R-regularity of set-valued mappings under the relaxed constant positive linear dependence constraint qualification with applications to parametric and bilevel optimization

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The presence of Lipschitzian properties for solution mappings associated with nonlinear parametric optimization problems is desirable in the context of stability analysis or bilevel optimization. An example of such a Lipschitzian property for set-valued mappings, whose graph is the solution set of a system of nonlinear inequalities and equations, is R-regularity. Based on the so-called relaxed constant positive linear dependence constraint qualification, we provide a criterion ensuring the presence of the R-regularity property. In this regard, our analysis generalizes earlier results of that type which exploited the stronger Mangasarian–Fromovitz or constant rank constraint qualification. Afterwards, we apply our findings in order to derive new sufficient conditions which guarantee the presence of R-regularity for solution mappings in parametric optimization. Finally, our results are used to derive an existence criterion for solutions in pessimistic bilevel optimization and a sufficient condition for the presence of the so-called partial calmness property in optimistic bilevel optimization.

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1 Introduction

Lipschitzian properties of implicitly given set-valued mappings are of essential importance in order to study the stability of optimization problems, see e.g. Gfrerer and Outrata (2016); Luderer et al. (2002); Mordukhovich (2006) and the references therein. Particularly, such stability is desirable in the context of bilevel optimization where a function has to be minimized over the graph of a solution mapping associated with a given parametric optimization problem, see Bard (1998); Dempe (2002); Dempe et al. (2015) or Section 4.2 for details. Indeed, in order to infer existence results, optimality conditions, or solution algorithms in bilevel programming, one generally has to assume the presence of certain properties of this solution map. However, it is often not easy to verify such properties. In this paper, we focus on the derivation of sufficient criteria for the presence of so-called R-regularity of set-valued mappings, see Definition 2.4. This property, in turn, is beneficial in order to study Lipschitzian properties of marginal (or optimal value) functions and solution mappings in parametric optimization, see Bednarczuk et al. (2019); Luderer et al. (2002); Minchenko and Stakhovski (2011b), and these features possess some extensions to bilevel optimization as well.

In this paper, we investigate set-valued mappings $\Gamma : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ of the form

$$\forall x \in \mathbb{R}^n : \Gamma(x) := \left\{ y \in \mathbb{R}^m \mid h_i(x,y) \leq 0 \quad i \in I \atop h_i(x,y) = 0 \quad i \in J \right\} \tag{1}$$

where $I = \{1, \ldots , \ell \}$ and $J := \{ \ell + 1, \ldots , p \}$ are index sets and $h_1, \ldots , h_p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are functions such that the mappings $h_1(x,\cdot), \ldots , h_p(x,\cdot) : \mathbb{R}^m \to \mathbb{R}$ are continuous for each $x$ coming from the domain of $\Gamma$. It is a well known that the presence of R-regularity for mappings of this type is guaranteed under validity of the Mangasarian–Fromovitz constraint qualification, see Borwein (1986); Luderer et al. (2002). More recently, this result has been extended to situations where relaxed versions of the constant rank constraint qualification hold at the underlying reference points, see Bednarczuk et al. (2019); Minchenko and Stakhovski (2011b). However, in some situations, these qualification conditions may turn out to be too selective in order to guarantee applicability of the obtained results in order to investigate the presence of R-regularity for solution mappings, see e.g. Remark 4.12. That is why we aim for a generalization of these findings in the presence of the so-called relaxed constant positive linear dependence constraint qualification, introduced in Andreani et al. (2012), which is generally weaker than the aforementioned qualification conditions. Our main results Theorems 3.3 and 3.6 depict that this is indeed possible. With these new sufficient conditions for the presence of R-regularity for the mapping $\Gamma$ at hand, we are in position to state new criteria ensuring local Lipschitz continuity of the marginal function and R-regularity of the solution mapping associated with nonlinear parametric optimization problems whose feasible region is modelled with the aid of $\Gamma$. Afterwards, we use these findings in order to study the existence of so-called pessimistic solutions as well as the presence of the celebrated partial calmness property in bilevel optimization. The latter, introduced in Ye and Zhu (1995), is one of the key assumptions one generally postulates on the optimal value reformulation of an
optimistic bilevel optimization problem in order to infer necessary optimality conditions and solution algorithms, see Section 4.2 for details and suitable references.

The remaining parts of this manuscript are organized as follows: In Section 2, we provide the fundamental notation exploited in this paper. Furthermore, we recall some important constraint qualifications from nonlinear programming as well as the underlying fundamentals of set-valued analysis. Section 3 is dedicated to the study of the relaxed constant positive linear dependence constraint qualification as a sufficient condition for R-regularity of the mapping $\Gamma$. In Section 4, we investigate some applications of our findings. First, we apply the obtained results to nonlinear parametric optimization problems in order to state new sufficient conditions for the local Lipschitz continuity of the associated optimal value function as well as R-regularity of the associated solution mapping in Section 4.1. Afterwards, we employ these results in the context of bilevel optimization in order to formulate criteria ensuring the existence of pessimistic solutions as well as the presence of partial calmness in Section 4.2. In Section 5, we close the paper with the aid of some final comments.

2 Notation and preliminaries

In this paper, we mainly make use standard notation. The tools of set-valued analysis we exploit here can be found, e.g., in Bank et al. (1983); Mordukhovich (2006); Rockafellar and Wets (1998).

2.1 Basic notation

Throughout the paper, we equip $\mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$. For some point $x \in \mathbb{R}^n$ and a scalar $\varepsilon > 0$, we use

$$U_\varepsilon(x) := \{y \in \mathbb{R}^n \mid \|y - x\| < \varepsilon\} \quad \mathbb{B}_\varepsilon(x) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq \varepsilon\}$$

in order to denote the open and closed $\varepsilon$-ball around $x$, respectively. For brevity, we make use of $\mathbb{B} := \mathbb{B}_1(0)$. For a nonempty and closed set $A \subset \mathbb{R}^n$, we use

$$\text{dist}(x, A) := \inf\{\|y - x\| \mid y \in A\} \quad \Pi(x, A) := \text{argmin}\{\|y - x\| \mid y \in A\}$$

to denote the distance of $x$ to $A$ and the set of projections of $x$ onto $A$, respectively. It is well known that the distance function $\text{dist}(\cdot, A) : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitz continuous with Lipschitz modulus 1. Generally, we call a map $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ locally Lipschitz continuous at $x$ w.r.t. $\Omega \subset \mathbb{R}^n$ whenever there are $\delta > 0$ and $L > 0$ such that

$$\forall y, y' \in U_\delta(x) \cap \Omega: \quad \|\phi(y) - \phi(y')\| \leq L\|y - y'\|$$

holds. Note that this notion is only reasonable in the situation $x \in \text{cl}\Omega$. For $A := \mathbb{R}^n$, we recover the classical definition of local Lipschitz continuity.

Let $I_1$ as well as $I_2$ be finite index sets and let $(a^i)_{i \in I_1} \subset \mathbb{R}^n$ as well as $(b^i)_{i \in I_2} \subset \mathbb{R}^n$ be two given families of vectors. We call the pair of families $((a^i)_{i \in I_1}, (b^i)_{i \in I_2})$ positive-linearly dependent whenever there are scalars $\alpha_i \geq 0$, $i \in I_1$, and $\beta_i$, $i \in I_2$, which are
not all vanishing such that
\[ \sum_{i \in I_1} \alpha_i a^i + \sum_{i \in I_2} \beta_i b^i = 0. \]

Otherwise, we refer to this pair of families as positive-linearly independent.

The following lemma follows from (Andreani et al., 2012, Lemma 1).

**Lemma 2.1.** Let \( v^1, \ldots, v^{r+s} \in \mathbb{R}^n \) be given vectors such that the family \( (v^i)_{i=1}^{r+s} \) is linearly independent. Furthermore, let \( z \in \mathbb{R}^n \setminus \{0\} \) be given as \( z = \sum_{i=1}^{r+s} \alpha_i v^i \) for reals \( \alpha_1, \ldots, \alpha_{r+s} \) satisfying \( \alpha_{r+1}, \ldots, \alpha_{r+s} > 0 \). Then, there exist an index set \( I \subset \{r+1, \ldots, r+s\} \) and reals \( \bar{\alpha}_i, i \in \{1, \ldots, r\} \cup I \), satisfying \( \bar{\alpha}_i > 0 \) for all \( i \in I \), such that the family \( (v^i)_{i \in \{1, \ldots, r\} \cup I} \) is linearly independent and
\[ z = \sum_{i \in \{1, \ldots, r\} \cup I} \bar{\alpha}_i v^i. \]

### 2.2 Constraint qualifications in nonlinear programming

Supposing that \( \Gamma \) models the feasible region of a given parametric optimization problem, certain constraint qualifications need to be imposed on the images of \( \Gamma \) in order to ensure that the associated Karush–Kuhn–Tucker conditions provide a necessary optimality condition. In this regard, we postulate the following assumption which may hold throughout the section.

**Assumption 2.2.** Let us fix a reference parameter \( \bar{x} \in \mathbb{R}^n \) and some point \( \bar{y} \in \Gamma(\bar{x}) \). Furthermore, let all the functions \( h_1, \ldots, h_p \) be continuous as well as continuously differentiable w.r.t. \( y \) in a neighbourhood of \( \{\bar{x}\} \times \Gamma(\bar{x}) \).

Let us now introduce the qualification conditions of our interest. Therefore, we will exploit the set of indices associated with inequality constraints active at \((\bar{x}, \bar{y})\) which is defined as stated below:
\[ I(\bar{x}, \bar{y}) := \{ i \in I \mid h_i(\bar{x}, \bar{y}) = 0 \}. \]

**Definition 2.3.** We say that

(a) the Mangasarian–Fromovitz constraint qualification (MFCQ) holds at \((\bar{x}, \bar{y})\) whenever the pair of families
\[ \left( (\nabla_y h_i(\bar{x}, \bar{y}))_{i \in I(\bar{x}, \bar{y})}, (\nabla_y h_i(\bar{x}, \bar{y}))_{i \in J} \right) \]

is positive-linearly independent,

(b) the relaxed constant rank constraint qualification (RCRCQ) holds at \((\bar{x}, \bar{y})\) (w.r.t. \( \Omega \subset \mathbb{R}^n \)) whenever there is a neighbourhood \( U \) of \((\bar{x}, \bar{y})\) such that for each set \( K \subset I(\bar{x}, \bar{y}) \), the family \( (\nabla_y h_i(x, y))_{i \in K \cup J} \) has constant rank on \( U \) (on \( U \cap (\Omega \times \mathbb{R}^m) \)),

(c) the relaxed constant positive linear dependence constraint qualification (RCPLD) holds at \((\bar{x}, \bar{y})\) (w.r.t. \( \Omega \subset \mathbb{R}^n \)) whenever there is a neighbourhood \( U \) of \((\bar{x}, \bar{y})\) and an index set \( S \subset J \) such that the following conditions hold:
(i) \( \{ \nabla y h_i(\bar{x}, \bar{y}) \mid i \in I \} \) is a basis of the span of \( \{ \nabla y h_i(\bar{x}, \bar{y}) \mid i \in J \} \),

(ii) the family \( (\nabla y h_i(x, y))_{i \in J} \) has constant rank on \( U \) (on \( U \cap (\Omega \times \mathbb{R}^m) \)), and

(iii) for each set \( K \subset I(\bar{x}, \bar{y}) \) such that the pair of families

\[
\left( (\nabla y h_i(\bar{x}, \bar{y}))_{i \in K}, (\nabla y h_i(\bar{x}, \bar{y}))_{i \in S} \right)
\]

is positive-linearly dependent, the family \( (\nabla y h_i(x, y))_{i \in K \cup S} \) is linearly dependent for each point \( (x, y) \in U \) (for each point \( (x, y) \in U \cap (\Omega \times \mathbb{R}^m) \)).

While MFCQ is a well-known constraint qualification, RCRCQ and RCPLD are less popular. Let us mention that RCRCQ, which has been introduced in Minchenko and Stakhovski (2011a), is a less restrictive constraint qualification than the classical constant rank constraint qualification, see Janin (1984). On the other hand, RCPLD dates back to Andreani et al. (2012) and generalizes the classical constant positive linear dependence constraint qualification, see Andreani et al. (2005); Qi and Wei (2000). Checking these references, one can observe that both MFCQ and RCRCQ individually imply validity of RCPLD. However, neither does MFCQ imply validity of RCRCQ nor vice versa. Let us mention that RCPLD is stable in the sense that whenever it is valid at some reference point, then it also holds in a neighbourhood of this point. In order to see this, one may adapt the proof of (Andreani et al., 2012, Theorem 4), which is stated in the non-parametric setting, to the situation at hand. Finally, we would like to mention that the notion of RCPLD can be extended to non-smooth constraint systems as well as complementarity-type feasible regions, and, thus, applies to mathematical programs with complementarity constraints and different reformulations of bilevel optimization problems, see Chieu and Lee (2013); Guo and Lin (2013); Xu and Ye (2020) for details.

### 2.3 Properties of set-valued mappings

Let \( \Upsilon : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) be a set-valued mapping. We refer to the sets

\[
gph \Upsilon := \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in \Upsilon(x) \} \quad \text{dom} \Upsilon := \{ x \in \mathbb{R}^n \mid \Upsilon(x) \neq \emptyset \}
\]

as graph and domain of \( \Upsilon \), respectively. Let us fix a point \( \bar{x} \in \text{dom} \Upsilon \). We call \( \Upsilon \) locally bounded at \( \bar{x} \) whenever there are a bounded set \( B \subset \mathbb{R}^m \) and a neighbourhood \( U \subset \mathbb{R}^n \) of \( \bar{x} \) such that \( \Upsilon(x) \subset B \) holds for all \( x \in U \). One calls \( \Upsilon \) upper semicontinuous at \( \bar{x} \) whenever for each open set \( O \subset \mathbb{R}^m \) which satisfies \( \Upsilon(\bar{x}) \subset O \), there exists a neighbourhood \( U \subset \mathbb{R}^n \) of \( \bar{x} \) such that \( \Upsilon(x) \subset O \) holds for all \( x \in U \). Recall that \( \Upsilon \) is called lower semicontinuous at \( \bar{x} \) (w.r.t. \( \Omega \subset \mathbb{R}^n \)) whenever for each open set \( O \subset \mathbb{R}^m \) with \( \Upsilon(\bar{x}) \cap O \neq \emptyset \), there is a neighbourhood \( U \subset \mathbb{R}^n \) of \( \bar{x} \) such that \( \Upsilon(x) \cap O \neq \emptyset \) holds for all \( x \in U \) (for all \( x \in U \cap \Omega \)). We call \( \Upsilon \) inner semicontinuous at some point \( (\bar{x}, \bar{y}) \in \text{gph} \Upsilon \) (w.r.t. \( \Omega \)) whenever for each sequence \( \{ x^k \}_{k \in \mathbb{N}} \subset \mathbb{R}^n \) (\( \{ x^k \}_{k \in \mathbb{N}} \subset \Omega \)) converging to \( \bar{x} \), there exists a sequence \( \{ y^k \}_{k \in \mathbb{N}} \subset \mathbb{R}^m \) which converges to \( \bar{y} \) and satisfies \( y^k \in \Upsilon(x^k) \) for sufficiently large \( k \in \mathbb{N} \). Note that \( \Upsilon \) is lower semicontinuous at \( \bar{x} \) (w.r.t. \( \Omega \)) if and only if it is inner semicontinuous at each point from \( \{ \bar{x} \} \times \Upsilon(\bar{x}) \) (w.r.t. \( \Omega \)). The situation \( \Omega := \text{dom} \Upsilon \) will be of particular interest in this manuscript.
In the theory of set-valued analysis, there exist several different notions of Lipschitzian-
ity. Recall that \( \Upsilon \) possesses the Aubin property at some point \((\bar{x}, \bar{y})\) in \( \text{gph } \Upsilon \) (w.r.t. \( \Omega \)) whenever there exist neighbourhoods \( U \) and \( V \) of \( \bar{x} \) and \( \bar{y} \), respectively, as well as a constant \( \kappa > 0 \) such that
\[
\forall x, x' \in U (\forall x, x' \in U \cap \Omega): \quad \Upsilon(x) \cap V \subset \Upsilon(x') + \kappa \|x - x'\| \mathbb{B}
\]
holds. One can easily check that whenever \( \Upsilon \) possesses the Aubin property at \((\bar{x}, \bar{y})\) (w.r.t. \( \Omega \)), then it is inner semicontinuous (w.r.t. \( \Omega \)) at this point. Using the concept of coderivatives which is based on the limiting normal cone from variational analysis, one can formulate a necessary and sufficient condition for the presence of the Aubin property for set-valued mappings with closed graphs, see (Mordukhovich, 2006, Theorem 4.10). In (Mordukhovich, 2006, Corollary 4.39), one can find a characterization of the Aubin property of \( \Gamma \) from (1) at some point of its graph under validity of an MFCQ-type assumption. Let us, however, note that MFCQ from Definition 2.3 is only sufficient but not necessary for the presence of the Aubin property. A recent study on the presence of the Aubin property for implicitly defined set-valued mappings of more general form can be found in Gfrerer and Outrata (2016).

Let us now focus on the particular mapping \( \Gamma \) from (1) in more detail. In this manuscript, we are interested in the property of \( \Gamma \) being so-called R-regular at a point of its graph, see (Luderer et al., 2002, Section 6.2).

**Definition 2.4.** Fix \((\bar{x}, \bar{y})\) in \( \text{gph } \Gamma \) and some set \( \Omega \subset \mathbb{R}^n \). Then, \( \Gamma \) is called R-regular at \((\bar{x}, \bar{y})\) (w.r.t. \( \Omega \)) whenever there exist a constant \( \kappa > 0 \) and a neighbourhood \( U \) of \((\bar{x}, \bar{y})\) such that the condition
\[
\forall (x, y) \in U (\forall (x, y) \in U \cap (\Omega \times \mathbb{R}^n)): \quad \text{dist}(y, \Gamma(x)) \leq \kappa \max\{0, \max\{h_i(x, y) | i \in I\}, \max\{|h_i(x, y)| | i \in J\}\}
\]
holds.

The notion of R-regularity can be traced back to Fedorov (1979); Ioffe (1979) where it has been exploited as a constraint qualification. Following Bosch et al. (2004); Fabian et al. (2010); Robinson (1976), one might be tempted to say that the presence of R-regularity is equivalent to the validity of a local error bound condition at some reference point of the constraint system induced by \( \Gamma \) provided the latter does not depend on the parameter. In this regard, R-regularity of a parametric constraint system is a generalization of the concept of error bounds. Let us note that due to (Borwein, 1986, Theorem 3.2), R-regularity of \( \Gamma \) at a given reference point is implied by validity of MFCQ at the latter. We would like to point out that R-regularity can be interpreted as a variant of metric regularity, see Ioffe (2000) and the references therein.

Invoking (Bednarczuk et al., 2019, Theorem 5.1), one can easily check that whenever \( \Gamma \) is R-regular at \((\bar{x}, \bar{y})\) w.r.t. \( \Omega \) while all the functions \( h_1, \ldots, h_p \) are locally Lipschitz continuous at this point, then \( \Gamma \) possesses the Aubin property w.r.t. \( \Omega \) at this point. By means of simple examples, one can check that the converse statement does not hold in general even if the data functions are continuously differentiable and, thus, locally
Lipschitzian, see (Minchenko and Stakhovski, 2011b, Example 1). The following result even holds in the absence of local Lipschitz continuity of the data functions.

Lemma 2.5. Let $\Gamma$ be R-regular at some point $(\bar{x}, \bar{y}) \in \text{gph } \Gamma$ w.r.t. $\text{dom } \Gamma$ and let the functions $h_1, \ldots, h_p$ be continuous at $(\bar{x}, \bar{y})$. Then, $\Gamma$ is inner semicontinuous at $(\bar{x}, \bar{y})$ w.r.t. $\text{dom } \Gamma$.

Proof. The assumptions of the lemma particularly imply the existence of a constant $\kappa > 0$ and some $\delta > 0$ such that

$$\text{dist}(\bar{y}, \Gamma(x)) \leq \kappa \max\{0, \max\{h_i(x, \bar{y}) \mid i \in I\}, \max\{|h_i(x, \bar{y})| \mid i \in J\}\}$$

holds for all $x \in U_\delta(\bar{x}) \cap \text{dom } \Gamma$. Thus, for each sequence $\{x^k\}_{k \in \mathbb{N}} \subset \text{dom } \Gamma$ with $x^k \to \bar{x}$, the estimate

$$\|\bar{y} - y^k\| \leq \kappa \max\{0, \max\{h_i(x^k, \bar{y}) \mid i \in I\}, \max\{|h_i(x^k, \bar{y})| \mid i \in J\}\}$$

holds for sufficiently large $k \in \mathbb{N}$ where $y^k \in \Pi(\bar{y}, \Gamma(x^k))$ is arbitrarily chosen. Note that $\Pi(\bar{y}, \Gamma(x^k))$ is nonempty for each $k \in \mathbb{N}$ since $\Gamma(x^k)$ is nonempty and closed by continuity of $h_1(x^k, \cdot), \ldots, h_p(x^k, \cdot)$ and the choice $x^k \in \text{dom } \Gamma$. Exploiting the continuity of $h_1, \ldots, h_p$ at $(\bar{x}, \bar{y})$, we find $\|\bar{y} - y^k\| \to 0$ as $k \to \infty$, i.e., $\Gamma$ is inner semicontinuous at $(\bar{x}, \bar{y})$ w.r.t. $\text{dom } \Gamma$. □

By definition, R-regularity of a set-valued mapping at a given reference point is stable in the sense that it extends to points in sufficiently small neighbourhood. However, we get the following even stronger stability property from (Luderer et al., 2002, Lemma 6.19) which shows that the modulus of R-regularity is uniformly bounded in a neighbourhood of a compact set of points where a given set-valued mapping is R-regular.

Lemma 2.6. Let $C \subset \text{gph } \Gamma$ be compact and assume that $\Gamma$ is R-regular w.r.t. $\text{dom } \Gamma$ at each point from $C$. Then, there exist a constant $\kappa > 0$ and an open set $U \supset C$ such that (2) holds with $\Omega := \text{dom } \Gamma$, i.e., there is a uniform modulus $\kappa$ of R-regularity on $C$.

3 A sufficient condition for R-regularity

If not stated otherwise, we assume that Assumption 2.2 holds throughout the section. Furthermore, we will, at some instances, exploit the following additional assumptions.

(A1) For each $x \in \mathbb{R}^n$, the functions $h_i(x, \cdot): \mathbb{R}^m \to \mathbb{R}$, $i \in I$, are convex while the functions $h_i(x, \cdot): \mathbb{R}^m \to \mathbb{R}$, $i \in J$, are affine.

(A2) The set-valued mapping $\Gamma$ is locally bounded at $\bar{x} \in \text{dom } \Gamma$.

Subsequently, we will first derive a sequential characterization of R-regularity which holds under validity of the aforementioned conditions. Afterwards, we will relate this sequential characterization with the validity of the constraint qualification RCPLD.
3.1 A sequential characterization of R-regularity

For some parameter $x \in \text{dom} \Gamma$ and $\nu \notin \Gamma(x)$, $\Pi(\nu, \Gamma(x))$ equals the solution set of

$$\min_y \{ \| y - \nu \| \mid y \in \Gamma(x) \}$$

since $\Gamma(x)$ is a closed set by continuity of $h_1(x, \cdot), \ldots, h_p(x, \cdot)$. Due to $\nu \notin \Gamma(x)$, the objective function of the above problem is continuously differentiable in a neighbourhood of all points from $\Pi(\nu, \Gamma(x))$. Thus, it is reasonable to investigate the associated Lagrange multiplier set

$$\Lambda_{\nu}(x, y) := \left\{ \lambda \in \mathbb{R}^p \left| \frac{y - \nu}{\| y - \nu \|} + \sum_{i=1}^{p} \lambda_i \nabla y h_i(x, y) = 0, \forall i \in I: \lambda_i \geq 0, \lambda_i h_i(x, y) = 0 \right. \right\}$$

for each $y \in \Pi(\nu, \Gamma(x))$ as long as the pair $(x, y)$ is close to $(\bar{x}, \bar{y})$ where $\bar{y} \in \Gamma(\bar{x})$. For some constant $M > 0$, we make use of

$$\Lambda^M_{\nu}(x, y) := \{ \lambda \in \Lambda_{\nu}(x, y) \mid \sum_{i=1}^{p} |\lambda_i| \leq M \}.$$  

Let us note that under validity of (A1), the image sets of $\Gamma$ are convex which yields that the associated projection sets from above are actually singletons.

Using this notation, we obtain the following technical lemma.

**Lemma 3.1.** Let (A1) and (A2) hold. Assume that there exist a constant $M > 0$ as well as sequences $\{x^k\}_{k \in \mathbb{N}} \subset \text{dom} \Gamma$, $\{\nu^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m$, and $\{y^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m$ satisfying $x^k \to \bar{x}$, $\nu^k \to \bar{\nu}$, as well as $\nu^k \notin \Gamma(x^k)$ and $y^k \in \Pi(\nu^k, \Gamma(x^k))$ for all $k \in \mathbb{N}$ such that the set $\Lambda^M_{\nu^k}(x^k, y^k)$ is nonempty for sufficiently large $k \in \mathbb{N}$. Then, we have $y^k \to \bar{y}$ and

$$\text{dist}(\nu^k, \Gamma(x^k)) \leq M \max\{0, \max \{h_i(x^k, \nu^k) \mid i \in I\}, \max \{|h_i(x^k, \nu^k)| \mid i \in J\} \} \quad (3)$$

for sufficiently large $k \in \mathbb{N}$.

**Proof.** Due to (A2), $\{y^k\}_{k \in \mathbb{N}}$ is bounded. Fix an arbitrary convergent subsequence $\{y^{k_s}\}_{s \in \mathbb{N}}$ with limit $\bar{y} \in \mathbb{R}^m$. By continuity of $h_1, \ldots, h_p$ at $(\bar{x}, \bar{\nu})$, we obtain $\bar{y} \in \Gamma(\bar{x})$. By assumption, for all sufficiently large $s \in \mathbb{N}$, we find $\Lambda^M_{\nu^k_s}(x^{k_s}, y^{k_s})$. Exploiting (A1) and the definition of the set $\Lambda^M_{\nu^k_s}(x^{k_s}, y^{k_s})$, we obtain

$$\|y^{k_s} - \nu^{k_s}\| = \sum_{i=1}^{p} \lambda_i^{k_s} \nabla y h_i(x^{k_s}, y^{k_s})^T (\nu^{k_s} - y^{k_s})$$

$$\leq \sum_{i=1}^{p} \lambda_i^{k_s} (h_i(x^{k_s}, \nu^{k_s}) - h_i(x^{k_s}, y^{k_s})) = \sum_{i=1}^{p} \lambda_i^{k_s} h_i(x^{k_s}, \nu^{k_s})$$

$$\leq \sum_{i \in I} \lambda_i^{k_s} \max\{0, h_i(x^{k_s}, \nu^{k_s})\} + \sum_{i \in J} |\lambda_i^{k_s}| |h_i(x^{k_s}, \nu^{k_s})|$$

$$\leq M \max\{0, \max \{h_i(x^{k_s}, \nu^{k_s}) \mid i \in I\}, \max \{|h_i(x^{k_s}, \nu^{k_s})| \mid i \in J\} \}$$

for sufficiently large $s \in \mathbb{N}$. Taking the limit $s \to \infty$ yields $\|\bar{y} - \bar{\nu}\| \leq 0$, i.e., $\bar{y} = \bar{\nu}$. Particularly, the bounded sequence $\{y^k\}_{k \in \mathbb{N}}$ possesses the unique accumulation point $\bar{y}$ which must be its limit. Reprising the above arguments, we infer the second statement of the lemma from $\text{dist}(\nu^k, \Gamma(x^k)) = \|y^k - \nu^k\|$. \hfill \Box
Next, we exploit Lemma 3.1 in order to characterize R-regularity of \( \Gamma \) under validity of (A1) and (A2). This result is related to (Bednarczuk et al., 2019, Theorem 3.2) and (Minchenko and Stakhovski, 2011b, Theorems 2 and 3) where these assumptions are replaced by some a-priori inner semicontinuity of \( \Gamma \). Here, we follow the ideas used for the proof of (Minchenko and Stakhovski, 2011b, Theorem 2).

**Theorem 3.2.** Let (A1) and (A2) hold. Then, the following statements are equivalent.

(a) The mapping \( \Gamma \) is R-regular at \((\bar{x}, \bar{y})\) w.r.t. \( \text{dom} \Gamma \).

(b) There exists a constant \( M > 0 \) such that for each sequences \( \{x^k\}_{k \in \mathbb{N}} \subset \text{dom} \Gamma \), \( \{\nu^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m \), and \( \{y^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m \) satisfying \( x^k \to \bar{x}, \nu^k \to \bar{y}, \) and \( \nu^k \notin \Gamma(x^k) \) as well as \( y^k \in \Pi(\nu^k, \Gamma(x^k)) \) for all \( k \in \mathbb{N} \), the set \( \Lambda^M_{\nu^k}(x^k, y^k) \) is nonempty for sufficiently large \( k \in \mathbb{N} \).

**Proof.** We show both implications separately.

(a) \( \implies \) (b): Let \( \Gamma \) be R-regular at \((\bar{x}, \bar{y})\) w.r.t. \( \text{dom} \Gamma \). Then, we find \( \kappa > 0 \) and \( \gamma, \delta > 0 \) such that

\[
\forall x \in \Upsilon_{\gamma}(\bar{x}) \cap \text{dom} \Gamma \forall y \in U_{\delta}(\bar{y}):
\quad \text{dist}(y, \Gamma(x)) \leq \kappa \max\{0, \max\{h_i(x, y) \mid i \in I\}, \max\{|h_i(x, y)| \mid i \in J\}\}
\]

(4)
holds. Furthermore, let \( \{x^k\}_{k \in \mathbb{N}} \subset \text{dom} \Gamma \), \( \{\nu^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m \), and \( \{y^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m \) be sequences which satisfy the requirements in (b). We first show \( y^k \to \bar{y} \). Indeed, we have

\[
\|y^k - \bar{y}\| \leq \|y^k - \nu^k\| + \|\nu^k - \bar{y}\| = \text{dist}(\nu^k, \Gamma(x^k)) + \|\nu^k - \bar{y}\|,
\]

and the term on the right tends to zero as \( k \to \infty \) by R-regularity of \( \Gamma \) at \((\bar{x}, \bar{y})\) and continuity of \( h_1, \ldots, h_p \) at \((\bar{x}, \bar{y})\).

Fix \( k \in \mathbb{N} \) and define mappings \( \Phi_k, \Psi_k : \mathbb{R}^m \to \mathbb{R} \) by means of

\[
\forall w \in \mathbb{R}^m : \quad \Phi_k(w) := \|w - \nu^k\| \quad \Psi_k(w) := \Phi_k(w) + 2\text{dist}(w, \Gamma(x^k)).
\]

Observing that \( \Phi_k \) is globally Lipschitz continuous with Lipschitz modulus 1 while \( \Gamma(x^k) \) is nonempty and closed, Clarke’s principle of exact penalization, see (Clarke, 1983, Proposition 2.4.3), implies that \( y^k \) is a global minimizer of \( \Psi_k \).

For sufficiently large \( k \in \mathbb{N} \), we have \( x^k \in \Upsilon_{\gamma}(\bar{x}) \) and \( y^k \in U_{\delta/2}(\bar{y}) \). Consider such \( k \in \mathbb{N} \) and an arbitrary vector \( v \in \text{dom} \Gamma \). Then, the above considerations and (4) yield the estimate

\[
\Phi_k(y^k) = \Psi_k(y^k) \leq \Phi_k(w) = \Phi_k(w) + 2\text{dist}(w, \Gamma(x^k))
\]

\[
\leq \Phi_k(w) + 2\kappa \max\{0, \max\{h_i(x^k, w) \mid i \in I\}, \max\{|h_i(x^k, w)| \mid i \in J\}\}
\]

\[
= \max\left\{ \Phi_k(w) + \sum_{i=1}^p \lambda_i h_i(x^k, w) \left| \forall i \in I : \lambda_i \geq 0, \lambda_i \min\{0, h_i(x^k, w)\} = 0 \right. \right\}
\]

\[
\sum_{i=1}^p |\lambda_i| \leq 2\kappa
\]

Using the function \( \mathcal{L}_k : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R} \) and the set \( \bar{\Lambda}_k(w) \) given by

\[
\mathcal{L}_k(w, \lambda) := \Phi_k(w) + \sum_{i=1}^p \lambda_i h_i(x^k, w)
\]
\[ \tilde{\Lambda}^k(w) := \left\{ \lambda \in \mathbb{R}^p \mid \sum_{i=1}^{p} |\lambda_i| \leq 2\kappa, \forall i \in I: \lambda_i \geq 0, \lambda_i \min\{0, h_i(x^k, w)\} = 0 \right\} \]

for all \( w \in \cup_{\delta/2}(y^k) \) and \( \lambda \in \mathbb{R}^p \), we have

\[ \forall w \in \cup_{\delta/2}(y^k): \quad \Phi_k(y^k) \leq \max\{ \mathcal{L}_k(w, \lambda) \mid \lambda \in \tilde{\Lambda}^k(w) \}. \]

By continuity of the functions \( h_i(x^k, \cdot), i = 1, \ldots, \ell \), we find \( \delta_k' \in (0, \delta/2] \) such that

\[ \forall w \in \cup_{\delta_k'}(y^k): \quad \Phi_k(y^k) \leq \max\{ \mathcal{L}_k(w, \lambda) \mid \lambda \in \tilde{\Lambda}^k(y^k) \}. \tag{5} \]

Defining \( Q_k: \cup_{\delta_k'}(y_k) \to \mathbb{R} \) by means of

\[ \forall w \in \cup_{\delta_k'}(y^k): \quad Q_k(w) := \max\{ \mathcal{L}_k(w, \lambda) \mid \lambda \in \tilde{\Lambda}^k(y^k) \}, \]

we obtain \( \Phi_k(y^k) \leq Q_k(w) \) for all \( w \in \cup_{\delta_k'}(y^k) \) from (5). Furthermore, \( \Phi_k(y^k) = Q_k(y^k) \) holds which is why \( y^k \) is a global minimizer of \( Q_k \). For sufficiently large \( k \in \mathbb{N} \), \( \mathcal{L}_k \) is continuously differentiable at \( y^k \). Noting that \( \tilde{\Lambda}^k(y^k) \) is a compact polyhedron, \( Q_k \) is directionally differentiable at \( y^k \), and the directional derivative can be approximated from above by means of

\[ \forall d \in \mathbb{R}^m: \quad Q_k'(y^k, d) \leq \max\{ \nabla_y \mathcal{L}_k(y^k, \lambda)^\top d \mid \lambda \in \tilde{\Lambda}^k(y^k) \} \]

which follows from Danskin’s theorem, see (Bertsekas, 1999, Proposition B.25), due to validity of (A1). Recalling that \( y^k \) is a global minimizer of \( Q_k \), we have \( Q_k'(y^k, d) \geq 0 \) for all \( d \in \mathbb{R}^m \). Defining a polytope \( P \subset \mathbb{R}^m \) by means of

\[ P := \{ \nabla_y \mathcal{L}_k(y^k, \lambda) \mid \lambda \in \tilde{\Lambda}^k(y^k) \}, \]

we find \( \max\{ \xi^\top d \mid \xi \in P \} \geq 0 \) for all \( d \in \mathbb{R}^m \). This yields \( 0 \in P \). By definition of \( P \), \( \mathcal{L}_k \), and \( \Lambda_k \), \( \Lambda_k^M(x^k, y^k) \neq \emptyset \) follows. Since the above arguments apply to all sufficiently large \( k \in \mathbb{N} \), (b) holds.

(b) \( \implies \) (a) Let (b) hold and assume that \( \Gamma \) is not R-regular at \( (\bar{x}, \bar{y}) \) w.r.t. dom \( \Gamma \). Then, we find sequences \( \{x^k\}_{k \in \mathbb{N}} \subset \text{dom} \Gamma \) and \( \{\nu^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m \) such that \( x^k \to \bar{x}, \nu^k \to \bar{y}, \) and

\[ \text{dist}(\nu^k, \Gamma(x^k)) \geq k \max\{0, \max\{h_i(x^k, \nu^k) \mid i \in I\}, \max\{|h_i(x^k, \nu^k)| \mid i \in J\} \} \tag{6} \]

as well as \( \nu^k \notin \Gamma(x^k) \) hold for all \( k \in \mathbb{N} \). For each \( k \in \mathbb{N} \), we fix \( y^k \in \Pi(\nu^k, \Gamma(x^k)) \). Due to validity of (b), the set \( \Lambda^M_k(x^k, y^k) \) is nonempty for sufficiently large \( k \in \mathbb{N} \). By means of (A1) and (A2), Lemma 3.1 yields a contradiction since (3) and (6) are incongruous. \( \square \)

### 3.2 R-regularity under RCPLD

In this section, we want to exploit the sequential characterization of R-regularity obtained in Theorem 3.2 in order to show that validity of RCPLD is a sufficient criterion for R-regularity in the presence of (A1) and (A2). This generalizes (Bednarczuk et al., 2019, Theorem 4.2) and (Minchenko and Stakhovski, 2011b, Theorem 4) where a-priori inner semicontinuity of \( \Gamma \) at the reference point as well as RCRCQ were the necessary ingredients to come up with a related result in the absence of (A1) and (A2).
Theorem 3.3. Let (A1) and (A2) hold. Suppose that RCPLD holds at each point from \( \{ \bar{x} \} \times \Gamma(\bar{x}) \) w.r.t. \( \text{dom} \Gamma \). Then, \( \Gamma \) is R-regular at each point from \( \{ \bar{x} \} \times \Gamma(\bar{x}) \) w.r.t. \( \text{dom} \Gamma \).

Proof. Suppose that there exists \( \bar{y} \in \Gamma(\bar{x}) \) such that \( \Gamma \) is not R-regular at \( (\bar{x}, \bar{y}) \) w.r.t. \( \text{dom} \Gamma \). Due to Theorem 3.2, this shows that for each \( \sigma \in \mathbb{N} \), there exist sequences \( \{x_{\sigma}^k\}_{k \in \mathbb{N}} \subset \text{dom} \Gamma \), \( \{\nu_{\sigma}^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m \), and \( \{y_{\sigma}^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m \) satisfying \( x_{\sigma}^k \to \bar{x}, \nu_{\sigma}^k \to \bar{y}, \nu_{\sigma}^k \notin \Gamma(x_{\sigma}^k) \) as well as \( \mu_{\sigma}^k \in \Pi(\nu_{\sigma}^k, \Gamma(x_{\sigma}^k)) \) for all \( k \in \mathbb{N} \), and \( \Lambda_{\nu_{\sigma}^k}^\sigma(x_{\sigma}^k, y_{\sigma}^k) = \emptyset \) for all \( s \in \mathbb{N} \), i.e., the latter holds at least on a subsequence. Performing a standard diagonal sequence argument, we, thus, find sequences \( \{x^\sigma\}_{\sigma \in \mathbb{N}} \subset \text{dom} \Gamma \), \( \{\nu^\sigma\}_{\sigma \in \mathbb{N}} \subset \mathbb{R}^m \), and \( \{y^\sigma\}_{\sigma \in \mathbb{N}} \) satisfying \( x^\sigma \to \bar{x}, \nu^\sigma \to \bar{y} \), as well as \( \nu^\sigma \notin \Gamma(x^\sigma), y^\sigma \in \Pi(\nu^\sigma, \Gamma(x^\sigma)) \), and \( \Lambda_{\nu^\sigma}^\sigma(x^\sigma, y^\sigma) = \emptyset \) for all \( \sigma \in \mathbb{N} \).

Invoking (A2) and the continuity of \( h_1, \ldots, h_p \) at each point from \( \{ \bar{x} \} \times \Gamma(\bar{x}) \), we obtain that for each \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( \Gamma(\bar{x}) \subset \Gamma(\bar{x} + \epsilon \mathbb{B}) \) holds for all \( x \in \text{U}_\delta(\bar{x}) \) since \( \Gamma \) is upper semicontinuous at \( \bar{x} \), see (Rockafellar and Wets, 1998, Theorem 5.19) as well. Thus, recalling that RCPLD is locally stable, it needs to hold at the points \( (x^\sigma, y^\sigma) \) for sufficiently large \( \sigma \in \mathbb{N} \). Exploiting the fact that RCPLD is, actually, a constraint qualification, this implies \( \Lambda_{\nu^\sigma}(x^\sigma, y^\sigma) \neq \emptyset \). Since we have \( \Lambda_{\nu^\sigma}(x^\sigma, y^\sigma) = \emptyset \) from above, we conclude that each sequence \( \{\mu^\sigma\}_{\sigma \in \mathbb{N}} \) with \( \mu^\sigma \in \Lambda_{\nu^\sigma}(x^\sigma, y^\sigma) \) for all \( \sigma \in \mathbb{N} \) satisfies \( \|\mu^\sigma\| \to \infty \) as \( \sigma \to \infty \). Choose such a sequence. Recall that this means

\[
0 = \frac{y^\sigma - \nu^\sigma}{\|y^\sigma - \nu^\sigma\|} + \sum_{i=1}^p \mu_i^\sigma \nabla g_i(x^\sigma, y^\sigma), \quad (7a)
\]

\[
\forall i \in I(x^\sigma, y^\sigma): \quad \mu_i^\sigma \geq 0, \quad (7b)
\]

\[
\forall i \in I \setminus I(x^\sigma, y^\sigma): \quad \mu_i^\sigma = 0 \quad (7c)
\]

for all \( \sigma \in \mathbb{N} \) sufficiently large.

Clearly, (A2) guarantees that \( \{y^\sigma\}_{\sigma \in \mathbb{N}} \) is locally bounded and, thus, converges along a subsequence (without relabeling) to some \( \bar{y} \in \Gamma(\bar{x}) \) by continuity of \( h_1, \ldots, h_p \) at each point from \( \{ \bar{x} \} \times \Gamma(\bar{x}) \). Since RCPLD holds at \( (\bar{x}, \bar{y}) \) w.r.t. \( \text{dom} \Gamma \), we find a neighbourhood \( U \) of this point as well as an index set \( S \subset J \) satisfying the requirements (i), (ii), and (iii) from part (c) of Definition 2.3. Particularly, the family \( (\nabla g_i(x,y))_{i \in S} \) needs to be linearly independent while the vectors from \( (\nabla g_i(x,y))_{i \in J \setminus S} \) need to be linearly dependent on the family \( (\nabla g_i(x,y))_{i \in S} \) for all \( (x,y) \in U \cap (\text{dom} \Gamma \times \mathbb{R}^m) \). For sufficiently large \( \sigma \in \mathbb{N} \), \( (x^\sigma, y^\sigma) \in U \cap (\text{dom} \Gamma \times \mathbb{R}^m) \) holds true. The above arguments lead to the existence of \( \tilde{\mu}_i^\sigma, i \in J \), such that

\[
\sum_{i \in J \setminus S} \mu_i^\sigma \nabla g_i(x^\sigma, y^\sigma) = \sum_{i \in S} \tilde{\mu}_i^\sigma \nabla g_i(x^\sigma, y^\sigma), \quad (8a)
\]

\[
\forall i \in J \setminus S: \quad \tilde{\mu}_i^\sigma = 0 \quad (8b)
\]

holds for sufficiently large \( \sigma \in \mathbb{N} \) where, additionally, the family \( (\nabla g_i(x^\sigma, y^\sigma))_{i \in S} \) is linearly independent. Now, (8a) allows to rewrite (7a) as

\[
0 = \frac{y^\sigma - \nu^\sigma}{\|y^\sigma - \nu^\sigma\|} + \sum_{i \in J \setminus S} \mu_i^\sigma \nabla g_i(x^\sigma, y^\sigma) + \sum_{i \in S} \tilde{\mu}_i^\sigma \nabla g_i(x^\sigma, y^\sigma)
\]
for sufficiently large $\sigma \in \mathbb{N}$. Observing that there are only finitely many subsets of $I$, we may pass to a subsequence (without relabelling) in order to guarantee $I(x^\sigma, y^\sigma) = J$ for all $\sigma \in \mathbb{N}$ and some set $J \subset I$. Now, we apply Lemma 2.1 to the situation at hand. Thus, for each sufficiently large $\sigma \in \mathbb{N}$, we find a set $I^\sigma \subset J$ as well as reals $\lambda^\sigma_i, i \in I^\sigma \cup S$, satisfying $\lambda^\sigma_i > 0$ for all $i \in I^\sigma$, such that the family $(\nabla_y h_i(x^\sigma, y^\sigma))_{i \in I^\sigma \cup S}$ is linearly independent while

$$0 = \frac{y^\sigma - \nu^\sigma}{\|y^\sigma - \nu^\sigma\|} + \sum_{i \in I^\sigma \cup S} \lambda^\sigma_i \nabla_y h_i(x^\sigma, y^\sigma)$$

holds for all $\sigma \in \mathbb{N}$. By passing once more to a subsequence (without relabelling), we may ensure that $I^\sigma = J$ holds for all $\sigma \in \mathbb{N}$ and some index set $I \subset I$. Let us set $\hat{\lambda}^\sigma := 0$ for all $i \in (I \setminus \mathcal{I}) \cup (J \setminus S)$ in order to rewrite the above equation as

$$0 = \frac{y^\sigma - \nu^\sigma}{\|y^\sigma - \nu^\sigma\|} + \sum_{i = 1}^p \lambda^\sigma_i \nabla_y h_i(x^\sigma, y^\sigma). \quad (9)$$

Thus, we have shown $\hat{\lambda}^\sigma \in \Lambda^\sigma(x^\sigma, y^\sigma)$. The above arguments show the convergence $\|\hat{\lambda}^\sigma\| \to \infty$ as $\sigma \to \infty$. Consequently, dividing (9) by $\|\hat{\lambda}^\sigma\|$ and taking the limit $\sigma \to \infty$, we infer

$$0 = \sum_{i = 1}^p \lambda_i \nabla_y h_i(\bar{x}, \bar{y}),$$

$$\forall i \in \mathcal{I}: \lambda_i \geq 0,$$

$$\forall i \in (I \setminus \mathcal{I}) \cup (J \setminus S): \lambda_i = 0$$

for some non-vanishing multiplier $\lambda \in \mathbb{R}^p$ by the assumed continuity of the derivatives $\nabla_y h_1, \ldots, \nabla_y h_p$ at $(\bar{x}, \bar{y})$. Thus, the pair of families $((\nabla_y h_i(\bar{x}, \bar{y}))_{i \in \mathcal{I}}, (\nabla_y h_i(\bar{x}, \bar{y}))_{i \in S})$ is positive-linearly dependent. On the other hand, we have already shown above that the families $(\nabla_y h_i(x^\sigma, y^\sigma))_{i \in \mathcal{I} \cup S}$ are linearly independent. This, however, contradicts the validity of RCPLD at $(\bar{x}, \bar{y})$ and, thus, completes the proof.

As a consequence of the above theorem and Lemma 2.5, we obtain the following corollary.

**Corollary 3.4.** Let the assumptions of Theorem 3.3 hold. Then, $\Gamma$ is lower semicontinuous w.r.t. $\text{dom} \Gamma$ at $\bar{x}$.

Inspecting the proofs of Lemma 3.1 as well as Theorems 3.2 and 3.3, the following remark is at hand.

**Remark 3.5.** Observe that the proofs of Lemma 3.1 as well as Theorems 3.2 and 3.3 remain true in the following setting which is slightly more general than the one of Assumption 2.2: For each $i \in \{1, \ldots, p\}$, there exist functions $g_i: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $t_i: \mathbb{R}^n \to \mathbb{R}$ such that $h_i(x, y) = g_i(x, y) + t_i(x)$ holds true for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Furthermore, $g_i$ is continuous as well as continuously differentiable w.r.t. $y$ in a neighbourhood of $\{\bar{x}\} \times \Gamma(\bar{x})$. Finally, we have $|t_i(x)| < \infty$ for all $x \in \text{dom} \Gamma$ from a neighbourhood of $\bar{x}$ and $t_i$ is continuous at $\bar{x}$.
Observe that the assertion of Theorem 3.3 is essentially different from the one of (Bednarczuk et al., 2019, Theorem 4.2). In Bednarczuk et al. (2019), the authors claimed validity of inner semicontinuity and RCRCQ at one point from the graph of \( \Gamma \) in order to obtain R-regularity at the reference point. Here, however, we postulate (A1) and assume validity of RCPLD at all points from \( \{ \bar{x} \} \times \Gamma(\bar{x}) \) in order to deduce R-regularity of \( \Gamma \) at all these points. Thus, in this setting, one may interpret the statement of Theorem 3.3 as a sufficient condition for lower semicontinuity of \( \Gamma \) as well, see Corollary 3.4. Observe that we cannot modify the statement of Theorem 3.3 in such a way that assuming validity of RCPLD at one reference point \((\bar{x}, \bar{y}) \in \Gamma \) ensures R-regularity of \( \Gamma \) at the same point without adding inner semicontinuity of \( \Gamma \) at \((\bar{x}, \bar{y}) \) while relying on the provided proof. However, we obtain the following result which generalizes (Bednarczuk et al., 2019, Theorem 4.2).

**Theorem 3.6.** Assume that \( \Gamma \) is inner semicontinuous at \((\bar{x}, \bar{y}) \) w.r.t. \( \text{dom} \Gamma \) and let RCPLD hold at this point w.r.t. \( \text{dom} \Gamma \). Then, \( \Gamma \) is R-regular at \((\bar{x}, \bar{y}) \) w.r.t. \( \text{dom} \Gamma \).

**Proof.** We follow the lines of the proof of Theorem 3.3 while respecting the following changes: First, the role of \( \tilde{y} \) is played by \( \bar{y} \). Second, inner semicontinuity of \( \Gamma \) at \((\bar{x}, \bar{y}) \) w.r.t. \( \text{dom} \Gamma \) ensures validity of the sequential characterization of R-regularity from Theorem 3.2 in the absence of (A1) and (A2), see (Bednarczuk et al., 2019, Theorem 3.2). Third, inner semicontinuity of \( \Gamma \) at \((\bar{x}, \bar{y}) \) w.r.t. \( \text{dom} \Gamma \) can be used to infer the convergence \( y^\sigma \to \bar{y} \) without presuming validity of (A2). Fourth, the relation \( \Lambda_{\nu}^\sigma(x^\sigma, y^\sigma) \neq \emptyset \) follows for sufficiently large \( \sigma \in \mathbb{N} \) directly from local stability of RCPLD. 

Let us point out that in case where \( \Gamma \) does not depend on the parameter \( x \), Theorem 3.6 provides a sufficient condition for the presence of an error bound at some reference point of a nonlinear constraint system. For a similar result under slightly stronger assumptions, we refer the interested reader to (Andreani et al., 2012, Theorem 7). Furthermore, we would like to mention (Chieu and Lee, 2013, Theorem 4.2) where this result has been obtained in the context of mathematical problems with complementarity constraints.

The upcoming example, which closes this section, shows that the statements of Theorems 3.3 and 3.6 do not need to hold in the absence of the convexity assumption (A1) or the inner semicontinuity of \( \Gamma \) at the reference point, respectively.

**Example 3.7.** We consider the mapping \( \Gamma : \mathbb{R} \to \mathbb{R} \) given by

\[
\forall x \in \mathbb{R} : \quad \Gamma(x) := \{ y \in \mathbb{R} \mid x - y \leq 0, y - y^2 \leq 0, y - 1 \leq 0 \}.
\]

A simple calculation reveals

\[
\forall x \in \mathbb{R} : \quad \Gamma(x) = \begin{cases} [x, 0] \cup \{1\} & x \in (-\infty, 0], \\ \{1\} & x \in [0, 1], \\ \emptyset & x \in (1, \infty). \end{cases}
\]
We study the point $\bar{x} := 0$ as well as the associated images $\bar{y} := 0$ and $\tilde{y} := 1$ in $\Gamma(\bar{x})$. Note that $\Gamma$ is inner semicontinuous at $(\bar{x}, \bar{y})$ but not at $(\bar{x}, \tilde{y})$. Thus, $\Gamma$ cannot be R-regular at $(\bar{x}, \tilde{y})$ due to Lemma 2.5.

Observe that the family $(-1, 1-2y)$ is positive-linearly dependent around $\tilde{y}$ while the family $(1-2y, 1)$ is positive-linearly dependent around $\tilde{y}$. Thus, RCPLD is valid at $(\bar{x}, \tilde{y})$ and $(\bar{x}, \bar{y})$, respectively. This shows that the statement of Theorem 3.3 does not generally hold in the absence of (A1) while the assertion of Theorem 3.6 is not generally true if $\Gamma$ is not inner semicontinuous at the reference point.

4 Applications

4.1 Parametric optimization

For a function $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, we investigate the parametric optimization problem

$$\min_y \{f(x, y) \mid y \in \Gamma(x)\} \quad (P(x))$$

where $\Gamma: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the set-valued mapping given in (1). Associated with the problem $(P(x))$ are the solution mapping $S: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by

$$\forall x \in \mathbb{R}^n: \quad S(x) := \arg\min_y \{f(x, y) \mid y \in \Gamma(x)\}$$

as well as the optimal value (or marginal) function $\varphi: \mathbb{R}^n \to \mathbb{R}$ defined via

$$\forall x \in \mathbb{R}^n: \quad \varphi(x) := \inf_y \{f(x, y) \mid y \in \Gamma(x)\}.$$  

Clearly, we have the relation

$$\forall x \in \mathbb{R}^n: \quad S(x) = \{y \in \Gamma(x) \mid f(x, y) \leq \varphi(x)\}$$

which is why $S$ can be interpreted as a solution mapping associated with a parametric system of nonlinear inequalities and equations. It is well known that under comparatively weak assumptions, the optimal value function $\varphi$ is continuous at a given reference point, see e.g. Bank et al. (1983). Keeping Remark 3.5 in mind, we are thus in position to apply the theory from Section 3 to this representation of $S$ in order to infer its R-regularity at a given reference point under suitable assumptions. This way, we also obtain new sufficient criteria for the presence of the Aubin property of $S$ or its inner semicontinuity at a given reference point. For the sake of brevity and consistency, we define $h_0: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ by means of

$$\forall x \in \mathbb{R}^n, \forall y \in \mathbb{R}^m: \quad h_0(x, y) := f(x, y) - \varphi(x)$$

and see that $S$ possesses the representation

$$\forall x \in \mathbb{R}^n: \quad S(x) = \left\{ y \in \mathbb{R}^m \mid \begin{array}{ll}
    h_i(x, y) \leq 0 & i \in I \cup \{0\} \\
    h_i(x, y) = 0 & i \in J 
\end{array} \right\}. \quad (10)$$
This representation of \( S \) can be addressed with the theory from Section 3. In this section, we will refer to the parametric constraint systems induced by \( \Gamma \) and \( S \). In this regard, we will exploit the notions RCPLD\(_{\Gamma} \) and RCPLD\(_{S} \) in order to avoid any confusion.

Let us emphasize that, if not stated otherwise, we will include the constraint function \( h_0 \) as an inequality constraint when considering \( S \), i.e., we exploit the representation of \( S \) from (10) in most of the cases. However, it is also possible to exploit \( h_0 \) in terms of an equality constraint.

**Remark 4.1.** We also have the representation

\[
\forall x \in \mathbb{R}^n: \quad S(x) = \left\{ y \in \mathbb{R}^m \left| \begin{array}{l}
 h_i(x, y) \leq 0 \quad i \in I \\
 h_i(x, y) = 0 \quad i \in J \cup \{0\} 
\end{array} \right. \right\},
\]

and, in some situations, it might be beneficial to apply the theory of Section 3 to this representation of \( S \) instead of the one from (10).

We postulate the following standing assumption throughout the section.

**Assumption 4.2.** The functions \( f \) and \( h_1, \ldots, h_p \) are continuously differentiable.

Note that by continuity of \( h_1, \ldots, h_p \), we already know that \( \text{gph} \Gamma \) is closed. Particularly, the image sets of \( \Gamma \) are closed. By continuity of \( f \), we even know that the image sets of \( S \) are closed.

Finally, we will exploit the following modified version of (A1) in some situations:

**(A1’)** For each \( x \in \mathbb{R}^n \), the functions \( f(x, \cdot): \mathbb{R}^m \rightarrow \mathbb{R} \) and \( h_i(x, \cdot): \mathbb{R}^m \rightarrow \mathbb{R}, \ i \in I \), are convex while the functions \( h_i(x, \cdot): \mathbb{R}^m \rightarrow \mathbb{R}, \ i \in J \), are affine.

We note that (A1’) is the counterpart of (A1) which addresses the representation of \( S \) from (10). In case where one aims to exploit the representation of \( S \) from Remark 4.1, the convexity of \( f(x, \cdot): \mathbb{R}^m \rightarrow \mathbb{R} \) for each \( x \in \mathbb{R}^n \) has to be replaced by the property of this mapping to be affine.

### 4.1.1 Continuity properties of marginal functions

In the subsequent lemma, we collect some results regarding the continuity properties of the function \( \varphi \). The proof is stated for the reader’s convenience.

**Lemma 4.3.** Fix a point \( \bar{x} \in \text{dom} \ \Gamma \) where (A2) is valid. Then, the following assertions hold.

(a) The function \( \varphi \) is lower semicontinuous at \( \bar{x} \).

(b) Assume that there exists \( \bar{y} \in \Gamma(\bar{x}) \) such that \( \Gamma \) is inner semicontinuous at \( (\bar{x}, \bar{y}) \) w.r.t. \( \text{dom} \ \Gamma \). Then, \( \varphi \) is continuous at \( \bar{x} \) w.r.t. \( \text{dom} \ \Gamma \).

(c) Assume that \( \Gamma \) possesses the Aubin property at each point from \( \{\bar{x}\} \times S(\bar{x}) \). Then, \( \varphi \) is locally Lipschitz continuous at \( \bar{x} \).
(d) Assume that there exists \( \bar{y} \in S(\bar{x}) \) such that \( \Gamma \) possesses the Aubin property at \((\bar{x}, \bar{y})\) while \( S \) is inner semicontinuous at this point. Then, \( \phi \) is locally Lipschitz continuous at \( \bar{x} \).

Proof. (a) By continuity of the functions \( h_1, \ldots, h_p \) and validity of (A2), we obtain upper semicontinuity of \( \Gamma \) at \( \bar{x} \). Thus, the desired assertion can be distilled from (Bank et al., 1983, Theorem 4.2.1) since \( f \) is continuous.

(b) Consulting the proof of (Bank et al., 1983, Theorem 4.2.1), inner semicontinuity of \( \Gamma \) at \((\bar{x}, \bar{y})\) is enough to guarantee that \( \phi \) is upper semicontinuous at \( \bar{x} \) since \( f \) is continuous. Combining this with (a), the desired result follows.

(c) Due to validity of (A2), the solution mapping \( S \) is locally bounded at \( \bar{x} \) as well. Particularly, \( S \) possesses bounded images in a neighbourhood of \( \bar{x} \). Due to \( \bar{x} \in \text{dom} \; \Gamma \), we have \( \Gamma(\bar{x}) \neq \emptyset \) and, thus, \( S(\bar{x}) \neq \emptyset \) by Weierstrass' theorem. Since \( \Gamma \) possesses the Aubin property at each point from \( \{\bar{x}\} \times S(\bar{x}) \), \( \Gamma \) is inner semicontinuous at each point \((\bar{x}, y)\) \( \in \text{gph} \; S \) and, thus, possesses nonempty image sets in a neighbourhood of \( \bar{x} \). Thus, we deduce that \( S \) possesses bounded and nonempty image sets in a neighbourhood of \( \bar{x} \). Furthermore, \( \phi \) is lower semicontinuous at \( \bar{x} \) by (a). Thus, the statement follows from (Mordukhovich and Nam, 2005, Theorem 5.3(ii))

(d) This follows directly from (Mordukhovich and Nam, 2005, Theorem 5.3(i)) while observing that \( \phi \) is continuous at \( \bar{x} \) by inner semicontinuity of \( S \) at \((\bar{x}, \bar{y})\) and continuity of \( f \).

We would like to mention that statement (d) of Lemma 4.3 holds even true in the absence of (A2) since the latter has not been used in the proof.

As a corollary of Theorems 3.3 and 3.6 as well as Lemma 4.3, we obtain the following result as a consequence of the local Lipschitz continuity of the functions \( h_1, \ldots, h_p \).

Corollary 4.4. Fix some point \( \bar{x} \in \text{dom} \; \Gamma \). Let one of the following additional assumptions be valid.

(a) Let (A1) and (A2) hold. Furthermore, let \( \text{RCPLD}_\Gamma \) hold at each point from \( \{\bar{x}\} \times \Gamma(\bar{x}) \) and assume that \( \bar{x} \) is an interior point of \( \text{dom} \; \Gamma \).

(b) Let \( \bar{y} \in S(\bar{x}) \) be chosen such that \( S \) is inner semicontinuous at \((\bar{x}, \bar{y})\) while \( \text{RCPLD}_\Gamma \) holds at this point.

Then, \( \phi \) is locally Lipschitz continuous at \( \bar{x} \).

Let us mention that in the presence of (A1), the validity of MFCQ at one point from \( \{\bar{x}\} \times \Gamma(\bar{x}) \) implies that Slater’s constraint qualification is valid for the set \( \Gamma(\bar{x}) \), and the latter implies that MFCQ and, thus, \( \text{RCPLD}_\Gamma \) hold at each point from \( \{\bar{x}\} \times \Gamma(\bar{x}) \). Thus, the assumptions in the first statement of Corollary 4.4 are weaker than postulating validity of MFCQ at one point from \( \{\bar{x}\} \times \Gamma(\bar{x}) \), and the latter is a classical assumption.
in the literature to guarantee local Lipschitz continuity of marginal functions, see, e.g., (Klatte and Kummer, 1985, Theorem 1).

We would like to point out that the assumption on \( \bar{x} \) in the first statement of Corollary 4.4 to be an interior point of \( \text{dom} \Gamma \) is, in general, indispensable in order to infer the local Lipschitz continuity of \( \varphi \) at this point since Theorem 3.3 only provides \( R \)-regularity, and, thus, the Aubin property, of \( \Gamma \) w.r.t. \( \text{dom} \Gamma \). Observe that the assumptions of the second statement of Corollary 4.4 already imply that \( \bar{x} \) is an interior point of \( \text{dom} S \).

Example 4.5. Let us consider the simple parametric optimization problem

\[
\min_y \{ y \mid 0 \leq y \leq x \}.
\]

Observing that all involved functions are fully linear, \( \text{RCPLD}_\Gamma \) holds at each point of \( \text{gph} \Gamma \) in this example. Nevertheless, the associated optimal value function \( \varphi \) is discontinuous at \( \bar{x} := 0 \) which is a boundary point of \( \text{dom} \Gamma = [0, \infty) \). However, we note that \( \varphi \) is Lipschitz continuous w.r.t. \( \text{dom} \Gamma \).

It is also possible to obtain Lipschitzian properties of the optimal value function \( \varphi \) w.r.t. \( \text{dom} \Gamma \) without relying on the fundamentals of variational analysis, which were used in Mordukhovich and Nam (2005), but exploiting the concept of \( R \)-regularity directly.

Lemma 4.6. Fix some point \( \bar{x} \in \text{dom} \Gamma \). Let one of the following additional assumptions be valid.

(a) Let (A2) hold and assume that \( \Gamma \) is \( R \)-regular at each point from \( \{ \bar{x} \} \times S(\bar{x}) \) w.r.t. \( \text{dom} \Gamma \).

(b) Assume that there exists \( \bar{y} \in S(\bar{x}) \) such that \( \Gamma \) is \( R \)-regular at \( (\bar{x}, \bar{y}) \) w.r.t. \( \text{dom} \Gamma \) while \( S \) is inner semicontinuous at this point w.r.t. \( \text{dom} \Gamma \).

Then, \( \varphi \) is locally Lipschitz continuous at \( \bar{x} \) w.r.t. \( \text{dom} \Gamma \).

Proof. (a) Due to \( \bar{x} \in \text{dom} \Gamma \) and validity of (A2), we indeed know \( S(\bar{x}) \neq \emptyset \). Additionally, the set \( S(\bar{x}) \) is closed, i.e., \( \{ \bar{x} \} \times S(\bar{x}) \) is compact. Thus, we can apply Lemma 2.6 in order to find constants \( \kappa, \gamma > 0 \) as well as an open set \( O \supset S(\bar{x}) \). Similar as in the proof of statement (c) of Lemma 4.3, we can ensure \( S(x) \neq \emptyset \) for all \( x \in U_\gamma(\bar{x}) \cap \text{dom} \Gamma \) if only \( \gamma \) is small enough. Moreover, due to Lemma 2.5, we know that \( \Gamma \) is inner semicontinuous at each point from \( \{ \bar{x} \} \times S(\bar{x}) \) w.r.t. \( \text{dom} \Gamma \). Thus, we can apply statement (b) of Lemma 4.3 in order to see that \( \varphi \) is continuous at \( \bar{x} \) w.r.t. \( \text{dom} \Gamma \). Combining this with the local boundedness of \( S \) and the continuity of \( h_1, \ldots, h_p \), we obtain that \( S \) is upper semicontinuous at \( \bar{x} \). Thus, we can even choose \( \gamma \) so small that \( S(x) \subset O \) holds for all \( x \in U_\gamma(\bar{x}) \cap \text{dom} \Gamma \). Clearly, \( \Gamma \) is upper semicontinuous at \( \bar{x} \) as well which is why we find an open set \( O' \supset \Gamma(\bar{x}) \) which satisfies \( O' \supset O \) and \( \Gamma(x) \subset O' \) for all \( x \in U_\gamma(\bar{x}) \cap \text{dom} \Gamma \) if only \( \gamma \) is sufficiently small. By continuous differentiability of the functions \( f \) and \( h_1, \ldots, h_p \), these functions are Lipschitz continuous on \( B_\gamma(\bar{x}) \times \text{cl} O' \). Let \( L_f > 0 \) and \( L_1, \ldots, L_p > 0 \) be the associated Lipschitz moduli.
Now, fix $x^1, x^2 \in U_\gamma(\bar{x}) \cap \text{dom } \Gamma$. Then, we find $y^1, y^2 \in O$ such that $y^1 \in S(x^1)$ and $y^2 \in S(x^2)$. We exploit (Clarke, 1983, Proposition 2.4.3) in order to see that $y^j$ is a global minimizer of that map $O' \ni y \mapsto f(x^j, y) + 2L_f \text{dist}(y, \Gamma(x^j)) \in \mathbb{R}$ for $j = 1, 2$ as well. Particularly, we obtain

$$\varphi(x^j) = f(x^j, y^j) \leq f(x^j, y^{3-j} + 2L_f \text{dist}(y^{3-j}, \Gamma(x^j)), \quad j = 1, 2.$$  

Now, we exploit (2) in order to obtain

$$\varphi(x^1) \leq f(x^1, y^2) + 2L_f \text{dist}(y^2, \Gamma(x^1))$$

$$\leq f(x^2, y^2) + f(x^1, y^2) - f(x^2, y^2)$$

$$+ 2L_f \kappa \max\{0, \max\{h_1(x^1, y^2) \mid i \in I\}, \max\{|h_1(x^1, y^2) \mid i \in J\}\}$$

$$\leq \varphi(x^2) + f(x^1, y^2) - f(x^2, y^2)$$

$$+ 2L_f \kappa \max\{0, \max\{h_1(x^1, y^2) - h_1(x^2, y^2) \mid i \in I\},$$

$$\max\{|h_1(x^1, y^2) - h_1(x^2, y^2) \mid i \in J\}\}\}$$

$$\leq \varphi(x^2) + L_f \|x^1 - x^2\| + 2L_f \kappa \max\{L_i \mid i \in I \cup J\}\|x^1 - x^2\|$$

$$\leq \varphi(x^2) + L_f (1 + 2\kappa \max\{L_i \mid i \in I \cup J\})\|x^1 - x^2\|.$$  

Changing the roles of the pairs $(x^1, y^1)$ and $(x^2, y^2)$ yields the local Lipschitz continuity of $\varphi$ w.r.t. $\text{dom } \Gamma$.

\[(b)\] The proof can be carried out in a similar way as in (a). The postulated $R$-regularity of $\Gamma$ at $(\bar{x}, \bar{y})$ yields the existence of constants $\kappa > 0$ as well as $\gamma, \delta > 0$ such that (4) holds. By inner semcontinuity of $S$ at $(\bar{x}, \bar{y})$ w.r.t. $\text{dom } \Gamma$, we can choose $\gamma$ and $\delta$ so small such that we have

$$\forall x \in U_\gamma(\bar{x}) \cap \text{dom } \Gamma: \quad U_{\delta/2}(\bar{y}) \cap S(x) \neq \emptyset.$$  

Furthermore, we note that by continuous differentiability of the functions $f$ and $h_1, \ldots, h_p$, these functions are Lipschitz continuous on $B_\gamma(\bar{x}) \times B_{2\delta}(\bar{y})$ with Lipschitz moduli $L_f > 0$ and $L_1, \ldots, L_p > 0$.

Now, fix $x^1, x^2 \in U_\gamma(\bar{x}) \cap \text{dom } \Gamma$. The above arguments yield the existence of $y^1, y^2 \in U_{\delta/2}(\bar{y})$ such that $y^1 \in S(x^1)$ and $y^2 \in S(x^2)$ hold. Exploiting (Clarke, 1983, Proposition 2.4.3), we find

$$\varphi(x^j) = f(x^j, y^j) \leq f(x^j, y^{3-j} + 2L_f \text{dist}(y^{3-j}, \Gamma(x^j) \cap B_{2\delta}(\bar{y})), \quad j = 1, 2.$$  

Due to $y^j \in \Gamma(x^j) \cap U_{\delta/2}(\bar{y})$, we even have

$$\text{dist}(y^{3-j}, \Gamma(x^j) \cap B_{2\delta}(\bar{y})) = \text{dist}(y^{3-j}, \Gamma(x^j)), \quad j = 1, 2,$$

and, thus, the rest of the proof can be carried out as in statement (a).
Let us briefly mention that the first statement of the above lemma may be interpreted as an adjustment of (Bednarczuk et al., 2019, Theorem 5.4) whose set of assumptions is not complete. Indeed, in the proof of this theorem, the authors exploit the presence of R-regularity at each point from \( \{ \bar{x} \} \times S(\bar{x}) \) which is not covered by the assumptions stated there.

We obtain the following corollary from Theorems 3.3 and 3.6 as well as Lemma 4.6.

**Corollary 4.7.** Fix some point \( \bar{x} \in \text{dom} \Gamma \). Let one of the following additional assumptions be valid.

(a) Let (A1) and (A2) hold. Furthermore, let RCPLD_\Gamma w.r.t. dom \Gamma hold at each point from \( \{ \bar{x} \} \times \Gamma(\bar{x}) \).

(b) Let \( \bar{y} \in S(\bar{x}) \) be chosen such that \( S \) is inner semicontinuous at \( (\bar{x}, \bar{y}) \) w.r.t. dom \( \Gamma \) while RCPLD_\Gamma w.r.t. dom \( \Gamma \) holds at this point.

Then, \( \varphi \) is locally Lipschitz continuous at \( \bar{x} \) w.r.t. dom \( \Gamma \).

**4.1.2 R-regularity of solution mappings**

The following theorem provides a sufficient criterion for R-regularity of the solution mapping \( S \).

**Theorem 4.8.** Fix a point \( \bar{x} \in \text{dom} \Gamma \). Then, the following assertions hold.

(a) Let (A1’) and (A2) hold. Furthermore, let RCPLD_\( S \) hold at each point from \( \{ \bar{x} \} \times S(\bar{x}) \). Finally, let \( \varphi \) be continuous at \( \bar{x} \). Then, \( S \) is R-regular at each point from \( \{ \bar{x} \} \times S(\bar{x}) \).

(b) Let \( \bar{y} \in S(\bar{x}) \) be chosen such that \( S \) is inner semicontinuous at \( (\bar{x}, \bar{y}) \) w.r.t. dom \( \Gamma \) while RCPLD_\( S \) holds at this point. Then, \( S \) is R-regular at \( (\bar{x}, \bar{y}) \). Moreover, \( S \) possesses the Aubin property at this point.

**Proof.** We show both statements separately.

(a) This statement follows directly from Theorem 3.3 keeping Remark 3.5 in mind. The continuity of \( \varphi \) at \( \bar{x} \) yields that \( \bar{x} \) is an interior point of dom \( S \).

(b) Let us first note that validity of RCPLD_\( S \) at \( (\bar{x}, \bar{y}) \in \text{gph} S \) implies validity of RCPLD_\( \Gamma \) there. Particularly, the second statement of Corollary 4.4 shows that \( \varphi \) is locally Lipschitz continuous at \( \bar{x} \). Thus, the representation of the mapping \( S \) from (10) satisfies the continuity requirements from Assumption 2.2. We can now apply Theorem 3.6 to infer R-regularity of \( S \). Observing that \( \varphi \) is locally Lipschitz continuous at \( \bar{x} \), the data functions used for the description of \( S \) are locally Lipschitz continuous at the points of interest. Thus, validity of the Aubin property of \( S \) at the reference points follows naturally.

\( \square \)
The next examples indicate that the continuity assumption in the first statement of the above theorem is, unluckily, indispensable in general since it may not follow from the postulated assumptions.

**Example 4.9.** Once more, let us investigate the parametric optimization problem from Example 4.5 which satisfies (A1') and (A2). There, we have

\[
\forall x \in \mathbb{R}: S(x) = \begin{cases} \emptyset & x \in (-\infty, 0), \\ \{0\} & x \in [0, \infty), \end{cases} \quad \varphi(x) = \begin{cases} +\infty & x \in (-\infty, 0), \\ 0 & x \in [0, \infty). \end{cases}
\]

Observing that all data functions used for the modelling of the given parametric optimization problem are fully linear, RCPLDS holds at each point from \( \text{gph} \ S \), particularly at \((\bar{x}, \bar{y}) := (0, 0)\). However, \( \varphi \) is discontinuous at \( \bar{x} \), and for \( x_k := -\frac{1}{k}, k \in \mathbb{N} \), we obtain

\[
\text{dist}(\bar{y}, S(x^k)) = +\infty > \kappa/k = \kappa \max\{0, \bar{y} - \varphi(x^k), -\bar{y}, \bar{y} - x^k\}
\]

for each \( \kappa > 0 \) and each \( k \in \mathbb{N} \), i.e., \( S \) cannot be R-regular at \((\bar{x}, \bar{y})\).

**Example 4.10.** We consider the parametric optimization problem

\[
\min_y \{y_1 | -1 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, xy_1 - y_2 = 0\}.
\]

We see that this problem inherently satisfies (A1') and (A2). The associated solution mapping \( S \) and the associated marginal function \( \varphi \) take the following form:

\[
\forall x \in \mathbb{R}: S(x) = \begin{cases} \{(1/x, 1)\} & x \in (-\infty, -1), \\ \{(-1, -x)\} & x \in [-1, 0], \\ \{(0, 0)\} & x \in (0, \infty), \end{cases} \quad \varphi(x) = \begin{cases} 1/x & x \in (-\infty, -1), \\ -1 & x \in [-1, 0], \\ 0 & x \in (0, \infty). \end{cases}
\]

We fix the reference points \( \bar{x} := 0 \) and \( \bar{y} := (-1, 0) \). Clearly, \( \varphi \) is not continuous at \( \bar{x} \).

One can check that RCPLDS is violated at \((\bar{x}, \bar{y})\) when using the representation of \( S \) from (10). However, keeping Remark 4.1 in mind, we may also consider the representation

\[
\forall x \in \mathbb{R}: S(x) = \{(y_1, y_2) | -1 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, xy_1 - y_2 = 0, y_1 - \varphi(x) = 0\}
\]

of \( S \) in order to address the proof of Theorem 4.8 since this representation still possesses the necessary convex structure w.r.t. \( y \). One can easily check that RCPLD holds for this mapping at \((\bar{x}, \bar{y})\) since the family

\[
\left( \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)
\]

associated with the equality constraints has already constant rank 2 in a neighbourhood of \((\bar{x}, \bar{y})\). However, as observed above, \( \varphi \) is not continuous at \( \bar{x} \), i.e., one cannot use Theorem 3.3 and Remark 3.5 in order to infer R-regularity of the solution mapping at the reference point.
The subsequent theorem now strengthens the first assertion of Theorem 4.8

**Theorem 4.11.** Fix a point \( \bar{x} \in \text{dom}\Gamma \). Let (A1') and (A2) hold. Furthermore, let RCPLD_S hold at each point from \( \{\bar{x}\} \times S(\bar{x}) \). Finally, let RCPLD_F hold at each point from \( \{\bar{x}\} \times \Gamma(\bar{x}) \), and let \( \bar{x} \) be an interior point of \( \text{dom}\Gamma \). Then, \( S \) is R-regular at each point from \( \{\bar{x}\} \times S(\bar{x}) \). Additionally, \( S \) possesses the Aubin property at all these points.

**Proof.** The proof follows the lines of the proof associated with the second statement of Theorem 4.8 exploiting first statement of Corollary 4.4 and Theorem 3.3. \( \square \)

Fix some point \( \bar{x} \in \text{dom}S \). The crucial requirement in Theorem 4.11 is clearly the validity of RCPLD_S at each point from \( \{\bar{x}\} \times S(\bar{x}) \). As mentioned earlier, validity of MFCQ at one point from \( \{\bar{x}\} \times \Gamma(\bar{x}) \) is already enough to make sure that RCPLD_F holds at all these points. Let us mention that, by definition of \( \varphi \), there is no \( y \in S(\bar{x}) \) such that \( h_0(\bar{x}, y) < 0 \), i.e., Slater’s constraint qualification cannot hold for the set \( S(\bar{x}) \). Consequently, we cannot guarantee validity of RCPLD_S at each point from \( \{\bar{x}\} \times S(\bar{x}) \) by the simple arguments used above.

A related comment can be found in the subsequent remark.

**Remark 4.12.** Fix some point \((\bar{x}, \bar{y}) \in \text{gph}S\). It is well known that this guarantees the validity of the so-called Fritz–John conditions, i.e., we find \( \lambda_0, \lambda_1, \ldots, \lambda_p \in \mathbb{R} \) which do not all vanish at the same time such that

\[
\lambda_0 \nabla_y f(\bar{x}, \bar{y}) + \sum_{i=1}^p \lambda_i \nabla_y h_i(\bar{x}, \bar{y}) = 0,
\forall i \in I \cup \{0\} : \lambda_i \geq 0,
\forall i \in I : \lambda_i h_i(\bar{x}, \bar{y}) = 0
\]

holds, see (Bertsekas, 1999, Proposition 3.3.5). This, however, shows that the constraint qualification MFCQ w.r.t. the representation (10) of the mapping \( S \) cannot hold at \((\bar{x}, \bar{y})\) since the pair of families

\[
\left( (\nabla_y h_i(\bar{x}, \bar{y}))_{i \in I(\bar{x}, \bar{y}) \cup \{0\}} : (\nabla_y h_i(\bar{x}, \bar{y}))_{i \in J} \right)
\]

is positive-linearly dependent. Thus, versions of Theorem 4.8 and Theorem 4.11 which exploit MFCQ w.r.t. \( S \) instead of RCPLD_S would not be reasonable at all. On the other hand, simple examples reveal that RCPLD_S can hold at \((\bar{x}, \bar{y})\), see Example 4.19 as well.

The subsequent remark comments on a way which allows a slight generalization of the second statement of Theorem 4.8 as well as Theorem 4.11.

**Remark 4.13.** Let \( S \) be R-regular at some point \((\bar{x}, \bar{y}) \in \text{gph}S\) w.r.t. \( \text{dom}S \). Inspecting the proof of (Bednarczuk et al., 2019, Theorem 5.1), one only needs local Lipschitz continuity of all data functions at \((\bar{x}, \bar{y})\) w.r.t. the set \( \text{dom}S \times \mathbb{R}^m \) in order to infer validity of the Aubin property of \( S \) at \((\bar{x}, \bar{y})\) w.r.t. \( \text{dom}S \).

Thus, the second assertion of Theorem 4.8 as well as the assertion of Theorem 4.11 remain true if all stated assumptions and assertions are stated w.r.t. \( \text{dom}\Gamma \) since this is enough to ensure local coincidence of \( \text{dom}S \) and \( \text{dom}\Gamma \). Particularly, relying on Corollary 4.7, the assumption on \( \bar{x} \) to be an interior point of \( \text{dom}\Gamma \) can be removed from the associated counterpart of Theorem 4.11.
Keeping Lemma 2.5 and Remark 4.13 in mind, the following corollary is a direct consequence of Theorem 4.11. Indeed, this is not surprising in the light of Corollary 3.4.

**Corollary 4.14.** Fix a point \( \bar{x} \in \text{dom } \Gamma \). Let \((A1')\) and \((A2)\) hold. Furthermore, let \( \text{RCPLD}_S \) w.r.t. \( \text{dom } \Gamma \) hold at each point from \( \{ \bar{x} \} \times S(\bar{x}) \). Finally, let \( \text{RCPLD}_\Gamma \) be valid at each point from \( \{ \bar{x} \} \times \Gamma(\bar{x}) \) w.r.t. \( \text{dom } \Gamma \). Then, \( S \) is lower semicontinuous at \( \bar{x} \) w.r.t. \( \text{dom } S \).

### 4.2 Bilevel optimization

Let us now consider the bilevel optimization problem

\[
\min_{x} \left\{ F(x, y) \mid x \in X, y \in S(x) \right\}
\]

(BPP)

where \( F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is a continuously differentiable mapping, \( X \subset \mathbb{R}^n \) is a closed set, and \( S: \mathbb{R}^n \Rightarrow \mathbb{R}^m \) is the solution mapping associated with \( (P(x)) \). The model \( (BPP) \) dates back to v. Stackelberg (1934) where it has been stated first in the context of economical game theory. The quotation marks in \( (BPP) \) emphasize that this problem is not necessarily well-determined. Indeed, whenever there is some \( x \in X \cap \text{dom } S \) where \( S(x) \) is not a singleton, then the decision maker in \( (BPP) \) cannot determine the associated objective value and, thus, classical minimization is not applicable. In order to avoid this shortcoming, one often replaces \( (BPP) \) by its so-called optimistic or pessimistic version which are given by

\[
\min_{x} \{ \varphi_o(x) \mid x \in X \} \quad \text{and} \quad \min_{x} \{ \varphi_p(x) \mid x \in X \},
\]

respectively, where the functions \( \varphi_o, \varphi_p: \mathbb{R}^n \to \mathbb{R} \) are defined as follows:

\[
\forall x \in \mathbb{R}^n : \quad \varphi_o(x) := \inf_y \{ F(x, y) \mid y \in S(x) \} \quad \varphi_p(x) := \sup_y \{ F(x, y) \mid y \in S(x) \}.
\]

This way, the optimistic and pessimistic reformulation of \( (BPP) \) reflect a cooperative behaviour and a worst-case scenario between the decision makers in \( (BPP) \) and \( (P(x)) \), respectively.

Due to numerous underlying applications e.g. from finance, chemistry, or logistics, bilevel optimization is one of the hot topics in mathematical programming. On the other hand, \( (BPP) \) is an inherently difficult problem. Besides the above observation that it might not be well-defined, it suffers from inherent non-convexity, irregularity, and the implicit character of its feasible set. That is why numerous publications dealing with the derivation of problem-tailored optimality conditions, constraint qualifications, and solution algorithms appeared during the last three decades. We refer the interested reader to the monographs Bard (1998); Dempe (2002); Dempe et al. (2015) for a detailed introduction to bilevel optimization.

Let us take a look back at the optimistic and pessimistic version of \( (BPP) \) first. Under not too restrictive assumptions, the solution mapping \( S \) is upper semicontinuous, and this property implies lower semicontinuity of \( \varphi_o \), i.e., in case where \( X \) is compact, the
optimistic version of (BPP) is likely to possess a global minimizer. On the other hand, in order to guarantee lower semicontinuity of $\varphi_p$, one has to assume that $S$ is lower semicontinuous w.r.t. $\text{dom} \ S$. This is quite a restrictive assumption, but our result from Corollary 4.14 depicts that it can be valid in particular problem settings. In this regard, the subsequent theorem follows from our aforementioned result and (Dempe, 2002, Theorem 5.3).

Theorem 4.15. Let (A1') hold. Furthermore, assume that $X \subset \text{dom} \Gamma$ holds true and that $\Gamma$ is locally bounded at each point from $X$. Additionally, let RCPLD$_S$ w.r.t. $\text{dom} \Gamma$ hold at each point from $\text{gph} \ S \cap (X \times \mathbb{R}^m)$. Moreover, let RCPLD$_\Gamma$ w.r.t. $\text{dom} \Gamma$ hold at each point from $\text{gph} \ \Gamma \cap (X \times \mathbb{R}^m)$. Finally, let $X$ be nonempty and compact. Then, there exists a pessimistic solution of (BPP).

The crucial requirement in the above theorem obviously is the validity of RCPLD$_S$ w.r.t. $\text{dom} \Gamma$ at each point from $\text{gph} \ S \cap (X \times \mathbb{R}^m)$. However, let us note that this is inherent for lower level problems of type

$$\min_y \{c^T y \mid By \leq b(x)\}$$

where $c \in \mathbb{R}^m$ and $B \in \mathbb{R}^{\ell \times m}$ are matrices while $b: \mathbb{R}^n \to \mathbb{R}^\ell$ is a continuous function. In this situation, RCPLD$_\Gamma$ w.r.t. $\text{dom} \Gamma$ holds at all points from $\text{gph} \ \Gamma \cap (X \times \mathbb{R}^m)$ as well. This means that (BPP) with the special lower level problem (11) is likely to possess a pessimistic solution.

Observing that the optimistic and pessimistic version of (BPP) might be interpreted as a three-level decision process, the derivation of optimality conditions via these models is quite challenging, see e.g. Dempe et al. (2012, 2014). In the literature, it is a common approach to consider

$$\min_{x,y} \{F(x, y) \mid x \in X, y \in S(x)\} \quad (\text{BPP}')$$

instead. This well-defined optimization problem is closely related to the optimistic version of (BPP), see (Dempe et al., 2012, Proposition 6.9) for details. Furthermore, by definition of the optimal value function, one can easily check that (BPP') is fully equivalent to the single-level optimization problem

$$\min_{x,y} \{F(x, y) \mid x \in X, f(x, y) - \varphi(x) \leq 0, y \in \Gamma(x)\} \quad (\text{OVR})$$

which is commonly referred to as the optimal value reformulation or value function transformation of (BPP). Although this problem is still quite challenging due to the implicit character of $\varphi$, the general non-smoothness of $\varphi$, and its inherent irregularity, it has been exploited intensively for the derivation of necessary optimality conditions and solution algorithms, see, e.g., Dempe et al. (2007); Dempe and Franke (2015, 2016); Dempe and Zemkoho (2013); Fischer et al. (2019); Mordukhovich et al. (2012); Ye and Zhu (1995, 2010) and the references therein. The key idea in all these papers is to use a partial penalization argument in order the shift the crucial constraint $f(x, y) - \varphi(x) \leq 0$ from the feasible set of (OVR) to its objective function. Whenever this penalization is locally exact, this approach is reasonable in theory and numerical practice. Following Ye and Zhu (1995), we refer to this property as partial calmness.
Definition 4.16. Let \((\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m\) be a locally optimal solution of \((\text{BPP}'')\). We say that this program is partially calm at \((\bar{x}, \bar{y})\) if there exist a neighbourhood \(U\) of \((\bar{x}, \bar{y}, 0)\) and some constant \(\kappa > 0\) such that we have \(F(x, y) - F(\bar{x}, \bar{y}) + \kappa |u| \geq 0\) for each triplet \((x, y, u) \in U\) which satisfies

\[ x \in X, \quad f(x, y) - \varphi(x) \leq u, \quad y \in \Gamma(x). \]

Indeed, (Ye and Zhu, 1995, Proposition 3.3) shows that \((\text{BPP}'')\) is partially calm at one of its local minimizers \((\bar{x}, \bar{y})\) if and only if there is some \(\kappa > 0\) such that \((\bar{x}, \bar{y})\) is a local minimizer of

\[ \min_{x,y} \{ F(x, y) + \kappa (f(x, y) - \varphi(x)) \mid x \in X, y \in \Gamma(x) \} \]

for each \(\bar{k} \geq \kappa\). Noting that the latter optimization problem may satisfy standard constraint qualifications, the presence of partial calmness indeed opens a way to the derivation of necessary optimality conditions for \((\text{BPP}'')\) since the potential non-smoothness of \(\varphi\) now can be simply handled with suitable subdifferential constructions from variational analysis.

In (Mehlitz et al., 2020, Section 3), the authors provide an overview of conditions which are sufficient for the presence of partial calmness in bilevel optimization. Our particular interest here lies in a result which can be distilled from (Mehlitz et al., 2020, Lemmas 3.4, 3.5, and 3.6).

**Proposition 4.17.** Let \((\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m\) be a local minimizer of \((\text{BPP}'')\) such that \(S\) is R-regular at \((\bar{x}, \bar{y})\) w.r.t. \(\text{dom } S\). Furthermore, assume that the sets \(\text{dom } \Gamma\) and \(\text{dom } S\) coincide locally around \(\bar{x}\). Then, \((\text{BPP}'')\) is partially calm at \((\bar{x}, \bar{y})\).  

We would like to mention that a related result can be found in (Bednarczuk et al., 2019, Theorem 6.1).

We are now in position to apply Theorems 4.8 and 4.11 as well as Remark 4.13 in order to infer new sufficient conditions for the validity of partial calmness.

**Theorem 4.18.** Let \((\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m\) be a local minimizer of \((\text{BPP}'')\). Additionally, let one of the following additional conditions hold.

(a) Let \((A1')\) and \((A2)\) be valid. Furthermore, let \(\text{RCPLD}_{S} \text{ w.r.t. dom } \Gamma\) hold at each point from \(\{\bar{x}\} \times S(\bar{x})\). Finally, let \(\text{RCPLD}_{\Gamma} \text{ w.r.t. dom } \Gamma\) hold at each point from \(\{\bar{x}\} \times \Gamma(\bar{x})\).

(b) Let \(S\) be inner semicontinuous at \((\bar{x}, \bar{y})\) w.r.t. \(\text{dom } \Gamma\) and let \(\text{RCPLD}_{S} \text{ w.r.t. dom } \Gamma\) hold at this point.

Then, \((\text{BPP}'')\) is partially calm at \((\bar{x}, \bar{y})\).

As we already observed above, the crucial assumption \(\text{RCPLD}_{S}\) is generally valid for lower level problems of type (11) which is why the local minimizers of the associated bilevel optimization problem \((\text{BPP}'')\) are always partially calm. This observation already has been made in (Minchenko and Berezhnov, 2017, Lemma 2.1) and...
(Mehlitz et al., 2020, Theorem 4.1). However, we would like to point out that our result from Theorem 4.18 may address far more general situations as demonstrated with the aid of the subsequent example.

**Example 4.19.** Let us consider the bilevel optimization problem

$$\min_{x,y} \{(x - 1)^2 + y^2 \mid y \in S(x)\} \quad (12)$$

where $S: \mathbb{R} \rightrightarrows \mathbb{R}$ is the solution mapping of the parametric optimization problem

$$\min_y \{(x + y - 2)^2 \mid y^2 - x \leq 0, y \geq 0\}.$$  

Some computations show

$$\forall x \in \mathbb{R}: \quad S(x) = \begin{cases} \emptyset & x \in (-\infty, 0), \\ \{\sqrt{x}\} & x \in [0, 1], \\ \{2-x\} & x \in [1, 2], \\ \{0\} & x \in [2, \infty), \end{cases} \quad \varphi(x) = \begin{cases} \infty & x \in (-\infty, 0), \\ (x + \sqrt{x} - 2)^2 & x \in [0, 1], \\ 0 & x \in [1, 2], \\ (x - 2)^2 & x \in [2, \infty). \end{cases}$$

We observe that $S$ is a single-valued and continuous map w.r.t. its domain. Particularly, it is inner semicontinuous w.r.t. $\text{dom} S$ at each point of its graph. Furthermore, $\text{dom} S = \text{dom} \Gamma$ holds. Using the above formula for $S$, one can easily check that (12) possesses the uniquely determined global minimizer $(\tilde{x}, \tilde{y}) := (3/2, 1/2)$ while there is another local minimizer at $(\bar{x}, \bar{y}) := (1/2, \sqrt{2}/2)$.

Next, we consider the point $(\tilde{x}, \tilde{y})$. Here, the set of lower level active constraints is empty and the gradient of the lower level objective function vanishes but, clearly, does not generally vanish in a neighbourhood of $(\tilde{x}, \tilde{y})$. Thus, $\text{RCPLD}_S$ is violated at $(\tilde{x}, \tilde{y})$, i.e., (12) is partially calm at this point by Theorem 4.18.

We observe that each subsystem of the family $(2(x + y - 2), 2y)$ possesses constant rank around the reference point $(\tilde{x}, \tilde{y})$, and this is sufficient for the validity of $\text{RCPLD}_S$ at $(\tilde{x}, \tilde{y})$, i.e., (12) is partially calm at this point by Theorem 4.18.

5 Conclusions

In this manuscript, we have shown that the validity of the constraint qualification $\text{RCPLD}$ is sufficient to infer the presence of $\text{R-regularity}$ for set-valued mappings of type (1). Our results generalize similar considerations which exploit the constraint qualifications $\text{MFCQ}$ or $\text{RCRCQ}$ for that purpose, see Bednarczuk et al. (2019); Luderer et al.
We applied our finding in order to study nonlinear parametric optimization problems and bilevel optimization problems. First, we inferred new criteria ensuring Lipschitz continuity of optimal value functions as well as R-regularity and lower semicontinuity of solution mappings in parametric programming. As we have seen, a similar analysis w.r.t. the solution mapping is not possible under MFCQ. Second, these results were exploited in order to state a criterion for the existence of solutions in pessimistic bilevel optimization as well as a sufficient condition for the validity of the partial calmness property in optimistic bilevel optimization. Throughout the manuscript, simple examples illustrate applicability but also the limits of our findings.

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