Special positions of body-and-cad frameworks

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Abstract

A recent result provides a combinatorial characterization of the generic rigidity for the majority of Computer Aided Design (CAD) structures. However, an algorithm based on this result will incorrectly classify a design as well-constrained if it is in a special (non-generic) position allowing an internal motion. Since, in practice, CAD users often rely on highly organized structural elements and design patterns which may exhibit this non-generic behavior, we seek an approach to determine whether a design is in a special position. We present a combinatorial approach for finding the factors of the polynomial whose vanishing indicates a special position. For certain structures, we further find geometric properties determining when factors of the polynomial vanish by using the Grassmann-Cayley algebra and present case studies demonstrating our approach.

1. Introduction

Constraint-based Computer Aided Design (CAD) software, such as the popular SolidWorks program, allows engineers to create designs using intuitive geometric constraints. Often, the engineer is seeking to create a well-constrained, or minimally rigid, design that does not allow internal motion. A recent characterization [8] describes a combinatorial condition for the minimal rigidity of body-and-cad frameworks, which model the majority of CAD structures, and we present an algorithm for this condition in Section 3.

However, any combinatorial characterization in rigidity theory is only valid for generic structures with given combinatorics. For body-and-cad frameworks, a structure is generic if a specific polynomial, the so-called “pure condition,” does not vanish. The pure condition is the determinant of the Jacobian of the constraint equations. For a fixed combinatorial design, the set of non-generic realizations has measure zero, making generic characterizations robust for many applications in, e.g., physics (unstructured glasses, jamming in dense-soft systems), sensor networks (localization), or structural biology (protein flexibility).

CAD, however, has a different character. Designs often feature repeated motifs, parallel components, and symmetries. In this scenario, genericity is not a well-founded assumption. Formally, the reason is that these special elements are the result of the geometry satisfying equations not implied by the combinatorics, which is the essence of non-generic situations.

The topic of this paper is detecting when a generically rigid structure is actually in a special position that allows an internal motion, which must be present for geometric, not combinatorial,
reasons. One approach to this problem is to evaluate the pure condition, but this can suffer from numerical instability, making it difficult to tell whether a structure is in or simply near a special position. We present a combinatorial-algebraic approach to factoring the pure condition, reducing the problem of analyzing it to one of lower degree.

However, simply detecting the presence of an unexpected infinitesimal motion does not give an explanation for the geometric conditions causing it. Therefore, we also present algorithms for automatically deriving intuitive, geometric conditions that imply both non-genericity and the existence of infinitesimal motions. These so-called “synthetic” statements take forms such as “these three planes being parallel and mated to the same point are over-defining the design,” which can then be presented as usable feedback to users of CAD systems. While such geometric interpretations may not always exist, we provide three case studies of designs demonstrating our approach.

1.1 Contributions

We present an end-to-end system for identifying special positions of body-and-cad frameworks and extracting geometric information algebraically.

A body-and-cad framework $G(p)$ is a collection of finitely many full-dimensional rigid bodies and a finite set of pairwise geometric coincidence, angular, and distance constraints among affine linear spaces affixed to the bodies. A multigraph $G$ captures the combinatorics of the structure, and a collection of vectors $p(e)$, one for each $e \in E(G)$, coordinatize its geometry.
The central object of study is $C_G$, the pure condition\(^1\) of $G$, which is the determinant of a matrix $M(G(x))$ whose nonzero entries are formal indeterminates replacing the entries of $p$. Evaluated for a specific assignment of vectors $p(e)$, the kernel of $M(G(p))$ contains the internal infinitesimal motions of $G(p)$. Our contributions are in three parts:

- An efficient pebble game algorithm for detecting when the pure condition vanishes identically and the minimal obstructions to its non-vanishing (i.e., “infinitesimal rigidity”); see Section 3.

- A combinatorial algorithm that computes the irreducible factors based on a new graph-theoretical characterization of these factors and a polynomial-space/polynomial-delay enumeration scheme; see Section 4.

- A algebraic-geometric approach based on White’s Cayley factorization algorithm for extracting Rota-type synthetic invariants; see Section 5.

In addition, Section 5 validates our approach with a series of case studies that compare the present work with the existing state-of-the-art in commercial software, using the popular SolidWorks software. As a preview, consider Figure 1: SolidWorks indicates that it is “Fully Defined,” yet is able to find a motion numerically. Our methods are able to predict this unexpected behavior.

1.2 Results in context

While we present, for the first time, a complete, automated, “end-to-end” approach to analyzing arbitrary body-and-cad designs, the tools we build on have appeared in various theoretical and applied contexts. Combinatorially, our $[a, b]$-graphs, are part of a widely-studied family of “sparse matroidal graphs” that are fundamental in generic rigidity theory (see [20, Appendix], which reports work of White and Whiteley). Of particular relevance is [7, 12] and the pebble game algorithms there; pebble games are, by now, the standard for checking generic rigidity. The extension to circuit finding is, perhaps, less well-known, but appears before in [7, Section 6] and has been used in [11].

The foundations of body-and-cad rigidity theory were established in [5] through the construction of the rigidity matrix and corresponding combinatorial model. A combinatorial characterization of body-and-cad rigidity incorporating 20 of these 21 geometric constraints (omitting point-point coincidences) was proven in [8]. The development of body-and-cad rigidity theory follows the algebraic approach of White and Whiteley [17], requiring special treatment for angular constraints. Within the body-and-bar setting, White and Whiteley establish the relationship between factors of the pure condition $C_G$ and rigid subgraphs of $G$; a set of examples were presented that demonstrate potential for geometric interpretations of the vanishing of $C_G$ via Cayley factorization.

Singularity analysis plays a central role in engineering applications, where specific examples, such as the Stewart Platform [3] have been intensively studied from a variety of perspectives, including that of synthetic geometry [9].

A numerical approach for determining generic rigidity relies on testing the system at a “witness” corresponding to a specific solution of the CAD constraints in which parameters dimensioning lengths and angles may differ from the design’s values, but remain fixed [10]. Under the

\(^1\)This terminology is from White and Whiteley [17].
approach’s main assumption that special positions are a set of measure zero, this witness will have the properties of a generic solution with probability 1. However, real-world designs are not chosen at random from the set of all realizations, leading to the study of special positions.

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2. Background

In this section, we provide combinatorial preliminaries, then give background for the development of the pure condition for body-and-cad rigidity. We conclude with relevant concepts and definitions from the Grassmann-Cayley algebra, which we will use to provide geometric interpretations of the vanishing of the pure condition. For a more thorough treatment, see [17] for body-and-bar rigidity theory; the body-and-cad rigidity theory [5,8] follows a similar development.

2.1 Combinatorics

Combinatorial counting properties arise as necessary, and sometimes sufficient, conditions in rigidity theory. For integers \( k, \ell \), a multigraph (with self-loops allowed) \( G = (V, E) \) is \((k, \ell)\)-sparse if every subset of \( n' \) vertices spans at most \( kn' - \ell \) edges; if, in addition, \( G \) has exactly \( kn - \ell \) edges, it is called \((k, \ell)\)-tight. For brevity, \((k, \ell)\)-tight graphs will be called \((k, \ell)\)-graphs. A subset of vertices of \( G \) that induces a \((k, \ell)\)-graph is a \((k, \ell)\)-block. The Nash-Williams and Tutte Theorem [11,15] states that \( G \) is a \((k, k)\)-graph if and only if \( G \) is the edge-disjoint union of \( k \) spanning trees. When \( \ell \in [0, 2k) \), \((k, \ell)\)-graphs form the bases of a matroid [7], which we call the \((k, \ell)\)-matroid.

2.2 Body-and-cad frameworks

Given \( n \) bodies and geometric constraints between affine linear spaces affixed to pairs of them, the first-order behavior of the system can be studied by associating a set of “angular” (affecting only rotational degrees of freedom) and “blind” (generically nonzero in each coordinate) linearized constraints. The combinatorics are encoded via a bi-colored graph \( G = (V, E = R \cup B) \) on vertex set \([n] = \{1, \ldots, n\}\) with a vertex for each rigid body, a “red” edge (in \( R \)) for each “angular” constraint, and a “black” edge (in \( B \)) for each “blind” constraint. The linearized equations are
collected into a “rigidity matrix” whose kernel corresponds to infinitesimal motions satisfying the constraints; we generalize this with the following definition.

**Definition 2.1.** For integers $a, b$, let $k = a + b$. We define an $[a, b]$-frame $G(p)$ to be a bi-colored graph $G = (V, E = R \cup B)$ with $kn - k$ edges, along with a function $p : E \to \mathbb{R}^k$. The function $p$ labels each edge with a $k$-vector, which is zero in the last $b$ entries if the edge is in $R$. The generic $[a, b]$-frame $G(x)$ has formal indeterminates replacing the nonzero coordinates of $p$.

A 3D body-and-cad framework is a $[3, 3]$-frame where the coordinates of the 6-vectors are in correspondence with the coordinates of instantaneous screws (or twists) whose first 3 coordinates encode rotational motion. When $b = 0$, we will refer to the $[k, 0]$-frame as simply a $k$-frame. A 3D body-and-bar framework contains only point-point distance constraints (called bars) and is a 6-frame in which the 6-vectors are the Plücker coordinates of the lines lying along the bars.

To any $[a, b]$-frame $G(p)$, we define a rigidity matrix $M(G(p))$. This matrix has $k$ columns for each vertex $i$ and one row for each edge of $G$. In the row corresponding to an edge $e$ with endpoints $i$ and $j$ (where $i < j$), we have $p(e)$ in the columns corresponding to $i$, $-p(e)$ in the columns corresponding to $j$, and zeroes in all other entries.

If $G(p)$ corresponds to a body-and-cad framework, then the kernel of $M(G(p))$ is the space of infinitesimal motions that preserve the constraints. This space always contains the trivial motions of Euclidean space that assign the same twist to each body. We can eliminate these trivial motions by “tying” down a body, or equivalently, appending to $M(G(p))$ a $k \times kn$ matrix whose only nonzero entries are given by the identity matrix in the first $k$ columns. We denote the tied down rigidity matrix by $M_T(G(p))$. With these preliminaries we can now formally define the pure condition associated to $G$.

**Definition 2.2.** Given an $[a, b]$-frame, the pure condition $C_G$ is the determinant of the matrix $M_T(G(x))$ associated with the generic $[a, b]$-frame $G(x)$.

We remind the reader that an $[a, b]$-frame is infinitesimally minimally rigid if its pure condition is nonzero and generically minimally rigid if the pure condition of $G(x)$ (as a polynomial) is nonzero. In Section 4, we will generalize the tie down, as the vanishing and non-vanishing of $C_G$ is generically independent of its choice (Theorem 7 below).

We can compute the determinant of $M_T(G(x))$ by Laplace expansion about groups of $k$ columns corresponding to each vertex. Thus, it is natural to write the determinant as a sum of products of $k \times k$ minors, which are each the determinant of $k$ edge labels. If $v_1, \ldots, v_k$ are $k$-vectors, then the bracket $[v_1 \cdots v_k]$ is the determinant of the matrix whose columns are $v_1, \ldots, v_k$ in order. Hence, we can write $C_G$ as a polynomial in brackets. The representation of $C_G$ as a bracket polynomial is not unique. In fact, bracket polynomials in a fixed set of vectors are elements of the homogeneous coordinate ring of a Grassmannian, and writing $C_G$ as a bracket polynomial corresponds to choosing a coset representative in this quotient ring. For more detail, see [13].

A result from [8] gives a combinatorial characterization of when the pure condition (as a polynomial) is nonzero:

**Theorem 1.** An $[a, b]$-frame with underlying graph $G = (V, E = R \cup B)$ is generically minimally rigid if and only if $\exists B' \subseteq B$ such that:

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2 The $[a, b]$-frame defined here is equivalent to the $(a + b, a)$-frame defined in [8].
• \((V, R \cup B')\) is the edge-disjoint union of a trees, and
• \((V, B \setminus B')\) is the edge-disjoint union of b trees

2.3 Cayley factorization

We use the Grassmann-Cayley algebra to express the linearized constraints and provide background for the factorization concepts and algorithm.

2.3.1. The Grassmann-Cayley algebra

The Grassmann-Cayley algebra of a vector space \(V = \mathbb{R}^k\), denoted \(\wedge(V)\), was defined by Rota as a way of encoding geometric statements about linear subspaces of \(V\) algebraically, and a comprehensive introduction is given in [13]. A linear subspace with basis \(a_1, \ldots, a_\ell\) is represented up to a nonzero scalar by the extensor, \(a_1 \vee \cdots \vee a_\ell\). Here, the \(\vee\) denotes the usual exterior product of vectors, and in this context is called the join, as the exterior product of two extensors represents the join, or sum, of the corresponding subspaces. We will usually suppress the \(\vee\)'s and write \(a_1 \cdots a_\ell\) for the join of the \(\ell\) vectors.

The Grassmann-Cayley algebra also has another operation, the meet, which is meant to encode the intersection of subspaces. We require that \(c + d \geq k\), and define

\[
(a_1 \cdots a_c) \wedge (b_1 \cdots b_d) = \sum_{\sigma} \text{sign}(\sigma)[a_{\sigma(1)} \cdots a_{\sigma(k-d)}] b_1 \cdots b_d a_{\sigma(k-d+1)} \cdots a_{\sigma(c)},
\]

where \(\sigma\) is a shuffle of \(\{1, \ldots, c\}\), i.e., a permutation which preserves \(\sigma(1) < \cdots < \sigma(k-d)\) and \(\sigma(k-d+1) < \cdots < \sigma(c)\). The bracket \([a_{\sigma(1)} \cdots a_{\sigma(k-d)}] b_1 \cdots b_d\) appearing in the definition of the meet operation is a scalar obtained by taking the determinant of the matrix with columns given by the vectors \(a_i\).

A Grassmann-Cayley algebra expression is simple if it only contains the meet and join operations (and not the vector space sum operation); such expressions encode geometric information. For example, suppose that \(V = \mathbb{R}^3\) and \(a, b, c, d \in V\). Then the expression \((ab) \wedge (cd) = [acd]b - [bcd]a\) is a vector along the line in which the planes \(ab\) and \(cd\) meet. A simple Grassmann-Cayley expression can be expanded into a bracket polynomial if it lies in the degree zero piece of the Grassmann-Cayley algebra.

2.3.2. White’s Cayley factorization algorithm

The pure condition of a body-and-cad framework is a multilinear bracket polynomial. We can reverse the process described in Section 2.3.1 using White’s Cayley factorization algorithm. (See [13] [18].) The goal is to produce a simple Grassmann-Cayley expression from a bracket polynomial with the hopes of using the geometric interpretation of the Grassmann-Cayley expression to detect special flexible embeddings of our framework.

The algorithm proceeds by first finding the atomic extensors, i.e., the extensors that would appear in a factorization if it exists. Then, using the straightening law in the bracket ring, brackets are stripped off, meets between atomic extensors are determined, and the remaining bracket polynomials are factored recursively.
3. Pebble games for CAD

We begin by giving a combinatorial algorithm for deciding whether a body-and-cad framework is generically minimally rigid. We provide a pebble game that works with the following class of graphs; see Figure 2 for an example.

**Definition 3.1.** Let $G = (V, E = R \cup B)$ be a bicolored graph, $a, b$ be positive integers, and $k = a + b$. Then $G$ is $[a, b]$-sparse if there exists $B' \subseteq B$ such that:

- $(V, R \cup B')$ is $(a, a)$-sparse,
- $(V, B \setminus B')$ is $(b, b)$-sparse

If, in addition, $G$ has exactly $kn - k$ total edges, then $G$ is $[a, b]$-tight, or an $[a, b]$-graph.

By Tutte and Nash-Williams [11, 15], the $[a, b]$-tight property is equivalent to there existing $B' \subseteq B$ such that $(V, R \cup B')$ is the edge-disjoint union of $a$ spanning trees and $(V, B \setminus B')$ is the edge-disjoint union of $b$ spanning trees.

Rigidity of body-and-cad structures is characterized by $[1, 2]$-graphs in 2D and $[3, 3]$-graphs in 3D (omitting point-point coincidence constraints [8]). That this class is matroidal follows from the Matroid Union Theorem [2] Prop. 7.6.14. Therefore, for an $[a, b]$-sparse graph $G = (V, E)$ and an edge $e$, we will say that $e$ is independent in $G$ if $G' = G + e$ is also $[a, b]$-sparse and dependent otherwise.

3.1 Algorithm

The $[a, b]$-pebble game Algorithm proceeds in the same way as the colored $(k, \ell)$-pebble game from [12], with a few extra steps. For completeness, we describe the entire process (see Algorithm 1), italicizing the prominent modifications. As in the $(k, \ell)$-pebble game algorithm with components in [7], it is straightforward to adapt Algorithm 1 to maintain and detect components (subgraphs that are $(a + b, a + b)$-tight and maximal with respect to vertices).

![Figure 2](image-url)
Algorithm 1 The \([a, b]\)-pebble game algorithm.

**Input:** A bicolored graph \(G = (V, E = R \sqcup B)\), with \(R\) a set of red edges and \(B\) a set of black edges.

**Output:** \([a, b]\)-sparsity property: tight, sparse, dependent and contains spanning tight, or dependent.

**Setup:** Initialize an empty directed graph \(H\) on vertex set \(V\). On each vertex, place \(a\) aqua pebbles and \(b\) black pebbles.

**Valid moves:**

- **Add red edge** \(ij\):
  - Precondition: \(\geq a + b + 1\) total pebbles on \(i\) and \(j\), at least \(a + 1\) of which are aqua
  - add the new edge, cover it with an aqua pebble from \(i\) (there is one by the precondition)
  - orient \(ij\) out of \(i\)

- **Add black edge** \(ij\):
  - Precondition: \(\geq a + b + 1\) total pebbles on \(i\) and \(j\)
  - add the new edge; cover it with a pebble from \(i\) using black (if there are \(b + 1\) black) or aqua (otherwise, there are \(a + 1\) aqua)
  - orient \(ij\) out of \(i\)

- **Red edge reversal**:
  - Precondition: vertex \(j\) has an aqua pebble on it and a red in-edge \(ij\)
  - reverse the edge by taking the aqua pebble off \(j\), putting it on \(ji\) and returning the (aqua) pebble originally covering \(ij\) onto \(i\)

- **Black edge reversal**:
  - Precondition: vertex \(j\) has a pebble on it and a black in-edge \(ij\)
  - reverse the edge by returning the pebble covering \(ij\) onto \(i\) and covering \(ji\) with the pebble from \(j\)

- **Pebble swap at vertex** \(i\):
  - Precondition: vertex \(i\) has a black out-edge \(e\) covered by an aqua pebble and another black out-edge \(f\) covered by a black pebble
  - Exchange the pebbles on \(e\) and \(f\)

- **Pebble flip at vertex** \(i\):
  - Precondition: vertex \(i\) has a black out-edge and a pebble of the opposite color on it
  - Exchange the pebble covering the out-edge with the one on the vertex

**Method:** Attempt to add each edge \(ij \in E\) to \(H\). If there are not enough pebbles on \(i\) and \(j\), attempt to collect them. Use DFS with (1) the restriction that traversal on a red edge into a vertex with \(a\) red out-edges may explore along only those \(a\) red edges, and (2) the modification that each vertex contains two marks: “red-visited” and “black-visited,” for incoming red and black edges, respectively. If a path is found, use the moves, including **pebble swap** and **flip** where necessary, to reverse it and collect a pebble. If enough pebbles can be collected, **add** the edge; otherwise, **reject** it.

If every edge is added: output **tight** if there are \(a + b\) pebbles remaining and **sparse** otherwise. Else, there were rejected edges: output **dependent and contains spanning tight** if there are \(a + b\) pebbles remaining and **dependent** otherwise.

### 3.2 Complexity analysis

The running time of Algorithm 1 is the same as for the basic pebble game [Algorithm 1](#), since the modified depth-first search needs to cross each edge at most two times.
3.3 Correctness

In this section we prove the correctness of Algorithm [1]

**Theorem 2.** A bicolored graph is \([a, b]\)-sparse if and only if it can be constructed with the pebble game.

The matroidal property of \([a, b]\)-graphs implies that the order in which we add the edges is irrelevant. Here are some invariants that hold throughout the run of the pebble game.

**Proposition 3.2.** Let \(X\) be the subgraph induced on edges covered by aqua pebbles and \(Y\) the subgraph induced on edges covered by black ones. Let \(k = a + b\).

For any subset \(V'\) of \(n'\) vertices, let \(p'_a\) and \(p'_b\) be the number of aqua and black pebbles on vertices of \(V'\), \(m'_a\) and \(m'_b\) the number of edges covered by aqua and black pebbles induced by \(V'\), and \(o'_a\) and \(o'_b\) the number of edges with tail in \(V'\) and head in \(V \setminus V'\) covered by aqua and black pebbles.

The following invariants hold throughout Algorithm [1]: (i) \(p'_a + p'_b + m'_a + m'_b + o'_a + o'_b = kn'\); (ii) \(p'_a + m'_a + o'_a = an'\); (iii) \(p'_b + m'_b + o'_b = bn'\); (iv) Every red edge is in \(X\); (v) \(X\) is \((a, a)\)-sparse; and (vi) \(Y\) is \((b, b)\)-sparse.

**Proof.** These follow directly from the proof of [7] Lemma 10. The pebble swap and pebble flip moves maintain the sum of out-edges and pebbles at a vertex, required by invariants (i)-(iii). □

**Proof of Theorem 2.** The final two invariants in Proposition 3.2 imply that, at all times, the graph being built by the pebble game is \([a, b]\)-sparse, giving one direction.

What remains to be proven is that at any stage of the pebble game, if \(H\) the underlying \([a, b]\)-sparse graph and \(H + e\) is also \([a, b]\)-sparse for some edge \(e = ij\), then \(e\) will be added.

By Proposition 3.2, if we let \(X\) be the edges in \(H\) covered by aqua pebbles and \(Y\) be the edges in \(H\) covered by black pebbles, then \(X\) is \((a, a)\)-sparse and \(Y\) is \((b, b)\)-sparse. By the same argument as in [7] Proof of Lemma 13, applied to \(X\) and \(Y\) separately, Proposition 3.2 shows that we can collect \(a\) aqua pebbles and \(b\) black ones on \(i\). Thus, we may assume the given configuration has \(k\) pebbles on \(i\) and hence the out degree of \(i\) is 0.

We first assume that \(e\) is red. We will show that, if we do not already have \(a + 1\) aqua pebbles on \(i\) and \(j\) together, then we can collect an extra aqua pebble onto \(j\).

It is possible that even though \(H + e\) is \([a, b]\)-sparse, \(X + e\) is not \((a, a)\)-sparse if there are black edges covered by aqua pebbles. However, since \(e\) is independent, there is a partition \(\{X', Y'\}\) of the edges of \(H\) such that \(X'\) is \((a, a)\)-sparse and contains all the red edges, \(Y'\) is \((b, b)\)-sparse, and \(X'\) does not contain an \((a, a)\)-block spanning \(i\) and \(j\).

Given such an \(X'\) and \(Y'\), one can produce a pebble game configuration satisfying the invariants of Proposition 3.2 as follows. For each vertex \(i\), start with \(a\) aqua and \(b\) black pebbles; then, using the edge orientations of the current configuration, for each out-edge \(ij\), cover \(ij\) with an aqua pebble if \(ij \in X'\) and a black pebble otherwise. Since the orientation is the same as \(H\), the only difference with the current configuration may be locally at each vertex: the color of pebbles that are free or covering black out-edges may need to be changed. Therefore, the pebble swap and pebble flip moves can convert \(\{X, Y\}\) to \(\{X', Y'\}\).

The moves changing \(\{X, Y\}\) to \(\{X', Y'\}\) preserve directed paths. Therefore, upon completion of these moves, by the single-colored pebble game applied to \(X'\), there must be a directed path \(P\) contained in \(X'\) starting at \(j\) and ending at a different vertex with an aqua pebble; reversing this path brings an aqua pebble as desired.
To complete the proof, observe that, in the original configuration \( \{X, Y\} \), the path \( P \) must be allowed, since at any vertex with a red edge pointed out, \( X' \) and \( X \) must coincide (and no other vertices place restrictions on the path). Furthermore, the specified depth-first search will find \( P \); the only restriction on allowed paths comes from the color of an incoming edge, and the search procedure will try both at each vertex when possible. This concludes the proof for when \( e \) is red.

The case in which \( e \) is black is symmetric.

3.4 Finding circuits

If the pebble game rejects an edge, we can extract more information. In particular, we show:

**Theorem 3.** Let \( H \) be a pebble game configuration and \( e = ij \) be an edge rejected by the pebble game. Collect \( k \) pebbles on \( i \) and define \( H' \) to be the subgraph induced by vertices reachable via allowed paths from \( j \). Then \( H' + e \) is a circuit.

**Proof.** For convenience, define \( H'' \) to be the graph \( H' + e \). After removing any edge from \( H' \), a pebble appears on a vertex in \( H' \) not equal to \( i \). By hypothesis, it can be collected on \( j \), implying that it is possible to add the edge \( e \). From this and Theorem 2, it follows that \( H'' \) is minimally dependent.

4. Factoring the pure condition

The \([a, b]\)-pebble game can be used to determine whether a body-and-cad framework is generically minimally rigid. However, even in this case, the specific embedding may actually be in a special position, witnessed by the vanishing of the pure condition. In this section, we show how to compute the irreducible factors of the pure condition, as these allow us to reduce the problem of understanding special positions to smaller instances.

4.1 Body-and-bar frameworks

We begin by considering only bar constraints to provide the groundwork for body-and-cad frameworks. A body-and-bar framework is generically minimally rigid if and only if it is a \( k \)-frame whose underlying graph is \((k, k)\)-tight \([14] \).

4.1.1. Preliminaries

Let \( G \) be a \((k, k)\)-graph and therefore the edge-disjoint union of \( k \) spanning trees \( T_1, T_2, \ldots, T_k \); the \( T_i \) are called a tree decomposition \( T \). We say that \( G \) is “tied down” by the addition of (any) \( k \) loops \( \ell_1, \ell_2, \ldots, \ell_k \); a standard tie down puts these loops at one vertex. A looped tree decomposition of \( G \) assigns the loop \( \ell_i \) to the tree \( T_i \) to partition the edges of \( G \). A \( k \)-fan diagram \( F \) is an out-degree exactly \( k \) orientation of the edges of \( G \), where loops are taken to always point out of the vertex where they are located; define \( F_i \) to be the set of edges oriented out of vertex \( i \).

Now let \( G(x) \) be a generic \( k \)-frame. We fix an ordering on the vertices by identifying them with the set \([n]\), and use this to induce a “base orientation” on the edges by orienting \( ij \) from \( i \) to \( j \) when \( i < j \). We also fix a “base ordering” of the edges, which induces an order on any subset, and, when converting unordered subsets of the edges to ordered ones, we choose the one from the base ordering. The two orderings capture the ordering of rows and columns in the rigidity
(a) Two of 36 tree decompositions and associated tree monomials.

(b) The only two $k$-fan diagrams and associated fan monomials.

Figure 3: A $(3,3)$-graph with a standard tie down on vertex 1 (denoted by double-outline).

matrix; the base orientation of the edges is consistent with the convention of putting the “plus” entries to the left of the “minus” entries from Section 2.

For a spanning tree along with a single loop on some root vertex, we define the sign $\epsilon(T)$ to be $(-1)^t$, where $t$ is the number of edges disagreeing with the base orientation when all edges are oriented towards the root. For a tree decomposition $\mathcal{T}$ of a tied-down $(k,k)$-graph, the tree decomposition monomial $\{\mathcal{T}\}$ is then defined as

$$\text{sgn}(\tau) \text{sgn}(\sigma) \prod_{i \in [k]} \epsilon(T_i) \prod_{e \in T_i} x(e)_i$$

where $\sigma$ is the permutation of the edges that puts them in the order $T_1, T_2, \ldots, T_k$, and $\tau$ is the fixed permutation that shuffles the columns of $M(G(x))$ so they are ordered by coordinate and then by vertex. Figure 3(a) shows an example of the correspondence. For a $k$-fan diagram $\mathcal{F}$, define $\epsilon(\mathcal{F})$ as $(-1)^t$, where $t$ is the number of edges disagreeing with the base orientation. Then the fan bracket monomial $\{\mathcal{F}\}$ is defined as

$$\text{sgn}(\sigma) \epsilon(\mathcal{F}) \prod_{i \in [n]} [F_i]$$

where $\sigma$ is the permutation putting the edges in the order $F_1, F_2, \ldots, F_n$. Figure 3(b) shows the correspondence between $k$-fan diagrams and bracket monomials.

4.2 Combinatorial formulas for the pure condition

With this terminology, we can state an important theorem which can be extracted from 17 and 19.

**Theorem 4.** Let $G$ be a tied down $(k,k)$-graph. Then, the pure condition $C_G$ is given by the following
formulas:

\[ C_G = \sum_{\text{tree \ decomp. } T} \{ T \} \quad \text{in } \mathbb{R}[x] \quad (1) \]

\[ C_G = \sum_{\text{k-fan \ diagrams } F} \{ F \} \quad \text{in the bracket ring} \quad (2) \]

where Equation (2) is a coset representative associated with the tie down. In any tie down, the vectors \( x(\ell_i) \) labeling the loops factor out of Equation (1); in a standard tie down, they factor out of Equation (2). In particular, the vanishing of \( C_G \) is independent of any tie down by Equation (1).

Figure 4 shows how the vectors labeling the loops in a generalized tie down appear in the pure condition in its bracket ring representation. These vectors factor out of the polynomial form, since the self loops may be associated with the trees in an arbitrary way.

4.3 Overview of the approach

Our general scheme for finding the irreducible factors of the pure condition is based on the following result of White and Whiteley:

**Theorem 5** ([17, Remark 4.13]). Let \( G \) be a \((k,k)\)-graph. Then the pure condition of \( G \) is irreducible (in either the polynomial ring or bracket ring) if and only if \( G \) contains no proper block.

In light of Theorem 5, we define a \((k,k)\)-graph with no proper block to be an irreducible \((k,k)\)-graph. An alternative characterization of irreducible \((k,k)\)-graphs is that they are circuits in the \((k,k+1)\)-matroid [7]. A refinement of Theorem 5 may be extracted from [17]. We give a proof for completeness.

**Theorem 6** ([17]). Let \( G \) be a \((k,k)\)-graph with a standard tie down and let \( H \) be a proper block of \( G \). Then \( C_G = \pm C_H \cdot C_{G/H} \).

A slightly more complicated statement holds for generalized tie downs. That \( C_{G/H} \) is well-defined follows from a basic fact about \((k,k)\)-graphs that we record for later use:

**Lemma 4.1** ([17]). Let \( G \) be a \((k,k)\)-graph and \( H \) a proper block. The contracted graph \( G/H \) is a \((k,k)\)-graph. \( \square \)
Proof of Theorem 6. According to Equation (1), each monomial of \( C_G \) corresponds bijectively with a tree decomposition. Because \( H \) is a block, i.e., a smaller \((k,k)\)-graph, any tree decomposition of \( G \) induces a decomposition of the edges of \( H \) into \( k \) edge-disjoint spanning trees (as opposed to just \( k \) forests, which happens for an arbitrary subgraph) of \( H \). It then follows from Lemma 4.1 that the tree decompositions of \( G \) can be parameterized by choosing, independently, a tree decomposition of \( H \) and a tree decomposition of \( G/H \), since the operation of contracting \( H \) contracts a single subtree of each of the \( T_i \) in any tree decomposition of \( G \).

4.3.1. Identifying the factor graphs. By iteratively applying Theorem 6, we can find a set of \((k,k)\)-graphs that correspond to the irreducible factors of the pure condition. We call these the factor graphs of \( G \). The factor graphs can be found with Algorithm 2.

Algorithm 2 Finding body-and-bar factor graphs.

**Input:** A \((k,k)\)-graph \( G \).

**Output:** A set \( \mathcal{G} \) of \((k,k)\)-graphs corresponding to the irreducible factors of \( C_G \).

**Method:**

1. Use the \((k,k+1)\)-pebble game to find a maximal \((k,k+1)\)-sparse subgraph \( G' \) of \( G \).
2. For each edge \( e \in G \setminus G' \), use the \((k,k+1)\)-pebble game to find the fundamental \((k,k+1)\)-circuit \( G_e \) of \( e \) in \( G' \).
3. Add all the \( G_e \) to \( \mathcal{G} \).
4. Contract each of the subgraphs \( G_e \) to obtain a graph \( G'' \).
5. If \( G'' \) has more than one vertex, set \( G := G'' \) and continue with step (i).

Correctness follows from Theorem 5 and Theorem 6; each identified factor graph is a \((k,k+1)\)-circuit, and the recursion ends when there are no more edges. Since the recursion is \( O(n) \) levels deep, and each level takes \( O(n^2) \) time with the pebble game [7], the total running time is \( O(n^3) \).

As an example, consider the Algorithm 2 applied to the \((3,3)\)-graph \( G_0 \) shown in Figure 5(a). The first step of the algorithm identifies each copy of the doubled triangle as a \((3,4)\)-circuit in \( G_0 \) (see Figure 5(b)). These are then contracted to produce the residual graph \( G_1 \) (see Figure 5(c)).
4.4 Producing bracket polynomials

To compute the pure condition of an irreducible \((k, k)\)-graph, Equation (2) implies that we only need to enumerate all the \(k\)-fan diagrams: once we have the bracket form, we can evaluate the bracket expressions to get the polynomial form. To do this, we will work with generalized tie downs, in which \(G\) is a subgraph of a \((k, k)\)-graph and more loops are allowed. The loops need to be placed so that \(G\) has a decomposition into \(k\) forests, with one loop assigned to each connected component of each forest. The definition of fan diagrams extends in the natural way.

Given \(G\) and a \(k\)-fan diagram, a polygon is a directed cycle; a polygon reversal is the operation of reversing a directed cycle. It is straightforward that polygon reversals produce an orientation that is another \(k\)-fan diagram. Algorithm 3 produces all \(k\)-fan diagrams in a tied down \((k, k)\)-graph \(G\) using polygon reversals and an idea from \([16]\).

**Algorithm 3** Enumerating \(k\)-fans

**Input:** A generalized tied down graph \(G\), and a \(k\)-fan diagram \(F\).

**Output:** A listing of the \(k\)-fans in \(G\).

**Method:**

1. If \(G\) has no polygon, output \(F\).
2. Otherwise \(G\) has a polygon \(P\). Let \(e = ij\) be a directed edge on \(P\).
3. Let \(G_1\) be the graph obtained by replacing \(e\) with a loop on \(i\) with the same label as \(e\). Recurse on \(G_1\).
4. Let \(G_2\) be the graph obtained by reversing \(P\) and then replacing \(e\) with a loop on \(j\) with the same label as \(e\). Recurse on \(G_2\).

The proof of correctness is analogous to that of a similar algorithm \([16]\, “Basic Algorithm”\]) for enumerating bipartite matchings. We use induction on the number of edges in \(G\). It is easy to see that if \(F\) has no polygon, then it is the only \(k\)-fan diagram on the generalized tied down graph \(G\). This establishes a base case for the recursion. Otherwise, there is an edge \(e\) on some polygon \(P\), making steps (ii)–(iv) well-defined. Moreover, in any \(k\)-fan diagram on \(G\), the edge \(e\) is oriented exactly one way, so the \(k\)-fan diagrams of the graphs \(G_1\) and \(G_2\) are disjoint sets. By induction, the algorithm will list all of them exactly once.

Since each internal node requires linear time, the total time used in the enumeration is \(O(n) \cdot N\), where \(N\) is the number of \(k\)-fans. However, the depth-first nature of the recursion implies that only \(O(n)\) cells of working memory are required.

4.5 Body-and-cad frameworks

We now turn to the general case of body-and-cad frameworks. Let \(G = (V, R \cup B)\) be an \([a, b] \)-graph; then \(\exists B' \subseteq B\) such that \((V, R \cup B')\) is the edge-disjoint union of \(a\) spanning trees and \((V, B \setminus B')\) is the edge-disjoint union of \(b\) spanning trees. For convenience, we will index the trees of an \([a, b] \)-tree decomposition \(T\) by \(A_1, A_2, \ldots, A_a, B_1, B_2, \ldots, B_b\). The definitions of generalized tie downs and \(k\)-fan diagrams extend in the natural way for \([a, b] \)-fan diagrams, with a new restriction that all red edges and loops can take part only in an “aqua” set of \(a\) outgoing edges at
4.6 The body-and-cad pure condition

From [8], we can extract a generalization of Theorem 4.

**Theorem 7.** Let $G$ be a tied down $[a, b]$-graph. Then, the pure condition $C_g$ has the expansions

$$C_G = \sum_{[a, b]-\text{tree}} \{T\} \quad \text{in } \mathbb{R}[x] \quad (3)$$

$$C_G = \sum_{[a, b]-\text{fan}} \{F\} \quad \text{in the bracket ring} \quad (4)$$

In particular, Equation (3) shows that the vanishing of $C_G$ is independent of any tie down.

4.7 Irreducible $[a, b]$-graphs

Let $G$ be an $[a, b]$-graph. We define $G$ to be irreducible (as a graph), if, for each edge $e$, the graph $G + e$, obtained by adding a copy of $e$ with the same color is a circuit in the $[a, b]$-matroid. Observe that, if all the edges are black, the the notions of irreducibility defined here and for body-and-bar frameworks coincide.

In the Appendix, we prove:

**Theorem 8.** Let $G$ be an $[a, b]$-graph. The pure condition $C_G$ is irreducible if and only if $G$ is irreducible.

The main challenge of extending the ideas from Section 4.1 is that an $[a, b]$-graph may be irreducible as an uncolored $(k, k)$-graph, but have a reducible pure condition. Figure 2 shows one such example with a $[2, 2]$-graph with one factor associated to the pair of edges $c$ and $d$ forming a red $(2, 2)$-block; the circuit of a copy of either of $c$ or $d$ is not the whole graph (it is just three parallel edges).

This phenomenon requires us to work with generalized tie downs to find all the factor graphs. Instead of contracting only proper $(k, k)$-blocks, we must separate out subgraphs corresponding to circuits of copies of edges, then tie down the remaining graph. These situations arise because of the more complicated circuit structure in the $[a, b]$-matroid. Combinatorial descriptions for circuits include the following: (i) graphs that are, in terms of their underlying uncolored edge set, $(k, k)$-circuits; (ii) graphs with all red edges that are $(a, a)$-circuits; and (iii) graphs that are constructed from a spanning all red $(a, a)$-graph and an all black $(b, b)$-circuit on a proper subset of the vertices. We suspect this is a complete classification, which would simplify the algorithm, but a proof remains an open problem.

The structural results, which follow readily from Theorem 7 and Theorem 8, that we need are:

**Lemma 4.2.** Let $G$ be an $[a, b]$-graph and let $H$ be a subgraph of $G$ that, as an uncolored graph, is a $(k, k)$-block. Then $C_G = \pm C_{G/H} \cdot C_H$. \hfill $\square$
Lemma 4.3. Let $H$ be a subgraph of an $[a, b]$-graph with a generalized tie down. If, for all edges $e$ of $H$, adding a copy of $e$ produces a graph that is an $[a, b]$-circuit (as opposed to simply containing one), then $C_H$ is irreducible.

4.8 Finding the factor graphs

By Lemma 4.2, we can use Algorithm 2 to reduce the problem to the case in which our $[a, b]$-graph $G$ is irreducible as a $(k, k)$-graph. For the leaves of the recursion tree, we then use Lemma 4.3 to extract generalized tie downs that have irreducible pure conditions. Algorithm 4 gives the overall structure.

Algorithm 4 Finding body-and-cad factor graphs

Input: A generalized tied down $[a, b]$-graph $G$ that is irreducible as a $(k, k)$-graph.
Output: The collection $G$ of the irreducible factor graphs.
Method:

1. For each edge $e$ of $G$, check, using the $[a, b]$-pebble game, whether the fundamental circuit of $e$ in $G$ is all of $G + e$. If so, $G$ is irreducible. Add it to $G$ and stop.

2. Otherwise, there is an edge $e$ and a proper subgraph $H$ of $G$ such that $H + e$ is a circuit.

3. Find an $[a, b]$-fan diagram $\mathcal{F}$ of $G$.

4. Form the graph $G'$ by removing all the edges of $H$ from $G$ and replacing them them by a loop on the tail.

5. Form the graph $H'$ by removing all the edges of $G \setminus H$ from $G$ and replacing them with a loop on the tail.

6. Recurse on $G'$ and $H'$.

To find the pure condition in the polynomial ring, we can compute the pure condition of each factor graph, remove the variables corresponding to loops, and then multiply the resulting polynomials together.

4.9 Enumerating $[a, b]$-fans

What remains to be shown is how to compute the pure condition for a generalized tie down of a subgraph of an $[a, b]$-graph. This is very similar to Algorithm 2 except that we must preserve the $[a, b]$-fan diagram property. To do this, we define an admissible polygon in an $[a, b]$-fan diagram to be a directed cycle such that every red edge into a vertex with a red edges pointing out is followed by a red edge. To reverse an admissible polygon, we first modify our $[a, b]$-fan diagram so that every red edge is followed by an edge in the aqua part and then reverse the directed cycle. It is straightforward to check that:

Lemma 4.4. An admissible polygon reversal transforms an $[a, b]$-fan diagram into another $[a, b]$-fan diagram. Moreover, if there is no admissible polygon, then there is exactly one $[a, b]$-fan diagram.
Since we can find an admissible polygon with a small modification to depth-first search, we have completed the generalization of Algorithm 2.

5. Case studies

The algorithms from Section 4 give us a way of producing factors of $C_G$. The vanishing of any factor indicates a special position, allowing the detection of when a framework will exhibit non-generic behavior. This can provide useful feedback in identifying a minimal set of constraints that are over-defined. Ultimately, though, we seek a method for providing more informative feedback, particularly focusing on describing geometric conditions leading to the special position.

In this section, we initiate the study of finding such geometric conditions using the Grassmann-Cayley algebra. We provide three case studies of generically minimally rigid structures with irreducible pure conditions admitting a Cayley factorization and construct designs in special positions that admit internal motions. Using the popular SolidWorks CAD application, we observe behaviors indicating that even commercial software does not have robust methods for dealing with special positions of generically rigid structures.

These case studies motivate the need for future work and show that the Grassmann-Cayley algebra is a powerful language for connecting the geometry of a structure to its pure condition. At present, we are not concerned with how to obtain a Cayley factorization or which structures admit one. The examples that we present are simple enough that pure conditions may be factored by hand, though we also used an implementation of the Cayley factorization algorithm coded in Macaulay 2 \cite{2}. Instead we focus on the geometric interpretations of the given factorizations.

5.1 The doubled triangle: bodies and bars in 2D

We start with a simple body-and-bar example in 2D and investigate its behavior in the Sketch environment of SolidWorks. Consider the doubled triangle (Example 3.1 in \cite{17}). We see in Figure 3(b) that if we tie down vertex 1 there are only two possible 3-fan diagrams. These diagrams imply that the pure condition is $[abd][cef] - [abc][def]$ and it has Cayley factorization $(cd \wedge ab) \wedge ef$.

To understand the geometric interpretation of the Grassmann-Cayley factorization, suppose that we have an embedding of $G$ in $\mathbb{P}^2$. Then $a, b, c, d, e, f$ are the Plücker coordinates of lines along the bars in this embedding of $G$, which are elements of $(\mathbb{P}^2)^*$, a copy of $\mathbb{P}^2$ dual to our original. Hence, the geometric interpretation of $(cd \wedge ab) \wedge ef$ in $(\mathbb{P}^2)^*$ is that we find the intersection $p$ of the lines $cd$ and $ab$ and ask if $p$ meets the line $ef$. In general, $p$ will not be on $ef$, and together they will span all of $(\mathbb{P}^2)^*$. In this case, $(cd \wedge ab) \wedge ef$ is the scalar given by evaluating the bracket polynomial at our points. If $p$ does lie on $ef$, then they fail to span $(\mathbb{P}^2)^*$, and the pure condition vanishes.

This example has a natural interpretation in terms of the geometry of the bars in the embedding of $G$ in our original $\mathbb{P}^2$. Here, we have $(c \wedge d) \vee (a \wedge b) \vee (e \wedge f)$, and the interpretation is that for each pair of vertices we find the intersection of the pair of lines between them and then take the span of these three intersection points. Either the three points span all of $\mathbb{P}^2$ or they are collinear, and the pure condition vanishes.

Using this insight, we construct an embedding of the doubled triangle with an internal motion. We embed the body-and-bar framework in the 2D Sketch environment of SolidWorks as a
bar-and-joint framework in which we represent our 3 rigid bodies by rigid subgraphs of $G$. We first confirm the behavior of a generic embedding, found to be correctly classified as “Fully Defined” by SolidWorks; see Figures 6(a) and 6(b) (the 3 bodies are highlighted in blue, green and red). Two of the rigid bodies (blue and green) consist of a bar joining a pair of vertices, while the third body (red) is a 4-bar cycle with a diagonal brace.

We then create two sketches of the special position, making the bars $a$, $b$, $c$ and $d$ parallel and the same length: in one sketch, we create the bars with dimensioned line segments, and in the other with equality relations. Since we have made the pair of bars corresponding to each doubled edge parallel, the three intersection points $(e \land f), (a \land b), \text{and } (c \land d)$, are all on the line at infinity. Hence, we know that they do not span all of $\mathbb{P}^2$, and the pure condition $C_G$ vanishes for this embedding. Indeed, both designs are correctly found to be “Under Defined” by SolidWorks (see Figures 6(c) and 6(d)); however, SolidWorks does not find a motion for the the dimensioned-bars sketch even though the framework is clearly flexible.

5.2 4 bodies in the plane: bodies and bars in 2D

We now turn to a slightly more complicated example in the plane. In Figure 7 we depict an irreducible 3-frame on 4 vertices together with the four 3-fan diagrams obtained when tying down vertex 1.

The Cayley factorization of $C_G$ is $(cd \land ab) \land ((ef \land gh) \lor i)$. The geometric interpretation in $(\mathbb{P}^2)^*$ is that we find the point in the intersection of lines $cd$ and $ab$, and meet it with the line spanned by the point $i$ and $ef$ intersect $gh$. 

Figure 6: Modeling the doubled triangle in SolidWorks. 

(a) Doubled triangle graph. (b) Generic embedding is “Fully Defined.”

(c) Special position when using dimensioned bars does not move, even though it is “Under Defined.” (d) Special position when using equality constraints does move and is “Under Defined.”
The dual expression is

$$(c \land d)(a \land b)(((e \land f)(g \land h)) \land i).$$

(5)

It represents the span of three points: the intersection of the lines $c$ and $d$, the intersection of the lines $a$ and $b$, and the intersection of the line $i$ with the line spanned by the two points $e \land f$ and $g \land h$. If we make the bars corresponding to a doubled edge parallel, then the lines along these bars intersect at a point on the line at infinity in $\mathbb{P}^2$. So let $a||b$, $c||d$, $e||f$, and $g||h$. This implies that $(e \land f)(g \land h)$ is the line at infinity. Hence, $(e \land f)(g \land h) \land i$ is a point on the line at infinity, and the three points in Equation 5 are collinear. Since they do not span all of $\mathbb{P}^2$, the pure condition vanishes.

As in Section 5.1, we embed the body-and-bar framework as a bar-and-joint framework in the plane and investigate the behavior using the SolidWorks Sketch environment. Figure 8(b) shows a generic embedding in which each body is constructed out of a 4-bar cycle with a diagonal brace; SolidWorks correctly classifies this as “Fully Defined.”

However, SolidWorks exhibited problematic behavior when presented with a special position satisfying the geometric conditions from the Cayley factorization. Embed $G$ so that the 4 bars in each set $\{a, b, f, e\}$ and $\{c, d, g, h\}$ are parallel and equal in length. In each set we dimension one bar and set the other 3 to be equal to that one in length; we then dimension the long diagonal bar.

At the bottom of Figure 8(c) we see that SolidWorks reports that the design is “Fully Defined,” which should indicate that it is rigid. However, SolidWorks admits an internal motion, as depicted. This may indicate that SolidWorks uses a combinatorial algorithm to determine when a structure is “Fully Defined.” Such an algorithm would detect that a generic embedding of this combinatorial structure is rigid. However, a numerical solver should allow a motion due to the special embedding.

When the structure in Figure 8(c) is saved and subsequently rebuilt, SolidWorks behaves in an unexpected fashion, suddenly identifying the structure as over constrained highlighting all the bars as part of a system that cannot be solved.

Even when the special position is designed using dimensioned line segments, SolidWorks still incorrectly labels it as “Fully Defined” while allowing a motion (refer back to Figure 1). However, it does save and rebuild successfully. This inconsistent behavior indicates that even commercial CAD software has difficulty with these special positions.
Figure 8: Modeling the single-braced, doubled 4-cycle in SolidWorks: (a) the underlying graph is the doubled 4-cycle with a single diagonal brace; (b) a generic embedding results in a “Fully Defined” system; (c) a special position for the graph has a motion (gray and black lines indicate two configurations along the path of the motion), even though SolidWorks still reports “Fully Defined”; (d) after saving, the rebuild process throws an alert that the system cannot be solved.
5.3 The quadrupled triangle: bodies and bars in 3D

We present a final example of a generically rigid body-and-bar framework in 3D, using the Assembly environment of SolidWorks to investigate the corresponding behavior. Let \( G \) be the quadrupled triangle depicted in Figure 9. \( G \) is a generically minimally rigid 6-frame. If we tie down a vertex, then there are six 6-fan diagrams corresponding the number of ways of choosing 2 of the 4 remaining (undirected) edges to direct from vertex 2 to vertex 3; see Figure 9(b).

![6-frame](image)

**Figure 9:** The quadrupled triangle.

The pure condition has a Cayley factorization \((hijk) \land (abcd) \land (efgh)\). The points \(a, b, c, d, e, f, g, h, i, j, k\) are the Plücker coordinates of lines in \(\mathbb{P}^3\); hence, they are points in \(\mathbb{P}^5\) lying on the Grassmannian \(Gr(1,3)\), which is a quadric hypersurface. Therefore, the geometric interpretation of the Cayley expression is an intersection of the 3-plane \(hijk\) with the 3-plane \(abcd\) to obtain a line. This line is then intersected with \(efgh\). We expect that a line and a 3-plane in \(\mathbb{P}^5\) will span all of \(\mathbb{P}^5\) and not intersect. If that is the case, then the Cayley expression evaluates to the value of the bracket polynomial evaluated at our points. Otherwise, the Cayley expression is zero, indicating a special position.

One way to make the Cayley expression vanish is to require that the 4 points \(a, b, c, d\) are collinear and span a line \(L\). Then the line \(L\) intersects \(Gr(1,3)\) in at least 4 points. Since \(Gr(1,3)\) is a quadric, this implies that \(L\) must lie on \(Gr(1,3)\). But by Exercise 6.4 in [6], this implies that the lines \(a, b, c, d\) are concurrent and lie in a plane. Moreover, if all 4 lines are parallel, then they are concurrent at a point on the line at infinity in this plane.

We can apply the same reasoning to the two other sets of quadrupled edges. Thus, an embedding of \(G\) in which the quadrupled edges are parallel bars is a special embedding for which the pure condition vanishes. Figure 10 shows an assembly in which the bars corresponding to each quadrupled edge are parallel and equal in length; each bar is a distinct part, and SolidWorks successfully explores the internal motion. When the bar constraints are specified as point-point distances instead of explicit parts (design not shown), SolidWorks successfully detects the dependencies, reporting that the design is “Over Defined.”

Interestingly, though, as the design moves towards a generic embedding, some unexpected behavior occurs. Figure 11(a) depicts a 3D assembly with the 12 point-point distance constraints;
this special position does admit an internal motion (which SolidWorks correctly explores). However, we would expect the design to be “Over Defined,” as for the previous special position, but it is labeled “Under Defined.” Modifying one constraint results in a generic embedding that is correctly identified as “Fully Defined” (see Figure 11(b)).

Figure 11: Embeddings of the quadrupled triangle.

6. Open questions

The case studies in Section 5 demonstrate the potential that Cayley factorization has for providing geometric interpretations of conditions for special positions. However, in general, the pure condition of a body-and-cad structure may not have a Cayley factorization. For such cases, it would be interesting to understand if any information could be obtained from auxiliary data produced by the Cayley factorization algorithm. For example, the algorithm produces a list of atomic
extensors of $C_G$, which are extensors that are guaranteed to appear in some Cayley factorization of $C_G$ if one exists. If $C_G$ is Cayley factorable, then the vanishing of an atomic extensor implies that the pure condition vanishes. It is natural to ask if the vanishing of an atomic extensor implies that the pure condition vanishes regardless of whether $C_G$ has a Cayley factorization.

Even when a Cayley factorization does exist, it is nontrivial to extract geometric information the original framework from it. A Cayley factorization of $C_G$ tells us about the linear geometry of the points that appear as labels of edges in $G$. In the body-and-bar world, these vectors are the Plücker coordinates of lines along bars. Geometric interpretations are relatively straightforward in the 2D body-and-bar setting as the vectors along edges of the graph lie in a space dual to the space in which $G$ is embedded. However, in the 3D body-and-bar setting, the Plücker coordinates of lines in $\mathbb{P}^3$ lie on the Grassmannian $G(1,3)$ which is a quadric surface in $\mathbb{P}^5$. We saw in Section 5 that it is substantially more difficult to translate geometric conditions about points on $G(1,3)$ into the geometry of lines in $\mathbb{P}^3$.

In the body-and-cad case, the vectors labeling the edges of a graph $G$ arise in many different ways and cannot all be interpreted as the vector of Plücker coordinates of lines. Moreover, since there are geometric constraints (e.g., a line-line coincidence) that impose multiple linear constraints, we may have to consider conditions on a set of vectors.

Finally, we must remember that $C_G$ vanishes when $G(p)$ is infinitesimally flexible; then the special positions that we find may not be truly flexible. However, an infinitesimally flexible structure carries an internal stress, which points towards weaknesses potentially still of interest to a CAD user. Moreover, we may be able to combine conditions implying a special position together to create degenerate embeddings with true motions.

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A. The pure condition of an irreducible graph

This appendix provides auxiliary results about the support of a factor of $C_G$ and how a stress corresponding to an irreducible factor is carried on the edges of $G$ before turning to a proof of Theorem 8.

A.1 Factors and their support

Let $G$ be an $[a, b]$-graph with a generalized tie down and let $f$ be an irreducible factor of the pure condition $C_G$. We define $\mathcal{V}(f)$ to be the set of $p$ such that $f(p) = 0$; a point $p$ is generic on $\mathcal{V}(f)$ if, for any polynomial $h$ vanishing at $\vec{p}$, $h = g \cdot f$ for some polynomial $g$. In Section 4.2 of [17], White and Whiteley observe that as $C_G$ is multilinear (and in fact every term of $C_G$ is divisible by a coordinate of $p(e)$ for each edge $e$), there exist real generic points of irreducible factors of the pure condition of a body-and-bar framework, and this carries over to the body-and-cad setting. An edge $e$ of $G$ is defined to be in the support of an irreducible factor $f$ if any of the coordinates of $p(e)$ appear in $f$. Then Equation (3) implies that:

**Lemma A.1.** For any irreducible factor $f$, either all the coordinates of $p(e)$ appear in $f$ or none of them do. In particular, an edge is in the support of exactly one irreducible factor.

We can sharpen this statement in two directions. First, we focus on the internal structure of $f$:

**Lemma A.2.** Let $e$ be the support of an irreducible factor $f$. Then every monomial in $f$ contains some coordinate of $p(e)$.

**Proof.** We suppose the contrary and fix some term ordering on the polynomial ring. Let $f_e$ be the largest term of $f$ that does not contain a coordinate of $p(e)$. Let $g = C_G / f$, and let $\text{in}(g)$ be the largest term of $g$. Then $f_e \cdot \text{in}(g)$ is larger than any other term of $f \cdot g$ that does not contain a coordinate of $p(e)$. So, $f_e \cdot \text{in}(g)$ cannot cancel in the expansion and simplification of $f \cdot g$. Therefore, $f \cdot g = C_G$ would contain a term that does not contain a coordinate of $p(e)$, which is a contradiction to Equation (3).

Second, we focus on edges with the same endpoints:

**Lemma A.3.** Let $e$ and $e'$ be edges with the same endpoints. Then $e$ and $e'$ are in the support of the same irreducible factor.

**Proof.** Suppose, for a contradiction, that $e$ is in the support of an irreducible factor $f$ and $e'$ is in the support of an irreducible factor $g \neq f$. It is clear that if $p(e) = p(e')$ then the pure condition vanishes. On the other hand, the irreducibility of $f$ and $g$ imply that the intersections $\mathcal{V}(f) \cap \{p(e) = s\}$ and $\mathcal{V}(g) \cap \{p(e') = s\}$ are proper, closed subsets for generic $s \in \mathbb{R}^k$. In particular, we can find a $p$ with $p(e) = p(e')$ so that the pure condition does not vanish, which is a contradiction.
A.2 Stresses

A stress $\omega \in \mathbb{R}^m$ is a linear relation on the rows of the rigidity matrix; i.e., $\omega M(G(p)) = 0$. We use the notation $\omega_p(e)$ for the entries of $\omega$. Viewed combinatorially, the stress puts weights on the edges of $G$. The support of a stress is the set of edges assigned nonzero weight by the stress; a stress is non-trivial when it has non-empty support. If $G$ is an $[a,b]$-graph, then for generic $p$, $G(p)$ does not have a non-trivial stress. Conversely, because the pure condition vanishes exactly when the rank of the rigidity matrix drops, we see that:

**Lemma A.4.** Let $p$ be any point in $\mathcal{V}(f)$. Then $G(p)$ has at least one stress.

Generically, there is only one stress.

**Lemma A.5.** Let $G$ be an $[a,b]$-graph. Let $p$ be generic in $\mathcal{V}(f)$. Then $G(p)$ has exactly one stress.

**Proof.** The condition for having $t > 1$ stresses is Zariski closed in $\mathcal{V}(f)$, so we only need to find a single $p \in \mathcal{V}(f)$ with exactly one stress to show that the property holds generically. To do this, first take $q$ generic, so that $G(q)$ is infinitesimally rigid; then define $p$ to be equal to $q$ in all coordinates except in those corresponding to some edge $e$ in the support of $f$; set the $k$ coordinates corresponding to $e$ to 0. Since $G(q)$ was infinitesimally rigid and only one row was changed, $M(G(p))$ has exactly one row dependence, and hence one stress. By Lemma A.2, $p \in \mathcal{V}(f)$, since every monomial in $f$ evaluates to zero at $p$. 

In light of Lemma A.5, we may now refer to the generic stress $\omega_p$ of a factor $f$; the notation suppresses the dependence on $f$, but the reader should keep in mind that it is there.

A.3 The support of the generic stress

We start by showing that the generic stress of an irreducible factor $f$ is generically nonzero on the support of $f$. We prove a slightly stronger statement:

**Lemma A.6.** Let $e$ be an edge in the support of an irreducible factor $f$. Then $\omega_p(e)$ is a non-constant rational function of $p$.

**Proof.** If $\omega_p(e)$ is constant on an open subset of $\mathcal{V}(f)$, then whether $f$ vanishes is independent of $p(e)$, contradicting Lemma A.2. That the dependence is a rational function follows from Cramer’s Rule.

As an immediate corollary we obtain

**Lemma A.7.** Let $p \in \mathcal{V}(f)$ be generic. Then the generic stress of $G(p)$ is non-zero on the support of $f$.

If $G$ is irreducible, we can say more.

**Lemma A.8.** Let $p \in \mathcal{V}(f)$ be generic. If $G$ is irreducible, then the support of $\omega_p$ contains all of the edges of $G$. 

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Proof. Suppose the contrary, so that \( \omega_p(d) = 0 \) for some edge \( d \). By Lemma A.7, \( \omega_p(e) \neq 0 \) for some edge \( e \) in the support of \( f \). Construct the graph \( H := G - d + e' \) by removing \( d \) and adding a copy \( e' \) of \( e \). Construct a realization \( p' \) of \( H \) by letting \( p'(e') \in \mathbb{R}^k \) be generic and otherwise \( p' = p \).

Observe that \( \omega_p \) induces a stress on \( H(p') \) because \( d \) was not in its support, and, in particular that \( f \) vanishes at \( p' \) and that \( p' \) is generic for \( f \). On the other hand, because \( G \) is irreducible, \( H \) is a generalized tie down, so the polynomial \( C_H(x) \neq 0 \). Since \( C_H(p') = 0 \) and \( p' \) is generic for \( f \), we see that \( f \) is a factor of \( C_H \) (as well as of \( C_G \)). By Lemma A.2, the coordinates of \( p(e') \) are in \( f \), which is a contradiction.

A.4 Proof of Theorem 8

If \( C_G \) is irreducible, then the irreducibility of \( G \) follows easily. Therefore, let \( G \) be an irreducible generalized tie down and suppose, for a contradiction, that there is an irreducible factor \( f \) and an edge \( e \) outside the support of \( f \). Let \( g \) be the irreducible factor with \( e \) in its support. Select a \( p \) generic in \( V(f) \). This, in particular, implies that \( p \notin V(g) \). Moreover, we can set \( p(e) \) to anything without moving \( p \) off of \( V(f) \). By Lemma A.8, \( \omega_p(e) \) is nonzero and, by genericity and Lemma A.5, \( (G - e)(p) \) is independent.

Now we use the hypothesis that \( G \) was irreducible as a graph: adding a copy \( e' \) of \( e \) produces a new graph \( G' \) that is an \([a, b]\)-circuit. Create \( p' \) by assigning \( p'(e') \) generically and otherwise \( p' = p \). The combinatorial hypothesis then implies that, for any edge \( d \) in the support of \( f \), \( (G' - d)(p') \) is infinitesimally rigid. But this implies the same about \( G'(p') \), and subsequently about \( (G' - e)(p') \). Since this final graph is isomorphic to \( G \) and all we have done is changed the coordinates of \( e \), we have \( p' \in V(f) \). The infinitesimal rigidity of \( (G' - e)(p') \) is then a contradiction, and the theorem is proved.