A SEPARABLE FRÉCHET SPACE OF ALMOST UNIVERSAL DISPOSITION

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ABSTRACT. The Gurarii space is the unique separable Banach space $G$ which is of almost universal disposition for finite-dimensional Banach spaces, which means that for every $\varepsilon > 0$, for all finite-dimensional normed spaces $E \subseteq F$, for every isometric embedding $e: E \to G$ there exists an $\varepsilon$-isometric embedding $f: F \to G$ such that $f \upharpoonright E = e$. We show that $G^N$ with a special sequence of semi-norms is of almost universal disposition for finite-dimensional graded Fréchet spaces. The construction relies heavily on the universal operator on the Gurarii space, recently constructed by Garbulińska-Węgrzyn and the third author. This yields in particular that $G^N$ is universal in the class of all separable Fréchet spaces.

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1. Introduction

A remarkable result of Banach and Mazur [13] states that the separable Banach space $C[0,1]$ is universal for separable Banach spaces. The above theorem has been extended by Mazur and Orlicz to the

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A class of separable Fréchet spaces, i.e., metrizable and complete locally convex spaces: They proved that the separable Fréchet space $C(\mathbb{R})$ is universal in the class of all separable Fréchet spaces, see again [13]. An essential progress of the research around the Banach-Mazur theorem is due to Gurariǐ. The Gurariǐ space constructed by Gurariǐ [8] in 1965, is the separable Banach space $G$ of “almost universal disposition for finite-dimensional spaces” that is:

(G) For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every isometric embedding $e: E \to G$ there exists an $\varepsilon$-isometric embedding $f: F \to G$ such that $f \upharpoonright E = e$.

Moreover, if $Y$ is any other separable Banach space fitting this definition, then there exists a linear isomorphism $u: G \to Y$ with $\|u\| \cdot \|u^{-1}\|$ arbitrarily close to 1. Lusky [11] proved that all separable Banach spaces of almost universal disposition are isometric, see also [10] for a simpler proof. Recall, that a linear operator $f: E \to F$ between Banach spaces $E$ and $F$ is an $\varepsilon$-isometric embedding for $\varepsilon > 0$ if

$$(1 + \varepsilon)^{-1} \cdot \|x\| \leq \|f(x)\| \leq (1 + \varepsilon) \cdot \|x\|, \quad x \in E \setminus \{0\}$$

Recall also that Gurariǐ has already observed in [8] that no separable Banach space $E$ is of universal disposition, i.e., satisfies condition (G) with removing $\varepsilon$.

Being motivated by recent developments in the theory of Fréchet spaces we will study the concrete separable Fréchet space $G^N$ endowed with the product topology and generated by two natural sequences of semi-norms, where $G$ is the Gurariǐ space. We prove that $G^N$ is universal in the class of all separable Fréchet spaces although (as we show) that there is no separable Fréchet space which is of universal disposition for finite-dimensional Fréchet spaces. Our main result states however that $G^N$ is the unique separable Fréchet space which is of almost universal disposition for finite-dimensional Fréchet spaces.

2. Preliminaries

Recall that a Fréchet space is a metrizable locally convex linear topological space which is complete with respect to its canonical uniformity. It is well-known that a complete separated locally convex topological vector space is a Fréchet space if and only if it satisfies the following condition: There exists a sequence $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ of semi-norms in $X$ such that $B_i(v, r) = \{x \in X : \max_{i \leq n} \|x - v\|_n < r\}$, where $i \in \mathbb{N}$, $v \in X$, $r > 0$, generate the topology of $X$. Fréchet spaces have played an important role in functional analysis for a very long time. Many vector
spaces of holomorphic, differentiable or continuous functions which appear in analysis and its applications carry a natural Fréchet topology. We refer the reader to an excellent survey of some recent developments in the theory of Fréchet spaces and of their duals, see [2]. We refer also to [13] for other fundamental facts and concepts related with Fréchet spaces.

In this paper Fréchet spaces $E$ are considered with a fixed sequence of semi-norms. In the case of an increasing sequence, i.e. if $\| \cdot \|_1 \leq \| \cdot \|_2 \leq \ldots$, we call (following Vogt [14]) $E$ endowed with this sequence a graded Fréchet space.

Graded Fréchet spaces have been studied in the context of the inverse function theorem of Nash and Moser, see [9], and in the context of tame Fréchet spaces, see e.g. [14, 15, 12]. Note however that the concept of the category of graded Fréchet spaces considered in [5] differs from the notion of graded Fréchet spaces used in this article. For a recent application of graded Fréchet spaces, we refer the interested reader to [4].

In this section we construct an increasing sequence of semi-norm on $G^N$ under which $G^N$ is a graded Fréchet space of almost universal disposition for finite-dimensional Fréchet spaces. Our construction uses the following result being a special case of Theorem 6.5 in [3]; property (2) below appears also as condition (‡) on page 765 in [3]. It turns out that condition (2) determines $\pi$ uniquely up to linear isometries, although this fact was not proved in [3].

**Theorem 2.1.** There exists a non-expansive linear operator $\pi: G \to G$ with the following properties.

1. For every non-expansive operator $T: X \to G$ with $X$ a separable Banach space, there exists an isometric embedding $i: X \to G$ such that $T = \pi \circ i$.

2. For every $\epsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every non-expansive linear operator $T: F \to G$, for every isometric embedding $e: E \to G$ such that $T \mid E = \pi \circ e$, there exists an $\epsilon$-isometric embedding $f: F \to G$ such that

$$\| f \mid E - e \| \leq \epsilon \quad \text{and} \quad \| \pi \circ f - T \| \leq \epsilon.$$  

Furthermore, $\ker \pi$ is linearly isometric to $G$.

The last proposition may be applied to obtain the following useful

**Proposition 2.2.** The operator $\pi$ from Theorem 2.1 is a projection and it satisfies the following condition:

1. For every $\epsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every non-expansive linear operator $T: E \to G$, for
every isometric embedding $e: E \to G$ such that $T \restriction E = \pi \circ e$, there exists an $\varepsilon$-isometric embedding $f: F \to G$ such that $f \restriction E = e$ and $\pi \circ f = T$.

Proof. Taking $T = \text{id}_E$ in Theorem \[2.1\], we see that $\pi$ is a projection. From now on, we shall identify the range of $\pi$ with a suitable subspace of its domain. In order to show (2'), we need to use the following fact which is easy to prove (see the proof of \[6\, \text{Theorem 2.7}\] for more details).

Claim 2.3. Let $F$ be a finite-dimensional normed space and let $S = \{v_0, \ldots, v_n\}$ be a linear basis of $F$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every pair of linear operators $f, g: F \to X$ into an arbitrary Banach space $X$, the following implication holds:

$$\max_{v \in S} \|f(v) - g(v)\| \leq \delta \implies \|f - g\| \leq \varepsilon.$$ 

Now assume that $E \subseteq F$ and $S \cap E$ is a linear basis of $E$. Fix $\varepsilon > 0$ and let $\delta > 0$ be as mentioned in the claim. We apply property (2) from Theorem \[2.1\] with $\delta$ instead of $\varepsilon$. This provides a map $f: F \to G$. Define $f'$ so that $f' \restriction E = e$ and $f'(v) = f(v)$ for every $v \in S \setminus E$. These conditions specify $f'$ uniquely and by the claim we have that $f'$ is a $2\varepsilon$-isometric embedding. Furthermore, we have that $\|\pi \circ f' - T\| \leq 2\varepsilon$. We have proved that for every $r > 0$ there is an $r$-isometric embedding $f: F \to G$ extending $e$ and such that $\|\pi \circ f - T\| \leq r$. Let us use this fact for $r = \delta$, where $\delta$ and $\varepsilon$ are as before. We obtain a $\delta$-isometric embedding $f: F \to G$ extending $e$ and satisfying $\|\pi \circ f - T\| \leq \delta$. Fix $v \in S \setminus E$. Then the vector $w_v = \pi(f(v)) - T(v)$ has norm $\leq \delta$. Define $f': F \to G$ so that $f' \restriction E = f$ and

$$f'(v) = f(v) - w_v, \quad v \in S \setminus E.$$ 

Note that $\|f'(v) - f(v)\| \leq \delta$ for $v \in S$, therefore $\|f' - f\| \leq \varepsilon$. Finally, $\pi f'(v) = \pi f(v) - \pi f(v) + \pi T(v) = T(v)$. It follows that $\pi f' = T$. □

Now we are ready to construct a graded sequence of semi-norms on $G^N$. Having in mind the last two results mentioned above, we conclude that there is a norm $\| \cdot \|_2$ on the product $G \times G \cong (\text{im} \, \pi) \times (\ker \, \pi)$ satisfying

$$\|x\|_G = \|(x, y)\|_1 \leq \|(x, y)\|_2$$

since $x = \pi(x, y)$ and $\pi$ is non-expansive. In addition, $(G^2, \| \cdot \|_2)$ is isometric to $G$. We identify $G \times G$ with $G$ and use the notation $\pi_2$ for $\pi$ in this case to stress that we consider it as mapping from $G^2$ to $G$. Inductively, for all $n \in \mathbb{N}$, we get a norm $\| \cdot \|_n$ on $G^n$ satisfying $\|x\|_1 \leq \cdots \leq \|x\|_{n-1} \leq \|x\|_n$ for all $x \in G^n$ and $(G^n, \| \cdot \|_n)$ is isometric.
to $G$. Therefore this construction provides an increasing sequence of semi-norms on $G^N$ as claimed. We use the notation $\pi_n : G^n \to G^{n-1}$ for the universal operator $\pi$ if we want to stress that we consider it as projection from $G^n$ to $G^{n-1}$, i.e., a projection onto the first $n-1$ components.

In order to formulate a condition similar to (G) for Fréchet spaces we need to define the corresponding concept of $\varepsilon$-isometries for Fréchet spaces.

**Definition 2.4.** Let $(X, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ and $(Y, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ be Fréchet spaces with fixed sequences of semi-norms. A mapping $f : X \to Y$ is called an $\varepsilon$-isometric embedding iff it is an embedding and

\[
(1 + \varepsilon)^{-1}\|x\|_i \leq \|f(x)\|_i \leq (1 + \varepsilon)\|x\|_i
\]

holds for all $i \in \mathbb{N}$ and all $x \in X$. The mapping $f$ is called an isometric embedding iff

\[
\|f(x)\|_i = \|x\|_i
\]

holds for all $i \in \mathbb{N}$ and all $x \in X$.

Now we are ready to formulate the analogue of condition (G) for Fréchet spaces.

**Definition 2.5.** A Fréchet space $E$ is of almost universal disposition for finite-dimensional Fréchet spaces if for all $\varepsilon > 0$ and for all finite-dimensional Fréchet spaces $X \subseteq Y$ with an isometric embedding $f_0 : X \to E$ there exists an $\varepsilon$-isometric embedding $f : Y \to E$ satisfying $f \upharpoonright E = f_0$.

### 3. A graded Fréchet space of almost universal disposition

We show that the space $G^N$ equipped with the graded sequence $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ of semi-norms defined above (coming from the universal operator $\pi$) is of almost universal disposition for finite-dimensional graded Fréchet spaces. We need the following

**Lemma 3.1.** Let $(X, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ and $(Y, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ be Fréchet spaces with fixed semi-norms and $i : X \to Y$ an isometric embedding. Then for all $i \in \mathbb{N}$ the mapping

\[
i_i : X / \ker \|\cdot\|_i \to Y / \ker \|\cdot\|_i, \overline{x} \mapsto \overline{i(x)}
\]

is a well-defined isometric embedding. Moreover, the diagram
\[
\prod X / \ker \parallel \cdot \parallel_i \xrightarrow{\iota} \prod Y / \ker \parallel \cdot \parallel_i
\]

is commutative.

**Proof.** As \(\iota\) is an isometric embedding, we have \(\|\iota(x)\|_i = \|x\|_i\). Hence \(x \in \ker \parallel \cdot \parallel_i\) iff \(\iota(x) \in \ker \parallel \cdot \parallel_i\), i.e. \(\iota_i\) is well-defined. By definition we have
\[
\|\iota_i(\bar{x})\|_i = \|\iota(x)\|_i = \|x\|_i = \|\bar{x}\|_i
\]
and hence \(\iota_i\) is an isometric embedding. The commutativity of the diagram directly follows from the definition of \(\iota_i\). \(\square\)

In addition, we also need the following technical

**Lemma 3.2.** Let \(X \subseteq Y\) and \(A\) be finite-dimensional Banach spaces, \(Z\) a Banach space, \(e: X \to A\) an isometric embedding, \(T: Y \to Z\) a linear operator with \(\|T\| \leq r\), \(r > 1\), and \(\pi: A \to Z\) a non-expansive operator such that the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\pi} & Z \\
e & \downarrow & \downarrow T \\
X & \xleftarrow{\pi} & Y
\end{array}
\]
commutes. There exists a finite-dimensional Banach space \(C\), an isometric embedding \(i_A: A \to C\), an \((r - 1)\)-isometric embedding \(i_Y: Y \to C\) and a non-expansive operator \(\pi': C \to Z\) such that the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{i_A} & C & \xrightarrow{\pi'} & Z \\
e & \downarrow i_A & \downarrow i_Y & \downarrow T \\
X & \xleftarrow{\pi} & C & \xleftarrow{T} & Y
\end{array}
\]
is commutative.

**Proof.** We define \(C := (A \oplus Y)/\Delta\) with \(\Delta = \{(e(x),-x): x \in X\}\) equipped with the norm
\[
\|(a,y)\| = \inf_{x \in X} \{\|a - e(x)\|_A + r\|x + y\|_Y\}.
\]
First we show that \( i_A \) is an isometry. For \( a \in A \) we obtain
\[
\|i_A(a)\| = \inf_{x \in X} \{\|a - e(x)\|_A + r\|x\|_Y\}
\]
\[
= \inf_{x \in X} \{\|a - e(x)\|_A + \|e(x)\|_A + (r - 1)\|x\|_Y\}
\]
\[
\geq \inf_{x \in X} \{\|a\|_A + (r - 1)\|x\|_Y\} = \|a\|_A
\]
and, by setting \( x = 0 \), \( \|i_A(a)\| \leq \|a\|_A \). For \( y \in Y \) we get
\[
\|i_Y(y)\| = \|(0, y)\| = \inf_{x \in X} \{\| - e(x)\|_A + r\|x + y\|_Y\} \leq r\|y\|_Y
\]
by setting \( x = 0 \), and
\[
\|i_Y(y)\| = \inf_{x \in X} \{\| - e(x)\|_A + r\|x + y\|_Y\}
\]
\[
= \inf_{x \in X} \{\| - x\|_Y + \|y + x\|_Y + (r - 1)\|x + y\|_Y\}
\]
\[
\geq \inf_{x \in X} \{\|y\|_Y + (r - 1)\|x + y\|_Y\} \geq \|y\|_Y \geq \frac{1}{r}\|y\|_Y,
\]
again using the triangle inequality. The linear operator \( \pi' : A \to Z \) can be defined as
\[
\pi'(a, y) = \pi(a - e(x)) + T(x + y) = \pi(a) - \pi(e(x)) + T(x) + T(y)
\]
\[
= \pi(a) + T(y).
\]
It satisfies
\[
\|\pi'(a, y)\|_Z = \|\pi'(a - e(x), x + y)\|_Z
\]
\[
\leq \|\pi(a - e(x))\|_Z + \|T(x + y)\|_Z
\]
\[
\leq \|a - e(x)\|_A + r\|x + y\|_Y
\]
for all \( x \in X \) and hence \( \|\pi'(a, y)\| \leq \|(a, y)\| \). \( \Box \)

**Theorem 3.3.** The space \( \mathbb{G}^N \) equipped with the graded sequence
\[
\{\| \|_n\}_{n \in \mathbb{N}}
\]
of semi-norms defined above is of almost universal disposition for finite-dimensional graded Fréchet spaces.

**Proof.** Given an isometric embedding \( f : X \to \mathbb{G}^N \) and \( \varepsilon > 0 \), we choose a sequence \( (\varepsilon_i)_{i \in \mathbb{N}} \) satisfying \( \prod_{i=1}^{\infty} (1 + \varepsilon_i) < 1 + \varepsilon \). We will use the notation \( X_i := X/\ker \| \|, Y_i := Y/\ker \| \|. \) By \( f_i : X_i \to \mathbb{G}^i \) we denote the mapping induced by the isometric embedding \( f : X \to \mathbb{G}^N \).

As a first step, we can use that \( X_1 \subseteq Y_1 \) are finite-dimensional Banach spaces and \( f_1 : X_1 \to \mathbb{G} \) is an isometric embedding to obtain an \( \varepsilon_1 \)-isometric embedding \( g_1 : Y_1 \to \mathbb{G} \) which extends \( f_1 \).
Now assume that we already have constructed a $\delta$-isometric embedding
$$g_{n-1}: Y_{n-1} \to \mathbb{G}^{n-1},$$
where $1 + \delta = \prod_{i=1}^{n-1}(1 + \varepsilon_i)$, extending $f_{n-1}: X_{n-1} \to \mathbb{G}^{n-1}$.

We consider the diagram
$$\begin{array}{c}
X_n & \xrightarrow{f_n} & Y_n \\
\downarrow & & \downarrow \\
\pi \circ f_n(X_n) & \xrightarrow{\pi} & \pi' \circ f_n(X_n) \\
\downarrow & & \downarrow \\
\pi' \circ f_n(X_n) & \xrightarrow{\pi'} & \pi' \circ f_n(X_n) \\
\downarrow & & \downarrow \\
C & \xrightarrow{g_n} & \mathbb{G}^n \\
\end{array}$$

where $p_{n-1}^n: Y_n \to Y_{n-1}$ denotes the canonical projection. Note that it is commutative since $\pi \circ f_n = f_{n-1}$. From Lemma 3.2 we deduce the existence of a finite-dimensional Banach space $C$ such that the diagram
$$\begin{array}{c}
\pi \circ f_n(X_n) & \xrightarrow{\pi'} & \pi' \circ f_n(X_n) \\
\downarrow & & \downarrow \\
\pi' \circ f_n(X_n) & \xrightarrow{\pi'} & \pi' \circ f_n(X_n) \\
\downarrow & & \downarrow \\
C & \xrightarrow{g_n} & \mathbb{G}^n \\
\end{array}$$

commutes and $\pi'$ is non-expansive. From Proposition 2.2 we may now conclude that there is an $\varepsilon_n$-isometric embedding $\tilde{g}_n: C \to \mathbb{G}^n$ which extends both $f_n$ and $\pi'$. Hence $g_n = \tilde{g}_n \circ \iota_Y$ is an $\delta'$-isometric embedding, where $1 + \delta' = \prod_{i=1}^{n}(1 + \varepsilon_i)$, which extends both $f_n$ and $g_{n-1}$.

Therefore by induction we get an $\varepsilon$-isometric embedding $g: Y \to \mathbb{G}^N$ extending the embedding $f: X \to \mathbb{G}^N$. \hfill $\square$

4. **Uniqueness and Universality**

The aim of this section is to prove universality and the following uniqueness result for the space $\mathbb{G}^N$.

**Proposition 4.1.** Let $E$ and $F$ be graded Fréchet spaces which are of almost universal disposition for finite-dimensional graded Fréchet spaces, $\varepsilon > 0$, $X \subseteq E$ a finite-dimensional subspace and $f: X \to F$ an $\varepsilon$-isometric embedding. Then there exists a bijective isometry $h: E \to F$ satisfying $\|h(x) - f(x)\|_i < 4\varepsilon\|x\|_i$ for all $i \in \mathbb{N}$.

First we need to show some additional lemmata used for the proof. Let $(X, \{\|\cdot\|_i\}_{i \in \mathbb{N}})$ and $(Y, \{\|\cdot\|_i\}_{i \in \mathbb{N}})$ be Fréchet spaces with a fixed sequence of semi-norms and $f: X \to Y$ a linear mapping with the property that for all $i \in \mathbb{N}$ there exists a constant $C_i > 0$ such that
∥f(x)∥_i ≤ C_i ∥x∥_i holds for all x ∈ X. We can do this since we only want to consider mappings which are isometries or at least ε-isometries.

We can now define

∥f∥_i = sup_{∥x∥_i=1} ∥f(x)∥_i

for all i ∈ N. Note that the above condition on f is stronger than the continuity of f.

We need the following Lemma motivated by [7, Lemma 2.2] for Banach spaces.

**Lemma 4.2.** Let \((X, \{∥·∥_{X,i}\}_{i∈N})\) and \((Y, \{∥·∥_{Y,i}\}_{i∈N})\) be finite-dimensional graded Fréchet spaces and let ε > 0 and \(f: X → Y\) be an ε-isometric embedding. There exists a finite-dimensional graded Fréchet space Z and isometric embeddings \(ι: X → Z\) and \(j: Y → Z\) such that \(∥j ∘ f − ι∥_i ≤ ε\) holds for all \(i ∈ N\).

**Proof.** We set \(Z = X ⊕ Y\) equipped with the semi-norms

\[
∥(x, y)∥_i = \inf\{∥u∥_{X,i} + ∥v∥_{Y,i} + ε∥w∥_{X,i}: x = u + w, \\
y = v - f(w), u, w ∈ X, v ∈ Y\}.
\]

We have

\[
∥j(f(x)) - ι(x)∥_i = ∥(x, -f(x))∥_i ≤ ε∥x∥_{X,i}
\]

by taking \(u = 0, v = 0\) and \(w = x\). Hence \(∥j ∘ f − ι∥_i ≤ ε\) for all \(i ∈ N\). From

\[
\frac{1}{1 + ε} ≥ 1 - ε ⇔ 1 ≥ 1 - ε^2
\]

we may deduce \(∥f(x)∥_{Y,i} ≥ (1 + ε)^{-1}∥x∥_{X,i} ≥ (1 - ε)∥x∥_{X,i}\) which we will need in the following. Now we show that ι is an isometric embedding. For \(x ∈ X\) we have

\[
∥ι(x)∥_i = ∥(x, 0)∥_i ≤ ∥x∥_{X,i}.
\]

Setting \(x = u + w\) and \(0 = v - f(w)\), we obtain

\[
∥u∥_{X,i} + ∥v∥_{Y,i} + ε∥w∥_{X,i} = ∥u∥_{X,i} + ∥f(w)∥_{Y,i} + ε∥w∥_{X,i}
\]

\[
≥ ∥u∥_{X,i} + (1 - ε)∥w∥_{X,i} + ε∥w∥_{X,i}
\]

\[
= ∥u∥_{X,i} + ∥w∥_{X,i} ≥ ∥u + w∥_{X,i} = ∥x∥_{X,i}
\]

and hence \(∥ι(x)∥_i = ∥(x, 0)∥_i ≤ ∥x∥_{X,i}\). Therefore ι is an isometric embedding. Analogously to above, we have \(∥j(y)∥_i = ∥(0, y)∥_i ≤ ∥y∥_{Y,i}\).
Setting \(0 = u + w\) and \(y = v - f(w)\), we obtain
\[
\|u\|_{X,i} + \|v\|_{Y,i} + \varepsilon \|w\|_{X,i} = \|w\|_{X,i} + \|v\|_{Y,i} + \varepsilon \|w\|_{X,i} = (1 + \varepsilon)\|w\|_{X,i} + \|v\|_{Y,i} \geq \|f(w)\|_{Y,i} + \|v\|_{Y,i} = \|y\|_{Y,i}
\]
and hence \(\|j(y)\|_i = \|(0, y)\|_i \geq \|y\|_{Y,i}\), i.e. \(j\) is also an isometric embedding.

We need also the next

**Lemma 4.3.** Let \(E\) be a graded Fréchet space which is of almost universal disposition for finite-dimensional graded Fréchet spaces, \(X \subseteq E\) a finite-dimensional subspace and \(\varepsilon > 0\). Given an \(\varepsilon\)-isometry \(f : X \to Y\), for all \(\delta > 0\) there exists a \(\delta\)-isometry \(g : Y \to E\) with the property \(\|g \circ f - \text{id}_X\|_{X,i} < 2\varepsilon\) for all \(i \in \mathbb{N}\).

**Proof.** Choose \(0 < \delta' < \min\{\delta, 1\}\). By Lemma 4.2 there exists a finite-dimensional Fréchet space \(Z\) and isometric embeddings \(i : X \to Z\) and \(j : Y \to Z\) such that \(\|j \circ f - i\|_i \leq \varepsilon\) for all \(i \in \mathbb{N}\). As \(E\) is of almost universal disposition for finite-dimensional Fréchet spaces, there exists a \(\delta'\)-isometric embedding \(h : Z \to E\) with the property \(h \circ i | X = \text{id}_X\). Setting \(g = h \circ j\), we get that \(g\) is a \(\delta\)-isometric embedding as it is the composition of a \(\delta\)-isometric embedding with an isometric embedding. Additionally for \(x \in X\) we obtain
\[
\|g(f(x)) - x\|_i = \|h(j(f(x))) - h(i(x))\|_i \leq (1 + \delta')\|j(f(x)) - i(x)\|_i \\
\leq \varepsilon(1 + \delta')\|x\|_i < 2\varepsilon\|x\|_i
\]
for all \(i \in \mathbb{N}\). □

Now we can use these results to show that \(G^N\) is, up to isometry, uniquely determined by the property of being of almost universal disposition for finite-dimensional Fréchet spaces.

**Proof of Proposition 4.1.** Let \(\{\varepsilon_n\}_{n \in \mathbb{N}}\) be a fixed sequence of positive real numbers satisfying a decay condition which will be specified later in the proof.

We pick \(\varepsilon_0 = \varepsilon\) and set \(X_0 = X, Y_0 = Y\) and \(f_0 = f\). By assumption the mapping \(f_0 : X_0 \to Y_0\) is an \(\varepsilon_0\)-isometric embedding.

Now assume that \(X_i, Y_i\) and \(f_i\) have already been constructed for \(i \leq n\) and that also the mappings \(g_i\) have been constructed for \(i < n\). Using Lemma 4.3 we obtain an \(\varepsilon_{n+1}\)-isometric embedding \(g_n : Y_n \to X_{n+1}\) satisfying
\[
\|g_n(f_n(x)) - x\|_i \leq 2\varepsilon_n\|x\|_i
\]
for all $i \in \mathbb{N}$. Here the space $X_{n+1}$ is defined as an appropriately
equilonged $g_n[Y_n]$ such that $Y_{n-1} \subseteq Y_n$ and $\bigcup_{n \in \mathbb{N}} X_n$ is dense in $E$. Again
by using Lemma 4.3 we get an $\varepsilon_{n+1}$-isometric embedding $f_{n+1}: X_{n+1} \to Y_{n+1}$ where $Y_{n+1}$ is chosen analogously to $X_{n+1}$. This mapping satisfies
\[(4) \quad \|f_{n+1}(g_n(y)) - y\|_i \leq 2\varepsilon_{n+1}\|y\|_i\]
for all $i \in \mathbb{N}$.

Now for fixed $n$ and $x \in X_n$ we get
\[
\|f_{n+1}(g_n(f_n(x))) - f_n(x)\|_i \leq 2\varepsilon_{n+1}\|f_n(x)\|_i \leq 2\varepsilon_{n+1}(1 + \varepsilon_n)\|x\|_i
\]
and
\[
\|f_{n+1}(g_n(f_n(x))) - f_{n+1}(x)\|_i \leq (1 + \varepsilon_{n+1})\|f_n(x) - x\|_i
\leq 2\varepsilon_n(1 + \varepsilon_{n+1})\|x\|_i.
\]
Using the triangle inequality, we obtain
\[
\|f_{n+1}(x) - f_n(x)\|_i \leq (\varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1})\|x\|_i
\]
from the inequalities above. Now we assume that
\[
\varepsilon_0 + 2\varepsilon_0\varepsilon_1 + \varepsilon_1 + \sum_{n=1}^{\infty} (\varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1}) < 2\varepsilon
\]
which implies that \{f(x_n)\}_{n \in \mathbb{N}} is a Cauchy sequence in $F$.

For $x \in \bigcup_{n \in \mathbb{N}} X_n$ we define $h(x) = \lim_{m \geq n} f_n(x)$ where $m$ is chosen
such that $x \in X_m$. Then $h$ is an $\varepsilon_n$-isometry for all $n \in \mathbb{N}$ and hence an
isometry which can be uniquely extended to an isometry on $E$, which
will be denoted by $h$ as well. From the inequalities above we deduce
\[
\|f(x) - h(x)\|_i \leq 2\varepsilon \sum_{n=0}^{\infty} (\varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1}) < 4\varepsilon.
\]
In order to show that $h$ is bijective, we repeat the above procedure to
show that \{g_n(y)\}_{n \geq m} is a Cauchy sequence for all $y \in Y_m$. Again we
may obtain an isometry $g_{\infty}: F \to E$. From the conditions (3) and (4) we
may conclude $g_{\infty} \circ h = \text{id}_F$ and $h \circ g_{\infty} = \text{id}_F$. \hfill \Box

We conclude this section by showing that every graded Fréchet space
can be embedded isometrically into $\mathbb{G}^\mathbb{N}$.

*Theorem 4.4.* The space $\mathbb{G}^\mathbb{N}$ is universal for separable Fréchet spaces.

*Proof.* Let $X$ be a separable Fréchet space and \{\| \cdot \|_i\}_{i \in \mathbb{N}} its fixed se-
quency of semi-norms. For all $i \in \mathbb{N}$ we denote by $X_i$ the normed space
$X/\ker \| \cdot \|_i$, equipped with the norm $\| \cdot \|_i$ and by $\tilde{X}_i$ its completion.
From the universality of $\mathbb{G}$ we may deduce the existence of an isometric
embedding \( f_1 : \tilde{X}_1 \to G \). Now assume that we have an isometric embedding \( f_i : \tilde{X}_i \to G^i \). Note that since \( \|x\|_i \leq \|x\|_{i+1} \) for all \( x \in X \), the composition \( T \) of \( f_i \) with the canonical mapping \( \text{can}_{i+1} : X_{i+1} \to X_i \) is non-expansive. As \( G^{i+1} \), equipped with the semi-norm \( \|\cdot\|_{i+1} \), is isometric to \( G \), we can use property (1) of the universal projection \( \pi_{i+1} \) given in Theorem 2.1 to find an isometric embedding \( f_{i+1} : \tilde{X}_{i+1} \to G^{i+1} \) so that the diagram

\[
\begin{array}{ccc}
G^{i+1} & \xrightarrow{\pi_{i+1}} & G^i \\
\downarrow{f_{i+1}} & & \downarrow{f_i} \\
X_{i+1} & \xrightarrow{\text{can}_{i+1}} & X_i
\end{array}
\]

is commutative. Hence,

\[
f : X \to \mathbb{G}^N, \quad x \mapsto ((f_i(\text{can}_i(x)))_{i \in \mathbb{N}}
\]

is an isometric embedding. \( \square \)

## 5. Final remarks

Note that by [13, Proposition V.5.4] the space \( C(\mathbb{R}) \) is universal for all separable Fréchet spaces. The following shows that, like in the case of \( C([0, 1]) \) for Banach spaces, the space \( C(\mathbb{R}) \) is not of almost universal disposition for finite-dimensional Fréchet spaces.

**Proposition 5.1.** Let \( X \) be a hemicompact space and \( \{K_i\}_{i \in \mathbb{N}} \) be a sequence of compact sets satisfying \( K_i \subseteq K_{i+1} \) and \( \bigcup_{i \in \mathbb{N}} K_i = X \). The space \( C(X) \) equipped with the sequence of semi-norms \( \|\cdot\|_i = \sup_{x \in K_i} |f(x)| \) is not of almost universal disposition for finite-dimensional Fréchet spaces.

**Proof.** Assume for a contradiction that \( C(X) \) is of almost universal disposition for finite-dimensional Fréchet spaces. By Proposition 4.1 we deduce that \( C(X) \cong \mathbb{G}^N \) holds isometrically. Hence for all \( i \in \mathbb{N} \) we obtain \( C(X)/\ker \|\cdot\|_i \cong \mathbb{G} \) isometrically. Observe that \( C(X)/\ker \|\cdot\|_i = C(K_i) \). This shows that \( C(K) \)-space is a Gurariĭ space, a contradiction with [1, Corollary 5.4]. \( \square \)

**Proposition 5.2.** There is no separable Fréchet space which is of universal disposition for finite-dimensional Fréchet spaces.

**Proof.** Assume, for a contradiction, that \( F \) is a separable Fréchet space of universal disposition for finite-dimensional Fréchet spaces. Hence \( F \) is also of almost universal disposition for finite-dimensional Fréchet spaces, and hence by Proposition 4.1 isometrically isomorphic to \( \mathbb{G}^N \). Therefore it is sufficient to show that \( \mathbb{G}^N \) is not of universal disposition.
Let $X \subseteq Y$ be finite-dimensional Banach spaces. Setting $\| \cdot \|_{X,i} := \| \cdot \|_X$ and $\| \cdot \|_{Y,i} := \| \cdot \|_Y$ for all $i \in \mathbb{N}$, we obtain two finite-dimensional Fréchet spaces. Assume, for a contradiction, that $G^\mathbb{N}$ is of universal disposition for finite-dimensional Fréchet spaces. Given an isometric embedding $f: (X, \| \cdot \|_X) \to G$ the product mapping $X \to G^\mathbb{N}, x \mapsto \{f(x)\}_{i \in \mathbb{N}}$ is an isometric embedding of Fréchet spaces. Hence there would exist an isometric extension $g: Y \to G^\mathbb{N}$. Therefore also the mapping $Y \to G, y \mapsto (g(y))_i$ is an isometry since $G = G^\mathbb{N}/\ker \| \cdot \|_1$ and it extends $f$. This would mean that $G$ is a separable Banach space of universal disposition for finite-dimensional Banach spaces, in contradiction to [6, Proposition 5.1].

We conclude the paper with the construction of a sequence of semi-norms on $G^\mathbb{N}$ under which it is of almost universal disposition for Fréchet spaces with a fixed but not necessarily increasing sequence of semi-norms.

For this we can use the semi-norms coming from the coordinates, namely, for each $n \in \mathbb{N}$ define $\|x\|'_n = \|x(n)\|_G$, $x \in G^\mathbb{N}$, where $\| \cdot \|_G$ is the norm of the Gurari˘ı space. We will not prove that this space is unique and universal as these proofs follow the lines of the corresponding ones for graded Fréchet spaces.

In order to shorten the notation, we denote by $X_i := X/\ker \| \cdot \|_i$ equipped with the norm $\| \cdot \|_i$ and by $Y_i$ the corresponding quotient of $Y$.

In order to show that $G^\mathbb{N}$ with these semi-norms is of almost universal disposition, we need the following

**Lemma 5.3.** Let $f_0: X \to G^\mathbb{N}$ be an ($\varepsilon$-)isometric embedding. For all $i \in \mathbb{N}$ the mapping $f_0^i: X_i \to G, \bar{x} \mapsto (f_0(x))_i$

is an ($\varepsilon$-)isometric embedding and the diagram

$$
\begin{array}{ccc}
X & \overset{f_0}{\longrightarrow} & G^\mathbb{N} \\
\Pi (X/\ker \| \cdot \|_i) \downarrow & & \downarrow \Pi f_0^i \\
X_i & & G^\mathbb{N}
\end{array}
$$

is commutative.

**Proof.** First we show that $f_0^i: X_i \to G, \bar{x} \mapsto (f_0(x))_i$. \hfill $\square$
is well-defined. Let \( x_1, x_2 \in X \) such that \( \| x_1 - x_2 \|_i = 0 \). Then

\[
\|(f_0(x_1))_i - (f_0(x_2))_i\| = \|(f_0(x_1 - x_2))_i\| \leq (1 + \varepsilon)\|x_1 - x_2\|_i = 0
\]

and hence \( (f_0(x_1))_i = (f_0(x_2))_i \), i.e. the mapping is well-defined. It is by definition an \((\varepsilon\text{-})\)isometric embedding. Finally note that the identity \( (f_0(x))_i = f_0(\bar{x}_i)\|\cdot\|_i \) for \( i \in \mathbb{N} \) follows analogously. □

**Proposition 5.4.** Let \((X, \{\| \cdot \|_i\}_i \in \mathbb{N})\) and \((Y, \{\| \cdot \|_i\}_i \in \mathbb{N})\) be finite-dimensional Fréchet spaces with fixed semi-norms, \( \varepsilon > 0 \), \( \iota: X \to Y \) and \( f_0: X \to \mathbb{G}^\mathbb{N} \) isometric embeddings. Then there is an \( \varepsilon \)-isometric embedding \( f: Y \to \mathbb{G}^\mathbb{N} \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f_0} & \mathbb{G}^\mathbb{N} \\
\downarrow{\iota} & & \downarrow{f} \\
Y & & \\
\end{array}
\]

is commutative.

**Proof.** As \( X \) and \( Y \) are finite-dimensional, the quotient spaces \( X_i \) and \( Y_i \) are finite-dimensional Banach spaces for all \( i \in \mathbb{N} \). By Lemma 3.1 the mappings \( \iota_i: X_i \to Y_i \) are isometric embeddings. The same is true for the mappings \( f_0^i: X_i \to \mathbb{G} \) by Lemma 5.3. As the Gurarii space is of almost universal disposition for finite-dimensional Banach spaces, there is an \( \varepsilon \)-isometric embedding \( f_i: Y_i \to \mathbb{G} \) making the diagram

\[
\begin{array}{ccc}
X_i & \xrightarrow{f_0^i} & \mathbb{G} \\
\downarrow{\iota_i} & & \downarrow{f_i} \\
Y_i & & \\
\end{array}
\]

commutative. For \( y \in Y \), we set \( f(y) = \{f_i(\|y\|_i)\}_{i \in \mathbb{N}} \), i.e. \( f \) is defined so that

\[
\prod Y_i \xrightarrow{\prod f_i} \mathbb{G}^\mathbb{N} \\
\downarrow{f} \\
Y
\]

is commutative. Since \( f_i \) is an \( \varepsilon \)-isometric embedding for all \( i \in \mathbb{N} \), we get

\[
\|f(y)\|_i = \|(f(y))_i\| = \|f_i(\|y\|_i)\| \leq (1 + \varepsilon)\|y\|_i = (1 + \varepsilon)\|y\|_i
\]

and by an analogous computation \( \|f(y)\|_i \geq (1 + \varepsilon)^{-1}\|y\|_i \), i.e. \( f \) is an \( \varepsilon \)-isometric embedding. Now let \( x \in X \), we have

\[
(f(\iota(x)))_i = f_i(\|\iota(x)\|_i) = f_i(\iota_i(x)) = f_0^i(\bar{x}) = (f_0(x))_i.
\]

Hence \( f(\iota(x)) = f_0(x) \), i.e. \( f \upharpoonright X = f_0 \). □
Remark 5.5. Note that in both cases $G^N/\ker \cdot \|_n = G^n$. Therefore all neighbourhoods of zero contain straight lines. This means in other words that there is no continuous norm on the space $G^N$ equipped with either of the sequences of semi-norms.

References

[1] A. Avilés, F. Cabello Sánchez, J. M. F. Castillo, M. González, and Y. Moreno. Banach space of universal disposition. *J. Funct. Anal.*, 261(9):2347–2361, 2011.
[2] K. D. Bierstedt and J. Bonet. Some aspects of the modern theory of Fréchet spaces. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 97(2):159–188, 2003.
[3] F. Cabello Sánchez, J. Garbulińska-Węgrzyn, and W. Kubiś. Quasi-Banach spaces of almost universal disposition. *J. Funct. Anal.*, 267(3):744–771, 2014.
[4] S. Dave. Rapidly converging approximations and regularity theory. *Monatsh. Math.*, 170(2):121–145, 2013.
[5] P. Domański and D. Vogt. A splitting theorem for the space of smooth functions. *J. Funct. Anal.*, 153(2):203–248, 1998.
[6] J. Garbulińska and W. Kubiś. Remarks on Gurarii spaces. *Extracta Math.*, 26(2):235–269, 2011.
[7] J. Garbulińska-Węgrzyn. *Universal structures in Banach spaces*. PhD thesis, Jagiellonian University, 2013.
[8] V. I. Gurarii. Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces. *Sibirsk. Mat. Ž.*, 7:1002–1013, 1966.
[9] R. S. Hamilton. The inverse function theorem of nash and moser. *Bull. Amer. Math. Soc. (N.S.)*, 7(1):65–222, 1982.
[10] W. Kubiś and S. Solecki. A proof of uniqueness of the Gurarii space. *Israel J. Math.*, 195(1):449–456, 2013.
[11] W. Lusky. The Gurarij spaces are unique. *Arch. Math. (Basel)*, 27(6):627–635, 1976.
[12] M. Poppenberg and D. Vogt. A tame splitting theorem for exact sequences of Fréchet spaces. *Math. Z.*, 219(1):141–161, 1995.
[13] S. Rolewicz. *Metric linear spaces*. PWN-Polish Scientific Publishers, Warsaw, 1972. Monografie Matematyczne, Tom. 56.
[14] D. Vogt. Operators between Fréchet spaces. Analysis Conference Manila, 1987.
[15] D. Vogt. Tame spaces and power series spaces. *Math. Z.*, 196(4):523–536, 1987.