Time evolution of quantum systems with time-dependent non-Hermitian Hamiltonian and the pseudo Hermitian invariant operator

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Abstract

We study the time evolution of quantum systems with a time-dependent non-Hermitian Hamiltonian given by a linear combination of SU(1,1) and SU(2) generators. With a time-dependent metric, the pseudo-Hermitian invariant operator is constructed in the same manner as for both the SU(1,1) and SU(2) systems. The exact common solutions of the Schrödinger equations for both the SU(1,1) and SU(2) systems are obtained in terms of eigenstates of the pseudo-Hermitian invariant operator.

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1 Introduction

The use of invariants theory to solve quantum systems, whose Hamiltonian is an explicit function of time, has the advantage to offer an exact solution for problems solved by the traditional time-dependent perturbation theory. The existence of invariants (constants of the motion or first integral) introduced by Lewis \cite{11} and Lewis- Riesenfeld \cite{12} is a factor of central importance in the study of such systems. The invariants method is very simple due to the relationship between the eigenstates of the invariant operator and the solutions of the Schrödinger equation by means of the phases; in this case the problem is reduced to find the explicit form of the invariant operator and the phases. In most cases, use is made of the Lewis–Riesenfeld quadratic invariant to study two archetypal examples. One of these is the time-dependent generalized harmonic oscillator,
the Hamiltonian of which is a time-dependent function of the SU(1,1) generator and the other is the spin in a time-dependent varying magnetic field with Hamiltonian consisting of the SU(2) generator. In [3, 4, 5, 6], the SU(1, 1) and SU(2) time-dependent systems are exactly integrated and the time evolution operator are obtained thanks to the invariant Hermitian operator onto the unitary transformation approach.

There is a growing interest in the study of non-Hermitian Hamiltonian operators due to the fact that these operators may constitute valid quantum mechanical systems [7, 9, 10, 11, 12, 13], because under certain conditions, non-Hermitian Hamiltonians may have a real spectrum and therefore may describe realistic physical systems. It has been clarified [11, 12, 13] that a non-Hermitian Hamiltonian having all eigenvalues real is connected to its Hermitian conjugate through a linear, Hermitian, invertible and bounded metric operator

\[
\eta = \rho + \rho^+ \rho
\]

with a bounded inverse, satisfying

\[
H^+ = \eta H \eta^{-1} \quad \text{i.e.} \quad H \text{ is Hermitian with respect to a positive definite inner product defined by } \langle \cdot, \cdot \rangle_\eta = \langle \cdot | \eta | \cdot \rangle
\]

and called as \( \eta \)-pseudo-Hermitian. Essentially the same idea had appeared previously under the name of “quasi-Hermiticity” by Scholtz et al [7]. It is also established [11, 12, 13] that the non Hermitian Hamiltonian \( H \) can be transformed to an equivalent Hermitian one given by \( h = \rho H \rho^{-1}, \) where \( h \) is the equivalent Hermitian analog of \( H \) with respect to the standard inner product \( \langle \cdot, \cdot \rangle. \)

All these efforts have been devoted to study time-independent non-Hermitian systems. Whereas the treatment for systems with time-dependent non-Hermitian Hamiltonians with time-independent metric operators have been extensively studied [14, 15], the generalization to time-dependent metric operators is quite controversial [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Recent contributions [25, 26] have advanced the grounds for treating time-dependent non-Hermitian Hamiltonians through time dependent Dyson maps. It has been argued that it is incompatible to maintain unitary time evolution for time-dependent non-Hermitian Hamiltonians when the metric operator is explicitly time dependent.

In light of the above discussion, one important question motivates our work here: How can we treat a non-Hermitian SU(1, 1) and SU(2) time-dependent quantum problems and investigate the possibility of finding the exact solution of the Schrödinger equation in terms of eigenstates of the pseudo- invariant operator as well the real associated phases?

Let’s first briefly recall the pseudo-Hermitian invariants theory [27, 31]. The invariant operator \( I_{PH}(t) \) is said to be pseudo-Hermitian with respect to \( \eta(t) \) if

\[
I_{PH}^\dagger(t) = \eta(t)I_{PH}(t)\eta^{-1}(t) \quad \Leftrightarrow \quad I_h(t) = \rho(t)I_{PH}(t)\rho^{-1}(t) = I_{h}^\dagger(t),
\]

where \( \eta(t) = \rho^+(t)\rho(t) \) is a linear time dependent Hermitian invertible operator. Thus \( I_{PH}(t) \) may be mapped to the Hermitian invariant operator \( I_h(t) \), by a similarity transformation \( \rho(t) \).

The solutions of the time-dependent Schrödinger equation ( \( \hbar = c = 1 \) are used throughout)

\[
i \frac{\partial}{\partial t} \left| \Phi(t) \right\rangle = H(t) \left| \Phi(t) \right\rangle,
\]

for the non- Hermitian Hamiltonian \( H(t) \) can be found with the aid of the quantum pseudo invariant method of Lewis and Riesenfeld. A pseudo-Hermitian invariant operator \( I_{PH}(t) \) for a given non-Hermitian Hamiltonian \( H(t) \) is defined to satisfy

\[
\frac{\partial I_{PH}(t)}{\partial t} = i \left[ I_{PH}(t), H(t) \right],
\]
where $I^{\text{PH}}(t)$ has a finite number of nondegenerate eigenstates $|\phi_n^H(t)\rangle$ satisfying

$$I^{\text{PH}}(t) |\phi_n^H(t)\rangle = \lambda_n |\phi_n^H(t)\rangle,$$

and

$$\langle \phi_m^H(t) | \eta(t) | \phi_n^H(t) \rangle = \delta_{m,n}$$

with time-independent eigenvalues $\lambda_n$. Since the Hermitian invariant $I^h(t)$ and the non-Hermitian invariant $I^{\text{PH}}(t)$ are related by a similarity transformation, they belong to the same similarity class and therefore have the same eigenvalues. The reality of the eigenvalues $\lambda_n$ is guaranteed, since one of the invariants involved, i.e. $I^h(t)$, is Hermitian.

Now, if the exact invariant $I^{\text{PH}}(t)$ (constant of motion) exists and does not contain any time derivative operators, we can write the solutions of the Schrödinger equation in terms of the eigenfunctions $|\phi_n^H(t)\rangle$ of $I^{\text{PH}}(t)$,

$$|\Phi_n^H(t)\rangle = e^{i\varphi_n(t)} |\phi_n^H(t)\rangle,$$

the phase functions $\varphi_n(t)$ are derived from the equation:

$$\frac{d\varphi_n(t)}{dt} = \langle \phi_n^H(t) | \eta(t) \left[ i\hbar \frac{\partial}{\partial t} - H(t) \right] |\phi_n^H(t)\rangle.$$

In Eq. (7), the first term is parallel to a familiar non-adiabatic geometrical phase, but the second term, representing effects due to a time-dependent Hamiltonian, is a dynamical phase. The sum of these two terms that can ensure that the phase functions $\varphi_n(t)$ are real.

The general solution of the Schrödinger equation for the system with non-Hermitian time-dependent Hamiltonians $H(t)$ and pseudo-Hermitian invariant operator are readily obtained as follows:

$$|\Phi^H(t)\rangle = \sum_n C_n e^{i\alpha_n(t)} |\phi_n^H(t)\rangle$$

where the $C_n = \langle \phi_n^H(0) | \eta(0) |\Phi^H(0)\rangle$ are time-independent coefficients.

Let’s note that, it then follows immediately by direct substitution of (11) into (3) that the two invariants operators $I^{\text{PH}1}(t)$ and $I^h(t)$ satisfy the following equations:

$$\frac{\partial I^{\text{PH}1}(t)}{\partial t} = i \left[ I^{\text{PH}1}(t), \eta(t) H(t) \eta^{-1}(t) + i\dot{\eta}(t) \eta^{-1}(t) \right],$$

$$\frac{\partial I^h(t)}{\partial t} = i \left[ I^h(t), \rho(t) H(t) \rho^{-1}(t) + i\dot{\rho}(t) \rho^{-1}(t) \right],$$

which show that the non-Hermitian Hamiltonian $H(t)$ is related to its Hermitian conjugate $H^\dagger(t)$ as

$$H^\dagger(t) = \eta(t) H(t) \eta^{-1}(t) + i\dot{\eta}(t) \eta^{-1}(t),$$

and the Hermitian Hamiltonian $h(t)$ is linked to the non-Hermitian Hamiltonian $H(t)$ by the time-dependent Dyson equation

$$h(t) = \rho(t) H(t) \rho^{-1}(t) + i\dot{\rho}(t) \rho^{-1}(t),$$
The above equations have been obtained by Fring and Moussa [25, 26] by assuming that the two solutions \(|\Phi^{H}(t)\rangle\) and \(|\Psi^{h}(t)\rangle\) of the two time-dependent Schrödinger equations ruled by \(H(t)\) and \(h(t)\) respectively, are related by a time-dependent invertible operator \(\eta(t)\) as \(|\Psi^{h}(t)\rangle = \eta(t) |\Phi^{H}(t)\rangle\). Then, they argued that the time-dependent quasi-Hermiticity relation and the time-dependent Dyson equation can be solved consistently in such scenario for a time-dependent Dyson map and time-dependent metric operator, respectively.

Our approach is different from Fring’s and Moussa’s one, because we resolve the standard quasi-Hermiticity relation and the standard Dyson equation (1) for a time-dependent invariant operator with time-dependent \(\eta(t)\) and a time-dependent similarity transformation \(\rho(t)\). While the key feature in Fring’s and Moussa’s approach is that the relation (11) is stated as the time-dependent quasi-Hermiticity relation. We believe that the resolution of the time-dependent Dyson equation and the time-dependent quasi-Hermiticity relation stated by Fring and Moussa become more difficult due to the presence of the last term in equations (11) and (12).

In this paper, we answer this question from a new perspective by studying the time-dependent non-Hermitian Hamiltonian systems given by a linear combination of SU(1, 1) and SU(2) generators using a pseudo-invariant operator theory which is constructed in a manner as for both the SU(1, 1) and SU(2) systems. An advantage of the pseudo-invariant operator is that it allows to obtain the exact solution of the Schrödinger equation in terms of eigenstates of the invariant operator as well as the time-evolution operator.

## 2 Evolution of non-Hermitian SU(1, 1) and SU(2) time-dependent systems

The SU(1, 1) and SU(2) time-dependent systems that we consider are described by the non-Hermitian Hamiltonian

\[
H(t) = 2\omega(t)K_0 + 2\alpha(t)K_- + 2\beta(t)K_+,
\]

(13)

where \((\omega(t), \alpha(t), \beta(t)) \in C\) are arbitrary functions of time. \(K_0\) is a Hermitian operator, while \(K_+ = (K_-)^{\dagger}\). The commutation relations between these operators are

\[
\begin{align*}
[K_0, K_+] &= K_+ \\
[K_0, K_-] &= -K_- \\
[K_+, K_-] &= DK_0
\end{align*}
\]

(14)

The Lie algebra of SU(1, 1) and SU(2) consists of the generators \(K_0, K_-\) and \(K_+\) corresponding to \(D = -2\) and 2 in the commutation relations (14), respectively.

In what follows, we investigate the quantum dynamics of our time-dependent systems (2) associated with the Hamiltonian (13). To this end, we consider the most general invariant \(I^{PH}(t)\) in the form

\[
I^{PH}(t) = 2\delta_1(t)K_0 + 2\delta_2(t)K_- + 2\delta_3(t)K_+,
\]

(15)

where \(\delta_1(t), \delta_2(t), \delta_3(t)\) are time dependent real parameters. The invariant (15) is of course manifestly non-Hermitian when \(\delta_2(t) \neq \delta_3(t)\).

As is well known [32, 33, 34] an element of the group of SU(1, 1) or SU(2) can be obtained by exponentiation of an element of the corresponding algebra. It is also well known that we can
The transformed invariant operator $I^h(t)$ is defined as:

$$
I^h(t) = \rho(t) I^P(t) \rho^{-1}(t)
$$

where

$$\rho(t) = \exp \left\{ 2 \left[ \epsilon(t) K_0 + \mu(t) K_- + \mu^*(t) K_+ \right] \right\},$$

and

$$\vartheta^\pm(t) = \frac{2 \mu^* \sinh \theta}{\theta \cosh \theta - \epsilon \sinh \theta} = -\zeta(t)e^{-i\varphi(t)},$$

$$\vartheta_0(t) = \left( \cosh \theta - \frac{\epsilon}{\theta} \sinh \theta \right)^{-2} = - \frac{D}{2} \zeta^2(t) - \chi(t),$$

$$\vartheta_-(t) = \frac{2 \mu \sinh \theta}{\theta \cosh \theta - \epsilon \sinh \theta} = -\zeta(t)e^{i\varphi(t)},$$

$$\chi(t) = -\frac{\epsilon}{\theta} + \frac{\mu}{\theta} \frac{\cosh \theta + \frac{\epsilon}{\theta} \sinh \theta}{\cosh \theta - \frac{\epsilon}{\theta} \sinh \theta}, \quad \theta = \sqrt{\epsilon^2 + 2D |\mu|^2}.$$}

This factorization is valid for SU(1, 1) ($D = -2$) and for SU(2) ($D = 2$).

The key point of our method is to solve the standard quasi-Hermiticity relation $[\text{eq} \ref{eq:1}]$ by making, for simplicity, the Hermitian ansatz $[\text{eq} \ref{eq:16}]$ for time-dependent invertible operator $\rho(t)$. Let us solve the standard quasi-Hermiticity relation $[\text{eq} \ref{eq:1}]$ by making the following general and, for simplicity, Hermitian ansatz for a time dependent metric $\rho(t)$. We obtain, after some algebra, the transformed invariant operator $I^h(t) = \rho(t) I^P(t) \rho^{-1}(t)$.

The derivation of equation $[\text{eq} \ref{eq:18}]$ is made of the following identities:

$$\begin{align*}
\exp \left[ \vartheta_- K_- \right] K_0 \exp \left[ -\vartheta_- K_- \right] &= K_0 + \vartheta_- K_- \\
\exp \left[ \vartheta_+ K_+ \right] K_0 \exp \left[ -\vartheta_+ K_+ \right] &= K_0 - \vartheta_+ K_+ \quad \text{[eq\ref{eq:19}]},
\end{align*}
$$

$$\begin{align*}
\exp \left[ \ln \vartheta_0 K_0 \right] K_- \exp \left[ -\ln \vartheta_0 K_0 \right] &= \frac{K_0}{\vartheta_0} \\
\exp \left[ \vartheta_+ K_+ \right] K_- \exp \left[ -\vartheta_+ K_+ \right] &= K_- + D \vartheta_+ K_0 - \frac{D}{2} \vartheta_+^2 K_+ \quad \text{[eq\ref{eq:20}]},
\end{align*}
$$

$$\begin{align*}
\exp \left[ \ln \vartheta_0 K_0 \right] K_+ \exp \left[ -\ln \vartheta_0 K_0 \right] &= \vartheta_0 K_+ \\
\exp \left[ \vartheta_- K_- \right] K_+ \exp \left[ -\vartheta_- K_- \right] &= K_+ - D \vartheta_- K_0 - \frac{D}{2} \vartheta_-^2 K_+ \quad \text{[eq\ref{eq:21}]},
\end{align*}
$$

For $I^h(t)$ to be Hermitian ($I^h(t) = I^{h*}(t)$) we require the coefficient of $K_0$ is real, and the coefficients of $K_-$ and $K_+$ are complex conjugate of one another. Using these two requirements, we have:
from the first constraint we derive the equality
\[ \delta_2 = \delta_3 \chi, \] (23)
while the other two constraints leads to
\[ \delta_1 = \left( \frac{D^2 \vartheta_+ - \chi}{\vartheta_-} \right) \delta_3, \]
\[ \delta_1 = \left( \frac{D^2 \vartheta_- - \chi}{\vartheta_+} \right) \delta_3. \] (24)

From the equations (24), it follows that \( \vartheta_+(t) = \vartheta_-(t) \equiv -\zeta(t) \) implying that the time dependent parameter \( \mu(t) \) must be real, i.e. \( \mu(t) = \mu^*(t) \). Finally the similarity transformation (16) maps the non-Hermitian quadratic invariant (15) into \( I^h(t) \) given by
\[ I^h(t) = \frac{2}{\vartheta_0} \left[ \left( \frac{D^2 \zeta^2}{2} - \chi \right) \delta_1 - 2D \chi \zeta \delta_3 \right] K_0. \] (25)

Let \( |\psi_n^h\rangle \) be the eigenstate of \( K_0 \) with eigenvalue \( k_n \) i.e.
\[ K_0 |\psi_n^h\rangle = k_n |\psi_n^h\rangle. \] (26)
The eigenstates of \( I^h(t) \) (25) are obviously given by
\[ I^h(t) |\psi_n^h(t)\rangle = \frac{2}{\vartheta_0} \left[ \left( \frac{D^2 \zeta^2}{2} - \chi \right) \delta_1 - 2D \chi \zeta \delta_3 \right] k_n |\psi_n^h\rangle, \] (27)
because of the time-dependence, the invariant \( I^h(t) \) is a conserved quantity whose eigenvalues are real constants. However, without loss of generality, the factor \( [(D^2 \zeta^2/2 - \chi) \delta_1 - 2D \chi \zeta \delta_3] / \vartheta_0 \) can be taken equal to 1. It follows that the eigenstate \( |\phi_n^H(t)\rangle \) of \( I^{PH}(t) \) can be directly deduced from the basis \( |\psi_n^h\rangle \) of its Hermitian counterpart \( I^h(t) \) through the similarity transformation \( |\phi_n^H(t)\rangle = \rho^{-1}(t) |\psi_n^h\rangle \) with time-independent eigenvalue \( k_n \).

According to the above discussions, the problem is reduced to find a pseudo Hermitian invariant operator and the suitable real phases of its eigenfunctions to take them as a solution for the Schrödinger equation. In a first step, we will determine the real parameters \( \delta_1, \delta_2, \delta_3 \) so that our invariant operator \( I^{PH}(t) \) (15) is pseudo Hermitian. Imposing the quasi-Hermiticity condition (1) on \( I^h(t) \), we get
\[ I^{PH}(t) = \rho^+(t) I^h(t) \rho^{-1+}(t) = 2\delta_1 \dot{K}_0 + 2\delta_3 \dot{K}_- + 2\delta_2 \dot{K}_+ \]
\[ = \frac{2}{\dot{\vartheta}_0} \left[ \left( \frac{D}{2} \zeta^2 - \chi \right) \dot{K}_0 - \zeta \dot{K}_- - \chi \zeta \dot{K}_+ \right]. \quad (28) \]

From the above equation the real parameters \( \delta_1, \delta_2, \delta_3 \) follow straightforwardly:
\[ \delta_1 = \frac{\left( \frac{D}{2} \zeta^2 - \chi \right)}{\vartheta_0}, \quad \delta_2 = -\frac{\chi \zeta}{\vartheta_0}, \quad \delta_3 = -\frac{\zeta}{\vartheta_0}. \quad (29) \]

Therefore, the pseudo Hermitian invariant operator \( I^{PH}(t) \) is written in the following form
\[ I^{PH}(t) = \frac{2}{\vartheta_0} \left[ \left( \frac{D}{2} \zeta^2 - \chi \right) K_0 - \chi \zeta K_- - \zeta K_+ \right]. \quad (30) \]

The second step in the method is imposing for \( I^{PH}(t)(30) \) the invariance condition \( (3) \) which lead to the following relations:
\[ \dot{\vartheta}_0 = \frac{2\vartheta_0}{\zeta} \left[ -2\zeta |\omega| \sin \varphi_\omega + |\alpha| \sin \varphi_\alpha + \left( \chi - \frac{D}{2} \zeta^2 \right) |\beta| \sin \varphi_\beta \right], \quad (31) \]
\[ \dot{\zeta} = -2\zeta |\omega| \sin \varphi_\omega + 2 |\alpha| \sin \varphi_\alpha - D \zeta^2 |\beta| \sin \varphi_\beta, \quad (32) \]
\[ \chi |\beta| \cos \varphi_\beta = |\alpha| \cos \varphi_\alpha \]
\[ \left( \chi - \frac{D}{2} \zeta^2 \right) |\alpha| \cos \varphi_\alpha = \chi \zeta |\omega| \cos \varphi_\omega, \quad (33) \]
\[ \zeta |\omega| \cos \varphi_\omega = \left( \chi - \frac{D}{2} \zeta^2 \right) |\beta| \cos \varphi_\beta \]

here, \( \varphi_\omega, \varphi_\alpha, \) and \( \varphi_\beta \) are the polar angles of \( \omega, \alpha, \) and \( \beta, \) respectively.

The final step consists in determining the Schrödinger solution \( (13) \) which is an eigenstate of the pseudo Hermitian invariant \( (30) \) multiplied by a time-dependent factor \( (7) \)
\[ \frac{d\varphi_n(t)}{dt} = \langle \phi_n^H(t) | \eta(t) \left[ i \frac{\partial}{\partial t} - H(t) \right] | \phi_n^H(t) \rangle \]
\[ = \langle \psi_n^h | i\rho \dot{\rho}^{-1} - \rho H \rho^{-1} | \psi_n^h \rangle. \quad (34) \]

Using the non-Hermitian Hamiltonian \( H(t)(13) \) and then deriving the transformed Hamiltonian \( [i\rho \dot{\rho}^{-1} - \rho H \rho^{-1}] \) through the metric operator \( \rho(t)(13) \), we further identify this transformed Hamiltonian as
\[ i\rho \dot{\rho}^{-1} - \rho H \rho^{-1} = 2W(t) K_0 + 2U(t) K_- + 2V(t) K_+, \quad (35) \]

where the coefficient functions are
\[ W(t) = -\frac{1}{\vartheta_0} \left[ \omega \left( \frac{D}{2} \zeta^2 - \chi \right) - D \zeta (\alpha + \beta \chi) + \frac{i}{2} \left( \dot{\vartheta}_0 + D \zeta \dot{\zeta} \right) \right], \quad (36) \]
\[ U(t) = \frac{1}{\vartheta_0} \left[ \omega \zeta - \alpha + \frac{D}{2} \beta \zeta^2 + \frac{i}{2} \dot{\zeta} \right], \quad (37) \]
\[ V(t) = \frac{1}{\vartheta_0} \left[ \omega \chi \zeta + \frac{D}{2} \alpha \zeta^2 - \beta \chi^2 - \frac{i}{2} \left( \zeta \dot{\vartheta}_0 - \vartheta_0 \dot{\zeta} + \frac{D}{2} \vartheta_0 \dot{\zeta} \right) \right]. \quad (38) \]
By using Eqs. (33), the above time-dependent coefficients $U$, $V$ are identically equal to zero ($U = V = 0$), whereas the coefficients $W$ is reduced to

$$W(t) = -\frac{1}{\vartheta_0} \left\{ \left( \frac{D}{2} \zeta^2 - \chi \right) |\omega| \cos \varphi_\omega - 2D\zeta |\alpha| \cos \varphi_\alpha - i\frac{\vartheta_0}{\zeta} \left[ |\omega| \sin \varphi_\omega - |\alpha| \sin \varphi_\alpha - \chi |\beta| \cos \varphi_\beta \right] \right\}.$$  

(39)

Knowing that the phase $\varphi_n(t)$ (34) must be real, we need to impose that the frequency $W(t)$ is real. Then, we obtain the exact phase of the eigenstate

$$\varphi_n(t) = -2k_n \int_0^t \frac{1}{\vartheta_0} \left[ \left( \frac{D}{2} \zeta^2 - \chi \right) |\omega| \cos \varphi_\omega - 2D\zeta |\alpha| \cos \varphi_\alpha \right] dt'.$$  

(40)

Therefore, the general solution (8) of the Schrödinger equation is given by

$$|\Phi^H(t)\rangle = \sum_n C_n(0) \exp \left( -ik_n \int_0^t \frac{2}{\vartheta_0} \left[ |\omega| \left( \frac{D}{2} \zeta^2 - \chi \right) \cos \varphi_\omega - 2D\zeta |\alpha| \cos \varphi_\alpha \right] dt' \right) |\phi_n^H(t)\rangle.$$  

(41)

3 Few special examples

3.1 Generalized time dependent non-Hermitian Swanson Hamiltonian

We now consider the SU(1,1) case first where $D = -2$. The SU(1,1) Lie algebra has a realization in terms of boson creation and annihilation operators $a^+$ and $a$ such that

$$K_0 = \frac{1}{2} \left( a^+ a + \frac{1}{2} \right), \quad K_- = \frac{1}{2} a^2, \quad K_+ = \frac{1}{2} a^{+2}.$$  

(42)

When the Hamiltonian (13) is expressed in terms of position $x$ and momentum $p$, it describes the generalized quadratic time-dependent non-Hermitian harmonic oscillator. The celebrated model of a non-Hermitian PT-symmetric Hamiltonian quadratic in position and momentum was studied first by Ahmed [35] and made popular by Swanson [36] when it was expressed in terms of the usual harmonic oscillator creation $a^+$ and annihilation $a$ operators with $\omega, \alpha$ and $\beta$ time-independent real parameters, such that $\alpha \neq \beta$ and $\omega^2 - 4\alpha\beta > 0$. This Hamiltonian has been studied extensively in the literature by several authors [37, 38, 39, 40, 41, 42].

We construct here, by employing the Lewis-Riesenfeld method of invariants, the solutions for the generalized version of the non-Hermitian Swanson Hamiltonian with time-dependent coefficients [26]

$$H(t) = \omega(t) \left( a^+ a + \frac{1}{2} \right) + \alpha(t) a^2 + \beta(t) a^{+2},$$  

(43)
where \((\omega(t), \alpha(t), \beta(t)) \in C\) are time-dependent parameters. The form for \(I_{PH}(t)\), which is both convenient for calculations, is

\[
I_{PH}(t) = \exp\left[\frac{\zeta}{2}a^2\right] \exp\left[-\frac{\ln \theta_0}{2}\left(a^+a + \frac{1}{2}\right)\right] \exp\left[\frac{\zeta}{2}a^{+2}\right] \left(a^+a + \frac{1}{2}\right) \times \exp\left[-\frac{\zeta}{2}a^{+2}\right] \exp\left[\frac{\ln \theta_0}{2}\left(a^+a + \frac{1}{2}\right)\right] \exp\left[-\frac{\zeta}{2}a^2\right],
\]

which brings out the Hamiltonian \(2K_0 = (a^+a + \frac{1}{2})\) of the usual harmonic oscillator whose eigenstates \(|n\rangle\) \((n = 0, 1, 3, \ldots)\) and eigenvalues \((n + \frac{1}{2})\) are well known. As eigenstates of \(I_{PH}(t)\) one can then take

\[
|\phi_n^H(t)\rangle = \exp\left[\frac{\zeta}{2}a^2\right] \exp\left[-\frac{\ln \theta_0}{2}\left\{(a^+a + \frac{1}{2}) - \left(n + \frac{1}{2}\right)\right\}\right] \exp\left[\frac{\zeta}{2}a^{+2}\right] |n\rangle,
\]

the corresponding phase \((40)\) \(\varphi_n(t)\) is

\[
\varphi_n(t) = (n + \frac{1}{2}) \int_0^t \frac{1}{\theta_0} \left(\zeta^2 + \chi\right) |\omega| \cos \varphi_\omega - 4\zeta |\alpha| \cos \varphi_\alpha \right] dt'.
\]

### 3.2 A spinning particle in a time-varying magnetic field

Now, we consider \(D = 2\) where the Hamiltonian \((13)\) and the invariant \((30)\) possess the symmetry of the dynamical group \(SU(2)\). There is substantial literature on the time evolution of two-level system governed by a non-Hermitian Hamiltonian \(H(t) = B(t)\sigma\) \([43, 44, 45, 46, 47, 48, 49]\), where \(\sigma\) is the vector of Pauli and the components of the field \(B(t)\) are complex.

Knowing that, the ferromagnetic materials like Cobalt and Iron produce magnetic fields whose magnitudes are measured by real numbers. Imaginary or complex fields are, however, essential in the fundamental theory that underlies the statistical physics of phase transitions, such as those associated with the onset of magnetization. Long thought to be merely mathematical constructs, a realization of these imaginary fields has now been observed in magnetic resonance experiments performed on the spins of a molecule \([50]\), following an earlier theoretical proposal. A spin in a time-varying complex magnetic field is a practical example for the case \(D = 2\). Let

\[
K_0 = J_z, \quad K_- = J_-, \quad K_+ = J_+,
\]

the Hamiltonian and the invariant are

\[
H(t) = 2\left[\omega(t)J_z + \alpha(t)J_- + \beta(t)J_+\right],
\]

\[
I_{PH}(t) = \frac{2}{\theta_0} \left[(\zeta^2 - \chi) J_z - \chi \zeta J_- - \zeta J_+\right],
\]

where \(\mathbf{J}\) is the spin angular momentum of the particle. The form for \(I_{PH}(t)\), which is both convenient for calculations, is

\[
I_{PH}(t) = \exp[\zeta J_-] \exp[-\ln \theta_0 J_z] \exp[\zeta J_+] J_z \exp[-\zeta J_z] \exp[\ln \theta_0 J_z] \exp[-\zeta J_-].
\]
The instantaneous eigenstates of $I^{PH}(t)$ can be written in terms of the eigenstates of $J_z$ denoted by $|m\rangle$, as

$$|\phi^H_m(t)\rangle = \exp[\zeta J_-] \exp[-\ln \vartheta_0 (J_z - m)] \exp[\zeta J_+] |m\rangle,$$

the corresponding eigenvalues are $m$. With the factor of $\exp[\ln \vartheta_0]$ included in the definition of $|\phi^H_m(t)\rangle$, the vector potential is singular only at the south pole.

For this case, the phase (40) $\varphi_m(t)$ is easy to calculate and is given by

$$\varphi_m(t) = -m\int_0^t \frac{2}{\vartheta_0} \left[(\zeta^2 - \chi) |\omega| \cos \varphi_\omega - 4\zeta |\alpha| \cos \varphi_\alpha \right] dt'.$$

(51)

Before concluding this paper, we give a particular case when the parameters of $H(t)$ are reals; i.e., $(\omega(t), \alpha(t), \beta(t)) \in \mathbb{R}$.

### 3.3 The Hamiltonian $H(t)$ with real coefficients $\omega(t), \alpha(t), \beta(t)$

When considering the time-dependent coefficients $\omega(t), \alpha(t), \beta(t)$ to be real functions instead of complex ones, the polar angles $\varphi_\omega, \varphi_\alpha, \varphi_\beta$ of $\omega, \alpha,$ and $\beta$, vanish. By imposing that $\varphi_\omega = \varphi_\alpha = \varphi_\beta = 0$, the Eqs.(31-33) are simplified to

$$\dot{\vartheta}_0 = 0,$$

(52)

$$\dot{\zeta} = 0,$$

(53)

$$\chi |\beta| = |\alpha|,$$

$$\zeta \omega_0 = \left(\chi - \frac{D}{2} \zeta^2\right) |\alpha| = \chi \zeta |\omega|.$$  

(54)

As one can see from the last equations that the metric parameters (17) $\zeta, \vartheta_0$ are constants. Thus, the time-dependent real coefficients $\omega(t), \alpha(t), \beta(t)$ of $H(t)$ provide a time-independent metric and consequently the gaugelike term $i\hbar \dot{\eta}(t) \eta^{-1}(t)$ in the quasi-Hermiticity relation (11) disappears and the standard quasi-Hermiticity relation $\eta(t) H(t) = H^\dagger(t) \eta(t)$ for the Hamiltonian $H(t)$ itself is recovered in complete analogy with the time-independent scenario. Thus $H(t)$ is self-adjointed operator and therefore observable and can be written in the following simple form

$$H(t) = 2\frac{\omega(t)}{\left(\frac{D}{2} \zeta^2 - \chi\right)} \left\{ \left(\frac{D}{2} \zeta^2 - \chi\right) K_0 - \chi \zeta K_- - \zeta K_+ \right\} = \frac{\omega(t)\vartheta_0}{\left(\frac{D}{2} \zeta^2 - \chi\right)} I^{PH}(t),$$

(55)

which reveal its self-adjointed character and therefore its observability. From Eqs.(54), we derive the metric parameter $\zeta$ in terms of parameters of the Hamiltonian $H(t)$

$$\zeta = \frac{1}{2 |\beta|} \left( -\frac{D}{2} |\omega| \pm \sqrt{|\omega|^2 + 2D |\alpha| |\beta|} \right).$$
4 Conclusion

The results we have presented offer a general and comprehensive treatment of the non-Hermitian dynamics of SU(1,1) and SU(2) quantum systems. Non-Hermitian Hamiltonian operators have been the subject of considerable interest during the last years within the framework of the PT symmetry and pseudo-Hermiticity theories.

Recently, It has demonstrated that a time-dependent metric operator cannot ensure the unitarity of the time evolution simultaneously with the observability of the Hamiltonian and thus the general framework for a description of a time evolution for time-dependent non-Hermitian Hamiltonians has been stated [25, 26]. A well-known method based on a time-dependent unitary transformation for the treatment of time-dependent Hermitian Hamiltonians [6, 51], has been adapted by the authors of Ref. [26] to solve the the time-dependent Dyson and the time-dependent quasi-Hermiticity relations for non-Hermitian Swanson Hamiltonian with time-dependent coefficients, where the time-dependent unitary transformation is replaced by a non-unitary transformation to conform to non-Hermitian Hamiltonians.

In this work, using the standard quasi-Hermiticity relation (11) between a non-Hermitian invariant operator $I^{PH}(t)$ and a Hermitian one $I^{H}(t)$, we have considered the dynamical behavior of SU(1,1) and SU(2) non-Hermitian time-dependent quantum systems by presenting an alternative approach to solve it. We investigated in detail the main frames of time-dependent non-Hermitian SU(1,1) and SU(2) systems in the framework of the Lewis and Riesenfeld method which ensures that a solution of the Schrödinger equation governed by a time-dependent non-Hermitian Hamiltonian is an eigenstate of an associated pseudo-Hermitian invariant operator $I^{PH}(t)$ with a time-dependent global real phase factor $\varphi_n(t)$.

The properties derived here help us to understand better systems described by time-dependent non-Hermitian Hamiltonians and should play a central role in time-dependent non-Hermitian quantum mechanics. After going through these properties, we then have presented two illustrative examples: the generalized Swanson model and a spinning particle in a time-varying magnetic field. When a time dependent parameters $\omega(t), \alpha(t), \beta(t)$ are supposed real, the standard quasi-Hermiticity relation for the time dependent Hamiltonian $H(t)$ occur and the metric operator become time-independent.

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