RESEARCH NOTES

ON A DENSITY PROBLEM OF ERDŐS

SAFWAN AKBIK

(Received 13 April 1998 and in revised form 10 June 1998)

ABSTRACT. For a positive integer $n$, let $P(n)$ denote the largest prime divisor of $n$ and define the set $S(x) = \{ n \leq x : n$ does not divide $P(n)! \}$. Paul Erdős proposed that $|S| = o(x)$ as $x \to \infty$, where $|S|$ is the number of $n \in S$. This was proved by Ilias Kastanas. In this paper we will show the stronger result that $|S| = O(xe^{-a\sqrt{\log x}})$.

Keywords and phrases. Number theory, primes, factorial, density, divisibility.

1991 Mathematics Subject Classification. 11B05, 11N25.

Introduction. For a positive integer $n$, let $P(n)$ denote the largest prime divisor of $n$ and define the set

$$S(x) = S = \{ n \leq x : n \text{ does not divide } P(n)! \}. \quad (1)$$

Paul Erdős [1] proposed that $|S| = o(x)$ as $x \to \infty$, where $|S|$ is the number of $n \in S$. A solution [3] was provided by Ilias Kastanas. There was also [3] a claim of proving that $|S(x)| = O(x/\log x)$. In this paper, we show the stronger result.

Theorem. For some constant $a > 0$, we have

$$|S| = O \left( xe^{-a\sqrt{\log x}} \right). \quad (2)$$

In fact, $a = 1/4$ suffices.

Lemma 1. Let $\nu(n)$ be the number of distinct prime divisors of $n$. Define

$$S_1 = \{ n \leq x : \nu(n) > 4k \log \log x, k \geq 1 \}. \quad (3)$$

Then

$$|S_1| = O \left( \frac{x}{(\log x)^k} \right) \quad (4)$$

uniformly in $k$.

Proof. It is well known [2] that if $d(m)$ is the number of divisors of $m$, then $\sum_{m \leq x} d(m) = O(x \log x)$. Since $d(m) \geq 2^\nu(m)$,

$$O(x \log x) \geq \sum_{m \leq x} 2^\nu(m) \geq \sum_{m \in S_1} 2^\nu(m) \geq \sum_{m \in S_1} (\log x)^{(4\log 2)k} \geq |S_1| (\log x)^{k+1}, \quad (5)$$

and the lemma follows. \qed
Lemma 2. Let \( C(x) = C = (\log x)^k \), where \( k = k(x) \) will be chosen later. Define
\[
S_2 = \{ n \leq x : p^2 \mid n \text{ for some prime } p > C \}. \tag{6}
\]
Then
\[
|S_2| = O\left( \frac{x}{(\log x)^k} \right). \tag{7}
\]

Proof. Since \( n \in S_2 \) if and only if \( n = tp^2 \) for some \( C < p \leq \sqrt{x} \) and some \( t \leq x/p^2 \),
\[
|S_2| = \sum_{C < p \leq \sqrt{x}} \left[ \frac{x}{p^2} \right] \leq \sum_{C < p \leq \sqrt{x}} \frac{x}{p^2} = O\left( \frac{x}{C} \right) = O\left( \frac{x}{(\log x)^k} \right). \tag{8}
\]
The first big \( O \) in (8) follows since \( \sum_{p > C} 1/p^2 \leq \int_{[C]}^\infty du/u^2 = 1/[C] = O(1/C) \), for \( C \geq 1 \).

Lemma 3. Let
\[
S_3 = \{ n \leq x : p^\alpha \mid n \text{ for some } \alpha \geq T \text{ and some prime } P \leq C \}, \tag{9}
\]
where \( T = 2\log C \). Then
\[
|S_3| = O\left( \frac{x}{(\log x)^k} \right). \tag{10}
\]

Proof.
\[
|S_3| = \sum_{\alpha \geq T} \left[ \frac{x}{p^\alpha} \right] \leq 2 \sum_{p \leq C} \frac{x}{p^T} \leq 2 \left( \frac{x}{2^T} \right) \sum_{p} \frac{4}{p^2} = O\left( \frac{x}{2^T} \right)
\]
\[
= O\left( \frac{x}{C^2 \log^2} \right) \leq O\left( \frac{x}{C} \right) = O\left( \frac{x}{(\log x)^k} \right). \tag{11}
\]
The first inequality in (11) is valid because \( \sum_{\alpha \geq T} 1/p^\alpha \leq 1/p^T + 1/p^{T+1} + \cdots \leq 2/p^T \) and the second is valid because \( p^T = 2^T (p/2)^T \geq 2^T (p/2)^2 \) for \( T \geq 2 \).

Lemma 4. Let \( S'(x) = S - (S_1 \cup S_2 \cup S_3) \), then, for any \( n \in S' \), we have
\[
P(n) \leq 2CT. \tag{12}
\]

Proof. Let \( n \in S' \). Then \( n \in S \) and so \( n \) does not divide \( P(n)! \). There exists a prime \( p_0 \) dividing \( n \) such that
\[
\nu_{p_0}(n) > \nu_{p_0}(P(n)!), \tag{13}
\]
where \( \nu_p(m) \) denotes the largest integer \( t \) such that \( p^t \) divides \( m \). Since \( \nu_{p_0}(P(n)!) \geq 1 \), (13) implies that \( \nu_{p_0}(n) \geq 2 \). Since \( n \notin S_2 \), it follows that \( p_0 \leq C \). Also \( n \notin S_3 \), so that \( T \geq \nu_{p_0}(n) \). Hence,
\[
T \geq \nu_{p_0}(n) > \nu_{p_0}(P(n)!) \geq \frac{P(n)}{2p_0} \geq \frac{P(n)}{2C} \tag{14}
\]
which implies (12). Note that the third inequality in (14) is true because \( \nu_p(m!) \geq \left\lfloor \frac{m}{p} \right\rfloor > \frac{m}{p} - 1 \geq \frac{m}{2p} \), \( p \neq m \), (15)
and \( \lfloor m/p \rfloor = 1 \geq m/2p \) if \( p = m \).

**Proof of the Theorem.** For \( n \in S' \), we have \( n \notin S_1 \cup S_2 \cup S_3 \). Thus, \( \nu_p(n) \leq 4k \log \log x \). Also, if \( p^\alpha \) is any prime power dividing \( n \), then one of the following two possibilities must occur:

(a) \( p \leq C \) and \( \alpha \leq T \),
(b) \( p > C \) and \( \alpha = 0 \) or 1.

Case (a) generates at most \( C(T + 1) \leq 2CT \) prime powers. For Case (b), the number of prime powers \( p^\alpha \) with \( p > C \) and \( \alpha \leq 1 \) is at most \( P(n) \). By Lemma 4, this is at most \( 2CT \). Hence, the number of possible prime powers \( p^\alpha \) that divide an \( n \in S' \) is at most \( 4CT \).

Therefore,
\[
|S'| \leq (4CT)^{4k \log \log x} = (8C \log C)^{4k \log \log x}.
\]

Choosing
\[
k = \frac{1}{4}, \frac{\sqrt{\log x}}{\log \log x},
\]
(17)

(16) gives \( |S'| \leq e^{(1/2)\log x} = x^{1/2} \). Hence,
\[
S' = O \left( xe^{-\sqrt{\log x}} \right).
\]
(18)

From (17), we have \( x/(\log x)^k = xe^{-1/2\sqrt{\log x}} \). Lemmas 1, 2, and 3 imply that
\[
|S_1| = O \left( xe^{-1/2\sqrt{\log x}} \right), \quad i = 1, 2, 3.
\]
(19)

Finally, \( S = S' \cup (S \cap (S_1 \cup S_2 \cup S_3)) \). Hence, (18) and (19) yield
\[
|S| \leq |S'| + |S_1| + |S_2| + |S_3| = O \left( xe^{-(1/4)\sqrt{\log x}} \right),
\]
(20)
and (2) follows with \( a = 1/4 \).

**Remark.** If \( \pi(x) \) is the number of prime integers that are less than or equal to \( x \), an early version of the prime numbers theorem asserts that
\[
\pi(x) = \int_2^x \frac{du}{\log u} + O \left( xe^{-a\sqrt{\log x}} \right),
\]
(21)
for some constant \( a \). Although the big \( O \) terms in (19) and (2) are similar, there is no apparent relationship between the PNT and (2).
References

[1] P. Erdős, A Proposed Problem, Amer. Math. Monthly 98 (1991), 965.
[2] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 5th ed., The Clarendon Press, Oxford University Press, New York, 1979. MR 81i:10002. Zbl 423.10001.
[3] I. Kastanas, The smallest factorial that is a multiple of n, Amer. Math. Monthly 101 (1994), 179.

AKBIK: DEPARTMENT OF MATHEMATICS, HOFSTRA UNIVERSITY, HEMPSTEAD, NY 11550, USA
Submit your manuscripts at http://www.hindawi.com