Noncommutative Deformation of Instantons

Yoshiaki Maeda, Akifumi Sako

Department of Mathematics, Faculty of Science and Technology, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan

Mathematical Research Centre, University of Warwick
Coventry, CV4 7AL, United Kingdom

Department of General Education, Kushiro National College of Technology
Otanoshike-Nishi 2-32-1, Kushiro 084-0916, Japan

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Abstract

We construct instanton solutions on noncommutative Euclidean 4-space which are deformations of instanton solutions on commutative Euclidean 4-space. We show that the instanton numbers of these noncommutative instanton solutions coincide with the commutative solutions and conjecture that the instanton number in $\mathbb{R}^4$ is preserved for general noncommutative deformations. We also study noncommutative deformation of instanton solutions on a $T^4$ with twisted boundary conditions.

1 Introduction

Gauge theory originated in physics as a convenient framework for electromagnetic fields and their generalizations to e.g. the Yang-Mills theories. In mathematics, gauge theory has been highly developed to study the topology of 4-manifolds, with Donaldson’s construction of a new obstruction to the smoothability of 4-manifolds, which produced a series of examples of exotic differentiable structures. For these problems, it is important to study the moduli spaces of anti-self dual connections or instantons. These moduli spaces have an algebro-geometric interpretation. In particular, anti-selfdual connections are classified by their instanton numbers.

Many authors have worked on extending gauge theory to noncommutative geometry. Several authors have treated the ADHM construction on noncommutative 

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Euclidean 4-manifolds and have shown that the instanton number is given by an integer which does not depend on the noncommutative parameter \([4, 5, 6, 7, 8]\). We note that the relation between these noncommutative instantons and deformed solutions from the commutative ADHM construction \([12]\) is unknown.  

In the paper \([9]\), we constructed a noncommutative vortex solution which is a deformation of Taubes’s vortex solution and showed that its vortex number is undeformed, i.e., independent of the deformation parameter. It is therefore natural to construct a deformed instanton solution via the ADHM construction from the commutative one and to see if the corresponding instanton number is deformed.  

In this paper, we construct a noncommutative formal instanton solution which is a deformation of the commutative instanton solution. Our construction starts with a commutative instanton solution, which is determined by its ADHM data, and then solves the infinite systems of elliptic PDE equations with decay conditions term by term in the noncommutative parameter \(\hbar\). We study the (noncommutative) instanton number for this noncommutative instanton solution and show that it is independent of \(\hbar\) (Theorem 5.1). This result supports our conjecture on the independence of the noncommutative instanton number for noncommutative \(\mathbb{R}^4\). We also study noncommutative deformations of instantons and their corresponding instanton numbers on \(T^4\) with a twisted bundle.

## 2 Notations

Noncommutative Euclidean 4-space is given by the following commutation relations:

\[
[x^\mu, x^\nu]_* = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4,
\]

(2.1)

where \((\theta^{\mu\nu})\) is a real, \(x\)-independent, skew-symmetric matrix, called the noncommutative parameters. \(\star\) is known as the Moyal product \([10]\). The Moyal product (or star product) is defined on functions by

\[
\begin{align*}
  f(x) \star g(x) & := f(x) \exp \left( \frac{i}{2} \partial_\mu \theta^{\mu\nu} \partial_\nu \right) g(x) \\
  & = f(x)g(x) + \sum_{n=1}^{\infty} \frac{1}{n!} f(x) \left( \frac{i}{2} \partial_\mu \theta^{\mu\nu} \partial_\nu \right)^n g(x).
\end{align*}
\]

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1. Noncommutative instanton solutions were constructed with the ADHM method in \([1]\). After that, many authors have constructed noncommutative instantons in the similar way. See for example \([2]\) and their bibliography.

2. There are few noncommutative instanton solutions whose commutative limits are clarified, and they are constructed without using the ADHM method \([3]\).
Here \( \overrightarrow{\partial}_\mu \) and \( \overrightarrow{\partial}_\nu \) are partial derivatives with respect to \( x^\mu \) for \( f(x) \) and to \( x^\nu \) for \( g(x) \), respectively.

We define a Lie algebra by
\[
[T_a, T_b] = i f_{abc} T_c, \tag{2.2}
\]
where the generators \( T_a \) are given by Hermitian matrices. We consider a compact simply connected Lie group with this Lie algebra. The covariant derivative is defined by
\[
D_\mu := \partial_\mu + i A_\mu, \quad A_\mu = A_\mu^a T_a. \tag{2.3}
\]
The curvature two form \( F \) is defined by
\[
F := \frac{1}{2} F_{\mu\nu} dx^\mu \wedge \ast dx^\nu = dA + A \wedge \ast A \tag{2.4}
\]
where \( \wedge \ast \) is defined by
\[
A \wedge \ast A := \frac{1}{2} (A_\mu \ast A_\nu) dx^\mu \wedge dx^\nu. \tag{2.5}
\]

To consider smooth noncommutative deformations, we introduce a parameter \( \hbar \) and a fixed constant \( \theta_{0}^{\mu\nu} < \infty \) with
\[
\theta^{\mu\nu} = \hbar \theta_{0}^{\mu\nu}. \tag{2.6}
\]
We define the commutative limit by letting \( \hbar \to 0 \).

3 Noncommutative Instantons

Instanton solutions or anti-selfdual connections satisfy the (noncommutative) instanton equation
\[
F^+ = \frac{1}{2} (1 + \ast) F = 0 , \tag{3.1}
\]
where \( \ast \) is the Hodge star operator. Formally we expand the connection as
\[
A_\mu = \sum_{l=0}^{\infty} A_\mu^{(l)} h^l. \tag{3.2}
\]
Then,
\[
A_\mu \ast A_\nu = \sum_{l,m,n=0}^{\infty} h^{l+m+n} \frac{1}{l! m! n!} A_\mu^{(m)} (\overrightarrow{\Delta})^l A_\mu^{(n)} \tag{3.3}
\]
\[
\overrightarrow{\Delta} \equiv \frac{i}{2} \overrightarrow{\partial}_\mu \theta_{0}^{\mu\nu} \overrightarrow{\partial}_\nu.
\]
We introduce the selfdual projection operator $P$ by

$$P := \frac{1 + *}{2} ; \quad P_{\mu\nu,\rho\tau} = \frac{1}{2} (\delta_{\mu\rho} \delta_{\nu\tau} - \delta_{\nu\rho} \delta_{\mu\tau} + \epsilon_{\mu\nu\rho\tau}). \quad (3.4)$$

Then the instanton equation is

$$P_{\mu\nu,\rho\tau} F^{\rho\tau} = 0. \quad (3.5)$$

In the noncommutative case, the $l$-th order equation of (3.5) is given by

$$P_{\mu\nu,\rho\tau} \left( \partial_\rho A^{(l)}_\tau - \partial_\tau A^{(l)}_\rho + i[A^{(0)}_\mu, A^{(l)}_\rho] + i[A^{(l)}_\rho, A^{(0)}_\tau] + C^{(l)}_{\rho\tau} \right) = 0, \quad (3.6)$$

$$C^{(l)}_{\rho\tau} := \sum_{(p; m, n) \in I(l)} \frac{1}{p!} (A^{(m)}_p (\Delta^p) A^{(n)}_\rho - A^{(m)}_\rho (\Delta^p) A^{(n)}_p),$$

$$I(l) \equiv \{(p; m, n) \in \mathbb{Z}^3 | p + m + n = l, p, m, n \geq 0, m \neq l, n \neq l\}.$$

Note that the 0-th order equation is the commutative instanton equation with solution $A^{(0)}_\mu$ a commutative instanton. The asymptotic behavior of commutative instanton $A^{(0)}_\mu$ is given by

$$A^{(0)}_\mu = gdg^{-1} + O(|x|^{-2}), \quad gdg^{-1} = O(|x|^{-1}), \quad (3.7)$$

where $g \in G$ and $G$ is a gauge group. We introduce covariant derivatives associated to the commutative instanton connection by

$$D^{(0)}_\mu f := \partial_\mu f + i[A^{(0)}_\mu, f], \quad D_{A^{(0)}} f := d f + A^{(0)} \wedge f \quad (3.8)$$

Using this, (3.6) is given by

$$P^{\mu\nu,\rho\tau} (D^{(0)}_\rho A^{(l)}_\tau - D^{(0)}_\tau A^{(l)}_\rho + C^{(l)}_{\rho\tau}) = 0,$$

$$P(D_{A^{(0)}} A^{(l)} + C^{(l)}) = 0. \quad (3.9)$$

In the following, we fix a commutative anti-selfdual connection $A^{(0)}$. We impose the following condition for $A^{(l)} (l \geq 1)$ \[11\]

$$A - A^{(0)} = D^*_A A^{(0)} B, \quad B \in \Omega^2_+, \quad (3.10)$$

where $D^*_A A^{(0)}$ is defined by

$$(D^*_A A^{(0)})^{\mu\nu}_\rho B_{\mu\nu} = \delta^\nu_\rho \partial^\mu B_{\mu\nu} - \delta^\mu_\rho \partial^\nu B_{\mu\nu} + i \delta^\nu_\rho [A^\mu, B_{\mu\nu}] - \delta^\mu_\rho [A^\nu, B_{\mu\nu}] - \delta^\mu_\rho D^{(0)} B_{\mu\nu} - \delta^\nu_\rho D^{(0)} B_{\mu\nu}. \quad (3.11)$$
We expand $B$ in $\hbar$ as we did with $A$. Then $A^{(l)} = D^*_{A^{(0)}} B^{(l)}$. In this gauge, (3.9) is given by

$$PD_{A^{(0)}} D^*_{A^{(0)}} B^{(l)} + PC^{(l)} = 0.$$  

(3.12)

Using the fact that the $A^{(0)}$ is an anti-selfdual connection, (3.12) simplifies to

$$2D^2_{(0)} B^{(l)\mu\nu} + P^{\mu\nu,\rho\tau} C^{(l)}_{\rho\tau} = 0,$$

(3.13)

where

$$D^2_{(0)} \equiv D^\rho A^{(0)} D_A^{(0)}\rho.$$

4 Green’s Functions

In this section, we derive some properties of the Green’s function of $D^2_{(0)}$ in preparation for Theorem 5.1. To apply results from the ADHM construction, we restrict ourselves to $U(n)$ gauge theory.

We consider the Green’s function for $D^2_{(0)}$:

$$D^2_{(0)} G_0(x, y) = \delta(x - y),$$

where $\delta(x - y)$ is a four dimensional delta function. Here $D^2_{(0)} \equiv D^\rho_{A^{(0)}} D_{A^{(0)}}\rho$, and this $A^{(0)}$ is an instanton in commutative $\mathbb{R}^4$. Instantons in commutative $\mathbb{R}^4$ are given by the ADHM construction [12], and arbitrary commutative instantons are in one-to-one correspondence with ADHM data. $G_0(x, y)$ has been constructed in [13] (see also [14, 15]):

$$G_0(x, y) = \frac{[v_1(x) \otimes v_2(x)]^T (1 - \mathfrak{M}) [v_1(y) \otimes v_2(y)]}{4\pi^2 (x - y)^2}.$$  

(4.1)

Here $\mathfrak{M}$ and $v_1, v_2$ are determined by the ADHM data and $v_i$ is a bounded function. Using this Green’s function, we solve the equation (3.13) as

$$B^{(l)\mu\nu} = -\frac{1}{2} \int_{\mathbb{R}^4} G_0(x, y) P^{\mu\nu,\rho\tau} C^{(l)}_{\rho\tau}(y) d^4 y$$

(4.2)

and the noncommutative instanton $A = \sum A^{(l)}$ is given by

$$A^{(l)} = D^*_{A^{(0)}} B^{(l)}.$$  

(4.3)

The key fact used in the following proposition is that the asymptotic behavior of Green’s function of $D^2_{(0)}$ is given by

$$G_0(x, y) = O(|x - y|^{-2}) , |x - y| > 1.$$  

(4.4)

We now list some features of Green’s functions like $G_0$.
Proposition 4.1. Let $G(x, y)$ be a Green’s function on $\mathbb{R}^4$ written as

$$G(x, y) = \frac{b(x, y)}{|x - y|^2}, \quad (4.5)$$

where $b(x, y)$ is a bounded function. Let $f(x)$ be a function such that $|f(x)| < \frac{C}{1 + |x|^4}$ where $C$ is some constant. We define $F(x)$ by

$$F(x) := \int_{\mathbb{R}^4} G(x, y) f(y) d^4 y. \quad (4.6)$$

Then $F(x) = O(|x|^{-2})$.

Lemma (3.3.35) in [16] contains a more general formula but with a rougher estimate, so we give a proof of this proposition.

[Proof] We introduce two balls whose radii are $\frac{1}{2}|x|$ with centers at the origin and $x$ in $\mathbb{R}^4$. Let $B_0$ and $B_1$ denote these balls respectively, and let $C$ be their complement. Then

$$F(x) = \int_{\mathbb{R}^4} G(x, y) f(y) d^4 y \quad (4.7)$$

The first term is estimated as follows.

$$\int_{B_0} G(x, y) f(y) d^4 y < \int_{|y| \leq \frac{1}{2}|x|} \frac{C}{|x - y|^2(1 + |y|^4)} d^4 y$$

$$\leq \int_{|y| \leq \frac{1}{2}|x|} \frac{C}{|x|^2(1 + |y|^4)} d^4 y \quad \text{(because } |x - y| \geq \frac{1}{2}|x|\text{)}$$

$$= 4\pi^2 C|x|^{-2} \int_0^{\frac{1}{2}|x|} \frac{r^3}{1 + r^4} dr$$

$$= \pi^2 C|x|^{-2} [\log(1 + r^4)]_0^{\frac{1}{2}|x|} = O(|x|^{-2}) \quad (4.8)$$

The second term is estimated as follows.

$$\int_{B_1} G(x, y) f(y) d^4 y \leq 2\pi^2 \lim_{\epsilon \to 0} \int_{\epsilon \leq |y - x| \leq \frac{1}{2}|x|} \frac{r^3}{r^2(|x| - r)^4}$$

$$= -2\pi^2 \lim_{\epsilon \to 0} \left\{ \left[ \frac{1}{6} \right] \left[ \frac{1}{3} \right] \frac{1}{(|x| - \epsilon)^2} + |x| \left[ \frac{1}{3} (r - |x|)^{-3} \right]_\epsilon \right\}$$

$$= O(|x|^{-2}) \quad (4.9)$$
where \( y = x + r \omega, r \in \mathbb{R}_{\geq 0}, |\omega| = 1 \) and we use the fact that \(|y| \geq |x| - r \geq |x| - r\).

To estimate the last term in (4.7) we introduce \( D_1 \) and \( D_2 \) by

\[
D_1 := \{ y \mid |y| \geq \frac{1}{2}|x|, |y - x| \geq \frac{1}{2}|x|, |y| \geq |y| \} \tag{4.10}
\]

\[
D_2 := \{ y \mid |y| \geq \frac{1}{2}|x|, |y - x| \geq \frac{1}{2}|x|, |y - x| \leq |y| \}.
\]

Then,

\[
\int_C G(x, y) f(y) d^4 y < C \int_{D_1} \frac{1}{|x - y|^2} d^4 y
\]

\[
< C \int_{D_1} \frac{d^4 y}{|y|^6} + C \int_{D_2} \frac{d^4 y}{|y - x|^6} = O(|x|^{-2}). \tag{4.11}
\]

We introduce the notation \( O'(|x|^{-m}) \) as in [16]. If \( s \) is a function of \( \mathbb{R}^4 \) which is \( O(|x|^{-m}) \) as \( |x| \to \infty \) and \( |D^k_{(0)} s| = O(|x|^{-m-k}) \), then we denote this natural growth condition by \( s \in O'(|x|^{-m}) \).

Examine the proof of Proposition 4.1, and keeping track of estimates for higher derivatives, we have the following (see Lemma 3.3.36 in [16]).

**Proposition 4.2.** If \( f(x) \in O(|x|^{-m}) \) and \( |D^2_{(0)} f(x)| = O'(|x|^{-m-2}) \), then \( f(x) \in O'(|x|^{-m}) \).

We apply these propositions to our case.

**Theorem 4.3.** If \( C^{(l)} \in O(|x|^{-4}) \), then \( B^{(k)} < O'(|x|^{-2}) \)

**Proof** It follows easily from the construction of \( G_0 \) [13] and the choice of ADHM data that our Green’s function can be written as

\[
G_0(x, y) = \frac{b(x, y)}{|x - y|^2}, \tag{4.12}
\]

where \( b(x, y) \) is a bounded function. If \( C^{(l)}_{pr} \) is \( O'(|x|^{-4}) \) Proposition 4.1 implies that \( B^{(k)} = O(|x|^{-2}) \). It follows from Proposition 4.2 that \( B^{(k)} = O'(|x|^{-2}) \)

In our case, \( C^{(l)}_{pr} = O'(|x|^{-4}) \) by (3.7), and so \( |B^{(1)}| < O'(|x|^{-2}) \) from Theorem 4.3 and \( |A^{(1)}| < O'(|x|^{-3}) \) as \( A^{(l)} = D^*_{A^{(0)}} B^{(l)} \). Repeating the argument \( l \) times, we get

\[
|A^{(l)}| < O'(|x|^{-3+\epsilon}), \quad \forall \epsilon > 0. \tag{4.13}
\]
5 Instanton Number

The first Pontrjagin number is defined by

$$ I_\hbar := \frac{1}{8\pi^2} \int tr F \wedge \star F. $$

(5.1)

We rewrite (5.1) as

$$ \frac{1}{8\pi^2} \int tr F \wedge \star F = \frac{1}{8\pi^2} \int tr (A \wedge \star dA + \frac{2}{3}A \wedge \star A \wedge \star A) + \frac{1}{8\pi^2} \int tr P^* $$

(5.2)

where

$$ P^* = \frac{1}{3} \{ F \wedge \star A \wedge \star A + 2A \wedge \star F \wedge \star A + A \wedge \star A \wedge \star F + A \wedge \star A \wedge \star A \wedge \star A \}. $$

(5.3)

$\int tr P^*$ is 0 in the commutative limit, but does not vanish in noncommutative space. The cyclic symmetry of trace is broken by the noncommutative deformation.

The trace of the first three terms in (5.3) equals

$$ tr \{ F \wedge \star A \wedge \star A + 2A \wedge \star F \wedge \star A + A \wedge \star A \wedge \star F \} $$

(5.4)

$$ = tr \sum_{k=1}^{\infty} \sum_{l=1}^{2} \frac{i^k}{2^k k!} \theta^{\mu_1 \nu_1} \ldots \theta^{\mu_k \nu_k} \{ (\partial_{\mu_1} \ldots \partial_{\mu_k} P_l) \wedge (\partial_{\nu_1} \ldots \partial_{\nu_k} Q_l) $$

$$ + (\partial_{\mu_1} \ldots \partial_{\mu_k} P_l) \wedge (\partial_{\nu_1} \ldots \partial_{\nu_k} Q_l) \}, $$

where

$$ P_1 = A \wedge \star F, \; Q_1 = A, \; P_2 = A, \; Q_2 = F \wedge \star A $$

The trace of the last term in (5.3) is

$$ tr A \wedge \star A \wedge \star A \wedge \star A $$

(5.5)

$$ = \frac{1}{2} tr \sum_{k=1}^{\infty} \frac{i^k}{2^k k!} \theta^{\mu_1 \nu_1} \ldots \theta^{\mu_k \nu_k} \{ (\partial_{\mu_1} \ldots \partial_{\mu_k} P_3) \wedge (\partial_{\nu_1} \ldots \partial_{\nu_k} Q_3) $$

$$ + (\partial_{\mu_1} \ldots \partial_{\mu_k} P_3) \wedge (\partial_{\nu_1} \ldots \partial_{\nu_k} Q_3) \}, $$

where

$$ P_3 = A \wedge \star A \wedge \star A, \; Q_3 = A. $$
We discuss a more general case in the following. Let $P$ and $Q$ be an $n$-form and a $(4 - n)$-form ($n = 0, \ldots, 4$), respectively, and let $P \wedge Q$ be $O(h^k)$. Consider

$$\int_{\mathbb{R}^d} tr(P \wedge \ast Q - (-1)^{n(4-n)} Q \wedge \ast P).$$

(5.6)

Note that (5.4) and (5.5) are sums of the form (5.6). The lowest order term in $h$ vanishes because of the cyclic symmetry of the trace, i.e. $\int tr(P \wedge Q - (-1)^{n(4-n)} Q \wedge P) = 0$. The term of order $h$ is given by

$$\frac{i}{2} \int_{\mathbb{R}^4} tr\{i\hbar^{\mu\nu}(\partial_\mu P \wedge \partial_\nu Q)\}$$

(5.7)

$$= \frac{i}{2} \int_{\mathbb{R}^4} d^4 x \hbar^{\mu\nu}(n!(4-n))! \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} tr\{\partial_\mu P_{\mu_1 \ldots \mu_n} \partial_\nu Q_{\mu_{n+1} \ldots \mu_4}\}$$

$$= \frac{i}{2} \int_{\mathbb{R}^4} (n!(4-n))! \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} tr\left\{\left(\frac{1}{4} \epsilon_\rho \epsilon_\sigma \epsilon^{\rho \sigma \nu} dx^{\mu} \wedge dx^{\nu}\right) \wedge (\partial_\nu P_{\mu_1 \ldots \mu_n} \partial_\rho Q_{\mu_{n+1} \ldots \mu_4} dx^{\sigma} \wedge dx^{\nu})\right\}$$

$$= \frac{i}{2} \int_{\mathbb{R}^4} (n!(4-n))! \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} tr\left\{\{\ast \theta\} \wedge d(P_{\mu_1 \ldots \mu_n} dQ_{\mu_{n+1} \ldots \mu_4})\right\}$$

where $\ast \theta = \epsilon_\rho \epsilon_\sigma \epsilon^{\rho \sigma \nu} dx^{\mu} \wedge dx^{\nu} \wedge dx^{4}/4$. These integrals are zero if $P_{\mu_1 \ldots \mu_n} dQ_{\mu_{n+1} \ldots \mu_4}$ is $O'(|x|^{-(4-1+\epsilon)})$ ($\epsilon > 0$). Similarly, higher order terms in $h$ in (5.6) can be written as total divergences and hence vanish under the decay hypothesis. This fact and (4.13) imply that $\int tr P_{\ast} = 0$.

From the above discussion and (4.13),

$$\frac{1}{8\pi^2} \int tr F \wedge \ast F = \frac{1}{8\pi^2} \int tr d(A \wedge \ast A + \frac{2}{3} A \wedge \ast A \wedge \ast A) + \frac{1}{8\pi^2} \int tr P_{\ast}$$

$$= \frac{1}{8\pi^2} \int tr F^{(0)} \wedge F^{(0)},$$

(5.8)

where $F^{(0)}$ is the curvature two form of $A^{(0)}$. Thus the instanton number is not deformed under noncommutative deformation.

Summarizing the above discussions, we get following theorems.

**Theorem 5.1.** Let $A^{(0)}_{\mu}$ be a commutative instanton solution in $\mathbb{R}^4$ given by the ADHM construction. There exists a formal noncommutative instanton solution $A_{\mu} = \sum_{i=0}^{\infty} A^{(i)}_{\mu} h^i$ such that the instanton number $I_h$ defined by (5.7) is independent of the noncommutative parameter $h$.
6 Instanton Numbers on Noncommutative Torus

In the proofs in the previous sections, the key point is that the volume of the space is infinite. Therefore it is natural to expect that instanton number depends on the noncommutative parameter for noncommutative deformations of a finite volume space. To study this phenomena, in this section we consider noncommutative deformation of instantons on $T^4$.

We first consider instantons with twisted boundary condition on a commutative $T^4$ (see [17, 18, 19, 20]). The twisted boundary conditions for covariant derivatives $D_\mu(x)$ are given by

$$D_\mu(x_\nu + 2\pi, x_\rho(\rho \neq \nu)) = \Omega_\nu(x)D_\mu(x_\nu, x_\rho(\rho \neq \nu))\Omega_\nu^\dagger(x).$$

(6.1)

For the simplicity, we chose $\Omega_\mu(x)$ by

$$\Omega_1(x_2) = e^{i\frac{\pi}{N}x_2}U_1, \quad \Omega_2(x_1) = V_1,$n
$$\Omega_3(x_4) = e^{i\frac{\pi}{N}x_4}U_2, \quad \Omega_2(x_1) = V_2,$n

for unitary matrices $U_i, V_i$ satisfying

$$U_iV_j = (1 - \delta_{ij}(1 - e^{-2\pi i \frac{1}{N}}))V_jU_i,$n
$$U_iU_j = U_jU_i, \quad V_iV_j = V_jV_i \quad (i, j = 1, 2).$$

These $\Omega_\mu$ satisfy the consistency conditions,

$$\Omega_1(x_2 + 2\pi)\Omega_2(x_1) = \Omega_2(x_1 + 2\pi)\Omega_1(x_2),$$n
$$\Omega_3(x_4 + 2\pi)\Omega_4(x_3) = \Omega_4(x_3 + 2\pi)\Omega_3(x_4).$$

(6.2)

In [17], a $k^2$ instanton solution with these twisted boundary conditions for $U(N^2)$ gauge theory is given by

$$D_1 = \partial_1, \quad D_2 = \partial_2 + \frac{k}{2N}(x_11_N) \otimes 1_N,$n
$$D_3 = \partial_3, \quad D_4 = \partial_4 - \frac{k}{2N}(x_31_N) \otimes 1_N,$n

where $1_N$ is the identity matrix of degree $N$. These covariant derivatives are valid operators under the consistency conditions. For this connection,

$$F_{12} = -F_{34} = -\frac{i}{2\pi N}k1_N \otimes 1_N, \quad F_{13} = F_{24} = F_{14} = F_{23} = 0,$n

(6.4)

which obviously satisfies the instanton equation. The instanton number is given by $k^2$. 

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Let us deform this solution to a noncommutative instanton. For simplicity, we chose the commutation relations as

\[ [x_1, x_2]_\theta = 2\pi i \theta, \quad [x_3, x_4]_\theta = 2\pi i \theta, \]  

(6.5)

with all other commutators zero. After this noncommutative deformation,

\[ D_1 = \partial_1, \quad D_2 = \partial_2 + f(x_1 1_N) \otimes 1_N, \]
\[ D_3 = \partial_3, \quad D_4 = \partial_4 - f(x_3 1_N) \otimes 1_N \]

still satisfy the noncommutative instanton equation for an arbitrary constant \( f \). But the consistency conditions (6.2) restrict \( f \) to

\[ f = \frac{k}{2\pi(N - k \theta)}. \]

Thus, covariant derivatives can be deformed smoothly from those of the commutative torus. The instanton number is also deformed to

\[ \frac{1}{8\pi^2} \int_{T^4} tr F \wedge *F = \frac{k^2 N^2}{(N - k \theta)^2}. \]  

(6.7)

From this observation and Theorem 5.1, a new question arises: “Which instantons preserve their instanton number under noncommutative deformation?” This question is left as an open problem.

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References

[1] N. Nekrasov and A. S. Schwarz, “Instantons on noncommutative \( R^4 \) and (2,0) superconformal six dimensional theory,” Commun. Math. Phys. 198, 689 (1998) \texttt{hep-th/9802068}.

[2] K. Y. Kim, B. H. Lee and H. S. Yang, “Comments on instantons on noncommutative \( R^4 \),” J. Korean Phys. Soc. 41, 290 (2002) \texttt{hep-th/0003093}.

K. Furuuchi, “Equivalence of projections as gauge equivalence on noncommutative space,” Commun. Math. Phys. 217, 579 (2001) \texttt{hep-th/0005199}.

N. A. Nekrasov, “Noncommutative instantons revisited,” Commun. Math.
Phys. **241**, 143 (2003) [hep-th/0010017].

K. Furuuchi, “Dp-D(p+4) in noncommutative Yang-Mills,” JHEP **0103**, 033 (2001) [hep-th/0010119].

N. A. Nekrasov, “Trieste lectures on solitons in noncommutative gauge theories,” [hep-th/0011095](hep-th/0011095).

D. H. Correa, G. S. Lozano, E. F. Moreno and F. A. Schaposnik, “Comments on the U(2) noncommutative instanton,” Phys. Lett. B **515**, 206 (2001) [hep-th/0105085].

O. Lechtenfeld and A. D. Popov, “Noncommutative multi-solitons in 2+1 dimensions,” JHEP **0111**, 040 (2001) [hep-th/0106213](hep-th/0106213).

T. Ishikawa, S. I. Kuroki and A. Sako, “Elongated U(1) instantons on noncommutative \( \mathbb{R}^4 \),” JHEP **0111**, 068 (2001) [arXiv:hep-th/0109111](arXiv:hep-th/0109111).

S. Parvizi, “Non-commutative instantons and the information metric,” Mod. Phys. Lett. A **17**, 341 (2002) [hep-th/0202025](hep-th/0202025).

N. A. Nekrasov, “Lectures on open strings, and noncommutative gauge fields,” [hep-th/0203109](hep-th/0203109).

Y. Tian and C. J. Zhu, “Instantons on general noncommutative \( \mathbb{R}^4 \),” Commun. Theor. Phys. **38**, 691 (2002) [hep-th/0205110](hep-th/0205110).

D. H. Correa, E. F. Moreno and F. A. Schaposnik, “Some noncommutative multi-instantons from vortices in curved space,” Phys. Lett. B **543**, 235 (2002) [hep-th/0207180](hep-th/0207180).

F. Franco-Sollova and T. A. Ivanova, “On noncommutative merons and instantons,” J. Phys. A **36**, 4207 (2003) [hep-th/0209153](hep-th/0209153).

Y. Tian and C. J. Zhu, “Comments on noncommutative ADHM construction,” Phys. Rev. D **67**, 045016 (2003) [hep-th/0210163](hep-th/0210163).

M. Hamanaka, “Noncommutative solitons and D-branes,” [hep-th/0303256](hep-th/0303256).

J. Broedel, T. A. Ivanova and O. Lechtenfeld, “Construction of noncommutative instantons in 4k dimensions,” Mod. Phys. Lett. A **23** (2008) 179 [hep-th/0703009](hep-th/0703009).

[3] O. Lechtenfeld and A. D. Popov, “Noncommutative ’t Hooft instantons,” J. High Energy Phys. **03** (2002) 040 [hep-th/0109209](hep-th/0109209).

Z. Horvath, O. Lechtenfeld and M. Wolf, “Non-commutative instantons via dressing and splitting approaches”, J. High Energy Phys. 0212 (2002) 060 [hep-th/0211041](hep-th/0211041).

[4] T. Ishikawa, S. Kuroki and A. Sako, “Instanton number on noncommutative \( R^4 \),” [hep-th/0201196](hep-th/0201196) “Calculation of the Pontrjagin class for U(1) instantons on noncommutative \( R^{4n} \),” JHEP **0208**, 028 (2002).

[5] A. Sako, “Instanton number of noncommutative U(N) Gauge Theory”, JHEP **0304**, 023 (2003) [hep-th/0209139](hep-th/0209139).
[6] K. Furuuchi, “Instantons on noncommutative $R^4$ and projection operators”, Prog. Theor. Phys. 103, 1043, (2000) hep-th/9912047.

[7] K. Furuuchi, “Topological charge of U(1) instantons”, hep-th/0010006.

[8] Y. Tian, C. Zhu and X. Song, “Topological charge of noncommutative ADHM instanton”, hep-th/0211225.

[9] Y. Maeda, A. Sako, “Are vortex numbers preserved?”, To appear in J.Geom. Phys. math-ph/0612041.

[10] J. E. Moyal, “Quantum mechanics as a statistical theory”, Proc. Cambridge Phil.Soc. 45, 99 (1949).

[11] D. S. Freed and K. K. Uhlenbeck, “Instantons and Four-Manifolds,” New York, USA: Springer (1984) 232 P. (Mathematical Sciences Research Institute Publications, 1)

[12] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, “Construction of instantons,” Phys. Lett. A 65, 185 (1978).

[13] E. Corrigan, P. Goddard and S. Templeton, “Instanton Green’s functions and tensor products,” Nucl. Phys. B 151, 93 (1979).

[14] E. Corrigan, D. B. Fairlie, S. Templeton and P. Goddard, “A Green’s function for the general selfdual gauge field,” Nucl. Phys. B 140, 31 (1978).

[15] N. H. Christ, E. J. Weinberg and N. K. Stanton, “General self-dual Yang-Mills solutions,” Phys. Rev. D 18, 2013 (1978).

[16] S.K. Donaldson and P.B. Kronheimer, “The Geometry of Four-Manifolds,” Oxford Math. Monographs, Oxford Univ. Press, 1990.

[17] M. Hamanaka and H. Kajiura, “Gauge fields on tori and T-duality,” Phys. Lett. B 551, 360 (2003) hep-th/0208059.

[18] P. M. Ho, “Twisted bundle on quantum torus and BPS states in matrix theory,” Phys. Lett. B 434, 41 (1998) hep-th/9803166.

[19] B. Morariu and B. Zumino, “Super Yang-Mills on the noncommutative torus,” hep-th/9807198.

[20] O. J. Ganor, S. Ramgoolam and W. Taylor, “Branes, fluxes and duality in M(atrix)-theory,” Nucl. Phys. B 492, 191 (1997) hep-th/9611202.