DYNAMICS OF SEMIGROUPS OF ENTIRE MAPS IN $\mathbb{C}^k$

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Abstract. The goal of this paper is to study some basic properties of the Fatou and Julia sets for a family of holomorphic endomorphisms of $\mathbb{C}^k$, $k \geq 2$. We are particularly interested in studying these sets for semigroups generated by various classes of holomorphic endomorphisms of $\mathbb{C}^k$, $k \geq 2$. We prove that if the Julia set of a semigroup $G$ which is generated by endomorphisms of maximal generic rank $k$ in $\mathbb{C}^k$ contains an isolated point, then $G$ must contain an element that is conjugate to an upper triangular automorphism of $\mathbb{C}^k$. This generalizes a theorem of Fornaess–Sibony. Secondly, we define recurrent domains for semigroups and provide a description of such domains under some conditions.

1. Introduction

The purpose of this note is to study the Fatou–Julia dichotomy, not for the iterates of a single holomorphic endomorphism of $\mathbb{C}^k$, $k \geq 2$, but for a family $\mathcal{F}$ of such maps. The Fatou set of $\mathcal{F}$ will be by definition the largest open set where the family is normal, i.e., given any sequence in $\mathcal{F}$ there exists a subsequence which is uniformly convergent or divergent on all compact subsets of the Fatou set, while the Julia set of $\mathcal{F}$ will be its complement.

We are particularly interested in studying the dynamics of families that are semigroups generated by various classes of holomorphic endomorphisms of $\mathbb{C}^k$, $k \geq 2$. For a collection $\{\psi_\alpha\}$ of such maps let

$$G = \langle \psi_\alpha \rangle$$

denote the semigroup generated by them. The index set to which $\alpha$ belongs is allowed to be uncountably infinite in general. The Fatou set and Julia set of this semigroup $G$ will be henceforth denoted by $F(G)$ and $J(G)$ respectively. The $\psi_\alpha$'s that will be considered in the sequel will belong to one of the following classes:

- $\mathcal{E}_k$: The set of holomorphic endomorphisms of $\mathbb{C}^k$ which have maximal generic rank $k$.
- $\mathcal{I}_k$: The set of injective holomorphic endomorphisms of $\mathbb{C}^k$.
- $\mathcal{V}_k$: The set of volume preserving biholomorphisms of $\mathbb{C}^k$.
- $\mathcal{P}_k$: The set of proper holomorphic endomorphisms of $\mathbb{C}^k$.

The main motivation for studying the dynamics of semigroups in higher dimensions comes from the results of Hinkkanen–Martin and Fornaess–Sibony. While [3] considers the dynamics of semigroups generated by rational functions on the Riemann sphere, [2] puts forth several basic results about the dynamics of the iterates of a single holomorphic endomorphism of $\mathbb{C}^k$, $k \geq 2$. Under such circumstances, it seemed natural to us to study the dynamics of semigroups in higher dimensions.

Section 2 deals with basic properties of $F(G)$ and $J(G)$ when $G$ is generated by elements that belong to $\mathcal{E}_k$ and $\mathcal{P}_k$. The main theorem in Section 3 states that if $J(G)$ contains an isolated point, then $G$ must contain an element that is conjugate to an upper triangular automorphism of $\mathbb{C}^k$. Finally we define recurrent domains for semigroups in Section 4 and provide a description of

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such domains under some conditions. All these results generalize the corresponding statements of Fornaess–Sibony \[ 2 \] for the iterates of a single holomorphic endomorphism of \( \mathbb{C}^k \), \( k \geq 2 \).

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2. **Properties of the Fatou set and Julia set for a semigroup \( G \)**

In this section we will prove some basic properties of the Fatou set and the Julia set for semigroups.

**Proposition 2.1.** Let \( G \) be a semigroup generated by elements of \( \mathcal{E}_k \) where \( k \geq 2 \) and for any \( \phi \in G \) define

\[ \Sigma_\phi = \{ z \in \mathbb{C}^k : \det \phi(z) = 0 \}. \]

Then for every \( \phi \in G \)

(i) \( \phi(F(G) \setminus \Sigma_\phi) \subset F(G) \).

(ii) \( J(G) \cap \phi(\mathbb{C}^k) \subset \phi(J(G)) \), if \( G \) is generated by elements of \( \mathcal{P}_k \) or \( \mathcal{I}_k \).

**Proof.** Note that \( \phi \in G \) is an open map at any point \( z \in F(G) \setminus \Sigma_\phi \). Since for any sequence \( \psi_n \in G \), the sequence \( \psi_n \circ \phi \) has a convergent subsequence around a neighbourhood of \( z \) (say \( V_z \)), \( \psi_n \) also has a convergent subsequence on the open set \( \phi(V_z) \) containing \( \phi(z) \).

Now if \( G \) is generated by elements of \( \mathcal{P}_k \) or \( \mathcal{I}_k \) then \( \phi \) is an open map at every point in \( \mathbb{C}^k \). Then the Fatou set is forward invariant and hence the Julia set is backward invariant in the range of \( \phi \).

A family of endomorphisms \( \mathcal{F} \) in \( \mathbb{C}^k \) is said to be locally uniformly bounded on an open set \( \Omega \subset \mathbb{C}^k \) if for every point there exists a small enough neighbourhood of the point (say \( V \subset \Omega \)) such that \( \mathcal{F} \) restricted to \( V \) is bounded i.e.,

\[ \| f \|_V = \sup_{V} |f(z)| < M \]

for some \( M > 0 \) and for every \( f \in \mathcal{F} \).

**Proposition 2.2.** Let \( G = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle \), where each \( \phi_j \in \mathcal{E}_k \) and let \( \Omega_G \) be a Fatou component of \( G \) such that \( G \) is locally uniformly bounded on \( \Omega_G \). Then for every \( \phi \in G \) the image of \( \Omega_G \) under \( \phi \) i.e., \( \phi(\Omega_G) \) is contained in Fatou set of \( G \).

**Proof.** Let \( K \subset \subset \Omega_G \), i.e., \( K \) is a relatively compact subset of \( \Omega_G \), then

Claim:- \( \Omega_K \) is a Runge domain i.e., \( \hat{K} \subset \Omega_K \) where

\[ \hat{K} := \{ z \in \mathbb{C}^k : |P(z)| \leq \sup_K |P| \text{ for every polynomial } P \}. \]

Let \( K_\delta = \{ z \in \mathbb{C}^k : \text{dist}(z, K) \leq \delta \} \). Choose \( \delta > 0 \) such that \( K_\delta \subset \subset \Omega_G \). Now note that \( \hat{K}_\delta \subset \subset \mathbb{C}^k \), \( \hat{K}_\delta \supset \hat{K} \) and \( G \) is uniformly bounded on \( K_\delta \). Pick \( \phi \in G \) then there exists a polynomial endomorphism \( P_\phi \) in \( \mathbb{C}^k \) such that

\[ |\phi(z) - P_\phi(z)| \leq \epsilon \text{ for every } z \in \hat{K}_\delta \]

i.e.,

\[ |P_\phi(z)| - \epsilon \leq |\phi(z)| \leq |P_\phi(z)| + \epsilon. \]

Hence

\[ |\phi(z)| \leq |P_\phi(z)| + \epsilon \leq \sup_{K_\delta} |P_\phi(z)| + \epsilon \leq \sup_{K_\delta} |\phi(z)| + 2\epsilon \leq M + 2\epsilon \]

for every \( z \in \hat{K}_\delta \) and some constant \( M > 0 \). So \( G \) is uniformly bounded on \( \hat{K}_\delta \) and \( \hat{K} \subset \Omega_G \).
Let
\[ \Sigma_i = \{ z \in \mathbb{C}^k : \det \phi_i(z) = 0 \} \]
for every \( 1 \leq i \leq n \) and
\[ \Sigma = \bigcup_{i=1}^{n} \Sigma_i. \]

Thus \( \phi_i \) for every \( i \), where \( 1 \leq i \leq n \) is an open map in \( \Omega_G \setminus \Sigma \). Hence \( \phi_i(\Omega_G \setminus \Sigma) \) is contained inside a Fatou component say \( \Omega_i \) and \( G \) is locally uniformly bounded on each of \( \Omega_i \) for every \( 1 \leq i \leq n \) i.e., each \( \Omega_i \) is a Runge domain.

Now, pick \( p \in \Omega_G \cap \Sigma \). Since \( \Sigma \) is a set with empty interior, there exists a sufficiently small disc centered at \( p \) say \( \Delta_p \) such that \( \Sigma \setminus \{p\} \subset \Omega_G \setminus \Sigma \). Then \( \phi_i(\Sigma \setminus \{p\}) \subset \Omega_i \) for every \( 1 \leq i \leq n \) and since, each \( \Omega_i \) is Runge \( \phi_i(p) \in \Omega_i \) i.e., \( \phi_i(\Omega_G) \) is contained in the Fatou set for every \( 1 \leq i \leq n \).

Now for any \( \phi \in G \) there exists a \( m > 0 \) such that
\[ \phi = \phi_{n_1} \circ \phi_{n_2} \circ \ldots \circ \phi_{n_m} \]
where \( 1 \leq n_j \leq n \) for every \( 1 \leq j \leq m \). Thus applying the above argument repeatedly for each \( \phi_{n_j}(\Omega_j) \) where \( G \) is locally uniformly bounded on \( \Omega_j \) it follows that \( \phi(\Omega_G) \) is contained in the Fatou set of \( G \).

**Proposition 2.3.** If \( G = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle \) where each \( \phi_i \in \mathcal{E}_k \) for every \( 1 \leq i \leq n \) and let \( \Omega_G \) be a Fatou component of \( G \). Then for any \( \phi \in G \) there exists a Fatou component of \( G \), say \( \Omega_\phi \) such that \( \phi(\Omega_G) \subset \Omega_\phi \) and
\[ \partial \Omega_G \subset \bigcup_{i=1}^{n} \phi_i^{-1}(\partial \Omega_\phi). \]

**Proof.** Let \( \phi \in G \) and let \( \Sigma_\phi \) denote the set of points in \( \mathbb{C}^k \) where the Jacobian of \( \phi \) vanishes. Since \( \Omega_G \setminus \Sigma_\phi \) is connected it follows that \( \phi(\Omega_G \setminus \Sigma_\phi) \subset \Omega_\phi \) where \( \Omega_\phi \) is a Fatou component of \( G \) and by continuity \( \phi(\Omega_G) \subset \Omega_\phi \).

Pick \( p \in \partial \Omega_G \) such that \( p \notin \partial \Omega_{\phi_i} \) for every \( 1 \leq i \leq n \). Since \( \phi_i(\Omega_G) \subset \Omega_{\phi_i} \), \( \phi_i(p) \in \Omega_{\phi_i} \) for every \( 1 \leq i \leq n \). So there exists \( V_{\phi_i} \) an open neighbourhood of \( \phi_i(p) \) in \( \Omega_{\phi_i} \) for every \( i \). Let \( V_p \) be a neighbourhood of \( p \) such that
\[ \bar{V}_p \subset \bigcap_{i=1}^{n} \phi_i^{-1}(V_{\phi_i}). \]

Let \( \{\psi_n\} \) be a sequence in \( G \) and without loss of generality it can be assumed that there exists a subsequence such that \( \psi_n = f_n \circ \phi_1 \). Now \( \phi_1(V_p) \) is a compact subset in \( \Omega_1 \) and \( f_n \) has a subsequence which either converges uniformly on \( \phi_1(V_p) \) or diverges to infinity. Thus \( V_p \) is contained in the Fatou set of \( G \) which is a contradiction!

The next observation is an extension of the fact that if \( \phi \in \mathcal{P}_k \), then \( F(\phi) = F(\phi^n) \) for every \( n > 0 \) for the case of semigroups.

**Definition 2.4.** Let \( G \) be a semigroup generated by endomorphisms of \( \mathbb{C}^k \). A sub semigroup \( H \) of \( G \) is said to have finite index if there is a finite collection of elements say \( \psi_1, \psi_2, \ldots, \psi_{m-1} \in G \) such that
\[ G = \left( \bigcup_{i=1}^{m-1} \psi_i \circ H \right) \cup H. \]

The index of \( H \) in \( G \) is the smallest possible number \( m \).
Definition 2.5. A sub semigroup $H$ of a semigroup $G$ of endomorphisms of $\mathbb{C}^k$ is of co–finite index if there is a finite collection of elements say $\psi_1, \psi_2, \ldots, \psi_{m-1} \in G$ such that either 

$$\psi \circ \psi_j \in H \text{ or } \psi \in H$$

for every $\psi \in G$ and for some $1 \leq j \leq m - 1$. The index of $H$ in $G$ is the smallest possible number $m$.

Proposition 2.6. Let $G$ be a semigroup generated by proper holomorphic endomorphisms of $\mathbb{C}^k$ and $H$ be a sub semigroup of $G$ which has a finite (or co–finite) index in $G$. Then $F(G) = F(H)$ and $J(G) = J(H)$.

Proof. From the definition itself it follows that $F(G) \subset F(H)$. To prove the other inclusion, pick any sequence $\{\phi_n\} \subset G$. Since $H$ has a finite index in $G$, there exists $\psi_i, 1 \leq i \leq m - 1$ such that 

$$G = \left( \bigcup_{i=1}^{m-1} \psi_i \circ H \right) \cup H.$$ 

So without loss of generality one can assume that there exists a subsequence say $\phi_{n_k}$ with the property 

$$\phi_{n_k} = \psi_1 \circ h_{n_k}$$

where $\{h_{n_k}\}$ is a sequence in $H$. Now on $F(H)$, the sequence $\{h_{n_k}\}$ has a convergent subsequence. Hence, so do $\{\phi_{n_k}\}$ and $\{\phi_n\}$ as $\psi_1$ is a proper map in $\mathbb{C}^k$. 

Let $G$ be a semigroup 

$$G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$$

where $\phi_i \in \mathcal{P}_k$, for every $1 \leq i \leq m$ and each of these $\phi_i$’s commute with each other, i.e., $\phi_i \circ \phi_j = \phi_j \circ \phi_i$ for $i \neq j$. Let $H$ be a sub semigroup of $G$ defined as 

$$H = \langle \phi_{l_1}, \phi_{l_2}, \ldots, \phi_{l_m} \rangle$$

where $l_i > 0$ for every $1 \leq i \leq m$. Then $H$ has a finite index in $G$ and hence by Proposition 2.6 $F(G) = F(H)$.

Corollary 2.7. Let $\phi_i$ be elements in $\mathcal{P}_k$ for $1 \leq i \leq m$, $l = (l_1, l_2, \ldots, l_m)$ a $m$–tuple of positive integers and $G_l = \langle \phi_{l_1}, \phi_{l_2}, \ldots, \phi_{l_m} \rangle$. Then $F(G_l)$ and $J(G_l)$ are independent of the $m$–tuple $l$, if $\phi_i \circ \phi_j = \phi_j \circ \phi_i$ for every $1 \leq i, j \leq m$, i.e., given two $m$–tuples $p$ and $q$, $F(G_p) = F(G_q)$.

Proof. Since $G_l$ has a finite index in $G$ for every $m$–tuple $l = (l_1, l_2, \ldots, l_m)$, it follows that $F(G_l) = F(G)$ and $J(G_l) = J(G)$. 

Example 2.8. Let $G = \langle f, g \rangle$ where $f(z_1, z_2) = (z_1^2, z_2^2)$ and $g(z_1, z_2) = (z_1^2/a, z_2^2)$ where $a \in \mathbb{C}$ such that $|a| > 1$. Then it is easy to check that 

$$J(f) = \{|z_1| = 1\} \times \{|z_2| \leq 1\} \cup \{|z_1| \leq 1\} \times \{|z_2| = 1\}$$

and 

$$J(g) = \{|z_1| = |a|\} \times \{|z_2| \leq 1\} \cup \{|z_1| \leq |a|\} \times \{|z_2| = 1\}.$$ 

Now consider the bidisc $\{|z_1| < 1, |z_2| < 1\}$. Clearly this domain is forward invariant under both $f$ and $g$. This shows that $\{|z_1| < 1, |z_2| < 1\} \subset F(G)$. Similarly observe that 

$$\{|z_2| > 1\} \cup \{|z_1| > |a|\} \subset F(G).$$ 

We claim that 

$$\{1 \leq |z_1| \leq |a|\} \times \{|z_2| \leq 1\} \subset J(G).$$ 

Note that $\{|z_1| = |a|, |z_2| \leq 1\}$ is contained inside $J(G)$ and since $J(G)$ is backward invariant it follows that 

$$\{|z_1| = |a|^{1/2}, |z_2| \leq 1\} \subset f^{-1}(\{|z_1| = |a|, |z_2| \leq 1\}) \subset J(G).$$
So inductively we get that
\[ \{ |z_1| = |a|^i, |z_2| \leq 1 \} \subset J(G) \]
for any \( t = k2^{-n} \) where \( 1 \leq k \leq 2^n \) and \( n \geq 1 \). As \( \{ k2^{-n} : 1 \leq k \leq 2^n, n \geq 1 \} \) is dense in \([0,1]\), it follows that \( \{ 1 \leq |z_1| \leq |a| \} \times \{ |z_2| \leq 1 \} \subset J(G) \). Thus the Julia set of the semigroup \( G \) is not forward invariant and clearly from the above observations one can prove that
\[ J(G) = \{ |z_1| \leq 1 \} \times \{ |z_2| = 1 \} \cup \{ 1 \leq |z_1| \leq |a| \} \times \{ |z_2| \leq 1 \} . \]

Example 2.9. Let \( T_0(z) = 1, \ T_1(z) = z \) and \( T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z) \) for \( n \geq 1 \) and \( G = \langle f_0, f_1, f_2, \ldots \rangle \), with \( f_i(z_1, z_2) = (T_i(z_1), z_2^2) \) for \( i \geq 0 \). Consider
\[ G_1 = \langle T_0(z_1), T_1(z_1), T_2(z_1), \ldots \rangle, \ G_2 = \langle z_2^2 \rangle. \]

Since any sequence in \( G_1 \) is uniformly unbounded on the complement of \([-1,1]\) it follows that
\[ J(G) = [-1,1] \times \{ |z_2| \leq 1 \}. \]

Also as \( J(G_1) \subset \mathbb{C} \) is completely invariant so is \( J(G) \).

3. ISOLATED POINTS IN THE JULIA SET OF A SEMIGROUP \( G \).

Proposition 3.1. Let \( G = \langle \phi_1, \phi_2, \ldots \rangle \) where each \( \phi_i \in E_k \). If the Julia set \( J(G) \) contains an isolated point (say \( a \)) then there exists a neighbourhood \( \Omega_a \) of \( a \) such that \( \Omega_a \setminus \{ a \} \subset F(G) \) and \( \psi \in G \) which satisfy the following properties:

(i) \( \Omega_a \subset \subset \psi(\Omega_a) \).
(ii) If \( G \) is a semigroup generated by proper maps, then \( \psi^{-1}(a) = a \).

Proof. Assume \( a = 0 \) is an isolated point in the Julia set \( J(G) \). Then there exists a sufficiently small ball \( B(0, \epsilon) \) around \( 0 \) such that \( B(0, \epsilon) \setminus \{ 0 \} \) is contained \( F(G) \). Let
\[ A := \{ z : \epsilon/2 \leq |z| \leq \epsilon \}. \]

Then \( A \subset F(G) \).

Claim: There exists a sequence \( \phi_n \in G \) such that \( \phi_n \) diverges to infinity on \( A \).

Suppose not. Then for every sequence \( \{ \phi_n \} \in G \), there exists a subsequence \( \{ \phi_{n_k} \} \) which converges to a finite limit in \( A \). By the maximum modulus principle
\[ ||\phi_{n_k}||_{B(0,\epsilon)} < M. \]

By the Arzelá–Ascoli Theorem it follows that \( \phi_{n_k} \) is equicontinuous on \( B(0, \epsilon) \), which contradicts that \( 0 \in J(G) \).

By the same reasoning as above there exists a sequence \( \{ \phi_n \} \in G \) such that it diverges uniformly to infinity on \( A \) but does not diverge uniformly to infinity on \( B(0, \epsilon) \), since it would again imply that \( B(0, \epsilon) \) is contained in the Fatou set of \( G \). Thus there exists a sequence of points \( x_n \) in \( B(0, \epsilon) \) such that \( \phi_n(x_n) \) is bounded i.e.,
\[ |\phi_n(x_n)| < M \]

for some large \( M > 0 \). So we can choose a subsequence of this \( \{ \phi_n \} \) and relabel it as \( \{ \tilde{\phi}_n \} \) again such that it satisfies the following condition:
\[ \tilde{\phi}_n(x_n) \to q \text{ and } x_n \to p \]

where \( p \in \overline{B(0,\epsilon)} \).

Claim: \( p = 0 \).

Suppose not. Then \( \tilde{\phi}_n(p) \) is bounded. Let \( \tilde{A} = \{ z : \min(|p|, \epsilon/2) \leq |z| \leq \epsilon \} \). Then \( \tilde{A} \supset A \).

Now \( \tilde{\phi}_{n_k}(p) \) converges on \( \tilde{A} \), then \( \phi_{n_k} \) on \( \tilde{A} \) converges to a finite limit, and hence on \( A \) by the maximum modulus principle. This is a contradiction!
Since $\phi_n|_{\partial B(0,\epsilon)} \to \infty$ for large $n$

Thus for a sufficiently large $R > 0$ and $n$

$$B(0,|q| + R) \cap \phi_n(B(0,\epsilon)) \neq \emptyset.$$ 

Now, if $B(0,\epsilon) \not\subseteq \phi_n(B(0,\epsilon))$, then $B(0,|q| + R) \not\subseteq \phi_n(B(0,\epsilon))$ since $B(0,\epsilon) \subset B(0,|q| + R)$ for large $R > 0$. Then there exists $y_n \in \partial B(0,\epsilon)$ such that $|\phi_n(y_n)| < |q| + R$, which is not possible. Hence $B(0,\epsilon) \subset \phi_n(B(0,\epsilon))$ for sufficiently large $n$. Relabel this $\phi_n$ as $\psi$ and consider the neighbourhood $\Omega_0$ as $B(0,\epsilon)$.

Since $0 \in B(0,\epsilon) \subset \psi(B(0,\epsilon))$, there exists $\alpha \in B(0,\epsilon)$ such that $\psi(\alpha) = 0$. From Proposition 2.1 it follows that $\alpha = 0$.

**Theorem 3.2.** Let $G = \{\phi_1, \phi_2, \ldots\}$ where each $\phi_i \in \mathcal{I}_k$. If the Julia set $J(G)$ contains an isolated point, say $a$ then there exists an element $\psi \in G$ such that $\psi$ is conjugate to an upper triangular automorphism.

**Proof.** Without loss of generality we can assume that $a = 0$. Now by Proposition 3.1 it follows that there exists a sufficiently small ball $B(0,\epsilon)$ around 0 and an element $\psi \in G$ such that $B(0,\epsilon) \subset \psi(B(0,\epsilon))$. Since $\psi$ is injective map in $\mathbb{C}^k$, $\psi(B(0,\epsilon))$ is biholomorphic to $B(0,\epsilon)$ and hence we can consider the inverse i.e.,

$$\psi^{-1} : \psi(B(0,\epsilon)) \to B(0,\epsilon).$$

Note that $\psi(B(0,\epsilon))$ is bounded and $B(0,\epsilon)$ is compactly contained in $\psi(B(0,\epsilon))$. Therefore there exists an $\alpha > 1$ such that the map defined by

$$\psi_\alpha = \alpha \psi^{-1}(z)$$

is a self map of the bounded domain $\psi(B(0,\epsilon))$ with a fixed point at 0. Then by the Carathéodory–Cartan–Kaup–Wu Theorem (See Theorem 11.3.1 in [3]) it follows that all the eigenvalues of $\psi_\alpha$ are contained in the unit disc. Hence 0 is a repelling fixed point for $\psi$ and also is an isolated point in the Julia set of $\psi$.

Since $B(0,\epsilon) \setminus \{0\} \in J(G)$, $B(0,\epsilon) \setminus \{0\}$ is also contained in the Fatou set of $\psi$ and using the same argument as in the Proposition 3.1 there exists a subsequence (say $n_k$) such that

$$\|\psi^{n_k}\|_{\partial B(0,\epsilon)} \to \infty$$

uniformly. Thus for any given $R > 0$ there exists $k_0$ large enough such that $B(0,R) \subset \psi^{n_{k_0}}(B(0,\epsilon))$. Hence $\psi$ is an automorphism of $\mathbb{C}^k$ and the basin of attraction of $\psi^{-1}$ at 0 is all of $\mathbb{C}^k$. Now by the result of Rosay–Rudin ([5]) $\psi$ is conjugate to an upper triangular map.

**Remark 3.3.** The proof here shows that there exists a sequence $\phi_n \in G$ such that each $\phi_n$ is conjugate to an upper triangular map.

Recall that a domain $\omega$ is holomorphically homotopic to a point in a domain $\Omega$ if there exists a continuous map $h : [0,1] \times \mathbb{C} \to \Omega$ with $h(1,z) = z$ and $h(0,z) = p$ where $p \in \omega$ and $h(t,*)$ is holomorphic in $\omega$ for every $t \in [0,1]$.

**Proposition 3.4.** Let $\phi$ be a non-constant endomorphism of $\mathbb{C}^k$ such that on a bounded domain $U \subset F(\phi)$, the map $\phi$ is proper onto its image, $U \subset \subset \phi(U)$ and $U$ is holomorphically homotopic to a point in $\phi(U)$ then

(i) $\phi$ has a fixed point, say $p$ in $U$.

(ii) $\phi$ is invertible at its fixed points.

(iii) The backward orbit of $\phi$ at the fixed point in $U$ is finite i.e., $O^-\phi(p) \cap U$ is finite where

$$O^-\phi(p) = \{z \in \mathbb{C}^k : \phi^n(z) = p, n \geq 1\}.$$
**Proof.** That the map \( \phi \) has a fixed point \( p \) in \( U \) follows from Lemma 4.3 in [2]. Without loss of generality we can assume \( p = 0 \). Consider \( \psi(z) = \phi(p + z) - p \) and \( \Omega = \{ z - p : z \in U \} \). Then \( \psi \) is the required map with the properties \( \Omega \subset \subset \psi(\Omega) \) and 0 is a fixed point for \( \psi \).

Suppose \( \psi \) is not invertible at 0, i.e., \( A = D\psi(0) \) has a zero eigenvalue. Let \( \lambda_i, 1 \leq i \leq k \) be the eigenvalues of \( A \). Therefore there exist an \( \alpha \) such that \( 0 < \alpha < 1 \) and \( 1 < m \leq k \) such that \( 0 = |\lambda_i| < \alpha \) for \( 1 \leq i \leq m \) and \( |\lambda_i| > \alpha \) for \( m < i \leq k \). Choose \( \delta > 0 \) such that

\[
0 < \|D_G \psi(z) - A\| < \epsilon_0 = \min \left\{ \alpha, |\lambda_i| - \alpha \right\}
\]

for \( z \in B(0, \delta) \) and \( m < i \leq k \). Let \( \Psi \) be a Lipschitz map in \( \mathbb{C}^k \) such that

\[
Lip(\Psi) = \|A\| + \epsilon_0
\]

and

\[
\Psi \equiv \psi \text{ on } B(0, \delta).
\]

Now

\[
W_s^\psi := \{ z \in \mathbb{C}^k : |\alpha^n \Psi^n(z)| \text{ is bounded } \}
\]

can be realized as a graph of a continuous function (See [6]) \( G_\psi : \mathbb{C}^m \to \mathbb{C}^{k-m} \) such that \( G_\psi(0) = 0 \). Since

\[
W_s^\psi = W_s^\psi \text{ on } B(0, \delta/2)
\]

\( W_s^\psi \cap \Omega \) is an infinite non-empty set containing 0. Also \( \psi^{n_k} \mid _{\Omega} \to \psi_0 \) for some sequence \( n_k \) and \( \psi_0 \) is holomorphic on the component (say \( F_0 \)) of \( F(\psi) \) containing \( \Omega \). Let

\[
W_1^\psi = \{ z \in F_0 : \psi^{n_k}(z) \to 0 \text{ as } k \to \infty \}.
\]

Then \( W_s^\psi \cap F_0 \subset W_1^\psi \) and

\[
W_1^\psi = \bigcap_{i=1}^{k} \psi_{0,i}^{-1}(0)
\]

where \( \psi_{0,i} \) is the \( i \)–th coordinate function of \( \psi_0 \). If \( W_1^\psi \cap \partial \Omega = \emptyset \) then \( W_1^\psi \cap \Omega \) and hence \( W_s^\psi \cap \Omega \) will have to be finite which is not true. Thus there exists a positive integer \( n_0 \) such that \( \psi^{n_0}(\partial \Omega) \cap \Omega \neq \emptyset \) but by assumption it follows that \( \Omega \subset \subset \psi^{n}(\Omega) \) for all \( n \geq 1 \), i.e., \( \psi^n(\partial \Omega) \cap \Omega = \emptyset \) for all \( n > 0 \). This proves that \( A \) has no zero eigenvalues.

Note that this observation also reveals that \( W_1^\psi \cap \Omega \) has to be a finite set, and since

\[
O^{-}(0) \subset W_1^\psi
\]

the backward orbit of 0 under \( \psi \) is finite. \( \square \)

Now we can state and prove Theorem 3.2 for semigroups generated by the elements of \( \mathcal{E}_k \).

**Theorem 3.5.** Let \( G = \langle \phi_1, \phi_2, \ldots \rangle \) where each \( \phi_i \in \mathcal{E}_k \). If the Julia set \( J(G) \) contains an isolated point (say \( a \)) then there exists a \( \psi \in G \) such that \( \psi \) is conjugate to an upper triangular automorphism.

**Proof.** Assume \( a = 0 \). Then as before by Proposition 3.1 there exists a map \( \psi \in G \) and a domain \( \Omega \) such that \( \Omega \subset \subset \psi(\Omega) \).

If 0 is in the Julia set of \( \psi \) then 0 is an isolated point in \( J(\psi) \) and by applying Theorem 4.2 in [2] \( \psi \) is conjugate to an upper triangular automorphism.

Suppose \( \Omega \subset F(\psi) \) and from Proposition 3.4 \( \psi \) has a fixed point in \( \Omega \) i.e., \( \{ \psi^n \} \) has a convergent subsequence in \( \Omega \).

**Case 1:** Suppose that \( G = \langle \phi_1, \phi_2, \ldots \rangle \) where each \( \phi_i \in \mathcal{P}_k \).
Applying Proposition 3.1 we have that \( \psi^{-1}(0) = 0 \) and there exists \( \psi \in G \) such that
\[
\Omega \subset B(0, R) \subset \psi(\Omega)
\]
where \( \Omega \) is a sufficiently small ball at \( 0 \) and \( R > 0 \) is a sufficiently large number. Now let \( \omega \) is the component of \( \psi^{-1}(B(0, R)) \) in \( \Omega \) containing the origin. Also from Proposition 3.4 it follows that 0 is a regular point of \( \psi \), which implies that \( \psi \) is a biholomorphism on \( \psi(\omega) \)
\[
\Psi_\beta(z) = \beta \psi^{-1}(z)
\]
and note that \( \Psi_\beta \) is a self map of \( B(0, R) \) for some \( \beta > 1 \) with a fixed point at 0. Then the eigenvalues of \( D\psi_\beta(0) \) are in the closed unit disc, i.e.,
\[
\beta |\lambda_i^{-1}| \leq 1
\]
where \( \lambda_i \) are eigenvalues of \( A \). Hence 0 is a repelling fixed point for the map \( \psi \) and \( 0 \notin F(\psi) \).
Since 0 is an isolated point in the Julia set of \( \psi \), by Theorem 4.2 in [2] \( \psi \) is conjugate to an upper triangular automorphism of \( \mathbb{C}^k \).

Case 2: Suppose that \( G = \langle \phi_1, \phi_2, \ldots \rangle \) where each \( \phi_i \in \mathcal{E}_k \).
As before by Proposition 3.1 there exists \( \psi \in G \) such that
\[
\Omega \subset B(0, R) \subset \psi(\Omega)
\]
and let \( \omega \) be a component of \( \psi^{-1}(B(0, R)) \subset \Omega \). Then \( \omega \) satisfies all the condition of Proposition 3.1 and hence there exists a fixed point \( p \) of \( \psi \) in \( \omega \) and \( O_\psi(p) \cap \omega \) is finite.

Claim: \( \psi^{-1}(p) = p \) in \( \omega \).

Suppose not i.e.,
\[
\#\{\psi^{-1}(p)\} = \text{the cardinality of } \psi^{-1}(p) = m
\]
and \( m \geq 2 \). Let \( a_1 \in \psi^{-1}(p) \setminus \{p\} \) in \( \omega \) and define
\[
S_1 = O_\psi^{-1}(a_1) \cap \omega.
\]
Then \( S_1 \subset O_\psi^{-1}(p) \cap \omega \). Now choose inductively \( a_n \in \psi^{-1}(a_{n-1}) \setminus \{a_{n-1}\} \) for \( n \geq 2 \) and define
\[
S_n = O_\psi^{-1}(a_n) \cap \omega.
\]
Then
\[
S_n \subset S_{n-1} \text{ and } \bigcup_{i=1}^n S_i \subset O_\psi^{-1}(p) \cap \omega
\]
for every \( n \geq 2 \). Note that \( a_n \notin S_n \), otherwise there is a positive integer \( k_n > 0 \) such that \( \psi^{k_n}(a_n) = a_n \) i.e., \( a_n \) is a periodic point of \( \psi \), and
\[
\psi^{k_n+m}(a_n) = p
\]
for any \( m > n \). Since \( O_\psi^{-1}(p) \cap \omega \) is finite it follows that \( S_n \) has to be empty for large \( n \). This implies that there exists a \( n_0 \geq 1 \) such that \( \psi^{-1}(a_{n_0}) = a_{n_0} \) and \( a_{n_0} \in \omega \). But by Proposition 3.4 \( \psi \) is invertible at its fixed points which means that \( a_{n_0} \) is a regular value of \( \psi \) and
\[
\#\{\psi^{-1}(a_{n_0})\} = m \geq 2
\]
which is a contradiction! Hence the claim.

Now by similar arguments as in the case of proper maps it follows that \( \psi \) is a biholomorphism from \( \omega \) to \( B(0, R) \) and \( p \) is a repelling fixed point of \( \psi \) and hence lies in \( J(\psi) \subset J(G) \). Since \( \omega \cap J(G) = \{0\} \), we have \( p = 0 \) which is an isolated point in the Julia set of \( \psi \) and hence \( \psi \) is conjugate to an upper triangular automorphism. \( \square \)
4. Recurrent and Wandering Fatou Components of a Semigroup $G$.

As discussed in Section [1] we will be studying the properties of recurrent and wandering Fatou components of semigroup generated by entire maps of maximal generic rank on $\mathbb{C}^k$. The wandering and the recurrent Fatou components for a semigroup $G$ are defined as:

**Definition 4.1.** Let $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ where each $\phi_i \in \mathcal{E}_k$ for every $1 \leq i \leq m$. Given a Fatou component $\Omega$ of $G$ and $\phi \in G$, let $\phi \Omega$ be the Fatou component of $G$ containing $\phi(\Omega \setminus \Sigma_\phi)$ where $\Sigma_\phi$ is the set where the Jacobian of $\phi$ vanishes. A Fatou component is *wandering* if the set $\{\phi \Omega : \phi \in G\}$ contains infinitely many elements.

**Definition 4.2.** Let $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ where each $\phi_i \in \mathcal{E}_k$ for every $1 \leq i \leq m$. A Fatou component $\Omega$ of $G$ is *recurrent* if for any sequence $\{g_j\}_{j \geq 1} \subset G$, there exists a subsequence $\{g_{jm}\}$ and a point $p \in \Omega$ (the point $p$ depends on the chosen sequence) such that $g_{jm}(p) \to p_0 \in \Omega$.

**Remark 4.3.** Note that we assume here a stronger definition of recurrence than the existing definition for the case of iterations of a single holomorphic endomorphism of $\mathbb{C}^k$. We define it in this way in order to ensure that a recurrent Fatou component is not wandering.

The next lemma gives us an alternative description for the recurrent Fatou components of $G$.

**Lemma 4.4.** A Fatou component $\Omega$ is recurrent if and only if for any sequence $\{\phi_j\} \subset G$, there exists a compact set $K \subset \Omega$ and a subsequence $\{\phi_{jm}\}$ such that $\phi_{jm}(p_{jm}) \to p_0 \in \Omega$ for a sequence $\{p_{jm}\} \subset K$.

**Proof.** Take any sequence $\{\phi_j\} \subset G$. Let there exist a subsequence $\{\phi_{jm}\}$ and points $\{p_{jm}\} \subset K$ with $K$ compact in $\Omega$ such that

$$\phi_{jm}(p_{jm}) \to p_0 \in \Omega.$$  

Without loss of generality we assume $p_{jm} \to p_0 \in K$. It is easy to show that $\phi_{jm}(p_0) \to p_0 \in \Omega$ using the fact that any sequence of $G$ is normal on the Fatou set of $G$. \(\square\)

**Proposition 4.5.** Let $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ where each $\phi_i \in \mathcal{E}_k$ for every $1 \leq i \leq m$. If $\Omega$ is a recurrent Fatou component of $G$, then $G$ is locally bounded on $\Omega$. Moreover $\Omega$ is pseudoconvex and Runge.

**Proof.** Assume $G$ is not locally bounded on $\Omega$. Then there exists a compact set $K \subset \Omega$ and $\{g_r\} \subset G$ such that $|g_r(z_r)| > r$ with $z_r \in K$ for every $r \geq 1$. Clearly this can not be the case since $\Omega$ is a recurrent Fatou component, so we can always get a subsequence $\{g_{r_j}\}$ from the sequence $\{g_r\} \subset G$ such that it converges to a holomorphic function uniformly on compact set in $\Omega$ and in particular on $K$. From the proof of Proposition 2.2 it follows that local boundedness of $G$ on $\Omega$ implies that $\Omega$ is polynomially convex. So $\Omega$ is pseudoconvex. \(\square\)

**Theorem 4.6.** Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$. Assume that $\Omega$ is a recurrent Fatou component for $G$. Let $\phi \in G$ be such that $\phi(\partial \Omega) \subset \partial \Omega$. Then one of the following is true

(i) There is an attracting fixed point $p$ for $\phi$.

(ii) There exists a closed connected complex submanifold $M_\phi \subset \Omega$ of dimension $r$ with $1 \leq r \leq (k - 1)$ and an integer $l \geq 1$ such that $\phi^l$ is an automorphism of $M_\phi$.

(iii) There exists a subsequence $\{\phi^{m_l}\}$ converging uniformly on compact sets of $\Omega$ to the identity.

**Proof.** Pick any $\phi \in G$. Let $\Omega_\phi$ be the recurrent Fatou component of $\phi$ that contains $\Omega$ and $l \geq 1$ be the smallest integer for which $\phi^l(\Omega_\phi) \cap \Omega_\phi \neq \emptyset$. If $l > 1$, then $\phi^{l+1}(\Omega_\phi) \cap \Omega_\phi = \emptyset$ for all $k \geq 1$. This contradicts the fact that $\Omega$ is a recurrent Fatou component for $G$. Hence $\phi(\Omega_\phi) \cap \Omega_\phi \neq \emptyset$. 


Consider the set of all maps $h : \Omega \to \Omega$ with $h(p) = p$ for some $p \in \Omega$ and $h = \lim \phi^{m_j}$ for some sequence $m_j$ such that $\{p_j\}$ and $\{q_j\}$ where

$$p_j = m_{j+1} - m_j \text{ and } q_j = m_{j+1} - 2m_j$$

both diverge to infinity as $j \to \infty$.

**Claim:** This set is non-empty.

Since $\Omega$ is recurrent, there exists $q \in \Omega$ and a sequence $m_j$ such that $\phi^{m_j}(q) \to q_0 \in \Omega$. If $h$ is a limit of the subsequence $\{\phi^{m_j+1-m_j}\}$, then $h(q_0) = q_0$.

Fix one of these maps $h$ of maximal generic rank $r$. If $r = 0$, then $h(\Omega) = p$. Consequently $h(\mathcal{O}_\phi) = p$. Moreover since $h(p) = p$ and $\phi(\partial \mathcal{O}_\phi) \subset \partial \mathcal{O}_\phi$ we have $\phi(p) \in \mathcal{O}_\phi$. So $\phi(p) = \phi(h(p)) = \lim \phi^{m_j}(\phi(p)) = h(\phi(p)) = p$. Since some iterates of $\phi$ converges to a constant map, all the eigenvalues of $\phi$ at $p$ must have modulus strictly less than one. Therefore $p$ is an attracting fixed point for $\phi$ when $r = 0$.

Assume $r \geq 1$. Let $\Delta$ be a small polydisc centered at $p$. Further choose a small enough polydisc $\Delta' \subset \Delta$ such that $\omega = h(\Delta')$ is a smooth $r$-dimensional manifold in $\Delta$. If $h$ to be a limit of $\phi^p$, we get $h = \text{Id}$ on $\omega$ where we assumed $h$ is a limit of $\phi^m$. So if the rank of $h$ is $k$, there exists a subsequence i.e., $\{\phi^p\}$ converging uniformly on compact subsets of $\omega$ to the identity. Define

$$M := \{x \in \Omega : \tilde{h}(x) = x\}.$$ 

**Claim:** $M$ is a closed complex submanifold of $\Omega$.

To prove this claim we will first show that the map $\tilde{h}$ obtained is a holomorphic retraction in $\Omega$, i.e., a map which satisfies the condition

$$\tilde{h} \circ \tilde{h} = \tilde{h} \text{ on } \Omega.$$ 

by Lemma 2.1.28 in [1] it will follow that $M$ is a closed complex submanifold of $\Omega$.

Let $h_1$ be the limit function of the sequence $\{\phi^p\}$ then

$$h_1 \circ h_1 = \tilde{h} \text{ and,}$$

$$h_1 \circ \tilde{h} = \tilde{h} \circ h.$$ 

Now

$$\tilde{h} \circ \tilde{h} = \tilde{h} \circ h \circ h_1 = h \circ h_1 = \tilde{h}.$$ 

Let $M_\phi$ be the irreducible component of $M$ containing $\omega$ which is a closed connected submanifold of $\Omega$.

**Claim:** $\phi^j(M_\phi) \subset \Omega$ for all $j \geq 0$.

Let $q \in M_\phi$. This implies $\tilde{h}(q) = q$ with $q \in \Omega$. Since $\tilde{h} = \lim \phi^p$, therefore $\phi^p(q) \to q$ as $j \to \infty$. Since $\phi(\partial \Omega) \subset \partial \Omega$, we get $\phi^n(q)$ is in $\Omega$ for all $n \geq 0$.

Hence

$$\tilde{h} \circ \phi^n = \phi^n \circ \tilde{h}$$

on $M_\phi$. Therefore $\tilde{h} = \text{Id}$ on $\phi^n(M_\phi)$ which gives $\phi^n(M_\phi) \subset M$ for all $n \geq 0$.

**Claim:** There exists some $i \geq 1$ for which $\phi^i(M_\phi) \subset M_\phi$.

If not, then for any sufficiently small $r$-dimensional polydisc $\Delta'' \subset \Delta'$, there exists $w_i \in h(\Delta'')$ such that $\phi^i(w_i) \notin M_\phi$. Let $w_0$ be a limit point of $\{w_i\}_{i \geq 1}$ in $\omega$. Then $\phi^p(w_{p_i}) \to w_0$ since $\tilde{h} = \lim \phi^p = \text{Id}$ on $\omega$ which is a contradiction.

Take smallest $i \geq 1$ such that $\phi^i(M_\phi) \subset M_\phi$. Since $\tilde{h} = \lim \phi^p = \text{Id}$ on $M$, $\lim \phi^r(\phi^i) \to \text{Id}$ for some $0 \leq r < i$.

**Claim:** $r = 0$. 

Since $G$ is locally bounded on $\Omega$, without loss of generality we take $\psi = \lim \phi^{k_i}$. Then we have $\phi^i \circ \psi = \text{Id}$ on $M$. Pick any $a \in M_\phi$. Because of the fact that $M_\phi$ is invariant under $\phi^i$, we have $\psi(m) \in M_\phi$. But $\phi^i(M_\phi) \cap M_\phi = \emptyset$ for $1 \leq r \leq (i - 1)$. This shows that $r = 0$.

Taking $\psi_1 = \lim \phi^{(l-1)i}$, we have

$$\phi^i \circ \psi_1 = \text{Id}, \quad \psi_1 \circ \phi_i = \text{Id}$$

with $\psi_1(M_\phi) \subset M_\phi$. This implies that $\phi^i \in \text{Aut}(M_\phi)$. \hfill $\square$

**Proposition 4.7.** Let $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ where each $\phi_i \in \mathcal{V}_k$ for every $1 \leq i \leq m$ and let $\Omega$ be an invariant Fatou component of $G$. Then either $\Omega$ is recurrent or there exists a sequence $\{\phi_n\} \subset G$ converging to infinity.

**Proof.** If $\Omega$ is not recurrent, then there exists a sequence $\{\phi_n\} \subset G$ such that $\{\phi_n\} \rightarrow \partial \Omega \cup \{\infty\}$ uniformly on compact sets of $\Omega$. Assume $\{\phi_{n_k}\}$ converges to a holomorphic function $f$ on $\Omega$. This implies that $f(\Omega) \subset \partial \Omega$ contradicting the assumption that each $\phi_{n_k}$ is volume preserving. Hence $\{\phi_{n_k}\}$ diverges to infinity uniformly on compact subsets of $\Omega$. \hfill $\square$

**Proposition 4.8.** Let $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ where each $\phi_i \in \mathcal{V}_k$ for every $1 \leq i \leq m$ and let $\Omega$ be a wandering Fatou component of $G$. Then there exists a sequence $\{\phi_n\} \subset G$ converging to infinity.

**Proof.** Since $\Omega$ is wandering, one can choose a sequence $\{\phi_n\} \subset G$ so that

$$(4.1) \quad \phi_n \Omega \cap \phi_m \Omega = \emptyset$$

for $n \neq m$. If this sequence $\{\phi_n\}$ does not diverge to infinity uniformly on compact subsets, some subsequence $\{\phi_{n_k}\}$ will converge to a holomorphic function $h$ on $\Omega$. By abuse of notation, we denote $\{\phi_{n_k}\}$ still by $\{\phi_n\}$. Fix $z_0 \in \Omega$. Then for any given $\epsilon$, there exists $\delta$ such that

$$(4.2) \quad |\phi_{n_0}(z) - \phi_n(z)| < \epsilon$$

for all $n \geq n_0$ and for all $z \in B(z_0, \delta)$. From (4.2) it follows that $\text{vol}(\cup_{n \geq n_0} \phi_n(B(z_0, \delta)))$ is finite. On the other hand, since each $\phi_n$ is volume preserving and (4.1) holds, we get

$$\text{Vol}\left( \bigcup_{n \geq n_0} \phi_n(B(z_0, \delta)) \right) = +\infty.$$ 

Hence we have proved the existence of a sequence in $G$ converging to infinity. \hfill $\square$

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