Dispersionless Fermionic KdV

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Abstract

We analyze the dispersionless limits of the Kupershmidt equation, the SUSY KdV-B equation and the SUSY KdV equation. We present the Lax description for each of these models and bring out various properties associated with them as well as discuss open questions that need to be addressed in connection with these models.
1 Introduction:

In the last two decades, integrable models [1-3] have drawn a lot of attention from various points of view. More recently, however, research in a particular class of integrable models, known as the dispersionless limits of integrable models, has become quite active. These involve equations of hydrodynamic type [4-8] and include systems like the Riemann equation [8-10], the polytropic gas dynamics [8,11], the chaplygin gas, the Born-Infeld model [12-13] etc. These are models which can be obtained from a “classical” limit [6,14] of integrable models where the dispersive terms are absent. Thus, the dispersionless limit of the KdV equation [15], for example, corresponds to the Riemann equation and so on. The interesting thing about these models (and the “classical” limit) is that, if we know the Lax operator description of a given model, then, the dispersionless limit also has a Lax description, in terms of a Lax function on the phase space, where the commutator is replaced by a Poisson bracket. The Lax function can, in fact, be determined from the Lax operator in a systematic manner [7]. (Of course, there are also dispersionless models whose integrable, dispersive counterparts are not known.)

While a lot is known about the dispersionless limits of bosonic theories, nothing is known about the fermionic theories (with or without supersymmetry). The main difficulty lies in the fact that, in dealing with fermionic models, we have to deal with a phase space with both bosonic and fermionic coordinates. The classical fermionic variables are nilpotent and, consequently, a power series representation that is so crucial in a Lax description seems to fail. Thus, for example, the supersymmetric KdV equation [16] is described in terms of a Lax operator which involves powers of the supercovariant derivative $D$. This is a fermionic operator whose square equals $\partial^2$. On the other hand, a phase space description of the model in the dispersionless limit would naively seem to require using a fermionic phase space variable which would give the same action as $D$ through a Poisson bracket relation. Such an object, however, would be nilpotent and there cannot be a power series in this variable. Such difficulties have hindered the understanding of such fermionic theories so far.

In this paper, we make the first attempt towards understanding such models. We analyze the dispersionless limits of fermionic KdV equations which include the Kupershmidt equation [17], the SUSY KdV-B equation [16,18] as well as the SUSY KdV equation [16]. All these equations can be thought of as fermionic extensions (with or without supersymmetry) of the Riemann equation which are integrable (Of course, we only concern ourselves with $N=1$ supersymmetry.). We obtain a “classical” Lax description for each of these models. However, the Lax functions and the Lax equations are obtained by brute force (with a lot of hard work) and we do not yet understand a systematic way in which the phase space Lax functions (for the fermionic models) can be constructed from the corresponding Lax operators. This remains an open question. From the Lax description, we construct all the local conserved quantities for the models in the standard manner. However, as we emphasize in the paper, fermionic models, in particular, the supersymmetric ones contain nonlocal conserved charges [19-20] as well, and it is not at all clear how a phase space Lax description can generate such quantities from a “Trace” of the Lax function. This, too, is an interesting open question. While the Lax description of the bosonic models in the dispersionless limit does give
the first Hamiltonian structure from a generalized Gelfand-Dikii bracket [10], as we discuss, except for the SUSY KdV-B equation, we are unable to obtain the first Hamiltonian structure from a generalized Gelfand-Dikii bracket for the other two models. This is in spite of a Dirac analysis which we have carried out for the constrained form of the Lax functions [21]. In the case of SUSY KdV, this may be understood as signifying that the first Hamiltonian structure of SUSY KdV vanishes in this “classical” limit. However, for the dispersionless Kupershmidt equation, there does exist a first Hamiltonian structure and it is unclear how to obtain this from the Lax description itself. Our paper is organized as follows. In section 2, we recall, very briefly, the essential features of the dispersionless KdV equation. In section 3, we present the Lax description for the dispersionless Kupershmidt equation and discuss all its properties in detail. In section 4, we discuss the analogous results for the dispersionless SUSY KdV-B equation. Here, there are nonlocal conserved charges as well, and we present the algebra of the charges which takes a particularly simple form. In section 5, we discuss results for the dispersionless SUSY KdV equation. Here, too, there are nonlocal charges and we present the algebra of the charges in a closed form. In section 6, we present a short conclusion emphasizing the open questions within the context of such models.

2 Dispersionless Limit of KdV Equation:

In this section, we will briefly review the known features of the dispersionless KdV equation [8,10]. As is well known, the KdV equation [15]

\[
\frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}
\]

can be described by the Lax equation

\[
\frac{\partial L}{\partial t} = 4[(L^{3/2})_+, L]
\]

where the Lax operator

\[ L = \partial^2 + u(x) \]

with \( \partial \) representing \( \frac{\partial}{\partial x} \), \( u \) the KdV variable and \((,)_+ \) denoting the differential part of a pseudo-differential operator. The third derivative term on the right hand side of eq. (1) represents a dispersive term and the dispersionless limit of this equation (namely, the equation where the dispersive term is absent) is obtained as follows. Let

\[
\frac{\partial}{\partial t} \rightarrow \epsilon \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x} \rightarrow \epsilon \frac{\partial}{\partial x}
\]

without rescaling the dynamical variable. Then, in the limit, \( \epsilon \rightarrow 0 \), the KdV equation reduces to

\[
\frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x}
\]
which is the Riemann equation and has no dispersive term. This is known as the dispersionless limit of the KdV equation and it has a Lax description as well. Consider the Lax function on the classical phase space

$$L = p^2 + u(x)$$  \hspace{1cm} (6)

where $p$ is the momentum variable of the phase space. Then, with the canonical Poisson bracket relations, it is easy to check that

$$\frac{\partial L}{\partial t} = -4\{(L^{3/2})_+, L\}$$  \hspace{1cm} (7)

with $()_+$ now representing terms with non-negative powers of $p$, gives the Riemann equation or the dispersionless limit of the KdV equation. Thus, one can think of the dispersionless limit as sort of a “classical” limit where the Lax operator goes into a Lax function ($\partial \rightarrow p$) and an operator Lax equation involving a commutator goes into a Lax equation involving Poisson brackets. Furthermore, the conserved quantities and the Hamiltonian structures (at least, the first structure) can be obtained from this Lax function [10]. (The difference in sign between the operator equation (2) and the phase space equation (7) results from the fact that $[\partial, f] = \frac{df}{dx}$ while $\{p, f\} = -\frac{df}{dx}$. This would be reflected in all the cases that we discuss.)

3 Dispersionless Limit of Kupershmidt Equation:

The Kupershmidt equation [17] is a nontrivial fermionic extension of the KdV equation which is integrable. The dynamical equations, in this case, involve a bosonic variable $u$ as well as a fermionic variable $\psi$ and are given by the coupled equations

$$u_t = u_{xxx} + 6uu_x - 12\psi\psi_{xx}$$  \hspace{1cm} \psi_t = 4\psi_{xxx} + 3u_x\psi + 6u\psi_x$$  \hspace{1cm} (8)

where the subscripts denote differentiation with respect to the particular variable. This is not a supersymmetric system, but gives a nontrivial, coupled boson-fermion system (reducing to the KdV equation when $\psi = 0$) which has a bi-Hamiltonian structure and is integrable.

The Kupershmidt equation can be described by a Lax equation. Consider a Lax operator of the form

$$L = \partial^2 + u + \psi\partial^{-1}\psi$$  \hspace{1cm} (9)

Unlike the KdV equation, this Lax operator is truly a pseudo-differential operator involving the fermionic variable $\psi$. Once again, it can be easily checked that the Lax operator equation

$$\frac{\partial L}{\partial t} = 4\{(L^{3/2})_+, L\}$$  \hspace{1cm} (10)

gives the Kupershmidt equations.
In trying to derive the dispersionless limit of the Kupershmidt equation, one runs into various problems. First, we note that, under the scaling \( \partial \rightarrow \epsilon \partial \),
\[
\partial^{-1} \rightarrow \epsilon^{-1} \partial \tag{11}
\]
so that in the limit \( \epsilon \rightarrow 0 \), the term in the Lax operator containing the fermionic variables would appear to diverge. Second, if we naively let \( \partial \rightarrow p \), then, of course, the fermionic term in the Lax operator would vanish. An alternate approach is to recognize that \( \partial^{-1} \) is really an operator which can be taken to the right with the help of the Leibnitz rule, giving an infinite series of terms in which one can let \( \partial \rightarrow p \) and the terms involving the fermionic variables would no longer vanish. However, a short calculation shows that such a procedure leads to an inconsistent Lax equation. Therefore, finding a Lax function and a Lax description for the dispersionless limit of the Kupershmidt equation genuinely poses a challenge.

The solution to this problem comes as follows. Consider the Lax function
\[
L = p^2 + u - p^{-2} \psi \psi_x \tag{12}
\]
Namely, the term containing the fermionic variables corresponds only to the first nontrivial term in taking the operator \( \partial^{-1} \) to the right and setting \( \partial = p \). It is now easy to check that the Lax equation (once again, note the difference in sign)
\[
\frac{\partial L}{\partial t} = -4\{(L^{3/2})_+, L\} \tag{13}
\]
gives rise to the equations
\[
\begin{align*}
    u_t &= 6uu_x - 12\psi \psi_{xx} \\
    \psi_t &= 3u_x \psi + 6u\psi_x 
\end{align*} \tag{14}
\]
which indeed represent the dispersionless limit of the Kupershmidt equation in (8). (It can be checked that the “classical” limit involves the scaling \( \partial \rightarrow \epsilon \partial \) and \( \psi \rightarrow \epsilon^{-1/2} \psi \) without which the fermion terms would not be present in the boson equation. This is indeed very different from the classical limit in bosonic theories. We will see this in all the fermionic models we describe.) The passage from the Lax operator description of the Kupershmidt equation to the Lax function description of the corresponding dispersionless equation clearly is not as straightforward as the bosonic KdV equation and it is not at all clear how the Lax function of eq. (12) could have been deduced from that of eq. (9).

The existence of a Lax description, as we know, gives, in a simple way, many of the interesting properties of the integrable system. Thus, for example, from the Lax function, we can define
\[
H_n = \text{Tr} \, L^{2n+1} = (-1)^{n+1} \left( \frac{2n+1}{n+1} \right) \int dx \left[ u^{n+1} - 2n(n+1)u^n \psi \psi_x \right] \tag{15}
\]
where “Tr” represents the integral of the coefficient of the \( p^{-1} \) term in the expression. We can, in fact, easily check, using the equations of motion, that the \( H_n \)'s are conserved. Furthermore, these
can be identified with the limit $\epsilon \to 0$ of the conserved charges for the Kupershmidt equation under the scaling

$$\partial \to \epsilon \partial, \quad \psi \to \epsilon^{-1/2} \psi$$

(16)

This is again a manifestation of the necessity for scaling the fermion variables to obtain the dispersionless limit (without the scaling, there would be no fermion terms).

The Lax description, for the bosonic KdV equation, of course, gives the Hamiltonian structures (at least the first one) naturally through a generalization of the Gelfand-Dikii bracket [10]. A similar analysis fails in this case, in spite of a careful treatment (Dirac analysis) of the constrained nature of the Lax function [21-22]. The derivation of the Hamiltonian structure for the Kupershmidt equation from the Lax description, therefore, remains an open question. However, the Hamiltonian structures for this system are not hard to obtain directly. In fact, it can be checked easily that

$$D_1 = \left( \begin{array}{cc} \partial & 0 \\ 0 & -\frac{1}{4} \end{array} \right) \delta(x-y)$$

$$D_2 = \left( \begin{array}{cc} \partial u + u \partial & \frac{1}{2} \partial \psi + \psi \partial \\ \frac{1}{2} \psi \partial + \partial \psi & -\frac{1}{2} u \end{array} \right) \delta(x-y)$$

(17)

give the first two Hamiltonian structures of the system which can also be derived from the first two Hamiltonian structures of the Kupershmidt equation under appropriate scaling (see (16)). Once we know the Hamiltonian structures, it is straightforward to show that the conserved quantities are in involution. For example, with the first Hamiltonian structure in (17), we have

$$\{H_n, H_m\}_1 = -2m(n+1)(m+1)(m-1) \int dx \left( u^{n+m-2} \psi \psi_x \right)_x$$

$$= 0$$

(18)

with the usual assumptions on asymptotic fall off of the dynamical variables. This proves that the model remains integrable in the dispersionless limit.

4 Dispersionless Limit of SUSY KdV-B Equation:

The supersymmetric KdV-B equations correspond to a trivial supersymmetrization of the KdV equation [16,18] and are given by

$$u_t = u_{xxx} + 6uu_x$$

$$\psi_t = \psi_{xxx} + 6u\psi_x$$

(19)

This is a set of simple equations where the boson equation does not depend on the fermionic variable. However, the pair of equations in eq. (19) are invariant under the supersymmetry transformations

$$\delta \psi = \lambda u$$

$$\delta u = -\lambda \psi_x$$

(20)
with $\lambda$ a constant Grassmann parameter of the transformation. It is this particular supersymmetrytization of the KdV equation which shows up in the supersymmetric one matrix model.

The SUSY KdV-B has a Lax description and, being a supersymmetric system, the proper description for it is in superspace. Let us define a fermionic superfield

$$\Phi(x, \theta) = \psi(x) + \theta u(x)$$  \hspace{1cm} (21)

where $\theta$ represents the Grassmann coordinate of the superspace. Let us now define a Lax operator

$$L = D^4 + (D\Phi)$$  \hspace{1cm} (22)

where the supercovariant derivative is defined as

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}, \quad D^2 = \partial$$  \hspace{1cm} (23)

It is now straightforward to check that the Lax equation

$$\frac{\partial L}{\partial t} = -4\{\frac{L^3}{2} + (L^3/2)_+, L\}$$  \hspace{1cm} (24)

where $(.)_+$ refers to non-negative powers of $D$, leads to

$$\Phi_t = \Phi_{xxx} + 6(D\Phi)\Phi_x$$  \hspace{1cm} (25)

which contains both bosonic and fermionic components of the SUSY KdV-B equation.

This Lax operator is in many ways reminiscent of the Lax operator for the KdV equation and finding the dispersionless limit of this system, therefore, does not pose too much difficulty. However, there are some interesting features that arise in the case of the dispersionless equation which we will describe. First, let us note that although the Lax operator for the SUSY KdV-B system is written in terms of the supercovariant derivative $D$, we could have written it equally well in terms of $\partial$ because of the relation between the two. Keeping this in mind, let us look at the Lax function

$$L = p^2 + (D\Phi)$$  \hspace{1cm} (26)

It is then, easy to check that the Lax equation

$$\frac{\partial L}{\partial t} = -4\{(L^{3/2})_+, L\}$$  \hspace{1cm} (27)

gives the dispersionless limit of the SUSY KdV-B equation, namely,

$$\Phi_t = 6(D\Phi)\Phi_x$$  \hspace{1cm} (28)

which, in components, takes the form

$$u_t = 6uu_x$$

$$\psi_t = 6u\psi_x$$  \hspace{1cm} (29)
These equations are indeed the dispersionless limits of the SUSY KdV-B equations in (19) and we can think of them as the trivial supersymmetrization of the Riemann equation.

The Lax description, as we have seen, gives us the conserved quantities and it is easy to check that

\[
H_{n+1} = \frac{2^{n+2} n!}{(2n + 3)!!} \text{Str} \frac{L^{2n+3}}{2}
\]

\[
= \frac{1}{(n+1)(n+2)} \int dz (D\Phi)^{n+2}
\]

\[
= \frac{1}{n+1} \int dx u^{n+1} \psi_x
\]

(30)

are conserved for \(n = 0, 1, 2, \cdots\). Here “\text{Str}” stands for the integration of the coefficient of \(p^{-1}\) over the entire superspace with \(dz = dx d\theta\). We have also chosen a particular normalization for later purposes. There are several things to note about these conserved charges. First, since they are expressed as integrals over the superspace, they are automatically invariant under supersymmetry transformations. However, we see that these charges are fermionic in nature and this suggests that the Hamiltonian structure for the system should be odd (anti-bracket structure). This can, in fact, be readily verified, namely, let us define the dual

\[
Q = q_0 + q_1 p^{-1}
\]

(31)

so that

\[
\text{Str} \ LQ = \int dz q_1 (D\Phi) = - \int dz (Dq_1) \Phi
\]

(32)

Then, a generalization of the Gelfand-Dikii definition of the first Hamiltonian structure [10]

\[
\{\text{Str} \ LQ, \text{Str} \ LV\} = \text{Str} \ L \{Q, V\}
\]

(33)

yields

\[
\{\Phi(z), \Phi(z')\} = -2 \delta(z - z')
\]

(34)

which is, indeed, the correct Hamiltonian structure for the system (up to normalization) and is an odd structure. This is a special feature of the SUSY KdV-B system [23] (as well as the SUSY TB-B [24] system).

One way to understand the odd Hamiltonian structure is as follows. Let us consider the Lagrangian

\[
L = \int dz \left[ \frac{1}{2} \Phi (\Phi_t - \frac{1}{(n+1)(n+2)} (D\Phi)^{n+2}) \right]
\]

\[
= \int dx \left[ \frac{1}{2} (u \psi_t - \psi u_t) - \frac{1}{n+1} u^{n+1} \psi_x \right]
\]

(35)

This Lagrangian describes the dispersionless limit of the SUSY KdV-B hierarchy as its Euler-Lagrange equations and is clearly fermionic. It is then, easy to work out from the special structure
of this Lagrangian that there are constraints in such a theory which modify the canonical even
Poisson brackets to Dirac brackets which are odd.

The first Hamiltonian structure can be written as a $2 \times 2$ matrix in terms of the components
as (up to a normalization)

$$D_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(36)

It is also easy to check from the explicit forms of the conserved quantities in eq. (30) that the
recursion operator (function) for the system is given by

$$R = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} = u \cdot I$$

(37)

so that

$$\left( \begin{array}{c} \frac{\delta H_{n+2}}{\delta u} \\ \frac{\delta H_{n+2}}{\delta \psi} \end{array} \right) = R \left( \begin{array}{c} \frac{\delta H_{n+1}}{\delta u} \\ \frac{\delta H_{n+1}}{\delta \psi} \end{array} \right)$$

(38)

The recursion operator, then, leads to the hierarchy of Hamiltonian structures for the system given
by

$$D_{n+1} = R^n D_1 = \begin{pmatrix} 0 & -u^n \\ u^n & 0 \end{pmatrix}$$

(39)

and all of them are indeed odd structures.

In addition to the fermionic conserved charges, $H_{n+1}$, the dispersionless SUSY KdV-B equation
also has bosonic conserved charges. It is easy to check that the charges

$$Q_{n+1} = \int dz \Phi (D\Phi)^n$$

$$= \int dx \left( u^{n+1} - n u^{n-1} \psi \bar{\psi} \right)$$

(40)

are conserved. Furthermore, these charges are supersymmetric and can be obtained from the
conserved bosonic charges of the SUSY KdV-B system (which are obtained by taking “super Trace”
of multiples of quartic powers of the Lax operator). However, the charges, in the present case, cannot
be obtained from the Lax function of the system. This is, in fact, a puzzling general feature of such
systems. Namely, we know that supersymmetric integrable systems possess nonlocal conserved
charges and they arise, say in the case of SUSY KdV, from the trace of quartic powers of the Lax
operator such as $s\text{Tr} L^{2n+1}$. In the case of operators, this is meaningful because the supercovariant
derivative, $D$, is an operator square root of $\partial$. In a classical phase space, however, no such relation
exists and it is not clear how to obtain the analog of the nonlocal charges in such a case. Thus, for
example, in the dispersionless SUSY KdV-B system, it is easy to see that the quantities

$$q_n = \int dx u^n = \int dz D^{-1} (D\Phi)^n$$

(41)

are conserved. However, these are nonlocal quantities and it is not clear how to obtain such
conserved quantities from the Lax function. Let us note that $q_n$’s are purely bosonic, and are not
invariant under supersymmetry transformations (even though they can be written as an integral over superspace, they involve $D^{-1}$ which leads to violation of supersymmetry).

Finally, let us note that we have a set of bosonic and fermionic conserved charges, in this theory, which are invariant under supersymmetry. Therefore, it is meaningful to investigate the algebra of these charges. With the first Hamiltonian structure, for example, we can easily calculate the algebra which has the form

$$\{H_n, H_m\} = 0$$
$$\{Q_n, H_m\} = 0$$
$$\{Q_n, Q_m\} = (n - m)(n - 1)(m - 1)H_{n+m-1}$$  \hspace{1cm} (42)

The first equation simply says that the conserved Hamiltonians are in involution so that the system is integrable. The second is an expression of the fact that the $Q_n$’s are conserved under all the flows of the hierarchy. The nontrivial and really interesting one is the last one which is reminiscent of supersymmetry algebras. The important thing to remember in this algebra is the fact that $Q_n$’s are bosonic, $H_n$’s fermionic and the bracket is odd.

5 Dispersionless Limit of SUSY KdV Equation:

The supersymmetric KdV-B equation represents a trivial supersymmetrization of the KdV equation and hence there was not much difficulty in taking its dispersionless limit. The $N=1$ supersymmetric KdV equation, on the other hand, is a case where we expect some challenge in taking the dispersionless limit, just like the Kupershmidt equation. Let us recall that the equation [16]

$$\Phi_t = (D^6\Phi) + 3D^2(\Phi(D\Phi))$$  \hspace{1cm} (43)

which in components has the form

$$u_t = u_{xxx} + 6uu_x - 3\psi\psi_{xx}$$
$$\psi_t = \psi_{xxx} + 3(u\psi)_x$$  \hspace{1cm} (44)

represents the $N=1$ supersymmetric KdV equation which is known to be integrable. This equation is invariant under the supersymmetry transformations of eq. (20) and has a Lax description of the following form. Let us consider a Lax operator of the form

$$L = D^4 + D\Phi$$  \hspace{1cm} (45)

Then, it is easy to check that (in this case, there is a degeneracy and the Lax operator $L = D^4 - \Phi D$ works equally well) the Lax equation

$$\frac{\partial L}{\partial t} = 4[(L^{3/2})_+, L]$$  \hspace{1cm} (46)
gives us the SUSY KdV equation. It is here that we see the fermion nature of the problem coming into play. Unlike the case of SUSY KdV-B equation, here the Lax operator cannot be written completely in terms of the bosonic \( \partial \). One has to understand how to take the “classical” limit of the operator \( D \). Of course, classically, we would have a phase space which would involve both bosonic and fermionic coordinates. If \( p \) and \( p_\theta \) represent the bosonic and the fermionic momenta respectively, then, we can define a fermionic function

\[
\Pi = -(p_\theta + \theta p)
\]

which would generate covariant differentiation through [25]

\[
\{\Pi, A\} = (DA)
\]

for any arbitrary superfield \( A \), and would also satisfy

\[
\{\Pi, \Pi\} = -2p
\]

reminiscent of the operator relation (23). However, \( \Pi \) is a fermionic function and, therefore, nilpotent, which is not quite the behavior of the operator \( D \).

Furthermore, if we look at the supersymmetry transformations of eq. (20), it is clear that the scaling \( \partial \rightarrow \epsilon \partial \) would lead to (in the limit \( \epsilon \rightarrow 0 \))

\[
\delta \psi = \lambda u, \quad \delta u = 0
\]

which gives rise to a nilpotent algebra. This is also clear from an analysis of the supersymmetric algebra, namely, if it is only the generator of spatial translation which scales, then, the supersymmetry algebra would become nilpotent. To have supersymmetry as we know it, in the “classical” limit, we need, therefore, to scale the Grassmann coordinates and the fermionic variables as

\[
\theta \rightarrow \epsilon^{-1/2} \theta, \quad \psi \rightarrow \epsilon^{-1/2} \psi
\]

This is another indication that, unlike the bosonic variables, fermion variables need to scale for a consistent “classical” limit.

From the structure of the Lax operator in eq. (45), we note that if we introduce, in analogy with the KdV equation, the Lax function

\[
L = p^2 - \Phi \Pi
\]

where \( \Pi \) is defined in eq. (47), it is easy to verify that the Lax equation leads to inconsistencies. Thus, \textit{a priori}, it is not clear how to proceed in finding a Lax description and the dispersionless limit. A little bit of analysis shows that the structure of the Lax function, in this case, is likely to be an infinite series of the form

\[
L = p^2 + \sum_{n=0}^{\infty} (A_n + \Pi B_n)p^{-n}
\]
where $A_n$ and $B_n$ are superfields to be determined recursively. An infinite series with undetermined coefficients, of course, would not be very useful in studying the properties of the system.

Surprisingly, however, we have found that the finite order Lax function

$$L = p^2 + \frac{1}{2} (D\Phi) + \frac{p^{-2}}{16} ((D\Phi)^2 - 2\Phi \Phi_x) - \frac{p^{-4}}{32} \Phi (D\Phi) \Phi_x$$

(54)

leads to the dispersionless limit of the SUSY KdV equation

$$\Phi_t = 3D^2(\Phi(D\Phi))$$

(55)

through the Lax equation

$$\frac{\partial L}{\partial t} = -4\{(L^{3/2})_+, L\}$$

(56)

This is quite a nontrivial Lax function and it is interesting to note that it does not involve the fermionic variable \(\Pi\). It is described completely in powers of \(p\), and it is not clear how one could have deduced this particular form of the Lax function from the Lax operator of SUSY KdV in eq. (45).

On the other hand, since we have a Lax description of the dispersionless SUSY KdV, we can determine the conserved quantities of this theory. It is easy to check that the quantities

$$H_n = \text{Tr} L^{\frac{2n+1}{2}} = \frac{1}{2} \left( \frac{2n+1}{n+1} \right) \int dz \Phi(D\Phi)^n$$

(57)

are conserved under the evolution of the system. It is interesting to note that, although the conserved charges are obtained as traces, they can actually be written in a completely supersymmetric manner and they are bosonic unlike the case in SUSY KdV-B. Unfortunately, the Gelfand-Dikii brackets do not give a meaningful first Hamiltonian structure for this system. While this may seem reminiscent of the Kupershmidt equation, in this case, it is easy to check that the first Hamiltonian structure of SUSY KdV vanishes under the scaling (with \(\epsilon = 0\)), namely, in the “classical” limit. On the other hand, it is not hard to obtain a Hamiltonian structure of the theory directly (as far as we know, it is only the first Hamiltonian structure which is known so far to be derivable from the Gelfand-Dikii brackets in the dispersionless limit of bosonic theories) and has the structure

$$D = -\frac{1}{2} \left( 3\Phi D^2 + (D\Phi)D + 2(D^2\Phi) \right) \delta(z - z')$$

(58)

This, indeed, coincides with the “classical” limit of the second Hamiltonian structure of SUSY KdV.

Once we have the Hamiltonian structure, it is easy to check that

$$\{H_n, H_m\} = \int \frac{\delta H_n}{\delta \Phi} D \frac{\delta H_m}{\delta \Phi} = 0$$

(59)
Namely, all the conserved charges are in involution as we would expect for an integrable system. In addition to these local conserved quantities, the dispersionless SUSY KdV also has two sets of nonlocal conserved quantities, namely, one can check directly that

\[ F_n = \int dz (D^{-1}\Phi)^n \]
\[ G_n = \int dz \Phi (D^{-1}\Phi)^n \]

are conserved under the evolution. It is worth noting that while \( F_n \)'s are fermionic, \( G_n \)'s are all bosonic and that

\[ G_0 = 4H_0 \]

Once again, it is not clear how one can derive such nonlocal charges from a Lax function description. The algebra of these charges is tedious, but can be calculated with the Hamiltonian structure of the system in eq. (58) and has the form

\[
\begin{align*}
\{H_n, H_m\} &= 0 = \{F_n, H_m\} = \{G_n, H_m\} \\
\{F_n, G_m\} &= 0 = \{G_n, G_m\} \\
\{F_n, F_m\} &= nm G_{n+m-2}
\end{align*}
\]

Vanishing of the first three brackets is simply a statement that the Hamiltonians, \( H_n \) are in involution and that \( F_n \) and \( G_n \) are conserved. However, from the other relations, we see that \( F_n \) and \( G_n \) together define a supersymmetry algebra. In particular, it is not difficult to check that \( F_1 \) corresponds to the generator of supersymmetry of the system (up to normalization) with \( G_0 \) \((= 4H_0)\) representing the Hamiltonian.

### 6 Conclusion

Although a lot is known about the structure of dispersionless limits of bosonic integrable systems, the dispersionless limits of fermionic integrable models have proven difficult for a variety of reasons. In this paper, we have made the first attempt towards understanding such models. We have studied the dispersionless limits of the Kupershmidt equation, the SUSY KdV-B equation as well as the SUSY KdV equation all of which can be thought of as fermionic extensions (with or without supersymmetry) of the Riemann equation. We have obtained the Lax function description for each of these systems and constructed the Hamiltonians for such systems from them. However, there is as yet no systematic understanding of how such Lax functions can be deduced from the Lax operator description of the original theory. We have presented the Hamiltonian structures, the nonlocal conserved charges for such systems, as well as the algebra of conserved charges in closed form. However, it is not clear how a Lax function description can give rise to nonlocal conserved quantities nor is a generalization of the Gelfand-Dikii brackets for the Kupershmidt and SUSY KdV equations is known. While we have discussed many interesting features of such fermionic models, we have also brought out several interesting open questions that deserve further study.
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