Abstract

The main purpose of these lectures is to discuss briefly recent methods of calculation of statistical properties of quantum eigenvalues for chaotic systems based on semi-classical trace formulas. Under the assumption that periodic orbit actions are non-commensurable it is demonstrated by a few different methods that the spectral statistics of chaotic systems without time-reversal invariance in the universal limit agrees with statistics of the Gaussian Unitary Ensemble of random matrices. The methods used permit to obtain not only the limiting statistics but also the way the spectral statistics of dynamical systems tends to the universal limit. The statistics of the Riemann zeta function zeros is considered in details.
1 Generalities

1.1 Trace formulas

The quantum density of states for a chaotic dynamical systems

\[ d(E) = \sum_n \delta(E - E_n) \] (1)

under quite general conditions can be expressed through a sum over classical periodic orbits by the Gutzwiller trace formula [1], [2] (plus corrections if necessary)

\[ d(E) = \bar{d}(E) + d^{(osc)}(E), \] (2)

where the smooth part, \( \bar{d}(E) \), for a \( f \)-dimensional system with a Hamiltonian \( H(\vec{p}, \vec{q}) \) is given by the Thomas-Fermi formula

\[ \bar{d}(E) = \int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^f} \Theta(E - H(\vec{p}, \vec{q})), \] (3)

and the (most interesting) oscillating part of the level density, \( d^{(osc)}(E) \), has the following form

\[ d^{(osc)}(E) = \frac{1}{\pi^2} \sum_{\text{ppo}} T_p \sum_{n=1}^{\infty} \left| \det\left(M_p^n - 1\right) \right|^{1/2} \cos\left(n \frac{S_p}{\hbar} - \frac{\pi}{2} \mu_p \right). \] (4)

Here the first summation is done over primitive classical periodic orbits (ppo) with energy \( E \) and the second one is performed over all repetitions of a given ppo. \( S_p \) is the classical action calculated for a ppo labeled by \( p \), \( M_p \) is the monodromy matrix for this trajectory, and \( \mu_p \) is its Maslov index [1], [2].

Similar formulas can be written also for other spectral functions.

In particular the staircase function has the following form

\[ N(E) = \int_{-\infty}^{E} d(E')dE' = \bar{N}(E) + N^{(osc)}(E), \] (5)

where

\[ \bar{N}(E) = \int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^f} \Theta(E - H(\vec{p}, \vec{q})), \] (6)

and

\[ N^{(osc)}(E) = \frac{1}{\pi} \sum_{\text{ppo}} \sum_{n=1}^{\infty} \frac{1}{n \left| \det\left(M_p^n - 1\right) \right|^{1/2}} \sin\left(n \frac{S_p}{\hbar} - \frac{\pi}{2} \mu_p \right). \] (7)
We shall need also the **dynamical zeta function** which (for 2-dimensional systems) equals the following product over \( p p o \)

\[
Z(E) = \prod_{p p o} \prod_{m=0}^{\infty} \left( 1 - e^{i S_p / \hbar - i \mu / 2} |\Lambda_p|^{1/2} \Lambda_p^m \right). \tag{8}
\]

Here \( \Lambda_p \) is the largest eigenvalue of the monodromy matrix (\( |\Lambda_p| > 1 \))

\[
M_p u = \Lambda_p u. \tag{9}
\]

This zeta function is connected with the density of states and the staircase function by the following relations

\[
d^{(osc)}(E) = -\frac{1}{2\pi i} \left( \frac{Z'(E)}{Z(E)} - \text{c.c.} \right), \tag{10}
\]

and

\[
N^{(osc)}(E) = -\frac{1}{2\pi i} (\ln Z(E) - \text{c.c.}). \tag{11}
\]

The trace formulas exist not only for dynamical systems but also for the **Riemann zeta function** \( \zeta(s) \) (and others number-theoretical zeta functions as well).

The Riemann zeta function is defined as

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \tag{12}
\]

where the product is taken over prime numbers. This function converges only when \( \text{Re} s > 1 \) but it is well known that it can be analytically continued in the whole complex \( s \)-plane, the only singularity being the pole at \( s = 1 \) with unit residue.

According to the famous Riemann conjecture all nontrivial zeros of this function have the form \( s_n = \frac{1}{2} + i E_n \) and the density of these zeros \( d(E) = \sum_n \delta(E - E_n) \) can be expressed by the following ‘trace’ formula

\[
d(E) = \tilde{d}(E) + d^{(osc)}(E), \tag{13}
\]

where

\[
\tilde{d}(E) = \frac{1}{2\pi} \ln \frac{E}{2\pi}, \tag{14}
\]
and
\[ d^{(\text{osc})}(E) = -\frac{1}{\pi} \sum_p \sum_{n=1}^{\infty} \ln p \cos(En \ln p), \] (15)
where the summation is performed over all prime numbers.

The staircase function for the Riemann zeros has a similar form
\[ N(E) = \bar{N}(E) + N^{(\text{osc})}(E), \] (16)
where
\[ \bar{N}(E) = \frac{E}{2\pi} (\ln \frac{E}{2\pi} - 1) + \frac{7}{8}, \] (17)
and
\[ N^{(\text{osc})}(E) = -\frac{1}{\pi} \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{n/2}} \sin(En \ln p). \] (18)

By comparing Eq. (4) and Eq. (15) one observes a remarkable correspondence between different quantities in these trace formulas [4].

Periodic orbits of chaotic systems ↔ primes,
periodic orbit period ↔ \( \ln p \).

The convergence properties of both formulas are also quite similar. The number of periodic orbits with period less than \( T \) for chaotic systems is asymptotically (see e.g. [5])
\[ N(T_p < T) = \frac{e^{hT}}{hT}, \] (19)
where constant \( h \) is called the topological entropy. The number of prime numbers less than \( x \) is given by the prime number theorem [3]
\[ N(p < x) \equiv \pi(x) = \frac{x}{\ln x}. \] (20)
As \( \ln p \equiv T_p \) this expression has the form similar to (19) with \( h = 1 \)
\[ N(T_p < T) = \frac{e^T}{T}. \] (21)

Due to these similarities number-theoretical zeta functions play the role of simple (but non-trivial) models of quantum chaos and in these lectures we consider in parallel both dynamical systems and zeta functions and shall check our methods first on the Riemann case.
1.2 Random matrix theory

Wigner and Dyson in the fifties had proposed to describe complicated (and mostly unknown) Hamiltonian of heavy nuclei by a member of an ensemble of random matrices (see [6]) and they argued that the type of this ensemble depends only on the symmetry of the Hamiltonian.

For systems without time-reversal invariance the relevant ensemble is the Gaussian Unitary Ensemble (GUE), for systems invariant with respect to time-reversal the ensemble should be the Gaussian Orthogonal Ensemble (GOE) and for systems with time-reversal invariance but with half-integer spin energy levels should be described according to the Gaussian Symplectic Ensemble (GSE) of random matrices [6, 7]. For these classical ensembles all correlation functions can be written explicitly [7]. In particular 2-point correlation functions

\[ R_2(x) = 1 + \delta(x) - Y_2(x), \]  \hspace{1cm} (22)

have the following form [7], [10].

For GOE

\[ Y_2(x) = (\frac{\sin \pi x}{\pi x})^2 - (\text{Si}(\pi x) - \frac{\pi}{2} \epsilon(x))\left(\frac{\cos \pi x}{\pi x} - \frac{\sin \pi x}{(\pi x)^2}\right). \]  \hspace{1cm} (23)

Here

\[ \text{Si}(x) = \int_0^x \frac{\sin y}{y} dy, \]  \hspace{1cm} (24)

and \( \epsilon(x) = \text{sgn}(x) \).

For GUE

\[ Y_2(x) = (\frac{\sin \pi x}{\pi x})^2. \]  \hspace{1cm} (25)

For GSE

\[ Y_2(x) = (\frac{\sin 2\pi x}{2\pi x})^2 - \text{Si}(2\pi x)(\frac{\cos 2\pi x}{2\pi x} - \frac{\sin 2\pi x}{(2\pi x)^2}). \]  \hspace{1cm} (26)

2-point correlation form factor is the Fourier transform of the 2-point correlation function

\[ K(t) = \int_{-\infty}^{\infty} R_2(x) e^{2\pi itx} dx, \]  \hspace{1cm} (27)

and for three classical ensembles it has the following forms.
For GOE

\[
K(t) = \begin{cases} 
2t - t \ln(1 + 2t) & 0 < t < 1 \\
2 - t \ln((2t + 1)/(2t - 1)) & t > 1 
\end{cases} 
\]  
(28)

For GUE

\[
K(t) = \begin{cases} 
t & 0 < t < 1 \\
1 & t > 1 
\end{cases} 
\]  
(29)

For GSE

\[
K(t) = \begin{cases} 
\frac{1}{2}t - \frac{1}{4}t \ln(|1 - t|) & 0 < t < 2 \\
1 & t > 2 
\end{cases} 
\]  
(30)

The nearest neighbor distributions for classical ensembles can be expressed through solutions of certain integral equations and numerically they are close to the Wigner surmises

\[
\text{GOE : } p(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right). 
\]  
(31)

\[
\text{GUE : } p(s) = \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi} s^2\right). 
\]  
(32)

\[
\text{GSE : } p(s) = \frac{2^{18}}{3^6 \pi^3} s^4 \exp\left(-\frac{64}{9\pi} s^2\right). 
\]  
(33)

Later it was understood that the same conjectures can be applied not only for heavy nuclei but also for simple dynamical systems and to-day standard accepted conjectures are the following

- The energy levels of classically integrable systems on the scale of the mean level density behave as independent random variables and their distribution is close to the Poisson distribution (Berry, Tabor [8]).

- The energy levels of classically chaotic systems are not independent but on the scale of the mean level density they are distributed as eigenvalues of random matrix ensembles depending only on symmetry properties of the system considered (Bohigas, Giannoni, Schmit [9]).

- For systems without time-reversal invariance the distribution of energy levels should be close to the distribution of the Gaussian Unitary Ensemble (GUE) characterized by quadratic level repulsion.
For systems with time-reversal invariance the corresponding distribution should be close to that of the Gaussian Orthogonal Ensemble (GOE) with linear level repulsion.

And for systems with time-reversal invariance but with half-integer spin energy levels should be described according to the Gaussian Symplectic Ensemble (GSE) of random matrices with quartic level repulsion.

These conjectures are very well confirmed by numerical calculations (see e.g. [10], [11]). The main purpose of these lectures is the developing of methods which permit to prove analytically one part of these conjectures, namely that in the universal limit (i.e. on the scale of the mean level density) the 2-point correlation function for systems without time-reversal invariance agrees with Eq. (25). For further references it is convenient to rewrite this expression in the dimensional form

\[ \tilde{R}_2(\epsilon) = d^2 R_2(d\epsilon). \]  

It gives

\[ \tilde{R}_2(\epsilon) = d^2 + \tilde{R}_2(\epsilon) + R_2^{(osc)}(\epsilon), \]  

where the smooth part

\[ \tilde{R}_2(\epsilon) = -\frac{1}{2\pi^2\epsilon^2}, \]  

and the oscillating part

\[ R_2^{(osc)}(\epsilon) = \frac{e^{2\pi i d\epsilon} + e^{-2\pi i d\epsilon}}{4\pi^2\epsilon^2}. \]  

The plan of the paper is the following. In Section 2 we relate through the trace formulas correlation functions of quantum eigenvalues (and of Riemann zeros) with periodic orbit sums and in Section 2.1 the simplest approximation of evaluating these sums called the diagonal approximation is discussed. But as shown in Section 2.2 this approximation is valid only for large energy difference and to find correlation functions in the full range new methods are needed. In Section 3 a method specific for the Riemann zeta function is discussed. To calculate off-diagonal terms we use the Hardy-Littlewood conjecture for distribution of pairs of primes and in Section 3.1 it is demonstrated that this conjecture leads to interesting formulas for the 2-point correlation.
function of the Riemann zeros. In the universal limit this correlation function tends to the GUE results but it also permits to investigate how the correlation function tends to the universal result. In Section 3.2 we briefly mention a certain model where a generalization of the Hardy-Littlewood conjecture also permits to calculate the 2-point correlation function. But the Hardy-Littlewood conjecture can be applied only for primes and it cannot be generalized for dynamical systems. In Section 4 a certain method is proposed which overcomes this difficulty. The main ingredient of this method is an artificial construction of approximate density of states which has the correct analytical properties but requires the knowledge only of finite number of periodic orbits. In Section 4.1 it is shown that for the case of the Riemann zeros the method gives exactly the same result as was obtained by using the Hardy-Littlewood conjecture. In Section 4.2 this method is applied to dynamical systems and it was demonstrated that under the assumption that all periodic orbit actions are non-commensurable it is possible to prove that spectral statistics of generic dynamical systems without time-reversal invariance in the universal limit tends to the GUE statistics. In Section 5 an other method of calculation of spectral statistics is proposed. The method is based on the universality of different random matrix ensembles and it is shown that it gives exactly the same expressions for 2-point correlation functions for the Riemann zeros and dynamical systems as have been obtained in previous Sections by different methods. Finally in Section 6 one more method is discussed which is based on the Riemann-Siegel type resummation of the trace formulas which also leads to the same expressions for 2-point correlation functions. Though the exact mathematical proof of the above results is not known and all our methods should be considered as heuristics or ‘physical’ proofs the agreement between different methods strongly indicates the correctness of the results.

2 Correlation functions

Formally $n$-point correlation functions of energy levels are defined as the probability of having $n$ energy levels at given positions and they are connected to the density of states by the relations

$$R_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = \langle d(E + \epsilon_1) d(E + \epsilon_2) \ldots d(E + \epsilon_n) \rangle, \quad (38)$$
where the brackets $< \ldots >$ denote the smoothing over an appropriate energy window

$$
<f(E)> = \int f(E')\sigma(E - E')dE',
$$

(39)

with

$$
\int \sigma(E)dE = 1.
$$

(40)

Function $\sigma(E)$ is assumed centered near zero and has a width $\Delta E$ which obeys inequalities

$$
\Delta E_q \ll \Delta E \ll \Delta E_{cl} \ll E.
$$

(41)

Here $\Delta E_q$ is of the order of the mean level separation, $\Delta E_q \approx 1/d$, and $\Delta E_{cl}$ denotes the energy scale at which classical dynamics changes noticeable.

Let us write the trace formula for the density of states (4) in the form

$$
d(E) = \tilde{d}(E) + \sum_{p,n} T_p A_{p,n} e^{i n S_p(E)/\hbar} + c.c.,
$$

(42)

where

$$
A_{p,n} = \frac{1}{2\pi\hbar|\det(M_p^n - 1)|^{1/2}} e^{-\pi i n \mu_p/2}.
$$

(43)

Substituting this expression to the formula for 2-point correlation function one gets

$$
R_2(\epsilon_1, \epsilon_2) = \tilde{d}^2 + \sum_{p_1,p_2} T_{p_1} T_{p_2} A_{p_1,n_1} A^*_{p_2,n_2} < \exp \frac{i}{\hbar}(S_{p_1}(E + \epsilon_1) - S_{p_2}(E + \epsilon_2)) > + c.c.,
$$

(44)

and the terms with the sum of actions are assumed to be washed out by the smoothing procedure.

Expanding the actions and taking into account that $\partial S(E)/\partial E = T(E)$ where $T(E)$ is the classical period of motion one finds

$$
R_2(\epsilon_1, \epsilon_2) = \tilde{d}^2 + \sum_{p_1,n_1} A_{p_1,n_1} A^*_{p_2,n_2} < \exp \frac{i}{\hbar}(n_1 S_{p_1}(E) - n_2 S_{p_2}(E)) > \\
\times \exp \frac{i}{\hbar}(n_1 T_{p_1}(E)\epsilon_1 - n_2 T_{p_2}(E)\epsilon_2) + c.c.
$$

(45)
2.1 Diagonal approximation

Berry [4] proposed to estimate this sum by taking into account terms only with exactly the same actions having in mind that terms with different values of actions should be small after the smoothing. Therefore in this approximation (called the diagonal approximation)

\[ R_2^{(\text{diag})}(\epsilon) = \sum_{p,n} T_p^2 |A_{p,n}|^2 e^{i n T_p(E) \epsilon} + \text{c.c.}, \]  

(46)

Here \( \epsilon = \epsilon_1 - \epsilon_2 \) and the sum is taken over all periodic orbits with exactly the same action.

Introducing the classical zeta function \([5]\)

\[ Z_{cl}(s) = \prod_{ppo} (1 - e^{s T_p})^{-1}, \]  

(47)

one can rewrite the 2-point correlation function in the form

\[ R_2^{(\text{diag})}(\epsilon) = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial \epsilon^2} \ln \Delta(\epsilon), \]  

(48)

where

\[ \Delta(\epsilon) = |Z_{cl}(i\epsilon)|^2 \Phi^{(\text{diag})}(\epsilon), \]  

(49)

and function \( \Phi^{(\text{diag})}(\epsilon) \) is defined as the following convergent sum over periodic orbits

\[ \Phi^{(\text{diag})}(\epsilon) = \exp \left( \sum_{ppo} \sum_{m=1}^{\infty} \frac{1}{m|A_p|^m} \left( \frac{1}{m(1 - \Lambda_p^{-m})^2} - 1 \right) e^{i m T_p(E)} + \text{c.c.} \right). \]  

(50)

Another useful form of these relations is the expression for the diagonal approximation of 2-point form factor defined in Eq. (27)

\[ K^{(\text{diag})}(t) = 2\pi \sum_{p,n} T_p^2 |A_{p,n}|^2 \delta(2\pi t - n T_p(E)) + \text{c.c.} \]  

(51)

\(^1\)Often the classical zeta function is introduced in such a manner that

\[ \sum_{ppo} \sum_{k=1}^{\infty} \frac{e^{i T_p k}}{k |\det(A_p^k - 1)|} = \ln Z_{cl}, \]  

which gives \( Z_{cl}(s) = \prod_{ppo} \prod_{m=0}^{\infty} \left( 1 - \frac{e^{i T_p}}{|A_p|^{m+1}} \right)^{-(m+1)}. \)

As the convergent properties of both zeta functions are the same we chose the simplest definition sufficient for our purposes.
According to the Hannay-Ozorio de Almeida sum rule for ergodic systems \[12\] one can compute such sums by substituting the local density of periodic orbits

\[
\rho_p = \frac{|\det(M_p - 1)|}{T_p},
\]

and consequently

\[
K^{(\text{diag})}(t) = \frac{g}{2\pi} \int T_p \delta(2\pi t - T_p) dT_p = gt,
\]

where \(g\) is the mean multiplicity of periodic orbits (i.e. the number of periodic orbits with exactly the same action). For generic systems without time-reversal invariance there is no reasons for equality of actions for different periodic orbits and \(g = 1\) but for systems with time-reversal invariance each orbits can be passed in two directions therefore in general for such systems \(g = 2\). Comparing (53) with Eqs. (28) and (29) one concludes that the diagonal approximation reproduces the correct small-\(t\) behavior of form-factors of classical ensembles.

Unfortunately, \(K^{(\text{diag})}(t)\) grows with increasing of \(t\) but the exact form-factor for systems without spectral degeneracy should tends to \(\bar{d}\) for large \(t\) which reflects the existence of the delta function in \(R_2(\epsilon)\) for small \(\epsilon\) \[4\]

\[
R_2(\epsilon) \to \bar{d}\delta(\epsilon), \quad \text{when } \epsilon \to 0,
\]

or

\[
K(t) \to \bar{d}, \quad \text{when } t \to \infty.
\]

This evident contradiction clearly indicates that the diagonal approximation cannot be correct for all values of \(t\) and more complicated tools are needed to obtain the full form-factor.

### 2.2 Criterion of applicability of diagonal approximation

One can give a (pessimistic) estimate till what time the diagonal approximation can be correct by the following method. The main ingredient of the diagonal approximation is the assumption that after smoothing all off-diagonal terms give negligible contribution. This condition is almost the same as the condition of the absence of quantum interference. But it is known that the
quantum interference is not important for times smaller than the Ehrenfest time which is of the order of

\[ t_E \approx \frac{1}{\lambda} \ln(1/\hbar), \quad (56) \]

where \( \lambda \) is a (classical) constant of the order of the Lyapunov exponent defined in such a way that the mean splitting of two nearby trajectories for the time \( t \) grows as \( \exp(\lambda t) \). For billiards \( \lambda = k\lambda' \) where \( k \) is the momentum and \( \lambda' \) determines the deviation of two trajectories with the length \( L = kt \), and the role of \( \hbar \) plays \( k^{-1} \).

\[ t_E \approx \frac{1}{\lambda'k} \ln(k). \quad (57) \]

More refined estimates can be done as follows. The off-diagonal terms can be neglected if

\[ \langle \exp \left( \frac{i}{\hbar} (S_{p_1}(E) - S_{p_2}(E)) \right) \rangle \ll 1. \quad (58) \]

But this quantity is small provided the difference of period of two orbits \( \Delta T = T_{p_1} - T_{p_2} \) times the energy window \( \Delta E \) used in the definition of smoothing procedure is large

\[ \frac{1}{\hbar} (T_{p_1} - T_{p_2}) \Delta E \gg 1. \quad (59) \]

For billiards \( T_p = L_p/k \) and this condition means that one has to consider all periodic orbits such that their difference of lengths is

\[ L_{p_1} - L_{p_2} \gg \frac{\hbar k}{\Delta E}. \quad (60) \]

But the number of periodic orbits with the length \( L \) grows exponentially

\[ N(L_p < L) = \frac{e^{hL}}{hL}, \quad (61) \]

where \( h \) is of the order of the Lyapunov exponent \( \lambda' \). Therefore in the interval \( L, L + \delta l \) there is \( e^{hL} \delta l / L \) orbits and the mean difference of lengths between orbits with the length \( < L \) is of the order of

\[ \Delta L = L \exp(-hL). \quad (62) \]
To fulfilled the above condition one has to restrict the maximum length of periodic orbits, $L_m$, by

$$L_m \exp(-hL_m) \approx \frac{k\hbar}{\Delta E}$$

(63)

In the limit of large $L_m$ it gives

$$L_m \approx \frac{1}{\hbar} \ln \frac{\Delta E}{k\hbar h},$$

(64)

which corresponds to the Ehrenfest estimate above.

Let us denote $\hbar = 1/l_0$ where $l_0$ has the dimensionality of the length. Then the above estimate can be transform into the following form

$$L_m \approx l_0 \ln \frac{\Delta E}{E_T},$$

(65)

and $E_T$ is an analog of the Thouless energy

$$E_T = \frac{\hbar}{\tau_T} \quad \text{and} \quad \tau_T = \frac{l_0}{k}.$$

(66)

Note that the Heisenberg time is

$$t_H = 2\pi \bar{d},$$

(67)

and for billiards because $\bar{d}$ is a constant

$$t_E \ll t_H.$$  

(68)

For the Riemann zeta function the situation is better because the role of ‘energy’ in this case plays the ‘momentum’ and the density of states for the Riemann zeta function is $(\ln(E/2\pi))/(2\pi)$. As in this case $\hbar = 1$

$$t_E = t_H,$$

(69)

and the diagonal approximation for the Riemann zeta function is valid till the Heisenberg time [13], [14].

This type of estimates directly indicates that the diagonal approximation for dynamical systems can not, strictly speaking, be used to obtain an information about the form-factor for large value of $t$. (Note that for GUE systems the diagonal approximation gives the expected answer till the Heisenberg time but it just signifies that one has to find special reasons why all other terms cancel.)
Beyond the diagonal approximation

The simplest and the most natural way of semi-classical computation of 2-point correlation functions is to find a method of calculating off-diagonal terms. We shall discuss here this type of computation on the example of the Riemann zeta function where much more information than for dynamical systems is available.

The trace formula for the Riemann zeta function may be rewritten in the form

\[ d^{(osc)}(E) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \Lambda(n) \cos(E \ln n), \]  

(70)

where

\[ \Lambda(n) = \begin{cases} \ln p, & \text{if } n = p^k, \\ 0, & \text{otherwise}. \end{cases} \]  

(71)

The connected 2-point correlation function of the Riemann zeros, \( R^{(c)}_2 = R_2 - d^2 \), is

\[ R^{(c)}_2(\epsilon_1, \epsilon_2) = \frac{1}{4\pi^2} \sum_{n_1, n_2} \frac{\Lambda(n_1)\Lambda(n_2)}{\sqrt{n_1n_2}} < e^{i(E_1+\epsilon_1)\ln n_1- i(E_2+\epsilon_2)\ln n_2} > +c.c.. \]  

(72)

The diagonal approximation corresponds to taking into account terms with \( n_1 = n_2 \)

\[ R^{(diag)}_2(\epsilon_1, \epsilon_2) = \frac{1}{4\pi^2} \sum_n \frac{\Lambda^2(n)}{n} (e^{i(\epsilon_1-\epsilon_2)\ln n} + c.c.) \]

\[ = \frac{1}{4\pi^2} \sum_{p, m} \frac{\ln^2 p}{p^m} (e^{i(\epsilon_1-\epsilon_2)m\ln p} + c.c.). \]  

(73)

This expression may be transform as follows

\[ R^{(diag)}_2(\epsilon) = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial \epsilon^2} \ln \Delta(\epsilon), \]  

(74)

where

\[ \Delta(\epsilon) = |\zeta(1 + i\epsilon)|^2 \Phi^{(diag)}(\epsilon), \]  

(75)

and function \( \Phi^{(diag)}(\epsilon) \) is given by a convergent sum over prime numbers

\[ \Phi^{(diag)}(\epsilon) = \exp \left( \sum_p \sum_{m=1}^{\infty} \frac{1 - m}{m^2 p^m} e^{i m \ln p \epsilon} + c.c. \right). \]  

(76)
In the limit $\epsilon \to 0 \ \zeta(1 + i\epsilon) \to (i\epsilon)^{-1}$ and $\Phi^{(\text{diag})}(\epsilon) \to \text{const}$. Therefore in this limit

$$R_2^{(\text{diag})}(\epsilon) \to -\frac{1}{2\pi^2 \epsilon^2},$$

which agrees with the smooth part of the GUE result (36).

The off-diagonal contribution has the form

$$R_2^{(\text{off})}(\epsilon_1, \epsilon_2) = \frac{1}{4\pi^2} \sum_{n_1 \neq n_2} \frac{\Lambda(n_1)\Lambda(n_2)}{\sqrt{n_1 n_2}} (\epsilon_1 \ln(n_1/n_2) + \epsilon_2 \ln n_2) > +\text{c.c.})$$

The term $\exp(iE \ln(n_1/n_2))$ oscillates quickly if $n_1$ is not close to $n_2$. Denoting

$$n_1 = n_2 + d$$

and expanding all smooth functions on $d$ one gets

$$R_2^{(\text{off})}(\epsilon) = \frac{1}{4\pi^2} \sum_{n,d} \frac{\Lambda(n)\Lambda(n+d)}{n} (\epsilon \ln(n/d) + \epsilon \ln n) > +\text{c.c.})$$

where $\epsilon = \epsilon_1 - \epsilon_2$.

The main problem is clearly seen here. The function

$$F(n, d) = \Lambda(n)\Lambda(n+d)$$

is quite a wild function as it is nonzero only when both $n$ and $n+d$ are power of prime numbers. As we have assumed that $d \ll n$ it is quite natural to assume that the dominant contribution to the 2-point correlation function will come from the mean value of this function over all $n$, i.e. one has to substitute into $R_2^{(\text{off})}(\epsilon)$ instead of $F(n, d)$ its mean value

$$\alpha(d) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Lambda(n)\Lambda(n+d).$$

### 3.1 The Hardy-Littlewood conjecture

Fortunately the explicit expression for this function comes from the famous Hardy–Littlewood conjecture [15]. There are two different forms of this conjecture which, of course, can be mutually transformed

$$\alpha(d) = C_2 \prod_{p|d} \frac{p - 1}{p - 2},$$

15
where the product is taken over all prime divisors of \( d \) bigger than 2 and \( C_2 \) is the so-called twin prime constant

\[
C_2 = 2 \prod_{p>2} \left( 1 - \frac{1}{(p-1)^2} \right) \approx 1.32032 \ldots
\] (84)

The other form which will be useful for us is expressed through the so-called singular series

\[
\alpha(d) = \sum_{(p,q)=1} \exp\left( 2\pi i \frac{p}{q} d \right) \left( \frac{\mu(q)}{\psi(q)} \right)^2,
\] (85)

where the sum is taken over all natural \( q \) and all integer \( p \) co-prime to \( q \) (\( p < q \)). Function \( \mu(n) \) is the Mobius function defined through the factorization of \( n \) on prime factors

\[
\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \ldots p_k \\ 0 & \text{if } n \text{ is divisible on } p^2 \\ \end{cases}.
\] (86)

Function \( \psi(n) \) is the Euler function which counts the number of integers smaller than \( n \) and co-prime to \( n \)

\[
\psi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),
\] (87)

where the product is taken over all prime divisors of \( n \).

Taking the above formulae as granted we get

\[
R_2^{(off)}(\epsilon) = \frac{1}{4\pi^2} \sum_n \frac{1}{n} e^{i \epsilon \ln n} \sum_d \alpha(d) e^{i E \frac{d}{n}} + c.c.
\] (88)

After substitution the formula for \( \alpha(d) \) and performing the sum over all \( d \) one obtains

\[
R_2^{(off)}(\epsilon) = \frac{1}{4\pi^2} \sum_n \frac{1}{n} e^{i \epsilon \ln n} \sum_{(p,q)=1} \left( \frac{\mu(q)}{\psi(q)} \right)^2 \delta\left( p \frac{E}{2\pi n} - q \right) + c.c.,
\] (89)

where the summation is taken over all pairs of mutually co-prime positive integers \( p \) and \( q \) (without the restriction \( p < q \)).
Changing the sum over \( n \) to the integral permits to transform this expression to the sum over that values of \( n \) where
\[
\frac{p}{q} - \frac{E}{2\pi n} = 0,
\]
and in this approximation

\[
R_2^{(\text{off})}(\epsilon) = \frac{1}{4\pi^2} e^{i\epsilon \ln \frac{\bar{d}}{2\pi}} \sum_{(p,q)=1} \left( \frac{\mu(q)}{\psi(q)} \right)^2 \left( \frac{q}{p} \right)^{1+i\epsilon} + \text{c.c.} \tag{90}
\]

Using the formula
\[
\sum_{(p,q)=1} f(p) = \sum_{k=1}^{\infty} \sum_{\delta|q} f(k\delta) \mu(\delta), \tag{91}
\]
and taking into account that \( 2\pi \bar{d} = \ln(E/2\pi) \) we obtain

\[
R_2^{(\text{off})}(\epsilon) = \frac{1}{4\pi^2} |\zeta(1+i\epsilon)|^2 e^{2\pi i \bar{d} \epsilon} \Phi^{(\text{off})}(\epsilon) + \text{c.c.}, \tag{92}
\]
where function \( \Phi^{(\text{off})}(\epsilon) \) is given by a convergent product over primes

\[
\Phi^{(\text{off})}(\epsilon) = \prod_p \left( 1 - \frac{(1-p^{i\epsilon})^2}{(p-1)^2} \right), \tag{93}
\]
and \( \Phi^{(\text{off})}(0) = 1 \).

In the limit of small \( \epsilon \)

\[
R_2^{(\text{off})}(\epsilon) = \frac{1}{(2\pi \epsilon)^2} \left( e^{2\pi i \bar{d} \epsilon} + e^{-2\pi i \bar{d} \epsilon} \right), \tag{94}
\]
which exactly corresponds to the GUE results for the oscillating part of the 2-point correlation function \( \mathcal{B} \).

The above calculations demonstrates how one can compute the 2-point correlation function through the knowledge of pair-correlation function of periodic orbits. For the Riemann case one can prove under the same conjectures that all \( n \)-point correlation functions of Riemann zeros tend to corresponding GUE results \( \mathcal{C} \).
The interesting consequence of the above formula is the expression for the 2-point form-factor

\[ K^{(\text{off})}(t) = \frac{1}{4\pi^2} \sum_{(p,q)=1} \left( \frac{\mu(q)}{\psi(q)} \right)^2 \left( \frac{q}{p} \right)^2 \delta(t - \bar{d} - \frac{1}{2\pi} \ln \frac{q}{p}). \]  

(95)

This formula means that the off-diagonal 2-point form factor is a sum over \( \delta \)-functions in special points which are situated in a vicinity of the Heisenberg time plus a difference of periods of two pseudo-orbits (= logarithm of the difference between two integers). This set of \( \delta \)-functions is dense but the largest peaks correspond to the shortest pseudo-orbits. Similarly the 2-point diagonal form factor is the sum of \( \delta \)-functions in the positions of periodic orbits

\[ K^{(\text{diag})}(t) = \frac{1}{4\pi^2} \sum_{p,m} \frac{\ln^2 p}{p^m} \delta(t - \frac{m}{2\pi} \ln p). \]  

(96)

The smooth values corresponding to the random matrix predictions appears only after a smoothing of these functions over a suitable interval of \( t \).

### 3.2 Arithmetical systems

Similar behavior has been observed in a completely different model, namely for distribution of eigenvalues of the Laplace–Beltrami operator for the modular domain \([17]\). Using a generalization of the Hardy-Littlewood method it was shown that in this model the 2-point correlation form factor can be written in the following form

\[ K(t) = \frac{1}{\pi^3 k} \sum_{(p,q)=1} \left| \frac{q}{p} \beta(p,q) \right|^2 \delta(t - t_{p,q}). \]  

(97)

where

\[ t_{p,q} = \frac{2}{k} \ln \frac{kq}{\pi p}, \]  

(98)

and

\[ \beta(p,q) = \frac{S(p,p;q)}{q^2 \prod_{\omega|q} (1 - \omega^{-2})}. \]  

(99)
Here the product is taken over all prime divisors of $q$ and $S(p,p;q)$ is the Kloosterman sums

$$S(n,m;c) = \sum_{(d,c)=1} \exp \left( \frac{2\pi}{c} (nd + md^{-1}) \right). \quad (100)$$

This model belongs to the so-called arithmetical models which are models on the constant curvature surfaces generated by discrete arithmetic groups. For all these models due to the exponential multiplicity of periodic orbits one expects [18] that the spectral statistics will tend to the Poisson distribution though from classical point of view all these models are the best known examples of classically chaotic motion. Using the above expression one can prove this statement for the modular domain.

4 Construction of the density of states from finite number of periodic orbits

The main difficulty in using the trace formulas is their divergent character. They cannot converge on the real axis as they have to produce $\delta$-functions singularities there. Usually they are defined by an analytical continuation from a region in the complex plane of energy which do not touch the real axis. E.g. the Riemann zeta function converges only when $\text{Re } s > 1$ but zeros are assumed to lie on the axis $\text{Re } s = 1/2$. The diagonal approximation consists in some sense on the computing the density of states from a sum over a finite number of periodic orbits but such sums can never have $\delta$-function singularities. In this section we shall discuss a special method [19] which permits to avoid this difficulty and produces an (artificial) expression for the density of states with required singularities from the knowledge of finite number of periodic orbits.

Let us write the semi-classical formula not for the density of states but for the staircase function

$$N_{T^*}(E) = \tilde{N}(E) + N_{T^*}^{(osc)}(E), \quad (101)$$

where the oscillating part of this function is truncated (smoothly if necessary) so to include periodic orbits with period up to a fixed period $T^*$, where $T^*$
is a parameter to be fixed later
\[ N_{T^*}^{(osc)}(E) = 2 \sum_{T_p < T^*} \sum_{n=1}^{\infty} \tilde{A}_{p,n} \sin(n\frac{S_p}{\hbar} - \frac{\pi}{2}\mu_p), \] (102)

and for dynamical systems the pre-factor is
\[ \tilde{A}_{p,n} = \frac{1}{2\pi |\det(M_p^n - 1)|^{1/2}}, \] (103)

and for the Riemann zeta function
\[ \tilde{A}_{p,n} = \frac{-1}{2\pi np^{n/2}}, \] (104)

In the limit \( T^* \to \infty \) function \( N_{T^*}(E) \) should have a unit jump each time when \( E \) equals an eigenvalue. For finite value of \( T^* \) one can at the best have a smooth increase at these points. The idea of the proposed method is to define semi-classical eigenvalues \( E_n \) according to the following ‘quantization condition’ [21]
\[ N_{T^*}(E_n) = n + \frac{1}{2}. \] (105)

The main advantage of this method is that it cannot miss any one level simply because lines \( N = n + 1/2 \) will in any case cross the curve \( N(E) \) and will produce (approximate) semi-classical energy levels. In principle, one can obtain additional levels if the curve \( N_{T^*}(E) \) has decreasing parts. But for not too big number of included orbits it is not the case and this method is quite efficient for numerical computations of semi-classical levels [20].

The other important point (which explain why we take into account the infinite sum over repetitions of periodic orbit) is that
\[ \exp(2\pi i N_{T^*}(E)) = e^{2\pi i \tilde{N}_{T^*}(E)} \tilde{z}_{T^*}(E), \] (106)

where
\[ \tilde{z}_{T^*}(E) = \prod_{T_p < T^*} \prod_{m=0}^{\infty} (1 - e^{iS_p/\hbar - i\pi\mu_p/2}), \] (107)

is a truncated product over periodic orbits and the quantization condition (105) is equivalent to calculation of semi-classical energy levels \( E_n \) from the condition
\[ Z_{T^*}(E_n) = 0, \] (108)
where \( Z_{T^*}(E) \) is a special form of the dynamical zeta function

\[
Z_{T^*}(E) = z_{T^*}(E) + e^{2\pi i \bar{N}} z_{T^*}^*(E).
\]  

(109)

The above expression has a Riemann–Siegel form \([23]\) and automatically obeys the important functional equation for the zeta function

\[
Z(E) = e^{2\pi i \bar{N}} Z^*(E),
\]  

(110)

which is crucial in proving the correct analytical properties of the zeta function.

Let us define a new bootstrapped density

\[
D_{T^*}(E) = \sum_n \delta(E - E_n),
\]  

(111)

where instead of exact energy levels we use the semi-classical ones defined as solutions of quantization condition (105) (or (108). Rewriting \( D_{T^*}(E) \) in the form

\[
D_{T^*}(E) = d_{T^*}(E) \sum_n \delta(N_{T^*}(E) - n - 1/2),
\]  

(112)

where \( d_{T^*}(E) = dN_{T^*}(E)/dE \) and using the Poisson summation formula one gets

\[
D_{T^*}(E) = d_{T^*}(E) \sum_{k=-\infty}^{\infty} (-1)^k \exp(2\pi i k N_{T^*}(E)).
\]  

(113)

We called it a bootstrapped density as it has all Fourier harmonics and not only the ones with period \( T < T^* \), the role of the effective orbits introduced beyond \( T^* \) being to generate the correct analytical properties associated with the discreteness of quantum spectrum.

Substituting this expression to the formula for the 2-point correlation function one gets

\[
R_2(\epsilon_1, \epsilon_2) = < d_{T^*}(E + \epsilon_1) d_{T^*}(E + \epsilon_2) \sum_{k_1,k_2} (-1)^{k_1-k_2} \\
\times \exp(2\pi i (k_1 N_{T^*}(E + \epsilon_1) - N_{T^*}(E + \epsilon_2))) > .
\]  

(114)

Consider first the \( k_1 = k_2 = 0 \) term, which we write in the form

\[
< d_{T^*}(E + \epsilon_1) d_{T^*}(E + \epsilon_2) >= d^2 + R_{2(diag)}^{(diag)}(\epsilon_1, \epsilon_2),
\]  

(115)

21
where

\[ R_2^{(diag)}(\epsilon_1, \epsilon_2) = \frac{1}{4\pi^2} \frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} \ln \Delta(\epsilon_1, \epsilon_2), \] (116)

and

\[ \ln \Delta(\epsilon_1, \epsilon_2) = 4\pi^2 < N_T^{(osc)}(E + \epsilon_1)N_T^{(osc)}(E + \epsilon_2) >. \] (117)

Because only a finite number of orbits enters this expression we assume that the resulting sum over pairs of periodic orbits can be replaced by the diagonal terms, for which only orbits with exactly the same actions will contribute. The result can be expressed in the form

\[ \Delta(\epsilon_1, \epsilon_2) = |Z_g(i\epsilon)|^2, \] (118)

where

\[ Z_g(s) = \prod_{T_p < T^*} \prod_{n=1}^{\infty} \exp\left(\frac{4\pi^2}{n^2} g_p |A_{p,n}|^2 e^{i n T_p s}\right), \] (119)

and \( g_p \) is the number of orbits with period \( T_p \) (multiplicity of periodic orbit).

In generic systems the multiplicity \( g_p \) is the same for almost all orbits and so, if its value is denoted \( g \)

\[ Z_g(s) = Z^g(s), \] (120)

where \( Z(s) \) is defined by the same formula but with \( g_p = 1 \). For systems without time-reversal invariance \( g = 1 \), and for systems whose dynamics is time-reversal symmetric \( g = 2 \).

Since the \( k_1 = k_2 = 0 \) term corresponds to the usual diagonal approximation, the other terms are representing the off-diagonal contributions, \( R_2^{(off)} \).

To evaluate them we note that the Taylor expansion in powers of \( \epsilon \) of the mean staircase function leads to a term \( (k_1 - k_2)N(E) \) in the phase which is of the order of \( \hbar^{-1} \). Hence the energy average renders negligible any contributions with \( k_1 \neq k_2 \), and therefore

\[ R_2^{(off)}(\epsilon) = \frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} \sum_{k \neq 0} \frac{1}{(2\pi k)^2} \exp(2\pi i \tilde{k}(\epsilon_1 - \epsilon_2)) \Phi_k(\epsilon_1, \epsilon_2), \] (121)

where

\[ \Phi_k(\epsilon_1, \epsilon_2) = < \exp(2\pi i k N_T^{(osc)}(E + \epsilon_1) - N_T^{(osc)}(E + \epsilon_2)) >. \] (122)
We now make the key assumption that in generic systems the orbits up to period $T^*$ are non-commensurable modulo exact degeneracies, and hence that the energy average of any smooth function of $\exp(iS_p)$

$$< f > = < f(e^{iS_1(E)}, e^{iS_2(E)}, \ldots, e^{iS_M(E)}) >,$$

(123)
can be calculated using

$$< f > = \int_0^{2\pi} \ldots \int_0^{2\pi} f(e^{i\phi_1}, \ldots, e^{i\phi_M}) \prod_j \frac{d\phi_j}{2\pi}.$$

(124)

This is essentially equivalent to random phase approximation, or to ergodic theorem for quasi-periodic functions with non-commensurate periods, or to strict diagonal approximation.

Using the trace formula for the staircase function one can thus perform the energy average by evaluating these integrals. Exact results for certain cases will be discussed later. First, for clarity, we consider a simple leading order approximation to $\Phi_k$ based on the relation

$$< \exp(iG(E)) > \approx \exp(-\frac{1}{2} < G^2(E) >).$$

(125)

This is an identity if $G$ is a Gaussian random function with zero mean. But if one ignores all terms with repetitions of the same periodic orbits the formula for $N^{(osc)}(E)$ will be equal to a sum of a big number of terms with non-commensurable frequencies which can be considered as independent random variables therefore the resulting distribution of $N^{(osc)}(E)$ being a sum over many independent random variables should a Gaussian distributed random function. Consequently this type of approximation is expected to be a good approximation for generic systems.

From it one obtains

$$\Phi_k(\epsilon_1, \epsilon_2) = < \exp(-2\pi^2k^2 < (N^{(osc)}_{T^*}(E + \epsilon_1) - N^{(osc)}_{T^*}(E + \epsilon_2))^2) >$$

$$= \left(\frac{\Delta(\epsilon)}{L^2}\right)k^2,$$

(126)

where $\Delta(\epsilon)$ was defined above and

$$L = \exp(2\pi^2 < (N^{(osc)}(E))^2 >).$$

(127)
Using the Hannay-Ozorio de Almeida sum rule \[12\], it may be shown that
\[ L \approx (T^*)^g. \] (128)
Since we anticipate taking \( T^* \approx T_H \), it then follows that the terms in the \( k \) sum decrease rapidly as \( (\bar{d})^{-2g^2} \). Hence in the leading semi-classical order as \( \epsilon \bar{d} \to \infty \), we may retain just the \( k = \pm 1 \) contributions and when deriving over \( \epsilon \) take into account only terms with \( \exp(2\pi i \bar{d} \epsilon) \). Finally one gets
\[
R_2^{(off)}(\epsilon_1, \epsilon_2) = d^2 e^{2\pi i \bar{d} \epsilon} \begin{pmatrix} z_{T^*}^*(E + \epsilon_1)z_{T^*}^*(E + \epsilon_2) \\ z_{T^*}^*(E + \epsilon_1)z_{T^*}^*(E + \epsilon_2) \end{pmatrix} + c.c. \] (129)
The last step consists in performing the exact average in this formula. We shall consider the case of the Riemann zeta function first.

### 4.1 Off-diagonal terms for the Riemann zeta function

For the Riemann zeta function the the ratio of four truncated zeta functions in the above formula has the form
\[
< \frac{z_{T^*}^*(E + \epsilon_1)z_{T^*}^*(E + \epsilon_2)}{z_{T^*}^*(E + \epsilon_1)z_{T^*}^*(E + \epsilon_2)} > = \prod_{\ln p < T^*} R_p, \] (130)
where
\[
R_p(\phi_p) = \frac{(1 - A_p e^{i\phi_p + i\gamma_p^{(1)}})(1 - A_p e^{-i\phi_p - i\gamma_p^{(2)}})}{(1 - A_p e^{-i\phi_p - i\gamma_p^{(1)}})(1 - A_p e^{i\phi_p + i\gamma_p^{(2)}})}, \] (131)
and\[B\]
\[
A_p = \frac{1}{\sqrt{p}}, \quad \phi_p = E \ln p, \quad \gamma_p^{(i)} = \epsilon_i \ln p. \] (132)
As the logarithms of prime numbers are non-commensurable, \( e^{i\phi_p} \) act as independent random variables for each \( p \) and
\[
< \prod_{\ln p < T^*} R_p > = \prod_{\ln p < T^*} < R_p > . \] (133)
The mean value of an individual \( R_p \) is
\[
< R_p > = \int_0^{2\pi} R_p(\phi) \frac{d\phi_p}{2\pi} = \frac{1}{2\pi i} \int (1 - A_p z e^{i\gamma_p^{(1)}})(1 - A_p z^{-1} e^{-i\gamma_p^{(2)}}) \frac{dz}{(1 - A_p z^{-1} e^{-i\gamma_p^{(1)}})(1 - A_p z e^{i\gamma_p^{(2)}})}, \] (134)
where the integral is taken over a unit circle in the complex $z$-plane.

The last integral is easily computed by the residues. There is two poles inside the contour. The first is at $z = 0$ and the second one is at $z = A_p e^{-i\tau_p^{(1)}}$. Simple algebra gives

$$< R_p > = \frac{(1 - A_p^2)^2}{|1 - A_p^2 e^{i\tau_p^0}|^2} \left(1 - \frac{A_p^4 (1 - e^{i\tau_p^0})^2}{(1 - A_p^2)^2}\right), \quad (135)$$

where $\tau_p = \tau_p^{(1)} - \tau_p^{(2)}$.

The total contribution equals the product over all primes up to $T^*$

$$R_2(\epsilon) = C^2 \exp(2\pi i \bar{d} \epsilon) |\zeta(1 + i\epsilon)|^2 \Phi^{(off)}(\epsilon), \quad (136)$$

where

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \quad \Phi^{(off)}(\epsilon) = \prod_p (1 - \frac{(1 - p^{i\epsilon})^2}{(p - 1)^2}), \quad (137)$$

and

$$C = \bar{d} \prod_p (1 - \frac{1}{p}). \quad (138)$$

All products in these expressions include prime numbers up to $\ln p = T^*$. The two first products converge when $T^* \to \infty$ and only the last one require regularisation. But our parameter $T^*$ has not yet been fixed. Let us choose it in such a way that

$$C \equiv \bar{d} \prod_{\ln p < T^*} (1 - \frac{1}{p}) = \frac{1}{2\pi}. \quad (139)$$

It is easy to check [3] that asymptotically this equation leads

$$T^* = 2\pi \bar{d} e^\gamma, \quad (140)$$

where $\gamma$ is the Euler constant. Therefore, our $T^*$ is of the order of the Heisenberg time as was expected. Note that exactly the same factor $e^\gamma$ often appears in the statistical approach to prime numbers (see discussion in [13]) and can be considered as a renormalisation of formally divergent sums.

After this renormalisation we get exactly the same formula as has been derived in the previous section using the Hardy-Littlewood conjecture about the pairwise distribution of prime numbers. Note that in a present derivation no analogous conjectures have been assumed.


4.2 Off-diagonal contribution for dynamical systems

Let us now compute off-diagonal contribution to dynamical systems. Our starting point will be the same expression as for the Riemann zeta function

\[
R_2^{(off)}(\epsilon_1, \epsilon_2) = \partial^2 e^{2\pi i \delta^2} < \frac{z_{T^*}(E + \epsilon_1)z_{T^*}(E + \epsilon_2)}{z_{T^*}(E + \epsilon_1)z_{T^*}(E + \epsilon_2)} > .
\] (141)

The only difference with Riemann case is that the truncated zeta function

\[
z_{T^*}(E) = \prod_{T_p < T^*} \prod_{m=1}^{\infty} \left( 1 - \frac{e^{iS_p/h - i\pi/2\mu_p}}{|\Lambda_p|^{1/2}\Lambda_p^{m}} \right).
\] (142)

As above we shall assume that all periods of primitive periodic orbits are non-commensurable. Therefore \( e^{iT_p \epsilon} \) can be considered as independent random variables but one cannot ignore the existence of products over \( m \) in Eq. (142)

\[
< \frac{z_{T^*}(E + \epsilon_1)z_{T^*}(E + \epsilon_2)}{z_{T^*}(E + \epsilon_1)z_{T^*}(E + \epsilon_2)} > = \prod_{T_p < T^*} < R_p(\phi_p) > ,
\] (143)

where

\[
< R_p > = \int_0^{2\pi} R_p(\phi_p) \frac{d\phi_p}{2\pi},
\] (144)

and

\[
R_p(\phi_p) = \prod_{m=1}^{\infty} \frac{(1 - A_{p,m}e^{-i\phi_p - i\tau_p^{(1)}})(1 - A_{p,m}e^{i\phi_p + i\tau_p^{(2)}})}{(1 - A_{p,m}e^{i\phi_p + i\tau_p^{(1)}})(1 - A_{p,m}e^{-i\phi_p + i\tau_p^{(2)}})},
\] (145)

and

\[
A_{p,m} = |\Lambda_p|^{-1/2}\Lambda_p^{-m}, \quad \phi_p = S_p(E)/h - \pi\mu_p/2, \quad \tau_p^{(i)} = T_p\epsilon_i.
\] (146)

Though it is possible to calculate the integral \(< R_p >\) by the residues as has been done in the previous Section it is more convenient to use the so-called \(q\)-binomial theorem ([22] p.7)

\[
\sum_{n=0}^{\infty} (a; q)_n z^n = \frac{(az; q)_\infty}{(z; q)_\infty}.
\] (147)

Here

\[
(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \ldots (1 - aq^{n-1}),
\] (148)
and

\[ (a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \quad (149) \]

Denoting

\[ q = \Lambda_p^{-1}, \quad z = |\Lambda_p|^{-1/2} e^{-i\phi_p - ir_p^{(2)}}, \quad a = e^{i\tau_p^{(2)} - i\tau_p^{(1)}}, \quad (150) \]

one obtains

\[ \prod_{m=0}^{\infty} \frac{1 - A_{p,m} e^{-i\phi_p - ir_p^{(1)}}}{1 - A_{p,m} e^{-i\phi_p - ir_p^{(2)}}} = (az; q)_\infty \quad (z; q)_\infty. \quad (151) \]

Similarly

\[ \prod_{m=0}^{\infty} \frac{1 - A_{p,m} e^{i\phi_p + ir_p^{(2)}}}{1 - A_{p,m} e^{i\phi_p + ir_p^{(1)}}} = (az'; q)_\infty \quad (z'; q)_\infty, \quad (152) \]

where \( z' = |\Lambda_p|^{-1/2} e^{i\phi_p + ir_p^{(2)}} \).

Using the \( q \)-binomial theorem these expressions can be represented as follows

\[ \prod_{m=0}^{\infty} \frac{1 - A_{p,m} e^{-i\phi_p - ir_p^{(1)}}}{1 - A_{p,m} e^{-i\phi_p - ir_p^{(2)}}} = \sum_{n=0}^{\infty} (a; q)_n z^n, \quad (153) \]

and

\[ \prod_{m=0}^{\infty} \frac{1 - A_{p,m} e^{i\phi_p + ir_p^{(2)}}}{1 - A_{p,m} e^{i\phi_p + ir_p^{(1)}}} = \sum_{n=0}^{\infty} (a; q)_n z'^n. \quad (154) \]

Now \( < R_p > \) is the integral of the product of these two expressions and in the diagonal approximation only terms with the same power will contribute and finally

\[ < R_p > = \sum_{n=0}^{\infty} \frac{(a; q)_n^2}{(q; q)_n^2} y^n, \quad (155) \]

where \( y = |\Lambda_p|^{-1} e^{i(\tau_p^{(1)} - \tau_p^{(2)})} \). This sum may be expressed through the so-called \( q \)-hypergeometric function \[\text{[22]}\]

\[ < R_p > = _2 \phi_1(e^{-iT_p}; e^{-iT_p}; \Lambda^{-1}_p; \Lambda^{-1}_p, |\Lambda_p|^{-1} e^{iT_p}), \quad (156) \]

where we take into account that \( \tau_p^{(1)} - \tau_p^{(2)} = T_p(\epsilon) \).
The total mean value equals the product over all periodic orbits till $T^*$. The divergent part comes only from $n = 1$ term

\[ \prod_{T_p < T^*} (1 + \frac{1}{|\Lambda_p|}(e^{iT_p\epsilon} + e^{-iT_p\epsilon} - 2)). \]  

But the divergence of this product coincides with the divergent part of the expression

\[ \prod_{T_p < T^*} \left| \frac{Z_p(0)}{Z_p(i\epsilon)} \right|^2, \]  

where

\[ Z_p(s) = 1 - \frac{e^{T_ps}}{|\Lambda_p|}. \]  

The product

\[ \prod_{T_p < T^*} <R_p> \left| \frac{Z_p(i\epsilon)}{Z_p(0)} \right|^2 \]  

converges when $T^* \to \infty$ and

\[ R_2(\epsilon) = \frac{e^{2\pi i \epsilon}}{4\pi^2} |\gamma|^{-1} Z_{cl}(i\epsilon)^2 \Phi^{(osc)}(\epsilon) + c.c., \]  

where

\[ \Phi^{(osc)}(\epsilon) = \prod_{p} <R_p> \left| \frac{Z_p(i\epsilon)}{Z_p(0)} \right|^2. \]  

and $Z_{cl}(s)$ is a classical zeta function (see the footnote at page [10]) defined when Res $< 0$ by the product over all ppo

\[ Z_{cl}(s) = \prod_{ppo}(1 - e^{T_ps}/|\Lambda_p|)^{-1}. \]  

Let us fix the maximal period $T^*$ from the condition

\[ d \prod_{T_p < T^*} Z_p(0) = \frac{1}{2\pi |\gamma|}, \]  

where $\gamma$ is the residue of $Z_{cl}(s)$ at $s = 1$ (which existence is a consequence of Hannay-Ozorio de Almeida sum rule)

\[ \gamma = \lim_{s \to 0} sZ_{cl}(s). \]  

As above this renormalisation defines $T^*$ of the order of $T_H$ and ensures that when $\epsilon \to 0$ $R_2(\epsilon)$ tends to the oscillatory part of the GUE result (37)

$$R_2^{(off)}(\epsilon) = \frac{1}{4\pi^2\epsilon^2} (e^{2\pi i \epsilon} + e^{-2\pi i \epsilon}).$$  (166)

In principle any zeta function whose divergent part at real $\epsilon$ coincides with $Z_{cl}(i\epsilon)$ can be used in the above expressions.

5 Random matrix universality

This section is devoted to another method of semi-classical calculation of off-diagonal part of correlation function based on a different idea.

In is well known [7] that the standard ensembles of random matrices correspond to the following choice of measure in the ensemble

$$P(M) = \exp(-\text{Tr} V(M)), $$  (167)

where $V(M)$ is an arbitrary function. The important universality in the random matrices ensembles is the fact that the unfolded distribution does not depend on the explicit form of this function [23] provided that it corresponds to the so-called definite momentum problem [24].

For unitary ensembles all correlation functions have the form [7]

$$R_n(x_1, \ldots , x_n) = \det |K_N(x_i, x_j)|_{i,j=1,\ldots,n}, $$  (168)

where the kernel

$$K_N(x, y) = e^{-\frac{V(x)}{2} - \frac{V(y)}{2}} \sum_{n=1}^N p_n(x)p_n(y),$$  (169)

and $p_n(x)$ are polynomials of degree $n - 1$ orthogonal with respect to the measure $\exp(-V(x))$

$$\int p_n(x)p_m(x)e^{-V(x)}dx = \delta_{nm}. $$  (170)

The ‘semi-classical’ asymptotic of orthogonal polynomials leads to the following expression for the kernel $K(x, y) = K_N(x, y)$ when $N \to \infty$

$$K(x, y) = \frac{\sin \pi(N(x) - N(y))}{\pi(x - y)}, $$  (171)
where \( N(x) = \int_x^x \rho(x')dx' \) is a mean staircase function related to \( V(x) \) by the Dyson equation
\[
\int \frac{\rho(t)}{x-t}dt = \frac{1}{2}V'(x). \tag{172}
\]
The mean density \( \rho(x) \) does depend on the form of \( V(x) \) but if
\[
\epsilon = x - y \ll \text{characteristic scale of the potential } V(x)
\]
one can expand the difference \( N(x) - N(y) \) into powers of \( \epsilon, \) \( N(x) - N(y) = \bar{d}\epsilon, \) and all correlation functions will be functions of \( \bar{d}\epsilon \) which ensures their universality after rescaling.

Now we shall use this fact in addition to the trace formula.

Let us assume that we know all periodic orbits up to period \( T^* \) but, contrary to what was used in the previous sections, this maximal period will be much smaller than the Heisenberg time but still much bigger than pure classical time
\[
t_{cl} \ll T^* \ll t_H. \tag{173}
\]
Below we assume that the maximal period is of the order of the Ehrenfest time
\[
T^* \approx t_E. \tag{174}
\]
According to the trace formula the density of states has the form
\[
d(E) = \tilde{d}(E) + \eta(E), \tag{175}
\]
where as above
\[
\tilde{d}(E) = \bar{d} + \sum_{T_p < T^*} \sum_{n=1}^\infty \frac{T_p}{h} A_{p,n} e^{\frac{\mu_p}{h} - \pi \mu_p/2} + c.c., \tag{176}
\]
and \( \eta(E) \) is (unknown) part of the density constructed from high-period orbits.

Let us now try to construct a random matrix ensemble which has the mean density exactly equals \( \tilde{d}(E) \). In principle the necessary potential can be computed from the Dyson equation (172). But the explicit form of this potential is irrelevant as all correlation functions depend only on the mean staircase function \( \bar{N}(x) = \int_x^x \tilde{d}(x')dx' \) which is supposed to be known.
Hence, the 2-point correlation function in such ensemble takes the form

\[ \tilde{R}_2(x, y) = \tilde{d}(E + x)\tilde{d}(E + y) - \frac{\sin^2(\pi(\tilde{N}(E + x) - \tilde{N}(E + y)))}{(\pi(x - y))^2}. \]  

(177)

\(\tilde{d}(E)\) in this formula is the mean (over constructed ensemble) density of eigenvalues and it is not a constant but includes periodic orbits with period up to \(T^*\) and consequently it has oscillations in the scale smaller or equal \(T^*\). To be consistent with the standard definition of the 2-point correlation function (38) it is necessary to perform an additional smoothing over an energy window, \(\Delta E\), much bigger than \(\bar{\hbar}/T^*\)

\[ \frac{T^*}{\hbar} \Delta E \gg 1, \]

(178)

Denoting this average by the brackets \(<\ldots>\), the 2-point correlation function

\[ R_2(x, y) = <\tilde{d}(E + x)\tilde{d}(E + y) - \frac{\sin^2(\pi\tilde{N}(E + x) - \pi\tilde{N}(E + y))}{\pi^2(x - y)^2}>. \]

(179)

Separating the smooth and oscillatory parts one gets

\[ R^{(\text{diag})}_2(x, y) = <\tilde{d}(E + x)\tilde{d}(E + y) - \frac{\sin^2(\pi\tilde{N}(E + x) - \pi\tilde{N}(E + y))}{\pi^2(x - y)^2}> = \tilde{d}^2 + \frac{1}{4\pi^2} \frac{\partial^2}{\partial x \partial y} \ln \Delta(x, y), \]

(180)

where

\[ \Delta(x, y) = \frac{1}{(x - y)^2} \exp(-4\pi^2 <\tilde{N}(E + x)\tilde{N}(E + y)>), \]

(181)

and

\[ R^{(\text{off})}_2(x, y) = \frac{1}{4\pi^2(x - y)^2} <e^{2\pi i(\tilde{N}(E + x) - \tilde{N}(E + y))}> + c.c. \]

(182)

As above we assume that all periodic orbit actions in the explicit expression for \(\tilde{d}(E)\) are non-commensurable. Because we chose \(T^*\) to be of the order of the Ehrenfest time, one can perform the smoothing over energy window in the
strict diagonal approximation. But all the expressions which have to smooth are exactly the same as were above. The only difference is that the limiting period $T^*$ is of the order not of the Heisenberg time but of the Ehrenfest time and there is a factor $(x - y)^{-2}$. Using the same transformations as above one concludes that

$$
\Delta(\epsilon) = \frac{1}{\epsilon^2} |Z_{cl}(i\epsilon, T^*)|^2 \Phi^{(\text{diag})}(\epsilon), \quad (183)
$$

and

$$
R_2^{(\text{off})}(\epsilon) = \frac{1}{4\pi^2 \epsilon^2} \left| \frac{Z_{cl}(i\epsilon, T^*)}{Z_{cl}(0, T^*)} \right|^2 \Phi^{(\text{off})}(\epsilon). \quad (184)
$$

Here $\epsilon = x - y$ and functions $\Phi^{(\text{diag})}(\epsilon)$ and $\Phi^{(\text{off})}(\epsilon)$ are defined exactly as Eqs. (163) and (162) except that they are truncated at $T^*$. Because these products converge and the value of $T^*$ is assumed to obey inequalities (173), one can go to the limit $T^* \to \infty$ and the resulting expressions will coincide with Eqs. (50) and (162).

The function $Z_{cl}(s, T^*)$ is the truncated product

$$
Z_{cl}(s, T^*) = \prod_{T_p < T^*} \left( 1 - \frac{e^{sT_p}}{|\Lambda_p|} \right)^{-1}. \quad (185)
$$

Because $T^*$ is such that this product includes a large number of terms one can consider the behavior of the product in the limit $T^* \to \infty$.

Let us perform the following formal transformations (which can be done rigorously by adding a small imaginary part to $\epsilon$

$$
\frac{Z_{cl}(i\epsilon, T^*)}{\epsilon Z_{cl}(0, T^*)} \equiv \frac{1}{\epsilon} \prod_{T_p < T^*} \frac{1 - \frac{1}{|\Lambda_p|}}{1 - e^{i\epsilon T_p}} = \frac{1}{\gamma} Z_{cl}(i\epsilon), \quad (186)
$$

where $Z_{cl}(i\epsilon)$ is the classical zeta function (163) defined as the product over ppo with arbitrary period and

$$
\gamma = \epsilon \prod_{T_p < T^*} \frac{1}{1 - e^{i\epsilon T_p}}. \quad (187)
$$

But

$$
\prod_{T_p < T^*} \frac{1}{1 - \frac{1}{|\Lambda_p|}} = \lim_{s \to 0} \prod_{T_p < T^*} \frac{1}{1 - e^{isT_p}}
$$

$$
= \lim_{s \to 0} Z_{cl}(is) \left( \prod_{T_p > T^*} \left( 1 - \frac{e^{isT_p}}{|\Lambda_p|} \right) \right). \quad (188)
$$
Therefore

$$\gamma = \epsilon \lim_{s \to 0} Z_{cl}(is) \prod_{T_p > T^*} \frac{(1 - \frac{e^{isT_p}}{|\Lambda_p|})}{(1 - \frac{e^{i\epsilon T_p}}{|\Lambda_p|})}.$$  (189)

When $T^* \to \infty$

$$\prod_{T_p > T^*} (1 - \frac{e^{isT_p}}{|\Lambda_p|}) \to \exp(- \sum_{T_p > T^*} \frac{e^{isT_p}}{|\Lambda_p|}).$$  (190)

Using Hannay-Ozorio de Almeida sum rule [12] one concludes that when $T^* \to \infty$ this expression is close to

$$\exp(- \int_{T^*}^{\infty} \frac{e^{isT}}{T} dT).$$  (191)

Finally for large $T^*$

$$\gamma = \epsilon \lim_{s \to 0} Z_{cl}(is) \exp(- \int_{T^*}^{\infty} \frac{e^{iT}}{T} dT + \int_{T^*}^{\infty} \frac{e^{i\epsilon T}}{T} dT).$$  (192)

As the limiting period has been chosen such that $\epsilon T^* \ll 1$ (and of course $sT^* \ll 1$) one can use the approximation

$$\int_{T^*}^{\infty} \frac{e^{iT}}{T} dT \to \ln(T^*\epsilon) + \text{const},$$  (193)

Therefore

$$\gamma = \lim_{s \to 0} sZ_{cl}(is),$$  (194)

and as only $|\gamma|$ is important Eq. (184) coincides with Eq. (161) derived by completely different method.

6 Riemann-Siegel form of density of states

In this Section we propose one more method of computation spectral statistics based on ideas different from the ones discussed in the previous Sections. For simplicity we consider only the Riemann case.

It is known that the best method of calculation of the Riemann zeta function is the using the Riemann-Siegel representation of it [8], [25] (called in simple cases the approximative functional equation).

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Heuristically it can be derived as follows. Let us divide the sum over integers in the definition of the Riemann zeta function into two contributions

\[ \zeta(1/2 - iE) = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} e^{iE \ln n} = \sum_{n=1}^{N} \frac{1}{n^{1/2}} e^{iE \ln n} + \sum_{n=N+1}^{\infty} \frac{1}{n^{1/2}} e^{iE \ln n}. \]  

(195)

In the first sum the summation is done over integers up to a (large) integer \( N \) and the second sum includes all terms with \( n > N \). Using the Poisson summation formula one can rewrite the latter in the form

\[ \sum_{n=N+1}^{\infty} \frac{1}{n^{1/2}} e^{iE \ln n} = \sum_{m=-\infty}^{\infty} \int_{N}^{\infty} \frac{1}{n^{1/2}} e^{-2\pi m n + iE \ln n} \, dn, \]  

(196)

and it is naturally to apply the stationary phase method to compute the integral. The dominant contribution comes from vicinities of points \( n_{sp} \) obeying the saddle-point equation

\[ 2\pi m = \frac{E}{n_{sp}}, \]  

(197)

or

\[ n_{sp} = \frac{E}{2\pi m}, \]  

(198)

but this contribution exists only if this stationary point lies between \( N \) and \( \infty \) which is equivalent to the following restriction on \( m \)

\[ 0 < m < N^*, \]  

(199)

where

\[ N^* = \frac{E}{2\pi N}. \]  

(200)

Computing the second derivative of the phase and performing the integration in the Gaussian approximation one gets

\[ \sum_{n=N+1}^{\infty} \frac{1}{n^{1/2}} e^{iE \ln n} = e^{2\pi i \bar{N}(E)} \sum_{n=1}^{N^*} \frac{1}{n^{1/2}} e^{-iE \ln n}, \]  

(201)

where \( \bar{N}(E) \) is the mean staircase function given by Eq. (17). (On more careful analysis see [23].)
The same result can be obtained without calculation (but less rigorously) by using the functional equation of the Riemann zeta function. It is known that this function obeys the relation

\[ \zeta(1/2 - iE) = e^{2\pi i N(E)} \zeta(1/2 + iE), \]  

(202)

which is a necessary condition that the complex valued function \( \zeta(s) \) has zeros on a line. Substituting the formal expansion

\[ \zeta(1/2 \pm iE) = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} e^{\pm i E \ln n} \]  

(203)

into this relation and comparing terms in two parts one concludes that

\[ \sum_{\ln n > T_1} \frac{1}{n^{1/2}} e^{i E \ln n} = e^{2\pi i N(E)} \sum_{\ln n < T_2} \frac{1}{n^{1/2}} e^{-i E \ln n}, \]  

(204)

provided \( T_1 + T_2 = T_H \) where the Heisenberg time \( T_H = 2\pi \tilde{d}(E) \). One can also get a slightly more general relation

\[ \sum_{n=N_1}^{N_2} \frac{1}{n^{1/2-iE}} = e^{2\pi i N(E)} \sum_{n=E/(2\pi N_2)}^{E/(2\pi N_1)} \frac{1}{n^{1/2+iE}}. \]  

(205)

This type of bootstrap where a sum of large integers is proportional to a sum of the short ones is the cornerstone of the Riemann-Siegel method and it is very useful in calculation of zeta functions. Our purpose will be to use it to obtain an information about sums of primes.

Let us split the density of Riemann zeros \([13]\) into two contributions

\[ d^{(osc)}(E) = d_1(E) + d_2(E). \]  

(206)

The first one includes the sum over all prime up to a certain integer \( N \)

\[ d_1(E) = -\frac{1}{2\pi} \sum_{p<N} \ln p \sum_{n=1}^{\infty} \frac{1}{p^{n/2}} e^{i E n \ln p} + c.c. \]  

(207)

and the second term contains all primes bigger than \( N \)

\[ d_2(E) = -\frac{1}{2\pi} \sum_{p>N} \ln p \sum_{n=1}^{\infty} \frac{1}{p^{n/2}} e^{i E n \ln p} + c.c. \]  

(208)
The main ingredient of following discussion is the sieve representation of prime numbers. Namely we shall use the following relation

\[
\sum_{p > N} f(p) = \sum_{n > N} f(n) - \sum_{p_j, n > N/p_j} f(p_j n) + \sum_{p_j, p_k: n > N/(p_j p_k)} f(p_j p_k n) + \ldots
\]

where \(\mu(k)\) is the Euler function.

This relation is just the manifestation of the fact that when one substitute instead of a sum over prime numbers the sum over integers one has first to subtract from this sum all numbers which are proportional to short primes but all numbers which are proportional to product of two primes are subtracted twice, therefore, it is necessary to add integers proportional to the product of two primes etc. This type of inclusion-exclusion principle is exact provided one is restricted to a finite collection of primes.

Applying this formula to \(f(n) = 1/n^{1/2-iE}\) one gets

\[
\sum_{p > N} \frac{1}{p^{1/2-iE}} = \sum_{k=1}^{\infty} \sum_{n > N/k} \frac{\mu(k)}{n^{1/2-iE} k^{1/2-iE}}.
\]

Using the approximate functional equation (204) for the sums over \(n\) in this formula one gets

\[
\sum_{p = N_1}^{N_2} \frac{1}{p^{1/2-iE}} = e^{2\pi i \tilde{N}(E)} \sum_{k=1}^{\infty} \sum_{n = E_k/(2\pi N_2)}^{E_k/(2\pi N_1)} \frac{\mu(k)}{n^{1/2+iE} k^{1/2+iE}}.
\]

Taking into account that the total ‘period’ of an individual term in the double sum is \(t = \ln k - \ln n\) this expression can be transformed into the following form

\[
\sum_{p = N_1}^{N_2} \frac{1}{p^{1/2-iE}} = e^{2\pi i \tilde{N}(E)} \left( \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu(k)}{n^{1/2+iE} k^{1/2+iE}} \right)_{T_H - \ln N_2 < t < T_H - \ln N_1},
\]

where the symbol \(t_1 < t < t_2\) means that one has to expand the expression in the brackets into the Fourier series on \(\exp(-iEt)\) and to take into account only terms with period \(t_1 < t < t_2\). It is important to note that if \(\ln N_1\) is
of the order of $T_H$ the summation in the above formula includes only small number of terms. But

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k^{1/2-iE}} = \frac{1}{\zeta(1/2-iE)},$$

(213)

and these sum can be rewritten as follows

$$\sum_{N_1 < p < N_2} \frac{1}{p^{1/2-iE}} = e^{2\pi i \bar{N}(E)} \left( \frac{\zeta(1/2+iE)}{\zeta(1/2-iE)} \right)_{T_H-\ln N_2 < t < T_H-\ln N_1}.\quad (214)$$

Using the prime number theorem the main part of the sum of large primes in the trace formula for the density of the Riemann zero can be rewritten in the form

$$\sum_{p=N_1}^{N_2} \frac{\ln p}{p^{1/2-iE}} = \frac{N_2 - N_1}{\sum_{N_1 < p < N_2} 1} \sum_{N_1 < p < N_2} p^{-1/2+iE}.\quad (215)$$

Let us use the inclusion-exclusion principle for the denominator

$$\sum_{p=N_1}^{N_2} 1 = (N_2 - N_1)(1 - \sum_{p} \frac{1}{p} + \sum_{p_1,p_2} \frac{1}{p_1 p_2} + \ldots) = (N_2 - N_1) \prod_{p} (1 - \frac{1}{p}).\quad (216)$$

Therefore

$$d_2(E) = -\frac{e^{2\pi i \bar{N}(E)}}{2\pi} \left( \prod_{p} \frac{(1 - p^{-1/2+iE})}{(1 - p^{-1/2-iE})(1 - p^{-1})} \right)_{0 < t < T_H-\ln N} + \text{c.c.},\quad (217)$$

and the density of Riemann zeros takes the form

$$d(E) = -\frac{1}{2\pi} \sum_{p < N} \ln p \sum_{n=1}^{\infty} \frac{1}{p^{n/2}} e^{i E n \ln p}$$

$$-\frac{e^{2\pi i \bar{N}(E)}}{2\pi} \left( \prod_{p} \frac{(1 - p^{-1/2+iE})}{(1 - p^{-1/2-iE})(1 - p^{-1})} \right)_{0 < t < T_H-\ln N} + \text{c.c.}\quad (218)$$

This formula can be considered as a Riemann-Siegel look-like ressumation of the trace formula. It expresses the (unknown) sum over large primes through small ones.
In particular this formula is useful for the computation of statistics of Riemann zeros. Direct application of the definition (38) gives

\[
R^2(\epsilon) = R^2_{\text{diag}}(\epsilon) + \frac{e^{\pi i \bar{d} \epsilon}}{4\pi^2} \prod_p \frac{(1 - p^{-1/2 + i(E + \epsilon_1)})(1 - p^{-1/2 - i(E + \epsilon_2)})}{(1 - p^{-1/2 - i(E + \epsilon_1)})(1 - p^{-1/2 + i(E + \epsilon_2)})(1 - p^{-1})^2}.
\]

Here \(R^2_{\text{diag}}\) is the same as in Eq. (73) except that the summation includes primes up to \(N\). Because \(N\) is assumed to be large and the sum in (73) converges conditionally at real \(\epsilon\) one can go to the limit \(N \to \infty\) and Eqs. (74)-(76) will be recovered. The second term, corresponding to \(R^2_{\text{osc}}\), comes from the summation over large primes and it has the form as in Eq. (130) (but with correct pre-factor). Repeating the same calculation as in Section 4.1 and taking into account that the final product now converges one will get Eq. (136) with the correct value (139) of the pre-factor.

7 Conclusion

The above-discussed methods demonstrate how, in principle, the existence of the trace formula and certain natural conjectures about the distribution of periodic orbits (or primes) combine together to produce universal local statistics. For systems without the time-reversal invariance the assumption that periods for all (or at least for most) periodic orbits are non-commensurable leads to the GUE statistics (for 2-point correlation function).

In all cases the main formulas may be written in the following forms

\[
R^2(\epsilon) = R^2_{\text{diag}}(\epsilon) + R^2_{\text{off}}(\epsilon),
\]

where the diagonal part

\[
R^2_{\text{diag}}(\epsilon) = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial \epsilon^2} \ln(|Z_{cl}(\epsilon)|^2 \Phi^{\text{diag}}(\epsilon)),
\]

and the off-diagonal part

\[
R^2_{\text{off}}(\epsilon) = \frac{1}{4\pi^2} e^{\pi i \bar{d} \epsilon} \frac{1}{\gamma} |Z_{cl}(\epsilon)|^2 \Phi^{\text{off}}(\epsilon).
\]
Here $Z_{cl}(\epsilon)$ is a classical zeta function and $\Phi^{(diag)}(\epsilon)$ and $\Phi^{(off)}(\epsilon)$ are certain convergent products over periodic orbits.

For dynamical systems

$$Z_{cl}(\epsilon) = \prod_p \left( 1 - \exp(i\epsilon T_p)/|\Lambda_p| \right)^{-1}. \quad (223)$$

For the Riemann case

$$Z_{cl}(\epsilon) = \zeta(1 - iE). \quad (224)$$

In all cases $Z_{cl}(\epsilon) \to \gamma/\epsilon$ when $\epsilon \to 0$ and $\Phi^{(off)}(0) = 1$ which ensures the GUE value for small $\epsilon$.

The close relation between diagonal and off-diagonal terms (first observed for disordered systems by Andreev and Altshuler [26]) suggests the existence of a certain unified principle. The best candidate for it is the ‘unitarity’ property of the trace formula, namely, that the distribution of periodic orbits should be such that the corresponding eigenvalues will be real. In some sense certain long-period orbits are connected to the short ones and the investigation of this connection may clarify the origin of universal spectral statistics.

The interesting question remains what conjectures about periodic orbits are necessary to obtain correlation functions for systems with time-reversal invariance where almost all periodic orbits appear in pairs with exactly the same action.

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