Dual forms of the orthogonality relations of some classical $q$-orthogonal polynomials

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Abstract

In this paper, by introducing new matrix operations and using a specific inverse relation, we establish the dual forms of the orthogonality relations for some well-known discrete and continuous $q$-orthogonal polynomials from the Askey-scheme such as the little and big $q$-Jacobi, $q$-Racah, (generalized) $q$-Laguerre, as well as the Askey-Wilson polynomials. As one of the most interesting results, we show that the Askey-Wilson $q$-beta integral represented in terms of the VWP-balanced $\phi_7$ series is just a dual form of the orthogonality relation of the Askey-Wilson polynomials.

Keywords: $q$-orthogonal polynomials; dual form; inverse relation; transformation; $(f,g)$-inversion formula; Askey-scheme.

MSC Classification: 33D15, 05A30

1. Introduction

In the past decades various methods of finding and proving summation and transformation formulas of hypergeometric series have been proposed. One of typical methods is the inverse relations investigated systematically by J. Riordan [20] at first time and developed thoroughly by Ch. Krattenthaler, L. C. Hsu and W. C. Chu, as well as others. The reader might consult [3, 4, 5, 8, 9, 13, 17, 10] and related references therein. Of these, it is particularly noteworthy that a series of research

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2Supported by the Natural Science Foundation of Zhejiang Province under (Grant No. LY24A010012) and the National Natural Science Foundation of China (Grant No. 12001492)
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works [5] by W. C. Chu displays that the inverse relations, as the celebrated WZ-method [19] does, is one of powerful tools to hypergeometric series. Indeed, in the rich world of summation and transformation formulas, many results seem totally different in form but are equivalent to each others up to inverse relations.

As we will see later, similar equivalences occur often in the orthogonality relations of orthogonal polynomials. Nevertheless, it has been lacking of systematic and full investigation so far. It is the problem that the present paper will attack. For a more general development of the theory of orthogonal polynomials, the reader might consult [11, 6, 2] by M. E. H. Ismail. Regarding applications of $q$-series to orthogonal polynomials, the reader is referred to [7, Chapter 7] for a good survey.

Our aim of this paper is to associate the orthogonality relations of discrete and continuous $q$-orthogonal polynomials listed in the $q$-Askey-scheme [12] with some summation and transformation formulas of basic hypergeometric series via the use of inverse relations, regardless of the cost of tedious calculations involved. As rewarding, the latter would at least, even if it is not always new to us, offer a possibly novel insight into discrete and continuous $q$-orthogonal polynomials.

### 1.1. Some preliminaries on inverse relations

At this stage, we had better give some explanations on the method of inverse relations. Generally speaking, the inverse relations is always related to a pair of reciprocal (or inverse) relations. Recall that a inverse relation or a (matrix) inversion formula in the context of combinatorial analysis, following up [14], is usually defined to be a pair of infinite-dimensional lower-triangular (in short, ILT) matrices $F = (f_{n,k})_{n,k \in \mathbb{N}}$ and $G = (g_{n,k})_{n,k \in \mathbb{N}}$, denoted by $(F,G)$, over the complex field $\mathbb{C}$ such that

$$\sum_{n \geq i \geq k} f_{n,i} g_{i,k} = \sum_{n \geq i \geq k} g_{n,i} f_{i,k} = \delta_{n,k} \text{ for all } n, k \in \mathbb{N},$$

(1.1)

where $\delta$ denotes the usual Kronecker delta, $\mathbb{N}$ is the set of nonnegative integers. Basic application of such a pair of inverse matrices is that it provides a standard technique for deriving new summation formulas from known ones. More precisely, assume that $(f_{n,k})$ and $(g_{n,k})$ are inverses of each other, then of course the transformation of two sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$

$$\sum_{k=0}^{n} f_{n,k} a_k = b_n \text{ is equivalent to } \sum_{k=0}^{n} g_{n,k} b_k = a_n.$$  

(1.2)

In other words, once a relation in (1.2) is known, then another comes true as its dual form. That is the routine way how to use the inverse relations.

Needless to say, finding the inverse $(g_{n,k})$, often denoted by $(f_{n,k})^{-1}$, enables us to establish even more summation formulas, which is the one of fundamental roles that
the inverse relations play in the theory of hypergeometric series. This problem, as pointed out by Ir. Gessel and D. Stanton [8, p.175, §2], is equivalent to the Lagrange inversion formula. As of today, there has been a more general inverse relation called the \((f,g)\)-inversion formula [15], since it contains all previously known results such as the Gould-Hsu inversion formula [9], the Krattenthaler inversion formula [14], as well as the Warnaar inversion formula [23] as special cases.

**Theorem 1.1.** [15, Theorem 1.3] Let \(f(x,y)\) and \(g(x,y)\) be two arbitrary functions over \(\mathbb{C}\) in variables \(x, y\) and \(\{b_i\}_{i \in \mathbb{N}}\) be such a sequence that none of the terms \(g(b_i, b_j)\) and \(f(x_i, b_j)\) in the denominator of the right side of (1.3a) and (1.3b) vanish. Suppose that \(g(x,y)\) is antisymmetric, i.e., \(g(x,y) = -g(y,x)\). Let \(F = (f_{n,k})_{n,k \in \mathbb{N}}\) and \(G = (g_{n,k})_{n,k \in \mathbb{N}}\) be two ILT matrices with the entries given by

\[
\begin{align*}
  f_{n,k} &= \frac{\prod_{i=k}^{n-1} f(x_i, b_k)}{\prod_{i=k+1}^{n} g(b_i, b_k)} \quad \text{and} \quad (1.3a) \\
  g_{n,k} &= \frac{f(x_k, b_k) \prod_{i=k+1}^{n} f(x_i, b_n)}{f(x_n, b_n) \prod_{i=k}^{n-1} g(b_i, b_n)} \quad \text{, respectively. (1.3b)}
\end{align*}
\]

Then \(F = (f_{n,k})_{n,k \in \mathbb{N}}\) and \(G = (g_{n,k})_{n,k \in \mathbb{N}}\) is a pair of inverse relations if and only if \(f(x,y)\) and \(g(x,y)\) satisfy that for any \(a, b, c, x \in \mathbb{C}\),

\[
g(b,c)f(x,a) + g(c,a)f(x,b) + g(a,b)f(x,c) = 0. \quad (1.4)
\]

Up to now, there have been found many pairs of \(f(x,y)\) and \(g(x,y)\) satisfying (1.4), each pair of which in turn yields an \((f,g)\)-inversion formula.

### 1.2. A special \((f,g)\)-inversion formula

To facilitate our discussions, we need an equivalent statement of Theorem 1.1. The reader is referred to [22] for its detailed proof.

**Lemma 1.2.** Let \(\{x_i\}_{i \in \mathbb{N}}\) and \(\{b_i\}_{i \in \mathbb{N}}\) be arbitrary sequences over \(\mathbb{C}\) such that \(b_i's\) are pairwise distinct, \(f(x,y)\) and \(g(x,y)\) are subject to (1.4). Then

\[
F_n = \sum_{k=0}^{n} G_k f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, b_n)}{\prod_{i=1}^{k} f(x_i, b_n)} \quad (1.5)
\]

if and only if

\[
G_n = \sum_{k=0}^{n} F_k \frac{\prod_{i=0}^{n-1} f(x_i, b_k)}{\prod_{i=0, i \neq k}^{n} g(b_i, b_k)}. \quad (1.6)
\]

This lemma covers a very useful inverse relation on which our argument relies heavily.
Proposition 1.3. Let $N(a)$ be the ILT matrix with the $(n, k)$-th entry given by

$$[N(a)]_{n,k} = \frac{(q^{-n}, aq^n; q)_k}{(q, aq; q)_k} q^k. \tag{1.7a}$$

Then the $(n, k)$-th entry of its inverse $N^{-1}(a)$ must be

$$[N^{-1}(a)]_{n,k} = \frac{(a, q^{-n}; q)_k}{(q, aq^{1+n}; q)_k} \frac{1 - aq^{2k}}{1 - a} q^{kn}. \tag{1.7b}$$

Proof. It suffices to take $f(x, y) = x - y, g(x, y) = x - y, b_i = q^{-i}, x_i = a q^i$ in Lemma 1.2. Note that $g(x, y) = x - y$ and $f(x, y) = x - y$ satisfy (1.4). \hfill \blacksquare

We remark that Proposition 1.3 is originally due to Carlitz [4] and appeared in a different version in the study of Bailey chains due to G. E. Andrews [1, Chapter 3], which has now been known as a $q$-analogue of the famous Gould-Hsu inversion formula [9].

1.3. Sketch of main idea and notation

As mentioned earlier, we will apply Proposition 1.3 to find any possibly new form of the orthogonality relation of discrete or continuous $q$-orthogonal polynomials arising mainly from the Askey-scheme [12]. Our motivation is built on a common feature: the orthogonality relations of many $q$-orthogonal polynomials listed in the Askey-scheme [12] are always identified with certain summation or transformation formulas. This universal phenomenon would not be transparent, as far as we are aware, unless viewed from the inverse relation (1.1).

Briefly, our argument can be summarized as follows: first to write the orthogonality relation of the given discrete orthogonal polynomials as the identity of the $(n, m)$-entries of certain matrix equation

$$[A(v_i \delta_{i,j}) A^T]_{n,m} = [(h_i \delta_{i,j})]_{n,m}. \tag{1.8}$$

It is more often than not that the matrix $A$ can be expressed as

$$A = CX,$$

while $C$ is of the form $D_1 Y D_2$, $Y \in \{N(a), N^{-1}(a)\}$, $N(a)$ is given by Proposition 1.3, $D_i (i = 1, 2)$ are diagonal. Thanks to this inverse relation, we are able to transform the matrix equation (1.8) by the inverse relations to

$$X (v_i \delta_{i,j}) X^T = C^{-1} (h_i \delta_{i,j}) C^{-T},$$

hereafter, the superscript $T$ denotes matrix transpose as usual. Accordingly, we have the relation often taken in form of hypergeometric series

$$[X (v_i \delta_{i,j}) X^T]_{n,m} = [C^{-1} (h_i \delta_{i,j}) C^{-T}]_{n,m}. \tag{1.9}$$
This is precisely what we want. Henceforth, (1.9) is called the \textit{dual form} of (1.8), since they are equivalent to each other.

The present paper, as further study of the \((f, g)\)-inversion formula \([15, 16]\), is organized as follows. In Section 2 we will apply the inverse relation \((N(a), N^{-1}(a))\) to find some dual forms of the orthogonality relations of some discrete \(q\)-orthogonal polynomials, among include the little \(q\)-Jacobi, \(q\)-Racah, and \(q\)-Laguerre orthogonal polynomials. Section 3 is devoted to dual forms of the orthogonality relations of some continuous \(q\)-orthogonal polynomials, such as the big \(q\)-Jacobi and the Askey-Wilson orthogonal polynomials. To that end, we need to introduce a kind of new matrix operations. In both cases, some new summation and transformation formulas are obtained as byproducts.

\textbf{Notations and conventions.} Throughout this paper we will use the standard notation and terminology for basic hypergeometric series as in G. Gasper and M. Rahman \([7]\). For instance, given a (fixed) complex number \(q\) with \(|q| < 1\), a complex number \(a\) and a nonnegative integer \(n\), denote the rising shifted factorial by

\[
(a; q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k), \quad (a)_n := \frac{(a; q)_\infty}{(aq^n; q)_\infty},
\]

\[
(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n.
\]

The \textit{basic hypergeometric series} with the base \(q\) and the argument \(z\) is defined by

\[
{}_{r+1}\phi_r \left[ \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array} ; q, z \right] := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} t_n (1 + s - r) z^n,
\]

where the nonnegative integers \(r \leq s + 1\) and \(t_n(k) = (-1)^{nk} q^{kn(n-1)/2}\). In particular, the \(r+1\phi_r\) series is said to be very-well-poised (in brief, VWP), if all parameters \(a_i\) and \(b_i\) are subject to the conditions that

\[
a_1 q = a_2 b_1 = a_3 b_2 = \cdots = a_{r+1} b_r; \quad a_2 = q\sqrt{a_1}, \quad a_3 = -q\sqrt{a_1}.
\]

Such \(r+1\phi_r\) series is often denoted by the more compact notation

\[
{}_{r+1}W_r(a_1; a_4, \ldots, a_{r+1}; q, z).
\]

In addition, for any sequence \(\{A_n\}_{n \in \mathbb{Z}}\), we will employ the convention of defining (cf. \([7, 3.6.12]\))

\[
\prod_{i=k}^{n} A_i := \begin{cases} A_k A_{k+1} \cdots A_n, & \text{if } n \geq k; \\ 1, & \text{if } n = k - 1; \\ 1/(A_{n+1} A_{n+2} \cdots A_{k-1}), & \text{if } n \leq k - 2 \end{cases}
\]

over the set of integers \(\mathbb{Z}\), the notation \(A = (a_{n,k})\) to denote the infinite matrix \(A\) with the \((n, k)\)-th entry \(a_{n,k}\), or in short, \([A]_{n,k} = a_{n,k}\).
2. Dual forms of the orthogonality relations of discrete $q$-Jacobi, $q$-Racah, and $q$-Laguerre polynomials

As planned, in this part we will apply Proposition 1.3 to pursuit for any possibly new equivalent forms of the orthogonality relations of discrete $q$-orthogonal polynomials from the Askey-scheme [12]. We content ourselves only with displaying a few typical discrete $q$-orthogonal polynomials from [12] as good examples to illustrate our idea described in Section 1.3.

2.1. The little $q$-Jacobi polynomials

Recall that the little $q$-Jacobi polynomials [12, (3.12.1)] are defined, for $a \not\in \{q^m : m \in \mathbb{Z}, m < 0\}$, by

\[ p_n(x; a, b; q) := \phi_1 \begin{array}{c} q^{-n}, abq^{n+1} \vspace{0pt} \end{array} \begin{array}{c} aq \vspace{0pt} \end{array} ; xq. \]  

(2.1)

**Lemma 2.1.** (cf. [12, (3.12.2)]) The little $q$-Jacobi polynomials satisfy the orthogonality relation

\[ \sum_{k=0}^{\infty} a^k q^k (bq; q)_k \frac{p_n(q^k; a, b; q)p_m(q^k; a, b; q)}{(q; q)_k} = h_n \delta_{n,m}, \]  

(2.2)

where

\[ h_n = \frac{(abq^2; q)^n}{(aq; q)^n} \frac{(aq; abq; q)_n}{(aq; q)_{n+1}} (1 - abq)(aq)^n. \]

We refer the reader to [7] for an alternate proof of (2.2) by appealing to the nonterminating $q$-Saalschütz summation formula ([7, (II.24)]). From the viewpoint of inverse relations, it is equivalent to

**Theorem 2.2.** The dual form of the orthogonality relation (2.2) is the terminating $6\phi_5$ summation formula ([7, (II.20)])

\[ 6W_5 \left( abq; bq, q^{-n}, q^{-m}, q, aq^{1+m+n} \right) = \frac{(abq^2; q)_n(abq^2; q)_m(aq; q)_n}{(aq; q)_n(aq; q)_m(abq^2; q)_{n+m}}. \]  

(2.3)

**Proof.** It suffices to express (2.2) in terms of matrices. For this, we introduce three infinite matrices $A, B, X$ with the entries

\[ [A]_{n,k} = p_n(q^k; a, b; q), [B]_{n,k} = a^k q^k (bq; q)_k \delta_{n,k}, [X]_{n,k} = \frac{(abq^2; q)_n}{(aq; q)_n} q^{nk}. \]

By doing so, we can recast the orthogonality relation (2.2) in terms of these matrices as

\[ ABA^T = \begin{pmatrix} h_n \delta_{n,m} \end{pmatrix}. \]
From (2.1) it is clear that

\[ A = N(abq)X. \]  \hspace{1cm} (2.4)

A substitution of (2.4) yields

\[ N(abq) \left\{ XBX^T \right\} N(abq)^T = (h_n \delta_{n,m}). \]

By using the inverse relation \((N(a), N^{-1}(a))\), we get

\[ XBX^T = N^{-1}(abq) (h_n \delta_{n,m}) (N^{-1}(abq))^T. \]  \hspace{1cm} (2.5)

Upon equating the \((n, m)\)-th entries on both sides of (2.5), we obtain

\[
\sum_{k \geq 0} \frac{(abq^2; q)_n}{(aq; q)_n} q^{kn} a^k q^k (bq; q)_k \frac{(abq^2; q)_m}{(aq; q)_m} q^{km} \times (q, bq, q)_k \frac{(q^{-m}; q)_k}{(aq, bq; q)_k} \frac{(abq; q)_k}{(aq; q)_k} 1 - abq^{2k+1} \]

\[ = \sum_{k \geq 0} \frac{(q^{-n}; q)_k}{(abq^{2+n}; q)_k} \frac{(abq; q)_k}{(aq; q)_k} 1 - abq^{2k+1} \frac{(abq^2; q)_\infty (1 - abq)(aq)^k}{(aq; q)_\infty 1 - abq^{2k+1}} \times \frac{(q, bq, q)_k}{(aq, bq; q)_k} \frac{(q^{-m}; q)_k}{(aq, bq; q)_k} \frac{(abq; q)_k}{(aq; q)_k} 1 - abq^{2k+1} \]

After series rearrangement and simplification, noting that the sum on the left-hand side of the above identity can be evaluated in closed form by the \(q\)-binomial theorem \([7, (II.3)]\), it reduces to

\[
\sum_{k \geq 0} \frac{(q^{-n}; q^{-m}, abq, bq; q)_k}{(abq^{2+n}; abq^{2+m}, q, aq; q)_k} \frac{(1 - abq^{2k+1}) (aq^{1+n+m})^k}{1 - abq} \]

\[ = \frac{(abq^2; q)_n (abq^2; q)_m (aq; q)_n (aq; q)_m (abq^2; q)_n (abq^2; q)_m}{(aq; q)_n (aq; q)_m (abq^2; q)_n (abq^2; q)_m}. \]

It is consistent with (2.3) after recast in standard notation of basic hypergeometric series.

2.2. The \(q\)-Racah polynomials

Another kind of orthogonal polynomials deserving our attention is the \(q\)-Racah polynomials. Recall that the \(q\)-Racah polynomials \([7, (7.2.17)]\) is defined, for \(N \geq m, n\), by

\[ R_n(x; a, b, c, N; q) := 4\phi_3 \left[ \begin{array}{c} q^{-n}, \abq^{n+1}, q^{-x}, \cq^{x-N}, \bcq \\ aq, q^{-N}, bcq \\ \end{array} \right]. \]  \hspace{1cm} (2.6)

Lemma 2.3. (cf. \([7, (7.2.18)-(7.2.20)]\)) The \(q\)-Racah polynomials satisfy the orthogonality relation

\[
\sum_{k=0}^{\infty} \rho(k; q) R_n(k; a, b, c, N; q) R_m(k; a, b, c, N; q) = \frac{\delta_{n,m}}{h_n}, \hspace{1cm} (2.7)
\]
where
\[ \rho(k; q) = \frac{(cq^{-N}; q)_k (1 - cq^{2k-N})}{(q; q)_k (1 - cq^{-N})} \frac{(aq, bcq, q^{-N}; q)_k}{(ca^{-1}q^{-N}, b^{-1}q^{-N}, cq; q)_k} (abq)^{-k} \]
and
\[ h_n = \frac{(bq, aq/c; q)_N}{(abq^2, 1/c; q)_N} \frac{(1 - abq^{2n+1})}{(1 - abq)} \frac{(abq, aq, bcq, q^{-N}; q)_n}{(abq, aq/c, abq^{N+2}; q)_n} \frac{\left(\frac{q^N}{c}\right)^n}{(abq^{1+n+m})}. \]

As is to be expected, this orthogonality relation leads us to a new transformation between two terminating \( s\phi_7 \) series, stated as below.

**Theorem 2.4.** The dual form of the orthogonality relation (2.7) is
\[ sW_7(abq; bq, aq/c, abq^{N+2}, q^{-n}, q^{-m}; q, cq^{m+n-N}) = \frac{(cq^{n-N}/a, q^{n-N}/b; q)_{N-n} (abq^2, q^{-n}, cq^{n-N}; q)_m}{(q^{-1-N}/ab, cq^{2n-N+1}; q)_{N-n} (aq, q^{-N}, bcq; q)_m} \]
\[ \times sW_7 \left( cq^{2n-N}; cq^{m+n-N}, q^{1+n}, aq^{1+n}, bcq^{1+n}, q^{-N}; q, \frac{1}{abq^{1+n+m}} \right), \]
where \( N \geq n \geq m > 0 \).

**Proof.** Observe first that the orthogonality relation of the \( q \)-Racah polynomials, as already given by (2.7), can be reformulated in terms of matrix operations as
\[ ABA^T = \frac{\delta_{n,m}}{h_n}, \]
where, we introduce three infinite matrices \( A, B, X \) such that
\[ [A]_{n,k} = R_n(k; a, b, c, N; q), [B]_{n,k} = \rho(k; q)\delta_{n,k}, [X]_{n,k} = \frac{(abq^2, q^{-k}, cq^{k-N}; q)_n}{(aq, q^{-N}, bcq; q)_n}. \]

Using these matrices and the ILT matrix \( N(a) \), we easily restate (2.6) by the matrix identity
\[ A = N(abq)X. \]

In the sequel, we are able to reformulate the orthogonality relation (2.7) as
\[ XBX^T = N^{-1}(abq) \left( \frac{\delta_{n,m}}{h_n} \right) N^{-T}(abq). \]

Upon equating the \((n, m)\)-th entries on both sides, we obtain
\[ \sum_{k \geq 0} \frac{(abq^2, q^{-k}, cq^{k-N}; q)_n}{(aq, q^{-N}, bcq; q)_n} \frac{(abq^2, q^{-k}, cq^{k-N}; q)_m}{(aq, q^{-N}, bcq; q)_m} \]
\[ \times \frac{(cq^{-N}, aq, bcq, q^{-N}; q)_k}{(q, ca^{-1}q^{-N}, b^{-1}q^{-N}, cq; q)_k} \frac{1 - cq^{-N}}{1 - cq^{-N}} \]
\[ = \sum_{k \geq 0} \frac{(q^{-n}; q)_k}{(abq^{2+n}; q)_k} \frac{(abq; q)_k 1 - abq^{2k+1}}{(aq; q)_k} \frac{q^{kn}}{1 - abq} \frac{(q^{-m}; q)_k}{(abq^{2m}; q)_k} \frac{(abq; q)_k 1 - abq^{2k+1}}{(aq; q)_k} \frac{q^{km}}{1 - abq} \]
\[ \times \frac{(abq^2, 1/c; q)_N}{(bq, aq/c; q)_N} \frac{(aq, bcq, q^{-N}; q)_k}{(aq, bcq, q^{-N}; q)_k} \frac{(1 - abq)}{(1 - abq^{2k+1})} \frac{\left(\frac{c}{q^N}\right)^k}{(abq^{1+n+m})}. \]
On considering the factors \((q^{-k}; q)_n(q^{-k}; q)_m = 0\) for \(k < n\) or \(k < m\), we now assume that \(k \geq n \geq m\) without any loss of generality. In this case, change the index of summation from \(k\) to \(k + n\) on the left-hand and reformulate the last identity in the form

\[
\sum_{k \geq 0} (q^{-k-n}, cq^{-n}; q)_n (q^{k-n}, cq^{k-n}; q)_m (1 - cq^{2k-2n-N})
\]

\[= \frac{(abq^2, 1/c; q)_N}{(bq, aq/c; q)_N} \sum_{k \geq 0} \frac{(q^{-n}; q)_k}{(q^{k-n}; q)^2} \frac{1}{(1 - abq)(1 - abq^{2k+1})} \frac{(q; q)_k}{(1 - abq)(1 - abq^{2k+1})} \frac{(aq, bcq, q^{-N}; q)_k}{(cq^{m+n-N}; q)^k}.
\]

At this stage, in view of the basic relation \([7, (1.13)]\)

\[
(xq^{-k}; q)_n = \frac{(x; q)_n(q/x; q)_k}{(q^{-n}/x; q)_k} q^{-nk},
\]

it is easy to check that for integer \(n \geq 0\), there hold

\[
(q^{-k-n}; q)_n = \frac{(q^{-n}; q)_n(q^{1+n}; q)_k}{(q; q)_k} q^{-nk}
\]

and

\[
(cq^{k+n-N}; q)_n = \frac{(cq^{-n-N}; q)_n(q^{2n-N}; q)_k}{(cq^{-N}; q)_k}.
\]

Simplify the preceding identity by these expressions. The result is

\[
\frac{(abq^2, q^{-n}, cq^{-n-N}; q)_m (abq^2; q)_n}{(aq, q^{-N}, bcq; q)_m (ca^{-1}q^{-N}, b^{-1}q^{-N}; q)_n} \frac{(cq^{1-N}; q)_2n(-ab)^{-n}q^{-(n+3)/2}}{(cq; q)_n}
\]

\[= \frac{(abq^2, 1/c; q)_N}{(bq, aq/c; q)_N} \sum_{k \geq 0} \frac{(abq, q^{-m-n}, bq, aq/c, abq^{N+2}; q)_k}{(cq^{m+n-N}; q)_k} \frac{1 - abq^{2k+1}}{1 - abq} \frac{(aq, bcq, q^{-N}; q)_k}{(cq^{m+n-N}; q)^k},
\]

which, written out in standard notation of basic hypergeometric series, is recognized to be

\[
\text{_{8}W_{7}} (abq; bq, aq/c, abq^{N+2}, q^{-n}, q^{-m}; q, cq^{m+n-N})
\]

\[= C_{0} \text{_{8}W_{7}} \left( \left. cq^{2n-N}; cq^{m+n-N}, q^{1+n}, aq^{1+n}, bcq^{1+n}, q^{n-N}; q, \frac{1}{abq^{1+n+m}} \right) \right),
\]

(2.12)
For integers $N \geq n \geq m > 0$, there holds
\[ 4\phi_3 \left[ \begin{array}{c} q^{-m}, aq/c, q^{-n}, q^{1-n}/b \\ q^{-N}, aq, q^{-m-n}/bc \\ q, q \end{array} \right] = \frac{(abq^2, bcq^{1+n}; q)_m}{(bcq^{1+n}, q^{-N}, aq; q)_m} 4\phi_3 \left[ \begin{array}{c} q^{-m}, aq^{1+n}, bcq^{1+n}, q^{n-N} \\ q^{1+n-m}, q^{n-N}, abq^{2+n}; q, q \end{array} \right]. \] (2.14)

Proof. As indicated above, by applying Watson’s transformation \cite[(III.17)]{7} to both sides of (2.8) simultaneously, we obtain
\[ \frac{(abq^2, bcq^{1+n}; q)_m}{(bcq, abq^{2+n}; q)_m} 4\phi_3 \left[ \begin{array}{c} q^{-m}, aq/c, q^{-n}, q^{1-n}/b \\ q^{-N}, aq, q^{-m-n}/bc \\ q, q \end{array} \right] = C_0 \left( \frac{cq^{1+2n-N}, q^{1-N}/ab; q}_{cq^{n-N}/a, q^{n-N}/b; q} \right)_{N-n} 4\phi_3 \left[ \begin{array}{c} q^{-m}, aq^{1+n}, bcq^{1+n}, q^{n-N} \\ q^{1+n-m}, q^{n-N}, abq^{2+n}; q, q \end{array} \right], \]
where $C_0$ is given by (2.13). To simplify further, we reformulate it as
\[ 4\phi_3 \left[ \begin{array}{c} q^{-m}, aq/c, q^{-n}, q^{1-n}/b \\ q^{-N}, aq, q^{-m-n}/bc \\ q, q \end{array} \right] = C_1 4\phi_3 \left[ \begin{array}{c} q^{-m}, aq^{1+n}, bcq^{1+n}, q^{n-N} \\ q^{1+n-m}, q^{n-N}, abq^{2+n}; q, q \end{array} \right]. \] (2.15)

by defining
\[ C_1 = \frac{(bcq, abq^{2+n}; q)_m (cq^{1+2n-N}, q^{1-N}/ab; q)_{N-n}}{(abq^2, bcq^{1+n}; q)_m (cq^{n-N}/a, q^{n-N}/b; q)_{N-n}} \times \frac{(cq^{n-N}/a, q^{n-N}/b; q)_{N-n}(abq^2, q^{-n}, cq^{n-N}; q)_m}{(q^{1-N}/ab, cq^{2n-N+1}; q)_{N-n}(aq, q^{-N}, bcq; q)_m}. \]

A direct simplification yields
\[ C_1 = \frac{(abq^{2+n}, q^{-n}, cq^{n-N}; q)_m}{(bcq^{1+n}, q^{-N}, aq; q)_m}. \]

Thus the transformation (2.14) follows from (2.15).
2.3. The q-Laguerre polynomials

A well-known version of the q-Laguerre polynomials [12, (3.21.1)] is given, for \( \alpha > -1 \), by

\[
L_n^{(\alpha)}(x; q) := \frac{1}{(q; q)_n} \phi_1 \left[ q^{-n}, -x ; q, q^{n+\alpha+1} \right].
\] (2.16)

**Lemma 2.6.** (cf. [12, (3.21.3)]) The q-Laguerre polynomials satisfy the following discrete orthogonality relation

\[
\sum_{k=-\infty}^{\infty} \frac{q^{k+\alpha}}{(-cq^k; q)_\infty} L_n^{(\alpha)}(cq^k; q) L_m^{(\alpha)}(cq^k; q) = h_n \delta_{n,m},
\] (2.17)

where \( c > 0 \) and

\[
h_n = \frac{(q, -cq^{\alpha+1}, -c^{-1}q^{-\alpha}; q)_\infty (q^{\alpha+1}; q)_n}{(q^{\alpha+1}, -c, -c^{-1}q; q)_\infty (q; q)_n} q^{-n}.
\] (2.18)

Evidently, this orthogonality relation is distinct from all preceding cases. Viewed in light of inverse relations, it implies

**Theorem 2.7.** For \( \alpha > -1, c > 0 \), the dual form of the orthogonality relation (2.17) is the basic identity

\[
\sum_{k=0}^{n} \binom{n}{k}_q y^{n-k} x^k (y)_k = \sum_{k=0}^{n} \frac{(q^{-n}, x, y; q)_k}{(q; q)_k} q^k,
\] (2.19)

where \((x, y) = (q^{-m}, q^{\alpha+1})\), the q-binomial coefficient

\[
\binom{n}{k}_q := \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}.
\]

**Proof.** Proceeding as before, in order to express (2.17) in terms of matrix operations, now we define three infinite matrices \( A, B, X \) with the \((n, k)\)-th entries

\[
[A]_{n,k} = L_n^{(\alpha)}(cq^k; q), [B]_{n,k} = \frac{q^{k+\alpha}}{(-cq^k; q)_\infty} \delta_{n,k}, [X]_{n,k} = (cq^k; q)_n q^{(\alpha+1)n}.
\]

Note that in this case, the index \( k \in \mathbb{Z} \). From (2.16) it is clear that

\[
A = \left( \frac{\delta_{n,k}}{(q; q)_n} \right) N^{-1}(0) X.
\]

Now, with these facts and \( N(a) \) given by (1.7a), the orthogonality relation (2.17) of the q-Laguerre polynomials can be reformulated as

\[
X B X^T = N(0) ((q; q)_n \delta_{n,k}) (h_n \delta_{n,m}) ((q; q)_n \delta_{n,k}) N(0)^T.
\] (2.20)
By equating the \((n,m)\)-th entries on both sides of (2.20) gives
\[
\sum_{k=0}^{\infty} (cq^k; q)_n q^{(a+1)n} (q^{k+a}; q)_\infty (cq^k; q)_m q^{(a+1)m} = \sum_{k=0}^{\infty} (q^{-n}; q)_k (q^{-m}; q)_k h_k q^{2k}.
\]
Upon substituting (2.18) for \(h_n\) and simplifying, we obtain
\[
q^{(a+1)(n+m)} \sum_{k=0}^{\infty} (-cq^k; q)_n (q^{-a}; q)_\infty \sum_{k=0}^{\min\{m, n\}} \frac{(q^{-n}, q^{-m}, q^{a+1}; q)_k}{(q; q)_k} q^k.
\]
Next, by applying the q-binomial theorem [7, (II.3)] to \((-cq^k; q)_n\) on the left-hand side of (2.21) and interchanging the order of summation, we obtain
\[
\text{LHS of (2.21)} = q^{(a+1)(n+m)} \sum_{k=0}^{\infty} (-cq^k, c)_\infty \sum_{i=0}^{n} \sum_{i=0}^{\infty} (-cq^m; q)_k q^{i+1} i + \sum_{i=0}^{\infty} (-cq^m; q)_k q^{i+1} i + \sum_{i=0}^{\infty} (-cq^m; q)_k q^{i+1} i + \sum_{i=0}^{\infty} (-cq^m; q)_k q^{i+1} i.
\]
The last equality comes from Ramanujan’s \(\psi_1\) summation formula [7, (II.29)] with \(b = 0\). Some routine simplifications on this expression finally reduce (2.21) to the basic identity
\[
\sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k (y; q)_k = \sum_{k=0}^{n} \frac{(q^{-n}, x, y; q)_k}{(q; q)_k} q^k.
\]
It is just a limiting case of [7, (III.13)]. Hence the theorem is proved. \(\blacksquare\)

It may come as a surprise that by the above argument, we can extend the orthogonality relation (2.17) to the following

**Theorem 2.8.** For \(|y| < 1\), we redefine the q-Laguerre polynomials by
\[
L_n(x, y; q) := \frac{1}{(q; q)_n} \phi_1 \left[ \frac{q^{-n}, -x}{0}; q, y^q \right].
\]
Then, for \(c > 0\), \(L_n(x, y; q)\) satisfies the following orthogonality relation:
\[
\sum_{k=-\infty}^{\infty} \frac{y^k}{(-cq^k; q)_\infty} L_n(cq^k, y; q) L_m(cq^k, y; q) = \frac{(q, -cy, y/cy; q)_\infty (y, q)_n}{(y, -c, -c^{-1}y; q)_\infty (q, q)_n} q^{-n} \delta_{n,m}.
\]
Evidently, \(L_n(x, y; q)\) is polynomial in two variables \(x\) and \(y\). Thus we may guess that it should be an orthogonal polynomial in \(y\) with respect to certain measure, too.
3. Dual forms of the orthogonality relations of the continuous big $q$-Jacobi and Askey-Wilson polynomials

As a matter of fact, the foregoing argument to find dual forms of the orthogonality relations by the inverse relations is also valid for continuous $q$-orthogonal polynomials. For this purpose, we need a few new concepts which are analogue to the usual matrix algebras or the incidence algebras over partially ordered set [21].

3.1. New matrix operations

Definition 3.1. Given two finite or infinite index sets $\mathbb{K}_1, \mathbb{K}_2$ and the number field $\mathbb{K}$, any binary mapping

$$f : \mathbb{K}_1 \times \mathbb{K}_2 \to \mathbb{K}$$

is called a two-dimensional matrix over $\mathbb{K}_1 \times \mathbb{K}_2$. The set of such matrices is denoted by

$$\Omega^{\mathbb{K}_1 \times \mathbb{K}_2} := \{ A = (f(x, y)) \mid x \in \mathbb{K}_1, y \in \mathbb{K}_2 \}.$$  

As customary, we employ the notation $[A]_{x,y}$ for its entry $f(x, y)$.

Definition 3.2. For arbitrary matrices $X \in \Omega^{Z \times C}$, $Y \in \Omega^{C \times Z}$, and $A \in \Omega^{Z \times Z}$, we further introduce the following matrix operations.

(i) $\circ : \Omega^{Z \times Z} \times \Omega^{Z \times C} \to \Omega^{Z \times C}$ by defining

$$[A \circ X]_{m,x} := \sum_k [A]_{m,k}[X]_{k,x};$$

(ii) $\bullet_{(a,b)} : \Omega^{Z \times C} \times \Omega^{C \times Z} \to \Omega^{Z \times Z}$ by defining

$$[X \bullet_{(a,b)} Y]_{m,n} := \int_a^b W(x)[X]_{m,x}[Y]_{x,n}dx;$$

(iii) $\bullet_{(q;a,b)} : \Omega^{Z \times C} \times \Omega^{C \times Z} \to \Omega^{Z \times Z}$ by defining

$$[X \bullet_{(q;a,b)} Y]_{m,n} := \int_a^b W(x)[X]_{m,x}[Y]_{x,n}d_qx;$$

(iv) the transpose $X^T$ of $X$ by $[X^T]_{x,m} := [X]_{m,x}$.

In the above, $W(x)$ is the general weight function, and the $q$-integral is defined by (cf.[7, (1.11.2)-(1.11.3)])

$$\int_b^a f(t)d_qt := \int_0^a f(t)d_qt - \int_0^b f(t)d_qt,$$

$$\int_0^a f(t)d_qt := a(1-q) \sum_{k=0}^\infty f(aq^k)q^k.$$
A full and rigorous study on Definition 3.2 will be given in our forthcoming paper. Here, we only point out that the multiplications \( \circ \), \( \bullet_{(a,b)} \) and \( \bullet_{(q;a,b)} \) obey the associative law, namely,

**Proposition 3.3.** Under the assumption of Definition 3.2, there hold

\[
A \circ (X \bullet_{(a,b)} Y) = (A \circ X) \bullet_{(a,b)} Y, \quad A \circ (X \bullet_{(q;a,b)} Y) = (A \circ X) \bullet_{(q;a,b)} Y.
\]

With the help of these new matrix operations, we are ready to consider two kinds of continuous \( q \)-orthogonal polynomials, as further illustrations of our idea.

### 3.2. Dual form of the orthogonality relation of the big \( q \)-Jacobi polynomials

It is well-known that the big \( q \)-Jacobi polynomials (cf. [12, (3.5.1)] or [7, (7.3.10)]) are defined by

\[
P_n(x; a, b, c; q) := \phi_2[q^{-n}, abq^{n+1}, x; aq, cq].
\]

(3.1)

From [12, (3.5.2)], it is clear

**Lemma 3.4.** The big \( q \)-Jacobi polynomials \( P_n(x; a, b, c; q) \) satisfy the orthogonality relation

\[
\int_{aq}^{cq} \frac{(x/a, x/c; q)_{\infty}}{(x/bx/c; q)_{\infty}} P_n(x; a, b, c; q) P_m(x; a, b, c; q) d_q x = h_n(a, b, c; q) \delta_{n,m},
\]

(3.2)

where

\[
h_n(a, b, c; q) = M \frac{1 - abq}{1 - abq^{2n+1}} \frac{(q, bq, abq/c; q)_n}{(aq, cq, abq; q)_n} (-aq^2)^n q^{n(n-1)/2}
\]

(3.3)

and

\[
M = \frac{aq(1-q)(q, c/a, aq/c, abq^2; q)_{\infty}}{(aq, bq, cq, abq/c; q)_{\infty}}.
\]

(3.4)

A proof of (3.2) can be found in [7] with the \( q \)-Gauss sum [7, (II.8)] involved. Instead here, in light of inverse relations, we can show

**Theorem 3.5.** The dual form of the orthogonality relation (3.2) is the following transformation of \( \phi_2 \) series

\[
\phi_2 \left[ a, b, c \begin{array}{l} d, e \end{array} ; q, de/abc \right] = \frac{(e/a, e/b, e/c; q)_{\infty}}{(e, e/\alpha; de/abc; q)_{\infty}} \phi_2 \left[ d/a, d/b, d/c \begin{array}{l} d, dq/e \end{array} ; q, q \right]
\]

(3.5)

\[+ \frac{(d/a, d/b, d/c; q)_{\infty}}{(d, d/\alpha; de/abc; q)_{\infty}} \phi_2 \left[ e/a, e/b, e/c \begin{array}{l} e, eq/d \end{array} ; q, q \right],
\]

where all parameters subject to convergent conditions of the sums.
Proof. As indicated above, in order to express (3.2) in terms of matrix operations, we now define the $q$-integral for any matrix $(a_{i,j}(t))$ to be

$$\int_0^a (a_{i,j}(t)) \, dq \, t := \left( \int_0^a a_{i,j}(t) \,dq \right).$$ \hfill (3.6)

Besides, we need yet to introduce three matrices $A, X \in \Omega^{N \times C}, B \in \Omega^{C \times C}$ such that

$$[A]_{n,t} = P_n(t; a, b, c; q), [X]_{n,t} = \frac{(t, abq^2; q)_n}{(aq, cq; q)_n},$$

and

$$[B]_{t,s} = \frac{(t/a, t/c; q)_{\infty}}{(t, bt/c; q)_{\infty}} \delta_{t,s},$$

where the (generalized) Kronecker delta $\delta_{t,s} = 1$ for $t = s$ and 0 otherwise. As such, (3.1) is now restated equivalently as

$$A = N(abq) \circ X.$$ \hfill (3.7)

Next, taking (3.6) and the inverse relation $(N(a), N^{-1}(a))$ into account, we now restate the orthogonality relation (3.2) of the big $q$-Jacobi polynomials as (in this case, $W(x) = 1$)

$$A \bullet_{(q;\omega, aq)} B \bullet_{(q;\omega, aq)} A^T = (h_n(a, b, c; q) \delta_{n,m}),$$ \hfill (3.8)

which, after (3.7) inserted, becomes

$$N(abq) \circ \{ X \bullet_{(q;\omega, aq)} B \bullet_{(q;\omega, aq)} X^T \} \circ N(abq)^T = (h_n(a, b, c; q) \delta_{n,m}).$$

By inverting $N(abq)$ and $N(abq)^T$, we obtain

$$X \bullet_{(q;\omega, aq)} B \bullet_{(q;\omega, aq)} X^T = N^{-1}(abq) (h_n(a, b, c; q) \delta_{n,m}) N^{-T}(abq).$$ \hfill (3.9)

Next, substituting (3.3) for $h_n(a, b, c; q)$ and equating the $(n, m)$-th entries on both sides of the identity obtained, then we find immediately

\begin{align*}
\int_{cq}^{aq} \frac{(t/a, t/c; q)_{\infty}}{\frac{t}{bt/c; q)_{\infty}} \frac{aq, cq; q}_n \frac{aq, cq; q)_m}{(aq, cq; q)_n \frac{aq, cq; q)_m}} \, dq \, t \\
= M \sum_{k=0}^{\infty} \frac{(q^{-n}; q)_k (abq; q)_k 1 - abq^{2k+1}}{(aq, cq; q)_k (q; q)_k 1 - abq^{2k+1}} q^{kn} (abq; q)_k \frac{(q^{-m}; q)_k 1 - abq^{2k+1}}{(aq, cq; q)_k (aq, cq, abq; q)_k} \frac{1 - abq^{2k+1}}{1 - abq^{2k+1}} q^{-k} (-acq^2)^k q^{k(k-1)/2},
\end{align*}

where $M$ is the same as in (3.4). After some rearrangement and simplification, it reduces to

\begin{align*}
\lim_{d \to 0} \left[ W_T \left( \frac{abq; bq, abq/c, acq^2/d, x, y; q, \frac{d}{xy}}{x = q^{-m}, y = q^{-n}} \right) \right] \\
= \frac{aq(1 - q)(q,c/a, qq/c, abq^2; q)_\infty \frac{abq^2; q}_n (abq^2; q)_m}{aq(1 - q)(q,c/a, qq/c, abq^2; q)_\infty \frac{abq^2; q}_n (aq, cq; q)_n (aq, cq; q)_m} \\
\times \int_{cq}^{aq} \frac{(t/t; q)_n (t/c; q)_m \frac{t/a, t/c; q)_{\infty}}{(t, bt/c; q)_{\infty}}} \, dq \, t. \hfill (3.10)
\end{align*}
In this case, we can apply Watson’s $q$-Whipple transformation (III.17) of [7] to the $8\phi_7$ series on the left side of (3.10). The result is

\[
8W_7 \left( \frac{abq}{bq}, \frac{abq}{c}, \frac{acq^2}{d}, x, y; q, \frac{d}{xy} \right) \bigg|_{x=q^{-m}, y=q^{-n}} = \frac{(abq^2, bdq^n/c; q)_m}{(bd/c, abq^{n+2}; q)_m} 4\phi_3 \left[ \frac{c/b}{aq}, \frac{acq^2}{aq}, \frac{q^{-m}}{cq}, \frac{cq^{1-n-m}/bd}{q}; q \right].
\]

We therefore get

\[
\int_{cq}^{aq} \frac{(t; q)_n(t; q)_m(t/a, t/c; q)_\infty}{(t, bt/c; q)_\infty} d_q t \quad = \quad \frac{aq(1 - q)(q, c/a, aq/c; q)_\infty(abq^{2+m+n}; q)_\infty(aq, cq; q)_m}{(aq^{1+n}, bq, cq^{1+n}, abq/c; q)_\infty} \times 3\phi_2 \left[ \frac{c/b}{aq}, \frac{q^{-m}}{cq}; q, abq^{2+n+m} \right].
\] (3.11)

It is easy to check that (3.11) is equivalent to the following product of three analytic functions in variables $y, z$

\[
F(y, z) = G(y, z)H(y, z)
\] (3.12)

at the point $(y, z) = (q^n, q^m)$, $m, n \geq 0$, where $F(y, z), G(y, z)$ and $H(y, z)$ are defined, respectively, by

\[
F(y, z) = \frac{aq(1 - q)(q, c/a, aq/c, abyq^2, aq, cq; q)_\infty}{(aqy, aqz, cqy, cqz, bq, abq/c; q)_\infty},
\]

\[
G(y, z) = \frac{(aq, c/a, abq^{2+n}; yz; q)_\infty}{(aq, a/c, abq^{2+n}; yz; q)_\infty} \quad \text{and}
\]

\[
H(y, z) = 3\phi_2 \left[ \frac{c/b}{aq}, \frac{1/y}{cq}, \frac{1/z}{cq}; q, abyq^2 \right].
\]

By analytic continuation, we find that (3.12) holds as a nonterminating version of (3.11) on a disk around $(0, 0)$. Reformulate $F(y, z)$ on the left side of (3.12) in notation of hypergeometric series, divide both sides of the resulting identity by $G(y, z)$, and finally make the replacement $(y, z) \rightarrow (1/y, 1/z)$. Then we obtain

\[
3\phi_2 \left[ \frac{aq/y, aq/z}{aq, aq/c}; q, q \right] = \frac{(cq/y, cq/z, bq; q)_\infty}{(cq, c/a, abq^{2+n}/yz; q)_\infty} 3\phi_2 \left[ \frac{aq/y, aq/z, abq/c}{aq, aq/c}; q, q \right] \quad + \quad \frac{(aq/y, aq/z, abq/c; q)_\infty}{(aq, a/c, abq^{2+n}/yz; q)_\infty} 3\phi_2 \left[ \frac{cq/y, cq/z, bq}{cq, cq/a}; q, q \right].
\] (3.13)

As a last step, by making the simultaneous replacement of parameters

\[(a, b, c, y, z, q) \rightarrow (d/q, e/aq, e/q, b, c, q)\]
in (3.13), (3.5) follows at once. The theorem is now proved.

More interestingly, by putting $b = c$ in (3.13), we recover the nonterminating form of the $q$-Chu-Vandermonde formula.

**Corollary 3.6.** (cf. [7, (II.23)]) For $a/c, y/c, z/c \not\in \{q^{-k} : 1 \leq k \in \mathbb{N}\}$, it holds

$$2\phi_1 \left[ \frac{aq/y}{aq/c}, \frac{aq/z}{aq/c} ; q, q \right] + \frac{(c/a, acq/y, acq/z ; q)_\infty}{(c/a, cq/y, cq/z ; q)_\infty} 2\phi_1 \left[ \frac{cq/y}{cq/a}, \frac{cq/z}{cq/a} ; q, q \right]$$

$$= \frac{(c/a, acctq^2/yz ; q)_\infty}{(cq/y, cq/z ; q)_\infty},$$

(3.14)

provided that the sums are convergent.

Another two special cases that $z = cq^{1+n}$ and $(y, z) = (aq^{1+m}, cq^{1+n})$, $m, n \geq 0$, of (3.13) are worth mentioning.

**Corollary 3.7.** For $a, b, c \not\in \{q^{-k} : 1 \leq k \in \mathbb{N}\}$ and $m, n \geq 0$, there hold

(i) $$3\phi_2 \left[ \frac{c/b}{aq}, \frac{y}{cq}, \frac{cq^{1+n}}{aq} ; q, abq^{1-n}/yc \right] = \frac{(aq/y, abq/c ; q)_\infty}{(abq/y, acq ; q)_\infty} \left( \frac{yc}{ab} \right)^n 3\phi_2 \left[ \frac{q^{-n}}{bc}, \frac{cq/y}{cq}, \frac{cq^{1+n}}{cq} ; q, q \right];$$

(ii) $$3\phi_2 \left[ \frac{c/b}{aq}, \frac{aq^{1+m}}{aq}, \frac{cq^{1+n}}{cq} ; q, bq^{-m-n}/c \right] = 0.$$ 

(3.15)

### 3.3. Dual form of the orthogonality relation of the Askey-Wilson polynomials

In what follows, we will consider the orthogonality relation of the famous Askey-Wilson polynomials. As a member on the top of the Askey-scheme [12], the Askey-Wilson polynomials in a variable $x = \cos \theta$ is defined by [12, (3.1.1)]

$$W_n(x; a, b, c, d|q) := \frac{(ab, ac, ad; q)_n}{a^n} 4\phi_3 \left[ \frac{q^{-n}}{ab}, \frac{abcdq^{n-1}}{ac}, \frac{ae^{i\theta}}{ad} ; q, q \right].$$

(3.16)

It is well known that

**Lemma 3.8.** (cf. [12, (3.1.2)]) The Askey-Wilson polynomials satisfy the orthogonality relation

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{w(x)}{\sqrt{1-x^2}} W_n(x; a, b, c, d|q)W_m(x; a, b, c, d|q)dx = \kappa_n \delta_{n,m},$$

(3.17)

where

$$w(x) = \frac{h(x, 1)h(x, -1)h(x, q^{1/2})h(x, -q^{1/2})}{h(x, a)h(x, b)h(x, c)h(x, d)},$$

(3.18)

$$h(x, y) = (ye^{i\theta}, ye^{-i\theta}; q)_\infty,$$

(3.19)

$$\kappa_n = \frac{(abcdq^{n-1}; q)_n(abcdq^{2n}; q)_\infty}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty}.$$
Now we are able to present the dual form of (3.17) as the following theorem, whose derivation may be regarded as a new proof of (3.17) via the integral representation of VWP series \[7, \S 6.3\].

**Theorem 3.9.** For \(\max\{|a|, |b|, |c|, |d|\} < 1\) and \(a, b, c, d \in \mathbb{C}\), the dual form of the orthogonality relation (3.17) is the Askey-Wilson \(q\)-beta integral

\[
\frac{1}{2\pi} \int_{-1}^{1} \frac{w(x)}{\sqrt{1-x^2}} (ae^{i\theta}, ae^{-i\theta}; q)_n (ae^{i\theta}, ae^{-i\theta}; q)_m dx = K (ab, ac, ad; q)_m (ab, ac, ad; q)_n s W_7 \left( abcd/q, bc, bd, cd, q^{-m}, q^{-n}, q, a^2 q^{m+n} \right),
\]

where

\[K = (abcd; q)_\infty / (q, ab, ac, ad, bc, bd, cd; q)_\infty.\]

**Proof.** It suffices to define three matrices \(A, D,\) and \(X,\) respectively, by

\[[A]_{n,x} = W_n(x; a, b, c, d|q), [D]_{n,k} = \frac{(ab, ac, ad; q)_n}{a^n} \delta_{n,k},\]

and

\[[X]_{n,x} = \frac{(abcd, ae^{i\theta}, ae^{-i\theta}; q)_n}{(ab, ac, ad; q)_n}.\]

Further, we choose the weight function

\[W(x) = \frac{1}{2\pi} \frac{w(x)}{\sqrt{1-x^2}},\]

where \(x \in [-1, 1]\). In light of the matrix operation \(\circ\), it is easily found that (3.16) is just equivalent to

\[A = D \times N(abcd/q) \circ X,\]

where “\(\times\)” denotes the usual matrix multiplication. Thus the orthogonality relation (3.17), by use of the matrix operation \(\bullet_{(-1,1)}\), can still be restated equivalently as

\[A \bullet_{(-1,1)} A^T = (\kappa_n \delta_{n,m}),\]

viz.,

\[D \times N(abcd/q) \circ X \bullet_{(-1,1)} X^T \circ N(abcd/q)^T \times D^T = (\kappa_n \delta_{n,m}),\]

from which we may deduce

\[X \bullet_{(-1,1)} X^T = N^{-1}(abcd/q) \times \left( \frac{a^{2n} \kappa_n}{(ab, ac, ad; q)_n^2} \delta_{n,m} \right) \times N(abcd/q)^{-T}.\]

Thus, by equating the \((n, m)\)-th entries on both sides of (3.22), we obtain (3.20).
It is worth pointing out that (3.20) is in agreement with (6.3.8) of [7] under the simultaneous replacement of parameters \((a, b, c, d, f, g) \rightarrow (b, c, d, aq^m, aq^n, a)\).

From a quick glance over [18], we know that Nassrallah and Rahman proved (6.3.8) of [7] via the direct use of \(q\)-Askey-Wilson beta integral without appeal to the orthogonality relation of the Askey-Wilson polynomials \(W_n(x; a, b, c, d|q)\). It is also of interest to see that the limitation \(m, n \rightarrow \infty\) of Theorem 3.9 yields

**Corollary 3.10.** Let \(w(x)\) and \(h(x, a)\) be given by (3.18) and (3.19), respectively. Then

\[
\int_{-1}^{1} \frac{w(x)}{\sqrt{1-x^2}} h^2(x, a) dx = \frac{2\pi(ab, ac, ad; q)_\infty}{(q, bc, bd, cd, abcd; q)_\infty} \times \sum_{k=0}^{\infty} \frac{1 - abcdq^{2k-1}}{1 - abcd/q} (abcd/q, bc, bd, cd; q)_k a^{2k} q^{k(k-1)}. \tag{3.23}
\]

4. Concluding remarks

So far, we have applied the inverse relation \((N(a), N^{-1}(a))\) to find the dual forms of the orthogonality relations for some discrete and continuous \(q\)-orthogonal polynomials. Apart from \(N(a)\), there are many inverse relations can be used. For instance, let \(M(a)\) and \(K(a, c)\) be two ILT matrices with the entries given by

\[
[M(a)]_{n,k} = \frac{(q^n, 1/a; q)_k q^k}{(q, q^{1-n}/a; q)_k q^k} \quad \text{and} \quad [K(a, c)]_{n,k} = \frac{(aq^n, q^{-n}, q^{\sqrt{c}}, -q^{\sqrt{c}}, c/a, c; q)_k q^k}{(cq^{1-n}/a, cq^{n+1}, q, \sqrt{c}, -\sqrt{c}, aq; q)_k q^k},
\]

respectively. Then it is easy to verify by Theorem 1.1 that

\[
M^{-1}(a) = M(1/a), \quad K^{-1}(a, c) = K(c, a).
\]

We remark that \(K^{-1}(a, c)\) is consistent with the Krattenthaler inversion formula (cf. [14, (1.5)(1)/(1.5)(2)]). Without any doubt, the potential applications of these two inverse relations similar to those of \(N(a)\) deserve further investigation. Generally speaking, inverse relations are helpful to reveal possibly association of orthogonality relations like (2.22)/(2.23) of discrete and continuous \(q\)-orthogonal polynomials with summation and transformation of hypergeometric series, even more in solving the problem of connection coefficients between \(q\)-orthogonal polynomials.

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