Quantum $K$-theory and $q$-difference Equations

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Dedicated to Professor Banghe Li on His 80th Birthday

Abstract This is a set of lecture notes for the first author’s lectures on the difference equations in 2019 at the Institute of Advanced Study for Mathematics at Zhejiang University. We focus on explicit computations and examples. The convergence of local solutions is discussed.

Keywords Quantum $K$-theory, $q$-difference equation

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1 Introduction

The linear differential equation appears naturally in quantum cohomology and mirror symmetry as the Picard–Fuch equation of period integrals. It has been studied extensively throughout history. Its close cousin, the difference equation, is an ancient topic of mathematics as old as the linear differential equation. However, it did not receive nearly enough attention. During the last decade, it appears analogous to the Picard–Fuch equation in quantum $K$-theory. The latter enjoys a revival due to the recent realization that quantum $K$-theory is a 3d TQFT. It was well-known that quantum cohomology is a 2d TQFT.

Compared to its more famous cousin, there is a lack of literature on difference equations, which slows down the development of the subject. This set of notes is an attempt to improve the situation. Instead of developing the general theory, we focus on the explicit computation of the solutions. We claimed no originality of material and did not attempt to complete references, for which the authors apologize.

1.1 Linear Difference Equation

Our main consideration is the following equation

$$\sum_{i=0}^{n} a_i(z)f(q^iz) = 0,$$

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in which the \( a_i(z) \) are known functions of the complex variable \( z, q \) and \( |q| > 1 \). It is called the homogeneous linear ordinary \( q \)-difference equation of the \( n \)-th order.

In early 2000s, Givental [8] and Lee [12] introduced the \( K \)-theoretic Gromov–Witten theorem, in which the \( q \)-difference equations play the role of Picard–Fuchs equation in quantum cohomology. More precisely, the quantum \( K \)-theoretic \( I \)-function satisfies certain \( q \)-difference equations.

For example, let’s consider projective space \( \mathbb{P}^N \), the modified \( K \)-theoretic \( I \)-function is of the form

\[
\tilde{I}^\mathbb{P}^N (z,q) = P^\ell_q(z) \sum_{d=0}^\infty \frac{z^d}{\prod_{k=1}^{d} (1 - Pq^k)^{N+1}},
\]

where \( P = \mathcal{O}(-1) \) on \( \mathbb{P}^N \). Denote by \( \sigma_q := q^{2\ell_q} \) the difference operator shifting \( z^k \) by \( q^k z^k \).

Since \( (1 - P)^{N+1} = 0 \) in \( K(\mathbb{P}^N) \), then the \( K \)-theoretic \( I \)-function of \( \mathbb{P}^N \) satisfies the following degree \( N + 1 \) difference equation

\[
[(1 - \sigma_q)^{N+1} - z] \tilde{I}^\mathbb{P}^N(q,z) = 0.
\]

In 2018, Ruan–Zhang [15] introduced a key new feature in quantum \( K \)-theory, i.e., the level structure, which is now well-understood to correspond to Chern–Simons term in so-called 3d \( \mathcal{N} = 2 \) theory in physics [11, 19]. It plays an essential role in 3d theories [14] and affects the difference equation as follows.

Still taking projective space \( \mathbb{P}^N \) as an example, let’s consider the level structure with respect to the standard representation of \( \mathbb{C}^* \) of level \( l \), see [15] for details. Then the modified \( I \)-function with level structure is

\[
\tilde{I}^\mathbb{P}^N_{l} (z,q) = P^\ell_q(z) \sum_{d=0}^\infty \frac{(P^d q^{d(d-1)})^{l} z^d}{\prod_{k=1}^{d} (1 - Pq^k)^{N+1}},
\]

and it satisfies

\[
[(1 - \sigma_q)^{N+1} - z\sigma_q^l] \tilde{I}^\mathbb{P}^N_{l} (z,q) = 0. \tag{1.1}
\]

Moreover, we could consider hypersurfaces inside projective space. Among them, the quintic 3-fold plays an essential role. Let’s denote the Fermat quintic threefold by \( X \), which could be realized as a degree 5 hypersurface in the projective space \( \mathbb{P}^4 \). By using the quantum Lefschetz hyperplane theorem [9], we have

\[
\tilde{I}^X (z,q) = P^\ell_q(z) \sum_{d=0}^\infty \frac{\prod_{k=1}^{5d} (1 - P^5 q^k)}{\prod_{k=1}^{d} (1 - Pq^k)^{5(N+1)}} z^d, \tag{1.3}
\]

where we still use \( P \) to denote \( \mathcal{O}(-1) \) on \( \mathbb{P}^4 \). Since \( (1 - P)^5 = 0 \) in \( K(\mathbb{P}^4) \), then (1.3) satisfies the following difference equation

\[
[(1 - \sigma_q)^5 - z \prod_{k=1}^{5} (1 - q^5 \sigma_q^k)] \tilde{I}^X (q,z) = 0. \tag{1.4}
\]

It’s a degree 25 difference equation!

Note that the difference equation (1.2) behaves well when level \( 0 \leq l \leq N+1 \). In these cases, the difference equations are regular singular (Definition 2.12). And for difference equations (1.2)
with \( l \geq N + 1 \) and the difference equation (1.4), they are called irregular singular (Definition 2.12), their solutions are not as good as regular singular cases.

Finding the difference equations that \( I \)-functions satisfy is significant. One formulation of 3d mirror symmetry is to interchange the quantum/equivariant parameters of mirror pair [3, 6, 14]. Usually, \( I \)-functions are very complicated, making it challenging to compute them directly. One often accomplishes it by analyzing its \( q \)-difference equation.

In this paper, our primary goal is to find solutions to \( q \)-difference equations via the Frobenius method. We will use many concrete examples to demonstrate how it works in both regular singular cases and irregular singular cases. We focus on computations rather than general results. Among these examples, we will see a lot of modular forms! The paper is organized as follows. Section 2 reviews some basic definitions of difference equations and some general results about regular singular cases. Section 3 introduces the Frobenius method and applies it to some examples. We end Section 3 with a discussion about the convergence of the solutions. We will show that all the power-series solutions in regular singular cases are convergent under certain conditions. Section 4 deals with irregular singular cases; we start by showing how the general technique works and then apply it to some examples, including the difference equations for the Fermat quintic threefold and projective space with level structures. We also end up with a discussion about the convergence of the specific solutions in irregular cases.

2 A Brief Review of General Theory

In this section, we review some basics in the theory of \( q \)-difference equations. The main references are [13, 16, 18].

Notations Here are some standard notations of general use:
- \( z \) and \( q \) are complex variables and \( |q| > 1 \),
- \( \mathbb{C}\{z\} \) is the field of meromorphic germs at 0, is the quotient field of \( \mathbb{C}\{z\} \),
- \( \mathcal{M}(\mathbb{C}^*) \) is the field of meromorphic functions on \( \mathbb{C}^* \),
- \( \mathcal{M}(\mathbb{C}^*, 0) \) is the ring of germs at punctured neighborhood of \( z = 0 \),
- \( \mathcal{M}(E_q) \) is the field of meromorphic functions on elliptic curve \( E_q = \mathbb{C}^*/q\mathbb{Z} \), i.e., the field of elliptic functions,
- \( (a; q)_d = (1 - a)(1 - qa) \cdots (1 - q^{d-1}a) \) for \( d \in \mathbb{N} \cup \{+\infty\} \) is the \( q \)-Pochhammer symbol.

Definition 2.1 A difference field is a pair \((K, \sigma)\), where \( K \) is a field, and \( \sigma \) is a field automorphism of \( K \).

Example 2.2 We will focus on the fields in the above notations, \( \mathcal{M}(\mathbb{C}^*) \subset \mathcal{M}(\mathbb{C}^*, 0) \), they are all endowed with the \( q \)-shift operator \( \sigma_q := q^{\partial z} : f(z) \mapsto f(qz) \). Let \( K = \mathcal{M}(\mathbb{C}^*) \) or \( \mathcal{M}(\mathbb{C}^*, 0) \). Usually, we denote the field of constants of the difference field \((K, \sigma_q)\) as \( K^{\sigma_q} \). For example, \( \mathcal{M}(\mathbb{C}^*)^{\sigma_q} = \mathcal{M}(E_q) \). This is the main reason the modular form, such as the elliptic function, appears naturally in the theory of the \( q \)-difference equation.

2.1 Difference Equations

The \( q \)-difference equation is as follows:
\[
a_n(z, q)(\sigma_q)^n f + a_{n-1}(z, q)(\sigma_q)^{n-1} f + \cdots + a_0(z, q) f = 0,
\]
where \( a_i(z, q) \) are meromorphic functions. Let \( P := a_n(z, q)(\sigma_q)^n + a_{n-1}(z, q)(\sigma_q)^{n-1} + \cdots + a_0(z, q) \). Then the vectorization trick shows that:

\[
P \cdot f = 0 \iff \sigma_q X_f = A_P X_f,
\]

where

\[
X_f := \begin{pmatrix}
f(z, q) \\
\sigma_q f(z, q) \\
\vdots \\
(\sigma_q)^{n-1} f(z, q)
\end{pmatrix} \quad \text{and} \quad A_P := \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 1 & -a_{n-1}/a_n
\end{pmatrix}.
\]

For this reason, the study of scalar linear difference equations boils down to that of difference systems. In the following, we focus on the study of difference systems. In the next section, we will focus on the linear difference equations to apply the Frobenius method.

**Remark 2.3** We see how to obtain a difference system from a linear difference equation. In some sense, the converse is true.

**Definition 2.4** Let \( \sigma_q X_q(z) = A_q(z)X_q(z) \) be a \( q \)-difference system. Consider a matrix \( P_q \in \text{GL}_n(K) \). The gauge transform of the matrix \( A_q \) by the gauge transformation \( P_q \) is the matrix

\[
P_q \cdot [A_q] := (\sigma_q P_q) A_q P_q^{-1}.
\]

A second \( q \)-difference system \( \sigma_q X_q(z) = B_q(z)X_q(z) \) is said to be equivalent by gauge transform to the first one if there exists a matrix \( P_q \in \text{GL}_n(K) \) such that

\[
B_q = P_q \cdot [A_q].
\]

**Proposition 2.5** ([18, Theorem 2.4.8] (Cyclic vector lemma)) Every \( A \in \text{GL}_n(K) \), where \( K \) is a field, is equivalent over \( K \) to some \( A_P \), where \( P \) is a difference operator.

**Definition 2.6** Let \( (E_q) : \sigma_q X_q(z) = A_q(z)X_q(z) \) be a \( q \)-difference system, with \( A_q \in \text{GL}_n(K) \). We define the solution space of this \( q \)-difference equation by

\[
\text{Sol}(E_q) = \{ X_q \in K^n \mid \sigma_q X_q(z) = A_q(z)X_q(z) \}.
\]

**Remark 2.7** From now on, we will focus on the local solutions at \( z = 0 \). The results will also hold for \( z = \infty \). The reason why we don’t consider solutions at other singular points is that: if a function \( f(z) \) is a solution of a \( q \)-difference equation \( \sigma_q f(z) = a(z)f(z) \) and has a singularity at some \( z_0 \neq 0, \infty \), then \( f(z) \) has a singularity at any complex number \( z_0 q^k \).

**Proposition 2.8** ([18, Theorem 2.3.1, p.118]) Let \( (E_q) : \sigma_q X_q(z) = A_q(z)X_q(z) \) be a \( q \)-difference system. Then, we have

\[
\dim_{K^{\sigma_q}}(\text{Sol}(E_q)) \leq \text{rank}(A_q).
\]

Even more, if we consider solutions in a bigger space, i.e., the extension of the difference field \( (K, \sigma_q) \) (is a difference field \( (K', \sigma_q') \) such that \( K \subseteq K' \) and \( \sigma_q'|_K = \sigma_q \)), we still have

\[
\dim_{(K')^{\sigma_q}}(\text{Sol}(E_q, K')) \leq \text{rank}(A_q).
\]
Lemma 2.9 ([18, Lemma 2.3.3] (Wronskian lemma)) Let $f_1, \ldots, f_n \in K$ and denote the determinant of their wronskian matrix as

$$w_n := \det W_n(f_1, \ldots, f_n),$$

where the Wronskian matrix is defined as

$$W_n(f_1, \ldots, f_n) := \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ \sigma_q f_1 & \sigma_q f_2 & \cdots & \sigma_q f_n \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_q^{n-1} f_1 & \sigma_q^{n-1} f_2 & \cdots & \sigma_q^{n-1} f_n \end{pmatrix}.$$ 

Then $w_n(f_1, \ldots, f_n) = 0$ if and only if $f_1, \ldots, f_n$ are linearly dependent over $K^{\sigma_q}$.

Definition 2.10 Let $\sigma_q X_q(z) = A_q(z)X_q(z)$ be a $q$-difference system. The fundamental solution is a family $X_q = (f_1, \ldots, f_n)$ in $\text{Sol}(E_q)$, such that the determinant of Wronskian matrix is not zero.

Proposition 2.11 ([18, Proposition, 2.4.4, p.120]) Let $X_q \in \text{GL}_n(K)$ be a fundamental matrixial solution of (2.1). Then:

$$\text{Sol}(E_q) = \{X_q C \mid C \in (K^{\sigma_q})^n\}.$$ 

2.2 Regular Singular $q$-difference Equations

Usually, we say global study of $q$-difference equations if we take $K = \mathbb{C}(z)$, local analytic study if we take $K = \mathbb{C}(\{z\})$ and formal study if we take $K = \mathbb{C}(\!(z)\!)$. In the following, we shall start from the local analytic study, i.e., taking field $\mathbb{C}(\{z\})$, and then look for solutions in the field $K = \mathcal{M}(\mathbb{C}^*)$ or $\mathcal{M}(\mathbb{C}^*, 0)$ due to many consequences of the shape of the analytical theory.

Let us introduce the regular singular $q$-difference equations.

Definition 2.12 Let $A_q \in \text{GL}_n(K)$, a system $q^z \partial_z X_q(Q) = A_q(z)X_q(z)$ is said to be regular singular at $z = 0$ if there exists a $q$-gauge transform $P_q \in \text{GL}_n(K)$ such that the matrix $(P_q \cdot [A_q])(0)$ is well-defined and invertible: $P_q \cdot [A_q](0) \in \text{GL}_n(\mathbb{C})$. Otherwise, we say the system is irregular singular.

Let us give criteria for when a $q$-difference equation is regular singular at $z = 0$.

Proposition 2.13 ([13, Proposition V.2.1.14]) Let $P = \sum_k a_k(z, q)(\sigma_q)^k$ be a $q$-difference operator. As we stated before that the $q$-difference equation $P \cdot f(z) = 0$ can be vectorised to a $q$-difference system $\sigma_q X_q(z) = A_q(z)X_q(z)$ where $A_q(z)$ is the companion matrix of the operator $P$. The resulting $q$-difference system is

$$\sigma_q \begin{pmatrix} f(z) \\ \sigma_q f(z) \\ \vdots \\ (\sigma_q)^{n-1} f(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 1 \\ -a_0/a_n & -a_1/a_n & \cdots & -a_{n-1}/a_n \end{pmatrix} \begin{pmatrix} f(z) \\ \sigma_q f(z) \\ \vdots \\ (\sigma_q)^{n-1} f(z) \end{pmatrix}.$$ 

We denote by $\text{val}_a(a_k)$ the $z$-adic valuation of the polynomial $a_k$, i.e., the lowest integer $\alpha \in \mathbb{Z} \cup \{+\infty\}$ such that $(z^{-\alpha}a_k(z)) \big|_{z=0} \neq 0$. The $q$-difference system associated to the $q$-difference
equation \( P(\sigma_q) f(z) = 0 \) is regular singular if and only if \( \text{val}_0(a_0(z)) - \text{val}_0(a_n(z)) = 0 \), and for every \( k \in \{1, \ldots, n-1\} \), \( \text{val}_0(a_k(z)) - \text{val}_0(a_n(z)) \geq 0 \).

Let’s introduce some special functions which are needed to solve \( q \)-difference equations. The Jacobi’s theta function is defined as follows

\[
\theta_q(z) = \sum_{d \in \mathbb{Z}} q^{\frac{d(d+1)}{2}} z^d.
\]

This function satisfies the \( q \)-difference equation \( \sigma_q \theta_q(z) = z \theta_q(z) \). And it has a famous Jacobi’s triple identity

\[
\theta_q(z) = (q^{-1}; q^{-1})_\infty (-q^{-1}z; q^{-1})_\infty (-z^{-1}; q^{-1})_\infty.
\]

We define the following two special functions using Jacobi’s theta function.

**Definition 2.14** Let \( \lambda_q \in \mathbb{C}^* \). The \( q \)-character associated to \( \lambda \) is the function \( e_{q,\lambda_q} \in \mathcal{M}(\mathbb{C}^*) \) defined by

\[
e_{q,\lambda_q}(z) = \frac{\theta_q(z)}{\theta_q(z/\lambda_q)} \in \mathcal{M}(\mathbb{C}^*).
\]

The function \( e_{q,\lambda_q} \) satisfies the \( q \)-difference equation \( \sigma_q e_{q,\lambda_q}(z) = \lambda_q e_{q,\lambda_q}(z) \).

**Definition 2.15** The \( q \)-logarithm is the function \( \ell_q \in \mathcal{M}(\mathbb{C}^*) \) defined by

\[
\ell_q(z) = z \frac{\theta_q'(z)}{\theta_q(z)}.
\]

Since \( \theta_q(qz) = z \theta_q(z) \), then

\[
\frac{\partial}{\partial z} \theta_q(qz) = q \theta_q'(qz) = \theta_q(z) + z \theta_q'(z).
\]

So

\[
q \theta_q'(qz) = z \frac{\theta_q'(z)}{\theta_q(qz)} + z \frac{\theta_q(z)}{\theta_q(qz)} = z \frac{\theta_q'(z)}{\theta_q(z)} + 1,
\]

i.e.,

\[
\sigma_q \ell_q(z) = \ell_q(z) + 1.
\]

**Remark 2.16** In the literature [1], one also considered \( \log z \log_q \), which satisfies

\[
\sigma_q \left( \frac{\log z}{\log q} \right) = \frac{\log z}{\log q} + 1,
\]

but it is multi-value. In this article, we prefer the single-valued function \( \ell_q(z) \). For the Picard–Fuchs equation, one can read off the monodromy directly from the local solution. On the other hand, we can have both single-valued and multi-valued solutions for the \( q \)-difference equation. This is related to the fact that the \( q \)-difference equation has a rather large field of constants. The issue of monodromy for \( q \)-difference equation is rather subtle [17].

Now we can state the existence of a fundamental solution for regular singular \( q \)-difference equations under specific conditions.
Definition 2.17 Consider a regular singular q-difference system $\sigma_q X_q(z) = A_q(z)X_q(z)$ and denote by $(\lambda_i)$ the eigenvalues of the matrix $A_q(0)$. This q-difference system is said to be non-(q-)resonant if for every $i \neq j$, we have $\frac{\lambda_i}{\lambda_j} \notin q\mathbb{Z}\setminus\{0\}$, where $q\mathbb{Z}\setminus\{0\} := \{q^k \mid k \in \mathbb{Z}\setminus\{0\}\} \subset \mathbb{C}$.

For a non-resonant system, we can recursively build a gauge transform $F_q \in \text{GL}_n(\mathbb{C}\{z\})$ which sends the matrix $A_q(0)$ to the constant matrix $A_q(z)$, for details, see [18]. Then taking the Jordan–Chevalley decomposition of $A_q(0) = A_sA_u$ where $A_s$ is semi-simple, $A_u$ is unipotent and $A_s$, $A_u$ commute.

Since $N = A_u - I_n$ is nilpotent, we can define

$$A_u^{\ell_q} := (I_n + N)^{\ell_q} := \sum_{k \geq 0} \binom{\ell_q}{k} N^k,$$  \hspace{1cm} (2.3)

where

$$\binom{\ell_q}{k} := \frac{\ell_q(\ell_q - 1) \cdots (\ell_q - (k - 1))}{k!}.$$

Note that (2.3) is a finite sum and $A_u^{\ell_q}$ is unipotent, then we have

$$\sigma_q A_u^{\ell_q} = A_u A_u^{\ell_q} = A_u^{\ell_q} A_u.$$  

Thus we set

$$e_{q,A_u} := A_u^{\ell_q}.$$  

Take a basis change $P$ to diagonalise $A_s = P^{-1}\text{diag}(\lambda_i)P$. We define

$$e_{q,A_s} := P^{-1} \text{diag}(e_{q,\lambda_i}(z)) P,$$  \hspace{1cm} (2.4)

which satisfies

$$\sigma_q e_{q,A_s} = A_s e_{q,A_s} = e_{q,A_s} A_s.$$  

Then one can check that the product $F_q \cdot e_{q,A_s} \cdot e_{q,A_u} =: X_q(z)$ is a fundamental solution of the q-difference system $\sigma_q X_q(z) = A_q(z)X_q(z)$. We arrive at the following theorem.

Proposition 2.18 ([18, Theorem 3.3.1]) The q-difference system $\sigma_q X_q(z) = A_q(z)X_q(z)$, regular singular at $z = 0$, admits a fundamental matricial solution $X := Me_{q,C} \in \text{GL}_n(\mathbb{M}(\mathbb{C}^*, 0))$, where $C \in \text{GL}_n(\mathbb{C})$ and where $M \in \text{GL}_n(\mathbb{C}(\{z\}))$. The $e_{q,C}$ is defined by the Jordan–Chevalley decomposition of $C$ as above.

Remark 2.19 Let $A, P \in \text{GL}_n(\mathbb{C})$. One can check that $e_{q,PAP^{-1}} = Pe_{q,A}P^{-1}$. Thus, (2.4) is independent of the choice of $P$.

3 Local Solutions for Regular Singular Cases

In the last section, we introduced some general results about the solutions of the regular singular q-difference system. However, obtaining an explicit formula for a solution is still hard. In this section, we will use the Frobenius method to construct the solutions. The Frobenius method for linear ordinary q-difference equation could be dated back to [1] and [4].

In the following, we focus on the computations and use concrete examples to show how the Frobenius method works rather than giving a general result.
3.1 Frobenius method

Let’s consider the equation

$$\sum_{i=0}^{n} a_i(z)(\sigma_q)^i f(z) = 0,$$

with

$$a_i(z) = a_{i0} + a_{i1} z + a_{i2} z^2 + \cdots.$$

Since it is regular singular, then coefficients $$a_i(z)$$ satisfy conditions in Proposition 2.13, so we could assume that $$a_{00}, a_{n0} \neq 0$$.

Let’s consider the following equation

$$a_{n0} x^n + a_{n-1} x^{n-1} + \cdots + a_{10} x + a_{00} = 0. \quad (3.1)$$

It is called the characteristic equation, which plays an important role in constructing solutions.

- **Non-resonant case:** Suppose the $$n$$ roots $$\{c_1, \ldots, c_n\}$$ of (3.1) are all distinct and

$$c_i/c_j \notin q^Z, \quad \forall i \neq j. \quad (3.2)$$

Then there exists a set of $$n$$ power-series solutions of the form:

$$S_i(z, q) = e_{q, c_i}(z) \cdot F_i(z, q), \quad \text{where } F_i(z, q) = \sum_{k=0}^{\infty} f_{ik}(q) z^k, \quad (3.3)$$

and for $$i = 1, \ldots, n$$.

- **Resonant case:** Suppose the $$n$$ roots are as follows

$$c_i \cdot q^{-m_{ij}}, \quad \text{for } i = 1, \ldots, r, j = 0, \ldots, k_i, \quad (3.4)$$

such that

1. $$c_i/c_j \notin q^Z,$$
2. $$0 = m_{i0} \leq m_{i1} \leq \ldots \leq m_{ik_i}, \text{ and } \sum_{i=1}^{r}(k_i + 1) = n.$$

Then there are $$n$$ power-series solutions of the following form

$$S_{i0}(z, q) = e_{q, c_i}(z) F_{i0}(z, q), \quad (3.5)$$

$$S_{i1}(z, q) = \ell_q(z) S_{i0}(z, q) + e_{q, c_i q^{-m_{i1}}}(z) F_{i1}(z, q), \quad (3.6)$$

$$\ldots$$

$$S_{ik_i}(z, q) = \ell_q(z) S_{i, k_i-1}(z, q) + e_{q, c_i q^{-m_{ik_i}}}(z) F_{ik_i}(z, q), \quad (3.7)$$

where $$i = 1, \ldots, r$$ and $$j = 0, \ldots, k_i$$, and

$$F_{ij}(z, q) = \sum_{l=0}^{\infty} f_{ijl}(q) z^l.$$

Thus, in the regular singular case, there exists a set of $$n$$ power-series solutions. In the following, we use concrete examples to show how the Frobenius method works. Furthermore, at the end of this section, we show that these $$n$$ power-series solutions provide a complete set of solutions analytic in the vicinity of the origin under certain conditions.
Example 3.1 Consider the following degree 2 regular singular difference equation

\[ [a(z, q)\sigma_q^2 + b(z, q)\sigma_q + d(z, q)] f = 0. \]

From Proposition 2.13, we know

\[ \text{val}_0(a(z, q)) - \text{val}_0(d(z, q)) = 0 \quad \text{and} \quad \text{val}_0(b(z, q)) - \text{val}_0(a(z, q)) \geq 0. \]  \hfill (3.8)

For simplicity, we assume \( a(z, q) = 1 \) and \( b(z, q) = \sum_{n=0}^{\infty} b_n(q)z^n, d(z, q) = \sum_{n=0}^{\infty} d_n(q)z^n, d_0 \neq 0. \)

If we are looking for a solution to the form

\[ \sum_{n=0}^{\infty} f_n(q)z^{n+r}, \]

then we have

\[ \sum_{n=0}^{\infty} f_n q^{2(n+r)}z^{n+r} + \sum_{n=0}^{\infty} b_n z^n \sum_{n=0}^{\infty} f_n q^{n+r}z^{n+r} + \sum_{n=0}^{\infty} d_n z^n \sum_{n=0}^{\infty} f_n z^{n+r} = 0. \]

The coefficient of \( z^{n+r} \) equals 0, i.e.,

\[ f_n(q^{2(n+r)} + q^{n+r}b_0(q) + d_0(q)) + \sum_{k=1}^{n-1} f_k(q^{k+r}b_{n-k} + d_n-k) = 0, \]

with initial condition

\[ f_0(q^{2r} + q^r b_0(q) + d_0(q)) = 0. \]

Then we obtain a necessary condition for a non-zero solution, i.e.,

\[ q^{2r} + q^r b_0(q) + d_0(q) = 0. \]

We call

\[ x^2 + b_0(q)x + d_0(q) = 0, \] \hfill (3.9)

the characteristic equation as (3.1). Furthermore, if \( q^{2(n+r)} + q^{n+r}b_0(q) + d_0(q) \neq 0, \) i.e., \( q^{n+r} \) is not a solution of characteristic equation for \( n \geq 1, \) then

\[ f_n = \frac{-1}{q^{2(n+r)} + q^{n+r}b_0(q) + d_0(q)} \left( \sum_{k=1}^{n-1} f_k(q^{k+r}b_{n-k} + d_{n-k}) \right) \]

**Conclusion:** Suppose \( q^{r_1}, q^{r_2} \) are the \( q^r \)-solutions of the characteristic equation (3.9),

- Case 1: If \( r_1 - r_2 \notin \mathbb{Z}, \) then there exist two solutions of the form
  
  \[ z^{r_1} F_1, \quad z^{r_2} F_2, \]

  for \( F_1 \) and \( F_2 \) are all power series.

- Case 2: If \( r_1 - r_2 = n \in \mathbb{Z}_+ \), then there exists a solution of the form
  
  \[ z^{r_1} F_1, \]

  where \( F_1 \) is a power series.
To construct the second solution in Case 2, we have to use the $q$-logarithm defined in Definition 2.15. And we will require an intermediary result on it.

**Lemma 3.2** ([1] and [13], Lemma VI.1.1.10) Let $N \in \mathbb{Z}_{\geq 0}$. The family consisting of the functions $\ell_q(z)^i \in \mathcal{M}(\mathbb{C}^*)$ for $i \in \{0, \ldots, N\}$ is linearly independent over field $\mathcal{M}(E_q)$.

**Example 3.3** Let’s come back to degree two difference equation

$$[\sigma^2_q + b(z, q)\sigma_q + d(z, q)] f = 0, \quad (3.10)$$

then the characteristic equation is

$$x^2 + b(0, q)x + d(0, q) = 0.$$ 

Suppose there are two roots (not necessary $q^r$-roots)

$$c_1, \quad c_2.$$ 

**Case 1** if $c_1/c_2 \notin q^\mathbb{Z}$, then there are two solution of the form

$$e_{q, c_1}(z) \sum_{n=0}^{\infty} f_n z^n, \quad e_{q, c_2}(z) \sum_{n=0}^{\infty} g_n z^n.$$

The computation is similar to Example 3.1.

**Case 2** if $c_1 = c_2 q^{n_0}$, $n_0 \in \mathbb{Z}_+$, then the first solution is of the form

$$e_{q, c_1}(z) \sum_{n=0}^{\infty} f_n z^n.$$

Now let’s construct the second solution of the following form

$$\ell_q(z)e_{q, c_1}(z) \sum_{n=0}^{\infty} f_n z^n + e_{q, c_2}(z) \sum_{n=0}^{\infty} g_n z^n.$$

Substituting into (3.10), we have

$$0 = (\ell_q(z) + 2)c_1^2 \cdot e_{q, c_1}(z) \sum_{n=0}^{\infty} f_n q^{2n} z^n + (\ell_q(z) + 1)c_1 \cdot e_{q, c_1}(z) \sum_{n=0}^{\infty} f_n q^n z^n \sum_{n=0}^{\infty} b_n(q) z^n$$

$$+ \ell_q(z)e_{q, c_1}(z) \sum_{n=0}^{\infty} f_n z^n \sum_{n=0}^{\infty} d_n(q) z^n$$

$$+ c_2 \cdot e_{q, c_2}(z) \sum_{n=0}^{\infty} g_n q^{2n} z^n + c_2 \cdot e_{q, c_2}(z) \sum_{n=0}^{\infty} g_n q^n z^n \sum_{n=0}^{\infty} b_n(q) z^n$$

$$+ e_{q, c_2}(z) \sum_{n=0}^{\infty} g_n q^n z^n \sum_{n=0}^{\infty} d_n(q) z^n.$$

- The $\ell_q(z)$-term:

$$0 = \ell_q(z) \left[ c_1^2 \cdot e_{q, c_1}(z) \sum_{n=0}^{\infty} f_n q^{2n} z^n + c_1 \cdot e_{q, c_1}(z) \sum_{n=0}^{\infty} f_n q^n z^n \sum_{n=0}^{\infty} b_n z^n$$

$$+ e_{q, c_1}(z) \sum_{n=0}^{\infty} f_n z^n \sum_{n=0}^{\infty} d_n z^n \right]$$
\[= \ell_q(z) \cdot e_{q,c_1}(z) \cdot \sum_{n=0}^{\infty} \left( (c_1^2 q^{2n} + c_1 b_0 q^n + d_0) f_n + \sum_{k=0}^{n-1} (q^k b_{n-k} + d_{n-k}) f_k \right) z^n,\]

with initial condition
\[(c_1^2 + b_0 c_1 + d_0) f_0 = 0.\]

It is automatically satisfied, so \(f_0\) is arbitrary.

- The remaining term:
\[
0 = 2c_2^2 \cdot e_{q,c_1}(z) \sum_{n=0}^{\infty} f_n q^{2n} z^n + c_1 \cdot e_{q,c_1}(z) \sum_{n=0}^{\infty} f_n q^n z^n \cdot \sum_{n=0}^{\infty} b_n(q) z^n
\]
\[+ c_2^2 \cdot e_{q,c_2}(z) \sum_{n=0}^{\infty} g_n q^{2n} z^n + c_2 \cdot e_{q,c_2}(z) \sum_{n=0}^{\infty} g_n q^n z^n \cdot \sum_{n=0}^{\infty} b_n(q) z^n
\]
\[+ e_{q,c_2}(z) \sum_{n=0}^{\infty} g_n z^n \sum_{n=0}^{\infty} d_n(q) z^n.\]

Since \(c_1 = c_2 q^{n_0}\), then
\[
e_{q,c_1}(z) = q^{n_0(n_0-1)} z^{n_0} e_{q,c_2}(z).
\]

So it becomes
\[
0 = 2c_2^2 \cdot q^{n_0(n_0-1)} z^{n_0} e_{q,c_2}(z) \sum_{n=0}^{\infty} f_n q^{2n} z^n + c_1 \cdot q^{n_0(n_0-1)} z^{n_0} e_{q,c_2}(z) \sum_{n=0}^{\infty} f_n q^n z^n \cdot \sum_{n=0}^{\infty} b_n(q) z^n
\]
\[+ c_2^2 \cdot e_{q,c_2}(z) \sum_{n=0}^{\infty} g_n q^{2n} z^n + c_2 \cdot e_{q,c_2}(z) \sum_{n=0}^{\infty} g_n q^n z^n \cdot \sum_{n=0}^{\infty} b_n(q) z^n
\]
\[+ e_{q,c_2}(z) \sum_{n=0}^{\infty} g_n z^n \sum_{n=0}^{\infty} d_n(q) z^n
\]
\[
e_{q,c_2}(z) \left\{ \sum_{n=0}^{\infty} \left[ (2c_2^2 \cdot q^{n_0(n_0-1)} q^{2n} + b_0 c_1 q^{n_0(n_0-1)} q^n) f_n + \sum_{k=0}^{n-1} \frac{n_0(n_0-1)}{2} c_1 f_k q^k b_{n-k} \right] z^{n+n_0} \right. \]
\[+ \sum_{n=0}^{\infty} z^n \left[ (2c_2 q^{2n} + c_2 g_n + d_0) g_n + \sum_{k=0}^{n-1} (c_2 q^k b_{n-k} + d_{n-k}) g_k \right] \right\}.
\]

Thus for \(n < n_0\), \((c_2 q^n)^2 + b_0 c_2 q^n + d_0 \neq 0\). We obtain
\[
g_n = \frac{n_0(n_0-1)}{2} \frac{1}{(c_2 q^n)^2 + b_0 c_2 q^n + d_0} \left[ \sum_{k=0}^{n-1} (c_2 q^k b_{n-k} + d_{n-k}) g_k \right].
\]

For \(n > n_0\), one could see that \(g_n\) is determined by \(\{f_0, \ldots, f_{n-n_0}\}\). For \(n = n_0\), we have
\[
0 = q^{n_0(n_0-1)} \left( (2c_1^2 + b_0 c_1) f_0 + \sum_{k=0}^{n_0-1} (c_2 q^k b_{n-k} + d_{n-k}) g_k \right).
\]

(3.11)

Since \(c_1, c_2\) are two roots of
\[x^2 + b_0 x + d_0 = 0,
\]
then
\[b_0 = -(c_1 + c_2), \quad d_0 = c_1 \cdot c_2.\]
So

\[ 2c_1 + b_0c_1 = 2c_1^2 - (c_1 + c_2)c_1 = c_1^2 - c_1c_2. \]

If \( c_1 \neq c_2 \), i.e., \( n_0 \neq 0 \), then from (3.11) we have

\[ f_0 = -\frac{q^{n_0(2n-1)}}{c_1(c_1 - c_2)} \sum_{k=0}^{n_0-1} (c_2q^k b_{n-k} + d_{n-k})g_k, \]  
(3.12)

and \( f_0 \) is determined by \( g_0 \). Thus we have two free parameters \( g_0 \) and \( g_{n_0} \).

**Case 3**  
if \( c = c_1 = c_2 \), i.e.,

\[ 0 = x^2 + b_0(q)x + d_0(q) = (x - c)^2. \]

Then \( b_0 = -2c \) and \( d_0 = c^2 \). Now consider the solution of the form

\[ \ell_q(z)e_{q,c}(z) \sum_{n=0}^{\infty} f_n z^n + e_{q,c}(z) \sum_{n=0}^{\infty} g_n z^n. \]

Substituting into (3.10), we have

\[
\begin{align*}
(\ell_q(z) & + 2)c^2 e_{q,c}(z) \sum_{n=0}^{\infty} f_n(q) q^{2n} z^n + (\ell_q(z) + 1)c \cdot e_{q,c}(z) \sum_{n=0}^{\infty} f_n(q) q^n z^n \sum_{n=0}^{\infty} b_n(q) z^n \\
+ \ell_q(z) e_{q,c}(z) & \sum_{n=0}^{\infty} f_n(q) z^n \sum_{n=0}^{\infty} d_n(q) z^n + c^2 e_{q,c}(z) \sum_{n=0}^{\infty} g_n(q) q^{2n} z^n \\
+ c \cdot e_{q,c}(z) \sum_{n=0}^{\infty} g_n(q) q^n z^n \sum_{n=0}^{\infty} b_n(q) z^n + e_{q,c}(z) \sum_{n=0}^{\infty} g_n(q) z^n \sum_{n=0}^{\infty} d_n(q) z^n.
\end{align*}
\]

- The \( \ell_q(z) \)-term:

\[ 0 = \ell_q(z)e_{q,c}(z) \cdot \sum_{n=0}^{\infty} \left[ (q^{2n} c^2 + cq^n b_0 + d_0)f_n + \sum_{k=0}^{n-1} (cq^k b_{n-k} + d_{n-k}) f_k \right] z^n, \]  
(3.13)

with initial condition

\[ (c^2 + b_0c + d_0)f_0 = 0. \]

So \( f_0 \) is arbitrary. Note that \( (cq^n)^2 + b_0(cq^n) + d_0 \neq 0 \) for \( n \geq 1 \). From (3.13) we have

\[ f_n = \frac{-1}{q^{2n} c^2 + cq^n b_0 + d_0} \sum_{k=0}^{n-1} (cq^k b_{n-k} + d_{n-k}) f_k. \]

- The remaining term:

\[
\begin{align*}
0 &= e_{q,c}(z) \sum_{n=0}^{\infty} \left[ (2c^2q^{2n} + cq^n b_0) f_n + \sum_{k=0}^{n-1} (cq^k b_{n-k} f_k \\
+ (c^2 q^{2n} + cq^n b_0 + d_0) g_n + \sum_{k=0}^{n-1} (rq^k b_{n-k} + d_{n-k}) g_k \right] z^n
\end{align*}
\]

with initial condition

\[ 2c^2 + b_0 \cdot c = 0. \]
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So $g_0$ is arbitrary. Then $g_n$ is determined by

$$\{g_0, \ldots, g_{n-1}\} \quad \text{and} \quad \{f_0, \ldots, f_n\}.$$  

Conclusion

- If $c_1/c_2 \notin q\mathbb{Z}$, then there are two solutions of the form
  $$e_{q,c_1}(z) \sum_{n=0}^{\infty} f_n z^n, \quad e_{q,c_2}(z) \sum_{n=0}^{\infty} g_n z^n.$$  

- If $c_1 = c_2 q^{n_0}$, $n_0 \in \mathbb{Z}_{\geq 0}$, there are two solutions of the form
  $$e_{q,c_1}(z) \sum_{n=0}^{\infty} f_n z^n, \quad e_{q,c_2}(z) \sum_{n=0}^{\infty} g_n z^n.$$  

3.2 The $q$-hypergeometric Equation

Consider the following degree 2 difference equation

$$[(1 - \sigma q)(1 - q^{r} \sigma q) - z(1 - q^\alpha \sigma q)(1 - q^\beta \sigma q)] f = 0.$$  

Suppose $r \notin \mathbb{Z}$, then the characteristic equation is

$$(1 - x)(1 - q^r x) = 0,$$  

then we have two $q\mathbb{Z}$-roots

$$x = q^0 \quad \text{or} \quad x = q^{-r}.$$  

For the first root $x = q^0$, let

$$f = \sum_{n=0}^{\infty} f_n z^n,$$  

then

$$\sum_{n=0}^{\infty} f_n(1 - q^n)(1 - q^{n+r})z^n - \sum_{n=0}^{\infty} f_n(1 - q^{n+\alpha})(1 - q^{n+\beta})z^{n+1} = 0.$$  

We obtain

$$f_n(1 - q^n)(1 - q^{n+r}) = f_{n-1}(1 - q^{n-1+\alpha})(1 - q^{n-1+\beta}), \quad (3.14)$$  

with initial condition

$$f_0(1 - q^0)(1 - q^r) = 0.$$  

So $f_0$ is arbitrary. From (3.14), we obtain

$$\frac{f_n}{f_{n-1}} = \frac{(1 - q^{\alpha+n-1})(1 - q^{\beta+n-1})}{(1 - q^n)(1 - q^{r+n})}.$$  

Assuming $f_0 = 1$, then

$$f_n = \frac{f_n}{f_{n-1}} \ldots \frac{f_1}{f_0} = \frac{(1 - q^\alpha)(1 - q^{\alpha+1})(1 - q^{\alpha+2}) \cdots (1 - q^{\alpha+n-1})}{(1 - q)(1 - q^{\beta+1})(1 - q^{\beta+2}) \cdots (1 - q^{\beta+n-1})}$$  

$$= \frac{(q^\alpha; q)_n (q^{\beta}; q)_n}{(q; q)_n (q^{r+1}; q)_n}.$$  

So we obtain the first solution
\[ F_1 = \sum_{n=0}^{\infty} \frac{(q^\alpha; q)_n (q^\beta; q)_n}{(q; q)_n (q^{r+1}; q)_n} z^n. \]
This is the \( q \)-hypergeometric series.

For another root \( x = q^{-r} \), similarly, we obtain a solution as follows:
\[ F_2 = \sum_{n=0}^{\infty} \frac{(q^{\alpha-r}; q)_n (q^{\beta-r}; q)_n}{(q; q)_n (q^{1-r}; q)_n} z^n. \]

**Remark 3.4** The generalized \( q \)-hypergeometric series is as follows:
\[
\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n} \cdot ((-1)^n q^{\frac{n(n-1)}{2}})^{1+s-r} z^n.
\]

It satisfies the following difference equation
\[
\left[ \prod_{i=1}^{s} (1 - b_i q^\sigma) - z (-q^\sigma)^{1+s-r} \prod_{i=1}^{r} (1 - a_i q^\sigma) \right] G = 0.
\]
This kind of difference equation often appears in quantum K-theory, especially for \( I \)-function with level structures; see [15] for more details.

### 3.3 Difference equation for \( \mathbb{P}^1 \) with standard level structure (0 \( \leq \) \( l \) \( \leq \) 2)

As we mentioned in the introduction, the modified \( I \)-function of \( \mathbb{P}^1 \) with level structure is
\[
\tilde{I}^{K,l}_{\mathbb{P}^1}(z, q) = P^l_q(z) \sum_{d=0}^{\infty} \frac{(P^d_q q^{(d-1)/2})^l \cdot d^d}{\prod_{k=1}^{d} (1 - P^k q)^2}.
\]
satisfying the following difference equation
\[
[(1 - \sigma^l_q)^2 - z \sigma^l_q] \tilde{I}^{K,l}_{\mathbb{P}^1}(z, q) = 0.
\]
If 0 \( \leq \) \( l \) \( \leq \) 2, the above difference equation is regular singular.

The characteristic equation is
\[
(1 - x)^2 = 0,
\]
than we have a double root
\[ x_1 = x_2 = q^0. \]
From (3.5)–(3.7) we know there exist two solutions of the following form
\[
F_1(z, q), \quad \ell_q(z) F_1(z, q) + F_2(z, q),
\]
where
\[
F_i(z, q) = \sum_{k=0}^{\infty} f_{ik}(q) z^k, \quad \text{with} \quad f_{i0} = 1, \quad i = 1, 2.
\]
As before, one can easily find that
\[
F_1(z, q) = \sum_{d=0}^{\infty} \frac{(q^{\frac{d(d-1)}{2}})^l \cdot d^d}{\prod_{k=1}^{d} (1 - q^k)^2}.
\]
Substituting the second solution into the difference equation, we have

\[ [2\sigma_q(\sigma_q - 1) - l z \sigma_q^l] F_1(z, q) = [z \sigma_q^l - (1 - \sigma_q)^2] F_2(z, q). \]

Then we obtain the following recursive formula for \( F_2(z, q) \):

\[ f_{2,d}(q) = \frac{q^{ld(d-1)}}{(1 - q^d)^2} f_{2,d-1}(q) + \frac{q^{ld(d-1)/2}}{\prod_{k=1}^{d} (1 - q^k)^2} \left( \frac{2q^d}{1 - q^d} + l \right). \]

So we have

\[ \ell_q(z) F_1(z, q) + F_2(z, q) = \sum_{d=0}^{\infty} \frac{q^{ld(d-1)}/2}{\prod_{k=1}^{d} (1 - q^k)^2} \left( \ell_q(z) - \sum_{k=1}^{d} \frac{2q^d}{1 - q^d} \right) + \sum_{d=0}^{\infty} \frac{ld \cdot q^{ld(d-1)}/2}{\prod_{k=1}^{d} (1 - q^k)^2}. \]

### 3.4 Convergence of Solutions

In this subsection, we prove the convergence of solutions in regular singular cases. Here we follow [1].

Let’s consider the regular singular equation

\[ \sum_{k=0}^{n} a_k(z) (\sigma_q)^k f(z) = 0, \]

with

\[ a_k(z) = a_{k0} + a_{k1}z + a_{k2}z^2 + \cdots, \]

i.e., the coefficients \( a_k(z) \) satisfy conditions in Proposition 2.13.

**Assumptions:**

(†) We assume that \( a_{00}, a_{n0} \neq 0 \), without loss of generality we may assume \( a_n(z) \equiv 1 \).

(||) It will be assumed further that all of the power series \( a_k(z) \) to be analytic at the origin and have a radius of convergence \( > 1 \).

To see the convergence of solutions, it’s sufficient to show the following two solutions in the resonant case:

\[ S_{i0}(z, q) = e_{q,c_i}(z) F_{i0}(z, q), \quad \text{(3.15)} \]
\[ S_{i1}(z, q) = \ell_q(z) S_{i0}(z, q) + e_{q,c_i,q^{-m_1}}(z) F_{i1}(z, q), \quad \text{(3.16)} \]

are convergent. We begin by proving the convergence of the first one directly.

Without loss of generality, suppose

\[ F_{i0}(z, q) = \sum_{d=0}^{\infty} f_{i,0,d}(q) z^d, \quad f_{i,0,0} = 1. \]

Since \( \sigma_q e_{q,c_i}(z) = c_i \cdot e_{q,c_i}(z) \), then

\[ \sum_{k=0}^{n} c_i^k \cdot a_k(z)(\sigma_q)^k F_{i0}(z, q) = 0, \]

that is,

\[ \sum_{k=0}^{n} c_i^k \cdot \left( \sum_{j=0}^{\infty} a_{kj}z^j \right) \left( \sum_{d=0}^{\infty} q^{kd} \cdot f_{i,0,d} \cdot z^d \right) = 0. \]
Then the coefficient of $z^m$-term is
\[ \sum_{j+d=m} \sum_{k=0}^n (c_i^k \cdot q^k \cdot a_{kj}) f_{i,0,d} = 0. \]

Let
\[ L_{ijd} = \sum_{k=0}^n (c_i^k \cdot q^k \cdot a_{kj}). \]

Then the coefficient $f_{i,0,m}$ is determined by the relation
\[ f_{i,0,m} = -\frac{\sum_{j=1}^m L_{i,j,m-j} \cdot f_{i,0,m-j}}{L_{i,0,m}}, \quad m > 0. \]

Considering first the numerator of this quotient, by the assumption (†), $a_n(z) \equiv 1$, then
\[ |L_{i,0,m}| = |(c_i q^m)^n| \cdot \left| 1 + \frac{a_{n-1,0}}{c_i q^m} + \cdots + \frac{a_{00}}{(c_i q^m)^n} \right|. \]

Let $m = m_1$ be large enough that the second factor on the right is $> \frac{1}{2}$. For $1 \leq m < m_1$, we have $|L_{i,0,m}| = A_m|(c_i q^m)^n|$, where $A_m \neq 0$. Setting
\[ A = \min\{A_1, \ldots, A_{m_1-1}, \frac{1}{2}\}. \]

And by the assumption (††), $a_i(z)$ are convergent for $z = 1$, therefore, we have
\[ |a_{ij}| < M_1. \]

The sets $|c_i^k|$, for $k = 0, 1, \ldots, n$ have an upper bound $M_2$, then for $j \geq 1$, we have
\[ L_{ijm} = \sum_{k=0}^n ((c_i q^m)^k \cdot a_{kj}) \leq n M_1 M_2 |q|^{m(n-1)}. \]

Then we have
\[ |f_{i,0,m}| < \frac{n M_1 M_2 |q|^{(m-1)(n-1)} \sum_{j=0}^{m-1} |f_{i,0,j}|}{A|(c_i q^m)^n|}. \]

Defining $\tilde{f}_{i,0,0}$ as $|f_{i,0,0}|$, we obtain the following upper bound for $f_{i,0,m}$:
\[ |f_{i,0,m}| < \frac{n M_1 M_2 \sum_{j=0}^{m-1} |\tilde{f}_{i,0,j}|}{A|c_i^n||q|^{n+m-1}} =: \tilde{f}_{i,0,m}. \]

As $m$ becomes infinite, the limit of
\[ \frac{\tilde{f}_{i,0,m}}{\tilde{f}_{i,0,m+1}} = \frac{|q| \sum_{j=0}^{m-1} |\tilde{f}_{i,0,j}|}{\tilde{f}_{i,0,m} + \sum_{j=0}^{m-1} |\tilde{f}_{i,0,j}|} = \frac{|q|}{n M/|q|^{n+m-1} + 1} \]
is $|q|$, where $M = M_1 M_2/(A|c_i^n|)$. It implies the convergence of the power-series solution (3.15).

Now let’s consider the second solution (3.16),
\[ S_{i1}(z, q) = \ell_q(z) e_{q,c_i} F_{i0}(z, q) + e_{q,c,q^{-m_1}} F_{i1}(z, q), \]
and
\[ e_{q,c,q^{-m_1}}(z) = q^{\frac{m_1(m_1-1)}{2}} z^{-m_1} e_{q,c_i}(z). \]
Setting \( q_0 = q^{-\frac{m_i(m_i+1)}{2}} \), then from the difference equation, we obtain

\[
\sum_{k=0}^{n} \left( \sum_{j=0}^{\infty} a_{kj} z^j \right) k c_i^k \left( \sum_{d=0}^{\infty} q^{kd} f_{i,0,d} z^d \right) + q_0 \sum_{k=0}^{n} \left( \sum_{j=0}^{\infty} a_{kj} z^j \right) c_i^k \left( \sum_{d=0}^{\infty} q^{k(d+m_i)} f_{i,1,d} z^{d+m_i} \right) = 0.
\]

Let

\[
L'_{i,j,d} = \sum_{k=0}^{n} (k c_i^k \cdot q^{kd} \cdot a_{kj}).
\]

We find that the \( f_{i,1,m} \) satisfy the relations

\[
f_{i,1,m-m_{i1}} = -\frac{L'_{i,0,m} \cdot f_{i,0,m} + \sum_{j=1}^{m-1} \left[ L'_{i,j,m-j} \cdot f_{i,0,m-j} + q_0 L_{i,j,m-j} \cdot f_{i,1,m-j-m_{i1}} \right]}{q_0 L_{i,0,m}}.
\]

Here we use the notation that \( f_{i,1,m} = 0 \) for \( m < 0 \). Proceeding as before, we find

\[
|f_{i,1,m-m_{i1}}| < \frac{n M \sum_{j=1}^{m-1} |f_{i,1,j-m_{i1}}|}{|q|^{n+m-1}} + \frac{n(n-1) M \sum_{j=1}^{m-1} |f_{i,0,j}|}{|q|^{n+k-1}} + \frac{n(n+1) M |f_{i,0,m}|}{2},
\]

\[
< \frac{n^2(n+1) M^2 \sum_{j=0}^{m-1} [f_{i,1,j-m_{i1}} + 2f_{i,0,j}]}{|q|^{n+m-1}}.
\]

To obtain the last relation, we use the inequality for \( f_{i,0,m} \). Similarly, defining \( \tilde{f}_{i,1,0} \) as \( |f_{i,1,0}| \) and \( \tilde{f}_{i,1,m} \) as before, then we have

\[
\frac{\tilde{f}_{i,1,m+1-m_{i1}}}{\tilde{f}_{i,1,m-m_{i1}}} = \frac{1}{|q|} \left[ \frac{n^2(n+1) M^2}{|q|^{n+m-1}} + \frac{2n M}{|q|^{n+1}} \sum_{j=0}^{m-1} \tilde{f}_{i,0,j} + \sum_{j=0}^{m-1} \tilde{f}_{i,1,j-m_{i1}} + 2 \tilde{f}_{i,0,j} \right].
\]

The series \( \sum_{j=0}^{m-1} \tilde{f}_{i,0,j} \) is convergent. As \( m \) tends to be infinite, \( \sum_{i=0}^{m-1} \tilde{f}_{i,1,j-m_{i1}} \) must either approach a limit or become infinite, in either case the limit of the above quotient is \( 1/|q| \). Hence it proves the convergence of the second solution.

### 4 Local Solutions for Irregular Singular Cases

In the last section, we show how to construct solutions in the regular singular case. In this section, we focus on irregular cases. Following the method of [2], we show how to construct formal series solutions in some irregular singular degree 2 difference equations.

Let’s consider the following degree 2 difference equation

\[
[a_2(z,q) \sigma_q^2 + a_1(z,q) \sigma_q + a_0(z,q)] f = 0,
\]

with

\[
a_i(z,q) = a_{i0} + a_{i1} z + a_{i2} z^2 + \cdots, \quad i = 0, 1, 2.
\]

From Proposition 2.13, we know that irregularity implies that one of \( a_{20}, a_{00} \) equals 0 or both of them vanish.

**Example 4.1 (Ramanujan Equation)** Let’s consider the following Ramanujan equation

\[
[qz \sigma_q^2 - \sigma_q + 1] f = 0,
\]

(4.1)
whose characteristic equation is

\[-x + 1 = 0, \quad \text{i.e., } x = q^0.\]

Consider a solution of the form

\[F = \sum_{n=0}^{\infty} f_n(q)z^n.\]

Substituting into (4.1), we have

\[0 = qz \sum_{n=0}^{\infty} f_nq^{2n}z^n - \sum_{n=0}^{\infty} f_nq^n z^n + \sum_{n=0}^{\infty} f_n z^n\]

\[= \sum_{n=1}^{\infty} f_{n-1}q^{2n-1}z^n - \sum_{n=0}^{\infty} f_nq^n z^n + \sum_{n=0}^{\infty} f_n z^n.\]

Therefore, we have

\[f_{n-1}q^{2n-1} - (q^n - 1)f_n = 0,\]

with initial condition \((1 - q^0)f_0 = 0\). So \(f_0\) is a free parameter and

\[\frac{f_n}{f_{n-1}} = \frac{q^{2n-1}}{1 - q^n} = \frac{q^{2(n-1)+1}}{1 - q^n}.\]

Thus, assume \(f_0 = 1\), we have

\[f_n = \frac{f_n}{f_{n-1}} \cdot \frac{f_{n-1}}{f_{n-2}} \cdots \frac{f_1}{f_0} = \frac{-q^{2n-1}}{1 - q^n} \cdot \frac{-q^{2n-3}}{1 - q^{n-1}} \cdots \frac{-q}{1 - q} = \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n}q^n.\]

Then the first solution is of the form

\[F = \sum_{n=0}^{\infty} \left(\frac{q^{n(n-1)/2}}{(q;q)_n}\right)^2 (-qz)^n = 0\phi_1(-, 0; q; -q x),\]

where

\[r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q; x) := \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n} (-1)^n q^{n(n-1)/2}(1 + s - r x^n).\]

Let’s consider another solution of the form

\[\theta^{-1}_q(z) \sum_{n=0}^{\infty} f_n(q)z^n.\]

Substituting into (4.1), we obtain

\[0 = qz \cdot q^{-1} z^{-2} \theta^{-1}_q(z) \sum_{n=0}^{\infty} f_n(q)q^{2n}z^n - z^{-1} \theta^{-1}_q(z) \sum_{n=0}^{\infty} f_n q^n z^n + \theta^{-1}_q(z) \sum_{n=0}^{\infty} f_n z^n\]

\[= \theta^{-1}_q(z) \sum_{n=0}^{\infty} f_n(q)q^{2n}z^{n-1} - \theta^{-1}_q(z) \sum_{n=0}^{\infty} f_n q^n z^{n-1} + \theta^{-1}_q(z) \sum_{n=0}^{\infty} f_n z^n.\]
\[= \theta^{-1}_q(z) \sum_{n=0}^{\infty} (f_{n+1}(q)q^{2(n+1)} - f_{n+1}(q)q^{n+1} + f_n)z^n,\]

with initial condition

\[f_0(q^0 - q^0) = 0.\]

For \(n > 0\), we have

\[f_{n+1}(q^{2(n+1)} - q^{n+1}) + f_n = 0,\]

then

\[
\frac{f_{n+1}}{f_n} = \frac{1}{q^{n+1} - q^{2(n+1)}} = \frac{1}{q^{n+1}(1 - q^{n+1})},
\]

so

\[f_n = f_0 \frac{1}{q^n(1 - q^n)} \cdots \frac{1}{q(1 - q)} = \frac{\left(\frac{q^{n(n-1)}}{q^2}\right)^{-1}}{(q; q)_n} \frac{1}{q^n}.
\]

Thus, we obtain the second solution (assume \(f_0 = 1\))

\[F_2 = \theta^{-1}_q(z) \cdot _2\phi_0\left(0, 0; -; q; -\frac{x}{q}\right)\]

Here we use Jacobi’s theta function introduced in Subsection 2.2. This is not a coincidence; in the following, we show how to do it in general.

4.1 General Technique: Newton Polygon

Let’s consider the equation

\[\sum_{i=0}^{n} a_i(z)(\sigma_q)^i f(z) = 0,\]

with

\[a_i(z) = a_{i0} + a_{i1}z + a_{i2}z^2 + \cdots .\]

Denote by \(a_{i,j}\), the first nonzero coefficient in \(a_i(z)\), and choosing \(i\)- and \(j\)-axes as horizontal and vertical axes respectively, plot the points \((n - i, j_i)\). Construct a broken line, convex downward, so both ends of each line segment are points of the set \((n - i, j_i)\). Then we obtain a Newton polygon as follows:
Note that the horizontal segment corresponds to the characteristic equation
\[ a_{k,0}x^k + a_{k-1,0}x^{k-1} + \cdots + a_{d,0}x^d = 0. \]

The degree of the above characteristic equation is 1 less than the number of points that lie on or above that segment.

Then the general technique to construct solutions is as follows

- **Horizontal segment**: As mentioned above, it corresponds to the characteristic equation. Using the non-zero roots, we could construct the associated solutions as regular singular cases.

- **Non-horizontal segment**: For each non-horizontal segment of slope \( \mu \), a rational number.
  - If \( \mu = r \) is an integer; we consider a formal series solution of the form
    \[ \theta_r^q(z) \sum_{n=0}^{\infty} f_n(q)z^n. \]
  - If \( \mu = t/s \) is a rational number with \( s \) positive, then we consider a formal series solution of the form
    \[ \theta_{t/s}^q(z^{t/s}) \sum_{n=0}^{\infty} f_n(q)z^{n/s}, \]

where \( \theta_{t/s}^q(z^{t/s}) \) satisfies
\[ q^{-\partial_z} \theta_{t/s}^q(z^{t/s}) = \theta_{t/s}^q((qz)^{t/s}) = q^{t/s} \theta_{t/s}^q(z^{t/s}). \]

**Example 4.2** (Slope \( \mu = -1 \)) Consider the following difference equation
\[ [z^2\sigma_q^2 + z\sigma_q + 1]f = 0. \]

The Newton polygon is

```
(0,2) -- (2,0)
```

with slope \( \mu = -1 \), then consider the solution of the form
\[ \theta_{-1}^q(z)F(z). \]

Then we obtain a new difference equation for \( F(z) \) as follows:
\[ [q^{-1}\sigma_q^2 + \sigma_q + 1]F(z) = 0, \]

with characteristic equation
\[ q^{-1}x^2 + x + 1 = 0. \]
Suppose the roots are \( c_1 \) and \( c_2 \), so new difference equation could be rewritten as

\[
(\sigma_q - c_1)(\sigma_q - c_2)F = 0.
\]

It’s easy to construct two solutions

\[
\theta_q^{-1}(z)e_{q,c_1}(z), \quad \theta_q^{-1}(z)e_{q,c_2}(z).
\]

**Example 4.3** (Slope \( \mu = -2 \))  Consider the following difference equation

\[
\left[z^2\sigma_q^2 - \sigma_q + 1\right] f = 0. \tag{4.2}
\]

The Newton polygon of (4.2) is as follows:

![Newton polygon](image)

The first segment’s slope is \(-2\), and the second horizontal segment corresponds to the characteristic equation

\[-x + 1 = 0.
\]

Then let’s first consider the solution of the following form

\[
F_1 = \sum_{n=0}^{\infty} f_n z^n.
\]

Substituting into (4.2), we obtain

\[
0 = \sum_{n=0}^{\infty} f_n q^{2n} z^{n+2} - \sum_{n=0}^{\infty} f_n q^n z^n + \sum_{n=0}^{\infty} f_n z^n
\]

\[
= \sum_{n=2}^{\infty} f_{n-2} q^{2n-4} z^n + \sum_{n=0}^{\infty} f_n (1 - q^n) z^n, \tag{4.3}
\]

with initial conditions

\[
f_0(1 - q^0) = 0, \quad f_1(1 - q^1) = 0,
\]

i.e., \( f_1 = 0 \) and \( f_0 \) is arbitrary. For \( n \geq 1 \), from (4.3) we have

\[
\frac{f_{2n}}{f_{2n-2}} = -\frac{q^{4(n-1)}}{1 - q^{2n}}.
\]

Thus

\[
f_{2n} = -\frac{q^{4(n-1)}}{1 - q^{2n}} \cdot -\frac{q^{4(n-2)}}{1 - q^{2(n-1)}} \cdot \cdots \cdot -\frac{1}{1 - q^2} \cdot f_0
\]

\[
= (\frac{-1}{{q^2}_n})^{2n(n-1)} \cdot f_0.
\]
Taking \( f_0 = 1 \), we finally obtain
\[
F_1(z) = \sum_{n=0}^{\infty} \frac{q^{2n(n-1)}}{(q^2; q^2)_n} (-1)^n z^{2n}.
\]

Now let’s construct the second solution. Consider the solution of the following form
\[
F_2 = \theta_q^{-2}(z) F(z).
\]
Substituting into (4.2), we obtain
\[
0 = z^2 \sigma^2_q (\theta_q^{-2}(z) F(z)) - \sigma_q (\theta_q^{-2}(z) F(z)) + \theta_q^{-2}(z) F(z)
\]
\[
= z^2 q^{-2} z^{-4} \theta_q^{-2}(z) \sigma^2_q F(z) - z^{-2} \theta_q^{-2}(z) \sigma_q F(z) + \theta_q^{-2}(z) F(z)
\]
\[
= z^{-2} \theta_q^{-2}(z) \cdot [q^{-2} \sigma^2_q - \sigma_q + z^2] F(z).
\]

So we obtain a new difference equation
\[
[q^{-2} \sigma^2_q - \sigma_q + z^2] F(z) = 0, \tag{4.4}
\]
with characteristic equation
\[
q^{-2} x^2 - x = 0.
\]
The roots are \( x = 0 \) or \( x = q^2 \). For \( x = q^2 \), consider the following solution
\[
F(z) = \sum_{n=0}^{\infty} f_n z^{n+2}. \tag{4.5}
\]
Substituting (4.5) into (4.4), we have
\[
0 = \sum_{n=0}^{\infty} f_n q^{2n+2} z^{n+2} - \sum_{n=0}^{\infty} f_n q^{n+2} z^{n+2} + \sum_{n=0}^{\infty} f_n z^{n+4}.
\]
Thus we obtain
\[
\frac{f_{n+2}}{f_n} = \frac{1}{q^{n+4} (1 - q^{n+2})},
\]
with the initial conditions
\[
f_0(q^2 - q^2) = 0, \quad f_1(q^4 - q^3) = 0.
\]
So we know \( f_{2n+1} = 0 \) and
\[
f_{2n} = \frac{q^{-2}}{q^{2n} (1 - q^{2n})} \cdots \frac{q^{-2}}{q^2 (1 - q^2)} \cdot f_0.
\]
Taking \( f_0 = 1 \), we conclude that the solution (4.5) is as follows:
\[
F(z) = \sum_{n=0}^{\infty} \frac{q^{-n(n+3)}}{(q^2; q^2)_n} z^{2n+2}.
\]

**Remark 4.4** (1) In general, for a root \( x = c \), we consider the solution of the form
\[
F(z) = e_{q,c}(z) \sum_{n=0}^{\infty} f_n z^n,
\]
as we did in the regular cases. If \( c \) is a \( q^n \)-root, say \( q^n \), then
\[
e_{q,q^n}(z) = q^{\frac{n(n-1)}{2}} z^n.
\]
(2) For another root \( x = 0 \), we can not construct a new solution from it. Indeed, if we consider a solution of the form

\[
G = \sum_{n=0}^{\infty} g_n z^n,
\]

substituting into (4.4), we obtain

\[
0 = \sum_{n=0}^{\infty} g_n q^{2n-2} z^n - \sum_{n=0}^{\infty} g_n q^n z^n + \sum_{n=0}^{\infty} g_n z^n
\]

\[
= \sum_{n=0}^{\infty} q^n (q^{n-2} - 1) g_n z^n = \sum_{n=2}^{\infty} g_{n-2} z^n,
\]

with initial conditions

\[
q^0(q^{-2} - 1)g_0 = 0, \quad q^1(q^{-1} - q)g_1 = 0 \quad \text{and} \quad q^2(q^0 - 1)g_2 = 0.
\]

So \( g_0 = g_1 = 0 \) and \( g_2 \) could be arbitrary. Then for \( n \geq 2 \), we have

\[
\frac{g_{2n+2}}{g_{2n}} = \frac{1}{q^{2n+2}(1 - q^{2n})},
\]

which is the same as the case when root \( x = q^2 \).

In summary, for the degree 2 difference equation

\[
[z^2 \sigma_q^2 - \sigma_q + 1] f = 0,
\]

we can construct two solutions

\[
F_1(z) = \sum_{n=0}^{\infty} q^{2n(n-1)}(q^2; q^2)_n (-1)^n z^{2n},
\]

\[
F_2(z) = \theta_q^{-2}(z) \sum_{n=0}^{\infty} q^{-n(n+3)}(q^2; q^2)_n z^{2n+2}.
\]

Example 4.5 (Slope \( \mu = -1/2 \)) Consider the following difference equation

\[
[z \sigma_q^2 - 1] f = 0,
\]

then the associated Newton polygon is as follows:

with only one segment of slope \(-\frac{1}{2}\), then consider the solution of the form

\[
\theta_q^{-1/2}(z^{-1/2}) F(z^{1/2}).
\]

Substituting into (4.6), we have

\[
z(q^{-1/2}z^{-1} \theta_q^{-1/2}(z^{-1/2}) \sigma_q^2 F(z^{1/2})) - \theta_q^{-1/2}(z^{-1/2}) F(z^{1/2}) = 0.
\]
So we obtain a new difference equation
\[ [q^{-\frac{1}{2}} \sigma_q^2 - 1] F(z^{1/2}) = 0, \]
with the characteristic equation
\[ q^{-\frac{1}{2}} x^2 - 1 = 0. \]
The two roots are \( x = q^{\frac{1}{4}} \) and \( x = -q^{\frac{1}{4}} \). Then one could consider the solutions of the form
\[ F_1 = e_{q,q^{\frac{1}{4}}}(z) \sum_{n=0}^{\infty} f_n z^{n/2}, \quad F_2 = e_{q,-q^{\frac{1}{4}}}(z) \sum_{n=0}^{\infty} f_n z^{n/2}, \]
by a little bit of computation, one could find these solutions are
\[ F_1 = e_{q,q^{\frac{1}{4}}}(z), \quad F_2 = e_{q,-q^{\frac{1}{4}}}(z). \]
Then the solutions to the original equation are
\[ \theta_{q^{\frac{1}{2}}}(z^{1/2}) e_{q,q^{\frac{1}{4}}}(z), \quad \theta_{q^{\frac{1}{2}}}(z^{1/2}) e_{q,-q^{\frac{1}{4}}}(z). \]

4.2 Difference Equation for \( \mathbb{P}^1 \) with Level Structure \((l \geq 2)\)

As we mentioned in subsection 3.3, the modified \( I \)-function of \( \mathbb{P}^1 \) with level structure is
\[ \tilde{I}_{K,l}^{P_1} = P_{\ell}(q) \sum_{d=0}^{\infty} \frac{(P^{d}q^{d(1-1)}l)_{d} z^{d}}{\prod_{k=1}^{d}(1-Pq^{k})^{2}}, \]
satisfying the following difference equation
\[ [(1 - \sigma_q) z^{l} - 2 \sigma_q^{l}] \tilde{I}_{K,l}^{P_1} = 0. \quad (4.7) \]
Here we consider \( l > 2 \); the above difference equation is irregular singular.

The associated Newton polygon is as follows. There are two segments, one is of slope \(-1/(l - 2)\), and another one is horizontal.

The characteristic equation to the horizontal segment is
\[ (1 - x)^2 = 0, \]
which is the same as the \( 0 \leq l \leq 2 \) (regular singular) case in Subsection 3.3. Thus there are two solutions as before:
\[ F_1(z, q) = \sum_{d=0}^{\infty} \frac{(q^{d(d-1)}l)_{d} z^{d}}{\prod_{k=1}^{d}(1 - q^k)^2}, \]
and
\[ \ell_q(z) F_1(z, q) + F_2(z, q) = \sum_{d=0}^{\infty} \frac{q^{d(d-1)}l_{d} z^{d}}{\prod_{k=1}^{d}(1 - q^k)^2} \left( \ell_q(z) - \sum_{k=1}^{d} \frac{2q^k}{1 - q^k} \right) + \sum_{d=0}^{\infty} \frac{ld \cdot q^{l(d-1)}l_{d} z^{d}}{\prod_{k=1}^{d}(1 - q^k)^2}. \]
For the segment of slope $\mu = -1/(l-2)$, we consider solutions of the form
\[ \theta_{q^{-1/(l-2)}}(z^{-1/(l-2)}) F(z^{1/(l-2)}, q). \]
Let $Q = z^{1/(l-2)}$, $p = q^{1/(l-2)}$ and $\sigma_p = p^l \partial q$. Then one finds that $F(Q, p)$ satisfies
\[ [\sigma_p^l - \sigma_p^2 + 2Q - Q^2] F(Q, p) = 0, \]
with a new Newton polygon

The characteristic equation is
\[ x^2(x^{l-2} - 1) = 0. \]
So each $(l-2)$-th root of unity $\zeta$, we could construct a solution of the form
\[ e_{p,\zeta}(Q) \sum_{d=0}^{\infty} f_d(\zeta, p) Q^d \quad \text{with} \quad f_0(\zeta, p) = 1. \]
Then we obtain a relation of the coefficient $f_d(\zeta, p)$ as follows,
\[ \sum_{d \geq 0} [f_d(\zeta, p) \cdot \zeta^2(\zeta^l - p^{2d}) Q^d + 2f_d(\zeta, p)Q^{d+1} - f_d(\zeta, p)Q^{d+2}] = 0, \]
that is
\[ \zeta^2(\zeta^l - p^{2d})f_d(\zeta, p) = 2f_{d-1}(\zeta, p) - f_{d-2}(\zeta, p), \quad d \geq 1, \]
where we set $f_{-1}(\zeta, p) = 0$. Thus we construct $l$ solutions for the difference equation (4.7).

### 4.3 Difference Equation for Quintic 3-fold

As we introduced in the introduction, the modified $I$-function of quintic is
\[ \tilde{I}_X^K(z, q) = P_{\ell_q}(z) \sum_{d=0}^{\infty} \prod_{k=1}^{\lfloor d/5 \rfloor} \frac{(1 - P^k q^5)^5}{z^d}, \]
satisfying the following difference equation of degree 25
\[ \left[ (1 - \sigma_q)^5 - z \prod_{k=1}^{5} (1 - q^k \sigma_q^5) \right] \tilde{I}_X^K(q, z) = 0, \]
whose associated Newton polygon is
There are two segments, one is of slope $\mu = -1/20$, and another one is horizontal. We could do the same as the last subsection. For the non-horizontal segment, we have

**Proposition 4.6** ([20, Proposition 3.1]) Let $\xi$ be the 20th root of unity and $p = q^{1/20}$. Setting $y = z^{1/20}$ and $\sigma_p = p\sigma_0$. There are 20 solutions associated with the segment of slope $\mu = -1/20$ of the form

$$
eq_p, \xi p^{-1/2}F(y) = \sum_{n \geq 0} f_n y^n$$

where $F(y)$ satisfies

$$[(y - \xi p^{-\frac{2}{20}})(y - \xi p^{-\frac{12}{20}})(y - \xi p^{-\frac{20}{20}})(y - \xi p^{-\frac{2}{2}})(y - \xi p^{-\frac{4}{2}})(y - \xi p^{-\frac{6}{2}})(y - \xi p^{-\frac{8}{2}})(y - \xi p^{-\frac{10}{2}})]F(y) = 0.$$  

**Remark 4.7** In this notes, we require $|q| > 1$. However, in [20], to do the analytic continuation, it needs to require $|q| < 1$ for the convergence reason. And see [7] for additional discussion on $q$-deformed Picard–Fuchs equation and Frobenius method for quintic 3-fold.

### 4.4 Convergent Solutions for Irregular Cases

In this subsection, we prove the convergence of certain solutions in irregular singular cases. Here we follow [1, 2, 18].

Let $P = a_0 + \cdots + a_n \sigma_q^n$ be the standard form, suppose $a_0 a_n \neq 0$ and at least one $a_i(0)$ is nonzero. Setting

$$a_k(z) = \sum_{j=0}^{\infty} a_{kj} z^j,$$

and

$$P_j(\sigma_q) = \sum_{k=0}^{n} a_{kj} \cdot \sigma_q^k.$$  

Then we have

**Lemma 4.8** ([18, Lemma 5.2.8]) Assume that the lowest slope of $P$ is 0 and $P_0(1) = 0$, $P_0(q^k) \neq 0$ for $k \geq 1$. Then the unique formal solution of

$$P \cdot f(z, q) = [a_0 + \cdots + a_n \sigma_q^n] f(z, q) = 0$$

in the form

$$f = 1 + f_1 z + \cdots \in \mathbb{C}[z]$$

converges if $|q|^n > 2$.

**Proof** It almost follows the procedure in Subsection 3.4, here we give a brief proof. There exist a constant $A > 0$ such that

$$|P_0(q^k)| \geq A |q|^{kn}, \quad \forall k \geq 1.$$  

Let $R$ be strictly bounded above by the radius of convergence of $a_0(z), \ldots, a_n(z)$. Then there is a $B > 0$ such that

$$\forall i \in \{0, \ldots, n\}, \quad \forall j \geq 0, \quad |a_{ij}| \leq BR^{-j}.$$
Then

$$|P_j(q^i)| \leq DR^{-j}|q|^{ni}.$$  

From the recursive relation, we have for $m \geq 1$

$$|f_m| \leq \frac{D}{A} \sum_{i=0}^{m-1} \frac{R^{-i(m-i)}|q|^{ni}}{|q|^{nm}}|f_i|.$$  

Quotient by $(R^{-1}|q|^{-n})^m$ on both sides, then we obtain

$$\frac{|f_m|}{(R^{-1}|q|^{-n})^m} \leq \frac{D}{A} \sum_{i=0}^{m-1} \left(\frac{|f_i|}{(R^{-1}|q|^{-n})^i}\right),$$

that is

$$|f_m| \leq \frac{2^{m-1}D}{A(R|q|^n)^m}|f_0|.$$  

The convergence of $f$ follows.

Remark 4.9  
• The above argument fails if 0 is not the lowest slope of $P$. For example, one could check our Examples 4.2 and 4.3 that the solutions associate with horizontal segment do not converge.

• If the lowest slope of $P$ is $\mu \in \mathbb{Z}_{<0}$, then there is a solution of the form

$$\theta^\mu_q(z) F(z, q) = \theta^\mu_q(z) \sum_{d=0}^{\infty} f_d(q) \cdot z^d,$$

and $F(z, q)$ is analytic at the origin. The proof is almost the same since the prefactor $\theta^\mu_q(z)$ will make the lowest slope of the new difference equation for $F(z, q)$ to be 0.

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References
[1] Adams, C. R.: On the linear ordinary $q$-difference equation. *Ann. of Math.* (2), 30(1/4), 195–205 (1928)
[2] Adams, C. R.: On the irregular cases of the linear ordinary difference equation. *Trans. Amer. Math. Soc.*, 30(3), 507–541 (1928)
[3] Aganagic, M., Hori, K., Karch, A., Tong, D.: Mirror symmetry in $2+1$ and $1+1$ dimensions. *J. High Energy Phys.*, 07, 022 (2001)
[4] Carmichael, R. D.: The general theory of linear $q$-difference equations. *Amer. J. Math.*, 34(2), 147–168 (1912)
[5] Chiodo, A., Ruan, Y.: Landau–Ginzburg/Calabi–Yau correspondence for quintic three-folds via symplectic transformations. *Invent. Math.*, 182, 117–165 (2010)
[6] Dorey, N., Tong, D.: Mirror symmetry and toric geometry in three-dimensional gauge theories. *J. High Energy Phys.*, 5, Paper No. 018, 16 pp. (2000)
[7] Garoufalidis, S., Scheidegger, E.: On the quantum $K$-theory of the quintic. *SIGMA*, 18, Paper No. 021, 20 pp. (2022)
[8] Givental, A.: On the WDVV equation in quantum $K$-theory, Dedicated to William Fulton on the occasion of his 60th birthday. *Michigan Math. J.*, 48, 295–304 (2000)
[9] Givental, A.: Permutation-equivariant quantum K-theory V. Toric \textit{q}-hypergeometric functions, arXiv:1509.03903 (2015)
[10] Iritani, H., Milanov, T., Tonita, V.: Reconstruction and convergence in quantum K-theory via difference equations. \textit{Int. Math. Res. Not. IMRN}, 11, 2887–2937 (2015)
[11] Jockers, H., Mayr, P.: A 3d gauge theory/quantum k-theory correspondence. \textit{Adv. Theor. Math. Phys.}, 24, 327–457 (2020)
[12] Lee, Y.-P.: Quantum K-theory I: Foundations. \textit{Duke Math. J.}, 121(3), 389–424 (2004)
[13] Roquefeuil, A.: Confluence of quantum K-theory to quantum cohomology for projective spaces, arXiv:1911.00254 (2019)
[14] Ruan, Y., Wen, Y., Zhou, Z.: Quantum K-theory of toric varieties, level structures, and 3d mirror symmetry, arXiv:2011.07519 (2020)
[15] Ruan, Y., Zhang, M.: The level structure in quantum K-theory and mock theta functions, arXiv:1804.06552 (2018)
[16] Sauloy, J.: Systèmes aux \textit{q}-différences singuliers réguliers: classification, matrice de connexion et monodromie. \textit{Ann. Inst. Fourier (Grenoble)}, 50(4), 1021–1071 (2000)
[17] Sauloy, J.: Galois theory of fuchsian \textit{q}-difference equations. \textit{Ann. Sci. École Norm. Sup. (4)}, 36(6), 925–968 (2003)
[18] Sauloy, J.: Analytic study of \textit{q}-difference equations. In: Galois Theories of Linear Difference Equations: An Introduction, Math. Surveys Monogr., Vol. 211, Amer. Math. Soc., Providence, RI, 2016, 103–171
[19] Ueda, K., Yoshida, Y.: 3d $\mathcal{N} = 2$ Chern–Simons-matter theory, Bethe ansatz, and quantum K-theory of Grassmannians. \textit{J. High Energ. Phys.}, 8, Paper No. 157, 43 pp. (2020)
[20] Wen, Y.: Difference equation for quintic 3-fold. \textit{SIGMA}, 18, Paper No. 043, 25 pp. (2022)