Mechanism of High Temperature Superconductivity in a striped Hubbard Model

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It is shown, using asymptotically exact methods, that the two dimensional repulsive Hubbard model with strongly modulated interactions exhibits “high temperature superconductivity”. Specifically, the explicit modulation, which has the same symmetry as period 4 bond-centered stripes, breaks the system into an alternating array of more and less heavily hole doped, nearly decoupled two-leg ladders. It is shown that this system exhibits a pairing scale determined by the spin-gap of the undoped two-leg ladder, and a phase ordering temperature proportional to a low positive power of the inter-ladder coupling.

Much has been written concerning the mechanism of high temperature superconductivity (HTC) since the discovery of the cuprate superconductors in 1986, and indeed even before that. However, what is meant by “the mechanism” is rarely defined, and clearly evokes different images for different authors. The BCS mechanism, in which pairing is a consequence of a weak induced attraction produced by the exchange of phonons between well defined quasiparticles, is not only consistent with a remarkable number of experimental facts in conventional superconductors, it is also of well established theoretical validity in simple models. Because it is a weak coupling theory, even the mean-field estimate of $T_c$ (which is exponentially small, $T_c \propto \exp(-1/g)$ where $g$ is the induced attraction) is known to be quantitatively reliable. However, there are many well known reasons to believe that the BCS mechanism always leads to low $T_c$’s as recently reviewed in Ref. 3.

An alternative idea, which has been the focus of much of the theoretical effort in the field, is that in a doped Mott insulator, high temperature superconductivity arises directly from the repulsive interactions between electrons. However, even as a point of principle, the validity of a mechanism of this sort has not been well established for any simple model.

In this note we demonstrate the existence of a “high temperature superconducting” phase of the Hubbard and $t-J$ models on a square lattice with periodically modulated parameters—see Eq. (1). In particular, we show that a period 2 modulation can produce superconductivity with a relatively low $T_c$ in a restricted doping range, while a period 4 modulation produces higher critical temperatures on a broader range of doping. Specifically, we consider a caricature of a stripe ordered state consisting of a quasi-one dimensional array of two leg Hubbard ladders weakly coupled to each other with a hopping matrix element $\delta t$. For a range of electron densities per site, $< n > \equiv 1 - x$, it has been well established9,10 that the two leg ladder exhibits a Luther-Emery liquid11 phase, with a large spin-gap, $\Delta_s \sim J/2$, and a divergent superconducting susceptibility for $T \ll \Delta_s$,

$$\chi_{SC}(T) \sim \Delta_s / T^{2-K^{-1}},$$  \quad (1)

where $K$ is the charge Luttinger parameter, and $T$ is the temperature. This sounds like a promising start. However, even as a non-zero $T_c$ is impossible in one dimension (1D), so to have a chance of a high transition temperature, inter-ladder couplings must be taken into account. If all the ladders are equivalent (a caricature of a period 2 stripe ordered or column state12,13), we shall see that this coupling leads to a superconducting state in a restricted range of small $x$ with rather low $T_c$. For more substantial values of $x$, it inevitably leads to an insulating, incommensurate charge density wave (CDW) state with (in units in which the lattice constant is $a = 1$) an ordering wave number $P = 2\pi x$. (It is customary to call this the $4k_F$-CDW since, despite the fact that there is a spin-gap and hence no Fermi surface whatever, $P = 4k_F$, where $k_F$ is the Fermi momentum of a one-dimensional non-interacting electron gas at the same electron density.) That the superconducting transition is so easily preempted by CDW order follows from the fact that the CDW susceptibility of the Luther-Emery liquid diverges as

$$\chi_{CDW}(P,T) \sim \Delta_s / T^{2-K},$$  \quad (2)

Under most circumstances for repulsive interactions $K < 1$, and hence $\chi_{CDW}$ of Eq. (2) is more strongly divergent than $\chi_{SC}$ of Eq. (1). However, if we consider an alternating array of A and B type ladders (with different electron affinities) then the tendency to CDW order is greatly suppressed due to the mismatch between ordering vectors, $P_A$ and $P_B$, on neighboring ladders14,15. We shall show that, so long as the exponent inequalities

$$2 > K_A^{-1} + K_B^{-1} - K_A; \quad 2 > K_A^{-1} + K_B^{-1} - K_B$$  \quad (3)

are satisfied, the superconducting instability wins out. (If the Luttinger parameter is the same for both ladders, these inequalities reduce to $K > K_c \equiv (\sqrt{3} - 1) \approx 0.8$.)
Under these circumstances, the superconducting (Kosterlitz-Thouless) transition temperature can be reliably estimated by treating the 1D fluctuations exactly but the inter-ladder Josephson coupling $J$ in mean-field approximation\textsuperscript{16,17}.

$$T_c \sim \Delta_{\sigma} \left( \frac{J}{W} \right)^\alpha ; \quad \alpha = \frac{2KAKB}{4AKA_B - KA - KB} \quad (4)$$

where $J$ is an effective coupling and $W$ is a microscopic energy which we will discuss in detail below; typically, we find $J \sim \delta t^2/J$ and $W \sim J$. Although $T_c$ is small for small $J$, it is only power law small. In fact typically $\alpha \sim 1$. Because of the mean-field character of this estimate for $T_c$, one expects this to be an upper bound to the actual $T_c$. One also generally expects $T_c$ to be somewhat suppressed by phase fluctuations but typically by no more than a factor of 2. Indeed, a perturbative renormalization-group treatment for small $J$ yields the same power law dependence as Eq. (4) suggesting that this expression is asymptotically exact for $J << W$. This fact is supported in Appendix \textsuperscript{A} where the accuracy of interchain mean-field estimates is discussed for related models.

Since we expect $T_c$ to be smooth function of $\delta t/J$, it is reasonable to extrapolate Eq. (4) to the case in which $\delta t$ is a substantial fraction of $J$. This suggests a maximum $T_c$ of order $\Delta_{\sigma}$, and so can easily account for relatively high transition temperatures\textsuperscript{16,18}. This is in contrast to the case of an exponentially small $T_c$ as obtained, for example, in a BCS-like mechanism.

I. THE STRIPED HUBBARD MODEL

While the results obtained in this paper are quite robust in the sense that they apply for a broad range of microscopic interactions, to establish their validity it is useful to consider an explicit model. The model we study is the striped Hubbard model:

$$H = - \sum_{<\vec{r},\vec{r}',\sigma>}$ \eps_{\vec{r},\sigma} c_{\vec{r},\sigma}^\dagger c_{\vec{r},\sigma} + h.c. \quad (5)$$

$$+ \sum_{\vec{r},\sigma} \left[ \eps_{\vec{r}} c_{\vec{r},\sigma}^\dagger c_{\vec{r},\sigma} + (U/2) c_{\vec{r},\sigma}^\dagger c_{\vec{r},-\sigma}^\dagger c_{\vec{r},-\sigma} c_{\vec{r},\sigma} \right]$$

where $<\vec{r},\vec{r}'>$ designates nearest-neighbor sites, $c_{\vec{r},\sigma}^\dagger$ creates an electron on site $\vec{r}$ with spin polarization $\sigma = \pm 1$ and satisfies canonical anticommutation relations, and $U > 0$ is the repulsion between two electrons on the same site. In the limit of strong repulsions, $U \gg t_{\vec{r},\vec{r}'}$, this model reduces approximately to the corresponding $t - J$ model, which operates in the subspace of no double occupied sites, but with an exchange coupling, $J_{\vec{r},\vec{r}'} = 4|t_{\vec{r},\vec{r}'}|^2/U$ between neighboring spins. Our results only depend on the low-energy physics of the ladder and, thus, apply equally to the $t - J$ and Hubbard models.

In the translationally invariant Hubbard model, $t_{\vec{r},\vec{r}'} = t$ and $\eps_{\vec{r}} = 0$. The striped version of this model is still translationally invariant along the stripe direction (which we take to be the $y$ axis), so $t_{\vec{r},\vec{r}'+x} = t$. However, perpendicular to the stripes the hopping matrix takes on alternately large and small values: $t_{\vec{r},\vec{r}'+x} = t'$ for $r_x$ is even, and $t_{\vec{r},\vec{r}'+x} = \delta t = t'$ for $r_x$ is odd. This defines a “period 2 striped Hubbard model,” as shown in Fig. 1. For the “period 4 striped Hubbard model,” we include a modulated site energy, $\eps_{\vec{r}} = \sqrt{2t} \cos(\pi r_y/2 - \pi/4)$, which has site energy $\eps$ and $-\eps$ respectively on every other 2-leg ladder, with $\eps \gg \delta t$.

![FIG. 1: Schematic representation of the striped Hubbard model analyzed in this paper.](image)

II. ISOLATED 2-LEG LADDER

For $\delta t = 0$, the model breaks up into a series of disconnected 2-leg ladders. Considerable analytic and numerical effort has gone into studying the properties of 2-leg $t - J$ and Hubbard ladders, and much is known about them. For $x = 0$, the undoped two leg ladder has a unique, fully gapped state, referred to as COS0 in the notation of Ref. 8, meaning 0 gapless charge and 0 gapless spin modes. In the large $U$ limit, the magnitude of the spin-gap of the undoped\textsuperscript{9,20} ladder is $\Delta_{\sigma} \approx J/2$. Then, for a substantial range of $x$ ($0 < x < x_c$) the ladder exhibits a Luther-Emery or C1S0 phase, with a spin-gap that drops smoothly\textsuperscript{21} with increasing $x$, and vanishes at a critical value of the doping, $x = x_c$. (This particular Luther-Emery liquid is known\textsuperscript{5,6,7,8,9} to have “d-wave-like” superconducting correlations, in the sense that the pair-field operator has opposite signs along the edge of the ladder ($y$ direction) and on the rungs ($x$ direction).) For $x > x_c$, the numerical results are scarce, nor is there uniform agreement concerning the number of phases; there may\textsuperscript{10,11} or may not\textsuperscript{18} be narrow ranges of C2S1 and C2S2 phases for $x$ slightly larger than $x_c$. At any rate, for $x$ large enough, $x_c < x < 1$, the ladder manifestly enters a Luttinger liquid C1S1 phase, and finally, a trivial COS0 phase when $x = 1$ ($< n > = 0$).

For the purposes of the present paper, we will confine ourselves to the range of parameters where both A and B type ladders are in the Luther-Emery phase. The low
energy physics (at all energies less than $\Delta_s$) of the two-leg ladder in the Luther-Emery phase is contained in the free bosonic Hamiltonian for the collective charge degrees of freedom,

$$H = \int dy \frac{v_c}{2} \left[ K (\partial_y \theta)^2 + \frac{1}{K} (\partial_y \phi)^2 \right] + \ldots$$  \hspace{1cm} (6)

where $\phi$ is the CDW phase and $\theta$ is the superconducting phase; these two fields are dual to each other, and so satisfy the canonical commutation relations, $[\phi(y'), \partial_y \theta(y)] = i\delta(y - y')$. This effective Hamiltonian is general and physical; the precise $x$ dependence of the spin-gap, $\Delta_s$, the charge Luttinger exponent, $K$, the charge velocity, $v_c$, and the chemical potential, $\mu(x)$, depends on details such as the values of $U/t$ and $v'/t$. For certain cases these have been accurately computed in Monte-Carlo studies, and these studies could be straightforwardly extended to other values of the parameters.

The ellipsis in Eq. (6) represent cosine potentials, which we will not explicitly exhibit here, that produce forwardly extended to other values of the parameters. Thus, we need to reintroduce gapped fields $\phi_s$ to represent the spin-degrees of freedom. Since this is standard, we will not belabor the point; the appropriate continuum fermionic fields are

$$\Psi_{\pm, \sigma} \propto \exp \left\{ \sqrt{\pi/2} [\theta \pm \phi + \sigma \theta_s \pm \sigma \phi_s] \pm iPy/2 \right\}$$  \hspace{1cm} (7)

where $\pm$ refer to left and right going fermions with momentum near $\pm P/2$, respectively, $\sigma = \pm 1$ represents the spin polarization. It is important to stress that for strongly interacting problems, such as the present one, there is no simple relation between the original lattice fermions and the continuum fermion fields which describe the “physical” $\Psi$-fermions of Eq. (7).

In particular, what appears as a $2k_F$ CDW expressed in terms of $\Psi$-fermions, would be considered a $4k_F$ CDW in terms of the original, lattice Fermions. In terms of these $\Psi$-fields, the component of the charge density operator which varies with wave numbers near $P$ is

$$\hat{\rho}_P(y) = \sum_{\sigma} \Psi_{L, \sigma}^\dagger \Psi_{R, \sigma} \propto \exp[iPy + i\sqrt{2\pi} \phi(y)]$$  \hspace{1cm} (8)

while the singlet pair creation operator,

$$\hat{\Phi}(y) = [\Psi_{L, \uparrow}^\dagger \Psi_{R, \downarrow}^\dagger + \Psi_{R, \downarrow}^\dagger \Psi_{L, \uparrow}^\dagger] \propto \exp[i\sqrt{2\pi} \theta],$$  \hspace{1cm} (9)

where in the right-most expressions we have again suppressed the dependence on the spin fields.

Before leaving the single ladder problem, it is worth mentioning a useful intuitive caricature of its electronic properties. We picture a singlet pair of electrons on neighboring sites as being a hard-core bosonic “dimer.” The undoped ladder can be thought of as a Mott insulating state of these dimers, with one dimer per rung of the ladder, i.e. a “valence bond crystal” with lattice spacing one. To remove one electron from the system, we need to destroy one dimer and remove one electron, leaving behind a single electron with spin 1/2 and charge $e$. However, when we remove a second electron from the system, we have the choice of either breaking another dimer, thus producing two quasiparticles with the quantum numbers of an electron, or of removing the unpaired electron left behind by the first removal, thus producing a new boson - a missing dimer - with charge $2e$ and spin 0. The persistence of the spin-gap upon doping the ladder can thus be interpreted as implying that the energy needed to break a dimer (of order $\Delta_s$) is sufficiently large that one charge $2e$ boson costs less than two charge $e$ quasiparticles. At finite $x$, the missing dimers can be treated as a dilute gas of hardcore bosons. That the elementary excitations of the undoped ladder can be constructed in this simple manner reflects the fact that this is a confining phase, not a spin liquid.

### III. INTER-LADDER INTERACTIONS

We now address the effect of a small, but non-zero coupling $(i.e. $ single-particle hopping$)$ between ladders, $\delta t > 0$. Because of the spin-gap, $\delta t$ is an irrelevant perturbation in the renormalization group sense, and so does not directly affect the thermodynamic state of the system. However, second order processes result in various induced interactions between neighboring ladders. These consist of marginal forward scattering interactions, which are negligible for small $\delta t$, and potentially relevant Josephson tunnelling and back-scattering density-density interactions.
The important (possibly relevant) low energy pieces of these latter interactions are most naturally expressed in terms of the bosonic collective variables defined above:

\[ H' \equiv - \sum_j \int dy \left\{ J \cos[\sqrt{2\pi}(\theta_j - \theta_{j+1})] + V \cos[(P_j - P_{j+1})y + \sqrt{2\pi}(\phi_j - \phi_{j+1})] \right\}, \]

where \( P_j = 2\pi x_j \), with \( x_j \) the concentration of doped holes on ladder \( j \), and \( \phi_j \) and \( \theta_j \) are the charge field and its dual on each ladder. Here, again, the form of the low energy interactions between two Luther Emery liquids is entirely determined by symmetry considerations, but the magnitude of the Josephson coupling \( J \) and the induced interaction between CDW’s, \( \Delta_s \), are obtained, given the state of DMRG calculations, from microscopics; they are renormalized parameters which result from “integrating” out the high energy degrees of freedom with energies between the bandwidth \( W \sim 4t \) and the renormalized cutoff, \( \Delta_s \), or with wavelengths between \( a \) and \( \xi_s = v_s/\Delta_s \) where \( v_s \) is the spin-wave velocity. Thus, the dimensionless measure of the inter-ladder couplings, which for instance enter the expressions for \( T_c \), are \( J/\tilde{W} \) and \( V/\tilde{W} \) where \( \tilde{W} = \Delta_s/\xi_s \). (So long as \( x \) is not too near \( x_c \), \( \Delta_s \sim J \), and hence \( \tilde{W} \sim J \).)

Quantitative estimates of \( J \) and \( V \) could certainly be obtained, given the state of DMRG calculations, from studies of four-leg ladders consisting of two weakly coupled 2-leg ladders. However, such calculations have not, yet, been carried out. Fortunately, our qualitative conclusions are not very sensitive to the values of \( V \) and \( J \), which can anyway be estimated with reasonable accuracy from bosonization, as discussed in Ref. 16. The subtlety here is that the inter-ladder hopping is expressed in terms of microscopic lattice fermions, whereas our low energy theory is expressed in terms of the \( \Psi \)-fermions of Eq. 7. However, since these have the same quantum numbers as an electron, and operate on the scale of \( \Delta_s \), which is large with respect to \( \delta t \), and \( \delta t/\Delta_s \), there is no reason to expect any large renormalization of the hopping parameters. If we assume that the inter-ladder hopping can be approximated as \( \delta t \) times an operator representing the hopping amplitude for \( \Psi \)-fermions, then from second order perturbation theory we obtain

\[ J \approx V \sim A (\delta t)^2 / J \]

where \( A \) is the dimensionless function of \( \Delta_s / J \)

\[ A = \int dy dr \left\{ e^{i\sqrt{2\pi}\left[\theta_s(r) - \theta_s(0) + \phi_s(r) - \phi_s(0)\right]} \right\}^2 \]

\( r = (y, \tau) \) and \( \tau \) denotes imaginary time, the expectation value is taken with respect to the spin-fields on the decoupled ladders, and in deriving this expression we have assumed that the charge fields are slowly varying compared to the spin-fields. Simple scaling arguments of the sort discussed in Ref. 16 suggest that \( A \sim 1 \) as \( \Delta_s / J \to 0 \). (For further discussion see footnote 26.) In any case, so long as \( x \) is not too close to \( x_c \), \( \Delta_s \) is of order of the exchange coupling, \( J \), so it is reasonable to assume \( A \sim 1 \).

The only aspects of this estimate which matter qualitatively for our present purposes is that the two couplings are comparable in size, \( J \sim V \) and both are small in proportion to \( \delta t^2 \).

IV. RENORMALIZATION-GROUP ANALYSIS AND INTER-LADDER MEAN FIELD THEORY

The effect of these inter-chain couplings can be deduced from an analysis of the lowest order perturbative renormalization group equations in powers of the couplings \( V \) and \( J \). However, equivalent results are obtained from inter-ladder mean-field theory, which is conceptually simpler. These equations are the analogue of the BCS gap equations applied to this model, and are expected to give a quantitatively accurate estimate of \( T_c \) for small \( \delta t / \Delta_s \) for precisely the same reason. A discussion of the accuracy of interchain mean-field theory is given in Appendix A. In the present two-dimensional system, \( T_c \) should be interpreted as the onset of quasi-long range order, i.e. as a Kosterlitz-Thouless transition.

To implement this mean-field theory, we need to compute the expectation value \( M_j(h_j) = \langle \cos[\sqrt{2\pi}\theta_j] \rangle \) of the pair creation operator on an isolated ladder, where the expectation value is taken with respect to the mean-field Hamiltonian

\[ H_{MF} = H_j - h_j \int dy \cos[\sqrt{2\pi}\theta_j] \]

in which \( H_j \) is the effective Hamiltonian in Eq. 10 with parameters appropriate to ladder \( j \), and \( h_j \) represents the mean-field due to the neighboring ladders, and so satisfies the self-consistency condition

\[ h_j = J [M_{j+1} + M_{j-1}] \]

The expression for the mean-field transition temperature can be expressed in terms of the corresponding susceptibility, \( \chi^{(SC)} = \partial M_j(h)/\partial h \bigg|_{h=0} \), which is related to the superconducting susceptibility in Eq. 11 by a proportionality constant which depends on the expectation value of the spin-fields. In the case in which all the ladders are equivalent, this yields the implicit relation

\[ 2J \chi^{(SC)}(T_c) = 1 \]

For an alternating array of \( M \) and \( B \) type ladders, the expression for the superconducting \( T_c \) is easily seen to be

\[ (2J)^2 \chi^{(A)}(T_c) \chi^{(B)}(T_c) = 1 \]

Notice that in the case in which the \( A \) and \( B \) type ladders are identical Eq. 12 reduces properly to the equivalent ladders. The expression for \( \chi^{(SC)} \) from Eq. 11 can be used to invert Eq. 13 to obtain the estimate for \( T_c \) given in Eq. 4.
The mean-field equations for the CDW order are obtained similarly. The expression for the transition temperature for CDW order with wave-vector $P$ is

$$(2V)^2 \chi_{CDW}^{(A)}(P, T_c) \chi_{CDW}^{(B)}(P, T_c) = 1 \quad (15)$$

where the notation is the obvious extension of that used in the superconducting case. The best ordering vector is that which maximizes $T_c$. For $P = P_A$, $\chi_{CDW}^{(A)}(P_A, T)$ diverges with decreasing temperature as in Eq. (2), but $\chi_{CDW}^{(B)}(P_A, T)$ saturates to a finite, low temperature value when $T \sim v_c|P_A - P_B|$. Thus, even if $\chi_{CDW}^{(A)}(P_A, T)$ diverges more strongly with decreasing temperature than $\chi_{SC}^{(A)}$, there are two divergent susceptibilities in the expression for the superconducting $T_c$, and only one for the CDW $T_c$; so long as the inequalities in Eq. (4) are satisfied, the superconducting transition preempts the CDW transition.

V. THE $x \to 0$ LIMIT

Since $K \to 2$ as $x \to 0$, there is necessarily a regime of small $x$ in which the superconducting susceptibility on the isolated ladder is more divergent than the CDW susceptibility. Here, in the presence of weak inter-ladder coupling, even the period 2 striped Hubbard model (i.e. with $\epsilon = 0$) is superconducting. However, care must be taken in this limit, since, as mentioned above, the range of energies over which $H$ in Eq. (6) is applicable vanishes in proportion to $x^2$. Fortunately, a complementary treatment of the problem, which takes into account the additional terms, the ellipsis in Eq. (6), can be employed in this limit. The small $x$ problem can be mapped onto a problem of dilute, hard-core charge 2e bosons (with concentration $x$ per rung) with an anisotropic dispersion, $E(\vec{k}) = \tilde{t}k_y^2 - \tilde{\delta} \cos[2k_x]$. (The 2 reflects the ladder periodicity.) Consequently, for small $x$,

$$T_c \approx 2\pi \sqrt{2J t} x F(x) \sim |\delta t| x \quad (16)$$

where $F(x) \sim 1/\ln \ln(1/x)$ is never far from 1, and the logarithm reflects the fact $d = 2$ is the marginal dimension for Bose condensation. (This result is not substantially different for the period 4 striped Hubbard model, so long as $\epsilon$ is not too large.) There is a complicated issue of order of limits when both $\delta t$ and $x$ are small; roughly, we expect that $T_c$ will be determined by whichever expression, Eq. (14) or Eq. (15), gives the higher $T_c$, but with the understanding that $\chi_{SC}$ must be computed taking into account the terms represented by the ellipsis in Eq. (6) which cause the susceptibility to vanish as $x \to 0$.

The period 2 striped Hubbard or $t-J$ model indeed has a superconducting phase at small $x$, because this phase is confined to rather small $x \lesssim 0.1$, where $T_c$ is small in proportion to both $\delta t$ and $x$. Moreover, this may still not be enough to establish a mechanism of high temperature superconductivity. The situation looks even worse when the effects of weak disorder are considered - when the disorder strength is greater than the intra-ladder energy scale $E_F = 2\pi t x^2$, it is unlikely that any sort of superconducting coherence will survive.

For an array of alternating ladders, the range of $x$ for which superconductivity dominates is much extended. This means that the maximum $T_c$ is much greater, and the superconductivity much more robust to disorder for the period 4 than the period 2 striped Hubbard model.

VI. OPTIMAL DEGREE OF INHOMOGENEITY FOR SUPERCONDUCTIVITY

Here we have established that in a strongly striped Hubbard model, superconductivity is produced directly by the repulsive interactions between electrons. The resulting $T_c$ is proportional to a positive power of $\delta t/t$, and so rises as the stripe order becomes less strong. It is thus natural to ask: Is the stripe order introduced in the present paper simply a calculational crutch which permits us obtain well controlled results or is inhomogeneity essential to the mechanism of high temperature superconductivity, as has been suggested in several previous studies?

The answer to this question turns on the issue of whether or not the uniform Hubbard model, and its strong coupling relative the $t-J$ model, by themselves support high temperature superconductivity. This question has been the focus of much theoretical research since the discovery of superconductivity in the cuprates. To this date this is not a settled issue. Nor is it the purpose of the present paper to review this extensive literature. Variational calculations have been interpreted both as giving evidence in support and against superconductivity in Hubbard and $t-J$-type models. There is also considerable evidence, from several numerical techniques and high temperature expansions, that the canonical $t-J$ and Hubbard models on a square lattice most likely do not support high temperature superconductivity; instead they show clear evidence for other types of order which compete with superconductivity.

Assuming that the uniform model does not support high-temperature superconductivity, it follows from the arguments given in the previous sections that there is an optimal degree of inhomogeneity (an optimal degree of stripe order) for a strongly correlated system to exhibit superconductivity. Probably this occurs when $\delta t \sim \Delta_s$. An analogous result was established recently in the weakly interacting limit of the 4-leg ladder (itself a caricature of a single unit cell of the present model). We should also note that there is nothing essential about having period 4. In fact, the longer the period the more the CDW instability is suppressed and the larger the range of superconductivity.


VII. RELATION TO SUPERCONDUCTIVITY IN THE CUPRATES

While the main purpose of the present paper was to establish, as a point of principle, that the striped Hubbard model analyzed here exhibits high temperature superconductivity, a few comments are in order concerning the more general implications of the present results for the mechanism of superconductivity in the cuprates.

Firstly, the explicit striped inhomogeneities introduced here are a caricature of the spontaneous symmetry breaking in a charge striped phase. However, the model possesses a large spin-gap, and so does not contain any of the physics of low energy incommensurate spin-fluctuations which are the principle experimental signatures to date of stripe correlations in the cuprates. Secondly, although the superconducting state is “d-wave-like” in the sense that the order parameter changes sign under rotations by $\pi/2$, since the striped Hamiltonian explicitly breaks this symmetry, there is no precise symmetry distinction between d-wave and s-wave superconductivity. Thirdly, this symmetry, there is no precise symmetry distinction between d-wave and s-wave superconductivity. Thirdly, the superconducting state is not even truly adiabatically connected to the superconducting state observed in the cuprates, because the existence of a spin-gap implies the absence of gapless “nodal” quasiparticles in the superconducting state. However, the transition between a node-less and nodal d-wave-like state was studied in Ref. 35,36, where it was found to be a mean field (Lifshitz) transition with relatively little effect on $T_c$. Moreover, using the same lines of reasoning employed in that article, it is possible to make compelling (although not entirely rigorous) arguments that upon heavier doping, the present model, too, will exhibit a nodal superconducting state. We are currently working to obtain a more complete treatment of the phase diagram of the present model.

The present model realizes the idea that the pairing scale, in this case the spin-gap, can be inherited from a parent Mott insulating state. Moreover, like the underdoped cuprates, the gap scale in the present model is a decreasing function of increasing $x$, while the actual superconducting transition occurs at a $T_c$ much smaller than $\Delta_s/2$, and is determined by the phase ordering temperature rather than the pairing scale. Hence, for $x$ not too close to $x_c$, this model exhibits a pseudogap regime for temperatures between $T_c$ and $T^* \sim \Delta_s/2$, reminiscent of that seen in underdoped cuprates. However, $T_c$ is always bounded from above by $\Delta_s$ and so tends to zero as $x \to x_c$.

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APPENDIX A: ACCURACY OF THE INTERCHAIN MEAN-FIELD THEORY ESTIMATES

In this section, we discuss the accuracy of the interchain mean-field theory. Although no general proof exists (to the best of our knowledge), we believe that it is asymptotically exact in the present case, at least to logarithmic accuracy (as defined in Eq. 11 below). The latter conclusion also follows, as mentioned in the text, by comparison with perturbative RG calculations.

Quite generally, using an argument based on Griffiths inequalities, one knows that the exact $T_c$ of a general anisotropic ferromagnetic system (not necessarily an Ising model) will obey the bounds: $T_c(J_x) \leq T_c \leq T_c(J_y)$, for $0 < J_x \leq J_y$, where $T_c(J)$ is the $T_c$ of an isotropic system of coupling constant $J$. However, for specific systems it is possible to establish more precise estimates of $T_c$.

As our first example, consider the 2D anisotropic Ising model on a square lattice, with couplings $J_x$ and $J_y \leq J_x$ in the x and y directions, respectively. For the case of the 2D Ising model, it is also known that the exact $T_c$ is the solution to the equation:

$$\sinh(2J_x/T_c) \sinh(2J_y/T_c) = 1.$$  \hspace{1cm} (A1)

Interchain mean-field theory for the same model gives the familiar expression for the mean-field transition temperature, $T_0$:

$$2J_y \chi_{1D}(T_0) = 1$$ \hspace{1cm} (A2)

which is analogous to Eq. 3, and where

$$\chi_{1D}(T) = T^{-1} \exp[2J_x/T]$$ \hspace{1cm} (A3)

is the susceptibility of the 1D Ising model. In the limit of small $J_y/J_x$, it thus follows that the ratio

$$\frac{T_0}{T_c} = 1 + \frac{\ln 2}{\ln[J_y/J_x]} + \ldots$$ \hspace{1cm} (A4)

tends to 1 as $J_y/J_x \to 0$, i.e. the interchain mean-field theory is asymptotically exact without any apologies.

Before leaving the Ising example, it is interesting to see how well the interchain mean-field theory works when extrapolated to the isotropic case $J_x = J_y = J$. It is easy to verify that $T_0 = 3.53J$ and $T_c = 2J/\ln[1 + \sqrt{2}]$, so

$$\frac{T_0}{T_c} = 1.55 \quad \text{for } J_y/J_x = 1.$$ \hspace{1cm} (A5)

In general, $T_0/T_c$ rises monotonically from 1 for increasing $J_y/J_x$, but $T_0$ gives a reasonably good estimate of $T_c$ over the entire range of parameters. (Note, ordinary mean field theory gives $T_0' = 2(J_x + J_y)$, which is not much worse than interchain mean-field theory in the isotropic limit, but $T_0'/T_c \to \infty$ as $J_y/J_x \to 0$.)

Now, we move to the 2D classical XY model on a square lattice - the case of most direct relevance to the
estimates of $T_c$ made in the text. The susceptibility of an isolated chain can easily be seen to be

$$\chi_{1D}(T) = \frac{1}{2T} \left( \frac{I_0(J_x/T) + I_1(J_x/T)}{I_0(J_x/T) - I_1(J_x/T)} \right) \quad (A6)$$

where $I_n(x)$ is a Bessel function. For $J_y/J_x \ll 1$, Eq. (A3) and Eq. (A6) yield the following estimate of the critical temperature

$$T_0 = 2\sqrt{J_x J_y} \left[ 1 + \mathcal{O}\left( \frac{J_y}{J_x} \right) \right], \quad (A7)$$

while $T_0 = 1.755J$ in the isotropic limit.

Unlike the Ising case, no exact results exist for the 2D XY model. Extensive Monte-Carlo work has been done on the isotropic 2D XY model, form which we know that the Kosterlitz-Thouless transition occurs at $T_c = 0.89J$, so in this limit $T_0/T_c \approx 2$. In the limit of extreme anisotropy, the 2D classical XY model can be mapped onto the familiar 1D quantum XY (rotor) model with Hamiltonian

$$H = \sum_n \left[ \frac{I_n^2}{2} - \frac{\lambda}{2} \cos(\theta_{n+1} - \theta_n) \right] \quad (A8)$$

where the coupling constant is $\lambda = 2J_x J_y/T^2$ (see Ref. [30]). The critical value of the coupling of the quantum rotor model, $\lambda_c$, has been computed quite accurately using a Padé-Borel resummation of the strong-coupling series[30,31]. Using the notation of these papers, an accurate estimate for the critical coupling to be $\lambda_c = 1.8 \pm 0.5$ is obtained. By carefully inverting this mapping, we get

$$T_c = A\sqrt{J_x J_y} \left[ 1 + \mathcal{O}\left( \frac{J_y}{J_x} \right) \right] \quad (A9)$$

where $A = 1.05 \pm 0.1$. Thus, we see that

$$\frac{T_0}{T_c} \to (2/A) \quad \text{as} \quad \frac{J_y}{J_x} \to 0. \quad (A10)$$

It seems unlikely that the error bars on $A$ are sufficient to be consistent with a limit of 1. The interchain mean-field theory is therefore found to be asymptotically exact only to logarithmic accuracy, i.e.

$$\frac{\ln T_0}{\ln T_c} \to 1 \quad \text{as} \quad \frac{J_y}{J_x} \to 0. \quad (A11)$$

This, we believe, is the generically true of interchain mean-field theory as applied in the present paper. Nonetheless, in all cases where the exact answers are known, interchain mean-field theory gives estimates of $T_c$ that are within a factor of 2 of the exact results. This is certainly sufficiently accurate for present purposes.

Finally, it is worth mentioning that the 2D XY model is something of a worst-case example, because 2D is the lower critical dimension and hence fluctuation effects are anomalously large. If we consider an anisotropic 3D XY model with couplings $J_x \geq J_y \geq J_z$ in the three directions, the mean-field transition temperature can still be readily computed according to $2(J_y + J_z)\chi_{1D}(T_0) = 1$. Monte-Carlo results exist[12] for the $T_c$ of layered models, $g = J_z/J_x$ for various values of $J_z/J_x$. For instance,

$$T_c = 1.1J \quad \frac{T_0}{T_c} = 1.60 \quad \text{for} \quad \frac{J_z}{J_x} = 0.01$$
$$T_c = 1.324J \quad \frac{T_0}{T_c} = 1.41 \quad \text{for} \quad \frac{J_z}{J_x} = 0.1$$
$$T_c = 2.2J \quad \frac{T_0}{T_c} = 1.29 \quad \text{for} \quad \frac{J_z}{J_x} = 1.0.$$

Clearly, even a very small amount of interplane coupling can be expected to greatly improve the accuracy of our $T_c$ estimates. (Interplane mean-field theory, of course, is still more accurate, as shown in Ref. [12].)

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1. J. G. Bednorz and K. A. Müller, Z. Phys. B 64, 189 (1986).
2. R. Brout, Phys. Rev. 118, 1009 (1960).
3. For a review, with extensive references to the original literature, see E. W. Carlson, V. J. Emery, S. A. Kivelson and D. Orgad, to be published (soon, we hope) in “The Physics of Conventional and Unconventional Superconductors” edited by K. H. Bennemann and J. B. Ketterson (Springer-Verlag), arXiv:cond-mat/0206217. In particular, this paper reviews reasons to believe that the BCS mechanism does not produce high temperature superconductivity, reviews the numerical searches for superconductivity in the uniform $t – J$ and Hubbard models, reviews the relevant aspects of the theory of the one dimensional electron gas, especially the Luther-Emery liquid, and issues concerning the competition between CDW and SC order in quasi-1D systems.
4. H. Tsunetsugu, M. Troyer and T. M. Rice, Phys. Rev. B 51, 16456 (1995), studied arrays of two-leg $t – J$ ladders as a way to understand the physics of the translationally invariant 2D system. Although the model they studied nominally corresponds to the period 2 case we discuss below, and some of their discussion prefigures the present analysis, the questions asked by these authors were quite different. In particular they did not consider the mechanism of superconductivity in inhomogeneous 2D systems which we discuss here.
5. S. White, I. Affleck, and D. J. Scalapino, Phys. Rev. B 65, 165122 (2002), arXiv:cond-mat/0111320. Note that the normalization convention on the fields used in the present paper differs from that of White and coworkers, so that our $K$ is the same as their $2K_{c,t}$.
6. R. M. Noack, S. R. White and D. J. Scalapino, Phys. Rev. B 56, 7102 (1997), arXiv:cond-mat/9601047.
7. T. Siller, M. Troyer, T. M. Rice and S. R. White, Phys.
A mechanism is also related to the original RVB ideas. This mechanism bears a family resemblance to the interlayer tunnelling mechanism of S. Chakravarty, A. Sudbo, P. W. Anderson, and S. Strong, Science 261, 337 (1993), although many aspects are quite different.

This mechanism is also related to the original RVB ideas of P. W. Anderson, Science 235, 1196 (1987), and S. A. Kivelson, D. Rokhsar and J. P. Sethna, Phys. Rev. B 35, 865 (1987), in that the pairing correlations (and scale) are inherited from the correlated insulator.

For a restricted range of $x$, the authors of Ref. show numerical evidence indicating that the spin gap decreases smoothly with increasing $x$. We are not aware of any published studies that carefully trace the spin gap as a function of $x$, and in particular ones that accurately determine the critical doping, $x_c$, at which it vanishes.

In deriving the estimates of $\mathcal{J}$ and $\mathcal{V}$, we have made a crude approximation concerning the spectrum of the isolated ladder at energies above $\Delta_s$. Specifically: 1) We have neglected the possibility that there are additional important excitations, requiring extra bosonic fields, which might be vestiges of the two-bands present in weak coupling. For instance, we can define a gap, $\Delta_0$, to the lowest lying excited state which is odd under exchange of the legs of the ladders. As far as we know, no numerical studies have measured this gap, although this should be straightforward. So long as $\Delta_0 > \Delta_s$, we don’t expect the existence of such additional modes will affect the estimates of these parameters qualitatively. However, it would be interesting in future numerical studies to evaluate $\Delta_0$. In particular, if there are intermediate C2S1 and/or C2S2 phases between the C1S0 phase at $x < x_c$ and the C1S1 phase at $x > x_c$ with $x_c > x_c$, then $\Delta_0$ must vanish as $x \to x_c$, while if there are no such intermediate phases, $\Delta_0$ must remain finite at $x = x_c$. 2) We have neglected spin-charge coupling. Although such couplings are irrelevant in the renormalization group sense, this only justifies their neglect at low energies, whereas the states we are integrating out to derive $\mathcal{J}$ are high energy states. Indeed, it is clear from the intuitive dimer picture that at $x = 0$, there are (gapped) elementary excitations that carry both spin and charge. In the conceptually interesting case of $x$ near $x_c$, where $\Delta_s/J$ is small, spin-charge coupling may seriously affect the value of $\mathcal{J}$, but in the typical case, $\Delta_s/J \sim 1$, no interesting renormalizations are possible, anyway.

D. S. Fisher and P. C. Hohenberg, Phys. Rev. B 37, 4936 (1988).

E. Orignac and T. Giamarchi, Phys. Rev. B 56, 7167 (1997), arXiv:cond-mat/9704064.

S. Chakravarty, M. Gelland, and S. A. Kivelson, Science 254, 970 (1991).

V. J. Emery, S. A. Kivelson, and O. Zachar, Phys. Rev. B 56, 6120 (1997), arXiv:cond-mat/9610004.

A model of an inhomogeneous strongly correlated system with attractive interactions was studied in mean field theory by Ivar Martin, Gerardo Ortiz, A. V. Balatsky and A. R. Bishop, Europhys. Lett. 56, 849 (2001), arXiv:cond-mat/000331, and an inhomogeneous $t-J$-model with magnetic anisotropy was studied numerically in small systems by J. Eroles, G. Ortiz, A.V. Balatsky and A.R. Bishop, Europhysics Letters 50, 540 (2000), arXiv:cond-mat/0003321.

S. Sorella, G. B. Martins, F. Becca, C. Gazza, L. Capriotti, A. Parolla and E. Dagotto, Phys. Rev. Lett. 88, 117002 (2002), arXiv:cond-mat/0110460.

S. Sorella, A. Parola, F. Becca, L. Capriotti, C. Gazza, E. Dagotto and G. Martins, Phys. Rev. Lett. 89, 279703 (2003); M. Calandra and S. Sorella, Phys. Rev. B 61, R11894 (2000), arXiv:cond-mat/9911478.

S. Zhang, J. Carlson and J. Gubernatis, Phys. Rev. Lett. 78, 4486 (1997), arXiv:cond-mat/9703017.

T. K. Lee, C. T. Shih, Y. C. Chen and H. Q. Lin, Phys. Rev. Lett. 89, 279702 (2003); C.T. Shih, Y.C. Chen, H.Q. Lin and T.K. Lee, Phys. Rev. Lett. 81, 1294 (1998), arXiv:cond-mat/9807027.

L. Pryadko, S. A. Kivelson, and O. Zachar, Phys. Rev. Lett. 92, 067002 (2004), arXiv:cond-mat/0306342.

M. Granath, V. Oganessian, S. A. Kivelson, E. Fradkin and V. J. Emery, Phys. Rev. Lett. 87, 167011 (2001), arXiv:cond-mat/010350.

S. Sachdev, Physica A 313, 252 (2002), and references therein.

T. Schultz, D.C. Mattis and E. Lieb, Rev. Mod. Phys. 36, 856 (1964).

R. Gupta and C.F. Baillie, Phys. Rev. B 45, 2883 (1992).

E. Fradkin and L. Susskind, Phys. Rev. D 17, 2637 (1978).

C. J. Hamer, J. B. Kogut and L. Susskind, Phys. Rev. Lett. 41, 1337 (1978).

C. J. Hamer and J. B. Kogut, Phys. Rev. B20, 3859 (1979).

E. W. Carlson, S. A. Kivelson, V. J. Emery and E. Manousakis, Phys. Rev. Lett. 83, 612 (1999).