Travelling waves and light-front approach in relativistic electrodynamics

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Abstract

We briefly report on a recent proposal [1] for simplifying the equations of motion of charged particles in an electromagnetic (EM) field $F^{\mu\nu}$ that is the sum of a plane travelling wave $F_i^{\mu\nu}(ct-z)$ and a static part $F_s^{\mu\nu}(x,y,z)$; it adopts the light-like coordinate $\xi = ct-z$ instead of time $t$ as an independent variable. We illustrate it in a few cases of extreme acceleration, first of an isolated particle, then of electrons in a plasma in plane hydrodynamic conditions: the Lorentz-Maxwell & continuity PDEs can be simplified or sometimes even completely reduced to a family of decoupled systems of ordinary ones; this occurs e.g. with the impact of the travelling wave on a vacuum-plasma interface (what may produce plasma waves or the slingshot effect).

1 Introduction

The equation of motion of a charged particle in an external EM field

$$\begin{align*}
\dot{p}(t) &= qE[t, x(t)] + \frac{p(t)}{\sqrt{m^2c^2 + p^2(t)}} \wedge qB[t, x(t)], \\
\dot{x}(t) &= \frac{cp(t)}{\sqrt{m^2c^2 + p^2(t)}} 
\end{align*}$$

in its general form is non-autonomous and highly nonlinear in the unknowns $x(t), p(t)$. Here $m, q, x, p$ are the rest mass, electric charge, position and relativistic momentum of the particle, $E = -\partial_t A/c - \nabla A^0$ and $B = \nabla \wedge A$ are the electric and magnetic field,
$(A^\mu) = (A^0, -A)$ is the electromagnetic (EM) potential 4-vector $(E^i = F^{0i}, B^1 = F^{32}, \text{etc.})$; we use Gauss CGS units. We decompose $x = x i + y j + z k = x^i + z k$, etc., in the cartesian coordinates of the laboratory frame, and often use the dimensionless variables $\beta \equiv v/c = \dot{x}/c$, $\gamma \equiv 1/\sqrt{1-\beta^2} = \sqrt{1+u^2}$ and the 4-velocity $u = (u^0, u) \equiv (\gamma, \gamma \beta)$, i.e. the dimensionless version of the 4-momentum $p$. Usually, (1) is simplified assuming:

1. $E, B$ are constant or vary “slowly” in space/time; or
2. $E, B$ are “small” (so that nonlinear effects in $E, B$ are negligible); or
3. $E, B$ are monochromatic waves, or slow modulations of; or
4. the motion of the particle keeps non-relativistic.

The on-going, astonishing developments of laser technologies today allow the construction of compact sources of extremely intense coherent EM waves, possibly concentrated in very short laser pulses. *Chirped Pulse Amplification* [2, 3] allows the production of pulses of intensity up to $10^{23}$ Watt per square centimeter and duration down to $10^{-15}$ seconds. Huge investments in new technologies (thin film compression, relativistic mirror compression, etc. [4, 5]) will soon allow to produce even more intense/short (or cheaper) pulses. For instance, 850 MEuro have been allocated for the *Extreme Light Infrastructure* (ELI) program within the European Union ESFRI roadmap, with three of the planned four sites already under construction in Czech Republic, Hungary, Romania. One major motivation is the quest for table-top particle accelerators based on Laser Wake Field Acceleration (LWFA) [6] in plasmas. Among the possible applications of such accelerators we mention:

- **Medicine**: inspection (PET,...), cancer therapy by accelerated particles (electrons, protons, ions) or radioisotope production,...;
- **Research**: particle physics, materials science, structural biology, (inertial) nuclear fusion, X-ray free electron laser,...;
- **Industry**: atomic scale lithography, surface treatment of materials, sterilization, energy efficient manufacturing, detection systems,...;
- **Environmental remediation**: flue gas cleanup, petroleum cracking, transmutation of nuclear wastes,...

These and other applications of small accelerators were discussed e.g. at the “Big Idea Summit” organized by the US Department of Energy (Washington, 2016). In Europe the large network of research centers “European Plasma Research Accelerator with eXcellence In Applications” (EUPRAXIA) has been recently created to develop the associated technologies. Extremely intense and rapidly varying electromagnetic fields are present also in several violent astrophysical processes (see e.g. [5] and references therein). In either case the effects are so fast, huge, highly nonlinear, ultra-relativistic that conditions 1-4 are not fulfilled. Alternative simplifying approaches are therefore welcome.
Here we summarize an approach \[1\] that systematically applies the light-front formalism \[7\]; it is especially fruitful if in the spacetime region of interest (where we wish to follow the particles’ worldlines) \(E, B\) are the sum of static parts and plane transverse travelling waves propagating in the \(z\) direction:

\[
E(t, x) = e^+(ct-z) + E_s(x), \quad B(t, x) = k \wedge e^+(ct-z) + B_s(x). \tag{2}
\]

The starting point is: as no particle can reach the speed of light \(c\), then \(\tilde{\xi}(t) = ct - z(t)\) is strictly growing and we can make the change \(t \mapsto \xi = ct - z\) of independent parameter along the worldline \(\lambda\) (fig. 1) of the particle; then the term \(\epsilon^+ [ct-z(t)]\), where the unknown \(z(t)\) is in the argument of the highly nonlinear and rapidly varying \(\epsilon^+\), becomes the known forcing term \(\epsilon^+(\xi)\). We apply the approach first to an isolated particle (sections 2, 3), then to a cold diluted plasma initially at rest and hit by a plane EM wave (section 4).

The fields (2) can be obtained from an EM potential of the same form, \(A^\mu(x) = \alpha^\mu(ct-z) + A_s^\mu(x)\); in the Landau gauges (\(\partial_\mu A^\mu = 0\)) \(A_s\) must fulfill the Coulomb gauges (\(\nabla \cdot A_s = 0\)), and it must be \(\alpha^\prime = \alpha^0\), \(\epsilon^+ = -\alpha^+\), \(E_s = -\nabla A_s^0\), \(B_s = \nabla \wedge A_s\). We shall set \(\alpha^\prime = \alpha^0 = 0\), as they appear neither in the observables \(E, B\) nor in the equations of motion. Assuming only that \(\epsilon^+ (\xi)\) is piecewise continuous and

\[\alpha^+ (\xi) \equiv -\int_{-\infty}^{\xi} dy \epsilon^+ (y); \tag{4}\]

in case a) \(\alpha^+(\xi) = 0\) if \(\xi \leq 0\), \(\alpha^+(\xi) = \alpha^+(l)\) if \(\xi \geq l\). We can treat on the same footing all \(\epsilon^+\) fulfilling (3) regardless of their Fourier analysis, in particular:

1. A modulated monochromatic wave:

\[
\epsilon^+ (\xi) = \epsilon (\xi) \left[ i a_1 \cos(k\xi + \varphi) + i a_2 \sin(k\xi) \right] \tag{5}
\]

(with \(a_1^2 + a_2^2 = 1\)). Under rather general assumptions

\[
\alpha^+ (\xi) = -\frac{\epsilon (\xi)}{k} \epsilon^+_p (\xi) + O \left( \frac{1}{k^2} \right) \simeq -\frac{\epsilon (\xi)}{k} \epsilon^+_p (\xi), \tag{6}
\]

\(\epsilon^+_p (\xi) := \epsilon^+_o (\xi + \pi/2k); \) in the appendix we recall upper bounds for the remainder \(O(1/k^2)\). For slow modulations (i.e. \(|\epsilon'| \ll |k\epsilon|\) - like the ones characterizing most conventional applications (radio broadcasting, ordinary laser beams, etc.) - the right estimate is very good.

2. A superposition of waves of type 1.

3. An ‘impulse’ (few, one, or even a fraction of oscillation) \[8 \[4\].
2 Set-up and general results for a single particle

Let \( \dot{x}(\xi) \) be the position as a function of \( \xi \); it is determined by \( \dot{x}(\xi) = x(t) \). More generally, for any given function \( f(t, x) \) we denote \( \dot{f}(\xi, \dot{x}) \equiv f[(\xi + \xi)/c, \dot{x}] \), abbreviate \( \dot{f} \equiv df/dt \), \( \dot{f}' \equiv d\dot{f}/d\xi \) (total derivatives). Also the change of dependent (and unknown) variable \( u^z \mapsto s \) is convenient, where the \textit{s-factor} [1]

\[
\begin{align*}
\hat{s} &\equiv \gamma - u^z = u^- = \gamma(1 - \beta^2) = \frac{\gamma d\xi}{c\, dt} > 0 \quad (7)
\end{align*}
\]

is the light-like component of \( u \) (the dimensionless version of \( p \)), as well as the Doppler factor of the particle. In fact, \( \gamma, \textbf{u}, \beta \) are \textit{rational} functions of \( \hat{u}^z, s \):

\[
\begin{align*}
\gamma &= \frac{1 + u^z + s^2}{2s}, \quad \text{and} \quad u^z = \frac{1 + u^z - s^2}{2s}, \quad \beta = \frac{u}{\gamma} \quad (8)
\end{align*}
\]

(these relations hold also with the carets); so, replacing \( d/dt \mapsto (cs/\gamma)d/d\xi \) and putting carets on all variables [1] becomes \textit{rational} in the unknowns \( \hat{u}^z, \hat{s} \):

\[
\begin{align*}
\dot{x}' &= \frac{\hat{u}}{\hat{s}}, \\
\dot{u}'^z &= \frac{q}{mc^2} \left[ \frac{\hat{E} + \hat{u} \wedge \hat{B}}{\hat{s}} \right]^z, \\
\dot{\hat{s}} &= \frac{q}{mc^2} \left[ \frac{\hat{u}^z \hat{E}^z - \dot{\hat{E}}^z - (\hat{u} \wedge \hat{B}^z)^z}{\hat{s}} \right]
\end{align*}
\]

(9)
with \( \hat{u}^\xi, \hat{\gamma} \) expressed as in (8). These equations amount [1] to the Euler-Lagrange equations

\[
\frac{d}{\partial t} \frac{\partial L}{\partial \dot{\sigma}} \Bigg|_{\dot{\sigma} = \hat{\sigma}} = \frac{\partial L}{\partial \sigma},
\]

that are obtained applying Hamilton’s principle to the action functional \( S(\lambda) \) with \( \lambda \) parametrized by \( \xi \) (instead of \( t \)), as well as to the Hamilton equations

\[
\dot{x}' = \frac{\partial H}{\partial \Pi}, \quad \dot{\Pi}' = -\frac{\partial H}{\partial x},
\]

where the Hamiltonian

\[
\hat{H}(\hat{x}, \hat{\Pi}; \xi) = mc^2 \left( \frac{1 + \gamma^2 + \hat{u}^{\perp2}}{2 \hat{s}} \right) + q\hat{A}^0(\xi, \hat{x}), \quad \text{with}
\]

\[
\begin{cases}
\hat{u}^\parallel = \hat{\Pi}^\perp - q\hat{A}^\perp(\xi, \hat{x}) \\
\hat{s} = -\frac{\hat{\Pi}^\perp + q[\hat{A}^0, \hat{A}^\perp](\xi, \hat{x})}{mc^2},
\end{cases}
\]

(10)

is obtained by Legendre transform from \( L \) and again is rational in \( \hat{\Pi} \equiv \frac{\partial L}{\partial \dot{x}} \), or equivalently in \( \hat{u}^\parallel, \hat{s} \). Along the solutions \( \hat{H} \) gives the particle energy as a function of \( \xi \), and

\[
\frac{d\hat{H}}{d\xi} = \frac{\partial \hat{H}}{\partial \xi}.
\]

Under the EM field (2) equations (9) amount to

\[
\begin{align*}
\dot{x}' &= \frac{\hat{u}}{\hat{s}}, \\
\dot{u}' &= \frac{q}{mc^2} \left[ (1 + \gamma^2)\hat{E}_s^\perp + \hat{x}'^\perp \hat{B}_s^\perp + \epsilon^\perp(\xi) \right], \\
\dot{s}' &= -\frac{q}{mc^2} \left[ \hat{E}_s^\perp \cdot \hat{x}'^\perp \cdot \hat{E}_s^\perp + (\hat{x}'^\perp \cdot \hat{B}_s^\perp)^2 \right],
\end{align*}
\]

(12)

while the energy gain (normalized to the rest energy \( mc^2 \)) in the interval \([\xi_0, \xi_1]\)

\[
\mathcal{E} := \frac{\hat{H}(\xi_1) - \hat{H}(\xi_0)}{mc^2} = \int_{\xi_0}^{\xi_1} d\xi \frac{q\epsilon^\perp \cdot \hat{u}^\perp}{mc^2 \hat{s}}(\xi).
\]

(13)

In particular, under assumption (3a) we obtain the total energy gain choosing \( \xi_0 = 0, \xi_1 = l \), which are the values of the lightlike coordinate at the beginning and at the end of the interaction, see fig. [1]. If we used parameter \( t \), to compute \( \mathcal{E} \) we should first determine the time \( t_f \) when the pulse-particle interaction finishes. Once solved (12), analytically or numerically, to obtain the solution as a function of \( t \) we just need to invert \( \hat{t}(\xi) = \xi + \hat{z}(\xi) \) and set \( \hat{x}(t) = \hat{x}[\hat{t}(t)] \).

Contrary to (12), (1) is not rational in \( u \), and the unknown \( z(t) \) appears in the argument of the rapidly varying functions \( \epsilon^\perp, \alpha^\perp \) in (1a), which now reads:

\[
\frac{mc}{q} u(t) = E_s + \frac{u \cdot B_s}{\sqrt{1 + u^2}} + \frac{u \cdot \epsilon^\perp [ct - z(t)]}{\sqrt{1 + u^2}} k + \left( 1 - \frac{u^z}{\sqrt{1 + u^2}} \right) \epsilon^\perp [ct - z(t)].
\]

\( H(x, P, t) \) is not rational in \( P := \frac{\partial L}{\partial \dot{x}} \), and also determining \( \mathcal{E}(t) \) is more complicated.

### 2.1 Dynamics under a \( A^\mu \) independent of the transverse coordinates

Eq. (9) are further simplified if \( A^\mu = A^\mu(t, z) \). This applies in particular if \( E_s = E_s^\perp(z) k, B_s = B_s^\perp(z) \), choosing \( A^0 = -\int^z d\zeta E_s^\perp(\zeta), A^\perp = \alpha^\perp - k \int^z d\zeta B^\perp(\zeta), A^z \equiv 0 \). As \( \partial \hat{H}/\partial \hat{x}^\perp = 0 \),
we find $\dot{\mathbf{H}} = q\mathbf{K} = \text{const}$, i.e. the known result $\frac{m^2 c^2}{q} \ddot{u}^+ = \mathbf{K} - \dot{\mathbf{A}}^+(\xi, \hat{z})$. Setting $v := \ddot{u}^+/2$ and replacing in $(12)$ we obtain
\begin{equation}
\begin{aligned}
\dot{z}' &= \frac{1 + \dot{\nu}}{2s^2} - \frac{1}{2}, \\
\dot{s}' &= -\frac{2}{mc^2} E_s^x(\hat{z}) - \frac{1}{2s} \frac{\partial \nu}{\partial \hat{z}}.
\end{aligned}
\end{equation}

Once solved system (14) for $\hat{s}(\xi), \hat{s}(\xi)$, the other unknowns are obtained from
\begin{equation}
\begin{aligned}
\hat{x}(\xi) &= x_0 + \int_{\xi_0}^{\xi} dy \frac{\hat{u}(y)}{s(y)}.
\end{aligned}
\end{equation}

[the $z$-component of (15) amounts to (14a) with initial condition $\hat{z}(\xi_0) = z_0$. If in addition $B_s \equiv 0$, then $A_s \equiv 0$, implying that $\ddot{u}^+(\xi) = \frac{q}{mc^2} [\mathbf{K} - \alpha^+(\xi)]$ and $\dot{\nu} = \ddot{u}^+ / 2$ are already known. The system (14) to be solved simplifies to
\begin{equation}
\begin{aligned}
\dot{z}' &= \frac{1 + \dot{\nu}}{2s^2} - \frac{1}{2}, \\
\dot{s}' &= -\frac{2}{mc^2} E_s^x(\hat{z}).
\end{aligned}
\end{equation}

**Remarks.** Some remarkable properties of the corresponding solutions are [1]:

1. Where $\epsilon^+(\xi) = 0$ then $\dot{\nu}(\xi) = v_c = \text{const}$, $\bar{H}$ is conserved, (16) is solved by quadrature.

2. In case (3a) the final transverse momentum is $mc \ddot{u}^+(l)$. If $\epsilon$ of (5) varies slowly and $\ddot{u}^+(0) = 0$, then by (6) $\ddot{u}^+(l) \approx 0$.

3. Fast oscillations of $\epsilon^+$ make $\dot{z}(\xi)$ oscillate much less than $x^+(\xi)$, and $\dot{s}(\xi)$ even less: as $\dot{s} > 0$, $\dot{\nu} = \ddot{u}^+ / 2 \geq 0$, integrating (16a) averages the fast oscillations of $u^+$ to yield much smaller relative oscillations of $\hat{z}$, while integrating (16b) averages the residual small oscillations of $E_s^x[\hat{z}(\xi)]$ to yield an essentially smooth $\dot{s}(\xi)$. On the contrary, $\gamma(\xi), \beta(\xi), \ddot{u}(\xi), ..., $ which are recovered via (6), oscillate fast, and so do also $\gamma(t), \beta(t), u(t), ...$. See e.g. fig. 2 4 6.

4. If $u^+(0) = 0$ and the EM wave is a slowly modulated (5) - (3a), integrating (13) by parts across $[0, l]$ and using (6) we find $E \simeq \int_0^l d\xi \dot{v}(\xi) \dot{s}'(\xi) / 2 \hat{s}^2(\xi)$: the energy gain will be automatically positive (resp. negative) if $\dot{s}(\xi)$ is growing (resp. decreasing) in all of $[0, l]$. Correspondingly, the interaction with the EM wave can be used to accelerate (resp. decelerate) the particle.

### 3 Some solutions in closed form under constant $B_s, E_s$

Assume $B_s, E_s$ are constants, and let $b := qB_s/mc^2$, $e := qE_s/mc^2$. Upon integration over $\xi$ and use of (12a) equations (12b-c) yield
\begin{equation}
\begin{aligned}
\dot{u}^x &= (e^+ - b^y) \dot{z} + b^z \dot{y} + w^x(\xi), \\
\dot{u}^y &= (e^y + b^x) \dot{z} - b^z \dot{x} + w^y(\xi), \\
\dot{s} &= (e^+ - b^y) \dot{x} + (e^y + b^x) \dot{y} - w^z(\xi),
\end{aligned}
\end{equation}
where \( w(\xi) \equiv q[K - \alpha^+(\xi) + \xi E_s]/mc^2 \) (\( w \) is known and dimensionless), and \( K \) is an integration constant. For any \( E_x^{\pm}, B_y^{\pm}, E_z^{\pm} \), if \( B_z^{\pm} = k \wedge E_x^{\pm} \), then \( e^x = b^y, e^y = -b^x \), and (17c) is solved:
\[
\hat{s} = -w^z(\xi).
\]
Then we solve in closed form the rest of the system (17), (12a) first for \( \hat{x}^z(\xi) \), then for \( \hat{u}^z(\xi), \hat{u}^z(\xi), \hat{z}(\xi) \). Assuming for simplicity the initial conditions \( x(0) = 0 = u(0) \) we find
\[
(\hat{x} + i\hat{y})(\xi) = (1 - e^z\xi)^{ib^{\mp}/e^z} \int_0^\xi d\zeta \left( \frac{w^x + iu^y}{(1 - e^z\xi)^1 + ib^{\mp}/e^z} \right),
\]
\[
\hat{z}(\xi) = \frac{1}{2} \left[ \frac{1}{(1 - e^z\xi)^2} + \hat{x}^{i\prime z}(\xi) - 1 \right], \quad \hat{s}(\xi) = 1 - e^z\xi,
\]
\[
\hat{u}^z(\xi) = (1 - e^z\xi) \hat{x}^{i\prime z}(\xi), \quad \hat{\gamma}(\xi) = 1 - e^z\xi + \hat{u}^z(\xi)
\]
\[
\hat{u}^z(\xi) = \frac{1}{2(1 - e^z\xi)} + (1 - e^z\xi) \frac{\hat{x}^{i\prime z}(\xi) - 1}{2}
\]
if \( e^z \neq 0 \) and
\[
(\hat{x} + i\hat{y})(\xi) = \int_0^\xi d\zeta e^{-ib(\xi - \zeta)}(w^x + iu^y)(\zeta), \quad \hat{u}^z = \hat{x}^{i\prime z},
\]
\[
\hat{z} = \hat{\gamma} = \hat{u}^{i\prime 2}(\xi) = \gamma(\xi) - 1, \quad \hat{z}(\xi) = \int_0^\xi d\xi \frac{\hat{u}^{i\prime 2}(\xi)}{2}.
\]
if \( e^z = 0 \). As fas as we now, such general solutions have not appeared in the literature before Ref. [1]. We next analyze a few special cases (the first two have already appeared in the literature).

### 3.1 Case \( E_s = B_s = 0 \) (zero static fields). Then (18) becomes [9, 10]:

\[
\hat{s} \equiv 1, \quad \hat{u}^z = \frac{-q\alpha^+}{mc^2}, \quad \hat{u}^{i\prime 2} = \hat{z}, \quad \hat{\gamma} = 1 + \hat{u}^z
\]
\[
\hat{z}(\xi) = \int_0^\xi dy \frac{\hat{u}^{i\prime 2}(y)}{2}, \quad \hat{x}^{i\prime z}(\xi) = \int_0^\xi dy \hat{u}^z(y).
\]

The solutions (20) induced by two \( x \)-polarized pulses and the corresponding electron trajectories in the \( zx \) plane are shown in fig. 2. Note that:

- The maxima of \( \gamma, \alpha^+ \) coincide (and approximately also of \( e(\xi), \) if \( e(\xi) \) is slowly varying).
- Since \( u^z \geq 0 \), the \( z \)-drift is nonnegative-definite. If we rescale \( e^z \mapsto a e^z \) then \( \hat{x}^z, \hat{u}^z \) scale like \( a \), whereas \( \hat{z}, \hat{u}^z \) scale like \( a^2 \); hence the trajectory goes to a straight line in the limit \( a \to \infty \). This is due to magnetic force \( q\beta \wedge B \).
- **Corollary [1]** The final \( u \) and energy gain read
\[
u^z_f = \hat{u}^z(\infty), \quad u^z_f = E_f = \frac{1}{2} u^{i\prime 2}_f = \gamma_f - 1
\]
Figure 2: Solutions (20) (up and center) and corresponding electron trajectories in the $zx$ plane (down) induced by two $x$-polarized pulses with carrier wavelength $\lambda = 0.8 \mu m$, gaussian modulation $\epsilon(\xi) = a \exp\left[-\xi^2/2 \sigma^2\right]$, $\sigma = 20 \mu m^2$, $|q|a\lambda/mc^2 = 4, 15$ (left, right).
[in case (3a) it is also $u_+^j = \hat{u}_+^j(l)$]. By (6), both are very small if the pulse modulation $\epsilon$ is slow [extremely small if $\epsilon \in S(\mathbb{R})$ or $\epsilon \in C_c^\infty(\mathbb{R})$]. This can be seen as a rigorous version of the Lawson-Woodward Theorem [11, 12, 13, 14] (an outgrowth of the original Woodward-Lawson Theorem [15, 16]): this theorem states that, in spite of large energy variations during the interaction, the final energy gain $E_f$ of a charged particle interacting with an EM field is zero if:

i) the interaction occurs in $\mathbb{R}^3$ vacuum (no boundaries);

ii) $E_0 = B_0 = 0$ and $\epsilon^+$ is slowly modulated;

iii) $v^z \simeq c$ along the whole acceleration path;

iv) nonlinear (in $\epsilon^+$) effects $q\beta \wedge B$ are negligible;

v) the power radiated by the particle is negligible.

Our Corollary, as Ref. [17], states that the same result holds if we relax iii), iv), but the EM field is a plane travelling wave. To obtain a non-zero $E_f$ one has to violate some other conditions of the theorem, as e.g. we consider in next cases.

### 3.2 Case $E_0 = 0$, $B_0 = B_z^2 k$. Then the solution (18) becomes (see fig. 3)

\begin{equation}
(\hat{x} + i\hat{y})(\xi) = \int_0^\xi d\zeta e^{ib(\xi-\zeta)}(w^x + iw^y)(\zeta), \quad \hat{u}_+ = \hat{x}^+, \quad \hat{s} = 1, \quad \hat{u}_z = \hat{z}_z = 1, \quad \hat{z}(\xi) = \int_0^\xi \frac{d\zeta}{2} 2W(\zeta).
\end{equation}

For monochromatic $\epsilon^+$ it reduces to the solution of [18, 19, 20] and leads to cyclotron autoresonance if $-b = k \gg \frac{1}{2}$: assuming for simplicity circular polarization $[a_1 = a_2 = 1$ in (5)], by (6) it is $w^x(\xi) + iw^y(\xi) \simeq e^{i\xi}w(\xi)$, whence

\begin{equation}
(\hat{x} + i\hat{y})(\xi) \simeq iW(\xi)e^{i\xi}, \quad \hat{z}(\xi) \simeq \int_0^\xi d\zeta \frac{k^2 W^2(\zeta)}{2}, \quad W(\xi) = \int_0^\xi q_\epsilon(\zeta) \frac{k m c^2}{2} > 0;
\end{equation}

clearly $W(\xi)$ grows with $\xi$. In particular if $\epsilon^+(\xi) = 0$ for $\xi \geq l$, then for such $\xi$

\begin{equation}
\hat{z}'(\xi) \simeq \frac{k^2}{2} W^2(l) \simeq 2E_f, \quad \frac{|\hat{x}^+(\xi)|}{\hat{z}'(\xi)} \simeq \frac{2}{kW(l)} \ll 1;
\end{equation}

the final energy gain is noteworthy by the first formula, the final collimation is very good by the second.

### 3.3 Case $E_0 = E_z^2 k$, $B_0 = 0$. The solution (18) reduces to

$\hat{s}(\xi) = 1 - e^z \xi$,

\begin{equation}
(\hat{x} + i\hat{y})(\xi) = \int_0^\xi dy \frac{(w^x + iw^y)(y)}{1 - e^z y}, \quad \hat{z}(\xi) = \int_0^\xi \frac{dy}{2} \left\{ \frac{1 + \hat{v}(y)}{1 - e^z y^2} - 1 \right\};
\end{equation}

where
by Remark 2.4 if $\epsilon^\perp$ is slowly modulated the energy gain $E_f$ is negative if $e^z \equiv qE^z / mc^2 > 0$, is positive if $e^z < 0$, and has a unique maximum at some point $e^z_M < 0$ if $\epsilon(\xi)$ fulfills (3a) with a unique maximum. An acceleration device based on this solution would consist of the following: at $t = 0$ the particle is initially at rest with $z_0 \lesssim 0$, just at the left of a metallic grating $G$ contained in the $z = 0$ plane and set at zero electric potential (see fig. 4); another metallic plate $P$ contained in a plane $z = z_p > 0$ is set at electric potential $V = V_p$. A short laser pulse $\epsilon^\perp$ travelling in the positive $z$-direction hits and boosts the particle into the latter region (see section 3.1); choosing $qV_p > 0$ implies $e^z < 0$, and a backward longitudinal electric force $qE^z_s$. If $qV_p$ is large enough, then $z(t)$ reaches a maximum smaller than $z_p$, then is accelerated backwards and exits the grating with energy $E_f$ and negligible transverse momentum. A large $E_f$ requires extremely large $|V_p|$, far beyond the material breakdown threshold, what prevents its realization by a static potential (sparks between $G, P$ would arise and rapidly reduce $|V_p|$). A way out is to make the pulse itself generate such large $|E^z_s|$ within a plasma just at the right time, so as to induce the slingshot effect, as sketchily explained at the end of next section.

4 Plane plasma problems

Assume that the plasma is initially in hydrodynamic conditions with all initial data [Eulerian velocities $v^h$ and densities $n^h$ of the $h$-th fluid, EM fields of the form (2); $h$ enumerates electrons and kinds of ions composing the plasma, $q_h, m_h$ are their charge, mass] not de-
Figure 4: Left: the motion (23) induced by a linearly polarized modulated EM wave (5) with wavelength $\lambda = 2\pi/k = 0.8\mu m$, gaussian enveloping amplitude $\epsilon(\xi) = a \exp[-\xi^2/2\sigma]$ with $\sigma = 20\mu m^2$ and $|q|a\sqrt{2}/kmc^2 = 6.6$, trivial initial conditions, $B_s = 0$, $E_s = kE_z^M$, where $E_M^z q \approx 37 GeV/m$ (this yields the maximum energy gain $E_f \approx 1.5$ with such a wave). Right: the corresponding trajectory in the $zx$ plane within an hypothetical acceleration device based on a laser pulse and metallic gratings $G, P$ at potentials $V = 0, V_p$, with $qV_p/\lambda \approx 37 GeV/m$. 

Depending on $x^+$, then also the solutions of the Lorentz-Maxwell and continuity equations for $B, E, u_h, n_h$ do not depend on $x^+$, nor the displacements $\Delta x_h \equiv x_h(t, X) - X$ on $X^+$. Here $x_h(t, X)$ is the position at $t$ of the material element of the $h$-th fluid with initial position $X \equiv (X, Y, Z)$; $X_h(t, x)$ is the inverse of $x_h(t, X)$ (at fixed $t$); $\beta_h = v_h/c$, etc. More specifically, we consider (fig. 5) a very short and intense EM plane wave (3a) hitting normally a cold plasma (or a gas that is locally ionized into a plasma by the very high electric field of the pulse itself) initially in equilibrium, possibly in a static and uniform magnetic field $B_s$; the initial conditions are:

$$n_h(0, x) = 0 \text{ if } z \leq 0, \quad u_h(0, x) = 0, \quad j^0(0, x) = \sum_h q_h n_h(0, x) = 0,$$

$$E(0, x) = \epsilon^+(-z), \quad B(0, x) = k \wedge \epsilon^+(-z) + B_s,$$  

whence the 4-current density $j = (j^0, j) = \sum_h q_h n_h(1, \beta_h)$ is zero at $t = 0$. Then the Maxwell equations $\nabla \cdot E = 4\pi j^0, \partial_t E^z/c + 4\pi j^z = (\nabla \wedge B)^z = 0$ imply

$$E^z(t, z) = 4\pi \sum_h q_h \tilde{N}_h[Z_h(t, z)], \quad \tilde{N}_h(Z) := \int_0^Z d\zeta n_h(0, \zeta);$$

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Within short time intervals \([0,t']\) (to be determined \textit{a posteriori}) we can thus: approximate \(A^\perp(t,z)\approx\alpha^\perp(\text{ct}-z)+\left(\frac{B^\perp}{c}\wedge\mathbf{x}\right)^{-1}\); also neglect the motion of ions with respect to the motion of (much lighter) electrons. Hence it is \(z_p(t,Z)\equiv Z\), and the proton density \(n_p\) (due to ions of all kinds) equals the initial one and therefore the initial electron density \(\tilde{n}_0(z):=n_e(0,z)\), by the initial electric neutrality of the plasma. Then the equations (6) & initial conditions for the electron fluid amount to

\[
mc^2\tilde{s}_e^z(\xi,Z)=4\pi e^2\left[\tilde{N}(\tilde{z}_e)-\tilde{N}(Z)\right]+e(\Delta\tilde{x}_e^z\wedge\tilde{B}_e^z)^z,
\]

\[
mc^2\tilde{u}_e^z(t,\mathbf{X})=e\alpha^\perp(t)\tilde{E}(\mathbf{X})-e(\Delta\tilde{x}_e^z\wedge\tilde{B}_e^z)^z,
\]

\[
\Delta\tilde{x}_e^z(0,\mathbf{X})=0, \quad \tilde{u}_e^z(0,\mathbf{X})=0 \quad \Rightarrow \quad \tilde{s}_e(0,\mathbf{X})=1.
\]
(28) is a family parametrized by $Z$ of decoupled ODEs in the unknowns $\Delta \dot{\mathbf{x}}, \dot{s}, \dot{u}_e^+,$ which can be solved numerically. The approximation on $A^+(t,z)$ is acceptable as long as the so determined motion makes $|\text{rhs}(27)| \ll |\alpha^+ + \frac{B_2}{2} \wedge \mathbf{x}|$; otherwise rhs(27) determines the first correction to $A^+$; and so on.

If $B_s = 0$, again (28b) is solved by $\dot{u}_e^+(\xi) = e \alpha^+(\xi)/mc^2$, while, setting $v = \dot{u}_e^+ \equiv (\dot{u}_e^{+})$, and the $z$-component of (28c) take \[21, 22\] the form of (16).

\begin{align*}
\Delta \ddot{z}_e + \frac{1+v}{2 \dot{s}^2} \dot{z}_e & - \frac{1}{2} \dot{z}_e = \frac{4 \pi e^2}{mc^2} \left\{ \tilde{N}[z_\alpha] - \tilde{N}(Z) \right\}. & (30)
\end{align*}

If $n_e(0, X) = n_0 \theta(Z)$ (with a constant electron density $n_0$), then as long as $\dot{z}_e(\xi, Z) > 0$ (30), (29) reduce to the same Cauchy problem for all $Z$:

\begin{align*}
\Delta' &= \frac{1+v}{2 \dot{s}^2} - \frac{1}{2}, & s' &= M \Delta, & M &:= \frac{4 \pi e^2 n_0}{mc^2} \equiv \frac{\omega_p^2}{c^2}, & (31)
\Delta(0) &= 0, & s(0) &= 1. & (32)
\end{align*}

These are the equations of motion of a relativistic harmonic oscillator with a forcing term $v$. In fig. 6 we depict the solution corresponding to the pulse of fig. 5-right (with $l \approx 27 \mu m$) and to $n_0 = 2 \times 10^{18}$ cm$^{-3}$; $s(\xi)$ is indeed insensitive to the fast oscillations of $\epsilon^+$ (see remark 23). $\Delta(\xi)$ grows positive for small $\xi$. The other unknowns are obtained through (15). After the pulse is passed the solution becomes periodic with period $\xi_H \approx 49 \mu m$. These $l, n_0$ fulfill

\[2l \lesssim \xi_H = ct_H,\]

where $t_H$ is the plasma period associated to $n_0$ (recall that $t_H \geq t_H^v \equiv 2\pi/\omega_p$, the non-relativistic limit of $t_H^v$). For all layers of electrons with initial $Z > \Delta_M$ ($\Delta_M$ is the oscillation amplitude) it is $\ddot{z}_e(\xi, Z) = Z + \Delta(\xi)$ for all $\xi$ (because this keeps positive for all $\xi$), $u(t, z) = \dot{u}(ct - z)$, and similarly for all other Eulerian fields: a plasma wave with spacial period $\xi_H$ and velocity $c$ travels the pulse [23] [24]. On the other hand, if $Z < \Delta_M$ then $\ddot{z}_e(\xi, Z) = Z + \Delta(\xi)$ becomes negative at some $\xi = \xi_e$, namely the layers of electrons with such initial $Z$ exit the plasma bulk; in the $\xi$-intervals where $Z + \Delta(\xi) < 0$ the ruling equation (30b) becomes $\dot{s}_e'(\xi, Z) = -MZ$. Condition (33) secures both that the pulse is completely inside the bulk before any electron gets out of it, and that the spacial period of the plasma wave is larger than the pulse length.

Replacing these solutions in the rhs(27) we find that $A^+ \simeq \alpha^+$ is indeed verified at least for $t < t_c \approx 5 \xi_H / c$. On the other hand we find [23] [24] that, while the map $z_e(t, \cdot) : Z \mapsto z$ is indeed one-to-one everywhere for $t < t_c$, at later times wave-breaking [25] (due to crossing of different $Z$-layers) occurs near the vacuum-plasma interface $Z \approx 0$. This implies that the hydrodynamic description is globally self-consistent for $t < t_c$, whereas the use of kinetic theory (i.e of a statistical description in phase space taking collisions into account, e.g. by BGK [26] equations or effective linear inheritance relations [27]) is necessary if $t > t_c$, starting from a region near the vacuum-plasma interface. But as its effects can propagate only with

\footnote{When $\dot{v} = 0$ then \[31\] implies $\Delta'' = -M \Delta / \dot{s}^3$. In the nonrelativistic regime $\dot{s} \approx 1$, $\dot{\xi}(t) \approx ct$, $cd/d\xi \approx d/dt$, and this becomes the nonrelativistic harmonic equation $\Delta = -\omega_p^2 \Delta$.}
a velocity smaller than $c$, they will not affect the plasma wave trailing the pulse with phase velocity $c$.

The above predictions are based on idealizing the laser pulse as a plane EM wave. In a more realistic picture the laser pulse is cylindrically symmetric around the $\vec{z}$-axis and has a finite spot radius $R$. Using causality and heuristic arguments we can compute [21] rough $R < \infty$ corrections to the above predictions: as a result, the impact of a very short and intense laser pulse on the surface of a cold low-density plasma (or gas, ionized into a plasma by the pulse itself), as considered e.g. in fig. 5-right, may induce [for carefully tuned $R, \tilde{n}_0(Z)$], beside a plasma traveling-wave propagating behind the pulse, also the slingshot effect [21, 22, 28], i.e. the backward acceleration and expulsion from the plasma of some surface electrons (those with smallest $Z$ and closest to the $\vec{z}$-axis) with remarkable energy. For reviews see also [29, 30, 31].

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5 Appendix: some useful estimates of oscillatory integrals

Given a function \( f \in S(\mathbb{R}) \), integrating by parts we find for all \( n \in \mathbb{N} \)

\[
\int_{-\infty}^{\xi} d\zeta f(\zeta)e^{ik\zeta} = -\frac{i}{k}f(\xi)e^{ik\xi} + R_1^f(\xi)
\]

\[
= ... = -\sum_{h=0}^{n-1} \left( \frac{i}{k} \right)^{h+1} f^{(h)}(\xi)e^{ik\xi} + R_n^f(\xi), \quad \text{where}
\]

\[
R_1^f(\xi) := \frac{i}{k} \int_{-\infty}^{\xi} d\zeta f'(\zeta)e^{ik\zeta} = \left( \frac{i}{k} \right)^2 \left[ -f'(\xi)e^{ik\xi} + \int_{-\infty}^{\xi} d\zeta f''(\zeta)e^{ik\zeta} \right],
\]

\[
R_n^f(\xi) := \left( \frac{i}{k} \right)^n \int_{-\infty}^{\xi} d\zeta f^{(n)}(\zeta)e^{ik\zeta} = \left( \frac{i}{k} \right)^{n+1} \left[ -f^{(n)}(\xi)e^{ik\xi} + \int_{-\infty}^{\xi} d\zeta f^{(n+1)}(\zeta)e^{ik\zeta} \right].
\]

Hence we find the following upper bounds for the remainders \( R_n^f \):

\[
|R_1^f(\xi)| \leq \frac{1}{|k|^2} \left[ |f'(\xi)| + \int_{-\infty}^{\xi} d\zeta |f''(\zeta)| \right] \leq \frac{||f'||_{\infty} + ||f''||_1}{|k|^2}, \tag{37}
\]

\[
|R_n^f(\xi)| \leq \frac{1}{|k|^{n+1}} \left[ f^{(n)}(\xi) + \int_{-\infty}^{\xi} d\zeta |f^{(n+1)}(\zeta)| \right] \leq \frac{||f^{(n)}||_{\infty} + ||f^{(n+1)}||_1}{|k|^n}, \tag{38}
\]

It follows \( R_1^f = O(1/k^2) \), and more generally \( R_n^f = O(1/k^{n+1}) \), so that \( (35) \) are asymptotic expansions in \( 1/k \). All inequalities in \( (37), (38) \) are useful: the left inequalities are more stringent, while the right ones are \( \xi \)-independent.

Equations \( (34), (37) \) and \( R_1^f = O(1/k^2) \) hold also if \( f \in W^{2,1}(\mathbb{R}) \) (a Sobolev space), in particular if \( f \in C^2(\mathbb{R}) \) and \( f, f', f'' \in L^1(\mathbb{R}) \), because the previous steps can be done also under such assumptions. Equations \( (34) \) will hold with a remainder \( R_1^f = O(1/k^2) \) also under weaker assumptions, e.g. if \( f' \) is bounded and piecewise continuous and \( f, f', f'' \in L^1(\mathbb{R}) \), but \( R_1^f \) will be a sum of contributions like \( (36) \) for every interval in which \( f' \) is continuous. Similarly, \( (35), (38) \) and/or \( R_n^f = O(1/k^{n+1}) \) hold also under analogous weaker conditions.

Letting \( \xi \to \infty \) in \( (34), (37) \) we find for the Fourier transform \( \hat{f}(k) = \int_{-\infty}^{\infty} d\zeta f(\zeta)e^{-ik\zeta} \) of \( f(\xi) \)

\[
|\hat{f}(k)| \leq \frac{||f'||_{\infty} + ||f''||_1}{|k|^2}, \tag{39}
\]

hence \( \hat{f}(k) = O(1/k^2) \) as well. Actually, for functions \( f \in S(\mathbb{R}) \) the decay of \( \hat{f}(k) \) as \( |k| \to \infty \) is much faster, since \( \hat{f} \in S(\mathbb{R}) \) as well. For instance, for the gaussian \( f(\xi) = \exp[-\xi^2/2\sigma] \) it is \( \hat{f}(k) = \sqrt{\pi\sigma} \exp[-k^2\sigma/2] \).

To prove approximation \( (6) \) now we just need to choose \( f = \epsilon \) and note that every component of \( \alpha^\perp \) will be a combination of \( (35) \) and \( (35)_{k \to -k} \).
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