FUNCTIONS WITH ISOLATED SINGULARITIES ON SURFACES

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Abstract. Let $M$ be a smooth connected compact surface, $P$ be either the real line $\mathbb{R}$ or the circle $S^1$, and $f : M \rightarrow P$ be a smooth mapping. In a previous series of papers for the case when $f$ is a Morse map the author calculated the homotopy types of stabilizers and orbits of $f$ with respect to the right action of the diffeomorphisms group of $M$. The present paper extends those calculations to a large class of maps $M \rightarrow P$ with degenerate singularities satisfying certain set of axioms.

1. Introduction

Let $M$ be a smooth compact connected surface and $P$ be either the real line $\mathbb{R}$ or the circle $S^1$. Then the group $\mathcal{D}(M)$ of diffeomorphisms of $M$ naturally acts from the right on the space $\mathcal{C}^\infty(M, P)$ by the formula:

$$h \cdot f = f \circ h, \quad h \in \mathcal{D}(M), \ f \in \mathcal{C}^\infty(M, P).$$

This action is one of the main objects in singularities theory. For the case of surfaces it was extensively studied in recent years, see e.g. [4, 3, 27, 31, 33, 32, 35, 36, 18, 19, 11, 39].

For $f \in \mathcal{C}^\infty(M, P)$ let $\Sigma_f$ be the set of critical points of $f$ and

$$\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\},$$

$$\mathcal{S}(f) = \{h \mid f = f \circ h, \ h \in \mathcal{D}(M)\}$$

be respectively the orbit and the stabilizer of $f$. We will endow $\mathcal{D}(M)$ and $\mathcal{C}^\infty(M, P)$ with the corresponding topologies $\mathcal{C}^\infty$. Then these topologies induce certain topologies on $\mathcal{O}(f)$ and $\mathcal{S}(f)$. Let $\mathcal{D}_{\text{id}}(M)$ and $\mathcal{S}_{\text{id}}(f)$ be the identity path-component of $\mathcal{D}(M)$ and $\mathcal{S}(f)$, and $\mathcal{O}_f(f)$ the path-component of $f$ in $\mathcal{O}(f)$ with respect to topologies $\mathcal{C}^\infty$.

In [18, 19] the author calculated the homotopy types of $\mathcal{S}_{\text{id}}(f)$ and $\mathcal{O}_f(f)$ for all Morse maps $f : M \rightarrow P$. These calculations are essentially based on the description of homotopy types of groups of orbits

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preserving diffeomorphisms for certain classes of vector fields obtained in [16, 24]. In a series of papers [17, 23, 20, 22] the classes of vector fields were extended and using these results it was then announced in [21] that calculations of [18, 19] can be done for a large class of smooth maps $M \to P$ with isolated “homogeneous” singularities.

The aim of this paper is to show that the technique used in [18, 19] can be formalized and thus extended to classes of isolated singularities even larger than homogeneous ones, see Theorems 5.1 and 5.2.

We will introduce three types of isolated critical points $S$, $P$, and $N$ for a germ of smooth maps $f : M \to P$. These points will be discussed in §4 and now we only note that $S$-points are saddles while $P$- and $N$-points are local extremes. All these points can be degenerate however they satisfy certain “non-degeneracy” conditions formulated in the terms of shift map of the corresponding local Hamiltonian vector field of $f$. In particular, class of $S$-points ($P$-points) have properties similar to non-degenerate saddles (local extremes) of Morse functions and include such points, while $N$-points behave like degenerate local extremes of homogeneous polynomials, see Lemma 5.6. These $N$-points bring new effects in comparison with Morse functions.

Now we put following three axioms on $f$:

**Axiom (A1).** $f$ is constant at each connected component of $\partial M$ and $\Sigma_f \subset \text{Int}M$.

**Axiom (A2).** Every critical point of $f$ is either a $S$- or a $P$- or an $N$-point.

**Axiom (A3).** The natural map $p: \mathcal{D}(M) \to \mathcal{O}(f)$ defined by $p(h) = f \circ h^{-1}$ is a Serre fibration with fiber $S(f)$ in the corresponding topologies $\mathcal{C}^\infty$.

The following theorem describes the homotopy types of $S_{id}(f)$ and $O_f(f)$ for a generic situation. Detailed formulations are given in Theorems 5.1 and 5.2 below.

**Theorem 1.1.** Suppose $f$ satisfies axioms (A1)-(A3) and has at least one $S$-point. Let also $n$ be the total number of critical points of $f$. Then $S_{id}(f)$ is contractible, $O_f(f)$ is weakly homotopy equivalent to a CW-complex of dimension $\leq 2n - 1$. Moreover, $\pi_i O_f(f) = \pi_i M$ for $i \geq 3$, $\pi_2 O_f(f) = 0$, and for $\pi_1 O_f(f)$ we have the following exact sequence:

$$1 \to \pi_1 \mathcal{D}(M) \oplus \mathbb{Z}^k \to \pi_1 O_f(f) \to G \to 1,$$

where $G$ is a certain finite group and $k \geq 0$.

1The symbols $P$ and $N$ stand for periodicity and non-periodicity of shift map.
1.2. Structure of the paper. The exposition of the paper follows the line of [18]. The principal new feature of the paper is that we consider N-points. This requires additional arguments almost everywhere, therefore in many places we repeat the arguments of [18] with necessary modifications.

In §2 we recall some results concerning the shift map along the orbits of vector fields. §3 describes two constructions related to a smooth function $f$ on a surface: foliation $\Delta_f$ by connected components of level-sets of $f$ and the Kronrod-Reeb graph $\Gamma(f)$ of $f$. In §4 we introduce three types of critical points S, P, and N. Further in §5 we formulate main results of the paper Theorems 5.1 and 5.2 and also discuss sufficient conditions for axiom (A3). In §§6 we put on the Kronrod-Reeb graph of $f$ additional data which describe combinatorial behavior of diffeomorphisms $h \in S(f)$ near N-points. §7 contains the proof of Theorems 5.1. The proof follows the line of [18, Th. 1.3]. The rest of the paper is devoted to the proof of Theorem 5.2.

Remark 1.3. I must warn the reader that the paper [18] contains the following “dangerous” places which are also corrected in the present and in previous papers by the author.

1) The calculation of homotopy types of stabilizers given in [18, Th. 1.3] is essentially based on the principal result of another paper of mine [16] which unfortunately contains some mistakes. The corrections to [16] are given in [22, 24], where it is also shown that for the case described in [18] the results of [16] holds true, see Theorem 2.12. Thus [18, Th. 1.3] remains valid,

2) [18, Eq. (8.6)] is not true in general, see Remark 3.6 and Example 3.7 for details. This changes the meaning of the group $G$ in [18, Th. 1.5]: it remains finite however now it is a group of automorphisms of a more complicated object than the Kronrod-Reeb graph of $f$, which takes to account orientations of level-sets of $f$, see §6.6. A correct formulation of [18, Eq. (8.6)] is given in Lemma 6.10.

3) In the proof of [18, Th. 1.5] it was claimed without explanations that a certain central extension of $\pi_1 D_3(M)$ with a free abelian group $J_0$ is just a direct sum. In general, central extensions of abelian groups even with free abelian groups are not trivial. We will show in Theorem 8.1 that in our case that extension is trivial.

1.4. Notations. If $M$ is a non-orientable surface, then $p: \tilde{M} \to M$ will always denote the oriented double covering of $M$ and $\xi: \tilde{M} \to \tilde{M}$ be a $C^\infty$ involution generating the group $\mathbb{Z}_2$ of deck transformations. For a function $f : M \to P$ we put $\tilde{f} = p \circ f : \tilde{M} \to P$. 
2. Shift maps along orbits of flows

2.1. $r$-homotopies. Let $M, N$ be smooth manifolds and $0 \leq r \leq \infty$. Say that a map $\Omega : M \times I \to N$ is an $r$-homotopy if the corresponding map $\omega : I \to C^r(M, N)$ defined by $\omega(t)(x) = \Omega(x, t)$ is continuous from the standard topology of $I$ to the weak topology $C^r$ of $C^r(M, N)$. In other words, all partial derivatives of $\Omega$ in $x \in M$ up to order $r$ continuously depends on $t \in I$. If in addition $\Omega_t : M \to N$ is an embedding for every $t \in I$, then $\Omega$ will be called an $r$-isotopy, see [20].

Thus a usual homotopy is a 0-homotopy. Moreover, every $C^r$-map $M \times I \to N$ is an $r$-homotopy, but not wise verse.

2.2. Local flow. Let $F$ be a smooth vector field on a smooth manifold $M$ tangent to $\partial M$. Then for every $x \in M$ its integral trajectory with respect to $F$ is a unique mapping $o_x : \mathbb{R} \supset (a_x, b_x) \to M$ such that $o_x(0) = x$ and $\dot{o}_x = F(o_x)$, where $(a_x, b_x) \subset \mathbb{R}$ is the maximal interval on which a map with the previous two properties can be defined. Then the following set $\text{dom}(F) = \bigcup_{x \in M} x \times (a_x, b_x)$, is an open neighbourhood of $M \times 0$ in $M \times \mathbb{R}$, and the local flow of $F$ is defined by

$$F : M \times \mathbb{R} \supset \text{dom}(F) \longrightarrow M, \quad F(x, t) = o_x(t).$$

If $M$ is compact, then $F$ is defined on all of $M \times \mathbb{R}$, e.g. [29]. The set of zeros of $F$ will be denoted by $\Sigma_F$.

2.3. Shift map. Let $V \subset M$ be a submanifold (possibly with boundary or even with corners) such that $\text{dim} V = \text{dim} M$. Denote by $\text{func}(F, V)$ the subset of $C^\infty(V, \mathbb{R})$ consisting of functions $\alpha$ whose graph $\Gamma_\alpha = \{(x, \alpha(x)) : x \in V\}$ is included in $\text{dom}(F)$. Then we can define the following map:

$$\varphi(\alpha)(x) = F(x, \alpha(x)), \quad \alpha \in \text{func}(F, V), \quad x \in V. \quad (2.1)$$

We will call $\varphi$ the shift map along orbits of $F$ on $V$ and denote its image in $C^\infty(V, M)$ by $\text{Sh}(F, V)$.

**Definition 2.4.** Let $h : V \to M$ be a smooth map, $V' \subset V$ a submanifold and $\alpha : V' \to \mathbb{R}$ a smooth function. We will say that $\alpha$ is a shift function for $h$ on $V'$ if $h(x) = F(x, \alpha(x))$ for all $x \in V'$.

2.5. Shift map at a singular point. Let $z \in V \cap \Sigma_F$. Denote by $\hat{D}(F, z)$ the space of at $z$ germs of orbit preserving diffeomorphisms $h : (M, z) \to (M, z)$. Thus if $h$ is defined on some neighbourhood $W$ of $z$, then $h(W \cap o) \subset o$ for every orbit $o$ of $F$. 
Let also $C^\infty_\infty(M)$ be the space of germs of $C^\infty$ functions $\alpha : M \to \mathbb{R}$ at $z$. Since $z$ is a singular point for $F$ we have a well-defined *shift map*

$$\hat{\varphi}_z : C^\infty_\infty(M) \to \hat{\mathcal{D}}(F,z), \quad \hat{\varphi}_z(\alpha)(x) = F(x, \alpha(x)).$$

Denote by $\hat{Sh}(F,z) \subset \hat{\mathcal{D}}(F,z)$ the image of $\hat{\varphi}_z$. Then $\hat{Sh}(F,z)$ is a *subgroup* of $\hat{\mathcal{D}}(F,z)$, see [16] Eqs. (8),(9) or [23] Lm. 3.1.

There is a natural homomorphism $J_z(\hat{h}) : \hat{\mathcal{D}}(F,z) \to \text{Aut}(T_z M)$ associating to each $\hat{h} \in \hat{\mathcal{D}}(F,z)$ the corresponding linear automorphism $T_z h$ of the tangent space at $z$.

Choose local coordinates $(x_1, \ldots, x_n)$ at $z$. Then we can regard $J_z$ as a map $J_z : \hat{\mathcal{D}}(F,z) \to \text{GL}(n, \mathbb{R})$ associating to each $\hat{h} \in \hat{\mathcal{D}}(F,z)$ its Jacobi matrix at $z$.

Let also $F = (F_1, \ldots, F_n)$ be the coordinate functions of $F$. Then the following matrix

$$\nabla F(z) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(z) & \cdots & \frac{\partial F_1}{\partial x_n}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(z) & \cdots & \frac{\partial F_n}{\partial x_n}(z) \end{pmatrix}$$

will be called the *linear part* of $F$ at $z$.

It is easy to show, [23] Lm. 5.3, that if $\alpha \in C^\infty(V, \mathbb{R})$ then $J_z(\varphi_V(\alpha)) = J_z(F_\alpha(z)) = e^{\nabla F(z) t}$, whence

$$J_z(\hat{Sh}(F,z)) = J_z(\{F_t\}_{t \in \mathbb{R}}) = \{e^{\nabla F(z) t}\}_{t \in \mathbb{R}}.$$

**2.6. Kernel of shift map.** The set $\ker(\varphi_V) = \varphi_V^{-1}(i_V)$ will be called the *kernel* of $\varphi_V$. It consists of all $C^\infty$ functions $\alpha : V \to \mathbb{R}$ such that $F(x, \alpha(x)) = x$ for all $x \in V$.

**Lemma 2.7.** [16] Let $\alpha, \beta \in \text{func}(F,V)$. Then $\varphi_V(\alpha) = \varphi_V(\beta)$ iff $\alpha - \beta \in \ker(\varphi_V)$. In other words

$$F(x, \alpha(x)) \equiv F(x, \beta(x)) \iff F(x, \alpha(x) - \beta(x)) \equiv x.$$

Suppose $V$ is connected and the set $\Sigma_F$ of singular points of $F$ is nowhere dense in $V$. Then one of the following conditions holds true:

**Nonperiodic case:** $\ker(\varphi_V) = \{0\}$ and $\varphi_V : \text{func}(F,V) \to Sh(F,V)$ is a bijection. This holds for instance if $F$ has at least one non-closed orbit, or for some singular point $z$ of $F$ the linear part of $F$ at $z$ vanishes, i.e. $\nabla F(z) = 0$; or

**Periodic case:** $\ker(\varphi_V) = \{n\theta\}_{n \in \mathbb{Z}}$ for some $C^\infty$ strictly positive function $\theta : V \to (0, +\infty)$. In this case

- every $x \in V \setminus \Sigma_F$ is periodic so $\text{func}(F,V) = C^\infty(V, \mathbb{R})$,
- there exists an open and everywhere dense subset $Q \subset V \setminus \Sigma_F$ such that $\theta(x) = \text{Per}(x)$ for all $x \in Q$;
\[ \varphi_V^{-1} \circ \varphi_V(\alpha) = \{ \alpha + n\theta \} \text{ for every } \alpha \in C^\infty(V, \mathbb{R}). \]

Let \( \mathcal{E}(F, V) \) be the subset of \( C^\infty(V, M) \) consisting of maps \( h : V \to M \) such that

(i) \( h(\omega) \subset \omega \) for every orbit \( \omega \) of \( F \);

(ii) \( h \) is a local diffeomorphism at each singular points of \( F \).

Let also \( \mathcal{D}(F, V) \) be the subset of \( \mathcal{E}(F, V) \) consisting of immersions \( V \to M \).

For \( 0 \leq r \leq \infty \) denote by \( \mathcal{E}_{id}(F, V)^r \) (resp. \( \mathcal{D}_{id}(F, V)^r \)) the path component of the identity inclusion \( i_V : V \subset M \) in \( \mathcal{E}_{id}(F, V)^r \) (resp. \( \mathcal{D}_{id}(F, V)^r \)) with respect to the topology \( C^r \). It consists of maps \( h \in \mathcal{E}_{id}(F, V)^r \) (resp. \( h \in \mathcal{D}_{id}(F, V)^r \)) which are \( r \)-homotopic to \( i_V \) in \( \mathcal{E}(F, V) \) (resp. \( \mathcal{D}(F, V) \)).

If \( V = M \), we will omit \( V \) from notations. Moreover, we will also often omit superscript \( \infty \) and denote \( \mathcal{E}_{id}(F, V)^\infty \) and \( \mathcal{D}_{id}(F, V)^\infty \) simply by \( \mathcal{E}_{id}(F, V) \) and \( \mathcal{D}_{id}(F, V) \).

**Lemma 2.8.** \cite{16} \cite{22} Let \( H_t : V \times I \to M \) be an \( r \)-homotopy such that \( H_0 = i_V \) and \( H_t \in \mathcal{E}(F, V) \). Then there exists a unique \( r \)-homotopy \( \Lambda : (V \setminus \Sigma_F) \times I \to \mathbb{R} \) such that \( \Lambda_0 = 0 \), \( \Lambda_t : V \setminus \Sigma_F \to \mathbb{R} \) is \( C^\infty \), and \( H_t(x) = F(x, \Lambda_t(x)) \) for all \( x \in V \setminus \Sigma_F \) and \( t \in I \).

In particular, for every \( h \in \mathcal{E}_{id}(F, V)^0 \) there exists a smooth shift function on \( V \setminus \Sigma_F \).

For \( \alpha \in \text{func}(F, V) \) and \( z \in V \) we will denote by \( F(\alpha) \) the Lie derivative of \( \alpha \) along \( F \) at \( z \). Then, \cite{16} Theorem 19, \( \varphi_V(\alpha) \) is a local diffeomorphism at \( z \) iff \( F(\alpha)(z) \neq -1 \). Put

\[
\Gamma^+_V = \{ \alpha \in \text{func}(F, V) : F(\alpha) > -1 \}.
\]

Evidently, \( \Gamma^+_V \) is \( S^1 \)-open and convex subset of \( \text{func}(F, V) \). It also follows from \cite{16} Theorem 25 that

\[
\Gamma^+_V = \varphi_V^{-1}(\mathcal{D}_{id}(F, V)^\infty).
\]

**Lemma 2.9.** \cite{20} The following inclusions hold true:

\[
\text{Sh}(F, V) \subset \mathcal{E}_{id}(F, V)^\infty \subset \cdots \subset \mathcal{E}_{id}(F, V)^r \subset \cdots \subset \mathcal{E}_{id}(F, V)^0.
\]

\[
\varphi_V(\Gamma^+_V) \subset \mathcal{D}_{id}(F, V)^\infty \subset \cdots \subset \mathcal{D}_{id}(F, V)^r \subset \cdots \subset \mathcal{D}_{id}(F, V)^0.
\]

If \( \text{Sh}(F, V) = \mathcal{E}_{id}(F, V)^r \) for some \( r \geq 0 \), then \( \varphi_V(\Gamma^+_V) = \mathcal{D}_{id}(F, V)^r \).

**2.10. Openness of shift map.** We recall here the principal results obtained in \cite{24}, see Theorem 2.12 below. A subset \( V \subset M \) will be called a \( D \)-submanifold, if \( V \) is a connected submanifold with boundary (possibly with corners) of \( M \) and \( \dim V = \dim M \). We will also say that \( V \) is a \( D \)-neighbourhood for each \( z \in \text{Int}V \).
Lemma 2.11. Endow $\func(F, V)$ and $\sh(F, V)$ with topologies $C^\infty$. If $\Sigma_F \cap V = \emptyset$ then the shift map $\varphi_V$ is locally injective, whence the following conditions are equivalent:

1. $\varphi_V : \func(F, V) \to \sh(F, V)$ is an open map.
2. $\varphi_V : \func(F, V) \to \sh(F, V)$ is a local homeomorphism.

Suppose these conditions hold true. Consider the restriction $\varphi_V|_{\Gamma^+_V} : \Gamma^+_V \to \varphi_V(\Gamma^+_V)$.

If $\varphi_V$ is non-periodic, then $\varphi_V$ and $\varphi_V|_{\Gamma^+_V}$ are homeomorphisms onto their images. In particular, $\sh(F, V)$ and $\varphi_V(\Gamma^+_V)$ are contractible.

Suppose $\varphi_V$ is periodic. Then the maps $\varphi_V$ and $\varphi_V|_{\Gamma^+_V}$ are $\mathbb{Z}$-covering maps onto their images, and $\sh(F, V)$ and $\varphi_V(\Gamma^+_V)$ are homotopy equivalent to the circle $S^1$.

We will be interesting in establishing (1) and (2) of Lemma 2.11 for the shift map $\varphi$, i.e. for the case $V = M$. It is convenient to formulate this as the following condition.

(Z) The shift map $\varphi : C^\infty(M, \mathbb{R}) \to \sh(F)$ is a local homeomorphism with respect to topologies $C^\infty$.

We will now recall sufficient conditions for (Z) obtained in [24]. Let $V$ be a compact $D$-submanifold, and $U$ be an open neighbourhood of $V$. Then the restriction $F|_U$ of $F$ to $U$ generates a local flow $F_U : U \times \mathbb{R} \ni \dom(F_U) \to U$.

The corresponding shift map of $F|_U$ will be denoted by $\varphi_{U,V}$. Thus $\varphi_{U,V} : \func(F|_U, V) \to \sh(F|_U, V)$.

Let us introduce the following conditions for $V$ and $U$.

(A) The shift map $\varphi_V : \func(F, V) \to \sh(F, V)$ is $C^\infty$-open, that is open between the corresponding topologies $C^\infty$;

(B) The shift map $\varphi_{U,V} : \func(F|_U, V) \to \sh(F|_U, V)$ is $C^\infty$-open.

(C) The set $\sh(F|_U, V)$ is $C^\infty$-open in $\sh(F, V)$.

Finally for each point $z \in M$ consider the following properties.

(A1) There exists a base $\beta_z = \{V_j\}_{j \in J}$ at $z$ consisting of compact $D$-neighbourhoods of $z$ such that every $V \in \beta_z$ satisfies (A).

(B1) There exist an open neighbourhood $U$ of $z$ and a base $\beta_z = \{V_j\}_{j \in J} \subset U$ at $z$ consisting of compact $D$-neighbourhoods such that every $V \in \beta_z$ satisfies (B).
(C1) There exists a neighbourhood $U$ of $z$ every compact $D$-submanifold $V \subset U$ satisfies (C).

(R1) $z$ is non-periodic and non-recurrent.

(R2) $z$ is periodic and the Poincaré return map of the orbit $o_z$ of $z$ is the identity.

(S1) $z \in \Sigma_F$ and there exists an $F$-invariant neighbourhood $W$ of $z$.

(S2) $z \in \Sigma_F$ and there exists a neighbourhood $W$ of $z$ with the following property: if $x \in \partial W = \overline{W} \setminus W$ then there exists a neighbourhood $\gamma \subset o_x$ of $x$ in the orbit $o_x$ such that $\gamma \cap \partial W = \{x\}$.

**Theorem 2.12.** [24] The following implications hold true:

1) $(A) \iff (B) \& (C)$;
2) $(A1)$ for every $z \in M \Rightarrow (Z)$;
3) $(R1) \lor (R2) \Rightarrow (A1)$;
4) $(S1) \lor (S2) \Rightarrow (C1)$.

In particular, suppose that every regular point $z \in M \setminus \Sigma_F$ of $F$ satisfies either of the conditions (R1), (R2), and every singular point $z \in \Sigma_F$ of $F$ satisfies (S1), (S2), and (B1). Then the shift map $\varphi : C^\infty(M, \mathbb{R}) \to Sh(F)$ is either a homeomorphism or a $\mathbb{Z}$-covering map between topologies $C^\infty$.

**Remark 2.13.** Statement 1) of Theorem 2.12 in particular implies that if the map $\varphi_V$ is open (condition (A)), then the map $\varphi_{U,V}$ is open for any open neighbourhood $U$ of $V$ (condition (B)).

3. Functions on surfaces

In this section we will assume that $M$ is a compact surface and $f : M \to P$ a $C^\infty$ map satisfying the following conditions:

(i) $f$ takes constant value on each connected component of $\partial M$

(ii) all critical points of $f$ are isolated and contained in Int$M$.

Let $z \in \text{Int}M$. Then in some local charts at $z$ in $M$ and at $f(z)$ in $P$ we can regard $f$ as a function

$$
(M, z) \supset (\mathbb{C}, 0) \xrightarrow{\bar{f}} (\mathbb{R}, 0) \subset (P, f(z))
$$

such that $z = 0 \in \mathbb{C}$ and $\bar{f}(0) = 0 \in \mathbb{R}$.

We say that $z$ is a local extreme for $f$ if it is a local extreme for $\bar{f}$. Moreover, if we fix an orientation of $P$ and restrict ourselves to representations (3.1) in which the embedding $\mathbb{R} \subset P$ preserves orientation, then $z$ will be called a local maximum (minimum) of $f$ whenever so is $0 \in \mathbb{C}$ for $\bar{f}$. 
3.1. **Isolated critical points.** Suppose that $z \in \text{Int}M$ is an isolated critical point of $f$. Then there exists a germ of a homeomorphism $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$f \circ h(z) = \begin{cases} \pm |z|^2, & \text{if 0 is a local extremum}, \\ \text{Re}(z^n) & \text{for some } n \in \mathbb{N}, \text{otherwise}. \end{cases}$$

If 0 is not a local extreme for $f$, then the number $n$ does not depend on a particular choice of $h$, and in this case $z$ will be called a (generalized) $n$-saddle.

The topological structure of the foliation $\Delta_f$ near local extremes, 1-, and 3-saddles is illustrated in Figure 3.1. The corresponding critical components of level-set of $f$ are designed in bold.

![Figure 3.1. Isolated critical points](image)

3.2. **Foliation $\Delta_f$ of $f$.** Notice that $f$ defines on $M$ a certain one-dimensional foliation $\Delta_f$ with singularities in the following way: a subset $\omega \subset M$ is a leaf of $\Delta_f$ if and only if $\omega$ is either a critical point of $f$ or a connected component of the set $f^{-1}(c) \setminus \Sigma_f$ for some $c \in P$. Thus the leaves of $\Delta_f$ are 1-dimensional submanifolds of $M$ and singular points of $f$.

Denote by $\Delta_f^{\text{reg}}$ the union of all leaves of $\Delta_f$ homeomorphic to the circle and by $\Delta_f^{\text{cr}}$ the union of all other leaves. It follows from (i) that $\partial M \subset \Delta_f^{\text{reg}}$.

The leaves in $\Delta_f^{\text{reg}}$ (resp. $\Delta_f^{\text{cr}}$) will be called regular (resp. critical). Similarly, connected components of $\Delta_f^{\text{reg}}$ (resp. $\Delta_f^{\text{cr}}$) will be called regular (resp. critical) components of $\Delta_f$.

Evidently, every critical leaf of $\Delta_f^{\text{cr}}$ either homeomorphic to an open interval or is a singular point of $f$. If $f$ has at least one critical point or $\partial M = \emptyset$, then every regular component of $\Delta_f$ is diffeomorphic with $S^1 \times (0, 1)$.

Denote by $\mathcal{D}(\Delta_f)$ the group of diffeomorphisms $h$ of $M$ such that $h(\omega) = \omega$ for every leaf $\omega$ of $\Delta_f$, and let $\mathcal{D}^+(\Delta_f)$ be its subgroup consisting of diffeomorphisms of $M$ preserving orientations of all 1-dimensional leaves of $\Delta_f$. 
For each critical point $z \in \Sigma_f$ we will denote by $\hat{D}(\Delta_f, z)$ the group of germs of diffeomorphisms $h : (M, z) \to (M, z)$ at $z$ preserving leaves of $\Delta_f$. More precisely, let $V$ be a neighbourhood of $z$ and $h : V \to M$ be an embedding such that $h(z) = z$. Then $h \in \hat{D}(\Delta_f, z)$ if and only if $h(\omega \cap V) \subset \omega$ for each leaf $\omega$ of $\Delta_f$.

Lemma 3.3. [18, Lm. 3.4] If $f$ satisfies (i) and (ii), then
\[ D^r_{\text{id}}(\Delta_f) = D^r_{\text{id}}(\Delta_f) = S^r(f), \quad 0 \leq r \leq \infty. \]

Proof. In [18, Lm. 3.4] the proof was given for the case $r = 0$. But literally the same arguments hold for all $r$ if we replace everywhere in [18, Lm. 3.4] the word “isotopy” with “$r$-isotopy”. □

3.4. KR-graph of $f$. Let $\Gamma(f)$ be the Kronrod-Reeb graph (KR-graph) of $f$, e.g. [12, 3, 35, 18]. This graph is obtained from $M$ by shrinking every connected component of every level-set of $f$ to a point, see Figure 3.2. Then we have the following decomposition of $f$:
\[ f = f_{\Gamma} \circ p_f : M \xrightarrow{p_f} \Gamma(f) \xrightarrow{f_{\Gamma}} P, \]
where $p_f$ is a factor-map and $f_{\Gamma}$ is the induced function which will be called KR-function for $f$. Evidently, the edges (resp. vertexes) of $\Gamma(f)$ correspond to regular (resp. critical) components of $\Delta_f$.

![Figure 3.2. Foliation $\Delta_f$ and KR-graph $\Gamma(f)$ of $f$](image_url)

3.5. The action of $\mathcal{S}(f)$ on $\Gamma(f)$. Notice that every $h \in \mathcal{S}(f)$ interchanges the leaves of $\Delta_f$ and even yields a homeomorphism of $\Gamma(f)$. So we have a natural homomorphism $\lambda : \mathcal{S}(f) \to \text{Aut}(\Gamma(f))$.

Evidently, $h \in \ker(\lambda)$ iff $h$ preserves every regular leaf of $\Delta_f^{\text{reg}}$. On the other hand it is easy to construct examples when $h \in \ker(\lambda)$ interchanges critical leaves of $\Delta_f$. It follows that
\[ D^+(\Delta_f) \subset D(\Delta_f) \subset \ker(\lambda). \]
Remark 3.6. I should warn the reader that [18, Eq. (8.6)] wrongly claims that \( D^+ (\Delta_f) \cap D_{id} (M) = \ker (\lambda) \cap D_{id} (M) \). In fact, the proof contains a misprint and a mistake: it refers to [18, Lm. 3.6(2)] which does not exist instead of [18 Lm. 3.5(2)]. Nonetheless, [18, Lm. 3.5(2)] is not applicable since it claims about \( h \in S_{id} (f) \) but not about \( h \in D^+ (\Delta_f) \cap D_{id} (M) \): the difference is that every \( h \in S_{id} (f) \) is isotopic to \( id_M \) via an \( f \)-preserving isotopy, while \( h \in D^+ (\Delta_f) \cap D_{id} (M) \) also preserves \( f \) and is isotopic to \( id_M \) but the isotopy is not assumed to be \( f \)-preserving. On the other hand, it follows from [18, Lm. 3.5(1)] that [18 Eq. (8.6)] holds for orientable surfaces. The following example shows that a difference between \( D^+ (\Delta_f) \cap D_{id} (M) \) and \( \ker (\lambda) \cap D_{id} (M) \) appears indeed.

Example 3.7. Let \( \mathbb{R}P^2 \) be a real projective plane and \( f : \mathbb{R}P^2 \to \mathbb{R} \) be a Morse function having exactly one minimum \( x \), one saddle \( y \) and one maximum \( z \). Regard \( \mathbb{R}P^2 \) as a space obtained from unit 2-disk \( D^2 \) by identifying the opposite points on its boundary \( \partial D^2 \). The foliation \( \Delta_f \) on \( D^2 \) and the KR-graph \( \Gamma (f) \) of \( f \) are shown in Figure 3.3. The vertexes of \( \Gamma (f) \) are denoted by the same letters as the corresponding critical points of \( f \) and the up-down arrows show how to glue the boundary of \( D^2 \).

Let \( h : D^2 \to D^2 \) be a mirror symmetry with respect to the horizontal line passing through \( x \), see Figure 3.3. Then \( h \) trivially acts on \( \Gamma (f) \) though it changes orientations of all regular leaves. On the other hand, since \( D (\mathbb{R}P^2) \) is connected, it follows that \( h \) is isotopic to \( id_{\mathbb{R}P^2} \). Thus \( h \in (\ker (\lambda) \setminus D^+ (\Delta_f)) \cap D_{id} (\mathbb{R}P^2) \).

![Figure 3.3.](image)

4. Special critical points

In this section we introduce three types of critical point which will play a key role throughout the paper.
Definition 4.1. (Special critical points) Let $z \in \Sigma_f$ be an isolated critical point of $f$. We will say that $z$ is special for $f$ if there exists a neighbourhood $U$ of $z$ and a vector field $F$ on $U$ with the following properties:

(SP1) $df(F) \equiv 0$ and $z$ is a unique singular point of $F$;
(SP2) there is a base $\beta_z = \{V_j\}_{j \in I}$ of compact connected $D$-neighbourhoods of $z$ such that for each $V \in \beta_z$ the shift mapping $\varphi_V : \text{func}(F,V) \to \text{Sh}(F,V)$ is a local homeomorphism with respect to topologies $C^\infty$.

The corresponding vector field $F$ will be called special as well.

Let $z \in \Sigma_f$ be a special critical point of $f$ and $F$ be a vector field at $z$ satisfying (SP1) and (SP2). Then by (SP1) the orbits of $F$ coincide with the level-curves of $f$ near $z$, whence

$$(4.1) \quad \hat{D}(F,z) = \hat{D}(\Delta_f,z).$$

Moreover, it follows from Remark 2.13 that in Definition 4.1 a neighbourhood $U$ can be taken arbitrary small.

Definition 4.2 (S-point). Say that $z$ is an S-point for $f$ if $z$ is not a local extreme for $f$, (i.e. a saddle point) and it has a vector filed $F$ satisfying (SP1) and (SP2) and such that $\hat{\text{Sh}}(F,z) = \hat{\text{D}}(F,z)$.

Such a vector field will be called special for $z$.

Now let $z$ be a local extreme of $f$. Then we can assume that $U$ is connected and $F$-invariant, see 

It also follows that all the orbits of $F$ except for $z$ are periodic and wrap around $z$. Let $\theta : U \setminus z \to (0, +\infty)$ be the function associating to each $x \in U \setminus z$ its period $\text{Per}(x)$. Then it easily follows from smoothness of the Poincaré’s first return map that $\theta$ is $C^\infty$ on $U \setminus z$ but it can be even discontinuous at $z$. We will call $\theta$ the period function for $F$.

It is shown in [25] that after some linear change of coordinates the linear part $\nabla F(z)$ of $F$ at $z$, see (2.2), can be reduced to one of the following forms:

$$(4.2) \quad a) \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \quad b) \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad c) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. By (2.3) the image $J_z(\hat{\text{Sh}}(F,z))$ of $\hat{\text{Sh}}(F,z)$ under $J_z$ coincides with $\{e^{\nabla F(z) t}\}_{t \in \mathbb{R}}$, whence we obtain the following three possibilities for $J_z(\hat{\text{Sh}}(F,z))$:

$$(4.3) \quad a) \text{SO}(2), \quad b) \{(1 \ 0), t \in \mathbb{R}\}, \quad c) \{(1 \ 0), (0 \ 1)\}.$$
Definition 4.3 (P-point). Let $z$ be a special local extreme of $f$. Say that $z$ is a P-point for $f$ if there exists a vector field $F$ on some neighbourhood $U$ of $z$ satisfying (SP1) and (SP2) and such that the corresponding period function $\theta : U \setminus z \to (0, +\infty)$ smoothly extends to all of $U$.

In this case $F$ will also be called special.

Lemma 4.4. \cite{38, 25} Let $z \in \Sigma_f$ be a local extreme of $f$. Choose some local coordinates $(x, y) \in \mathbb{R}^2$ at $z$ in which $z = 0$. Let $F = (F_1, F_2)$ be a $C^\infty$ vector field near $z$ such that $F(f) \equiv 0$ and $z$ is a unique singular point of $F$. Then the following conditions (P1)-(P6) are equivalent.

(P1) $z$ is a P-point for $f$ and $F$ is the corresponding special vector field for $z$;
(P2) The eigen values of $\nabla F(z)$ are non-zero purely imaginary, so $\nabla F(z)$ can be reduced to the form a) of \cite{12}.
(P3) There exist germs at $z$ of $C^\infty$ functions $\beta, X, Y : U \to \mathbb{R}$ such that $\beta(z) \neq 0$, and $X$ and $Y$ are flat at $z$, and $F$ is $C^\infty$ equivalent to the following vector field

$$\beta(x^2 + y^2) \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}\right) + X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y}.$$ 

(P4) There exist a connected smooth 2-submanifold $V \subset U$ such that $z \in \text{Int}V$ and the shift map $\varphi_V$ is periodic, i.e. $\ker(\varphi_V) \neq \{0\}$.
(P5) For every connected smooth 2-submanifold $V \subset U$ the shift map $\varphi_V$ is periodic.
(P6) $J_z(\text{Sh}(F, z))$ is conjugate in $\text{GL}(2, \mathbb{R})$ to $\text{SO}(2)$.

In these cases $\text{Sh}(F, z) = \tilde{D}^+(F, z)$, $\ker(\varphi_V) = \{n\theta\}_{n \in \mathbb{Z}}$ for any connected 2-submanifold $V \subset U$.

Moreover, there are $C^\infty$ germs $g, \mu : U \to \mathbb{R}$ at $z$ such that $\mu$ is flat at $z = 0$ and $f$ is $C^\infty$ equivalent to the function $g(x^2 + y^2) + \mu(x, y)$.

If either of conditions (P1)-(P6) is violated, then $\lim_{x \to z} \theta(z) = +\infty$.

Proof. Equivalence (P1)$\iff$(P2)$\iff$(P3) was established in \cite{25}. In fact F. Takens \cite{38} described normal forms for vector fields satisfying (P2). He proved that there are two types of such forms. The first one is described by (P3). Moreover, it was observed in \cite{25} that vector fields of the second type have non-closed orbits near $z$, whence in our case $F$ can belong only to the first type. This implies (P2)$\Rightarrow$(P3). The proof of (P3)$\Rightarrow$(P1)$\Rightarrow$(P2) is one of the main results of \cite{25}.

(P4)$\Rightarrow$(P1). Suppose $\ker(\varphi_V) = \{n\nu\}_{n \in \mathbb{Z}}$ for some $C^\infty$ function $\nu : V \to (0, +\infty)$. Then by Lemma \cite{27} $\nu = \text{Per} \equiv \theta$ on open and every where dense subset $Q \subset V \setminus \{z\}$, whence $\nu = \theta$ on all of $V \setminus \{z\}$, and thus $\theta$ extends to a $C^\infty$ function $\nu$ near $z$. 

\section*{Functions with Isolated Singularities on Surfaces}
(P1)⇒(P5). Suppose θ is $C^\infty$ on all of $U$ and let $V \subset U$ be a 2-submanifold. Then $F(x,\theta(x)) = x$ for all $x \in V$, whence $\theta|_V \in \ker(\varphi_V) \neq \{0\}$. By the previous arguments, $\ker(\varphi_V) = \{n\theta|_V\}_{n\in\mathbb{Z}}$.

(P5)⇒(P4) is evident, and (P2)⇒(P6) follows from (4.3).

All other statements are also proved in [25].

**Definition 4.5 (N-point).** Let $z$ be a special local extreme of $f$. Say that $z$ is an **N-point** for $f$ if there exists a vector field $F$ near $z$ satisfying (SP1) and (SP2) and such that

(i) the corresponding period function $\theta : U \setminus \{z\} \to (0, +\infty)$ can not be continuously extended to all of $U$, so by Lemma 4.4

\[ \lim_{x \to z} \theta(x) = +\infty; \]

(ii) $\ker(J_z) \subset \hat{S}h(F, z)$;

(iii) if $\nabla F(z) = 0$, then $J_z(\hat{D}(F, z))$ is finite.

Again, such a vector field will be called **special**.

Condition (ii) means that for each $h \in \hat{D}(F, z)$ with $J_z(h) = \text{id}$ there exists a germ of a $C^\infty$ function $\alpha \in C^\infty_\infty(M)$ such that $h(x) = F(x, \alpha(x))$ for all $x$ sufficiently close to $z$.

Consider the following subsets of $GL(2, \mathbb{R})$:

\[ \mathcal{A}_{++} = \{(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}) | t \in \mathbb{R}\}, \quad \mathcal{A}_{--} = \{(\begin{smallmatrix} -1 & t \\ 0 & -1 \end{smallmatrix}) | t \in \mathbb{R}\}, \]

\[ \mathcal{A}_{+-} = \{(\begin{smallmatrix} 1 & t \\ 0 & -1 \end{smallmatrix}) | t \in \mathbb{R}\}, \quad \mathcal{A}_{-+} = \{(\begin{smallmatrix} -1 & t \\ 0 & 1 \end{smallmatrix}) | t \in \mathbb{R}\}, \]

\[ \mathcal{A}_+ = \mathcal{A}_{++} \cup \mathcal{A}_{--}, \quad \mathcal{A} = \mathcal{A}_{++} \cup \mathcal{A}_{--} \cup \mathcal{A}_{+-} \cup \mathcal{A}_{-+}. \]

Then $\mathcal{A}_+ \subset \mathcal{A}$ are subgroups of $GL(2, \mathbb{R})$, $\mathcal{A}_{++}$ is the unity component of both $\mathcal{A}$ and $\mathcal{A}_+$, $\mathcal{A}_+ / \mathcal{A}_{++} \approx \mathbb{Z}_2$, and $\mathcal{A} / \mathcal{A}_{++} \approx \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Lemma 4.6.** [26] Let $z$ be an **N-point** of $f$ and $F$ be a special vector field for $z$. Then the following statements hold.

(N1) $\nabla F(z)$ is nilpotent, whence it is similar to one of the matrices b) or c) of (1.2).

(N2) If $\nabla F(z) = (\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix})$, see b) of (1.2), then

\[ (4.4) \quad J_z(\hat{S}h(F, z)) = \mathcal{A}_{++}, \]

\[ (4.5) \quad \hat{S}h(F, z) = J_z^{-1}(\mathcal{A}_{++}), \]

\[ (4.6) \quad J_z(\hat{D}^+(F, z)) \subseteq \mathcal{A}_+, \]

\[ (4.7) \quad J_z(\hat{D}(F, z)) \subseteq \mathcal{A}. \]

Hence either $\hat{S}h(F, z) = \hat{D}^+(F, z)$ or $\hat{D}^+(F, z)/\hat{S}h(F, z) \approx \mathbb{Z}_2$, and $\hat{D}^+(F, z)/\hat{S}h(F, z)$ is isomorphic with a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. 


(N3) If $\nabla F(z) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, see c) of (4.2), then $\ker(J_z) = \hat{\text{Sh}}(F, z)$, and the image $J_z(\hat{D}^+(F, z))$ is a finite cyclic subgroup of $\text{GL}(2, \mathbb{R})$ of some order $n_z$, so $\hat{D}^+(F, z)/\hat{\text{Sh}}(F, z) \approx \mathbb{Z}_{n_z}$. If in addition $\hat{D}(F, z) \neq \hat{D}^+(F, z)$, then $\hat{D}^+(F, z)/\hat{\text{Sh}}(F, z)$ is a dihedral group $\mathbb{D}_{n_z}$.

**Proof.** (N1) follows from (P6) and Eq. (4.2).

(N2). Eq. (4.4) follows from (2.3). To establish Eq. (4.5) consider $h \in J_{z}^{-1}(\mathcal{A}_{++})$. Since $J_{z}(\hat{\text{Sh}}(F, z)) = \mathcal{A}_{++}$, there exists $g \in \hat{\text{Sh}}(F, z)$ such that $J_{z}(g) = J_{z}(h)$. Hence $g^{-1} \circ h \in \ker(J_{z}) \subset \hat{\text{Sh}}(F, z)$, whence $h \in \hat{\text{Sh}}(F, z)$ as well.

Eq. (4.7) is proved in [25], whence $J_{z}(\hat{D}^+(F, z)) \subset A \cap \text{GL}^+(2, \mathbb{R}) = A$. This proves Eq. (4.6).

(N3) follows from the well-known fact that every finite subgroup of $\text{GL}^+(n, \mathbb{R})$ is cyclic. □

Due to (N2) and (N3) of Lemma 4.6 we will distinguish two types of $\mathbb{N}$-points.

**Definition 4.7.** An $\mathbb{N}$-point $z$ of $f$ will be called an **NN-point** if $\nabla F(z)$ is non-zero nilpotent. Otherwise, $\nabla F(z) = 0$ and we will call $z$ an **NZ-point**.

**4.8. Examples.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial without multiple linear factors, that is

$$f = L_1 \cdots L_a \cdot Q_{q_1}^{q_1} \cdots Q_{q_b}^{q_b},$$

where $L_i$, $(i = 1, \ldots, a)$, is a non-zero linear function, $Q_j$, $(j = 1, \ldots, b)$, is an irreducible over $\mathbb{R}$ (definite) quadratic form, $q_j \geq 1$, $L_i/L_i' \neq \text{const}$ for $i \neq i'$, and $Q_j/Q_j' \neq \text{const}$ for $j \neq j'$. Put

$$D = Q_1^{q_1-1} \cdots Q_b^{q_b-1}.$$

Then $f = L_1 \cdots L_a \cdot Q_1 \cdots Q_b \cdot D$ and it is easy to see that $D$ is the greatest common divisor of the partial derivatives $f'_x$ and $f'_y$. The following polynomial vector field on $\mathbb{R}^2$:

$$F(x, y) = -(f'_y/D) \frac{\partial}{\partial x} + (f'_x/D) \frac{\partial}{\partial y}$$

will be called the **reduced Hamiltonian** vector field of $f$. In particular, if $f$ has no multiple factors, i.e. $l_i = q_j = 1$ for all $i, j$, then $D \equiv 1$ and $F$ is the usual Hamiltonian vector field of $g$.

Notice that if $f$ had multiple linear factors, then $0 \in \mathbb{R}^2$ would not be an isolated critical point.
Lemma 4.9. The origin $0 \in \mathbb{R}^2$ is a special critical point of $f$ belonging to one of the types $S$, $P$, or $N$, and $F$ is the corresponding special vector field. More precisely,

1. If $a > 0$, then $0$ is an $S$-point for $f$;
2. If $a = 0$ and $b = 1$, i.e. $f = Q_1^{q_1}$, and thus in some local coordinates $f(x, y) = (x^2 + y^2)^{q_1}$, then $0 \in \mathbb{R}^2$ is a $P$-point of $f$;
3. Otherwise, when $a = 0$ and $b \geq 2$, so $f = Q_1^{q_1} \cdots Q_b^{q_b}$, then $0 \in \mathbb{R}^2$ is an $N$-point (even an $NZ$-point) of $f$.

Proof. The assumption (SP1) of the Definition 4.1 that $F(f) \equiv 0$ is evident, while (SP2) is proved in [24]. Hence $0 \in \mathbb{R}^2$ is special.

1. If $a > 0$, then the identity $\hat{S}h(F, z) = \hat{D}^+(F, z)$ follows from [17, 23], see [23, Th. 11.1].

2. Suppose $a = 0$ and $b = 1$. Then we can assume that $f(x, y) = (x^2 + y^2)^{q_1}$, whence $F(x, y) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$. Hence by (P2) of Lemma 4.4 $0$ is a $P$-point for $f$. Moreover, the assumption (SP2) of the Definition 4.1 is independently reproved in [25].

3. Suppose $a = 0$ and $b \geq 2$. Then (i) of Definition 4.5 is established in [25], and (ii) and (iii) in [17, 23], see [23, Th. 7.1 & 11.1]. It is also easy to see that $\nabla F(0) = 0$, so $0$ is an $NZ$-point. □

4.10. Extension of shift functions. In this paragraph we prove two lemmas about extensions of shift functions.

Lemma 4.11. Let $z$ be a special critical point of $f$ being a local extreme, $F$ be the corresponding special vector field defined on some neighbourhood $U$ of $z$, $W \subset V$ be two open connected $F$-invariant neighbourhoods of $z$ such that $W \subset V$, and $h : V \rightarrow V$ be a diffeomorphism preserving orientation and orbits of $F$.

1. Let $Y \subset V \setminus \{z\}$ be any open connected subset and $\alpha : Y \rightarrow \mathbb{R}$ be any shift function for $h$ on $Y$. If $z$ is a $P$-point for $f$, then $\alpha$ uniquely extends to a unique $C^\infty$ shift function for $h$ on all of $V$.

2. Any shift function $\alpha : W \rightarrow \mathbb{R}$ for $h$ on $W$ uniquely extends to a $C^\infty$ shift function for $h$ on all of $V$.

Proof. Since $h$ preserves orientation, it is not hard to show that $h$ has a $C^\infty$ shift function $\beta : V \setminus \{z\} \rightarrow \mathbb{R}$, see [23]. Such a function is defined up to a summand $n\theta$, $(n \in \mathbb{Z})$. In particular, any shift function for $h$ defined on $Y \subset V \setminus \{z\}$ coincides with $\beta + n\theta$ for some $n \in \mathbb{Z}$.

1. Suppose $z$ is a $P$-point, so $\theta$ extends to $C^\infty$ function on $V$. As just noted $\alpha = \beta + n\theta$ on $Y$ for some $n$. 
We claim that $\beta$ extends to a $C^\infty$ function on $V$. Indeed, by Lemma 4.4 $h \in \tilde{D}(F, z) = \tilde{S} h(F, z)$, so $h$ has a $C^\infty$ shift function $\beta'$ on some neighbourhood of $z$. Then $\beta' = \beta + k\theta$ for some $k \in \mathbb{Z}$. Since $\beta'$ and $\theta$ are $C^\infty$ near $z$, so is $\beta$.

Hence $\alpha$ also uniquely extends to a $C^\infty$ function $\beta + n\theta$ on $V$.

(2) We have that $\alpha = \beta + n\theta$ on $W \setminus \{z\}$ for some $n$, though $\beta$ and $\theta$ are even not defined at $z$. On the other hand they are $C^\infty$ on $V \setminus \{z\}$, so we define $\alpha$ on $V \setminus W$ by $\alpha = \beta + n\theta$. Then $\alpha$ is $C^\infty$ on all of $V$. □

**Lemma 4.12.** Let $z \in \Sigma_f$ be a critical point of $f$ of either of types $S$ or $P$ or $N$, $F$ be a special vector field for $z$ defined on some neighbourhood $U$ of $z$ containing no other critical points of $f$, $V \subset U$ be an open neighbourhood of $z$, and $\tilde{V} = V \setminus \{z\}$. Let also $H : V \times I \to U$ be an $r$-homotopy in $E(F, V)$ such that $H_0 = i_V : V \subset U$, and $\Lambda : \tilde{V} \times I \to \mathbb{R}$ be a unique $r$-homotopy such that $\Lambda_0 = 0$ and $\Lambda_t$ is a smooth shift function for $H_t$ on $\tilde{V}$ for every $t \in I$, see Lemma 2.8. If $z$ is an $N$-point suppose in addition that $r \geq 1$. Then for each $t \in I$ the function $\Lambda_t$ extends to a $C^\infty$ function on all of $V$.

**Proof.** First we show that $H_t \in \hat{S} h(F, z)$ for each $t \in I$, i.e. $H_t$ has a $C^\infty$ shift function $\alpha_t$ on some neighbourhood $W$ of $z$ in $V$.

Indeed, since $H_0 = i_V$, it follows the germ of $H_t$ at $z$ belongs to $\hat{D}^+(F, z)$. Hence if $z$ is either an $S$- or a $P$-point, then by Definitions 4.2 and 4.3 $H_t \in \hat{S} h(F, z)$.

Suppose $z$ is an $N$-point. Then by assumption $r \geq 1$, so the matrix $J_z(H_t)$ continuously depends on $t$. If $\nabla F(z)$ is non-zero nilpotent, then we get from (N3) of Lemma 4.6 that $J_z(H_t)$ belongs to the unity component $A_{+, +}$ of the group $A$ as well as $J_z(H_0) = id$. Hence $H_t \in J_z^{-1}(A_{+, +}) = \hat{S} h(F, z)$.

Similarly, if $\nabla F(z) = 0$, then the set of possible values for $J_z(H_t)$ is finite, whence $J_z(H_t) = J_z(H_0) = id$ for all $t \in I$. Therefore by (ii) of Definition 4.3 $H_t \in \ker(J_z) \subset \hat{S} h(F, z)$ as well.

Thus $\Lambda_t$ and $\alpha_t$ are shift functions for $H_t$ on some neighbourhood connected neighbourhood $W$ of $z$, whence $\Lambda_t - \alpha_t \in \ker(\varphi_{\tilde{W}})$, where $\tilde{W} = W \setminus \{z\}$.

If $z$ is an $S$-point, then $\tilde{W}$ contains non-closed orbits of $F$, whence $\ker(\varphi_{\tilde{W}}) = \{0\}$. Therefore $\Lambda_t = \alpha_t$ is $C^\infty$ on $\tilde{W}$. Thus if we put $\Lambda_t(z) = \alpha_t(z)$ then $\Lambda_t$ will become $C^\infty$ on all of $V$.

Suppose that $z$ is a $P$-point. Then $\ker(\varphi_{\tilde{W}}) = \{n\theta\}_{n \in \mathbb{Z}}$, so $\Lambda_t - \alpha_t = n\theta$ for some $n \in \mathbb{Z}$. But by Definition 4.3 $\theta$ smoothly extends to all of $W$, whence so does $\Lambda_t = \alpha_t + n\theta$. 


Suppose that $z$ is an $N$-point, so $\lim_{x \to z} \theta(x) = +\infty$. This implies that the shift map $\varphi_W$ is non-periodic, so $\alpha$ is a unique shift function for $H_t$ on $W$. Notice that by assumption $r \geq 1$, so the restriction $H : W \times I \to U$ is a 1-homotopy in $Sh(F,W)$ having a shift function $\Lambda : \widehat{W} \times I \to \mathbb{R}$ such that $\Lambda_0 = 0$ is $C^\infty$ on $W$. Then the main result of [22] is applicable to this situation and claims that $\Lambda_t = \alpha_t$ on $\widehat{W}$ for all other $t \in I$. Hence $\Lambda_t$ smoothly extends to all of $V$. □

5. Main results

The sets of $S$-, $P$-, and $N$-points of $f$ will be denoted by $\Sigma_S f$, $\Sigma_P f$, and $\Sigma_N f$ respectively. The following Theorems 5.1 and 5.2 extend the results of [18, 19].

Theorem 5.1. c.f. [18, Th. 1.3]. Suppose that $f : M \to P$ satisfies axioms (A1) and (A2). Then

\begin{equation}
(5.1) \quad S_{\text{id}}(f)^\infty = \cdots = S_{\text{id}}(f)^2 = S_{\text{id}}(f)^1.
\end{equation}

Moreover, each of the following conditions implies $S_{\text{id}}(f)^1 = S_{\text{id}}(f)^0$:

(a) $f$ has no critical points of type $N$, i.e. $\Sigma_N f = \emptyset$;
(b) $M$ is a 2-disk, $\Sigma_f$ consists of a unique critical point $z$ which is of type $N$ with $n_z = 1$;
(c) $M$ is a 2-sphere, $\Sigma_f$ consists of exactly two critical points $z'$ and $z$ such that $z'$ is of type $P$, and $z$ is of type $N$ with $n_z = 1$.

The space $S_{\text{id}}(f) := S_{\text{id}}(f)^\infty$ is contractible if and only if one of the following conditions holds true:

- $f$ has at least one critical point of type $S$ or $N$;
- $M$ is non-orientable.

In all other cases $S_{\text{id}}(f)$ is homotopy equivalent to $S^1$.

Theorem 5.2. c.f. [18, Th. 1.5], [19]. Suppose that $f : M \to P$ satisfies axioms (A1)-(A3) and $f$ has at least $S$-point. Let $n$ be the total number of critical points of $f$. Then $O_f(f)$ is weakly homotopy equivalent to a CW-complex of dimension $\leq 2n - 1$. Moreover, $\pi_i O_f(f) = \pi_i M$ for $i \geq 3$, $\pi_2 O_f(f) = 0$, and for $\pi_1 O_f(f)$ we have the following exact sequence:

\begin{equation}
(5.2) \quad 1 \to \pi_1 D(M) \oplus \mathbb{Z}^k \to \pi_1 O_f(f) \to \mathcal{G} \to 1,
\end{equation}

where $\mathcal{G}$ is a certain finite group and $k \geq 0$. 


5.3. **Discussion of axioms.** Axioms (A1) and (A2) put restrictions only on the foliation $\Delta_f$ of $f$. For instance suppose that $f : M \to \mathbb{R}$ satisfies them and has a global minimum $\min f = 0$. Then for every $n \geq 2$ the function $f^n$ also satisfies these axioms. Replacing $f$ by its $n$-th degree makes that global minimum of $f$ “more degenerate” but does not changes the foliation $\Delta_f$.

On the other hand, if $f$ satisfies (A3), then usually the orbit $\mathcal{O}(f^n)$ of $f^n$ for $n \geq 2$ has infinite codimension in $C^\infty(M, \mathbb{R})$ and the verification of (A3) for $f^n$ is not an easy problem, see [30].

5.4. **Sufficient condition for (A3).** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, $z \in \mathbb{R}^2$, and $f(z) = 0$. Denote by $A$ the algebra of germs of smooth functions $\mathbb{R}^2 \to \mathbb{R}$. Then the Jacobi ideal of $f$ at $z$ is the ideal $\Delta(f, z)$ in $A$ generated by germs partial derivatives $f_x'(z)$ and $f_y'(z)$ of $f$ at $z$.

The codimension $\mu_\mathbb{R}(f, z)$ of a $f$ at $z$ (or a real Milnor number of $z$) is the codimension of the Jacobi ideal $\Delta(f, z)$ in $A$:

$$
\mu_\mathbb{R}(f, z) = \dim_{\mathbb{R}}[A/\Delta(f, z)],
$$

see [34].

**Lemma 5.5.** c.f. [34], [18 § 11.30]. Let $C^\infty_0(M, P)$ be the subspace of $C^\infty(M, \mathbb{R})$ consisting of maps satisfying axiom (A1), and let $f \in C^\infty_0(M, P)$. Suppose that $\mu_\mathbb{R}(f, z) < \infty$ for each critical point $z \in \Sigma_f$. Then $\mathcal{O}(f)$ is a Fréchet submanifold of $C^\infty_0(M, P)$ of finite codimension and the map $p : D(M) \to \mathcal{O}(f)$ is a principal locally trivial $S(f)$-fibration. In particular, $p$ is a Serre fibration, so $f$ satisfies (A3).

**Lemma 5.6.** c.f. [21]. Suppose that $f : M \to P$ satisfies (A1) and has the following property: for every critical point $z$ of $f$ there exists a local presentation $f_z : \mathbb{R}^2 \to \mathbb{R}$ of $f$ in which $z = (0, 0)$ and $f_z$ is a homogeneous polynomial without multiple factors. Then $f$ satisfies all other axioms (A2), (A3).

**Proof.** Axiom (A2) follows from results of [17, 23], see also [24].

For the axiom (A3) it suffices establish finiteness of $\mu_\mathbb{R}(f_z, z)$. Regard $f_z$ as a polynomial two complex variables with real coefficients. Since $f_z$ has no multiple factors, it follows that $z = (0, 0)$ is an isolated critical point of $f_z$, whence the complex Milnor number defined analogously with respect to the algebra of germs of analytical functions $(\mathbb{C}^2, 0) \to \mathbb{C}$ is finite: $\mu_\mathbb{C}(f_z, z) < \infty$, see [2]. It remains to note the following inequality $\mu_\mathbb{R}(f_z, z) \leq 2\mu_\mathbb{C}(f_z, z)$, which can be easily proved. □
6. Framings

In this section we will assume again that \( f \) satisfies conditions (i) and (ii) at the beginning of \( \S 3.2 \). The results of this section will be used for the proof of Theorem 5.2 only.

6.1. Framings for \( N \)-points. Suppose \( z \) is an \( N \)-point of \( f \) and let \( F \) be the corresponding special vector field defined on some neighbourhood \( U \) of \( z \). Recall that the group \( G_z = \hat{D}^+(F,z) / \hat{Sh}(F,z) \) is cyclic and we have denoted its order by \( n_z \).

For each non-zero tangent vector \( v \in T_z M \) put

\[
F_z(v) := \{ \pm T_z h(v) \mid h \in \hat{D}^+(F,z) \}.
\]

Thus \( F_z(v) \) is the union of orbits of \( v \) and \(-v\) under the natural action of \( \hat{D}^+(F,z) \) on \( T_z M \). Equivalently, we may put

\[
F_z(v) := \{ \pm \xi(v) \mid \xi \in J_z(\hat{D}^+(F,z)) \}.
\]

Definition 6.2. The set \( F_z(v) \) will be called a framing at \( z \) if it is finite and invariant with respect to the whole group \( \hat{D}(F,z) \).

Lemma 6.3. (1) Framings exist.
(2) Let \( F_z(v) \) be any framing. If \( n_z \) is odd (resp. even), then \( F_z(v) \) consists of \( 2n_z \) (resp. \( n_z \)) elements.
(3) The kernel of the action of \( \hat{D}^+(F,z) \) on any framing \( F_z(v) \) is \( \hat{Sh}(F,z) \). Hence \( \hat{D}^+(F,z) \) induces a free action of \( G_z \) on \( F_z(v) \).
(4) Let \( F_z(v) \) be a framing at \( z \) and \( h \in S(f) \). Then \( z := h(z) \in \Sigma_N^f \) and

\[
F_z'(v) := T_z h(F_z(v)) \subset T_z'M
\]

is a framing at \( z' \).
(5) If \( g \in S(f) \) is such that \( g(z) = h(z) \), then

\[
T_z g(F_z(v)) = T_z h(F_z(v)).
\]

Proof. (1)-(3). First suppose that \( z \) is an \( NN \)-point, so we can choose local coordinates at \( z \) in which \( \nabla F(z) = (0,0,1) \). Then by Lemma 4.6

\[
J_z(\hat{Sh}(F,z)) = A_{++}, \quad J_z(\hat{D}^+(F,z)) \subset A_+, \quad J_z(\hat{D}(F,z)) \subset A.
\]

Let \( v = (a,0) \in T_z M \) be any vector with respect to these coordinates such that \( a \neq 0 \). Evidently, the orbit of \( v \) with respect to each of the groups \( A_+ \) and \( A \) is \( \{ \pm v \} \), see Figure 6.1h), while \( A_{++} \) is the stabilizer of \( v \) with respect to \( A_+ \). This implies that \( F_z(v) = \{ \pm v \} \), this set is invariant with respect to \( \hat{D}(F,z) \), and \( \hat{Sh}(F,z) \) is the kernel of the action of \( \hat{D}^+(F,z) \).
If \( \hat{S}h(F, z) = \hat{D}^+(F, z) \), i.e. \( n_z = 1 \), then \( |F_z(v)| = 2 = 2n_z \). Otherwise, \( \hat{D}^+(F, z)/\hat{S}h(F, z) = \mathbb{Z}_2 \), so \( n_z = 2 = |F_z(v)| \).

On the other hand, if \( w \in T_zM \) is another tangent vector which is not collinear to \( v \), then \( w = (b, c) \) with \( c \neq 0 \) and \( F_z(w) = \{(t, \pm c) \mid t \in \mathbb{R} \} \) is a pair of lines. Thus \( F_z(w) \) is not a framing, see Figure 6.1b).

Suppose now that \( z \) is an NZ-point, i.e. \( \nabla F(z) = 0 \). Then by Lemma 4.6 \( \hat{S}h(F, z) \) is the kernel of \( J_z\),

\[
J_z(\hat{D}^+(F, z)) = \hat{D}^+(F, z)/\hat{S}h(F, z) = G_z
\]
is a finite cyclic group of order \( n_z \). Whence for any non-zero \( v \in T_zM \) the set \( F_z(v) \) is finite and \( \hat{S}h(F, z) \) is the kernel of the action of \( \hat{D}^+(F, z) \) on \( F_z(v) \).

It remains to choose \( v \) for which \( F_z(v) \) is invariant with respect to \( \hat{D}(F, z) \). We can assume that \( \hat{D}(F, z) \neq \hat{D}^+(F, z) \), so \( J_z(\hat{D}(F, z)) \) is a dihedral group. Let \( h \in \hat{D}(F, z) \setminus \hat{D}^+(F, z) \). Then \( h \) changes orientation at \( z \), whence the linear map \( T_zh \) is a reflection with respect to a line \( \{tv \mid t \in \mathbb{R} \} \) determined by some non-zero vector \( v \in T_zM \). In particular, \( T_zh(v) = v \). Now it is evident that \( F_z(v) \) is invariant with respect to \( \hat{D}(F, z) \), so \( F_z(v) \) is a framing, see Figure 6.1c).

Statements (4) and (5) are easy and we leave them for the reader. □

**Figure 6.1.**

**Definition 6.4.** Say that a family of framings \( \{F_z(v_z) : z \in \Sigma_N^f \} \) is **compatible** if it is invariant with respect to \( S(f) \), i.e.

\[
T_zh(F_z) = F_{h(z)}, \quad \forall h \in S(f).
\]

**Corollary 6.5.** Compatible framings exist.

**Proof.** Since \( h(\Sigma_N^f) = \Sigma_N^f \) for every \( h \in S(f) \), we can divide \( \Sigma_N^f \) by orbits with respect to \( S(f) \). Let \( O = \{z_1, \ldots, z_l\} \subset \Sigma_N^f \) be one of these orbits. It suffices to define a compatible framing on \( \hat{O} \).

Fix any framing \( F_{z_1} \) for \( z_1 \). Let \( z_i \in O \) and \( h \in S(f) \) be such that \( h(z_1) = z_i \). Then we set \( F_z = T_z h(F_{z_1}) \). By Lemma 6.3 this definition does not depend on a particular choice of such \( h \). □
6.6. Framed KR-graph $\widehat{\Gamma}(f)$ of $f$. Let $\Gamma(f)$ be the KR-graph of $f$, $p_f : M \to \Gamma(f)$ be the factor map, and $w$ be a vertex of $\Gamma(f)$. Say that

- $w$ is a $\partial$-vertex if $p_f^{-1}(w)$ is a connected component of $\partial M$;
- $w$ is an $S$-vertex if $p_f^{-1}(w)$ contains $S$-points of $f$ (in this case $p_f^{-1}(w)$ is a critical component of $\Delta_f$ being not local extreme of $f$);
- $w$ is $P$-vertex (resp. $N$-vertex) if $p_f^{-1}(w)$ is a $P$-point (resp. $N$-point) of $f$.

Let $w$ be an $N$-vertex of $\Gamma(f)$ and $z = p_f^{-1}(w)$ be the corresponding $N$-point of $f$. Then by cyclic order of $w$ we will call the corresponding cyclic order $n_z$ of $z$ and denote it by $n_w := n_{p_f^{-1}(w)} = n_z$.

Let $k_z$ be the total number of vectors in (any) framing at $z$, see Lemma 6.3. We will denote this number by $k_z$ and $k_w$ as well. Thus $k_z = n_z$ for even $n_z$ and $k_z = 2n_z$ for odd $n_z$.

To each $N$-vertex $w$ let us glue $k_w$ edges $e_w(1), \ldots, e_w(k_w)$ as shown in Figure 6.2. We will call them edges tangent to $w$. The set of tangent edges to $w$ will be denoted by $T_w$. They should be thought as “lying in the plane orthogonal to the edge incident to $w$”.

![Figure 6.2. Tangent edges to a vertex $w$](image-url)

Denote the obtained graph by $\widehat{\Gamma}(f)$ and call it the framed KR-graph of $f$. Thus $\Gamma(f)$ is a subgraph of $\widehat{\Gamma}(f)$ and we can extend the KR-function $f_\Gamma : \Gamma(f) \to P$ to all of $\widehat{\Gamma}(f)$ to be constant on tangent edges: $f_\Gamma(e_w(i)) = f_\Gamma(w)$.

6.7. Automorphisms of $\widehat{\Gamma}(f)$. Let $E(\Gamma(f))$ be the set of edges of $\Gamma(f)$. By an automorphism of $\widehat{\Gamma}(f)$ we will mean a pair $(\nu, o)$, where

- $\nu : \widehat{\Gamma}(f) \to \widehat{\Gamma}(f)$ is a homeomorphism which preserves types of vertexes and the KR-function i.e. $f_\Gamma \circ \nu = f_\Gamma : \widehat{\Gamma}(f) \to P$, (in particular, $\nu$ sends tangent edges to tangent edges), and
- $o : E(\Gamma(f)) \to \mathbb{Z}_2$ is any function.

Define the composition of automorphisms by:

$$(\nu_1, o_1) \circ (\nu_2, o_2) = (\nu_1 \circ \nu_2, o_1 \circ \nu_2 \cdot o_2),$$
where $\cdot$ is a multiplication in $\mathbb{Z}_2 = \{\pm 1\}$.

Then it is easy to see that all the automorphisms of $\hat{\Gamma}(f)$ constitute a group. We will denote this group by $\text{Aut}(\hat{\Gamma}(f))$. The unit of $\text{Aut}(\hat{\Gamma}(f))$ is $(\text{id}_{\hat{\Gamma}(f)}, 1)$, where $1 : E(\hat{\Gamma}(f)) \to \mathbb{Z}_2$ is a constant map to $1 \in \mathbb{Z}_2$, and the inverse of $(\nu, o)$ is $(\nu^{-1}, -o \circ \nu^{-1})$.

6.8. **Action of $\mathcal{S}(f)$ on $\hat{\Gamma}(f)$**. Suppose $f$ has at least one critical point. We will now define a certain homomorphism $\mu : \mathcal{S}(f) \to \text{Aut}(\hat{\Gamma}(f))$.

Fix
- a compatible framing $\{F_z : z \in \Sigma_f^N\}$ for $\mathbb{N}$-points of $f$,
- for each $z \in \Sigma_f^N$ a bijection $\xi_z : F_z \to T_{p_f(z)}$ between the framing $F_z$ at $z$ and the set of tangent edges at the corresponding vertex $p_f(z)$ of $\hat{\Gamma}(f)$ (see Figure 6.3), and
- orientation of connected components of $\Delta_f^{\text{reg}}$ (this is possible since $\Sigma_f \neq \emptyset$, so all regular components of $\Delta_f$ are diffeomorphic to $S^1 \times (0, 1)$ and thus they are orientable surfaces).

![Figure 6.3.](image)

Now let $h \in \mathcal{S}(f)$. We have to associate to $h$ a pair $(\nu, o)$.

Notice that $h$ yields a certain automorphism $\nu = \lambda(h)$ of $\Gamma(f)$. To extend it to the set of tangent edges. Let $h \in \mathcal{S}(f)$, $z \in \Sigma_f^N$, $z' = h(z)$, $w = p_f(z)$ and $w' = p_f(z')$ be the corresponding vertexes on $\hat{\Gamma}(f)$. Then $z' \in \Sigma_f^N$ as well and $h$ yields a bijection between the framings $F_z$ and $F_{z'}$. So we define $\nu : T_w \to T_{w'}$ by

$$
\xi_{z'} \circ T_z h \circ \xi_z^{-1} : T_w \xrightarrow{\xi_z^{-1}} F_z \xrightarrow{T_z h} F_{z'} \xrightarrow{\xi_{z'}} T_{w'}.
$$

It remains to define a function $o : E(\Gamma(f)) \to \mathbb{Z}_2$. Let $e$ be an edge of $\Gamma(f)$ and $\gamma = p_f(e)$ be the corresponding connected component of $\Delta_f^{\text{reg}}$. Then $\gamma' = h(\gamma)$ is also a connected component of $\Delta_f^{\text{reg}}$. Notice that we fixed orientations of $\gamma$ and $\gamma'$. Therefore we define $o(e) = +1$ if $h : \gamma \to \gamma'$ preserves orientations and $o(e) = -1$ otherwise.

A direct verification shows that the correspondence $h \mapsto (\nu, o)$ is a well-defined homomorphism $\mu : \mathcal{S}(f) \to \text{Aut}(\hat{\Gamma}(f))$. 
6.9. **The kernel of \( \mu \) and the group \( \mathcal{D}^N(\Delta_f) \).** Denote by \( \mathcal{D}^N(M, \Sigma_f) \) the subgroup of \( \mathcal{D}(M) \) consisting of all diffeomorphisms \( h \) of \( M \) such that

- \( h(\Sigma_f) = \Sigma_f \);
- \( h \in \hat{\mathcal{S}} h(F_z, z) \) for each \( N \)-point \( z \in \Sigma^N_f \) and (any) special vector field \( F_z \) for \( z \). In other words, \( J_z(h) \in \mathcal{A}_{++} \) if \( \nabla F(z) = \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \), and \( J_z(h) = \text{id} \) if \( \nabla F(z) = 0 \).

Let \( \mathcal{O}^N(f) \) the orbit of \( f \) with respect to the action of \( \mathcal{D}^N(M, \Sigma_f) \). Put

\[
\mathcal{D}^N(\Delta_f) = \mathcal{D}^+(\Delta_f) \cap \mathcal{D}^N(M, \Sigma_f).
\]

Then it is easy to see that \( \mathcal{D}^N(\Delta_f) \) is a normal subgroup of \( \mathcal{S}(f) \) and

\[
\mathcal{D}^N(\Delta_f) \subset \ker(\mu).
\]

However in general \( \mathcal{D}^N(\Delta_f) \neq \ker(\mu) \). The difference is that each \( h \in \mathcal{D}^N(\Delta_f) \) is required to preserve all 1-dimensional leaves and their orientations, while each \( g \in \ker(\mu) \) must only preserve all regular leaves and their orientation. In fact, it is easy to construct an example of \( f \) and \( h \in \ker(\mu) \setminus \mathcal{D}^N(\Delta_f) \) which interchanges critical leaves of \( \Delta_f \), see e.g. [15] Lm. 6.9 & Fig. 6.1.

The following lemma repairs [18] Eq. (8.6), see Remark 3.6.

**Lemma 6.10.** \( \mathcal{D}^N(\Delta_f) \cap \mathcal{D}^N_{\text{id}}(M) = \ker(\mu) \cap \mathcal{D}^N_{\text{id}}(M) \).

*Proof.* Let \( h \in \ker(\mu) \cap \mathcal{D}^N_{\text{id}}(M) \). Then, \( h \) preserves all regular leaves of \( \Delta_f \) with their orientation and is isotopic to \( \text{id}_M \). Now by [18] Th. 7.1 \( h \) also preserves all critical leaves with their orientation. Moreover, \( h \in \hat{\mathcal{S}} h(F_z, z) \) for each \( z \in \Sigma^N_f \), whence \( h \in \mathcal{D}^N(\Delta_f) \). \( \square \)

**Lemma 6.11.** The intersections of the identity component \( \mathcal{D}^N_{\text{id}}(M, \Sigma_f) \) of \( \mathcal{D}^N(M, \Sigma_f) \) with each of \( \mathcal{D}^N(\Delta_f) \), \( \ker(\mu) \), and \( \mathcal{S}(f) \) coincide:

\[
\mathcal{D}^N(\Delta_f) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f) = \ker(\mu) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f) = \mathcal{S}(f) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f).
\]

*Proof.* The relation \( \mathcal{D}^N(\Delta_f) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f) = \ker(\mu) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f) \) follows from Lemma 6.10. For the second equality it suffices to establish that

\[
\ker(\mu) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f) \subset \mathcal{S}(f) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f).
\]

Let \( h \in \mathcal{S}(f) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f) \) and \( \lambda(h) \) be the automorphism of \( \Gamma(f) \) induced by \( h \). We have to show that \( \lambda(h) = \text{id}_{\Gamma(f)} \) and that \( h \) preserves orientation of all regular leaves on \( \Delta_f \).

First we show that \( \lambda(h) = \text{id}_{\Gamma(f)} \). The arguments are similar to the proof of [18] Eq. (8.8)]. Notice that \( h \) yields the identity automorphism of the first homology group \( H_1(\Gamma(f)) \). Indeed, there is a map \( s_f : \)
\( \Gamma(f) \to M \) such that \( s_f \circ p_f = \text{id}_{\Gamma(f)} \), see e.g. [13]. So we have the following commutative diagram:

\[
\begin{array}{ccc}
\Gamma(f) & \xrightarrow{id_{\Gamma(f)}} & \Gamma(f) \\
\downarrow_{s_f} & & \downarrow_{p_f} \\
\Gamma(f) & \rightarrow & \Gamma(f)
\end{array}
\]

Then for the first homology groups we have another commutative diagram in which rows are exact:

\[
\begin{array}{c}
0 \rightarrow H_1(\Gamma(f)) \xrightarrow{s_f} H_1(M, \Sigma_f) \xrightarrow{p_f} H_1(\Gamma(f)) \rightarrow 0 \\
\downarrow_{\lambda(h)} & & \downarrow_{h_1=\text{id}} & \downarrow_{\lambda(h)} \\
0 \rightarrow H_1(\Gamma(f)) \xrightarrow{s_f} H_1(M, \Sigma_f) \xrightarrow{p_f} H_1(\Gamma(f)) \rightarrow 0
\end{array}
\]

Since \( h \) is isotopic to \( \text{id}_M \) relatively \( \Sigma_f \), we obtain that \( h_1 = \text{id}_{H_1(M, \Sigma_f)} \) whence \( \lambda(h)_1 = \text{id}_{H_1(\Gamma(f))} \), and \( h \) preserves vertexes of the KR-graph \( \Gamma(f) \) of \( f \). This implies that \( \lambda(h) = \text{id}_{\Gamma(f)} \).

It remains to show that \( h \) preserves orientation of regular leaves of \( \Delta_f \). Indeed let \( z \) be any \( \mathbb{N} \)-point. Then \( h \) is isotopic to \( \text{id}_M \) via an isotopy fixed at \( z \), whence \( h \) preserves orientation at \( z \) and therefore it also preserves orientations of all leaves in a neighbourhood of \( z \). If \( M \) is orientable, the same is true for all 1-dimensional leaves of \( \Delta_f \) due to [18 Lm. 3.5(1)].

Suppose that \( M \) is non-orientable. Then \( h \) lifts to a unique symmetric orientation preserving diffeomorphism \( \tilde{h} \) of \( \tilde{M} \), that is \( p \circ \tilde{h} = h \circ p \). It easily follows that \( \tilde{h} \in S(\tilde{f}) \cap D_{\text{id}}^{N}(\tilde{M}, \Sigma_{\tilde{f}}) \). Indeed,

\[
\tilde{f} \circ \tilde{h} = f \circ p \circ \tilde{h} = f \circ h \circ p = f \circ p = \tilde{f},
\]

so \( \tilde{h} \in S(\tilde{f}) \). Moreover, any isotopy \( h_t : M \to M \) between \( h_0 = \text{id}_M \) and \( h_1 = h \) in \( D_{\text{id}}^{N}(M, \Sigma_f) \) lifts to an isotopy \( \tilde{h}_t : M \to M \) between \( \tilde{h}_0 = \text{id}_M \) and \( \tilde{h} = h_1 \) in \( D_{\text{id}}^{N}(\tilde{M}, \Sigma_{\tilde{f}}) \). Then \( \tilde{h} \) is a desired lifting of \( h \). Its uniqueness follows from Lemma 7.2.

Now by the orientable case \( \tilde{h} \) preserves orientation of all regular leaves of \( \Delta_f \), whence so does \( h \) for \( \Delta_f \). \( \square \)

**Lemma 6.12.** c.f. [18 Lm. 2.7] If \( f \) has at least one \( \mathcal{S} \)-point and satisfies axiom (A2), then \( D_{\text{id}}(M, \Sigma_f) \) and \( D_{\text{id}}^{N}(M, \Sigma_f) \) are contractible.

**Proof.** Contractibility of \( D_{\text{id}}(M, \Sigma_f) \). Let \( b \) be the total number of connected components of \( \partial M \) and \( n \) be the total number of critical
points of $f$. It follows from the description of homotopy types of the groups of diffeomorphisms of compact surfaces, see [6, 8, 10], that the group $D_{\text{id}}(M, \Sigma_f)$ is contractible if and only if $\chi(M) < n + b$.

By the assumption $n \geq 1$. Therefore if $\chi(M) \leq 0 < 1 \leq n$, our statement is evident. Suppose that $M = S^2$ or $\mathbb{R}P^2$. Then $b = 0$. Moreover, in these cases every smooth map $f : M \rightarrow P$ is null-homotopic, whence $f$ has also maximum and minimum. Therefore $n \geq 3 > 2 \geq \chi(M)$.

Contractibility of $D_{\text{id}}^N(M, \Sigma_f)$. Denote by $D(M, \Sigma_f, \text{id})$ the subgroup of $D(M, \Sigma_f)$ consisting of all diffeomorphisms $h$ such that $J_z(h) = \text{id}$ for each $z \in \Sigma_f$. Denote by $s, p, a$, and $b$ respectively the total number of critical points of $f$ of types $S$, $P$, $\text{NN}$, and $\text{NZ}$, and put $n = s + p + a + b$. Then we can order all critical points $z_1, \ldots, z_n$ of $f$ in such a way that the first $s$ points are of type $S$, the next $p$ points are of type $P$, the next $a$ points are of type $\text{NN}$, and the last $b$ ones are of type $\text{NZ}$. For every $z_i \in \Sigma_f$ fix local coordinates $(x, y)$ in which $z_i = 0$. Then we have a natural map

$$\xi : D(M, \Sigma_f) \rightarrow \prod_{i=1}^n \text{GL}(2, \mathbb{R}), \quad \xi(h) = (J_{z_1}(h), \ldots, J_{z_n}(h)).$$

It is easy to show that $\xi$ is a locally trivial fibration with fiber $D(M, \Sigma_f, \text{id})$. Notice that $\text{GL}(2, \mathbb{R})$ has the homotopy type of $O(2)$ being a disjoint union of two circles. Moreover, as just proved $D_{\text{id}}(M, \Sigma_f)$ is contractible. Then from exact sequence of homotopy groups we obtain $\pi_i D(M, \Sigma_f, \text{id}) = 0$ for $i \geq 1$, and thus the boundary homomorphism $\partial_1$ is an isomorphism:

$$0 \rightarrow \pi_1 \prod_{i=1}^n \text{GL}(2, \mathbb{R}) \xrightarrow{\partial_1} \pi_0 D(M, \Sigma_f, \text{id}) \rightarrow 0,$$

whence $\pi_0 D(M, \Sigma_f, \text{id}) = \mathbb{Z}^n$. Consider the following subset

$$Q := (\text{GL}(2, \mathbb{R}))^{s+p} \times (A_+)^a \times (\text{id})^b \subset \prod_{i=1}^n \text{GL}(2, \mathbb{R}).$$

Then $D^N(M, \Sigma_f) = \xi^{-1}(Q)$, whence the restriction $\xi : D^N(M, \Sigma_f) \rightarrow Q$ is fibration with the same fiber $D^N(M, \Sigma_f)$. Since $A_+$ is diffeomorphic to $\mathbb{R}$, we see that $Q$ is homotopy equivalent to $(S^1)^{s+p}$. Therefore from exact sequence of homotopy groups we get $\pi_i D^N(M, \Sigma_f) = \pi_i Q = 0$ for $i \geq 2$ and also obtain the following exact sequence:

$$0 = \pi_1 D(M, \Sigma_f) \rightarrow \pi_1 D^N(M, \Sigma_f) \rightarrow \pi_1 Q \xrightarrow{\partial_1} \pi_0 D(M, \Sigma_f).$$

Evidently, the inclusion $Q \subset \prod_{i=1}^n \text{GL}(2, \mathbb{R})$ induces a monomorphism on $\pi_1$-groups, whence by (6.1) we have that in (6.2) the map $\partial_1$ is a
monomorphism. Therefore $\pi_1\mathcal{D}_\Sigma f = 0$. Thus all the homotopy groups of $\pi_1\mathcal{D}_\Sigma f$ vanish.

Lemma 6.13. If $f$ satisfies (A3), then

1. $p^{-1}(\mathcal{O}(f)) = \mathcal{D}(M, \Sigma f)$, whence the map
   \[ p : \mathcal{D}(M, \Sigma f) \rightarrow \mathcal{O}(f), \quad p(h) = f \circ h^{-1} \]
   is a Serre fibration as well;

2. $\mathcal{O}(f)$ (resp. $\mathcal{O}^N(f)$) is the orbit of $f$ with respect to the action
   of $\mathcal{D}_\Sigma f$ (resp. $\mathcal{D}_\Sigma^N f$), c.f. [19];

3. the boundary map $\partial_1 : \pi_1 \mathcal{O}(f) \rightarrow \pi_0 \mathcal{S}(f)$ is a homomorphism
   and the image of the induced map $p_1 : \pi_1 \mathcal{D}(M) \rightarrow \pi_1 \mathcal{O}(f)$ is
   included in the center of $\pi_1 \mathcal{O}(f)$, [18, Lemma 2.2.].

7. Proof of Theorem 5.1

Orientable case. Suppose $M$ is orientable and $f$ satisfies axioms (A1) and (A2). For each $z \in \Sigma f$ let $F_z$ be the corresponding special vector field defined on some neighbourhood $U_z$ of $z$.

Lemma 7.1. There exists a vector field $F$ on $M$ such that

1. $df(F) \equiv 0$ and $F(z) = 0$ iff $z \in \Sigma f$.
2. $F = \pm F_z$ near $z$ for every $z \in \Sigma f$.

Moreover, for any vector field $F$ satisfying (i) and (ii) the following conditions holds true:

3. The shift map $\varphi : C^\infty(M, \mathbb{R}) \rightarrow \mathcal{Sh}(F)$ is either a homeomorphism or a $\mathbb{Z}$-covering map with respect to topologies $C^\infty$. Hence by Lemma 2.7 so is the restriction $\varphi|_{\Gamma^+} : \Gamma^+ \rightarrow \varphi(\Gamma^+)$, and both spaces $\mathcal{Sh}(F)$ and $\varphi(\Gamma^+)$ are either contractible or homotopy equivalent to $S^1$.
4. $\mathcal{Sh}(F) = \mathcal{E}_\Sigma^1, \varphi(\Gamma^+) = \mathcal{D}_\Sigma(F)^1$.
5. If $f$ has no $\mathbb{N}$-points then $\mathcal{Sh}(F) = \mathcal{E}_\Sigma^0, \varphi(\Gamma^+) = \mathcal{D}_\Sigma(F)^0$. Suppose $f$ has no $\mathbb{S}$-points and only one $\mathbb{N}$-point $z$, though it may have $\mathbb{P}$-points. If in addition $n_z = 1$, then $\mathcal{Sh}(F) = \mathcal{E}_\Sigma^0$ and $\varphi(\Gamma^+) = \mathcal{D}_\Sigma(F)^0$. In this case $f$ satisfies one of the conditions (b) or (c) of Theorem 5.1.
6. $\mathcal{D}_\Sigma(\Delta f)^r = \mathcal{D}_\Sigma(F)^r = \mathcal{S}_\Sigma(F)^r$ for all $r = 0, \ldots, \infty$.
7. If $\mathcal{Sh}(F) = \mathcal{E}_\Sigma(F)^r$, then $\mathcal{D}_\Sigma(\Delta f)^r = \mathcal{S}_\Sigma(F)^r$.

Suppose Lemma 7.1 is proved. Then by (iv) and (vii) we get
\[ \varphi(\Gamma^+) = \mathcal{S}_\Sigma(f)^\infty = \cdots = \mathcal{S}_\Sigma(f)^1, \]
which proves (5.1.1). Moreover, the case (a) of Theorem 5.1 corresponds to (v), while (b) and (c) correspond to (vii) of Lemma 7.1 and in these cases $\varphi(\Gamma^+) = \mathcal{S}_\Sigma(f)^1 = \mathcal{S}_\Sigma(f)^0$. 

Further, by (iii) of Lemma 7.1, $\varphi(\Gamma^+) = S_{id}(f)^\infty$ is either contractible or homotopy equivalent to the circle.

If $f$ has a critical point $z$ of type S or N, then there exists a neighbourhood $V$ of $z$ such that $\varphi_V$ is non-periodic, hence for each $h \in S_{id}(f)$ there may exist at most one $C^\infty$ shift function on $V$. This implies that $\varphi$ is non-periodic as well, whence $S_{id}(f)$ is contractible.

On the other hand, suppose all critical points of $f$ are of type P. Then $f$ has at most two critical points and it is not hard to prove that $\varphi$ is periodic, whence $S_{id}(f)$ is homotopy equivalent to the circle. This case is analogous to [18, Th. 1.9]. Orientable case of Theorem 5.1 is completed modulo Lemma 7.1.

Proof of Lemma 7.1 (i), (ii). Since $M$ is orientable, it has a symplectic structure and we can construct the corresponding Hamiltonian vector field $F'$ of $f$. By definition this vector field satisfies (i) and also is parallel to $F_z$ near each $z \in \Sigma_f$. Changing the sign of $F_z$ (if necessary) we may assume that $F'$ and $F_z$ has the same directions near $z$. Then using partition unity technique we can glue $F'$ with all of $F_z$ so that the resulting vector field $F$ on $M$ would satisfy (i) and (ii).

(iii). We have to show that the map $\varphi : C^\infty(M, \mathbb{R}) \to Sh(F)$ is either a homeomorphism or a $\mathbb{Z}$-covering map. Due to Theorem 2.12 it suffices to prove that every regular point $z \in M \setminus \Sigma_F$ satisfies either of the conditions (R1), (R2), and every singular point $z \in \Sigma_F$ of $F$ satisfies (S1), (S2), and (B1).

Let $z$ be a regular point of $f$. Then $z$ is also non-singular for $F$. Denote by $\omega$ the connected component of a level-set $f^{-1}(z)$ containing $z$.

If $\omega$ contains critical points of $f$, then the orbit $o_z$ of $z$ is either “an arc connecting two critical points” or a “loop at some critical point” of $f$, see points $z_1$ and $z_2$ in Figure 7.1. In both cases the limit sets of the orbit $o_z$ of $z$ is finite, whence $z$ is non-recurrent, and thus it satisfies (R1).

Otherwise $\omega$ contains no critical points and therefore is diffeomorphic to $S^1$. Hence $\omega$ is a periodic orbit of $F$, see Figure 7.1. Then it is easy to see that the first return map of such orbit is the identity, whence $z$ has property (R2).

Let $z$ be a critical point of $f$ and $U$ be a neighbourhood of $z$ on which $F = F_z$. Then by condition (SP2) of Definition 4.1 there exists a base $\beta_z = \{V_j\}_{j \in J} \subset U$ of $D$-neighbourhoods of $z$ such that for each $V \in \beta_z$ the shift map of $F_z$

$$\varphi_{U,V} : \text{func}(F_z|_U, V) \to Sh(F_z|_U, V)$$

(7.1)
is a local homeomorphism between the corresponding topologies $\mathcal{C}^\infty$. Since $F = F_z$ on $U$, the map (7.1) is the same as the following one:

$$
\varphi_{U,V} : \func(F|_U, V) \to Sh(F|_U, V).
$$

In particular, $\varphi_{U,V}$ is open as well. This implies (B1) for $z$.

Suppose $z$ is a local extreme of $f$, then it has arbitrary small $F$-invariant neighbourhoods, see Figure 7.2a), i.e. satisfies (S1).

Suppose $z$ is a saddle. Then there exists arbitrary small 2-disk $U$ such that $\partial U$ is smooth and transversal to orbits everywhere except for finitely many points $x_1, \ldots, x_k$, and for each $x_i$ there exists an open arc $\gamma_i$ on $\partial x_i$ containing $x_i$ such that $\gamma_i \cap \partial U = \{x_i\}$, see Figure 7.2b). This implies property (S2) for $z$.

Figure 7.2. A neighbourhood $U$

(iv), (v). We have to identify $Sh(F)$ with $\mathcal{E}_{id}(F)^r$ for $r = 0$ or 1. Let $h \in \mathcal{E}_{id}(F)^r$, so there exists an $r$-homotopy $H : M \times I \to M$ such that $H_0 = id_M$, $H_1 = h$, and $H_t \in \mathcal{E}(F)$ for all $t \in I$. Then by Lemma 2.8 there exists an $r$-homotopy $\Lambda : (M \setminus \Sigma_F) \times I \to \mathbb{R}$ such that $\Lambda_0 \equiv 0$ and $H_t(x) = F(x, \Lambda_t(x))$ for all $x \in M \setminus \Sigma_F$.

If $r = 1$ or if $r = 0$ but $f$ has no $N$-points, then by Lemma 4.12 (applied to each $z \in \Sigma_f$) the function $\Lambda_t$, $t \in (0, 1]$, extends to a $\mathcal{C}^\infty$ function on all of $M$. Hence $H_t \in Sh(F)$. In particular, $h = H_1 \in Sh(F)$, and so $Sh(F) = \mathcal{E}_{id}(F)^1$.

(vi). Suppose that $f$ has no $S$-points and only one $N$-point $z$ which also satisfies $n_z = 1$. Thus $f$ has only local extremes and one of them is
Since $M$ is connected there may exist at most two such points. Thus either $\Sigma_f = \{z, z'\}$, where $z'$ is a P-point, $M = S^2$, and $f$ satisfies (b) of Theorem 5.1 or $\Sigma_f = \{z\}$, $M = D^2$, and $f$ satisfies (c) of Theorem 5.1.

Notice that the orbits of $F$ on $M \setminus \Sigma_f$ are closed, the shift map $\varphi_{M \setminus \Sigma_f}$ is periodic, and the function $\theta : M \setminus \Sigma_f \to (0, +\infty)$ associating to every $x \in M \setminus \Sigma_f$ its period $\operatorname{Per}(x)$ generates the kernel $\ker(\varphi_{M \setminus \Sigma_f})$. Moreover, if $\Sigma_f = \{z, z'\}$ and $z'$ is a P-point then by Definition 4.3 $\theta$ smoothly extends to some neighbourhood of $z'$. Thus we can assume that $\theta$ is $C^\infty$ on $M \setminus \{z\}$.

Now let $h \in \mathcal{E}_{id}(F)^0$. We have to verify that $h \in Sh(F)$.

By Lemma 2.3 there exists a smooth shift function $\alpha : M \setminus \Sigma_f \to \mathbb{R}$ for $h$ on $M \setminus \Sigma_f$. Moreover, if $\Sigma_f = \{z, z'\}$ where $z'$ is a P-point, then by Lemma 4.12 $\alpha$ smoothly extends to a $C^\infty$ function near $z'$. Therefore we can assume that $\alpha$ is $C^\infty$ on $M \setminus \{z\}$ as well as $\theta$. Then for each $k \in \mathbb{Z}$ the function $\alpha + k\theta$ is also a $C^\infty$ shift function for $h$ on $M \setminus \{z\}$.

By definition $h$ is a local diffeomorphism at $z$ and since $h \in \mathcal{E}_{id}(F)^0$ it follows that $h$ preserves orientation at $z$, so $h \in \widehat{D}(F, z)$. Then the assumption $n_z = 1$, means that $J_z(\widehat{D}(F, z)) = \text{id}$, so

$$h \in \widehat{D}(F, z) \subset \ker(J_z) \subset \widehat{Sh}(F, z).$$

Hence there exists a $C^\infty$ shift function $\beta$ for $h$ on some neighbourhood $W$ of $z$.

Therefore $\alpha$ and $\beta$ are $C^\infty$ shift functions for $h$ on $W \setminus \{z\}$, whence $\beta - \alpha = n \theta$ for some $k \in \mathbb{Z}$. Hence $\alpha + n \theta$ is $C^\infty$ shift function for $h$ on all of $M$, so $h \in Sh(F)$.

(vii). It follows from (i) that the foliation $\Delta_f$ coincides with the foliation by orbits of $F$, whence $\mathcal{D}(\Delta_f) = \mathcal{D}(F)$, and by Lemma 3.3 $\mathcal{D}_{id}(\Delta_f)^r = \mathcal{D}_{id}(F)^r \subset \mathcal{S}_{id}(f)^r$.

(viii). Suppose $Sh(F) = \mathcal{E}_{id}(F)^r$ for some $r \geq 0$. Then

$$Sh(F) = \mathcal{E}_{id}(F)^r \supset \mathcal{D}_{id}(F)^r = \mathcal{S}_{id}(f)^r = \mathcal{D}_{id}(\Delta_f)^r \supset \mathcal{D}_{id}^N(\Delta_f)^r.$$

In particular, each $h \in \mathcal{S}_{id}(f)^r$ has a $C^\infty$ shift function on $M$ with respect to $F$. This implies that $J_z(h) \in \widehat{Sh}(F, z)$ for each S-point $z$ of $f$, whence by definition $h \in \mathcal{D}^N(\Delta_f)$. Thus $\mathcal{S}_{id}(f)^r \subset \mathcal{D}^N(\Delta_f)$, and therefore $\mathcal{S}_{id}(f)^r \subset \mathcal{D}_{id}^N(\Delta_f)^r$. \hfill $\square$

**Non-orientable case.** Suppose $M$ is non-orientable. Let $p : \widetilde{M} \to M$ be the oriented double covering of $M$ and $\xi : \widetilde{M} \to \widetilde{M}$ be a $C^\infty$ involution generating the group $\mathbb{Z}_2$ of deck transformations.
A vector field $F$ on $\tilde{M}$ tangent to $\partial\tilde{M}$ (as well as its flow $F$) will be called skew-symmetric if $\xi^*(F) = -F$, i.e., $F \circ \xi = -T\xi \circ F$. This is equivalent to the requirement that $F_{\xi} = \xi \circ F_{-t}$ for all $t \in \mathbb{R}$.

A smooth map $\tilde{h} : \tilde{M} \to \tilde{M}$ will be called symmetric if $\tilde{h} \circ \xi = \xi \circ \tilde{h}$. Denote by $\widetilde{D}^+(\tilde{M})$ the group of all orientation preserving symmetric diffeomorphisms of $\tilde{M}$.

**Lemma 7.2.** Every $h \in D(M)$ has a unique lifting $\tilde{h} \in \tilde{D}^+(\tilde{M})$, so $p \circ \tilde{h} = h \circ p$. Moreover, the correspondence $h \mapsto \tilde{h}$ is a homeomorphism $\eta : D(M) \to \tilde{D}^+(\tilde{M})$ with respect to each topology $C^r$, ($0 \leq r \leq \infty$).

**Proof.** Notice that each $h \in D(M)$ has exactly two symmetric liftings. If $\tilde{h} \in \tilde{D}(\tilde{M})$ is a lifting of $h$, then another one is given by $\xi \circ \tilde{h}$. Since $\xi$ reverses orientation of $\tilde{M}$, we can assume that $\tilde{h}$ preserves orientation of $\tilde{M}$, i.e. $\tilde{h} \in \tilde{D}^+(\tilde{M})$. Then it is easy to see that the correspondence $h \mapsto \tilde{h}$ is a bijection between $D(M)$ and $\tilde{D}^+(\tilde{M})$. The verification that $\eta$ is a homeomorphism with respect to the topology $C^r$ for each $0 \leq r \leq \infty$ is direct and we left it for the reader.

Suppose $f$ satisfies axioms (A1) and (A2). Since $p$ is a local diffeomorphism, it follows that the map $\tilde{f} = f \circ p : \tilde{M} \to P$ also satisfies these axioms.

Let $y \in \Sigma_f$, $G_y$ be a special vector field near $y$, and $z, z' \in \Sigma_{\tilde{f}}$ be critical points of $\tilde{f}$ such that $p^{-1}(y) = \{z, z'\}$. Define vector fields $F_z$ and $F_{z'}$ near $z$ and $z'$ respectively as pullbacks of $G_y$ via $p$:

$$F_z = p^*G_y, \quad F_{z'} = -p^*G_y.$$ 

Let $F'$ be any vector field on $\tilde{M}$ satisfying (i) and (ii). Then the vector field $F = \frac{1}{2}(F' - \xi^*F')$ is skew-symmetric and also satisfies (i) and (ii) and therefore all other statements of Lemma 7.1 see [18 Lm. 5.1] for details.

**Lemma 7.3.** c.f. [18 Lm. 4.9 & 5.1]. The following conditions hold true:

(ix) Let $\tilde{D}(\Delta_f) = D(\Delta_f) \cap \tilde{D}^+(\tilde{M})$ be the group of symmetric diffeomorphisms preserving foliation $\Delta_f$. Then $\eta(D(\Delta_f)) = \tilde{D}(\Delta_f)$, see Lemma 7.2. In particular, for all $r = 0, \ldots, \infty$ we have homeomorphisms $\tilde{D}_{id}(\Delta_f)^r \approx \tilde{D}_{id}(\Delta_{f})^r$.

(x) Put $E_0 = \{\alpha \in C^\infty(\tilde{M}, \mathbb{R}) \mid \alpha \circ \xi = -\alpha\}$. Then the shift map $\varphi$ of $F$ yields a homeomorphism $\varphi : E_0 \cap \Gamma^+ \to \tilde{D}(\Delta_f)$ with respect to topologies $C^\infty$. Since $E_0 \cap \Gamma^+$ is convex, $\tilde{D}(\Delta_f)$ is contractible.
Proof. (ix) follows from Lemma 7.2 and (x) from [15, Lm. 4.9]. □

Now we can complete non-orientable case of Theorem 5.1. By (x) of Lemma 7.3, \(\varphi(\Gamma^+ \cap E_0) = \tilde{D}_{id}(\Delta_f)^1\), and \(\varphi(\Gamma^+ \cap E_0) = \tilde{D}_{id}(\Delta_f)^0\) if \(f\) and therefore \(\tilde{f}\) have no \(\mathbb{N}\)-points. Moreover for all \(r = 0, \ldots, \infty\) we have the following identifications

\[
\tilde{D}_{id}(\Delta_f)^r \overset{\text{Lm. 7.2}}{=} \tilde{D}_{id}(\Delta_f)^r \overset{\text{Lm. 5.3}}{=} S_{id}(f)^r.
\]

This implies that \(S_{id}(f) = \cdots = S_{id}(f)^1\) and this space is contractible. Moreover, \(S_{id}(f)^1 = S_{id}(f)^0\) whenever \(f\) has no \(\mathbb{N}\)-points.

Theorem 5.1 is proved. □

8. Proof of Theorem 5.2

Suppose \(f: M \to P\) satisfies (A1)-(A3) and has at least one \(S\)-point. Statement about higher homotopy groups of \(O(f)\) is a simple consequence of Theorem 5.1 and (A3).

Indeed, choose \(id_M\) to be a base point in \(S(f)\) and \(D(M)\), and \(f\) to be a base point in \(O(f)\). Then axiom (A3) implies that there exists an exact sequence of homotopy groups of the fibration \(p:\)

\[
\cdots \to \pi_k S(f) \overset{i_k}{\to} \pi_k D(M) \overset{p_k}{\to} \pi_k O(f) \overset{\partial_k}{\to} \pi_{k-1} S(f) \to \cdots,
\]

where \(i_k\) is induced by the inclusion \(i: S(f) \subset D(M)\), and \(\partial_k\) is the boundary homomorphism.

Recall that \(\pi_i D(M) = \pi_i M\) for \(i \geq 3\), and \(\pi_2 D(M) = 0\), see [6, 7, 8, 10]. Since by Theorem 5.1 \(S_{id}(f)\) is contractible, we obtain isomorphisms \(\pi_k O(f) \approx \pi_k D(M)\) for \(k \geq 2\) and also the following exact sequence:

\[
1 \to \pi_1 D(M) \overset{p_1}{\to} \pi_1 O(f) \overset{\partial_1}{\to} \pi_0 (S(f) \cap D_{id}(M)) \to 1,
\]

where \(\pi_0 (S(f) \cap D_{id}(M))\) can be regarded as the kernel of the induced map \(i_0: \pi_0 S(f) \to \pi_0 D(M)\).

It remains to establish the short exact sequence (5.2) for \(\pi_1 O(f)\) and show that \(O_f(f)\) is weakly homotopy equivalent to some finite dimensional CW-complex.

Proof of (5.2). The arguments are similar to [18, §9]. Recall that we have a homomorphism \(\mu: \pi_0 S(f) \to \text{Aut}(\tilde{\Gamma}(f))\). Let \(G = \mu(\ker i_0)\) be the image of the subgroup \(\ker i_0 = \pi_0 (S(f) \cap D_{id}(M))\) of \(\pi_0 S(f)\) under \(\mu\), and \(J_0\) be the kernel of the restriction of \(\mu\) to \(\ker i_0\). Thus

\[
J_0 := \pi_0 (\ker(\mu) \cap D_{id}(M)) \overset{\text{Lm. 6.10}}{=} \pi_0 (D^N(\Delta_0) \cap D_{id}(M)).
\]
Since $\mathcal{J} = \pi_0(\mathcal{D}^N(\Delta_f)) \cong \mathbb{Z}^k$ is a free abelian group, we see that so is its subgroup $\mathcal{J}_0$. Thus $\mathcal{J}_0 \cong \mathbb{Z}^l$ for some $l \geq 0$.

Then we have the following commutative diagram in which all vertical and horizontal lines are exact:

$$
\begin{array}{cccccc}
1 & 1 \\
\downarrow & \downarrow \\
\pi_1\text{Id}(M) & \mathcal{H} & \mathcal{J}_0 & \cong & \mathbb{Z}^l \\
\downarrow & \downarrow & \downarrow \\
\pi_1\text{Id}(M) & \pi_1\mathcal{O}(f) & \ker(i_0) & \cong & \mathbb{Z}^l \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1
\end{array}
$$

where $\mathcal{H} = \partial_1^{-1}(\mathcal{J}_0)$. To show that the left vertical sequence coincides with [5.2] we have to prove the following:

**Theorem 8.1.** A short exact sequence

(8.1)

$$
1 \rightarrow \pi_1\text{Id}(M) \xrightarrow{p} \mathcal{H} \xrightarrow{\partial_1} \mathcal{J}_0 \rightarrow 1
$$

admits a section $s : \mathcal{H} \rightarrow \mathcal{J}_0$ such that $\partial_1 \circ s = \text{id}_{\mathcal{J}_0}$. Since this sequence is a central extension, see (3) of Lemma 6.13 it splits, so $\mathcal{H} \cong \pi_1\text{Id}(M) \oplus \mathbb{Z}^l$.

This statement was claimed without explanations in the proof of [18, Th. 1.5]. But in general there are central extensions that do not split. Therefore we will present a proof of Theorem 8.1 in §11. Of course, the statement of Theorem 8.1 is non-trivial only for the surfaces with $\pi_1\text{Id}(M) \neq 0$ or (which is equivalent) with $\chi(M) \geq 0$.

To show finiteness of homotopy dimension of $\mathcal{O}_f(f)$ we have to calculate the homotopy type of $\mathcal{O}^N_f(f)$. The statement here is the following theorem

**Theorem 8.2.** c.f. [18 Eq. (8.8)]. If $f$ satisfies (A1) and (A2), then

$$
\mathcal{S}(f) \cap \mathcal{D}^N_{\text{Id}}(M, \Sigma_f) = \mathcal{S}_{\text{Id}}(f).
$$

Hence $\mathcal{S}^N(f) \cap \mathcal{D}^N_{\text{Id}}(M, \Sigma_f) = \mathcal{S}_{\text{Id}}(f)$, and thus

$$
\pi_0(\mathcal{S}^N(f) \cap \mathcal{D}^N_{\text{Id}}(M, \Sigma_f)) = 0.
$$

The proof of this theorem will be given in §10.

**Corollary 8.3.** $\pi_i\mathcal{O}^N_f(f) = 0$ for all $i \geq 0$. 

Proof. By (2) of Lemma 6.13 the map \( p : \mathcal{D}^N(M, \Sigma_f) \to \mathcal{O}^N(f) \) is a Serre fibration. Moreover, \( \mathcal{S}^N_{\text{id}}(f) = \mathcal{S}_{\text{id}}(f) \) is contractible, and by Lemma 6.12 so is \( \mathcal{D}^N_{\text{id}}(M, \Sigma_f) \). Hence \( \pi_i \mathcal{O}^N(f) = 0 \) for all \( i \geq 2 \) and we also obtain an isomorphism

\[
\pi_1 \mathcal{O}^N(f) \cong \pi_0 (\mathcal{S}^N(f) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f)) \cong 0,
\]

where \( \pi_0 (\mathcal{S}^N(f) \cap \mathcal{D}^N_{\text{id}}(M, \Sigma_f)) \) is the kernel of the induced homomorphism \( j_0 : \pi_0 \mathcal{S}^N(f) \to \pi_0 \mathcal{D}^N(M, \Sigma_f) \) induced by the inclusion map \( j : \mathcal{S}^N(f) \subset \mathcal{D}^N(M, \Sigma_f) \).

The proof that \( \mathcal{O}_f(f) \) has the weak homotopy type of a CW-complex of dimension \( \leq 2n-1 \) is similar to [19]. For the completeness we briefly recall the principal arguments.

Let \( n \) be the total number of critical points of \( f \),

\[
\mathcal{P}_n(\text{Int}M) = \{(x_1, \ldots, x_n) \mid x_i \in \text{Int}M, x_i \neq x_j \text{ for } i \neq j \} \subset \prod_n \text{Int}M,
\]

and \( P_n \) be the \( n \)-th symmetrical group of all permutations of \( n \) distinct symbols. Evidently, \( P_n \) freely acts on \( \mathcal{P}_n(\text{Int}M) \) by permutations of coordinates. The factor space \( \mathcal{F}_n(\text{Int}M) = \mathcal{P}_n(\text{Int}M)/P_n \) is the \( n \)-th configuration space of the interior \( \text{Int}M \), i.e. the space on unordered \( n \)-tuples of mutually distinct points of \( \text{Int}M \).

Then we can define the following map \( k : \mathcal{O}_f(f) \to \mathcal{F}_n(\text{Int}M) \) associating to every \( g \in \mathcal{O}_f(f) \) the set \( \Sigma_g \) of its critical points. It can be shown similarly to [19] that \( k \) is a locally trivial fibration such that every connected component of a fiber is homeomorphic with \( \mathcal{O}_f^N(f) \) and therefore is weakly contractible. From the exact sequence of homotopy groups of this fibration we obtain that \( k \) is a monomorphism on \( \pi_1 \) and isomorphism on all \( \pi_i \) for \( i \geq 2 \). Let \( \mathcal{F}(f) \) be the covering space of \( \mathcal{F}_n(\text{Int}M) \) corresponding to the subgroup \( k(\pi_1 \mathcal{O}_f(f)) \subset \pi_1 \mathcal{F}_n(\text{Int}M) \). Then \( k \) lifts to the map \( \tilde{k} : \mathcal{O}_f(f) \to \mathcal{F}(f) \) which induces isomorphisms between all the corresponding homotopy groups and therefore is a homotopy equivalence. It remains to note that \( \mathcal{F}(f) \) is a manifold (usually non-compact) and \( \dim \mathcal{F}(f) = 2n \), whence it is hootopy equivalent to a \((2n-1)\)-dimensional CW-complex.

This completes Theorem 5.2 modulo Theorems 8.2 and 8.1. To prove them we have to calculate the group \( \pi_0 \mathcal{D}^N(\Delta_f) \).

9. Group \( \pi_0 \mathcal{D}^N(\Delta_f) \)

In this section we calculate the group \( \pi_0 \mathcal{D}^N(\Delta_f) \). The exposition is similar to [18, Th. 6.2] but we take to account \( \mathbb{N} \)-points.
Let \(\hat{\Gamma}(f)\) be the framed KR-graph of \(f\). An edge \(e\) of \(\hat{\Gamma}(f)\) will be called external, if it is either tangent to some N-point, or is incident either to a P- or to a \(\partial\)-vertex. Otherwise, \(e\) is internal.

Moreover, an internal edge \(e\) will be called an N-edge (resp. an S-edge) if at least one of its vertexes is an N-vertex (resp. both vertexes of \(e\) are S-vertexes).

Suppose that \(f : M \to P\) satisfies (A1) and (A2). Let \(\gamma\) be a regular leaf of \(\Delta_f\), so \(\gamma\) is homeomorphic to \(S^1\). Denote by \(K_\gamma\) the connected component of \(\Delta_f^{\operatorname{reg}}\) containing \(\gamma\). Then there exists a Dehn twist \(\tau_\gamma\) along \(\gamma\) preserving \(\Delta_f\) and being identity outside arbitrary small neighbourhood \(U_\gamma \subset K_\gamma\) of \(\gamma\) consisting of full leaves of \(\Delta_f\), see [18, §6] and Figure 9.1. In particular, we have that \(\tau_\gamma \in \mathcal{D}_N(\Delta_f)\).

Notice that the image of \(K_\gamma\) in \(\hat{\Gamma}(f)\) is a certain edge \(e\). We will say that \(\gamma\) (as well as \(K_\gamma\)) is internal (external) if so is \(e\), and that \(\tau_\gamma\) is a twist around \(e\).

Now let \(e_1, \ldots, e_k\) be all the internal edges of \(\hat{\Gamma}(f)\). For each \(i\) choose any internal leaf \(\gamma_i\) corresponding to \(e_i\) and take any Dehn twist \(\tau_i \in \mathcal{D}_N(\Delta_f)\) along \(\gamma_i\). Put

\[
T = \bigcup_{i=1}^k \operatorname{supp} \tau_i.
\]

Let \(\mathcal{J} = \langle \tau_1, \ldots, \tau_k \rangle \subset \mathcal{D}_N(\Delta_f)\) be a subgroup generated by the internal Dehn twists. Since \(\operatorname{supp} \tau_i \cap \operatorname{supp} \tau_j = \emptyset\) for \(i \neq j\), we see that \(\mathcal{J}\) is a free abelian group with basis \(\langle \tau_1, \ldots, \tau_k \rangle\), so \(\mathcal{J} \approx \mathbb{Z}^k\).

**Theorem 9.1.** c.f. [18, Th. 6.2]. Suppose that \(f : M \to P\) satisfies axioms (A1) and (A2), and has at least one critical point of type S. Then the inclusion \(\zeta : \mathcal{J} \subset \mathcal{D}_N(\Delta_f)\) is a homotopy equivalence. In particular, we have an isomorphism

\[
\zeta_0 : \mathcal{J} \equiv \pi_0 \mathcal{J} \to \pi_0 \mathcal{D}_N(\Delta_f),
\]

so \(\pi_0 \mathcal{D}_N(\Delta_f) \approx \mathbb{Z}^k\) is freely generated by the isotopy classes of internal Dehn twists.

The proof is similar to [18, Th. 6.2]. It suffices to establish the following statement:
Lemma 9.2. 1) Every $h \in \mathcal{D}^N(\Delta_f)$ is isotopic in $\mathcal{D}^N(\Delta_f)$ to a product of internal Dehn twists, whence $\zeta : \mathcal{J} \to \pi_0\mathcal{D}^N(\Delta_f)$ is surjective.

2) If $h = \tau_{i_1}^{m_1} \circ \cdots \circ \tau_{k}^{m_k} \in S_d(f)$ for some $m_i \in \mathbb{Z}$, then $m_i = 0$ for all $i$ and thus $h = \text{id}_M$, whence $\zeta : \mathcal{J} \to \pi_0\mathcal{D}^N(\Delta_f)$ is a monomorphism.

Proof. First suppose that $M$ is orientable. Let $F$ be a vector field on $M$ satisfying assumptions of Lemma 9.1.

Claim 9.3. For every $h \in \mathcal{D}^N(\Delta_f)$ there exists a unique $C^\infty$ function $\sigma : M \setminus T \to \mathbb{R}$ being shift function for $h$ with respect to $F$. If $h$ is fixed on $M \setminus T$, then $\sigma = 0$ on $M \setminus T$.

Proof. a) Let $z$ be an $N$-point of $f$. Then $h \in \ker(sh_z) = \hat{S}h(F,z)$, so $h$ has a (unique) $C^\infty$ shift function $\sigma_z$ defined on some $F$-invariant closed neighbourhood $V_z$ of $z$ containing no other critical points of $f$. We can assume that $V_z \cap V_{z'} = \emptyset$ for $z \neq z'$. Then the functions $\sigma_z$ define a unique shift function $\sigma_N$ for $h$ on the following set $U_N := \bigcup_{z \in \Sigma_f^N} V_z$.

b) Since $f$ has at least one $S$-point, there exists a critical component $K$ of $\Delta_f$ (that is a connected component of $\Delta_f^g$) which is not a local extreme. Let $z \in K \cap \Sigma_f$. Then $z$ is an $S$-point of $f$. The assumption $h \in \mathcal{D}^N(\Delta_f)$ means that $h \in \hat{S}h(F,z)$, so there exists a neighbourhood $V_z$ of $z$ and a unique $C^\infty$ function $\sigma_z : V_z \to \mathbb{R}$ such that $h(x) = F(x,\sigma_z(x))$ for all $x \in V_z$. Notice that $K \setminus \Sigma_f$ is a disjoint union of intervals, these functions extend to a unique $C^\infty$ function $\sigma_K$ defined on some neighbourhood $V(K)$ of $K$ such that $h(x) = F(x,\sigma_K(x))$ for all $x \in V(K)$, see [13, Lm. 6.4].

We can assume that $V(K)$ is $F$-invariant, so $\partial V(K)$ consists of regular leaves of $\Delta_f$.

Put $U_S := \cup_K V(K)$, where $K$ runs over all critical components of $\Delta_f$ that are not local extremes. We can also assume that

$$V(K) \cap V(K') = V(K) \cap U_N = \emptyset$$

for distinct critical components $K$ and $K'$. Then the functions $\sigma_K$ define a unique shift function $\sigma_S$ for $h$ on $U_S$.

c) Notice that the set $M \setminus U_S$ is a union of cylinders and 2-disks. Moreover, every cylinder contains no critical point of $f$, while every 2 disk contains a unique critical point of $f$ and this point is a local extreme of $f$. Let $B_1, \ldots, B_k$ be all the connected components of $M \setminus U_S$ having one or the following properties: either

- $B_i$ is a cylinder such that $B_i \cap \partial M \neq \emptyset$, or
- $B_i$ is a 2 disk containing a $P$-point of $f$.  

In particular, \( B_i \cap U_N = \emptyset \). Denote \( U_{\partial,P} = \cup_{i=1}^{k} B_i \). We claim that \( \sigma_S \) extends to a \( C^\infty \) function on \( U_S \cup U_{\partial,P} \).

Indeed, suppose \( B_i \approx S^1 \times I \) is a cylinder. Then \( B_i \cap U_S \) is a connected \( F \)-invariant neighbourhood of one of the connected components of \( \partial B_i \) and this neighbourhood does not contain another component of \( \partial B_i \). So we can assume that \( B_i \cap U_S = S^1 \times [0,\varepsilon] \). Then the function \( \sigma_S \) on \( B_i \cap U_S \) extends to a \( C^\infty \) shift function for \( h \) on all of \( B_i \), see \[18\] Lm. 4.12(2)]. Denote this function by \( \beta_i : B_i \to \mathbb{R} \).

Suppose \( B_i \) is a 2-disk containing a \( P \)-point \( z \). Then \( Y_i := B_i \cap U_S \) is a neighbourhood of \( \partial B_i \) and by (1) of Lemma \[4.11\] \( \sigma_S \) extends to a unique \( C^\infty \) shift function \( \beta_i : B_i \to \mathbb{R} \) for \( h \) on all of \( B_i \).

Denote \( U := U_N \cup U_S \cup U_{\partial,P} \). Then the functions

\[
\sigma_N : U_N \to \mathbb{R}, \quad \sigma_S : U_S \to \mathbb{R}, \quad \beta_i : B_i \to \mathbb{R}, \quad (i = 1, \ldots, k),
\]

define a unique \( C^\infty \) shift function \( \sigma : U \to \mathbb{R} \) for \( h \). Notice that \( M \setminus U \) is contained in the union of internal components of \( \Delta_f^{reg} \). Therefore we can assume that in fact \( M \setminus U \subset T \), so \( U \supset M \setminus T \).

If \( h \) is fixed on \( M \setminus U \subset T \), then it follows from uniqueness of \( \sigma_N, \sigma_S \) and uniqueness of extensions \( \beta_i \) of \( \sigma_S \) to \( U_{\partial,P} \), that \( \sigma \equiv 0 \).

In order to complete statement 1) it remains to construct an isotopy of \( h \) in \( D^N(\Delta_f) \) to a diffeomorphism fixed on \( M \setminus T \). Let \( \mu : M \to [0,1] \) be a \( C^\infty \) function constant on leaves of \( \Delta_f \) and such that \( \mu = 1 \) on some neighbourhood of \( U \). Define the following map:

\[
H : M \times I \to M, \quad H_t(x) = F(x, t \cdot \mu(x) \cdot \sigma_h(x)).
\]

Then \( H \) is \( C^\infty \), \( H_0 = \text{id}_M \), and \( H_1 = h \) on some neighbourhood of \( U \), see for details \[18\] Lemma 4.14. In particular, every \( H_t \in D^N(\Delta_f) \). Hence the following isotopy \( G_t = h \circ H_t^{-1} \) deforms \( G_0 = h \) in \( D^N(\Delta_f) \) to a diffeomorphism \( G_1 \) fixed on some neighbourhood of \( U \). In other words \( \text{supp} \ G_1 \subset T \), whence \( G_1 \) is isotopic in \( D^N(\Delta_f) \) to a product of some internal Dehn twists.

2) Let \( h = \tau_{i_1}^{m_1} \circ \cdots \circ \tau_{i_k}^{m_k} \in \mathcal{S}_{id}(f) \). We have to show that \( m_i = 0 \) for all \( i = 1, \ldots, k \). For each \( i \) let \( U_i \) be a cylinder neighbourhood of \( \gamma_i \) containing \( \text{supp} \tau_i \), see Figure \[9\]. Then \( h|_{U_i} = \tau_i^{m_i}|_{U_i} \).

Since \( h \in \mathcal{S}_{id}(f) \) and \( f \) has at least one \( S \)-point, it follows from (iv) of Lemma \[7.1\] that there exists a unique \( C^\infty \) shift function \( \alpha : M \to \mathbb{R} \) for \( h \), so \( h(x) = F(x, \alpha(x)) \) for all \( x \in M \). Therefore an isotopy of \( h \) to \( \text{id}_M \) in \( D^N(\Delta_f) \) can be given by \( h_t(x) = F(x, t\alpha(x)) \), \( t \in I \).

On the other hand by Claim \[9.3\] there exists a unique \( C^\infty \) function \( \sigma : M \to \mathbb{R} \) being shift function for \( h \) on \( M \setminus T \), whence \( \alpha = \sigma \) on \( M \setminus T \). Moreover, as \( h \) is fixed on \( M \setminus T \), we have that \( \alpha = \sigma = 0 \).
on $M \setminus T$. Hence $h_t$ is fixed on $M \setminus T$ for all $t \in I$. This implies that $h_i|_{U_i} = \tau^{m_i}|_{U_i}$ is isotopic to $id_{U_i}$ relatively some neighbourhood of $\partial U_i$. But this is possible if and only if $m_i = 0$. This proves Theorem 9.1 for orientable case.

Suppose $M$ is non-orientable. Let $p : \tilde{M} \to M$ be the oriented double covering of $M$, and $\xi$ be $p$-equivariant involution of $\tilde{M}$. Then we have a function $\tilde{f} = f \circ \tilde{\gamma} : \tilde{M} \to P$. Since every internal leaf $\gamma_i$ is two-sided, it follows that $p^{-1}(\gamma_i)$ consists of two connected components $\gamma_{i1}$ and $\gamma_{i2}$ being internal leaves of the foliation $\Delta_{\tilde{f}}$ of $\tilde{f}$. Then there are Dehn twists $\tau_{i1}, \tau_{i2} \in \tilde{D}^N(\Delta_{\tilde{f}})$ along $\gamma_{i1}$ and $\gamma_{i2}$ respectively such that $\tau_{i1} = \xi \circ \tau_{i1} \circ \xi$ and $\tau_i := \tau_{i2} \circ \tau_{i1}$ is a lifting of $\tau_i$. Evidently, each $\tau_{ij}$ is internal, and by the oriented case the group $\pi_0D^N(\Delta_{\tilde{f}})$ is generated by $\tau_{ij}$ for $i = 1, \ldots, k$ and $j = 1, 2$.

Consider the subgroup $\tilde{D}^N(\Delta_{\tilde{f}})$ of $D^N(\Delta_{\tilde{f}})$ consisting of symmetric $h$, i.e. $h \circ \xi = \xi \circ h$. Then similarly to [18 Claim 6.5.1] it can be shown that the isotopy classes of $\tilde{\tau}_i$, $(i = 1, \ldots, k)$ generate $\pi_0\tilde{D}^N(\Delta_{\tilde{f}})$, whence $\pi_0\tilde{D}^N(\Delta_{\tilde{f}}) \approx \mathbb{Z}^k$. Moreover, it follows from (ix) of Lemma 7.3 that there is a natural homeomorphism between $D^N(\Delta_{\tilde{f}})$ and $\tilde{D}^N(\Delta_{\tilde{f}})$. Hence $\pi_0D^N(\Delta_{\tilde{f}}) \approx \mathbb{Z}^k$ and this group is generated by the isotopy classes of internal Dehn twists. Theorem 9.1 is completed. □

9.4. Action of $J$ on $H_1(M, \Sigma_f)$. Let $J_N$ and $J_S$ be the subgroups of $J$ generated by internal N-twists and S-twists respectively. Evidently, $J = J_N \oplus J_S$. Notice that $J$ naturally acts on the first relative homology group $H_1(M, \Sigma_f)$ with integer coefficients. Let $\nu : J \to Aut(H_1(M, \Sigma_f))$ be the corresponding homomorphism.

Lemma 9.5. If $M$ is orientable, then $\ker(\nu) = J_N$.

Proof. Evidently, every N-internal Dehn twist $\tau \in J_N$ is isotopic to $id_M$ relatively to $\Sigma_f$, so $\ker(\nu) \supset J_N$.

To prove the converse inclusion consider an intersection form $\langle \cdot, \cdot \rangle$ on $H_1(M, \Sigma_f)$. Evidently, the action of $J_S$ on $H_1(M, \Sigma_f)$ is given by

$$\prod_{i=1}^{a} \tau_i^{m_i} \cdot x = x + \sum_{i=1}^{a} m_i(\langle x, \gamma_i \rangle \gamma_i, \gamma_i)$$

see also [18 Eq. (6.1)]. Notice that S-internal curves represent linearly independent cycles in $H_1(M, \Sigma_f)$. This implies that $\nu|_{J_S}$ is a monomorphism, so $\ker(\nu) = J_N$. □
10. **Proof of Theorem 8.2**

We have to show that $S(f) \cap D^N_{id}(M, \Sigma_f) = S_{id}(f)$. By Lemma 6.10 $S(f) \cap D^N_{id}(M, \Sigma_f) = D^N(\Delta_f) \cap D^N_{id}(M, \Sigma_f)$, moreover $D^N_{id}(\Delta_f) = S_{id}(f)$. Hence it suffices to prove the following proposition:

**Proposition 10.1.** c.f. [18] Pr. 8.5 $D^N(\Delta_f) \cap D^N_{id}(M, \Sigma_f) = D^N_{id}(\Delta_f)$.

**Proof.** Evidently, $D^N(\Delta_f) \cap D^N_{id}(M, \Sigma_f) \supset D^N_{id}(\Delta_f)$.

Conversely let $h \in D^N(\Delta_f) \cap D^N_{id}(M, \Sigma_f)$. Then by Theorem 9.1 $h$ is isotopic in $D^N(\Delta_f)$ to some $h' \in J$, whence we can assume that $h \in J$ itself. Write $h = h_N \circ h_S$, where $h_N \in J_N$ and $h_S \in J_S$. We have to show that $h_S = h_N = id_M$.

*Proof that $h_S = id_M$. If $M$ is orientable, then by Lemma 9.5 $h$ trivially acts on $H_1(M, \Sigma_f)$, i.e. $h \in \ker(\nu) = h_N$. Hence $J_S = id_M$.

Suppose $M$ is non-orientable. Then $h_S$ lifts to a diffeomorphism

$$\tilde{h}_S \in \tilde{D}^N(\Delta_f) \cap \tilde{D}^N_{id}(\tilde{M}, \Sigma_f) \subset D^N(\Delta_f) \cap D^N_{id}(\tilde{M}, \Sigma_f)$$

of the oriented double covering $\tilde{M}$ of $M$. By the orientable case we have that $\tilde{h}_S = id_{\tilde{M}}$, whence $h_S = id_M$ as well.

*Proof that $h_N = id_M$. Let $\tau_1, \ldots, \tau_\nu$ be the internal Dehn twists generating $J_N$, so we can write $h_N = \tau_1^{m_1} \circ \tau_\nu^{m_\nu}$ for some $m_i \in \mathbb{Z}$. Then supp $\tau_i$ is contained in some closed 2-disk $B_i \subset M$ which in turn contains a unique critical point $z_i$ of $f$ being an N-point and such that supp $\tau_i \cap (z_i \cup \partial B_i) = \emptyset$.

**Claim 10.2.** There exists an isotopy $h_t : M \rightarrow M$ between $h_0 = id_M$ and $h_1 = h_N$ fixed on some neighbourhood of $\bigcup_{i=1}^\nu \{z_i\}$. Hence $m_i = 0$ for all $i$, and thus $h_N = id_M$.

**Proof.** Let $h_t : M \rightarrow M$ be a 1-isotopy between $h_0 = id_M$ and $h_1 = h_N$ in $D^N_{id}(M, \Sigma_f)$. Then $J_{z_i}(h_t)$ continuously depends on $t$ for each $i$.

There exists another isotopy $h'_t$ between $h_0 = id_M$ and $h_1 = h_N$ in $D^N_{id}(M, \Sigma_f)$ such that $J_{z_i}(h'_t) = id$ for all $t \in I$ and $i = 1, \ldots, \nu$.

Indeed, if $z_i$ is an NZ-point, then the set of possible values for $J_{z_i}(h_t)$ is finite, whence $J_{z_i}(h_t) = J_{z_i}(h_0) = id$, so there is nothing to do.

Suppose $z_i$ is an NN-point. Then in some local coordinates at $z_i$ in which $\nabla F(z) = (0, \lambda, 0)$ we have that

$$J_{z_i}(h_t) = \begin{pmatrix} 1 & \lambda(t) \\ 0 & 1 \end{pmatrix},$$

where $\lambda : I \rightarrow \mathbb{R}$ is a continuous function with $\lambda(0) = \lambda(1) = 0$. Then there exists a 1-isotopy $g_t : M \rightarrow M$ fixed outside some neighbourhood of $\Sigma_f$ and such that $g_0 = g_1 = id_M$, $g_t(z_i) = z_t$, and $J_{z_i}(g_t) = J_{z_i}(h_t)$.
for all \( t \in I \) and \( i = 1, \ldots, v \). Such an isotopy \((g_t)\) can be constructed by introducing a parameter in [28 Lm. 5.4].

Evidently, the isotopy \( h'_t = g_t^{-1} \circ h_t \) deforms \( \text{id}_M \) to \( h_t(\mathcal{D}_{\text{id}}(M, \Sigma_f)) \) and satisfies \( J_z(h_t) = \text{id} \).

Now similarly to the proof of [28 Lm. 5.2] we can change \( h'_t \) to another isotopy \( h''_t \) between \( \text{id}_M \) and \( h_N \) such that \( h''_t \) is fixed on some neighbourhood \( V \) of \( \bigcup_{i=1}^{v} \{ z_i \} \) for each \( t \in I \). Then \( (h''_t) \) satisfies the statement of Claim 10.2. \( \square \)

Proposition 10.1 and therefore Theorem 8.2 are completed. \( \square \)

11. Proof of Theorem 8.1

As noted above, we have to prove our theorem only for the case \( \chi(M) \geq 0 \).

Let \( \{ g_\alpha \}_{\alpha \in A} \) be any set of generators for \( \mathcal{J}_0 \). For each \( \alpha \in A \) let \( \omega_\alpha : I \to \mathcal{D}_{\text{id}}(M) \) be a path such that \( \omega_\alpha(0) = \text{id}_M \) and \( \omega_\alpha(1) = g_\alpha \), see Figure 11.1a). Since \( g_\alpha \in \mathcal{J}_0 \subset \mathcal{S}(f) \), we see that the map \( \nu : I \to \mathcal{O}(f) \) defined by \( \nu(t) = f \circ \omega_\alpha(t) \) is a loop, i.e. \( \nu(0) = \nu(1) = f \).

Let also \( \mathcal{F} \) be a free group generated by symbols \( \{ \hat{g}_\alpha \}_{\alpha \in A} \). Then there exists a unique epimorphism \( \eta : \mathcal{F} \to \mathcal{J}_0 \) such that \( \eta(\hat{g}_\alpha) = g_\alpha \), and a unique homomorphism \( \psi : \mathcal{F} \to \pi_1 \mathcal{O}(f) \) defined by \( \psi(\hat{g}_\alpha) = [f \circ \omega_\alpha] \) for all \( \alpha \in A \). Moreover we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{s} & \mathcal{J}_0 \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{\psi} & \mathcal{J}_0 \\
\end{array}
\]

Evidently, if \( \ker \eta \subset \ker \psi \), then \( \partial_1 \) admits a section \( s : \mathcal{J}_0 \to \mathcal{H} \), whence \( \mathcal{H} \approx \pi_1 \mathcal{D}_{\text{id}}(M) \times \mathcal{J}_0 \). Thus for the proof of Theorem 8.1 we have to find conditions when \( \ker \eta \subset \ker \psi \).
First we present exact formulas for $\psi$. Let $\hat{h} = (\hat{g}_1)^{\varepsilon_1} \cdots (\hat{g}_a)^{\varepsilon_a} \in \mathcal{F}$, where $\varepsilon_i \in \mathbb{Z}$. Put $h = \eta(\hat{h}) = g_1^{\varepsilon_1} \cdots g_a^{\varepsilon_a} \in \mathcal{J}_0$ and define the following path $\kappa_{\hat{h}} : [0, a] \to \mathcal{D}_{\text{id}}(M)$ by

$$
(11.1) \quad \kappa_{\hat{h}}(t) = \begin{cases} 
\omega_1(t)^{\varepsilon_1}, & t \in [0, 1], \\
g_1^{\varepsilon_1} \circ \omega_2(t - 1)^{\varepsilon_2}, & t \in [1, 2], \\
g_1^{\varepsilon_1} \circ g_2^{\varepsilon_2} \circ \omega_3(t - 2)^{\varepsilon_3}, & t \in [2, 3], \\
\cdots \cdots \cdots, & t \in [a - 1, a], \\
g_1^{\varepsilon_1} \circ \cdots \circ g_a^{\varepsilon_a} \circ \omega_a(t - a + 1)^{\varepsilon_a}, & t \in [a - 1, a],
\end{cases}
$$

see Figure 11.1b). Evidently, $\kappa_{\hat{h}}(0) = \text{id}_M$ and $\kappa_{\hat{h}}(1) = h \in \mathcal{J}_0 \subset \mathcal{S}(f)$. Hence the map $\nu_{\hat{h}} : I \to \mathcal{O}(f)$ defined by $\nu_{\hat{h}}(t) = f \circ \kappa_{\hat{h}}(t)$ is a loop. Then it is easy to see that $\psi(\hat{h}) = [\nu_{\hat{h}}]$. Suppose $\hat{h} \in \ker \eta$, so $h = \eta(\hat{h}) = \text{id}_M$. Then $\kappa_{\hat{h}}$ is a loop in $\mathcal{D}_{\text{id}}(M)$.

**Lemma 11.1.** Let $\{g_\alpha\}_{\alpha \in A}$ be a set of generators for $\mathcal{J}_0$, $\mathcal{F}$ be a free group generated by symbols $\{\hat{g}_\alpha\}_{\alpha \in A}$, and $H = \{\hat{h}_\beta\}_{\beta \in B} \subset \mathcal{F}$ be the subset whose normal closure coincides with $\ker \eta$, so

$$
\mathcal{J}_0 = \langle \{g_\alpha\}_{\alpha \in A} \mid \{\hat{h}_\beta\}_{\beta \in B} \rangle
$$

is a presentation for $\mathcal{J}_0$. For each $\alpha \in A$ take a path $\omega_\alpha : I \to \mathcal{D}_{\text{id}}(M)$ such that $\omega_\alpha(0) = \text{id}_M$ and $\omega_\alpha(1) = g_\alpha$. Then each of the following conditions implies that $\ker \eta \subset \ker \psi$.

1) For each $\beta \in B$ the loop $\kappa_{\hat{h}_\beta}$ is null-homotopic in $\mathcal{D}_{\text{id}}(M)$.

2) There exists a subset $Q \subset M$ consisting of $k > \chi(M)$ points and such that $\omega_\alpha(t)$ is fixed on $Q$ for all $\alpha \in A$ and $t \in I$.

**Proof.** Statement 1) is trivial.

2) Let $x_1, \ldots, x_k \in Q$ be $k$ distinct points, $\mathcal{D}(M, k)$ be the group of diffeomorphisms of $M$ that fix each of these points, and $\mathcal{D}_{\text{id}}(M, k)$ be the identity path component of $\mathcal{D}(M, k)$. Then for each $\hat{h}_\beta$ the loop $\kappa_{\hat{h}_\beta}$ is contained in $\mathcal{D}_{\text{id}}(M, k)$. Since $\chi(M) < k$, it follows that $\mathcal{D}_{\text{id}}(M, k)$ is contractible, whence $\kappa_{\hat{h}_\beta}$ is null-homotopic in $\mathcal{D}_{\text{id}}(M, k) \subset \mathcal{D}_{\text{id}}(M)$. Therefore $\psi(\hat{h}_\beta) = [f \circ \kappa_{\hat{h}_\beta}]$ is null-homotopic in $\mathcal{H}$, so $\hat{h}_\beta \in \ker \psi$. □

Thus for the proof of Theorem 11.1 it suffices to show that for every surface $M$ with $\chi(M) \geq 0$ one of the conditions (1) or (2) of Lemma 11.1 is satisfied.

**Lemma 11.2.** Suppose $M$ is one of the surfaces $S^2$, $\mathbb{RP}^2$, $D^2$, $M$ or $S^1 \times I$. Then there exists even infinite subset $Q \subset M$ such that every internal Dehn twist is fixed on $Q$ and isotopic to $\text{id}_M$ relatively.
In particular, so does every $h \in J_0$, whence by 2) of Lemma [11.1] $\ker \eta \subset \ker \psi$.

Proof. Suppose $M$ is either 2-disk $D^2$ or Möbius band $M\delta$, or a cylinder $S^1 \times I$. Put $Q = \partial M$ if $M$ is either $D^2$ or $M\delta$, and $Q = S^1 \times 0$ if $M = S^1 \times I$. Then every internal Dehn twist is fixed near $Q$ and is isotopic to $\text{id}_M$ relatively to $Q$, see e.g. [1], [9, Th. 3.4 & 5.2], [37].

Let $M$ be either 2-sphere $S^2$ or a projective plane $\mathbb{R}P^2$. In this case $f$ always has a local extreme. Denote this point by $z$ and let $Q \subset M \mathbb{R}P^2$ be a closed neighbourhood of $z$ diffeomorphic to 2-disk. Then $M \setminus Q$ is either a 2-disk (if $M = S^2$) or a Möbius band (if $M = \mathbb{R}P^2$). Moreover, every internal Dehn twist $\tau$ is fixed on some neighbourhood of $Q$. Hence $\tau$ is isotopic to the identity relatively $Q$. □

Lemma 11.3. Let $M$ be either a 2-torus $T^2$ or a Klein bottle $\mathbb{K}$. Suppose that every internal leaf $\gamma_i$, $(i = 1, \ldots, k)$, separates $M$. Then there exists a subset $Q \subset M$ satisfying 2) of Lemma [11.1].

Proof. Let $U_i$ be an open neighbourhood of $\gamma_i$ diffeomorphic to $S^1 \times (0, 1)$ such that $\partial U_i$ consists of two regular leafs of $\Delta f$ and supp $\tau_i \subset U_i$. Since $\gamma_i$ separates $M$, it follows that $M \setminus U_i$ consists of two connected components $B_i$ and $C_i$. Moreover, as $M$ is either a 2-torus $T^2$ or a Klein bottle $\mathbb{K}$, we have that one of the components, say $B_1$, is either a 2-disk or a Möbius band.

a) Suppose $B_1$ is a Möbius band. Then $M$ is a Klein bottle, and $C_1$ as well as $C_1 \cup U_1$ are Möbius bands. Put $Q = \partial U_1 \cap B_1$. Then $Q$ is a simple closed curve “parallel” to $\gamma_1$ and also separating $M$ into two Möbius bands. Moreover, every internal Dehn twist $\tau_i$ is fixed on some neighbourhood $Q$, and therefore it is isotopic to $\text{id}_M$ relatively $Q$.

b) Suppose that neither of $B_i$ or $C_i$ is a Möbius band. Then $B_i$ is a 2-disk. Renumbering $\gamma_i$ (if necessary) we can assume that there exists $r \in \{1, \ldots, k\}$ such that

- if $i = 1, \ldots, r$, then $B_i$ is not contained in any of $B_j$ for $j = 1, \ldots, k$, and $j \neq i$, so $B_i$ is “maximal”;
- for $j = r + 1, \ldots, k$ every $B_j$ is contained in some $B_i$ for $i = 1, \ldots, r$.

Put

\[ Q := \bigcap_{i=1}^r C_i = \bigcap_{i=1}^r M \setminus (B_i \cup U_i) = M \setminus \bigcap_{i=1}^r (B_i \cup U_i). \] (11.2)

Then $Q$ is connected as a complement to a disjoint union of closed 2-disks on a connected surface. It is also easy to see that $Q$ does not contain any internal curve. Indeed, suppose $\gamma_j \subset Q$. Since $\gamma_j \subset U_j$, we obtain from (11.2) that $j \in \{r + 1, \ldots, k\}$, whence $B_j \subset B_i$ for some
i = 1, \ldots, r. On the other hand \( \gamma_j \) separates \( Q \) into two non-empty components \( D_j \) and \( D_j' \) such that one of them, say \( D_j \), is contained in \( B_j \). Hence \( D_j \subset B_j \subset B_i \subset M \setminus Q \) which contradicts to the assumption that \( D_j \subset Q \).

Thus \( Q \) contains no internal curves and therefore every \( \tau_i \) is fixed on some neighbourhood of \( Q \). As \( M \setminus Q \) is a union of 2-disks, we see that \( g \) is isotopic to the identity relatively \( Q \). Hence so does every \( g \in J_0 \). □

Lemma 11.4. Let \( M \) be either a 2-torus \( T^2 \) or a Klein bottle \( \mathbb{K} \). Suppose that \( \gamma_1 \) does not separate \( M \).

(i) If \( M \) is a 2-torus \( T^2 \), then there exists a subset \( Q \) satisfying 2) of Lemma 11.1, whence \( \ker \eta \subset \ker \psi \).

(ii) If \( M \) is a Klein bottle \( \mathbb{K} \), then \( \ker \eta \subset \ker \psi \) as well.

Proof. Since all \( \gamma_i \) are mutually disjoint simple closed curves on a 2-torus or a Klein bottle, we can assume that for some \( a = 1, \ldots, k \) the curves \( \gamma_1, \ldots, \gamma_a \) are non-separating and isotopic each other, while each of \( \gamma_{a+1}, \ldots, \gamma_k \) separate \( M \) so that one of the components of \( M \setminus \gamma_i \) is a 2-disk. It follows that \( \tau_i \) is isotopic to \( \tau_1 \) for \( i = 1, \ldots, a \), and to \( \text{id}_M \) for \( i = a + 1, \ldots, k \).

Let \( Q \) be a regular leaf of \( \Delta_f \) contained in \( U_1 \setminus \text{supp} \tau_1 \), see Figure 11.2. Then every \( \tau_j \) is fixed on some neighbourhood of \( Q \) in \( U_1 \setminus \text{supp} \tau_1 \).

Now let \( h \in J_0 \). Thus \( h = \tau_1^{i_1} \circ \cdots \circ \tau_a^{i_a} \circ \tau_{a+1}^{i_{a+1}} \circ \cdots \circ \tau_k^{i_k} \) for some \( i_j \in \mathbb{Z} \) and \( h \) is isotopic to \( \text{id}_M \). Put \( d = i_1 + \cdots + i_a \). Since \( M \setminus Q \) is a cylinder, we see that \( h \) is isotopic to \( \tau_1^d \) relatively to \( Q \).

(i) Suppose \( M \) is a 2-torus \( T^2 \). As \( h \) is isotopic to \( \text{id}_{T^2} \), we have that \( d = 0 \), whence \( \tau_1^d \) (and thus \( h \) itself) is isotopic to \( \text{id}_{T^2} \) relatively to \( Q \).

(ii) Let \( M \) be a Klein bottle \( \mathbb{K} \). It is well known that \( \tau_1^2 \) is isotopic to \( \text{id}_{\mathbb{K}} \), see [14, Lm. 5]. Moreover, we can assume \( Q \) is invariant (but not fixed!) under such an isotopy, see Figure 11.2. Since \( h \) is isotopic to \( \text{id}_{\mathbb{K}} \), we obtain that \( d \) is even and \( h \) is also isotopic to \( \text{id}_{\mathbb{K}} \) via an isotopy which leaves \( Q \) invariant.

Figure 11.2. Isotopy of \( \tau_1^2 \) to \( \text{id}_{\mathbb{K}} \)
Let \( \{g_\alpha\}_{\alpha \in A} \) be a set of generators for \( J_0 \), \( F \) be a free group generated by symbols \( \{\hat{g}_\alpha\}_{\alpha \in A} \), and \( H = \{\hat{h}_\beta\}_{\beta \in B} \subset F \) be the subset whose normal closure coincides with \( \ker \eta \). By the previous arguments for each \( g_\alpha \) there exists a path \( \omega_\alpha : I \rightarrow D_{id}(\mathbb{K}) \) between \( id_\mathbb{K} \) and \( g_\alpha \) such that \( \omega_\alpha(t)(Q) = Q \) for all \( \alpha \in A \) and \( t \in I \).

Let \( \beta \in B \) and \( \kappa_{\hat{h}_\beta} : I \rightarrow D_{id}(\mathbb{K}) \) be the corresponding loop in \( D_{id}(\mathbb{K}) \). Then \( \kappa_{\hat{h}_\beta}(t)(Q) = Q \) for all \( t \in I \) as well. Now it is easy to see that \( \kappa_{\hat{h}_\beta} \) is null-homotopic, so the assumption (2) of Lemma 11.1 holds.

Indeed, let \( \sigma \subset \mathbb{K} \) be a simple closed curve shown in Figure 11.2. Then \( \mathbb{K} \setminus \sigma \) consists of two Möbius bands. It is well-known that \( \pi_1 D_{id}(\mathbb{K}) = \mathbb{Z} \) and this group is generated by the isotopy which rotates \( \mathbb{K} \) twice along \( \sigma \), see e.g. [10]. In particular, if \( \kappa : I \rightarrow D_{id}(\mathbb{K}) \) is a loop being not null-homotopic, then \( \kappa(t)(Q) \neq Q \) for some \( t \in I \). Therefore \( \kappa_{\hat{h}_\beta} \) is null-homotopic for all \( \hat{h}_\beta \in \ker \eta \).

\[ \square \]

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