vector $\mathbf{m}$. 
optimization process. To calculate the yield function in the used equations, the stress ratio $\alpha$ was used to control the stress condition, for example, $\alpha = 0$ for uniaxial stress and $\alpha = 1$ for equi-biaxial stress. The obtained minimum values were used to show the initiation of bifurcation as a possible fracture onset in the fracture limit diagram in the $\sigma/h$ plane, as shown in Figs. 2 and 3.

Two calculations were performed to investigate the characteristics of the proposed method. The first was performed to investigate the effect of the hardening coefficient of the applied material on fracture limit prediction. The result is shown in Fig. 2. The hardening term in the functional was evaluated at different strains, $\varepsilon_h = 0.1, 0.5, \text{ and } 1.0$, to show the independence of the hardening property from the fracture limit exhibited in the $\sigma/h$ plane. As shown in this figure, there is almost no difference among these calculation results. This supports our assumption that the hardening property does not affect the search for the minimum and shows that the linearization procedure explained above is valid. The second calculation was performed to observe the effect of the $K_C$ value. Generally, as $K_C$ increases, the fracture limit deteriorates because of the growing stress-rate dependence. As shown in Fig. 3, the fracture limit lines descend with increasing $K_C$. This means that the bifurcation analysis carried out in this study was valid and shows physically reasonable behavior.

![Figure 2. SHFLD for different work-hardening evaluation points $\varepsilon_h$.](image1)

![Figure 3. SHFLD for different $K_C$ values.](image2)

4. Conclusion

In this paper, a novel fracture limit prediction framework based on non-associated flow rule and stress-rate-dependent constitutive equation was presented and analyzed. The proposed concept of SHFLD, which was introduced on the basis of 3D bifurcation theory, proved to be effective through some numerical experiments. However, the tested combinations of the variables were not sufficient; therefore, further investigation is required.

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