EXACT CONSTRUCTION OF THE ELECTROMAGNETIC CURRENT OPERATOR FOR RELATIVISTIC COMPOSITE SYSTEMS

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Abstract:

The electromagnetic current operator of a composite system must be a relativistic vector operator satisfying current conservation, cluster separability and the condition that interactions between the constituents do not renormalize the total electric charge. Assuming that these interactions are described in the framework of relativistic quantum mechanics of systems with a fixed number of particles we explicitly construct the current operators satisfying the above properties in cases of two and three particles and prove that solutions exist for any number of particles. The consideration is essentially based on the method of packing operators in the point form of relativistic dynamics developed by Sokolov. Using the method developed by Sokolov and Shatny we also construct the current operators in the instant and front forms and prove that the corresponding results are physically equivalent. The paper is self-contained what makes it possible for the reader to learn both, relativistic quantum mechanics of systems with a fixed number of particles and the problem of constructing the electromagnetic current operator for such systems.
Chapter 1

Introduction

1.1 The statement of the problem

The present and future experiments on powerful electron accelerators will yield an important information about nuclear forces at small distances and the quark structure of matter. However to extract this information from the experimental data it is necessary to know the structure of the electromagnetic current operator (ECO) for the system under consideration. The problem of constructing the ECO is essentially model-dependent since the ECO depends on the interactions between the constituents and on the way of introducing these interactions into the ECO. Nevertheless any model for the ECO must necessarily be such that the ECO is a relativistic vector operator satisfying current conservation, cluster separability and the condition that interactions between the constituents comprising our system do not renormalize the total electric charge. Let us formulate these conditions mathematically.

The wave function of any relativistic system must transform according to a unitary representation of the Poincare group in some Hilbert space $H$. Let $\hat{U}(a) = \exp(i\hat{P}_\mu a^\mu)$ be the representation operator corresponding to the displacement of the origin in spacetime translation of Minkowski space by the 4-vector $a$. Here $\hat{\mathbf{P}} = (\hat{P}^0, \hat{\mathbf{P}})$ is the operator of the 4-momentum, $\hat{P}^0 = \hat{E}$ is the Hamiltonian, and $\hat{\mathbf{P}}$ is the operator of ordinary momentum. Let also $\hat{U}(l)$ be the representation operator corresponding to $l \in SL(2, C)$ and $L(l)$ be the Lorentz
transformation corresponding to \( l \). Then the ECO \( \hat{J}^\mu(x) \) must be the selfadjoint relativistic vector operator such that

\[
\hat{U}(a)^{-1} \hat{J}^\mu(x) \hat{U}(a) = \hat{J}^\mu(x - a),
\]

\[
\hat{U}(l)^{-1} \hat{J}^\mu(x) \hat{U}(l) = L(l)^\mu_\nu \hat{J}^\nu(L(l)^{-1}x)
\]

(1.1)

where a sum over repeated indices \( \mu, \nu = 0, 1, 2, 3 \) is assumed. Since at least some of the operators \( \hat{U}(a) \) and \( \hat{U}(l) \) depend on interactions in the system under consideration, the immediate consequence of Eq. (1.1) is that \( \hat{J}^\mu(x) \) also depends on these interactions and thus \( \hat{J}^\mu(x) \) cannot be written only as a sum of the constituent ECO’s. This fact was first pointed out by Siegert [1].

Another important condition is the continuity equation or current conservation

\[
\frac{\partial \hat{J}^\mu(x)}{\partial x^\mu} = 0
\]

(1.2)

The cluster separability condition for \( \hat{J}^\mu(x) \) can be formulated along with this condition for the representation operators. Namely, it is assumed that the representation space is the same in all cases (not depending on whether interactions are present or not) and if \( \alpha_1, ... \alpha_n \) are arbitrary subsystems comprising the system under consideration and all interactions between these subsystems are turned off then the representation \( \hat{U} \) must become the tensor product of the representations \( \hat{U}_{\alpha_i} \) describing the subsystems and \( \hat{J}^\mu(x) \) must become a sum of the ECO’s \( \hat{J}^\mu_{\alpha_i}(x) \) for the subsystems. In other words the cluster separability condition for the current operator is formulated in full analogy with this condition for the generators of the representation \( \hat{U} \) (for more details see refs. [2, 3, 4, 5, 6]).

Let

\[
\hat{Q} = \int \hat{J}^\mu(x) d\sigma_\mu(x)
\]

(1.3)

be the system electric charge operator where \( d\sigma_\mu(x) = \lambda_\mu \delta(\lambda x - \tau) d^4x \) is the volume element of the space-like hypersurface defined by the time-like vector \( \lambda \) \((\lambda^2 = 1)\) and the evolution parameter \( \tau \). Then the important physical condition is that the interactions do not renormalize the electric charge, i.e. \( \hat{Q} \) does not depend on the choice of \( \lambda \) and \( \tau \).
and has only one eigenvalue equal to the sum of electric charges of constituents. It is well known that Eq. (1.2) ensures that $\hat{Q}$ does not depend on $\tau$ and $\lambda$ but this condition does not ensure that $\hat{Q}$ has the same value as for noninteracting particles.

In addition, if we consider the problem of constructing the ECO for a system of strongly interacting particles, then we must require that

$$\hat{U}_P(\hat{J}^0(x^0, x), \hat{\mathbf{J}}(x^0, \mathbf{x})){\hat{U}}_P^{-1} = (\hat{J}^0(x^0, -\mathbf{x}), -\hat{\mathbf{J}}(x^0, -\mathbf{x})),$$

$$\hat{U}_R\hat{J}^\mu(x){\hat{U}}_R^{-1} = \hat{J}^\mu(-x)$$

(1.4)

where $\hat{U}_P$ is the unitary operator corresponding to the space reflection $P$ and $\hat{U}_R$ is the antiunitary operator corresponding to the space-time reflection $R = PT$. The operator $\hat{U}_R$ must be antiunitary according to Wigner [7]; an alternative definition of $\hat{U}_R$ proposed by Schwinger [8] involves transposed operators, but we shall not discuss this question.

### 1.2 Brief review of the literature on constructing the ECO and related topics

In the framework of local quantum field theory there exist the well known methods of introducing electromagnetic interactions into the system of strongly interacting quantized fields (for example, the minimal substitution method). Each electromagnetic process is described by many Feynman diagrams including those obtained from some diagrams of the strong interaction theory by coupling the virtual or real photons to all possible places of the latter diagrams. To explicitly construct the ECO in such an approach it is necessary to explicitly construct first the underlying theory of strong interactions. Therefore a realistic construction of the ECO can be carried out only at some additional assumptions.

A vast amount of literature is devoted to the problem of constructing the ECO for few nucleon systems assuming that the underlying theory of strong interactions is the meson-nucleon theory. It is clear that the most detailed study can be carried out for two-nucleon
systems. As it has been shown by several authors (see for example, refs. [9, 10, 11, 12, 13, 14], the corresponding ECO satisfying the properties specified in Sec. 1.1 can be constructed assuming that the interaction between the nucleons is described by the Bethe-Salpeter (BS) or other covariant equation. This is achieved by choosing the longitudinal part of the ECO in the form satisfying the two-particle Ward-Takahashi identity. It has been also shown that the knowledge of the kernel of the two-body equation is not sufficient for a unique determination of the ECO since the transversal part of the ECO remains unconstrained by current conservation.

The ECO considered in the above references cannot be written only as a sum of one-nucleon ECO’s and necessarily contains the contribution of interaction currents usually associated with meson exchange currents (MEC). These currents may give contributions to different observables even in the nonrelativistic approximation. The first detailed study of MEC was carried out apparently in ref. [15]. The present status of nonrelativistic MEC is described in refs. [16, 17] and in references cited therein. The role of MEC in three and four-nucleon systems is discussed in refs. [18, 19] and in references cited therein. The problem of reformulating the theory of MEC in the framework of constituent quark model is discussed in ref. [20] and in references cited therein.

In the nonrelativistic case the interaction between nucleons is described by a nucleon-nucleon potential. By analogy with QED and other realistic theories it is reasonable to assume that the potential description of nucleon-nucleon interactions exists not only in the nonrelativistic approximation but at least to order \((v/c)^2\). A detailed investigation of MEC in first order in \((v/c)^2\) was carried out in ref. [21]. The present status of this theory has been described in refs. [22] and in references cited therein. A fully covariant approach to MEC has been considered in refs. [23, 24]. A detailed investigation of all possible contributions to the deuteron photodisintegration to order \((v/c)^3\) has been carried out in refs. [25, 26, 27] and in references cited therein. The latest development in the description of the deuteron electrodisintegration is described in refs. [28, 29, 30] and in references cited therein. The photo
and electrodisintegration of three-nucleon systems have been considered so far only in the plane wave impulse approximation \[31, 32\].

The approach in which the problem of constructing the ECO is tackled starting from the underlying theory of strong interactions is of course most fundamental. However this approach encounters considerable practical difficulties. For example, using the two-body BS equation we have either to assume some dependence of the kernel of this equation on the relative time (or relative energy) or to use some three-dimensional reduction of this equation. Meanwhile it is not clear how to control the validity of approximations used to solve the BS equation and even the crucial property of relativistic invariance may be lost at some approximations. Let us note that in local quantum field theory any description of an interacting system can be consistent only in terms of infinite number of degrees of freedom since the number of particles is not the conserving physical quantity. In particular the Hilbert space \(H\) necessarily contains subspaces with all possible numbers of particles, and the representation operators of the Poincare group necessarily have transitions between these subspaces. Therefore any truncation of the Hilbert space leaving only a finite number of particles necessarily breaks relativistic invariance.

For systems of three and more particles the description in terms of the BS equation becomes much more complex. In particular the cluster separability condition is not trivial in this case. One might think that in nuclear physics different observables at intermediate energies can be reliably calculated using the \(v/c\) expansion, though, as noted by some authors (see, for example, refs. \([33, 34]\)), the convergence of the \(v/c\) expansion may be poor. However, as shown in refs. \([35, 36]\), the unified description of the meson and baryon spectra is possible only in fully relativistic constituent quark models. Let us also note that for systems of constituent quarks the underlying theory of strong interactions is ”soft” QCD, i.e. the theory which is not elaborated so far. For this reason even in calculations of the electromagnetic form factor of the pion (considered as a system of two constituent quarks) the interaction currents are usually not taken into account (see, for example,
refs. [37, 38, 39, 40]) and the electromagnetic properties of baryons as systems of three constituent quarks are also described assuming that the baryon ECO is the sum of the quark ECO's [27].

It is often reasonable to assume that interactions between constituents can be reliably described in the framework of relativistic quantum mechanics (RQM) of systems with a fixed number of particles. This theory is expected to be a reasonable approach in different applications. The well known examples are the atom consisting of a fixed number of electrons, the nucleus consisting of a fixed number of nucleons, and the meson and baryon consisting of a fixed number of constituent quarks. The first results in RQM was obtained in the work by Dirac [41] and the important contribution was made by Coester, Sokolov and other scientists. The reviews of RQM can be found, for example, in refs. [42, 43]. The practical advantage of RQM is that in models based on this theory such fundamental properties as relativistic invariance and cluster separability are satisfied automatically.

The statement of the problem considered in the present work can now be formulated as follows. We have to investigated whether the ECO satisfying the properties described above can be explicitly constructed in the framework of RQM. In the remainder of this section we discuss some results of RQM and the literature on constructing the ECO in this theory.

The problem of constructing the operator $\hat{J}^\mu(x)$ in the framework of RQM was investigated by several authors. The general properties of the ECO and the constraints imposed on the matrix element of $\hat{J}^\mu(x)$ by the above conditions has been studied in detail in Ref. [44]. As noted by different authors [45, 46, 47, 44], the above conditions are not sufficient for a unique determination of $\hat{J}^\mu(x)$, (in agreement with the results obtained in the framework of local quantum field theory) but these conditions are necessary, since a consideration of electromagnetic processes using the current operator not satisfying these conditions is, generally speaking, inconsistent. The above conditions are also necessary in order to be sure that the low-energy theorem for the Compton scattering and the Drell-Hearn-Gerasimov sum rules are
satisfied\cite{48, 49, 50}.

In the literature there exists a rather complete investigation of the problem under consideration in first order in $1/c^2$ \cite{51, 52, 53, 54, 55}. For example, it has been known for a long time that the nontrivial effect in this order is the appearance of the nonadditive Foldy-Wouthuysen corrections. It has been also shown by Osborn and Foldy\cite{56} and Coester and Ostebee\cite{46} that (in agreement with the results by Gross\cite{57} obtained in the framework of quantum field theory) interaction terms indeed may be present already in the nonrelativistic current-density operator as was first noted in ref. \cite{1}.

In the nonrelativistic approximation the charge-density operator $\hat{J}^0(x)$ is written in zero order in $1/c$ and the current-density operator $\hat{J}(x)$ in first order. In order $1/c^2$ the first relativistic corrections to $\hat{J}^0(x)$ are taken into account. As noted above, this approximation is insufficient to describe the existing data in nuclear and particle physics, and in the literature there have been proposed different approaches for calculating relativistic effects in electromagnetic processes to orders more high than $1/c^2$. For example, in refs.\cite{58, 59, 60, 61} the deuteron electromagnetic form factors at large momentum transfer were calculated using the front form of dynamics and the assumption that some matrix elements can be calculated with a good accuracy using the additivity of the $+$ components of the current operator. A similar approach was used for deriving the light-front sum rules \cite{32} and for calculations of the pion and nucleon electromagnetic form factors and the photoproduction of the delta isobar in the framework of constituent quark model \cite{63, 64, 65}.

The approach of refs.\cite{58, 59, 60, 61} was criticized in refs.\cite{66} since, as shown in these references, the expressions for the form factors used in refs.\cite{58, 59, 60, 61} contained the admixture of nonphysical terms depending on the direction in which the observer passes to the infinite-momentum-frame. It was proposed to use instead the expressions which do not contain this dependence but again no interaction terms were taken into account. As the result, the authors of refs.\cite{66} have come to the prescription proposed in ref.\cite{67}. To substantiate
neglecting the interaction terms in their expressions the authors of refs. [66] consider some class of Feynman diagrams and show that these terms can be present only in the contribution of multiparticle states while the contribution of the two-nucleon component in the deuteron does not contain the interaction terms. However, as noted above, it is difficult to justify the two-nucleon approximation in the diagrammatic approach.

In the just described calculations no expansion in powers of $1/c$ was used. On the other hand, in refs. [68, 69, 70, 71, 72, 73, 74] the elastic electromagnetic nucleon form factors, the electro and photo-production of nucleon resonances and the nucleon Compton scattering were calculated in constituent quark model taking into account some interaction terms, but the current operator was expanded only to order $1/c^3$. In ref. [75] the problem has been solved to order $1/c^4$ but the resulting expressions have turned out to be very cumbersome.

In the present paper we explicitly find a class of exact solutions for systems of two and three particles and prove that solutions exist for any number of particles. Therefore the results can be applied to the description of various electromagnetic processes involving the lightest nuclei, mesons (as systems of two constituent quarks) and baryons (as systems of three constituent quarks).

As pointed out by Dirac [41], any physical system can be described in different forms of relativistic dynamics. Let $\hat{M}^{\mu\nu}$ ($\hat{M}^{\mu\nu} = -\hat{M}^{\nu\mu}$) be the generators of the Lorentz group. We use $P$ and $M^{\mu\nu}$ to denote the 4-momentum operator and the generators of the Lorentz group in the case when all interactions are turned off. By definition, the description in the point form implies that the operators $\hat{U}(l)$ are the same as for noninteracting particles, i.e. $\hat{U}(l) = U(l)$ and $\hat{M}^{\mu\nu} = M^{\mu\nu}$, and thus interaction terms can be present only in the 4-momentum operators $\hat{P}$ (i.e. in the general case $\hat{P}^{\mu} \neq P^{\mu}$ for all $\mu$). The description in the instant form implies that the operators of ordinary momentum and angular momentum do not depend on interactions, i.e. $\hat{P} = P$, $\hat{M} = M$ ($\hat{M} = (\hat{M}^{23}, \hat{M}^{31}, \hat{M}^{12})$) and therefore interactions may be present only in $\hat{E}$ and the generators of the Lorentz boosts $\hat{N} = (\hat{M}^{01}, \hat{M}^{02}, \hat{M}^{03})$. In
the front form with the marked \( z \) axis we introduce the + and - components of the 4-vectors as
\[
p^+ = \left( p^0 + p^z \right)/\sqrt{2}, \quad p^- = \left( p^0 - p^z \right)/\sqrt{2}.
\]
Then we require that the operators \( \hat{P}^+, \hat{P}^j, \hat{M}^{12}, \hat{M}^{+-}, \hat{M}^{+j} (j = 1, 2) \) are the same as the corresponding free operators and therefore interaction terms may be present only in the operators \( \hat{M}^{-j} \) and \( \hat{P}^- \). We see that the front form contains 3 generators depending on interactions while the point and instant form contain 4 such generators. However, in the front form the operators \( \hat{U}_P \) and \( \hat{U}_R \) necessarily depend on interactions while in the point and instant forms we can choose representations with \( \hat{U}_P = U_P \) and \( \hat{U}_R = U_R \).

The most popular forms are the instant and front ones since it is clear how these forms are related to quantum field theory (see, for example, refs. [42, 43]). In the case of the point form the relation is less obvious [76]. However from the group theory point of view the point form looks as most natural. In the literature different authors argue in favor of one or other form. However, as proved by Sokolov and Shatny [77] all the three basic forms are unitarily equivalent (unfortunately this important result is not widely known). Therefore the choice of the form is only the matter of convenience but not the matter of principle.

Recently Karmanov [78] and Fuda [79] considered in detailed such a version of the front form where the direction of the motion of the infinite momentum frame is defined by the arbitrary unit vector \( \xi \). In ref. [79] it is proved that the ”\( \xi \)-picture” is unitarily equivalent to the ordinary front form at least for systems of two particles and there are all grounds to believe that this is also the case for any number of particles.

We shall see below that the natural solution for the ECO exists in the point form. The method of packing operators was first developed by Sokolov also in the point form [5]. Then this method was used in other forms and the full solution of the N-body problem in RQM was first given by Coester and Polyzou [4] and Mutze [5] in the instant form. Therefore, using the construction of ref. [77] it can be shown that the full solution of the N-body problem in RQM and the solution for the ECO exist in all forms.
In Ref. [3] the explicit expressions for the three-particle packing operators were demonstrated on the example when one of the particles has spin 1/2 and two others are spinless. The explicit expressions for particles with arbitrary spins in the point form were derived in ref. [6] (see also ref. [13]), but the derivation was rather complicated since the explicit expressions for the Lorentz group generators in the light cone variables were used. Let us note that in refs. [4, 5, 80] the expressions for the packing operators were also derived using the expressions for the Lorentz group generators.

1.3 The outline of this paper

In Secs. 2.2 and 2.3 following the ideas of ref. [3] we show that the expressions for the three-particle packing operators in the point form for particles with arbitrary spins can be obtained in a much more simple way than in previous publications if one works directly with the operators $U(l)$ and not with the generators of the Lorentz group. The main results of Chap. 2 are derived explicitly, and therefore this chapter can be used even for the first acquaintance with RQM. The general properties of the ECO are analyzed in Chap. 3, and in Chap. 4, using the results of Chaps. 2 and 3 we explicitly describe a class of solutions when the ECO satisfies the properties specified in Sec. 1.1. In Chap. 5, using the results of ref. [77] we construct the ECO in the instant form assuming that the ECO in the point form has been already constructed. In Chap. 6 a similar procedure is used for constructing the ECO in the front form.

As it has been already mentioned, the explicit expressions for the representation generators are not needed to solve the problem. Nevertheless it is useful to write down these expressions in some cases. In the general case the commutation relations for the representation generators can be realized in the form

\[
[\hat{P}^\mu, \hat{P}^\nu] = 0, \quad [\hat{M}^{\mu\nu}, \hat{P}^\rho] = -i(\eta^{\mu\rho} \hat{P}^\nu - \eta^{\nu\rho} \hat{P}^\mu),
\]

\[
[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = -i(\eta^{\mu\rho} \hat{M}^{\nu\sigma} + \eta^{\nu\sigma} \hat{M}^{\mu\rho} - \eta^{\mu\sigma} \hat{M}^{\nu\rho} - \eta^{\nu\rho} \hat{M}^{\mu\sigma}) \quad (1.5)
\]
where $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$, the metric tensor in Minkowski space has the nonzero components $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = 1$, and we use the system of units with $\hbar = c = 1$. 
Chapter 2

Systems with a fixed number of particles in the point form of relativistic dynamics

2.1 Systems of two particles

To describe a relativistic system of interacting particles it is necessary to choose first the explicit form of the unitary irreducible representation (UIR) of the Poincare group describing an elementary particle of mass $m > 0$ and spin $s$. There are many equivalent ways to construct an explicit realization of such a representation [81, 82]. For our purposes it is convenient to choose the realization in the following form.

Let $p$ be the particle 4-momentum, $g = p/m$ be the particle 4-velocity, $s$ be the spin operator, $D(s)$ be the space of the UIR of the group SU(2) with the spin $s$, $||...||$ be the norm in $D(s)$ and

$$d\rho(g) = \frac{d^3 g}{2(2\pi)^3 g^0} = \frac{d^3 p}{2(2\pi)^3 m^2 \omega(p)}$$  \hspace{1cm} (2.1)

where $\omega(p) = (m^2 + p^2)^{1/2}$ and $g^0 = (1 + g^2)^{1/2}$ (since $g^2 = 1$, only three components of $g$ are independent). Then the Hilbert space $H$ can be chosen as the space of functions $\varphi(g)$ with the range in $D(s)$ and such that

$$(\varphi, \varphi) = \int ||\varphi(g)||^2 d\rho(g) < \infty$$  \hspace{1cm} (2.2)
Let $\alpha(g) \in \text{SL}(2,\mathbb{C})$ be the matrix
\[
\alpha(g) = \frac{g^0 + 1 + \sigma g}{2(g^0 + 1)^{1/2}} \tag{2.3}
\]
where $\sigma$ are the Pauli matrices, $D[s; u]$ is the representation operator of the group $\text{SU}(2)$ corresponding to the element $u \in \text{SU}(2)$ for the representation with the generators $s$, and $g' = L(l)^{-1}g$. Then if an element of the Poincare group is defined by $a$ and $l$ then the corresponding representation operator acts as follows
\[
U(a, l)\varphi(g) = \exp(\imath mg'a)D[s; \alpha(g)^{-1}l\alpha(g')]\varphi(g') \tag{2.4}
\]
(it is easy to verify that $\alpha(g)^{-1}l\alpha(g') \in \text{SU}(2)$).

The generators of the UIR given by Eq. (2.4) have the form
\[
P = mg, \quad M = l(g) + s, \quad N = -\imath g^0 \frac{\partial}{\partial g} + \frac{s \times g}{1 + g^0} \tag{2.5}
\]
where $g$ is the operator of multiplication by $g$ and $l(g) = -\imath g \times (\partial/\partial g)$ is the orbital angular-momentum operator.

Let us now consider a system of two free particles with the masses $m_i > 0$ and the spin operators $s_i$ ($i = 1, 2$). As follows from cluster separability (see Sec. 1.1) the representation describing the free system $(1,2)$ must be the tensor product of the single-particle representations for particles 1 and 2. Therefore the representation space $H$ is the space of functions $\varphi(g_1, g_2)$ with the range in the tensor product $\mathcal{D}(s_1) \otimes \mathcal{D}(s_2)$ and such that
\[
\int ||\varphi(g_1, g_2)||^2 d\rho(g_1) d\rho(g_2) < \infty \tag{2.6}
\]
where $||...||$ is the norm in $\mathcal{D}(s_1) \otimes \mathcal{D}(s_2)$ and, as follows from Eq. (2.4), the operators of this representation act as
\[
U(a, l)\varphi(g_1, g_2) = \exp[\imath(m_1 g_1' + m_2 g_2')a] \cdot \prod_{i=1}^{2} D[s_i; \alpha(g_i)^{-1}l\alpha(g_i')]\varphi(g_1', g_2') \tag{2.7}
\]
As follows from Eq. (2.4) the total 4-momentum is now equal to $P = m_1 g_1 + m_2 g_2$. By analogy with the single-particle case we define
the system mass and 4-velocity as $M = |P|$ and $G = PM^{-1}$ respectively where the modulus of a 4-vector is taken in the the Lorentz metric, i.e. as $|P| = (P^2)^{1/2}$. It is easy to see that $M > 0$. Let us also define the 4-vectors $q_i$ ($i = 1, 2$) as

$$q_i = L[\alpha(G)]^{-1}m_i g_i$$  \hspace{1cm} (2.8)

It is easy to verify that $q_1 = -q_2$, i.e. $q_i$ are the momenta of the particles in their c.m. frame.

The above expressions define $G$ and $q \equiv q_1$ as the functions of $g_i$ ($i = 1, 2$). Conversely, the $g_i$ are uniquely defined by $G$ and $q$:

$$m_i g_i = q_i + G \omega_i(q) + \frac{(G q_i) G}{1 + G^0}$$ \hspace{1cm} (2.9)

where $\omega_i(q) = (m_i^2 + q^2)^{1/2}$, $G^0 = (1 + G^2)^{1/2}$. It is easy to see that $M$ as the function of $q$ is equal to $M = \omega_1(q) + \omega_2(q)$. A direct calculation using Eq. (2.9) shows that

$$d\rho(g_1) d\rho(g_2) = d\rho(G) d\rho(int),$$

$$d\rho(int) = \frac{M(q)^3 d^3 q}{2(2\pi)^3 \omega_1(q) \omega_2(q) m_1^2 m_2^2}$$ \hspace{1cm} (2.10)

Let us define the ”internal” Hilbert space $H_{int}$ as the space of functions $\chi = \chi(q)$ with the range in $D(s_1) \otimes D(s_2)$ and such that

$$\|\|\chi\|\|^2 = \int \|\|\chi(q)\|\|^2 d\rho(int) < \infty$$ \hspace{1cm} (2.11)

We also define the Hilbert space $\tilde{H}$ as the space of functions $\tilde{\varphi}(G)$ with the range in $H_{int}$ and such that

$$\int \|\|\tilde{\varphi}(G)\|\|_{int}^2 d\rho(G) < \infty$$ \hspace{1cm} (2.12)

As easily follows from Eqs. (2.11) and (2.12), the operator

$$U_{12} = U_{12}(G, q) = \prod_{i=1}^{2} D[s_i; \alpha(g_i)^{-1} \alpha(G) \alpha(q_i/m_i)]$$ \hspace{1cm} (2.13)

is the unitary operator from $\tilde{H}$ to $H$.

Our nearest goal is to determine the explicit form of the unitary representation $\tilde{U}$ in $\tilde{H}$ such that $U(a, l) = U_{12} \tilde{U}(a, l) U_{12}^{-1}$. The
form of the operators $\tilde{U}(a)$ corresponding to the spacetime translations is obvious from Eq. (2.7) since $U_{12}$ commutes with $U(a)$:

$$\tilde{U}(a)\varphi(G) = \exp[iM(Ga)]\varphi(G) \quad (2.14)$$

To determine the form of the operators $\tilde{U}(l)$ let us note that if $g'_i = L(l)^{-1}g_i$, then, as follows from the definition of $G$ as the function of $g_1$ and $g_2$: $G(g'_1, g'_2) \equiv G' = L(l)^{-1}G$. Taking into account that $\alpha(G)^{-1}l\alpha(G') \in \text{SU}(2)$ we get from Eq. (2.8) that the vector functions $q' \equiv q(g'_1, g'_2)$ and $q = q(g_1, g_2)$ are related to each other by a three-dimensional rotation:

$$q' = L[\alpha(G')^{-1}l^{-1}\alpha(G)]q \quad (2.15)$$

Let us define the unitary representation of the group $\text{SU}(2)$ in $H_{\text{int}}$ as follows. If $u \in \text{SU}(2)$ then

$$R(u)\chi(q) = \chi[L(u)^{-1}q] \quad (2.16)$$

It is well known that $l(q)$ are just the generators of the representation defined by Eq. (2.16) and therefore $R(u) = D[l(q); u]$. If $\varphi = \varphi(g_1, g_2) \in H$, $\tilde{\varphi} = \tilde{\varphi}(G) \in \tilde{H}$ and $\varphi = U_{12}\tilde{\varphi}$, then, as follows from Eqs. (2.7), (2.13), (2.15) and (2.16)

$$U(l)\varphi(g_1, g_2) = U_{12} \prod_{i=1}^{2} D[s_i; \alpha(q_i/m_i)^{-1}\alpha(G)^{-1} \cdot l\alpha(G')\alpha(q'_i/m_i)]D[l(q); \alpha(G)^{-1}l\alpha(G')]\tilde{\varphi}(G') \quad (2.17)$$

Now we take into account the well known property (which can be verified directly) that if $u \in \text{SU}(2)$ and $g$ is some 4-vector such that $g^2 = 1$ then

$$u\alpha(g) = \alpha(L(u)g)u \quad (2.18)$$

Using this property and taking into account Eq. (2.17) we derive from Eq. (2.17) the final expression for the action of $\tilde{U}(l)$ in $\tilde{H}$:

$$\tilde{U}(l)\tilde{\varphi}(G) = D[s; \alpha(G)^{-1}l\alpha(G')]\tilde{\varphi}(G') \quad (2.19)$$

where $S = l(q) + s_1 + s_2.$
Comparing Eqs. (2.14) and (2.19) on the one hand and Eq. (2.4) on the other we see that indeed $M$ can be interpreted as the mass of the two-particle system, $G$ as its 4-velocity and $H_{int}$ as the internal space of the two-particle system since the operators $\tilde{U}(a,l)$ in $\tilde{H}$ have the same form as the single-particle generators in Eq. (2.4) if $m$ is replaced by $M$, $g$ by $G$ and $s$ by $S$. Therefore the generators $\tilde{\Gamma}^i$ $(i = 1, 2...10)$ of the representation $\tilde{U}$ in $\tilde{H}$ can be immediately written by analogy with Eq. (2.5)

$$\tilde{P} = MG, \quad \tilde{M} = I(G) + S, \quad \tilde{N} = -\partial G^0 \quad \frac{\partial}{\partial G} + \frac{S \times G}{1 + G^0} \quad (2.20)$$

Here $M$ is the operator of multiplication by the function $M(q)$. This operator and the operator $S$ act only through the variables of the space $H_{int}$.

Since the representation $U$ in $H$ is the tensor product of the single-particle representations, the generators of $U$ are sums of the corresponding single-particle generators. We conclude that the sums of the single-particle generators can be written in the form of Eq. (2.20) only for the case of spinless particles since only in this case $U_{12} = 1$. In the general case, if $\Gamma^i$ are the generators of the representation $U$ then

$$\Gamma^i = U_{12} \hat{\Gamma}^i U_{12}^{-1} \quad (2.21)$$

If particles 1 and 2 interact with each other then cluster separability (see Sec. 1.1) implies that the representation $\hat{U}$ describing the system $(1,2)$ should be constructed in such a way that it becomes $U$ when the interaction is turned off. The condition that $\hat{U}$ is the representation of the Poincare group is equivalent to the condition that the generators $\hat{\Gamma}^i$ $(i = 1, 2...10)$ of this representation satisfy the commutation relations in the form of Eq. (1.5). In the point form the simplest way of introducing the interaction into the two-particle system is to write $\hat{\Gamma}^i = U_{12} \hat{\tilde{\Gamma}}^i U_{12}^{-1}$ where $\hat{\tilde{\Gamma}}^i$ are the generators defined by Eq. (2.20) but with $M$ replaced by a selfadjoint operator $\hat{M}^{int}$ in $H$ such that $\hat{M}^{int}$ commutes with $S$ and becomes $M$ when the interaction is turned off. Therefore, as follows from Eq. (2.20), the representation generators in
\[ \hat{P} = \hat{M}G, \quad \hat{M} = U_{12}[I(G) + S]U_{12}^{-1}, \]
\[ \hat{N} = U_{12}[-iG^0 \frac{\partial}{\partial G} + \frac{S \times G}{1 + G^0}]U_{12}^{-1} \]

(2.22)

where \( \hat{M} = U_{12}\hat{M}^{int}U_{12}^{-1} \).

A similar way of introducing the interaction into the two-particle system was first proposed by Bakamdjian and Thomas [83] in the instant form. The Bakamdjian-Thomas (BT) procedure has been discussed in detail in refs. [13, 84] where it has been shown that in the general case interaction terms should be also introduced into the operator \( U_{12} \), i.e. we should replace \( U_{12} \) by a unitary operator \( \hat{U}_{12} = A_{12}U_{12} \) while the choice \( A_{12} = 1 \) can be substantiated only for particles of equal mass. Since the most interesting applications of the results concerning the ECO are expected to be in cases when constituents have equal masses (in nuclear physics we usually can neglect the difference of masses between the proton and the neutron and in constituent quark models the same can be assumed for the masses of the \( u \) and the \( d \) quarks), we shall consider in this paper only the case \( A_{12} = 1 \).

We can write \( \hat{M}^{int} = M + v \), and if the interaction operator \( v \) is an integral operator then its action can be written as (see Eq. (2.10))

\[ v\chi(q) = \int v(q,q')\chi(q')\frac{M(q')^3d^3q'}{2(2\pi)^3\omega_1(q')\omega_2(q')} \]  

(2.23)

where the kernel \( v(q,q') \) is an operator acting only through the spin variables.

### 2.2 Sokolov’s method of packing operators for systems of three particles

Let us consider a system of three particles with the masses \( m_i > 0 \) and the spin operators \( s_i \) (\( i = 1, 2, 3 \)). The three-particle Hilbert space \( H \) is the tensor product of the single-particle spaces \( H_i \). Now we supply the operators relating to the subsystem \( ij \) with the
corresponding index, for example \( \hat{P}_{ij}, \hat{M}_{ij}, q_{ij} \) etc. We use \( P \) to denote the 4-momentum operator when all the three particles do not interact with each other, \( P_{(ij)k} \) to denote this operator when only particles \( i \) and \( j \) interact with each other \( (i \neq j \neq k) \) and \( \hat{P} \) to denote this operator when all the three particles interact with each other. The corresponding mass operators will be written as \( M, M_{(ij)k} \) and \( \hat{M} \) and the corresponding 4-velocity operators as \( G, G_{(ij)k} \) and \( \hat{G} \). As follows from cluster separability and Eqs. (2.3) and (2.22), \( P_{(ij)k} = \hat{M}_{ij} G_{ij} + m_k g_k \) and therefore

\[
M_{(ij)k} = |\hat{M}_{ij} G_{ij} + m_k g_k|,
G_{(ij)k} = \frac{\hat{M}_{ij} G_{ij} + m_k g_k}{|\hat{M}_{ij} G_{ij} + m_k g_k|}
\] (2.24)

while

\[
M = |M_{ij} G_{ij} + m_k g_k| = |m_i g_i + m_j g_j + m_k g_k|,
G = \frac{M_{ij} G_{ij} + m_k g_k}{|M_{ij} G_{ij} + m_k g_k|} = \frac{m_i g_i + m_j g_j + m_k g_k}{|m_i g_i + m_j g_j + m_k g_k|}
\] (2.25)

In the point form the operators \( \hat{U}(a) = \exp[i(\hat{P}a)] \) and \( U(l) \) define a unitary representation of the Poincare group or in other words, the operator \( \hat{P} \) and the generators of the representation \( l \to U(l) \) must form the system of generators satisfying the commutation relations (1.3). In addition, as follows from cluster separability, the operator \( \hat{P} \) must become \( P_{(ij)k} \) when all interactions involving particle \( k \) are turned off.

By analogy with the way of introducing the interaction in Sec. 2.1 and with ordinary quantum mechanics one might think that a possible way of writing the three-body mass operator is to represent it in the form

\[
\hat{M} = M + (M_{(12)3} - M) + (M_{(13)2} - M) + (M_{(23)1} - M)
\] (2.26)

since this operator indeed becomes \( M_{(ij)k} \) when all interactions involving any particle \( k \) are turned off. However Eq. (2.26) cannot be correct since it leads to the breaking of the commutation relations (1.3). The
matter is that $M$ commutes with $G$, $M_{(ij)k}$ commutes with $G_{(ij)k}$ and all the operators $G$, $G_{(12)3}$, $G_{(13)2}$, $G_{(23)1}$ differ each other.

The idea of the Sokolov method of packing operators [3] is that the composition of interactions in the three-body mass operator should involve only mass operators commuting with one and the same 4-velocity operator, for example with $G$. To realize this idea Sokolov notes first that, as follows from physical considerations, all the operators $G$ and $G_{(ij)k}$ should have the same spectrum and therefore these operators should be unitarily equivalent to each other even if the operators $\hat{M}_{ij}$ and $M_{ij}$ have different spectra. Let $A_{ij,k}$ be such a unitary operator that

$$G_{(ij)k} = A_{ij,k} G A_{ij,k}^{-1} \quad (2.27)$$

Then the operators

$$M_{ij,k} = A_{ij,k}^{-1} M_{(ij)k} A_{ij,k} \quad (2.28)$$

commute with $G$. Suppose also that the unitary operator $A$ is constructed from the operators $A_{12,3}$, $A_{13,2}$ and $A_{23,1}$ in such a way that it becomes $A_{ij,k}$ when all interactions involving particle $k$ are turned off. Then we can construct the mass operator $\hat{M}$ as

$$\hat{M} = A M_{123} A^{-1}, \quad M_{123} = M + (M_{12,3} - M) + (M_{13,2} - M) + (M_{23,1} - M) + V_{123} \quad (2.29)$$

where $V_{123}$ is a three-particle interaction operator which commutes with $G$, with $U(l)$ and becomes zero if all interactions involving any of three particles are turned off.

The difference between Eqs. (2.26) and (2.29) is that the "auxiliary" mass operators $M_{ij,k}$ entering into the expression for the "auxiliary" mass operator $M_{123}$ commute with one and the same 4-velocity operator $G$. The operator $\hat{M}$ in Eq. (2.29) satisfies cluster separability since if all interactions involving any particle $k$ are turned off then $M_{123}$ becomes $M_{ij,k}$, $A$ becomes $A_{ij,k}$ and therefore, as follows from Eq. (2.28), $\hat{M}$ becomes $M_{(ij)k}$. At the same time, if the operators $A_{ij,k}$ and $A$ commute with $U(l)$ then the commutation relations and cluster separability will be satisfied if

$$\hat{P} = \hat{M} \hat{G}, \quad \hat{G} = A G A^{-1} \quad (2.30)$$
This statement can be proved if we note the following. The set 
\((P_{ij}k, M, N)\), where \(M\) and \(N\) are the generators of the three-particle representation \(U(l)\), satisfies the commutation relations \((1.3)\) by construction. Therefore the set \((P_{ij,k} = M_{ij,k}G, M, N)\) also satisfies these relations as follows from Eqs. \((2.27), (2.28)\) and the fact that \(A_{ij,k}\) commutes with \(M\) and \(N\). In turn this means that \(M_{ij,k}\) commutes with \(G, M\) and \(N\). Therefore, as follows from Eq. \((2.29)\), \(M_{123}\) also commutes with \(G, M\) and \(N\). In turn this means that the set \((P_{123} = M_{123}G, M, N)\) satisfies the conditions \((1.3)\). Therefore the set \((\hat{P} = \hat{M}G, M, N)\) also satisfies these conditions since \(A\) commutes with \(M\) and \(N\). This set satisfies cluster separability by construction since when all interactions involving any particle \(k\) are turned off then \(\hat{P}\) becomes \(P_{(ij)k}\).

We see that the sense of the operators \(A_{ij,k}\) is that they ”pack” the operators \(M_{(ij)k}\) to the operators \(M_{ij,k}\) (see Eq. \((2.28)\)). The latter enter into the expression determining the composition of interactions in Eq. \((2.29)\). That is why Sokolov referred to his method as that of packing operators.

Following Sokolov [3] we shall seek the operators \(A_{ij,k}\) in the form

\[
A_{ij,k} = U_{ij}B_{ij,k}(\hat{M}_{ij}^{int})^{-1}B_{ij,k}(M_{ij})U_{ij}^{-1} \tag{2.31}
\]

where the unitary operators \(B_{ij,k}(\hat{M}_{ij}^{int})\) and \(B_{ij,k}(M_{ij})\) commute with the operators of the tensor product \(\hat{U}_{ij} \otimes U_k\) reduced on \(\text{SL}(2,\mathbb{C})\). As usual, the dependence on a selfadjoint operator is understood in the sense of the spectral decomposition. Namely, if \(\hat{e}_{ij}(m)\) is the spectral function of the operator \(\hat{M}_{ij}^{int}\) and \(e_{ij}\) is the spectral function of the operator \(M_{ij}\) then

\[
B_{ij,k}(\hat{M}_{ij}^{int}) = \int B_{ij,k}(m)\hat{e}_{ij}^{int}(m),
\]
\[
B_{ij,k}(M_{ij}) = \int B_{ij,k}(m)e_{ij}(m) \tag{2.32}
\]

Here the integrals are understood as the strong limits of the corresponding Riemann sums. The operators \((2.32)\) are correctly defined and are unitary if the operators \(B_{ij,k}(m)\) are unitary and commute with \(\hat{e}_{ij}^{int}(m')\) and \(e_{ij}(m')\) for all \(m\) and \(m'\) belonging to the spectra of the operators...
\( \hat{M}_{ij}^{int} \) and \( M_{ij} \).

As follows from Eqs. \((2.22), (2.24)\) and \((2.25)\)

\[
G(G_{ij}, g_k, m) = U_{ij} \{ \int G(G_{ij}, g_k, m) d\hat{c}^{int}_{ij}(m) \} U_{ij}^{-1},
\]

\[
G = U_{ij} \{ \int G(G_{ij}, g_k, m) d\epsilon_{ij}(m) \} U_{ij}^{-1},
\]  
(2.33)

where

\[
G(G_{ij}, g_k, m) = \frac{mG_{ij} + m_k g_k}{|mG_{ij} + m_k g_k|}
\]  
(2.34)

Then, as follows from Eqs. \((2.31-2.33)\), Eq. \((2.27)\) will be satisfied if the operator

\[
C(G_{ij}, g_k) = B_{ij,k}(m) G(G_{ij}, g_k, m) B_{ij,k}(m)^{-1}
\]  
(2.35)

does not depend on \( m \). In refs. \[3\] there were explicitly considered

the cases \( C(G_{ij}, g_k) = g_k \) and \( C(G_{ij}, g_k) = G_{ij} \) and in ref. \[3\] there was explicitly considered an infinite number solutions corresponding to different choices of \( C(G_{ij}, g_k) \). It has been shown that only the solution with \( C(G_{ij}, g_k) = G_{ij} \) has physical sense. Therefore the problem of finding the operators \( A_{ij,k} \) will be solved if we succeed in finding the unitary operators \( B_{ij,k}(m) \) commuting with \( \hat{e}^{int}_{ij}(m') \), \( e_{ij}(m') \), with the operators \( \hat{U}_{ij} \otimes U_k \) reduced on \( SL(2, C) \) and such that

\[
B_{ij,k}(m)^{-1} C(G_{ij}, g_k) B_{ij,k}(m) = G(G_{ij}, g_k, m)
\]  
(2.36)

### 2.3 Explicit expressions for the packing operators and the operators \( M_{ij,k} \)

As follows from Eqs. \((2.7)\) and \((2.19)\), the tensor product of the representations \( \hat{U}_{ij} \) and \( U_k \) reduced on \( SL(2, C) \) is realized in the space of functions \( \varphi(G_{ij}, g_k) \) with the range in \( H_{ij}^{int} \otimes \mathcal{D}(s_k) \) and such that

\[
\int ||\varphi(G_{ij}, g_k)||^2 d\rho(G_{ij}) d\rho(g_k) < \infty
\]  
(2.37)

where the norm is taken in the space \( H_{ij}^{int} \otimes \mathcal{D}(s_k) \). The operators of this representation act as

\[
[\hat{U}_{ij}(l) \otimes U_k(l)] |\varphi(G_{ij}, g_k) = D[S_{ij}; \alpha(G_{ij})^{-1} l \alpha(G'_{ij})] \cdot
\]
\[ D[s_k; \alpha(G_k)^{-1}l\alpha(G'_k)] \varphi(G'_{ij}, g'_k) \]  

where \( G'_{ij} = L(l)^{-1}G_{ij}, g'_k = L(l)^{-1}g_k \) and \( S_{ij} \) is the spin operator of the system \( ij \) defined in Sec. 2.1.

We introduce the 4-vector \( K_{ij}(m) = K_{ij}(G_{ij}, g_k, m) = L[\alpha(G(m))]^{-1}mG_{ij} \)  

where for simplicity we write \( G(m) \) instead of \( G(G_{ij}, g_k, m) \). The vector defined by Eq. (2.39) has the sense of the 4-momentum of the system \( ij \) with the mass \( m \) in the c.m. frame of the system consisting of \( ij \) as the subsystem and particle \( k \) as the other subsystem (compare with Eq. (2.7)). We use \( K'_{ij}(m) \) to denote the quantity \( K_{ij}(G'_{ij}, g'_k, m) \). By analogy with Eq. (2.15) it follows from Eq. (2.39) that the quantities \( K_{ij}(m) \) and \( K'_{ij}(m) \) are related to each other by a usual three-dimensional rotation:

\[ K_{ij}(m) = L[\alpha(G(m))]^{-1}l\alpha(G'(m))]K'_{ij}(m) \quad (2.40) \]

where \( G'(m) = L(l)^{-1}G(m) \).

Instead of \( G_{ij} \) and \( g_k \) we can choose as independent variables \( G(m) \) and \( g_k \). Then a direct calculation using Eq. (2.34) yields

\[ d\rho(G(m)) = J(G_{ij}, g_k, m)d\rho(G_{ij}), \]

\[ J(G_{ij}, g_k, m) = \frac{m^3[m + m_k(G_{ij}, g_k, m)]}{|mG_{ij} + m_kg_k|^4} \quad (2.41) \]

Now using Eqs. (2.39)–(2.41) it is easy to verify that the operator \( B_{ij,k}(m)^{-1} \) defined as

\[ B_{ij,k}(m)^{-1} \varphi(G_{ij}, g_k) = J(G_{ij}, g_k, m)^{1/2}D[S_{ij}; \alpha(G_{ij})]^{-1} \cdot \alpha(G(m))\alpha(K_{ij}(m)/m)]\varphi(G(m), g_k) \]

is indeed unitary and commutes with the operators \( \tilde{U}_{ij}(l) \otimes U_k(l) \) defined by Eq. (2.38).

Let us introduce the 4-vectors

\[ H(m) = H(G_{ij}, g_k, m) = \frac{1}{m}\{G_{ij}[m_k(G_{ij}, g_k) +
\]

\[ + (m^2 + m_k^2(G_{ij}, g_k)^2 - m_k^2)^{1/2} - m_kg_k]\}, \]

\[ R(m) = R(G_{ij}, g_k, m) = L[\alpha(G_{ij})]^{-1}mH(m) \quad (2.43) \]
The operations $G_{ij} \rightarrow H(m)$ and $G_{ij} \rightarrow G(m)$ are inverse to each other since it is easy to verify that

$$H(G(m), g_k, m) = G(H(m), g_k, m) = G_{ij}.$$ 

Therefore using Eq. (2.42) it is easy to verify that

$$B_{ij,k}(m)\varphi(G_{ij}, g_k) = J(H(m), g_k, m)^{-1/2} D[S_{ij}; \alpha(R(m)/m)]^{-1} \varphi(G_{ij}, g_k) = G_{ij}^{-1} \alpha(H(m)/m)\varphi(H(m), g_k)$$  \hspace{1cm} (2.44) 

Finally, it is easy to verify that the operators defined by Eqs. (2.42) and (2.44) satisfy Eq. (2.36). The action of these operators through the variables of the space $H_{ij}^{int}$ is fully defined by the operator $S_{ij}$ and the quantities $G_{ij}$ and $g_k$. Since $S_{ij}$, $G_{ij}$ and $g_k$ commute with $\hat{M}_{ij}^{int}$ and $M_{ij}$ (see Sec. 2.1) then $B_{ij,k}(m)$ indeed commutes with $\hat{e}_{ij}^{int}(m')$ and $e_{ij}(m')$ for all $m$ and $m'$. Therefore we conclude that the operators $B_{ij,k}(m)^{-1}$ and $B_{ij,k}(m)$ defined by Eqs. (2.42) and (2.44) satisfy all the necessary conditions.

The three-particle representation of the Poincare group is realized in the tensor product of the three-particle spaces, i.e. the Hilbert space $H$ is now the space of functions $\varphi(g_1, g_2, g_3)$ with the range in $D(s_1) \otimes D(s_2) \otimes D(s_3)$ and such that

$$\int \left\||\varphi(g_1, g_2, g_3)||^2 \prod_{i=1}^3 d\rho(g_i) \right| < \infty$$  \hspace{1cm} (2.45) 

Instead of the variables $g_1, g_2, g_3$ we introduce the variables $G, k_1, k_2, k_3$, where $G$ is defined by Eq. (2.25) and the $k_i$ ($i = 1, 2, 3$) are formally defined as $q_i$ in Eq. (2.8) but for the case of three-particles:

$$q_i = L[\alpha(G)]^{-1} m_i g_i,$$  \hspace{1cm} (2.46) 

while Eq. (2.8) should be rewritten as

$$k_i^{(ij)} = L[\alpha(G_{ij})]^{-1} m_i g_i$$  \hspace{1cm} (2.47) 

and $q_i^{(ij)}$ should be understood as the spatial part of $q_i^{(ij)}$. Now $q_i^{(ij)}$ has the sense of the 4-momentum of particle $i$ in the c.m. frame of the system $ij$ and the quantities $k_i$ are the 4-momenta in the c.m. frame of
the three-particle system. These 4-momenta are not independent since \( k_1 + k_2 + k_3 = 0 \) as it should be. A direct calculation using Eqs. (2.25) and (2.46) yields

\[
\prod_{i=1}^{3} d\rho(g_i) = d\rho(G)d\rho(int), \quad \rho(int) = 2(2\pi)^3 M^3.
\]

\[
\cdot \delta^{(3)}(k_1 + k_2 + k_3) \prod_{i=1}^{3} d\rho(k_i/m_i)
\]  (2.48)

where the mass of the three-particle system is expressed in terms of the \( k_i \) as \( M = \omega_1(k_1) + \omega_2(k_2) + \omega_3(k_3) \).

In the three-particle case we introduce \( H_{int} \) as the space of functions \( \chi = \chi(k_1, k_2, k_3) \) with the range in \( D(s_1) \otimes D(s_2) \otimes D(s_3) \) and such that

\[
||\chi||^2 = \int ||\chi(k_1, k_2, k_3)||^2 d\rho(int) \quad < \infty
\]  (2.49)

We also introduce the Hilbert space \( \tilde{H} \) as the space of functions \( \tilde{\varphi}(G) \) with the range in \( H_{int} \) and such that

\[
\int ||\tilde{\varphi}(G)||^2_{int} d\rho(G) \quad < \infty
\]  (2.50)

(formally this expression looks as Eq. (2.12), but it is clear that in the cases of two and three particles the spaces \( H_{int} \) are different). Then, as easily follows from Eqs. (2.45) and (2.48)-(2.50), the operator

\[
U_{123} = \prod_{i=1}^{3} D[s_i; \alpha(g_i)^{-1} \alpha(G) \alpha(k_i/m_i)]
\]  (2.51)

is the unitary operator from \( \tilde{H} \) to \( H \).

Since the vectors \( k_i \) are not independent, it is often convenient to choose in \( H_{int} \) any two independent vectors, for example \( K_{ij} \) and \( k_{ij} \) where \( K_{ij} \) is the spatial part of the 4-vector \( K_{ij} = k_i + k_j \) and \( k_{ij} \) is the spatial part of the 4-vector

\[
k_{i}^{(ij)} = L[\alpha(K_{ij}/M_{ij})]^{-1}k_i
\]  (2.52)

As follows from Eq. (2.46), \( K_{ij} \) coincides with the 4-vector introduced in Eq. (2.39) for the case \( m = M_{ij} \). A direct calculation shows that
instead of Eq. (2.48) the volume element $d\rho(int)$ in the variables $K_{ij}$ and $k_{ij}$ has the form

$$
d\rho(int) = \frac{M^3 M_{ij} d^3 K_{ij} d^3 k_{ij}}{4(2\pi)^6 E_{ij} \omega_i(k_{ij}) \omega_j(k_{ij}) \omega_k(K_{ij}) m_1^2 m_2^2 m_3^2}$$

(2.53)

where $M_{ij} = \omega_i(k_{ij}) + \omega_j(k_{ij})$, $E_{ij} = (M_{ij}^2 + K_{ij}^2)^{1/2}$, $M = E_{ij} + \omega_k(K_{ij})$.

Let us note that the dependence of $M_{ij}$ on $k_{ij}$ is the same as on $q_{ij}$ (see Sec. 2.1) since, as follows from Eqs. (2.46), (2.47) and (2.52), these vectors are related to each other by the three-dimensional rotation:

$$
q_{ij} = L[\alpha(G_{ij})^{-1} \alpha(G) \alpha(K_{ij}/M_{ij})]k_{ij}
$$

(2.54)

Let $u \to R(u)$ be the representation of the group SU(2) in $H_{int}$ which acts as

$$
R(u)\chi(k_1, k_2, k_3) = \chi(L(u)^{-1} k_1, L(u)^{-1} k_2, L(u)^{-1} k_3)
$$

(2.55)

It is easy to show that the generators of this representation can be written as $l(K_{ij}) + l(k_{ij})$ for any $ij$. Then by analogy with Eq. (2.19) it is easy to show that if $U(a, l) = U_{123}\tilde{U}(a, l)U_{123}^{-1}$ then the action of $\tilde{U}(l)$ in $\tilde{H}$ is given by

$$
\tilde{U}(l)\tilde{\varphi}(G) = D[S; \alpha(G)^{-1} l\alpha(G')]\tilde{\varphi}(G')
$$

(2.56)

where $G' = L(l)^{-1} G$ and $S = l(K_{ij}) + l(k_{ij}) + s_1 + s_2 + s_3$. Comparing Eqs. (2.4) and (2.56) we see that the operator $S$ has the sense of the spin operator for the three-particle system, and therefore the operators $\tilde{M}$ and $\tilde{N}$ for the representation defined by Eq. (2.56) indeed have the canonical form given by Eq. (2.20) but now $G$ is the three-particle 4-velocity and $S$ is the three-particle spin operator.

Let us begin to calculate the operator $M_{ij,k}$. As follows from Eqs. (2.22), (2.24), (2.31), (2.42) and (2.44)

$$
M_{ij,k} = U_{ij} B_{ij,k}(M_{ij})^{-1} B_{ij,k}(\hat{M}_{ij}^{int}) |\hat{M}_{ij}^{int} G_{ij} + m_k g_k| \cdot
\cdot B_{ij,k}(\hat{M}_{ij}^{int})^{-1} B_{ij,k}(M_{ij})U_{ij}^{-1}
$$

(2.57)

and, as follows from Eqs. (2.43) and (2.44)

$$
B_{ij,k}(\hat{M}_{ij}^{int}) |\hat{M}_{ij}^{int} G_{ij} + m_k g_k| B_{ij,k}(\hat{M}_{ij}^{int})^{-1} = m_k (G_{ij}, g_k) +
+[(\hat{M}_{ij}^{int})^2 + m_k^2 (G_{ij}, g_k)^2 - m_k^2]^{1/2}
$$

(2.58)
If $\tilde{M}_{ij}^\text{int} = M_{ij} + v_{ij}$ then we introduce the operator

$$v_{ij}^B = U_{ij}B_{ij,k}(M_{ij})^{-1}v_{ij}B_{ij,k}(M_{ij})U_{ij}^{-1}$$

and, as follows from Eqs. (2.42) and (2.44)

$$M_{ij,k} = \omega_k(K_{ij}) + \left[(M_{ij} + v_{ij}^B)^2 + K_{ij}^2\right]^{1/2} \quad (2.59)$$

where we have taken into account that, as follows from Eq. (2.46) and the definition of $K_{ij}$

$$\frac{m_k[M_{ij}(G_{ij}, g_k) + m_k]}{|M_{ij}G_{ij} + m_kg_k|} = \omega_k(K_{ij}) \quad (2.60)$$

The action of $M_{ij,k}$ in $\tilde{H}$ is determined by the operator $\tilde{M}_{ij,k}$ such that

$$\tilde{M}_{ij,k} = U_{123}M_{ij,k}U_{123}^{-1}.$$ Let us introduce the operator $\tilde{v}_{ij}$ such that

$$\tilde{v}_{ij} = U_{123}\tilde{v}_{ij}U_{123}^{-1}.$$ Then as follows from Eq. (2.59)

$$\tilde{M}_{ij,k} = \omega_k(K_{ij}) + \left[(M_{ij} + \tilde{v}_{ij})^2 + K_{ij}^2\right]^{1/2} \quad (2.61)$$

The operators of multiplication by $\omega_k$ and $M_{ij}$ do not depend on the variable $G$ and act only through the variables of the space $H_{\text{int}}$. A direct calculation using the definitions of the operators $v_{ij}^B$, $\tilde{v}_{ij}$ and Eqs. (2.13), (2.23), (2.42), (2.44) (2.51) shows that $\tilde{v}_{ij}$ has the same properties and its action in $H_{\text{int}}$ is given by

$$\tilde{v}_{ij}\chi(K_{ij}, k_{ij}) = \int \prod_{l=i,j} D[s_l; \alpha(k_l/m_l)\alpha(k_l^{(ij)}/m_l)] \cdot$$

$$v_{ij}(k_{ij}, k'_{ij})\prod_{l=i,j} D[s_l; \alpha(k_l^{(ij)}/m_l)^{-1}\alpha(K_{ij}/M_{ij}^{-1})^{-1}\alpha(k_l^{(ij)}/m_l)] \cdot$$

$$\left(\frac{M'}{M}\right)^{3/2}\left(\frac{K_{ij}^0}{K_{ij}^0}\right)^{1/2}\chi(K_{ij}, k'_{ij}) \cdot$$

$$\frac{M_{ij}M_{ij}^d d^3k_{ij}}{2(2\pi)^3m_i^2m_j^2\omega_i(k'_{ij})\omega_j(k'_{ij})} \quad (2.62)$$

where $K_{ij}$ is the 4-vector $((M_{ij}^2 + K_{ij}^2)^{1/2}, K_{ij})$ and in all the primed functions of $K_{ij}$ and $k_{ij}$ the argument $k_{ij}$ is replaced by $k'_{ij}$. We conclude that $\tilde{M}_{ij,k}$ actually acts only in $H_{\text{int}}$ and therefore we can write $M_{ij,k}^\text{int}$ instead of $\tilde{M}_{ij,k}$. This result can be also proved when $A_{ij} \neq 1$ (see Sec. 2.2).
Using Eq. (2.53) one can verify that \( \tilde{v}_{ij} \) is the Hermitian operator in \( H_{\text{int}} \), and, as easily seen from Eq. (2.53), \( [\tilde{v}_{ij}, S] = 0 \). Comparing Eqs. (2.23) and (2.62) we see that while \( v_{ij} \) does not act through \( G_{ij} \) and the variables of particle \( k \) and acts only through \( q_{ij} \) and the spin variables of particles \( i \) and \( j \) (i.e. acts only in \( H^\text{int}_{ij} \), the operator \( \tilde{v}_{ij} \) does not act through \( G \) and the spin variables of particle \( k \), commutes with \( K_{ij} \) and acts only through \( k_{ij} \) and the spin variables of particles \( i \) and \( j \). In the general case, if \( O_{ij} \) is an operator which acts only in \( H^\text{int}_{ij} \), does not act through the other variables and

\[
\tilde{O}_{ij} = U_{123}^{-1}B_{ij,k}(M_{ij})^{-1}O_{ij}B_{ij,k}(M_{ij})U_{ij}^{-1}U_{123} \tag{2.63}
\]

Then by analogy with Eq. (2.62) it can be shown that

\[
\tilde{O}_{ij} = \frac{(K_{ij})^{1/2}M_{ij}}{M^{3/2}}\left\{ \prod_{l=i,j} D[s_l; \alpha(k_l/m_l)^{-1}\alpha(K_{ij}/M_{ij}) \cdot \alpha(k_{ij}(i)/m_l)] \right\} F(O_{ij}) \left\{ \prod_{l=i,j} D[s_l; \alpha(k_l/m_l)^{-1} \cdot \alpha(K_{ij}/M_{ij})\alpha(k_{ij}(i)/m_l)] \right\}^{-1} \frac{M^{3/2}}{(K_{ij})^{1/2}M_{ij}} \tag{2.64}
\]

where the operator \( F(O_{ij}) \) acts through \( k_{ij} \) in the same manner as \( O_{ij} \) acts through \( q_{ij} \).

The operator \( V_{123} \) in Eq. (2.29) will satisfy all the needed properties if

\[
V_{123} = U_{123}v_{123}U_{123}^{-1}
\]

and \( v_{123} \) is an operator which acts only in \( H_{\text{int}} \) and commutes with \( S \). This follows from the fact that, as pointed out above, the generators of the representation defined by Eq. (2.56) has the form of Eq. (2.20). Therefore the operator \( M_{123} \) can be written as \( M_{123} = U_{123}M^\text{int}_{123}U_{123}^{-1} \)

where

\[
M^\text{int}_{123} = M + (M^\text{int}_{123} - M) + (M^\text{int}_{132} - M) + (M^\text{int}_{231} - M) + v_{123} \tag{2.65}
\]

If we are interested in calculating only the spectrum of the three-body system and the internal three-body wave function, we can
consider only the eigenvalue problem for the operator $M_{123}^{\text{int}}$ in $H^{\text{int}}$ since $M_{123}^{\text{int}}$ is unitarily equivalent to the three-particle operator $\hat{M}$ by construction. Using Eqs. (2.61) and (2.62) it can be easily shown that the operator defined by Eq. (2.65) coincides with the three-body mass operator in the instant form derived in ref. [84] if $A_{ij} = 1$ and the normalization in both cases is chosen to be the same. In turn, as shown in ref. [43], such a mass operator is unitarily equivalent to the mass operators derived in refs. [85, 3, 86, 87, 88, 4, 5, 6, 80] from different considerations in different forms of dynamics, and the choice of such a solution for the three-body mass operator has a physical substantiation.

To completely describe the representation of the Poincaré group for the three-body system we have to choose an explicit expression for the operator $A$ in terms of $A_{ij,k}$. This can be done in different ways [3, 4, 5]. However, as we shall see in Sec. 3.3, the physical observables for the three-particle system do not depend on the choice of the expression for $A$ in terms of $A_{12,3}$, $A_{13,2}$ and $A_{23,1}$.

The results of Secs. 2.2 and 2.3 can be summarized as follows. The operators $\hat{U}(a)$ and $U(l)$ describing the representation of the Poincaré group for the three-particle system can be written as

$$\hat{U}(a) = AU_{123}\hat{U}(a)U_{123}^{-1}A^{-1}, \quad U(l) = U_{123}\hat{U}(l)U_{123}^{-1}$$

where $A$ is a unitary operator commuting with $U(l)$. The ”auxiliary” representation defined by the operators $\hat{U}(a)$ and $\hat{U}(l)$ has the same form as the single-particle representation but the role of the ”external” variable is played by $G$, the role of the internal space—by the space $H^{\text{int}}$, and the role of the mass and spin operators—by the operators $M_{123}^{\text{int}}$ and $S$ which act only in $H^{\text{int}}$ and commute with each other. Therefore the three-particle generators in the three-particle Hilbert space $H$ can be written as (compare with Eq. (2.22))

$$\hat{P} = AU_{123}M_{123}^{\text{int}}GU_{123}^{-1}A^{-1}, \quad M = U_{123}[l(G) + S]U_{123}^{-1},$$

$$N = U_{123}[-\imath G^0 \frac{\partial}{\partial G} + \frac{S \times G}{1 + G^0}]U_{123}^{-1}$$

(2.66)
2.4 On the problem of constructing the packing operators for systems with any number of particles

The representation of the Poincare group for a system of $N$ particles with the masses $m_i > 0$ and the spin operators $s_i (i = 1, ... N)$ is realized in the space of functions $\varphi(g_1, ... g_N)$ with the range in $\mathcal{D}(s_1) \otimes ... \otimes \mathcal{D}(s_N)$ and such that

$$\int ||\varphi(g_1, ... g_N)||^2 N \prod_{i=1}^{N} d\rho(g_i) < \infty$$  \hspace{1cm} (2.67)

Instead of the variables $g_1, ... g_N$ we introduce the variables $G, k_1, ... k_N$ where $G = (m_1 g_1 + ... + m_N g_N)/|m_1 g_1 + ... + m_N g_N|$ and the $k_i$ are formally defined as in Eq. \((2.40)\). Then by analogy with Eq. \((2.48)\) one can show that

$$\prod_{i=1}^{N} d\rho(g_i) = d\rho(G)d\rho(int), \quad d\rho(int) = 2(2\pi)^3 M^3 \cdot \delta^{(3)}(k_1 + ... + k_N) \prod_{i=1}^{N} d\rho(k_i/m_i)$$  \hspace{1cm} (2.68)

where $M = \omega_1(k_1) + ... + \omega_N(k_N)$. The ”internal” space $H_{int}$ can be defined by analogy with Eq. \((2.49)\), i.e. as the space of functions $\chi(k_1, ... k_N)$ with the range in $\mathcal{D}(s_1) \otimes ... \otimes \mathcal{D}(s_N)$ and such that

$$||\chi||^2 = \int ||\chi(k_1, ... k_N)||^2 d\rho(int) < \infty$$  \hspace{1cm} (2.69)

and the space $\tilde{H}$ can be defined as the space of functions $\tilde{\varphi}(G)$ with the range in $H_{int}$ and such that Eq. \((2.50)\) is satisfied. Then by analogy with Eq. \((2.51)\) one can show that

$$\mathcal{U} = \prod_{i=1}^{N} D[s_i; \alpha(g_i)^{-1}\alpha(G)\alpha(k_i/m_i)]$$  \hspace{1cm} (2.70)

is the unitary operator from $\tilde{H}$ to $H$.

Our goal is to construct a unitary operator $A$ in $H$ such that $A$ commutes with $U(l)$, the generators $\tilde{\Gamma}^i$ of the representation under
consideration \((i = 1, \ldots, 10)\) have the form \(\hat{\Gamma}^i = A U \hat{\Gamma}^i U^{-1} A^{-1}\), where the operators \(\hat{\Gamma}^i\) in \(\tilde{H}\) have the following ”canonical” form (compare with Eqs. (2.22) and (2.66))

\[
P = \hat{M}_{int} G, \quad M = [l(G) + S], \quad N = -iG^0 \frac{\partial}{\partial G} + \frac{S \times G}{1 + G^0}, \quad (2.71)
\]

\(\hat{M}_{int}\) and \(S\) act only in \(H_{int}\), \(S\) satisfies the commutation relation for the spin operators and \([\hat{M}_{int}, S] = 0\).

As explicitly shown above, such a construction does exist for systems of two and three particles. In the case of four and more particles a possible way of constructing the system of generators in the form of Eq. (2.71) is the following.

First it is necessary to find unitary operators \(A_{\alpha\beta}\) such that (compare with Eqs. (2.24) and (2.27)) \(\hat{G}_{\alpha\beta} = A_{\alpha\beta} G A_{\alpha\beta}^{-1}\) where \(\alpha\) and \(\beta\) are any subsystems comprising our system and

\[
\hat{G}_{\alpha\beta} = \frac{\hat{M}_\alpha \hat{G}_\alpha + \hat{M}_\beta \hat{G}_\beta}{|\hat{M}_\alpha \hat{G}_\alpha + \hat{M}_\beta \hat{G}_\beta|} \quad (2.72)
\]

where \(\hat{M}_\alpha\) and \(\hat{G}_\alpha\) are the mass and the 4-velocity operators of the system \(\alpha\) and analogously for \(\hat{M}_\beta\) and \(\hat{G}_\beta\). If we know the operators \(A_{\alpha\beta}\) for any partition of our system into two noninteracting subsystems then we also know the operators \(A_{\alpha_1\ldots\alpha_n}\) for the case when our system is partitioned into \(n\) noninteracting subsystems \(\alpha_1, \ldots, \alpha_n\). These operators satisfy the property

\[
A_{\alpha_1\ldots\alpha_n} G A_{\alpha_1\ldots\alpha_n}^{-1} = \hat{G}_{\alpha_1\ldots\alpha_n} \quad (2.73)
\]

where \(\hat{G}_{\alpha_1\ldots\alpha_n}\) is the 4-velocity in the case when the subsystems \(\alpha_1, \ldots, \alpha_n\) do not interact with each other (in this case \(\hat{G}_{\alpha_1\ldots\alpha_n}\) depends only on interactions inside the subsystems \(\alpha_1, \ldots, \alpha_n\)).

Let \(M_{(\alpha_1), \ldots, (\alpha_n)}\) be the system mass operator in this case. According to the idea of the Sokolov method of packing operators, we can introduce the ”auxiliary” mass operators

\[
M_{\alpha_1\ldots\alpha_n} = U^{-1} A_{\alpha_1\ldots\alpha_n}^{-1} M_{(\alpha_1), \ldots, (\alpha_n)} A_{\alpha_1\ldots\alpha_n} U \quad (2.74)
\]
which act in $H_{\text{int}}$ and commute with $G$ by construction. Then we construct the operator $\hat{M}_{\text{int}}$ from the operators $M_{\alpha_1...\alpha_n}$ and the unitary operator $A$ from the operators $A_{\alpha_1...\alpha_n}$ in such a way that $\hat{M}_{\text{int}}$ becomes $M_{\alpha_1...\alpha_n}$ and $A$ becomes $A_{\alpha_1...\alpha_n}$ when all interactions between the subsystems $\alpha_1, ..., \alpha_n$ are turned off. If this program is carried out then the generators given by Eq. (2.71) will satisfy all the required properties.

This program has been proposed and partially carried out by Sokolov [89, 90]. The major technical difficulty in realizing the program is the following. The operators $A_{\alpha\beta}$ should be constructed in such a way that the operators $A_{\alpha_1,...\alpha_n}$ do not depend on the order in which the interactions between the subsystems are turned off (see the discussion in refs. [90, 4, 5, 91, 43]. If this is the case then the operator $\hat{M}_{\text{int}}$ can be constructed as (see refs. [90, 4, 5, 91, 43] for details)

$$\hat{M}_{\text{int}} = \sum_{k=2}^{N} (-1)^k (k-1)! M_{(k)}^{\text{int}} + v_N, \quad M_{(k)}^{\text{int}} = \sum_{\alpha_1...\alpha_k} M_{\alpha_1...\alpha_k}^{\text{int}} \quad (2.75)$$

where $v_N$ is some ”$N$-particle” interaction and the sum in the second expression is taken over all partitions of the indices 1,2,...,N into $k$ groups $\alpha_1...\alpha_k$. In particular, this expression becomes Eq. (2.65) if $N = 3$.

One might try to seek explicit solutions for the operators $A_{\alpha\beta}$ not leaving the frames of the point form. The operators satisfying Eq. (2.73) have been found by Sokolov in ref. [89], but these operators do not satisfy the symmetry condition mentioned above if $N > 3$. Nevertheless the construction by Sokolov and Shatny [77] which establishes the unitary equivalence of three basic forms for any number of particles makes it possible to find the packing operators in the point form if they are found in some other form. Since the $N$-body problem is solved by Coester and Polyzou [4] and Mutze [5] in the instant form (see also refs. [80, 42, 43]), then the solution for the operators $A_{\alpha\beta}$ exist also in the instant form. For this reason, considering the problem of constructing the ECO we shall assume that for any $N$ the generators of the Poincare group can be written in the form of Eq. (2.71). Some aspects of the unitary equivalence of three basic forms are considered in Secs. 5.2 and 6.2.

In this chapter we did not consider the conditions imposed
by the P and T invariance, but it is easy to show [4, 12, 3, 6, 43] that they are not too restrictive. Therefore considering the problem of constructing the ECO we shall assume that RQM can be constructed in such a way that it is invariant under the extended Poincare group containing the operations P and T. As noted in Sec. 1.2, the operators $\hat{U}_P$ and $\hat{U}_R$ can be chosen as for the case of free particles and therefore the action of these operators in $\mathcal{H}$ can be written in the form [82]

$$U_P\varphi(g_1, \ldots g_N) = (\prod_{i=1}^{N} \eta_{iP})\varphi(-g_1, \ldots -g_N),$$

$$U_R\varphi(g_1, \ldots g_N) = \{\prod_{i=1}^{N} \eta_{iR}D[s_i; C]\}\overline{\varphi}(g_1, \ldots g_N) \quad (2.76)$$

Here $\eta_{iP}$ and $\eta_{iR}$ are the P and R parities of the i-th particle, the bar means the complex conjugation, $C = \sigma_2$ and we write $g_i$ instead of $g_i$ in the arguments of the function $\varphi$. It is easy to show that the operators $U_P$ and $U_R$ reduced onto the space $\mathcal{H}_{int}$ have the form

$$\tilde{U}_P\chi(k_1, \ldots k_N) = (\prod_{i=1}^{N} \eta_{iP})\chi(-k_1, \ldots -k_N),$$

$$\tilde{U}_R\chi(k_1, \ldots k_N) = \{\prod_{i=1}^{N} \eta_{iR}D[s_i; C]\}\overline{\chi}(k_1, \ldots k_N) \quad (2.77)$$
Chapter 3

General properties of the electromagnetic current operator

3.1 Relativistic invariance and current conservation

Let \( x = 0 \) be the origin in Minkowski space. Then, as follows from Eq. (1.1)

\[
\hat{J}^\mu(x) = \exp(i \hat{P} x) \hat{J}^\mu(0) \exp(-i \hat{P} x)
\]

(3.1)

and, as follows from Eqs. (1.1-1.3) and (3.1), the operator \( \hat{J}^\mu(0) \) must satisfy the properties

\[
U(l)^{-1} \hat{J}^\mu(0) U(l) = L(l)^\mu_\nu \hat{J}^\nu(0), \quad [\hat{P}_\mu, \hat{J}^\mu(0)] = 0
\]

(3.2)

while the charge operator is expressed in terms of \( \hat{J}^\nu(0) \) as

\[
\hat{Q} = \int \exp(i \hat{P} x) \hat{J}^\mu(0) \exp(-i \hat{P} x) d\sigma_\mu(x)
\]

(3.3)

On the contrary, suppose we have found the operator \( \hat{J}^\mu(0) \) satisfying Eq. (3.2), cluster separability and the condition that the operator defined by Eq. (3.3) does not depend on \( \lambda \) and \( \tau \) and has only one eigenvalue equal to the sum of the electric charges of constituents. In this case we treat Eq. (3.1) as the definition of \( \hat{J}^\mu(x) \). Then, using the well known properties of representations of the Poincare group, it is easy to verify that the operator \( \hat{J}^\mu(x) \) defined in such a way indeed satisfies all the necessary conditions. We see that the problem of seeking the
operator $\hat{J}^\mu(x)$ can be reduced to the problem of seeking the operator $\hat{J}^\mu(0)$.

The action of $\hat{J}^\mu(0)$ in $\tilde{H}$ is defined by the operator $\hat{J}^\mu$ such that

$$\hat{J}^\mu(0) = A U \hat{J}^\mu U^{-1} A^{-1}$$  \hspace{1cm} (3.4)

It will be convenient to define the action of $\hat{J}^\mu$ in $\tilde{H}$ as follows

$$\hat{J}^\mu \tilde{\varphi}(G) = 2 \int \hat{M}_{int}^{3/2} \hat{J}^\mu(G, G') \hat{M}_{int}^{3/2} \tilde{\varphi}(G') d\rho(G')$$  \hspace{1cm} (3.5)

where the kernel $\hat{J}^\mu(G, G')$ is an operator in $H_{int}$ for any fixed values of $G$ and $G'$. Since $\hat{J}^\mu$ is the selfadjoint operator in $\tilde{H}$, the kernel must satisfy the property $\hat{J}^\mu(G, G')^* = \hat{J}^\mu(G', G)$ where $^*$ means the Hermitian conjugation in $H_{int}$ (in the general case the property of an operator to be selfadjoint is stronger than to be Hermitian but we shall not discuss this question).

It is obvious that the first expression in Eq. (3.2) will be satisfied if

$$\tilde{U}(l)^{-1} \hat{J}^\mu \tilde{U}(l) = L(l)^\mu_\nu \hat{J}^\nu$$  \hspace{1cm} (3.6)

The action of the operators $\tilde{U}(l)$ is defined by Eq. (2.56) and therefore, using Eq. (3.5), one can show that Eq. (3.6) is satisfied if

$$\hat{J}^\mu(G, G') = L(l)^\mu_\nu D[S; \alpha(G)^{-1} l \alpha(L(l)^{-1} G)] \hat{J}^\nu(L(l)^{-1} G, L(l)^{-1} G') D[S; \alpha(G')^{-1} l \alpha(L(l)^{-1} G')]^{-1}$$  \hspace{1cm} (3.7)

Note that in contrast with the notations of Chap. 2 the quantities $G$ and $G'$ in this expression are arbitrary and $l$ is an arbitrary element of the group SL(2,C).

We use $\alpha(G, G')$ to denote $\alpha((G + G')/|G + G'|) \in$SL(2,C) and $L(G, G')$ to denote the Lorentz transformation $L[\alpha(G, G')]$. We also introduce the 4-vectors

$$f = L(G, G')^{-1} G, \quad f' = L(G, G')^{-1} G'$$  \hspace{1cm} (3.8)

These 4-vectors are constructed as the c.m.frame 4-velocities of two particles with unit masses and the 4-velocities $G$ and $G'$ (compare with Eq. (2.8)). Let us note that this is only a formal construction since
\(G\) and \(G'\) in Eq. (3.5) have the sense of the 4-velocities of one and the same system in the final and initial states. Nevertheless, as follows from Eq. (3.8), the 4-vectors \(f\) and \(f'\) are such that
\[
f^2 = f'^2 = 1, \quad f + f' = 0, \quad f^0 = f'^0 = (1 + f^2)^{1/2} \tag{3.9}
\]
Therefore the 4-vectors \(f\) and \(f'\) are fully determined by one three-dimensional vector \(h = f/f^0\). Note that \(h < 1\).

If the operator \(\hat{J}^\mu\) satisfies Eq. (3.6) then, as follows from Eq. (3.7),
\[
\hat{J}^\mu(G, G') = L(G, G')\rho D[S; \alpha(G)^{-1}\alpha(G, G')\alpha(f)]j^\nu(h) \cdot D[S; \alpha(G')^{-1}\alpha(G, G')\alpha(f')^{-1}] \tag{3.10}
\]
where we use \(j^\nu(h)\) to denote \(\hat{J}^\mu(f, f')\). As follows from Eq. (3.8), the condition for the Hermiticity of the operator \(\hat{J}^\mu(G, G')\) (see the remark after Eq. (3.5)) will be satisfied if and only if \(j^\nu(h)^* = j^\nu(-h)\). We see that if Eq. (3.6) is satisfied then the kernel of the operator \(\hat{J}^\mu\) is fully determined by an operator the action of which in \(H_{int}\) depends only on \(h\). As follows from Eq. (3.7) and the fact that \(u\alpha(G) = \alpha(L(u)G)u\) if \(u \in SU(2)\), the operator \(j^\nu(h)\) has the property
\[
\hat{J}^\nu(h) = L(u)^\nu_\rho D[S; u]\hat{J}^\rho(L(u)^{-1}h)D[S; u]^{-1} \tag{3.11}
\]
In particular, since \(L(u)\) is the three-dimensional rotation corresponding to \(u \in SU(2)\), Eq. (3.11) yields
\[
\hat{J}^0(h) = D[S; u]\hat{J}^0(L(u)^{-1}h)D[S; u]^{-1}, \quad \hat{J}(h) = L(u)D[S; u]\hat{J}(L(u)^{-1}h)D[S; u]^{-1} \tag{3.12}
\]
On the contrary, let \(j^\nu(h)\) be an operator satisfying Eq. (3.12) and Eq. (3.10) be treated as the definition of \(\hat{J}^\mu(G, G')\). Then we have to verify that Eq. (3.7) is satisfied. This can be done directly taking into account that the 4-vectors \(f(l)\) and \(f'(l)\) which are constructed from \(L(l)^{-1}G\) and \(L(l)^{-1}G'\) in the same manner as \(f\) and \(f'\) are constructed from \(G\) and \(G'\), are connected with \(f\) and \(f'\) by the three-dimensional rotation:
\[
f = L[\alpha(G, G')^{-1}l\alpha(L(l)^{-1}G, L(l)^{-1}G')]f(l),
\]
\[
f' = L[\alpha(G, G')^{-1}l\alpha(L(l)^{-1}G, L(l)^{-1}G')]f'(l) \tag{3.13}
\]
We conclude that the first expression in Eq. (3.2) will be satisfied if\( \hat{J}^\mu(G, G') \) is expressed in terms of \( \hat{j}^\nu(h) \) according to Eq. (3.10) and \( \hat{j}^\nu(h) \) satisfies Eq. (3.12).

Let us now consider the second expression in Eq. (3.2) which is the consequence of the continuity equation. As easily follows from Eq. (3.5), this expression is satisfied if and only if

\[
\hat{M}_{\text{int}} G^\mu \hat{J}^\mu(G, G') - \hat{J}^\mu(G, G') \hat{M}_{\text{int}} G'^\mu = 0 \quad (3.14)
\]

In turn, as follows from Eqs. (3.3) and (3.8-3.10), Eq. (3.14) is satisfied if and only if

\[
[\hat{M}_{\text{int}}, \hat{j}^0(h)] = h\{\hat{M}_{\text{int}}, \hat{j}(h)\} \quad (3.15)
\]

where we use curly brackets to denote the anticommutator. Therefore Eq. (3.13) is the sufficient condition ensuring the continuity equation.

Let us consider some simple consequences of Eq. (3.15). If \( h = 0 \) then

\[
[\hat{M}_{\text{int}}, \hat{j}^0(0)] = 0 \quad (3.16)
\]

It will be shown in Sec. 3.3 that \( \hat{j}^0(0) = e \) where \( e \) is the electric charge of the system under consideration. Therefore \( \hat{j}^0(0) \) must indeed commute with \( \hat{M}_{\text{int}} \).

Taking the derivative of Eq. (3.15) over \( h \) at \( h = 0 \) we get

\[
[\hat{M}_{\text{int}}, \partial_\h \hat{j}^0(0)] = \{\hat{M}_{\text{int}}, \hat{j}(0)\} \quad (3.17)
\]

Let \( \Pi_i \) be the projector onto the subspace in \( H_{\text{int}} \) corresponding to the discrete eigenvalue \( m_i \) of the operator \( \hat{M}_{\text{int}} \). Then as follows from Eq. (3.17)

\[
\Pi_i \hat{j}(0) \Pi_j = \frac{m_i - m_j}{m_i + m_j} \Pi_i \frac{\partial \hat{j}(0)}{\partial h} \Pi_j \quad (3.18)
\]

In particular

\[
\Pi_i \hat{j}(0) \Pi_i = 0 \quad (3.19)
\]

Assuming that \( \hat{j}^0(h) \) and \( \hat{j}(h) \) are analytic functions of \( h \) in the vicinity of the point \( h = 0 \) we can expand both parts of Eq. (3.13)
in powers of $h$ and equate the terms containing equal powers of $|h|$. Then we get

$$[\hat{M}^{\text{int}}, \frac{\partial^{n+1} \hat{j}_0(0)}{\partial h_{i_1} \cdots \partial h_{i_{n+1}}}] = \{\hat{M}^{\text{int}}, \sum_{l=1}^{n} \frac{\partial^{n} \hat{j}_l(0)}{\partial h_{i_1} \cdots \partial h_{i_{l-1}} \partial h_{i_{l+1}} \cdots \partial h_{i_{n+1}}}\}$$

(3.20)

where $i_1, \ldots, i_{n+1}=1,2,3$, $n = 0, 1, \ldots$ To investigate the consequences of Eq. (3.15) in more details it is convenient to use the spectral decomposition of the ECO. This is done in the next section.

### 3.2 Spectral decomposition of the ECO

Let $\hat{\varepsilon}_{\text{int}}(m)$ be the spectral function of the operator $\hat{M}^{\text{int}}$. In the general case $\hat{J}(G, G')$ does not commute with $\hat{M}^{\text{int}}$ and therefore we cannot write the spectral decomposition of $\hat{J}(G, G')$ by analogy with Eq. (2.32). However we can define $\hat{J}(G, G')$ using a set of operators $\hat{J}(G, m; G', m')$ such that

$$\hat{J}(G, G') = \int \int d\hat{\varepsilon}_{\text{int}}(m) \hat{J}(G, m; G', m') d\hat{\varepsilon}_{\text{int}}(m')$$

(3.21)

where the integrals are understood as the strong limits of the corresponding Riemann sums constructed as follows. Let $S$ be the spectrum of the operator $\hat{M}^{\text{int}}$ and $\{S_\alpha\}$ be a partition of $S$ such that

$$\bigcap_{\alpha} S_\alpha = \emptyset, \quad \bigcup_{\alpha} S_\alpha = S$$

(3.22)

Let $\Pi_\alpha$ be the projector onto the subspace of $H_{\text{int}}$ corresponding to $S_\alpha$. Then $\hat{J}(G, G')$ can be approximated by the sums

$$\hat{J}(G, G') = \sum_{\alpha, \beta} \Pi_\alpha \hat{J}(G, m_\alpha; G', m_\beta) \Pi_\beta$$

(3.23)

where $m_\alpha \in S_\alpha$.

It is easy to see that Eq. (3.14) will be satisfied if

$$(mG_\mu - m'G'_\mu) \hat{J}(G, m; G', m') = 0$$

(3.24)

Analogously we can write the spectral decomposition of $\hat{j}^\nu(h)$ in the form

$$\hat{j}^\nu(h) = \int \int d\hat{\varepsilon}_{\text{int}}(m) \hat{j}^\nu(h; m, m') d\hat{\varepsilon}_{\text{int}}(m')$$

(3.25)
and the relation between $\hat{J}^\mu(G, m; G', m')$ and $\hat{j}^\nu(h; m, m')$ is obvious from Eq. (3.10). As follows from Eqs. (3.10), (3.13) and (3.24), the continuity equation will be satisfied if

$$(m - m')\hat{j}^0(h; m, m') = (m + m')\mathbf{h}\hat{j}(h; m, m')$$ (3.26)

and this result is also obvious from Eq. (3.13).

The operators $\hat{j}^\nu(h; m, m')$ are simply related to $\hat{j}^\nu(h)$ if $m$ and $m'$ belong to the discrete spectrum of the operator $\hat{M}^\text{int}$:

$$\hat{j}^\nu(h; m_i, m_j) = \prod_i \hat{j}^\nu(h) \prod_j$$ (3.27)

The analogous relation takes place for the operator $\hat{J}^\mu(G, m; G', m')$.

However if $m$ and $m'$ belong to the continuous spectrum of the operator $\hat{M}^\text{int}$, the operators $\hat{J}^\mu(G, m; G', m')$ and $\hat{j}^\nu(h; m_i, m_j)$ should be handled with care since these operators are the operatorial distributions. For example, as follows from Eq. (3.26)

$$(m - m')\hat{j}^0(0; m, m') = 0$$ (3.28)

(compare with Eq. (3.16)) but this does not necessarily imply that $\hat{j}^0(0; m, m') = 0$ since, as follows from Eq. (3.25)

$$\int \int d\mathbf{e}^{\text{int}}(m)\hat{j}^0(0; m, m')d\mathbf{e}^{\text{int}}(m') = e$$ (3.29)

We see that $\hat{j}^0(0; m, m')$ is the operatorial analog of $e\delta(m - m')$.

Assuming that $\hat{j}^\nu(h; m, m')$ is the analytic function of $h$ we obtain from Eq. (3.26) (compare with Eq. (3.20))

$$\frac{m - m'}{m + m'} \frac{\partial^{n+1}\hat{j}^0(0; m, m')}{\partial h_{i_1} \cdots \partial h_{i_{n+1}}} = \sum_{l=1}^n \frac{\partial^n \hat{j}_{i_l}(0; m, m')}{\partial h_{i_1} \cdots \partial h_{i_{l-1}} \partial h_{i_{l+1}} \cdots \partial h_{i_{n+1}}}$$ (3.30)

In particular (compare with Eq. (3.17))

$$\frac{m - m'}{m + m'} \frac{\partial \hat{j}^0(0; m, m')}{\partial h} = \hat{j}(0; m, m')$$ (3.31)

If $m$ and $m'$ belong to the discrete spectrum of the operator $\hat{M}^\text{int}$, this expression is equivalent to Eq. (3.18) if we take into account Eq. (3.27).
However Eq. (3.31) does not necessarily imply that $\hat{j}(0; m, m') = 0$ if $m$ belongs to the continuous spectrum of the operator $\hat{M}^{\text{int}}$. The analog of such a situation is the equality $(m - m') \frac{d\delta (m - m')}{dm'} = \delta (m - m')$. We shall see in Sec. 4.2 that for systems of two free particles

$$\hat{j}(0; m, m') = 0 \text{ if } m \neq m';$$

$$\int \int d\hat{e}^{\text{int}}(m) \hat{j}(0; m, m') d\hat{e}^{\text{int}}(m') = \hat{j}(0) \quad (3.32)$$

The theory of operatorial integrals is given in several monographs (see, for example, refs. [92, 93]). To avoid singularities in practical calculations of the sums defined by Eq. (3.23) it is desirable to choose the points $m_\alpha$ and $m_\beta$ in such a way that $m_\alpha \neq m_\beta$ when $m_\alpha$ and $m_\beta$ belong to the continuous spectrum of the operator $\hat{M}^{\text{int}}$.

### 3.3 Charge operator and cluster separability

Our next goal is to express the action of the charge operator in $\tilde{H}$ in terms of $\hat{j}''(h)$. As follows from Eqs. (3.3), (3.5) and (3.10) this action is given by

$$\tilde{Q}\tilde{\varphi}(G) = 2 \int \int \hat{M}^{3/2}_{\text{int}} \exp[i\hat{M}_{\text{int}}(Gx)]L(G, G')^\mu \Gamma^\nu[L^\lambda (\lambda x - \tau) d^4x d\rho(G')]$$

where for brevity we do not write the arguments of the D-operators. Instead of $x$ we introduce the 4-vector $y = L(G, G')^{-1}x$ and use $z = z(G, G')$ to denote the 4-vector $L(G, G')^{-1}\lambda$. Then using Eqs. (3.8), (3.9) and (3.25) we can integrate over $y$ and obtain

$$\tilde{Q}\tilde{\varphi}(G) = 2(2\pi)^3 \int \int (mm')^{3/2} \exp[i(m - m') f^0 \frac{\tau}{z^0}] \cdot$$

$$\delta^3[(m - m') f^0 \frac{z}{z^0} - (m + m') f]D[...] d\hat{e}^{\text{int}}(m) [\hat{j}''(h; m, m')]$$

$$\frac{z}{z^0} \hat{\varphi}(h; m, m') d\hat{e}^{\text{int}}(m') D[...]^{-1} \varphi(G') d\rho(G') \quad (3.34)$$

Note that $\hat{e}_{\text{int}}(m)$ and $D[...]$ commute with each other since $\hat{M}_{\text{int}}$ commutes with $S$. 

It is easy to see that only those \( m' \) contribute to Eq. (3.34) which are in the infinitely small vicinity of the point \( m' = m \). Indeed, if \( m' \neq m \) then the argument of the delta-function can be equal to zero only if \( z/z^0 = (m + m')h/(m - m') \), and the integrand in Eq. (3.34) becomes zero as follows from Eq. (3.26). We see that the exponent with \( \tau \) in Eq. (3.34) can be dropped and indeed one of the consequence of the continuity equation in the form of Eq. (3.26) is that \( \hat{Q} \) does not depend on \( \tau \). The second consequence is that only an infinitely small vicinity of the point \( h \) contributes to Eq. (3.34). Since \( G' \) can be expressed as the function of \( G \) and \( h \) and it is obvious from Eq. (3.8) that \( G = G' \) if \( h = 0 \), we can replace \( G' \) by \( G \) in the arguments of the D-operators, the function \( \varphi \) and the 4-vector \( z \) entering into the integrand of Eq. (3.34). Then, as follows from Eq. (3.10), the D-operators in Eq. (3.34) can be dropped and \( z \) becomes the 4-vector \( z = z(G) = L[\alpha(G)]^{-1}\lambda \).

By analogy with Eq. (2.9) we can write neglecting the order-\( o(h) \) terms
\[
G' = -h + \frac{G + G'}{2} - \frac{(Gh)G}{1 + G^0} \tag{3.35}
\]
where we have taken into account that \( |G + G'| = 2(1 + f^0) \approx 2 \). As easily follows from this expression, we can replace \( d\rho(G') \) by \( 4d^3h/(2\pi)^3 \) in Eq. (3.34). Therefore Eq. (3.34) can be written in the form
\[
\tilde{Q}\tilde{\varphi}(G) = \int \int d\hat{e}^{int}(m)[j^0\left(\frac{m - m'}{m + m'}z^0(G); m, m'\right) - \frac{z(G)}{z^0(G)} \hat{j}\left(\frac{m - m'}{m + m'}z^0(G); m, m'\right)d\hat{e}^{int}(m')d^3h\tilde{\varphi}(G) \tag{3.36}
\]
We again see that the integrand is equal to zero if \( m \neq m' \) as easily follows from Eq. (3.26).

The contribution of the discrete spectrum to the integral (3.36) can be easily determined since, as follows from Eqs. (3.19) and (3.27), each point \( m_i \) of the discrete spectrum contributes \( \Pi_i j^0(0)\Pi_i \) to this integral. In the general case, expanding \( j^0 \) and \( j \) in Eq. (3.30) in powers of the first argument and using Eq. (3.30) we see that only \( j^0(0; m, m') \) survives in this expansion while all other terms cancel. Therefore we
conclude that
\[ \tilde{Q} = \int \int d\hat{e} \int \nu \left( m, m' \right) d\hat{e} \int \nu \left( m' \right) = \tilde{\nu} \left( 0 \right) \] (3.37)

If the particles do not interact with each other then, as follows from cluster separability (see Sec. 1.1), the operator \( \hat{J}^{\mu} \left( x \right) \) is a sum of the constituent ECO’s and therefore, as follows from Eq. (1.3), the system electric charge \( e \) is the sum of the electric charges of constituents as it should be. Therefore, as follows from Eq. (3.37), the operator \( \hat{j}^{\nu} \left( \mathbf{h} \right) \) must depend on interactions only in such a way that \( \hat{j}^{\nu} \left( 0 \right) = j^{\nu} \left( 0 \right) = e \).

Let us now consider the conditions imposed on \( \hat{j}^{\nu} \left( \mathbf{h} \right) \) by cluster separability in the general case. If our system consists of \( n \) noninteracting subsystems \( \alpha_1, \ldots, \alpha_n \) and the ECO’s \( \hat{J}^{\mu} \left( x \right) \) for them are known then, as follows from cluster separability, the ECO for the whole system is equal to
\[ \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \left( x \right) = \sum_{i=1}^{n} \hat{J}^{\mu} \left( \alpha_i \right) \] (3.38)
Therefore the operator \( \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \left( 0 \right) \) is also known and the action of this operator in \( \tilde{H} \) is defined by the operator \( \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \) such that (see Eq. (3.4))
\[ \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \left( 0 \right) = A_{\alpha_1, \ldots, \alpha_n} U \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) U^{-1} A_{\alpha_1, \ldots, \alpha_n}^{-1} \] (3.39)
In turn, if \( \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \) is known, we can determine the operator
\[ \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \left( G, G' \right) \]
such that (see Eq. (3.3))
\[ \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \left( G \right) = 2 \int \left( M^{int} \right)_{\alpha_1, \ldots, \alpha_n}^{3/2} \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \left( G, G' \right) \cdot \left( M^{int} \right)_{\alpha_1, \ldots, \alpha_n}^{3/2} \varphi \left( G' \right) d\rho \left( G' \right) \] (3.40)
where \( M_{\alpha_1, \ldots, \alpha_n} \) is defined by Eq. (2.74). Finally, the operator
\[ \hat{J}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \left( G, G' \right) \]
determines the operator \( \hat{j}^{\nu} \left( \alpha_1, \ldots, \alpha_n \right) \left( \mathbf{h} \right) \) such that (see Eq. (3.10))
\[ \hat{j}^{\mu} \left( \alpha_1, \ldots, \alpha_n \right) \left( G, G' \right) = L \left( G, G' \right) D \left[ S; \alpha \left( G \right)^{-1} \alpha \left( G', G' \right) \alpha \left( f' \right) \right] \cdot \hat{j}^{\nu} \left( \alpha_1, \ldots, \alpha_n \right) \left( \mathbf{h} \right) D \left[ S; \alpha \left( G' \right)^{-1} \alpha \left( G, G' \right) \alpha \left( f' \right) \right]^{-1} \] (3.41)
We see that if the ECO’s for all subsystems are known then it is possible to determine the operators \( \hat{J}^\nu_{\alpha_1...\alpha_n}(h) \) corresponding to the case when the subsystems \( \alpha_1, ..., \alpha_n \) do not interact with each other. Let us now compare Eqs. (3.4), (3.5) and (3.10) on the one hand and Eqs. (3.39-3.41) on the other. If all interactions between the subsystems \( \alpha_1, ..., \alpha_n \) are turned off then \( A \) becomes \( A_{\alpha_1...\alpha_n} \) and \( \hat{M}_{\text{int}} \) becomes \( M_{\alpha_1...\alpha_n} \) by construction (see Sec. 2.4); therefore cluster separability will be satisfied if \( \hat{J}^\nu(h) \) becomes \( \hat{J}^\nu_{\alpha_1...\alpha_n}(h) \).

### 3.4 General method of constructing the ECO

The results of Secs. 3.1 and 3.3 can be formulated as follows. The operator \( \hat{J}^\mu(x) \) satisfies all the conditions described in Sec. 1.1 if the operator \( \hat{J}^\nu(h) \) satisfies Eqs. (3.12) and (3.15), \( \hat{J}^0(0) \) is the operator of multiplication by \( e \), \( \hat{J}^\nu(h) \) becomes \( \hat{J}^\nu_{\alpha_1...\alpha_n}(h) \) if for any set \( (\alpha_1, ..., \alpha_n) \) the interactions between the subsystems \( \alpha_1, ..., \alpha_n \) are turned off and \( \hat{J}^\nu(h)^* = \hat{J}^\nu(-h) \) (in some cases it is convenient to use Eq. (3.15) in the form of Eq. (3.20)). Therefore the problem of constructing the operator \( \hat{J}^\mu(x) \) will be solved if we succeed in constructing the operator \( \hat{J}^\nu(h) \) satisfying the above properties.

In turn the latter problem can be reduced as follows. First, using Eqs. (3.16) and (3.17), we can write Eq. (3.15) in the form

\[
\begin{align*}
[\hat{M}_{\text{int}}^0, \hat{J}^0(0)] - \hat{J}^0(0) - h\left(\partial\hat{J}^0(0)/\partial h\right) &= h\left(\hat{M}_{\text{int}}^1, \hat{J}(h) - \hat{J}(0)\right) \quad (3.42)
\end{align*}
\]

Since \( \hat{J}(h) - \hat{J}(0) \) becomes zero when \( h \to 0 \), we can uniquely decompose this operator into longitudinal and transverse parts:

\[
\hat{J}(h) = \hat{J}(0) + \frac{h}{|h|}\hat{J}_{\parallel}(h) + \hat{J}_{\perp}(h) \quad (3.43)
\]

where \( h\hat{J}_{\perp}(h) = 0 \) and

\[
\begin{align*}
\hat{J}_{\parallel}(h) &= \frac{1}{|h|}(h, \hat{J}(h) - \hat{J}(0)), \\
\hat{J}_{\perp}(h) &= \hat{J}(h) - \hat{J}(0) - \frac{h}{|h|^2}(h, \hat{J}(h) - \hat{J}(0)) \quad (3.44)
\end{align*}
\]
Now, as follows from Eqs. (3.42) and (3.43), Eqs. (3.15) and (3.26) can be written in the form

\[ [\hat{M}^{\text{int}}, \hat{j}^0(h) - j^0(0) - h \frac{\partial \hat{j}^0(0)}{\partial h}] = h\{\hat{M}^{\text{int}}, \hat{j}(h)\}, \]

\[ \hat{j}_{||}(h; m, m') = \frac{(m - m')}{(m + m')} |h| [\hat{j}^0(h; m, m') - \frac{\partial \hat{j}^0(0; m, m')}{\partial h}] \]

Equations (3.31) and (3.45) show that if \( \hat{j}^0(h) \) is known then \( \hat{j}(0) \) and \( \hat{j}_{||}(h) \) can be uniquely determined from the continuity equation while, as follows from Eqs. (3.43) and (3.45), the continuity equation does not impose any constraint on the transverse part \( \hat{j}_{\perp}(h) \) of the operator \( \hat{j}(h) \).

We conclude that the problem of constructing the operators \( \hat{J}^\mu(x) \) will be solved if we succeed in constructing the operators \( \hat{j}^0(h) \) and \( \hat{j}_{\perp}(h) \) such that (see Eqs. (3.12) and (3.37))

\[ \hat{j}^0(h) = D[S; u] \hat{j}^0(L(u)^{-1}h) D[S; u]^{-1}, \]

\[ \hat{j}_{\perp}(h) = L(u) D[S; u] \hat{j}_{\perp}(L(u)^{-1}h) D[S; u]^{-1}, \]

\( \hat{j}^0(0) = e \), the operators \( \hat{j}^0(h) \) and \( \hat{j}_{\perp}(h) \) satisfy cluster separability in the sense described above, and \( \hat{j}^0(h)^* = \hat{j}^0(-h), \hat{j}_{\perp}(h)^* = \hat{j}_{\perp}(-h) \).

In addition, it is easy to show that if the theory should be P and T invariant then, as follows from Eq. (1.4)

\[ \tilde{U}_P(\hat{j}^0(h), \hat{j}_{\perp}(h)) \tilde{U}_P^{-1} = (\hat{j}^0(-h), -\hat{j}_{\perp}(-h)), \]

\[ \tilde{U}_R \hat{j}^\nu(h) \tilde{U}_R^{-1} = \hat{j}^\nu(h) \]

where the operators \( \tilde{U}_P \) and \( \tilde{U}_R \) are defined by Eq. (2.17).

In Chap. 4 we consider how the problem of constructing such operators can be solved in the cases of one, two, three and many particles.
3.5 Matrix elements of the ECO

Let us define the one-particle states with a definite 4-momentum \( p' = mg' \) and the spin projections \( \sigma' \). In the scattering theory such states are usually normalized as

\[
\langle p'', \sigma'' | p', \sigma' \rangle = 2(2\pi)^3 \omega(p') \delta^{(3)}(p'' - p') \delta_{\sigma'' \sigma'}
\]

where \( \delta_{\sigma'' \sigma'} \) is the Kronecker symbol. Therefore if the scalar product is chosen according to Eq. (2.2) then the state \( |p', \sigma'\rangle \) depends on \( g' \) and \( \sigma \) as follows

\[
|p', \sigma'\rangle = \frac{2}{m} (2\pi)^3 g^0 \delta^{(3)}(g - g') \delta_{\sigma\sigma'}
\]

To define the scattering states in the N-particle case we have to solve the eigenvalue problem for the operator \( \hat{M}_{int} \) in the space \( H_{int} \). Let \( \chi' \in H_{int} \) be the internal wave function of a bound state with the mass \( M' \) and \( G' \) be the 4-velocity of this bound state. Then, as follows from Eq. (2.71), the wave function of such a state in the space \( H \) can be written as

\[
|P', \chi'\rangle = A \left( \frac{2}{M'} \right) (2\pi)^3 G^0 \delta^{(3)}(G - G') \chi'
\]

where \( P' = M'G' \). If \( P'' = M''G'' \) and \( \chi'' \) are the total 4-momentum and the internal wave function in the final state then it is obvious from Eq. (3.50) that (compare with Eq. (3.48))

\[
\langle P'', \chi'' | P', \chi' \rangle = 2(2\pi)^3 (M'^2 + P'^2)^{1/2} \delta^{(3)}(P'' - P') \langle \chi'' | \chi' \rangle
\]

Here the last scalar product is taken in the space \( H_{int} \). It is obvious that \( \langle \chi'' | \chi' \rangle = 0 \) if \( M'' \neq M' \).

In the general case the particles in the initial or final state can be not only bound but the system under consideration may consist of \( n \) bound subsystems \( (1 \leq n \leq N) \). Then the system wave function can be also written in the form of Eq. (3.50) but \( \chi' \) should be a generalized eigenfunction of the operator \( \hat{M}_{int} \). The construction of such functions (\(|\text{in}\rangle \) or \(|\text{out}\rangle \) states) requires a special consideration in each concrete case, and we shall not discuss this problem.
Since the functions defined by Eq. (3.50) form a (generalized) basis in $H$, all matrix elements of the operator $\hat{J}^\mu(x)$ can be expressed in terms of the quantities $\langle P^\prime, \chi^\prime | \hat{J}^\mu(x) | P^\prime, \chi^\prime \rangle$. As follows from Eqs. (3.1), (3.4), (3.5), (3.10) and (3.50)

\[
\langle P^\prime, \chi^\prime | \hat{J}^\mu(x) | P^\prime, \chi^\prime \rangle = 2(M^\prime M^\mu)^{1/2} \exp(i\Delta x) L(G^\prime, G')^{\mu}\nu \cdot \\
\cdot \langle \chi^\prime | D[S; \alpha(G^\prime)^{-1}\alpha(G^\prime, G')\alpha(f)] j^\nu(h) \cdot \\
\cdot D[S; \alpha(G^\prime)^{-1}\alpha(G^\prime, G')\alpha(f')^{-1}] \chi^\prime \rangle 
\] 

(3.52)

where $\Delta = P^\prime - P'$, $f$ and $f'$ are defined by Eq. (3.8) with $G$ replaced by $G''$, and the matrix element on the right-hand-side must be calculated only in the space $H_{int}$.

One of the consequences of Eq. (3.52) is that the matrix elements of the ECO in the N-particle case do not depend on the way of constructing the N-particle operator $A$. In particular, the matrix elements in the three-particle case indeed do not depend on the way of constructing the operator $A$ in terms of $A_{123}, A_{132}$ and $A_{231}$ (see Sec. 2.3). However this does not imply that the operators $A$ play only a formal role since the operator $\hat{M}_{int}$ for the N-particle system depends on the operators $A$ for its subsystems.

The main conclusion of this section is that the matrix elements of the ECO are fully defined by the operator $\hat{j}^\nu(h)$ which must satisfy the properties mentioned in Sec. 3.4. Let us also note that the matrix element given by Eq. (3.52) has especially simple form if $G'' + G' = 0$ and in this case

\[
\langle P^\prime, \chi^\prime | \hat{J}^\mu(x) | P^\prime, \chi^\prime \rangle = 2(M^\prime M^\mu)^{1/2} \exp(i\Delta x) \langle \chi^\prime | \hat{j}^\mu(h) | \chi^\prime \rangle 
\] 

(3.53)

It is interesting to note that the Breit frame is defined by the condition $P'' + P' = 0$ which is not equivalent to $G'' + G' = 0$ if $M'' \neq M'$. 


Chapter 4

Explicit construction of the electromagnetic current operator for different systems

4.1 One-particle ECO

In the one-particle case the method of constructing the operator $J_\mu(x)$ given in Sec. 3.4 leads to the well-known results. Nevertheless we briefly describe some of these results since they are used in the following.

Now the role of $H_{int}$ is played by $\mathcal{D}(s)$ (see Sec. 2.1). Therefore $j^\nu(h)$ is the operator which acts only through the spin variables. Since the spectrum of the mass operator consists only of one point corresponding to the particle mass $m$, it follows from Eqs. (3.19) and (3.45) that $j(0) = j_\parallel(h) = 0$ and, as follows from Eq. (3.43) $j(h) = j_\perp(h).

The number of independent functions determining the structure of the operator $j^\nu(h)$ depends on whether in addition to Poincare invariance the P and T invariance are also assumed (see ref. [44] for details). We suppose that this is the case.

For the spinless particle the most general choice of $j^\nu(h)$ under the above conditions is the following

$$
j^0(h) = \frac{e}{(1 - h^2)^{1/2}}F_E\left(-\frac{4m^2h^2}{(1 - h^2)^{1/2}}\right), \quad j(h) = 0 \quad (4.1)
$$

where $F_E$ is some real function such that $F_E(0) = 1$. Then as follows from Eqs. (2.2), (3.4), (3.5) and (3.10) the action of the operator
\( J^\mu(0) = J^\mu \) is given by

\[
J^\mu \varphi(g) = em^2 \int F_E((p - p')^2)(p + p')^\mu \varphi(g')d\rho(g')
\]  (4.2)

where \( p = mg \) and we take into account that, as follows from Eqs. (2.8), (2.9) and (3.9)

\[
|g + g'| = \frac{2}{(1 - h^2)^{1/2}}, \quad (p - p')^2 = -\frac{4m^2h^2}{(1 - h^2)^{1/2}}
\]  (4.3)

In turn, as follows from Eqs. (2.2), (3.2), (3.48) and (4.2)

\[
\langle p'', \sigma'' | J^\mu(x) | p', \sigma' \rangle = eF_E(\Delta^2)e^{i\Delta x}((p'' + p')^\mu
\]  (4.4)

where \( \Delta = p'' - p' \). This is the well known form of the matrix element of the ECO in the spinless case and it is well-known that \( F_E \) has the sense of the electric form factor.

If the spin of the particle under consideration is equal to 1/2, then the most general choice of \( j''(h) \) is the following

\[
j^0(h) = eF_E(-\frac{4m^2h^2}{(1 - h^2)^{1/2}}),
\]

\[
j(h) = -\frac{2ie}{(1 - h^2)^{1/2}}F_M(-\frac{4m^2h^2}{(1 - h^2)^{1/2}})(h \times s)
\]  (4.5)

where \( F_E \) and \( F_M \) are some real functions and \( F_E(0) = 1 \).

Using Eqs. (2.2), (3.4), (3.5) and (3.10) one can verify that instead of Eq. (4.2)

\[
J^\mu \varphi(g) = 2m^3 \int J^\mu(g, g') \varphi(g')d\rho(g')
\]  (4.6)

where \( J^\mu(g, g') \) acts in \( D(s) \) as

\[
J^\mu(g, g')\chi(\sigma) = \sum_{\sigma'} \bar{u}(p, \sigma)[(F_E((p - p')^2) - F_M((p - p')^2)) \cdot
\]

\[
\frac{(p + p')^\mu}{(p + p')^2} + \frac{1}{2m}F_M((p - p')^2)\gamma^\mu]u(p', \sigma')\chi(\sigma'),
\]  (4.7)

\( u(p, \sigma) \) is the Dirac bispinor describing the particle with the 4-momentum \( p \) and the spin projection \( \sigma \), \( \gamma^\mu \) is the Dirac \( \gamma \) matrix and
\[ \bar{u} = u^* \gamma^0. \] In turn, as follows from Eqs. (2.2), (3.2), (3.48), (4.6) and (4.7)

\[
\langle p^\prime, \sigma^\prime | J^\mu(x) | p', \sigma' \rangle = e \cdot exp(i \Delta x) \bar{u}(p^\prime, \sigma^\prime)[2m(F_E(\Delta^2) - F_M(\Delta^2))\langle p^\prime + p' \rangle^\mu] + F_M(\Delta^2)\gamma^\mu u(p', \sigma') \tag{4.8}
\]

and, since this is the well known expression, we conclude that \( F_E \) and \( F_M \) are just the Sachs electric and magnetic form factors. Therefore \( F_M(0) \) is the particle magnetic moment in units \( e/2m \).

For the particle with unit spin a possible choice of \( j''(h) \) is

\[
\begin{align*}
j^0(h) &= \frac{e}{(1 - h^2)^{1/2}} \left\{ F_E(-\frac{4m^2h^2}{(1 - h^2)^{1/2}}) + \frac{2}{(1 - h^2)^{1/2}}F_Q(-\frac{4m^2h^2}{(1 - h^2)^{1/2}})\left[\frac{2}{3}h^2 - (sh)^2\right]\right\}, \\
j(h) &= -\frac{ie}{1 - h^2}F_M(-\frac{4m^2h^2}{(1 - h^2)^{1/2}})(h \times s) \tag{4.9}
\end{align*}
\]

where \( F_E, F_M \) and \( F_Q \) are some real functions and \( F_E(0) = 1 \). Using Eqs. (2.2), (3.2), (3.4), (3.5), (3.10), (3.48) and (4.9) it can be shown that

\[
\begin{align*}
\langle p^\prime, \sigma^\prime | J^\mu(x) | p', \sigma' \rangle &= e \cdot exp(i \Delta x)E'^\alpha E'_\beta \cdot \\
&\cdot \left\{-\langle p^\prime + p' \rangle^\mu\left[F_0(\Delta^2)\eta^{\alpha\beta} + \frac{1}{2m^2}F_2(\Delta^2)\Delta^\alpha \Delta^\beta \right] + \right. \\
&\left. + F_1(\Delta^2)(\eta^{\alpha\mu} \Delta^\beta - \eta^{\beta\mu} \Delta^\alpha) \right\} \tag{4.10}
\end{align*}
\]

where \( \alpha, \beta, \mu = 0, 1, 2, 3, \) \( E' \) is the polarization vector in the initial state, \( E'' \) is the polarization vector in the final state and the quantities \( (F_0, F_1, F_2) \) are expressed in terms of \( (F_E, F_M, F_Q) \) as follows:

\[
\begin{align*}
F_E &= \frac{1}{3}(3 + 2\eta)F_0 - 2\eta F_1 - 2\eta(1 + \eta)F_2, \quad F_M = F_1, \\
F_Q &= F_0 - F_1 - (1 + \eta)F_2 \tag{4.11}
\end{align*}
\]

where \( \eta = h^2/(1 - h^2) \). Equations (4.10) and (4.11) are well known, and it is well known that \( F_M(0) \) is the particle magnetic moment in units \( e/2m \) and \( F_Q(0) \) is the particle quadrupole moment in units \( e/m^2 \).
4.2 ECO for systems of two particles

Let us first consider the case when the particles do not interact with each other. Then as follows from Eq. (3.38)

\[ J^\mu(0) = J^\mu_1(0) + J^\mu_2(0) \]  

(4.12)

where \( J^\mu_i(0) \) \((i = 1, 2)\) is the one-particle ECO for particle \( i \). The action of \( J^\mu_1(0) \) in the two-particle Hilbert space \( H \) (see Sec. 2.1) is given by (see Eqs. (3.5))

\[ J^\mu_1(0) \varphi(g_1, g_2) = 2m_1^3 \int J^\mu_1(g_1, g'_1)\varphi(g'_1, g_2)d\rho(g'_1) \]  

(4.13)

and the analogous expression can be written for the action of \( J^\mu_2(0) \).

As follows from Eqs. (2.1) and (2.10), the integral in this expression can be written in the form

\[ \int J^\mu_1(g_1, g'_1)\varphi(g'_1, g_2)d\rho(g'_1) \cdot 2(2\pi)^3g'_2^0\delta^{(3)}(g_2 - g'_2)d\rho(g'_1)d\rho(g'_2) = \int \int J^\mu_1(g_1, g'_1)\varphi(G', q') \cdot g'_2^0\delta^{(3)}(g_2 - g'_2)\frac{M(q')^3d^3q'}{\omega_1(q')\omega_2(q')m_1^2m_2^2}d\rho(G') \]  

(4.14)

where the relations between the variables \( G', q' \) and \( g'_1, g'_2 \) have the same form as the relations between the variables \( G, q \) and \( g_1, g_2 \) given by Eqs. (2.8) and (2.9). It is assumed in the last integral that \( g_i \) \((i = 1, 2)\) are functions of \( G, q \) and \( g'_i \) are functions of \( G', q' \). To integrate over \( q' \) we use Eq. (2.3) for calculating the Jacobian

\[ |\frac{\partial^3g'_2}{\partial^3q'}| = \frac{g'_2^0}{m_2\omega_2(q')} \]  

(4.15)

and therefore, as follows from Eqs. (4.13-4.15), the action of \( J^\mu_1(0) \) in the variables \( G, q \) can be written in the form

\[ J^\mu_1(0)\varphi(G, q) = 2m_1\int J^\mu_1(g_1, g'_1)\varphi(G', q')\frac{M(q')^3}{\omega_1(q')}d\rho(G') \]  

(4.16)

where it is assumed that \( g_1 \) depends on \( G, q \) and \( g'_1 \) depends on \( G', q' \) as above, but \( q' \) depends on \( G, G' \) and \( q \). This dependence can be found
from the condition \( g_2 = g'_2 \) (see Eq. (4.14)) and therefore, as follows from Eq. (2.7)
\[
q'_2 = \alpha(G')^{-1} \alpha(G) q_2
\]
(4.17)
where the 4-vector \( q_2 \) has the components \((\omega_2(q), -q)\) and the 4-vector \( q'_2 \) has the components \((\omega_2(q'), -q')\).

In the case considered in Sec. 2.1 the operator \( A \) entering into Eq. (3.4) is equal to unity and the operator \( U \) is equal to \( U_{12} \) given by Eq. (2.13). Therefore, as follows from Eqs. (2.8), (3.4) and (4.16), the contribution of \( J_1^\mu(0) \) to the operator \( J^\mu \) acting in the two-particle space \( H_{int} \) is given by

\[
J_1^\mu \tilde{\varphi}(G, q) = 2m_1 \int U_{12}(G, q)^{-1} J_1^\mu(L[\alpha(G)] \frac{q_1}{m_1}, L[\alpha(G') \frac{q'_1}{m_1}]) \cdot U_{12}(G, q') \frac{M(q')^3}{\omega_1(q')} \tilde{\varphi}(G', q') d\rho(G')
\]
(4.18)

Then the contribution of \( J_1^\mu(0) \) to the two-particle operator \( j^\nu(h) \) can be easily determined from Eqs. (3.5) and (3.10) and analogously we can determine the contribution of \( J_2^\mu(0) \).

The final results for the operator \( j^\nu(h) \) corresponding to the operator given by Eq. (4.12) can be summarized as follows. Let \( d_1 \) and \( d_2 \) be the vectors defined by the condition that the Lorentz transformation \( L[\alpha(f)^{-1} \alpha(f)] \) transfers the 4-vector \((\omega_2(q), -q)\) to \((\omega_2(d_1), -d_1)\) and \((\omega_1(q), q)\) to \((\omega_1(d_2), d_2)\) (compare with Eq. (4.17)). Using Eq. (2.13) it is easy to show that the explicit expressions for \( d_1 \) and \( d_2 \) in terms of \( q \) and \( h \) are the following:

\[
d_1 = q - \frac{2h}{1 - h^2} [\omega_2(q) - hq], \quad d_2 = q + \frac{2h}{1 - h^2} [\omega_1(q) + hq]
\]
(4.19)

Let \( d_1 = (\omega_1(d_1), d_1) \), \( d_2 = (\omega_2(d_2), -d_2) \) and \( I_i(h) (i = 1, 2) \) be the operators defined by the conditions \( I_i(h) \chi(q) = \chi(d_i) \). Then

\[
j^\nu(h) = \sum_{i=1}^{2} U_{12}(f, q)^{-1} J_i^\nu(L[\alpha(f)] \frac{q_i}{m_i}, L[\alpha(f')] \frac{d_i}{m_i}) \cdot U_{12}(f', d_i) \frac{m_i}{\omega_i(d_i)} [\frac{M(d_i)^3}{M(q)}]^{3/2} I_i(h)
\]
(4.20)
If the particles do not interact with each other, the operator defined by Eq. (4.20) automatically satisfies all the conditions mentioned in Sec. 3.4 since each operator $J_i^\mu(0)$ satisfies these conditions and $J_i^\mu(0)$ acts only through the variables of particle $i$.

Using Eqs. (2.13) and (3.10) it is possible to express $j^\nu(h)$ directly in terms of the single-particle operators $j^\nu_i(h)$. Let $f_i$ and $f_i'$ be the 4-vectors constructed by means of Eq. (3.8) with $G$ replaced by $L[\alpha(f)]q_i/m_i$ and $G'$ replaced by $L[\alpha(f')]d_i/m_i$, and $h_i = f_i/f_i^0$. Then

$$j^\nu(h) = \frac{2}{\omega_i(d_i)} \left[ M(d_i) \right]^{3/2} I_i(h)$$

where $k = 2$ if $i = 1$ and $k = 1$ if $i = 2$.

If $h = 0$ then (see Eq. (4.19)) $d_1 = d_2 = q$ and therefore $I_i(0) = 1$, $h_i = 0$. Since $j_i^0(0) = e_i$ and $j_i(0) = 0$ where $e_i$ is the electric charge of particle $i$ (see Sec. 4.1), it follows from Eq. (4.21) that

$$j_i^0(0) = e_1 + e_2,$$

$$j(0) = q\left(\frac{e_1}{\omega_1(q)} - \frac{e_2}{\omega_2(q)}\right)$$

The first expression obviously follows from Eq. (3.37) and the second expression shows that $j(0)$ commutes with the free mass operator and therefore satisfies Eq. (3.32).

When the particles interact with each other the results of Sec. 3.4 imply that to solve the problem of constructing the operator $\hat{J}^\mu(x)$ we should find the operators $w^0(h)$ and $w_\perp(h)$ such that

$$\hat{j}(h) = j^0(h) + w^0(h), \quad \hat{j}_\perp(h) = j_\perp(h) + w_\perp(h),$$

$$w^0(h) = D[S; u]w^0(L(u)^{-1}h)D[S; u]^{-1},$$
\[ w^0(h)^* = w^0(-h), \quad w^0(0) = 0, \]
\[ w_\perp(h) = L(u) D[S; u] w_\perp(L(u)^{-1}h) D[S; u]^{-1}, \]
\[ w_\perp(h)^* = w_\perp(-h), \]
\[ (4.24) \]

and the operators \( w^0(h) \) and \( w_\perp(h) \) become zero when the interaction is turned off. In addition, it is easy to show that if the theory should be P and T invariant, then, as follows from Eq. (3.47)
\[
\tilde{U}_P(w^0(h), w_\perp(h)) \tilde{U}_P^{-1} = (w^0(-h), -w_\perp(-h)),
\]
\[
\tilde{U}_R(w^0(h), w_\perp(h)) \tilde{U}_R^{-1} = (w^0(h), w_\perp(h)) \quad (4.25)
\]

Then the operator \( \hat{J}^\mu(x) \) is defined by Eqs. (3.1), (3.4), (3.5), (3.10), (3.31), (3.43) and (3.45).

We see that a possible choice for \( w^0(h) \) and \( w_\perp(h) \) is \( w^0(h) = w_\perp(h) = 0 \) and therefore \( \hat{j}^0(h) = j^0(h), \hat{j}_\perp(h) = j_\perp(h) \). Let us note that even in this case \( \hat{j}(0) \neq j(0) \) and \( \hat{j}_\parallel(h) \neq j_\parallel(h) \) since, as follows from Eqs. (3.31) and (3.45), \( \hat{j}(0) \) and \( \hat{j}_\parallel(h) \) depend not only on \( \hat{j}^0(h) \) but also on \( \hat{M}_{int} \). Another reasons for the operator \( \hat{J}^\mu(x) \) to differ from \( J^\mu(x) \) are the presence of \( \hat{M}_{int} \) in Eq. (3.3) and the presence of \( \hat{P} \) in Eq. (3.1).

We conclude that there exists an infinite number of solutions for the two-particle operator \( \hat{J}^\mu(x) \) since there exists an infinite number of sets \( w^0(h) \) and \( w_\perp(h) \) satisfying Eq. (4.24). In particular, a possible solution corresponds to the choice \( w^0(h) = w_\perp(h) = 0 \), but there are no physical grounds to prefer this solution in comparison with the others.

### 4.3 ECO for systems of three particles

If particles \( \alpha \) and \( \beta \) interact with each other and particle \( \gamma \) is free \( (\alpha, \beta, \gamma = 1, 2, 3, \alpha \neq \beta \neq \gamma) \) then, as follows from Eqs. (2.31), (3.38) and (3.39), the action of the three-particle ECO in the three-particle space \( \tilde{H} \) is given by
\[
J^\mu_{\alpha\beta, \gamma} = U_{123}^{-1} U_{\alpha\beta} B_{\alpha\beta, \gamma} (M_{\alpha\beta})^{-1} B_{\alpha\beta, \gamma} (\hat{M}_{int})^{-1} [\hat{J}^\mu_{\alpha\beta} + J^\mu_{\gamma}] B_{\alpha\beta, \gamma} (\hat{M}_{int})^{-1} B_{\alpha\beta, \gamma} (M_{\alpha\beta}) U_{\alpha\beta}^{-1} U_{123} \quad (4.26)
\]
where \( \hat{J}_{\alpha\beta}^\mu \) is the ECO of the interacting system \( \alpha\beta \) in the two-particle space \( \tilde{H}_{\alpha\beta} \). In turn, as follows from Eqs. (3.40) and (3.41), this action is fully determined by the operators \( j^\nu_{\alpha\beta,\gamma}(h) \) such that
\[
J_{\alpha\beta,\gamma}(G, G') = 2 \int (M_{\alpha\beta,\gamma}^{int})^{3/2} J_{\alpha\beta,\gamma}^\mu(G, G') (M_{\alpha\beta,\gamma}^{int})^{3/2} \tilde{\varphi}(G') d\rho(G')
\] (4.27)
\[
\begin{align*}
J_{\alpha\beta,\gamma}(G, G') &= L(G, G')^\mu \cdot D[S; \alpha(G)^{-1} \alpha(G, G') \alpha(f)] \\
\alpha(G')^{-1} \alpha(G, G') \alpha(f')^{-1} &- 1 (4.28)
\end{align*}
\]
where \( S \) is the three-particle spin operator (see Sec. 2.3).

Analogously, if all the three particles do not interact with each other, the action of the three-particle ECO in the three-particle space \( \tilde{H} \) is fully determined by the operator \( j^\nu(h) \) such that
\[
J^\mu = U_{123}^{-1} \sum_{i=1}^{3} J_i^\mu U_{123},
\]
\[
\begin{align*}
J^\mu \tilde{\varphi}(G) &= 2 \int M^{3/2} J^\mu(G, G') M^{3/2} \tilde{\varphi}(G') d\rho(G') \\
J^\mu(G, G') &= L(G, G')^\mu \cdot D[S; \alpha(G)^{-1} \alpha(G, G') \alpha(f)] j^\nu(h) \\
D[S; \alpha(G')^{-1} \alpha(G, G') \alpha(f')^{-1} &- 1 (4.29)
\end{align*}
\]
It is obvious that \( j^\nu_{\alpha\beta,\gamma}(h) \) becomes \( j^\nu(h) \) when the interaction between particles \( \alpha \) and \( \beta \) is turned off and therefore cluster separability for the three-particle operator \( \hat{J}^\nu(h) \) will be satisfied if \( \hat{J}^\nu(h) \) becomes \( \hat{J}^\nu_{\alpha\beta,\gamma}(h) \) when all interactions involving any particle \( \gamma \) are turned off (see Sec. 3.4). Therefore to explicitly construct the operator \( \hat{J}^\nu(h) \) we have to determine first the explicit expressions for the operators \( j^\nu_{\alpha,\beta,\gamma}(h) \) and \( j^\nu(h) \).

As follows from Eqs. (4.28-4.29),
\[
j^\nu_{\alpha,\beta,\gamma}(h) = j^\nu_{\alpha,\beta}(h) + j^\nu_{\gamma}(h), \quad j^\nu(h) = \sum_{i=1}^{3} \nu^\nu_i(h) \quad (4.30)
\]
where \( j^\nu_{\alpha,\beta}(h) \) is the contribution of \( \hat{J}_{\alpha,\beta}^\mu \) to \( j^\nu_{\alpha,\beta,\gamma}(h) \), \( j^\nu_{\gamma}(h) \) is the contribution of \( J_{\gamma}^\mu \) to \( j^\nu_{\alpha,\beta,\gamma}(h) \) and \( \nu^\nu_i(h) \) is the contribution of \( J_i^\mu \) to \( j^\nu(h) \). It is important to note that all the operators in Eq. (4.30) act in the three-particle Hilbert space \( H_{int} \). We suppose that the single-particle
and two-particle ECO’s have been already constructed as described in Secs. 4.1 and 4.2, and our goal is to express the action of the operators in Eq. (4.30) in terms of the single-particle operators \(j_\gamma(\mathbf{h})\) and two-particle operators \(\hat{J}_{\alpha\beta}(\mathbf{h})\) acting in the internal single-particle and two-particle spaces respectively.

The calculation of the operators entering into Eq. (4.30) is tedious but all preparatory expressions have been already derived above. Let us briefly describe the calculation of \(j_\gamma(\mathbf{h})\) which is most cumbersome.

We have to calculate the action of \(J_{\alpha\beta,\gamma}(\mathbf{G})\) (see Eq. (4.26)) on the function \(\tilde{\varphi}(\mathbf{G})\in \tilde{H}\) which depends additionally on \(\mathbf{K}_{\alpha\beta}, \mathbf{k}_{\alpha\beta}\) and the spin variables of the three particles (see Sec 2.3). The action of

\[
B_{\alpha\beta,\gamma}(\hat{M}_{\alpha\beta})^{-1}B_{\alpha\beta,\gamma}(\hat{M}_{\alpha\beta})U_{\alpha\beta}^{-1}U_{123}
\]

on this function can be calculated using Eqs. (2.13), (2.32), (2.42), (2.44) and (2.51). Then to calculate the action of \(J_{\alpha\beta}\) we use Eq. (3.3) and write the kernel of this operator in the form (see Eq. (3.21))

\[
\hat{J}_{\alpha\beta}(\mathbf{G}, \mathbf{G'}) = \int \int d\tilde{e}_{\alpha\beta}(m) \hat{J}_{\alpha\beta}(G, m; G', m') d\tilde{e}_{\alpha\beta}(m') \tag{4.31}
\]

We change the integration variable from \(\mathbf{G'}\) to \(\mathbf{G'} = \mathbf{G'}(m') = (m'\mathbf{G'}_{\alpha\beta} + m_\gamma g_\gamma)/|m'\mathbf{G'}_{\alpha\beta} + m_\gamma g_\gamma|\) using Eq. (2.41). Then we again use Eqs. (2.13), (2.32), (2.42), (2.44) and (2.51) in order to calculate the action of

\[
U_{123}^{-1}U_{\alpha\beta}B_{\alpha\beta,\gamma}(\hat{M}_{\alpha\beta})^{-1}B_{\alpha\beta,\gamma}(\hat{M}_{\alpha\beta})
\]

Expressing all the variables in terms of \(\mathbf{G}, \mathbf{K}_{\alpha\beta}, \mathbf{k}_{\alpha\beta}\) and using Eqs. (2.61) and (3.3) we can determine the contribution of \(\hat{J}_{\alpha\beta}\) to

\[
J_{\alpha\beta}(\mathbf{G}, \mathbf{G'})
\]

It is very convenient to use the relation between the operators \(O_{\alpha\beta}\) in \(H^{\text{int}}_{\alpha\beta}\) and \(\tilde{O}_{\alpha\beta}\) in \(\tilde{H}\) (see Eqs. (2.63) and (2.64)). Finally, as follows from Eq. (3.10), \(j_{\alpha\beta}(\mathbf{h}) = J_{\alpha\beta}(\mathbf{f}, \mathbf{f'})\).

The result of such a calculation is the following. We use \(k_\gamma\) and \(k'_\gamma\) to denote the 4-vectors

\(\left(\omega_\gamma(\mathbf{K}_{\alpha\beta}), -\mathbf{K}_{\alpha\beta}\right)\) and \(\left(\omega_\gamma(\mathbf{K}'_{\alpha\beta}), -\mathbf{K}'_{\alpha\beta}\right)\)
respectively where $K'_{\alpha\beta}$ is defined by the condition (compare with Eq. (4.17))

$$K'_{\alpha\beta} = K_{\alpha\beta} - \frac{2\hbar}{1 - \hbar^2} [\omega_{\gamma}(K_{\alpha\beta}) - \hbar K_{\alpha\beta}]$$  \hspace{1cm} (4.32)

We use $K_{\alpha\beta}(m)$ and $K'_{\alpha\beta}(m')$ to denote the 4-vectors

\[
((m^2 + K^2_{\alpha\beta})^{1/2}, K_{\alpha\beta}) \quad \text{and} \quad ((m'^2 + K'^2_{\alpha\beta})^{1/2}, K'_{\alpha\beta})
\]

respectively and use $G_{\alpha\beta}(m)$ and $G'_{\alpha\beta}(m')$ to denote the 4-vectors

$L[\alpha(f)]K_{\alpha\beta}(m)/m$ and $L[\alpha(f')]K'_{\alpha\beta}(m')/m'$

respectively (compare with Eq. (2.39)). Then

$$j^\mu_{(\alpha\beta)}(h)\chi(K_{\alpha\beta}, k_{\alpha\beta}) = D[s_\gamma; \alpha(\frac{k_{\gamma}}{m_\gamma})^{-1} \alpha(f)^{-1} \alpha(f') \alpha(\frac{k'_{\gamma}}{m_\gamma})] \cdot$$

\[
\int \int \left[ \frac{mm'}{K^0_{\alpha\beta}(m)K^0_{\alpha\beta}(m')} \right]^{1/2} D[\tilde{S}_{\alpha\beta}; \alpha(K_{\alpha\beta}(m)/m)^{-1} \alpha(f)^{-1} \alpha(G_{\alpha\beta}(m))] \cdot d\tilde{c}_{\alpha\beta}(m)\tilde{J}^\mu_{\alpha\beta}(G_{\alpha\beta}(m), m; G'_{\alpha\beta}(m'), m')d\tilde{c}_{\alpha\beta}(m') \cdot
\]

$$D[\tilde{S}_{\alpha\beta}; \alpha(G'_{\alpha\beta}(m'))^{-1} \alpha(f') \alpha(\frac{K'_{\alpha\beta}(m')}{m'})] \chi(K'_{\alpha\beta}, k_{\alpha\beta})$$  \hspace{1cm} (4.33)

In practical computations one may take into account that if $u \in SU(2)$ then

$$D[\tilde{S}_{\alpha\beta}; u]\chi(K_{\alpha\beta}, k_{\alpha\beta}) = (\prod_{i=\alpha, \beta} D[s_i; u])\chi(K_{\alpha\beta}, L[\alpha(u)]^{-1}k_{\alpha\beta})$$  \hspace{1cm} (4.34)

Using Eq. (3.10) it is possible to express $j^\mu_{(\alpha\beta)}(h)$ directly in terms of $j^\mu_{(\alpha\beta)}(h, m, m')$. Let $f_{\alpha\beta}(m, m')$ and $f'_{\alpha\beta}(m, m')$ be the 4-vectors constructed by means of Eq. (3.8) with $G$ replaced by $G(m)$ and $G'$ replaced by $G'(m')$, and

$$h_{\alpha\beta}(m, m') = \frac{f_{\alpha\beta}(m, m')}{f^0_{\alpha\beta}(m, m')}$$

Then

$$j^\mu_{(\alpha\beta)}(h)\chi(K_{\alpha\beta}, k_{\alpha\beta}) = D[s_\gamma; \alpha(\frac{k_{\gamma}}{m_\gamma})^{-1} \alpha(f)^{-1} \alpha(f') \alpha(\frac{k'_{\gamma}}{m_\gamma})] \cdot$$

\[
\int \int \left[ \frac{mm'}{K^0_{\alpha\beta}(m)K^0_{\alpha\beta}(m')} \right]^{1/2} D[\tilde{S}_{\alpha\beta}; \alpha(K_{\alpha\beta}(m)/m)^{-1} \alpha(f)^{-1} \alpha(f') \alpha(G_{\alpha\beta}(m))] \cdot d\tilde{c}_{\alpha\beta}(m)\tilde{J}^\mu_{\alpha\beta}(G_{\alpha\beta}(m), m; G'_{\alpha\beta}(m'), m')d\tilde{c}_{\alpha\beta}(m') \cdot
\]

$$D[\tilde{S}_{\alpha\beta}; \alpha(G'_{\alpha\beta}(m'))^{-1} \alpha(f') \alpha(\frac{K'_{\alpha\beta}(m')}{m'})] \chi(K'_{\alpha\beta}, k_{\alpha\beta})$$  \hspace{1cm} (4.33)
\[
\int \int D[\tilde{\mathbf{S}}_{\alpha\beta}; \alpha(\frac{K_{\alpha\beta}(m)}{m})]^{-1} \alpha(f)^{-1} \alpha(G_{\alpha\beta}(m), G'_{\alpha\beta}(m')) \cdot \\
\alpha(f_{\alpha\beta}(m, m'))] \tilde{e}_{\alpha\beta}(m) L[G_{\alpha\beta}(m), G'_{\alpha\beta}(m')]_{\mu}^{\nu} \cdot \\
\tilde{f}_{\alpha\beta}(h_{\alpha\beta}(m, m'); m, m') \tilde{e}_{\alpha\beta}(m') D[\tilde{\mathbf{S}}_{\alpha\beta}; \alpha(f_{\alpha\beta}(m, m'))]^{-1} \cdot \\
\alpha(G_{\alpha\beta}(m), G'_{\alpha\beta}(m'))^{-1} \alpha(f') \alpha(\frac{K'_{\alpha\beta}(m')}{m'}) \cdot \\
\frac{m m'}{K_{\alpha\beta}(m) K'_{\alpha\beta}(m')}^{1/2} \chi(K'_{\alpha\beta}, k_{\alpha\beta}) 
\] (4.35)

The result for \( j^\mu_{(\gamma)}(h) \) is the following. We again use \( k_\gamma, k'_\gamma, K_{\alpha\beta}(m) \) and \( K'_{\alpha\beta}(m) \) to denote the quantities expressed in terms of \( K_{\alpha\beta} \) and \( K'_{\alpha\beta} \) as above, but now \( K'_{\alpha\beta} = K'_{\alpha\beta}(m) \) is defined by the condition \( L[\alpha(f)] K_{\alpha\beta}(m)/m = L[\alpha(f')] K'_{\alpha\beta}(m')/m' \) and therefore (compare with the second expression in Eq. (4.19))

\[
K'_{\alpha\beta}(m) = K_{\alpha\beta} + \frac{2h}{1-h^2}[(m^2 + K_{\alpha\beta}^2)^{1/2} + h K_{\alpha\beta}] 
\] (4.36)

Then

\[
\int \tilde{d}_{\alpha\beta}(m) D[\tilde{\mathbf{S}}_{\alpha\beta}; \alpha(\frac{k_{\gamma}(m)}{m_{\gamma}})]^{-1} \alpha(f)^{-1} \alpha(L[\alpha(f)] \frac{k_{\gamma}}{m_\gamma}]) \cdot \\
\frac{m_{\gamma}}{\omega_{\gamma}(K'_{\alpha\beta})(m)} \frac{K_{\alpha\beta}(m)}{K'_{\alpha\beta}(m)}^{1/2} \int j^\mu_{(\gamma)}(L[\alpha(f)] k_{\gamma}/m_\gamma, L[\alpha(f')] k'_{\gamma}(m)/m_\gamma) \\
D[s_{\gamma}; \alpha(L[\alpha(f')] \frac{k'_{\gamma}(m)}{m_\gamma})]^{-1} \alpha(f') \alpha(\frac{k'_{\gamma}(m)}{m_\gamma}) \chi(K'_{\alpha\beta}, k_{\alpha\beta}) 
\] (4.37)

Using Eq. (3.10) it is possible to express \( j^\mu_{(\gamma)}(h) \) directly in terms of \( j^\mu_{(\gamma)}(h) \). Let \( f_{\gamma}(m) \) and \( f'_\gamma(m) \) be the 4-vectors constructed by means of Eq. (3.3) with \( G \) replaced by \( L[\alpha(f)] k_{\gamma}/m_\gamma \) and \( G' \) replaced by \( L[\alpha(f')] k'_{\gamma}(m)/m_\gamma \), and \( h_\gamma(m) = f_{\gamma}(m)/f_{\gamma}^0(m) \). Then

\[
\int \tilde{d}_{\alpha\beta}(m) L(L[\alpha(f)] \frac{k_{\gamma}}{m_\gamma}, L[\alpha(f')] \frac{k'_{\gamma}(m)}{m_\gamma})^\nu 
\]
\[ D[s_\gamma; \alpha\left(\frac{k_\gamma}{m_\gamma}\right)^{-1}\alpha(f)^{-1}\alpha(L[\alpha(f)]\frac{k_\gamma}{m_\gamma}, L[\alpha(f')][\frac{k'_\gamma(m)}{m_\gamma}]\alpha(f_\gamma(m)) \right) \cdot \]

\[ D[\tilde{S}_{\alpha\beta}; \alpha\left(\frac{K_\alpha\beta(m)}{m}\right)^{-1}\alpha(f)^{-1}\alpha(f')\alpha\left(\frac{K'_\alpha\beta(m)}{m}\right)]j^\nu_\gamma(h_\gamma(m)) \cdot \]

\[ D[s_\gamma; \alpha(f'_\gamma(m))^{-1}\alpha(L[\alpha(f)]\frac{k_\gamma}{m_\gamma}, L[\alpha(f')][\frac{k'_\gamma(m)}{m_\gamma}]^{-1}\alpha(f') \cdot \]

\[ \alpha\left(\frac{k'_\gamma(m)}{m_\gamma}\right)\frac{m_\gamma}{\omega_{\gamma}(K'_{\alpha\beta}(m))} \left[ \frac{K^0_{\alpha\beta}(m)}{K^0_{\alpha\beta}(m)} \right]^{1/2} \chi(K'_{\alpha\beta}(m), k_{\alpha\beta}) \] (4.38)

The result for \( \nu^\mu_\gamma(h) \) follows from Eq. (4.38):

\[ (4.38) \]

\[ D[s_\gamma; \alpha\left(\frac{k_\gamma}{m_\gamma}\right)^{-1}\alpha(f)^{-1}\alpha(L[\alpha(f)]\frac{k_\gamma}{m_\gamma}, L[\alpha(f')][\frac{k'_\gamma(m)}{m_\gamma}]\alpha(f_\gamma(m)) \right) \cdot \]

\[ D[\tilde{S}_{\alpha\beta}; \alpha\left(\frac{K_\alpha\beta(m)}{M_{\alpha\beta}}\right)^{-1}\alpha(f)^{-1}\alpha(f')\alpha\left(\frac{K'_\alpha\beta(m)}{M_{\alpha\beta}}\right)]j^\nu_\gamma(h_\gamma(m)) \cdot \]

\[ D[s_\gamma; \alpha(f'_\gamma(m))^{-1}\alpha(L[\alpha(f)]\frac{k_\gamma}{m_\gamma}, L[\alpha(f')][\frac{k'_\gamma(m)}{m_\gamma}]^{-1}\alpha(f')\alpha\left(\frac{k'_\gamma(m)}{m_\gamma}\right) \cdot \]

\[ \frac{m_\gamma}{\omega_{\gamma}(K'_{\alpha\beta}(m))} \left[ \frac{K^0_{\alpha\beta}(m)}{K^0_{\alpha\beta}(m)} \right]^{1/2} \chi(K'_{\alpha\beta}(m), k_{\alpha\beta}) \] (4.39)

where the quantities \( K'_{\alpha\beta}, k'_\gamma, K_\alpha\beta, K'_\alpha\beta, f_\gamma \) and \( f'_\gamma \) depend on \( M_{\alpha\beta} = M_{\alpha\beta}(k_{\alpha\beta}) \) in the same manner as they depend on \( m \) in Eq. (4.38).

Equations (4.30), (4.34), (4.35), (4.36), (4.38) and (4.39) explicitly describe the operators \( j^\nu_\alpha\beta_\gamma(h) \) and \( j^\nu_\gamma(h) \). As follows from the results of Sec. 3.3, \( j^0_\alpha\beta_\gamma(0) = j^0(0) = e_\alpha + e_\beta + e_\gamma \). This can be verified directly, as follows from Eqs. (3.13), (4.38) and (4.39), \( j^0_\gamma(0) = \nu^0_\gamma(0) = e_\gamma \) since \( j^0_\gamma(0) = e_\gamma \), and if \( h = 0 \) then \( K'_{\alpha\beta} = K_{\alpha\beta} \) in Eq. (4.33) and

\[ j^0_{(\alpha\beta)}(0) = \int \int \left[ \frac{mm'}{K^0_{\alpha\beta}(m)K^0_{\alpha\beta}(m')} \right]^{1/2} \tilde{e}^\int_{\alpha\beta} \left( \frac{K_{\alpha\beta}(m)}{m} \right) \tilde{J}^0_{\alpha\beta} \left( \frac{K_{\alpha\beta}(m)}{m} \right) dm; \]

\[ \tilde{K}_{\alpha\beta}(m') \cdot m' \tilde{e}^\int_{\alpha\beta}(m') \] (4.40)
As follows from Eq. (3.24),

$$\hat{j}_0^{\alpha\beta}(K_{\alpha\beta}(m), m; \frac{K'_{\alpha\beta}(m')}{m'}, m') = 0$$

if \(m \neq m'\). Therefore only the values of \(m'\) in an infinitely small vicinity of the point \(m\) may contribute to Eq. (4.40). Then Eq. (4.40) is a special case of Eq. (3.33) for \(\tau = 0\) and \(\lambda = (1, 0, 0, 0)\). Therefore \(j_0^{(\alpha\beta)}(0) = e_\alpha + e_\beta\) as it should be.

As follows from the results of Sec. 3.4, we can seek the three-particle operators \(\hat{j}_0^{(\alpha\beta)}(h)\) and \(\hat{j}_\perp^{(\alpha\beta)}(h)\) in the form (compare with Eq. (2.65))

$$\hat{j}_0^{(\alpha\beta)}(h) = j_{12,3}^{(\alpha\beta)}(h) + j_{13,2}^{(\alpha\beta)}(h) + j_{23,1}^{(\alpha\beta)}(h) - 2j_0^{(\alpha\beta)}(h) + w_0^{(\alpha\beta)}(h) =$$

$$= j_{12}^{(\alpha\beta)}(h) + j_{13}^{(\alpha\beta)}(h) + j_{23}^{(\alpha\beta)}(h) + j_{11}^{(\alpha\beta)}(h) + j_{12}^{(\alpha\beta)}(h) + j_{13}^{(\alpha\beta)}(h) -$$

$$- 2s_1^{(\alpha\beta)}(h) - 2s_2^{(\alpha\beta)}(h) - 2s_3^{(\alpha\beta)}(h) + w_0^{(\alpha\beta)}(h)$$

(4.41)

$$\hat{j}_\perp^{(\alpha\beta)}(h) = j_{12,3}^{(\alpha\beta)}(h) + j_{13,2}^{(\alpha\beta)}(h) + j_{23,1}^{(\alpha\beta)}(h) - 2j_\perp^{(\alpha\beta)}(h) + w_\perp^{(\alpha\beta)}(h) =$$

$$= j_{12}^{(\alpha\beta)}(h) + j_{13}^{(\alpha\beta)}(h) + j_{23}^{(\alpha\beta)}(h) + j_{11}^{(\alpha\beta)}(h) + j_{12}^{(\alpha\beta)}(h) +$$

$$+ j_{13}^{(\alpha\beta)}(h) - 2s_1^{(\alpha\beta)}(h) - 2s_2^{(\alpha\beta)}(h) - 2s_3^{(\alpha\beta)}(h) + w_\perp^{(\alpha\beta)}(h)$$

(4.42)

where the operators \(w_0^{(\alpha\beta)}(h)\) and \(w_\perp^{(\alpha\beta)}(h)\) describe the contribution of three-body interactions to \(j_0^{(\alpha\beta)}(h)\) and \(j_\perp^{(\alpha\beta)}(h)\). In other words, \(w_0^{(\alpha\beta)}(h)\) and \(w_\perp^{(\alpha\beta)}(h)\) must be zero if any particle does not interact with the others. These operators must satisfy the relations which can be written in the form of Eq. (4.24).

We conclude that a possible choice of \(w_0^{(\alpha\beta)}(h)\) and \(w_\perp^{(\alpha\beta)}(h)\) in Eqs. (4.41) and (4.42) is \(w_0^{(\alpha\beta)}(h) = w_\perp^{(\alpha\beta)}(h) = 0\), but in the general case there exists an infinite number of solutions satisfying the conditions described in Sec. 1.1.
4.4 ECO for systems with any number of particles

As follows from the results of Sec. 3.4, we can seek the operators \( \hat{j}^0(h) \) and \( \hat{j}_\perp(h) \) in the form (compare with Eq. (2.75))

\[
\hat{j}^0(h) = \sum_{k=2}^{N} (-1)^k (k-1)! j^0_{(k)}(h) + w^0(h),
\]

\[
\hat{j}_\perp(h) = \sum_{k=2}^{N} (-1)^k (k-1)! j_\perp_{(k)}(h) + w_\perp(h),
\]

where \( w^0(h) \) and \( w_\perp(h) \) describe the contribution of \( N \)-body interactions to \( \hat{j}^0(h) \) and \( \hat{j}_\perp(h) \) respectively. The operators \( w^0(h) \) and \( w_\perp(h) \) must satisfy the relations which can be written in the form of Eq. (4.24).

We conclude that a possible choice for \( w^0(h) \) and \( w_\perp(h) \) in Eqs. (4.43) and (4.44) is \( w^0(h) = w_\perp(h) = 0 \), but in the general case there exists an infinite number of solutions satisfying the conditions described in Sec. 4.4.

To explicitly describe the ECO for a system of \( N \) particles we have to calculate the explicit expressions for the operators \( j^\mu_{\alpha_1...\alpha_k}(h) \). For this purpose we have to know the explicit expressions for the operators \( A_{\alpha_1...\alpha_k} \) (see Sec. 3.3). As noted in Sec. 2.4, the problem of constructing such operators is solved in principle, but explicit expressions for the operators \( A_{\alpha_1...\alpha_k} \) in the point form have not been written so far for the case \( N > 3 \).

4.5 Example: relativistic correction to the magnetic moment of the deuteron

It is well known that in the nonrelativistic approximation the magnetic moment of the deuteron is given by

\[
\mu_d^{nr} = \mu_p + \mu_n - \frac{3}{2} P_D(\mu_p + \mu_n - \frac{1}{2})
\]
where $\mu_d$, $\mu_p$, and $\mu_n$ are the magnetic moments of the deuteron, proton and neutron in nuclear magnetons, "nr" means nonrelativistic and $P_D$ is the probability of the D-state in the deuteron. For most realistic nucleon-nucleon potentials the value of $\mu_d^{nr}$ is less than the experimental value $\mu_d^{exp} = 0.857$. For example, $\mu_d^{nr}$ is equal to 0.843, 0.845, 0.847 and 0.849 for the Reid soft core, Argonne, Paris and Nijmegen potentials respectively[94]. For different versions of the Bonn potential the value of $\mu_d^{nr}$ is in the range 0.852-0.856 [94], but the attitude of physicists to the Bonn potential is controversial, and we shall not discuss this question.

In our approach the magnetic moment of the deuteron can be calculated using the fact that $F_M(0)$ in Eq. (4.9) is the deuteron magnetic moment in units $e/2m_d$ where $e$ and $m_d$ are the deuteron electric charge and mass respectively (see Sec. 4.1). Therefore we have to calculate the matrix element $\langle \chi''(q)|\hat{j}_\perp(h)|\chi'(q)\rangle$ between two deuteron states in first order in $h$ and compare the result with the quantity

$$-ieF_M(0)\langle \chi''(q)|h\times S|\chi'(q)\rangle.$$

Let us choose for $\hat{j}_\perp(h)$ the solution corresponding to $w_\perp(h) = 0$ (see Sec. 4.2). Then $\hat{j}_\perp(h) = j_\perp(h)$ and $j_\perp(h)$ is given by Eq. (4.20). We use $m$ to denote the nucleon mass and neglect the difference between the masses of the proton and neutron. Since the deuteron wave function is symmetrical under the interchange of spatial and spin variables of the proton and neutron, we can calculate the contribution of only the first term in Eq. (4.20) where $J'_1(g_1, g'_1)$ is taken from Eq. (4.7) but for the nucleon form factors $F_E$ and $F_M$ in these expressions we must take the sum of the corresponding quantities for the proton and neutron at zero momentum transfer, i.e. $F_E^0 = 1$, $F_M^0 = \mu_p + \mu_n$.

As follows from the "minimal relativity principle" [95, 96, 97], we can use for $\chi'(q)$ and $\chi''(q)$ the deuteron wave functions calculated for the usual phenomenological potentials (see also refs. [98, 94]). Therefore we shall use the functions normalized as

$$\int ||\chi(q)||^2 d^3q = \int [\varphi_0(q)^2 + \varphi_2(q)^2] d^3q = 1 \quad (4.46)$$
where $\varphi_0(q)$ and $\varphi_2(q)$ are the radial wave function of the S and D states in momentum representation. Then a direct calculation using Eqs. (2.3), (2.10), (2.13), (4.7), (4.19) and (4.20) shows that in first order in $\hbar$ the action of $j(\hbar)$ is given by

$$j(\hbar)\chi(q) = \frac{1}{\omega} \left\{ -4i F^0_M (\hbar \omega - \frac{q(q\hbar)}{\omega + m}) \times s_1 - \frac{\omega q}{m(\omega + m)} ((\hbar \times q) s_1) \right\} + 2 F^0_E (q - \hbar \omega) [1 - i (\hbar \times q)\omega (s_1 - s_2) - 2i \omega (\hbar \times q) s_1] \chi(q - 2\hbar \omega)$$

(4.47)

where $\omega = \omega(q) = (m^2 + q^2)^{1/2}$. We use $t_1$ and $t_2$ to denote the spin projections of the proton and neutron. Then the internal deuteron wave function describing the deuteron with the polarization vector $e$ has the form

$$\chi(q, t_1, t_2) = \frac{1}{\sqrt{2}} [\varphi_0(q) \delta_{ik} - \frac{1}{\sqrt{2}} (\delta_{ik} - 3 \frac{q_i q_k}{q^2}) \varphi_2(q)] e_k (\sigma_i \sigma_2)_{t_1 t_2}$$

(4.48)

where $q = |q|$ and a sum over repeated indices $i, k = 1, 2, 3$ is assumed.

The result of our calculation is the following. If $\epsilon$ is the deuteron binding energy and $\mu_d = \mu^{nr}_d + \delta \mu_d$ where $\delta \mu_d$ is the relativistic correction to the deuteron magnetic moment in nuclear magnetons then

$$\delta \mu_d = \frac{\epsilon}{m_d} \mu^{nr}_d - \frac{2m}{3m_d} \int \frac{q^2 d^3 q}{\omega(q) + m} [\varphi_0(q)^2 + \frac{1}{\sqrt{2}} \varphi_0(q) \varphi_2(q) - \varphi_2(q)^2] \left[ \frac{F^0_M}{\omega(q)} + \frac{F^0_E - F^0_M}{m} \right]$$

(4.49)

This quantity is negligible for all the realistic potentials mentioned above. For example, $\delta \mu_d = 6 \cdot 10^{-4}$ for the Reid soft core potential.

The relativistic correction to the deuteron magnetic moment was considered by many authors. Usually $\delta \mu_d$ turned out to be negative, but in the framework of approach considered in refs. [98, 94] $\delta \mu_d$ is of about a half of the quantity $\mu^{exp}_d - \mu^{nr}_d$. However, the ECO’s used in these approaches do not satisfy the properties given by Eqs. (1.1) and
The result given by Eq. (4.49) is obtained by using the ECO which does satisfy these properties but among the solutions discussed in Sec. 4.2 the special solution with \( w_\perp(h) = 0 \) has been chosen. We conclude that this choice also does not solve the problem of the deuteron magnetic moment.
Chapter 5

Electromagnetic current operator in the instant form of dynamics

5.1 Systems of interacting particles in the instant form of dynamics

In the instant form it is convenient to use the realization of UIR’s of the Poincare group in the space of functions \( \phi(g) \) with the range in \( \mathcal{D}(s) \) and such that

\[
(\phi, \phi) = \int \|\phi(p)\|^2 \frac{d^3p}{(2\pi)^3} < \infty \tag{5.1}
\]

Then the generators of the UIR have the well known form (see, for example, refs. [81, 82, 4, 5])

\[
P = p, \quad E = (m^2 + p^2)^{1/2}, \quad M = l(p) + s, \quad N = -i(m^2 + p^2)^{1/4} \frac{\partial}{\partial p} (m^2 + p^2)^{1/4} + \frac{s \times p}{m + (m^2 + p^2)^{1/2}} \tag{5.2}
\]

The Hilbert space \( H_I \) for the representations describing systems of \( N \) free or interacting particles can be realized as the space of functions \( \phi(p_1, \ldots, p_N) \) with the range in \( \mathcal{D}(s_1) \otimes \ldots \otimes \mathcal{D}(s_N) \) and such that

\[
\int \|\phi(p_1, \ldots, p_N)\|^2 \prod_{i=1}^N d\rho_i^I(p_i) < \infty \tag{5.3}
\]
where (compare with Eq. (2.1))
\[
d\rho^I_i(p_i) = \frac{d^3p_i}{2(2\pi)^3\omega_i(p_i)} = m^2d\rho_i(g_i) \tag{5.4}
\]
(the subscript "I" means "instant").

Instead of the variables \(p_1, \ldots, p_N\) we introduce the variables \(P, k_1, \ldots, k_N\) where \(P = p_1 + \cdots + p_N\) and \(k_i\) is the spatial part of the 4-vector \(k_i = L[\alpha(P/M)]^{-1}p_i\) where \(p_i\) is the momentum of the \(i\)-th particle, \(P = p_1 + \cdots + p_N\) and \(M = |P|\). It is obvious that \(k_i\) defined in such a way is the same as in Eq. (2.46).

By analogy with Eq. (2.68) we can show that
\[
\prod_{i=1}^{N} d\rho^I_i(p_i) = \frac{d^3P}{2(2\pi)^3(1 + P^2/M^2)^{1/2}} d\rho^I(int),
\]
\[
d\rho^I(int) = (2\pi)^3\delta(3)(k_1 + \cdots + k_N) \prod_{i=1}^{N} d\rho_i^I(k_i) \tag{5.5}
\]

Let us define the "internal" space \(H^I_{int}\) as the space of functions \(\chi(k_1, \ldots, k_N)\) with the range in \(\mathcal{D}(s_1) \otimes \cdots \otimes \mathcal{D}(s_N)\) and such that
\[
||\chi||^2 = \int ||\chi(k_1, \ldots, k_N)||^2 d\rho^I(int) < \infty \tag{5.6}
\]
and the space \(\tilde{H}_I\) as the space of functions \(\tilde{\phi}(P)\) with the range in \(H^I_{int}\) and such that
\[
\int ||\tilde{\phi}(P)||^2_{int} d^3P < \infty \tag{5.7}
\]

Let \(U\) be the same operator as in Eq. (2.70) but with \(g_i\) replaced by \(p_i/m_i\). Then \(U^I = (1 + P^2/M^2)^{1/4}U\) is the unitary operator from \(\tilde{H}_I\) to \(H_I\).

The method of Sokolov packing operators in the instant form implies that the generators \(\hat{\Gamma}^i_I\) \((i = 1, \ldots, 10)\) of the representation describing a system of \(N\) interacting particles should be written in the form \(\hat{\Gamma}^i_I = A_I\hat{U}_I\hat{\Gamma}^i\hat{U}_I^{-1}A_I^{-1}\) where the packing operator \(A_I\) commutes with \(P\) and \(M\) and the generators \(\hat{\Gamma}^i_I\) in \(\tilde{H}_I\) have the following "canonical" form (compare with Eq. (5.2))
\[
P_I = P, \quad \hat{E}_I = ((\hat{M}_{int})^2 + P^2)^{1/2}, \quad M_I = l(P) + S,
\]
\[ \hat{N}_I = -\imath((\hat{M}_{int})^2 + P^2)^{1/4} \frac{\partial}{\partial P}((\hat{M}_{int})^2 + P^2)^{1/4} + \frac{S \times P}{\hat{M}_{int} + ((\hat{M}_{int})^2 + P^2)^{1/2}} \]  

(5.8)

Here the first expression implies that the momentum operator is equal to the operator of multiplication by the variable \( P \) defined above, the spin operator \( S \) is the same as in Eq. (2.71) and the mass operator \( \hat{M}_{int} \) in the instant form acts only in \( H_{int} \) and commute with \( S \). As already noted, the problem of constructing the operators \( A_I \) and \( \hat{M}_{int} \) has been solved by Coester and Polyzou [4] and Mutze [5] (see also refs. [42, 80, 99, 43]).

### 5.2 Unitary equivalence of the point and instant forms of dynamics

As already noted, the unitary equivalence of the three basic forms of dynamics has been proved by Sokolov and Shatny [77]. The key element of their approach is the construction of unitary operators relating the "canonical" forms of the generators in these forms of dynamics. In particular, to prove the unitary equivalence of the point and instant forms we have to construct a unitary operator \( \Theta \) from \( \hat{H} \) to \( \hat{H}_I \) which transform the generators defined by Eq. (2.71) to the generators defined by Eq. (5.8). Following ref. [77] we shall seek \( \Theta \) in the form

\[ \Theta = U_{IP} \xi(M)^{-1} \xi(\hat{M}_{int}) \]  

(5.9)

where the unitary operators \( \xi(M) \) and \( \xi(\hat{M}_{int}) \) in \( \hat{H} \) are defined by the spectral integrals (compare with Eq. (2.32))

\[ \xi(\hat{M}_{int}) = \int \xi(m)d\hat{e}_{int}(m), \quad \xi(M) = \int \xi(m)de(m) \]  

(5.10)

over the spectral measures of the operators \( \hat{M}_{int} \) and \( M \) respectively and \( U_{IP} \) is the unitary operator from \( \hat{H} \) to \( \hat{H}_I \) defined as

\[ \tilde{\phi}(P) = U_{IP} \tilde{\phi}(G) = \frac{\tilde{\phi}(P/M)}{(1 + P^2/M^2)^{1/4}m_1 \cdot \cdot \cdot m_N} \]  

(5.11)
The fact that $U_{IP}$ is unitary easily follows from Eqs. (2.69), (5.5-5.7). The operator $U_{IP}^{-1}$ is given by

$$\tilde{\varphi}(G) = U_{IP}^{-1} \tilde{\varphi}(P) = m_1 \cdots m_N (1 + G^2)^{1/4} \tilde{\phi}(MG)$$  \hspace{1cm} (5.12)$$

As follows from Eqs. (2.71), (5.8), (5.9), (5.11) and (5.12), the condition $\Theta \hat{M}_{int} G \Theta^{-1} = P$ implies that

$$\xi(\hat{M}_{int}) \hat{M}_{int} G \xi(\hat{M}_{int}) = \xi(M) MG \xi(M)^{-1}$$  \hspace{1cm} (5.13)$$

This expression will be satisfied if for all $m$ belonging to the spectra of the operators $\hat{M}_{int}$ and $M$

$$\xi(m) m G \xi(m)^{-1} = m_0 G$$  \hspace{1cm} (5.14)$$

where $m_0$ is some constant [77]. This condition can be satisfied if the actions of the operators $\xi(m)$ and $\xi(m)^{-1}$ are given by [77]

$$\xi(m) \tilde{\varphi}(G) = \left( \frac{m}{m_0} \right)^{3/2} \left( \frac{1 + G^2}{1 + m_0^2 G^2 / m^2} \right)^{1/4} \tilde{\phi}(\frac{m_0}{m} G)$$

$$\xi(m)^{-1} \tilde{\varphi}(G) = \left( \frac{m}{m_0} \right)^{3/2} \left( \frac{1 + G^2}{1 + m_0^2 G^2 / m^2} \right)^{1/4} \tilde{\phi}(\frac{m}{m_0} G)$$  \hspace{1cm} (5.15)$$

As follows from Eqs. (2.69), (5.5) and (5.15), the operators $\xi(m)$ and $\xi(m)^{-1}$ are unitary and commute with the operators $\hat{e}_{int}(m')$ and $e(m')$. Therefore the operators defined by Eq. (5.10) are unitary.

Let $O$ be an operator in $H_{int}$. We symbolically represent the action of $O$ in the form

$$O \tilde{\varphi}(G, int) = \int O(int, int') \tilde{\varphi}(G, int') d\rho(int')$$  \hspace{1cm} (5.16)$$

where $O(int, int')$ is the kernel of the operator $O$. Let us introduce the operator

$$F_I\{O\} \equiv U_{IP} \xi(M)^{-1} O \xi(M) U_{IP}^{-1}$$  \hspace{1cm} (5.17)$$

Then, as follows from Eqs. (2.69), (5.5-5.7), (5.13) and (5.16), $F_I\{O\}$ is the operator in $H_{int}^I$. If we write symbolically

$$F_I\{O\} \tilde{\phi}(P, int) = \int F_I\{O\}(int, int') \tilde{\phi}(P, int') d\rho^I(int')$$  \hspace{1cm} (5.18)$$
then the kernels of the operators \( O \) and \( F_I\{O\} \) are related as

\[
F_I\{O\}(int, int') = \frac{2(M M')^{3/2}}{m_1^2 \cdots m_N^2} O(int, int') \tag{5.19}
\]

Since \( \xi(\hat{M}_{int}) \) obviously commutes with \( \hat{M}_{int} \), then as follows from Eqs. (5.9) and (5.17), the mass operators in the point and instant forms are related as

\[
\hat{M}^I_{int} = \Theta \hat{M}_{int} \Theta^{-1} = F_I\{\hat{M}_{int}\} \tag{5.20}
\]

In particular, using Eq. (5.19) we can verify the fact mentioned in Sec. 2.3 that the mass operators found in that section and in ref.[84] are unitarily equivalent if \( A_{ij} = 1 \).

Using Eqs. (5.9-5.12), (5.15) and (5.20) one can explicitly verify that not only the momentum generators, but also the remaining 7 generators from the sets defined by Eqs. (2.71) and (5.8) are related as

\[
\hat{E}_I = \Theta \hat{E} \Theta^{-1}, \quad M_I = \Theta M \Theta^{-1}, \quad \hat{N}_I = \Theta \hat{N} \Theta^{-1} \tag{5.21}
\]

As shown in ref.[77], if the operators \( A, \hat{M}_{int}, \) and \( \Theta \) are known then it is possible to determine \( A_I \) and conversely, if \( A_I, \hat{M}^I_{int}, \) and \( \Theta \) are known then it is possible to determine \( A \). Therefore the operator \( \hat{U}_{IP} = A_I \hat{U} \Theta \hat{U}^{-1} A^{-1} \) realizes the unitary equivalence of the point and instant forms.

### 5.3 Explicit construction of the ECO in the instant form of dynamics

If \( \hat{J}_I^\mu(x) \) is the ECO in the instant form, then, by analogy with Eq. (3.4) we can write

\[
\hat{J}_I^\mu(x) = \exp(i\hat{P}_I x)\hat{J}_I^\mu(0)\exp(-i\hat{P}_I x) \tag{5.22}
\]

and, by analogy with Eq (3.2), one can easily show that \( \hat{J}_I^\mu(0) \) must satisfy the properties

\[
\hat{U}_I(l)^{-1}\hat{J}_I^\mu(0)\hat{U}_I(l) = L(l)^\mu_\nu\hat{J}_I^\nu(0), \quad [\hat{P}_{IJ}, \hat{J}_I^\mu(0)] = 0 \tag{5.23}
\]
In turn, by analogy with Eq. (3.4), we shall seek $\hat{J}_I^\mu(0)$ in the form

$$\hat{J}_I^\mu(0) = A_I U_I \hat{J}_I^\mu U_I^{-1} A_I^{-1}$$  \hspace{1cm} (5.24)

where $\hat{J}_I^\mu$ acts in $\tilde{H}_I$. Then it is obvious from the results of Sec. 5.2 that the operator $\hat{J}_I^\mu(x)$ will satisfy all the properties described in Sec. 1.1 if

$$\hat{J}_I^\mu = \Theta \hat{J}^\mu \Theta^{-1}$$ \hspace{1cm} (5.25)

The action of $\hat{J}_I^\mu$ can be written in the form

$$\hat{J}_I^\mu \tilde{\phi}(P) = \int \hat{J}_I^\mu(P, P') \tilde{\phi}(P') \frac{d^3P'}{(2\pi)^3}$$ \hspace{1cm} (5.26)

where the kernel $\hat{J}_I^\mu(P, P')$ is an operator in $H_{\text{int}}^I$ for each fixed values of $P$ and $P'$. In turn, by analogy with Eq. (3.21), the operator $\hat{J}_I^\mu(P, P')$ can be defined by the set of operators $\hat{J}_I^\mu(P, m; P', m')$ such that

$$\hat{J}_I^\mu(P, P') = \int \int d\hat{e}_\text{int}^I(m) \hat{J}_I^\mu(P, m; P', m') d\hat{e}_\text{int}^I(m')$$ \hspace{1cm} (5.27)

where $\hat{e}_\text{int}^I(m)$ is the spectral function of the operator $\hat{M}_\text{int}^I$.

A direct calculation using Eqs. (3.3), (3.21), (5.9-5.12), (5.15), (5.17), (5.20) and (5.25-5.27) shows that if the operator

$$\hat{J}^\mu(G, m; G', m')$$

in the point form is known then

$$\hat{J}_I^\mu(P, m; P', m') = \frac{(mm')^{1/2}}{[(m^2 + P^2)(m'^2 + P'^2)]^{1/4}} \cdot F_I \left\{ \hat{J}^\mu \left( \frac{P}{m}, \frac{P'}{m'}; m, m' \right) \right\}$$ \hspace{1cm} (5.28)

Therefore Eqs. (5.19), (5.27) and (5.28) make it possible to explicitly determine the ECO in the instant form if the problem of constructing the ECO is solved in the point form.

### 5.4 Matrix elements of the ECO in the instant form of dynamics

Since we require that one-particle states with a definite momentum should be normalized as in Eq. (3.48), then, as follows from Eq. (5.1),...
such states should be chosen in the form

\[ |p', \sigma'\rangle_I = (2\pi)^3[2\omega(p')^{1/2}]\delta^{(3)}(p - p')\delta_{\sigma\sigma'} \quad (5.29) \]

To define the scattering states in the N-particle case we have to solve the eigenvalue problem for the operator \( \hat{M}_{int}^I \) in the space \( H_{int}^I \). Let \( \chi'_I \in H_{int}^I \) be the internal wave function of a bound state with the mass \( M' \) and \( P' \) be the momentum of this bound state. Then, as follows from Eq. (5.8), the wave function of such a state in the space \( H_I \) can be written as

\[ |P', \chi'\rangle_I = A_I U_I (2\pi)^3(2E')^{1/2}\delta^{(3)}(P - P')\chi'_I \quad (5.30) \]

where \( P'^0 = E' = (M'^2 + P'^2)^{1/2} \) is the total energy of the bound state. It is clear from Eq. (5.1) that such states will be normalized as in Eq. (3.51). As in the point form, Eq. (5.30) can be written not only if \( \chi'_I \) is a bound state, but also if \( \chi'_I \) is a generalized eigenfunction of the operator \( \hat{M}_{int}^I \) with the eigenvalue \( M' \). Therefore we conclude that all matrix elements of the operator \( \hat{J}^\mu_I(x) \) can be expressed in terms of the quantities \( I\langle P'', \chi''|\hat{J}^\mu_I(x)|P', \chi'_I\rangle_I \).

As follows from Eqs. (5.22), (5.24), (5.26), and (5.30)

\[ i\langle P'', \chi''|\hat{J}^\mu_I(x)|P', \chi'_I\rangle_I = 2[(M'')^2 + P'')^2](M'^2 + P'^2)]^{1/2} \cdot \exp(i\Delta x)\langle \chi''|\hat{J}^\mu_I(P'', \chi')|\chi'_I\rangle_I \quad (5.31) \]

where the matrix element on the right-hand-side must be calculated only in the space \( H_{int}^I \). If the problem of constructing the ECO is solved in the point form then, as follows from Eq. (5.28), we can calculate the matrix elements of the ECO in the instant form. On the other hand, as follows from Eqs. (5.9) and (5.20), if \( \chi' \) is an eigenfunction of the operator \( \hat{M}_{int} \) with the eigenvalue \( M' \) then \( \chi'' = U_{IP}\xi(M)^{-1}\chi' \) is an eigenfunction of the operator \( \hat{M}_{int}^I \) with the same eigenvalue. Therefore, as follows from Eqs. (5.17) and (5.28)

\[ I\langle \chi''|\hat{J}^\mu_I(P'', P')|\chi'_I\rangle_I = \frac{(M''M')^{1/2}}{[(M'')^2 + P'')^2(M'^2 + P'^2)]^{1/2}} \cdot \langle \chi''|\hat{J}^\mu_I(P'', P')|\chi' \rangle \quad (5.32) \]
Expressing $\hat{J}^{\mu}(G'', \mathbf{G}')$ in terms of $j^{\nu}(\mathbf{h})$ according to Eq. (3.10) and comparing Eq. (3.52) with Eqs. (5.31) and (5.32) we conclude that

$$\langle P''', \chi'' | \hat{J}^{\mu}(x) | P', \chi' \rangle = \langle P''' | \hat{J}^{\mu}_{I}(x) | P', \chi' \rangle_{I}$$

(5.33)

This result shows that the matrix elements of the ECO do not depend on the choice of the form of dynamics as it should be.
Chapter 6

Electromagnetic current operator in the front form of dynamics

6.1 Systems of interacting particles in the front form of dynamics

In the front form it is convenient to use the realization of UIR’s of the Poincare group in the space of functions $\phi(p_\perp, p^+)\) with the range in $D(s)$ and such that

$$
(\phi, \phi) = \int \|\phi(p_\perp, p^+)\|^2 dp^F(p_\perp, p^+) < \infty,
$$

$$
dp^F(p_\perp, p^+) = \frac{d^2p_\perp dp^+}{2(2\pi)^3 p^+}
$$

(6.1)

where the superscript ”$F$” means ”front”. The spin variables in the front form are defined assuming that the boosts are described not by the matrices $\alpha(p/m)$ (see Eq. (2.3)) but by the matrices $\beta(p/m) = \alpha(p/m)v(p/m)$ where

$$
v(p/m) = \exp\left(\frac{2i\epsilon_{jl}p^j s^l}{p_\perp} \arctg \frac{p_\perp}{m + \omega(p) + p^z}\right)
$$

(6.2)

is the Melosh matrix [100]. Here $p_\perp = |p_\perp|$, a sum over $j, l = x, y$ is assumed and $\epsilon_{jl}$ has the components $\epsilon_{xy} = -\epsilon_{yx} = 1, \epsilon_{xx} = \epsilon_{yy} = 0$. Then the generators of the UIR have the well known form (see, for
example, refs. [101, 102, 103, 104])

\[ \begin{aligned} P_F^+ & = p^+, & P_{F\perp} & = p_\perp, & P_F^- & = p^- = \frac{m^2 + p_\perp^2}{2p^+}, \\
M_F^{+-} & = ip^+ \frac{\partial}{\partial p^+}, & M_F^{+\perp} & = -ip^+ \frac{\partial}{\partial p^\perp}, & M_F^{x\perp} & = l^z(p_\perp) + s^z, \\
M_F^{-\perp} & = -i(p^+ \frac{\partial}{\partial p^+} + p^- \frac{\partial}{\partial p^-}) - \frac{\epsilon_{ij}}{p^+}(ms^l + p^l s^z) 
\end{aligned} \] (6.3)

The Hilbert space \( H_F \) for the representation of the Poincare group describing a system of \( N \) free or interacting particles is realized in the space of functions \( \phi(p_{1\perp}, p_1^+, \ldots p_{N\perp}, p_N^+) \) with the range in \( \mathcal{D}(s_1) \otimes \cdots \otimes \mathcal{D}(s_N) \) and such that

\[ \int ||\phi(p_{1\perp}, p_1^+, \ldots p_{N\perp}, p_N^+)||^2 N \prod_{i=1}^N d\rho_F^F(p_{i\perp}, p_i^+) < \infty \] (6.4)

Instead of the variables \( p_{1\perp}, p_1^+ , \ldots p_{N\perp}, p_N^+ \) we introduce the variables \( P_\perp, P^+, q_1, \ldots q_N \) where \( P_\perp = p_{1\perp} + \cdots + p_{N\perp}, P^+ = p_1^+ + \cdots + p_N^+ \) and \( q_i \) is the spatial part of the 4-vector \( q_i = L[\beta(P/M)]^{-1}p_i \) where \( P = p_1 + \cdots + p_N \) and \( M = |P| \). It is easy to see that \( k_i = L[v(P/M)]q_i \).

By analogy with Eq. (2.68) one can show that

\[ \prod_{i=1}^N d\rho_i^F(p_{i\perp}, p_i^+) = d\rho_F^F(P_\perp, P^+)d\rho_F^F(int), \]

\[ d\rho_F^F(int) = 2(2\pi)^3 M\delta(3)(q_1 + \cdots + q_N) \prod_{i=1}^N d\rho_i^F(q_i\perp, q_i^+) \] (6.5)

Let us define the ”internal’ space \( H_{int}^F \) as the space of functions \( \chi(q_1, \ldots q_N) \) with the range in \( \mathcal{D}(s_1) \otimes \cdots \otimes \mathcal{D}(s_N) \) and such that

\[ ||\chi||^2 = \int ||\chi(q_1, \ldots q_N)||^2 d\rho_F^F(int) < \infty \] (6.6)

and the space \( \tilde{H}_F \) as the space of functions \( \tilde{\phi}(P_\perp, P^+) \) with the range in \( H_{int}^F \) and such that

\[ \int ||\tilde{\phi}(P_\perp, P^+)||^2_{int}d\rho_F^F(P_\perp, P^+) < \infty \] (6.7)
The main reason for choosing $\beta(g)$ instead of $\alpha(g)$ (see, for example, ref. [103]) is that $\beta(g)$ satisfies the following important property

$$\beta(g)I\beta(L(l)^{-1}g) = 1$$  \hspace{1cm} (6.8)

Owing to this property the front analog of the operator $U$ given by Eq. (2.70) is equal to unity and the space $H_F$ coincides with $\tilde{H}_F$.

The method of Sokolov packing operators in the front form implies that the generators $\hat{\Gamma}_i^F$ ($i = 1, \ldots, 10$) of the representation describing a system of $N$ interacting particles should be written in the form $\hat{\Gamma}_i^F = A_F \hat{\Gamma}_i^F A_F^{-1}$ where the packing operator $A_F$ commutes with $P^+, P^j, M^{+-}, M^{+j}, M^{xy}$ ($j = x, y$), and the generators $\hat{\Gamma}_i^F$ in $H_F$ have the following ”canonical” form (compare with Eq. (6.3))

$$P_F^+ = P^+, \quad P_F^\perp = P_\perp, \quad \hat{P}_F^- = \frac{(\hat{M}_{int}^F)^2 + P_\perp^2}{2P^+},$$

$$M_{F+}^{+-} = iP^+ \frac{\partial}{\partial P^+}, \quad M_{F+}^{+j} = -iP^+ \frac{\partial}{\partial P^j}, \quad M_{F+}^{x+} = l^z(P_\perp) + S_z^F,$$

$$\hat{M}_{F-}^{+j} = -i(P^j \frac{\partial}{\partial P^+} + \hat{P}_F^- \frac{\partial}{\partial P^j}) - \frac{\epsilon_{jl}}{P^+} (\hat{M}_{int}^F S_F^l + P^l S_z^F).$$  \hspace{1cm} (6.9)

Here the first expression implies that the generator $P_F^+$ is equal to the operator of multiplication by the variable $P^+$ defined above and the second expression should be understood analogously.

The expressions for the system spin and mass operators $S_F$ and $\hat{M}_{int}^F$ differ from the corresponding expressions in the point and instant forms. The explicit expressions for $S_F$ and $\hat{M}_{int}^F$ have been first derived by Terentiev [102] for systems of two particles and by Berestetskii and Terentiev [106] for systems of three particles. As pointed out in ref. [86], the result of ref. [106] is inaccurate, and the authors of ref. [86] have derived a correct result (see also refs. [88, 43]). The explicit expressions defining the operator $A_F$ have been derived in refs. [91, 88, 43].

The investigation of the electromagnetic properties of mesons and baryons in the framework of constituent quark model in the front form was carried out by several authors (see, for example, refs. [107].
and references cited therein). However it was usually assumed that the ECO is the sum of the quark ECO’s.

### 6.2 Unitary equivalence of the point and front forms of dynamics

In Chap. 2 we have constructed the RQM in the point form assuming that the spin variables are defined using the matrix $\alpha(g)$ (see Eq. (2.3)) and the internal momenta are defined by Eq. (2.46). However it is possible to construct the description in the point form in terms of the light-front variables. This can be done by choosing the matrix $\beta(g)$ for the definition of the spin variables, the quantities $q_i$ for the definition of the internal momentum variables and the 4-velocity $G$ as the ”external” variable. In this case it is convenient to choose $(G^+, G_-)$ as three independent quantities defining $G$.

In particular, the UIR can be realized in the space of functions $\varphi(g_\perp, g^+)$ with the range in $\mathcal{D}(s)$ and such that

$$
(\varphi, \varphi) = \int ||\phi(g_\perp, g^+)||^2 d\rho^F(g_\perp, g^+) < \infty \quad (6.10)
$$

while the generators of the UIR have the form (compare with Eq. (5.3))

$$
P^+ = mg^+, \quad P_\perp = mg_\perp, \quad P^- = mg^- = m\frac{1 + g_\perp^2}{2g^+},
$$

$$
M^{+-} = ig^+ \frac{\partial}{\partial g^+}, \quad M^{+j} = -ig^+ \frac{\partial}{\partial g^+}, \quad M^{xy} = l^z(g_\perp) + s^z,
$$

$$
M^{-j} = -i(g^j \frac{\partial}{\partial g^+} + g^- \frac{\partial}{\partial g^j} - \frac{\epsilon_{jl}}{g^+} (s^l + g^l s^z)) \quad (6.11)
$$

As follows from Eq. (6.5)

$$
\prod_{i=1}^{N} d\rho_i^F(g_i, g_i^+) = d\rho^F(G, G^+)d\rho^P(int),
$$

$$
d\rho^P(int) = \frac{M^2}{m_1^2 \cdots m_n^2} d\rho^F(int) \quad (6.12)
$$
Let $H_{int}^P$ be the space of functions $\chi(q_1, \ldots, q_N)$ with the range in $D(s_1) \otimes \cdots \otimes D(s_N)$ and such that

$$||\chi||^2 = \int ||\chi(q_1, \ldots, q_N)||^2 d\rho^P(int) < \infty$$

(6.13)

and $\tilde{H}_P$ be the space of functions $\tilde{\varphi}(G_\perp, G^+)$ with the range in $H_{int}^P$ and such that

$$\int ||\varphi(G_\perp, G^+)||_{int}^2 d\rho^F(G_\perp, G^+) < \infty$$

(6.14)

Then by analogy with the results of the preceding section it can be shown that the space of the tensor product of the UIR’s describing particles $1, \ldots, N$ is just the space $H_P = \tilde{H}_P$.

In the given case the method of Sokolov packing operators implies that the generators $\hat{\Gamma}_i$ ($i = 1, \ldots, 10$) of the representation describing a system of $N$ interacting particles should be written in the form $\hat{\Gamma}_i = A_P \hat{\Gamma}_i A_P^{-1}$ where the packing operator $A_P$ commutes with $U(l)$ (as it should be in the point form) and the generators $\hat{\Gamma}_i$ in $H_P$ have the following ”canonical” form (compare with Eq. (6.9))

$$\hat{P}^+ = \hat{M}_{int}^P G^+, \quad \hat{P}_\perp = \hat{M}_{int}^P G_\perp, \quad \hat{P}^- = \hat{M}_{int}^P G^- =$$

$$\hat{M}_{int}^P \frac{1 + G_\perp^2}{2G^+}, \quad M^{++} = iG^+ \frac{\partial}{\partial G^+}, \quad M^{++} = -iG^+ \frac{\partial}{\partial G^+},$$

$$M^{xy} = l^z(G_\perp) + S^z_F, \quad M^{-j} = -i(G^j \frac{\partial}{\partial G^+} + G^- \frac{\partial}{\partial G^+}) -$$

$$\frac{\epsilon_{ij}}{G^+} (S^l_F + G^l S^z)$$

(6.15)

Here the spin operator $S_F$ has the same form as in Eq. (6.9) and $\hat{M}_{int}^P$ is the mass operator acting in $H_{int}^P$.

The operators $S_F$, $\hat{M}_{int}^P$ and $A_P$ must be unitarily equivalent to the operators $S$, $\hat{M}_{int}$ and $A$ respectively (see Sec. 2.4) since both these sets describe representations in the point form for different choices of the spin and momentum variables. The explicit expressions for the operators ($S_F$, $\hat{M}_{int}^P$, $A_P$) and the unitary operator relating the sets ($S_F$, $\hat{M}_{int}^P$) and ($S$, $\hat{M}_{int}$) in the cases $N = 2$ and $N = 3$ have been derived in refs. [6, 43].
The remainder of this section is an analog of Sec. 5.2. According to Sokolov and Shatny [77], the key element in proving the unitary equivalence of the point and front forms is the construction of the unitary operator $\Theta$ such that

$$\Theta \hat{\Gamma} \Theta^{-1} = \hat{\Gamma}_F$$  (6.16)

Let $U_{FP}$ be the unitary operator from $H_P$ to $H_F$ such that

$$U_{FP}\varphi(G_\perp, G^+) = \frac{\varphi(P_\perp/M, P^+/M)}{m_1 \cdots m_N},$$

$$U_{FP}^{-1}\phi(P_\perp, P^+) = m_1 \cdots m_N\tilde{\phi}(MG_\perp, MG^+)$$  (6.17)

The fact that $U_{FP}$ is unitary easily follows from Eqs. (6.6), (6.7), (6.12-6.14). By analogy with ref. [77] we shall seek $\Theta$ in the form

$$\Theta = U_{FP}\xi(M)^{-1}\xi(\hat{M}_{int}^P)$$  (6.18)

where the unitary operators $\xi(M)$ and $\xi(\hat{M}_{int}^P)$ in $H_P$ are defined by the spectral integrals analogous to those in Eq. (5.10). To avoid misunderstanding we note that the operators $\Theta$ and $\xi(m)$ in this section obviously differ from the corresponding operators in Sec. 5.2. However we use the same notations as in Sec. 5.2 in order to demonstrate that both construction are analogous.

Following ref. [77] we define the operators $\xi(m)$ and $\xi(m)^{-1}$ as (compare with Eq. (5.13))

$$\xi(m)\varphi(G_\perp, G^+) = \frac{m_0}{m}\varphi(\frac{m_0}{m}G_\perp, \frac{m_0}{m}G^+)$$

$$\xi(m)^{-1}\varphi(G_\perp, G^+) = \frac{m}{m_0}\varphi(\frac{m}{m_0}G_\perp, \frac{m}{m_0}G^+)$$  (6.19)

where $m_0 > 0$ is some constant. The fact that these operators are unitary easily follows from Eq. (5.12).

Let $O$ be an operator in $H_{int}$. We symbolically represent the action of $O$ in the form

$$O\varphi(G_\perp, G^+, int) = \int O(int, int')\varphi(G_\perp, G^+, int')d\rho^P(int')$$  (6.20)
where $O(int, int')$ is the kernel of the operator $O$. Let us introduce the operator

$$F_F\{O\} \equiv U_{FP}\xi(M)^{-1}O\xi(M)U_{FP}^{-1}$$

(6.21)

Then, as follows from Eqs. (6.6), (6.7), (6.12-6.14), (6.17) and (6.19-6.21), $F_F\{O\}$ is the operator in $H_{int}^F$. If we write symbolically

$$F_F\{O\}\phi(P_\perp, P^+, int) =$$

$$= \int F_F\{O\}(int, int')\phi(P_\perp, P^+, int')d\rho_F(int')$$

(6.22)

then the kernels of the operators $O$ and $F_F\{O\}$ are related as

$$F_F\{O\}(int, int') = \frac{MM'}{m_1^2 \cdots m_N^2}O(int, int')$$

(6.23)

Now using Eqs. (6.19) and (6.21) we can explicitly verify that Eq. (6.16) is satisfied if

$$\hat{M}_{int}^F = F_F\{\hat{M}_{int}^P\}$$

(6.24)

In other words, the operator defined by Eq. (6.18) is the operator realizing the unitary equivalence of the sets defined by Eqs. (6.15) and (6.9).

As shown in ref.[77], if the operators $A_P$, $\hat{M}_{int}^P$, and $\Theta$ are known then it is possible to determine $A_F$ and conversely, if $A_F$, $\hat{M}_{int}^F$, and $\Theta$ are known then it is possible to determine $A_P$. Therefore the operator $\hat{U}_{FP} = A_F\Theta A_P^{-1}$ realizes the unitary equivalence of the point and front forms.

The advantages of the front form in the Feynman diagram approach were first pointed out by Weinberg [110] (see also refs. [111, 112]). At present different physicists believe that important problems in the strong interaction theory can be solved by using quantum field theory in the front form (see, for example, refs. [113, 114, 115] and references cited therein). The major difference between this approach and conventional ones is in treating the vacuum problem.

In RQM the vacuum problem does not arise and there are no principal differences between the front form and other ones. One might think that the front form has technical advantages since the
generators for the two-body problem can be easily transformed to the form (6.9) without using Wigner rotations. However in cases of three and more particles the front form has serious practical disadvantages since the spin operators for the system as a whole necessarily depend on interactions (see, for example, ref. [43]). In any case, the results of ref. [77] show that in RQM the three basic forms are unitarily equivalent, and therefore we can stress once more that in the framework of RQM the choice of the form is only the matter of convenience but not the matter of principle.

6.3 Explicit construction of the ECO in the front form of dynamics

If \( \hat{J}_F^\mu(x) \) is the ECO in the front form, then, by analogy with Eq. (3.1) we can write

\[
\hat{J}_F^\mu(x) = \exp(i\hat{P}_F x)\hat{J}_F^\mu(0)\exp(-i\hat{P}_F x) \tag{6.25}
\]

and, by analogy with Eq. (3.2), one can easily show that \( \hat{J}_F^\mu(0) \) must satisfy the properties

\[
\hat{U}_F(l)^{-1}\hat{J}_F^\mu(0)\hat{U}_F(l) = L(l)^\mu_\nu \hat{J}_F^\nu(0), \quad [\hat{P}_{F\mu}, \hat{J}_F^\mu(0)] = 0 \tag{6.26}
\]

In turn, by analogy with Eq. (3.4), we shall seek \( \hat{J}_F^\mu(0) \) in the form

\[
\hat{J}_F^\mu(0) = A_F \hat{J}_F^\mu A_F^{-1} \tag{6.27}
\]

Then it is obvious from the results of Sec. 6.2 that the operator \( \hat{J}_F^\mu(x) \) will satisfy all the properties described in Sec. 1.1 if

\[
\hat{J}_F^\mu = \Theta \hat{J}^\mu \Theta^{-1} \tag{6.28}
\]

Let us note however that in order to explicitly calculate the ECO in the front form using such a prescription we have to determine first the action of the operator \( \hat{J}^\mu \) in the point form in terms of light front variables. It is easy to see that the scheme of constructing the operator \( \hat{J}^\mu \) described in Chaps. 3 and 4 can be also used for constructing the operator \( \hat{J}_F^\mu \) in these variables. Let us briefly discuss some aspects of such a construction.
The one-particle ECO in the light front variables can be constructed by analogy with the construction in the usual variables (see Sec. 4.1). For example, the expression defining the ECO for a spin 1/2 particle is the same as Eq. (4.7) but the usual Dirac bispinor \( u(p, \sigma) \) should be replaced by the light front bispinor \( w(p, \sigma) \). In the spinorial representation of the Dirac \( \gamma \) matrices \( u(p, \sigma) \) is the bispinor with the components \( \sqrt{m}(\alpha(p/m) \chi(\sigma), \alpha(p/m)^{-1}\chi(\sigma)) \) where \( \chi(\sigma) \) is the usual spinor with the \( z \) projection of the spin equal to \( \sigma \) while \( w(p, \sigma) \) is the bispinor with the components \( \sqrt{m}(\beta(p/m) \chi(\sigma), \beta(p/m)^{-1}\chi(\sigma)) \).

In the general case the action of \( \hat{J}^\mu \) can be again written in the form of Eq. (3.5) where \( \hat{M}_{\text{int}} \) is the mass operator in the light front variables and \( \hat{J}^\mu(G, G') = \hat{J}^\mu(G_\perp, G^+; G'_\perp, G'^+) \) (note that \( d\rho(G) = d\rho^F(G_\perp, G^+) \)). Then Eq. (3.7) will be valid if \( \alpha \) is replaced by \( \beta \). However since \( \beta(G) \) satisfies Eq. (6.8), Eq. (3.7) in the light front variables has a much more simple form

\[
\hat{J}^\mu(G, G') = L(l)^\mu_\nu \hat{J}^\nu(L(l)^{-1}G, L(l)^{-1}G')
\]

(6.29)

Now we use \( \beta(G, G') \) to denote \( \beta((G+G')/(|G+G'|)) \in \text{SL}(2, \mathbb{C}) \) and \( L_F(G, G') \) to denote the Lorentz transformation \( L[\beta(G, G')] \). Instead of \( f \) and \( f' \) defined by Eq. (3.8) we introduce

\[
f_F = L_F(G, G')^{-1}G, \quad f'_F = L_F(G, G')^{-1}G'
\]

(6.30)

Then it is easy to see that

\[
f_F^0 = f'_F^0 = f^0, \quad h_F = -h'_F = L[v(G + G')/(|G + G'|)]^{-1}h
\]

(6.31)

By analogy with Eq. (3.10) we can express \( \hat{J}^\mu(G, G') \) in terms of the operator \( \hat{j}^\mu(h_F) \) depending on one three-dimensional vector \( h_F \). However as follows from Eq. (6.29), the corresponding expression has a much more simple form than Eq. (3.10):

\[
\hat{J}^\mu(G, G') = L_F(G, G')^\mu_\nu \hat{J}^\nu(h_F)
\]

(6.32)
On the other hand, the choice of the light front variables has some serious practical disadvantages. In particular, the expressions describing the transformation of $\hat{j}^\nu(h_F)$ relative to the spatial rotations and the space reflection have a much more complicated form than Eqs. (3.12) and (3.47).

Let us note that the quantity $\hat{j}^0(0)$ in the light front variables also must be equal to the electric charge of the system under consideration, and a possible solution of the problem of constructing the ECO can be obtained assuming that the operators $w_0(h_F)$ and $w_\perp(h_F)$ in the light front variables are equal to zero. Here $w_\perp(h_F)$ is the part of $w(h_F)$ orthogonal to $h_F$.

Now we return to the front form. Here the action of $\hat{J}_F^{\mu}$ can be written as

$$\hat{J}_F^{\mu}\phi(P_\perp, P^+) = \int \hat{J}_F^{\mu}(P_\perp, P^+; P_\perp', P'^+)\phi(P_\perp', P'^+)d\rho^F(P_\perp', P'^+)$$

(6.33)

In turn, by analogy with Eq. (3.21), the operator

$$\hat{J}_F^{\mu}(P_\perp, P^+; P_\perp', P'^+)$$

can be defined by the set of operators

$$\hat{J}_F^{\mu}(P_\perp, P^+, m; P_\perp', P'^+, m')$$

such that

$$\hat{J}_F^{\mu}(P_\perp, P^+; P_\perp', P'^+) = \int \int d\hat{e}_F^{\text{int}}(m) \cdot \hat{J}_F^{\mu}(P_\perp, P^+, m; P_\perp', P'^+, m')d\hat{e}_F^{\text{int}}(m')$$

(6.34)

where $\hat{e}_F^{\text{int}}(m)$ is the spectral function of the operator $\hat{M}_F^{\text{int}}$.

On the other hand, by analogy with Eq. (3.21), the kernel

$$\hat{J}^{\mu}(G_\perp, G^+; G_\perp', G'^+)$$

can be defined by the set of operators

$$\hat{J}^{\mu}(G_\perp, G^+, m; G_\perp', G'^+, m')$$

such that

$$\hat{J}^{\mu}(G_\perp, G^+; G_\perp', G'^+) = \int \int d\hat{e}_\text{int}^{P}(m) \cdot \hat{J}^{\mu}(G_\perp, G^+, m; G_\perp', G'^+, m')d\hat{e}_\text{int}^{P}(m')$$

(6.35)
where $\hat{e}^P_{int}(m)$ is now the spectral function of the operator $\hat{M}^P_{int}$ in the light front variables. Then a direct calculation using Eqs. (6.12), (6.17-6.24), (6.28) and (6.33-6.35) shows that

$$\hat{J}^\mu_F(\mathbf{P}_\perp, P^+, m'; \mathbf{P}'_\perp, P'^+, m') = 2(mm')^{1/2} F_F\{\hat{J}^\mu(\frac{\mathbf{P}_\perp}{m}, \frac{P^+}{m}; \frac{\mathbf{P}'_\perp}{m'}, \frac{P'^+}{m'}, m')\}$$ (6.36)

We conclude that if the ECO in the point form is constructed in terms of the light front variables then Eqs. (6.25), (6.27), (6.33), (6.34), and (6.36) (where $F_F$ is defined by Eq. (6.23)) make it possible to explicitly construct the ECO in the front form.

### 6.4 Matrix elements of the ECO in the front form of dynamics

The one-particle states in the light front variables are usually chosen in the form

$$|p', \sigma'\rangle_F = 2(2\pi)^3 p'^+ \delta^{(2)}(\mathbf{p}_\perp - \mathbf{p}'_\perp) \delta(p^+ - p'^+) \delta_{\sigma\sigma'}$$ (6.37)

Then as follows from Eq. (6.1)

$$F\langle p'', \sigma''|p', \sigma'\rangle_F = 2(2\pi)^3 p'^+ \delta^{(2)}(\mathbf{p}_\perp'' - \mathbf{p}'_\perp) \delta(p''^+ - p'^+) \delta_{\sigma''\sigma'}$$ (6.38)

It is easy to see that this normalization is equivalent to the normalization in the usual variables given by Eq. (5.46).

To define the scattering states in the N-particle case we have to solve the eigenvalue problem for the operator $\hat{M}^F_{int}$ in the space $H^F_{int}$. Let $\chi'_F \in H^F_{int}$ be the internal wave function describing a scattering state. If $\chi'_F$ is the eigenfunction of the operator $\hat{M}^F_{int}$ with the eigenvalue $M'$ and the scattering state is the eigenstate of the operators $P^\perp$ and $P^+$ with the eigenvalues $P'_\perp$ and $P'^+$ respectively then, as follows from Eq. (6.9), the wave function of such a state in the space $H_F$ can be written as

$$|P', \chi'_F\rangle_F = A_F 2(2\pi)^3 P'^+ \delta^{(2)}(\mathbf{P}_\perp - \mathbf{P}'_\perp) \delta(P^+ - P'^+) \chi'_F$$ (6.39)
It is obvious from Eq. (6.9) that this state is the eigenstate of the operator $\hat{P}^-$ with the eigenvalue $P^- = (M'^2 + P'^2_\perp)/2P'^+$. As follows from Eq. (6.33), the normalization condition for the wave function given by Eq. (6.39) has the form

$$F\langle P'', \chi''_F | P', \chi'_F \rangle_F = 2(2\pi)^3 P'^+ \delta(2) (P'' - P'_\perp) \cdot \delta(P'^+ - P'_\perp) F\langle \chi''_F | \chi'_F \rangle_F$$

(6.40)

where the last scalar product should be calculated only in the space $H_{int}^F$.

By analogy with the considerations in Secs. 3.5 and 5.4 we conclude that all matrix elements of the operator $\hat{J}_\mu^F(x)$ can be expressed in terms of the quantities $F\langle P'', \chi''_F | \hat{J}_\mu^F(x) | P', \chi'_F \rangle_F$. As follows from Eqs. (6.25-6.28), (6.33) and (6.34)

$$F\langle P'', \chi''_F | \hat{J}_\mu^F(x) | P', \chi'_F \rangle_F = \exp(\Delta x) F\langle \chi''_F | \int \int d\hat{e}_{int}(m) \cdot \hat{J}_\mu^F(\mathbf{P}_\perp'', \mathbf{G}''^+, m; \mathbf{P}'_\perp, \mathbf{G}'^+, m') d\hat{e}_{int}(m') | \chi'_F \rangle_F$$

(6.41)

where $\Delta = P'' - P'$ and the matrix element on the right-hand-side must be calculated only in the space $H_{int}^F$.

On the other hand, as follows from Eqs. (3.3), (3.52) and (5.35), in the point form the matrix elements of the ECO in terms of the light front variables can be written as

$$P\langle P'', \chi''_P | \hat{J}_\mu^P(x) | P', \chi'_P \rangle_P = 2(M'' M')^{1/2} \exp(\Delta x) \cdot P\langle \chi''_P | \int \int d\hat{e}_{int}(m) \hat{J}_\mu^P(\mathbf{G}_\perp'', \mathbf{G}''^+, m; \mathbf{G}_\perp', \mathbf{G}'^+, m') \cdot d\hat{e}_{int}(m') | \chi'_P \rangle_P$$

(6.42)

If $\chi'_P \in H_{int}^P$ is the eigenfunction of the operator $\hat{M}_\perp^P$ with the eigenvalue $M'$ then, as follows from Eqs. (6.21) and (6.24),

$$\chi'_R = U_{FP} \xi(M)^{-1} \chi'_P$$

is the eigenfunction of the operator $\hat{M}_\perp^P$ with the same eigenvalue. Taking into account the definition of the operation $F_F$ (see Eq. (6.21)) and Eq. (6.36) we conclude that

$$F\langle P'', \chi''_F | \hat{J}_\mu^F(x) | P', \chi'_F \rangle_F = P\langle P'', \chi''_P | \hat{J}_\mu^P(x) | P', \chi'_P \rangle_P$$

(6.43)
This result shows that the matrix elements of the ECO do not depend on the choice of the form of dynamics as it should be.
Chapter 7

Discussion and conclusions

Let us discuss the main results of the present paper.

In Chap. 2 we explicitly construct the description of systems of two and three particles in the point form of relativistic dynamics. For this purpose we derive explicit expressions for the Sokolov packing operators in the case of particles with arbitrary spin. We also note that it is possible to explicitly describe systems with any number of particles.

In Chap. 3 the problem of constructing the operator \( \hat{J}^\mu(x) \) is reduced to the problem of constructing the operator \( \hat{j}^\nu(h) \). The results of this chapter are based on the assumption that the representation of the Poincare group for the system under consideration is realized in the point form. However we do not assume that the system is described in the framework of RQM. Therefore the results can be in principle applied also to the problem of constructing the ECO for systems of quantized fields.

It has been known for a long time that the matrix elements of the one-particle ECO become especially simple in the Breit frame, i.e. in the reference frame where the momenta of the initial and final states \( p' \) and \( p'' \) satisfy the relation \( p'' + p' = 0 \). This relation can also be written as \( g'' + g' = 0 \) where \( g' \) and \( g'' \) are the 4-velocities in the initial and final states. There exists an obvious analogy between the Breit frame and the c.m.frame of two interacting particles. The latter is defined by the condition \( p_1 + p_2 = 0 \), where \( p_i \) is the momentum of particle \( i \), and the interaction between the particles has the simplest
form just in the c.m.frame. The difference between the Breit frame
and the c.m.frame is that the former is defined for one and the same
particle in the initial and final states while the latter is defined for two
different particles.

If the system under consideration has the mass spectrum con-
sisting of more than one point then the conditions $P'' + P' = 0$ and
$G'' + G' = 0$ are not equivalent. The Lorentz transformation which
transfers the 4-vectors $P'$ and $P''$ to the reference frame where $P'' + P' = 0$
depends not only on $P'$ and $P''$ but also on the masses $M'$ and $M''$ in
the initial and final states. Therefore for different pairs $(M', M'')$ there
exist different Lorentz transformations transferring $P'$ and $P''$ to the
reference frame where $P'' + P' = 0$. On the other hand, the Lorentz
transformations transferring $G'$ and $G''$ to the reference frame where
$G'' + G' = 0$ depends only on $G'$ and $G''$. Taking also into account
that the representation operators of the Lorentz group do not depend
on interactions in the point form we conclude that the transition from
the operator $\hat{J}^{\mu}(G', G'')$ discussed in Sec. 3.1 is
much more simple than the analogous transitions in the instant and
front forms.

The vector $f$ defined by Eq. (3.8) is the velocity analog of the
momentum in the c.m.frame. We have seen in Chap. 2 that the role
of the external variables in the point form is played by velocities, not
momenta, but the role of the internal variables may play momenta, as
usual. It is possible to consider such a version of the point form where
the role of the internal variables is also played by velocities. Such a
version was first considered by Ruijgrok and coauthors [116, 117]. It is
easy to show that the ordinary approach to RQM can be formulated in
terms of the Ruijgrok approach and vica versa. However we see that
in the problem of constructing the ECO the analog of the Ruijgrok
approach has considerable advantages.

Combining the results of Chaps. 2 and 3 we derive in Chap. 4
the exact solution for the ECO in the framework of RQM. This solution
is not unique since it is easy to write down a variety of operators $w^0(h)$
and $w_{\perp}(h)$ satisfying Eq. (4.24) and cluster separability. The fact
that relativity and current conservation do not impose considerable restrictions on the operator $\hat{J}^\mu(x)$ was noted by several authors [9, 10, 13, 44]. For example, in ref. [44] the problem of constructing the operator $\hat{J}^\mu(x)$ was studied on the language of matrix elements of this operator and it was shown that after taking into account the constraints imposed by relativity and current conservation, there would exist a minimal (usually infinite) set of unconstrained matrix elements.

Though the solution of the problem under consideration is not unique, the very fact that the ECO satisfying the properties specified in Sec. 1.1 can be explicitly constructed seems very important. In particular we expect that for various electromagnetic processes the theoretical predictions obtained by using a correct ECO will considerably differ from the predictions obtained by using ECO’s not satisfying the above properties. Our solution for the ECO is most general in the sense that if we assume that interactions between the constituents can be described in the framework of RQM then any model for the ECO should correspond to a certain choice of the operators $w^0(h)$ and $w_\perp(h)$. At the same time it is necessary to investigate what additional constraints should be imposed on the ECO in order to make the solution unique.

Our solution for the ECO is described in the point form of dynamics while, as noted in Sec. 1.2, the most popular forms are the instant and front ones. One might try to construct the ECO in these forms using only the representation operators of the Poincaré group in the corresponding form. However, as argued above, such a construction is expected to be more difficult than in the point form. In Chaps. 5 and 6 we show that these difficulties can be bypassed with the help of the results by Sokolov and Shatny [77] who have proved that all three basic forms of dynamics are unitarily equivalent. Namely, using the unitary operators relating these forms we can explicitly construct the ECO in the instant and the front forms if the solution in the point form has been already constructed. Analogously, using the unitary operators of ref. [79] relating the $\xi$-picture and the front form we can construct the ECO in the $\xi$-picture considered in ref. [79].

We conclude that the results of this paper can be the basis
for systematic calculations of different electromagnetic observables in nuclear physics and constituent quark models. Such calculations are necessary for explaining the existing experimental data and planning future experiments on powerful electron accelerators.

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