Projective integrable mechanical billiards

Airi Takeuchi* and Lei Zhao

University of Augsburg, Augsburg, Germany
E-mail: airi1.takeuchi@uni-a.de

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Abstract
In this paper, we use the projective dynamical approach to integrable mechanical billiards as in (Zhao 2021 Commun. Contemp. Math. 24 2150085) to establish the integrability of natural mechanical billiards with the Lagrange problem, which is the superposition of two Kepler problems and a Hooke problem, with the Hooke center at the middle of the Kepler centers, as the underlying mechanical systems, and with any finite combinations of confocal conic sections with foci at the Kepler centers as the reflection wall, in the plane, on the sphere, and in the hyperbolic plane. This covers many previously known integrable mechanical billiards, especially the integrable Hooke, Kepler and two-center billiards in the plane, as has been investigated in (Takeuchi and Zhao 2021 arXiv:2110.03376), as subcases. The approach of (Takeuchi and Zhao 2021 arXiv:2110.03376) based on conformal correspondence has been also applied to integrable Kepler billiards in the hyperbolic plane to illustrate their equivalence with the corresponding integrable Hooke billiards on the hemisphere and in the hyperbolic plane as well.

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1. Introduction
A two-dimensional mechanical billiard system \((M, g, U, B)\) is defined on a two-dimensional Riemannian manifold \((M, g)\) with a piecewise smooth curve \(B \subset M\) playing the role of a reflection wall and with \(U : M \rightarrow \mathbb{R}\) an openly and densely defined smooth force function on \(M\) determining a natural mechanical system whose equation of motion is

* Author to whom any correspondence should be addressed.
\begin{equation}
\n\nabla \dot{q} = \nabla U(q).
\end{equation}

A particle moves according to the underlying force field and gets reflected elastically at \( B \), i.e. at the point of reflection the tangential component of the velocity does not change while the normal component change its signs. The kinetic energy \( K(q, \dot{q}) = \frac{1}{2} g(q, \dot{q}) \) is invariant under elastic reflections, and thus the total energy \( E(q, \dot{q}) = K(q, \dot{q}) - U(q) \) as well.

A two-dimensional mechanical billiard is called integrable, if there exists an additional first integral, i.e. a first integral of the underlying mechanical system invariant under the reflections, independent of the total energy \( E \). Note that to address the problem of integrability, we do not insist on that the billiard mappings is always well-defined.

Some examples of integrable mechanical billiards are known:

For the free motion in the plane \( \mathbb{R}^2 \), the billiards with a circular or an elliptic reflection wall have well-defined billiard mappings and are integrable. The integrability of the circular case is very easy to check since the angle of reflection is preserved. The integrability of the elliptic case has been shown by Birkhoff [3]. This integrability result can be extended also to the billiard system on the two-dimensional sphere and the two-dimensional hyperbolic plane [32, 37]. Additionally, the Birkhoff–Portitsky conjecture states that any closed convex reflection wall of an integrable billiard system in the plane is an ellipse [26]. This conjecture is still open, but there have been many important progresses recently [15, 16, 20, 21]. Note that when the arcs of confocal conics and line segments were concatenated, then the billiard systems may not be well-defined at the corners, and the dynamics may be only of pseudo-integrable type [9–11].

There are also known integrable billiard examples in the presence of non-constant force functions, the most studied systems are those defined with the Hooke or the Kepler problems.

The Hooke problem and the Kepler problem in the plane \( \mathbb{R}^2 \) refer to the case when \( U = f r^2 \) and \( U = m/r \) respectively, where \( r \) is the distance of the particle from the fixed centre \( O \in \mathbb{R}^2 \) and \( f, m \) are parameters which we assume can take both signs: The force may be either attractive or repulsive.

For the Hooke problem in \( \mathbb{R}^2 \), it is rather direct to check that the systems with any line as a reflection wall are integrable. Also the one with a centred conic section as a reflection wall is integrable, in which the case of centred ellipse follows directly from the classical work of Jacobi [18]. The geometry and dynamics of the integrable Hooke billiards were studied in [27–29] for \( \mathbb{R}^2 \) and in [12, 19] for \( \mathbb{R}^n \).

For the Kepler problem in \( \mathbb{R}^2 \), the billiard systems with a line reflection wall which is not passing the centre were proposed by Boltzmann [4]. The integrability of such systems has been established recently by Gallavotti and Jauslin [14] with an analysis on the geometry of ellipses, with alternative proofs by [13, 38]. In [35], we establish that any conic sections focused at the centre and any confocal combination of them are also integrable, by using the classical Hooke–Kepler correspondence.

As compared to the Hooke and the Kepler problems, Euler’s two-centre problem in \( \mathbb{R}^2 \) are not super-integrable and the associated billiard problems were less studied. In [22], Kozlov showed the integrability of the billiard system in the ellipse with some class of potentials including the two-centre potential by using the method of separation of variables. Kozlov’s work has been further extended in [5–8]. In [35], the integrability of such billiards with certain extensions was studied based on conformal transformations. The integrable billiard systems defined in constant curvature spaces, some with the presence of potentials, were studied in [6, 7, 34, 37].
In this paper, we explain that certain integrable mechanical billiards in the two-dimensional plane and constant curvature surfaces are related by projective correspondences. This allows us to yet extend some previous results from [22, 35], concerning integrable mechanical billiards in the plane to surfaces of constant curvatures.

Our main methodology in this paper is based on the projective correspondence between mechanical billiards. This means that in addition to the projective correspondence of the underlying natural mechanical systems, also the laws of reflection are in correspondence to each other, so that a billiard trajectory in one system is projected to a billiard trajectory in the other system. The energies of the systems then give rise to a pair of independent first integrals for both of the two billiard systems. With this method, the projective correspondence between integrable planar and spherical Kepler billiards with a line or centred circle reflection wall was presented in [38]. The method can be thought of as an adaptation of the projective method for geodesic flows and free billiards as in [24, 31–34] to the case of mechanical billiards. For separable potentials, the integrability of the associated mechanical billiard systems has been established in [7] using geodesic equivalence of Riemannian metrics.

In this paper we consider the billiard systems defined through the Lagrange problem in the plane with $U = m_1/r_1 + m_2/r_2 + f r^2$, on a sphere with $U = m_1 \cot \theta_1 + m_2 \cot \theta_2 + f \tan^2 \theta_{\text{elas}}$, and in a hyperbolic plane with $U = m_1 \coth \theta_1 + m_2 \coth \theta_2 + f \tanh^2 \theta_{\text{elas}}$, which are the problems of adding an elastic force to the two-centre problem defined on such a space centred at the middle of the two centres. The precise definitions of the notations are given in sections 3 and 5. This integrable system has been identified by Lagrange [23] in the planer case. Note that such a system is singular at the Kepler centres, as well as a singular set created by the elastic force on the sphere, and is regular elsewhere.

By setting some of the mass factors to zero we get several systems as particular cases including the two-centre problem, the Kepler problem, and the Hooke problem in the plane, on a sphere, and in a hyperbolic plane. By confocal conic sections we shall mean those with the two Kepler centres as foci.

Theorem 1. The mechanical billiard problems defined in the plane, on a sphere and in a hyperbolic plane with the Lagrange problem and with any combination of confocal conic sections with foci at the two Kepler centres as reflection wall, are integrable.

The energy of the mechanical system under consideration is always a first integral. The additional first integral for each system is derived from the energy of a corresponding system. The explicit description of them is given in (16) for the planar case, in (14) for the spherical case, and in (24) for the hyperbolic case.

In $\mathbb{R}^2$, with an elliptic reflection wall, the result appears as a particular case of formula (4.3) of [22], a further generalization with the two-centre potential to the sphere was briefly indicated as well.

In the plane, the billiard problems defined through the Hooke, the Kepler, and the two-centre problems with combinations of confocal conic sections are therefore subcases of theorem 1 and thus their integrability directly follows. These have been previously discussed via a different method, based on conformal transformations, in [35]. Theorem 1 provides an alternative proof of their integrability as well as extensions to the sphere and the hyperbolic plane.

Note that somehow in contrast to the conformal transformation used in [35], this projective method can be directly applied to the case of higher dimensional problems, and will always provide two first integrals for the Lagrange problems. These higher dimensional problems are discussed in a subsequent work [36]. Restricting to dimension 2 there raises the question of whether some of the integrable systems can be indeed also related by conformal transformations. Toward the end of this article we shall present such links of integrable Kepler billiards in
the hyperbolic plane, and the integrable Hooke billiards defined on the sphere and the hyperbolic plane. In proposition 10, we also show that a family of confocal focused hyperbolic conic sections are transformed into a family of confocal centred spherical/hyperbolic conic sections by the complex square mapping in conformal charts, which might have an independent geometrical interest.

We organize this paper as follows:

In section 2, we explain the settings and the principle properties of projective dynamics and define projective correspondences of billiard systems. In section 3, we recall projective properties of the Hooke and the Kepler problems and their spherical/hyperbolic analogous systems. We get the projective property of Lagrange problems as has been discovered in [1]. In section 4, we prove theorem 1 for the planar and the spherical cases, and we discuss some subcases. In section 5, we briefly discuss the hyperbolic case and establish theorem 1 for this case. The conformal correspondences among the Hooke and the Kepler billiards in the hyperbolic space and the Hooke billiard on the hemisphere are discussed in section 6.

2. General principles of projective dynamics and projective billiards

We start with a brief discussion on our main approach.

Let \((M, g, U)\) be a natural mechanical system. The system possesses a corresponding system if there exists another natural mechanical system \((M', g', U')\) such that they have the same orbits in \(M\) up to time-parametrizations. In this case, any first integral of \((M, g, U)\) is also a first integral of \((M', g', U')\) and vice versa. In particular, the energy \(E'\) of the second system \((M', g', U')\) is a first integral of \((M, g, U)\). When \(E'\) is functional independent from the energy \(E\) of \((M, g, U)\) we have an additional first integral of the system \((M, g, U)\).

The procedure of the projection from \(M\) to \(M'\) by a diffeomorphism \(\phi: M \rightarrow M'\) with a time reparametrization factor \(\rho: M' \rightarrow \mathbb{R}\) is that a force field \(F\) on \(M\) is projected into the force field \(F' := \rho \cdot \phi_* F\).

In words:
The projection of the force field of a system is the force field given by the push-forward of the projection multiplied with a time reparametrization factor.

Now arguing with force fields defined on a manifold, we have the following principle of superposition:

The projection of superposition of the force fields is the superposition of the projections of the force fields.

When the force fields are derived from potentials, then so is their superposition. In general, the projections of these force fields are no longer derived from potentials. However this indeed holds for special systems that we are going to address in this article, which provide corresponding systems to the original systems.
We now comment on billiard correspondences. For this it seems convenient to identify the base manifold $M$ by a diffeomorphism and consider the law of reflection in $M$ with respect to the two metrics. The tangential direction is free from the choice of the metric but the normal direction depends on the metric, and therefore a priori the elastic laws of reflections with respect to different metrics are different. We say that there is a billiard correspondence when the elastic laws of reflection agree in addition to the correspondence of underlying natural mechanical systems. As we can see, this depends on the choice of metric and the shape of the reflection wall. When there is a billiard correspondence, then the billiard trajectories, ignoring time parametrizations, correspond to each other by projection and therefore their billiard mappings are equivalent. Conserved quantities of one system are thus transformed into conserved quantities of the other system, and therefore the integrability of the billiard system also carries over. We refer to [1, 2, 38] for further, more detailed presentations of projective dynamics.

3. Projective properties of the Hooke, Kepler and Lagrange problems

In this section, we discuss some projective properties of the Hooke and the Kepler problems and their spherical/hyperbolic analogous systems that we need. Then we shall show that the Lagrange problem also has spherical/hyperbolic analogous systems by the principle of superposition of projective dynamics, which then gives to each of these systems a pair of independent first integrals including their own energies.

3.1. The hemisphere-plane projection

We set $V = \mathbb{R}^2 \times \{-1\} \subset \mathbb{R}^3$ and $S \subset \mathbb{R}^3$ the unit sphere in $\mathbb{R}^3$. The central projection from the origin of $\mathbb{R}^3$ projects the open south-hemisphere $S_{SH}$ onto the plane $V$. We equip $S$ and $S_{SH}$ with their induced round metrics from $\mathbb{R}^3$, while on $V$ we allow an affine change of metric. A force field $F_V$ on $V$ is carried to a force field $\tilde{F}_S$ on $S_{SH}$ by the push-forward of the central projection, which is consequently reparametrized into another force field $F_S$ with the factor of time change uniquely determined by the projection.

The Euclidean norm of $\mathbb{R}^3$ as well as its restriction to $V$ is denoted by $\| \cdot \|$.

Let $q \in S_{SH}$ be projected to $\tilde{q} \in V$ by the central projection:

$$q = \|\tilde{q}\|^{-1}\tilde{q}.$$ 

We write $\dot{\cdot} := \frac{d}{dt}$ the time derivative. We start by the force field $F_V$ in $V$ and deduce the corresponding force field $F_S$ on $S_{SH}$ which is equivalent to the other way around but the computation simplifies. The equation of motion of the system in $V$ is

$$\ddot{\tilde{q}} = F_V(\tilde{q}).$$

We compute

$$\dot{\tilde{q}} = \|\tilde{q}\|^{-2}(\dot{q}\|\tilde{q}\| - \langle \nabla\|\tilde{q}\|, \dot{\tilde{q}} \rangle \tilde{q}).$$

We now take a new time variable $\tau$ for the system on $S_{SH}$, and $\dot{\cdot} := \frac{d}{d\tau}$ such that

$$\frac{d}{d\tau} = \|\tilde{q}\|^2 \frac{d}{dt}. \tag{3}$$
We thus have
\[
q' = \left( \hat{q}||\hat{q}|| - \langle \nabla ||\hat{q}||, \hat{q} \rangle \hat{q} \right).
\]
and
\[
q'' = ||\hat{q}||^2 \left( \hat{q}||\hat{q}|| - \langle \nabla ||\hat{q}||, \hat{q} \rangle \hat{q} \right).
\]
Consequently we have
\[
q'' = ||\hat{q}||^2 (F_V(\hat{q}||\hat{q}||) - \lambda(\hat{q}, \hat{q}, \hat{q})). \tag{4}
\]
in which we have set \(\lambda(\hat{q}, \hat{q}, \hat{q}) = \langle \nabla ||\hat{q}||, \hat{q} \rangle + \langle \langle \nabla ||\hat{q}||, \hat{q} \rangle \hat{q} \rangle\).

We observe that the first term of the right hand side of this equation depends only on \(\hat{q}\) and consequently depends only on \(q \in S_{SH}\) by central projection, while the second term is radial. Projecting both sides of this equation to the tangent space \(T_q S_{SH}\) we get the equation of motion on \(S_{SH}\), assuming the form
\[
\nabla_q q' = F_S(q).
\]

For our purpose, we would like to have natural mechanical systems which are centrally projected to natural mechanical systems, i.e. the question is, when we start from a natural mechanical system on \(S_{SH}\) resp. \(V\), then whether the projected system on \(V\) resp. \(S_{SH}\) is also derived from a potential and is thus also a natural mechanical system. As we would expect this does not hold in general. Nevertheless, it actually holds for some important systems.

### 3.2. Projective properties of the Hooke and Kepler problems

We consider a central force problem \(F_S\) on \(S\) with a distinguished centre \(Z \in S_{SH}\). By assumption the force field is invariant under the \(SO(2)\)-action by rotations around \(Z\) on \(S\). The projected force field \(F_V\) on \(V\) is in general not derived from a potential. In the same way, a central force problem \(F_V\) in \(V\) with a centre \(Z\) might not project to a system derived from a potential on \(S_{SH}\).

There are special cases that this does hold. The first is relatively easy to see: when \(Z = (0,0,-1)\), the projected force field \(F_V\) is also invariant under the \(SO(2)\)-action by rotations in \(V\) as inherited from rotations around the vertical axis in \(\mathbb{R}^3\), and therefore \(F_V\) is derived from a potential. The second case is maybe not as easy to see: The point \(Z \in S_{SH}\) can be chosen arbitrary, and \(F_V\) will be derived from a potential when \(F_S\) is the force field of the Kepler–Serret problem on the sphere [30], and in this case \(F_V\) itself is the force field of a Kepler problem in \(V\) for a proper choice of an affine metric. Also, among the problems belonging to the first case, the Hooke problems have the property that \(F_V\) is derived from a potential for any affine metrics in \(V\).

#### 3.2.1. The Kepler problems. We first discuss the case of the Kepler problems. The Kepler-Serret problem, or the spherical Kepler problem, is the natural mechanical system \((S, g_{st}, \dot{m} \cot \theta_Z)\), in which \(g_{st}\) is the round metric on the sphere, \(\dot{m} \in \mathbb{R}\) is the mass-factor and \(\theta_Z\) is the central angle the moving particle made with \(Z\). The system naturally restricts to a natural mechanical system \((S_{SH}, g_{st}, \dot{m} \cot \theta_Z)\) by restriction. In the case that \(Z\) is vertical, \(Z = (0,0,-1)\), it is not hard to see by a direct computation that the spherical Kepler problem is projected to the planar Kepler problem \((V, \| \cdot \|, \dot{m} / ||\hat{q}||)\). Consequently the orbits of the spherical Kepler problem are all conic sections on the sphere by means of orbital correspondence.
and analytic extension. A special property of the Kepler problem is that this remains true when \( Z \) is not vertical, up to a change of metric and of the mass factor [17]. See also [2, 38].

To normalize the situation we set \( Z = (0, \frac{a}{\sqrt{1 + a^2}}, -\frac{1}{\sqrt{1 + a^2}}) \in S_{SH} \) for \( a \in \mathbb{R} \), and \( \tilde{Z} = (0, a, -1) \) the projection point of \( Z \) in \( V \). For \( \tilde{q} = (\tilde{x}, \tilde{y}, -1) \in V \) we define

\[
\| \tilde{q} \|_a = \sqrt{\tilde{x}^2 + \frac{\tilde{y}^2}{1 + a^2}}
\]

(5)

which is an affine change of norm from the induced norm on \( V \) with origin at \( (0, 0, -1) \) of the standard Euclidean norm \( \| \cdot \| \) in \( \mathbb{R}^3 \).

**Proposition 1.** The spherical Kepler problem \((S_{SH}, \hat{g}_{SH}, m\cot\theta_Z)\) projects to the Kepler problem \((V, \| \cdot \|_a, m/\|\tilde{q} - \tilde{Z}\|_a)\) such that \( m = \frac{\tilde{m}}{\sqrt{1 + a^2}} \).

**Proof.** With the procedure explained in section 3.1, we arrive from a planar force field \( F_V \) to a spherical force field \( F_S \).

We now consider the Kepler problem on \( V \):

\[
(V, \| \cdot \|_a, m/\|\tilde{q} - \tilde{Z}\|_a^{-1})
\]

(6)

which determines the force field

\[
F_V(\tilde{q}) := -m\|\tilde{q} - \tilde{Z}\|_a^{-3} (\tilde{q} - \tilde{Z})
\]

(7)

on \( V \).

We now plug (7) into the right hand side of (4) and compute its projection to the tangent space of \( T_\theta S_{SH} \). We may effectively forget the second term in the right hand side of (4) since it projects to zero in \( T_\theta S_{SH} \). As for the first term in the right hand side of (4), we see that it is again central on \( S_{SH} \) by the central projection. Therefore it is enough to compute its norm to determine the corresponding \( F_S \) on \( S_{SH} \).

For this purpose, we restrict the system to (oriented) planes passing through the centres \( Z, \tilde{Z} \) as well as the centre \( O = (0, 0, 0) \) of \( S_{SH} \). These planes in \( \mathbb{R}^3 \) form an \( S^1 \)-family. We compute the restricted force field on any of these planes.

We fix such a plane \( W \), which necessarily intersects \( V \) by construction. Let \( \ell \) be the intersection line. Let \( G \) be the point on \( \ell \) such that \( OG \) is perpendicular to \( \ell \). Let \( \phi \) be the angle between \( \ell \) and the intersection line \( \{(0, y, -1)\} \) of the \( yz \)-plane and \( V \). The restriction to \( \ell \) of the function \( \|\tilde{q} - \tilde{Z}\|_a \) can be written as

\[
\|\tilde{q} - \tilde{Z}\|_a = \|\tilde{q} - \tilde{Z}\| \sqrt{\sin^2 \phi + \frac{\cos^2 \phi}{1 + a^2}} = \|\tilde{q} - \tilde{Z}\| \sqrt{\frac{1 + a^2 \sin^2 \phi}{1 + a^2}}.
\]

Thus the force filed \( F_V(\tilde{q}) \) restricted to \( \ell \) is given by

\[
F_V(\tilde{q}) = -m \left( \frac{1 + a^2}{1 + a^2 \sin^2 \phi} \right)^{3/2} \|\tilde{q} - \tilde{Z}\|^{-3} (\tilde{q} - \tilde{Z})
\]

The line \( \ell \) passes through the two points \( \tilde{q} \) and \( \tilde{Z} \) and the equation of \( \ell \) is given by

\[y = \cot \phi \cdot x + a.\]
Let $Z_0 = (0, 0, -1)$, then $G$ can be obtained as the point on $\ell$ such that $Z_0G$ is perpendicular to $\ell$ and computed as

$$G := (-a \sin \phi \cos \phi, -a \sin^2 \phi, -1).$$

By (4), the corresponding force field on $S_{SH}$ is determined by the projection of $||\tilde{q}||^3 F_V(\tilde{q})$ to $T_\tilde{q}S_{SH}$, which is computed as

$$||\tilde{q}||^2 \sqrt{1 + a^2 \sin^2 \phi} \cdot \frac{\sqrt{1 + a^2 \sin^2 \phi}}{||\tilde{q}||} = ||\tilde{q}||^2 \sqrt{1 + a^2 \sin^2 \phi} \cdot F_V(\tilde{q}),$$

where $\theta_G = \angle \tilde{q}OG$.

We now compute its norm as

$$|m||\tilde{q}|^2 ||\tilde{q} - Z||^{-2} (1 + a^2)^{3/2} (1 + a^2 \sin^2 \phi)^{-1}$$
$$= \sqrt{1 + a^2} |m||\tilde{q}|^2 ||Z||^2 ||\tilde{q} - Z||^{-2} (1 + a^2 \sin^2 \phi)^{-1}$$
$$= |\tilde{m}| ||\tilde{q}|^2 ||Z||^2 ||\tilde{q} - Z||^{-2}||G||^{-2}$$
$$= |\tilde{m}| \sin^{-2} \theta_Z$$

if we set $m = \frac{\tilde{m}}{\sqrt{1 + a^2}}$. For the last equality we applied the law of sines for the triangle $\tilde{q}OZ$.

The computation is illustrated in figure 1.

So after this computation we conclude that $F_S$ is the central force field on $S_{SH}$ with strength $|\tilde{m}| \sin^{-2} \theta_Z$ in which $\theta_Z$ is the central angle of $q$ to $Z$, pointing toward $Z$ or its antipodal point according to the sign of $\tilde{m}$. This force field can be extended to the whole $S$ which is singular only at $Z$ and its antipodal point, and is invariant under rotations along the line $OZ$. Restricting to a great circle passing through the point $Z$ we conclude that this system is derived from the force function $\tilde{m} \cot \theta_Z$.

\[\square\]
Note that among all homogeneous central force problems, this property of being projective invariant is unique for the Kepler problem \[2\].

### 3.2.2. The Hooke problems

The spherical Hooke problem is the system \((S_{SH}, g_{st}, f \tan^2 \theta Z)\) with \(f \in \mathbb{R}\). This is seen to be the analytic extension of the projection \((S_{SH}, g_{st}, f \tan^2 \theta Z)\) of the Hooke problem in the plane \((V, \| \cdot \|, f \| \tilde{q} \|^2)\). A special projective property of the Hooke problem is summarized in the following proposition. In contrast to the Kepler case, here we assume that the centre for the Hooke problem is vertical i.e. \(Z = (0, 0, -1)\).

**Proposition 2.** The spherical Hooke problem \((S_{SH}, g_{st}, f \tan^2 \theta Z)\) with \(Z = (0, 0, -1)\) projects to any of the Hooke problems in \(V\) of the form \((V, \| \cdot \|, f \tilde{q}^2)\) for any \(a \in \mathbb{R}\).

**Proof.** The Hooke problem in \(V\) with respect to a norm \(\| \cdot \|_a\) is the system \((V, \| \cdot \|_a, f \tilde{q})\).

The corresponding force field is given by

\[F_V(\tilde{q}) := 2f(\tilde{q} - Z).\]

A simple property which nevertheless worths to be mentioned, is that this force field is independent of \(a\), i.e. this force field corresponds to any Hooke system of the above form.

The corresponding force field \(F_S\) on \(S_{SH}\) is again determined by the central projection, and we obtain a central force field on \(S_{SH}\) centred at \(Z\), with the sign of \(f\) determines whether \(Z\) is attractive or repulsive just as in the planar case. Again, we just have to determine the norm of \(F_S\).

By again restricting to a great circle passing through the centre \(Z\), we get that this system has the force function \(f \tan^2 \theta Z\).

### 3.3. The Lagrange problems in the plane and on the sphere

The Lagrange problem in the plane \(\mathbb{R}^2\) is the system

\[
(\mathbb{R}^2, \| \cdot \|, m_1/\|q - Z_1\| + m_2/\|q - Z_2\| + f/\|q - (Z_1 + Z_2)/2\|^2),
\]

with \(m_1, m_2, f \in \mathbb{R}\), which is the superposition of two Kepler problems and a Hooke problem, with the Kepler centres placed symmetrically with respect to the Hooke centre.

Similarly, we define the Lagrange problem on the sphere as the system

\[
(S, g_{st}, \tilde{m}_1 \cot \theta Z_1 + \tilde{m}_2 \cot \theta Z_2 + f \tan^2 \theta_{z_{mid}}),
\]

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for which we assume that \( Z_2 \notin \{ Z_1, -Z_1 \} \). \( \theta_p \) central angle of the moving particle to a point \( P \in S \), \( Z_{mid} \) middle point of \( Z_1 \) and \( Z_2 \).

Based on the previous propositions 1 and 2, we see that the following remarkable theorem holds

**Theorem 2.** (Albouy [1]) In the case \( Z_{mid} \) is vertical, then the spherical Lagrange problem on \( S_{SH} \) with masses \( m_1, m_2, f \) is projected to a planar Lagrange problem in \( V \), with the projections of the Kepler and Hooke centres as its own Kepler and Hooke centres, with the affine norm \( \| \cdot \|_a \) and parameters \( m_1, m_2, f \) as determined by proposition 1.

**Proof.** We assume that the \( Z_{mid} \) is vertical i.e. \( Z_{mid} = (0, 0, -1) \). Additionally, for the normalization purpose, we set \( Z_1 = (0, a, -1) \) and \( Z_2 = (0, -a, -1) \). We then define the norm in \( V \) as

\[
\| q \|_a = \sqrt{x^2 + \frac{y^2}{1 + a^2}}
\]

for \( q = (x, y, -1) \). The affine norm \( \| \cdot \|_a \) in \( V \) was chosen as common for all the three central force problems, two Kepler problems and a Hooke problem. By the principle of superposition, we may thus superpose them and the conclusion follows from the previous propositions 1 and 2.

As a consequence to theorem 2, we have

**Proposition 3.** The energy of the spherical Lagrange problem induces an additional first integral for the planar Lagrange problem independent of its energy. Vice versa, the energy of the planar Lagrange problem induces an additional first integral for the spherical Lagrange problem independent of its energy.

**Proof.** The conservation of the energy of the planar problem in the spherical problem as well as the conservation of the energy of the spherical problem in the planar problem both follow from the fact that these systems are in correspondence, so their orbits in the configuration spaces are equivalent up to a time reparametrization.

To show their independence, we give their explicit expressions in a common chart as in [38]. To normalize our situation, we here again assume that \( Z_{mid} = (0, 0, -1) \), \( \tilde{Z}_1 = (0, a, -1) \), and \( \tilde{Z}_2 = (0, -a, -1) \). Then the planar energy for the Lagrange problem in \( V \) is described as

\[
E_{pl} = \frac{\| \tilde{q} \|_a^2}{2} - f\| \tilde{q} \|_a^2 - \frac{m_1}{\| \tilde{q} - \tilde{Z}_1 \|_a} - \frac{m_2}{\| \tilde{q} - \tilde{Z}_2 \|_a}
\]

\[
= \frac{1}{2} \left( \tilde{x}^2 + \frac{\tilde{y}^2}{1 + a^2} \right) - \frac{m_1}{\sqrt{\tilde{x}^2 + \frac{(\tilde{y} - a)^2}{1 + a^2}}} - \frac{m_2}{\sqrt{\tilde{x}^2 + \frac{(\tilde{y} + a)^2}{1 + a^2}}}
\]

where \( \tilde{q} = (\tilde{x}, \tilde{y}) \in V \) and \( (\tilde{x}, \tilde{y}) \in T_q V \).

We now write the energy of the spherical problem in the gnomonic chart \( V \). In \( \mathbb{R}^3 \) the spherical kinetic energy is given by

\[
\frac{x'^2 + y'^2 + z'^2}{2},
\]

where \( q = (x, y, z) \in S_{SH} \) and \((x', y', z') \in T_q S_{SH} \). Let \( q = (x, y, z) \in S_{SH} \) and \( \tilde{q} = (\tilde{x}, \tilde{y}, -1) \in V \) be corresponded via the central projection as

\[
x = \frac{\tilde{x}}{\sqrt{\tilde{x}^2 + \tilde{y}^2 + 1}}, \quad y = \frac{\tilde{y}}{\sqrt{\tilde{x}^2 + \tilde{y}^2 + 1}}, \quad z = -\frac{1}{\sqrt{\tilde{x}^2 + \tilde{y}^2 + 1}}.
\]

\[
(10)
\]
Then the corresponding push-forward transformation from $T_{\tilde{q}}V$ to $T_{\tilde{q}}S_{SI}$ is given by

$$
\begin{pmatrix}
\dot{x}' \\
\dot{y}' \\
\dot{z}'
\end{pmatrix} = \begin{pmatrix}
\frac{x^2 + 1}{(x^2 + y^2 + 1)^{3/2}} & \frac{x y}{(x^2 + y^2 + 1)^{3/2}} & \frac{x z}{(x^2 + y^2 + 1)^{3/2}} \\
\frac{y x}{(x^2 + y^2 + 1)^{3/2}} & \frac{y^2 + 1}{(x^2 + y^2 + 1)^{3/2}} & \frac{y z}{(x^2 + y^2 + 1)^{3/2}} \\
\frac{z x}{(x^2 + y^2 + 1)^{3/2}} & \frac{z y}{(x^2 + y^2 + 1)^{3/2}} & \frac{z^2 + 1}{(x^2 + y^2 + 1)^{3/2}}
\end{pmatrix}
\begin{pmatrix}
\dot{\tilde{x}}' \\
\dot{\tilde{y}}' \\
\dot{\tilde{z}}'
\end{pmatrix}.
$$

Using this, the projection of the spherical kinetic energy is represented as

$$
\frac{(1 + \tilde{y}^2)\tilde{x}^2 - 2\tilde{x}\tilde{y}\tilde{y}'}{2(\tilde{x}^2 + \tilde{y}^2 + 1)^2}
$$

at $\tilde{q} = (\tilde{x}, \tilde{y}, -1) \in V$. Remember that $(\cdot)'$ is the time derivative with respect to the time parameter $\tau$ defined as (3). From this, the spherical kinetic energy in the gnomonic chart has an expression

$$
K_{sp} := \frac{(1 + \tilde{y}^2)\tilde{x}^2 - 2\tilde{x}\tilde{y}\tilde{y} + (1 + \tilde{x}^2)\tilde{y}^2}{2(\tilde{x}^2 + \tilde{y}^2 + 1)^2} = \frac{\tilde{x}^2 + \tilde{y}^2 + (\tilde{x}\tilde{y} - \tilde{y})^2}{2}
$$

at $(\tilde{x}, \tilde{y}, -1) = (-x/z, -y/z, -1)$ in $V$, which can be seen as the combination of the planar kinetic energy and the squared angular momentum.

The spherical potential consists of the terms $-f\tan^2\theta_{Z_{ext}}$, $-\dot{m}_1\cot\theta_{Z_I}$, and $-\dot{m}_2\cot\theta_{Z_I}$. They are expressed in the gnomonic chart $V$ as

$$
-f(\tilde{x}^2 + \tilde{y}^2),
$$

$$
-\dot{m}_1\frac{a\tilde{y} + 1}{\sqrt{(\tilde{y} - a)^2 + (1 + a^2)\tilde{x}^2}},
$$

and

$$
-\dot{m}_2\frac{-a\tilde{y} + 1}{\sqrt{(\tilde{y} + a)^2 + (1 + a^2)\tilde{x}^2}},
$$

respectively.

Combining these, we get the following expression of the spherical energy of the Lagrange problem in the gnomonic chart:

$$
E_{sp} = \frac{(1 + \tilde{y}^2)\tilde{x}^2 - 2\tilde{x}\tilde{y}\tilde{y} + (1 + \tilde{x}^2)\tilde{y}^2}{2} - \dot{m}_1\frac{a\tilde{y} + 1}{\sqrt{(\tilde{y} - a)^2 + (1 + a^2)\tilde{x}^2}} - \dot{m}_2\frac{-a\tilde{y} + 1}{\sqrt{(\tilde{y} + a)^2 + (1 + a^2)\tilde{x}^2}} - f(\tilde{x}^2 + \tilde{y}^2).
$$

The functional independence of $E_{pl}$ and $E_{sp}$ now follows from these expressions. Indeed, one can check that the Jacobi matrix

$$
J := \begin{pmatrix}
\frac{dE_{pl}}{dx} & \frac{dE_{pl}}{dy} & \frac{dE_{pl}}{dz} & \frac{dE_{sp}}{dx} & \frac{dE_{sp}}{dy} & \frac{dE_{sp}}{dz}
\end{pmatrix}
$$
has rank 2. To see this, it suffices to observe that the $2 \times 2$ submatrix

$$\begin{pmatrix}
\frac{df}{dx} & \frac{df}{dy} \\
\frac{dg}{dx} & \frac{dg}{dy}
\end{pmatrix} = \begin{pmatrix}
\dot{x} & \dot{y} \\
\frac{\dot{y}^2 + 1}{1 + a^2} & \frac{\dot{x}^2 + 1}{1 + a^2}
\end{pmatrix}$$

has rank 2.

Therefore we get an additional first integral for the planar problem from its corresponding spherical problem.

Similarly, the same argument equips the spherical problem in $S_{SH}$ with an additional first integral.

We now show that the projected planar energy to $S_{SH}$ extends to $S$ in an analytical way, outside of its singularities, thus the integrability extends to the problem on $S$.

We first consider the kinetic energy and we provide a differently, more direct argument as in [38]. The planar kinetic energy at $\tilde{q} = (\tilde{x}, \tilde{y}, 1)$ on $V$ is given by

$$\frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 \right)$$

for which we have taken the affine change of norm given by (5) into account. We now change the time parameter according to (3), then the above quantity can be rewritten into

$$\frac{1}{2} \left( \dot{x}'^2 + \dot{y}'^2 \right) \left( \dot{x}^2 + \dot{y}^2 + 1 \right)^{-2}.$$

Let $q = (x, y, z) \in S_{SH}$ be the centrally projected point of $\tilde{q} = (\tilde{x}, \tilde{y}, 1) \in V$ on $S_{SH}$. We have

$$\tilde{x} = -\frac{x}{z}, \quad \tilde{y} = -\frac{y}{z}.$$

Then the push-forward transformation from $T_q S_{SH}$ to $T_{\tilde{q}} V$ is given by

$$\begin{pmatrix}
\dot{x}' \\
\dot{y}'
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{z} & 0 \\
0 & -\frac{1}{z}
\end{pmatrix} \begin{pmatrix}
\dot{x}' \\
\dot{y}'
\end{pmatrix}.$$

Using this, we obtain the transformed expression of the planer kinetic energy (11) defined on $S_{SH}$ given by

$$\frac{\left( (1 + a^2)x^2 + y^2 \right) z^2 - 2z' \left( (1 + a^2)xx' + yy' \right) z + z'^2 \left( (1 + a^2)x^2 + y^2 \right)}{2(1 + a^2)}.$$

at $q = (x, y, z) \in S_{SH}$. Realize that this expression (12) of the planer kinetic energy can be analytically extended to the whole sphere $S$.

For the potential

$$-\frac{m_1}{\sqrt{x'^2 + (\frac{\dot{y}}{1 + a^2})^2}} - \frac{m_2}{\sqrt{x'^2 + (\frac{\dot{x}}{1 + a^2})^2}} - f \left( \frac{x^2 + \dot{y}^2}{1 + a^2} \right)$$

of the planer Lagrange problem in $V$, just as in [38] we apply the change of coordinates

$$\tilde{x} = -\frac{x}{z}, \quad \tilde{y} = -\frac{y}{z}.$$
which is derived from the central projection: \( V \ni (\tilde{x}, \tilde{y}, -1) \mapsto (x, y, z) \in S_{SH} \), and obtain the projected representation

\[
\begin{align*}
- \frac{m_1}{\sqrt{x^2 + \frac{(x/y-a)^2}{1+a^2}}} - \frac{m_2}{\sqrt{y^2 + \frac{(y/z+a)^2}{1+a^2}}} &= -f \left( \frac{(a^2 + 1) x^2 + y^2}{(a^2 + 1) z^2} \right) \\
\end{align*}
\]

(13)
deﬁned on \( S_{SH} \). This quantity can be analytically extended to the whole unit sphere \( S \), outside its singularities, which are the Kepler centres and their antipodal points and the horizontal equator \( \{ (x, y, z) \in S \mid z = 0 \} \), when the corresponding mass parameter is not zero. Thus, we obtain the additional ﬁrst integral

\[
\begin{align*}
\left((1 + a^2)x^2 + y^2\right)z^2 - 2z^2 \left((1 + a^2)xx' + yy'\right)z + z^2 \left((1 + a^2)x^2 + y^2\right) \nonumber \end{align*}
\]

(14)

\[
\begin{align*}
- \frac{m_1}{\sqrt{x^2 + \frac{(x/y-a)^2}{1+a^2}}} - \frac{m_2}{\sqrt{y^2 + \frac{(y/z+a)^2}{1+a^2}}} &= -f \left( \frac{(1 + a^2) x^2 + y^2}{(1 + a^2) z^2} \right) \\
\end{align*}
\]

for the spherical Lagrange problem.

Note that via the afﬁne transformation, we can normalize the norm in \( V \) to the standard one of \( \mathbb{R}^2 \) as described in [38, section 5]. After the normalization, the planer Lagrange problem its centres at \( \tilde{Z}_0, \tilde{Z}_1, \tilde{Z}_2 \) has its own energy

\[
\begin{align*}
\frac{\|\dot{\tilde{q}}\| - f\|\tilde{q}\|}{2} &= \frac{m_1}{\|\tilde{q} - Z_1\|} - \frac{m_2}{\|\tilde{q} - Z_2\|} \\
= \frac{\dot{x}^2 + \dot{y}^2}{2} - f(x^2 + y^2) - \frac{m_1}{\sqrt{x^2 + (y-a)^2}} - \frac{m_2}{\sqrt{x^2 + (y+a)^2}} \\
\end{align*}
\]

(15)

and the additional ﬁrst integral

\[
\begin{align*}
\frac{1}{2} \left( \dot{x}^2 + \left(1 + a^2\right) \left( \dot{x}^2 \dot{y}^2 - 2\dot{x}\dot{y}^2 + \dot{y}^2 \dot{y}^2 + \dot{y}^2 \right) \right) - f \left( x^2 + \left(1 + a^2\right) \dot{y}^2 \right) \\
- \frac{m_1}{\sqrt{(\sqrt{1+a^2}y - a)^2 + (1 + a^2)x^2}} - \frac{m_2}{\sqrt{(\sqrt{1+a^2}y + a)^2 + (1 + a^2)x^2}} \\
\end{align*}
\]

(16)

4. Integrable Lagrange billiards

4.1. Billiard correspondence at confocal conic sections

In this subsection, we consider the problem of projective correspondence of a reﬂection wall \( \tilde{B} \) in \( V \) and its corresponding reﬂection wall \( B \) in \( S_{SH} \). Recall that in this case a projective correspondence refers to the property that the laws of reﬂection in \( V \) and on \( S_{SH} \) correspond to each other via the central projection. When this holds, then the billiard trajectories correspond to each other. This property does not hold for general reﬂection wall \( \tilde{B} \subset V \). In this section we show that this nevertheless holds for any conic sections in \( V \) centred at \((0,0,-1)\), with respect to a compatible \( \| \cdot \|_* \), in \( V \), meaning that the \( \| \cdot \|_* \)-distance of the foci of the conic section, deﬁned with respect to \( \| \cdot \|_* \), equals \( 2a \).
Proposition 4. Any centred confocal conic section $\tilde{B} \subset V$ is projected to a centred confocal conic section $B \subset S_{SH}$. The foci of $B$ are the projection of the foci of $\tilde{B}$ by the central projection. The law of reflection at $\tilde{B}$ with respect to a compatible $\| \cdot \|_a$ and the law of reflection at $B \subset S_{SH}$ correspond to each other.

Proof. Since spherical Kepler problems in $S_{SH}$ and planer Kepler problems are in correspondence as described in proposition 1, their orbits are projected to each other up to some time parametrization. Any connected component of confocal conic sections in a plane/on a sphere is an orbit of the planer/spherical Kepler problem with the centre at one of the foci. Indeed any confocal ellipse and branch of any confocal hyperbola are orbits of Kepler problems with positive mass-factor, and for hyperbolas, the other branch is obtained as an orbit of Kepler problem with negative mass-factor. Each connected component of a confocal conic sections is projected to a connected component of a conic section with a focus at the projected centre which is an orbit of the spherical/planer Kepler problem with the corresponding projected centre. We now look at the other focus and its correspondence. For this purpose, we regard the same conic section as an orbit of the planer/spherical Kepler but with the centre at the other focus. Then from the same projective argument, one can see that the other focus is also projected from the corresponding focus.

We will now check the projective correspondence between the laws of reflection at confocal conic sections in $V$ and on $S_{SH}$. We first construct such reflection walls in $V$ and on $S_{SH}$.

For the normalization purpose, we set two foci $\tilde{Z}_1 = (0, a, -1)$, and $\tilde{Z}_2 = (0, -a, -1)$ in $V$, then the norm $\| \cdot \|_a$ in $V$ should be chosen as (5).

We consider a centred elliptic cone given by

$$\frac{x^2}{\tan^2 \alpha} + \frac{y^2}{\tan^2 \beta} - z^2 = 0,$$

with $\alpha, \beta \in [0, \pi/2]$ such that

$$1 + a^2 = \frac{\tan^2 \beta + 1}{\tan^2 \alpha + 1}.$$  

The intersection of $V$ and the cone (17) gives a centred ellipse

$$\tilde{F} := \frac{x^2}{\tan^2 \alpha} + \frac{y^2}{\tan^2 \beta} - 1 = 0$$

defined in $V$. The foci $(0, c), (0, -c)$ of the ellipse $\tilde{F} = 0$ depends on the involved norm, and is computed as

$$\frac{c^2}{1 + a^2} = \frac{\tan \beta^2}{1 + a^2} - \tan^2 \alpha \Leftrightarrow c^2 = a^2.$$ 

This means the foci are at two centres $\tilde{Z}_1$ and $\tilde{Z}_2$, thus the ellipse $\tilde{F} = 0$ is confocal.

From the first and the second statement of this proposition, the intersection of this elliptic cone (17) and $S_{SH}$ is again a confocal ellipse on $S_{SH}$ and is given by the equation

$$F := \frac{x^2}{\sin^2 \alpha} + \frac{y^2}{\sin^2 \beta} - 1 = 0.$$  

(19)
To see the projective correspondence of elastic reflections, we show that velocities before and after the reflection at $F = 0$ on $S_{SH}$ is projected to velocities before and after the reflection at $\tilde{F} = 0$ in $V$. Unfortunately we have not found a geometrical way to see this. Here we provide a proof with a direct computation.

Set

$$q := (x, y, z) = \left( \sin \alpha \cos \theta, \sin \beta \sin \theta, -\sqrt{1 - \sin^2 \alpha \cos^2 \theta - \sin^2 \beta \sin^2 \theta} \right)$$

which lies in a confocal ellipse $F = 0$ on $S_{SH}$. The tangent vector to the ellipse at the point $q$ is given by

$$s := \left( -\sin \alpha \sin \theta, \sin \beta \cos \theta, \frac{(\sin^2 \alpha - \sin^2 \beta) \sin \theta \cos \theta}{\sqrt{1 - \sin^2 \alpha \cos^2 \theta - \sin^2 \beta \sin^2 \theta}} \right),$$

and the normal vector is given by

$$n := \left( \frac{\sin \alpha \cos \theta}{\tan \alpha}, \frac{\sin \beta \sin \theta}{\tan \beta}, \frac{1}{\sqrt{1 - \sin^2 \alpha \cos^2 \theta - \sin^2 \beta \sin^2 \theta}} \right).$$

When the velocity vectors before the reflection at $q$ is given as

$$v = k_1 \cdot s + k_2 \cdot n,$$

where $k_1, k_2 \in \mathbb{R}$ are coefficients, then the reflected vector becomes as

$$w = k_1 \cdot s - k_2 \cdot n.$$

Clearly, tangent vectors are projected to tangent vectors along the reflection walls. To see that $v$ and $w$ are projected to velocities before and after the elastic reflection at the corresponding point $\tilde{q}$ in $\tilde{F} = 0$, we observe that it suffices to check that the normal vector $n$ is projected to the corresponding normal vector at $\tilde{q} \in V$ with respect to the corresponding metric on $V$, since then $v$ and $w$ are projected to vectors in $V$ having the same tangential component and opposite normal components.

The point $q$ lying in $F = 0$ is projected to the point

$$\tilde{q} := \left( \frac{x}{-x}, \frac{y}{-y}, -1 \right) = \left( \frac{\sin \alpha \cos \theta}{\sqrt{1 - \sin^2 \alpha \cos^2 \theta - \sin^2 \beta \sin^2 \theta}}, \frac{\sin \beta \sin \theta}{\sqrt{1 - \sin^2 \alpha \cos^2 \theta - \sin^2 \beta \sin^2 \theta}}, -1 \right)$$

lying in $\tilde{F} = 0$.

The corresponding push-forward transformation from $T_q S_{SH}$ to $T_{\tilde{q}} V$ is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{(\cos^2 \alpha - \cos^2 \beta) \cos^2 \theta + \cos^2 \beta}} & 0 & \sin \alpha \cos \theta \\ 0 & 1 & \frac{\cos^2 \alpha - \cos^2 \beta \cos^2 \theta + \cos^2 \beta}{\sqrt{(\cos^2 \alpha - \cos^2 \beta) \cos^2 \theta + \cos^2 \beta}} \\ \frac{\sin \beta \cos \theta}{\sqrt{(\cos^2 \alpha - \cos^2 \beta) \cos^2 \theta + \cos^2 \beta}} & \frac{\cos^2 \alpha - \cos^2 \beta \cos^2 \theta + \cos^2 \beta}{\sqrt{(\cos^2 \alpha - \cos^2 \beta) \cos^2 \theta + \cos^2 \beta}} & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$
Using this, the tangent vector \( s \) is projected to the (tangent) vector
\[
\tilde{s} = \frac{1}{(\cos^2 \alpha - \cos^2 \beta) \cos^2 \theta + \cos^2 \beta}(-\sin \alpha \cos^2 \beta \sin \theta, \sin \beta \cos^2 \alpha \cos \theta)
\]
and the normal vector \( n \) is projected to the vector
\[
\tilde{n} = \frac{1}{\sin \alpha \sin \beta (\cos^2 \alpha - \cos^2 \beta) \cos^2 \theta + \cos^2 \beta} \left( \sin \beta \cos \theta, \sin \alpha \sin \theta \right).
\]
We ignore the factors and take
\[
\hat{s} = \left( -\sin \alpha \cos^2 \beta \sin \theta, \sin \beta \cos^2 \alpha \cos \theta \right), \]
\[
\hat{n} = \left( \sin \beta \cos \theta, \sin \alpha \sin \theta \right).
\]
Their inner product with respect to \( || \cdot ||_a \) is
\[
\langle \hat{s}, \hat{n} \rangle_a = -\sin \alpha \sin \beta \cos^2 \beta \sin \theta \cos \theta + \frac{\tan^2 \alpha + 1}{\tan^2 \beta + 1} \sin \alpha \sin \beta \cos^2 \alpha \sin \theta \cos \theta = 0.
\]
They are thus orthogonal. Hence, the projection \( \hat{n} \) of \( n \) is indeed a normal vector at \( \tilde{q} \in \{ F = 0 \} \) in \( V \).

Thus, the law of reflection at centred confocal ellipses in \( V \) and the law of reflection at centred confocal ellipses on \( \mathcal{S}_{SH} \) correspond to each other. The case of reflections at centred confocal hyperbolae is completely analogous.

4.2. Integrability of Lagrange billiards with confocal conic section reflection walls

We now prove theorem 1 for the planar and spherical problems.

**Proof.** From proposition 4, we know the spherical and planer law of reflection at centred confocal conic sections are in correspondence, meaning that the incoming and the outgoing velocity vectors of an elastic reflection against such reflection walls in the plane are projected again to the incoming and outgoing velocity vectors of an elastic reflection against the corresponding reflection walls on the sphere, up to a time change which depends only on the point of reflection. Therefore the billiard trajectories on the sphere are projected to billiard trajectories in the plane in our situation, in which the underlying mechanical systems are in correspondence. As a consequence, the energy of the spherical system, written in the gnomonic chart \( V \), is invariant under the reflections at a corresponding confocal conic section in \( V \). Also, the energy of the planar system, while being expressed on \( \mathcal{S}_{SH} \) and further extended to \( \mathcal{S} \), is invariant under the reflections on \( \mathcal{S} \) at a corresponding confocal conic section on the sphere. We get additional first integrals for both billiard systems independent of their energies. The proof is completed.

4.3. Subcases of integrable Lagrange billiards

4.3.1. The integrable free billiards. The case \( m_1 = m_2 = f = 0 \) of the system (8), and the case \( \hat{m}_1 = \hat{m}_2 = f = 0 \) of the system (9) correspond respectively to the cases of free motions in the plane and on the sphere. We recover the classical theorem of Birkhoff in the planar and spherical case.

**Corollary 1.** The free billiards in the plane and on the sphere with any finite combination of confocal conic section reflection walls are integrable.
4.3.2. The integrable Hooke billiards. The case $m_1 = m_2 = 0, f \neq 0$ of the system (8), and the case $\hat{m}_1 = \hat{m}_2 = 0, f \neq 0$ of the system (9) correspond respectively to the Hooke problems in the plane and on the sphere. In this case we recover the following theorem:

**Corollary 2.** The Hooke billiards in the plane and on the sphere with any finite combination of confocal conic section reflection walls centred at the Hooke centre are integrable.

4.3.3. The integrable Kepler billiards. The case $m_1 = 0, m_2 = 0$ of the system (8), and the case $\hat{m}_1 = \hat{m}_2 = 0$ of the system (9) correspond respectively to the Kepler problems in the plane and on the sphere. In this case we recover the following theorem:

**Corollary 3.** The Kepler billiards in the plane and on the sphere with any finite combination of confocal conic section reflection walls focused at the Kepler centre are integrable.

4.3.4. The integrable two-centre billiards. The case $m_1, m_2 = 0, f = 0$ of the system (8), and the case $\hat{m}_1, \hat{m}_2 = 0, f = 0$ of the system (9) correspond respectively to the two-centre problems in the plane and on the sphere. In this case we recover the following theorem:

**Theorem 3.** The billiards defined with the two-centre problems in the plane and on the sphere with any finite combination of confocal conic section reflection walls focused at the two centres are integrable.

4.3.5. The integrable billiards with superposition of Hooke and Kepler Problems. The case $m_1, f \neq 0, m = 0$ of the system (8), and the case $\hat{m}_1, f \neq 0, \hat{m}_2 = 0$ of the system (9) correspond respectively to the superposition of a Hooke and a Kepler problems in the plane and on the sphere. In this case we recover the following theorem:

**Corollary 4.** The billiards defined with the superposition of a Hooke and a Kepler problems in the plane and on the sphere with any finite combination of confocal conic section reflection walls focused the Kepler centre and centred at the Hooke centre are integrable.

5. The plane-hyperboloid projection and integrable Lagrange billiards in the hyperbolic plane

We now discuss the projection between the plane and the hyperbolic space, with the hyperboloid model for the latter.

5.1. The hyperboloid-plane projection

We consider the Minkowski space $\mathbb{R}^{2,1}$, equipped with the pseudo-Riemannian metric

$$\mathrm{d}t^2 + \mathrm{d}y^2 - \mathrm{d}z^2. \quad (20)$$

Consider the embedded two-sheeted hyperboloid given by the equation

$$\mathcal{H} := \{(x, y, z) \in \mathbb{R}^{2,1} \mid x^2 + y^2 - z^2 = -1\}$$

and its lower sheet

$$\mathcal{H}_S := \{(x, y, z) \in \mathcal{H} \mid z < 0\}.$$
The restriction of the pseudo-Riemannian metric $dx^2 + dy^2 - dz^2$ to $\mathcal{H}$ is Riemannian, and equipped both sheets of $\mathcal{H}$ with a hyperbolic metric. The space $\mathcal{H}_S$ equipped with this hyperbolic metric is called the hyperboloid model of the hyperbolic plane.

We consider the plane $V_H = \{z = -1\} \subset \mathbb{R}^{2,1}$ which is tangent to $\mathcal{H}_S$ at its pole $(0,0,-1)$. The central projection from the origin of $\mathbb{R}^{2,1}$ projects the lower sheet of hyperboloid $\mathcal{H}_S$ onto the unit disc $D := \{(x,y) \in V_H \mid x^2 + y^2 < 1\}$ in $V$, and equips $D$ with an induced hyperbolic metric, making it the Klein disc model for the hyperbolic plane.

We denote by $\| \cdot \|_{\mathcal{H}}$ the Minkowski norm in $\mathbb{R}^{2,1}$. Just as in the case of spherical-plane correspondence in section 3.1, a force field $F_H$ on $\mathcal{H}_S$ is carried to a force field $F_V$ on $V$ by the central projection.

Indeed, in this setting, a point $q \in \mathcal{H}_S$ is centrally projected to the point $\tilde{q} \in V_H$:

$$q = \|\tilde{q}\|_{\mathcal{H}}^{-1} \tilde{q}.$$ 

Suppose we have a natural mechanical system in $V_H$ with the equations of motion

$$\ddot{\tilde{q}} = F_V(\tilde{q}).$$

Thus we have

$$\dot{q} = \|\tilde{q}\|_{\mathcal{H}}^{-2} (\dot{\tilde{q}}\|\tilde{q}\|_{\mathcal{H}} - \langle \nabla \|\tilde{q}\|_{\mathcal{H}}, \dot{\tilde{q}} \rangle \tilde{q}).$$

Again, we take a new time variable $\tau$ for the system on $\mathcal{H}_S$, and write $' := \frac{d}{d\tau}$ so that

$$\frac{d}{d\tau} = \|\tilde{q}\|_{\mathcal{H}}^2 \frac{d}{d\tau}$$

and consequently

$$q' = (\dot{\tilde{q}}\|\tilde{q}\|_{\mathcal{H}} - \langle \nabla \|\tilde{q}\|_{\mathcal{H}}, \dot{\tilde{q}} \rangle \tilde{q}),$$

$$q'' = \|\tilde{q}\|_{\mathcal{H}}^2 \left(\ddot{\tilde{q}}\|\tilde{q}\|_{\mathcal{H}} - \left(\langle \nabla \|\tilde{q}\|_{\mathcal{H}}, \dot{\tilde{q}} \rangle + \langle \nabla \|\tilde{q}\|_{\mathcal{H}}, \dot{\tilde{q}} \rangle \right) \tilde{q}\right).$$

We thus have

$$q'' = \|\tilde{q}\|_{\mathcal{H}}^2 (F_V(\tilde{q}) \|\tilde{q}\|_{\mathcal{H}} - \lambda (\tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) \tilde{q})$$

in which we have set $\lambda (\tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) = \langle \nabla \|\tilde{q}\|_{\mathcal{H}}, \dot{\tilde{q}} \rangle + \langle \nabla \|\tilde{q}\|_{\mathcal{H}}, \ddot{\tilde{q}} \rangle$. The gradient and the inner product are defined with respect to the pseudo-Riemannian metric (20).

The Levi–Civita connection of a pseudo-Riemannian manifold projects to the Levi-Civita connection of its embedded submanifold. In our case, $\mathcal{H}_S$ is Riemannian with the induced metric from $\mathbb{R}^{2,1}$. So again by projecting both sides of this equation to the tangent space $T_q \mathcal{H}_S$, we get the equations of motion of the form

$$\nabla_q q' = F_H(q).$$

We see that to switch from the plane-sphere correspondence as in section 3.1 to the plane-hyperboloid correspondence with our setting, it is enough to properly change some signs in proper places while the others are completely similar. We shall make use of this similarity in the sequel to omit certain details.
5.2. Projective properties of the Hooke and Kepler problems in the hyperbolic plane

5.2.1. The Kepler problems and the Hooke problems. We first discuss the case of the Kepler problems. The hyperbolic Kepler problem is the natural mechanical system \((\mathcal{H}_S, g_H, \hat{m} \coth \theta_Z)\), in which \(g_H\) is the induced hyperbolic metric on \(\mathcal{H}_S\), \(\hat{m} \in \mathbb{R}\) is the mass-factor and the angle \(\theta_Z\) is the central hyperbolic angle the moving particle made with the centre \(Z \in \mathcal{H}_S\).

Without loss of generality, we set \(Z = \left(0, \frac{a}{\sqrt{1-a^2}}, -\frac{1}{\sqrt{1-a^2}}\right) \in \mathcal{H}_S\) for \(a \in (-1, 1)\), and \(\hat{Z} = (0, a, -1)\) the projection point of \(Z\) in \(D \subset V_H\). For \(\hat{q} = (x, y, -1) \in V_H\) we define

\[
\|\hat{q}\|_a = \sqrt{\hat{x}^2 + \frac{\hat{y}^2}{1-a^2}}. \tag{23}
\]

Similar to the case of plane-spherical correspondence, we have

**Proposition 5.** The hyperbolic Kepler problem \((\mathcal{H}_S, g_H, \hat{m} \coth \theta_Z)\) projects to the planar Kepler problem \((V_H, \| \cdot \|_a, m/\|q - Z\|_a)\) such that \(m = \frac{\hat{m}}{\sqrt{1-a^2}}\).

By analyticity, a proof of this proposition follows from proposition 1 by formally substituting \((x, y, a, z)\) by \((ix, iy, i, z)\) and argue with the equations of motion. The geometric proof of proposition 1 also carries over to this hyperbolic case, but now using hyperbolic geometry.

Our second case is the Hooke problems. The hyperbolic Hooke problem is the natural mechanical systems given by \((\mathcal{H}_S, g_H, f \tanh^2 \theta_Z)\) with the mass-factor \(f \in \mathbb{R}\).

Analogously as in the Kepler case, we get the following correspondences between Hooke systems.

**Proposition 6.** The hyperbolic Hooke problem \((\mathcal{H}_S, g_H, f \tanh^2 \theta_Z)\) with \(Z = (0, 0, -1)\) projects to any of the Hooke problems in \(V\) of the form \((V_H, \| \cdot \|_a, f \|\hat{q}\|_a^2)\) for any \(a \in \mathbb{R}\).

In contrast to the Kepler case, we can freely choose the parameter \(a\) in the affine changed norm \(\| \cdot \|_a\) for the Hooke problems.

5.3. The Lagrange problems in the plane and in the hyperbolic plane

By superposing two hyperbolic Kepler problems and a hyperbolic Hooke problem, we obtain the hyperbolic Lagrange problem

\[
(\mathcal{H}, g_H, \hat{m}_1 \coth \theta_{Z_1} + \hat{m}_2 \coth \theta_{Z_2} + f \tanh^2 \theta_{Z_{mid}}),
\]

for which we assume that \(Z_1\) and \(Z_2\) are in the same sheet of two-sheeted hyperboloid \(\mathcal{H}\). Here, \(\theta_j\) is a hyperbolic central angle of the moving particle to a point \(P \in \mathcal{H}\).

By combining the previous propositions 5 and 6, we get the following correspondence on the Lagrange problems in the plane and in the hyperbolic plane as an analogy of the spherical case.

**Theorem 4.** In the case \(Z_{mid}\) is vertical, then the hyperbolic Lagrange problem on \(S_H\) with masses \(m_1, m_2, f \in \mathbb{R}\) is projected to the planer Lagrange problem in \(V_H\), with the projections of the Kepler and the Hooke centres as its own Kepler and Hooke centres, with the affine norm \(\| \cdot \|_a\) and parameters \(m_1, m_2, f\) as determined by proposition 5.

From this theorem, we get the following proposition as a consequence.

**Proposition 7.** The energy of the hyperbolic Lagrange problem induces an additional first integral for the planer Lagrange problem independent of its energy. Vice versa, the energy of
the planer Lagrange problem induces an additional first integral for the hyperbolic Lagrange problem independent of its energy.

The additional first integral for the hyperbolic Lagrange problem induced from the planer energy is given by

\[
\left((1 - a^2) x^2 + y^2\right) z^2 - 2z'\left((1 - a^2) xx' + yy'\right) z + z'^2 \left((1 - a^2) x^2 + y^2\right)
\]

\[
- \frac{m_1}{\sqrt{\frac{z}{1 - a^2}}} - \frac{m_2}{\sqrt{\frac{z}{1 - a^2}}} - f\left((1 - a^2) x^2 + y^2\right)ight),
\]

(24)

where \(m_1 = \frac{\partial u}{\sqrt{1 - a^2}}\) and \(m_2 = \frac{\partial u}{\sqrt{1 - a^2}}\).

5.4. Integrable Lagrange billiards in the hyperbolic plane

We here consider the presence of a confocal conic section reflection wall \(\tilde{B}\) in \(V_H\) and its corresponding reflection wall \(B\) in \(H_S\). The proof goes analogously as in the case of plane-spherical correspondence.

Proposition 8. Any confocal conic section \(\tilde{B} \subset V_H\) is projected to a confocal conic section \(B \subset H_S\). The foci of \(B\) are the projection of the foci of \(\tilde{B}\) by the central projection. The law of reflection at \(\tilde{B}\) with respect to a compatible \(\| \cdot \|_+\) and the law of reflection at \(B \subset H_S\) correspond each other.

5.5. Proof of theorem 1 in the hyperbolic case and the subcases

With all these ingredients, the proof of theorem 1 for the spherical and planar case from section 4.2 carries directly to the hyperbolic case as well, which completes the proof of theorem 1 in all cases.

Also, the subcases as listed in section 4.3 carries to integrable systems defined on the hyperbolic plane as well.

6. The complex square mapping and Hooke–Kepler correspondence in the hyperbolic space and on the sphere

The classical conformal correspondence between the planar Hooke and Kepler problems via the complex square mapping has been generalized to conformal correspondences among the Hooke problems defined on the sphere, in the hyperbolic plane, and the Kepler problem defined in the hyperbolic plane by Nersessian and Pogosyan [25]. We explain that these conformal correspondences extend to integrable billiards defined with these natural mechanical systems.

We take the plane \(\{z = 0\}\) as a stereographic chart from the North pole \((0, 0, 1)\) of the unit sphere \(S\). For the hyperbolic plane we take the Poincaré disc model in the unit disc in the plane \(\{z = 0\}\), seen as projection of the hyperboloid model from the ‘North pole’ \((0, 0, 1)\). We identify the plane \(\{z = 0\}\) with \(\mathbb{C}\) in which the Poincaré disc is \(D := \{w \in \mathbb{C} | |w| < 1\}\).

The round metric on \(S\) is represented in the stereographic chart as

\[
\frac{4}{(1 + |q|^2)^2} dq dq.
\]

(25)
Analogously the Poincaré disk \( \mathcal{D} \) is equipped with the hyperbolic metric

\[
\frac{4}{(1 - |q|^2)} dq \, d\bar{q}.
\]  

(26)

The spherical kinetic energy in the stereographic chart is thus

\[
\frac{(1 + |q|^2)^2 |p|^2}{8}
\]

by using the cometric of (25). In this stereographic chart, the force functions of the spherical Hooke and spherical Kepler problems are given respectively as

\[
-\frac{4f |q|^2}{(1 - |q|^2)^2}
\]

and

\[
\hat{m} \frac{1 - |q|^2}{2|q|},
\]

respectively, with \( f, \hat{m} \in \mathbb{R} \). Analogously, the hyperbolic kinetic energy in the Poincaré disk \( \mathcal{D} \) is

\[
\frac{(1 - |q|^2)^2 |p|^2}{8},
\]

with the force functions of the hyperbolic Hooke and hyperbolic Kepler problems

\[
-\frac{4f |q|^2}{(1 + |q|^2)^2}
\]

and

\[
\hat{m} \frac{1 + |q|^2}{2|q|},
\]

respectively, with \( f, \hat{m} \in \mathbb{R} \).

**Proposition 9.** (Nersessian–Pogosyan [25]) The spherical Hooke problem, the hyperbolic Hooke problem, and the hyperbolic Kepler problem are mutually in conformal correspondence.

**Proof.** We start with the Hamiltonian of the spherical/hyperbolic Hooke problem

\[
\frac{(1 \pm |z|^2)^2 |w|^2}{8} + \frac{4f |z|^2}{(1 \mp |z|^2)^2} - \hat{m} = 0
\]

restricted to its \( \hat{m} \)-energy hypersurface. The signs determine whether it is the spherical or the hyperbolic problem we are considering. By multiplying both sides by \( \frac{1 \mp |z|^2}{|z|^2} \), we get

\[
\frac{(1 - |z|^4)^2 |w|^2}{8|z|^2} + 4f - \hat{m} \left( \frac{1 \mp |z|^2}{|z|^2} \right)^2 = 0.
\]
We now apply the conformal transformation \((z, w) \mapsto (z^2, w/2\bar{z}) : = (p, q)\) and the transformed Hamiltonian becomes
\[
\frac{(1-|q|^2)^2}{8}|p|^2 + 4f - m \frac{(1+|q|^2)^2}{|q|} = 0
\]
after a proper time change. As we can rewrite this system into
\[
\frac{(1-|q|^2)^2}{8}|p|^2 + 4f - m \frac{1+|q|^2}{|q|} \pm 2\hat{m} = 0,
\]
this is the Hamiltonian of a hyperbolic Kepler problem restricted to the energy level with energy \(-(4f \pm 2\hat{m})\).

The same trick, with a multiplicative factor of \(\frac{(1+|q|^2)^2}{(1-|q|^2)^2}\) gives a transformation between the spherical and hyperbolic Hooke problems restricted to energy levels.

**Corollary 5.** In the Poincaré disc in the plane \(\{z = 0\} \cong \mathbb{C}\), the curve representing a branch of a conic section on the hyperboloid model focused at the ‘South pole’ \((0, 0, -1)\) is transformed via the complex square mapping \(\mathbb{C} \to \mathbb{C} : z \mapsto z^2\) into a curve simultaneously representing a conic section centred at the ‘South pole’ on the hyperboloid model, and part of a conic section defined on the hemisphere \(S^+\) centred at the South pole.

**Proof.** This follows from proposition 9, which implies that an orbit of the hyperbolic Kepler problem is sent to an orbit of the spherical/hyperbolic Hooke problem up to a time parametrization. Thus the conclusion of the corollary follows.

We now show that any confocal family of centred spherical/hyperbolic conic sections is transformed into a confocal family of focused hyperbolic conic sections by this series of conformal transformations.

**Proposition 10.** A family of confocal focused hyperbolic conic sections on \(H_S\), expressed in the Poincaré disc \(D\) are transformed into a family of confocal centred spherical/hyperbolic conic sections in the stereographic chart/Poincaré disc in the plane \(\{z = 0\} \cong \mathbb{C}\) via the complex square mapping \(\mathbb{C} \to \mathbb{C} : z \mapsto z^2\).

**Proof.** We start with a family of confocal focused hyperbolic conic section on \(H_S\). Choose a parameter \(0 < a < 1\), and suppose that a family of such hyperbolic conic sections has common centres at \((0, \frac{a\sqrt{1-a^2}}{\sqrt{1-a^2}}, -\frac{1}{\sqrt{1-a^2}})\).

We take a new set of orthogonal coordinates in the Minkowski space \(\mathbb{R}^{2,1}\) as
\[
\begin{align*}
u &= x, \quad \nu = \frac{y + az}{\sqrt{1-a^2}}, \quad w = \frac{ay + z}{\sqrt{1-a^2}}.
\end{align*}
\]
The pseudo-Riemannian metric defined by (20) is expressed in these new coordinates as
\[
d\nu^2 + d\nu^2 - dw^2.
\]
In this coordinates, the two-sheeted hyperboloid is given by the equation
\[
\mathcal{H} = \{(u, v, w) \in \mathbb{R}^{2,1} \mid u^2 + v^2 - w^2 = -1\}.
\]
The plane \( \dot{V} := \{ w = -1 \} \) is tangent to the hyperboloid \( H \) at the point \((u, v, w) = (0, 0, -1)\). We equip \( V \) with the norm \( \| \cdot \|_v \) defined as
\[
\| (\tilde{u}, \tilde{v}) \|_v^2 = \tilde{u}^2 + \frac{\tilde{v}^2}{1 - a^2}
\]
for \((\tilde{u}, \tilde{v}) \in \dot{V}\). We now consider the family of confocal centred ellipses in \( \dot{V} \) with foci at \((0, -a, -1)\) and \((0, a, -1)\) (with respect to \( \| \cdot \|_v \)) given by the equation
\[
\frac{\tilde{u}^2}{R^2-A} + \frac{\tilde{v}^2}{B^2} = 1 = 0,
\]
where \( B > a \) is a positive parameter.

We now project this family of confocal centred ellipses in \( \dot{V} \) to the hyperboloid by the central projection. Let \((u, v, w) \in H_3\) be the centrally projected point of \((\tilde{u}, \tilde{v}, -1)\). Then we have
\[
\tilde{u} = -\frac{u}{w}, \quad \tilde{v} = -\frac{v}{w}
\]
and the transformed expression of the family of confocal ellipses is given by
\[
\frac{u^2}{w^2(1-a^2)} + \frac{v^2}{w^2B^2} - 1 = 0.
\]
(28)

As an implication of the projective correspondence of the hyperbolic Kepler problem and the planer Kepler problem, the central projection projects the hyperbolic conic sections to conic sections in the plane and projects foci to foci when they are centred. Thus, the projected conic sections on \( H_3 \) is again confocal.

In the original coordinates \((x, y, z) \in \mathbb{R}^{2,1}\), the equation (28) can be written as
\[
\frac{x^2}{(ay+z)^2(B^2-a^2)} + \frac{(y+az)^2}{(ay+z)^2B^2} - 1 = 0.
\]

We now rewrite this in the coordinates \((q_1, q_2)\) in the Poincaré disc \( D \) with the stereographic projection
\[
x = \frac{2q_1}{1-q_1^2-q_2^2}, \quad y = \frac{2q_2}{1-q_1^2-q_2^2}, \quad z = -\frac{1+q_1^2+q_2^2}{1-q_1^2-q_2^2},
\]
which transforms the equation of the confocal focused hyperbolic conic sections in the Poincaré disc \( D \) into
\[
\frac{4(1-a^2)q_1^2}{(B^2-a^2)(2aq_2-q_1^2-q_2^2-1)^2} + \frac{(-2q_2+a(q_1^2+q_2^2+1))^2}{B^2(2aq_2-q_1^2-q_2^2-1)^2} - 1 = 0.
\]

We now apply the complex square mapping. Set
\[
q_1 + iq_2 = (z_1 + iz_2)^2
\]
and the above equation is now
\[
\frac{4(1-a^2)^2((-z_1^4-2z_1^2z_2^2-z_2^4+4az_1z_2^2-1)^2+(-4z_1z_2+a(z_1^4+2z_1^2z_2^2+z_2^4+1))^2}{B^2(-z_1^4-2z_1^2z_2^2-z_2^4+4az_1z_2^2-1)^2} - 1 = 0.
\]
Suppose that \((z_1, z_2) \in \mathcal{D}\) corresponds to the point \((x, y, z) \in S_{24}\) via stereographic projection:

\[
z_1 = -\frac{x}{z}, \quad z_2 = -\frac{y}{z}.
\]

Then the above equation can be equivalently written as

\[
4 \left(1 - a^2\right)^2 \left(1 + z^2\right)^2 (x^2 + y^2)^2
\]

\[
(B + a) (B - a) \left(z^4 - 4z^3 + (-4xya + 6)z^2 + (8xya - 4)z + x^4 + 2x^2y^2 + y^4 - 4axy + 1\right)^2
\]

\[
+ \left(z^4a - 4z^3a + (-4xy + 6a)z^2 + (8xy - 4a)z + (x^4 + 2x^2y^2 + y^4 + 1)a - 4xy\right)^2
\]

\[
+ \frac{B^2 (z^4 - 4z^3 + (-4xy + 6)z^2 + (8xya - 4)z + x^4 + 2x^2y^2 + y^4 - 4axy + 1)^2}{2} - 1 = 0.
\]

In order to see that this equation determines spherical conic sections with common centres at the ‘South pole’ and common foci, we project them to the plane \(V = \{z = -1\}\) by the central projection and examine their images therein. In the gnomonic chart \(V\), the above equation is expressed with coordinates \((\hat{x}, \hat{y}, -1) \in V\) as

\[
x = \frac{\hat{x}}{\sqrt{\hat{x}^2 + \hat{y}^2 + 1}}, \quad y = \frac{\hat{y}}{\sqrt{\hat{x}^2 + \hat{y}^2 + 1}}, \quad z = -\frac{1}{\sqrt{\hat{x}^2 + \hat{y}^2 + 1}}.
\]

By using Maple, this can be factorized into

\[
4 \left(4\hat{x}^2 + 4\hat{y}^2 + 8\right) \sqrt{\hat{x}^2 + \hat{y}^2 + 1} + \hat{x}^4 + (2\hat{y}^2 + 8)\hat{x}^2 + \hat{y}^4 + 8\hat{y}^2 + 8 \right) (1 + \hat{x}^2 + \hat{y}^2)
\]

\[
\times \left( -\frac{(-a^2 + B) (B + 1) \hat{x}^2}{2} + a\hat{y} (B - 1) (B + 1)\hat{x} - \frac{(-a^2 + B) (B + 1) \hat{y}^2}{2} - B^2 + a^2 \right).
\]

\[
\times \left( -\frac{(a^2 + B) (B - 1) \hat{x}^2}{2} + a\hat{y} (B - 1) (B + 1)\hat{x} - \frac{(a^2 + B) (B + 1) \hat{y}^2}{2} - B^2 + a^2 \right) = 0
\]

The factors in the first line only take positive value. Thus, we only consider the last two factors:

\[
G_1 := -\frac{(-a^2 + B) (B + 1) \hat{x}^2}{2} + a\hat{y} (B - 1) (B + 1)\hat{x} - \frac{(-a^2 + B) (B + 1) \hat{y}^2}{2} - B^2 + a^2
\]

and

\[
G_2 := -\frac{(a^2 + B) (B - 1) \hat{x}^2}{2} + a\hat{y} (B - 1) (B + 1)\hat{x} - \frac{(a^2 + B) (B + 1) \hat{y}^2}{2} - B^2 + a^2.
\]

In the rotated coordinates \(\hat{X} = \frac{\hat{x} + \hat{y}}{\sqrt{2}}, \hat{Y} = \frac{\hat{x} - \hat{y}}{\sqrt{2}}\), they can be rewritten into

\[
G_1 = \frac{\hat{X}^2}{2(\hat{B} - \hat{a})} + \frac{\hat{Y}^2}{2(\hat{B} + \hat{a})} - 1
\]

and

\[
G_2 = \frac{\hat{X}^2}{2(\hat{B} + \hat{a})} + \frac{\hat{Y}^2}{2(\hat{B} - \hat{a})} - 1.
\]
Notice that $G_1 = 0$ contains no real points, since the coefficients of $\tilde{X}^2, \tilde{Y}^2$ are both negative. Hence, only $G_2 = 0$ determines centred conic sections in $V$. We now compute the positions of their foci by taking the affine change of the norm on $V$ into account. Suppose that the foci of $G_2 = 0$ are located at $(\tilde{X}, \tilde{Y}) = (\pm c, 0)$, then the norm $\| \cdot \|_c$ in $V$ which depends on the positions of foci is necessarily defined as

$$\| (\tilde{X}, \tilde{Y}) \|^2_c = \frac{\tilde{X}^2}{1 + c^2} + \tilde{Y}^2.$$ 

This means we have the following equation in terms of $c$:

$$\frac{c^2}{1 + c^2} = \frac{2(b^2 - a^2)}{(a-1)(B-1)(B-a)} - \frac{-2(B^2 - a^2)}{(a+1)(B-1)(B+a)}.$$

By solving this with respect to $c$, we obtain

$$c = \pm \frac{2\sqrt{a}}{1 - a}$$

which depends only on $a$. Therefore, the equation $G_2 = 0$ determines a family of confocal central conic sections in $V$. From this fact and the projective correspondence of the spherical Kepler problem and the planer Kepler problem, we conclude that the equation (29) determines confocal centred spherical conic sections on $S_{3H}$.

Should we start from a family of confocal centred hyperbolae in $\hat{V}$ instead of ellipses, then we get the same type of results in a similar way. we thus conclude that a family of confocal focused hyperbolic conic sections are transformed into a family of confocal centred spherical conic sections.

Analogously, one can show that a family of confocal focused hyperbolic conic sections are transformed into a family of confocal centred hyperbolic conic sections.

Combining these results, we obtain the following proposition.

**Proposition 11.** The hyperbolic Kepler billiards with any finite combination of branches of confocal conic sections focused at the 'South pole' $(0, 0, -1)$ on the hyperboloid model as reflection wall are conformally transformed into the hemispherical/hyperbolic Hooke billiards with the corresponding combination of confocal conic sections reflection wall centred at the ‘South pole’ on the hemisphere/hyperboloid. Therefore their integrabilities are equivalent by [35].

**Data availability statement**

No new data were created or analysed in this study.

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ORCID ID

Airi Takeuchi 🐢 https://orcid.org/0000-0002-5213-2094

References

[1] Albouy A 2013 There is a projective dynamics EMS Newsl. 89 37–43
[2] Albouy A 2008 Projective dynamics and classical gravitation Regul. Chaot. Dyn. 13 525–42
[3] Birkhoff G D 1927 On the periodic motions of dynamical systems Acta Math. 50 359–79
[4] Boltzmann L 1868 Lösung eines mechanischen Problems Wiener Ber. 58 1035–44 Wissenschaftliche Abhandlungen vol 1 pp 97–105
[5] Dragovich V I 1998 Integrable perturbations of a Birkhoff billiard inside an ellipse J. Appl. Math. Mech. 62 159–62
[6] Dragović V 2002 The Appell hypergeometric functions and classical separable mechanical systems J. Phys. A: Math. Gen. 35 2213–21
[7] Dragović V, Jovanović B and Radnović M 2003 On elliptical billiards in the Lobachevsky space and associated geodesic hierarchies J. Geom. Phys. 47 221–34
[8] Dragović V and Radnović M 2006 Geometry of integrable billiards and pencils of quadrics J. Math. Pures Appl. 85 758–90
[9] Dragović V and Radnović M 2014 Bicentennial of the great Poncelet theorem (1813–2013): current advances Bull. Am. Math. Soc. 51 373–445
[10] Dragović V and Radnović M 2014 Pseudo-integrable billiards and arithmetic dynamics J. Mod. Dyn. 8 109–32
[11] Dragović V and Radnović M 2015 Periods of pseudo-integrable billiards Arnold Math. J. 1 69–73
[12] Fedorov Y N 2001 An ellipsoidal billiard with a quadratic potential Func. Anal. Appl. 35 199–208
[13] Felder G 2021 Poncelet property and quasi-periodicity of the integrable Boltzmann system Lett. Math. Phys. 111 1–19
[14] Gallavotti G and Jauslin I 2020 A theorem on ellipses, an integrable system and a theorem of Boltzmann (arXiv:2008.01955)
[15] Glutsyuk A A 2018 On two-dimensional polynomially integrable billiards on surfaces of constant curvature Dokl. Math. 98 382–5
[16] Glutsyuk A A 2020 On polynomially integrable Birkhoff billiards on surfaces of constant curvature J. Eur. Math. Soc. 23 995–1049
[17] Halphen G H 1878 Sur les lois de Kepler Bull. Soc. Phil. Paris 7 89–91
[18] Jacobi C G J 1866 Vorlesungen über Dynamik (Verlag von Georg Reimer)
[19] Jovanović B 2013 The Jacobi-Rosochatius problem on an ellipsoid: the Lax representations and billiards Arch. Ration. Mech. Anal. 210 101–31
[20] Kaloshin V and Sorrentino A 2018 On the local Birkhoff conjecture for convex billiards Ann. Math. 188 315–80
[21] Kozlov V V and Treshchëv D V 1991 Billiards. A Genetic Introduction to the Dynamics of Systems With Impacts (Translation of Mathematical Monographs vol 89) (American Mathematical Society) (https://doi.org/10.1090/mmono/089)
[22] Kozlov V V 1995 Some integrable extensions of Jacobi’s Problem of geodesics on an ellipsoid J. Appl. Math. Mech. 59 1–7
[23] Lagrange J L 1766–1769 Recherches sur le mouvement d’un corps qui est attiré vers deux centres fixes Second mémoire, VIII Miscellanea Taurinensia, t. IV (Œuvres Complètes Tome 2) pp 67–121
[24] Matveev V S and Topalov P 1998 Geodesic equivalence and integrability MPIM Preprint Series No. 74
[25] Nersessian A and Pogosyan G 2001 Relation of the oscillator and Coulomb systems on spheres and pseudospheres Phys. Rev. A 63 020103
[26] Porsitsky H 1950 The billiard ball problem on a table with a convex boundary—An illustrative dynamical problem Ann. Math. 51 446–70
[27] Pustovoitov S E 2019 Topological analysis of a billiard in an elliptic ring in a potential field Fundam. Prikl. Mat. 22 201–25
[28] Pustovoitov S E 2021 Topological analysis of a billiard bounded by confocal quadrics in a potential field Sb. Math. 212 211–33
[29] Radnović M 2015 Topology of the elliptical billiard with the Hooke’s potential Theor. Appl. Mech. 42 1–9
[30] Serret P 1860 Théorie Nouvelle Géométrique et Mécanique des Lignes à Double Courbure (Mallet-Bachelier)
[31] Tabachnikov S 1997 Exact transverse line fields and projective billiards in a ball Geom. Funct. Anal. GAFA 7 594–608
[32] Tabachnikov S 1997 Introducing projective billiards Ergod. Theory Dyn. Syst. 17 957–76
[33] Tabachnikov S 1999 Projectively equivalent metrics, exact transverse linefields and the geodesic flow on the ellipsoid Comment. Math. Helv. 74 306–21
[34] Tabachnikov S 2002 Ellipsoids, complete integrability and hyperbolic geometry Mosc. Math. J. 2 185–98
[35] Takeuchi A and Zhao L 2021 Conformal transformations and integrable mechanical billiards Adv. Math. (arXiv:2110.03376)
[36] Takeuchi A and Zhao L 2023 Integrable mechanical billiards in higher dimensional space forms (arXiv:2303.12443)
[37] Veselov A P 1990 Confocal surfaces and integrable billiards on the sphere and in the Lobachevsky space J. Geom. Phys. 7 81–107
[38] Zhao L 2021 Projective dynamics and an integrable Boltzmann billiard model Commun. Contemp. Math. 24 2150085