The Ising universality class in dimension three: corrections to scaling

P. H. Lundow
Department of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden

I. A. Campbell
Laboratoire Charles Coulomb (L2C), UMR 5221 CNRS-Université de Montpellier, Montpellier, F-France.
(Dated: March 13, 2022)

Simulation data are analyzed for four 3D spin-1/2 Ising models: on the FCC lattice, the BCC lattice, the SC lattice and the Diamond lattice. The observables studied are the susceptibility, the reduced second moment correlation length, and the normalized Binder cumulant. From measurements covering the entire paramagnetic temperature regime the corrections to scaling are estimated. We conclude that a correction term having an exponent which is consistent within the statistics with the bootstrap value of the universal subleading thermal confluent correction exponent, $\theta_2 \sim 2.454(3)$, is almost always present with a significant amplitude. In all four models, for the normalized Binder cumulant the leading confluent correction term has zero amplitude. This implies that the universal ratio of leading confluent correction amplitudes $a_\chi/a_\chi = 2$ in the 3D Ising universality class.

I. INTRODUCTION

The development of the "conformal bootstrap" approach has led to a major step forward in understanding the canonical Ising universality class in dimension three [1-6]. The principle universal critical exponents: the correlation length exponent $\nu = 0.62999(5)$, the anomalous dimension $\eta = 0.03631(3)$ (and so the susceptibility exponent $\gamma = (2 - \eta)\nu = 1.23710(12)$) and the specific heat exponent $\alpha = D\nu - 2 = 0.11003(15)$) and the leading confluent correction exponent $\omega = 0.8303(18)$ (and so the thermal confluent correction $\theta = \omega\nu = 0.5231(11)$) are established to high precision. Earlier high temperature scaling expansion (HTSE) [7,8] and numerical simulation [9,10] values are slightly less accurate than but fully consistent with the bootstrap results.

The subleading conformal correction exponent, which is also universal, was however never directly estimated in earlier work but was simply assumed to be $\omega_2 = 1.67(11)$ (and so the thermal confluent correction exponent $\theta_2 = \omega_2\nu = 1.05(7)$) following Ref. [11]. For this exponent the bootstrap estimates are dramatically higher: $\omega_2 \sim 4.3$, and so $\theta_2 \sim 2.7$ [1,2]; the most recent bootstrap calculation, Ref. [12] Table 2, provides a high precision estimate for the $\epsilon''$ stable operator parameter $\Delta = 6.8956(43)$ which can be translated to give the second thermal conformal correction exponent $\theta_2 = 2.454(3)$. Here we examine extensive numerical and HTSE data on the 3D Ising universality class in the light of this result.

Historically, the standard thermal scaling variable in Ising models was $\tau = (1 - \beta/\beta_c)$ where $\beta$ is the inverse temperature $1/T$. $\tau$ was used initially by Donb and Sykes in 1962 [13], by Wegner for his RGT expansion in 1972 [14], and has been in continuous use in HTSE analyses ever since (see for instance Ref. [2]). $\tau$ varies from 0 at criticality to 1 at infinite temperature, with no divergence, so by using $\tau$ as the thermal scaling variable HTSE and simulation data can be analysed in detail from criticality to infinite temperature with empirical scaling analyses in terms of temperature dependent effective exponents such as $\gamma(\tau)$, as has already been demonstrated in various specific Ising and ISG models, e.g. [7,17,19].

The Wegner expansion for the susceptibility can be written (see for instance Ref. [20])

$$\chi(\tau) = C_\chi \tau^{-\gamma} \left( 1 + a_\tau^0 + b_\tau + c_\tau^{\theta_2} + \cdots \right)$$

where only the three leading correction terms are written explicitly: the exponents are universal but the critical amplitude $C_\chi$ and the correction amplitudes $a_\tau, b_\tau$, $c_\tau$ are not, though conformal correction amplitude ratios such as $a_\chi/a_\xi$ are universal [21,22]. The first correction term is the leading confluent correction, the second is the leading analytic correction, and the third one is the subleading confluent correction. Equivalent expressions can be written for other thermodynamic variables $Q(\tau)$ [7]. There is little in the way of a priori guidelines as to expected critical amplitudes or correction amplitudes for specific models. A forbidding list of further potential correction terms is indicated by Privman et al. [20], with "minor" terms in $\tau^2$, $\tau^{\theta+1}$, $\tau^{\theta+2}$, etc. As Eq. (1) can be written

$$\chi(\tau) = C_\chi \tau^{-\gamma} \left( 1 + a_\tau^\theta[1 + a_1 \tau^\theta + a_2 \tau + \cdots] + b_\tau[1 + b_1 \tau^\theta + b_2 \tau + \cdots] + c_\tau \tau^\theta[1 + c_1 \tau^\theta + c_2 \tau + \cdots] + \cdots \right)$$

all the "minor" terms can be considered as corrections to corrections. Leading correction amplitudes $a_\tau, b_\tau, c_\tau$ turn out to be typically 0.10 or less, so a plausible assumption is that the "correction to correction" terms have very small amplitudes, $\sim 0.001$. Indeed Ref. [8] states "several ["minor"] corrections ... apparently conspire to give a uniformly small correction", and Ref. [10] states "We estimate the error caused by ["minor"] correction terms that are not included by comparing the results obtained by using different ansätze and ... by fitting different quantities." In the present three-term analyses including only the three leading correction terms in Eq. (1) with
exponents $\theta = 0.523, 1$, and $\theta_2 = 2.45$, the "minor" correction having exponent $\tau^{20} \sim 1.04$ would be confused with the leading analytic correction term to give an effective amplitude. If any neglected "minor" contributions from corrections having exponents of the order of $\sim 1.5$ have significant amplitudes they would be visible as perturbations to the three-term fits. No evidence for such perturbations has been seen in the fits described below, so "minor" correction terms will be considered negligible.

Although the Wegner expression was initially introduced for improving asymptotic scaling analyses "near the critical point", the temperature dependence of effective exponents such as $\gamma(\tau, L) = -\partial \ln \chi(\tau, L)/\partial \ln \tau$ can be readily measured up to infinite temperatures by HTSE or numerically, with HTSE data coming essentially exact for high $\tau$ if $\beta_c$ is well known (see [7]). A Wegner correction term with an exponent considerably higher than 1 will influence the data significantly only at high $\tau$ so in practice the "near the critical point" condition must be relaxed. Thus, in Ref. [3] Figs. 14 and 16 all the effective susceptibility exponent $\gamma(\tau)$ curves for SC and BCC lattices and for spins from $S = 1/2$ to $S = \infty$ can be seen by inspection to have high temperature upturns, consistently indicative of a negative high exponent correction term of strong amplitude.

II. SCALING ANALYSES

Traditional analyses of simulation data in general focus either on finite size scaling (FSS) at the critical temperature, or on scaling as a function of temperature using the thermal scaling parameter $t = (T - T_c)/T_c$. This approach follows the choice made by K. Wilson who expressed Renormalization Group Theory (RGT) in terms of $t$. However $t$ diverges at high temperatures, so $t$ scaling can obviously only be used in the near-critical regime. In consequence, according to the conventional wisdom critical exponents and intrinsic correction terms can only be estimated from numerical measurements using high precision simulations close to $T_c$ for large sample sizes $L$. As high exponent correction terms only become important at temperatures well above criticality, the $t$ scaling approach can be ruled out for obtaining numerical or HTSE evidence concerning subleading conformal corrections with exponent $\theta_2 \sim 2.45$.

(\text{It is underlined in Ref. [5] that it can be possible to modify a model Hamiltonian so as to give an "improved" Hamiltonian, where the amplitude of the leading conformal correction term in $\tau^\theta$ becomes zero, for all thermodynamic variables. All the "minor" correction terms containing factors $\tau^\theta$ will then also be suppressed simultaneously, again for all observables.})

The Wegner expansion is for infinite samples, but holds also for finite-size samples in the regime where $L \gg \xi(\tau)$; in practice the rule $L > 7 \xi(\tau)$ is sufficient (see for instance Ref. [23]). When this condition holds, observable $Q(\tau, L)$ data for all $L$ correspond to the infinite-$L$ limit $Q(\tau, \infty)$ and so data for all $L$ coincide. Explicit comparisons between $L$ and $\xi(\tau)$ are generally not needed to establish where the ThL limit holds, as the ThL regime can be recognized by inspection of data plots for $\chi(\tau, L)$ against $\tau$, or equivalent plots for other observables $Q(\tau, L)$. Once the ThL plots for $Q(\tau, \infty)$ and $\xi(\tau, \infty)$ including the thermal scaling corrections have been established, $Q(\tau, L)$ data for all $L$ and all $\tau$ can be concatenated through the Privman-Fisher finite-size scaling rule [24], $Q(\tau, L)/Q(\tau, \infty) = F[L/\xi(\tau, \infty)]$.

It can be noted that at infinite temperature for spin $S = 1/2$ models the susceptibility $\chi(\tau = 1, L) \equiv 1$. It was pointed out in Ref. [15] that Wegner expressions for other observables $Q(\tau)$ only take up strictly the susceptibility form with non-diverging correction amplitudes if the observable is normalized such that at infinite temperature $Q_n(\tau = 1) = 1$. (Note that with three correction terms this rule imposes the closure condition $C(1 + a + b + c) = 1$). As well as the susceptibility we will study data on the near-neighbor second-moment correlation length $\xi(\tau)$ and on the Binder cumulant $g(\tau)$. $\xi(\tau)$ always tends to $\xi(\tau = 1) = 0$ at infinite temperature; from the general HTSE series [25] the leading $S = 1/2$ series term for the nearest-neighbor second-moment correlation is $\mu_2(\beta) = z \beta$ where $z$ is the number of near neighbors. So when $\xi(\tau)$ is defined appropriately for the lattice being considered, the reduced correlation length $\xi(\tau)/\beta^{1/2}$ will have an exact high temperature limit $\xi(\tau)/\beta^{1/2} = 1$, and an unaltered critical exponent, as carefully explained in Refs. [15, 16]. This reduced correlation length has a Wegner temperature dependence

$$\xi(\tau)/\beta^{1/2} = C\xi^\nu/\beta^{1/2} (1 + a_2\tau^\theta + b_2\tau + c_2\tau^2 + \cdots)$$

with the universal critical exponent $\nu$ and with correction amplitudes which, as will be seen below, turn out to be weak so the effective ThL reduced correlation-length exponent $\nu(\tau) = \partial \ln[\xi(\tau, L)/\beta^{1/2}]/\partial \ln \tau$ varies little over the entire paramagnetic temperature range.

Assuming hyperscaling, the critical exponent for the second field derivative of the susceptibility $\chi_4(\tau)$ (also called the non-linear susceptibility) is Ref. [7]

$$\gamma_4 = \gamma + 2\Delta_{\text{gap}} = D\nu + 2\gamma$$

$\chi_4$ in a cubic lattice is directly related to the Binder cumulant through

$$2g(\tau, L) = \frac{-\chi_4}{L^D}\frac{m^2}{\langle m^4 \rangle} = \frac{3\langle m^2 \rangle^2 - \langle m^4 \rangle}{\langle m^2 \rangle^2}$$

see Eq. (10.2) of Ref. [20]. Thus in the ThL regime the normalized Binder cumulant $L^Dg(\tau, L) \equiv -\chi_4(\tau, L)/(2\chi(\tau, L)^2)$ scales with a critical exponent $(D\nu + 2\gamma) - 2\gamma = D\nu$. In any $S = 1/2$ Ising system the infinite-temperature (i.e. independent spin) limit for the Binder cumulant is $g(0, N) \equiv 1/N$, where $N$ is the number of spins. As $N = L^D$ for a cubic lattice with $L$ defined
appropriately, at infinite temperature $L^3 g(\tau, L) \equiv 1$. Thus the 3D normalized Binder cumulant $L^3 g(\tau)$ also obeys the high-temperature limit rule for normalized observables introduced above, and the appropriate Wegner expression is

$$L^3 g(\tau, L) = C g \tau^{-3\nu} \left(1 + a_g \tau^\theta + b_g \tau + c_g \tau^{\theta_2} + \cdots\right) \quad (6)$$

### III. SIMULATIONS AND ANALYSES

We will present data measured over the entire range from criticality to infinite temperature for spin $S = 1/2$ Ising models on cubic, body centered cubic, simple cubic, and diamond lattices presented in decreasing order of the number of near neighbors. Most of the data were originally generated for the critical regime analyses of Refs. 26, 27, where the critical temperatures and critical exponents were estimated. The susceptibility up to high temperatures for these lattices (together with others) was presented in Ref. 17 where it was shown that the "crossover" behavior to a high-temperature scaling regime claimed in Refs. 28, 29 was an artefact due to the use by these authors of

$$Q(\tau) = C g \tau^{-\lambda_4} \left(1 + a_4 \tau^\theta + b_4 \tau + c_4 \tau^{\mu_4}\right) \quad (7)$$

which is the generalization of Eq. (1), with $\lambda_4$ standing for the known bootstrap critical exponents $\gamma$, $\nu$ and $3\nu$ respectively, and $\mu_4$ is the bootstrap subleading conformal correction exponent from Ref. 12. The amplitudes are estimated from the fits. All "minor" correction terms are assumed to have negligible amplitudes. In view of the number of fit parameters we do not attempt to estimate the errors in the individual correction amplitudes. Our final aim is to show that the data are consistent with the presence in all the data sets of correction terms having a unique value for $\mu$ (identified with $\theta_2 \sim 2.45$) and significant amplitudes. (Exceptionally there might be evidence for a further high-order correction term which could be ascribed to corrections with the further exponent $\theta_3$.)

To estimate critical amplitudes and the corrections to scaling, simulation data and HTSE data when available can be displayed over the entire paramagnetic temperature regime as $y(\tau) = Q(\tau, L) \tau^{\lambda_4}$ against $x(\tau) = \tau^\theta$. When the leading correction term is the confluent correction with exponent $\tau^\theta$ this plot is linear at small $x(\tau)$ and the second, analytic, term is nearly proportional to $x(\tau)^2$. This display is appropriate for all the susceptibility and normalized correlation length data. However, we will see that in this Ising universality class, for the normalized Binder cumulant the leading confluent correction term is missing so the appropriate plot becomes $y(\tau)$ against $\tau$. In addition for all observables the data can be displayed in the form of effective temperature-dependent exponents $\lambda_q(\tau) = -\partial \ln Q(\tau, L)/\partial \ln \tau$ (see Ref. 6).

In the latter form of display with the three correction term expression above one has

$$\lambda_q(\tau) = \lambda_q - \frac{a_q \theta \tau^\theta + b_q \tau + c_q \mu \tau^\mu}{1 + a_q \tau^\theta + b_q \tau + c_q \tau^\mu} \quad (8)$$

so the limit values are exact : at criticality $\lambda_q(0)$ is by definition equal to the critical exponent : $\gamma$, $\nu$ or $D\nu$ for the susceptibility, the reduced correlation length, and the normalized Binder cumulant respectively. Close to criticality the leading confluent correction amplitude can be estimated from the initial slope of the $\lambda_q(\tau)$ against $\tau$ plot, as

$$\lambda_q(\tau) = -\partial \ln Q(\tau)/\partial \ln \tau = \lambda_q - a_q \theta \tau^\theta \quad (9)$$

In practice this limiting slope is hard to estimate accurately. As a result the values of the universal ratios such as $a_\gamma/a_\xi$ are only known approximately. For the normalized Binder cumulant $a_q = 0$ (see below) in which case

$$\lambda_q(\tau) = -\partial \ln Q(\tau)/\partial \ln \tau = \lambda_q - b_q \theta \tau \quad (10)$$

With the three correction-term expression above, one has at infinite temperature the limiting value

$$\lambda_q(\tau = 1) = \lambda_q - \frac{a_q \theta + b_q + c_q \mu}{1 + a_q + b_q + c_q} \quad (11)$$

At infinite temperature, from the leading HTSE series terms 22 one also knows that the $S = 1/2$, $\lambda_q(\tau = 1)$ limiting values are equal to $z\beta_c$, $(z/2)\beta_c$ and $2z\beta_c$ for the susceptibility, the reduced correlation length, and the normalized Binder cumulant respectively where $z$ is the number of near neighbors. These exact limit values are indicated by red arrows in each of the effective exponent plots. These two relations provide an additional closure condition for each observable on the fit correction-term amplitudes together with the fit value for the exponent $\mu$. In particular for the normalized Binder cumulant data with $a_q = 0$ (see below) the parameters $C_g$ and $b_q$ can be read off the critical limit plots; then as $C_g(1 + b_g + c_g) = 1$, the infinite temperature limit condition

$$2z\beta_c = 3\nu - \frac{b_g + c_g \mu}{1 + b_g + c_g} \quad (12)$$

leaves $\mu$ fixed.
FIG. 1. (Color on line) FCC lattice. Susceptibility data in the form $\chi(\tau, L)\tau^\gamma$ against $\tau^\theta$. Data for $L = 128, 64, 32, 16, 8$ from left to right. Green curve is the fit to the ThL data, see Eq. (13).

IV. FACE CENTERED CUBIC LATTICE

In this lattice each site has 12 near neighbors and 4 sites per unit cell. The critical inverse temperature is $\beta_c = 0.102069(1)$. The ThL critical amplitudes for the susceptibility and the normalized correlation length $\chi(\tau, L)$ and $\xi(\tau)/\beta^{1/2}$ are both close to 1. The susceptibility data can be fitted satisfactorily with three weak correction terms only: the leading confluent correction $a\tau^\theta$, the leading analytic correction $b\tau$ and the further term $c\tau^\mu$

$$\chi(\tau) = 1.023\tau^{-\gamma} (1 - 0.080\tau^\theta + 0.0595\tau - 0.0022\tau^\mu)$$

with $\mu \sim 2.45$, see Figs. 1 and 2.

The reduced correlation length in the ThL regime can also be fitted with three terms only:

$$\frac{\xi(\tau)}{\beta^{1/2}} = 1.0071\tau^{-\nu} (1 - 0.0655\tau^\theta + 0.0635\tau - 0.0043\tau^\mu)$$

again with $\mu \sim 2.45$, see Figs. 3 and 4. The effective exponents $\gamma(\tau)$ and $\nu(\tau)$ each vary by only about 1% over the entire temperature range from criticality to infinity.

The value $\mu \sim 2.45$ for the tiny third correction term exponents is only rough.

The results for the normalized Binder parameter are more remarkable. The usual leading confluent correction turns out to have zero amplitude and the only visible correction term is the analytic term which is strong and linear in $\tau$. All further higher-order correction terms have negligible amplitudes also, so

$$L^3g(\tau, L) = 1.614\tau^{-3\nu} (1 - 0.380\tau)$$

As the normalized Binder cumulant is equal to $-\chi_4(\tau)/2\chi(\tau)^2$, the absence of the leading confluent correction term implies that the $\chi_4(\tau)$ and $\chi(\tau)$ confluent correction amplitudes have a ratio $a_{\chi_4}/a_{\chi} = 2$. Because confluent correction amplitude ratios $a(Q_i)/a(Q_j)$ are universal, the normalized Binder parameter leading confluent correction amplitude will be zero for all models in the 3D Ising universality class. This is indeed confirmed below from the data for the other models studied.
FIG. 4. (Color on line) FCC lattice. The temperature dependent effective correlation length exponent $\frac{\partial \ln [\xi(\tau, L)]}{\partial \ln \tau} / \beta^{1/2}$ against $\tau^\theta$. Data for $L = 48, 32, 24, 16, 12, 8$ from left to right. Green curve is the fit to the ThL data, calculated from Eq. (14).

V. BODY CENTERED CUBIC LATTICE

In this lattice each site has 8 near neighbors and 2 sites per unit cell. The critical inverse temperature $\beta_c = 0.1573725(5)$ [7, 27, 30]. Extensive lists of exact HTSE terms for this lattice are given in Ref. [25]; we have used these tables to calculate HTSE values for the observables in the high-temperature range where the HTSE sums are essentially exact. Accurate effective exponents to lower temperatures can be obtained by appropriate extrapolation (see Ref. [7]). The ThL susceptibility $\chi(\tau, L)$ from simulations and HTSE data can be fitted satisfac-

torily with three correction terms:

$$\chi(\tau) = 1.0377\tau^{-\gamma} \left( 1 - 0.0771\tau^\theta + 0.054\tau - 0.0137\tau^\mu \right)$$

with $\mu \sim 2.45$, see Figs. 7 and 8. The latter is essentially identical to the curve shown in Ref. [7], Fig. 14.

The fit values for the critical amplitude and the confluent correction amplitude can be compared to those estimated in Ref. [7], $C_\chi = 1.0404(1), a_\chi = -0.129(3)$. Because of the opposite signs of the various correction term amplitudes the temperature dependent effective exponent $\gamma(\tau)$ changes slope twice. The upturns in $\gamma(\tau)$ at high temperatures in both plots correspond to the
Data from Ref. [25]. Green curve is the fit to the ThL data, see Eq. (17).

The HTSE curve is a 23 term sum of data from Ref. [25]. Green curve is the fit to the ThL data, calculated from \( \chi(\tau) \) in Eq. (16).

The present reduced correlation length critical amplitude can be compared to estimates 0.468(3) and \(-0.100(4)\) in Ref. [7]. The ThL normalized Binder parameter behaves slightly differently from the FCC model. The dominant correction term is again the strong analytic term linear in \( \tau \); however there is a further weak term having an exponent \( \mu \sim 2.45 \). All other terms are missing including the normal leading correction term in \( \tau^0 \) and ”minor” correction terms, so:

\[
L^3 g(\tau, L) = 1.597 \tau^{-3\nu} (1 - 0.3657 \tau - 0.0068 \tau^\mu) \tag{18}
\]

with \( \mu \sim 2.65 \), see Figs. 11 and 12.

**VI. SIMPLE CUBIC LATTICE**

In this lattice each site has 6 near neighbors and 1 site per unit cell. The critical inverse temperature is \( \beta_c = 0.221654(2) \) [7] [26, 31]. Extensive lists of exact HTSE terms for this lattice are given in Ref. [23]. The ThL susceptibility \( \chi(\tau) \) data and the high-temperature HTSE data can be fitted satisfactorily with three correction terms:

\[
\chi(\tau) = 1.120\tau^{-\gamma} (1 - 0.112\tau^\theta + 0.021\tau - 0.019\tau^\mu) \tag{19}
\]

with \( \mu \sim 2.45 \), see Figs. 13 and 14. The fit value for the critical amplitude can be compared to that estimated in Ref. [7], \( C_\chi = 1.14(1) \). The temperature-dependent ThL effective exponent \( \gamma(\tau) \) in Fig. 14 is very similar to the \( \gamma(\tau) \) curve for the same model shown in Ref. [7] Fig. 16.

The reduced correlation length in the ThL regime can be fitted by

\[
\xi(\tau) = 1.073\tau^{-\nu} (1 - 0.107\tau^\theta + 0.048\tau - 0.010\tau^\mu) \tag{20}
\]
FIG. 11. (Color on line) BCC lattice. Normalized Binder cumulant data in the form $L^3 g(\tau, L)^{\beta_3}$ against $\tau$. Data for $L = 48, 32, 16, 12, 8$ and HTSE from left to right. The HTSE curve is a 23 term sum of data from Ref. [25]. Green curve is the fit to the ThL data, see Eq. (15).

FIG. 12. (Color on line) BCC lattice. The temperature dependent effective normalized Binder cumulant exponent in the form $\partial \ln [L^3 g(\tau, L)] / \partial \ln \tau$ against $\tau$. Data for $L = 48, 32, 24, 16, 12$ and HTSE from left to right. The HTSE curve is a 23 term sum of data from Ref. [25]. Green curve is the fit to the ThL data, calculated from Eq. (18).

FIG. 13. (Color on line) SC lattice. Susceptibility data in the form $\chi(\tau, L)^{\gamma}$ against $\tau$. Data for $L = 64, 48, 16, 8$ and HTSE from left to right. The HTSE curve is a 23 term sum of data from Ref. [25]. Green curve is the fit to the ThL data, see Eq. (19).

FIG. 14. (Color on line) SC lattice. The temperature dependent effective susceptibility exponent $\partial \ln \chi(\tau, L) / \partial \ln \tau$ against $\tau$ against $\tau^\theta$. Data for $L = 48, 32, 16, 8$ and HTSE from left to right. The HTSE curve is a 23 term sum of data from Ref. [25]. Green curve is the fit to the ThL data, calculated from $\chi(\tau)$ in Eq. (19).

with $\mu \sim 2.45$, see Figs. [15] and [16]. The fit value for the critical amplitude can be compared to that estimated in Ref. [7], equivalent to $C_\xi = 1.077(12)$.

The ratios $a_\chi/a_\xi$ should be identical for these three models. The values estimated above are 1.22 for the FCC model, 1.32 for the BCC and 1.05 for the SC. The BCC model values for different spins $S$ estimated in Ref. [7] were all close to 1.28. The present variations reflect the difficulties in extrapolating precisely so as to estimate the initial critical slopes.

For the normalized Binder parameter, as for the other lattices the standard leading correction term in $\tau^\theta$ is missing. There is a strong analytic correction term linear in $\tau$, accompanied by another strong term proportional to $\tau^\mu$ with $\mu$ close to 2.45. All other terms are negligible so that

$$L^3 g(\tau, L) = 1.565 \tau^{-3\nu} (1 - 0.282 \tau - 0.081 \tau^\mu)$$  \hspace{1cm} (21)

with $\mu \sim 2.65$, see Figs. [17] and [18].

We identify this $\mu$ with the subleading confluent correction exponent, so we estimate $\theta_2 = 2.45(5)$. All the fits to
FIG. 15. (Color on line) SC lattice. Normalized correlation length data in the form \(\xi(\tau, L)/\beta^{1/2}\) against \(\tau^\theta\). Data for \(L = 48, 32, 16, 8, \) HTSE from left to right. The HTSE curve is a 23 term sum of data from Ref. [25]. Green curve is the fit to the ThL data, see Eq. (20).

The data sets for this and the other models are compatible with a universal correction term being present having approximately this exponent. Obviously when the high exponent correction term amplitude is very weak, as the case for instance for the SC \(\chi(\tau)\) and \(\xi(\tau)/\beta^{1/2}\) data sets, the estimate for the corresponding \(\mu\) is much more approximate. Nevertheless acceptable fits to these data sets also can only be made when a high-exponent correction term is included.

FIG. 16. (Color on line) SC lattice. The temperature dependent effective correlation length exponent \(\partial \ln \xi(\tau, L)/\partial \ln \tau\) against \(\tau^\theta\). Data for \(L = 48, 32, 16, 12, 8, 4\) from left to right. Green curve is the fit to the ThL data, calculated from Eq. (14).

FIG. 17. (Color on line) SC lattice. Normalized Binder cumulant data in the form \(L^3g(\tau, L)^{3\nu}\) against \(\tau\). Data for \(L = 48, 32, 16, 12, 8, 6, \) HTSE and 4 from left to right. The HTSE curve is a 23 term sum of data from Ref. [25]. Green curve is the fit to the ThL data, see Eq. (21).

FIG. 18. (Color on line) SC lattice. The temperature dependent effective normalized Binder cumulant exponent in the form \(\partial \ln \xi(\tau, L)/\beta^{1/2}/\partial \ln \tau\) against \(\tau^\theta\). Data for \(L = 16, 12, 8, 6, \) HTSE and 4 from left to right. The HTSE curve is a 23 term sum of data from Ref. [25]. Green curve is the fit to the ThL data, calculated from Eq. (21).

VII. DIAMOND LATTICE

In this lattice each site has 4 near neighbors and 8 sites per unit cell. The critical inverse temperature \(\beta_c = 0.3697398(1)\) [3 27]. For technical reasons it is more difficult to equilibrate and obtain accurate numerical data for this model. The ThL susceptibility \(\chi(\tau)\) can be fitted satisfactorily with three correction terms:

\[
\chi(\tau) = 1.250\tau^{-\gamma} (1 - 0.147\tau^\theta - 0.011\tau - 0.04\tau^\mu) \quad (22)
\]
with $\mu \sim 2.45$, see Figs. 19 and 20. We do not dispose of sufficient data to analyse the normalized correlation length in this model.

The normalized Binder parameter behaves in much the same way as in the SC model. The leading correction term is strong and linear in $\tau$, with a strong second term proportional to $\tau^\mu$ with $\mu \sim 2.7$ so

$$L^3 g(\tau, L) = 1.535\tau^{-3\nu} (1 - 0.157\tau - 0.186\tau^\mu), \quad (23)$$

see Figs. 21 and 22. The $\mu$ correction term amplitude is even stronger than for the SC model. Unfortunately no HTSE data are available for this model. The estimate for $\mu$ is marginally higher than the SC and BCC model Binder cumulant analyses but this may be due to technical difficulties with this model.

VIII. CONCLUSION

We measure the susceptibility, reduced second-moment correlation length, and normalized Binder-cumulant data for the 3D spin-1/2 FCC, BCC, SC and diamond Ising models, covering the entire paramagnetic temperature range. We treat the bootstrap values for the principle
critical exponents as exact and carry out three term (or two term for the normalized Binder cumulant) fits adjusting the critical amplitudes and the correction-term amplitudes, including a high-order term with exponent approximately equal to the bootstrap $\theta_2$ value. Our principal conclusion is that for the models and observables studied, there systematically exist correction terms of exponent consistent with the bootstrap subleading conformal correction term value $\theta_2 = 2.454(3)$ [12], and with significant amplitudes. For all three observables these high order correction term $c r^{2.45}$ amplitudes are always negative and pass progressively from almost negligible for the FCC lattice to strong for the SC and Diamond lattices. This evolution is particularly notable for the normalized Binder cumulant.

All the critical amplitudes and correction-term amplitudes evolve regularly from one model to the next as functions of the numbers of nearest neighbors, with the susceptibility and correlation-length critical amplitudes becoming systematically stronger as the number of neighbors drops. Amplitude ratios for the leading confluent correction term value $a_\tau/\tau$ is absent to within the statistical uncertainty. As the normalized Binder parameter is equal to $-\chi_4(\tau)/(2\chi(\tau)^2)$ the correction Binder-cumulant amplitude ratio being equal to zero is equivalent to $a_\chi/a_\chi = 2$ for the 3D Ising universality class. The normalized Binder cumulant analytic corrections $b\tau$ are always strong but decrease progressively as the number of neighbors drops. All "minor" correction term amplitudes appear to be negligible in all cases.

ACKNOWLEDGMENTS

We would like to thank D. Simmons-Duffin, P. Butera, S. Rychkov and Y. Nakayama for helpful comments. The computations were performed on resources provided by the Swedish National Infrastructure for Computing (SNIC) at Chalmers Centre for Computational Science and Engineering (C3SE).

[1] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, Phys. Rev. D 86, 025022 (2012).
[2] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, J. Stat. Phys. 157, 869 (2014).
[3] F. Gliozzi, P. Liendo, M. Meineri, A. Rago, JHEP 05, 036 (2015).
[4] Y. Nakayama, Phys. Rev. Lett. 116, 141602 (2016).
[5] F. Kos, D. Poland and D. Simmons-Duffin, JHEP 11, 109 (2014).
[6] D. Simmons-Duffin, JHEP 06, 174 (2015).
[7] P. Butera and M. Comi, Phys. Rev. B, 65, 144431 (2002).
[8] M. Campostrini, M. Hasenbusch, A. Pelissetto, and E. Vicari, Phys. Rev. B 74, 144506 (2002).
[9] Y. Deng and H. W. J. Blöte, Phys. Rev. E 68, 036125 (2003).
[10] M. Hasenbusch Phys. Rev. B 82, 174433 (2010).
[11] K. E. Newman and E. K. Riedel, Phys. Rev. B 30, 6615 (1984).
[12] D. Simmons-Duffin, JHEP 03, 86 (2017).
[13] C. Domb and M.F. Sykes, Phys. Rev. 128, 168 (1962).
[14] F. Wegner, Phys. Rev. B 5, 4529 (1972).
[15] I. A. Campbell, K. Hukushima, and H. Takayama, Phys. Rev. Lett. 97, 117202 (2006).
[16] I. A. Campbell and P. Butera, Phys. Rev. B 78, 024435 (2008).
[17] P. H. Lundow and I. A. Campbell, Phys. Rev B 83, 184408 (2011).
[18] P. H. Lundow and I. A. Campbell, Phys. Rev B 83, 014411 (2011).
[19] B. Berche, C. Chatelain, C. Dhall, R. Kenna, R. Low, and J.C. Walter, J. Stat. Mech. (2008) P11010.
[20] V. Privman, P. C. Hohenberg and A. Aharony, Universal Critical-Point Amplitude Relations, in Phase Transitions and Critical Phenomena (Academic, NY, 1991), eds. C. Domb and J. L. Lebowitz, 14, 1.
[21] M. Ferer, Phys. Rev. B 16, 419 (1977).
[22] M.C. Chang and A. Houghton, Phys. Rev. B 21, 1881 (1980).
[23] J.-K. Kim, A. J. F. de Souza and D. P. Landau, Phys. Rev. E 54, 2291 (1996).
[24] V. Privman and M. E. Fisher, Phys. Rev. B 30, 322 (1984).
[25] P. Butera and M. Comi, arXiv:0204007 (2002) (unpublished).
[26] R. Häggkvist, A. Rosengren, P. H. Lundow, K. Markström, D. Andrén, and P. Kundrotas, Adv. Phys. 56, 653 (2007).
[27] P. H. Lundow, K. Markström, and A. Rosengren, Phil. Mag. 89, 22 (2009).
[28] E. Luijten, H. W. J. Blöte, and K. Binder, Phys. Rev. Lett. 79, 561 (1997).
[29] E. Luijten, Phys. Rev. E 59, 4997 (1999).
[30] Y. Murase and N. Ito, J. Phys. Soc. Jpn. 77, 014002 (2008).
[31] H. W. J. Blöte, E. Luijten and J. R. Heringa, J. Phys. A: Math. Gen. 28, 6289 (1995).