Renormalization Transformations of the 4D BFYM Theory

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Abstract

We study the most general renormalization transformations for the first-order formulation of the Yang–Mills theory. We analyze, in particular, the trivial sector of the BRST cohomology of two possible formulations of the model: the standard one and the extended one. The latter is a promising starting point for the interpretation of the Yang–Mills theory as a deformation of the topological BF theory. This work is a necessary preliminary step towards any perturbative calculation, and completes some recently obtained results.

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1 Introduction

The confinement problem in QCD has been recently studied by reformulating the Yang–Mills action through the first-order formalism [1]. This model, named “gaussian” BFYM, is described by the action

$$S = \text{Tr} \int_{\mathbb{R}^4} \left( iB \wedge F + g^2 B \wedge \ast B \right) ,$$

where $F$ is the field-strength and $B$ is a 2-form. It is easy to see that this action is on-shell equivalent to the classical YM action. This formulation [1, 2] allows the introduction of ’t Hooft-like order-disorder parameters [3], and leads to an explicit realization of the ’t Hooft picture of the vacuum as a dual superconductor, in which the BFYM theory appears to be in the confining phase [1].

Another formulation of the theory is possible. Its construction relies on the observation that the zero coupling limit of (1.1) is the topological pure BF theory [4, 5], whose topological properties are closely related to a further symmetry, named “topological”, besides the gauge invariance. Then, by introducing a 1-form $\eta$, another first-order formulation may be defined, which is again on-shell equivalent to the YM theory but has a further symmetry that acts on $A$ and $B$ like the topological symmetry of the pure BF theory [6]:

$$S = \text{Tr} \int_{\mathbb{R}^4} \left[ iB \wedge F + g^2 \left( B - dA\eta \right) \wedge \ast \left( B - dA\eta \right) \right] ;$$

we will call this formulation “extended”. Then the model can be interpreted as a deformation of the BF topological theory [6].

The question arises if the classical equivalence of these two models with the Yang-Mills extends also at the quantum level, and of what are their perturbative properties. A study of the 3D case was performed in [7], where the quantum equivalence was explicitly demonstrated together with the absence of anomalies by algebraic and power-counting methods, and the complete renormalization transformations were given. The anomalies and the quantum equivalence in 4D have been treated in [8] and a generalization to all dimensions can be found in [9], where no power-counting argument is used. The equivalence has been studied also in [10], where an explicit one-loop computation of the $\beta$-function in the gaussian formulation has been carried out showing that it is equal to the YM case, and in [11] with background field methods.

In this letter we wish to complete the algebraic analysis of [8, 9] by studying the trivial sector of the BRST cohomology at ghost number zero, i.e. the counterterms that can be written as BRST variations of some field functional. Following [7] we will give the complete renormalization transformations needed to absorb all the invariant counterterms, and will study the restriction arising in the Landau gauge for both formulations. We will also show that a certain class of counterterms, though algebraically allowed to appear, have vanishing coefficient to all orders. In section 2 we will study the gaussian formulation and in section 3 we will address the more involved extended formulation.

We will adopt the following notation. The field strength is $F = dA + A \wedge A$, the covariant derivative is $d_A \omega = d\omega + [A, \omega]$. All the fields are in the adjoint representation...
of the gauge group and the generators are taken to be antihermitean. The exterior product of a p-form with a q-form is
\[ \omega \wedge \lambda = \frac{1}{p! q!} \varepsilon_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_{p+q}} \, dx_{\mu_1} \ldots dx_{\mu_{p+q}} \]
the Hodge dual of a p-form is
\[ *\omega = \frac{1}{(4-p)!} \varepsilon_{\mu_1 \ldots \mu_{4-p} \mu_{5-p}} \omega_{\mu_1 \ldots \mu_p} \, dx_{\mu_{p+1}} \ldots dx_{\mu_4} \].
Moreover \([\omega, \lambda]\) will indicate the graded commutator between the two forms \(\omega\) and \(\lambda\), and we will omit the wedge product between forms.

## 2 Gaussian formulation

The gaussian model (1.1) is invariant with respect to the gauge symmetry
\[ \delta_g A = - d_A \varepsilon \]
\[ \delta_g B = - [B, \varepsilon] , \]
where \(\varepsilon\) is a local Grassmann-odd adjoint-valued zero-form. Following the BRST quantization procedure we introduce a couple of ghost and antighost \((c, \bar{c})\) and the auxiliary field \(h_A\) and we define the BRST transformation \(s\):
\[ s A = - d_A c , \quad s B = - [B, c] , \]
\[ s c = \frac{1}{2} [c, c] , \]
\[ s \bar{c} = h_A , \]
\[ s h_A = 0 , \]
which is off-shell nilpotent. Then we define the gauge-fixing lagrangean, choosing the covariant Landau gauge \(d^\dagger A = 0\):
\[ S_{gf} = s \text{ Tr } \int_{\mathbb{R}^4} \bar{c} * d^\dagger A . \]

Furthermore we introduce a set of external sources coupled to the nonlinear BRST transformations:
\[ S_{ext} = \text{ Tr } \int_{\mathbb{R}^4} \left( \Omega_A * s(A) + \Omega_B * s(B) + \Omega_c * s(c) \right) . \]

Eventually, the tree-level action is
\[ \Sigma[A, B, c, \bar{c}, h_A, \Omega_A, \Omega_B, \Omega_c] = S_{BFYM} + S_{gf} + S_{ext} = \]
\[ = \text{ Tr } \int_{\mathbb{R}^4} \left[ i BF + g^2 B * B + \bar{c} * d^\dagger d_A c + h_A * d^\dagger A + \right. \]
\[ - \Omega_A * d_A c - \Omega_B * [B, c] + \Omega_c * \frac{1}{2} \{c, c\} \] .

The dimensions, ghost-number and space-time inversion parity of the fields are shown in table [i].

\[ \text{2}\text{The main motivation for this choice is the richer algebraic structure present in this gauge, and its convenience when performing perturbative calculations.} \]
Due to the gauge invariance (2.1), the action (2.5) satisfies the Slavnov–Taylor condition:

\[ S(\Sigma) = 0, \tag{2.6} \]

where

\[ S(\Sigma) = \text{Tr} \int d^4x \left( \frac{\delta \Sigma}{\delta A_\mu} \frac{\delta \Sigma}{\delta \Omega_{A\mu}} + \frac{\delta \Sigma}{\delta B_{\mu\nu}} \frac{\delta \Sigma}{\delta \Omega_{B\mu\nu}} + \delta A_\mu \frac{\delta \Sigma}{\delta \bar{c}} + \frac{\delta \Sigma}{\delta c} \frac{\delta \Sigma}{\delta \Omega_{c}} \right). \tag{2.7} \]

Moreover it satisfies the following constraints:

\[ \frac{\delta \Sigma}{\delta h_A} = \partial_\mu A_\mu, \tag{2.8} \]

\[ \check{\mathcal{G}} \Sigma = \frac{\delta \Sigma}{\delta \bar{c}} + \partial_\mu \frac{\delta \Sigma}{\delta \Omega_{A\mu}} = 0, \tag{2.9} \]

\[ \mathcal{G} \Sigma = \int d^4x \left( \frac{\delta \Sigma}{\delta c} + \left[ \bar{c}, \frac{\delta \Sigma}{\delta h_A} \right] \right) = \int d^4x \left( \left[ \Omega_{A\mu}, A_\mu \right] + \left[ \Omega_{B\mu\nu}, B_{\mu\nu} \right] - \left[ \Omega_c, c \right] \right), \tag{2.10} \]

\[ W^{\text{rig}} \Sigma = \int d^4x \sum_\varphi \left[ \varphi, \frac{\delta \Sigma}{\delta \varphi} \right] = 0 \quad \varphi = \text{all fields}. \tag{2.11} \]

The constraint (2.9) is obtained by commuting (2.8) with the S.T. identity (2.6), and the constraint (2.11) is obtained by commuting (2.10), which is a peculiarity of the Landau gauge \[12\], with (2.6). As a consequence of (2.9) the antighost enter in the action only through the combination:

\[ \check{\Omega}_{A\mu} = \Omega_{A\mu} + \partial_\mu \bar{c}. \tag{2.12} \]

We can define the reduced action

\[ \tilde{\Sigma}[A, B, c, \check{\Omega}_A, \Omega_B, \Omega_c] = \Sigma[A, B, c, \bar{c}, h_A, \Omega_A, \Omega_B, \Omega_c] - \int d^4x h_A \partial_\mu A_\mu, \tag{2.13} \]

so that with respect to these new variables, the constraints (2.8-2.11) become

\[ \frac{\delta \tilde{\Sigma}}{\delta h_A} = 0, \tag{2.14} \]

\[ \frac{\delta \tilde{\Sigma}}{\delta \bar{c}} = 0, \tag{2.15} \]

\[ \int \frac{\delta \tilde{\Sigma}}{\delta c} = \int d^4x \left( \left[ \check{\Omega}_{A\mu}, A_\mu \right] + \left[ \Omega_{B\mu\nu}, B_{\mu\nu} \right] - \left[ \Omega_c, c \right] \right), \tag{2.16} \]

\[ W^{\text{rig}} \tilde{\Sigma} = \int d^4x \sum_\varphi \left[ \varphi, \frac{\delta \tilde{\Sigma}}{\delta \varphi} \right] = 0 \quad \varphi = \text{all fields}. \tag{2.17} \]
By introducing the linearized S.T. operator
\[ \hat{B}_\Sigma = \text{Tr} \int d^4x \left( \frac{\delta \hat{\Sigma}}{\delta A_\mu} \frac{\delta}{\delta \Omega_{A\mu}} + \frac{\delta \hat{\Sigma}}{\delta \Omega_{A\mu}} \frac{\delta}{\delta A_\mu} + \frac{\delta \hat{\Sigma}}{\delta B_{\mu\nu}} \frac{\delta}{\delta \Omega_{B_{\mu\nu}}} + \frac{\delta \hat{\Sigma}}{\delta B_{\mu\nu}} \frac{\delta}{\delta \Omega_{B_{\mu\nu}}} + \frac{\delta \hat{\Sigma}}{\delta c} \frac{\delta}{\delta \Omega_c} + \frac{\delta \hat{\Sigma}}{\delta \Omega_c} \frac{\delta}{\delta c} \right), \]
(2.18)
the S.T. identity \( \Sigma \) is rewritten as
\[ S(\Sigma) = \frac{1}{2} \hat{B}_\Sigma \hat{\Sigma} = 0, \]
which, in turn, implies the nilpotency of the S.T. operator:
\[ \hat{B}_\Sigma \hat{B}_\Sigma = 0. \]
(2.20)

As it is well known \([13]\) the anomalies and the counterterms are described by the local cohomology of the S.T. operator \emph{modulo} total derivatives. In fact, both the possible breaking \( \Delta \) of the S.T. identity and the invariant counterterms have to satisfy the consistency condition
\[ \hat{B}_\Sigma \Delta + d\Delta = 0 \]
(2.21)
where \( \Delta \) is a local 4-form of ghost number 1 and 0 respectively, and dimension 4 thanks to the QAP \([14]\), and \( \Delta \) is a 3–form of ghost number 2 and dimension 3. The solution of these two cohomology problems have been worked out in \([8, 9]\), whose result is that the cohomology of the S.T. operator \( \Sigma \) is isomorphic to the YM case. Then the only anomaly allowed by the theory is the ABBJ one \([15]\), which is absent in our case since all the fields are in the adjoint representation. Moreover, the only physical renormalization is the coupling constant one, the related counterterm being \( \Delta = \frac{1}{g^2} F \ast F \) which is \( \hat{B}_\Sigma \) \emph{modulo} \( d \)-equivalent to a multiple of \( g^2 B \ast B \).

It remains to analyze the trivial counterterms; they are of the form
\[ \hat{B}_\Sigma \text{Tr} \int_{\mathbb{R}^4} \tilde{\Delta}, \]
(2.22)
where \( \tilde{\Delta} \) is a local 4-form of ghost number \(-1\) and dimension 4. Let us introduce the following notation:
\[ N_\varphi = \int_{\mathbb{R}^4} \varphi \ast \frac{\delta}{\delta \varphi}; \quad N_\varphi \ast \omega = \int_{\mathbb{R}^4} \omega \ast \frac{\delta}{\delta \varphi}. \]
(2.23)
Then the trivial counterterms can be expressed as
\[ \hat{B}_\Sigma (\text{Tr} \int \Omega_A \ast A) = (N_A - N_\tilde{\Omega}_A) \tilde{\Sigma} \equiv N_A \tilde{\Sigma}, \]
\[ \hat{B}_\Sigma (\text{Tr} \int \Omega_B \ast B) = (N_B - N_\tilde{\Omega}_B) \tilde{\Sigma} \equiv N_B \tilde{\Sigma}, \]
\[ \hat{B}_\Sigma (\text{Tr} \int \Omega_B \circ dA) = (N_{B \ast \circ dA} + N_\tilde{\Omega}_A \ast d\Omega_B) \tilde{\Sigma} \equiv N_{\ast \circ \partial} \tilde{\Sigma}, \]
\[ \hat{B}_\Sigma (\text{Tr} \int \Omega_B [A, A]) = (N_{B \ast \circ [A, A]} + 2N_\tilde{\Omega}_A \ast [A, \Omega_B]) \tilde{\Sigma} \equiv N_{\ast \circ \partial} \tilde{\Sigma}, \]
\[ \hat{B}_\Sigma (\text{Tr} \int \Omega_B \ast B) = (N_{B \ast \circ B} - N_{\Omega_B \ast \circ \Omega_B}) \tilde{\Sigma} \equiv N_{\ast \circ \partial} \tilde{\Sigma}, \]
\[ \hat{B}_\Sigma (\text{Tr} \int \Omega_B \circ dA) = (N_{B \ast \circ dA} - N_\tilde{\Omega}_A \ast d\Omega_B) \tilde{\Sigma} \equiv N_{\ast \circ \partial} \tilde{\Sigma}, \]
\[ \hat{B}_\Sigma (\text{Tr} \int \Omega_B [A, A]) = (N_{B \ast \circ [A, A]} - 2N_\tilde{\Omega}_A \ast [A, \Omega_B]) \tilde{\Sigma} \equiv N_{\ast \circ \partial} \tilde{\Sigma}. \]
(2.24)
The $\mathcal{N}_c\hat{\Sigma}$ counterterm is furthermore excluded by the ghost equation (2.10). Then all the counterterms can be absorbed at the order $\hbar^n$ by the following renormalization transformation, assuming that the renormalization process has been carried out till the order $\hbar^{n-1}$ (details can be found in [7], which deals with the 3D case):

$$
\begin{align*}
A &= A_R + \hbar^n z_A A_R , \\
B &= B_R + \hbar^n z_B B_R + \hbar^n z_{BA} * dA_R + \hbar^n z_{BAA} * \frac{1}{2} [A_R, A_R] + \hbar^n z_B^* B_R + \hbar^n z_{BA}^* dA_R + \hbar^n z_{BAA}^* [A_R, A_R] , \\
c &= c_R , \\
\hat{\Omega}_A &= \hat{\Omega}_{AR} - \hbar^n z_A \hat{\Omega}_{AR} + \hbar^n z_{BA} * d\Omega_{BR} + \hbar^n z_{BAA} * [A_R, \Omega_{BR}] + \hbar^n z_B^* B_R + \hbar^n z_{BAA}^* [A_R, \Omega_{BR}] , \\
\Omega_B &= \Omega_{BR} - \hbar^n z_B \Omega_{BR} - \hbar^n z_B^* \Omega_{BR} , \\
\Omega_c &= \Omega_{cR} , \\
g &= g_R + \hbar^n z_g g_R .
\end{align*}
$$

We note that the terms containing $F$ get contributions not only from the rescaling of $g$ and $A$, as it happens in the YM case, but also from the rotation of the $B$-field, that gives a priori two different weights to $dA$ and $[A, A]$.

Let us comment about the terms of coefficient $z^*$. These terms produce counterterms of the type $\epsilon_{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma}$, $B_{\mu\nu} \partial_{\mu} A_{\nu}$ and $B_{\mu\nu} [A_{\mu}, A_{\nu}]$ which alter the tensorial structure of the 1PI functions $\Gamma_{BB}$, $\Gamma_{BA}$ and $\Gamma_{BAA}$ with respect to the corresponding Feynman rules. But it is possible to show that starting from the Feynman rules stemming from the action (2.5) these structure are not generated at any order of perturbation, meaning that all the renormalization constants $z^*$ are equal to zero to all orders. Indeed if we consider, for example, any given diagram of the two point function $\Gamma_{BA}$ we see that the sum of the number of vertices $BAA$ and the number of propagators $BA$, that are the only Feynman rules proportional to the totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$, is always an odd number so that the net result is that every diagram is proportional to this tensor and the structures coming from the $z^*$ terms do not appear.

The situation is as follows. To obtain the equivalence with the Yang-Mills theory we could have equally well started from the first-order lagrangean $iB^*F + g^2 B^*B^*$, whose $g \to 0$ limit is again symmetric with respect to the topological-like symmetry $B \to B + *dA\xi$ allowing for the extended formulation that will be discussed in the next section. The algebraic structure of each model allows for the generation of the counterterms corresponding to the other theory, but its Feynman rules do not generate them: the two theories, then, do not mix. They would do if we would have started from a Lagrangian of the type $iBF \pm iB^*F + g^2 B^*B$, which is equivalent on shell to the Yang–Mills in the self-dual (anti self-dual) gauge, and whose $g \to 0$ limit is again symmetric with respect to $B \to B + dA\xi \mp *dA\xi$. 


3 Extended formulation

The analysis of this formulation proceeds in the same way as the former one, so that we will shorten the discussions; details can be found in [8].

The action (1.2) is invariant with respect to the gauge symmetry \( \delta_g \), the topological symmetry \( \delta_t \) and the further local symmetry \( \delta' \) [6, 8]

\[
\begin{align*}
\delta_g A &= -d_A \varepsilon , & \delta_t A &= 0 , & \delta' A &= 0 , \\
\delta_g B &= -[B, \varepsilon] , & \delta_t B &= d_A \theta , & \delta' B &= [F, \sigma] , \\
\delta_g \eta &= -[\eta, \varepsilon] , & \delta_t \eta &= \theta , & \delta' \eta &= d_A \sigma ,
\end{align*}
\]

where \( \varepsilon, \sigma \) are local 0-forms and \( \theta \) is a local 1-form. We remark that the topological symmetry \( \delta_t \) is reducible, requiring a second ghost generation in order to correctly quantize the theory and that \( \delta' \) is not independent of the topological symmetry as it can be seen by choosing \( \theta = d_A \sigma \).

The BRST quantization requires for each classical symmetry the introduction of a couple of ghost and antighost and a Lagrange multiplier, called respectively \( (c, \bar{c}, h_A) \), \( (\psi, \bar{\psi}, h_B) \) and \( (\rho, \bar{\rho}, h_\eta) \). We have also to introduce the second generation ghosts and multiplier \( (\phi, \bar{\phi}, h_\psi) \) and also the pair of fields \( (u, h_\bar{\psi}) \) to fix a further degeneracy related to \( \bar{\psi} \). Their dimension, ghost number and parity are shown in table 2. Then, we define the BRST transformation

\[
\begin{align*}
s A &= -d_A c , & s B &= -[B, c] + d_A \psi + [F, \rho] , & s \eta &= -[\eta, c] + \psi + d_A \rho , \\
s c &= \frac{1}{2}[c, c] , & s \psi &= [\psi, c] + d_A \phi , & s \rho &= [\rho, c] - \phi , \\
s \bar{c} &= h_A , & s \bar{\psi} &= h_B , & s \bar{\rho} &= h_\eta , \\
sh_A &= 0 , & sh_B &= 0 , & sh_\eta &= 0 , \\
s \phi &= -[\phi, c] , & s \bar{\phi} &= h_\psi , \\
sh_\phi &= 0 , & su &= h_\bar{\psi} , \\
sh_\bar{\phi} &= 0
\end{align*}
\]

which is off-shell nihilpotent:

\[
s^2 = 0 ,
\]

Then we choose the gauge-fixing conditions:

\[
d^\dagger A = 0, \quad d^\dagger B = 0, \quad d^\dagger \eta = 0 ,
\]

and define the gauge-fixing lagrangean in the Landau gauge:

\[
S_{gf} = s \text{ Tr } \int_{\mathbb{R}^4} \left[ \bar{c} \ast d^\dagger A + \bar{\psi} \ast d^\dagger B + \bar{\rho} \ast d^\dagger \eta + d^\dagger \bar{\psi} \ast u + \bar{\phi} \ast d^\dagger \psi \right] .
\]
Finally, we introduce the external sources $\Omega$ coupled to the nonlinear BRST variations:

$$S_{ext} = \text{Tr} \int d^4x \left( - \Omega_A * d_A c + \frac{1}{2} \Omega_B * (d_A \psi - [B, c] + [F, \rho]) + \Omega_\eta * (\psi - [\eta, c] + d_A \rho) + \right.$$

$$\left. + \Omega_\phi * (d_A \phi + [\psi, c]) + \Omega_\rho * (-\phi + [\rho, c]) + \frac{1}{2} \Omega_c * [c, c] - \Omega_\phi * [\phi, c] \right).$$

(3.6)

The complete action

$$\Sigma = S_{BFYM} + S_{gf} + S_{ext}$$

(3.7)

satisfies the Slavnov-Taylor identity

$$S(\Sigma) = 0,$$

(3.8)

where

$$S(\Sigma) = \text{Tr} \int d^4x \left( \frac{\delta \Sigma}{\delta A_\mu} \frac{\delta A_\mu}{\delta \Omega_A} + \frac{1}{2} \frac{\delta \Sigma}{\delta B_{\mu\nu}} \frac{\delta B_{\mu\nu}}{\delta \Omega_{B_{\mu\nu}}} + \frac{\delta \Sigma}{\delta \eta_\mu} \frac{\delta \eta_\mu}{\delta \Omega_\eta} + \frac{\delta \Sigma}{\delta \psi_\mu} \frac{\delta \psi_\mu}{\delta \Omega_{\psi}} + \frac{\delta \Sigma}{\delta \rho} \frac{\delta \rho}{\delta \Omega_\rho} + \right.$$

$$\left. + \frac{\delta \Sigma}{\delta \psi} \frac{\delta \psi}{\delta \Omega_\phi} + \frac{\delta \Sigma}{\delta c} \frac{\delta c}{\delta \Omega_c} + \frac{\delta \Sigma}{\delta \phi} \frac{\delta \phi}{\delta \Omega_\phi} + h_A \frac{\delta \Sigma}{\delta \psi} \frac{\delta h_\psi}{\delta \psi_\mu} + h_{B\mu} \frac{\delta \Sigma}{\delta \psi} \frac{\delta h_\psi}{\delta \phi} + h_\psi \frac{\delta \Sigma}{\delta \psi} \frac{\delta h_\psi}{\delta \rho} + h_\eta \right),$$

(3.9)

and the following constraints:

- the gauge fixing conditions

$$\frac{\delta \Sigma}{\delta h_A} = \partial_\mu A_\mu , \quad \frac{\delta \Sigma}{\delta h_\psi} = \partial_\mu \psi_\mu ,$$

$$\frac{\delta \Sigma}{\delta h_B} = \partial_\mu B_{\mu\nu} - \partial_\nu u , \quad \frac{\delta \Sigma}{\delta h_\phi} = \partial_\mu \phi_\mu ,$$

$$\frac{\delta \Sigma}{\delta h_\eta} = \partial_\mu \eta_\mu , \quad \frac{\delta \Sigma}{\delta u} = \partial_\mu h_{B\mu} ;$$

(3.10)
the antighost equations
\[
\frac{\delta \Sigma}{\delta c} + \partial_\mu \frac{\delta \Sigma}{\delta \Omega_{A\mu}} = 0 , \quad \frac{\delta \Sigma}{\delta \bar{c}} + \partial_\mu \frac{\delta \Sigma}{\delta \Omega_{B\mu}} = \partial_\mu \bar{h}_\psi ,
\]
\[
\frac{\delta \Sigma}{\delta \bar{c}} + \partial_\mu \frac{\delta \Sigma}{\delta \Omega_{\eta\mu}} = 0 , \quad \frac{\delta \Sigma}{\delta \phi} - \partial_\mu \frac{\delta \Sigma}{\delta \Omega_{\psi\mu}} = 0 ;
\]

the ghost equations
\[
\int d^4x \left( \frac{\delta \Sigma}{\delta c} + [\bar{c}, \delta \frac{\delta \Sigma}{\delta h_A}] + [\bar{\psi}_\nu, \delta \frac{\delta \Sigma}{\delta h_{B\nu}}] + [u, \delta \frac{\delta \Sigma}{\delta h_\eta}] + [\bar{\rho}, \delta \frac{\delta \Sigma}{\delta h_\phi}] + [\bar{\phi}, \delta \frac{\delta \Sigma}{\delta h_\psi}] \right) = \Delta_{cl}^c ,
\]
\[
\int d^4x \left( \frac{\delta \Sigma}{\delta \phi} - [\bar{\phi}, \delta \frac{\delta \Sigma}{\delta h_A}] \right) = \Delta_{cl}^\phi ,
\]
\[
\int d^4x \left( \frac{\delta \Sigma}{\delta \bar{c}} + [A_\mu, \delta \frac{\delta \Sigma}{\delta \bar{c} A_\mu}] + [\bar{c}, \delta \frac{\delta \Sigma}{\delta \bar{c} \phi}] - [\Omega_{\psi\mu}, \delta \frac{\delta \Sigma}{\delta \Omega_{A\mu}}] + [\Omega_\phi, \delta \frac{\delta \Sigma}{\delta \Omega_c}] + 
\]
\[
+ [\bar{\phi}, \delta \frac{\delta \Sigma}{\delta \bar{c}}] - [h_\psi, \delta \frac{\delta \Sigma}{\delta h_A}] \right) = 0 ;
\]

where the classical breaking \( \Delta \) are
\[
\Delta_{cl}^c = \int d^4x \left( - [\Omega_{A\mu}, A_\mu] + \frac{1}{2} [\Omega_{B\mu\nu}, B_{\mu\nu}] + [\Omega_{\eta\mu}, \eta_\mu] - [\Omega_{\psi\mu}, \psi_\mu] - [\Omega_\rho, \rho] + 
\]
\[
+ [\Omega_\phi, \phi] - [\Omega_c, c] \right) ,
\]
\[
\Delta_{cl}^\phi = \int d^4x \left( [\Omega_{\psi\mu}, A_\mu] - \Omega_\rho + [\Omega_\phi, c] \right) ;
\]

since these breakings are linear in the quantum fields, they do not get radiative corrections;

the rigid gauge invariance
\[
W^{rig\Sigma} = \int d^4x \sum_\varphi \left[ \varphi, \frac{\delta \Sigma}{\delta \varphi} \right] = 0 \quad \varphi = \text{all fields} .
\]

Thanks to the antighost equations (3.11) and to the gauge-fixing conditions (3.10) we can redefine some sources:
\[
\hat{\Omega}_{A\mu} = \Omega_{A\mu} + \partial_\mu \bar{c} , \quad \hat{\Omega}_{\eta\mu} = \Omega_{\eta\mu} + \partial_\mu \bar{\rho} ,
\]
\[
\hat{\Omega}_{B\mu\nu} = \Omega_{B\mu\nu} + \partial_\mu \bar{\psi}_\nu , \quad \hat{\Omega}_{\psi\mu} = \Omega_{\psi\mu} - \partial_\mu \bar{\phi} ,
\]

and introduce the reduced action \( \hat{\Sigma} \):
\[
\hat{\Sigma}[A, B, \eta, c, \psi, \rho, \phi, \Omega, \bar{c}, \bar{\psi}, \bar{\rho}, \bar{\phi}, \bar{h}_\psi, \bar{h}_\rho, \bar{h}_\phi, \bar{h}_\psi, u, h_\psi, \Omega_A, \Omega_B, \Omega_\eta, \Omega_c, \Omega_\psi, \Omega_\rho, \Omega_\phi] =
\]
\[
= \Sigma[A, B, c, \bar{c}, h_A, \psi, \bar{\psi}, h_B, \rho, \bar{\rho}, h_\phi, \bar{h}_\psi, \psi, h_\psi, \Omega_A, \Omega_B, \Omega_\eta, \Omega_c, \Omega_\psi, \Omega_\rho, \Omega_\phi] + 
\]
\[
- \int \left( h_A \partial_\mu A_\mu + h_{B\mu\nu} \partial_\nu B_{\mu\nu} + h_\eta \partial_\mu \eta_\mu + h_\psi \partial_\mu \psi_\mu + h_\bar{\psi} \partial_\mu \bar{\psi}_\mu + uh_B \right) .
\]

The S.T. equation (3.8) becomes
\[
\tilde{B} \hat{\Sigma} = 0 ,
\]
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where

\[
\hat{B}_\Sigma = \text{Tr} \int d^4x \left( \frac{\delta \hat{\Sigma}}{\delta A_\mu} \frac{\delta}{\delta A_\mu} + \frac{\delta \hat{\Sigma}}{\delta \Omega_{A\mu}} \frac{\delta}{\delta \Omega_{A\mu}} + \frac{1}{2} \frac{\delta \hat{\Sigma}}{\delta \Omega_{B\mu \nu}} \frac{\delta}{\delta \Omega_{B\mu \nu}} + \frac{1}{2} \frac{\delta \hat{\Sigma}}{\delta \Omega_{B\mu \nu}} \frac{\delta}{\delta \Omega_{B\mu \nu}} + \frac{\delta \hat{\Sigma}}{\delta \eta_\mu} \frac{\delta}{\delta \Omega_{\eta_\mu}} + \frac{\delta \hat{\Sigma}}{\delta \psi} \frac{\delta}{\delta \phi} \right) + \\
\frac{\delta \hat{\Sigma}}{\delta \phi} \frac{\delta}{\delta \phi} + \frac{\delta \hat{\Sigma}}{\delta \Omega_\phi} \frac{\delta}{\delta \phi} + \frac{\delta \hat{\Sigma}}{\delta \eta_\phi} \frac{\delta}{\delta \phi} + \frac{\delta \hat{\Sigma}}{\delta \delta \Omega_\phi} \frac{\delta}{\delta \delta \Omega_\phi} + \frac{\delta \hat{\Sigma}}{\delta \delta \Omega_\phi} \frac{\delta}{\delta \delta \Omega_\phi} +
\right)
\] (3.21)

and

\[
\hat{B}_\Sigma \hat{B}_\Sigma = 0 .
\] (3.22)

The cohomology of \( \hat{B}_\Sigma \) is once again isomorphic to the YM case [8, 9], so that also this formulation of the BFYM is equivalent at quantum level to the Yang-Mills. Once again, the only anomaly term is the ABBJ one (equal to zero in our case) and the only physical renormalization is the coupling constant one through the \( \frac{1}{g^2} F^* F \) non trivial counterterm, which is equivalent to a multiple of \( g^2 (B - dA_\eta) \cdot (B - dA_\eta) \).

The analysis of the trivial counterterms in this case is more involved with respect to the gaussian formulation; indeed we have found that the counterterms allowed by the QAP are the BRST variations of the traces of the integrals of the following monomials, that are all parity-invariant:

| Monomial | Monomial | Monomial | Monomial |
|----------|----------|----------|----------|
| \( a_1 \hat{\Omega}_A * A \) | \( b_1 \hat{\Omega}_B * B \) | \( e_1 \Omega_c * c \) | \( m_1 \hat{\Omega}_\psi * \psi \) |
| \( a_2 \hat{\Omega}_A * \eta \) | \( b_2 \hat{\Omega}_B \) | \( e_2 \Omega_c * \rho \) | \( m_2 \hat{\Omega}_\psi * d\rho \) |
| \( b_3 \hat{\Omega}_B \) | \( b_4 \hat{\Omega}_B * d\eta \) | \( f_1 \Omega_\phi * \phi \) | \( m_3 \hat{\Omega}_\psi * [A, c] \) |
| \( d_1 \hat{\Omega}_\eta * \eta \) | \( b_5 \hat{\Omega}_B * [A, \eta] \) | \( f_2 \Omega_\phi * [c, c] \) | \( m_4 \hat{\Omega}_\psi * [\eta, c] \) |
| \( d_2 \hat{\Omega}_\eta * A \) | \( b_6 \hat{\Omega}_B * [A, \eta] \) | \( f_3 \Omega_\phi * [c, \rho] \) | \( m_5 \hat{\Omega}_\psi * d\eta \) |
| \( b_7 \hat{\Omega}_B B \) | \( f_4 \Omega_\phi * [\rho, \rho] \) | \( m_6 \hat{\Omega}_\psi * [A, \rho] \) |
| \( b_8 \hat{\Omega}_B * dA \) | \( f_5 \Omega_\phi * c \) | \( m_7 \hat{\Omega}_\psi * [\eta, \rho] \) |
| \( b_9 \hat{\Omega}_B * \frac{1}{2} [A, A] \) | \( l_1 \Omega_\rho * \rho \) |
| \( b_{10} \hat{\Omega}_B d\eta \) | \( l_2 \Omega_\rho * c \) |
| \( b_{11} \hat{\Omega}_B [A, \eta] \) |
| \( b_{12} \hat{\Omega}_B * [\eta, \eta] \) |

(3.23)

As in the gaussian formulation, we can extract a subset of counterterms that will get zero coefficient to all orders of perturbation by inspecting the tensorial structure of the 1PI functions. Indeed we can easily demonstrate that the number of \( \epsilon_{\mu\nu\rho} \) tensors in any 1PI function with \( E_A \) external legs \( A \), \( E_B \) legs \( B \) and \( E_\eta \) legs \( \eta \) is equal modulo 2 to \( E_B + E_\eta \). Therefore only the graphs with an odd number of legs \( B \) or \( \eta \) will be proportional to \( \epsilon_{\mu\nu\rho} \).
Therefore, as explained at the end of section 2, we get rid of a number of counterterms:

\[
b_7 = b_8 = b_9 = b_{10} = b_{11} = b_{12} = 0
\]
\[
a_2 = d_2 = m_4 = m_7 = 0
\]

(3.24)

at any order of perturbation. Another consequence is that the tensorial structure of the propagators and of the vertices present at the classical level is not changed by radiative corrections. Furthermore the ghost equations (3.12–3.14) exclude some of the remaining monomials requiring

\[
e_1 = e_2 = f_3 = f_4 = 0
\]

(3.25)

and impose the following conditions on the coefficients:

\[
\begin{aligned}
f_2 &= -l_2 = m_3 , \\
f_1 &= -l_1 , \\
a_1 + f_1 + m_1 - m_6 &= 0 ;
\end{aligned}
\]

(3.26)

hence the number of free parameters to be fixed by renormalization conditions is 14 (13 wave-function renormalizations and the coupling constant renormalization). By expressing the BRST variations of the remaining monomials in the same way as in (3.24) we can write at once the renormalization transformations, where the fields are understood to be renormalized till the order \( h^{n-1} \):

\[
A = A_R + a_1 h^n A_R ,
\]
\[
B = B_R + b_1 h^n B_R + b_2 h^n * dA_R + b_3 h^n * \frac{1}{2} [A_R, A_R] + b_4 h^n * d\eta_R +
\]
\[
+ b_5 h^n * [A_R, \eta_R] + b_6 h^n * [\eta_R, \eta_R] ,
\]
\[
\eta = \eta_R + d_1 h^n \eta_R ,
\]
\[
c = c_R ,
\]
\[
\psi = \psi_R - m_2 h^n dR - f_2 h^n [A_R, c_R] - m_3 h^n d\rho_R +
\]
\[
- (a_1 + f_1 + m_1) h^n [A_R, \rho_R] ,
\]
\[
\rho = \rho_R + f_1 h^n \rho_R + f_2 h^n c_R ,
\]
\[
\phi = \phi_R + f_1 h^n \phi_R + f_2 h^n [c_R, c_R] ,
\]
\[
\hat{\Omega}_A = \hat{\Omega}_{AR} - a_1 h^n \hat{\Omega}_{AR} - b_2 h^n * d\hat{\Omega}_{BR} + b_3 h^n * [A_R, \hat{\Omega}_{BR}] + b_4 h^n * [\eta_R, \hat{\Omega}_{BR}] +
\]
\[
- f_2 h^n [c_R, \hat{\Omega}_{BR}] - (a_1 + f_1 + m_1) h^n [\rho_R, \hat{\Omega}_{BR}] ,
\]
\[
\hat{\Omega}_B = \hat{\Omega}_{BR} - b_1 h^n \hat{\Omega}_{BR} ,
\]
\[
\hat{\Omega}_\eta = \hat{\Omega}_{\eta_R} - d_1 h^n \hat{\Omega}_{\eta_R} - b_4 h^n * d\hat{\Omega}_{BR} + b_5 h^n * [A_R, \hat{\Omega}_{BR}] + 2b_6 h^n * [\eta_R, \hat{\Omega}_{BR}] ,
\]
\[
\hat{\Omega}_c = \hat{\Omega}_{c_R} - f_2 h^n \hat{\Omega}_{c_R} + 2f_2 h^n [c, \hat{\Omega}_{c_R}] - m_2 h^n d\hat{\Omega}_{\psi_R} + f_2 h^n * [A_R, \hat{\Omega}_{\psi_R}] ,
\]
\[
\hat{\Omega}_\psi = \hat{\Omega}_{\psi_R} + m_1 h^n \hat{\Omega}_{\psi_R} ,
\]
\[
\Omega_\rho = \Omega_{\rho_R} - f_1 h^n \Omega_{\rho_R} - m_5 h^n d\hat{\Omega}_{\psi_R} + (a_1 + f_1 + m_1) h^n * [A_R, \hat{\Omega}_{\psi_R}] ,
\]
\[
\Omega_\phi = \Omega_{\phi_R} - f_1 h^n \Omega_{\phi_R} ,
\]
\[
g = g_R + h^n z_R g_R .
\]

(3.27)
4 Conclusions

We have analyzed the trivial counterterms of both the gaussian and the extended formulations of the BFYM theory in four dimensions, giving the full structure of the wave-function renormalizations and exploiting the restrictions arising in the Landau gauge. We have also found some restrictions to the tensorial structure of the 1PI functions in both formulations due to our choice of the classical Lagrangean, and consequently we have found a subclass of algebraically allowed counterterms that have nonetheless coefficients equal to zero to all orders of perturbation.

This ends the algebraic analysis of the perturbative renormalization of the theory, completing the results of [8, 9] who studied the anomaly and the physical parameters renormalization.

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