Global martingale and pathwise solutions and infinite regularity of invariant measures for a stochastic modified Swift–Hohenberg equation

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Abstract
We consider a 2D stochastic modified Swift–Hohenberg equations with multiplicative noise and periodic boundary. First, we establish the existence of local and global martingale and pathwise solutions in the regular Sobolev space $H^2_m$ for each $m \geq 1$. Associated with the unique global pathwise solution, we obtain a Markovian transition semigroup. Then, we show the existence of invariant measures and ergodic invariant measures for this Markovian semigroup on $H^2_{2m}$. At last, we improve the regularity of the obtained invariant measures to $H^2_{2m(\nu+1)}$. With appropriate conditions on the diffusion coefficient, we can deduce the infinite regularity of the invariant measures, which was conjectured by Glatt-Holtz et al in their situation (2014 J. Math. Phys. 55 277–304).

Keywords: stochastic modified Swift–Hohenberg model, global martingale solution, global pathwise solution, ergodic invariant measure, infinite regularity

Mathematics Subject Classification numbers: 60H15, 60G46, 37L55, 37A50

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In this paper, we investigate several stochastic problems, including the local and global existence of martingale and pathwise solutions, existence of ergodic invariant measures and infinite regularity of the invariant measures, for the following two-dimensional stochastic modified Swift–Hohenberg equation (MSHE for short) with multiplicative noise and periodic boundary,\[\begin{align*}
\frac{du}{dt} + (\Delta^2 u + 2\Delta u + a)u + b_1\varepsilon u^2 + u^3 + \phi(u)\,dW(t) &= 0, \\
\Delta &= \partial_{xx} + \partial_{yy}, \quad a, b \in \mathbb{R} \quad \text{and} \quad u_0(x, y) = u_0(x, y),
\end{align*}\]where $u(x, y, t)$ is the unknown amplitude function, $(x, y) \in \mathbb{R}^2$, $\Delta = \partial_{xx} + \partial_{yy}$, $a, b \in \mathbb{R}$ and $u_0$ is $2\pi$-periodic with respect to $x$ and $y$ respectively. The stochastic term $\phi(u)\,dW$ in (1.1) will be specified in section 2.2.

The Swift–Hohenberg equation was named after Swift and Hohenberg (see [43]), and arose from convective phenomena in the research of geophysical fluid flows in atmosphere, oceans and earth’s mantle. It is closely contacted with nonlinear Navier–Stokes equations coupled with the temperature equation. The Swift–Hohenberg equation has played a significant role in different branches of physics, ranging from hydrodynamics to nonlinear optics (see e.g. [25, 32, 38]). For this modified case, the cubic term $u^3$ in (1.1) is used as an approximation of a nonlocal integral term $\int |u|^2$ in (1.1) comes from the various pattern formation phenomena involving some kind of phase turbulence or phase transition [40], which eliminates the symmetry $u \rightarrow -u$.

There have been lots of research on the subject of MSHEs. Roughly speaking, these works mainly include three aspects: attractors [22, 36, 55, 57] and the regularity [41], bifurcations of solutions [5, 6, 56] and optimal control [12, 42]. What is more, for the nonautonomous MSHE, Wang et al presented in [49] a lower number of recurrent solutions by topological methods (see more in [34, 46–48]); Wang et al studied the existence of invariant measures and statistical solutions in [50]. Due to the delicate impact on the deterministic cases in real world, randomness has captured more and more concern over the development of evolution systems recently. It becomes centrally significant to take stochastic effects into account for...
mathematical models of complex phenomena in engineering and science. However, the study of stochastic MSHEs is still inadequate in the literature up to now, especially that of existence and regularity of the invariant measures.

In physics, invariant measures have performed a significant role in the research of turbulence [17]. In the field of dynamical systems, invariant measures are to the measure space for the phase space what invariant sets (even attractors) are to the phase space. Essentially, invariant measures also illustrate the long-time dynamical behaviors for a given dynamical system, but in view of measure. Particularly, for stochastic systems, the solution process is random. If we fix each random variable, the Wiener process $W$ would become a deterministic function. Then we can also study the attractor and its properties for the corresponding deterministic system. This treatment seems like straying from the essence of randomness. According to this observation, we are encouraged to consider the existence and regularity of invariant measures for the stochastic dynamical systems.

In recent decades, the research of (ergodic) invariant measures has witnessed the development of (stochastic) dynamical systems. References [3, 4, 26–31] studied the limit property or stochastic stability of some invariant measures for stochastic processes as the noise vanishes or perturbs. There have been more references concerning the existence of invariant measures, such as [2, 8, 13, 14, 19, 37, 45, 51, 52, 54] and their references. For the existence of invariant measures for stochastic partial differential equation, it often needs to assume that the drift terms to be dissipative. If the drift terms contains nonlinear ones, the nonlinear terms are often hoped to be globally Lipschitz or dissipative. In the book [8], Da Prato and Zabczyk provided basic theory on the existence of invariant measures for stochastic partial differential equations with globally Lipschitz or dissipative nonlinearity. Relatively, regularity of invariant measures has attracted little attention until now. Bogachev et al provided in [2] a method to improve the regularity of an invariant measure. By adopting this idea, Glatt-Holtz and Vicol in [19] presented another result, showing higher regularity of the invariant measures for the three dimensional stochastic primitive equations. The authors of [19] even conjecture that ‘the invariant measures for the 3D primitive equation are in fact supported on $C^\infty$ function’ with appropriate continuity and compatibility assumptions on the force.

Now, as presented in (1.1), the modified nonlinear terms lack the global Lipschitz continuity or dissipation, and the lack brings new challenges to our discussion. Moreover, what we study in this paper is the stochastic MSHE with a general multiplicative noise, which is allowed to be nonlinear. In this situation, we will not consider the mild solutions (studied in [36, 45, 54]), but discuss the weak and strong solutions in the stochastic sense, saying, the martingale and pathwise solutions, correspondingly, and establish their global existence. With the unique global pathwise solution in hand, we further explore the existence of ergodic invariant measures for the Markovian semigroup associated with the solution in spaces of high regularity. We also turn out that each invariant measure obtained is supported on $C^\infty \cap H$ (defined in section 2.1), which is said to be the infinite regularity of the invariant measure, under some appropriate assumption on the diffusion coefficient $\phi$. This means that we have actually proved the conjecture of Glatt-Holtz et al in our situation.

Since the equation (1.1) contains the modified nonlinearity, the existence of martingale and pathwise solutions is far from holding naturally without a detailed argument. We initiate our topics by introducing cut-off functions to treat the nonlinear terms, and then apply the Galerkin approximating methods to the cut-off system. By this setting, we are prompted to use the unique global existence of solutions [15] for finite-dimensional stochastic differential equation with globally Lipschitz nonlinearity to start our work.

To show the local existence of martingale and pathwise solutions in $H^m$ (the Sobolev space with periodic boundary given in section 2.1) for each $m \geq 1$, we adopt the classical Galerkin
method and Yamada–Wannabe theorem (see [9, 11, 20]) for stochastic partial differential equations. We actually conclude that the martingale solution (see theorem 4.2) or pathwise solution (see theorem 5.3) satisfies

\[ u(\cdot \wedge \tau) \in L^2(\Omega; C([0, \infty); H^{2m}) \cap L^2_{\text{loc}}(0, \infty; H^2(m+1))) \]

for some stopping time \( \tau \geq 0 \) or \( \tau > 0 \), where \( \Omega \) is a probability space that is described in section 2.2. Here it is only required that \( \phi \) is locally bounded (for martingale case) or locally Lipschitz (for pathwise case). As to the uniqueness of solutions, the usage of Skorohod embedding theorem cannot ensure the local uniqueness in the argument for existence of martingale solutions, albeit with Lipschitz condition of \( \phi \), which, however, is sufficient to imply the local uniqueness in the pathwise case, as the stochastic basis is chosen fixed in advance. The uniqueness enables us to extend every local pathwise solution to a maximal one. In particular, the maximal pathwise solution undergoes a blowup at the maximal living time.

For the global existence of martingale solutions, Dhariwal et al. in [10] considered a stochastic population cross-diffusion system and gave the existence of global martingale solutions, in the version that the solution depends on the time interval \([0, T]\), where \( T \) can be chosen arbitrarily. In this paper, we consider a new version that the global martingale is defined on the entire positive axis. In order to obtain the existence of such global martingale solutions, we need to assume \( |b| < 4 \) to constrain the modified term and \( \phi \) to be globally Lipschitz. Although infeasible to ensure uniqueness for martingale solutions, the Lipschitz condition makes the solutions coincide locally for the same finite-dimensional approximating equation (3.2) with the same initial datum but different cut-off functions. Based on this coincidence, we can then make good use of the diagonal method to pick a subsequence of the approximating solutions, which converges all over \( \tau \in [0, \infty) \). Thus, we successfully obtain a global martingale solution on \([0, +\infty)\) for (1.1)-(1.3). This process can be well applied to many other stochastic partial differential equations to construct a global martingale solution.

For the global existence of pathwise solutions, we need the same assumptions as the case for global martingale solutions, i.e. \( |b| < 4 \) and the global Lipschitz condition for \( \phi \). Here we prove a general and simple conclusion (lemma 5.5), similar to the dominated convergence theorem, but the limit function takes values at the infinity almost surely. The global unique existence follows immediately from this conclusion and a uniform estimate for the maximal pathwise solution given above by means of reduction to absurdity.

The unique existence of the global pathwise solution draws forth a Markovian transition semigroup \( \mathcal{P}_t \) on the Borel \( \sigma \)-algebra \( \mathcal{B}(H^{2m}) \) of \( H^{2m} \), which is proved to be Feller by a process referring to [45] with an additional condition (2.4) on \( \phi \). The existence of the invariant measures is a direct consequence of tightness for the measure family \( \{\nu_T\}_{T>0} \) with

\[ \nu_T(\cdot) = \frac{1}{T} \int_0^T \mathcal{P}_t(u, \cdot) dt, \]

by Krylov–Bogoliubov existence theorem and Prokhorov’s theorem (see [8]). To guarantee the existence of ergodic measures, we utilize the Krein–Milman Theorem [7] to find an extreme point of the set of invariant measures, and this extreme point is exactly an ergodic measure. Again we necessarily show the tightness of the set of all invariant measures, which is assured in nature by the uniform integrability of \( \|u\|_{H^{2m}} \), the square of \( H^{2m} \)-norm of \( u \), over \( H^{2m} \) with respect to each invariant measure. This uniform integrability is not an easy deduction. Thanks to Song et al’s work [41], in which they proved that the MSHE possesses a global attractor in the Sobolev space \( H^k \) for each \( k \geq 0 \) by using iteration procedure, we can as well use iteration procedure or induction method to raise the order \( k \in [0, m] \) one by one such that \( \|u\|_{H^k} \) is uniformly integrable over \( H^{2m} \) with respect to each fixed invariant measure.
For the rise of regularity of invariant measures, we consider to use the \textit{a posteriori} method by the idea propounded in [2], and first show that \( \|u\|_{H^{2m}} \) is uniformly integrable over \( H^{2m} \) with respect to each invariant measure under the same condition that ensures the existence of invariant measures on \( H^{2m} \). This adds one to the regularity of invariant measures. In this way, by assuming an appropriate stronger condition \((2.5)\) than \((2.3)\) and \((2.4)\), we can eventually obtain the infinite regularity of the invariant measures.

The rest part of this paper is arranged as follows. Section 2 provides the basic concepts, notions, settings and theorems that will be used frequently in the following sections. In section 3 we construct the cut-off system and Galerkin scheme of the original MSHE and give a uniform estimate for the solutions of Galerkin approximating equations. In section 4 we are devoted to proving the existence of local and global martingale solutions. And section 5 is for the demonstration of existence of local and global pathwise solutions. In section 6, we prove the existence of invariant measures and then ergodic invariant measures. In section 7, the regularity of the invariant measures is discussed and improved to the infinity under some appropriate conditions. Section 8 is about the summary and remarks for this work.

2. Preliminaries

We first present some basic notions and properties that will be used frequently in this paper.

2.1. On spaces and operators

For metric spaces \( X \) and \( Y \), we conventionally denote by \( \mathcal{C}(X, Y) \) \( (\mathcal{C}_b(X, Y)) \) the collection of continuous (and bounded) functionals from \( X \) to \( Y \). When \( Y = \mathbb{R} \), we simply use \( \mathcal{C}(X) \) \( (\mathcal{C}_b(X)) \) to represent \( \mathcal{C}(X, \mathbb{R}) \) \( (\mathcal{C}_b(X, \mathbb{R})) \).

Let \( I \) denote the interval \([0, 2\pi]\). The periodic boundary condition \((1.2)\) prompts us to study the modified Swift–Hohenberg problem \((1.1)–(1.3)\) on the two-dimensional torus \( T^2 \), saying, the quotient space \( I \times I / \sim \), where \( \sim \) is the equivalence relation such that
\[
(x_1, y_1) \sim (x_2, y_2) \quad \text{if and only if} \quad (x_1, y_1) = (x_2, y_2),
\]
or \( x_1 = x_2 \) and \( y_1, y_2 \in \{0, 2\pi\} \) or \( x_1, x_2 \in \{0, 2\pi\} \) and \( y_1 = y_2 \). This indicates that \((0, y) \sim (2\pi, y), \ (x, 0) \sim (x, 2\pi) \) for each \( x, y \in [0, 2\pi] \) and specifically \((0, 0) \sim (0, 2\pi) \sim (2\pi, 0) \sim (2\pi, 2\pi) \). Hence each function \( u \) on \( T^2 \) is defined on \( I \times I \) with \( u(0, y) = u(2\pi, y) \) and \( u(x, 0) = u(x, 2\pi) \). By this setting, the space \( L^2(T^2) \) is a Hilbert space with the scalar product
\[
(u, v) := \int_0^{2\pi} \int_0^{2\pi} u(x, y)v(x, y)\,dx\,dy,
\]
and the norm denoted by \( \| \cdot \| \).

Observe that (see also [56]) the eigenvalues of \(-\Delta: \mathcal{D}(-\Delta) \subset L^2(T^2) \rightarrow L^2(T^2) \) are non-negative integers \( n \) that satisfies \( n = k^2 + l^2 \) \( (k, l \in \mathbb{N}) \), with the corresponding eigenfunctions
\[
\cos(kx \pm ly), \ \sin(kx \pm ly), \ \cos(lx \pm ky) \ \text{and} \ \sin(lx \pm ky),
\]
for all possible \( k, l \in \mathbb{N} \). We give an order to these eigenvalues and denote them by new notations \( \{\lambda_i\}_{i \in \mathbb{N}} \) as follows
\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_i \leq \cdots \rightarrow \infty,
\]
with the unit eigenfunction of $\lambda_i$ denoted by $w_i$, where we have taken the multiplicities into account. In this way $\{w_i\}_{i \in \mathbb{N}}$ constitutes an orthonormal basis of $L^2(\mathbb{T}^2)$. The Sobolev space $H^\alpha(\mathbb{T}^2)$ ($\alpha \in \mathbb{R}$) is the Banach space such that
\[
u \in H^\alpha(\mathbb{T}^2) \quad \iff \quad \sum_{i \geq 0} (1 + \lambda_i)^\alpha |a_i|^2 < +\infty,
\]
which gives $u \in H^\alpha(\mathbb{T}^2)$ a norm
\[
\|u\|_\alpha := \left( \sum_{i \geq 0} (1 + \lambda_i)^\alpha |a_i|^2 \right)^{\frac{1}{2}}.
\]
Particularly, if $m$ is an integer, we also have
\[
\|(-\Delta + 1)^m u\| = \left( \sum_{i \geq 0} (1 + \lambda_i)^{2m} |a_i|^2 \right)^{\frac{1}{2}} = \|u\|_{2m}.
\]
This also makes $H^\alpha(\mathbb{T}^2)$ a Hilbert space for each $\alpha \in \mathbb{R}$ such that
\[(u, v)_{H^\alpha} := \sum_{i \geq 0} (1 + \lambda_i)^\alpha a_i b_i \quad \text{for} \quad u = \sum_{i \geq 0} a_i w_i, \quad v = \sum_{i \geq 0} b_i w_i.
\]
It is well known that the embedding of $H^\alpha$ into $H^\beta$ is dense and compact for all $\alpha > \beta$. Note that $L^2(\mathbb{T}^2) = H^0(\mathbb{T}^2)$. Denote also the norm of $L^p := L^p(\mathbb{T}^2)$ by $\| \cdot \|_p$ for all $2 < p \leq \infty$.

For notational simplicity, we denote
\[
H := L^2(\mathbb{T}^2), \quad H^\alpha := H^\alpha(\mathbb{T}^2), \quad L^p := L^p(\mathbb{T}^2), \quad W^{\alpha,p} := W^{\alpha,p}(\mathbb{T}^2)
\]
for $\alpha \in \mathbb{R}$ and $p \in [2, \infty]$, with norms of the latter two spaces denoted by $\| \cdot \|_p$ and $\| \cdot \|_{\alpha,p}$, respectively. Here we need to keep in mind that $L^2(\mathbb{T}^2) = L^2(I \times I)$, $W^{\alpha,p}(\mathbb{T}^2) = W^{\alpha,p}(I \times I)$ and $W^{\alpha,p}$ means the Sobolev space with the usual Sobolev norm, which is equivalent to the norm of $H^\alpha$ when $p = 2$. We also use $C^\infty$ to denote the set of all functions defined on $\mathbb{T}^2$ with derivatives of arbitrary order.

Let
\[
A = -\Delta + 1 : \mathcal{D}(A) \subset H \rightarrow H.
\]
We know the eigenvalues of $A$ are $\{\lambda_i + 1\}_{i \in \mathbb{N}}$ with the eigenfunctions $\{w_i\}_{i \geq 0}$, correspondingly. Recalling the basic knowledge of the fractional power of sectorial operator (see [24, 35]), we can define the fractional power $A^\alpha$ of $A$ as
\[
A^\alpha u = \sum_{i \geq 0} (\lambda_i + 1)^\alpha (u, w_i)w_i, \quad \text{for all} \quad u \in H^{2\alpha} \quad \text{and} \quad \alpha \in \mathbb{R}.
\]
We can deduce from (2.1) that $\mathcal{D}(A^\alpha) = H^{2\alpha}$ and $\|A^\alpha u\| = \|u\|_{2\alpha}$ for all $u \in H^{2\alpha}$.

In the process of completing our work, it is necessary to recall some spaces of fractional (in time) derivative and some compact embedding results (see [9, 16]). Let $X$ be a separable Hilbert space with the norm denoted by $\| \cdot \|_X$. For fixed $p > 1$ and $\alpha \in (0, 1)$, define
\[
W^{\alpha,p}(0, T; X) := \left\{ u \in L^p(0, T; X) : \int_0^T \int_0^T \frac{\|u(s) - u(\sigma)\|^p}{|s - \sigma|^{1+\alpha p}} d\sigma d\sigma < \infty \right\}
\]
with the norm
\[
\|u\|_{W^{\alpha,p}(0, T; X)} := \int_0^T \|u(s)\|^p_X ds + \int_0^T \int_0^T \frac{\|u(s) - u(\sigma)\|^p}{|s - \sigma|^{1+\alpha p}} d\sigma d\sigma.
\]
For the case when \( \alpha = 1 \), we take
\[
W^{1,p}(0,T;X) := \{ u \in L^p(0,T;X) : \frac{du}{dt} \in L^p(0,T;X) \}
\]
to be the classical Sobolev space with its usual norm
\[
\| u \|_{W^{1,p}(0,T;X)}^p := \int_0^T \left( \| u(s) \|_X^p + \left\| \frac{du(s)}{ds} \right\|_X^p \right) ds.
\]
Note that for \( \alpha \in (0,1) \),
\[
W^{1,p}(0,T;X) \subset W^{\alpha,p}(0,T;X) \text{ and } \| u \|_{W^{\alpha,p}(0,T;X)} \leq \| u \|_{W^{1,p}(0,T;X)}.
\] (2.2)
With these settings, we have the compact embeddings below [9, 16].

**Lemma 2.1.** (i) Suppose that \( X_2 \supset X_0 \supset X_1 \) are Banach spaces with \( X_2 \) and \( X_1 \) reflexive, and the embedding of \( X_1 \) into \( X_0 \) is compact. Then for each \( 1 < p < \infty \) and \( 0 < \alpha < 1 \), the embedding
\[
L^p(0,T;X_1) \cap W^{\alpha,p}(0,T;X_2) \subset \subset L^p(0,T;X_0)
\]
is compact (the notation \( \subset \subset \) is used to denote the compact embedding).

(ii) Suppose that \( Y_0 \supset Y \) are Banach spaces with \( Y \) compactly embedded in \( Y_0 \). Let \( \alpha \in (0,1] \) and \( p \in (1,\infty) \) be such that \( \alpha p > 1 \) then
\[
W^{\alpha,p}(0,T;Y) \subset \subset C([0,T];Y_0)
\]
and the embedding is compact.

2.2. On stochastic framework

As to determine the stochastic term \( \phi(u)\,dW \) in (1.1), we recall some basic knowledge of stochastic analysis (see [8] for more details).

Fix a stochastic basis \( \mathcal{S} := (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}, \{ W_t \}_{t \geq 0}) \), that is, a filtered probability space \( \Omega \) with a sequence \( \{ W_t \}_{t \geq 0} \) of independent standard one-dimensional Brownian motions relative to \( \mathcal{F}_t \). For the sake of avoiding unnecessary complications, we may assume that \( \mathcal{F}_t \) is complete and right continuous [8]. Fix a separable Hilbert space \( \mathcal{H} \) with an associated orthonormal basis \( \{ e_i \}_{i \geq 1} \). We may formally define \( W \) as
\[
W = \sum_{i \geq 0} W_i e_i,
\]
which is known as a 'cylindrical Brownian motion' over \( \mathcal{H} \).

Given another Hilbert space \( X \) with its associated inner product denoted by \( \langle \cdot, \cdot \rangle_X \). Let \( L_2(\mathcal{H}, X) \) be the separable Hilbert space of all Hilbert–Schmidt operators [8, appendix C] from \( \mathcal{H} \) to \( X \) with the inner product
\[
\langle \psi_1, \psi_2 \rangle = \sum_{i \geq 0} \langle \psi_1 e_i, \psi_2 e_i \rangle_X, \quad \text{for all } \psi_1, \psi_2 \in L_2(\mathcal{H}, X).
\]
To impose some conditions on the stochastic term, we introduce some new notations. Given each pair of Banach spaces \( X \) and \( Y \) with \( X \subset L^\infty \), we denote by \( \text{Bnd}_{1,\text{loc}}(X, Y) \), the collection of all continuous mappings \( \varphi : X \times [0, \infty) \rightarrow Y \) so that
\[
\| \varphi(x,t) \|_Y \leq \kappa(|x|_\infty)(1 + \| x \|_X), \quad x \in X, t \geq 0,
\]
where \(\kappa : \mathbb{R}^+ \to (0, \infty)\) is an increasing and locally bounded function, independently of \(t\). If, in addition,
\[
\|\varphi(x, t) - \varphi(y, t)\|_Y \leq \kappa(|x|_\infty + |y|_\infty)\|x - y\|_X,
\]
we say \(\varphi\) is in \(\text{Lip}_{\nu, \text{loc}}(X, Y)\). If the coefficient \(\kappa\) involved above is a constant, i.e. independent of \(t, x\) and \(y\), we denote \(\text{Bnd}_\nu(X, Y)\) and \(\text{Lip}_\nu(X, Y)\) instead, correspondingly. Obviously \(\text{Lip}_{\nu, \text{loc}}(X, Y) \subset \text{Bnd}_\nu(X, Y)\) and \(\text{Lip}_\nu(X, Y) \subset \text{Bnd}_\nu(X, Y)\).

Thus the stochastic term in (1.1) can be written as the formal expansion
\[
\phi(u)\,dW = \sum_{i \geq 0} \phi_i(u)\,dW_i, \quad \text{with} \quad \phi_i(u) = \phi(u)e_i.
\]

Let \(m \geq 1\). For the global existence of martingale and pathwise solutions in sections 4 and 5, we assume that the mapping \(\phi\) satisfies
\[
\phi \in \text{Lip}_\nu(H, L^2(\Omega, H)) \cap \text{Lip}_\nu(H^{2m}, L^2(\Omega, H^{2m})).
\]
For the existence of ergodic invariant measures in section 6, we assume further that
\[
\phi \in \text{Lip}_\nu(H^{2(m+1)}, L^2(\Omega, H^{2(m+1)})).
\]
For the infinite regularity of the invariant measures discussed in section 7, we need to require that
\[
\phi \in \bigcap_{m=0}^{\infty} \text{Lip}_\nu(H^{2m}, L^2(\Omega, H^{2m})).
\]

such as \(\phi(u) \in L^2(\Omega, L^2)^3\), \(u \in L^2\), (we use \((\cdot, \cdot)_\Omega\) the inner product of \(\Omega\)) so that
\[
\phi(u) = k \sum_{i=1}^{\infty} (u, w_i) w_i \otimes e_i\quad \text{and} \quad \phi(u) \circ W = k \sum_{i=1}^{\infty} (u, w_i)(e_i, W_{\Omega, w_i}),
\]
for \(W \in \Upsilon\), where \(\otimes\) is the tensor product between Hilbert spaces.

We also assume additionally
\[
\phi \in \text{Bnd}_{\nu, \text{loc}}(H^{2m}, L^2(\Omega, H^{2m})).
\]

or
\[
\phi \in \text{Lip}_{\nu, \text{loc}}(H^{2m}, L^2(\Omega, H^{2m})),
\]

for local consequences in sections 4 and 5.

Given an arbitrary separable Hilbert space \(X\), let
\[
\varphi \in L^2(\Omega, L^p_{\text{loc}}([0, \infty), L^2(\Omega, X)))
\]
be an \(X\)-valued predictable process. The Burkholder–Davis–Gundy inequality (BDG inequality for short) holds in the following form,
\[
\mathbb{E}\sup_{t \in [0, T]} \left\| \int_0^t \varphi\,dW \right\|_X^p \leq c\mathbb{E}\left( \int_0^T \|\varphi\|_{L^2(\Omega, X)}^2 \,dt \right)^{\frac{p}{2}}, \quad \text{for all} \ p \geq 1,
\]
where \(c\) only depends upon \(p\). Applying a variation of BDG inequality, for all \(p \geq 2\) and \(\alpha \in [0, 1/2]\), we also have [9, 16]
\[
\mathbb{E}\left\| \int_0^t \varphi\,dW \right\|_{W^{\alpha, p}(0, T, X)}^p \leq c\mathbb{E}\int_0^T \|\varphi\|^p_{L^2(\Omega, X)} \,dt,
\]
for all \(X\)-valued predictable \(\varphi \in L^2(\Omega, L^p_{\text{loc}}([0, \infty), L^2(\Omega, X)))\).
We consider the initial condition \( u_0 \) to be random in general if there is no special instructions. Let \( m \geq 1 \) and \( p \geq 2 \). When we study the case of martingale solutions (the definition of martingale and pathwise solutions mentioned below can be found in section 2.3), the stochastic basis is unknown for the problem, and hence we are only able to specify \( u_0 \) as an initial probability measure \( \mu_0 \) on \( H^{2m} \), i.e., \( \mu_0(\cdot) = \mathbb{P}(u_0 \in \cdot) \), such that
\[
\int_{H^{2m}} ||u||_{2m}^p d\mu_0(u) < +\infty, \quad \text{with} \quad q \geq p.
\] (2.9)

For the global existence of martingale solutions, we further assume that
\[
\int_{H^{2m}} \left( ||u||_{2m}^q + ||u||^q' \right) \mu_0(u) < +\infty, \quad \text{with} \quad q \geq p \quad \text{and} \quad q' \geq (2m + 3)p.
\] (2.10)

For the case of pathwise solutions where the stochastic basis \( \mathcal{S} \) is fixed, we assume that, relative to this basis, \( u_0 \) is an \( H^{2m} \)-valued random variable and \( \mathcal{F}_0 \) measurable. For some auxiliary results during the procedure, we assume further that
\[
\mathbb{E}(||u_0||_{2m}^q) < +\infty, \quad \text{with} \quad q \geq p.
\] (2.11)

Moreover, for the existence of global solutions and invariant measures, we sometimes also need to assume that \( u_0 \) satisfies
\[
\mathbb{E}(||u_0||_{2m}^q + ||u_0||^{q'}) < +\infty \quad \text{with} \quad q \geq p \quad \text{and} \quad q' \geq (2m + 3)p,
\] (2.12)
where \( q' \) will be specified when necessary in the sequel.

2.3. On martingale and pathwise solutions

In this subsection, we give the definitions of martingale and pathwise solutions for the stochastic modified Swift–Hohenberg problem (1.1)–(1.3) with multiplicative noise. By the settings above, the original problem (1.1)–(1.3) can be rewritten as
\[
du + [A^2u + f(u)]dt = \phi(u)dW, \quad t > 0; \quad u(0) = u_0,
\] (2.13)
where \( f \) is given by
\[
f(u) := (a + 3)u - 4Au + b|\nabla u|^2 + u^3.
\]
We present the definitions of local and global solutions of (2.13) in both martingale and pathwise senses below.

**Definition 2.2.** Suppose \( \mu_0 \) is a probability measure on \( H^{2m} \) \((m \geq 1)\) satisfying (2.9). Assume that \( \phi \) satisfies (2.6) or (2.3).

(i) A triple \( (\mathcal{S}, u, \tau) \) is a **local martingale solution** if
\[
\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)
\]
is a stochastic basis, \( \tau \) is a stopping time relative to \( \mathcal{F}_t \) and \( u(\cdot \land \tau) : \Omega \times [0, \infty) \rightarrow H^{2m} \) is an \( \mathcal{F}_t \) adapted process such that
\[
u(\cdot \land \tau) \in L^2(\Omega; C([0, \infty); H^{2m} )) \cap L^2_{\text{loc}}(0, \infty; H^{2(m+1)})
\] (2.14)
the law of \( u(0) \) is \( \mu_0 \) and \( u \) satisfies for almost every \((t, \omega) \in [0, \infty) \times \Omega,\)

\[
u(t \wedge \tau) + \int_{0}^{t \wedge \tau} [A^2 u + f(u)] \, ds = u(0) + \int_{0}^{t \wedge \tau} \phi(u) \, dW
\]

(2.15)

with the equality understood in \( H^{2(m-1)}. \)

(ii) We say that the martingale solution \((S, u, \tau)\) is **global** if \( \tau = \infty \), for almost surely \( \omega \in \Omega. \)

We recall a convergence lemma for stochastic integrals (see [9, lemma 2.1]).

**Lemma 2.3.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a fixed probability space, \(X\) a separable Hilbert space. Consider a sequence of stochastic bases \(\mathcal{F}_n = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W^n)\), that is a sequence so that each \(W^n\) is cylindrical Brownian motion (over \(\Omega\)) with respect to \(\mathcal{F}_n\). Assume that \(\{G^n\}_{n \geq 1}\) are a collection of \(X\)-valued \(\mathcal{F}_n\) predictable processes such that \(G^n \in L^2(0, T; L^2(\Omega, X))\) almost surely. Finally consider \(\mathcal{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)\) and \(G \in L^2(0, T; L^2(\Omega, X))\), which is \(\mathcal{F}_t\) predictable. If

\[
W^n \to W \text{ in probability in } C([0, T]; \Omega),
\]

\[
G^n \to G \text{ in probability in } L^2(0, T; L^2(\Omega, X)),
\]

then

\[
\int_{0}^{T} G^n \, dW^n \to \int_{0}^{T} G \, dW \text{ in probability in } L^2(0, T; X).
\]

**Definition 2.4.** Let \(S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)\) be a stochastic basis. Suppose that \(u_0\) is an \(H^{2m}\)-valued \((m \geq 1)\) random variable (relative to \(S\)) and \(\mathcal{F}_0\)-measurable and \(\phi\) satisfies (2.7) or (2.3).

(i) A pair \((u, \tau)\) is a **local pathwise solution** of (2.13) if \(\tau\) is a strictly positive stopping time and \(u : \Omega \times \mathbb{R}^+ \to H^{2m}\) is an \(\mathcal{F}_t\)-adapt process satisfying (2.14) and (2.15) for each \(t \geq 0\) and \(\mathbb{P}\)-almost surely, in \(H^{2m}\).

(ii) Pathwise solutions of (2.13) are called (pathwise) **unique**, if given two pathwise solutions \((u, \tau)\) and \((u', \tau')\) such that \(u(0) = u'(0)\) on a subset \(\Omega_0\) of \(\Omega\), then

\[
\mathbb{P}((u(t) - u'(t)) \chi_{\Omega_0} = 0, \text{ for all } t \in [0, \tau \wedge \tau']) = 1,
\]

where \(\chi_A\) is the characteristic function of \(A\), i.e. \(\chi_A(x) = 1\), for \(x \in A\) and \(\chi_A(x) = 0\), for \(x \notin A\).

(iii) Suppose that \(\{\tau_n\}_{n \geq 1}\) is a strictly increasing sequence of stopping times converging to a (possibly infinite) stopping time \(\tau_0\) and assume that \(u\) is a predictable process in \(H\). We say that the triple \((u, \tau_0, \{\tau_n\}_{n \geq 1})\) is a **maximal pathwise solution** if \((U, \tau_n)\) is a local pathwise solution for each \(n\) and

\[
\sup_{x \in [0, \tau_0]} \|u(x)\|_2^2 + \int_{0}^{\tau_0} \|A^2 u(x)\|_2^2 \, ds = \infty
\]

almost surely on the set \(\{\tau_0 < \infty\}\). If, moreover, as \(n\) tends to the infinity,

\[
\sup_{x \in [0, \tau_n]} \|u(x)\|_2^2 + \int_{0}^{\tau_n} \|A^2 u(x)\|_2^2 \, ds \to +\infty,
\]

(2.16)
If \((2.17)\)\n
A sequence of \(X\)-valued random variables such as line to line and even in the same line. When the constant \(\text{if and only if for every subsequence of joint probability laws,} \\)

\[\text{Let } X \text{ be an arbitrary Banach space with its Borel}\]

\[\text{function } \delta \nu \text{ be an invariant measure for} \]

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\[\nu \text{ is involved below) and } \kappa \text{ (only when (2.3) is satisfied), and may be different from line to line and even in the same line. When the constant } c \text{ depends on some extra parameters, such as } m, p, k \text{ and } l, \text{ we denote it with subscripts } c_m, c_p, c_m, p, c_k \text{ or } c_l \text{ instead for emphasis}.\]

\[\text{In order to obtain pathwise solutions from martingale ones, we adopt a convergence conclusion in probability given in [23]. Let } \{Y_n\}_{n \geq 0}\text{ be a sequence of } X\text{-valued random variables on a probability space } (\Omega, \mathcal{F}, \mathbb{P}) \text{. Let } \{\mu_{n,m}\}_{n,m \geq 0}\text{ be the collection of joint laws of } \{Y_n\}_{n \geq 0}, \text{ that is}\n
\[\mu_{n,m}(E) := \mathbb{P}(Y_n, Y_m) \in E), \quad E \in \mathcal{B}(X \times X).\]

The convergence conclusion reads as follows.

**Proposition 2.5.** A sequence of \(X\)-valued random variables \(\{Y_n\}_{n \geq 0}\) converges in probability if and only if for every subsequence of joint probability laws, \(\{\mu_{n,m}\}_{k \geq 0}\), there exists a further subsequence which converges weakly to a probability measure \(\mu\) such that

\[\mu(\{(x,y) \in X \times X : x = y\}) = 1. \quad (2.17)\]

2.4. On invariant measures

Let \(X\) be an arbitrary Banach space with its Borel \(\sigma\)-algebra \(\mathcal{B}(X)\). We use \(\mathcal{B}_b(X)\) to denote the space of bounded Borel measurable functions on \(X\). In the following we consider the \(X\)-valued stochastic process \(u(t; x)\) with the initial datum \(x \in X\). For a set \(\Gamma \in \mathcal{B}(X)\), we define the transition functions

\[\mathcal{P}_t(x, \Gamma) = \mathbb{P}(u(t; x) \in \Gamma) \quad \text{for all } t \geq 0.\]

The Markovian transition semigroup \(\mathcal{P}_t\) on \(\mathcal{B}_b(X)\) is defined as

\[\mathcal{P}_t \varphi(x) = \mathbb{E}_x \varphi(u(t; x)) = \int_X \varphi(y) \mathcal{P}_t(x, dy), \quad t \geq 0, \quad \varphi \in \mathcal{B}_b(X), \quad x \in X.\]

Note also that \(\mathcal{P}_t(x, \Gamma) = \mathcal{P}_t \chi_{\Gamma}(x)\). The dual semigroup \(\mathcal{P}^*_t\) of \(\mathcal{P}_t\) is defined on and into the set of Borel probability measures \(\nu\) on \(X\) by

\[\mathcal{P}^*_t \nu(\Gamma) = \int_X \mathcal{P}_t(x, \Gamma) \nu(dx), \quad \text{for each } \Gamma \in \mathcal{B}(X).\]

The Markovian transition semigroup \(\mathcal{P}_t\) (\(t \geq 0\)) is said to be Feller if for arbitrary \(\varphi \in \mathcal{C}_b(X)\) and \(t > 0\), the mapping \(x \mapsto \mathcal{P}_t \varphi(x)\) is continuous. An invariant measure for the stochastic process \(u(t; x)\) is a probability measure \(\nu\) on \(X\), which is a fixed point for \(\mathcal{P}^*_t\), that is to say,

\[\int_X \mathcal{P}_t(x, \Gamma) \nu(dx) = \nu(\Gamma), \quad \text{for each } \Gamma \in \mathcal{B}(X) \text{ and } t \in \mathbb{R}^+.\]

Let \(\nu\) be an invariant measure for \(\mathcal{P}_t\). We say that \(\nu\) is ergodic if

\[\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathcal{P}_t \varphi dt = \int_X \varphi(x) \nu(dx) \quad \text{for all } \varphi \in L^2(X, \mu).\]

In what follows, we denote \(c\) as an arbitrary positive constant, which only depends on the parameters of the original problem and the assumptions, i.e. \(a, b, T^2, N\) (only when the cut-off function \(\delta_T\) is involved below) and \(\kappa\) (only when (2.3) is satisfied), and may be different from line to line and even in the same line. When the constant \(c\) depends on some extra parameters, such as \(m, p, k\) and \(l\), we denote it with subscripts \(c_m, c_p, c_{m,p}, c_k\) or \(c_l\) instead for emphasis.
3. Estimates on the Galerkin scheme for the cut-off System

In this section, we introduce the Galerkin method to open the discussions in the following two sections.

We first apply the cut-off function to the original problem, so as to make use of the global existence of solutions for finite-dimensional differential equations with globally Lipschitz drift terms and diffusion coefficients when adopting the Galerkin scheme. Specifically, we consider the cut-off system for (2.13),

\[ du + [A^2 u + \delta_N(\|u\|_2) f(u)] dt = \delta_N(\|u\|_2) \phi(u) dW, \quad t > 0; \quad u(0) = u_0, \]

where \( \delta_N : \mathbb{R}^+ \to [0, 1] \), \( N \in \mathbb{N}^+ \), is a smooth cut-off function such that \( \delta_N(r) = 1 \) for \( r \in [0, N] \) and \( \delta_N(r) = 0 \) for \( r \geq N + 1 \).

For the Galerkin scheme we will use below, we introduce the subspace \( H_n \) of \( H \), with dimension \( n \geq 0 \), such that

\[ H_n = \text{span} \{ w_0, w_1, \ldots, w_n \} \]

and denote \( P_n \) to be the projection operator from \( H \) onto \( H_n \). Let

\[ u^n(t) = \sum_{i=0}^{n} a_i(t) w_i \quad \text{and} \quad u^n_0 = P_n u_0. \]

Note that all norms of \( H_n \) are equivalent. We consider the finite-dimensional stochastic differential equation

\[ du^n + [A^2 u^n + \delta_N(\|u^n\|_2) f_n(u^n)] dt = \delta_N(\|u^n\|_2) P_n \phi(u^n) dW, \]

with \( u^n(0) = u^n_0 \), where \( f_n : H_n \to H_n \) is defined as

\[ f_n(u^n) = (a + 3) u^n - 4 A u^n + b P_n |\nabla u^n|^2 + P_n (u^n)^3. \]

There is no hard to check that the drift term and diffusion coefficient of (3.2) are globally Lipschitz continuous in the finite-dimensional space \( H_n \) by the involvement of the cut-off function \( \delta_N(\|u^n\|_2) \) (2.6). Referring to [15, section 5.2], we know that there exists a unique global solution \( u^n \) to (3.2) in \( H_n \) for each initial datum \( u^n_0 \) satisfying (2.9).

Following this we need the uniform estimates to deduce the tightness of the law of \( u^n \). We recall the Gagliardo–Nirenberg inequality (see [39, 49]) for the discussion below.

**Lemma 3.1.** Let \( U \) be an open, bounded domain of the Lipschitz class in \( \mathbb{R}^n \). Assume that \( \tilde{p} \geq 1, 1 \leq \tilde{q}, \tilde{r} \leq \infty, 0 \leq k \leq m, 0 < \theta \leq 1 \) and that

\[ k - \frac{n}{\tilde{p}} \leq \theta \left( m - \frac{n}{\tilde{q}} \right) - \left( 1 - \theta \right) \frac{n}{\tilde{r}}. \]

Then there is a positive constant \( C \) such that

\[ \|u\|_{W^{k,\tilde{p}}(U)} \leq C \|u\|_{W^{m,\tilde{q}}(U)}^{\theta} \|u\|_{L^\tilde{r}(U)}^{1-\theta}, \quad \text{for all} \ u \in W^{m,\tilde{q}}(U). \]

**Lemma 3.2.** Let \( \mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W) \) be a stochastic basis \( p \geq 2 \) and \( T \geq 0 \). Suppose that \( \phi \) satisfies (2.6). Given every \( n \geq 0 \) and each \( u_0 \) satisfying (2.11), there exists a positive constant \( C_1 \) independent of \( n \), such that the solution \( u^n \) of (3.2) has the following uniform estimates,

\[ \mathbb{E} \left( \sup_{s \in [0,T]} \|A^m u^n(s)\|^p + \int_0^T \|A^m u^n\|^{p-2} \|A^{m+1} u^n\|^2 ds \right) \leq C_1. \]
We first show (3.4). First, we see that

\[ \mathbb{E}\left( \int_0^T \| A^{m+1} u^n(s) \|^2 ds \right)^{\frac{2}{p}} \leq C_1, \]

(3.4)

\[ \mathbb{E}\left( \left\| A^{m-1} u^n(t) - \int_0^t \delta_x(\| u^n \|_2) P_n A^{m-1} \phi(u^n(s)) ds \right\|^2 \right) \leq C_1, \]

(3.5)

and given \( \alpha \in (0, 1/2) \) additionally, it holds that

\[ \mathbb{E}\left( \int_0^T \delta_x(\| u^n \|_2) P_n A^{m} \phi(u^n) dW \right)^p \leq C_1. \]

(3.6)

**Proof.** We first show (3.3). Apply \( A^m \) to (3.2) and Itô’s Formula to \( \| A^m u^n \|_p \). We obtain that for all \( s \in [0, T] \),

\[ \frac{1}{p} d\| A^m u^n \|_p + \| A^m u^n \|_p^2 - \| A^{m+1} u^n \|_2^2 \]

\[ = -\delta_x(\| u^n \|_2) \| A^m u^n \|_p^2 \left( A^m u^n, A^m f_n(u^n) \right) ds \]

\[ + \frac{1}{2} \delta_x(\| u^n \|_2) \| A^m u^n \|_p^4 \left( (p-2)(A^m u^n, P_n A^m \phi(u^n)) \right)_{L^2(\Omega, \mathbb{R})} \]

\[ + \| A^m u^n \|_2^2 \| P_n A^m \phi(u^n) \|_2^2 \]

\[ + \delta_x(\| u^n \|_2) \| A^m u^n \|_p^2 \left( (A^m u^n, P_n A^m \phi(u^n)) dW \right) \]

\[ := (\tilde{P}_n^1 + \tilde{P}_n^2) ds + \tilde{P}_n^3 dW. \]

For arbitrary stopping times \( \tau' \) and \( \tau'' \) with \( 0 \leq \tau' \leq \tau'' \leq T \), we have

\[ \frac{1}{p} \| A^m u^n(s) \|_p + \int_{\tau'}^\tau \| A^m u^n \|_p^2 - \| A^{m+1} u^n \|_2^2 \]

\[ \leq \frac{1}{p} \| A^m u^n(\tau') \|_p + \int_{\tau'}^\tau \left( |\tilde{P}_n^1| + |\tilde{P}_n^2| \right) ds' + \left| \int_{\tau'}^\tau \tilde{P}_n^3 dW \right|. \]

(3.7)

Take the expectation of the supremum over \( s \in [\tau', \tau''] \) for (3.7), we have

\[ \mathbb{E}\left( \frac{1}{p} \sup_{s \in [\tau', \tau'']} \| A^m u^n(s) \|_p + \int_{\tau'}^\tau \| A^m u^n \|_p^2 - \| A^{m+1} u^n \|_2^2 \right) \]

\[ \leq \frac{1}{p} \mathbb{E}\| A^m u^n(\tau') \|_p + \mathbb{E} \int_{\tau'}^\tau \left( |\tilde{P}_n^1| + |\tilde{P}_n^2| \right) ds' \]

\[ + \mathbb{E} \sup_{s \in [\tau', \tau'']} \left| \int_{\tau'}^\tau \tilde{P}_n^3 dW \right|. \]

(3.8)

The key difficulty here lies in the estimation of \( (A^m u^n, A^m f_n(u^n)) \) in \( P_1 \). We are now devoted to addressing it. First, we see that

\[ |(A^m u^n, A^m f_n)| \leq \frac{2p - 1}{6p} \| A^{m+1} u^n \|^2 + c_p \| A^m u^n \|^2. \]

(3.9)

Observe by (2.1) that

\[ |(A^m u^n, A^m \nabla u^n)| = \| A^{m+1} u^n, A^{m-1} \nabla u^n \|^2 \]

\[ \leq \| A^{m+1} u^n \| \| A^{m-1} \nabla u^n \|^2 \]

(3.10)
for every $m \in \mathbb{N}^+$. We can expand $A^{m-1}|\nabla u|^2$ to be the sum of finitely many (set to be $l_1$) summands (with each coefficient being 1) of the following form

$$
\frac{\partial^m u^n}{\partial x^{m_1} \partial y^{m_2}} \frac{\partial^m u^n}{\partial x^{m_1} \partial y^{m_2}}
$$

with $m_1 + m_2 = m_i \geq 1, \quad i = 1, 2,$

$$
2 \leq m_j := m_1 + m_2 \leq 2m, \quad j = 1, \ldots, l_1.
$$

(3.11)

It is easy to see that $m_i \leq 2m - 1$ for $i = 1, 2$. By selecting $\theta = \frac{2m+1}{3(m+1)}$ in lemma 3.1, we can deduce that

$$
\left\| \frac{\partial^m u^n}{\partial x^{m_1} \partial y^{m_2}} \right\|_{L^1} \leq \|u^n\|_{m,m_4} \leq c_m \|u^n\|^{\frac{2m+1}{2(m+1)}} \|u^n\|^{\frac{4m-2m_i+1}{2(m+1)}}
$$

and hence with (3.11) in mind,

$$
\left\| \frac{\partial^m u^n}{\partial x^{m_1} \partial y^{m_2}} \right\| \leq \|u^n\|_{m_1, A} \|u^n\|_{m_2, A}
$$

$$
\leq c_m \|u^n\|^{\frac{4m-2m_i+1}{2(m+1)}} \|u^n\|^{\frac{4m-2m_i+1}{2(m+1)}}.
$$

Now we get back to (3.10) and immediately infer that

$$
\|A^{m+1}u^n\| |A^{m-1}|\nabla u|^2|
$$

$$
\leq c_m \sum_{l=1}^{l_1} \|A^{m+1}u^n\|^{\frac{2m+1}{2(m+1)}} \|u^n\|^{\frac{4m-2m_i+1}{2(m+1)}}
$$

$$
\leq \sum_{l=1}^{l_1} \left( \frac{2p - 1}{6l_1p} \|A^{m+1}u^n\|^2 + c_m \|u^n\|^{\frac{2(4m-2m_i+1)}{2(m+1)}} \right)
$$

$$
\leq \frac{2p - 1}{6p} \|A^{m+1}u^n\|^2 + c_m \left( \|u^n\|^{2(2m+3)} + 1 \right),
$$

(3.12)

where we have used $\varepsilon$-Young inequality and noticed that $\frac{4m-2m_i+1}{2m-m_i+1} \leq 2m + 3$. Next we treat the second nonlinear term $P_3(u^n)^3$ of $f_3(u^n)$. Similarly, we know

$$
|\left( A^{m}u^n, A^{m}(u^n)^3 \right) | = |\left( A^{m+1}u^n, A^{m-1}(u^n)^3 \right) | \leq \|A^{m+1}u^n\| \|A^{m-1}(u^n)^3\|.
$$

(3.13)

And then $A^{m-1}(u^n)^3$ can be expanded as the sum of $l_2$ summands (with each coefficient being 1) of the form

$$
\frac{\partial^m u^n}{\partial x^{m_1} \partial y^{m_2}} \frac{\partial^m u^n}{\partial x^{m_1} \partial y^{m_2}} \frac{\partial^m u^n}{\partial x^{m_1} \partial y^{m_2}}
$$

with $m_1 + m_2 = m_i$

and $\bar{m}_k := m_1 + m_2 + m_3 \leq 2m - 2, \quad i = 1, 2, 3, \quad k = 1, \ldots, l_2$. By choosing $\theta = \frac{3m+2}{6(m+1)}$ in lemma 3.1, one deduces that

$$
\left\| \frac{\partial^m u^n}{\partial x^{m_1} \partial y^{m_2}} \right\|_{L^1} \leq \|u^n\|_{m, A} \leq c_m \|u^n\|^{\frac{3m+2}{2(m+1)}} \|u^n\|^{\frac{6m-3m_i+4}{2(m+1)}}
$$

and

$$
\left\| \frac{\partial^m u^n}{\partial x^{m_1} \partial y^{m_2}} \right\| \leq c_m \|u^n\|^{\frac{3m+2}{2(m+1)}} \|u^n\|^{\frac{6m-3m_i+4}{2(m+1)}}.
$$
And hence

\[
\|A^{m+1}u^p\|\|A^{m-1}(u^p)^3\| \\
\leq \sum_{l=1}^{l^2} \|A^{m+1}u^p\|^{2m+4+\delta} \|u^p\|^{2m-\delta+4},
\]

\[
\leq \sum_{l=1}^{l^2} \left( \frac{2p-1}{6l^2p} \|A^{m+1}u^p\|^2 + c_{m,p} \|u^p\|^{2(2m+3)+1} \right),
\]

where the relation \( \frac{2m-\delta+4}{2m-\delta} \leq 2m+3 \) is used. Combining the inequalities from (3.9) to (3.14), we obtain

\[
|F_p^l| \leq \frac{2p-1}{2p} \|A^m u^p\|^{p-2} \|A^{m+1}u^p\|^2 \\
+ c_{m,p} \delta_N(\|u^p\|_2) \left( \|A^m u^p\|^p + \|A^m u^p\|^{p-2} \left( \|u^p\|^{2(2m+3)+1} + 1 \right) \right) \\
\leq \frac{2p-1}{2p} \|A^m u^p\|^{p-2} \|A^{m+1}u^p\|^2 \\
+ c_{m,p} \delta_N(\|u^p\|_2) \left( \|A^m u^p\|^p + \|u^p\|^{2(2m+3)p} + 1 \right).
\]

(3.15)

For \( F_2^l \), by (2.6), we have

\[
\|A^m \phi(u^p)\|_{L^1(\Omega)} \leq \kappa(\|u^p\|_\infty) (1 + \|A^m u^p\|)
\]

and hence

\[
\|A^m u^p, P_pA^m \phi(u^p)\|_{L^1(\Omega)}^2 \leq 2\kappa^2(\|u^p\|_\infty)\|A^m u^p\|^2 \left( 1 + \|A^m u^p\|^2 \right).
\]

Therefore, we obtain

\[
|F_2^l| \leq c\delta_N(\|u^p\|_2)\|A^m u^p\|^{p-2}\kappa^2(\|u^p\|_\infty)(1 + \|A^m u^p\|)^2 \\
\leq c_p \delta_N(\|u^p\|_2)\kappa^2(\|u^p\|_\infty)(1 + \|A^m u^p\|^p).
\]

(3.17)

For the stochastic term, using Hölder’s inequality, the BDG inequality and (3.16), we have

\[
E \sup_{t \in [\tau', \tau'')} \left| \int_{\tau'}^\tau F_2^l dW \right| \\
\leq c_p E \left( \int_{\tau'}^{\tau''} \delta_N(\|u^p\|_2)\|A^m u^p\|^{2(p-1)}\kappa^2(\|u^p\|_\infty)(1 + \|A^m u^p\|^2) ds' \right)^{\frac{1}{2}} \\
\leq c_p E \left( \sup_{t \in [\tau', \tau'')} \|A^m u^p(s)\|^p \right)^{\frac{1}{2}} \\
\times \left( \int_{\tau'}^{\tau''} \delta_N(\|u^p\|_2)\kappa^2(\|u^p\|_\infty)\|A^m u^p\|^{p-2}(1 + \|A^m u^p\|^2) ds' \right)^{\frac{1}{2}},
\]

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and hence
\[
E \sup_{s \in [\tau', \tau'']} \left| \int_{\tau'}^{\tau''} F^s_3 dW \right| \leq c_p E \int_{\tau'}^{\tau''} \delta_N(\|u^\alpha\|_2) \kappa^2(\|u^\alpha\|_\infty) \left( 1 + \|A^m u^\alpha\|^p \right) ds' + \frac{1}{2^p} E \sup_{s \in [\tau', \tau'']} \|A^m u^\alpha(s)\|^p .
\] (3.18)

It follows from (3.8), (3.15), (3.17), (3.18) that
\[
E \left( \sup_{s \in [\tau', \tau'']} \|A^m u^\alpha(s)\|^p + \int_{\tau'}^{\tau''} \|A^m u^\alpha\|^p - \|A^{m+1} u^\alpha\|^p \right) \leq c_m E \int_{\tau'}^{\tau''} \delta_N(\|u^\alpha\|_2) \left( 1 + \kappa^2(\|u^\alpha\|_\infty) \right) \left( 1 + \|A^m u^\alpha\|^p \right) + \|u^\alpha\|^{(2m+3)p} ds' + 2E \|A^m u^\alpha(\tau')\|^p .
\] (3.19)

In order that the expectation in (3.19) makes sense, we define
\[
\tau_R := \inf \{ t \geq 0 : \|A^m u^\alpha(t)\| > R \} .
\]

Since \(H^2 \subset L^\infty \subset H\), then \(\kappa\) and \(\|u^\alpha\|\) is bounded over the set \(\{ u^\alpha : \delta_N(\|u^\alpha\|_2) > 0 \}\), which excludes the set \(\{ u^\alpha : \|u^\alpha\|_\infty > \varepsilon(N + 1) \}\). Here we use \(\varepsilon\) to denote the embedding constant that \(\|u\|_\infty \leq \varepsilon\|u\|_2\) for each \(u \in H^2\). We thus note that the upper bounds of the terms
\[
\delta_N(\|u^\alpha\|_2) \left( 1 + \kappa^2(\|u^\alpha\|_\infty) \right) \text{ and } \delta_N(\|u^\alpha\|_2) \kappa^2(\|u^\alpha\|_\infty) |u^\alpha|^{(2m+3)p}
\]
only depend upon \(N\) (via \(\delta_N\) and \(\kappa\)), but independent of \(R\) and \(n\). Now we can apply the stochastic Gronwall’s inequality (see [21, 49]) to (3.19) and obtain
\[
E \left( \sup_{s \in [0, T \land \tau_R]} \|A^m u^\alpha(s)\|^p + \int_0^{T \land \tau_R} \|A^m u^\alpha\|^p - \|A^{m+1} u^\alpha\|^p \right) \leq C_1,
\] (3.20)

where \(C_1 = C'_1(a, b, p, T, N, u_0, \kappa, \Theta^2)\) is a positive constant independent of \(R\) and \(n\). Now let \(R \to \infty\). One sees \(T \land \tau_R \to T\) in (3.20) and hence (3.3) by the dominated convergence theorem.

Next, we show (3.4). By replacing \(\tau'\) and \(s\) by 0 and \(s\) respectively and setting \(p = 2\) in (3.7), we can obtain
\[
\int_0^\tau \|A^{m+1} u^\alpha\|^2 ds' \leq \frac{1}{2} \|A^m u^\alpha_0\|^2 + \int_0^\tau (|F_1^s| + |F_3^s|) ds' + \left| \int_0^\tau F_3^s dW \right|.
\]

By (3.15) with a slight adjustment to the parameters and (3.17), we can also get
\[
\int_0^\tau \|A^{m+1} u^\alpha\|^2 ds' \leq c_m \int_0^\tau \delta_N(\|u^\alpha\|_2) \left( 1 + \kappa^2(\|u^\alpha\|_\infty) \right) \left( 1 + \|A^m u^\alpha\|^2 \right) + |u^\alpha|^{(2m+3)} ds' + \|A^m u^\alpha_0\|^2 + \left| \int_0^\tau F_3^s dW \right|.
\] (3.21)

Similar to (3.18) and using BDG inequality, we know
\[
E \sup_{s \in [0, \tau]} \left| \int_0^\tau F_3^s dW \right|^\xi \leq c_p E \left( \int_0^\tau \delta_N(\|u^\alpha\|_2) \kappa^2(\|u^\alpha\|_\infty) \left( 1 + \|A^m u^\alpha\|^2 \right) ds' \right)^\xi
\]
\[
+ \frac{c_p}{2^p} E \left( \sup_{s \in [0, \tau]} \|A^m u^\alpha(s)\|^\xi \right)
\]

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\[ \leq c_p \mathbb{E} \sup_{s \in [0, T]} \| A^m u^p(s) \|^p \]
\[ + c_p \mathbb{E} \int_0^T \delta_N(\| u^p \|_2) \kappa^p(\| u^p \|_{\infty}) (1 + \| A^m u^p \|^p) \, ds'. \]

(3.22)

Then considering the \( p/2 \) power of (3.21) and taking the expected value of the supremum, we have by (3.22)

\[
\mathbb{E} \left( \int_0^T \| A^{m+1} u^p \|^2 \, ds' \right)^{\frac{p}{2}} \leq c_{m,p} \mathbb{E} \| A^m u_0^p \|^p + c_{p} \mathbb{E} \sup_{s \in [0, T]} \| A^m u^p(s) \|^p
\]
\[ + c_{m,p} \mathbb{E} \int_0^T \delta_N(\| u^p \|_2) \| u^p \|^{(2m+3)p} \, ds' \]
\[ + c_{m,p} \mathbb{E} \int_0^T \delta_N(\| u^p \|_2) (1 + \kappa^p(\| u^p \|_{\infty})) (1 + \| A^m u^p \|^p) \, ds'. \]

(3.23)

The inequality (3.4) follows immediately from (3.3), (3.23) and the properties of \( \delta_N \) and \( \kappa \).

Now we show (3.5). Applying (3.2), we have the following equation with stochastic integral in \( H \),

\[
A^{m-1} u^p(t) = \int_0^t \delta_N(\| u^p \|_2) P_n A^{m-1} \phi(u^p) \, dW
\]
\[ = A^{m-1} u_0^p - \int_0^t A^{m+1} u^p \, ds - \int_0^t \delta_N(\| u^p \|_2) P_n A^{m-1} f_n(u^p) \, ds
\]
\[ := J_1 + J_2 + J_3. \]

(3.24)

Note that

\[
\mathbb{E} \| J_1 \|^2 \leq \mathbb{E} \| A^{m-1} u_0 \|^2.
\]

(3.25)

By (3.3) with \( p = 2 \), we obtain

\[
\mathbb{E} \| J_2 \|_{W^{2,2}(0, T; H)}^2 = \mathbb{E} \int_0^T (\| J_2(s) \|^2 + \| A^{m+1} u^p \|^2) \, ds
\]
\[ \leq (T^2 + 1) \mathbb{E} \int_0^T \| A^{m+1} u^p \|^2 \, ds \leq C_2.
\]

(3.26)

According to the discussion from (3.10) to (3.14), we can similarly have

\[
\mathbb{E} \| J_3 \|_{W^{2,2}(0, T; H)}^2 \leq (T^2 + 1) \mathbb{E} \int_0^T \delta_N(\| u^p \|_2) \| A^{m-1} f_n(u^p(t)) \|^2 \, ds
\]
\[ \leq c(T^2 + 1) \mathbb{E} \int_0^T \delta_N(\| u^p \|_2) \left( \| A^{m+1} u^p \|^2 + \| u^p \|^2 + \| A^m u^p \|^2 \right) \, ds
\]
\[ \leq C_3'. \]

(3.27)

Then (3.5) follows immediately from (3.24) to (3.27).
To prove (3.6), since $\alpha \in (0,1/2)$, we adopt (2.8), (2.6) and (3.3) and obtain
\[
E \left\| \int_0^t \delta_N(||u^n||_2) \mathbb{P}(u^n, \phi(u^n(s)))dW \right\|_{W^{0,2}(0,T,H)}^p \leq c E \left( \int_0^t \delta_N(||u^n||_2) \kappa^p (||u^n||_\infty) (1 + \|A^{m}u^n\|^p)ds \right) \leq C_4',
\]
where we used the properties of $\delta_N$ and $\kappa$ that have been applied above. This is (3.6). The proof is complete.

4. Local and global existence of martingale solutions

4.1. Local existence of martingale solutions

Let $\mu_0$ be a given initial distribution on $H^2$. We fix a stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)$ upon which is defined an $\mathcal{F}_0$ measurable random element $u_0$ with distribution $\mu_0$. Consider the sequence of Galerkin approximations $(u^n)_{n \geq 1}$ solving (3.2) relative to this basis and initial condition. We consider the phase spaces
\[
\mathcal{X}_u = L^2(0,T;H^{2m}) \cap C([0,T];H^2), \quad \mathcal{X}_w = C([0,T];\Omega)
\]
and $\mathcal{X} = \mathcal{X}_u \times \mathcal{X}_w$, with $\mathcal{X}_u$ the space where the solution $u^n$ lives and $\mathcal{X}_w$ the set where the driving Brownian motions are defined. We consider the probability measures
\[
\mu^n_u(\cdot) = \mathbb{P}(u^n(\cdot)) \quad \text{and} \quad \mu^n_w(\cdot) = \mathbb{P}(W(\cdot)) \quad \text{in} \quad \mathcal{X}.
\]
This defines a sequence of probability measures $\mu^n := \mu^n_u \otimes \mu^n_w$ on the phase space $\mathcal{X}$. We now show that this sequence is tight as follows by using lemma 3.2.

Lemma 4.1. Suppose that $\phi$ satisfies (2.6) and $\mu_0$ satisfies (2.9). Then the sequence $(\mu^n)_{n \geq 1}$ is tight and weakly compact over the phase space $\mathcal{X}$.

Proof. Applying lemma 2.1 (i) by setting $X_2 = H^2$, $X_0 = H^{2m}$, $X_1 = H^{2(m+1)}$, $p = 2$ and $\alpha \in (0,1/2)$, we have that
\[
L^2(0,T;H^{2(m+1)}) \cap W^{0,2}(0,T;H^{2(m-1)}) \subset L^2(0,T;H^{2m})
\]
Given $R > 1$, define a set
\[
B_R^1 = \{ u \in L^2(0,T;H^{2(m+1)}) \cap W^{0,2}(0,T;H^{2(m-1)}): ||u^n||_{L^2(0,T;H^{2(m+1)})}^2 + ||u^n||_{W^{0,2}(0,T;H^{2(m-1)})}^2 \leq R^2 \}.
\]
Then $B^1_R$ is compact in $L^2(0,T;H^{2m})$. The Chebyshev inequality applies and we have
\[
\mu^n_u((B_R^1)^C) = \mathbb{P} \left( \left( ||u^n||_{L^2(0,T;H^{2(m+1)})}^2 + ||u^n||_{W^{0,2}(0,T;H^{2(m-1)})}^2 \right) \geq R^2 \right) \leq \mathbb{P} \left( \left( ||u^n||_{L^2(0,T;H^{2(m+1)})}^2 \geq \frac{R^2}{2} \right) \right) + \mathbb{P} \left( \left( ||u^n||_{W^{0,2}(0,T;H^{2(m-1)})}^2 \geq \frac{R^2}{2} \right) \right) \leq \frac{2}{R^2} E \left( \int_0^T \|A^{m+1}u^n(s)\|^2ds + \|A^{m}u^n\|^2_{W^{0,2}(0,T;H^{2(m-1)})} \right) \leq \frac{C_4'}{R^2},
\]

(4.4)
independent of \( n \), where we have used (3.3) with \( p = 2 \), (3.6), (3.5) and (2.2).

Take \( \alpha \in \left( \frac{3}{4}, \frac{5}{4} \right) \) with \( \alpha q > 1 \). By lemma 2.1 (ii) with \( Y_0 = H^{2(m-2)} \) and \( Y = H^{2(m-1)} \), we have the compact embeddings

\[
W^{1,2}(0, T; H^{2(m-1)}) \subset C([0, T]; H^{2(m-2)}),
\]

\[
W^{\alpha,q}(0, T; H^{2(m-1)}) \subset C([0, T]; H^{2(m-2)}).
\]

Given \( R > 1 \), let \( B_R^{21} \) and \( B_R^{22} \) be the closed balls of radius \( R \) in the spaces \( W^{1,2}(0, T; H^{2(m-1)}) \) and \( W^{\alpha,q}(0, T; H^{2(m-1)}) \) respectively. It follows that for \( R > 1 \), \( B_R := B_R^{21} + B_R^{22} \) is compact in \( C([0, T]; H^{2(m-2)}) \). Due to the inclusion

\[
\{ u^n \in B_R^2 \} \supset \left\{ u^n(t) - \int_0^t P_n \phi(u^n) dW \in B_R^{21} \right\} \cap \left\{ \int_0^t P_n \phi(u^n) dW \in B_R^{22} \right\},
\]

we obtain by Chebyshev inequality and (3.6), (3.5) that

\[
\mu^n_\nu(B_R^2)^C \leq \mathbb{P} \left( \left\| u^n(t) - \int_0^t P_n \phi(u^n) dW \right\|_{W^{2,2}(0, T; H^{2(m-1)})} \geq R^2 \right)
\]

\[
+ \mathbb{P} \left( \left\| \int_0^t P_n \phi(u^n) dW \right\|_{W^{\alpha,q}(0, T; H^{2(m-1)})} \geq R^1 \right) \leq \frac{C_6}{R^2}, \tag{4.5}
\]

also independent of \( n \).

It is trivial that \( B_R^2 \) is compact in \( L^2(0, T; H^{2m}) \cap C([0, T]; H^{2(m-2)}) \) for every \( R > 0 \). It follows from (4.4) and (4.5) that

\[
\mu^n_\nu(B_R^2)^C \leq \mu^n_\nu(B_R^2)^C + \mu^n_\nu(B_R^2)^C \leq \frac{C_5}{R^2} + \frac{C_6}{R^2}. \tag{4.6}
\]

Then for every \( \varepsilon > 0 \), we are allowed to pick a set \( A_\varepsilon \) (by increasing \( R \) for \( B_R^2 \)) such that

\[
\mu^n_\nu(A_\varepsilon) \geq 1 - \frac{\varepsilon}{2}, \quad \text{for all} \ n \geq 0.
\]

Now we consider the sequence \( \{\mu^n_\nu\}_{n \geq 0} \), which actually identically equals to \( \mu_W \), and is hence weakly compact. By Prokhorov’s Theorem [8], \( \{\mu^n_\nu\}_{n \geq 0} \) is surely tight. This helps find a compact set \( B_\varepsilon \) in \( C([0, T]; \mathbb{L}_0) \) such that for all \( n \geq 0 \),

\[
\mu^n_\nu(B_\varepsilon) \geq 1 - \frac{\varepsilon}{2}. \tag{4.7}
\]

Combining (4.6) and (4.7), we know that for every \( \varepsilon > 0 \), the compact set \( A_\varepsilon \times B_\varepsilon \) in \( \mathcal{X} \) satisfies that for all \( n \geq 0 \),

\[
\mu^n_\nu(A_\varepsilon \times B_\varepsilon) \geq 1 - \varepsilon,
\]

and therefore \( \{\mu^n_\nu\}_{n \geq 0} \) is tight in \( \mathcal{X} \) and finally weakly compact. The proof is finished.

Now we can deduce the following theorem to guarantee the existence of martingale solutions of (3.1).

**Theorem 4.2.** Suppose that \( \phi \) satisfies (2.6) and \( \mu_0 \) is a probability measure on \( H^{2m} \) satisfying (2.9). Then there exists a subsequence \( n_k \) and a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), on which there lies a sequence of \( \mathcal{X} \)-valued random variables \( \{\tilde{u}^n, \tilde{W}^n\} \) such that

(i) \( \{\tilde{u}^n, \tilde{W}^n\} \) converges almost surely, in the topology of \( \mathcal{X} \), to an element \( (\bar{u}, \bar{W}) \in \mathcal{X} \), the law of \( \{\tilde{u}^n, \tilde{W}^n\} \) is \( \mu^n \) for each \( k \) and \( \mu^n \) weakly converges to the law \( \mu \) of \( (\bar{u}, \bar{W}) \).
(ii) \( \tilde{W}^{u} \) is a cylindrical Wiener process, relative to the filtration \( \tilde{\mathcal{F}}^{u} \), given by the completion of
\[
\sigma((\tilde{u}^{u}(s), \tilde{W}^{u}(s)); s \leq t).
\]

(iii) every pair \((\tilde{u}^{u}, \tilde{W}^{u})\) satisfies (3.2) with only \( u^{\rho} \) and \( W \) replaced by \( \tilde{u}^{u} \) and \( \tilde{W}^{u} \) therein.

Let \( \tilde{S} = (\tilde{T}, \tilde{\mathcal{F}}, \{\tilde{F}_{t}\}_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{W}) \), with \( \tilde{\mathcal{F}} \), the completion of \( \sigma((\tilde{u}(s), \tilde{W}(s)); s \leq t) \). Then \((\tilde{S}, \tilde{u})\) is a global martingale solution of (3.1). As a result, by defining a stopping time
\[
\rho_{N} := \inf\{t \geq 0 : \|\tilde{u}(t)\|_{2} > N\}, \quad \text{for } N > 0, \tag{4.8}
\]
the triple \((\tilde{S}, \tilde{u}, \rho_{N})\) is a local martingale solution of (2.13).

**Remark 4.3.** Here the martingale solution of (3.1) is a slight modification of that of (2.13), i.e. definition 2.2. Similarly, the pathwise solution of (3.1) adopted in the next section is also a slight modification of definition 2.4.

**Proof of theorem 4.2.** The conclusion (i) is easily obtained by the Skorohod embedding theorem (see [8, theorem 2.4]). The conclusions (ii) and (iii) can be ensured by a similar procedure to that in [1, section 4.3.4].

Next we show that \((\tilde{S}, \tilde{u})\) is a global martingale solution of (3.1) through three steps.

**Step 1. Improvement of regularity on the phase space**

Observe from the conclusion (i) that
\[
\tilde{u}^{u} \to \tilde{u} \quad \text{in } \mathcal{X}_{\sigma} \text{ for } \mathbb{P} \text{ a.s. } \omega \in \tilde{\Omega}, \tag{4.9}
\]
\[
\tilde{W}^{u} \to \tilde{W} \quad \text{in } \mathcal{X}_{\rho} \text{ for } \mathbb{P} \text{ a.s. } \omega \in \tilde{\Omega}. \tag{4.10}
\]

According to the conclusion (iii) above, we know \( \tilde{u}^{u} \) has the same estimates as \( u^{\rho} \) stated in lemma 3.2. Using Banach–Alaoglu theorem, (3.3) and (3.6) for \( \tilde{u}^{u} \), we can obtain \( \overline{u} \in L^{2}(\tilde{\Omega}; L^{2}(0, T; H^{2(m+1)})) \) and \( \overline{u} \in L^{p}(\tilde{\Omega}; L^{\infty}(0, T; H^{2m})) \) such that
\[
\tilde{u}^{u} \to \overline{u} \quad \text{in } L^{2}(\tilde{\Omega}; L^{2}(0, T; H^{2(m+1)})) \text{ weakly, and} \tag{4.11}
\]
\[
\tilde{u}^{u} \to \overline{u} \quad \text{in } L^{p}(\tilde{\Omega}; L^{\infty}(0, T; H^{2m})) \text{ weakly star}, \tag{4.12}
\]
where the sequence \( \{\tilde{u}^{u}\} \) is perhaps chosen to be a subsequence. On account of (2.9) and with lemma 3.2 applied, we have
\[
\sup_{k \in \mathbb{N}} \mathbb{E} \sup_{t \in [0, T]} \|A^{m-2} \tilde{u}^{u}\|^p \leq c \sup_{k \in \mathbb{N}} \mathbb{E} \sup_{t \in [0, T]} \|A^{m} \tilde{u}^{u}\|^p < \infty.
\]

Then use the Vitali convergence theorem (see [18]) to the convergence in the conclusion (i) and base the estimation of (3.3), we have
\[
\tilde{u}^{u} \to \tilde{u} \quad \text{in } L^{p}(\tilde{\Omega}; L^{\infty}(0, T; H^{2(m-2)})). \tag{4.13}
\]

Now given each measurable subset \( R \) of \([0, T] \times \tilde{\Omega} \) and \( v \in H^{2(m+1)} \), by (4.11)–(4.13), we have
\[
\mathbb{E} \int_{0}^{T} \chi_{R} \langle \tilde{u}, v \rangle_{2m} ds = \mathbb{E} \int_{0}^{T} \chi_{R} \langle \overline{u}, v \rangle_{2m} ds = \mathbb{E} \int_{0}^{T} \chi_{R} \langle \tilde{u}, v \rangle_{2m} ds,
\]

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where $\langle \cdot, \cdot \rangle_{2m}$ means the dual product between $H^{-2m}$ and $H^{2m}$ and we have used the dense embeddings. This indicates that $\tilde{u} = \tilde{u} = \tilde{u}$ and therefore,

$$\tilde{u} \in L^p(\hat{\Omega}; L^\infty(0, T; H^{2m})) \cap L^2(\hat{\Omega}; L^2(0, T; H^{2(m+1)})).$$  \ (4.14)

**Step 2. Convergence to the integral form of (3.1)**

In this step, we show that for the limit $(\tilde{u}, \tilde{W})$, the following integral equation

$$\tilde{u}(t) + \int_0^t [A^2 \tilde{u} + \delta_N(||\tilde{u}||_2)f(\tilde{u})]ds = u_0 + \int_0^t \delta_N(||\tilde{u}||_2)\phi(\tilde{u})dW$$  \ (4.15)

for almost every $(t, \omega) \in [0, T] \times \hat{\Omega}$, holds in $H^{2(m+1)}$, for which, it suffices to show that (4.15) holds in $H$ by (4.14), (2.6) and (2.9).

We can follow an argument similar to that in [9, section 7] for this proof. But the nonlinear forcing and stochastic terms need to be discussed specifically in our situation. For sake of completeness and the reader’s convenience, we present the whole procedure in details.

Since $m \geq 1$, applying embeddings and Vitali convergence theorem again to the convergence in (4.9) and the fact that

$$\sup_{k \in \mathbb{N}} \mathbb{E} \left( \int_0^T ||A^{m+1}\tilde{u}^n(s)||^2 ds \right)^{\frac{p}{2}} < \infty \quad \text{by (3.6)},$$

one concludes that $\tilde{u}^n \to \tilde{u}$ in $L^p(\hat{\Omega}; L^2(0, T; H^{2(m+1)}))$. Then by picking a subsequence, we can further assume that

$$U_k := \tilde{u}^n - \tilde{u} \to 0 \quad \text{in } H^{2(m+1)} \text{ for almost surely } (t, \omega) \in [0, T] \times \hat{\Omega}. \quad (4.16)$$

Fix $v \in H^1$. We can infer from (4.14) and embeddings that

$$\left| \int_0^t \langle A^2 U_k, v \rangle_2 ds \right| \leq c||v||_2 \int_0^T ||U_k||_2 ds$$

and hence

$$\int_0^t \langle A^2 \tilde{u}^n, v \rangle_2 ds \to \int_0^t \langle A^2 \tilde{u}, v \rangle_2 ds,$$  \ (4.17)

for almost every $(t, \omega) \in [0, T] \times \hat{\Omega}$, where $\langle \cdot, \cdot \rangle_2$ is the dual product between $H^{-2}$ and $H^2$.

Then we address the nonlinear term $\delta_N(||\tilde{u}||_2)f(\tilde{u})$. Actually,

$$\left| \int_0^t \delta_N(||\tilde{u}^n||_2) f_n(\tilde{u}^n) - \delta_N(||\tilde{u}||_2)f(\tilde{u}), v \right|_2 ds \right|$$

$$\leq \int_0^t \delta_N(||\tilde{u}^n||_2) \langle f_n(\tilde{u}^n) - f(\tilde{u}), v \rangle_2 ds$$

$$+ \int_0^t \delta_N(||\tilde{u}^n||_2) \langle f(\tilde{u}), v \rangle_2 ds$$

$$\leq \int_0^t \delta_N(||\tilde{u}^n||_2) \langle f(\tilde{u}^n) - f(\tilde{u}), P_{\mu}v \rangle_2 ds$$

$$+ \int_0^t \delta_N(||\tilde{u}^n||_2) \langle f(\tilde{u}), (1 - P_{\mu})v \rangle_2 ds$$

$$+ \int_0^t \delta_N(||\tilde{u}^n||_2) \langle f(\tilde{u}), P_{\mu}v \rangle_2 ds$$

$$:= K_1^n + K_2^n + K_3^n. \quad (4.18)$$
For the estimation of $K_1$, first we see that
\[
\|\langle f(\tilde{u}^n) - f(\tilde{u}), P_m v \rangle_2 \| \leq \| (a + 3) \langle U_k, P_m v \rangle_2 + 4 \langle AU_k, P_m v \rangle_2 + |b| |\nabla \tilde{u}^n|^2 - |\nabla \tilde{u}|^2, P_m v \rangle_2 \| + |\langle (\tilde{u}^n)^3 - (\tilde{u})^3, P_m v \rangle_2 | \leq c \| U_k \|_2 \| v \|_2 + c \| \nabla U_k \|^2, P_m v \rangle_2 | + c \| \nabla U_k \cdot \nabla \tilde{u}^n, P_m v \rangle_2 | + c \| (\tilde{u}^n)^2 U_k, P_m v \rangle_2 | + c \| (\tilde{u}^n)^2 U_k, P_m v \rangle_2 |.
\] (4.19)

Note that all the components in dual products in (4.19) are actually in $H$ by embeddings. Hence by using lemma 3.1, it follows with no hard from (4.19) and embeddings that
\[
\|\langle f(\tilde{u}^n) - f(\tilde{u}), P_m v \rangle_2 \| \leq c \| U_k \|_2 \| v \|_2 (1 + \| \tilde{u}^n \|^2 + \| U_k \|^2),
\]
and thus by the property of $\delta_N$ and (4.16),
\[
K_1 \leq c \| v \|_2 \int_0^1 \| U_k \|_2 (1 + \| U_k \|^2) ds \to 0, \quad \text{as } k \to \infty.
\] (4.20)

For $K_2$, by embeddings and (4.14), we have as $k \to \infty$,
\[
K_2 \leq c \| (1 - P_m) v \|_2 \int_0^1 \| f(\tilde{u}) \| ds \leq c \| (1 - P_m) v \|_2 \int_0^1 (\| \tilde{u} \|^2 + \| \tilde{u} \|^2) ds \to 0.
\] (4.21)

As to $K_3$, we need to notice that $H^{2(m+1)} \subset H^2$, which with (4.16) implies that $\tilde{u}^n - \tilde{u} \to 0$ in $H^2$ for a.s. $(t, \omega) \in [0, T] \times \tilde{\Omega}$ and hence
\[
\tilde{u}^n \to \tilde{u} \quad \text{in } H^2 \text{ for } t \in [0, T] \text{ in measure and a.s. } \omega \in \tilde{\Omega}.
\] (4.22)

Note also by (4.16) that for a.s. $\omega \in \tilde{\Omega}$,
\[
\|\langle f(\tilde{u}(t)), v \rangle_2 \| \leq c \| v \|_2 \sup_{t \in [0, T]} (\| \tilde{u}(t) \|^2 + \| \tilde{u}(t) \|^2) < \infty.
\] (4.23)

The facts (4.22) and (4.23) guarantee that for each $\varepsilon > 0$, there exists a subset $E_{\varepsilon} \subset [0, T]$ with its Lebesgue measure so small that
\[
\int_{E_{\varepsilon}} |\delta_N(\| \tilde{u}^n \|^2) - \delta_N(\| \tilde{u} \|^2)\| (f(\tilde{u}), v_2) ds < \frac{\varepsilon}{2}.
\] (4.24)

and for a.s. $\omega \in \tilde{\Omega}$,
\[
\tilde{u}^n(t) \to \tilde{u}(t), \quad \text{in } H^2 \text{ uniformly in } t \in [0, T] \setminus E_{\varepsilon} \text{ as } k \to \infty.
\] (4.25)

Moreover, $\delta_N$ is obviously uniformly continuous on $\mathbb{R}^+$. Finally, we can infer from (4.24) and (4.25) that for all $\varepsilon > 0$,
\[
K_3 \leq \left( \int_{E_{\varepsilon}} + \int_{[0, T] \setminus E_{\varepsilon}} \right) |\delta_N(\| \tilde{u}^n \|^2) - \delta_N(\| \tilde{u} \|^2)\| (f(\tilde{u}), v_2) ds < \varepsilon.
\] (4.26)
when \( k \) is sufficiently large. Now the conclusions (4.18), (4.20), (4.21) and (4.26) imply that as \( k \to \infty \),

\[
\int_0^t \langle \delta_N(||\tilde{u}^n||_2) P_n, \phi(\tilde{u}^n) \rangle \, ds \to \int_0^t \langle \delta_N(||\tilde{u}||_2) f(\tilde{u}), \phi(\tilde{u}) \rangle \, ds. \tag{4.27}
\]

For the initial datum \( u_0 \), it is easy to see that as \( k \to \infty \)

\[
\tilde{u}^n(0) = u_0 \to u_0 \quad \text{in} \quad H^2.
\tag{4.28}
\]

Next we cope with the convergence for the stochastic term. First we have

\[
\| \delta_N(||\tilde{u}^n||_2) P_n \phi(\tilde{u}^n) - \delta_N(||\tilde{u}||_2) \phi(\tilde{u}) \|_{L^2(\Omega, L^2)} \\
\leq \| \delta_N(||\tilde{u}^n||_2) P_n \phi(\tilde{u}^n) - \delta_N(||\tilde{u}||_2) \phi(\tilde{u}) \|_{L^2(\Omega, L^2)} \\
+ \| \delta_N(||\tilde{u}||_2) (1 - P_n) \phi(\tilde{u}) \|_{L^2(\Omega, L^2)}.
\]

By the continuity of \( \delta_N \) and \( \phi \) (by (2.6)), the embedding \( H^2 \subset H \) and the inequality,

\[
\| \delta_N(||\tilde{u}||_2) \phi(\tilde{u}) \|_{L^2(\Omega, L^2)} \leq \delta_N(||\tilde{u}||_2) \epsilon(||\tilde{u}||_\infty)(1 + ||\tilde{u}||),
\]

it yields from (4.16) that

\[
\| \delta_N(||\tilde{u}^n||_2) P_n \phi(\tilde{u}^n) - \delta_N(||\tilde{u}||_2) \phi(\tilde{u}) \|_{L^2(\Omega, L^2)} \to 0, \quad \text{as} \quad k \to \infty,
\]

for almost every \((t, \omega) \in [0, T] \times \tilde{\Omega}\). Applying (2.6) again, we observe from (3.6) that

\[
\sup_{k \in \mathbb{N}} \mathbb{E} \int_0^T \| \delta_N(||\tilde{u}^n||_2) P_n \phi(\tilde{u}^n) \|^2_{L^2(\Omega, L^2)} \, ds \\
\leq c \mathbb{E} \int_0^T \delta_N(||\tilde{u}||_2) \epsilon(||\tilde{u}^n||_\infty)(1 + ||\tilde{u}^n||^2) \, ds \leq C.
\]

By the dominated convergence theorem, we know

\[
\delta_N(||\tilde{u}^n||_2) P_n \phi(\tilde{u}^n) \to \delta_N(||\tilde{u}||_2) \phi(\tilde{u}) \quad \text{in} \quad L^2(\tilde{\Omega}; L^2(0, T; L^2(\Omega, H))).
\]

This also deduces the following convergence

\[
\delta_N(||\tilde{u}^n||_2) P_n \phi(\tilde{u}^n) \to \delta_N(||\tilde{u}||_2) \phi(\tilde{u})
\]

in probability in \( L^2(0, T; L^2(\Omega, H)) \). Then using the conclusion (i) and lemma 2.3, we obtain

\[
\int_0^t \delta_N(||\tilde{u}^n||_2) P_n \phi(\tilde{u}^n) \, d\tilde{W}^n \to \int_0^t \delta_N(||\tilde{u}||_2) \phi(\tilde{u}) \, d\tilde{W}
\]

in probability in \( L^2(0, T; H) \), and hence we can assume

\[
\int_0^t \langle \delta_N(||\tilde{u}^n||_2) P_n \phi(\tilde{u}^n), \nu \rangle_2 \, ds \to \int_0^t \langle \delta_N(||\tilde{u}||_2) \phi(\tilde{u}), \nu \rangle_2
\tag{4.29}
\]

for almost every \((t, \omega) \in [0, T] \times \tilde{\Omega}\), by picking a subsequence if necessary.

As a final result, by (4.16), (4.17), (4.27), (4.28), (4.29) and the conclusion (iii) that \((\tilde{u}^n, \tilde{W}^n)\) satisfies (3.2) with only \( u^a \) and \( W \) replaced by \( \tilde{u}^n \) and \( \tilde{W}^n \) therein, we can infer that for all \( \nu \in H^2 \) and almost every \((t, \omega) \in [0, T] \times \tilde{\Omega}\),
According to (4.14), we have (by [8, chapter 8])

\[ z \in L^2(\tilde{\Omega};C([0,T];H^{m+1})) \cap L^2(\tilde{\Omega};L^2(0,T,H^{2m+1})) \].

Then define \( U = \tilde{u} - z \). By (3.1) and (4.31), we have

\[ \frac{dU}{dt} = -A^2U - \delta_N(\|U + z\|_2)f(U + z), \quad U(0) = 0. \] (4.33)

According to (4.32) and (4.14), we obtain that \( U \in L^2(\tilde{\Omega};L^2(0,T,H^{2m+1})) \). Each term on the right hand of (4.33) belongs to \( L^2(\tilde{\Omega};L^2(0,T,H^{2m+1})) \). Actually, we have proved that

\[ A^mU \in L^2(\tilde{\Omega};L^2(0,T,H^2)) \quad \text{and} \quad \frac{d}{dt}A^mU \in L^2(\tilde{\Omega};L^2(0,T,H^{-2})) \]

Utilizing [44, chapter 3, lemma 1.2], we know for almost every \( \omega \in \tilde{\Omega} \), \( U \in C([0,T];H^{2m}) \). Then combining this with (4.14) and (4.32), we infer that \( \tilde{u} \in L^2(\tilde{\Omega};C([0,T];H^{2m})) \).

Now we have shown that \((\mathcal{F},\tilde{u})\) is a global martingale solution of (3.1). By the definition (4.8), we know that \( \rho_N \) is a stopping time. One sees that

\[ \int_0^{t_\rho_N} \delta_N(\|\tilde{u}\|_2)f(\tilde{u}) \, ds = \int_0^{t_\rho_N} f(\tilde{u}) \, ds \]

and

\[ \int_0^{t_\rho_N} \delta_N(\|\tilde{u}\|_2)\phi(\tilde{u}) \, d\tilde{W} = \int_0^{t_\rho_N} \phi(\tilde{u}) \, d\tilde{W}. \]
Taking (4.30) into consideration, we infer that \((\tilde{S}, \tilde{u}, \rho_0)\) is a local martingale solution of (2.13).

The proof is finally finished.

### 4.2. Global existence of martingale solutions

With a further assumption on \(\phi, b\) and \(u_0\), we actually can obtain the existence of global martingale solutions for (2.13). For the proof to this consequence, after an estimate on the Galerkin scheme for (2.13), we will repeat the procedure given above in section 4.1 to obtain the existence of local martingale solutions first.

To discuss the existence of solutions of this equation, we still need to introduce the cut-off function. Based on the settings in section 3, we consider the related equations,

\[
\begin{align*}
    & \, du + [A^2 u + \delta_N(||u||_2) f(u)] dt = \phi(u) dW \\
    & \, du^a + [A^2 u^a + \delta_N(||u^a||_2) f_n(u^a)] dt = P_n \phi(u^a) dW \quad \text{and} \quad u^o(0) = u_0^o.
\end{align*}
\]

The problem (4.34) has a unique global solution \(u^a\) in \(H_n\) for every \(u_0^o\) satisfying (2.9). Now we give renewed estimates for the solution of (4.34).

**Lemma 4.4.** Let \(S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)\) be a stochastic basis and \(T \geq 0\). Suppose that \(\phi\) satisfy (2.3). Given every \(n \geq 0\) and each initial datum \(u_0\) satisfying (2.12), there exists a positive constant \(C_2\) independent of \(n\) and \(N\), such that the solution \(u^a\) of (3.2) has the following uniform estimates,

\[
\begin{align*}
    & \mathbb{E} \left( \sup_{t \in [0, T]} \|A^n u^n(s)\|^p + \int_0^T \|A^n u^n\|^{p-2} \|A^{n+1} u^n\|^2 ds \right) \leq C_2, \\
    & \mathbb{E} \left( \int_0^T \|A^{n+1} u^n(s)\|^2 ds \right)^{\frac{p}{2}} \leq C_2, \\
    & \mathbb{E} \left\| A^{n-1} u^n(t) - \int_0^t \delta_N(||u^n||_2) P_n A^{n-1} \phi(u^n(s)) ds \right\|^2_{W^{\alpha,2}(0,T,H)} \leq C_2,
\end{align*}
\]

and given \(\alpha \in (0, 1/2)\) additionally, it holds that

\[
\mathbb{E} \left\| \int_0^t \delta_N(||u^n||_2) P_n A^{n-1} \phi(u^n) dW \right\|^2_{W^{\alpha,2}(0,T,H)} \leq C_2.
\]

**Proof.** For the estimate discussed later, we first estimate \(\mathbb{E} \sup_{t \in [0, T]} \|u^n(t)\|^{q_0} \) with \(q_0 := (2m + 3)\). Applying Itô’s Formula to \(\|u^n\|^{q_0}\), for all stopping times \(\tau, \tau'\) with \(0 \leq \tau \leq s \leq \tau' \leq T\), we can deduce that

\[
\begin{align*}
    & \frac{1}{q_0} \|u^n(s)\|^{q_0} + \int_{\tau}^{\tau'} d\|u^n\|^{q_0-2} \left( \|A u^n\|^2 + \delta_N(||u^n||_2) ||u^n||_2^2 \right) ds' \\
    &= \frac{1}{q_0} \|u^n(\tau')\|^{q_0} \int_{\tau}^{\tau'} d\delta_N(||u^n||_2) \|u^n\|^{q_0-2} \left( \|A u^n\|^2, (a+3)u^n - 4A u^n + b \nabla u^n \right) ds' \\
    &\quad + \frac{1}{2} \int_{\tau}^{\tau'} d\|u^n\|^{q_0-4} \left( q_0 - 2 \|u^n, P_n \phi(u^n)\|_{L^1(\Omega,\mathbb{R}^n)}^2 + \|u^n\|^2 \|P_n \phi(u^n)\|^2_{L^2(\Omega,H)} \right) ds' \\
    &\quad + \int_{\tau}^{\tau'} d\|u^n\|^{q_0-2} \left( u^n, P_n \phi(u^n) dW \right).
\end{align*}
\]
Observe by lemma 3.1, integration by parts, Hölder inequality and Young’s inequality that

\[
\left| (u^n, (\alpha + 3)u^n + 4Au^n + b|\nabla u^n|^2) \right|
\leq c\|u^n\|^2 + 4\|u^n, Au^n\| + \frac{|b|}{2} \left( \left( u^n \right)^2, \Delta u^n \right)
\leq c\|u^n\|^2 + 4\|u^n, Au^n\| + \frac{|b|}{4} \left( |u^n|^2 + (\Delta u^n, \Delta u^n) \right)
= \frac{|b|}{4} \left( |u^n|^2 + \|Au^n\|^2 \right) + c \left( \|u^n\|^2 + \|Au^n\| \|u^n\| \right)
\leq \frac{|b|}{4} \left( |u^n|^2 + \|Au^n\|^2 \right) + c \left( \|u^n\|^2 + \|Au^n\|^2 + |u^n|^2 \right).
\]

(4.40)

Combining (4.39), (4.40) and (3.17) with \(\delta_N(\|u^n\|_2), \kappa(\|u^n\|_\infty)\) and \(m\) therein replaced by \(1, \kappa\) and 0, correspondingly, we have by the property of polynomials and \(|b| < 4\) that

\[
\frac{1}{q_0} \|u^n(s)\|^{q_0} + \frac{1}{2} \left( 1 - \frac{|b|}{4} \right) \int_{\tau'}^{\tau''} \|u^n\|^{q_0-2} \left( \|Au^n\|^2 + \delta_N(\|u^n\|_2) |u^n|_4^2 \right) ds'
\leq \frac{1}{q_0} \|u^n(\tau')\|^{q_0} + \int_{\tau'}^{\tau''} \|u^n\|^{q_0-2} \left[ c \|Au^n\|^2 - \frac{1}{2} \left( 1 - \frac{|b|}{4} \right) \|Au^n\|^2 \right] ds'
+ \int_{\tau'}^{\tau''} \delta_N(\|u^n\|_2) \|u^n\|^{q_0-2} \left[ c \left( |u^n|^2_4 + |u^n|^2_{\infty} \right) - \frac{1}{2} \left( 1 - \frac{|b|}{4} \right) |u^n|^2_4 \right] ds'
+ c_{m,p} \int_{\tau'}^{\tau''} (1 + \|u^n\|^{q_0}) ds' + \int_{\tau'}^{\tau''} \|u^n\|^{q_0-2} (u^n, P_n \phi(u^n)) dW
\leq \frac{1}{q_0} \|u^n(\tau')\|^{q_0} + c_{m,p} \int_{\tau'}^{\tau''} (1 + \|u^n\|^{q_0}) ds'
+ c(s - \tau') + \int_{\tau'}^{\tau''} \|u^n\|^{q_0-2} (u^n, P_n \phi(u^n)) dW.
\]

(4.41)

Taking the expectation of the supremum over \(s \in [\tau', \tau''']\) for (4.41) and by (3.18) with \(\delta_N(\|u^n\|_2), \kappa(\|u^n\|_\infty)\) and \(m\) therein replaced by \(1, \kappa\) and 0, correspondingly, we obtain

\[
\mathbb{E} \sup_{s \in [\tau', \tau''']} \|u^n(s)\|^{q_0} + \left( 1 - \frac{|b|}{4} \right) q_0 \mathbb{E} \int_{\tau'}^{\tau''} \|u^n\|^{q_0-2} \|Au^n\|^2 ds
\leq 2 \mathbb{E} \|u^n(\tau')\|^{q_0} + c_{m,p} \mathbb{E} \int_{\tau'}^{\tau''} (1 + \|u^n\|^{q_0}) ds + c(\tau''' - \tau').
\]

(4.42)

Note here that (4.42) has no business with \(N\) except \(u^n\) itself. Now similar to the argument for (3.3), we can infer by using stopping times and the stochastic Gronwall’s inequality that

\[
\mathbb{E} \sup_{s \in [0,T]} \|u^n(s)\|^{q_0} + \mathbb{E} \int_0^T \|u^n\|^{q_0-2} \|Au^n\|^2 ds \leq C_j^t,
\]

(4.43)

where \(C_j^t = C_j^t(a, b, m, p, T, \kappa, u_0, \mathbb{T}^2)\).
Next we estimate \( \mathbb{E} \sup_{t \in [0, 1]} \| A^m u^n \|^p \). Following the similar estimation in the proof of lemma 3.2 and replacing \( \delta_{N} \left( \| u^n \|_2 \right) \) and \( \kappa(\| u^n \|_\infty) \) related to the stochastic term by 1 and \( \kappa \), correspondingly, we have an estimate similar to (3.19),

\[
\begin{align*}
\mathbb{E} \left( \sup_{t \in [\tau', \tau'']} \| A^m u^n(s) \|^p + \int_{\tau'}^{\tau''} \| A^m u^n \|^p \| A^{m+1} u^n \|^2 ds' \right) \\
\leq 2 \mathbb{E} \| A^m u^n(\tau') \|^p + c_{m, p} \mathbb{E} \int_{\tau'}^{\tau''} (1 + \| A^m u^n \|^p) ds' \\
+ c_{m, p} \mathbb{E} \int_{\tau'}^{\tau''} \| u^n \|^{(2m+3)p} ds'.
\end{align*}
\]

(4.44)

Also (4.44) is independent of all \( n \) and \( N \). With a similar argument, (4.43) and the fact that \( u_0 \) satisfies (2.12), we easily conclude (4.35). The inequalities (4.36)–(4.38) can be similarly obtained, we omit the detailed argument. \( \square \)

Next we need the local uniqueness of the solution in the following sense.

**Lemma 4.5.** Suppose the conditions of lemma 4.4 holds. Let \( N_1, N_2 \in \mathbb{N}^+ \) with \( N_2 > N_1 \) and \( u_i^n \) be the unique solution of (4.34) with \( N \) replaced by \( N_i \), \( i = 1, 2 \). Define a stopping time \( \zeta \) so that

\[
\zeta = \inf \{ t \geq 0 : \| u_1^n(t) \|_2 \vee \| u_2^n(t) \|_2 > N_1 \}.
\]

Then

\[
P(\zeta^t = u_2^n(t); \forall t \in [0, \zeta]) = 1.
\]

(4.45)

**Proof.** Let \( \varphi = u_2^n - u_1^n \). We know \( \varphi(0) = 0 \) and that \( \varphi \) satisfies the following equation

\[
\begin{align*}
d\varphi + [A^2 \varphi + (\delta_{N_1}(\| u_2^n \|_2)f_n(u_2^n) - \delta_{N_2}(\| u_1^n \|_2)f_n(u_1^n))] dt \\
= P_n(\phi(u_2^n) - \phi(u_1^n)) dW.
\end{align*}
\]

(4.46)

Define a stopping time for each \( R > 0 \),

\[
\varphi = \inf \{ t \geq 0 : \| u_2^n(t) \|_{2m} + \| u_2^n(t) \|_{2m} > R \} \wedge \zeta.
\]

Apply \( A^m \) to (4.46) and use Itô’s Formula to \( \| A^m \varphi \|^p \). It yields for all stopping times \( \tau', \tau'' \) with \( 0 \leq \tau' \leq s \leq \tau'' \leq \varphi \) that

\[
\begin{align*}
&\frac{1}{p} \| A^m \varphi(s) \|^p + \int_{\tau'}^{\tau''} \| A^m \varphi \|^{p-2} \| A^{m+1} \varphi \|^2 ds' \\
\leq &\frac{1}{p} \| A^m \varphi(\tau') \|^p + \frac{p-1}{2} \int_{\tau'}^{\tau''} \| A^m \varphi \|^{p-2} \| A^m (\phi(u_2^n) - \phi(u_1^n)) \|^2_{L^2(\Omega, \mathbb{H})} ds' \\
&- \int_{\tau'}^{\tau''} \| A^m \varphi \|^{p-2} (A^m \varphi, \delta_{N_i}(\| u_i^n \|_2)A^n f(u_i^n) - \delta_{N_j}(\| u_i^n \|_2)A^n f(u_i^n)) ds' \\
&+ \int_{\tau'}^{\tau''} \| A^m \varphi \|^{p-2} (A^m \varphi, P_n A^m (\phi(u_2^n) - \phi(u_1^n)) dW) \\
:= &\frac{1}{p} \| A^m \varphi(\tau') \|^p + M_1 + M_2 + M_3.
\end{align*}
\]

(4.47)
Note that when \( s \leq \varrho \), \( \delta N(u_i'(s)) = 1 \), \( i = 1, 2 \). Hence by

\[
|M_1| = \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} \left( A^m \mathbf{v}, A^m (f(u_2'') - f(u_1'')) \right) |ds'|
\]

\[
\leq c \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p} ds' + \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} (A^m \mathbf{v}, A^{m+1} \mathbf{v}) |ds'|
\]

\[
+ \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} \left( \langle A^{m+1} \mathbf{v}, A^{m-1} (b(\nabla u_2^2 - \nabla u_1^2) \rangle \right) |ds'|
\]

\[
+ \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} \left( \langle A^{m+1} \mathbf{v}, A^{m-1} ((u_2^2)^3 - (u_1^2)^3) \rangle \right) |ds'|
\]

\[
\leq \frac{2p-1}{2p} \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} \|A^{m+1} \mathbf{v}\|^2 ds' + c_m \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p} ds'
\]

\[
+ c_p \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} \left\| A^{m-1} (\nabla (u_2^2 + u_1^2 + (u_1^2)^2)) \right\|^2 |ds'|
\]

\[
\leq \frac{2p-1}{2p} \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} \|A^{m+1} \mathbf{v}\|^2 ds' + c_m \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p} ds' + c_p \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} \left\| A^{m-1} (\nabla (u_2^2 + u_1^2 + (u_1^2)^2)) \right\|^2 |ds'.
\]

(4.48)

Since by the discussions to (3.10) and (3.13) and the embeddings,

\[
\|A^{m-1} (\nabla (u_2^2 + u_1^2)) \|^2 \leq c_m \|A^m \mathbf{v}\|^2 (\|u_1^2\|_{L^m}^2 + \|u_2^2\|_{L^m}^2),
\]

(4.49)

\[
\|A^{m-1} (\nabla (u_2^2 + u_1^2 + (u_1^2)^2)) \|^2 \leq c_m \|A^m \mathbf{v}\|^2 (\|u_1^2\|_{L^m}^2 + \|u_2^2\|_{L^m}^2),
\]

(4.50)

we have

\[
|M_1| \leq \frac{2p-1}{2p} \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} \|A^{m+1} \mathbf{v}\|^2 ds' + c_m \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p} ds' + c_p \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p-2} \left\| A^{m-1} (\nabla (u_2^2 + u_1^2 + (u_1^2)^2)) \right\|^2 |ds'.
\]

(4.51)

By (2.3), we see

\[
|M_2| \leq c_p \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p} ds'.
\]

(4.52)

For \( M_3 \), we use BDG inequality and obtain that

\[
\mathbb{E} \sup_{s \in [\tau', \tau'']} |M_3| \leq c \mathbb{E} \left( \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{2p} ds \right)^{\frac{1}{2}}
\]

\[
\leq c \mathbb{E} \left( \sup_{s \in [\tau', \tau'']} \|A^m \mathbf{v}(s)\|^{p} \right) \left( \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p} ds \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2p} \mathbb{E} \sup_{s \in [\tau', \tau'']} \|A^m \mathbf{v}(s)\|^{p} + c_p \mathbb{E} \int_{\tau'}^{\tau''} \|A^m \mathbf{v}\|^{p} ds.
\]

(4.53)

Consider the expectation of the supremum of (4.47) over \( s \in [\tau', \tau''] \) and combine (4.51), (4.52) and (4.53), we get
Following the same procedure as given in the proof of Theorem 4.6, every pair \( N \) and \( \hat{N} \) satisfy (4.57)

Suppose that \( \hat{N} \) only depends on \( N_1, N_2 \) and \( n \). Observe that \( g_B \to \zeta \) as \( R \to +\infty \). We see that (4.55) indicates (4.45). The proof is complete.

Now we are ready to show the global existence of martingale solutions for (2.13). Let \( n, N \in \mathbb{N}^+ \), and \( U^{n,N} \) be the solution of (4.34). We repeat the argument presented in section 4.1 and follow the settings given there. Let

\[
\hat{\mu}^{n,N} \left( U^{n,N} \right) = \mathbb{P} \left( U^{n,N} \subset \cdot \right) \quad \hat{\nu}^{n,N} \left( \cdot \right) = \hat{\mu}^{n,N} \left( \cdot \right)
\]

and \( \hat{\mu}^{N} = \hat{\mu}^{n,N} \times \hat{\nu}^{n,N} \). For each fixed \( N \in \mathbb{N}^+ \), we have the same conclusion as Lemma 4.1, saying, for each \( T > 0 \), as long as \( t_0 \) satisfies (2.9), the sequence \( \{ \hat{\mu}^{n,N} \}_{n \geq 1} \) is tight in the phase space \( X \). As a final result, we infer the following consequence.

**Theorem 4.6.** Suppose that \( \mu_0 \) is a probability measure on \( H^m \) satisfying (2.10), \( \phi \) satisfies (2.3) and \( |b| < 4 \). Then there exists a global martingale solution to (2.13).

**Proof.** Following the same procedure to that given in the proof of Theorem 4.2, we know that for each \( T > 0 \) and \( N \in \mathbb{N}^+ \), there exists a subsequence \( n_k \) and a probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\), on which there is a sequence of \( X \)-valued random variables \((\hat{U}^{n_k,N}, \hat{W}^{n_k,N})\) such that

(i) \((\hat{U}^{n_k,N}, \hat{W}^{n_k,N})\) converges almost surely, in the topology of \( X \), to an element \((\hat{U}^N, \hat{W}^N) \in X \),

the law of \((\hat{U}^{n_k,N}, \hat{W}^{n_k,N})\) is \( \hat{\mu}^{n_k,N} \) for each \( k \) and \( \hat{\mu}^{n_k,N} \) weakly converges to the law \( \hat{\mu}^{N} \) of \((\hat{U}^N, \hat{W}^N)\).

(ii) \( \hat{W}^{n_k,N} \) is a cylindrical Wiener process, relative to the filtration \( \hat{\mathcal{F}}^N_t \), given by the completion of \( \sigma(\{\hat{U}^{n_k,N}(s), \hat{W}^{n_k,N}(s) ; s \leq t\}) \).

(iii) every pair \((\hat{U}^{n_k,N}, \hat{W}^{n_k,N})\) satisfies (4.34) with only \( u^n \) and \( W \) replaced by \( \hat{U}^{n_k,N} \) and \( \hat{W}^{n_k,N} \) therein.

For each solution \( \hat{U}^{n_k,N} \) and \( R \in \mathbb{R}^+ \), define the stopping time \( \zeta^{n_k,N}_R \) such that

\[
\zeta^{n_k,N}_R = \inf \{ t \geq 0 : \| \hat{U}^{n_k,N}(t) \|_2 > R \}.
\]

Note here in (4.57) \( n \) denotes the dimension of \( H_n \) and \( N \) is implied in the cut-off function \( \delta_N \).

Next we adopt the diagonal method to extend the local martingale solution to a global one.

Firstly, when \( N = 1 \), we have a subsequence \( n_k^1 \) of \( n_k \) such that

\((\hat{U}^{n_k^1,N}, \hat{W}^{n_k^1,N})\) converges a.s. to \((\hat{U}^1, \hat{W}^1)\),

and \( \hat{\mu}^{n_k^1,N} \) weakly converges to \( \hat{\mu}^1 \).
Then by theorem 4.2, we know \((\hat{S}^1, \hat{U}^1, \hat{\rho}_1)\) is a local martingale solution of (2.13) with 
\[
\hat{\rho}_1 = \inf \{t \geq 0 : \|\hat{U}^1\|_2 > 1\}, 
\]
\[
\hat{S}^1 = (\hat{\Omega}, \hat{\mathcal{F}}^1, \{\hat{\mathcal{F}}^1_t\}_{t \geq 0}, \hat{\mathbb{P}}, \hat{\mathbb{W}}) \text{ and } \hat{\mathcal{F}}^1 \text{ the completion of } \sigma((\hat{U}^1(s), \hat{W}^1(s)); s \leq t). \]
For \(N = 2\), we further have a subsequence \(n^2_k\) of \(n^1_k\) such that 
\[
(\hat{U}^{n^1_k,2}, \hat{W}^{n^1_k,2}) \text{ converges a.s. to } (\hat{U}^2, \hat{W}^2), 
\]
and \(\hat{\mu}^{n^1_k,2}\) weakly converges to \(\hat{\mu}^2\).

The triple \((\hat{S}^2, \hat{U}^2, \hat{\rho}_2)\) is thus also a local martingale of (2.13) with \(\hat{\rho}_2 = \inf \{t \geq 0 : \|\hat{U}^2\|_2 > 2\}, \hat{S}^2 = (\hat{\Omega}, \hat{\mathcal{F}}^2, \{\hat{\mathcal{F}}^2_t\}_{t \geq 0}, \hat{\mathbb{P}}, \hat{\mathbb{W}}) \text{ and } \hat{\mathcal{F}}^2 \text{ being the completion of } \sigma((\hat{U}^2(s), \hat{W}^2(s)); s \leq t). \) In this sense, for each \(N \in \mathbb{N}^+\), we have a subsequence \(n^{N+1}_k\) of \(n^N_k\) such that 
\[
(\hat{U}^{n^{N+1}_k, N+1}, \hat{W}^{n^{N+1}_k, N+1}) \text{ converges a.s. to } (\hat{U}^{N+1}, \hat{W}^{N+1}), 
\]
and \(\hat{\mu}^{n^{N+1}_k, N+1}\) weakly converges to \(\hat{\mu}^{N+1}\).

And we find for each \(N \in \mathbb{N}^+, \ (\hat{S}^N, \hat{U}^N, \hat{\rho}_N)\) is a local martingale of (2.13) with \(\hat{\rho}_N = \inf \{t \geq 0 : \|\hat{U}^N\|_2 > N\}, \hat{S}^N = (\hat{\Omega}, \hat{\mathcal{F}}^N, \{\hat{\mathcal{F}}^N_t\}_{t \geq 0}, \hat{\mathbb{P}}, \hat{\mathbb{W}}) \text{ and } \hat{\mathcal{F}}^N \text{ being the completion of } \sigma((\hat{U}^N(s), \hat{W}^N(s)); s \leq t). \)

Now we pick the sequence \(\{\hat{U}^{n^N_k, k}, \hat{W}^{n^N_k, k}\}_{k \in \mathbb{N}^+}\). Given each \(N \in \mathbb{N}^+, \) let 
\[
\zeta_N = \lim \inf_{k \to \infty} c^{n^N_k}_{\hat{\mathcal{F}}^N} \quad \text{and} \quad \zeta_N^d = \sum_{j=N}^{\infty} c^{n^N_k}_{\hat{\mathcal{F}}^N}. \tag{4.58}
\]

Since obviously \(\zeta_N^{d,j} \leq c^{n^N_k}_{\hat{\mathcal{F}}^N}\), we have \(c^{n^N_k}_{\hat{\mathcal{F}}^N} \leq \zeta_N^{d,j}\), and hence \(\zeta_N \leq \zeta_{N+1}\). We will show that for each \(N \in \mathbb{N}^+, \) as \(k \to \infty, \) the sequence \((\hat{U}^{n^N_k, k}, \hat{W}^{n^N_k, k})\) converges almost surely to the pair \((\hat{U}^N, \hat{W}^N)\) on \([0, \zeta_N \wedge T]\), and correspondingly, its law sequence \(\hat{\mu}^{n^N_k, k}\) weakly converges to \(\hat{\mu}^N\).

By lemma 4.5, Skorohod’s Theorem and (4.56), we know whenever \(1 \leq N < N' \leq k, \) for almost surely \(t \in [0, \zeta_N^{d,j} \wedge T], \) it holds that 
\[
(\hat{U}^{n^N_k, N}(t), \hat{W}^{n^N_k, N}(t)) = (\hat{U}^{n^N_k, N'}(t), \hat{W}^{n^N_k, N'}(t)). \tag{4.59}
\]

Since the right side of (4.59) is indeed a subsequence of \((\hat{U}^{n^N_k, N}(t), \hat{W}^{n^N_k, N}(t))\), by the uniqueness of almost surely limit and picking all \(k \in \mathbb{N}^+, \) we conclude that 
\[
(\hat{U}^N(t), \hat{W}^N(t)) = (\hat{U}^{N'}(t), \hat{W}^{N'}(t)) \quad \text{for all } N' > N
\]
for almost every \(t \in [0, \zeta_N \wedge T] \) and \(\omega \in \hat{\Omega}\) due to the almost surely equality we also have that \(\hat{\mu}^N = \hat{\mu}^{N'}\) for \(N' > N\). These results have already implied the two convergences we desire.

We now claim that 
\[
\zeta_N \to \infty \quad \text{almost surely as } N \to +\infty. \tag{4.60}
\]

By assuming the claim, one can easily extend the pair \((\hat{U}^{N}, \hat{W}^{N})\) on \([0, \zeta_N \wedge T] \) to \((\hat{U}, \hat{W})\) defined on \([0, T]\) with \((\hat{U}(t), \hat{W}(t)) = (\hat{U}^N(t), \hat{W}^N(t))\) on \([0, \zeta_N \wedge T]\) for each \(N \in \mathbb{N}^+\) and \(T > 0\). Correspondingly, for each \(T > 0, \) the law \(\hat{\mu}\) of \((\hat{U}, \hat{W})\) is equal to \(\hat{\mu}^N\) for each \(N \in \mathbb{N}^+, \) and the stochastic basis \(\hat{\mathcal{S}} = (\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}}, \hat{\mathbb{W}})\) can be similarly obtained by extending each \(\hat{\mathcal{F}}^N\) to \(\hat{\mathcal{F}}\). Since \((\hat{S}^N, \hat{U}^N, \hat{\rho}_N)\) is a local martingale solution of (2.13) and \(\hat{\rho}_N \leq \zeta_N, \) then
(\hat{S}, \hat{U}, \zeta_N) = (\hat{S}^N, \hat{U}^N, \zeta_N)$ is also a local martingale solution. Due to (4.60), we see that $(\hat{S}, \hat{U})$ is a global martingale solution of (2.13).

Eventually, it is sufficient to show the claim (4.60). We show it by contradiction and assume there is $\hat{\Omega}_0 \subset \hat{\Omega}$ with $\mathbb{P}(\hat{\Omega}_0) > 0$ such that $\zeta_N(\omega)$ is bounded for all $\omega \in \hat{\Omega}_0$ and $N \in \mathbb{N}^+$. Observe that

$$\hat{\Omega}_0 = \bigcup_{k \in \mathbb{N}^+} \hat{\Omega}_k \quad \text{with} \quad \hat{\Omega}_k := \{\omega \in \hat{\Omega}_0 : \sup_{N \in \mathbb{N}^+} \zeta_N(\omega) \leq k\}.$$

Surely, there exists $k_0 \in \mathbb{N}^+$ so that $\epsilon_0 := \mathbb{P}(\hat{\Omega}_0) > 0$. Fix each $N \in \mathbb{N}^+$. By the first definition in (4.58), we can pick a subsequence $\{k_i\}_{i \geq 1}$ so that $\zeta_{N k_i}^\omega < k_0 + 1$ almost surely in $\hat{\Omega}_0$. Then by the second definition in (4.58), we deduce that for some $N_{k_i} \geq N$,

$$\sup_{x \in [0,k_0+1]} ||\hat{U}_{N k_i}^\omega(x)||_2 \geq N \quad \text{occurs almost surely on } \hat{\Omega}_0.$$

As a result, it follows from (4.35) and the embeddings that for $i = \mathbb{N}^+$ and $p \geq 2$,

$$N^p \epsilon_0 \leq \int_{\hat{\Omega}_0} \sup_{x \in [0,k_0+1]} ||\hat{U}_{N k_i}^\omega(x)||_2^p P(\omega) \leq \mathbb{E} \sup_{x \in [0,k_0+1]} ||\hat{U}_{N k_i}^\omega(x)||_2^p \leq C_p,$$

(4.61)

where $C_p > 0$ is independent of all $k_i$ and $N_{k_i}$. However, the $N$ on the left hand of (4.61) can be chosen arbitrarily, which contradicts the boundedness of the right hand. The proof is finished.

**Remark 4.7.** The procedure we adopt in this subsection can be applied to much more examples of stochastic partial differential equations with nonlinear forcing terms and globally Lipschitz diffusion coefficients, to establish the global existence of martingale solutions. The equation is required to be dissipative to some extent. More importantly, we need not to be aware of the existence of global pathwise solutions here.

5. Local and global existence of pathwise solutions

We are now ready to study the existence of pathwise solutions of (2.13). Recalling proposition 2.5, we will show that the sequence of solutions $u^\theta$ of (3.1) converges almost surely in $L^2(0,T;\mathcal{H}^{m+1}) \cap \mathcal{C}([0,T];\mathcal{H}^m)$ relative to initial stochastic basis. We first deal with the condition (2.17), which can be interpreted by pathwise uniqueness.

5.1. Local existence of pathwise solutions

We will adopt the Yamada–Wannabe theorem (see e.g. [20]) to give the local existence of pathwise solutions. Accordingly, The first result to be proved is the local pathwise uniqueness for an arbitrary pair of solutions of the cut-off system (3.1).

**Lemma 5.1.** Assume that $\phi$ satisfies (2.7). Suppose $(S,u^{(1)})$ and $(S,u^{(2)})$ are two global solutions of (3.1) given by theorem 4.2, relative to the same stochastic basis $S := \{\hat{\Omega}, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P}, \mathcal{W}\}$. Define

$$\Omega_0 = \{u^{(1)}(0) = u^{(2)}(0)\},$$

(5.1)
Then $u^{(1)}$ and $u^{(2)}$ are indistinguishable on $\Omega_0$, i.e.
\[ P\left((u^{(1)}(t) - u^{(2)}(t))\chi_{\Omega_0} = 0; \text{for all } t \geq 0\right) = 1. \tag{5.2} \]

**Proof.** Define $V = u^{(1)} - u^{(2)}$ and $\bar{V} = \chi_{\Omega_0} V$. By definition 2.2, we know that
\[ \bar{V} \in C([0, \infty); H^{2m}) \cap L^2_{\text{loc}}(0, \infty; H^{2(m+1)}) \text{ almost surely}. \]

Define a stopping time $\bar{\tau}_R$ such that
\[ \bar{\tau}_R = \inf\{t \geq 0 : \|u^{(1)}(t)\|_{2m} + \|u^{(2)}(t)\|_{2m} > R\}. \]

Trivially, it can be seen that $\bar{\tau}_R \rightarrow +\infty$, as $R \rightarrow +\infty$. And the uniqueness follows immediately from the conclusion
\[ E\sup_{s \in [0,T]} \|\bar{V}(s)\|^p_{2m} = 0. \tag{5.3} \]

Now we are devoted to proving (5.3). Subtracting the equation (3.1) for $u^{(2)}$ from that for $u^{(1)}$, we get the following equation for $V$,
\[ dV + [A^2 V + \Phi_1(u^{(1)}), u^{(2)})]dt = \Phi_2(u^{(1)}, u^{(2)})dW, \tag{5.4} \]
where
\[ \Phi_1(u^{(1)}, u^{(2)}) := \delta_N(\|u^{(1)}\|_2 f(u^{(1)}) - \|u^{(2)}\|_2 f(u^{(2)})), \]
\[ \Phi_2(u^{(1)}, u^{(2)}) := \delta_N(\|u^{(1)}\|_2 \phi(u^{(1)}) - \|u^{(2)}\|_2 \phi(u^{(2)})). \]

Apply $A^m$ to (5.4). It yields from Itô’s Formula for $\|A^mV\|^p$ that for each fixed $T > 0$ and $\tau'$, $\tau''$ with $0 \leq \tau' < s \leq \tau'' \leq T \land \bar{\tau}_R$, we have
\[ \frac{1}{p}\|A^m V(s)\|^p + \int_{\tau'}^{s} \|A^m V\|^{p-2} \|A^{m+1} V\|^2 ds' \leq \frac{1}{p}\|A^m V(\tau')\|^p - \int_{\tau'}^{s} \|A^m V\|^{p-2} \left( A^m V, A^m \Phi_1(u^{(1)}, u^{(2)}) \right) ds' \]
\[ + \frac{p-1}{2} \int_{\tau'}^{s} \|A^m V\|^{p-2} \|A^m \Phi_2(u^{(1)}, u^{(2)})\|^2 dz_{\tau}(s,d) ds' \]
\[ + \int_{\tau'}^{s} \|A^m V\|^{p-2} \left( A^m V, A^m \Phi_2(u^{(1)}, u^{(2)})dW \right) \]
\[ := \frac{1}{p}\|A^m V(\tau')\|^p + M'_1 + M'_2 + M'_3. \tag{5.5} \]

Noting that $\delta_N$ is globally Lipschitz (depending on $N$) and similar to (4.48), we deduce that
\[ |M'_1| \leq \int_{\tau'}^{s} \|A^m V\|^{p-2} \left( \|A^{m+1} V, (\delta_N(\|u^{(1)}\|_2) - \delta_N(\|u^{(2)}\|_2))A^{m-1} f(u^{(1)}) \right) \| ds' \]
\[ + \int_{\tau'}^{s} \|A^m V\|^{p-2} \left( \|A^{m+1} V, A^{m-1} (f(u^{(1)}) - f(u^{(2)})) \right) \| ds' \]
\[ \leq \frac{2p-1}{2p} \int_{\tau'}^{s} \|A^m V\|^{p-2} \|A^{m+1} V\|^2 ds' + c_{m,p} \int_{\tau'}^{s} \|A^m V\|^p \|A^{m-1} f(u^{(1)})\|^2 ds'. \]
\[
+ c \int_{\tau}^{\tau'} \|A^m V\|^{p} ds' + c_p \int_{\tau}^{\tau'} \|A^m V\|^{p-2} \left\| A^{m-1} \left( b \nabla V \cdot \nabla (u^{(1)} + u^{(2)}) \right) \right\|^2 ds'.
\]

For the norm \( \|A^{m-1} f(u^{(1)})\| \), we recall the discussions to (3.10) and (3.13) and consider the embeddings into \( H^{2m} \). Then we know that
\[
\|A^{m-1} f(u^{(1)})\| \leq c \left( \|u^{(1)}\|_{2m} + \|u^{(1)}\|_{2m}^2 + \|u^{(1)}\|_{2m}^2 \right).
\]

Using similar estimates to (4.49) and (4.50), we obtain that
\[
|M'_1| \leq \frac{2p-1}{2p} \int_{\tau}^{\tau'} \|A^m V\|^{p-2} \|A^{m+1} V\|^2 ds'
+ c_{m,p} \int_{\tau}^{\tau'} \|A^m V\|^{p} \left( ||u^{(1)}||_{2m}^4 + ||u^{(2)}||_{2m}^4 + 1 \right) ds'. \tag{5.6}
\]

For \( M'_2 \) and \( M'_3 \), we necessarily notice the following estimate,
\[
\|\delta \nu (\|u^{(1)}\|_2) A^m \phi (u^{(1)}) \|
\leq c \left( \|\delta \nu (\|u^{(1)}\|_2) A^m \phi (u^{(2)}) \|_{L^2(U,H)} \right)^2
+ c \|A^m V\|^2 \kappa^2 (\|u^{(1)}\|_{\infty}) \left( 1 + \|u^{(1)}\|_{2m}^2 + c\kappa^2 (\|u^{(1)}\|_{\infty} + \|u^{(2)}\|_{\infty}) \|A^m V\|^2 \ight)
\]
\[
\leq c \|A^m V\|^2 \left( \kappa^2 (\|u^{(1)}\|_{\infty}) \left( 1 + \|u^{(1)}\|_{2m}^2 + \kappa^2 (\|u^{(1)}\|_{\infty} + \|u^{(2)}\|_{\infty}) \right) \right)
:= c \|A^m V\|^2 \Psi (u^{(1)}, u^{(2)}).
\]

Then
\[
|M'_2| \leq c_p \int_{\tau}^{\tau'} \|A^m V\|^p \Psi (u^{(1)}, u^{(2)}) ds'. \tag{5.7}
\]

For \( M'_3 \), we use BDG inequality as before and obtain that
\[
E \left( \chi_{\Omega_0} \sup_{s \in [\tau', \tau'']} |M'_3(s)| \right) \leq c \left( \int_{\tau}^{\tau'} \|A^m V\|^{2p} \Psi (u^{(1)}, u^{(2)}) ds \right)^{\frac{1}{2}}
\]
\[
\leq c \left( \sup_{s \in [\tau', \tau'']} \|A^m \hat{V}(s)\|^p \right)^{\frac{1}{2}} \left( \int_{\tau}^{\tau'} \|A^m \hat{V}\|^p \Psi (u^{(1)}, u^{(2)}) ds \right)^{\frac{1}{2}}
\]
\[
\leq \frac{1}{2p} \left( \sup_{s \in [\tau', \tau'']} \|A^m \hat{V}(s)\|^p \right)^{\frac{1}{2}} \left( \int_{\tau}^{\tau'} \|A^m \hat{V}\|^p \Psi (u^{(1)}, u^{(2)}) ds \right)
\]. \tag{5.8}

Taking the expected value of the product of \( \chi_{\Omega_0} \) and the supremum of (5.5) over \( s \in [\tau', \tau''] \), and combining (5.6)–(5.8), we have
Suppose that

The proof is a slight modification of that for lemma

The tightness of the sequence

tells us the uniqueness of local pathwise solutions

Suppose that

use Skorohod’s Theorem to attain the existence of a probability space

\[ \mu \]

one

and

Then we similarly have the following tightness for

Indeed, lemma 5.1 tells us the uniqueness of local pathwise solutions \((u, \rho_N)\) (defined in definition 2.4), where \(\rho_N\) is defined in (4.8) with \(u\) therein replaced by \(u\).

Next we continue following proposition 2.5, for which we recall the Galerkin solutions \(u^\varepsilon\) of (3.2) with the given stochastic basis \(S\). Let \(\mu_{u^\varepsilon}^m\) be the collection of joint distributions given by \((u^\varepsilon, u^\varepsilon)\) and define the extended phase space

\[ X_u^T = X_u \times X_u, \quad X^T = X_u^T \times X_W \]

with \(X_u\) and \(X_W\) defined in (4.1). Corresponding to \(X_u^T\) and \(X^T\) and following the definitions (4.2) and (4.3), we ulteriorly define

\[ \mu_u^{m,n} = \mu_u^n \times \mu_u^m, \quad \mu^{m,n} = \mu_u^{m,n} \times \mu_W. \]

Then we similarly have the following tightness for \(\{\mu^{m,n}\}_{n,m \geq 0}\).

**Lemma 5.2.** Suppose that \(\phi\) satisfies (2.7) and \(u_0\) satisfies (2.11). Then the collection \(\{\mu^{m,n}\}_{n,m \geq 0}\) is tight on \(X^T\).

**Proof.** The proof is a slight modification of that for lemma 4.1. After determining the sets \(B_k^T\) and \(B_k\), we can take compact \(A_\varepsilon\) and \(B_\varepsilon\) in \(X_u\) and \(X_W\) respectively such that \(\mu_u^n(A_\varepsilon) \geq 1 - \frac{\varepsilon}{4}\) and \(\mu_u^n(B_\varepsilon) \geq 1 - \frac{\varepsilon}{4}\) for each \(n \in \mathbb{N}\). Then the set \(A_\varepsilon \times A_\varepsilon \times B_\varepsilon\) is compact in \(X^T\) and

\[ \mu^{m,n}(A_\varepsilon \times A_\varepsilon \times B_\varepsilon) \geq \left(1 - \frac{\varepsilon}{4}\right)^2 \left(1 - \frac{\varepsilon}{2}\right) \geq 1 - \varepsilon \]

for all \(\varepsilon > 0\) and \(m,n \in \mathbb{N}\). This is the tightness.

With the tightness result, we are prepared to show the existence of local pathwise solution of (2.13).

**Theorem 5.3.** Suppose that \(\phi\) satisfies (2.7). Then the problem (2.13) possesses a unique local pathwise solution. Moreover, each local pathwise solution can be extended to be a maximal one \((u, \tau_0, \{\tau_n\}_{n \geq 1})\), and the sequence \(\{\tau_n\}_{n \geq 1}\) announces any finite time blowup.

**Proof.** The tightness of the sequence \(\{\mu^{m,n}\}_{n,m \geq 0}\) enables us to choose a subsequence so that \(\mu^{m,n}\) converges weakly to an element \(\mu\) by Prohorov’s Theorem. Similarly as above, we can use Skorohod’s Theorem to attain the existence of a probability space \((\Omega, \mathcal{F}, \tilde{\mathbb{P}})\), on which there
defines a sequence of random elements $(\tilde{u}^n, \tilde{\mu}^n, \tilde{W})$ converging almost surely to an element $(\tilde{u}, \tilde{\mu}^*, \tilde{W})$ with their laws satisfying
\[ \tilde{P}(\tilde{u}^n, \tilde{\mu}^n, \tilde{W}) = \mu^{n,\mu_n}, \quad \tilde{P}(\tilde{u}, \tilde{\mu}^*, \tilde{W}) = \mu. \]

For the almost surely convergences $(\tilde{u}^n, \tilde{\mu}^n, \tilde{W}) \to (\tilde{u}, \tilde{\mu}^*, \tilde{W})$, by similar argument in the proof of theorem 4.2, we infer that both $\tilde{u}$ and $\tilde{\mu}^*$ are global martingale solutions of (3.1) relative to the same stochastic basis $\tilde{S} = \{ \tilde{\Omega}, \tilde{\mathcal{F}}, \{ \tilde{\mathcal{F}}_t \}_{t \geq 0}, \tilde{P}, \tilde{W} \}$ with $\tilde{\mathcal{F}}_t$ the completion of $\sigma$-algebra generated by $\{ (\tilde{u}(s), \tilde{\mu}^*(s), \tilde{W}(s)) : s \leq t \}$. Note that $\mu^{n,\mu_n}$ converges weakly to the measure $\mu^*$ on $\mathcal{X}$ with $\mu^*(\cdot) = \tilde{P}(\tilde{u}, \tilde{\mu}^*) \in \cdot$. Since the choice of $u^*$ and $\mu^n$ has ensured the equality $\tilde{u}(0) = \tilde{\mu}^*(0) = u_0$ by the convergence, by lemma 5.1, we deduce that $\tilde{u} = \tilde{\mu}^*$ in $\mathcal{X}_u$ almost surely, saying,
\[ \mu^t(\{(x, y) \in \mathcal{X}_u : x = y\}) = \tilde{P}(\tilde{u} = \tilde{\mu}^* \in \mathcal{X}_u) = 1. \]

This indeed indicates that the original sequence $u^n$ given in section 3 defined on the initial probability space $\Omega, \mathcal{F}, \mathbb{P}$ converges to an element $u$, in the topology of $\mathcal{X}_u$. Then $u$ is a global pathwise solution of (3.1).

Based on this result, we choose a appropriate strictly positive stopping time $\tau$ to construct the local pathwise solution of (2.13). Note that $u_0 \in H^2$ almost surely. First we assume $\|u_0\|_{2m} \leq \mathcal{M}$ for some deterministic $\mathcal{M} > 0$. Then by embeddings, $\|u_0\|_1 \leq c\mathcal{M}$ for some $c > 0$. Let $u$ be the global pathwise solution of (3.1), which is in $L^2(\Omega; C([0, \infty); H^2))$ by embeddings. To deduce the maximal pathwise solution, we recall the stopping time $\rho_N$ defined in (4.8). Since $u \in C([0, +\infty); H^2)$, if we take $N > c\mathcal{M}$, we can easily see that $\rho_N > 0$. Thus let $\tau = \rho_N$ and the pair $(u, \tau)$ is just the local pathwise solution of (2.13).

For the general case when $u_0$ is a $H^p$-valued random variable and $\mathcal{F}_0$-measurable, we proceed as what appeared in [20, section 4.2]. Define, for each $k \geq 0$, $u^{(k)}_0 = u_0 \chi \{ k \leq \|u_0\|_{2m} < k+1 \}$. By the discussion above, we get a corresponding local pathwise solution $(\mu^{(k)}, \tau^{(k)})$. Thus we obtain a local pathwise solution $(u, \tau)$ for the initial datum $u_0$ by defining
\[ u = \sum_{k \geq 0} u^{(k)} \chi \{ k \leq \|u_0\|_{2m} < k+1 \}, \quad \tau = \sum_{k \geq 0} \tau^{(k)} \chi \{ k \leq \|u_0\|_{2m} < k+1 \}. \]

Next, for every fixed $u_0 \in H^p$, we go on to extend the solution $(u, \tau)$ to a maximal time (referring the method in [20, 21]). Let $\mathcal{T}$ be the set of all stopping times $\tau$ for a local pathwise solution of (2.13). Take $\tau_0 = \sup \mathcal{T}$. By the uniqueness of pathwise solutions we can obtain a process $u$ defined on $[0, \tau_0)$ such that $(u, \tau)$ is a local pathwise solution for each stopping time $\tau \in (0, \tau_0)$.

In the following, we show the existence of the strictly positive increasing stopping time sequence $\{ \tau_n \}_{n \geq 1}$ such that the triple $(u, \tau, \{ \tau_n \}_{n \geq 1})$ is a maximal pathwise solution of (2.13). For each $n \in \mathbb{N}^+$, take
\[ \tilde{\rho}_n = \inf \{ t \geq 0 : \|u(t)\|_2 > n \} \wedge \tau_0. \]

(5.10)

The continuity of $u$ on $H^2$ ensures that $\tilde{\rho}_n$ is a well-defined stopping time. Moreover, by uniqueness we know that $(u, \tilde{\rho}_n)$ is a local pathwise solution on $\{ \tilde{\rho}_n > 0 \}$ for each $\omega \in \Omega$ and $n > 0$ (Note that each $\omega$ has sufficiently large $N$ so that $\tilde{\rho}_n(\omega) > 0$). Suppose by contradiction that, for some $n, T > 0$, we have $\tilde{P}(\tilde{\tau}_n = \tilde{\rho}_n \wedge T) > 0$. Since $\tau_0 > 0$, we see that $(u, \tilde{\rho}_n \wedge T)$ is a local pathwise solution. By the unique existence of local pathwise solutions established above, we have another stopping time $\tau’ > \tilde{\rho}_n \wedge T$ and a process $u’$ such that $(u’, \tau’)$ is a local pathwise solution corresponding to $u(\tilde{\rho}_n \wedge T)$ and then also to $u_0$. This denies the maximality of $\tau_0$. 

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We have shown that \( \mathbb{P}(\tau_0 = \tilde{\rho}_n \land T) = 0 \), for all \( n, T > 0 \). Actually, on the set \( \{ \tau_0 < \infty \} \), it can be seen that \( \tilde{\rho}_n < \tau_0 \) for all \( n > 0 \) almost surely by appropriate choices of \( T \). By (5.10), we infer that \( \sup_{s \in [0, \tilde{\rho}_n]} \|u(s)\|_2 = n \) and so as \( n \to \infty \),

\[
\sup_{s \in [0, \tilde{\rho}_n]} \|u(s)\|_2^2 + \int_0^{\tilde{\rho}_n} \|A^2 u(s)\|_2^2 \, ds \to +\infty.
\]

Now let \( \tilde{\tau}_n \) be a strictly positive increasing stopping time with \( \tilde{\tau}_n < \tau_0 \) and \( \tilde{\tau}_n \to \infty \) as \( n \to \infty \). We have known \( (u, \tilde{\tau}_n) \) is a local pathwise solution for each \( n \). Then the sequence \( \tau_n := \tilde{\rho}_n \lor \tilde{\tau}_n \) is the sequence we require. We have finally accomplished the proof now. \( \square \)

5.2. Global existence of pathwise solutions

In this subsection we establish the global existence of pathwise solution of (2.13). To this aim, we need to assure that for the maximal pathwise solution \( (u, \tau_0, \{\tau_n\}_{n \geq 1}) \), the probability \( \mathbb{P}(\tau_0 = \infty) = 1 \). This is stated in the following theorem.

**Theorem 5.4.** Let \( S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W) \) be a stochastic basis. Suppose that \( u_0 \) is an \( H^m \)-valued \( (m \geq 1) \) random variable (relative to \( S \)) and \( \mathcal{F}_0 \)-measurable, \( \phi \) satisfies (2.3) and \( |b| < 4 \). Let \( (u, \tau_0, \{\tau_n\}_{n \geq 1}) \) be the maximal pathwise solution of (2.13). Then \( \mathbb{P}(\tau_0 = \infty) = 1 \) and \( (u, \tau_n) \) is naturally a unique global pathwise solution of (2.13) corresponding to the initial datum \( \mu_0 \).

To show theorem 5.4, we need a small result as follows.

**Lemma 5.5.** Let \( (\Omega, \mu) \) be a positive measure space and \( \Omega_0 \subset \Omega \) be a measurable subset of \( \Omega \) with \( \mu(\Omega_0) < \infty \). Suppose that \( \{f_k\}_{k \geq 1} \) is a sequence of nonnegative \( \mu \)-integrable functions defined on \( \Omega \) such that \( f_k(\omega) \to +\infty \) as \( k \to \infty \) for almost every \( \omega \in \Omega_0 \). Then

\[
\int_{\Omega_0} f_k(\omega) \mu(\,d\omega) \to +\infty, \quad \text{as } k \to \infty. \tag{5.11}
\]

**Proof.** It is sufficient to show that for each \( R > 0 \), there exists \( K > 0 \) so that when \( k \geq K \),

\[
\int_{\Omega_0} f_k(\omega) \mu(\,d\omega) > R. \quad \text{Indeed, if we set } g_k(\omega) = \arctan f_k(\omega) \text{ for each } \omega \in \Omega_0, \text{ we know that } g_k(\omega) \to \frac{\pi}{2} \text{ from below for a.e. } \omega \in \Omega_0. \quad \text{By Egorov’s theorem, we know that there is a measurable subset } \Omega'_0 \text{ of } \Omega \text{ with } \mu(\Omega'_0) = \mu(\Omega_0)/2, \text{ such that } \{g_k\}_{k \geq 1} \text{ uniformly converges to } \frac{\pi}{2} \text{ when restricted to } \Omega'_0. \text{ This means that we can find } K > 0 \text{ such that whenever } k \geq K, \quad g_k(\omega) = \arctan \frac{2R}{\mu(\Omega_0)}, \quad \text{for every } \omega \in \Omega'_0,
\]

and hence \( f_k(\omega) > \frac{2R}{\mu(\Omega_0)} \). As a result, we obtain that

\[
\int_{\Omega_0} f_k(\omega) \mu(\,d\omega) \geq \int_{\Omega'_0} f_k(\omega) \mu(\,d\omega) \geq \frac{2R}{\mu(\Omega_0)} \cdot \frac{\mu(\Omega_0)}{2} = R.
\]

The proof is finished. \( \square \)

**Remark 5.6.** Lemma 5.5 is a simple result in measure theory and should have appeared somewhere, but we have not found any one in other references. For the completeness of this paper, we give a concise proof above.

Now we go on to verify theorem 5.4.
Proof of theorem 5.4. The uniqueness of local pathwise solution has ensured the uniqueness of the global pathwise solution. We hence only need to show $P(\tau_0 = \infty) = 1$.

Since $(u, \tau_n)$ is a local pathwise solution (2.13) for each $n \in \mathbb{N}^+$, we can apply the estimation method in the proof of lemma 4.4 so that for each $T > 0$, $n \in \mathbb{N}^+$ and $0 \leq \tau' \leq \tau'' \leq \tau_n \wedge T$

$$E \sup_{x \in [\tau', \tau'']} ||u(s)||^{2(2m+3)} \leq 2E||u(\tau')||^{2(2m+3)} + c_{m,E} \int_{\tau'}^{\tau''} \left(1 + ||u||^{2(2m+3)}\right) ds + c(\tau'' - \tau'),$$

similar to (4.42) by observing that essentially $\delta_N$ does not involve the estimation in that proof except the maximum of $\delta_N$ being 1. Define a subspace $D_r$ of $\Omega$ for each $r > 0$ such that

$$D_r := \{\omega \in \Omega : ||u_0||^2 + ||u_0||^{2(2m+3)} \leq r\}.$$

Using stochastic Gronwall’s inequality, we similarly obtain a positive constant $C_{11} = C_{11}(r)$ independent of $n$, such that

$$E \sup_{x \in [0, \tau_n \wedge T]} ||u(s)||^{2(2m+3)} \chi_{D_r} \leq C_{11}' \quad \text{and}$$

$$E \left(\sup_{x \in [0, \tau_n \wedge T]} ||u(s)||^2 + \int_0^{\tau_n \wedge T} ||A^2 u||^2 ds\right) \chi_{D_r} \leq C_{11}'.$$

Now we show the conclusion $P(\tau_0 = \infty) = 1$ by contradiction and suppose the contrary. Then there is a subset $\Omega_0 \subset \Omega$ with $P(\Omega_0) > 0$ such that for all $\omega \in \Omega_0$, $\tau_0(\omega) < \infty$. Note that

$$\Omega_0 = \bigcup_{k \in \mathbb{N}^+} \Omega_k \cap \{\tau_0(\omega) \leq k\}.$$

We immediately have $r_0 \in \mathbb{R}^+$ and $k_0 \in \mathbb{N}^+$ so that $\varepsilon_0 := P(\Omega_0 \cap D_{r_0}) > 0$. Apparently, $\tau_n(\omega) \leq k_0$ for all $\omega \in \Omega_0 \cap D_{r_0}$ and $n \in \mathbb{N}^+$.

According to (2.16) in the definition of maximal pathwise solution, we can pick $T = k_0$ and obtain that for almost every $\omega \in \Omega_0$, (2.16) holds with $\tau_n$ therein replaced by $\tau_n \wedge T$. Then taking $n \to \infty$, we see by lemma 5.5 the left side of the second inequality tends to the infinity, but $C_{11}'$ is independent of $n$, which causes a contradiction and finishes the proof.

6. Existence of ergodic invariant measures

In this section, we consider the existence of ergodic invariant measures for the two-dimensional stochastic modified Swift–Hohenberg periodic problem (1.1)–(1.3). In what follows in this section, let $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)$ be a stochastic basis. We assume that $|b| < 4$ and $\phi$ satisfies (2.3) and (2.4). This is sufficient to guarantee the unique existence of global pathwise solutions by theorem 5.4 for each $H^{2m}$-valued random variable $u_0$ measured by $\mathcal{F}_0$.

Let $u(t; u_0)$ be the unique global pathwise solution in $H^{2m}$ of the problem (2.13). This defines a stochastic process $u(t; \cdot)$ on the space $H^{2m}$. For notational convenience, we often use $u(t)$ to denote stochastic process $u(t; \cdot)$. Let $\mathcal{P}_t$ be the Markovian transition semigroup associated with the stochastic process $u(t)$ on $B(H^{2m})$. In order to prove the existence of invariant measures for the semigroup $\mathcal{P}_t$, we first present some auxiliary consequences of the estimates of moment bounds and some related probabilities for the solution $u(t; u_0)$. 2691
6.1. Estimates on moment bounds of solutions

In the following, we let the initial data \( u_0 \) be an \( H^{2m} \)-valued random variable. For each \( u_0 \) given above and \( T > 0 \), recall the deduction of (5.12) and (5.13). In that \( u(t; u_0) \) is a given global pathwise solution of (2.13), we can repeat the procedure from (4.39) to (4.44) by replacing \( u^p \) and \( \delta_N \) therein by \( u \) and \( l \). Thus we can obtain similar estimates to (4.43) and (4.35) as follows, for each \( t \in [0, T] \) and \( u_0 \) satisfying (2.12) with \( p = 2 \),

\[
\mathbb{E} \left( \sup_{s \in [0, t]} \| u(s) \|^2(2m+3) + \int_0^t \| u(r) \|^2(2m+2) \| A u(r) \|^2 \, ds \right) \leq C_d \left( \mathbb{E} \| u_0 \|^2(2m+3) + t \right),
\]

(6.1)

\[
\mathbb{E} \left( \sup_{s \in [0, t]} \| A^m u(s) \|^2 + \int_0^t \| A^{m+1} u(r) \|^2 \, ds \right) \leq C_d \left( \mathbb{E} \| u_0 \|^2_{2m} + \mathbb{E} \| u_0 \|^2(2m+3) + t \right),
\]

(6.2)

where \( C_d = C_d(T) \) is a positive constant.

Define a stopping time \( \xi_r(u_0) \) for each \( r > 0 \) and \( H^{2m} \)-valued stochastic variable \( u_0 \) so that

\[
\xi_r(u_0) := \inf \{ t \geq 0 : \| A^m u(t; u_0) \|^2 > r \}.
\]

(6.3)

In order to obtain a description of the instant parabolic regularization for the solutions of (2.13), we define another stopping time \( \eta_r(u_0) \) for each \( r > 0 \) and \( H^{2m} \)-valued stochastic variable \( u_0 \) such that

\[
\eta_r(u_0) = \inf \{ t \geq 0 : t \| A^{m+1} u(t; u_0) \|^2 > r \}.
\]

To estimate the probability concerning \( \eta_r \), we first give a suitable local moment bound on the \( H^{2(m+1)} \)-norm of the solutions.

**Lemma 6.1.** Let \( \phi \) satisfy (2.3) and (2.4). For each \( T > 0 \) and \( u_0 \) satisfying (2.12) with \( p = 2 \) and \( q' > 2(2m + 5) \), it holds that

\[
\mathbb{E} \sup_{s \in [0, t]} \langle s \| A^{m+1} u(s) \|^2 \rangle \leq C_5 \left( \mathbb{E} \| u_0 \|^2_{2m} + \mathbb{E} \| u_0 \|^2(2m+5) + t \right),
\]

(6.4)

for all \( t \in [0, T] \), where \( C_5 := C_5(T) \) is a positive constant.

**Proof.** Apply \( A^{m+1} \) to (2.13) and Itô’s Formula to \( s \| A^{m+1} u(s) \|^2 \), we deduce that for discount times \( 0 \leq \tau' \leq \xi \leq s \leq \tau'' \leq t \leq T \),

\[
\begin{align*}
\| A^{m+1} u(s) \|^2 &+ 2 \int_\xi^{t'} s' \| A^{m+2} u \|^2 \, ds' \\
&= \xi \| A^{m+1} u(\xi) \|^2 - \int_\xi^{t'} \left[ 2s' \langle A^{m+1} u, A^{m+1} f(u) \rangle - \| A^{m+1} u \|^2 \right] \, ds' \\
&\quad + \int_\xi^{t'} s' \| A^{m+1} \phi(u) \|^2_{L_2(\Omega, L_d)} \, ds' + 2 \int_\xi^{t'} s' \langle A^{m+1} u, A^{m+1} \phi(u) \rangle \, dW.
\end{align*}
\]

(6.5)

Similar to the treatment in the proof of Lemma 3.2, we have by (2.4) that

\[
\begin{align*}
\| A^{m+1} u, A^{m+1} f(u) \| &\leq \| A^{m+2} u \|^2 + c \left( \| A^{m+1} u \|^2 + \| u \|^2(2m+5) + 1 \right) \\
\| A^{m+1} \phi(u) \|^2_{L_2(\Omega, L_d)} &\leq c \left( \| A^{m+1} u \|^2 + 1 \right).
\end{align*}
\]

(6.6) \hspace{1cm} (6.7)
The estimates from (6.5) to (6.7) indicate that
\[ s\|A^{m+1}u(s)\|^2 \]
\[ \leq c\|A^{m+1}u(\varsigma)\|^2 + c\int_{\varsigma}^{s'} s\left(\|A^{m+1}u\|^2 + \|u\|^{2(2m+5)} + 1\right) ds' \]
\[ + \int_{\varsigma}^{s'} \|A^{m+1}u\|^2 ds' + 2\left|\int_{\varsigma}^{s'} s'(A^{m+1}u, A^{m+1}\phi(u)dW)\right|. \quad (6.8) \]
Integrating (6.8) with respect to \( \varsigma \) over \([\tau', s]\) and dividing it by \( s - \tau' \), we have
\[ s\|A^{m+1}u(s)\|^2 \]
\[ \leq \frac{1}{s - \tau'} \int_{\tau'}^{s'} s\|A^{m+1}u(\varsigma)\|^2 d\varsigma + c\int_{\tau'}^{s'} s\left(\|A^{m+1}u\|^2 + \|u\|^{2(2m+5)} + 1\right) ds' \]
\[ + \int_{\tau'}^{s'} \|A^{m+1}u\|^2 ds' + 2\sup_{\varsigma \in [\tau', s]}\left|\int_{\varsigma}^{s'} s'(A^{m+1}u, A^{m+1}\phi(u)dW)\right|. \quad (6.9) \]
Note also by BDG inequality that
\[ \mathbb{E}\sup_{s \in [\tau', \tau'']} \sup_{\varsigma \in [\tau', \tau]} \left|\int_{\varsigma}^{s'} s'(A^{m+1}u, A^{m+1}\phi(u)dW)\right| \]
\[ = \mathbb{E}\sup_{s \in [\tau', \tau'']} \left(\left(\int_{\varsigma}^{s'} s'(A^{m+1}u, A^{m+1}\phi(u)dW)\right)\right) \]
\[ \leq 2\mathbb{E}\sup_{s \in [\tau', \tau'']} \left|\int_{\varsigma}^{s'} s'(A^{m+1}u, A^{m+1}\phi(u)dW)\right| \]
\[ \leq c\mathbb{E}\left(\int_{\varsigma}^{s'} (s')^2\|A^{m+1}u\|^2 (1 + \|A^{m+1}u\|^2) ds'\right)^{\frac{1}{2}} \]
\[ \leq c\mathbb{E}\left(\sup_{s' \in [\tau', \tau'']} (s')\|A^{m+1}u(s')\|^2\right)^{\frac{1}{2}} \left(\int_{\varsigma}^{s'} s' \left(1 + \|A^{m+1}u(s')\|^2\right) ds'\right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{4} \mathbb{E}\sup_{s \in [\tau', \tau'']} (s)\|A^{m+1}u(s)\|^2 + c\mathbb{E}\int_{\tau'}^{s'} s' \left(\|A^{m+1}u\|^2 + \|u\|^{2(2m+5)} + 1\right) ds'. \]
Thus, by considering the supremum of (6.9) over \( s \in [\tau', \tau''] \), and then taking the expected value, we get
\[ \mathbb{E}\sup_{s \in [\tau', \tau'']} (s)\|A^{m+1}u(s)\|^2 \]
\[ \leq c\mathbb{E}\sup_{s \in [\tau', \tau'']} \frac{1}{s - \tau'} \int_{\tau'}^{s} s\|A^{m+1}u(\varsigma)\|^2 d\varsigma + c\mathbb{E}\int_{\tau'}^{s'} \|A^{m+1}u\|^2 ds' \]
\[ + c\mathbb{E}\int_{\tau'}^{s'} s' \left(\|A^{m+1}u\|^2 + \|u\|^{2(2m+5)} + 1\right) ds'. \quad (6.10) \]
By (6.1) and (6.2), we know the right side of (6.10) makes sense even when \( \tau' = 0 \) and \( \tau'' = t \) and hence we immediately have
\[ E \sup_{s \in [0,t]} (s\|A^{m+1}u(s)\|^2) \leq cE \sup_{s \in [0,t]} \frac{1}{s} \int_0^s \|A^{m+1}u(\zeta)\|^2 \, d\zeta + cE \int_0^t \|A^{m+1}u\|^2 \, ds' + cE \int_0^t s' \left( \|A^{m+1}u\|^2 + \|u\|^2(2m+5) + 1 \right) \, ds'. \]

which implies (6.4) by (6.1) and (6.2). The proof is over now. \[ \square \]

**Remark 6.2.** Note that \( s\|A^{m+1}u(s)\|^2 \) is not defined for \( s = 0 \). This prompts us to integrate (6.8) for \( \zeta \) over \([\tau', s]\) and then we successfully avoid the abuse of \( \left( s\|A^{m+1}u(s)\|^2 \right)_{s=0}^{\tau} \).

Now we give the estimate of the probabilities of \( \xi_\zeta \) and \( \eta_\tau \) in the following lemma.

**Lemma 6.3.** For each \( t \in (0, T) \), \( r > 0 \) and \( u_0 \) satisfying (2.12) with \( p = 2 \) and \( q' \geq 2(2m+5) \),

\[ P(\xi_\zeta(u_0) < t) \leq \frac{C_4}{r} \left( E\|u_0\|_{2m}^2 + E\|u_0\|^{2(2m+3)} + t \right), \]

\[ (6.11) \]

and

\[ P(\eta_\tau(u_0) < t) \leq \frac{C_5}{r} \left( E\|u_0\|_{2m}^2 + E\|u_0\|^{2(2m+5)} + t \right), \]

\[ (6.12) \]

where \( C_4 \) and \( C_5 \) are given by (6.2) and (6.4), respectively.

**Proof.** By applying Markovian inequality, we see that for each \( t \in (0, T] \),

\[ P(\xi_\zeta(u_0) < t) = P \left( \sup_{s \in [0,t]} \|A^m u(s)\|^2 > r \right) \leq \frac{1}{r} E \sup_{s \in [0,t]} \|A^m u(s)\|^2 \leq \frac{C_4}{r} \left( E\|u_0\|_{2m}^2 + E\|u_0\|^{2(2m+3)} + t \right), \]

which is exactly (6.11). The inequality (6.12) follows in the same way. \[ \square \]

### 6.2. Feller property

In this subsection we apply ourselves to confirming the Feller property of \( \mathcal{P}_r \), for which we first extend the stopping times \( \xi_\zeta \) and \( \eta_\tau \) to relying on two initial data. For arbitrary deterministic initial data \( u_{10} \) and \( u_{20} \), denote

\[ \xi_\zeta(u_{10}, u_{20}) := \xi_\zeta(u_{10}) \land \xi_\zeta(u_{20}) \quad \text{and} \quad \eta_\tau(u_{10}, u_{20}) := \eta_\tau(u_{10}) \land \eta_\tau(u_{20}). \]

**Lemma 6.4.** For each fixed deterministic \( T > 0 \), \( r > 0 \) and \( u_{10}, u_{20} \in H^{2m} \), it holds that

\[ E \sup_{s \in [0,T]} \|u(s, u_{10}) - u(s, u_{20})\|_{2m}^2 \leq C_6 \|u_{10} - u_{20}\|_{2m}^2, \]

\[ (6.13) \]

for all \( t \in [0, T] \), where \( C_6 = C_6(T, r) \) is a positive constant.

**Proof.** Let \( t \in [0, T] \), \( u_i(t) := u(t, u_0) \), \( i = 1, 2 \) and \( Z(t) = u_1(t) - u_2(t) \), for \( t \in [0, T] \). Then by (2.13) we have

\[ dZ + [A^2 Z + \Phi_1(u_1, u_2)] \, dt = \hat{\Phi}_2(u_1, u_2) \, dW, \]

\[ (6.14) \]
where
\[ \tilde{\phi}_1(u_1, u_2) := f(u_1) - f(u_2) \] and \( \tilde{\phi}_2(u_1, u_2) := \phi(u_1) - \phi(u_2) \).

Applying \( A^m \) to (6.14) and Itô’s Formula to \( \|A^mZ(t)\|^2 \), and similar to the argument in the proof of lemma 4.5, we have for all stopping times \( 0 \leq \tau' \leq s \leq \tau'' \leq t \land \xi(u_{t0}, u_{20}) \),

\[
\frac{1}{2} \|A^mZ(s)\|^2 + \int_{\tau'}^{s} \|A^{m+1}Z\|^2 \, ds' \\
\leq \frac{1}{2} \|A^mZ(\tau')\|^2 + \left| \int_{\tau'}^{s} \left( A^mZ, A^m\tilde{\phi}_1(u_1, u_2) \right) \, ds' \right| \\
+ \frac{1}{2} \int_{\tau'}^{s} \|A^m\tilde{\phi}_2(u_1, u_2)\|^2 \, ds' + \left| \int_{\tau'}^{s} \left( A^mZ, A^m\tilde{\phi}_2(u_1, u_2) dW \right) \right| \\
\leq \frac{1}{2} \|A^mZ(\tau')\|^2 + \int_{\tau'}^{s} \|A^{m+1}Z\|^2 \, ds' + \left| \int_{\tau'}^{s} \left( A^mZ, A^m\tilde{\phi}_2(u_1, u_2) dW \right) \right| \\
+ c \int_{\tau'}^{s} \|A^mZ\|^2 \left( \|u_1\|_{2m}^2 + \|u_2\|_{2m}^2 + 1 \right) \, ds'.
\]

Combining the estimate

\[
E \sup_{s \in [\tau', \tau'')} \left| \int_{\tau'}^{s} \left( A^mZ, A^m\tilde{\phi}_2(u_1, u_2) dW \right) \right| \\
\leq \frac{1}{2} E \sup_{s \in [\tau', \tau'')} \|A^mZ(s)\|^2 + c E \int_{\tau'}^{\tau''} \|A^mZ\|^2 \, ds',
\]

we similarly obtain

\[
E \sup_{s \in [\tau', \tau'')} \|A^mZ(s)\|^2 \leq 2 E \|A^mZ(\tau')\|^2 \\
+ c E \int_{\tau'}^{\tau''} \|A^mZ\|^2 \left( \|u_1\|_{2m}^2 + \|u_2\|_{2m}^2 + 1 \right) \, ds'.
\]

The stochastic Gronwall’s inequality and embeddings apply and we obtain (6.13). The proof is finished. \( \square \)

Now we can show the Feller property of the semigroup \( \mathcal{P}_t \).

**Theorem 6.5.** For each \( \varphi \in C_b(H^{2m}) \) and \( t \geq 0 \), the mapping \( u \mapsto \mathcal{P}_t \varphi(u) \) is continuous.

**Proof.** Pick \( u_{t0}, u_{20} \in H^{2m} \) with \( u_{t0} \) fixed. Let \( T > t \) be fixed and \( r_1, r_2 > 0 \). Due to the embedding of \( H^{2m} \) into \( H \), we can fix \( r_3 > 0 \) such that when \( \|u_{t0} - u_{20}\|_{2m} < r_3 \),

\[
\|u_{20}\|_{2m}^2 \leq \|u_{10}\|_{2m}^2 + 1 \quad \text{and} \quad \|u_{20}\|_{2m+3}^2 \leq \|u_{10}\|_{2m+5}^2 + 1.
\]

Now we divide \( |\mathcal{P}_t \varphi(u_{10}) - \mathcal{P}_t \varphi(u_{20})| \) into three parts as follows,

\[
|\mathcal{P}_t \varphi(u_{10}) - \mathcal{P}_t \varphi(u_{20})| = \left| E \left( \varphi(u(t, u_{10})) - \varphi(u(t, u_{20})) \right) \right| \\
\leq \left| E \left( \varphi(u(t, u_{10})) - \varphi(u(t, u_{20})) \right) \chi_{\{t(\eta_{r_1}(u_{t0}, u_{20}) < t)\}} \right| \\
+ \left| E \left( \varphi(u(t, u_{10})) - \varphi(u(t, u_{20})) \right) \chi_{\{t(\eta_{r_2}(u_{t0}, u_{20}) < t)\}} \right| \\
+ \left| E \left( \varphi(u(t, u_{10})) - \varphi(u(t, u_{20})) \right) \chi_{\{t(\eta_{r_1}(u_{t0}, u_{20}) < t)\} \cap \{t(\eta_{r_2}(u_{t0}, u_{20}) < t)\}} \right| \\
:= E_1 + E_2 + E_3. \quad (6.15)
\]
Let \( \|\varphi\|_\infty := \sup_{u \in B_r^m} |\varphi(u)| \) be the sup-norm of \( C_b(H^{2m}) \). We first consider \( E_1 \). By lemma 6.3, we have
\[
E_1 \leq 2\|\varphi\|_\infty \Pr(\xi_1(u_{10}, u_{20}) < t) \leq 2\|\varphi\|_\infty (\Pr(\xi_1(u_{10}) < t) + \Pr(\xi_1(u_{20}) < t))
\]
\[
\leq \frac{2C_4}{r_1} \|\varphi\|_\infty \left( \|u_{10}\|_m^2 + \|u_{10}\|^{2(m+3)}_m + \|u_{20}\|_m^2 + \|u_{20}\|^{2(m+3)}_m + t \right)
\]
\[
\leq \frac{4C_4}{r_1} \|\varphi\|_\infty \left( \|u_{10}\|_m^2 + \|u_{10}\|^{2(m+3)}_m + T + 1 \right). \tag{6.16}
\]
For \( E_2 \), by lemma 6.3, we have
\[
E_2 \leq 2\|\varphi\|_\infty \Pr(\eta_2(u_{10}, u_{20}) < t) \leq 2\|\varphi\|_\infty (\Pr(\eta_2(u_{10}) < t) + \Pr(\eta_2(u_{20}) < t))
\]
\[
\leq \frac{4C_5}{r_2} \|\varphi\|_\infty \left( \|u_{10}\|_m^2 + \|u_{10}\|^{2(m+3)}_m + T + 1 \right). \tag{6.17}
\]
For each given \( \varepsilon > 0 \), by taking \( r_1, r_2 \) sufficiently large and fixed in (6.16) and (6.17), we can ensure that
\[
E_1 + E_2 < \frac{\varepsilon}{2}. \tag{6.18}
\]
Next we address \( E_3 \). For this, we approximate \( \varphi \) by a Lipschitz function \( \tilde{\varphi} \) (to be chosen below). Observe that on the set \( \{ \eta_2(u_{10}, u_{20}) \geq t \} \), \( \|A^{(m)} u(t, u_{10})\|^2 \leq r_2/t \). Then it yields that
\[
E_3 \leq 2 \sup_{u \in B} |\varphi(u) - \varphi(u)| + |\mathbb{E}(\tilde{\varphi}(u(t, u_{10})) - \varphi(u(t, u_{20})))\chi_{\{\xi_1(u_{10}, u_{20}) > t\}}|
\]
\[
\leq 2 \sup_{u \in B} |\varphi(u) - \varphi(u)| + L_{\tilde{\varphi}} \mathbb{E}\left( \|u(t, u_{10}) - u(t, u_{20})\|_{2m} \right) \chi_{\{\xi_1(u_{10}, u_{20}) > t\}} \tag{6.19}
\]
where \( B := \mathbb{B}_2((u_{20}, u_{20})^{(m+1)}) \), \( \mathbb{B}_2(r) \) is the closed ball in \( H^\alpha \) centered at 0 with radius \( r \), and \( L_{\tilde{\varphi}} \) is the Lipschitz constant of \( \tilde{\varphi} \). Since \( H^{2(m+1)} \) is compactly embedded into \( H^{2m} \), we know that \( B \) is compact in \( H^{2m} \). By the density of Lipschitz functions in \( C_b(B) \), we can find a Lipschitz function \( \tilde{\varphi} \in C_b(B) \) such that
\[
\sup_{u \in B} |\varphi(u) - \varphi(u)| < \frac{\varepsilon}{8}. \tag{6.20}
\]
The choice of \( \tilde{\varphi} \) determines \( L_{\tilde{\varphi}} \). By Jensen’s inequality and lemma 6.4, we have
\[
L_{\tilde{\varphi}} \mathbb{E}\left( \|u(t, u_{10}) - u(t, u_{20})\|_{2m} \right) \chi_{\{\xi_1(u_{10}, u_{20}) > t\}} \leq L_{\tilde{\varphi}} \left( \mathbb{E}\left( \sup_{s \in [0, t] \cap \xi_1(u_{10}, u_{20})} \|u(s, u_{10}) - u(s, u_{20})\|_{2m}^2 \right) \right)^{1/2}
\]
\[
\leq C_6 L_{\tilde{\varphi}} \|u_{10} - u_{20}\|_{2m} \tag{6.21}
\]
Hence, for the above \( \varepsilon \), we have \( r_4 > 0 \) such that when \( \|u_{10} - u_{20}\|_{2m} < r_4 \),
\[
C_6 L_{\tilde{\varphi}} \|u_{10} - u_{20}\|_{2m} < \frac{\varepsilon}{4}. \tag{6.22}
\]
Combining the estimations from (6.15) to (6.22), we see that for each \( \varepsilon > 0 \), whenever \( \|u_{10} - u_{20}\|_{2m} < \min\{r_3, r_4\} \),
\[
|\mathcal{P}_1 \varphi(u_{10}) - \mathcal{P}_1 \varphi(u_{20})| < \varepsilon,
\]
which proves the continuity. \( \square \)
6.3. Existence of ergodic invariant measures

In this subsection, we first apply the classical procedure—Krylov–Bogoliubov existence theorem [8, corollary 11.8] and Prokhorov’s theorem [8, theorem 2.3] to verify the the existence of the invariant measures for the semigroup \( u(t) \) on \( H^{2m} \). And then we give the existence of ergodic invariant measures for \( P_t \). We present the classical procedure in our situation here.

**Lemma 6.6.** Assume that \( P_t \) is Feller on \( C_0(H^{2m}) \). For each \( u \in L^2(\Omega, H^{2m}) \), define a family of Borel probability measures on \( H^{2m} \),

\[
\nu_T(\cdot) := \frac{1}{T} \int_0^T P_t(u, \cdot) dt, \quad T > 0.
\]  

(6.23)

Suppose that the family \( \{\nu_T\}_{T > 0} \) is tight. Then each sequence \( \{\nu_{T_n}\}_{n \in \mathbb{N}} \) with \( T_n \to \infty \) as \( n \to \infty \) has a weakly convergent subsequence and the corresponding weak limit is an invariant measure for \( P_t \).

Now we show the existence of the invariant measures.

**Theorem 6.7.** Suppose that \( |b| < 4 \) and \( \phi \) satisfies (2.3) and (2.4). Then there exists an invariant measure on \( H^{2m} \) for the semigroup \( P_t \).

**Proof.** Following theorem 6.5 and lemma 6.6, we only need to show the tightness of the \( \nu_T \) defined as (6.23) for some suitable initial datum \( u_0 \), saying, given some \( H^{2m} \)-valued \( \mathcal{F}_0 \)-measurable random variable \( u_0 \), for arbitrary \( \varepsilon > 0 \), there exists a compact set \( K_c \subset H^{2m} \) such that

\[
\nu_T(K_c) \geq 1 - \varepsilon, \quad \text{for all } T > 0,
\]

(6.24)

Indeed, if we set \( u_0 = 0 \), then due to the compact embedding of \( H^{2m} \) into \( H^{2m} \) and the fact \( u(t; 0) \in H^{2(m+1)} \) for all \( t > 0 \), by Chebyshev’s inequality and (6.2) and choosing \( K_c = \overline{B}_{2(m+1)}(\mathcal{R}) \), we have

\[
\nu_T(H^{2m} \setminus \overline{B}_{2(m+1)}(\mathcal{R})) = \nu_T(\bigcup_{n \in \mathbb{N}} A^n(\mathcal{R})) \\
= \frac{1}{T} \int_0^T \mathbb{P}(\|A^{m+1}u(t, 0)\| > \mathcal{R}) dt \\
\leq \frac{1}{T \mathcal{R}^2} \mathbb{E} \int_0^T \|A^{m+1}u(t, 0)\|^2 dt \leq \frac{C_4}{\mathcal{R}^2} < \varepsilon,
\]

as long as \( \mathcal{R} \) is sufficiently large. This asserts the existence of \( K_c \) for (6.24). Therefore, lemma 6.6 implies the existence of invariant Borel probability measures.

The second task now is to investigate the existence of ergodic invariant measures for the Markovian semigroup \( P_t \). For this goal, it is necessary for us to show the tightness of \( \mathcal{I} \), the set of all the invariant measures for \( u(t) \) on \( H^{2m} \). We first prove the uniform boundedness of some integrations over \( H^{2m} \) with respect to each invariant measure.

**Lemma 6.8.** Let \( \nu \in \mathcal{I} \) and \( q \geq 2 \). Then we have \( \tilde{p}_0^q > 0 \) such that

\[
\int_{H^{2m}} \|u\|^q \nu(du) \leq \tilde{p}_0^q.
\]

(6.25)

**Proof.** We utilize the property of invariant measures to give the proof. We again start with the equation (2.13). Apply Itô’s Formula to \( \|u\|^q \) and then we obtain
\[
\begin{align*}
\text{d}||u(\zeta)||^q + ||u(\zeta)||^q\text{d}\zeta + q||u||^{q-2}\langle Au, u\rangle\text{d}\zeta \\
= \frac{q}{2}||u||^{q-4}\left(r + 2\langle u, \phi(u)\rangle_{L^2(\Omega, \mathbb{R})} + ||u||^2\langle \phi(u), \phi(u)\rangle_{L^2(\Omega, \mathbb{R})}\right)\text{d}\zeta \\
+ ||u(s)||^q\text{d}\zeta - q||u||^{q-2}\langle u, f(u)\rangle\text{d}\zeta + q||u||^{q-2}\langle u, \phi(u)dW\rangle. 
\end{align*}
\] (6.26)

Let \(\xi^0_t(u_0)\) be a stopping time such that \(\xi^0_t(u_0) = \inf\{t \geq 0 : ||u(t)||^2 > r\}\). Then multiply (6.26) by \(e^t\) and integrate the obtained equality over \(\zeta \in [0, s]\) with \(s \in (0, t \cup \xi^0_t(u_0))\). Similar to the treatment of (4.39) and by noting that it is not necessary to take the abstract value for the stochastic integral, we deduce that

\[
\begin{align*}
||u(s)||^q + \frac{q}{2} \int_0^s e^{s-t}||u||^{q-2}\left(||Au||^2 + ||u||^2\right)\text{d}\zeta \\
\leq ||u_0||^q e^{-t} + q \int_0^s e^{s-t}||u||^{q-2}\left[e||Au||^2 - \frac{1}{2} \left(1 - \frac{|b|}{4}\right)||Au||^2\right]\text{d}\zeta \\
+ q \int_0^s e^{s-t}||u||^{q-2}\left[e \left(1 + |u|^2 + |u|^2\right) - \frac{1}{2} \left(1 - \frac{|b|}{4}\right)|u|^2\right]\text{d}\zeta \\
+ q \int_0^s e^{s-t}||u||^{q-2}\langle u, \phi(u)dW\rangle \\
\leq ||u_0||^q e^{-t} + \bar{\rho}_0 + q \int_0^s e^{s-t}||u||^{q-2}\langle u, \phi(u)dW\rangle, 
\end{align*}
\] (6.27)

where \(\bar{\rho}_0 = \bar{\rho}_0(q)\) is a positive constant independent of \(u\) and \(s\). Let \(R > 0\) and consider the set

\[D_R = \{\omega \in \Omega : ||u_0||^q \leq R\} .\]

Taking \(s = t \cup \xi^0_t(u_0)\), multiplying (6.27) by \(\chi_{D_R}\) and taking the expected value, we have

\[E||u(t)\cap \xi^0_t(u_0)||^q \chi_{D_R} \leq E||u_0||^q e^{-t\cap \xi^0_t(u_0)} \chi_{D_R} + \bar{\rho}_0 \leq E||u_0||^q \chi_{D_R} + \bar{\rho}_0, \]

by the property of stochastic integrals [8], which annihilates the stochastic integral. Fix \(R > 0\). Applying the dominated convergence theorem, we let \(t\) tend to the infinity, and obtain

\[E||u(t)||^q \chi_{D_R} \leq e^{-t}E||u_0||^q \chi_{D_R} + \bar{\rho}_0. \]

For each fixed \(R > 0\), we can find \(t_R > 0\) such that

\[E||u(t_R)||^q \chi_{D_R} \leq 2\bar{\rho}_0. \]

By the property of invariant measures, we know that the distribution of \(||u(t_R)||^q\) is \(\nu\). Hence

\[E||u(t_R)||^q \chi_{D_R} = \int_{H^m} ||u||^q \chi_{u(x,D_R)}\nu(du) = \int_{H^m} ||u||^q \nu(du), \]

as \(R \to +\infty\). Note that \(\bar{\rho}_0\) is independent of \(R\). By taking \(\bar{\rho}_0 = 2\bar{\rho}_0\) and setting \(R \to +\infty\), we obtain (6.25) and complete the proof.

**Lemma 6.9.** Let \(\nu \in \mathcal{I}\). Then for the \(m\) given in this section, we have \(\bar{\rho}_m > 0\) such that

\[\int_{H^m} ||u||^2 \nu(du) \leq \bar{\rho}_m. \] (6.28)
**Proof.** First we follow (6.27), and deduce a more concise form
\[ \|u(s)\|_q^q + \int_0^s e^{s-t}\|u\|_q^q \|Au\|_2^2 \, \text{d}c \]
\[ \leq \|u_0\|_q^q e^{-t} + \hat{p}_0 + c_\theta \int_0^s e^{s-t}\|u\|_q^q \|u(\phi(u))\| \, \text{d}W, \]  
for \( s \in (0, t] \), where \( \hat{p}_0 \) is independent of \( u \) and \( s \). For \( r > 0 \) and each \( 0 \leq k \leq m \), we define
\[ \overline{\xi}_k(u_0) := \inf \left\{ t \geq 0 : \inf_{0 \leq \phi \leq k} \|A^\phi u(t)\|^2 > r \right\}. \]
After (6.29) we claim that for each \( s \in (0, t] \), with \( t > 0 \)
\[ \|A^k u(s)\|^2 + \int_0^s e^{s-t}\|A^k+1 u\|^2 \, \text{d}c \]
\[ \leq c_k \left( \|A^k u_0\|^2 + Q_k(u_0) e^{-s} + \hat{p}_k + \Theta_k(s) \right), \]
for each \( 0 \leq k \leq m \) and some deterministic \( \hat{p}_k \) independent of \( u \) and \( s \), where \( Q_k(s, u_0) \equiv 0 \) and for \( k \geq 1 \),
\[ Q_k(u_0) := \sum_{i=1}^k \|u_0\|^2(2i+3), \]
\( \Theta_k(s) \) is the sum of the terms for the stochastic integral with respect to \( W \) and hence \( \mathbb{E}[\Theta_k(t \wedge \tau)] = 0 \), for each stopping time \( \tau \geq 0 \).

Now we show the claim by the induction method. Firstly, the inequality (6.29) has implied the claim for the case when \( k = 0 \). Secondly, we show that when the claim holds for \( k = l - 1 \) with \( l \in \{1, 2, \ldots, m\} \), then it also holds for \( k = l \).

Still apply \( A^l \) to (2.13), use Itô’s Formula to \( \|A^l u\|^2 \) and add \( \|A^l u(\phi)\|^2 \) to both sides of the obtained equality. We have
\[ d\|A^l u(\phi)\|^2 + \int_0^s e^{s-t}\|A^l u(\phi)\|^2 \, \text{d}c \]
\[ = \left\| \int_0^s e^{s-t}\|A^l u(\phi)\|^2 \, \text{d}c \right\|^2 \]
\[ + 2 \left( A^l u, A^l \phi(u) \right) \right). \]  
According to the treatment of (3.7), we notice that
\[ \|A^l \phi(u)\|^2 - 2\left( A^l u, A^l \phi(u) \right) \leq \|A^{l+1} u\|^2 + c_l \left( \|A^l u\|^2 + \|u\|^{2(l+3)} + 1 \right). \]
We multiply (6.32) with \( e^s \), integrate (6.28) over \( \zeta \in [0, s] \) with \( s \in (0, t] \), and obtain
\[ \|A^l u(s)\|^2 + \int_0^s e^{s-t}\|A^{l+1} u\|^2 \, \text{d}c \]
\[ \leq \|A^l u_0\|^2 e^{-s} + \int_0^s e^{s-t} \left( \|A^l u\|^2 + \|u\|^{2(l+3)} + 1 \right) \, \text{d}c \]
\[ + 2 \int_0^s e^{s-t} \left( A^l u, A^l \phi(u) \right) \right). \]
By (6.30) with \( k = l - 1 \) and (6.29), it follows by embeddings that
\[
\|A'u(s)\|^2 + \int_0^s e^{c-s}\|A'\|^2 \, d\zeta \\
\leq [\|A'u_0\|^2 + c_1 (\|A'\| + Q_{l-1}(u_0) s)] e^{-s} + c_1 \rho_{l-1} + c_1 \Theta_{l-1}(s) \\
+ c_1 \int_0^s e^{c-s} (\|u_0\|^2 + (\rho_{0,2(2l+3)} + 1)) \, d\zeta \\
+ c_1 \int_0^s e^{c-s} \int_0^s e^{c-s} \|u\|^2 (u, \phi(u) dW(s')) \, d\zeta \\
+ 2 \int_0^s e^{c-s} (A'u, A'\phi(u) dW(s)) \, d\zeta.
\]

By setting
\[
\rho' = c_1 (\rho_{l-1} + \rho_{0,2(2l+3)} + 1) \\
The(s) = c_1 \Theta_{l-1}(s) + c_1 \int_0^s e^{c-s} \|u\|^2 (u, \phi(u) dW(s')) \, d\zeta \\
+ 2 \int_0^s e^{c-s} (A'u, A'\phi(u) dW(s)) \, d\zeta,
\]
we see by Fubini theorem that for each stopping time \( \tau \geq 0 \),
\[
E\Theta(t \wedge \tau) = c_1 E \int_0^{t \wedge \tau} (t \wedge \tau - \zeta') e^{c-s} \|u\|^2 (u, \phi(u) dW(s')) = 0,
\]
and (6.33) indicates the claim for \( k = l \).
By the claim we have
\[
\|A^m u(t \wedge \xi^m(u_0))\|^2 + \int_0^{t \wedge \xi^m(u_0)} e^{c-s} \|A^{m+1} u\|^2 \, d\zeta \\
\leq c_m \left[ \|A^m u_0\|^2 + Q_m(u_0) \right] e^{-c} + \rho_m' + \Theta_m(t \wedge \xi^m(u_0)).
\]

Now let \( R > 0 \) and define
\[
D_R := \{ \omega \in \Omega : \|A^m u_0\|^2 + Q_m(u_0) \leq R \}.
\]
Similarly multiply (6.34) with \( \chi_{D_R} \) and take the expected value of the obtained inequality. One sees that
\[
E \|A^m u(t \wedge \xi^m(u_0))\|^2 \chi_{D_R} \\
\leq c_m E \left[ \|A^m u_0\|^2 + Q_m(u_0) \right] e^{-c} + \rho_m' + \Theta_m(t \wedge \xi^m(u_0)) \chi_{D_R} + \rho_m'.
\]

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Fix $R > 0$, use the dominated convergence theorem, and let $r$ tend to the infinity. We obtain that
\[ E\|A^m u(t)\|^2 \chi_{D_{X}} \leq C_{m, E} \|A^m u_0\|^2 + Q_m(u_0)t \chi_{D_{X}} e^{-t} + \tilde{\rho}_m. \]
For each fixed $R > 0$, we have a $t'_R > 0$ such that
\[ E\|A^m u(t)\|^2 \chi_{D_{X}} \leq 2\tilde{\rho}_m. \]
Again, using the property of invariant measures, we infer that as $R \to +\infty$,
\[ E\|A^m u(t)\|^2 \chi_{D_{X}} = \int_{H^n} \|A^m u\|^2 \chi_{D_{X}}(du) \nu(du) \to \int_{H^n} \|A^m u\|^2 \nu(du). \]
Since $\tilde{\rho}_m'$ is independent of $R$, we set $\tilde{\rho}_m = 2\tilde{\rho}_m'$ and $R \to +\infty$ in (6.36) and get (6.28).

We now can present the tightness of $\mathcal{I}$ as follows.

**Lemma 6.10.** The invariant measure set $\mathcal{I}$ is tight.

**Proof.** According to the definition, for every $\varepsilon > 0$, we look for a compact subset $B_{2(m+1)}(R)$ with a sufficiently large $R$ such that
\[ \nu\{H^{2m} \setminus B_{2(m+1)}(R)\} < \varepsilon \text{ for all } \nu \in \mathcal{I}. \]
Indeed, for some fixed $t > 0$ and by the invariance and lemma 6.1, we have
\begin{align*}
\nu(H^{2m} \setminus B_{2(m+1)}(R)) &= \int_{H^n} P_t(\tilde{u}, H^{2m} \setminus B_{2(m+1)}(R)) \nu(du) \\
&= \int_{H^n} P(t\|A^{m+1} u(t, \tilde{u})\|^2 > rR^2) \nu(du) \\
&\leq \int_{H^n} C_s \frac{r}{R^2} \left( \|\tilde{u}\|^2_{2m} + \|\tilde{u}\|^2 (2m+5) + r \right) \nu(du),
\end{align*}
where $C_s = C_s(t)$ is given by (6.5). By lemmas 6.8 and 6.9, we know the integral with respect to $\nu$ over $H^{2m}$ in (6.38) is bounded only by a deterministic positive constant depending on $t$, i.e.
\[ \nu(H^{2m} \setminus B_{2(m+1)}(R)) \leq C_s(t) \frac{r}{R^2}. \]
Thus, for each $\varepsilon > 0$, if we choose $R$ sufficiently large, we are able to ensure (6.37) and complete the proof.

Now we are prepared to verify the existence of ergodic invariant measures.

**Theorem 6.11.** Suppose that $|b| < 4$ and $\phi$ satisfies (2.3) and (2.4). Then there exists an ergodic invariant measure on $H^{2m}$ for the Feller Markovian semigroup $P_t$.

**Proof.** We adopt a similar routine to [45, theorem IV.13] to address this proof. Since an invariant measure is ergodic if and only if it is an extreme point of $\mathcal{I}$ (see [8, proposition 11.12]), we only need to show the existence of extreme points of $\mathcal{I}$, for which, it suffices to show that $\mathcal{I}$ is nonempty, compact and convex, by the Krein–Milman Theorem (see [7, theorem 7.4, chapter V]).
The existence of invariant measures implies that \( \mathcal{I} \neq \emptyset \). The linearity of \( \mathcal{P}_t^\nu \) with respect to \( \nu \) implies that \( \mathcal{I} \) is convex. For the compactness, we first show that \( \mathcal{I} \) is closed.

Let \( \{ \nu_n \}_{n \in \mathbb{N}} \) be a given weakly convergent sequence in \( \mathcal{I} \) such that \( \nu_n \) weakly converges to \( \nu \), as \( n \to \infty \). By the invariance, we know that for all \( \Gamma \in \mathcal{B}(H^{2m}) \), when \( n \) tends to the infinity,

\[
\nu(\Gamma) = \lim_{n \to \infty} \nu_n(\Gamma) = \lim_{n \to \infty} \int_H \mathcal{P}_t(u, \Gamma) \nu_n(du) = \int_H \mathcal{P}_t(u, \Gamma) \nu(du) = \mathcal{P}_t^\nu(\Gamma).
\]

By the uniqueness of weak limit, we know that \( \nu \) is also an invariant measure. Hence the closedness.

The closedness and lemma 6.10 guarantee the compactness of \( \mathcal{I} \), which eventually proves this theorem.

\[\blacksquare\]

### 7. Infinite regularity of invariant measures

In this section we are to give the infinite regularity of the invariant measures obtained in section 6. Let \( m \geq 1 \) and \( \nu \) be the invariant measure for the Markovian semigroup \( \mathcal{P}_t \) on \( H^{2m} \) given in theorem 6.7. First we increase the regularity of \( \nu \) to the smoother space \( H^{2(m+1)} \).

For this aim, recall the discussion in section 6. We note that if it could hold that (stated in theorem 7.1 below)

\[
\int_{H^{2m}} \|u\|^{2(m+1)} \nu(du) < +\infty,
\]

we can conclude \textit{a posteriori} that the support of \( \nu \) is contained in \( H^{2(m+1)} \) (using the ideas in [2, 19]). Unfortunately, this can not come true in the way presented in the proof of lemma 6.9, since \( A^{m+1}u(t) \) makes no sense at \( t = 0 \). However, we can make it by reusing the method given in the proof of lemma 6.1.

**Theorem 7.1.** Let \( |b| < 4 \) and \( \phi \) satisfy (2.3) and (2.4). Then (7.1) holds true. Moreover, the support of \( \nu \) is contained in \( H^{2(m+1)} \).

**Proof.** We only show (7.1). Similar to the estimation in the proof of lemma 6.1, we apply Itô’s Formual to \( \|A^{m+1}u\|^2 \) and deduce like the argument in the proof of lemma 6.8 that for stopping times \( 0 \leq \zeta \leq s \leq t \cap \xi^{m+1}_r(u_0) \), with \( \xi^{m+1}_r(u_0) \) defined for each \( r > 0 \) as

\[
\xi^{m+1}_r(u_0) := \inf \left\{ t \geq 0 : \inf_{0 \leq t' \leq m+1} \|A^{t'} u(t)\|^2 > r \right\},
\]

it holds that

\[
\|A^{m+1}u(s)\|^2 \leq \|A^{m+1}u(\zeta)\|^2 e^{\zeta-t} + c \int_{\zeta}^{t} e^{s-t} \left( \|A^{m+1}u\|^2 + \|u\|^{2(2m+5)} + 1 \right) ds' + 2 \int_{\zeta}^{t} e^{s-t}(A^{m+1}u, A^{m+1}\phi(u)dW).
\]

Integrating (7.2) over \( \zeta \in [0, s] \) and recalling (6.29) and the claim (6.30), we infer that

\[
s\|A^{m+1}u(s)\|^2 \leq \int_{0}^{s} \|A^{m+1}u(\zeta)\|^2 e^{\zeta-t} d\zeta + cs \int_{0}^{s} e^{t-s} \left( \|A^{m+1}u\|^2 + \|u\|^{2(2m+5)} + 1 \right) ds'.
\]

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\[ + \int_0^t \int_\mathbb{R} e^{s-t} (A^{m+1} u_{t}, A^{m+1} \phi(u) dW(s')) ds \cdot \]
\[ \leq (1 + cs) (c_m \|A^m u_0\|^2 + Q_m(u_0) s) e^{-s} + \tilde{\rho}^+_m + \Theta_m(s) \]
\[ + cs \int_0^s e^{s-t} \left( \|u_0\|^2 \left(2(2m+5)\right) e^{-t} + \tilde{\rho}^+_m \right) \Gamma \left(2(2m+5) + 1\right) ds' \]
\[ + c_m \int_0^t \int_\mathbb{R} e^{s-t} \|u\|^2 (u, \phi(u) dW(s')) ds' \]
\[ + 2s'd e^{-s-t} (A^{m+1} u_t, A^{m+1} \phi(u) dW(s')) \]
\[ \leq c_m (1 + s) \left( \|A^m u_0\|^2 + Q_m(u_0) s \right) e^{-s} + c(1 + s) \tilde{\rho}^+_m + \Theta_m(s). \]

where \(Q_m(u_0)\) is given in (6.31), \( \tilde{\rho}^+_m = \tilde{\rho}^+_m + \tilde{\rho}^+_0(2(2m+5) + 1) \) and
\[ \Theta_m(s) = (1 + cs) \Theta_m(s) + c_m \int_0^s (s - c') e^{-c'} \|u\|^2 (u, \phi(u) dW(s')) \]
\[ + 2 \int_0^s s'e^{-c'} (A^{m+1} u_t, A^{m+1} \phi(u) dW(s')). \]

Similar to the proof of lemmas 6.8 and 6.9, we again define
\[ \mathcal{D}_\mathcal{R} = \{ \omega \in \Omega : \|A^m u_0\|^2 + \|u_0\|^2 \left(2(2m+5) + \right) \mathcal{R} \}. \]

Picking \( s = t \wedge \tilde{\xi}^{m+1}_t(u_0) \) in (7.3), multiplying the obtained inequality by \( \chi_{\mathcal{D}_\mathcal{R}} \) and taking the expectation of what is just obtained, we annihilate the stochastic term \( \Theta_m(t \wedge \tilde{\xi}^{m+1}_t(u_0)) \) and easily obtain
\[ E \left( t \wedge \tilde{\xi}^{m+1}_t(u_0) \right) \|A^{m+1} u(t \wedge \tilde{\xi}^{m+1}_t(u_0)) \|^2 \chi_{\mathcal{D}_\mathcal{R}} \]
\[ \leq c_m E \left( \|A^m u_0\|^2 + Q_m(u_0) \right) + c(1 + t) \tilde{\rho}^+_m. \]

by the boundedness of \( s^k e^{-s} \) with \( s \geq 0 \) and \( k \in \mathbb{N}^+ \). Fix \( \mathcal{R} > 0 \). Using the dominated convergence theorem to (7.4) and considering \( r \to +\infty \), we can obtain that
\[ E \|A^{m+1} u(t)\|^2 \chi_{\mathcal{D}_\mathcal{R}} \]
\[ \leq c_m \left( 1 + \frac{1}{t} \right) e^{-t} E \left( \|A^m u_0\|^2 + Q_m(u_0) \right) \chi_{\mathcal{D}_\mathcal{R}} + c \left( 1 + \frac{1}{t} \right) \tilde{\rho}^+_m. \]

Given each fixed \( \mathcal{R} > 0 \), it is easy to know that there exists \( t_\mathcal{R} > 0 \) such that
\[ E \|A^{m+1} u(t)\|^2 \chi_{\mathcal{D}_\mathcal{R}} \leq 2e \tilde{\rho}^+_m. \]

Then as \( \mathcal{R} \to +\infty \), we obtain
\[ E \|A^{m+1} u(t)\|^2 \chi_{\mathcal{D}_\mathcal{R}} = \int_{\mathcal{D}_\mathcal{R}} \|A^{m+1} u\|^2 \chi_{\mathcal{D}_\mathcal{R}} \nu(du) \to \int_{\mathcal{D}_\mathcal{R}} \|A^{m+1} u\|^2 \nu(du). \]

Owing to the independence of \( \mathcal{R} \) for \( \tilde{\rho}^+_m \), we immediately obtain (7.1) by letting \( \mathcal{R} \to +\infty \) in (7.5). The proof is complete. \( \square \)

Based on the regularity of \( \nu \) obtained in theorem 7.1, the infinite regularity of \( \nu \) is a trivial consequence by induction and embeddings.
Theorem 7.2. Let $|b| < 4$. Suppose that $\phi$ satisfies (2.5). Then each invariant measure supported on $H^2$ for the Markovian semigroup $P_t$ is supported on $C^\infty \cap H$.

8. Summary and remarks

In this paper, we aim to obtain the existence and regularity of invariant measures for the stochastic MSHE with multiplicative noise and periodic boundary. To this end, we first confirm the unique existence of global pathwise solutions, and by Yamada–Wannabe theorem, we are also required to provide the existence of martingale solutions in advance. With a further study, we also obtain the global existence of martingale solutions.

In our situation, the concept of global martingale solution is stronger a bit than that in a previous work [37], and we use a new method to obtain this kind of global martingale solutions. Moreover, our new method can be applied extensively to other stochastic nonlinear partial differential equations to get a global martingale solution, once the drift is dissipative somehow and the diffusion coefficient is globally Lipschitz, or more weakly, the solutions coincide locally in time for Galerkin approximating cut-off systems with different cut-off functions. However, we are still unaware whether the global martingale solution is unique.

In the proof of existence of ergodic invariant measures in $H_2^m$, we actually follow the estimation methods used frequently in attractor theory to study the existence of the global attractor in spaces of high regularity (see [41, 50, 53]). Correspondingly, to improve the regularity of the invariant measures, we also use the regularity estimation methods for deterministic systems. All in all, we eventually obtain the infinite regularity of invariant measures for stochastic MSHE with multiplicative noise and periodic boundary under appropriate conditions on the diffusion coefficient $\phi$, which is just what Glatt-Holtz expected in [19] for the stochastic primitive equation.

Combining the research on invariant measures and attractors in Chen et al [3, 4] and Huang et al [26–31], we have some ideas about the relation between invariant measures for stochastic systems and attractors for deterministic systems as follows.

Consider the following abstract deterministic differential equation
\[ u_t = Au + f(u), \quad t > 0, \tag{8.1} \]
and the corresponding stochastic differential equation with multiplicative noise
\[ du = [Au + f(u)]dt + \phi(u)dW, \quad t > 0, \tag{8.2} \]
where $u : [0, +\infty) \to X$ is the unknown function, $X$ is a Hilbert (or Banach) space, $A : D(A) \subset X \to X$ is a linear operator, $f$ is the external force, $\phi(u)dW$ is the stochastic term and $W$ is a Wiener process.

Firstly, we conjecture that, if (8.1) possesses a unique global strong solution in $X$ for each initial datum, then (8.2) possesses a unique global pathwise solution for a given stochastic basis under some appropriate global Lipschitz condition on $\phi$. Secondly, we conjecture that, if the dynamical system generated by (8.1) has a global attractor $\mathcal{A}$ in $X$, then the Markovian transition semigroup associated with (8.2) has an invariant measure $\mu$ on $X$ under some appropriate global Lipschitz condition on $\phi$. Moreover, if $\mathcal{A}$ is contained in a subspace $Y$ with higher regularity of $X$, then $\mu$ is supported on $Y$ as well, under some appropriate global Lipschitz condition on $\phi$. Thirdly, if the conjectures mentioned above hold true, perhaps the converse of these conclusions are also hopeful to be valid under some appropriate conditions.

The proof of these conjectures may need very elaborate investigations, but at least, there should be many examples indicating these conclusions. What is more, we are also interested
in the existence of (ergodic) invariant measures when the drift is not dissipative, for example, when $|b| \geq 4$ for the stochastic MSHE (1.1).

**Data availability statement**

No new data were created or analysed in this study.

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