Universality in Quasi-normal Modes of Neutron Stars

L.K. Tsui and P.T. Leung

Physics Department, The Chinese University of Hong Kong, Shatin, Hong Kong SAR, China.

8 November 2018

ABSTRACT

We study the universality in gravitational waves emitted from non-rotating neutron stars characterized by different equations of state (EOS). We find that the quasi-normal mode frequencies of such waves, including the $w$-modes and the $f$-mode, display similar universal scaling behaviours that hold for most EOS. Such behaviours are shown to stem from the mathematical structure of the axial and the polar gravitational wave equations, and the fact that the mass distribution function can be approximated by a cubic-quintic polynomial in radius. As a benchmark for other realistic neutron stars, a simple model of neutron stars is adopted here to reproduce the pulsation frequencies and the generic scaling behaviours mentioned above with good accuracy.

Key words: gravitational waves - quasi-normal modes of compact stars - equation of state - stars: neutron - methods: analytical.

1 INTRODUCTION

The search for gravitational wave has been the goal of endeavour for generations of physicists since its existence was predicted in the theory of general relativity (see, e.g. a review by Thorne in Hawking & Israel 1987, chap. 9). In the beginning of a new millennium, the breakthroughs in technology and endless efforts of researchers strongly convince us that the first detection of gravitational waves emitted from violent stellar collapses will be realized within several years (see e.g. Hughes 2003, Mason 2004, and references therein). Gravitational wave detectors of different designs, including ground-based interferometers (e.g. LIGO and VIRGO), and resonant antennas (e.g. EXPLORER and NIBOSE), are either already in operation or in their final testing stages (Hughes 2003, Mason 2004). These detectors, together with the space-based interferometers LISA, which is currently under construction and is expected to be launched around 2011, can detect gravitational waves in different frequency ranges. As a consequence, gravitational-wave astronomy that enables researchers to probe astronomical activities at extremely large distances from gravitational waves detected emerges as a new subject of interest (Hughes 2003, Mason 2004).

Among all the possible sources of gravitational waves, pulsating neutron stars may probably be the most interesting one (Thorne & Campolattaro 1967). It is believed that gravitational waves emitted in binary neutron mergers are plausibly detectable with the current technology and the frequency of detection could be as high as several hundreds per year after upgrading the present detectors (Hughes 2003, Belczynski et al. 2001). Besides, waves generated in the formation of neutron stars via stellar core collapses might also be strong enough for detection provided that the collapse is sufficiently asymmetric (Fryer et al. 2002, Lindblom et al. 1998). Most interestingly, such gravitational waves are likely to carry the information about the internal structure of neutron stars from which they are emitted. As the structure of a neutron star, such as its radius and density distribution, certainly reflects the properties of nuclear and quark matters, a close examination of the wave signals is deemed rewarding (Anderson & Kokkotas 1996, 1998, Benhar et al. 1999, Kokkotas et al. 2001).

Gravitational waves are commonly analyzed in terms of quasi-normal modes (QNMs), which properly describe damped harmonic pulsations. A QNM oscillation has a time dependence $\exp(\omega t)$ and is characterized by a complex eigenfrequency $\omega = \omega_r + i\omega_i$ (Press 1977, Leaver 1985, Ching et al. 1996, Kokkotas & Schmidt 1999). The QNM frequencies of gravitational waves of a neutron star with a fixed mass $M$ are generally model-sensitive and significantly depend on the EOS adopted in the stellar model. However, some universal behaviours in the frequency $\omega_r$ and the damping time $\tau = 1/\omega_i$ of the leading gravitational wave $w$-mode of non-rotating neutron stars have recently been observed (Anderson & Kokkotas 1998, Benhar et al. 1999).
which can be summarized by the following pair of formulas:

\[
\omega_f \approx \frac{1}{R} \left[ a_1 \left( \frac{M}{R} \right) + b_1 \right],
\]

\[
\frac{1}{\tau} \approx \frac{1}{M} \left[ a_0 \left( \frac{M}{R} \right)^2 + b_0 \left( \frac{M}{R} \right) + c_1 \right].
\]

Here \( R \) is the radius of the star, \( a_1, b_1, a_0, b_0 \) and \( c_1 \) are model-independent constants determined from curve fitting. It is worthy of remark that such simple scaling formulas apply to both axial and polar gravitational wave \( w \)-modes of neutron stars [Andersson & Kokkotas 1998, Benhar et al. 1999].

On the other hand, QNM frequencies of the fluid \( f \)-mode are discovered to follow approximately another pair of universal formulas [Andersson & Kokkotas 1998]:

\[
\omega_f \approx \alpha_f \left( \frac{M}{R} \right)^{1/2} + \beta_f,
\]

\[
\frac{1}{\tau} \approx \frac{M^2}{R^2} \left[ a_0 \left( \frac{M}{R} \right) + b_0 \right],
\]

where again \( \alpha_f, \beta_f, a_0, \) and \( b_0 \) are model-independent constants chosen to yield best fit. Based on the above-mentioned universal behaviours, it has been shown that the radius and the mass of a neutron star can be inferred from the pulsation frequencies of its fundamental fluid \( f \)-mode and the first \( w \)-mode, and the equation of state (EOS) can in turn be identified [Andersson & Kokkotas 1998, Benhar et al. 1999].

These universal behaviours are intriguing in their own right. Why can Eqs. (1-4) successfully reproduce the QNM frequencies for stars with different EOS? What are their physical interpretations? Is there any exception to them? In this paper, we seek the physical mechanism underlying the above-mentioned universality in the gravitational wave of non-rotating neutron stars. By properly scaling the axial and polar wave equations, we show that the scaled complex eigenfrequencies \( M\omega \) of axial and polar \( w \)-modes, and \( f \)-mode oscillations, to a good approximation, depend only on the compactness \( M/R \). Moreover, to interpret such EOS-independent generic behaviour, we use a simple stellar model — the Tolman VII model (TVIIM) — to approximate the mass distribution inside realistic neutron stars [Tolman 1934], and find that TVIIM is indeed a good approximation. We then look for the QNMs of TVIIM with the scaled wave equation. Interestingly enough, such QNMs manifestly depend only on the compactness of the star and, in addition, demonstrate the scaling behaviour discovered in realistic neutron stars. Therefore, the universality in the QNMs \( (w \text{-mode and } f \text{-mode}) \) is ascribable to the fact that that the mass distribution of most realistic neutron stars can be well approximated by TVIIM.

The organization of our paper is as follows. In Sect. 2, we show how the frequencies (including the real and the imaginary parts) of QNMs of neutron stars with different EOS can be captured by a single formula. In Sect. 3 we briefly review the equilibrium configuration of a neutron star and introduce TVIIM as an approximation of realistic neutron stars. Sections 4 and 5 study the scaling behaviours of the axial and the polar pulsation modes, respectively. In Sect. 6 we discuss potential application of our finding and consider some possible exceptional cases where obvious deviations from the universality are expected. We then conclude our paper in Sect. 7. Unless otherwise stated, geometrized units in which \( G = c = 1 \) are adopted in the following discussion.

## 2 Generic Behaviour of QNMs

The theory of non-radial oscillations of relativistic stars was pioneered by Thorne & Campolattaro [1967]. Later, Lindblom & Detweiler [1983], and Chandrasekhar & Ferrari [1991] further simplified the equations describing gravitational wave generation and fluid motion, which will be detailed in the later part of this paper. To display the universality in the oscillation modes, we numerically integrate the gravitational wave equations for neutron stars, and impose the outgoing wave boundary condition at spatial infinity to locate relevant QNMs. Eight different EOS, including models A (Pandharipande 1971a) and C (Pandharipande 1971b) proposed by Pandharipande, three models (AU, UU and UT) proposed by Wiringa et al. [1988], models APR1 and APR2 proposed by Akmal et al. [1998], and model GM24 (Glendenning 1997, p. 244) are considered in our papers. Figures 4 and 5 show the relation between \( M\omega \) and the compactness \( M/R \) for the first axial \( w \)-mode, the first polar \( w \)-mode and the \( f \)-mode, respectively. It is clearly shown that both \( M\omega_{i} \) and \( M\omega_{w} \) depend on the compactness in a universal way. In particular, as shown by the solid line in these figures, a quadratic function in \( M/R \), namely:

\[
M\omega = a \left( \frac{M}{R} \right)^2 + b \left( \frac{M}{R} \right) + c
\]

(5)

successfully reproduces the dependence on the compactness in each case. Here \( a, b \) and \( c \) are complex constants determined by regression and their values for individual cases are tabulated in Table 1 for reference. While reproducing the observed universality of the \( w \)-mode as described in 4 and 5, Eq. (5) is different from 4 and 5 discovered previously [Andersson & Kokkotas].
Universality in Quasi-normal Modes of Neutron Stars

Table 1. Values of $a$, $b$ and $c$ in (2).

| Wave type     | Data            | $a$     | $b$     | $c$     |
|---------------|-----------------|---------|---------|---------|
| 1st axial $w$-mode | realistic stars | $-3.9\pm15.6$ | $2.8\pm11.6$ | $-0.030\pm0.12$ |
|               | TVIIM           | $-4.4\pm16.3$ | $3.1\pm11.9$ | $-0.072\pm0.098$ |
| 2nd axial $w$-mode | realistic stars | $-13\pm18.1$  | $6.1\pm12.8$  | $0.034\pm0.11$   |
|               | TVIIM           | $-14\pm19.3$  | $6.7\pm13.3$  | $-0.049\pm0.060$ |
| 1st polar $w$-mode | realistic stars | $-8.2\pm110$  | $3.9\pm13.3$  | $0.055\pm0.043$  |
|               | TVIIM           | $-8.6\pm111$  | $4.2\pm13.6$  | $0.0080\pm0.014$ |
| 2nd polar $w$-mode | realistic stars | $-18\pm18.6$  | $8.0\pm13.1$  | $0.031\pm0.11$   |
|               | TVIIM           | $-20\pm19.9$  | $8.8\pm13.6$  | $-0.077\pm0.058$ |
| polar $f$-mode  | realistic stars | $0.15\pm15.8$ E-4 | $0.56\pm16.7$ E-4 | $-0.020\pm16.2$ E-5 |
|               | TVIIM           | $0.14\pm16.9$ E-4 | $0.60\pm17.2$ E-4 | $-0.027\pm16.8$ E-5 |

In fact, for nonzero $\beta$, Eq. (3) leads to explicit mass-depence of $M\omega$, which is not observed in our study. On the other hand, we note that $p$-mode QNM frequencies (not shown in the present paper), especially the imaginary part, do not demonstrate analogous behaviour. However, as shown in Fig. 4 and 5, such universality also exists in non-leading axial and polar $w$-modes.

The neutron stars considered here are constructed from nuclear matters with different compositions (e.g. pure neutron matter and hyperon matter) and different stiffness. It is well known that the mass-radius curve for neutron stars reveals marked EOS dependence. Yet the scaled QNM eigenfrequency, $M\omega$, can display an EOS-independent generic behaviour. Two questions arise naturally from our finding. First, why is the scaled eigenfrequency, $M\omega$, EOS-insensitive? Second, does the universal curve in Eq. (5) has any physical meaning? In the ensuing discussion we will work out the answers to these questions.

3 EQUILIBRIUM CONFIGURATION AND TVIIM

To look for a simple yet robust model for neutron stars, we first briefly review the equilibrium configuration of non-rotating neutron stars. In spherical coordinates, $(t, r, \theta, \varphi)$, the geometry of spacetime around a non-rotating neutron star is given by the line element:

$$ds^2 = -e^{\nu(r)}dr^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

where the metric coefficient $e^{\lambda(r)}$ is determined by the mass distribution function $m(r)$, the mass inside circumferential radius $r$, as follows:

$$e^{-\lambda(r)} = 1 - \frac{2m(r)}{r}.$$ (7)

On the other hand, the metric coefficient $e^{\nu(r)}$, the density of mass-energy, $\rho(r)$, and the pressure $P(r)$ obey the Tolman-Oppenheimer-Volkoff (TOV) equations (Tolman 1939, Oppenheimer & Volkoff 1939):

$$\frac{d\nu}{dr} = \frac{2m + 8\pi r^3 P}{r(r - 2m)},$$ (8)

$$\frac{dm}{dr} = 4\pi r^2 \rho,$$ (9)

$$\frac{dP}{dr} = -\frac{1}{2}(\rho + P)\frac{d\nu}{dr}.$$ (10)

These four equations, together with an EOS, $P = P(\rho)$, and the following boundary conditions at the surface of the star where $r = R$:

$$P(r = R) = 0,$$ (11)

$$e^{-\lambda(R)} = e^{\nu(R)} = 1 - \frac{2M}{R},$$ (12)

suffice to determine the equilibrium configuration of a neutron star of mass $M$ and the spacetime around it as well.

For a given physical EOS and a given central density $\rho(r = 0) = \rho_0$, this system of equations can be solved by numerical integration. However, the problem can also tackled by expanding relevant quantities as power series of $r$ about the origin (Chandrasekhar & Ferrari 1991):

$$\rho(r) = \rho_0 + \rho_2 r^2 + \ldots,$$ (13)

$$m(r) = 4\pi \left(\frac{\rho_0 r^3}{3} + \frac{\rho_2 r^5}{5} + \ldots\right),$$ (14)

$$P(r) = P_0 + P_2 r^2 + \ldots.$$ (15)
Similarly, the series expansions of the metric coefficients are readily obtainable from (7) and (8). Direct substitution of these expansion into the TOV equations and the EOS yields formulas expressing all the expansion coefficients in terms of the central density $\rho_0$ (Chandrasekhar & Ferrari 1991).

Motivated by this expansion scheme that works nicely about the origin $r = 0$, we consider a mass distribution:

$$m_c(r) = M \left[ \frac{5}{2} \frac{r}{R} \right]^3 - \frac{3}{2} \frac{r}{R}^5,$$

which was first introduced by Tolman (1939) and is commonly referred to as Tolman VII model (TVIIM). In addition to being a good approximation near the origin, TVIIM satisfies the boundary condition $\rho(r = R) = 0$ at the stellar surface, which is correct for most EOS as $P(r = R) = 0$. Therefore, it is reasonable to conjecture that TVIIM is able to provide a good global approximation to the exact mass distribution. The conjecture is indeed confirmed by the results shown in Fig. 6, where the normalized mass distribution function $m(r)/M$ is plotted against the scaled radius $r/R$ for stars described by different EOS with a common compactness $M/R = 0.2$. It is manifest that $m(r)$ obtained from stars with different EOS are close to that of TVIIM, and similar conclusion has also been drawn recently by Lattimer & Prakash (2001). This discovery suggests that the QNMs of TVIIM could be close to those of realistic neutron stars.

Moreover, TVIIM has a distinct feature that gives rise to the universal behaviours observed in the $w$-mode QNMs. Introducing the scale transformations:

$$\tilde{r} = r/M, \quad \tilde{m}(\tilde{r}) = m(r)/M, \quad \tilde{P}(\tilde{r}) = M^2 P(r), \quad \tilde{\rho}(\tilde{r}) = M^2 \rho(r), \quad \tilde{\nu}(\tilde{r}) = \nu(r),$$

we rewrite the TOV equations as follows:

$$\frac{d\tilde{m}}{d\tilde{r}} = 4\pi \tilde{r}^2 \tilde{\rho}, \quad \frac{d\tilde{\nu}}{d\tilde{r}} = \frac{2\tilde{m} + 8\pi \tilde{r}^3 \tilde{P}}{\tilde{r}(\tilde{r} - 2\tilde{m})}, \quad \frac{d\tilde{P}}{d\tilde{r}} = -\frac{1}{2} (\tilde{\rho} + \tilde{P}) \frac{d\tilde{\nu}}{d\tilde{r}}.$$  

These scaled TOV equations have exactly the same forms as the original ones. In particular, for TVIIM the scaled mass distribution function is given by:

$$\tilde{m}_c(\tilde{r}) \equiv m_c(r)/M = \frac{5}{2} \left( \frac{M}{R} \right) \tilde{r}^3 - \frac{3}{2} \left( \frac{M}{R} \right)^5 \tilde{r}^5.$$  

It is clear that the function $\tilde{m}_c(\tilde{r})$ is solely characterized by the compactness $M/R$ and is independent of the mass $M$. Consequently, other scaled functions (e.g. $\tilde{\rho}(\tilde{r})$, $\tilde{P}(\tilde{r})$ and $\tilde{\nu}(\tilde{r})$) associated with TVIIM depend only on $M/R$ as well. In fact, this statement holds as long as $\tilde{m}(\tilde{r})$ does not contain any explicit mass-dependence. However, as shown in Fig. 6, TVIIM provides the simplest, yet robust, global approximation to $m(r)$. Besides, TOV equation for TVIIM is exactly solvable (Tolman 1939), thus further simplifying relevant calculations. In the subsequent discussion, we will show that both the axial and the polar gravitational wave equations also exhibit similar scaling behaviour, thus leading to the universality in the QNMs.

4 SCALING BEHAVIOUR OF AXIAL QNMS

The equation of motion for relativistic stars can be decomposed in terms of spherical harmonics and in turn classified into axial and polar oscillations (Thorne & Campolattaro 1967, Lindblom & Detweiler 1983, Chandrasekhar & Ferrari 1991). The equation for the axial oscillations of neutron stars is given by a Regge-Wheeler-type equation (Chandrasekhar & Ferrari 1991):

$$\frac{d^2}{dr_*^2} + \omega^2 - V(r_*) \psi(r_*) = 0,$$

where the tortoise coordinate $r_*$ and the potential $V$ are defined by

$$r_* = \int_0^r e^{-(\nu + \lambda)/2} dr,$$

and
Universality in Quasi-normal Modes of Neutron Stars

\[ V(r_*) = \frac{e^\nu}{r^2} \left\{ l(l + 1)r + 4\pi r^3 (\rho - P) - 6m(r) \right\}, \]  

where \( r_* = r + 2M \ln \left( \frac{r}{2M} - 1 \right) + C, \) respectively. Outside the star, the pressure and the density vanish and hence the tortoise radial coordinate there can be simplified to

\[ V_{tw}(r_*) = \left( 1 - \frac{2M}{r} \right) \left[ \frac{l(l + 1)}{r^2} - \frac{6M}{r^3} \right]. \]

The axial mode wave equation can be scaled by multiplying \( M^2 \) to (28), yielding

\[ \frac{d^2}{dr_*^2} + \tilde{\omega}^2 - \tilde{V}(r_*) \tilde{\psi}_n(r_*) = 0, \]

where

\[ \tilde{\omega} = M\omega, \quad \tilde{r}_* = r_*/M, \quad \tilde{\psi}_n(r_*) = \psi(r_*), \]

and

\[ \tilde{V}(r_*) = \left\{ \begin{array}{ll}
\tilde{\rho} \tilde{\pi} - 6\tilde{m} & , \quad \tilde{r} \leq R/M;
\left( 1 - 2\tilde{\gamma}^{-1} \right) \left[ \frac{l(l + 1)}{r^2} - \frac{6\tilde{r}^{-3}}{r} \right] & , \quad \tilde{r} > R/M.
\end{array} \right. \]

For TVIIM, \( \tilde{m}(\tilde{r}) = \tilde{m}_c(\tilde{r}), \tilde{\rho} \) and \( \tilde{\pi} \) are universal functions of \( \tilde{r} \) with the compactness being the unique parameter. As a consequence, QNMs of TVIIM depend only on the compactness. For other realistic stars whose mass profiles are similar to that of TVIIM, it is plausible to expect that the scaled QNM frequency \( \tilde{\omega} \) is close to that of TVIIM with the same compactness. Hence, we argue that \( \tilde{\omega} \) of realistic stars has a trend similar to that of TVIIM. This conjecture is confirmed in Fig. 14 where QNMs of TVIIM (solid-circles) are shown to be close to those of other realistic stars. It is amazing that the dotted line, the best quadratic fit to QNMs of TVIIM, almost coincides with the solid line that is obtained by fitting a quadratic expression in \( M/R \) to \( \tilde{\omega} \) for stars with eight different EOS. Obviously, the universal behaviour in QNMs of neutron stars, including realistic ones and TVIIM, is well approximated by the dotted line. For reference and comparison, we record the values of the regression parameters \( a, b, \) and \( c \) (see (39)) of the dotted line in Table 1. One can see that the parameters \( a \) and \( b \) of the two lines are indeed quite close, which agrees with our observation from Fig. 14. On the other hand, there is a disparity between the parameters \( c \) for the two cases. However, such a disparity is obviously unimportant and does not affect the agreement between the two curves.

5 SCALING BEHAVIOUR OF POLAR QNMS

The equation of motion for polar case allows pulsating modes of various kinds, including the fundamental fluid \( f \)-mode, the gravitational wave \( u \)-mode, the pressure \( p \)-mode and the gravity \( g \)-mode (see, e.g., a recent review by Kokkotas & Schmidt 1999). To investigate why QNMs of \( f \)-mode and polar \( u \)-mode oscillations display generic behaviour as discussed above, we adopt the notation of Lindblom & Detweiler (1983) and consider the set of polar wave equations governing the variation of the three metric perturbation functions, \( H_0, H_1, \) and \( K, \) and the two fluid perturbation functions, \( W \) and \( V, \) inside the star \(( r \leq R \) \) (Lindblom & Detweiler 1983):

\[ H'_1 = -\frac{1}{r} \left\{ l + 1 + \frac{2m\lambda}{r} + 4\pi r^2 e^\lambda (P - \rho) \right\} H_1 + \frac{1}{r} e^\lambda [H_0 + K - 16\pi(p + P)V]. \]  

\[ K' = -\frac{1}{r} H_0 + \frac{1}{2r} [l(l + 1) - \frac{\nu^2}{2}] K - \frac{8\pi(p + P)\nu^{\lambda/2}}{r} W, \]  

\[ W' = -\frac{l(l + 1)}{r} W + \frac{r e^{\lambda/2}}{\gamma P} \left[ r e^{\nu/2} X - \frac{l(l + 1)}{r^2} V + \frac{1}{2} H_0 + K \right]. \]  

\[ X' = -\frac{1}{r} X + (p + P) e^{\nu/2} \left\{ \frac{1}{2} \left( \frac{1}{r} - \frac{\nu'}{2} \right) H_0 + \frac{1}{2} \left( \frac{r e^{\lambda/2}}{r^2} \right) [H_1 + \frac{1}{2} \left( \frac{3}{2} \nu' - \frac{1}{r} \right) K - \frac{1}{2} \left( \frac{l(l + 1)}{r^2} \right) \nu' V - \frac{1}{2} \left( \frac{e^{-\lambda/2} \nu'}{r^2} \right)] W \right\}. \]
Here $\gamma$ is the adiabatic index:
\[
\gamma = \left(1 + \frac{\rho}{P}\right) \frac{dP}{d\rho},
\]
and $X$ is defined as:
\[
X = \omega^2 (\rho + P) e^{-\nu/2} V - \frac{1}{r} P' e^{(\nu-\lambda)/2} W + \frac{1}{2} (\rho + P) e^{\nu/2} H_0,
\]
which is related to the Lagrangian perturbation in pressure, $\Delta p$, through the equation $X = -e^{\nu/2} \Delta p$ [Lindblom & Detweiler 1983]. These four first order ODE’s, together with the constraint equation, completely determine the dynamics inside the star.

Outside the star, the whole problem could be described by only two metric perturbation functions, namely $H_0$ and $K$. A linear combination of these two functions defines the Zerilli wave function $Z(r_*)$ that satisfies the Zerilli wave equation (Zerilli 1970):
\[
\left[\frac{d^2}{dr_*^2} + \omega^2 - V_2(r_*)\right] Z(r_*) = 0,
\]
with $V_2(r_*)$ being the Zerilli potential,
\[
V_2(r_*) = \frac{1 - 2M/r}{r^3 (nr^2 + 3M)^2} \left[2n^2 (n + 1)r^3 + 6n^2 Mr^2 + 18nM^2 r + 18M^3\right],
\]
and $n = (l - 1)(l + 2)/2$.

A close examination of the polar wave equations inside the star reveals that the scaling behaviour is likely to be marred. The culprit is the presence of the adiabatic index $\gamma$, which directly depends on $P$ (instead of $\tilde{P}$) and is hence not amenable to scaling. In fact, it has been shown QNMs of $p$-mode oscillations (especially the damping rate) do not follow the universal behaviour. However, as $X$ is proportional to the Lagrangian perturbation in the pressure, it should be negligibly small for $w$-mode and $f$-mode waves. To verify this issue, we solved the polar wave equations with the assumption $X = 0$ and found that eigenfrequencies of $w$-mode and $f$-mode waves are almost unaffected by such assumption (see Fig. 7 for the result in the $w$-mode case). This shows that the term $X/\gamma$ in (35) can be omitted as far as $w$-mode and $f$-mode waves are concerned.

Under the approximation $X = 0$, which is valid for $w$-mode and $f$-mode oscillations, we can show that the universality observed in the axial case also exists in the polar case. Again we have to introduce more scale transformations:
\[
\tilde{W} = W/M^2,
\]
\[
\tilde{X} = M^2 X,
\]
\[
\tilde{V} = V/M^2,
\]
while for $H_0$, $H_1$, $K$, $\lambda$, the transformed quantities are identical to the original ones. In terms of these scaled variables, Eqs. (45), (47) and (48) then become
\[
\tilde{H}_1 = \frac{1}{\tilde{r}} \left[l + 1 + \frac{2\tilde{n}\tilde{e}^{\lambda}}{\tilde{r}} + 4\pi\tilde{r}^2 e^{\lambda} (\tilde{\dot{P}} - \tilde{\rho})\right] \tilde{H}_1 + \frac{1}{\tilde{r}} e^{\lambda} [\tilde{H}_0 + \tilde{K} - 16\pi (\tilde{\rho} + \tilde{\dot{P}}) \tilde{V}],
\]
\[
\tilde{K}' = \frac{1}{\tilde{r}} \tilde{H}_0 + \frac{1}{2\tilde{r}} (l + 1) \tilde{H}_1 - \frac{8(\tilde{\rho} + \tilde{\dot{P}})^{\lambda/2}}{\tilde{r}} \tilde{W},
\]
\[
\tilde{W}' = -\frac{3(l + 1)}{\tilde{r}} \tilde{W} + \tilde{r} e^{\lambda/2} \left[l \tilde{W} + 12 \tilde{\dot{H}}_0 + 12\tilde{K}\right],
\]
where a prime in these transformed equations indicates differentiation with respect to $\tilde{r}$. Note that there is no need to consider $\omega_0$ because of the approximation $X = 0$. Besides, Eq. (44) can be rewritten accordingly and manifest scale invariance.

Similarly, we define the transformed Zerilli wave function $\tilde{Z} = Z/M$, which satisfies a scaled wave equation outside the star:
\[
\left[\frac{d^2}{d\tilde{r}^2} + \tilde{\omega}^2 - \tilde{V}_2(\tilde{r}_*)\right] \tilde{Z}(\tilde{r}_*) = 0,
\]
with the scaled potential $\tilde{V}_2(\tilde{r}_*)$ given by:
\[
\tilde{V}_2(\tilde{r}_*) = \frac{1 - 2\tilde{r}^2}{\tilde{r}^3 (n\tilde{r}^2 + 3)^2} \left[2n^2 (n + 1)\tilde{r}^3 + 6n^2 \tilde{r}^2 + 18n\tilde{r} + 18\right].
\]

It is clear that the scaled polar oscillation equations are explicitly mass-independent. Hence, the scaled polar QNM frequencies $\tilde{\omega}$ of TVIIM could also display a similar universality as those of the axial oscillations, save cases in which perturbation in pressure becomes non-negligible, e.g. the pressure $p$-mode. Since the mass profiles of other realistic stars are close to that of
TVIIM, their QNMs should follow the same trend. In Fig. 4 and Fig. 5 the solid circles show the scaled frequencies of polar \( w \)-mode and \( f \)-mode QNMs of TVIIM, which clearly illustrate the scaling behaviour of these modes. Besides, they are also well approximated by the solid (dotted) line obtained from the best quadratic fit to QNMs of realistic neutron stars (TVIIM).

The similarity between these lines in turn establishes independent corroboration of our theory.

We have shown here that the scaled polar QNM frequencies \( \tilde{\omega} \) of realistic neutron stars are approximately given by a universal function of the compactness. Our finding is particularly interesting for the \( f \)-mode oscillation because Eq. (3) is different from the empirical formulas (3) and (4) discovered previously (Andersson & Kokkotas 1998). As mentioned in Sect. I, Eq. (3) in general leads to explicit mass-dependence of the compactness. Our finding is particularly interesting for the universal function of the compactness. Our finding is particularly interesting for the

As TVIIM successfully captures the universality in QNMs of \( w \)- and \( f \)-modes, it can be used to set up a benchmark for realistic stars. Bases on this, a viable scheme to infer the mass and the radius of a neutron star from its \( w \)- and \( f \)-modes QNMs is proposed here. In Fig. 6 we plot \( 10 \times \text{Re} \omega / \text{Re} \omega_a \) (short dashed-line); \( 10 \times \text{Re} \omega / \text{Re} \omega_p \) (long dashed-line); \( \text{Re} \omega_a / \text{Im} \omega_a \) (solid-line); and \( \text{Re} \omega_p / \text{Im} \omega_p \) (dot-dashed-line) against the compactness, where \( \omega_a, \omega_p \) and \( \omega_i \) are QNM frequencies of the leading axial and polar \( w \)-modes, and \( f \)-mode of TVIIM, respectively. The curves are all monotonically increasing function of \( M/R \) in the regime \( 0.1 < M/R < 0.3 \) where physical neutron stars are stable. Depending on experimental data available, we can use any one of these ratios to approximately infer the compactness of an unknown neutron star. Once the compactness is known, we can readily read the approximate mass of the star from Figs. 1 – 5 if the real (or imaginary) part of relevant QNM frequency is obtainable in gravitational wave detection. Since both the real and imaginary parts of the scaled frequencies of \( w \)- and \( f \)-modes are well approximated by quadratic functions of \( M/R \), implementation of the above-mentioned procedure is straightforward. Still, feasibility of this scheme hinges on availability of accurate experimental data, which is beyond the scope of the present paper (see, e.g. [Kokkotas et al. 2001] for a discussion on this issue).

Here we illustrate our method with the following example. Consider a neutron star constructed from APR2 model (Akmal et al. 1998), whose mass and radius are 1.596M⊙ and 11.77 km, respectively. The first polar \( w \)-mode QNM of this star has a angular frequency 64.57 kHz and a damping time of 2.666 \times 10^{-5} s. Therefore, the ratio \( \text{Re} \omega_p / \text{Im} \omega_p \) is equal to 1.722. If such wave signals are detected by us, we can use TVIIM as a benchmark for this star and infer its approximate mass and radius from the wave data. From the ratio \( \text{Re} \omega_p / \text{Im} \omega_p = 1.722 \) and the dot-dashed-line in Fig. 6, the compactness of the corresponding TVIIM star is found to be 0.2025 (in geometrized units). Hence, the scaled frequency can be obtained from Fig. 2 By comparing the scaled frequency with unscaled one, the mass of the star can be readily found, which is 1.581M⊙ in this case. Similarly, the radius of the star can be found from the compactness given above and is equal to 11.51 km. Both the values of the radius and the mass inferred from TVIIM star are close to those of APR2 star. From this example one can see that TVIIM can in fact serve as a template for other realistic stars. In this regard, the role of TVIIM in gravitational wave analysis is reminiscent of that of the hydrogen atom in atomic spectroscopy.

Before we can put this scheme into practice, we have to further examine whether the universality demonstrated in Figs. 1 – 5 is truly generic. As discussed above, the universality is closely tied with the validity of TVIIM, which is developed under the assumption that the density function \( \rho(r) \) is expandable in series of \( r \). Despite that the assumption is appealing, there is one well-known exception to it, namely the strange star (Witten 1984). One salient feature of the EOS of strange quark matter, say the MIT bag model (see, e.g. [Glendenning 1997], chap. 8), is that the density is non-zero at zero pressure. Therefore, the density of a bare quark star (i.e. one without a normal nuclear matter shell) does not vanish at the surface of the star. Obviously, TVIIM fails to mimic such stars as there is a discontinuity in the density on the stellar surface. As shown in Fig. 6 the QNMs of bare strange quark stars deviate marked from the university demonstrated by other stars that are described by continuous density profiles. To pinpoint the cause of the deviation, the QNMs of strange quark stars with a normal nuclear matter shell are also shown in Fig. 6 It is obvious the QNMs of such hypothetically “dressed” quark
star, which have no discontinuities in their density profiles, also display the universality discussed in the present paper. We conclude that the scheme proposed above works properly as long as the star in consideration is characterized by a continuous density profile. Furthermore, by virtue of the TOV equations, the pressure is always a continuous function of \( r \) and the density profile is continuous if relevant EOS \( P(\rho) \) is a continuous function of \( \rho \). As a result, deviation from the universal curve might indicate a discontinuous (or rapidly varying) EOS. This effect can indeed be observed from the QNM frequencies of GM24 shown in Fig. 1 and 2. Neutron stars constructed from GM24 will become unstable if \( M/R > 0.21 \) owing to emergence of phase transition that leads to discontinuous (or rapidly varying) EOS. Correspondingly, QNM frequencies of GM24 stars no longer follow the those of TVIIM as its compactness approaches that limit.

7 CONCLUSION

In this paper we have unveiled some universal scaling behaviours prevailing in QNM frequencies of \( w \)- and \( f \)-modes. In addition, we have shown that these behaviours are attributable to the generic form of the mass distribution function. Based on our discovery, a scheme is proposed here to infer the mass and the radius of an unknown star from its gravitational wave spectrum. Despite that there might be some exceptional cases, e.g. bare strange stars, the scheme does provide reasonable estimate of relevant values in normal situation. As it has been shown that the mass and the radius of a neutron star can be used to infer the EOS of stellar matter (Lindblom 1992), our discussion will also lead to determination of EOS from gravitational wave signals.

The TVIIM considered here is a two-parameter model that is characterized by the mass and the radius of a star. While reproducing the main features of realistic stars, it cannot capture the fine details in individual cases. Should it be possible to gather more than one QNM frequency of a pulsating star, we could, in principle, construct more refined template for it. We are currently working along this direction and relevant result will be reported in due course.

Our analysis sketched above is based on assumption that QNMs of gravitational waves are clearly discernible. However, it is well known that gravitational wave signals are usually overwhelmed by noises of various kinds, including thermal, quantum and gravity-gradient noises (see, e.g. Liu & Thorne 2000, Thorne & Weinstein 1999, Santamore & Levin 2001, and references therein). As yet there are no viable methods to completely eliminate these noises such that clear and reliable gravitational wave signals can be recorded. However, with the continuous efforts of researchers in this field, we will surely get rid of these noisy gremlins at a later date. Thus, gravitational wave astronomy will open up a new channel to observe distant compact stars and the scheme outlined here will be applicable in the near future.

ACKNOWLEDGMENTS

We thank J Wu for discussions and assistance in preparation of the manuscript. Our work is supported in part by the Hong Kong Research Grants Council (grant No: CUHK4282/00P) and a direct grant (Project ID: 2060260) from the Chinese University of Hong Kong.

REFERENCES

Akmal A., Pandharipande V. R., Ravenhall D. G., 1998, Phys. Rev. C, 58, 1804
Andersson N., Kokkotas K., 1996, Phys. Rev. Lett., 77, 20
Andersson N., Kokkotas K. D., 1998, MNRAS, 299, 1059
Belczynski K., Kalogera V., Bulik T., 2001, ApJ, 572, 407
Benhar O., Berti E., Ferrari V., 1999, MNRAS, 310, 797
Chandrasekhar S., Ferrari V., 1991, Proc. R. Soc. A, 432, 247
Ching E. S. C., Leung P. T., Suen W. M., Young K., 1996, Phys. Rev. D, 54, 3778
Fryer C. L., Holz D. E., Hughes S. A., 2002, ApJ, 565, 430
Glendenning N. K., 1997, Compact Stars - Nuclear Physics, Particle Physics, and General Relativity. Springer, NY
Hawking S., Israel W., eds, 1987, 300 Years of Gravitation. Cambridge University Press
Hughes S., 2003, Ann. Phys., 303, 142
Kokkotas K., Apostolatos T., Andersson N., 2001, MNRAS, 320, 307
Kokkotas K., Schmidt B., 1999, Living Rev. Rel., 2, 2
Lattimer J. M., Prakash M., 2001, ApJ, 550, 426
Leaver E., 1986, Phys. Rev. D, 34, 384
Lindblom L., 1992, ApJ, 398, 56
Lindblom L., Detweiler S., 1983, ApJ, 53, 73

© 2004 RAS, MNRAS 000, 1–
Lindblom L., Owen B. J., Morsink S. M., 1998, Phys. Rev. Lett., 80, 4843
Liu Y. T., Thorne K. S., 2000, Phys. Rev. D, 62, 122002
Mason J., ed. 2004, Astrophysics Update. Springer-Praxis, pp 281–310
Oppenheimer J. R., Volkoff G. M., 1939, Phys. Rev., 55, 374
Pandharipande V., 1971a, Nucl. Phys A, 174, 641
Pandharipande V., 1971b, Nucl. Phys A, 178, 123
Press W., 1971, ApJ, 170, L105
Regge T., Wheeler J., 1957, Phys. Rev., 108, 1063
Santamore D. H., Levin Y., 2001, Phys. Rev. D, 64, 042002
Thorne K., Campolattaro A., 1967, ApJ, 149, 591
Thorne K. S., Weinstein C. J., 1999, Phys. Rev. D, 60, 082001
Tolman R. C., 1939, Phys. Rev., 55, 364
Wiringa R. B., Fiks V., Fabrocini A., 1988, Phys. Rev. C, 38, 1010
Witten E., 1984, Phys. Rev. D, 30, 272
Zerilli F., 1970, Phys. Rev. Lett., 24, 737
Figure 1. (a) The real and (b) the imaginary parts of the scaled frequency $M\omega$ of the least-damped axial $w$-mode are plotted against the compactness $M/R$ for neutron stars described by different EOS, including APR1, APR2, AU, GM24, Models A and C, UT, UU, and also TVIIM. The solid line represents the best quadratic fit in $M/R$ to the scaled frequencies of the eight realistic stars. Likewise, the dotted-line is the best quadratic fit to those of TVIIM.
Universality in Quasi-normal Modes of Neutron Stars

Figure 2. (a) The real and (b) the imaginary parts of the scaled frequency $M \omega$ of the least-damped polar $\nu$-mode are plotted against the compactness $M/R$ for neutron stars described by different EOS (see the caption of Fig. 1). The solid and dotted-lines represent the best quadratic fit in $M/R$ to the scaled frequencies of the eight realistic stars and TVIIM, respectively.
Figure 3. (a) The real and (b) the imaginary parts of the scaled frequency $M\omega$ of the fluid $f$-mode are plotted against the compactness $M/R$ for neutron stars described by different EOS (see the caption of Fig. 1). The solid and dotted-lines represent the best quadratic fit in $M/R$ to the scaled frequencies of the eight realistic stars and TVIIM, respectively.
Figure 4. Same as Fig. 1 for the second-least damped axial $w$-mode.
Figure 5. Same as Fig. 2 for the second-least damped polar $\omega$-mode.

Figure 6. The mass distribution functions of neutron stars characterized by eight realistic different EOS and with a common compactness $M/R = 0.2$ are shown. The mass distribution function of TVIIM, the solid line, is shown as a benchmark for realistic stars.
Figure 7. The complex eigenfrequencies of the least damped polar $\omega$-mode for stars constructed from seven realistic EOS (unfilled symbols) and TVIM (solid-circle), which have a common compactness $M/R = 0.2$, are shown. The grey (or dark-grey for the case of TVIM) symbols indicate the corresponding results obtained under the approximation $X = 0$.

Figure 8. (a) $10 \times \text{Re}\omega_a / \text{Re}\omega_a$ (short dashed-line); (b) $10 \times \text{Re}\omega_t / \text{Re}\omega_p$ (long dashed-line); (c) $\text{Re}\omega_a / \text{Im}\omega_a$ (solid-line); and (d) $\text{Re}\omega_p / \text{Im}\omega_p$ (dot-dashed-line) are plotted against the compactness $M/R$. 
Figure 9. (a) The real and (b) the imaginary parts of the scaled frequency $M\omega$ of the least-damped polar $w$-mode are plotted against the compactness $M/R$ for neutron stars described by different EOS (see caption of Fig. 1). The solid and dotted-lines respectively show the results of “dressed” and bare strange quark stars.