Cosmological disformal transformations to the Einstein frame and gravitational couplings with matter perturbations

Shinji Tsujikawa

1Department of Physics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

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The disformal transformation of metric $g_{\mu\nu} \rightarrow \Omega^2(\phi)g_{\mu\nu} + \Gamma(\phi,X)\partial_\mu\phi\partial_\nu\phi$, where $\phi$ is a scalar field with the kinetic energy $X = \partial_\mu\phi\partial^\mu\phi/2$, preserves the Lagrangian structure of Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories (which is the minimum extension of Horndeski theories). In the presence of matter, this transformation gives rise to a kinetic-type coupling between the scalar field $\phi$ and matter. We consider the Einstein frame in which the second-order action of tensor perturbations on the isotropic cosmological background is of the same form as that in General Relativity and study the role of couplings at the levels of both background and linear perturbations. We show that the effective gravitational potential felt by matter perturbations in the Einstein frame can be conveniently expressed in terms of the sum of a General Relativistic contribution and couplings induced by the modification of gravity. For the theories in which the transformed action belongs to a class of Horndeski theories, there is no anisotropic stress between two gravitational potentials in the Einstein frame due to a gravitational de-mixing. We propose a concrete dark energy model encompassing Brans-Dicke theories as well as theories with the tensor propagation speed $c_t$ different from 1. We clarify the correspondence between physical quantities in the Jordan/Einstein frames and study the evolution of gravitational potentials and matter perturbations from the matter-dominated epoch to today in both analytic and numerical approaches.

I. INTRODUCTION

The large-distance modification of gravity has been under active study in connection to the dark energy problem [1, 2]. Modifications of the Einstein-Hilbert term $R/(16\pi G)$ in the Lagrangian of General Relativity (GR), where $R$ is the Ricci scalar and $G$ is the Newton gravitational constant, generally give rise to a radiative scalar degree of freedom $\phi$ [3, 4]. Provided that the fifth force mediated by this new degree of freedom is suppressed in the solar system through Vainshtein [5] or chameleon [6] mechanisms, the same scalar field can potentially be the source for the late-time cosmic acceleration.

Horndeski theories [7] are known as the most general scalar-tensor theories with one scalar degree of freedom whose equations of motion are kept up to second-order in time and spatial derivatives (see also Refs. [5]). Many of dark energy models proposed in the literature—such as $f(R)$ gravity [9, 10], Brans-Dicke (BD) theory [11, 12], kinetic braiding [13], and Galileons [14, 15]—belong to a sub-class of Horndeski theories. In the presence of additional matter, the authors in Ref. [16] derived linear perturbation equations of motion on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background to confront dark energy models in the framework of Horndeski theories with the observations of large-scale structures, weak lensing, and Cosmic Microwave Background (CMB) (see also Refs. [17, 22]).

In BD theories, a scalar degree of freedom $\phi$ is coupled to the Ricci scalar $R$ of the form $\phi R$ [22]. The frame in which matter fields are minimally coupled to the metric is dubbed the Jordan frame (JF). The standard interpretation of measurements is usually performed in this frame. In the JF of BD theories, the scalar field $\phi$ mediates a fifth force with matter through its gravitational interaction with the metric. This interaction can be clearly seen in the Einstein frame (EF) where the Lagrangian is described by the Einstein-Hilbert term plus a canonical scalar field [23]. Provided that the fifth force $\phi R$ mediates a scalar degree of freedom [7], the same scalar field can potentially be the source for the late-time cosmic acceleration. In the JF of BD theories, the scalar field $\phi$ mediates a fifth force with matter through its gravitational interaction with the metric. This interaction can be clearly seen in the Einstein frame (EF) where the Lagrangian is described by the Einstein-Hilbert term plus a canonical scalar field [23]. Provided that the fifth force $\phi R$ mediates a scalar degree of freedom [7], the same scalar field can potentially be the source for the late-time cosmic acceleration.

The disformal transformation of metric $g_{\mu\nu} \rightarrow \Omega^2(\phi)g_{\mu\nu} + \Gamma(\phi,X)\partial_\mu\phi\partial_\nu\phi$, where $\phi$ is a scalar field with the kinetic energy $X = \partial_\mu\phi\partial^\mu\phi/2$, preserves the Lagrangian structure of Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories (which is the minimum extension of Horndeski theories). In the presence of matter, this transformation gives rise to a kinetic-type coupling between the scalar field $\phi$ and matter. We consider the Einstein frame in which the second-order action of tensor perturbations on the isotropic cosmological background is of the same form as that in General Relativity and study the role of couplings at the levels of both background and linear perturbations. We show that the effective gravitational potential felt by matter perturbations in the Einstein frame can be conveniently expressed in terms of the sum of a General Relativistic contribution and couplings induced by the modification of gravity. For the theories in which the transformed action belongs to a class of Horndeski theories, there is no anisotropic stress between two gravitational potentials in the Einstein frame due to a gravitational de-mixing. We propose a concrete dark energy model encompassing Brans-Dicke theories as well as theories with the tensor propagation speed $c_t$ different from 1. We clarify the correspondence between physical quantities in the Jordan/Einstein frames and study the evolution of gravitational potentials and matter perturbations from the matter-dominated epoch to today in both analytic and numerical approaches.
tions $\gamma_{ij}$ holds under the disformal transformation (1.1) (see also Refs. 35–37 for related works). For appropriate choices of $\Omega$ and $\Gamma$, it is possible to transform the action to that in the EF where the second-order action of tensor perturbations is of the same form as that in GR [14, 51, 52]. This property is useful for the computation of primordial scalar and tensor power spectra generated during inflation [14]. Since the leading-order tensor power spectrum in GLPV theories is proportional to the Hubble parameter squared $H^2$ in the EF, the detection of primordial gravitational waves can determine the energy scale $H$ during inflation.

In GLPV theories, even though matter is minimally coupled to gravity in the JF, there is a mixture between the JF and the EF under the disformal transformation (1.1), allowing to encompass GLPV theories as general in that it is not restricted to the transformation relevant to redshift-space distortions [57, 58]. This comes from a kinetic-type mixing associated with the presence of higher-order derivatives beyond the Horndeski domain, which is weighed by a parameter $\alpha_H$ characterizing the deviation from Horndeski theories [53, 54, 55]. The disformal transformation (1.1), which contains higher-order derivatives, is helpful to understand the origin of such a kinetic mixing affecting the scalar and matter sound speeds [43].

If the scalar degree of freedom $\phi$ in Horndeski theories is responsible for dark energy, the tensor propagation speed $c_t$ is typically close to 1 during the early cosmological epoch [50]. This is not the case for GLPV theories, in which the deviation from $c_t = 1$ is allowed due to the absence of extra conditions Horndeski theories obey. Recently, dark energy models with constant $c_t$ [53] and varying $c_t$ [54] have been proposed in the framework of GLPV theories. In particular, the latter provides an interesting possibility of realizing weak gravity for the perturbations relevant to redshift-space distortions [57, 58].

In GLPV theories with $c_t$ different from 1, the disformal transformation (1.1) to the EF should allow us to understand the structure of the matter-scalar couplings mentioned above. For the models proposed in Refs. [53, 54], the anisotropy parameter $\eta = -\Phi/\Psi$ between two gravitational potentials $\Psi$ and $\Phi$ deviates from 1 in the JF, but we will see that it is possible to de-mix the gravitational potentials and the scalar field in such a way that there is no anisotropic stress between $\Psi$ and $\Phi$ in the EF. Moreover, we will show that the effective gravitational coupling with matter can be well understood in the EF due to the separation of a GR-like contribution and modifications arising from $\alpha_H$.

In this paper we obtain relations of physical quantities between the JF and the EF under the disformal transformation (1.1), and study roles of gravitational couplings with matter at the levels of both background and linear perturbations. A similar prescription was taken in Ref. [58] with the disformal transformation $\hat{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu} + \Gamma(\phi)\partial_\mu \phi \partial_\nu \phi$, but our treatment is more general in that it is not restricted to the transformation between Horndeski theories alone. We employ the approach of effective field theory of cosmological perturbations [52, 56], allowing to encompass GLPV theories as a specific case. We propose a new dark energy model in the framework of GLPV theories, which accommodates models with $c_t^2 \neq 1$ as well as models based on BD theories (which lead to "coupled quintessence" models [67] in the EF).

Our paper is organized as follows. In Sec. II we briefly review GLPV theories and in Sec. III we show how the GLPV action is transformed under the disformal transformation (1.1) in the presence of matter. In Sec. IV we present linear perturbation equations of motion on the flat FLRW background in both the JF and the EF. In Sec. V we consider the transformation to the EF and discuss the matter-scalar coupling in the EF. In Sec. VI we propose a new dark energy model belonging to GLPV theories and study the correspondence of physical quantities between the JF and the EF in detail. Section VII is devoted to conclusions.

II. GLPV THEORIES IN THE PRESENCE OF MATTER

GLPV theories [39] are the generalizations of Horndeski theories written in terms of ADM scalar quantities defined below [68]. We begin with the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

(2.1)

where $N$ is the lapse function, $N^i$ is the shift vector, and $h_{ij}$ is the three-dimensional spatial metric. We express the three-dimensional Ricci tensor on the constant time hyper-surfaces $\Sigma_t$, as $R_{\mu\nu} = (3) R_{\mu\nu}$. The extrinsic curvature is defined by $K_{\mu\nu} = h^\lambda_{\mu} n_{\nu;\lambda}$, where $n_{\mu} = (-N, 0, 0, 0)$ is a unit vector orthogonal to $\Sigma_t$. We introduce a number of geometric scalar quantities, as

$$K = K^\mu_{\mu}, \quad S = K_{\mu\nu} K^{\mu\nu},$$

$$R = R^\mu_{\mu\nu}, \quad U = R_{\mu\nu} K^{\mu\nu}. $$

(2.2)

Horndeski theories, which have one scalar degree of freedom $\phi$, can be reformulated by using the above geometric scalars with the choice of unitary gauge

$$\phi = \phi(t),$$

(2.3)

under which $\phi$ depends on the cosmic time $t$ alone. On the flat FLRW background with the scale factor $a(t)$, the Lagrangian of Horndeski theories can be expressed in the form

$$L = A_2(N, t) + A_3(N, t) K + A_4(N, t)(K^2 - S) + B_4(N, t) U + A_5(N, t) K_3 + B_5(N, t) (U - \frac{1}{2} K R),$$

(2.4)

where $K_3 \equiv K_3 = 3K K_{\mu\nu} K^{\mu\nu} + 2K_{\mu\nu} K^{\mu\lambda} K_{\nu\lambda}$, and $A_i$, $B_i$ are functions of $N$ and $t$ satisfying the two conditions

$$A_4 = 2X B_{4,X} - B_4, \quad A_5 = -\frac{1}{3} X B_{5,X},$$

(2.5)
where $B_{i,X} \equiv \partial B_i/\partial X$ with $X \equiv g^\mu\nu \partial_\mu \phi \partial_\nu \phi$. In the unitary gauge we have $X = -\dot{\phi}^2(t)/N^2$, where a dot represents a derivative with respect to $t$. Hence the dependence on $N$ and $t$ translates to that on $X$ and $\phi$.

Violation of the conditions (2.5) can generally give rise to derivatives higher than second order, but it was shown in Refs. [39, 41, 42] that there is only one scalar propagating degree of freedom on the flat FLRW background. GLPV theories are described by the Lagrangian (2.4) without having the two conditions (2.5). In this paper we focus on GLPV theories in the presence of a matter field $\Psi_m$ described by the Lagrangian $L_m$. Then, we consider the following action

$$S = \int d^4x \sqrt{-g} L(N, K, S, \mathcal{R}, \mathcal{U}; t) + \int d^4x \sqrt{-g} L_m(g_{\mu\nu}, \Psi_m), \quad (2.6)$$

where $g$ is the determinant of metric $g_{\mu\nu}$, and $L$ is given by Eq. (2.4). The matter field $\Psi_m$ is assumed to be a barotropic perfect fluid, which can be modeled by a k-essence Lagrangian $P(Y)$ depending on the kinetic term $Y = g^\mu\nu \partial_\mu \chi \partial_\nu \chi$ of a scalar field $\chi$. The term $K_3$ in Eq. (2.4) can be expressed as $K_3 = 3H(\mathcal{H}^2 - 2\mathcal{H}K - K^2 - \mathcal{S})$ up to second order in the perturbations [68], where $H$ is the Hubble parameter defined later in Eq. (3.11). Hence the Lagrangian (2.4) of GLPV theories depends on $N, K, S, \mathcal{R}, \mathcal{U}$, and $t$ up to linear order in the perturbations.

We assume that, in the JF, the matter field $\Psi_m$ is minimally coupled to the metric $g_{\mu\nu}$. The matter energy-momentum tensor following from $L_m$ is given by

$$T^\mu\nu = \frac{2}{\sqrt{-g}} \frac{\delta(-\sqrt{-g} L_m)}{\delta g_{\mu\nu}}. \quad (2.7)$$

We can derive the background and linear perturbation equations of motion by varying the action (2.6) up to first and second orders in the perturbations, respectively [53, 58, 70].

### III. DISFORMAL TRANSFORMATIONS

In this section we discuss how background/perturbed quantities and the action (2.6) are mapped under the disformal transformation (1.1).

#### A. Transformation of background and perturbed quantities

In the unitary gauge (2.8), the line element $ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$ in the transformed frame reads [35, 44]

$$ds^2 = -\hat{N}^2 dt^2 + \hat{h}_{ij}(dx^i + \hat{N}^i dt)(dx^j + \hat{N}^j dt), \quad (3.1)$$

where

$$\hat{N} = N \alpha, \quad \hat{h}_{ij} = \Omega^2 h_{ij}, \quad (3.2)$$

with

$$\alpha \equiv \sqrt{1 + \Gamma X}. \quad (3.3)$$

In the JF, let us consider the linearly perturbed line element on the flat FLRW background,

$$ds^2 = -(\hat{N}^2 + 2A)dt^2 + 2\psi_{ij} dx^i dx^j + a^2(t)[1 + 2\zeta]\delta_{ij} + 2E_{ij} + \gamma_{ij}] dx^i dx^j, \quad (3.4)$$

where $\hat{N}$ is the background value of the lapse; $A, \psi, \zeta, E$ are the scalar metric perturbations; $\gamma_{ij}$ is the tensor perturbation, and the lower index “$i$” denotes the covariant derivative with respect to the three-dimensional metric $h_{ij}$. Comparing Eq. (2.1) with Eq. (3.4), we have the relations $2A = N^2 - h_{ij} N^i N^j - N^2$, $\psi_{ij} = h_{ij} N^j$, and

$$h_{ij} = a^2(t) [(1 + 2\zeta)\delta_{ij} + 2E_{ij} + \gamma_{ij}]. \quad (3.5)$$

Introducing the scale factor $\hat{a}(t)$ in the transformed frame, as

$$\hat{a}(t) = (\Omega(t)a(t), \quad (3.6)$$

the three-dimensional metric $h_{ij}$ in Eq. (3.2) reduces to

$$\hat{h}_{ij} = \hat{a}^2(t) [(1 + 2\hat{\xi})\delta_{ij} + 2\hat{E}_{ij} + \hat{\gamma}_{ij}], \quad (3.7)$$

where

$$\hat{\xi} = \zeta, \quad \hat{E} = E, \quad \hat{\gamma}_{ij} = \gamma_{ij}. \quad (3.8)$$

In what follows we use an over-hat for quantities in the transformed frame. From Eq. (3.8), the perturbations $\zeta$, $E$, and $\gamma_{ij}$ are invariant under the disformal transformation (1.1) [44]. From Eq. (3.2) and the relation $\hat{N}^i = N^i$ we also obtain

$$\delta \hat{N} = \frac{1}{\beta} \delta N, \quad \hat{\psi} = \Omega^2 \psi, \quad (3.9)$$

where an over-bar represents background quantities, and

$$\beta \equiv \sqrt{\Omega^2 + \Gamma X \over \Omega^2 - X^2 \Gamma X}. \quad (3.10)$$

Since $\phi = \phi(t)$ and $X = -\dot{\phi}^2(t)/N^2$ in the unitary gauge, the quantities $\Gamma$ and $X$ in Eq. (3.10) contain the information of perturbations through the lapse function $N$.

#### B. Transformation of the action

The disformal transformation of the action $S_a = \int d^4x \sqrt{-g} L$, where $L$ is the Lagrangian (2.4) of GLPV theories, was already discussed in Refs. [43, 44]. First of all, the volume element $\sqrt{-g}$ transforms as

$$\sqrt{-g} = \sqrt{-g} \Omega^3 \alpha. \quad (3.11)$$
In the unitary gauge, the intrinsic and extrinsic curvatures obey the transformation laws

\[ \hat{\mathcal{R}}_{ij} = \mathcal{R}_{ij}, \quad \hat{K}_{ij} = \frac{\Omega^2}{\alpha} \left( K_{ij} + \frac{\omega}{N} h_{ij} \right), \quad (3.12) \]

where

\[ \omega \equiv \frac{\dot{\Omega}}{\Omega}. \quad (3.13) \]

The action in the transformed frame \( \mathcal{S}_m = \int d^4x \sqrt{-\hat{g}} \hat{L}_m \) preserves the structure of original GLPV action, such that

\[ \hat{L} = \hat{A}_2(N, t) + \hat{A}_3(N, t) \hat{K} + \hat{A}_4(N, t)(\hat{K}^2 - \hat{S}) + \hat{B}_4(N, t) \hat{\mathcal{R}} + \hat{A}_5(N, t) \hat{K} + \hat{B}_5(N, t) \left( \dot{\mathcal{U}} - \frac{1}{2} \hat{K} \hat{R} \right), \quad (3.14) \]

where

\[ \hat{A}_2 = \frac{1}{\Omega^3} \left( A_2 - \frac{3\omega}{N} A_3 + \frac{6\omega^2}{N^2} A_4 - \frac{6\omega^3}{N^3} A_5 \right), \quad (3.15) \]
\[ \hat{A}_3 = \frac{1}{\Omega^3} \left( A_3 - \frac{4\omega}{N} A_4 + \frac{6\omega^2}{N^2} A_5 \right), \quad (3.16) \]
\[ \hat{A}_4 = \frac{\alpha}{\Omega^3} \left( A_4 - \frac{3\omega}{N} A_5 \right), \quad (3.17) \]
\[ \hat{B}_4 = \frac{1}{\Omega^2} \left( B_4 + \frac{\omega}{2N} B_5 \right), \quad (3.18) \]
\[ \hat{A}_5 = \frac{\alpha^2}{\Omega^3} A_5, \quad (3.19) \]
\[ \hat{B}_5 = \frac{1}{\Omega} B_5. \quad (3.20) \]

The matter action in the transformed frame is given by \( \mathcal{S}_m = \int d^4x \sqrt{-\hat{g}} \hat{L}_m \), where

\[ \hat{L}_m = \frac{1}{\Omega^4 \alpha} L_m, \quad (3.21) \]

where \( L_m \) depends on the JF metric \( g_{\mu\nu} \) and the matter field \( \Psi_m \). By expressing \( g_{\mu\nu} \) in terms of the metric \( \hat{g}_{\mu\nu} \) in the transformed frame from Eq. (1.1), it contains the contribution of \( \phi \) and its derivative. Hence the scalar field \( \phi \) is (kinetically) coupled to matter in the transformed frame.

From Eq. (2.7) the transformation law of \( T_{\mu\nu} \) is given by

\[ T^{\mu\nu} = \sqrt{-\hat{g}} \frac{\delta T}{\delta \hat{g}_{\mu\nu}} \frac{\delta g_{\mu\nu}}{\sqrt{-g}}. \quad (3.22) \]

On using Eqs. (1.1) and (3.11), it follows that

\[ T^{\mu\nu} = \hat{T}^{\gamma\rho} \Omega^3 \alpha \left( \Omega^2 \delta^\gamma_\rho \delta^\nu_\mu - \Gamma^\gamma_{\rho\mu} \phi \partial^\rho \phi \partial^\nu \phi \right). \quad (3.23) \]

The transformed metric with upper indices is given by

\[ \hat{g}^{\mu\nu} = \frac{1}{\Omega^2} \left( g^{\mu\nu} - \frac{\Gamma}{\alpha^2} \phi \partial^\mu \phi \partial^\nu \phi \right). \quad (3.24) \]

From Eqs. (3.23) and (3.24), the mixed energy-momentum tensor obeys the transformation law:

\[ T^\lambda_\rho = \hat{T}^\gamma_\sigma \Omega^3 \alpha \left[ \Omega^2 \delta^\gamma_\sigma \left( \delta^\rho_\lambda - \frac{\Gamma}{\alpha^2} \partial^\rho \phi \partial^\lambda \phi \right) \right. \]
\[ \left. - \frac{\Omega^2 \Gamma}{\alpha^2} \partial_\gamma \phi \partial^\sigma \phi \partial^\lambda \phi \partial^\rho \phi \right]. \quad (3.25) \]

For the choice of the unitary gauge, we obtain the relations

\[ T^0_0 = \hat{T}^0_0 \Omega^3 \beta, \quad T^i_0 = \hat{T}^i_0 \Omega^3 \alpha, \quad T^i_j = \hat{T}^i_j \Omega^3 \alpha. \quad (3.26) \]

We decompose the energy-momentum tensor into the background and perturbed parts, as \( T^0_0 = -\rho - \delta \rho, \quad T^i_0 = \partial_\lambda \delta q, \quad T^i_j = (P + \delta P) \delta j^i \), where \( T^0_0 \) is a perturbed quantity itself. The background energy density \( \rho \) and the pressure \( P \) are subject to the transformations

\[ \dot{\rho} = \Omega^3 \alpha \delta \rho, \quad \dot{P} = \Omega^3 \alpha \delta P. \quad (3.27) \]

For the linear perturbations, we have

\[ \delta \rho = \frac{\delta \rho}{\Omega^3 \alpha} \delta \rho + \frac{\nu}{\Omega^3 \alpha} \delta N, \quad (3.28) \]
\[ \delta q = \frac{\delta q}{\Omega^3 \alpha} \delta q, \quad (3.29) \]
\[ \delta P = \frac{\delta P}{\Omega^3 \alpha} (\delta P - \mu P \delta N), \quad (3.30) \]

where

\[ \mu \equiv \frac{1}{\alpha \partial N} \bigg|_{N=N}, \quad \nu \equiv \frac{1}{\alpha \partial N} \bigg|_{N=\hat{N}}. \quad (3.31) \]

The quantity \( \mu \) is related to \( \hat{\alpha} \) and \( \hat{\beta} \), as

\[ \mu \hat{N} = \frac{1}{\hat{\alpha} \hat{\beta}} - 1. \quad (3.32) \]

In summary, the action in the transformed frame reads

\[ S = \int d^4x \sqrt{-\hat{g}} \hat{L}(\hat{N}, \hat{K}, \hat{S}, \hat{R}, \hat{U}; t) \]
\[ + \int d^4x \sqrt{-\hat{g}} \hat{L}_m(\hat{g}_{\mu\nu}(\phi, \partial_\mu \phi), \Psi_m), \quad (3.33) \]

where \( \hat{L} \) and \( \hat{L}_m \) are given, respectively, by Eqs. (3.11) and (3.21).

### IV. EQUATIONS OF MOTION IN THE JF AND THE TRANSFORMED FRAME

In this section we present linear perturbation equations of motion on the flat FLRW background for the theories described by the action (2.4). We then study how they are transformed under the disformal transformation (1.1).
A. Equations of motion in the JF

The equations of motion in the JF were already derived in Refs. \([13, 53, 55]\) for the action \((2.6)\). We consider the background line element \(ds^2 = -N^2 dt^2 + a^2(t) \delta_{ij} dx^i dx^j\) without setting \(\bar{N} = 1\). Defining the Hubble parameter

\[
H \equiv \dot{a} / Na ,
\]

the background values of extrinsic and intrinsic curvatures are given, respectively, by

\[
\bar{K}_{\mu\nu} = H \bar{h}_{\mu\nu} , \quad \bar{R}_{\mu\nu} = 0 ,
\]

and hence \(\bar{K} = 3H, \bar{S} = 3H^2\), and \(\bar{R} = \ddot{\bar{U}} = 0\).

The background and perturbation equations of motion follow from first-order and second-order Lagrangians, respectively, by expanding the action \((2.3)\) up to quadratic order in perturbations. The perturbations of \(N, K, S\) are given by \(\delta N = N - \bar{N}, \delta K = K - 3H\), and \(\delta S = 2H \delta K + \bar{K} \delta R\). We write the intrinsic curvature as \(R = \delta_1 R + \delta_2 R\), where \(\delta_1 R\) and \(\delta_2 R\) are first-order and second-order perturbations, respectively. For the perturbation \(U\), we have the relation

\[
\int d^4x \sqrt{-g} \lambda(t) U = \int d^4x \sqrt{-g} [\lambda(t) \bar{R} / 2 + \lambda(t) \bar{L} / (2N)]
\]

up to a boundary term, where \(\lambda(t)\) is an arbitrary function with respect to \(t\) [53].

We consider the perturbed metric [34] with the choice of unitary gauge

\[
\delta \phi = 0 , \quad E = 0 , \quad (4.3)
\]

under which the temporal and spatial coordinate transformation vectors are fixed, respectively.

1. Background equations

Expanding the gravitational action \(S_g = \int d^4x \sqrt{-g} L\) up to first order in the scalar perturbations, it follows that [43]

\[
\delta S_g = \int d^4x \left[ a^3 \left( L + \bar{N} L_{,N} - 3HF \right) \delta N + 3a^2 \bar{N} \left( \ddot{L} - \frac{\ddot{\bar{F}}}{N} - 3HF \right) \delta a \right] ,
\]

where

\[
\bar{F} \equiv L_{,K} + 2HL_{,S} ,
\]

and we dropped a boundary term irrelevant to the dynamics. Here and in the following, the coefficients in front of perturbed quantities [such as those in front of \(\delta N\) and \(\delta a\) in Eq. (4.1)] should be evaluated on the background.

Variation of the matter energy-momentum tensor \(\delta S_m = \int d^4x \sqrt{-g} T_{\mu\nu} \delta g_{\mu\nu}/2\) reads

\[
\delta S_m = \int d^4x \left( -a^3 \rho \delta N + 3a^2 \bar{N} \rho \delta a \right) ,
\]

From the variational principle \(\delta S_g + \delta S_m = 0\), we obtain the background equations of motion

\[
\bar{L} + \bar{N} L_{,N} - 3HF = \rho ,
\]

\[
\bar{L} - \frac{\ddot{\bar{F}}}{N} - 3HF = -P .
\]

Since the matter component is not directly coupled to the field \(\phi\) in the JF, it obeys the standard continuity equation

\[
\frac{\dot{\bar{N}}}{N} + 3H (\rho + P) = 0 . \quad (4.9)
\]

2. Perturbation equations

Expanding the action \((2.6)\) with the Lagrangian \((2.4)\) up to second order in scalar perturbations and taking the variation with respect to \(\delta N, \partial^2 \psi, \zeta\), and the field perturbation \(\delta \chi\) associated with the matter Lagrangian \(\bar{P}(Y)\), the resulting perturbation equations of motion in the presence of a barotropic perfect fluid are given, respectively, by [43, 53, 55]

\[
(2\bar{N} L_{,N} + \bar{N}^2 L_{,NN} - 6H \bar{N} W + 12L_{,S} H^2) \frac{\delta N}{N}
\]

\[
+ \left( 3\zeta - \frac{\partial^2 \psi}{a^2} \right) W - 4(\bar{N} \bar{D} + E) \frac{\partial^2 \zeta}{a^2} = \delta \rho ,
\]

\[
W \frac{\delta N}{N} - \frac{4L_{,S}}{N^2} \zeta = -\delta q ,
\]

\[
\frac{1}{a^3 \bar{N}} \frac{d}{dt} \left( a^3 \bar{N} \chi \right) + 4(\bar{N} \bar{D} + E) \frac{\partial^2 \delta N}{a^2 N} + \frac{4E}{a^2} \partial^2 \chi
\]

\[
- 3(\rho + P) \frac{\delta N}{N} = 3\delta P ,
\]

\[
\frac{\delta \rho}{N} + 3H (\delta \rho + \delta P)
\]

\[
= - (\rho + P) \left( \frac{3\zeta}{N} - \frac{\partial^2 \psi}{a^2 N} \right) - \bar{N} \frac{\partial^2 \delta q}{a^2} ,
\]

where

\[
\bar{D} \equiv L_{,RN} - \frac{L_{,\mu}}{2N^2} + HL_{,NN} ,
\]

\[
\mathcal{E} \equiv L_{,R} + \frac{L_{,\mu}}{2N} + \frac{3}{2} HL_{,\mu} ,
\]

\[
\bar{W} \equiv L_{,KN} + 2HL_{,SN} + \frac{4HL_{,S} N}{N} ,
\]

\[
\gamma \equiv 4L_{,S} \frac{\partial^2 \psi}{a^2} - 3\delta q .
\]

The momentum perturbation \(\delta q\) obeys

\[
\frac{1}{N} \frac{d}{dt} (\bar{N} \delta q) + 3H \bar{N} \delta q = -(\rho + P) \frac{\delta N}{N} - \delta P .
\]
Substituting Eq. (4.17) into Eq. (4.12) and using Eq. (4.13), it follows that
\[
\left( \frac{\dot{L}_S}{N} + H L_S - \frac{\dot{N}^2}{N} L_S \right) \psi + L_S \frac{\dot{\psi}}{N^2} + (\ddot{N} D + \mathcal{E}) \frac{\delta N}{N} + \delta \zeta = 0, \tag{4.19}
\]
where we set the integration constant 0. We define the gauge-invariant Bardeen potentials \[71\]
\[\Psi \equiv \frac{\delta N}{N} + \frac{1}{N} \frac{d}{dt} \left( \frac{\psi}{N} \right), \quad \Phi \equiv \zeta + H \frac{\psi}{N}, \tag{4.20}\]
and the anisotropy parameter
\[\eta \equiv -\frac{\Phi}{\Psi}. \tag{4.21}\]
The effective gravitational potential associated with the deviation of light rays in CMB and weak lensing observations is given by \[72\]
\[\Phi_{\text{eff}} = \frac{1}{2} (\Psi - \Phi) = \frac{1}{2} (1 + \eta) \Psi. \tag{4.22}\]
One can write Eq. (4.19) in the form
\[\Psi + \Phi = \frac{L_S}{NH L_S} (\zeta - \Phi) - \left( \frac{c_s^2}{N^2} - 1 \right) \zeta - \alpha_H \frac{\delta N}{N}, \tag{4.23}\]
where \(c_s^2 = \tilde{N}^2 \mathcal{E}/L_S\) is the tensor propagation speed squared discussed later in Sec. V and
\[\alpha_H \equiv \frac{\tilde{N} D + \mathcal{E}}{L_S} - 1 = \frac{c_s^2}{N^2} - 1 + \frac{\tilde{N} D}{L_S}. \tag{4.24}\]
The parameter \(\alpha_H\) characterizes the deviation from Horndeski theories. Provided that one of the conditions \(L_S \neq 0, c_s^2 \neq N^2,\) and \(\alpha_H \neq 0\) is satisfied, the anisotropic stress does not generally vanish \((\eta \neq 1)\).

We define the gauge-invariant matter density contrast, as
\[\delta_m \equiv \delta - 3V_m, \tag{4.25}\]
where \(\delta \equiv \delta \rho/\rho\) and \(V_m \equiv \tilde{N} H \delta q/\rho.\) Taking the time derivative of Eq. (4.13) and using Eq. (4.18) in Fourier space, we obtain
\[\dot{\delta}_m + 2H \delta_m + \frac{k^2}{a^2} \Psi = -3 \tilde{B} - 6 H \dot{B}, \tag{4.26}\]
where \(B \equiv \zeta + V_m,\) and \(k\) is a coming wave number. We define the effective gravitational coupling \(G_{\text{eff}}\), as
\[\frac{k^2}{a^2} \Psi = -4\pi G_{\text{eff}} \rho \delta_m. \tag{4.27}\]
The gravitational potential \(\Psi\) contains the information of gravitational coupling with the scalar field \(\phi.\) The modified gravitational interaction affects the evolution of matter perturbations through Eq. (4.20). The evolution of \(\Psi\) and \(G_{\text{eff}}\) is known by solving the coupled Eqs. (4.10)- (4.13) with Eq. (4.18).

On using Eqs. (4.10) and (4.11), the second-order action of scalar perturbations can be expressed in terms of \(\zeta, \delta \phi,\) and its derivatives. Assuming that the matter sector does not correspond to a ghost mode, the scalar ghost is absent under the condition \[53, 68\]
\[q_s = \frac{2L_S (4L_S w_s + 3 \tilde{N}^2 W^2)}{N^3 W^2} > 0, \tag{4.28}\]
where \(w_s = 2 \tilde{N} L_N + \tilde{N}^2 L_{NN} - 6 H \tilde{N} W + 12 L_S H^2.\)

In GLPV theories there is a mixing between the scalar propagation speed \(c_s\) and the matter sound speed \(c_m.\) For non-relativistic matter characterized by \(P = +0\) and \(\delta P = 0,\) we have \(c_s^2\) is +0 in the small-scale limit, while \(c_s^2\) is given by \[39, 53, 56\]
\[c_s^2 = \frac{2N}{q_s} \left[ \frac{M}{N} + H M - \mathcal{E} - \frac{4 L_S^2 \rho}{N^2 W^2} (1 + 2 \alpha_H) \right], \tag{4.29}\]
where
\[M \equiv \frac{4L_S (\tilde{N} D + \mathcal{E})}{NW}. \tag{4.30}\]
This shows that the deviation from Horndeski theories \((\alpha_H \neq 0)\) modifies the scalar sound speed. We require \(c_s^2 > 0\) to avoid Laplacian instabilities.

**B. Equations of motion in the transformed frame**

In the transformed frame described by the action \[333\], we also derive the background and perturbation equations of motion. The cosmic time \(\dot{t}\) in the transformed frame is related to \(t\) in the JF, as
\[\dot{t} = \int \dot{\mathcal{N}} dt. \tag{4.31}\]

**1. Background equations**

Following the same procedure as that in the JF, we obtain the background equations of motion
\[\dot{\mathcal{N}} \dot{\mathcal{L}} - 3 \mathcal{H} \mathcal{F} = \rho, \tag{4.32}\]
\[\dot{\mathcal{N}} \dot{\mathcal{L}} - \mathcal{F} = 3 \mathcal{H} \mathcal{F} = -\dot{P}, \tag{4.33}\]
where \(\mathcal{F} \equiv \mathcal{L}_K + 2 \mathcal{H} \mathcal{L}_S\) and
\[\dot{\mathcal{H}} \equiv \frac{\dot{\alpha}}{N \dot{\alpha}} = \frac{1}{a} \frac{d}{dt}. \tag{4.34}\]
The relations of the quantities \(\mathcal{N}, \mathcal{H}, \mathcal{F}, \mathcal{L}_K, \mathcal{L}_S\) with those in the JF are given in Appendix A (see also Ref. \[44\]).
Using these relations as well as the background equations (4.7)-(4.8) in the JF and the correspondence (3.27), we can also derive Eqs. (4.10)-(4.12) in the JF by using the relations (3.8), (3.27), (3.32), and (A1), as well as the background equations (4.1)-(4.2) in the JF. This modification comes from an explicit coupling between matter and the scalar degree of freedom in the transformed frame.

From Eqs. (4.39) and (4.40) it follows that

$$
\left( \frac{d\tilde{L}}{dt} + \tilde{H}\tilde{\dot{s}} - \frac{1}{N} \frac{d\tilde{N}}{dt} \tilde{L}\tilde{\dot{s}} \right) \frac{\dot{\psi}}{\tilde{N}} + \frac{\tilde{L}}{\tilde{N}} d\tilde{\psi} 
+ (\tilde{N}\tilde{D} + \tilde{E}) \frac{\tilde{N}}{\tilde{N}} \tilde{\dot{\zeta}} = 0.
$$

We introduce the gauge-invariant Bardeen potentials in the transformed frame, as

$$
\dot{\Psi} \equiv \frac{\delta\tilde{N}}{\tilde{N}} + \frac{d}{dt} \left( \frac{\psi}{\tilde{N}} \right), \quad \dot{\Phi} \equiv \hat{\zeta} + \hat{H}\frac{\psi}{\tilde{N}}.
$$

From Eq. (4.29) we obtain the relation similar to Eq. (4.28) with additional over-hats to each quantity. As we see in Sec. V it is possible to find a metric frame in which some of the terms generating the anisotropic stress vanish.

We also introduce the gauge-invariant matter density contrast, as

$$
\delta m \equiv \hat{\delta} - 3\hat{V}_m, \quad (4.44)
$$

where $\hat{\delta} \equiv \hat{\delta}/\hat{\rho}$ and $\hat{V}_m \equiv \hat{N}\hat{H}\hat{\delta}\tilde{q}/\hat{\rho}$. From Eqs. (4.40) and (4.41) we can derive the second-order equation for $\delta m$ analogous to Eq. (4.28) in the JF. If we transform to the EF, the effective gravitational coupling with matter becomes particularly transparent. We shall address this issue in Sec. V.

V. EINSTEIN FRAME

We define the EF in which the second-order action of tensor perturbations $\gamma_{ij}$ is of the same form as that in GR [44]. In the following we discuss the transformation of the action in the JF frame to that in the EF.

A. Transformation to the EF

Expanding the action (2.6) with the Lagrangian (2.4) in terms of tensor perturbations $\gamma_{ij}$, the resulting quadratic action of $\gamma_{ij}$ in the JF is given by

$$
S_2^{(b)} = \int d^4x \, a^3 q_i \delta^{ik} \delta^l \left( \gamma_{ij} \gamma_{kl} - \frac{c_i^2}{a^2} \partial_i \gamma_{ij} \partial_k \gamma_{kl} \right), \quad (5.1)
$$

where

$$
q_i \equiv \frac{L_s}{4N}, \quad c_i^2 \equiv \frac{\tilde{N}^2 e}{L_s}. \quad (5.2)
$$
In Eq. (5.1) the quantities \( q_t \) and \( c_t^2 \) should be evaluated on the background, such that the kinetic term \( X \) appearing in \( \dot{L}_S = -A_4 - 3HA_5 \) and \( \dot{E} = B_4 + \ddot{B}_5/(2\tilde{N}) \) corresponds to the time derivative \( \dot{X}(t) = -\dot{\phi}^2/2(2\tilde{N}^2) \). We require the conditions \( q_t > 0 \) and \( c_t^2 > 0 \) to avoid ghosts and Laplacian instabilities. In GR we have \( L = M^2_{pl}R/2 = -(M^2_{pl}/2)(R^2 - \mathcal{S}) + (M^2_{pl}/2)R_\text{c} \), in which case \( q_t = M^2_{pl}/(8\tilde{N}) \) and \( c_t^2 = \tilde{N}^2 \) (where \( M_{pl} \) is the reduced Planck mass).

Under the disformal transformation \( 144 \), the tensor perturbation is invariant, see Eq. (3.8). Hence the second-order tensor action in the transformed frame reads

\[
S^{(h)}_2 = \int d^4x \, \delta \hat{q}_\gamma \delta^{ijkl} \left( \dot{\gamma}_{ik} \dot{\gamma}_{jl} - \frac{c_t^2}{d} \delta_{ij} \partial_{\gamma_{ik}} \partial_{\gamma_{jl}} \right), \tag{5.3}
\]

where \( \hat{q}_t = \hat{\dot{L}}_S/(4\tilde{N}) \) and \( \hat{c}_t^2 = \hat{\tilde{N}}^2 \). The tensor action \( (5.1) \) in the JF can be transformed to that in the EF for the choice

\[
\Omega^2 = \frac{8\hat{q}_t \hat{c}_t}{M^2_{pl}}, \quad \Gamma = \frac{8\hat{q}_t \hat{c}_t}{M^2_{pl}} \left( \frac{\hat{c}_t^2}{\tilde{N}^2} - 1 \right) \frac{1}{X}. \tag{5.5}
\]

The quantities \( \hat{q}_t \) and \( \hat{c}_t^2 \) in the action \( (5.1) \) depend on the time \( t \) alone. Then, the factor \( \Gamma \) in Eq. (5.5) has the dependence \( \Gamma(\phi, X) = \gamma(\phi)/X \) in the unitary gauge, where \( \gamma(\phi) = (8\hat{q}_t \hat{c}_t/M^2_{pl})(\hat{c}_t^2/\tilde{N}^2 - 1) \).

For the choices \( (5.10) \), the terms \( \alpha \) and \( \beta \) in Eqs. (3.9) and (3.10) are given by

\[
\alpha = \frac{1}{\beta} = \Omega \frac{\hat{c}_t}{\tilde{N}}, \tag{5.6}
\]

Since both \( \alpha \) and \( \beta \) are functions of \( t \), we have \( \mu = 0 = \nu \) from Eq. (3.31). Then, the coupling (4.36) reduces to

\[
Q = -\frac{\omega}{N} \left( \dot{\hat{\rho}} - 3\hat{\dot{P}} \right) - \frac{C_t}{NC_t} \hat{\dot{\rho}}, \tag{5.7}
\]

where

\[
C_t \equiv \frac{\hat{c}_t}{\tilde{N}}. \tag{5.8}
\]

If \( \hat{c}_t = \tilde{N} \), then \( \Gamma = 0 \) and \( Q = -\omega(\dot{\hat{\rho}} - 3\hat{\dot{P}}) \). This case corresponds to the well-known conformal transformation arising e.g., in BD theory \( 67 \). For radiation \( (\dot{\rho} = 3P) \) the coupling \( Q \) vanishes, but for non-relativistic matter \( (\dot{P} = 0) \), we have that \( Q = -\omega(\dot{\hat{\rho}}) \). If \( C_t \) varies in time, the last term on the rhs of Eq. (5.7) does not vanish even for radiation.

### B. Background equations in the EF

The choice \( (5.5) \) corresponds to the conditions

\[
\dot{L}_S = -\dot{A}_4 - 3\dot{H}\dot{A}_5 = \frac{\dot{M}^2_{pl}}{2}, \tag{5.9}
\]

\[
\dot{E} = \dot{B}_4 + \frac{1}{2} \frac{d\dot{B}_5}{dt} = \frac{\dot{M}^2_{pl}}{2}. \tag{5.10}
\]

Using the relation (5.9), the background Eqs. (4.32) and (4.33) in the EF can be written in the following forms:

\[
3\dot{M}^2_{pl} \dot{H}^2 = \dot{\rho}_{DE} + \dot{\rho}, \tag{5.11}
\]

\[
-2\dot{M}^2_{pl} \frac{d\dot{H}}{dt} = \dot{P}_{DE} + \dot{P} + \dot{P}, \tag{5.12}
\]

where

\[
\dot{\rho}_{DE} = -\dot{A}_2 - 6\dot{H}^3\dot{\dot{A}}_5 \tag{5.13}
\]

\[
-\dot{\tilde{N}} \left( \dot{A}_{2,N} + 3\dot{H} A_{3,N} - 12\dot{H}^3 A_{5,N} \right), \tag{5.13}
\]

\[
\dot{P}_{DE} = \dot{A}_2 + 6\dot{H}^3 \dot{\dot{A}}_5 \tag{5.14}
\]

\[
- \left( \frac{d\dot{A}_4}{dt} + 12\dot{H} \frac{d\dot{A}_5}{dt} \right) \frac{d\dot{A}_5}{dt} \right) \tag{5.14}
\]

From Eqs. (5.27) and (5.4), the matter equation of state \( w = P/\rho \) is invariant under the transformation to the EF, i.e., \( P/\rho = \tilde{P}/\tilde{\rho} \). The energy density \( \dot{\rho}_{DE} \) and the pressure \( \dot{P}_{DE} \) obey the equation of motion

\[
\frac{d\dot{\rho}_{DE}}{dt} + 3\dot{H} \left( \dot{\rho}_{DE} + \dot{P}_{DE} \right) = -Q, \tag{5.15}
\]

where \( Q \) is given by Eq. (5.7). Comparing Eq. (4.38) with Eq. (5.15), the scalar field and matter interact with each other in the EF.

The background equations of motion in the JF do not contain the terms \( B_4 \) and \( B_5 \). The theories with same values of \( A_{2,3,4,5} \) but with different \( B_{4,5} \) cannot be distinguished from each other at the background level \( 54,70 \). In other words, two theories with different values of \( c_t^2 \) lead to the same background dynamics for given \( A_{2,3,4,5} \).

This implies that the coupling \( -C_t/(\tilde{N}C_t) \rho \) appearing in Eq. (5.7) does not essentially modify the background physics even for the theories in which \( C_t \) varies in time. In Sec. VII we shall confirm this property for a concrete dark energy model.

### C. Perturbations in the EF

Substituting the relations \( \dot{L}_S = \dot{E} = M^2_{pl}/2 \) into Eqs. (4.37)-(4.42), we obtain the perturbation equations of motion in the EF. From Eq. (4.42) the gauge-invariant Bardeen potentials obey the relation

\[
\dot{\Psi} + \dot{\Phi} = -a_H \frac{\delta \tilde{N}}{N}, \tag{5.16}
\]
where $\dot{\alpha}_H = 2\ddot{N}\dot{H}/M_{pl}^2$ is the parameter characterizing the departure from Horndeski theories. Since $\dot{L}_S = 0$ and $\dot{\alpha}_H^2 = \ddot{N}^2$, the first and second terms present on the rhs of Eq. (4.22) in the JF vanish in the EF.

The full GLPV action cannot be mapped to the full Horndeski action under the disformal transformation [43, 44], so the parameter $\dot{\alpha}_H$ in Eq. (5.19) does not generally vanish. It is, however, possible to transform part of the GLPV action to the action belonging to Horndeski theories, in which case $\dot{\alpha}_H = 0$ and hence there is no anisotropic stress in the EF.

Let us consider perturbations of non-relativistic matter characterized by $P = 0$ and $\delta P = 0$. In the EF, Eqs. (4.40) and (4.41) reduce, respectively, to

$$\frac{1}{H} \frac{d\dot{V}_m}{dt} + \left( \frac{1}{H\beta} \frac{d\dot{\Omega}}{dt} + \frac{1}{H^2} \frac{d\dot{H}}{dt} \right) \dot{V}_m = -\frac{\delta N}{N},$$

$$\frac{d\delta}{dt} + 3 \frac{\dot{\zeta}}{\dot{N}} \frac{k^2}{a^2} \dot{\Psi}_N - \frac{1}{N^2} \hat{C}_t^2 \frac{a^2}{H} \frac{d\dot{H}}{dt} = 0,$$

where we used the background Eq. (4.35). Taking the $\dot{t}$ derivative of Eq. (5.18) and employing Eq. (5.17), the matter density contrast (4.44) obeys

$$\frac{d^2\delta_m}{dt^2} + \left( 2\dot{H} - \frac{1}{\Omega} \frac{d\dot{\Omega}}{dt} + \frac{1}{C_t} \frac{dC_t}{dt} \right) \frac{d\dot{\delta}_m}{dt} + \frac{k^2}{\hat{C}_t^2} \Psi_g = -3 \frac{d^2\dot{B}}{dt^2} - 3 \left( 2\dot{H} - \frac{1}{\Omega} \frac{d\dot{\Omega}}{dt} + \frac{1}{C_t} \frac{dC_t}{dt} \right) \frac{d\dot{B}}{dt},$$

where $\dot{B} \equiv \dot{\zeta} + \dot{V}_m$, and

$$\hat{\Psi}_g \equiv \hat{\Psi} - \left( \frac{1}{\Omega} \frac{d\dot{\Omega}}{dt} - \frac{1}{C_t} \frac{dC_t}{dt} \right) \frac{\dot{\psi}}{N} + \frac{1}{N^2} \frac{\delta N}{\hat{C}_t^2} = \frac{\delta N}{N}.$$

The effective potential $\hat{\Psi}_g$ characterizes the strength of gravitational coupling with matter. In the EF, it is clear that $\hat{\Psi}_g$ is expressed in terms of the sum of the gravitational potential $\hat{\Psi}$ and contributions from the variations of $\Omega$ and $C_t$ as well as the difference of $\delta N$ from 1. For the theories with $C_t = 1$, which is the case for BD theories, the variation of the conformal factor $\Omega$ gives rise to the modification to $\hat{\Psi}$. The deviation of $\delta N$ from 1 and the variation of $C_t$ occur for the theories studied in Refs. [53, 54], in which case the gravitational interaction is modified as well.

On using the correspondence $\delta N/\ddot{N} = \delta N/\dot{N}$ and $\dot{\psi}/\dot{N} = \Omega \psi/(\ddot{N} C_t)$, the gravitational potential $\hat{\Psi}$ can be expressed by using $\hat{\Psi}$ in the JF. Then, we obtain the simple relation

$$\hat{\Psi}_g = \frac{\hat{\Psi}}{\hat{C}_t^2}.$$

This shows that the effective potential $\hat{\Psi}_g$ is directly related to $\hat{\Psi}$ appearing in the matter perturbation Eq. (4.20) in the JF. Once we find the evolution of $\hat{\Psi}_g$ in the EF, the potential $\hat{\Psi}$ and the effective gravitational coupling $G_{\text{eff}}$ in the JF are known accordingly.

D. Model belonging to Horndeski theories in the EF

One example of realizing $\dot{\alpha}_H = 0$ in the EF is the model described by the JF Lagrangian

$$L = A_2(N, t) + A_3(N, t)K + A_4(t)(K^2 - S) + B_4(t)R,$$

In this case the tensor propagation speed squared $\hat{C}_t^2$ divided by $N^2$ reads

$$\hat{C}_t^2 = -\frac{B_4}{A_4}.$$

In Horndeski theories we have $-A_1 = B_4$ from the first condition of Eq. (5.23), but in GLPV theories there is no such restriction and hence $\hat{C}_t^2$ generally differs from 1. Since $D = 0$ and $\alpha_H = \hat{C}_t^2 - 1$ for the model (5.22), it follows that

$$\Psi + \Phi = \frac{\dot{A}_1}{NHA_4}(\zeta - \Phi) - (\hat{C}_t^2 - 1) \left( \frac{\zeta}{N} + \frac{\delta N}{\hat{C}_t^2} \right),$$

from Eq. (5.24). For $A_4$ depending on $t$ and for $\hat{C}_t^2$ different from 1, the anisotropic stress is present in the JF.

Under the disformal transformation with the factors

$$\Omega^2 = \frac{2\sqrt{-A_4B_4}}{M_{pl}^2},$$

$$\Gamma = \frac{2\sqrt{-A_4B_4}}{M_{pl}^2} \left( -\frac{B_4}{A_4} - 1 \right) \frac{1}{X},$$

the Lagrangian (5.22) is transformed to

$$\tilde{L} = \tilde{A}_2(\tilde{N}, t) + \tilde{A}_3(\tilde{N}, t)K + \frac{M_{pl}^2}{2} \left( \tilde{S} - \tilde{K}^2 + \tilde{R} \right),$$

where $\tilde{A}_2 = (A_2 - 3\omega A_4/N + 6\omega^2 A_4/N^2)/(\Omega^3 \alpha)$ and $\tilde{A}_3 = (A_3 - 4\omega A_4/N)/\Omega^3$ with $\alpha = (\sqrt{2}/M_{pl})(-B_4/A_4)^{1/4}$ and $\omega = (A_4/A_4 + B_4/B_4)/4$. The last term of Eq. (5.27) corresponds to the Einstein-Hilbert term $M_{pl}^4\bar{R}/2$, where $\bar{R}$ is the four-dimensional Ricci scalar. Since $\dot{\alpha}_H = 0$ for the Lagrangian (5.27), it follows that

$$\Psi + \tilde{\Phi} = 0.$$

Thus, for the Lagrangian (5.22), the disformal transformation allows one to de-mix the gravitational potentials in the EF, such that the anisotropy parameter $\eta = -\Phi/\Psi$ is equivalent to 1.

While the gravitational potentials are de-mixed in the EF, the matter perturbation $\delta_m$ is subject to gravitational mixing described by the effective potential (5.20) mediated by the scalar field $\phi$. Multiplying the term $3\dot{N}\dot{H}$ for Eq. (4.38) and taking the sum with Eq. (4.37), one can relate $\delta_m$ with metric perturbations. Let us employ the sub-horizon approximation [73] under which the dominant contributions to the lhs of Eqs. (4.37)
and (4.38) are those involving $\partial^2 \hat{\omega} / \hat{a}^2$ and $\partial^2 \hat{\zeta} / \hat{a}^2$. In the EF the terms $\hat{\varepsilon}$ and $\hat{\mathcal{W}}$ are given, respectively, by $\hat{\varepsilon} = M_{\text{pl}}^2 / 2$ and $\hat{\mathcal{W}} = 2 \hat{H} M_{\text{pl}} / \hat{N} + \hat{A}_{3,\hat{N}}$. Provided that $2 \hat{H} M_{\text{pl}}^2 / \hat{N} \gg |A_{3,\hat{N}}|$, we obtain the Poisson equation

$$\frac{k^2}{a^2} \Phi \simeq \frac{1}{2M_{\text{pl}}} \delta_m^e. \quad (5.29)$$

On using Eq. (5.22) and introducing the gravitational constant as $G = (8\pi M_{\text{pl}}^2)^{-1}$, it follows that

$$\frac{k^2}{a^2} \Psi \simeq -4\pi G \delta_m^e. \quad (5.30)$$

This shows that the gravitational coupling associated with $\Psi$ is simply given by $G$ under the condition $2 \hat{H} M_{\text{pl}}^2 / \hat{N} \gg |A_{3,\hat{N}}|$. Hence, the modified gravitational interaction from GR arises from the terms on the rhs of Eq. (5.24) other than $\Psi$.

In Sec. VI we consider a concrete model belonging to the Lagrangian (5.22) and study the correspondence of physical quantities between the JF and the EF in detail.

### VI. CONCRETE MODEL

We study a dark energy model described by the action (2.26), where the Lagrangian $L$ is given by Eq. (5.22). We consider the following functions

$$A_2 = -\frac{1}{2} e(\phi) X - V(\phi), \quad A_3 = -M_{\text{pl}}^2 \sqrt{-X} F_1, \quad A_4 = -\frac{1}{2} M_{\text{pl}}^2 F_1, \quad B_4 = \frac{1}{2} M_{\text{pl}}^2 F_2, \quad (6.1)$$

where $e(\phi), V(\phi), F_1(\phi)$, and $F_2(\phi)$ are functions that depend on $\phi$, i.e., on $t$ in the unitary gauge. The dependence of $A_2$ and $A_3$ on $N$ arises in the kinetic term $X = -\partial^2 / N^2$. To accommodate BD theory (2.26) as well as theories recently proposed in Refs. [55, 56], we choose the functions

$$F_1(\phi) = e^{-2q_1 \phi / M_{\text{pl}}}, \quad F_2(\phi) = c_{q_2}^2 e^{-2q_2 \phi / M_{\text{pl}}}, \quad e(\phi) = (1 - 6q_1^2) F_1(\phi), \quad (6.2)$$

where $q_1, q_2, c_{q_1}$ are positive constants. We assume that $q_1, q_2 \ll 1$ for the compatibility with observations [67, 73]. In Horndeski theories, the first condition of Eq. (2.5) demands that $F_2(\phi)$ is equivalent to $F_1(\phi)$. The original BD theory without the field potential corresponds to the case $V(\phi) = 0$ and $F_2(\phi) = F_1(\phi)$ with the BD parameter $\omega_{\text{BD}} = (1 - 6q_1^2) / (4q_1^2)$ [12].

Let us first consider theoretical consistent conditions in the JF. In the following we set the background value of the lapse $\bar{N}$ to be 1. From Eqs. (4.24) and (4.28) the conditions for avoiding tensor and scalar ghosts are given, respectively, by

$$q_1 = \frac{1}{8} M_{\text{pl}}^2 F_1 > 0, \quad (6.3)$$
$$q_2 = \frac{M_{\text{pl}}^2 \sqrt{\phi}}{2(H - q_1 \phi)^2} > 0, \quad (6.4)$$

which are satisfied for $F_1 > 0$. The tensor and scalar propagation speed squares are given, respectively, by

$$c_T^2 = \frac{F_2}{F_1} = c_{q_1}^2 e^{2(q_1 - q_2) \sqrt{\phi} / M_{\text{pl}}}, \quad (6.5)$$
$$c_s^2 = c_T^2 + \frac{(1 - c_T^2)\Omega_m}{2x_1^2} + \frac{2(q_1 - q_2) e^{2\sqrt{\phi} - 6q_1 x_1} 3x_1}{3x_1}, \quad (6.6)$$

where

$$x_1 = \frac{\phi}{\sqrt{6H M_{\text{pl}}}}, \quad \Omega_m = \frac{\rho}{3M_{\text{pl}}^2 H^2 F_1}. \quad (6.7)$$

Under the no-ghost condition $F_1 > 0$, the condition (6.3) is satisfied for $F_2 > 0$. For the theories with $q_1 = q_2$, $c_T^2$ is constant ($c_T^2 = c_{q_1}^2$). Since $c_T^2 = c_T^2 + (1 - c_T^2)\Omega_m / (2x_1^2)$ in this case, $c_s^2$ is positive for $0 < c_T^2 < 1$, while $c_s^2$ can be negative for $c_T^2 > 1$. In Ref. [55] the authors studied the cosmology for the specific case with $q_1 = q_2 = 0$. If $q_1 \neq q_2$, then $c_s^2$ varies in time. The variation of $c_s^2$ gives rise to a contribution to $c_s^2$, i.e., the last term on the rhs of Eq. (6.6). The cosmology with $q_1 = 0$ and $q_2 \neq 0$ was recently studied in Ref. [56] as a model of realizing weak gravity on scales relevant to large-scale structures.

For dark energy models in which the ratio $\Omega_m / x_1^2$ decreases with time, $c_s^2$ grows to be very much larger than 1 as we go back to the past. This behavior can be avoided for the scaling model characterized by the potential $V(\phi) = V_1 e^{-\lambda_1 \phi / M_{\text{pl}}} + V_2 e^{-\lambda_2 \phi / M_{\text{pl}}}, \quad (6.8)$

where $V_1, V_2, \lambda_1, \lambda_2$ are positive constants. Provided the first potential on the rhs of Eq. (6.3) dominates over the second one, the scaling solution with the constant ratio $\Omega_m / x_1^2$ is realized during radiation and matter eras [77].

The solution exits from the scaling regime to the epoch of cosmic acceleration due to the existence of the second potential. We shall consider the situation in which the slopes $\lambda_1$ and $\lambda_2$ are in the range

$$\lambda_1 \gtrsim 10, \quad \lambda_2 \lesssim 1, \quad (6.9)$$

for consistency with the big-bang nucleosynthesis [78] and the late-time cosmic acceleration [1]. There are seven free parameters ($q_1, q_2, c_{q_1}, V_1, V_2, \lambda_1, \lambda_2$) in our model.

---

1. There are some models like kinetic braiding [13] in which the dependence of $A_3$ on $N$ modifies the gravitational coupling; see Refs. [14, 17].
A. Transformation to the Einstein frame

For the theories given by the functions (5.16), the two factors (5.25) and (5.26) transforming to the EF are given, respectively, by

\[ \Omega^2(\phi) = \sqrt{F_1(\phi)F_2(\phi)} = F_1(\phi)c_i(\phi), \]
\[ \Gamma(\phi, X) = \sqrt{F_1(\phi)F_2(\phi)} \left( \frac{F_2(\phi)}{F_1(\phi)} - 1 \right) \frac{1}{X}. \]

We also have

\[ \alpha = \frac{1}{\beta} = \sqrt{F_1(\phi)c_i^2(\phi)}. \]

Then the action in the EF is given by Eq. (3.33) with the Lagrangian (5.27), where

\[ A_2 = \frac{\dot{\phi}^2}{2N^2} \left[ 1 - \frac{3}{2}(q_1 - q_2)^2 \right] - \hat{V}(\phi), \]
\[ A_3 = \frac{M_p\dot{\phi}}{N}(q_1 - q_2), \]

and the EF frame potential

\[ \hat{V}(\phi) = \frac{V(\phi)}{\sqrt{F_1(\phi)F_2(\phi)}} = \frac{V(\phi)}{c_i^3} \left( q_1 + 3q_2 \right) \frac{\phi}{M_p}. \]

For the JF potential (6.8) it follows that

\[ \hat{V}(\phi) = \hat{V}_1 e^{-\mu_1\phi/M_p} + \hat{V}_2 e^{-\mu_2\phi/M_p}, \]

where \( \hat{V}_1 = V_1/c_{i1} \), \( \hat{V}_2 = V_2/c_{i1} \), and

\[ \mu_1 = \lambda_1 - q_1 - 3q_2, \quad \mu_2 = \lambda_2 - q_1 - 3q_2. \]

For the theories with \( q_1 = q_2 \) (i.e., \( c_i^2 \) constant) the term \( \hat{A}_3 \) vanishes, in which case the Lagrangian (5.27) corresponds to that of the canonical scalar field \( \phi \) coupled to matter.

B. Background dynamics

From Eqs. (4.7) and (4.9) the background equations of motion in the JF are given by

\[ 3M_p^2H^2F_1 = \frac{\epsilon}{2}\phi^2 + V - 3M_p^2H\phi F_1, \phi + \rho, \]
\[ -2M_p^2\dot{H}F_1 = \epsilon\phi^2 + M_p^2(\dot{F}_1 - H\ddot{F}_1) + \rho + P, \]
\[ \dot{\rho} + 3H(\rho + P) = 0. \]

These equations depend on the function \( F_1(\phi) \) but not on \( F_2(\phi) \), so the quantities \( q_2 \) and \( c_{i2}^2 \) are irrelevant to the background dynamics. This means that the theories with same \( q_1 \) and different \( q_2 \) (i.e., same \( A_4 \) and different \( B_4 \)) cannot be distinguished from each other for the background cosmology in the JF (5.27).

The disformal transformation to the EF corresponds to the change of tensor propagation speed squared \( c_t^2 = -B_4/A_4 \) to \( \tilde{c}_t^2 = \tilde{N}^2 \), so the quantity \( q_2 \) arises in the EF. Nevertheless we are dealing with the same physics in the two frames, so any physical condition (such as the stability of fixed points) should not be subject to change. In what follows we shall study the correspondence of background quantities between the EF and the JF.

1. Einstein frame

The background equations of motion in the EF are given by Eqs. (5.11)–(5.12), where

\[ \dot{\rho}_r = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 \left[ 1 - \frac{3}{2}(q_1 - q_2)^2 \right] + \hat{V}(\phi), \]
\[ \dot{P}_r = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 \left[ 1 - \frac{3}{2}(q_1 - q_2)^2 \right] - \hat{V}(\phi), \]
\[ \dot{\rho} = \frac{q_1 + q_2}{2M_p} \left( \rho - 3\bar{P} \right) \frac{d\phi}{dt} - \frac{q_1 - q_2}{M_p} \frac{d\phi}{dt}. \]

The matter fluid and the scalar field \( \phi \) obey Eqs. (6.15) and (5.15), respectively, with

\[ Q = \frac{q_1 + q_2}{2M_p} \left( \rho - 3\bar{P} \right) \frac{d\phi}{dt} - \frac{q_1 - q_2}{M_p} \frac{d\phi}{dt}. \]

The first term on the rhs of Eq. (6.23) arises for the standard coupled dark energy scenario characterized by \( q_1 = q_2 \) and \( c_{i1}^2 = 1 \). The second term on the rhs of Eq. (6.23) does not vanish for the theories with \( q_1 \neq q_2 \) (i.e. time-varying \( c_{i1}^2 \)).

To discuss the background dynamics in the EF, it is convenient to introduce the dimensionless quantities

\[ \hat{x}_1 \equiv \frac{1}{\sqrt{6M_pH}} \frac{d\phi}{dt}, \quad \hat{x}_2 \equiv \frac{\sqrt{\hat{V}}}{\sqrt{3M_pH}}, \]
\[ \hat{\Omega}_r \equiv \frac{\rho_r}{3M_p^2H^2}, \quad \hat{\Omega}_m \equiv \frac{\rho}{3M_p^2H^2}, \]
\[ w \equiv \frac{\bar{P}}{\rho}, \quad \mu \equiv -\frac{M_p\dot{\bar{V}}}{\bar{V}}, \quad \hat{\epsilon}_H \equiv -\frac{\ddot{H}}{H}. \]

where a prime represents a derivative with respect to \( \hat{N} = \int \ddot{H} dt \). Recall that the matter equation of state \( w \) is invariant under the disformal transformation to the EF. In the following we assume that \( w \) is constant. From Eq. (5.11) there is the relation \( \hat{\Omega}_r + \hat{\Omega}_m = 1 \).

The variables \( \hat{x}_1 \) and \( \hat{x}_2 \) obey the differential equations

\[ \frac{d\hat{x}_1}{dt} = 1/\sqrt{6M_pH}, \quad \frac{d\hat{x}_2}{dt} = \sqrt{\hat{V}}. \]
\[
\hat{x}' = \frac{\sqrt{6}}{4}[2q_1(3w - 1) - 2\sqrt{6}(1 + q_1(q_1 - q_2)(3w - 1))\hat{x}_1 + q_1\{3(q_1 - q_2)^2 - 2\}(3w - 1)\hat{x}_1^2 \\
+ 2\{3q_2 + \mu - q_1(2 + 3w)\}x_1 + \hat{x}_1\epsilon_H , \tag{6.25}
\]

\[
\hat{x}_2' = -\frac{\sqrt{6}}{2}\mu\hat{x}_1 + \hat{x}_2\epsilon_H , \tag{6.26}
\]

\[
\hat{\epsilon}_H = \frac{3}{2}(1 + w - \hat{x}_2^2) - \frac{3}{2}q_1(q_1 - q_2)(3w - 1) + \frac{3}{2}\sqrt{6}(1 - q_2)\{1 - w + q_1(q_1 - q_2)(3w - 1)\}\hat{x}_1 \\
- \frac{3}{4}\{3(q_1 - q_2)^2 - 2\}(1 - w + q_1(q_1 - q_2)(3w - 1))\hat{x}_1^2 - \frac{3}{4}[w - (q_1 - q_2)\{(q_1(2 + 3w) - 3q_2 - \mu)\}x_1^2 , \tag{6.27}
\]

where \(\hat{\epsilon}_H\) is related to the effective equation of state \(\hat{w}_{\text{eff}}\) of the system, as \(\hat{w}_{\text{eff}} = -1 + 2\hat{\epsilon}_H/3\). The field density parameter and the equation of state are given, respectively, by

\[
\dot{\Omega}_{\text{DE}} = 1 - \Omega_m = \frac{\hat{x}_1}{2}\left[2\sqrt{6}(q_1 - q_2) + \{2 - 3(q_1 - q_2)^2\}\hat{x}_1\right] + \hat{x}_2^2 , \tag{6.28}
\]

\[
\dot{\hat{\omega}}_{\text{DE}} = \frac{\dot{\hat{P}}_{\text{DE}}}{\hat{\rho}_{\text{DE}}} = \frac{3[2 - 3(q_1 - q_2)^2]\hat{x}_1^2 - 6\hat{x}_2^2 - 2\sqrt{6}(q_1 - q_2)\{(\hat{x}_1 - \hat{\epsilon}_H\hat{x}_1)\}}{3[2 - 3(q_1 - q_2)^2]\hat{x}_1^2 + 6\hat{x}_2^2 + 6\sqrt{6}(q_1 - q_2)\hat{x}_1} . \tag{6.29}
\]

If \(\mu\) is constant, which is the case for the exponential potential \(\hat{V}(\phi) = V_0 e^{-\mu\phi}/M_p^4\), there are five fixed points characterized by constant \(\hat{x}_1\) and \(\hat{x}_2\). They are summarized in Table I.

For the fixed point (a), the field density parameter reads

\[
\dot{\Omega}_{\text{DE}} = \frac{q_1(3w - 1)[q_1\{4 - 3(q_1 - q_2)^2\} - 6q_2 + 3\{3q_1(q_1 - q_2)^2 + 2q_2\}w]}{3[1 - w + q_1(q_1 - q_2)(3w - 1)]^2} , \tag{6.30}
\]

so that both \(\dot{\Omega}_{\text{DE}}\) and \(\hat{x}_1\) vanish for radiation \((w = 1/3)\).

If \(q_2 = q_1\), it follows that \(\hat{x}_1 = \sqrt{6}q_1(3w - 1)/[3(1 - w)]\), \(\hat{w}_{\text{eff}} = [2q_1^2(1 - 3w)^2 + 3w(1 - w)]/[3(1 - w)]\), and \(\dot{\Omega}_{\text{DE}} = 2q_1^2(3w - 1)^2/[3(1 - w)]^2\) for the point (a). When \(w = 0\), this corresponds to the \(\phi\)-matter-dominated era \((\phi\text{MDE})\) characterized by \(\hat{w}_{\text{eff}} = \dot{\Omega}_{\text{DE}} = 2q_1^2/3\). Provided that \(w \neq 1/3\), the point (a) is a kind of scaling solution with a constant ratio \(\hat{\Omega}_m/\dot{\Omega}_{\text{DE}}\).

Since the effective equations of state \(\hat{w}_{\text{eff}}\) for the points (b1) and (b2) are 1, they are neither relevant to radiation/matter eras nor the late-time cosmic acceleration.

The point (c) is the scalar-field dominated point \((\dot{\Omega}_{\text{DE}} = 1)\) relevant to dark energy. When \(q_2 = q_1\), we have \(\hat{x}_1 = \mu/\sqrt{6}, \hat{x}_2 = \sqrt{1 - \mu^2/6},\) and \(\hat{w}_{\text{eff}} = -1 + \mu^2/3\), so the cosmic acceleration occurs for \(\mu^2 < 2\). For \(q_2 \neq q_1\), \(\hat{w}_{\text{eff}}\) is close to \(-1\) provided that \(q_1, q_2, \mu\) are smaller than the order of 1.

The point (d) corresponds to the scaling solution with the field density parameter

\[
\dot{\Omega}_{\text{DE}} = \frac{q_1^2(9w^2 - 30w - 23) + 2q_1[8\mu + 3q_2\{9 + w(4 + 3w)\}]}{[2\mu - q_1 + 3q_2 - 3w(q_1 + q_2)]^2} (\hat{x}_1 + 4/\mu^2) \tag{6.31}
\]

by considering small perturbations \(\delta\hat{x}_1\) and \(\delta\hat{x}_2\) about each of them. The linearized version of Eqs. (6.25) and (6.26) can be written in the form

\[
\begin{pmatrix}
\delta\hat{x}_1' \\
\delta\hat{x}_2'
\end{pmatrix} = \mathcal{M}
\begin{pmatrix}
\delta\hat{x}_1 \\
\delta\hat{x}_2
\end{pmatrix} , \tag{6.32}
\]

where \(\mathcal{M}\) is a \(2 \times 2\) matrix. If the two eigenvalues \(\hat{\kappa}_{1,2}\) of \(\mathcal{M}\) are negative (or imaginary with negative real parts), then the corresponding fixed point is stable.
Table I. The fixed points in the EF and corresponding values of $\hat{w}_{\text{eff}}$ and $\Omega_{\text{DE}}$ for the system characterized by the autonomous Eqs. (6.2a)–(6.2c) with Eq. (6.27). The scaling radiation and matter points (d) are stable for $\mu \geq 10$ and $q_1, q_2 \ll 1$, whereas the accelerated point (c) is stable for $\mu \leq 1$ and $q_1, q_2 \ll 1$. The potential (6.16) allows for the transition from the matter point (d) with $\mu \simeq \mu_1 \geq 10$ to the point (c) with $\mu \simeq \mu_2 \leq 1$.

In the presence of radiation ($w = 1/3$), the eigenvalues of the point (d) are given by

$$\hat{\kappa}_{1,2}^{(d)} = -\frac{\mu - 3q_1 + 3q_2 \pm \sqrt{64 - 15(\mu - 3q_1 + 3q_2)^2}}{2(\mu - q_1 + q_2)}.$$  (6.33)

For $\mu \geq 10$ and $q_1, q_2 \ll 1$, $\hat{\kappa}_{1,2}^{(d)}$ are imaginary with negative real parts, so the point (d) is a stable spiral. For non-relativistic matter ($w = 0$), the eigenvalues of the point (d) read

$$\hat{\kappa}_{1,2}^{(d)} = -\frac{3(\mu - q_1 + 3q_2) \pm \sqrt{D_{(d)}}}{2(2\mu - q_1 + 3q_2)},$$  (6.34)

where $D_{(d)} = 9(\mu - q_1 + 3q_2)^2 - 24[3 - 4q_1^2 + 2q_1(\mu + 3q_2)]/[\mu + 3q_2]^2 - 5q_1(\mu + 3q_2) - 3 + 6q_1^2]$. For $\mu \geq 10$ and $q_1, q_2 \ll 1$, the eigenvalues (6.34) are again imaginary with negative real parts. Thus, the first potential on the rhs of Eq. (6.10) leads to the scaling radiation and matter eras driven by the fixed point (d) with $\mu = \mu_1$.

The point (a) can be potentially relevant to radiation and matter eras, but one of the eigenvalues is positive, i.e., $\hat{\kappa}_1^{(a)} = 2$ for $w = 1/3$ and $\hat{\kappa}_1^{(a)} = (3 + 2q_1 + 6q_2 - 4q_1^2)/(2(1 + q_1 + q_2))$ for $w = 0$. Hence the solutions are attracted by the scaling solution (d) rather than the point (a). For $\mu$ smaller than the order of 1, the stable scaling matter solution (d) with $\Omega_{\text{DE}} < 1$ does not exist [79], in which case the matter era is replaced by the $\phi$MDE (a) [67]. In our model, we do not consider this latter case to avoid very large values of $c_s^2$ in the early cosmological epoch.

After the dominance of the second potential on the rhs of Eq. (6.10), the solutions exit from the scaling matter era (d) to the epoch of cosmic acceleration driven by the point (c). In the presence of non-relativistic matter, the eigenvalues of the point (c) are given by

$$\hat{\kappa}_1^{(c)} = \frac{6 - \mu - 3q_1 + 3q_2^2}{2 + (q_1 - q_2)(\mu - 3q_1 + 3q_2)},$$  (6.35)

$$\hat{\kappa}_2^{(c)} = \frac{6 - 2(\mu - 3q_1 + 3q_2)(\mu - 2q_1 + 3q_2)}{2 + (q_1 - q_2)(\mu - 3q_1 + 3q_2)}.$$  (6.36)

For $\mu \leq 1$ and $q_1, q_2 \ll 1$, it is clear that both $\hat{\kappa}_1^{(c)}$ and $\hat{\kappa}_2^{(c)}$ are negative. Hence, the solutions finally approach the accelerated attractor (c) with $\hat{w}_{\text{eff}}$ close to −1 for $\mu_2 \ll 1$ (see Table I for the value of $\hat{w}_{\text{eff}}$).

In summary, for the potential (6.10) with $\mu_1 \geq 10$ and $\mu_2 \leq 1$, the background cosmological sequence in the EF is as follows: (i) scaling radiation point (d) with $w = 1/3$ and $\mu = \mu_1$, (ii) scaling matter point (d) with $w = 0$ and $\mu = \mu_1$, (iii) accelerated point (c) with $\mu = \mu_2$.

2. Jordan frame

The background dynamics in the JF can be known by using relations for physical quantities between the two frames. We define the dimensionless quantities

$$x_1 \equiv \frac{\dot{\phi}}{\sqrt{6M_pH}}, \quad x_2 \equiv \frac{\sqrt{V}}{\sqrt{3F_1M_pH}},$$

$$\lambda \equiv -\frac{M_pV}{\rho}, \quad \Omega_m \equiv -\frac{\rho}{3M_p^2H^2F_1},$$  (6.37)

where the field density parameter is given by $\Omega_{\text{DE}} = 1 - \Omega_m$. On using Eqs. (6.27), (6.10), (6.12), and (A1), we obtain the following correspondence

$$x_1 = (1 + \omega_H)x_1', \quad x_2 = (1 + \omega_H)x_2',$$

$$\Omega_m = (1 + \omega_H)^2\Omega_m',$$  (6.38)

where

$$\omega_H \equiv \frac{\dot{\Omega}_m}{H\Omega_m} = -\frac{\sqrt{6}}{2}(q_1 + q_2)x_1 = -\frac{\sqrt{6}(q_1 + q_2)x_1}{2 + \sqrt{6}(q_1 + q_2)x_1}.$$  (6.39)

The slow-roll parameter $\epsilon_H \equiv -\dot{H}/H^2$, which is associated with the effective equation of state $w_{\text{eff}}$ in the JF as $w_{\text{eff}} = -1 + 2\epsilon_H/3$, satisfies the relation

$$\epsilon_H = (1 + \omega_H)\left[\frac{\dot{\epsilon}_H - \frac{\sqrt{6}}{2}(q_1 + q_2)\dot{x}_1}{H(1 + \omega_H)}\right].$$  (6.40)
The slope $\lambda$ defined in Eq. (6.37) is related to the slope $\mu$ in the EF, as

\[ \mu = \lambda - q_1 - 3q_2. \]  

Using the above correspondence, one can readily translate the background cosmological dynamics in the EF to that in the JF. In the JF, the fixed point (d) in Table II corresponds to

\[ x_1^{(d)} = \frac{\sqrt{6}(1 + w)}{2\lambda}, \]

\[ x_2^{(d)} = \frac{3(1 - w^2) + 2q_1(1 - 3w)\lambda - 3q_1(1 + w)}{\sqrt{2}\lambda}, \]  

with

\[ w^{(d)}_{\text{eff}} = w - 2(1 + w)\frac{q_1}{\lambda}, \]

\[ \Omega^{(d)}_{\text{DE}} = 3(1 + w)(1 - 4q_1^2) + q_1\lambda(7 + 3w) \lambda^2. \]  

The fixed point (c) translates to

\[ x_1^{(c)} = \frac{\lambda - 4q_1}{\sqrt{6(1 - 4q_1^2 + q_1 \lambda)}}, \]

\[ x_2^{(c)} = \frac{\sqrt{1 - (\lambda - 4q_1)^2}/6}{1 - 4q_1^2 + q_1 \lambda}, \]  

with

\[ w^{(c)}_{\text{eff}} = \frac{3 - \lambda^2 - 20q_1^2 + 9q_1 \lambda}{3(1 - 4q_1^2 + q_1 \lambda)}, \]

\[ \Omega^{(c)}_{\text{DE}} = 1. \]  

The quantity $q_2$ disappears after the transformation from the EF to the JF, which reflects the fact that the factor $F_2$ is absent in the background equations (6.18)-(6.20).

The stability of fixed points should be independent of the values of $q_2$. Substituting Eq. (6.41) into Eqs. (6.39)-(6.40), it follows that the numerators of the eigenvalues do not contain the term $q_2$. In the denominators there are still $q_2$-dependent terms, but they are simply associated with the transformation of the number of $e$-foldings, i.e.,

\[ \frac{dN}{dN} = 1 - \frac{\sqrt{6}}{2}(q_1 + q_2)x_1. \]  

The evolution of homogenous perturbations $\delta x_j \propto e^{\kappa_j N}$ ($j = 1, 2$) in the EF translates to the JF evolution proportional to $e^{\kappa_j N}$, where $\kappa_j = \tilde{k}_j [1 - \sqrt{6}(q_1 + q_2)x_1/2]$. On using the index $j$, the $q_2$ dependence in the denominators of $\tilde{k}_j$ vanishes identically. Provided the rhs of Eq. (6.43) is positive, which is the case for $q_1, q_2 \ll 1$ and $|x_1| \ll 1$, the stability conditions of fixed points are identical to each other in the two frames.

The above discussion shows that, in the JF, the scaling radiation fixed point (d) with $w = 1/3$ and $\lambda = \lambda_1$ is followed by the scaling matter point (d) with $w = 0$ and $\lambda = \lambda_1$, and then the solutions finally approach the point (c) with $\lambda = \lambda_2$. During this sequence, the effective equation of state and the field density parameter evolve as (i) $w_{\text{eff}} = 1 + 3 - 8q_1/(3\lambda_1), \Omega_{\text{DE}} = 4(1 - 4q_1^2 + 2q_1 \lambda_1)/\lambda_1^2$ (radiation era), $\rightarrow$ (ii) $w_{\text{eff}} = -2q_1/\lambda_1$, $\Omega_{\text{DE}} = [3(1 - 4q_1^2) + 7q_1 \lambda_1]/\lambda_1^2$ (matter era), $\rightarrow$ (iii) $w_{\text{eff}} = -(3 - \lambda_2^2 - 20q_1^2 + 9q_1 \lambda_2)/[3(1 - 4q_1^2 + q_1 \lambda_2)], \Omega_{\text{DE}} = 1$ (accelerated era).

C. Perturbations and matter-scalar couplings

We consider the evolution of cosmological perturbations and the resulting matter-scalar coupling in the presence of non-relativistic matter satisfying $P = 0$ and $\delta P = 0$. In what follows we shall focus on the case where

\[ q_1 = q_2 \equiv q, \]  

under which $c_s^2$ is constant. Then, at the background level, the coupling (6.23) reduces to

\[ Q = \frac{\dot{\rho}}{\rho} \frac{d \phi}{d t}. \]  

The quantity $\omega_H$ in Eq. (6.39) reads

\[ \omega_H = \frac{\dot{F}_1}{2HF_1} = \frac{\dot{\alpha}}{H \alpha} = - \frac{\dot{\beta}}{H \beta} = - \frac{\dot{q} \phi}{HM_{\text{pl}}}. \]  

In the EF, the gauge-invariant matter density contrast is defined by Eq. (4.41). Since $\delta = \delta$ and $\dot{V}_m = (1 + \omega_H)V_m$, $\dot{\delta}_m$ is not equivalent to $\delta_m$. For the perturbation deep inside the Hubble radius the velocity potential $\dot{V}_m$ is much smaller than $\delta$, so the difference between $\delta_m$ and $\dot{\delta}_m$ is small. We introduce the effective gravitational coupling $G_{\text{eff}}$ in the EF, as

\[ \frac{k^2}{a^2} \dot{\Psi}_g = -4\pi G_{\text{eff}} \rho \dot{\delta}_m, \]  

where $\dot{\Psi}_g$ is given by Eq. (5.20). For the choice (6.49), $\dot{\Psi}_g$ reduces to

\[ \dot{\Psi}_g = \dot{\Psi} + \frac{q_0}{M_{\text{pl}}} \dot{\chi} + \left( \frac{1}{c_s^2} - 1 \right) \frac{\delta N}{N}, \]  

where

\[ \dot{\chi} \equiv \frac{\dot{H} \dot{\psi}}{N}. \]  

Using the relation (5.21) as well as the approximation $\delta_m \simeq \dot{\delta}_m$ for the perturbations deep inside the Hubble radius, we can rewrite Eq. (6.52) of the form $(k^2/a^2)\dot{\Psi} \simeq -4\pi (G_{\text{eff}}/F_1)\rho \delta_m$. Hence, the effective gravitational coupling $G_{\text{eff}}$ in the JF is related to $G_{\text{eff}}$, as

\[ G_{\text{eff}} \simeq \frac{\dot{G}_{\text{eff}}}{F_1}. \]  

The above discussion shows that, once \( \hat{\Psi} \) is known by solving the perturbation equations in the EF, the gravitational potential \( \Psi \) and the resulting matter-scaler coupling \( G_{\text{eff}} \) in the JF are determined accordingly.

Since \( \dot{\Omega}_H = 0 \) in the EF, we have the relation

\[
\dot{\Phi} = -\hat{\Psi}.
\]

(6.56)

On using the relations \( \delta N/\hat{N} = \delta N/\bar{N}, \hat{\zeta} = \zeta, \) and \( \chi \equiv H\psi/\bar{N} = c_t^2 \hat{\chi}/(1 + \omega_H) \) with \( \bar{N} = 1 \), the gravitational potentials \( \Psi \) and \( \Phi \) in the JF are related to \( \hat{\Psi} \) and \( \hat{\Phi} \) in the EF, as

\[
\Psi = \hat{\Psi} + (c_t^2 - 1) \frac{d}{dt} \left( \frac{\hat{\chi}}{H} \right) - c_t^2 \frac{\omega_H}{1 + \omega_H} \hat{\chi},
\]

(6.57)

\[
\Phi = \hat{\Phi} + \left( \frac{c_t^2}{1 + \omega_H} - 1 \right) \hat{\chi}.
\]

(6.58)

The anisotropy parameter \( \eta \) in the JF generally differs from 1 due to the presence of the perturbation \( \hat{\chi} \).

1. Einstein frame

Let us study the evolution of perturbations in the EF during the scaling matter and accelerated epochs. From Eqs. (4.37)-(4.42) we obtain the perturbation equations

\[
\dot{\zeta}' = \frac{\delta N}{\bar{N}} + \frac{3}{2} \Omega_m \dot{V}_m, \quad (6.59)
\]

\[
\dot{\chi}' = -(\epsilon_H + 1)\dot{\chi} - \frac{\delta N}{\bar{N}}, \quad (6.60)
\]

\[
\delta' = -3\dot{\zeta}' - \bar{K}^2 \left( \hat{\chi} - \frac{\dot{V}_m}{c_t^2} \right), \quad (6.61)
\]

\[
\dot{V}_m' = - \left( \epsilon_H + \sqrt{6} q_1 \right) V_m - \frac{\delta N}{\bar{N}}, \quad (6.62)
\]

\[
\frac{\delta N}{\bar{N}} = \frac{3\bar{\Omega}_m \delta_m - 2\bar{K}^2 (\hat{\chi} + \hat{\zeta})}{6x_1^2}, \quad (6.63)
\]

where \( \bar{K} \equiv k/(\hat{a} \hat{H}) \). Taking the \( \dot{N} \) derivative of Eq. (6.62) and using other equations of motion, the velocity potential \( \dot{V}_m \) obeys

\[
\dot{V}_m'' + C_1 \dot{V}_m' + \left[ \frac{(1 - c_t^2)\bar{\Omega}_m}{2c_t^2 x_1^2} \bar{K}^2 + C_2 \right] \dot{V}_m = - \left[ \hat{\chi} + \frac{\sqrt{6} q_1}{3x_1} (\hat{\chi} + \hat{\zeta}) \right] \bar{K}^2,
\]

(6.64)

where

\[
C_1 = \frac{2(3\bar{x}_1 - \sqrt{6} q_1)(\bar{x}_1^2 - 1)}{\bar{\Omega}_m \{3\bar{x}_1 - 2\sqrt{6} (q + \mu)\}} / (2\bar{x}_1),
\]

(6.65)

\[
C_2 = \frac{3[12\bar{x}_1^2 - 4\sqrt{6} \mu \bar{x}_1^2 - 2\bar{x}_1^2(9 + 2\mu^2 + 3q - 6\bar{\Omega}_m)}{\bar{x}_1^2 \{2\mu + \bar{\Omega}_m(\bar{\Omega}_m - 1) + 2q(q + \mu)\}}
\]

\[- \sqrt{6} \bar{x}_1^2 \{4q - 4\mu + (3q + 5\mu)\bar{\Omega}_m\}
\]

\[- \sqrt{6} (q + \mu)\bar{\Omega}_m(\bar{\Omega}_m - 1) / (2\bar{x}_1). \]

(6.66)

The general solution to Eq. (6.64) can be written in the form \( \dot{V}_m = V_m^{(s)} + V_m^{(h)} \), where \( V_m^{(s)} \) is the special solution and \( V_m^{(h)} \) is the homogenous solution derived by setting the rhs of Eq. (6.64) to be zero. For the sub-horizon perturbations satisfying \( \bar{K} \gg 1 \), the homogenous solution induces the rapid oscillation of the velocity potential with a frequency associated with the term \( (1 - c_t^2)\bar{\Omega}_m \bar{K}^2 / (2c_t^2 \bar{x}_1^2) \).

For the theories with \( q = 0 \), as long as the homogenous solution is initially suppressed relative to the special solution, it was found in Ref. [53] that the perturbations \( \chi, \zeta, \) and \( V_m \) in the JF stay nearly constant during the scaling matter epoch. As we confirm later numerically, this is also the case for \( q \neq 0 \). At the scaling fixed point (d) with \( w = 0 \) and \( \mu = \mu_1 \), the ratio \( \bar{\Omega}_m / \bar{x}_1^2 \) is constant. We consider the sub-horizon perturbations satisfying the condition \( (1 - c_t^2)\bar{\Omega}_m \bar{K}^2 / (2c_t^2 \bar{x}_1^2) \gg |C_2| \). Then, the special solution to Eq. (6.64) is given by

\[
\dot{V}_m^{(s)} \simeq - \left[ \hat{\chi} + \frac{\sqrt{6} q_1}{3x_1} (\hat{\chi} + \hat{\zeta}) \right] \frac{2c_t^2 \bar{x}_1^2}{(1 - c_t^2)\bar{\Omega}_m}. \quad (6.67)
\]

Let us derive solutions along which \( \hat{\zeta} \) and \( \hat{\chi} \) stay nearly constant in the scaling matter era. Setting \( \hat{\zeta}' \simeq 0 \) and \( \hat{\chi}' \simeq 0 \) in Eqs. (6.59) and (6.60), respectively, we obtain

\[
\hat{\chi} \simeq \frac{3[1 - c_t^2(1 - \sqrt{6} q_1)]\bar{\Omega}_m \delta_m}{2\bar{K}^2[3c_t^2 \bar{x}_1^2 + (1 - c_t^2)\bar{\epsilon}_H]}, \quad (6.68)
\]

\[
\hat{\zeta} \simeq \frac{3[1 + \bar{\epsilon}_H](1 - c_t^2) + 2c_t^2 \bar{x}_1(3\bar{x}_1 + \sqrt{6} q_1)\bar{\Omega}_m}{2\bar{K}^2[3c_t^2 \bar{x}_1^2 + (1 - c_t^2)\bar{\epsilon}_H]}, \quad (6.69)
\]

\[
\dot{V}_m^{(s)} \simeq \frac{c_t^2 \bar{x}_1(3\bar{x}_1 - \sqrt{6} q_1\bar{\epsilon}_H)}{\bar{K}^2[3c_t^2 \bar{x}_1^2 + (1 - c_t^2)\bar{\epsilon}_H]} \delta_m, \quad (6.70)
\]

\[
\frac{\delta N}{\bar{N}} \simeq \frac{3c_t^2 \bar{x}_1(3\bar{x}_1 - \sqrt{6} q_1\bar{\epsilon}_H)}{2\bar{K}^2[3c_t^2 \bar{x}_1^2 + (1 - c_t^2)\bar{\epsilon}_H]} \delta_m, \quad (6.71)
\]

where, in the denominators of Eqs. (6.68) and (6.70), we have neglected the terms without containing \( \bar{K}^2 \). The gravitational potentials \( \Psi \) and \( \Phi \), which are defined by Eq. (4.43), obey

\[
\ddot{\Phi} = -\dot{\Phi} \simeq \frac{3\bar{\Omega}_m \bar{\delta}_m}{2\bar{K}^2}, \quad (6.72)
\]

which is the equivalent to Eq. (5.28) with Eq. (5.22). Since \( \bar{A}_3 = 0 \) for \( q_1 = q_2 \), the condition \( 2\bar{H} \bar{M}_{pl}^2 / \bar{N} \gg |\bar{A}_3, \bar{\delta}_{\bar{N}}| \)
used for the derivation of Eq. (6.30) is automatically satisfied.

Substituting Eqs. (6.68), (6.71), and (6.72) into Eq. (6.53), we obtain

\[ \dot{\Psi}_g \simeq -\frac{3}{2\sqrt{2}} \Omega_m \hat{\chi} \left[ 1 + 2q^2 \frac{1 - c_1^2}{\left( \sqrt{6} \hat{x}_1 + 2q \right) \left( \sqrt{6} \hat{x}_1 - 2q \hat{\epsilon}_H \right)} \right], \]

(6.73)

where \( \hat{x}_1 = \sqrt{6}/[2(\mu + q)] \) and \( \hat{\epsilon}_H = 3\mu/[2(\mu + q)] \) for the fixed point (d) with \( w = 0 \). The effective gravitational coupling \( G_{\text{eff}} \) during the scaling matter epoch reads

\[ \frac{G_{\text{eff}}}{G} \simeq 1 + 2q^2 + \left( 1 - c_1^2 \right) \frac{\left( \sqrt{6} \hat{x}_1 + 2q \left( \sqrt{6} \hat{x}_1 - 2q \hat{\epsilon}_H \right) \right)}{2(3c^2_1 \hat{x}_1^2 + (1 - c_1^2) \hat{\epsilon}_H)}, \]

(6.74)

where the term \( 2q^2 \) arises in BD theories after the conformal transformation to the EF [67]. The last term on the rhs of Eq. (6.74) does not vanish for \( c_1^2 \) different from 1. Since we are now considering the case where \( c_1^2 \) is constant, the variation of \( c_1^2 \) does not appear in the expression of Eq. (6.74).

For the perturbations deep inside the Hubble radius, the rhs of Eq. (5.19) can be neglected relative to the lhs of it. Then, during the scaling matter era, the matter perturbation obeys

\[ \ddot{\delta}_m + \frac{\mu - 2q}{2(\mu + q)} \dot{\delta}_m - \frac{3}{2} \frac{G_{\text{eff}}}{G} \Omega_m \dot{\chi} \simeq 0. \]

(6.75)

Provided that \( \mu \gg q \), there is a growing-mode solution to Eq. (6.76),

\[ \hat{\delta}_m \propto \hat{a}^p, \quad p = \frac{\mu - 2q}{4(\mu + q)} \left[ \sqrt{1 + \frac{24(\mu + q)^2}{(\mu - 2q)^2} \hat{g} - 1} \right], \]

(6.76)

and \( \hat{g} \equiv (G_{\text{eff}}/G) \hat{\Omega}_m \). For \( q/\mu \to 0 \) and \( \hat{g} \to 1 \), the matter density contrast evolves as \( \delta_m \propto \hat{a} \). In this case, the quantity \( (\hat{a} \hat{H})^2 \hat{\delta}_m \), which appears in Eqs. (6.68)-(6.71), remains constant, so the perturbations \( \hat{\chi}, \hat{\xi}, \hat{V}_m^{(s)} \), and \( \delta N/\hat{N} \) do not vary in time.

For \( q \neq 0 \) and \( c_1^2 \neq 1 \), the quantity \( \hat{g} \) is different from 1. As long as \( q \ll 1 \) and \( \mu \gg q \), the deviation of \( \hat{g} \) from 1 is small, so the analytic formulas (6.68)-(6.72) are accurately valid in the scaling matter era. The perturbations \( \hat{\chi} \) and \( \hat{\xi} \) start to vary after the Universe enters the epoch of cosmic acceleration, in which regime the analytic solutions (6.68)-(6.71) are no longer valid.

In Fig. 1 we plot the evolution of perturbations \( \hat{\xi}, \hat{V}_m, \delta N/\hat{N}, \) and \( -\hat{\Psi} \) (= \( \hat{\Phi} \)) for the model parameters \( q = 0.1, c_1^2 = 0.5, \lambda_1 = 10, \lambda_2 = 0.5, \) and \( V_2/V_1 = 10^{-6} \) (i.e., \( \mu_1 = 9.6, \mu_2 = 0.1, V_2/V_1 = 10^{-6} \)). The background initial conditions are chosen to start from the scaling matter fixed point (d), i.e., \( \hat{x}_1 = \sqrt{6}/[2(\mu_1 + q)] \) and \( \hat{x}_2 = \sqrt{(3 + 2q^2 + 2q\mu_1)/[2(\mu_1 + q)^2]} \). We choose the initial value of \( \hat{V}_m \) close to the special solution (6.67) with \( \hat{\chi}' \simeq 0 \) and \( \hat{\xi}' \simeq 0 \) for the normalized wave number \( \hat{K} = 30 \).

All the perturbations shown in Fig. 1 stay nearly constant during the scaling matter era. We also confirmed that the analytic formulas (6.68)-(6.72) are in good agreement with numerically integrated solutions in the scaling regime. For sub-horizon perturbations (\( \hat{K} \gg 1 \)), the choice of different wave numbers \( k \) only modifies the amplitudes of perturbations \( \hat{\chi}, \hat{\xi}, \hat{V}_m^{(s)}, \) and \( \delta N/\hat{N} \). There is a simple relation \( \delta N/\hat{N} = -(3\hat{\Omega}_m/2)\hat{V}_m^{(s)} \) from Eqs. (6.70) and (6.71). The perturbations \( \delta N/\hat{N} \) and \( \hat{V}_m^{(s)} \) are suppressed relative to \( \hat{\chi} \) and \( \hat{\xi} \) because of the conditions \( \hat{x}_1 \ll 1 \) and \( q \ll 1 \). The Universe finally enters the epoch of cosmic acceleration driven by the fixed point (c) with \( \mu_2 = \lambda_2 - 4q \). As we see in Fig. 1 the perturbations start to vary after the onset of cosmic acceleration.

Numerically, we also compute the gravitational potential \( \Psi_g \) and the resulting effective gravitational coupling \( G_{\text{eff}} \) from the definition (6.52). In Fig. 2 the evolution of \( G_{\text{eff}}/G \) is plotted for four different values of \( q \) and \( c_1^2 \). We confirmed that the analytic estimation (6.74) of \( G_{\text{eff}} \) is in good agreement with the numerical result in the scaling
Figure 2. Evolution of the effective gravitational coupling \(\hat{G}_{\text{eff}}\) normalized by the gravitational constant \(G\) in the EF for the four cases: (i) \(q = 0\), \(c_t^2 = 0.5\), (ii) \(q = 0.05\), \(c_t^2 = 0.5\), (iii) \(q = 0.1\), \(c_t^2 = 0.5\), and (iv) \(q = 0.1\), \(c_t^2 = 0.99\), with the model parameters \(\lambda_1 = 10\), \(\lambda_2 = 0.5\), and \(V_2/V_1 = 10^{-6}\). The initial conditions are chosen in the same way as those explained in the caption of Fig. 1.

matter regime. The growth of \(\hat{G}_{\text{eff}}\) starts to occur after the scaling matter era.

If we compare the cases (ii) and (iii) with the case (i) in Fig. 2, we find that the existence of coupling \(q\) leads to the value of \(\hat{G}_{\text{eff}}/G\) smaller than that for \(q = 0\). Provided \(c_t^2\) is not close to 1, the analytic formula (6.74) implies that \(\hat{G}_{\text{eff}}/G\) approaches 1 in the limit where \(q \gg \hat{x}_1\) and \(\hat{x}_1 \ll 1\). Hence, for a given value of \(c_t^2\) different from 1, \(\hat{G}_{\text{eff}}/G\) tends to decrease with increasing \(q\). For larger \(q\), the variation of \(\hat{G}_{\text{eff}}/G\) occurs at a later cosmological epoch.

In the limit that \(c_t^2 \to 1\), Eq. (6.74) reduces to the value \(\hat{G}_{\text{eff}}/G \to 1 + 2q^2\). The case (iv) in Fig. 2 is close to such an example, in which case the variation of \(\hat{G}_{\text{eff}}/G\) is small even after the onset of cosmic acceleration. This property can be clearly distinguished from the model with \(q = 0\) and \(c_t^2 \neq 1\).

If we choose initial conditions where \(\hat{V}_m\) is not close to the special solution \(\hat{V}_m^{(s)}\), the homogenous solution \(\hat{V}_m^{(h)}\) gives rise to the oscillation of \(\hat{V}_m\). This oscillation continues by the time when the amplitude of \(\hat{V}_m^{(h)}\) decreases sufficiently relative to that of \(\hat{V}_m^{(s)}\). This situation is analogous to what was found for the case \(q = 0\) [55].

2. Jordan frame

The evolution of perturbations in the EF can translate to that in the JF by using the correspondence

\[
\chi = \frac{c_t^2}{1 + \omega_H} \hat{\chi}, \quad \zeta = \hat{\zeta}, \quad V_m = \frac{\hat{V}_m}{1 + \omega_H},
\]

\[
\delta N = \frac{\delta N}{N}, \quad \delta = \hat{\delta}, \quad \mathcal{K} = \frac{K + \omega_H}{c_t}, \quad (6.77)
\]

where \(\mathcal{K} \equiv k/(aH)\) and \(\omega_H = -\sqrt{6q}\hat{x}_1/(1 + \sqrt{6q}\hat{x}_1)\). The gauge-invariant matter perturbation \(\delta_m\) is related to \(\hat{\delta}_m\), as

\[
\delta_m = \hat{\delta}_m + 3\omega_H \frac{V_m}{1 + \omega_H} \hat{V}_m. \quad (6.78)
\]

At the background level, we also have the relations (6.38) and (6.40). Using the analytic solutions (6.68)-(6.71) in the EF during the scaling matter era, the perturbations \(\chi, \zeta, V_m, \) and \(\delta N\) in the JF can be expressed in terms of \(\delta_m, \mathcal{K}, \Omega_m, c_t^2, x_1, \) and \(\epsilon_H\).

The gravitational potentials \(\Psi, \Phi, \) and \(\hat{\chi}\) by using Eqs. (6.52) and (6.53). Substituting the solutions (6.68) and (6.72) into Eqs. (6.57) and (6.58) in the scaling matter regime and employing the approximation \(\delta_m \approx \delta_m\) for the perturbations deep inside the Hubble radius, it follows that

\[
\Psi \simeq -\frac{3(1 - c_t^2)\epsilon_H(1 + 2\omega_H) + 3x_1^2 + \omega_H^2}{2\mathcal{K}^2[(1 - c_t^2)(\epsilon_H + \omega_H)(1 + \omega_H) + 3x_1^2]} \Omega_m \delta_m, \quad (6.79)
\]

\[
\Phi \simeq \frac{3c_t^2(1 + 2\omega_H) + (1 + \omega_H)[1 + \epsilon_H + 2\omega_H - c_t^2(2 + \epsilon_H + 3\omega_H)] + 3c_t^2x_1^2}{2c_t^2K^2[(1 - c_t^2)(\epsilon_H + \omega_H)(1 + \omega_H) + 3x_1^2]} \Omega_m \delta_m, \quad (6.80)
\]

\[
\eta \simeq 1 + \frac{[(1 - c_t^2)(1 + \epsilon_H) + 2\omega_H][1 - c_t^2 + \omega_H(1 - 2c_t^2)]}{c_t^2(1 - c_t^2)\epsilon_H(1 + 2\omega_H) + 3x_1^2 + \omega_H^2}, \quad (6.81)
\]

\[
\hat{G}_{\text{eff}} \simeq \frac{(1 - c_t^2)\epsilon_H(1 + 2\omega_H) + 3x_1^2 + \omega_H^2}{(1 - c_t^2)(\epsilon_H + \omega_H)(1 + \omega_H) + 3c_t^2x_1^2 F_1} G, \quad (6.82)
\]

where \(\omega_H = -\sqrt{6q}\hat{x}_1\) and \(x_1 = \sqrt{6}/(2\lambda_1) \gg |\omega_H|\) for \(q \ll 1\). The perturbation \(\hat{\chi}\) induces the anisotropic stress
in the JF, so the anisotropy parameter $\eta$ is different from 1.

In the limit $c_2^2 \to 1$, the formulas \((6.79)\), \((6.82)\) give $\Psi \simeq -3(1 + 2q^2)\Omega_m\delta_m/(2K^2)$, $\Phi \simeq 3(1 - 2q^2)\Omega_m\delta_m/(2K^2)$, and

$$\eta \simeq \frac{1 - 2q^2}{1 + 2q^2}, \quad (6.83)$$

$$G_{\text{eff}} \simeq (1 + 2q^2)\frac{G}{F_1}. \quad (6.84)$$

In the limit $q \to 0$ (i.e., $\omega_H \to 0$), we use the approximations $\epsilon_H \simeq 3/2$ and $\Omega_m \simeq 1$ during the scaling regime and eliminate the term $x_1^2$ on account of Eq. \((6.6)\). This process leads to

$$\eta \simeq 1 + \frac{5(1 - c_2^2)(c_2^2 - c_1^2)}{3c_2^2(1 + c_2^2 - c_1^2)}, \quad (6.85)$$

$$G_{\text{eff}} \simeq \left(1 + \frac{1 - c_1^2}{c_1^2}\right)G, \quad (6.86)$$

which match those derived in Ref. \[55\] without referring to the EF. Since $c_2^2$ can be much greater than 1 for $c_2^2 \ll 1$, the parameter $\eta$ exhibits the large deviation from 1. In this case, $G_{\text{eff}}$ is slightly larger than $G$.

If $c_2^2 \neq 1$ and $q \neq 0$, the difference between the gravitational potentials $-\Psi$ and $\Phi$ depends on the magnitudes of the terms $1 - c_2^2$ and $\omega_H$. Provided $|1 - c_1^2| \ll \{\omega_H, x_1^2\}$, the parameter $\eta$ of Eq. \((6.81)\) is close to the value \((6.83)\) whereas, for $|1 - c_1^2| \gg \{\omega_H, x_1^2\}$, $\eta$ is close to the value \((6.85)\).

In Fig. 3 we illustrate the evolution of $-\Psi$, $\Phi$, and $-\Phi_{\text{eff}} = (\Phi - \Psi)/2$ in the JF versus $1 + z = 1/a$ for the same model parameters and initial conditions as those given in Fig. 1. The present epoch ($z = 0$) is identified by the condition $\Omega_m = 0.3$. The vertical dot-dashed line represents the onset at which the cosmic acceleration ($\ddot{a} > 0$) sets in ($z \simeq 0.63$). Numerically, we integrate the background and perturbation equations in the EF and then compute $\Psi$ and $\Phi$ by using the transformation laws \((6.17)\) and \((6.18)\).

In Fig. 4 we plot the evolution of $f(z)\sigma_8(z)$ versus the redshift $z$ in the JF for $\lambda_1 = 10$, $\lambda_2 = 0.5$, and $V_2/V_1 = 10^{-6}$. From the top to the bottom, each curve corresponds to (i) $q = 0$, $c_2^2 = 0.5$, (ii) $q = 0.05$, $c_2^2 = 0.5$, (iii) $q = 0.1$, $c_2^2 = 0.5$, and (iv) $q = 0.1$, $c_2^2 = 0.99$, respectively. The initial conditions are chosen in the similar way to those explained in the caption of Fig. 1. For a given value of $c_2^2$ ($\neq 1$), the growth rate of matter perturbations tends to be smaller with increasing $q$ due to the decrease of $G_{\text{eff}}$. The black points with error bars correspond to the data from the recent observations of $f(z)\sigma_8(z)$ by 2dFGRS \[80\], 6dFGRS \[81\], WiggleZ \[82\], SDSSLRG \[83\], BOSSCMASS \[84\], and VIPERS \[58\] surveys.
dynamics in galaxy clusterings. Note that the growth rate \( f(z) \sigma_8(z) \) can be also measured by using only the peculiar motions of galaxies in the low redshift range. The initial condition of \( \delta_m \) is chosen such that its amplitude today is equivalent to \( \sigma_8(0) = 0.82 \). During the scaling matter era, we have numerically checked that \( G_{\text{eff}} \) is well described by Eq. (6.82). When \( q = 0 \) and \( c_f^2 = 0.5 \) we have \( G_{\text{eff}} \approx 1.03G \) from Eq. (6.85), whereas, for \( q = 0.1 \) and \( c_f^2 \approx 1 \), \( G_{\text{eff}} \approx 1.02G/F_1 \) from Eq. (6.84).

For larger \( q \) with a given value of \( c_f^2 (\neq 1) \), the onset of growth of \( G_{\text{eff}} \) occurs at later cosmological epochs. In fact, Fig. 3 shows that the values of \( f(z) \sigma_8(z) \) for \( c_f^2 = 0.5 \) tend to be smaller with increasing \( q \) in the low-redshift regime. For \( c_f^2 \) close to 1, the variation of \( G_{\text{eff}} \) is small by the present epoch (see Fig. 2). In this case, \( G_{\text{eff}} \) is approximately given by Eq. (6.83) even around today.

In Fig. 4 we also plot the recent data of \( f(z) \sigma_8(z) \) constrained by the redshift-space distortion (RSD) measurements. Using the bound on \( \sigma_8(0) \) from the Planck data [57], the RSD data tend to favor the growth rate of \( \delta_m \) lower than that predicted by GR [57]. In Fig. 4 such a property can also be observed in our model where \( G_{\text{eff}} \) is slightly larger than \( G \). The recent 6dF galaxy surveys using only the peculiar motions of galaxies provided the constraint \( f(0) \sigma_8(0) = 0.415 \pm 0.065 \), which is consistent with the four cases shown in Fig. 4. It remains to see how future RSD and peculiar velocity measurements will pin down the error bars of \( f(z) \sigma_8(z) \).

We have thus clarified the evolution of observables associated with the measurements of CMB, redshift-space distortions, and weak lensing by transforming back from the EF to the JF. Since the EFTCAMB code [8] for modified gravity theories is written in the JF, our results in this section can be used to place observational constraints on the model [57,22] with the functions [61].

Since the staring point of coupled scalar-field models (including coupled quintessence [57,41], chameleons [43], and disformally coupled models [52]) is usually assumed to be the EF, our analysis in the EF is also useful to constrain coupled dark energy models with \( c_f^2 \) different from 1. We leave observational constraints on such models for a future work.

VII. CONCLUSIONS

In the presence of matter, we have studied cosmological disformal transformations in a generalized class of Horndeski theories (GLPV theories). In these theories there is one propagating scalar degree of freedom \( \phi \) coupled to the metric \( g_{\mu\nu} \) in the JF on the flat FLRW background. Even if matter is minimally coupled to gravity in the JF, the matter sector feels the modification of gravity through the change of gravitational potentials mediated by the field \( \phi \).

The structure of the Lagrangian in GLPV theories, which is given by Eq. (2.1) in the unitary gauge, is preserved under the disformal transformation (1.1), while the matter Lagrangian contains a coupling with the field \( \phi \) and its derivatives in the transformed frame. Thus, the matter-scalar interaction becomes explicit after the disformal transformation. In Sec. III we clarified how the energy-momentum tensor of matter and associated background/perurbed quantities are mapped under the disformal transformation.

In Sec. IV we have derived the background and linear perturbation equations of motion in both the JF and the transformed frame. In the transformed frame, the coupling \( Q \) in Eq. (1.36) arises for the matter continuity Eq. (1.35) at the background level. The matter perturbation Eq. (1.13) in the JF is also transformed to the more involved Eq. (1.14) due to the matter-scalar interaction, while the structure of other perturbation equations is not subject to change.

In Sec. V we discussed the transformation from the JF to the EF in which the second-order action of tensor perturbations is of the same form as that in GR. Under the choice (5.5) of the factors \( \Omega \) and \( \Gamma \), the Bardeen potentials \( \Psi \) and \( \Phi \) in the EF obey the “de-mixed” relation (5.10). If the action in the EF belongs to a class of Horndeski theories (\( \tilde{\alpha}_H = 0 \)), there is no anisotropic stress between \( \Psi \) and \( \Phi \).

The non-relativistic matter density contrast \( \delta_m \) obeys the differential Eq. (6.19) in the EF, where \( \tilde{\Psi}_g \) is the effective gravitational potential given by Eq. (6.20). In the EF, it becomes transparent that the variations of \( \Omega \) and \( \Gamma \), as well as the deviation of \( \tilde{C}_f^2 \) from 1 lead to the modification of gravitational interactions with matter perturbations. The gravitational potential \( \Psi \) in the JF is simply related to \( \tilde{\Psi}_g \), as \( \Psi = \tilde{C}_f^2 \tilde{\Psi}_g \).

In Sec. VI we proposed a concrete model of dark energy in which the coupling between matter and the scalar degree of freedom \( \phi \) is manifest after the disformal transformation to the EF. For the field potential (6.8) with \( \lambda_1 \gtrsim 10 \) and \( \lambda_2 \lesssim 1 \), there exist scaling solutions corresponding to radiation and matter eras followed by an attractor with the cosmic acceleration. At the background level, the disformal transformation to the EF gives rise to the term \( q_2 \) associated with the function \( B_4 \), but we showed that the stability of fixed points is independent of \( q_2 \). This reflects the fact that the background equations in the JF do not contain the function \( B_4 \).

We also studied the evolution of linear cosmological perturbations from the matter era to today for the case \( q_1 = q_2 \). In the EF we derived the second-order equation of the velocity potential \( \tilde{V}_m \) and identified the special solution \( \tilde{V}_m^{(s)} \) on scales deep inside the Hubble radius. For the initial conditions satisfying \( |\tilde{V}_m^{(s)}| \gg |\tilde{\tilde{V}}_m^{(H)}| \), we obtained analytic solutions of perturbations where \( \tilde{\zeta} \) and \( \tilde{\chi} \) stay nearly constant. On using these solutions, we derived the effective gravitational potential \( \tilde{\Psi}_g \) of the form (6.73) during the scaling matter era. The coupling \( q \) and the deviation of \( \tilde{C}_f^2 \) from 1 lead to the gravitational coupling \( \tilde{G}_{\text{eff}} \) modified from that in GR.

Once the evolution of perturbations is known in the
EF, it is straightforward to transform it back to that in the JF by using the correspondence $\Phi = -\Psi$ in the EF for the Lagrangian (5.22). While $\hat{\Phi}$ and $\hat{\Psi}$ are related, the field $\hat{\chi}$ generates the anisotropic stress in the JF such that $\eta = -\hat{\Phi}/\hat{\Psi} \neq 1$. For subhorizon perturbations the effective gravitational coupling $G_{\text{eff}}$ in the JF is related to $G_{\text{eff}}$ in the EF as $G_{\text{eff}} \simeq G_{\text{eff}}/F_1$, so the growth rate of matter perturbations in the JF is known accordingly.

We have analytically estimated $\eta$ and $G_{\text{eff}}$ during the scaling matter era and confirmed that they are in good agreement with numerical results before the onset of cosmic acceleration. We also numerically computed the evolution of $f(z)\sigma_8(z)$ by transforming back from the EF to the JF. As we see in Fig. 4 it is possible to distinguish between the models with different values of $q$ and $c_{\text{I}}$ observationally.

We have thus shown that the disformal transformation is useful for understanding gravitational interactions with matter mediated by the scalar field. After transforming to the EF, the background and perturbation dynamics in the JF are readily known by using the correspondence of physical quantities between the two frames. We can apply our results to observational constraints on dark energy models in the framework of GLPV theories. Moreover, our analysis in the EF is useful for constraining coupled dark energy models in which the starting point is the EF Lagrangian with matter-scalar couplings.

While we focused on the cosmological set up, it is of interest to extend the disformal transformation to general space-time including the spherically symmetric background. This should help us to understand the nature of matter-scalar couplings in local regions of the Universe. In particular, the derivation of the effective gravitational coupling around a compact body (like the Sun) will be important to place constraints on theories beyond Horndeski from local gravity experiments.

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Appendix A: Correspondence between the two frames

We show relations for the quantities connected through the disformal transformation. The background quantities are transformed as

$$\hat{N} = \bar{\alpha} \bar{N}, \quad \hat{H} = \frac{1}{\bar{\alpha}} \left( H + \frac{\omega}{N} \right),$$

$$\hat{\mathcal{F}} = \frac{F}{\Omega^3}, \quad \hat{\bar{L}} = \frac{1}{\Omega^3} \bar{L},$$

$$\hat{\bar{N}} \hat{L}_{\bar{N}} = \frac{\beta}{\Omega^3} \left[ \bar{L} + \bar{N} L_{\bar{N}} - 3 HF - \frac{1}{\alpha^2} \left( \bar{L} - 3 HF - \frac{3\omega F}{N} \right) \right]. \tag{A1}$$

For the quantities associated with perturbations, we have

$$\hat{\mathcal{W}} = \frac{\beta}{\Omega^3} \mathcal{W}, \quad \hat{\bar{L}} \hat{\mathcal{S}} = \frac{\bar{\alpha}}{\Omega^3} \bar{L} \mathcal{S}, \quad \hat{\mathcal{Y}} = \frac{1}{\alpha^3 \bar{\alpha}} \mathcal{Y},$$

$$\hat{\mathcal{E}} = \frac{1}{\Omega^3} \mathcal{E}, \quad \hat{\bar{N}} \hat{D} + \hat{\bar{\mathcal{E}}} = \frac{\beta}{\Omega} \left( \bar{N} D + \mathcal{E} \right),$$

$$2 \hat{\bar{L}} \hat{N} + \hat{\bar{N}} \hat{L}_{\bar{N}} = \frac{12 \bar{L}_{\bar{S}} H^2}{\bar{N}} + \frac{12 \bar{L}_{\bar{S}} H^2}{\bar{N}} + \frac{12 \bar{L}_{\bar{S}} H^2}{\bar{N}} + \frac{\nu \beta \rho}{\Omega^3}. \tag{A2}$$

[1] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006) [hep-th/0603057].
[2] A. Silvestri and M. Trodden, Rept. Prog. Phys. 72, 066901 (2009) [arXiv:0904.0024 [astro-ph.CO]]; S. Tsujikawa, Lect. Notes Phys. 800, 99 (2010) [arXiv:1101.0191 [gr-qc]]; T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Rept. Phys. 513, 1 (2012) [arXiv:1106.2476 [astro-ph.CO]].
A. Joyce, B. Jain, J. Khoury and M. Trodden, Rept. Phys. 568, 1 (2015) [arXiv:1407.0050 [astro-ph.CO]].
[3] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010) [arXiv:0805.1726 [gr-qc]].
[4] A. De Felice and S. Tsujikawa, Living Rev. Rel. 13, 3 (2010) [arXiv:1002.4928 [gr-qc]].
[5] A. I. Vainshtein, Phys. Lett. B 39, 393 (1972).
[6] J. Khoury and A. Weltman, Phys. Rev. Lett. 93, 171104 (2004) [astro-ph/0309300]; J. Khoury and A. Weltman, Phys. Rev. D 69, 044026 (2004) [astro-ph/0309411].
[7] G. W. Horndeski, Int. J. Theor. Phys. 10, 363-384 (1974).
[8] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, Phys. Rev. D 84, 064039 (2011) [arXiv:1103.3260 [hep-th]]; T. Kobayashi, M. Yamaguchi and J. i. Yokoyama, Prog. Theor. Phys. 126, 511 (2011) [arXiv:1105.5723 [hep-th]]; C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saf-
E. Macaulay, I. K. Wehus and H. K. Eriksen, Phys. Rev. Lett. 111, 161301 (2013) [arXiv:1303.6583 [astro-ph.CO]].

[58] S. de la Torre et al., Astron. Astrophys. 557, A54 (2013) [arXiv:1303.2622 [astro-ph.CO]].

[59] S. Weinberg, Phys. Rev. D 77, 123541 (2008) [arXiv:0804.4291 [hep-th]].

[60] M. Park, K. M. Zurek and S. Watson, Phys. Rev. D 81, 124008 (2010) [arXiv:1003.1722 [hep-th]].

[61] J. K. Bloomfield and E. E. Flanagan, JCAP 1210, 039 (2012) [arXiv:1112.0303 [gr-qc]]; J. K. Bloomfield, E. Flanagan, M. Park and S. Watson, JCAP 1308, 010 (2013) [arXiv:1211.7054 [astro-ph.CO]]; J. Bloomfield, JCAP 1312, 044 (2013) [arXiv:1304.6712 [astro-ph.CO]].

[62] R. A. Battye and J. A. Pearson, JCAP 1207, 019 (2012) [arXiv:1203.0398 [hep-th]].

[63] E. M. Mueller, R. Bean and S. Watson, Phys. Rev. D 87, 083504 (2013) [arXiv:1209.2706 [astro-ph.CO]].

[64] G. Gubitosi, F. Piazza and F. Vernizzi, JCAP 1302, 032 (2013) [arXiv:1210.0201 [hep-th]].

[65] F. Piazza and F. Vernizzi, Class. Quant. Grav. 30, 214007 (2013) [arXiv:1307.4350]; F. Piazza, H. Steigerwald and C. Marinoni, JCAP 1405, 043 (2014) [arXiv:1312.6111 [astro-ph.CO]].

[66] N. Frusciante, M. Raveri and A. Silvestri, JCAP 1402, 026 (2014) [arXiv:1310.6026 [astro-ph.CO]].

[67] L. Amendola, Phys. Rev. D 62, 043511 (2000) [astro-ph/9904323]; L. Amendola, Phys. Rev. D 60, 043501 (1999) [astro-ph/9904120].

[68] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, JCAP 1308, 025 (2013) [arXiv:1304.4840 [hep-th]].

[69] D. Giamakis and W. Hu, Phys. Rev. D 72, 063502 (2005) [astro-ph/0501423]; F. Arroja and M. Sasaki, Phys. Rev. D 81, 107301 (2010) [arXiv:1002.1376 [astro-ph.CO]].

[70] R. Kase and S. Tsujikawa, Int. J. Mod. Phys. D 23, no. 13, 1443008 (2015) [arXiv:1409.1984 [hep-th]].

[71] J. M. Bardeen, Phys. Rev. D 22, 1882 (1980).

[72] C. Schimd, J. P. Uzan and A. Riazuelo, Phys. Rev. D 71, 083512 (2005) [astro-ph/0412120]; L. Amendola, M. Kunz and D. Sapone, JCAP 0804, 013 (2008) [arXiv:0704.2421 [astro-ph]].

[73] B. Boisseau, G. Esposito-Farese, D. Polarski and A. A. Starobinsky, Phys. Rev. Lett. 85, 2366 (2000) [gr-qc/0001066]; S. Tsujikawa, Phys. Rev. D 76, 023514 (2007) [arXiv:0705.1032 [astro-ph]]; S. Nesseris, Phys. Rev. D 79, 044015 (2009) [arXiv:0811.4292 [astro-ph]]; A. De Felice, S. Mukohyama and S. Tsujikawa, Phys. Rev. D 82, 023524 (2010) [arXiv:1006.0281 [astro-ph.CO]].

[74] R. Kimura and K. Yamamoto, JCAP 1104, 025 (2011) [arXiv:1011.2006 [astro-ph.CO]].