ISOMORPHISM PROPERTIES OF TOEPLITZ OPERATORS AND PSEUDO-DIFFERENTIAL OPERATORS BETWEEN MODULATION SPACES

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Abstract. We investigate the lifting property of modulation spaces and construct explicit isomorphisms between them. For each weight function \( \omega \) and suitable window function \( \varphi \), the Toeplitz operator (or localization operator) \( T_{\varphi}(\omega) \) is an isomorphism from \( M_{p,q}^{\omega_0} \) onto \( M_{p,q}^{\omega_0/\omega} \) for every \( p, q \in [1, \infty] \) and arbitrary weight function \( \omega_0 \). The methods involve the pseudo-differential calculus of Bony and Chemin and the Wiener algebra property of certain symbol classes of pseudo-differential operators.

0. Introduction

Many families of function spaces possess a lifting property. This fundamental property ensures that spaces of similar type, but with respect to different weights, are isomorphic, and the isomorphism is usually an explicit and natural operator that is intrinsic in the definition of the function spaces. We study isomorphisms of pseudo-differential operators and Toeplitz operators between modulation spaces with different weights.

The simplest example of the lifting property occurs for the family of weighted \( L^p \)-spaces. Let \( \omega > 0 \) be a weight function on \( \mathbb{R}^d \) and \( L^p_{(\omega)}(\mathbb{R}^d) \) be the weighted \( L^p \)-space defined by the norm \( \|f\|_{L^p_{(\omega)}} = \| f \omega \|_{L^p} \). Then \( L^p(\mathbb{R}^d) \) and \( L^p_{(\omega)}(\mathbb{R}^d) \) are isomorphic, and the isomorphism is given explicitly by the multiplication operator \( f \mapsto f \omega^{-1} \) from \( L^p \) onto \( L^p_{(\omega)}(\mathbb{R}^d) \).

Perhaps the best known example of a lifting property is related to the family of Besov spaces on \( \mathbb{R}^d \) (cf. [43]). For fixed \( p, q \in (0, \infty) \) the (homogeneous) Besov spaces \( B^{p,q}_s(\mathbb{R}^d) \) with smoothness parameter \( s \in \mathbb{R} \) are all isomorphic, and the isomorphism is given by a power of the Laplacian. Precisely, the operator \((-\Delta)^{-r/2}\) is an isomorphism from \( B^{p,q}_s(\mathbb{R}^d) \) onto \( B^{p,q}_{s+r}(\mathbb{R}^d) \) (see [43]).

In this paper, we study the lifting property for modulation spaces. Whereas with Besov spaces the smoothness is measured with derivatives and differences, the norm of a modulation space measures the smoothness of a function by means of its phase-space distribution. Precisely, let \( \varphi \in \mathcal{S}(\mathbb{R}^d) \setminus 0 \) be fixed. Then the short-time Fourier transform (STFT) of \( f \in \mathcal{S}(\mathbb{R}^n) \) with respect to the window function \( \varphi \) is defined as

\[
V_{\varphi}f(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \overline{\varphi(y-x)} e^{-i(y,\xi)} \, dy.
\] (0.1)
Writing \( \varphi_{x,\xi}(y) = \varphi(y-x)e^{i(y,\xi)} \) and \( V_\varphi f(x,\xi) = (f, \varphi_{x,\xi}) \), where \((\cdot,\cdot)\) denotes the scalar product on \( L^2(\mathbb{R}^d) \), the definition of the STFT can be extended to a continuous map from \( \mathcal{S}'(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}) \).

For \( 1 \leq p, q \leq \infty \) and a non-negative weight function \( \omega > 0 \) on \( \mathbb{R}^{2d} \), the \textit{modulation space} \( M_{p,q}^{\omega}(\mathbb{R}^d) \) is defined as the set of all \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that

\[
\|f\|_{M_{p,q}^{\omega}} \equiv \left( \int \left( \int |V_g f(x,\xi)\omega(x,\xi)|^p \, dx \right)^{q/p} \, d\xi \right)^{1/q} < \infty, \tag{0.2}
\]

(with obvious modifications when \( p = \infty \) or \( q = \infty \)). If \( \omega \equiv 1 \), we write \( M_{p,q}^{\omega} \) instead of \( M_{p,q}^{\omega}(\mathbb{R}^d) \). See [19] for a systematic exposition of modulation spaces and [15] for a survey.

From abstract arguments it is known that for fixed \( p, q \) each \( M_{p,q}^{\omega}(\mathbb{R}^d) \) isomorphic to the unweighted modulation space \( M_{p,q}(\mathbb{R}^d) \) [19, Ch. 13]. However, so far concrete isomorphisms are known only for few special weight functions. For instance, if \( \omega(x,\xi) = (1 + |\xi|^2)^r = \langle \xi \rangle^r \), then the operator \( (1 - \Delta)^{-r/2} \) is an isomorphism between \( M_{p,\omega}^{(\langle \xi \rangle^r)} \) and \( M_{p,\omega}^{(\langle \xi \rangle^{r+r})} \). This fact resembles the corresponding result for Besov spaces and was already established by Feichtinger [12]. More generally, if the weight function \( \omega \) depends only on one variable only, then it is not difficult to come up with explicit isomorphisms. If \( \omega(x,\xi) = m(x) \) and \( m \) is sufficiently smooth, then the multiplication operator \( f \mapsto fm^{-1} \) establishes an isomorphism between \( M_{p,\omega}^{(m)} \) and \( M_{p,q}^{\omega} \), which is in complete analogy with the family of weighted \( L^p \)-spaces. If \( \omega(x,\xi) = \mu(\xi) \), then the isomorphism is given by the corresponding Fourier multipliers, in analogy with the example of Besov spaces. (See [34, 37].)

In this paper we investigate the isomorphism property for modulation spaces with general weights depending on both variables. For the general case the lifting property has been established only for a few types of weights. The difficulty is that of understanding and characterizing the range of certain pseudo-differential operators, so-called Toeplitz operators. This is clearly a much harder problem than the one encountered for multiplication operators or Fourier multipliers.

The first hint about the concrete form of the isomorphism between modulation spaces with different weights comes from the theory of the Shubin classes [29]. These space are by definition the range of a so-called Toeplitz operator on phase-space. Informally, the Toeplitz operator with symbol \( \omega \) and “window” \( \varphi \) is defined to be

\[
T_{\varphi,\omega}f = \int_{\mathbb{R}^{2d}} \omega(x,\xi) V_\varphi f(x,\xi) \varphi_{x,\xi} \, dx \, d\xi. \tag{0.3}
\]

For precise definitions and boundedness results we refer to Section 1, in particular Propositions 1.5 and 1.6. Toeplitz operators arise in pseudo-differential calculus [18, 28], in the theory of quantization (Berezin quantization [2]), and in signal processing [10] (under the name of time-frequency localization operators or STFT multipliers).

The Shubin-Sobolev space \( Q_s \) is defined as the range of the Toeplitz operator \( T_{\varphi,\omega} \) with weight \( \omega(x,\xi) = (1 + |x|^2 + |\xi|^2)^{s/2} \) and window \( \varphi(x) = e^{-|x|^2/2} \). Thus \( Q_s \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that \( T_{\varphi,\omega}f \in L^2(\mathbb{R}^d) \). It was shown
recently [3, 4] that Sobolev-Shubin space $Q_{\omega_s}(\mathbb{R}^d)$ coincides with the modulation space $M_{\omega_s}^{2,2}(\mathbb{R}^d)$. In other words, the Toeplitz operator $T_{\varphi}(\omega_s)$ provides an isomorphism between $L^2$ and $M_{\omega_s}^{2,2}(\mathbb{R}^d)$. Thus the modulation spaces of Hilbert type possess the lifting property.

The lifting property between Sobolev-Shubin spaces was extended to more general modulation spaces in [5]. If $\omega$ is smooth and strictly hypoelliptic and if $\varphi$ is a Schwartz function, then $T_{\varphi}(\omega)$ is an isomorphism from $M_{\omega}^{p,q}$ onto $M_{\omega}^{p,q}$. In particular, $f \in M_{\omega}^{p,q}$, if and only if $T_{\varphi}(\omega)f \in M_{\omega}^{p,q}$. The key step in [5] involves the Fredholm theory for elliptic operators. Thus the assumptions on $\omega$ and $\varphi$ are essential in this approach.

It was conjectured [9] (last remark) that the smoothness of the symbol and the window are unnecessary for the lifting property. In fact, the following observation in [9] gave some plausibility to this conjecture: if $0 < A \leq \omega(x, \xi) \leq B$ for all $(x, \xi) \in \mathbb{R}^{2d}$, then $T_{\varphi}(\omega)$ is an isomorphism on $M_{\omega}^{p,q}$. The key argument requires the Wiener algebra property of certain symbol classes.

In this paper we give a complete solution of the isomorphism problem between modulation spaces for arbitrary moderate weight functions. A weight on $\mathbb{R}^{2d}$ is called moderate, if there exist constants $C, N \geq 0$ such that $\omega(X_1 + X_2) \leq C \langle X_1 \rangle^N \omega(X_2)$ for all $X_1, X_2 \in \mathbb{R}^{2d}$. This property is equivalent to the invariance of the modulation space $M_{\omega}^{p,q}$ under the time-frequency shifts $f \rightarrow f_{x, \xi}$, and is therefore no restriction.

Our main result can be stated as follows.

**Theorem 0.1.** Assume that $\omega$ is a moderate weight function and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then the Toeplitz operator $T_{\varphi}(\omega)$ is an isomorphism from $M_{\omega}^{p,q}$ onto $M_{\omega/\omega_0}^{p,q}$ for every moderate weight $\omega_0$ and every $p, q \in [1, \infty]$.

In other words, the family of modulation spaces possesses the lifting property.

We will establish several versions of this result. Firstly, the window function may be chosen in certain modulation spaces that are much larger than the Schwartz class. Secondly, the theorem holds for a more general family of modulation spaces that includes the $M_{\omega}^{p,q}$. Thirdly, we will also establish isomorphisms given by pseudo-differential operators rather than Toeplitz operators.

Our proofs use three different types of results: (1) boundedness properties of pseudo-differential operators and Toeplitz operators between modulation spaces, (2) the pseudo-differential calculus of Bony and Chemin in [6], and (3) the Wiener algebra property of appropriate symbol classes (cf. [1, 20–22, 31]).

The boundedness results serve to show that all operators involved are well-defined between the respective modulation spaces. The application of these results is the hairy technical part of our paper.

The pseudo-differential calculus of Bony and Chemin [6] enables us to reduce the isomorphism problem to a problem involving only operators of order 0. In particular, [6] provides a useful set of isomorphisms between modulation spaces of Hilbert type.
The Wiener algebra property states that certain symbol classes, in particular the Hörmander class $S^0_{0,0}$ and the generalized Sjöstrand classes $M^{\infty,1}_{(\omega)}$, are preserved under inversion. This property is crucial to ensure the correct boundedness properties of the inverse of the Toeplitz operator $T_{\varphi}(\omega)$.

Let us mention that our theory covers only weights of polynomial growth. It is possible to prove general results for arbitrary subexponential weights, but the arguments are rather different and will be pursued elsewhere.

The paper is organized as follows: In Section 1 we collect the prerequisites about modulation spaces, Toeplitz operators, and pseudo-differential calculus. In Section 2 we prove a first abstract isomorphism theorem for pseudo-differential operators between modulation spaces.

In Section 3 we will prove that $T_{\varphi}(\omega)f \in M^{p,q}_{(\omega_0)}$ if and only if $f \in M^{p,q}_{(\omega_0)}$ for arbitrary $\omega$ and $\omega_0$ (without any hypoelliptic assumptions on the weights) and with $\varphi$ belonging to an appropriate modulation space. If in addition $\omega$ is smooth, then the isomorphism theorem holds for a broader class of modulation spaces.

In Section 4 we construct explicit isomorphisms between weighted modulation spaces. We show that for an appropriate Gauss function $\Phi$ and arbitrary moderate weights $\omega_0$ and $\omega$, the Toeplitz operator $T_{\varphi}(\omega_0 \ast \Phi)$ is an isomorphism from $M^{p,q}_{(\omega)}$ onto $M^{p,q}_{(\omega/\omega_0)^*}$.

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1. Preliminaries

In this section we recall some concepts from time-frequency analysis and discuss some basic results. For details we refer to the books [18, 19].

**The Short-Time Fourier Transform.** The short-time Fourier transforms (STFT) is defined by (0.1), whenever $\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$. After writing $U$ for the map $F(x,y) \mapsto F(y,y-x)$ and $\mathcal{F}_2$ for the partial Fourier transform of $F(x,y)$ with respect to the $y$-variable, we note that

$$V_{\varphi}f(x,\xi) = \mathcal{F}_2(U(f \otimes \varphi))(x,\xi) = \mathcal{F}(f \overline{\varphi(\cdot-x)})(\xi). \tag{1.1}$$

The Fourier transform $\mathcal{F}$ is the linear and continuous map on $\mathcal{S}'(\mathbb{R}^d)$, which takes the form

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y)e^{-i(y,\xi)} \, dy,$$

when $f \in \mathcal{S}(\mathbb{R}^d)$. We write $\hat{f}(x) = f(-x) = \mathcal{F}^2f(x)$.

Each of the operators $U$ and $\mathcal{F}_2$ is an isomorphism on $\mathcal{S}(\mathbb{R}^{2d})$ and possesses a unique extension to an isomorphism on $\mathcal{S}'(\mathbb{R}^{2d})$ and to a unitary operator on $L^2(\mathbb{R}^{2d})$. If $\varphi \in \mathcal{S}'(\mathbb{R}^d) \setminus 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$, then we may define $V_{\varphi}f$ by $V_{\varphi}f = \mathcal{F}_2(U(f \otimes \varphi))$. Since $\mathcal{F}_2$ and $U$ are unitary bijections on $L^2(\mathbb{R}^{2d})$, it follows that $V_{\varphi}f \in L^2(\mathbb{R}^{2d})$, if and only if $f, \varphi \in L^2(\mathbb{R}^d)$, and

$$\|V_{\varphi}f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)}\|\varphi\|_{L^2(\mathbb{R}^d)}.$$
The short-time Fourier transform is similar to the Wigner distribution, which is defined as the tempered distribution
\[ W_{f,g}(x, \xi) = \mathcal{F} \left( f(x + \cdot/2) g(x - \cdot/2) \right)(\xi) \]
for \( f, g \in \mathcal{S}'(\mathbb{R}^d) \). By straight-forward computations it follows that
\[ V_g f(x, \xi) = 2^{-d} e^{-i(x,\xi)/2} W_{f,g}(-x/2, \xi/2). \]
If in addition \( f, g \in L^2(\mathbb{R}^d) \), then \( W_{f,g} \) is given by integral form
\[ W_{f,g}(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x + y/2) g(x - y/2) e^{-i(y,\xi)} dy. \]

Later on we shall use Weyl calculus, and then it is convenient to use the symplectic Fourier transform and symplectic STFT which are defined on tempered distributions on the phase space \( \mathbb{R}^{2d} \). The (standard) symplectic form on \( \mathbb{R}^{2d} \) is defined by the formula \( \sigma(X,Y) = \langle y,\xi \rangle - \langle x,\eta \rangle \), where \( X = (x,\xi) \in \mathbb{R}^{2d} \) and \( Y = (y,\eta) \in \mathbb{R}^{2d} \). The symplectic Fourier transform \( \mathcal{F}_\sigma \) is the linear, continuous and bijective mapping on \( \mathcal{S}(\mathbb{R}^{2d}) \), defined by
\[ \mathcal{F}_\sigma b(Y) = \pi^{-d} \int_{\mathbb{R}^{2d}} b(Z) e^{2i\sigma(Y,Z)} dZ, \quad b \in \mathcal{S}(\mathbb{R}^{2d}). \]

Again, \( \mathcal{F}_\sigma \) extends to a continuous bijection on \( \mathcal{S}'(\mathbb{R}^{2d}) \), and to unitary operator on \( L^2(\mathbb{R}^{2d}) \). For each \( \varphi \in L^2(\mathbb{R}^{2d}) \) and \( a \in \mathcal{S}(\mathbb{R}^{2d}) \), the symplectic STFT is defined by \( \mathcal{V}_\varphi a(X, Y) = \mathcal{F}_\sigma(a(\varphi(\cdot - X))(Y)) \). We note that \( \mathcal{V}_\varphi a \) is given by
\[ \mathcal{V}_\varphi a(X, Y) = \pi^{-d} \int_{\mathbb{R}^{2d}} a(Z) \overline{\varphi(Z - X)} e^{2i\sigma(X,Z)} dZ, \]
when \( a \in \mathcal{S}(\mathbb{R}^{2d}) \).

**Weight Functions.** A weight function is a non-negative function \( \omega \in L^\infty_{\text{loc}}(\mathbb{R}^d) \setminus \{0\} \). Given two weights \( \omega, v \in L^\infty_{\text{loc}}(\mathbb{R}^d) \), \( \omega \) is called \( v \)-moderate if
\[ \omega(x_1 + x_2) \leq C \omega(x_1) v(x_2) \quad \text{for all } x_1, x_2 \in \mathbb{R}^d \tag{1.2} \]
for some constant \( C > 0 \) independent of \( x_1, x_2 \in \mathbb{R}^d \).

We note that for \( C \) fixed, then the smallest choice of \( v \) which fulfills \( (1.2) \) is submultiplicative in the sense that \( v(x_1 + x_2) \leq C v(x_1) v(x_2) \) for \( x_1, x_2 \in \mathbb{R}^d \) (cf. [36]). Throughout we will assume that the submultiplicative weights are even and non-zero. Then \( (1.2) \) implies that \( \omega(x) > 0 \) for all \( x \in \mathbb{R}^d \) and so \( 1/\omega \) is always well-defined.

If \( v \) in \( (1.2) \) can be chosen as a polynomial, then \( \omega \) is called polynomially moderate. We let \( \mathcal{P}(\mathbb{R}^d) \) be the set of all polynomially moderate weight functions. Furthermore, we let \( \mathcal{P}_0(\mathbb{R}^d) \) be the set of all \( \omega \in \mathcal{P}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \) such that \( (\partial^s \omega)/\omega \in L^\infty \).
Modulation Spaces. We use the general definition of modulation spaces taken from [15, 16, 19].

Assume that $\mathcal{B}$ is a Banach space of complex-valued measurable functions on $\mathbb{R}^d$ and that $v \in \mathcal{P}(\mathbb{R}^d)$. Then $\mathcal{B}$ is called a (translation) invariant BF-space on $\mathbb{R}^d$ (with respect to $v$), if there is a constant $C$ such that the following conditions are fulfilled:

1. $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subseteq \mathcal{S}'(\mathbb{R}^d)$ (continuous embeddings).
2. If $x \in \mathbb{R}^d$ and $f \in \mathcal{B}$, then $\tau_x f = f(\cdot - x) \in \mathcal{B}$, and
   \[ \|\tau_x f\|_{\mathcal{B}} \leq C v(x) \|f\|_{\mathcal{B}}. \] (1.3)
3. If $f, g \in L^1_{\text{loc}}(\mathbb{R}^d)$, $g \in \mathcal{B}$, and $|f| \leq |g|$ almost everywhere, then $f \in \mathcal{B}$ and
   \[ \|f\|_{\mathcal{B}} \leq C \|g\|_{\mathcal{B}}. \]

It follows that if $f \in \mathcal{B}$ and $h \in L^\infty$, then $f \cdot h \in \mathcal{B}$, and
\[ \|f \cdot h\|_{\mathcal{B}} \leq C \|f\|_{\mathcal{B}} \|h\|_{L^\infty}. \] (1.4)

Remark 1.1. Assume that $\omega_0, v, v_0 \in \mathcal{P}(\mathbb{R}^d)$ are such that $\omega$ is $v$-moderate, and assume that $\mathcal{B}$ is a translation-invariant BF-space on $\mathbb{R}^d$ with respect to $v_0$. Also let $\mathcal{B}_0$ be the Banach space which consists of all $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that $\|f\|_{\mathcal{B}_0} \equiv \|f \omega\|_{\mathcal{B}}$ is finite. Then $\mathcal{B}_0$ is a translation invariant BF-space with respect to $v_0 v$.

Definition 1.2. Assume that $\mathcal{B}$ is a translation invariant BF-space on $\mathbb{R}^{2d}$, $\omega \in \mathcal{P}(\mathbb{R}^{2d})$, and that $\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Then the modulation space $M(\omega) = M(\omega)(\mathcal{B})$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that
\[ \|f\|_{M(\omega)(\mathcal{B})} \equiv \|V_\varphi f \omega\|_{\mathcal{B}} \] (1.5)
is finite. For $\omega = 1$ we write $M(\mathcal{B})$ instead of $M(1)(\mathcal{B})$.

In Definition 1.2 we may assume without loss of generality that $\omega$ and $v$ belong to $\mathcal{P}_0$, because there exists always a weight $\omega_0 \in \mathcal{P}_0(\mathbb{R}^{2d})$ such that $C^{-1} \omega_0 \leq \omega \leq C \omega_0$ for some constant $C > 0$ (see e.g., Corollary 1.3 [1] or [36].) Therefore $M(\omega)(\mathcal{B}) = M(\omega_0)(\mathcal{B})$ with equivalent norms.

Assume that $\omega \in \mathcal{P}(\mathbb{R}^{2d})$, $p, q \in [1, \infty]$, and let $L^{p,q}(\mathbb{R}^{2d})$ be the mixed-norm space of all $F \in L^1_{\text{loc}}(\mathbb{R}^{2d})$ such that
\[ \|F\|_{L^{p,q}(\omega)} \equiv \left( \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} |F(x, \xi)|^p \omega(x, \xi) dx \right)^{q/p} d\xi \right)^{1/q} < \infty \]
(with obvious modifications when $p = \infty$ or $q = \infty$). If $\omega = 1$, then we set $L^{p,q} = L^{p,q}$. Choosing $\mathcal{B} = L^{p,q}$, we obtain the standard modulation spaces
\[ M^{p,q}(\mathbb{R}^d) = M(L^{p,q}(\omega)(\mathbb{R}^{2d})) = M(\omega)(L^{p,q}(\mathbb{R}^{2d})) \].

(See also (1.2).) For convenience we use the notation $M^p(\omega)$ instead of $M^{p,p}(\omega)$. Furthermore, for $\omega = 1$ we set
\[ M(\mathcal{B}) = M(\omega)(\mathcal{B}), \quad M^{p,q} = M^{p,q}(\omega), \quad \text{and} \quad M^p = M^p(\omega). \]
We use the notation $\mathcal{M}_{\omega}^{p,q}$ instead of $M_{\omega}^{p,q}$, if the symplectic STFT is used in the definition of modulation space norm. That is, if $\varphi \in \mathcal{S}(\mathbb{R}^{2d}) \setminus \{0\}$ and $\omega \in \mathcal{P}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$, then $M_{\omega}^{p,q}(\mathbb{R}^{d})$ consists of all $a \in \mathcal{S}^\prime(\mathbb{R}^{2d})$ such that
\[
\|a\|_{M_{\omega}^{p,q}} \equiv \left( \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^{2d}} |V_{\varphi}a(X,Y)\omega(X,Y)|^p dX \right)^{q/p} dY \right)^{1/q} < +\infty.
\]
The symplectic definition of modulation spaces does not yield any new spaces. In fact, setting $\omega(X,Y) = \omega_0(X,-2q,2y)$ for $X \in \mathbb{R}^{2d}$ and $Y = (y,\eta) \in \mathbb{R}^{2d}$, it follows from the definition that $M_{\omega}^{p,q} = M_{\omega_0}^{p,q}$ with equivalent norms.

Finally, we note that Definition \[\text{(1.2)}\] also include the amalgam spaces $W_{\omega}^{p,q}(\mathbb{R}^{d})$ defined by the norm
\[
\|f\|_{W_{\omega}^{p,q}} \equiv \left( \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} |V_{\varphi}f(x,\xi)\omega(x,\xi)|^q d\xi \right)^{p/q} dx \right)^{1/p} < \infty.
\]

Properties of Modulations Spaces. In the following proposition we list some well-known properties of modulation spaces. See [19, Ch. 11] for proofs.

**Proposition 1.3.** Let $p,q,p_j,q_j \in [1,\infty]$ for $j = 1,2$, and $\omega,\omega_1,\omega_2,v \in \mathcal{P}(\mathbb{R}^{2d})$. Assume that $\omega$ is $v$-moderate and $\omega_2 \leq C\omega_1$ for some constant $C > 0$. Let $\mathcal{B}$ be a translation-invariant BF-space with respect to $v$. Then the following are true:

1. The modulation space $M_{\omega}(\mathcal{B})$ is a Banach space with the norm \[\text{\textnormal{(1.5)}}\]. Let $\psi \in M_{\omega}^{1}(\mathbb{R}^{d}) \setminus \{0\}$. Then $f \in M_{\omega}(\mathcal{B})$, if and only if $V_{\psi}f\omega \in \mathcal{B}$. Moreover, $f \mapsto \|V_{\psi}f\omega\|_{\mathcal{B}}$ is an equivalent norm on $M_{\omega}(\mathcal{B})$.

2. If $p_1 \leq p_2$ and $q_1 \leq q_2$, then
\[
\mathcal{S}(\mathbb{R}^{d}) \hookrightarrow M_{\omega_1}^{p_1,q_1}(\mathbb{R}^{d}) \hookrightarrow M_{\omega_2}^{p_2,q_2}(\mathbb{R}^{d}) \hookrightarrow \mathcal{S}^\prime(\mathbb{R}^{d}) .
\]

3. The $L^2$ product $(\cdot,\cdot)$ on $\mathcal{S}$ extends to a continuous map from $M_{\omega}^{p,q}(\mathbb{R}^{n}) \times M_{\omega}^{p',q'}(\mathbb{R}^{d})$ to $\mathcal{C}$, and
\[
f \mapsto \sup \|(f,g)\|
\]
is an equivalent norm on $M_{\omega}^{p,q}(\mathbb{R}^{d})$. Here the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^{d})$ such that $\|g\|_{M_{\omega}^{p',q'}} \leq 1$.

4. If $p,q < \infty$, then $\mathcal{S}(\mathbb{R}^{d})$ is dense in $M_{\omega}^{p,q}(\mathbb{R}^{d})$ and the dual space of $M_{\omega}^{p,q}(\mathbb{R}^{d})$ can be identified with $M_{(1/\omega)}^{p',q'}(\mathbb{R}^{d})$, through the form $(\cdot,\cdot)_{L^2}$. Moreover, $\mathcal{S}(\mathbb{R}^{d})$ is weakly dense in $M_{\omega}^{p,q}(\mathbb{R}^{d})$.

**Remark 1.4.** For modulation spaces of the form $M_{\omega}^{p,q}$ with fixed $p,q \in [1,\infty]$ the norm equivalence in Proposition \[\text{\textnormal{(1.3)}}\] can be extended to a larger class of windows. In fact, assume that $\omega, v \in \mathcal{P}(\mathbb{R}^{2d})$ with $\omega$ being $v$-moderate and
\[
1 \leq r \leq \min(p,p',q,q').
\]

Let $\varphi \in M_{\omega}^{1}(\mathbb{R}^{d}) \setminus \{0\}$. Then a tempered distribution $f \in \mathcal{S}(\mathbb{R}^{d})$ belongs to $M_{\omega}^{p,q}(\mathbb{R}^{d})$, if and only if $V_{\varphi}f \in L_{\omega}^{p,q}(\mathbb{R}^{2d})$. Furthermore, different choices of $\varphi \in \mathcal{S}$ yield equivalent norms on $M_{\omega}^{p,q}(\mathbb{R}^{d})$. The proof of this follows from the fact that $\varphi \in L_{\omega}^{p,q}(\mathbb{R}^{2d})$ if $\varphi \in M_{\omega}^{1}(\mathbb{R}^{d}) \setminus \{0\}$.
\[ M^\nu_{(\nu)}(\mathbb{R}^d) \setminus \{0\} \text{ in } \|V\varphi f\|_{L^p_{\omega}} \text{ give rise to equivalent norms. (Cf. Proposition 3.1 in [41].)} \]

Proposition 1.5 and Remark 1.4 allow us to be rather vague concerning the particular choice of \( \varphi \in M^\nu_{(\nu)} \setminus \{0\} \) in (1.2) and \( \varphi \in M^1_{(\nu)} \setminus \{0\} \) in (1.5).

For future references we remark that modulation spaces can be arbitrarily close to \( \mathcal{S} \) and \( \mathcal{S}' \) in the sense that
\[
\mathcal{S}(\mathbb{R}^d) = \bigcap_{s \in \mathbb{R}} M^{p,q}_{(v_s)}(\mathbb{R}^d), \quad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{s \in \mathbb{R}} M^{p,q}_{(v_s)}(\mathbb{R}^d), \quad v_s(X) = \langle X \rangle^s, \quad (1.6)
\]

and \( M^{p,q}_{(v_s)} \subseteq M^{p,q}_{(v_t)} \) as \( s \leq t \), and \( X = (x, \xi) \in \mathbb{R}^{2d} \).

**Toeplitz Operators.** Fix a symbol \( a \in \mathcal{S}(\mathbb{R}^{2d}) \) and a window \( \varphi \in \mathcal{S}(\mathbb{R}^d) \). Then the Toeplitz operator \( T_{\varphi,a} \) is defined by the formula
\[
(T_{\varphi,a} f_1, f_2)_{L^2(\mathbb{R}^{2d})} = (a \varphi f_1, V\varphi f_2)_{L^2(\mathbb{R}^{2d})}, \quad (1.7)
\]
when \( f_1, f_2 \in \mathcal{S}(\mathbb{R}^d) \). Obviously, \( T_{\varphi,a} \) is well-defined and extends uniquely to a continuous operator from \( \mathcal{S}'(\mathbb{R}^d) \) to \( \mathcal{S}(\mathbb{R}^d) \).

The definition of Toeplitz operators can be extended to more general classes of windows and symbols by using appropriate estimates for the short-time Fourier transforms in (1.7).

We state two possible ways of extending (1.7). The first result follows from [8, Corollary 4.2] and its proof, and the second result is a special case of [42, Theorem 3.1]. We use the notation \( \mathcal{L}(V_1, V_2) \) for the set of linear and continuous mappings from the topological vector space \( V_1 \) into the topological vector space \( V_2 \). We also set
\[
\omega_{0,t}(X, Y) = v_1(2Y)^{t-1} \omega_0(X) \quad \text{for } X, Y \in \mathbb{R}^{2d}. \quad (1.8)
\]

**Proposition 1.5.** Let \( 0 \leq t \leq 1 \), \( p, q \in [1, \infty] \), and \( \omega, \omega_0, v_0, v_1 \in \mathcal{S}(\mathbb{R}^{2d}) \) be such that \( v_0 \) and \( v_1 \) are submultiplicative, \( \omega_0 \) is \( v_0 \)-moderate and \( \omega \) is \( v_1 \)-moderate. Set
\[
v = v_1^{t}v_0 \quad \text{and} \quad \vartheta = \omega_0^{1/2},
\]
and let \( \omega_{0,t} \) be as in (1.8). Then the following are true:

1. The definition of \((a, \varphi) \mapsto T_{\varphi,a}\) from \( \mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)) \) extends uniquely to a continuous map from \( \mathcal{M}^{\infty}_{(\omega_{0,t})}(\mathbb{R}^{2d}) \times M^1_{(\nu)}(\mathbb{R}^d) \) to \( \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)) \).
2. If \( \varphi \in M^1_{(\nu)}(\mathbb{R}^d) \) and \( a \in \mathcal{M}^{\infty}_{(\omega_{0,t})}(\mathbb{R}^{2d}) \), then \( T_{\varphi,a} \) extends uniquely to a continuous map from \( M^{p,q}_{(\omega_0)}(\mathbb{R}^d) \) to \( M^{p,q}_{(\omega_\vartheta)}(\mathbb{R}^d) \).

**Proposition 1.6.** Let \( \omega, \omega_1, \omega_2, v \in \mathcal{S}(\mathbb{R}^{2d}) \) be such that \( \omega_1 \) is \( v \)-moderate, \( \omega_2 \) is \( v \)-moderate and \( \omega = \omega_1/\omega_2 \). Then the following are true:

1. The mapping \((a, \varphi) \mapsto T_{\varphi,a}\) extends uniquely to a continuous map from \( L^\infty_{(\omega)}(\mathbb{R}^{2d}) \times M^2_{(\nu)}(\mathbb{R}^d) \) to \( \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)) \).
2. If \( a \in L^\infty_{(1/\omega)}(\mathbb{R}^{2d}) \) and \( \varphi \in M^2_{(\nu)}(\mathbb{R}^d) \), then \( T_{\varphi,a} \) extends uniquely to a continuous operator from \( M^2_{(\omega)}(\mathbb{R}^d) \) to \( M^2_{(\omega_2)}(\mathbb{R}^d) \).
Pseudo-Differential Operators. The definition of Toeplitz operators can be extended even further by using pseudo-differential calculus. Assume that \( a \in \mathcal{S}(\mathbb{R}^{2d}) \), and fix \( t \in \mathbb{R} \). Then the pseudo-differential operator \( \text{Op}_t(a) \) is the linear and continuous operator on \( \mathcal{S}(\mathbb{R}^d) \) defined by the formula

\[
\text{Op}_t(a)f(x) = (2\pi)^{-d} \int \int a((1 - t)x + ty, \xi) e^{i(x-y, \xi)} dyd\xi.
\]

For general \( a \in \mathcal{S}'(\mathbb{R}^{2d}) \), the pseudo-differential operator \( \text{Op}_t(a) \) is defined as the continuous operator from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \) with distribution kernel

\[
K_{t,a}(x, y) = (2\pi)^{-d/2}(\mathcal{F}^{-1}a)((1 - t)x + ty, x - y).
\]

This definition makes sense, since the mappings \( \mathcal{F}_2 \) and \( F(x, y) \mapsto F((1 - t)x + ty, y - x) \) are isomorphisms on \( \mathcal{S}'(\mathbb{R}^{2d}) \). Furthermore, by the Schwartz kernel theorem the map \( a \mapsto \text{Op}_t(a) \) is a bijection from \( \mathcal{S}'(\mathbb{R}^{2d}) \) onto \( \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)) \).

Symbol Classes. The standard symbol classes in pseudo-differential calculus were introduced by Hörmander and are defined by appropriate polynomial decay conditions of the partial derivatives of the symbol. (See [25, 26] and the references therein.) In particular the class \( S^0_{0,0} \) consists of all symbols \( a \in C^\infty(\mathbb{R}^{2d}) \) all of whose derivative are bounded. For \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \) we also consider the weighted symbol class \( S(\omega) (\mathbb{R}^{2d}) \) which consists of all \( a \in C^\infty(\mathbb{R}^{2d}) \) such that \((\partial^\alpha a)/\omega \in L^\infty(\mathbb{R}^{2d})\).

In fact, Hörmander introduced in [25, 26] a much broader family of symbol classes with smooth symbols containing \( S(\omega) \) (and the standard classes \( S^s_{\rho,\delta} \)). Each symbol class \( S(\omega, g) \) is parameterized by an appropriate weight function \( \omega \) and an appropriate Riemannian metric \( g \) on the phase space. In [25, 26], Hörmander also proved several continuous results, for example that \( \text{Op}^w(a) \) is continuous on \( \mathcal{S}(\mathbb{R}^d) \), and extends uniquely to a continuous operator on \( \mathcal{S}'(\mathbb{R}^d) \) when \( a \in S(\omega, g) \).

The theory was extended and improved in several ways by Bony, Chemin and Lerner (cf. e.g. [6, 7]). Bony and Chemin introduced in [6] a family of Hilbert spaces of Sobolev type where each space \( H(\omega, g) \) depends on the weight \( \omega \) and metric \( g \). These spaces fit the Weyl calculus well, because for appropriate \( \omega \) and \( \omega_0 \) and \( a \in S(\omega, g) \) the operator \( \text{Op}^w(a) \) is continuous from \( H(\omega_0, g) \) to \( H(\omega_0/\omega, g) \). Furthermore, they proved that for appropriate \( \omega \), there are \( a \in S(\omega, g) \) and \( b \in S(1/\omega, g) \) such that

\[
\text{Op}^w(a) \circ \text{Op}^w(b) = \text{Op}^w(b) \circ \text{Op}^w(a) = \text{Id}_{\mathcal{S}'},
\]
the identity operator on \( \mathcal{S}'(\mathbb{R}^d) \).

The composition of Weyl operators corresponds to the \textit{Weyl product} (also called the twisted product in the literature) of the involved operator symbols. For \( a, b \in \mathcal{S}'(\mathbb{R}^{2d}) \) the Weyl product \( a \# b \) is defined as the Weyl symbol of \( \text{Op}_w(a) \circ \text{Op}_w(b) \), whenever this composition makes sense as a continuous operator from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \). In short, this relation can be written as

\[
c = a \# b \iff \text{Op}_w(c) = \text{Op}_w(a) \circ \text{Op}_w(b).
\]

We remark that \( (a, b) \mapsto a \# b \) is a well-defined and continuous mapping from \( \mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^{2d}) \) to \( \mathcal{S}(\mathbb{R}^d) \), since \( \text{Op}_w(a) \circ \text{Op}_w(b) \) makes sense as a continuous operator from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}(\mathbb{R}^d) \).

An important question in the calculus concerns of finding convenient unique extensions of the Weyl product to larger spaces. For example, for an appropriate metric \( g \) and appropriate weight functions \( \omega_1, \omega_2 \) on \( \mathbb{R}^{2d} \) the Weyl product on \( \mathcal{S} \) is uniquely extendable to a continuous mapping from \( S(\omega_1, g) \times S(\omega_2, g) \) to \( S(\omega_1 \omega_2, g) \) [26, Thm. 18.5.4]. Consequently, if \( a_1 \in S(\omega_1, g) \) and \( a_2 \in S(\omega_2, g) \), then \( a_1 \# a_2 \in S(\omega_1 \omega_2, g) \), or

\[
S(\omega_1, g) \# S(\omega_2, g) \subseteq S(\omega_1 \omega_2, g).
\]  

(1.11)

For the case of constant metric, we will use the following proposition.

\begin{proposition}
Assume that \( \omega_j \in \mathcal{P}(\mathbb{R}^{2d}) \) for \( j = 0, 1, 2 \), \( s, t \in \mathbb{R} \), and set \( \omega_{0,N}(X, Y) = \omega_0(X)\langle Y \rangle^{-N} \) when \( N \geq 0 \) is an integer. Then the following is true:

\begin{enumerate}
    \item If \( a_1, a_2 \in \mathcal{S}'(\mathbb{R}^{2d}) \) satisfy \( \text{Op}_{s}(a_1) = \text{Op}_{t}(a_2) \), then \( a_1 \in S(\omega_0)(\mathbb{R}^{2d}) \) if and only if \( a_2 \in S(\omega_0)(\mathbb{R}^{2d}) \).
    \item \( S(\omega_1) \# S(\omega_2) \subseteq S(\omega_1 \omega_2) \).
    \item \( S(\omega_0) = \bigcap_{N \geq 0} \mathcal{M}_{1/\omega_{0,N}}^{\infty,1} \cap \bigcap_{N \geq 0} \mathcal{M}_{1/\omega_{0,N}}^{\infty,1} \).
\end{enumerate}
\end{proposition}

\begin{proof}
The assertion (1) is an immediate consequence of Theorem 18.5.10 in [26], the assertion (2) follows from (1.11) for the case of constant metric, and (3) is proved in [24] (cf. (2.21) in [24]).
\end{proof}

In time-frequency analysis one also considers mapping properties for pseudodifferential operators between modulation spaces or with symbols in modulation spaces. Especially we need the following two results, where the first one is a generalization of [33, Theorem 2.1] by Tachizawa, and the second one is a weighted version of [19, Theorem 14.5.2]. We refer to [40, Theorem 2.2] for the proof of the first proposition and to [39] for the proof of the second one.

\begin{proposition}
Assume that \( t \in \mathbb{R} \), \( \omega, \omega_0 \in \mathcal{P}(\mathbb{R}^{2d}) \), \( a \in S(\omega_0)(\mathbb{R}^{2d}) \), \( t \in \mathbb{R} \), and that \( \mathcal{B} \) is an invariant BF-space on \( \mathbb{R}^{2d} \). Then \( \text{Op}_t(a) \) is continuous from \( M(\omega_0)(\mathcal{B}) \) to \( M(\omega_0)(\mathcal{B}) \), and also continuous on \( \mathcal{S}(\mathbb{R}^d) \) and \( \mathcal{S}'(\mathbb{R}^d) \)
\end{proposition}
Proposition 1.9. Assume that $p, q \in [1, \infty]$, that $\omega \in \mathcal{P}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ and $\omega_1, \omega_2 \in \mathcal{P}(\mathbb{R}^{2d})$ satisfy
\[
\frac{\omega_2(X - Y)}{\omega_1(X + Y)} \leq C \omega(X, Y), \quad X, Y \in \mathbb{R}^{2d},
\]  
for some constant $C$. If $a \in \mathcal{M}^{\omega_1,1}(\mathbb{R}^{2d})$, then $\text{Op}^w(a)$ extends uniquely to a continuous map from $M^{\omega_1,1}(\mathbb{R}^{d})$ to $M^{\omega_2,1}(\mathbb{R}^{d})$.

Finally we need the following result concerning mapping properties of modulation spaces under the Weyl product. The result is a special case of Theorem 0.3 in [24]. See also [32] for related results.

Proposition 1.10. Assume that $\omega_j \in \mathcal{P}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ for $j = 0, 1, 2$ satisfy
\[
\omega_0(X, Y) \leq C \omega_1(X - Y + Z, Z) \omega_2(X + Z, Y - Z),
\]  
for some constant $C > 0$ independent of $X, Y, Z \in \mathbb{R}^{2d}$. Then the map $(a, b) \mapsto a\#b$ from $\mathcal{P}(\mathbb{R}^{2d}) \times \mathcal{P}(\mathbb{R}^{2d})$ to $\mathcal{P}(\mathbb{R}^{2d})$ extends uniquely to a mapping from $M^{\omega_1,1}(\mathbb{R}^{2d}) \times M^{\omega_2,1}(\mathbb{R}^{2d})$ to $M^{\omega_1,1}(\mathbb{R}^{2d})$.

In the proof of our main theorem we will need the following consequence of Proposition 1.10.

Proposition 1.11. Assume that $\omega_0, \omega, v_0, v_1 \in \mathcal{P}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$, that $\omega_0$ is $v_0$-moderate and $\omega$ is $v_1$-moderate. Set $\vartheta = \omega^{1/2}_0$, and
\[
\omega_1(X, Y) = \frac{v_0(2Y)^{1/2}v_1(2Y)}{\vartheta(X + Y)\vartheta(X - Y)},
\]
\[
\omega_2(X, Y) = \vartheta(X - Y)\vartheta(X + Y)v_1(2Y),
\]
\[
v_2(X, Y) = v_1(2Y).
\]  
Then
\[
S_{(1/\vartheta)} \# \mathcal{M}^{\infty,1}_{(\omega_1)} \# S_{(1/\vartheta)} \subset \mathcal{M}^{\infty,1}_{(v_2)},
\]
\[
S_{(1/\vartheta)} \# \mathcal{M}^{\infty,1}_{(v_2)} \# S_{(1/\vartheta)} \subset \mathcal{M}^{\infty,1}_{(\omega_2)}.
\]  
Proof. Since $S_{(1/\vartheta)} = \bigcap_{N \geq 0} \mathcal{M}^{\infty,1}_{(\vartheta_N)}$ with $\vartheta_N(X, Y) = \vartheta(X)Y^N$ (Proposition 1.7(3)), it suffices to argue with the symbol class $\mathcal{M}^{\infty,1}_{(\vartheta_N)}$ for some sufficiently large $N$ instead of $S_{(1/\vartheta)}$.

Introducing the intermediate weight
\[
\omega_3(X, Y) = \frac{v_1(2Y)\vartheta(X + Y)}{\omega_0(X - Y)},
\]
we will show that for suitable $N$
\[
\omega_3(X, Y) \leq C \omega_1(X - Y + Z, Z) \vartheta_N(X + Z, Y - Z),
\]
\[
v_1(2Y) \leq C \vartheta_N(X - Y + Z, Z) \omega_3(X + Z, Y - Z).
\]
Proposition 1.10 applied to (1.17) shows that $\mathcal{M}_{(\omega_1)}^{\infty,1} \# S_{(1/\vartheta)} \subseteq \mathcal{M}_{(\omega_3)}^{\infty,1}$, and (1.18) implies that $S_{(1/\vartheta)} \# \mathcal{M}_{(\omega_3)}^{\infty,1} \subseteq \mathcal{M}_{(\omega_2)}^{\infty,1}$, and (1.15) now follows.

Since $\vartheta$ is $v_0^{1/2}$-moderate and $v_0$ grows at most polynomially, we have $\vartheta(X - Y)^{-1} \leq v_0(2Z)^{1/2} \vartheta(X - Y + 2Z)^{-1}$ and $\vartheta(X + Y) \leq \vartheta(X + Z)(Y - Z)^N$ for suitable $N \geq 0$. Using these inequalities repeatedly in the following, a straight-forward computation yields

$$\omega_3(X, Y) = \frac{v_1(2Y)\vartheta(X + Y)^{1/2}}{\vartheta(X - Y)^2}$$

$$\leq C_1 \frac{v_0(2Z)^{1/2}v_1(2Z)\vartheta(X + Y)(Y - Z)^N}{\vartheta(X - Y + 2Z)\vartheta(X - Y)}$$

$$= C_1 \omega_1(X - Y + Z, Z)\vartheta_N(X + Z, Y - Z),$$

for some $C_1 > 0$ and $N > 0$.

Likewise we obtain

$$v_1(2Y) = \frac{\vartheta(X - Y)v_1(2Y)\vartheta(X - Y)}{\vartheta(X - Y)^2}$$

$$\leq C_1 \frac{\vartheta(X - Y)v_0(2Y)^{1/2}v_1(2Y)\vartheta(X + Y)}{\vartheta(X - Y)^2}$$

$$\leq C_2 \frac{\vartheta(X - Y + Z)(Z)^Nv_0(2Y - Z)^{1/2}v_1(2Y - Z)\vartheta(X + Y)}{\vartheta(X - Y + 2Z)^2}$$

$$= C_2 \vartheta_N(X - Y + Z, Z)\omega_3(X + Z, Y - Z).$$

The twisted convolution relation (1.16) is proved similarly. Let

$$\omega_4(X, Y) = \vartheta(X - Y)v_1(2Y)$$

be the intermediate weight. Then the inequality

$$\omega_4(X, Y) = \vartheta(X - Y)v_1(2Y) \leq C\omega_0(X - Y + Z)(Z)^Nv_1(2Y - Z))$$

$$= C\vartheta_N(X - Y + Z, Z)v_2(, X + Z, Y - Z)$$

implies that $S_{(1/\vartheta)} \# \mathcal{M}_{(\omega_2)}^{\infty,1} \subseteq \mathcal{M}_{(\omega_4)}^{\infty,1}$.

Similarly we obtain

$$\omega_2(X, Y) \leq C\vartheta(X - Y)v_1(2Z)\vartheta(X + Z)(Z - Y)^N$$

$$= C\omega_4(X - Y + Z, Z)\vartheta(X + Z)(Z - Y)^N$$

$$= C\omega_3(X - Y + Z, Z)\vartheta_N(X + Z, Y - Z),$$

and thus $\mathcal{M}_{(\omega_4)}^{\infty,1} \# S_{(1/\vartheta)} \subseteq \mathcal{M}_{(\omega_2)}^{\infty,1}$.  \hfill \Box
The Wiener Algebra Property. Proposition 1.8 generalizes parts of Calderon-Vaillancourt’s theorem. In fact, if we let \( \omega = \omega_0 = 1 \) and \( \mathcal{B} = L^2 \), then Proposition 1.8 asserts that \( \text{Op}_t(a) \) is continuous on \( L^2 \) as \( a \in S^0 \). In the same way, by letting \( p = q = 2, \omega_1 = \omega_2 = 1 \) and \( \omega(x, \xi, \eta, y) = v(\eta, y) \), where \( v \) is submultiplicative, then Proposition 1.9 shows that \( \text{Op}_t(a) \) is \( L^2 \)-continuous when \( a \in M^\infty_v \).

As a further crucial tool in our study of the isomorphism property of Toeplitz operators we need to combine these continuity results with convenient invertibility properties. These properties are the so-called Wiener algebra property of certain symbol classes, and asserts that the inversion of a pseudo-differential operator preserves the symbol class and is often referred to as the spectral invariance of a symbol class.

Theorem 1.12. Assume that \( t \in \mathbb{R} \) and that \( v \in \mathcal{P}(\mathbb{R}^{4d}) \) is submultiplicative and depends only on the second variable \( v(X, Y) = v_0(Y) \). Then the following are true:

1. If \( a \in S^0_{0,0}(\mathbb{R}^{2d}) \) and \( \text{Op}_t(a) \) is invertible on \( L^2(\mathbb{R}^d) \), then the inverse of \( \text{Op}_t(a) \) is equal to \( \text{Op}_t(b) \) for some \( b \in S^0_{0,0}(\mathbb{R}^{2d}) \).

2. If \( a \in M^\infty_v(\mathbb{R}^{2d}) \) and \( \text{Op}_t(a) \) is invertible on \( L^2(\mathbb{R}^d) \), then the inverse of \( \text{Op}_t(a) \) is equal to \( \text{Op}_t(b) \), for some \( b \in M^\infty_v(\mathbb{R}^{2d}) \).

The assertion (1) in Theorem 1.12 is a classical result of Beals (cf. [1]), while (2) is proved in the unweighted case in [31]. We refer to [20, Corollary 5.5] or [21] for the general case of (2), and to [22, 23] for further generalizations. We also remark that Beals’ result is an immediate consequence of (2) because \( S^0_{0,0} \) is the intersection of all \( M^\infty_{v_s} \), with \( v_s(X, Y) = \langle Y \rangle^s \).

Weyl formulation of Toeplitz operators. We finish this section by recalling some important relations between Weyl operators, Wigner distributions, and Toeplitz operators. For instance, the Weyl symbol of a Toeplitz operator is the convolution between the Toeplitz symbol and a Wigner distribution. More precisely, if \( a \in \mathcal{S}(\mathbb{R}^{2d}) \) and \( \varphi \in \mathcal{S}(\mathbb{R}^d) \), then

\[
T_{\varphi}(a) = (2\pi)^{-d/2} \text{Op}^w(a \ast W_{\varphi,\varphi}).
\]

(1.19)

Our analysis of Toeplitz operators is based on the pseudo-differential operator representation, given by (1.19). Furthermore, any extension of the definition of Toeplitz operators to cases which are not covered by Propositions 1.5 and 1.6 are based on this representation. Here we remark that this leads to situations where certain mapping properties for the pseudo-differential operator representation make sense, while similar interpretations are difficult or impossible to make in the framework of (1.7) (see Remark 3.7 in Section 3). We refer to [40] or Section 3 for extensions of Toeplitz operators in context of pseudo-differential operators.

2. Identifications of modulation spaces

In this section we show that for each \( \omega \) and \( \mathcal{B} \), there are canonical ways to identify the modulation space \( M(\omega)(\mathcal{B}) \) with \( M(\mathcal{B}) \) by means of convenient bijections in the
Theorem 2.1. Assume that $\omega \in \mathcal{P}(\mathbb{R}^{2d})$ and $t \in \mathbb{R}$. Then the following are true:

1. There exist $a \in S(\omega)(\mathbb{R}^{2d})$ and $b \in S(1/\omega)(\mathbb{R}^{2d})$ such that
\[ \text{Op}_t(a) \circ \text{Op}_t(b) = \text{Op}_t(b) \circ \text{Op}_t(a) = \text{Id}_{\mathcal{S}'(\mathbb{R}^d)}. \] (2.1)

Furthermore, $\text{Op}_t(a)$ is an isomorphism from $M(\omega_0)(\mathcal{B})$ onto $M(\omega_0/\omega)(\mathcal{B})$, for every $\omega_0 \in \mathcal{P}(\mathbb{R}^{2d})$ and invariant BF-space $\mathcal{B}$.

2. If $a \in S(\omega)(\mathbb{R}^{2d})$ and $\text{Op}_t(a)$ is an isomorphism from $M^2(\omega_1)(\mathbb{R}^{2d})$ to $M(\omega_1/\omega)(\mathbb{R}^{2d})$ for some $\omega_1 \in \mathcal{P}(\mathbb{R}^{2d})$, then $\text{Op}_t(a)$ is an isomorphism from $M(\omega_2)(\mathcal{B})$ to $M(\omega_2/\omega)(\mathcal{B})$, for every $\omega_2 \in \mathcal{P}(\mathbb{R}^{2d})$ and every invariant BF-space $\mathcal{B}$.

We need some preparations for the proof. As a first step we prove that modulation spaces of Hilbert type agree with certain types of Bony-Chemin spaces (cf. Section 1).

If $g$ is the (standard) Euclidean metric on $\mathbb{R}^{2d}$, then the definition of Bony-Chemin spaces (cf. Section 1 and Appendix) specializes to the following condition. Let $\psi \in C_0^\infty(\mathbb{R}^{2d}) \setminus \{0\}$ be non-negative, even and supported in a ball of radius $1/4$, and set $\tau_Y \psi = \psi(\cdot - Y)$. Then $H(\omega, g) = H(\omega)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^{2d})$ such that
\[ \|f\|_{H(\omega)} = \left( \int_{\mathbb{R}^{2d}} \omega(Y)^2 \|\text{Op}^w(\tau_Y \psi)f\|_{L^2}^2 dY \right)^{1/2} \] (2.2)
is finite.

Proposition 2.2. Assume that $\omega \in \mathcal{P}(\mathbb{R}^{2d})$. Then $H(\omega) = M^2(\omega)(\mathbb{R}^{2d})$ with equivalent norms.

We need two lemmas for the proof. Recall that a linear operator $T$ is of trace-class if
\[ \sup \sum |(Tf_j, g_j)| < \infty, \]
where the supremum is taken over all orthonormal sequences $(f_j)$ and $(g_j)$ in $L^2(\mathbb{R}^{2d})$.

The following result is an immediate consequence of the spectral theorem for compact operators. We refer to [27] or Lemma 1.3 and Proposition 1.10 in [35] for the proof of the first part. The second part follows from the first part and straightforward computations.

Lemma 2.3. If $\psi \in \mathcal{S}(\mathbb{R}^{2d})$, then $\text{Op}^w(\psi)$ is a trace-class operator, and there exist two orthonormal sequences $(f_j)$ and $(g_j)$ in $\mathcal{S}(\mathbb{R}^{2d})$ and a non-negative non-increasing sequence $(\lambda_j) \in \ell^1$ such that
\[ \text{Op}^w(\psi)f = \sum_{j=1}^\infty \lambda_j (f, f_j)g_j, \]
when \( f \in L^2(\mathbb{R}^d) \). Moreover, set \( f_{j,Y}(x) = e^{i(x,y)}f_j(x-y) \) and \( g_{j,Y}(x) = e^{i(x,y)}g_j(x-y) \) for \( Y = (y,\eta) \), then

\[
\text{Op}^w(\tau_Y \psi)f = \sum_{j=1}^\infty \lambda_j(f, f_{j,Y}) g_{j,Y}.
\] (2.3)

We also need the following lemma. Since it is difficult to find a proof in the literature, we include its short proof.

**Lemma 2.4.** Assume that \( f \in \mathcal{S}(\mathbb{R}^{d_1+d_2}) \). Then there are \( f_0 \in \mathcal{S}(\mathbb{R}^{d_1+d_2}) \) and strictly positive rotation-invariant function \( g \in \mathcal{S}(\mathbb{R}^{d_1}) \) such that

\[
f(x_1, x_2) = f_0(x_1, x_2)g(x_1).
\]

**Proof.** We only prove the result for \( d_1 = d \) and \( d_2 = 0 \). The general case is similar and left to the reader. For each integer \( j \geq 1 \) define the set

\[
\Omega_j = \{ x \in \mathbb{R}^d ; \sum_{|\alpha|,|\beta| \leq 2^j} |x^\alpha D^\beta f(x)| \geq 2^{-2j} \langle x \rangle^{-2j} \}.
\]

Since \( f \in \mathcal{S}(\mathbb{R}^{d_1}) \), \( \Omega_j \) is compact, and \( \Omega_j \subseteq \Omega_{j+1} \) for all \( j \).

Let \( R_0 = -1 \) and \( R_j = j + \sup\{|x|; x \in \Omega_j\} \) for \( j \geq 1 \), and let \( (\varphi_j)_{j=0}^\infty \) be a bounded set in \( C_0^\infty(\mathbb{R}) \) such that \( \varphi_j \geq 0 \),

\[
\text{supp} \varphi_j \subseteq \{ r ; R_j - 1 \leq r \leq R_{j+1} + 1 \}
\]

and

\[
\sum_{j=0}^\infty \varphi_j(r) = 1 \quad \text{when} \quad r \geq 0.
\]

Now set

\[
g(x) = \sum_{j=0}^\infty \varphi_j(|x|)2^{-j} \langle x \rangle^{-j}, \quad \text{and} \quad f_0(x) = f(x)/g(x).
\]

Then \( f_0, g \in \mathcal{S}(\mathbb{R}^d) \) and \( f = f_0g \). \( \square \)

**Proof of Proposition 2.2.** Let \( \psi \in C_0^\infty(\mathbb{R}^{2d}) \setminus \{0\} \) be as in (2.2), and let

\[
G(x, z) = (\mathcal{F}_2 \psi)\left(\frac{x+z}{2}, z-x\right),
\]

which belongs to \( \mathcal{S}(\mathbb{R}^{2d}) \). By definition, \( \omega \) is \( v \)-moderate for some \( v \in \mathcal{S}(\mathbb{R}^{2d}) \), and by Lemma 2.4 we may choose \( G_1 \in \mathcal{S}(\mathbb{R}^{2d}) \) and a strictly positive \( \varphi \in \mathcal{S}(\mathbb{R}^d) \), such that \( G(x, z) = G_1(x, z)\varphi(z) \). Using formula (1.9) for \( f \in \mathcal{S}(\mathbb{R}^d) \) and \( Y = \ldots \)
\((y, \eta) \in \mathbb{R}^{2d}\), the expression for the \(H(\omega)\)-norm of \(f\) becomes
\[
(2\pi)^{2d} \|f\|_{H(\omega)}^2 = \iint \left| \omega(Y) \iint \psi\left(\frac{x + z}{2} - y, \xi - \eta\right) f(z) e^{i(x - z, \xi)} \, dzd\xi \right|^2 \, dxdY
\]
\[
= \iint \left| \omega(Y) \iint \psi\left(\frac{x + z}{2}, \xi\right) f(z + y) e^{-i(z, \eta)} e^{i(x - z, \xi)} \, dzd\xi \right|^2 \, dxdY
\]
\[
= \iint \left| \omega(Y) \int G(x, z) f(z + y) e^{-i(z, \eta)} \, dz \right|^2 \, dxdY
\]
\[
= \iint \left| \omega(Y) \int G_1(x, z) \varphi(z) f(z + y) e^{-i(z, \eta)} \, dz \right|^2 \, dxdY.
\]
In the second equality we have taken \(z - y, \xi - \eta, x - y\) and \(Y\) as the new variables of integration. Since the inner integral on the right-hand side is the Fourier transform of the product \(G_1(x, z) \cdot (\varphi(z) f(z + y))\) with respect to the variable \(z\), we obtain
\[
(2\pi)^{2d} \|f\|_{H(\omega)}^2
\]
\[
= \iint \left| \omega(Y) \left( |\mathcal{F}_2(G_1)(x, \cdot)| \ast |\mathcal{F}(\varphi f(\cdot + y)))\right)\right| \, dxdY
\]
\[
= \iint \left| \omega(Y) \left( |\mathcal{F}_2(G_1)(x, \cdot)| \ast |V_{\varphi} f(y, \cdot)\right)\right| \, dxdY
\]
\[
\leq C_1 \iint \left| \left( |\mathcal{F}_2(G_1)(x, \cdot)| \ast |V_{\varphi} f(y, \cdot)\right)\right| \, dxdY
\]
\[
\leq C_2 \|V_{\varphi} f \omega\|_{L^2}^2 = C_2 \|f\|_{M_{\omega}^2}^2,
\]
for some constants \(C_1\) and \(C_2\). Here the last inequality follows from Young’s inequality. Hence \(M_{\omega}^2(\mathbb{R}^{2d}) \subseteq H(\omega)\).

For the reverse inclusion we note that \((f_{j, Y})\) and \((g_{j, Y})\) in the spectral representation \((2.3)\) are orthonormal sequences for each fixed \(Y \in \mathbb{R}^{2d}\). Hence \((2.3)\) and Bessel’s inequality give
\[
\| \text{Op}^w(\tau_Y \psi) f \|_{L^2} = \left( \sum_{j=1}^{\infty} \lambda^2_j |\langle f, f_{j, Y} \rangle_{L^2}|^2 \right)^{1/2} \geq \lambda_1 |\langle f, f_{1, Y} \rangle_{L^2}| = \lambda_1 |V_{f_1} f(y, \eta)|.
\]
A combination of these estimates gives
\[
\|f\|_{H(\omega)}^2 = \int \omega(Y)^2 \| \text{Op}^w(\tau_Y \psi) f \|_{L^2}^2 \, dY
\]
\[
\geq \lambda_1^2 \int \omega(y, \eta)^2 |V_{f_1} f(y, \eta)|^2 \, dxdY = \|f\|_{M_{\omega}^2}^2,
\]
where the last identity follows from Proposition \[1.3(1)\] on norm equivalence, and the fact that \(f_1\) belongs to \(\mathcal{S}\). Consequently \(H(\omega) \subseteq M_{\omega}^2(\mathbb{R}^{2d})\).
Summing up, we have shown that \( H(\omega) = M^2_{(\omega)}(\mathbb{R}^{2d}) \) with equivalent norms, and the proof is complete. \( \square \)

**Proof of Theorem 2.1.** We first remark that by Proposition 1.7(1) the statements are independent of the pseudo-differential calculus used. Hence we may assume that \( t = 1/2 \) and use the Weyl calculus.

To prove (1), we use a fundamental result of Bony and Chemin [6]. According to [6, Corollary 6.6] and Proposition 1.8 there exist \( a \in S_{(\omega)}(\mathbb{R}^{2d}) \) and \( b \in S_{(1/\omega)}(\mathbb{R}^{2d}) \) such that both \( \text{Op}^w(a) \circ \text{Op}^w(b) \) and \( \text{Op}^w(b) \circ \text{Op}^w(a) \) are equal to the identity operator on \( H(\omega_0) = M^2_{(\omega_0)}(\mathbb{R}^{2d}) \) for every \( \omega_0 \in \mathcal{P}(\mathbb{R}^{2d}) \). Since the symbol \( a \# b \) of \( \text{Op}^w(a) \circ \text{Op}^w(b) \) is in \( S_{(1)} = S^0_{0,0} \) by Proposition 1.7(2), the boundedness result of Proposition 1.8 is applicable and shows that both \( \text{Op}^w(a) \circ \text{Op}^w(b) \) and \( \text{Op}^w(b) \circ \text{Op}^w(a) \) are equal to the identity operator on \( M_{(\omega_0)}(\mathcal{B}) \) for arbitrary \( \omega_0 \in \mathcal{P}(\mathbb{R}^{2d}) \) and invariant BF-space \( \mathcal{B} \).

Since \( \text{Op}^w(a) \) maps \( M_{(\omega_0)}(\mathcal{B}) \) to \( M_{(\omega_0)/\omega}(\mathcal{B}) \) and \( \text{Op}^w(b) \) maps \( M_{(\omega_0)/\omega}(\mathcal{B}) \) to \( M_{(\omega_0)}(\mathcal{B}) \), the factorization of the identity operator implies that these mappings are one-to-one and onto. Hence \( \text{Op}^w(a) \) is an isomorphism between \( M_{(\omega_0)}(\mathcal{B}) \) and \( M_{(\omega_0)/\omega}(\mathcal{B}) \).

(2) By (1), we may find

\[
\begin{align*}
a_1 & \in S_{(\omega_1)}, \quad b_1 \in S_{(1/\omega_1)}, \quad a_2 \in S_{(\omega_1/\omega)}, \quad b_2 \in S_{(\omega_1/\omega)}
\end{align*}
\]

satisfying the following properties:

- \( \text{Op}^w(a_j) \) and \( \text{Op}^w(b_j) \) are inverses to each others on \( \mathcal{S}''(\mathbb{R}^d) \) for \( j = 1, 2 \);
- For arbitrary \( \omega_2 \in \mathcal{P}(\mathbb{R}^{2d}) \), each of the mappings

\[
\begin{align*}
\text{Op}^w(a_1) : M^2_{(\omega_2)} & \to M^2_{(\omega_2/\omega_1)}, \\
\text{Op}^w(b_1) : M^2_{(\omega_2)} & \to M^2_{(\omega_2\omega_1)}, \\
\text{Op}^w(a_2) : M^2_{(\omega_2)} & \to M^2_{(\omega_2\omega_1/\omega)}, \\
\text{Op}^w(b_2) : M^2_{(\omega_2)} & \to M^2_{(\omega_2\omega_1/\omega)}
\end{align*}
\]

is an isomorphism.

In particular, \( \text{Op}^w(a_1) \) is an isomorphism from \( M^2_{(\omega_1)} \) to \( L^2 \), and \( \text{Op}^w(b_1) \) is an isomorphism from \( L^2 \) to \( M^2_{(\omega_1)} \).

Now set \( c = a_2 \# a \# b_1 \). Then by Proposition 1.7(2), the symbol \( c \) satisfies

\[
c = a_2 \# a \# b_1 \in S_{(\omega_1/\omega)} \# S_{(\omega)} \# S_{(1/\omega_1)} \subseteq S_{(1)} = S^0_{0,0}.
\]

Furthermore, \( \text{Op}^w(c) \) is a composition of three isomorphisms and consequently \( \text{Op}^w(c) \) is boundedly invertible on \( L^2 \).

By the Wiener algebra property of \( S^0_{0,0} \) with respect to the Weyl product (cf. Proposition 1.12 or [1, 22]), the inverse of \( \text{Op}^w(c) \) is equal to \( \text{Op}^w(c_1) \) for some \( c_1 \in S^0_{0,0} \). Hence, by (1) it follows that \( \text{Op}^w(c) \) and \( \text{Op}^w(c_1) \) are isomorphisms on \( M_{(\omega_2)}(\mathcal{B}) \), for each \( \omega_2 \in \mathcal{P}(\mathbb{R}^{2d}) \). Since \( \text{Op}^w(c) \) and \( \text{Op}^w(c_1) \) are bounded on every
$M(\omega)(B)$, the factorization of the identity $Op^w(c)Op^w(c_1) = \text{Id}$ is well-defined on every $M(\omega)(B)$. Consequently, $Op^w(c)$ is an isomorphism on $M(\omega)(B)$.

Using the inverses of $a_2$ and $b_1$, we now find that

$$Op^w(a) = Op^w(a_1) \circ Op^w(c) \circ Op^w(b_2)$$

is a composition of isomorphisms from the domain space $M(\omega_2)(B)$ unto the target space $M(\omega_2/\omega)(B)$ (factoring through some intermediate spaces) for every $\omega_2 \in \mathcal{P}(\mathbb{R}^{2d})$ and every translation invariant BF-space $B$. The proof is complete. \hfill \Box

3. Mapping properties for Toeplitz operators

In this section we study the bijection properties of Toeplitz operators between modulation spaces. We first state results for Toeplitz operators that are well-defined in the sense of (1.7) and Propositions 1.5 and 1.6. Then we state and prove more general results for Toeplitz operators that are defined only in the framework of pseudo-differential calculus.

We start with the following result about Toeplitz operators with smooth symbols.

**Theorem 3.1.** Assume that $\omega, v \in \mathcal{P}(\mathbb{R}^{2d})$, $\omega_0 \in \mathcal{P}_0(\mathbb{R}^{2d})$, and that $\omega_0$ is $v$-moderate. If $\varphi \in M^1(v)(\mathbb{R}^{d})$ and $B$ is a translation invariant BF-space, then $Tp_\varphi(\omega_0)$ is an isomorphism from $M(\omega)(B)$ to $M(\omega/\omega_0)(B)$.

In the next result we relax our restrictions on the weights but impose more restrictions on the modulation spaces.

**Theorem 3.2.** Assume that $0 \leq t \leq 1$, $p, q \in [1, \infty]$, and $\omega, \omega_0, v_0, v_1 \in \mathcal{P}(\mathbb{R}^{2d})$ are such that $\omega_0$ is $v_0$-moderate and $\omega$ is $v_1$-moderate. Set $v = v_1v_0$, $\vartheta = \omega_0^{1/2}$ and let $\omega_{0,t}$ be the same as in (1.8). If $\varphi \in M^1(v)(\mathbb{R}^{d})$ and $\omega_0 \in M^\infty_{(1/\omega_0,t)}(\mathbb{R}^{2d})$, then $Tp_\varphi(\omega_0)$ is an isomorphism from $M^{p,q}_{(p,q)}(\mathbb{R}^{d})$ to $M^{p,q}_{(p,q)}(\mathbb{R}^{d})$.

Before the proofs of Theorems 3.1 and 3.2 we state the following consequence of Theorem 3.2 which was the original goal of our investigations.

**Corollary 3.3.** Assume that $\omega, \omega_0, v_1, v_0 \in \mathcal{P}(\mathbb{R}^{2d})$ and that $\omega_0$ is $v_0$-moderate and $\omega$ is $v_1$-moderate. Set $v = v_1v_0$ and $\vartheta = \omega_0^{1/2}$. If $\varphi \in M^1(v)(\mathbb{R}^{d})$, then $Tp_\varphi(\omega_0)$ is an isomorphism from $M^{p,q}_{(p,q)}(\mathbb{R}^{d})$ to $M^{p,q}_{(p,q)}(\mathbb{R}^{d})$ simultaneously for all $p, q \in [1, \infty]$.

**Proof.** Let $\omega_1 \in \mathcal{P}_0(\mathbb{R}^{2d})$ be such that $C^{-1} \leq \omega_1/\omega_0 \leq C$, for some constant $C$. Hence, $\omega/\omega_0 \in L^\infty \subseteq M^\infty$. By Theorem 2.2 in [40], it follows that $\omega = \omega_1 \cdot (\omega/\omega_1)$ belongs to $M^\infty_{(\omega_2)}(\mathbb{R}^{2d})$, when $\omega_2(x, \xi, \eta, y) = 1/\omega_0(x, \xi)$. The result now follows by setting $t = 1$ and $q_0 = 1$ in Theorem 3.2. \hfill \Box

In the proofs of Theorems 3.1 and 3.2 we consider Toeplitz operators as defined by an extension of the form (1.7). Later on we present extensions of these theorems (cf. Theorems 3.1 and 3.2 below) for those readers who accept to use pseudo-differential calculus to extend the definition of Toeplitz operators. For the proofs of Theorems 3.1 and 3.2 we therefore refer to the proofs of these extensions.

We need some preparations and start with the following lemma.
Lemma 3.4. Let $\omega, v \in \mathcal{P}(\mathbb{R}^{2d})$ be such that $\vartheta = \omega^{1/2}$ is $v$-moderate. Assume that $\varphi \in M^2_{(v)}$. Then $T_{p,\varphi}(\omega)$ is an isomorphism from $M^2_{(\vartheta)}(\mathbb{R}^d)$ onto $M^2_{(1/\vartheta)}(\mathbb{R}^d)$.

Proof. Recall from Remark 1.4 that for $\varphi \in M^2_{(\omega)}(\mathbb{R}^d) \setminus \{0\}$ the expression $\|V_{\varphi}f \cdot \vartheta\|_{L^2}$ defines an equivalent norm on $M^2_{(\vartheta)}$. Thus the occurring STFTs with respect to $\varphi$ are well defined.

Since $T_{p,\varphi}(\omega)$ is bounded from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ by Proposition 1.6, the estimate

$$\| T_{p,\varphi}(\omega)f \|_{M^2_{(1/\vartheta)}} \leq C \| f \|_{M^2_{(\vartheta)}}$$

holds for all $f \in M^2_{(\vartheta)}$.

Next we observe that

$$(T_{p,\varphi}(\omega)f, g)_{L^2(\mathbb{R}^d)} = (\omega V_{\varphi}f, V_{\varphi}g)_{L^2(\mathbb{R}^d)} = (f, g)_{M^2_{(\omega)}}$$

for $f, g \in M^2_{(\omega)}(\mathbb{R}^d)$ and $\varphi \in M^2_{(\omega)}(\mathbb{R}^d)$. The duality of modulation spaces (Proposition 3.3) now yields the following identity:

$$\| f \|_{M^2_{(\vartheta)}} = \sup_{\| g \|_{M^2_{(\vartheta)}} = 1} \| (f, g)_{M^2_{(\omega)}} \| = \sup_{\| g \|_{M^2_{(\vartheta)}} = 1} | (T_{p,\varphi}(\omega)f, g)_{L^2} | = \| T_{p,\varphi}(\omega)f \|_{M^2_{(1/\vartheta)}}.$$  (3.3)

A combination of (3.1) and (3.3) shows that $\| f \|_{M^2_{(\vartheta)}}$ and $\| T_{p,\varphi}(\omega) \|_{M^2_{(1/\vartheta)}}$ are equivalent norms on $M^2_{(\vartheta)}$.

In particular, $T_{p,\varphi}(\omega)$ is one-to-one on $M^2_{(\vartheta)}$ and has closed range in $M^2_{(1/\vartheta)}$. Since $T_{p,\varphi}(\omega)$ is self-adjoint with respect to $L^2$, it follows by duality that $T_{p,\varphi}(\omega)$ has dense range in $M^2_{(1/\vartheta)}$. Consequently, $T_{p,\varphi}(\omega)$ is onto $M^2_{(1/\vartheta)}$. By Banach’s theorem $T_{p,\varphi}(\omega)$ is an isomorphism from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. $\Box$

We need a further generalization of Proposition 1.5 to more general classes of symbols and windows. Set

$$\omega_1(X,Y) = \frac{v_0(2Y)^{1/2}v_1(2Y)}{\omega_0(X + Y)^{1/2}\omega_0(X - Y)^{1/2}}.$$  (3.4)

Proposition 1.5. Let $0 \leq t \leq 1$, $p,q,q_0 \in [1, \infty]$, and $\omega, \omega_0, v_0, v_1 \in \mathcal{P}(\mathbb{R}^{2d})$ be such that $v_0$ and $v_1$ are submultiplicative, $\omega_0$ is $v_0$-moderate and $\omega$ is $v_1$-moderate. Set

$$\omega_{0,t} = \frac{2q_0}{(2q_0 - 1)}, \quad v = v_1^t v_0 \quad \text{and} \quad \vartheta = \omega_0^{1/2},$$

and let $\omega_{0,t}$ and $\omega_1$ be as in (1.8) and (3.4). Then the following are true:

1. The definition of $(a, \varphi) \mapsto T_{p,\varphi}(a)$ from $\mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^d)$ to $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ extends uniquely to a continuous map from $M_{v_0}^{\infty,q_0}(\mathbb{R}^{2d}) \times M_{(v)}^{r_0}(\mathbb{R}^d)$ to $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$.

2. If $\varphi \in M_{(\omega)}^{\infty,q_0}(\mathbb{R}^{2d})$ and $a \in M_{(\omega_0,t)}^{\infty,q_0}(\mathbb{R}^{2d})$, then $T_{p,\varphi}(a) = \text{Op}^w(a_0)$ for some $a_0 \in M_{(\omega_1)}^{r_0}(\mathbb{R}^{2d})$, and $T_{p,\varphi}(a)$ extends uniquely to a continuous map from $M_{(\omega_0,t)}^{p,q}(\mathbb{R}^{2d})$ to $M_{(\omega_1)}^{p,q}(\mathbb{R}^d)$. 

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Lemma 3.5. Assume that \( q_0, r_0 \in [1, \infty] \) satisfy \( r_0 = 2q_0/(2q_0 - 1) \). Also assume that \( v \in \mathcal{P}(\mathbb{R}^{2d}) \) is submultiplicative, and that \( \kappa, \kappa_0 \in \mathcal{P}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}) \) satisfy
\[
\kappa_0(X_1 + X_2, Y) \leq C \kappa(X_1, Y) v(Y + X_2)v(Y - X_2) \quad X_1, X_2, Y \in \mathbb{R}^{2d},
\]
for some constant \( C > 0 \). Then the map \((a, \varphi) \mapsto T_{p_a}(a)\) from \( \mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^{d}) \) to \( \mathcal{L}(\mathcal{S}(\mathbb{R}^{d}), \mathcal{S}'(\mathbb{R}^{d})) \) extends uniquely to a continuous mapping from \( M_{(\omega)}^{\infty} \) to \( \mathcal{L}(\mathcal{S}(\mathbb{R}^{d}), \mathcal{S}'(\mathbb{R}^{d})) \). Furthermore, if \( \varphi \in M_{(\omega)}^{r_0} \) and \( a \in M_{(\omega)}^{r_0} \), then \( T_{p_a}(a) = Op^v(b) \) for some \( b \in M_{(\kappa)}^{\infty} \).

Proof of Proposition 1.5. We show that the conditions on the involved parameters and weight functions satisfy the conditions of Lemma 3.5.

First we observe that
\[
v_j(2Y) \leq C v_j(Y + X_2)v_j(Y - X_2), \quad j = 0, 1
\]
for some constant \( C \) which is independent of \( X_2, Y \in \mathbb{R}^{2d} \), because \( v_0 \) and \( v_1 \) are submultiplicative. Referring back to (3.4) this gives
\[
\omega_1(X_1 + X_2, Y) = \frac{v_0(2Y)^{1/2}v_1(2Y)}{\omega_0(X_1 + X_2 + Y)^{1/2}\omega_0(X_1 + X_2 - Y)^{1/2}} \leq C_1 \frac{v_0(2Y)^{1/2}v_1(2Y)v_0(X_2 + Y)^{1/2}v_0(X_2 - Y)^{1/2}}{\omega_0(X_1)}
\]
\[
= C_1 v_1(2Y)^{1-t}v_0(2Y)^{1/2}v_1(2Y)^tv_0(X_2 + Y)^{1/2}v_0(X_2 - Y)^{1/2} \omega_0(X_1)
\]
\[
\leq C_2 v_1(2Y)^{1-t}v_1(X_2 + Y)^t v_1(X_2 - Y)^t v_0(X_2 + Y)v_0(X_2 - Y) \omega_0(X_1).
\]

Hence
\[
\omega_1(X_1 + X_2, Y) \leq C \frac{v_1(2Y)^{1-t}v_0(X_2 + Y)v_0(X_2 - Y)}{\omega_0(X_1)}.
\]

By letting \( \kappa_0 = \omega_1 \) and \( \kappa = 1/\omega_{0,t} \), it follows that (3.6) agrees with (3.5). The result now follows from Lemma 3.5.

In the remaining part of the paper we interpret \( T_{p_a}(a) \) as the extension of a Toeplitz operator provided by Proposition 1.5. (See also Remark 3.7 below for more comments.)

Proposition 1.5 can also be applied on Toeplitz operators with smooth weight.

Proposition 3.6. Assume that \( \omega_0 \in \mathcal{P}_0(\mathbb{R}^{2d}) \), that \( v \in \mathcal{P}(\mathbb{R}^{2d}) \) is submultiplicative, and that \( \omega_0^{1/2} \) is \( v \)-moderate. If \( \varphi \in M_{(\omega)}^2 \), then \( T_{p_a}(\omega_0) = Op^v(b) \) for some \( b \in S_{(\omega_0)}(\mathbb{R}^{2d}) \).

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Proof. By Proposition 1.7 we have $\omega_0 \in \mathcal{M}^{\infty,1}_{(1/\omega_0,N)}(\mathbb{R}^{2d})$ for every $N \geq 0$, where $\omega_{0,N}(X,Y) = \omega_0(X)\langle Y \rangle^{-N}$. Furthermore,

$$\omega_1(X,Y) = \frac{\langle Y \rangle^N v(2Y)^{1/2}}{\omega_0(X+Y)^{1/2} \omega_0(X-Y)^{1/2}} \geq C\langle Y \rangle^{N-N_0}/\omega_0(X),$$

for some constants $C$ and $N_0$ which are independent of $N$. Proposition 1.5 implies that existence of some $b \in \mathcal{M}^{\infty,1}_{(1/\omega_0,N)}$, such that $T_{p_\varphi}(\omega_0) = \text{Op}_v^w(a)$. Applying Proposition 1.7 (3) once again, we find that $b \in \bigcap_{N \geq 0} \mathcal{M}^{\infty,1}_{(1/\omega_0,N)} = S(\omega)(\mathbb{R}^{2d})$. \hfill $\square$

Remark 3.7. As remarked and stated before, there are different ways to extend the definition of a Toeplitz operator $T_{p_\varphi}(a)$ (from $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $a \in \mathcal{S}(\mathbb{R}^{2d})$) to more general classes of symbols and windows. For example, Propositions 1.5 and 1.6 are based on the “classical” definition (1.7) of such operators and a straightforward extension of the $L^2$-form on $\mathcal{S}$. Proposition 3.6 interprets $T_{p_\varphi}(\omega)$ as a pseudo-differential operator. Let us emphasize that in this context the bilinear form (1.7) may not be well defined, even when $\varphi \in M^2_{(v)}(\mathbb{R}^d)$ and $\omega \in \mathcal{P}_0(\mathbb{R}^{2d})$.

To shed some light on this subtlety, consider a window $\varphi \in L^2 \setminus M^1$ with normalization $\|\varphi\|_{L^2} = 1$ and the symbol $\omega \equiv 1$. Then the corresponding Toeplitz operator $T_{p_\varphi}(\omega)$ is the identity operator. This is nothing but the inversion formula for the short-time Fourier transform, e.g., [19]. Clearly the identity operator is an isomorphism on every space. However, the Toeplitz operator in (1.7), $T_{p_\varphi}(\omega)$ is not defined on $M^\infty$ because it is not clear what $(1 \cdot V_\varphi f, V_\varphi g)$ from (1.7) means for $\varphi \in L^2$, $f \in M^\infty$ and $g \in M^1$.

In Theorems 3.1 and 3.2 below, we will extend the definition of Toeplitz operators within the framework of pseudo-differential calculus and we interpret Toeplitz operators as pseudo-differential operators. With this understanding, the stated mapping properties are well-defined.

The reader who is not interested in full generality or does not accept Toeplitz operators that are not defined directly by an extension of (1.7) may only consider the case when the windows belong to $M^1_{(v)}$. For the more general window classes in Theorems 3.1 and 3.2 below, however, one should then interpret the involved operators as “pseudo-differential operators that extend Toeplitz operators”.

The following generalization of Theorem 3.1 is an immediate consequence of Theorem 2.1 and Proposition 3.6.

Theorem 3.1. Assume that $\omega, v \in \mathcal{P}(\mathbb{R}^{2d})$, $\omega_0 \in \mathcal{P}_0(\mathbb{R}^{2d})$ and that $\omega_0$ is $v$-moderate. If $\varphi \in M^2_{(v)}(\mathbb{R}^d)$ and $B$ is a translation invariant BF-space, then $T_{p_\varphi}(\omega_0)$ is an isomorphism from $M_{(\omega)}(B)$ to $M_{(\omega/\omega_0)}(B)$.

Theorem 3.1 holds only for smooth weight functions. In order to relax the conditions on the weight function $\omega_0$, we use the Wiener algebra property of $\mathcal{M}^{\infty,1}_{(v)}$ instead of $S^0_{0,0}$. On the other hand, we have to restrict our results to modulation spaces of the form $M^{p,q}_{(\omega)}$ instead of $M_{(\omega)}(B)$.

Theorem 3.2. Assume that $0 \leq t \leq 1$, $p, q, q_0 \in [1, \infty]$ and $\omega, \omega_0, v_0, v_1 \in \mathcal{P}(\mathbb{R}^{2d})$ are such that $\omega_0$ is $v_0$-moderate and $\omega$ is $v_1$-moderate. Set $r_0 = 2q_0/(2q_0 - 1)$,
v = v_1 v_0, \vartheta = \omega_0^{1/2} and let \omega_{0,t} be the same as in (1.8). If \varphi \in M_{(v)}^{p,q}(\mathbb{R}^d) and \omega_0 \in \mathcal{M}_{(1/\omega,t)}^{\infty} \mathcal{M}_{(1/\omega,t)}^{\infty}, then \text{Tr}_\varphi(\omega_0) is an isomorphism from \text{M}^{p,q}_{(\omega_0)}(\mathbb{R}^d) to \text{M}^{p,q}_{(\omega_0)}(\mathbb{R}^d).

\textbf{Proof.} First we note that the Toeplitz operator \text{Tr}_\varphi(\omega_0) is an isomorphism from \text{M}_2(\vartheta) to \text{M}_1(\vartheta) in view of Lemma [3.3]. With \omega_1 defined in (3.4), Proposition 1.15 implies that there exist \beta \in \mathcal{M}_{(\omega_1)}^{\infty,1} and \gamma \in \mathcal{M}_{(\omega_1)}^{\infty,1} such that

\text{Tr}_\varphi(\omega_0) = \text{Op}^w(\beta) \quad \text{and} \quad \text{Tr}_\varphi(\omega_0)^{-1} = \text{Op}^w(\gamma).

Let \omega_2 be the “dual” weight defined as

\omega_2(X,Y) = \vartheta(X - Y)\vartheta(X + Y) v_1(2Y). \quad (3.7)

We will prove that \gamma \in \mathcal{M}_{(\omega_2)}^{\infty,1}(\mathbb{R}^{2d}). Let us assume for now that we have already proved the existence of such a symbol \gamma. Then we may proceed as follows.

After checking (1.12), we can apply Proposition 1.9 and find that each of the

\text{Op}^w(\beta) : \text{M}^{p,q}_{(\omega_0)} \rightarrow \text{M}^{p,q}_{(\omega_0)} \quad \text{and} \quad \text{Op}^w(\gamma) : \text{M}^{p,q}_{(\omega_0)} \rightarrow \text{M}^{p,q}_{(\omega_0)} \quad (3.8)

is well-defined and continuous.

In order to apply Proposition 1.10, we next check condition (1.13) for the weights \omega_1, \omega_2, and

\omega_3(X,Y) = \frac{\vartheta(X + Y)}{\vartheta(X - Y)}. \quad \text{In fact, for some constant } C_1 > 0 \text{ we have}

\omega_1(X - Y + Z, Z)\omega_2(X + Z, Y - Z)

= \left( \frac{v_0(2Z)^{1/2}v_1(2Z)}{\vartheta(X - Y + 2Z)\vartheta(X - Y)} \right) \cdot (\vartheta(X - Y + 2Z)\vartheta(X + Y) v_1(2(Y - Z))) \quad \vartheta(X - Y) \quad \vartheta(X + Y)

\geq C_2 \frac{\vartheta(X + Y)}{\vartheta(X - Y)} = C_2 \omega_3(X,Y).

Therefore Proposition 1.10 shows that the Weyl symbol of \text{Op}^w(\beta) \circ \text{Op}^w(\gamma) belongs to \mathcal{M}_{(\omega_3)}^{\infty,1}(\mathbb{R}^{2d}), or equivalently, \gamma \beta \in \mathcal{M}_{(\omega_3)}^{\infty,1}. Since \text{Op}^w(\beta) is an isomorphism from \text{M}_2(\vartheta) to \text{M}_1(\vartheta) with inverse \text{Op}^w(\gamma), it follows that \beta \gamma = 1 and that the constant symbol 1 belongs to \mathcal{M}_{(\omega_3)}^{\infty,1}. By similar arguments it follows that \gamma \beta = 1. Therefore the identity operator \text{Id} = \text{Op}^w(\beta) \text{Op}^w(\gamma) on \text{M}^{p,q}_{(\omega_0)} factors through \text{M}^{p,q}_{(\omega_0)} \text{M}^{p,q}_{(\omega_0)}, and thus \text{Op}^w(\beta) = \text{Tr}_\varphi(\omega_0) is an isomorphism from \text{M}^{p,q}_{(\omega_0)} onto \text{M}^{p,q}_{(\omega_0)} with inverse \text{Op}^w(\gamma). This proves the assertion.

It remains to prove that \gamma \in \mathcal{M}_{(\omega_2)}^{\infty,1}(\mathbb{R}^{2d}). Using once again the basic result of Bony and Chemin [6], we choose \alpha \in S_{(1/\vartheta)}(\mathbb{R}^{2d}) and \gamma \in S_{(\vartheta)}(\mathbb{R}^{2d}) such that the map

\text{Op}^w(\alpha) : L^2(\mathbb{R}^d) \rightarrow \text{M}^{p,q}_{(\vartheta)}(\mathbb{R}^d)
is an isomorphism with inverse $\text{Op}^w(c)$. By Proposition 1.7, $\text{Op}^w(a)$ is also bijective from $M^2_{(1/\varrho)}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Furthermore, by Theorem 2.1 it follows that $a \in \mathcal{M}^{\infty,1}_{(\varrho_N)}$ for each $N \geq 0$, where
\[ \vartheta_N(X,Y) = \vartheta(X)\langle Y \rangle^N. \]
Let $b_0 = a \# b \# a$. From Proposition 1.11 we know that
\[ b_0 \in \mathcal{M}^{\infty,1}_{(v_2)}(\mathbb{R}^d), \quad \text{where} \quad v_2(X,Y) = v_1(2Y) \quad (3.9) \]
is submultiplicative and depends on $Y$ only. Since $\text{Op}^w(b)$ is bijective from $M^2_{(\varrho)}$ to $M^2_{(1/\varrho)}$ by Lemma 3.4 (2), $\text{Op}^w(b_0)$ is bijective and continuous on $L^2$.

Since $v_2$ is submultiplicative and in $\mathcal{P}(\mathbb{R}^d)$, $\mathcal{M}^{\infty,1}_{(v_2)}$ is a Wiener algebra by Theorem 1.12. Therefore, the bijective operator $\text{Op}^w(b_0)$ on $L^2$ possesses an inverse $\text{Op}^w(d_0)$ for some $d_0 \in \mathcal{M}^{\infty,1}_{(v_2)}(\mathbb{R}^d)$.

Since
\[ \text{Op}^w(d_0) = \text{Op}^w(b_0)^{-1} = \text{Op}^w(a)^{-1} \text{Op}^2(b)^{-1} \text{Op}^w(a)^{-1}, \]
we find that
\[ \text{Op}^w(d) = \text{Op}^w(b)^{-1} = \text{Op}^w(a) \text{Op}^w(d_0) \text{Op}^w(a), \]
or equivalently,
\[ d = a \# d_0 \# a, \quad \text{where} \quad a \in S_{(1/\varrho)} \text{ and } d_0 \in \mathcal{M}^{\infty,1}_{(v_2)}. \quad (3.10) \]
The definitions of the weights are chosen such that Proposition 1.11 implies that $d \in \mathcal{M}^{\infty,1}_{(\varrho_4)}$. With this fact, the proof is now complete.

4. Examples on bijective pseudo-differential operators on modulation spaces

In this section we construct explicit isomorphisms between modulation spaces with different weights. Applying the results of the previous sections, these may be in the form of pseudo-differential operators or of Toeplitz operators.

In fact, the following two propositions are immediate consequences of (1.19), and Theorems 3.1 and 3.2.

**Proposition 4.1.** Assume that $\omega, v \in \mathcal{P}(\mathbb{R}^d)$, $\omega_0 \in \mathcal{P}_0(\mathbb{R}^d)$ and that $\omega_0$ is $v$-moderate. If $\varphi \in M^2_{(v)}(\mathbb{R}^d)$ and $\mathcal{B}$ is a translation invariant BF-space, then $\text{Tp}_\varphi(\omega_0)$ is an isomorphism from $M_{(\omega)}(\mathcal{B})$ to $M_{(\omega/\omega_0)}(\mathcal{B})$.

**Proposition 4.2.** Assume that $0 \leq t \leq 1$, $p,q,t_0 \in [1, \infty]$ and $\omega, \omega_0, v_0, v_1 \in \mathcal{P}(\mathbb{R}^d)$ are such that $\omega_0$ is $v_0$-moderate and $\omega$ is $v_1$-moderate. Set $\nu_0 = 2q_0/(2q_0 - 1)$, $v = v_1^tv_0$ and let $\omega_{0,t}$ be the same as in (1.8). If $\varphi \in M^2_{(v)}(\mathbb{R}^d)$ and $\omega_0 \in \mathcal{M}^{\infty,qt_0}_{(1/\omega_{0,t})}$, then $\text{Op}^w(W_{\varphi,v_0^*}\omega_0)$ is an isomorphism from $M^p_{(\omega/\omega_0)}(\mathbb{R}^d)$ to $M^p_{(\omega/\omega_0)}(\mathbb{R}^d)$.

**Corollary 4.3.** Assume that $p,q \in [1, \infty]$, $\omega_0, \omega \in \mathcal{P}(\mathbb{R}^d)$, and let $\mathcal{B}$ be a translation invariant BF-space on $\mathbb{R}^d$. For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2_+$ let $\Phi_\lambda$ be the Gaussian
\[ \Phi_\lambda(x, \xi) = C e^{-(\lambda_1|x|^2 + \lambda_2|\xi|^2)}. \]
(1) The weight \( \omega_0 \ast \Phi_\lambda \) is in \( \mathcal{P}_0(\mathbb{R}^{2d}) \) for all \( \lambda \in \mathbb{R}^+ \) and

\[
C^{-1}\omega_0 \leq \omega_0 \ast \Phi_\lambda \leq C\omega_0,
\]

for some constant \( C > 0 \).

(2) If \( \lambda_1 \cdot \lambda_2 < 1 \), then there exists a \( \nu \in \mathbb{R}^2_+ \) and \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) such that \( \text{Op}^w(\omega_0 \ast \Phi_\lambda) = T_\varphi(\omega_0 \ast \Phi_\nu) \) is bijective from \( M(\omega)(\mathcal{B}) \) to \( M(\omega/\omega_0)(\mathcal{B}) = M(\omega/\omega_0)(\mathcal{B}) \) for all \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \).

(3) If \( \lambda_1 \cdot \lambda_2 \leq 1 \) and in addition \( \omega_0 \in \mathcal{P}_0(\mathbb{R}^{2d}) \), then \( \text{Op}^w(\omega_0 \ast \Phi_\lambda) = T_\varphi(\omega_0) \) is bijective from \( M(\omega)(\mathcal{B}) \) to \( M(\omega/\omega_0)(\mathcal{B}) = M(\omega/\omega_0)(\mathcal{B}) \) for all \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \).

Proof. The assertion (1) follows easily from the definitions.

(2) Choose \( \mu_j > \lambda_j \) such that \( \mu_1 \cdot \mu_2 = 1 \). Then the Gaussian \( \Phi_\mu \) is a multiple of a Wigner distribution, precisely \( \Phi_\mu = cW(\varphi, \varphi) \) with \( \varphi(x) = e^{-\mu_1|x|^2/2} \). By the semigroup property of Gaussian functions (cf. e.g., [18, 19]) there exists another Gaussian, namely \( \Phi_\nu \), such that \( \Phi_\lambda = \Phi_\mu \ast \Phi_\nu \). Using (1.19), this factorization implies that the Weyl operator with symbol \( \omega_0 \ast \Phi_\lambda \) is in fact a Toeplitz operator, namely

\[
\text{Op}^w(\omega_0 \ast \Phi_\lambda) = \text{Op}^w(\omega_0 \ast \Phi_\mu \ast \Phi_\nu) = \text{Op}^w(\omega_0 \ast \Phi_\nu \ast cW(\varphi, \varphi)) = c(2\pi)^{d/2}T_\varphi(\omega_0 \ast \Phi_\nu)
\]

By (1) \( \omega_0 \ast \Phi_\nu \in \mathcal{P}_0(\mathbb{R}^{2d}) \) is equivalent to \( \omega_0 \). Hence Proposition 4.1 shows that \( \text{Op}^w(\omega_0 \ast \Phi_\lambda) \) is bijective from \( M(\omega)(\mathcal{B}) \) to \( M(\omega/\omega_0)(\mathcal{B}) \). This proves (2).

(3) follows from (2) in the case \( \lambda_1 \cdot \lambda_2 < 1 \). If \( \lambda_1 \cdot \lambda_2 = 1 \), then as above \( \Phi_\lambda = cW(\varphi, \varphi) \) for \( \varphi(x) = e^{-\lambda_1|x|^2/2} \) and thus

\[
\text{Op}^w(\omega_0 \ast \Phi_\lambda) = T_\varphi(\omega_0)
\]

is bijective from \( M(\omega)(\mathcal{B}) \) to \( M(\omega/\omega_0)(\mathcal{B}) \), since \( \omega_0 \in \mathcal{P}_0(\mathbb{R}^{2d}) \). The proof is complete. \qed

APPENDIX

Bony and Chemin [6, Section 5] give a definition of the Sobolev-type spaces \( H(\omega, g) \) for a general class of metrics \( g \) and weight functions. This norm is rather complicated (cf. [6, Section 5] for strict definition). For example, the definition of the \( H(\omega, g) \)-norm in formula (5.1) of [6] involves a sum of expressions that are similar to the right-hand side of (2.2). However, when \( g \) is the usual Euclidean metric on \( \mathbb{R}^{2d} \), then the functions \( \varphi_Y, \psi_{Y,\nu} \) and \( \theta_{Y,\nu} \) in [6, Definition 5.1] can be chosen in the following way.

Let \( 0 \leq \theta \in C_0^\infty(\mathbb{R}^{2d}) \) \( \setminus 0 \) be even and supported in the ball with center at origin and radius 1/4. Then it follows that \( \varphi = \theta \ast \sigma \cdots \ast \theta \in C_0^\infty(\mathbb{R}^{2d}) \) \( \setminus 0 \) is even and non-negative. Here \( \ast \sigma \) is the twisted convolution, defined by the formula

\[
(a \ast \sigma b)(x, \xi) = (2/\pi)^{d/2} \int \int_{\mathbb{R}^{2d}} a(x - y, \xi - \eta)b(y, \eta)e^{2i(y, \xi - (y, \eta))} dyd\eta.
\]
Now let $\varphi = c\tilde{\varphi}$, where $c > 0$ is chosen such that $\|\varphi\|_{L^1} = 1$. From Lemma 1.5 and Proposition 1.6 in [35] we have

$$\tilde{\varphi} = \theta *_{\sigma} \theta *_{\sigma} \theta = (2\pi)^{-d} \theta \# \theta \# \theta = (2\pi)^{-d} \theta \# \theta \# \theta.$$ 

By letting

$$\varphi_Y = \varphi(\cdot - Y), \quad \psi_{Y,0} = \theta_Y = \theta(\cdot - Y),$$

$$\theta_{Y,\nu} = \psi_{Y,\nu} = 0, \quad \nu \geq 1,$$

it follows that all the required properties in [6, Definition 5.1] are fulfilled. Consequently, (2.2) defines a norm for $H(\omega)$.

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