Multiples of integral points on Mordell curves

by

AMIR GHADERMARZI (Tehran)

1. Introduction. A well-known theorem of Siegel states that there are only finitely many integral points on an elliptic curve $E/\mathbb{Q}$. Therefore, it is clear that if $P \in E(\mathbb{Q})$ is a non-torsion point, then there are at most finitely many integral points among the multiples $[n]P$ of $P$. It is an easy exercise to show that if $P$ is a non-torsion, non-integral point on an elliptic curve defined over $\mathbb{Z}$, it has no integral multiples. A natural question is: is there a bound on the number of integral multiples of a point on a given elliptic curve? One can keep rescaling the coordinates of a point on the curve to construct more and more integral points. This artificial construction can be avoided if we only consider minimal curves. In the case of minimal curves, it seems plausible that there exists a uniform bound. Indeed, Lang [15, p. 140] conjectured that the number of integral points on a quasi-minimal model of an elliptic curve $E(\mathbb{Q})$ is bounded above by the rank of the curve. Hindry and Silverman [10] proved Lang’s conjecture by assuming Szpiro’s conjecture. Silverman [19, Theorem A] proved this result unconditionally for all curves with integral $j$-invariant, and more generally for all curves with $j$-invariant non-integral for at most a fixed number of primes. Gross and Silverman [9] obtained an explicit version of Silverman’s result. Based on their work, as a special case, the number of integral points on any rank 1 quasi-minimal elliptic curve $E/\mathbb{Q}$ with integral $j$-invariant is bounded by $3.28 \times 10^{33}$. We use the method of Ingram [11] to obtain much better results for Mordell curves of rank 1.

The methods utilizing lower bounds for linear forms in elliptic logarithms give effective bounds on the value of $n$ such that $[n]P$ is integral, but the bounds depend on the height of the curve and the point $P$. Ingram used division polynomials of elliptic curves to make this dependence more explicit.

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Then he applied a gap principle to show that there exists a constant $C$ such that at most one multiple $[n]P$ is integral for $n > C$. Stange [21] modified Ingram’s argument to obtain a bound that depends on the height ratio $h(E)/\hat{h}(P)$. Note that Lang [15, p. 92] conjectured that there is a uniform constant such that for all elliptic curves $E/\mathbb{Q}$ in minimal Weierstrass form, the ratio $h(E)/\hat{h}(P)$ is bounded above by that constant. Lang’s conjecture is known to be true for all elliptic curves with integral $j$-invariant and for the ones in which the denominator of the $j$-invariant is divisible by a fixed number of primes [10] [20].

In this paper, we consider the question of integral multiples of points on the Mordell curves $E_B : y^2 = x^3 + B$. To avoid the artificial construction of many integral multiples by rescaling, we consider the Mordell curves in quasi-minimal Weierstrass equation. Let $E/\mathbb{Q}$ be an elliptic curve. A short Weierstrass equation for $E$ is called a quasi-minimal (Weierstrass) equation of $E$ if the discriminant of $E$ is minimal, subject to the condition of $E$ being in short Weierstrass form with integral coefficients. These models are minimal at all primes $p > 3$. In the case of Mordell curves, the equation $y^2 = x^3 + B$ is a quasi-minimal model if and only if $B$ is a sixth-power-free integer. Note that the equation $y^2 = x^3 + B$ is a global minimal Weierstrass equation of $E$ if and only if $B$ is a sixth-power-free integer and $B \not\equiv 16 \pmod{64}$. If $B \equiv 16 \pmod{64}$, it is minimal at any odd prime $p$ [23] Lemma 4.7. Throughout, we assume that $B$ is a sixth-power-free integer, and we call the short Weierstrass form $y^2 = x^3 + B$ a quasi-minimal Mordell curve. We aim to bound the number of $n$’s for which $[n]P$ is integral, where $P$ is a non-torsion integral point on the quasi-minimal curve $E_B$.

In Section 3, we investigate small values of $n$ such that $[n]P$ is integral. We use some elementary arguments, $p$-adic valuation of the division polynomials, height estimations, and known results on primitive divisors of elliptic divisibility sequences to prove the following theorem:

**Theorem 1.1.** Let $P$ be a non-torsion integral point on a quasi-minimal curve $E_B$, and suppose $[n]P$ is integral. If $n > 5$, then $n$ has no prime factor less than 11. Save for some exceptions, there are at most two values $1 < n < 11$ such that $[n]P$ is integral. The exceptions are the points $P = (6, \pm 18)$ on the curve $E_{108}$, $P = (60, \pm 450)$ on $E_{-13500}$, $P = (4, \pm 12)$ on $E_{80}$, and $P = (84, \pm 756)$ on $E_{-21168}$. In all these exceptions, the point $P$ has exactly three integral multiples with $n > 1$. There are infinitely many pairs $(E, P)$ of integral points $P$ on quasi-minimal Mordell curves $E$ such that $[n]P$ is integral for two values of $1 < n < 5$.

In the proof, we explicitly construct all integral points $P$ on the quasi-minimal Mordell curves such that $[n]P$ is integral for two values of $n < 11$.

Concerning larger values of $n$, first we will show that if $n \geq 11$ and $[n]P$ is integral, then $n$ is prime. We apply Ingram’s techniques [11] to prove a
gap principle for integral multiples of $P$ and apply it to prove the following theorem in Section 4.

**Theorem 1.2.** Let $P$ be a non-torsion integral point on a quasi-minimal curve $E_B$. Then $[n]P$ is integral for at most one $n \geq 29$.

In Section 5, based on Theorems 1.1 and 1.2 our gap principle and careful consideration of infinite families mentioned in Theorem 1.1 we prove the following result:

**Theorem 1.3.** Let $P$ be a non-torsion integral point on a quasi-minimal curve $E_B$. Then $[n]P$ is integral for at most three values of $n > 1$.

In [2], based on an extensive database, we conjectured that the number of integral points on the quasi-minimal model of a Mordell curve of rank 1 is at most 12. In the last section, we prove some results in this direction for the quasi-minimal Mordell curves $E_B$ with trivial rational torsion subgroups and curves with a rational torsion point of order 2 (for all the sixth-power-free, non-square $B$ values).

### 2. Preliminaries.

Let $E$ be an elliptic curve defined over $\mathbb{Q}$, $P$ be a point on $E(\mathbb{Q})$, and let $x ([n]P) = A_n/D_n^2$ be in lowest terms with $D_n > 0$. The sequence $(D_n^2)$ is called the elliptic divisibility sequence defined by $E$ and $P$. We are looking for all the terms $D_k$ such that $D_k = 1$. In order to find them, we consider a closely related divisibility sequence.

Let $K$ be a number field with ring of integers $R$. Consider an elliptic curve $E$ defined over $K$ by a general Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in R.$$  

Then there exist polynomials $\phi_m, \psi_m, \omega_m \in \mathbb{Z}[a_1, \ldots, a_6, x, y]$, defined recursively as in [18] Exercise III.3.7, such that $\phi_m, \psi_m$ are relatively prime polynomials in $\mathbb{Z}[x]$. Moreover, if $P = (x_0, y_0)$ is a point of $E(K)$, and $m$ an integer, then $[m]P = (\phi_m(x_0), \omega_m(x_0, y_0), \psi_m(x_0))$. For more information about division polynomials, their properties, and their applications in determining $S$-integral points on elliptic curves see [1], [6] Chapter 9 and [3].

For Mordell curves $E_B$, the above polynomials are defined as follows:

\[
\begin{align*}
\psi_1 &= 1, \quad \psi_2 = 2y, \\
\psi_3 &= 3x^4 + 12Bx, \\
\psi_4 &= 4y(x^6 + 20Bx^3 - 8B^2), \\
\psi_{2m+1} &= \psi_m + 2\psi_m^3 - \psi_{m-1}\psi_{m+1}, \\
2y\psi_{2m} &= \psi_m(\psi_{m+2}\psi_{m-1} - \psi_{m-2}\psi_{m+1}), \\
\phi_1 &= x, \\
\phi_m &= x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \\
4y\omega_m &= \psi_m + 2\psi_m^2 - \psi_{m-1}\psi_{m+1}.
\end{align*}
\]

(2.1)
With the notation as above,\
\[
\frac{A_n}{D_n^2} = x([n]P) = \frac{\phi_n(P)}{\psi_n(P)^2}.
\]

Since \(\psi_n\) vanishes exactly at all the non-trivial \(n\)-torsion points, we have\
\[
\text{div}(\psi_n) = \sum_{Q \in E[n]} (Q) - n^2(O).\]

Hence, since \(\psi_n^2\) is a polynomial in \(x\) with leading coefficient \(n^2\) [18, Exercise III.3.7], we obtain the following key equation:

\[
\psi_n(P)^2 = n^2 \prod_{Q \in E[n]\{O\}} |x(P) - x(Q)|.
\]

If we assign the weights\
\[
w(x) = 2, \quad w(4B) = 6,
\]

then by induction, for odd (respectively, even) values of \(m\), \(\psi_m\) (respectively, \(\psi_m/2y\)) is a binary form of degree \(m^2 - 1\) (respectively, \(m^2 - 4\)) in \(x^3\) and \(4B\). When \(m\) is odd, \(\psi_m\) is a polynomial in \(\mathbb{Z}[x]\) with leading coefficient \(m\), and the constant term is zero when \(3 \mid m\) and \(\pm (4B)^{(m^2 - 1)/6}\) otherwise; and \(\phi_m\) is always a monic polynomial of degree \(m^2\) in \(x\).

The resultant of \(\phi_m\) and \(\psi_m^2\) is \((432B^2)^d\), where \(d = \frac{1}{6}m^2(m^2 - 1)\) [17 Claim 1, p. 477]. For a given prime number \(p\), let \(\text{ord}_p(n)\) denote the \(p\)-adic valuation of \(n\), that is, the exponent of the highest power of \(p\) dividing \(n\). If \([n]P\) is integral, then any prime \(p\) that divides \(\psi_n(P)\) divides \(6B\), and

\[
2 \text{ord}_p(\psi_n(P)) \leq \text{ord}_p(\phi_n(P)).
\]

By [1, Theorem A], these are exactly the primes \(p\) for which \(P \pmod{p}\) is singular.

Throughout this paper, we need lower bounds for the canonical height and the difference between the canonical and absolute logarithmic height of points on a quasi-minimal Mordell curve \(E_B\). Fortunately, Voutier and Yabuta [23] proved some sharp height estimates for Mordell curves. For a rational non-torsion point \(P\) on a quasi-minimal Mordell curve \(E_B\), Theorem 1.2 of [23] implies

\[
\hat{h}(P) > \begin{cases} \\
\frac{1}{36} \log |B| - 0.2247 & \text{if } B < 0, \\
\frac{1}{36} \log |B| - 0.2262 & \text{if } B > 0.
\end{cases}
\]

In order to fully exploit their result, we use some of the estimates they established for possible values of \(B\) modulo powers of 2 and 3. These estimates enable us to have stronger bounds for particular values of \(B\). The following two lemmas are based on [23, Tables 4 and 5].
Lemma 2.1. Let $P$ be a rational non-torsion point on a quasi-minimal Mordell curve $E_B$. Then $\hat{h}(P) > \frac{1}{36} \log |B| - C$ where

$$C = \begin{cases} 
0.2262 & \text{if } B \equiv 15120, 3024 \text{ or } 1296 \pmod{15552}, \\
0.1347 & \text{if } B \equiv 80 \text{ or } 208 \pmod{576}, \\
0.1347 & \text{if } B \equiv 13392 \text{ or } 9936 \pmod{15552}, \\
0.1107 & \text{if } B \equiv 6372, 2052 \text{ or } 324 \pmod{7776}, \\
0.1107 & \text{if } B \equiv 108 \text{ or } 540 \pmod{3888}, \\
0.1107 & \text{if } B \equiv 1809 \text{ or } 297 \pmod{1944}, \\
0.1107 & \text{if } B \equiv 144 \pmod{1728}, \\
0.0431 & \text{otherwise.} 
\end{cases}$$

Proof. The proof is a long but straightforward calculation based on [23, proof of Theorem 1 and Tables 4 and 5]. We present the proof for $B \equiv 13392 \text{ or } 9936 \pmod{15552}$; the other cases follow similarly. For a complete proof containing all the details, we refer the reader to Section 7 of the earlier version of this paper (arXiv:2204.10950).

Let $B \equiv 13392 \text{ or } 9936 \pmod{15552}$ and let $P$ be a rational non-torsion point on $E_B$. We denote by $c_p$ the Tamagawa index of $E_B$ at the rational prime $p$, that is, the order of the component group, $E_B(Q_p)/E^0_B(Q_p)$, of $E_B$ at $p$, where $E^0_B(Q_p)$ is the connected component of the identity in $E_B(Q_p)$. Based on [23, Tables 4 and 5], we have:

(a) If $c_p = 1$ for all primes $p > 3$, then

$$\hat{h}(P) > \frac{1}{6} \log |B| - 1.2921.$$ 

(b) If $c_p | 4$ for all primes $p > 3$ and $2 | c_p$ for at least one such $p$, then

$$\hat{h}(P) > \frac{1}{24} \log |B| - 0.1176.$$ 

(c) If $c_p | 3$ for all primes $p > 3$ and $c_p = 3$ for at least one such $p$, then

$$\hat{h}(P) > \frac{1}{18} \log |B| - 0.4182.$$ 

(d) If $c_p | 12$ for all primes $p > 3$, $2 | c_p$ for at least one such $p$, and $c_q | 3$ for at least one other such prime $q$, then

$$\hat{h}(P) > \frac{1}{36} \log |B| - 0.1347.$$ 

We compare the lower bounds in each of the above cases with the one mentioned in the lemma.

For curves in (a), $\frac{1}{36} \log |B| - 0.1347 < \frac{1}{6} \log |B| - 1.2921$ unless $|B| \leq 4160$. $B = -2160$ is the only integer that satisfies the stated congruence conditions and $|B| \leq 4160$. The Mordell–Weil group of $E_{-2160}(\mathbb{Q})$ is a torsion-free group of rank 1, with generator $P = (24, 108)$. Using PARI/GP [16], one can show...
that
\[ \hat{h}(P) > \frac{1}{36} \log(2160) + 0.01718 > \frac{1}{36} \log(2160) - 0.1347. \]
Note that our definition of the canonical height is as in [18] and is half that resulting from the height function, ellheight, in PARI.

The result clearly holds for any rational non-torsion point on the curves in part (b).

Next, we consider rational points on curves in (c). For all values of $|B| > 27066$, we observe that
\[ \frac{1}{36} \log |B| - 0.1347 < \frac{1}{18} \log |B| - 0.4182. \]
Thus, it suffices to check rational points on $E_B$ with $B \in \{-21168, -17712, -5616, -2160, 9936, 13392, 25488\}$. By factoring the $B$-values in this set and referring to [2, Table 1], we find that the only curve satisfying the conditions of (c) is $E_{-21168}$. The Mordell–Weil group of $E_{-21168}(\mathbb{Q})$ is a torsion-free group of rank 1, and it has $P = (84, 756)$ as a generator. Using PARI, we verify that
\[ \hat{h}(P) > \frac{1}{36} \log(21168) - 0.1277 > \frac{1}{36} \cdot 21168 - 0.1347, \]
as desired.

When $B$ is a sixth-power-free cube, based on [23, Table 1], the Tamagawa number $c_p$ of $E_B$ at any prime $p > 3$ is either 1 or 3. Therefore, by the same argument as in the proof of Lemma 2.1 and referring to [23, Tables 4 and 5], we have the following lemma (see Section 7.4 of arXiv:2204.10950).

**Lemma 2.2.** Let $P$ be a rational non-torsion point on a quasi-minimal Mordell curve $E_B$ where $B$ is a cube. Then $\hat{h}(P) > \frac{1}{24} \log |B| - C$ where
\[
C = \begin{cases} 
0.002 & \text{if } B \text{ is odd}, \\
-0.2290 & \text{if } B \text{ is even}.
\end{cases}
\]

Moreover, [23] provides some bounds on the difference between absolute logarithmic and canonical height of an arbitrary non-torsion rational point on the quasi-minimal curve $E_B$:

\[
(2.4) \quad \frac{1}{2} h(P) - \hat{h}(P) > \begin{cases} 
-0.28 & \text{if } B < 0, \\
-\frac{1}{6} \log |B| - 0.299 & \text{if } B > 0.
\end{cases}
\]

In the course of the proof of Theorem 1.1, we consider the curves with a non-trivial rational torsion subgroups separately. This will be particularly beneficial in Section 6.2 when we study the number of integral points on quasi-minimal Mordell curves with a rational torsion point of order 2. It is not difficult to determine the torsion subgroup of all Mordell curves at once. We have the following known lemma to categorize the rational torsion subgroup of Mordell curves.
we will prove that non-torsion integral point curves with a non-trivial rational torsion subgroups separately. the examples with three integral multiples less than 11. We also consider the and it happens for infinitely many points. We will explicitly determine all this, we consider all the values is integral and
\[ \text{Lemma 2.3 ([14] Theorem 5.3]). Let } E_B \text{ be a quasi-minimal Mordell curve. Then} \]
\[ E_B(\mathbb{Q})_{\text{tors}} = \begin{cases} \mathbb{Z}/6\mathbb{Z} & \text{if } B = 1, \\ \mathbb{Z}/3\mathbb{Z} & \text{if } B = -432, \\ \mathbb{Z}/3\mathbb{Z} & \text{if } B \neq 1 \text{ is a square,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } B \neq 1 \text{ is a cube,} \\ O & \text{otherwise.} \end{cases} \]

Remark 2.4. Let } B = B_0^2 \text{ be a square different from 1. Then } E_B(\mathbb{Q})_{\text{tors}} = \{[0, B_0], [0, -B_0], O\}, \text{ and if } B = B_3^2 \text{ is a cube different from 1, then } E_B(\mathbb{Q})_{\text{tors}} = \{[B_0, 0], O\}. \]

3. Proof of Theorem [11]. This section investigates the integral multiples \([n]P\) of a non-torsion integral point \(P\) on a quasi-minimal curve \(E_B : y^2 = x^3 + B\) where \(n\) has a divisor less than 11. We will show that if \([n]P\) is integral and \(n > 5\), then \(n\) has no prime factor less than 11. To achieve this, we consider all the values \(n = mc\) with \(2 \leq m \leq 10\). We will show that besides a few exceptions, \(P\) has at most two integral multiples less than 11, and it happens for infinitely many points. We will explicitly determine all the examples with three integral multiples less than 11. We also consider the curves with a non-trivial rational torsion subgroups separately.

3.1. \(n = 2m\). In this subsection, we study integral multiples \([2m]P\) of a non-torsion integral point \(P\) on a quasi-minimal curve \(E_B\). We will show that \([2m]P\) is not integral when \(m > 10\) or \(8 \mid m\). In the subsequent subsections, we will prove that \([2m]P\) can be integral only when \(m = 1\) or 2.

There are infinitely many points \(P\) on \(E_B\) such that \([2]P\) is integral (see the construction below). Let \(P = (a, b)\) be a non-torsion integral point on \(E_B\), so by Remark 2.3 \(ab \neq 0\). Then \(x([2]P) = (\frac{3a^2}{2b})^2 - 2a\). Therefore, \([2]P\) is integral if and only if \(2b \mid 3a^2\). Hence, if \([2]P\) is integral, then \(a\) is even, and any prime \(p \neq 3\) that divides \(b\) must divide \(a\) and \(B\).

Let \(p \neq 3\) be a prime divisor of \(b\). Since \(B\) is a sixth-power-free integer, from the equation \(b^2 = a^3 + B\), we have \(\text{ord}_p(b^2) \neq \text{ord}_p(a^3)\). Therefore, \(\text{ord}_p(B) = \min(\text{ord}_p(a^3), \text{ord}_p(b^2))\).

If \(\min(\text{ord}_p(a^3), \text{ord}_p(b^2)) = \text{ord}_p(a^3) = \text{ord}_p(B)\), then \(\text{ord}_p(B) = 3\), \(\text{ord}_p(a) = 1\), and \(\text{ord}_p(b) \geq 2\). But, since \(\text{ord}_p(2b) \leq \text{ord}_p(a^2)\), this case is not possible when \(p = 2\). For odd values of \(p\) in this case we have \(\text{ord}_p(b) = 2\), which means \(\text{ord}_p(B) = \frac{3}{2} \text{ord}_p(b)\).

If \(\min(\text{ord}_p(a^3), \text{ord}_p(b^2)) = \text{ord}_p(b^2) = \text{ord}_p(B)\), then \((\text{ord}_p(b), \text{ord}_p(B))\) is \((1, 2)\) or \((2, 4)\). Either way from \(3 \text{ord}_p(a) > 2 \text{ord}_p(b)\), we conclude that \(\text{ord}_p(a) \geq \text{ord}_p(b)\).
If \( p = 3 \mid b \) and \( \text{ord}_p(b) \leq 2 \text{ord}_p(a) \), then exactly the same argument works. The only remaining case is when \( p = 3 \mid b \) and \( \text{ord}_p(b) > 2 \text{ord}_p(a) \). A direct 3-adic analysis of the equation \( b^2 = a^3 + B \) shows that this can happen only if \( (\text{ord}_p(a), \text{ord}_p(b), \text{ord}_p(B)) \) is \((0, 1, 0)\) or \((1, 3, 3)\).

From the above \( p \)-adic consideration of divisors of \( b \), we can categorize all possible non-torsion points \( P \) on quasi-minimal Mordell curves \( E_B \) with \([2]P \) integral as follows:

\[
(3.1) \quad \begin{cases} 
P = (MNt, M^2N), \\
B = M^3N^2K, \\
M = Nt^3 + K, \\
x([2]P) = -2tNM + \left(\frac{3t^2N}{2}\right)^2, 
\end{cases}
\quad \text{or} \quad \begin{cases} 
P = (MNt, 3M^2N), \\
B = M^3N^2K, \\
9M = Nt^3 + K, \\
x([2]P) = -2tNM + \left(\frac{3t^2N}{2}\right)^2, 
\end{cases}
\]

where, \( M, N, t \) and \( K \) are non-zero integers, \((M, Nt) = 1, (K, MNt) = 1, Nt \) is even, and \( M^3N^2K \) is sixth-power-free.

Considering \((3.1)\), it is easy to construct infinitely many pairs \((E_B, P)\) of quasi-minimal Mordell curves \( E_B \) and non-torsion points \( P \) on them such that \([2]P \) is integral.

The next step is to consider the possibility of such a construction when \( E_B \) has a non-trivial torsion subgroup. Let \( B \) be a sixth-power-free integer, not equal to 1 or \(-432\), such that \( E_B \) has a non-trivial rational torsion subgroup.

Then from Lemma \((2.3)\), either \( B \) is a cube \((N = 1, K = k^3 \) for some integer \( k \)) or \( B \) is a square \((M = 1, K = k^2 \) for some integer \( k \)). By \((3.1)\), in the former case, finding \( E_B \) with an integral point divisible by 2 is equivalent to finding all square-free numbers \( M \) such that \( M \) or \( 9M \) can be written as a sum of cubes. However, the latter case clearly contains infinitely many curves with integral points divisible by 2. To see this, for any third-power-free \( K = k^2 \), write \( 1 - k^2 \) (respectively, \( 9 - k^2 \)) as \( t^3N \) where \( N \) is third-power-free. Then \( P = (Nt, N) \) (respectively, \( P = (N, 3N) \)) is a point on \( E_{N^2k^2} \) with \([2]P \) integral.

**Lemma 3.1.** Let \( P \) be a non-torsion integral point on a quasi-minimal Mordell curve \( E_B \) such that \([2]P \) is integral. Then

\[
\hat{h}(P) < \frac{7}{18} \log |B| + 0.68277.
\]

**Proof.** Assume \( P = (x, y) \). Then by \((3.1)\), \(|y|^3 \) divides \( 3^3B^2 \). Therefore, \(|x^3| = |B - y^2| < 9|B^{4/3}| + |B| < 10|B|^{4/3} \). So if \([2]P \) is integral, we get an upper bound on the height of \( P \):

\[
h(P) < \frac{4}{9} \log |B| + \frac{1}{3} \log 10,
\]

and from \((2.4)\),

\[
\hat{h}(P) < \frac{7}{18} \log |B| + 0.68277. \quad \blacksquare
\]
Lemma 3.2. Let $P$ be a non-torsion integral point on a quasi-minimal Mordell curve $E_B$. If $[2m]P$ is integral, then $m < 11$.

Proof. Assume $[2m]P$ is integral, and let $Q = [m]P$. From Lemma 2.1, 
\[ \hat{h}(Q) > m^2 \left( \frac{1}{36} \log |B| - C \right), \]
where $C$ depends on the congruence conditions on $B$ and is given in Lemma 2.1.

On the other hand, since $[2]Q = [2m]P$ is integral, so is $Q$. Therefore, by Lemma 3.1, 
\[ \hat{h}(Q) < \frac{7}{18} \log |B| + 0.68277. \]

Comparing these two inequalities for $m \geq 11$, we obtain
\[ 121 \left( \frac{1}{36} \log |B| - C \right) < \frac{7}{18} \log |B| + 0.68277. \]

This means that
\[ B < \begin{cases} 
12566 & \text{if } B \equiv 15120, 3024 \text{ or } 1296 \pmod{15552}, \\
303 & \text{if } B \equiv 80 \text{ or } 208 \pmod{576}, \\
303 & \text{if } B \equiv 13392 \text{ or } 9936 \pmod{15552}, \\
115 & \text{if } B \equiv 6372, 2052 \text{ or } 324 \pmod{7776}, \\
115 & \text{if } B \equiv 108 \text{ or } 540 \pmod{3888}, \\
115 & \text{if } B \equiv 1809 \text{ or } 297 \pmod{1944}, \\
115 & \text{if } B \equiv 144 \pmod{1728}, \\
8 & \text{otherwise}. 
\end{cases} \]

There are only a few possible $B$ values in each of the above cases. If, for any of those values, $E_B$ has an integral point $P$ with $[2m]P$ integral and $m \geq 11$, then $E_B$ will have at least eight non-torsion integral points $(\{ \pm P, \pm [2]P, \pm [m]P, \pm [2m]P \})$. Moreover, the largest canonical height of integral points on $E_B$ will be at least 484 times the smallest canonical height of such points. By applying these two criteria, we can easily check the integral points on the corresponding Mordell curves $E_B$, using the data in [2]. A quick search reveals that none of these curves has an integral point $P$ such that $[2m]P$ is integral with $m \geq 11$. In fact, for integral points $P$ on these curves, there is no integral multiple $[n]P$ with $n > 5$. \[\blacksquare\]

3.1.1. $n = 4m$. In this subsection, we will identify all points $P$ on quasi-minimal Mordell curves $E_B$ such that $[4]P$ is integral. The first step is to find necessary and sufficient conditions on $P$ for $[4]P$ to be integral. Assume $P = (a, b)$ is a non-torsion integral point on $E_B$. Then $[2]P = (a', b')$ where
\[
(3.2) \quad a' = \frac{a(9a^3 - 8b^2)}{4b^2}, \quad b' = \frac{-27a^6 + 36a^3b^2 - 8b^4}{8b^3} = \frac{a^6 + 20Ba^3 - 8B^2}{8b^3}. \]

By the arguments in Section 3.1, $[4]P$ is integral if and only if $2b \mid 3a^2$, $a'$ and $b'$ are integers, and $2b' \mid 3a'^2$. 
Lemma 3.3. Let $P = (a, b)$ be a non-torsion integral point on a quasi-minimal Mordell curve $E_B$. If $(\text{ord}_2(a), \text{ord}_2(b)) = (1, 1)$, then $[4]P$ is not integral.

Proof. If $(\text{ord}_2(a), \text{ord}_2(b)) = (1, 1)$, then by (3.2), $a'$ is odd, and so $2b' \nmid 3a'^2$. 

Now we can state necessary and sufficient conditions on $P$ for $[4]P$ to be integral.

Lemma 3.4. Let $P = (a, b)$ be a non-torsion integral point on a quasi-minimal Mordell curve $E_B$ and $[2]P = (a', b')$. Then $[4]P$ is integral if and only if

(i) $(\text{ord}_2(a), \text{ord}_2(b)) \neq (1, 1)$,
(ii) $b'$ is an integer,
(iii) any prime that divides $b'$ divides $b$.

Proof. Assume $[4]P$ is integral. Then $[2]P$ is also integral, and by Lemma 3.3 $(\text{ord}_2(a), \text{ord}_2(b)) \neq (1, 1)$. To prove that prime divisors of $b'$ divide $b$, we examine various prime factors of $b'$.

Let $p \geq 5$ be a prime divisor of $b'$. Since $2b' \mid 3a'^2$, from (3.2) we have $-27a^6 + 36a^3b^2 - 8b^4 \equiv 0 \pmod{p}$ and $a(9a^3 - 8b^2) \equiv 0 \pmod{p}$. If $p$ divides $a$, then by the first congruence $p$ divides $b$. So assume $9a^3 \equiv 8b^2 \pmod{p}$. Plugging this congruence into the first congruence, we conclude that $p \mid b$.

Suppose $p = 2$ divides $b'$. Since $a$ is even, if $\text{ord}_2(b) = 0$ then by (3.2),

$$\text{ord}_2(b') = \text{ord}_2 \left( \frac{-27a^6 + 36a^3b^2 - 8b^4}{8b^3} \right) = 0.$$ 

Suppose $p = 3$ divides $b'$. If $3 \mid b'$, then $-27a^6 + 36a^3b^2 - 8b^4 \equiv 0 \pmod{3}$. Therefore, $3 \mid b$.

Conversely, assume that $P = (a, b)$ and $[2]P = (a', b')$ are integral points on $E_B$, and any prime that divides $b'$ divides $b$. To prove that $[4]P$ is integral, it is sufficient to show that for any prime divisor $p$ of $2b$, $\text{ord}_p(2b') \leq \text{ord}_p(3a'^2)$.

Suppose $p$ divides $b$. Since $\text{ord}_p(2b) \leq \text{ord}_p(3a'^2)$, $p$ divides $a$ unless $(\text{ord}_3(a), \text{ord}_3(b)) = (0, 1)$, in which case $\text{ord}_p(b') = 0$. So, assume $(\text{ord}_3(a), \text{ord}_3(b)) \neq (0, 1)$. Since $b^2 = a^3 + B$ and $B$ is a sixth-power-free integer, $\text{ord}_p(b^2) \neq \text{ord}_p(a^3)$. Therefore, $\text{ord}_p(B) = \min(2\text{ord}_p(b), 3\text{ord}_p(a))$.

First, assume that $\min(2\text{ord}_p(b), 3\text{ord}_p(a)) = 2\text{ord}_p(b) = \text{ord}_p(B)$. If $p$ is an odd prime, then

$$\text{ord}_p(b') = \text{ord}_p \left( \frac{a^6 + 20Ba^3 - 8B^2}{8b^3} \right) = 2\text{ord}_p(B) - 3\text{ord}_p(b) = \text{ord}_p(b).$$
On the other hand,
\[ \text{ord}_p(a') = \text{ord}_p \left( a \frac{(9a^3 - 8b^2)}{4b^2} \right) = \text{ord}_p(a). \]

Hence, the desired result follows. If \( p = 2 \), since we assumed \((\text{ord}_2(a), \text{ord}_2(b)) \neq (1, 1)\), we have \(\text{ord}_2(a) \geq 2\). With the same argument, \(\text{ord}_2(b') = \text{ord}_2(b)\) and \(\text{ord}_2(a') \geq \text{ord}_2(a)\), and the result follows.

Next, assume that \(\min(2 \text{ord}_p(b), 3 \text{ord}_p(a)) = 3 \text{ord}_p(a) = \text{ord}_p(B)\). Since \(B\) is sixth-power-free and \(\text{ord}_p(2b) \leq \text{ord}_p(3a^2)\), we have \(\text{ord}_p(a) = 1\), \(\text{ord}_p(B) = 3\) and \(p\) is odd. If \(p \geq 5\), then \(\text{ord}_p(b) = 2\) and so
\[ \text{ord}_p(b') = \text{ord}_p \left( -\frac{27a^6 + 36a^3b^2 - 8b^4}{8b^3} \right) = 0, \]
and the result follows. Finally, if \(p = 3\), then either \(\text{ord}_3(b) = 2\), from which we can conclude that \(\text{ord}_3(b') = 2\) and \(\text{ord}_3(a') = 1\), or \(\text{ord}_3(b) = 2\), in which case \(\text{ord}_p(b') = \text{ord}_p(a') = 0\).

Let \(P = (a, b)\) be a point on a quasi-minimal Mordell curve \(E_B\) such that \([4]P\) is integral. In (3.1) we classified all points \(P\) such that \([2]P\) is integral. With the notation of (3.1), assume that \(P = (a, b) = (MNt, 3^\alpha M^2 N)\) (\(\alpha = 0\) or \(1\)). Let \([2]P = (a', b')\). By the argument in the proof of Lemma 3.4, \(b' = \pm N\) or \(\pm 9N\) and \(a' = a \cdot l\) for some integer \(l\). As immediate consequences, we prove the following two lemmas.

**Lemma 3.5.** Let \(E_B\) be a quasi-minimal Mordell curve with a non-trivial rational torsion subgroup. If \(P\) is a non-torsion point on \(E_B\), then \([4]P\) is not integral.

**Proof.** From Lemma 2.3, we can assume \(B \neq 1, -432\) is a sixth-power-free cube or square. Let \(P = (a, b)\) be a point on \(E_B\) such that \([4]P\) is integral and \(Q = (a', b') = [2]P\). By (3.1), \(b = (MNt, 3^\alpha M^2 N)\) (\(\alpha = 0\) or \(1\)), and \(B = M^3 N^2 K\). From the above argument \(b' = \pm N\) or \(\pm 9N\).

Let \(B = B_1^3\) be a cube. In the above notation, \(N = 1\). Therefore, \(b' = \pm 1\) or \(\pm 9\). Hence, \(a'^3 + B_2^3 = 1\) or \(a'^3 + B_3^3 = 81\). The first equation leads to torsion points \((0, \pm 1)\) on \(E_1\), and the second one has no solutions.

If \(B = B_2^3\) is a square, then in the above notation, \(M = 1\). Therefore, \(b' = \pm b\) or \(\pm b/3\). If \(b' = \pm b\), then \(P\) is a torsion point. The case \(b' = \pm b/3\) happens only when \(P = (Nt, 3N)\). Hence, in this case, we have \(Q = (Ntl, N)\) and the following system of equations holds:

\[
\begin{align*}
9N^2 &= N^3 t^3 + B, \\
N^2 &= N^3 t^3 l^3 + B.
\end{align*}
\]

This means \(l = 0\) and \(P\) is a torsion point.
With the same argument as above, we can show that \([8]P\) is never integral.

**Lemma 3.6.** For any non-torsion point \(P\) on a quasi-minimal curve \(E_B\), \([8]P\) is not integral.

**Proof.** Let \(P\) be an integral point on \(E_B\) such that \([8]P\) is integral. With the notation of (3.1), \(y(P) = 3^\alpha M^2N\) (where \(\alpha = 0\) or \(1\)) and \(B = M^3N^2K\). Then for \(Q = [2]P\), we have \(|y(Q)| = N\) or \(9N\). Since \([4]Q = [8]P\) is an integral point, by the same argument as in the proof of Lemma 3.5 we have \(y([2]Q) = \pm y(Q)\). Therefore, \(Q\) and so \(P\) are torsion points. 

In order to identify all non-torsion points \(P\) on quasi-minimal Mordell curves \(E_B\) such that \([4]P\) is integral, we need another version of Lemma 3.4 that can be checked more easily.

**Lemma 3.7.** Let \(P = (a, b)\) be a non-torsion point on a quasi-minimal Mordell curve \(E_B\). Then \([4]P\) is integral if and only if \(2b \mid 3a^2\), \((\text{ord}_p(a), \text{ord}_p(b)) \neq (1, 1)\), and there exist non-negative integers \(\alpha\) and \(\gamma_i\) such that

\[
f(a, B) = a^6 + 20Ba^3 - 8B^2 = 2^\alpha \prod_{p \mid b} p_i^{\gamma_i}.
\]

**Proof.** The condition \(2b \mid 3a^2\) is equivalent to condition (ii) of Lemma 3.4. Since \(y([2]P) = \frac{a^6 + 20Ba^3 - 8B^2}{8b^3} = \frac{-27a^6 + 36a^3b^2 - 8b^4}{8b^3}\), equation (3.4) holds if and only if condition (iii) in Lemma 3.4 holds.

Let \(P = (a, b)\) be a point on a quasi-minimal curve \(E_B\) that satisfies the conditions of Lemma 3.7. Consider (3.4) as a binary form of degree 2 in \(a^3\) and \(B\). So \(F(X, Y) = X^2 + 20XY - 8Y^2\). Let

\[
A = \frac{a^3}{\gcd(a^3, B)}, \quad B = \frac{B}{\gcd(a^3, B)}.
\]

Then for any point \(P\) such that \([4]P\) is integral, we obtain the polynomial \(F(A, B)\). With this notation, we have:

**Lemma 3.8.** Let \(P = (a, b)\) be an integral point on a quasi-minimal Mordell curve \(E_B\) such that \([4]P\) is integral. Then \(F(A, B) = \pm 2^3 \cdot 3^\gamma\), where \(\gamma \in \{0, 2, 3\}\).

**Proof.** Note that if a prime \(p \neq 3\) divides \(b\), it divides both \(a\) and \(B\).

Let \(p \geq 5\) be a prime divisor of \(b\). If \(\text{ord}_p(a^3) \neq \text{ord}_p(B)\), then exactly one of \(A\) and \(B\) is divisible by \(p\). Therefore, \(p \nmid F(A, B)\). If \(\text{ord}_p(a^3) = \text{ord}_p(B)\), then since we have assumed \(B\) is sixth-power-free, we have \(\text{ord}_p(a^3) = \text{ord}_p(B) = 3\), and from the equation \(a^3 + B = b^2\) we obtain \(A \equiv -B \not\equiv 0 \pmod{p}\). Therefore, \(F(A, B) \equiv -27A^2 \not\equiv 0 \pmod{p}\).
Let $p = 2$. Since $a$ is even, and we ruled out the case $\text{ord}_2(a) = \text{ord}_2(b) = 1$, it is easy to see that $\text{ord}_2(a^3) \geq \text{ord}_2(B) + 2$. Thus, $\text{ord}_2(\mathcal{F}(A, B)) = 3$.

Let $p = 3$. Again, if $\text{ord}_3(a^3) \neq \text{ord}_3(B)$, then exactly one of $A$ and $B$ is divisible by 3, and therefore $3 \nmid \mathcal{F}(A, B)$. So assume $\text{ord}_3(a^3) = \text{ord}_3(B)$. Then we write $\mathcal{F}(A, B)$ as $((A + B) + 9B)^2 - 108B^2$. Therefore, $\gamma = \text{ord}_3(\mathcal{F}(A, B))$ can be easily found based on $\text{ord}_3(A + B)$. To be more precise, if $(\text{ord}_3(a), \text{ord}_3(b), \text{ord}_3(B)) = (0, 0, 0)$, then $\gamma = 0$; if $(\text{ord}_3(a), \text{ord}_3(b), \text{ord}_3(B)) = (1, 2, 3)$, then $\gamma = 2$; and if $(\text{ord}_3(a), \text{ord}_3(b), \text{ord}_3(B)) = (0, 1, 0)$ or $(\text{ord}_3(a), \text{ord}_3(b), \text{ord}_3(B)) = (1, 3, 3)$, then $\gamma = 3$. ■

We now rewrite Lemma 3.8 in the notation of (3.1). For a point $P = (a, b)$ such that $[4]P$ is integral, we obtain $A = Nt^3$ and $B = K$. Moreover, $Nt^3$ is even, so the equation in Lemma 3.8 becomes

\[
\left( \frac{Nt^3}{2} + 5K \right)^2 - 3(3K)^2 = \pm 2 \cdot 3^\gamma.
\]

To summarize, we can categorize all points $P$ on curves $E_B$ where $B$ is sixth-power-free and $[4]P$ is integral as follows:

\[
\begin{align*}
P &= (MNT, M^2N), \\
B &= M^3N^2K, \\
M &= Nt^3 + K, \\
\left( \frac{Nt^3}{2} + 5K \right)^2 - 3(3K)^2 &= \pm 2, \\
3 &\nmid M.
\end{align*}
\]

or

\[
\begin{align*}
P &= (MNT, 3M^2N), \\
B &= M^3N^2K, \\
9M &= Nt^3 + K, \\
\left( \frac{Nt^3}{2} + 5K \right)^2 - 3(3K)^2 &= \pm 54, \\
3 &\nmid N.
\end{align*}
\]

Here $(K, MNt) = (M, Nt) = (1, Nt)$ is always even, and $t$ is even unless $\text{ord}_2(x(P)) = \text{ord}_2(y(P)) = 2$.

As a final remark, assuming $[4]P$ is integral, we can rewrite $[2]P$ in the notation of (3.1) as $y([2]P) = N$, $x([2]P) = Nt$. So $h([2]P) > \frac{1}{3} \log B - \frac{1}{3} \log 2$. On the other hand, $|x([4]P)| = |x([2]P)(\frac{9t^2}{4}(x([2]P)) \pm 2)|$. Since $|x([2]P)| \geq 2$, we have $|x([4]P)| > \frac{2}{3} x([2]P)^2$. Therefore,

\[
h([4]P) > \frac{2}{3} \log B - 0.239.
\]

\subsection{n = 3m}

In this subsection, we investigate integral multiples $[3m]P$ of a non-torsion integral point $P$ on a quasi-minimal curve $E_B$. We will show that $[3m]P$ is not integral when $m > 10$ or when $m$ is even or divisible by 3.
In the next section, we will show that \([3m]P\) is not integral when \(m = 5\) or 7. Therefore, \([3m]P\) can be integral only when \(m = 1\).

Let \(P = (a, b)\) be a non-torsion integral point on \(E_B\). It is clear that \([3]P\) is integral if and only if for any prime \(p\) that divides \(\psi_3(P)\), \(2 \text{ord}_p(\psi_3(P)) \leq \text{ord}_p(\phi_3(P))\). Therefore, to identify all points \(P = (a, b)\) on \(E_B\) such that \([3]P\) is integral, we investigate the \(p\)-adic order of \(\psi_3(P)\) and \(\phi_3(P)\) for all primes that divide \(\psi_3(P)\). The resultant of \(\phi_m\) and \(\psi_m^2\) is \((432B^2)^d\) where \(d = \frac{1}{6}m^2(m^2 - 1)\) \([17]\). Hence, these primes are among the primes that divide \(6B\) and are exactly the primes \(p\) for which \(P \pmod{p}\) is singular \([\Pi]\) Theorem A).

Let \(P = (a, b)\) be a point on \(E_B\) such that \([3]P\) is integral. Note that by \((2.1)\),

\[
\psi_3(P) = 3a(a^3 + 4B), \quad \phi_3(P) = a^9 - 96Ba^6 + 48B^2a^3 + 64B^3.
\]

Let \(p > 3\) be a prime divisor of \(B\). From the equation of \(\psi_3\), \(p\) divides \(\psi_3(P)\) if and only if it divides \(a\). Let \(p > 3\) be such a prime with \(\text{ord}_p(B) = m\) and \(\text{ord}_p(a) = n > 0\). We claim that \(m > n\). Indeed, assume that \(m \leq n\). Then \(\text{ord}_p(\psi_3(P)) = m + n\). On the other hand, \(\text{ord}_p(\phi_3(P)) = 3m\). Thus, \(\text{ord}_p(\phi_3(P)) < 2 \text{ord}_p(\psi_3(P))\), and therefore \([3]P\) is not integral. Since \(m < 6\), considering \(m > n > 0\) and the equation \(b^2 = a^3 + B\), there are only a few possible values for \(\text{ord}_p(a)\) and \(\text{ord}_p(B)\). With a little work on these possible values, \(2 \text{ord}_p(\psi_3(P)) \leq \text{ord}_p(\phi_3(P))\) if and only if \((\text{ord}_p(a), \text{ord}_p(B)) = (1, 2), (1, 3), (2, 4)\). Correspondingly, \((\text{ord}_p(x([3]P)), \text{ord}_p(B)) = (0, 2), (1, 3), (0, 4)\).

Let \(p = 2\). Again 2 divides \(\psi_3(P)\) if and only if 2 divides \(a\). However, unlike the previous case, 2 does not necessarily divide \(B\). By a similar argument to the above, if \(\text{ord}_2(a) = n\) and \(\text{ord}_2(B) = m\), then \(m > n\) unless \(m = n = 2\) or \((n, m) = (1, 0)\). Again by checking the possible values of \(\text{ord}_2(a)\) and \(\text{ord}_2(B)\), we see that \(2 \text{ord}_2(\psi_3(P)) \leq \text{ord}_2(\phi_3(P))\) if and only if \((\text{ord}_2(a), \text{ord}_2(B)) = (2, 2), (1, 2), (1, 3), (2, 4), (3, 4), (1, 0)\). Correspondingly, \((\text{ord}_2(x([3]P)), \text{ord}_2(B)) = (0, 2), (1, 2), (1, 3), (0, 4), (0, 4), (0, 0)\). Note that if \(a\) is not even then \(x([3]P)\) is not even, and if \(x([3]P)\) is even then \(\text{ord}_2(y([3]P)) \geq 2\).

Let now \(p = 3\). In contrast to the previous case, 3 always divides \(\psi_3(P)\), but may not divide \(a\). By the same argument as above, if \(\text{ord}_2(a) = n\) and \(\text{ord}_2(B) = m\), then \(m > n\) unless \(m = n = 0\). By checking the possible values of \(\text{ord}_3(a), \text{ord}_3(b), \text{and} \text{ord}_3(B)\), we find that \(2 \text{ord}_3(\psi_3(P)) \leq \text{ord}_3(\phi_3(P))\) if and only if \((\text{ord}_3(a), \text{ord}_3(b), \text{ord}_3(B)) = (0, \geq 1, 0), (1, 2, 3), (1, \geq 3, 3)\). The corresponding 3-adic valuations of \([3]P = (a', b')\) are as follows: \((\text{ord}_3(a'), \text{ord}_3(b'), \text{ord}_3(B)) = (0, \geq 1, 0), (1, \geq 2, 3), (1, \geq 3, 3)\).

So far, we have investigated the prime divisors of \(\psi_3(P)\) that divide \(6B\). The final step is to prove necessary and sufficient conditions that guarantee
no other prime divides $\psi_3(P)$. Note that if a prime $p$ divides $a$, then it divides $\psi_3(P)$. Therefore, if $[3]P$ is integral, any odd prime $p$ that divides $a$ must divide $B$. Define

$$F(P) = \frac{\psi_3(P)}{3a \gcd(a^3, B)}.$$ 

Since $a$ has no prime divisor outside the set of divisors of $6B$, $\psi_3(P)$ has no prime divisor that does not divide $6B$ if and only if any prime divisor of $F(P)$ divides $6B$. Hence, $F(P) = \prod_{p \mid 6B} p^{\gamma_p}$ where $\gamma_p$'s are non-negative integers. With careful consideration of the $p$-adic values of $F(P)$ based on the acceptable possible congruences introduced before, we have $F(P) = 3 \cdot 2^\alpha$ where $\alpha \in \{0, 3\}$. The case $\alpha = 3$ happens only when $(\ord_2(a), \ord_2(B)) = (2, 4)$.

We summarize the above argument in the following lemma.

**Lemma 3.9.** All the pairs $(P, E_B)$ of non-torsion points $P$ on a quasi-minimal Mordell curve $E_B$ such that $[3]P$ is integral can be categorized as in Table 1, where $M$, $N$, and $K$ are relatively prime odd integers, $(3, NK) = 1$, and $M^3N^2K$ is a sixth-power-free integer.

**Proof.** As discussed above, $[3]P$ is integral if and only if all prime divisors $p$ of $\psi_3(P)$ divide $6B$, and for these primes, $2\ord_p(\psi_3(P)) \leq \ord_p(\phi_3(P))$. The conditions in columns 1 and 2 of Table 1 guarantee that any prime divisor of $a$ is a prime divisor of $2B$. This condition is necessary for $[3]P$ to be integral, as shown above. The numbers in the first two columns meet all the introduced congruence conditions that warrant $2\ord_p(\psi_3(P)) \leq \ord_3(\phi_3(P))$ for prime divisors of $6B$. By the condition in the third column, there exists an integral point $P$ on $E_B$ with given values $x(P)$ and $B$. As proved above, the condition in the last column means that any prime divisors of $\psi_3(P)$ divide $6B$.

**Table 1.** Integral points $P$ on elliptic curves $E_B$ with $[3]P$ integral; here “$= \square$” means “is a square”.

| Type | $x(P)$ | $B$ | Condition for $P \in E_B$ | Congruence for $\psi_3(P)$ condition |
|------|--------|-----|-----------------------------|-------------------------------------|
| Type I | $2MN$ | $M^3N^2K$ | $M(8N + K) = \square$ | $2N + K = \pm 3$ |
| Type II | $MN$ | $M^3N^2K$ | $M(N + K) = \square$ | $N + 4K = \pm 3$ |
| Type III | $4MN$ | $4M^3N^2K$ | $M(16N + K) = \square$ | $4N + K = \pm 3$ |
| Type IV | $2MN$ | $4M^3N^2K$ | $M(2N + K) = \square$ | $N + 2K = \pm 3$ |
| Type V | $2MN$ | $8M^3N^2K$ | $2M(N + K) = \square$ | $N + 4K = \pm 3$ |
| Type VI | $4MN$ | $16M^3N^2K$ | $M(4N + K) = \square$ | $N + K = \pm 6$ |
| Type VII | $8MN$ | $16M^3N^2K$ | $M(32N + K) = \square$ | $8N + K = \pm 3$ |
Using Table 1, it is easy to find infinitely many pairs \((P, E_B)\) such that \([3]P\) is integral. As an example, let us introduce infinitely many examples of type II. Let \(l\) be a non-zero integer. Take \(M = 1, N = 12l^2 - 1,\) and \(K = 1 - 3l^2\). Note that for infinitely many values of \(l,\) \(M^3N^2K\) is sixth-power-free \(\text{[3]}\). Hence, \(K, M, N\) satisfy the conditions of Lemma 3.9. Let \(P = (12l^2 - 1, 3(12l^2 - 1)l)\). Then \(P\) is a non-torsion point on the Mordell curve \(E_{(12l^2-1)^2(1-3l^2)}\), where \([3]P\) is integral with \(x([3]P) = 576l^6 - 288l^4 + 36l^2 - 1\).

However, if we assume \(E_B\) has a non-trivial rational torsion subgroup then it is a different matter.

**Lemma 3.10.** Let \(P = (a, b)\) be a non-torsion point on a quasi-minimal Mordell curve \(E_B\) with a non-trivial rational torsion subgroup. Then \([3]P\) is not integral.

**Proof.** Since \(E_B\) has a non-trivial rational torsion subgroup, by Lemma 2.3 we can assume \(B \neq 1, -432\) is a cube or a square.

Let \(B\) be a cube, \(P\) be a point on \(E_B\), and \([3]P\) be integral. In the notation of Table 1, \(N = \pm 1, K = k^3,\) and \(P\) is a point of type I, II or V. Based on the condition in the last column of Table 1 in each type there will be at most one value for \(K\) corresponding to \(N = \pm 1\). In view of the third column and the possible values of \(M,\) this process only leads to torsion points.

Let \(B\) be a square, \(P\) be a point on \(E_B\), and suppose \([3]P\) is integral. In the notation of Table 1, \(M = \pm 1, K = k^2,\) and \(P\) is a point of type I, II, III, IV, V, or VI. In each type, we found \(N\) in terms of \(K\) and rewrote the condition in the third column in terms of \(K = k^2\) and \(M\). This only yields torsion points \(P\).

**Lemma 3.11.** There is no integral non-torsion point \(P = (a, b)\) on a quasi-minimal Mordell curve \(E_B\) such that \([3m]P\) is integral and \(m\) is even, or \(m\) is divisible by 3, or \(m > 10\).

**Proof.** Let \(P\) be a non-torsion integral point on a quasi-minimal Mordell curve \(E_B\) such that \([3m]P\) is integral.

Assume \(m\) is even. Then \([2]P, [3]P,\) and \([6]P\) are all integral. Since \([2]P\) and \([2]([3]P)\) are integral, \(a\) and \(x([3]P)\) are even. However, we have already seen that if \([3]P\) is integral then \(\text{ord}_2(x([3]P)) \leq 1,\) and if \(x([3]P)\) is even, then \(\text{ord}_2(y([3]P)) \geq 2.\) Thus, \(2(y([3]P)) \nmid 3(x([3]P))^2,\) and \([6]P\) is not integral.

Next, assume \(m\) is divisible by 3. Then \([3]P\) and \([9]P\) are integral. Let \(Q = [3]P = (a', b').\) Since \([3]Q\) is integral, by Lemma 3.9 any prime divisor of \(a'\) must divide \(2B.\) Recall that \(a' = \frac{\phi_3(P)}{\psi_3(P)^2}\) and \(\phi_3(P)\) is a binary form in \(a^3\) and \(4B,\) with leading term \(a^9,\) so any prime that divides \(2B\) and \(a'\) must divide \(a.\)
In the notation of Table \([1]\) let \(a = 2^\alpha MN\), where \(\alpha\) is a non-negative integer less than 5. According to the argument before Lemma \([3.9]\), we obtain \(a' = 2^\beta M\) and \(\beta = 0\) or 1. Now if we assume \([3]Q\) is integral, we can locate \(Q\) as type I, II, IV or V in Table \([1]\) with corresponding new value \(N = \pm 1\). However, the conditions in the third and fourth columns of the table with \(N = \pm 1\) only lead to torsion points.

Finally, assume \(m > 10\). Let \(Q\) be a non-torsion point such that \([3]Q\) is integral. By Table \([1]\), \(x(Q)^2 < 4B\), and so

\[
h(Q) < \frac{\log |B| + \log(4)}{2}.
\]

Therefore, by inequality \((2.4)\),

\[
\hat{h}(Q) < \frac{5}{12} |B| + 0.646.
\]

Since \([3m]P = [3]([m]P)\) is integral, we can take \(Q\) to be \([m]P\) in the above inequality. Hence, \(\hat{h}([m]P)\) satisfies the above inequality. On the other hand, if \(m \geq 11\), then from Lemma \([2.1]\)

\[
\hat{h}([m]P) > 121 \left( \frac{1}{36} \log |B| - C \right),
\]

where \(C\) depends on the congruence condition of \(B\) and is given in Lemma \([2.1]\). Comparing these two inequalities, if \([3m]P\) is integral for \(m \geq 11\) then

\[
121 \left( \frac{1}{36} \log |B| - C \right) < \frac{5}{12} \log |B| + 0.646.
\]

This means that

\[
|B| < \begin{cases} 
13562 & \text{if } B \equiv 15120, 3024 \text{ or } 1296 \pmod{15552}, \\
317 & \text{if } B \equiv 80 \text{ or } 208 \pmod{576}, \\
317 & \text{if } B \equiv 13392 \text{ or } 9936 \pmod{15552}, \\
118 & \text{if } B \equiv 6372, 2052 \text{ or } 324 \pmod{7776}, \\
118 & \text{if } B \equiv 108 \text{ or } 540 \pmod{3888}, \\
118 & \text{if } B \equiv 1809 \text{ or } 297 \pmod{1944}, \\
118 & \text{if } B \equiv 144 \pmod{1728}, \\
8 & \text{otherwise}.
\end{cases}
\]

There are only a few possible values of \(B\) in each of the above cases. We have already checked integral points on \(E_B\) corresponding to these values in Lemma \([3.1]\). The integral points \(P\) on these curves have no integral multiple \([n]P\) with \(n > 5\). ■

3.3. \(n = mc, m = 5, 7\). The question of finding integral multiples of a point \(P\) is indeed a weaker version of the question about the primitive divisor in the elliptic divisibility sequence related to \(P\). So we can use the idea of \([12]\). For \(n \geq 5\), the idea is to embed the points \(P\) such that \([n]P\)
is integral into the set of solutions of finitely many Thue or Thue–Mahler equations.

3.3.1. $m = 5$. Let $P = (a,b)$ and $|5|P$ be integral points on a quasi-minimal Mordell curve $E_B$. By [1] Corollaire A, $ψ_5(a,b) = ± \prod_{p \mid 5B} p^{γ_p}$ for some non-negative integers $γ_p$. We have

$$ψ_5(x, B) = 5x^{12} + 380Bx^9 - 240B^2x^6 - 1600B^3x^3 - 256B^4.$$ 

Consider $ψ_5(x, B)$ as a binary form of degree 4 in $x^3$ and $4B$ with integer coefficients. Let $X = \frac{x^3}{(4B, x^3)}$ and $B = \frac{4B}{(4B, x^3)}$. With this notation we obtain

$$Ψ_5(\mathcal{X}, \mathcal{B}) = 5\mathcal{X}^4 + 95B\mathcal{X}^3 - 15B^2\mathcal{X}^2 - 25B^3\mathcal{X} - B^4.$$ 

Lemma 3.12. Let $P = (a, b)$ be a non-torsion point on a quasi-minimal elliptic curve $E_B$ such that $|5|P$ is integral. Let $\mathcal{X} = \frac{a^3}{(a^3, 4B)}$ and $\mathcal{B} = \frac{4B}{(a^3, 4B)}$. Then

$$Ψ_5(\mathcal{X}, \mathcal{B}) = ±3^α5^γ,$$ where $α \in \{0, 4, 6\}$ and $γ \in \{0, 1\}$.

Proof. Since $|5|P$ is integral by [1] Theorem A, the only primes that might divide $ψ_5(a, B)$, and therefore $Ψ_5(\mathcal{X}, \mathcal{B})$, are the prime divisors of $6B$.

Let $p \geq 5$ divide $B$. If ord$_p(a^3) \neq$ ord$_p(4B)$, then in the binary form $Ψ_5(\mathcal{X}, \mathcal{B})$ exactly one of $\mathcal{X}$ or $\mathcal{B}$ is divisible by $p$. Therefore, if $p > 5$, or $p = 5$ and $5 \nmid \mathcal{B}$, all the terms but one on the right hand side of (3.7) are divisible by $p$. Hence $p \nmid Ψ_5(\mathcal{X}, \mathcal{B})$. If $p = 5$ and $5 \mid \mathcal{B}$ then $5\mathcal{X}^4$ is the term with minimum 5-adic valuation in $Ψ_5(\mathcal{X}, \mathcal{B})$. So ord$_5(Ψ_5(\mathcal{X}, \mathcal{B})) = 1$. If ord$_p(a^3) \neq$ ord$_p(4B)$, then none of $\mathcal{X}$ or $\mathcal{B}$ is divisible by $p$. Hence, from $b^2 = a^3 + B$, we have $\mathcal{X} ≡ -\mathcal{B} \pmod{p}$. Therefore, $Ψ_5(\mathcal{X}, \mathcal{B}) ≡ 81\mathcal{X}^4 \pmod{p}$, and $p ∤ Ψ_5(\mathcal{X}, \mathcal{B})$.

Let $p = 2$. If ord$_2(a^3) \neq$ ord$_2(4B)$, then exactly one of $\mathcal{X}$ or $\mathcal{B}$ is odd, so by (3.7), $Ψ_5(\mathcal{X}, \mathcal{B})$ is odd. If ord$_2(a^3) =$ ord$_2(4B)$, then both $\mathcal{X}$ and $\mathcal{B}$ are odd. Therefore $Ψ_5(\mathcal{X}, \mathcal{B})$ is odd.

Let $p = 3$. In Lemma 4.1 we will explicitly determine ord$_3(ψ_n(P))$ for odd $n$ not divisible by 3. Based on that lemma, ord$_3 Ψ_5(\mathcal{X}, \mathcal{B}) ∈ \{0, 4, 6\}$. ■

From this lemma, it is possible to find all points $P$ with $|5|P$ integral:

Lemma 3.13. The only non-torsion points $P$ on a quasi-minimal Mordell curves $E_B$ such that $|5|P$ is integral are the points $±P$ on the curve $E_{108}$ where $P = (6, 18)$. The only integral multiples of $P$ with $n > 1$ are $|2|P = (-3, 9), |3|P = (-2, -10)$ and $|5|P = (366, 7002)$.

Proof. Note that since we assumed $B$ is sixth-power-free, any values of $\mathcal{X}$ and $\mathcal{B}$ give rise to one point on $E_B$. From the previous lemma, the set of points such that $|5|P$ is integral is embedded into the set of solutions of a finite set of Thue equations. To be more precise, the possible pairs $(P, E_B)$ of non-torsion points $P$ on the quasi-minimal Mordell curves $E_B$ such that $|5|P$
is integral are embedded in solutions of the following 12 Thue equations:
\[5x^4 + 95yx^3 - 15y^2x^2 - 25yx^3 - y^4 = \pm 3^\alpha 5^\gamma,\]
where \(\alpha \in \{0, 4, 6\}\) and \(\gamma \in \{0, 1\}\).

We have solved these Thue equations using PARI and double checked with Magma. The only solutions related to a non-torsion point are \([1, 2]\) and \([-1, -2]\) and correspond to the equation \(\Psi_5(\mathcal{X}, \mathcal{B}) = -81\), which gives rise to the points \((6, \pm 18)\) on \(E_{108}\).

**Remark 3.14.** In [12], there is a typo in determining \(\psi_5(X, Y)\). As a result, the author missed the points \((6, \pm 18)\) on \(E_{-81}\) and concluded that the 5th terms in the elliptic divisibility sequence defined by a point \(P\) on a quasi-minimal Mordell curve fail to have a primitive divisor. However, this is the only example missed by the author in studying the 5th term of the corresponding elliptic divisibility sequence of points on quasi-minimal Mordell curves.

**3.3.2.** \(m = 7\). Ingram [12] proved that there are no elliptic divisibility sequences arising from curves on quasi-minimal Mordell curves where the 7th term has no primitive divisor when \(\alpha > 0\). Indeed, by that result [7] \(P\) is not integral for any point \(P\) on a quasi-minimal Mordell curve \(E_B\). However, for the sake of completeness and since there are some small typos/unproven statements in his proof, we present the proof with more details. The idea is very similar to the case \(m = 5\). Let \(P = (a, b)\) be a non-torsion point on a quasi-minimal Mordell curve \(E_B\) such that \([7]P\) is integral. By [1, Corollaire A], \(\psi_7(a, b) = \pm \prod_{p|6B} p^{\gamma_p}\) for some non-negative integers \(\gamma_p\). We have
\[\psi_7(x, B) = 7x^{24} + 3944Bx^{21} - 42896B^2x^{18} - 829696B^3x^{15} - 928256B^4x^{12} - 1555456B^5x^9 - 2809856B^6x^6 - 802816B^7x^3 + 65536B^8.\]
Consider \(\psi_7(x, B)\) as a binary form of degree 8 in \(x^3\) and \(4B\) with integer coefficients. Let \(\mathcal{X} = \frac{x^3}{(4B, x^3)}\) and \(\mathcal{B} = \frac{4B}{(4B, x^3)}\). With this notation we obtain
\[(3.8) \quad \Psi_7(\mathcal{X}, \mathcal{B}) = 7\mathcal{X}^8 + 986B\mathcal{X}^7 - 2681\mathcal{B}^2\mathcal{X}^6 - 12964\mathcal{B}^3\mathcal{X}^5 - 3626\mathcal{X}^4\mathcal{B}^4 - 1519\mathcal{X}^3\mathcal{B}^5 - 686\mathcal{X}^2\mathcal{B}^6 - 49\mathcal{X}\mathcal{B}^7 + \mathcal{B}^8.\]

**Lemma 3.15.** Let \(P = (a, b)\) be a non-torsion point on a quasi-minimal elliptic curve \(E_B\) such that \([7]P\) is integral. Let \(\mathcal{X} = \frac{a^3}{(a^3, 4B)}\) and \(\mathcal{B} = \frac{4B}{(a^3, 4B)}\). Then
\[\Psi_7(\mathcal{X}, \mathcal{B}) = \pm 3^\alpha 7^\gamma, \quad \text{where } \alpha \in \{0, 8, 12\} \text{ and } \gamma \in \{0, 1\}.\]

**Proof.** The proof parallels that of Lemma 3.12. Note that \(\gamma = 1\) if and only if \(7|\mathcal{B}\), and \(\alpha > 0\) if and only if 3 does not divide any of \(\mathcal{X}\) and \(\mathcal{B}\), which in this case yields \(\mathcal{X} \equiv -\mathcal{B} \pmod{3}\). \(\blacksquare\)
The difference between the cases \( m = 7 \) and \( m = 5 \) is that unlike \( \Psi_5(P) \), \( \Psi_7(P) \) is not irreducible over \( \mathbb{Q} \): we have

\[
\Psi_7(\mathcal{X}, \mathcal{B}) = \mathcal{F}_7(\mathcal{X}, \mathcal{B})\mathcal{G}_7(\mathcal{X}, \mathcal{B}),
\]

where

\[
\mathcal{F}_7(\mathcal{X}, \mathcal{B}) = \mathcal{X}^6 + 141\mathcal{X}^5\mathcal{B}^3 - 363\mathcal{X}^4\mathcal{B}^4 - 1924\mathcal{X}^3\mathcal{B}^5 - 741\mathcal{X}^2\mathcal{B}^6 - 48\mathcal{X}\mathcal{B}^7 + \mathcal{B}^6,
\]

\[
\mathcal{G}_7(\mathcal{X}, \mathcal{B}) = 7\mathcal{X}^2 - \mathcal{X}\mathcal{B} + \mathcal{B}^2.
\]

**Lemma 3.16.** Let \( P = (a, b) \) be a non-torsion integral point on a quasi-minimal elliptic curve \( E_B \) such that \( [7]P \) is integral. Then

\[
\mathcal{F}_7(\mathcal{X}, \mathcal{B}) = \pm 3^{\alpha_1}, \quad \mathcal{G}_7(\mathcal{X}, \mathcal{B}) = 3^{\alpha_2}7^\gamma,
\]

where \( \alpha_1 \) is non-zero if and only if \( \alpha_2 \) is non-zero, \( \alpha_1 \in \{0, 2, 3\} \), \( \alpha_1 + \alpha_2 \in \{0, 8, 12\} \), and \( \gamma \in \{0, 1\} \).

**Proof.** As \( 7x^2 - xy + y^2 \) is a positive definite binary form, \( \mathcal{G}_7(\mathcal{X}, \mathcal{B}) \) is always a positive integer. By Lemma 3.15, the only primes that might divide \( \mathcal{G}_7(\mathcal{X}, \mathcal{B}) \) are 3 and 7. Moreover, \( \text{ord}_7(\mathcal{F}_7(\mathcal{X}, \mathcal{B}) \cdot \mathcal{G}_7(\mathcal{X}, \mathcal{B})) \leq 1 \), and it is 1 if and only if \( 7 \mid \mathcal{B} \). But if \( 7 \mid \mathcal{B} \), it is clear that \( 7 \mid \mathcal{G}_7(\mathcal{X}, \mathcal{B}) \). Therefore, \( \text{ord}_7(\mathcal{G}_7(\mathcal{X}, \mathcal{B})) \leq 1 \) and \( 7 \mid \mathcal{F}_7(\mathcal{X}, \mathcal{B}) \).

Next, consider the 3-adic valuation of \( \mathcal{G}_7(\mathcal{X}, \mathcal{B}) \). By the argument in the proof of Lemma 3.15, we can assume \( \mathcal{X} \equiv -\mathcal{B} \not\equiv 0 \pmod{3} \), since otherwise \( \alpha_1 = \alpha_2 = 0 \). Therefore, \( \min(\text{ord}_3(\mathcal{X} + \mathcal{B}), \text{ord}_3(2\mathcal{X} - \mathcal{B})) = 1 \). From the equation

\[
\mathcal{G}_7(\mathcal{X}, \mathcal{B}) = (\mathcal{X} + \mathcal{B})^2 + 3\mathcal{X}(2\mathcal{X} - \mathcal{B}),
\]

we obtain \( \text{ord}_3(\mathcal{G}_7(\mathcal{X}, \mathcal{B})) \geq 2 \). If \( \text{ord}_3(\mathcal{X} + \mathcal{B}) > 1 \) or \( \text{ord}_3(2\mathcal{X} - \mathcal{B}) > 1 \), then \( \text{ord}_3(\mathcal{G}_7(\mathcal{X}, \mathcal{B})) = 2 \). If \( \text{ord}_3(\mathcal{X} + \mathcal{B}) = \text{ord}_3(2\mathcal{X} - \mathcal{B}) = 1 \), then by the equation

\[
\mathcal{G}_7(\mathcal{X}, \mathcal{B}) - 3(\mathcal{X} + \mathcal{B})(2\mathcal{X} - \mathcal{B}) = (\mathcal{X} - 2\mathcal{B})^2
\]

we have \( \text{ord}_3(\mathcal{G}_7(\mathcal{X}, \mathcal{B})) = 2 \) or 3. The last statement of the lemma follows from Lemma 3.15.

**Lemma 3.17.** Let \( P = (a, b) \) be a rational non-torsion point on a quasi-minimal Mordell curve \( E_B \). Then \( [7]P \) is not integral.

**Proof.** From Lemmas 3.15 and 3.16, pairs \( (P, E_B) \) of non-torsion points \( P \) on quasi-minimal Mordell curves \( E_B \) such that \( [7]P \) is integral are embedded in the set of non-trivial primitive solutions of the system

\[
\begin{cases}
x^6 + 141x^5y - 363x^4y^2 - 1924x^3y^3 - 741x^2y^4 - 48xy^5 + y^6 = \pm 3^{\alpha_1}, \\
7x^2 - xy + y^2 = 3^{\alpha_2}7^\gamma,
\end{cases}
\]

where \( (\alpha_1, \alpha_2) \in \{(0, 0), (5, 3), (6, 2), (9, 3), (10, 2)\} \) and \( \gamma = 0 \) or 1.

The only non-trivial primitive solutions are \([1, -1]\) and \([-1, 1]\), which correspond to the system defined by \((\alpha_1, \alpha_2, \gamma) = (6, 2, 0)\) with a positive
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sign in the first equation, and the solutions $[2, 1], [-2, -1], [1, -4]$ and $[-1, 4]$ correspond to the system defined by $(\alpha_1, \alpha_2, \gamma) = (9, 3, 0)$ with a negative sign in the first equation. None of these solutions lead to a non-torsion point on $E_B$.

In order to find the solutions to the above system, we can apply two different approaches. The first one is to consider the first equation as a Thue equation, determine the primitive solutions using Magma or PARI, and then check whether they satisfy the second equation. The other approach is to solve the second equation by using the algorithm described in [5, p. 75] and then check whether the solutions satisfy the first equation.

3.4. Points with more than two integral multiples

$[n]P$, $2 \leq n \leq 10$. So far, we have proved there is no point $P$ on a quasi-minimal Mordell curve $E_B$ with integral multiple $[n]P$, $6 \leq n \leq 10$. When $n = 5$, there are only two points $P$ with $[5]P$ integral. To complete the proof of Theorem 1.1 we proceed by investigating the points $P$ for which $[2]P$ and $[3]P$ are integral. Subsequently, we examine whether $[4]P$ is also integral. This approach allows us to identify all the possible points $P$ on quasi-minimal Mordell curves with more than two integral multiples $[n]P$, where $2 \leq n \leq 10$, in addition to the aforementioned ones. To begin, we determine all points that satisfy (3.1) and the conditions in Table 1 simultaneously. In Proposition 3.18 we will see that there are infinitely many pairs $(P, E_B)$ of points $P$ on quasi-minimal Mordell curves $E_B$ that satisfy both sets of conditions. We can categorize all these pairs and use (3.5) to examine if, for any one of them, $[4]P$ is integral. The following proposition summarizes this process. Note that the first two families correspond to $t = 2$ in (3.1).

**Proposition 3.18.** Let $M$, $N$, and $K$ be non-zero pairwise relatively prime integers such that $M$ and $K$ are odd, $3 \nmid NK$, and $M^3N^2K$ is sixth-power-free. Then all non-torsion points $P$ on quasi-minimal curves $E_B$ with $[2]P$ and $[3]P$ integral at the same time can be categorized as follows:

- The first family ($t = 2$ and $3 \mid M$):

\[
\begin{aligned}
P &= (2MN, \pm M^2N), \\
B &= M^3N^2K, \\
2N + K &= \pm 3, \\
8N + K &= M.
\end{aligned}
\]

\[
\begin{aligned}
x &= 6(2N + 1)N, \\
y &= \pm 9(2N + 1)^2N, \\
B &= 27(2N + 1)^3N^2(-2N + 3), \\
x([3]P) &= 16N^4 - 16N^3 - 12N^2 + 2N + 1.
\end{aligned}
\]

Considering (3.5), the only point in this family with $[4]P$ integral corresponds to $N = 2$. This yields the points $\pm P$ on $E_{-13500}$ where $P = (60, 450)$. The only integral multiples of these points are $\pm P$, $\pm [2]P$, $\pm [3]P$, and $\pm [4]P$.

In the above equations, $N \equiv 2 \pmod{3}$, and if $N \neq 2$, then $h([3]P) > 0.57 \log B$. Moreover, always $x([3]P) > 2|B|^{1/3}$.
The second family \((t = 2\) and \(\text{ord}_3(y(P)) > \text{ord}_3(x(P)^2))\):

\[
\begin{align*}
\begin{cases}
P = (2MN, \pm 3M^2N), \\
B = M^3N^2K, \\
2N + K = \pm 3, \\
8N + K = 9M.
\end{cases}
\end{align*}
\]

By checking (3.5), we obtain no non-torsion integral point \(P\) with \([4]P\) integral in this family. For \(P\) in this family, \(h([3]P) > 0.66 \log B\) and \(x([3]P) > 2|B|^{1/3}\).

The third family \((t = 1, \text{ord}_2(N) = 1, \text{and} \text{ord}_3(M) = 1)\):

\[
\begin{align*}
\begin{cases}
P = (MN, M^2N), \\
B = M^3N^2K, \\
N + 4K = \pm 6, \\
N + K = M.
\end{cases}
\end{align*}
\]

By the conditions on the 2-adic valuation of \(x(P)\) and \(B\), \([4]P\) cannot be integral. In these equations \(K \equiv 1 \pmod{6}\), and if \(N \neq 1\), \(h([3]P) > 0.59 \log B\). Moreover, if \(N \neq 1\), then \(x([3]P) > 2|B|^{1/3}\). Note that \(N = 1\) corresponds to \(E_{108}\).

The fourth family \((t = 1, \text{ord}_3(y(P)) > \text{ord}_3(x(P)^2), \text{and} \text{ord}_2(N) = 1)\):

\[
\begin{align*}
\begin{cases}
P = (MN, 3M^2N), \\
B = M^3N^2K, \\
N + 4K = \pm 6, \\
N + K = 9M.
\end{cases}
\end{align*}
\]

By the conditions on the 2-adic valuation of \(x(P)\) and \(B\), \([4]P\) cannot be integral. For \(P\) in this family, \(h([3]P) > 0.66 \log B\) and \(x([3]P) > 2|B|^{1/3}\).

The fifth family \((t = 1, \text{ord}_3(M) = 1, \text{and} \text{ord}_2(N) = 2)\):

\[
\begin{align*}
\begin{cases}
P = (MN, M^2N), \\
B = M^3N^2K, \\
N + 4K = \pm 24, \\
N + K = M
\end{cases}
\end{align*}
\]

By checking (3.5), we obtain no non-torsion integral point \(P\) with \([4]P\) integral. We can take \(K \equiv 1 \pmod{6}\). Taking \(K = 1\) gives rise to the point \(\pm P\) where \(P = (420, 8820)\) on \(E_{370440}\) where the only integral multiples are \(\pm [2]P\).
and \( \pm [3]P \). Taking \( K = 7 \) gives rise to the points \( \pm P \) where \( P = (-12, 36) \) on \( E_{3024} \) where the only integral multiples are \( \pm [2]P, \pm [3]P \). If \( K \neq 1, 7 \), then \( x([3]P) > 2B^{1/3} \) and \( h([3]P) > 0.44 \log |B| \).

- The sixth family (\( t = 1, \text{ord}_3(y(P)) > \text{ord}_3(x^2) \) and \( \text{ord}_2(N) = 2 \)):

\[
\begin{align*}
P &= (2MN, 3M^2N), \\
B &= M^3N^2K \\
N + 4K &= \pm 24, \\
N + K &= 9M
\end{align*}
\]

By checking (3.5), the only points in this family with \([4]P \) integral correspond to \( c = 0 \) and \( c = -1 \). This leads to the points \( \pm P \) where \( P = (4, 12) \) on \( E_{80} \) and \( Q = (84, 756) \) on \( E_{-21168} \). The only integral multiples of these points are \( \{\pm P, \pm [2]P, \pm [3]P, \pm [4]P \} \) and \( \{\pm Q, \pm [2]Q, \pm [3]Q, \pm [4]Q \} \). If \( C \neq 0, 1 \), then \( x([3]P) > 2B^{1/3} \) and \( h([3]P) > 0.51 \log |B| \).

**Remark 3.19.** Based on the above calculations, let \( P \) be a non-torsion integral point on \( E_B \) such that \([2]P \) and \([3]P \) are integral. If \([n]P \) is integral for some \( n > 10 \), then \( x([3]P) > 2B^{1/3} \) and \( h([3]P) > 0.44 \log |B| \).

**Corollary 3.20.** There are infinitely many pairs \((P, E_B)\) of non-torsion points \( P \) on quasi-minimal Mordell curves \( E_B \) such that \( P \) has at least two integral multiples \([n]P \) with \( n > 1 \).

**Proof.** By a theorem of Erdős [8], each of the families in Proposition 3.18 contains infinitely many sixth-power-free values of \( B \).

**Corollary 3.21.** The points \( P = (60, \pm 450) \) on \( E_{-13500} \), \( P = (4, \pm 12) \) on \( E_{80} \), and \( P = (84, \pm 756) \) on \( E_{-21168} \) are the only non-torsion points on quasi-minimal Mordell curves \( E_B \) that have more than two integral multiples \([n]P \) with \( 1 < n < 5 \).

**Remark 3.22.** All examples of points \( P \) mentioned in Theorem 1.1 with three integral multiples \([n]P \), \( n < 11 \), have exactly three integral multiples.

Theorem 1.1 follows from Lemmas 3.2, 3.6, 3.11, 3.13, 3.17, Corollaries 3.20 and 3.21, and Remark 3.22.

4. **Proof of Theorem 1.2.** This section follows Ingram’s techniques [11] to investigate integral multiples \([n]P \) when \( n > 10 \). First, we consider the case where \(|B| \leq 75 \). Among the 27 curves in this range with four or more non-torsion integral points, none of them possesses an integral point \( P \) with \([n]P \) integral for some \( n > 5 \). So, for the entirety of this section, we assume \( B > 75 \). From Theorem 1.1, any such integer \( n \) has no prime factor less than 11. Therefore, throughout this section, we assume \( n \) has no prime
factor less than 11. Under the assumption that \([n]P\) is integral, we use (2.2) to find an upper bound on the height of \(P\).

**Lemma 4.1.** Let \(n > 10\), and \(P = (a, b)\) be a point on a quasi-minimal Mordell curve \(E_B\) such that \([n]P\) is integral. Then

\[
|\psi_n(P)| < (3^{3/2}2^2 |B|)^{(n^2-1)/6}.
\]

**Proof.** By Theorem 1.1, \(n\) is odd and not divisible by 3. Therefore, by the argument following (2.2), for any point \(P = (a, b)\) on \(E_B\), \(\psi_n(P)\) is a binary form of degree \((n^2 - 1)/6\) in terms of \(a^3\) and \(4B\). The coefficient of \((a^3)(n^2-1)/6\) in \(\psi_n(P)\) is \(n\), and the coefficient of \((4B)(n^2-1)/6\) is \(\pm 1\). Recall that

\[
x([n]P) = \frac{\phi_n(P)}{\psi_n(P)}.
\]

Since \(x([n]P)\) is integral, \(\gcd(\psi_n(P), \phi_n(P)) = \psi_n(P)\). As we have seen before, it means that \(\psi_n(P) = \pm \prod_{p \mid 6B} p^{\gamma_p}\) for some non-negative integers \(\gamma_p\)'s. To bound \(|\psi_n(P)|\), we find upper bounds on the \(p\)-adic valuations of \(\psi_n(P)\) for various prime divisors \(p\) of \(6B\).

Let \(p > 3\) be a prime divisor of \(B\) and \(\psi_n(P)\). Since \(p \mid \phi_n(P)\) and \(\phi_n(P)\) is monic in \(a^3\), we have \(p \mid a\). As \(B\) is sixth-power-free, from the equation \(b^2 = a^3 + B\) we have \(\text{ord}_p(B) \leq \text{ord}_p(a^3)\). First assume \(\text{ord}_p(B) < \text{ord}_p(a^3)\). Considering \(\psi_n(P)\) as a binary form in \(a^3\) and \(4B\), the term with minimum \(p\)-adic valuation is \((\pm 4B)^{(n^2-1)/6}\). Hence, in this case, \(\text{ord}_p(\psi_n(P)) = \frac{n^2 - 1}{6} \text{ord}_p(B)\). Next, assume \(\text{ord}_p(a^3) = \text{ord}_p(B) = 3\). Let \(a_1 = a^3/p^3\) and \(B_1 = B/p^3\). Since \(\text{ord}_p(a^3 + B)\) is even, we have \(a_1 \equiv -B_1 \pmod{p}\). Therefore, \(\psi_n(P)/p^{3(n^2-1)/6} \equiv a_1^{(n^2-1)/6} \psi_n(1, -1) \pmod{p}\). But, \(\psi_n(1, -1)\) is a power of 3 (see [12]). Hence, \(\text{ord}_p(\psi_n(P)) = \frac{n^2 - 1}{6} \text{ord}_p(B)\).

To evaluate \(\text{ord}_2(\psi_n(P))\), again consider \(\psi_n(P)\) as a binary form in \(4B\) and \(a^3\) of degree \((n^2 - 1)/6\). The leading term of \(\psi_n(P)\) in terms of \(a^3\) is \(n\), and it is 1 in terms of \(4B\). If \(\text{ord}_2(a^3) \neq \text{ord}_2(4B)\), then \(\text{ord}_2(\psi_n(P)) = \frac{n^2 - 1}{6} (\min(\text{ord}_2(a^3), \text{ord}_2(4B)))\). If \(\text{ord}_2(a^3) = \text{ord}_2(4B) = 6\), then by induction and the recursive formula in (2.1) we have

\[
\text{ord}_2(\psi_n(P)) \begin{cases} 
= n^2 - 1 & \text{if } 3 \nmid n, \\
\geq n^2 & \text{if } 3 \mid n.
\end{cases}
\]

Finally, we find \(\text{ord}_3(\psi_n(P))\). If \(3\) divides only one of \(a\) or \(B\), or if \(3\) divides both of them but \(\text{ord}_3(a^3) \neq \text{ord}_3(B)\), then since \(3 \nmid n\) and \(\psi_n(a^3, B)\) is a binary form in \(a^3\) and \(B\), by the same argument as above we have \(\text{ord}_3(\psi_n(P)) = \frac{n^2 - 1}{6} \min(\text{ord}_3(a^3), \text{ord}_3(B))\). For the remaining cases, we use simple induction and (2.1) to calculate the 3-adic valuation of \(\psi_n(P)\).
Multiples of integral points on Mordell curves

• If $3 \nmid aB$, then
  $$\text{ord}_3(\psi_n(P)) \begin{cases} 
  \geq 0 & \text{if } 3 \mid n, \\
  = 0 & \text{otherwise}. 
  \end{cases}$$

• If $3 \nmid aB$ and $3 \mid b$, then
  $$\text{ord}_3(\psi_n(P)) \begin{cases} 
  = \frac{n^2-1}{4} & \text{if } n \text{ is odd,} \\
  \geq \frac{n^2}{4} & \text{if } n \text{ is even.} 
  \end{cases}$$

• If $\text{ord}_3(a^3) = \text{ord}_3(B) = 3$ and $\text{ord}_3(b) = 2$, then
  $$\text{ord}_3(\psi_n(P)) \begin{cases} 
  \geq \frac{2}{3}n^2 & \text{if } 3 \mid n, \\
  = \frac{2}{3}(n^2-1) & \text{if } 3 \nmid n. 
  \end{cases}$$

• If $\text{ord}_3(a^3) = \text{ord}_3(B) = 3$ and $\text{ord}_3(b) > 2$, then
  $$\text{ord}_3(\psi_n(P)) \begin{cases} 
  \geq \frac{3}{4}n^2 & \text{if } n \text{ is even,} \\
  = \frac{3}{4}(n^2-1) & \text{if } n \text{ is odd.} 
  \end{cases}$$

This completes the proof. □

The next step to fully exploit equation (2.2) is to find an upper bound on $|x(Q)|$ when $Q$ is a non-identity torsion point of order dividing $n$ on $E(\mathbb{C})$. David [4] determined a bound in the general case, but for a better result we use the idea of [11].

**Lemma 4.2.** Let $Q$ be a non-identity, torsion point of order dividing $n \geq 11$ on $E_B(\mathbb{C})$. Then $|x(Q)| < \left(\frac{n^2}{7}\right)|B|^{1/3}$.

**Proof.** We know that all curves $E_B$ are isomorphic over $\mathbb{C}$. Recall the isomorphism

$$E_B(\mathbb{C}) \cong E_1(\mathbb{C}), \quad (x, y) \mapsto (xB^{-1/3}, yB^{-1/2}).$$

Therefore, it is sufficient to prove the lemma for $E_1$, which has complex multiplication; the corresponding lattice is $\Lambda = \omega \mathbb{Z}[\rho]$, where $4.2065 < \omega < 4.2066$ is the real period of $E_1$ and $\rho = e^{2\pi i/3}$. Recall that (for any lattice $\Lambda \subset \mathbb{C}$) the related Weierstrass $\wp$-function is defined by

$$\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda, \omega \neq 0} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

and there is an isomorphism

$$\mathbb{C}/\Lambda \to E_1(\mathbb{C}), \quad z \mapsto (\wp(z; \Lambda), \frac{1}{2}\wp'(z; \Lambda)).$$
Choose a representative for $z$ in $\mathbb{C}/\Lambda$ such that $|\text{Re}(z)|, |\text{Im}(z)| \leq \omega/2$. Then $|u - z| \geq |u|/2$ for all $u \in \Lambda$. Therefore,

$$\left| \sum_{\substack{u \in \Lambda \\ u \neq 0}} \left( \frac{1}{(u - z)^2} - \frac{1}{u^2} \right) \right| \leq 2|z| \sum_{\substack{u \in \Lambda \\ u \neq 0}} \frac{4}{|u|^3} + |z|^2 \sum_{\substack{u \in \Lambda \\ u \neq 0}} \frac{4}{|u|^4}.$$ 

We also recall that (for any lattice) if $\sigma$ is the minimum distance between points in $\Lambda$ and $c = 8\pi/\sigma^2$, then for any $k > 2$,

$$\sum_{\substack{\omega \in \Lambda \\ |\omega| \geq 1}} \frac{1}{|\omega|^k} \leq c\zeta(k - 1).$$ 

Therefore,

$$|\wp(z; A)| < |z|^{-2} + 8|z|^2 \frac{2\pi}{|\omega^2|} \zeta(2) + 4|z|^2 \frac{2\pi}{\omega^2} \zeta(3).$$

Since $|z| < \omega/\sqrt{2}$ and $4.2065 < \omega < 4.2066$, we obtain

$$|\wp(z; A)| < z^{-2} + 9.2325.$$ 

If $z$ is a non-zero torsion point of order dividing $n$, then $|z| > \omega/n$. Hence,

$$|\wp(z; \lambda)| < n^2/\omega^2 + 9.2325.$$ 

Since we assumed $n \geq 11$, we have

$$|\wp(z; A)| < n^2/7,$$

and the result follows from the first isomorphism mentioned in the proof. ■

Lemmas 4.2 and 4.1 enable us to bound the height of integral points $P$ for which $[n]P$ is integral.

**Lemma 4.3.** Let $P$ be a non-torsion rational point on a quasi-minimal Mordell curve $E_B$ such that $[n]P$ is integral for some integer $n \geq 11$. Then

$$\hat{h}(P) < \begin{cases} \log n + \frac{1}{6} \log B - 0.617 & \text{if } B < 0, \\ \log n + \frac{1}{3} \log B - 0.597 & \text{if } B > 0. \end{cases}$$

**Proof.** Suppose $[n]P$ is integral. By (2.2) and Lemma 4.1, we have

$$\min_{[n]Q=O \neq O} |x(P) - x(Q)|^{n^2 - 1} < 3^{3/2} \cdot 4 \cdot B^{(n^2 - 1)/3}.$$ 

From the bound on $x(Q)$ in Lemma 4.2 and the assumption $n \geq 11$, we obtain

$$|x(P)| < \frac{n^2}{6} |B|^{1/3}, \quad \text{so} \quad h(P) < 2 \log n + \frac{1}{3} \log |B| - \log 6.$$ 

The result follows from (2.4). ■
This upper bound will be crucial in our next steps. As the first application, we prove the following lemma:

**Lemma 4.4.** Let $P$ be a non-torsion point on a quasi-minimal Mordell curve $E_B$ such that $[n]P$ is integral for some integer $n > 10$. Then $n$ is prime.

**Proof.** Assume, to the contrary, that $n = q_1 q_2$ where $1 < q_1 \leq q_2$. Then $[q_1]([q_2]P)$ is integral, so by Lemma 4.3, we have
\[
\hat{h}([q_2]P) < \log q_1 + \frac{1}{3} \log B - 0.597.
\]
On the other hand, $\hat{h}([q_2]P) > q_2^2 \hat{h}(P) > q_1^2 \hat{h}(P)$, so we have
\[
q_2^2 \left( \frac{1}{36} \log B - C \right) < \log q_1 + \frac{1}{3} \log B - 0.597,
\]
where the value $C$ depends on the congruence conditions on $B$ and is given in Lemma 2.1. By Theorem 1.1, $q_1 \geq 11$, which means that
\[
|B| < \begin{cases} 
15283 & \text{if } B \equiv 15120, 3024 \text{ or } 1296 \text{ (mod 15552)}, \\
395 & \text{if } B \equiv 80 \text{ or } 208 \text{ (mod 576)}, \\
395 & \text{if } B \equiv 13392 \text{ or } 9936 \text{ (mod 15552)}, \\
152 & \text{if } B \equiv 6372, 2052 \text{ or } 324 \text{ (mod 7776)}, \\
152 & \text{if } B \equiv 108 \text{ or } 540 \text{ (mod 3888)}, \\
152 & \text{if } B \equiv 1809 \text{ or } 297 \text{ (mod 1944)}, \\
152 & \text{if } B \equiv 144 \text{ (mod 1728)}, \\
11 & \text{otherwise.}
\end{cases}
\]

There are only a few possible values of $B$ in each of the above cases. We have already checked integral points on $E_B$ corresponding to most of these values in Lemma 3.1. The integral points $P$ on these curves have no integral multiple $[n]P$ where $n > 5$. For the few remaining values, we applied the same method used in Lemma 3.1 to investigate integral multiples of rational points on $E_B$. Similarly, for the integral points $P$ on these curves, we found no integral multiple $[n]P$, where $n > 5$.

**4.1. Linear forms in elliptic logarithms and upper bound on $n$.**

We use the theory of linear forms in elliptic logarithms to derive an upper bound on $n$ such that $[n]P$ is integral. Assume $[n]P$ is integral. Let $\omega$ be the real period of the elliptic curve $E$, and $z$ be the principal value of the elliptic logarithm of $P$ (the point in related fundamental domain such that $P = (\wp(z), \frac{1}{2} \wp'(z))$). Determine $m$ such that the linear form
\[
L_{N,m}(z, \omega) = nz + m\omega
\]
is the principal value of the elliptic logarithm $[n]P$. The following lemma [22] gives an upper bound for this linear form.
Lemma 4.5. Let $Q$ be a rational point on a Mordell curve $E_B$, and $z$ be the principal value of the elliptic logarithm of $Q$. If $x(Q) \geq 2\sqrt[3]{|B|}$, then
\[ \log |z| \leq \frac{3}{2} \log 2 - \frac{1}{3} \log |x(Q)|. \]

As immediate consequences, we have the following lemmas:

Lemma 4.6. Let $P$ be a non-torsion point on a quasi-minimal Mordell curve $E_B$. Suppose $[n]P$ is integral for some $n \geq 11$, and let $L_{n,m}(z, \omega) = nz + m\omega$ be the principal value of the elliptic logarithm of $[n]P$. Then
\[ \log |L_{n,m}(z, \omega)| < -n^2\left(\frac{1}{36} \log |B| - C\right) + \frac{1}{6} \log |B| + 1.339, \]
where $C$ depends on the congruence conditions on $B$ and is given in Lemma 2.1.

Proof. To apply Lemma 4.5 to $[n]P$, first, we will show that $x([n]P) > 2|B|^{1/3}$. Since $n \geq 11$,
\[ \hat{h}([n]P) \geq 121\left(\frac{1}{36} \log |B| - C\right). \]
Hence, by inequalities (2.4),
\[ h([n]P)/2 > 121\left(\frac{1}{36} \log |B| - C\right) - \frac{1}{6} \log |B| - 0.299 \]
if $h([n]P) < \log(2|B|^{1/3})$. Comparing the upper bound and the lower bound of $h([n]P)$, we obtain
\[
|B| < \begin{cases} 26736 & \text{if } B \equiv 15120, 3024 \text{ or } 1296 \text{ (mod } 15552), \\ 440 & \text{if } B \equiv 80 \text{ or } 208 \text{ (mod } 576), \\ 440 & \text{if } B \equiv 13392 \text{ or } 9936 \text{ (mod } 15552), \\ 150 & \text{if } B \equiv 6372, 2052 \text{ or } 324 \text{ (mod } 7776), \\ 150 & \text{if } B \equiv 108 \text{ or } 540 \text{ (mod } 3888), \\ 150 & \text{if } B \equiv 1809 \text{ or } 297 \text{ (mod } 1944), \\ 150 & \text{if } B \equiv 144 \text{ (mod } 1728), \\ 8 & \text{otherwise.} \end{cases}
\]
But, by the same method as in the proof of Lemma 3.2, for none of the values $B$ above does $E_B$ have a point $P$ with $[n]P$ integral, with $n > 5$. Therefore, $h([n]P) > \log(2|B|^{1/3})$. On the other hand, since $[n]P$ is integral, $h([n]P) = \log |x(P)|$. Since $[n]P$ is a point on $E_B$, $x(P) > -|B|^{1/3}$. Therefore, $x([n]P) > 2|B|^{1/3}$. We can now apply Lemma 4.5 to $[n]P$ to obtain the result.

Lemma 4.7. Let $P$ be a non-torsion point on a quasi-minimal Mordell curve $E_B$. Suppose $[4]P$ is integral, and let $L_{4,m}(z, \omega) = 4z + m\omega$ be the principal value of the elliptic logarithm of $[4]P$. Then
\[ \log |L_{4,m}(z, \omega)| < -\frac{1}{3} \log |B| + 0.921. \]
Proof. From (3.6), we have \( h([4]P) > \frac{2}{3} \log |B| - 0.239 \). For \( |B| > 16 \) we obtain \( x([4]P) > 2|B|^{1/3} \). The curves \( E_B \) with \( |B| \leq 16 \) have no non-torsion point \( P \) with \([4]P \) integral. The result follows from Lemma 4.5. 

Combining Remark 3.19 and Lemma 4.5 we obtain the following lemma:

**Lemma 4.8.** Let \( P \) be a non-torsion point on a quasi-minimal Mordell curve \( E_B \). Assume \([2]P \) and \([3]P \) are integral, and \( L_{3,m}(z, \omega) = 3z + m\omega \) is the principal value of the elliptic logarithm of \([3]P \). Then

\[
\log |L_{3,m}(z, \omega)| < -0.22\log |B| + \frac{3}{2} \log 2.
\]

Now that we have established an upper bound for \( \log |L_{n,m}(z, \omega)| \), we prove a lower bound by applying David’s lower bounds on linear forms in elliptic logarithms [11].

**Lemma 4.9.** Let \( E/\mathbb{Q} \) be an elliptic curve, and let \( \omega \) and \( \omega' \) be the real and complex periods of \( E \), chosen such that \( \tau = \omega'/\omega \) is in the fundamental region

\[ \{ z \in \mathbb{C} : |z| \geq 1, \text{Im}(z) > 0, \text{and } |\text{Re}(z)| \leq 1/2 \} \]

of the action of \( \text{SL}_2(\mathbb{Z}) \) on the upper half-plane. Let \( P, z, \) and \( L_{n,m} \) be defined as in Lemma 4.6, and let \( B, V_1, \) and \( V_2 \) be positive real numbers chosen such that

\[
\log V_2 \geq \max \left\{ h(E), \frac{3\pi}{\text{Im}(\tau)} \right\},
\]

\[
\log V_1 \geq \max \left\{ 2h(P), h(E), \frac{3\pi|z|^2}{|\omega|^2 \text{Im}(\tau)}, \log V_2 \right\},
\]

\[
\log B \geq \max \{ \text{eh}(E), \log |n|, \log |m|, \log V_1 \}.
\]

Then either \( L_{n,m}(z, \omega) = 0 \), or else

\[
\log |L_{n,m}(z, \omega)| \geq -C(\log B + 1)(\log \log B + h(E) + 1)^3 \log V_1 \log V_2
\]

where \( C \) is some large absolute constant (we may take \( C = 4 \cdot 10^{41} \)).

Note that \( L_{n,m}(z, \omega) = 0 \) if \( P \) is a torsion point.

Let \( P \) be a non-torsion point on a quasi-minimal Mordell curve \( E_B \) such that \([n]P \) is integral for some \( n \geq 11 \). Let \( L_{n,m}(z, \omega) \) be the principal value of the elliptic logarithm of \([n]P \). Assume that \( |B| > 8 \) so that \( h(E) = \log V_2 = \log(4|B|) \). We derive a lower bound on \( \log |L_{n,m}(z, \omega)| \) by applying the above lemma. To simplify the outcome bound, we consider several cases:

Assume \( 2 \log n < \frac{1}{3} \log |B| \). Then we can take

\[
\log V_2 = \log(4|B|), \quad \log V_1 = \log(4|B|), \quad \log B = e \log(4|B|).
\]

So we obtain

\[
(4.1) \quad \log |L_{n,m}(z, \omega)| > -5.944 \cdot 10^{42} \log(4|B|)^6.
\]
Assume \( \frac{1}{3} \log |B| < 2 \log n < (e - \frac{2}{3}) \log |B| \). Then we can take
\[
\log V_2 = \log (4|B|), \quad \log V_1 = e \log (|B|) - 1.199, \quad \log B = e \log (4|B|).
\]
So we obtain
\[
\log |L_{n,m}(z, \omega)| > -5.562 \cdot 10^{42} \log (4|B|)^6.
\]
Assume \( (e - \frac{2}{3}) \log |B| < 2 \log n \). Then we can take
\[
\log V_2 = \log (4|B|), \quad \log V_1 = \frac{6e}{3e - 2} \log n - 1.199,
\]
\[
\log B = e \left( \frac{6}{3e - 2} \log n + \log 4 \right).
\]
So we obtain
\[
\log |L_{n,m}(z, \omega)| > -4.144 \cdot 10^{42} (\log n)^5 \log (4|B|).
\]

**Lemma 4.10.** Let \( P \) be a non-torsion point on a quasi-minimal curve \( E_B \). Assume \([n]P\) is integral. Then
\[ n < \max \{ 6 \cdot 10^{22} \cdot (\log |B|)^{5/2}, 7.511 \cdot 10^{26} \}.
\]

**Proof.** We can assume \( |B| > 75 \) and \( n > 10 \). If \( 2 \log n < (e - \frac{2}{3}) \log |B| \), then by Lemma 4.6 and inequalities (4.1) and (4.2), we have
\[ n^2 \left( \frac{1}{36} \log |B| - C \right) < 5.944 \cdot 10^{42} \log (4|B|)^6 + \frac{1}{6} \log |B| + 1.339.
\]
For various values of \( C \) in (2.1), we check all the curves \( E_B \) where \( \log (4|B|)/(\frac{1}{36} \log |B| - C) > 150 \). None of these curves has a point \( P \) such that \([n]P\) is integral for some \( n > 5 \). In this case, if we assume \( \log (4|B|)/(\frac{1}{36} \log |B| - C) < 150 \), we conclude that \( n < 6 \cdot 10^{22} \cdot (\log |B|)^{5/2} \).

If \( (e - \frac{2}{3}) \log |B| < 2 \log n \) then by Lemma 4.6 and inequality (4.3), we have
\[ n^2 \left( \frac{1}{36} \log |B| - C \right) < 4.144 \cdot 10^{42} (\log n)^5 \log (4|B|) + \frac{1}{6} \log |B| + 1.339,
\]
and with the same argument as above, we conclude that \( n < 7.511 \cdot (10)^{26} \).

The next step is to prove a gap principle between larger values of \( n \) where \([n]P\)'s are integral.

**Lemma 4.11.** Let \( P \) be a non-torsion point on a quasi-minimal Mordell curve \( E_B \), and \( z \) be the principal value of the elliptic logarithm of the integral point \( P \). Assume \([n]P\) is integral and \( nz + m \omega \) is the principal value of \([n]P\). If \( n \geq 11 \), then \( m \neq 0 \).

**Proof.** Again, the idea is to bound the elliptic logarithm of the point \([n]P\), this time assuming \( m = 0 \). If \( m = 0 \), then \( \log nz = \log z + \log n \). Therefore, from Lemma 4.6, we have
\[ n^2 \left( \frac{1}{36} \log |B| - C \right) - \frac{1}{6} \log |B| - 1.339 < - \log |z| - \log n; \]
on the other hand,

\[ - \log |z| = - \log \left| \frac{1}{2} \int_{x_p}^{\infty} \frac{dt}{\sqrt{t^3 - N^2 t}} \right| \leq \frac{3}{2} \log 2 + \frac{1}{2} \log \max \{|x(P)|, 2|B|^{1/3}\}. \]

Assume \( x(P) > 2|B|^{1/3} \). Then by comparing the bound obtained in the proof of (4.3) with the lower bound of \( \log(nz) \), we get

\[ n^2 \left( \frac{1}{36} \log |B| - C \right) < \frac{1}{3} \log |B| + \frac{3}{2} \log 2 - \frac{1}{2} \log 6 + 1.339. \]

This means that

\[ |B| < \begin{cases} 14035 & \text{if } B \equiv 15120, 3024 \text{ or } 1296 \pmod{15552}, \\ 363 & \text{if } B \equiv 80 \text{ or } 208 \pmod{576}, \\ 363 & \text{if } B \equiv 13392 \text{ or } 9936 \pmod{15552}, \\ 139 & \text{if } B \equiv 6372, 2052 \text{ or } 324 \pmod{7776}, \\ 139 & \text{if } B \equiv 108 \text{ or } 540 \pmod{3888}, \\ 139 & \text{if } B \equiv 1809 \text{ or } 297 \pmod{1944}, \\ 139 & \text{if } B \equiv 144 \pmod{1728}, \\ 10 & \text{otherwise.} \end{cases} \]

There are only a few possible values of \( B \) in each of the above cases. For these values, we have already investigated the multiples of integral points on the corresponding curves \( E_B \) in Lemma 4.4. As we have seen, integral points \( P \) on these curves have no integral multiples \([n]P \) with \( n > 5 \).

If \( x(P) < 2|B|^{1/3} \), then

\[ n^2 \left( \frac{1}{36} \log |B| - C \right) < \frac{1}{3} \log |B| + 2 \log 2 - \log n + 1.339. \]

This is again impossible for \( n \geq 11 \). \( \blacksquare \)

Now we are ready to prove a gap principle

**Lemma 4.12.** Let \( n_1 < n_2 \) be two integers greater than 10. Assume \( P \) is a non-torsion point on a quasi-minimal Mordell curve \( E_B \) such that both \([n_1]P \) and \([n_2]P \) are integral. Then

\[ n_1^2 \left( \frac{1}{36} \log |B| - C \right) + \log \omega_1 - \frac{1}{3} \log |B| - 1.399 < \log n_2, \]

where \( C \) is as defined in Lemma 2.1 and \( \omega_1 \) is the real period of \( E_1 \).

**Proof.** Let \( L_{n_1,m_1}(z, \omega_B), L_{n_2,m_2}(z, \omega_B) \) be the principal values of the elliptic logarithms of \([n_1]P \) and \([n_2]P \). We have

\[ \omega |n_2 m_1 - n_1 m_2| \leq n_2 |n_1 z + m_1 \omega| + n_1 |n_2 z + m_2 \omega|. \]

Assume \( n_2 m_1 - n_1 m_2 = 0 \). Then since \( n_2 \) and \( m_2 \) are non-zero, we have \( n_2 \mid m_2 n_1 \). As \( n_1 \) and \( n_2 \) are both prime, we have \( n_2 \mid m_2 \), but it is easy to
see that $m_2 \leq \frac{1}{2}(n_2 + 1)$. Since by Lemma 4.11, $m_2 \neq 0$, it follows that $|n_2m_1 - n_1m_2| \geq 1$. Therefore, from Lemma 4.6 we obtain
\[
|B|^{-1/6} \omega_1 \leq n_2 \exp(-n_1^2(\frac{1}{36} \log |B| - C) + \frac{1}{6} \log |B| + 1.339) \\
\quad + n_1 \exp(-n_2^2(\frac{1}{36} \log |B| - C) + \frac{1}{6} \log |B| + 1.339).
\]
With the assumption $|B| > 75$ and the congruence relations, we can see that the first summand on the left-hand side is greater than the second one, and we obtain
\[
\frac{1}{2} |B|^{-1/6} \omega_1 \leq n_2 \exp(-n_1^2(\frac{1}{36} \log |B| - C) + \frac{1}{6} \log |B| + 1.339).
\]
The results follow by taking logarithms of both sides. □

**Remark 4.13.** Note that in the proof, we only used $m_2 \neq 0$, so the above argument also works when $m_1 = 0$.

We are now ready to prove Theorem 1.2. Let $P$ be a non-torsion point on a quasi-minimal Mordell curve $E_B$ such that both $[n_1]P$ and $[n_2]P$ are integral with $n_1, n_2 \geq 11$. Combining Lemma 4.10 with the gap principle in Lemma 4.12, we obtain
\[
(4.5) \quad n_1^2(\frac{1}{36} \log |B| - C) + \log \omega_1 - \frac{1}{3} \log |B| - 1.399 < \log n_2 \\
\quad < \log \max \{7.511 \cdot 10^{26}, 6 \cdot 10^{22}(\log |B|)^{5/2}\}.
\]
If $n_1 \geq 29$, then we have
\[
|B| < \begin{cases} 
56776 & \text{if } B \equiv 15120, 3024 \text{ or } 1296 \pmod{15552}, \\
2791 & \text{if } B \equiv 80 \text{ or } 208 \pmod{576}, \\
2791 & \text{if } B \equiv 13392 \text{ or } 9936 \pmod{15552}, \\
846 & \text{if } B \equiv 6372, 2052 \text{ or } 324 \pmod{7776}, \\
846 & \text{if } B \equiv 108 \text{ or } 540 \pmod{3888}, \\
846 & \text{if } B \equiv 1809 \text{ or } 297 \pmod{1944}, \\
846 & \text{if } B \equiv 144 \pmod{1728}, \\
71 & \text{otherwise}.
\end{cases}
\]

As we have already checked the case $|B| < 75$, any of the above conditions holds only for a few $B$ values. With a similar argument to the proof of Lemma 3.2, Based on the database in [2] none of the corresponding Mordell curves $E_B$ has an integral point $P$ with $[n]P$ integral and $n > 5$. This completes the proof of Theorem 1.2.

With exactly the same argument, for $n_1 = 11$, we can also prove the following lemma for larger values of $|B|$: 

**Lemma 4.14.** Let $|B| > 6.2675 \cdot 10^{12}$, and $P$ be a non-torsion point on a quasi-minimal Mordell curve $E_B$. Then there is at most one value $n > 10$ such that $[n]P$ is integral.
5. **Proof of Theorem 1.3.** Let \( P \) be a torsion point on a quasi-minimal curve \( E_B \). In this section, we will show that \( P \) has at most three integral multiples \([n]P\) with \( n > 1 \). So far, we have talked about the number of integral multiples of \( P \) when \( n \leq 10 \) (Theorem 1.1), when \( n \geq 29 \) (Theorem 1.2), and also when \( n \geq 11 \) for larger values of \( B \) (Lemma 4.14).

For values \( 10 < n < 29 \), we prove the following lemma.

**Lemma 5.1.** Let \( P \) be a non-torsion point on a quasi-minimal curve \( E_B \). Then \([n]P\) is integral for at most one value \( 10 < n < 29 \).

**Proof.** Assume \([n_1]P\) and \([n_2]P\) are integral for \( 10 < n_1 < n_2 < 29 \). By Lemma 4.4, \( n_1 \) and \( n_2 \) are both prime so they are both less than 24. We can apply the gap principle in Lemma 4.12 for \( n_1 \) and \( n_2 \) to obtain

\[
121 \left( \frac{1}{36} \log |B| - C \right) + \log \omega_1 - \frac{1}{3} \log |B| - 1.399 < \log 23.
\]

Therefore if both \([n_1]P\) and \([n_2]P\) are integral, we have

\[
|B| < \begin{cases} 
23456 & \text{if } B \equiv 15120, 3024 \text{ or } 1296 \pmod{15552}, \\
606 & \text{if } B \equiv 80 \text{ or } 208 \pmod{576}, \\
606 & \text{if } B \equiv 13392 \text{ or } 9936 \pmod{15552}, \\
233 & \text{if } B \equiv 6372, 2052 \text{ or } 324 \pmod{7776}, \\
233 & \text{if } B \equiv 108 \text{ or } 540 \pmod{3888}, \\
233 & \text{if } B \equiv 1809 \text{ or } 297 \pmod{1944}, \\
233 & \text{if } B \equiv 144 \pmod{1728}, \\
16 & \text{otherwise}.
\end{cases}
\]

But we have already seen that non-torsion points on the quasi-minimal Mordell curves \( E_B \) with the above values of \( B \) do not have any integral multiples \([n]P\) with \( n > 5 \).

To summarize, by Theorems 1.1 and 1.2 and Lemmas 4.4, 4.14 and 5.1, Theorem 1.3 holds in the following cases: (I) \(|B| > 6.2675 \cdot 10^{12}\), (II) \( P \) has either fewer or more than two integral multiples \([n]P\) with \( 2 \leq n \leq 10 \), (III) \( P \) has no integral multiple \([n]P\) with \( 11 \leq n \leq 23 \). Therefore, to complete the proof, it is sufficient to prove that if \( P \) is a non-torsion point on a quasi-minimal Mordell curve \( E_B \) with \(|B| < 6.2675 \cdot 10^{12}\) and two integral multiples \([n]P\), \( 1 < n < 11 \), then \([n]P\) is not integral for any \( 11 \leq n \leq 23 \).

By Theorem 1.1 and Lemma 3.13, if \( P \) has two integral multiples \([n]P\) with \( 1 < n < 11 \), then either \([2]P\) and \([3]P\) are integral or \([4]P\) is integral.

Assume \([2]P\) and \([3]P\) are integral. Recall Proposition 3.18 in which we categorized all the points \( P \) with \([2]P\) and \([3]P\) integral. For each family of quasi-minimal curves \( E_B \) mentioned in Proposition 3.18, there are only finitely many curves \( E_B \) with \(|B| < 6.2675 \cdot 10^{12}\). In each family, we determine the points \( P \) on these curves with \([2]P\) and \([3]P\) integral, and calculate
\([nP \text{ for } n = 11, 13, 17, 19, 23.} \) (By Lemma 4.4, we only need to check prime values of \(n\)). We have used Sage to perform these calculations. Based on our calculations, there is no integral multiple among the calculated points.

Finally, assume \([4]P\) is integral and \(|B| < 6.2675 \cdot 10^{12}\). All the points with \([4]P\) integral can be determined by solving some Pell equations. To be more precise, we look for solutions corresponding to the Pell equations associated with equations in (3.5), with the restriction \(|B| < 6.2675 \cdot 10^{12}\). All the non-torsion integral points on the quasi-minimal curve \(E_B\) with this restriction are given in Table 2. None of these points has an integral multiple \([n]P\) with a prime integer \(n\), \(11 \leq n \leq 23\). This completes the proof of Theorem 1.3.

**Table 2.** Integral points \(P\) on elliptic curves \(E_B\) with \([4]P\) integral and \(|B| < 6.2675 \cdot 10^{12}\).

| \(B\)            | \(P\)              |
|------------------|---------------------|
| −13500           | [60, 450]           |
| −21168           | [84, 756]           |
| −2743600         | [380, 7220]         |
| 80               | [4, 12]             |
| −1124695         | [286, 4719]         |
| 59400            | [60, 900]           |
| −83338860528     | [11928, 1270332]    |
| 513              | [6, 27]             |
| −197137217456    | [15908, 1956684]    |
| 30371652         | [228, 6498]         |
| 74088784         | [308, 10164]        |
| 7301384400       | [1420, 100820]      |
| 5322709227600    | [12780, 2722140]    |
| 3086626985       | [1066, 6559]        |

**Remark 5.2.** An alternative of proving Theorem 1.3 is to use the gap principle (Lemma 4.12) and Lemmas 4.7 and 4.8 to find a lower bound on the smallest integral multiple greater than 4 of \(P\), assuming \([4]P\) or \([2]P\) and \([3]P\) are integral. One can prove such a result for values \(B\) greater than some constant and treat the remaining cases as above.

**6. Integral points on Mordell curves of rank 1**

**6.1. Torsion-free quasi-minimal Mordell curves.** Let \(E_B\) be a torsion-free quasi-minimal Mordell curve of rank 1. From Theorem 1.3, it is clear that \(E_B\) has at most eight integral points. This bound is sharp in the sense that all the curves \(E_{108}, E_{80}, E_{−13500},\) and \(E_{−21168}\) mentioned in Theorem 1.1 are quasi-minimal Mordell curves of rank 1 with eight integral
points. However, as we will explain, it is plausible that these are the only torsion-free rank 1 quasi-minimal Mordell curves with more than six integral points. Let $B$ be a sixth-power-free integer not equal to 108, 80, −13500, or −21168, and $E_B$ be a torsion-free Mordell curve of rank 1 with more than six integral points. Let $P$ be a generator of $E_B$’s Mordell–Weil group. Then $P$ has three integral multiples $[n]P$ with $n > 1$. Let $N$ be the largest integer such that $[n]P$ is integral. By Theorems 1.1, 1.2, Lemmas 4.14, 5.1, and the argument in the proof of Theorem 1.3, $N \geq 29$ if $|B| < 6.2675 \cdot 10^{12}$, and $N \geq 11$ otherwise. Since $[N]P$ is integral, from inequalities (2.3) and (2.4) we obtain

\[
\frac{1}{2} \log(x([N]P)) > N^2 \left( \frac{1}{36} \log |B| - 0.2262 \right) - \frac{1}{6} \log |B| - 0.299.
\]

On the other hand, regarding upper bounds on the height of integral points on Mordell curves, we have the following well-known conjecture of Marshal Hall.

**Conjecture 6.1 (Hall).** Given $\epsilon > 0$, there exists a positive constant $C_\epsilon$ such that

\[
|x(P)| < C_\epsilon |B|^{2+\epsilon}
\]

for all integral points $P$ on the Mordell curve $E_B$.

In [2] the authors list all the Mordell curves encountered with Hall measure $x(P)^{1/2}/|B|$ exceeding 1, with the largest value being less than 1.41. There are some lists of integral points on Mordell curves with relatively large value of $x(P)^{1/2}/|B|$ [7, 13]. The largest known value is about 46.60, and the second largest is less than 17. However, if $B > 8000$, and inequality (6.1) holds, the value of $x([N]P)^{1/2}/|B|$ would be greater than 7575. So it is likely that any torsion-free rank 1 subgroup of rational points on Mordell curves $E_B$ bar the above mentioned exceptions has at most six integral points. Note that we introduce infinitely many examples of quasi-minimal Mordell curves with six integral points on a torsion-free subgroup of rank 1 of their Mordell–Weil group (Proposition 3.18 and Corollary 3.20). From the above argument in connection to Theorem 1.1, we can expect that a point $P$ on a quasi-minimal Mordell curve $E_B$, at least for larger values of $|B|$, has no integral multiples $[n]P$ with $n > 5$. However, the methods we used in this paper only give a reasonable upper bound on the second largest integral multiple of a point $P$ on the quasi-minimal Mordell curve $E_B$. The largest integral multiple is still based on linear forms in logarithms, which is far from expected.

**6.2. Quasi-minimal Mordell curves with $\mathbb{Z}/2\mathbb{Z}$ as the torsion subgroup.** Let $B$ be a sixth-power-free integer. Assume the Mordell curve $E_B$ has a rational torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$. By Lemma 2.3, $B = B_0^3$ is a cube different from 1. Since $B \not\equiv 16 \pmod{64}$, the equation $y^2 = x^3 + B$
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is a global minimal model for $E_B$ over the field of rational numbers. From Remark 2.4, $T = (-B_0, 0)$ is the non-trivial torsion point on $E_B$.

**Lemma 6.2.** Let $B$ be a sixth-power-free cube not equal to 1, and $P$ be a non-torsion point on $E_B$.

(a) If $|B| > 5.333 \cdot 10^5$ is odd, then $[n]P$ is integral for at most two values $n > 1$.

(b) If $|B| > 1408$ is even, then $[n]P$ is integral for at most one value $n > 1$.

**Proof.** By Theorem 1.1 and Lemmas 3.13, 3.10, and 3.5, $[n]P$ is not integral for any $3 \leq n \leq 10$. Moreover, if $B$ is even, from (3.1), $[2]P$ is not integral. Now assume $[n]P$ integral for two values $n_1, n_2 > 10$. Note that by Theorem 1.1, $n_1, n_2$ are both odd. By Lemma 4.5, we have

$$11^2 \left( \frac{1}{24} \log |B| - C \right) + \log \omega_1 - \frac{1}{3} \log |B| - 1.399 < \log n_2$$

$$< \log \max \{7.511 \cdot 10^{26}, 6 \cdot 10^{22} (\log |B|)^{5/2}\},$$

where from inequalities (2.2), we can take $C = 0.002$ when $B$ is odd, and $C = -0.2290$ when $B$ is even. The above inequality holds when $B$ is odd and $|B| < 5.333 \cdot 10^5$, or when $B$ is even and $|B| < 1409$. ■

Let $P$ be a rational point on $E_B$. To see that $[4]P + T$ can be integral, we recall a result of Ayad [1]. Note that this result holds for elliptic curves in a global minimal model.

**Lemma 6.3.** Let $M = (a/d^2, b/d^3)$ be a non-torsion rational point on a minimal elliptic curve $E$ defined over $\mathbb{Z}$, and $L = (u, v)$ be a rational 2-torsion point on $E$. Assume the set $S$ contains all rational primes $p$ where $M \pmod{p}$ is singular. Let $m \in \mathbb{Z}$, $m \neq 0$. Then $L + mM$ is an $S$-integer if and only if

$$\hat{\phi}_m - ud^2 \hat{\psi}_m^2 = \pm \prod_{p \in S} p^{\epsilon_p} \quad \text{if } u \in \mathbb{Z},$$

and

$$4(\hat{\phi}_m - ud^2 \hat{\psi}_m^2) = \pm \prod_{p \in S} p^{\epsilon_p} \quad \text{if } u \notin \mathbb{Z},$$

where $\hat{\phi}_m = d^{m^2} \phi_m(M)$ and $\hat{\psi}_m = d^{m^2-1} \psi_m(M)$.

**Lemma 6.4.** Let $P$ be non-torsion rational point on a Mordell curve $E_B$, where $B = B_3^3$ is a sixth-power-free cube and $T = (-B_0, 0)$. Then $[4]P + T$ is not an integral point.

**Proof.** Let $P = (a/d^2, b/d^3)$. Note that $T$ is a rational 2-torsion point on $E_B$, and the equation $y^2 = x^3 + B$ is a global minimal model for $E_B$ over the field of rational numbers. From [1] Theorem A, $P \pmod{p}$ is singular if
and only if $p = 2$ or $p$ divides both $B$ and $a$. Applying the above lemma and (2.1), if $[4]P + T$ is integral, then

\begin{equation}
(6.2) \quad f(x, y) = (x^8 + 8x^7y - 32x^6y^2 - 16x^5y^3 - 56x^4y^4 - 64x^3y^5 + 64x^2y^6 - 64xy^7 - 32y^8)^2
\end{equation}

\[= \pm \prod_{p | 2B_0} p^{f_p},\]

where $x = a$ and $y = B_0d^2$. Equation (6.2) is a Thue–Mahler equation which has finitely many solutions. Here the goal is to reduce it to finitely many Thue equations. Let

\[A = \frac{a}{\gcd(a, B_0d^2)} \quad \text{and} \quad B = \frac{B_0d^2}{\gcd(a, B_0d^2)}.\]

Let $p > 3$ be a divisor of $B_0d^2$ and $a$. If $\text{ord}_p(a) \neq \text{ord}_p(Bd^2)$ then clearly $p \mid f(A, B)$. Assume $\text{ord}_p(a) = \text{ord}_p(Bd^2)$ so that $\text{ord}_p(A) = \text{ord}_p(B) = 0$. Then since $b^2 = a^3 + (B_0d^2)^3$, we have $\text{ord}_p(A^3 + B^3) \geq 1$. Let $g(x, y) = x^5 + 8yx^4 - 32y^2x^3 - 17y^3x^2 - 64y^4x - 32y^5$. Then

\[f(A, B) - g(A, B)(A^3 + B^3) = 81A^6B^2.\]

Therefore, unless $p = 3$, in this case, $p \nmid f(A, B)$. If $p = 3$ and $\text{ord}_3(A) = \text{ord}_3(B) = 0$, then $\text{ord}_3(A^3 + B^3) \geq 3$ and $\text{ord}_3(g(A, B)) \geq 2$. Hence $\text{ord}_3(f(A, B)) = 4$. If $p = 2$, then it is easy to see that $\text{ord}_2(f(A, B)) = 0$ when $\text{ord}_2(A) = 0$, while $\text{ord}_2(f(A, B)) = 5$ when $\text{ord}_2(A) > 0$.

Hence the solutions of (6.2) embed into the solutions of the Thue equations $f(x, y) = \pm 2^\alpha 3^\beta$ where $\alpha \in \{0, 5\}$ and $\beta \in \{0, 4\}$. We have checked the solutions of these Thue equations using Magma. The only solutions are $(0, \pm 1), (\pm 1, 0), (\pm 1, \pm 1)$, and $(\pm 2, \pm 1)$, which correspond to torsion points. So there is no rational non-torsion point $P$ on $E_B$ with $[4]P + T$ integral. \hfill \qed

**Remark 6.5.** The same steps for $[2]P + T$ lead to some Pellian equations with infinitely many integral solutions.

**Lemma 6.6.** Let $E_B$ be a minimal Mordell curve of rank 1 with a rational torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Then $E_B$ has at most 14 integral points when $B$ is odd and at most 12 integral points when $B$ is even.

**Proof.** First assume $|B| < 10^6$. We have directly checked the number of integral points on the corresponding quasi-minimal Mordell curves $E_B$ using the data in [2]. For elliptic curves $E_B$ with rank 1 and $B$ even in this range, the elliptic curve $E_8$ has at most seven integral points. For elliptic curves $E_B$ with rank 1 and $B$ odd in this range, the elliptic curve $E_{-343}$ has at most nine integral points. So we might assume $|B| > 10^6$.
By the assumption, $B = B_0^3$ is a sixth-power-free cube not equal to 1. According to Remark 2.4, $T = (-B_0, 0)$ is a non-trivial torsion point on $E_B$. Let $P$ be a generator of its Mordell Weil group. The group of rational points is generated by $P$ and $T$.

By Lemmas 6.2 and 6.4, when $|B| > 5 \cdot 10^5$ and $B$ is odd, the set of integral points on $E_B$ is a subset of
$$\{ \pm P, \pm (P + T), \pm [2]P, \pm ([2]P + T), \pm [m_1]P, \pm ([m_2]P + T), \pm ([2m_3]P + T) \}.$$ Here the $m_i$’s are odd integers larger than 10.

By the same argument, when $|B| > 1409$ is even, the set of integral points is a subset of
$$\{ \pm P, \pm (P + T), \pm ([2]P + T), \pm [m_1]P, \pm ([m_2]P + T), \pm ([2m_3]P + T) \}.$$ Again the $m_i$’s are odd integers larger than 10. ■

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Amir Ghadermarzi
School of Mathematics, Statistics and Computer Science
College of Science
University of Tehran
Tehran, Iran
and
School of Mathematics
Institute of Research in Fundamental Science (IPM)
Tehran, Iran
E-mail: a.ghadermarzi@ut.ac.ir