Fully Distributed Continuous-Time Algorithm for Nonconvex Optimization over Unbalanced Directed Networks

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Abstract—This paper investigates the distributed continuous-time nonconvex optimization problem over unbalanced directed networks. The objective is to cooperatively drive all the agent states to an optimal solution that minimizes the sum of the local cost functions. Based on the topology balancing technique and adaptive control approach, a novel fully distributed algorithm is developed for each agent with neither prior global information concerning network connectivity nor convexity of local cost functions. By viewing the proposed algorithm as a perturbed system, its input-to-state stability with a vanishing perturbation is first established, and asymptotic convergence of the decision variables toward the optimal solution is then proved under the relaxed condition. A key feature of the algorithm design is that it removes the dependence on the smallest strong convexity constant of local cost functions, and the left eigenvector corresponding to the zero eigenvalue of the Laplacian matrix of unbalanced directed topologies. The effectiveness of the proposed fully distributed algorithm is illustrated with two examples.

Index Terms—Fully distributed, nonconvex optimization, continuous-time optimization, unbalanced directed networks, adaptive control.

I. INTRODUCTION

DISTRIBUTED optimization problem (DOP) has experienced significant advances in the past decade because of its great potential in a wide range of applications. Typical examples of application include resource allocation, sensor networks, and power systems [1], [2]. In the typical DOP of large-scale networks, each agent is often endowed with an individual local cost function. Seminal works on this topic primarily focus on discrete-time cases with convex local cost functions, which can be traced back to [3], [4]. For more recent developments on distributed discrete-time convex optimization, one may refer to [5]–[7] and references therein.

In parallel, the distributed continuous-time convex optimization problem has also been extensively studied (see, for example, [8]–[12]) because many practical systems (e.g., unmanned vehicles and robots) operate in a continuous-time setting [13]. A pioneering distributed gradient-based control approach was developed in [8] to address the DOP of multi-agent systems with single integrator dynamics over undirected graphs. In the subsequent work [9], the restriction on the communication network topologies was relaxed to balanced digraphs. By virtue of the proportional-integral control approach, a modified distributed Lagrangian-based algorithm was then proposed in [10] at the cost of special initialization so that communication and computation can be reduced. Other works that involve distributed continuous-time convex optimization, over either undirected or balanced directed networks, can be found in [14], [15] and references therein.

A central and standard assumption for the analysis of gradient-based minimization method in convex optimization is the convexity of the corresponding local cost functions [10], [16], which is exploited to not only refrain from the existence of local optima but also facilitate convergence analysis [17]. However, it cannot be satisfied in a broad class of practical applications such as sparse approximations of images, matrix factorization, and compressed sensing [18]. In fact, cost functions in engineering practice are often nonconvex. More recently, there are increasing number of studies on distributed discrete-time nonconvex optimization providing that local cost functions satisfy additional but relaxed conditions [19]–[21], such as the Polyak-Łojasiewicz (P-Ł) condition [22], the ρ-weakly convex condition [23], the μ-gradient dominated condition [24], and the second order sufficiency condition [25] among others. On the contrary, few works focus on developing continuous-time algorithms for distributed nonconvex optimization problems. Based on the canonical duality theory, a continuous-time algorithm was proposed in [26] for a class of nonconvex optimization, but it can only be applied when undirected networks are considered.

The above-mentioned works, whether studying convex optimization or nonconvex optimization, assume that the concerned network topologies are undirected or balanced directed. It is of much more theoretical and practical significance to study unbalanced directed topologies as the information exchange between neighboring agents may be unidirectional due to limited bandwidth or other physical constraints. In the discrete-time case, by employing a row or a column stochastic matrix, several consensus-based strategies were proposed in [5], [6] to tackle unbalanced digraphs in distributed convex optimization problem. These strategies are then extended to address the more challenging scenario of distributed nonconvex optimization over unbalanced digraphs [20], [23]. However, the agents in those works were required to know their out-degree [5], [20], which is a form of global information.

In the continuous-time case, a new distributed control strategy was developed based on the topology balancing technique to tackle the distributed convex optimization over unbalanced directed networks [27]. Nevertheless, it cannot be adopted when the left eigenvector corresponding to the zero eigenvalue of the concerned Laplacian matrix is not available in advance. For the same problem, a distributed estimator was designed in [28] to remove the explicit dependence on the
left eigenvector, and the gradient term therein was divided by the state of the distributed estimator. However, the control gains of the algorithm proposed in [28] still involve certain global information concerning the network connectivity such as the second smallest eigenvalue of the Laplacian matrix, and the smallest strong convexity constant of local cost functions. To the best of our knowledge, no distributed continuous-time algorithm has been proposed to address the distributed nonconvex optimization over unbalanced directed networks until now. To sum up, the existing algorithms that tackle unbalanced digraphs more or less rely on global information concerning the network connectivity or/and cost functions, and are thus not fully distributed.

Motivated by the above observations, this paper aims at developing a fully distributed continuous-time algorithm to address the distributed nonconvex optimization over unbalanced directed networks. The main challenge lies in establishing asymptotic convergence of the agent states in the absence of symmetric Laplacian matrix and the convexity of local cost functions. A novel algorithm is developed over unbalanced directed network topologies based on the topology balancing technique [28], [29] and adaptive control approach [27], [30]. The developed continuous-time algorithm is fully distributed in the sense that it does not depend on any global information about the network connectivity or the local cost functions. The main contributions of this paper are summarized as follows.

1) In contrast with those works addressing the DOP for undirected graphs or balanced digraphs [8]–[10], [14], [15], [31], this work considers unbalanced digraphs that are more general and also more challenging. Contrary to the algorithms in [20], [23], [27], [28] that deal with unbalanced digraphs, our distributed adaptive continuous-time algorithm does not depend on any global information concerning network connectivity or cost functions, and can be applied in a fully distributed manner. Specifically, the left eigenvector corresponding to the zero eigenvalue of the Laplacian matrix, which plays a crucial role in [27], no longer needs to be known a priori. The algorithm proposed in this work is thus expected to be adopted in a wider range of applications.

2) The requirement on the convexity of the local cost functions, which are exploited to facilitate the convergence analysis in [8]–[10], [14], [15], [28], is removed. Instead, in this work, the local cost functions are allowed to be nonconvex. Such a relaxed condition greatly broadens the application scope of distributed optimization.

The layout of this paper is as follows. Section II reviews some preliminaries on graph theory and convex analysis, and then formulates the problem.

II. Preliminaries and Problem Formulation

In this section, we first present some preliminaries on graph theory and convex analysis, and then formulate the problem.

A. Graph Theory

A directed graph (in short, a digraph) of order $N$ can be described by a triplet $G = (V, \mathcal{E}, A)$, where $V = \{1, \ldots, N\}$ is a set of nodes, $\mathcal{E} \subseteq V \times V$ is a collection of edges, and an adjacency matrix $A$. For $i, j \in V$, the ordered pair $(j, i) \in \mathcal{E}$ refers to an edge from $j$ to $i$. A directed path in a digraph is an ordered sequence of nodes, in which any pair of consecutive nodes is a directed edge. A digraph is said to be strongly connected if for each node, there exists a directed path from any other node to itself. The adjacency matrix is defined as $A = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ii} = 0$ for all $i$, $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ associated with the digraph $G$ is defined as $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. A digraph $G$ is called balanced if and only if $1_N^T L = 0_N^T$, otherwise it is called unbalanced. One can consult [29] for more details.

Lemma 1. (see [32], [33]) Let $L$ be the Laplacian matrix associated with a strongly connected digraph $G$. Then the following statements hold.

i) There exists a left eigenvector $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T$ associated with the zero eigenvalue of the Laplacian matrix such that $\sum_{i=1}^{N} \xi_i = 1$, $\xi_i > 0$, $i = 1, 2, \ldots, N$, and $\xi^T L = 0_N^T$.

ii) Define $\bar{L} = RL + L^T R$ with $R = \text{diag}(\xi_1, \xi_2, \ldots, \xi_N)$. Then $\min_{x^T x = 0, x \neq 0} x^T \bar{L} x > \frac{\lambda_2(L)}{\xi_1^2} x^T x$ for any positive vector $c$, where $\lambda_2(L)$ represents the second smallest eigenvalue of $L$.

iii) $e^{-tL}$ is a nonnegative matrix with positive diagonal entries for all $t > 0$, and $\lim_{t \to \infty} e^{-tL} = 1_N \xi^T$.

B. Convex Analysis

This subsection reviews the definitions of convexity and Lipschitz continuity. One can refer to [16], [34] for more details.

A subset $\Omega$ of $\mathbb{R}^n$ is called convex if, for all $\alpha \in [0, 1]$,

$$\alpha x + (1 - \alpha) y \in \Omega, \quad \forall x, y \in \Omega.$$  

A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is called convex if, for all $\alpha \in [0, 1]$,

$$f(\alpha x + (1 - \alpha) y) \leq \alpha f(x) + (1 - \alpha) f(y), \quad \forall x, y \in \Omega, \quad (1)$$

Notation: Let $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{N \times N}$ be the sets of real numbers, $n$-order real vectors and $N$-dimensional real square matrices, respectively. $I_n$ refers to the $n$-dimensional identity matrix. Let $0_n$ and $1_n$, or simply $0$ and $1$, represent the $n$-dimensional column vector in which all entries are equal to 0 and 1, respectively. $A_i$ and $A_{ij}$ denote the $i$th row elements and the $(i, j)$ entry of matrix $A$, respectively. The transpose of vector $x$ and matrix $A$ are denoted by $x^T$ and $A^T$, respectively. $\| \cdot \|$ represents the Euclidean norm of vectors or induced 2-norm of matrices. The Kronecker product of matrices $A$ and $B$ is represented by $A \otimes B$. col$(x_1, x_2, \ldots, x_n)$ and diag$(x_1, x_2, \ldots, x_n)$ represent a column vector and a diagonal matrix, respectively, with $x_1, x_2, \ldots, x_n$ being their elements.
otherwise it is called nonconvex. A convex function \( f : \Omega \to \mathbb{R} \) is called strictly convex if, for all \( \alpha \in (0, 1) \), the inequality (1) is strict for all \( x, y \in \Omega \) with \( x \neq y \). A convex function \( f : \Omega \to \mathbb{R} \) is called strongly convex if there exists a positive constant \( m \) such that \( (x - y)^T (\nabla f(x) - \nabla f(y)) \geq m \|x - y\|^2 \) for all \( x, y \in \Omega \), where \( \nabla f \) denotes the gradient.

A function \( g : \mathbb{R}^n \to \mathbb{R}^n \) is said to be Lipschitz continuous, or simply Lipschitz, if there exists a constant \( l > 0 \) such that the following Lipschitz condition is satisfied,

\[
\| g(x) - g(y) \| \leq l \| x - y \|, \quad \forall x, y \in \mathbb{R}^n.
\]  

(2)

C. Problem Formulation

Consider a multi-agent system composed of \( N \) identical agents over an unbalanced directed network. Each agent \( i \) is attached with a private local cost function \( f_i : \Omega \to \mathbb{R} \), where \( s \in \mathbb{R}^n \) represents the local decision variable. The global cost function and the corresponding optimal solution are defined as \( f(s) = \sum_{i=1}^{N} f_i(s) \) and \( s^* \), respectively. In this work, the objective is to design a fully distributed continuous-time algorithm in the sense that neither strong convexity of local cost functions nor global information about network connectivity is required, such that the following DOP can be solved,

\[
\min_{s \in \mathbb{R}^n} f(s).
\]  

(3)

To solve the problem, necessary assumptions are introduced.

Assumption 1. The digraph \( G \) is strongly connected.

Remark 2. In most existing works that study the DOP [8]–[10], [14], [15], [31] that address the distributed continuous-time optimization for undirected graphs or balanced digraphs, this paper concentrates on the more general and also more challenging case of unbalanced directed communication topologies. Unbalanced topologies bring challenges to the convergence establishment of the agent states toward the exact optimal solution because consensus cannot be achieved with the unweighted gradient information transmitted by the asymmetric topologies.

Assumption 2. Each local cost function \( f_i \) is differentiable, and its gradient \( \nabla f_i \) is globally Lipschitz on \( \mathbb{R}^n \) with constant \( l_i \). The optimal solution \( s^* \in \mathbb{R}^n \) to the DOP in (3) exists and is unique.

Remark 3. The existence and uniqueness of the optimal solution can be guaranteed when the global cost function is strictly convex or the set of global minimizers is a singleton [16], [18]. The differentiability of local cost functions is required only to facilitate the convergence analysis. Illustrative Example 2 shows that our proposed adaptive algorithm can still solve the concerned problem even if the cost function is nondifferentiable.

III. MAIN RESULT

In this section, we propose a fully distributed adaptive algorithm to solve the DOP in (3) over unbalanced directed networks without any global information concerning the network connectivity or local cost functions.

A. Fully Distributed Algorithm

In this subsection, based on adaptive control approach, a fully distributed algorithm is designed for each agent \( i, i = 1, 2, \ldots, N \) as follows,

\[
\begin{align*}
\dot{x}_i &= -\frac{\nabla f_i(x_i)}{w_i} - (\sigma_i + \rho_i) \sum_{j=1}^{N} a_{ij}(x_i - x_j) - \sum_{j=1}^{N} a_{ij}(v_i - v_j), \\
\dot{v}_i &= (\sigma_i + \rho_i) \sum_{j=1}^{N} a_{ij}(x_i - x_j), \\
\dot{w}_i &= -\sum_{j=1}^{N} a_{ij}(w_i - w_j), \\
\dot{\sigma}_i &= \left( \sum_{j=1}^{N} a_{ij}(x_i - x_j) \right)^T \left( \sum_{j=1}^{N} a_{ij}(x_i - x_j) \right).
\end{align*}
\]  

(4)

where \( x_i \in \mathbb{R}^n \) is the state of agent \( i \), \( v_i \in \mathbb{R}^n \) and \( w_i \in \mathbb{R}^N \) are two auxiliary variables, \( w_i \) is the \( i \)th component of \( w_i \), and the initial value \( w_i(0) \) satisfies \( w_i(0) = 0 \), otherwise \( w_i(0) = 0 \) for all \( k \neq i \); \( \sigma_i \) is an adaptive gain with initial condition \( \sigma_i(0) > 0 \), and the dynamic gain \( \rho_i \) is designed as

\[
\rho_i = \left( \sum_{j=1}^{N} a_{ij}(x_i - x_j) \right)^T \left( \sum_{j=1}^{N} a_{ij}(x_i - x_j) \right).
\]

Define \( w = \text{col}(w_1, w_2, \ldots, w_N) \). It follows that \( \dot{w} = - (\mathcal{L} \otimes I_N) w \). By referring to \( w_i(0) = 1 \), and \( w_i(0) = 0 \) for all \( k \neq i \) as well as iii) of Lemma 1, one has

\[
\begin{align*}
w_i^1(t) &= (e^{- (\mathcal{L} \otimes I_N)^t})_{(i-1)N+i} \cdot w(0) \nonumber \\
&\quad \quad - (e^{- (\mathcal{L} \otimes I_N)^t})_{(i-1)N+i} \cdot w_i^1(0) \nonumber > 0,
\end{align*}
\]  

(5)

for all \( t \geq 0 \). Therefore, the term \( -\nabla f_i(x_i) \) in algorithm (4) is well defined. Moreover, by applying iii) of Lemma 1 and recalling the initial condition of \( w_i(0) \), we can obtain that

\[
\lim_{t \to \infty} w_i(t) = \lim_{t \to \infty} e^{- (\mathcal{L} \otimes I_N)^t} w(0) = (1_N e^T \otimes I_N) w(0) = 1_N \otimes \xi.
\]  

(6)

This implies that \( w_i^1(t), i = 1, 2, \ldots, N \) are bounded for all \( t > 0 \).

Remark 4. Compared to the algorithm proposed in [27], we use \( \nabla f_i(x_i) \) instead of \( \nabla f_i(x_i) \) in the adaptive algorithm (4) so that it is able to not only tackle the imbalance caused by unbalanced directed topologies but also eliminate the restrictive requirement on the exact value of the left eigenvector corresponding to the zero eigenvalue of the Laplacian matrix. The algorithm (4) does not depend on any global information concerning network connectivity or cost functions, and is thus fully distributed.
Remark 5. The design of algorithm (4) takes the modified Lagrange structure in [10] with an adaptive control scheme. By using the adaptive gain $\sigma_i$ and the dynamic gain $\rho_i$ instead of constant gains as in [10], [28], some global information concerning cost functions and network connectivity is no longer needed, such as the smallest strong convexity constant of local cost functions, and the second smallest or largest eigenvalue of the Laplacian matrix.

Define
\[ x = \text{col}(x_1, x_2, \ldots, x_N), \quad v = \text{col}(v_1, v_2, \ldots, v_N), \]
\[ C = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_N), \quad B = \text{diag}(\rho_1, \rho_2, \ldots, \rho_N), \]
\[ \nabla \tilde{f}(x) = \text{col}(\nabla f_1(x_1), \nabla f_2(x_2), \ldots, \nabla f_N(x_N)), \]
\[ W^{-1} = \text{diag}(\frac{1}{w_1^1}, \frac{1}{w_2^2}, \ldots, \frac{1}{w_N^N}), \quad e_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j). \]

It can be seen from (5) that the matrix $W^{-1}$ is well defined. Given any $\sigma_i(0) > 0$, it can be proved that the adaptive gain $\sigma_i(t)$ remains to be positive for all $t > 0$. Thus, the dynamics of $(x, v, w, \sigma_i)$ can be written in the following form,
\[ \dot{x} = -(W^{-1} \otimes I_n)\nabla \tilde{f}(x) - ((C + B)L \otimes I_n)x - (L \otimes I_n)v, \]
\[ \dot{v} = ((C + B)L \otimes I_n)x, \]
\[ \dot{w} = -(L \otimes I_n)w, \quad \sigma_i = e_i^T e_i, \]
(7)

In what follows, our goal is to show that the agent states $x_i, i = 1, 2, \ldots, N$ of (4) converge to the optimal solution $s^*$ of the DOP in (3).

B. Convergence Analysis

To proceed, a preliminary result on the optimality condition will be first established. Define $\mathcal{R}^{-1} = \text{diag}(\frac{1}{\xi_1^1}, \frac{1}{\xi_2^2}, \ldots, \frac{1}{\xi_n^n})$, where $\xi_i, i = 1, 2, \ldots, N$ is the $i$th component of the left eigenvector $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T$. The following lemma reveals the optimality condition in terms of a set of equations that the optimal solution $s^*$ satisfies.

Lemma 2. Under Assumptions 1–2, suppose that the point $(\bar{x}, \bar{v})$ satisfies the following equations,
\[ 0 = -(\mathcal{R}^{-1} \otimes I_n)\nabla \tilde{f}(\bar{x}) - ((C + B)L \otimes I_n)\bar{x} - (L \otimes I_n)\bar{v}, \]
(8a)
\[ 0 = ((C + B)L \otimes I_n)\bar{x}. \]
(8b)
Then, one has $\bar{x} = 1_N \otimes s^*$, with $s^*$ being the optimal solution of the DOP in (3).

Proof. Define $\bar{x} = \text{col}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)$. It can be seen from (8b) that $\bar{x} = 1_N \otimes q$ holds for some vector $q \in \mathbb{R}^n$. On the one hand, by pre-multiplying both sides of equation (8a) with $\xi^T \otimes I_n$, it follows from i) of Lemma 1 that $\sum_{i=1}^{N} \nabla f_i(\bar{x}_i) = 0_n$. On the other hand, under Assumption 2, the optimality condition $\sum_{i=1}^{N} \nabla f_i(s^*) = 0_n$ is satisfied. Therefore, it can be obtained that $\bar{x} = 1_N \otimes s^*$. \hfill \Box

With Lemma 2 in hand, to prove that the agent states $x_i, i = 1, 2, \ldots, N$ of (4) converge to the optimal solution $s^*$ of the DOP in (3), it is sufficient to show that $(x, v)$ of (7) converges to $(\bar{x}, \bar{v})$ in Lemma 2.

To proceed, define $\tilde{x} = x - \bar{x}$ and $\tilde{v} = v - \bar{v}$. By subtracting the equations (7) from (8), and noting that $W^{-1} \neq \mathcal{R}^{-1}$, the dynamics of $(\tilde{x}, \tilde{v}, w, \sigma_i)$ can be written in the following form,
\[ \dot{\tilde{x}} = -(W^{-1} \otimes I_n)h + ((\mathcal{R}^{-1} - W^{-1}) \otimes I_n)\nabla \tilde{f}(\bar{x}) \]
\[ - ((C + B)L \otimes I_n)\bar{x} - (L \otimes I_n)\bar{v}, \]
(9a)
\[ \dot{\tilde{v}} = ((C + B)L \otimes I_n)\tilde{x}, \]
(9b)
\[ \dot{w} = -(L \otimes I_n)w, \quad \sigma_i = e_i^T e_i, \]
(9c)
where $h = \nabla \tilde{f}(\tilde{x} + \bar{x}) - \nabla \tilde{f}(\bar{x})$. Therefore, to show that $\lim_{t \to \infty} x_i(t) = s^*$, by $\tilde{x} = x - \bar{x}$ in (9) and $\tilde{x} = 1_N \otimes s^*$ in Lemma 2, we only need to prove that the state $\tilde{x}$ of (9) converges to zero as time tends to infinity.

However, the origin $(\bar{x}, \bar{v}) = (0, 0)$ is not the equilibrium point of (9) as $\mathcal{R}^{-1} \neq W^{-1}$. This brings some extra challenge to the convergence analysis. To tackle this issue, we need to introduce some coordinate transformations. Define two new variables $\zeta = (L \otimes I_n)\bar{x}$ and $\eta = (L \otimes I_n)\bar{v}$. In what follows, we will first prove that $\lim_{t \to \infty} \zeta(t) = 0$ and $\lim_{t \to \infty} \eta(t) = 0$, which are followed by $\lim_{t \to \infty} \tilde{x}(t) = 1_N \otimes \tau_1$ and $\lim_{t \to \infty} \tilde{v}(t) = 1_N \otimes \tau_2$, for two constant vectors $\tau_1, \tau_2 \in \mathbb{R}^n$. Then, we will show that $\tau_1 = 0$ and $\tau_2 < \infty$ by seeking a contradiction.

Now we are ready to present the main result of this work.

Theorem 1. Suppose Assumptions 1–2 hold. For $i = 1, 2, \ldots, N$, let $\sigma_i(0) > 0$, $w_i^0(0) = 1$, and $w_i^0(0) = 0$ for all $k \neq i$. Then, for any initial conditions $x_i(0)$ and $v_i(0)$, the DOP in (3) is solved by the fully distributed algorithm (4).

Proof. To prove Theorem 1, it is sufficient to prove that the state $\tilde{x}$ of (9) will converge to zero as time tends to infinity in the case of $\mathcal{R}^{-1} \neq W^{-1}$. The proof is composed of the following two parts.

Part 1. Show that $\lim_{t \to \infty} \zeta(t) = 0$ and $\lim_{t \to \infty} \eta(t) = 0$.

To proceed, let $\zeta_i \in \mathbb{R}^n, i = 1, 2, \ldots, N$ be a column vector stacked from the $((i-1) \times n+1)$th element to the $(i \times n)^{th}$ element of vector $\zeta$. Recalling that $e_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j)$, simple derivation gives $e_i = \zeta_i$. Then, the dynamics of $(\zeta, \eta, w, \sigma_i)$ can be written as follows,
\[ \dot{\zeta} = -(LV^{-1} \otimes I_n)h + (L(\mathcal{R}^{-1} - W^{-1}) \otimes I_n)\nabla \tilde{f}(\bar{x}) \]
\[ - (L(C + B) \otimes I_n)\zeta - (L \otimes I_n)\eta, \]
(10)
\[ \dot{\eta} = (L(C + B) \otimes I_n)\zeta, \]
\[ \dot{w} = -(L \otimes I_n)w, \quad \sigma_i = e_i^T e_i. \]

Define $\chi = \text{col}(\zeta, \eta)$. Then the dynamics of $\chi$ can be rewritten as
\[ \dot{\chi} = \varphi(\chi) + \phi(t), \]
(11)
where $\varphi(\chi)$ and $\phi(t)$ are defined in (12) on the next page. At first, we will establish the asymptotical convergence to the origin of $\chi$ in system (11) with $\mathcal{R}^{-1} \neq W^{-1}$ and $\phi(t) \neq 0$.

Under Assumption 2, $h = \nabla \tilde{f}(\tilde{x} + \bar{x}) - \nabla \tilde{f}(\bar{x})$ and thus $\varphi(\chi)$ are locally Lipschitz in $\chi \in \Omega$ for any compact subset $\Omega \subset \mathbb{R}^n$. In addition, it follows from (6) that $\phi(t)$ is bounded for all $t \geq 0$. According to Theorem 4.19 in [34], to establish the asymptotical convergence to the origin of $\chi$ in (11), global uniform asymptotical stability of the unperturbed system $\dot{\chi} = \varphi(\chi)$...
\[ \varphi(\chi) = \left(-\mathcal{L}W^{-1} \otimes I_N \right) h - \left(\mathcal{L}(C + B) \otimes I_N\right) \zeta - \left(\mathcal{L} \otimes I_N\right) \eta \right), \quad \phi(t) = \left(\mathcal{L}(\mathcal{R}^{-1} - W^{-1}) \otimes I_N\right) \nabla \tilde{f}(\tilde{x}) \right) \] (12)

\( \varphi(\chi) \) should be established at first, and input-to-state stability (ISS) of the system (11) will be presented in turn. The proof can be accomplished by two steps.

**Step 1.** Show that the equilibrium point \( \chi = 0 \) of the unperturbed system \( \dot{\chi} = \varphi(\chi) \) is globally uniformly asymptotically stable. By referring to [35], one can obtain that \( \text{rank}(\mathcal{L}^T \mathcal{L}) = \text{rank}(\mathcal{L}) = N - 1 \). Thus, zero is a simple eigenvalue of matrix \( \mathcal{L}^T \mathcal{L} \). Note that \( \zeta_i \) and \( \sigma_i \) are positive for all \( i = 1, 2, \ldots, N \). Consider the following Lyapunov function candidate,

\[ V = V_1 + V_2 + \frac{33N \lambda_N(\mathcal{L}^T \mathcal{L})}{\lambda_2} V_3, \] (13)

where \( \lambda_N(\mathcal{L}^T \mathcal{L}) \) denotes the largest eigenvalue of \( \mathcal{L}^T \mathcal{L} \), \( \lambda_2(\mathcal{L}) \) denotes the second smallest eigenvalue of \( \mathcal{L} = \mathcal{R}^2 \), and

\[
\begin{align*}
V_1 &= \frac{1}{2} \sum_{i=1}^{N} (\sigma_i - \sigma_0)^2, \\
V_2 &= \sum_{i=1}^{N} \xi_i (2\sigma_i + \rho_i) \zeta_i^T \zeta_i, \\
V_3 &= \frac{1}{2}(\zeta + \eta)^T(\mathcal{R} \otimes I_N)(\zeta + \eta),
\end{align*}
\]

with \( \sigma_0 \) being a positive constant to be determined later.

The derivatives of \( V_1 \) and \( V_2 \) along the trajectories of the unperturbed system \( \dot{\chi} = \varphi(\chi) \) satisfy

\[
\begin{align*}
\dot{V}_1 &= \sum_{i=1}^{N} \zeta_i^T(\sigma_i - \sigma_0) \zeta_i, \\
\dot{V}_2 &= 2\sum_{i=1}^{N} \xi_i(\sigma_i + \rho_i) \zeta_i^T \zeta_i + \sum_{i=1}^{n} \xi_i \rho_i \zeta_i^T \zeta_i.
\end{align*}
\] (14)

By combining (14)–(15), and recalling \( \varphi(\chi) \) in (12), we can obtain that

\[
\dot{V}_1 + \dot{V}_2 \leq -2 \zeta^T((C + B)\mathcal{R}\mathcal{L}(C + B) \otimes I_N) \zeta + \zeta^T((C + RB - \sigma_0 I_N) \otimes I_N) \zeta - 2 \zeta^T((C + B)\mathcal{R}\mathcal{L}W^{-1} \otimes I_N) h
\]

Thus, it follows from ii) of Lemma 1 that

\[
-2 \zeta^T((C + B)\mathcal{R}\mathcal{L}(C + B) \otimes I_N) \zeta = -\zeta^T((C + B)(\mathcal{R}\mathcal{L} + \mathcal{L}^T \mathcal{R})(C + B) \otimes I_N) \zeta \leq -\lambda_2(\mathcal{L}) N \zeta^T((C + B)^2 \otimes I_N) \zeta.
\]

Since it is proved in (5)–(6) that \( w_i^j(t) > 0, \ i = 1, 2, \ldots, N \) are bounded for all \( t > 0 \), \( \bar{w} = \min \{ \inf_{t>0} w_i^j(t), \ i = 1, 2, \ldots, N \} \) is well defined. Then, the following two inequalities are satisfied,

\[
-2 \zeta^T((C + B)\mathcal{R}\mathcal{L}W^{-1} \otimes I_N) h \leq \lambda_2(\mathcal{L}) N \zeta^T((C + B)^2 \otimes I_N) \zeta + \frac{4N \lambda_N(\mathcal{L}^T \mathcal{L})}{\lambda_2} \| h \|^2, \\
-2 \zeta^T((C + B)\mathcal{R}\mathcal{L} \otimes I_N) \eta \leq \lambda_2(\mathcal{L}) N \zeta^T((C + B)^2 \otimes I_N) \zeta + \frac{4N \lambda_N(\mathcal{L}^T \mathcal{L})}{\lambda_2} \| \eta \|^2.
\]

For convenience, we subsequently abbreviate \( \lambda_2(\mathcal{L}) \) and \( \lambda_N(\mathcal{L}^T \mathcal{L}) \) as \( \lambda_2 \) and \( \lambda_N \), respectively. Then, (16) can be rewritten as follows,

\[
\dot{V}_1 + \dot{V}_2 \leq -\frac{\lambda_2}{2N} \zeta^T((C + B)^2 \otimes I_N) \zeta + \frac{4N \lambda_N(\mathcal{L}^T \mathcal{L})}{\lambda_2} \| h \|^2 + \frac{4N \lambda_N}{\lambda_2} \| \eta \|^2.
\] (17)

The derivative of \( V_3 \) along the trajectories of the unperturbed system \( \dot{\chi} = \varphi(\chi) \) is given as follows,

\[
\dot{V}_3 = \zeta^T(\mathcal{L} \otimes I_N)(\zeta + \eta) \leq \zeta^T(\mathcal{L} \otimes I_N) \zeta - \eta^T(\mathcal{L} \otimes I_N) \eta.
\] (18)

Applying ii) of Lemma 1 leads to

\[
-\eta^T(\mathcal{L} \otimes I_N) \eta \leq \frac{\lambda_2}{2N^2} \| \eta \|^2.
\] (19)

Moreover, it can be verified that the following inequalities hold,

\[
\begin{align*}
-\zeta^T(\mathcal{L} \otimes I_N) \eta &\leq \frac{\lambda_2}{8N} \| h \|^2 + \frac{2\lambda_2}{\lambda_2} \| \eta \|^2, \\
-\eta^T(\mathcal{L} \otimes I_N) \eta &\leq \frac{\lambda_2}{2} \| \eta \|^2 + \frac{3\lambda_2}{\lambda_2} \| h \|^2, \\
-\zeta^T(\mathcal{L} \otimes I_N) \eta &\leq \| | h \|^2 + \frac{\lambda_2}{4N^2} \| \eta \|^2.
\end{align*}
\] (20)

Then, substituting inequalities (19)–(20) into (18) yields

\[
\dot{V}_3 \leq -\frac{\lambda_2}{2N^2} \| \eta \|^2 + \frac{2\lambda_2}{\lambda_2} \| \eta \|^2 + \frac{3\lambda_2}{\lambda_2} \| h \|^2.
\] (21)

By combining (17) and (21), the derivative of \( V \) in (13) along the trajectories of the unperturbed system \( \dot{\chi} = \varphi(\chi) \) satisfies the following inequality,

\[
\dot{V} \leq \zeta^T \left(\left(-\frac{\lambda_2}{2N^2}(C + B)^2 + (C + RB - \sigma_0 I_N)\right) \otimes I_N \right) \zeta - \frac{17N \lambda_N}{4N^2} \| \eta \|^2 + \left(\frac{4N \lambda_N}{\lambda_2} + \frac{33N \lambda_N(8 + \lambda_2)}{4N^2 \lambda_2^2}\right) \| h \|^2.
\] (22)

By the Lipschitz condition of the gradients in Assumption 2, we have \( h = \nabla \tilde{f}(\tilde{x}) + \nabla f(\bar{x}) \leq \bar{f} \| \tilde{x} \|, \) where \( \bar{f} = \max\{|l_1, l_2, \ldots, l_N|\} \).

Denote the second smallest eigenvalue of matrix \( \mathcal{L}^T \mathcal{L} \) by \( \lambda_2(\mathcal{L}) \), or simply \( \lambda_2 \). Recall that \( \zeta = (\mathcal{L} \otimes I_N) \tilde{x} \) and \( \mathcal{L} I_N = 0_N \). It can be obtained from the symmetry of \( \mathcal{L}^T \mathcal{L} \) and \( \lambda_2(\mathcal{L}^T \mathcal{L}) > 0 \) that \( \| \zeta \|^2 = \tilde{x}^T((\mathcal{L} \otimes I_N) \tilde{x}) \geq \lambda_2 \| \tilde{x} \|^2 \). Thus, one has \( \| h \|^2 \leq l^2 / \lambda_2 \| \zeta \|^2 \). Define \( \omega_1 = \frac{33 \lambda_N}{\lambda_2} \) and
Therefore, it is proved that the equilibrium point $\chi = 0$ of the unperturbed system $\dot{\chi} = \varphi(\chi)$ is globally uniformly asymptotically stable, and the adaptive control gains $\sigma_i$, $i = 1, 2, \ldots, N$ converge to some finite positive constants.

**Step 2.** Show that the perturbed system (11) is ISS, and the state $\chi(t)$ converges to zero as $t \to \infty$. Reconsider the Lyapunov function candidate $V$ in (13), but with $\sigma_0$ being another positive constant to be specified. Similarly, by referring to (23), the derivative of $V$ along the trajectories of (11) can be given as follows,

$$
\dot{V} \leq - \zeta^T \left( \frac{\lambda}{2N} \left( (C + B) - \frac{N}{\lambda} I_N \right) \otimes I_n \right) \zeta - \left( \sigma_0 - \omega_1 - \omega_2 - \frac{N}{\lambda} \right) \|\zeta\|^2 - \frac{17N\lambda x}{4\lambda_2} \|\eta\|^2.
$$

Choose $\sigma_0 = 1 + \omega_1 + \omega_2 + \frac{N}{\lambda}$. One thus has

$$
\dot{V} \leq - \frac{\lambda}{2N} \|\zeta\|^2 - \frac{17N\lambda x}{4\lambda_2} \|\eta\|^2.
$$

(24)

Therefore, it is proved that the equilibrium point $\chi = 0$ of the unperturbed system $\dot{\chi} = \varphi(\chi)$ is globally uniformly asymptotically stable, and the adaptive control gains $\sigma_i$, $i = 1, 2, \ldots, N$ converge to some finite positive constants.

Remark 6. The adaptive gain $\sigma_i$ in (4) is updated based on relative state errors so that it will always increase as long as the consensus is not achieved, eventually rendering the state consensus of the agents. By adopting the adaptive control approach, the requirement on the smallest strong convexity constant of local cost functions is no longer needed to generate a negative term related to agent states. Thus, the proposed adaptive algorithm is able to solve the DOP when local cost functions are nonconvex.

Remark 7. The DOP over unbalanced digraphs is investigated in [28]. The advantages of the distributed adaptive algorithm (4) in this work over the algorithm in [28] can be stated in the following two aspects. First, the control gains in [28] rely on some prior global information concerning network connectivity and cost functions, such as the second smallest eigenvalue of the Laplacian matrix, the smallest strong convexity constant of local cost functions as well as the largest Lipschitz constant of their gradients. On the contrary, the algorithm (4) developed in this work does not require those information and is thus fully distributed. Second, to guarantee the convergence of their algorithm, sufficient large control gains are needed in [28]. A potential problem with high-gain feedback is that it may result in some undesirable issues, such as increased sensitivity to unmodeled dynamics and noise, oscillations, and even instability. The algorithm developed in this work can avoid this problem by adopting an adaptive control mechanism.

IV. ILLUSTRATIVE EXAMPLES

In this section, the effectiveness of the developed fully distributed algorithm (4) over unbalanced directed networks is illustrated by two examples.
A. Example 1

Consider five networked agents with their communication network topology being described by the unbalanced digraph \( G \) in Fig. 1. It can be verified that this digraph is strongly connected, and Assumption 1 is thus satisfied. Suppose that agent \( i, i = 1, \ldots, 5 \), are endowed with the following local cost functions respectively:

\[
\begin{align*}
  f_1 &= 5 \sin \left( \|s + [4, 5]^T\| \right), \\
  f_2 &= 10 \cos \left( \ln\left(\|s + [8, 10]^T\|\right) \right), \\
  f_3 &= 4 \times \|s + [2, 3]^T\|^2, \\
  f_4 &= 2 \times \|s - [3, 5]^T\|^2, \\
  f_5 &= \|s + [1, 2]^T\|^2 / \sqrt{\|s + [1, 2]^T\|^2 + 2},
\end{align*}
\]

where \( s \in \mathbb{R}^2 \). Note that the local cost functions \( f_1(\cdot) \) and \( f_2(\cdot) \) are nonconvex. However, it can be verified that the global cost function \( f(s) = \sum_{i=1}^5 f_i(s) \) is strictly convex, which implies that the global minimizer \( s^* \) is unique. Moreover, it can be verified that the gradients of \( f_i(\cdot), i = 1, \ldots, 5 \), are globally Lipschitz on \( \mathbb{R} \), and Assumption 2 is thus satisfied.

We now present the convergence performance of the fully distributed algorithm (4) under the relaxed condition that the global cost function is strictly convex while local cost functions may be nonconvex. For \( i = 1, \ldots, 5 \), let the initial values \( x_i(0) \) and \( v_i(0) \) be arbitrarily chosen, and the initial values of the adaptive gains \( \sigma_i \)'s be chosen as \( \sigma_i(0) = 1 \). The simulation results are shown in Fig. 2. It can be observed that the trajectories of agent states \( x_i = [x_{i1}, x_{i2}]^T, i = 1, \ldots, 5 \), converge to the optimal solution \( s^* = [1.4136, 2.53658]^T \), which minimizes the global cost function \( f(s) = \sum_{i=1}^5 f_i(s) \). Therefore, without either strong convexity of local cost functions or prior global information of network connectivity, it has been shown that the fully distributed algorithm (4) solves the DOP in (3).

B. Example 2

In this example, we apply the fully distributed algorithm (4) to solve the distributed parameter estimation problem, which is one of the most important research topics in wireless sensor networks. The objective is to estimate a parameter or function based on large amounts of data collected by sensors from the environment. The distributed parameter estimation problem can be reformulated as a DOP. Moreover, the estimator is typically the optimal solution of the corresponding optimization problem. Please refer to [36] and the references therein for more details.

More specifically, consider a group of \( N \) sensors over general directed networks that cooperatively estimate a parameter \( s \in \mathbb{R}^n \) by collecting private measurements. Suppose that each sensor \( i \) collects \( n_i \) measurements \( Q_{ij} \in \mathbb{R}^n, j = 1, \ldots, n_i \). Let \( Q_i = \{Q_{ij}, j = 1, \ldots, n_i\} \) denote the private data set of sensor \( i \). Then the local cost function of sensor \( i \) can be defined as follows,

\[
f_i(s) = \left\| \sum_{Q_{ij} \in Q_i} H(Q_{ij}, s) \right\|_1,
\]

where \( H(\cdot, \cdot) \) represents the Huber loss function. It is noted that \( f_i(s) \) is nondifferentiable. In particular, the Huber loss function for one-dimensional variable is defined as follows,

\[
H(Q_{ij}, s) = \begin{cases} 
\frac{(Q_{ij} - s)^2}{2}, & \text{for } |Q_{ij} - s| \leq \varsigma, \\
\varsigma |Q_{ij} - s| - \frac{\varsigma^2}{2}, & \text{for } |Q_{ij} - s| > \varsigma,
\end{cases}
\]

where \( \varsigma \) is a positive constant. Compared with the typical least squares loss function, the Huber loss function takes a smaller value when the difference between the measurements and the
parameter to be estimated exceeds the tolerance \( \epsilon \). Therefore, it is less sensitive to outliers and improves the robustness of the estimators. The corresponding DOP can be formulated as follows,

\[
\min_{s \in \mathbb{R}^n} f(s) = \sum_{i=1}^{N} \left\| \sum_{Q_{ij} \in Q_i} H(Q_{ij}, s) \right\|_1. \tag{28}
\]

To proceed, consider a sensor network with communication topology depicted by Fig. 1. Let the parameter to be estimated be taken as \( s = [1, 2, 3]^T \). Assume that the data set of each sensor \( i \) contains 500 measurements. Besides, the obtained measurements satisfy the independent identically Gaussian distribution \( \mathcal{N}(\mu, \sigma_i) \) with mean vector \( \mu = [1, 2, 3]^T \) and covariance matrix \( \sigma_i = 0.011I_3 \), \( i = 1, \ldots, 5 \). Let the Huber loss parameter \( \epsilon = 0.5 \). We apply the proposed fully distributed algorithm (4) to solve the distributed parameter estimation problem in (28). Let the initial values of adaptive gains \( \sigma_i \) be chosen as \( \sigma_i(0) = 1 \), for \( i = 1, \ldots, 5 \). The initial values of agent states \( x_i \) and auxiliary variables \( \psi_i \) are randomly chosen.

The simulation result is shown in Fig. 3. It can be observed that \( |x_1 - 1|, |x_2 - 2| \) and \( |x_3 - 3|, i = 1, 2, 5 \) are all upper bounded by 0.02 before 100s. In other words, the trajectories of states \( x_i, i = 1, \ldots, 5 \) converge to a small neighborhood of \( s = [1, 2, 3]^T \), which is the parameter to be estimated. Therefore, it is shown that the distributed parameter estimation problem over unbalanced directed networks can be solved by the fully distributed algorithm (4).

V. CONCLUSION

In this paper, a novel fully distributed continuous-time algorithm has been developed to solve the distributed nonconvex optimization problem over unbalanced directed networks. The proposed adaptive algorithm design does not need any prior global information concerning network connectivity or convexity of local cost functions. Under mild assumptions, the proposed algorithm guarantees that the agent states converge to the optimal solution that minimizes the sum of the local cost functions. The effectiveness of the developed fully distributed algorithm has been illustrated by two simulation examples. Future work will focus on solving the fully distributed optimization problem for systems with more general agent dynamics.

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