Non-classical correlations in reducible quantum electrodynamics

Jan Naudts
Universiteit Antwerpen, Belgium
E-mail: Jan.Naudts@uantwerpen.be

Keywords: photon entanglement, reducible representations, spectral mode reconstruction, Bell inequality, DGCZ inequality, one-photon states

Abstract
The question is discussed whether the momentum of a photon has a quantum uncertainty or whether it is a classical quantity. The latter assumption is the main characteristic of reducible Quantum Electrodynamics (QED). Recent experiments in Quantum Optics may resolve the question. The non-classical correlation of quantum noise in color-entangled beams cannot be explained by reducible QED without modification of the standing explanation. On the other hand, reducible QED explains uncertainty of the momentum of a single photon when it is entangled with a quantum spin residing in its environment. The explanation of the historical experiment with equally-polarized pairs of photons, showing violation of the Bell inequalities, invokes the argument of collapse of the wave function, also in reducible QED.

1. Introduction
In Fock space a superposition of a horizontally polarized photon with wave vector \( k \) and a vertically polarized photon with wave vector \( k' \) is described by

\[
\frac{1}{\sqrt{2}} (a_{H}^{+}|0,0\rangle + a_{V}^{+}|0,0\rangle).
\]

Here, \(|0,0\rangle\) denotes the vacuum state and \(a_{H}^{+}, a_{V}^{+}\) are the creation operators for a horizontally, respectively vertically polarized photon. Such a superposition with equal vectors, \( k = k' \), is for instance needed to describe circularly polarized states. The existence of superpositions of the form

\[
\frac{1}{\sqrt{2}} (a_{H}^{+}|0,0\rangle + a_{H'}^{+}|0,0\rangle), \quad k = k',
\]

involving a single photon in superposition of two distinct wave vectors, is required in the analysis of color entanglement as found in [1, 2]. The existence of color entanglement has been experimentally established in a convincing manner [3–10]. The present paper investigates the question whether the assumption of existence of the superposition (2) is a necessity for explaining the experimental observations.

The superpositions (1, 2) with \( k = k' \) describe photon states whose energy and momentum are undetermined. They are not allowed in reducible Quantum Electrodynamics (QED) and are replaced by a single wave function which depends on the wave vector, i.e. \( \psi_{k} = a_{H}^{+}|0,0\rangle \) and \( \psi_{k'} = a_{V}^{+}|0,0\rangle \), respectively

\( \psi_{k'} = a_{H}^{+}|0,0\rangle \). The present paper questions whether states of the form (1) or (2) do occur in Nature. If they don’t then a modified theory of QED, such as reducible QED, is needed.

The original picture behind QED is that Euclidean space is filled with two-dimensional quantum harmonic oscillators the excitations of which are photons. Marek Czachor suggested [11] that the frequency of these oscillators should be quantized as well. Together with his collaborators he developed a non-canonical theory of QED. See [11–18] and references quoted in these papers. One of the main results of the theory is that QED after renormalization is recovered as a limiting case, without the need to fall back on ad hoc procedures like renormalization. On the other hand, an attempt to force the formalism into a mathematically rigorous framework was not successful [19].

© 2019 The Author(s). Published by IOP Publishing Ltd
The formalism of Czachor is still compatible with the existence of superpositions of the form (1), (2). In fact, it emphasizes the role of these superpositions by assuming that the frequency of the oscillators, and hence the momentum of the photons, is undergoing quantum fluctuations. Recently [20, 21], the present author restored the historical assumption of the frequency being a parameter and added some simplifying assumptions to the Czachor formalism. The resulting theory is a reducible version of QED. It is shortly introduced in the next section.

Section 3 investigates whether reducible QED is compatible with experiments on multicolor beams. Section 4 analyzes the historical experiments demonstrating the entanglement of a photon pair. At the end follows a section Discussion and Conclusions.

2. Reducible QED

In reducible QED a quantum electromagnetic field is described by a normalized wave function \( \psi_k \) in the Hilbert space \( \mathcal{H} \) of the two-dimensional quantum harmonic oscillator. The wave function \( \psi_k \) depends on the wave vector \( k \) which is a non-vanishing vector in the three-dimensional Euclidean space \( \mathbb{R}^3 \). A relativistic description is obtained by adding \( |k| \) as the zeroth component of the 4-momentum.

A special role is reserved for the coherent electromagnetic fields. Let \( F_H(k) \) and \( F_V(k) \) be two complex functions of the wave vector \( k \). Let \( [F_H(k), F_V(k)]^c \) denote the corresponding coherent wave function of the two-dimensional quantum harmonic oscillator. It describes a coherent electromagnetic field. A short calculation shows that its total energy equals

\[
\langle H \rangle = \epsilon^H \int dk /\hbar |k| \left[ |F_H(k)|^2 + |F_V(k)|^2 \right].
\]

This gives \( \epsilon^H |F_H(k)|^2 \) and \( \epsilon^V |F_V(k)|^2 \) the meaning of the density of horizontally respectively vertically polarized photons with wave vector \( k \).

Details of the formalism of reducible QED are found in the appendices of [21].

3. Fluctuations of the electric field operator

In reducible QED the electric field operator at space-time position \( x \) is given by

\[
E_{\alpha,k}(x) = c |k| \frac{\lambda}{2N_0(k)} \left[ e^{(H)}(k) \right] i \left[ e^{-ik_x x^\alpha} a_H - e^{ik_x x^\alpha} a_H^\dagger \right]
+ c |k| \frac{\lambda}{2N_0(k)} \left[ e^{(V)}(k) \right] i \left[ e^{-ik_x x^\alpha} a_V - e^{ik_x x^\alpha} a_V^\dagger \right],
\]

It differs from the expression found in standard QED by the integration over the wave vector which is missing.

The intensity of a light beam is measured with a photo detector. It produces an electric signal proportional to the intensity. The quantum uncertainties \( \Delta E_H^\alpha(x) \) and \( \Delta E_V^\alpha(x) \) of the two quadratures of the electric field result in a contamination of the electric signal with noise. In the case of a coherent field this is shot noise and its intensity is referred to as the quantum noise level. The noise is a direct evidence of the quantum nature of light. In particular, squeezing of light results in a reduction of noise in one of the quadratures. Reid [1] proposed to measure the noise levels of the two quadratures in order to demonstrate the Einstein-Podolsky-Rosen (EPR) paradox.

Expressions for the autocorrelations of a polarized beam are derived in appendix A within the formalism of reducible QED. A beam consisting of two subbeams with distinct wave vectors \( k_1 \neq k_2 \) is considered. Because superpositions are not allowed the covariance matrix of the subbeams cannot violate the DGCZ inequality [22]. However, entanglement with the environment can never be excluded. It is shown in appendix B that when the two subbeams are entangled with a single Pauli spin then it is possible to describe the situation that the two subbeams together contain a single photon the wave vector of which depends on the state of the Pauli spin. Sufficient conditions for violation of the DGCZ inequality are derived.

In principle it is possible to reconstruct the quantum state of a two-color photon beam by spectrum analysis of the time-dependent photon intensity. It is shown in appendix C that the analysis of [23, 24] carries over to the context of reducible QED.

4. Absorption of a photon

Let us finally discuss some aspects of non-locality in Quantum Mechanics. The historical experiments [25, 26] violating the Bell inequalities [27] can only be understood if one assumes that the detection of a photon causes a
collapse of the wave function. The collapse prevents a subsequent measurement to detect the same photon once again.

The analysis of the original experiment can be adapted to reducible QED. See the appendix D. The main assumption is that it suffices to consider the isolated system consisting of two photons which can be distinguished for instance by their polarization or by their average wave vector. It then follows in a straightforward manner that annihilation of one of the two photons leaves only one photon to be detected by the second observer. The resulting statistics is the same as in the analysis of [28].

5. Discussion and conclusions

Let A and B denote two non-intersecting regions in the space of wave vectors \( \mathbf{k} \). They define two subbeams with different ‘color’. Experimental papers on color entanglement observe a non-classical correlation between the noise signals which shows up when the two subbeams are measured each with their own photo detector. The quantum nature of the correlations is established by demonstrating the violation of a DGCZ inequality [22]. This theoretical explanation cannot be reproduced here without modification because in reducible QED the density matrices \( \rho_\mathbf{k} \) and \( \rho_\mathbf{k'} \), as given by (3) in appendix A, are separable right from the start.

At first sight one may conclude that reducible QED is not compatible with experiment. An easy way out to conciliate both is to take into account that entanglement with the environment is unavoidable. As proved in appendix B, entanglement with a quantum spin suffices to allow violation of the DGCZ inequalities under certain circumstances.

The quantum electromagnetic field of multicolor beams can be reconstructed from experimental data [23, 24]. In principle, these reconstructions should allow to decide whether superpositions of photons with distinct wave vector are allowed. However, the calculations of appendix C show that reducible QED reproduces the same results as those obtained in [24].

The standard explanation of the historical experiments [25, 29], demonstrating the entanglement of a pair of equally polarized photons, relies on the probabilistic interpretation of quantum mechanics and on the assumption of a collapse of the wave function. A similar reasoning is used in appendix D. The assumption is made that the measurement of a photon by one detector destroys the photon and modifies the wave function by the action of the annihilation operator \( a \). A subsequent measurement by the second detector yields a correlated result.

6. Conclusions

In mainstream QED two photons with distinct wave vectors \( \mathbf{k} \neq \mathbf{k'} \) are treated as distinct particles. This is a consequence of the canonical commutation relations, which postulate that the commutator of creation and annihilation operators is proportional to a Dirac delta function \( \delta^{(3)}(\mathbf{k} - \mathbf{k'}) \). As a consequence it is meaningful to study the bi-partite entanglement of photons and to interpret the observed quantum correlations in terms of entanglement. In reducible QED the entanglement of photons with different wave vectors is not allowed. Never the less, a bipartite experiment can reveal non-classical correlations. This is the main result of the present paper.

Some difficulties remain. The analysis of the historical two-photon experiment [25, 29] invokes the argument of the collapse of the wave function. The same explanation works in the context of reducible QED. However, a more in depth analysis requires the use of measurement theory, which is not considered here. The main question of the present paper, whether superpositions of photon states with unequal wave vectors are physical, and hence whether reducible QED, as presented here, is a complete description of quantum Electrodynamics, is left open.

Appendix A. The covariance matrix

Here and in the next appendix we focus on a polarized light beam. To simplify the notations all references to the polarization are omitted.

In the presence of interactions with the environment, the electromagnetic field is described in reducible QED by a wave vector-dependent density matrix \( \rho_\mathbf{k} \) instead of a wave function \( \psi_\mathbf{k} \). Fix two wave vectors \( \mathbf{k} \) and \( \mathbf{k'} \) and consider measurements at wavelength \( \mathbf{k} \) to be independent from measurements at wavelength \( \mathbf{k'} \). Any observable \( A \) at wavelength \( \mathbf{k} \), respectively \( \mathbf{k'} \), maps onto an operator \( A \otimes \mathbb{1} \), respectively \( \mathbb{1} \otimes A \). Because the representation is reducible the density matrix \( \rho_{\mathbf{k}\mathbf{k'}} \) of the product space is separable and can be written as
\[ \rho_{\mathbf{k},\mathbf{k}'} = \sum_n p_n \sigma_n \otimes \tau_n, \]

with density matrices \( \sigma_n, \tau_n \) and classic probabilities \( p_n \geqslant 0, \sum_n p_n = 1 \). They satisfy

\[ \rho_{\mathbf{k}} = \sum_n p_n \sigma_n \quad \text{and} \quad \rho_{\mathbf{k}'} = \sum_n p_n \tau_n. \]

Averages are given by

\[ \begin{align*}
\langle A \rangle_{\mathbf{k}} & \equiv \text{Tr} \rho_{\mathbf{k}} A = \sum_n p_n \text{Tr} \sigma_n A, \\
\langle B \rangle_{\mathbf{k}'} & \equiv \text{Tr} \rho_{\mathbf{k}'} B = \sum_n p_n \text{Tr} \tau_n B, \\
\langle A \otimes B \rangle & = \sum_n p_n (\text{Tr} \sigma_n A)(\text{Tr} \tau_n B). 
\end{align*} \]

Following [24] we consider a column vector \( \mathbf{X} \) with 4 elements given by

\[ \mathbf{X}^T = (p \otimes \mathbb{I} \ q \otimes \mathbb{I} \ I \otimes p \ \mathbb{I} \otimes q). \]

The operators \( p \) and \( q \) are defined by

\[ q = \frac{1}{\sqrt{2}} (a + a^*) \quad \text{and} \quad p = \frac{i}{\sqrt{2}} (a^* - a) \]

and satisfy the commutation relation \([p, q] = -i\). A covariance matrix is then defined by

\[ \Sigma_{ij}(\mathbf{k}, \mathbf{k}') = \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle. \]  

One has

\[ \begin{pmatrix} \langle p^2 \rangle_{\mathbf{k}} & \langle pq \rangle_{\mathbf{k}} & \langle p \otimes p \rangle & \langle p \otimes q \rangle \\ \langle qp \rangle_{\mathbf{k}} & \langle q^2 \rangle_{\mathbf{k}} & \langle q \otimes p \rangle & \langle q \otimes q \rangle \\ \langle p \otimes p \rangle & \langle q \otimes p \rangle & \langle p^2 \rangle_{\mathbf{k}'} & \langle pq \rangle_{\mathbf{k}'} \\ \langle p \otimes q \rangle & \langle q \otimes q \rangle & \langle pq \rangle_{\mathbf{k}'} & \langle q^2 \rangle_{\mathbf{k}'} \end{pmatrix} \]

and

\[ \langle \mathbf{X}^T \rangle = (\langle p \rangle_{\mathbf{k}} \ \langle q \rangle_{\mathbf{k}} \ \langle p \rangle_{\mathbf{k}'} \ \langle q \rangle_{\mathbf{k}'}). \]

The criterion used in the Literature to decide whether the two subbeams are entangled is based on violation of the DGCZ inequality [22]. Note that the density matrix \( \rho(\mathbf{k}, \mathbf{k}') \), given by (3), is separable. Therefore the inequality cannot be violated by this density matrix.

Introduce for instance [5] operators \( p_{\pm} = (p \otimes \mathbb{I} \pm \mathbb{I} \otimes p) / \sqrt{2} \) and \( q_{\pm} = (q \otimes \mathbb{I} \pm \mathbb{I} \otimes q) / \sqrt{2} \). They satisfy \([q_+, p_-] = i\). Then the inequality reads

\[ \Delta^2 p_+ + \Delta^2 q_+ \geqslant 1. \]

A1. The coherent case

Consider the case that the density matrices \( \sigma_n \) and \( \tau_n \) are orthogonal projections onto coherent states \( |F_n(\mathbf{k}) + iG_n(\mathbf{k})\rangle \), respectively \( |F_n(\mathbf{k}') + iG_n(\mathbf{k}')\rangle \), where \( F_n(\mathbf{k}) \) and \( G_n(\mathbf{k}) \) are real functions of the wave vector \( \mathbf{k} \). Then one calculates

\[ \langle X \rangle = \sqrt{2} \sum_n p_n Y_n \]

with

\[ Y_n^T = (G_n(\mathbf{k}) \quad F_n(\mathbf{k}) \quad G_n(\mathbf{k}') \quad F_n(\mathbf{k}')). \]

and

\[ \begin{pmatrix} \langle X_i X_j \rangle \end{pmatrix}_{ij} = \Sigma^{(0)} + 2 \sum_n p_n Y_n Y_n^T, \]

with

\[ \Sigma^{(0)} = \frac{1}{2} \begin{pmatrix} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \end{pmatrix}. \]
One concludes that in this case the covariance matrix $\Sigma(k, k')$ equals $\Sigma^{(0)}$. In particular, the two subbeams are not correlated. From

$$\langle \Delta^2 p_+ \rangle = \frac{1}{2} \Sigma_{11} + \frac{1}{2} \Sigma_{33} - \Sigma_{13},$$

$$\langle \Delta^2 q_+ \rangle = \frac{1}{2} \Sigma_{22} + \frac{1}{2} \Sigma_{44} + \Sigma_{24}$$

(4)

it follows that in the case of a superposition of coherent fields one has $\Delta^2 p_+ = \Delta^2 q_+ = 1/2$. Hence, in this case the DGCZ inequality is actually an equality, as expected.

### Appendix B. Entanglement with the environment

The field of density matrices $\rho_k$, introduced in the previous Appendix, cannot explain all experimental data. Let us therefore consider an explicit example of a photon field $\psi_k$ entangled with a Pauli spin in its environment. The state of the system is described by

$$\mathcal{X} \psi_k \otimes |\uparrow\rangle + \lambda \psi_k \otimes |\downarrow\rangle$$

with $\mathcal{X}$ and $\lambda$ complex numbers satisfying $|\mathcal{X}|^2 + |\lambda|^2 = 1$. The quantum expectation of a field operator $A$ is given by

$$\langle A \otimes |\rangle\rangle_k = |\mathcal{X}|^2 \langle \psi_k^A | A \psi_k^A \rangle + |\lambda|^2 \langle \psi_k^A | A \psi_k^A \rangle.$$ 

Repeat the construction of the previous Appendix. Consider the wave vectors $k$ and $k'$ as belonging to independent subbeams. Then the quantum expectation of an observable of the form $A \otimes B \otimes |\rangle\rangle_k, k'$, where $A$ refers to the wave vector $k$ and $B$ to $k'$, is given by

$$\langle A \otimes B \otimes |\rangle\rangle_k, k' = |\mathcal{X}|^2 \langle \psi_k^A | A \psi_k^A \rangle \langle \psi_k^B | B \psi_k^B \rangle + |\lambda|^2 \langle \psi_k^A | A \psi_k^A \rangle \langle \psi_k^B | B \psi_k^B \rangle.$$ 

#### B1. Coherent case

Calculate the covariance matrix $\Sigma(k, k')$, assuming coherent wave functions

$$|\psi_k^A\rangle = |F^A(k) + iG^A(k)\rangle,$$

with $F^A(k), G^A(k), G^B(k)$ real functions of the wave vector $k$. One finds

$$\langle X \rangle = |\mathcal{X}|^2 \sqrt{2} Y^1 + |\lambda|^2 \sqrt{2} Y^1$$

with

$$(Y^1)^T = (G^A(k) \quad F^A(k) \quad G^B(k) \quad F^B(k'))$$

and a similar definition for $Y^1$. The second order expression is

$$\langle (X_i X_j) \rangle_{k,d} = \Sigma^{(0)} + 2|\mathcal{X}|^2 Y^1(Y^1)^T + 2|\lambda|^2 Y^1(Y^1)^T.$$ 

One obtains

$$\Sigma(k, k') = \Sigma^{(0)}$$

$$+ 2|\mathcal{X}|^2 (1 - |\mathcal{X}|^2) Y^1(Y^1)^T$$

$$+ 2|\lambda|^2 (1 - |\lambda|^2) Y^1(Y^1)^T$$

$$- 2|\mathcal{X}|^2 |\lambda|^2 [Y^1(Y^1)^T + Y^1(Y^1)^T].$$

Now calculate

$$\langle \Delta^2 p_+ \rangle = \frac{1}{2} + |\mathcal{X}|^2 [G^A(k) - G^A(k')]^2 + |\lambda|^2 [G^B(k) - G^B(k')]^2$$

$$- (|\mathcal{X}|^2 [G^A(k) - G^A(k')]^2 + |\lambda|^2 [G^B(k) - G^B(k')]^2) \geq \frac{1}{2}.$$
The inequality follows because the function \( f(x) = x^2 \) is convex and \( |\lambda|^2 + |\lambda|^2 = 1 \). Similarly is
\[
\langle \Delta^2 q_+ \rangle = \frac{1}{2} + |\lambda|^2 [F^+(k) + F^+(k')]^2 + |\lambda|^2 |F^-(k) + F^-(k')|^2
- (|\lambda|^2 |F^+(k) + F^+(k')| + |\lambda|^2 |F^-(k) + F^-(k')|^2) \geq \frac{1}{2}.
\]
One concludes that the DGCZ inequality is not violated. This is not unexpected. See the comments made in [30].

**B2. One-photon states**

In the other extreme case the wave function is a one-photon state entangled with a spin state. It is described by
\[
|c^+(k)|1\rangle + \gamma^+(k)|0\rangle \otimes |\uparrow\rangle + \frac{1}{\sqrt{2}}|c^+(k)|1\rangle + \gamma^+(k)|0\rangle \otimes |\downarrow\rangle ,
\]
where \( c^+(k) \) and \( c^+(k) \) are complex functions satisfying \( |c^+(k)| \leq 1 \), and
\[
\gamma^+(k) = \sqrt{1 - |c^+(k)|^2},
\]
with a similar definition of \( \gamma^+(k) \). For the sake of simplicity the two spin states have the same weight.

Let \( X \) be as before. A tedious calculation yields the DGCZ inequality
\[
\frac{1}{2} |c^+(k)|^2 + \frac{1}{2} |c^+(k')|^2 + \frac{1}{2} |c^-(k)|^2 + \frac{1}{2} |c^-(k')|^2
+ \frac{1}{2} |c^+(k')|^2 + \frac{1}{2} |c^+(k')|^2 + \frac{1}{2} |c^-(k')|^2 + \frac{1}{2} |c^-(k')|^2
\geq \gamma^+(k) \gamma^+(k') \left[ \text{Re } c^+(k) \text{Re } c^+(k') + \text{Im } c^+(k) \text{Im } c^+(k') \right]
+ \gamma^+(k') \gamma^+(k') \left[ \text{Re } c^+(k') \text{Re } c^+(k) + \text{Im } c^+(k') \text{Im } c^+(k') \right]
+ \gamma^+(k) \gamma^+(k') \left[ \text{Re } c^+(k) \text{Re } c^+(k') - \text{Im } c^+(k) \text{Im } c^+(k') \right]
+ \gamma^+(k) \gamma^+(k') \left[ \text{Re } c^+(k') \text{Re } c^+(k) - \text{Im } c^+(k) \text{Im } c^+(k') \right].
\]

Take for instance
\[
\text{Re } c^+(k) = \text{Re } c^+(k') = \text{Re } c^+(k') = \text{Re } c^+(k') = u,
\]
\[
\text{Im } c^+(k) = \text{Im } c^+(k') = -\text{Im } c^+(k') = -\text{Im } c^+(k') = v.
\]
The DGCZ inequality becomes
\[
2u^2 + 2v^2 + 2(u^2 + v^2) \geq 4(1 - u^2 - v^2)(u^2 + v^2),
\]
which is equivalent with \( u^2 + v^2 \geq 1/3 \). One concludes that for small intensities the inequality can be violated.

**Appendix C. Reconstruction of spectral modes**

The photon current \( I(x) \) measured by a detector at space-time position \( x \) is proportional to the expectation \( \langle E^{(+)} E^{(+)} \rangle \), where \( E^{(+)} \) are the positive and negative frequency parts of the electric field operator. For a given field \( \psi \) one obtains
\[
I(x) \sim \epsilon^3 \int dk \int dk' \sum_{\alpha} \langle \psi_k | E^{(+)}_{\alpha,k}(x) E^{(+)}_{\alpha,k}(x) | \psi_k \rangle = \sum_{\alpha} \langle \theta_{\alpha} (x) | \theta_{\alpha} (x) \rangle
\]
with
\[
|\theta_{\alpha} (x)\rangle = \epsilon^{3/2} \int dk E^{(+)}_{\alpha,k}(x) |\psi_k\rangle = \epsilon^{3/2} \int dk \frac{|k| \lambda}{2N_0(k)} e^{-ikx} \left[ \phi^{(+)}_{\alpha}(k) a_{\alpha1} |\psi_k\rangle + e^{(V)}(k) a_{\alpha2} |\psi_k\rangle \right].
\]
Consider now a signal field with wave functions \( \psi_{k}^{s} \). It is the field which one wants to investigate by letting it interfere with a coherent field. The superposition of both fields is described by wave functions
\[
|\psi_k\rangle = \sqrt{1 - \epsilon^2} |F_{\text{H}}(k), F_{\text{V}}(k)\rangle + \epsilon |\psi_{k}^{s}\rangle,
\]
where $F_1(k)$ and $F_2(k)$ are complex functions and $0 < \epsilon \ll 1$ is a small constant. One obtains

$$
|\theta_\alpha(x)\rangle = \epsilon^{3/2} \int dk \frac{c|k|\lambda}{2N_0(k)} e^{-ikx}\left[\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)\right]|F_1(k), F_2(k)\rangle + \epsilon^{3}\int dk \frac{c|k|\lambda}{2N_0(k)} e^{-ikx}\left[\varepsilon^{(H)}_\alpha(k)a_\alpha^\dagger|\psi^H_k\rangle + \varepsilon^{(V)}_\alpha(k)a_\alpha|\psi^V_k\rangle\right] + O(\epsilon^2).
$$

The photon current becomes

$$
I(x) \sim \epsilon^3 \int dk \frac{c|k|\lambda}{2N_0(k)} e^{ikx}e^{ikx} \int dk' \frac{c|k'|\lambda}{2N_0(k')} e^{-ik'x} \sum_\alpha \left[|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2 + \epsilon^2|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2\right]
$$

$$
\times \left(\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle + \epsilon^2|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2\right)
$$

$$
\times \left(\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle + \epsilon^2|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2\right)
$$

$$
\times \left(\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle + \epsilon^2|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2\right)
$$

$$
\times \left(\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle + \epsilon^2|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2\right) + O(\epsilon^2).
$$

Relevant information is obtained by spectral analysis of the time dependence of $I(x)$. Let $A$ denote the set of wave vectors $k$ for which $|F_1(k), F_2(k)|^2 = 0, 0$ and $B$ the set of wave vectors $k$ for which $|F_1(k), F_2(k)|^2 = 0, 0$. Assume that $A$ and $B$ do not overlap and select a frequency $\omega$ such that $|k| = |k| + \omega/\epsilon = 0$ for any pair of wave vectors $k, k'$ in $A$. Then the contribution of leading order in $\epsilon$ vanishes and one obtains

$$
\bar{I}(\omega) \equiv \frac{1}{2\pi\epsilon} \int d\omega e^{i\omega t} I(x)
$$

$$
\sim 2 \Re \epsilon^3 \int_A dk \frac{c|k|\lambda}{2N_0(k)} \int_B dk' \frac{c|k'|\lambda}{2N_0(k')} e^{-i(k-k')x}\left(\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle + \epsilon^2|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2\right)
$$

$$
\times \left(\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle + \epsilon^2|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2\right)
$$

$$
\times \left(\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle + \epsilon^2|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2\right)
$$

$$
\times \left(\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle + \epsilon^2|\varepsilon^{(H)}_\alpha(k)F_1(k) + \varepsilon^{(V)}_\alpha(k)F_2(k)|^2\right) + O(\epsilon^2).
$$

Experimentally, it is feasible to measure at once this spectrum and the one with $F(k)$ replaced by $iF(k)$ [24]. In this way also the imaginary part of $\bar{I}(\omega)$ is obtained. By varying experimental parameters one can obtain values for the functions $\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle$ and $\langle F_1(k), F_2(k)\rangle \langle F_1(k'), F_2(k')\rangle$. The coherent states form an overcomplete basis of the Hilbert space of wave functions. One concludes that, in principle, a reconstruction of the wave vectors $a_\alpha^\dagger|\psi^H_k\rangle$ and $a_\alpha|\psi^V_k\rangle$ is feasible. This shows that the results of [24] translate to the context of reducible QED.

**Appendix D. Entanglement of polarized photon states**

Assume here for convenience that horizontally polarized photons are independent from vertically polarized photons. Then the wave function $|\psi//\rangle$ of the two-dimensional harmonic oscillator is said to be separable when it can be written in the form $|\psi//\rangle \otimes |\psi\rangle$.

Consider a two-photon field of the form

$$
|\psi_k\rangle = c_{00}|0, 0\rangle + c_{20}|2, 0\rangle + c_{11}|1, 1\rangle + c_{02}|0, 2\rangle,
$$

with wave vector-dependent complex coefficients $c_{ij}$ satisfying $\sum |c_{ij}|^2 = 1$. Assume two sites $x, y$ in space-time where it is possible to detect a photon. The measurement at position $x$ destroys a photon of either horizontal or vertical polarization, using one of the following two annihilation operators

$$
a_\phi^\dagger = \cos(\phi)a_{11} + \sin(\phi)a_\alpha \quad \text{ detection of } H,
$$

$$
a_\phi^\dagger = -\sin(\phi)a_{11} + \cos(\phi)a_\alpha \quad \text{ detection of } V.
$$

The angle $\phi$ takes into account that the basis of polarization can be rotated. The wave function $|\psi_k\rangle$ can be written in terms of these rotated operators as

---

1. de Broglie and Andrade e Silva [31] argue that the intensity of the electric field determines the probability of detecting a photon. The strength of the electron-photon interaction in reducible QED depends indeed on the intensity of the electric field [21].
\[ |\psi_k\rangle = \left\{ d_{00} + \frac{1}{\sqrt{2}}d_{20}(a_H^0)^2 + d_{11}(a_H^0)(a_V^0) + \frac{1}{\sqrt{2}}d_{02}(a_V^0)^2 \right\} |0, 0\rangle, \]

with

\[
\begin{align*}
d_{00} &= c_{0,0}, \\
d_{20} &= c_{2,0} \cos^2(\phi) + \frac{1}{\sqrt{2}}c_{1,1} \sin(2\phi) + c_{0,2} \sin^2(\phi), \\
d_{11} &= c_{1,1} \cos(2\phi) - \frac{1}{\sqrt{2}}[c_{2,0} - c_{0,2}] \sin(2\phi), \\
d_{02} &= c_{0,2} \cos^2(\phi) - \frac{1}{\sqrt{2}}c_{1,1} \sin(2\phi) + c_{2,0} \sin^2(\phi).
\end{align*}
\]

If a horizontally polarized photon is taken out then the remaining field is

\[ \psi_k^H \rightarrow [d_{11}(a_V^0)^2 + d_{20}(a_H^0)^2 + \text{scalar}] |0, 0\rangle. \]

The arrow \( \rightarrow \) indicates the result of the measurement. In the original basis the above expression reads

\[
\begin{align*}
\psi_k^H &\rightarrow (\sin(\phi)d_{11} + \cos(\phi)d_{20}) |1, 0\rangle \\
&\quad + (\cos(\phi)d_{11} + \sin(\phi)d_{20}) |0, 1\rangle + \text{scalar}|0, 0\rangle.
\end{align*}
\]

Without restriction assume that the second measurement occurs in the original basis of the polarization vectors. Then (5) implies that the probability \( P(H, H) \) that the second measurement returns a horizontal polarization is proportional to

\[ P(H, H) \sim |\sin(\phi)d_{11} + \cos(\phi)d_{20}|^2, \]

while the probability \( P(H, V) \) of a vertical polarization equals

\[ P(H, V) \sim |\cos(\phi)d_{11} + \sin(\phi)d_{20}|^2. \]

In the case that \( d_{11} = 0 \), this result coincides with that discussed in [28]. This case is realized for instance when \( c_{1,1} = 0 \) and \( c_{2,0} = c_{0,2} \), which means that the two photons contributing to \( \psi_k \) have the same polarization and both polarizations H and V have the same weight.

**ORCID iDs**

Jan Naudts @ https://orcid.org/0000-0002-4646-1190

**References**

[1] Reid M D 1989 Demonstration of the Einstein-Podolsky-Rosen paradox using nondegenerate parametric amplification Phys. Rev. A 40 913

[2] Schori C, Serensen J L and Polzik E S 2002 Narrow-band frequency tunable light source of continuous quadrature entanglement Phys. Rev. A 66 053802

[3] Ou Z Y, Pereira S F, Kimble H J and Peng K C 1992 Realization of the Einstein-Podolsky-Rosen paradox for continuous variables Phys. Rev. Lett. 69 2636–43

[4] Kwiat P G, Mattle K, Weinfurter H, Zeilinger A, Sergienko A V and Shih Y 1995 New High-Intensity source of polarization-entangled photon pairs Phys. Rev. Lett. 75 4337–41

[5] Villar A S, Cruz L S, Cassemiarto K N, Martinelli M and Nussenzveig P 2005 Generation of bright two-color continuous variable entanglement Phys. Rev. Lett. 95 243603

[6] Villar A S, Martinelli M, Fabre C and Nussenzveig P 2006 Direct production of tripartite pump-signal-idler entanglement in the above-threshold optical parametric oscillator Phys. Rev. Lett. 97 140504

[7] Boyer V, Marino A M, Pooser R C and Lett P D 2008 Entangled images from four-wave mixing Science 321 544–7

[8] Grosse N B, Assad S, Mehmet M, Schnabel R, Symul T and Lam P K 2008 Observation of entanglement between two light beams spanning an octave in optical frequency Phys. Rev. Lett. 100 243601

[9] Coelho A S, Barbosa F A S, Cassemiarto K N, Villar A S, Martinelli M and Nussenzveig P 2009 Three-color entanglement Science 326 823–6

[10] Bruzace de Andrade R 2018 An optical parametric oscillator for a light-atomic media interface PhD Thesis University of São Paulo

[11] Czachor M 2000 Non-canonical quantum optics J. Phys. A 33 S081

[12] Czachor M 2003 States of light via reducible quantization Phys. Lett. A 313 380–8

[13] Czachor M 2004 Reducible representations of CAR and CCR with possible applications to field quantization J. Nonlin. Math. Phys. 11S1 78–84

[14] Czachor M and Naudts J 2007 Regularization as quantization in reducible representations of CCR Int. J. Theor. Phys. 46 73

[15] Czachor M and Włocłowski M 2007 Cavity-qed tests of representations of canonical commutation relations employed in field quantization Int. J. Theor. Phys. 46 1215–28

[16] Czachor M and Wrzask K 2009 Automatic regularization by quantization in reducible representations of CCR: point-form quantum optics with classical sources Int. J. Theor. Phys. 48 2511–49

[17] Włocłowski M and Czachor M 2009 Theory versus experiment for vacuum Rabi oscillations in lossy cavities Phys. Rev. A 79 033856
[18] Wilczewski M and Czachor M 2009 Theory versus experiment for vacuum Rabi oscillations in lossy cavities (II): direct test of uniqueness of vacuum Phys. Rev. A 80 013802
[19] Kuna M and Naudts J unpublished
[20] Naudts J 2017 On the emergence of the coulomb forces in quantum electrodynamics Adv. High En. Phys. 2017 7232798
[21] Naudts J 2019 Emergent coulomb forces in reducible quantum electrodynamics to appear in Found. Sci.
[22] Duan L-M, Giedke G, Cirac J I and Zoller P 2000 Inseparability criterion for continuous variable systems Phys. Rev. Lett. 84 2722–5
[23] Barbosa F A S, Coelho A S, Cassemiro K N, Nussenzveig P, Fabre C, Martinelli M and Villar A S 2013 Beyond spectral homodyne detection: complete quantum measurement of spectral modes of light Phys. Rev. Lett. 111 200402
[24] Barbosa F A S, Coelho A S, Cassemiro K N, Nussenzveig P, Fabre C, Villar A S and Martinelli M 2013 Quantum state reconstruction of spectral field modes: homodyne and resonator detection schemes Phys. Rev. A 88 052113
[25] Aspect A, Grangier P and Roger G 1981 Experimental tests of realistic local theories via Bell’s theorem Phys. Rev. Lett. 47 460
[26] Grangier P, Roger G and Aspect A 1986 Experimental evidence for a photon anticorrelation effect on a beam splitter: a new light on single-photon interferences Europhysics Lett. 1 173–9
[27] Bell J 1964 On the einstein podolsky rosen paradox Physics 1 195–200
[28] Horne M A and Zeilinger A 1986 Einstein–podolsky–rosen interferometry Amr. N. Y. Acad. Sc. 480 469–74
[29] Clauser J F, Horne M A, Shimony A and Holt R A 1969 Proposed experiment to test local hidden-variable theories Phys. Rev. Lett. 23 880
Clauser J F, Horne M A, Shimony A and Holt R A 1970 Proposed experiment to test local hidden-variable theories Phys. Rev. Lett. 24 549 Erratum
[30] Mandel L 1982 Tests of quantum mechanics based on interference of photons Phys. Lett. 89A 325–6
[31] de Broglie L and Andrade e Silva J 1968 Interpretation of a recent experiment on interference of photon beams Phys. Rev. 172 1284–5