Collisional Penrose process in Kerr–Taub–NUT spacetime

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Abstract Maximum efficiency of collisional Penrose process with spinning and non-spinning particles in Kerr–Taub–NUT spacetime has been studied. We consider three cases in detail: two massive particles collide near the horizon, one of the resulting massive particles escapes to infinity, and the other massive particle falls into the black hole; a massless particle collides with a massive particle, then the massless daughter particle escapes from the black hole to infinity, and the massive daughter particle falls into the black hole (Compton scattering); a massive particle collides with a massless particle, the massive daughter particle escapes from the black hole to infinity and the massless daughter particle falling into the horizon (inverse Compton scattering). We find that for these cases, regardless of whether particles are spinning or not, the maximum energy extraction efficiency of the collisional Penrose process always decreases as the NUT charge increases, and the energy extraction efficiency in the spinning case is always higher than that in the non-spinning case.

1 Introduction

Penrose process is a well-known theoretical assumptions in general relativity first proposed by Penrose in 1969 [1]. As Penrose discovered, when a particle is divided into two parts in the ergosphere of a rotating black hole, one part will be able to carry negative energy (measured by an observer at infinity) and falls into the event horizon. In contrast, the other part will be able to carry positive energy (by the principle of conservation of energy) and escape from the black hole to infinity. The Penrose process allows us to extract the energy from a rotating black hole. Initially, Wald investigated the energy extraction efficiency of this process and found that for the case of a single massive particle decaying into two photons, the maximum efficiency ($\eta_{\text{max}} = \frac{\text{output energy}}{\text{input energy}}$) is 121% [2]. After that, Piran et al. [3] proposed an alternative Penrose process in which two falling particles collide in the ergosphere. One of the two particles generated by the collision falls into the black hole, whereas the other particle escapes from the black hole. This process is called the collisional Penrose process (CPP).

In 2009, Bañados et al. [4] found that when two particles are in free fall from infinity, if either particle (but not both) has the critical angular momentum, their center-of-mass energy can reach arbitrarily high levels when they collide near an extreme Kerr black hole. This discovery is known as the BSW mechanism and has been the subject of many studies since it was proposed [5–9]. However, as the arbitrarily large center-of-mass energy is generated near the horizon, it has to be considered whether the resulting particle can escape to infinity [10]. Thus, it is also critical to study how large is the efficiency of the energy extraction from a black hole. Bejger et al. [10] considered two massive particles colliding near the horizon of a rotating black hole and converting into two photons. In this case, the maximum efficiency of CPP is about 129%. Schnittman [11] found that for Compton scattering between a photon and a massive particle near the horizon, the efficiency could reach nearly 1400%. Leiderschneider and Piran [12] analyzed the collision on the equatorial plane and more general off-plane orbits. They found that the maximum efficiency is obtained on the equatorial plane.

Particles falling into the black hole are spinning (rather than spinless) is a more realistic assumption, and the question of how the particles’ spin affect the efficiency of the CPP is worth considering. Unlike free-falling non-spinning particles, free-falling spinning particles do not move along the geodesic due to timelike conditions. Specifically, the trajectory of free-falling spinning particles is described by the Mathisson–Papapetrou–Dixon equations [13–15] with its 4-momentum not always parallel to its 4-velocity. When investigating the CPP with spinning particles in Kerr spacetime, Mukherjee [16] showed that the energy extraction is more
significant than the non-spinning case. Maeda et al. [17] discussed the collision between massive and massless spinning particles in detail. They found that the Compton scattering obtains the maximal efficiency and is about 2700%, which is remarkably greater than the 1400% of spinless case.

Kerr–Taub–NUT (KTN) spacetime is a remarkable stationary and axisymmetric solution of Einstein field equations, which describes the gravity of a rotating source equipped with a gravitomagnetic monopole moment [18]. Compared with the Kerr metric, the KTN metric contains not only mass parameter $M$ and spin parameter $a$ (angular speed per unit mass) but also NUT charge $n$ (i.e., gravitomagnetic monopole moment). If the NUT charge $n$ vanishes, the KTN spacetime will return to the well-known Kerr spacetime. Due to the presence of the NUT charge, the KTN spacetime possesses several novel properties compared to the Kerr spacetime. For example, KTN spacetime is asymptotically non-flat [19], and it does not contain curvature singularity, but there is a conical singularity on the symmetry axis [20]. Despite some undesirable physical properties, various physical phenomena and processes in this spacetime have been extensively studied, including relativistic accretion flow [21], frame-dragging phenomenon [22], the shadow of the KTN black hole [23,24] and circular orbits of particles [25,26], etc. In the light of the BSW mechanism, Liu et al. [27] studied the collision of two particles near the KTN black hole. They found that the unlimited center-of-mass energy can also be reached, and the presence of the NUT charge will modify the restrict conditions for the angular momentum parameter of the black hole, which is different from the Kerr black hole. The research related to the original Penrose process is also in progress. Pradhan [26] found that, for the case of the original Penrose process that happened on the equatorial plane, the value of energy extraction from the KTN black hole negatively correlates with its NUT charge. Thus, a natural question is what impact will the existence of the NUT charge have on the CPP in KTN spacetime? And does this conclusion agree with the original Penrose process?

In this paper, we concentrate on the maximum energy extraction efficiency of the CPP for the spinning and non-spinning particles in KTN spacetime and specifically investigates the effect of NUT charge on the maximum energy extraction efficiency. Also, in this spacetime, we will determine whether the presence of particles’ spin still positively affects the CPP. The outline of this paper is as follows. In Sect. 2, we discuss the equations of motion for spinning particles in the KTN spacetime. In Sect. 3, we discuss the maximum energy extraction efficiency of the collissional Penrose process with spinning and non-spinning particles in the extreme KTN black hole in three cases, analyze the effects of the NUT charge and spin on it. Conclusion was given in Sect. 4. Throughout the paper, we adopt the geometrical unit ($c = G = 1$).

2 Basic equation

2.1 Kerr–Taub–NUT black hole

The Kerr–Taub–NUT spacetime is a vacuum rotating solution of the Einstein field equation. Due to the existence of the NUT charge, it is asymptotically non-flat. In Boyer–Lindquist coordinates $(t, r, \theta, \phi)$, the Kerr–Taub–NUT metric can be expressed as follows:

$$ds^2 = -\frac{\Delta}{\Sigma} [dt + (2n \cos \theta - a \sin^2 \theta) d\phi]^2 + \frac{\Sigma}{\Delta} dr^2$$

$$+ \frac{\sin^2 \theta}{\Sigma} [adr - (r^2 + a^2 + n^2) d\phi]^2 + \Sigma d\theta^2$$

(2.1)

with

$$\Delta = r^2 - 2Mr + a^2 - n^2,$$

(2.2)

$$\Sigma = r^2 + (n + a \cos \theta)^2,$$

(2.3)

where $M$ is the black hole’s mass parameter, $a$ is its rotation angular momentum and $n$ is the NUT charge, which represents the gravitomagnetic monopole moment of the source. By vanishing $n$, this metric can be reverted to the Kerr metric. Using Eq. (2.2), we can determine the radius of the horizon of the KTN black hole as

$$r_{\pm} = M \pm \sqrt{M^2 + n^2 - a^2},$$

(2.4)

where $r_+ \equiv r_H$ denotes event horizon, $r_-$ denotes Cauchy horizon. If $M^2 + n^2 - a^2 = 0$, this two horizons coincide and locates at $r_{\pm} = M$. In this case, it is an extreme black hole.

2.2 Equations of motion of spinning test particles

As opposed to non-spinning particles, when we discuss spinning particles, which are more natural in the universe, moving in the curved spacetime, we often need to consider their internal structure and assume them as extended bodies with multiple moments. So, the geodesic equation describing the motion of non-spinning particles in the past must be replaced by the Mathisson–Papapetrou–Dixon equations [13–15] in the present, that is

$$\frac{D p^\mu}{d\tau} = -\frac{1}{2} R^\mu_{\nu\rho\sigma} v^\nu S^\rho\sigma,$$

(2.5)

$$\frac{DS^\mu\nu}{d\tau} = D^\mu v^\nu - p^\mu v^\nu,$$

(2.6)

where

$$v^\mu = \frac{dz^\mu(\tau)}{d\tau}$$

(2.7)
is a tangent vector of the center-of-mass world line \( z^\mu (\tau) \) with affine parameter \( \tau \), \( R^\mu_{\nu\rho\sigma} \) denotes the Riemann tensor of the KTN spacetime, and \( p^\mu \) is the 4-momentum of the spinning particle. Thus, the mass of a particle may be defined as

\[
m^2 = -p^\mu p_\mu ,
\]  

(2.8)

which is conserved along the orbit. Then, \( S^{\mu\nu} \) is the antisymmetric spin tensor of the particle, and the magnitude of the spin \( S \) is also conserved, as shown by

\[
S^{\mu\nu} S_{\mu\nu} = 2S^2 = 2m^2 s^2 ,
\]  

(2.9)

where \( s \) is the spin of the given particle, respectively.

We would like to point out that to completely determine the trajectory of a spinning particle, Eqs. (2.5) and (2.6) are not sufficient. In this regard, we may employ the spin supplementary condition \([28,29]\) as

\[
S^{\mu\nu} p_\mu = 0 .
\]  

(2.10)

We also introduce a normalized specific 4-momentum:

\[
u^\mu = \frac{p^\mu}{m} u^\nu u_\mu = -1 ,
\]  

(2.11)

which is not parallel to the 4-velocity \( v^\mu \). Thus, the affine parameter \( \tau \) of the world line is normalized as \([30]\)

\[
u^\mu v_\mu = -1 .
\]  

(2.12)

Therefore, the detailed relation between the 4-velocity \( v^\mu \) and the specific 4-momentum \( u^\mu \) can be obtained from the above equations as

\[
u^\mu - u^\mu = \frac{S^{\mu\nu} R_{\nu\rho\sigma\lambda} \nu^\rho S^\sigma\lambda}{2(m^2 + 1/2 R_{\alpha\beta\rho\sigma} S^\alpha\beta S^\rho S^\sigma)}.
\]  

(2.13)

2.3 Conserved quantities

The conserved quantity of a free-falling spinning particle in curved spacetime is given by

\[
Q_\xi = p^\mu \xi_\mu + \frac{1}{2} S^{\mu\nu} \nabla_\nu \xi_\mu ,
\]  

(2.14)

where \( \xi_\mu \) is the Killing vector field. In KTN spacetime, there will always be a timelike Killing vector \( \xi^{(1)}_\mu \) and an axial Killing vector \( \xi^{(2)}_\mu \). By introducing a set of tetrad bases:

\[
e^{(0)}_\mu = \sqrt{\frac{\Delta}{\Sigma}} (dt + (2n \cos \theta - a \sin^2 \theta) d\phi) ,
\]

\[
e^{(1)}_\mu = \sqrt{\frac{\Sigma}{\Delta}} d\tau ,
\]

\[
e^{(2)}_\mu = \sqrt{\frac{\Sigma}{\Delta}} d\theta ,
\]

\[
e^{(3)}_\mu = \sqrt{\frac{\sin \theta}{\Sigma}} (-a dt + (r^2 + a^2 + n^2) d\phi) ,
\]

(2.15)

these two Killing vectors can be written as

\[
\xi^{(1)}_\mu = -\sqrt{\frac{\Delta}{\Sigma}} (a \sin \theta - 2n \cos \theta) e^{(3)}_\mu ,
\]

(2.16)

\[
\xi^{(2)}_\mu = \frac{\sin \theta}{\Sigma} (r^2 + a^2 + n^2) e^{(3)}_\mu .
\]

(2.17)

Using \( e^{(a)}_\mu = e^{(b)}_\nu g^{\nu\mu} g_{(a)(b)} \) to raise and lower the indices, we have \( p^\mu = p^{(a)} e^{(a)}_\mu \) and \( S^{\mu\nu} = S^{(a)(b)} e^{(a)}_\mu e^{(b)}_\nu \).

In the rest of this paper, we only concentrate on the particle’s motion in the equatorial plane (\( \theta = \pi/2 \)) for convenience. Combining Eqs. (2.14), (2.15) and (2.16), the particle’s energy \( E \) can be described as

\[
E = -Q_\xi^{(0)} = p^{(0)} \sqrt{\frac{\Delta}{r^2 + n^2}} + p^{(3)} \frac{a}{\sqrt{r^2 + n^2}} + S^{(0)(1)} M (r^2 - n^2) + 2n^2 r
\]

\[
- S^{(0)(2)} (n^2 r^2 - 2M r) + S^{(2)(3)} \frac{2n (n^2 + M r)}{r^2 + n^2} .
\]

(2.18)

Similarly, combining Eqs. (2.14), (2.15) and (2.17) can obtain the particle’s \( z \) component of the total angular momentum \( J \):

\[
J = Q_\xi^{(0)} = p^{(0)} \sqrt{\frac{\Delta}{r^2 + n^2}} + p^{(3)} \frac{r^2 + n^2 + a^2}{\sqrt{r^2 + n^2}} + S^{(0)(1)} a (3n^2 r^2 + r^3 + M (r^2 - n^2))
\]

\[
- S^{(0)(2)} \frac{n \sqrt{\Delta}}{r^2 + n^2} + S^{(1)(3)} \frac{r \sqrt{\Delta}}{r^2 + n^2}
\]

\[
+ S^{(2)(3)} \frac{2an (n^2 + M r)}{r^2 + n^2} .
\]

(2.19)

2.4 Equations of motion in the equatorial plane

A specific spin vector \( s^{(a)} \) may be introduced as

\[
S^{(a)} = -\frac{1}{2m} e^{(a)}_{(b)(c)(d)} u^{(b)} S^{(c)(d)} ,
\]

(2.20)

where \( e^{(a)(b)(c)(d)} \) is the totally antisymmetric tensor with component \( e^{(0)(1)(2)(3)} = 1 \). As a result of particle orbit constraints, we find that the spin direction is always perpendicular to the equatorial plane of the black hole \([30]\). Therefore, the only nontrivial component of \( s^{(a)} \) is \( s^{(2)} = -s \). The particle’s spin will be parallel to the rotation of the black hole if \( s > 0 \) and antiparallel if \( s < 0 \). Accordingly, the components of spin tensor with non-zero coefficients can be described as follows:

\[
S^{(0)(1)} = -s p^{(3)} , \quad S^{(0)(3)} = s p^{(1)} , \quad S^{(1)(3)} = s p^{(0)} .
\]

(2.21)
Combining Eqs. (2.18), (2.19), and (2.21), then we can obtain the components of 4-momentum in the tetrad representation:

\[ p^{(0)} = \frac{N}{\sqrt{\mathcal{A} \Delta}} \left[ E \left( a^2 N^3/2 + N^{5/2} + a \right) \right. \]
\[ \left. + s \left( r^3 + 3 n^2 r + M (r^2 - n^2) \right) \right] \]
\[ - J \left( 2 n^2 r s + a N^{3/2} + M s (r^2 - n^2) \right), \] (2.22)

\[ p^{(3)} = N^2 \left( J \sqrt{\mathcal{N}} - E (a \sqrt{\mathcal{N}} + r s) \right), \] (2.23)

where

\[ \mathcal{N} = r^2 + n^2, \quad \mathcal{A} = N^3 - rs^2 (2 n^2 r + M (r^2 - n^2)). \] (2.24)

There also should be a normalization condition for the 4-momentum:

\[ p^{\mu} p_{\mu} = p^{(a)} p_{(a)} = \rho, \] (2.25)

where \( \rho = -m^2 \) corresponds to the massive particle and \( \rho = 0 \) corresponds to the massless particle. With regard to the previous definition of the specific 4-momentum \( u^{(a)} = p^{(a)}/m \), for the massive particles, we have

\[ u^{(1)} = \sigma \sqrt{E^2 - (u^{(3)})^2} - (u^{(3)})^2 - 1, \] (2.26)

where \( \sigma = \pm 1 \), the “+” denotes that the particle is outgoing, and the “−” denotes that the particle is ingoing, respectively. Combining Eqs. (2.1), (2.13), and (2.21), the relation between the 4-velocity and the specific 4-momentum in the tetrad representation is given by

\[ v^{(0)} = u^{(0)} \frac{1 + \zeta}{1 + \zeta (1 + 3 (u^{(3)})^2)}, \] (2.27)

\[ v^{(1)} = u^{(1)} \frac{1 + \zeta}{1 + \zeta (1 + 3 (u^{(3)})^2)}, \] (2.28)

\[ v^{(3)} = u^{(3)} \frac{1 - 2 \zeta}{1 + \zeta (1 + 3 (u^{(3)})^2)}, \] (2.29)

where

\[ \zeta = \frac{s^2 (a^2 + 3 n^2 (M - r) r - M r^3)}{N^3}. \] (2.30)

Hence, we obtain the components of 4-velocity in the frame representation as follows:

\[ \frac{dt}{d\tau} = v^{(0)} = \frac{a (2 + N) (1 + \zeta) u^{(0)}}{\sqrt{\mathcal{N} \Delta} \left[ 1 + \zeta \left( 1 + 3 (u^{(3)})^2 \right) \right]} \]
\[ + \frac{a (1 - 2 \zeta) u^{(3)}}{\sqrt{\mathcal{N} \left[ 1 + \zeta \left( 1 + 3 (u^{(3)})^2 \right) \right]}}, \] (2.31)

\[ \frac{d\phi}{d\tau} = v^{(1)} = \frac{\mathcal{A} (1 + \zeta) u^{(1)}}{\sqrt{\mathcal{N} \left[ 1 + \zeta \left( 1 + 3 (u^{(3)})^2 \right) \right]}}, \] (2.32)

\[ \frac{dr}{d\tau} = v^{(3)} = \frac{1}{\sqrt{\mathcal{N} \Delta} \left[ 1 + \zeta \left( 1 + 3 (u^{(3)})^2 \right) \right]} \]
\[ + \frac{a (1 + \zeta) u^{(0)}}{\sqrt{N \left[ 1 + \zeta \left( 1 + 3 (u^{(3)})^2 \right) \right]}}, \] (2.33)

In what follows, for the sake of convenience, we introduce the dimensionless variables as

\[ E = \frac{E}{m}, \quad J = \frac{J}{m}, \quad \zeta = \frac{s}{M}, \quad \bar{r} = \frac{r}{M}, \quad \bar{\tilde{r}} = \frac{\tilde{r}}{M}, \quad \bar{n} = \frac{n}{M}. \] (2.34)

Therefore,

\[ \bar{\Delta} = \frac{\Delta}{M^2} = \bar{r}^2 + \bar{\tilde{r}}^2 - 2 \bar{r} - \bar{n}^2, \quad \bar{N} = \frac{N}{M^2} = \bar{r}^2 + \bar{n}^2. \] (2.35)

Throughout the rest paper, unless otherwise indicated, we omit the tilde “∼” for brevity.

### 2.5 Constraints on the orbits

Since the particle’s collision position is closer to the event horizon, the energy extraction of the collision Penrose process is more efficient [17]. Therefore, our goal is to find an orbit of the spinning particle which can reach the horizon \( r_H = 1 + \sqrt{1 + n^2 - a^2} \). Following this line, the Eq. (2.32) must be real solutions. Combining this constraint with the Eqs. (2.23), (2.22) and (2.26), we have

\[ - N^2 \left[ E (a^2 \sqrt{N^3} + \sqrt{N} s (r^3 + 3 m^2 r + r^2 - n^2)) \right] \]
\[ - J \left( a \sqrt{N^3} + 2 m^2 r s + s (r^2 - n^2) \right) \]
\[ \geq \Delta \left[ N^2 + N^4 \left( J \sqrt{N} - E (a \sqrt{N} + r s) \right)^2 \right]. \] (2.37)

An impact parameter \( b \equiv J/E \) is introduced to describe the radial turning point (the position where particles bounce back) \( dr/d\tau = 0 \) of different orbits. Taking variable \( r \) in Eq. (2.37) to horizon radius \( r_H \), we can obtain the critical impact parameter \( b_{cr} \) as follows:

\[ b_{cr} = \frac{1}{a \sqrt{(r_H^2 + n^2)^3 + 2 m^2 r s + s (r_H^2 - n^2)}} \]
\[ \times \left[ a^2 \sqrt{(r_H^2 + n^2)^3 + \sqrt{(r_H^2 + n^2)^5}} \right] \]
\[ + a s (r_H^3 + 3 m^2 r_H + r_H^2 - n^2). \] (2.38)

Therefore, in order for particles to reach the horizon, condition \( b \geq b_{cr} \) must be satisfied. On the other hand, massive
Fig. 1 a The timelike condition region for the spin $s$ and energy $E$ with different values of $n$. b The comparison of the maximal and minimum value of spin $s$ with $s_c$ and $-s_c$ for the different values of $n$.

3 Collision of particles

In this section, we begin to discuss the collisional Penrose process that occurs near the horizon of an extreme KTN black hole in detail. In our model, particles 1 and 2 are free to fall into the ergosphere from infinity at rest with 4-momentum $p_1^\mu$ and $p_2^\mu$, respectively. After colliding near the horizon, the debris consists of particles 3 and 4. Particle 3 with 4-momentum $p_3^\mu$ escapes from the black hole to infinity, while particle 4 with 4-momentum $p_4^\mu$ is trapped by the black hole and falls into the horizon.

It is reasonable to assume that the spins and the 4-momentums of the system are conserved throughout the process, i.e.,

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu,$$

$$S_1^{\mu \nu} + S_2^{\mu \nu} = S_3^{\mu \nu} + S_4^{\mu \nu}.$$  

As mentioned above, the KTN black hole, like the general rotating black hole, also has a timelike Killing vector and an axial Killing vector. Their corresponding physical quantities should also be conserved. That is

$$E_1 + E_2 = E_3 + E_4,$$

$$J_1 + J_2 = J_3 + J_4.$$  

Therefore, we can obtain the conservation relation of the radial components of 4-momentum and the spin of particles during the collision as follows:

$$m_1s_1 + m_2s_2 = m_3s_3 + m_4s_4,$$

$$p_1^{(1)} + p_2^{(1)} = p_3^{(1)} + p_4^{(1)}.$$  

Due to the collision of particles taking place close to the horizon of extreme KTN black hole, so the position of collision in the radial direction can be set at $r_c = 1/(1 - \epsilon)$, where $\epsilon \to 0^+$. Combining this fact with the Eqs. (2.22), (2.23) and (2.26), we have

$$p_0^{(1)} = \frac{(1 + n^2) \left| J - 2\sqrt{1 + n^2}E \right|}{\epsilon (1 + n^2 - s)} + O(\epsilon^0).$$
As we know, the impact parameter \( b = \frac{\xi}{2} \) can be used to classify admissible orbits. Along the same line of [17,31], we assume that the orbit of particle 1 is critical \((J_1 = b_c, E_1)\), while the particle 2 is non-critical \((J_2 < b_c, E_2)\) that means the particle 2 must be ingoing \(\sigma_2 = -1\) [17], and the particle 3 is near-critical \((J_3 = b_c, E_3 + O(\epsilon))\). In this case, we have \(b_c = 2\sqrt{1 + n^2}\). Thus, the total angular momentum \(J\) and energy \(E\) are related in the following way:

\[
J_1 = 2\sqrt{1 + n^2} E_1, \quad (3.8)
\]

\[
J_2 = 2\sqrt{1 + n^2} E_2(1 + \xi), \quad (3.9)
\]

\[
J_3 = 2\sqrt{1 + n^2} E_3(1 + \alpha_3 \epsilon + \beta_3 \epsilon^2 + \cdots), \quad (3.10)
\]

where \(\xi < 0\) with \(\xi = O(\epsilon^0)\), \(\alpha_3\) and \(\beta_3\) are parameters of \(O(\epsilon^0)\). Using the above conservation laws, we can obtain

\[
J_4 = 2\sqrt{1 + n^2} E_4 \left(1 + \frac{E_2}{E_4} \xi + \cdots \right). \quad (3.11)
\]

Combining the above assumptions about orbits with the Eqs. (3.7) and (3.6), we have

\[
\left| J_2 - 2\sqrt{1 + n^2} E_2 \left(\frac{\sigma_4}{1 + n^2 - s_4} - \frac{\sigma_2}{1 + n^2 - s_2}\right)\right| = O(\epsilon^1). \quad (3.12)
\]

In order to balance the two sides of this equation, we find that \(\sigma_4 = \sigma_2\) and \(s_4 = s_2\). Thus, the conservation of angular momentum requires that \(s_1 = s_3\).

In what follows, along the same line of [31], three collision cases are considered. In the first case \(\text{MMM}\), particle 1, 2 and daughter particle 3, 4 are all massive particles; the second case \(\text{PMP}\) is Compton scattering, which means particle 1 and daughter particle 3 are massless (photon), while particle 2 and daughter particle 4 is massive; the third case \(\text{MPM}\) is inverse Compton scattering, i.e., particle 1 and daughter particle 3 is massive as well as particle 2 and daughter particle 4 is massless. Here, the letters \(\text{M}\) and \(\text{P}\) denote a massive particle and a massless particle, respectively. There is no need to include particle 4 in the abbreviation since it does not contribute to the energy extraction efficiency.

### 3.1 Maximum efficiency in case MMM

In this case, we assume that \(m_1 = m_2 = m_3 = m_4 = m\), then the conservation laws can be reduced as follows:

\[
E_1 + E_2 = E_3 + E_4, \quad (3.13)
\]

\[
J_1 + J_2 = J_3 + J_4, \quad (3.14)
\]

\[
s_1 + s_2 = s_3 + s_4, \quad (3.15)
\]

\[
u_1^{(1)} + u_2^{(1)} = u_3^{(1)} + u_4^{(1)}. \quad (3.16)
\]

Using the Eqs. (2.22), (2.23) and (2.26) to calculate the radial specific 4-momentum of particles, and combine it with the relations between total angular momentum \(J\) and energy \(E\): Eqs. (3.8)–(3.11), we can get the results into the power series forms of \(\epsilon\) as follows [31]:

\[
u_1^{(1)} = \lambda_{11} \epsilon^{-1} + \lambda_{12} \epsilon^0 + \lambda_{13} \epsilon^1 + \cdots, \quad (3.17)
\]

\[
u_2^{(1)} = \lambda_{21} \epsilon^{-1} + \lambda_{22} \epsilon^0 + \lambda_{23} \epsilon^1 + \cdots, \quad (3.18)
\]

\[
u_3^{(1)} = \lambda_{31} \epsilon^{-1} + \lambda_{32} \epsilon^0 + \lambda_{33} \epsilon^1 + \cdots, \quad (3.19)
\]

\[
u_4^{(1)} = \lambda_{41} \epsilon^{-1} + \lambda_{42} \epsilon^0 + \lambda_{43} \epsilon^1 + \cdots. \quad (3.20)
\]

We have obtained the constraints on \(\sigma_4, \sigma_2\) and the spin angular momentum \(s\) through the leading order equation of \(\epsilon^{-1}\) i.e., the Eqs. (3.7) and (3.12). Now, we continue to calculate the next leading order of \(\epsilon^0\), that is

\[
\lambda_{12} + \lambda_{22} = \lambda_{32} + \lambda_{42}. \quad (3.21)
\]

It is convenient for us to let \(N = 1 + n^2\), then

\[
\lambda_{12} = \sigma_1 \sqrt{-1 + \frac{E_2^2 k(s_1, 0, n)}{(N^2 - s_1^2)^2}}, \quad (3.22)
\]

\[
\lambda_{22} = -\frac{E_2 \sqrt{N} h(s_2, \xi, n)}{(N - s_2)(N + s_2)}, \quad (3.23)
\]

\[
\lambda_{32} = \sigma_3 \sqrt{-1 + \frac{E_2^2 k(s_1, \alpha_3, n)}{(N^2 - s_2^2)^2}}, \quad (3.24)
\]

\[
\lambda_{42} = -\frac{\sqrt{N}}{(N - s_2)(N + s_2)} \left(E_1 h(s_2, 0, n) + E_2 h(s_2, \xi, n) + E_3 (N - s_2) f(s_2, \alpha_3, n)\right), \quad (3.25)
\]

where

\[
k(s_1, \alpha_3, n) = N \left(N^2(3 - n^2 - 8N\alpha_3 + 4N^2\alpha_3^2)
+ s_1^2(n^2(-3 + n^2) + 4\alpha_3(-1 + n^4)
+ 4N^2\alpha_3^2) + 2s_1N(3 - n^2 + 4N^2\alpha_3^2)
+ 2\alpha_3(-3 - 2n^2 + n^4)\right), \quad (3.26)
\]

\[
h(s_2, \xi, n) = 2N^2(1 + \xi(2 + n^2)) - s_2N(1 - 2\xi
+ n^2(1 + 4\xi)) + s_2^2(-1 + n^2(-1 - 4\xi)), \quad (3.27)
\]

\[
f(s_2, \alpha_3, n) = 2\alpha_3N(s_2 + N) - 2N - s_2(1 - n^2). \quad (3.28)
\]

Thus, the Eq. (3.21) can be simplified as follows:

\[
\lambda_{12} + \lambda_{22} = \lambda_{32} + \lambda_{42}. \quad (3.21)
\]
Fig. 2 The contour maps of $E_3$ in terms of $\alpha_3$ and $s_2$. a $n = 0$, b $n = 0.3$ and c $n = 0.5$ show that energy $E_3$ reaches its maximum value when both $s_2$ and $\alpha_3$ are sufficiently small.

Fig. 3 The contour maps of $E_3$ in terms of $s_1$ and $s_2$. The timelike condition for the particle 3 is satisfied in the green shaded region. The maximum value of $E_3$ for the spinning case is marked at the red point, and for the non-spinning case is marked at the orange point ($s_1 = 0, s_2 = 0$). a When $n = 0$, which corresponds to the Kerr case, for the spinning case, $E_{3\text{max}} \approx 30.02$ at ($s_1 = 0.01378, s_2 = -0.2679$); for the non-spinning case, $E_{3\text{max}} \approx 12.66$. b When $n = 0.3$, for the spinning case, $E_{3\text{max}} \approx 21.78$ at ($s_1 = 0.02184, s_2 = -0.2623$); for the non-spinning case, $E_{3\text{max}} \approx 11.29$. c When $n = 0.5$, for the spinning case, $E_{3\text{max}} \approx 14.95$ at ($s_1 = 0.04065, s_2 = -0.2433$); for the non-spinning case, $E_{3\text{max}} \approx 9.319$.

Fig. 4 a The admissible region of the $\xi$ and spin $s$ with different values of NUT charge $n$ when $E_2 = 1$. b The relation between $\xi$ and $\beta_3$ for different values of $n$ when $E_2 = 1$. c The relation between maximum efficiency $\eta_{\text{max}} = E_{3\text{max}}/2$ and NUT charge $n$ for the spinning and non-spinning cases where

$$A = \frac{N^2 f(s_2, \alpha_3, n)^2}{(N^2 - s_2^2)^2} - \frac{k(s_1, \alpha_3, n)}{(N^2 - s_1^2)^2},$$

$$B = \frac{E_1 N^2 f(s_2, \alpha_3, n) f(s_2, 0, n)}{(N^2 - s_2^2)^2} - \frac{\sigma_1 \sqrt{N} g(s_1, E_1, n) f(s_2, \alpha_3, n)}{(N^2 - s_2^2)(N^2 - s_1^2)},$$
\[ C = 1 + \frac{E_0^2 N f(s_2, 0, n)^2}{(N^2 - s_0^2)^2} + \frac{g^2(s_1, E_1, n)}{(N^2 - s_1^2)^2} - \frac{2\sigma_1 E_1 \sqrt{N} f(s_2, 0, n) g(s_1, E_1, n)}{(N^2 - s_2^2)(N^2 - s_1^2)}, \] (3.30)

with

\[ g(s_1, E_1, n) = \sqrt{E_1^2 k(s_1, 0, n) - (N^2 - s_1^2)^2}. \] (3.31)

Given the results above, the \( E_3 \) in the Eq. (3.29) has two roots, and in light of the following definition of energy extraction efficiency:

\[ \eta = \frac{E_3}{E_1 + E_2}. \] (3.32)

An increase in \( E_3 \) can improve the efficiency. Therefore, we may choose the larger root \( E_3 = E_{3+}. \) Furthermore, we can conclude from Eq. (3.30) that the value of \( E_3 \) is not affected by \( \sigma_3. \) Here, referring to the conclusions discussed in [17], we take \( \sigma_3 = -1. \) In conclusion, we get the expression of \( E_3 \) as

\[ E_{3+} = \frac{\sqrt{B^2 - AC + B}}{A}, \] (3.33)

which is decoupled with the parameters \( \xi, \beta_3 \) and the energy \( E_2. \) Next, we consider the leading order of \( \epsilon^1: \)

\[ \lambda_{13} + \lambda_{23} = \lambda_{33} + \lambda_{43}, \] (3.34)

where

\[ \lambda_{13} = \frac{E_0^2 k_1(s_1, n)}{(N^2 - s_1^2)^2 g(s_1, E_1, n)}, \] (3.35)

\[ \lambda_{23} = \frac{(N - s_2)^4(N + s_2)^2 + E_0^2 N h_1(s_2, \xi, n)}{4\xi E_2 \sqrt{N^3}(s_2 - N)^3(N + s_2)^2}, \] (3.36)

\[ \lambda_{33} = \frac{E_0^2 \lambda_{32}}{-4(N^2 - s_1^2)^2 - E_0^2 N(N - s_1)^2 + E_0^2 N^2 f(s_1, \alpha_3, n)^2} \times \left[ \frac{N - s_1)^2 k_1(s_1, \alpha_3, n)}{N + s_1} + \frac{E_2^2 k_{12}(s_1, \alpha_3, n) f^2(s_1, \alpha_3, n)}{N^2 - s_2^2} \times \frac{N f(s_1, \alpha_3, n) k_{13}(s_1, \alpha_3, \beta_3, n)}{N^2 - s_2^2} \right], \] (3.37)

\[ \lambda_{43} = \frac{1}{4E_2^2 \sqrt{N^3}(N - s_2)^3(N + s_2)^2} \times \left[ \frac{-E_2^2 N h_1(s_2, \xi, n) - (N - s_2)^4}{(N - s_2)^4}) \times \left( N^2(E_0^2 + E_0^2) + (N + s_2)^2 \times -2E_4 N (N - s_2) \times (E_2 h_{15}(s_2, 0, 0, \xi, n) - E_3 N(N - s_2)^2) \times + E_2 E_0 N (N - s_2) h_{15}(s_2, \alpha_3, \beta_3, \xi, n) \right]. \] (3.38)

where \( k_1(s_1, n_1), k_1(s_2, \xi, n) \) and so on are shown in the appendix. After simplifying, the energy \( E_2 \) can be set as

\[ E_2 = \frac{\sqrt{N} (N - s_2)(E_1 - E_3)^2}{P_1}, \] (3.39)

where

\[ P_1 = 4 \xi (N + s_2)^2 (P_{11} + P_{12} + P_{13}) \] (3.40)

with \( P_{11}, P_{12} \) and \( P_{13}, \) which are also shown in the appendix.

In the previous discussion, we have determined the directions of the particles 2, 3, and 4, that is \( \sigma_2 = \sigma_3 = \sigma_4 = -1. \) Here, along the same line of [17], we choose that the particle 1 is outgoing when it collides with the particle 2, i.e., \( \sigma_1 = 1. \)

The definition equation (3.32) of the energy extraction efficiency states that, in addition to increasing the energy \( E_3 \) of the escaped particle 3, decrease the energy \( E_1 \) and \( E_2 \) of the plunging particles 1, 2 can also improve the energy extraction efficiency. Since we consider the two falling particles are from infinity at rest, that means \( E_1 \geq 1 \) and \( E_2 \geq 1. \) Without loss of generality, we normalize the energy \( E_1 \) as \( E_1 = 1. \)

We have known that the \( E_3 \) is independent of the parameters \( \xi, \beta_3 \) and \( E_2. \) And as shown in Fig. 1a, larger energy always requires smaller spin angular momentum. So, we assume \( s_1 = 0 \) to find the relation between \( E_3 \) and \( \alpha_3, \) which is shown in Fig. 2. From this figure, we find that the maximum value of energy \( E_3 \) will only be obtained when \( \alpha_3 \) tends to zero. Hence, we choose \( \alpha_3 = 0^+ \) in the following discussion. Accordingly, the contour maps of energy \( E_3 \) with the spin \( s_1 \) and \( s_2 \) are shown in Fig. 3, and we also consider the case of spinless by vanishing the spin. From this figure, we can see that, in both cases, the maximum value of \( E_3 \) will decrease as \( n \) increases. No matter what value \( n \) takes, the results obtained in the spinning case are always greater than those obtained in the spinless case.

The energy \( E_1 \) has been normalized. Now we consider whether the energy \( E_2 \) can also be normalized as \( E_2 = 1. \) It may be noted that \( E_2 \) is a function of the parameters \( \beta_3 \) and \( \xi \) while \( E_3 \) is not. Combining the Eqs. (2.42) and (3.9), we can find the constraints of \( \xi \) for different values of \( s \) and \( n \) when we set \( E_1 = 1, \) which is shown in the Fig. 4a. Furthermore, we substitute the values of the parameters \( (\alpha_3, \sigma_1, \text{etc.)} \) that can reach the maximum efficiency into the Eq. (3.39), then we can get the relation between \( \beta_3 \) and \( \xi \) for different values of \( n \) which gives \( E_2 = 1, \) which is shown in Fig. 4b. Therefore, we could normalize the \( E_2 \) within the above constraints.

Finally, we can obtain the relation between the maximum efficiency \( \eta_{max} \) and the NUT charge \( n \) shown in the Fig. 4c. From this figure, we find that as NUT charge \( n \) increases, the maximum efficiency of energy extraction \( \eta_{max} \) decreases in both spinning and non-spinning cases, and the spin always plays a positive role in efficiency. Once \( n \) reaches zero, the result returns to that of Kerr spacetime [17].
massless particles are non-spinning, and assume a massless particle, known as Compton scattering. Due to the In this case, we consider a massive particle colliding with a

3.2 Maximum efficiency in case PMP

In this case, we consider a massive particle colliding with a massless particle, known as Compton scattering. Due to the massless particles are non-spinning, and assume \( m_2 = m_4 = m \), the conservation laws which described by Eqs. (3.15), (3.16) can be reduced as

\[
\begin{align*}
    s_2 &= s_4, \\
    p_1^{(1)} + p_2^{(1)} &= p_3^{(1)} + p_4^{(1)}. 
\end{align*}
\]

\[
\mathcal{O} = \frac{s_2(-1 + n^2)N + \sigma_1 s_2^2 \sqrt{3 - n^2} - N^2(2 + \sqrt{3} - n^2)}{s_2N(-1 + n^2 + 2\alpha_3 N) + \sigma_3 s_2^2 \sqrt{N^2 - 3k(0, \alpha_3, n)} - 2N^2 + 2\alpha_3 N^3 - \sigma_3 \sqrt{Nk(0, \alpha_3, n)}},
\]

As indicated by the Eq. (2.25), in the extreme KTN spacetime, the radial component of the normalized 4-momentum for a massless particle can be written as

\[
p^{(1)} = \sigma \sqrt{-4EJ(n^2 + r)\sqrt{1 + n^2} + J^2(n^2 - r(r - 2)) + E^2(4n^4 + n^2(3 + 2r + 3r^2) + r(2 + r + r^3))} / (r - 1)^2(n^2 + r^2). \]  

Then the coefficients of \( \epsilon^0 \) and \( \epsilon^1 \) for particle 1 and particle 3 can be rewritten as follows:

\[
\begin{align*}
    \lambda_{12} &= \sigma_1 E_1 \sqrt{3 - n^2} / N, \ (3.44) \\
    \lambda_{13} &= \sigma_1 E_1 \sqrt{3n^2 - 1} / \sqrt{N^3(3 - n^2)}. \ (3.45)
\end{align*}
\]

\[
E_2 = \frac{N^2(s_2 - N)(E_1 - E_3)^2}{(N + s_2)^2 P_2}, \]

where \( P_2 \) is shown in the Appendix.

From the Eq. (3.48), in the case of Compton scattering, the energy extraction efficiency \( \eta \) will be written as
\[ \eta = \frac{E_3}{E_1 + E_2} = \frac{\mathcal{O}}{1 + E_2 E_1}. \]  

(3.51)

Here, for the two colliding particles falling from infinity at rest, the energy of massive particle 2 is constrained as \( E_2 \geq 1 \), while the energy of massless particle 1 is constrained as \( E_1 \geq 0 \). Since the result of \( E_2/E_1 \) depends on the parameters \( \alpha_3, \beta_3, \xi, s_2, n, \) and is not affected by \( E_1 \) and \( E_2 \) itself. From the Eq. (3.50), we can get the asymptotic behavior of the \( E_2/E_1 \) as follows:

\[ E_2 \approx \frac{(N - s_2)^3(1 - \mathcal{O}^2)}{8\beta_3 \mathcal{O} N (s_2^3 + s_2^2 N - s_2 N^2 + s_2^2 - N^3)}. \]  

(3.52)

Since the particle 2 is massive and its orbit is non-critical \((J < b, \xi, E_2)\), the constraints of \( \xi \) and \( \beta_3 \), which have been discussed in the case MMM, can be directly used. Considering that \( \mathcal{O} \neq 0 \), and under the assumption that the denominator of Eq. (3.52) is not equal to zero. So that make \( \beta_3 \to -\infty \) can result in \( \beta_3 \xi \to -\infty \), i.e., \( E_2/E_1 \to 0 \). Therefore, we can obtain the maximum efficiency \( \eta_{max} = \mathcal{O}_{max} \), which is shown in Fig. 5. From this figure, we can see that the increasing \( n \) can decrease the maximum efficiency \( \eta_{max} \) in both spinning and non-spinning cases, and the efficiency of the spinning case is always larger than the non-spinning case, which is similar to the case MMM.

### 3.3 Maximum efficiency in case MPM

The last case is the so-called inverse Compton scattering. Here, we assume that particle 1 is a massive particle with a spin, whereas particle 2 is a massless particle without a spin. So, the conservation laws can be rewritten as follows:

\[ s_1 = s_3, \]

\[ p_1^{(1)} + p_2^{(1)} = p_3^{(1)} + p_4^{(1)}. \]  

(3.53)

(3.54)

The radial components of the 4-momentum of massless particles has already been given in Eq. (3.43). Thus, the coefficients of \( \epsilon^0 \) and \( \epsilon^1 \) for particle 2 and particle 4 simplify as follows:

\[ \lambda_{22} = \frac{2E_2(1 + 2\xi + n^2\xi)}{\sqrt{N}}, \]

\[ \lambda_{23} = \frac{-E_2(N + 8n^2(\xi + \xi^2) - 4\xi^2)}{4\xi \sqrt{N}^3}, \]

\[ \lambda_{42} = \frac{-2}{\sqrt{N}}(E_1 + E_2 - E_3 + 2\xi E_2 + n^2\xi E_2 + E_3 \alpha_3 + n^2 \alpha_3 E_3), \]

\[ \lambda_{43} = \frac{-1}{4\xi E_2 \sqrt{N}^3}(E_2^2(1 - 4\xi^2 + n^2(1 + 8\xi + 8\xi^2)) + E_2^2 N + 2E_1(E_2(N + 4\xi n^2) - E_3 N + NE_2^3 - 2E_2 E_3 N^{-3}N_{14}(\alpha_3, \beta_3, \xi, n))\].  

(3.58)

The radial components of the 4-momentum of massive particles is identical to those of the case MMM. So, the expressions of \( E_3 \) and \( E_2 \) can be set as

\[ E_3 = \frac{\sqrt{B^2 - AC} + B}{A}, \]

\[ E_2 = \frac{\sqrt{N(N - s_2)(E_1 - E_3)^2}}{\mathcal{P}_1}, \]

(3.59)

(3.60)

where \( A, B, C \) and \( \mathcal{P}_1 \) are given by Eqs. (3.30) and (3.40) with \( s_2 = 0 \).

Similarly to the case PMP, the particle 1 and particle 2 are also falling from infinity at rest, which means the energy of massive particle 1 is constrained by \( E_1 \geq 1 \), and the energy of massless particle 2 is constrained by \( E_2 \geq 0 \). After normalizing the energy \( E_1 \) as \( E_1 = 1 \), as we did previously, the contour map of \( E_3 \) in terms of \( \alpha_3 \) and \( s_1 \) for different values of \( n \) has been shown in Fig. 6.

It is necessary to consider whether the energy of particle 2 can reach zero to achieve maximum energy extraction efficiency. Rewriting \( \mathcal{P}_1 \) in Eq. (3.60) as asymptotic form:

\[ \mathcal{P}_1 \mid_{s_2=0} \approx 8\xi \beta_3 E_3 N^2 \left( 1 - \frac{2(1 - \alpha_3 N)}{\sqrt{3 - n^2 - 8\alpha_3 N + 4\alpha_3^2 N^2}} \right). \]  

(3.61)

From the Eqs. (3.61) and (3.60), we can find that the \( \beta_3 \xi \to +\infty \) can lead to the \( E_2 \to 0 \). Combining the Eqs. (2.42) and (3.9), we have

\[ -\frac{\mathcal{H}(E, s, n)}{2\sqrt{N}} + \frac{s - N}{2N} < \xi < \frac{\mathcal{H}(E, s, n)}{2\sqrt{N}} + \frac{s - N}{2N}. \]  

(3.62)

with

\[ \mathcal{H}(E, s, n) = \frac{(1 + 2n^2 + n^4 - s^2)(1 + n^4 + 2n^2 + n^2 s^2 - s^2)}{s E \sqrt{3(1 - n^4)}(2 + 4n^2 + 2n^2 s^2 - n^2 s^2)}. \]

(3.63)

Since the particle 2 is massless with non-spinning, and it has been realized that the \( \xi < 0 \), the Eq. (3.62) can be reduced as \(-\infty < \xi < 0 \). The Eq. (3.10) shows that the \( \beta_3 \) is arbitrary as long as \( \alpha_3 > 0 \). Thus, \( \beta_3 \xi \to +\infty \) can be realized, i.e., \( E_2 \to 0 \) is possible.

Therefore, the maximum efficiency is \( \eta_{max} = E_{3max}/2 \). Similar to the case MMM and the case PMP, the maximum
efficiency $\eta_{\text{max}}$, in this case, also decreases with the increase of the NUT charge $n$.

4 Conclusion

In this paper, we have studied the collisional Penrose process with spinning and non-spinning particles in the extreme Kerr–Taub–NUT spacetime and investigated the effects of NUT charge on the maximum efficiency. We assume that the collision occurs as close as possible to the horizon. One of the daughter particles with negative energy falls into the horizon, and the other with positive energy escapes to infinity.

We specifically consider three cases. Firstly, we focus on a collision between two massive particles. One of the daughter particles falls into the black hole, and the other escapes to infinity. Secondly, we consider the Compton scattering, which occurs when a massless particle 1 collides with a massive particle 2, resulting in a massless particle 3 and a massive particle 4. The massless daughter particle 3 escapes to infinity with higher energy than the sum of the original particles, while the massive daughter particle 4 falls into the horizon. Lastly, there is the inverse Compton scattering, which means a massive particle 1 collides with a massless particle 2 to result in a massive particle 3 and a massless particle 4, from which the massive daughter particle 3 with a greater energy escapes from the black hole to infinity and the massless daughter particle 4 falls into the horizon.

After comparing the non-spinning cases with the spinning cases, it can be found that a significant improvement in the efficiency of the CPP can be achieved when the spin is taken into account in the extreme KTN spacetime. This conclusion is in agreement with the findings of previous research on CPP with spinning particles in Kerr and Kerr–Sen spacetimes [16,17,31]. Moreover, when the NUT charge $n$ increases, the maximum efficiency of energy extraction $\eta_{\text{max}}$ decreases in all of the above cases. However, in the extreme KTN spacetime, the maximal center-of-mass energy for the collision of two spinless particles is proportional to the NUT charge [27]. It is also worth pointing out that, for the original Penrose process with spinless particles in the KTN spacetime, its relation between energy extraction efficiency and the NUT charge is consistent with the above CPP case. But for the extreme KTN black hole, its extracted energy from the original Penrose process is independent of the NUT charge [26].

It should be noted that when the value of the NUT charge is equal to zero, the results regarding the maximum efficiency of both spinning and non-spinning cases are consistent with those obtained within the Kerr spacetime [17] since the Kerr–Taub–NUT metric returns to the general Kerr metric.

In conclusion, decreasing the NUT charge $n$ will always increase the maximum efficiency of the collisional Penrose process in the extreme Kerr–Taub–NUT spacetime.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: Since what I do is purely theoretical calculation, there is no data.]

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Listed below are the formulas mentioned in this paper:

\[
\begin{align*}
\mathcal{P}_3 & = \frac{\sigma_3 E_3^2 \lambda_{32} (N^2 - s_1^2)^2}{(N^2 - s_1^2)^2 + E_3^2 N^2 (N - s_1)^2 - E_3^2 N f(s_1, \alpha_3, n)^2} \times \left(\frac{k_{11}(s_1, \alpha_3, n)}{(N - s_1)(N + s_1)^3} + \frac{k_{12}(s_1, n) f(s_1, \alpha_3, n)}{(N^2 - s_1^2)^3} + \frac{N f(s_1, \alpha_3, n) k_{13}(s_1, \alpha_3, \beta_3, n)}{(N^2 - s_1^2)^2}\right), \\
\mathcal{P}_2 & = \frac{2E_1 N h_15(s_2, 0, 0, \xi, n) - 2E_3 N h_15(s_2, \alpha_3, \beta_3, \xi, n)}{(N^2 - s_1^2)^2} + \frac{4 \sigma_1 E_1 \xi (1 - 3n^2)}{\sqrt{3 - n^2}} + \frac{\sigma_3 E_3 \xi}{\sqrt{3 - n^2 - 8 \sigma_3 N^2 + 4 \alpha_3 N^2} - 4 \alpha_3 N^2 (2 + n^2) + 4 \beta_3 N^2)} \times (1 - 3n^2 + 4 \alpha_3 N^2 (2 + n^2) + 4 \beta_3 N^2 - 4 \alpha_3 N^2 (2 + \beta_3 N^2)).
\end{align*}
\]

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