General Solution of a Fractional Diffusion-Advection Equation for Solar Cosmic-Ray Transport

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Abstract

In this effort we exactly solve the fractional diffusion-advection equation for solar cosmic-ray transport proposed in [15] and give its general solution in terms of hypergeometric distributions. Also, we regain all the results and approximations given in [15] as particular cases of our general solution.

Keywords:
1 Introduction

There is a considerable body of evidence, from data collected by spacecrafts like Ulysses and Voyager 2, indicating that the transport of energetic particles in the turbulent heliospheric medium is superdiffusive \[1, 2\]. Considerable effort has been devoted in recent years to the development of superdiffusive models for the transport of electrons and protons in the heliosphere \[3, 4, 5\]. This kind of transport regime exhibits a power-law growth of the mean square displacement of the diffusing particles, \(\langle \Delta x^2 \rangle \propto t^\alpha\), with \(\alpha > 1\) (see, for instance, \[6\]). The special case \(\alpha = 2\) is called ballistic transport. The limit case \(\alpha \to 1\) corresponds to normal diffusion, described by the well-known Gaussian propagator. The energetic particles detected by the aforementioned probes are usually associated with violent solar events like solar flares. These particles diffuse in the solar wind, which is a turbulent environment than can be assumed statistically homogeneous at large enough distances from the sun \[1\]. This implies that the propagator \(P(x, x', t, t')\), describing the probability of finding at the space time location \((x, t)\) a particle that has been injected at \((x', t')\), depends solely on the differences \(x - x'\) and \(t - t'\). In the superdiffusive regime the propagator \(P(x, x', t, t')\) is not Gaussian, and exhibits power-law tails. It arises as solution a non local diffusive process governed by an integral equation that can be recast under the guise of a diffusion equation where the well-known Laplacian term is replaced by a term involving fractional derivatives \[7\]. Diffusion equations with fractional derivatives have attracted considerable attention recently (see \[8, 9, 10, 11, 12\] and references therein) and have lots of potential applications \[13, 14\]. In particular, the observed distributions of solar cosmic ray particles are often consistent with power-law tails, suggesting that a superdiffusive process is at work.

A proper understanding of the transport of energetic particles in space is a vital ingredient for the analysis of various important phenomena, such as the propagation of particles from the Sun to our planet or, more generally, the acceleration and transport of cosmic rays. The superdiffusion of particles in interplanetary turbulent environments is often modelled using asymptotic expressions for the pertinent non-Gaussian propagator, which have a limited range of validity. A first step towards a more accurate analytical treatment of this problem was recently provided by Litvinenko and Effenberger (LE) in \[15\]. LE considered solutions of a fractional diffusion-advection equation describing the diffusion of particles emitted at a shock front that propagates at a constant upstream speed \(V_{sh}\) in the solar wind rest frame. The shock front
is assumed to be planar, leading to an effectively one-dimensional problem. Each physical quantity depends only on the time $t$ and on the spatial coordinate $x$ measured along an axis perpendicular to the shock front. In the present contribution we re-visit the fractional diffusion-advection equation previously studied by LE, providing a closed analytical solution.

## 2 Formulation of the Problem

The authors of [15] advanced the equation

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^\alpha f}{\partial |x|^{\alpha}} + a \frac{\partial f}{\partial x} + \delta(x), \quad (2.1)$$

where $t > 0$ and $f(x, t)$ is the distribution function for solar cosmic-rays transport. Here the fractional spatial derivative is defined as

$$\frac{\partial^\alpha f}{\partial |x|^{\alpha}} = \frac{1}{\pi} \sin \left( \frac{\pi \alpha}{2} \right) \Gamma(\alpha + 1) \int_0^\infty \frac{f(x + \xi) - 2f(x) + f(x - \xi)}{\xi^{\alpha+1}} \, d\xi. \quad (2.2)$$

(See [15] and references therein).

To solve this equation the authors use the Green function governed by the equation:

$$\frac{\partial G}{\partial t} = \kappa \frac{\partial^\alpha G}{\partial |x|^{\alpha}} + \delta(x)\delta(t). \quad (2.3)$$

With this Green function, the solution of (2.1) can be expressed as

$$f(x, t) = \int_0^t G(x + at', t') \, dt'. \quad (2.4)$$

In this work we obtain the solutions of Eqs. (2.1) and (2.3) using distributions as main tools [16]. Also, we re-obtain all results and approximations obtained in [15], but as particular cases of our general solutions of (2.1) and (2.2).

For our task we use, as a first step, the solution obtained in [15] for the Green function through the use of the Fourier Transform given by

$$\hat{G}(k, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, t)e^{-ikx} \, dx, \quad (2.5)$$
from which we obtain for $\hat{G}$:

$$\hat{G}(k, t) = -\kappa |k|^\alpha \hat{G}(k, t) + \frac{1}{2\pi} \delta(t), \quad (2.6)$$

whose solution is

$$\hat{G}(k, t) = \frac{H(t)}{2\pi} e^{-\kappa |k|^\alpha t}, \quad (2.7)$$

where $H(t)$ is the Heaviside’s step function.

### 3 General Solution of the Equations

From (2.7) we have for $\hat{G}$

$$\hat{G}(k, t) = \frac{H(t)}{2\pi} e^{-\kappa |k|^\alpha t} = \frac{H(t)}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n k^\alpha t^n}{n!}, \quad (3.1)$$

and, invoking the inverse Fourier transform

$$G(x, t) = \frac{H(t)}{2\pi} \int_{-\infty}^{\infty} e^{-\kappa |k|^\alpha t} e^{ikx} \, dk =$$

$$\frac{H(t)}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n t^n}{n!} \left[ \int_{0}^{\infty} k^\alpha e^{ikx} \, dx + \int_{0}^{\infty} k^\alpha e^{-ikx} \, dx \right]. \quad (3.2)$$

Fortunately, we can find in the classical book of [16] the results for the two integrals of (3.2). We obtain

$$G(x, t) = \frac{H(t)}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n t^n}{n!} \Gamma(\alpha n + 1) \left[ \frac{e^{i\frac{\pi}{2}(\alpha n + 1)}}{(x + i0)^{\alpha n + 1}} + \frac{e^{-i\frac{\pi}{2}(\alpha n + 1)}}{(x - i0)^{\alpha n + 1}} \right]. \quad (3.3)$$

Using now (2.4) we have for $f$

$$f(x, t) = \int_{0}^{t} G(x + a t', t') \, dt'.$$
so that one can write

\[
    f(x, t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n}{n!} \Gamma(\alpha n + 1) \times \\
    \int_0^t \left[ \frac{e^{i\pi/2(\alpha n + 1)}}{(x + at' + i0)^{\alpha n + 1}} + \frac{e^{-i\pi/2(\alpha n + 1)}}{(x + at' - i0)^{\alpha n + 1}} \right] t^n \, dt'.
\]  

(3.4)

According to Eq. (A.1) of the Appendix, where \( t > 0 \), we now obtain for \( f \), invoking hypergeometric functions \( F(\alpha n + 1, 2; 3; z) \) and Beta functions \( B(1, n + 1) \),

\[
    f(x, t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n t^{n+1}}{n!} \Gamma(\alpha n + 1) B(1, n + 1) \times \\
    \left[ \frac{e^{i\pi/2(\alpha n + 1)}}{(x + i0)^{\alpha n + 1}} F\left(\alpha n + 1, n + 1; n + 2; -\frac{at}{x + i0}\right) + \\
    \frac{e^{-i\pi/2(\alpha n + 1)}}{(x - i0)^{\alpha n + 1}} F\left(\alpha n + 1, n + 1; n + 2; -\frac{at}{x - i0}\right) \right].
\]  

(3.5)

This is the general solution of Eq. (2.1) for the initial condition \( f(x, 0) = 0 \).

In the next section we will see that all results and approximations obtained in [15] can be regarded as particular cases of the general solution (3.5).

4 Weak Diffusion Approximation

Following LE, we shall now consider a weak diffusion approximation. Within this approximation we can treat \( \kappa \) as a small parameter and develop \( f \) up to order one [15]. Thus, we can write:

\[
    f(x, t) = f_0(x, t) + f_1(x, t),
\]  

(4.1)

where i) \( f_0 \) is independent of \( \kappa \) and ii) in \( f_1 \) the corresponding power of \( \kappa \) is unity.

Eq. (3.5) entails that we have, for \( f_0 \) (\( n = 0 \) in (3.5)),

\[
    f_0(x, t) = \frac{it}{2\pi} \left[ (x + i0)^{-1} F\left(1, 1; 2; -\frac{at}{x + i0}\right) - \\
    \right.
\]
\[(x - i0)^{-1} F \left( 1, 1; 2; \frac{at}{x - i0} \right) \]. \hspace{1cm} (4.2)

Recourse to the celebrated Tables of \[17\] allows us to write
\[F(1, 1; 2; -z) = \frac{1}{z} \ln(1 + z),\] \hspace{1cm} (4.3)
and we obtain for \(f_0\)
\[f_0(x, t) = \frac{1}{a} \left[ H(-x) - H(-x - at) \right] = \frac{1}{2a} \left[ \text{Sgn}(x + at) - \text{Sgn}(x) \right].\] \hspace{1cm} (4.4)

When we take \(n = 1\) in (3.5), \(f_1\) is defined as
\[f_1(x, t) = -\frac{i \kappa t^2}{4\pi} \Gamma(\alpha + 1) \left[ \frac{e^{i \frac{\pi}{2} \alpha}}{(x + i0)^{\alpha + 1}} F \left( \alpha + 1, 1, 2; \frac{at}{x + i0} \right) + \frac{e^{-i \frac{\pi}{2} \alpha}}{(x - i0)^{\alpha + 1}} F \left( \alpha + 1, 1, 2; \frac{at}{x - i0} \right) \right].\] \hspace{1cm} (4.5)

Now, from (A.1) of Appendix we have, for the hypergeometric function,
\[F(\alpha + 1, 2; 3; z) = \frac{2}{\alpha(\alpha - 1)z^2} \left[ 1 + \frac{\alpha}{(1 - z)^{\alpha}} \right],\]
so that, using this result, \(f_1\) adopts the form
\[f_1(x, t) = \frac{i \kappa t^2}{2\pi a^2} \left\{ (x + \alpha at) \left[ \frac{e^{i \frac{\pi}{2} \alpha}}{(x + at + i0)^{\alpha}} - \frac{e^{-i \frac{\pi}{2} \alpha}}{(x + at - i0)^{\alpha}} \right] + \frac{e^{-i \frac{\pi}{2} \alpha}}{(x - i0)^{\alpha - 1}} - \frac{e^{i \frac{\pi}{2} \alpha}}{(x + i0)^{\alpha - 1}} \right\}.\] \hspace{1cm} (4.6)

Using at this point (4.1), (4.3), and (4.6), the final result for \(f\), up to first order in \(\kappa\), reads, invoking the sign function \(\text{Sgn}(x)\),
\[f(x, t) = \frac{1}{2a} \left[ \text{Sgn}(x + at) - \text{Sgn}(x) \right] + \]
\[\frac{i \kappa t^2}{2\pi a^2} \left\{ (x + \alpha at) \left[ \frac{e^{i \frac{\pi}{2} \alpha}}{(x + at + i0)^{\alpha}} - \frac{e^{-i \frac{\pi}{2} \alpha}}{(x + at - i0)^{\alpha}} \right] + \]
\[\frac{e^{-i \frac{\pi}{2} \alpha}}{(x - i0)^{\alpha - 1}} - \frac{e^{i \frac{\pi}{2} \alpha}}{(x + i0)^{\alpha - 1}} \right\}.\]
\[
\frac{e^{-i\frac{\pi}{2}\alpha}}{(x - i0)^{\alpha - 1}} - \frac{e^{i\frac{\pi}{2}\alpha}}{(x + i0)^{\alpha - 1}}.
\] 

(4.7)

From this expression for \( f \), we will obtain all approximate results reported in [15]. Thus, for \( x > 0 \) (4.7) becomes
\[
f(x, t) = \frac{\kappa \sin(\frac{\pi \alpha}{2}) \Gamma(\alpha - 1)}{\pi a^{2}} \left[ \frac{1}{x^{\alpha - 1}} - \frac{x + \alpha at}{(x + at)^{\alpha}} \right].
\] 

(4.8)

We have to distinguish two limiting cases. The first one is the asymptotic situation \( x >> at \). In this case,
\[
f(x, t) = \frac{1}{2\pi} \sin(\frac{\pi \alpha}{2}) \Gamma(\alpha + 1) \frac{\kappa t^{2}}{x^{\alpha + 1}}.
\] 

(4.9)

The second case limiting case is \( 0 < x << at \). The corresponding expression for \( f \) becomes
\[
f(x, t) = \frac{1}{\pi} \sin(\frac{\pi \alpha}{2}) \Gamma(\alpha - 1) \frac{\kappa}{a^{2}} x^{1 - \alpha}.
\] 

(4.10)

We consider signs now. When \( x + at < 0 \), from (4.7) we have
\[
f(x, t) = \frac{\kappa \sin(\frac{\pi \alpha}{2}) \Gamma(\alpha - 1)}{\pi a^{2}} \left[ \frac{1}{|x|^{\alpha - 1}} + \frac{x + \alpha at}{|x + at|^{\alpha}} \right]
\] 

(4.11)

Again, two special cases must be considered. One is for \( x << -at \) for which
\[
f(x, t) = \frac{1}{2\pi} \sin(\frac{\pi \alpha}{2}) \Gamma(\alpha + 1) \frac{\kappa t^{2}}{|x|^{\alpha + 1}}.
\] 

(4.12)

The other special situation is \( x < 0, x + at > 0, x >> -at \). Here,
\[
f(x, t) = \frac{1}{a} + \frac{1}{\pi} \sin(\frac{\pi \alpha}{2}) \Gamma(\alpha - 1) \frac{\kappa}{a^{2}} |x|^{1 - \alpha}.
\] 

(4.13)

At this stage, we have re-obtained all approximations given in [15], but using a more general procedure. More specifically, all approximations have been obtained from only one relation: Eq. (4.7), which, in turn, is deduced from our general formula (3.5).
5 Change of Variables

We assume that in the solar wind rest frame the particles’ transport is described by the fractional-diffusion equation with no advection term (that is, with $a = 0$ in (2.1)). The shock front (which started at $x_0 = -V_{sh}t_0$, moves with constant speed $V_{sh}$, and is regarded as highly localized in the $x$-coordinate) constitutes the source of the particles. Consequently, we have a fractional-diffusion equation with a uniformly moving Dirac’s delta source of the form $\delta(x - V_{sh}t)$. In order to have a stationary delta source we need to perform an appropriate change of coordinates, re-casting our problem in a reference frame where the shock front is stationary. We also change the origin of time so that the source starts being active at $t = 0$. In this new reference frame the transport equation has an advection term with advection velocity $a = V_{sh}$, and a stationary source $\delta(0)$ that starts at $t = 0$. After solving the diffusion-advection equation in this new frame (which is what we have done in the previous sections) we re-express the solution in terms of the original coordinates associated with the solar wind rest frame. This last step is succinctly described by the three correspondences $a \rightarrow v_{sh}$, $t \rightarrow t + t_0$, and $x \rightarrow x - v_{sh}t$, after which Eq. (3.5) adopts the appearance

$$f(x, t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n(t + t_0)^{n+1}}{n!} \Gamma(\alpha n + 1) B(1, n + 1) \times$$

$$\left[ \frac{e^{i\frac{\pi}{2}(\alpha n+1)}}{(x - v_{sh}t + i0)^{\alpha n+1}} F\left(\alpha n + 1, n + 1; n + 2; -\frac{v_{sh}(t + t_0)}{x - v_{sh}t + i0}\right) + \right.\
\left. \frac{e^{-i\frac{\pi}{2}(\alpha n+1)}}{(x - v_{sh}t - i0)^{\alpha n+1}} F\left(\alpha n + 1, n + 1; n + 2; -\frac{v_{sh}(t + t_0)}{x - v_{sh}t - i0}\right) \right]. \quad (5.1)$$

Accordingly, in the weak diffusion approximation (4.7) we have

$$f(x, t) = \frac{1}{2v_{sh}} [Sgn(x + v_{sh}t_0) - Sgn(x - v_{sh}t)] +$$

$$\frac{i\kappa\Gamma(\alpha - 1)}{2\pi V_{sh}^2} \left\{ (x + (\alpha - 1)v_{sh}t + v_{sh}t_0) \left[\frac{e^{i\frac{\pi}{2}\alpha}}{(x + v_{sh}t_0 + i0)^\alpha} - \right.\right.$$

$$\left. \frac{e^{-i\frac{\pi}{2}\alpha}}{(x + v_{sh}t_0 - i0)^\alpha} \right] + \left. \frac{e^{-i\frac{\pi}{2}\alpha}}{(x - v_{sh}t - i0)^\alpha} - \right.$$

$$\left. \frac{e^{i\frac{\pi}{2}\alpha}}{(x - v_{sh}t + i0)^\alpha} \right\} \quad (5.2)$$
6 Conclusions

We have provided an explicit analytical solution for and advection-diffusion equation involving fractional derivatives in the diffusion term. This equation governs the diffusion of particles in the solar wind injected at the front of a shock that travels at a constant upstream speed $V_{sh}$ in the solar wind rest frame. The shock is assumed to have a planar front, leading to a problem with an effective one dimensional geometry, where all the relevant quantities depend solely on time and on the coordinate $x$ measured along an axis perpendicular to the front.

We obtained the exact solution of the above mentioned equation in the $x$-configuration space (besides the associated formal solution in the $k$-space related to the previous one via a Fourier transform). Our solution allow us to obtain in a unified and systematic way all the relevant approximations that were previously discussed by LE, each one in a separated way.
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Appendix: Some properties of Hypergeometric Function

Using data from [18] we have

\[
\int_{0}^{t} \frac{t^n}{(x + at' \pm i0)\alpha n + 1} \, dt' = \frac{t^{n+1}}{(x \pm i0)\alpha n + 1} B(1, n + 1) \times \\
F\left( \alpha n + 1, n + 1, n + 2; -\frac{at}{x \pm i0} \right). \tag{A.1}
\]

Now we appeal to the transformation formula given in [19] for the hypergeometric function

\[
F(\alpha + 1, 2; 3; z) = \frac{2\Gamma(1 - \alpha)}{\Gamma(2 - \alpha)} (-1)^{\alpha + 1} z^{-\alpha - 1} F\left( \alpha + 1, \alpha - 1; \alpha; \frac{1}{z} \right) + \\
\frac{2\Gamma(\alpha - 1)}{\Gamma(\alpha + 1)} z^{-2} \, F\left( 2, 0; 2 - \alpha; \frac{1}{z} \right), \tag{A.2}
\]

with

\[
F(a, 0; c; z) = 1. \tag{A.3}
\]

In these circumstance we obtain

\[
F(\alpha + 1, 2; 3; z) = \frac{2\Gamma(1 - \alpha)}{\Gamma(2 - \alpha)} (-1)^{\alpha + 1} z^{-\alpha - 1} F\left( \alpha + 1, \alpha - 1; \alpha; \frac{1}{z} \right) + \\
\frac{2\Gamma(\alpha - 1)}{\Gamma(\alpha + 1)} z^{-2}
\] \tag{A.4}

Now we invoke the transformation formula [20]

\[
F\left( \alpha + 1, \alpha - 1; \alpha; \frac{1}{z} \right) = \left( 1 - \frac{1}{z} \right)^{-\alpha} F\left( -1, 1; \alpha; \frac{1}{z} \right), \tag{A.5}
\]

or

\[
F\left( \alpha + 1, \alpha - 1; \alpha; \frac{1}{z} \right) = \frac{z^\alpha}{(z - 1)^{\alpha}} \frac{(az - 1)}{\alpha z}. \tag{A.6}
\]

At this stage, we have, finally,

\[
F(\alpha + 1, 2; 3; z) = \frac{2}{\alpha(\alpha - 1)z^2} \left[ 1 + \frac{\alpha z - 1}{(1 - z)^\alpha} \right]. \tag{A.7}
\]