Abstract

Let $G$ be a non-trivial finite group. The well-known Dold’s theorem states that: There is no simplicial $G$-equivariant map from an $n$-connected simplicial $G$-complex to a free simplicial $G$-complex of dimension at most $n$. In this paper, we give a new generalization of Dold’s theorem, by replacing ”dimension at most $n”$ with a sharper combinatorial parameter. Indeed, a generalization of $G$-Tucker’s lemma, a combinatorial consequence of Dold’s theorem, will be presented as well.

Keywords: Borsuk-Ulam theorem, Chromatic number, Dold’s theorem, Tucker’s lemma, Compatibility graph

1. Introduction

From the viewpoint of transformation groups, the famous Borsuk-Ulam theorem states that: There is no $\mathbb{Z}_2$-equivariant map from the $n$-sphere $S^n$ to the $m$-sphere $S^m$, whenever $n > m$. It is known that the Borsuk-Ulam theorem has a lot of generalizations and various applications in many directions; see for examples [8, 11]. Probably the best generalization of the classical Borsuk-Ulam theorem is Dold’s theorem, as most of the recent progress in topological combinatorics are due to this theorem.

Theorem 1 (Dold’s theorem [6]). Let $G$ be a non-trivial finite group, $K$ an $n$-connected simplicial $G$-complex, and $L$ a free simplicial $G$-complex of dimension at most $n$. Then there is no simplicial $G$-equivariant map from $K$ to $L$.

The Borsuk-Ulam theorem has also a handy combinatorial consequence, called octahedral Tucker’s lemma [9]. Several extensions of the octahedral Tucker lemma with interesting applications in various area such as graph colorings, and fair division problems are known. We refer the interested reader to [5, 7, 14] for generalizations, and [1, 5, 9] for applications. Recently, in [3], a new generalization of octahedral Tucker’s lemma, called $G$-Tucker’s lemma, was introduced. Moreover, as an application of that generalization, a new method for
constructing graphs with high chromatic number and small clique number was given. To recall the lemma, we need to make some definitions and conventions.

From now on, $G$ stands for a non-trivial finite group with $0 \not= G$. Furthermore, its identity element will be denoted by $e$, and we define $g \cdot 0 = 0$ for all $g \in G$. Consider the $G$-poset (for definition, see the preliminaries section) $G \times \{1, \ldots, n+1\}$ with natural $G$-action, $h \cdot (g, i) \rightarrow (hg, i)$, and the order defined by $(h, x) \prec_1 (g, y)$ if $x < y$ (in $\mathbb{N}$). Also, let $(G \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$ be the $G$-poset whose action is $g \cdot (x_1, \ldots, x_n) = (g \cdot x_1, \ldots, g \cdot x_n)$, and the order relation is given by:

$$x = (x_1, \ldots, x_n) \preceq y = (y_1, \ldots, y_n),$$

if for every $i \in \{1, \ldots, n\}$, $x_i \not= 0$ implies $x_i = y_i$. We are now in a position to recall $G$-Tucker’s lemma.

**Lemma 2** ($G$-Tucker’s lemma [3]). Suppose that $n$ is a positive integers, $G$ is a non-trivial finite group, and

$$\lambda : (G \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \rightarrow G \times \{1, \ldots, (n-1)\}$$

is a map such that $\lambda(g \cdot x) = g \cdot \lambda(x)$ for all $g \in G$ and all $x \in (G \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. Then there exist two elements $x, y \in (G \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$, and $e \not= g \in G$ such that $x \prec y$ and $\lambda(x) = g \cdot \lambda(y)$.

It is worth mentioning that octahedral Tucker’s lemma can be simply derived from $G$-Tucker’s lemma by putting $G = \mathbb{Z}_2$.

In this paper, we give a generalization of Dold’s theorem. Furthermore, in an example, we will see that one can hope to infer much more information from the generalized one than ordinary Dold’s theorem. Finally, we will present a generalization of $G$-tucker’s lemma as well. Our machinery here is based on an adapted version of compatibility graphs, called strong compatibility graphs, and the methods that we used in [4]. Generally speaking, the compatibility graph $C_P$ is a graph assigned to a $G$-poset $P$; a partially ordered set equipped with a group action $G$. We refer the interested reader to [3, 4] for more information on compatibility graphs and its applications. It is worth noting that some variant of this concept, compatibility graph, for $G = \mathbb{Z}_2$ were defined before by several authors [2, 13, 15].

2. Preliminaries

In this section, we collect some definitions and auxiliary results needed later in the paper. We assume that the reader is familiar with standard definitions and concepts of simplicial complexes. For background on simplicial complexes we refer the reader to [8]. Throughout the paper, the geometric realization of a simplicial complex $\mathcal{K}$ will be denoted by $||\mathcal{K}||$.

**Graphs:** In this paper, all graphs are finite, simple and undirected. The chromatic number $\chi(H)$ of a graph $H$ is the smallest number of colors needed to color the vertices of $H$ so that no two adjacent vertices share the same color.
**Posets and G-posets:** A pair of elements $a, b$ of a partially ordered set $(P, \preceq)$ (poset for short) are called comparable if either $a \preceq b$ or $b \preceq a$. A subset of a poset in which each two elements are comparable is called a chain. To any poset $(P, \preceq)$ we associate its order complex $\Delta(P)$, whose simplices are given by chains in $P$. A $G$-poset is a poset together with a $G$-action on its elements that preserves the partial order, i.e., $x < y \Rightarrow g \cdot x < g \cdot y$. A $G$-poset $P$ is called free $G$-poset, if for all $x$ in $X$, $g \cdot x = x$ implies $g = e$. One can see that, if $P$ is a free $G$-poset then its order complex $\Delta(P)$ is a free simplicial $G$-complex.

**Connectivity and $G$-index:** Let $k \geq 0$. A topological space $X$ is $k$-connected if its homotopy groups $\pi_0(X), \pi_1(X), \ldots, \pi_k(X)$ are all trivial. Also, for convenience, we make conventions that $(-1)$-connected means nonempty, and the empty set is $-\infty$-connected. The largest $k$, if it exists, that $X$ is $k$-connected is called the connectivity of $X$, and denoted by $\text{conn}(X)$. A simplicial complex $K$ is called $k$-connected if $||K||$ is $k$-connected. Similarly, a poset $P$ is called $k$-connected if $||\Delta(P)||$ is $k$-connected.

For an integer $n \geq 0$ and a group $G$, an $\mathbb{E}_n G$ space is the geometric realization of an $(n - 1)$-connected free $n$-dimensional simplicial $G$-complex. For a $G$-space $X$, we define

$$\text{ind}_G X = \min\{n | \text{there is a } G\text{-equivariant map } X \to \mathbb{E}_n G\}.$$ 

It is worth pointing out that the value of $\text{ind}_G X$ is independent of which $\mathbb{E}_n G$ space is chosen, because any of them $G$-equivariantly maps into any other, see [8, section 6.2] for details. For a concrete example, one can see that the geometric realization of order complex of $(G \times \{1, \ldots, n + 1\}, \preceq_{1})$, $||\Delta(G \times \{1, 2, \ldots, n + 1\})||$, is an example of $\mathbb{E}_n G$ space. It is worth noting that $\Delta(G \times \{1, 2, \ldots, n + 1\})$ is the standard $(n + 1)$-fold join $G * G * \cdots * G$.

Let us finish this section by listing some basic properties of $\text{ind}_G X$.

**Proposition 3 ([8]).** Let $G$ be a non-trivial finite group, and let $X, Y$ be $G$-spaces.

1. If there is $G$-map from $X$ to $Y$, we have $\text{ind}_G X \leq \text{ind}_G Y$.
2. For any $\mathbb{E}_n G$ spaces, $\text{ind}_G \mathbb{E}_n G = n$.
3. If $X$ is $k$-connected, then $k + 1 \leq \text{ind}_G X$.

**3. Strong compatibility graphs and Dold’s Theorem**

Let us begin by recalling compatibility graphs from [4].

**Definition 4 (Compatibility graph).** Let $P$ be a $G$-poset. The compatibility graph of $P$, denoted by $\tilde{C}_P$, is the graph $\tilde{C}_P$ with vertex set $P$, and two elements $x, y \in P$ are adjacent if there is an element $g \in G \setminus \{e\}$ such that $x$ and $g \cdot y$ are comparable in $P$.

The main idea of this paper was inspired from the following theorem [4].

**Theorem 5.** If $P$ is a finite free $G$-poset, then

$$\text{ind}_G ||\Delta(P)|| + |G| \leq \chi(C_P).$$
Actually, above theorem shows us a connection between the connectivity of a $G$-poset and the chromatic number of its compatibility graph, as for any finite $G$-space we always have $\text{Ind}_G X \geq \text{conn}(X) + 1$. So it was natural to think that there might be a relation between Dold’s theorem and above statement. At first we thought we could replace “dimension of at most $n$”, in the statement of Dold’s theorem, with the chromatic number of the compatibility graph of a suitable $G$-space. But, there were two issues with that: One is that, the difference between $\chi(\tilde{C}_P)$ and $\text{ind}_G|\|P||$ becomes larger, as the size of $G$ increased. The other one is that, it can also be an arbitrary large gap between the dimension of a $G$-poset and the chromatic number of its compatibility graph. For example, the compatibility graph of $(G \times \{1, \ldots, n\}, \preceq_1)$, $\tilde{C}_{G \times \{1, \ldots, n\}}$, is isomorphic with the complete graph $K_{|G|}$. Thus $\chi(\tilde{C}_{G \times \{1, \ldots, n\}}) = |G|$, which can be arbitrary larger than $\text{dim}(G \times \{1, \ldots, n\}) = n - 1$.

Therefore, for our purpose, a new version of the compatibility graph is needed. Let us define strong compatibility graph as follows.

**Definition 6** (Strong compatibility graph). Let $P$ be a $G$-poset. The strong compatibility graph of $P$, denoted by $\tilde{C}_P$, is the graph $\tilde{C}_P$ with vertex set $P$, and two elements $x, y \in P$ are adjacent if there is an element $g \in G \setminus \{e\}$ such that $x$ and $g \cdot y$ are comparable in $P$ and $y \notin [x]$, where $[x] = \{g \cdot x : g \in G\}$.

Despite of compatibility graphs, in the next lemma, we will see that the chromatic number of strong compatibility graph of a finite $G$-poset has a good connection to its dimension.

**Lemma 7.** If $P$ is a finite $G$-poset, then

$$\chi(\tilde{C}_P) \leq \text{dim}(P) + 1.$$  

**Proof.** Define

$$c : \tilde{C}_P \to \{1, \ldots, \text{dim}(P) + 1\}$$

$$p \mapsto \max\{|p \prec p_1 \prec \cdots \prec p_m : p_i \in P\}.$$  

We claim that $c$ is a proper coloring of $\tilde{C}_P$. If $p$ and $q$ are connected in $\tilde{C}_P$, then there is $e \neq g \in G$ such that $p$ and $g \cdot q$ are comparable in $P$. Note that $p \neq g \cdot q$, since by definition of $\tilde{C}_P$, $p \notin \{h \cdot q : h \in G\}$. Without loss of generality assume that $p < g \cdot q$. Also, let $c(q) = t + 1$ which means there is a chain of length $t$ in $P$ of the following form

$$q < q_1 < \cdots < q_t.$$  

By multiplying the previous chain by $g$, we get

$$g \cdot q < g \cdot q_1 < \cdots < g \cdot q_t.$$  

On the other hand, we have $p < g \cdot q$. Now, by the transitivity of $\prec$

$$p < g \cdot q < g \cdot q_1 < \cdots < g \cdot q_t,$$

and so $c(p) < c(q)$, hence $p$ and $q$ are not adjacent in $\tilde{C}_P$. Therefore, $c$ is a proper coloring of $\tilde{C}_P$.

Let $p \in P$ be arbitrary. By definition of $\text{dim}(P)$, there is a chain $p < p_1 < \cdots < p_{\text{dim}(P)}$ in $P$. Now, consider the chain $p < g \cdot p_1 < \cdots < g \cdot p_{\text{dim}(P)}$ in $\tilde{C}_P$, and note that $c(p) = \text{dim}(P)$ and $c(g \cdot p_{\text{dim}(P)}) = \text{dim}(P) + 1$. Hence

$$c(p) \leq \max\{|p \prec p_1 \prec \cdots \prec p_m : p_i \in P\} = \text{dim}(P) + 1.$$  

Therefore, $\chi(\tilde{C}_P) \leq \text{dim}(P) + 1$, as claimed.
which means \( c(p) > c(q) \). Therefore \( c \) is a proper coloring of \( \tilde{C}_p \). Thus,

\[
\chi(\tilde{C}_p) \leq \dim(P) + 1.
\]

The following inequality is an analogue of the inequality given in Theorem 5 with this advantage that the size of \( G \) is replaced by 1.

**Theorem 8.** If \( P \) is a finite free \( G \)-poset, then

\[
\text{ind}_G|\Delta(P)| + 1 \leq \chi(\tilde{C}_p).
\]

**Proof.** Let \( c: \tilde{C}_p \to \{1, \ldots, m\} \) be a proper coloring of \( \tilde{C}_p \) with \( m \) colors. In the following, we will show that this coloring induces a simplicial \( G \)-equivariant map

\[
\lambda: \Delta(P) \to \Delta(G \times \{1, \ldots, m\}),
\]

\[
x \mapsto (\lambda_1(x), \lambda_2(x)).
\]

First, we divide \( P \) into equivalence classes, the orbits under the \( G \)-action, where for every \( x \in P \) each class \([x]\) contains all elements \( g \cdot x \), where \( g \in G \), i.e., \([x] = \{g \cdot x : g \in G\}\). We pick one element from each class \([x]\) as a representative, say \( x' \), and we set \( \lambda(x') = (e, c(x')) \). Then, we extend \( \lambda \) on the remaining elements of each class in the only possible way that it preserves the \( G \)-action and \( \lambda_2(x) = c(x') \) for each \( x \in [x'] \). In other words, we define \( \lambda(g \cdot x') = (g, c(x')) \) for each \( g \in G \). Let us to verify that \( \lambda \) is a well-defined function. For this purpose, we need to show that any point in \([x']\), say \( x \), can be uniquely represented as \( x = g \cdot x' \) for some \( g \in G \). Now, suppose that \( x = g \cdot x' = h \cdot x' \) for some \( g, h \in G \). Then \( (h^{-1}g) \cdot x' = x' \). So \( h^{-1}g = e \), as \( P \) is a free \( G \)-poset. Thus, \( h = g \).

Next, we show that \( \lambda \) is a simplicial \( G \)-equivariant map. Clearly, by the definition of \( \lambda \), this map preserves the \( G \)-action, i.e., for each \( g \in G \) and \( x \in P \), \( \lambda(g \cdot x) = g \cdot \lambda(x) \). So, to prove our claim we just need to show that \( \lambda \) is a simplicial map, i.e., takes any simplex to a simplex. Note that \( \sigma \subseteq G \times \{1, \ldots, m\} \) is a simplex of \( \Delta(G \times \{1, \ldots, m\}) \) if and only if it contains no two different elements with the same second entries. Therefore, \( \lambda \) is a simplicial map if and only if for all comparable elements \( x, y \) in \( P \), if \( \lambda_2(x) = \lambda_2(y) \), then \( \lambda_1(x) = \lambda_1(y) \).

Now, let \( x \) and \( y \) be distinct elements of \( P \) with \( x \prec y \). Moreover, assume that \( \lambda(x) = (g, c(x')) \) and \( \lambda(y) = (h, c(y')) \), where \( x' \) and \( y' \) are representatives of classes \([x]\) and \([y]\), respectively, and \( c(x') = c(y') \). To finish the proof, we need to show that \( g = h \). Suppose, contrary to our claim, that \( g \neq h \). By definition of \( \lambda \), \( x' = g^{-1} \cdot x \) and \( y' = h^{-1} \cdot y \). So,

\[
g^{-1} \cdot x \prec g^{-1} \cdot y = (g^{-1}h) \cdot h^{-1} \cdot y.
\]

Therefore, taking into account that \( g^{-1}h \neq e \), we will conclude that \( x' \) and \( y' \) are adjacent in \( \tilde{C}_p \), as soon as we prove that \( y' \notin [x'] \). And, this contradicts the fact that \( c \) is a proper
coloring of $\tilde{C}_P$. To obtain a contradiction, suppose that $y' \in [x']$. This implies that $y \in [x]$ as well. So, there is a nontrivial element $s \in G$ such that $y = s \cdot x$. Thus $x \prec s \cdot x$, as $x \prec y$. By multiplying both sides of previous inequality by $e, s, \ldots, s^{[G]-1}$, respectively, we get

$$x \prec s \cdot x$$
$$s \cdot x \prec s^2 \cdot x$$
$$\vdots$$
$$s^{[G]-1} \cdot x \prec s^{[G]} \cdot x = x.$$

Now, by the transitivity of $\prec$, $x \prec x$, which is impossible. In summary, until yet, we have concluded that $\lambda$ is a $G$-simplicial map. This map naturally induces a $G$-equivariant map from $||\Delta(P)||$ to $||\Delta(G \times \{1, \ldots, m\})||$. Therefore, according to Proposition 3,

$$\text{ind}_G||\Delta(P)|| \leq \text{ind}_G||\Delta(G \times \{1, \ldots, m\})|| = m - 1.$$

Consequently, $\text{ind}_G||\Delta(P)|| + 1 \leq \chi(\tilde{C}_P)$. This is the desired conclusion. □

As corollaries of Lemma 7 and Theorem 8, we will state and prove a generalization of Dold’s theorem, and $G$-Tucker’s lemma. Before proceeding, let us recall a definition. The face poset of a simplicial complex $K$ is the poset $P(K)$, which is the set of all nonempty simplices of $K$ ordered by inclusion. Moreover, if $K$ is a free $G$-simplicial complex, then $G$ induces a free action on the poset $P(K)$, and consequently turns it to a free $G$-poset. Now, we are in a position to state and prove a generalization of Dold’s theorem.

**Theorem 9** (A generalization of Dold’s theorem). Let $G$ be a non-trivial finite group. Let $K$ be an $n$-connected simplicial $G$-complex, and $\mathcal{L}$ a free simplicial $G$-complex such that $\chi(\tilde{C}_P) \leq n + 1$. Then there is no simplicial $G$-equivariant map from $K$ to $\mathcal{L}$.

**Proof.** Suppose, contrary to our claim, that there is a simplicial $G$-equivariant map $\Psi : K \rightarrow \mathcal{L}$. Therefore, by Proposition 3 and Theorem 8

$$n + 1 = \text{conn}||K|| + 1 \leq \text{Ind}_G||K|| \leq \text{Ind}_G||\mathcal{L}|| = \text{Ind}_G||\Delta(P(\mathcal{L}))|| \leq \chi(\tilde{C}_P) - 1 \leq n,$$

which is impossible. □

Lemma 7 ensure us that Dold’s theorem is a consequence of the previous theorem. Furthermore, next example tells us, one can hope to infer much more information from the generalized one than ordinary Dold’s theorem.

**Example 10.** In this example, we equip $G \times \{1, \ldots, n\}$ with another ordering, called $\preceq_2$, rather than $\preceq_1$. Let $R = G \times \{1, \ldots, n\}$ be the $G$-poset whose the action is given $h \cdot (g, i) \rightarrow (hg, i)$, and the order is defined by $(g, x) \preceq_2 (h, y)$ if $x < y$ and $g = h$. This action turns
**Theorem 11** (A generalization of G-Tucker lemma). Let $G$ be a non-trivial finite group. Let $P$ be an $n$-connected $G$-poset, and let $Q$ be a free $G$-poset such that $\chi(\tilde{C}_Q) \leq n + 1$. If $\lambda : P \to Q$ is a $G$-map, i.e., $\lambda(g \cdot x) = g \cdot \lambda(x)$, then there exist comparable elements $x, y$ in $P$ such that $\lambda(x)$ and $\lambda(y)$ are not comparable in $Q$.

**Proof.** Suppose the assertion of the lemma is false. Then, $\lambda$ induces a simplicial $G$-equivariant map from $\Delta(P)$ to $\Delta(Q)$. Applying Proposition 3 and Theorem 8, we have

$$n + 1 = \text{conn}(|\Delta(P)||) + 1 \leq \text{Ind}_G(|\Delta(P)||) \leq \text{Ind}_G(|\Delta(Q)||) \leq \chi(\tilde{C}_Q) - 1 \leq n,$$

which is a contradiction. \(\square\)

In the light of the following facts, one can easily check that Theorem 11 is a generalization of G-Tucker’s lemma.

1. The space $|\Delta((G \cup \{0\})^n \setminus \{(0, \ldots, 0)\})|$ has the homotopy type of a wedge of $(n-1)$-dimensional spheres [8, section 6.2]. Therefore $|\Delta((G \cup \{0\})^n \setminus \{(0, \ldots, 0)\})|$ is $(n-2)$-connected.
2. The strong compatibility graph of \((G \times \{1, \ldots, (n-1)\}, \preceq_1)\), \(\tilde{C}_{G \times \{1,\ldots,(n-1)\}}\), is isomorphic with the complete \((n-1)\)-partite graph \(K_{\left|G\right|, \ldots, \left|G\right|, n-1}\). Thus,

\[
\chi\left(\tilde{C}_{G \times \{1,\ldots,(n-1)\}}\right) = n - 1 \quad \text{as} \quad \chi\left(K_{\left|G\right|, \ldots, \left|G\right|, n-1}\right) = n - 1.
\]

3. Distinct elements \((g, x)\) and \((h, y)\) are comparable in \((G \times \{1, \ldots, (n-1)\}, \preceq_1)\) if and only if \(x \neq y\).

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