A NOTE ON THE $M^\ast$-LIMITING CONVOLUTION BODY

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Abstract. We introduce the mixed convolution bodies of two convex symmetric bodies. We prove that if the boundary of a body $K$ is smooth enough then as $\delta$ tends to 1 the $\delta-M^\ast$-convolution body of $K$ with itself tends to a multiple of the Euclidean ball after proper normalization. On the other hand we show that the $\delta-M^\ast$-convolution body of the $n$-dimensional cube is homothetic to the unit ball of $\ell_1^n$.

1. Introduction

Throughout this note $K$ and $L$ denote convex symmetric bodies in $\mathbb{R}^n$. Our notation will be the standard notation that can be found, for example, in [2] and [4]. For $1 \leq m \leq n$, $V_m(K)$ denotes the $m$–th mixed volume of $K$ (i.e. mixing $m$ copies of $K$ with $n - m$ copies of the Euclidean ball $B_n$ of radius one in $\mathbb{R}^n$). Thus if $m = n$ then $V_n(K) = \text{vol}_n(K)$ and if $m = 1$ then $V_1(K) = w(K)$ the mean width of $K$.

For $0 < \delta < 1$ we define the $m$–th mixed $\delta$–convolution body of the convex symmetric bodies $K$ and $L$ in $\mathbb{R}^n$:

**Definition 1.2.** The $m$–th mixed $\delta$–convolution body of $K$ and $L$ is defined to be the set,

$$C_m(\delta; K, L) = \{ x \in \mathbb{R}^n : V_m(K \cap (x + L)) \geq \delta V_m(K \cap L) \}.$$

It is a consequence of Brunn–Minkowski inequality for mixed volumes that these bodies are convex.

If we write $h(u)$ for the support function of $K$ in the direction $u \in S^{n-1}$ then we have,

$$w(K) = 2M_1^\ast K = 2 \int_{S^{n-1}} h(u)d\nu(u),$$

where $\nu$ is the Lebesgue measure of $\mathbb{R}^n$ restricted on $S^{n-1}$ and normalized so that $\nu(S^{n-1}) = 1$. In this note we study the limiting behavior of $C_1(\delta; K, K)$ (which we will abbreviate with $C_1(\delta)$) as $\delta$ tends to 1 and $K$ has a $C^2_+$ boundary. For simplicity we will call $C_1(\delta)$ the "$\delta-M^\ast$–convolution body of $K$".

We are looking for suitable $\alpha \in \mathbb{R}$ so that the limit

$$\lim_{\delta \to 1^-} \frac{C_1(\delta)}{(1 - \delta)^\alpha}$$

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exists (convergence in the Hausdorff distance). In this case we call the limiting body “the limiting $M^*$–convolution body of $K$”.

We prove that for a convex symmetric body $K$ in $\mathbb{R}^n$ with $C^2_+$ boundary the limiting $M^*$–convolution body of $K$ is homothetic to the Euclidean ball. We also get a sharp estimate (sharp with respect to the dimension $n$) of the rate of the convergence of the $\delta$–$M^*$–convolution body of $K$ to its limit. By $C^2_+$ we mean that the boundary of $K$ is $C^2$ and that the principal curvatures of $bd(K)$ at every point are all positive.

We also show that some smoothness condition on the boundary of $K$ is necessary for this result to be true, by proving that the limiting $M^*$–convolution body of the $n$–dimensional cube is homothetic to the unit ball of $\ell^1_n$.

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2. THE CASE “$bd(K)$ IS A $C^2_+$ MANIFOLD”

In this section we prove the following:

**Theorem 2.1.** Let $K$ be a convex symmetric body in $\mathbb{R}^n$ so that $bd(K)$ is a $C^2_+$ manifold. Then for all $x \in S^{n-1}$ we have,

$$\left\| x \right\| \frac{C_1(\delta)}{\sqrt{s}} - \frac{c_n}{M_K} \leq C \frac{c_n}{M_K} (M^*_K n(1 - \delta))^2,$$

where $c_n = \int_{S^{n-1}} |\langle x, u \rangle| d\nu(u) \sim 1/\sqrt{n}$ and $C$ is a constant independent of the dimension $n$. In particular,

$$\lim_{\delta \to 1^-} \frac{C_1(\delta)}{1 - \delta} = \frac{M^*_K}{c_n} B_n.$$

Moreover the estimate (2.1.1) is sharp with respect to the dimension $n$

By “sharp” with respect to the dimension $n$ we mean that there are examples (for instance the $n$–dimensional Euclidean ball) for which the inequality (2.1.1) holds true if “$\leq$” is substituted with “$\geq$” and the constant $C$ is adjusted by a (universal) constant factor.

Before we proceed with the proof we will need to collect some standard notation which can be found in [4]. We write $p : bd(K) \to S^{n-1}$ for the Gauss map $p(x) = N(x)$ where $N(x)$ denotes the unit normal vector of $bd(K)$ at $x$. $W_x$ denotes the Weingarten map, that is, the differential of $p$ at the point $x \in bd(K)$. $W_x^{-1}$ is the reverse Weingarten map at $u \in S^{n-1}$ and the eigenvalues of $W_x$ and $W_u^{-1}$ are respectively the principal curvatures and principal radii of curvature of the manifold $bd(K)$ at $x \in bd(K)$ and $u \in S^{n-1}$. We write $\|W\|$ and $\|W^{-1}\|$ for the quantities: $\sup_{x \in bd(K)} \|W_x\|$ and $\sup_{u \in S^{n-1}} \|W_u^{-1}\|$ respectively. These quantities are finite since the manifold $bd(K)$ is assumed to be $C^2_+$.

For $\lambda \in \mathbb{R}$ and $x \in S^{n-1}$ we write $K_\lambda$ for the set $K \cap (\lambda x + K)$. $p_\lambda^{-1} : S^{n-1} \to bd(K_\lambda)$ is the reverse Gauss map, that is, the affine hyperplane $p_\lambda^{-1}(u) + [u]_1$ is tangent to $K_\lambda$ at $p_\lambda^{-1}(u)$. The normal cone of $K_\lambda$ at $x$ is denoted by $N(K_\lambda, x)$ and similarly for $K$. The normal cone is a convex set (see [4]). Finally $h_\lambda$ will denote the support function of $K$. 
It is not difficult to see that the cosine of the angle of the vectors
than the largest principal curvature of
bd
inverses are Lipschitz and they all have the same Lipschitz constant
are equipped with an atlas whose charts are functions which are Lipschitz, their
Y
 Consequently we can write, (2.1.2)
M^*_K = \delta M^*_K.

We estimate now M^*_Kλ. Let u ∈ S^{n−1}. We need to compare h_λ(u) and h(u). Set
Y_λ = bd(K) ∩ bd(λx + K).

Case 1. p^{-1}_λ(u) ∉ Y_λ.
In this case it is easy to see that
h_λ(u) = h(u) − |⟨λx, u⟩|.

Case 2. p^{-1}_λ(u) ∈ Y_λ.
Let y_λ = p^{-1}_λ(u) and y'_λ = y_λ − λx ∈ bd(K). The set N(K_λ, y_λ) ∩ S^{n−1} defines a curve γ which we assume to be parametrized on [0, 1] with γ(0) = N(K, y_λ) and γ(1) = N(K, y'_λ). We use the inverse of the Gauss map p to map the curve γ to a curve \tilde{γ} on bd(K) by setting \tilde{γ} = p^{-1}γ. The end points of \tilde{γ} are y_λ (label it with A) and y'_λ (label it with B). Since u ∈ γ we conclude that the point p^{-1}(u) belongs to the curve \tilde{γ} (label this point by Γ). Thus we get:

0 ≤ h(u) − h_λ(u) = |⟨A\tilde{γ}, u⟩|.

It is not difficult to see that the cosine of the angle of the vectors A\tilde{γ} and u is less than the largest principal curvature of bd(K) at Γ times |A\tilde{γ}|, the length of the vector A\tilde{γ}. Consequently we can write,

0 ≤ h(u) − h_λ(u) ≤ ||W|||A\tilde{γ}|^2.

In addition we have,

|A\tilde{γ}| ≤ length (\tilde{γ}|_A^Γ) ≤ length (\tilde{γ}|_A^B)
= \int_0^1 |d_t \tilde{γ}| dt = \int_0^1 |d_t p^{-1}γ| dt
≤ ||W||length(γ) ≤ \frac{2}{\pi}||W||||p(y_λ) − p(y'_λ)||,

where |. . | is the standard Euclidean norm. Without loss of generality we can assume that the points y_λ and y'_λ belong to the same chart at y_λ. Let \varphi be the chart mapping \mathbb{R}^{n−1} to a neighborhood of y_λ on bd(K) and ψ the chart mapping \mathbb{R}^{n−1} on a neighborhood of N(K, y_λ) in S^{n−1}. We assume, as we may, that the graph of γ is contained in the range of the chart ψ. It is now clear from the above series of inequalities that

|A\tilde{γ}| ≤ c||W||^{-1}||ψ^{-1}ψ(\tilde{γ}) − ψ^{-1}ψ(\lambda)|.
where $t$ and $s$ are points in $\mathbb{R}^{n-1}$ such that $\varphi(t) = y_\lambda$ and $\varphi(s) = y'_\lambda$ and $c_0 > 0$ is a universal constant. Now the mean value theorem for curves gives,

$$|\tilde{A}| \leq C\|W^{-1}\|\|W\|\|t - s\|$$

$$\leq C\|W^{-1}\|\|W\|\|y_\lambda - y'_\lambda\|$$

$$= C\|W^{-1}\|\|W\|\lambda,$$

where $C$ may denote a different constant every time it appears. Thus we have,

$$0 \leq h(u) - h_\lambda(u) \leq C\|W\| (\|W^{-1}\|\|W\|)^2 \lambda^2.$$

Consequently,

$$\int_{S^{n-1}\setminus p_\lambda(Y_\lambda)} (h(u) - |\langle \lambda x, u \rangle|) \, d\nu(u) + \int_{p_\lambda(Y_\lambda)} (h(u) - C\lambda^2) \, d\nu(u)$$

$$\leq M_{\star K_\lambda} = \delta M^\star_K \leq C\|W\| \lambda,$$

where $C$ now depends on $\|W\|$ and $\|W^{-1}\|$. Rearranging and using $c_n$ for the quantity $\int_{S^{n-1}} |\langle x, u \rangle| \, d\nu(u)$ and the fact $\lambda = 1/\|x\|_{C_1(i)}$ we get:

$$\left\| \frac{x}{c_n} \right\|_{C_1(i)/1-\delta} - \frac{c_n}{M^\star_K} \leq \frac{c_n}{M^\star_K} \left( \frac{\int_{p_\lambda(Y_\lambda)} |\langle x, u \rangle| \, d\nu(u)}{c_n} + C\lambda \frac{\mu(p_\lambda(Y_\lambda))}{c_n} \right).$$

We observe now that for $u \in p_\lambda(Y_\lambda)$, $|\langle x, u \rangle| \leq \text{length}(\gamma)/2 \leq \|W\|\lambda$. Using this in the last inequality and the fact that $p_\lambda(Y_\lambda)$ is a “band” around an equator of $S^{n-1}$ of width at most $\text{length}(\gamma)/2$ we get,

$$\left\| \frac{x}{c_n} \right\|_{C_1(i)/1-\delta} \leq \frac{c_n}{M^\star_K} Cn^2 \lambda^2$$

$$\leq \frac{c_n}{M^\star_K} Cn \frac{(1 - \delta)^2}{\|x\|^2_{C_1(i)/1-\delta}}.$$

Our final task is to get rid of the norm that appears on the right side of the latter inequality. Set

$$T = \frac{\|x\|_{C_1(\delta)/1-\delta}}{c_n/M^\star_K}.$$

We have shown that

$$T^2 |T - 1| \leq C \frac{M^\star_K}{c_n} n (1 - \delta)^2.$$

If $T \geq 1$ then we can just drop the factor $T^2$ and we are done. If $T < 1$ we write $T^2 |T - 1|$ as $(1 - (1 - T))^2 (1 - T)$ and we consider the function

$$f(x) = (1 - x)^2 x : (-\infty, 1) \to \mathbb{R}.$$
This function is strictly increasing thus invertible on its range, that is, \( f^{-1} \) is well defined and increasing in \((-\infty, \frac{4}{27})\). Consequently if

\[
(2.1.3) \quad C \frac{M^*_K}{c_n} n(1-\delta)^2 \leq \frac{4}{27},
\]

we conclude that,

\[
0 \leq 1 - T \leq f^{-1}\left(C \frac{M^*_K}{c_n} n(1-\delta)^2\right) \leq C \frac{M^*_K}{c_n} n(1-\delta)^2.
\]

The last inequality is true since the derivative of \( f^{-1} \) at zero is 1. Observe also that the convergence is “essentially realized” after (2.1.3) is satisfied. \( \square \)

We now proceed to show that some smoothness conditions on the boundary of \( K \) are necessary, by proving that the limiting \( M^* \)-convolution body of the \( n \)-dimensional cube is homothetic to the unit ball of \( \ell^n \). In fact we show that the \( \delta-M^* \)-convolution body of the cube is already homothetic to the unit ball of \( \ell_n^1 \).

**Example 2.3.** Let \( P = [-1,1]^n \). Then for \( 0 < \delta < 1 \) we have,

\[
C_1(P) = \frac{C_1(\delta; P, P)}{1-\delta} = n^{3/2} \text{vol}_{n-1}(S^{n-1}) B_{\ell^n_1}.
\]

**Proof.** Let \( x = \sum_{j=1}^{n} x_j e_j \) where \( x_j \geq 0 \) for all \( j = 1, 2, \ldots, n \) and \( e_j \) is the standard basis of \( \mathbb{R}^n \). Let \( \lambda > 0 \) be such that \( \lambda x \in bd(C_1(\delta)) \). Then,

\[
P \cap (\lambda x + P) = \{ y \in \mathbb{R}^n : y = \sum_{j=1}^{n} y_j e_j, -1 + \lambda x_j \leq y_j \leq 1, j = 1, 2, \ldots, n \}.
\]

The vertices of \( P_\lambda = P \cap (\lambda x + P) \) are the points \( \sum_{j=1}^{n} \alpha_j e_j \) where \( \alpha_j \) is either 1 or \( -1 + \lambda x_j \) for all \( j \). Without loss of generality we can assume that \(-1 + \lambda x_j < 0 \) for all the indices \( j \). Put \( \text{sign} \alpha_j = \alpha_j / |\alpha_j| \) when \( \alpha_j \neq 0 \) and \( \text{sign} 0 = 0 \). Fix a sequence of \( \alpha_j \)'s so that the point \( v = \sum_{j=1}^{n} \alpha_j e_j \) is a vertex of \( P_\lambda \). Clearly,

\[
N(P_\lambda, v) = N\left(P, \sum_{j=1}^{n} (\text{sign} \alpha_j) e_j \right).
\]

If \( u \in S^{n-1} \cap N(P_\lambda, v) \) then,

\[
h_\lambda(u) = h(u) - \left| \sum_{j=1}^{n} (\alpha_j - \text{sign} \alpha_j) e_j, u \right|.
\]

If \( \text{sign} \alpha_j = 1 \) then \( \alpha_j - \text{sign} \alpha_j = 0 \) otherwise \( \alpha_j - \text{sign} \alpha_j = \lambda x \).

Let \( A \subseteq \{1, 2, \ldots, n\} \). Consider the “\( A \)-orthant”:

\[
\mathcal{Q}_A = \{ y \in \mathbb{R}^n : \langle y, e_j \rangle < 0, \text{ if } j \in A \text{ and } \langle y, e_j \rangle > 0 \text{ if } j \notin A \}.
\]
Then $O_{A} = N \left( P, \sum_{j=1}^{n} (\text{sign} \, \alpha_j) e_j \right)$ if and only if $\text{sign} \, \alpha_j = 1$ exactly for every $j \notin A$. Hence,

$$h_{\lambda}(u) = h(u) - \left| \sum_{j \in A} \lambda x_j e_j, u \right|,$$

for all $u \in O_{A} \cap S^{n-1}$. Hence using the facts $M_{P,\lambda}^{*} = \delta M_{P}^{*}$ and $\lambda = 1/\|x\|_{C_{1}(\delta)}$ we get,

$$\|x\|_{C_{1}(\delta)} - \delta = \frac{1}{M_{P}^{*}} \sum_{A \subseteq \{1,2,\ldots,n\}} \sum_{j \in A} x_j \int_{O_{A} \cap S^{n-1}} \langle e_j, u \rangle d\nu(u),$$

which gives the result since

$$\int_{O_{A} \cap S^{n-1}} \langle e_j, u \rangle d\nu(u) = \frac{1}{2^{n-1}} \int_{S^{n-1}} |\langle e_1, u \rangle| d\nu(u).$$

□

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