Light-cone SU(2) Yang-Mills theory and conformal mechanics

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Abstract

We examine the mechanical matrix model that can be derived from the SU(2) Yang-Mills light-cone field theory by restricting the gauge fields to depend on the light-cone time alone. We use Dirac’s generalized Hamiltonian approach. In contrast to its well-known instant-time counterpart the light-cone version of SU(2) Yang-Mills mechanics has in addition to the constraints, generating the SU(2) gauge transformations, the new first and second class constraints also. On account of all of these constraints a complete reduction in number of the degrees of freedom is performed. It is argued that the classical evolution of the unconstrained degrees of freedom is equivalent to a free one-dimensional particle dynamics. Considering the complex solutions to the second class constraints we show at this time that the unconstrained Hamiltonian system represents the well-known model of conformal mechanics with a “strength” of the inverse square interaction determined by the value of the gauge field spin.
1 Introduction

Nowadays the correspondence between gauge theories in various dimensions and integrable systems has become a subject of intensive study. After the pioneering work by Seiberg and Witten [1], demonstrating that $N = 2$ supersymmetric Yang-Mills theory in four dimensions is exactly solvable in the low-energy limit, considerable progress in the understanding of these relations has been marked. In scope of the correspondence to the underlying integrable systems several properties of the Seiberg-Witten theory have been investigated using the approach proposed in [2]. In particular, it was shown that the low-energy effective action can be described in terms of different one-dimensional integrable many-body systems ranging from the classical Toda-chain model in the case of supersymmetric Yang-Mills theory without matter, to elliptic Calogero-Moser model when adjoint matter is added and to classical spin XXX chain for theory with fundamental matter included (for comprehensive reviews of these studies see e.g. [3–5]).

At the same time similar relationships have been observed also for non-supersymmetric gauge theories. It was recognized that the XXX Heisenberg spin chains are related to other physically interesting limits in QCD. Namely, an equivalence was found between the Hamiltonian describing the Regge asymptotic behavior of hadron-hadron scattering amplitudes in QCD and the Hamiltonian of the $SL(2, \mathbb{C})$ XXX Heisenberg magnet [6]. Furthermore it turns out that the logarithmic evolution of the composite operators in QCD on the light-cone is similar to the dynamics of $SL(2, \mathbb{R})$ XXX Heisenberg spin chain [7]. Based on this hidden integrability of the effective theories of QCD a kind of stringy/brane picture was developed recently [8].

There is also a physically very important regime when finite-dimensional system arises in the context of gauge field theory. The long-wavelength approximation in the dynamics of gauge fields effectively leads to the so-called dimensional $(1 + 0)$ reduction of the field theory and at first has been intensively studied for the non-supersymmetric Yang-Mills theory, both from physical as well as from a purely mathematical point of view (see e.g [9]-[19] and references therein). In the middle of 1980's analogous supersymmetric mechanical models with more than four supersymmetries were constructed from the corresponding super Yang-Mills theory [20–22]. In particular, in [20] the maximally supersymmetric $N = 16$ gauge mechanics was considered. The recent renewed interest in the supersymmetric version of Yang-Mills mechanics is motivated by the observation that the Hamiltonian of $D = 1$ $SU(n)$ super Yang-Mills theory in the large $n$ limit describes the dynamics of $D = 11$ supermembrane [23] (for a review, see, e.g., [24] and references therein) and claims to the role of M-theory Hamiltonian [25]. This conjecture and the fact that the low-energy limit of the M-theory is described by eleven-dimensional supergravity pose the important question of existence of zero-energy normalizable eigenfunctions. Using a complete set of gauge invariant variables and generalization of the Born-Oppenheimer approximation the simplest case of $SU(2)$ matrix theory has been investigated and an asymptotic form of the ground state was proposed [26]. Even the simplest of these dimensionally reduced models are still rather complicated and possesses non-trivial dynamics. It was found [9,10,29] that

$^{1}$The case of the $SU(n)$ group with arbitrary $n \geq 2$ was considered in [27,28].
the classical non-supersymmetric $SU(2)$ Yang-Mills mechanics exhibits chaotic behavior when the dynamics takes place on a special invariant submanifold. It was proved that on this submanifold there is no analytical integral of motion except the energy integral, and thus the Yang-Mills mechanics represents a non-integrable system [30]. A similar investigation of the classical dynamics of bosonic membrane matrix model yielded again chaotic behavior. However, recently, in [31,32] the supersymmetric $SU(2) \times SO(2)$ matrix model was investigated in detail and it was demonstrated that there exists a chaos-order transition depending on the value of the angular momentum.

In the present paper we shall continue the study of models obtained from the $SU(2)$ Yang-Mills field theory under the supposition of fields homogeneity. We consider the model of light-cone $SU(2)$ Yang-Mills classical mechanics and address the problem of its complete Hamiltonian reduction and integrability. Analogously to the instant form of Yang-Mills mechanics, the light-cone version follows from the light-cone Yang-Mills field theory when the gauge fields depend on the light-cone time only. Both dynamical systems, obtained under such suppositions, contain a finite number of degrees of freedom and inherit in a specific form the gauge invariance of the original Yang-Mills theory. In a recent article we outlined such a difference of the light-cone version of Yang-Mills mechanics to its instant form counterpart even in the character of the local gauge invariance [33]. Now we present a result of the Hamiltonian reduction of the light-cone $SU(2)$ Yang-Mills mechanics and demonstrate that after elimination of all ignorable coordinates the corresponding unconstrained Hamiltonian system represents a simple integrable system.

We start with the formulation of the $SU(2)$ light-cone mechanics as a degenerate Lagrangian model for a matrix valued variable $A$, perform the standard Hamiltonian analysis proving that the presence of constraints force the classical dynamics to develop on the subspace of matrices with rank$||A|| = 1$. Using the adapted coordinates frame we show that it is equivalent to the dynamics of a free particle in one dimension. We also study the complex solutions to the second class constraints and demonstrate that in this case the reduced system coincides with the well-known model of so-called conformal mechanics, introduced by V. de Alfaro, S. Fubini and G. Furlan [34].

After Dirac’s famous paper [35] on different forms of relativistic dynamics it has been recognized that the different choice of the time evolution parameter can drastically change the content and interpretation of the theory. The present study shows that the long-wavelength approximation in instant and light-front formulation leads to the models that differ drastically even in sense of their classical integrability. The question whether models with the higher order gauge groups as well as after inclusion of an additional supersymmetry stay integrable is still open. It is also interesting to study the question of their correspondence to the known superconformal generalizations of conformal mechanics. \(^2\) Here we note, that the quantum mechanical model with periodicity in light-cone time, obtained by the dimensional reduction of the light-cone version of $N = 1$ super $SU(n)$

\(^2\)The $N = 2$ supersymmetric extension of conformal mechanics was generalized in [38,39] to an $SU(1,1 \mid 1)$ invariant superconformal mechanics. Soon after, $N = 4$ extension of conformal mechanics with $SU(1,1 \mid 1)$ superconformal symmetry was elaborated [40,41] and using the geometric method the superconformal mechanics was formulated in a manifestly invariant manner for an arbitrary even $N$ [40].
Yang-Mills theory was studied in [37]. It was shown that this model is integrable in the sense that its partition function is a tau-function of the Toda hierarchy and only in the large $n$ limit can be solved exactly.

The fact that a finite dimensional system obtained by dimensional reduction inherits the conformal symmetry of the original field theory is not quite unexpected. One-dimensional conformally invariant systems already appeared in black hole physics [42] and cosmology [43]. However, to our knowledge, their relation to the light-cone Yang-Mills theory has not been pointed out yet.

The organization of the rest of the paper is as follows. In Section 2 we start with the Lagrangian formulation of the light-cone model and give the standard analysis of Hamiltonian constraints including their separation into the first and second class constraints sets. Then in Section 3 the Hamiltonian reduction is performed. First the constraints generating the $SU(2)$ gauge transformations are eliminated using the coordinates adapted to the gauge symmetry. Further to this the reduction due to the remaining first and second class constraints is carried out exploiting the new convenient set of coordinates. Section 4 gives our final conclusions and comments. The appendix is devoted to the derivation of the Lagrangian equations of motion of the unconstrained system starting from the Lagrangian equations of motion for the light-cone $SU(2)$ Yang-Mills mechanics by elimination of all Lagrangian constraints.

2 Light-cone model and analysis of constraints

In this Section we give the formulation of the $SU(2)$ light-cone Yang-Mills mechanics, calculate all constraints and separate them into the first and second class ones.

2.1 Model formulation

We start with the action of Yang-Mills field theory in four-dimensional Minkowski space $M_4$, endowed with a metric $\eta$ and represented in the coordinate free form

$$I := \frac{1}{g^2} \int_{M_4} \text{tr} F \wedge *F ,$$

where $g$ is a coupling constant and the $su(2)$ algebra valued curvature two-form

$$F := dA + A \wedge A$$

is constructed from the connection one-form $A$. The connection and curvature, as Lie algebra valued quantities, are expressed in terms of the antihermitian $su(2)$ algebra basis $\tau^a = \sigma^a / 2i$ with the Pauli matrices $\sigma^a$, $a = 1, 2, 3$,

$$A = A^a \tau^a , \quad F = F^a \tau^a .$$
The metric $\eta$ enters the action through the dual field strength tensor defined in accordance with the Hodge star operation

$$\ast F_{\mu\nu} = \frac{1}{2} \sqrt{\eta} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \quad (4)$$

To formulate the light-cone version of the theory let us introduce the basis vectors in the tangent space $T_P(M_4)$

$$e_{\pm} := \frac{1}{\sqrt{2}} (e_0 \pm e_3), \quad e_{\perp} := (e_k, k = 1, 2). \quad (5)$$

The first two vectors are tangent to the light-cone and the corresponding coordinates are referred usually as the light-cone coordinates $x^\mu = (x^+, x^-, x^\perp)$

$$x^\pm := \frac{1}{\sqrt{2}} (x^0 \pm x^3), \quad x^\perp := x^k, \quad k = 1, 2. \quad (6)$$

The non-zero components of the metric $\eta$ in the light-cone basis $(e_+, e_-, e_k)$ are

$$\eta_{+-} = -\eta_{-+} = -\eta_{11} = -\eta_{22} = 1. \quad (7)$$

The connection one-form in the light-cone basis is given as

$$A = A_+ dx^+ + A_- dx^- + A_k dx^k. \quad (8)$$

By definition the Lagrangian of light-cone Yang-Mills mechanics follows from the corresponding Lagrangian of Yang-Mills theory if one supposes that the components of the connection one-form $A$ in (8) depend on the light-cone “time variable” $x^+$ alone

$$A_\pm = A_\pm(x^+), \quad A_k = A_k(x^+). \quad (9)$$

Substitution this ansatz into the classical action (1) defines the Lagrangian of light-cone Yang-Mills mechanics

$$L = \frac{1}{2g^2} \left( F_{++}^a F_{--}^a + 2 F_{+k}^a F_{-k}^a - F_{12}^a F_{12}^a \right), \quad (10)$$

where the light-cone components of the field-strength tensor are given by

$$F_{++}^a = \frac{\partial A_+^a}{\partial x^+} + \epsilon^{abc} A_+^b A_-^c, \quad (11)$$

$$F_{+k}^a = \frac{\partial A_+^a}{\partial x^k} + \epsilon^{abc} A_k^b A_-^c, \quad (12)$$

$$F_{-k}^a = \epsilon^{abc} A_k^b A_+^c, \quad (13)$$

$$F_{ij}^a = \epsilon^{abc} A_j^b A_i^c, \quad i, j, k = 1, 2. \quad (14)$$

Hence, the Yang-Mills light-cone mechanics is a finite dimensional system with configuration coordinates $A_\pm, A_k$ whose evolution with respect to the time $\tau$

$$\tau := x^+ \quad (15)$$

is determined by the Lagrangian (10).
### 2.2 Generalized Hamiltonian dynamics

Performing the Legendre transformation,

\[
\pi_+^a = \frac{\partial L}{\partial \dot{A}_+^a} = 0 , \tag{16}
\]

\[
\pi_-^a = \frac{\partial L}{\partial A_-^a} = \frac{1}{g^2} \left( A_-^a + \epsilon^{abc} A_+^b A_-^c \right) , \tag{17}
\]

\[
\pi^k_a = \frac{\partial L}{\partial A^a_k} = \frac{1}{g^2} \epsilon^{abc} A_-^b A^c_k , \tag{18}
\]

we obtain the canonical Hamiltonian

\[
H_C = \frac{g^2}{2} \pi_-^a \pi_-^a - \epsilon^{abc} A_-^b \left( A_-^c \pi_-^a + A^c_k \pi^k_a \right) + V(A_k) \tag{19}
\]

with a potential term

\[
V(A_k) = \frac{1}{2g^2} \left[ \left( A_1^a A_1^b \right) \left( A_2^c A_2^d \right) - \left( A_1^a A_2^b \right) \left( A_1^c A_2^d \right) \right] . \tag{20}
\]

The non-vanishing Poisson brackets between the fundamental canonical variables are

\[
\{ A_\pm^a , \pi^\pm_a \} = \delta^a_b , \tag{21}
\]

\[
\{ A^a_k , \pi^k_a \} = \delta^a_k \delta^a_b . \tag{22}
\]

The Hessian of the Lagrangian system \([10]\) is degenerate, \( \det \left| \frac{\partial^2 L}{\partial A \partial A} \right| = 0 \), and as a result there are primary constraints

\[
\varphi_a^{(1)} := \pi_+^a = 0 , \tag{23}
\]

\[
\chi^k_a := g^2 \pi^a_k + \epsilon^{abc} A_-^b A^c_k = 0 , \tag{24}
\]

satisfying the following Poisson brackets relations

\[
\{ \varphi^{(1)}_a , \varphi^{(1)}_b \} = 0 , \tag{25}
\]

\[
\{ \varphi^{(1)}_a , \chi^b_k \} = 0 , \tag{26}
\]

\[
\{ \chi^a_i , \chi^b_j \} = -2 g^2 \epsilon^{abc} A_-^c \eta_{ij} . \tag{27}
\]

According to the Dirac prescription, the presence of primary constraints affects the dynamics of the degenerate system. Now the generic evolution is governed by the total Hamiltonian

\[
H_T = H_C + U_a(\tau) \varphi_a^{(1)} + V_k^a(\tau) \chi^a_k , \tag{28}
\]

\(^3\)To simplify the formulas we shall use overdot to denote derivative of a function with respect to light-cone time \( \tau \). Further, we shall treat in equal footing the up and down isotopic indexes denoted with \( a, b, c, d \).
where $U_a(\tau)$ and $V^a_k(\tau)$ are unspecified functions of the light-cone time $\tau$. Using this Hamiltonian the dynamical self-consistence of the primary constraints may be checked. From the requirement of conservation of the primary constraints $\varphi^{(1)}_a$ it follows

$$0 = \dot{\varphi}^{(1)}_a = \{\pi^+_a, H_T\} = \epsilon^{abc} (A^b_c \pi^-_c + A^b_k \pi^k_c) .$$  \hspace{1cm} (29)

Therefore there are three secondary constraints $\varphi^{(2)}_a$

$$\varphi^{(2)}_a := \epsilon^{abc} (A^b_c \pi^-_c + A^b_k \pi^k_c) = 0 ,$$  \hspace{1cm} (30)

which obey the $so(3, \mathbb{R})$ algebra

$$\{\varphi^{(2)}_a , \varphi^{(2)}_b \} = \epsilon^{abc} \varphi^{(2)}_c .$$  \hspace{1cm} (31)

The same procedure for the primary constraints $\chi^a_k$ gives the following self-consistency conditions

$$0 = \dot{\chi}^a_k = \{\chi^a_k , H_C\} - 2 \rho^2 \epsilon^{abc} V^b_k A^c_\perp .$$  \hspace{1cm} (32)

The analysis of these equations depends on the properties of the matrix $C_{ab} = \epsilon^{abc} A^c_\perp$. This matrix is degenerate with a rank varying from 0 to 2 depending on the point of the configuration space. If its rank is 2 then among the six primary constraints $\chi^a_k$ there are two first class constraints and a maximum of four Lagrange multipliers $V^b_k$ can be determined from (32). When the rank of the matrix $C_{ab}$ is minimal, the locus points are $A^a_\perp = 0$ and all six constraints $\chi^a_k$ are Abelian ones. For such an exceptional configuration the constrained system reduces to the dynamically trivial one. Hereinafter we shall consider the subspace of configuration space where rank$|C| = 2$. For those configurations we are able to introduce the unit vector

$$N^a = \frac{A^a}{\sqrt{(A^a_\perp)^2 + (A^a_\parallel)^2 + (A^a_\parallel)^2}} ,$$  \hspace{1cm} (33)

which is a null vector of the matrix $\|\epsilon^{abc} A^c_\perp\|$, and to decompose the set of six primary constraints $\chi^a_k$ as

$$\psi_k := N^a \chi^a_k ,$$  \hspace{1cm} (34)

$$\chi^a_k \perp := \chi^a_k - (N^b \chi^b_k) N^a .$$  \hspace{1cm} (35)

In this decomposition the first two constraints $\psi_k$ are functionally independent and satisfy the Abelian algebra

$$\{\psi_i , \psi_j \} = 0 ,$$ \hspace{1cm} (36)

while the constraints $\chi^a_k \perp$ are functionally dependent due to the conditions

$$N^a \chi^a_k \perp = 0 .$$  \hspace{1cm} (37)

Choosing among them any four independent constraints we can determine four Lagrange multipliers $V^k_{b\perp}$. 

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The Poisson brackets of the constraints $\psi_k$ and $\varphi_{a}^{(2)}$ with the total Hamiltonian vanish after projection on the constraint surface (CS) defined by equations $\psi_k = 0$ and $\varphi_{a}^{(2)} = 0$

$$\{\psi_k, H_T\} |_{CS} = 0, \quad (38)$$
$$\{\varphi_{a}^{(2)}, H_T\} |_{CS} = 0. \quad (39)$$

and thus there are no ternary constraints.

Summarizing, we arrive at the set of constraints $\varphi_{a}^{(1)}, \psi_k, \varphi_{a}^{(2)}, \chi_k^b$. The Poisson brackets algebra of the first three is

$$\{\varphi_{a}^{(1)}, \varphi_{a}^{(1)}\} = 0, \quad (40)$$
$$\{\psi_i, \psi_j\} = 0, \quad (41)$$
$$\{\varphi_{a}^{(2)}, \varphi_{b}^{(2)}\} = \epsilon_{abc} \varphi_{c}^{(2)}, \quad (42)$$
$$\{\varphi_{a}^{(1)}, \psi_k\} = \{\varphi_{a}^{(1)}, \varphi_{b}^{(2)}\} = \{\psi_k, \varphi_{a}^{(2)}\} = 0. \quad (43)$$

The constraints $\chi_k^b$ satisfy the relations

$$\{\chi_{i \perp}^a, \chi_{j \perp}^b\} = -2 g^2 \epsilon^{abc} A^c_{\perp} \eta_{ij}, \quad (44)$$

and the Poisson brackets between these two sets of constraints are

$$\{\varphi_{a}^{(2)}, \chi_k^b\} = \epsilon^{abc} \chi_{k \perp}^c, \quad (45)$$
$$\{\varphi_{a}^{(1)}, \chi_{i \perp}^b\} = \{\psi_i, \chi_{j \perp}^b\} = 0. \quad (46)$$

From these relations we conclude that the model has 8 first-class constraints $\varphi_{a}^{(1)}, \psi_k, \varphi_{a}^{(2)}$ and 4 second-class constraints $\chi_{k \perp}^a$. Counting the degrees of freedom taking into account all these constraints, we obtain that instead of 24 constrained phase space degrees of freedom there are $24 - 2(5 + 3) - 4 = 4$ unconstrained degrees of freedom, in contrast to the instant form of Yang-Mills mechanics where the number of the unconstrained canonical variables is 12.

### 3 Unconstrained version of light-cone mechanics

Now we shall perform a Hamiltonian reduction of the degrees of freedom starting with an elimination of the gauge degrees of freedom associated to the $SU(2)$ constraints $\varphi_{a}^{(2)}$. The purpose of the present part of the paper is to rewrite the theory in terms of special coordinates adapted to the action of this gauge symmetry.

#### 3.1 Polar decomposition

Let us organize the configuration variables $A_{a}^i$ and $A_{a}^\alpha$ in one $3 \times 3$ matrix $A_{ab}$ whose entries of the first two columns are $A_{a}^i$ and third column is composed by the elements $A_{a}^\alpha$

$$A_{ab} := \|A_{1a}^i, A_{2a}^i, A_{a}^\alpha\|, \quad (47)$$
and the momentum variables similarly
\[ \Pi_{ab} := \| \pi^a_1, \pi^a_2, \pi^a_3 \| . \] (48)

In order to find an explicit parametrization of the orbits with respect to the gauge symmetry action, it is convenient to use a polar decomposition [47] for the matrix \( A_{ab} \)
\[ A = OS, \] (49)
where \( S \) is a positive definite \( 3 \times 3 \) symmetric matrix, \( O(\phi_1, \phi_2, \phi_3) = e^{\phi_1 J_3} e^{\phi_2 J_1} e^{\phi_3 J_3} \) is an orthogonal matrix parameterized by the three Euler angles \( (\phi_1, \phi_2, \phi_3) \). The matrices \((J_a)_{ij} = \epsilon_{iaj}\) are the \( SO(3, \mathbb{R}) \) generators in adjoint representation.

It is in order to make a few remarks on the change of variables in (49). It is well-known that the polar decomposition is valid for an arbitrary matrix. However, the orthogonal matrix in (49) is uniquely determined only for an invertible matrix \( A \).
\[ O = AS^{-1}, \quad S = \sqrt{AA^T}. \] (50)

The non-degenerate \( 3 \times 3 \) matrices can be identified with an open set of the \( \mathbb{R}^9 \) using the entries of the matrix \( A_{ab} \) as corresponding Cartesian coordinates and in this case the polar decomposition (49) is a uniquely invertible transformation from these Cartesian coordinates to a new set of coordinates, the entries of positive matrix \( S \) and the angles parameterized the orthogonal matrix \( O \). For degenerate matrices a more sophisticated analysis is necessary. Here we note only that the set of \( n \times n \) matrices with rank \( k \) is a manifold with dimension \( k(2n - k) \), but in contrast the no-degenerate case the manifold atlas now necessarily contains several charts. Hence, for degenerate matrices \( A \) the representation [49] has to be replaced by a more elaborated construction.

Now we shall limit ourselves to the subspace of non-degenerate matrices and hence one can treat the polar decomposition (49) as a uniquely invertible transformation from the configuration variables \( A_{ab} \) to a new set of Lagrangian variables: six coordinates \( S_{ij} \) and three coordinates \( \phi_i \). It is worth to note here that in virtue of the constraints (24) the determinant of the matrix \( A \) is related to the third component of the gauge field spin
\[ 2 \det A - g^2 \epsilon_{3ik} A^a_k \pi^a_i = 0. \] (51)

The polar decomposition (49) induces the point canonical transformation from the coordinates \( A_{ab} \) and \( \Pi_{ab} \) to new canonical pairs \( (S_{ab}, P_{ab}) \) and \( (\phi_a, P_a) \) with the following non-vanishing Poisson brackets
\[ \{ S_{ab}, P_{cd} \} = \frac{1}{2} (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) , \] (52)
\[ \{ \phi_a, P_b \} = \delta_{ab} . \] (53)

The expression of the old \( \Pi_{ab} \) as a function of the new coordinates is [48, 49]
\[ \Pi = O(P - k_a J_a) , \] (54)
where
\[ k_a = \gamma_{ab}^{-1} \left( \eta^L_b - \varepsilon_{bmn} (SP)_{mn} \right), \]  
\( \gamma_{ik} = S_{ik} - \delta_{ik} \text{tr} S \) and \( \eta^L_a \) are three left-invariant vector fields on the \( SO(3, \mathbb{R}) \) group

\[ \eta^L_1 = \frac{\sin \phi_3}{\sin \phi_2} P_1 + \cos \phi_3 P_2 - \cot \phi_2 \sin \phi_3 P_3, \]
\[ \eta^L_2 = \frac{\cos \phi_3}{\sin \phi_2} P_1 - \sin \phi_3 P_2 - \cot \phi_2 \cos \phi_3 P_3, \]
\[ \eta^L_3 = P_3. \]

In terms of the new variables the constraints take the form
\[ \varphi^{(2)}_a = O_{ab} \eta^L_b, \]
\[ \chi_{am} = O_{ab} (P_{bm} + \epsilon_{bmc} k_c + \epsilon_{bijd} S_{i3} S_{jm}). \]

Thus one can pass to the equivalent set of constraints
\[ \eta^L_a = 0, \]
\[ \tilde{\chi}_{ai} = P_{ai} + \epsilon_{aij} \gamma^{-1}_{jk} \epsilon_{kmn} (SP)_{mn} + \epsilon_{amn} S_{m3} S_{ni} = 0 \]

with vanishing Poisson brackets
\[ \{ \eta^L_a, \tilde{\chi}_{ai} \} = 0. \]

Using the polar decomposition [49] and [51] we separate the variables \((S_{ab}, P_{ab})\), invariant under gauge transformations generated by Gauss law constraints \( \varphi^{(2)}_a \), from the gauge variant ones \((\phi_a, P_a)\). Now in order to eliminate all gauge degrees of freedom related to this symmetry it is enough to project to the constraint shell described by condition of nullity of the Killing vector fields \( \eta^L_a \). After projection the corresponding cyclic degrees of freedom, the angles \( \phi_a \), automatically disappear from the projected Hamiltonian.

### 3.2 Main-axes decomposition

In order to proceed further in resolution of the remaining constraints [52] we introduce the main-axes decomposition for the symmetric \( 3 \times 3 \) matrix \( S \)

\[ S = R^T(\chi_1, \chi_2, \chi_3) \left( \begin{array}{ccc} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{array} \right) R(\chi_1, \chi_2, \chi_3), \]

with orthogonal matrix \( R(\chi_1, \chi_2, \chi_3) = e^{\chi_1 J_3} e^{\chi_2 J_1} e^{\chi_3 J_3} \), parameterized by three Euler angles \((\chi_1, \chi_2, \chi_3)\). The Jacobian of this transformation is

\[ \frac{\partial (S_{i<j})}{\partial (q_a, \chi_b)} \sim \prod_{a \neq b}^3 |q_a - q_b|. \]
Therefore equation (64) can be used as definition of new configuration variables: three “diagonal” variables \((q_1, q_2, q_3)\), eigenvalues of the matrix \(S\), and three angular variables \((\chi_1, \chi_2, \chi_3)\), if and only if all eigenvalues of the matrix \(S\) are different, \(q_1 \neq q_2 \neq q_3\). The eigenvalues \(q_a\) parameterize the orbits of the adjoint action of \(SO(3, \mathbb{R})\) group in the space of \(3 \times 3\) symmetric matrices and the configurations with \(q_1 < q_2 < q_3\) represent the so-called principle orbit. Our consideration given below is correct for this type of orbits whereas the treatment of orbits with coinciding eigenvalues of the matrix \(S\), the singular orbits [50], requires different and more elaborated treatment that is beyond the scope of the present paper.

The momenta \(p_a\) and \(p_{\chi_a}\), canonically conjugated to the diagonal \(q_a\) and angular variables \(\chi_a\), can be found using the canonical invariance of the symplectic one-form

\[
\sum_{a,b=1}^{3} P_{ab} dS_{ab} = \sum_{a=1}^{3} p_a dq_a + \sum_{a=1}^{3} p_{\chi_a} d\chi_a .
\] (66)

The original momenta \(P_{ab}\), expressed in terms of the new canonical variables, read

\[
P = R^T \sum_{a=1}^{3} (p_a \bar{\alpha}_a + \mathcal{P}_a \alpha_a) R .
\] (67)

Here \(\bar{\alpha}_a\) and \(\alpha_a\) denote the diagonal and off-diagonal basis elements of the space of symmetric matrices with orthogonality relations

\[
\text{tr} (\bar{\alpha}_a \bar{\alpha}_b) = \delta_{ab} , \quad \text{tr} (\alpha_a \alpha_b) = 2\delta_{ab} , \quad \text{tr} (\bar{\alpha}_a \alpha_b) = 0
\] (68)

and

\[
\mathcal{P}_a = -\frac{1}{2} \frac{\xi^R_a}{q_b - q_c} \quad \text{(cyclic permutations} a \neq b \neq c) .
\] (69)

The \(\xi^R_a\) are three \(SO(3, \mathbb{R})\) right-invariant vector fields given in terms of the angles \(\chi_a\) and their conjugated momenta \(p_{\chi_a}\) via

\[
\xi^R_a = M^{-1}_{ba} p_{\chi_b} ,
\] (70)

where the matrix \(M\) is given by

\[
M_{ab} = -\frac{1}{2} \text{tr} \left( J_a \frac{\partial R}{\partial \chi_b} R^T \right) .
\] (71)

The explicit form of the three \(SO(3, \mathbb{R})\) right-invariant Killing vector fields is

\[
\xi^R_1 = -\sin \chi_1 \cot \chi_2 p_{\chi_1} + \cos \chi_1 p_{\chi_2} + \frac{\sin \chi_1}{\sin \chi_2} p_{\chi_3} ,
\] (72)

\[
\xi^R_2 = \cos \chi_1 \cot \chi_2 p_{\chi_1} + \sin \chi_1 p_{\chi_2} - \frac{\cos \chi_1}{\sin \chi_2} p_{\chi_3} ,
\] (73)

\[
\xi^R_3 = p_{\chi_1} .
\] (74)
Using these formulas the constraints $\tilde{\chi}$ in (62) may be rewritten in terms of the main-axes variables as

$$\tilde{\chi} = \sum_{a=1}^{3} R^T \left[ \pi_a \alpha_a - \frac{1}{2} \rho_a \alpha_a + \frac{1}{2} \rho_a^+ J_a \right] R,$$

(75)

where

$$\rho_a^\pm = \frac{\xi_a R}{q_b \pm q_c} \pm \frac{1}{g^2} q_a n_a (q_b \pm q_c),$$

(76)

and $n_a = R_{a3}$.

Note that the constraint (51) on the determinant of the matrix $A$ now takes the form

$$2 q_1 q_2 q_3 - g^2 \xi^L_3 = 0,$$

(77)

where $\xi^L_3$ is the third left-invariant Killing vector field, $\xi^L_a = R_{ab} \xi^R_b$.

$$\xi^L_1 = \frac{\sin \chi_3}{\sin \chi_2} p_{\chi_1} + \cos \chi_3 p_{\chi_2} - \cot \chi_2 \sin \chi_3 p_{\chi_3},$$

(78)

$$\xi^L_2 = \frac{\cos \chi_3}{\sin \chi_2} p_{\chi_1} - \sin \chi_3 p_{\chi_2} - \cot \chi_2 \cos \chi_3 p_{\chi_3},$$

(79)

$$\xi^L_3 = p_{\chi_3}.$$  

(80)

As was shown above the constraints $\chi^a_i$ represent the mixed system of first and second class constraints $\psi_i$ and $\chi^a_i$. To perform the reduction to the constraint shell it is useful at first to introduce the gauge fixing condition and eliminate the two first class constraints $\psi_i$. The expression (34) for the Abelian constraints $\psi_i$ dictates the appropriate gauge fixing condition

$$\overline{\psi}_i := N^a_i A^a_i = 0,$$

(81)

which is the canonical one in the sense that

$$\{ \overline{\psi}_i, \psi_j \} = \delta_{ij}.$$  

(82)

The constraints $\psi_i = 0$ rewritten in terms of the main-axes variables may be identified with the nullity of the momenta

$$p_{\chi_1} = 0, \quad p_{\chi_2} = 0,$$

(83)

while the canonical gauge-fixing condition (81) fixes the corresponding angular variables $\chi_1$ and $\chi_2$

$$\chi_1 = \frac{\pi}{2}, \quad \chi_2 = \frac{\pi}{2}.$$  

(84)

Introduction of the gauge fixing conditions (81) means that all constraints are now second class ones and therefore the reduction to unconstrained variables can now be achieved by the projection of canonical Hamiltonian onto the constraint shell with simultaneously replacement of the canonical Poisson brackets by the Dirac ones.
Projection of the canonical Hamiltonian (19) to the surface described by constraints (83) and (84) gives

\[ H_{LC} := H_C(\chi_1 = \frac{\pi}{2}, p_{\chi_1} = 0, \chi_2 = \frac{\pi}{2}, p_{\chi_2} = 0) = \frac{g^2}{2} \left( p_1^2 + \frac{q_2^2 q_3^2}{g^4} \right). \]  

(85)

Furthermore, taking into account the constraint (77) the projected Hamiltonian (85) may be rewritten as

\[ H_{LC} \bigg|_{2q_1, q_2, q_3 - g^2 \xi_L^3 = 0} = \frac{g^2}{2} \left( p_1^2 + \left( \frac{\xi_L^3}{2q_1} \right)^2 \right). \]  

(86)

But it is not the end of the reduction procedure. Two further steps are required. First, it is necessary to examine all four second class constraints \( \chi_{i, \perp} \) and to verify whether (86) is indeed the expression for the reduced Hamiltonian describing the dynamics of unconstrained variables. Second, it is necessary to calculate the fundamental Dirac brackets between unconstrained variables in order to determine the correct equation of motion.

It may be checked that the constraints \( \chi_{i, \perp} \) lead to the conditions on the “diagonal” canonical pairs \( (q_i, p_i) \). Namely, the canonical momenta \( p_2 \) and \( p_3 \) are vanishing

\[ p_2 = 0, \quad p_3 = 0, \]  

(87)

while the corresponding coordinates \( q_2 \) and \( q_3 \) are subject to the constraint

\[ q_2^2 + q_3^2 = 0 \]  

(88)

as well the constraint (77). The real solution of the equation (88) is the trivial one \( q_2 = q_3 = 0 \). For this solution according to the constraint (77) \( \xi_L^3 \) turns to be zero and thus the Hamiltonian (86) reduces further to a Hamiltonian of free one-dimensional particle motion.

Here it is in order to make an explanatory comment, because we arrived at certain contradiction to our initial assumptions. The reduced Hamiltonian system obtained here contains only two degrees of freedom, while according to the counting given at the end of the Section 2, we were expected to obtain a 4-dimensional unconstrained system. In order to explain this contradiction note that this counting was based on the assumption that the configuration space of the initial Lagrangian system is 12-dimensional or in another words the \( 3 \times 3 \) matrix \( A \) in (47) is non-degenerate. However the constraint (88) states that \( \text{det}(||A|| = \text{det}(||S|| = 0) \), and rigorously speaking our consideration shows only that the dynamics of the unconstrained system develops on the subspace with \( \text{rank}(||A|| \leq 2 \). Therefore it is necessary to consider the configuration space of the initial Lagrangian system consisting from the degenerate matrices with rank less than maximal and perform the whole analysis again. \(^4\)

\(^4\)For example, the counting of the degrees of freedom is now as follows: the dimension of subspace of \( 3 \times 3 \) matrices with rank \( k = 2 \) is \( 2 \times (2 \times 3 - 2) = 8 \). So, the configuration space of the initial system is not 12-dimensional, but 11-dimensional and thus the reduced Hamiltonian system contains only \( 22 - 2(5 + 3) - 4 = 2 \) degrees of freedom.
However instead of an explicit parametrization of the configuration space with rank $|A| = 2$ and rank $|A| = 1$ we use the following trick. Let us consider the analytic continuation of the constraint (88) into a complex domain and explore its complex solution

$$q_2 = \pm i q_3.$$  

(89)

Expressing $q_3$ from equation (77)

$$q_3 = \frac{1 \mp i}{2} \sqrt{\frac{g^2 \xi_3}{q_1}},$$

(90)

we find that $(q_1, p_1)$ and $(\chi_3, p_{\chi_3})$ remain real unconstrained variables whose Dirac brackets are the canonical ones

$$\{q_1, p_1\}_D = 1, \quad \{\chi_3, p_{\chi_3}\}_D = 1.$$  

(91)

Therefore the dynamics of the unconstrained pairs $(q_1, p_1)$ and $(\chi_3, p_{\chi_3})$ is given by the standard Hamilton equations with the Hamiltonian (86). Remarking that the $\xi_3^L$ is conserved we conclude that (86) coincides with the Hamiltonian of conformal mechanics

$$H = \frac{g^2}{2} \left( p_1^2 + \frac{\kappa^2}{q_1^2} \right),$$  

(92)

with "coupling constant" $\kappa^2 = (\xi_3^L/2)^2$ determined by the value of the gauge spin, while the gauge field coupling constant $g$ controls the scale for the evolution parameter.

From equation (90) it follows that the quantity $\kappa$ is the parameter which measures the deviation from the real classical trajectories. They all are lying in subspace with $\det |A| = 0$ and are described as the integral curves of the Hamiltonian (86) with vanishing coupling constant $\kappa = 0$, and therefore indeed correspond to a free particle motion.

### 4 Concluding remarks

To conclude, we have considered the light-cone $SU(2)$ Yang-Mills field theory supposing that the gauge potentials in the classical action are functions only of the light-cone time. As we have demonstrated this ansatz effectively reduces the field theory to a degenerate Lagrangian mechanical system whose unconstrained version significantly differs from the corresponding well-known instant time Yang-Mills mechanics. Comparing with the instant form dynamics, the light-cone version of Yang-Mills mechanics has a more complicated description considered as a constrained system. Applying the Dirac Hamiltonian method, we found that now the constraint content of the theory is richer: there is, apart from the expected constraints which are generators of the $SU(2)$ gauge transformations, a

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5 As justification, in the appendix we give an alternative derivation directly from the Lagrangian equation of motion by solving the corresponding Lagrangian constraints.
new set of first and second class constraints. The presence of the new constraints leads to an essential decrease of the number of the “true” degrees of freedom and finally to its integrability.

In the present paper we have studied the Hamiltonian reduction of the degenerate light-cone Yang-Mills mechanical system but have left open several related questions such as analysis of the symmetries, both gauge and rigid ones. The knowledge of symmetries allows to understand the roots of the classical integrability of the system and we plan to give the detailed presentation of these investigations elsewhere.

We end this section with a remark about the possible link between the classical integrability of the model obtained in long-wavelength approximation of light-cone theory and the properties of the corresponding vacuum. The nonintegrability and chaotic nature of the instant form Yang-Mills mechanics is usually treated as the manifestation of existence of the non-trivial structure of the physical vacuum of gauge theories (see e.g. [14]). On the other hand it is well-known that owing to purely kinematical reasons the physical light-cone vacuum of the theory coincides with the free Fock vacuum [51]. Therefore it seems that the integrability of the light-cone Yang-Mills mechanics opposite to its instant counterpart model is in accordance with the different vacuum structures in these two forms of dynamics.

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A Appendix: The Euler-Lagrange Equations

The first variation of the Lagrangian (10) with respect to the variables $A^a_+, A^a_k$ and $A^a_-$ gives the constraints, equations containing only first order derivatives, and the proper equations of motion. Among the constraints there are the Gauss Law equations (summation over $k = 1, 2$)

$$\dot{A}_- \times A_+ + A_- (A_k \cdot A_k) - A_k (A_k \cdot A_-) - A_+ (A_- \cdot A_-) + A_- (A_+ \cdot A_-) = 0, \quad (93)$$
as well as the additional constraints \((\text{no summation over} \; i, k = 1, 2 \; \text{and} \; i \neq k)\)

\[
2\dot{A}_k \times A_- - \dot{A}_- \times A_k - A_k (A_i \cdot A_i) + A_i (A_i \cdot A_k) + 2A_k (A_+ \cdot A_-) - A_- (A_+ \cdot A_k) - A_+ (A_- \cdot A_k) = 0. \tag{94}
\]

The true equations of motion, containing second order derivatives of the variables \(A_-\) are (\text{summation over} \; k = 1, 2)

\[
\ddot{A}_- + \dot{A}_+ \times A_- + \dot{A}_- \times A_+ - A_k \times \dot{A}_k + A_+ ((A_+ \cdot A_-) - (A_k \cdot A_k)) - A_- (A_+ \cdot A_+) + A_k (A_+ \cdot A_k) = 0. \tag{95}
\]

Here we have introduced the vector notation for the isotopic components of the vector potential \(A_\pm = (A_1^\pm, A_2^\pm, A_3^\pm)\) and \(A_k = (A_1^k, A_2^k, A_3^k)\). The standard definitions for dot and cross product of three dimensional isotopic vectors are used as well.

The aim of this Appendix is to show how to pass from this system of nonlinear Euler-Lagrange equations \((93)-(95)\) to the one-dimensional equation of motion of a free particle for \(\text{em real solutions and to conformal mechanics for the case of complex solution to the Lagrangian constraints.}\)

To be close to the Hamiltonian consideration given in the main text let us introduce in the isotopic space a positively oriented orthonormal frame of unit vectors, \((l, m, n)\),

\[
l \cdot l = 1, \quad m = n \times l, \tag{96}
\]

\[
m \cdot m = 1, \quad m = n \times l, \tag{97}
\]

\[
n \cdot n = 1, \quad n = l \times m, \tag{98}
\]

and start with the following \textit{ansatz} for the gauge potential

\[
A_- = x n, \tag{99}
\]

\[
A_k = S_{1k} l + S_{2k} m. \tag{100}
\]

Note that this \textit{ansatz} corresponds to the polar decomposition \((49)\) when the only nonvanishing element in the third row and the third column of the matrix \(S\) is \(S_{33} = x\) and moreover it is supposed in \((100)\) that

\[
S_{12} = S_{21}. \tag{101}
\]

By construction the \textit{ansatz} is such that the potentials \((99)\) and \((100)\) obey four equations

\[
l \cdot A_- = 0, \quad m \cdot A_- = 0, \quad n \cdot A_k = 0. \tag{102}
\]

Note that these equations are equivalent to the two primary Abelian constraints \((34)\) and to the two gauge-fixing conditions \((81)\) imposed on the gauge potential in the main text.

Now we shall demonstrate that the system of equations \((93)-(95)\) admits a separation into three subsets. The first one establish the connection between \(A_+\) component of the gauge potential and the frame \((l, m, n)\), the second one consists of the equations for
the variables \( S_{1k} \) and \( S_{2k} \) and the third one represents only one second order differential equation for the variable \( x \) with a parameter, whose value is the first integral of the equations for the variables \( S_{1k} \) and \( S_{2k} \).

Let us start with the Gauss law constraints and try to resolve it against the variable \( A_+ \). But, because the vector \( A_- \) is the zero mode of these equations, only two components of \( A_+ \), transverse to \( A_- \), can be fixed uniquely. Indeed, in our parametrization (99), (100), when the \( A_- \) direction coincides with the \( n \) direction, projection of the Gauss law equations by the transverse vectors (1 and \( m \)) yields

\[
x \left[ (l \cdot h) - (l \cdot A_+) \right] = 0, \tag{103}
\]

\[
x \left[ (m \cdot h) - (m \cdot A_+) \right] = 0, \tag{104}
\]

while its projection to the third \( n \)-component results in constraint on the variables \( S_{1k} \)

\[
S_{11}^2 + S_{12}^2 + S_{21}^2 + S_{22}^2 = 0. \tag{105}
\]

In (103) and (104) the helicity vector \( h = \dot{n} \times n \) has been introduced. So, from (103) and (104), supposing \( x \neq 0 \), it follows that

\[
A_+ = h + f n, \tag{106}
\]

with unspecified function \( f \).

With this \( A_+ \) one can rewrite the equation of motion (95) for the \( A_- \) component as (summation over \( k = 1, 2 \))

\[
n \ddot{x} = A_k \times \dot{A}_k - A_k (h \cdot A_k). \tag{107}
\]

Using (94) the projection of the equation (107) onto the \( n \) direction looks as

\[
\ddot{x} = \frac{1}{x^2} ((A_1 \cdot A_1)(A_2 \cdot A_2) - (A_1 \cdot A_2)(A_2 \cdot A_1)), \tag{108}
\]

while its projections onto \( l \) and \( m \) direction give again the constraint (105).

Consider now the equations (95), projection to \( l \) and \( m \) directions results in equations (no summation \( i \neq k \))

\[
\dot{x} S_{2k} + 2x \dot{S}_{2k} + 2x [(1 \cdot \dot{m}) - f] S_{1k} = S_{1k} (S_{11}^2 + S_{21}^2) - S_{1i} (S_{1i} S_{1k} + S_{2i} S_{2k}), \tag{109}
\]

\[
-\dot{x} S_{2k} - 2x \dot{S}_{2k} + 2x [(1 \cdot \dot{m}) - f] S_{2k} = S_{2k} (S_{11}^2 + S_{21}^2) - S_{2i} (S_{1i} S_{1k} + S_{2i} S_{2k}), \tag{110}
\]

while projection onto the third direction \( n \) gives

\[
(n \cdot \dot{m}) S_{1k} - (n \cdot \dot{l}) S_{2k} = (h \cdot \dot{l}) S_{1k} + (h \cdot \dot{m}) S_{2k}. \tag{111}
\]

The last two equations (111) are identically satisfied when the relations

\[
h \cdot \dot{l} = n \cdot \dot{m}, \quad h \cdot m = -n \cdot \dot{l} \tag{112}
\]
are taken into account. One can make the equations (109) and (110) independent of the frame \((l, m, n)\) if the function \(f\) in the \(A_+\) decomposition is chosen as
\[
f = l \cdot \dot{m} + \rho, \tag{113}
\]
where the function \(\rho\) is still unspecified.

With this identification the following system of differential equations for the unknown functions \(S_{11}, S_{12}, S_{21}, S_{22}\) arises
\[
\begin{align*}
\dot{s}S_{11} + 2x\dot{s}S_{11} - 2x\rho S_{21} &= -S_{11}(S_{11}S_{22} - S_{12}S_{21}), \tag{114} \\
\dot{s}S_{12} + 2x\dot{s}S_{12} - 2x\rho S_{22} &= S_{11}(S_{11}S_{22} - S_{12}S_{21}), \tag{115} \\
\dot{s}S_{21} + 2x\dot{s}S_{21} + 2x\rho S_{11} &= -S_{22}(S_{11}S_{22} - S_{12}S_{21}), \tag{116} \\
\dot{s}S_{22} + 2x\dot{s}S_{22} + 2x\rho S_{12} &= S_{21}(S_{11}S_{22} - S_{12}S_{21}). \tag{117}
\end{align*}
\]

Introduction of the new functions
\[
S_{ij} \sqrt{x} = Y_{ij}
\]
removes the derivatives of the function \(x\)
\[
\begin{align*}
\dot{Y}_{11} - \rho Y_{21} &= \frac{Y_{12}}{2x^2} (Y_{11}Y_{22} - Y_{12}Y_{21}), \tag{118} \\
\dot{Y}_{12} - \rho Y_{22} &= -\frac{Y_{11}}{2x^2} (Y_{11}Y_{22} - Y_{12}Y_{21}), \tag{119} \\
\dot{Y}_{21} + \rho Y_{11} &= \frac{Y_{22}}{2x^2} (Y_{11}Y_{22} - Y_{12}Y_{21}), \tag{120} \\
\dot{Y}_{22} + \rho Y_{12} &= -\frac{Y_{21}}{2x^2} (Y_{11}Y_{22} - Y_{12}Y_{21}). \tag{121}
\end{align*}
\]

Now one can specify the function \(\rho\). Due to the symmetry condition (101), \(Y_{12} = Y_{21}\), this leads to the relation
\[
\rho = \frac{1}{2x^2} (Y_{11}Y_{22} - Y_{12}Y_{21}). \tag{122}
\]
Moreover, because the system of equations (118)-(121) possesses the following first integral
\[
Y_{11}Y_{22} - Y_{12}Y_{21} = \mu, \quad \mu := \text{constant}, \tag{123}
\]
one can express \(\rho\) solely in terms of the \(x\) variable
\[
\rho = \frac{\mu}{2x^2}. \tag{124}
\]
Therefore, finally the equations (118)-(121) reduce to a system of three differential equations
\[
\begin{align*}
\dot{Y}_{11} &= \frac{\mu}{x^2} Y_{12}, \tag{125} \\
\dot{Y}_{12} &= \frac{\mu}{2x^2} (Y_{22} - Y_{11}), \\
\dot{Y}_{22} &= -\frac{\mu}{x^2} Y_{12}.
\end{align*}
\]
These equations should be solved together with the algebraic constraint (105) which states
\[ Y_{11}^2 + 2Y_{12}^2 + Y_{22}^2 = 0. \] (126)
The last equation has only trivial real solutions
\[ Y_{11} = Y_{12} = Y_{22} = 0 \] (127)
that lead to the free equation of motion for the \( x \) variable.

However, one can consider the analytic continuation of our variables \( Y \) in a complex domain. So, relaxing the reality conditions, we use the following parametrization for the complex \( Y \)-functions
\[ Y_{12} = \nu(i - 1) \sin \chi \cos \chi, \quad Y_{11} = \nu(\cos^2 \chi + i \sin^2 \chi), \] (128)
\[ Y_{21} = \nu(i - 1) \sin \chi \cos \chi, \quad Y_{22} = \nu(\sin^2 \chi + i \cos^2 \chi). \] (129)
Here the parameter \( \nu \) is expressed through the first integral constant \( \nu^2 = -i\mu \).

Within the parametrization (128) and (129) all equations (125) are satisfied if the angular variable \( \chi \) obeys the equation
\[ \dot{\chi} = \frac{\mu}{2x^2}. \] (130)
Now in order to insure the self consistency of the solution it may be checked that the right hand side of the equation (108) evaluated with \( Y \) given in (128) reduces to
\[ \ddot{x} = \frac{\mu^2}{x^3}. \] (131)
This completes the proof, since (131) is the equation of motion for an one-dimensional system with the Lagrange function
\[ L_{CM} := \frac{1}{2} \left( \dot{x}^2 - \frac{\mu^2}{x^2} \right). \] (132)

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