ERROR OF TIKHONOV’S REGULARIZATION FOR
INTEGRAL CONVOLUTION EQUATIONS

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Abstract. Let \( \varphi \) be a nontrivial function of \( L^1(\mathbb{R}) \). For each \( s \geq 0 \) we put

\[
p(s) = -\log \int_{|t| \geq s} |\varphi(t)| dt.
\]

If \( \varphi \) satisfies

\[
(0.1) \quad \lim_{s \to \infty} \frac{p(s)}{s} = \infty,
\]

we obtain asymptotic estimates of the size of small-valued sets 
\[ B_\epsilon = \{ x \in \mathbb{R} : |\hat{\varphi}(x)| \leq \epsilon, |x| \leq R_\epsilon \} \]

of Fourier transform 
\[ \hat{\varphi}(x) = \int_{-\infty}^{\infty} e^{-ixt} \varphi(t) dt, \ x \in \mathbb{R}, \]

in terms of \( p(s) \) or in terms of its Young dual function
\[ p^*(t) = \sup_{s \geq 0} [st - p(s)], \ t \geq 0. \]

Applying these results, we give an explicit estimate for the error of Tikhonov’s regularization for the solution \( f \) of the integral convolution equation

\[
\int_{-\infty}^{\infty} f(t-s)\varphi(s)ds = g(t),
\]

where \( f, g \in L^2(\mathbb{R}) \) and \( \varphi \) is a nontrivial function of \( L^1(\mathbb{R}) \) satisfying condition (0.1), and \( g, \varphi \) are known non-exactly. Also, our results extend some results of [4] and [5].

1. Introduction

Let \( \varphi \) be a nontrivial function of \( L^1(\mathbb{R}) \). Then its Fourier transform is defined by

\[
\hat{\varphi}(x) = \int_{-\infty}^{\infty} e^{-ixt} \varphi(t) dt, \ x \in \mathbb{R}.
\]
If \( \varphi \) is of compact support, there associates an entire function of exponential type of Laplace transform type
\[
\Phi(z) = \int_{-\infty}^{\infty} e^{zt} \varphi(t) dt.
\]
Generally, if \( \varphi \in L^1(\mathbb{R}) \) satisfies
\[
\int_{-\infty}^{\infty} e^{s|t|} |\varphi(t)| dt < \infty,
\]
for all \( s > 0 \) then \( \Phi \) in (1.1) is defined on all over the complex plane \( \mathbb{C} \) and is an entire function, but its order may be any positive number. If \( \varphi \geq 0 \) then condition (1.2) is also necessary for \( \Phi \) to be an entire function. Indeed, in this case we have, for every \( R > 0 \),
\[
\max_{|z| \leq R} |\Phi(z)| \geq \frac{1}{2} (\Phi(R) + \Phi(-R)) \geq \frac{1}{2} \int_{-\infty}^{\infty} e^{R|t|} \varphi(t) dt.
\]
As we will show in Section 2, the growth of \( \Phi(z) \) is related to the function
\[
p(s) = -\log \int_{|t| \geq s} |\varphi(t)| dt, \ s \geq 0.
\]
In fact, Theorem 3 show that Young dual function \( p^*(s) \) of \( p(t) \), defined by
\[
p^*(s) = \sup_{t \geq 0} \left[ st - p(t) \right], \ s \geq 0,
\]
is an appropriate quantity to estimate the growth of \( \log |\Phi(z)| \). It turns out that \( \varphi \) satisfies (1.2) if and only if it satisfies
\[
\lim_{s \to \infty} \frac{p(s)}{s} = \infty.
\]
Properties of Fourier and Laplace transforms are of great interest because these functions often occur in different areas of calculus. As a typical application, Fourier transforms give us the direct solution of integral convolution equations. These problems, so-called deconvolution problems (see, e.g., [7]), arise in applications: an input signal \( f \) generates an output signal \( g \) by the formula
\[
Af = \int_D \varphi(x, \gamma) f(\gamma) d\gamma = g(x), \ x \in D.
\]
Often one has
\[
Af = \int_{-\infty}^{\infty} \varphi(t-s)f(s) ds = g(t), \ t \in \mathbb{R},
\]
where \( \varphi(t) \in L^1(\mathbb{R}) \). The deconvolution problem consists of finding \( f \), given \( g \) and \( \varphi(x, \gamma) \). For (1.5), another problem, so-called the identification problem, is of practical interest: given \( f \) and \( g \), find \( \varphi(t) \). The function \( \varphi(t) \) characterizes the linear system which generates the output \( g(t) \) given the input \( f(t) \).
Now if $f$ is the solution of (1.5), applying Fourier transform to both sides of (1.5) we get

\[(1.6) \quad \hat{f} = \frac{\hat{g}}{\varphi}.\]

Then apply the inverse Fourier transform we find $f$. We may see easily from equation (1.6) that $f$ depends nonlinearly on $\varphi$.

As easily seen, in case $\varphi$ has compact support, the set of zeros of $\hat{\varphi}(z) \equiv \Phi(-iz)$ effects largely the recovering the function $f$ from its Fourier transform. Since $\hat{\varphi}(z)$ is a function of class Cartwright, that is

\[\int_{-\infty}^{\infty} \frac{\log^+|\hat{\varphi}(t)|}{1 + t^2} \, dt \leq \int_{-\infty}^{\infty} \frac{\log^+ \|\varphi\|_{L^1(\mathbb{R})}}{1 + t^2} \, dt < \infty,\]

the distribution of its zeros is well known (see, e.g., [6]). In a recent paper [9], Sedletskii obtained some detailed properties of the set of zeros of $\Phi$ for a subclass of functions $f$ that is of $L^1(0,1)$ and positive (see also references in [9]).

However, for really computing the solution $f$ and for the regularization of equation (1.5), we must know more about the structure of the small-valued sets of $\Phi$. This problem goes back to the well-known theorem of Cartan about the size of the small-valued sets $B_\epsilon = \{z \in C : |P(z)| \leq \epsilon\}$ where $P(z)$ is a polynomial. He proved that $B_\epsilon$ is contained in a finite number of disks whose sum of radii is less than $C_1 \epsilon^{\frac{1}{n}}$ where $n$ is the degree of $P(z)$ and $C_1$ is a constant that depends only on the leading coefficient of $P(z)$ and $n$ (see Theorem 3 of §11.3 in [6]). In particular we have

\[(1.7) \quad \lim_{\epsilon \to 0} m(B_\epsilon) = 0,\]

where $m(\cdot)$ is the Lebesgue’s measure. For general case of entire functions, the conclusion (1.7) does not hold, for even simple functions such as $e^z$, instead of many results representing the structure of $B_\epsilon$ (see, e.g., [6]).

Recently, in seeking regularization schemes for the problem of determination of heat source in one and two dimensional, the authors in [4] and [5] proved that $\Phi$ in (1.1) has only finitely many zeros on the positive real-axis if $\varphi$ is of a fairly wide subclass of $L^2(0,1)$. Indeed, as easily seen, the results in [4] and [5] depend only on the estimate of the size of the set

\[B_\epsilon = \{x \in \mathbb{R} : |\Phi(x)| \leq \epsilon, |x| \leq R_\epsilon\},\]

where $\gamma, \Phi$ are as above and $R_\gamma > 0$ depends on $\gamma$.

In this paper, we will give some asymptotic estimates for the small-valued sets

\[B_\epsilon = \{x \in \mathbb{R} : |\hat{\varphi}(x)| \leq \epsilon, |x| \leq R_\epsilon\},\]

to functions $\varphi$ satisfying condition (1.4) and apply these estimates to the Tikhonov’s regularization of Problem (1.5). Also, our results extend some
results in [4] and [5]. For convenience, we recall briefly Tikhonov’s regularization before stating results.

The difficulty of applying direct formula (1.6) arises in two aspects: one, in reality we can not have the exact data \((\varphi_0, g_0)\) but only the measured data \((\varphi_\epsilon, g_\epsilon)\), and two, even for the case in which we have the exact data, the inverse Fourier transform may not be efficiently computed if \(\hat{\varphi}_0\) has zeros on the real axis. Hence a regularization is needed.

Let \(\epsilon > 0\), we denote by \((\varphi_0, g_0)\) the exact data and \((\varphi_\epsilon, g_\epsilon)\) the measured data with the tolerable error \(\epsilon\), that is

\[
\epsilon \geq \max\{|\varphi_0 - \varphi_\epsilon|_{L^1(\mathbb{R})}, |g_0 - g_\epsilon|_{L^2(\mathbb{R})}\}.
\]

Tikhonov’s regularization applied to equation (1.5) is to construct an approximation \(f_\epsilon\) whose Fourier transform is

\[
\hat{f}_\epsilon = \frac{\hat{g}_\epsilon \varphi_\epsilon}{\delta_\epsilon + |\varphi_\epsilon|^2},
\]

where \(\delta_\epsilon > 0\) is a regularization parameter appropriately chosen depending on \(\epsilon\).

We have the following general estimate for the error \(||f_0 - f_\epsilon||_{L^2(\mathbb{R})}\) (see also [2] and Chapter 10 in [3] for similar estimates).

**Theorem 1.** Let \(\beta\) be a constant in \((0, \frac{1}{3})\). Let \(f_0 \in L^2(\mathbb{R})\) be the exact solution of (1.5) corresponding to \(\varphi_0 \in L^1(\mathbb{R})\), \(g_0 \in L^2(\mathbb{R})\). Let \(\epsilon > 0\) be small, \(\varphi_\epsilon, g_\epsilon\) be as in (1.8) and let \(f_\epsilon\) be as in (1.9), where \(\delta_\epsilon = \sqrt{\frac{C_1 C_2}{\epsilon^2(1+3\beta)^2}}\) and

\[
C_1 = 4(1 + ||g_0||^2_{L^2(\mathbb{R})} + ||\varphi_0||^2_{L^1(\mathbb{R})}),
\]

\[
C_2 = 1 + ||g_0||^2_{L^2(\mathbb{R})}.
\]

Let \(R_\epsilon > 0\) be a sequence such that

\[
\lim_{\epsilon \to 0} R_\epsilon = \infty.
\]

Then

\[
||f_0 - f_\epsilon||^2_{L^2(\mathbb{R})} \leq 3\int_{|\lambda| > R_\epsilon, |\varphi_0(\lambda)| < \epsilon^3} |F(f_0)|^2 d\lambda + \int_{|\lambda| < R_\epsilon, |\varphi_0(\lambda)| < \epsilon^3} |F(f_0)|^2 d\lambda + 2\sqrt{C_1 C_2 \epsilon^{1-3\beta}},
\]

where \(F(f_0)(\lambda) = \int_{-\infty}^{\infty} f_0(x)e^{-i\lambda x} dx\) is the Fourier transform of \(f_0\) and \(\hat{\varphi}_0(\lambda) = \int_{-\infty}^{\infty} \varphi_0(x)e^{-i\lambda x} dx\) is the Fourier transform of \(\varphi_0\).

If \(\varphi_0\) satisfies condition (1.2), we can show that the bound of \(||f_0 - f_\epsilon||_{L^2(\mathbb{R})}\) in above Theorem decreases to zero as \(\epsilon\) decreases to zero. This
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assertion comes easily from

\[
\lim_{\epsilon \to 0} \int_{|\lambda| > R_\epsilon} |\varphi_0(\lambda)|^2 \lambda + \int_{|\lambda| < R_\epsilon} |\varphi_0(\lambda)|^2 \lambda = 0,
\]

by Lebesgue’s dominated convergence theorem.

However, to get a more explicit estimate for \( ||f_0 - f_\epsilon||_{L^2(\mathbb{R})} \) some a-priori information about \( f_0 \) must be assumed. For example, in Chapter 10 of [3] (see also [2]), the author gave estimates for \( ||f_0 - f_\epsilon||_{L^2(\mathbb{R})} \) in the case the kernel \( \varphi_0 \) is of two special types of \( L^1 \) functions (see conditions (10.4) and (10.10) in [3]) and in additionally, the solution \( f_0 \) satisfies the condition

(1.10) \( |\hat{f}_0(\lambda)|^2 \leq C_1(1 + |\lambda|^2)^{-q}, \lambda \in \mathbb{R}, \)

for some \( q > \frac{1}{2} \).

Condition (10.4) in [3] is a set of four conditions and condition (10.10) in [3] is that

(1.11) \( |\hat{\varphi}_0(\lambda)|^2 \geq C_0 e^{-a|\lambda|}, \lambda > 0, \)

where \( a, C > 0 \) are constants. While condition (1.10) imposed on \( f_0 \) is natural and is easily verified (for example, if \( f_0 \in W^{1,1}(\mathbb{R}) \) we have \( |\hat{f}_0(\lambda)|^2 \leq C_0(1 + |\lambda|^2)^{-1} \)), conditions (10.4) and (10.10) above may not hold for a general kernel \( \varphi_0 \in L^1(\mathbb{R}) \) and fairly difficult to verify for a concrete kernel \( \varphi_0 \) in particularly in the case \( \varphi_0 \) is known not-exactly.

In this paper, we will give estimates for error of Tikhonov’s regularization for Problem (1.5) for the case the solution \( f_0 \) satisfies (1.10) and the kernel \( \varphi_0 \) satisfies (1.4). As we will show in Section 2, this class consists of functions satisfying condition (1.2).

For each \( \epsilon > 0 \) we put

(1.12) \( s_\epsilon = \inf \{ s > 0 : e^{-p(s)} \leq \epsilon \}, \)

where \( p(s) \) is as in (1.1).

Our main result is the following theorem. The estimate of our result in the case \( \varphi_0 \) has compact support is comparable to the estimate \( ||f_0 - f_\epsilon||_{L^2(\mathbb{R})} \leq C(\log \frac{1}{\epsilon})^{-q + \frac{1}{2}} \) gave in Theorem 10.8 in [3] (which uses additionally condition (1.11) in its proof).

**Theorem 2.** Let assumptions be as in Theorem 1. In additional, assume that \( f_0 \) satisfies condition (1.10) and \( \varphi_0 \) satisfies condition (1.4). For \( \epsilon > 0 \) small enough choose \( s_\epsilon \) as in (1.12) and \( R_\epsilon > 0 \) satisfying

\[
[(q + \frac{1}{2}) \log R_\epsilon + \log(15e^3)] \log ||\varphi_0||_{L^1(\mathbb{R})} + 2es_\epsilon R_\epsilon = - \log(e^3 + \epsilon).
\]

Then

\[
\lim_{\epsilon \to 0} R_\epsilon = \infty,
\]
and there exists a constant $C_3 > 0$ independent of $\varepsilon$ such that
\[
\|f_0 - f_\varepsilon\|_{L^2(\mathbb{R})} \leq C_3 R^{q + \frac{1}{2}},
\]
for $\varepsilon > 0$ small enough. In particular, in case $\varphi_0$ is of compact support then $s_\varepsilon$ is bounded for $\varepsilon > 0$ small enough, hence
\[
\lim_{\varepsilon \to 0} \frac{\log R_\varepsilon}{\log \log \frac{1}{\varepsilon}} = 1.
\]

Proof. The existence of $R_\varepsilon$ is easily seen by considering the function
\[
f(R) = \left[ (q + \frac{1}{2}) \log R + \log(15e^3) \right] \log ||\varphi_0||_{L^1(\mathbb{R})} + 2e\varepsilon R, \quad R \geq 0.
\]
Condition (1.4) gives
\[
\lim_{\varepsilon \to 0} R_\varepsilon = \infty.
\]
By Theorem 1 and condition (1.10), there exists $C_4 > 0$ such that
\[
\|f_0 - f_\varepsilon\|_{L^2(\mathbb{R})} \leq C_4 (R^{-q + \frac{1}{2}} + m(B_\varepsilon) + \varepsilon^{1 - 3\beta}),
\]
where $m(B_\varepsilon)$ is the Lebesgue’s measure of the set
\[
B_\varepsilon = \{ x \in \mathbb{R} : |\widehat{\varphi_0}(x)| \leq \varepsilon^\beta, |x| \leq R_\varepsilon \}.
\]
Now apply Theorem 5 we get the conclusion of Theorem 2. \hfill \Box

The rest of this paper consists of three sections. In Section 2 we explore some properties of entire functions of Laplace transform type. In Section 3 we state and prove estimates of the size of small-valued sets for entire functions of Laplace transform type and for Fourier transforms. In Section 4 we prove Theorem 1.

2. Properties of Laplace transforms

In this section we explore some properties of entire functions of type (1.1), where $\varphi \in L^1(\mathbb{R})$ satisfies condition (1.2).

First we consider the case $\varphi$ has a compact support. Without loss of generality, we assume the support of $\varphi$ is contained in $[0, 1]$. So the function $\Phi$ in (1.1) take a simpler form
\[
\Phi(z) = \int_0^1 e^{zt} \varphi(t) dt,
\]
where $\varphi(t)$ is a nontrivial function of $L^1(0, 1)$.

We put
\[
\sigma = \inf \{ a \in [0, 1] : \varphi|_{[a, 1]} = 0 \ a.e. \},
\]
\[
\mu = \sup \{ a \in [0, 1] : \varphi|_{[0, a]} = 0 \ a.e. \}.
\]
Lemma 1. (On order and type of $\Phi$)

i) Order of $\Phi$ is unity.

ii) Type of $\Phi$ is $\sigma$.

iii) We have

\[
\limsup_{R \to \infty} \frac{\log |\Phi(R)|}{R} = \sigma,
\]
\[
\limsup_{R \to \infty} \frac{\log |\Phi(-R)|}{R} = -\mu.
\]

Before proving Lemma 1 we recall that (see e.g. [6]) if $\Phi(z)$ is an entire function then its order $\rho$ (or $\text{ord}(\Phi)$ for short) and type $\sigma$ are defined by

\[
\rho = \limsup_{r \to \infty} \frac{\log \log M_\Phi(r)}{\log r},
\]
\[
\sigma = \limsup_{r \to \infty} \frac{\log M_\Phi(r)}{r^\rho},
\]

where

\[
M_\Phi(r) = \max_{|z| \leq r} |\Phi(z)|.
\]

Proof. i) We have

\[
|\Phi(z)| \leq Ce^{\rho|z|},
\]

for $z \in C$ and $C = ||\varphi||_{L^1(\mathbb{R})}$. It follows that $\text{ord}(\Phi) \leq 1$.

Now we show $\text{ord}(\Phi) = 1$. Assume by contradiction that $\text{ord}(\Phi) < 1$.

Applying Theorem 1 in §6.1 in [6] for domains $\{-\frac{\pi}{2} < \arg z < \frac{\pi}{2}\}$ and $\{ \frac{\pi}{2} < \arg z < \frac{3\pi}{2}\}$, noting that $\Phi$ is bounded on the imaginary-axis, we have that $\Phi$ is bounded on $C$. Thus $\Phi$ is a constant, so $\Phi'(z) \equiv 0$. Thus we have $t\varphi \equiv 0$ or equivalently $\varphi \equiv 0$, since

\[
\Phi'(z) = \int_0^1 e^{zt}\varphi(t)dt,
\]

which is a contradiction.

ii) We have the type of $\Phi$ is less than or equal to $\sigma$ because

\[
|\Phi(z)| = |\int_0^\sigma e^{zt}\varphi(t)dt| \leq e^{\sigma|z|} \int_0^\sigma |\varphi(t)|dt.
\]

We prove an equivalent result: If $a \in [0, 1]$ is so that $\varphi$ is not identity to zero in $[a, 1]$ then the type of $\Phi$ is greater than or equal to $a$.

We assume by contradiction that the type of $\Phi$ is less than $a$.

We consider the function

\[
\Gamma(z) = e^{-az} \int_a^1 e^{zt}\varphi(t)dt.
\]
If \( z > 0 \) we have
\[
|\Gamma(z)| = |e^{-az}\Phi(z) - e^{-az} \int_0^a e^{zt}\varphi(t)| \\
\to 0
\]
when \( z \to +\infty \) on \( R \). Indeed, because of our contradiction assumption that the type of \( \Phi \) is less than \( a \) it follows
\[
\lim_{z \to \infty} e^{-az}\Phi(z) = 0,
\]
while by Lebesgue’s dominated convergence theorem applied to the sequence \( g_z(t) = e^{z(t-a)}\varphi(t) \) on \([0, a] \) it follows that
\[
\lim_{z \to \infty} e^{-az} \int_0^a e^{zt}\varphi(t)dt = \lim_{z \to \infty} \int_0^a g_z(t)dt = 0.
\]
If \( z < 0 \), by Lebesgue’s dominated convergence theorem as above we get
\[
|\Gamma(z)| = e^{a|z|} \int_0^a e^{-|z|t}\varphi(t)dt \\
\to 0
\]
when \( z \to -\infty \) on \( R \).

If \( z \) is purely imaginary, we have
\[
|\Gamma(z)| \leq C.
\]
Applying Theorem 1 in §6.1 in [6] for domains \( \{0 < \arg z < \frac{\pi}{2}\}, \{\frac{\pi}{2} < \arg z < \pi\}, \{\pi < \arg z < \frac{3\pi}{2}\} \) and \( \{\frac{3\pi}{2} < \arg z < 2\pi\} \) we get \(|\Gamma(z)| \leq C\) for all \( z \in C \), which implies that \( \Gamma(z) = \text{constant} \), a contradiction (qed).

iii) We prove similarly to the proof of ii). We prove first that
\[
\limsup_{R \to \infty} \frac{\log |\Phi(-R)|}{R} = -\mu.
\]
If \( R > 0 \) we have
\[
|\Phi(-R)| = |\int_{-\mu}^1 e^{-Rt}\varphi(t)dt| \leq e^{-R\mu}\|\varphi\|_{L^1(0,1)},
\]
hence
\[
\limsup_{R \to \infty} \frac{\log |\Phi(-R)|}{R} \leq -\mu.
\]
Thus, in order to prove the equality required, we need to show only that:
if \( a > \mu \) then
\[
\limsup_{R \to \infty} \frac{\log |\Phi(-R)|}{R} \geq -a.
\]
Assume by contradiction that
\[
\limsup_{R \to \infty} \frac{\log |\Phi(-R)|}{R} < -a,
\]
for some \( a > \mu \). We consider

\[
\Gamma(z) = e^{-az} \int_0^a e^{zt} \varphi(t) dt.
\]

By arguments in ii) we have

\[
\lim_{R \to \infty} |\Gamma(R)| = 0,
\]

\[
\lim_{R \to \infty} |\Gamma(-R)| = \lim_{R \to \infty} |e^{aR} \Phi(-R) - e^{aR} \int_1^1 e^{-Rt} \varphi(t) dt| = 0.
\]

Thus as in ii) we have \( \Gamma(z) \) is a constant, which is a contradiction. The proof of

\[
\limsup_{R \to \infty} \frac{\log |\Phi(R)|}{R} = \sigma
\]

is similar and indeed was contained in the proof of ii).

\[\square\]

**Lemma 2.** (On zeros of \( \Phi \))

\[
\lim_{R \to \infty} n(R) = \infty
\]

and

\[
\lim_{R \to \infty} \frac{n(R)}{R} = \frac{d}{\pi} := \frac{\sigma - \mu}{\pi},
\]

where \( n(R) = \{ z : |z| \leq R, \ \Phi(z) = 0 \} \).

**Proof.** First we show that \( \Phi \) has infinitely many zeros. Indeed, if \( \Phi \) has only finitely many zeros, since \( \Phi \) is of order 1, by Hadamard’s theorem we may write

\[
\Phi(z) = e^{az} P(z),
\]

where \( P \) is a polynomial.

So if we differentiate \( e^{-az} \Phi(z) \) \( m \) times, where \( m \) is greater than the order of \( P(z) \) we obtain

\[
\int_0^1 e^{z(t-a)} (t-a)^m \varphi(t) dt \equiv 0,
\]

so \( \varphi \equiv 0 \), a contradiction.

We have \(|\Phi(it)| \leq C = ||\varphi||_{L^1(R)}\) if \( t \in R \). So

\[
\int_{-\infty}^{\infty} \frac{\log^+ |\Phi(it)|}{1 + t^2} dt \leq \int_{-\infty}^{\infty} \frac{\log^+ C}{1 + t^2} dt < \infty.
\]

Thus \( \Phi(iz) \) is of class Cartwright (see Lecture 16 in [6]).

Applying Theorem 1 of §17.2 in [6], we have

\[
\lim_{R \to \infty} \frac{n(R)}{R} = \frac{d}{\pi}
\]
where $d$ is the width of the indicator diagram of $\Phi(iz)$, i.e.

$$d = \limsup_{R \to \infty} \frac{\log |\Phi(R)|}{R} + \limsup_{R \to \infty} \frac{\log |\Phi(-R)|}{R}.$$

By Lemma 1 iii) we have

$$\limsup_{R \to \infty} \frac{\log |\Phi(R)|}{R} = \sigma,$$

$$\limsup_{R \to \infty} \frac{\log |\Phi(-R)|}{R} = -\mu.$$

So $d = \sigma - \mu$. \hfill \Box

**Lemma 3.** (On representation of $\Phi$). Denote $C^+ = \{z \in C : \Im z > 0\}$, i.e. $C^+$ is the upper half-plane. Then

$$\Phi(z) = Cz^m e^{\frac{\sigma + \mu}{2}z} \prod_{z_i \in \mathbb{R}, |z_i| \leq r} (1 - \frac{z}{z_i}) \prod_{z_i \in C^+, |z_i| \leq r} (1 - z(1 - \frac{z}{z_i}) + \frac{z^2}{|z_i|^2}),$$

where $C \in \mathbb{R}$.

**Proof.** Applying Theorem 1 of §17.2 in [6] for $\Phi(iz)$, if \{z_k\}, $z_k \neq 0$ are zeros of $\Phi$ then

$$\sum_k |Re \frac{1}{z_k}| < \infty.$$

Applying Hadamard’s theorem, noting that $\Phi(\overline{z}) = \overline{\Phi(z)}$, we may write

$$\Phi(z) = Cz^m e^{az} \lim_{r \to \infty} \prod_{z_i \in \mathbb{R}, |z_i| \leq r} (1 - \frac{z}{z_i}) \prod_{z_i \in C^+, |z_i| \leq r} (1 - \frac{z}{z_i})(1 - \frac{z}{z_i} e^{2i(\frac{1}{z_i} + \frac{1}{z_i})}).$$

Since $ord(\Phi) = 1$ we have

$$\sum_{z_i} \frac{1}{|z_i|^2} < \infty.$$

This, combined with the event that $\frac{1}{z_i} + \frac{1}{z_i} = 2Re \frac{1}{z_i}$ and the inequality above

$$\sum_k |Re \frac{1}{z_k}| < \infty,$$

allows us to write

$$\Phi(z) = Cz^m e^{bz} \prod_{z_i \in \mathbb{R}} (1 - \frac{z}{z_i}) \prod_{z_i \in C^+} (1 - \frac{z}{z_i})(1 - \frac{z}{z_i}).$$

That $C, b \in \mathbb{R}$ is easy to see. Now we show that $b = \frac{\sigma + \mu}{2}$. 
If \( z_i \notin \mathbb{R} \) and \( z \in C \) we have
\[
|(1 - \frac{z}{z_i})(1 - \frac{1}{z_i})| = |1 - z(\frac{1}{z_i} + \frac{1}{z}) + \frac{z^2}{|z_i|^2}| \leq 1 + |z||\frac{1}{z_i} + \frac{1}{z}| + \frac{|z|^2}{|z_i|^2} \leq (1 + |z||\frac{1}{z_i} + \frac{1}{z}|)(1 + \frac{|z|^2}{|z_i|^2}).
\]

Thus, if \( z \in C \) we have
\[
|\Phi(z)| \leq |e^{bz}|F_1(|z|)F_2(|z|)F_3(|z|)
\]
where
\[
F_1(z) = |C|z^m \prod_{z_i \in \mathbb{R}} (1 + \frac{z}{|z|})
\]
\[
F_2(z) = \prod_{z_i \in \mathbb{C}^+} (1 + z|\frac{1}{z_i} + \frac{1}{z}|)
\]
\[
F_3(z) = \prod_{z_i \in \mathbb{C}^+} (1 + \frac{z^2}{|z_i|^2}).
\]

Since
\[
\sum_k |\text{Re} \frac{1}{z_k}| < \infty,
\]
we have that \( F_1 \) and \( F_2 \) are of minimal type. Moreover it is easy to see that
\[
\lim_{R \to \infty} \frac{\log |F_1(R)|}{R} = \lim_{R \to \infty} \frac{\log |F_2(R)|}{R} = 0.
\]

Now if we arrange \( \{z_k\}, 0 < |z_1| = |z_2| \leq \cdots \leq |z_3| = |z_3| \ldots \) and put \( \lambda_k = |z_k| \) then Theorem 1 of §17.2 in [6] gives
\[
\lim_{n \to \infty} \frac{n}{\lambda_n} = \frac{d}{2\pi} = \frac{\sigma - \mu}{2\pi}.
\]

So we can apply Theorem 2 of §12.1 in [6] for \( F_3(iz) \) to get
\[
\lim_{R \to \infty} \frac{\log |F_3(R)|}{R} = \frac{d}{2} = \frac{\sigma - \mu}{2}.
\]

Now if \( R > 0 \) we have
\[
\frac{\log |\Phi(R)|}{R} \leq b + \frac{\log |F_1(R)|}{R} + \frac{\log |F_2(R)|}{R} + \frac{\log |F_3(R)|}{R},
\]
\[
\frac{\log |\Phi(-R)|}{-R} \geq b + \frac{\log |F_1(R)|}{-R} + \frac{\log |F_2(R)|}{-R} + \frac{\log |F_3(R)|}{-R}.
\]
Hence
\[\sigma = \limsup_{R \to \infty} \frac{\log |\Phi(R)|}{R} \leq \lim_{R \to \infty} \left( b + \frac{\log |F_1(R)|}{R} + \frac{\log |F_2(R)|}{R} + \frac{\log |F_3(R)|}{R} \right)\]
\[= b + \frac{\sigma - \mu}{2},\]
\[\mu = \liminf_{R \to \infty} \frac{\log |\Phi(-R)|}{-R} \geq \lim_{R \to \infty} \left( b + \frac{\log |F_1(R)|}{-R} + \frac{\log |F_2(R)|}{-R} + \frac{\log |F_3(R)|}{-R} \right)\]
\[= b - \frac{\sigma - \mu}{2}.\]

From above two inequalities we get
\[b = \frac{\sigma + \mu}{2}.\]

Now we consider the general case of kernel \(\varphi \in L^1(\mathbb{R})\) satisfying condition (1.2).

Lemma 4. A function \(\varphi \in L^1(\mathbb{R})\) satisfies condition (1.2) if and only if it satisfies condition (1.4). In this case \(\Phi(z)\) defined by (1.1) is an entire function.

Proof. If \(\varphi\) satisfies condition (1.2), for each \(n \in \mathbb{N}\) there is \(A_n < \infty\) such that
\[\int_{-\infty}^{\infty} e^{n|t|} |\varphi(t)| dt \leq A_n.\]

Then
\[0 \geq \limsup_{s \to \infty} \frac{\log \int_{|t| \geq s} e^{n|t|} |\varphi| dt}{s} \geq \limsup_{s \to \infty} \frac{\log \int_{|t| \geq s} e^{sn} |\varphi| dt}{s} = n + \limsup_{s \to \infty} \frac{\log \int_{|t| \geq s} |\varphi| dt}{s}.\]

Because \(n\) is arbitrary, we have
\[\lim_{s \to \infty} \frac{\log \int_{|t| \geq s} |\varphi(t)| dt}{s} = -\infty.\]

So \(\varphi\) satisfies (1.4).

Conversely, if \(\varphi\) satisfies (1.4), we take \(p(s)\) be as in (1.3), that is
\[p(s) = -\log \int_{|t| \geq s} |\varphi(t)| dt, \quad s \geq 0.\]

Consider the function
\[H(s) = \int_{-\infty}^{\infty} e^{|t|s} |\varphi(t)| dt.\]
Because \( \frac{d}{ds} e^{-p(s)} = -(|\varphi(s)| + |\varphi(-s)|) \) and \( \varphi \) satisfies (1.4), using integration by parts we get
\[
H(s) = -e^{ts - p(t)} |_{s=0}^{s=\infty} + s \int_{0}^{\infty} e^{ts - p(t)} dt \\
= \|\varphi\|_{L^1(\mathbb{R})} + s \int_{0}^{\infty} e^{ts - p(t)} dt < \infty
\]
for any \( s \geq 0 \), so \( \varphi \) satisfies (1.4).

When \( \varphi \) satisfies (1.4), \( \Phi(z) \) is clearly an entire function and
\[
|\Phi(z)| \leq H(|z|).
\]

As Lemma 4 shows, the growth of \( \Phi(z) \) is determined by the function (2.1)
\[
G(s) = \int_{0}^{\infty} e^{ts - p(t)} dt.
\]

From now on we assume that \( \varphi \) satisfies (1.2). For the sake of simplicity, we assume also that \( \varphi \) has a non-compact support, so \( p(s) < \infty \) for all \( s \geq 0 \).

It turns out that the growth of \( G(s) \) in (2.1) is related to Young dual function \( p^*(s) \) of \( p(s) \), which is defined by

\[
p^*(s) = \sup_{t \geq 0} [st - p(t)], \quad s \geq 0.
\]

It is easy to verify that \( p^* \) is convex and satisfies
\[
p(t) + p^*(s) \geq st, \quad s, t \geq 0,
\]
\[
\lim_{s \to \infty} \frac{p^*(s)}{s} = \infty.
\]
(see for e.g. Lecture 25 in [6]). So we can define Young dual function \( p^{**} \) of \( p^* \). This function satisfies \( p^{**}(t) \leq p(t) \) for all \( t \geq 0 \).

We have the following result

**Theorem 3.** Let \( \varphi \) be a function of \( L^1(\mathbb{R}) \) with non-compact support. Define \( p(s) \) as in (1.3) and Young dual function \( p^*(s) \) and Young double-dual function \( p^{**}(s) \) as above. Define \( G(s) \) by (2.1). Then
\[
\lim_{s \to \infty} \inf \frac{\log G(s)}{p^*(s)} \geq 1,
\]
and for any \( \kappa > 0 \)
\[
\lim_{s \to \infty} \sup \frac{\log G(s)}{p^*(s + \kappa)} \leq 1.
\]
If additionally we have one of the following two conditions
\[
(2.2) \quad \exp\{-p(t) + \frac{1}{\gamma}p(t\gamma)\} \in L^1(\mathbb{R}),
\]
for all $1 > \gamma > 0$ close enough to 1, or

\begin{equation}
\exp\{-p^{\ast\ast}(t) + \frac{1}{\gamma}p^{\ast\ast} (t \gamma)\} \in L^1(\mathbb{R}),
\end{equation}

for all $1 > \gamma > 0$ close enough to 1, then

$$\lim_{s \to \infty} \frac{\log G(s)}{p^*(s)} = 1.$$ 

**Proof.** Because of properties of $p(s)$, for each $s > 0$ large enough we can find a $t_s > 0$ such that

$$st_s \geq p(t_s) + p^*(s) - \frac{1}{2},$$

and

$$\lim_{s \to \infty} t_s = \infty.$$

We have

$$G(s) = \int_0^\infty e^{ts-p(t)} dt \geq \int_{t_{s-1}}^{t_s} e^{ts-p(t)} dt.$$ 

Now fixed $\gamma$ be any number less than 1. By above inequality we have

$$e^{-\gamma p^*(s)} G(s) \geq e^{(1-\gamma)p^*(s)} \int_{t_{s-1}}^{t_s} e^{ts-p(t)-p^*(s)} dt.$$ 

Because $p(t)$ in (1.3) is increasing, for $t_s - 1 \leq t \leq t_s$ we have

$$ts - p(t) - p^*(s) \geq s(t_s - 1) - p(t_s) - p^*(s) \geq -s - \frac{1}{2}.$$ 

So

$$e^{-\gamma p^*(s)} G(s) \geq e^{(1-\gamma)p^*(s)-s-\frac{1}{2}}.$$ 

Because of

$$\lim_{s \to \infty} \frac{p^*(s)}{s} = \infty,$$

we get that

$$\liminf_{s \to \infty} \frac{\log G(s)}{p^*(s)} \geq 1.$$ 

Now for any $\kappa > 0$ we have $p(t) + p^*(s + \kappa) \geq st + \kappa t$. Hence

$$e^{-p^*(s+\kappa)} G(s) = \int_0^\infty e^{ts-p(t)-p^*(s+\kappa)} dt$$

$$\leq \int_0^\infty e^{-\kappa t} dt = \frac{1}{\kappa}.$$ 

Thus

$$\limsup_{s \to \infty} \frac{\log G(s)}{p^*(s + \kappa)} \leq 1.$$
Now assume that (2.2) or (2.3) is satisfied. Fixed 1 > γ > 0 close enough to 1. We have
\[
\sup_{s \geq 0} \left[ st - \frac{1}{\gamma} p^*(s) \right] = \frac{1}{\gamma} \sup_{s \geq 0} \left[ s.(\gamma t) - p^*(s) \right] = \frac{1}{\gamma} p^{**}(\gamma t).
\]
Hence for all \( t \geq 0 \) we have
\[
st - p(t) - \frac{1}{\gamma} p^*(s) \leq -p(t) + \frac{1}{\gamma} p^{**}(\gamma t).
\]
Because \( p^{**}(t) \leq p(t) \) from above inequality we get
\[
st - p(t) - \frac{1}{\gamma} p^*(s) \leq \min\{-p(t) + \frac{1}{\gamma} p(\gamma t), -p^{**}(t) + \frac{1}{\gamma} p^{**}(\gamma t)\}.
\]
Hence
\[
e^{st-p(t)-\frac{1}{\gamma} p^*(s)} \leq \min\{e^{-p(t)+\frac{1}{\gamma} p(\gamma t)}, e^{-p^{**}(t)+\frac{1}{\gamma} p^{**}(\gamma t)}\},
\]
for all \( s \geq 0 \).
We have
\[
\lim_{s \to \infty} e^{st-p(t)-\frac{1}{\gamma} p^*(s)} = 0,
\]
for all \( t \geq 0 \), so we can apply Lebesgue’s dominated convergence theorem to get
\[
\lim\sup_{s \to \infty} \frac{\log G(s)}{p^*(s)} \leq \frac{1}{\gamma},
\]
for all 1 > γ > 0 close enough to 1, so it follows
\[
\lim\sup_{s \to \infty} \frac{\log G(s)}{p^*(s)} \leq 1.
\]
Because we proved before that
\[
\lim\inf_{s \to \infty} \frac{\log G(s)}{p^*(s)} \geq 1,
\]
we get that
\[
\lim_{s \to \infty} \frac{\log G(s)}{p^*(s)} = 1.
\]

\[\square\]

**Remark 1.** If \( \varphi \geq 0 \) then \( \Phi(z) \) in (1.1) satisfies
\[
\max_{|z| \leq \bar{s}} |\Phi(z)| \geq \frac{1}{2} H(s),
\]
where \( H(s) \) is the function defined in the proof of Lemma 4. So Theorem 3 shows that \( p^*(|z|) \) is an appropriate quantity to estimate the growth of \( \log |\Phi(z)| \).
Remark 2. If $\varphi$ has the support in $[0, 1]$ and $\sigma = \inf \{ a \in [0, 1] : \varphi|_{[a, 1]} = 0 \, \text{a.e.} \}$, then it is easy to see that
\[
\lim_{s \to \infty} \frac{p^*(s)}{\sigma s} = 1,
\]
so by Theorem 3 we obtain again the familiar result
\[
\limsup_{|z| \to \infty} \frac{\log |\Phi(z)|}{|z|} \leq \sigma.
\]

Remark 3. The class of functions $p(t)$ satisfying conditions (1.4) and (2.2) are fairly large. For example we can take $p(t) = t \log t$ or $p(t) = t^\gamma$ for any $\gamma > 1$.

3. The size of small-valued sets

In this section, we estimate the size of small-valued sets of Fourier transforms, i.e., estimate the Lebesgue measure of the sets
\[
B_\epsilon = \{ x \in \mathbb{R} : |\hat{\varphi}(x)| \leq \epsilon, |x| \leq R_\epsilon \},
\]
where $\varphi$ is a nontrivial function of $L^1(\mathbb{R})$ satisfying condition (1.4), $R_\epsilon$ depends on small numbers $\epsilon$ and
\[
\hat{\varphi}(x) = \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt, \quad x \in \mathbb{R}.
\]

We will estimate these small-valued sets in terms of the function $p(s)$ defined by (1.3) or in terms of its Young dual function $p^*(s)$.

First we obtain estimates in terms of $p(s)$. We define
\[
\varphi^\epsilon(t) = \begin{cases} 
\varphi(t) & \text{if } |t| \leq s_\epsilon \\
0 & \text{if } |t| > s_\epsilon.
\end{cases}
\]
and
\[
\Phi^\epsilon(z) = \int_{-\infty}^{\infty} e^{zt} \varphi^\epsilon(t) dt \equiv \int_{-s_\epsilon}^{s_\epsilon} e^{zt} \varphi(t) dt,
\]
\[
\hat{\varphi}^\epsilon(z) = \int_{-\infty}^{\infty} e^{-izt} \varphi^\epsilon(t) dt \equiv \int_{-s_\epsilon}^{s_\epsilon} e^{-izt} \varphi(t) dt,
\]
where $s_\epsilon$ is defined by (1.12).

It is easy to see that $\Phi^\epsilon(-iz) = \hat{\varphi}^\epsilon(z)$ are entire functions of order 1 and type $\leq s_\epsilon$. More explicitly
\[
|\hat{\varphi}^\epsilon(z)| \leq ||\varphi||_{L^1(\mathbb{R})} e^{s_\epsilon |z|}.
\]

By the definition of $s_\epsilon$, we see that if $x \in \mathbb{R}$ then
\[
|\hat{\varphi}^\epsilon(x) - \hat{\varphi}(x)| \leq \epsilon.
\]

First we estimate the size of small-valued sets of $\Phi^\epsilon$. We have the following result
Theorem 4. Let $s_\epsilon, \Phi^\epsilon$, be as above. Let $q > \frac{1}{2}$, $\beta > 0$ be constants. For $\epsilon > 0$ small enough we choose $R_\epsilon$ to satisfy

$$[(q + \frac{1}{2}) \log R_\epsilon + \log(15e^3)]\log ||\varphi||_{L^1(\mathbb{R})} + 2es_\epsilon R_\epsilon = -\log(\epsilon^\beta + \epsilon).$$

If $\epsilon$ small enough then

$$m(B_{\epsilon^{\beta+\epsilon}}) \leq R_\epsilon^{-q + \frac{1}{2}},$$

where

$$B_{\epsilon^{\beta+\epsilon}} = \{z \in \mathbb{C} : |\Phi^\epsilon(z)| \leq \epsilon^\beta + \epsilon, |z| \leq R_\epsilon\}.$$

Proof. Because $||\varphi||_{L^1(\mathbb{R})} \neq 0$, there exists $x_0 \in \mathbb{R}$ such that $\hat{\varphi}(x_0) \neq 0$. Then there exists a constant $C_0 > 0$ such that

$$|\hat{\varphi}(x_0)| \geq C_0,$$

if $\epsilon$ is small enough. Changing variable if necessarily, we may assume that $|\Phi^\epsilon(0)| \geq C_0$ if $\epsilon$ is small enough. If we choose

$$\eta_\epsilon = R_\epsilon^{-q + \frac{1}{2}},$$

then by Theorem 4 of §11.3 in [6] if $|z| \leq R_\epsilon$ then

$$|\Phi^\epsilon(z)| \geq \exp\{-[(q + \frac{1}{2})R_\epsilon + \log(15e^3)]\log ||\varphi||_{L^1(\mathbb{R})} + s_\epsilon R_\epsilon\} = \epsilon^\beta + \epsilon,$$

except a set of disks whose sum of radii is less than

$$\eta_\epsilon R_\epsilon \equiv R_\epsilon^{-q + \frac{1}{2}}.$$

Because $|\hat{\varphi}(x)| \leq \epsilon^\beta + \epsilon$ if $|\hat{\varphi}(x)| \leq \epsilon^\beta$, by Theorem 4 we have immediately

Theorem 5. Let assumptions and choose $R_\epsilon$ as in Theorem 4. If $\epsilon$ small enough then

$$m(B_{\epsilon^\beta}) \leq R_\epsilon^{-q + \frac{1}{2}},$$

where

$$B_{\epsilon^\beta} = \{x \in \mathbb{R} : |\hat{\varphi}(x)| \leq \epsilon^\beta, |x| \leq R_\epsilon\}.$$

From Theorem 3 we have

$$(3.1) \quad |\Phi(z)| \leq ||\varphi||_{L^1(\mathbb{R})} + |z|e^{p_\epsilon(|z|+1)},$$

for all $z \in \mathbb{C}$. So applying Theorem 4 of §11.3 in [6] directly to $\Phi(z)$ we get the following result.
Theorem 6. Let ϕ have a non-compact support. Let \(q > \frac{1}{2}\) be a constant. For \(\epsilon > 0\) small enough, define \(R_\epsilon\) by

\[
[(q + \frac{1}{2})R_\epsilon + \log(15e^3)][1 + \log(R_\epsilon) + p^*(2R_\epsilon + 1)] = -\log \epsilon.
\]

Then

\[m(B_\epsilon) \leq R_\epsilon^{-q + \frac{1}{2}},\]

where \(m(B_\epsilon)\) is the Lebesgue’s measure of the set

\[B_\epsilon = \{z \in \mathbb{C} : |\Phi(z)| \leq \epsilon, |z| \leq R_\epsilon\}.
\]

Moreover we have

\[\lim_{\epsilon \to 0} \frac{\log p^*(2R_\epsilon + 1)}{\log \log \frac{1}{\epsilon}} = 1.
\]

Proof. The conclusion about the size of \(B_\epsilon\) is obtained similarly that of Theorems 4 and 5. Because

\[
\lim_{s \to \infty} \frac{p^*(s)}{s} = \infty,
\]

in view of the definition of \(R_\epsilon\) we get

\[\lim_{\epsilon \to 0} \frac{\log p^*(2R_\epsilon + 1)}{\log \log \frac{1}{\epsilon}} = 1.
\]

\[\Box\]

4. Proof of Theorem 1

Because \(f_0, f_\epsilon \in L^2(\mathbb{R})\) we have

\[
||f_0 - f_\epsilon||^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}||F(f_0) - F(f_\epsilon)||^2_{L^2(\mathbb{R})},
\]

where \(F(f_0)(\lambda) = \int_{-\infty}^{\infty} f_0(x)e^{-i\lambda x}dx = \hat{f}_0\), \(F(f_\epsilon)(\lambda) = \int_{-\infty}^{\infty} f_\epsilon(x)e^{-i\lambda x}dx = \hat{f}_\epsilon\).

We have

\[
||F(f_0) - F(f_\epsilon)||^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \left|\frac{\hat{g}_0\overline{\varphi_0}}{|\varphi_0|^2} - \frac{\hat{g}_\epsilon\overline{\varphi_\epsilon}}{|\varphi_\epsilon|^2}\right|^2.
\]

Now

\[
\left|\frac{\hat{g}_0\overline{\varphi_0}}{|\varphi_0|^2} - \frac{\hat{g}_\epsilon\overline{\varphi_\epsilon}}{|\varphi_\epsilon|^2}\right|^2 \leq \left|\frac{\hat{g}_0\overline{\varphi_0}}{|\varphi_0|^2} - \frac{\hat{g}_\epsilon\overline{\varphi_\epsilon}}{|\varphi_\epsilon|^2}\right|^2 + \left|\frac{\hat{g}_0\overline{\varphi_0}}{|\varphi_0|^2} - \frac{\hat{g}_\epsilon\overline{\varphi_\epsilon}}{|\varphi_\epsilon|^2}\right|^2 + \frac{\delta_\epsilon |\hat{g}_0\overline{\varphi_0}|}{|\varphi_0|^2(\delta_\epsilon + |\varphi_\epsilon|^2)} + \frac{\delta_\epsilon |\hat{g}_0\overline{\varphi_0}|}{(\delta_\epsilon + |\varphi_\epsilon|^2)(\delta_\epsilon + |\varphi_\epsilon|^2)}
\]

\[
+ \frac{\delta_\epsilon |\hat{g}_0\overline{\varphi_0}|}{(\delta_\epsilon + |\varphi_\epsilon|^2)(\delta_\epsilon + |\varphi_0|^2)}.
\]
Now using Cauchy’s inequality we have

\[
\frac{\delta_e |\hat{g}_0| \overline{\varphi_0}}{|\varphi_0|^2(\delta_e + |\varphi_0|^2)} \leq \min\{\frac{\hat{g}_0}{\varphi_0} = |F(f_0)|, \frac{\delta_e |\hat{g}_0|}{|\varphi_0|^3}\},
\]

\[
\frac{\delta_e |\hat{g}_0\varphi_0 - \hat{g}_c \varphi_c|}{(\delta_e + |\varphi_0|^2)(\delta_e + |\varphi_0|^2)} \leq \frac{|\hat{g}_0\varphi_0 - \hat{g}_c \varphi_c|}{\delta_e}.
\]

Hence

\[
\int_\mathbb{R} |F(f_0) - F(f_c)|^2 d\lambda \leq 3\int_\mathbb{R} \left( \frac{\delta_e |\hat{g}_0| \overline{\varphi_0}}{|\varphi_0|^2(\delta_e + |\varphi_0|^2)} \right)^2 + \int_\mathbb{R} \left( \frac{\delta_e |\hat{g}_0\varphi_0 - \hat{g}_c \varphi_c|}{(\delta_e + |\varphi_0|^2)(\delta_e + |\varphi_0|^2)} \right)^2
\]

\[
+ \int_\mathbb{R} \left( \frac{\delta_e |\hat{g}_0\varphi_0 - \hat{g}_c \varphi_c|}{\delta_e} \right)^2
\]

\[
= 3\int_\mathbb{R} \left( \frac{\delta_e |\hat{g}_0| \overline{\varphi_0}}{|\varphi_0|^2(\delta_e + |\varphi_0|^2)} \right)^2 + \frac{1}{\delta_e^2} \int_\mathbb{R} |\hat{g}_0\varphi_0 - \hat{g}_c \varphi_c|^2
\]

\[
+ \frac{1}{\delta_e^2} \int_\mathbb{R} |\hat{g}_0\varphi_0 - \hat{g}_c \varphi_c|^2.
\]

Now we write

\[
\int_\mathbb{R} \left( \frac{\delta_e |\hat{g}_0| \overline{\varphi_0}}{|\varphi_0|^2(\delta_e + |\varphi_0|^2)} \right)^2
\]

\[
= \int_{|\lambda| > R_c, |\varphi_0(\lambda)| < \epsilon^a} \left( \frac{\delta_e |\hat{g}_0| \overline{\varphi_0}}{|\varphi_0|^2(\delta_e + |\varphi_0|^2)} \right)^2 + \int_{|\varphi_0(\lambda)| > \epsilon^a} \left( \frac{\delta_e |\hat{g}_0| \overline{\varphi_0}}{|\varphi_0|^2(\delta_e + |\varphi_0|^2)} \right)^2
\]

\[
+ \int_{|\lambda| < R_c, |\varphi_0(\lambda)| < \epsilon^a} \left( \frac{\delta_e |\hat{g}_0| \overline{\varphi_0}}{|\varphi_0|^2} \right)^2
\]

\[
\leq \int_{|\lambda| > R_c, |\varphi_0(\lambda)| < \epsilon^a} |\hat{g}_0| |\varphi_0|^2 + \int_{|\varphi_0(\lambda)| > \epsilon^a} \left( \frac{\delta_e |\hat{g}_0| |\varphi_0|}{|\varphi_0|^3} \right)^2
\]

\[
+ \int_{|\lambda| < R_c, |\varphi_0(\lambda)| < \epsilon^a} \left( \frac{\delta_e |\hat{g}_0| |\varphi_0|}{|\varphi_0|^2} \right)^2
\]

\[
\leq \int_{|\lambda| > R_c, |\varphi_0(\lambda)| < \epsilon^a} |F(f_0)|^2 + \frac{\delta_e^2}{(\epsilon^3)^2} \int_{|\varphi_0(\lambda)| > \epsilon^a} |\hat{g}_0|^2
\]

\[
+ \int_{|\lambda| < R_c, |\varphi_0(\lambda)| < \epsilon^a} |F(f_0)|^2
\]

\[
\leq \int_{|\lambda| > R_c, |\varphi_0(\lambda)| < \epsilon^a} |F(f_0)|^2 + C_2 \frac{\delta_e^2}{(\epsilon^3)^2} + \int_{|\lambda| < R_c, |\varphi_0(\lambda)| < \epsilon^a} |F(f_0)|^2.
\]
We have $||\hat{\varphi} - \hat{\varphi}_\epsilon||_{L^\infty(\mathbb{R})} \leq ||\varphi - \varphi_\epsilon||_{L^1(\mathbb{R})}$, a standard procedure gives easily that $\lim_{\epsilon \to 0} ||\hat{g}_0\hat{\varphi} - \hat{g}_\epsilon\hat{\varphi}_\epsilon||_{L^2(\mathbb{R})} = 0$. Indeed, we have

$$||\hat{g}_0\hat{\varphi} - \hat{g}_\epsilon\hat{\varphi}_\epsilon||_{L^2(\mathbb{R})}^2 \leq 2(||\hat{\varphi}_\epsilon||_{L^\infty(\mathbb{R})}||\hat{g}_0 - \hat{g}_\epsilon||_{L^2(\mathbb{R})} + ||\hat{g}_0||_{L^2(\mathbb{R})}||\hat{\varphi}_\epsilon - \varphi_\epsilon||_{L^\infty(\mathbb{R})})^2 \leq 2(||\varphi_0||_{L^1(\mathbb{R})} + \epsilon)^2 + ||g_0||_{L^2(\mathbb{R})}^2 \epsilon^2$$

Similarly we have

$$||\hat{g}_0\hat{\varphi} - \hat{g}_\epsilon\hat{\varphi}_\epsilon||_{L^2(\mathbb{R})}^2 \leq 2(||\varphi_0||_{L^1(\mathbb{R})}^2 + ||g_0||_{L^2(\mathbb{R})}^2 + 1)\epsilon^2.$$

So we have

$$\frac{1}{\delta^2} \int_{\mathbb{R}} |\hat{g}_0\hat{\varphi}_\epsilon - \hat{g}_\epsilon\hat{\varphi}_\epsilon|^2 + \frac{1}{\delta^2} \int_{\mathbb{R}} |\hat{g}_0\hat{\varphi}_\epsilon - \hat{g}_\epsilon\hat{\varphi}_\epsilon|^2 \leq C_1 \frac{\epsilon^2}{\delta^2}.$$

Combining estimates above we get the conclusion of Theorem.

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ERROR OF TIKHONOV’S REGULARIZATION FOR INTEGRAL CONVOLUTION EQUATIONS

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