POLE PLACEMENT FOR OVERDETERMINED 2D SYSTEMS

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ABSTRACT. We formulate and solve a pole placement problem by state feedback for overdetermined 2D systems modeled by commutative operator vessels. In this setting, the transfer function of the system is given by a meromorphic bundle map between two holomorphic vector bundles of finite rank over the normalization of a projective plane algebraic curve. An obstruction for a solution is given by an existence of a certain meromorphic bundle map on the input bundle. Reducing to the 1D case, this gives a functional obstruction which is equivalent to the classical pole placement theorem. Our result gives a new approach to pole placement even in the classical case, and answers a question of Ball and Vinnikov.

0. INTRODUCTION

An overdetermined 2D continuous-time time-invariant linear input-state-output system is a linear system described by the following system of equations:

\[
\Sigma : \begin{cases}
\frac{\partial x}{\partial t_1}(t_1, t_2) = A_1 x(t_1, t_2) + B_1 u(t_1, t_2) \\
\frac{\partial x}{\partial t_2}(t_1, t_2) = A_2 x(t_1, t_2) + B_2 u(t_1, t_2) \\
y(t_1, t_2) = C x(t_1, t_2) + D u(t_1, t_2).
\end{cases}
\]

Here, \( u, x \) and \( y \) represents the input, state, and output signals, respectively. The input space is denoted by \( E \), the state space by \( \mathcal{H} \) and the output space by \( E^* \). All spaces are assumed to be Hilbert spaces over the complex numbers. The operators \( A_1, A_2, B_1, B_2, C \) and \( D \) act as follows:

\[
A_1, A_2 : \mathcal{H} \to \mathcal{H} \\
B_1, B_2 : E \to \mathcal{H} \\
C : \mathcal{H} \to E^* \\
D : E \to E^*.
\]

Experience showed that a good model to study these kind of systems is a notion called a Livšic-Kravitsky commutative two-operator vessel. We recall the definition and most important properties of this model in Section 1 below. The purpose of this article is to initiate the development of a theory of state feedback for these kinds of systems. The next quote is taken from [11, Page 14]:

"The Pole-Shifting Theorem is central to linear systems theory and is itself the starting point for more interesting analysis." 

2010 Mathematics Subject Classification: primary: 93B55, secondary: 14H60, 47N70, 93B25, 93B27.
The main result of this paper is a generalization of the pole placement theorem to the setting of operator vessels. The transfer function of an operator vessel is given by a meromorphic bundle map between two vector bundles over a compact Riemann surface given by the normalization of a plane algebraic curve. Interpolation problems for such functions are far from being trivial, and the classical approach to pole shifting using an explicit construction of the feedback operator directly from the prescribed pole datum seems difficult to achieve.

In view of this difficulty, we propose in this paper a new approach for pole placement. We will show (Proposition 2.1.3) that whenever a closed loop system of an operator vessel is formed by state feedback, its transfer function factors as a composition of the transfer function of the open loop system, and the transfer function of another system, called the controller system associated to the state feedback operator. The controller system has a simpler structure, and is thus easier to construct. As far as we know this construction gives a new approach for pole placement even for classical multidimensional linear systems.

We were led to the definition of the controller system by the rigidity of the vessel conditions. Thus, this work serves as a demonstration for the principle that developing system theoretic ideas in the more complicated overdetermined 2D setting might shed new light on the classical one dimensional case. Here is the main result of this text:

**Theorem.** Let $B$ be an operator vessel (see Definition 1.1.1) satisfying the assumptions (1.1.14), (1.1.10), (1.2.3) and (1.3.5). Denote by $X$ the compact Riemann surface associated to it. Let $E_{in}$ and $E_{out}$ be the input and output holomorphic vector bundles over $X$ associated to $B$, and denote by $S : E_{in} \to E_{out}$ the transfer function of $B$.

Given a meromorphic bundle map $T : E_{in} \to E_{out}$, there exist an admissible state feedback operator $F$, such that the closed loop system obtained from $B$ by applying the feedback operator $F$ has a transfer function equal to $T$, if and only if the left zero divisor of $T$ is contained in the left zero divisor of $S$, and $T$ is equal to $S$ at all points of $X$ which lie over the line at infinity.

This is repeated as Theorem 2.3.2 in the body of the paper. It answers a question of Ball and Vinnikov (see [4, Section 4]).

1. OPERATOR VESSELS AND STATE FEEDBACK

1.1. **Operator vessels and their associated compact Riemann surface.** We begin by recalling the definition of an operator vessel, a notion which serves as a useful model for studying overdetermined 2D systems as in equation (0.1). We refer the reader to [4,8,9,10,12] and their references for more background about these objects.

**Definition 1.1.1.** A Livščik-Kravitsky commutative two-operator vessel (abbreviated to operator vessel, or simply a vessel) is a collection of linear operators and spaces of the form:

$$
\mathcal{B} = (A_1, A_2, \tilde{B}, C, D, \tilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_1^*, \sigma_2^*, \gamma^*; \mathcal{H}, \mathcal{E}, \tilde{\mathcal{E}}, \mathcal{E}^*, \tilde{\mathcal{E}}^*)
$$

Here, the vector spaces $\mathcal{H}, \mathcal{E}, \tilde{\mathcal{E}}, \mathcal{E}^*$ and $\tilde{\mathcal{E}}^*$ are finite dimensional vector spaces over $\mathbb{C}$, and there are equalities $\dim \mathcal{E} = \dim \tilde{\mathcal{E}}, \dim \mathcal{E}^* = \dim \tilde{\mathcal{E}}^*$.

The operators act as follows: $A_1, A_2 : \mathcal{H} \to \mathcal{H}, \tilde{B} : \tilde{\mathcal{E}} \to \mathcal{H}, \sigma_1, \sigma_2, \gamma : \mathcal{E} \to \tilde{\mathcal{E}}, C : \mathcal{H} \to \mathcal{E}^*, D : \mathcal{E} \to \mathcal{E}^*, \tilde{D} : \mathcal{E}^* \to \tilde{\mathcal{E}}^*$ and $\sigma_{1*}, \sigma_{2*}, \gamma_{*} : \mathcal{E}^* \to \tilde{\mathcal{E}}^*$. 

It is assumed that the operators $D$ and $\tilde{D}$ are invertible, and that the following conditions, called the vessel conditions, hold:

$$(A1) \quad A_1 A_2 = A_2 A_1$$
$$(A2) \quad A_1 \tilde{B} \sigma_2 - A_2 \tilde{B} \sigma_1 + \tilde{B} \gamma = 0$$
$$(A3) \quad \sigma_2 \gamma A_1 - \sigma_1 \gamma A_2 + \gamma C = 0$$
$$(A4) \quad \sigma_1 D = D \sigma_1, \quad \sigma_2 D = \tilde{D} \sigma_2$$
$$\gamma D = \tilde{D} \gamma + \sigma_1 \gamma B \sigma_2 - \sigma_2 \gamma B \sigma_1.$$  \hfill (1.1.2)

1.1.3. Given an operator vessel
$$B = (A_1, A_2, \tilde{B}, C, D, \tilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_1 \gamma, \sigma_2 \gamma, \gamma; H, E, \tilde{E}, E_s, \tilde{E}_s),$$
we define two polynomials in two complex variables:

$$p_{\text{in}}(\lambda_1, \lambda_2) = \det(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma)$$
and

$$p_{\text{out}}(\lambda_1, \lambda_2) = \det(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma).$$  \hfill (1.1.3)

1.1.4. We will make the following assumption on the polynomials $p_{\text{in}}$ and $p_{\text{out}}$: we assume that

$$p_{\text{in}}(\lambda_1, \lambda_2) = (p_1(\lambda_1, \lambda_2))^r,$$
and

$$p_{\text{out}}(\lambda_1, \lambda_2) = (p_2(\lambda_1, \lambda_2))^s$$
for some irreducible polynomials $p_1, p_2 \in \mathbb{C}[\lambda_1, \lambda_2]$. We define the following plane algebraic curves:

$$C_1 = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 | p_1(\lambda_1, \lambda_2) = 0\}, \quad C_2 = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 | p_2(\lambda_1, \lambda_2) = 0\}.$$  \hfill (1.1.4)

By abuse of notation we will also denote the extensions of these affine curves to $\mathbb{P}^2 \mathbb{C}$ by $C_1, C_2$. Denoting by $L_\infty$ the line at infinity of $\mathbb{P}^2 \mathbb{C}$, we will make the assumption that for any $p \in C_1$ (respectively $p \in C_2$), such that $p \in L_\infty$, the intersection number of $C_1$ (resp. $C_2$) and $L_\infty$ at $p$ is equal to 1.

1.1.7. Given a plane algebraic curve $C = \{(\lambda_1, \lambda_2) \mid f(\lambda_1, \lambda_2) = 0\}$, for some $f \in \mathbb{C}[x, y]$, and given some $p \in C$, we denote by $\mu_p(C)$ the of multiplicity of $p$ on $C$. By definition, this is the smallest integer $n$, such that all partial derivatives of $f$ of degrees $< n$ vanish at $p$, and at least one partial derivative of $f$ or order $n$ does not vanish at $p$. Note that $C$ is smooth at $p$ if and only if $\mu_p(C) = 1$.

1.1.8. For any $(\lambda_1, \lambda_2) \in C_1$, we consider the set
$$E_{\text{in}}(\lambda_1, \lambda_2) = \ker(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma).$$
Similarly, for $(\lambda_1, \lambda_2) \in C_2$, we define
$$E_{\text{out}}(\lambda_1, \lambda_2) = \ker(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma).$$
By [10] Proposition 10.5.1], for any $p \in C_1$, and any $q \in C_2$ one has inequalities
$$\dim E_{\text{in}}(p) \leq \mu_p(C_1) \cdot r, \quad \dim E_{\text{out}}(q) \leq \mu_q(C_2) \cdot s.$$  \hfill (1.1.5)

Here, $r$ and $s$ are as in (1.1.5) and (1.1.6). Note that $E_{\text{in}}$ and $E_{\text{out}}$ have the structure of torsion free sheaves over $C_1, C_2$. 


1.1.10. We will further make the maximality assumption, namely, that the two inequalities of (1.1.9) are equalities at all points of \( C_1 \) and \( C_2 \). We also make a somewhat stronger assumption that \( E_{in} \) and \( E_{out} \) are fully saturated (see [6 Section 4], [7 Section 2.4.5] or [13 Page 340] for discussions about this notion). The most important thing to note about this assumption, as explained in [4], is that it is satisfied if \( C_1 \) and \( C_2 \) are smooth algebraic curves.

1.1.11. As explained in [4] Section 1.2, the assumptions (1.1.4) and (1.1.10) and the fact that the operator \( D \) is invertible imply that there is some constant \( \mu \in \mathbb{C}^\times \) such that \( p_{out}(\lambda_1, \lambda_2) = \mu \cdot p_{in}(\lambda_1, \lambda_2) \). Thus, under these assumptions, to any vessel \( B \) there is an associated plane algebraic curve

\[
C = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid p_{in}(\lambda_1, \lambda_2) = 0\} = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid p_{out}(\lambda_1, \lambda_2) = 0\}.
\]

Denote by \( X \) the associated compact Riemann surface obtained from the normalization of \( C \). According to [2] Theorem 2.1], the torsion free sheaves \( E_{in} \) and \( E_{out} \) lift to holomorphic vector bundles over \( X \). By abuse of notation, we will also denote them by \( E_{in} \) and \( E_{out} \). Vector bundles that arise in such a way are called vector bundles which have a determinantal representation.

1.2. The transfer function of an operator vessel. To discuss the transfer function associated to the vessel \( B \) we first recall the notion of a joint spectrum:

1.2.1. Let \( A_1, A_2 \in M_n(\mathbb{C}) \) be two square matrices. We say that \( A_1, A_2 \) are commuting if \( A_1 \cdot A_2 = A_2 \cdot A_1 \). In this case, their joint spectrum \( \text{Spec}(A_1, A_2) \) is defined to be the set of all pairs \((\lambda_1, \lambda_2) \in \mathbb{C}^2\), such that \( A_1 \cdot v = \lambda_1 \cdot v \) and \( A_2 \cdot v = \lambda_2 \cdot v \) for some non-zero vector \( v \in \mathbb{C}^n \). The following easy fact from linear algebra characterizes the joint spectrum: for any \((\lambda_1, \lambda_2) \in \mathbb{C}^2\), there are \( \xi_1, \xi_2 \in \mathbb{C} \) such that \( \xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2) \) is invertible if and only if \((\lambda_1, \lambda_2) \notin \text{Spec}(A_1, A_2)\).

1.2.2. The transfer function of \( B \) is defined as follows: given \((\lambda_1, \lambda_2) \in \mathbb{C}\), such that

\[
(\lambda_1, \lambda_2) \notin \text{Spec}(A_1, A_2),
\]

let \( \xi_1, \xi_2 \in \mathbb{C} \) be such that

\[
\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2)
\]

is invertible. For any \( v \in E_{in}(\lambda_1, \lambda_2) \), we define:

\[
S_B(\lambda_1, \lambda_2)v = (D + C(\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2)))^{-1}B(\xi_1 \sigma_1 + \xi_2 \sigma_2)v.
\]

It was shown in [4] that this is independent of the choices of \( \xi_1, \xi_2 \), and that

\[
S_B(\lambda_1, \lambda_2)v \in E_{out}(\lambda_1, \lambda_2).
\]

This means that \( S_B \) is a bundle map, defined outside the finite set \( \text{Spec}(A_1, A_2) \). This map may be lifted to a meromorphic bundle map \( E_{in} \to E_{out} \) over \( X \) which is holomorphic outside points which lie over \( \text{Spec}(A_1, A_2) \) (and may or may not have poles at the points above the joint spectrum). Another useful property of \( S_B \) which follows from this definition is that \( S_B \) is equal to \( D \) when restricted to the point of \( X \) which lies over \( L_\infty \).

1.2.3. In view of the above discussion, we will make the following additional assumption on the vessel \( B \): every point \( \lambda \) in the joint spectrum \( \text{Spec}(A_1, A_2) \) is a smooth point of \( C \). This ensures that the singularities of \( S_B \) lie all over the smooth points of \( C \), and there are no poles at singular points.

We next discuss a class of functions that the transfer function belongs to.
1.2.4. Let $X$ be a compact Riemann surface, and let $\pi_E : E \to X, \pi_F : F \to X$ be two holomorphic vector bundles over $X$. In particular, $E$ and $F$ are complex manifolds, so it makes sense to talk about holomorphic and meromorphic functions between them. A map $T : E \to F$ is called a **meromorphic bundle map** if it is a meromorphic map which is also a bundle map, that is: $\pi_F \circ T = \pi_E$, and $T$ is linear over each fiber in which it is defined.

1.2.5. The transfer function of an operator vessel is an example of a meromorphic bundle map. For vector bundles which have determinantal representations, the converse is also true: every meromorphic bundle map between such bundles is the transfer function of some operator vessel (see [2, 3, 5] for a proof of this fact).

To discuss zero and pole data of meromorphic bundle maps, we follow the local case, as in [1]. Given $p \in \mathbb{C}$, we denote by $\mathcal{O}_p$ the ring of germs of holomorphic functions at $p$, and by $\mathcal{O}_p^\times$ its subset consisting of germs $\phi$ such that $\phi(p) \neq 0$.

1.2.6. Let $A(z)$ be a square matrix of rational functions, such that $\det A(z)$ is not identically zero. This implies that $A^{-1}$ is also a rational matrix function. Given $\phi \in \mathcal{O}_{z_0}^\times$, we say that $A$ has a left zero at the point $z_0$ in direction $\phi$ of order $n$, if $\phi(z)A(z) = z^n \psi(z)$ for some $\psi \in \mathcal{O}_{z_0}^\times$. We say that $A(z)$ has a left pole at $z_0$ in direction $\phi(z)$ of order $n$ if $A^{-1}(z)$ has a left zero at $z_0$ in direction $\phi$ of order $n$.

These definitions are generalized to the global case of meromorphic bundle maps by replacing holomorphic germs by germs of holomorphic sections. We define the divisor datum of a meromorphic bundle map as follows:

**Definition 1.2.7.** Let $X$ be a compact Riemann surface, and let $T : E \to F$ be a meromorphic bundle map between two holomorphic vector bundles over $X$.

(1) The left zero set of $T$ is the set
$$LZ(T) = \{(\phi, n, z_0) \mid T \text{ has a left zero at } z_0 \text{ of order } \geq n \text{ at direction } \phi\}.$$

(2) The left pole set of $T$ is the set
$$LP(T) = \{(\phi, n, z_0) \mid T \text{ has a left pole at } z_0 \text{ of order } \geq n \text{ at direction } \phi\}.$$

1.2.8. Note that by definition, a left zero of $T : E \to F$ is a triple $(\phi, n, z_0)$, where $\phi$ is a germ of an holomorphic section of the bundle $F^\ast$, the dual of the bundle $F$. Similarly, a left pole of $T$ is a triple $(\phi, n, z_0)$, where $\phi$ is a germ of an holomorphic section of the bundle $E^\ast$.

1.3. **Controllability and Observability of operator vessels.**

1.3.1. A one dimensional linear system $\Sigma = (A, B, C, D; \mathcal{H}, \mathcal{E}, \mathcal{E}_e)$ is called controllable if the pair $(A, B)$ is controllable. Explicitly, this means that
$$\sum_{n=0}^{\infty} \text{Im } A^n B = \mathcal{H}.$$ 

Similarly, $\Sigma$ is called observable if the pair $(C, A)$ is observable. That is,
$$\bigcap_{n=0}^{\infty} \ker (CA^n) = \{0\}.$$

These linear algebra definitions are equivalent to the usual system-theoretic definitions of these terms.
Similarly, for operator vessels, we define:

**Definition 1.3.2.** Let $\mathcal{B} = (A_1, A_2, \tilde{B}, C, D, \tilde{D}, \sigma_1, \sigma_2, \gamma; \sigma_1^*, \sigma_2^*, \gamma^*; H, E, \tilde{E}, \tilde{E}, \tilde{E}_s)$ be an operator vessel.

1. We say that $\mathcal{B}$ is controllable if
\[
\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \Im A_1^{n_1} A_2^{n_2} \tilde{B} = H.
\]

2. We say that $\mathcal{B}$ is observable if
\[
\bigcap_{n_1=0}^{\infty} \bigcap_{n_2=0}^{\infty} \ker (CA_1^{n_1} A_2^{n_2}) = \{0\}.
\]

3. The operator vessel $\mathcal{B}$ is called minimal if it is both controllable and observable.

**1.3.3.** In [4, Proposition 1.11], it was shown that as in the one dimensional case, one may give system-theoretic definitions to these terms, imitating the usual ones in terms of the controllable subspace and unobservable subspace, and that they are equivalent to Definition 1.3.2. As we will not need these in this paper, we omit recalling them.

**1.3.4.** Given a vessel $\mathcal{B} = (A_1, A_2, \tilde{B}, C, D, \tilde{D}, \sigma_1, \sigma_2, \gamma; \sigma_1^*, \sigma_2^*, \gamma^*; H, E, \tilde{E}, \tilde{E}, \tilde{E}_s)$ we say that a direction $(\xi_1, \xi_2) \in \mathbb{P}^2 \mathbb{C}$ is a regular direction for $\mathcal{B}$ if the operator $\sigma_{\xi} := \xi_1 \sigma_1 + \xi_2 \sigma_2$ is invertible. By the vessel condition (A4), this implies that the operator $\sigma_{\xi}^* = \xi_1 \sigma_1^* + \xi_2 \sigma_2^*$ is also invertible.

**1.3.5.** We will make the following assumption: all vessels in this paper have regular directions. Equivalently, the function $\det(\sigma_{\xi})$ (equivalently, $\det(\sigma_{\xi})$) is not identically zero.

**1.3.6.** Given a direction $\xi = (\xi_1, \xi_2)$, we will shorten notation and set
\[
A_{\xi} = \xi_1 A_1 + \xi_2 A_2, \quad B_{\xi} = \tilde{B}(\xi_1 \sigma_1 + \xi_2 \sigma_2).
\]

Using these notations, we define:
\[
(1.3.7) \quad S_{\xi}(\lambda) = D + C(\lambda I - A_{\xi})^{-1} B_{\xi}
\]

The function $S_{\xi}(\lambda)$ is rational matrix function, called the restricted transfer function of $\mathcal{B}$ at direction $(\xi_1, \xi_2)$.

**1.3.8.** Given a rational matrix function $S(\lambda)$, a system theoretic realization of $S$ is a presentation:
\[
S(\lambda) = D + C(\lambda I - A)^{-1} B.
\]

Such a presentation is called minimal if the square matrix $A$ has minimal size along all possible realizations of $S$. By [1] Theorem 4.1.4, this happens if and only if the pair $(C, A)$ is observable, and the pair $(A, B)$ is controllable.

**Proposition 1.3.9.** Let $\mathcal{B}$ be a minimal vessel, and let $(\xi_1, \xi_2) \in \mathbb{P}^2 \mathbb{C}$ be a regular direction for $\mathcal{B}$. Then the realization (1.3.7) of the restricted transfer function $S_{\xi}$ is minimal.

This follows from the next two lemmas:

**Lemma 1.3.10.** Suppose $\mathcal{B}$ is an observable vessel, and suppose that $\xi$ is a regular direction for $\mathcal{B}$. Then
\[
\bigcap_{n=0}^{\infty} \ker CA_{\xi}^n = \{0\}
\]
Proof. Since $B$ is observable, we have that
\[
\bigcap_{n_1=0}^{\infty} \bigcap_{n_2=0}^{\infty} \ker( CA_1^{n_1} A_2^{n_2} ) = \{ 0 \}
\]
Since $\xi$ is regular, it follows that $\sigma_\xi \xi$ is invertible. By the vessel condition (A3) we have that
\[
\sigma_2 \sigma_\xi CA_1 - \sigma_1 \sigma_\xi CA_2 + \gamma_\xi C = 0
\]
Multiplying both sides of this equation by $\xi_1 \xi_2$ and rearranging we get
\[
CA_1 = \frac{1}{\xi_1} \sigma_\xi^{-1} (\xi_1 \sigma_1 \sigma_\xi C - \xi_1 \xi_2 \gamma_\xi C)
\]
and similarly
\[
CA_2 = \frac{1}{\xi_2} \sigma_\xi^{-1} (\xi_2 \sigma_2 \sigma_\xi C + \xi_1 \xi_2 \gamma_\xi C)
\]
so that both $CA_1$ and $CA_2$ are of the form $E_1 + E_2 \sigma_\xi C$ for some matrices $E_1, E_2$. We now claim that for all $n_1 \geq 0, n_2 \geq 0$ one may write $CA_1^{n_1} A_2^{n_2} = \sum_{k=0}^{n_1+n_2} E_k C A_\xi^k$ for some matrices $E_0, \ldots, E_{n_1+n_2}$. We prove this by induction. By symmetry and since $A_1$ and $A_2$ commute, it is enough to show that if it is true for $CA_1^{n_1} A_2^{n_2}$ then it is true for $CA_1^{n_1+1} A_2^{n_2+1}$. Let $CA_2 = M_0 C + M_1 CA_\xi$. Write
\[
CA_1^{n_1} A_2^{n_2} = \sum_{k=0}^{n_1+n_2} E_k C A_\xi^k
\]
and multiply this by $A_2$. Then
\[
CA_1^{n_1} A_2^{n_2+1} = \sum_{k=0}^{n_1+n_2} E_k C A_\xi^k A_2
\]
However, $A_2$ and $A_\xi$ commute, so we may write each term as:
\[
E_k C A_\xi^k A_2 = E_k C A_\xi^k A_2 = E_k (M_0 C + M_1 CA_\xi) A_\xi^k = E_k M_0 C A_\xi^k + E_k M_1 C A_\xi^k + 1
\]
so the entire sum has the required form. Thus, we obtain that for all $n_1 \geq 0, n_2 \geq 0$ we have that
\[
\bigcap_{k=0}^{n_1+n_2} \ker( CA_\xi^k ) \subseteq \ker( CA_1^{n_1+1} A_2^{n_2+1} )
\]
so the result follows. □ □

Dually, and using the vessel condition (A2) one has that

**Lemma 1.3.11.** Suppose $B$ is a controllable vessel, and suppose that $\xi$ is a regular direction for $B$. Then
\[
\sum_{n=0}^{\infty} \text{Im} A_\xi^n B_\xi = \mathcal{H}
\]
1.4. State feedback for operator vessels. Following [4, Example 1.20], we now introduce state feedback for operator vessels. Because of the centrality of this construction to this paper, we verify the following in details, even though it is a bit tedious.

**Proposition 1.4.1.** Let

\[ \mathcal{B} = (A_1, A_2, \tilde{B}, C, D, \tilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_1*, \sigma_2*, \gamma*; \mathcal{H}, \mathcal{E}, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_*) \]

be a vessel, and let \( F : \mathcal{H} \to \mathcal{E} \) be a linear operator. Suppose that \( F \) satisfies the following two conditions

\[ \sigma_2 F A_1 - \sigma_1 F A_2 + \gamma F = 0, \]  
\[ \sigma_1 F \tilde{B} \sigma_2 - \sigma_2 F \tilde{B} \sigma_1 = 0 \]

then the collection \( \mathcal{B}_F^{\text{CL}} = \)  

\[ (A_1 + \tilde{B} \sigma_1 F, A_2 + \tilde{B} \sigma_2 F, \tilde{B}, C + DF, D, \tilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_1*, \sigma_2*, \gamma*; \mathcal{H}, \mathcal{E}, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_*) \]

is an operator vessel. We say that \( F \) is an admissible state feedback operator, and that \( \mathcal{B}_F^{\text{CL}} \) is the closed loop system formed by applying the state feedback operator \( F \).

**Proof.** Suppose \( F \) satisfies equations (1.4.2) and (1.4.3). We have to verify the four vessel conditions.

Condition (A1):

\[
\begin{align*}
(A_1 + \tilde{B} \sigma_1 F)(A_2 + \tilde{B} \sigma_2 F) - (A_2 + \tilde{B} \sigma_2 F)(A_1 + \tilde{B} \sigma_1 F) & = \\
& = (A_1 A_2 + A_1 \tilde{B} \sigma_2 F + \tilde{B} \sigma_1 F A_2 + \tilde{B} \sigma_1 F \tilde{B} \sigma_2 F) \\
& - (A_2 A_1 + A_2 \tilde{B} \sigma_1 F + \tilde{B} \sigma_2 F \tilde{B} \sigma_1 F + \tilde{B} \sigma_2 F A_1) \\
& = (A_1 A_2 - A_2 A_1) + (\tilde{B} \sigma_1 F \tilde{B} \sigma_2 F - \tilde{B} \sigma_2 F \tilde{B} \sigma_1 F) + \\
& + (A_1 \tilde{B} \sigma_2 F + \tilde{B} \sigma_1 F A_2 - A_2 \tilde{B} \sigma_1 F - \tilde{B} \sigma_2 F A_1)
\end{align*}
\]

The first term vanishes because of vessel condition (A1) satisfied by \( \mathcal{B} \). The second term vanishes because of (1.4.3). For the third term:

\[
A_1 \tilde{B} \sigma_2 F + \tilde{B} \sigma_1 F A_2 - A_2 \tilde{B} \sigma_1 F - \tilde{B} \sigma_2 F A_1 = \\
A_1 \tilde{B} \sigma_2 F - A_2 \tilde{B} \sigma_1 F + \tilde{B}(\sigma_1 F A_2 - \sigma_2 F A_1)
\]

From (1.4.2), we have \( \sigma_1 F A_2 - \sigma_2 F A_1 = \gamma F \), so the last term is equal to

\[ A_1 \tilde{B} \sigma_2 F - A_2 \tilde{B} \sigma_1 F + \tilde{B} \gamma F = (A_1 \tilde{B} \sigma_2 - A_2 \tilde{B} \sigma_1 + \tilde{B} \gamma) F = 0 \]

where the last equality follows from the vessel condition (A2) for the vessel \( \mathcal{B} \). This establishes (A1).

Condition (A2):

\[
\begin{align*}
(A_2 + \tilde{B} \sigma_2 F) \tilde{B} \sigma_1 - (A_1 + \tilde{B} \sigma_1 F) \tilde{B} \sigma_2 - \tilde{B} \gamma & = \\
& = (A_2 \tilde{B} \sigma_1 - A_1 \tilde{B} \sigma_2 - \tilde{B} \gamma) + (\tilde{B} \sigma_2 F \tilde{B} \sigma_1 - \tilde{B} \sigma_1 F \tilde{B} \sigma_2) = 0
\end{align*}
\]

where the first term vanishes because of the vessel condition (A2) of \( \mathcal{B} \), and the vanishing of the second term follows from (1.4.3).
Condition (A3):
\[(1.4.7)\]
\[
\begin{align*}
\sigma_2(C + DF)(A_1 + \bar{B}\sigma_1 F) - \sigma_1(C + DF)(A_2 + \bar{B}\sigma_2 F) + \gamma_5(C + DF) &= \\
= \sigma_2(CA_1 + C\bar{B}\sigma_1 F + DFA_1 + DFB\sigma_1 F) \\
- \sigma_1(CA_2 + C\bar{B}\sigma_2 F + DFA_2 + DFB\sigma_2 F) + \gamma_5C + \gamma_5DF &= \\
= (\sigma_2CA_1 - \sigma_1CA_2 + \gamma_5C) + (\sigma_2CB\sigma_1 F - \sigma_1CB\sigma_2 F + \gamma_5DF) + \\
+ (\sigma_2DFA_1 - \sigma_1DFA_2 + \sigma_2DF\bar{B}\sigma_1 F - \sigma_1DF\bar{B}\sigma_2 F)
\end{align*}
\]

The first term vanishes because of condition (A3) for the vessel $B$. For the third term, using the equation $\sigma_iD = \bar{D}\sigma_i$ (condition (A4) for $B$), we have
\[(1.4.8)\]
\[
\begin{align*}
\sigma_2DFA_1 - \sigma_1DFA_2 + \sigma_2DF\bar{B}\sigma_1 F - \sigma_1DF\bar{B}\sigma_2 F &= \\
= \bar{D}\sigma_2FA_1 - \bar{D}\sigma_1FA_2 + \bar{D}\sigma_2\bar{B}\sigma_1 F - \bar{D}\sigma_1\bar{B}\sigma_2 F &= \\
= \bar{D}(\sigma_2FA_1 - \sigma_1FA_2) + (\sigma_2\bar{B}\sigma_1 F - \sigma_1\bar{B}\sigma_2 F) &= -\bar{D}\gamma F
\end{align*}
\]

where the last equation follows from (1.4.2) and (1.4.3). Hence, equation (1.4.7) becomes
\[(1.4.9)\]
\[
(\sigma_2C\bar{B}\sigma_1 F - \sigma_1C\bar{B}\sigma_2 F + \gamma_5DF) - \bar{D}\gamma F
\]

using the relation $\bar{D}\gamma = \gamma_5C - \sigma_1C\bar{B}\sigma_2 + \sigma_2C\bar{B}\sigma_1$ (condition (A4) for the vessel $B$), the equation (1.4.9) becomes
\[(1.4.10)\]
\[
(\sigma_2C\bar{B}\sigma_1 F - \sigma_1C\bar{B}\sigma_2 F + \gamma_5DF) - (\gamma_5C - \sigma_1C\bar{B}\sigma_2 + \sigma_2C\bar{B}\sigma_1)F = 0
\]

This establishes (A3).

Condition (A4):
The equations $\sigma_iD = \bar{D}\sigma_i$ are satisfied because of the vessel condition (A4) of $B$. We now verify the last equation of (A4):
\[(1.4.11)\]
\[
\begin{align*}
\gamma_5C - \bar{D}\gamma - \sigma_1(C + DF)\bar{B}\sigma_2 + \sigma_2(C + DF)\bar{B}\sigma_1 &= \\
= (\gamma_5C - \bar{D}\gamma - \sigma_1C\bar{B}\sigma_2 + \sigma_2C\bar{B}\sigma_1) + (\sigma_2DF\bar{B}\sigma_1 - \sigma_1DF\bar{B}\sigma_2)
\end{align*}
\]

The vanishing of the first term follows from condition (A4) of $B$. For the second term, using the relation $\sigma_iD = \bar{D}\sigma_i$ we obtain
\[(1.4.12)\]
\[
\begin{align*}
\sigma_2DF\bar{B}\sigma_1 - \sigma_1DF\bar{B}\sigma_2 &= \bar{D}\sigma_2\bar{B}\sigma_1 - \bar{D}\sigma_1\bar{B}\sigma_2 = \bar{D}(\sigma_2\bar{B}\sigma_1 - \sigma_1\bar{B}\sigma_2) &= 0
\end{align*}
\]

where the last equality follows from (1.4.3).

Hence, $B^CL$ satisfies (A1)-(A4), so it is indeed a vessel.

We may now state the question this paper answers: Let $B$ be a minimal vessel. Which transfer functions may be obtained as transfer functions of closed loop systems obtained from $B$ by state feedback? The next section will be dedicated to answer this question.

We finish this section with the following construction: state space similarity for vessels. We omit the proof which is a trivial verification, similar to the above, but easier.

**Proposition 1.4.13.** Let $B = (A_1, A_2, \bar{B}, C, D, \bar{D}, \sigma_1, \sigma_2, \gamma_5; \mathcal{H}, \mathcal{E}, \mathcal{F}, \mathcal{E}_s)$ be a vessel. Given an isomorphism $N : \mathcal{H}' \rightarrow \mathcal{H}$, the collection $N^{-1}BN$ given by
\[
(N^{-1}A_1N, N^{-1}A_2N, N^{-1}\bar{B}, CN, D, \bar{D}, \sigma_1, \sigma_2, \gamma; \mathcal{H}_s; \mathcal{E}, \mathcal{F}, \mathcal{E}_s)
\]

is an operator vessel, and $S_{N^{-1}BN} = S_B$. 
2. THE POLE PLACEMENT THEOREM

2.1. The controller vessel. The following is the main tool used in the proof of the main result of this paper.

Proposition 2.1.1. Consider an operator vessel
\[ \mathcal{B} = (A_1, A_2, B, C, D, \tilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*; \mathcal{H}, \mathcal{E}, \tilde{\mathcal{E}}, \mathcal{E}_*, \tilde{\mathcal{E}}) \]
and a linear operator \( F : \mathcal{H} \to \mathcal{E} \). Then \( F \) is an admissible state feedback operator if and only if the collection
\[ \mathcal{B}^\text{Ctrl} \]
\[
(1.4.2)
\]
is an operator vessel. The vessel \( \mathcal{B}^\text{Ctrl} \) is called the controller vessel of the state feedback operator \( F \).

Proof. Conditions (A1) and (A2) are exactly the two conditions of equations (1.4.2) and (1.4.3). Assuming \( F \) is an admissible state feedback operator, we leave the (easy) verification of the vessel conditions (A3)-(A4) to the reader.

\( \square \)

The next proposition explains the importance of the controller vessel for the pole placement problem.

Proposition 2.1.3. Let \( \mathcal{B} \) be a vessel, and let \( F \) be an admissible state feedback. Then the transfer function \( S_{\mathcal{B}} \) associated to \( \mathcal{B} \) factors as follows:
\[ \begin{array}{c}
\text{E}_{\text{in}} \xrightarrow{S_{\mathcal{B}}} S_{\mathcal{B}} \xrightarrow{S_{\mathcal{B}}^\text{Ctrl}} \text{E}_{\text{out}} \\
\text{E}_{\text{in}} \xrightarrow{S_{\mathcal{B}}^\text{Ctrl}} \text{E}_{\text{out}}
\end{array} \]

Proof. Let \( C \) be the plane curve associated to \( \mathcal{B} \), and let \( \lambda = (\lambda_1, \lambda_2) \in C \). Suppose further that \( \lambda \) is not a pole of either of the three transfer functions above. Let us set \( S(\lambda) = S_{\mathcal{B}}(\lambda) \), \( T = S_{\mathcal{B}}^\text{Ctrl}(\lambda) \), and \( R = S_{\mathcal{B}}^\text{Ctrl}(\lambda) \). By continuity, it is enough to show that \( T(\lambda) = S(\lambda)R^{-1}(\lambda) \).

We have that
\[ S(\lambda) = D + C(\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2))^{-1} \tilde{B}(\xi_1 \sigma_1 + \xi_2 \sigma_2) \]
and
\[ T(\lambda) = D + (C + DF)(\xi_1(\lambda_1 I - A_1 - \tilde{B} \sigma_1 F) + \xi_2(\lambda_2 I - A_2 - \tilde{B} \sigma_2 F))^{-1} \tilde{B}(\xi_1 \sigma_1 + \xi_2 \sigma_2) \]
By [4] Theorem 1.16, we have that:
\[ R^{-1}(\lambda) = I + F(\xi_1(\lambda_1 I - A_1 - \tilde{B} \sigma_1 F) + \xi_2(\lambda_2 I - A_2 - \tilde{B} \sigma_2 F))^{-1} \tilde{B}(\xi_1 \sigma_1 + \xi_2 \sigma_2) \]
where we have chosen a direction \((\xi_1, \xi_2)\) so that all of these will be well defined. This is possible because for every \( \lambda \) (which is not a pole) there are only finitely many choices of directions \((\xi_1, \xi_2)\) in which the above expressions are not well defined. To shorten notation, let us set \( V = \tilde{B}(\xi_1 \sigma_1 + \xi_2 \sigma_2), \quad N = (\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2)) \), and \( M = (\xi_1(\lambda_1 I - A_1 - \tilde{B} \sigma_1 F) + \xi_2(\lambda_2 I - A_2 - \tilde{B} \sigma_2 F)) \). Under these notations, we have
\[ S = D + CN^{-1}V, \quad T = D + (C + DF)M^{-1}V \text{ and } R^{-1} = I + FM^{-1}V. \]
Hence,
\[ SR^{-1} = (D + CN^{-1}V)(I + FM^{-1}V) = D + DFM^{-1}V + CN^{-1}V + CN^{-1}VFM^{-1}V \]
Thus, we must show that
\[ CN^{-1}V + CN^{-1}VFM^{-1}V = CM^{-1}V, \]
or that
\[ (CN^{-1} + CN^{-1}VFM^{-1})V = CM^{-1}V. \]
For this it is enough to show that \( CN^{-1}M + CN^{-1}VF = C \) which is equivalent to \( CN^{-1}M + CN^{-1}VF = C \). To show this it is enough to show that \( N^{-1}M + N^{-1}VF = I \), which is equivalent to \( M + VF = N \). Since this is true, the result follows. \( \square \)

2.2. Left pole datum of operator vessels.

2.2.1. It is a well known fact in classical system theory that given a minimal realization
\[ S(\lambda) = D + C(\lambda I - A)^{-1}B. \]
of a rational matrix function \( S \) one can read the left pole data of \( S \) from the pair \((A, B)\). See [1, Theorem 4.2.1(iii)] for a precise statement of this idea.

The next lemma generalizes this fact to the setting of operator vessels, by showing that the triple \((A_1, A_2, \tilde{B})\) contains the data of the left pole set of the transfer function of a minimal operator vessel.

**Lemma 2.2.2.** Let
\[ \mathcal{B} = (A_1, A_2, \tilde{B}, C, D, \check{D}, \sigma_1, \sigma_2, \gamma, \sigma_1^*, \sigma_2^*, \gamma^*; \mathcal{H}, \mathcal{E}, \check{\mathcal{E}}, \mathcal{E}^*, \check{\mathcal{E}}^*) \]
be a minimal operator vessel, and let \( S = S_{\mathcal{B}} : E_{in} \to E_{out} \) be its transfer function. Let \( R : E_{in} \to E_{in} \) be a meromorphic bundle map, such that \( R|_{L_{\infty}} = 1 \). Then
\[ \text{LP}(R) \subseteq \text{LP}(S) \]
if and only if the following holds: there exists an operator \( K : \mathcal{H} \to \mathcal{E} \) such that the collection
\[ \mathcal{V} = (A_1, A_2, \tilde{B}, K, I, I, \sigma_1, \sigma_2, \gamma, \sigma_1^*, \sigma_2^*, \gamma^*; \mathcal{H}, \mathcal{E}, \check{\mathcal{E}}, \mathcal{E}^*, \check{\mathcal{E}}^*) \]
is an operator vessel such that \( R = S_{\mathcal{V}} \).

**Proof.** Suppose first \( R = S_{\mathcal{V}} \), where
\[ \mathcal{V} = (A_1, A_2, \tilde{B}, K, I, I, \sigma_1, \sigma_2, \gamma, \sigma_1^*, \sigma_2^*, \gamma^*; \mathcal{H}, \mathcal{E}, \check{\mathcal{E}}, \mathcal{E}^*, \check{\mathcal{E}}^*). \]
Let \((\lambda_1, \lambda_2) \in \mathbb{C} \). We will show that if \( R \) has a pole at this point then \( S \) also has a pole at this point, at the same direction, of at least the same order. Choose a direction \((\xi_1, \xi_2) \in \mathbb{P}^2 \mathbb{C} \) such that
\[ \xi_1(\lambda_1^* I - A_1) + \xi_2(\lambda_2^* I - A_2) \]
is invertible for all \((\lambda_1^*, \lambda_2^*) \) in some open punctured neighborhood \( U \) of \((\lambda_1, \lambda_2) \), and such that \((\xi_1, \xi_2) \) is a regular direction, making \( (\lambda_1^*, \lambda_2^*) \to \xi_1 \lambda_1^* + \xi_2 \lambda_2^* \) a local coordinate of \( \mathbb{C} \) near \((\lambda_1, \lambda_2) \).

To shorten notation, let us set
\[ A_\xi = \xi_1 A_1 + \xi_2 A_2, \quad B_\xi = \tilde{B}(\xi_1 \sigma_1 + \xi_2 \sigma_2). \]
and \( \lambda_\xi = \xi_1 \lambda_1 + \xi_2 \lambda_2 \). Using this notation, we have that
\[ S_\xi(\lambda_\xi) = D + C(\lambda_\xi I - A_\xi)^{-1}B_\xi \]
and
\[ R_\xi(\lambda_\xi) = I + K(\lambda_\xi I - A_\xi)^{-1}B_\xi. \]
Since $\xi$ is a regular direction, Proposition 1.3.9 says that this realization of $S_\xi$ is minimal. Hence, according to [1, Theorem 4.2.1(iii)], the pair $(A_\xi, B_\xi)$ is a global left pole pair of $S_\xi$. Since $R_\xi$ shares the same $(A_\xi, B_\xi)$, it follows that every left pole of $R_\xi$ is a left pole of $S_\xi$. Since in a neighborhood of $(\lambda_1, \lambda_2)$, $S$ and $R$ are obtained by restrictions of $S_\xi$ and $R_\xi$, the result follows.

Conversely, assume $\text{LP}(R) \subseteq \text{LP}(S)$. First, note that there is some minimal vessel

$$\mathcal{V'} = (A'_1, A'_2, \bar{B}', K', I, \sigma_1, \sigma_2, \gamma, \sigma_1, \sigma_2, \gamma; \mathcal{H}', \mathcal{E}, \bar{\mathcal{E}}, \tilde{\mathcal{E}})$$

such that $S_{\mathcal{V'}} = R$. This follows from [2, Theorem 3.4] if all the left poles of $R$ are simple poles, and from [5, Theorem 4.1] in the general case. See also [3, Theorem 6.1].

The proof of either of these statements constructs the triple

$$(A'_1, A'_2, \bar{B}')$$

directly from the pole data, and our assumption that $\text{LP}(R) \subseteq \text{LP}(S)$ implies from these constructions that there is some subspace $\mathcal{H}'' \subset \mathcal{H}$, and a linear isomorphism $T : \mathcal{H}' \to \mathcal{H}''$, such that the following holds: $A_1$ and $A_2$ are $\mathcal{H}''$-invariant, and there are equalities $TB = \bar{B}'$, $T^{-1}A_1|_{\mathcal{H}''} T = A'_1$ and $T^{-1}A_2|_{\mathcal{H}''} T = A'_2$.

The vessel $\mathcal{V}$ will now be obtained by enlarging the state-space of the vessel $T^{-1}\mathcal{V}'T$, obtained from $\mathcal{V}'$ using state space isomorphism as in Proposition 1.4.13. Explicitly: Let $\mathcal{H}''$ be some complementary subspace of $\mathcal{H}'$ in $\mathcal{H}$, and define $\tilde{K} : \mathcal{H} \to \mathcal{E}$ as follows: $K|_{\mathcal{H}''} = K'T$, and let $K|_{\mathcal{H}''} = 0$. Then we get that the collection

$$\mathcal{V} = (A_1, A_2, \bar{B}, K, I, \sigma_1, \sigma_2, \gamma, \sigma_1, \sigma_2, \gamma; \mathcal{H}, \mathcal{E}, \bar{\mathcal{E}}, \tilde{\mathcal{E}})$$

is an operator vessel, and that

$$S_{\mathcal{V}} = S_{T^{-1}\mathcal{V}'T} = S_{\mathcal{V}'} = R.$$

This proves the claim. \hfill \Box

2.3. The pole placement theorem.

Lemma 2.3.1. Let $X$ be a compact Riemann surface, and let $\pi_E : E \to X, \pi_F : F \to X$ be two holomorphic vector bundles over $X$. Let $S, T : E \to F$ be two meromorphic bundle maps, and let $R = T^{-1} \circ S : E \to E$. Then $\text{LP}(R) \subseteq \text{LP}(S)$ if and only if $\text{LZ}(T) \subseteq \text{LZ}(S)$.

Note that the condition $\text{LP}(R) \subseteq \text{LP}(S)$ makes sense, because by (1.2.8), elements of each of these sets are triples $(\phi, n, z_0)$ where $\phi$ is a germ of an holomorphic section of $E^n$.

Proof. As this is a local question, we may assume that $X = \mathbb{C}$, and that $E, F$ are trivial. Suppose $\text{LP}(R) \subseteq \text{LP}(S)$. Assume $T$ has a left zero of order $k$ at direction $\phi$ at the origin. Let $\psi \in \mathcal{O}_0^\times$, such that $\phi(z)T(z) = z^k\psi(z)$. Hence, $\phi(z)S(z) = \phi(z)T(z)R(z) = z^k\psi(z)R(z)$. Assume $\psi(z)R(z) = z^l\alpha(z)$. If $l \geq 0$, then $S$ has a zero in direction $\phi$ of order greater or equal to $k$, which proves the claim. Otherwise, if $l < 0$, then setting $m = -l$, we see that $R$ has a left pole of order $m$ at direction $\alpha$. Hence, since $\text{LP}(R) \subseteq \text{LP}(S)$, it follows that for some $n \geq m$, there is a local section $\beta$ near 0, such that $\beta(z)S(z) = z^{-n}\alpha(z)$. On the other hand, the above calculation shows that $\phi(z)S(z) = z^{k-n}\alpha(z)$. Since $m - k < n$, we get a contradiction, so $l \geq 0$, and the claim follows. The converse is proved similarly. \hfill \Box

Here is the main result of this paper.
Theorem 2.3.2. Let \( \mathcal{B} \) be a minimal operator vessel. Let \( T : E_{in} \to E_{out} \) be a meromorphic bundle map whose poles do not lie over the singularities of \( \mathbb{C} \). Then there is an admissible state feedback operator \( F : \mathcal{H} \to \mathcal{E} \) such that \( T \) is the transfer function of the closed loop system \( \mathcal{B}_{F}^{Ctr} \) if and only if \( LZ(T) \subseteq LZ(\mathcal{S}_{\mathcal{B}}) \) and \( T|_{L_{\infty}} = S_{\mathcal{B}|L_{\infty}} \).

Proof. Suppose first that \( LZ(T) \subseteq LZ(\mathcal{S}_{\mathcal{B}}) \) and \( T|_{L_{\infty}} = S_{\mathcal{B}|L_{\infty}} \). Let

\[
R = T^{-1} \circ S : E_{in} \to E_{in}.
\]

Clearly, \( R|_{L_{\infty}} = 1 \). Moreover, by Lemma 2.3.1 we have that \( \text{LP}(R) \subseteq \text{LP}(S) \). Hence, by Lemma 2.2.2 there is an operator \( K : \mathcal{H} \to \mathcal{E} \) such that the collection

\[
\mathcal{V} = (A_{1}, A_{2}, \hat{B}, K, I, I, \sigma_{1}, \sigma_{2}, \gamma, \sigma_{1}, \sigma_{2}, \gamma; \mathcal{H}, \mathcal{E}, \hat{E}, \mathcal{E}, \hat{E})
\]

is an operator vessel, and such that \( R = S_{\mathcal{V}} \). Let \( F = -K \). Note that by definition of the controller vessel, \( \mathcal{V} = B_{F}^{Ctr} \). In particular, by Proposition 2.1.1 the operator \( F \) is an admissible state feedback operator. Hence, by Proposition 2.1.3 we have that

\[
S_{\mathcal{B}_{F}^{Ctr}} = S_{\mathcal{B}} \circ R^{-1}.
\]

so that \( T = S_{\mathcal{B}_{F}^{Ctr}} \). Conversely, suppose \( T = S_{\mathcal{B}_{F}^{Ctr}} \) for some admissible state feedback operator \( F : \mathcal{H} \to \mathcal{E} \). Note that the controller vessel \( B_{F}^{Ctr} \) is of the form required in Lemma 2.2.2 hence the map \( R = S_{\mathcal{B}_{F}^{Ctr}} \) satisfies: \( R|_{L_{\infty}} = 1 \) and \( \text{LP}(R) \subseteq \text{LP}(S) \). By Proposition 2.1.3 \( T = S \circ R^{-1} \). The result now follows from Lemma 2.3.1. \( \square \)

3. Pole placement over line bundles

3.1. General theory. In this final section, we analyze further the pole placement problem under the assumption that the vector bundles \( E_{in} \) and \( E_{out} \) are vector bundles of rank 1, that is, line bundles. In this case, we will show below precisely how the geometry of the compact Riemann surface \( X \) dictates the solution of the pole placement problem.

3.1.1. Recall that we assumed that the polynomials \( p_{in}(\lambda_{1}, \lambda_{2}) \) and \( p_{out}(\lambda_{1}, \lambda_{2}) \) are given by

\[
p_{in}(\lambda_{1}, \lambda_{2}) = (p_{1}(\lambda_{1}, \lambda_{2}))^{r}, \quad p_{out}(\lambda_{1}, \lambda_{2}) = (p_{2}(\lambda_{1}, \lambda_{2}))^{s}
\]

for some irreducible polynomials \( p_{1}, p_{2} \) and some \( r, s \geq 1 \). We shall now make the additional assumption (that holds generically) that \( r = s = 1 \). This implies that \( E_{in} \) and \( E_{out} \) are line bundles over \( X \), so that the directional information of pole and zero data degenerate. In other words, under these assumptions, \( \text{LP}(S_{\mathcal{B}}) \) and \( LZ(S_{\mathcal{B}}) \) are ordinary effective divisors on \( X \).

3.1.2. For any line bundle \( E \) on \( X \), recall that there is a natural isomorphism

\[
\text{Hom}_{\mathcal{C}_{X}}(E, E) \cong \mathcal{M}_{X}
\]

between the \( \mathbb{C} \)-algebra of meromorphic bundle maps \( E \to E \) and the \( \mathbb{C} \)-algebra of meromorphic functions on \( X \).

The above facts allow us to derive more explicit data from Theorem 2.3.2 in the line bundle case:

Corollary 3.1.3. Let \( \mathcal{B} \) be a minimal operator vessel as above, such that its input and output bundles are line bundles. Let \( Z = LZ(S_{\mathcal{B}}) \in \text{Div}_{\geq 0}(X) \) and \( P \in \text{Div}_{\geq 0}(X) \) be effective divisors. Then there is an admissible state feedback operator \( F : \mathcal{H} \to \mathcal{E} \) such that \( \text{LP}(S_{\mathcal{B}_{F}^{Ctr}}) = P \) if and only if there is some meromorphic function \( f \in \mathcal{M}_{X} \), such that \( \text{div}(f) = Z - P \), and such that \( f|_{L_{\infty}} = 1 \).
Proof. Given such a function $f \in \mathcal{M}_X$, the isomorphism \ref{lem:isomorphism} provides us with a meromorphic bundle map $R : E_{in} \to E_{in}$ with the same divisor data as $f$, and with $R(\mathcal{L}_\infty) = 1$. Letting $T = S_B \circ R^{-1}$, one sees that $LP(T) = P$, and by Theorem \ref{lem:feedback_dimension}, there is an admissible state feedback operator $F$ such that $S_{BC} = T$. Conversely, given an admissible state feedback operator $F$ such that $LP(S_{BC}) = P$, let $T = S_{BC}$, and let $R = T^{-1} \circ S$. Then one sees that $R : E_{in} \to E_{in}$, and that the meromorphic function $f$ corresponding to it satisfies $\text{div}(f) = Z - P$, and $f|_{\mathcal{L}_\infty} = 1$.

It follows that under the assumption that the input and output bundles are line bundles, the pole placement problem reduces to a classical interpolation problem over a compact Riemann surface $X$:

**Problem 3.1.4.** Let $C$ be a projective plane algebraic curve of degree $m$ over $\mathbb{C}$ such that its intersection with the line at infinity contains $m$ different points. Let $X$ be the compact Riemann surface associated to its normalization, and let $Z$ be an effective divisor on $X$. For which effective divisors $P$ on $X$, there is a meromorphic function $f \in \mathcal{M}_X$ such that $\text{div}(f) = Z - P$, and such that $f|_{\mathcal{L}_\infty} = 1$?

3.1.5. As usual in the Riemann-Roch formalism, given a divisor $D$ over $X$, we denote by $L(D)$ the vector space $L(D) = \{f \in \mathcal{M}_X : \text{div}(f) \geq -D\} \cup \{0\}$. We also set $\ell(D) = \dim_{\mathbb{C}}(L(D))$.

Denote by $D_{\mathcal{L}_\infty}$ the effective divisor of the points of $X$ over the points of $C$ at infinity.

3.1.6. Using the Riemann-Roch formalism, we may parametrize the functions $f \in \mathcal{M}_X$ that appear in Problem \ref{lem:feedback_dimension} as follows: given $g \in L(Z - D_{\mathcal{L}_\infty})$, by definition we have that $g(x) = 0$ for all $x \in L_{\mathcal{L}_\infty}$, and the pole divisor of $g$ is contained in $Z$. Hence, the function $f = \frac{1}{g}$ satisfies that $f(x) = 1$ for all $x \in L_{\mathcal{L}_\infty}$, and the zero divisor of $f$ is contained in $Z$. Conversely, if $f|_{\mathcal{L}_\infty} = 1$, and the zero divisor of $f$ is contained in $Z$, then $g = \frac{1}{f} - 1 \in L(Z - D_{\mathcal{L}_\infty})$.

3.1.7. Our minimality assumption on the vessel $B$ implies that $\text{deg}(Z) = \dim_{\mathbb{C}}(\mathcal{H})$. Let us denote this number, the dimension of the state space, by $n$. Similarly, in the line bundle case we consider in this section, we have that $\text{deg}(D_{\mathcal{L}_\infty}) = \dim_{\mathbb{C}}(\mathcal{E})$, the dimension of the input space. Let us denote it by $m$. Then $\text{deg}(Z - D_{\mathcal{L}_\infty}) = n - m$.

3.1.8. Let $K$ be a canonical divisor of $X$. Thus, $K$ is the divisor of some meromorphic 1-form on $X$. Denote by $g$ the genus of $X$. Applying the Riemann-Roch theorem to the divisor $Z - D_{\mathcal{L}_\infty}$ implies that

$$\ell(Z - D_{\mathcal{L}_\infty}) = \ell(K - Z + D_{\mathcal{L}_\infty} + \text{deg}(Z - D_{\mathcal{L}_\infty}) - g + 1 = \ell(K - Z + D_{\mathcal{L}_\infty}) + n - m - g + 1.$$ 

Let us denote this number by $\text{fb. dim}(B) \in \mathbb{N}$. We call this number the feedback dimension of $B$. The above equality implies that

$$\text{fb. dim}(B) \geq n - m - g + 1 = \dim(\mathcal{H}) - \dim(\mathcal{E}) - g + 1.$$ 

3.1.9. Let $r = \text{fb. dim}(B)$, and let $f_1, \ldots, f_r$ be some basis of the vector space $L(Z - D_{\mathcal{L}_\infty})$. Denote by $X_0$ the non-compact Riemann surface $X - \mathcal{L}_\infty$, obtained from $X$ by deleting all the points that lie over the line at infinity of $C$. Consider the complex manifold

$$\mathcal{F} \subset X_0 \times X_0 \times \cdots \times X_0$$

$r$ times
given by all the $r$-tuples $p = (p_1, \ldots, p_r)$, such that for $1 \leq i < j \leq r$, we have that $p_i \neq p_j$. We define functions $\mathcal{M} : \mathcal{V} \to M_r(\mathbb{C})$ and $\mathcal{P} : \mathcal{V} \to \mathbb{C}$ as follows: for each $p = (p_1, \ldots, p_r) \in \mathcal{V}$, define a square matrix $\mathcal{M}(p) \in M_r(\mathbb{C})$ by $\mathcal{M}(p) = (b_{i,j})$, where $b_{i,j} = f_i(p_j)$, and let $\mathcal{P}(p) = \det(\mathcal{M}(p))$.

**Lemma 3.1.10.** The function $\mathcal{P} : \mathcal{V} \to \mathbb{C}$ is a meromorphic function. Moreover, assuming that $r = \text{fb. dim}(\mathcal{B}) > 0$, it is not identically zero.

**Proof.** Since each of $f_1, \ldots, f_r$ is a meromorphic function $X_0 \to \mathbb{C}$, we see that $\mathcal{M} : \mathcal{V} \to M_r(\mathbb{C})$ is also a meromorphic function, and hence $\mathcal{P}$ is also a meromorphic function. Assume now that $r > 0$. Clearly, if $r = 1$ the claim holds. Let us assume by induction that each minor of $\mathcal{M}$ is not identically zero. Writing $\mathcal{P}$ as the Laplace expansion of $\mathcal{M}$ along the last column, we may write

$$\mathcal{P}(p_1, \ldots, p_r) = \sum_{i=1}^{r} a_i(p_1, \ldots, p_{r-1}) \cdot f_i(p_r)$$

where each $a_i$ is a meromorphic function

$$a_i : X_0 \times X_0 \times \cdots \times X_0 \to \mathbb{C} \quad \text{r - 1 times}$$

which is not identically zero. Let us choose some

$$(p_1, \ldots, p_{r-1}) \in X_0 \times X_0 \times \cdots \times X_0 \quad \text{r - 1 times}$$

where for $1 \leq i < j \leq r - 1$, we have that $p_i \neq p_j$, such that $a_1(p_1, \ldots, p_{r-1}) \neq 0$. Then the fact that $f_1, \ldots, f_r$ are linearly independent implies that for infinitely many $p \in X_0$, it holds that

$$\sum_{i=1}^{r} a_i(p_1, \ldots, p_{r-1}) \cdot f_i(p) \neq 0.$$ 

Hence, the $r$-tuple $(p_1, \ldots, p_{r-1}, p)$ satisfies $\mathcal{P}(p_1, \ldots, p_{r-1}, p) \neq 0$, as claimed. \qed

**3.1.11.** Let us set

$$\mathcal{NF} = \{(p_1, \ldots, p_r) \in \mathcal{V} \mid \mathcal{P}(p_1, \ldots, p_r) = 0\}$$

This set, the No-Feedback set, is a codimension 1 subset of the $r$-dimensional complex manifold $\mathcal{V}$. In particular it is of measure 0. Note that this set is independent of the chosen basis $f_1, \ldots, f_r$ of the vector space $L(Z - D_{L_{\infty}})$. Note that $(p_1, \ldots, p_r) \notin \mathcal{NF}$ if and only if given $f \in L(Z - D_{L_{\infty}})$ such that $f(p_1) = f(p_2) = \cdots = f(p_r) = 0$ it holds that $f \equiv 0$. Hence, $(p_1, \ldots, p_r) \notin \mathcal{NF}$ if and only if

$$l(Z - D_{L_{\infty}} - \sum_{i=1}^{r} p_i) = 0.$$ 

**3.1.13.** Given an $r$-tuple $(p_1, \ldots, p_r) \notin \mathcal{NF}$, the fact that $\mathcal{P}(p_1, \ldots, p_r) \neq 0$, implies that there are $a_1, \ldots, a_r \in \mathbb{C}$ such that

$$\sum_{i=1}^{r} a_i \cdot f_i(p_j) = -1$$
for all $1 \leq j \leq r$. Hence, letting

$$g = \sum_{i=1}^{r} a_i \cdot f_i \in L(Z - D_{L_\infty}),$$

and applying the construction of \ref{3.1.6} to $g$, we obtain that $f = \frac{1}{x - 1}$ is a meromorphic function on $X$, which satisfies $f(x) = 1$ for all $x \in L_\infty$, its zero divisor is contained in $Z$, and $f$ has a pole in each of the points $p_1, \ldots, p_r$. Further, note that since for $(p_1, \ldots, p_r) \notin \mathcal{NF}$ the matrix $M(p_1, \ldots, p_r)$ is invertible, the function $g$ from \ref{3.1.14} is unique. Hence, there is a unique meromorphic function $f$ on $X$ which satisfies $f(x) = 1$ for all $x \in L_\infty$, its zero divisor is contained in $Z$, and $f$ has a pole in each of the points $p_1, \ldots, p_r$.

To summarize the above discussion, we have proved the following theorem:

**Theorem 3.1.15.** Consider a minimal operator vessel

$$\mathcal{B} = (A_1, A_2, \bar{B}, C, D, \bar{D}, \sigma_1, \sigma_2, \gamma, \sigma_1 \cdot, \sigma_2 \cdot, \gamma \cdot; H, \mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$$

as above, and assume that the input and output bundles of $\mathcal{B}$ are line bundles over the compact Riemann surface $X$. Denote by $g$ the genus of $X$, let $Z = LZ(S_B) \in \text{Div}(X)$, and let $K \in \text{Div}(X)$ be a canonical divisor on $X$. Then for $r = \text{fb. dim}(\mathcal{B}) = \ell(K - Z + D_{L_\infty}) + \dim(\mathcal{H}) - \dim(\mathcal{E}) - g + 1 \geq \dim(\mathcal{H}) - \dim(\mathcal{E}) - g + 1$, given any $r$ distinct points $p_1, \ldots, p_r$ of $X$, except possibly tuples belonging to a codimension 1 subset of measure 0

$$\mathcal{NF} \subseteq X_0 \times X_0 \times \cdots \times X_0$$

there exists an admissible state feedback operator $F : \mathcal{H} \to \mathcal{E}$, such that the closed loop system $\mathcal{B}^{\text{CL}}_F$ has a pole at $p_1, \ldots, p_r$. If moreover $(p_1, \ldots, p_r) \notin \mathcal{NF}$, then the rest of the poles of $\mathcal{B}^{\text{CL}}_F$ are determined uniquely by the choice of the $r$ poles $p_1, \ldots, p_r$.

### 3.2. Examples of small genus.

**3.2.1. Genus 0.** Assume that the compact Riemann surface $X$ is of genus 0. In this particular case, the Riemann-Roch theorem says that

$$\text{fb. dim}(\mathcal{B}) = \ell(Z - D_{L_\infty}) = \deg(Z - D_{L_\infty}) + 1 = \dim(\mathcal{H}) - \dim(\mathcal{E}) + 1 = n - m + 1.$$  

or $\text{fb. dim}(\mathcal{B}) = 0$ if this number is $\leq 0$. Setting $r = \text{fb. dim}(\mathcal{B})$, and choosing any $r$ distinct points $p_1, \ldots, p_r$, note that

$$\ell(Z - D_{L_\infty} - \sum_{i=1}^{r} p_i) = \deg(Z - D_{L_\infty}) - r + 1 = 0$$

Hence, it follows from \ref{3.1.12} that in this case $\mathcal{NF} = \emptyset$. Thus, the pole placement theorem says in this case that given $r = \dim(\mathcal{H}) - \dim(\mathcal{E}) + 1$ distinct points $p_1, \ldots, p_r$ in $X$, there is a closed loop system obtained by state feedback such that $p_1, \ldots, p_r$ are poles of this system. The rest of the poles of the closed loop system are then uniquely determined by the choice of these $r$ poles. Specializing further to the case where $m = 1$, so that $X$ is simply $\mathbb{P}^1_C$. In this case, the vessel $\mathcal{B}$ represents a classical 1D continuous-time-invariant linear system, and we recover the classical pole placement theorem: for any choice of $n = \dim(\mathcal{H})$ points, one can construct a closed loop system whose poles are the prescribed points.
3.2.2. Genus 1. Assume that the compact Riemann surface \( X \) is of genus 1. Let us choose some specific point \( c_\infty \in L_\infty \). Using this choice, \( X \) has the structure of an elliptic curve over \( \mathbb{C} \). In particular, its points have the structure of an abelian group. Denote this group operation by \( \oplus \), and let \( \Phi : \text{Div}(X) \to (X, \oplus) \) be the canonical group homomorphism. In this genus one case, the Riemann-Roch theorem states that

\[
\text{fb. dim}(B) = \ell(Z - D_{L_\infty}) = \deg(Z - D_{L_\infty}) = \dim(\mathcal{H}) - \dim(\mathcal{E}) = n - m.
\]

assuming this number is positive. If this number is negative, then \( \text{fb. dim}(B) = 0 \). If \( n = m \), then this number is either 0 or 1. It is 1 if and only if \( Z - D_{L_\infty} \) is a principal divisor, equivalently, if \( \Phi(Z - D_{L_\infty}) = c_\infty \).

Let \( r = \text{fb. dim}(B) \), and suppose that \( r > 0 \) and that \( n > m \). Given \( r \) distinct points \( p_1, \ldots, p_r \) in \( X \), we have that

\[
\deg(Z - D_{L_\infty} - \sum_{i=1}^{r} p_i) = 0.
\]

Hence, we have

\[
\ell(Z - D_{L_\infty} - \sum_{i=1}^{r} p_i) = 0
\]

if and only if \( \Phi(Z - D_{L_\infty} - \sum_{i=1}^{r} p_i) \neq c_0 \). Using (3.1.12), we obtain the following characterization of \( \mathcal{N} \mathcal{F} \): for any \( r - 1 \) distinct points \( p_1, \ldots, p_{r - 1} \), there is a unique point \( p_r \in X_0 \) such that \( (p_1, \ldots, p_r) \in \mathcal{N} \mathcal{F} \). This point is given by

\[
(3.2.1) \quad p_r = \Phi(Z - D_{L_\infty} - \sum_{i=1}^{r-1} p_i).
\]

In particular, \( \mathcal{N} \mathcal{F} \neq \emptyset \).

To summarize the genus 1 case, the pole placement theorem in this case states that given \( r = \dim(\mathcal{H}) - \dim(\mathcal{E}) \) distinct points \( p_1, \ldots, p_r \), such that the point \( p_r \) is not the unique point that satisfies (3.2.1), there is a closed loop system obtained by state feedback such that \( p_1, \ldots, p_r \) are poles of this system, and the rest of the poles of the closed loop system are then uniquely determined by the choice of these \( r \) poles.

3.2.3. Higher genus. Suppose now that \( X \) is a compact Riemann surface of genus \( g > 1 \). As the genus of \( X \) increases, it becomes more difficult to make a precise analysis of \( \text{fb. dim}(B) \) and of the set \( \mathcal{N} \mathcal{F} \). If however the dimension of the state space \( \mathcal{H} \) is large enough compared to the dimension of the input space \( \mathcal{E} \), we know the following from the Riemann-Roch theorem: assuming that

\[
(3.2.2) \quad \dim(\mathcal{H}) - \dim(\mathcal{E}) > 2 \cdot g - 2,
\]

there is an equality

\[
\text{fb. dim}(B) = \ell(Z - D_{L_\infty}) = \deg(Z - D_{L_\infty}) - g + 1 = \dim(\mathcal{H}) - \dim(\mathcal{E}) - g + 1.
\]

It follows, that, under the assumption (3.2.2), it is possible to place, generically,

\[
\dim(\mathcal{H}) - \dim(\mathcal{E}) - g + 1
\]

poles, except possibly if these points belong to the measure zero set \( \mathcal{N} \mathcal{F} \) introduced in (3.1.11).
REFERENCES

[1] Ball, J. A., Gohberg, I., & Rodman, L. (1990). Interpolation of Rational Matrix Functions. Operator Theory: Advances and Applications, 45.
[2] Ball, J. A., & Vinnikov, V. (1996). Zero-pole interpolation for meromorphic matrix functions on an algebraic curve and transfer functions of 2D systems. Acta Applicandae Mathematicae, 45(3), 239-316.
[3] Ball, J. A., & Vinnikov, V. (1999). Zero-Pole Interpolation for Matrix Meromorphic Functions on a Compact Riemann Surface and a Matrix Fay Trisecant Identity. American Journal of Mathematics, 841-888.
[4] Ball, J. A., & Vinnikov, V. (2003). Overdetermined Multidimensional Systems: State Space and Frequency Domain Methods. In Mathematical Systems Theory in Biology, Communications, Computation, and Finance (pp. 63-119). Springer, New York, NY.
[5] Ball, J. A., & Vinnikov, V. (2017) Discrete-time 2D overdetermined linear systems: System-theoretic properties and transfer-function Hankel realization for meromorphic bundle maps on a compact Riemann surface. Preprint.
[6] Kerner, D., & Vinnikov, V. (2010). Decomposability of local determinantal representations of hypersurfaces. arXiv preprint arXiv:1009.2517.
[7] Kerner, D., & Vinnikov, V. (2012). Determinantal representations of singular hypersurfaces in $\mathbb{P}^n$. Advances in Mathematics, 231(3), 1619-1654.
[8] Kravitsky, N. (1983). Regular colligations for several commuting operators in Banach space. Integral Equations and Operator Theory, 6(1), 224-249.
[9] Livšic, M. S. (1986). Commuting nonselfadjoint operators and mappings of vector bundles on algebraic curves. Operator Theory and Systems, 19, 255-279.
[10] Livšic, M. S., Kravitsky, N., Markus, A. S., & Vinnikov, V. (2013). Theory of commuting nonselfadjoint operators (Vol. 332). Springer Science & Business Media.
[11] Sontag, E. D. (1998). Mathematical Control Theory: Deterministic Finite Dimensional Systems (Vol. 6). Springer Science & Business Media.
[12] Vinnikov, V. (1998). Commuting operators and function theory on a Riemann surface. Holomorphic spaces (Berkeley, CA, 1995), 33, 445-476.
[13] Vinnikov, V. (2012). LMI representations of convex semialgebraic sets and determinantal representations of algebraic hypersurfaces: past, present, and future. Mathematical methods in systems, optimization, and control, 222, 325-349.

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