Lie Bracket Approximation of Extremum Seeking Systems

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Abstract

Extremum seeking feedback is a powerful method to steer a dynamical system to an extremum of a partially or completely unknown map. It often requires advanced system-theoretic tools to understand the qualitative behavior of extremum seeking systems. In this paper, a novel interpretation of extremum seeking is introduced. We show that the trajectories of an extremum seeking system can be approximated by the trajectories of a system which involves certain Lie brackets of the vector fields of the extremum seeking system. It turns out that the Lie bracket system directly reveals the optimizing behavior of the extremum seeking system. Specifically, we establish a theoretical foundation and prove that uniform asymptotic stability of the Lie bracket system implies practical uniform asymptotic stability of the corresponding extremum seeking system. We use the established results in order to prove local and semi-global practical uniform asymptotic stability of the extrema of a certain map for multi-agent extremum seeking systems.

1. Introduction

In diverse engineering applications one faces the problem of finding an extremum of a map without knowing its explicit analytic expression. Suppose, for example, one vehicle tries to minimize the distance to another vehicle. The only information available, is the distance to the other vehicle. Clearly, the distance does not provide a direction in which the vehicle has to move. However, it is intuitively clear that one can obtain a direction by using multiple measurements of the distance. Extremum seeking feedback exploits this procedure in a systematic way and can be used for steering dynamical systems to the extremum of an unknown map. Extremum seeking has a long history and has found many applications to diverse problems in control and communications (see \cite{13} and references therein).

In this paper, we provide a novel methodology to analyze extremum seeking systems which differs from commonly used techniques. Specifically, this work contains three main contributions.

First, we provide a novel view on extremum seeking by identifying the sinusoidal perturbation in the extremum seeking system as artificial inputs and by writing it in a certain input-affine form. Based on this input-affine form, we derive an approximate system which captures the behavior of the trajectories of the original extremum seeking system. It turns out that the approximate system can be represented by certain Lie brackets of the vector fields in the extremum seeking system which we call Lie bracket system. The proposed methodology is essentially different from results in the existing literature (see e.g. \cite{8}, \cite{9} and \cite{26}).

Second, we establish a theoretic foundation which is based on this novel viewpoint. We prove that the trajectories of a class of input-affine systems with certain inputs are approximated by the trajectories of the Lie bracket systems. Similar results concerning sinusoidal inputs are covered

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in [10] and were extended in [5, 11] to the class of periodic inputs. In [24] and [25] convergence of trajectories of a class of input-affine systems to the trajectories of more general Lie bracket systems was established. These results are closely related to our results. Furthermore, we prove under mild assumptions that semi-global (local) practical uniform asymptotic stability of a class of input-affine systems follows from global (local) uniform asymptotic stability of the corresponding Lie bracket systems. These results are based on [15] and [16]. Summarizing, to the authors best knowledge, the generality of the setup proposed herein was not addressed in the literature before.

Third, we apply the foregoing results to analyze the behavior and the stability properties of extremum seeking vehicles with single-integrator and unicycle dynamics and with static maps. We formulate a multi-agent setup consisting of extremum seeking systems where the individual nonlinear maps of the agents satisfy a certain relationship which assures the existence of a potential function. We use the established theoretical results to show that the set of extrema of the potential function is (locally or semi-globally) practically uniformly asymptotically stable for the multi-agent system. This multi-agent setup is strongly related to game theory and potential games (see [14]). In the single-agent case, this potential function coincides with the individual nonlinear map. Similar extremum seeking vehicles were analyzed in [29] and [30] by using averaging theory (see [7] and [19]). The authors proposed various extremum seeking feedbacks for different vehicle dynamics and provided a local stability analysis for quadratic maps. Using sinusoidal perturbations with vanishing gains, the authors of [22] and [23] were able to extend these results to prove almost sure convergence in the case of noisy measurements of the map. In a slightly different setup the authors of [26] considered feedbacks which stabilize the extremum of a scalar, dynamic input-output map and established semi-global practical stability of the overall system under some technical assumptions. Multi-agent extremum seeking setups which use similar game-theoretic approaches can be found in [20] and [21], where the agents seek a Nash equilibrium (see [17]). The authors proved almost sure convergence of the scheme but without explicit consideration of the global stability properties. A closely related result, which considers the local stability of Nash equilibrium seeking systems, can be found in [4].

Preliminary results of this paper were published in [2] and [3].

1.1. Organization

The remainder of this paper is structured as follows. In Section 2 we illustrate the main idea using a simple example. In Section 3 we present theoretical results which relate the stability properties of an input-affine system to its Lie bracket system. In Section 4 we apply these results to analyze stability properties of some typical multi-agent extremum seeking systems. Finally, in Section 5 we illustrate the results with examples and give a conclusion in Section 6.

1.2. Notation

\( \mathbb{N}_0 \) denotes the set of positive integers including zero. \( \mathbb{Q}_{++} \) denotes the set of positive rational numbers. The intervals of real number are denoted by \((a, b) = \{ x \in \mathbb{R} : a < x < b \}\), \([a, b) = \{ x \in \mathbb{R} : a \leq x < b \}\) and \([a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}\). Let \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k \), then we write \( f(\cdot, y) \) if we consider \( f \) as a function of the first argument only and for all \( y \in \mathbb{R}^m \). We denote by \( C^n \) with \( n \in \mathbb{N}_0 \) the set of \( n \) times continuously differentiable functions and by \( C^\infty \) the set of smooth function. The norm \( \|\cdot\| \) denotes the Euclidian norm. The Jacobian of a continuously differentiable function \( b \in C^1 : \mathbb{R}^n \to \mathbb{R}^m \) is denoted by

\[
\frac{\partial b(x)}{\partial x} := \begin{bmatrix}
\frac{\partial b_1(x)}{\partial x_1} & \cdots & \frac{\partial b_1(x)}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial b_m(x)}{\partial x_1} & \cdots & \frac{\partial b_m(x)}{\partial x_n}
\end{bmatrix}
\]
and the gradient of a continuously differentiable function \( J \in C^1 : \mathbb{R}^n \rightarrow \mathbb{R} \) is denoted by \( \nabla_x J(x) := \left[ \frac{\partial J(x)}{\partial x_1}, \ldots, \frac{\partial J(x)}{\partial x_n} \right]^T \). The Lie bracket of two vector fields \( f, g : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) with \( f(t, \cdot), g(t, \cdot) \) being continuously differentiable is defined by \([f, g](t, x) := \frac{\partial g(t, x)}{\partial t} f(t, x) - \frac{\partial f(t, x)}{\partial t} g(t, x)\). The \( a \)-neighborhood of a set \( S \subseteq \mathbb{R}^n \) with \( a \in (0, \infty) \) is denoted by \( U^a_S := \{ x \in \mathbb{R}^n : \inf_{y \in S} |x - y| < a \} \). \( \overline{U^a_S} \) denotes the closure of \( U^a_S \). A function \( u : \mathbb{R} \rightarrow \mathbb{R} \) is called measurable if it is Lebesgue-measurable. We use \( s \in C \) for the complex variable of the Laplace transformation if not indicated otherwise.

2. Main Idea

One simple extremum seeking feedback for static maps is shown in Fig. 1 (see also [8] and [30]). Suppose that the function \( f \in C^2 : \mathbb{R} \rightarrow \mathbb{R} \) admits a local, strict maximum at \( x^* \) and \( \alpha, \omega \in (0, \infty) \).

\[ \dot{x} = \alpha \sqrt{\omega} \cos(\omega t) + f(x) \sqrt{\omega} \sin(\omega t). \]  

The overall system can be written as

\[ \dot{x} = \alpha \sqrt{\omega} \cos(\omega t) + f(x) \sqrt{\omega} \sin(\omega t). \]  

The main idea is to identify \( \sin(\omega t) \) and \( \cos(\omega t) \) as artificial inputs, i.e. \( u_1(\omega t) := \cos(\omega t) \) and \( u_2(\omega t) := \sin(\omega t) \). Thus, we obtain an input-affine system of the form

\[ \dot{x} = b_1(x) \sqrt{\omega} u_1(\omega t) + b_2(x) \sqrt{\omega} u_2(\omega t) \]  

with \( b_1(x) = \alpha \) and \( b_2(x) = f(x) \). Interestingly, if one computes the so called Lie bracket system involving \([b_1, b_2]\), i.e.

\[ \dot{z} = \frac{1}{2} [b_1, b_2](z) = \alpha \nabla f(z), \]  

one sees that this system maximizes \( f(z) \). Having in mind, that trajectories resulting from sinusoidal inputs in \([1]\) can be approximated by trajectories of \([3]\) (see [3], [10], [11], [25]) allows us to establish a novel methodology to analyze extremum seeking systems.

The goal of this paper is to generalize this viewpoint to a larger class of extremum seeking systems. We derive a methodology which allows to analyze a broad class of extremum seeking systems by calculating their respective Lie bracket systems. The procedure can be summarized as follows: Write the extremum seeking system in input-affine form, calculate its corresponding Lie bracket system, prove its asymptotic stability and conclude practical asymptotic stability for the extremum seeking system.

3. Lie Bracket Approximation for a Class of Input-Affine Systems

In this section we consider a class of input-affine systems depending on a parameter and deliver general results for approximating the trajectories of such systems by the trajectories of their
respective Lie bracket systems. First, we state the definition of practical stability of a compact, invariant set for this class of systems. Second, we prove that their trajectories are approximated by the trajectories of their corresponding Lie bracket system for large values of the parameter. Third, we show how the stability properties of the input-affine system and the Lie bracket system are related. The results in this section, rely on a combination of results in [5], [10], [11], [25] and [15], [16].

3.1. Practical Stability

In the following, we define the notion of practical stability which is closely related to Lyapunov stability and applies to differential equations depending on a parameter. Throughout the paper, we denote this parameter as $\omega$. For related literature on this concept we refer to [15], [26], [27] and references therein.

Let $x(t) := x(t; t_0, x_0, \omega)$ denote the solution of the differential equation

$$\dot{x} = f_\omega(t, x)$$

through $x(t_0) = x_0$, where the vector field $f_\omega : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ depends on $\omega \in (0, \infty)$.

**Definition 1.** A compact set $S \subseteq \mathbb{R}^n$ is said to be practically uniformly stable for (4) if for every $\epsilon \in (0, \infty)$ there exists a $\delta \in (0, \infty)$ and $\omega_0 \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $\omega \in (\omega_0, \infty)$

$$x(t_0) \in U_\delta^S \Rightarrow x(t) \in U_\epsilon^S, t \in [t_0, \infty).$$

**Definition 2.** Let $\delta \in (0, \infty)$. A compact set $S \subseteq \mathbb{R}^n$ is said to be $\delta$-practically uniformly attractive for (4) if for every $\epsilon \in (0, \infty)$ there exists a $t_f \in [0, \infty)$ and $\omega_0$ such that for all $t_0 \in \mathbb{R}$ and all $\omega \in (\omega_0, \infty)$

$$x(t_0) \in U_\delta^S \Rightarrow x(t) \in U_\epsilon^S, t \in [t_0 + t_f, \infty).$$

**Definition 3.** A compact set $S \subseteq \mathbb{R}^n$ is said to be locally practically uniformly asymptotically stable for (4) if it is practically uniformly stable and there exists a $\delta \in (0, \infty)$ such that it is $\delta$-practically uniformly attractive.

**Definition 4.** Let $S \subseteq \mathbb{R}^n$ be a compact set. The solutions of (4) are said to be practically uniformly bounded if for every $\delta \in (0, \infty)$ there exists an $\epsilon \in (0, \infty)$ and $\omega_0 \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $\omega \in (\omega_0, \infty)$

$$x(t_0) \in U_\delta^S \Rightarrow x(t) \in U_\epsilon^S, t \in [t_0, \infty).$$

**Definition 5.** A compact set $S \subseteq \mathbb{R}^n$ is said to be semi-globally practically uniformly asymptotically stable for (4) if it is practically uniformly stable and for every $\delta \in (0, \infty)$ it is $\delta$-practically uniformly attractive. Furthermore the solutions of (4) must be practically uniformly bounded.

For the notion of stability in the sense of Lyapunov, we refer to e.g. [7], [15] and [18].

3.2. Lie Bracket Approximation

Throughout the paper, we consider the class of input-affine systems which can be written in the following form

$$\dot{x} = b_0(t, x) + \sum_{i=1}^m b_i(t, x)\sqrt{\omega}u_i(t, \omega t)$$

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with \( x(t_0) = x_0 \in \mathbb{R}^n \) and \( \omega \in (0, \infty) \). Next, we define a differential equation, which we call the Lie bracket system corresponding to (8)

\[
\dot{z} = b_0(t, z) + \sum_{i=1}^{m} [b_i, b_j](t, z) \nu_j(t) \tag{9}
\]

with

\[
\nu_j(t) = \frac{1}{T} \int_0^T u_j(t, \theta) \int_0^\theta u_i(t, \tau) d\tau d\theta. \tag{10}
\]

**Remark 1.** If \( u_i \) can be decomposed as \( u_i(t, \omega t) = r_i(t) \hat{u}_i(\omega t) \), \( i = 1, \ldots, m \), then (8) yields \( \dot{x} = b_0(t, x) + \sum_{i=1}^{m} \hat{b}_i(t, x) \sqrt{\omega} \hat{u}_i(\omega t) \) with \( \hat{b}_i(t, x) = b_i(t, x)r_i(t) \). Hence, in this case explicit dependence of \( u_i \) on the first argument can be omitted, which is usually done in the existing literature (see e.g. [10], [25]).

We impose the following assumptions on \( b_i \) and \( u_i \):

**A1** \( b_i \in C^2 : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( i = 0, \ldots, m \).

**A2** For every compact set \( K \subseteq \mathbb{R}^n \) there exist \( A_1, \ldots, A_6 \in [0, \infty) \) such that \( |b_i(t, x)| \leq A_1 \), \( |\partial b_i(t, x)| \leq A_2 \), \( |\partial^2 b_i(t, x)| \leq A_3 \), \( |\partial^2 b_i(t, x)| \leq A_4 \), \( |\partial^3 b_i(t, x)| \leq A_5 \), \( |\partial^4 b_i(t, x)| \leq A_6 \) for all \( x \in K, t \in \mathbb{R}, i = 0, \ldots, m, j = 1, \ldots, m, k = j, \ldots, m \).

**A3** \( u_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, \ldots, m \) are measurable functions. Moreover, for every \( i = 1, \ldots, m \) there exist constants \( L_i, M_i \in (0, \infty) \) such that \( |u_i(t_1, \theta) - u_i(t_2, \theta)| \leq L_i |t_1 - t_2| \) for all \( t_1, t_2 \in \mathbb{R} \) and such that \( \sup_{t, \theta \in \mathbb{R}} |u_i(t, \theta)| \leq M_i \).

**A4** \( u_i(t, \cdot) \) is \( T \)-periodic, i.e. \( u_i(t, \theta + T) = u_i(t, \theta) \), and has zero average, i.e. \( \int_0^T u_i(t, \theta) d\tau = 0 \), with \( T \in (0, \infty) \) for all \( t, \theta \in \mathbb{R}, i = 1, \ldots, m \).

**Remark 2.** Assumption A1 states a regularity assumption on the vector fields, which are usually assumed to be smooth in the case of extremum seeking systems, see [9] and [20].

**Remark 3.** Assumption A2 states that expressions involving \( b_i, i = 0, \ldots, m \) and their derivatives must be bounded uniformly in \( t \). A similar assumption was made in Eq. (2.2), Section 2 in [10].

**Remark 4.** Assumption A3 imposes measurability on \( u_i, i = 1, \ldots, m \) which is necessary to establish existence of solutions of (8) (see Theorem 3 in Appendix B). Alternatively, one could have imposed that the inputs \( u_i, i = 1, \ldots, m \) are continuous functions and argue using the existence and uniqueness theorem of Picard-Lindelöf (see [11]). However, this does not cover the case of piecewise continuous inputs, which might be interesting for further applications, i.e. replacing the sinusoids with piecewise constant functions in the extremum seeking systems.

**Remark 5.** Similarly as in [13] we impose in Assumption A4 the \( T \)-periodicity and zero average of \( u_i, i = 1, \ldots, m \), which is common in the averaging literature but also in the literature dealing with Lie brackets.

Finally, we introduce the set \( B \) of initial conditions for (9) which have uniformly bounded solutions, i.e.

\[
B = \{ z \in \mathbb{R}^n : \text{there exists an } A \in (0, \infty) \text{ such that for all } t_0 \in \mathbb{R} : |z(t)| \leq A, \quad t \in [t_0, \infty) \text{ with } z(t_0) = z \}. \tag{11}
\]
The set $\mathcal{B}$ is used in the proof of the following theorems and is crucial in order to assure existence of trajectories uniformly in $t_0$.

In the following, we state the main theorems which relate stability properties of the systems in (8) and (9). First, we state that trajectories of (8) are approximated by trajectories of (9) formulated in the sense of Hypothesis 2 in [15]. Related results are also presented in [5], [11] and [27]. However, we show for a larger class of inputs that the time interval can be made arbitrary large by choosing $\omega$ sufficiently large. Similar as in [15], we extend this result to infinite time-intervals and prove that the semi-global (local) practical uniform asymptotic stability of the input-affine system (8) directly follows from the global (local) uniform asymptotic stability of the corresponding Lie bracket system (9).

**Theorem 1.** Let Assumptions A1–A4 be satisfied. Then for every compact set $\mathcal{K} \subseteq \mathcal{B}$, for every $D \in (0, \infty)$ and for every $t_f \in (0, \infty)$, there exists an $\omega_0 \in (0, \infty)$ such that for every $\omega \in (\omega_0, \infty)$, for every $t_0 \in \mathbb{R}$ and every $x_0 \in \mathcal{K}$ there exist solutions $x, z : \mathbb{R} \to \mathbb{R}^n$ of (8) and (9) through $x(t_0) = z(t_0) = x_0$ which satisfy

$$|x(t) - z(t)| < D, \quad t \in [t_0, t_0 + t_f].$$

(12)

The proof of Theorem 1 goes along similar lines as indicated in B.3, p. 1941 in [16] but considers the more general case of inputs, which are characterized by Assumptions A3 and A4. It can be found in Appendix C.

**Theorem 2.** Let Assumptions A1–A4 be satisfied and suppose that a compact set $\mathcal{S}$ is locally uniformly asymptotically stable for (9). Then $\mathcal{S}$ is locally practically uniformly asymptotically stable for (8).

The proof can be found in Appendix D.

**Theorem 3.** Let Assumptions A1–A4 be satisfied and suppose that a compact set $\mathcal{S}$ is globally uniformly asymptotically stable for (9). Then $\mathcal{S}$ is semi-globally practically uniformly asymptotically stable for (8).

We omit the proof of Theorem 3 since it is already covered in [15] for the case of $\mathcal{S}$ being the origin. The proof can be directly applied here by replacing the Euclidian norm with the distance to the set $\mathcal{S}$.

**Remark 6.** The results above only capture stability and not performance and do not deliver a systematic way for choosing $\omega$. The notion of practical stability only requires the existence of $\omega_0$ without explicitly considering a specific value. As indicated by Theorem 1 the choice of $\omega$ depends on the set of initial conditions $\mathcal{K}$, the distance $D$ and the time $t_f$.

This is the basis for the result of the next section, which generalizes the ideas of the example in Section 2.

4. Lie Bracket Approximation of Extremum Seeking Systems

In this section, we show how the results from the previous section can be applied to multi-agent extremum seeking systems. As indicated in Section 2, the procedure is to write the extremum seeking system in the input-affine form, calculate the corresponding Lie bracket system and conclude using Theorems 2 and 3 the respective stability properties from the stability properties of the Lie bracket system.
In the following, we define a suitable framework for multi-agent extremum seeking systems. Given is a group of $N$ agents in the plane, which are meant to achieve a common goal formalized as seeking an extremum of a common map $F$. Specifically, let us enumerate the agents using the superscript $i$. The position of agent $i$ is denoted by $\bar{x}^i = [x_1^i, x_2^i] \in \mathbb{R}^2$. We define furthermore $\bar{x} := [x_1^1, x_2^1, \ldots, x_1^N, x_2^N] \top$ as the position vector of the overall system. Every agent is equipped with a specific extremum seeking feedback, which is defined below. We do not assume that all agents are seeking the extremum of the same map, but rather that each agent is equipped with an individual map $f^i : \mathbb{R}^{2N} \rightarrow \mathbb{R}$, $i = 1, \ldots, N$, which also depends on the states of the other agents and satisfies

\begin{align*}
B1 \quad & f^i \in C^2, 
\end{align*}

Furthermore, the individual maps have to satisfy the following assumption

\begin{align*}
B2 \quad & \text{There exists a function } F \in C^1 : \mathbb{R}^{2N} \rightarrow \mathbb{R} \text{ such that } \nabla_{\bar{x}} f^i(\bar{x}) = \nabla_{\bar{x}} F(\bar{x}), i = 1, \ldots, N, \bar{x} \in \mathbb{R}^{2N}.
\end{align*}

These conditions implies, that if every agent moves into the direction of the gradient of its individual map $f^i$ then it also moves in the direction of the gradient of $F$. We call this a potential function. The goal of the multi-agent system is to find the minimum (maximum) of the common map $F$ by only seeking the minimum (maximum) of the individual map $f^i$.

The following assumptions guarantee the existence of local (global) maxima of the potential function

\begin{align*}
B3 \quad & \text{There exists a nonempty and compact set } S_{loc} \subseteq \mathbb{R}^{2N} \text{ of strict local maxima and a } \delta \in (0, \infty) \text{ such that } F(\bar{x}^*) > F(\bar{x}) \text{ for all } \bar{x}^* \in S_{loc} \text{ and all } \bar{x} \in U^S_{loc} \setminus S_{loc}. \text{ Furthermore, } \nabla_{\bar{x}} F(\bar{x}) = 0 \text{ implies } \bar{x} \in S_{loc} \text{ for all } \bar{x} \in U^S_{loc}.
\end{align*}

\begin{align*}
B4 \quad & \text{There exists a nonempty and compact set } S_{glob} = \{ \bar{x} \in \mathbb{R}^{2N} : \bar{x} = \arg \max_{\bar{x} \in \mathbb{R}^{2N}} F(\bar{x}) \} \text{ of global maxima. Furthermore, } F(\bar{x}) \rightarrow -\infty \text{ for } |\bar{x}| \rightarrow \infty \text{ and } \nabla_{\bar{x}} F(\bar{x}) = 0 \text{ implies } \bar{x} \in S_{glob} \text{ for all } \bar{x} \in \mathbb{R}^{2N}.
\end{align*}

This framework originates from game theory, where Assumption B2 formally defines a potential game with potential function $F$. We refer to [14] for an introduction to potential games.

Remark 7. Under the assumptions above, the common goal can be formalized as the minimization (maximization) of the potential function $F$. There exist powerful tools to construct meaningful individual maps for a given potential function (see e.g. the approach using the so-called Wonderful Life Utility in [28]). The design should be done such that an optimization of the individual maps leads to an optimization of $F$, see [14]. For this case, even though the utility functions are designed, they usually depend on some parameters or functions (e.g. environmental conditions, individual agents’ properties) which are unknown a priori. A typical example for this scenario is the coverage control problem formulated as a potential game in [12] and [2]. These aspects justify the usage of extremum seeking in this setup. For a specific application of the extremum seeking in a potential game framework we refer to [2].

In the next subsection, we show how the framework above can be combined with extremum seeking agents. We saw in Section 2 that the trajectories of the extremum seeking system can be approximated by the trajectories of its corresponding Lie bracket system, which moves into the gradient direction of its individual map. We generalize this to the multi-agent case. If each agent is equipped with an extremum seeking feedback which drives it into the gradient direction of its individual map $f^i$, we expect with Assumption B2 that the overall system practically converges to an extremum of $F$. This is shown in the next subsection.
4.1. Multi-Agent Extremum Seeking

In the following, we show how extremum seeking can be applied to the above framework assuming single-integrator agent dynamics.

Consider the system in Fig. 2 which is motivated by a similar extremum seeking feedback in [30]. Since the agents move in the plane, there are two extremum seeking loops, one for each dimension. The perturbations are chosen to be sinusoidal, whose frequencies are chosen for each agent individually, as specified below. The high-pass filters \( G_i(s) = \frac{s}{s + 1}, \ i = 1, \ldots, N \) are introduced since they provide better transition behavior by removing possible constant offsets of the individual maps \( f^i, \ i = 1, \ldots, m. \) They introduce an additionally degree of freedom, but do not influence the stability of the overall system, as it can be seen in the proof of the following theorems.

Define \( \bar{x}_c := [x^1_c, \ldots, x^N_c]^\top \) and \( x := [\bar{x}_c^\top, \bar{x}_c^\top]^\top \) with \( x^i_c \) denoting the state of the filter \( G_i(s) = \frac{s}{s + 1}, \ i = 1, \ldots, N, \) i.e. in state space form we have \( \dot{x}^i_c = -x^i_c h^i + u^i \) and \( y^i = -x^i_c h^i + w^i \) with \( u^i = f^i(\bar{x}). \)

The differential equations describing the dynamics of agent \( i \) are given by

\[
\begin{align*}
\dot{x}^1_c &= c^i (f^i(\bar{x}) - x^i_c h^i) \sqrt{\omega^i u^i_1(\omega^i t) + \alpha^i \sqrt{\omega^i u^i_2(\omega^i t) + \alpha^i \sqrt{\omega^i u^i_1(\omega^i t)}}} \\
\dot{x}^2_c &= -c^i (f^i(\bar{x}) - x^i_c h^i) \sqrt{\omega^i u^i_2(\omega^i t) + \alpha^i \sqrt{\omega^i u^i_1(\omega^i t)}} \\
\dot{x}^i_c &= -x^i_c h^i + f^i(\bar{x}) \quad (13)
\end{align*}
\]

with \( u^i_1(\omega^i t) = \sin(\omega^i t), \ u^i_2(\omega^i t) = \cos(\omega^i t). \)

We need an additional assumption for the multi-agent case concerning the parameter \( \omega. \) We see in the proof of the next theorem that if the following assumption is satisfied, then some of the \( \nu^i_j \) in [10] vanish in the corresponding Lie bracket system. This can be assured by assuming

B5 \( \omega^i = a^i \omega \) and \( a^i \neq a^j, \ i \neq j, \ a^i \in \mathbb{Q}_{++}, \ \omega \in (0, \infty), \ h^i, \alpha^i, c^i \in (0, \infty), \ i, j = 1, \ldots, N. \)

Since the high-pass filter \( \frac{s}{s + 1}, \) introduces an additional state \( x^i_c, \) which has also to be taken into account in the analysis, we denote by

\[
\mathcal{E}^S := \{ \bar{x}_c \in \mathbb{R}^N : \bar{x}_c = [\frac{f^1(\bar{x})}{h^1}, \ldots, \frac{f^N(\bar{x})}{h^N}]^\top, \bar{x} \in S \} \tag{14}
\]
with $S$ is either $S_{\text{loc}}$ or $S_{\text{glob}}$, the set which is shown to be attractive for the filter states $x^i_e$, $i = 1, \ldots, N$.

**Theorem 4.** Consider a multi-agent system with $N$ agents, each one having dynamics given by (13). Let Assumptions B1 to B3 and B5 be satisfied, then the set $S_{\text{loc}} \times E^{S_{\text{loc}}}$ is locally practically uniformly asymptotically stable for the overall system with state $[\bar{x}^T, \bar{x}_e^T]^T$.

**Proof.** The proof can be split up into three steps. In the first step, we rewrite the system in the input-affine form. In the second step, we calculate the corresponding Lie bracket system and in the third step, we prove uniform asymptotic stability of the Lie bracket system. Theorem 2 then allows to conclude practical asymptotic stability for the original system.

In the first step, we rewrite the overall system with state $x = [\bar{x}^T, \bar{x}_e^T]^T$, where each component is described by the differential equations given in (13), as input-affine system of the form

$$
\dot{x} = \sum_{i=1}^{N} b^0_i(x) + b^i_1(x)\sqrt{\omega^i}\sin(\omega^i t) =: u^i_1(\omega^i t)
$$

with $b^0_i, b^i_1, b^i_2$ having non-zero entries only at positions corresponding to agent $i$ and zeros elsewhere, i.e. $b^0_i(x) = [0, \ldots, 0, -x^i_x + f^i(\bar{x}), 0, \ldots, 0]^T$, $b^i_1(x) = [0, \ldots, 0, c^i(f^i(\bar{x}) - x^i_h^i), \alpha^i, 0, 0, \ldots, 0]^T$, $b^i_2(x) = [0, \ldots, 0, \alpha^i, -c^i(f^i(\bar{x}) - x^i_h^i), 0, 0, \ldots, 0]^T$.

Note that due to Assumption B5 we have that $a_i$ can be written as $a_i = \frac{p_i}{q_i}$ with $p_i, q_i \in \mathbb{N}$ and define $q := \prod_{i=1}^{N} q_i$ and $\bar{\omega} = \frac{\omega}{q}$. Thus, $a_i \omega = \frac{p_i}{q_i} \omega = p_i \prod_{j \neq i} q_j \bar{\omega} = n^i \bar{\omega}$, $i = 1, \ldots, N$ and $j = 1, 2$ and for $n^i := p_i \prod_{j \neq i} q_j \in \mathbb{N}$. We rewrite (15) as follows

$$
\dot{x} = \sum_{i=1}^{N} b^0_i(x) + b^i_1(x)\sqrt{n^i \bar{\omega}} u^i_1(n^i \bar{\omega} t)
$$

(16)

It can directly be seen that $u^i_1(n^i \theta) \in \{\sin(n^i \theta), \cos(n^i \theta)\}$ are also $2\pi$-periodic in $n^i \bar{\omega} t$ for $i = 1, \ldots, N$ and $k = 1, 2$ and for $n^i \in \mathbb{N}$.

In the second step, we calculate the corresponding Lie bracket system as defined in (9). Define $\bar{z} := [z^1_1, z^1_2, \ldots, z^1_N]^T$, $\bar{e}_c := [z^1_c, \ldots, z^N_c]^T$ and $z := [\bar{z}^T, \bar{e}_c^T]^T$ and $\nu_1^{i,j} = \frac{1}{2\pi} \int_0^{2\pi} u^i_k(n^i \tau) \int_0^\tau u^j_l(n^j \theta) d\theta d\tau$ which are constant for all $i, j = 1, \ldots, N$ and $k, l = 1, 2$.

The crucial point now is that some Lie bracket in the differential equation of the overall system vanish due to the choice of different parameters $\omega^i$ for the agents. We obtain using Lemma A.2 (see Appendix A) that $\nu_1^{i,j} = -\frac{1}{2n^i}$ for all $n^i = n^j$ and $k = l$ and $\nu_1^{i,j} = 0$ otherwise. Thus, the Lie bracket system simplifies to

$$
\dot{z} = \sum_{i=1}^{N} b^0_i(z) - \frac{1}{2n^i} [\sqrt{n^i} b^i_1, \sqrt{n^i} b^i_2](z)
$$

(17)

$$
= \sum_{i=1}^{N} b^0_i(z) - \frac{1}{2} [b^i_1, b^i_2](z).
$$
Explicitly, for the states of agent $i$ we obtain

$$
\dot{z}_i = \frac{1}{2} \left( c^i \alpha^i \nabla_{z_i} f^i(\bar{z}) - c^{i2} \nabla_{z_i} f^{i}(\bar{z})(f^{i}(\bar{z}) - z_i^e h) \right)
$$

$$
\dot{z}_2 = \frac{1}{2} \left( c^i \alpha^i \nabla_{z_i} f^i(\bar{z}) + c^{i2} \nabla_{z_i} f^{i}(\bar{z})(f^{i}(\bar{z}) - z_i^e h) \right)
$$

$$
\dot{z}_e = - z_i^e h + f^i(\bar{z}).
$$

(18)

In the third step, we prove uniform asymptotic stability of the set $\mathcal{S}_{loc} \times \mathcal{E}_{loc}$ for (17). We first need to show existence of the solutions of (17) on $[t_0, \infty)$ for all $t_0 \in \mathbb{R}$. Note that the vector field in (17) is independent of $t$ and continuously differentiable in $z$. The existence and uniqueness theorem by Picard-Lindelöf (see [1]) guarantees that there exist a time $t_f \in (0, \infty)$ and a solution of $z : \mathbb{R} \to \mathbb{R}^{3N}$ defined on $[t_0, t_0 + t_f]$ for all $t_0 \in \mathbb{R}$. Note furthermore, that with $h^i \in (0, \infty)$ in Assumption B5, the differential equation for $z_e^i$, i.e. $\dot{z}_e^i = -h^i z_e^i + u$ with $u = f^i(\bar{z})$ in (18) is linear and its origin is exponentially stable for $u = 0$. Thus if $f^i(\bar{z}(t))$ is bounded then $z_e^i$ exists and is bounded with gain $\frac{1}{h^i}$ for all $i = 1, \ldots, N$, for all $t_0 \in \mathbb{R}$ and for all $t \in [t_0, \infty)$. Suppose now that $\mathcal{S}_{loc}$ is uniformly asymptotically stable for $\bar{z}$, then it can be shown that the set $\mathcal{E}_{loc}$ is uniformly asymptotically stable for $z_e^i$, $i = 1, \ldots, N$. Therefore, the set $\mathcal{S}_{loc} \times \mathcal{E}_{loc}$ is uniformly asymptotically stable for the overall system $[\bar{z}^T, z_e^T]^T$.

It is left to show that the set $\mathcal{S}_{loc}$ is uniformly asymptotically stable for $\bar{z}$. Choose $V := -F$ which is due to Assumption B3 a valid Lyapunov function in $\mathcal{U}_{\bar{z}}$. Observe that due to Assumption B2 we have that $\nabla_{\bar{z}} f^i(\bar{z}) = \nabla_{\bar{z}} F(\bar{z}), i = 1, \ldots, N$ and thus

$$
\dot{V} = - \sum_{i=1}^{N} \frac{c^i \alpha^i}{2} \left( \nabla_{z_i} F(\bar{z})^T \nabla_{z_i} F(\bar{z}) \right)
$$

$$
+ \nabla_{z_i} F(\bar{z})^T \nabla_{z_i} F(\bar{z}).
$$

(19)

Due to $c^i, \alpha^i \in (0, \infty), i = 1, \ldots, N$ in Assumption B5, we know that $V(\bar{z}(t))$ is decreasing along the trajectories of $\bar{z}(t)$ for all $\bar{z}(t_0) \in \mathcal{U}_{\bar{z}}$, all $t_0 \in \mathbb{R}$ and all $t \in [t_0, t_0 + t_f)$. We conclude that $|\bar{z}(t)|$ is bounded and therefore all $f^i(\bar{z}(t)), i = 1, \ldots, N$, are bounded for all $\bar{z}(t_0) \in \mathcal{U}_{\bar{z}}$, all $t_0 \in \mathbb{R}$ and all $t \in [t_0, \infty)$. Thus, $z(t) = [\bar{z}(t)^T, \bar{z}_e(t)^T]^T$ exists for all $t_0 \in \mathbb{R}$, for all $z(t_0) \in \mathcal{U}_{\bar{z}}$ and for all $t \in [t_0, \infty)$. Furthermore, we conclude with (19) and Assumption B3 that the set $\mathcal{S}_{loc}$ is locally uniformly asymptotically stable for the subsystem $\bar{z} = [z_1^e, z_2^e, \ldots, z_N^e, z_e^N]^T$ in (17).

Note that due to Assumption B1 and the fact that $u_{k}^i(n^i \hat{\theta}) \in \{\sin(n^i \hat{\theta}), \cos(n^i \hat{\theta})\}$ for $i = 1, \ldots, N$ and $k = 1, 2$ we conclude that Assumptions A1 to A4 are satisfied. Thus, with Theorem 2 the set $\mathcal{S}_{loc} \times \mathcal{E}_{loc}$ is locally practically uniformly asymptotically stable for the overall system with state $[\bar{z}^T, z_e^T]^T$. 

\textbf{Theorem 5.} Consider a multi-agent system with $N$ agents, each one having dynamics given by (13). Let Assumptions B1, B2, B4 and B5 be satisfied, then the set $\mathcal{S}_{loc} \times \mathcal{E}_{glob}$ is semi-globally practically uniformly asymptotically stable for the overall system with state $[\bar{x}^T, \bar{x}_e^T]^T$.

\textbf{Proof.} If Assumption B4 is satisfied then $\mathcal{S}_{glob}$ is a connected set containing the global maximum of $F$. Furthermore, $F$ is radially unbounded and with (19) we see that if $\dot{V}(\bar{z}) = 0$ implies $\bar{z}(t) \in \mathcal{S}_{glob}$. Thus, we conclude that $\mathcal{S}_{loc} \times \mathcal{E}_{glob}$ is globally uniformly asymptotically stable for (17) and thus with Theorem 3, it is semi-globally practically uniformly asymptotically stable for the overall system with state $[\bar{x}^T, \bar{x}_e^T]^T$. 

\hfill $\square$
In the following, we analyze the same setup but replace the single-integrator dynamics with unicycle dynamics as shown in Fig. 3. The setup is motivated by [29].

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{unicycle_dynamics.png}
\caption{Unicycle Dynamics}
\end{figure}

Let us consider the unicycle model for each agent given by the equations

\begin{align}
\dot{x}_1^i &= u^i \cos(x^i_\theta), \quad \dot{x}_2^i = u^i \sin(x^i_\theta), \quad \dot{x}_\theta^i = v^i. \tag{20}
\end{align}

The extremum seeking feedback controls only the forward velocity of the vehicle, whereas the angular velocity is constant, so that the inputs to each vehicle are

\[ u^i(t, x) = (c^i f^i(\bar{x}) - x^i_e h^i)\sqrt{\omega^i} \sin(\omega^i t) + \alpha^i \sqrt{\omega^i} \cos(\omega^i t) + \bar{x}^i \]

\[ v^i = \Omega^i. \]

We assume that \( x^i_\theta(t_0) = 0 \) and for all \( i = 1, \ldots, N \) and \( B6 \quad \Omega^i = d^i \Omega \) with \( d^i \in \mathbb{Q}_{++}, \Omega \in \mathbb{R} \setminus \{0\} \).

**Remark 8.** It becomes clear in the proof that the corresponding vector field of Lie bracket system is time-varying and vanishes at discrete points in time. Assumption B6 assures that the vector field is periodic, so that a LaSalle-like argument can be used in order to prove uniform asymptotic stability. Note, that the \( \Omega^i \)'s can be equal, whereas the \( \omega^i \)'s must be different for all agents.

By substituting the expressions for the inputs into (20) and replacing \( x^i_\theta(t) = \Omega^i t \) we obtain

\begin{align}
\dot{x}_1^i &= \left( c^i (f^i(\bar{x}) - x^i_e h^i)\sqrt{\omega^i} u^i_1(\omega^i t) \right. \\
&\quad + \left. \alpha^i \sqrt{\omega^i} u^i_2(\omega^i t) \right) \cos(\Omega^i t) \\
\dot{x}_2^i &= \left( c^i (f^i(\bar{x}) - x^i_e h^i)\sqrt{\omega^i} u^i_1(\omega^i t) \right. \\
&\quad + \left. \alpha^i \sqrt{\omega^i} u^i_2(\omega^i t) \right) \sin(\Omega^i t) \\
\dot{x}_e^i &= -x^i_e h^i + f^i(\bar{x})
\end{align}

with \( u^i_1(\omega^i t) = \sin(\omega^i t), \ u^i_2(\omega^i t) = \cos(\omega^i t). \)

**Theorem 6.** Consider a multi-agent system with \( N \) agents, each one having dynamics given by (21). Let Assumptions B1 to B3, B5 and B6 be satisfied, then the set \( S_{\text{loc}} \times E_{\text{loc}} \) is locally practically uniformly asymptotically stable for the overall system with state \( [\bar{x}^T, x_e^T]^T \).

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\textbf{Proof.} The proof goes along the same lines as the proof of Theorem 4. In the first step, we rewrite the overall system as input-affine system

\begin{equation}
\dot{x} = \sum_{i=1}^{N} b^i_0(x) + b^i_1(t, x)\sqrt{\omega^i}u^i_1(\omega^i t) + b^i_2(t, x)\sqrt{\omega^i}u^i_2(\omega^i t)
\end{equation}

where \( b^i_0, b^i_1, b^i_2 \) have non-zero entries only at the positions corresponding to agent \( i \) and zeros elsewhere, i.e. \( b^i_0(x) = [0, \ldots, 0, -x^i_h \dot{\alpha}, f^i(\dot{x}), 0, \ldots, 0]^\top, \) \( b^i_1(t, x) = [0, \ldots, (c^i(f^i(\dot{x}) - x^i_h \dot{\alpha})) \cos(\Omega^i t), (c^i(f^i(\dot{x}) - x^i_h \dot{\alpha})) \sin(\Omega^i t), 0, 0, \ldots, 0]^\top \) and \( b^i_2(t, x) = [0, \ldots, \alpha^i \cos(\Omega^i t), \alpha^i \sin(\Omega^i t), 0, 0, \ldots, 0]^\top. \)

Note that due to Assumption B5 \( a_i(t) \) can be written as \( a_i(t) = \frac{p_i}{q_i} \) with \( p_i, q_i \in \mathbb{N} \) and define \( q := \prod_{i=1}^{N} q_i \) and \( \tilde{\omega} = \frac{\omega}{q} \). Thus, \( a_i \omega = \frac{p_i}{q_i} \omega = p_i \prod_{j \neq i} q_j \tilde{\omega} = n^i \tilde{\omega}, i = 1, \ldots, N \) and \( j, k \in \mathbb{N} \). We rewrite (22) as follows

\begin{equation}
\dot{x} = \sum_{i=1}^{N} b^i_0(x) + b^i_1(t, x)\sqrt{n^i}\sqrt{\tilde{\omega}}u^i_1(n^i \tilde{\omega} t) + b^i_2(t, x)\sqrt{n^i}\sqrt{\tilde{\omega}}u^i_2(n^i \tilde{\omega} t).
\end{equation}

In the second step, we calculate the corresponding Lie bracket system as it was defined in (9),

\begin{equation}
\dot{z} = \sum_{i=1}^{N} b^i_0(z) - \frac{1}{2}[b^i_1, b^i_2](t, z).
\end{equation}

By the same reasoning as in the proof of Theorem 4, this yields for the state of agent \( i \)

\begin{equation}
\begin{aligned}
\dot{z}_1^i &= \frac{1}{2}(c^i \alpha^i \nabla_{z^1} f^i(\dot{z}) \cos^2(\Omega^i t) + c^i \alpha^i \nabla_{z^1} f^i(\dot{z}) \sin(\Omega^i t)) \\
\dot{z}_2^i &= \frac{1}{2}(c^i \alpha^i \nabla_{z^2} f^i(\dot{z}) \sin^2(\Omega^i t) + c^i \alpha^i \nabla_{z^2} f^i(\dot{z}) \sin(\Omega^i t)) \\
\dot{z}_v^i &= -z_v^i h^i + f^i(\dot{z}).
\end{aligned}
\end{equation}

In the third step, we prove uniform asymptotic stability of the set \( \mathcal{S}_{loc} \times \mathcal{E}_{loc} \) for the Lie bracket system of (23). Due to Assumption B3 we exploit the function \( V := -F \) as a Lyapunov function candidate which is valid in \( \mathcal{U}_{loc} \times \mathcal{E}_{loc} \). Observe that due to Assumption B2 we have that \( \nabla_{z^i} f^i(\dot{z}) = \nabla_{z^i} F(\dot{z}), i = 1, \ldots, N \), and thus

\begin{equation}
\begin{aligned}
\dot{V} &= -\sum_{i=1}^{N} \frac{c^i \alpha^i}{2} (\nabla_{z^1} F(\dot{z}) \cos(\Omega^i t) + \nabla_{z^2} F(\dot{z}) \sin(\Omega^i t))^\top \\
&\quad \cdot (\nabla_{z^1} F(\dot{z}) \cos(\Omega^i t) + \nabla_{z^2} F(\dot{z}) \sin(\Omega^i t)).
\end{aligned}
\end{equation}

We have that \( c^i, \alpha^i \in (0, \infty), i = 1, \ldots, N \) from Assumption B5, and thus \( \dot{V} \) is negative semi-definite. Observe that the vector field in (25) is time-varying and there are time-instances where
\( \dot{z}(t) = 0 \), but which are not steady-states for the system. Next, we make use of Assumption B6, which assures the existence of \( k^i, l^i \in \mathbb{N}, i = 1, \ldots, N \) such that \( d^i = \frac{k^i}{l^i} \). One can verify that the vector field of the overall system (24) consisting of \( N \) agents with system equations as in (25), is \( T \)-periodic with \( T = \frac{2\pi}{\Omega N} \prod_{i=1}^{N} l^i \). We can now use Theorem 1.3, p. 50 in [18] which is a LaSalle-like argument to conclude uniform asymptotic stability for periodic vector fields. It is left to show that no trajectory of (24) can stay identically in the set where \( \dot{V}(\bar{z}) = 0 \) except for \( \bar{z} \in S_{loc} \). To see this, observe that the summands of \( \dot{V} \) can only be equal to zero if \( \nabla_{z_1} F(\bar{z}(t)) \cos(\Omega t) + \nabla_{z_2} F(\bar{z}(t)) \sin(\Omega t) = 0 \), \( i = 1, \ldots, N \). On the set \( \dot{V}(\bar{z}) = 0 \) the differential equation yields \( \dot{z}_1 = \dot{z}_2 = 0 \) and therefore \( \dot{z}_1(t) = \text{const.} \) and \( \dot{z}_2(t) = \text{const.} \). Thus \( \nabla_{z_1} F(\bar{z}(t)) = \text{const.} \) and \( \nabla_{z_2} F(\bar{z}(t)) = \text{const.} \). But there are no constants \( a, b \in \mathbb{R} \) such that \( a \cos(\Omega t) + b \sin(\Omega t) = 0 \) for all \( t \in [t_0, \infty) \) except \( a = b = 0 \) and therefore \( \nabla_{z_1} F(\bar{z}(t)) = \nabla_{z_2} F(\bar{z}(t)) = 0 \). We conclude that the set \( S_{loc} \) is locally uniformly asymptotically stable for the subsystem \( \bar{z} \) in (25). Observe furthermore, that due to \( h^i \in (0, \infty) \) in Assumption B5, the differential equation for \( z^i \), i.e. \( \dot{z}^i = -h^i z^i + u \) with \( u = f^i(\bar{z}) \) in (18), is linear and its origin is exponentially stable for \( u = 0 \). Thus if \( f^i(\bar{z}(t)) \) is bounded then \( z^i(t) \) exists and is bounded with gain \( \frac{1}{h^i} \) for all \( i = 1, \ldots, N \), for all \( t_0 \in \mathbb{R} \) and for all \( t \in [t_0, \infty) \). Therefore, the set \( S_{loc} \times E_{Slab} \) is uniformly asymptotically stable for the overall system \( [\bar{x}^T, \bar{x}^e]^T \).

Note that due to Assumption B1 and the fact that \( u_k^i(n^i \bar{\theta}) \in \{ \sin(n^i \bar{\theta}), \cos(n^i \bar{\theta}) \} \) for \( i = 1, \ldots, N \) and \( k = 1, 2 \) we conclude that Assumptions A1 to A4 are satisfied. Thus, with Theorem 2 the set \( S_{loc} \times E_{Slab} \) is locally practically uniformly asymptotically stable for the overall system with state \( [\bar{x}^T, \bar{x}^e]^T \).

**Theorem 7.** Consider a multi-agent system with \( N \) agents, each one having dynamics given by (21). Let Assumptions B1, B2 and B4 to B6 be satisfied, then the set \( S_{loc} \times E_{Slab} \) is semi-globally practically uniformly asymptotically stable for the overall system with state \( [\bar{x}^T, \bar{x}^e]^T \).

The proof uses the same argumentation as the proof of Theorem 5.

**Remark 9.** It is important to observe that the amplitudes and frequencies of the sinusoids of the extremum seeking feedbacks in Fig. 2 and Fig. 3 are different, compared to the amplitudes in the corresponding schemes in the existing literature [20], [29] and [26]. The choice of \( \sqrt{\omega} \) for the amplitudes in combination with \( \omega \) for the frequency is crucial in order to obtain the Lie bracket system as approximation of the input-affine system. This is also pointed out on p. 241 in [10].

Comparing Theorem 7 to Theorem 10.5 on p. 417 in [7] we see that there is a close relationship between the results herein and averaging theory. The Lie bracket system in (9) can be seen as the average system of (8) (see [11]). In [30] and [29] the amplitudes of the perturbations are chosen to be \( \omega \) and 1, respectively, whereas the frequencies are chosen to be \( \omega \). Even though the schemes differ only in the choice of the amplitudes, the observation above let us expect that the average systems of the corresponding extremum seeking systems in [30] and [29] differ from the Lie bracket systems obtained here. A similar reasoning applies to [26] concerning the results on static maps, where the parameters do not influence the frequencies of the perturbations but only their amplitudes.

For results concerning averaging theory and practical stability, we refer to [27] and references therein.

**Remark 10.** Theorem 4 and Theorem 7 state local and semi-global practical uniform asymptotic stability for a group of \( N \) agents with single-integrator and unicycle dynamics. A special case is a single-agent extremum seeking system for which we have \( N = 1 \) and \( f^1 = F \). Furthermore, a similar analysis can be adopted in a straightforward fashion to the case of extremum seeking in one dimension by removing one feedback loop in Fig. 2.
Remark 11. In the presented schemes in Fig. 2, Fig. 3 and Fig. 4 it is not essential that the
perturbation signals are sinusoidal. Theorem 5 and Theorem 3 can be applied to analogous schemes
where the sinusoidal perturbations are replaced with other appropriately defined periodic signals
as long as they satisfy Assumptions A3 and A4. This also includes discontinuous and/or non-
differentiable signals such as square, triangle or sawtooth waveforms (see also Remark 4 above).

In the following, we show further advantages of the Lie bracket approximation using numerical
experiments.

5. Examples

In this section, we show numerical examples which illustrate the main results. First, we
compare for different values of \( \omega \) the trajectories of the single-integrator system of (15) with
its corresponding Lie bracket system (17). Second, using the Lie bracket system, we explain a
characteristic effects which we denote as characteristic points and which arise in the extremum
seeking with unicycle dynamics.

We consider a system of \( N = 3 \) agents and enumerate them with \( a, b, c \). We assign each agent
the following maps

\[
\begin{align*}
\dot{x}^a &= -\frac{1}{2}(x^a_1 - 1)^2 - \frac{1}{2}(x^a_2 - 1)^2 + x^b_1 - \frac{1}{2}x^b_2 + e^{-x^b_1 - x^b_2} - 10, \\
\dot{x}^b &= -\frac{1}{2}(x^b_1 + 1)^2 - \frac{1}{2}(x^b_2 + 1)^2 + \sin(x^a_1 + x^a_2) - 10, \\
\dot{x}^c &= -\frac{1}{2}(x^c_1 + 1)^2 - \frac{3}{2}(x^c_2 - 1)^2 + 10.
\end{align*}
\]

We choose the parameters \( h = h^a = h^b = h^c = 1, \alpha^a = \alpha^b = \alpha^c = 1, \epsilon^a = \epsilon^b = \epsilon^c = 0.3 \) and
the initial conditions \( \bar{x}_0^i, \bar{x}_w^i \) of the respective other agents.

Furthermore, we consider the quadratic function

\[
F(x) = -\frac{1}{2}(\bar{x} - \bar{x}^*)^\top Q(\bar{x} - \bar{x}^*)
\]  

(30)

where \( \bar{x}^* = [1, 1, -1, -1, -1, 1]^\top \) and the diagonal matrix \( Q = \text{diag}(1, 1, 1, 1, 1, 1) \). We can verify
that \( \nabla_{\bar{x}} f^i(\bar{x}) = \nabla_{\bar{x}} F(\bar{x}), i = a, b, c \) and we see that \( F \) is quadratic and attains its maximal value
at \( \bar{x}^* \). We expect from Theorems 3 and 7 that \( [(\bar{x}^*)^\top, \dot{f}_R(\bar{x}^*), \dot{f}_R(\bar{x}^*), \dot{f}_R(\bar{x}^*)]^\top \) is semi-globally
practically uniformly asymptotically stable for the extremum seeking systems.

In Fig. 4 the trajectories of the original and the Lie bracket systems are depicted with \( \omega = 10 \)
and \( \omega^a = \omega, \omega^b = 2\omega, \omega^c = 3\omega \). The trajectories of the Lie bracket system captures the qualitative
evolution of the trajectories of the original system. In Fig. 4 we see a simulation with the same
parameters but with \( \omega = 100 \).

These examples illustrate two properties. First, the trajectories of the original system approach
those of the Lie bracket system for large values of \( \omega \). This observation points up the result of
Theorem 1. Second, we deduce from Fig. 4 and Fig. 5 that even though each of the \( f^i \)'s, \( i = a, b, c \)
contains highly nonlinear terms depending on the states of the other agents, the overall system
practically converges even for small values of \( \omega \) to the expected extremum.
Figure 4: Comparison of trajectories of a three-agent single-integrator system and its respective Lie Bracket system, for $\omega = 10$

Figure 5: Comparison of trajectories of a three-agent single-integrator system and its respective Lie Bracket system, for $\omega = 100$
The same result can be observed in the case of unicycle dynamics and the same choice of parameters as above, with additionally $\Omega^a = 1$, $\Omega^b = 2$, $\Omega^c = 3$. In Fig. 6 the trajectories of the original and the Lie bracket systems are depicted for $\omega = 80$. Observe that the overall system practically converges as expected to the extremum. The trajectory of the extremum seeking system contains characteristic points, which also appear in the trajectory of the Lie bracket system. Apparently the vector field changes its direction abruptly. This can be explained by regarding the differential equation of the Lie bracket system in (25), which is time-varying and vanishes at the zero-crossing instances of the sinusoids.

6. Conclusion

In this work we developed a methodology, which led to a novel interpretation as well as to novel stability results for extremum seeking systems. By identifying the sinusoidal perturbations of the extremum seeking as artificial inputs, we were able to rewrite the system in a certain input-affine form and to relate this system to the so-called Lie bracket system, which nicely reveals the optimizing behavior of extremum seeking. The Lie bracket system viewpoint of extremum seeking allowed us to establish strong stability results for extremum seeking systems. We proved that the trajectories of systems belonging to a certain class of input-affine systems can be approximated by the trajectories of their corresponding Lie bracket system. Furthermore, we showed that global (local) uniform asymptotic stability of the Lie bracket system implies semi-global (local) practical uniform asymptotic stability of the input-affine system. We applied these results to a multi-agent extremum seeking system consisting of agents with either single-integrator or unicycle dynamics. Finally, the results are illustrated using numerical examples.

Appendix A. Existence and Uniqueness

Consider the differential equation

\[ \dot{x} = f(t, x) \]  

(A.1)
with \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) and with initial condition \( x(t_0) = x_0 \in \mathbb{R}^n \). If there exist a \( t_c \in (0, \infty) \) and a function \( x : \mathbb{R} \to \mathbb{R}^n \) such that
\[
x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau))d\tau \tag{A.2}
\]
and \( \dot{x}(t) = f(t, x(t)) \) for \( t \in [t_0, t_0 + t_c) \) except on a set of measure zero, then \( x(\cdot) = x(\cdot; t_0, x_0) \) is said to be a solution of \( \text{(A.1)} \). Thus, for every \( t \in \mathbb{T} \) through \( x(t_0) = x_0 \) defined on \([t_0, t_0 + t_c)\).

**Theorem 8** (see [1, 6]). Consider \( \text{(A.1)} \) and suppose for every compact sets \( \mathcal{T} \subseteq \mathbb{R} \) and \( \mathcal{X} \subseteq \mathbb{R}^n \) there exist measurable functions \( M, L : \mathbb{R} \to \mathbb{R} \) such that
\[
|f(t, x)| \leq M(t),
|f(t, x_1) - f(t, x_2)| \leq L(t)|x_1 - x_2|,
\]
\( t \in \mathcal{T}, x, x_1, x_2 \in \mathcal{X} \). Then for every \( t_0 \in \mathcal{T} \) and \( x(t_0) \in \mathcal{X} \) there exist a \( t_c \in (0, \infty) \) and a unique solution \( x : \mathbb{R} \to \mathbb{R}^n \) through \( x(t_0) \), which is defined on \([t_0, t_0 + t_c)\).

**Appendix B. Preliminary Lemmas**

**Lemma 1.** Let
\[
\nu_{ij} = \frac{1}{2\pi} \int_{0}^{2\pi} u^i(n^i \tau) \int_{0}^{\tau} u^j(n^j \theta) d\theta d\tau \tag{B.1}
\]
with \( n^i, n^j \in \mathbb{N}, u^i(n^i \tau) \in \{\sin(n^i \tau), \cos(n^i \tau)\} \), then
\[
\nu_{ij} = \begin{cases} 
\frac{1}{2\pi}, & n^i = n^j, u^i(n^i t) = \sin(n^i t), \\
\frac{1}{2\pi}, & n^i = n^j, u^i(n^i t) = \cos(n^i t), \\
\frac{1}{2\pi}, & n^i = n^j, u^i(n^i t) = \cos(n^i t), \\
0, & \text{else}
\end{cases} \tag{B.2}
\]

*Proof.* The result follows by a direct calculation. \( \square \)

**Lemma 2.** Let \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfy Assumption \[ \text{A.3} \]. Furthermore, \( u(t, \cdot) \) is \( T \)-periodic, i.e. \( u(t, \theta + T) = u(t, \theta) \) for some \( T \in (0, \infty) \) and all \( t, \theta \in \mathbb{R} \). Then, there exist \( k_1, k_2 \in [0, \infty) \) such that the inequality
\[
\left| \int_{t_0}^{t} \left( u(\tau, \omega \tau) - \frac{1}{T} \int_{0}^{T} u(\tau, \theta) d\theta \right) d\tau \right| \leq \frac{k_1 (t - t_0) + k_2}{\omega} \tag{B.3}
\]
is satisfied for all \( t_0 \in \mathbb{R} \) and all \( t \in [t_0, \infty) \). Furthermore, \( k_2 = 0 \) if \( \omega (t - t_0) \) is an integer multiple of \( T \), i.e. there exists an \( n \in \mathbb{N}_0 \) such that \( \omega (t - t_0) = T n \).

*Proof.* Using the fact that \( u(\tau, \omega \tau) = \frac{1}{T} \int_{0}^{T} u(\tau, \omega \tau) d\theta \) and applying the change of variables \( r = \omega \tau, dr = \omega d\tau \), the expression in
the norm of left hand-side in (B.3) yields
\[
\frac{1}{T \omega} \int_0^T \int_0^{\omega t} u\left(\frac{r}{\omega}, r\right) - u\left(\frac{r}{\omega}, \theta\right) dr d\theta. \tag{B.4}
\]

Since \( T \in (0, \infty) \) we can divide \([\omega t_0, \omega t]\) into \( n \in \mathbb{N}_0 \) pieces of length \( T \) such that
\[
\omega (t - t_0) = T n + \delta
\]
with \( 0 \leq \delta < T \) being the leftover piece. We obtain for (B.4)
\[
\frac{1}{T \omega} \sum_{k=0}^{n-1} \int_{0}^{T} \int_{\omega t_0 + T k}^{\omega t_0 + T (k+1)} u\left(\frac{r}{\omega}, r\right) - u\left(\frac{r}{\omega}, \theta\right) dr d\theta + R_1, \tag{B.5}
\]
where we introduced the left-over piece
\[
R_1 := \frac{1}{T \omega} \int_0^T \int_{\omega t_0 + T n}^{\omega t_0 + T n + \delta} u\left(\frac{r}{\omega}, r\right) - u\left(\frac{r}{\omega}, \theta\right) dr d\theta, \tag{B.6}
\]
which is considered later.

The integration interval in (B.5) is now shifted by introducing the change of variable \( s = r - \omega t_0 - Tk \), \( ds = dr \)
\[
\frac{1}{T \omega} \sum_{k=0}^{n-1} \int_0^T \int_0^T u\left(\frac{h_k(s)}{\omega}, h_k(s)\right) - u\left(\frac{h_k(s)}{\omega}, \theta\right) ds d\theta + R_1, \tag{B.7}
\]
with \( h_k(s) := s + \omega t_0 + Tk \). Since \( u(t, \cdot) \) is \( T \)-periodic, it follows that \( u\left(\frac{h_k(s)}{\omega}, h_k(s)\right) = u\left(\frac{h_k(s)}{\omega}, h_0(s)\right) \).
Thus, this simplifies to
\[
\frac{1}{T \omega} \sum_{k=0}^{n-1} \int_0^T \int_0^T u\left(\frac{h_k(s)}{\omega}, h_0(s)\right) - u\left(\frac{h_k(s)}{\omega}, \theta\right) ds d\theta + R_1 \tag{B.8}
\]
Note, that since the integration with respect to \( s \) and with respect to \( \theta \) is performed from 0 to \( T \)
and due to the periodicity of \( u(t, \cdot) \), we can add \( \int_0^T u\left(\frac{h_k(0)}{\omega}, \theta\right) d\theta \) and subtract \( \int_0^T u\left(\frac{h_k(0)}{\omega}, h_0(s)\right) ds \) which sums up to zero. We obtain
\[
\frac{1}{T \omega} \sum_{k=0}^{n-1} \int_0^T \int_0^T u\left(\frac{h_k(s)}{\omega}, h_0(s)\right) - u\left(\frac{h_k(0)}{\omega}, h_0(s)\right) \quad + \quad u\left(\frac{h_k(s)}{\omega}, \theta\right) - u\left(\frac{h_k(0)}{\omega}, \theta\right) ds d\theta + R_1. \tag{B.9}
\]
Assumption A3 yields the existence of \( L \in (0, \infty) \) such that the above expression can be bounded from above as follows \( |u\left(\frac{h_k(s)}{\omega}, h_0(s)\right) - u\left(\frac{h_k(0)}{\omega}, h_0(s)\right)| \leq \frac{L}{\omega} |s| \) and \( |u\left(\frac{h_k(0)}{\omega}, \theta\right) - u\left(\frac{h_k(s)}{\omega}, \theta\right)| \leq \frac{L}{\omega} |s| \).
Thus, (B.9) can be upper bounded by
\[
\frac{1}{T\omega} \sum_{k=0}^{n-1} \int_0^T \int_0^T 2L_\omega |s| ds d\theta + |R_1| \\
= \frac{T^2L_1}{\omega^2} n + |R_1|. \tag{B.10}
\]

We now consider the expression $R_1$ in (B.6). Assumption A3 yields the existence of $M \in (0, \infty)$ such that it can be upper bounded as follows
\[
|R_1| \leq \frac{1}{T\omega} \int_0^T \int_{\omega t_0 + Tn + \delta}^{\omega t_0 + Tn + \delta} 2M d\tau d\theta = \frac{2M\delta}{\omega}. \tag{B.11}
\]
Therefore, using the definition of $n = \frac{\omega(t-t_0)-\delta}{T}$ we obtain
\[
\frac{T^2L_1}{\omega^2} n + \frac{2M\delta}{\omega} = \frac{T^2L_1 \omega(t-t_0) - \delta}{\omega^2} + \frac{2M\delta}{\omega} \leq \frac{T[L(t-t_0) + 2M\delta]}{\omega}. \tag{B.12}
\]
Choosing $k_1 := TL$ and $k_2 := 2M\delta$ proves the first claim. If $\omega(t-t_0) = Tn$ then $\delta = 0$ and therefore, $k_2 = 0$ which proves the second claim.

**Lemma 3.** Let $u_i, u_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy Assumptions A3 and A4. Furthermore, let
\[
\tilde{u}_{ij}(t, \theta) := u_i(t, \theta) \int_0^\theta u_j(t, r) dr
\]
then there exist $M_{ij}, L_{ij} \in (0, \infty)$ such that
1. $\tilde{u}_{ij}(t, \theta)$ is $T$-periodic in $\theta$, i.e. $\tilde{u}_{ij}(t, \theta + T) = \tilde{u}_{ij}(t, \theta)$,
2. $\sup_{t \in \mathbb{R}} |\tilde{u}_{ij}(t, \omega t)| \leq M_{ij}$,
3. $|\tilde{u}_{ij}(t_1, \theta) - \tilde{u}_{ij}(t_2, \theta)| \leq L_{ij}|t_1 - t_2|$.

**Proof.** To (1): Consider $\tilde{u}_{ij}(t, \theta + T)$. Performing a change of variables $s = r - T$ and $ds = dr$ yields
\[
u_i(t, \theta + T) \int_0^{\theta + T} u_j(t, r) dr = u_i(t, \theta) \int_{-T}^\theta u_j(t, s + T) ds. \tag{B.14}
\]
Due to Assumption A4, $u_j(t, \cdot)$ has zero average. Thus, the expression above yields
\[
u_i(t, \theta) \int_0^\theta u_j(t, r) dr - u_i(t, \theta) \int_0^T u_j(t, r) dr = 0. \tag{B.15}
\]
To (2): Since $T \in (0, \infty)$ we can divide $[0, \theta]$ into $n \in \mathbb{N}_0$ pieces of length $T$ such that $\theta = Tn + \delta$ with $0 \leq \delta < T$ being the leftover piece. Due to Assumption A4, the first pieces are zero. Thus, we obtain

$$
|\tilde{u}_{ij}(t, \theta)| = |u_i(t, \theta) \sum_{k=0}^{n-1} \int_{kT}^{(k+1)T} u_j(t, r)dr = 0 + u_i(t, \theta) \int_{nT}^{nT+\delta} u_j(t, r)dr| 
$$

$$
\leq M_i M_j (\theta - nT) \leq M_i M_j T, \quad := M_{ij} 
$$

where the last step follows from Assumption A3.

To (3): Using the definition of $\tilde{u}_{ij}$ in (B.13) we can add and subtract the term $u_i(t_1, \theta) \int_0^\theta u_j(t_2, r)dr$ which yields

$$
|\tilde{u}_{ij}(t_1, \theta) - \tilde{u}_{ij}(t_2, \theta)| = |u_i(t_1, \theta) \int_0^\theta (u_j(t_1, r) - u_j(t_2, r)dr) + (u_i(t_1, \theta) - u_i(t_2, \theta)) \int_0^\theta u_j(t_2, r)dr|.
$$

Since $T \in (0, \infty)$ we can divide $[0, \theta]$ into $n \in \mathbb{N}_0$ pieces of length $T$ such that $\theta = Tn + \delta$ with $0 \leq \delta < T$ being the leftover piece. We obtain for the expression above

$$
= |u_i(t_1, \theta) \sum_{k=0}^{n-1} \int_{kT}^{(k+1)T} (u_j(t_1, r) - u_j(t_2, r)dr) + u_i(t_1, \theta) \int_{nT}^{nT+\delta} (u_j(t_1, r) - u_j(t_2, r)dr) + (u_i(t_1, \theta) - u_i(t_2, \theta)) \sum_{k=0}^{n-1} \int_{kT}^{(k+1)T} u_j(t_2, r)dr + (u_i(t_1, \theta) - u_i(t_2, \theta)) \int_{nT}^{nT+\delta} u_j(t_2, r)dr|.
$$

The first and third line in (B.18) sum up to zero due to Assumption A4. Furthermore, due to Assumptions A3 we obtain

$$
\leq |u_i(t_1, \theta)| \int_{nT}^{nT+\delta} L_j |t_1 - t_2| dr + L_i |t_1 - t_2| \int_{nT}^{nT+\delta} |u_j(t_2, r)| dr \leq (M_i L_j + L_i M_j) \delta |t_1 - t_2| \leq (M_i L_j + L_i M_j) T |t_1 - t_2|.
$$

\[=: L_{ij}\]
This was the last property we had to prove.

Lemma 4. Let \( u_i, u_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfy Assumptions \( A3 \) and \( A4 \). Then there exist \( k_1, k_2, k_3, k_4 \in [0, \infty) \) such that the following inequality

\[
\left| \int_{t_0}^{t} \left( \omega u_i(\tau, \omega \tau) \int_{t_0}^{\tau} u_j(s, \omega s) ds \right) d\tau \right| \\
- \frac{1}{T} \int_{0}^{T} \left[ u_i(\tau, \theta) \int_{0}^{\theta} u_j(\tau, r) dr \right] d\theta \\
\leq k_1 \frac{(t - t_0)^2}{\omega} + k_2 \frac{t - t_0}{\omega} + k_3 \frac{1}{\omega} + k_4 \frac{1}{\omega^3}
\]  

(B.20)
is satisfied for all \( t_0 \in \mathbb{R} \) and all \( t \in (t_0, \infty) \).

Proof. In order to use Lemma 2, we add and subtract \( \int_{t_0}^{t} \tilde{u}_{ij}(\tau, \omega \tau) d\tau = \int_{t_0}^{t} (u_i(\tau, \omega \tau) \int_{0}^{\omega \tau} u_j(\tau, r) dr) d\tau \) (see (B.13)) in the norm on the left-hand side of (B.20). Thus, it can be written as

\[
\int_{t_0}^{t} \left( \tilde{u}_{ij}(\tau, \omega \tau) - \frac{1}{T} \int_{0}^{T} \tilde{u}_{ij}(\tau, \theta) d\theta \right) d\tau + R 
\]  

(B.21)

with

\[
R := \int_{t_0}^{t} \left( \omega u_i(\tau, \omega \tau) \int_{t_0}^{\tau} u_j(s, \omega s) ds \right) \\
- \tilde{u}_{ij}(\tau, \omega \tau) \right) d\tau. 
\]  

(B.22)

Due to Lemma 3, the expression \( \tilde{u}_{ij} \) in (B.21) satisfies all assumptions needed in Lemma 2, which can now be applied in order to establish the existence of \( \tilde{k}_1, \tilde{k}_2 \in [0, \infty) \) such that

\[
\left| \int_{t_0}^{t} \tilde{u}_{ij}(\tau, \omega \tau) - \frac{1}{T} \int_{0}^{T} \tilde{u}_{ij}(\tau, \theta) d\theta d\tau \right| \\
\leq \tilde{k}_1 (t - t_0) + \tilde{k}_2 \frac{1}{\omega}. 
\]  

(B.23)

In the following we establish an upper bound for \( R \). We first split up the third integration interval in (B.22), i.e. \( \int_{0}^{\omega \tau} u_j(\tau, r) dr = \int_{0}^{\omega t_0} u_j(\tau, r) dr + \int_{\omega t_0}^{\omega \tau} u_j(\tau, r) dr \) and obtain

\[
R = \int_{t_0}^{t} \left( \omega u_i(\tau, \omega \tau) \int_{t_0}^{\tau} u_j(s, \omega s) ds \right) \\
- \frac{1}{\omega} \int_{0}^{\omega \tau} u_j(\tau, r) dr \right) d\tau + R_1, 
\]  

(B.24)

where we introduced

\[
R_1 := - \int_{t_0}^{t} \left( u_i(\tau, \omega \tau) \int_{0}^{\omega t_0} u_j(\tau, r) dr \right) d\tau. 
\]  

(B.25)
By the changes of variables \( p = \omega \tau, \, dp = \omega d\tau \) and \( q = \omega s, \, dq = \omega ds \) we obtain

\[
R = \frac{1}{\omega} \int_{t_0}^{t} \left( u_i \left( \frac{p}{\omega}, p \right) \int_{t_0}^{p} u_j \left( \frac{q}{\omega}, q \right) dq \right. \\
\left. - \int_{t_0}^{p} u_j \left( \frac{p}{\omega}, r \right) dr \right) dp + R_1.
\] (B.26)

Since the integration intervals with respect to \( r \) and \( q \) are now equal, we combine the two inner integrals and introduce \( I(q, p) := u_j \left( \frac{q}{\omega}, q \right) - u_j \left( \frac{p}{\omega}, q \right) \). Furthermore, we divide \([\omega t_0, \omega t]\) into \( n \in \mathbb{N}_0 \) pieces of length \( T \) such that \( \omega(t - t_0) = Tn + \delta \) with \( 0 \leq \delta < T \) being the leftover piece. Thus, we have

\[
R = \frac{1}{\omega} \sum_{k=0}^{n-1} \int_{\omega t_0 + Tk}^{\omega t_0 + T(k+1)} \left[ u_i \left( \frac{p}{\omega}, p \right) \int_{t_0}^{p} I(q, p) dq \right] dp \\
+ \frac{1}{\omega} \int_{\omega t_0 + Tn}^{\omega t_0 + Tn + \delta} \left[ u_i \left( \frac{p}{\omega}, p \right) \int_{t_0}^{p} I(q, p) dq \right] dp \\
+ R_1.
\] (B.27)

For reasons which become clear later, we again split up the integration interval \( \int_{t_0}^{p} I(q, p) dq = \int_{t_0}^{t_0 + Tk} I(q, p) dq + \int_{t_0 + Tk}^{p} I(q, p) dq, \, k = 1, \ldots, n \) and obtain \( R = R_1 + R_2 + R_3 \), where we define

\[
R_2 := \frac{1}{\omega} \sum_{k=0}^{n-1} \int_{\omega t_0 + Tk}^{\omega t_0 + T(k+1)} \left[ u_i \left( \frac{p}{\omega}, p \right) \int_{t_0}^{p} I(q, p) dq \right] dp \\
+ \frac{1}{\omega} \int_{\omega t_0 + Tn}^{\omega t_0 + Tn + \delta} \left[ u_i \left( \frac{p}{\omega}, p \right) \int_{t_0}^{p} I(q, p) dq \right] dp
\] (B.28)

and

\[
R_3 := \frac{1}{\omega} \sum_{k=0}^{n-1} \int_{\omega t_0 + Tk}^{\omega t_0 + T(k+1)} \left( u_i \left( \frac{p}{\omega}, p \right) \int_{t_0}^{p} I(q, p) dq \right) dp \\
+ \frac{1}{\omega} \int_{\omega t_0 + Tn}^{\omega t_0 + Tn + \delta} \left( u_i \left( \frac{p}{\omega}, p \right) \int_{t_0}^{p} I(q, p) dq \right) dp.
\] (B.29)

Each part is now treated separately.

For \( R_1 \) we split up the second integration interval \([0, \omega t_0]\) by introducing \( l \in \mathbb{N}_0 \) such that \( \omega t_0 = Tl + \epsilon \) with \( 0 \leq \epsilon < T \) being the left-over piece. We know from Assumption A4 that \( u_j(t, \cdot) \) has zero average. Thus (B.25) simplifies to

\[
R_1 = - \int_{t_0}^{t} \left( u_i \left( \tau, \omega \tau \right) \int_{Tl}^{Tl+\epsilon} u_j \left( \tau, s \right) ds \right) d\tau
\] (B.30)

and with Assumption A3 i.e. \( |\int_{Tl}^{Tl+\epsilon} u_j \left( \tau, s \right) ds| \leq M_j \epsilon \) the expression \( \bar{u}_j(\tau, \theta) := u_i(\tau, \theta) \int_{Tl}^{Tl+\epsilon} u_j(\tau, s) ds \) is bounded, \( \bar{u}_j(\tau, \cdot) \) is \( T \)-periodic with zero mean and \( \bar{u}_j(\cdot, \theta) \) is Lipschitz continuous which follows from the same reasoning as in the proof of part (3) of Lemma 3 (i.e. (B.17) to (B.19)). Thus it satisfies all assumptions of Lemma 2. We conclude with the first statement of Lemma 2 that there
exist \( \tilde{k}_3, \tilde{k}_4 \in [0, \infty) \) such that

\[
|R_1| \leq \frac{\tilde{k}_3 (t - t_0) + \tilde{k}_4}{\omega}.
\]

We now turn to \( R_2 \). Since \( u_j(t, \cdot) \) is \( T \)-periodic with zero mean, we have that \( \int_{\omega t_0}^{\omega t_0 + T} u_j\left(\frac{r}{\omega}, q\right) dq = 0 \) and therefore \( \int_{\omega t_0}^{\omega t_0 + T} I(q, p) dq = \int_{\omega t_0}^{\omega t_0 + T} u_j\left(\frac{r}{\omega}, q\right) dq k = 1, \ldots, n \). The crucial point now is that this integral does not depend on \( p \) anymore. Thus the expression \( R_2 \) can be written as

\[
R_2 = \frac{1}{\omega} \sum_{k=0}^{n-1} \int_{\omega t_0}^{\omega t_0 + T(k+1)} u_i\left(\frac{p}{\omega}, p\right) dp \int_{\omega t_0}^{\omega t_0 + T} u_j\left(\frac{q}{\omega}, q\right) dq
+ \frac{1}{\omega} \int_{\omega t_0}^{\omega t_0 + T + \delta} u_i\left(\frac{p}{\omega}, p\right) dp \int_{\omega t_0}^{\omega t_0 + T + \delta} u_j\left(\frac{q}{\omega}, q\right) dq.
\]

Substituting \( r = \frac{p}{\omega}, dr = \frac{dp}{\omega} \) and \( s = \frac{q}{\omega}, ds = \frac{dq}{\omega} \) yields

\[
R_2 = \omega \sum_{k=0}^{n-1} \int_{t_0}^{t_0 + T(k+1)} u_i(r, \omega r) dr \int_{t_0}^{t_0 + T} u_j(s, \omega s) ds
+ \omega \int_{t_0}^{t_0 + T + \delta} u_i(r, \omega r) dr \int_{t_0}^{t_0 + T + \delta} u_j(s, \omega s) ds.
\]

Since \( u_i \) is bounded by \( M_i \in (0, \infty) \) and both \( u_i, u_j \) satisfy the conditions of Lemma \( 2 \) and \( \omega(t_0 + T_k - t_0 - T(k+1)) = T \) as well as \( \omega(t_0 + T_k - t_0) = T k, k = 1, \ldots, n \), we obtain with the second statement of Lemma \( 2 \) that there exist \( \tilde{k}_5, \tilde{k}_6, \tilde{k}_7 \in [0, \infty) \) such that

\[
|R_2| \leq \omega \sum_{k=0}^{n-1} \tilde{k}_5 \frac{T \tilde{k}_6 T_k}{\omega^2} + M_i \delta \frac{\tilde{k}_7 T}{\omega^2}
\leq \frac{\tilde{k}_5 \tilde{k}_6}{\omega} \left( \frac{(t - t_0)^2}{\omega} + \frac{\delta^2}{\omega^2} \right) + \frac{\tilde{k}_7 M_i T(t - t_0)}{\omega},
\]

where we have made use of \( 0 \leq \delta < T \) and the definition of \( n = \frac{\omega(t-t_0)-\delta}{T} \) above.

For \( R_3 \) we proceed as follows. Note that due to Assumption A3, we have that \( |I(q, p)| \leq \frac{L_j}{\omega} |q - p| \) and furthermore, \( |u_i(t, \theta)| \leq M_i \), for all \( t, \theta \in \mathbb{R} \) and all \( i, j = 1, \ldots, m \). Thus, we obtain for \( |R_3| \)

\[
|R_3| \leq \frac{1}{\omega} \sum_{k=0}^{n-1} \int_{\omega t_0}^{\omega t_0 + T(k+1)} M_i \int_{\omega t_0 + T k}^{p} \frac{L_j}{\omega} |q - p| dq dp
+ \frac{1}{\omega} \int_{\omega t_0}^{\omega t_0 + T + \delta} M_i \int_{\omega t_0 + T + \delta}^{p} \frac{L_j}{\omega} |q - p| dq dp.
\]

The crucial point now is that the lower integration limits of both integrations are equal. One can verify that after the substitutions \( s = q - \omega t_0 - T k, ds = dq \) and \( r = p - \omega t_0 - T k, dr = dp \),
that for every compact set \( K \subseteq \mathbb{R} \) and \( k \) bound for the left hand-side of (B.20) with \( k \).

Appendix C. Proof of Theorem 1

Consider the vector field \( f_\omega(t,x) = b_0(t,x) + \sum_{i=1}^{m} b_i(t,x) \sqrt{\omega u_i(t,\omega t)} \) in (8) and note that due to Assumptions A1 and A2 \( f_\omega(t,\cdot) \) is continuously differentiable and \( f_\omega(\cdot,x) \) is measurable. Furthermore, with Assumption A2 we have that for every compact set \( K \subseteq \mathbb{R}^n \) and every \( \omega \in (\omega_0,\infty) \) there exist an \( M, L \in [0,\infty) \) such that \[ |b_0(t,x) + \sum_{i=1}^{m} b_i(t,x) \sqrt{\omega u_i(t,\omega t)}| \leq M \text{ and such that } |b_0(t,x_1) + \sum_{i=1}^{m} b_i(t,x_1) \sqrt{\omega u_i(t,\omega t)} - b_0(t,x_2) - \sum_{i=1}^{m} b_i(t,x_2) \sqrt{\omega u_i(t,\omega t)}| \leq |b_0(t,x_1) - b_0(t,x_2)| + M_1 \sqrt{\omega} \sum_{i=1}^{m} (b_i(t,x_1) - b_i(t,x_2))| \leq L|x_1 - x_2|, \quad t \in \mathbb{R}, x_1, x_2 \in K. \] Thus, with Theorem 8 in Appendix A, there exist a \( t_\epsilon \in (0,\infty) \) and a unique solution \( x : \mathbb{R} \to \mathbb{R}^n \) of (8) that can be written as

\[
x(t) = x_0 + \int_{t_0}^{t} b_0(\tau, x) + \sum_{i=1}^{m} b_i(\tau, x) \sqrt{\omega u_i(\tau, \omega \tau)} d\tau \quad (C.1)
\]

with \( t \in [t_0, t_0 + t_\epsilon] \) and \( x_0 = x(t_0) \). Since, \( x(t) \) is absolutely continuous on \([t_0, t_0 + t_\epsilon]\) we can perform a partial integration for each \( b_i(\tau, x) \sqrt{\omega u_i(\tau, \omega \tau)}, i = 1, \ldots, m \) with derivative \( \frac{\partial b_i(\tau, x)}{\partial \tau} = \frac{\partial b_i(\tau, x)}{\partial x} \frac{\partial x}{\partial \tau} + \frac{\partial b_i(\tau, x)}{\partial \tau} \) and obtain

\[
x(t) = x_0 + \int_{t_0}^{t} \left[ b_0(\tau, x) - \sqrt{\omega} \sum_{i=1}^{m} \left( \frac{\partial b_i(\tau, x)}{\partial x} \frac{\partial x}{\partial \tau} + \frac{\partial b_i(\tau, x)}{\partial \tau} \right) U_i(t_0, \tau) \right] d\tau + \sqrt{\omega} \sum_{i=1}^{m} b_i(t, x(t)) U_i(t_0, t) \quad (C.2)
\]
with $U_i(t_0, t) := \int_{t_0}^t u_i(r, \omega r)dr$. Since $\dot{x}(t) = b_0(t, x(t)) + \sum_{i=1}^m b_i(t, x(t))\sqrt{\omega}u_i(t, \omega t)$ for almost all $t$, we obtain

$$x(t) = x_0 + \int_{t_0}^t \left[ b_0(\tau, x) - \omega \sum_{i,j=1}^m \frac{\partial b_i(\tau, x)}{\partial x} b_j(\tau, x)u_j(\tau, \omega \tau)U_i(t_0, \tau) \right] d\tau + R_1 + R_2$$

where we introduced

$$R_1 := -\sqrt{\omega} \int_{t_0}^t \sum_{i=1}^m \left( \frac{\partial b_i(\tau, x)}{\partial x} b_0(\tau, x) \right) U_i(t_0, \tau) d\tau$$

$$R_2 := \sqrt{\omega} \sum_{i=1}^m b_i(t, x(t)) U_i(t_0, t).$$

Adding and subtracting the expression $\omega \int_{t_0}^t \sum_{i=1}^m \sum_{j=i+1}^m \frac{\partial b_j(\tau, x)}{\partial x} b_i(\tau, x)u_j(\tau, \omega \tau)U_i(t_0, \tau) d\tau$ yields

$$x(t) = x_0 + \int_{t_0}^t \left[ b_0(\tau, x) + \omega \sum_{i=1}^m \sum_{j=i+1}^m [b_i, b_j](\tau, x)u_j(\tau, \omega \tau)U_i(t_0, \tau) \right] d\tau + R_1 + R_2 + R_3 + R_4$$

with

$$R_3 := -\omega \int_{t_0}^t \sum_{i=1}^m \frac{\partial b_i(\tau, x)}{\partial x} b_i(\tau, x) \frac{1}{2} \frac{\partial U_i(t_0, \tau)^2}{\partial \tau} d\tau$$

$$R_4 := -\omega \int_{t_0}^t \sum_{i=1}^m \sum_{j=1}^{i-1} \frac{\partial b_i(\tau, x)}{\partial x} b_j(\tau, x) \cdot \frac{\partial U_i(t_0, \tau)U_j(t_0, \tau)}{\partial \tau} d\tau$$

and by using $\frac{\partial U_i(t_0, \tau)U_j(t_0, \tau)}{\partial \tau} = u_i(\tau, \omega \tau)U_j(t_0, \tau) + u_j(\tau, \omega \tau)U_i(t_0, \tau)$, $i = 1, \ldots, m$, $j = 1, \ldots, m$. Note that, $R_3$ and $R_4$ contain the rest terms after relabeling the indices. Furthermore, $R_3$ contains the terms where $i = j$, which is treated as a special case.

We now turn to (9). By assumption, the solution $z : \mathbb{R} \to \mathbb{R}^n$ of (9) exists and $z(t)$ is bounded
for \( t = [t_0, \infty) \) and for all \( z(t_0) = z_0 \in \mathcal{B} \). Thus, \( z(t) \) that can be written as

\[
z(t) = z_0 + \int_{t_0}^{t} b_0(\tau, z) + \sum_{j=1}^{m} [b_i, b_j](\tau, z)\nu_j(\tau)d\tau \tag{C.9}
\]

with \( t \in [t_0, \infty) \), \( z(t_0) = z_0 \) and \( \nu_j(t) \) as defined in [9].

In the following, we show that the distance between \( x(t) \) and \( z(t) \) with \( z(t_0) = x(t_0) = x_0 \) can be made arbitrary small on a finite time interval with \( \omega \) chosen sufficiently large. Choose \( z(t_0) = x(t_0) = x_0 \in \mathcal{K} \) and since \( \mathcal{K} \subseteq \mathcal{B} \) is compact and since solutions initialized in \( \mathcal{B} \) stay uniformly bounded, there exists a compact set \( \mathcal{M} \subseteq \mathbb{R}^n \) such that for all \( t_0 \in \mathbb{R} \) and all \( z(t_0) \in \mathcal{K} \) we have \( z(t) \in \mathcal{M} \), \( t \in [t_0, \infty) \). Next, define a tubular set around \( z(t) \), i.e. \( \mathcal{O}(t) = \{ x \in \mathbb{R}^n : |x - z(t)| \leq D \} \), \( t \in [t_0, \infty) \). Suppose now that \( t_e < t_f \) where \( [t_0, t_0 + t_e) \) is the maximal interval of existence of \( x \). Thus \( |x(t)| \to \infty \) for \( t \to t_e \) and by the compactness of \( \mathcal{O}(t) \), \( t \in [t_0, \infty) \) there exists a time \( t_D < t_e \) where \( x(t) \) leaves \( \mathcal{O}(t) \). Notice that \( x(t) \) is contained in \( \overline{\mathcal{U}}_{D}^{M} \) up to time \( t_0 + t_D \) (see Fig. C.7). We show now that \( t_D \geq t_f \) for \( \omega \) sufficiently large.

Consider the distance between \( x(t) \) and \( z(t) \) through \( z(t_0) = x(t_0) \) for \( t \in [t_0, t_0 + t_D] \). We add and subtract the expression \( \int_{t_0}^{t} [b_i, b_j](\tau, x)\nu_j(\tau)d\tau \) and obtain

\[
x(t) - z(t) = \int_{t_0}^{t} b_0(\tau, x) - b_0(\tau, z) \\
+ \sum_{j=1}^{m} \left( [b_i, b_j](\tau, x) - [b_i, b_j](\tau, z) \right)\nu_j(\tau)d\tau \\
+ R_1 + R_2 + R_3 + R_4 + R_5 \tag{C.10}
\]

with

\[
R_5 := \sum_{i=1}^{m} \sum_{j=i+1}^{m} \int_{t_0}^{t} [b_i, b_j](\tau, x)V_{ji}(\tau, \omega \tau)d\tau. \tag{C.11}
\]

and \( V_{ji}(\tau, \omega \tau) = \omega u_j(\tau, \omega \tau)U_j(t_0, \tau) - \nu_j(\tau) \).

Figure C.7: \( x(t) \) stays in \( \overline{\mathcal{U}}_{D}^{M} \) for all \( t \in [t_0, t_0 + t_D] \)

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Suppose for that there exist \( k \in [0, \infty) \) and \( \omega_0^* \in (0, \infty) \) such that for every \( \omega \in (\omega_0^*, \infty) \) we have \( \sum_{i=1}^{5} |R_i| \leq \frac{k}{\sqrt{\omega}} \). Note that with Assumption A3 we have that \( |\nu_{ji}(\tau)| \leq \frac{1}{T} \int_0^T \sum_{i=1}^{M} [b_i, b_j](\tau, \theta) du_i(\tau, \theta) \leq \frac{1}{2} M L_2 M_1 T \) and thus, with Assumptions A1 we have that for every compact set \( \bar{U}_M^2 \) an there exists an \( L \in (0, \infty) \) such that \( |b_0(\tau, x) - b_0(\tau, z) + \sum_{j=i+1}^{m} \sum_{j=i+1}^{m} (b_i, b_j)(\tau, x) - (b_i, b_j)(\tau, z))| \leq L |x(\tau) - z(\tau)| \) and therefore

\[
|x(t) - z(t)| \leq \int_{t_0}^{t} L |x(\tau) - z(\tau)| d\tau + \frac{k}{\sqrt{\omega}} \tag{C.12}
\]

with \( x(\tau), z(\tau) \in \bar{U}_M^2, t \in [t_0, t_0 + t_D] \) and \( \omega \in (\omega_0^*, \infty) \). Using the Lemma of Gronwall-Bellman we get

\[
|x(t) - z(t)| \leq \frac{k}{\sqrt{\omega}} e^{L(t-t_0)}. \tag{C.13}
\]

Suppose now that \( t_D < t_f \) and choose \( \omega_0 = \max\{\omega_0, \omega_0^*, \omega_0^*\} \), which yields with (C.13) \( |x(t) - z(t)| \leq \frac{D}{2} e^{L(t-t_0-t_f)} \). Thus \( |x(t) - z(t)| < D \) for all \( t \in [t_0, t_0 + t_D] \) and all \( \omega \in [\omega_0, \infty) \) which contradicts the assumption that \( t_D \) is the time when \( x(t) \) leaves the tube \( \mathcal{O}(t) \). Furthermore, \( |x(t) - z(t)| < D \) for all \( t \in [t_0, t_0 + t_f] \) which contradicts \( t_D < t_f \). We conclude that for every compact set \( \mathcal{K} \subseteq \mathcal{B} \), for every \( D \in (0, \infty) \) and every \( t_f \in (0, \infty) \) there exist an \( \omega_0 \in (0, \infty) \) such that for every \( \omega \in [\omega_0, \infty) \), for every \( t_0 \in \mathbb{R} \) and for every \( x_0 \in \mathcal{K} \) there exist solutions \( x, z : \mathbb{R} \rightarrow \mathbb{R}^n \) through \( x(t_0) = z(t_0) = x_0 \) which satisfy \( |x(t) - z(t)| \leq D, t \in [t_0, t_0 + t_f] \).

It remains to show that there exist \( k \in [0, \infty) \) and \( \omega_0^* \in (0, \infty) \) such that for every \( \omega \in (\omega_0^*, \infty) \) we have \( \sum_{i=1}^{5} |R_i| \leq \frac{k}{\sqrt{\omega}} \). Following the same lines as in (16) the expressions \( |R_i|, i = 1, \ldots, 5 \) decay uniformly to zero with \( \omega \to \infty \) on compact sets. Due to space limitations, this is shown only for \( R_5 \). The procedure is similar for \( R_1 \) to \( R_4 \).

Due to Assumption A1 the vector fields \( b_i, i = 1, \ldots, m \) are twice continuously differentiable and thus we can perform a partial integration which yields for \( R_5 \)

\[
R_5 = \sum_{j=1}^{m} \sum_{j=1}^{m} \left[ \int_{t_0}^{t} V_{ji}(\tau, \omega \theta) d\tau \right]
\]

\[
- \int_{t_0}^{t} \left[ \left( \frac{\partial[b_i, b_j](\tau, x)}{\partial x} \right) \dot{x} + \frac{\partial[b_i, b_j](\tau, x)}{\partial \tau} \right] \int_{t_0}^{T} V_{ji}(\theta, \omega \theta) d\theta d\tau. \tag{C.14}
\]
Substituting \( \dot{x}(\tau) = b_0(x, x(\tau)) + \sum_{i=1}^{m} b_i(x, x(\tau)) \sqrt{\omega} u_i(\tau, \omega) \) yields

\[
R_5 = \sum_{i=1}^{m} |b_i| \int_{t_0}^{t} V_j(\tau, \omega) d\tau
\]

\[
-\int_{t_0}^{t} \left[ \left( \frac{\partial b_i}{\partial x}(\tau, x) \right) \left( b_0(x, x) + \sum_{i=1}^{m} b_i(x, x) \sqrt{\omega} u_i(\tau, \omega) \right) \right] + \int_{t_0}^{t} V_j(\theta, \omega) d\theta ] d\tau.
\]

(C.15)

For every \( \bar{U}_D^M \) there exist due to Assumptions A1, A2 and A3 constants \( C_1, \ldots, C_4 \in [0, \infty) \) such that \( |b_i| \leq C_1, |\frac{\partial b_i}{\partial x}| \leq C_2, |b_0(x, x) + \sum_{i=1}^{m} b_i(x, x) \sqrt{\omega} u_i(\tau, \omega)| \leq C_3 \) and \( |\sum_{i=1}^{m} b_i(x, x) u_i(\tau, \omega)| \leq C_4 \) for every \( \tau, t \in \mathbb{R} \) and every \( x \in \bar{U}_D^M \). This yields

\[
|R_5| \leq \sum_{i=1}^{m} C_1 \left| \int_{t_0}^{t} V_j(\tau, \omega) d\tau \right|
\]

\[
+ \int_{t_0}^{t} C_2(C_3 + \sqrt{\omega} C_4) \left| \int_{t_0}^{T} V_j(\theta, \omega) d\theta \right| d\tau.
\]

(C.16)

Furthermore, the functions \( u_i \), \( i = 1, \ldots, m \) satisfy the assumptions of Lemma 4 and thus, there exist \( k_1^{j_1}, k_2^{j_2}, k_3^{j_3}, k_4^{j_4} \in [0, \infty) \) such that \( |\int_{t_0}^{t} V_j(\tau, \omega) d\tau| \leq k_1^{j_1}(t-t_0)^{\frac{3}{2}} + k_2^{j_2}(t-t_0)^{\frac{3}{2}} + k_3^{j_3}(t-t_0) + k_4^{j_4}(t-t_0)^{\frac{3}{2}} \) and also \( \int_{t_0}^{t} |\int_{t_0}^{T} V_j(\tau, \omega) d\theta| d\tau \leq k_1^{j_1}(t-t_0)^{\frac{3}{2}} + k_2^{j_2}(t-t_0)^{\frac{3}{2}} + k_3^{j_3}(t-t_0) + k_4^{j_4}(t-t_0)^{\frac{3}{2}} \). From these estimates it becomes clear that for every compact set \( \bar{U}_D^M \subseteq \mathbb{R}^n \), for every \( t_0, t_D \in \mathbb{R} \) there exist \( k_{0.5} \in [0, \infty) \) and \( \omega_{0.5} \in (0, \infty) \) such that for all \( \omega \in (\omega_{0.5}, \infty) \) and for all \( t \in [t_0, t_0 + t_D] \) we have \( |R_5| \leq k_{0.5}^{j_1} \). Estimates for \( R_1 \) and \( R_2 \) follow immediately from Assumptions A1 to A4 and Lemma 2. For the expressions \( R_3 \) and \( R_4 \) a partial integration yields a similar result. Thus, there exist \( k_{0.1}, j_{0.1} \), such that \( |R_3| \leq \frac{k_{0.1}^{j_1}}{\sqrt{\omega}}, \omega \in (\omega_{0.1}, \infty), i = 1, \ldots, 5 \) respectively. Summarizing, for every compact set \( \bar{U}_D^M \subseteq \mathbb{R} \), for every \( t_0, t_D \in \mathbb{R} \) there exist \( w_0 = \max_i \omega_0 \) and \( k = \max_i \{ k_{0.1} \} \) such that for all \( \omega \in (\omega_0, \infty) \) and for all \( t \in [t_0, t_0 + t_D] \)

\[
\sum_{i=1}^{5} |R_i| \leq \frac{k}{\sqrt{\omega}}.
\]

(C.17)

Appendix D. Proof of Theorem 2

This follows the same argumentation as in [15] but extends it to the stability of a compact set.

**Practical uniform stability** We show now that \( S \) is locally practically uniformly stable for [3], see Definition 1. Take an arbitrary \( \epsilon \in (0, \infty) \) and let \( B_2 \in (0, \epsilon) \). First observer that, since the system [3] is uniformly stable with respect to \( S \), there exists a \( \delta \in (0, \delta) \) with \( \bar{U}_D^S \) being contained in the region of attraction of \( S \) for [3] such that for all \( t_0 \in \mathbb{R} \)

\[
z(t_0) \in \bar{U}_D^S \Rightarrow z(t) \in \bar{U}_D^{S_B}, \ t \in [t_0, \infty).
\]

(D.1)

Second observer that, since the set \( S \) is uniformly attractive for [3] and \( 0 < \delta < \bar{\delta} \) we have that for
every $B_1 \in (0, \delta)$ there exists a time $t_f \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$

$$z(t_0) \in U^S_\delta \Rightarrow z(t) \in U^S_{t_f}, \; t \in [t_0 + t_f, \infty), \quad (D.2)$$

which is true due to the fact, that $0 < \delta < \bar{\delta}$. Let $D = \min\{\delta - B_1, \epsilon - B_2\}$, $K = \bar{U}^S_\delta$ and $t_f$ determined above. Due to Theorem 1, there exists an $\omega_{0,1} \in (0, \infty)$ such that for all $\omega \in (\omega_{0,1}, \infty)$ and all $x(t_0) \in \bar{U}^S_\delta$, $|x(t) - z(t)| < D$, $t \in [t_0, t_0 + t_f]$. This together with (D.1) and (D.2) yields for all $\omega \in (\omega_{0,1}, \infty)$

$$x(t_0) \in U^S_\delta \Rightarrow x(t) \in U^S_{t_f}, \quad t \in [t_0, t_0 + t_f], \quad (D.3)$$

Since $x(t_0 + t_f) \in U^S_\delta$ a repeated application of the procedure with another solution $z(t)$ of (9) through $z(t_0 + t_f)$ yields for all $t_0 \in \mathbb{R}$, for all $\omega \in (\omega_{0,1}, \infty)$

$$x(t_0) \in U^S_\delta \Rightarrow x(t) \in U^S_{t_f}, \; t \in [t_0, \infty). \quad (D.4)$$

**Practical uniform attractiveness** We show now that there exists a $\delta \in (0, \infty)$ such that $S$ is $\delta$-practically uniformly attractive for (8), see Definition 2. Choose some $\delta \in (0, \bar{\delta})$ with $U^S_\delta$ being contained in the region of attraction of $S$ for (9) and choose some $\epsilon \in (0, \infty)$. By practical uniform stability proven above, there exist $C_3 \in (0, \infty)$ and $\omega_{0,1} \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $\omega \in (\omega_{0,1}, \infty)$

$$x(t_0) \in U^S_{C_3} \Rightarrow x(t) \in U^S_{t_f}, \; t \in [t_0, \infty). \quad (D.5)$$

Let $B_3 \in (0, C_3)$. Since the set $S$ is uniformly attractive for (9), there exists a $t_f \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$

$$z(t_0) \in U^S_\delta \Rightarrow z(t) \in U^S_{B_3}, \; t \in [t_0 + t_f, \infty). \quad (D.6)$$

Due to Theorem 1 with $K = \bar{U}^S_\delta$, $D = C_3 - B_3$ and $t_f$ defined above yields the existence of $\omega_{0,2} \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $\omega \in (\omega_{0,2}, \infty)$ and all $x(t_0) \in \bar{U}^S_\delta$ we have that $|x(t) - z(t)| < D$, $t \in [t_0, t_0 + t_f]$. This estimate together with (D.6) yield for all $t_0 \in \mathbb{R}$ and for all $\omega \in (\omega_{0,2}, \infty)$

$$x(t_0) \in U^S_\delta \Rightarrow x(t_0 + t_f) \in U^S_{C_3}, \quad (D.7)$$

This, together with (D.5), leads for all $t_0 \in \mathbb{R}$, for all $\omega \in (\omega_{0,2}, \infty)$ where $\omega_0 = \max\{\omega_{0,1}, \omega_{0,2}\}$ to

$$x(t_0) \in U^S_\delta \Rightarrow x(t) \in U^S_{t_f}, \; t \in [t_0 + t_f, \infty). \quad (D.8)$$

This is the last property we had to prove.

**References**

[1] E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. McGraw-Hill Book Company, Inc., 1955.

[2] H. B. Dürr, M. S. Stanković, and K. H. Johansson. Distributed positioning of autonomous.
mobile sensors with application to coverage control. In Proceedings of the 2011 American Control Conference, San Francisco, pages 4822 – 4827, 2011.

[3] H. B. Dürr, M. S. Stanković, and K. H. Johansson. A Lie bracket approximation for extremum seeking vehicles. In Proceedings of the 18th IFAC World Congress, Milano, pages 11393 – 11398, 2011.

[4] P. Frihauf, M. Krstic, and T. Basar. Nash equilibrium seeking in noncooperative games. IEEE Transactions on Automatic Control, 57(5):1192 – 1207, 2012.

[5] L. Gurvits. Averaging approach to nonholonomic motion planning. In In Proceedings of the IEEE International Conference on Robotics and Automation, volume 2, pages 2541 – 2546, 1992.

[6] J. K. Hale. Ordinary Differential Equations. volume XXI of Pure and Applied Mathematics. Wiley-Interscience, 1969.

[7] H. K. Khalil. Nonlinear systems. Prentice Hall, Upper Saddle River, N.J., 3rd edition, 2002.

[8] M. Krstic and K. B. Ariyur. Real-Time Optimization by Extremum-Seeking Control. Wiley-Interscience, 2003.

[9] M. Krstic and H. H. Wang. Stability of extremum seeking feedback for general nonlinear dynamic systems. Automatica, 36(4):595 – 601, 2000.

[10] J. Kurzweil and J. Jarník. Limit processes in ordinary differential equations. Journal of Applied Mathematics and Physics, 38:241 – 256, 1987.

[11] Z. Li and L. Gurvits. Smooth time-periodic solutions for non-holonomic motion planning. In Z. Li and J. F. Canny, editors, Nonholonomic Motion Planning, pages 53 – 108. Kluwer Academic Publishers, 1992.

[12] J. R. Marden, G. Arslan, and J. S. Shamma. Cooperative control and potential games. IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics, 39(6):1393 – 1407, Dec. 2009.

[13] W. H. Moase, C. Manzie, D. Nesic, and I.M.Y. Mareels. Extremum seeking from 1922 to 2010. In 29th Chinese Control Conference, pages 14 – 26, July 2010.

[14] D. Monderer and L. S. Shapley. Potential games. Games and Economic Behavior, 14(1):124 – 143, 1996.

[15] L. Moreau and D. Aeyels. Practical stability and stabilization. IEEE Transactions on Automatic Control, 45(8):1554 – 1558, 2000.

[16] L. Moreau and D. Aeyels. Trajectory-based local approximations of ordinary differential equations. SIAM Journal on Control and Optimization, 41(6):1922 – 1945, 2003.

[17] J. Nash. Non-cooperative games. The Annals of Mathematics, 54(2):286 – 295, 1951.

[18] N. Rouche, P. Habets, and M. Laloy. Stability Theory by Liapunov’s Direct Method, volume 22 of Applied Mathematical Sciences. Springer Berlin / Heidelberg, 1977.

[19] J. A. Sanders, F. Verhulst, and J. A. Murdock. Averaging methods in nonlinear dynamical systems. Springer, 2nd edition, 2007.
[20] M. S. Stanković, K. H. Johansson, and D. M. Stipanović. Distributed seeking of Nash equilibria in mobile sensor networks. In Proceedings of the 49th IEEE Conference Decision and Control, Atlanta, pages 5598 – 5603, 2010.

[21] M. S. Stanković, K. H. Johansson, and D. M. Stipanović. Distributed seeking of Nash equilibria with applications to mobile sensor networks. IEEE Transactions on Automatic Control, 57(4):904 – 919, 2012.

[22] M. S. Stanković and D. M. Stipanović. Discrete time extremum seeking by autonomous vehicles in a stochastic environment. In Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, pages 4541 – 4546, 2009.

[23] M. S. Stanković and D. M. Stipanović. Extremum seeking under stochastic noise and applications to mobile sensors. Automatica, 46:1243 – 1251, 2010.

[24] H. J. Sussmann and W. Liu. Limits of highly oscillatory controls and approximation of general paths by admissible trajectories. In Proceedings of the 30th IEEE Conference on Decision and Control, pages 437 – 442, 1991.

[25] H. J. Sussmann and W. Liu. Lie bracket extensions and averaging: The single-bracket case. In Z. Li and J. F. Canny, editors, Nonholonomic Motion Planning, pages 109 – 147. Kluwer Academic Publishers, 1992.

[26] Y. Tan, D. Nešić, and I. Mareels. On non-local stability properties of extremum seeking control. Automatica, 42:889 – 903, 2006.

[27] A. R. Teel, J. Peuteman, and D. Aeyels. Global asymptotic stability for the averaged implies semi-global practical asymptotic stability for the actual. In Proceedings of the 37th IEEE Conference on Decision and Control, volume 2, pages 1458 – 1463, 1998.

[28] D. H. Wolpert. Theory of collective intelligence. Technical report, June 21 2003. NASA Ames Research Center, Moffett Field, CA 95033.

[29] C. Zhang, D. Arnold, N. Ghods, A. Siranosian, and M. Krstić. Source seeking with nonholonomic unicycle without position measurement and with tuning of forward velocity. Systems and Control Letters, 56(3):245 – 252, 2007.

[30] C. Zhang, A. Siranosian, and M. Krstić. Extremum seeking for moderately unstable systems and for autonomous vehicle target tracking without position measurements. Automatica, 43(10):1832 – 1839, 2007.