Analytic Discs, Global Extremal Functions and Projective Hulls in Projective Space

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Abstract

Using a recent result of Lárusson and Poletsky regarding plurisubharmonic subextensions we prove a disc formula for the quasiplurisubharmonic global extremal function for domains in $\mathbb{P}^n$. As a corollary we get a characterization of the projective hull for connected compact sets in $\mathbb{P}^n$ by the existence of analytic discs.

1 INTRODUCTION

The global extremal function, also called the Siciak-Zahariuta extremal function, has proven very useful for pluripotential theory in $\mathbb{C}^n$, see [5, §13] and [6, §5] for an overview of the applications. We are however most interested in its counterpart in the theory of quasiplurisubharmonic functions on compact manifolds. The quasiplurisubharmonic global extremal function was defined by Guedj and Zeriahi [3] and has already proven useful, most notably in connection with projective hulls [4]. But the projective hull of a compact set in $\mathbb{P}^n$ is the natural generalization of the polynomial hull in $\mathbb{C}^n$.

We start by looking at a recent result of Lárusson and Poletsky [7] regarding plurisubharmonic subextensions for domains in $\mathbb{C}^n$. There we make a small observation regarding their results (Corollary 2.5), and we also define a disc structure for sets in $\mathbb{C}^{n+1}\setminus\{0\}$ with some nice properties. This is done in Section 2.

In Section 3 we turn our attention to quasiplurisubharmonic function, or $\omega$-plurisubharmonic functions, on $\mathbb{P}^n$ which we denote by $\mathcal{PSH}(\mathbb{P}^n, \omega)$. The current $\omega$ is here the Fubini-Study Kähler form. Using the results from the Section 2 we prove a disc formula for the global extremal function for a
domain $W \subset \mathbb{P}^n$ (Theorem 3.3),

$$\sup\{ u(x); u \in \mathcal{P}(\mathbb{P}^n, \omega), u|_{W} \leq \varphi \} = \inf \left\{ \frac{-1}{2\pi} \int_{D} \log |\cdot| f^\ast \omega + \int_{\mathbb{T}} \varphi \circ f \, d\sigma \ ; \ f \in A^W_{\mathbb{P}^n}, f(0) = x \right\}.$$  

In Section 4 we show an applications of these results (Theorem 4.5). There we show that the points in the projective hull $\hat{K}$ of connected compact sets $K \subset \mathbb{P}^n$ can be characterized by the existence of analytic discs with specific properties. That is, for $\Lambda > 0$ and a connected compact subset $K \subset \mathbb{P}^n$ the following is equivalent for a point $x \in \mathbb{P}^n$:

(A) $x \in \hat{K}(\Lambda)$

(B) For every $\varepsilon > 0$ and every neighbourhood $U$ of $K$ there exists a disc $f \in A^U_{\mathbb{P}^n}$ such that $f(0) = x$ and

$$-\frac{1}{2\pi} \int_{D} \log |\cdot| f^\ast \omega < \Lambda + \varepsilon.$$  

Here, $\hat{K}(\Lambda), \Lambda > 0$, are specific subsets of $\hat{K}$ with $\hat{K} = \cup_{\Lambda} \hat{K}(\Lambda)$ which are defined using the best constant function for $K$ (see [4, §4]).

Finally, in Section 5 we see how the methods presented in Section 2 and 3 work also for other currents, in particular for the current of integration for the hyperplane at infinity $H_\infty \subset \mathbb{P}^n$. This gives rise to a disc formula for the Siciak-Zahariuta extremal function for a domain $W \subset \mathbb{C}^n$,

$$\sup\{ u(x); u \in \mathcal{L}, u|_{W} \leq \varphi \} = \inf \left\{ -\sum_{a \in f^{-1}(H_\infty)} \log |a| + \int_{\mathbb{T}} \varphi \circ f \, d\sigma \ ; \ f \in A^W_{\mathbb{P}^n}, f(0) = x \right\},$$  

(1)

where $\mathcal{L}$ is the Lelong-class of plurisubharmonic functions of logarithmic growth. This formula was first proved in [8], when $\varphi = 0$, and in [11], when $\varphi$ is upper semicontinuous. For the case when $W$ is not connected see [7].

We now must establish some notation. We assume $X$ is a complex manifold, here the role of $X$ will either be played by subsets of affine space or projective space. Let $A_X$ denote the family of closed analytic discs in $X$, that is continuous maps $f : \mathbb{D} \to X$ which are holomorphic on the unit disc $\mathbb{D}$. Assume $W \subset X$, then $A^W_X$ is the subset of discs in $A_X$ which map the unit circle $\mathbb{T}$ into $W$. 

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If $H$ is a disc functional, that is a function from a subset of $A_X$ to $[-\infty, +\infty]$, then its envelope with respect to the family $C \subset A_X$ is defined as the function

$$E_C H(x) = \inf \{ H(f); f \in C, f(0) = x \}.$$  

The domain of $E_C H$ is all the points $x \in X$ such that $\{ f \in C; f(0) = x \}$ is non-empty.

For convenience we write $E$ for $E_{A_X}$ and $E_W$ for $E_{A_W}$. The standard example of a disc functional is the Poisson disc functional $H_\varphi : A_X \to [-\infty, +\infty]$ for a function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$. It is defined as $H_\varphi(f) = \int_T \varphi \circ f \, d\sigma$. The measure $\sigma$ is the arclength measure on $T$ normalized to one. Other disc functional we will use are the Poisson disc functional for the class of $\omega$-plurisubharmonic functions

$$H_{\omega,\varphi}(f) = -\frac{1}{2\pi} \int_D \log |\cdot| \cdot f^{\ast} \omega + \int_T \varphi \circ f \, d\sigma,$$

where $f^{\ast} \omega$ is the pullback of $\omega$ by $f$. In our case the $(1,1)$-current $\omega$ will be the Fubini-Study Kähler form on $\mathbb{P}^n$. However, in Section 5 we look briefly at other currents on $\mathbb{P}^n$, specially the case when the current is the current of integration for the hyperplane at infinity.

If $\varphi$ is a function defined on $W \subset X$ then we let

$$F_\varphi = \{ u \in \mathcal{PS}\mathcal{H}(X); u|_W \leq \varphi \},$$

$$F_{\omega,\varphi} = \{ u \in \mathcal{PS}\mathcal{H}(X,\omega); u|_W \leq \varphi \}.$$  

**Remark:** Although it is more traditional to look at analytic discs which are holomorphic in a neighbourhood of the closed unit disc we are only assuming the discs are continuous to the boundary. This does in fact not alter the results obtained here since every disc holomorphic in a neighbourhood of $\mathbb{D}$ is clearly in $A_X$, and conversely if $f \in A_X$ then $f(r \cdot)$, $r < 1$ is a family of discs in holomorphic in a neighbourhood of $\mathbb{D}$ such that $H_{\omega,\varphi}(f(r \cdot)) \to H_{\omega,\varphi}(f)$, when $r \to 1^\circ$. The reason for this is that the authors of [7] applied a results of Forstnerič [2] which uses discs which are only continuous up to the boundary.

2 PLURISUBHARMONIC SUBEXTENSIONS

We now turn our attentions to the work done by Lárusson and Poletsky [7].
Their setting is the following. For domains $W \subset X \subset \mathbb{C}^n$ and an upper semicontinuous function $\varphi : W \to \mathbb{R} \subset \{-\infty\}$, they consider the function
\[
\sup F_{\varphi}(x) = \sup \{u(x); u \in F_{\varphi}\},
\]
which is largest plurisubharmonic function on $X$, dominated by $\varphi$ on $W$. Under sufficient conditions on $W$ and $X$ they prove a disc formula for this function, namely that
\[
\sup F_{\varphi} = E_W H_{\varphi}, 	ext{ or if we write it out }
\sup \{u(x); u \in \mathcal{PSH}(X), u(T) \subset W\} = \inf \left\{ \int_T \varphi \circ f \, d\sigma; f \in A^W_X, f(0) = x \right\}.
\]
Before we look at this formula in more detail we need the following definitions.

**Definition 2.1.** We say that two discs $f_0$ and $f_1$ in $A^W_X$ with $f_0(0) = f_1(0)$ are centre-homotopic if there is a continuous map $f : D \times [0, 1] \to X$ such that
- $f(\cdot, t) \in A^W_X$ for all $t \in [0, 1]$,
- $f(\cdot, 0) = f_0$ and $f(\cdot, 1) = f_1$,
- $f(0, t) = f_0(0) = f_1(0)$ for all $t \in [0, 1]$.

**Definition 2.2.** If $W \subset X$, then a $W$-disc structure on $X$ is a family $\beta = (\beta_\nu)_\nu$ of continuous maps $\beta_\nu : U_\nu \to A^W_X$, where $(U_\nu)_\nu$ is an open covering of $X$, such that
- $\beta_\nu(x)(0) = x$ for all $x \in U_\nu$ (i.e. $x$ is mapped to a disc centred at $x$),
- If $x \in U_\nu \cap U_\mu$ then $\beta_\nu(x)$ and $\beta_\mu(x)$ are centre-homotopic.

Furthermore, if there is $\mu$ such that $U_\mu = W$ and $\beta_\mu(w)(\cdot) = w$ for every $w \in W$ (i.e. $\beta_\mu(w)$ is the constant disc), then we say that the disc structure is **schlicht**.

For a $W$-disc structure $\beta$ we let $\mathcal{B} \subset A^W_X$ denote the family of discs $\mathcal{B} = \cup_\nu \beta_\nu(U_\nu)$.

**Lemma 2.3.** \cite[Lemma 2]{17} Let $W \subset X$ be domains in $\mathbb{C}^n$, and $\beta$ a $W$-disc structure on $X$. If $\varphi : W \to \mathbb{R} \subset \{-\infty\}$ is an upper semicontinuous function then
\[
E_W H_{\varphi} \leq E H_{E_\beta H_{\varphi}}.
\]
If $\beta$ is a $W$-disc structure on $X$ and $\varphi$ is upper semicontinuous then it follows easily from the continuity of the $\beta_i$’s that $E_B H_\varphi$ is an upper semicontinuous function on $X$.

**Theorem 2.4.** [17], Theorem 3] Let $W \subset X$ be domains in $\mathbb{C}^n$ and assume $\beta$ is a schlicht $W$-disc structure on $W$, then

$$\sup \mathcal{F}_\varphi = E_W H_\varphi.$$  

**Proof.** The formula follows from the following inequalities,

$$\sup \mathcal{F}_\varphi \leq E_W H_\varphi \leq E H_{E_B H_\varphi} \leq \sup \mathcal{F}_\varphi.$$  

The first inequality follows from the subaverage property of the subharmonic function $u \circ f$. If $u \in \mathcal{F}_\varphi$ and $f \in \mathcal{A}_X$, $f(0) = x$ then

$$u(x) = (u \circ f)(0) \leq \int_T u \circ f \, d\sigma \leq \int_T \varphi \circ f \, d\sigma = H_\varphi(f).$$  

Taking supremum on the left hand side over $u \in \mathcal{F}_\varphi$ and infimum on the right hand side over $f \in \mathcal{A}_X$ gives the inequality.

Lemma 2.3 gives the second inequality.

The last inequality follows from the fact that the function $E H_{E_B H_\varphi}$ is plurisubharmonic by Poletsky’s theorem [12, 13] and not greater than $\varphi$ since $\beta$ is schlicht. It is therefore in the class $\mathcal{F}_\varphi$ we take supremum over. 

Note that $E H_{E_B H_\varphi}$ is always plurisubharmonic when $\varphi$ is upper semicontinuous because then $E_B H_\varphi$ is upper semicontinuous. The only requirements for the last inequality are therefore that $E H_{E_B H_\varphi} \leq \varphi$ on $W$. From this small observation we get the following corollary.

**Corollary 2.5.** Let $W \subset X$ be domains in $\mathbb{C}^n$ and assume $\beta$ is a $W$-disc structure on $W$ such that $E_B H_\varphi \leq \varphi$ on $W$, then

$$\sup \mathcal{F}_\varphi = E_W H_\varphi.$$  

**Proof.** A fundamental property of the envelops of the Poisson disc functional is that $E H_\psi \leq \psi$ and therefore, with $\psi = E_B H_\varphi$,

$$E H_{E_B H_\varphi} \leq E_B H_\varphi \leq \varphi.$$  

The rest of the proof is then same the as in the proof of Theorem 2.4.  

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We will now construct a disc structure on a certain class of sets in $X = \mathbb{C}^m \setminus \{0\}$ satisfying the condition in Corollary 2.5. In Section 3 and 4 we let $m = n+1$ and look at $\mathbb{P}^n$ using homogeneous coordinates $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

**Definition 2.6.** A set $W \subset \mathbb{C}^m$ is a complex cone if $\lambda x \in W$ for every $\lambda \in \mathbb{C}$ and $x \in W$.

Later, when we talk about a complex cone in $\mathbb{C}^m \setminus \{0\}$ it is simply a complex cone in $\mathbb{C}^m$ with 0 removed.

**Definition 2.7.** Assume $W$ is a complex cone. A function $\varphi : W \to \mathbb{R} \cup \{-\infty\}$ is called logarithmically homogeneous if

$$
\varphi(\lambda x) = \varphi(x) + \log |\lambda|,
$$

for every $\lambda \in \mathbb{C}$ and $x \in W$.

Note that every function on a complex cone in $\mathbb{C}^m \setminus \{0\}$ which is logarithmically homogeneous extends automatically over 0 and takes the value $-\infty$ there.

**Lemma 2.8.** Assume $W \subset \mathbb{C}^m \setminus \{0\}$, $m \geq 2$ is a complex cone and a domain, and assume that $\varphi : W \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function which is logarithmically homogeneous. Then there exists a $W$-disc structure $\mathcal{B}$ in $\mathbb{C}^m \setminus \{0\}$ such that $E_{\mathcal{B}} \mathcal{H}_\varphi \leq \varphi$.

**Proof.** For each $w \in W$ let $U_w = \mathbb{C}^m \setminus \{\lambda w; \lambda \in \mathbb{C}\}$ and define the analytic discs $\beta_w(x)$ by

$$
f_{x,w}(t) = \beta_w(x)(t) = \left( \frac{\|x-w\|}{r} - \frac{r}{\|x-w\|} \right) tw + \left( 1 + \frac{r}{\|x-w\|} t \right) x
$$

where

$$
r = \min \left\{ \frac{\|x-w\|}{1 + \|x-w\|}, \frac{d(w, W^c)}{2} \right\},
$$

and $W^c$ is the complement of $W$ in $\mathbb{C}^m \setminus \{0\}$.

It is more convenient to write the formula for these discs in the following way

$$
f_{x,w}(t) = \begin{cases} 
(1 + \frac{\|x-w\|}{r}) w + \left( \frac{\|x-w\| + rt}{r + \|x-w\| t} \right)^{\frac{r}{\|x-w\|}} (x - w) & \text{if } t \neq \frac{r}{\|x-w\|} \\
(1 - \frac{r^2}{\|x-w\|^2}) (x - w) & \text{if } t = \frac{r}{\|x-w\|}
\end{cases}
$$
Then we see that 0 is mapped to x. Furthermore, the factor in the brackets $(\star)$ maps the closed unit disc into the complex line through x and w, and maps the unit circle into a circle with centre w and radius r. This can be seen from the fact that if $t \in T$ then $\left|\frac{\|x-w\|+rt}{r+\|x-w\|t}\right|=1$ and

$$\|w-(\star)\|=r<d(w, W^c).$$

This implies that for $t \in T$ the value $f_{x,w}(t) = (1 + \frac{\|x-w\|}{r}t)(\star)$ is also in W since W is a complex cone.

Note also that 0 is not in the image of $f_{x,w}$ because by the definition of $U_w$ the complex line through x and w does not include 0.

To show that this is a W-disc structure we need to show that every two discs with the same centre are centre-homotopic, that is $f_{x,w}$ and $f_{x,w'}$ are centre-homotopic for every $w, w' \in W$. Since W is connected the set $W \setminus \{\lambda x; \lambda \in \mathbb{C}\}$ is also connected and path connected. Therefore there is a path $\gamma : [0,1] \to W \setminus \{\lambda x; \lambda \in \mathbb{C}\}$ such that $\gamma(0) = w$ and $\gamma(1) = w'$. Define the map $f : D \times [0,1] \to \mathbb{C}^{n+1} \setminus \{0\}$ by

$$f(t, s) = f_{x,\gamma(s)}.$$ 

The function f clearly satisfies all the conditions in Definition 2.1, which means that we have defined a W-disc structure $B = \cup_{w \in W} \{f_{x,w}; x \in U_w\}$.

We now show that $E_BH_\varphi \leq \varphi$ on W. Fix $x \in W$ and $\varepsilon > 0$. Since $\varphi$ is upper semicontinuous there is an open neighbourhood $U$ of $x$ such that $\varphi|_U \leq \varphi(x) + \varepsilon/2$. Then select w close enough to x so that

- $\frac{1}{2\pi} \log(1 + \|x - w\|) < \varepsilon/2$,
- $r = \min \left\{ \frac{\|x-w\|}{1+\|x-w\|}, \frac{d(w, W^c)}{2} \right\}$ is equal to $\frac{\|x-w\|}{1+\|x-w\|}$,
- the disc on the complex line through x and w with centre w and radius r (defined as above) is in U.

Then by using the properties of $\varphi$, the properties of the term $(\star)$, and the
Riesz representation formula and we see that
\[ E_B H_\varphi(x) \leq \int_T \varphi \circ f_{x,w} \, d\sigma \]
\[ \leq \int_T \varphi \left( \left( 1 + \frac{\|x-w\|}{r} \right)^t \right) \, d\sigma \]
\[ \leq \int_T \varphi((\ast)) \, d\sigma + \int_T \log \left| 1 + \frac{\|x-w\|}{r} \right| \, d\sigma \]
\[ \leq \sup_{\mathcal{U}} \varphi - \frac{1}{2\pi} \log \left( 1 + \frac{\|x-w\|}{r} \right) \]
\[ \leq \varphi(x) + \frac{\varepsilon}{2} + \log(1 + \|x-w\|) \leq \varphi(x) + \varepsilon. \]
This holds for every \( \varepsilon > 0 \), hence \( E_B H_\varphi \leq \varphi \).

\[ \square \]

It should be noted here that this disc structure defined here is under heavy influence from the set of “good discs” used in both [8] and [11] for the original proof of (1).

3 DISC FORMULA FOR THE GLOBAL RELATIVE EXTREMAL FUNCTION IN \((\mathbb{P}^n, \omega)\)

We let \( \omega \) be the Fubini-Study Kähler form for \( \mathbb{P}^n \). Recall that an upper semi-continuous function \( u \) on \( \mathbb{P}^n \) is called \( \omega \)-plurisubharmonic (or quasiplurisubharmonic) if \( dd^c u + \omega \geq 0 \). We denote the family of \( \omega \)-plurisubharmonic function on \( \mathbb{P}^n \) by \( \mathcal{PSH}(\mathbb{P}^n, \omega) \).

If \( f \in \mathcal{A}_{\mathbb{P}^n} \) then there is a well defined pullback of \( \omega \) by \( f \), denoted \( f^* \omega \). It is defined locally by \( \Delta \psi \circ f \) where \( \psi \) is a local potential of \( \omega \), i.e. \( \psi \) is a plurisubharmonic function such that \( df^c \psi = \omega \). For a more details about \( \omega \)-plurisubharmonic functions and analytic discs see [10, §2].

The pullback of the current \( \omega \) to \( \mathbb{C}^{n+1} \setminus \{0\} \) satisfies
\[ \pi^* \omega = dd^c \log \| \cdot \|, \]
where \( \pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n \) is the projection
\[ \pi(z_0, z_1, \ldots, z_n) = [z_0 : z_1 : \cdots : z_n]. \]

This implies that if \( \tilde{f} \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{0\}} \) and we let \( f = \pi \circ \tilde{f} \in \mathcal{A}_{\mathbb{P}^n} \) then
\[ f^* \omega = \Delta \log \| \tilde{f} \|. \]
Proposition 3.1. There is a one to one correspondence between the class \( PSH(\mathbb{P}^n, \omega) \) and \( \{ u \in PSH(\mathbb{C}^{n+1} \setminus \{0\}); u \text{ logarithmically homogeneous} \} \).

Proof. If \( v \in PSH(\mathbb{P}^n, \omega) \) then \( u = v \circ \pi + \log \| \cdot \| \) is plurisubharmonic on \( \mathbb{C}^{n+1} \setminus \{0\} \) since
\[
  dd^c (v \circ \pi + \log \| \cdot \|) = \pi^* (dd^c v + \omega) \geq 0.
\]
Conversely, if \( u \) is in the later class, then for \( z \in \mathbb{C}^{n+1} \setminus \{0\} \), \( v([z]) = u(z) - \log \| z \| \) is a well defined function on \( \mathbb{P}^n \) since
\[
  u(\lambda z) - \log \| \lambda z \| = u(z) + \log |\lambda| - \log \| \lambda z \| = u(z) - \log \| z \|, \quad \lambda \in \mathbb{C}^*.
\]
Furthermore, \( v \) is \( \omega \)-plurisubharmonic by the same calculations as above. \( \square \)

Lemma 3.2. Assume \( X \) is a domain and a complex cone in \( \mathbb{C}^{n+1} \setminus \{0\} \). If \( \varphi : X \to \mathbb{R} \cup \{-\infty\} \) is upper semicontinuous and logarithmically homogeneous then the function \( \sup F_\varphi \),
\[
  \sup F_\varphi(x) = \sup \{ u(x); u \in PSH(\mathbb{C}^{n+1} \setminus \{0\}), u|_X \leq \varphi \}
\]
is also logarithmically homogeneous.

Proof. Since function \( \sup F_\varphi \) is dominated by \( \varphi \), then
\[
  \sup F_\varphi(\lambda x) + \log |\lambda| \leq \varphi(\lambda x) + \log |\lambda| = \varphi(x)
\]
and furthermore, since \( \sup F_\varphi \) is plurisubharmonic, the function \( x \mapsto \sup F_\varphi(\lambda x) \) is also plurisubharmonic and therefore in \( F_\varphi \). This implies, by the definition of \( \sup F_\varphi \), that
\[
  \sup F_\varphi(\lambda x) + \log |\lambda| \leq \sup F_\varphi(x), \quad \lambda \in \mathbb{C}^*.
\]
By setting \( \lambda^{-1} \) instead of \( \lambda \) and \( \lambda x \) instead of \( x \) we get the inverted inequality, hence \( \sup F_\varphi(\lambda x) + \log |\lambda| = \sup F_\varphi(x) \). \( \square \)

Now we prove the main result.

Theorem 3.3. Let \( \omega \) by the Fubini-Study Kähler form. If \( W \subset \mathbb{P}^n \) is a domain and \( \varphi : W \to \mathbb{R} \cup \{-\infty\} \) is an upper semicontinuous function then
\[
  \sup \{ u(x); u \in PSH(\mathbb{P}^n, \omega), u|_W \leq \varphi \} = 
  \inf \left\{ \frac{1}{2\pi} \int_D \log |\cdot| f^* \omega + \int_T \varphi \circ f \, d\sigma; \quad f \in A_W^\infty, f(0) = x \right\}. \quad (2)
\]
or, using the notation from Section \[7], \( \sup F_{\omega, \varphi} = E_W H_{\omega, \varphi} \).
Proof. Define the complex cone \( \tilde{W} = \pi^{-1}(W) \) and define the logarithmically homogeneous function \( \tilde{\varphi} : \tilde{W} \to \mathbb{R} \cup \{-\infty\} \) by
\[
\tilde{\varphi}(z) = \varphi([z_0 : z_1 : \cdots : z_n]) + \log \|z\|.
\]

Fix \( \tilde{f} \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{0\}}^{\tilde{W}} \) and let \( f = \pi \circ \tilde{f} \in \mathcal{A}_{\mathbb{C}^{n}}^{W} \). Let \( z = \tilde{f}(0) \) which implies \( \pi(z) = f(0) \). Then
\[
\tilde{\varphi} \circ \tilde{f} = \varphi \circ f + \log \|f\|
\]
By the Riesz representation formula for the function \( \log \|\tilde{f}\| \) at the point 0,
\[
\log \|\tilde{f}(0)\| = \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Delta \log \|\tilde{f}\|| d\sigma + \int_{\mathbb{T}} \log \|\tilde{f}\| d\sigma
\]
Since \( \tilde{f}^*(\pi^*\omega) = (\pi \circ \tilde{f})^*\omega = f^*\omega \), this shows that
\[
- \int_{\mathbb{T}} \log \|\tilde{f}\| d\sigma - \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Delta \log \|\tilde{f}\|| d\sigma = - \log \|\tilde{f}(0)\| = - \log \|z\|.
\]
Using the three previous equalities we derive that
\[
H_{\omega,\varphi}(f) = \int_{\mathbb{T}} \varphi \circ f d\sigma - \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Delta \log \|f\|| d\sigma - \frac{1}{2\pi} \int_{\mathbb{D}} \log |f^*\omega|
\]
\[
= \int_{\mathbb{T}} \tilde{\varphi} \circ \tilde{f} d\sigma - \int_{\mathbb{T}} \log \|\tilde{f}\| d\sigma - \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Delta \log \|f\|| d\sigma - \frac{1}{2\pi} \int_{\mathbb{D}} \log |f^*\omega|
\]
\[
= \int_{\mathbb{T}} \tilde{\varphi} \circ \tilde{f} d\sigma - \log \|z\|
\]
That is
\[
H_{\omega,\varphi}(f) = H_{\tilde{\varphi}}(\tilde{f}) - \log \|z\|. \tag{3}
\]
Now note that every disc \( \tilde{f} \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{0\}}^{\tilde{W}} \) gives a disc \( f = \pi \circ \tilde{f} \in \mathcal{A}_{\mathbb{C}^{n}}^{W} \), and conversely for every disc \( f = \pi \circ \tilde{f} \in \mathcal{A}_{\mathbb{C}^{n}}^{W} \) there is a disc \( \tilde{f} \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{0\}}^{\tilde{W}} \) such that \( f = \pi \circ \tilde{f} \).

Hence, by taking the infimum over \( f \) on the left hand side of (3) corresponds to taking infimum over \( \tilde{f} \) on the right hand side. This shows that
\[
E_{\tilde{W}}H_{\omega,\varphi}(\pi(z)) = E_{\tilde{W}}H_{\tilde{\varphi}}(z) + \log \|z\|. \tag{4}
\]
By Lemma 2.8, \( \hat{\mathcal{W}} \) admits a \( \hat{\mathcal{W}} \)-disc structure \( \beta \) such that \( E_{\mathcal{W}} H_{\varphi} \leq \hat{\varphi} \), and by Corollary 2.5 we have

\[
\sup \mathcal{F}_{\hat{\varphi}} = E_{\hat{\mathcal{W}}} H_{\hat{\varphi}}
\]

in \( \mathbb{C}^{n+1} \setminus \{0\} \).

Finally, by (4), (5), and Lemma 3.2 we can show that \( E_{\mathcal{W}} H_{\omega,\varphi} = \sup \mathcal{F}_{\omega,\varphi} \):

\[
E_{\mathcal{W}} H_{\omega,\varphi}(\pi(z)) = E_{\hat{\mathcal{W}}} H_{\hat{\varphi}}(z) + \log \| z \| = \sup \mathcal{F}_{\hat{\varphi}}(z) + \log \| z \| = \sup \mathcal{F}_{\omega,\varphi}(\pi(z)).
\]

4 APPLICATIONS TO PROJECTIVE HULLS

The case when \( \varphi = 0 \) in Theorem 3.3 is interesting in its own way since it gives the global extremal function for the set \( \mathcal{W} \), [3, §5 and 6].

**Definition 4.1.** Let \( E \) be a Borel subset in \( \mathbb{P}^n \). The *global extremal function* for \( E \) is defined as

\[
\Lambda_E(x) = \sup \{ u(x); u \in \mathcal{PSH}(\mathbb{P}^n, \omega), u|_E \leq 0 \}.
\]

In this case the when \( E \) is a domain Theorem 3.3 gives the following formula

\[
\Lambda_E(x) = \inf \{-\frac{1}{2\pi} \int_D \log | \cdot | f^* \omega; f \in A^E_{\mathbb{P}^n}, f(0) = x \}.
\]

**Definition 4.2.** Let \( K \) be a compact subset of \( \mathbb{P}^n \). The *projective hull* of \( K \), denoted \( \hat{K} \) is defined as all the points \( x \in \mathbb{P}^n \) for which there exists a constant \( C_x \) such that

\[
\| P(x) \| \leq C_x^d \sup_K \| P \|, \quad \text{for all } P \in H^0(\mathbb{P}^n, \mathcal{O}(d)).
\]

Just as the polynomial hull can be characterized by the Siciak-Zahariuta extremal function, the projective hull can be characterized using the global extremal function.

**Proposition 4.3.** [4 §4] If \( K \subset \mathbb{P}^n \) is compact then

\[
\hat{K} = \{ x \in \mathbb{P}^n; \Lambda_K(x) < +\infty \}.
\]

Furthermore, or each \( x \) the value \( \exp(\Lambda_K(x)) \) is equal to the infimum of all \( C_x \) such that (6) holds.
For a constant $\Lambda \geq 0$ we let

$$\hat{K}(\Lambda) = \{ x \in \mathbb{P}^n; \Lambda_K(x) \leq \Lambda \}.$$

The projective hull can then be written as a union of the sets $\hat{K}(\Lambda)$.

The disc formula proved in Section \[3\] is only for domains in $\mathbb{P}^n$ but not compact sets. This forces us to take a sequence of open neighbourhoods of $K$, and the following proposition allows us to take a limit to obtain $\Lambda_K$.

**Proposition 4.4.** Assume $K \subset \mathbb{P}^n$ is a compact set and $(U_j)_j$ is a decreasing sequence of open subsets in $\mathbb{P}^n$ such that $\cap_j U_j = K$. Then

$$\Lambda_K = \lim_{j \to \infty} \Lambda_{U_j}.$$

*Proof.* Note first that since $U_j$ is a decreasing sequence then the sequence $\Lambda_{U_j}$ is increasing, in particular $\lim_{j \to \infty} \Lambda_{U_j}$ exists.

Since each function $\Lambda_{U_j}$ is in $\mathcal{PSh}(\mathbb{P}^n, \omega)$ (see [3, Theorem 5.2 and Proposition 5.6]) and is 0 on $U_j \supset K$, then $\Lambda_K \geq \Lambda_{U_j}$ and therefore $\Lambda_K \geq \lim_{j \to \infty} \Lambda_{U_j}$.

Let $\varepsilon > 0$. Since each function $u \in \mathcal{PSh}(\mathbb{P}^n, \omega)$, $u|_K \leq 0$ is upper semicontinuous there is a neighbourhood $U$ of $K$ such that $u|_U \leq \varepsilon$. Find $U_{j_0}$ such that $U_{j_0} \subset U$, then for $x \in X$,

$$u(x) - \varepsilon \leq \Lambda_{U_{j_0}}(x) \leq \lim_{j \to \infty} \Lambda_{U_j},$$

which implies, by taking supremum over $u$ and letting $\varepsilon \to 0$, that $\Lambda_K \leq \lim_{j \to \infty} \Lambda_{U_j}$. \[\square\]

By combining the disc formula for the global extremal function with Proposition 4.3 we can get a new characterization of the projective hull for connected sets. The characterization is quantitative, that is it uses $\hat{K}(\Lambda)$, just as the characterization by existence of currents [4, Theorem 11.1].

**Theorem 4.5.** Let $\Lambda > 0$. For a point $x$ in a connected compact subset $K \subset \mathbb{P}^n$ the following is equivalent

(A) $x \in \hat{K}(\Lambda)$

(B) For every $\varepsilon > 0$ and every neighbourhood $U$ of $K$ there exists a disc $f \in \mathcal{A}^U_{\mathbb{P}^n}$ such that $f(0) = x$ and

$$-\frac{1}{2\pi} \int_D \log |f^* \omega| < \Lambda + \varepsilon.$$
Proof. First assume \( x \in \hat{K}(\Lambda) \). By Proposition 4.4 there is a domain \( V \) such that \( K \subset V \subset U \) and \( \Lambda_V(x) < \Lambda_K(x) + \varepsilon/2 \). By Theorem 3.3 there is a disc \( f \in \mathcal{A}_V^p \) such that \( f(0) = x \) and
\[
-\frac{1}{2\pi} \int_D \log | \cdot | f^* \omega < \Lambda_V(x) + \frac{\varepsilon}{2}
\]
Then
\[
-\frac{1}{2\pi} \int_D \log | \cdot | f^* \omega \leq \Lambda_V(x) + \frac{\varepsilon}{2} \leq \Lambda_K(x) + \varepsilon \leq \Lambda + \varepsilon.
\]

Conversely, assume (B) holds. Now let \( U_j \) decreasing sequence of domains such that \( \cap_j U_j = K \) and \( \lim_{j \to \infty} \Lambda_{U_j} = \Lambda_K \). The sets \( U_j \) can be chosen connected because \( K \) is always contained in one connected component of an open neighbourhood of \( K \) (otherwise \( K \) would not be connected).

For each \( j \) there is a disc \( f_j \in \mathcal{A}_{U_j}^p \) such that \( f_j(0) = x \) and
\[
\Lambda_{U_j}(x) \leq -\frac{1}{2\pi} \int_D \log | \cdot | f_j^* \omega < \Lambda + \frac{1}{j}
\]
which implies \( \Lambda_K(x) = \lim_{j \to \infty} \Lambda_{U_j}(x) \leq \Lambda \), that is \( x \in \hat{K}(\Lambda) \).

4.1 Characterizations of Drnovšek and Forstnerič

In [1] Drnovšek and Forstnerič gave several characterizations of the projective and polynomial hulls using the existence of analytic disc, both for connected sets and not connected sets. They used analytic disc in \( \mathbb{P}^n \) with the bounded lifting property obtained from Poletsky’s disc formula and disc in \( \mathbb{C}^{n+1} \setminus \{0\} \) derived from the disc formula of the Siciak-Zahariuta extremal function [8].

The result from [1] regarding connected sets which is best compatible with Theorem 4.5 is the following.

**Theorem 4.6.** ([1], Theorem 5.1) Let \( K \) be a compact connected set in \( \mathbb{P}^n \). A point \( p \in \mathbb{C}^{n+1} \setminus \{0\} \) belongs to the polynomial hull of the set \( S_K \subset \mathbb{C}^{n+1} \), and hence \( x = \pi(p) \in \mathbb{P}^n \) belongs to the projective hull of \( K \), if and only if there exists a sequence of analytic discs \( F_j : \mathbb{B} \to \mathbb{C}^{n+1} \setminus \{0\} \) such that \( F_j(0) = p \) and
\[
\lim_{j \to \infty} \max_{t \in [0, 2\pi]} \text{dist}(F_j(e^{it}, S_K)) = 0.
\]

The set \( S_K \) is the lifting of \( K \) restricted to the unit sphere in \( \mathbb{C}^{n+1} \),
\[
S_K = \pi^{-1}(K) \cap \{z \in \mathbb{C}^{n+1}; \|z\| = 1\}.
\]

Restating the theorem above without the limit and focusing on the point \( x \in \mathbb{P}^n \) we can state it as follows.
Theorem 4.7. For a point \( x \) in a connected compact subset \( K \subset \mathbb{P}^n \) the following is equivalent.

(A') \( x \in \hat{K} \).

(B') There exists \( p \in \pi^{-1}(x) \) such that for every \( \varepsilon > 0 \) and every neighbourhood \( \tilde{U} \) of \( S_K \) there exists a disc \( \tilde{f} \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{0\}} \) such that \( \tilde{f}(0) = p \).

Note that \( x \) is in \( \hat{K} \) if and only if there exists a point \( p \in \pi^{-1}(x) \) which is in the polynomial hull of \( S_K \) in \( \mathbb{C}^{n+1} \) by [4, Corollary 5.3]. Furthermore, if we define

\[
\rho_K(x) = \sup\{\|p\|; p \in \pi^{-1}(x) \text{ and } p \text{ is in the polynomial hull of } S_K\},
\]

then we have following connection between the global extremal function \( \Lambda_K \), the best constant function \( C_K \) and the function \( \rho_K \), see [4, Proposition 5.2],

\[
\Lambda_K = \log C_K = - \log \rho_K.
\] (7)

The definition of \( \rho_K \) implies that condition (B') holds for \( p \) such that \( \|p\| \leq \rho_K(x) \).

Since both condition (B) and (B') characterize the projective hull they are clearly equivalent. But for completeness we show how one can be derived from the other.

(B') implies (B): Let \( x \in \hat{K}(\Lambda) \), where \( \Lambda > \Lambda_K(x) \). Fix \( \varepsilon > 0 \) and let \( U \) be an open neighbourhood of \( K \) in \( \mathbb{P}^n \). From the definition of \( \rho_K \) and (7) there is \( p \in \pi^{-1}(x) \) such that \(-\log \|p\| < \Lambda \). Then by (B') there is a disc \( \tilde{f} \) such that \( \tilde{f}(0) = p \) and

\[
\max_{t \in T} \log \|\tilde{f}(t)\| \leq \varepsilon.
\]

That is, we let \( \tilde{U} = \{z \in \mathbb{C}^{n+1}; \|z\| < \exp(\varepsilon)\} \). Now define \( f = \pi \circ \tilde{f} \). Using the Riesz representation theorem for the function \( \log \|\tilde{f}\| \) at the point 0 we get

\[
\log \|p\| = \frac{1}{2\pi} \int_{\mathbb{D}} \log |\cdot| \Delta \log \|\tilde{f}\| + \int_T \log \|\tilde{f}\| d\sigma
\]

Using the inequality \(-\log \|p\| < \Lambda \) and the fact that \( f^*\omega = \Delta \log \|\tilde{f}\| \) we derive the following.

\[
-\frac{1}{2\pi} \int_{\mathbb{D}} \log |\cdot| f^*\omega = -\frac{1}{2\pi} \int_{\mathbb{D}} \log |\cdot| \Delta \log \|\tilde{f}\|
\]

\[
= -\log \|p\| + \int_T \log \|\tilde{f}\| d\sigma
\]

\[
\leq \Lambda + \max_{t \in T} \log \|\tilde{f}(t)\|
\]

\[
\leq \Lambda + \varepsilon.
\]
(B) implies (B'): Given a neighbourhood $\tilde{U}$ of $S_K$. Let $U = \pi(\tilde{U})$ and $f \in A^{D_n}_{\mathbb{C}}$ be a disc as in (B). Then there is a lifting $\tilde{f}_0 \in A^{C_{n+1} \setminus \{0\}}$ such that $f = \pi \circ \tilde{f}_0$, see e.g. [10, §3]. Note that $\tilde{f}_0(\mathbb{T}) \subset \pi^{-1}(U)$ since $f(\mathbb{T}) \subset U$.

Now let $u$ be the solution of the Dirichlet problem on the unit disc with boundary values $\log \|\tilde{f}_0\|$, and let $v$ be the corresponding harmonic conjugate. Note that $v$ is harmonic on $\mathbb{D}$ but not necessarily continuous up to the boundary. However, since

$$\frac{\tilde{f}_0(t)}{\exp(u(t))} = \frac{\tilde{f}_0(t)}{f(t)} = 1, \quad \text{for all } t \in \mathbb{T},$$

there is $r < 1$ such that $\tilde{f}_0(t)/ \exp(u(rt)) \in \tilde{U}$ for all $t \in \mathbb{T}$. If we then define the closed analytic disc $\tilde{f}$ by

$$\tilde{f}(t) = \frac{\tilde{f}_0(t)}{\exp(u(rt) + iv(rt))},$$

we see that $\tilde{f}$ maps the unit circle into $\tilde{U}$ and $p = \tilde{f}(0) \in \pi^{-1}(x)$. Since $r < 1$ the function $\exp(u(r \cdot) + iv(r \cdot))$ is continuous on $\mathbb{D}$ and therefore $\tilde{f} \in A^{C_{n+1} \setminus \{0\}}$.

Furthermore, if $x \in \hat{K}(\Lambda)$, $\Lambda > 0$ and $\varepsilon > 0$, then we can select $f$ such that

$$-\frac{1}{2\pi} \int_{\mathbb{D}} \log | \cdot | f^* \omega < \Lambda + \varepsilon,$$

and we can select $r$ close enough to $1$ so that

$$-\varepsilon < \int_{\mathbb{T}} \log \| \tilde{f}(t) \| \ d\sigma < \varepsilon.$$

This implies, by similar calculations as before, that

$$-\log \|p\| = -\frac{1}{2\pi} \int_{\mathbb{D}} \log | \cdot | \Delta \log \|\tilde{f}\| - \int_{\mathbb{T}} \log \| \tilde{f} \| \ d\sigma$$

$$\leq -\frac{1}{2\pi} \int_{\mathbb{D}} \log | \cdot | f^* \omega + \varepsilon$$

$$\leq \Lambda + 2\varepsilon$$

$$\leq \Lambda_K(x) + 2\varepsilon = -\log \rho_K(x) + 2\varepsilon. \quad (8)$$

By definition we have $\|p\| \leq \rho_K(x)$. This, along with inequality (8) above, shows that

$$e^{2\varepsilon} \rho_K(x) \leq \|p\| \leq \rho_K(x). \quad (9)$$

In other words, we can choose $p$ such that $\|p\|$ is arbitrary close to $\rho_K(x).$
Remark: There are two details that should be pointed out here:

- The results in [1] and Theorem 4.7 are not quantitative, in other words they do not use \( \Lambda \) and \( \hat{K}(\Lambda) \) as Theorem 4.5 does.
- Theorem 4.6 implies that for every \( p \) in the polynomial hull of \( S_K \) there exists a disc with the properties in (B’) and center \( p \). On the other hand, Theorem 4.5 does not allow us to select the center \( p \) specifically. It only allows us to find discs such that the modulus of its center is as close to \( \rho_K(x) \) as we wish, as seen from (9).

The author wishes to thank Franc Forstnerič and Josip Globevnik for referring him to the results in [1] and asking about the connections between the characterizations of the projective hull there and the one in Theorem 4.5.

5 SICIAK-ZAHARIUTA EXTREMAL FUNCTION AND OTHER GLOBAL EXTREMAL FUNCTIONS

There are other quasihomolomorphic functions of interest in \( \mathbb{P}^n \). The most known are those when the current is the current of integration \([H_\infty]\) for the hyperplane at infinity \( H_\infty \). Then we are looking at \( \mathbb{P}^n \) as the union of \( \mathbb{C}^n \) and \( H_\infty \). The class of quasihomolomorphic functions \( \mathcal{PSH}(\mathbb{P}^n, [H_\infty]) \) then becomes the Lelong-class

\[
\mathcal{L} = \{ u \in \mathcal{PSH}(\mathbb{C}^n); u(x) \leq \log \|x\| + C_u \}.
\]

The potential for the pullback of this current to \( \mathbb{C}^{n+1} \setminus \{0\} \), denoted \( \pi^*[H_\infty] \), then has a global potential and can be written as \( \pi^*[H_\infty] = dd^c \log |z_0| \), assuming \( H_\infty = \pi(\{z \in \mathbb{C}^{n+1} \setminus \{0\}; z_0 = 0\}) \).

The Siciak-Zahariuta extremal function for a set \( W \) is defined as

\[
\sup\{u(x); u \in \mathcal{L}, u|_W \leq 0\},
\]

and the weighted version as

\[
\sup\{u(x); u \in \mathcal{L}, u|_W \leq \varphi\},
\]

where \( \varphi : W \to \mathbb{R} \) is a function.

If \( W \subset \mathbb{C}^n \) is a domain and \( \varphi : W \to \mathbb{R} \cup \{-\infty\} \) is an upper semicontinuous function then there is a disc formula for the Siciak-Zahariuta function
\[ \sup \mathcal{L}_\varphi(x) = \inf \left\{ -\sum_{a \in f^{-1}(H_\infty)} \log |a| + \int_T \varphi \circ f \, d\sigma ; \ f \in A^W_{\mathbb{P}^n}, f(0) = x \right\} . \]

There is also a formula when \( W \) is not connected [9], but it is a little bit different and somewhat unwieldier.

The formula above can be proven easily by the same methods as the formula in Theorem 3.3 by replacing \( \log \|z\| \) with \( \log |z_0| \).

We only have to note two things. First, a function \( u \in \mathcal{L} \) extends to a plurisubharmonic and logarithmically homogeneous function \( \tilde{u} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R} \cup \{-\infty\} \), by

\[ \tilde{u}(z_0, z_1, \ldots, z_n) = u \left( \frac{z_1, \ldots, z_n}{z_0} \right) + \log |z_0| \]  

Secondly, if \( f = [f_0 : f_1 : \cdots : f_n] \in A_{\mathbb{P}^n} \), then

\[ \int_D \log | \cdot | f^*[H_\infty] = \sum_{a \in f^{-1}(H_\infty)} m_a \log |a| , \]

where \( m_a \) is the multiplicity of the zero of \( f_0 \) at \( a \). However, by Proposition 1 in [8] the multiplicity \( m_a \) can be omitted, because for a disc \( f \) with zero of order \( m_a \) at \( a \) there is disc with \( m_a \) different simple zeros sufficiently close to \( a \). This implies that the multiplicity \( m_a \) can be omitted in the disc formula above.

**Remark:** The methods described here actually apply to every current \( \tilde{\omega} \) on \( \mathbb{P}^n \), such that the pullback to \( \mathbb{C}^{n+1} \setminus \{0\} \), \( \pi^* \tilde{\omega} \), has a logarithmically homogeneous potential \( \psi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R} \cup \{-\infty\} \) with \( dd^c \psi = \pi^* \tilde{\omega} \).

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