Volterra operators and Hankel forms on Bergman spaces of Dirichlet series

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Abstract
For a Dirichlet series $g$, we study the Volterra operator $T_g f(s) = - \int_s^{+\infty} f(w)g'(w) \, dw$, acting on a class of weighted Hilbert spaces $\mathcal{H}^2_w$ of Dirichlet series. We obtain sufficient / necessary conditions for $T_g$ to be bounded (resp. compact), involving BMO and Bloch type spaces on some half-plane. We also investigate the membership of $T_g$ in Schatten classes. Moreover, we show that if $T_g$ is bounded, then $g$ is in $\mathcal{H}^p_w$, the $L^p$-version of $\mathcal{H}^2_w$, for every $0 < p < \infty$. We also relate the boundedness of $T_g$ to the boundedness of a multiplicative Hankel form of symbol $g$, and the membership of $g$ in the dual of $\mathcal{H}^1_w$.

Keywords Volterra operator · Dirichlet series · Hankel forms

Mathematics Subject Classification Primary 31B10 · 32A36; Secondary 30B50 · 30H20

1 Introduction

Dirichlet series are functions of the form

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}, \text{ with } s \in \mathbb{C}. \quad (1.1)$$
For a real number $\theta$, $\mathbb{C}_\theta$ stands for the half-plane $\{s, \Re s > \theta\}$, and $\mathbb{D}$ for the unit disk. $\mathcal{D}$ denotes the class of functions $f$ of the form (1.1) in some half-plane $\mathbb{C}_\theta$, and $\mathcal{P}$ is the space of Dirichlet polynomials.

The increasing sequence of prime numbers will be denoted by $(p_j)_{j \geq 1}$, and the set of all primes by $\mathbb{P}$. Given a positive integer $n$, $n = p^\kappa$ will stand for the prime number factorization $n = p_1^{\kappa_1}p_2^{\kappa_2} \cdots p_d^{\kappa_d}$, which associates uniquely to $n$ the finite multi-index $\kappa(n) = (\kappa_1, \kappa_2, \ldots, \kappa_d)$. The number of prime factors in $n$ is denoted by $\Omega(n)$ (counting multiplicities), and by $\omega(n)$ (without multiplicities).

The space of eventually zero complex sequences $c_00$ consists in all sequences which have only finitely many non zero elements. We set $\mathbb{D}_\text{fin}^\infty = \mathbb{D}^\infty \cap c_00$ and $\mathbb{N}_0^\infty \cap c_00$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of non-negative integers.

Let $F : \mathbb{D}_\text{fin}^\infty \to \mathbb{C}$ be analytic, i.e. analytic at every point $z \in \mathbb{D}_\text{fin}^\infty$ separately with respect to each variable. Then $F$ can be written as a convergent Taylor series

$$F(z) = \sum_{\alpha \in \mathbb{N}_0^\infty \text{fin}} c_\alpha z^\alpha, \quad z \in \mathbb{D}_\text{fin}^\infty.$$ 

The truncation $A_m F$ of $F$ onto the first $m$ variables is defined by

$$A_m F(z) = F(z_1, \ldots, z_m, 0, 0, \ldots).$$

For $z, \chi$ in $\mathbb{D}^\infty$, we set $z.\chi := (z_1 \chi_1, z_2 \chi_2, \ldots)$, and $p^x := (p_1^x, p_2^x, \ldots)$ for a real number $x$.

The Bohr lift [11] of the Dirichlet series $f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$ is the power series

$$Bf(\chi) = \sum_{n=1}^{+\infty} a_n \chi^{\kappa(n)} = \sum_{\alpha \in \mathbb{N}_0^\infty \text{fin}} \tilde{a}_\alpha \chi^\alpha, \quad \text{where } \tilde{a}_\alpha = a_{p^\alpha}, \chi \in \mathbb{D}_\text{fin}^\infty,$n

with the multiindex notation $\chi^\alpha = \chi_1^{\alpha_1} \chi_2^{\alpha_2} \cdots$.

Given a sequence of positive numbers $w = (w_n)_n = (w(n))_n$, one considers the Hilbert space (see [21,23])

$$\mathcal{H}_w^2 := \left\{ \sum_{n=1}^{+\infty} a_n n^{-s} : \sum_{n=1}^{+\infty} \frac{|a_n|^2}{w_n} < +\infty \right\}.$$ 

The choice $w_n = 1$ corresponds to the space $\mathcal{H}_w^2$, introduced in [19].

The weights considered in this article satisfy $w_n = O(n^\epsilon)$ for every $\epsilon > 0$; from the Cauchy-Schwarz inequality, Dirichlet series in $\mathcal{H}_w^2$ absolutely converge in $\mathbb{C}_{1/2}$.

We are interested in the Volterra operator $T_g$ of symbol $g(s) = \sum_{n=1}^{+\infty} b_n n^{-s}$, defined by

$$T_g f(s) := -\int_{s}^{+\infty} f(w) g'(w) dw, \quad \Re s > \frac{1}{2}. \quad (1.2)$$
On the unit disk $\mathbb{D}$, the Volterra operator, whose symbol is an analytic function $g$, is given by

$$J_g f(z) := \int_{0}^{z} f(u) g'(u) du, \quad z \in \mathbb{D}. \quad (1.3)$$

Pommerenke [26] showed that $J_g$ (1.3) is bounded on the Hardy space $H^2(\mathbb{D})$ if and only if $g$ is in $BMOA(\mathbb{D})$. Let $\sigma$ be the Haar measure on the unit circle $\mathbb{T}$. Fefferman's duality Theorem states that $BMOA(\mathbb{D})$ is the dual space of $H^1(\mathbb{D})$. Thus the boundedness of $J_g$ is equivalent to the boundedness of the Hankel form

$$H_g(f,h) := \int_{\mathbb{T}} f(u) h(u) \overline{g(u)} d\sigma(u), \quad f, h \in H^2(\mathbb{D}). \quad (1.4)$$

Let $V$ be the Lebesgue measure on $\mathbb{C}$, normalized such that $V(\mathbb{D}) = 1$. Many authors, in particular [2], have studied Volterra operators on Bergman spaces of $\mathbb{D}$. The classical Bergman space $A^2_\gamma(\mathbb{D})$, $\gamma > 0$, is associated to the measure $d\tilde{m}_\gamma(z) := \gamma (1 - |z|^2)^{\gamma-1} dV(z)$. $J_g$ is bounded on $A^2_\gamma(\mathbb{D})$ if and only if $g$ is in the Bloch space, which is the dual of $A^1_\gamma(\mathbb{D})$.

The Bergman space of the finite polydisk $A^2_\gamma(\mathbb{D}^d)$, $d \geq 1$, corresponds to the measure

$$d\tilde{\nu}_\gamma(z) := d\tilde{m}_\gamma(z_1) \times \cdots \times d\tilde{m}_\gamma(z_d).$$

The boundedness of the Hankel form

$$H_g(f,h) := \int_{\mathbb{D}^d} f(z) h(z) \overline{g(z)} d\tilde{\nu}_\gamma(z), \quad f, h \in A^2_\gamma(\mathbb{D}^d), \quad (1.5)$$

is equivalent to the membership of $g$ to the Bloch space (see [17]), defined by

$$\text{Bloch}(\mathbb{D}^d) := \left\{ f : \mathbb{D}^d \rightarrow \mathbb{C} \text{ holomorphic} : \max_{\kappa \in \mathcal{I}_d} \sup_{z \in \mathbb{D}^d} \left| \partial^\kappa f(\kappa.z) \right| (1 - |z|)^K < +\infty \right\},$$

where $\mathcal{I}_d$ denotes the set of multi-indices $\kappa = (\kappa_1, \ldots, \kappa_d)$, with entries in $\{0, 1\}$, and

$$z = (z_1, \ldots, z_d), \quad \partial^\kappa = \partial_{z_1}^{\kappa_1} \cdots \partial_{z_d}^{\kappa_d}, \quad (1 - |z|)^K = (1 - |z_1|)^{\kappa_1} \cdots (1 - |z_d|)^{\kappa_d}.$$
The norm in the space $\mathcal{H}^\infty := H^\infty(\mathbb{C}_0) \cap \mathcal{D}$ is
\[
\|f\|_{\mathcal{H}^\infty} = \sup_{s \in \mathbb{C}_0} |f(s)|.
\]

Let $H^\infty(\mathbb{D}^\infty)$ be the space of series $F$ which are finitely bounded, i.e.
\[
\|F\|_{H^\infty(\mathbb{D}^\infty)} = \sup_{m \in \mathbb{N}_0, z \in \mathbb{D}^\infty} |A_m F(z)| < \infty.
\]

Via the Bohr isomorphism, we have \[16,19\]
\[
\|f\|_{\mathcal{H}^\infty} = \|B f\|_{H^\infty(\mathbb{D}^\infty)}.
\]

Several abscissae are related to a function $g$ in $\mathcal{D}$, of the form $g(s) = \sum_{n=1}^{+\infty} b_n n^{-s}$:

- the abscissa of convergence $\sigma_c = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{+\infty} b_n n^{-\sigma} \text{ converges} \right\}$;
- the abscissa of absolute convergence $\sigma_a = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{+\infty} |b_n| n^{-\sigma} \text{ converges} \right\}$;
- the abscissa of uniform convergence $\sigma_u = \inf \left\{ \theta \in \mathbb{R} : \sum_{n=1}^{+\infty} b_n n^{-s} \text{ converges uniformly in } \mathbb{C}_\theta \right\}$.

The abscissa of regularity and boundedness, denoted by $\sigma_b$, is the infimum of those $\theta$ such that $g(s)$ has a bounded analytic continuation, to the half-plane $\Re(s) > \theta + \epsilon$, for every $\epsilon > 0$.

We have $-\infty \leq \sigma_c \leq \sigma_u \leq \sigma_a \leq +\infty$, and, if any of the abscissae is finite $\sigma_a - \sigma_c \leq 1$. Moreover, it is known that $\sigma_b = \sigma_u$ \[11\], and $\sigma_a - \sigma_u \leq \frac{1}{2}$.

Volterra operators (1.2) on the spaces $\mathcal{H}_w^2$ have been investigated in \[13\]. Our aim is to study similar questions for the spaces $\mathcal{H}_w^2$, associated to specific weights $w$ in the class $W$ defined below.

**Definition 1** Let $\beta > 0$. A sequence $w$ belongs to $\mathcal{W}$ if it has one of the following forms:

1. $w_n = [d(n)]^\beta$, where $d(n)$ is the number of divisors of the integer $n$. Then $\mathcal{H}_w^2 := B_2^\beta$.
2. $w_n = d_{\beta+1}(n)$, where $d_{\gamma}(n)$ are the Dirichlet coefficients of the power of the Riemann zeta function, namely $\zeta^{\gamma}(s) = \sum_{n=1}^{+\infty} d_{\gamma}(n)n^{-s}$. Then $\mathcal{H}_w^2 := A_2^\beta$.

As in the case of $\mathcal{H}^2$ \[13\], we obtain sufficient/necessary conditions for $T_g$ to be bounded on the Hilbert spaces $\mathcal{H}_w^2$. However, due to the lack of information of the behavior of the symbols in the strip $0 < \Re s < 1/2$, it seems difficult to get an “if and only if” condition. In the Hardy space setting, it is shown that $T_g$ is bounded on $\mathcal{H}^2$ provided that $g$ in $BMOA(\mathbb{C}_0)$. Since the spaces $A_2^\beta$ and $B_2^\beta$ (see Sect. 2) locally behave like Bergman spaces of the half plane $\mathbb{C}_0$, we would expect that the membership of $g$ in Bloch($\mathbb{C}_0$) (resp. Bloch0($\mathbb{C}_0$)) would imply the boundedness (resp.
compactness) of $T_g$ on $\mathcal{H}_w^2$. We obtain such a sufficient condition when $Bg$ depends on a finite number of variables $z_1, \ldots, z_d$. However, our method specifically uses that $d$ is finite, and we do not know whether the same result holds if $Bg$ is a function of infinitely many variables.

Let $\mathcal{N}_d$ be the set of positive integers which are multiples of the primes $p_1, \ldots, p_d$,

$$\mathcal{D}_d := \left\{ f \in \mathcal{D} : f(s) = \sum_{n \in \mathcal{N}_d} a_n n^{-s} \right\}, \text{ and } \mathcal{H}_{d,w}^p := \mathcal{H}_w^p \cap \mathcal{D}_d.$$

One of our main results is the following.

**Theorem 1** Let $T_g$ be the operator defined by (1.2) for some Dirichlet series $g$ in $\mathcal{D}$.

(a) If $g(s) = \sum_{n=2}^{+\infty} b_n n^{-s}$ is in $\mathcal{D}_d \cap \text{Bloch}(\mathbb{C}_0)$, then $T_g$ is bounded on $\mathcal{H}_w^2$ and

$$\| T_g \|_{\mathcal{L}(\mathcal{H}_w^2)} \lesssim \| g \|_{\text{Bloch}(\mathbb{C}_0)}.$$

(b) If $g$ is in $\text{BMOA}(\mathbb{C}_0)$, then $T_g$ is bounded on $\mathcal{H}_w^2$ and

$$\| T_g \|_{\mathcal{L}(\mathcal{H}_w^2)} \lesssim \| g \|_{\text{BMOA}(\mathbb{C}_0)}.$$

(c) If $T_g$ is bounded on $\mathcal{H}_w^2$, then $g$ is in $\text{Bloch}(\mathbb{C}_{1/2})$ and

$$\| g \|_{\text{Bloch}(\mathbb{C}_{1/2})} \lesssim \| T_g \|_{\mathcal{L}(\mathcal{H}_w^2)}.$$

Via the Bohr lift, $\mathcal{H}_w^2$ are $L^2$-spaces of functions on the polydisk $\mathbb{D}^\infty$. Precisely, there exists a probability measure $\mu_w$ on $\mathbb{D}^\infty$ such that

$$\| f \|_{\mathcal{H}_w^2}^2 = \int_{\mathbb{D}^\infty} |B f(z)|^2 \, d\mu_w(z).$$

Analogously to the spaces $\mathcal{H}_w^p$, we define the space $\mathcal{H}_w^p$, $0 < p < \infty$ (see Sect. 2), as the closure of Dirichlet polynomials under the norm (quasi-norm if $0 < p < 1$)

$$\| f \|_{\mathcal{H}_w^p} = \| B f \|_{L^p(\mathbb{D}^\infty, \mu_w)}.$$

Let $\mathcal{X}_w = \mathcal{X}(\mathcal{H}_w^2)$ be the space of symbols $g$ giving rise to bounded operators $T_g$ on $\mathcal{H}_w^2$. Our study provides the following strict inclusions:

$$\text{BMOA}(\mathbb{C}_0) \cap \mathcal{D} \subsetneq \mathcal{X}_w \subsetneq \cap_{0 < p < \infty} \mathcal{H}_w^p.$$
on the weak product $\mathcal{H}_w^2 \odot \mathcal{H}_w^2$. As in the case of $\mathcal{H}^2$ [13], we only get partial results. For Dirichlet series involving $d$ primes, we have

$$D_d \cap \text{Bloch}(\mathbb{C}_0) \subset D_d \cap X_w \subsetneq B^{-1}\text{ Bloch}(\mathbb{D}^d).$$

The paper is organized as follows. Section 2 starts by presenting some properties of the spaces $\mathcal{H}_w^2$. As a space of analytic functions on the half-plane $\mathbb{C}_{1/2}$, $\mathcal{H}_w^2$ is continuously embedded in a space of Bergman type of $\mathbb{C}_{1/2}$. In view of the Bohr lift, the norm of $\mathcal{H}_w^2$ can be expressed in terms of a probability measure $\mu_w$ on the polydisk. For $0 < p < \infty$, we consider the Bohr–Bergman space $\mathcal{H}_w^p$, and derive equivalent norms for these spaces.

In Sect. 3, we present some properties of the Dirichlet series which belong to a BMO or Bloch space of some half-plane $\mathbb{C}_\theta$. In particular, we relate the Carleson measures for both spaces of Dirichlet series and Bergman type spaces.

Section 4 is devoted to the proof of Theorem 1. First we consider the case when $g$ is a function of $p_1^{-s}, \ldots, p_d^{-s}$. To prove (b), we observe that the boundedness of $T_g$ on $\mathcal{H}_w^2$ implies the boundedness of $T_g$ on $\mathcal{H}_w^2$. On another hand, combining the fact that $\mathcal{H}_w^2$ is embedded in a Bergman type space of the half-plane $\mathbb{C}_{1/2}$ with some characterizations of Carleson measures, we establish that

$$X_w \subset \text{Bloch}(\mathbb{C}_{1/2}).$$

Compactness and Schatten classes are considered in Sects. 5 and 6.

In Sect. 7, we consider some specific symbols: fractional primitives of translates of a “weighted zeta”-function and homogeneous symbols. These examples will be used in Sect. 8.

In Sect. 8, we investigate the relationship between the boundedness of the Volterra operator $T_g$, the boundedness of the Hankel form

$$H_g(fh) = (fh, g)_{\mathcal{H}_w^2},$$

and the membership of $g$ in the dual of $\mathcal{H}_w^1$. In particular, we study examples of Hankel forms on Bergman spaces of Dirichlet series, which are the counterparts of the Hilbert multiplicative matrix [12].

Additionally, we show the strictness of the inclusions derived previously

$$BMOA(\mathbb{C}_0) \cap D \subsetneq X_w \subsetneq \cap_{0 < p < \infty} \mathcal{H}_w^p,$$

and compare the space $D_d \cap X_w$ with Bloch spaces.

For two functions $f, g$, the notation $f = O(g)$ or $f \preceq g$, means that there exists a constant $C$ such that $f \leq Cg$. If $f = O(g)$ and $g = O(f)$, we write $f \asymp g$. 

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2 The Bohr–Bergman spaces $\mathcal{B}^2_{\beta}$, $\mathcal{A}^2_{\beta}$

2.1 The spaces $\mathcal{B}^2_{\beta}$, $\mathcal{A}^2_{\beta}$

These spaces are related to number theory. The number of divisors of the integer $n$, $d(n)$, is $d(n) = (\kappa_1 + 1) \cdots (\kappa_d + 1)$ when $n = p^k$. We consider the following scale of Hilbert spaces

$$\mathcal{B}^2_{\beta} = \left\{ f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} : \|f\|_{\mathcal{B}^2_{\beta}} = \left( \sum_{n=1}^{+\infty} \frac{|a_n|^2}{(d(n))^\beta} \right)^{\frac{1}{2}} < \infty \right\}, \text{ for } \beta > 0.$$  

The case $\beta = 0$ corresponds to the Hardy space $\mathcal{H}^2$. The reproducing kernels of $\mathcal{B}^2_{\beta}$ are

$$K_{\mathcal{B}^2_{\beta}}(s, u) = \zeta_{\beta}(s + u), \text{ where } \zeta_{\beta}(s) = \sum_{n=1}^{+\infty} [d(n)]^\beta n^{-s}.$$  

It is shown in [30] that there exists $\phi_{\beta}(s)$, an Euler product which converges absolutely in $\mathbb{C}_{1/2}$, such that

$$\zeta_{\beta}(s) = [\zeta(s)]^{2\beta} \phi_{\beta}(s), \text{ and } \phi_{\beta}(1) \neq 0.$$  

Another family of spaces arises from the so-called generalized divisor function. For $\gamma > 0$, the numbers $d_{\gamma}(n)$ are defined by the relation

$$\zeta_{\gamma}(s) = \sum_{n=1}^{+\infty} d_{\gamma}(n) n^{-s}.$$  

A computation involving Euler products shows that we have

$$d_{\gamma}(p^r) = \frac{\gamma(\gamma + 1) \cdots (\gamma + r - 1)}{r!}, \text{ for } p \in \mathbb{P}, \text{ and any integer } r.$$  

From its definition, $d_{\gamma}$ is a multiplicative function, i.e. $d_{\gamma}(k l) = d_{\gamma}(k) d_{\gamma}(l)$ if $k$ and $l$ are relatively prime. Thus, $d_{\gamma}(n)$ can be computed explicitly from the decomposition $n = p^k$.

We define the spaces

$$\mathcal{A}^2_{\beta} = \left\{ f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} : \|f\|_{\mathcal{A}^2_{\beta}} = \left( \sum_{n=1}^{+\infty} \frac{|a_n|^2}{d_{\beta+1}(n)} \right)^{\frac{1}{2}} < \infty \right\}, \text{ for } \beta > 0.$$
with reproducing kernels \(K^{\mathcal{H}^2_w}(s, u) = \zeta^{\beta+1}(s + \overline{u})\).

Notice that, in each case, the reproducing kernel has the form

\[
K^{\mathcal{H}^2_w}(s, u) = Z_w(s + \overline{u}),
\]

where \(Z_w(s) := \sum_{n=1}^{+\infty} w_n n^{-s}\) has a singularity at \(s = 1\), with an estimate of the type

\[
Z_w(s) = C_w(s - 1)^{-(\beta+1)} \left[1 + O(1)\right]. \tag{2.1}
\]

2.2 Bohr–Bergman spaces on \(\mathbb{D}^\infty\)

The Bohr correspondence is an isometry between \(\mathcal{H}^2_w\) and the weighted Bergman space of the infinite polydisk

\[
H^2_w(\mathbb{D}^\infty) = \left\{ \sum_{v \in \mathbb{N}_0^\infty} a_v z^v : \sum_v |a_v|^2 w_v < \infty \right\}, \quad \text{where } w_v = \prod_j w_{v_j}.
\]

In particular, the space \(\mathcal{H}^2\) is identified with the Hardy space \(H^2(\mathbb{T}^\infty)\) [19]. Let us consider the following probability measures on the unit disk \(\mathbb{D}\),

\[
dm_w(z) := M(|z|^2) dV(z),
\]

where \(M(r) = \left\{ \begin{array}{ll} \frac{1}{\Gamma(\beta)} \left(\log \frac{1}{r}\right)^{-1}, & \text{if } w_n = [d(n)]^\beta, \\
\beta(1-r)^{-1}, & \text{if } w_n = d_{\beta+1}(n) \end{array} \right. \beta > 0.
\]

On the finite polydisk \(\mathbb{D}^d\) \((d \in \mathbb{N})\), the corresponding Bergman spaces \(H^2_w(\mathbb{D}^d)\) - specifically \(B^2_\beta(\mathbb{D}^d)\) and \(A^2_\beta(\mathbb{D}^d)\) - are the \(L^2\)–closures of polynomials with respect to the norm

\[
\|f\|_{H^2_w(\mathbb{D}^d)} := \left( \int_{\mathbb{D}^d} |f(z_1, \ldots, z_d)|^2 dm_w(z_1) \times \cdots \times dm_w(z_d) \right)^{1/2}
\]

If \(f(z) = \sum_{n \in \mathbb{N}^d} a_n z^n\) is defined on \(\mathbb{D}^d\), we have

\[
\|f\|^2_{B^2_\beta(\mathbb{D})} = \sum_{n \in \mathbb{N}} |a_n|^2 \frac{n!}{(n+1)\beta} \quad \text{and} \quad \|f\|^2_{A^2_\beta(\mathbb{D})} = \sum_{n \in \mathbb{N}} |a_n|^2 \frac{n!}{(\beta+1)(\beta+2)\cdots(\beta+n)}. \tag{2.2}
\]

When \(d\) is finite, the estimate

\[
\frac{n!}{(\beta+1)(\beta+2)\cdots(\beta+n)} \asymp (1+n)^{-\beta}
\]
Volterra operators and Hankel forms on Bergman spaces…

yields that, the spaces $B^2_{\beta}(\mathbb{D}^d)$ and $A^2_{\beta}(\mathbb{D}^d)$ coincide as sets, with equivalent norms. However, the norms are no longer equivalent in the case of infinitely many variables.

The $H^2_w$-norm will be computed via the rotation invariant probability measure

$$d\mu_w(\chi) = dm_w(\chi_1) \times dm_w(\chi_2) \times dm_w(\chi_3) \times \ldots \text{ on } \mathbb{D}^\infty.$$  

Applying the Bohr lift to a Dirichlet series $f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$, and using (2.2) for each variable, one obtains the following formula (see [5] in the case of $B^2_{\beta}$)

$$\int_{\mathbb{D}^\infty} |B f(\chi)|^2 d\mu_w(\chi) = \sum_{n=1}^{+\infty} \frac{|a_n|^2}{w_n} = \|f\|^2_{H_w^2}.$$  

**Definition 2** For $0 < p < \infty$, the Bohr–Bergman spaces of Dirichlet series $B^p_{\beta}$ and $A^p_{\beta}$ - denoted by $H^p_w$ - are the completions of the Dirichlet polynomials in the norm (quasi norm when $0 < p < 1$)

$$\|f\|^p_{H^p_w} := \int_{\mathbb{D}^\infty} |B f(\chi)|^p d\mu_w(\chi).$$  

The Kronecker flow of the point $\chi = (\chi_1, \chi_2, \ldots) \in \mathbb{C}^\infty$ is given by

$$T_t(\chi) = \left(2^{-it}\chi_1, 3^{-it}\chi_2, 5^{-it}\chi_3, \ldots\right), \quad t \in \mathbb{R},$$  

which defines an ergodic flow on $\mathbb{T}^\infty$ by Kronecker’s theorem.

Therefore, it follows from Fubini’s Theorem that, for any rotation invariant probability measure $d\nu$ on $\mathbb{D}^\infty$ and any probability measure $d\lambda$ on $\mathbb{R}$, we have

$$\|f\|^p_{L^p(\mathbb{D}^\infty, d\nu)} = \int_{\mathbb{D}^\infty} \int_{\mathbb{R}} |(B f)(T_t(\chi))|^p d\lambda(t) d\nu(\chi). \quad (2.3)$$

### 2.3 On the half-plane $\mathbb{C}_{1/2}$

For $\theta \in \mathbb{R}$, let $\tau_\theta$ be the following mapping from $\mathbb{D}$ to $\mathbb{C}_\theta$,

$$\tau_\theta(z) = \theta + \frac{1 + z}{1 - z}. \quad (2.4)$$

For $\delta > 0$, the conformally invariant Bergman space $A_{i,\delta}(\mathbb{C}_{1/2})$ is the space of those functions $f$ which are analytic in $\mathbb{C}_{1/2}$, and such that

$$\|f\|^2_{A_{i,\delta}(\mathbb{C}_{1/2})} := \|f \circ \tau_{1/2}\|^2_{A^2_{\delta}(\mathbb{D})} = 4^\delta \delta \int_{\mathbb{C}_{1/2}} |f(s)|^2 \frac{(\sigma - \frac{1}{2})^{\delta-1}}{|s + \frac{1}{2}|^{2\delta+2}} dm(s) < \infty.$$
The weights $w$ of the class $W$ satisfy a Chebyshev-type estimate

$$\sum_{n \leq x} w_n \asymp x (\log x)^{\delta}, \quad \text{where} \quad \delta = \delta(w) := \begin{cases} 2^\beta - 1 & \text{if} \ w_n = \lfloor d(n) \rfloor^\beta, \\ \beta & \text{if} \ w_n = d_{\beta+1}(n). \end{cases} \quad (2.5)$$

For any real number $\tau$, set $S_\tau = \left[ \frac{1}{2}, 1 \right] \times [\tau, \tau + 1]$. As mentioned in the introduction, the Dirichlet series which belong the $H_{w}$ absolutely converge in $\mathbb{C}_{1/2}$. The space $H_w$ is locally embedded in $A_{i,\delta(w)}(\mathbb{C}_{1/2})$ [23,25], which means

$$\sup_{\tau \in \mathbb{R}} \int_{S_\tau} |f(s)|^2 \left( \frac{\sigma - \frac{1}{2}}{|s + \frac{1}{2}|^{\frac{1}{2} + \frac{1}{2}}} \right) dm(s) \leq c \left( H_w^2 \right) \| f \|_{H_w^2}^2.$$

Since functions in $H_{w}$ are uniformly bounded in $\mathbb{C}_{1}$, these embeddings are global (see [5,8]).

**Lemma 1** Let $\delta = \delta(w)$ be defined in (2.5). Then $H_{w}$ is continuously embedded in $A_{i,\delta}(\mathbb{C}_{1/2})$.

### 2.4 Generalized vertical limits

Every $\chi = (\chi_1, \chi_2, \ldots)$ in $\mathbb{C}^\infty$ defines a completely multiplicative function by the formula $\chi(n) = \chi^k$, where $n = p^k$. For $f$ of the form (1.1), the twisted Dirichlet series [5,6], is defined by

$$f_{\chi}(s) = \sum_{n=1}^{+\infty} a_n \chi(n)n^{-s}. \quad (2.6)$$

Notice that if $\chi \in T^\infty$, $f_{\chi}$ is the vertical limit of $f$, introduced in [19].

We also consider the translations $f_{\delta}(s) = f(s + \delta)$, $\delta \in \mathbb{R}$. For those $\chi \in D^\infty$ and $s = \sigma + it$ for which the series (2.6) converges, we have

$$f_{\chi}(s) = (B f_{\sigma} T_t)(\chi). \quad (2.7)$$

When $f$ is in $H_{w}^2$, the Cauchy-Schwarz inequality implies that (2.7) holds whenever $s \in \mathbb{C}_{1/2}$ and $\chi \in D^\infty$. By the Rademacher-Menchov Theorem (see [22]), (2.7) can be extended in the following way (the argument given in [5] for $B_{\beta}^2$ remains true for $A_{\beta}^2$).

**Lemma 2** If $f$ is in $H_{w}^2$, the Dirichlet series $f_{\chi}$ as defined in (2.6) converges in $\mathbb{C}_0$ for almost every $\chi \in D^\infty$, with respect to $\mu_w$.

Recall that $\tau_{\theta}$, $\theta \in \mathbb{R}$, is the conformal mapping defined in (2.4). For $0 < p < \infty$, the conformally invariant Hardy space $H_{p}^p(\mathbb{C}_{\theta})$, is the space of those functions $f$
such that \( f \circ \tau_\theta \) is in \( H^p(\mathbb{T}) \), the usual Hardy space of the unit disk. Setting \( d\lambda(t) = \pi^{-1}(1 + t^2)^{-1}dt \), we get

\[
\| f \|_{H^p_t(\mathbb{C}_0)}^p = \int_{\mathbb{R}} |f(\theta + it)|^p d\lambda(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f \circ \tau_\theta(u)|^p du, \quad \text{for } f \in H^p_t(\mathbb{C}_0).
\]

Let \( f \) be in \( \mathcal{H}^p_w \). In view of relation (2.3), and using the same argument as in [6,19], one can prove that for almost all \( \chi \), with respect to \( \mu_w \), \( f \chi \) can be extended analytically on \( \mathbb{C}_0 \) to an element of \( H^p_t(\mathbb{C}_0) \). The norm of \( f \) in \( \mathcal{H}^p_w \) can be expressed as

\[
\| f \|^p_{\mathcal{H}^p_w} = \int_{\mathbb{D}^\infty} \| f\chi \|^p_{H^p_t(\mathbb{C}_0)} d\mu_w(\chi). \tag{2.8}
\]

### 2.5 A Littlewood–Paley formula

We now derive another expression for the norm in \( \mathcal{H}^p_w \).

**Proposition 1** Let \( \lambda \) be a probability measure on \( \mathbb{R} \), and \( p \geq 1 \).

(a) If \( f \in \mathcal{H}^p_w \), then \( \| f \|^p_{\mathcal{H}^p_w} \asymp I_p(f) \), where

\[
I_p(f) := |f(+\infty)|^p + 4 \int_{\mathbb{D}^\infty} \int_{\mathbb{R}} \int_0^{+\infty} |f\chi(y + it)|^{p-2} |f\chi'(y + it)|^2 y dy d\lambda(t) d\mu_w(\chi).
\]

When \( p = 2 \), we have \( \| f \|^2_{\mathcal{H}^2_w} = I_2(f) \).

(b) Let \( f \in \mathcal{D} \), \( f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} \), such that \( f \) and \( f\chi \) converge on \( \mathbb{C}_0 \) for a.a. \( \chi \in \mathbb{D}^\infty \). If \( I_p(f) < \infty \), then \( f \in \mathcal{H}^p_w \).

**Proof** Since the real variable \( t \) corresponds to a rotation in each variable of \( \mathbb{D}^\infty \), the rotation invariance of \( \mu_w \) entails that \( I_p(f) \) does not depend on the choice of the probability measure \( \lambda \). For general \( p \geq 1 \), we prove (a), by using (2.8). We adapt the argument from [10] (for \( H^p \)), by integrating over the polydisk \( \mathbb{D}^\infty \) instead of the polytorus \( \mathbb{T}^\infty \).

Suppose \( f \) is in \( \mathcal{H}^2_w \), and take \( y > 0 \). From (2.3) and the rotation invariance, we obtain

\[
\int_{\mathbb{R}} \int_{\mathbb{D}^\infty} |f\chi'(y + it)|^2 d\mu_w(\chi) d\lambda(t) = \int_{\mathbb{D}^\infty} \left| B f\chi'(\chi) \right|^2 d\mu_w(\chi)
\]

\[
= \sum_{n=1}^{+\infty} \frac{|a_n|^2}{w_n} (\log n)^2 n^{-2y}.
\]

Integration against \( y \) on \( (0, +\infty) \) gives the formula (see details in [7] for the case of \( \mathcal{H}^2 \)).
If \( f \) is as in (b), the integrand in \( I_p(f) \) is measurable. For \( \chi \in \mathbb{D}^\infty \), the change of variables \( s = y + it = \omega(z) = 2 \frac{1 + z}{1 - z} \) transfers the Littlewood–Paley formula from \( \mathbb{D} \) to \( \mathbb{C}_0 \),

\[
\int_{\mathbb{R}} |f_\chi(it)|^p \frac{2}{\pi(2^2 + t^2)} \, dt \\
\leq |f_\chi(2)|^p \\
+ \int_{\mathbb{D}} \left( 1 - |z|^2 \right) |f_\chi(\omega(z))|^{p-2} \left| f_\chi'(\omega(z)) \right|^2 |\omega'(z)|^2 \, dV(z) \\
\leq |f_\chi(2)|^p \\
+ \int_0^\infty \int_{\mathbb{R}} \frac{2y}{(y+2)^2 + t^2} \left| f_\chi(y+it) \right|^{p-2} \left| f_\chi'(y+it) \right|^2 \, dt \, dy \\
\lesssim \| f^* \|_{L^\infty(\mathbb{C}_2)}^p \\
+ \int_0^\infty \int_{\mathbb{R}} \frac{y}{1 + t^2} \left| f_\chi(y+it) \right|^{p-2} \left| f_\chi'(y+it) \right|^2 \, dt \, dy,
\]

where \( f^*(s) := \sum_{n=1}^{+\infty} |a_n| n^{-s} \) is bounded on \( \mathbb{C}_2 \).

Integrating on \( \mathbb{D}^\infty \) with respect to \( \mu_w \), and using (2.3), we get that

\[
\| B f \|_{L^p(\mathbb{D}^\infty, \mu_w)} \lesssim \| f^* \|_{L^\infty(\mathbb{C}_2)}^p + I_p(f) < \infty.
\]

Therefore, \( B f \in L^p(\mathbb{D}^\infty, \mu_w) \). The martingale \( (A_m B f)_m \) (with respect to the increasing sequence of \( \sigma \)-algebras of the sets \( \mathbb{D}^m \times \{0\} \)) converges in \( L^p(\mathbb{D}^\infty, \mu_w) \) to \( B f \).

Polynomial approximation in the Bergman spaces of the finite polydisks \( \mathbb{D}^m \) shows that \( B f \) is in \( B^p_{H^p_w} \).

\[\square\]

3 Spaces of symbols of Volterra operators in half-planes

If \( g \) is in \( \mathcal{D} \), the definition (1.2) of \( T_g \) shows that we can assume that \( g(+\infty) = 0 \), i.e.

\[
g(s) = \sum_{n=2}^{+\infty} b_n n^{-s}.
\]

As in the study of Volterra operators on Bergman spaces the unit disk [2], and on the space of Dirichlet series \( \mathcal{H}^2 \) [13], the boundedness of \( T_g \) on \( \mathcal{H}^2_w \) will be related to Carleson measures, and to the membership of \( g \) to a BMO space or a Bloch space.

Let \( Y \) be either \( \mathcal{H}^2_w \) or the Bergman space \( A_{1,\delta}(\mathbb{C}_{1/2}) \), \( \delta > 0 \). A positive Borel measure \( \mu \) on \( \mathbb{C}_{1/2} \) is called a Carleson measure for \( Y \) if there exists a constant \( C \) such that,
\[ \int_{\mathbb{C}_{1/2}} |f|^2 \, d\mu \leq C \|f\|^2_Y \quad \text{for all } f \in Y. \]

The smallest such constant, denoted by \( \|\mu\|_{CM(Y)} \), is called the Carleson constant for \( \mu \) with respect to \( Y \). A Carleson measure \( \mu \) is a vanishing Carleson measure for \( Y \) if we have

\[ \lim_{k \to \infty} \int_{\mathbb{C}_{1/2}} |f_k|^2 \, d\mu = 0, \]

for every weakly compact sequence \((f_k)_k\) in \( Y \) (which means that \( \|f_k\|_Y \) is bounded and \( f_k(s) \to 0 \) on every compact set of \( \mathbb{C}_{1/2} \)).

### 3.1 BMO spaces of Dirichlet series

The space \( BMOA(\mathbb{C}_\theta) \) consists of holomorphic functions \( g \) in the half-plane \( \mathbb{C}_\theta \) which satisfy

\[ \|g\|_{BMO(\mathbb{C}_\theta)} := \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \left| \int_I g(\theta + it) - \frac{1}{|I|} \int_I g(\theta + i\tau) \, d\tau \right| \, dt < \infty. \]

Any \( g \) in \( \mathcal{D} \cap BMOA(\mathbb{C}_0) \) has an abscissa of boundedness \( \sigma_b \leq 0 \) (Lemma 2.1 of [13]).

The space \( VMOA(\mathbb{C}_0) \) consists in those functions \( g \) in \( BMOA(\mathbb{C}_0) \) such that

\[ \lim_{\delta \to 0^+} \sup_{|I| < \delta} \frac{1}{|I|} \left| \int_I f(it) - \frac{1}{|I|} \int_I f(i\tau) \, d\tau \right| \, dt = 0. \]

### 3.2 Bloch spaces of Dirichlet series

The Bloch space \( \text{Bloch}(\mathbb{C}_\theta) \) consists of holomorphic functions in the half-plane \( \mathbb{C}_\theta \) which satisfy

\[ \|g\|_{\text{Bloch}(\mathbb{C}_\theta)} := \sup_{\sigma + it \in \mathbb{C}_\theta} (\sigma - \theta) \left| f'(\sigma + it) \right|. \]

**Lemma 3** If \( g \) be in \( \mathcal{D} \cap \text{Bloch}(\mathbb{C}_0) \).

(a) Its abscissa of boundedness satisfies \( \sigma_b \leq 0 \).
(b) For every \( \chi \in \mathbb{D}^\infty \), \( g \chi \) is in \( \text{Bloch}(\mathbb{C}_0) \), and \( \|g \chi\|_{\text{Bloch}(\mathbb{C}_0)} \leq \|g\|_{\text{Bloch}(\mathbb{C}_0)} \).
(c) Suppose that \( y_0 > \frac{1}{2} \). Then there exists a constant \( C = C(y_0) \), such that,

\[ \left| g'_\chi(y + it) \right| \leq C 2^{-y} \|g\|_{\text{Bloch}(\mathbb{C}_0)}, \quad \text{for all } \chi \in \mathbb{D}^\infty, \ t \in \mathbb{R}, \ y \geq y_0. \]
Proof Let \( \epsilon > 0 \). If \( s = \sigma + it \) is in \( C_0 \), the definition of the Bloch-norm implies that
\[
\epsilon |g'(\epsilon + s)| \leq (\epsilon + \sigma) |g'(\epsilon + s)| \leq \|g\|_{\text{Bloch}(C_0)}.
\]
It follows that \( g' \), and then \( g \) is bounded in \( C_\epsilon \); (a) is proved.

Now fix \( \sigma > 0 \). Let \( m \geq 1 \) be an integer, and \( z = (z_1, \ldots, z_m, z_{m+1}, \ldots) \), \( \chi \) in \( D_\infty \). From the properties of \( \mathcal{H}^\infty \) and the proof of (a), we have
\[
\|A_mB(g'_\sigma)\chi(z)\| = |A_mB(g'_\sigma)(z,\chi)| \leq \|B_{g'_\sigma}\|_{\mathcal{H}^\infty(T_\infty)} = \|g'_\sigma\|_{\mathcal{H}^\infty},
\]
and \( \|g'_\sigma\chi\|_{\mathcal{H}^\infty} = \|B(g'_\sigma)\chi\|_{\mathcal{H}^\infty(T_\infty)} \leq \|g'_\sigma\|_{\mathcal{H}^\infty} \). Therefore, \( (g'_\sigma)\chi \) is in \( \mathcal{H}^\infty \); (b) holds, due to
\[
\sigma |g'_\chi(\sigma + it)| \leq \|g\|_{\text{Bloch}(C_0)}, \quad \text{for all } t \in \mathbb{R}, \chi \in T_\infty, \sigma > 0.
\]

If \( 0 < \delta < y_0 - \frac{1}{2} \), the Cauchy-Schwarz inequality and Parseval’s relation induce that
\[
|g'_\chi(y + it)|^2 \leq \left( \sum_{n=2}^{+\infty} |b_n| (\log n)n^{-y} \right)^2 \leq \left( \sum_{n=2}^{+\infty} |b_n| (\log n)n^{-\frac{\delta}{2}} n^{-\left(\frac{\delta}{2} + \frac{1}{2}\right) n^{-\left(\frac{\delta}{2} + \delta\right)}} \right)^2 \leq \xi (1 + \delta) 2^{-2\gamma} \|B_{g'_\sigma/2}\|_{H^2(T_\infty)}^2.
\]
We now get (c) from the chain of inequalities
\[
\|B_{g'_\sigma/2}\|_{H^2(T_\infty)} \leq \|B_{g'_\sigma/2}\|_{\mathcal{H}^\infty(T_\infty)} = \|g'_\sigma/2\|_{\mathcal{H}^\infty} \leq \frac{2}{\delta} \|g\|_{\text{Bloch}(C_0)},
\]
\( \square \)

Now, recall several characterizations of Bloch functions, which are extracted from [2,18].

Lemma 4 Assume \( \delta > 0 \). For \( g \) holomorphic in \( C_\theta \), the following are equivalent:
(a) \( g \in \text{Bloch}(C_\theta) \);
(b) \( h = g \circ \tau_\theta \in \text{Bloch}(\mathbb{D}) \);
(c) The measure \( d\mu_{C_\theta,g}(s) = |g'(\sigma + it)|^2 \frac{(\sigma-\theta)^{\delta+1}}{|s-\theta+1|^{\delta+2}} d\sigma dt \) is a Carleson measure for \( A_{s,\delta}(C_\theta) \);
(d) The measure \( d\mu_{\mathbb{D},h}(z) = |h'(z)|^2 (1 - |z|^2)^{\delta+1} dm_1(z) \) is a Carleson measure for \( A_{\delta}^2(\mathbb{D}) \);
(e) The operator \( J_h \), given by
\[
J_h f(z) = \int_0^z f(t)h'(t)dt,
\]
is bounded on \( A_{\delta}^2(\mathbb{D}) \).
Moreover, the quantities
\[ \| g \|_{Bloch(C_\theta)}, \| \mu_{C_\theta,g} \|_{CM(C_\theta)}, \| J_g \|_{\mathcal{L}(\lambda_1^2(\mathbb{D}))} \]
are comparable.

The little Bloch space is the space
\[ \text{Bloch}_0(C_\theta) = \{ f \in \text{Bloch}(C_\theta) : \lim_{\sigma \to \theta} (\sigma - \theta) \left| g'(s) \right| = 0 \} . \]

The membership in \( \text{Bloch}_0(C_\theta) \) is characterized by a little oh version of Lemma 4, involving vanishing Carleson measures.

We show that Dirichlet polynomials are dense in \( D \cap \text{Bloch}_0(C_0) \).

**Proposition 2** Let \( g \) be in \( \text{Bloch}_0(C_0) \cap D \), and \( \epsilon > 0 \). Then there exists \( P \) in \( D \) such that
\[ \| g - P \|_{\text{Bloch}(C_0)} \leq \epsilon . \]

If in addition \( g \) is in \( D_d \), \( P \) can be chosen in \( D_d \).

**Proof** For every \( \delta > 0 \), \( g_\delta = g(\delta + .) \) is also in \( \text{Bloch}_0(C_0) \). As \( \delta \) tends to 0, \( (g_\delta)_\delta \) converges to \( g \) uniformly on compact sets of \( C_0 \), and \( \lim_{\sigma \to 0^+} \sigma \left| g_\delta'(s) \right| = 0 \), uniformly with respect to \( \delta \in (0, 1) \). It then follows from [3] that \( \lim_{\delta \to 0^+} \| g - g_\delta \|_{\text{Bloch}(C_0)} = 0 \). Thus, we can choose \( \delta > 0 \) such that \( \| g - g_\delta \|_{\text{Bloch}(C_0)} \leq \frac{\epsilon}{2} \). Since \( \sigma_b(g) = \sigma_d(g) \leq 0 \), the partial sums \( (S_Ng)_N \) converge uniformly to \( g \) in \( C_\delta \), \( \lim_{N \to +\infty} \| S_Ng_\delta - g_\delta \|_{H^\infty} = 0 \). For large \( N \), the triangle inequality implies that
\[ \| g - S_Ng_\delta \|_{\text{Bloch}(C_0)} \leq \| g - g_\delta \|_{\text{Bloch}(C_0)} + \| g_\delta - S_Ng_\delta \|_{\text{Bloch}(C_0)} \leq \frac{\epsilon}{2} + 2 \| S_Ng_\delta - g_\delta \|_{H^\infty} \leq \epsilon . \]

\[ \square \]

### 3.3 Carleson measures on the half-plane \( C_{1/2} \)

On \( C_{1/2} \), we consider Carleson squares
\[ Q(s_0) = \left( \frac{1}{2}, \sigma_0 \right) \times \left[ t_0 - \frac{\epsilon}{2}, t_0 + \frac{\epsilon}{2} \right] , \]
where \( s_0 = \sigma_0 + it_0 \in C_{1/2} \)
is the midpoint of the right edge of the square and \( \epsilon = \sigma_0 - \frac{1}{2} \).

We need the following property (see Section 7.2 in [31]).
Lemma 5  Let $\delta > 0$ and let $\mu$ be a Borel measure on $\mathbb{C}_{1/2}$. Then $\mu$ is a Carleson measure for $A_{i,\delta} (\mathbb{C}_{1/2})$ if and only if, for every square $Q(s_0)$, with $s_0 = \sigma_0 + it_0$, we have

$$\mu (Q(s_0)) = O \left( (2\sigma_0 - 1)^{\delta+1} \right) \text{ as } \sigma_0 \to \left( \frac{1}{2} \right)^+.$$ 

In addition, $\mu$ is a vanishing Carleson measure for $A_{i,\delta} (\mathbb{C}_{1/2})$ if and only if, uniformly for $t_0$ in $\mathbb{R}$,

$$\mu (Q(s_0)) = o \left( (2\sigma_0 - 1)^{\delta+1} \right) \text{ as } \sigma_0 \to \left( \frac{1}{2} \right)^+.$$ 

By Lemma 1, $H^2_w$ is embedded in the Bergman-type space $A_{i,\delta} (\mathbb{C}_{1/2})$, the exponent $\delta = \delta(w)$ being defined in (2.5). Bounded Carleson measures for both spaces $H^2_w$ and $A_{i,\delta} (\mathbb{C}_{1/2})$ have been compared in [8,23,24]. We extend their results.

Lemma 6  Let $\mu$ be a positive Borel measure on $\mathbb{C}_{1/2}$.

(1) If $\mu$ is a Carleson measure (resp. vanishing Carleson measure) for $H^2_w$, then $\mu$ is a Carleson measure (resp. vanishing Carleson measure) for $A_{i,\delta} (\mathbb{C}_{1/2})$ and

$$\|\mu\|_{CM(A_{i,\delta}(\mathbb{C}_{1/2}))} \lesssim \|\mu\|_{CM(H^2_w)}.$$ 

(2) Assume that $\mu$ has bounded support. If $\mu$ is a Carleson measure (resp. vanishing Carleson measure) for $A_{i,\delta} (\mathbb{C}_{1/2})$, then $\mu$ is a Carleson measure (resp. vanishing Carleson measure) for $H^2_w$ and

$$\|\mu\|_{CM(H^2_w)} \lesssim \|\mu\|_{CM(A_{i,\delta}(\mathbb{C}_{1/2}))}.$$

Proof  Suppose that $\mu$ is a Carleson measure for $H^2_w$, and let $Q(s_0)$ be a small Carleson square in $\mathbb{C}_{1/2}$. For the test function $f_{s_0}(s) = K H^2_w (s, s_0)$, we have

$$\int_{Q(s_0)} |f_{s_0}|^2 \, d\mu \leq \int_{\mathbb{C}_{1/2}} |f_{s_0}|^2 \, d\mu \leq C(\mu) \left\| K H^2_w (., s_0) \right\|^2_{H^2_w} \lesssim Z_w (\Re s_0).$$ 

From the estimate of $Z_w$ (2.1) and Lemma 5, $\mu$ is a Carleson measure for $A_{i,\delta} (\mathbb{C}_{1/2})$, since

$$\left( \Re s_0 - \frac{1}{2} \right)^{-2(\delta+1)} \mu (Q(s_0)) \lesssim \left( \Re s_0 - \frac{1}{2} \right)^{-(\delta+1)}.$$ 

For $\mu$ a Carleson measure for $A_{i,\delta} (\mathbb{C}_{1/2})$ with bounded support, (2) holds [23,24].
As for vanishing Carleson measures, the reasoning used in [8] for \( B_\beta^2 \) can be transferred to the spaces \( A_{\beta}^2 \), with the test functions

\[
f_k(s) = \frac{K_{\mathcal{H}_w^2}(s, s_k)}{\|K_{\mathcal{H}_w^2}(\cdot, s_k)\|_{\mathcal{H}_w^2}},
\]

where \( s_k = 1/2 + \epsilon_k + i\tau_k \) is a sequence in \( \mathbb{C}_{1/2} \) such that \( \epsilon_k \to 0 \). \( \square \)

We also require an equivalent norm for \( A_{i, \delta}(\mathbb{C}_{1/2}) \), when \( \delta > 0 \). For Bergman spaces of the unit disk, recall the following consequence of Stanton’s formula [28,29]:

\[
\|h\|^2_{A_{i, \delta}(\mathbb{D})} \asymp |h(0)|^2 + \int_{\mathbb{D}} |h'(z)|^2 \left( 1 - |z|^2 \right)^{\delta + 1} dV(z), \quad \text{for } h \text{ holomorphic on } \mathbb{D}.
\]

Via the mapping \( \tau_{1/2} \), we obtain that, for any \( f \) holomorphic on \( \mathbb{C}_{1/2} \),

\[
\|f\|^2_{A_{i, \delta}(\mathbb{C}_{1/2})} \asymp \left| f\left( \frac{3}{2} \right) \right|^2 + \int_{\mathbb{C}_{1/2}} \left| f'(s) \right|^2 \frac{\left( \sigma - \frac{1}{2} \right)^{\delta + 1}}{|s + \frac{1}{2}|^{2\delta + 2}} dV(s). \quad (3.1)
\]

4 Boundedness of \( T_g \)

In this section, we characterize functions in \( X_w \), and prove Theorem 1.

4.1 Carleson measure characterization

The boundedness of \( T_g \) on \( \mathcal{H}_w^2 \) can be described in terms of Carleson measures. This generalizes the setting of the Hardy space \( \mathcal{H}_w^2 \) [13].

Recall that \( \mathcal{H}_w^2 \) is associated to the probability measure \( \mu_w \) on the polydisk \( \mathbb{D}^\infty \).

**Proposition 3** \( T_g \) is bounded on \( \mathcal{H}_w^2 \) if and only if there exists a constant \( C = C(g) \) such that

\[
\|T_g f\|^2_{\mathcal{H}_w^2} \asymp \int_{\mathbb{D}^\infty} \int_0^{+\infty} \int_0^{+\infty} |f_X(\sigma + it)|^2 \left| g'_X(\sigma + it) \right|^2 \frac{\sigma d\sigma dt}{1 + t^2} d\mu_w(\chi)
\]

\[
\leq C^2 \|f\|^2_{\mathcal{H}_w^2}, \quad (4.1)
\]

or, equivalently

\[
\int_{\mathbb{D}^\infty} \int_0^{+\infty} \left| f_X(\sigma) \right|^2 \left| g'_X(\sigma) \right|^2 \sigma d\sigma d\mu_w(\chi) \leq C^2 \|f\|^2_{\mathcal{H}_w^2}. \quad (4.2)
\]

The smallest constant \( C \) satisfying (4.1) is such that \( C \asymp \|T_g\|_{\mathcal{L}(\mathcal{H}_w^2)} \).
Proof Applying the Littlewood–Paley formula (Proposition 1) to the measure \( d\lambda(t) = \pi^{-1}(1 + t^2)^{-1}dt \) and the function \( T_g f \), we get (4.1).

The rotation invariance of the measure \( d\mu_w(\chi) \) gives (4.2).

\[ \square \]

4.2 Proof of Theorem 1 (a): \( Bg \) depends on a finite number of variables

For \( 1 \leq q \) and \( d \geq 1 \), recall that \( f \in \mathcal{H}^q_{d,w} \) if and only if \( f \) is in \( \mathcal{H}^q_w \) and \( Bf \) is a function of \( z_1, \ldots, z_d \).

When needed, we shall identify \( z = (z_1, \ldots, z_d) \in \mathbb{D}^d \) with \((z, 0) \in \mathbb{D}^{d+1} \times \{0\} \).

If \( g(s) = \sum_{n=2}^{+\infty} b_n n^{-s} \) is in \( \mathcal{H}^2_{d,w} \), we observe that for \( z \in \mathbb{D}^d \),

\[
Bg'(z) = \sum_{j=1}^{d} \log p_j \sum_{\alpha \in \mathbb{N}^d} \tilde{b}_{\alpha} \alpha_j z^\alpha = R Bg(z),
\]

where \( R \) is the operator

\[
RG(z_1, \ldots, z_d) = \sum_{j=1}^{d} (\log p_j) z_j \partial_j G(z_1, \ldots, z_d).
\]

We define the set

\[
\Delta_\epsilon := \{ z = (z_1, \ldots, z_d) \in \mathbb{D}^d, \forall j, |z_j| < p_j^{-\epsilon} \}, \quad \text{for } \epsilon > 0.
\]

Take \( x > 0 \), \( t \in \mathbb{R} \), and \( z \in \mathbb{D}^d \). By construction, \( z \in \bar{\Delta}_{\sigma(z)} \) and \( \sigma(p^{-x}z) \geq \sigma(z) + x \log p_1 \log p_d \).

For \( g \in \mathcal{D} \), we write \( g_z(x) = g(z,0)(x) = Bg_x(z) \). Since \( g \) is in \( \text{Bloch}(\mathbb{C}_0) \), we apply (1.6) to \( g'_x \), and get

\[
|g'_z(x + it)| = |Bg'_x(\mathcal{T}_t z)| \leq \sup_{\zeta \in \bar{\Delta}_{\sigma(p^{-x}z)}} |Bg'\zeta| \leq \sup_{s \in \mathbb{C}_{\sigma(p^{-x}z)}} |g'(s)| < \frac{\log p_d \|g\|_{\text{Bloch}(\mathbb{C}_0)}}{\log p_1} x + \sigma(z), \quad (4.3)
\]

Proof of Theorem 1(a) Let \( f(s) = \sum_{n \geq 1} a_n n^{-s} \) be in \( \mathcal{H}^2_w \), and, for \( \chi = (z, z') \in \mathbb{D}^d \times \mathbb{D}^{\infty} \),

\[
Bf(\chi) = \sum_{(\alpha,\alpha') \in \mathbb{N}^d \times \mathbb{N}_0^{\infty}} c_{\alpha,\alpha'} z^\alpha z'^{\alpha'} = \sum_{\alpha \in \mathbb{N}^d} c'_\alpha(z') z^\alpha, \quad \text{where } c'_\alpha(z') = \sum_{\alpha' \in \mathbb{N}_0^{\infty}} c_{\alpha,\alpha'} z'^{\alpha'}.
\]
In view of Proposition 3, we aim to estimate \( \| T_g f \|_{H^2_w}^2 \simeq I_1 + I_2 \), where

\[
I_1 := \int_{D^\infty} \int_0^1 \left| f_X(x) \right|^2 \left| g'_{X}(x) \right|^2 x dx d\mu_w(\chi),
\]
and

\[
I_2 := \int_{D^\infty} \int_1^{+\infty} \left| f_X(x) \right|^2 \left| g'_{X}(x) \right|^2 x dx d\mu_w(\chi).
\]

By (4.3), the rotation invariance and Fubini’s Theorem, we have

\[
I_1 \lesssim \| g \|_{\text{Bloch}(C_0)}^2 \int_0^1 x \int_{D^\infty} \int_{D^d} \frac{1}{[x + \sigma(z)]^2} \left| \sum_{\alpha \in \mathbb{N}^d} c_{\alpha}'(p^{-x} \cdot z') (z_1 p_1^{-x})^{\alpha_1} \cdots (z_d p_d^{-x})^{\alpha_d} \right|^2 d\mu_w(z, z') dx
\]

\[
\lesssim \| g \|_{\text{Bloch}(C_0)}^2 \int_0^1 x \sum_{\alpha \in \mathbb{N}^d} \left| c_{\alpha}'(p^{-x} \cdot z') \right|^2 I_{\alpha}(x) dx d\mu_w(z'),
\]

where

\[
I_{\alpha}(x) := \int_{D^d} \frac{1}{[x + \sigma(z)]^2} \left| z_1 p_1^{-x} \right|^{2\alpha_1} \cdots \left| z_d p_d^{-x} \right|^{2\alpha_d} d\mu_w(z).
\]

Using the rotation invariance again as well as the fact that \( p_j \geq 1 \), and setting \( J_{\alpha} := \int_0^1 x I_{\alpha}(x) dx \), we get

\[
I_1 \lesssim \sum_{\alpha, \alpha'} |c_{\alpha, \alpha'}|^2 J_{\alpha} \left( \int_{D^\infty} \left| c_{\alpha, \alpha'}(p^{-x} \cdot z') \right|^2 d\mu_w(z') \right)
\]

\[
\lesssim \sum_{\alpha, \alpha'} |c_{\alpha, \alpha'}|^2 J_{\alpha} \left( \int_{D^\infty} \left| z'^{\alpha'} \right|^2 d\mu_w(z') \right)
\]

\[
\lesssim \sum_{\alpha, \alpha'} \frac{|c_{\alpha, \alpha'}|^2 J_{\alpha}}{w(p_{\alpha + 1}^{d+1}) \cdots w(p_{\alpha r}^{d+1})}.
\]

For the moment, we admit that \( J_{\alpha} \leq C(d, w) \left[ \prod_{j=1}^d w(p_j^{\alpha_j}) \right]^{-1} \), which will be proved in Lemma 7. Hence,

\[
I_1 \lesssim \sum_{\alpha, \alpha'} \frac{|c_{\alpha, \alpha'}|^2}{w(p^{\alpha})} \lesssim \| g \|_{\text{Bloch}(C_0)}^2 \| f \|_{H^2_w}^2.
\]
Combining Lemma 3 with the following observation,

$$\int_{D^\infty} |f_\chi(x)|^2 \, d\mu_w(\chi) = \int_{D^\infty} \left| \sum_{n=p^\alpha} a_n n^{-x} \chi \right|^2 \, d\mu_w(\chi) = \sum_{n \geq 1} \frac{|a_n|^2 n^{-2x}}{w_n} \leq \|f\|_{L^2_w}^2,$$

we estimate $I_2$,

$$I_2 \lesssim \int_1^{+\infty} x \int_{D^\infty} \|g\|^2_{\text{Bloch}(\mathbb{C}_0)} 4^{-x} \left| f_\chi(x) \right|^2 \, d\mu_w(\chi) dx \lesssim \|g\|^2_{\text{Bloch}(\mathbb{C}_0)} \|f\|_{L^2_w}^2.$$

Recall that

$$I_\alpha(x) = \int_{D^d} \frac{1}{[x + \sigma(z)]^2} |z_1 p_1^{-x}|^{2\alpha_1} \cdots |z_d p_d^{-x}|^{2\alpha_d} \, d\mu_w(z), \quad \alpha \in \mathbb{N}_d, \quad 0 < x < 1.$$

**Lemma 7** There exists a constant $C = C(w, d)$, such that

$$J_\alpha := \int_0^1 x I_\alpha(x) dx \leq C \prod_{j=1}^d \frac{1}{w(p_j^{\alpha_j})}.$$

The proof of Lemma 7 relies on technical computations (Lemma 8).

**Lemma 8** For $0 < T < 1$, and a real number $p \geq 2$, set $L := -\frac{\log T}{2 \log p}$ and $K = \min(1, L)$. There exists a constant $C = C(p, w) > 0$, such that

$$J(p, T) := (\log T)^{-2} \int_0^K x M\left(T p^{2x}\right) \, dx \lesssim C \begin{cases} M(T) & \text{if } \beta \geq 1 \text{ or } (\beta < 1, p^{-2} < T < 1), \\ M(T p^2) & \text{if } \beta < 1, 0 < T \leq p^{-2}. \end{cases}$$

**Proof** When $p^{-2} < T < 1$, the change of variables $u = T p^{2x}$ gives

$$J(p, T) = (\log T)^{-2} \int_T^{1} \frac{1}{(2 \log p)^2} \int_T^{1} \log \frac{u}{T} M(u) \frac{du}{u}.$$

Since $\log \frac{u}{T} \leq \log \frac{1}{T}$ and $1 \leq \frac{1}{u} \leq \frac{1}{T} < p^2$,

$$J(p, T) \leq (\log T)^{-2} \left( \frac{1}{2 \log p} \right)^2 \int_T^{1} \log \frac{1}{T} M(u) \frac{1}{u} \, du \lesssim M(T).$$
Next suppose that $0 < T \leq p^{-2}$. Since $(\log T)^2 \geq 4(\log p)^2$, we notice that

$$J(p, T) \lesssim \int_0^1 x M(T p^{2x}) dx \lesssim \begin{cases} \int_0^1 M(T) dx & \text{if } \beta \geq 1, \\ \int_0^1 M(T p^2) dx & \text{if } \beta < 1. \end{cases}$$

\[ \square \]

**Proof of Lemma 7** Resorting to polar coordinates, and using changes of variables, we have

$$\mathcal{J}_\alpha \lesssim \int_Q \frac{xt^\alpha}{x + \sigma(p_1^x \sqrt{t_1}, \ldots, p_d^x \sqrt{t_d})^2} \left( \prod_{k=1}^d M(p_k^2 t_k) p_k^{2x} \right) dx dt_1 \cdots dt_d,$$

where $Q = \{(x, t) \in (0, 1) \times (0, 1)^d, \forall k = 1 \ldots d, 0 < t_k < p_k^{-2x}\}$.

For $t = (t_1, \ldots, t_d) \in (0, 1)^d$, set

$$l_k(t) := -\frac{\log t_k}{2 \log p_k}, \quad K_k := \min(1, l_k), \quad 1 \leq k \leq d,$$

$$l(t) := \min_{1 \leq k \leq d} l_k(t), \quad K := \min(1, l).$$

We observe that $Q = \{(x, t) \in (0, 1) \times (0, 1)^d, 0 < x < K(t)\}$. Now, for $1 \leq k \leq d$, we set $Q_k := \{(x, t), t \in (0, 1)^d, l(t) = l_k(t), 0 < x < K_k(t)\}$.

Let $(x, t)$ be in $Q_k$. We have

$$0 < t_l \leq T_{k,l} := t_k^{\log p_l / \log p_k} < 1, \quad \text{for } 1 \leq l \leq d. \quad (4.4)$$

In addition, since $0 < x < l_k(t)$, (4.4) implies $p_1^x \sqrt{t_l} < p_1^{l_k(t)} \sqrt{t_l} \leq 1$, and we see that $\frac{1}{\sqrt{t_l}} p_1^{l_k(t) - x} \geq p_1^{l_k(t) - x}$. Thus

$$(\log p_d) \sigma(p_1^x \sqrt{t_1}, \ldots, p_d^x \sqrt{t_d}) = \log \min_{1 \leq l \leq d} \left( \frac{1}{\sqrt{t_l}} p_1^{l_k(t) - x} \right) \geq \log p_1 (l_k(t) - x),$$

and $x + \sigma(p_1^x \sqrt{t_1}, \ldots, p_d^x \sqrt{t_d}) \gtrsim -\log t_k$.

Set $d\hat{t}_k = dt_1 \cdots dt_{k-1} dt_{k+1} \cdots dt_d$, and

$$\hat{Q}_k := \{(x, t), 0 < t_k < 1, 0 < t_l < T_{k,l} \text{ for } l \neq k, 0 < x < K_k(t)\}.$$

It follows that $\mathcal{J}_\alpha \lesssim \sum_{k=1}^d \mathcal{J}_{\alpha,k}$, where

$$\mathcal{J}_{\alpha,k} = \int_{\hat{Q}_k} \frac{xt^\alpha}{x + \sigma(p_1^x \sqrt{t_1}, \ldots, p_d^x \sqrt{t_d})^2} \left( \prod_{l=1}^d M(p_l^{2x} t_l) \right) dx dt.$$
We will obtain the Lemma by showing that
\[
J_{\alpha,k} \lesssim \prod_{l=1}^{d} \left[w\left(p_l^{\alpha_l}\right)\right]^{-1}. \tag{4.5}
\]

When \(\beta \geq 1\), we use that, for \((x,t) \in \tilde{Q}_k\), and \(l \neq k\), \(M\left(p_l^{2x}t_l\right) \leq M\left(t_l\right)\), altogether with Lemma 8. We derive (4.5) from
\[
J_{\alpha,k} \lesssim \int_{0 < t_k < 1} \left( \int_{\prod_{j \neq k} (0,T_{k,j})} t_l^{\alpha_l} \int_0^{K_{k(t)}} x (\log t_k)^{-2} M\left(p_k^{2x}t_k\right) dx \prod_{l \neq k} M(t_l) dt_l \right) dt_k
\]
\[
\lesssim \int_{0 < t_k < 1} t_l^{\alpha_l} M\left(t_k\right) \left( \prod_{j \neq k} \int_0^{T_{k,j}} \right. t_j^{\alpha_j} M\left(t_j\right) dt_j \left. \right) dt_k \lesssim \prod_{l=1}^{d} \int_0^{1} t_l^{\alpha_l} M\left(t_l\right) dt_l.
\]

Next, suppose \(0 < \beta < 1\). If \((x,t) \in \tilde{Q}_k\), notice that, for \(l \neq k\), \(t_l p_l^{2x} \leq t_l p_l^{2k(t_l)} \leq 1\); this shows that \(M\left(p_l^{2x}t_l\right) \leq M\left(p_l^{2k(t_l)}t_l\right)\). Hence, we see that \(J_{\alpha,k} \lesssim J_1 + J_2\), where, by Lemma 8 and the relation \(p_l^{2k(t_l)} = T_{k,l}^{-1}\),
\[
J_1 \lesssim \int_{0 < t_k < p_k^{2x}} t_l^{\alpha_l} M\left(p_k^{2x}t_k\right) \left( \prod_{j \neq k} \int_0^{T_{k,j}} t_j^{\alpha_j} M\left(t_j\right) dt_j \right) dt_k,
\]
\[
J_2 \lesssim \int_{p_k^{2x} < t_k < 1} t_l^{\alpha_l} M\left(t_k\right) \left( \prod_{j \neq k} \int_0^{T_{k,j}} t_j^{\alpha_j} M\left(t_j\right) dt_j \right) dt_k.
\]

A change of variables provides the desired estimate.

\[
\square
\]

4.3 Proof of Theorem 1(b) and (c)

If \(f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}\) and \(g(s) = \sum_{n=1}^{+\infty} b_n n^{-s}\), we have
\[
T_n f(s) = \sum_{n=2}^{\infty} 1 \log n \left( \sum_{k|n,k < n} a_k b_{n/k} \right) n^{-s}.
\]

As in the case of \(H^2\), the operator
\[
a_1 + \sum_{n=2}^{\infty} a_n n^{-s} \mapsto a_1 + \sum_{n=2}^{\infty} a_n (\log n)^{-1} n^{-s}
\]
is compact on $\mathcal{H}_w$. Thus, set $b_1 = 1$, and our study will be unchanged if we replace $T_g$ by
\[
\tilde{T}_g f(s) = \sum_{n=2}^{\infty} \frac{1}{\log n} \left( \sum_{k|n} a_k b_{n/k} \right) n^{-s}.
\]

**Lemma 9** If $T_g$ is bounded on $\mathcal{H}_2^w$, then $g$ is in $X_w$, and the operator norms satisfy
\[
\| T_g \|_{L(H_2^w)} \leq \| T_g \|_{L(\mathcal{H}_2^w)}.
\]

**Proof** If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is in $\mathcal{H}_w^2$, the function $\tilde{f}(s) = \sum_{n=1}^{\infty} a_n w_n^{-1/2} n^{-s}$ is in $\mathcal{H}_2^w$ and $\| f \|_{\mathcal{H}_2^w} = \| \tilde{f} \|_{\mathcal{H}_2^w}$. Since $w_k \leq w_{kl}$ for any integers $k, l$, the Lemma is proven by the inequality
\[
\| T_g f \|_{\mathcal{H}_2^w} \leq \sum_{n=2}^{\infty} (\log n)^{-2} \left( \sum_{k|n, k < n} w_k^{-1/2} a_k b_{n/k} \right)^2 = \| T_g \tilde{f} \|_{\mathcal{H}_2^w}^2.
\]

We will also use the sufficient condition proved in Theorem 2.3 in [13], stating that if $g$ is in $BMOA(\mathbb{C}_0) \cap \mathcal{D}$, then $T_g$ is bounded on $\mathcal{H}_2^w$, with
\[
\| T_g \|_{\mathcal{H}_2^w} \lesssim \| g \|_{BMOA(\mathbb{C}_0)}. \tag{4.6}
\]

**Proof of Theorem 1(b) and (c)** If $g$ is in $BMOA(\mathbb{C}_0)$, $T_g$ is bounded on $\mathcal{H}_2$, and (b) is a consequence of (4.6) and Lemma 9.

To prove (c), we use that $(T_g f)' = fg'$, and that $\mathcal{H}_w^2$ is embedded in $A_{i, \delta}(\mathbb{C}_1/2)$, with $\delta = \delta(w) > 0$. We set
\[
d\nu_g(s) = |g'(s)|^2 \left( \frac{\sigma - \frac{1}{2}}{\delta + \frac{1}{2}} \right)^{\delta+1} dV(s).
\]

Now formula (3.1), the boundedness of $T_g$ on $\mathcal{H}_w^2$ and Lemma 1 induce that
\[
\int_{\mathbb{C}_1/2} |f(s)|^2 d\nu_g(s) \lesssim \| T_g f \|_{A_{i, \delta}(\mathbb{C}_1/2)}^2 \leq c(w) \| T_g f \|_{\mathcal{H}_w^2}^2 \leq c(w) \| T_g \|_{L(\mathcal{H}_w^2)}^2 \| f \|_{\mathcal{H}_w^2}^2.
\]

Thus, $\nu_g$ is a Carleson measure for $\mathcal{H}_w^2$ and $\| \nu_g \|_{CM(\mathcal{H}_w^2)} \lesssim \| T_g \|_{L(\mathcal{H}_w^2)}^2$. By Lemma 6, $\nu_g$ is also a Carleson measure for $A_{i, \delta}(\mathbb{C}_1/2)$ and
\[
\| \nu_g \|_{CM(A_{i, \delta}(\mathbb{C}_1/2))} \lesssim \| T_g \|_{L(\mathcal{H}_w^2)}^2.
\]
We conclude by the characterization of the Bloch space given in Lemma 4.

We get a result which is in agreement with the situation for Hardy spaces [15], Bergman spaces [2] or the Hardy space of Dirichlet series $H^2$ [13], with the same proof.

**Corollary 1** If $g$ is in $X_w$, then $g$ is in $\cap_{0<p<\infty}H^p_w$, and there exists $c>0$, such that the function $e^{c|Bg|}$ is integrable on $\mathbb{D}^\infty$, with respect to $d\mu_w$.

## 5 Compactness

We now present a little oh version of Theorem 1.

If the symbol is a vector of the standard orthonormal basis of $H^2_w$, that is

$$g(s) = e_{w,n}(s) := w_n^{1/2} n^{-s},$$

the operator $T^*_g T_g$ is diagonal, and its eigenvalues

$$\lambda_{n,k}^2 = \frac{w_n w_k}{wn_k} \left( \frac{\log n}{\log n + \log k} \right)^2$$

tend to 0 as $k \to +\infty$. Thus $T_g$ is compact. It follows that every Dirichlet polynomial generates a compact Volterra operator on $H^2_w$.

### 5.1 Case when $B g$ depends on a finite number of variables

We approximate a symbol $g$ which is in Bloch$_0(\mathbb{C}_0) \cap D_d$ by a Dirichlet polynomial $P$ in the Bloch($\mathbb{C}_0$)-norm. From Theorem 1(a), $T_g$ is approximated in the operator norm by the compact operator $T_P$.

**Theorem 2** If $g$ is in $\text{Bloch}_0(\mathbb{C}_0) \cap D_d$, then $T_g$ is compact on $H^2_w$.

### 5.2 Sufficient/necessary conditions for compactness

In general, if the symbol $g(s) = \sum_{n \geq 2} b_n n^{-s}$ satisfies an inequality of the form

$$\|T_g\|^2_{L(H^2_w)} \leq \sum_{n \geq 2} |b_n|^2 W(n) < \infty,$$

we approximate $T_g$ in the operator norm by the compact operator $T_{SN g}$. Therefore, $T_g$ is compact (see [13]).

The little oh version of Theorem 1 is related to the properties of $VMOA(\mathbb{C}_0) \cap D$, and with the concept of vanishing Carleson measures.

**Theorem 3** Let $g$ be in $D$.

1. If $g$ is in $VMOA(\mathbb{C}_0) \cap D$, then $T_g$ is compact on $H^2_w$.
2. If $T_g$ is compact on $H^2_w$, then $g$ is in $\text{Bloch}_0(\mathbb{C}_{1/2})$. 

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**Proof** In order to prove (1), we use that \( \text{VMOA}(\mathbb{C}_0) \cap D \) is the closure of Dirichlet polynomials in \( \text{BMOA}(\mathbb{C}_0) \) (see [13]), and that, from Theorem 1, we have \( \left\| T_g \right\|_{L(H_w^2)} \lesssim \left\| g \right\|_{\text{BMOA}(\mathbb{C}_0)}. \)

Recall that \( H_w^2 \) is embedded in \( A_{i,\delta}(\mathbb{C}_{1/2}), \delta = \delta(w) \) being defined in (2.5). Assume that \( T_g \) is compact on \( H_w^2 \), and consider the measure

\[
d\nu_g(s) = \left| g'(s) \right|^2 \frac{(\sigma - \frac{1}{2})^{\delta+1}}{|s + \frac{1}{2}|^{2(\delta+1)}} dV(s).
\]

Let \( (f_k)_k \) be a weakly compact sequence in \( H_w^2 \). Formula (3.1), and Lemma 1 imply that

\[
\int_{\mathbb{C}_{1/2}} |f_k(s)|^2 d\nu_g(s) \lesssim \left\| T_g f_k \right\|_{A_{i,\delta}(\mathbb{C}_{1/2})}^2 \lesssim \left\| T_g f_k \right\|_{H_w^2}^2.
\]

By the compactness of \( T_g \), \( \nu_g \) is a vanishing Carleson measure for \( A_{i,\delta}(\mathbb{C}_{1/2}) \), with

\[
\lim_{k \to \infty} \int_{\mathbb{C}_{1/2}} |f_k(s)|^2 d\nu_g(s) = 0.
\]

Now, \( g \) is in \( \text{Bloch}_0(\mathbb{C}_{1/2}) \), by the characterization of vanishing Carleson measures (Lemma 5).

\( \Box \)

### 6 Membership in Schatten classes

Let \( g \) be a non constant symbol. As in the case of \( H^2 \), the Volterra operator \( T_g \) on \( H_w^2 \) does not belong to any Schatten class.

**Theorem 4** If the Dirichlet series \( g(s) = \sum_{n \geq 2} b_n n^{-s} \) is not 0, then \( T_g : H_w^2 \to H_w^2 \) is not in the Schatten class \( S_p \), for any \( 0 < p < \infty \).

**Proof** Recall that \( (e_{w,n})_n \) is an orthonormal basis of \( H_w^2 \). We follow the reasoning of Theorem 7.2 [13]. Using that \( w_N \lesssim w_N w_n \), we see that, for \( N \geq n \),

\[
\left\| T_g e_{w,n} \right\|_{H_w^2}^2 = \sum_{k=2}^{+\infty} \frac{|b_k|^2 (\log k)^2}{(\log (kn))^2} \frac{w_n}{w_{kn}} \geq \frac{|b_N|^2 (\log N)^2}{(2 \log n)^2} \frac{w_n}{w_N} \gtrsim \frac{|b_N|^2 (\log N)^2}{(2 \log n)^2} \frac{1}{w_N}.
\]

For \( p \geq 2 \), we obtain

\[
\left\| T_g \right\|_{S_p}^p \geq \sum_{n=N}^{+\infty} \left\| T_g e_{w,n} \right\|_{H_w^2}^p = +\infty.
\]

Therefore \( T_g \) is not in \( S_p \) for \( p \geq 2 \), neither for \( 0 < p < \infty \). \( \Box \)
7 Examples

In this section, we study the boundedness of $T_g$ on $\mathcal{H}^2_w$, for specific symbols $g$. We consider fractional primitives of translates of the weighted Zeta function $Z_w$ and homogeneous symbols, which are the counterparts of the symbols presented in [13] in the $\mathcal{H}^2$ setting. The techniques of proof, as well as the results are similar to theirs, and we omit the details.

7.1 Fractional primitives of translates of $Z_w$

**Proposition 4** With the notation of (2.5), take $1/2 \leq a < 1$, $2\gamma > \delta(w) - 1$. If

$$g(s) = \sum_{n=2}^{\infty} w_n n^{-a} \frac{n^{-s}}{(\log n)^{\gamma+1} n^{-s}},$$

then $T_g$ is unbounded on $\mathcal{H}^2_w$.

**Proof** Abel summation and the Chebyshev estimate induce that $g$ is in $\mathcal{H}^2_w$. If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, and $g(s) = \sum_{n=2}^{\infty} \frac{b_n}{\log n} n^{-s}$, we set $A_n = \sum_{k|n} a_n/b_k$, so that

$$\|\tilde{T}_g f\|_{\mathcal{H}^2_w}^2 = \sum_{n=2}^{\infty} \frac{1}{(w_n \log n)^2} A_n^2.$$

We adapt the test functions of [13], and take $f_J(s) = \prod_{j=1}^{J} \left(1 + \frac{1}{2} \frac{w_2}{p_j - a}\right)$, for $J \geq 1$. By construction, it satisfies $\|f_J\|_{\mathcal{H}^2_w} \asymp 2^{1/2}$. Now, for $\mathcal{J}$ a non-empty subset of $\{1, \ldots, J\}$, we set $n_{\mathcal{J}} = \prod_{j \in \mathcal{J}} p_j$, and observe that

$$A_{n_{\mathcal{J}}} \asymp \sum_{1 \leq k \leq \mathcal{J}, \{p_{j_1}, \ldots, p_{j_k}\} \subset \mathcal{J}} w_2^{\frac{\mathcal{J}}{2} \frac{1 - k}{}} \left[\log (p_{j_1} \cdots p_{j_k})\right]^{-\gamma} w_2^k (p_{j_1} \cdots p_{j_k})^{-a} + w_2^{\frac{1}{2}}.$$

First assume that $\gamma \geq 0$. From the prime number Theorem, we obtain that

$$A_{n_{\mathcal{J}}} \asymp w_2^{\frac{\mathcal{J}}{2}} \left[J \log J\right]^{-\gamma} \left[1 + \sum_{1 \leq k \leq \mathcal{J}, \{p_{j_1}, \ldots, p_{j_k}\} \subset \mathcal{J}} w_2^{k/2} (p_{j_1} \cdots p_{j_k})^{-a}\right].$$

Therefore, it follows again from the prime number Theorem that

$$\|\tilde{T}_g f_J\|_{\mathcal{H}^2_w}^2 \asymp \sum_{\mathcal{J} \subset \{1, \ldots, J\}, |\mathcal{J}| \geq J/2} \frac{1}{(\log n_{\mathcal{J}})^2} \left[J \log J\right]^{-2\gamma} \prod_{j \in \mathcal{J}} \left(1 + \frac{w_2^{1/2} p_j - a}{2}\right)^2 \asymp 2^{J-1} \left[J \log J\right]^{-2\gamma} \min_{|\mathcal{J}| \geq J/2} \frac{1}{(\log n_{\mathcal{J}})^2} \prod_{j \in \mathcal{J}} \left(1 + \frac{w_2^{1/2} p_j - a}{2}\right)^2.$$
\[ \geq e^{cJ^{1-a}(\log J)^{-a}} \| f \|_{\mathcal{H}^2_w}^2, \]

for some constant \( c > 0 \), and \( T_g \) is unbounded. The case when \( \gamma < 0 \) is similar. \( \square \)

### 7.2 Homogeneous symbols

An \( m \)-homogeneous Dirichlet series has the form

\[ g(s) = \sum_{\Omega(n) = m} b_n n^{-s}. \]

We extend Theorem 4.2 in [13] to the spaces \( \mathcal{H}^2_w \).

**Proposition 5** There exist weights \( W_m(n) \) such that for \( g(s) = \sum_{\Omega(n) = m} b_n n^{-s} \),

\[ \| T_g \|_{\mathcal{L}(\mathcal{H}^2_w)} \leq \left( \sum_{\Omega(n) = m} |b_n|^2 W_m(n) \right)^{1/2}. \tag{7.1} \]

Precisely, there exist absolute constants \( C_m \) for which

\[ W_m(n) = \begin{cases} C_1 \log n & \text{for } m = 1, \\ C_2 \log \log n & \text{for } m = 2, \\ C_m \frac{n^{m-3}}{\log n^{m-2}} & \text{for } m \geq 3. \end{cases} \]

Moreover, when \( m = 2 \), \( \log_2 n \) cannot be replaced in (7.1) by \( (\log_2 n)^{1+\varepsilon} \) for any \( \varepsilon > 0 \).

**Proof** If a linear symbol \( (m = 1) \) \( g(s) = \sum_{p \in \mathbb{P}} b_p p^{-s} \) belongs to \( \mathcal{H}^2 \), we observe that \( \| g \|_{\mathcal{H}^2}^2 = 2^\beta \| g \|_{B_p^2}^2 = (\beta + 1) \| g \|_{A_p^2}^2 \). Hence, it follows from Theorem 4.1 in [13] and Lemma 9 that \( T_g \) is bounded on \( \mathcal{H}^2_w \) and \( \| T_g \|_{\mathcal{L}(\mathcal{H}^2_w)} \leq \| T_g \|_{\mathcal{L}(\mathcal{H}^2)} \). One can choose \( C_1 = \max \left( (\beta + 1)^{-1}, 2^{-\beta} \right) \).

(7.1) is a consequence of Theorem 4.2 in [13] and Lemma 9. We now prove the sharpness of the factor \( \log_2 n \). We assume that for some \( \varepsilon > 0 \), every 2-homogeneous Dirichlet series \( g \) satisfies

\[ \| T_g \|_{\mathcal{L}(\mathcal{H}^2_w)} \leq C_2 \left( \sum_{\Omega(n) = m} |b_n|^2 \frac{\log n}{(\log_2 n)^{1+\varepsilon}} \right)^{1/2}. \tag{7.2} \]
For $x$ a large real number, and $q \sim e^x$ a prime number, the symbol considered in [13] is

$$g_x(s) = \sum_{x/2 < p \leq x} \frac{(\log_2(pq))^{1+\varepsilon/2}}{p} (pq)^{-s}.$$  

We take as test functions

$$f_x(s) = \sum_{n=1}^{+\infty} a_n n^{-s} = \prod_{x/2 < p \leq x} \left(1 + \frac{w^{1/2}}{p} p^{-s}\right).$$

If $S_x$ denotes the set of square-free integers generated by the primes $x/2 < p \leq x$, we have \( \| f_x \|^2_{\mathcal{H}_w^2} \asymp |S_x| = 2^{N(x)} \), where $N(x) := \pi(x) - \pi(x/2)$. Now,

$$\left\| T_{g_x} f_x \right\|^2_{\mathcal{H}_w^2} \gtrsim \frac{1}{|S_x|} \sum_{n \in S_x} w^{-1}_n (\log(nq))^{-2} \left| \sum_{pq|nq} \log(pq) \frac{(\log_2(pq))^{1+\varepsilon/2}}{p} a_{n/p} \right|^2.$$  

If $n \in S_x$, and $p | n$, we have $a_{n/p} = w^{1/2}_2 \omega(n)^{-1}$, $w_n = w^{\omega(n)}_2$, and $w_{nq} = w_n w_q$. Thus,

$$\left\| T_{g_x} f_x \right\|^2_{\mathcal{H}_w^2} \gtrsim \frac{1}{|S_x|} \frac{(\log x)^{2+\varepsilon}}{x^2} \sum_{n \in S_x} \omega(n)^2.$$  

Now $\sum_{n \in S_x} \omega(n)^2 = \sum_{k=1}^{N(x)} \binom{N(x)}{k} k^2 \asymp N(x)^2 2^{N(x)}$, and (7.2) does not hold, due to

$$\left\| T_{g_x} f_x \right\|^2_{\mathcal{H}_w^2} \gtrsim (\log x)^{\varepsilon}.$$  

\[\square\]

We will exhibit a homogeneous symbol $g$ which is in $\mathcal{H}_w^2 \cap \text{Bloch}_0(\mathbb{C}_{1/2})$, but not in $\mathcal{X}_w$. In fact, we observe that $g$ is in every $\mathcal{H}_w^p$.

**Lemma 10** If $g$ is an $m$-homogeneous Dirichlet series in $\mathcal{H}_w^2$, then $g$ is in $\cap_{0 < p < \infty} \mathcal{H}_w^p$ and, for any $0 < p < \infty$, there exists $c = c(m, p)$ such that

$$\| g \|_{\mathcal{H}_w^p} \leq c \| g \|_{\mathcal{H}_w^2}. \quad (7.3)$$

**Proof** It is enough to consider the case $p \geq 2$. We first prove the inequality for $p = 2^k$, $k$ being a positive integer, by an induction argument.

Obviously, it holds for $k = 1$.  

\[\square\]
Our proof is inspired of Lemma 8 in [27]. For any integer $m$, there exists a constant $C(m)$, such that max $(w_n, d(n)) \leq C(m)$, whenever $\Omega(n) = m$.

If $f(s) = \sum_n a_n n^{-s}$ is $m$-homogeneous, then $f^2(s) = \sum_n b_n n^{-s}$ is $2m$-homogeneous, and $|b_n|^2 \leq d(n) \sum_{k|n} |a_k|^2 |a_{n/k}|^2$. Since $w_n \geq \sqrt{w_k \sqrt{w_n/k}}$,

$$\| f \|^4_{\mathcal{H}^2_w} = \left\| f^2 \right\|^2_{\mathcal{H}^2_w} \leq \sum_{\Omega(n) = 2m} d(n) w_n^{-1} \left( \sum_{k|n} |a_k|^2 |a_{n/k}|^2 \right) \leq C(2m) \sum_{\Omega(n) = 2m} \left( \sum_{k|n} \frac{|a_k|^2}{\sqrt{w_k}} \frac{|a_{n/k}|^2}{\sqrt{w_{n/k}}} \right) \leq C(2m) \left( \sum_k \frac{|a_k|^2}{\sqrt{w_k}} \right)^2 \leq C(2m) C(m) \| f \|^4_{\mathcal{H}^2_w}.$$ 

Now, suppose that, for some $k$, an $m$-homogeneous Dirichlet series $h$ satisfies

$$\| h \|_{\mathcal{H}^k_w} \leq K(m, k) \| h \|_{\mathcal{H}^k_w}^2$$ 

for any $m$.

We obtain that

$$\| f \|_{\mathcal{H}^{k+1}_w} = \left\| f^2 \right\|_{\mathcal{H}^k_w} \leq K(2m, k) \left\| f^2 \right\|_{\mathcal{H}^k_w} = K(2m, k) \| f \|_{\mathcal{H}^k_w} \leq K(2m, k) \left[ C(2m) C(m) \| f \|^4_{\mathcal{H}^2_w} \right]^{2^{k+1}-1}.$$ 

For general $p$, (7.3) is a consequence of Hölder’s inequality. $\square$

For our construction, we need two technical Lemmas.

**Lemma 11** Assume that $0 < \delta < 1$ and $0 < \eta$. For $j = 1..3$, we set $h_j(s) = \sum_{p \geq 3} \alpha_{j, p} p^{-s}$, where

$$\alpha_{1, p} = (\log_2 p)^{-\delta}, \quad \alpha_{2, p} = \log_2 p, \quad \alpha_{3, p} = \log p (\log_2 p)^{-\eta}.$$ 

For a real number $\sigma > 1$, set $\sigma' := \frac{1}{\sigma - 1}$. Then we have

$$h_1(\sigma) \propto (\log \sigma')^{1-\delta}; \quad h_2(\sigma) \asymp \log_2 (\sigma'); \quad h_3(\sigma) \asymp \sigma' (\log \sigma')^{-\eta}, \quad \text{as } \sigma \to 1^+.$$ 

(7.4)

**Proof** These asymptotics will follow from computations inspired by [4,20]. Recall that

$$A_1(t) := \sum_{3 \leq p \leq t} \frac{1}{p} = \log_2 t + O(1).$$ 

(7.5)
Setting \( f_1(t) = \frac{t^{-(\sigma-1)}}{(\log_2 t)^\delta} \), we have

\[
h_1(\sigma) = \sum_{p \geq 3} \frac{p^{-(\sigma-1)}}{p (\log_2 p)^\delta} = - \int_3^{+\infty} A_1(t) f_1(t) dt + O(1)
\]

\[
= (\sigma - 1) \int_3^{+\infty} (\log_2 t)^{1-\delta} t^{-\sigma} dt
\]

\[
= (\sigma - 1) \left( \int_{\log 3}^{\sigma'} + \int_{\sigma'}^{+\infty} \right) (\log x)^{1-\delta} e^{-(\sigma-1)x} dx.
\]

Using integration by parts (for the first integral), and a change of variable (for the second one), we obtain

\[
h_1(\sigma) \asymp (\sigma - 1) \int_{\log 3}^{\sigma'} (\log x)^{1-\delta} dx + \int_{1}^{+\infty} (\log y + \log \sigma')^{1-\delta} e^{-y} dy
\]

\[
\asymp (\sigma - 1) \left[ x (\log x)^{1-\delta} \right]_{x=\sigma'} + \int_{1}^{+\infty} \left[ (\log y)^{1-\delta} + (\log \sigma')^{1-\delta} \right] e^{-y} dy
\]

\[
\asymp (\log \sigma')^{1-\delta}.
\]

The functions \( h_2 \) and \( h_3 \) are handled similarly. For \( x \geq 3 \), summation by parts and (7.5) induce that,

\[
A_2(x) := \sum_{3 \leq p \leq x} \frac{1}{p \log_2 p} = \frac{A_1(x)}{\log_2 x} + \int_3^x \frac{A_1(t)}{t \log t (\log_2 t)^2} dt + O(1) \asymp \log_3 x.
\]

Set \( f_2(t) := t^{-(\sigma-1)} \). Then,

\[
h_2(\sigma) \asymp - \int_3^{+\infty} A_2(t) f_2(t) dt + O(1) \asymp (\sigma - 1) \int_3^{+\infty} (\log_3 t)t^{-\sigma} dt
\]

\[
= (\sigma - 1) \left( \int_{\log 3}^{e\sigma'} + \int_{e\sigma'}^{+\infty} \right) (\log_2 x)e^{-(\sigma-1)x} dx.
\]

Now

\[
(\sigma - 1) \int_{\log 3}^{e\sigma'} (\log_2 x)e^{-(\sigma-1)x} dx \asymp (\sigma - 1) \int_{\log 3}^{e\sigma'} (\log_2 x)dx
\]

\[
\leq (\sigma - 1)e\sigma' (\log_2 (e\sigma')) \lesssim \log_2 \sigma'.
\]

We perform a change of variable in the integral over \([e\sigma', +\infty)\).

\[
I_{2,2} := (\sigma - 1) \int_{e\sigma'}^{+\infty} (\log_2 x)e^{-(\sigma-1)x} dx = \int_{e}^{+\infty} \left[ \log (\log y + \log \sigma') \right] e^{-y} dy
\]
\[ \geq (\log_2 \sigma') \int_e^{+\infty} e^{-y} dy \gtrsim \log_2 \sigma'. \]

Since \( \log(a + b) \leq \log a \log b + 1 \), for \( a \geq e \) and \( b \geq e \), we obtain

\[ I_{2,2} \leq \int_e^{+\infty} \left[ (\log_2 y)(\log_2 \sigma') + 1 \right] e^{-y} dy \lesssim \log_2 \sigma', \]

and \( I_{2,2} \asymp \log_2 \sigma' \). It follows that \( h_2(\sigma) \asymp \log_2 \sigma' \).

We now focus on \( h_3 \). By Mertens’ first Theorem, \( A_3(x) := \sum_{3 \leq p \leq x} \frac{\log p}{p} = \log x + O(1) \), and putting \( f_3(t) := t^{-(\sigma-1)} (\log_2 t)^{-\eta} \), we see that

\[ h_3(\sigma) = -\int_3^{+\infty} A_3(t) f_3'(t) dt + O(1) \]

\[ \asymp (\sigma - 1) \int_3^{+\infty} (\log t) t^{-\sigma} (\log_2 t)^{-\eta} dt \]

\[ \asymp (\sigma - 1) \left( \int_{\log 3}^{\sigma'} + \int_{\sigma'}^{+\infty} \right) xe^{-(\sigma-1)x} (\log x)^{-\eta} dx. \]

Integration by parts gives that

\[ I_{3,1} := (\sigma - 1) \int_{\log 3}^{\sigma'} xe^{-(\sigma-1)x} (\log x)^{-\eta} dx \]

\[ \asymp (\sigma - 1) \int_{\log 3}^{\sigma'} x (\log x)^{-\eta} dx \asymp \sigma' (\log \sigma')^{-\eta}. \]

Next, (7.4) is a consequence of

\[ I_{3,2} := (\sigma - 1) \int_{\sigma'}^{+\infty} xe^{-(\sigma-1)x} (\log x)^{-\eta} dx \]

\[ = \frac{1}{\sigma - 1} \int_1^{+\infty} ye^{-y} (\log y + \log \sigma')^{-\eta} dy \]

\[ \lesssim \sigma' \int_1^{+\infty} ye^{-y} (\log \sigma')^{-\eta} dy. \]

\[ \square \]

**Lemma 12** If \( 2\eta > 1 \) and \( \delta + \eta > 1 \), we have

\[ S := \sum_{p_1, p_2, p_3 \in \mathbb{P}, p_j \geq 3} \frac{1}{p_1 p_2 p_3 (\log_2 p_1)^{2\delta} (\log_2 p_2)^2} \times \]

\[ \frac{(\log p_3)^2}{(\log_2 p_3)^{2\eta} (\log(p_1 p_2 p_3))^2} < \infty. \]
Proof For $p_1, p_2 \geq 3$, we set $L := \log(p_1 p_2)$ and $S_3(p_1, p_2) := \sum_{p_3} \frac{1}{p_3 (\log p_3)^2 (\log p_3 + L)^2}$. Then, we have

$$S = \sum_{p_1, p_2, p_3} \frac{1}{p_1 p_2 (\log p_1)^2 (\log p_2)^2} S_3(p_1, p_2).$$

Under the condition $2\eta > 1$, the sum $S_3(p_1, p_2)$ converges, and

$$S_3(p_1, p_2) = -\int_3^{+\infty} A_1(t) \frac{d}{dt} \left[ \frac{(\log t)^2}{(\log_2 t)^{2\eta} (\log t + L)^2} \right] dt + \frac{O(1)}{L^2} \lesssim \frac{O(1)}{L^2} + \int_3^{+\infty} \frac{\log t}{t (\log_2 t)^{2\eta} (\log t + L)^2} dt \lesssim \frac{O(1)}{L^2} + \left( \int_{\log 3}^L + \int_{L}^{+\infty} \right) \frac{x dx}{(\log x)^{2\eta} (x + L)^2}.$$

Integration by parts gives

$$I_{3,1} := \int_{\log 3}^L \frac{x dx}{(\log x)^{2\eta} (x + L)^2} \times \frac{1}{L^2} \int_{\log 3}^L \frac{x dx}{(\log x)^{2\eta}} \asymp (\log L)^{-2\eta}.$$

We handle the second integral via a change of variable:

$$I_{3,2} := \int_L^{+\infty} \frac{x dx}{(\log x)^{2\eta} (x + L)^2} = \left( \int_1^L + \int_L^{+\infty} \right) \frac{y dy}{(1 + y)^2 (\log y + \log L)^{2\eta}} \lesssim \frac{1}{(\log L)^{2\eta}} \int_1^L \frac{dy}{y} + \int_L^{+\infty} \frac{dy}{y (\log y)^{2\eta}} \asymp (\log L)^{1-2\eta}.$$

Therefore

$$S_3(p_1, p_2) \lesssim (\log L)^{1-2\eta}, \ L = \log(p_1 p_2).$$

We next put $M = \log p_1$, and deal with

$$S_2(p_1) := \sum_{p_2} \frac{1}{p_2 (\log_2 p_2)^2} S_3(p_1, p_2) \lesssim \sum_{p} \frac{1}{p (\log_2 p)^2 [\log (\log p + M)]^{2\eta-1}}.$$

With the notation $f_2(t) := \left[ (\log_2 t)^2 [\log (\log t + M)]^{2\eta-1} \right]^{-1}$, we obtain that

$$S_2(p_1) = \frac{O(1)}{(\log M)^{2\eta-1}} - \int_3^{+\infty} A_1(t) f_2'(t) dt \lesssim \frac{O(1)}{(\log M)^{2\eta-1}} + I_{2,1} + I_{2,2}.\)
where
\[
I_{2,1} := \int_{3}^{+\infty} \frac{dt}{t \log t \left( \log_2 t \right)^2 \left[ \log (\log t + M) \right]^{2\eta-1}};
\]
\[
I_{2,2} := \int_{3}^{+\infty} \frac{dt}{t \left( \log_2 t \right) \left( \log t + M \right) \left[ \log (\log t + M) \right]^{2\eta}}.
\]

We derive
\[
I_{2,1} = \left( \int_{\log 3}^{M} + \int_{M}^{+\infty} \right) \frac{dx}{x \left( \log x \right)^2 \left[ \log (x + M) \right]^{2\eta-1}}
\approx \frac{1}{\left( \log M \right)^{2\eta-1}} \int_{\log 3}^{M} \frac{dx}{x \left( \log x \right)^2} + (\log M)^{1-2\eta} \int_{M}^{+\infty} \frac{dx}{x \left( \log x \right)^2} \lesssim (\log M)^{1-2\eta}.
\]

The second integral is estimated in the same way:
\[
I_{2,2} = \left( \int_{\log 3}^{M} + \int_{M}^{+\infty} \right) \frac{dx}{(x + M) \left( \log x \right) \left[ \log (x + M) \right]^{2\eta}}
\approx \frac{1}{M \left( \log M \right)^{2\eta}} \int_{\log 3}^{M} \frac{dx}{\log x} + \frac{1}{\left( \log M \right)^{2\eta-1}} \int_{M}^{+\infty} \frac{dx}{x \left( \log x \right)^2}
\times \frac{1}{M \left( \log M \right)^{2\eta}} \left( \int_{\log 3}^{M} \frac{dx}{x \left( \log x \right)} + \int_{\log 3}^{M} \frac{x^2 \left( \log x \right)^{-2} \left( \log x \right)^{-1}}{x^2} dx \right)
\approx \frac{1}{\left( \log M \right)^{2\eta}} \times \frac{1}{\left( \log M \right)^{2\eta}}.
\]

Therefore, we have
\[
S_2(p_1) \lesssim \frac{1}{\left( \log M \right)^{2\eta-1}}, \ M = \log p_1.
\]

It follows that
\[
S \lesssim \sum_{p_1} \frac{1}{p_1 \left( \log_2 p_1 \right)^{2\eta}} S_2(p_1) \lesssim \sum_{p \geq 3} \frac{1}{p \left( \log_2 p \right)^{\epsilon}}, \ \epsilon := 2\delta + 2\eta - 1.
\]

Again, partial summation gives that when \( \epsilon > 1 \),
\[
\sum_{3 \leq p} \frac{1}{p (\log p)^\varepsilon} \ll \varepsilon \int_{3}^{+\infty} \frac{\log t}{t (\log t) (\log_2 t)^{\varepsilon + 1}} dt < \infty.
\]

Proposition 6  There exists a 3-homogeneous function \( g \) which is in \( \bigcap_{0 < p < \infty} \mathcal{H}^p_w \) \( \cap \) \( \text{Bloch}_0(\mathbb{C}_{1/2}) \), such that \( T_g \) is unbounded on \( \mathcal{H}^2_w \).

Proof Using Lemma 11, we see that, if \( g' = -(h_1 h_2 h_3)_{1/2} \), \( g' \) is convergent on \( \mathbb{C}_{1/2} \), and its estimate near the line \( \Re s = \frac{1}{2} \) is determined by the behavior of the functions \( h_j \) near the line \( \Re s = 1 \). Then \( g \) is in \( \text{Bloch}_0(\mathbb{C}_{1/2}) \), because of

\[
|g'(\sigma)| \asymp \frac{1}{\sigma - \frac{1}{2}} \left( \frac{\log 1/\sigma - 1/2}{\sigma - 1/2} \right)^{1-\delta-\eta} \left( \log_2 \frac{1}{\sigma - 1/2} \right), \quad \text{as } \sigma \to 1/2^+.
\]

On another hand, the 3-homogeneous function

\[
g(s) = \sum_n b_n n^{-s} = \sum_{p_1, p_2, p_3} \frac{\alpha_{1,p_1} \alpha_{2,p_2} \alpha_{3,p_3}}{\log(p_1 p_2 p_3)} (p_1 p_2 p_3)^{-s}
\]

is in \( \mathcal{H}^2_w \) by Lemma 12, since \( \|g\|^2_{\mathcal{H}^2_w} = \sum_n |b_n|^2 \times \sum_n |b_n|^2 \times S < \infty \).

By Lemma 10, \( g \) is in \( \bigcap_{0 < p < \infty} \mathcal{H}^p_w \).

It remains to prove that \( T_g \) is unbounded on \( \mathcal{H}^2_w \). We again choose as test functions (cf the proof of Proposition 5)

\[
f_x(s) := \prod_{\frac{5}{2} < p \leq x} \left( 1 + w_2^{1/2} p^{-s} \right) = \sum_{n \geq 1} a_n n^{-s}.
\]

\( S_x \) is the set of square free integers generated by \( \frac{5}{2} < p \leq x \). Set \( V_x = \{ n \in S_x, \omega(n) \geq \frac{N(x)}{2} \} \).

For \( n \in V_x \), set

\[
A_n := \sum_{p_1 p_2 p_3 | n} b_{p_1 p_2 p_3} (\log(p_1 p_2 p_3)) a_n^{p_1 p_2 p_3}
\]

The coefficients in \( A_n \) satisfy

\[
b_{p_1 p_2 p_3} (\log(p_1 p_2 p_3)) \gtrsim \frac{\log x}{x^{3/2} (\log_2 x)^{\eta+\delta+1}}.
\]

Since \( \|f_x\|^2_{\mathcal{H}^2_w} \asymp |V_x| \), we see that

\[
\|T_g f_x\|^2_{\mathcal{H}^2_w} \geq \sum_{n \in V_x} w_n^{-1} (\log n)^{-2} A_n^2
\]
\[
\sum_{n \in V_x} w_2^{-\omega(n)} (\omega(n) \log x)^{-2} \times \\
\left[ \frac{\log x}{x^{3/2} \left( \log_2 x \right)^{\eta+\delta+1}} \left( \frac{\omega(n)}{3} \right) \left( w_2^{1/2} \right)^{\omega(n)-3} \right]^2 \\
\geq \| f_x \|_{\mathcal{H}_w^2}^2 \left( \frac{x}{\log x} \right)^4 \frac{1}{x^3 \left( \log_2 x \right)^{2(\delta+\eta+1)}},
\]
and the proof is complete. \(\square\)

8 Comparison of \(\mathcal{X}_w\) with other spaces of Dirichlet series

The previous results enable us to derive some inclusions involving \(\mathcal{X}_w\).

In the context of the unit disk, the space of symbols \(g\) for which the Volterra operator \(J_g\) (1.3) is bounded on \(A^2_\alpha(D)\) is Bloch \((D)\). It coincides with the space of holomorphic \(g\) such that the Hankel form (1.5) is bounded, and with the dual space of \(A^1_\alpha(D)\).

We shall study the counterparts of these facts for \(\mathcal{X}_w\).

8.1 Bounded Hankel forms

The Hilbert space \(\mathcal{H}_w^2\) is equipped with the inner product \(\langle ., . \rangle_{\mathcal{H}_w^2}\). The Hankel form of symbol \(g \in \mathcal{D}\) is defined on \(\mathcal{H}_w^2\) by

\[
H_g(fh) := \langle fh, g \rangle_{\mathcal{H}_w^2}.
\]

We say that \(H_g\) is bounded on \(\mathcal{H}_w^2 \times \mathcal{H}_w^2\) if there is a constant \(C\) such that

\[
|H_g(fh)| \leq C \| f \|_{\mathcal{H}_w^2} \| h \|_{\mathcal{H}_w^2} \quad \text{for } f, h \in \mathcal{H}_w^2.
\]

The weak product \(\mathcal{H}_w^2 \odot \mathcal{H}_w^2\) is the Banach space defined as the closure of all finite sums \(F = \sum_k f_k h_k\), where \(f_k, h_k \in \mathcal{H}_w^2\), under the norm

\[
\| F \|_{\mathcal{H}_w^2 \odot \mathcal{H}_w^2} := \inf \sum_k \| f_k \|_{\mathcal{H}_w^2} \| h_k \|_{\mathcal{H}_w^2}.
\]

Here the infimum is taken over all finite representations of \(F\) as \(F = \sum_k f_k h_k\).

Let \(\mathcal{Y}\) be a Banach space of Dirichlet series in which the space of Dirichlet polynomials \(\mathcal{P}\) is dense. We say that a Dirichlet series \(\phi\) is in the dual space \(\mathcal{Y}^*\) if the linear functional induced by \(\phi\) via the \(\mathcal{H}_w^2\)-pairing is bounded. In other words, \(\phi \in \mathcal{Y}^*\) if and only if

\[
v_\phi(f) = \langle f, \phi \rangle_{\mathcal{H}_w^2}, \quad f \in \mathcal{P},
\]
extends to a bounded functional on $\mathcal{Y}$.

From its definition, $H_g$ (8.1) is bounded on $\mathcal{H}^2_w$ if and only if $g \in (\mathcal{H}^2_w \odot \mathcal{H}^2_w)^\ast$. We aim to relate Hankel forms and Volterra operators. The primitive of $f \in \mathcal{D}$ with constant term 0 is denoted by

$$\partial^{-1}f(s) := -\int_s^{+\infty} f(u)du,$$

We observe that

$$H_g(fh) = f(\infty)h(\infty)g(\infty) + \left\langle \partial^{-1}(f'h),g \right\rangle_{\mathcal{H}^2_w} + \left\langle \partial^{-1}(fh'),g \right\rangle_{\mathcal{H}^2_w}.$$

The Banach space $\partial^{-1}(\partial \mathcal{H}^2_w \odot \mathcal{H}^2_w)$ is the completion of the space of Dirichlet series $F$ whose derivatives have a finite sum representation $F' = \sum_k f_k h'_k$, under the norm

$$\|F\|_{\partial^{-1}(\partial \mathcal{H}^2_w \odot \mathcal{H}^2_w)} := |F(\infty)| + \sum_k \|f_k\|_{\mathcal{H}^2_w} \|h_k\|_{\mathcal{H}^2_w},$$

where the infimum is taken over all finite representations. The product rule $(fg)' = f'g + fg'$ implies that

$$\mathcal{H}^2_w \odot \mathcal{H}^2_w \subset \partial^{-1}\left(\partial \mathcal{H}^2_w \odot \mathcal{H}^2_w\right),$$

and then

$$\left(\partial^{-1}\left(\partial \mathcal{H}^2_w \odot \mathcal{H}^2_w\right)\right)^\ast \subset \left(\mathcal{H}^2_w \odot \mathcal{H}^2_w\right)^\ast. \quad (8.2)$$

It has been shown in [14] that, for the space $\mathcal{H}^2_0 = \{ f \in \mathcal{H}^2 : f(\infty) = 0 \}$, the inclusion $\left(\partial^{-1}\left(\partial \mathcal{H}^2_0 \odot \mathcal{H}^2_0\right)\right)^\ast \subset \left(\mathcal{H}^2_0 \odot \mathcal{H}^2_0\right)^\ast$ is strict. As for the space $\mathcal{H}^2_w$, the question whether the inclusion is strict remains open.

The membership of $g$ in $\left(\partial^{-1}\left(\partial \mathcal{H}^2_w \odot \mathcal{H}^2_w\right)\right)^\ast$ is equivalent to the boundedness of the so-called “half-Hankel form”

$$(f, h) \mapsto \left\langle \partial^{-1}(f'h),g \right\rangle_{\mathcal{H}^2_w}. \quad (8.3)$$

As in the case of $\mathcal{H}^2$, the boundedness of $T_g$ implies the boundedness of $H_g$.

**Theorem 5** If the Volterra operator $T_g$ is bounded on $\mathcal{H}^2_w$, then the Hankel form $H_g$ is bounded.

**Proof** We adapt the proof of Corollary 6.2 in [13] to the framework of the polydisk $\mathbb{D}^\infty$. Polarizing the Littlewood–Paley formula (1), we get

$$(f, g)_{\mathcal{H}^2_w} = f(\infty)g(\infty) + 4 \int_{\mathbb{D}^\infty} \int_{\mathbb{R}} \int_{0}^{+\infty} f'_X(\sigma + it)g'_X(\sigma + it)\sigma d\sigma \frac{dt}{1 + t^2}d\mu_w(\chi).$$

$\square$ Springer
Then, we derive an expression of the half-Hankel form
\[
\langle a^{-1}(f'h), g \rangle_{H^2_w} = 4 \int \int_{\mathbb{D}^\infty} f'_X(\sigma + it) h_X(\sigma + it) g'_X(\sigma + it) \sigma d\sigma \int_0^{+\infty} \frac{dt}{1 + t^2} d\mu_w(\chi).
\]

Since $T_g$ is bounded on $H^2_w$, the Carleson measure characterization (4.1) induces that the form (8.3) is also bounded. Then $H_g$ is bounded on $H^2_w \odot H^2_w$ by the inclusion (8.2).

The previous Theorem states that we have
\[
X_w \subset (H^2_w \odot H^2_w)^*.
\]

The rest of the section is devoted to study the reverse inclusion.

Let $l^2_w$ denote the Hilbert space of complex sequences $a = (a_n)_n$ such that
\[
\|a\|_{l^2_w} := \left( \sum_{n \geq 1} \left| a_n \right|^2 w_n \right)^{1/2} < \infty.
\]

A sequence $(\rho_n)_n$ generates the following multiplicative Hankel form
\[
\rho(a, b) := \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} a_n b_m \frac{\rho_{mn}}{w_{mn}}, \quad a, b \in l^2_w. \tag{8.4}
\]

The symbol of the form is the Dirichlet series $g(s) = \sum_{n \geq 1} \overline{\rho_n} n^{-s}$. The form $\rho$ is said to be bounded if there is a constant $C$ such that
\[
|\rho(a, b)| \leq C \|a\|_{l^2_w} \|b\|_{l^2_w}.
\]

If $f$ and $h$ are Dirichlet series with coefficients $a$ and $b$, respectively, we have
\[
H_g(f'h) = \langle f'h, g \rangle_{H^2_w} = \rho(a, b).
\]

When the symbol $g$ has non negative coefficients, there is equivalence between the boundedness of $H_g$ and the half-Hankel form (8.3). In fact, the proof given for $H^2$ in [14] is valid for the spaces $H^2_w$.

**Proposition 7** Let $g(s) = \sum_{n \geq 1} \overline{\rho_n} n^{-s}$ be in $H^2_w$. The linear functional defined on $H^2_w$
\[
v_g(f) := \langle f, g \rangle_{H^2_w}.
\]
is bounded on $\partial^{-1}(\partial H^2_w \odot H^2_w)$ if and only if the weighted form

$$J_g(a, b) = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} a_m b_n \frac{\log n}{\log m + \log n} \frac{\rho_{mn}}{w_{mn}},$$

(where it is understood that for $m = n = 1$, the summand is 0) is bounded on $l^2_w \odot l^2_w$.

The norms are equivalent, i.e.

$$\|g\|_{(\partial^{-1}(\partial H^2_w \odot H^2_w))^*} \asymp \|v_g\| \asymp |\rho_1| + \|J_g\|.$$  

If $\rho_k \geq 0$ for all $k$, then $g \in (\partial^{-1}(\partial H^2_w \odot H^2_w))^*$ if and only if $g \in (H^2_w \odot H^2_w)^*$, with equivalent norms.

Proposition 7 will enable us to provide examples of symbols $g$ for which the Hankel form $H_g$ and the half-Hankel form (8.3) are bounded, but the Volterra operator $T_g$ is unbounded (see the proof of Proposition 9). This differs from the case of weighted Dirichlet spaces on the unit disk, for which the boundedness of $H_g$, the form (8.3) and $T_g$ are equivalent [1].

For convergence reasons, we will consider Hankel forms defined on Dirichlet series without constant term. So we will work on the space

$$\mathcal{H}^2_{w,0} = \left\{ f \in \mathcal{H}^2_w : f(+\infty) = 0 \right\}.$$

We have seen in Lemma 1 that the space $\mathcal{H}^2_w$ is embedded in a Bergman space of the form $A_{1,\delta}(\mathbb{C}_{1/2})$. For $\delta > 0$, it is thus natural to define the Hankel form

$$H^{(\delta)}(fh) := \int_{1/2}^{+\infty} f(\sigma)h(\sigma) \left( \sigma - \frac{1}{2} \right)^\delta d\sigma, \quad f, h \in \mathcal{H}^2_{w,0}. \quad (8.5)$$

Such multiplicative forms have been considered in the context of $\mathcal{H}^2$ [12] and on $A^2_{1,\delta}$ [9].

Since $K^{\mathcal{H}^2_w}(s, u) - 1 = \sum_{n \geq 2} w_n n^{-\eta} n^{-s}$ is the reproducing kernel of $\mathcal{H}^2_{w,0}$, we see that $H^{(\delta)}(fh) = (fh, \phi_\delta)_{\mathcal{H}^2_w}$, where

$$\phi_\delta(s) = \int_{1/2}^{+\infty} \left[ K^{\mathcal{H}^2_w}(s, \sigma) - 1 \right] \left( \sigma - \frac{1}{2} \right)^\delta d\sigma = \sum_{n=2}^{+\infty} \frac{w_n}{\sqrt{n} (\log n)^{\delta+1}} n^{-s}.$$  

**Proposition 8** Let $\delta > 0$ as in (2.5). Then $H^{(\delta)}$ defined in (8.5) is a multiplicative Hankel form with symbol $\phi_\delta$, which is bounded on $\mathcal{H}^2_{w,0} \odot \mathcal{H}^2_{w,0}$. 

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Proof The proof is similar to that of Theorem 13 in [9]. The Cauchy-Schwarz inequality ensures that

\[ |H^{(\delta)}(fh)| \leq \left( \int_{1/2}^{+\infty} |f(\sigma)|^2 \left( \sigma - \frac{1}{2} \right)^{\delta} d\sigma \right)^{1/2} \left( \int_{1/2}^{+\infty} |h(\sigma)|^2 \left( \sigma - \frac{1}{2} \right)^{\delta} d\sigma \right)^{1/2}. \]

If \( f(s) = \sum_{n=2}^{+\infty} a_n n^{-s} \), notice the pointwise estimate

\[ |f(\sigma)|^2 \leq \| f \|_{\mathcal{H}_w^2}^2 \left( \sum_{n=2}^{+\infty} w_n n^{-2\alpha} \right) \lesssim \| f \|_{\mathcal{H}_w^2}^2 \left( \log n \right)^{\delta+1} n^{-s}, \quad \text{for } \sigma \geq 1. \]

Since the bounded measure \( d\mu(\sigma + iT) = \chi \left( \frac{1}{2}, 1 \right) \left( \sigma - \frac{1}{2} \right)^{\delta} d\sigma \), supported on the real line, is Carleson for \( A_{i,\delta}(C_{1/2}) \), \( \mu \) is Carleson for \( \mathcal{H}_w^2 \) by Lemma 6, and

\[ \int_{1/2}^{+\infty} |f(\sigma)|^2 \left( \sigma - \frac{1}{2} \right)^{\delta} d\sigma = \left( \int_{1/2}^{1} + \int_{1}^{+\infty} \right) |f(\sigma)|^2 \left( \sigma - \frac{1}{2} \right)^{\delta} d\sigma \lesssim \| f \|_{\mathcal{H}_w^2}^2. \]

\[ \square \]

We next exhibit symbols giving rise to bounded Hankel forms and bounded half-Hankel forms, though the associated Volterra operator is unbounded.

Proposition 9 We have the strict inclusions

\[ \mathcal{X}(\mathcal{H}_{w,0}^2) \subsetneq \left( \mathcal{H}_{w,0}^2 \otimes \mathcal{H}_{w,0}^2 \right)^*; \]

\[ \mathcal{X}_w \subsetneq \left( \mathcal{H}_w^2 \otimes \mathcal{H}_w^2 \right)^*. \]

Proof It just remains to check the strictness of the inclusions. For the exponent \( \delta = \delta(w) \) and \( \frac{1}{2} \leq a < 1 \), consider the symbol in \( \mathcal{H}_{w,0}^2 \)

\[ g(s) = \sum_{n=2}^{+\infty} \frac{w_n}{n^a (\log n)^{\delta+1}} n^{-s}. \]

From Proposition 8 and the fact that the coefficients are positive, \( g \) is in \( \left( \mathcal{H}_{w,0}^2 \otimes \mathcal{H}_{w,0}^2 \right)^* \) for any \( \frac{1}{2} \leq a < 1 \). In fact, the half Hankel form corresponding to \( g \) is bounded. We have seen in Proposition 4 that \( T_g \) is not bounded on \( \mathcal{H}_w^2 \). Since \( T_g \mathbf{1} = g \), \( g \) does not belong to \( \mathcal{X}(\mathcal{H}_{w,0}^2) \).

In order to prove that \( g \in \left( \mathcal{H}_w^2 \otimes \mathcal{H}_w^2 \right)^* \), we consider the associated multiplicative form \( \rho \ (8.4) \). Let \( f, h \) be Dirichlet series with coefficients \( a, b \), belonging to \( \mathcal{H}_w^2 \). Since
\[ \rho(a, b) = \sum_{m,n \geq 2} a_m b_n \frac{\rho_{mn}}{w_{mn}} + a_1 \sum_{n=1}^{+\infty} b_n \frac{\rho_n}{w_n} + b_1 \sum_{m=1}^{+\infty} a_m \frac{\rho_m}{w_m} \]

\[ = H_g ((f - f(\infty))(g - g(\infty))) + f(\infty) \langle h, g \rangle_{\mathcal{H}_w^2} + g(\infty) \langle f, g \rangle_{\mathcal{H}_w^2}, \]

the first part of the proof entails that \( H_g \) is bounded on \( \mathcal{H}_w^2 \odot \mathcal{H}_w^2 \).

### 8.2 \( X_w \) and the dual of \( \mathcal{H}_w^1 \)

Keeping in mind the results known for Bergman spaces of the unit disk, it is natural to compare \( X_w \) and \((\mathcal{H}_w^1)^*\).

In general, the dual of \( \mathcal{H}_w^1 \) is not known. However, it is shown in [9] that

\[ \mathcal{K} \subset (A_1^1)^*, \quad (8.6) \]

where \( \mathcal{K} \) is the space of Dirichlet series \( f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} \) such that

\[ \sum_{n=1}^{+\infty} \frac{d_4(n)}{[d(n)]^2} |a_n|^2 < \infty. \]

The following consequence of this inclusion will stress upon the difference between the finite and infinite dimensional setting.

**Proposition 10** \((A_1^1)^* \) is not contained in \( X(A_1^2) \).

**Proof** By Abel summation and the Chebyshev estimate, the symbol

\[ g(s) = \sum_{n=2}^{+\infty} \frac{d(n)}{n^\alpha (\log n)^2} n^{-s}, \quad \text{for } \frac{1}{2} < \alpha < 1, \]

is in \( \mathcal{K} \), and thus in \((A_1^1)^*\). However, \( T_g \) is unbounded on \( A_1^2 \) (Proposition 4).

### 8.3 \( X_w \) and the spaces \( \mathcal{H}_w^p \)

It has been shown in [13] that \( BMOA(\mathbb{C}_0) \cap D \subset \neq X(\mathcal{H}_w^2) \subset \neq \cap_{0<p<\infty} \mathcal{H}_w^p \). We have an analogue for Bergman spaces of Dirichlet series.

**Theorem 6** We have the strict inclusions

\[ BMOA(\mathbb{C}_0) \cap D \subset \neq X_w \subset \neq \cap_{0<p<\infty} \mathcal{H}_w^p. \]

**Proof** The inclusions have been proved in Theorem 1 and Corollary 1. As observed in [13], the symbols \( g(s) = \sum_{n=2}^{+\infty} \frac{\psi(n)}{\log n} n^{-s} \), where \( \psi \) is the completely multiplicative function defined on the primes by \( \psi(p) := \lambda p^{-1} \log p, 0 < \lambda \leq 1 \), are in \( \mathcal{X}(\mathcal{H}_w^2) \), and satisfy

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\[
\sum_{n=1}^{+\infty} \psi(n)n^{-\sigma} \asymp \exp \left( \lambda \sum_{p} \frac{\log p}{p^{1+\sigma}} \right) \asymp \exp \left( \frac{1}{\sigma} \right), \quad \sigma > 0.
\]

Hence, they are not in \(BMOA(\mathbb{C}_0)\), though they belong to \(\mathcal{X}_w\) (Lemma 9).

The second inclusion is strict by Proposition 6. \(\Box\)

With the method of Proposition 4, one can show that \(g(s) = \sum_{n \geq 2} n^{-s} \log n n^{-s}, 1/2 \leq a < 1\), is not in \(\mathcal{X}_w\), though it belongs to \(BMOA(\mathbb{C}_1-a)\) [13]. Therefore, we have the strict inclusion

\[\mathcal{X}_w \subsetneq \text{Bloch}(\mathbb{C}_1/2).\]

### 8.4 \(\mathcal{X}_w \cap D_d\) and Bloch spaces

**Theorem 7** Let \(d\) be a positive integer. The following inclusions hold

\[D_d \cap \text{Bloch}(\mathbb{C}_0) \subset D_d \cap \mathcal{X}_w \subsetneq B^{-1} \text{Bloch}(\mathbb{D}^d).
\]

**Proof** The first inclusion has been shown in Theorem 1(a).

If \(g\) is in \(D_d \cap \mathcal{X}_w\), Theorem 5 implies that \(H_g\) is bounded on \(H^2_w\). Therefore, the form \(H_Bg\) (1.4) is bounded on the Bergman space \(H^2_w(\mathbb{D}^d)\). From [17], \(B^g\) is in \(\text{Bloch}(\mathbb{D}^d)\).

Here is a function \(g\) which is not in \(\mathcal{X}_w\), such that \(B^g\) is in \(\text{Bloch}(\mathbb{D}^2)\). Suppose that

\[g'(s) = \frac{1}{1 - 2^{-s}} \log \left( \frac{1}{1 - 3^{-s}} \right), \quad s \in \mathbb{C}_0.
\]

Straightforward computations show that \(B^g \in \text{Bloch}(\mathbb{D}^2)\). The norms \(\| \cdot \|_{A^2_{\lambda}(\mathbb{D}^2)}\) and \(\| \cdot \|_{B^2_{\lambda}(\mathbb{D}^2)}\) being equivalent, our setting will be the space \(A^2_{\lambda}(\mathbb{D}^2)\). Now, for

\[F(z) = \sum_{n=1}^{\infty} \frac{(n+1)^{\frac{\beta-1}{2}}}{\log(n+1)} z^n = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D},
\]

define \(f(s) = F(2^{-s})F(3^{-s})\), for \(s \in \mathbb{C}_0\). We have

\[\|f\|^2_{H^2_w} = \|F\|^4_{A^2_{\lambda}(\mathbb{D})} \asymp \left( \sum_{n=1}^{\infty} \frac{1}{(n+1)(\log(n+1))^2} \right) < \infty.
\]

Putting

\[h_1(z_1) = F(z_1) \frac{1}{1 - z_1} = \sum_{m=0}^{\infty} A_m z_1^m, \quad z_1 \in \mathbb{D},
\]
\( h_2(z_2) = F(z_2) \log \left( \frac{1}{1 - z_2} \right) = \sum_{n=0}^{\infty} B_n z_2^n, \quad z_2 \in \mathbb{D}, \)

we have \( A_m \gtrsim (m + 1)^{\frac{\beta + 1}{2}} \log (m + 1) \) and \( B_n \gtrsim (n + 1)^{\frac{\beta - 1}{2}} \). Therefore,

\[
\| T_g f \|_{H_w^2}^2 = \| R^{-1} (h_1 h_2) \|_{A_2^\beta (\mathbb{D})}^2 \sum_{m,n \geq 1} \frac{|A_m|^2 |B_n|^2}{(m + n + 1)^2 (m + 1)^{\beta} (n + 1)^{\beta}} \gtrsim \sum_{m \geq 1} \frac{m + 1}{(\log (m + 1))^2} \frac{\log (m + 1)}{(m + 1)^2} \frac{1}{(m + 1) \log (m + 1)} = +\infty,
\]

which proves the claim. \( \square \)

A consequence of Theorems 1 and 6 is that

\[
\text{Bloch}(\mathbb{C}_0) \cap D_d \subset \cap_{0 < p < \infty} H_{d,w}^p.
\]

This inclusion can be viewed as a counterpart of the situation of the disk, where \( \text{Bloch}(\mathbb{D}) \subset \cap_{0 < p < \infty} A_\beta^0 (\mathbb{D}) \).

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References

1. Aleman, A., Perfekt, K.M.: Hankel forms and embedding theorems in weighted Dirichlet spaces. Int. Math. Res. Not. IMRN 19, 4435–4448 (2012)
2. Aleman, A., Siskakis, A.G.: Integration operators on Bergman spaces. Indiana Univ. Math. J. 46, 337–356 (1997)
3. Anderson, J., Clunie, J., Pommerenke, C.: On Bloch functions and normal functions. J. Reine Angew. Math. 270, 12–37 (1974)
4. Apostol, T.M.: Introduction to Analytic Number Theory. Springer, New-York (1976)
5. Bailleul, M., Brevig, O.F.: Composition operators on Bohr–Bergman spaces of Dirichlet series. Ann. Acad. Sci. Fen. M. 41, 129–142 (2016)
6. Bailleul, M., Lefèvre, P.: Some Banach spaces of Dirichlet series. Stud. Math. 226(1), 17–55 (2015)
7. Bayart, F.: Compact composition operators on a Hilbert space of Dirichlet series. Ill. J. Math. 47(3), 725–743 (2003)
8. Bayart, F., Brevig, O.F.: Composition operators and embeddings theorems for some function spaces of Dirichlet series. Math. Z. 293(3–4), 989–1014 (2019)
9. Bayart, F., Brevig, O.F., Haimi, A., Ortega-Cerda, J., Perfekt, K.M.: Contractive inequalities for Bergman spaces and multiplicative Hankel forms. Trans. Am. Math. Soc. 371(1), 681–707 (2019)
10. Bayart, F., Queffélec, H., Seip, K.: Approximation numbers of composition operators on $H^p$ spaces of Dirichlet series. Ann. Inst. Fourier (Grenoble) 66(2), 551–588 (2016)
11. Bohr, H.: Über die Bedeutung der Potenzreihen unendlich vieler Variablen in der Theorie der Dirichlet-letschen reihen $\sum a_n/n^s$. Nachr. Ges. Wiss. Göttingen Math. Phys. Kl. 1913, 441–488 (1913)
12. Brevig, O.F., Perfekt, K.-M., Seip, K., Siskakis, A.G., Vukotic, D.: The multiplicative Hilbert matrix. Adv. Math. 302, 410–432 (2016)
13. Brevig, O.F., Perfekt, K.-M., Seip, K.: Volterra operators on Hardy spaces of Dirichlet series. J. Reine Angew. Math. (2019). https://doi.org/10.1515/crelle-2016-0069
14. Brevig, O.F., Perfekt, K.-M.: Weak product of Dirichlet series. Integral Equ. Oper. Theory 86(4), 453–473 (2016)
15. Cima, J.A., Schober, G.: Analytic functions with bounded mean oscillation and logarithms of $H^p$ functions. Math. Z. 151, 295–300 (1976)
16. Cole, B.J., Gamelin, T.W.: Representing measures and Hardy spaces for the infinite polydisk algebra. Proc. Lond. Math. Soc. 3(53), 112–142 (1986)
17. Constantin, O.: Weak product decompositions and Hankel operators on vector-valued Bergman spaces. J. Oper. Theory 59, 157–178 (2008)
18. Constantin, O.: Carleson embeddings and some classes of operators on weighted Bergman spaces. J. Math. Anal. Appl. 365, 668–682 (2010)
19. Hedenmalm, H., Lindqvist, P., Seip, K.: A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0; 1)$. Duke Math. J. 86, 1–37 (1997)
20. Ivic, A.: The Riemann Zeta-Function, Theory and Applications. Dover Publications Inc., New York (2003)
21. McCarthy, J.: Hilbert spaces of Dirichlet series and their multipliers. Trans. Am. Math. Soc. 356, 881–893 (2004)
22. Olevskii, A.M.: Fourier Series with Respect to General Orthonormal Systems. Springer, Berlin (1975)
23. Olsen, J.F.: Local properties of Hilbert spaces of Dirichlet series. J. Funct. Anal. 261, 2669–2696 (2011)
24. Olsen, J.F., Saksmann, E.: On the boundary behavior of the Hardy space of Dirichlet series and a frzme bound estimate. J. Reine Angew. Math. 663, 33–66 (2012)
25. Olsen, J.F., Seip, K.: Local interpolation in Hilbert spaces of Dirichlet series. Proc. Am. Math. Soc. 136, 203–212 (2008)
26. Pommerenke, C.: Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oscil- lation. Comment. Math. Helv. 52(4), 591–602 (1977)
27. Seip, K.: Zeros of functions in Hilbert spaces of Dirichlet series. Math. Z. 274(3-4), 1327–1339 (2013)
28. Smith, W.: Composition operators between Bergman spaces and Hardy spaces. Trans. Am. Math. Soc. 348, 2331–2348 (2013)
29. Stanton, C.S.: Counting functions and majorization for Jensen measures. Pac. J. Math. 125, 459–468 (1986)
30. Wilson, B.M.: Proofs of some formulae enunciated by Ramanujan. Proc. Lond. Math. Soc. 2(1), 235–255 (1923)
31. Zhu, K.: Operator Theory in Function Spaces. Mathematical Surveys and Monographs, vol. 138, 2nd edn. American Mathematical Society, Providence (2007)

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