A topological constraint for monotone Lagrangians in hypersurfaces of Kähler manifolds

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16th April 2014

Abstract

In this paper we establish a topological constraint for monotone Lagrangian embeddings in certain complex hypersurfaces of integral Kähler manifolds. As an application, we prove that it is impossible to embed a connected sum of $S^1 \times S^{2k}$s in $\mathbb{C}P^{2k+1}$ as a monotone Lagrangian.

1 Introduction and main result

This paper is concerned with a topological constraint on certain monotone Lagrangian submanifolds in symplectic hypersurfaces of Kähler manifolds, with the particular example of $\mathbb{C}P^n$ in mind. It is known that the existence of Lagrangian embeddings $L \hookrightarrow M$ imposes topological constraints on $L$: one may think for instance of Gromov’s celebrated theorem regarding the impossibility for a Lagrangian submanifold in $\mathbb{C}^n$ to be simply connected [Gro85], and more recently S. Nemirovski proved that Klein bottles do not admit a Lagrangian embedding in $\mathbb{C}^{2n}$ [Nem09]. We will consider, as in the two aforementioned results, a closed and connected Lagrangian. In our case, we find that under the right geometrical circumstances a $K(\pi,1)$ monotone, orientable Lagrangian must have some non-trivial element $g \in \pi$ whose centraliser is of finite index.

This finding actually echoes and generalises in the case of monotone Lagrangians a claim by Fukaya in [Fuk05]:

Let $L$ be a $K(\pi,1)$, spin, Lagrangian submanifold of $\mathbb{C}P^n$. Then there is some $A \in \pi_2(\mathbb{C}P^n, L)$ of Maslov index 2 such that the centraliser of $\partial A$ is of finite index in $\pi_1(L)$.

We will generalise this statement to the framework developed by P. Biran in [Bir01], of which $\mathbb{C}P^n \subset \mathbb{C}P^{n+1}$ is an example.
Theorem 1.1. Let $L^n$ be a monotone, compact, orientable and $K(\pi,1)$ Lagrangian submanifold of some symplectic manifold $(\Sigma^{2n}, \omega)$. Assume that $\Sigma$ is a complex hypersurface of a closed, integral Kähler manifold $(M^{2n+2}, \omega_M)$, that $[\Sigma] \in H_{2n}(M; \mathbb{Z})$ is Poincaré-dual to a multiple of $[\omega_M] \in H^2(M, \mathbb{Z})$, that $W = M \setminus \Sigma$ is a subcritical Weinstein domain, and that the first Chern number of $\Sigma$ is at least 2.

Then the Maslov number $N_L$ of $L$ is 2 and there exists some non-trivial $g \in \pi_1(L)$ such that its centraliser is of finite index.

By ever-so-slightly extending the main result, we also obtain, under the same hypothesis regarding $\Sigma$:

Theorem 1.2. Let $L^n$ be a monotone, compact, orientable Lagrangian submanifold of $(\Sigma^{2n}, \omega)$, such that all the odd-numbered cohomology groups of its universal cover $\tilde{L}$ vanish.

If $H^2(\Sigma, \mathbb{Z})$ is generated by $[\omega]$ or $H^2(L, \mathbb{Z}) = 0$, then the Maslov number $N_L$ of $L$ is 2 and there exists some non-trivial $g \in \pi_1(L)$ such that its centraliser is of finite index.

Corollary 1.3. Let $(L_i)_{i \in I}$ be a finite collection of compact, orientable, $2k+1$-dimensional manifolds such that all the odd-numbered cohomology groups of each universal cover $\tilde{L}_i$ vanish. Assume that either:

1. $\forall i \in I, H^2(L_i, \mathbb{Z}) = 0$ with $k > 1$, or
2. $H^2(\Sigma, \mathbb{Z})$ is generated by $[\omega_{\Sigma}]$. 

Then there is no Lagrangian monotone embedding of the connected sum $\sharp_{i \in I} L_i$ in $\Sigma$.

Corollary 1.4. Let $p > 1$, $k > 0$. There is no monotone embedding of $(S^1 \times S^{2k})^p$ in $\mathbb{C}P^{2k+1}$.

Outline of the proofs. Let $L$ be a closed, connected, $K(\pi,1)$ Lagrangian in a symplectic manifold $(\Sigma, \omega_\Sigma)$ as above. In the section 2, borrowing from P. Biran [Bir01], we will see how we can view most of $M$ as a complex line bundle over $\Sigma$. In the total space of this bundle we can associate to $L$ a circle bundle $\Gamma_L \rightarrow L$ by considering the points above $L$ of a given modulus. The resulting $\Gamma_L$ is a compact, orientable $K(\pi', 1)$, and a monotone Lagrangian submanifold of $W = M \setminus \Sigma$. Since $W$ is assumed to be subcritical, $\Gamma_L$ is also displaceable by an Hamiltonian isotopy.

In the section 3 we will recall some results obtained in [Dam12a, Dam12b] by M. Damian on precisely this type of Lagrangian. Namely, the Maslov number $N_{\Gamma_L}$ of $\Gamma_L$ is 2 and there exists some non-trivial $g \in \pi_1(\Gamma_L)$ such that its centraliser is of finite index. Furthermore, this element is the boundary of some pseudo-holomorphic disc with Maslov index 2.

Then, in section 4, we obtain a one-to-one correspondence between the pseudo-holomorphic discs on $(\Sigma, L)$ and those on $(W, \Gamma_L)$ with corresponding
boundary and Maslov index 2. To that end we use the techniques developed by Biran & Khanevsky [BK13] to project those discs in M down to Σ in a holomorphic way, involving some “stretching the neck”. This implies the theorem 1.1 on L.

This result actually has some interesting consequences, especially when translated to some looser condition on L. These corollaries are presented in section 5. Of particular note is the situation in \( \mathbb{C}P^n \): as its second cohomology group in generated by the symplectic form, the Euler class of \( \Gamma_L \) must vanish. It is thus a trivial \( S^1 \) bundle over L. Since the universal cover of \( \Gamma_L \) retracts to the one of L, those two covers have the same cohomology. In particular, if odd-numbered cohomology groups of the universal cover \( \tilde{L} \) vanish the same is true for \( \Gamma_L \). This condition on L is sufficient to apply the results from [Dam12b] and therefore, in \( \mathbb{C}P^n \), sufficient to obtain the same conclusion as in 1.1. In some examples, such as \( (S^1 \times S^{2k})^p \) in \( \mathbb{C}P^{2k+1} \), it is incompatible with the structure of the fundamental group, making it impossible to embed as a monotone Lagrangian.

Acknowledgements I want to thank Mihai Damian for his incredible patience as he guided me through the writing of this article. I also wish to thank Michael Khanevsky and Romain Ponchon for the time they took to discuss various parts of it. Thanks to Florian Delage for his thoughtful proofing.

2 The symplectic model

2.1 Standard symplectic bundle

Let \( (\Sigma, \tau) \) be a closed integral symplectic manifold, i.e. such that \( [\tau] \in H^2(\Sigma, \mathbb{Z}) \) is well defined. We now present the standard symplectic bundle, introduced by Paul Biran in [Bir01]. This will be our model to understand most of M as a symplectic manifold.

Let \( \mathcal{N} \) be a complex line bundle over \( \Sigma \) with \( [\tau] \) as its first Chern class. On \( \mathcal{N} \) choose an Hermitian metric \( |\cdot| \), an Hermitian connection \( \nabla \) and denote by \( H^\nabla \) the associated horizontal subbundle. The transgression 1-form \( \alpha \) is then defined out of the zero section by:

\[
\alpha|_{H^\nabla} = 0 \quad \alpha_{(p,u)}(u) = 0 \quad \alpha_{(p,u)}(iu) = \frac{1}{2\pi}
\]

where \( (p,u) \in \mathcal{N} \). Then \( d\alpha = -\pi^*\tau \) with \( \mathcal{N} \xrightarrow{\pi} \Sigma \). Designating by \( r \) the distance to the zero section induced by the metric, the standard symplectic form on \( \mathcal{N} \setminus 0_\Sigma \) is then:

\[
\omega_{\text{std}} = -d\left(e^{-r^2} \alpha\right) = e^{-r^2} \pi^*\tau + 2re^{-r^2} dr \wedge \alpha
\]

Remark that in the vertical direction, \( rdr \wedge d\alpha \) is the usual symplectic form on \( \mathbb{C} \). Hence each summand can be extended to the whole total space \( \mathcal{N} \) and is symplectic on the horizontal and vertical subbundle respectively.
The standard symplectic disc bundle is given by:

\[ E_r = \{(p, u) \in N, |u| \leq r\} \]

endowed with the restriction of \( \omega_{\text{std}} \).

**Definition 2.1.** An isotropic CW-complex in \( M \) is some subset that is homeomorphic to a CW-complex in such a way that the interior of each cell is isotropically embedded in \( M \).

**Theorem 2.2.** (Theorem 1.A in [Bir01]) Let \((M^{2n+2}, \omega)\) be a closed integral Kähler manifold and let \( \Sigma \subset M \) be a complex hypersurface whose homology class \([\Sigma] \in H_{2n}(M; \mathbb{Z})\) is Poincaré-dual to a multiple \( k [\omega] \) of \([\omega] \in H^2(M, \mathbb{Z})\). Then, there exists an isotropic CW-complex \( \Delta \subset M \) whose complement — the open dense subset \((M \setminus \Delta, \omega)\) — is symplectomorphic to the standard symplectic bundle \((N, \frac{1}{k} \omega_{\text{std}})\) over \( \Sigma \) pertaining to \( \tau = k|\omega|_{\Sigma} \).

In other words, there exists an embedding \( F : (N, \frac{1}{k} \omega_{\text{std}}) \hookrightarrow (M, \omega) \) such that

- The zero section is isomorphic to \( \Sigma \).
- \( \Delta = M \setminus F(N) \) is an isotropic CW-complex.
- \( \forall r > 0, \left( M \setminus F\left( \overset{\circ}{E}_r \right), \omega \right) \) is a Weinstein domain.

**Remark 2.3.** As an immediate consequence of it being isotropic the dimension of \( \Delta \) is at most half of \( M \)'s. This simple fact will be useful in the proof of proposition 4.1.

Subsequently we will perform a small abuse of notation and denote by \( \pi : M \setminus \Delta \to \Sigma \) the composition \( \pi \circ F^{-1} \).

**Example 2.4.** \( \mathbb{C}P^n \), seen as the hyperplane \( \{z_0 = 0\} \) in \( \mathbb{C}P^{n+1} \) constitutes an example with \( k = 1 \).

**Remark 2.5.** More recently, in [BK13], the notion of symplectic hyperplane section was introduced. The results of this paper should hold in that framework.

### 2.2 The Lagrangian circle fibration

Within the previous framework, we denote by \( P_{r_0} \) the \( S^1 \)-bundle over \( \Sigma \) given by elements of radius \( r_0 \), with the projection \( \pi_{r_0} : P_{r_0} \to \Sigma \). Then we define \( \Gamma_L \) as \( \pi_{r_0}^{-1}(L) \). Its tangent bundle can be locally decomposed as the tangent of \( L \) in \( H^\mathbb{C}_K \) and of a circle in \( \mathbb{C} \) vertically, hence it is a Lagrangian submanifold of \( M \) and of \( W = M \setminus \Sigma \).

We will now consider the morphism induced by \( \pi : (W \setminus \Delta, \Gamma_L) \to (\Sigma, L) \) on the second relative homotopy groups. For a Lagrangian submanifold \( \Lambda \) of a symplectic manifold \((V, \omega)\) we denote by \( \mu_K : \pi_2(V, K) \to \mathbb{Z} \) the Maslov index. Denote by \( \iota : W \setminus \Delta \to W \) the inclusion.
Proposition 2.6. (Proposition 4.1.A in [Bir06]) If \( \dim \Sigma > 1 \) or \( W \) is sub-critical:

1. The morphism \( \iota_* : \pi_2(W \setminus \Delta) \to \pi_2(W) \) induced by the inclusion is surjective. When \( \dim \Sigma > 2 \), it is an isomorphism.

2. for every \( B \in \pi_2(W \setminus \Delta, \Gamma_L) \),
   \[
   \mu_{\Gamma_L}(B) = \mu_L(\pi_* B)
   \]
   In particular, if \( L \subset \Sigma \) is monotone then \( \Gamma_L \subset W \) is monotone too, with the same minimal Maslov number.

3 Results from the lifted Floer homology

Recall that in our setup, \( L \subset \Sigma \) is monotone, hence \( \Gamma_L \subset W \) too. Since \( L \) is a compact \( K(\pi, 1) \) and \( \Gamma_L \) a circle bundle over \( L \) it is also a compact \( K(\pi', 1) \). In particular, odd-numbered cohomology groups of its universal cover \( \tilde{\Gamma}_L \) vanish. Given a volume form \( \nu \) on \( L \), we can construct a volume on \( \Gamma_L \) by pulling back \( \nu \) and wedging the result with \( \alpha \nabla \). Therefore, \( \Gamma_L \) is orientable. Besides, we assumed \( W \) to be subcritical: in particular every compact subset is Hamiltonian displaceable (see [Bir06]), and this applies to \( \Gamma_L \).

Theorem 3.1. [Dam12a] Let \( \Lambda \) be a monotone, compact, orientable, Hamiltonian displaceable, Lagrangian submanifold. Assume further that the odd-numbered cohomology groups of its universal cover \( \tilde{\Lambda} \) vanish. Then:

1. Its minimal Maslov number is \( N_\Lambda = 2 \).

2. For any generic almost complex structure \( J \) there exist \( p \in \Lambda \) and a non-trivial \( g \in \pi_1(\Lambda, p) \) such that the number of pseudo-holomorphic discs \( u \) evaluating in \( p \) and verifying \( [\partial u] = g \) and \( \mu_\Lambda(u) = 2 \) is (finite and) odd.

Proposition 3.2. [Dam12b] When some non-trivial \( g \in \pi_1(\Lambda) \) complies with the result numbered 2 in 3.1 as an hypothesis, then its centraliser is of finite order.

Now the purpose of the next part will be to repatriate this result down on \( L \) by constructing a one-to-one correspondence between the pseudo-holomorphic discs on \( (\Sigma, L) \) and those on \( (W, \Gamma_L) \) with corresponding boundary and Maslov index 2.

4 Proof of the main theorem

The crucial element in projecting the discs obtained by Damian’s theorem 3.1 is the obtention of a suitable almost complex structure on \( W \) through the procedure of “stretching the neck”. This procedure was first presented in [BEH+03], and refined for our setting in [BK13]. The idea is that by modifying the almost
complex structure on $W$ we can prevent pseudo-holomorphic discs of bounded energy – such as our Maslov-2 index ones in this monotone context – to thread their way too far from $P_{r_{0}}$ and in particular to approach $\Delta$. As a consequence, their projection will be well-defined.

4.1 Stretching the neck

Let us begin by taking a generic almost complex structure $J_{\Sigma}$ on $\Sigma$, which is tamed by $\omega_{\Sigma}$. We will here again use the notation

$$E_r = \{(p, u) \in \mathcal{N}, |u| \leq r\}$$

for the closed disc bundle of radius $r$ in $\mathcal{N}$.

Let us choose some $\epsilon > 0$ such that the restriction of $F$ (as defined by Biran’s theorem 2.2) to $E_{r_{0}+\epsilon}$ is a diffeomorphism, where $r_{0}$ is the radius used to define $\Gamma_{L}$ in subsection 2.2. Then the complement $U$ of $E_{r_{0}+\epsilon}$ in $M$ is a neighbourhood of $\Delta$ – and the part of $M$ we want the discs to avoid.

We set an almost complex structure $J_{N}$ on $N$, defined along the horizontal subbundle $H_{N}$ as the “pull-back” of $J_{\Sigma}$ by $\pi$:

$$\forall v \in H_{N}, J_{N}(v) = \left(T\pi\mid_{H_{N}}\right)^{-1} \circ J_{\Sigma} \circ T\pi (v)$$

and along the fibres as the multiplication by $i$. We then push it by $F$ on $E_{r_{0}+\epsilon} = M \setminus U$ and denote by $J_M$ a generic extension on $M$ taming $\omega$. $J_W$ will denote its restriction on $W$.

Recall that $P$ is the bundle over $\Sigma$ of $r_{0}$-radius circles in $N$ which – since it lies within $E_{r_{0}+\epsilon}$ – can be thought as being in either $M$ or $W$. We will thereafter consider the two connected components of respectively $M \setminus P$ and $W \setminus P$ with the following sign convention: $\Sigma \subset M^{+}$, $\Delta \subset U \subset W^{-} = M^{-}$. For $R > 0$ we put:

$$W^{R} = W^{-} \bigcup_{(-R) \times P} [-R, R] \times P \bigcup_{(R) \times P} W^{+}$$

On $W^{R}$ the almost complex structure is defined as $J_{W}$ on $W^{-}$ and $W^{+}$, and by translation invariance in the middle part. The resulting structure is only continuous on the glued boundaries but can be slightly deformed near them to a smooth almost complex structure which we denote by $J^{R}$. Furthermore, this smoothing can be achieved using only the radial coordinate $t \in [-R, R]$ and the angular one $\theta$ in the (circle) fibre of $P$, so as to be invisible once projected to $\Sigma$.

To push back $J^{R}$ on $W$, we make use of a (decreasing) diffeomorphism $\varphi_{R} : [-R - \epsilon, R] \to [r_{0}, r_{0} + \epsilon]$ such that its derivative satisfies $\varphi'_{R}(t) = -1$ near the boundary of $[-R - \epsilon, R]$. We set the diffeomorphism:

$$\lambda_{R} : W^{R} \to W$$
to be the identity on both $W^+$ and $U$, and between $[-R, R] \times P$ and $[r_0, r_0 + \epsilon] \times P$ to be induced by $\varphi_R$ on the first coordinate. Here we made use of the identification $W^- \backslash U \approx [r_0, r_0 + \epsilon] \times P$. Note that $\lambda_R$ preserves the projection and the angular coordinate wherever defined. Finally, we define $J_R$ on $W$ as the push-forward $(\lambda_R)_* J^R$; it happens to tame $\omega$. Besides $\pi$ is – by construction of $J_R - J_R$, $J^R$-holomorphic outside $U$.

**Proposition 4.1.** Suppose the minimal Chern number of $\Sigma$ is $N_\Sigma \geq 2$. Let $p$ be a point in $\Gamma_L$.

Then there exists $R_0 > 0$ such that for every $J_R$ as described above with $R > R_0$, every Maslov-$2$ $J_R$-holomorphic disc $u : (D, \partial D) \rightarrow (W, \Gamma_L)$ passing through $p$ is contained in the image $F(E_{r_0+\epsilon})$.

**Proof.** Below we will follow the reasoning of [BK13] and refer to the results of [BEH+03], which also hold for holomorphic curves with boundary on Lagrangian submanifolds.

We will assume by contradiction that for a generic almost complex structure $J_\Sigma$ on $\Sigma$, there exists a sequence $R_n > 0$ going to infinity with for every $n \in \mathbb{N}$ a Maslov-$2$ $J_{Rn}$-holomorphic disc $u_n'$ in $W$ with its boundary on $\Gamma_L$ that leaves the image of the $(r_0 + \epsilon)$-disk bundle. We will denote $J_{R_n}$ by $J_n$.

In [BEH+03] Bourgeois, Eliashberg, Hofer et al. give a sense to the idea of convergence for our sequence of pseudo-holomorphic discs. In addition, they establish the compactness of a moduli space where $(u'_n)$ lives, provided that its “total energy” is uniformly bounded.

Our sequence of discs leaving $E_{r_0+\epsilon}$ are the $u'_n : (D^2, \partial D) \rightarrow (W, \Gamma_L)$, and we denote by $u_n = \lambda_{R_n}^{-1} \circ u'_n : (D, \partial D) \rightarrow (W_{R_n}, \Gamma_L)$ the same discs seen in $W_{R_n}$. We wish to establish a uniform bound to the total energy of $(u_n)$. First, the $\omega$-energy of some $J_{R_n}$-holomorphic $u : (D, \partial D) \rightarrow (W_{R_n}, \Gamma_L)$ is:

$$E_\omega (u) = \int_{u^{-1}(W^+ \cup W^-)} u^* \omega + \int_{u^{-1}([-R,R] \times P)} u^* \pi^* \omega$$

where $\pi_P$ is the projection $[-R, R] \times P \rightarrow P$. We wish to compare this quantity to $\int_{u^{-1}([-R,R] \times P)} u^* \omega$. Since $\lambda_R$ on $[-R, R] \times P$ maps to $E_{r_0, r_0 + \epsilon}$ on which $\omega$ is canonical, we have:

$$\lambda_R^* \omega |_{[-R, R] \times P} = e^{-\varphi_n (t)^2} \pi^* \omega_{\Sigma} + 2re^{-r^2} dr \wedge \alpha^\nabla$$

Let us split $\int_{u^{-1}([-R, R] \times P)} u^* \omega$ as a sum and consider the first addend:

$$\int_{u^{-1}([-R, R] \times P)} u^* \left( 2re^{-r^2} dr \wedge \alpha^\nabla \right) = 2 \int_{u^{-1}([-R, R] \times P)} u^* \left( e^{-r^2} \right) u^* (rdv \wedge \alpha^\nabla)$$

is non-negative since $u'$ is $J_{R_n}$-holomorphic and $J_R$ is, on the fibre, the usual product by $i$: the part $u^* (rdv \wedge \alpha^\nabla)$ can be seen as a square norm. Meanwhile, $e^{-\varphi_n (t)^2} \geq e^{-(r_0+\epsilon)}$ implies

$$\int_{u^{-1}([-R, R] \times P)} u^* \left( e^{-\varphi_n (t)^2} \pi^* \omega_{\Sigma} \right) \geq \int_{u^{-1}([-R, R] \times P)} u^* \left( e^{-(r_0+\epsilon)} \pi^* \omega_{\Sigma} \right)$$

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since $\pi_\Sigma \circ u$ is $J_\Sigma$-holomorphic when defined, given our choice of $J_R$. Combining the two inequalities, we have

$$\int_{u^{-1}([-R,R] \times P)} u^* \omega \geq \int_{u^{-1}([-R,R] \times P)} u^* p^*_P \omega$$

And finally:

$$\int_{D^2} u^* \omega \geq E_\omega(u)$$

Let’s now remember that $\Gamma_L$ is monotone in $W$ and $u'_n$ has a Maslov index of 2, hence the integral of $\omega$’s pull-back by $(u'_n)$ is constant, which yields a bound on the $\omega$-energies of $(u_n)$.

Now by lemma 9.2 of [BEH+03], this bound implies one on the total energy of $(u_n)$. Hence its main theorem 10.6 holds and a subsequence of $(u_n)$ converges to a so-called stable holomorphic building $\bar{u}$. Abusing the notation, we will now refer to a converging subsequence by $(u_n)$.

To describe this limit, put $W^+ = \left[ \infty, 0 \right] \times P$ and $W^- = \left[ 0, \infty \right] \times P$ respectively glued on their boundaries. Extend $J_W$ on the cylindrical parts $\left[ -\infty, 0 \right] \times P \cup \left[ 0, \infty \right] \times P$ by invariance under translation and smooth it as $J_R$ was smoothed near the glued parts. The disjoint union endowed with this almost-complex structure will be denoted by $(W^\infty, J^\infty)$, and can be considered as the limit of $(W^R, J_R)$ as $R \to \infty$. On $\mathbb{R} \times P$ cylinders, $J^\infty$ is likewise defined by invariance under translation.

Now $\bar{u}$ is a disconnected $J^\infty$-holomorphic curve which consists of the following connected components:

- A base $J^\infty$-holomorphic map $u_+ : (S_+, \partial S_+) \to (W^+_\infty, \Gamma_L)$, where $S_+$ is a disc with one or more punctures. Near these punctures $u_+$ is asymptotically cylindrical and converges to a periodic orbit of the Reeb vector field of $(P, \alpha)$, where $\alpha$ is the transgression 1-form made explicit in 2.1. Given the choice of $\alpha$ the periodic orbits of the Reeb vector field are precisely the fibres of the circle bundle $P \to \Sigma$.

- A number of intermediate $J^\infty$-holomorphic maps $u_i : S_i \to \mathbb{R} \times P$ where each $S_i$ is a sphere with one or more punctures. Near those punctures, the $u_i$ are asymptotically cylindrical with Reeb orbits sections as well.

- Some capping $J^\infty$-holomorphic maps, each of the form $u_- : S_- \to W^-\infty$ where $S_-$ is a sphere with one or more punctures. $u_-$ is asymptotically cylindrical near each puncture in a similar way to $u_+$. To simplify the notation we will assume that there exists one such map; in the case there are many, the argument is the same.

Moreover, those components fit over the punctures, i.e. to each asymptotocal cylinder corresponds another, with the same base orbit, in the other direction on the $\mathbb{R}$ component. As such they can be glued, and the result remains a topological disc. We wish to compute, component by component, the Maslov index of $\bar{u}$.  

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By the definition of $J_\infty$ on $W_\infty^+$, the projection $\pi_+: W_\infty^+ \to \Sigma$ is $(J_\infty, J_\Sigma)$-holomorphic, hence $\pi_+$ sends $u_+$ to a punctured disc $\pi_+ \circ u_+: (S_+, \partial S_+) \to (\Sigma, L)$. The periodic orbits at infinity are projected by $\pi_+$ to single points in $\Sigma$ since they are the fibres of the circle bundle $P \to \Sigma$. Since the convergence near the puncture holds in the $C^1$ norm and the limit has bounded energy, they give rise to removable singularities on $\pi_+ \circ u_+$. Therefore $\pi_+ \circ u_+$ becomes a genuine $J_\Sigma$-holomorphic disc.

Likewise, the intermediate maps can be projected to $\Sigma$ by forgetting the $\mathbb{R}$ coordinate and projecting $P$. By the same argument, the singularities are removable and we obtain $J_\Sigma$-holomorphic spheres.

Alas, we cannot straightly use this method for $u_-$, for it may intersect the isotropic skeleton $\Delta$ and the projection, even where defined, has no reason to be holomorphic. Since $\text{codim} \, \Delta > 2 = \dim u_-$, we can at least perturb homotopically its part in $W^- \subset W_\infty^-$ so that the resulting $\tilde{u}_-$, while no more holomorphic, avoids $\Delta$. We can now project this perturbed curve to $\Sigma$; as before the singularities are removable and we obtain a sphere $v: S^2 \to \Sigma$. We claim that $v$ has a positive Chern number; since $\Sigma$ is monotone it suffices to show that its symplectic area is positive. We have:

$$\int_{S^2} u^* \omega_{\Sigma} = \int_{\tilde{u}_-^{-1}(W^-)} \tilde{u}_-^* \pi_+^* \omega_{\Sigma} = \int_{\tilde{u}_-^{-1}(W^- \setminus \Delta)} \tilde{u}_-^* (\pi_+^* \omega_{\Sigma} + \tilde{u}_-^* (W_\infty^- \setminus W^-))$$

Since $u_-$ is not perturbed on $W_\infty^- \setminus W^-$, where $\pi_-$ is besides $(J_\infty, J_\Sigma)$-holomorphic, the second addend is positive. For the first addend:

$$\int_{\tilde{u}_-^{-1}(W^-)} \tilde{u}_-^* \pi_+^* \omega_{\Sigma} = \int_{\tilde{u}_-^{-1}(W^- \setminus \Delta)} \tilde{u}_-^* (\pi_+^* \omega_{\Sigma})$$

$$= e^{(r_0 + \epsilon)^2} \int_{\partial \tilde{u}_-^{-1}(W^- \setminus \Delta)} \tilde{u}_-^* \left( -e^{-(r_0 + \epsilon)^2} \alpha \wedge \right.$$}

$$\left. \right)$$

$$= e^{(r_0 + \epsilon)^2} \int_{\tilde{u}_-^{-1}(W^- \setminus \Delta)} \tilde{u}_-^* \left( -e^{-r_0^2} \alpha \wedge \right.$$}

$$\left. \right)$$

$$= e^{(r_0 + \epsilon)^2} \int_{\tilde{u}_-^{-1}(W^- \setminus \Delta)} \tilde{u}_-^* \alpha$$

$$= e^{(r_0 + \epsilon)^2} \int_{\tilde{u}_-^{-1}(W^- \setminus \Delta)} \tilde{u}_-^* \omega$$

$$which is positive since $u_-$ is $J_\infty$-holomorphic. Finally, $\epsilon_1^2 ([v]) > 0$.

It suffices to use that:

$$2 = \mu_{\Gamma_\ell} (\tilde{u}) = \mu_{\ell} ([\pi_+ \circ u_+]) + \sum_i 2 \epsilon_1^2 ([\pi_+ \circ u_i]) + 2 \epsilon_1^2 ([v])$$
with all the Chern classes positive. Given that $\pi_+ \circ u_+$ is $J_\Sigma$-holomorphic, its Maslov index is non-negative; and $N_\Sigma \geq 2$ implies that each Chern class is at least 2. We have reached a contradiction.

### 4.2 Regularity of the almost complex structure

The result we use from [Dam12a] is derived from a flavour of Floer homology called the *lifted Floer homology*. As its older, non-lifted counterpart, it requires regularity of the almost complex structure in the sense given by McDuff and Salamon in [MS04].

Recall that the choice of almost complex structure $J_R$ on $W^R$ in 4.1 was only partially free. More specifically we could choose any structure taming $\omega$ on a neighbourhood $U$ of $W^-$, while on $E_{r_0+\epsilon}$ the definition was split between pulling back on the horizontal distribution $H^\Sigma$ some almost complex structure $J_\Sigma$ taming $\omega_\Sigma$, and multiplying by $i$ in the fibres.

We aim to establish the surjectivity of the linearization of the $\bar{\partial}$-operator $D_u$ at each $J_R$-holomorphic disk $u : (D, \partial D) \to (W^R, \Gamma_L)$. Since the almost complex structure on $U$ can be chosen arbitrarily, the general theory establishes regularity for discs going out of $E_{r_0+\epsilon}$.

To treat the discs staying within $E_{r_0+\epsilon}$, we first remark that $\lambda_R$ identifies $(E_{r_0+\epsilon}, J_R)$ with $(E'_R, J^R)$ where $E'_R = [-R, R] \times P \cup \{R\} \times P W^+$. We will reason within this later setting. Let us denote by $J_H$ the restriction of $J^R$ on $H^\Sigma$, which is essentially $J_\Sigma$ through the identification given by $T\pi$. The definition of $J^R$ can be written as: $(TE'_R, J^R) \simeq (H^\Sigma, J_H) \oplus \pi^*(\mathcal{N}, i)$. Let $u$ be a disc that stays inside $E'_R$, with $j$ being the complex structure on the unit disc. We have:

$$
\bar{\partial}_{J^R}(u) = \frac{1}{2} \left( T\pi|_{\overline{\mathcal{N}}} + (\pi^* i) \circ T\pi|_{\overline{\mathcal{N}}} \circ j \right) + \frac{1}{2} \left( T\pi|_{H^\Sigma} + J_H \circ T\pi|_{H^\Sigma} \circ j \right)
$$

We notice that

$$
T\pi|_{H^\Sigma} + J_H \circ T\pi|_{H^\Sigma} \circ j = \left( T\pi|_{H^\Sigma} \right)^{-1} (T(\pi \circ u) + J_\Sigma \circ T(\pi \circ u) \circ j)
$$

is zero if and only if $\pi \circ u$ is $J_\Sigma$-holomorphic.

Just as the operator $\bar{\partial}_{J_R}$, the vector bundle $\mathcal{E}$ of smooth $J_R$-anti-linear 1-forms over the smooth maps $(D^2, S^1) \to (W^R, \Gamma_L)$ splits as a direct sum:

$$
\mathcal{E}_u \approx \Omega^{0,1}_i (u^* \pi^* \mathcal{N}) \oplus \Omega^{0,1}_{J_H} (u^* H^\Sigma)
$$

The right addend effectively corresponds to the smooth $J_\Sigma$-anti-linear 1-forms over the smooth maps $(D^2, S^1) \to (\Sigma, L)$. To the right hand of the $\bar{\partial}_{J_R}$ we can associate a linearization which is surjective if and only if the linearization $\bar{\partial}_{J_\Sigma}$ is surjective. This last point is achieved thanks to the genericity of $J_\Sigma$.

To deal with the left addend, we can consider a holomorphic trivialisation $g : (\pi \circ u)^* \mathcal{N} \to D \times \mathbb{C}$. This yields the identifications:

$$
\Omega^{0,1}_i ((\pi \circ u)^* \mathcal{N}) \approx \Omega^{0,1}(\mathbb{C})
$$

10
\[
\frac{1}{2} \left( T_u[\pi^*N] + (\pi^*i) \circ T_u[\pi^*N] \circ j \right) = g^{-1} \circ \bar{\partial}
\]

Since this almost complex structure (multiplication by \(i\)) is regular we obtain the surjectivity on the left-hand part.

### 4.3 Uniqueness of the pseudo-holomorphic discs lifting

Let us first recall the lemma 7.1.1 of [BK13]:

**Proposition 4.2.** Let \( u : (D^2, S^1) \to (\Sigma, L) \) be a \( J_\Sigma \)-holomorphic disc. Given \( \xi \in S^1 \) and \( \tilde{p} \in \Gamma_L \cap \pi^{-1}(u(\xi)) \), there is a unique \( J_N \)-holomorphic lift \( \tilde{u} : (D^2, S^1) \to (N \setminus \Sigma, \Gamma_L) \) of \( u \) such that \( \tilde{u}(\xi) = \tilde{p} \).

We can actually check that the boundary of those lifted discs is as required, as we have more precisely:

**Proposition 4.3.** Let \( u : (D^2, S^1) \to (\Sigma, L) \) be a \( J_\Sigma \)-holomorphic disc such that \( \mu_L(u) = 2 \), \( [\partial u] = \pi_* \tilde{g} \) and \( u \) passes through \( p \in L \). If \( \tilde{u} \) is the pseudo-holomorphic lift of \( u \) passing through \( \tilde{p} \in \Gamma_L \cap \pi^{-1}(p) \), then \( \mu_{\Gamma_L}(\tilde{u}) = 2 \) and \( [\partial \tilde{u}] = \tilde{g} \).

**Proof.** We know there is an odd natural number, which is in particular positive, of \( J_N \)-holomorphic curves \( (D^2, S^1) \to (W, \Gamma_L) \) passing through \( \tilde{p} \), with Maslov index 2 and boundary in \( \tilde{g} \). Let us denote one by \( \tilde{u}_0 \).

Using proposition 2.6, we have that \( \mu_{\Gamma_L}(\tilde{u}_0) = \mu_L(u) = 2 \). By our main argument of neck-stretching 4.1 both \( \tilde{u}_0 \) and \( \tilde{u} \) avoid \( \Delta \).

Let us define

\[
\gamma : S^1 \to \Gamma_L
\]

so that \( \ker \pi_* = \langle [\gamma] \rangle \subset \pi_1(\Gamma_L) \). Since \( \pi_*[\partial \tilde{u}] = [\partial u] = g = \pi_* \tilde{g} \), there is some \( l \in \mathbb{Z} \) such that \( [\partial \tilde{u}] = \tilde{g}[\gamma]^l \). Furthermore, since \( \mu_{\Gamma_L}(\tilde{u}) = \mu_{\Gamma_L}(\tilde{u}_0) \) by monotonicity we have:

\[
\int_{D^2} \tilde{u}_0^* \omega = \int_{D^2} \tilde{u}^* \omega
\]

Since we avoid \( \Delta \), \( \omega = -d \left( e^{-r^2} \alpha^{\nabla} \right) \) and by applying the Stokes formula:

\[
e^{-r^2} \int_{S^1} \partial \tilde{u}_0^* \alpha^{\nabla} = e^{-r^2} \int_{S^1} \partial \tilde{u}^* \alpha^{\nabla}
\]

Hence

\[
\int_{S^1} \partial \tilde{u}_0^* \alpha^{\nabla} = \int_{S^1} \partial \tilde{u}^* \alpha^{\nabla} + l \int_{S^1} \gamma^* \alpha^{\nabla}
\]

Since \( \int_{S^1} \gamma^* \alpha^{\nabla} > 0 \), \( l = 0 \) so \( [\partial \tilde{u}] = \tilde{g} \).
5 Applications

This section will be dedicated to the proof of theorem 1.2, and corollaries 1.3 and 1.4, as stated in the introduction.

Recall that the minimal hypothesis for theorem 3.1 and proposition 3.2 concerning the topology of the Lagrangian is actually less stringent than being a $K(\pi, 1)$: it is enough for all of the odd-numbered cohomology groups of its universal cover to vanish. As the rest of our proof is not affected by this change, this is the condition we will look for henceforth.

In particular $L$ is not assumed to be a $K(\pi, 1)$ anymore unless specified.

5.1 On the triviality of $\Gamma_L$

Let us begin with a direct application of this weaker condition:

**Proposition 5.1.** If $\Gamma_L$ is a trivial circle bundle over $L$, and all the odd-numbered cohomology groups of its universal cover $\tilde{L}$ vanish, then the Maslov number $N_L$ of $L$ is 2 and there exists some non-trivial $g \in \pi_1(L)$ such that its centraliser is of finite index.

**Proof.** If $\Gamma_L$ is a trivial bundle, then $\tilde{L}$ is a retraction of $\tilde{\Gamma}_L = \tilde{L} \times \mathbb{R}$, and the cohomology of $\tilde{\Gamma}_L$ is exactly the same as $\tilde{L}$. In particular, the odd-numbered cohomology groups of $\tilde{\Gamma}_L$ vanish. The hypothesis of theorem 3.1 are now valid on $\Gamma_L$, hence the result is obtained there. The one-to-one correspondence built in section 4 between the pseudo-holomorphic discs on $(\Sigma, L)$ and those on $(W, \Gamma_L)$ with corresponding boundary and Maslov index 2 still exists, as we did not use any assumption on our Lagrangian topology in its proof. Therefore, we can apply proposition 3.2 to $L$.

**Lemma 5.2.** If $H^2(L, \mathbb{Z}) = 0$ or $H^2(\Sigma, \mathbb{Z})$ is generated by $[\omega_\Sigma]$ then $\Gamma_L$ is trivial. It is in particular the case for $\Sigma = \mathbb{C}P^n$.

**Proof.** In both cases we compute the Euler class $e_{\Gamma_L}$; if $H^2(L, \mathbb{Z}) = 0$ then it is trivially zero. In the other case, let us denote by $P$ the circle bundle over $\Sigma$ of same radius as $\Gamma_L$, such that $\Gamma_L = \iota^*P$ where $\iota : L \hookrightarrow \Sigma$ is the inclusion. By naturality of the Euler class, $e_{\Gamma_L} = \iota^*e_P$, but $e_P$ is collinear to $[\omega_\Sigma]$, and $\iota^*\omega_\Sigma = 0$ since $L$ is a Lagrangian submanifold.

The combination of those two points implies the theorem 1.2.

5.2 Connected sums

**Lemma 5.3.** Let $(G_i)_{i \in I}$ be a finite collection of groups, at least one being infinite and another non-trivial. Let $g \in \ast_{i \in I}G_i \setminus \{e\}$, where $e$ refers to the identity. Then its centraliser $Z(g)$ is not of finite index.

**Proof.** To simplify the notations we will assume that $I = \{1, 2\}$ with $G_1$ infinite and $G_2$ non-trivial. It is clear that any element in $(G_1 * G_2) \setminus \{e\}$ can be uniquely
written as a product of non-trivial elements of $G_1$ and $G_2$ alternately. This fact is the basis of a nice sub-lemma:

**Lemma 5.4.** Let $y \in G_i \setminus \{e\}$ and $x \in (G_1 * G_2) \setminus G_i$. Then $x$ and $y$ do not commute.

*Proof.* If $x$ is in the other group than $y$, obviously they do not commute. Let us then write $x = \prod_{k=1}^n x_k$ as a product of non-trivial elements of $G_1$ and $G_2$ alternately. Since $x$ is in neither $G_1$ nor $G_2$, $k > 1$. Therefore, if we write $xy$ in the same fashion, its leftmost factor is still $x_1$, even after simplification. Besides, $x_2$ is in the other group: to have $yx = xy$ we therefore need $x_1$ to be in $G_1 \setminus \{e, y^{-1}\}$. But now the leftmost factor of $yx$ is $(yx_1)$, which must be equal to the leftmost factor of $xy$, that is to say $x_1$. Since $y \neq e$, this is impossible. □

We can now use this result two ways: first, assume that $g \in G_i$. Then we have that its centraliser lie in $G_i$. We can pick some $h = h_1 h_2$, where each $h_i$ is some non-trivial element of $G_i$. Then for $n \in \mathbb{N}$, each $h^n Z(g)$ is distinct, otherwise it would imply that $h^k \in Z(g)$ for some $k \in \mathbb{N}$.

Now if $g \in (G_1 * G_2) \setminus (G_1 \cup G_2)$, we know that $Z(g) \cap G_i = \{e\}$ for $i = 1, 2$. In particular we see that taking a non-repeating sequence $(h_n)_{n \in \mathbb{N}}$ in $G_1$ gives infinitely many distinct classes $h_n Z(g)$.

In either case, the index of $Z(g)$ is infinite. □

**Remark 5.5.** On the other hand, it is reasonably straightforward to check that in $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle u, v \mid u^2, v^2 \rangle$, $Z(uv) = \langle uv \rangle = \{(uv)^k, k \in \mathbb{Z}\}$ is of finite index.

Combining this lemma 5.3 with the our main result as stated in the theorem 1.2, we obtain this corollary:

**Corollary 5.6.** Let $L$ be a compact, orientable manifold such that all the odd-numbered cohomology groups of its universal cover $\tilde{L}$ vanish. Assume that its fundamental group is the free product of a non-trivial group and an infinite group, and either:

1. $\forall i \in I, H^2(L_i, \mathbb{Z}) = 0$ or 
2. $H^2(\Sigma, \mathbb{Z})$ is generated by $[\omega_{\Sigma}]$.

Then $L$ cannot be embedded in $\Sigma$ as a monotone Lagrangian submanifold.

**Remark 5.7.** Let $G_1$ and $G_2$ be two non-trivial groups. Then for $i \in \{1, 2\}$, there exists some non-trivial $g_i \in G_i$, and $\{(g_1 g_2)^n, n \in \mathbb{N}\}$ clearly is an infinite subset of $G_1 * G_2$.

Hence, it suffices for the fundamental group to be the free product of three non-trivial groups.

We now prove corollary 1.3:
Corollary. Let \((L_i)_{i \in I}\) be a finite collection of compact, orientable, \(2k+1\)-dimensional manifolds such that all the odd-numbered cohomology groups of each universal cover \(\tilde{L}_i\) vanish. Assume that either:

1. \(\forall i \in I, H^2(L_i, \mathbb{Z}) = 0\) with \(k > 1\), or
2. \(H^2(\Sigma, \mathbb{Z})\) is generated by \([\omega_{\Sigma}]\).

Then there is no Lagrangian monotone embedding of the connected sum \(#_{i \in I}L_i\) in \(\Sigma\).

Proof. Since our \((L_i)_{i \in I}\) are compact manifolds, so are their universal covers whenever the fundamental group is finite. Yet their \(2k+1\)-cohomology groups vanish, so it is impossible.

Then, using the Mayer-Vietoris sequence, it is easy to see that the odd-numbered cohomology groups of each universal cover vanish also for the connected sum \(#_{i \in I}L_i\). The same reasoning shows that the assumption 1 is stable through connected sums. □

Corollary 1.4 is then a straightforward application.
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