Reducible Quantum Electrodynamics. III. The emergence of the Coulomb forces

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Abstract

The assumption is made that only transversely polarized photons are needed for a correct description of Quantum Electrodynamics. A simple mathematical transformation is used to introduce new field operators which satisfy the full Maxwell equations. In particular, they reproduce Coulomb forces between different regions of the charge field. The analogy with the polaron problem can give some insight in the physics underlying the transformation. In this context it is shown that the interaction of the electron field with a transversely polarized photon field can form bound states. The binding energy peaks for long wavelength photons.

1 Introduction

This is the third in a series of papers on reducible quantum electrodynamics (rQED). In the first paper [1], denoted (I) hereafter, the free electromagnetic field is studied. Free electron fields are the topic of the second paper [2], denoted (II) hereafter. The next step is to introduce interactions between free fields. In standard QED the interaction Hamiltonian involves additional states of the electromagnetic field which do not appear in the description of free fields. Associated with them are so-called longitudinal and scalar photons. Longitudinal and scalar photons do not appear in the present work. The obvious question is then how to quantize electrostatic fields. This is the topic of the present paper.

Michael Creutz [3] showed that a simple mathematical transformation can remove the static fields produced by electric charges. In the next section this idea in used in the opposite direction. It results in two different pictures of the same physics, a Heisenberg picture of interacting electron and photon fields, in absence of electrostatic fields, and an emergent picture in which the field operators satisfy the full Maxwell equations. The existence of this mathematical transformation suggests that the Coulomb forces are emergent forces, much in the same sense as the recent claims [4, 5] that gravity forces are emergent forces.

The remainder of the paper is an attempt to understand the physics behind this double description of electromagnetic fields. In Solid State Physics the polaron is a bound state of an electron and quantized lattice vibrations. An extensive body of knowledge about polarons exists — see for instance [6]. The Fröhlich Hamiltonian [7], which is used to describe polarons, is very similar to the Hamiltonian of QED. Hence, by analogy one expects that free electron fields can bind with the photon field.

Section 3 proves that transversely polarized photons can indeed lower the total energy of an electron field. Such a dressed electron field can be compared with a polaron. It is known that polarons can attract each other. This raises the question whether the Coulomb forces between different parts of the electron field can be explained as effects due to the dressing with photons. The final section discusses this point.

2 Gauss’ law

2.1 Hamiltonian

The use of the temporal gauge is obvious because of the assumption that longitudinal and scalar photons do not exist. In fact, what is used is sometimes called the transverse gauge. It is the combination of the
Coulomb gauge ($\nabla \cdot A = 0$) with the absence of any charges. The vector potential operator $\hat{A}^0$ vanishes identically. The three operators $\hat{A}_\alpha$ are not independent. They are defined by (see (I))

$$\hat{A}_\alpha(x) = \frac{1}{2} \lambda \epsilon^{(H)\alpha}(k^h) \left[ e^{-ik^h x^\alpha} \hat{a}_\alpha + e^{ik^h x^\alpha} \hat{a}_\alpha^\dagger \right] + \frac{1}{2} \lambda \epsilon^{(V)\alpha}(k^h) \left[ e^{-ik^h x^\alpha} \hat{a}_\alpha + e^{ik^h x^\alpha} \hat{a}_\alpha^\dagger \right],$$

(1)

with polarization vectors $\epsilon^{(H)\alpha}(k^h)$ and $\epsilon^{(V)\alpha}(k^h)$. The wave vector of the photon field is denoted $k^h$. The operators $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$ are the annihilation operators of horizontally, respectively vertically polarized photons. The constant $\lambda$ is there for dimensional reasons. A drawback of the temporal gauge is the loss of manifest Lorentz covariance.

The Dirac field operators are given by (see (II))

$$\hat{\psi}_{r,k}(x) = \sum_{s=1,2} u_s^{(x)}(k) \hat{\phi}_{s,k}^{(+)}(x) + \sum_{s=3,4} v_s^{(x)}(k) \hat{\phi}_{s,k}^{(-)}(x).$$

(2)

They are used to define the electric current operators

$$[j^\mu(x)\psi]_k = \frac{qc}{(2\pi)^3} \int dk^\prime j^\mu_{k,k^\prime}(x)\psi_{k^\prime}.$$

(3)

Here, the operators $j^\mu_{k,k^\prime}$ are defined by

$$j^\mu_{k,k^\prime}(x) = \frac{1}{2} \sum_{r,r'} \gamma^\mu_{rr'} \hat{\psi}_{r,k}(x) \hat{\psi}_{r',k^\prime}(x) - \frac{1}{2} \sum_{r,r'} \gamma_{rr'} \hat{\psi}_{r,k}(x) \hat{\psi}_{r',k^\prime}(x).$$

(4)

The Hamiltonian is of the usual form

$$\hat{H} = \hat{H}^{ph} + \hat{H}^{el} + \hat{H}^i$$

(5)

with

$$\hat{H}^{ph} = \hbar c |k^h| \left( \hat{a}_\alpha^\dagger \hat{a}_\alpha + \hat{a}_\alpha^\dagger \hat{a}_\alpha \right),$$

(6)

$$\hat{H}^{el} = \hbar \omega(k) \sum_{s=1}^4 \hat{\phi}_{s,k}^{(-)} \hat{\phi}_{s,k}^{(+)} + \hbar \omega(k) \hat{\phi}_{s,k} \hat{\phi}_{s,k}^\dagger,$$

(7)

$$\hat{H}^i = \int_{\mathbb{R}^3} dx \hat{J}^\mu(x,0) \hat{A}_\mu(x,0).$$

(8)

The interaction Hamiltonian can be written out as

$$\hat{H}^{\mu}_{\psi k^h,k} = \int dx \frac{qc}{(2\pi)^3} \int dk' \hat{A}_{\mu,k^h,k'}(x) \hat{J}^\mu_{k,k'}(x) |\psi_{k^{h},k'}\rangle_{k^h,k'}|_{\psi_{k^{h},k'}^0=0}.$$  

(9)

Its expectation satisfies

$$\langle \psi_{k^{h},k'} | \hat{H}^{\mu}_{\psi k^h,k} | \psi_{k^{h},k'} \rangle = \frac{qc}{(2\pi)^3} \int dx \int dk' \langle \psi_{k^h,k'} | \hat{A}_{\mu,k^h,k'}(x) \hat{J}^\mu_{k,k'}(x) | \psi_{k^{h},k'} \rangle_{k^h,k'}.$$  

(10)

To prove this use that the operators $\hat{A}_{\mu,k^h,k}$ and $\hat{J}^\mu_{k,k^\prime}$ commute with each other. The total charge

$$\hat{Q} = \frac{1}{c} \int dx \hat{J}^0(x)$$

(11)

is conserved because (see (II))

$$[\hat{J}^\mu_{k,k^\prime}(x),\hat{Q}^\dagger] = 0.$$  

(12)
2.2 The emergent picture

In [3] a unitary transformation $\hat{V}$ is defined by a generator $\hat{T}(x)$ through

$$\hat{V} = \exp \left( i \int d^3 x \hat{T}(x) \right).$$

(13)

The generator is of the form

$$\hat{T}(x) = \frac{q}{4\pi} \int d^3 y \mathbf{A}(x) \cdot \frac{x - y}{|x - y|^3} \hat{j}_0(y).$$

(14)

Here, $q$ is the elementary unit of charge. Bold characters are used to indicate three-vectors. The result of [3], in the context of standard QED, is that

$$\hat{V} \nabla \cdot \hat{E} \hat{V}^{-1} = \nabla \cdot \hat{E} - q \hat{j}_0,$$

(15)

where $\hat{E}(x)$ are the electric field operators. If they satisfy Gauss’s law in the presence of a charge distribution $\hat{j}_0(x)$ then $\hat{V} \nabla \cdot \hat{E} \hat{V}^{-1}$ satisfies Gauss’s law in absence of charges.

In the present work the trick of [3] is directly applied to define new electric field operators. They are defined by

$$\hat{E}''_\alpha(x) = \hat{E}'_\alpha(x) + \frac{\mu_0 c}{4\pi} \frac{1}{\partial x^0} \int d^3 y \frac{1}{|x - y|} \hat{U}(-x^0) \hat{j}^0(y,0) \hat{U}(x^0).$$

(16)

Here,

$$\hat{U}(x^0) = \exp(-ix^0\hat{H}/\hbar c)$$

(17)

is the time evolution of the interacting system. The new operators are marked with a double prime to distinguish them from the operators of the non-interacting system (without a prime) and those of the interacting system (with a single prime). One verifies immediately that Gauss’ law is satisfied

$$\sum_\alpha \frac{\partial}{\partial x^\alpha} \hat{E}''_\alpha(x) = -\mu_0 c \hat{j}''_0(x).$$

(18)

This follows because the Coulomb potential is minus the Green’s function of the Laplacian.

The second term in the r.h.s. of (16) is the Coulomb contribution to the electric field. The curl of this term vanishes. Hence it is obvious to take

$$\hat{B}''_\alpha(x) = \hat{B}'_\alpha(x).$$

(19)

This implies the second of the four equations of Maxwell, stating that the divergence of $\hat{B}''_\alpha(x)$ vanishes. Also the fourth equation, absence of magnetic charges, follows immediately because $\hat{E}''(x)$ and $\hat{E}'(x)$ have the same curl. Remains to write Faraday’s law as

$$(\nabla \times \hat{B}''(x))_\alpha - \frac{1}{\epsilon} \frac{\partial}{\partial x^0} \hat{E}''_\alpha(x) = -\mu_0 \hat{j}''_\alpha(x)$$

(20)

with

$$\hat{j}''_\alpha(x) = -\frac{1}{\mu_0 c} \frac{\partial}{\partial x^0} \left( \hat{E}''_\alpha(x) - \hat{E}'_\alpha(x) \right).$$

(21)

Finally, take $\hat{j}''_0(x) = \hat{j}_0(x)$. A short calculation shows that the newly defined current operators $\hat{j}''_\alpha(x)$ satisfy the continuity equation. Note that the operators $\hat{j}''_\alpha(x)$, $\alpha = 1, 2, 3$, are fully determined by the charge field $\hat{j}_0(x)$. 

3
2.3 Discussion

The operators $\hat{E}^\alpha_\mu(x), \hat{B}^\alpha_\mu(x), \hat{j}_0^\alpha(x)$, form what I call the emergent picture of QED. Their time evolution is the same as that of the operators in the original Heisenberg picture and is determined by the unitary operators (17). Because of $\hat{j}_0^\alpha(x) = \hat{j}_0(x)$ also the time-dependent charge field is the same in the two pictures.

Is there a physical interpretation underlying the mathematical equivalence of the two pictures? Here the analogy with the polaron comes into sight. A first step underpinning this analogy is the proof of the existence of bound states, which follows in the next section.

3 Bound states

3.1 Trial wave functions

Consider a wave function of the form

$$\psi_{k^b, k} = \sum_{m, n = 0}^{\infty} \tau_{m, n}(k^b, k) \sqrt{\rho(k^b) \rho^*(k)} m! n! \sqrt{Z_{el}(k)} \langle m, n \rangle \times |\{1\}\rangle + \sqrt{1 - \rho(k^b) \rho^*(k)} |0, 0\rangle \times |\emptyset\rangle$$

with $\tau_{m, n}(k^b, k)$ either 1 or 0, and with $\rho(k^b) \rho^*(k) \leq 1$ for all $k^b, k$. Let us assume that

$$\sum_{m, n = 0}^{\infty} \frac{1}{m! n!} \tau_{m, n}(k^b, k) = Z^a(k)$$

holds independently of the value of $k^b$ and that

$$l^3 \int dk^b \rho(k^b) = 1.$$  (24)

The wave function is properly normalized. It describes a superposition of vacuum with a spin-up electron field entangled with a photon field.

The different contributions to the total energy are as follows. The kinetic energy of the photon field is

$$\langle \hat{H}^b \rangle = l^6 \int dk^b \int dk^c \hbar c |k^b| \sum_{m, n = 0}^{\infty} \frac{m + n}{m! n!} \tau_{m, n}(k^b, k) \rho(k^b) \rho^*(k) \frac{\rho^a(k)}{Z^a(k)}$$

$$= l^3 \int dk^b \hbar c |k^b| \rho(k^b) Z^b(k^b)$$

with

$$Z^b(k^b) = l^3 \int dk \frac{\rho^a(k)}{Z^a(k)} \sum_{m, n = 0}^{\infty} \frac{m + n}{m! n!} \tau_{m, n}(k^b, k).$$

The kinetic energy of the electron field is

$$\langle \hat{H}^a \rangle = l^3 \int dk \hbar \omega(k) \rho^a(k).$$

The interaction energy equals

$$\langle \hat{H}^\iota \rangle = l^6 \int dk^b \int dk^c \int \langle \psi | \hat{H}^\iota | \psi \rangle$$

$$= \frac{qc}{(2\pi)^3} l^6 \int dk^b \rho(k^b) \int dk \int dk' \int dx \sqrt{\rho^a(k) \rho^a(k')} \frac{\sqrt{Z^a(k) Z^a(k')}}{Z^a(k) Z^a(k')}$$
Let us now make a simple choice for the coefficients

\[ U \]

follows that

The symmetries of these functions are important for what follows. From

\[ (34) \]

with

\[ w(k^b) = -\frac{\epsilon_3}{(2\pi)^3} \int dk \int dk' \sqrt{\rho(k)\rho(k')} \sqrt{Z^a(k)Z^a(k')} \sum_{m,m',n=0}^{\infty} \frac{\tau_{m,n}(k^b,k)\tau_{m',n'}(k^b,k')}{\sqrt{m!m'n!}} \]

\[ \times \sum_{\alpha} \langle m,n|\hat{A}_{\alpha,k^b}(x,0)|m',n'\rangle \{1\} |\hat{J}_{k,k'}^{\alpha}(x,0)|\{1\} \].

(28)

(29)

It is shown in the Appendix \[ \text{A} \] that the function \( w(k^b) \) can be written as

\[ w(k^b) = -\frac{\epsilon_3}{(2\pi)^3} \int dk U^{(H)}(k^b,k) \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \left\{ \frac{\tau_{m,n}(k^b,k)\tau_{m+1,n}(k^b,k+k^b)}{\sqrt{Z^a(k)Z^a(k+k^b)}} \right. \]

\[ \left. -\frac{\tau_{m,n}(k^b,-k)\tau_{m+1,n}(k^b,-k-k^b)}{\sqrt{Z^a(-k)Z^a(-k-k^b)}} \right\} \]

(30)

with functions \( U^{(H/V)}(k^b,k) \) given by

\[ U^{(H/V)}(k^b,k) = \frac{1}{2} qe\lambda \sqrt{\rho^2(k)\rho^2(k+k^b)} \sum_{\alpha} \epsilon_\alpha^{(H/V)}(k^b) \text{Re} \langle u^{(1)}(k+k^b)|\gamma^0\gamma^\alpha u^{(1)}(k)\rangle. \]

(31)

The symmetries of these functions are important for what follows. From \( \epsilon_\alpha^{(H)}(-k^b) = \epsilon_\alpha^{(H)}(k^b) \) and \( \epsilon_\alpha^{(V)}(-k^b) = -\epsilon_\alpha^{(V)}(k^b) \) and

\[ \langle u^{(1)}(-k^b)|\gamma^0\gamma^\alpha u^{(1)}(k)\rangle = -\langle u^{(1)}(k^b)|\gamma^0\gamma^\alpha u^{(1)}(k)\rangle \]

(32)

follows that \( U^{(H)}(-k^b,-k) = -U^{(H)}(k^b,k) \) and \( U^{(V)}(-k^b,-k) = -U^{(V)}(k^b,k) \). In addition is \( U^{(H)}(-k^b,k) = U^{(H)}(k^b,k-k^b) \) and \( U^{(V)}(-k^b,k) = -U^{(V)}(k^b,k-k^b) \). It is shown in the Appendix \[ \text{B} \] that

\[ U^{(H/V)}(k^b,k) \leq 0 \text{ if and only if } \sum_{\alpha} \epsilon_\alpha^{(H/V)}(k^b)k_\alpha \geq 0. \]

(33)

This implies in particular that \( U^{(H/V)}(k^b,k) \) and \( U^{(H/V)}(k^b,k \pm k^b) \) have the same sign.

### 3.2 A specific choice

Let us now make a simple choice for the coefficients \( \tau_{m,n} \)

\[ \tau_{1,0}(k^b,k) = \Theta(k \cdot \epsilon^{(H)}(k^b)), \]

(34)

\[ \tau_{0,0}(k^b,k) = 1 - \tau_{1,0}(k^b,k), \]

(35)

\[ \tau_{m,n}(k^b,k) = 0 \text{ otherwise.} \]

(36)
Then \( Z^a(k) = 1 \) for all \( k \). Note that \( \tau_{1,0}(k^a, k \pm k^b) = \tau_{1,0}(k^b, k) \). The expression \( \text{(30)} \) for \( w(k^b) \) becomes
\[
\begin{align*}
w(k^b) &= -l^3 \int dk U^{(H)}(k^b, k) \{1 - \tau_{1,0}(k^b, k)\} \{ \tau_{1,0}(k^b, k) - \tau_{1,0}(k^b, -k) \} \\
&= l^3 \int dk U^{(H)}(k^b, k) \tau_{1,0}(k^b, -k) \\
&= l^3 \int dk U^{(H)}(k^b, k) \tau_{1,0}(k^b, k).
\end{align*}
\]
(37)

Assume that \( \rho^a(k) \) has a Gaussian shape
\[
\rho^a(k) = \rho^a(0) \exp \left(-\frac{1}{2\sigma^2} |k|^2 \right).
\]
Then it is easy to evaluate the long wavelength limit of the function \( U^{(H)}(k^b, k) \) (see the Appendix \( \text{C} \)).

The result is
\[
U^{(H/V)}(k^b, k) = \frac{1}{2} \frac{q c \rho^a(k)}{\omega(k)} \left[1 - \frac{k \cdot k^b}{2\sigma^2} + O(|k^b|^2)\right] \\
\times \left[1 + \frac{1}{2} \frac{k \cdot k^b}{\sigma^2 + |k|^2} + O(|k^b|^2)\right] k \cdot \epsilon^{(H/V)}(k^b).
\]
(39)

There follows
\[
w(k^b) - \hbar c|k^b|Z^{ab}(k^b) = l^3 \int dk \Theta(k \cdot \epsilon^{(H)}(k^b)) \left[ \frac{1}{2} \frac{q c \rho^a(k)}{\omega(k)} \right] \left[1 + \frac{1}{2} \frac{k \cdot k^b}{\sigma^2 + |k|^2} + O(|k^b|^2)\right] \\
\times \left[-\hbar c|k^b|\rho^a(k)\right].
\]
(40)

Because of mirror symmetry in the direction of \( k^b \) the terms in \( k \cdot k^b \) drop. The result is
\[
w(k^b) - \hbar c|k^b|Z^{ab}(k^b) = l^3 \int dk \Theta(k \cdot \epsilon^{(H)}(k^b)) \rho^a(k) \\
\times \left[\frac{q c}{\omega(k)} \frac{1}{2} [k \cdot \epsilon^{(H)}(k^b)] - \hbar c|k^b| + O(|k^b|^2)\right].
\]
(41)

This suggests the following

**Lemma 3.1** There exists \( \epsilon > 0 \), independent of \( \rho^a(0) \), such that
\[
\text{if } w(k^a) - \hbar c|k^a|Z^{ab}(k^a) > 0 \quad \text{whenever} \quad |k^a|\sigma \exp(|k^a|^2/4\sigma^2) < \epsilon
\]
then \( w(k^a) - \hbar c|k^a|Z^{ab}(k^a) > 0 \).

A proof of the Lemma is found in the Appendix \( \text{D} \).

Use the Lemma to estimate
\[
\langle \hat{H}^{ab} \rangle + \langle \hat{H} \rangle = -l^3 \int dk^b \rho(k^b) [w(k^b) - \hbar c|k^b|Z(k^b)].
\]
(43)
Take \( \rho(k^b) = 0 \) when \( |k^b| > \delta \) where \( \delta \) is the solution of
\[
\delta \sigma \exp(\delta^2/4\sigma^2) = \epsilon.
\]
(44)

The constraints \( \text{(24)} \) and
\[
\rho(k^b)\rho^a(k) \leq 1 \quad \text{for all } k^b, k,
\]
(45)
can be satisfied by taking
\[ \rho(k^q) = \frac{3}{4\pi\delta^2 l^3}, \quad |k^q| < \delta, \]  
and requiring that
\[ 0 < \rho^i(0) \leq \frac{4\pi}{3} \delta^3 l^3. \]
Then the conditions of the Lemma are satisfied. One concludes that the integrand in the r.h.s. of (43) cannot be negative, and hence that the l.h.s. is strictly negative. This proves the following result.

**Theorem 3.2** Let \( \rho^i(k) = \rho^i(0) \exp(-|k|^2/2\sigma^2) \) with \( \rho^i(0) > 0 \) but small. There exist wave functions \( \psi \) for which
\[
\langle \hat{H}^m \rangle + \langle \hat{H}' \rangle < 0 \quad \text{and} \quad \int dk^q \langle \psi | \hat{H}^q \psi \rangle_{k^q,k} = \rho^i(k) \hbar \omega(k). \]

### 3.3 Discussion

The Theorem shows that the energy of an electron field can be lowered by binding with an electromagnetic field which is transversely polarized. Because of the conservation law the total energy remains at all times lower than the initial kinetic energy \( \langle \hat{H}^m \rangle \) of the electron field. However, one cannot exclude that part of the kinetic energy of the electron field is converted into asymptotically free photons, for instance by means of Bremsstrahlung. The open question is therefore whether the dressed electron field looses all of its dressing as time progresses, or keeps part of it. To answer this question the time evolution of the interacting system has to be studied. This is out of the scope of the present paper.

The proof of the Theorem uses a particular trial wave function. Of course, other choices are possible. The more general setup of Section 3.1 allows to construct further examples. However, a more systematic investigation is needed. In particular one wants to know the optimal choice of wave function and whether the total energy remains finite.

Note that only low photon numbers are relevant because the cost of creating an electromagnetic field increases linearly with the number of photons while the electromagnetic field strength is proportional to the square root. Note also that the choice of trial function depends on the spin of the electron field.

### 4 Final Remarks

In (I) electromagnetic potential operators \( \hat{A}_\alpha(x) \) are introduced. They describe free photon fields in the temporal gauge. In (II) the Dirac current operators \( \hat{j}^\mu(x) \) are defined for a non-interacting electron-positron field. Together, these operators \( \hat{A}_\alpha(x) \) and \( \hat{j}^\mu(x) \) determine the interaction Hamiltonian \( \hat{H}' \). It is given by (9). The operators of the interacting system are marked with a prime. The electric field operators \( \hat{E}^\alpha_0(x) \) still satisfy Gauss’ law in absence of a charge field. However, the simple transformation introduces new field operators which are such that Gauss’ law now reproduces the given charge field \( \hat{j}_0^\mu(x) \). This leads to the conclusion that two equivalent descriptions of the same time-dependent charge field exist, one in which Coulomb forces are absent, one in which they emerge as an effective description. This observation justifies the study of a version of QED in which no longitudinal or scalar photons exist.

Leaving out the longitudinal and scalar photons eliminates an important source of mathematical inconsistencies. The resulting theory can be formulated in a mathematically rigorous fashion. However, the use of excessive mathematical machinery has been avoided here to keep the paper readable for a larger audience.

An analogy with the polaron problem is made to explain the emergence of Coulomb forces as an effective description of the time dependence of the charge field. As a first step in revealing the nature of the interactions the existence of bound states is established. An analysis of the time evolution is still missing.
Many open questions remain. For instance, polarons do attract each other, while regions where the charge of the electron field has equal sign should repel. In addition, the forces should decay inversely proportional to the square of the distance. Hopefully in this respect is that the binding of the photon charge of the electron field has equal sign should repel. In addition, the forces should decay inversely proportional to the square of the distance. This supports at least qualitatively that the forces are long-ranged.

The next paper in this series of papers treats scattering theory in the context of reducible QED. In particular, it focuses on the consequences of omitting longitudinal and scalar photons.

Appendices

A Calculation of the interaction energy

Let us evaluate the function \( w(k^{\text{ph}}) \) as defined by (29). Use

\[
\langle m, n|\hat{A}_{\alpha, k^{\text{ph}}}(x, 0)|m', n'\rangle = \frac{1}{2}\lambda_\alpha^{(H)}(k^{\text{ph}})\langle m, n|e^{ik^{\text{ph}}\cdot\hat{a}_n} + e^{-ik^{\text{ph}}\cdot\hat{a}_n^\dagger}|m', n'\rangle + \frac{1}{2}\lambda_\alpha^{(V)}(k^{\text{ph}})\langle m, n|e^{ik^{\text{ph}}\cdot\hat{a}_\nu} + e^{-ik^{\text{ph}}\cdot\hat{a}_\nu^\dagger}|m', n'\rangle
\]

\[
= \frac{1}{2}\lambda_\alpha^{(H)}(k^{\text{ph}}) \left[ e^{ik^{\text{ph}}\cdot\sqrt{m + 1}\delta_{m+1,m'} + e^{-ik^{\text{ph}}\cdot\sqrt{m'} + 1}\delta_{m'+1,m}\right] \delta_{n,n'} + \frac{1}{2}\lambda_\alpha^{(V)}(k^{\text{ph}}) \delta_{m,m'} \left[ e^{ik^{\text{ph}}\cdot\sqrt{n + 1}\delta_{n+1,n'} + e^{-ik^{\text{ph}}\cdot\sqrt{n'} + 1}\delta_{n',n+1}\right].
\]

This gives

\[
w(k^{\text{ph}}) = -\frac{g\alpha}{2(2\pi)^3} \int dk \int dk' \int dx \frac{\sqrt{\rho^\dagger(k)\rho^\dagger(k')}}{\sqrt{Z^\dagger(k)Z(k')}} \times \left\{ \sum_{m,m',n,n'=0}^\infty \rho_{m,n}(k^{\text{ph}})\rho_{m',n'}(k^{\text{ph}},k') \right\} \langle \{1\}|\hat{A}_{k^{\text{ph}}}(x, 0)|\{1\}\rangle
\]

\[
= -\frac{g\alpha}{2(2\pi)^3} \int dk \int dk' \int dx \frac{\sqrt{\rho^\dagger(k)\rho^\dagger(k')}}{\sqrt{Z^\dagger(k)Z(k')}} \times \left\{ \sum_{m,n=0}^\infty \rho_{m,n}(k^{\text{ph}})\rho_{m+1,n}(k^{\text{ph}},k') \sum_{\alpha} \epsilon_\alpha^{(H)}(k^{\text{ph}})e^{ik^{\text{ph}}\cdot x} \right. \right.
\]

\[
+ \sum_{m',n'=0}^\infty \rho_{m',n'}(k^{\text{ph}},k)\rho_{m'+1,n}(k^{\text{ph}},k') \sum_{\alpha} \epsilon_\alpha^{(H)}(k^{\text{ph}})e^{-ik^{\text{ph}}\cdot x} \right.
\]

\[
+ \sum_{m,n=0}^\infty \rho_{m,n}(k^{\text{ph}})\rho_{m,n+1}(k^{\text{ph}},k') \sum_{\alpha} \epsilon_\alpha^{(V)}(k^{\text{ph}})e^{ik^{\text{ph}}\cdot x} \right.
\]

\[
+ \sum_{m',n'=0}^\infty \rho_{m',n'}(k^{\text{ph}},k)\rho_{m,n+1}(k^{\text{ph}},k') \sum_{\alpha} \epsilon_\alpha^{(V)}(k^{\text{ph}})e^{-ik^{\text{ph}}\cdot x} \right. \left. \right\} \langle \{1\}|\hat{A}_{k^{\text{ph}}}(x, 0)|\{1\}\rangle.
\]
Use that $J_{\alpha,k}^0(x) = J_{\alpha,k}^0(x)$ to obtain

$$w(k_{th}) = -\frac{qe\lambda}{(2\pi)^3} L^3 \int dk \int dk' \frac{\rho^e(k)\rho^e(k')}{\sqrt{Z^a(k)Z^a(k')}} \int dx \cos(k_{th} \cdot x) \sum_{m,n=0}^{\infty} \frac{1}{m!n!}$$

$$\times \left\{ \tau_{m,n}(k_{th}, k)\tau_{m+1,n}(k_{th}, k') \sum_\alpha \varepsilon_\alpha^H(k_{th})\langle 1 \rangle J_{\alpha,k}^0(x,0)\langle 1 \rangle + \tau_{m,n}(k_{th}, k)\tau_{m+1,n}(k_{th}, k') \sum_\alpha \varepsilon_\alpha^V(k_{th})\langle 1 \rangle J_{\alpha,k}^0(x,0)\langle 1 \rangle \right\}. \tag{52}$$

Now use that

$$\frac{1}{(2\pi)^3} \int dx e^{\pm ik_{th} \cdot x} \langle 1 \rangle J_{\alpha,k}^0(x,0)\langle 1 \rangle = \frac{1}{(2\pi)^3} \int dx e^{\pm ik_{th} \cdot x} \frac{1}{2} \sum_{s,t=1,2} \langle u^{(s)}(k)|\gamma^0 \gamma^\alpha u^{(t)}(k')\rangle \langle 1 \rangle \phi_{s,k}^0(x,0)\phi_{t,k}^0(x,0)\langle 1 \rangle$$

$$+ \frac{1}{(2\pi)^3} \int dx e^{\pm ik_{th} \cdot x} \frac{1}{2} \sum_{s,t=1,2} \langle u^{(s)}(k')|\gamma^0 \gamma^\alpha u^{(t)}(k)\rangle \langle 1 \rangle \phi_{s,k'}^0(x,0)\phi_{t,k'}^0(x,0)\langle 1 \rangle$$

$$= \frac{1}{(2\pi)^3} \int dx e^{\pm ik_{th} \cdot x} \delta(k - k + k_{th}) \langle u^{(1)}(k)|\gamma^0 \gamma^\alpha u^{(1)}(k')\rangle$$

$$+ \frac{1}{2} \delta(k - k + k_{th}) \langle u^{(1)}(k')|\gamma^0 \gamma^\alpha u^{(1)}(k)\rangle.$$

This gives

$$w(k_{th}) = -\frac{qe\lambda}{(2\pi)^3} L^3 \int dk \int dk' \frac{\rho^e(k)\rho^e(k')}{\sqrt{Z^a(k)Z^a(k')}} \left[ \delta(k - k' + k_{th}) + \delta(k - k' - k_{th}) \right]$$

$$\times \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \left\{ \tau_{m,n}(k_{th}, k)\tau_{m+1,n}(k_{th}, k') \sum_\alpha \varepsilon_\alpha^H(k_{th}) \text{Re} \langle u^{(1)}(k')|\gamma^0 \gamma^\alpha u^{(1)}(k)\rangle$$

$$+ \tau_{m,n}(k_{th}, k)\tau_{m+1,n}(k_{th}, k') \sum_\alpha \varepsilon_\alpha^V(k_{th}) \text{Re} \langle u^{(1)}(k')|\gamma^0 \gamma^\alpha u^{(1)}(k)\rangle \right\}$$

$$= -L^3 \int dk \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \left\{ \tau_{m,n}(k_{th}, k)\tau_{m+1,n}(k_{th}, k') \sum_\alpha \varepsilon_\alpha^H(k_{th}) \frac{U^H(k_{th}, k)}{\sqrt{Z^a(k)Z^a(k + k_{th})}}$$

$$+ \tau_{m,n}(k_{th}, k)\tau_{m+1,n}(k_{th}, k') \sum_\alpha \varepsilon_\alpha^V(k_{th}) \frac{U^V(k_{th}, k)}{\sqrt{Z^a(k)Z^a(k + k_{th})}}$$

$$+ \tau_{m,n}(k_{th}, k)\tau_{m+1,n}(k_{th}, k') \sum_\alpha \varepsilon_\alpha^V(k_{th}) \frac{U^V(k_{th}, k)}{\sqrt{Z^a(k)Z^a(k - k_{th})}} \right\}. \tag{54}$$

One can write this result as \ref{50}.

### B Estimates

One has

$$\langle u^{(1)}(k + k_{th})|\gamma^0 \gamma^\alpha u^{(1)}(k)\rangle = -c \frac{k_\alpha [\omega(k + k_{th}) + c\kappa] + (k_\alpha + k_{th})\omega(k) + c\kappa}{\sqrt{\omega(k + k_{th})\omega(k + k_{th}) + c\kappa} \sqrt{\omega(k)\omega(k) + c\kappa}}. \tag{55}$$

This implies

$$\sum_\alpha \varepsilon_\alpha^{H/V}(k_{th}) \text{Re} \langle u^{(1)}(k + k_{th})|\gamma^0 \gamma^\alpha u^{(1)}(k)\rangle = -\sum_\alpha \varepsilon_\alpha^{H/V}(k_{th}) [A_{\alpha} + B_{\alpha}k_{th}]. \tag{56}$$
From (59) and (31) then follows
\[ \sum_{\alpha} \gamma^{(H/V)}(k^\alpha)(k^\alpha) \Re \langle u^{(1)}(k + k^\alpha)|\gamma^{0,\alpha}u^{(1)}(k) \rangle = -A \sum_{\alpha} \gamma^{(H/V)}(k^\alpha)(k^\alpha) k_\alpha. \] (59)

One concludes that (33) holds.

\section{C Long wavelength limit}

Here, the long wavelength expansion of the function \( U^{(H)}(k^\alpha, k) \) is calculated.

From (59) and
\[ A = \frac{2}{\omega(k)} \left[ 1 + \frac{1}{2} \kappa^2 + |k|^2 + O(|k^\alpha|^2) \right] \] (60)

follows
\[ \sum_{\alpha} \gamma^{(H/V)}(k^\alpha) \Re \langle u^{(1)}(k + k^\alpha)|\gamma^{0,\alpha}u^{(1)}(k) \rangle = -\frac{2}{\omega(k)} \left[ 1 + \frac{1}{2} \kappa^2 + |k|^2 + O(|k^\alpha|^2) \right] \sum_{\alpha} \gamma^{(H/V)}(k^\alpha)(k^\alpha) k_\alpha. \] (61)

From the assumption that \( \rho^\alpha(k) \) has a Gaussian shape follows
\[ \rho^\alpha(k + k^\alpha) = \rho^\alpha(k) \left[ 1 - \frac{k \cdot k^\alpha}{\sigma^2} + O(|k^\alpha|^2) \right]. \] (62)

Combining this expression with (51) yields (39).

\section{D Proof of the Lemma}

Here a proof of Lemma (57) follows.
Assume \(|k| < \kappa\) and \(|k^\alpha| < \kappa\) and \(k \cdot \gamma^{(H)}(k^\alpha) > 0\). The quantity \(A\) defined by (57) satisfies
\[ A > \frac{4}{\sqrt{3}(1 + \sqrt{3}) \sqrt{2}} \frac{1}{\kappa} > \frac{1}{7\kappa}. \] (63)

From (59) and (31) then follows
\[ U^{(H)}(k^\alpha, -k) > \frac{1}{14\kappa} q c \lambda \sqrt{\rho^\alpha(k) \rho^\alpha(k - k^\alpha)} |k \cdot \gamma^{(H)}(k^\alpha)|. \] (64)

The estimate for \(w\) now follows from (37)
\[ w(k^\alpha) > l^3 \int d\kappa \Theta(k - |k|) \Theta(k \cdot \gamma^{(H)}(k^\alpha)) U^{(H)}(k^\alpha, -k) \]
\[ > \frac{1}{14\kappa} q c \lambda l^3 \int d\kappa \Theta(k - |k|) \Theta(k \cdot \gamma^{(H)}(k^\alpha)) \sqrt{\rho^\alpha(k) \rho^\alpha(k - k^\alpha)} |k \cdot \gamma^{(H)}(k^\alpha)| \]
\[ = \frac{1}{14\kappa} q c \lambda l^3 \int d\kappa \Theta(k - |k|) \Theta(k \cdot \gamma^{(H)}(k^\alpha)) |k \cdot \gamma^{(H)}(k^\alpha)| e^{- \frac{1}{\kappa^2} |k|^2 - \frac{1}{\kappa^2} |k - k^\alpha|^2} \] (65)
Introduce a new coordinate system with $k^{ph}$ in direction 3 and $\varepsilon^{(H)}(k^{ph})$ in direction 1. Then the above becomes

$$w(k^{ph}) > \frac{1}{14k} gc \rho^{\varepsilon}(0) e^{-\frac{1}{4\sigma^2} |k^{ph}|^2} l^3 \int_0^{\infty} dk_1 k_1 \int dk_2 k_2 \int dk_3 \Theta(k - |k|) \Theta(k) e^{-\frac{1}{2\sigma^2} |k|^2} e \frac{1}{2\sigma^2} k_3 [k^{ph}]$$

$$= \frac{1}{14k} gc \rho^{\varepsilon}(0) e^{-\frac{1}{4\sigma^2} |k^{ph}|^2} l^3 \int_0^{\infty} r^2 dr \int_{-\pi/2}^{\pi/2} \sin(\theta) d\theta \int_{-\pi/2}^{\pi/2} \cos(\phi) \cos(\phi) e^{-\frac{1}{2\sigma^2} r^2} e \frac{1}{2\sigma^2} r \sin \theta$$

$$= \frac{\pi}{14k} gc \rho^{\varepsilon}(0) e^{-\frac{1}{4\sigma^2} |k^{ph}|^2} l^3 \int_0^{\infty} r^2 dr e^{-\frac{1}{8\sigma^2} r^2} \int_{-\pi/2}^{\pi/2} \sin(\theta) d\theta \cos(\phi) e^{-\frac{1}{2\sigma^2} r \sin \theta}$$

$$= \frac{\pi}{14k} gc \rho^{\varepsilon}(0) e^{-\frac{1}{4\sigma^2} |k^{ph}|^2} l^3 \int_0^{\infty} r^2 dr e^{-\frac{1}{8\sigma^2} r^2} \int_{-\pi/2}^{\pi/2} \sin(\theta) d\theta \cos(\phi) \cosh\left(\frac{1}{2\sigma^2} r \sin \theta\right)$$

$$> \frac{\pi}{14k} gc \rho^{\varepsilon}(0) e^{-\frac{1}{4\sigma^2} |k^{ph}|^2} l^3 \int_0^{\infty} r^2 dr e^{-\frac{1}{8\sigma^2} r^2} \int_{0}^{\pi/2} \sin(\theta) d\theta \cos(\phi)$$

$$= \frac{\pi}{28k} gc \rho^{\varepsilon}(0) e^{-\frac{1}{4\sigma^2} |k^{ph}|^2} l^3 \int_0^{\infty} r^2 dr e^{-\frac{1}{8\sigma^2} r^2}$$

$$= l^3 \pi \sigma^2 gc \rho^{\varepsilon}(0) e^{-\frac{1}{4\sigma^2} |k^{ph}|^2} f\left(\frac{K}{\sigma}\right)$$

(66)

for $f$ given by

$$f(u) = \frac{1}{28} \int_0^u t^2 dt e^{-\frac{1}{2} t^2}.$$  

(67)

On the other hand is

$$hc|k^{ab}|Z^{ph}(k^{ph}) = \frac{hc|k^{ab}|l^3}{2} \int dk \Theta(k \cdot \varepsilon^{(H)}(k^{ph})) \rho^{\varepsilon}(k)$$

$$= \frac{hc|k^{ab}|l^3}{2} \int dk \Theta(k \cdot \varepsilon^{(H)}(k^{ph})) \exp\left(-\frac{1}{2\sigma^2} |k|^2\right)$$

$$= \frac{hc|k^{ab}|l^3}{2}\pi 2\pi \int_0^{\infty} r^2 dr e^{-\frac{1}{2\sigma^2} r^2}$$

$$= \frac{hc|k^{ab}|l^3}{2}\pi \sigma^3 \int_0^{\infty} t^2 dt e^{-\frac{1}{2} t^2}$$

$$= \frac{hc|k^{ab}|l^3}{2}\pi \sigma^3 \sqrt{2\pi}.$$  

(68)

One concludes that (42) holds for $\varepsilon$ given by

$$\varepsilon = \frac{1}{\sqrt{2\pi}} \lambda g f\left(\frac{K}{\sigma}\right).$$  

(69)

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