A stochastic control approach to robust duality in utility maximization

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Abstract

A celebrated financial application of convex duality theory gives an explicit relation between the following two quantities:

(i) The optimal terminal wealth $X^*(T) := X_{\varphi^*}(T)$ of the problem to maximize the expected $U$-utility of the terminal wealth $X_{\varphi}(T)$ generated by admissible portfolios $\varphi(t); 0 \leq t \leq T$ in a market with the risky asset price process modeled as a semimartingale;

(ii) The optimal scenario $\frac{dQ^*}{dP}$ of the dual problem to minimize the expected $V$-value of $\frac{dQ}{dP}$ over a family of equivalent local martingale measures $Q$, where $V$ is the convex conjugate function of the concave function $U$.

In this paper we consider markets modeled by Itô-Lévy processes. In the first part we extend the above relation in this setting, based on the maximum principle in stochastic control theory. We prove in particular that the optimal adjoint process for the primal problem coincides with the optimal density process, and that the optimal adjoint process for the dual problem coincides with the optimal wealth process. We get moreover an explicit relation between the optimal portfolio $\varphi^*$ and the optimal measure $Q^*$. We also obtain that the existence of an optimal scenario is equivalent to the replicability of a related $T$-claim. In the second part we present robust (model uncertainty) versions of the optimization problems in (i) and (ii), and we prove a relation between them. In particular, we show how to get from the solution of one of the problems to the other. We illustrate the results with explicit examples.

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1 Introduction

The purpose of this paper is to use stochastic control theory to obtain new results on the connections between the primal, utility maximization portfolio problem and its convex dual, both in the non-robust and the robust (worst case scenario/multiple-priors) setting. This approach allows us to get more detailed information about the connection between the primal and the dual problem. In particular, we show that the optimal wealth process of the primal problem coincides with the optimal adjoint process for the dual problem. This generalizes results that have been obtained earlier by using convex duality theory.

First, let us briefly recall the main results from the duality method in utility maximization, as presented in e.g. [9]: Let $U: [0, \infty) \to \mathbb{R}$ be a given utility function, assumed to be strictly increasing, strictly concave, continuously differentiable ($C^1$) and satisfying the Inada conditions:

$$U'(0) = \lim_{x \to 0^+} U'(x) = \infty$$
$$U'(\infty) = \lim_{x \to \infty} U'(x) = 0.$$ 

Let $S(t) = S(t, \omega); 0 \leq t \leq T, \omega \in \Omega$, represent the discounted unit price of a risky asset at time $t$ in a financial market. We assume that $S(t)$ is a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$. Let $\varphi(t)$ be an $\mathbb{F}$-predictable portfolio process, giving the number of units held of the risky asset at time $t$. If $\varphi(t)$ is self-financing, the corresponding wealth process $X(t) := X_\varphi^x(t)$ is given by

$$X(t) = x + \int_0^t \varphi(s)dS(s); 0 \leq t \leq T,$$

where $T \geq 0$ is a fixed terminal time and $x > 0$ is the initial value of the wealth. We say that $\varphi$ is admissible and write $\varphi \in \mathcal{A}$ if the integral in (1.1) converges and

$$X_\varphi^x(t) > 0 \text{ for all } t \in [0, T], \text{ a.s.}$$

The classical optimal portfolio problem is to find $\varphi^* \in \mathcal{A}$ (called an optimal portfolio) such that

$$u(x) := \sup_{\varphi \in \mathcal{A}} E[U(X_\varphi^x(T))] = E[U(X_{\varphi^*}^x(T))].$$

The duality approach to this problem is as follows: Let

$$V(y) := \sup_{x > 0} \{U(x) - xy\}; y > 0$$
be the convex conjugate function of $U$. Then it is well-known that $V$ is strictly convex, decreasing, $C^1$ and satisfies

$$V'(0) = -\infty, \ V'(\infty) = 0, \ V(0) = U(\infty) \text{ and } V(\infty) = U(0).$$  \hspace{1cm} (1.5) \hspace{1cm} \{eq1.5\}

Moreover,

$$U(x) = \inf_{y>0}\{V(y) + xy\}; \ x > 0,$$  \hspace{1cm} (1.6) \hspace{1cm} \{eq1.6\}

and

$$U'(x) = y \iff x = -V'(y).$$  \hspace{1cm} (1.7) \hspace{1cm} \{eq1.7\}

Let $\mathcal{M}$ be the set of probability measures $Q$ which are equivalent local martingale measures (ELMM), in the sense that $Q$ is equivalent to $P$ and $S(t)$ is a local martingale with respect to $Q$. We assume that $\mathcal{M} \neq \emptyset$, which means absence of arbitrage opportunities on the financial market. The dual problem to (1.3) is for given $y > 0$ to find $Q^* \in \mathcal{M}$ (called an optimal scenario measure) such that

$$v(y) := \inf_{Q \in \mathcal{M}} E \left[ V \left( y \frac{dQ}{dP} \right) \right] = E \left[ V \left( y \frac{dQ^*}{dP} \right) \right].$$  \hspace{1cm} (1.8) \hspace{1cm} \{eq1.8\}

One of the main results in [9] is that, under some conditions, $\varphi^*$ and $Q^*$ both exist and they are related by

$$U'(X^x\varphi^*(T)) = y \frac{dQ^*}{dP} \quad \text{with} \quad y = u'(x)$$  \hspace{1cm} (1.9) \hspace{1cm} \{eq1.9\}

i.e.

$$X^x\varphi^*(T) = -V' \left( y \frac{dQ^*}{dP} \right) \quad \text{with} \quad x = -v'(y).$$  \hspace{1cm} (1.10) \hspace{1cm} \{eq1.10\}

In this paper we will give a new proof of a result of this type by using stochastic control theory. We will work in the slightly more special market setting with a risky asset price $S(t)$ described by an Itô-Lévy process. This enables us to use the machinery of the maximum principle and backward stochastic differential equations (BSDE) driven by Brownian motion $B(t)$ and a compensated Poisson random measure $\tilde{N}(dt, d\zeta); \ t \geq 0; \ \zeta \in \mathbb{R}_0 := \mathbb{R}\setminus\{0\}$. (We refer to e.g. [14] for more information about the maximum principle). This approach gives a relation between the optimal scenario in the dual formulation and the optimal portfolio in the primal formulation. This is shown in Section 3 (see Theorem 3.1). We identify the solution of the primal problem with the solution of a Forward-Backward SDE (FBSDE) with constraints, and similarly with the dual problem. We prove moreover that the optimal adjoint process for the primal problem coincides with the optimal density process, and that the optimal adjoint process for the dual problem coincides with the optimal wealth process. As a step on the way, we prove in Section 2 a result of independent interest, namely that the existence of an optimal scenario is equivalent to the replicability of a related $T$-claim.

We then extend the discussion to robust (model uncertainty) optimal portfolio problems (Section 4). More precisely, we formulate robust versions of the primal problem (1.3) and of the dual problem (1.8) and we show explicitly how to get from the solution of one to the solution of the other.
This paper addresses duality of robust utility maximization problems entirely by means of stochastic control methods, but there are several papers of related interest based on convex duality methods, see e.g. the survey paper [5] and the references therein. We also refer the reader to [18] where the author uses convex duality to study utility maximization under model uncertainty (multiple priority) and obtains a BSDE characterization of the optimal wealth process in markets driven by Brownian motion. In [7], a robust dual characterization of the robust primal utility maximization problem is obtained by convex duality methods. The dual formulation obtained is similar to ours, but there is no BSDE connection.

2 Optimal portfolio, optimal scenario and replicability

We now specialize the setting described in Section 1 as follows:

Suppose the financial market has a risk free asset with unit price \( S_0(t) = 1 \) for all \( t \) and a risky asset with price \( S(t) \) given by

\[
\begin{align*}
    dS(t) &= S(t^-) \left( b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right) ; \ 0 \leq t \leq T \\
    S(0) &> 0
\end{align*}
\]  

(2.1) \{eq2.1\}

where \( b(t), \sigma(t) \) and \( \gamma(t, \zeta) \) are predictable processes satisfying \( \gamma > -1 \) and

\[
E \left[ \int_0^T \left\{ |b(t)| + \sigma^2(t) + \int_{\mathbb{R}} \gamma^2(t, \zeta)\nu(d\zeta) \right\} dt \right] < \infty.
\]  

(2.2) \{eq2.2\}

Here \( B(t) \) and \( \tilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)dt \) is a Brownian motion and an independent compensated Poisson random measure, respectively, on a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}, P) \) satisfying the usual conditions, \( P \) is a reference probability measure and \( \nu \) is the Lévy measure of \( N \).

Let \( \varphi(t) \) be a self financing portfolio and let \( X(t) := X^x_{\varphi}(t) \) be the corresponding wealth process given by

\[
\begin{align*}
    dX(t) &= \varphi(t)S(t^-) \left[ b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right] ; \ 0 \leq t \leq T \\
    X(0) &= x > 0.
\end{align*}
\]  

(2.3) \{eq2.3\}

**Definition 2.1 (Admissible Portfolios)** Let \( \varphi \) be an \( \mathbb{F} \)-predictable, \( S \)-integrable process. We say that \( \varphi \) is admissible if

\[
X^x_{\varphi}(t) > 0 \text{ for all } t \in [0, T], \text{ a.s.}
\]

\[
E \left[ \int_0^T \varphi(t)^2S(t)^2 \left\{ b(t)^2 + \sigma^2(t) + \int_{\mathbb{R}} \gamma^2(t, \zeta)\nu(d\zeta) \right\} dt \right] < \infty,
\]
there exists \( \epsilon > 0 \) such that

\[
E\left[ \int_0^T |X(t)|^{2+\epsilon} dt \right] < \infty \tag{2.4} \quad \text{[eq2.A1]}
\]

and

\[
E[U'(X(T))^{2+\epsilon}] < \infty. \tag{2.5} \quad \text{[eq2.A2]}
\]

We denote by \( \mathcal{A} \) the set of admissible portfolios. Conditions (2.4), (2.5) are needed for the application of the maximum principles. See Appendix A.

As in (1.3), for given \( x > 0 \), we want to find \( \varphi^* \in \mathcal{A} \) such that

\[
u(x) := \sup_{\varphi \in \mathcal{A}} E[U(X_{\varphi^*}(T))] = E[U(X_{\varphi^*}(T))]. \tag{2.6} \quad \text{[eq2.4]}
\]

The set \( \mathcal{M} \) of equivalent local martingale measures (ELMM) is represented now by means of the family of positive measures \( Q = Q_\theta \) of the form

\[
dQ_\theta(\omega) = G_\theta(T)dP(\omega) \text{ on } \mathcal{F}_T, \tag{2.7} \quad \text{[eq2.5]}
\]

where

\[
\begin{align*}
    dG_\theta(t) &= G_\theta(t^-) \left[ \theta_0(t)dB(t) + \int_{\mathbb{R}} \theta_1(t, \zeta) \tilde{N}(dt, d\zeta) \right]; \quad 0 \leq t \leq T \\
    G_\theta(0) &= y > 0,
\end{align*} \tag{2.8} \quad \text{[eq2.6]}
\]

and \( \theta = (\theta_0, \theta_1) \) is a predictable process satisfying the conditions

\[
E \left[ \int_0^T \left\{ \theta_0^2(t) + \int_{\mathbb{R}} \theta_1^2(t, \zeta) \nu(d\zeta) \right\} dt \right] < \infty, \quad \theta_1(t, \zeta) \geq -1 \quad \text{a.s.} \tag{2.9} \quad \text{[eq2.7]}
\]

and

\[
b(t) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta) \nu(d\zeta) = 0 ; \quad t \in [0, T]. \tag{2.10} \quad \text{[eq2.8]}
\]

If \( y = 1 \) this condition characterises \( Q_\theta \) as an ELMM for this market. See e.g. [14, Chapter 1].

We let \( \Theta \) denote the set of all \( \mathbb{F} \)-predictable processes \( \theta = (\theta_0, \theta_1) \) satisfying (2.9)-(2.10).

The dual problem corresponding to (1.8) is for given \( y > 0 \) to find \( \hat{\theta} \in \Theta \) and \( v(y) \) such that

\[
- v(y) := \sup_{\theta \in \Theta} E[-V(G_\theta^y(T))] = E[-V(G_{\hat{\theta}}^y(T))]. \tag{2.11} \quad \text{[eq2.9]}
\]

We will use two stochastic maximum principles for stochastic control to study the problem (2.11) and relate it to (2.6). We refer to Appendix A for a presentation of these principles and to [19] for more information about backward stochastic differential equations (BSDEs) with jumps.

We first prove two auxiliary results, the first of which may be regarded as a special case of Proposition 4.4 in [6].
Proposition 2.2 (Primal problem and associated constrained FBSDE) Let $\hat{\varphi}$ in $A$. Then $\hat{\varphi}$ is optimal for the primal problem (2.6) if and only if the (unique) solution $X, (\hat{p}_1, \hat{q}_1, \hat{r}_1)$ of the FBSDE consisting of the SDE (2.3) and the BSDE

$$\begin{cases}
    d\hat{p}_1(t) = \hat{q}_1(t)dB(t) + \int_{\mathbb{R}} \hat{r}_1(t, \zeta)\tilde{N}(dt, d\zeta); 0 \leq t \leq T \\
    \hat{p}_1(T) = U'(X_{\hat{\varphi}}^x(T))
\end{cases} \tag{2.12} \{\text{equa2.13}\}$$

satisfies the equation

$$b(t)\hat{p}_1(t) + \sigma(t)\hat{q}_1(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\hat{r}_1(t, \zeta)\nu(d\zeta) = 0; \ t \in [0, T]. \tag{2.13} \{\text{eq2.13b}\}$$

Proof. (i) The Hamiltonian corresponding to the primal problem is given by

$$H_1(t, x, \varphi, p, q, r) = \varphi S(t^-)(b(t)p + \sigma(t)q + \int_{\mathbb{R}} \gamma(t, \zeta)r(\zeta)\tilde{N}(dt, d\zeta)). \tag{2.14}$$

Assume $\hat{\varphi} \in A$ is optimal for the primal problem (2.6). Then by the necessary maximum principle (Theorem A.2), we have

$$\frac{\partial H_1}{\partial \varphi}(t, x, \varphi, \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t, \cdot)) \big|_{\varphi=\hat{\varphi}(t)} = 0,$$

where $(\hat{p}_1, \hat{q}_1, \hat{r}_1)$ satisfies (2.12), since $\frac{\partial H_1}{\partial x}(t, x, \varphi, \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t, \cdot)) = 0$. This implies (2.13).

(ii) Conversely, suppose the solution $(\hat{p}_1, \hat{q}_1, \hat{r}_1)$ of the BSDE (2.12) satisfies (2.13). Then $\hat{\varphi}$, with the associated $(\hat{p}_1, \hat{q}_1, \hat{r}_1)$ satisfies the conditions for the sufficient maximum principle (Theorem A.1) and hence $\hat{\varphi}$ is optimal. $\square$

We now turn to the dual problem (2.11). In the following we assume that

$$\sigma(t) \neq 0 \text{ for all } t \in [0, T]. \tag{2.15} \{\text{eq2.12}\}$$

This is more because of convenience and notational simplicity than of necessity. See Remark 2.7.

Remark 2.3 Equation (2.12) states that $\hat{p}_1$ is a martingale and $\hat{p}_1(t) = E[U'(X_{\hat{\varphi}}^x(T)) \mid \mathcal{F}_t]$ for all $t \in [0, T]$. Moreover $\hat{q}_1$ and $\hat{r}_1$ can be expressed in terms of the Malliavin derivatives of $\hat{p}_1$ as follows:

$$\hat{q}_1(t) = D_t \hat{p}_1(t) = \lim_{s \to t^-} D_s \hat{p}_1(t)$$
$$\hat{r}_1(t, \zeta) = D_{t, \zeta} \hat{p}_1(t) = \lim_{s \to t^-} D_{s, \zeta} \hat{p}_1(t)$$

where $D_s$ and $D_{s, \zeta}$ denote the Malliavin derivative at $s$ with respect to Brownian motion, and the Malliavin derivative at $s, \zeta$ with respect to Poisson random measure, respectively. See e.g. [4], Prop. 5.3 for the Brownian motion case and [2], Theorem 12.15 for the Poisson random measure case.
Remark 2.4 Note that by (2.12) we have \( \hat{p}_1(t) > 0 \), and if we divide equation (2.13) throughout by \( \hat{p}_1(t) \) we get

\[
b(t) + \sigma(t) \hat{\theta}_0(t) + \int_{RB} \gamma(t, \zeta) \hat{\theta}_1(t, \zeta) \nu(d\zeta) = 0 \quad ; \quad t \in [0, T],
\]

where

\[
\hat{\theta}_0(t) = \frac{\hat{q}_1(t)}{\hat{p}_1(t)} ; \quad t \in [0, T]. \tag{2.17}
\]

\[
\hat{\theta}_1(t, \zeta) = \frac{\hat{r}_1(t, \zeta)}{\hat{p}_1(t)} ; \quad t \in [0, T]. \tag{2.18}
\]

By the Girsanov theorem this is saying that if we define the measure \( Q(\hat{\theta}_0, \hat{\theta}_1) \) as in (2.7), (2.8) with \( y = 1 \), then \( Q(\hat{\theta}_0, \hat{\theta}_1) \) is an ELMM for the market described by (2.1).

Proposition 2.5 (Dual problem and associated constrained FBSDE) Let \( \hat{\theta} \in \Theta \). Then \( \hat{\theta} \) is an optimal scenario for the dual problem (2.11) if and only if the solution \( G_{\hat{\theta}}, (\hat{p}_2, \hat{q}_2, \hat{r}_2) \) of the FBSDE consisting of the FSDE (2.8) and BSDE

\[
\begin{cases}
d\hat{p}_2(t) = \frac{\hat{q}_2(t)}{\sigma(t)} b(t) dt + \hat{q}_2(t) dB(t) + \int_{\mathbb{R}} \hat{r}_2(t, \zeta) \tilde{N}(dt, d\zeta) ; & 0 \leq t \leq T \\
\hat{p}_2(T) = -V'(G_{\hat{\theta}}(T))
\end{cases}
\]

also satisfies

\[
- \frac{\hat{q}_2(t)}{\sigma(t)} \gamma(t, \zeta) + \hat{r}_2(t, \zeta) = 0 ; \quad 0 \leq t \leq T. \tag{2.20}
\]

Proof.

The Hamiltonian \( H_2 \) associated to (2.11) is, by (2.8)

\[
H_2(t, g, \theta_0, \theta_1, p, q, r) = g \theta_0 g + g \int_{\mathbb{R}} \theta_1(\zeta) r(\zeta) \nu(d\zeta). \tag{2.21}
\]

By (2.15) the constraint (2.10) can be written

\[
\theta_0(t) = \tilde{\theta}_0(t) = - \frac{1}{\sigma(t)} \left\{ b(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \theta_1(t, \zeta) \nu(d\zeta) \right\} ; \quad t \in [0, T]. \tag{2.22}
\]

Substituting this into (2.21) we get

\[
\tilde{H}_2(t, g, \theta_1, p_2, q_2, r_2) := H_2(t, g, \tilde{\theta}_0, \theta_1, p_2, q_2, r_2)
\]

\[
= g \left( - \frac{q_2}{\sigma(t)} \left\{ b(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \theta_1(\zeta) \nu(d\zeta) \right\} + \int_{\mathbb{R}} \theta_1(\zeta) r(\zeta) \nu(d\zeta) \right). \tag{2.23}
\]
Moreover, if (i) or (ii) holds, then

\[ F \text{ is a replicating portfolio for } \tilde{T} \]

Theorem 2.6

The following are equivalent:

\( H \) is a replicating portfolio for \( T \)

\[ \text{problem to the replication of a related } \]

\[ \text{are concave.} \]

\[ \text{The equation for the adjoint processes } (p_2, q_2, r_2) \text{ is thus the following BSDE:} \]

\[ dp_2(t) = \left[ \frac{q_2(t)}{\sigma(t)} b(t) + \int_{\mathbb{R}} \theta_1(t, \zeta) \left( \frac{q_2(t)}{\sigma(t)} \gamma(t, \zeta) - r_2(t, \zeta) \right) \nu(d\zeta) \right] dt \]

\[ + q_2(t) dB(t) + \int_{\mathbb{R}} r_2(t, \zeta) \tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \]

\[ p_2(T) = -V'(G_\theta(T)) \]  

(2.24) \{eq2.15\}

If there exists a maximiser \( \hat{\theta}_1 \) for \( \tilde{H}_2 \) then

\[ (\nabla_{\theta_1} \tilde{H}_2)_{\theta_1=\hat{\theta}_1} = 0, \]  

(2.25) \{eq2.16\}

i.e.

\[ - \frac{\hat{q}_2(t)}{\sigma(t)} \gamma(t, \zeta) + \hat{r}_2(t, \zeta) = 0 ; 0 \leq t \leq T, \]  

(2.26) \{eq2.17b\}

where \( (\hat{p}_2, \hat{q}_2, \hat{r}_2) \) is the solution of (2.24) corresponding to \( \theta = \hat{\theta} \) and \( \nabla_{\theta_1} \) denotes the Fréchet derivative with respect to \( \theta_1 \). We thus get (2.19) and this ends the necessary part.

The sufficient part follows from the fact that the functions \( g \to -V(g) \) and

\[ g \to \sup_{\theta_1} \tilde{H}_2(t, g, \theta_1, \hat{p}_2(t), \hat{q}_2(t), \hat{r}_2(t, \cdot)) = - \frac{\hat{q}_2(t)}{\sigma(t)} b(t) \]

are concave.

We deduce the following theorem which relates the existence of a solution of the dual problem to the replication of a related \( T \)-claim.

**Theorem 2.6** The following are equivalent:

(i) For given \( y > 0 \), there exists \( \hat{\theta} \in \Theta \) such that

\[ \sup_{\theta \in \Theta} E[-V(G_\theta^y(T))] = E[-V(G_{\hat{\theta}}^y(T))] < \infty. \]

(ii) For given \( y > 0 \), there exists \( \hat{\theta} \in \Theta \) such that the claim \( F := -V'(G_{\hat{\theta}}^y(T)) \) is replicable, with initial value \( x = \hat{p}_2(0) \), where \((\hat{p}_2, \hat{q}_2, \hat{r}_2)\) solves

\[ dp_2(t) = \frac{\hat{q}_2(t)}{\sigma(t)} \left[ b(t) dt + \sigma(t) dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \]

\[ \hat{p}_2(T) = -V'(G_{\hat{\theta}}^y(T)). \]  

(2.27) \{eq2.18\}

Moreover, if (i) or (ii) holds, then

\[ \phi(t) := \frac{\hat{q}_2(t)}{\sigma(t) S(t-)} \]

(2.28) \{eq221\}

is a replicating portfolio for \( F := -V'(G_{\hat{\theta}}^y(T)) \), where \((\hat{p}_2, \hat{q}_2, \hat{r}_2)\) is the solution of the linear BSDE (2.21).
Proof. (i) ⇒ (ii) Substituting (2.20) into (2.24) we get (2.27). This equation states that the contingent claim \( F := -V'(G_{\hat{\theta}}(T)) \) is replicable, with replicating portfolio \( \varphi(t) \) given by (2.28) and initial value \( x = \hat{p}_2(0) \). Note that \( \hat{p}_2(t) > 0 \) for all \( t \), since \( V \) is strictly decreasing so \( -V'(G_{\hat{\theta}}(T)) > 0 \).

(ii) ⇒ (i) Suppose that \((\hat{\theta}_0, \hat{\theta}_1) \in \Theta \) is such that \( F := -V'(G_{\hat{\theta}}(T)) \) is replicable with initial value \( x = \hat{p}_2(0) \), and let \( \varphi \in A \) be a replicating portfolio. Then \( X(t) = X_\varphi(t) \) satisfies the equation

\[
\begin{align*}
    dX(t) &= \varphi(t)S(t^-) \left[ b(t)dt + \sigma(t)dB(t) + \int_\mathbb{R} \gamma(t, \zeta)\tilde{N}(dt, d\zeta) \right]; \quad 0 \leq t \leq T \\
    X(T) &= -V'(G_{\hat{\theta}}(T)).
\end{align*}
\]

(2.29) \{eq2.20\}

Define

\[
\hat{p}(t) := X(t), \hat{q}(t) := \varphi(t)\sigma(t)S(t^-) \quad \text{and} \quad \hat{r}(t, \zeta) := \varphi(t)\gamma(t, \zeta)S(t^-). \quad \text{(2.30)} \quad \{eq2.21\}
\]

They satisfy the relation (2.20). By (2.29), \((\hat{p}, \hat{q}, \hat{r})\) satisfies the BSDE

\[
\begin{align*}
    d\hat{p}(t) &= \frac{\hat{q}(t)}{\sigma(t)}b(t)dt + \hat{q}(t)dB(t) + \int_\mathbb{R} \hat{r}(t, \zeta)\tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \\
    \hat{p}(T) &= -V'(G_{\hat{\theta}}(T)).
\end{align*}
\]

(2.31) \{eq2.22\}

and \( \hat{p}(0) = X(0) = x = \hat{p}_2(0) \). Hence (i) holds, by Proposition \ref{prop2.5}.

The last statement follows from (2.27) and (2.28). \( \square \)

Remark 2.7 Theorem \ref{thm2.6} still holds without assumption (2.15), if we replace (2.28) by

\[
\varphi(t) := \frac{q(t)}{\sigma(t)S(t^-)}\chi_{\sigma(t) \neq 0} + \frac{r(t, \zeta)}{\gamma(t, \zeta)S(t^-)}\chi_{\gamma(t, \zeta) \neq 0}. \quad \text{(3.2)} \quad \{eq2.37\}
\]

Proof. See Appendix \ref{appB}. \( \square \)

3 Relations between optimal scenario and optimal portfolio

We proceed to show that the method above actually gives a connection between an optimal scenario \( \hat{\theta} \in \Theta \) for the dual problem (2.11) and an optimal portfolio \( \hat{\varphi} \in A \) for the primal problem (2.6).

Theorem 3.1 a) Suppose \( \hat{\varphi} \in A \) is optimal for the primal problem (2.6). Let \((\hat{\varphi}_1(t), \hat{q}_1(t), \hat{r}_1(t, \zeta))\) be the associated adjoint processes, solution of the constrained BSDE (2.12)–(2.13). Define

\[
\hat{\theta}_0(t) = \frac{\hat{q}_1(t)}{\hat{p}_1(t^-)}, \quad \hat{\theta}_1(t, \zeta) = \frac{\hat{r}_1(t, \zeta)}{\hat{p}_1(t^-)}. \quad \text{(3.1)} \quad \{eq3.2\}
\]
Suppose
\[ E\left[ \int_0^T \{ \hat{\theta}_1(t) + \int_{\mathbb{R}} \hat{\theta}_2(t, \zeta) \nu(d\zeta) \} dt \right] < \infty; \quad \hat{\theta}_1 > -1. \]  
(3.2) \{eq.3.2b\}

Then \( \hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1) \in \Theta \) is optimal for the dual problem (2.11) with initial value \( y = \hat{p}_1(0) \). Moreover, with \( y = \hat{p}_1(0) \),
\[ G^y_{\hat{\theta}}(t) = \hat{p}_1(t); \quad t \in [0, T]. \] 
(3.3) \{eq.3.4\}

In particular
\[ G^y_{\hat{\theta}}(T) = U'(X_{\hat{\theta}}(T)). \] 
(3.4) \{eq.3.3\}

b) Conversely, suppose \( \hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1) \in \Theta \) is optimal for the dual problem (2.11). Let \((\hat{p}_2(t), \hat{q}_2(t), \hat{r}_2(t, \zeta))\) be the associated adjoint processes, solution of the BSDE (2.19) with the constraint (2.20). Suppose the portfolio
\[ \hat{\varphi}(t) := \frac{\hat{q}_2(t)}{\sigma(t)S(t^-)}; \quad 0 \leq t \leq T \] 
(3.5) \{eq.3.5\}

is admissible. Then \( \hat{\varphi} \) is an optimal portfolio for the primal problem (2.6) with initial value \( x = \hat{p}_2(0) \). Moreover, with \( x = \hat{p}_2(0) \),
\[ X^x_{\hat{\varphi}}(t) = \hat{p}_2(t); \quad t \in [0, T]. \] 
(3.6) \{eq.3.6\}

In particular
\[ X^x_{\hat{\varphi}}(T) = -V'(G^y_{\hat{\theta}}(T)). \] 
(3.7) \{eq.3.7\}

Proof.
a) Suppose \( \hat{\varphi} \) is optimal for problem (2.6) with initial value \( x \). Then, by Proposition 2.2, the adjoint processes \( \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t, \zeta) \) for Problem (2.6) satisfy (2.12)-(2.13). Consider the process \( \hat{\theta}(t) \) defined in (3.1) and suppose (3.2) holds. Then \( \hat{\theta} \in \Theta \) and (2.12) can be written
\[ \begin{cases} d\hat{p}_1(t) = \hat{p}_1(t^-) \left[ \hat{\theta}_0(t) dB(t) + \int_{\mathbb{R}} \hat{\theta}_1(t, \zeta) d\mathcal{N}(dt, d\zeta) \right] \\ \hat{p}_1(T) = U'(X^x_{\hat{\varphi}}(T)). \end{cases} \] 
(3.8) \{eq.3.8\}

Therefore \( \hat{p}_1(t) \equiv G^y_{\hat{\theta}}(t) \) (see (2.8)) if we put \( y := \hat{p}_1(0) > 0 \), and we have, by (1.7)
\[ U'(X^x_{\hat{\varphi}}(T)) = G^y_{\hat{\theta}}(T), \text{ i.e. } X^x_{\hat{\varphi}}(T) = -V'(G^y_{\hat{\theta}}(T)). \] 
(3.9) \{eq.3.9\}

Now define
\[ \hat{p}_2(t) := X^x_{\hat{\varphi}}(t), \hat{q}_2(t) := \hat{\varphi}(t)\sigma(t)S(t^-) \text{ and } \hat{r}_2(t, \zeta) := \hat{\varphi}(t)\gamma(t, \zeta)S(t^-). \] 
(3.10) \{eq.2.21b\}

Then \( (\hat{p}_2, \hat{q}_2, \hat{r}_2) \) satisfy the conditions of Proposition 2.5 which imply that \( \hat{\theta} \) is optimal for problem (2.11).
b) Suppose \( \hat{\theta} \in \Theta \) is optimal for problem (2.11) with initial value \( y \). Let \( \hat{p}_2(t), \hat{q}_2(t), \hat{r}_2(t) \) be the associated adjoint processes, solution of the BSDE (2.19) with (2.20). Then they satisfy the equation
\[
\begin{aligned}
    d\hat{p}_2(t) &= \frac{\hat{q}_2(t)}{\sigma(t)} \left[ b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right] \\
    \hat{p}_2(T) &= -V'(G_{\hat{\theta}}(T)).
\end{aligned}
\] (3.11)  \( \text{eq3.12} \)

Define
\[
\hat{\varphi}(t) := \frac{\hat{q}_2(t)}{\sigma(t)S(t^-)},
\] (3.12)  \( \text{eq3.13} \)

and assume \( \hat{\varphi}(t) \) is admissible. Then \( \hat{p}_2(t) \equiv X^x_{\hat{\varphi}}(t) \) for \( x = \hat{p}_2(0) \). In particular
\[
X^x_{\hat{\varphi}}(T) = -V'(G_{\hat{\theta}}(T)), \text{ i.e., } G_{\hat{\theta}}^y(T) = U'(X^x_{\hat{\varphi}}(T)).
\] (3.13)  \( \text{eq3.14} \)

Therefore \( G_{\hat{\theta}}^y(t) = G_{\hat{\theta}}(t) \) satisfies the equation
\[
\begin{aligned}
    dG_{\hat{\theta}}(t) &= G_{\hat{\theta}}(t^-) \left[ \hat{\theta}_0(t)dB(t) + \int_{\mathbb{R}} \hat{\theta}_1(t, \zeta) \tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \\
    G_{\hat{\theta}}(T) &= U'(X^x_{\hat{\varphi}}(T)).
\end{aligned}
\] (3.14)  \( \text{eq3.15} \)

Define now
\[
p_1(t) := G_{\hat{\theta}}(t), q_1(t) := G_{\hat{\theta}}(t)\hat{\theta}_0(t), r_1(t, \zeta) := G_{\hat{\theta}}(t)\hat{\theta}_1(t, \zeta).
\] (3.15)  \( \text{eq3.16} \)

Then by (3.14) \( (p_1, q_1, r_1) \) solves the BSDE
\[
\begin{aligned}
    dp_1(t) &= q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t, \zeta) \tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\
    p_1(T) &= U'(X^x_{\hat{\varphi}}(T)).
\end{aligned}
\] (3.16)  \( \text{eq3.17} \)

Moreover, since \( \hat{\theta} \in \Theta \), it satisfies (2.10), that is
\[
b(t) + \sigma(t)\hat{\theta}_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \hat{\theta}_1(t, \zeta) \nu(d\zeta) = 0 ; 0 \leq t \leq T
\] (3.17)  \( \text{eq3.18} \)

i.e., \( (p_1, q_1, r_1) \) satisfies the equation
\[
b(t) + \sigma(t) \frac{q_1(t)}{p_1(t)} + \int_{\mathbb{R}} \gamma(t, \zeta) \frac{r_1(t, \zeta)}{p_1(t)} \nu(d\zeta) = 0 ; 0 \leq t \leq T.
\] (3.18)  \( \text{eq3.19} \)

It follows from Proposition (2.2) that \( \hat{\varphi} := \hat{\varphi} \) is an optimal portfolio for problem (2.6) with initial value \( x = \hat{p}_2(0) \). \( \square \)
Example 3.1 As an illustration of Theorem 3.1 let us apply it to the situation when there are no jumps \( (N = 0) \) (A robust extension of this example will be given in Section 4.5.) Assume that there exists a constant \( C > 0 \) such that

\[
\frac{|b(t)|}{\sigma(t)} \leq C ; \ 0 \leq t \leq T. \tag{3.19} \]

Then \( \Theta \) has just one element \( \hat{\theta} \) given by

\[
\hat{\theta}(t) = -\frac{b(t)}{\sigma(t)}
\]

and

\[
G^y_\hat{\theta}(T) = y \exp \left( -\int_0^T \frac{b(s)}{\sigma(s)} dB(s) - \frac{1}{2} \int_0^T \frac{b^2(s)}{\sigma^2(s)} ds \right). \tag{3.20}
\]

Then, by Theorem 3.1b), if \( (p_2, q_2) \) is the solution of the BSDE

\[
\begin{cases}
    dp_2(t) = \frac{q_2(t)}{\sigma(t)} b(t) dt + q_2(t) dB(t) ; \ 0 \leq t \leq T \\
    p_2(T) = -V'(G^y_\hat{\theta}(T)),
\end{cases} \tag{3.21}
\]

then \( \hat{\varphi}(t) := \frac{q_2(t)}{\sigma(t)} \) is an optimal portfolio for the problem

\[
\sup_{\varphi \in A} E[U(X^x_\varphi(T))] \text{ with } x = p_2(0).
\]

In particular, if \( U(x) = \ln x \), then \( V(y) = -\ln y - 1 \) and \( V'(y) = -\frac{1}{y} \). So the BSDE (3.21) becomes

\[
\begin{cases}
    dp_2(t) = \frac{q_2(t)}{\sigma(t)} b(t) dt + q_2(t) dB(t) ; \ 0 \leq t \leq T \\
    p_2(T) = \frac{1}{y} \exp \left( \int_0^T \frac{b(s)}{\sigma(s)} dB(s) + \frac{1}{2} \int_0^T \frac{b^2(s)}{\sigma^2(s)} ds \right).
\end{cases} \tag{3.22}
\]

To solve this equation we try

\[
q_2(t) = p_2(t) \frac{b(t)}{\sigma(t)}. \tag{3.23}
\]

Then

\[
dp_2(t) = p_2(t) \left[ \frac{b^2(t)}{\sigma^2(t)} dt + \frac{b(t)}{\sigma(t)} dB(t) \right], \tag{3.24}
\]

which has the solution

\[
p_2(t) = p_2(0) \exp \left( \int_0^t \frac{b(s)}{\sigma(s)} dB(s) + \frac{1}{2} \int_0^t \frac{b^2(s)}{\sigma^2(s)} ds \right) ; \ 0 \leq t \leq T. \tag{3.25}
\]
Hence (3.22) holds with \( p_2(0) = \frac{1}{y} \) and we conclude by (3.6) that the optimal portfolio is

\[
\hat{\phi}(t) = p_2(t) \frac{b(t)}{\sigma^2(t)S(t^-)} = X^{\frac{1}{y}}(t) \frac{b(t)}{\sigma^2(t)S(t^-)}.
\]  

(3.26) \{eq3.27\}

This means that the optimal fraction of wealth to be placed in the risky asset is

\[
\hat{\pi}(t) = \frac{\hat{\phi}(t)S(t^-)}{X^{\frac{1}{y}}(t)} = \frac{b(t)}{\sigma^2(t)},
\]

(3.27) \{eq3.30\}

which agrees with the classical result of Merton.

**Remark 3.2** To check that \( \hat{\phi} \) is admissible, we have to verify that (2.4) and (2.5) hold for \( \varphi = \hat{\phi} \). To this end, we see that condition (3.19) suffices, in virtue of the Novikov condition for exponential martingales.

## 4 Robust duality

In this section we extend our study to a robust optimal portfolio problem and its dual.

### 4.1 Model uncertainty setup

To get a representation of model uncertainty, we consider a family of probability measures \( R = R^\kappa \sim P \), with Radon-Nikodym derivative on \( \mathcal{F}_t \) given by

\[
d\left(\frac{R^\kappa}{P} \bigg| \mathcal{F}_t\right) = Z^\kappa_t
\]

where, for \( 0 \leq t \leq T \), \( Z^\kappa_t \) is a martingale of the form

\[
dZ^\kappa_t = Z^\kappa_t [\kappa_0(t)dB_t + \int_{\mathbb{R}} \kappa_1(t, \zeta)\tilde{N}(dt, d\zeta)]; \quad Z^\kappa_0 = 1.
\]

Let \( \mathbb{K} \) denote a given set of admissible scenario controls \( \kappa = (\kappa_0, \kappa_1) \), \( \mathcal{F}_t \)-predictable, s.t. \( \kappa_1(t, z) \geq -1 + \epsilon \), and \( E[\int^T_t \{|\kappa_0^2(t)| + \int_{\mathbb{R}} \kappa_1^2(t, z)\nu(dz)\}dt] < \infty \).

By the Girsanov theorem, using the measure \( R^\kappa \) instead of the original measure \( P \) in the computations involving the price process \( S(t) \), is equivalent to using the original measure \( P \) in the computations involving the perturbed price process \( S_\mu(t) \) instead of \( P(t) \), where \( S_\mu(t) \) is given by

\[
\begin{cases}
dS_\mu(t) = S_\mu(t^-)[(b(t) + \mu(t)\sigma(t))dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta)] \\
S_\mu(0) > 0,
\end{cases}
\]

(4.1) \{eq5.1\}
with
\[\mu(t)\sigma(t) = -\sigma(t)\kappa_0(t) - \int_{\mathbb{R}} \gamma(t, \zeta)\kappa_1(t, \zeta)\nu(d\zeta)dt. \tag{4.2}\]

Accordingly, we now replace the price process \(S(t)\) in (2.1) by the perturbed process
\[
\begin{cases}
    dS_\mu(t) = S_\mu(t^-)[(b(t) + \mu(t)\sigma(t))dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta)] ; 0 \leq t \leq T \\
    S_\mu(0) > 0,
\end{cases}
\tag{4.3} \{eq5.1\}

for some perturbation process \(\mu(t)\), assumed to be predictable and satisfy
\[
E \left[ \int_0^T |\mu(t)\sigma(t)|dt \right] < \infty.
\]
Let \(\mathcal{M}\) denote this set of perturbation processes \(\mu\). Let \(X = X_{\varphi,\mu}^x\) be the corresponding wealth process given by
\[
\begin{cases}
    dX(t) = \varphi(t)S_\mu(t^-)[(b(t) + \mu(t)\sigma(t))dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta)] ; 0 \leq t \leq T \\
    X(0) = x > 0,
\end{cases}
\tag{4.4} \{eq5.4\}

where \(\varphi\) is an admissible portfolio, that is it belongs to the set \(\mathcal{A}\) of \(\mathbb{F}\)-predictable processes such that
\[
\left\{ \begin{array}{l}
(2.4) \text{ and } (2.5) \text{ hold,} \\
E \left[ \int_0^T \varphi(t)^2S_\mu(t)^2 \left\{ (b(t) + \mu(t)\sigma(t))^2 + \sigma^2(t) + \int_{\mathbb{R}} \gamma^2(t, \zeta)\nu(d\zeta) \right\} dt \right] < \infty, \\
X_{\varphi,\mu}(t) > 0 \text{ for all } t \in [0, T] \text{ a.s.}
\end{array} \right. \tag{4.5}
\]

for all \(\mu \in \mathcal{M}\).

4.2 The robust primal problem

Let \(\rho : \mathbb{R} \to \mathbb{R}\) be a convex penalty function, assumed to be \(C^1\), and \(U\) a utility function as in Section 1. We assume that \(\rho(\mu)\) has a minimum at \(\mu = 0\) and that \(\rho(0) = 0\). Then \(\rho(\mu)\) can be interpreted as a penalization for choosing \(\mu \neq 0\).

**Definition 4.1** The robust primal problem is, for given \(x > 0\), to find \((\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}\) such that
\[
\inf_{\mu \in \mathcal{M}} \sup_{\varphi \in \mathcal{A}} I(\varphi, \mu) = I(\hat{\varphi}, \hat{\mu}) = \sup_{\varphi \in \mathcal{A}} \inf_{\mu \in \mathcal{M}} I(\varphi, \mu), \tag{4.6} \{eq5.5\}
\]

where
\[
I(\varphi, \mu) = E \left[ U(X_{\varphi,\mu}^x(T)) + \int_0^T \rho(\mu(t))dt \right]. \tag{4.7} \{eq5.6\}
\]
The problem (4.6) is a stochastic differential game. To handle this, we use an extension of the maximum principle to games, as presented in e.g. [16]. We obtain the following characterization of a solution (saddle point) of (4.6):

**Proposition 4.2 (Robust primal problem and associated constrained FBSDE)** A pair \((\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}\) is a solution of the robust primal problem (4.6) if and only if the solution \(X(t), (p_1, q_1, r_1)\) of the FBSDE consisting of the SDE (4.4) and the BSDE

\[
\begin{align*}
    dp_1(t) &= q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t, \zeta)N(dt, d\zeta) ; 0 \leq t \leq T \\
    p_1(T) &= U'(X^{x}_{\hat{\theta}, \hat{\mu}}(T)) \quad \{eq5.11\}
\end{align*}
\]

satisfies

\[
    (b(t) + \hat{\mu}(t)\sigma(t))p_1(t) + \sigma(t)q_1(t) + \int_{\mathbb{R}} \gamma(t, \zeta)r_1(t, \zeta)\nu(d\zeta) = 0 ; t \in [0, T] \quad \{eq5.9\}
\]

\[
    \rho'(\hat{\mu}(t)) + \hat{\varphi}(t)S_{\mu}(t^-)\sigma(t)p_1(t) = 0 ; t \in [0, T]. \quad \{eq5.10\}
\]

**Proof.** Define the Hamiltonian by

\[
    H_1(t, x, \varphi, \mu, p, q, r) = \rho(\mu) + \varphi S_{\mu}(t^-) \left[ (b(t) + \mu \sigma(t))p + \sigma(t)q + \int_{\mathbb{R}} \gamma(t, \zeta)r(\zeta)\nu(d\zeta) \right]. \quad \{eq5.7\}
\]

The associated BSDE for the adjoint processes \((p_1, q_1, r_1)\) is (4.8).

The first order conditions for a maximum point \(\hat{\varphi}\) and a minimum point \(\hat{\mu}\), respectively, for the Hamiltonian are given by (4.9) and (4.10). Since \(H_1\) is concave with respect to \(\varphi\) and convex with respect to \(\mu\), these first order conditions are also sufficient for \(\hat{\varphi}\) and \(\hat{\mu}\) to be a maximum point and a minimum point, respectively. \(\square\)

**Remark 4.3** Equation (4.8) states that \(p_1\) is a martingale and \(p_1(t) = E[U'(X^{x}_{\hat{\theta}, \hat{\mu}}(T)) \mid \mathcal{F}_t]\) for all \(t\) in \([0, T]\). Moreover \(q_1\) and \(r_1\) can be expressed in terms of the Malliavin derivatives of \(p_1\). See Remark 2.3.

### 4.3 The robust dual problem

We now study a dual formulation of the robust primal problem (4.6). Let now \(\mathcal{M}\) be the family of positive measures \(Q = Q_{\theta, \mu}\) of the form

\[
    dQ_{\theta, \mu}(\omega) = G_{\theta, \mu}(T)dP(\omega) \text{ on } \mathcal{F}_T, \quad \{eq2.5a\}
\]

where \(G(t) = G_{\theta, \mu}(t)\) is given by

\[
    \begin{cases}
    dG(t) = G(t^-) \left[ \theta_0(t)dB(t) + \int_{\mathbb{R}} \theta_1(t, \zeta)N(dt, d\zeta) \right] ; 0 \leq t \leq T \\
    G(0) = y > 0
    \end{cases} \quad \{eq5.14\}
\]

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and \((\theta, \mu)\) is such that \(\mu \in \mathbb{M}\) and \(\theta = (\theta_0, \theta_1)\) is a predictable processes satisfying (2.9) and
\[
b(t) + \mu(t)\sigma(t) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) = 0 \; ; \; t \in [0, T]. \tag{4.14} \]

We let \(\Lambda\) denote the set of such processes \((\theta, \mu)\). If \(y = 1\), then the measure \(Q_{\theta,\mu}\) is an ELMM for the perturbed price process \(S_\mu\) in (4.3).

**Definition 4.4** The robust dual problem is for given \(y > 0\), to find \((\tilde{\theta}, \tilde{\mu}) \in \Lambda\) such that
\[
\sup_{(\theta, \mu) \in \Lambda} J(\theta, \mu) = J(\tilde{\theta}, \tilde{\mu}) \tag{4.15} \]
where
\[
J(\theta, \mu) = E \left[ -V(G_{\theta,\mu}^y(T)) - \int_0^T \rho(\mu(t))dt \right], \tag{4.16} \]
and \(V\) is the convex conjugate function of \(U\), as in Section 1.

**Proposition 4.5** (Robust dual problem and its associated constrained FBSDE.) A pair \((\tilde{\theta}, \tilde{\mu}) \in \Lambda\) is a solution of the robust dual problem (4.15)-(4.16) if and only the solution \(G(t), (p_2, q_2, r_2)\) of the FBSDE consisting of the FSDE (4.13) and the BSDE
\[
\begin{cases}
dp_2(t) = \frac{q_2(t)}{\sigma(t)}[b(t) + \tilde{\mu}(t)\sigma(t)]dt + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t, \zeta)\tilde{N}(dt, d\zeta) \; ; \; t \in [0, T] \\
p_2(T) = -V'(G_{\tilde{\theta},\tilde{\mu}}^y(T))
\end{cases} \tag{4.17} \]
satisfies
\[
-\frac{q_2(t)}{\sigma(t)}\gamma(t, \zeta) + r_2(t, \zeta) = 0, \tag{4.18} \]
\[
\rho'(\tilde{\mu}(t)) + G_{\tilde{\theta},\tilde{\mu}}^y(t)q_2(t) = 0. \tag{4.19} \]

**Proof.** Substituting
\[
\theta_0(t) = -\frac{1}{\sigma(t)} \left[ b(t) + \mu(t)\sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) \right] \tag{4.20} \]
into (4.13) we get
\[
\begin{cases}
dG(t) = G(t^-) \left( -\frac{1}{\sigma(t)} \left[ b(t) + \mu(t)\sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) \right] dB(t) \\
+ \int_{\mathbb{R}} \theta_1(t, \zeta)\tilde{N}(dt, d\zeta) \right) \; ; \; 0 \leq t \leq T \\
G(0) = y > 0.
\end{cases} \tag{4.21} \]
The Hamiltonian for the problem (4.15) then becomes
\[
H_2(t, g, \theta_1, \mu, p, q, r) = -\rho(\mu) - \frac{gq}{\sigma(t)} \left[ b(t) + \mu \sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \theta_1(\zeta) \nu(d\zeta) \right] + g \int_{\mathbb{R}} \theta_1(\zeta) \nu(d\zeta). \tag{4.22}
\]

The BSDE for the adjoint processes \((p_2, q_2, r_2)\) is
\[
\begin{cases}
p_2(t) = \left( \frac{q_2(t)}{\sigma(t)} \right) \left[ b(t) + \mu \sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \theta_1(t, \zeta) \nu(d\zeta) \right] - \int_{\mathbb{R}} \theta_1(t, \zeta) r_2(t, \zeta) \nu(d\zeta) + q_2(t) dB(t) + \int_{\mathbb{R}} r_2(t, \zeta) \tilde{N}(dt, d\zeta) ; \, 0 \leq t \leq T \\
p_2(T) = -V'(G_{\theta,\mu}(T)). \tag{4.23}
\end{cases}
\]

The first order conditions for a maximum point \((\tilde{\theta}, \tilde{\mu})\) for \(H_2\) are \(\nabla \theta_1 H_2 = 0\) and \(\frac{\partial H_2}{\partial \mu} = 0\) which are (4.18)-(4.19). Since \(H_2\) is concave w.r.t. \(\mu\) and \(\theta_1\), these necessary optimality conditions are also sufficient. Substituting (4.18) into (4.23) we get (4.17).

\[\square\]

4.4 Relations between robust primal and robust dual problems

We now use the characterizations above of the solutions \((\hat{\varphi}, \hat{\mu}) \in A \times M\) and \((\tilde{\theta}, \tilde{\mu}) \in \Lambda\) of the robust primal and the robust dual problem, respectively, to find the relations between them.

**Theorem 4.6** (i) From robust primal to robust dual.

Assume \((\hat{\varphi}, \hat{\mu}) \in A \times M\) is a solution of the robust primal problem and let \((p_1, q_1, r_1)\) be the associated adjoint processes solution of the FBSDE (4.4) & (4.8) and satisfying (4.9)- (4.10). Define
\[
\tilde{\mu} := \hat{\mu}, \quad \tilde{\theta}_0(t) := \frac{q_1(t)}{p_1(t)}; \quad \tilde{\theta}_1(t, \zeta) = \frac{r_1(t, \zeta)}{p_1(t)} \tag{4.24} \quad \text{and eq5.23}
\]
and suppose they satisfy (2.9). Then, they are optimal for the dual problem with initial value \(y = p_1(0)\). Moreover
\[
p_1(t) = G_{\tilde{\theta},\tilde{\mu}}(t) ; \, t \in [0, T]. \tag{4.26} \quad \text{eq5.27}
\]

In particular,
\[
U'(X_{\hat{\varphi},\hat{\mu}}(T)) = G_{\tilde{\theta},\tilde{\mu}}(T). \tag{4.27} \quad \text{eq5.28}
\]

(ii) From robust dual to robust primal Let \((\tilde{\theta}, \tilde{\mu}) \in \Lambda\) be optimal for the robust dual problem (4.15)-(4.16) and let \((p_2, q_2, r_2)\) be the associated adjoint processes satisfying (4.17)
with the constraints (4.18) and (4.19). Define
\[ \hat{\mu} := \tilde{\mu}, \]
\[ \hat{\varphi}(t) := \frac{q_2(t)}{\sigma(t)S_{\tilde{\mu}}(t^{-})}; \ t \in [0, T]. \] \[ \text{(4.28) \ \ \ {eq5.31}} \]
\[ \text{(4.29) \ \ \ {eq5.32}} \]

Assume that \( \hat{\varphi} \in \mathcal{A} \). Then \( (\hat{\mu}, \hat{\varphi}) \) are optimal for primal problem with initial value \( x = p_2(0) \).
Moreover,
\[ p_2(t) = X_{\hat{\varphi}, \hat{\mu}}(t) \quad t \in [0, T]. \] \[ \text{(4.30) \ \ \ {eq4.42a}} \]

In particular
\[ -V'(G_{\tilde{\theta}}(T)) = X_{\hat{\varphi}, \hat{\mu}}(T). \] \[ \text{(4.31) \ \ \ {eq4.42}} \]

**Proof.**

(i) Let \( (\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathbb{M} \) is a solution of the robust primal problem and let \( (p_1, q_1, r_1) \) be as in Proposition 4.2 i.e. assume that \( (p_1, q_1, r_1) \) solves the FBSDE (4.1) & (4.8) and satisfies (4.9)-(4.10).

We want to find the solution \( (\hat{\theta}, \tilde{\mu}) \in \Lambda \) of the robust dual problem. By Proposition 4.5 this means that we must find a solution \( (p_2, q_2, r_2) \) of the FBSDE (4.21) & (4.17) which satisfies (4.18)-(4.19). To this end, choose \( \hat{\mu}, \tilde{\theta}_0, \tilde{\theta}_1 \) given in (4.24)-(4.25). Then by (4.9) we have
\[ b(t) + \hat{\mu}(t)\sigma(t) + \sigma(t)\tilde{\theta}_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{\theta}_1(t, \zeta)\nu(d\zeta) = 0. \] \[ \text{(4.32) \ \ \ {eq5.25}} \]

Assume that (2.9) holds. Then \( (\hat{\mu}, \tilde{\theta}) \in \Lambda \). Substituting (4.25) into (4.8), we obtain
\[ \begin{cases}
    dp_1(t) = p_1(t^{-}) \left[ \tilde{\theta}_0(t)dB(t) + \int_{\mathbb{R}} \tilde{\theta}_1(t, \zeta)N(dt, d\zeta) \right] ; \ t \in [0, T] \\
    p_1(T) = U'(X_{\hat{\varphi}, \hat{\mu}}(T)).
\end{cases} \] \[ \text{(4.33) \ \ \ {eq5.26}} \]

Comparing with (4.13) we see that
\[ \frac{dG_{\tilde{\theta}, \tilde{\mu}}(t)}{G_{\tilde{\theta}, \tilde{\mu}}(t)} = \frac{dp_1(t)}{p_1(t)} \]
and hence, for \( y = G_{\tilde{\theta}, \tilde{\mu}}(0) = p_1(0) > 0 \) we get (4.26) and (4.27). Define
\[ p_2(t) := X_{\hat{\varphi}, \hat{\mu}}(t), q_2(t) := \hat{\varphi}(t)\sigma(t)S_{\tilde{\mu}}(t^{-}), r_2(t, \zeta) := \hat{\varphi}(t)\gamma(t, \zeta)S_{\tilde{\mu}}(t^{-}). \] \[ \text{(4.34) \ \ \ {eq5.29}} \]

Then by (4.4) and (4.27), combined with (1.7),
\[ \begin{cases}
    dp_2(t) = \hat{\varphi}(t)S_{\tilde{\mu}}(t^{-}) \left[ (b(t) + \hat{\mu}(t)\sigma(t))dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t, \zeta)N(dt, d\zeta) \right] \\
    = \frac{q_2(t)}{\sigma(t)} [b(t) + \hat{\mu}(t)\sigma(t)]dt + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t, \zeta)N(dt, d\zeta) ; \ 0 \leq t \leq T \\
    p_2(T) = X_{\hat{\varphi}, \hat{\mu}}(T) = -V'(G_{\tilde{\theta}, \tilde{\mu}}(T)).
\end{cases} \] \[ \text{(4.35) \ \ \ {eq5.30}} \]
Hence \((p_2, q_2, r_2)\) solves the BSDE (4.17), as requested. It remains to verify that (4.18) and (4.19) hold: By (4.34) we have
\[
-\frac{q_2(t)}{\sigma(t)}\gamma(t, \zeta) + r_2(t, \zeta) = -\dot{\varphi}(t)S_{\tilde{\mu}}(t^-)\gamma(t, \zeta) + \dot{\varphi}(t)S_{\tilde{\mu}}(t^-)\gamma(t, \zeta) = 0,
\]
which is (4.18). By (4.24), (4.26), (4.34) and (4.10),
\[
\rho'(\tilde{\mu}) + G_{\tilde{\theta}, \tilde{\mu}}(t)q_2(t) = \rho'(\tilde{\mu}) + p_1(t)\dot{\varphi}(t)\sigma(t)S_{\tilde{\mu}}(t^-) = 0,
\]
which is (4.19).

(ii) Next, assume that \((\tilde{\theta}, \tilde{\mu}) \in \Lambda\) is optimal for the robust dual problem (4.15)-(4.16) and let \((p_2, q_2, r_2)\) be as in Proposition 4.5. We will find \((\hat{\varphi}, \hat{\mu}) \in A \times M\) and \((p_1, q_1, r_1)\) satisfying Proposition 4.2. Choose \(\hat{\mu}\) and \(\hat{\varphi}\) given in (4.28)-(4.29) and assume that \(\hat{\varphi}\) is admissible. Then by (4.17) and (4.18),
\[
\begin{aligned}
dp_2(t) &= \dot{\varphi}(t)S_{\tilde{\mu}}(t^-) \left[ (b(t) + \hat{\mu}(t)\sigma(t))dt + \sigma(t)dB(t) + \int_\mathbb{R} \gamma(t, \zeta)\tilde{\nu}(dt, d\zeta) \right]; \quad 0 \leq t \leq T \\
p_2(T) &= -V'(G_{\tilde{\theta}, \tilde{\mu}}(T)).
\end{aligned}
\]
Hence, with \(x = p_2(0) > 0\), (4.30) holds. In particular
\[
X_{\hat{\varphi}, \hat{\mu}}(T) = p_2(T) = -V'(G_{\tilde{\theta}, \tilde{\mu}}(T)), \quad \text{ i.e. } G_{\tilde{\theta}, \tilde{\mu}}(T) = U'(X_{\hat{\varphi}, \hat{\mu}}(T)). \tag{4.36} \quad \{\text{eq5.34}\}
\]
We now verify that with \(\varphi = \hat{\varphi}, \mu = \hat{\mu}\), and \(p_1, q_1, r_1\) defined by
\[
p_1(t) := G_{\tilde{\theta}, \tilde{\mu}}(t), \quad q_1(t) := G_{\tilde{\theta}, \tilde{\mu}}(t)\tilde{\theta}_0(t), \quad r_1(t, \zeta) := G_{\tilde{\theta}, \tilde{\mu}}(t)\tilde{\theta}_1(t, \zeta), \tag{4.37} \quad \{\text{eq5.35}\}
\]
all the conditions of Proposition 4.2 hold: By (4.21) and (4.36),
\[
\begin{aligned}
\left\{ \begin{array}{l}
\left. dp_1(t) \right| = dG_{\tilde{\theta}, \tilde{\mu}}(t) = G_{\tilde{\theta}, \tilde{\mu}}(t^-) \left( -\frac{1}{\sigma(t)} \left[ b(t) + \hat{\mu}(t)\sigma(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{\theta}_1(t, \zeta)\tilde{\nu}(dt, d\zeta) \right] \right) dB(t) \\
+ \int_{\mathbb{R}} \tilde{\theta}_1(t, \zeta)\tilde{\nu}(dt, d\zeta) \right); \quad 0 \leq t \leq T \\
p_1(T) = G_{\tilde{\theta}, \tilde{\mu}}(T) = U'(X_{\hat{\varphi}, \hat{\mu}}(T)). \tag{4.38} \quad \{\text{eq5.36}\}
\end{array} \right.
\end{aligned}
\]
Hence (4.8) holds. It remains to verify (4.9) and (4.10). By (4.37) and (4.14) for \(\theta = \tilde{\theta}\), we get
\[
(b(t) + \hat{\mu}(t)\sigma(t))p_1(t) + \sigma(t)q_1(t) + \int_{\mathbb{R}} \gamma(t, \zeta)r_1(t, \zeta)\tilde{\nu}(dt, d\zeta)
\]
\[
= G_{\tilde{\theta}, \tilde{\mu}}(t) \left[ b(t) + \hat{\mu}(t)\sigma(t) + \sigma(t)\tilde{\theta}_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{\theta}_1(t, \zeta)\tilde{\nu}(dt, d\zeta) \right] = 0,
\]
which is (4.9). By (4.28), (4.29), (4.37) and (1.19) we get
\[
\rho'(\hat{\mu}(t)) + \dot{\varphi}(t)S_{\hat{\mu}}(t^-)\sigma(t)p_1(t) = \rho'(\hat{\mu}(t)) + q_2(t)G_{\tilde{\theta}, \tilde{\mu}}(t) = 0,
\]
which is (4.10). \(\square\)
4.5 Example

We consider a robust version of Example 3.1. We want to study
\[
\inf_{\mu \in \mathcal{M}} \sup_{\varphi \in \mathcal{A}} E \left[ U(X_{\varphi,\mu}(T)) + \int_0^T \rho(\mu(t))dt \right] \tag{4.39} \quad \{\text{eq5.5b}\}
\]
in the case with no jumps \((N = \gamma = 0)\). Then there is only one ELMM for the price process \(S_\mu(t)\) for each given \(\mu(t)\). So \(\theta = \theta_0 = -\frac{b(t)}{\sigma(t)} + \mu(t)\) and the corresponding robust dual problem simplifies to
\[
\sup_{\mu \in \mathcal{M}} E \left[ -V(G_{\tilde{\mu}}(T)) - \int_0^T \rho(\mu(t))dt \right], \tag{4.40} \quad \{\text{peq5.12}\}
\]
where
\[
\begin{cases}
    dG_{\mu}(t) = -G_{\mu}(t^-)[\frac{b(t)}{\sigma(t)} + \mu(t)]dB_t ; \ 0 \leq t \leq T \\
    G_{\mu}(0) = y > 0.
\end{cases} \tag{6.3} \{\text{6.3}\}
\]

The first order conditions for the Hamiltonian reduce to:
\[
\tilde{\mu}(t) = (\rho')^{-1}(-G_{\tilde{\mu}}(t)q_2(t)) \tag{4.42} \quad \{\text{equ4.44}\}
\]
which substituted into the adjoint BSDE equation gives:
\[
\begin{cases}
    dp_2(t) = q_2(t)[\frac{b(t)}{\sigma(t)} + (\rho')^{-1}(-G_{\tilde{\mu}}(t)q_2(t))]dt + q_2(t)dB_t ; \ t \in [0,T] \\
    p_2(T) = -V'(G_{\tilde{\mu}}(T)).
\end{cases} \tag{4.43} \{\text{6.5}\}
\]

We get that \(\tilde{\mu}\) is optimal for the robust dual problem if and only if there is a solution \((p_2, q_2, G_{\tilde{\mu}})\) of the FBSDE consisting of (4.43) and (4.41) with the constraint (4.42). Hence, the optimal \(\tilde{\mu}\) for the primal robust problem is given by \(\hat{\mu} := \tilde{\mu}\), and the optimal portfolio is
\[
\hat{\varphi}(t) = \frac{q_2(t)}{\sigma(t)S_{\tilde{\mu}}(t^-)}; \ t \in [0,T]. \tag{4.44} \quad \{\text{eq5.32bis}\}
\]

We have proved:

**Theorem 4.7** The solution \(\hat{\mu}, \hat{\varphi}\) of the robust primal problem (4.39) is given by (4.42) and (4.43), respectively, where \((G_{\tilde{\mu}}, p_2, q_2)\) is the solution of the constrained FBSDE (4.41)-(4.43).

Now assume that
\[
U(x) = \ln x \tag{4.45} \quad \{\text{ln}\}
\]
and
\[
\rho(x) = \frac{1}{2}x^2 \tag{4.46} \quad \{\text{ro}\}
\]
Then \(V(y) = -\ln y - 1\). If \(b(t)\) and \(\sigma(t)\) are deterministic, we can solve (4.40) by dynamic programming, and we get
\[
\tilde{\mu}(t) = -\frac{b(t)}{2\sigma(t)} ; \ t \in [0,T]. \tag{4.47} \quad \{\text{eq450}\}
\]
In view of this, it is natural to guess that (4.47) is the optimal choice of \( \mu \) also when \( b(t) \) and \( \sigma(t) \) are \( \mathcal{F}_t \)-adapted processes. To verify this we have to show that the system (4.41)-(4.43) is consistent. This system is now the following

\[
G_{\hat{\mu}}(t) = y \exp \left( - \int_0^t \frac{b(s)}{2\sigma(s)} dB(s) - \frac{1}{2} \left( \frac{b(s)}{2\sigma(s)} \right)^2 ds \right) \quad (4.48) \tag{55}
\]

\[
q_2(t) = \frac{1}{G_{\hat{\mu}}(t)} \frac{b(t)}{2\sigma(t)} \quad (4.49) \tag{451}
\]

\[
dp_2(t) = \frac{1}{G_{\hat{\mu}}(t)} \left[ \frac{b(t)}{2\sigma(t)} dB(t) + \left( \frac{b(t)}{2\sigma(t)} \right)^2 dt \right] ; \quad p_2(T) = \frac{1}{G_{\hat{\mu}}(T)} \quad (4.50) \tag{452}
\]

which gives

\[
\frac{1}{G_{\hat{\mu}}(t)} = \frac{1}{y} \exp \left( \int_0^t \frac{b(s)}{2\sigma(s)} dB(s) + \frac{1}{2} \left( \frac{b(s)}{2\sigma(s)} \right)^2 ds \right) \quad (4.51)
\]

i.e.

\[
d\left( \frac{1}{G_{\hat{\mu}}(t)} \right) = \frac{1}{G_{\hat{\mu}}(t)} \left[ \frac{b(t)}{2\sigma(t)} dB(t) + \left( \frac{b(t)}{2\sigma(t)} \right)^2 dt \right] . \quad (4.52) \tag{57}
\]

We see that (4.50) is in agreement with (4.52) with \( p_2(t) = \frac{1}{G_{\hat{\mu}}(0)} \), and this proves that \( \check{\mu}(t) \) given by (4.47) is indeed optimal also when \( b \) and \( \sigma \) are stochastic. The corresponding optimal portfolio for the robust utility maximization problem with initial value \( x = \frac{1}{y} \), by (4.29),

\[
\hat{\phi}(t) = \frac{b(t)}{G_{\hat{\mu}}(t)2\sigma^2(t)S_{\hat{\mu}}(t)} ; \quad t \in [0, T]. \quad (4.53) \tag{eq455}
\]

which means that the optimal fraction of wealth to be placed in the risky asset is

\[
\hat{\pi}(t) = \frac{\hat{\phi}(t)S_{\hat{\mu}}(t-)}{X(t)} = \frac{b(t)}{2\sigma^2(t)} \quad (4.54) \tag{eq4.54}
\]

We have proved:

**Theorem 4.8** Suppose (4.45)-(4.46) hold. Then the optimal scenario \( \hat{\mu} = \check{\mu} \) and optimal portfolio \( \hat{\phi} \) for the robust primal problem (4.39) are given by (4.47) and (4.53), respectively, with \( G_{\hat{\mu}}(\cdot) \) as in (4.48).

**Remark 4.9** Comparing (4.54) with (3.27) in Example 3.1 we see that the optimal fraction to be placed in the risky asset in the robust case is just half of the optimal fraction in the non-robust case.
A Maximum principles for optimal control

Consider the following controlled stochastic differential equation

\[
\begin{align*}
  dX(t) &= b(t, X(t), u(t), \omega)dt + \sigma(t, X(t), u(t), \omega)dB(t) \\
  & \quad + \int_{\mathbb{R}} \gamma(t, X(t), u(t), \omega, \zeta)\tilde{N}(dt, d\zeta) ; \ 0 \leq t \leq T \\
  X(0) &= x \in \mathbb{R}
\end{align*}
\]  

(A.1)

The performance functional is given by

\[
J(u) = E\left[ \int_0^T f(t, X(t), u(t), \omega)dt + \phi(X(T), \omega) \right]
\]  

(A.2)

where \( T > 0 \) and \( u \) is in a given family \( \mathcal{A} \) of admissible \( \mathcal{F} \)-predictable controls. For \( u \in \mathcal{A} \) we let \( X^u(t) \) be the solution of (A.1). We assume this solution exists, is unique and satisfies, for some \( \epsilon > 0 \),

\[
E[\int_0^T |X^u(t)|^{2+\epsilon}dt] < \infty.
\]  

(A.3)

We want to find \( u^* \in \mathcal{A} \) such that

\[
\sup_{u \in \mathcal{A}} J(u) = J(u^*). \tag{A.4}
\]

We make the following assumptions

\[
f \in C^1 \text{ and } E[\int_0^T |\nabla f|^2(t)dt] < \infty, \tag{A.5}
\]

\[
b, \sigma, \gamma \in C^1 \text{ and } E[\int_0^T (|\nabla b|^2 + |\nabla \sigma|^2 + \|\nabla \gamma\|^2)(t)dt] < \infty, \tag{A.6}
\]

where \( \|\nabla \gamma(t, \cdot)\|^2 := \int_{\mathbb{R}} \gamma^2(t, \zeta)\nu(d\zeta) \)

\[
\phi \in C^1 \text{ and for all } u \in \mathcal{A}, \exists \epsilon \text{ s.t. } E[\phi'(X(T))^{2+\epsilon}] < \infty. \tag{A.7}
\]

Let \( \mathbb{U} \) be a convex closed set containing all possible control values \( u(t); t \in [0, T] \).

The Hamiltonian associated to the problem (A.4) is defined by

\[
H : [0, T] \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \mapsto \mathbb{R}
\]

\[
H(t, x, u, p, q, r, \omega) = f(t, x, u, \omega) + b(t, x, u, \omega)p + \sigma(t, x, u, \omega)q + \int_{\mathbb{R}} \gamma(t, x, u, \zeta, \omega)r(t, \zeta)\nu(d\zeta).
\]

For simplicity of notation the dependence on \( \omega \) is suppressed in the following. We assume that \( H \) is Fréchet differentiable in the variables \( x, u \). We let \( m \) denote the Lebesgue measure on \([0, T]\).
The associated BSDE for the adjoint processes \((p,q,r)\) is
\[
\begin{aligned}
dp(t) &= -\frac{\partial H}{\partial x}(t) + q(t)dB(t) + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(dt,d\zeta); \quad 0 \leq t \leq T \\
p(T) &= \phi'(X(T)).
\end{aligned}
\]  
\tag{A.8} \{A8\}

Here and in the following we are using the abbreviated notation
\[\frac{\partial H}{\partial x}(t) = \frac{\partial}{\partial x}(t, X(t), u(t))\] etc.

We first formulate a sufficient maximum principle, with weaker conditions than in \([16]\).

**Theorem A.1 (Sufficient maximum principle)** Let \(\hat{u} \in A\) with corresponding solutions \(\hat{X}, \hat{p}, \hat{q}, \hat{r}\) of equations \(\text{(A.1) - (A.8)}.\) Assume the following:

- The function \(x \mapsto \phi(x)\) is concave
- (The Arrow condition) The function
  \[\mathcal{H}(x) := \sup_{v \in \mathcal{U}} H(t, x, v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))\]  
  is concave for all \(t \in [0, T]\).
- \[\sup_{v \in \mathcal{U}} H(t, \hat{X}(t), v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) = H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)); \ t \in [0, T].\]  
  \tag{A.10}

Then \(\hat{u}\) is an optimal control for the problem \(\text{(A.4)}\).

Next, we state a necessary maximum principle. For this, we need the following assumptions:

- For all \(t_0 \in [0, T]\) and all bounded \(\mathcal{F}_{t_0}\)-measurable random variables \(\alpha(\omega)\) the control
  \[\beta(t) := \chi_{[t_0, T]}(t)\alpha(\omega)\]
  belongs to \(A\).
- For all \(u, \beta \in A\) with \(\beta\) bounded, there exists \(\delta > 0\) such that the control
  \[\tilde{u}(t) := u(t) + a\beta(t); \ t \in [0, T]\]
  belongs to \(A\) for all \(a \in (-\delta, \delta)\).
• The derivative process

\[ x(t) := \frac{d}{da} X^{u+a\beta}(t) \bigg|_{a=0}, \]

exists and belongs to \( L^2(dm \times dP) \), and

\[
\begin{aligned}
    dx(t) &= \left\{ \frac{\partial b}{\partial x}(t)x(t) + \frac{\partial b}{\partial u}(t)\beta(t) \right\} dt + \left\{ \frac{\partial \sigma}{\partial x}(t)x(t) + \frac{\partial \sigma}{\partial u}(t)\beta(t) \right\} dB(t) \\
    &+ \int_{\mathbb{R}} \left\{ \frac{\partial \gamma}{\partial x}(t, \zeta)x(t) + \frac{\partial \gamma}{\partial u}(t, \zeta)\beta(t) \right\} \tilde{N}(dt, d\zeta) \\
    x(0) &= 0
\end{aligned}
\]  

(A.11) \{A23\}

**Theorem A.2 (Necessary maximum principle)** The following are equivalent

- \( \frac{d}{da} J(u + a\beta) \bigg|_{a=0} = 0 \) for all bounded \( \beta \in \mathcal{A} \)

- \( \frac{\partial H}{\partial u}(t) = 0 \) for all \( t \in [0, T] \).

For proofs of these results we refer to Section 3 of [17].

**B Proof of Remark 2.7:**

Let \( \lambda(t) \) be the Lagrange multiplier process and consider

\[
H_2^\lambda(\theta_0, \theta_1, \lambda) := g\theta_0q + g \int_{\mathbb{R}} \theta_1(\zeta)r(\zeta)\nu(d\zeta) \\
+ \lambda(t) \left( b(t) + \sigma(t)\theta_0 + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(\zeta)\nu(d\zeta) \right).
\]  

(B.1) \{eq2.28\}

Maximising \( H_2^\lambda \) over all \( \theta_0 \) and \( \theta_1 \) gives the following first order conditions

\[
q + \lambda(t)\sigma(t) = 0 \quad \text{(B.2)} \quad \{eq2.29\}
\]

\[
gr(\cdot) + \lambda(t)\gamma(t, \cdot) = 0. \quad \text{(B.3)} \quad \{eq2.30\}
\]

Since \( g = G_\theta(t) \neq 0 \), we can write these as follows:

\[
q(t) = -\frac{\lambda(t)}{G_\theta(t)}\sigma(t) \quad \text{(B.4)} \quad \{eq2.31\}
\]

\[
r(t, \zeta) = -\frac{\lambda(t)}{G_\theta(t)}\gamma(t, \zeta) \quad \text{(B.5)} \quad \{eq2.32\}
\]
The adjoint equations become:

\[
\begin{aligned}
dp(t) & = -\frac{\lambda(t)}{G_\theta(t)} \left\{ -\theta_0(t)\sigma(t) - \int_\mathbb{R} \theta_1(t, \zeta) \gamma(t, \zeta) \nu(d\zeta) \right\} dt \\
& \quad + \sigma(t) dB(t) + \int_\mathbb{R} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\
p(T) & = -V'(G_\theta(T))
\end{aligned}
\] (B.6) \{eq2.33\}

In view of (2.10) this can be written

\[
\begin{aligned}
dp(t) & = -\frac{\lambda(t)}{G_\theta(t)} \left\{ b(t) dt + \sigma(t) dB(t) + \int_\mathbb{R} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right\} ; 0 \leq t \leq T \\
p(T) & = -V'(G_\theta(T))
\end{aligned}
\] (B.7) \{eq2.34\}

Note that

\[
\begin{aligned}
\text{If } \sigma(t) \neq 0 & \text{ then } -\frac{\lambda(t)}{G_\theta(t)} = \frac{q(t)}{\sigma(t)} \quad \{eq2.35\} \\
\text{If } \gamma(t, \zeta) \neq 0 & \text{ then } -\frac{\lambda(t)}{G_\theta(t)} = \frac{r(t, \zeta)}{\gamma(t, \zeta)} \quad \{eq2.36\}
\end{aligned}
\]

If $\sigma(t) = \gamma(t, \zeta) = 0$, then by (B.4) and (B.5) we have $q(t) = r(t, \zeta) = 0$ and hence we have $dp(t) = 0$. Therefore we can summarize the above as follows: Define

\[
\varphi(t) = \frac{q(t)}{\sigma(t) S(t^-)} \chi_{\sigma(t) \neq 0} + \frac{r(t, \zeta)}{\gamma(t, \zeta) S(t^-)} \chi_{\sigma(t) = 0, \gamma(t, \zeta) \neq 0}.
\] (B.10) \{eq2.37b\}

Then by (B.7)

\[
\begin{aligned}
dp(t) & = \varphi(t) S(t^-) \left[ b(t) dt + \sigma(t) dB(t) + \int_\mathbb{R} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \\
p(T) & = -V'(G_\theta(T)).
\end{aligned}
\] (B.11) \{eq2.38\}

Therefore $-V'(G_\theta(T))$ is replicable, with replicating portfolio $\varphi(t)$ given by (2.32). Thus we see that Theorem 2.6 still holds without assumption (2.3), if we replace (2.28) by (B.10). □

**References**

[1] Bordigoni, G., Matoussi, A., Schweizer, M.: A stochastic control approach to a robust utility maximization problem. In: Benth, F.E. et al (eds): Stochastic Analysis and Applications. The Abel Symposium 2005, pp. 125-15, Springer (2007)

[2] Di Nunno, G., Øksendal, B., and Proske, F.: Malliavin Calculus for Lévy Processes with Applications to Finance. Springer, Corrected 2nd printing 2009.
[3] El Karoui, N. and Quenez, M.-C.: Dynamic programming and pricing of contingent claims in an incomplete market. SIAM J. Control and Optimization 33 (1995), 29-66.

[4] El Karoui, N., Peng, S. and Quenez, M.-C.: Backward stochastic differential equations in finance. Mathematical Finance 7 (1997), 1-71.

[5] Föllmer, H., Schied, A., Weber, S.: Robust preferences and robust portfolio choice, In: Mathematical Modelling and Numerical Methods in Finance. In: Ciarlet, P., Bensoussan, A., Zhang, Q. (eds): Handbook of Numerical Analysis 15, pp. 29-88 (2009)

[6] Fontana, C., Øksendal, B., Sulem, A.: Viability and martingale measures in jump diffusion markets under partial information. Preprint University of Oslo, 2, 2013, arXiv:1302.4254.

[7] Gushkin, A.: Dual characterization of the value function in the robust utility maximization problem. Theory Probab. Appl. 55 (2011), 611-630.

[8] Jeanblanc, M., Matoussi, A., Ngoupeyou, A.: Robust utility maximization in a discontinuous filtration, arXiv (2012)

[9] Kramkov, D. and Schachermayer, W.: Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab. 13 (2003), 1504-1516.

[10] Kreps, D.: Arbitrage and equilibrium in economics with infinitely many commodities. J. Math. Economics 8, 15-35 (1981)

[11] Lim, T., Quenez, M.-C.: Exponential utility maximization and indifference price in an incomplete market with defaults. Electronic J. Probability 16, 1434-1464 (2011)

[12] Loewenstein, M., Willard, G.: Local martingales, arbitrage, and viability. Economic Theory 16, 135-161 (2000)

[13] Maenhout, P.: Robust portfolio rules and asset pricing. Review of Financial Studies 17, 951-983 (2004)

[14] Øksendal, B., Sulem, A.: Applied Stochastic Control of Jump Diffusions. Second Edition, Springer (2007)

[15] Øksendal, B., Sulem, A.: Forward-backward stochastic differential games and stochastic control under model uncertainty. J. Optim. Theory Appl., DOI 10.1007/S10957-012-0166-7 (2012).

[16] Øksendal, B., Sulem, A.: Portfolio optimization under model uncertainty and BSDE games. Quantitative Finance 11(11), 1665-1674 (2011)

[17] Øksendal, B., Sulem, A.: Risk minimization in financial markets modeled by Itô-Lévy processes. Manuscript, August 2013.
[18] Quenez, M.-C.: Optimal portfolio in a multiple-priors model. In R.C. Dalang, M. Dozzi and F. Russo (editors): Seminar on Stochastic Analysis, Random Fields and Applications IV, Birkäuser 2004, pp. 291-321.

[19] Quenez, M.-C., Sulem, A.: BSDEs with jumps, optimization and applications to dynamic risk measures. Stochastic Processes and their Applications, 123 (2013), 3328–3357.

[20] Royer, M.: Backward stochastic differential equations with jumps and related non-linear expectations. Stochastic Processes and Their Applications 116, 1358–1376 (2006)

[21] Rockafellar, R.T.: Convex Analysis. Princeton University Press (1970)