THE Q-OPERATOR FOR THE QUANTUM NLS MODEL

N. M. Belousov* and S. E. Derkachov*

In this paper, we show that an operator introduced by A. A. Tsvetkov enjoys all the necessary properties of a Q-operator. It is shown that the Q-operator of the XXX spin chain with spin ℓ turns into Tsvetkov’s operator in the continuous limit as ℓ → ∞. Bibliography: 18 titles.

Dedicated to M. A. Semenov-Tian-Shansky on the occasion of his 70th birthday

1. Introduction

In the paper [1], A. A. Tsvetkov studied an infinite system of one-dimensional bosons with pair interaction. The Hamiltonian of the system is given by

\[ H = \int \partial_x \bar{\psi}(x) \partial_x \psi(x) \, dx + \int V(x - y) \bar{\psi}(x) \bar{\psi}(y) \psi(x) \psi(y) \, dxdy, \]

and the fields \( \bar{\psi}(x) \) and \( \psi(x) \) obey the canonical commutation relations

\[ [\bar{\psi}(x), \bar{\psi}(y)] = [\psi(x), \psi(y)] = 0, \quad [\psi(x), \bar{\psi}(y)] = \delta(x - y). \]

In [1], it was shown that under certain conditions on the potential \( V(x - y) \), the one-parameter family of operators

\[ A(\lambda) = :\exp \left( \frac{1}{\lambda} \int \bar{\psi}(x) \partial_x \psi(x) \, dx + \frac{1}{2\lambda^2} \int V(x - y) \bar{\psi}(x) \bar{\psi}(y) \psi(x) \psi(y) \, dxdy \right) : \]

turns out to be commutative: \([A(\lambda), A(\mu)] = 0\). By :: we denote the normal ordered form of an operator, in which all creation operators \( \bar{\psi} \) are to the left of all annihilation operators \( \psi \). For example, the potential \( V(x - y) = c\delta(x - y) \) satisfies these conditions. This potential corresponds to the quantum NLS model.

E. K. Sklyanin conjectured that the operator \( A(\lambda) \) introduced by A. A. Tsvetkov is the Q-operator for the quantum NLS model. The aim of this paper is to prove this conjecture.

The paper consists of two parts. In the first part, we consider the XXX spin chain of spin ℓ with the quantum space carrying infinite-dimensional \( sl_2 \)-representations. We discuss in detail the construction of the local Hamiltonian and the Q-operator for this model. It is well known [7,9] that the monodromy matrix and the Hamiltonian of the quantum NLS model can be obtained from the monodromy matrix and the local Hamiltonian of the XXX spin chain in the continuous limit as \( \ell \to \infty \). Here we investigate the continuous limit of the Q-operator for the XXX spin chain constructed in [11,13]. We show that A. A. Tsvetkov’s operator emerges naturally in this limit.

In the second part, we derive independently all the necessary identities for the Q-operator of the quantum NLS model. For completeness, we prove the main statements for the quantum NLS model, with more emphasis than usual on functional methods [14–16]. Also, we try to emphasize an analogy between the formulas from the spin chain and from the NLS model.

*St.Petersburg Department of Steklov Institute of Mathematics, St.Petersburg, Russia, e-mail: belousovnikita.m@gmail.com, derkach@pdmi.ras.ru.

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2. The XXX spin chain

2.1. The monodromy matrix and the algebraic Bethe ansatz. In the XXX spin chain of spin \( \ell \), the monodromy matrix is defined as the product of L-operators \([3,5,7,8]\):

\[
T(u) = L_n(u)L_{n-1}(u) \cdots L_2(u)L_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},
\]

\[
L_k(u) = u + \vec{\sigma} \otimes \vec{S}_k = \begin{pmatrix} u + S_k & S_k^- \\ S_k^+ & u - S_k \end{pmatrix} = \begin{pmatrix} u + \ell + a_k^+a_k & -a_k \\ a_k^+(a_k^+ + 2\ell) & u - \ell - a_k^+a_k \end{pmatrix}.
\]

Each L-operator in (2.1) acts as a matrix in the auxiliary linear space \( V \) of polynomials in one complex variable, and the vacuum vector is represented by the constant:

\[
|0\rangle = 0.
\]

In what follows, we assume that the spin parameter \( \ell \) is an arbitrary complex number. The entries of the monodromy matrix \( T(u) \), that is, the operators \( A(u), \ldots, D(u) \), act in the global quantum space \( V = V_n \otimes V_{n-1} \otimes \cdots \otimes V_1 \).

A convenient and especially natural choice is to take \( a_k^+ = z_k \), \( a_k = \partial_k \). Then \( V_k \) becomes the space \( \mathbb{C}[z_k] \) of polynomials in one complex variable, and the vacuum vector is represented by the constant: \( |0\rangle = 1 \). The global quantum space coincides with the space \( \mathbb{C}[z_1, \ldots, z_n] \) of polynomials in \( n \) variables, and the entries of the monodromy matrix are differential operators acting in this space.

The L-operators satisfy the following local relation:

\[
(u - v + \mathbb{P}_{12})(u + \vec{\sigma}_1 \otimes \vec{S}_k)(v + \vec{\sigma}_2 \otimes \vec{S}_k)(u - v + \mathbb{P}_{12}).
\]

Here, all operators act in the tensor product of three spaces \( V_1 \otimes V_2 \otimes V_k \), where \( V_1 = \mathbb{C}^2 \) and \( V_2 = \mathbb{C}^2 \) are auxiliary spaces, and \( V_k \) is the local Fock space. By \( \mathbb{P}_{12} \) we denote the permutation operator: \( \mathbb{P}_{12} \vec{x} \otimes \vec{y} = \vec{y} \otimes \vec{x} \) for \( \vec{x} \in V_1, \vec{y} \in V_2 \).

Clearly, from the local relation we can derive a global relation for the monodromy matrices

\[
T_1(u) = T(u) \otimes \mathbb{1} \quad \text{and} \quad T_2(v) = \mathbb{1} \otimes T(v):
\]

\[
(u - v + \mathbb{P}_{12})T_1(u)T_2(v) = T_2(v)T_1(u)(u - v + \mathbb{P}_{12}),
\]

\[
T_1(u) = \left( u + \vec{\sigma}_1 \otimes \vec{S}_n \right) \cdots \left( u + \vec{\sigma}_1 \otimes \vec{S}_1 \right),
\]

\[
T_2(v) = \left( v + \vec{\sigma}_2 \otimes \vec{S}_n \right) \cdots \left( v + \vec{\sigma}_2 \otimes \vec{S}_1 \right).
\]

Let us rewrite (2.4) in terms of \( 4 \times 4 \) matrices using the standard basis in the tensor product \( \mathbb{C}^2 \otimes \mathbb{C}^2 \):
In the chosen basis, the matrices $T_1(u)$ and $T_2(v)$ have the form
\[
T_1(u) = \begin{pmatrix} A(u) & 0 & B(u) & 0 \\ 0 & A(u) & 0 & B(u) \\ C(u) & 0 & D(u) & 0 \\ 0 & C(u) & 0 & D(u) \end{pmatrix}, \quad T_2(v) = \begin{pmatrix} A(v) & B(v) & 0 & 0 \\ C(v) & D(v) & 0 & 0 \\ 0 & 0 & A(v) & B(v) \\ 0 & 0 & C(v) & D(v) \end{pmatrix}.
\]
Similarly, the operator $\mathbb{P}_{12}$ is given by
\[
\mathbb{P}_{12} = \frac{1}{2}(\vec{\sigma} \otimes \vec{\sigma} + \mathbb{1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]
Substituting these formulas into (2.4), we get a system of quadratic relations for the operators $A(u), \ldots, D(u)$ and $A(v), \ldots, D(v)$. Note that, as follows from (2.4), the traces of the monodromy matrices, that is, the transfer matrices $t(u) = A(u) + D(u)$, commute: $t(u)t(v) = t(v)t(u)$. The eigenvectors of the transfer matrix $t(u)$ can be found within the framework of the algebraic Bethe ansatz method [3, 5, 7, 8, 17]. The following relations are at the heart of this method:
\[
\begin{aligned}
C(u)C(v) &= C(v)C(u), \\
A(u)C(v) &= \frac{u-v+1}{u-v}C(v)A(u) - \frac{1}{u-v}C(u)A(v), \\
D(u)C(v) &= \frac{u-v-1}{u-v}C(v)D(u) + \frac{1}{u-v}C(u)D(v).
\end{aligned}
\]
(2.6)
(2.7)
The algebraic Bethe ansatz method requires the existence of a vacuum vector $|0\rangle$ such that
\[
B(u)|0\rangle = 0, \quad A(u)|0\rangle = \Delta_+(u)|0\rangle, \quad D(u)|0\rangle = \Delta_-(u)|0\rangle,
\]
where $\Delta_{\pm}(u)$ are polynomials of degree $n$ in the spectral parameter $u$. In our case, each local quantum space $\mathbb{V}_k$ contains the vacuum vector $|0\rangle_k$. This means that $|0\rangle_k$ is an eigenvector for the operator $S_k = a_k^\dagger a_k + \ell$, while the operator $a_k$ annihilates it:
\[
L_k(u)|0\rangle_k = \begin{pmatrix} u + \ell \\ \vdots \\ u - \ell \end{pmatrix} |0\rangle_k.
\]
The global vector $|0\rangle$ is constructed from the local ones: $|0\rangle = |0\rangle_n \otimes |0\rangle_{n-1} \otimes \cdots \otimes |0\rangle_1$. So, we obtain explicit expressions for the functions $\Delta_{\pm}(u)$: $\Delta_{\pm}(u) = (u \pm \ell)^n$.
In these terms, the eigenvectors of the transfer matrix have the form
\[
|v_1 \ldots v_k\rangle = C(v_1) \cdots C(v_k)|0\rangle, \quad t(u)|v_1 \ldots v_k\rangle = \tau(u)|v_1 \ldots v_k\rangle.
\]
All information about the parameters $v_i$ is accumulated in the polynomial
\[
q(u) = (u - v_1) \cdots (u - v_k).
\]
Using (2.6) and (2.7), it can be shown (see [3, 5, 7, 8]) that the vector $|v_1 \ldots v_k\rangle$ is an eigenvector of the transfer matrix $t(u)$ with eigenvalue
\[
\tau(u) = \Delta_+(u)\frac{q(u+1)}{q(u)} + \Delta_-(u)\frac{q(u-1)}{q(u)}
\]
(2.8)
if the parameters $v_i$ obey the Bethe equations:
\[
\Delta_+(v_i)q(v_i + 1) + \Delta_-(v_i)q(v_i - 1) = 0, \quad i = 1, 2, \ldots, k.
\]
The Bethe equations have a simple interpretation: \( \tau(u) \) is a polynomial, but each term in the right-hand side of (2.8) has a pole at the point \( u = v_i \). Thus, the residues of these terms should cancel each other.

2.2. The local Hamiltonian. To construct the local Hamiltonian, we need the R-operator, which interchanges the spectral parameters in the product of L-operators [3]:

\[
R_{21}(u-v)L_2(u)L_1(v) = L_2(v)L_1(u)R_{21}(u-v).
\]

The R-operator can be represented in the following form [12] (here \( z_{ik} \equiv z_i - z_k \)):

\[
R(u) = \frac{\Gamma(z_{12}\partial_1 + 2\ell)}{\Gamma(z_{12}\partial_1 - u + 2\ell)} \frac{\Gamma(z_{21}\partial_2 + u + 2\ell)}{\Gamma(z_{21}\partial_2 + 2\ell)}.
\]

Calculating the derivatives with respect to \( u \) of both sides of (2.9) and putting \( v = u \), we get

\[
R'_{21}(0)L_2(u)L_1(u) + R_{21}(0)L_1(u) = L_2(u)R_{21}(0) + L_2(u)L_1(u)R'_{21}(0).
\]

At the same time, from the explicit formula for the R-operator we obtain

\[
R_{21}(0) = 1, \quad R'_{21}(0) = \psi(z_{12}\partial_1 + 2\ell) + \psi(z_{21}\partial_2 + 2\ell),
\]

where \( \psi(x) \) is the logarithmic derivative of the Euler gamma function. Finally, we obtain the commutation relation (see [3])

\[
[R'_{21}(0), L_2(u)L_1(u)] = L_2(u) - L_1(u).
\]

Let us introduce the operator \( H = R'_{1n}(0) + R'_{mn-1}(0) + \cdots + R'_{21}(0) \) acting in the global quantum space \( \mathbb{V} \). From the last expression it follows that this operator commutes with the transfer matrix \( t(u) \). Here we assume that the periodic boundary condition holds: \( k + n = k \). Nevertheless, it is especially convenient to work with the slightly modified operator

\[
H = \sum_{k=1}^{n} H_{k+1k}, \quad H_{k+1k} = \psi(z_{kk+1}\partial_k + 2\ell) + \psi(z_{k+1k}\partial_{k+1} + 2\ell) - 2\psi(2\ell).
\]

In this notation, the vacuum state \( |0\rangle \) corresponds to the zero eigenvalue of the pair Hamiltonian: \( H_{k+1k}|0\rangle = 0 \). Note that this Hamiltonian is local. In other words, each term in the sum describes the interaction between nearest neighbors only.

2.3. The Q-operator. Recall that all information about an eigenvector \( |v_1 \ldots v_k\rangle \) is accumulated in the polynomial \( q(u) = (u - v_1) \cdots (u - v_k) \). Also, the eigenvalues of the transfer matrix are expressed in terms of the function \( q(u) \). Baxter [15] suggested to interpret the polynomial \( q(u) \) as an eigenvalue of some operator, which we now call the Q-operator. In the general case,

\[
Q(u)|v_1 \ldots v_k\rangle = q(u)c(v)|v_1 \ldots v_k\rangle,
\]

where the constant \( c(v) \) depends on the parameters \( v_1, \ldots, v_k \) and does not depend on the spectral parameter \( u \). Usually, there exists a special point \( u_0 \) at which the Q-operator turns into the identity operator: \( Q(u_0) = 1 \). Therefore, \( c(v) = q^{-1}(u_0) \) and

\[
Q(u)|v_1 \ldots v_k\rangle = \frac{(u - v_1) \cdots (u - v_k)}{(u_0 - v_1) \cdots (u_0 - v_k)}|v_1 \ldots v_k\rangle.
\]

By definition, the Q-operator satisfies the following properties.

- **Commutativity:**
  \[
  [Q(u), Q(v)] = 0, \quad [Q(u), t(v)] = 0.
  \]

- **Finite-difference Baxter equation:**
  \[
  t(u)Q(u) = \Delta_+(u)Q(u + 1) + \Delta_-(u)Q(u - 1).
  \]

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These properties suffice to reproduce all formulas involving the polynomial $q(u)$ obtained in the framework of the algebraic Bethe ansatz. So, the Q-operator provides another approach to solving the model. Below, we construct the Q-operator for the case under consideration [11,13] using the most straightforward approach.

2.3.1. The R-operator. In the representation (2.3), the L-operator has a convenient factorized form:

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u_1 & -\partial \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}, \quad u_1 = u + \ell - 1, \quad u_2 = u - \ell. \quad (2.11)$$

The main building block in the construction of the Q-operator is the R-operator defined by

$$R_{21}(u_1, u_2|v_2)L_2(u_1, u_2)L_1(v_1, v_2) = L_2(u_1, v_2)L_1(v_1, u_2)R_{21}(u_1, u_2|v_2), \quad (2.12)$$

where

$$u_1 = u + \ell - 1, \quad u_2 = u - \ell, \quad v_1 = v + \ell - 1, \quad v_2 = v - \ell.$$ 

Note that this R-operator differs from the operator defined by (2.9) (the old one interchanges the parameters $u$ and $v$).

The R-operator acts in the space of polynomials in variables $z_1$ and $z_2$ and can be represented as a function with operator argument [12],

$$R_{21}(u_1, u_2|v_2) = \frac{\Gamma(z_{21}\partial_2 + u_1 - v_2 + 1)}{\Gamma(z_{21}\partial_2 + u_1 - u_2 + 1)},$$

or as an integral operator,

$$R_{21}(u_1, u_2|v_2)\Phi(z_2, z_1) = \frac{1}{\Gamma(v_2 - u_2)} \int_0^1 da \alpha^{u_1 - v_2}(1 - \alpha)^{v_2 - u_2 - 1}\Phi(\alpha z_{21} + z_1, z_1).$$

Two forms of the R-operator are connected through the integral representation of the Euler beta function

$$B(z_{21}\partial_2 + a, b - a) = \int_0^1 da \alpha^{a - 1 + z_{21}\partial_2(1 - \alpha)^{b-a-1}}$$

and an explicit expression for the action of the operator $\alpha z_{21}\partial_2 = e^{-z_1\partial_2}\alpha^{z_2\partial_2}e^{z_1\partial_2}$:

$$e^{-z_1\partial_2}\alpha^{z_2\partial_2}e^{z_1\partial_2}\Phi(z_2, z_1) = e^{-z_1\partial_2}\alpha^{z_2\partial_2}\Phi(z_2 + z_1, z_1) = e^{-z_1\partial_2}\Phi(\alpha z_{21} + z_1, z_1).$$

In our case, $a = u_1 - v_2 + 1, b = u_1 - u_2 + 1$.

2.3.2. The Baxter equation. The left-hand side of the Baxter equation (2.10) contains the product of the transfer matrix $t(u)$ and the Q-operator. The building blocks of the transfer matrix $t(u)$ are L-operators. In a similar way, we will construct the Q-operator using R-operators as its building blocks. A global relation involving the Q-operator and the transfer matrix will be derived from the corresponding local relation for their building blocks. The required local relation is the defining equation for the R-operator (2.12). Using the factorized form (2.11) of $L_2(u_1, u_2)$ and $L_1(v_1, v_2)$ and the commutativity $[R_{21}, z_1] = 0$, let us rewrite (2.12) as follows:

$$Z_2^{-1}R_{21}(u - v_2)L_2(u_1, u_2)Z_1 = \begin{pmatrix} u_1 & -\partial_2 \\ v_1 & 0 \end{pmatrix} Z_2^{-1}Z_1 \begin{pmatrix} u_1 & -\partial_1 \\ v_1 & 0 \end{pmatrix} R_{21}(u - v_2) \begin{pmatrix} v_1 & -\partial_1 \\ 0 & v_2 \end{pmatrix}^{-1}.$$
Note that the dependence of the R-operator on the parameter $v_2$ reduces to a shift of the spectral parameter $u$, and $Z_k \equiv \begin{pmatrix} 1 & 0 \\ z_k & 1 \end{pmatrix}$. Computing the product of the matrices in the right-hand side, we get

$$Z_k^{-1}R_{21}(u - v_2)L_2(u_1, u_2)Z_1$$

$$= \begin{pmatrix} R_{21}(u + 1 - v_2) + v_2R_{21}(u - v_2) & -R_{21}(u - v_2)\partial_2 \\ -v_2z_2R_{21}(u - v_2) & (u_1 - v_2)(u_2 - v_2)R_{21}(u - 1 - v_2) + v_2R_{21}(u - v_2) \end{pmatrix}.$$ 

Clearly, at the point $v_2 = 0$ the matrix in the right-hand side becomes upper triangular. So, we put $v_2 = 0$ in the derived matrix relation and slightly change it by choosing the second space to be the local quantum space at site $k$ and the first space to be the local quantum space at site $k - 1$:

$$Z_k^{-1}R_{kk-1}(u)L_k(u_1, u_2)Z_{k-1} = \begin{pmatrix} R_{kk-1}(u + 1) & -R_{kk-1}(u)\partial_{k-1} \\ 0 & u_1u_2R_{kk-1}(u - 1) \end{pmatrix}.$$ 

Actually, this particular local relation leads to the Baxter equation. Let us turn to global objects: adding one additional site $\zeta_0$, we take the product over all sites:

$$Z_n^{-1}R_{nm-1}(u) \cdots R_{21}(u)R_{10}(u)L_n(u) \cdots L_1(u)Z_0$$

$$= \begin{pmatrix} R_{nm-1}(u + 1) & -R_{nm-1}(u)\partial_n \\ 0 & u_1u_2R_{nm-1}(u - 1) \end{pmatrix} \cdots \begin{pmatrix} R_{10}(u + 1) & -R_{10}(u)\partial_1 \\ 0 & u_1u_2R_{10}(u - 1) \end{pmatrix}.$$ 

In this product, the matrices $Z_k$ and $Z_k^{-1}$ (for $k = 1, 2, \ldots, n - 1$) become neighbors and, therefore, cancel each other. Then we calculate the trace over the two-dimensional space $\mathbb{C}^2$, use the commutativity of all R-operators and $L_k$ with $\zeta_0$ in order to move $Z_0$ to the left, and, finally, identify sites 0 and $n$. In this way we obtain the Baxter equation

$$t(u)Q(u) = Q(u + 1) + (u_1u_2)^nQ(u - 1)$$

for the operator

$$Q(u) = R_{nm-1}(u) \cdots R_{21}(u)R_{10}(u)|_{\zeta_0 \rightarrow \zeta_n}.$$ 

One can also check the commutativity properties of $Q(u)$, see [11, 13]. Now, in order to visualize the constructed operator, we will derive an explicit formula for the action of $Q(u)$ on polynomials [13].

2.3.3. **An explicit formula for the action on polynomials.** Let us derive a very simple formula for the action of $Q(u)$ on the global generating function $(1 - x_nz_n)^{-2\ell} \cdots (1 - x_1z_1)^{-2\ell}$. This formula contains, in a transparent form, all information about the action of the operator $Q(u)$ on polynomials. The global problem reduces to the local one:

$$R_{nm-1}(u) \cdots R_{10}(u)(1 - x_nz_n)^{-2\ell} \cdots (1 - x_1z_1)^{-2\ell}$$

$$= R_{nm-1}(u)(1 - x_nz_n)^{-2\ell} \cdots R_{21}(u)(1 - x_2z_2)^{-2\ell}R_{10}(u)(1 - x_1z_1)^{-2\ell}.$$ 

An explicit formula for the action of the R-operator on the local generating function

$$R_{kk-1}(u)(1 - x_kz_k)^{-2\ell} = \frac{\Gamma(\ell + u)}{\Gamma(2\ell)}(1 - x_kz_k)^{-\ell - u}(1 - x_kz_{k-1})^{-\ell + u}$$

can be obtained using Feynman’s formula

$$\int_0^1 d\alpha \alpha^{\ell - 1}(1 - \alpha)^{b-1} \frac{1}{\alpha A + (1 - \alpha)B} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \frac{1}{A^aB^b}.$$
so the action on the global generating function is
\[
R_{mn-1}(u)\cdots R_{10}(u)(1 - x_nz_n)^{-2\ell} \cdots (1 - x_1z_1)^{-2\ell}
\]
\[
= \frac{\Gamma^n(\ell + u)}{\Gamma^n(2\ell)}(1 - x_nz_n)^{-\ell - u} \cdots (1 - x_1z_1)^{-\ell - u}.
\]

Finally, we put \( z_0 \to z_n \) and change the norm of the \( \Omega \)-operator:
\[
\Omega(u) \to \frac{\Gamma^n(2\ell)}{\Gamma^n(\ell + u)} \Omega(u).
\]
The action of the renormalized operator looks very simple:
\[
\Omega(u) : (1 - x_nz_n)^{-2\ell} \cdots (1 - x_1z_1)^{-2\ell}
\]
\[
\to (1 - x_nz_n)^{-\ell - u} \cdots (1 - x_1z_1)^{-\ell - u}.
\]

3. The continuous limit

3.1. The monodromy matrix in the continuous limit. The monodromy matrix of the continuous model satisfies the differential equation (4.1). In order to formulate a natural recipe for taking the \( \ell \to \infty \) limit [7,17,18], let us derive a finite-difference version of this differential equation. First, we extract the matrix \( \ell \sigma_3 \) from the \( \ell \)-operator (2.2),
\[
L_k(u) = (1 + \ell_k(u)) \ell \sigma_3,
\]
\[
\ell_k(u) = \left( \begin{array}{cc} \frac{u}{\ell} + \frac{1}{\ell} a_k^+ a_k & \frac{1}{\ell} a_k \\ \frac{2}{\ell} a_k^+ + \frac{1}{\ell} a_k^2 a_k & -\frac{u}{\ell} + \frac{1}{\ell} a_k^+ a_k \end{array} \right),
\]
and consider the difference
\[
T_n(u) - \ell \sigma_3 T_{n-1}(u) = (L_n(u) - \ell \sigma_3) T_{n-1}(u) = \ell_n(u)\ell \sigma_3 T_{n-1}(u).
\]
Hence, for the operator
\[
T_n(u) = (\ell \sigma_3)^{-n} T_n(u)
\]
we obtain a finite-difference equation of the form
\[
T_n(u) - T_{n-1}(u) = (\ell \sigma_3)^{-n} \ell_n(u) (\ell \sigma_3)^n T_{n-1}(u)
\]
\[
= \left( \begin{array}{cc} \frac{u}{\ell} + \frac{1}{\ell} a_n^+ a_n & \frac{(-1)^n}{\ell} a_n \\ 2(-1)^n a_n^+ & \frac{2}{\ell} a_n^+ + \frac{(-1)^n}{\ell} a_n^2 a_n & -\frac{u}{\ell} + \frac{1}{\ell} a_n^+ a_n \end{array} \right) T_{n-1}(u).
\]
Now, we introduce the renormalized creation and annihilation operators
\[
\psi_n = (-1)^n \frac{b}{2} a_n, \quad \bar{\psi}_n = \ell (-1)^n a_n^+,
\]
\[
\left[ \psi_n, \bar{\psi}_m \right] = \frac{\ell}{2} \left[ a_n, a_m^+ \right] = \frac{\ell}{2} \delta_{nm}
\]
in such a way that we can interpret the commutation relations (3.2) as a discrete version of the canonical commutation relations for the fields. Let $\Delta$ be the lattice spacing; then $x = n\Delta$, $y = m\Delta$ become continuous variables as $\Delta \to 0$, and

$$\psi_n \to \psi(x), \quad \psi_n \to \tilde{\psi}(x), \quad [\psi_n, \psi_m] = \frac{1}{\Delta} \delta_{nm} \to \left[\tilde{\psi}(x), \psi(y)\right] = \delta(x - y).$$

Taking into account (3.2), we get $\Delta = \frac{2}{\Delta u \lambda}$. The difference in the left-hand side of (3.1) turns into the derivative $T_n(u) - T_{n-1}(u) \to \Delta \partial_u T(x, u)$, so

$$\Delta \partial_u T(x, u) = \left( \frac{2}{\Delta u} \psi(x) + \frac{\Delta}{\ell} \psi^2(x) \psi(x) - \frac{u}{\ell} \Delta \tilde{\psi}(x) \psi(x) \right) T(x, u).$$

In the $\ell \to \infty$ limit, we obtain

$$\partial_u T(x, u) = \left( \frac{ab}{u} \frac{\partial \psi(x)}{\psi(x)} - \frac{ab}{u} \right) T(x, u).$$

This equation for the monodromy matrix coincides with (4.1) after the renormalization $u = \frac{1}{ab \lambda}$ of the spectral parameter.

**3.2. The local Hamiltonian in the continuous limit.** To find a natural interpretation of the parameter $ab$ appeared above, we will take the $\ell \to \infty$ limit in the local Hamiltonian $H = \sum_{k=1}^{n} H_{k+1k}$, see [6]:

$$H_{k+1k} = \psi \left( a_k^a a_k - a_{k+1}^a a_k + 2\ell \right) + \psi \left( a_{k+1}^a a_{k+1} - a_k^a a_{k+1} + 2\ell \right) - 2\psi(2\ell)$$

$$= \psi \left( \Delta \tilde{\psi}_k \psi_k + \Delta \tilde{\psi}_{k+1} \psi_k + 2\ell \right) + \psi \left( \Delta \tilde{\psi}_{k+1} \psi_{k+1} + \Delta \tilde{\psi}_k \psi_{k+1} + 2\ell \right) - 2\psi(2\ell).$$

First, recall that $x = k\Delta$ becomes a continuous variable as $\Delta \to 0$; second,

$$\psi_{k+1} \to \psi(x) + \Delta \partial \psi(x) + \frac{\Delta^2}{2} \partial^2 \psi(x) + \ldots,$$

and the same for $\tilde{\psi}_{k+1}$. Using these formulas and the asymptotic expansion of the logarithmic derivative of the gamma function

$$\psi(z) \to \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \ldots,$$

and replacing sums by integrals $\Delta \sum_k \to \int dx$, we get (see [6])

$$H = \sum_{k=1}^{n} H_{k+1k} \to \left( \frac{2}{\ell} + \frac{1}{2\ell^2} + \frac{1}{12\ell^3} \right)$$

$$\times \int dx \tilde{\psi}(x) \psi(x) - \Delta^2 \frac{1}{2\ell} \int dx \left( \partial \tilde{\psi}(x) \partial \psi(x) + ab \tilde{\psi}^2(x) \psi^2(x) \right) + \ldots,$$

where we have written all terms to order $\frac{1}{\ell^4}$. Terms with total derivatives $\partial \left( \tilde{\psi}(x) \psi(x) \right)$ cancel due to the periodic boundary conditions. Instead of the original local Hamiltonian, we can consider the operator

$$H' = \sum_{k=1}^{n} \left( H_{k+1k} - \Delta \left( \frac{2}{\ell} + \frac{1}{2\ell^2} + \frac{1}{12\ell^3} \right) \tilde{\psi}_k \psi_k \right)$$

(with $\Delta = \frac{2}{\Delta u \lambda}$), which also commutes with the transfer matrix and describes the interaction between nearest neighbors. By the same argument, we get the expansion

$$H' \to -\Delta^2 \frac{1}{2\ell} \int dx \left( \partial \tilde{\psi}(x) \partial \psi(x) + ab \tilde{\psi}^2(x) \psi^2(x) \right) + \ldots.$$
Thus, in the continuous limit we obtain the standard Hamiltonian of the continuous model [2, 9, 17]

\[ H = \int dx (\partial \bar{\psi}(x) \partial \psi(x) + ab \bar{\psi}^2(x) \psi^2(x)), \]

and \( ab = c \) is the coupling constant.

3.3. The \( Q \)-operator in the continuous limit. Knowing explicitly the connection between the discrete and continuous models, we can calculate the \( \ell \to \infty \) limit in the \( Q \)-operator

\[ Q(u) = \frac{\Gamma^n(2\ell)}{\Gamma^n(\ell + u)} R_{nn-1}(u) \cdots R_{10}(u) \bigg|_{z_0 \to z_n} \tag{3.3} \]

expressed in terms of the discrete fields \( \psi_k, \bar{\psi}_k \). Before we plunge into the calculation, recall that the eigenvalues of the operator \( Q(u) \) depend on the spin \( \ell \):

\[ Q(u)|v_1 \ldots v_l\rangle = \frac{u - v_1}{\ell - v_1} \cdots \frac{u - v_l}{\ell - v_l} |v_1 \ldots v_l\rangle. \]

So, before taking the limit, we should normalize the \( Q \)-operator appropriately. One way of doing this is

\[ \frac{(\ell/u)\sum z_k \partial_k Q(u)|v_1 \ldots v_l\rangle}{(\ell/u)\sum z_k \partial_k Q(u)|v_1 \ldots v_l\rangle} = \prod \frac{\ell - v_k}{u \ell - v_k} |v_1 \ldots v_l\rangle. \]

Hereafter, we mean that the summation is over all \( n \) sites of the chain. The operator \( \sum z_k \partial_k \) counts the number of \( v_l \)-parameters in a given state \( |v_1 \ldots v_l\rangle \), and we put the parameter \( u \) in the denominator to make the final result more convenient.

To take the continuous limit, we use the integral form of the R-operators:

\[ R_{kk-1}(u) = \frac{1}{\Gamma(\ell - u)} \int_0^1 d\alpha_k \alpha_k^{u+\ell-1} (1 - \alpha_k)^{\ell - u - 1} \alpha_k^{z_k-1} \partial_k. \]

The last factor in this expression acts nontrivially on the variable \( z_k \) only:

\[ \alpha_k^{z_k-1} \partial_k \Phi(z_k) = \Phi(\alpha_k z + (1 - \alpha_k)z_k). \]

Together with the normalization part from (3.3), the product of these operators acts on functions of all variables \( z_k, k = 1, \ldots, n \), as

\[ (\ell/u)\sum z_k \partial_k \left( \prod \alpha_k^{z_k-1} \partial_k \right) \bigg|_{z_0 \to z_n} \Phi(\ldots, z_k, \ldots) = \Phi(\ldots, (\ell/u)\alpha_k z_k + (\ell/u)(1 - \alpha_k)z_k, \ldots). \]

It is useful to rewrite this operator in normal ordered form. Consider the following operator:

\[ :e^{(c_1 z_k + c_2 z_k-1)\partial_k}: = \sum_{p=0}^{\infty} \frac{1}{p!} (c_1 z_k + c_2 z_k-1)^p \partial_k^p. \]

Under the normal ordering sign, the operators \( z_k \) and \( \partial_k \) commute. This operator acts on a monomial as

\[ :e^{(c_1 z_k + c_2 z_k-1)\partial_k}: z_k^p = z_k^p + p(c_1 z_k + c_2 z_k-1) z_k^{p-1} + \ldots + (c_1 z_k + c_2 z_k-1)^p = (c_1 + 1) z_k + c_2 z_k-1)^p, \]

and for an arbitrary function of \( z_k \) we get

\[ :e^{(c_1 z_k + c_2 z_k-1)\partial_k}: \Phi(z_k) = \Phi((c_1 + 1) z_k + c_2 z_k-1). \]

Note also that for the neighboring operators,

\[ :e^{(c_1 z_{k+1} + c_2 z_{k+1})\partial_{k+1}} : :e^{(c_1 z_k + c_2 z_k-1)\partial_k} : = :e^{(c_1 z_{k+1} + c_2 z_{k+1})\partial_{k+1}} e^{(c_1 z_k + c_2 z_k-1)\partial_k}, \]
where the factor in front of the integral equals

\[(\ell/u)\sum z_k \partial_k \left( \prod a_k^{z_{k-1} \partial_k} \right) \bigg|_{z_0 = z_n} = : \prod e^{(\ell/u-1)z_k \partial_k + (\ell/u)(1-\alpha_k)z_{k-1} \partial_k} : \]

Thus, we have the following formula for the renormalized Q-operator:

\[(\ell/u)\sum z_k \partial_k Q(u) = : G(u) :,\]

where

\[G(u) = \frac{\Gamma(2\ell)}{\Gamma(\ell+u) \Gamma(\ell-u)} \quad \times \prod_{k=1}^{n} \int_{0}^{1} d\alpha_k \alpha_k^{u+\ell-1} (1-\alpha_k)^{\ell-u-1} e^{(\ell/u-1)\alpha_k a_k (+ (\ell/u)(1-\alpha_k)(a_k^{\dagger} - a_k^{\dagger})a_k. \]

Under the normal ordering sign, the operators \(a_k^{\dagger} = z_k\) and \(a_k = \partial_k\) commute, hence in what follows all fields are treated as classical. The continuous model is defined on a finite interval. Denote by \(x\) and \(y\) the upper and lower endpoints of this interval, so \(x - y = n\Delta\). Since the length of the interval in the \(\ell \to \infty\) limit remains constant, the number \(n\) of chain sites satisfies \(n \sim \ell\). In other words, the number of integrals tends to infinity. Below, we will show that, in fact, in the continuous limit this multiple integral boils down to some functional integral.

Let us change the integration variables in each integral,

\[\alpha_k = \frac{1}{2} \left( 1 + \frac{\beta_k}{\ell} \right), \quad 1 - \alpha_k = \frac{1}{2} \left( 1 - \frac{\beta_k}{\ell} \right), \]

and pass from the original operators \(a_k^{\dagger}\) and \(a_k\) to the renormalized ones:

\[a_k^{\dagger} a_k = \Delta \bar{\psi} \psi, \quad (a_{k-1}^{\dagger} - a_k^{\dagger}) a_k = \Delta (\bar{\psi} - \bar{\psi}_{k-1}) \psi_k - 2\Delta \bar{\psi} \psi_k. \]

Then we have

\[G(u) = \mathcal{N} \int_{-\ell}^{\ell} d\beta_1 \ldots \int_{-\ell}^{\ell} d\beta_n \exp \left[ (\ell - 1) \sum \ln \left( 1 - \frac{\beta_k^2}{\ell^2} \right) + u \sum \ln \frac{1 + \beta_k}{1 - \beta_k} \right. \]

\[\left. + \Delta \sum \frac{\beta_k}{u-1} \bar{\psi} \psi_k + \frac{\ell}{2u} \Delta \sum \left( 1 - \frac{\beta_k}{\ell} \right) (\bar{\psi} - \bar{\psi}_{k-1}) \psi_k \right], \]

where the factor in front of the integral equals

\[\mathcal{N} = \frac{2}{4^\ell \ell B(\ell + u, \ell - u)} \]

Using Stirling’s formula for the beta function, it is easy to calculate its asymptotic behavior:

\[\mathcal{N} \sim \left[ \frac{1}{\sqrt{\pi \ell}} \right]^n \exp \left[ \frac{-u^2(x-y)c}{2} \right]. \]

The sums in the exponent can be interpreted as a discretized functional of the original fields \(\psi(t)\) and \(\bar{\psi}(t)\) and one auxiliary field \(\beta(t)\). In the \(\ell \to \infty\) limit, each sum turns into the
corresponding integral. For example, for the first sum we have

\[
(\ell - 1) \sum \ln \left( 1 - \frac{\beta^2}{\ell^2} \right) = -(\ell - 1) \sum \frac{\beta^2}{\ell^2} + \cdots \rightarrow -\frac{c}{2} \int_{y}^{x} dt \beta^2(t),
\]

where we have used the fact that \(1 = \frac{n}{\ell}\). For the remaining sums, everything is similar, so in the continuous limit we obtain

\[
G(u) = \int \mathcal{D}\beta \exp \left[ -\frac{u^2(x - y)c}{2} - \frac{c}{2} \int_{y}^{x} dt \beta^2(t) + cu \int_{y}^{x} dt \beta(t) + \int_{y}^{x} dt \left( \frac{1}{u} \beta(t) - 1 \right) \bar{\psi}(t)\psi(t) + \frac{1}{cu} \int_{y}^{x} dt \partial_t \bar{\psi}(t)\psi(t) \right].
\]

Here, we have introduced the following notation for the integration measure:

\[
\mathcal{D}\beta = \lim_{\ell \to \infty} \frac{d\beta_1}{\sqrt{\pi \ell}} \cdots \frac{d\beta_n}{\sqrt{\pi \ell}}.
\]

In order to check the normalization, we can consider a simpler integral:

\[
\int \mathcal{D}\beta \exp \left[ -\frac{c}{2} \int_{y}^{x} dt \beta^2(t) \right] = \lim_{\ell \to \infty} \prod_{k} \int \frac{d\beta_k}{\sqrt{\pi \ell}} \exp \left[ -\frac{1}{\ell^2} \beta_k^2 \right] = 1.
\]

Note that we can simplify the obtained formula by changing the integration variables:

\[
G(u) = \int \mathcal{D}\beta \exp \left[ -\frac{c}{2} \int_{y}^{x} dt \beta^2(t) + \frac{1}{u} \int_{y}^{x} dt \beta(t) \bar{\psi}(t)\psi(t) + \frac{1}{cu} \int_{y}^{x} dt \partial_t \bar{\psi}(t)\psi(t) \right] = \int \mathcal{D}\phi \exp \left[ -\frac{c}{2} \int_{y}^{x} dt \phi^2(t) + \frac{1}{u} \int_{y}^{x} dt \bar{\psi}(t)\phi(t) + \frac{1}{cu} \int_{y}^{x} dt \partial_t \bar{\psi}(t)\phi(t) \right].
\]

Moreover, this functional integral is Gaussian, so we can compute it:

\[
G(u) = \exp \left[ -\frac{1}{cu} \int_{y}^{x} dt \bar{\psi}(t)\partial_t \psi(t) + \frac{1}{2cu^2} \int_{y}^{x} dt \left( \bar{\psi}(t)\psi(t) \right)^2 \right].
\]

The additional terms appearing after the integration by parts in the first term cancel each other due to the periodic boundary conditions.

Recall that in the continuous limit we obtained the following equation for the monodromy matrix:

\[
\partial_x T(x, u) = \left( \begin{array}{cc}
\frac{ab}{x} u & a\psi(x) \\
\frac{b\psi(x)}{x} & -\frac{a}{2} u
\end{array} \right) T(x, u),
\]

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4.1. The monodromy matrix.

The classical monodromy matrix $T^x_y(\lambda)$ is defined by the differential equation (see [2,9])

$$
\partial_x T^x_y(\lambda) = L(x, \lambda)T^x_y(\lambda), \quad T^x_x(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L(x, \lambda) = \frac{\lambda}{2} \sigma_3 + a\psi(x)\sigma_+ + b\bar{\psi}(x)\sigma_- = \begin{pmatrix} \frac{\lambda}{2} & a\psi(x) \\ b\bar{\psi}(x) & -\frac{\lambda}{2} \end{pmatrix},
$$

which can be solved using the ordered exponential

$$
T^x_y(\lambda) = c\exp \int_{y}^{x} L(t, \lambda) dt = 1 + \int_{y}^{x} dt L(t, \lambda) + \int_{y}^{x} dt_1 L(t_1, \lambda) \int_{y}^{t_1} dt_2 L(t_2, \lambda) + \ldots. \quad (4.2)
$$

The entries of the monodromy matrix

$$
T^x_y(\lambda) = \begin{pmatrix} A^x_y(\lambda) & B^x_y(\lambda) \\ C^x_y(\lambda) & D^x_y(\lambda) \end{pmatrix}
$$

are functionals of the fields $\psi, \bar{\psi}$, hence the full notation for the monodromy matrix should be $T^x_y(\psi, \bar{\psi}, \lambda)$. To make formulas more readable, we omit the dependence on $\lambda$ in expressions containing functional derivatives with respect to the fields, and, similarly, drop the dependence on the fields in equations with no functional derivatives.

In the quantum NLS model [2,7,9,17,18], the fields obey the canonical commutation relations

$$
[\bar{\psi}(x), \bar{\psi}(y)] = [\psi(x), \psi(y)] = 0, \quad [\psi(x), \bar{\psi}(y)] = \delta(x - y).
$$

The global quantum space is the Fock space $\mathbb{F}$, and the vacuum vector is defined in a standard way:

$$
\psi(x)|0\rangle = 0.
$$

The Hamiltonian of the model is given by

$$
H = \int dx \left( \partial\psi(x)\partial\bar{\psi}(x) + c\bar{\psi}^2(x)\psi^2(x) \right),
$$

where $c = ab$ is the coupling constant.

The quantum monodromy matrix $T^x_y(\lambda)$ is defined as (see [2,7,17])

$$
T^x_y(\lambda) = \begin{pmatrix} A^x_y(\lambda) & B^x_y(\lambda) \\ C^x_y(\lambda) & D^x_y(\lambda) \end{pmatrix} = :T^x_y(\lambda): = \begin{pmatrix} :A^x_y(\lambda): & :B^x_y(\lambda): \\ :C^x_y(\lambda): & :D^x_y(\lambda): \end{pmatrix}. \quad (4.3)
$$

In other words, the normal (or Wick) symbol of the quantum monodromy matrix coincides with the classical monodromy matrix. The quantum monodromy matrix acts in the tensor product of two spaces, one auxiliary space $\mathbb{C}^2$ and the Fock space $\mathbb{F}$:

$$
T^x_y(\lambda): \mathbb{C}^2 \otimes \mathbb{F} \rightarrow \mathbb{C}^2 \otimes \mathbb{F}.
$$
This implies that the entries of the quantum monodromy matrix are operators acting in the Fock space, and \( T^x_y(\lambda) \) as a matrix acts nontrivially in the auxiliary space \( \mathbb{C}^2 \).

The commutation relations between the entries of the quantum monodromy matrices \( T^x_y(\lambda) \) and \( T^{x'}_{y'}(\mu) \) can be written compactly as (see [2, 7])

\[
\mathcal{R}_{12}(\lambda - \mu) \, T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) \mathcal{R}_{12}(\lambda - \mu).
\]

(4.4)

Here, all operators act in the tensor product of three spaces \( V_1 \otimes V_2 \otimes F \), where \( V_1 = \mathbb{C}^2 \) and \( V_2 = \mathbb{C}^2 \) are two auxiliary spaces. The operator \( T_1(\lambda) = T^x_y(\lambda) \otimes 1 \) as a matrix acts nontrivially in the first space \( V_1 \). Its matrix elements act in the Fock space. In the second auxiliary space \( V_2 \), the operator \( T_1(\lambda) \) acts as the identity matrix. In the same way, the operator \( T_2(\mu) = 1 \otimes T^{x'}_{y'}(\mu) \) acts nontrivially in \( V_2 \) and \( F \). Finally, the R-matrix is defined as

\[
\mathcal{R}_{12}(\lambda - \mu) = (\lambda - \mu) \mathbb{1} - e \mathbb{P}_{12},
\]

(4.5)

where by \( \mathbb{P}_{12} \) we denote the permutation operator (2.5). The proof of the commutation relations (4.4) is given in Appendix C.

### 4.2. The Q-operator

#### 4.2.1. The commutation relations with the monodromy matrix

Consider the operator (3.4):

\[
\mathcal{Q}^x_y(\lambda) := e^{S(\bar{\psi}, \psi, \lambda)} := e^{-\frac{1}{\lambda} \int_x^y \bar{\psi}(t) \partial t \psi(t) \, dt + \frac{ab}{2\lambda^2} \int_x^y (\bar{\psi}(t) \psi(t))^2 \, dt}.
\]

(4.6)

Below, we prove that this operator satisfies all the required properties of the Q-operator.

In what follows, we use the so-called universal notation [14] and omit the obvious integral symbol, the arguments of the fields, etc. An example is given in (4.6), where the exponent \( S(\bar{\psi}, \psi, \lambda) = -\frac{1}{\lambda} \int_x^y \bar{\psi}(t) \partial t \psi(t) \, dt + \frac{ab}{2\lambda^2} \int_x^y (\bar{\psi}(t) \psi(t))^2 \, dt \) is written using the universal notation, and below it is written in detailed form.

So, first let us prove that the operator \( \mathcal{Q}^x_y(\lambda) \) commutes with the transfer matrix. To do this, we want to rewrite the product of the Q-operator and the monodromy matrix in the normal ordered form. Representing the quantum monodromy matrix (4.3) as

\[
T^x_y(\mu) = : T^x_y(\bar{\psi}, \psi, \mu) := T^x_y(\frac{\delta}{\delta \bar{A}}, \frac{\delta}{\delta A}, \mu) : e^{A \bar{\psi}} e^{\bar{A} \psi} \bigg|_{A = \bar{A} = 0}
\]

and using the identity

\[
\mathcal{Q}^x_y(\lambda) e^{A \bar{\psi}} e^{\bar{A} \psi} = : \exp \left( S(\bar{\psi}, \psi, A, \lambda) + A \bar{\psi} + \bar{A} \psi \right) :,
\]

from (A.2), we get

\[
\mathcal{Q}^x_y(\lambda) T^x_y(\mu) = T^x_y(\frac{\delta}{\delta \bar{A}}, \frac{\delta}{\delta A}, \mu) : e^{S(\bar{\psi}, \psi, \lambda) + A \bar{\psi} + \frac{ab}{2\lambda^2} \bar{\psi}^2 + \frac{ab}{2\lambda^2} (\bar{A} \psi)^2} \bigg|_{A = \bar{A} = 0}
\]

\[
= T^x_y(\frac{\delta}{\delta \bar{A}}, \frac{\delta}{\delta A}, \mu) : e^{S(\bar{\psi}, \psi, \lambda) + A \bar{\psi} + \frac{ab}{2\lambda^2} \bar{\psi}^2 + \frac{ab}{2\lambda^2} (\bar{A} \psi)^2} \bigg|_{A = \bar{A} = 0}.
\]

Here, we use the universal notation again. As an example, we write two terms in detailed form,

\[
A \psi = \int_y^x A(t) \psi(t) \, dt, \quad (\bar{\psi} A)^2 = \int_y^x A^2(t) \bar{\psi}^2(t) \, dt,
\]

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and hope that in what follows the notation will be clear. Note that the underlined term
\( \frac{\delta}{\delta A_1} (\bar{\psi} A)^2 \) in the exponent leads to additional local terms corresponding to higher functional
derivatives, for instance,
\[
\frac{\delta^2}{\delta A(t_1) \delta A(t_2)} \rightarrow \ldots + \frac{2ab}{\lambda^2} \bar{\psi}^2(t_1) \delta(t_1 - t_2).
\]
In the expression (4.2) for the monodromy matrix, a strict order of the parameters \( t_i \) (such as \( t_1 > t_2 \), etc.) is required. Hence, the local terms do not contribute to the result.

Calculating the functional derivatives
\[
\frac{\delta}{\delta A(t)} \rightarrow \bar{\psi}(t) + \frac{1}{\lambda} \partial_t \bar{\psi}(t) + \frac{ab}{\lambda^2} \bar{\psi}^2(t) \psi(t), \quad \frac{\delta}{\delta A(t)} \rightarrow \psi(t),
\]
we obtain
\[
\mathcal{Q}_{xy}^x(\lambda) \mathcal{T}_y^x(\mu) = : e^{S(\bar{\psi}, \psi, \lambda)} \mathcal{T}_y^x(\mu) = : e^{S(\bar{\psi}, \psi, \lambda)} \mathcal{T}_y^x(\mu) :,
\]
where the monodromy matrix under the normal ordering sign is defined by the differential equation
\[
\partial_x \mathcal{T}_y^x(\lambda, \mu) = L(x, \lambda, \mu) \mathcal{T}_y^x(\lambda, \mu), \quad \mathcal{T}_y^x(\lambda, \mu) = 1,
\]
\[
L(x, \lambda, \mu) = \frac{\mu}{2} \sigma_3 + a \psi(x) \sigma_+ + b \left( \bar{\psi}(x) + \frac{1}{\lambda} \partial \bar{\psi}(x) + \frac{ab}{\lambda^2} \bar{\psi}^2(x) \psi(x) \right) \sigma_-. \tag{4.7}
\]
It is easy to check that the matrix \( L(x, \lambda, \mu) \) can be transformed into a much simpler matrix \( U(x, \lambda, \mu) \) by an appropriate gauge transformation:
\[
U(x, \lambda, \mu) = S^{-1}(x, \lambda) U(x, \lambda, \mu) S(x, \lambda) - S^{-1}(x, \lambda) \partial_x S(x, \lambda),
\]
\[
U(x, \lambda, \mu) = \left( \frac{\mu}{2} + \frac{ab}{\lambda} \bar{\psi}(x) \right) \sigma_3 + b \left( 1 - \frac{\mu}{\lambda} \right) \bar{\psi} \sigma_- + a \psi \sigma_+, \quad S(x, \lambda) = 1 - \frac{b}{\lambda} \bar{\psi}(x) \sigma_-.
\]
Then
\[
\mathcal{T}_y^x(\lambda, \mu) = S^{-1}(x, \lambda) F_y^x(\lambda, \mu) S(y, \lambda), \tag{4.8}
\]
where \( F_y^x(\lambda, \mu) \) is a solution of the equation
\[
\partial_x F_y^x(\lambda, \mu) = U(x, \lambda, \mu) F_y^x(\lambda, \mu), \quad F_y^x(\lambda, \mu) = 1,
\]
\[
U(x, \lambda, \mu) = \left( \frac{\mu}{2} + \frac{ab}{\lambda} \bar{\psi}(x) \psi(x) \right) \sigma_3 + \frac{ab}{\lambda} \bar{\psi}(x) \sigma_-.
\]
In the case of the periodic boundary conditions, \( \bar{\psi}(x) = \bar{\psi}(y), \bar{\psi}(x) = \bar{\psi}(y) \). Hence, if we take the trace of both sides in (4.8), the matrices \( S^{-1}(x, \lambda) \) and \( S(y, \lambda) \) cancel each other, and for the product of the Q-operator and the transfer matrix we get
\[
\mathcal{Q}_{xy}^x(\lambda) \text{tr} \mathcal{T}_y^x(\mu) = : e^{S(\bar{\psi}, \psi, \lambda)} \text{tr} F_y^x(\lambda, \mu) :,
\]
Now, consider the product of the Q-operator and the transfer matrix in the reverse order. As above, we use (A.2) and obtain a shift \( \bar{\psi} \rightarrow \bar{\psi} + \bar{A} \) of the field:
\[
e^{\bar{A} \bar{\psi}} e^{A \psi} \mathcal{Q}_{xy}^x(\lambda) = : \exp \left( S(\bar{\psi} + \bar{A}, \psi) + A \bar{\psi} + \bar{A} \psi \right) :
\]
\[
= : \exp \left( S(\bar{\psi}, \psi) + \bar{A} \left( \psi - \frac{1}{\lambda} \partial \psi + \frac{ab}{\lambda^2} \bar{\psi}^2 \psi \right) + \bar{A} \psi + \frac{ab}{\lambda^2} \left( \psi \bar{A} \right)^2 \right) :
\]
Differentiating with respect to the sources
\[
\frac{\delta}{\delta A(t)} \rightarrow \psi(t) - \frac{1}{\lambda} \partial_t \psi(t) + \frac{ab}{\lambda^2} \bar{\psi}^2(t) \bar{\psi}(t), \quad \frac{\delta}{\delta A(t)} \rightarrow \bar{\psi}(t),
\]
We use a gauge transformation again: 

\[ U \]

hence a similar formula holds for the matrices 

\[ A \]

As before, the matrices 

\[ B \]

To calculate the trace, we need the diagonal elements of the matrix 

\[ C \]

where 

\[ D \]

Note that the matrices 

\[ E \]

we find 

\[ F \]

We use a gauge transformation again: 

\[ G \]

Now, we derive the Baxter equation. Actually, we already have all the necessary formulæ. Thus, for the product of the Q-operator and the transfer matrix in the reverse order, we get 

\[ H \]

Note that the matrices 

\[ I \]

and 

\[ J \]

are connected by a similarity transformation, 

\[ K \]

hence a similar formula holds for the matrices 

\[ L \]

So, we have 

\[ M \]

This proves that the Q-operator (4.6) commutes with the transfer matrix: 

\[ N \]

As before, the matrices 

\[ O \]

and 

\[ P \]

in (4.10) cancel each other within the trace due to the periodic boundary conditions. Thus, for the product of the Q-operator and the transfer matrix, we get 

\[ Q \]

where 

\[ R \]

satisfies the differential equation 

\[ S \]

\[ T \]

\[ U \]

\[ V \]

\[ W \]

\[ X \]

\[ Y \]

\[ Z \]

We find 

\[ A \]

where the matrix 

\[ B \]

is a solution of the equation 

\[ C \]

\[ D \]

\[ E \]

\[ F \]

\[ G \]

\[ H \]

\[ I \]

\[ J \]

\[ K \]

\[ L \]

\[ M \]

\[ N \]

\[ O \]

\[ P \]

\[ Q \]

\[ R \]

\[ S \]

\[ T \]

\[ U \]

\[ V \]

\[ W \]

\[ X \]

\[ Y \]

\[ Z \]
and thus we derive the Baxter equation
\[ Q_y^r(\lambda) \text{tr} T_y^r(\lambda) = e^{(x-y)\lambda} :e^{S(\tilde{\psi},\psi,\lambda)+\frac{a}{2}\tilde{\psi}\psi} : + e^{-\frac{1}{2}(x-y)\lambda} :e^{S(\tilde{\psi},\psi,\lambda)-\frac{a}{2}\tilde{\psi}\psi} : \]
\[ = e^{(x-y)\lambda} \left( 1 + \frac{c}{\lambda} \right) \bar{Q}_y^r(\lambda + c) + e^{-\frac{1}{2}(x-y)\lambda} \left( 1 - \frac{c}{\lambda} \right) \bar{Q}_y^r(\lambda - c). \]

Here we have taken into account that \( ab = c \) and used (A.5).

Finally, we prove the commutation relation (see [1])
\[ Q_y^r(\lambda) Q_y^s(\mu) = Q_y^s(\mu) Q_y^r(\lambda). \]

Recall from the previous section that the Q-operator can be expressed as a functional integral:
\[ Q_y^r(\lambda) = \int D\phi e^{S(\tilde{\phi},\phi,\lambda)} = \int D\phi e^{-\frac{1}{2}\phi^2 - \frac{1}{\lambda}(\partial_t - c\phi(t)) \psi(t) dt} \]
\[ S(\phi, \tilde{\psi}, \psi, \lambda) = -\frac{c}{2} \int \phi^2(t) dt - \frac{1}{\lambda} \int \psi(t) (\partial_t - c\phi(t)) \psi(t) dt. \]

Using this formula and (A.3), we rewrite the product of the Q-operators in the normal ordered form as
\[ Q_y^r(\lambda) Q_y^s(\mu) = \int D\phi_1 \int D\phi_2 e^{-\frac{1}{2}(\phi_1^2+\phi_2^2)} e^{\tilde{\phi}_1 \left( \frac{1}{2} \partial_t - c\phi_1 \right) - \frac{1}{2} \partial_t \left( \phi_2 - c\phi_1 \right) - \frac{1}{2} \partial_t \left( \phi_2 - c\phi_1 \right) \psi}. \]

Clearly, the last expression is symmetric under the permutation \( \lambda \leftrightarrow \mu \).

So, we have shown in two ways, by taking the continuous limit as \( \ell \to \infty \) of the known Q-operator for the XXX spin chain and directly in the continuous model, that the Q-operator for the quantum NLS model is given by (4.6). It will be interesting to study the connection between the Q-operator and the Backlund transformation [4] for this model; hopefully, we will return to this in the future.

\section{Appendix A. The Normal Symbols of Operators}

All formulas in this appendix are written in the universal notation. The general formula for the product of two normal-ordered operators can be expressed in two forms: using functional derivatives,
\[ :F_1(\tilde{\psi},\psi) : F_2(\tilde{\psi},\psi) : = e^{\frac{\delta^2}{\delta \tilde{\phi} \delta \phi}} F_1(\tilde{\psi},\phi) F_2(\tilde{\phi},\psi) \bigg|_{\phi=\tilde{\psi}}, \]
or a functional integral,
\[ :F_1(\tilde{\psi},\psi) : F_2(\tilde{\psi},\psi) : = \int D\phi D\tilde{\phi} F_1(\tilde{\psi},\phi) F_2(\tilde{\phi},\psi) e^{-\phi - \tilde{\phi} \tilde{\phi}}. \] (A.1)

Applying any of these formulas, it is not difficult to show that
\[ e^{a\tilde{\psi}} e^{\tilde{\phi}} F(\tilde{\psi},\psi) : = e^{a\tilde{\psi} + \tilde{\phi}} F(\tilde{\psi} + \tilde{a},\psi) :, \quad F(\tilde{\psi},\psi) : e^{a\tilde{\psi}} e^{\tilde{\phi}} = e^{a\tilde{\psi} + \tilde{\phi}} F(\tilde{\psi},\psi + a). \] (A.2)

Similarly, the general relations
\[ :e^{\tilde{\psi} D_1 \psi} : e^{\tilde{\psi} D_2 \psi} : = e^{\tilde{\psi} (D_1 D_2 + D_1 + D_2) \psi} :, \] (A.3)
\[ :e^{a\tilde{\psi} \partial_t \psi + \beta(\tilde{\psi})^2 + \gamma \tilde{\psi} \psi} : = e^{\gamma \tilde{\psi} \psi} : e^{\frac{\beta}{\gamma+1} \hat{\tilde{\psi}} \hat{\tilde{\psi}} + \frac{\beta}{\gamma+1} \hat{\tilde{\psi}}^2} : (1 + \lambda) \hat{\tilde{\psi}} : = e^{\lambda \tilde{\psi}} : \] (A.4)
can be proved using (A.1). In the first formula, \( D_1, D_2 \) are any linear operators acting on the fields \( \psi(x) \) and \( \tilde{\psi}(x) \).
Putting $\alpha = -\frac{1}{X}$, $\beta = \frac{ab}{2\lambda X}$, and $\gamma = \pm \frac{ab}{X}$ in (A.4), we obtain formulas for the Q-operator:

\[ e^{-\frac{1}{X}\lambda\psi\partial\psi + \frac{ab}{2\lambda X}(\lambda\psi)^2} = e^{\frac{ab}{X}\psi\partial\psi + \frac{ab}{2(\lambda + ab)}(\lambda\psi)^2} = (1 \pm \frac{ab}{X}) e^{-\frac{1}{X}\lambda\psi\partial\psi + \frac{ab}{2(\lambda + ab)}(\lambda\psi)^2}. \quad (A.5) \]

**APPENDIX B. ORDERED EXponentials**

The fundamental solution of the differential equation

\[ \partial_x F(x,y) = U(x)F(x,y), \quad F(x,y)|_{x=y} = \mathbb{1} \quad (B.1) \]

can be expressed in terms of the ordered exponential

\[ F(x,y) = \exp \int_{y}^{x} U(t)dt = 1 + \int_{y}^{x} dt U(t) + \int_{y}^{x} dt_1 \int_{y}^{t_1} dt_2 U(t_1)U(t_2) + \ldots \quad (B.2) \]

The main formulas with ordered exponentials are

\[ F(x,y) F(y,z) = F(x,z), \quad F(y,x) = F^{-1}(x,y), \quad (B.3) \]

\[ F_V(x,y) = F(x,y) + \int_{y}^{x} dz F(x,z)V(z)F(z,y) \quad (B.4) \]

\[ + \int_{y}^{x} dz_1 \int_{y}^{z_1} dz_2 F(x,z_1)V(z_1)F(z_1,z_2)V(z_2,y) + \ldots . \]

The last formula defines the perturbation series for a solution of Eq. (B.1) with a slightly modified matrix, $U(x) \to U(x) + V(x)$; here, $F(x,y)$ is a solution of the original equation (B.1) with the matrix $U(x)$. In order to derive this result, consider an equation for the function $F_V(x,y)$:

\[ \partial_x F_V(x,y) = (U(x) + V(x)) F_V(x,y), \quad F_V(x,y)|_{x=y} = \mathbb{1}. \]

Let us look for a solution in the form $F_V(x,y) = F(x,y) f(x,y)$. Then for $f(x,y)$ we get

\[ \partial_x f(x,y) = F^{-1}(x,y)V(x)F(x,y) f(x,y), \quad f(x,y)|_{x=y} = \mathbb{1}, \]

hence the solution can be expressed in terms of (B.2) by substituting

\[ U(x) \to F^{-1}(x,y)V(x)F(x,y). \]

Using (B.3), we get

\[ f(x,y) = 1 + \int_{y}^{x} dt F(y,t)V(t)F(t,y) + \int_{y}^{x} dt_1 \int_{y}^{t_1} dt_2 F(y,t_1)V(t_1)F(t_1,t_2)V(t_2,y) + \ldots . \]

Multiplying both sides by $F(x,y)$, we obtain (B.4). Note also that the same formula (B.4) can be used to calculate the variation $\delta F(x,y) = F_{\delta U}(x,y) - F(x,y)$. Substituting $U \to U + \delta U$, 624
Using ordered exponentials, we can rewrite these formulas as

\[ \exp(-\lambda) = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots. \]

and so on.

B.1. Gauge transformations. Assume that we want to represent a solution of the differential equation

\[ B.1. \quad \frac{\partial}{\partial y} F(x, y) = U(x) F(x, y), \quad F(x, y)|_{x=y} = 1 \]

in the form \( F(x, y) = S(x) G(x, y) S^{-1}(y) \), where \( S(x) \) is some known matrix. One can easily derive an equation for \( G(x, y) \):

\[ B.1. \quad \frac{\partial}{\partial x} G(x, y) = \left( S^{-1}(x) U(x) S(x) - S^{-1}(x) \partial_x S(x) \right) G(x, y), \quad G(x, y)|_{x=y} = 1. \]

Using ordered exponentials, we can rewrite these formulas as

\[ \exp \int_y^x \left( S^{-1}(t) U(t) S(t) - S^{-1}(t) \partial_t S(t) \right) dt = S^{-1}(x) \left( \exp \int_y^x U(t) dt \right) S(y). \]

Appendix C. Commutation relations for the monodromy matrix in the continuous model

In this appendix, we prove relation (4.4). Note that this formula was proved by E. K. Sklyanin in his paper [2] in two ways. In fact, our proof repeats the main steps of E. K. Sklyanin’s proof, but with more emphasis on functional methods, and, of course, it is of purely methodological interest.

The proof is based on explicit formulas for products of normal-ordered operators. Let us rewrite the product of the monodromy matrices in the left-hand side of (4.4) in the normal ordered form:

\[ T_1(\lambda) T_2(\mu) = :T_1(\lambda): :T_2(\mu): = \frac{\delta T_1(\psi, \phi) T_2(\phi, \psi)}{\delta \phi(z_1) \cdots \delta \phi(z_n) \delta \psi(z_1) \cdots \delta \psi(z_n)} \bigg|_{\phi=\psi} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} \int d^2 \phi \int d^2 \psi \delta^n T_1(\psi, \phi) \delta^n T_2(\phi, \psi) \bigg|_{\phi=\psi} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} \int d^2 \phi \int d^2 \psi \delta^n T_1(\psi, \phi) \delta^n T_2(\phi, \psi) \bigg|_{\phi=\psi} \]
Note that (C.1) is symmetric under permutations of the variables \( z_k \), so we get rid of the factor \( \frac{1}{n!} \) by passing to integration over the fundamental domain \( \mathbb{Z}_n = \{ y < z_n < \ldots < z_1 < x \} \).

In order to compute the functional derivatives, substitute \( \delta U(x) = \phi(x) a_{\sigma_+} + \bar{\phi}(x) b_{\sigma_-} \) into (B.5). We obtain formulas similar to (B.6), (B.7):

\[
\begin{align*}
\frac{\delta T_1(\bar{\psi}, \phi)}{\delta \phi(z)} \bigg|_{\phi = \psi} &= T_z^x(\lambda) a_{\sigma_+} T_y^x(\lambda) \otimes 1, \\
\frac{\delta T_2(\bar{\phi}, \psi)}{\delta \phi(z)} \bigg|_{\phi = \bar{\psi}} &= 1 \otimes T_z^x(\mu) b_{\sigma_-} T_y^x(\mu).
\end{align*}
\]

Then we rewrite the product of these expressions in the following form:

\[
\left. \frac{\delta T_1(\bar{\psi}, \phi)}{\delta \phi(z)} \frac{\delta T_2(\bar{\phi}, \psi)}{\delta \phi(z)} \right|_{\phi = \psi = \bar{\psi}} = T_z^x(\lambda, \mu) (a_{\sigma_+} \otimes b_{\sigma_-}) T_y^x(\lambda, \mu),
\]

where \( T_z^x(\lambda, \mu) = T_z^x(\lambda) \otimes T_z^x(\mu) \). In the same way, we compute the higher derivatives, for example,

\[
\left. \frac{\delta^2 T_1(\bar{\psi}_1, \phi)}{\delta \phi(z_1) \delta \phi(z_2)} \frac{\delta^2 T_2(\bar{\phi}_1, \psi)}{\delta \phi(z_1) \delta \phi(z_2)} \right|_{\phi = \psi = \bar{\psi}} = T_{z_1}^x(\lambda, \mu) (a_{\sigma_+} \otimes b_{\sigma_-}) T_{z_2}^x(\lambda, \mu) (a_{\sigma_+} \otimes b_{\sigma_-}) T_y^x(\lambda, \mu).
\]

In fact, the sum in (C.1) has the same form as the perturbation series (B.4) with the substitutions \( F(x, y) \to T_x^y(\lambda, \mu) \) and \( V(x) \to a_{\sigma_+} \otimes b_{\sigma_-} \). The product \( T_y^x(\lambda, \mu) = T_x^y(\lambda) \otimes T_y^x(\mu) \) is a solution of the equation

\[
\partial_x T_y^x(\lambda, \mu) = (L(x, \lambda) \otimes 1 + 1 \otimes L(x, \mu)) T_y^x(\lambda, \mu)
\]

with initial condition \( T_y^x(\lambda, \mu) = 1 \). However, applying the normal ordering operation, we obtain the perturbation series (B.4) for \( T_y^x(\lambda, \mu) \), where \( T_y^x(\lambda, \mu) \) is a solution of the equation

\[
\partial_x T_y^x(\lambda, \mu) = (L(x, \lambda) \otimes 1 + 1 \otimes L(x, \mu) + a_{\sigma_+} \otimes b_{\sigma_-}) T_y^x(\lambda, \mu)
\]

with initial condition \( T_y^x(\lambda, \mu) = 1 \). Thus,

\[
\mathbb{T}_1(\lambda) \mathbb{T}_2(\mu) = : e^{\delta_{\lambda \bar{\psi}, \psi} T_1(\bar{\psi}, \phi) T_2(\bar{\phi}, \psi)} :_{\phi = \psi = \bar{\psi}} = : T_y^x(\lambda, \mu) :.
\]

Similarly, we derive a formula for the inverse product of the monodromy matrices:

\[
\mathbb{T}_2(\mu) \mathbb{T}_1(\lambda) = : e^{\delta_{\bar{\psi}, \psi} T_2(\bar{\psi}, \phi) T_1(\bar{\phi}, \psi)} :_{\phi = \psi = \bar{\psi}} = : T^{ux}_y(\mu, \lambda) :,
\]

where \( T^{ux}_y(\mu, \lambda) \) solves the equation

\[
\partial_x T^{ux}_y(\mu, \lambda) = (L(x, \lambda) \otimes 1 + 1 \otimes L(x, \mu) + a_{\sigma_+} \otimes b_{\sigma_-}) T^{ux}_y(\mu, \lambda)
\]

with initial condition \( T^{ux}_y(\mu, \lambda) = 1 \). So, the global relation (4.4) corresponds to the local relation

\[
\mathbb{R}_{12}(\lambda - \mu) (L(x, \lambda) \otimes 1 + 1 \otimes L(x, \mu) + a_{\sigma_+} \otimes b_{\sigma_-}) = (L(x, \lambda) \otimes 1 + 1 \otimes L(x, \mu) + a_{\sigma_+} \otimes b_{\sigma_-}) \mathbb{R}_{12}(\lambda - \mu),
\]

which can be easily proved. Note also that taking into account the explicit formula for the R-matrix (4.5), it actually remains to check the simpler relation

\[
\mathbb{R}_{12}(\lambda - \mu) (\lambda/2 \sigma_3 \otimes 1 + \mu/2 \sigma_3) = (\lambda/2 \sigma_3 \otimes 1 + \mu/2 \sigma_3).
\]

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