Timeseries thresholding and the definition of avalanche size

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Avalanches whose sizes and durations are distributed as power laws appear in many contexts. Here, we show that there is a hidden peril in thresholding continuous times series –either from empirical or synthetic data– for the detection of avalanches. In particular, we consider two possible alternative definitions of avalanche size used e.g. in the empirical determination of avalanche exponents in the analysis of neural-activity data. By performing analytical and computational studies of an Ornstein-Uhlenbeck process (taken as a guiding example) we show that if (i) relatively large threshold values are employed to determine the beginning and ending of avalanches and if (ii) –as sometimes done in the literature– avalanche sizes are defined as the total area (above zero) of the avalanche, then true asymptotic scaling behavior is not seen, instead the observations are dominated by transient effects. These can induce misinterpretations of the resulting scaling regimes as well as to a wrong assignation of universality classes.

I. INTRODUCTION

Episodic outburst of activity or “avalanches” of highly variable durations and sizes are observed in a large variety of scenarios in condensed matter physics (vortices of type II superconductors [1]), high-energy astrophysics (X-ray flares [2]), geophysics (earthquakes [3]), meteorology (rainfall [4]), neuroscience (neuronal avalanches [5]), and other biological processes (gene knock-out cascades [6]). The probability distributions of sizes and durations of such avalanches often exhibit “fat-tails” that can be fitted as power laws; i.e. the fingerprint of scaling behavior. Such scaling is often considered as evidence of underlying criticality and many of the above systems are claimed to operate at (either tuned or self-organized) critical points [7–12]. In particular, in the context of biology the idea that living systems (parts, aspects or groups of them) may extract important functional advantages from operating at criticality –i.e. at the edge of two different phases– has been deeply explored in recent years [12,13,21].

In this regard, groundbreaking experimental evidence by Beggs and Plenz [5], revealed the existence of scale-invariant episodes of electrochemical activity in neural tissues thereafter named neuronal avalanches. Subsequently, neuronal avalanches were robustly detected in a large number of experimental settings, tissues and species [5,14,20]. In particular, neuronal avalanche sizes, S, were observed to be distributed as a power-law $P(S) \sim S^{-\tau}$ with $\tau \approx 3/2$ up to some upper cutoff; similarly, avalanche durations $T$ were well fitted by $P(T) \sim T^{-\alpha}$ with $\alpha \approx 2$ up to some characteristic maximum time $\tau$. Furthermore, fundamental scaling relationships [21] were observed to be fulfilled: e.g. the averaged avalanche size scales as $\langle S \rangle \sim T^\gamma$ and the set of exponents obey $\gamma = (\alpha - 1)/(\tau - 1)$ [22].

This set of empirically reported exponent values is in agreement with that of the well-known critical (or “unbiased”) branching processes, also called Galton-Watson process, originally introduced to describe the statistics of the extinction of family names) [23,24]. Actually, the set of values $\tau = 3/2$, $\alpha = 2$ and $\gamma = 2$ are extremely universal as they are shared by many different propagation processes in high dimensions as well as in many types of networks. In particular, they are the mean-field exponents shared by models such as the contact process, directed and isotropic percolation, susceptible-infected-susceptible, and a large list of other prototypical models for spreading/propagation dynamics above their upper critical dimensions [21,23,24].

Thus, it was conjectured that neuronal systems might operate close to the edge of marginal propagation of neural (electro-chemical) activity [5,29], opening the door to exciting theoretical perspectives and some debate (see [12] for a recent review). However, as extensively discussed in the recent literature, diverse generative processes for the emergence of power-laws exist [30,32] and not all power-law distributions are a signature of criticality. In particular, it has been recently suggested that the origin of the observed power-law scaling in neural systems might stem from other types of criticality [33,34] or even be unrelated to critical behavior [35,36].

In the present brief paper, we contribute with an additional piece of information to the already controversial discussion about the statistics of neuronal avalanches. In particular, we show that some of the reported empirical evidence in favor of the value $\tau = 3/2$ –and thus, seemingly in favor of the existence of an underlying critical branching process– might be misleading as there is a technical problem in the way avalanches are measured, which hinders the observation of the true asymptotic behavior. More in general, we underline that particular attention needs to be taken when avalanches of activity –defined by thresholding– are inferred from a continuous time series of activity; this adds to the recent literature warning on the “perils” associated with thresholding in timeseries [35,39,40].
II. DEFINITION OF AVALANCHES

1. Avalanches in the Wiener and Ornstein-Uhlenbeck processes

Let us consider, for argument’s sake, a time series for a stochastic real variable \( x \) –as illustrated in Fig.1– generated by a Wiener Process, i.e. by a continuous-time unbiased random walk (RW) defined by the following Langevin equation [41]:

\[
\dot{x}(t) = \sigma \eta(t),
\]

where \( \eta(t) \) is a Gaussian white noise with zero mean and unit variance, and \( \sigma \) is the noise amplitude. For such a time series (which can be thought as describing the time course of the activity of some arbitrary system) the duration \( T \) of an avalanche is the amount of time for which \( x \) stays above a given threshold, i.e. an avalanche begins/ends when the activity signal crosses the threshold from below/above; the avalanche size \( S \) is the area covered between the walk trajectory and the threshold reference line (see Fig. 1). Observe that similarly, given the symmetry of the process, one could also define avalanches as excursions below threshold.

The probability distribution of avalanche durations \( T \) can be straightforwardly identified with the first-return time statistics of random walks (see Fig.1) which is well-known to scale with exponent \( \alpha = 3/2 \). Similarly, also the size-distribution exponent \( \tau = 4/3 \) and the remaining exponent \( \gamma = 3/2 \) are well-known for random walks (see Fig. 1 and Table I); a detailed derivation of these results, as well as a comparison with the branching process class Fig. 1 and Table I); a detailed derivation of these results, can be found in e.g. [42, 43]. Importantly, these results for the random walk do not depend of the value of the chosen threshold.

The probability distribution of avalanche sizes \( S \) are defined for an unbiased random walk. Up-to an upper scale controlled by \( 1/\alpha \).

In the more general case in which the walker is confined to hover around a given mean value, one can describe the problem, in first approximation, as an Ornstein-Uhlenbeck process [41]:

\[
\dot{x}(t) = -ax(t) + \sigma \eta(t),
\]

where there is an additional linear force term, \(-ax\) (corresponding to the negative derivative of the parabolic potential bounding the walker close to \( x = 0 \)). Such a force introduces an upper cutoff in the first-return times statistics of the unbiased RW (see e.g. [43] for a detailed derivation). Thus, avalanches intended as excursions above a given threshold in a process with a well-defined steady-state value, have power-law-distributed sizes and durations, with the exponents of the RW class, but only up to an upper scale controlled by \( 1/\alpha \).

As a corollary of all this, let us remark that many real time series describing (e.g. biological) problems in which some stochastic variable fluctuates symmetrically around a given mean value exhibit effective avalanching behavior that –up to certain scale of size and time– can be described by the exponent values of the RW.

2. On the definition of avalanche size

In many other circumstances time series exhibit asymmetric excursions around a mean value. An example of this is obtained when the variable under scrutiny is a density (e.g. of neural activity), which by definition is constrained to take positive values, \( x(t) > 0 \). In such cases, especially in the ones when there is always some lingering activity so that the zero-value is hardly reached (see Fig.2) a threshold \( \theta > 0 \) is often employed to define avalanches as periods during which the activity remains above such threshold. In these cases, two alternative possibilities are often used in the literature to measure the size of a so-defined avalanches [44].

(A) As done above, following the random walk analogy, for a given avalanche, one can define its size \( S \) as the area in between the time-series curve and the threshold \( \theta \) reference line.

(B) Alternatively, one can define the avalanche size \( \Sigma \) as the overall integral of the time series during the avalanche, i.e. above the reference line \( x = 0 \) (see e.g. [45] [46] but there are other works making this choice).

| \( P(S) \sim S^{-\alpha} \) | \( P(T) \sim T^{-\tau} \) | \( P(S|T) \sim T^{-\gamma} \) |
|---------------|---------------|---------------|
| BP \( \tau = 3/2 \) | \( \alpha = 2 \) | \( \gamma = 2 \) |
| RW \( \tau = 4/3 \) | \( \alpha = 3/2 \) | \( \gamma = 3/2 \) |

TABLE I. Summary of the avalanche (mean-field) exponents: size \( (\tau) \), duration \( (\alpha) \) and averaged avalanche size \( (\gamma) \) for the (un-biased) branching process (BP) and the (un-biased) random walk (RW); see e.g. [42].
imply a locally linear (i.e. “tent like”) shape of avalanches, same way, entailing that both sizes and durations scale in the same way, as determined employing criterion A (area above threshold, colored in orange) and T is its duration. On the other hand, using criterion B, Σ = S + s∗ (where s∗ is the area of the rectangle between zero and the threshold, colored in blueish color, with s∗ ∝ T) is an often-used alternative definition of avalanche size. As discussed in the text this definition may induce misleading interpretations of the resulting exponents.

The difference between the two criteria is sketched in Figure 2. In what follows, we compare the statistics of avalanches obtained using these two alternative definitions of size A and B for an Ornstein-Uhlenbeck process. This will serve as an illustration of a more general phenomenon that may also occur for other processes, such as the one sketched in Fig.2.

First we discuss computational results and then we employ scaling arguments to explain the findings. On the one hand, as already shown in Fig.1 using S, i.e. criterion A, one reproduces the expected theoretical results for all three avalanche exponents. On the other hand, as illustrated in Figure 3, the statistics of avalanche sizes, as determined employing Σ for an Ornstein-Uhlenbeck process is anomalous and does not match the expectations for the theoretically known values i.e. the measured value τ ≈ 3/2 does not coincide with the expected value τ = 4/3. In particular, the numerical observation of, τ ≈ 3/2 could be (wrongly) taken as evidence of branching process-like scaling [45, 46]. The fact that there is something suspicious with the definition (ii) can be noticed observing that both sizes and durations scale in the same way, entailing γ = (α − 1)/(τ − 1) = 1, which would imply a locally linear (i.e. “tent like”) shape of avalanches [47–49].

3. Scaling arguments

The correction s∗ for a given avalanche (such that Σ = S + s∗) is nothing but s∗ = θT, where T is the avalanche duration. The distribution of first-passage times for the Wiener process is given by P(T) ∼ T−3/2. Thus, Σ has a correction s∗ with respect to T that scales as the avalanche duration: P(s∗) ∼ s∗−3/2. Assuming that P(S|T) is a peaked function around the mean value (as usually occurs for avalanches [21]) and using the fact that ⟨S⟩ ∼ Tγ, with γ = 3/2, then

Σ(T) = S(T) + s∗(T) = ̃ct3/2 + θT, (3)

from where it follows that

dΣ = (ct1/2 + θ)dT (4)

Thus, we can readily write (e.g. using the implicit function theorem)

P(Σ(T)) = P(T) dT dΣ = N cT−3/2 + θ. (5)

From this, in the limit of vanishing θ in Eq. (4), one has

P(Σ) ≈ N cT−2 ∼ N c[Σ/c]2/3−2 ∼ Σ−4/3, (6)

which is the correct result for the avalanche size distribution of an Ornstein-Uhlenbeck process. On the other hand, for larger values of θ, and relatively small values of T (and, thus, also typically small values of Σ) one has

P(Σ) ≈ N θT−3/2 ≈ N θ[Σ/θ]−3/2 ∼ Σ−3/2, (7)
in agreement with numerical observations above. In other words, the additional contribution $s^*$ dominates the scaling behavior when $\theta$ is relatively large. It is important to emphasize that one should recover the correct asymptotic value, for large values of $\Sigma$, for the avalanche size exponent, $\tau = 4/3$, for any value of $\theta$ but this requires going to larger and larger avalanche sizes as $\theta$ is chosen larger and larger. In particular, Figure 3 illustrates that there is a crossover from the value $\tau \approx 3/2$ measured for small avalanche-sizes to the true asymptotic scaling $\tau = 4/3$, for larger sizes. The crossover point grows with $\theta$, so that the effect is not observed for $\theta \approx 0$, but may extend for many scales if $\theta$ is large. In particular, given that an upper cut-off to scaling may exist (controlled e.g. by $1/\alpha$ in the Ornstein-Uhlenbeck process or by the finite system size), the transient behavior usually extends all the way up to the cut-off, so that the true asymptotic behavior is never seen if large values of $\theta$ are considered.

Thus, summing up, considering criterion B for the definition of avalanche sizes together with relatively large threshold values or not sufficiently large statistics, may lead to the observation of an effective value $\tau \approx 3/2$; this may induce a misinterpretation of the scaling universality class, suggesting it is branching-process-like rather than what it actually is: a random-walk-like process.

Let us emphasize that all the previous discussion has been done for an Ornstein-Uhlenbeck process. However, it perfectly illustrates the problem associated with criterion B in more general circumstances, e.g. it also applies to asymmetric processes as the one sketched in Fig.2. In any case, criterion B mixes the scalings of actual sizes and times, leading to potential interpretation errors in general stochastic processes.

III. CONCLUSION

In this brief paper we have shown that, an inappropriate definition of avalanche sizes as measured as fluctuations in continuous time series can potentially lead to wrong conclusions. To illustrate this, we have studied a simple Wiener process (representing e.g. the time course of activity in a mesoscopic model of neural activity) and have measured avalanche sizes in two possible ways: (i) as the integrated activity $S$ over a given threshold and (ii) integrating the total activity signal in between two threshold crossings, as illustrated in Fig. 2. We have shown both computationally and using scaling arguments that this latest definition can induce strong biases in the determination of the avalanche-size exponent $\tau$.

In particular, if large values of the threshold $\theta$ are considered, then –for relatively small avalanche sizes– one observes the exponent value $\tau \approx 3/2$ which could lead to the erroneous interpretation that an effective un-biased branching process dynamics exists. On the other hand for sufficiently small threshold values and for sufficiently large avalanche sizes the correct scaling $\tau = 4/3$ is recovered. As discussed above the problem associated with criterion B extends to any type of stochastic process as it mixes up the scaling of actual sizes with that of durations.

We believe that this is the underlying reason why recent analyses of avalanches in mesoscopic models of neural activity which consider relatively large thresholds [45, 46] (see also [44]) obtain $\tau \approx 3/2$, a result that our analyses suggest is not asymptotic. Their corresponding underlying dynamics describe fluctuations around a given mean value and, thus, the associated avalanches should be related to excursions of random walkers and not to branching processes.

As a final remark, let us stress that considering the full set of avalanche exponents i.e. $\tau$, $\alpha$, and $\gamma$ as well as the scaling relations between them, would be important in order to avoid possible errors and misleading interpretations.
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