Cycles in Random Bipartite Graphs

Yilun Shang
University of Texas at San Antonio
Institute for Cyber Security
San Antonio, Texas 78249, USA
shylmath@hotmail.com

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Abstract
In this paper we study cycles in random bipartite graph $G(n,n,p)$. We prove that if $p \gg n^{-2/3}$, then $G(n,n,p)$ a.a.s. satisfies the following. Every subgraph $G' \subset G(n,n,p)$ with more than $(1 + o(1))n^2p/2$ edges contains a cycle of length $t$ for all even $t \in [4, (1 + o(1))n/30]$. Our theorem complements a previous result on bipancyclicity, and is closely related to a recent work of Lee and Samotij.

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1 Introduction
Given a complete bipartite graph $K_{n,n}$ and a real $p \in [0,1]$, let random bipartite graph model $G(n,n,p)$ be the probability space of subgraphs of $K_{n,n}$ obtained by taking each edge independently with probability $p$ (see e.g. [16]). For a given graph property $\mathcal{P}$, we say that $G(n,n,p)$ possesses property $\mathcal{P}$ asymptotically almost surely, or a.a.s. for brevity, if the probability that $G(n,n,p)$ possesses $\mathcal{P}$ tends to 1 as $n$ goes to infinity. In the previous work [17], we provided an edge condition for cycles in Hamiltonian subgraphs of $G(n,n,p)$:

**Theorem 1.** If $p \gg n^{-2/3}$, then $G(n,n,p)$ a.a.s. satisfies the following. Every Hamiltonian subgraph $G' \subset G(n,n,p)$ with more than $(1 + o(1))n^2p/2$ edges is bipancyclic (i.e., contains cycles of every possible even length).

Bipancyclicity of bipartite graphs is first studied by Schmeichel and Mitchem [15]. Theorem 1 can be viewed as an extension in random graph setting of a classical theorem of them [12], which says that every Hamiltonian bipartite balanced graph with $2n$ vertices and more than $n^2/2$ edges is bipancyclic. We note that Theorem 1 is best possible in two ways [17].
First, the range of $p$ is asymptotically tight. Second, the proportion $1/2$ of edges cannot be reduced.

In the present work, we will focus on the situation where the subgraph $G'$ is not necessarily Hamiltonian. We establish the following result.

**Theorem 2.** If $p \gg n^{-2/3}$, then $G(n,n,p)$ a.a.s. satisfies the following. Every subgraph $G' \subset G(n,n,p)$ with more than $(1 + o(1))n^2p/2$ edges contains a cycle of length $t$ for all even $t \in [4,(1 + o(1))n/30]$.

Seeking cycles of various lengths in random graph models is an interesting topic in probabilistic combinatorics and has since long attracted much research attention, see e.g. [1, 2, 3, 4, 7, 10, 11, 19]. Recently, Lee and Samotij [8] showed that if $p \gg n^{-1/2}$, then binomial random graph $G(n,p)$ a.a.s. satisfies the following: Every Hamiltonian subgraph $G' \subset G(n,p)$ with more than $(1/2 + o(1))n^2p/2$ edges is pancyclic (i.e., contains cycles of every possible length). In the arXiv version of their paper, they further proved the following result concerning cycles of short length.

**Theorem 3.** For $\varepsilon \in (0,1)$, there exists a constant $C$ such that if $p \geq Cn^{-1/2}$, then $G(n,p)$ a.a.s. satisfies the following. Every subgraph $G' \subset G(n,p)$ with more than $(1/2 + \varepsilon)n^2p/2$ edges contains a cycle of length $t$ for all $3 \leq t \leq \varepsilon n/2560$.

To prove our main result, we will roughly follow the line of the proof of Theorem 3. The bipartite structure of $G(n,n,p)$ entails some significant modifications which remarkably allow us to show the existence of longer cycles (i.e., cycles of linear length) with even smaller edge probability (See Theorem 2).

The rest of the paper is organized as follows. We will present some useful lemmas in Section 2 and prove the main result in Section 3.

## 2 Some lemmas

We begin with some notations. Let $G = (V_0, V_1, E)$ denote a bipartite graph with two classes of bipartition $V_0, V_1$ and edge set $E$. For a vertex $v$, we denote its $k$-th order neighborhood by $N^{(k)}(v)$, i.e., the set of vertices at distance $k$ from $v$. Let $N(v) = N^{(1)}(v)$ and $\text{deg}(v) = |N(v)|$ be the degree of $v$. The degree of set $X$ is defined as $\text{deg}(X) = \sum_{v \in X} \text{deg}(v)$. The first order neighborhood of set $X$ is defined as $N(X) = \{u \in V_0 \cup V_1 : u \in N(v) \text{ for some } v \in X\}$. The maximum degree of graph $G$ is defined as $\Delta(G)$. For a set $X$, let $E(X)$ be the set of edges in the induced subgraph $G[X]$, and let $e(X) = |E(X)|$. Analogously, the set of ordered pairs $(x,y) \in E$ with $x \in X$ and $y \in Y$ is denoted by $E(X,Y)$. Let $e(X,Y) = |E(X,Y)|$. When there are several graphs under consideration, we may use subscripts such as $N_G(v)$ to indicate the graph we are currently working with. Floor and ceiling signs are often omitted whenever they are not crucial.

The following concentration inequality (see e.g. [10, Corollary 2.3]) will be often used in the proof of main result.

**Lemma 2.1.** (Chernoff’s bound) Let $0 < \varepsilon \leq 3/2$. If $X$ is a binomial random variable with parameter $n$ and $p$, then

$$P(|X - \mathbb{E}(X)| \geq \varepsilon \mathbb{E}(X)) \leq 2e^{-\varepsilon^2 \mathbb{E}(X)/3},$$
where \( \mathbb{E} \) represents the expectation operator.

Lemma 2.2 can be found in [5] Proposition 1.2.2 and Lemma 2.3 is referred to as Pósa’s rotation-extension lemma ([14] and [9], Chapter 10, Problem 20).

**Lemma 2.2.** Let \( G \) be a graph on \( n \) vertices with at least \( dn \) edges. Then \( G \) contains a subgraph \( G' \subset G \) with minimum degree at least \( d \).

**Lemma 2.3.** Let \( G = (V, E) \) be a graph such that \( |N(X) \setminus X| \geq 2|X| - 1 \) for all \( X \subset V \) with \( |X| \geq t \). Then for any vertex \( v \in V \), there exists a path of length \( 3t - 2 \) in \( G \) which has \( v \) as an end point.

For a monotone increasing property \( \mathcal{P} \), the global resilience [13] of a graph \( G \) with respect to \( \mathcal{P} \) is defined as the minimum number \( r \) such that by deleting \( r \) edges from \( G \), one can obtain a graph not possessing \( \mathcal{P} \). The following result was stated in [17], and can be proved similarly as [7] Proposition 3.1 and [8] Proposition 2.7.

**Lemma 2.4.** Assume that \( 0 < p' \leq p \leq 1 \) and \( n^2p' \to \infty \) as \( n \to \infty \). If \( G(n, n, p') \) a.a.s. has global resilience at least \( (1/2 - \varepsilon/4)n^2p' \) with respect to a monotone increasing graph property, then \( G(n, n, p) \) a.a.s. has global resilience at least \( (1/2 - \varepsilon/2)n^2p \) with respect to the same property.

In the next lemma we establish the expansion property for subgraphs of \( G(n, n, p) \) with large minimum degree. A variant of this result for binomial random graph \( G(n, p) \) appeared in [7] Lemma 3.4.

**Lemma 2.5.** If \( p = Cn^{-2/3} \) for some \( C > 0 \) and \( \varepsilon' \in (0, 1) \), then a.a.s. every subgraph \( G' \subset G(n, n, p) \) with minimum degree at least \( \varepsilon'np \) satisfies the following expansion property. For all \( X \subset V_0 \cup V_1 \) with \( |X| \leq \varepsilon'n/15 \), we have \( |N_{G'}(X)| \geq 2|X| \).

**Proof.** Fix a subgraph \( G' \subset G(n, n, p) \) with minimum degree at least \( \varepsilon'np \). Assume to the contrary that there exists \( X \subset V_0 \cup V_1 \) such that \( |X| \leq \varepsilon'n/15 \) but \( |N_{G'}(X)| < 2|X| \). Let \( Y = X \cup N_{G'}(X) \). We have \( |Y| \leq 3|X| \leq \varepsilon'n/5 \). Since \( G' \) has minimum degree at least \( \varepsilon'np \),

\[
e_{G'}(Y) = \frac{1}{2}e_{G'}(Y, Y) \geq \frac{1}{2}e_{G'}(X, X) \geq \frac{1}{2}|X|\varepsilon'np \geq \frac{1}{6}|Y|\varepsilon'np.
\]

Denote \( |Y| = a \). Then we have \( \varepsilon'np \leq a \leq \varepsilon'n/5 \).

The probability that there exists a set of order \( a \) which spans at least \( a\varepsilon'np/6 \) edges is

\[
\sum_{b=1}^{a-1} \binom{n}{b} \binom{n}{a-b} \left( \frac{(a-b)b}{a\varepsilon'np/6} \right)^{a\varepsilon'np/6} p^{a\varepsilon'np/6} \\
\leq \sum_{b=1}^{a-1} \binom{en}{b} \binom{en}{a-b} \left( \frac{6(a-b)b}{a\varepsilon'np} \right)^{a\varepsilon'np/6} p^{a\varepsilon'np/6} \\
\leq \sum_{b=1}^{a-1} \frac{(en)^a}{b^b(a-b)^{a-b}} \left( \frac{3ea}{2\varepsilon'n} \right)^{a\varepsilon'np/6}.
\]  

(1)
An application of Young’s inequality yields
\[
\left( \frac{1}{b} \right)^b \left( \frac{1}{a - b} \right)^{a - b} \leq \frac{b}{a} \left( \frac{1}{b} \right)^a + \left( \frac{a - b}{a} \right) \left( \frac{1}{a - b} \right)^a.
\]
Therefore, the right-hand side of (1) is at most
\[
(\epsilon n)^a \left( \frac{3e\alpha}{2\epsilon n} \right)^{\alpha \epsilon' np/6} \cdot \sum_{b=1}^{a-1} \left( \frac{b^{1-a}}{a} + \left( \frac{1}{a - b} \right) \right)
\]
\[
\leq (1 + a)(\epsilon n)^a \left( \frac{3e\alpha}{2\epsilon n} \right)^{\alpha \epsilon' np/6} \ll \left( \frac{\epsilon}{3} \right)^{\alpha \epsilon' np/6} \ll n^{-2}.
\]
Summing over all \( \epsilon' np \leq a \leq \epsilon' n/5 \), we see that the probability that there is a set violating the assertion of the lemma is \( o(1) \). The proof is complete. 

\[ \blacksquare \]

3 Existence of cycles

In this section we will prove the following main theorem.

**Theorem 4.** For any \( \epsilon \in (0, 1) \), there exists a constant \( C \) such that if \( p \geq C n^{-2/3} \), then \( G(n, n, p) \) a.a.s. satisfies the following. Every subgraph \( G' \subset G(n, n, p) \) with more than \((1 + \epsilon)n^2p/2 \) edges contains a cycle of length \( t \) for all even \( t \in [4, (1 + \epsilon/2)n/30] \).

The following proposition is a key step towards the proof of Theorem 4. Different from [8], the second order neighborhood of any vertex in \( G(n, n, p) \) contains no edges. We will resort to a coupon collector argument to show the existence of a vertex with many edges between its first order and second order neighborhoods.

**Proposition 5.** For any \( \epsilon \in (0, 2/5] \), there exists a constant \( C_0 \) such that the following holds. If \( p = C_0 n^{-2/3} \) for some constant \( C \geq C_0 \), then \( G(n, n, p) \) a.a.s. satisfies the following. Every subgraph \( G' \subset G(n, n, p) \) with more than \((1 + \epsilon)n^2p/2 \) edges contains a vertex \( v_0 \) such that \( e(N_{G'}(v_0), B) \geq \epsilon(1 + \epsilon/2)n^3p^3/8 \) for some \( B \subset N_{G'}^{(2)}(v_0) \) satisfying \( |B \cup N_{G'}(v_0)| \leq \epsilon n^2p^2/2 \).

**Proof.** Let \( G \) be a graph drawn from \( G(n, n, p) \) and \( G' \) be a subgraph with \( e(G') > (1 + \epsilon)n^2p/2 \). By Lemma 2.1, we have a.a.s. \( \Delta(G') \leq \Delta(G) \leq (1 + \epsilon)np \). In what follows, we will show the proposition conditioned on the above event.

For \( i = 0, 1 \), denote by \( B_i \) the collection of the vertices in \( V_i \) which have degree at least \((1 + \epsilon/2)np/2 \) in \( G' \). Therefore,
\[
(1 + \epsilon)n^2p \leq 2e(G') = \deg_{G'}(V_0 \cup V_1) = \deg_{G'}(B_0 \cup B_1) + \deg_{G'}((V_0 \cup V_1) \setminus (B_0 \cup B_1)) \leq \deg_{G'}(B_0 \cup B_1) + (1 + \epsilon/2)n^2p,
\]
and hence
\[
\epsilon n^2p/2 \leq \deg_{G'}(B_0 \cup B_1) = \sum_{v \in V_0 \cup V_1} |N_{G'}(v) \cap (B_0 \cup B_1)|.
\]
A simple averaging argument implies that there exists a vertex, say $v_0 \in V_0$, such that $|N_{G'}(v_0) \cap B_1| \geq \varepsilon np/4$ (c.f. Fig. 1).

Consider a coupon collector’s problem where a customer tries, in several attempts, to collect a complete set of $|B_0|$ different coupons. At each attempt, the collector gets a coupon randomly chosen from $|B_0|$ kinds. We observe that the number $|B_0 \cap N_{G'}(N_{G'}(v_0) \cap B_1)|$ dominates the number $A$ of different coupons the collector obtains in $T = \alpha |B_0|$ attempts, where

$$\alpha = \frac{1}{|B_0|} \left( \frac{\varepsilon}{4} np \left( 1 + \frac{\varepsilon}{2} \right) \frac{np}{2} \right) = \frac{1}{8|B_0|} \left( 1 + \frac{\varepsilon}{2} \right) \varepsilon n^2 p^2.$$  

Note that $|B_0|$ satisfies $|B_0|(1 + \varepsilon)np + (n - |B_0|)(1 + \varepsilon/2)np/2 \geq (1 + \varepsilon)n^2 p/2$, which implies that $|B_0| \geq (\varepsilon/(2 + 3\varepsilon))n$. Therefore,

$$\alpha \leq \frac{(1 + \frac{\varepsilon}{2})(2 + 3\varepsilon)}{8\varepsilon} np^2 < 1,$$

for large enough $n$. Using Lemma 2.1, we can show that a.a.s. (see e.g. [13])

$$A \geq \left( 1 - \frac{\varepsilon}{2} \right) \left( 1 - \frac{1}{8} \left( 1 + \frac{\varepsilon}{2} \right) \frac{\varepsilon n^2 p^2}{|B_0|} \right) \frac{1}{8} \left( 1 + \frac{\varepsilon}{2} \right) \varepsilon n^2 p^2$$

$$\geq \frac{1}{8} \left( 1 - \frac{\varepsilon}{2} \right) \left( 1 + \frac{\varepsilon}{2} \right) \varepsilon n^2 p^2$$

$$> \frac{\varepsilon}{4} n^2 p^2.$$
where the last inequality holds since $\varepsilon \leq 2/5$. According to our above comment, we have

$$|B_0 \cap N_{G'}(N_{G'}(v_0) \cap B_1)| > \frac{\varepsilon n^2 p^2}{4}.$$ 

Now we choose a set $B \subseteq B_0 \cap N_{G'}(N_{G'}(v_0) \cap B_1) \subset N_{G'}^{(2)}(v_0)$ such that $|B| = \varepsilon n^2 p^2/4$. We obtain

$$e(N_{G'}(v_0), B) \geq \frac{(1 + \varepsilon/2) np}{2} |B| = \frac{\varepsilon}{8} \left(1 + \frac{\varepsilon}{2}\right) n^3 p^3,$$
and

$$|B \cup N_{G'}(v_0)| \leq (1 + \varepsilon)np + \frac{\varepsilon n^2 p^2}{4} \leq \frac{\varepsilon n^2 p^2}{2}$$
as desired. \hfill \blacksquare

Putting these together, we are now ready to show the main result.

**Proof of Theorem 4.** Lemma 2.4 implies that it suffices to show the theorem with $p = Cn^{-2/3}$ for some $C$ to be determined later. Without loss of generality, we may also assume that $\varepsilon \leq 2/5$. Let $G$ be a graph drawn from $G(n, n, p)$ and $G'$ be a subgraph with $e(G') > (1 + \varepsilon)n^2 p/2$.

It follows from Proposition 5, there exists a vertex, say $v_0 \in V_0$, such that $e(N_{G'}(v_0), B) \geq \varepsilon (1 + \varepsilon/2) n^3 p^3/8$ for some $B \subset N_{G'}^{(2)}(v_0)$ satisfying $|B \cup N_{G'}(v_0)| \leq \varepsilon n^2 p^2/2$. Consider the subgraph $G'[B \cup N_{G'}(v_0)]$. By Lemma 2.2, there exists a subset $D \subset B \cup N_{G'}(v_0)$ such that $G'[D]$ has minimum degree at least $(1 + \varepsilon/2)np/4$. Take an arbitrary vertex $v_1 \in D \cap V_1$, and take a vertex $v_2 \in D \cap V_0$ so that $v_0v_1v_2$ is a path of length 2 in $G'$. By Lemma 2.5, for all $X \subset V_0 \cup V_1$ of order $|X| \leq (1 + \varepsilon/2)n/60$, we have $|N_{G'[D]}(X) \setminus X| \geq 2|X|$. By Lemma 2.3, we can find a path of length $(1 + \varepsilon/2)n/30$ in $G'[D]$ which has $v_2$ as an end point. We call this path $v_2x_1x_2x_3\cdots x_{(1+\varepsilon/2)n/30}$, where $x_i \in N_{G'}(v_0)$ if $i$ is odd; $x_i \in B \subset N_{G'}^{(2)}(v_0)$ if $i$ is even.

![Figure 2: A depiction of case (i) and case (ii).](image)

Now we consider two cases (c.f. Fig. 2):

(i) $x_1 \neq v_1$. Then $v_0v_1v_2x_1x_2x_3\cdots x_sv_0$ forms a cycle of length $s + 3$ in $G'$ when $s$ is odd;

(ii) $x_1 = v_1$. Then $v_0x_1x_2x_3\cdots x_sv_0$ forms a cycle of length $s + 1$ in $G'$ when $s \geq 3$ and $s$ is odd.

This completes the proof of Theorem 4. \hfill \blacksquare
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