Operator Approach to Boundary Liouville Theory

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Abstract

We propose new methods for calculation of the discrete spectrum, the reflection amplitude and the correlation functions of boundary Liouville theory on a strip with Lorentzian signature. They are based on the structure of the vertex operator \(V = e^{-\phi}\) in terms of the asymptotic operators. The methods first are tested for the particle dynamics in the Morse potential, where similar structures appear. Application of our methods to boundary Liouville theory reproduces the known results obtained earlier in the bootstrap approach, but there can arise a certain extension when the boundary parameters are near to critical values. Namely, in this case we have found up to four different equidistant series of discrete spectra, and the reflection amplitude is modified respectively.

Keywords: Liouville theory, strings and branes, 2d conformal symmetry, boundary conditions, canonical quantization;

PACS: 11.10. Ef; 11.10. Kk; 11.10. Lm; 11.25. Hf; 11.25. Pm; 11.25. -w;
1 Introduction

Boundary Liouville theory (BLT) has been studied intensively [1] in the beginning of the 80’s in parallel to the periodic [2] and unbounded [3] cases. The main motivation for those investigations was Polyakov’s non-critical string theory, but due to the rich integrable structure of Liouville theory, the obtained results contributed to other areas (2d conformal field theory, 2d gravity, quantum groups, etc) as well. The interest in the boundary theory was renewed by the end of the 90’s in connection to the dynamics of branes. The new results of the last decade [4–8] were obtained mostly by the bootstrap method in the Euclidean formulation of the theory, whereas the approaches of the 80’s mainly concerned the space-time with Lorentzian signature.

In this paper we continue the investigation of BLT started in [9]. We study the theory given on a strip with Lorentzian signature and use the operator approach in the Heisenberg picture. In [9] we have constructed the vertex operator $V$, corresponding to the Liouville field exponential $e^{-\varphi}$. It is the simplest vertex operator, which is the basic building block in the vertex operators calculus. This operator can also be used in calculations of the scattering data of the theory. The operator $V$ was constructed similarly to the periodic case [2, 10] using a free-field parameterization, the conformal symmetry and the conditions of locality and Hermiticity. The parameterizing field in [9] was chosen to be the $\text{in}$-field of the theory. The corresponding Liouville field configurations describe scattering processes, and they belong to the hyperbolic monodromy.

For certain values of the boundary parameters the theory also contains bound states, which are absent in the periodic case. The monodromy for those field configurations switches to the elliptic one and the free-field parameterization fails there. More precisely, the free-field becomes complex with non-linearly related real and imaginary parts. This case was studied in [1] using a parameterization in terms of two fields, but the fields were not completely free and an exact description of the Hilbert space, like one has for the hyperbolic monodromy, is missing.

The aim of the present paper is to explore the structure of the vertex operator $V$ and to use it for the calculation of the reflection amplitude, the discrete spectrum and the correlations functions. These calculations can be compared with the results obtained by the bootstrap method in the Euclidean formulation of the theory [4–8]. The connection between the Euclidean and Minkowskian formulations of Liouville theory are well established in the periodic case [11–14] and it is a challenge to find the relation for the boundary theory as well. Due to the rich spectral picture of BLT, the operator approach seems to be appropriate and becomes indeed effective.

The paper is organized as follows: In Section 2 we first briefly summarize the main results of [9]. Then we introduce a generic vertex operator $V_\alpha$ given as an expansion in powers of the screening charges. We calculate the first few coefficients of this expansion and establish the relation between the parameters of the operator $V$ and the mass and boundary parameters of the theory. In Section 3 we consider particle dynamics in the Morse potential as a zero mode approximation to BLT. Here we develop a new scheme for quantum mechanical calculations. Namely, we show how from the structure of the operator $V$ one
can read off the discrete spectrum of the theory. We also present a method for calculation of
the phase of the reflection amplitude providing its integral representation. In addition, we
calculate correlation functions. These methods are applied to BLT in Section 4. First we
project the operator \( V \) on the vacuum sector, where it becomes similar to the corresponding
operator of the particle model in the Morse potential. We find the discrete spectrum of
the vacuum sector and calculate the reflection amplitude. Then we calculate the 1-point
function corresponding to our vertex operator, and compare our calculations to the results
obtained by the bootstrap method. In the last section we discuss open problems of the
operator approach, such as the quantization of the bound state sector, the S-matrix, etc.
Four appendices contain technical details and useful formulae.

2 Free-field description of BLT

2.1 The classical theory

The Liouville field \( \varphi(\tau, \sigma) \) is given on the strip \((\tau, \sigma) : \tau \in \mathbb{R}, \sigma \in (0, \pi)\) and its dynamics
is described by the action

\[
S = \frac{1}{2\pi} \int_0^\pi d\sigma \left[ (\partial_\tau \varphi)^2 - (\partial_\sigma \varphi)^2 - 4m^2 e^{2\varphi} \right] - \frac{2m}{\pi} \int d\tau \left[ l e^{\varphi(\tau,0)} + r e^{\varphi(\tau,\pi)} \right],
\]

(2.1)

where \( l \) and \( r \) are the boundary parameters in units of \( m \) \((m > 0)\). Note that the action
used in [9] differs from (2.1) by the factor \( \pi \).

The variation of (2.1) yields the Liouville equation and the boundary conditions

\[
\partial_\sigma \varphi|_{\sigma=0} = 2ml e^{\varphi}|_{\sigma=0} \quad \partial_\sigma \varphi|_{\sigma=\pi} = -2mr e^{\varphi}|_{\sigma=\pi}.
\]

(2.2)

These conditions provide \( \bar{T}(\bar{x}) = T(x) \) and \( T(\tau + 2\pi) = T(\tau) \), where \( T(x) = (\partial_x \varphi)^2 - \partial_{xx}^2 \varphi \)
and \( \bar{T}(\bar{x}) = (\partial_{\bar{x}} \varphi)^2 - \partial_{\bar{xx}}^2 \varphi \) are the chiral \((x = \tau + \sigma)\) and the anti-chiral \((\bar{x} = \tau - \sigma)\)
components of the energy-momentum tensor, respectively. Due to the periodicity of \( T(x) \)
the description of the boundary theory becomes similar to the periodic case. Namely, one
can classify the Liouville fields by the orbits of the group of diffeomorphisms of \( S^1 \) [15].

The Liouville field exponential \( V = e^{-\varphi} \) satisfies the equation \( \partial_{xx}^2 V = T(x)V \) and for a
constant energy-momentum tensor \( T(x) = T_0 \) it is given by

\[
V_0 = \frac{2m}{p \sinh \pi p} \left[ \sqrt{\Lambda(l, r; p)} \cosh p(\tau - \tau_0) + l \cosh p(\sigma - \pi) + r \cosh p\sigma \right], \quad (2.3)
\]

where \( \tau_0 \) is an arbitrary constant, \( p = 2\sqrt{T_0} \) and

\[
\Lambda(l, r; p) = l^2 + r^2 + 2lr \cosh p\sigma + \sinh^2 p\pi.
\]

(2.4)

For \( T_0 > 0 \) the solutions (2.3) describe a scattering process with \( \varphi \to -\infty \) at the time
asymptotics \( \tau \to \pm\infty \), whereas the case \( T_0 < 0 \) \((p = i\theta)\) corresponds to bound states
oscillating in time.
Necessary conditions for positivity of $V_0$ in the whole bulk $\sigma \in (0, \pi)$ are: $l \geq -1$ and $r \geq -1$. In addition, there are restrictions on allowed values of $T_0$. Here one has to distinguish two cases: $l + r \geq 0$ and $l + r < 0$. The first case describes only scattering processes: $T_0$ can not take negative values, but all positive values are allowed. Bound states come into the game only for $l + r < 0$. In this case $T_0 \geq -\frac{1}{4} \theta_*^2$, where $|\theta_*| \leq 1$ and it is obtained from $\Lambda(l, r; i\theta_*) = 0$ by

$$\cos \pi \theta_* = -lr + \sqrt{(1-l^2)(1-r^2)}.$$  \hfill (2.5)

The minimal allowed value $T_0 = -\frac{1}{4} (|\theta_*| = 1)$ is achieved for $l = r = -1$, and the corresponding Liouville field configuration $V_0 = 2m \sin \sigma$ is invariant under the $SL(2, \mathbb{R})$ conformal transformations.

The general $V$-field is obtained by the action of the 2d conformal group on (2.3)

$$V_0(x, \bar{x}) \rightarrow V(x, \bar{x}) = (\xi'(x)\xi'(-\bar{x}))^{-\frac{1}{2}} V_0(\xi(x), \xi(\bar{x})).$$  \hfill (2.6)

The functions $\xi(x)$ here are monotonic $\xi'(x) > 0$ and satisfy the condition $\xi(x + 2\pi) = \xi(x) + 2\pi$. The variable $\tau_0$ in (2.3) is absorbed by the zero mode of $\xi(x)$ and we find the following parameterization of the Liouville field in terms of $(T_0, \xi(x))$

$$V(x, \bar{x}) = \frac{m}{p \sinh \pi p} \left( (\sqrt{\Lambda(l, r; \xi)} - \frac{\xi'((\xi(x) + \xi(\bar{x})) + \sqrt{\Lambda(l, r; \xi)} e^{\frac{\pi}{p}(\xi(x) + \xi(\bar{x}))} + \right) \left( le^{-p\pi} + r \right) e^{\frac{\pi}{p}(\xi(x) - \xi(\bar{x}))} + \left( le^{-p\pi} + r \right) e^{\frac{\pi}{p}(\xi(x) - \xi(\bar{x}))} \right), \ p = 2\sqrt{T_0}. \hfill (2.7)$$

The action (2.1) defines the canonical 2-form of the theory, which in terms of $(T_0, \xi(x))$ reads

$$\omega = \delta T_0 \wedge \int_0^{2\pi} dx \xi'(x)\delta \xi(x) + T_0 \int_0^{2\pi} dx \delta \xi'(x) \wedge \delta \xi(x) + \frac{1}{4} \int_0^{2\pi} dx \frac{\delta \xi''(x)}{\xi'^2(x)} \wedge \delta \xi'(x).$$  \hfill (2.8)

The energy-momentum tensor calculated from (2.7) is given by

$$T(x) = \frac{\partial^2 V}{\partial^2 x} = T_0 \xi'^2(x) + \left( \frac{\xi''(x)}{2\xi'(x)} \right)^2 - \left( \frac{\xi''(x)}{2\xi'(x)} \right), \hfill (2.9)$$

and the symplectic form (2.8) provides the Poisson brackets \{\(T(x), \xi(y)\}\} = $\xi(y) \delta(x - y)$ and \{\(T(x), T_0\}\} = 0, which means that $T(x)$ is the generator of the conformal transformations.

From (2.8) one can also get the canonical Poisson bracket \{\(T_0, \xi_0\}\} = 1, where $\xi_0$ is the zero mode of $\xi(x)$: $\xi_0 = \int_0^{2\pi} dx [\xi(x) - x]$. If $T_0$ is negative, the solution (2.7) oscillates in time and the zero mode $\xi_0$ becomes cyclic. Therefore in quantum theory the negative values of $T_0$ have to be quantized. The semi-classical calculation of these levels is in accordance with the spectrum obtained by the bootstrap method [5,8].
The pair \((T_0, \xi(x))\) with \(T_0 > 0\) can be mapped to the chiral free-field
\[
\phi(x) = \frac{p\xi(x)}{2} + \frac{1}{2} \log \xi'(x) - \frac{1}{2} \log \frac{m \sqrt{\Lambda(l, r; p)}}{p \sinh \pi p},
\]
which can be expanded in the Fourier modes
\[
\phi(x) = \frac{1}{2} (q + px) + i \sum_{n \neq 0} a_n \frac{e^{-inx}}{2n},
\]
with \(p > 0\). The Poisson brackets of these variables are canonical
\[
\{p, q\} = 1, \quad \{a_n, a_m\} = i n \delta_{n+m, 0},
\]
since the symplectic form \((2.8)\) becomes canonical in terms of \(\phi(x)\), and the energy-momentum tensor \((2.9)\) takes the free-field form with an improvement term
\[
T(x) = \phi'^2(x) - \phi''(x).
\]

The map \((2.10)\) is invertible and by \((2.7)\) we obtain the free-field parameterization
\[
V(x, \bar{x}) = e^{-[\phi(x) + \phi(\bar{x})]} \left[1 + m B_p A(x) + m C_p A(\bar{x}) + m^2 D_p A(x) A(\bar{x})\right],
\]
with
\[
B_p = \frac{le^{-\pi p} + r}{\sinh \pi p}, \quad C_p = \frac{le^{\pi p} + r}{\sinh \pi p}, \quad D_p = \frac{\Lambda(l, r; p)}{\sinh^2 \pi p}
\]
and
\[
A(x) = \frac{e^{-\pi p}}{2 \sinh \pi p} \int_0^{2\pi} dy e^{2\phi(x+y)}.
\]

Here \(A(x)\) is the standard ‘screening charge’: \(A'(x) = e^{2\phi(x)}\). It is related to \(A_p(x)\) used in \([9]\) by the \(p\)-dependent factor \(A_p(x) = 2 \sinh \pi p A(x)\).

Since \(p > 0\), we find that \(\Phi(x, \bar{x}) = \phi(x) + \phi(\bar{x})\) is the in-field of the theory, and the term \(m^2 D_p A(x) A(\bar{x}) e^{-\Phi(x, \bar{x})}\) in \((2.14)\) corresponds to the out-field exponent.

It has to be mentioned that for \(T_0 < 0\) the field \(\phi(x)\) becomes complex and the free-field parameterization fails. Peculiarities of this case will be discussed in Section 5.

### 2.2 Quantization

We work in dimensionless variables and, to match with the notations used in the bootstrap approach, from now we denote the measure of non-commutativity of the canonical variables by \(2b^2\)
\[
[q, p] = i\hbar, \quad [a_m, a_n] = \hbar m \delta_{m+n, 0}, \quad \hbar = 2b^2.
\]
Note also that the Liouville field used in \([4, 5]\) is rescaled by a factor \(b^{-1}\) relative to our \(\varphi\).
Quantization of the system in the free-field variables assumes a realization of the commutation relations (2.17) in the Fock space with a \( p \)-dependent vacuum \( |p, 0\rangle \) \((p > 0)\), defined in a standard way
\[
a_n|p, 0\rangle = 0, \quad n > 0. \tag{2.18}
\]
We use the same notations for classical and corresponding quantum expressions, which, in general, are deformed in a consistent way to satisfy the principles of quantum theory.

The operators for chiral free-field exponentials \( e^{2\alpha\phi(x)} \) are defined in the normal ordered form
\[
e^{2\alpha\phi(x)} = e^{2\alpha\phi_0(x)} e^{2\alpha\phi_+(x)} e^{2\alpha\phi_-(x)}, \tag{2.19}
\]
where
\[
\phi_0(x) = \frac{q}{2} + \frac{px}{2}, \quad \phi_+(x) = -i \sum_{n>0} \frac{a_{-n}}{2n} e^{inx}, \quad \phi_-(x) = i \sum_{n>0} \frac{a_n}{2n} e^{-inx}, \tag{2.20}
\]
and the operator for the screening charge (2.16) reads
\[
A(x) = \frac{e^{-\pi p x^2}}{2 \sinh \pi p x^2} \int_0^{2\pi} dy \ e^{2\phi(x+y)}. \tag{2.21}
\]
The quantum energy-momentum tensor then becomes
\[
T(x) = \phi''(x) - (1 + b^2) \phi''(x), \tag{2.22}
\]
with a normal ordered \( \phi''(x) \)-term. The screening charge is a conformal scalar and the conformal dimension of \( e^{2\alpha\phi(x)} \) is (using the notation \( Q = b^{-1} + b \) for the background charge)
\[
\Delta_\alpha = \alpha (1 + b^2 - ab^2) = ab(Q - ab). \tag{2.23}
\]
The chiral operators \( e^{2\alpha\phi(x)} \) and \( A(x) \), together with \( p \)-dependent coefficients, are the building blocks for the vertex operators of the theory. They satisfy the following exchange relations
\[
f_p e^{2\alpha\phi(x)} = e^{2\alpha\phi(x)} f_{p-2iab^2}, \quad f_p A(x) = A(x) f_{p-2ib^2}, \tag{2.24}
\]
\[
e^{2\alpha\phi(x)} e^{2\beta\phi(y)} = e^{2\beta\phi(y)} e^{2\alpha\phi(x)} e^{-2i\pi\alpha\beta b^2 \epsilon(x-y)}, \tag{2.25}
\]
\[
A(x) e^{2\alpha\phi(y)} = e^{2\alpha\phi(y)} \left[ A(x) e^{-2i\pi\alpha b^2 \epsilon(x-y)} + A(y) \frac{i \sin 2\pi \alpha b^2 e^{\pi p (p-2iab^2 - ib^2)}}{\sinh \pi (p - 2iab^2 - ib^2)} \right], \tag{2.26}
\]
\[
A(x) A(y) e^{i\pi b^2 \epsilon(x-y)} = A(y) A(x) e^{-i\pi b^2 \epsilon(x-y)} + \frac{i \sin \pi b^2}{\sinh \pi (p + 2ib^2)} \left( e^{\pi (p+2ib^2) \epsilon(x-y)} A^2(y) - e^{-\pi (p+2ib^2) \epsilon(x-y)} A^2(x) \right), \tag{2.27}
\]
where \( \epsilon(x) \) is the stair-step function with \( \epsilon(x) = 1 \) in \( 0 < x < 2\pi \) and \( \epsilon(x + 2\pi) = \epsilon(x) + 2 \).

Eq. (2.24) and (2.25) follow directly from the canonical commutation relation (2.17). The
derivation of (2.26) for \( \alpha = 1/2 \) one can find in [9]. The method presented there is based on identities satisfied by the stair-step function \( \epsilon(x) \). That scheme can be easily generalized for arbitrary \( \alpha \) and it reproduces (2.27) as well. These exchange relations are helpful to verify the locality and Hermiticity properties of the vertex operators and to develop their algebraic calculus.

Another useful relation, which is usually used to bring the operator products to the normal ordered form, reads

\[
e^{2\alpha\phi-(x)} e^{2\beta\phi+(y)} = e^{2\beta\phi+(y)} e^{2\alpha\phi-(x)} F^{2\alpha\beta^2}(x - y),
\]

where \( F \) is the function, which describes the short distance singularities of the theory

\[
F(x) = \exp \left( \sum_{n>0} \frac{e^{-inx}}{n} \right) = \frac{1}{1 - e^{-ix}} = \frac{e^{\frac{i}{2}(x - \pi \epsilon(x))}}{2 \sin \frac{x}{2}},
\]

with real \( x \) understood as \( x - i0 \).

Applying this formula to the primary free-field exponential

\[
\Psi_\alpha(x, \bar{x}) = e^{-i\pi \alpha^2 b^2} e^{2\alpha \phi(x)} e^{2\alpha \phi(x)}
\]

(the phase factor in front guarantees Hermiticity), one finds its normal ordered form

\[
\Psi_\alpha(x, \bar{x}) = e^{4\alpha \phi_\alpha(r)} e^{2\alpha(\phi_+(\bar{x}) + \phi_+(x))} e^{2\alpha(\phi_-(\bar{x}) + \phi_-(x))} (2 \sin \sigma)^{-2\alpha^2 b^2}.
\]

Obviously, this operator is singular at the boundaries.

The simplest nontrivial vertex operator is \( V(x, \bar{x}) \), which is constructed on the basis of (2.14). It is given as a sum of four terms

\[
V(x, \bar{x}) = V_{in}(x, \bar{x}) + V_b(x, \bar{x}) + V_c(x, \bar{x}) + V_{out}(x, \bar{x}),
\]

where \( V_{in} \) and \( V_{out} \) correspond to the in and out-field exponentials:

\[
V_{in}(x, \bar{x}) = e^{-\frac{ixb^2}{4}} e^{-\phi(x)} e^{-\phi(x)},
\]

\[
V_b(x, \bar{x}) = m_b e^{-\frac{ixkb^2}{4}} B_p e^{-\phi(x)} e^{-\phi(x)} A(x),
\]

\[
V_c(x, \bar{x}) = m_b e^{-\frac{ixkb^2}{4}} C_p A(\bar{x}) e^{-\phi(\bar{x})} e^{-\phi(x)},
\]

\[
V_{out}(x, \bar{x}) = m_b^2 e^{-\frac{ixkb^2}{4}} A(\bar{x}) e^{-\phi(\bar{x})} D_p e^{-\phi(x)} A(x),
\]

with

\[
B_p = \frac{l_b e^{-\pi(p - ib^2)} + r_b}{\sinh \pi p}, \quad C_p = \frac{l_b e^{\pi(p + ib^2)} + r_b}{\sinh \pi p}, \quad D_p = \frac{\Lambda(l_b, r_b; p)}{\sinh \pi(p + ib^2)},
\]

and \( \Lambda(l_b, r_b; p) \) is given by (2.4) with the replacement \( l \to l_b, r \to r_b \). These \( p \)-dependent coefficients are the quantum analogs of (2.15) and the parameters \( (l_b, r_b, m_b) \) can be interpreted as ‘renormalized’ \( (l, r, m) \), respectively. The guiding principles for the construction...
of the vertex operator (2.32) are conformal symmetry, Hermiticity and locality. These
conditions fix \( V(x, \bar{x}) \) up to three free parameters \((l_b, r_b, m_b)\) and it is natural to look for
their exact relation to the physical parameters \((l, r, m)\) of the theory, which are defined
by the quantum version of the Liouville equation
\[
\partial_{xx}^2 \varphi(x, \bar{x}) + m^2 e^{2\varphi(x, \bar{x})} = 0
\]
and the boundary conditions (2.22). In the next sub-section we will get
\[
\frac{m_b}{m} = \frac{l_b}{l} = \frac{r_b}{r} = \sqrt{\frac{\sin \pi b^2}{\pi b^2}}.
\]
Note that the relation between \(m_b\) and \(m\) is the same as for the periodic Liouville theory.

### 2.3 Bulk and boundary vertex operators

The bulk Liouville vertex operator \( V_\alpha(x, \bar{x}) \) corresponding to the Liouville field exponential
\( e^{2\alpha \varphi(x, \bar{x})} \) is given by
\[
V_\alpha(x, \bar{x}) = \Psi_\alpha(x, \bar{x}) \mathcal{V}_\alpha(x, \bar{x}).
\]
Here \( \mathcal{V}_\alpha(x, \bar{x}) \) is a conformal scalar, which can be expanded in a formal powers series in \(m_b\)
and the screening charges
\[
\mathcal{V}_\alpha(x, \bar{x}) = \sum_{l=0}^\infty \frac{1}{m_b^l} \sum_{k=0}^l A^{l-k}(\bar{x}) A^k(x) c_{p}^{l,k}(\alpha) = 1 + m_b \left(A(\bar{x}) c_{p}^{1,0}(\alpha) + A(x) c_{p}^{1,1}(\alpha)\right)
\]
\[
+ m_b^2 \left(A^2(\bar{x}) c_{p}^{2,0}(\alpha) + A(\bar{x})A(x) c_{p}^{2,1}(\alpha) + A^2(x) c_{p}^{2,2}(\alpha)\right) + \ldots,
\]
with \(p\)-dependent coefficients \(c_{p}^{l,k}(\alpha)\), which we will calculate later. Note that the quadratic
term \(A(x)A(x)\) in (2.40) is eliminated by (2.27). Respectively, in the general term one can
choose the ordering where \(A(\bar{x})\) stands left to \(A(x)\).

The operator \( V(x, \bar{x}) \) given by (2.32)-(2.37), corresponds to \(\alpha = -\frac{i}{2}\), and by the exchange relations (2.24)-(2.26) it takes the form (2.40)-(2.41)
\[
V(x, \bar{x}) = e^{-i \pi b^2} e^{-\sqrt{\frac{\sin \pi b^2}{\pi b^2} \cdot \pi b^2}} e^{-\varphi(x)}\left[1 + m_b \left(A(\bar{x}) c_{p}^{1,0} + A(x) c_{p}^{1,1}\right) +
\right.
\]
\[
\left. m_b^2 \left(A(\bar{x})A(x) c_{p}^{2,1} + A^2(x) c_{p}^{2,2}\right)\right].
\]
Here we omit the index \(\alpha = -\frac{i}{2}\) on both sides of the equation. The expansion in \(m_b\) stops
at the second order and besides \(c_{p}^{2,0} = 0\) we find
\[
c_{p}^{1,0} = \frac{l_b e^{\pi p} + r_b e^{-i \pi b^2}}{\sinh \pi (p + ib^2)}, \quad c_{p}^{1,1} = \frac{l_b e^{-\pi p} + r_b e^{i \pi b^2}}{\sinh \pi (p + ib^2)},
\]
\[
c_{p}^{2,1} = \frac{\Lambda(l_b, r_b; p - ib^2) e^{-i \pi b^2}}{\sinh \pi p \sinh \pi (p - ib^2)} , \quad c_{p}^{2,2} = -\frac{i \sin \pi b^2 e^{-\pi (p-2ib^2)} \Lambda(l_b, r_b; p - ib^2)}{\sinh \pi (p - 2ib^2) \sinh \pi (p - ib^2) \sinh \pi p}.
\]
\(^1\)We keep the same letters as for the classical case.
For convenience, here we have chosen the ordering, where the primary free-field exponential stands to the left, the \( p \)-dependent coefficients to the right and the screening charge operators between them. Using the exchange relations (2.24)–(2.26), this structure can be preserved in algebraic calculations of the vertex operators (see Appendix A).

The coefficients of the expansion (2.41) can be obtained from the equation

\[
V_\alpha(x, -x) V(y, -y) = V(y, -y) V_\alpha(x, -x) , \quad (0 < y < x < \pi) , \tag{2.45}
\]

which corresponds to the vanishing equal time commutator between the \( V \) and \( V_\alpha \) fields. Expanding the left and right hand sides of eq. (2.45) in powers of \( m_b \) and using the above mentioned ordering of the screening charge operators, we get similar operator structures on both sides of the equation. This relates the \( p \)-dependent coefficients to each other and they can be obtained step by step. This scheme is described in Appendix A in more detail.

An alternative scheme is based on the calculation of \( V_{-\frac{n}{2}} \) operators, which correspond to the renormalized powers of (2.42). This scheme is also given in Appendix A, and both schemes lead to the same result. Here we consider the following three coefficients only

\[
c^1_p(0) = s(\alpha) \left. \frac{b e^{\pi(p - 4i\alpha b^2 - 2i\beta^2)} + r_b e^{-i\pi b^2}}{\sinh \pi(p - 4i\alpha b^2 - i\beta^2)} e^{2i\pi \alpha b^2 + i\pi b^2} \right|_{\alpha=0} , \tag{2.46}
\]

\[
c^1_p(\alpha) = s(\alpha) \frac{b e^{-\pi(p - 4i\alpha b^2 - 2i\beta^2)} + r_b e^{i\pi b^2}}{\sinh \pi(p - 4i\alpha b^2 - i\beta^2)} e^{-2i\pi \alpha b^2 - i\pi b^2} , \tag{2.47}
\]

\[
c^2_p(\alpha) = e^{-i\pi b^2} \left[ s(\alpha) \frac{\sinh \pi(p - 2i\alpha b^2 - 2i\beta^2)}{\sinh \pi(p - 4i\alpha b^2 - 2i\beta^2)} \right. \\
\left. + s^2(\alpha) \frac{b^2 + r_b^2 + 2l_b^2 r_b^2 \cosh \pi(p - 4i\alpha b^2 - 3i\beta^2)}{\sinh \pi(p - 4i\alpha b^2 - 2i\beta^2) \sinh \pi(p - 4i\alpha b^2 - 3i\beta^2)} \right] , \tag{2.48}
\]

where

\[
s(\alpha) = -\frac{\sin 2\pi \alpha b^2}{\sin \pi b^2} . \tag{2.49}
\]

These three coefficients are sufficient to establish the relations (2.39).

The Liouville field operator is defined by

\[
\varphi(x, \bar{x}) = \frac{1}{2} \partial_\alpha V_\alpha(x, \bar{x})|_{\alpha=0} , \tag{2.50}
\]

and the operator Liouville equation reads

\[
\partial_{xx} \varphi(x, \bar{x}) + m^2 V_1(x, \bar{x}) = 0 , \tag{2.51}
\]

where \( V_1(x, \bar{x}) \) is the vertex operator for \( \alpha = 1 \), with the conformal dimension equal to one.

The expansion of the Liouville field in powers of \( m_b \) by (2.41) leads to

\[
\varphi(x, \bar{x}) = \phi(x) + \phi(\bar{x}) + \frac{m_b}{2} \left( A(x) \partial_\alpha c^1_p(0) + A(\bar{x}) \partial_\alpha c^1_p(0) \right) + \\
\frac{m^2}{2} A(\bar{x}) A(x) \partial_\alpha c^2_p(0) + \ldots . \tag{2.52}
\]
Comparing then the mass square terms in (2.51) and using that $A'(x) = e^{2\phi(x)}$ (see (2.16)) we find

$$\frac{m_b^2}{2} \partial_\sigma c_p^{1,1}(0) = -m^2 e^{-i\pi b^2},$$

(2.53)

which by (2.48)-(2.49) provides the ratio $m_b/m$ given just by (2.39).

Now we consider the quantum boundary conditions. They involve the boundary vertex operators, which are defined as the boundary limit of the bulk operators after dividing out the short distance singularity relative to the boundary

$$V_{\alpha, l}(\tau) = \lim_{\sigma \to 0} V_{\alpha}(\tau, \sigma) (2\sigma)^{2\Delta_{\alpha}}, \quad V_{\alpha, r}(\tau) = \lim_{\sigma \to \pi} V_{\alpha}(\tau, \sigma) (2\pi - 2\sigma)^{2\Delta_{\alpha}}.$$  

(2.54)

Note that here the power of $2\sigma$ and $(2\pi - 2\sigma)$ is just $2\Delta_{\alpha} - \Delta_{2\alpha}$. The boundary behavior of the vertex operators $V_{\alpha}$ is governed by $\Psi_{\alpha}$ and from (2.31) we obtain

$$V_{\alpha, l}(\tau) = e^{4\alpha\phi(\tau)} \left[ 1 + m_b A(\tau) \left( c_p^{1,0}(\alpha) + c_p^{1,1}(\alpha) \right) + \ldots \right],$$

$$V_{\alpha, r}(\tau) = e^{4\alpha\phi(\tau-\pi)} e^{2\alpha\pi(p-i\alpha b^2)} \left[ 1 + m_b A(\tau-\pi) \left( c_p^{1,0}(\alpha) + c_p^{1,1}(\alpha) e^{2\pi(p-i\alpha b^2)} \right) + \ldots \right].$$

(2.55)

Here, for $V_{\alpha, r}(\tau)$, we have taken into account the monodromies

$$A(\tau + 2\pi) = A(\tau) e^{2\pi(p-i\alpha b^2)}, \quad e^{4\alpha\phi(\tau)} = e^{4\alpha\phi(\tau-\pi)} e^{2\alpha\pi(p-2i\alpha b^2)}. $$

(2.56)

The quantum version of the boundary conditions (2.2) assumes the form

$$\partial_\sigma \varphi(\tau, \sigma)|_{\sigma=0} = 2ml V_{1, l}(\tau), \quad \partial_\sigma \varphi(\tau, \sigma)|_{\sigma=\pi} = -2mr V_{1, r}(\tau).$$

(2.57)

Since $A'(x) = e^{2\phi(x)}$, the first order mass terms of these equations yield

$$\frac{m_b}{2} \left( \partial_\alpha c_p^{1,1}(0) - \partial_\alpha c_p^{1,0}(0) \right) = 2ml,$$

$$\frac{m_b}{2} \left( \partial_\alpha c_p^{1,1}(0) e^{2\pi(p-i\alpha b^2)} - \partial_\alpha c_p^{1,0}(0) \right) = 2mr e^{\pi(p-2i\alpha b^2)},$$

(2.58)

and by (2.46)-(2.47) we arrive at (2.39).

### 3 Dynamics in the Morse potential

The zero mode approximation of the periodic Liouville theory appears rather useful for understanding the analytical properties of correlation functions of the theory [16, 17]. This approximation corresponds to $\sigma$-independent field configurations, which are described by a particle dynamics in the exponential potential. The particle model is exactly solvable quantum mechanically [3, 18], and it allows a complete description of the zero mode approximation. It is natural to look for a similar approach to BLT. However, here the zero mode sector corresponds to the $\sigma$-dependent solutions (2.3). To proceed nevertheless in
this direction, we make the approximation on the level of the initial action (2.1) by putting there \( \varphi(\tau, \sigma) = y(\tau) \). In this way we find a mechanical system with the Lagrangean

\[
L = \frac{1}{2} \dot{y}^2 - \left[ 2m^2 e^{2y} + 2m\lambda e^y \right], \quad \lambda \equiv \frac{l + r}{\pi}.
\]  

(3.1)

This model describes the dynamics of a particle in the Morse potential

\[ \mathcal{M}(y) = 2m^2 e^{2y} + 2m\lambda e^y. \]  

(3.2)

As it is clear from Fig. 1, the case of negative \( \lambda \) contains both scattering and bound states like BLT.

![Fig.1 Typical form of the Morse potential \( \mathcal{M}(y) \) for positive and negative \( \lambda \).](image)

For quantum mechanical systems the reflection amplitude and the discrete spectrum are usually obtained from the analysis of the asymptotic behavior of the solutions of the Schrödinger equation. This equation for the Morse potential is exactly solvable [20] (see Appendix B). However, since a similar analysis of the Schrödinger equation in field theory is rather problematic, we use this toy model to introduce an alternative method, based on the structure of the Heisenberg operator \( V(\tau) = e^{-y(\tau)} \). This method is then shown to reproduce the exact results of Appendix B.

Classically \( V(\tau) \) satisfies the equation

\[ V\ddot{V} - \dot{V}^2 = 4m^2 + 2m\lambda V. \]  

(3.3)

Its general solution for positive energies \( E = \frac{1}{2} \dot{y}^2 + U(y) > 0 \) can be written in terms of asymptotic variables, similarly to (2.14) in BLT,

\[ V(\tau) = e^{-(q+p\tau)} + e^{q+p\tau} F_\lambda(p) + G_\lambda(p), \]  

(3.4)

with \( p = \sqrt{2E} > 0 \) and

\[ F_\lambda(p) = m^2 \frac{\lambda^2 + p^2}{p^4}, \quad G_\lambda(p) = \frac{2m\lambda}{p^2}. \]  

(3.5)
Negative energy solutions we get for imaginary $p = i\theta$. However, then to stay with real $V$ in (3.4), the integration constant $q$ generically no longer can be chosen real. Instead it has to have the form $q = -1/2 \log F_\lambda(i\theta) - i\beta$ with real $\beta$. Therefore, the general solution of (3.3) corresponding to negative energies we represent as

$$V(\tau) = f_\lambda(\theta) \left( e^{i(\beta-\theta\tau)} + e^{-i(\beta-\theta\tau)} \right) + g_\lambda(\theta) ,$$  

where $\theta = -\sqrt{2|E|}$ and $\beta$ is its canonically conjugated cyclic variable ($\beta + 2\pi \sim \beta$)

$$\{\theta, e^{i\beta}\} = ie^{i\beta} .$$  

(3.7)

In addition,

$$f_\lambda(\theta) = \frac{m}{\theta^2} \sqrt{\lambda^2 - \theta^2} , \quad g_\lambda(\theta) = -\frac{2m\lambda}{\theta^2} .$$  

(3.8)

The negative energy solutions obviously exist only for $\lambda < 0$ and $\theta$ is bounded by

$$-|\lambda| \leq \theta < 0 .$$  

(3.9)

The lower bound corresponds to the particle sitting at rest at the minimum of the potential. Due to the reflection symmetry ($\theta \mapsto -\theta$, $\beta \mapsto -\beta$) of (3.6), we take $\theta$ only negative to avoid double counting of solutions.

Eq. (3.4) defines a map from the asymptotic to the interacting variables $(p, q) \mapsto (p_y, y)$, which is a canonical transformation $dp_y \wedge dy = dp \wedge dq$. The parameters $p$ and $q$ in (3.4) correspond to the in-momentum and in-coordinate, respectively. From (3.4) one can read off the out-variables as well $p_{out} = -p$, $q_{out} = -q - \log F_\lambda(p)$. They are related to the in-variables by the canonical transformation, which is a combination of the $p$-dependent translation of $q$

$$(p, q) \mapsto (p, q + \log F_\lambda(p))$$  

(3.10)

and the reflection $(p, q) \mapsto (-p, -q)$. The quantum analog of the canonical transformation (3.10) is provided by the reflection amplitude. To foresee its structure we write (3.10) as an exponential action of Poisson brackets generated by a function $\gamma_\lambda(p)$: $(p, q) \mapsto e^{(\gamma_\lambda(p))} (p, q) = (p, q + \gamma'(p))$. Then this function is defined up to an integration constant $c$ as a solution of the equation

$$\partial_p \gamma_\lambda(p) = \log F_\lambda(p) ,$$  

$$\gamma_\lambda(p) = c + 2p + 2p \log m + p \log(p^2 + \lambda^2) - 4p \log p + 2\lambda \arctan(p/\lambda) .$$  

(3.11)

(3.12)

Quantizing this mechanical system we use the canonical commutator $[y, p_y] = i\hbar$. The quantization of the system in the Schrödinger picture is summarized in Appendix B. There we also derive that the Heisenberg operator $V(\tau)$ satisfies as its dynamical equation a deformed version of (3.3)

$$\frac{1}{2} \left( \dot{V} \ddot{V} + \ddot{V} V \right) - \dot{V}^2 + \frac{\hbar^2}{2} V^2 = 4m^2 + 2m\lambda V .$$  

(3.13)
The quantum version of (3.4) we represent in the form
\[ V(\tau) = e^{-q - p\tau} + e^{\frac{q}{2}} e^{p\tau} F_\lambda(p) e^{\frac{q}{2}} + G_\lambda(p) . \] (3.14)

This equation expresses the operator \( V(\tau) \) in terms of \((p, q)\), which are the operators for the \textit{in}-variables. In the \( p \)-representation \((p > 0)\) \( p \) acts as a multiplication and \( q = i\hbar \partial_p \).

\( F_\lambda(p) \) and \( G_\lambda(p) \) in (3.14) are the quantum analogs of (3.5) and they could have quantum deformations. Note that we have chosen the symmetric ordering of \( q \)-exponentials in the second term of (3.14). To fix the functions \( F_\lambda(p) \) and \( G_\lambda(p) \) in (3.14) we use the dynamical equation (3.13). With the help of the exchange relations \( e^{\pm q} f(p) = f(p \pm i\hbar) e^{\pm q} \) it provides
\[ F_\lambda(p) = m^2 \frac{\lambda^2 + p^2}{p^2 (p^2 + \frac{\hbar^2}{4})} , \quad G_\lambda(p) = -\frac{2m\lambda}{p^2 + \frac{\hbar^2}{4}} . \] (3.15)

3.1 The discrete spectrum in the Morse Potential

The operator \( V(\tau) \) in the sector of bound states is constructed in a similar way, but now on the basis of (3.6). The operator \( \theta \) has a discrete spectrum. Its eigenvalues \( \theta_n \) \((n = 0, 1, \ldots, N)\) are negative and bounded below, \( n = 0 \) corresponds to the ground state and the eigenstates \( |\theta_n\rangle \) are orthonormal. The operators for the exponentials \( e^{\pm i\beta} \) we denote by \( U_\pm \), which are the rising and lowering operators for \( \theta \), since they satisfy the commutation relations \( [\theta, U_\pm] = \pm \hbar U_\pm \) (see eq. (3.7)). A more general relation can be written in the exchange form
\[ f(\theta) U_\pm = U_\pm f(\theta \pm \hbar) , \] (3.16)

for an arbitrary \( f(\theta) \). In particular, \( e^{\mp i(\theta + \frac{\hbar}{2})\tau} U_\pm = U_\pm e^{\mp i(\theta \pm \frac{\hbar}{2})\tau} \). These operators are the quantum versions of \( e^{\pm i(\beta - \theta)\tau} \), since in the scattering sector we have a similar correspondence, due to
\[ e^{\pm(q + p\tau)} = e^{\pm q} e^{\pm(p + \frac{i\hbar}{2})\tau} = e^{\pm(p \pm \frac{i\hbar}{2})\tau} e^{\pm q} . \] (3.17)

After these remarks the operator \( V(\tau) \) is represented in the form
\[ V(\tau) = f_\lambda(\theta) e^{-i(\theta - \frac{\hbar}{2})\tau} U_+ + U_- e^{i(\theta - \frac{\hbar}{2})\tau} f_\lambda(\theta) + g_\lambda(\theta) , \] (3.18)

where \( f_\lambda(\theta) \) and \( g_\lambda(\theta) \) are the quantum versions of their classical counterparts in (3.6).

These functions are fixed again by eq. (3.13):
\[ f_\lambda(\theta) = m \sqrt{\frac{\lambda^2 - (\theta - \frac{\hbar}{2})^2}{\theta (\theta - \hbar) (\theta - \frac{\hbar}{2})^2}} , \quad g_\lambda(\theta) = -\frac{2m\lambda}{\theta^2 - \frac{\hbar^2}{4}} . \] (3.19)

The operator (3.18) for \( \tau = 0 \) acts on the basis vectors \( |\theta_n\rangle \) \((n < N)\) in the following way
\[ V|\theta_n\rangle = f_\lambda(\theta_n + \hbar) |\theta_n + \hbar\rangle + f_\lambda(\theta_n) |\theta_n - \hbar\rangle + g_\lambda(\theta_n) |\theta_n\rangle . \] (3.20)
Since $|\theta_0\rangle$ is the ground state, $f_\lambda(\theta_0) = 0$, and then by (3.19) $\theta_0 = \lambda + \frac{\hbar}{2}$. The function $f_\lambda(\theta)$ has no other zeros for negative $\theta$, and we get the equidistant spectrum

$$\theta_n = \lambda + (n + 1/2) \hbar ,$$  \hspace{1cm} (3.21)

which coincides with the exact result given in Appendix B. The eigenstates $|\theta_n\rangle$ correspond to the normalized wave functions (3.8) and one can check that (3.20) holds in the $y$-representation as well (see, for example, eq. (B.15)). This indicates the unitary equivalence of the $p$- and $y$-representations. The numbers $f_\lambda(\theta_n)$ and $g_\lambda(\theta_n)$ are real, therefore the operator $V$ is Hermitean. It is positive as well, only its action is not defined on the highest level state $|\theta_N\rangle$, like in the $y$-representation.

To establish the relation between the bound states and zeros of the reflection amplitude, we switch again to the scattering sector.

### 3.2 Reflection amplitude in the Morse potential

The first and the second terms in (3.14) are the $in$- and $out$-exponentials, respectively. Therefore, they are related by the $S$-matrix

$$e^{-(q+pr)} S = S e^{\frac{\hbar}{2} p^r} F_\lambda(p) e^{\frac{\hbar}{2}} .$$  \hspace{1cm} (3.22)

The operator $S$ can be represented in the form $S = \mathcal{P} R_\lambda(p)$, where $\mathcal{P}$ denotes the parity operation ($\mathcal{P} p = -p \mathcal{P}$, $\mathcal{P} q = -q \mathcal{P}$) and $R_\lambda(p)$ is the reflection amplitude, which by (3.22) satisfies the equation

$$R_\lambda(p + i\hbar/2) = R_\lambda(p - i\hbar/2) F_\lambda(p) .$$  \hspace{1cm} (3.23)

This type of equation for the reflection amplitude of the periodic Liouville theory was considered in [17, 19]. Here we propose a more constructive approach for its analysis.

Writing the unitary operator $R_\lambda(p)$ in exponential form $R_\lambda(p) = e^{-\frac{i}{\hbar} \gamma_\lambda(p)}$, from (3.23) we get the quantum version of (3.11)

$$\frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \partial_p \right) \gamma_\lambda(p) = \log F_\lambda(p) ,$$  \hspace{1cm} (3.24)

where the function $F_\lambda(p)$ is given by (3.15). So it is already deformed with respect to (3.5). The phase factor $\gamma_\lambda(p)$ is not defined uniquely from (3.24), since the operator $\sin \left( \frac{\hbar}{2} \partial_p \right)$ has zero modes $\psi_n(p) = e^{\frac{2\pi n}{\hbar} p}$. We first specify a special solution of (3.24), requiring analyticity in $\hbar$. For this purpose we write (3.24) just in the form (3.11) with a deformed (in powers of $\hbar$) right hand side

$$\partial_p \gamma_\lambda(p) = \hat{O}_h \log F_\lambda(p) ,$$  \hspace{1cm} (3.25)

with

$$\hat{O}_h = \frac{\hbar}{2} \partial_p = 1 + \frac{\hbar^2}{4 \cdot 3!} \partial_p^2 + \ldots .$$  \hspace{1cm} (3.26)
To calculate the deformed function we represent \( \log F_\lambda(p) \) as a Fourier type integral, where the integrand is given by the eigenfunctions of the operator \( \hat{O}_\hbar \). According to (3.15)

\[
\log F_\lambda(p) = \log m^2 + \log (p^2 + \lambda^2) - \log p^2 - \log (p^2 + \hbar^2/4),
\]

and using the integral representation (D.4) (see Appendix D), we find

\[
\log F_\lambda(p) = \log m^2 + \int_0^\infty dt \left[ A(t) e^{ipt} + B(t) e^{-ipt} + C(t) \right],
\]

with

\[
A(t) = B(t) = \frac{1 + e^{-\frac{1}{2} \hbar t} - e^{-|\lambda| t}}{t}, \quad C(t) = -\frac{2e^{-t}}{t}. \quad (3.29)
\]

Replacing the action of the operator \( \hat{O}_\hbar \) on (3.28) by its eigenvalues, from (3.25)-(3.26) we obtain

\[
\gamma_\lambda^I(p) = c + p \log m^2 + \int_0^\infty dt \left[ \tilde{A}(t) e^{ipt} + \tilde{B}(t) e^{-ipt} + pC(t) \right],
\]

where the ‘tilded’ functions are

\[
\tilde{A}(t) = \frac{\hbar}{2i \sinh \left( \frac{\hbar t}{2} \right)} A(t), \quad \tilde{B}(t) = -\frac{\hbar}{2i \sinh \left( \frac{\hbar t}{2} \right)} B(t),
\]

(3.31)
c is an integration constant and we have used the index \( I \) to distinguish this special solution of (3.24). We choose \( c = \pi \hbar \), providing \( e^{-\frac{i}{\hbar} \gamma_\lambda^I(0)} = -1 \). The integral in (3.30) splits into a sum of the three following terms

\[
I_1 = i\hbar \int_0^\infty \frac{dt}{t} \left[ \frac{e^{-(|\lambda| + \frac{1}{2} - ip)t} - e^{-(|\lambda| + \frac{1}{2} + ip)t}}{1 - e^{-\hbar t}} - \frac{2ip}{\hbar} e^{-t} \right],
\]

\[
I_2 = i\hbar \int_0^\infty \frac{dt}{t} \left( \frac{e^{-ipt} - e^{ipt}}{1 - e^{-\hbar t}} + \frac{4ip}{\hbar} e^{-t} \right),
\]

\[
I_3 = i\hbar \int_0^\infty \frac{dt}{t} \left( e^{ipt} - e^{-ipt} \right) = -\pi \hbar.
\]

The integrals \( I_1 \) and \( I_2 \) take a more familiar form if we rescale the integration variable in (3.32) and (3.33) by \( \hbar t \mapsto t \) and \( \hbar t \mapsto 2t \), respectively. Then, due to (D.5) and (D.6), eq. (3.30) becomes

\[
\gamma_\lambda^I(p) = 2p \log \left( \frac{4m}{\hbar} \right) + i\hbar \log \frac{\Gamma \left( \frac{2p}{\hbar} \right)}{\Gamma \left( \frac{1}{2} + \frac{|\lambda| - ip}{\hbar} \right)} \Gamma \left( \frac{1}{2} + \frac{|\lambda| + ip}{\hbar} \right).
\]

(3.35)

It is easy to check that the corresponding reflection amplitude \( R_\lambda^I(p) = e^{-\frac{i}{\hbar} \gamma_\lambda^I(p)} \) satisfies the initial equation (3.23). Note that \( F_\lambda(p) \) in this equation is an even function of \( \lambda \).
Therefore both $R_\lambda(p)$ and $R_{-\lambda}(p)$ satisfy this equation, though they have to be different. To specify $R_\lambda(p)$ we make for the time being two assumptions:

a) The phase $\gamma_\lambda(p)$ given by (3.35) corresponds to the phase of the reflection amplitude $R_\lambda(p)$ for $\lambda > 0$.

b) The reflection amplitude $R_\lambda(p)$ is analytic in $\lambda$.

From these assumptions follows that, in general, we have again (3.35), but using $\lambda$ instead of $|\lambda|$. In this way we obtain

$$R_\lambda(p) = e^{-\frac{i}{\hbar} \gamma_\lambda(p)} = \left(\frac{4m}{\hbar}\right)^{-\frac{2ip}{\hbar}} \frac{\Gamma\left(\frac{2ip}{\hbar}\right)}{\Gamma\left(-\frac{2ip}{\hbar}\right)} \frac{\Gamma\left(\frac{1}{2} + \frac{\lambda-i\nu}{\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda+i\nu}{\hbar}\right)},$$

(3.36)

which is just the reflection amplitude (B.7) derived in the Schrödinger picture. Obviously this $R_\lambda(p)$ also satisfies eq. (3.23). Considering $\gamma_\lambda(p)$ corresponding to (3.36), and using (D.1), we find

$$\gamma_\lambda(p) - \gamma_{-\lambda}(p) = 2\pi \lambda + i\hbar \log \left(1 + e^{2i\pi \lambda} e^{-2i\pi p}\right).$$

(3.37)

Thus, $\gamma_\lambda(p)$ and $\gamma_{-\lambda}(p)$ differ from each other by a term, which can be expanded in powers of $e^{-2i\pi p}$. Such non-analytic terms are just the zero modes of the differential operator in (3.24), hence their presence is not surprising.

Applying Stirling’s formula (D.9) to $\gamma_\lambda(p)$, defined from (3.36), we get the expansion in powers of $\hbar$

$$\gamma_\lambda(p) \sim \gamma_{\lambda,0}(p) + \sum_{n \geq 1} \gamma_{\lambda,n}(p) \hbar^n,$$

(3.38)

where $\gamma_{\lambda,0}(p)$ coincides with the classical solution (3.11) for $c = 0$, and the other expansion coefficients are even functions of $\lambda$ as well. As a result, $\gamma_\lambda(p)$ and $\gamma_{-\lambda}(p)$ have the same expansions in powers of $\hbar$, which can also be seen from (3.37). This sum is asymptotic, but it is Borel summable, and its Borel sum reconstructs the function (3.35). A direct check of this statement can be based on formula (D.8), which represents $\log \Gamma(z) - \log((z/e)^{z}(2\pi/z))$ as a Laplace transform.

Three remarks are in order concerning our assumptions:

a) The calculation of $\gamma_\lambda(p)$ by the integral representation corresponds to a spectral decomposition with a continuous spectrum. Since the case $\lambda > 0$ does not contain a discrete spectrum it is natural that $\gamma_\lambda(p)$ reproduces the correct answer for $\lambda > 0$.

b) The term $2m^2 e^{2y}$ of the Morse potential is dominant in the reflection wall and $2\lambda m e^y$ can be considered as a perturbation. Therefore, the analyticity in $\lambda$ is natural as well.

c) In addition, the comparison with the exact reflection amplitude (3.36) gives an independent proof that the assumptions are fully justified for the Morse potential.

Finally, we can discuss how for $\lambda < -\frac{\hbar}{2}$, the discrete spectrum (3.21) manifests itself in the reflection amplitude $R_\lambda(p) = e^{-\frac{i}{\hbar}\gamma_\lambda(p)}$. This function has zeros at $p = i\theta_n$ ($n \geq 0$), where $\theta_n = \lambda + n\hbar + \frac{\hbar}{2}$ are just the eigenstates (3.21). For $\lambda \geq -\frac{\hbar}{2}$ there are no bound states, since these zeros of $R_\lambda(p)$ now are on the positive imaginary axis.
3.3 Correlation functions in the Morse potential

At the end of this section we analyze the ‘ground state’ matrix elements of the operators $e^{2\alpha y}$, as a preparation for field-theoretical calculation of correlation functions. First we start with the straightforward calculations in the $y$-representation. The aim is to develop an equivalent method in the $p$-representation, with a possible generalization to BLT.

The ‘ket’ vectors $|p\rangle$ are the eigenstates of the Hamiltonian $H = p^2$. We choose a normalization, which in the $y$-representation implies the following asymptotic behavior at $y \to -\infty$

$$\langle y|p\rangle \sim e^{\frac{\lambda}{2}y} + R_\lambda(p)e^{-\frac{\lambda}{2}y} .$$

(3.39)

The wave functions $\langle y|p\rangle$ satisfy the Schrödinger equation (B.1) and also the relation

$$e^{-y}\langle y|p\rangle = \langle y|p+i\hbar\rangle + F_\lambda(p-i\hbar/2)\langle y|p-i\hbar\rangle + G_\lambda(p)\langle y|p\rangle ,$$

(3.40)

which can be obtained from (3.14) at $\tau = 0$, projecting it between the states $\langle y|$ and $|p\rangle$.

The analytical continuation of $\langle y|p\rangle$ in $p$, at $p = i\theta_0 = i(\lambda + \hbar/2)$ we denote by $\langle y|i\theta_0\rangle_c$. The state $|i\theta_0\rangle_c$ is a true ground state for $\lambda < -\hbar/2$. We call it ‘ground state’ for $\lambda \geq -\hbar/2$ as well, though it is not a state in standard quantum mechanical sense. A remarkable property of the ‘wave function’ $\langle y|i\theta_0\rangle_c$ is its simple form, which follows from vanishing of the reflection amplitude $R_\lambda(p)$ at $p = i\theta_0$. In some sense $|i\theta_0\rangle_c$ is an analog of the $SL(2,\mathbb{R})$ invariant vacuum of Liouville theory and we use it to calculate mean values of the operator $e^{2\alpha y}$. Due to (B.2) $\langle y|i\theta_0\rangle_c$ reduces to

$$\langle y|i\theta_0\rangle_c = e^{-(\lambda + \frac{1}{4})}\alpha e^{-\frac{\lambda}{\hbar}y} .$$

(3.41)

For $\lambda \geq -\hbar/2$ this function is not normalizable and for $\lambda < -\hbar/2$ we get

$$|i\theta_0\rangle_c = \left(\frac{4m}{\hbar}\right)^{\frac{1}{2}+\frac{1}{2}} \sqrt{\Gamma\left(-1 - \frac{2\lambda}{\hbar}\right)} |\theta_0\rangle ,$$

(3.42)

where $|\theta_0\rangle$ is the corresponding normalized state with unit norm.

The matrix element $\langle p'|e^{2\alpha y}|p\rangle$ is finite for real $p$, $p'$ and $\alpha > 0$ . The calculation of $\langle p'|e^{2\alpha y}|p\rangle$ in the $y$-representation by (B.2) leads to a rather complicated answer in terms of hypergeometric functions [21]. The answer is simplified for the state (3.41)

$$e^{i\theta_0}e^{2\alpha y}|i\theta_0\rangle_c = \int_{-\infty}^{\infty} dy \ e^{2\alpha y} e^{-\frac{4m}{\hbar}y} = \left(\frac{4m}{\hbar}\right)^{\frac{2\lambda}{\hbar} - 2\alpha + 1} \Gamma(2\alpha - \frac{2\lambda}{\hbar} - 1) .$$

(3.43)

Note that the integral in (3.43) is finite even for non-normalizable $|i\theta_0\rangle_c$, if $\alpha$ is large enough: $\alpha > \frac{\lambda}{\hbar} + \frac{1}{2}$. Then, the matrix element $e^{i\theta_0}e^{2\alpha y}|i\theta_0\rangle_c$ for generic $\alpha$ is understood as an analytical continuation of (3.43) in $\alpha$.

The correlation functions defined by the Heisenberg operators

$$U_\alpha = e^{i\theta_0}e^{2\alpha y}$$

(3.44)
are time independent and, therefore, they are given by (3.43). From this equation follows that 
\[ 4m^2 U_1 + 2m\lambda U_{\frac{1}{2}} = 0, \]
which is in accordance with the operator dynamical equation
\[ \ddot{y}(\tau) + 4m^2 e^{2y(\tau)} + 2m\lambda e^{y(\tau)} = 0, \quad (3.45) \]
even for non normalizable \( |i\theta_0\rangle_c \).

The analytical continuation of (3.43) in \( \alpha \), at \( \alpha = -\frac{1}{2} \) yields
\[ e\langle i\theta_0|V|i\theta_0\rangle_c = \left( \frac{4m}{\hbar} \right)^{\frac{2\lambda}{\hbar} + 2} \Gamma \left( -2 - \frac{2\lambda}{\hbar} \right) = \left( \frac{4m}{\hbar} \right)^{1 + \frac{2\theta_0}{\hbar}} \Gamma \left( -1 - \frac{2\theta_0}{\hbar} \right). \quad (3.46) \]

The same matrix element for a normalizable \( |i\theta_0\rangle_c \) can also be calculated by the operator (3.18). Indeed, from (3.18)-(3.20) we simply get
\[ \langle \theta_0|V|\theta_0\rangle = -\frac{2m}{\lambda + \hbar}, \quad (3.47) \]
which together with (3.42) reproduces (3.46). This equation for positive \( \lambda \) now can be treated as an analytical continuation in \( \lambda \) from the sector of bound states. Note that the norm of \( |i\theta_0\rangle_c \) can also be calculated by the formula: \[ |\langle i\theta_0|V|i\theta_0\rangle_c|^2 = -i\hbar \partial_p R_\lambda(p)|_{p=i\lambda+i\hbar/2} \] (see (3.13)), which for the reflection amplitude (3.36) is equivalent to (3.42).

Thus, the operator \( V \) in the sector of bound states provides a simple way for the calculation of the correlation function (3.46). Unfortunately this scheme has no direct generalization to BLT, since the ‘ground state’ there is given by the \( SL(2,\mathbb{R}) \) invariant vacuum, which is normalizable only for a certain value of the boundary parameters (see the next section) and one can not make a continuation, like here in \( \lambda \).

Now we consider an alternative scheme of derivation of (3.46) using the operator (3.14). This scheme is based on a regularization of non-normalizable states \( |i\theta_0\rangle_c \) in the \( p \)-representation.

Before starting its discussion, a remark is in order: The operator \( q = i\hbar \partial_p \) is not self-adjoint in the Hilbert space \( L^2(\mathbb{R}_+) \) and the exponentials \( e^{\pm q} \) are not even Hermitean. This problem is typical for Liouville theory, where the zero mode sector \((p, q)\) is restricted to the half-plane \( p > 0 \). On the other hand, the operator \( V = e^{-y} \) is obviously self-adjoint in the \( y \)-representation. We assume that the \( y \)- and \( p \)-representations are unitary equivalent, but the check of this statement or a direct verification of the self-adjointness of (3.14) needs additional labor. A proof of self-adjointness of the operator \( V \) for the particle dynamics in the Liouville potential \( (\lambda = 0) \) one can find in [17]. The proof is based on a continuation of wave functions \( \Psi(p) \) to the negative half-line, and then on the full line the Hermitean structure of \( V \) becomes crucial. This scheme can be easily generalized to the Morse potential. For this purpose we consider the space of square-integrable functions \( \Psi(p) \) given by
\[ \Psi(p) = e_\lambda(p) \psi(p), \quad (3.48) \]
where \( \psi(p) \) is an even \( (\psi(-p) = \psi(p)) \) holomorphic function and
\[ e_\lambda(p) = \left( \frac{4m}{\hbar} \right)^{\frac{\lambda}{2}} \frac{\Gamma \left( \frac{1}{2} + \frac{\lambda + ip}{\hbar} \right)}{\Gamma \left( \frac{2\lambda}{\hbar} \right)}. \quad (3.49) \]
This function satisfies the condition
\[
e^{\lambda(-p)} e^{\lambda(p)} = R_\lambda(p) ,
\]
and, therefore, the continuation of (3.48) to the negative half-line has the following reflection property
\[
\Psi(-p) = R_\lambda(p) \Psi(p) ,
\]
providing
\[
\int_0^\infty dp \, \Psi^*(p) V \Psi(p) = \frac{1}{2} \int_{-\infty}^\infty dp \, \Psi^*(p) V \Psi(p) ,
\]
which leads to the self-adjointness of the operator (3.14).

Let us regularize the ‘ground state’ \( |i\theta_0\rangle \) by \( e^{2\alpha y} |i\theta_0\rangle \), with a positive large enough parameter \( \alpha \), and consider the corresponding wave function in the \( p \)-representation
\[
\Psi_\alpha(p) = \langle p | e^{2\alpha y} |i\theta_0\rangle .
\]
Using the integral (D.11) we find that \( \Psi_\alpha(p) \) has the form (3.48) \( \Psi_\alpha(p) = e^{\lambda(p)} \psi_{\alpha,\lambda}(p) \) with
\[
\psi_{\alpha,\lambda}(p) = \left( \frac{4m}{\hbar} \right)^{-2\alpha + \frac{1}{4} + \frac{1}{2}} \frac{\Gamma \left( 2\alpha - \frac{1}{2} - \frac{\lambda + ip}{\hbar} \right) \Gamma \left( 2\alpha - \frac{1}{2} - \frac{\lambda - ip}{\hbar} \right)}{\Gamma(2\alpha)} .
\]
We represent the matrix element \( \langle \Psi_\alpha | V(0) | \Psi_\alpha \rangle \) as a sum \( V_1 + V_2 \), where \( V_1 \) corresponds to the contribution from the first two terms of (3.14) and \( V_2 \) reads
\[
V_2 = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} e(-p) e(p) \psi^2_{\alpha,\lambda}(p) G_\lambda(p) .
\]
Note that the extension of the integration to the negative half-line is allowed, due to the reflection \( p \leftrightarrow -p \) symmetry of the integrand, and the integration measure here corresponds to our normalization of \( |p\rangle \)-states. The term \( V_1 \) is simplified with the help of (3.50) and (3.23) in the form
\[
V_1 = \int_{-\infty}^{+\infty} \frac{dp}{\pi\hbar} e(-p) \psi_{\alpha,\lambda}(p) e(p - i\hbar) \psi_{\alpha,\lambda}(p - i\hbar) ,
\]
where the integration over the negative half-line now corresponds to the second term (the out-term) of (3.14). The integrals (3.55) and (3.56) are well defined due to the asymptotic behavior (D.12).

To remove the regularization (3.53) by sending \( \alpha \to 0 \), one cannot immediately use the integral representations (3.55)-(3.56), since they break down at \( 2\alpha = \frac{1}{2} + \frac{\lambda}{\hbar} \), due to singularities of the integrands. Therefore, we first shift the contour of integration
\[
\int_{-\infty}^{+\infty} dp \to \int_{ia-\infty}^{ia+\infty} dp + 2\pi i \sum \text{Res} ,
\]
with $a$ taken from the interval $\left(\frac{1}{2} + \frac{\lambda}{\hbar}, \frac{3}{2} + \frac{\lambda}{\hbar}\right)$. Now the integrals exist for all values of $\alpha$ down to zero and disappear in the limit, since the integrands tend to zero.

The pole of the integrand in (3.55) at $p = i(\lambda + \hbar/2) = i\theta_0$ provides the following residue term

$$
\frac{i}{\hbar} \text{Res} \left[ e(-p) e(p) \psi_{\alpha,\lambda}(p) G_\lambda(p) \right] |_{p=i\theta_0} = \left( \frac{4m}{\hbar} \right)^{-4\alpha+\frac{2\theta_0}{\hbar}} \frac{\Gamma^2 \left( 2\alpha - \frac{2\theta_0}{\hbar} \right)}{\Gamma \left( -\frac{2\theta_0}{\hbar} \right)} \frac{2m}{-\lambda - \hbar}. \quad (3.58)
$$

The other residue terms contain $\Gamma(2\alpha)$ in the denominator and they vanish in the limit $\alpha \to 0$. As a result, only the term (3.58) survives in the limit $\alpha \to 0$ and we end up with (3.46), which is given by

$$
e\langle i\theta_0 | V | i\theta_0 \rangle_c = G(i\theta_0) \left[ -i\hbar \partial_p R_\lambda(p) \right] |_{p=i\theta_0}, \quad (3.59)
$$

even for a non-normalizable ground state $|i\theta_0\rangle_c$.

Summarizing this section we conclude that the structure of the operator $V$ in terms of the asymptotic variables can be used for calculations of the discrete spectrum, the reflection amplitude and the correlation function in a rather simple way. We will apply these schemes to BLT in the next section.

4 The vacuum sector of BLT

Let us consider the projection of the operator (2.32) on the vacuum sector $\langle p', 0 | V | p, 0 \rangle$, where $|p, 0\rangle$ is the $p$-dependent vacuum (2.18). The matrix elements $\langle p', 0 | V | p, 0 \rangle$ yield an operator kernel in the $p$-representation of the zero mode sector. We denote the corresponding operator by $\hat{V}$. Then, using eqs. (2.28)-(2.29) to bring the operators (2.33)-(2.36) to the normal ordered form, we obtain

$$(2 \sin \sigma)^{\frac{b^2}{2}} \hat{V}_m = e^{-(q+pr)} \quad \text{(2.32)}$$

$$(2 \sin \sigma)^{\frac{b^2}{2}} \hat{V}_\text{out} = m_b^2 e^{\frac{b^2}{2}} \text{e}^{\sigma} \text{D}_p I_p e^{\frac{b^2}{2}}. \quad (4.1)$$

$$(2 \sin \sigma)^{\frac{b^2}{2}} \hat{V}_b = m_b B_p e^{(p-i\frac{b^2}{2})(\sigma-\pi)} J_p \quad \text{(2.33)}$$

$$(2 \sin \sigma)^{\frac{b^2}{2}} \hat{V}_c = m_b C_p e^{-(p+ib^2)(\sigma+\pi)} \bar{J}_p \quad (4.2)$$

where $I_p, J_p$ and $\bar{J}_p$ contain the integrals provided by the screening charges

$$I_p = \frac{e^{-2\pi p}}{4 \sinh^2 \frac{\pi}{p}} \int_0^{2\pi} dy \int_0^{2\pi} d\bar{y} e^{p(y+\bar{y})} \left( 1 - e^{iy} \right)^{b^2} \left( 1 - e^{-i\bar{y}} \right)^{b^2} \times \left( 1 - e^{2i\sigma} e^{iy} \right)^{b^2} \left( 1 - e^{2i\sigma} e^{-i\bar{y}} \right)^{b^2} \left( 1 - e^{2i\sigma} e^{i(y-\bar{y})} \right)^{-2b^2}, \quad (4.3)$$

$$J_p = \frac{1}{2 \sinh \frac{\pi}{p}} \int_0^{2\pi} dy e^{p(-ib^2)y} \left( 1 - e^{iy} \right)^{b^2} \left( 1 - e^{2i\sigma} e^{iy} \right)^{b^2}, \quad (4.4)$$

$$J_p = \frac{1}{2 \sinh \frac{\pi}{p}} \int_0^{2\pi} d\bar{y} e^{p(ib^2)\bar{y}} \left( 1 - e^{-iy} \right)^{b^2} \left( 1 - e^{2i\sigma} e^{-iy} \right)^{b^2}. \quad (4.5)$$

20
$I_{p}$ from (4.3) can be written with (2.29) in the form

$$I_{p}(\sigma) = \frac{e^{-2\pi p}}{4 \sinh^2 \frac{\pi p}{2}} \times$$

$$\int_{0}^{2\pi} dy \int_{0}^{2\pi} d\bar{y} e^{p(y+\bar{y})} \frac{|4 \sin \left(\frac{y}{2}\right) \sin \left(\frac{\bar{y}}{2}\right) \sin \left(\frac{y}{2} + \sigma\right) \sin \left(\frac{\bar{y}}{2} - \sigma\right)| y^2}{| \sin \left(\frac{y-\bar{y}}{2} + \sigma\right)| 2b^2} e^{-\pi b^2 \alpha(y,\bar{y},\sigma)},$$

where the phase of the last factor is given by

$$\alpha(y,\bar{y},\sigma) = \frac{1}{2} \epsilon(y + 2\sigma) - \frac{1}{2} \epsilon(\bar{y} - 2\sigma) - \epsilon(y - \bar{y} + 2\sigma).$$

This function takes only discrete values, $-1$, $0$ and $1$, as they are indicated in Fig. 2. The imaginary part of (4.6) can arise only due to the integration over the two small squares with the dash-line diagonals. But the modulus of the integrand is a symmetric function with respect to these diagonals, and, since the constant phases differ only by their sign, the imaginary contributions cancel each other. On the other hand, if we look at $I_{p}$ as a function of the complex variable $\zeta = e^{2i\sigma}$, it is obviously holomorphic inside the unit circle. But since $I_{p}$ is real on the unit circle, it is $\zeta$-independent. At $\zeta = 0$ the integral in (4.3) splits into the product of two integrals of the type (D.7) and, using (D.1), we find

$$I_{p} = \frac{\Gamma^2(1 + b^2) \Gamma(ip) \Gamma(-ip)}{\Gamma(1 + b^2 + ip) \Gamma(1 + b^2 - ip)}.$$  

Expanding the integral (4.4) in powers of $\zeta$: $J_{p} = \sum_{n \geq 0} j_{n} \zeta^{n}$, the coefficients $j_{n}$ are also obtained from (D.7)

$$j_{n} = \frac{\Gamma(n - b^2)}{\sinh \pi(p - b^2) \Gamma(-b^2) n!} \frac{\pi \Gamma(1 + b^2) e^{\pi(p + in - ib^2)}}{\Gamma(1 + ip - n + b^2) \Gamma(1 - ip + n)}.$$  

One observes that $j_{n}/j_{0}$ are the coefficients of the expansion for the hypergeometric function. The calculation of $J_{p}$ is similar and from (4.2) we get

$$(2 \sin \sigma)^{\frac{k^2}{2}} \hat{V}_{\beta} = b_{p} e^{\eta - ib^2} a_{2} F_{1}(-b^2, -b^2 - ip, 1 - ip, \zeta),$$  

Fig. 2 Values taken by the function $\alpha(y,\bar{y},\sigma)$ in the $(y,\bar{y})$-plane.
\[
(2 \sin \sigma) \frac{i^2}{\tau} \hat{V}_c = c_p e^{-(p+i\beta^2)\sigma} 2F_1(-b^2, -b^2 + ip, 1 + ip, \zeta), \tag{4.11}
\]
where \(b_p = m_b B_p j_0 e^{-\pi(p-i\beta^2)}, c_p = m_b C_p j_0 e^{-\pi(p+i\beta^2)},\) and by (2.37) they become
\[
b_p = c_{-p} = \frac{m_b}{\pi} \left( l_b e^{-\pi(p-i\beta^2)} + r_b \right) \Gamma(1 + b^2) \Gamma(ip) \Gamma(-ip - b^2). \tag{4.12}
\]
Thus, the operator \(\hat{V}\) has the form (3.14)
\[
\hat{V} = (2 \sin \sigma) \frac{i^2}{\tau} \left( e^{-(q+\pi \tau)} + e^{\pi \tau} F(p) e^{\pi \tau} + G(p, \sigma) \right), \tag{4.13}
\]
with \(F(p) = m_b^2 D_p I_p\) and \(G(p, \sigma)\) given as a sum of (4.10) and (4.11). The function \(F(p)\) in explicit form is obtained from (2.37), (4.8) and (2.4)
\[
F(p) = \frac{m_b^2}{\pi^2} \left( (1 + b^2) \frac{\Gamma(ip) \Gamma(-ip) \Gamma(1 - b^2 + ip) \Gamma(1 - b^2 - ip)}{\pi^2 (p^2 + b^4)} \right) \Lambda(l_b, r_b; p), \tag{4.14}
\]
while
\[
G(p, \sigma) = \frac{m_b}{\pi} \left( l_b e^{-\pi(p-i\beta^2)} + r_b \right) \Gamma(1 + b^2) \Gamma(ip) \Gamma(-ip - b^2) e^{(p-i\beta^2)\sigma} \times 2F_1(-b^2, -b^2 - ip; 1 - ip, e^{2i\sigma}) + (p \leftrightarrow -p). \tag{4.15}
\]
Note that both \(F\) and \(G\) are even functions of \(p\). Furthermore, \(F\) is manifestly real and the reality of \(G(p, \sigma)\) follows from the analytical properties of the hypergeometric functions (D.2). This provides a Hermitean structure for \(\hat{V}\) and gives some consistency check for the construction of \(V\). The positivity of the operator \(\hat{V}\), which is also related to the unitarity of the theory, puts certain restrictions on the parameters \(l_b\) and \(r_b\) like in the classical case. But, due to the quantum deformations, the corresponding analysis becomes more complicated. Note that in the limit \(b \to 0\) one reproduces the classical expressions corresponding to (2.14) for vanishing Fourier modes \(a_n = 0\)
\[
\frac{m_b^2}{p^2 \sinh^2 \pi p} \Lambda(l_b, r_b; p), \quad \frac{2m}{p \sinh \pi p} \left[ l \cosh p(\sigma - \pi) + r \cosh p\sigma \right]. \tag{4.16}
\]

The boundary parameters of the classical theory are given by the derivatives of the \(V\)-field at the boundaries (see (2.22))
\[
\partial_\sigma V(\tau, \sigma)|_{\sigma = 0} = -2ml, \quad \partial_\sigma V(\tau, \sigma)|_{\sigma = \pi} = 2mr. \tag{4.17}
\]
To find the corresponding quantum relation note that by (4.13)
\[
\partial_\sigma \left( (2 \sin \sigma) \frac{i^2}{\tau} \hat{V}(\tau, \sigma) \right) = \partial_\sigma G(p, \sigma). \tag{4.18}
\]
The boundary behavior of \(\partial_\sigma G(p, \sigma)\) can be obtained from (D.3)
\[
\partial_\sigma G(p, \sigma)|_{\sigma = 0} = -2m_b l_b \frac{\Gamma(1 + b^2) \Gamma(1 - 2b^2)}{\Gamma(1 - b^2)} (2\sigma)^2 e^{2b^2(1 + O(\sigma))}, \tag{4.19}
\]
\[
\partial_\sigma G(p, \sigma)|_{\sigma = \pi} = 2m_r r_b \frac{\Gamma(1 + b^2) \Gamma(1 - 2b^2)}{\Gamma(1 - b^2)} (2\pi - 2\sigma)^2 e^{2b^2(1 + O(\pi - \sigma))}. \tag{4.19}
\]
Then, taking into account (2.39), the quantum version of (1.17) becomes

$$
\lim_{\sigma \to 0} (2 \sin \sigma)^{-2b^2} \partial_\sigma \left[ (2 \sin \sigma)^{2b^2} \dot{V}(\tau, \sigma) \right] = -2ml \frac{\Gamma(1 - 2b^2)}{\Gamma^2(1 - b^2)},
$$

$$
\lim_{\sigma \to \infty} (2 \sin \sigma)^{-2b^2} \partial_\sigma \left[ (2 \sin \sigma)^{2b^2} \dot{V}(\tau, \sigma) \right] = 2mr \frac{\Gamma(1 - 2b^2)}{\Gamma^2(1 - b^2)}.
$$

This result reflects part of the structure which one would expect from a boundary OPE for $V$ at $\sigma \to 0$

$$
V(x, \bar{x}) = (2\sigma)^{\Delta_{-12\Delta_{-12}}} V_{-1,1}(\tau) + (2\sigma)^{-2\Delta_{-12}} Z(b) \mathbf{1} + \ldots .
$$

$V_{-1,1}(\tau)$ is defined in (2.51). Projecting (4.21) to the vacuum sector, using (2.23), and acting with $(2\sigma)^{-2b^2} \partial_\sigma (2\sigma)^{2b^2}$, a comparison with (1.20) gives for the bulk-boundary structure constant $Z(b)$

$$
Z(b) = - \frac{2ml}{b^2} \frac{\Gamma(-1 - 2b^2)}{\Gamma^2(-b^2)},
$$

which after identifying $\frac{ml}{\pi b^2}$ with $\mu_B$ (see below) coincides with those in [4].

Before starting calculations in the vacuum sector, let us make few remarks concerning the Virasoro generators and the $SL(2, \mathbb{R})$ invariant vacuum state. The Virasoro generators are the Fourier modes of the energy momentum tensor (2.22) with an appropriate normalization and a constant shift in $L_0$, to match the Virasoro algebra in the standard form

$$
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} (n^3 - n)\delta_{n+m,0}.
$$

Equation (2.22) defines the following normalization and the shift

$$
L_n = \frac{1}{2\pi b^2} \int_0^{2\pi} dx e^{inx} T(x) + \frac{Q^2}{4} \delta_{n,0}, \quad Q = b + \frac{1}{b},
$$

and the central charge is $c = 1 + 6Q^2$. Using the free-field modes, from (2.22) we find

$$
2b^2 L_0 = \frac{x^2 + (1 + b^2)^2}{2} + a_{-1} a_1 + a_{-2} a_2 + \ldots ,
$$

$$
2b^2 L_1 = (p + i(1 + b^2)) a_1 + a_{-1} a_2 + a_{-2} a_3 + \ldots ,
$$

$$
2b^2 L_{-1} = (p - i(1 + b^2)) a_{-1} + a_{-2} a_1 + a_{-3} a_2 + \ldots ,
$$

and similarly for other $L_n$’s. The $p$-dependent vacuum states $|p, 0\rangle$ defined by (2.18) are obviously annihilated by $L_n$ ($n > 0$) as well. The $SL(2, \mathbb{R})$ invariant vacuum is a special case, which is also annihilated by $L_0$ and $L_{-1}$. According to (4.23) this state corresponds to $p = i + ib^2$. As we will see below the corresponding state $|i(1 + b^2), 0\rangle$ usually is not normalizable, like the state (3.11) in the Morse potential for $\lambda > -\frac{1}{2}$. The $SL(2, \mathbb{R})$ invariant state becomes a bound state only for the special values of the boundary parameters $l_b = r_b = -\cos \pi b^2$. In this case due to the symmetry $p \to -p$, one could also
take the state with \( p = -i(1 + b^2) \). However, the state \( | -i(1 + b^2), 0 \rangle \) is not annihilated by \( L_{-1} \), if \( a_{-1} \) is treated as a standard creation operator in the Fock space. The key point for understanding of this subtlety is that \( \phi(x) \) is not any more a free-field, if \( p \) becomes imaginary; though the Fourier mode expansion (2.11) and the commutator relations (2.17) are still valid. This issue was mentioned above and it will be discussed in Section 5 in more detail.

### 4.1 The discrete spectrum of BLT

Now we investigate the discrete spectrum of the system by the same scheme as in subsection 3.1. The continuation of (4.13) to the sector of bound states is obtained similarly to (3.18)

\[
\hat{V} = (2\sin\sigma)^{-\frac{b^2}{2}} (f(\theta) e^{-i(\theta-b^2)\tau} U_+ + U_- e^{i(\theta-b^2)\tau} f(\theta) + g(\theta, \sigma)) \tag{4.26}
\]

where \( f(\theta) = \sqrt{F(i\theta - ib^2)} \), \( g(\theta, \sigma) = G(i\theta, \sigma) \) and \( U_\pm \) are the raising-lowering operators. We choose again negative \( \theta \). To get a \( \theta \)-spectrum bounded from below, candidates for ground states \( \theta_0 \) must fulfill the equation \( f(\theta_0) = 0 \). Furthermore, unitarity of the related Verma modules requires for the \( \theta \)-spectrum

\[-(1 + b^2) \leq \theta < 0 \tag{4.27}\]

where the lower bound corresponds to the \( SL(2, \mathbb{R}) \) invariant situation \((1 + b^2) = bQ\). In the classical limit this interval corresponds to \(-1 \leq \theta < 0\), as discussed in section 2.1.

Starting at such a ground state \( \theta_0 \) by acting with \( \hat{V} \), due to its raising part \( U_+ \), one reaches further states in integer steps of \( 2b^2 \). This generates a series with a finite number of discrete \( \theta \)-values in the interval (4.27). This series constitutes (a part of) the discrete spectrum iff there \( \hat{V} \) is Hermitian and positive. A necessary part of this condition is the reality of \( f \) and the positivity of \( g \) at all points of the series except the last one below the threshold at zero. In the following we analyze in some detail this necessary part. At the end one then can check the positivity of all eigenvalues of the matrix representing \( \hat{V} \) on such a series, at least numerically.

To find our candidates for ground states we have to start with the roots of the equation \( f(\theta) = 0 \). From (4.14) follows

\[
f^2(\theta) = \frac{m_b^2 \Gamma^2(1 + b^2)}{\pi^2} \Gamma(b^2 - \theta) \Gamma(\theta - b^2) \Gamma(-\theta) \Gamma(\theta - 2b^2) \Lambda(l_b, r_b; i\theta - ib^2) \tag{4.28}\]

and, therefore, the equation for the roots is reduced to \( \Lambda(l_b, r_b; i\theta - ib^2) = 0 \). Then, by (2.4) we find

\[
\cos \pi(\theta_0 - b^2) = -l_br_b \pm \sqrt{(1 - l_b^2)(1 - r_b^2)} \tag{4.29}
\]

In the classical case the lowest value of \( \theta \) is obtained similarly from the equation \( \Lambda(l, r; i\theta) = 0 \), which is quadratic with respect to \( \cos \pi\theta \), but one has to neglect the
smaller root and take only (2.5). The reason is that \( \Lambda(l, r; i\theta) \) becomes negative in the interval between the roots, which leads to a complex field in (2.3). From (2.5) \( \theta_* \) is defined uniquely, because classically it is bounded by \(-1 \leq \theta_* < 0\).

In the quantum case there are, in general, more possibilities, due to the discreteness of the spectrum and the deformations. To analyze these possibilities, it is convenient to introduce the following parameterization and notation

\[
l_b = \cos \pi \beta_l, \quad r_b = \cos \pi \beta_r; \quad \beta = \beta_l \pm \beta_r. \tag{4.30}
\]

To deal with the discrete spectrum we take the boundary parameters from the interval \( l_b, r_b \in [-1, 1] \). Then the parameters \( \beta \) are uniquely fixed if they are chosen out of the basic intervals

\[
0 \leq \beta_l, \beta_r \leq 1, \quad 0 \leq \beta_+ \leq 2, \quad -1 \leq \beta_- \leq 1. \tag{4.31}
\]

From (2.4) we get

\[
\Lambda(l_b, r_b; i\theta - ib^2) = 4 \cos \frac{\pi}{2} (\theta - b^2 + \beta_+) \cos \frac{\pi}{2} (\theta - b^2 - \beta_+) \times \cos \frac{\pi}{2} (\theta - b^2 + \beta_-) \cos \frac{\pi}{2} (\theta - b^2 - \beta_-). \tag{4.32}
\]

Now the roots are given by

\[
\theta_0 = 1 + b^2 - \beta_+ + 2k, \quad \text{or} \quad \theta_0 = 1 + b^2 + \beta_+ + 2k, \tag{4.33}
\]

with integer \( k \).

The candidate higher levels are obtained by the action of the raising operator \( U_+ \)

\[
\theta = 1 + b^2 - \beta_+ + 2k + 2nb^2, \quad \text{or} \quad \theta = 1 + b^2 + \beta_+ + 2k + 2nb^2, \quad (n \geq 0). \tag{4.34}
\]

The condition (4.27) imposes restrictions on the numbers \((k, n)\). Other restrictions come from the Hermiticity and positivity of \( \hat{V} \), mentioned above. The analysis of these conditions essentially depends on the value of \( b^2 \) and, in general, it is rather complicated. Note that for \( b^2 \geq 1 \) only the case \( n = 0 \) is allowed, and for \( b^2 \leq 1 \), \( k \) can not take more than one integer value, for each of the four series in (4.34).

We start with a discussion of the semi-classical situation \( b^2 \ll 1 \). Then one can use the classical expressions (4.16) for \( F \) and \( G \), what simplifies the analysis. In the classical theory the lowest allowed value for the boundary parameters is \(-1\). In quantum theory the corresponding cases when \( l_b \) and \( r_b \) are near (in the units of \( b^2 \)) to \(-1\) need a separate investigation, which will be done below. For the moment let us consider the case with \( b^2 \ll 1 - \beta_l, b^2 \ll 1 - \beta_r \) and \( \beta_+ > 1 \). On the basis of the above described classical picture for the allowed lowest state we deduce that from the spectrum candidates (4.34) only the series with \(-\beta_+ \) and \( k = 0 \) remains and gives the spectrum

\[
\theta_n = 1 + b^2 - \beta_+ + 2nb^2, \quad (n \geq 0). \tag{4.35}
\]
Let us note that the semi-classical calculation yields exactly the same answer \[9\].

We now want to get at least some flavor of the setting beyond the semi-classical case. The analytic structure of \(f^2\) in \(4.28\) is governed by the poles of the \(\Gamma\)-factors and the zeros of \(\Lambda(l_b, r_b; i\theta - ib^2)\) (see \(4.32\)). We restrict the following discussion to \(b^2 < 1/3\).

Then \(f^2\) for generic \(\beta-\) has only two poles in the interesting \(\theta\)-interval \(4.27\), namely at \(\theta = -1 + b^2\) and \(\theta = -1 + 2b^2\). In addition, for \(\beta_-= 0\) the pole at \(\theta = -1 + b^2\) is canceled by a double zero of \(\Lambda(l_b, r_b; i\theta - ib^2)\).

![Fig. 3(a)](image1.png)  
**Fig.3 (a)** The function \(f^2(\theta)\) for \(b^2 = 0.1, \beta_- = 0\) and \(\beta_+ = 1.75, 1.8\) (i.e. \(2 - 2b^2\)) and \(1.85, 1.9\) (i.e. \(2 - b^2\)), 1.92.  
(b) The function \(g(\theta, \sigma)\) for \(b^2 = 0.1, \beta_- = 0, \beta_+ = 1.85\).

Let us consider first the symmetric case \(l_b = r_b = \cos \pi \beta, \beta_- = 0, \beta_+ = 2\beta\) with \(0 \leq \beta \leq 1\). From a numerical analysis, for an example see fig. 3(b), we then conclude, that the function \(g(\theta, \sigma)\) is positive for all \(\sigma\) and \(-1 - b^2 < \theta < -b^2\). It has a pole at \(\theta = -b^2\) and is negative above this value. Since a point of the spectrum above \(-2b^2\) would be the last level below the continuum threshold, \(\hat{V}\) has not to be well defined there, anyway. In conclusion, we get no obstruction for possible spectral points from the properties of the function \(g\) and can turn to the analysis of \(f^2\). For \(\beta_- = 0, 2b^2 < \beta_+ < 2\) from \(4.34\) there remain only three candidate series (for \(0 < \beta_+ < 2b^2\) there is still another option taking \(k = -1\) in \(-\beta_+ + 2k\) in \(4.34\))

\[
\begin{align*}
\theta_n &= 1 + b^2 - 2\beta + 2nb^2, \\
\theta_n &= -1 + b^2 + 2nb^2, \\
\theta_n &= -3 + b^2 + 2\beta + 2nb^2.
\end{align*}
\]

These series can contribute to the spectrum only if \(\theta_0\) is in the interval \(4.27\) and if \(f^2 \geq 0\)
at $\theta_n$, $n > 1$. This implies

\begin{align}
    b^2 < \beta \leq \frac{1 + b^2}{2} & : \text{ no discrete spectrum,} \\
    \frac{1 + b^2}{2} < \beta < 1 - b^2 & : \text{ spectrum given by (4.36) only,} \\
    1 - b^2 \leq \beta < 1 & : \text{ all 3 series (4.36)-(4.38) can be in the spectrum.}
\end{align}

The described pattern is illustrated in fig.3(a) for $b^2 = 0.1$. As soon as $\beta_+ > 2 - 2b^2 = 1.8$ the function $f^2$ has three zeros, and starting at each zero, after a jump $\theta \rightarrow \theta + 2b^2 = \theta + 0.2$ one reaches positive $f^2$.

An important special situation one finds for $\beta = 1 - b^2$. Then $\theta_0 = -1 - b^2 = -bQ$, corresponding to a $SL(2, \mathbb{R})$ invariant state, plays the role of the ground state in the series (4.38). However, this state is invariant under the action of $\hat{V}$, since $f^2$ is again zero at $\theta_0 + 2b^2 = -1 + b^2$. This point just corresponds to the $\theta_0$-value for the series (4.37). Also the state at $-1 + b^2$ is invariant, because after a second $2b^2$-jump we reach the zero of $f^2$ responsible for the ground state of the series (4.36). Therefore, at $\beta = 1 - b^2$ we have several options: to take only the spectrum given by (4.36) or to take only one or both of the invariant states at $-1 - b^2$ and $-1 + b^2$ or to take (4.36) plus one or both of the invariant states. Note that at $\beta = 1$ all three series (4.36)-(4.38) coincide.

Closing the discussion of the $\beta_- = 0$, case let us mention, that also $\beta = 1 - b^2/2$ plays a special role. There $f^2$ has only 2 zeros and the pole at $\theta = -1 + 2b^2$ is absent.

---

**Fig.4** (a) The function $g(\theta, \sigma)$ for $b^2 = 0.1$, $\beta_- = 0.03$, $\beta_+ = 1.92$. Due to insufficient resolution the plot shows the singularity at $\theta = -0.9$ in a rudimentary form only. (b) The same function, zoom into the singularity at $\theta = -1 + b^2 = -0.9$.

---

2 For $f^2 = 0$ at $\theta_1$ the series would contain only $\theta_0$.
A single point series would arise also for $-2b^2 \leq \theta_0 < 0$.  

---

27
The situation becomes more involved, as soon as we switch on some asymmetry by choosing $\beta_- \neq 0$. Then the function $g$ is no longer positive definite in the whole range $-1 - b^2 < \theta < -b^2$, $0 < \sigma < \pi$ and develops a singularity at $\theta = -1 + b^2$, for an example see fig.4. The analysis has now to treat $f^2$ and $g$ in parallel and to take into account the fact that $g$ can change sign as a function of $\sigma$. By numerical study of several examples we found the remarkable possibility that for $\beta_+ > 2 - 2b^2$ one can reach situations where all 4 series in (4.34) can contribute to the spectrum. An example of this kind is shown in fig.5. Starting at any of the four zeros of $f^2$ jumping a distance $2b^2$ to the right one lands at positions with positive $f^2$ and $g$. At the end of these studies we mention, that for all examples plotted in the figures the positivity of all eigenvalues of $\hat{V}$ has been checked numerically.

When there are different options for the spectrum it is natural to assume that all of them are realized. This conjecture fixes the spectrum and, respectively, the $\hat{V}$-operator in the bound state sector. Let us note again that different options arise if values of the boundary parameters are near to the critical ones. In the next section we investigate how the different possibilities for the spectrum are realized in terms of the reflection amplitude.

4.2 The reflection amplitude of BLT

Similarly to (3.23), the reflection amplitude $R(p)$ associated with the operator (4.13) satisfies the equation

$$R(p + ib^2) = R(p - ib^2) F(p),$$

(4.40)

where $F(p)$ is given by (4.14). To find $R(p)$ we apply the scheme described in the subsection 3.2, which provides an integral representation for the phase $\gamma(p) = 2ib^2 \log R(p)$. At the first stage we have to find a Fourier type integral representation for $\log F(p)$. The
Γ-factors standing in \( F(p) \) can be represented with the help of (D.6), and for \( \Lambda(l_b, r_b; p) \) we use the continuation of (1.32) to the scattering sector

\[
\Lambda(l_b, r_b; p) = 4 \cos \frac{\pi}{2} (ip - \beta_+) \cos \frac{\pi}{2} (ip + \beta_-) \cos \frac{\pi}{2} (ip - \beta_-) \cos \frac{\pi}{2} (ip + \beta_+) .
\]  

(4.41)

Here \( \beta_\pm \) are the angle variables (1.30), which for \( l_b > 1 \) and \( r_b > 1 \) become pure imaginary. Writing then the cos-terms of (4.41) through Γ-functions by (D.1), we can apply again (D.6). Other technical details are given in Appendix C, and we obtain

\[
\gamma(p) = p \log \left( 4 \pi^2 m_b^2 \Gamma^2(1 + b^2) \right) + 
\int_0^\infty \frac{dt}{t} \left[ \frac{2ib^2 (e^{-ipt} - e^{ipt})}{(1 - e^{-b^2t})(1 - e^{-t})} - pe^{-t} \left( 4 + 2b^2 + \frac{4}{1 - e^{-t}} \right) \right] + 
\sum_{\varepsilon, \nu} \int_0^\infty \frac{dt}{t} \left[ \frac{2ib^2 (e^{-ipt} - e^{ipt}) e^{-(1+b^2-\varepsilon\beta_\nu)t}}{(1 - e^{-2b^2t})(1 - e^{-2t})} + p \left( \frac{2e^{-2t}}{1 - e^{-2t}} + (1 + \varepsilon\beta_\nu)e^{-2t} \right) \right] .
\]  

(4.42)

To represent these integrals in a familiar form, we rescale the integration variable for the term in the first line by \( t \to \frac{t}{b} \) and for the other four terms (which are given as a sum over \( \varepsilon = \pm 1 \) and \( \nu \equiv \pm \)) by \( t \to \frac{t}{2b} \). Then, using (C.10) and the integral (D.10), we express (4.42) through the Double Gamma function and the reflection amplitude becomes

\[
R(p) = R(s_+, s_-; P) \equiv \left[ m_b^2 \Gamma^2(b^2) b^{2-2b} \right]^{-\frac{1}{2b}} \frac{\Gamma_b(2ip)}{\Gamma_b(-2ip)} \left( \frac{\Gamma_b(\frac{Q}{2} - iP + s_+) \Gamma_b(\frac{Q}{2} - iP - s_-) \Gamma_b(\frac{Q}{2} - iP + s_-) \Gamma_b(\frac{Q}{2} + iP - s_+)}{\Gamma_b(\frac{Q}{2} + iP + s_+) \Gamma_b(\frac{Q}{2} + iP - s_-) \Gamma_b(\frac{Q}{2} - iP + s_-) \Gamma_b(\frac{Q}{2} - iP - s_+)} \right) ,
\]  

(4.43)

where \( Q \) is the background charge (1.24) and the variables \( P \) and \( s_\pm \) are obtained by the rescalings

\[
P = \frac{p}{2b} , \quad s_\pm = \frac{\beta_\pm}{2b} .
\]  

(4.44)

In this way we come to the notations of [4] and to have a complete correspondence, we also relate the Liouville fields and mass and boundary parameters, which are fixed by the normalization of the action (2.1)

\[
b\phi = \varphi , \quad \pi b^2 \mu = m^2 , \quad 2\pi b^2 \mu_{B_1} = 2ml , \quad 2\pi b^2 \mu_{B_2} = 2mr .
\]  

(4.45)

Here the left hand sides denote the corresponding quantities of [4]. In this paper we prefer to work with the initial action (2.1) without the rescalings (4.43). It helps us to control the semi-classical behavior of the theory and guides to fix results of quantum calculations.

To match eq. (2.25) of [4]: \( \cosh \pi b s_1 = \mu_{B_1} \mu^{-1/2} \sqrt{\sin \pi b^2} \) (and similarly for \( s_2 \)), we introduce \( s_1 \) and \( s_2 \) by: \( s_\pm = \frac{1}{2}(s_1 \pm s_2) \) and use (4.30) together with (4.44)-(4.45) and (2.33). As a result, \( R(p) \) given by (4.43) coincides with the reflection amplitude of BLT discussed in [4,5,8]. However, the integral representation (4.42), as it is derived in Appendix C, assumes \( b^2 < 1 \) and \( 1 - \varepsilon\beta_\nu > 0 \) for all four pairs \( (\varepsilon, \nu) \). The integrals in (4.42) are also well defined.
for pure imaginary $\beta$. $\lambda \sim (l_b > 1, r_b > 1)$. Both cases have no bound states and, therefore, the reflection amplitude \eqref{4.43} for them is exact. Then, similarly to the Morse potential, the reflection amplitude of BLT with discrete spectrum is obtained by a continuation of \eqref{4.43} in the boundary parameters, until they reach the critical value.

Let us discuss the continuation issue for the symmetric boundary conditions $l_b = r_b$. For $l_b \geq 1$, with the parameterization $l_b = \cosh \pi \rho$ ($\rho \geq 0$), eq. \eqref{4.43} becomes

$$R(p) = R(is, 0; P),$$

(4.46)

where $s = \frac{\rho}{b}$, $P$ is given by \eqref{4.42} and the function $R(is, 0; P)$ reads

$$R(is, 0; P) = \left[ m_b^2 \Gamma^2(b^2) b^{2\lambda} \right]^{-\frac{ip}{b}} \times$$

$$\frac{\Gamma_b(2iP) \Gamma_b(Q - iP + is) \Gamma_b(Q - iP - is) \Gamma_b(Q - iP)}{\Gamma_b(-2iP) \Gamma_b(Q + iP + is) \Gamma_b(Q + iP - is) \Gamma_b(Q + iP)}.$$  

(4.47)

In the next interval $0 \leq l_b < 1$, using the standard parameterization $l_b = \cos \pi \beta$ with $0 < \beta \leq \frac{1}{4}$, we have again \eqref{4.46}, but now with $is = \frac{\beta}{b}$. The $s$-dependent part in \eqref{4.47} is given by

$$\frac{\Gamma_b(Q - iP + is) \Gamma_b(Q - iP - is)}{\Gamma_b(Q + iP + is) \Gamma_b(Q + iP - is)} = \exp \left[ i \int_0^\infty \frac{dt}{t} \left( \frac{\sin(2Pt) \cos(2st)}{\sinh(bt) \sinh(t/b)} - \frac{2P}{t} \right) \right].$$

(4.48)

Note that the parameter $s$ does not change smoothly when we pass the point $l_b = 1$, but the dependence of \eqref{4.48} on $l_b$ is analytic. Farther continuation to the interval $-1 \leq l_b < 0$ corresponds to $\frac{1}{4} < \beta \leq 1$. The integral in \eqref{4.48} becomes divergent for pure imaginary $s$, if $|is| > \frac{Q}{2}$ ($\beta > \frac{1 + b^2}{2}$). Then one has to use the analytical continuation of \eqref{4.47} provided by \eqref{C.11}.

To get the discrete spectrum from the reflection amplitude one has to look for its zeros on the imaginary axis $p = i\theta$, with bounded $\theta$: $-1 - b^2 \leq \theta < 0$. The function $\Gamma_b(z)$ has no zeros, like usual $\Gamma$-function, but it has poles at $z = -\frac{2m}{b} - nb$ ($m \geq 0, n \geq 0$). One has to note that not all zeros of $R(i\theta)$ correspond to the discrete spectrum. For example, the reflection amplitude for the Morse potential \eqref{3.30} has zeros at $p = -\frac{in-h}{2}$ ($n \geq 0$), corresponding to the poles of $\Gamma\left(-\frac{2m}{b}\right)$, but they are not in the spectrum, and the spectrum is obtained by the poles of another factor in the denominator, which is $\lambda$ dependent. Similarly, in \eqref{4.47} one has to take only $s$-dependent zeros, corresponding to the poles of $\Gamma_b(Q/b + iP - is)$. They define exactly the spectrum \eqref{4.36}.

As it was mentioned in the previous subsection, there are other possibilities for the discrete spectrum near the critical value of the boundary parameter $\beta \geq 1 - b^2$. Some candidates for the corresponding reflection amplitude can be represented in the form \eqref{4.43}, with a suitable choice of the pair $(s_+, s_-)$. For example, the case $(s_+ = \frac{\beta}{b}, s_- = \frac{1}{b})$ corresponds to the unification of the spectra \eqref{4.36} and \eqref{4.37}. To unify \eqref{4.37} and \eqref{4.38}, one has to take $(s_+ = \frac{2}{b} - \frac{\beta}{b}, s_- = \frac{1}{b})$; and the case $(s_+ = \frac{2}{b} - \frac{\beta}{b}, s_- = 0)$ describes only the
spectrum \((4.38)\). But these examples do not cover the case with all three spectra \((4.36)-(4.38)\), since \((4.43)\) can not give more than two series of equidistant spectra. A modified reflection amplitude, which covers all three series \((4.36)-(4.38)\), is given by

\[
R(p) = R\left(\frac{\beta}{b} \frac{1-b}{b}; P\right) \frac{\sin \frac{\pi}{b}(Q/2 - iP - 2 \beta/b)}{\sin \frac{\pi}{b}(Q/2 - iP - 2 \beta/b)}. \tag{4.49}
\]

It has unit norm and satisfies eq. \((4.40)\), since the ratio of sin-functions is invariant under the shift \(P \mapsto P - ib\). Such a ratio of sin-functions modifies the reflection amplitude \((4.43)\) in a form compatible with the more general spectrum discussed in the previous sub-section.

For \(\beta = 1 - b^2\) the reflection amplitude \((4.49)\) becomes

\[
R(p) = R\left(\frac{1}{b} \frac{1}{b}; P\right) \frac{\sin \frac{\pi}{b}(Q/2 - iP)}{\sin \frac{\pi}{b}(Q/2 + iP)} = R\left(\frac{1}{b} - b, 0; P\right) \frac{\sin^2 \frac{\pi}{b}(Q/2 - iP)}{\sin^2 \frac{\pi}{b}(Q/2 + iP)}. \tag{4.50}
\]

Here, using the property \((C.11)\) of \(\Gamma_b\)-functions, we have shifted the argument of the function \(\mathcal{R}\) from \(\frac{1}{b}\) to 0. This creates an additional ratio of sin-functions. Note that the reflection amplitude \((4.50)\) indeed vanishes for \(P = -i\frac{Q}{2}\), which corresponds to the case with a normalizable \(SL(2,\mathbb{R})\) invariant vacuum.

### 4.3 1-point function of BLT

The 1-point function of BLT associated with the operator \(V\) is given by the matrix element of \(V\) between the \(SL(2,\mathbb{R})\) invariant vacuum states. The state \(|ibQ,0\rangle\), as a continuation of the states \(|p,0\rangle\) to \(p = i(1 + b^2) = ibQ\), is \(SL(2,\mathbb{R})\) invariant, but for generic \(l_b, r_b\) defined only as a singular element of the dual (to \(L^2(\mathbb{R}_+^\prime)\)) space, like the state \((3.41)\) for \(\lambda > -\frac{i}{2}\). Therefore, one has to be careful in giving a well defined meaning to the 1-point function.

#### 4.3.1 ZZ case, \(l_b = r_b = -\cos \pi b^2\)

In this case the state \(|ibQ,0\rangle\) is normalizable since, according to subsection 4.1, \(-bQ\) is in the \(\theta\)-spectrum. Let us call the corresponding normalized state by \(|\mathcal{O}\rangle\). Then we get with \((4.26)\) and \((4.15)\)

\[
\langle \mathcal{O}|V|\mathcal{O}\rangle = (2 \sin \sigma)^{-\frac{1}{2}} \frac{\Gamma(1 + b^2) \Gamma(1 - 2b^2)}{\Gamma(1 - b^2)} \sin \pi b^2. \tag{4.51}
\]

For the evaluation of \(G\) we note that the coefficient \(b_p\) vanishes for \(l_b = r_b\) and \(p = i + ib^2\), and \(c_p\) becomes (see \((4.12)\))

\[
c_p = -\frac{2i m_b}{\pi} \frac{b e^{i\pi b^2}}{\Gamma(1 + b^2) \Gamma(1 - 2b^2)} \sin \pi b^2. \tag{4.52}
\]

At the same time, the hypergeometric function standing in \((4.11)\) reduces to

\[
2F_1(-b^2, -1 - 2b^2, -b^2; e^{2i\sigma}) = (1 - e^{2i\sigma})^{1+2b^2}. \tag{4.53}
\]
As a result we obtain

\[
\langle O|V|O \rangle = -2m_b \cos \pi b^2 \frac{\Gamma(1 + b^2) \Gamma(-1 - 2b^2)}{\Gamma(-b^2)} (2 \sin \sigma)^{1 + \frac{3}{2} b^2} .
\] (4.54)

This result agrees with the 1-point function of [6] for the special case corresponding to our \( V \) and the 'basic' (1,1) vacuum. To make this manifest, one has to relate \( m_b \) via (4.45) and (2.39) to their mass parameter and to take into account also the relative rescaling of the Liouville field.

### 4.3.2 FZZT case, generic \( l_b = r_b \)

In this case the \( SL(2, \mathbb{R}) \) invariant vacuum is not normalizable and to calculate the 1-point correlation function corresponding to the operator (4.13) we need a regularization procedure like the one given at the end of sub-section 3.3. This procedure for BLT needs further investigation and will be discussed elsewhere.

Assuming for the time being that eq. (3.59), derived for the Morse potential, is analogously valid also in BLT for the operator (4.13), we get

\[
es \langle ibQ, 0|V|ibQ, 0 \rangle_c = (2 \sin \sigma)^{-\frac{b^2}{2}} G(ibQ, \sigma) \left[ -2ib^2 \partial_p R(p) \right]_{p=ibQ} .
\] (4.55)

The factor \((2 \sin \sigma)^{-\frac{b^2}{2}} G(ibQ, \sigma)\) is obtained from the r.h.s. of (4.54), after replacing the term \(-\cos \pi b^2\) by the generic boundary parameter \( l_b = \cosh \pi bs \). Switching to the parameters \( m, l \) (see (2.39)), this gives

\[
(2 \sin \sigma)^{-\frac{b^2}{2}} G(ibQ, \sigma) = Z(b) (2 \sin \sigma)^{1 + \frac{3}{2} b^2} ,
\] (4.56)

with \( Z(b) \) defined in (4.22) as a bulk-boundary structure constant.

Using the reflection amplitude (4.47), from (4.55) we find

\[
es \langle ibQ, 0|V|ibQ, 0 \rangle_c = (m_b^2 \Gamma^2(b^2))^{1 + \frac{1}{24b^2}} \Gamma(-2b^2) \Gamma \left( -2 - \frac{1}{2b^2} \right) \times
\]

\[
\left[ \cosh \pi s \left( 2b + \frac{1}{b} \right) - \cosh \pi s \left( 2b - \frac{1}{b} \right) \right] .
\] (4.57)

The r.h.s. of this equation can be compared with the 1-point function \( U(\alpha) \) of [4]

\[
U(\alpha) = \frac{2}{b} \left( \frac{\pi \mu \Gamma(b^2)}{\Gamma(1 - b^2)} \right)^{-\frac{2\alpha}{2b^2}} \Gamma(2\alpha/b^2) \Gamma(2\alpha/b - 1/b^2 - 1) \cosh \pi s(2\alpha - Q) .
\] (4.58)

at \( \alpha = -\frac{b}{2} \). The translations between the parameters (4.45) and (2.39) provide

\[
U(-b/2) = \frac{2}{b} \left( m_b^2 \Gamma^2(b^2) \right)^{1 + \frac{1}{24b^2}} \Gamma(-2b^2) \Gamma \left( -2 - \frac{1}{2b^2} \right) \cosh \pi s \left( 2b + \frac{1}{b} \right) .
\] (4.59)
The pre-factor $2/b$ in this equation could be related to the relative normalization of the states $|P\rangle$ and $|p\rangle$, but the existence of the second cosh-term in (4.57) indicates that the rule (4.57) yields a 1-point function not compatible with the conformal bootstrap result (4.58).

Furthermore, replacing (4.57) by the weaker assumption of some factorization

$$e^{ibQ,0|V|ibQ,0} = N (2 \sin \sigma)^{-\frac{\sqrt{2}}{2}} G(ibQ, \sigma),$$  \hspace{1cm} (4.60)

where $N$ denotes a regularized norm $e^{ibQ,0|ibQ,0}$, we can identify $N$ with $U(0)$, and then, due to (4.56), $Z(b)$ should be compared with

$$\frac{U(-b/2)}{U(0)} = m \sqrt{\frac{\sin \pi b^2}{\pi b^2} \cosh(b + Q)\pi s} \frac{\Gamma(1 + b^2)\Gamma(-1 - 2b^2)}{\Gamma(-b^2)}.$$  \hspace{1cm} (4.61)

Obviously, this does not agree with our $Z(b)$. However, there is an intriguing observation concerning the construction of $U(\alpha)$ as a solution of a functional equation in [4]. Replacing $Q$ in the argument of cosh in (4.58) by an arbitrary constant $c$ gives still a solution of this functional equation. Using the freedom to put $c = 0$ one would get for $U(-b/2)/U(0)$ just $1/2 \cdot Z(b)$. The choice $c = Q$ is forced by implementing the reflection relation as an additional input of the bootstrap approach.

Altogether, at the present status of our calculations, it is either not possible to make a comparison, due to the breakdown of (4.57), or the factorization (4.60), or there is a deeper problem in relating Euclidean bootstrap BLT to Lorentzian operator BLT.

5 Some open problems of the operator approach to BLT

Concerning open problems of the operator approach, first we refer to the expansion of the vertex operators in powers of the screening charges (2.41). The same expansion in the periodic case contains only even powers of $m_0$ and the general $p$-dependent coefficient is known in a closed form [10]. This enables one to verify that the operator Liouville equation (2.51) is fulfilled by the formal power series of $\varphi(x, \bar{x})$ and $V_1(x, \bar{x})$. However, in general, these expansions are asymptotic and, therefore, ineffective in practical calculations. But for $\alpha = -\frac{n}{2}$ the series becomes finite, as it is expected from the classical picture. $V_\alpha$ contains only $n + 1$ terms and one can obtain the corresponding correlation function $\langle p', 0|V_{-\frac{n}{2}}|p, 0 \rangle$ in a closed form [14]. The continuation of this expression to arbitrary $\alpha$ reproduces the 3-point correlation function of [11, 12]. We hope that this scheme of calculation of the correlation functions could work effectively for BLT as well, if one finds the coefficients $c_{p,k}^l(\alpha)$ in a closed form. Note that the operators $V_{-\frac{n}{2}}$ for BLT were constructed in [23], using a quantum group structure of the basic operators. But a representation of that result as a polynomial in powers of screening charges needs an additional combinatorial labor similar to the one given in Appendix A. The power series for the operators $V_\alpha$ can
also be used to find its compact integral representation \([13, 14]\), which is directly related to the correlation functions.

Another issue we would like to discuss is the \(S\)-matrix of Liouville theory, which is unknown in a closed form for the periodic case as well, though the expression for the out-field exponential in terms of the in-field is rather compact \([2.36]\). This relation provides the following equation for the \(S\)-matrix

\[
e^{-\phi(x)} e^{-\phi(x)} S = m^2 S A(x) e^{-\phi(x)} D_p e^{-\phi(x)} A(x) . \tag{5.62}
\]

On the basis of the classical picture, the \(S\)-matrix can be represented in the form \(S = \mathcal{P} R(p) S_p\), where \(\mathcal{P}\) is the parity operator in the zero mode sector as in \([3.22]\), \(R(p)\) is the reflection amplitude

\[
S |p, 0\rangle = R(p) | -p, 0\rangle , \tag{5.63}
\]

and \(S_p\) is the operator responsible for transitions in non-zero mode sectors. \(S_p\) contains \(p\) as a parameter and depends on the \(a_n\) operators. Inserting this ansatz in eq. \([5.62]\) and projecting it between the vacuum states, one finds that \(R(p)\) indeed satisfies \([4.40]\). Since \(R(p)\) is known, eq. \([5.62]\) reduces to a closed equation for \(S_p\)

\[
s(p) e^{-\hat{\phi}(x)} S_p = S_{p-\bar{a}^2} \int_0^{2\pi} dy e^{ip(y-\pi)} e^{2\hat{\phi}(x+y)} e^{-\hat{\phi}(x)} , \tag{5.64}
\]

where the function \(s(p)\) is the integrated short distance factor

\[
s(p) = \int_0^{2\pi} dy e^{ip(y-\pi)} (1 - e^{-ip}) b^2 = \frac{2\pi \Gamma(1 + b^2)}{\Gamma(1 - ip) \Gamma(1 + b^2 + ip)} , \tag{5.65}
\]

and \(e^{-\hat{\phi}(x)}\) and \(e^{2\hat{\phi}(x+y)}\) denote the normal ordered chiral free-field exponentials without the zero modes \(e^{-\hat{\phi}(x)} = e^{-\phi(x)} e^{-\phi.(x)}\). Note that in the periodic case the \(S\)-matrix has a similar form \(S = \mathcal{P} R(p) S_p \tilde{S}_p\), where the anti-chiral part \(\tilde{S}_p\) (depending on \(\bar{a}_n\)) is functionally identical to \(S_p\), and \(S_p\) satisfies the same equation \([5.64]\). From the analysis of this equation one can conclude that the transition amplitudes defined by \(S_p\) are non-zero only between equal levels of the Hamiltonian \(L_0\) and \(S_p\) has the following structure

\[
S_p = I + R^{1,1}_p a_{-1} a_1 + \left[ R^{01,01}_p a_{-2} a_2 + R^{01,20}_p a_{-2} a_1^2 + R^{20,01}_p a_{-2} a_2 + R^{20,20}_p (p) a_{-2}^2 a_1^2 \right] + \ldots , \tag{5.66}
\]

with some \(p\)-dependent coefficients \(R^{1,1}_p, R^{01,01}_p, \ldots\).

Projecting eq. \([5.64]\) between the states of the Fock space, one finds the \(p\)-dependent coefficients and transition amplitudes step by step. For examples, the projection between the vacuum and the first exited states provides

\[
S_p a_{-1} |p, 0\rangle = -\frac{p + i(1 + b^2)}{p - i(1 + b^2)} a_1^+ |p, 0\rangle . \tag{5.67}
\]
The aim is to find the matrix element of $S_p$ between arbitrary coherent states, which provides the normal symbol, or normal ordered form $S_p$. From eq. (5.64) one can find matrix elements between some coherent states, but a closed form of $S_p$ is still missing.

Finally we consider the problem of quantization of the sector of bound states, which corresponds to pure imaginary $p$. The Liouville field exponential (2.7) for $p = i\theta$ (negative $T_0$) can be written as

$$V(x, \bar{x}) = e^{-\phi(x)} e^{-\phi(x)} + e^{-\phi^*(x)} e^{-\phi^*(x)} - \frac{l e^{i\pi\theta} + r}{\sqrt{\Lambda}} e^{-\phi(x)} e^{-\phi^*(x)} - \frac{l e^{-i\pi\theta} + r}{\sqrt{\Lambda}} e^{-\phi^*(x)} e^{-\phi(x)},$$

where $\Lambda = l^2 + r^2 + 2lr \cos \pi\theta - \sin^2 \pi\theta$ and

$$e^{-\phi(x)} = \left( \frac{m \sqrt{\Lambda}}{\xi'(x) \sin \pi\theta} \right)^{\frac{1}{2}} e^{-i\theta \xi(x) - \frac{i\pi}{2}}.$$  

(5.69)

Here $\phi(x)$ is the same field as it stands in (2.10), but for imaginary $p$. This field is not real: $\phi(x) = \phi_1(x) + i\phi_2(x)$, and its real and imaginary parts are related by

$$\phi_2'(x) = \frac{m \sqrt{\Lambda}}{2\sin \pi\theta} e^{2\phi_1(x)}.$$  

(5.70)

It has to be mentioned that the time evolution for imaginary $p$ has an oscillating character and there are no asymptotic fields any more. But the continuation of in and out fields to the sector of bound states exist and they become complex conjugated to each other.

Quantization of this situation is a non-trivial problem indeed. At least we do not know global canonical variables, or an analog of the infinite dimensional translation symmetry to apply an alternative quantization scheme (like geometric, coherent state, etc). It is a typical situation for the elliptic monodromy. What remains still valid from the scattering picture (hyperbolic monodromy) is the Fourier mode expansion of $\phi(x)$ (2.11) and the Poisson brackets (2.12). But these brackets are not canonical, since the conjugation rule $a_n^* = a_{-n}$ is violated. It also remains the parameterization (2.14) and the formula for the screening charge (2.16) (with $p = i\theta$).

Quantization of the zero mode sector can be done by the scheme described above. The problem is in the non-zero mode sector. Here one can apply a formal algebraic quantization based on the commutation relations (2.17) between the $a_n$ operators. Choosing the normalized vacuum states by

$$a_n |\theta_m, 0\rangle = 0, \quad n > 0,$$

(5.71)

where $\theta_m$ are the discrete levels, the space of states can be constructed similarly to the Fock space, by the action of $a_n$, with $n < 0$ on $|\theta_m, 0\rangle$. Using the same ordering as in the hyperbolic case, formally one gets the desired structure (locality, conformal weight, etc) for the operator $V$. What remains to be specified to calculate matrix elements is the
scalar product. Taking into account the above mentioned conjugation between the ‘in’-‘out’ fields, one gets $a_n^c = b_n$, where $b_n$ are the Fourier modes for the ‘out-field’ operator. Thus, the scalar product in the sector of bound states is given by the continuation (in $p$) of the corresponding matrix elements of the operator $S_p$. It means that the construction of the $S$-matrix in a closed form could be the key point for a complete understanding of quantization of the elliptic sector.

6 Conclusions

We have completed the construction of the basic operator $V = e^{-\varphi}$ in terms of a free chiral field, started in our previous paper [9]. By going the first steps in the construction of generic $V_\alpha = e^{2\alpha \varphi}$, we were able to relate the mass and boundary parameters $m_b, l_b, r_b$ in our quantum version of $V$ to those $(m, l, r)$ appearing in the equation of motion and the boundary conditions for the Liouville field on operator level. Both equations keep the same form as in the classical case. For the boundary condition the necessary Liouville exponential has to be understood as a limit of the corresponding bulk operator after dividing out a short distance singularity.

Some new techniques for the extraction of the spectrum and the definition of expectation values in states which make sense only after some regularization and continuation have been developed and tested for a particle in the Morse potential. Here complete and exact results are available from the solutions of the corresponding Schrödinger equation. Due to the similarity of the potential to that in BLT, this particle model should share some qualitative properties with BLT. In particular, depending on the sign of the parameter $\lambda$, the Morse particle can have bound and scattering states or scattering states only.

The analog of $V$ in the scattering sector has been expressed in terms of a free canonical $(p, q)$ pair and then continued to the bound state sector. Then, by requiring Hermiticity and positivity for $V$, as well as an energy spectrum bounded from below, the ground state energy can be read off from the zero of a certain function appearing in this free field representation of $V$. Higher levels are obtained by the action of the raising operator $U_+$, which is a part of $V$. Identifying parts of $V$ as ingoing and outgoing exponentials of the particle position, a functional relation for the reflection amplitude has been derived and solved by turning it into an infinite order differential equation. The spectrum is then reproduced from the zeros of the reflection amplitude.

Furthermore, to mimic the problems encountered with the $SL(2, \mathbb{R})$ invariant state in BLT, we have shown that the norm of the state, obtained by continuing the $\delta$-function normalized scattering states to that imaginary momentum corresponding to the ground state energy, can be obtained also via a regularization and continuation process using either insertions of generic $V_\alpha$ or regularizations of the scattering states.

Turning back to BLT our interest concentrated on the issue of the spectrum of highest weight states for the Verma modules (spectrum of the zero mode contribution to the energy-momentum tensor) constituting BLT and on the calculation of correlation functions. For both issues the analysis is determined by the projection $\hat{V}$ of the operator $V$ to the vacuum
sector with respect to the free field oscillator Fock space.

This projection has been expressed in terms of exponentials of the zero mode \((p, q)\) pair and two functions of \(F(p)\) and \(G(p, \sigma)\). Besides the \(\sigma\)-dependence and the more complicated form of \(F\) and \(G\) this resembles to what has been done for the Morse particle.

Then from the analysis of \(F(p)\) and \(G(p, \sigma)\) we determined the wanted spectrum first for \(b^2 \ll 1\) and more general for \(b^2 < 1/3\). For values of the boundary parameters \(\beta_l, \beta_r < 1 - b^2\) only one series \(\theta_n\) with constant spacing \(h = 2b^2\) contributes, in agreement with [5]. However, for values above the critical \(1 - b^2\) three (for \(l = r\)) or even four (for \(l \neq r\)) such series may contribute.

The critical case \(l_b = r_b = -\cos \pi b^2\) turned out of particular interest. There the \(SL(2, \mathbb{R})\) invariant vacuum state at \(\theta_0 = -bQ\) is invariant under the action of \(\hat{V}\), and one has the option to restrict the theory to the corresponding Verma module (ZZ case [6]) or to include it as a ground state in the theory, which otherwise is nothing else than generic BLT (FZZT case) just at this special value of the boundary parameters.

The reflection amplitude for BLT has been derived from the knowledge of \(\hat{V}\) applying the technique demonstrated for the Morse particle. Thereby an integral representation for the phase of the reflection was obtained, which results in a representation in terms of the Barnes Double Gamma function. The result agrees with the boundary two point function obtained in the bootstrap approach [4]. In addition, it has been shown that making use of the freedom to add certain periodic factors, not fixed by the defining functional relation, modified reflection amplitudes can be constructed, whose pattern of zeros completely reproduces the spectrum obtained via the properties of \(\hat{V}\) before.

A last issue concerns the 1-point function for \(V\) with respect to the \(SL(2, \mathbb{R})\) invariant vacuum. In the critical case \(l_b = r_b = -\cos \pi b^2\) this quantity is a priori well defined, since the state is in the spectrum and therefore normalizable. There we get just the ZZ 1-point function [6]. In the generic case some regularization and continuation has to be implemented, which requires further analysis.

The main part of our results has been presented by G.J. at the conference “Conformal Field Theory and Integrability: from Condensed Matter Physics to String Theory” Yerevan and Tbilisi, 01.10.-09.10. 2007.

Acknowledgments
G.J. would like to thank the brothers Zamolodchikov for helpful discussion. He thanks ICTP Trieste for hospitality, where a part of his work was done, RFBR and also GNSF for the grant ST06/4-050. The research was supported in part by DFG with the grant DO 447-4/1.
Appendix A

Calculation of the coefficients $c^{l,k}_p(\alpha)$

In this Appendix we present two different schemes of calculation of $p$-dependent coefficients $c^{l,k}_p(\alpha)$, for the expansion (2.41). These calculations are used in Section 2 to find the relation between the parameters $(m_b, l_b, r_b)$ and $(m, l, r)$.

First we consider the scheme based on eq. (2.45). By (2.40), this equation reads

$$\Psi_\alpha(x, -x) V_\alpha(x, -x) \Psi_{\frac{1}{2}}(y, -y) V_{\frac{1}{2}}(y, -y) =$$

(A.1)

Using the exchange relations (2.24) - (2.27) and the locality for free field exponentials

$$\Psi_\alpha(x, -x) \Psi_{\frac{1}{2}}(y, -y) = \Psi_{\frac{1}{2}}(y, -y) \Psi_\alpha(x, -x),$$

(A.2)

one can cancel the free-field exponents in (A.1) and reduce the equation to

$$\tilde{V}_\alpha(x, -x; y, -y) V_{\frac{1}{2}}(y, -y) = \tilde{V}_{\frac{1}{2}}(y, -y; x, -x) V_\alpha(x, -x).$$

(A.3)

Here $\tilde{V}_\alpha(x, -x; y, -y)$ denotes the operator which comes from $V_\alpha(x, -x)$ after its exchange with $\Psi_{\frac{1}{2}}(y, -y)$. This procedure shifts the index $p$ of the coefficients $c_p^{l,k}(\alpha)$ by $p \rightarrow p+2ib^2$ and creates new screening charges $A(y)$ and $A(-y)$ according to (2.26). Similarly, the exchange of $V_{\frac{1}{2}}(y, -y)$ with $\Psi_\alpha(x, -x)$ creates $\tilde{V}_{\frac{1}{2}}(y, -y; x, -x)$. Thus, both sides of eq. (A.3) are power series in $A(x)$, $A(-x)$, $A(y)$ and $A(-y)$. Comparing the coefficients of the corresponding monoms we get equations for the coefficients $c_p^{l,k}(\alpha)$. For example, the terms linear in the screening charges provide four equations. One of them, which comes from the coefficients of $A(x)$, is

$$e^{2i\pi b^2} c_p^{1,1}(\alpha) = c_p^{1,1}(\alpha) + \mu_p(\alpha) \left( c_p^{1,0} + c_p^{1,1} \right).$$

(A.4)

Here $c_p^{1,0}$ and $c_p^{1,1}$ are given by (2.43) and $\mu_p(\alpha)$ is the coefficient of $A(y)$ in (2.26) for $\epsilon(x-y) = -1$:

$$\mu_p(\alpha) = \frac{i \sin(2\pi ab^2) e^{-\pi(p-2iab^2-ib^2)}}{\sinh \pi(p-2iab^2-ib^2)}.$$  

(A.5)

Eq. (A.4) is solved by (2.47). The comparison of the coefficients of $A(-x)$ leads to a similar equation for $c_p^{1,0}(\alpha)$, which is solved by (2.46). Then one checks that the two other equations, provided by the coefficients of $A(y)$ and $A(-y)$ are also solved by (2.46) - (2.47). Considering the coefficients of $A(-x)A(x)$, one finds the equation for $c_p^{2,1}(\alpha)$:

$$c_p^{2,1}(\alpha) = c_p^{2,1}(\alpha) + \nu_p(\alpha),$$

(A.6)

Here $\nu_p(\alpha)$ is calculated by the coefficients $c_p^{1,k}(k = 0, 1)$, $c_p^{2,k}(k = 1, 2)$ and $c_p^{1,k}(\alpha)$ $(k = 0, 1)$, which are already known from (2.43)-(2.41) and (2.46)-(2.47). The calculation
of $\nu_p(\alpha)$ is a rather long, but straightforward procedure. Finally, eq. (A.6) is solved by (2.34).

Note that the solutions of the equations (A.4) and (A.6), have a freedom in adding the terms $e^{\frac{\pi n b}{\nu_p}}$, with integer $n$. We neglect this freedom, requiring ‘smooth’ dependence on $h = b^2$. This question is discussed in Section 3 in more detail, where a method of solution of this type of equations is given as well.

Now we present an alternative scheme based on the construction of the vertex operators $V_{-\frac{a}{2}}(x, \bar{x})$. These operators are associated with the exponentials $e^{-\nu_p(x, \bar{x})}$. The operator $V_{-\frac{a}{2}}(x, \bar{x}) \equiv V(x, \bar{x})$ is given by (2.42) and others can be obtained step by step as its regularized powers

$$V_{-\frac{a}{2}}(x, \bar{x}) = \lim_{\epsilon \to 0} V(x + \epsilon, \bar{x} - \epsilon) V_{-\frac{a}{2}}(x, \bar{x}) \left| \epsilon \right|^{ni b^2}. \quad (A.7)$$

The screening charge operator $A(x)$ does not create short distance singularities in the operator products like (A.7), and the primary free-field exponentials (2.30) provide

$$\lim_{\epsilon \to 0} \Phi_\alpha(x, \bar{x}) \Phi_\beta(x + \epsilon, \bar{x} - \epsilon) \left| \epsilon \right|^{ni b^2} = \Phi_{\alpha+\beta}(x, \bar{x}). \quad (A.8)$$

This formula helps to calculate the regularized powers (A.7) by recurrence relations.

The operator $V_{-\frac{a}{2}}$ is a polynomial in $m_b$ of the order $2n$. Assuming that it can be written similarly to (2.42)

$$V_{-\frac{a}{2}}(x, \bar{x}) = e^{-\frac{\pi s n^2 b^2}{4}} e^{-n\phi(x)} e^{-n\phi(x)} \left[ 1 + m_b (A(\bar{x}) c_{n, p}^{1,0} + A(x) c_{n, p}^{1,1}) + m_b^2 (A^2(\bar{x}) c_{n, p}^{2,0} + A^2(x) c_{n, p}^{2,1} + A^2(x) c_{n, p}^{2,2}) + ... \right], \quad (A.9)$$

where, for convenience, we denote the expansion coefficients by $c_{n, p}^{l,k}$ instead of $c_{p}^{l,k}(\frac{-n}{2})$. Applying then the exchange relations (2.24)-(2.27) to (A.7), the operator $V_{-\frac{a}{2}}$ is reduced to the form (A.9), and one finds recurrence relations for the coefficients $c_{n, p}^{l,k}$.

For example, the coefficients of $A(\bar{x})$ and $A(x)$ provide

$$c_{n+1, p}^{1,0} = c_{n, p}^{1,0} + e^{-\pi n b^2} c_{p+2n b^2}^{1,0} + \mu_p^+(n/2) c_{p+2n b^2}^{1,1}, \quad (A.10)$$

and

$$c_{n+1, p}^{1,1} = c_{n, p}^{1,1} + e^{\pi n b^2} c_{p+2n b^2}^{1,1} + \mu_p^-(n/2) c_{p+2n b^2}^{1,0}, \quad (A.11)$$

respectively. Here $c_{p}^{1,0}$ and $c_{p}^{1,1}$ are given by (2.43), $\mu_p^-(n/2)$ is the coefficient (A.5) for $\alpha = -n/2$ and $\mu_p^+(n/2)$ differs from it only by the sign of the exponent in (A.5).

These recurrence relations are solved by

$$c_{n, p}^{1,0} = \frac{\sin \pi n b^2}{\sin \pi b^2} \frac{l_b e^{\pi n b^2} e^{i\pi(n-1)b^2} + r_b e^{-i\pi n b^2}}{\sinh \pi(p+2n b^2 - i b^2)}, \quad (A.12)$$

$$c_{n, p}^{1,1} = \frac{\sin \pi n b^2}{\sin \pi b^2} \frac{l_b e^{-\pi n b^2} e^{-i\pi(n-1)b^2} + r_b e^{i\pi n b^2}}{\sinh \pi(p+2n b^2 - i b^2)}. \quad (A.13)$$
The continuation of these expressions to real \( n \) and denoting \( n = -2\alpha \) leads to (2.46)-(2.47).

The same procedure for the coefficient \( c_{n,p}^{2,1} \) yields a similar recurrence relation

\[
\begin{align*}
c_{n+1,p}^{2,1} &= c_{n,p}^{2,1} + X_{n,p} \\
&= (-2\alpha) \quad \text{(A.14)}
\end{align*}
\]

where \( X_{n,p} \) is defined by the already known coefficient. Its calculation is similar to the calculation of \( \nu_p(\alpha) \) in (A.6). As a result, the solution of (A.14) coincides with (2.48) for \( \alpha = -n/2 \).

The higher order coefficients of the expansion (2.41) can also be calculated by the schemes presented here, but a closed form of the general coefficient \( c_{n,p}^{l,k} \) is still missing.

Appendix B

Particle in the Morse Potential

1. The eigenstates and the reflection amplitude

The Schrödinger equation in the Morse potential (3.2)

\[
-\frac{\hbar^2}{2} \Psi''_E(y) + \left[ 2m^2 e^{2y} + 2m \lambda e^y \right] \Psi_E(y) = E \Psi_E(y)
\]

(B.1)

can be reduced [20] to the equation for the confluent hypergeometric functions and the eigenstates for positive energies ( \( E > 0 \) ) read

\[
\begin{align*}
\Psi_E(y) &= \left( \frac{4m}{\hbar} \right)^{ik} \frac{\Gamma(-2ik)}{\Gamma(\nu + ik)} e^{-\frac{z}{2}} \Phi(\nu + ik, 1 + 2ik, z) e^{iky} \\
&\quad + \left( \frac{4m}{\hbar} \right)^{-ik} \frac{\Gamma(2ik)}{\Gamma(\nu + ik)} e^{-\frac{z}{2}} \Phi(\nu - ik, 1 - 2ik, z) e^{-iky} = W_{-\frac{\lambda}{\nu}, ik}(z) z^{-\frac{1}{2}}.
\end{align*}
\]

Here \( \Phi \) is the confluent hypergeometric function [21]

\[
\Phi(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha + 1)}{\gamma(\gamma + 1)} \frac{z^2}{2!} + \ldots,
\]

(B.3)

\( W_{-\frac{\lambda}{\nu}, ik}(z) \) is the Whittaker’s function [21] and we have used the notations

\[
z = \frac{4m}{\hbar} e^y, \quad k = \frac{\sqrt{2E}}{\hbar}, \quad \nu = \frac{1}{2} + \frac{\lambda}{\hbar}.
\]

(B.4)

The wave functions (B.2) vanish at \( y \to \infty \) and their behavior at \( y \to -\infty \) can be written as

\[
\Psi_E(y) \sim a^*(k)e^{iky} + a(k)e^{-iky},
\]

(B.5)

with

\[
a(k) = \left( \frac{4m}{\hbar} \right)^{-ik} \frac{\Gamma(2ik)}{\Gamma(\nu + ik)}.
\]

(B.6)
The ratio of out-going and in-coming coefficients defines the reflection amplitude

\[ r_\lambda(k) = \frac{a(k)}{a^*(k)} = \left(\frac{4m}{\hbar}\right)^{-2ik} \frac{\Gamma(2ik) \Gamma(\nu - ik)}{\Gamma(-2ik) \Gamma(\nu + ik)}. \]  

(B.7)

If \( \nu \) is negative there are, in addition, the bound states at \( k = ik_n \), where \( \kappa_n = \nu + n \). The second term in (B.2) vanishes for these imaginary values of \( k \) and, therefore, the normalized wave functions can be written as

\[ \psi_n = c_n \ e^{-(\nu+n)y} \ e^{-\frac{z}{2}} \left(1 + \sum_{m=1}^{n} \frac{n(n-1) \cdots (n-m+1)}{(2\nu+2n-1) \cdots (2\nu+2n-m)} \frac{z^m}{m!}\right), \]

(B.8)

where \( c_n \) are normalization coefficients. The number of the bound states is restricted by \( 0 \leq n < -\nu \). The coefficients \( c_n \) can be expressed through the derivatives of \( r_\lambda(k) \) at \( k = ik_n \), where this function has simple zeros. [22].

For this purpose let us introduce the function \( \Psi_k(y) = \frac{\Psi_E(y)}{a^*(k)} \), which has the following asymptotic behavior at \( y \to -\infty \)

\[ \Psi_k(y) \sim e^{iky} + r_\lambda(k) e^{-iky}. \]  

(B.9)

Its analytical continuation is related to the normalized wave functions (B.8) by

\[ \psi_n = c_n \Psi_{ik_n}. \]  

(B.10)

From the Schrödinger equation (B.2) one finds that

\[ \Psi_k''(y) \partial_k \Psi_k(y) - \partial_k \Psi_k''(y) \Psi_k(y) = 2k \Psi_k^2(y), \]  

(B.11)

and its integration leads to

\[ 2k \int_y^\infty \Psi_k^2(y_1) \, dy_1 = \partial_k \Psi_k(y) \Psi_k(y) - \Psi_k'(y) \partial_k \Psi_k(y). \]  

(B.12)

Taking here the limit \( y \to -\infty \) at \( k = ik_n \), we obtain

\[ \langle \Psi_{ik_n} | \Psi_{ik_n} \rangle = -i \partial_k r_\lambda(k)|_{k=ik_n}, \]

(B.13)

which by (B.7) provides

\[ c_n = \left(\frac{\hbar}{4m}\right)^{\nu+n} \sqrt{\frac{(-)^n \Gamma(2\nu+2n)}{n! \Gamma(-2\nu-2n) \Gamma(2\nu+n)}}. \]  

(B.14)

Finally, we present the normalized wave functions \( \psi_0 \) and \( \psi_1 \), written in terms of \( z \) (see (B.4))

\[ \psi_0 = \frac{1}{\sqrt{\Gamma(2|\nu|)}} \ z^{\nu} e^{-\frac{z}{2}}, \quad \psi_1 = \sqrt{\frac{2|\nu|-1}{\Gamma(2|\nu|-2)}} \ z^{\nu-1} e^{-\frac{z}{2}} \left(1 - \frac{z}{2|\nu|-1}\right). \]  

(B.15)
The dynamical equation for the operator $V(\tau)$

The Hamiltonian $H = \frac{1}{2} p_y^2 + 2m^2 e^{2y} + 2m\lambda e^y$ provides the following time evolution equations for the Heisenberg operators $V = e^{-y}$ and $p_y$

$$\dot{V} = -V \left( p_y + \frac{i\hbar}{2} \right), \quad \dot{p}_y = -\left( \frac{4m^2}{V^2} + \frac{2m\lambda}{V} \right).$$  \hspace{1cm} (B.16)

The second derivative of $V$ then becomes

$$\ddot{V} = V \left( p_y + \frac{i\hbar}{2} \right)^2 + \frac{4m^2}{V} + 2m\lambda,$$  \hspace{1cm} (B.17)

and expressing the momentum operator $p_y$ from (B.16) through $V$ and $\dot{V}$ one finds

$$\ddot{V} = \dot{V} \frac{1}{V} \dot{V} + \frac{4m^2}{V} + 2m\lambda.$$  \hspace{1cm} (B.18)

From (B.16) also follows the commutator $\dot{V}V - V\dot{V} = i\hbar V^2$, which together with (B.18) lead to the dynamical equation (3.13).

Appendix C

Integral representation of $\gamma(p)$

In this appendix we calculate the phase $\gamma(p) = i\hbar \log R(p)$ of the reflection amplitude of BLT. It satisfies the equation

$$\gamma'(p) = \hat{O}_h \log F(p),$$  \hspace{1cm} (C.1)

where $\hat{O}_h$ is the operator (3.26) and $\log F(p)$ is obtained from (4.14), (4.41). We follow the scheme described in Section 2, using $2b^2$ instead of $\hbar$.

The function $\log F(p)$ can be written as

$$\log F(p) = \log \left( 4\pi^2 m_h^2 \Gamma^2(1 + b^2) \right) - \log(p^2 + b^4) + \log F_1(p) - \log F_2(p),$$  \hspace{1cm} (C.2)

with

$$F_1(p) = \Gamma(ip) \Gamma(-ip) \Gamma(1 - b^2 + ip) \Gamma(1 - b^2 - ip)$$  \hspace{1cm} (C.3)

and

$$F_2(p) = \prod_{\varepsilon = \pm 1, \nu = \pm} \Gamma \left( \frac{1}{2} - \varepsilon \beta_\nu \frac{i}{2} + \frac{ip}{2} \right) \Gamma \left( \frac{1}{2} - \varepsilon \beta_\nu \frac{i}{2} - \frac{ip}{2} \right).$$  \hspace{1cm} (C.4)

Note that $F_2(p)$ corresponds to the product of four cos-terms in (4.41), expressed through the $\Gamma$-function by (D.1). The integral representations (D.4) and (D.6) then lead to

$$\log F(p) = \log \left( 4\pi^2 m_h^2 \Gamma^2(1 + b^2) \right) + \int_0^\infty dt \left[ A(t) e^{ipt} + B(t) e^{-ipt} + C(t) \right].$$  \hspace{1cm} (C.5)
The function \( A(t) \) is represented as a sum \( A(t) = A_0(t) + A_1(t) + A_2(t) \), where \( A_0(t) \), \( A_1(t) \) and \( A_2(t) \) correspond to the contributions from \( - \log(p^2 + b^4) \), \( \log F_1(p) \) and \( - \log F_2(p) \), respectively. Writing \( A_2(t) \) also as a sum \( A_2(t) = \sum_{\varepsilon, \nu} A_{2, \varepsilon, \nu}(t) \) corresponding to \( \log z \), we read off these functions by \((D.4)\) and \((D.6)\). We do the same for \( B(t) \) and \( C(t) \) and obtain

\[
A_0(t) = \frac{e^{-b^2t}}{t} = B_0(t) , \quad C_0 = -\frac{2e^{-t}}{t} , \quad (C.6)
\]

\[
A_1(t) = \frac{1 + e^{-(1 - i\beta)t}}{t (1 - e^{-t})} = B_1(t) , \quad C_1 = -\frac{2e^{-t}}{t} \left(1 + b^2 + \frac{2}{1 - e^{-t}}\right) , \quad (C.7)
\]

\[
A_{2, \varepsilon, \nu}(t) = -\frac{e^{-(1 - \varepsilon \beta)t}}{t (1 - e^{-2t})} = B_{2, \varepsilon, \nu}(t) , \quad C_{2, \varepsilon, \nu} = \frac{2e^{-2t}}{t (1 - e^{-2t})} + (1 + \varepsilon \beta) e^{-2t} . \quad (C.8)
\]

Then, similarly to \((3.30)\), we find a solution of \((C.1)\) in the form

\[
\gamma(p) = 2\pi b^2 + p \log \left(4\pi^2 m_b^2 \Gamma(1 + b^2)\right) + \int_0^\infty dt \left[A(t) e^{ipt} + \tilde{B}(t) e^{-ipt} + pC(t)\right] , \quad (C.9)
\]

where \( \tilde{A}(t) \) and \( \tilde{B}(t) \) are related to \( A(t) \) and \( B(t) \) of \((C.5)\) by \((3.31)\). The integral term in \((C.9)\) corresponding to \( \log F_1(p) - \log(p^2 + b^4) \) contains the integral \( I_3 \) of \((3.33)\), and it cancels the constant term \( 2\pi b^2 \). Finally \((C.9)\) leads to \((4.42)\).

The reflection amplitude \( R(p) = e^{-\frac{ip}{\hbar} \gamma(p)} \) corresponding to \((4.42)\) can be expressed through the Barnes Double Gamma function \([24] \Gamma_b(z) \) (see \((4.43)\)), since for \( \text{Re} \ z > 0 \) the following integral representation holds \([4]\)

\[
\log \Gamma_b(z) = \int_0^\infty \frac{dt}{t} \left[ e^{zt} - e^{\frac{1}{2} (b + \frac{1}{b}) t} \right] \frac{\left(b + \frac{1}{b} - 2z\right)^2}{8} e^{-t} - \frac{b + \frac{1}{b} - 2z}{2t} \right] . \quad (C.10)
\]

The continuation of this integral to \( \text{Re} \ z \leq 0 \) can be done by the relations

\[
\Gamma_b(z + b) = \sqrt{\frac{2\pi}{b}} \frac{b^z}{\Gamma(bz)} \Gamma_b(z) , \quad \Gamma_b(z + 1/b) = \frac{\sqrt{2\pi b}}{\Gamma(z/b) b^{z}} \Gamma_b(z) , \quad (C.11)
\]

which easily follow from \((C.10)\).

**Appendix D**

A list of useful formulas from \([21]\) used in the paper

\[
\Gamma\left(\frac{1}{2} + \theta\right) \Gamma\left(\frac{1}{2} - \theta\right) = \frac{\pi}{\cos \pi \theta} , \quad \Gamma(1 + ip) \Gamma(1 - ip) = \frac{\pi p}{\sinh \pi p} . \quad (D.1)
\]

\[
\begin{align*}
\text{I}_1(a,b,c; z) &= \frac{\Gamma(b - a)\Gamma(c)}{\Gamma(b)\Gamma(c - a)} (-z)^{-a} \text{I}_1(a,1-c+a,1-b+a;1/z) \\
&+ \frac{\Gamma(a - b)\Gamma(c)}{\Gamma(a)\Gamma(c - b)} (-z)^{-b} \text{I}_1(1-c+b,b,1-a+b;1/z) \quad (|\arg(z)| < \pi). \quad (D.2)
\end{align*}
\]
\[
\begin{align*}
2F_1(a, b, c; z) &= \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - a)\Gamma(c - b)} 2F_1(a, b, a + b + c + 1; 1 - z) \\
&\quad + \frac{\Gamma(a + b - c)\Gamma(c)}{\Gamma(a)\Gamma(b)} (1 - z)^{c-a-b} 2F_1(c - a, c - b, c - a - b + 1; 1 - z) \,.
\end{align*}
\]

\[
\log (p^2 + a^2) = -\int_0^\infty \frac{dt}{t} \left[ (e^{ipt} + e^{-ipt}) e^{-at} - 2e^{-t} \right] \quad (p > 0, \ a \geq 0).
\]

\[
\log a = \int_0^\infty \frac{dt}{t} (e^{-t} - e^{-at}) \quad (a > 0).
\]

\[
\log \Gamma(z) = \int_0^\infty \frac{dt}{t} \left[ \frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z - 1)e^{-t} \right] \quad (\text{Re} \ z > 0) .
\]

\[
\int_{2\pi}^{2\pi} dy e^{\rho y} (1 - e^{\pm iy})^\alpha = \frac{2\pi \Gamma(1 + \alpha) e^{\pi \rho}}{\Gamma(1 \pm i\rho)\Gamma(1 + \alpha \mp i\rho)} \quad (\alpha > -1) .
\]

\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \log \sqrt{2\pi} + \int_0^\infty \frac{dt}{t} e^{-zt} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \quad (\text{Re} \ z \geq 0) .
\]

\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \log \sqrt{2\pi} + \sum_{n \geq 1} \frac{B_{2n}}{2n(2n - 1)z^{2n - 1}} + O(z^{2m + 1}) \quad (\text{Re} \ z \geq 0) .
\]

\[
\int_0^\infty \frac{dt}{t} \left( \frac{2}{t} - \frac{2e^{-t}}{1 - e^{-t}} - e^{-t} \right) = \log(2\pi) .
\]

\[
\int_0^\infty \frac{dz}{z} e^{-\frac{z}{2}} z^\nu W_{a,b}(z) = \frac{\Gamma(\nu + \frac{1}{2} - b)\Gamma(\nu + \frac{1}{2} + b)}{\Gamma(\nu + 1 - a)} \quad (\text{Re} \ (\nu + 1/2 \pm b) > 0) .
\]

\[
\lim_{|y| \to \infty} |\Gamma(x + iy)| e^{\frac{\pi |y|}{2}} |y|^{\frac{1}{2} - x} = \sqrt{2\pi} \quad (x \ and \ y \ are \ real) .
\]

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