A note on a separating system of rational invariants for finite dimensional generic algebras

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Abstract. The paper deals with a construction of a separating system of rational invariants for finite dimensional generic algebras. In the process of dealing an approach to a rough classification of finite dimensional algebras is offered by attaching them some quadratic forms.

INTRODUCTION

In [1] we have offered an approach to classification problem of finite dimensional algebras with respect to basis changes. It has been shown that if one has a special map with some properties then he is able to classify, to list canonical representations, all algebras who’s set of structural constants , with respect to a fixed basis, do not nullify some polynomial. In this case he is also able to provide a separating system of rational invariants for those algebras. It was successfully applied in [2] to get a complete classification of all 2-dimensional algebras over algebraically closed fields.

Unfortunately, so far we have no example of such a map in 3-dimensional case. Therefore in the current paper we deal with a weaker problem, namely with a construction of separating system of rational invariants for finite dimensional generic algebras. The theoretical existence of such system of invariants is known [3]. By generic algebras we mean the set of all algebras who’s system of structural constants does not nullify a fixed nonzero polynomial in structural variables, over the basic field $F$. In process of dealing with the problem we show a way for a rough classification of finite dimensional algebras by attaching them some quadratic forms.

The next section contains the main results.

MAIN RESULTS

Further whenever $A = (a_{ij}) \in \text{Mat}(p \times q, F)$, $B \in \text{Mat}(p' \times q', F)$ we use $A \otimes B$ for the matrix

$$
\begin{pmatrix}
    a_{11}B & a_{12}B & \cdots & a_{1p}B \\
    a_{21}B & a_{22}B & \cdots & a_{2p}B \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{p1}B & a_{p2}B & \cdots & a_{pq}B
\end{pmatrix},
$$

where $F$ is a field of characteristic not 2.

Let us consider any $m$-dimensional algebra $A$ with multiplication $\cdot$ given by a bilinear map $(u, v) \mapsto u \cdot v$. If $e = (e_1, e_2, \ldots, e_m)$ is a basis for $A$ then one can represent the bilinear map by a matrix

$$
A_e = (A_{e_{jk}}^i)_{i,j,k=1,2,\ldots,m} \in \text{Mat}(m \times m^2; F),
$$
where \( e_j \cdot e_k = e_1 A_{e_{jk}}^1 + e_2 A_{e_{jk}}^2 + \ldots + e_m A_{e_{jk}}^m \), \( j, k = 1, 2, \ldots, m \), such that
\[
u \cdot v = e A_e(u \otimes v)
\]
for any \( u = eu, v = ev \), where \( u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \) are column vectors. So the algebra \( A \) (binary operation, bilinear map, tensor) is presented by the matrix \( A_e \in \text{Mat}(m \times m; F) \)-the matrix of structure constants (MSC) of \( A \) with respect to the basis \( e \).

If \( e' = (e'_1, e'_2, \ldots, e'_m) \) is also a basis for \( A, g \in \text{GL}(m, F) \), \( e'g = e \) then it is well known that
\[
A_{e'} = g A_e (g^{-1}) \otimes^2
\]
is valid. Further a basis \( e \) is fixed and therefore instead of \( A_e \) we use \( A \), we do not make difference between \( A \) and its matrix \( A \). Let \( X = (X_{jk})_{i,j,k=1,2,\ldots,m} \) stand for a variable matrix and \( T r_1(X), T r_2(X) \) stand for the row vectors
\[
\left( \sum_{i=1}^m X_{i1}, \sum_{i=1}^m X_{i2}, \ldots, \sum_{i=1}^m X_{im} \right); \quad \left( \sum_{i=1}^m X_{i1}, \sum_{i=1}^m X_{i2}, \ldots, \sum_{i=1}^m X_{im} \right)
\]
respectively.

We use \( \tau \) for the representation of \( GL(m, F) \) on the \( n = m^3 \) dimensional vector space \( V = \text{Mat}(m \times m; F) \) defined by
\[
\tau: (g, A) \mapsto B = g A (g^{-1} \otimes g^{-1}).
\]
For simplicity instead of "\( \tau \)-equivalent", "\( \tau \)-invariant" we use "equivalent" and "invariant".

We represent each MSC \( A \) as a row vector with entries from \( \text{Mat}(m, F) \) by parting it consequently into elements of \( \text{Mat}(m, F) \):
\[
A = (A_1, A_2, \ldots, A_m), \quad A_1, A_2, \ldots, A_m \in \text{Mat}(m, F).
\]

If \( C \) is a block matrix with blocks from \( \text{Mat}(m, F) \) we use notation \( C^\tau \), where \( * \) is the tensor product or transpose operation, to mean that the operation \( * \) with \( C \) is done "over \( \text{Mat}(m, F) \)" (not over \( F \)), for example for the above presented matrix \( A \)
\[
A^\tau = \begin{pmatrix} A_1 & \cdots & A_m \end{pmatrix} - \text{column vector over } \text{Mat}(m, F),
\]
\[
A^\otimes^2 = (A_1^2, A_1A_2, \ldots, A_1A_m, A_2A_1, \ldots, A_2A_m, \ldots, A_mA_1, A_mA_2, \ldots, A_m^2).
\]

One can see that the equality \( B = g A (g^{-1} \otimes g^{-1}) \) can be presented as
\[
B = (B_1, B_2, \ldots, B_m) = g A^\otimes (g^{-1} \otimes g^{-1}) = (g A_1 g^{-1}, g A_2 g^{-1}, \ldots, g A_m g^{-1}) (g^{-1} \otimes I),
\]
where \( I \) stands for \( m \times m \) size identity matrix. Moreover for any matrices \( C \) and \( D \) the equality
\[
(C \otimes I) \otimes (D \otimes I) = (C \otimes D) \otimes I
\]
holds true. Therefore the following equalities hold true.
\[
(B_1, B_2, \ldots, B_m) \otimes^k = (g A_1 g^{-1}, g A_2 g^{-1}, \ldots, g A_m g^{-1}) \otimes^k ((g^{-1} \otimes g^{-1}) \otimes I),
\]
\[
\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix} \otimes^k = \begin{pmatrix} B_1^1 & B_1B_2 & \cdots & B_1B_m \\ B_2^1 & B_2B_1 & \cdots & B_2B_m \\ \vdots & \vdots & \ddots & \vdots \\ B_m^1 & B_mB_1 & \cdots & B_mB_m \end{pmatrix} \otimes^k,
\]
\[
\begin{pmatrix} B_1^1 & B_1B_2 & \cdots & B_1B_m \\ B_2^1 & B_2B_1 & \cdots & B_2B_m \\ \vdots & \vdots & \ddots & \vdots \\ B_m^1 & B_mB_1 & \cdots & B_mB_m \end{pmatrix} = \begin{pmatrix} B_1^2 & B_1B_2 & \cdots & B_1B_m \\ B_2^2 & B_2B_1 & \cdots & B_2B_m \\ \vdots & \vdots & \ddots & \vdots \\ B_m^2 & B_mB_1 & \cdots & B_mB_m \end{pmatrix} \otimes^k.
\]
\[
(((g')^{-1})^{\otimes k} \otimes I) \begin{pmatrix}
  gA_1^2 g^{-1} & gA_1 A_2 g^{-1} & \cdots & gA_1 A_m g^{-1} \\
  gA_2 A_1 g^{-1} & gA_2^2 g^{-1} & \cdots & gA_2 A_m g^{-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  gA_m A_1 g^{-1} & gA_m A_2 g^{-1} & \cdots & gA_m^2 g^{-1}
\end{pmatrix} \cong_k ((g^{-1})^{\otimes k} \otimes I).
\]

Component-wise application of trace to this equality, which is denoted by \(\tilde{Tr}\), results in

\[
\tilde{Tr}(B_1^2 B_1 B_2 \cdots B_1 B_m) \cong_k \tilde{Tr}(B_2 B_1 B_2^2 \cdots B_2 B_m) = \cdots = \tilde{Tr}(B_m B_1 B_m B_2 \cdots B_m^2) \cong_k \tilde{Tr}(g^{-1})^{\otimes k}.
\]

as far as for any matrices \(C, D\) and \(E\), where \(D\) is a block matrix with blocks from \(\text{Mat}(m, F)\) and \((C \otimes I)D(E \otimes I)\) has meaning, the equality

\[
\tilde{Tr}((C \otimes I)D(E \otimes I)) = C\tilde{Tr}(D)E
\]

is valid. One can represent the above obtained matrix equality in the following compact form

\[
\tilde{Tr}((B^T B)^{\cong_k}) = \tilde{Tr}((A^T A)^{\cong_k})((g^{-1})^{\otimes k}).
\]

Note that \(\tilde{Tr}((A^T A)^{\cong_k})\) is a symmetric matrix. The obtained equality allows formulation of the following theorem.

**Theorem 1.** Invariants of the quadratic forms given by the matrix \(\tilde{Tr}((X^T X)^{\cong_k})\) are invariants of the \(m\)-dimensional algebras.

This result can be used for a rough classification of finite dimensional algebras: Two \(m\)-dimensional algebras \(A, B\) are rough equivalent if the quadratic forms given by matrices

\[
\tilde{Tr}(A^T A) = \begin{pmatrix}
  Tr(A_1^2) & Tr(A_1 A_2) & \cdots & Tr(A_1 A_m) \\
  Tr(A_2 A_1) & Tr(A_2^2) & \cdots & Tr(A_2 A_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  Tr(A_m A_1) & Tr(A_m A_2) & \cdots & Tr(A_m^2)
\end{pmatrix},
\]

\[
\tilde{Tr}(B^T B) = \begin{pmatrix}
  Tr(B_1^2) & Tr(B_1 B_2) & \cdots & Tr(B_1 B_m) \\
  Tr(B_2 B_1) & Tr(B_2^2) & \cdots & Tr(B_2 B_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  Tr(B_m B_1) & Tr(B_m B_2) & \cdots & Tr(B_m^2)
\end{pmatrix}
\]

are equivalent.

It is clear that entries of \(\tilde{Tr}(X^T X)\) are polynomials in components of \(X\) and there exists nonsingular matrix \(Q(X^T X)\) with rational entries in \(X\) such that the matrix

\[
\tilde{Tr}(X^T X) = (Q(X^T X)^{-1})^T \tilde{Tr}(X^T X)Q(X^T X)^{-1} = D(X)
\]

is a diagonal matrix and \(Q(g) = I\) whenever \(g\) is a nonsingular diagonal matrix, where \(X = \tau(Q(X^T X), X)\).
In algebraically closed field \( F \) case it means that one can define a nonempty invariant open subset \( V_0 \subset V \) such that \( Tr(\overline{A}) = D(A) \) and \( D(A) \) is nonsingular whenever \( A \in V_0 \).

**Theorem 2.** Two algebras \( A, B \in V_0 \) are equivalent (isomorphic) if and only if
\[
\mathcal{B} = \tau(g_0, \overline{A}) \text{ for some } g_0 \in GL(m, F) \text{ for which } g_0^t D(B) g_0 = D(A).
\]

**Proof.** If \( B = \tau(g_A) \) then \( \mathcal{B} = \tau(Q(B)^t B), \mathcal{B} = \tau(Q(B)^t B), \tau(g_A) = \tau(Q(B)^t B)g(A^t)A^{-1}, \tau(Q(A^t)A), A = \tau(Q(B)^t B)g(A^t)A^{-1}, \overline{A}, \)

and for \( g_0 = Q(B)^t B)g(A^t)A^{-1} \) one has
\[
g_0^t D(B) g_0 = (Q(B)^t B)g(A^t)A^{-1} D(B) (Q(B)^t B)g(A^t)A^{-1} =
\]
\[
(Q(A^t)A)^{-1} (g^t (Q(B)^t B)^t D(B) Q(B)^t B))g(A^t)A^{-1} =
\]
\[
(Q(A^t)A)^{-1} (g^t (Tr(B)^t B))g(A^t)A^{-1} = (Q(A^t)A)^{-1} Tr(A^t)A Q(A^t)A^{-1} = D(A).
\]

Visa versa if \( \mathcal{B} = \tau(g_0, \overline{A}) \) for some \( g_0 \) for which \( g_0^t D(B) g_0 = D(A) \) then for \( g = Q(B)^t B)gQ(A^t)A^{-1} \) one has
\[
\tau(g, A) = \tau(Q(B)^t B)^{-1} g_0 Q(A^t)A, A = \tau(Q(B)^t B)^{-1} g_0, \tau(Q(A^t)A), A =
\]
\[
\tau(Q(B)^t B)^{-1} g_0 \overline{A}, A = \tau(Q(B)^t B)^{-1}, \tau(g_0, \overline{A})) = \tau(Q(B)^t B)^{-1}, \overline{B} = B.
\]

Assume that there exists matrix \( P(X) \), with rational entries with respect to components of \( X \), such that \( P(\overline{A}) \) is nonsingular for any \( A \in V_0 \) and the equality
\[
P(\tau(g_0, \overline{A})) = P(\overline{A}) g^{-1} \text{ holds true whenever } g_0^{t} D(\tau(g_0, A)) g = D(A).
\]

**Theorem 3.** For \( A, B \in V_0 \) there exists \( g_0 \in GL(m, F) \) such that \( g_0^t D(B) g_0 = D(A) \) and \( \mathcal{B} = \tau(g_0, \overline{A}) \) if and only if
\[
\tau(P(B), \overline{A}) = (P(B)^{-1})^t D(B) P(B)^{-1} = (P(\overline{A})^{-1})^t D(A) P(\overline{A})^{-1}.
\]

**Proof.** If \( \mathcal{B} = \tau(g_0, \overline{A}) \) and \( g_0^t D(B) g_0 = D(A) \) then
\[
\tau(P(B), \overline{A}) = \tau(P(g_0, \overline{A}), \tau(g_0, \overline{A})) = \tau(P(\overline{A}) g_0^{-1}, \tau(g_0, \overline{A})) = \tau(P(\overline{A}), \overline{A})
\]

and \( (P(B)^{-1})^t D(B) P(B)^{-1} = ((P(\overline{A}) g_0^{-1})^{-1})^t D(B) P(\overline{A}) g_0^{-1})^{-1} =
\]
\[
((P(\overline{A})^{-1})^t g_0^t D(B) g_0 P(\overline{A})^{-1} = (P(\overline{A})^{-1})^t D(A) P(\overline{A})^{-1}.
\]

Visa versa, if equalities
\[
\tau(P(B), \overline{A}) = \tau(P(\overline{A}), \overline{A}), (P(B)^{-1})^t D(B) P(B)^{-1} = (P(\overline{A})^{-1})^t D(A) P(\overline{A})^{-1}
\]

are valid then for \( g_0 = P(B)^{-1} P(\overline{A}) \) one has \( g_0^t D(B) g_0 = D(A) \) and
\[
\tau(g_0, \overline{A}) = \tau(P(B)^{-1} P(\overline{A}), \tau(P(B)^{-1}, \tau(P(\overline{A}), \overline{A})) = \tau(P(B)^{-1}, \tau(P(\overline{B}), \overline{B})) = \overline{B}.
\]

So Theorems 2 and 3 imply that the system of entries of matrices
\[
\tau(P(X), \overline{X}), (P(X)^{-1})^t Tr(\overline{X} X) P(X)^{-1}
\]
is a separating system of rational invariants for algebras from \( V_0 \).

The above presented results show importance of construction of matrix \( P(X) \) with properties (1). Further we discuss a construction of such matrix by the use of rows \( r(\overline{A}) \) for which the equality
\[
r(\tau(g, A)) = r(\overline{A}) g^{-1}
\]
is valid, whenever \( g'D(\pi(g,A))g = D(A) \). To construct such rows one can use the following approach. Assume that the equalities
\[
\overline{B} = g\overline{A}(g^{-1})^{\otimes 2}, \quad \tilde{C} = gCg'
\]
are true, where \( C' = C \) and \( C \) is a nonsingular matrix. In this case
\[
\tilde{C}^{\otimes 2} = g^{\otimes 2}C^{\otimes 2}(g^{\otimes 2})', \quad \overline{B}\tilde{C}^{\otimes 2} = g\overline{A}C^{\otimes 2}(g^{\otimes 2})', \quad \tilde{C}^{\otimes 2}\overline{B} = g^{\otimes 2}C^{\otimes 2}\overline{A}g'
\]
and
\[
\overline{B}\tilde{C}^{\otimes 2}\overline{B} = g\overline{A}C^{\otimes 2}\overline{A}g'.
\]
On induction it is easy to see that for any natural \( k \) the equality
\[
\overline{B}^{\otimes 2^0}\overline{B}^{\otimes 2^1}...\overline{B}^{\otimes 2^{k-1}}\tilde{C}^{\otimes 2^k}(\overline{B}^{\otimes 2^0}\overline{B}^{\otimes 2^1}...\overline{B}^{\otimes 2^{k-1}})' \tilde{C}^{-1} =
\]
holds true. Therefore due to the equalities
\[
\overline{B}^{\otimes 2^0}\overline{B}^{\otimes 2^1}...\overline{B}^{\otimes 2^{k-1}}\tilde{C}^{\otimes 2^k}(\overline{B}^{\otimes 2^0}\overline{B}^{\otimes 2^1}...\overline{B}^{\otimes 2^{k-1}})' \tilde{C}^{-1} =
\]
\[
g\overline{A}^{\otimes 2^0}\overline{A}^{\otimes 2^1}...\overline{A}^{\otimes 2^{k-1}}C^{\otimes 2^k}(\overline{A}^{\otimes 2^0}\overline{A}^{\otimes 2^1}...\overline{A}^{\otimes 2^{k-1}})' \tilde{C}^{-1}g,\]
\[
\overline{B}^{\otimes 2^0}\overline{B}^{\otimes 2^1}...\overline{B}^{\otimes 2^{k-1}}\tilde{C}^{\otimes 2^k}(\overline{B}^{\otimes 2^0}\overline{B}^{\otimes 2^1}...\overline{B}^{\otimes 2^{k-1}})' \tilde{C}^{-1}\overline{B} =
\]
\[
g\overline{A}^{\otimes 2^0}\overline{A}^{\otimes 2^1}...\overline{A}^{\otimes 2^{k-1}}C^{\otimes 2^k}(\overline{A}^{\otimes 2^0}\overline{A}^{\otimes 2^1}...\overline{A}^{\otimes 2^{k-1}})' \tilde{C}^{-1}\overline{A}(g^{-1})^{\otimes 2}
\]
one has
\[
Tr_1(\overline{B}^{\otimes 2^0}\overline{B}^{\otimes 2^1}...\overline{B}^{\otimes 2^{k-1}}\tilde{C}^{\otimes 2^k}(\overline{B}^{\otimes 2^0}\overline{B}^{\otimes 2^1}...\overline{B}^{\otimes 2^{k-1}})' \tilde{C}^{-1}\overline{B}) =
\]
\[
Tr_1(\overline{A}^{\otimes 2^0}\overline{A}^{\otimes 2^1}...\overline{A}^{\otimes 2^{k-1}}C^{\otimes 2^k}(\overline{A}^{\otimes 2^0}\overline{A}^{\otimes 2^1}...\overline{A}^{\otimes 2^{k-1}})' \tilde{C}^{-1}\overline{A})g^{-1}, \quad i = 1, 2.
\]
The last equality shows that in our algebra case one can try to construct the needed matrix \( P(X) \) by the use of rows
\[
Tr_1(X^{\otimes 2^0}X^{\otimes 2}...X^{\otimes 2(k-1)}((\overline{Tr}(X^TX))^{-1})^{\otimes 2^k}(X^{\otimes 2^0}X^{\otimes 2}...X^{\otimes 2(k-1)})'\overline{Tr}(X^TX),
\]
where \( i = 1, 2, k = 0, 1, 2, ... \).

What is left unjustified here is that one should justify existence, in general, of a linear independent system consisting of \( m \) such rows.

**Remark.** After a rough classification one can classify further each case of the rough classification with respect to the corresponding stabilizer.

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**REFERENCES**

1. U. Bekbaev, (IOP Conf. Series: Journal of Physics: Conf. Series, 819), 2017, pp. 2-9.
2. H. Ahmed, U. Bekbaev, I. Rakhimov, (arXiv 1702.08616), 2017, pp. 1-11.
3. V. Popov, (arXiv: 1411.6570v2(math.AG)), 2014, pp. 1-20.