Existence of Solutions to a Phase-Field Model of 3D Grain Boundary Motion Governed by a Regularized 1-Harmonic Type Flow

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Abstract
In this paper, we propose a quaternion formulation for the orientation variable in the three-dimensional Kobayashi–Warren model for the dynamics of polycrystals. We obtain existence of solutions to the $L^2$-gradient descent flow of the constrained energy functional via several approximating problems. In particular, we use a Ginzburg–Landau-type approach and some extra regularizations. Existence of solutions to the approximating problems is shown by the use of nonlinear semigroups. Coupled with good a priori estimates, this leads to successive passages to the limit up to finally showing existence of solutions to the proposed model. Moreover, we also obtain an invariance principle for the orientation variable.

Keywords Parabolic system · Grain boundary motion · Orientations · Total variation

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1 Introduction

In Kobayashi and Warren (2005) and Pusztai et al. (2005, 2006), two very similar models for the dynamics of polycrystals in the three-dimensional space were introduced. The authors generalized the existing two-dimensional model by Kobayashi et al. (2000, 2000) to the case of 3D crystals. The 2D Kobayashi–Warren–Carter model (KWC) is one of the best-known diffuse interface models of grain boundary motion for polycrystals. The main features of KWC model are the following ones: First of all, it is frame independent. Secondly, it is a dual phase-field model; opposed to the multiphase case, in which there is an order parameter for each grain, it uses only two parameters. On the other hand, it has some drawbacks. Among the main ones, the fact that it is restricted to two-dimensional polycrystals and that only isotropic grain boundary energies can be modeled (see Admal et al. 2019 for a generalization to the three-dimensional case including anisotropies).

In essence, it consists in the $L^2$-gradient descent flow of the following energy functional:

$$[\eta, \theta] \in \{H^1(\Omega)\}^2 \mapsto \frac{1}{2} \int_{\Omega} |\nabla \eta|^2 \, dx + \int_{\Omega} G(\eta) \, dx + \int_{\Omega} \alpha(\eta)|\nabla \theta| \, dx + \frac{\kappa^2}{2} \int_{\Omega} |\nabla \theta|^2 \, dx \in [0, \infty].$$

In the model, $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$ is a bounded domain with a Lipschitz boundary $\Gamma$. The unknowns $\eta = \eta(t, x)$ and $\theta = \theta(t, x)$ represent, respectively, the “orientation order” and the “orientation angle” in a polycrystal. $\eta = 1$ corresponds to a completely ordered state, while $\eta = 0$ corresponds to the state where no meaningful value of mean orientation exists. $\alpha$ is a nonnegative function corresponding to the spatial mobility of grain boundaries, while $G$ is a single well potential ensuring that only the ordered state $\eta = 1$ is stable.

In order to generalize the model, one needs to consider orientations in 3D and misorientations, since the term $|\nabla \theta|$ represents the misorientation on a short scale. In 3D, orientations are elements of $SO(3)$, the special orthogonal group in $\mathbb{R}^3$. In Kobayashi and Warren (2005), the term $|\nabla \theta|$ is substituted by the corresponding Euclidean norm in $\mathbb{R}^9$; i.e., $||\nabla P||_{\mathbb{R}^9} := \left(\sum_{i,j=1}^{3} |\nabla p_{i,j}|^2\right)^{\frac{1}{2}}$ for $P = [p_{i,j}]_{i,j} \in SO(3)$. Then, one has to compute the gradient descent flow for the constrained energy, thus ensuring that the solutions for the orientation variable still belong to $SO(3)$.

In Pusztai et al. (2005, 2006), instead, a quaternion representation is used for $SO(3)$. Since quaternions can be identified as elements in the unit sphere in $\mathbb{R}^4$, i.e., $S^3$, the authors replaced the term $|\nabla \theta|$ by the Euclidean norm of the gradient of the quaternion: i.e., $|\nabla q| := \left(\sum_{i=0}^{3} |\nabla q^i|^2\right)^{\frac{1}{2}}$, for $q = (q^0, q^1, q^2, q^3) \in S^3$. 

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We take the point of view of Pusztai et al. (2005), Pusztai et al. (2006), and we consider the following energy functional, constrained to functions with values in the unit sphere of \( \mathbb{R}^M \) with \( 1 < M \in \mathbb{N} \), i.e., \( S^{M-1} \):

\[
[\eta, \mathbf{u}] \in L^2(\Omega) \times L^2(\Omega; \mathbb{R}^M) \mapsto \mathcal{F}(\eta, \mathbf{u})
\]

\[
:= \begin{cases} \\
\frac{1}{2} \int_\Omega |\nabla \eta|^2 \, dx + \int_\Omega G(\eta) \, dx + \int_\Omega \alpha(\eta) |\nabla \mathbf{u}| \, dx + \frac{\kappa^2}{2} \int_\Omega |\nabla \mathbf{u}|^2 \, dx, \\
\infty, & \text{if } \eta \in H^1(\Omega) \text{ and } \mathbf{u} \in H^1(\Omega; S^{M-1}),
\end{cases}
\]

subject to the initial and boundary conditions

\[
\nabla \eta \cdot \mathbf{n}_\Gamma = 0, \quad \left( \alpha(\eta) \frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|} + \kappa^2 \nabla \mathbf{u} \right) \mathbf{n}_\Gamma = 0 \quad \text{on} \quad (0, T) \times \partial \Omega,
\]

\[
\eta(0, x) = \eta_0(x), \quad \mathbf{u}(0, x) = \mathbf{u}_0(x), \quad x \in \Omega,
\]

where \( g(\eta) = \frac{d}{d\eta} G(\eta) \). We point out that the writing is purely formal since there are some undefined terms such as \( \frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|} \). The precise meaning of a solution to the system is given in Sect. 3.

A natural way to study this type of restricted functionals in the sphere is to relax the constraint via a Ginzburg–Landau approximation (see Bethuel et al. 1994); i.e., instead of considering \( \mathbf{u} \in H^1(\Omega; S^{M-1}) \), one lets \( \mathbf{u} \in H^1(\Omega; \mathbb{R}^M) \) and adds the following term to \( \mathcal{F} \):

\[
\frac{1}{4\delta} \int_\Omega (|\mathbf{u}|^2 - 1)^2 \, dx, \quad \text{with} \quad \delta > 0.
\]

After obtaining well posedness to the gradient descent flow, then the strategy is to let \( \delta \to 0^+ \) and to show convergence of a subsequence to the corresponding solution to the system (P).
We follow exactly this strategy, but, due to the non-differentiability of the Euclidean norm at the origin, we also need to approximate it by a sufficient smooth term; i.e., we replace

$$\int_{\Omega} \alpha(\eta)|\nabla u| \, dx \mapsto \int_{\Omega} \alpha(\eta)\sqrt{\varepsilon^2 + |\nabla u|^2} \, dx, \quad \text{for } \varepsilon > 0.$$ 

For a possible future study, we need to perform an extra approximation. One might study the limit problem when $\kappa \to 0^+$ in (1.1), i.e., the case in which $u \in BV(\Omega; SO(3))$. In this case, the term in the energy corresponding to $\int_{\Omega} \alpha(\eta)|\nabla u|$ needs to be replaced by its relaxed functional. This relaxed functional has a jump part that strongly depends on the metric considered in $SO(3)$ (see Giaquinta and Mucci 2008). By the considerations stated in Appendix B, and in order to uniquely identify a rotation as an element in $\mathbb{S}^3$, we need to restrict the solutions in the quaternion representation to lie in the open upper hemisphere $\mathbb{S}_+^3 := \{ p = (p_1, p_2, p_3, p_4) \in \mathbb{S}^3 : p_1 > 0 \}$.

For the sake of generality, we will consider the more general setting $u \in \mathbb{S}^{M-1}$ instead of $\mathbb{S}^3$. In Appendix B, we give an invariance principle which ensures that, if the initial datum is in a certain compact subset of $\mathbb{S}^{M-1}$, then the solution also does so. For the proof of this result, we need a technical restriction; namely, continuity of the solutions. Therefore, we need to perform an extra approximation to $F$, by adding the following term to the energy functional:

$$\frac{1}{N+1} \int_{\Omega} |v \nabla u|^{N+1} \, dx.$$ 

We stress that this extra regularization is only a technical tool to prove the invariance principle in Theorem 6 and it is not needed for any of the rest of the results in the present manuscript.

The plan of the paper is the following one:

First of all, in Sect. 2, we prescribe some notations, and recall some results about multi-vectors that are used in the paper. In Sect. 3, we set up our main assumptions and we state the Main Theorem, as the principal result of this paper. In Sect. 4, we consider the complete energy functional, which we call the free energy:

$$F_{\varepsilon, \nu, \delta}(\eta, u) := \begin{cases} 
\frac{1}{2} \int_{\Omega} |\nabla \eta|^2 \, dx + \int_{\Omega} G(\eta) \, dx + \int_{\Omega} \alpha(\eta)\sqrt{\varepsilon^2 + |\nabla u|^2} \, dx \\
+ \frac{1}{N+1} \int_{\Omega} |v \nabla u|^{N+1} \, dx + \frac{\kappa^2}{2} \int_{\Omega} |\nabla u|^2 \, dx \\
+ \frac{1}{4\delta} \int_{\Omega} (|u|^2 - 1)^2 \, dx, \\
\text{if } \eta \in H^1(\Omega) \text{ and } u \in H^1(\Omega; \mathbb{R}^M), \\
+\infty, \quad \text{otherwise.}
\end{cases}$$

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We define our notion of solution to its $L^2$- gradient descent flow, i.e., to the system denoted by $(P)^{\kappa}_{\epsilon, \nu, \delta}$. Then, we prove with the help of an auxiliary convex energy functional, that the system $(P)^{\kappa}_{\epsilon, \nu, \delta}$ admits a unique solution for sufficiently smooth initial data (see Theorem 2). Moreover, stability with respect to the parameter $\nu$ is also obtained.

Section 5 is devoted to the proof of Main Theorem, i.e., the proof of existence of solution to the $L^2$-gradient descent flow of $F$. First of all, an energy inequality and the corresponding uniform estimates (in the parameters $\epsilon$ and $\delta$) are obtained for solutions to $(P)^{\kappa}_{\epsilon, \nu, \delta}$. They lead to convergence, first with $\delta \to 0^+$ and up to subsequences, to a solution to the system $(P)^{\kappa}_{\epsilon, \nu}$, i.e., to the gradient descent flow of the restricted energy functional:

$$
F_{\epsilon, \nu}(\eta, u) := \begin{cases} 
\frac{1}{2} \int_{\Omega} |\nabla \eta|^2 \, dx + \int_{\Omega} G(\eta) \, dx + \int_{\Omega} \alpha(\eta) \sqrt{\epsilon^2 + |\nabla u|^2} \, dx \\
+ \frac{1}{N+1} \int_{\Omega} |\nu \nabla u|^{N+1} \, dx + \frac{\kappa^2}{2} \int_{\Omega} |\nabla u|^2 \, dx, \\
\text{if } \eta \in H^1(\Omega) \text{ and } u \in H^1(\Omega; S^{M-1}), \\
+\infty, \text{ otherwise.}
\end{cases}
$$

Moreover, solutions are shown to be continuous. Therefore, the invariance principle stated and proved in Appendix B applies, and we can proceed by letting $\nu \to 0^+$. Then, we obtain existence of solutions to the system $(P)_{\epsilon}$, i.e., to the gradient descent flow of $F_{\epsilon} := F_{\epsilon, 0}$. The final step is to let $\epsilon \to 0^+$, thus obtaining existence of solutions to the gradient descent flow of $F$, i.e., to $(P)$. We point out that since the successive convergences for the orientation variable (with $\nu \to 0^+$ and $\epsilon \to 0^+$) also hold a.e. in space time, it holds that the final solutions also satisfy the invariance principle.

Finally, we added two appendices. In the first one, we recall the concept of Mosco convergence and some results related to it that we use in the paper. The second one is devoted to the discussion about the relationship between rotations in $SO(3)$ and their representation as quaternions. It is there where we prove the invariance principle.

2 Preliminaries and Assumptions

2.1 Abstract Notations

For an abstract Banach space $X$, we denote by $\| \cdot \|_X$ the norm of $X$, and by $(\cdot, \cdot)_X$ the duality pairing between $X$ and its dual $X'$. In particular, when $X$ is a Hilbert space, we denote by $(\cdot, \cdot)_X$ the inner product of $X$. Moreover, when there is no possibility of confusion, we uniformly denote by $| \cdot |$ the Euclidean norm in $\mathbb{R}^d$, for any dimension $d \in \mathbb{N}$. We also write the inner product (scalar product) of $\mathbb{R}^d$, as follows:

$$
y \cdot \tilde{y} = \sum_{i=1}^{d} y_i \tilde{y}_i, \text{ for all } y = [y_1, \ldots, y_d], \quad \tilde{y} = [\tilde{y}_1, \ldots, \tilde{y}_d] \in \mathbb{R}^d.
$$
For any subset $A$ of a Banach space $X$, let $\chi_A : X \rightarrow \{0, 1\}$ be the characteristic function of $A$, i.e.,

$$
\chi_A : w \in X \mapsto \chi_A(w) := \begin{cases} 
1, & \text{if } w \in A, \\
0, & \text{otherwise}.
\end{cases}
$$

For Banach spaces $X_1, \ldots, X_d$, with $1 < d \in \mathbb{N}$, $X_1 \times \cdots \times X_d$ is the product Banach space endowed with the norm $||x_1 \times \cdots \times x_d|| := ||x_1|| + \cdots + ||x_d||$. However, in the case that all $X_1, \ldots, X_d$ are Hilbert spaces, $X_1 \times \cdots \times X_d$ denotes the product Hilbert space endowed with the inner product $(\cdot, \cdot)_{X_1 \times \cdots \times X_d} := (\cdot, \cdot)_{X_1} + \cdots + (\cdot, \cdot)_{X_d}$ and the norm $||x_1 \times \cdots \times x_d|| := \left( ||x_1||^2 + \cdots + ||x_d||^2 \right)^{\frac{1}{2}}$. In particular, when all $X_1, \ldots, X_d$ coincide with a Banach space $Y$, we write:

$$
[Y]^d := \underbrace{Y \times \cdots \times Y}_{d \text{ times}}.
$$

For a proper, lower semi-continuous (l.s.c.), and convex function $\Psi : X \rightarrow (-\infty, \infty]$ on a Hilbert space $X$, we denote by $D(\Psi)$ the effective domain of $\Psi$. Also, we denote by $\partial \Psi$ the subdifferential of $\Psi$. The subdifferential $\partial \Psi$ corresponds to a weak differential of $\Psi$, and it is a maximal monotone graph in the product space $X \times X$. The set $D(\partial \Psi) := \{z \in X \mid \partial \Psi(z) \neq 0\}$ is called the domain of $\partial \Psi$. We often use the notation $[[w_0, w_0^*]] \in \partial \Psi$ in $X \times X$,” to mean that “$w_0^* \in \partial \Psi(w_0)$ in $X$ for $w_0 \in D(\partial \Psi)$,” by identifying the operator $\partial \Psi$ with its graph in $X \times X$.

Next, for Hilbert spaces $X_1, \ldots, X_d$, with $1 < d \in \mathbb{N}$, and given a proper, l.s.c., and convex function on the product space $X_1 \times \cdots \times X_d$ $\widehat{\Psi} : w = [w_1, \ldots, w_d] \in X_1 \times \cdots \times X_d \mapsto \widehat{\Psi}(w) = \widehat{\Psi}(w_1, \ldots, w_d) \in (-\infty, \infty]$, for any $i \in \{1, \ldots, d\}$, we denote by $\partial_{w_i} \widehat{\Psi} : X_1 \times \cdots \times X_d \rightarrow X_i$ a set-valued operator, which maps any $w = [w_1, \ldots, w_i, \ldots, w_d] \in X_1 \times \cdots \times X_i \times \cdots \times X_d$ to a subset $\partial_{w_i} \widehat{\Psi}(w) \subset X_i$, as follows:

$$
\partial_{w_i} \widehat{\Psi}(w) = \partial_{w_i} \widehat{\Psi}(w_1, \ldots, w_i, \ldots, w_d) := \left\{ \tilde{w}_i^* \in X_i \left| \begin{array}{c}
\tilde{w}_i^* - w_i \in \partial \Psi(w_1, \ldots, \tilde{w}_i, \ldots, w_d) \\
\tilde{w}_i^* \leq \widehat{\Psi}(w_1, \ldots, \tilde{w}_i, \ldots, w_d), \text{ for any } \tilde{w}_i \in X_i
\end{array} \right. \right\}.
$$

As is easily checked,

$$
\partial \widehat{\Psi} \subset \left[ \partial_{w_1} \widehat{\Psi} \times \cdots \times \partial_{w_d} \widehat{\Psi} \right] \text{ in } [X_1 \times \cdots \times X_d]^2, \quad (2.1)
$$

where $[\partial_{w_1} \widehat{\Psi} \times \cdots \times \partial_{w_d} \widehat{\Psi}] : X_1 \times \cdots \times X_d \rightarrow 2^{X_1 \times \cdots \times X_d}$ is a set-valued operator, defined as:

$$
[\partial_{w_1} \widehat{\Psi} \times \cdots \times \partial_{w_d} \widehat{\Psi}](w) := \partial_{w_1} \widehat{\Psi}(w) \times \cdots \times \partial_{w_d} \widehat{\Psi}(w) \text{ in } X_1 \times \cdots \times X_d,
$$

for any $w = [w_1, \ldots, w_d] \in D[\partial_{w_1} \widehat{\Psi} \times \cdots \times \partial_{w_d} \widehat{\Psi}] := \bigcap_{i=1}^{d} D(\partial_{w_i} \widehat{\Psi})$.
It should be noted that the converse inclusion in (2.1) does not hold, in general.

2.2 Multi-vectors

Here, we recall some definitions and basic properties about multi-vectors that we need in our analysis. We refer to, e.g., Federer (1969, Chapter 1) and Darling (1994, Chapter 1) for details.

Let \( m \in \mathbb{N} \). The spaces \( \Lambda_0(\mathbb{R}^m) \) and \( \Lambda_1(\mathbb{R}^m) \) are defined as

\[
\Lambda_0(\mathbb{R}^m) := \mathbb{R} \quad \text{and} \quad \Lambda_1(\mathbb{R}^m) := \mathbb{R}^m,
\]

respectively.

For any integer \( 2 \leq k \leq m \), the \( k \)th exterior power of \( \mathbb{R}^m \), denoted by \( \Lambda_k(\mathbb{R}^m) \), is defined as a set spanned by generators, i.e., elements of the form

\[
\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k, \quad \mathbf{u}_i \in \mathbb{R}^m, \quad i = 1, \ldots, k.
\]

Generators are subject to the following rules:

\[
(a \mathbf{v} + b \mathbf{w}) \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k = a(\mathbf{v} \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k) + b(\mathbf{w} \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k);
\]

\[
\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \ \text{changes sign if two entries are transposed};
\]

for any basis \( \{ \mathbf{e}_1, \ldots, \mathbf{e}_m \} \) of \( \mathbb{R}^m \), the set

\[
\{ \mathbf{e}_\alpha := \mathbf{e}_{\alpha_1} \wedge \cdots \wedge \mathbf{e}_{\alpha_k} | \alpha = [\alpha_1, \ldots, \alpha_k] \in I(k, m) \}
\]

forms the basis of \( \Lambda_k(\mathbb{R}^m) \), where

\[
I(k, m) := \{ \alpha = [\alpha_1, \ldots, \alpha_k] \in \mathbb{Z}_+^k | 1 \leq \alpha_1 < \cdots < \alpha_k \leq m \}.
\]

The elements of \( \Lambda_k(\mathbb{R}^m) \) are called multi-vectors (or \( k \)-vectors), and \( \Lambda_k(\mathbb{R}^m) \) is a vector space of dimension \( \binom{m}{k} \). Given \( k, \ell \in \{0, \ldots, m\} \) with \( k + \ell \leq m \), there exists a unique bilinear map \( (\lambda, \mu) \rightarrow \lambda \wedge \mu \) from \( \Lambda_k(\mathbb{R}^m) \times \Lambda_\ell(\mathbb{R}^m) \) to \( \Lambda_{k+\ell}(\mathbb{R}^m) \), whose effect on generators is

\[
(\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k) \wedge (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_\ell) = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \wedge \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_\ell.
\]

Such a map satisfies

\[
\lambda \wedge \mu = (-1)^{-k\ell} (\mu \wedge \lambda) \quad \text{for} \quad \lambda \in \Lambda_k(\mathbb{R}^m), \quad \mu \in \Lambda_\ell(\mathbb{R}^m).
\]

(2.3)

There is an isomorphism between \( \Lambda_k(\mathbb{R}^m) \) and \( \Lambda_{m-k}(\mathbb{R}^m) \), called the Hodge-star operator:

\[
*(\cdot) : \Lambda_k(\mathbb{R}^m) \ni \lambda \rightarrow *\lambda \in \Lambda_{m-k}(\mathbb{R}^m),
\]
which is defined on the basis as
\[
*(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}) := e_{\alpha_{k+1}} \wedge \cdots \wedge e_{\alpha_m},
\]
for all permutations \([\alpha_1, \ldots, \alpha_m]\) of \([1, \ldots, m]\), having positive signature. \(2.4\)

In particular, in what follows we will systematically identify \(\Lambda_{m-1}(\mathbb{R}^m)\) with \(\mathbb{R}^m\) and \(\Lambda_m(\mathbb{R}^m)\) with \(\mathbb{R}\). We will use the following well-known formulas (see, e.g., Darling 1994, (1.64); Darling 1994, Table 1.2):
\[
*(*)\lambda = (-1)^{k(m-k)} \lambda \quad \text{for all} \quad \lambda \in \Lambda_k(\mathbb{R}^m), \quad (2.5)
\]
and
\[
a \wedge *(b \wedge c) = (a \cdot c) * b - (a \cdot b) * c \quad \text{in} \quad \mathbb{R}^m (= \Lambda_{m-1}(\mathbb{R}^m)), \quad (2.6)
\]
for all \(a, b, c \in \mathbb{R}^m (= \Lambda_1(\mathbb{R}^m))\). It follows from \(2.3\), \(2.6\), and \(2.5\) that
\[
|b|^2 a = (a \cdot b)b - *(a \wedge b) \wedge b \quad \text{in} \quad \mathbb{R}^m (= \Lambda_1(\mathbb{R}^m)), \quad (2.7)
\]
for all \(a, b \in \mathbb{R}^m (= \Lambda_1(\mathbb{R}^m))\). We next introduce the inner product on \(\Lambda_k(\mathbb{R}^m)\).
Given two generators \(\lambda = \lambda_1 \wedge \ldots \wedge \lambda_k, \mu = \mu_1 \wedge \ldots \wedge \mu_k\), we define
\[
\langle \lambda, \mu \rangle_k := \det \left( \langle \lambda_i, \mu_j \rangle_{i,j=1}^k \right). \quad (2.8)
\]
The inner product on \(\Lambda_k(\mathbb{R}^m)\) is an extension by linearity of this definition. Therefore, we easily see that
\[
\langle \lambda, \mu \rangle_k = \lambda \wedge *\mu. \quad (2.9)
\]
Moreover,
\[
|\lambda|_k := \langle \lambda, \lambda \rangle_k^{\frac{1}{2}} = \left( \sum_{\alpha \in I(k,m)} |\lambda_\alpha|^2 \right)^{\frac{1}{2}}, \quad \text{where} \quad \lambda = \sum_{\alpha \in I(k,m)} \lambda_\alpha e_\alpha. \quad (2.10)
\]
and using \(2.4\), it is immediate to see that
\[
|\ast \lambda|_{m-k} = |\lambda|_k \quad \text{for any} \quad \lambda \in \Lambda_k(\mathbb{R}^m). \quad (2.11)
\]
Finally, we recall that, given \(\lambda \in \Lambda_k(\mathbb{R}^m)\) and \(\eta \in \Lambda_\ell(\mathbb{R}^m)\) such that one of them is a generator, then (see Federer 1969, p. 32)
\[
|\lambda \wedge \eta|_{k+\ell} \leq |\lambda|_k |\eta|_\ell. \quad (2.12)
\]
2.3 Vector Valued Functions

Let $X$ be a Banach space with dual $X'$ and let $V \subset \mathbb{R}^d$ be a bounded open set endowed with the Lebesgue measure $\mathcal{L}^d$. A function $u : V \to X$ is called simple if there exist $x_1, \ldots, x_n \in X$ and $V_1, \ldots, V_n \in \mathcal{L}^m$-measurable subsets of $V$ such that $u = \sum_{i=1}^n x_i 1_{V_i}$. The function $u$ is called strongly measurable if there exists a sequence of simple functions $\{u_n\}$ such that $\|u_n(x) - u(x)\|_X \to 0$ as $n \to +\infty$ for almost all $x \in V$. If $1 \leq p < \infty$, then $L^p(V; X)$ stands for the space of (equivalence classes of) strongly measurable functions $u : V \to X$ with

$$\|u\|_{L^p(V; X)} := \left( \int_V \|u(x)\|_X^p \, dx \right)^{\frac{1}{p}} < \infty.$$  

Endowed with this norm, $L^p(V; X)$ is a Banach space. For $p = \infty$, the symbol $L^\infty(V; X)$ stands for the space of (equivalence classes of) strongly measurable functions $u : V \to X$ such that

$$\|u\|_{L^\infty(V; X)} := \operatorname{esssup}\{\|u(x)\|_X : x \in V\} < \infty.$$  

If $V = (0, T)$ with $0 < T \leq \infty$, we write $L^p(0, T; X) = L^p((0, T); X)$. For $1 \leq p < \infty$, $L^p(0, T; X') (\frac{1}{p} + \frac{1}{p'} = 1)$ is isometric to a subspace of $(L^p(0, T; X))'$, with equality if and only if $X'$ has the Radon–Nikodým property (see, for instance, Diestel and Uhl 1977).

We consider the vector space $D(V; X) := C_0^\infty(V; X)$, endowed with the topology for which a sequence $\varphi_n \to 0$ as $n \to +\infty$ if there exists $K \subset V$ compact such that $\operatorname{supp}(\varphi_n) \subset K$ for any $n \in \mathbb{N}$ and $D^\alpha \varphi_n \to 0$ uniformly on $K$ as $n \to +\infty$ for all multi-index $\alpha$. We denote by $D'(V; X)$ the space of distributions on $V$ with values in $X$; that is, the set of all linear continuous maps $T : D(V; X) \to \mathbb{R}$. As is well known, $L^p(V; X) \subset D'(V; X)$ through the standard continuous injection. Given $T \in D'(V; X)$, the distributional derivative of $T$ is defined by

$$\langle D_i T, \varphi \rangle := -\langle T, \partial_i \varphi \rangle \quad \text{for any} \quad \varphi \in D(V; X) \quad \text{and any} \quad i \in \{1, \ldots, d\}.$$  

(2.13)

General notations for matrices. Let $d, m \in \mathbb{N}$. If $A = [a^\ell_k] = [a^\ell_k]_{1 \leq \ell \leq m, 1 \leq k \leq d} \in \mathbb{R}^{md}$ is an $m \times d$ matrix, we write

$$a^\ell = [a^\ell_1, \ldots, a^\ell_d] \in \mathbb{R}^d \quad \text{for} \quad \ell = 1, \ldots, m,$$

$$a^k = [a^k_1, \ldots, a^k_m] \in \mathbb{R}^m \quad \text{for} \quad k = 1, \ldots, d.$$  

If $B = [b^\ell_k] = [b^\ell_k]_{1 \leq \ell \leq m, 1 \leq k \leq d} \in \mathbb{R}^{md}$ is also an $m \times d$ matrix, we let

$$A : B = \sum_{\ell=1}^m \sum_{k=1}^d a^\ell_k b^\ell_k \quad \text{and} \quad |A| = (A : A)^{\frac{1}{2}} = \left( \sum_{\ell=1}^m \sum_{k=1}^d (a^\ell_k)^2 \right)^{\frac{1}{2}}.$$
Given $A = \left[a_1, \ldots, a_m\right] \in \mathbb{R}^{d \times m}$ with $a_i \in \mathbb{R}^d$, $i = 1, \ldots, m$, and $b \in \mathbb{R}^d$, we let

$$A \land b := (a_1 \land b, \ldots, a_m \land b),$$

$$\ast(A \land b) := (\ast(a_1 \land b), \ldots, \ast(a_m \land b)).$$

### 2.4 Multi-vector Fields

Let $d, m \in \mathbb{N}$. Let $V \subset \mathbb{R}^d$ be a bounded open set. A multi-vector distribution in $U$ is a linear continuous map $\lambda \in \mathcal{D}^\prime(U; \Lambda_k(\mathbb{R}^m))$ (see §2.3). It may be expressed in terms of the basis (2.2) as

$$\lambda = \sum_{\alpha \in I(k, m)} \lambda_\alpha e_\alpha, \text{ with } \lambda_\alpha \in \mathcal{D}^\prime(V; \mathbb{R}^m) \text{ for any } \alpha \in I(k, m).$$

Thus, according to (2.13),

$$D_i \lambda = \sum_{\alpha \in I(k, m)} D_i \lambda_\alpha e_\alpha \text{ for any } i \in \{1, \ldots, d\}. \quad (2.14)$$

From (2.14), the following two identities are easily seen to hold for $k, \ell \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$:

$$D_i (\lambda \land \eta) = D_i \lambda \land \eta + \lambda \land D_i \eta \quad (2.15)$$

for any $\lambda \in L^2(V; \Lambda_k(\mathbb{R}^m))$ such that $D_i \lambda \in L^2(V; \Lambda_k(\mathbb{R}^m))$ and any $\eta \in L^2(V; \Lambda_\ell(\mathbb{R}^m))$ such that $D_i \eta \in L^2(V; \Lambda_\ell(\mathbb{R}^m))$;

$$\ast (D_i \lambda) = D_i (\ast \lambda) \text{ for any } \lambda \in \mathcal{D}^\prime(U; \Lambda_k(\mathbb{R}^m)). \quad (2.16)$$

For any $k \in \mathbb{N}$, $[\Lambda_k(\mathbb{R}^m)]^m$ is a Banach space with the norm

$$\|A\|_{[\Lambda_k(\mathbb{R}^m)]^m} := \left(\sum_{i=1}^m |A_i|_k^2\right)^{\frac{1}{2}} \text{ for } A = (A_1, \ldots, A_m)$$

with $|\cdot|_k$ given by (2.10).

### 3 Main Theorem

We begin with the assumptions that we use concerning system (P) and its approximating problems.

**(A0)** $1 < N \in \mathbb{N}$, $1 < M \in \mathbb{N}$, $\kappa > 0$ and $0 < T < \infty$. 
(A1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\Gamma := \partial \Omega$ and unit outer normal $\mathbf{n}$. We use the following notations:

$$
\Omega_T := (0, T) \times \Omega, \quad \Gamma_T := (0, T) \times \Gamma,
$$

$$
H := L^2(\Omega), \quad \mathcal{X} := L^2(\Omega; \mathbb{R}^M), \quad \mathcal{X} := H \times \mathcal{X}.
$$

$$
\Omega^1_T := (0, T) \times \partial \Omega_1, \quad \Gamma^1_T := (0, T) \times \partial \Omega^1.
$$

We use the following notations:

$$
\Omega^1_T := (0, T) \times \Omega^1, \quad \Gamma^1_T := (0, T) \times \Gamma^1,
$$

$$
H := H^1(\Omega), \quad \mathcal{X} := H^1(\Omega; \mathbb{R}^M), \quad \mathcal{X} := V \times \mathcal{W}.
$$

(A2) $g \in C^1(\mathbb{R})$ is a fixed Lipschitz function such that $g(0) \leq 0$ and $g(1) \geq 0$, with a potential $0 \leq G \in C^2(\mathbb{R})$, i.e., $G'(s) = \frac{d}{ds} G(s) = g(s)$ on $\mathbb{R}$.

(A3) $0 < \alpha \in C^2(\mathbb{R})$ is such that

- $\alpha'(0) = 0$, $\alpha'' \geq 0$ on $\mathbb{R}$, and $\alpha$ and $\alpha \alpha'$ are Lipschitz continuous on $\mathbb{R}$.

- $\alpha^* := \inf \alpha(\mathbb{R}) > 0$.

(A4) For any $\varepsilon \geq 0$, $f_\varepsilon : \mathbb{R}^{MN} \rightarrow [0, \infty)$ is a continuous convex function, defined as

$$
f_\varepsilon : W = [w^\ell_k]_{1 \leq \ell \leq M} \in \mathbb{R}^{MN} \mapsto f_\varepsilon(W) := \sqrt{\varepsilon^2 + |W|^2} \in \mathbb{R}.
$$

(A5) For any $\delta > 0$, $\Pi_\delta \in C^2(\mathbb{R}^M)$ is the following function:

$$
\Pi_\delta : w \in \mathbb{R}^M \mapsto \Pi_\delta(w) := \frac{1}{4\delta}(|w|^2 - 1)^2 \in \mathbb{R}.
$$

We let $\varpi_\delta \in C^1(\mathbb{R}^M; \mathbb{R}^M)$ be the gradient of $\Pi_\delta$, i.e.,

$$
\varpi_\delta : w \in \mathbb{R}^M \mapsto \varpi_\delta(w) := \nabla \Pi_\delta(w) = \frac{1}{\delta}(|w|^2 - 1) w \in \mathbb{R}^M.
$$

(A6) $U_0 := [\eta_0, u_0] \in \mathcal{W}$ is a fixed pair of initial data, with $0 \leq \eta_0 \leq 1$ a.e. in $\Omega$.

**Remark 1** The following setting:

$$
\eta(\eta) := \eta - 1, \quad G(\eta) := \frac{1}{2}(\eta - 1)^2, \quad \text{and} \quad \alpha(\eta) := \frac{\eta^2}{2} + \alpha^*, \quad \text{for all} \quad \eta \in \mathbb{R},
$$

provides a possible example of perturbation $g(\eta)$ and mobility $\alpha(\eta)$, fulfilling assumptions (A2) and (A3).

**Remark 2** From (A4), it immediately follows that $f_\varepsilon : \mathbb{R}^{MN} \rightarrow [0, \infty)$, $\varepsilon \geq 0$, are non-expansive over $\mathbb{R}^{MN}$. Also, if $\varepsilon > 0$, then $f_\varepsilon \in C^\infty(\mathbb{R}^{MN})$, and if $\varepsilon = 0$, then the corresponding function $f_0$ coincides with the (Euclidean) norm $\| \cdot \|_{\mathbb{R}^{MN}}$ on $\mathbb{R}^{MN}$. Additionally,

$$
\partial f_\varepsilon(W) = \begin{cases}
\frac{W}{\sqrt{\varepsilon^2 + |W|^2}} \left( = \left[ \nabla f_\varepsilon(W) \right] \right), & \text{if} \quad \varepsilon > 0, \\
\text{Sgn}^{M,N}(W), & \text{if} \quad \varepsilon = 0,
\end{cases}
$$
for all $W = [u_k^p]_{1 \leq k \leq M} \in \mathbb{R}^{MN}$, where $\text{Sgn}^{M,N} : \mathbb{R}^{MN} \rightarrow 2^{\mathbb{R}^{MN}}$ is the sign function on $\mathbb{R}^{MN}$, i.e.,

$$\text{Sgn}^{M,N}(W) := \begin{cases} \left\{ \frac{W}{|W|} \right\}, & \text{if } W \neq 0, \\ \{ \tilde{W} | W| \leq 1 \}, & \text{if } W = 0, \end{cases}$$

for all $W = [u_k^p]_{1 \leq k \leq M} \in \mathbb{R}^{MN}$.

Next, we define the notion of solution to our system (P).

**Definition 1** A pair of functions $U := [\eta, u] \in L^2(0, T; \mathcal{X})$ is called a solution to the system (P), if

$$U = [\eta, u] \in W^{1,2}(0, T; \mathcal{X}) \cap L^\infty(0, T; \mathcal{W}),$$

$$0 \leq \eta \leq 1 \text{ and } u \in S^{M-1}, \text{ a.e. in } \Omega_T,$$

$$\left( \partial_t \eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla u(t)|, \varphi \right)_H + \left( \nabla \eta(t), \nabla \varphi \right)_H = 0,$$  \hspace{1cm} (3.1)

for any $\varphi \in V$, a.e. $t \in (0, T)$, subject to $\eta(0) = \eta_0$ in $H$, and there exist functions $B^* \in L^\infty(\Omega_T; \mathbb{R}^{MN})$ and $\mu^* \in L^1(0, T; L^1(\Omega))$, such that a.e. in $\Omega_T$,

$$\begin{cases} B^* \in \text{Sgn}^{M,N}(\nabla u) \text{ in } \mathbb{R}^{MN}, \\ \mu^* := (\alpha(\eta)B^* + \kappa^2 \nabla u) : \nabla u, \end{cases}$$  \hspace{1cm} (3.3a)

and

$$\int_{\Omega} \partial_t u(t) \cdot \psi(t) \, dx + \int_{\Omega} \alpha(\eta(t))B^*(t) : \nabla \psi(t) \, dx = \int_{\Omega} \mu^*(t)u(t) \cdot \psi(t) \, dx,$$  \hspace{1cm} (3.3b)

for any $\psi \in C^1(\overline{\Omega}; \mathbb{R}^M)$, a.e. $t \in (0, T)$, subject to $u(0) = u_0$ in $\mathcal{X}$.

Now, we can state our main result.

**Main Theorem.** Let $U_0 = [\eta_0, u_0] \in \mathcal{W}$ with $u_0 \in S^{M-1}$ in $\Omega$. Then, the system (P) admits at least one solution $U = [\eta, u] \in L^2(0, T; \mathcal{X})$, such that

$$\mathcal{F}(U(s)) + \int_0^s \|\partial_t U(t)\|^2_{\mathcal{X}} \, dt \leq \mathcal{F}(U_0) \text{ for all } s \in (0, T),$$

where $\mathcal{F}$ is the free energy given in (1.1). Also, concerning the function $B^* \in L^\infty(\Omega_T; \mathbb{R}^{MN})$ in (3.3a), it holds that

$$\begin{cases} \text{div}(\alpha(\eta)B^* + \kappa^2 \nabla u) \in L^2(0, T; L^1(\Omega; \mathbb{R}^M)), \\ \text{div}(\alpha(\eta)B^* + \kappa^2 \nabla u) \wedge u \in L^2(0, T; L^2(\Omega; \Lambda_2(\mathbb{R}^M))) \end{cases}.$$
Moreover, if \( \mathbf{u}_0 \in \mathcal{S}^{M-1}_{+,r} \) in \( \Omega \) for some \( r \in (0, 1) \), then \( \mathbf{u} \in \mathcal{S}^{M-1}_{+,r} \) a.e. in \( \Omega_T \), where

\[
\mathcal{S}^{M-1}_{+,r} := \{ \mathbf{p} = (p_1, \ldots, p_M) \in \mathcal{S}^{M-1}_{+} \mid r \leq p_1 \}, \text{ for any } r \in (0, 1). \quad (3.4)
\]

## 4 Approximating Problems

In this section, we study the approximating problems to our system. Let us assume (A0)–(A6), and fix constants \( \epsilon \geq 0, \delta \geq 0, \) and \( \nu \geq 0 \). The approximating problems, denoted by \( (P)^{\kappa}_{\epsilon, \nu, \delta} \), are derived as gradient descent flows of a free energy, defined as

\[
\mathcal{F}^{\kappa}_{\epsilon, \nu, \delta} : U := [\eta, \mathbf{u}] \in \mathcal{X} \mapsto \mathcal{F}^{\kappa}_{\epsilon, \nu, \delta}(U) = \mathcal{F}^{\kappa}_{\epsilon, \nu, \delta}(\eta, \mathbf{u}) := \Psi_{0}(\eta) + \Psi^{\kappa}_{\epsilon, \nu, \delta}(U) = \Psi_{0}(\eta) + \Psi^{\kappa}_{\epsilon, \nu, \delta}(\eta, \mathbf{u}),
\]

with

\[
\Psi_{0} : \eta \in D(\Psi_{0}) := V \subset H \mapsto \Psi_{0}(\eta) := \frac{1}{2} \int_{\Omega} |\nabla \eta|^{2} \, dx + \int_{\Omega} G(\eta) \, dx,
\]

and

\[
\Psi^{\kappa}_{\epsilon, \nu, \delta} : U := [\eta, \mathbf{u}] \in D(\Psi^{\kappa}_{\epsilon, \nu, \delta}) := \left\{ [\tilde{\eta}, \tilde{\mathbf{u}}] \in \mathfrak{M} \biggm| \tilde{\mathbf{u}} \in L^{4}(\Omega; \mathbb{R}^{M}), \text{ and } \nu \tilde{\mathbf{u}} \in W^{1,N+1}(\Omega; \mathbb{R}^{M}) \right\}
\]

\[
\mapsto \Psi^{\kappa}_{\epsilon, \nu, \delta}(U) = \Psi^{\kappa}_{\epsilon, \nu, \delta}(\eta, \mathbf{u}) := \int_{\Omega} \alpha(\eta) f_{\epsilon}(\nabla \mathbf{u}) \, dx + \frac{\kappa^{2}}{2} \int_{\Omega} |\nabla \mathbf{u}|^{2} \, dx + \int_{\Omega} \Pi_{\delta}(\mathbf{u}) \, dx \in [0, \infty).
\]

More precisely,

**Problem \((P)^{\kappa}_{\epsilon, \nu, \delta}\):**

\[
\begin{align*}
\partial_{t} \eta - \Delta \eta + g(\eta) + \alpha'(\eta) f_{\epsilon}(\nabla \mathbf{u}) &= 0 \quad \text{in } \Omega_T, \\
\partial_{t} \mathbf{u} - \text{div}(\alpha(\eta) \partial f_{\epsilon}(\nabla \mathbf{u}) + \kappa^{2} \nabla \mathbf{u} + \nu |\nabla \mathbf{u}|^{N-1} \nabla \mathbf{u}) + \sigma(\mathbf{u}) &\equiv 0 \quad \text{in } \Omega_T,
\end{align*}
\]

subject to the initial and boundary conditions

\[
\begin{align*}
\nabla \eta \cdot \mathbf{n}_{\Gamma} &= 0, \quad (\alpha(\eta) \partial f_{\epsilon}(\nabla \mathbf{u}) + \kappa^{2} \nabla \mathbf{u} + \nu |\nabla \mathbf{u}|^{N-1} \nabla \mathbf{u}) \mathbf{n}_{\Gamma} &\equiv 0 \quad \text{on } \Gamma_T, \\
\eta(0, x) &= \eta_{0}(x), \quad x \in \Omega, \quad \mathbf{u}(0, x) = \mathbf{u}_{0}(x), \quad x \in \Omega.
\end{align*}
\]

**Definition 2** \( U := [\eta, \mathbf{u}] \in L^{2}(0, T; \mathcal{X}) \) is called a solution to the system \((P)^{\kappa}_{\epsilon, \nu, \delta}\), if

\[
U = [\eta, \mathbf{u}] \in W^{1,2}(0, T; \mathcal{X}) \cap L^{\infty}(0, T; \mathfrak{M}), \quad 0 \leq \eta \leq 1, \quad \text{a.e. in } \Omega_T;
\]

\[
(\partial_{t} \eta(t) + g(\eta(t)) + \alpha'(\eta(t)) f_{\epsilon}(\nabla \mathbf{u}(t)), \varphi)_{H} + (\nabla \eta(t), \nabla \varphi)_{H} = 0, \quad (4.1)
\]

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for any $\varphi \in V$, a.e. $t \in (0, T)$, subject to $\eta(0) = \eta_0$ in $H$, and there exists $B^* \in L^\infty(\Omega_T; \mathbb{R}^{MN})$, such that

$$B^* \in \partial f_\varepsilon(\nabla u) \text{ in } \mathbb{R}^{MN}, \text{ a.e. in } \Omega_T,$$

$$\left( \partial_t u(t) - \frac{1}{\delta} u(t), \psi \right) + \int_\Omega \left( \alpha(\eta(t))B^*(t) + \kappa^2 \nabla u \right) : \nabla \psi \, dx$$

$$+ v \int_\Omega |v \nabla u(t)|^{N-1} v \nabla u(t) : \nabla \psi \, dx + \frac{1}{\delta} \int_\Omega |u(t)|^2 u(t) \cdot \psi \, dx = 0,$$

for any $\psi \in W^{1,N+1}(\Omega; \mathbb{R}^M)$, a.e. $t \in (0, T)$, subject to $u(0) = u_0$ in $X$.

We note that $(P)_{\kappa,\varepsilon,\nu,\delta}$ can be reformulated in the form of a Cauchy problem of evolution equations in the Hilbert space $\mathcal{X}$, denoted by $(CP)_{\kappa,\varepsilon,\nu,\delta}$.

**Cauchy problem (CP)$_{\kappa,\varepsilon,\nu,\delta}$**:

$$\begin{cases}
U'(t) + \partial \Phi_{\varepsilon,\nu,\delta}(U(t)) + G_{\delta}(U(t)) \ni 0 \text{ in } \mathcal{X}, \ t \in (0, T), \\
U(0) = U_0 \text{ in } \mathcal{X}.
\end{cases}$$

Here, "$'$" denotes the time-derivative "$'\frac{d}{dt}$" of an $\mathcal{X}$-valued function (in time). For every $\kappa, \varepsilon, \nu, \delta > 0$, $\Phi_{\varepsilon,\nu,\delta} : \mathcal{X} \rightarrow [0, \infty]$ is a proper l.s.c. and convex function, defined as

$$\Phi_{\varepsilon,\nu,\delta} : U := [\eta, u] \in D(\Phi_{\varepsilon,\nu,\delta}) \subset \mathcal{X}$$

$$\iff \Phi_{\varepsilon,\nu,\delta}(U) := \frac{1}{2} \int_\Omega |\nabla \eta|^2 \, dx + \frac{1}{N+1} \int_\Omega |v \nabla u|^{N+1} \, dx$$

$$+ \frac{1}{2} \int_\Omega \left( \kappa f_\varepsilon(\nabla u) + \frac{1}{\kappa} \alpha(\eta) \right)^2 \, dx + \frac{1}{4\delta} \int_\Omega |u|^4 \, dx \in [0, \infty), \quad (4.2)$$

and $G_{\delta} : \mathcal{X} \rightarrow \mathcal{X}$ is a non-monotone perturbation, defined as:

$$G_{\delta} : U := [\eta, u] \in \mathcal{X} \iff G_{\delta}(U) := \left[ g(\eta) - \frac{1}{\kappa^2} \alpha(\eta) \alpha'(\eta), - \frac{1}{\delta} u \right] \in \mathcal{X}.$$

**Remark 3** Assumptions (A2) and (A3) guarantee the Lipschitz continuity of the perturbation $G_{\delta}$. Hence, the well posedness of the Cauchy problem $(CP)_{\kappa,\varepsilon,\nu,\delta}$ is immediately obtained by means of the general theory of nonlinear evolution equations (see Brézis 1973; Barbu 2010). Moreover, the continuous dependence of solution with respect to $\varepsilon \geq 0$, $\delta > 0$, $\nu \geq 0$, and $\kappa > 0$ is also obtained by means of the general theory of operator convergence (see Attouch 1984; Mosco 1969).

The goal of this section is to prove the following two results.

**Theorem 1** The system $(P)_{\kappa,\varepsilon,\nu,\delta}$ is equivalent to the Cauchy problem $(CP)_{\kappa,\varepsilon,\nu,\delta}$.

**Theorem 2** Given $u_0 \in L^4(\Omega; \mathbb{R}^M)$ and $\nu u_0 \in W^{1,N+1}(\Omega; \mathbb{R}^M)$, then
(I) $(P)_{\varepsilon, \nu, \delta}^\kappa$ admits a unique solution $U = [\eta, u] \in L^2(0, T; \mathcal{X})$, such that

$$0 \leq \eta \leq 1, \quad \text{a.e. in } \Omega_T$$

and

$$U = [\eta, u] \in W^{1,2}(0, T; \mathcal{X}) \cap L^\infty(0, T; \mathcal{W}),$$

$$v_n u \in L^{N+1}(0, T; W^{1,N+1}(\Omega; \mathbb{R}^M)).$$

(II) Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0, \infty)$, $\{\delta_n\}_{n=1}^\infty \subset (0, \infty)$, $\{\nu_n\}_{n=1}^\infty \subset [0, \infty)$, and $\{\kappa_n\}_{n=1}^\infty \subset (0, \infty)$ be sequences of constants, such that

$$[\kappa_n, \varepsilon_n, \nu_n, \delta_n] \to [\kappa, \varepsilon, \nu, \delta] \quad \text{in } \mathbb{R}^4, \quad \text{as } n \to \infty.$$

Let $\{U_{0,n}\}_{n=1}^\infty = \{[\eta_{0,n}, u_{0,n}]\}_{n=1}^\infty \subset \mathcal{W} \cap L^4(\Omega; \mathbb{R}^M)$ with $\{v_n u_{0,n}\} \subset W^{1,N+1}(\Omega; \mathbb{R}^M)$ be a sequence of initial data, such that

$$U_{0,n} = [\eta_{0,n}, u_{0,n}] \to U_0 = [\eta_0, u_0] \quad \text{in } \mathcal{X}, \quad \text{and weakly in } \mathcal{W},$$

and $v_n u_{0,n} \to v_0 u_0$ weakly in $W^{1,N+1}(\Omega; \mathbb{R}^M)$, as $n \to \infty.$

Let $U = [\eta, u] \in L^2(0, T; \mathcal{X})$ be the solution to the system $(P)_{\varepsilon, \nu, \delta}^\kappa$. Also, for any $n \in \mathbb{N}$, let $U_n = [\eta_n, u_n] \in L^2(0, T; \mathcal{X})$ be the solution to the system $(P)_{\varepsilon_n, \nu_n, \delta_n}^{\kappa_n}$ corresponding to the initial data $U_{0,n} = [\eta_{0,n}, u_{0,n}] \in \mathcal{W}$. Then, it holds that

$$U_n = [\eta_n, u_n] \to U = [\eta, u] \quad \text{in } C([0, T]; \mathcal{X}), \quad \text{in } L^2(0, T; \mathcal{W}),$$

weakly in $W^{1,2}(0, T; \mathcal{X})$, and weakly-* in $L^\infty(0, T; \mathcal{W})$, and

$$v_n u_n \to v_0 u_0 \quad \text{in } L^{N+1}(0, T; W^{1,N+1}(\Omega; \mathbb{R}^M)),$$

as $n \to \infty.$

4.1 Proofs of Theorems 1 and 2

Before proving Theorems 1 and 2, we need some auxiliary results.

Lemma 1 Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0, \infty)$, $\{\delta_n\}_{n=1}^\infty \subset (0, \infty)$, $\{\nu_n\}_{n=1}^\infty \subset [0, \infty)$ and $\{\kappa_n\}_{n=1}^\infty \subset (0, \infty)$ be sequences of constants, as in (4.5). Then, $(\Phi_{\varepsilon_n, \nu_n, \delta_n}^{\kappa_n})_{n=1}^\infty$ converge to $\Phi_{\varepsilon, \nu, \delta}^{\kappa}$ on $\mathcal{X}$, in the sense of Mosco (see Definition 3 for details), as $n \to \infty.$

Proof As is easily checked, the sequence $\left\{ \frac{1}{2} \left( \kappa_n f_{\varepsilon_n} + \frac{1}{\kappa_n} \alpha \right)^2 \right\}_{n=1}^\infty$ of continuous convex functions

$$\frac{1}{2} \left( \kappa_n f_{\varepsilon_n} + \frac{1}{\kappa_n} \alpha \right)^2 : [W, \xi] \in \mathbb{R}^{MN} \times \mathbb{R} \mapsto \frac{1}{2} \left( \kappa_n f_{\varepsilon_n}(W) + \frac{1}{\kappa_n} \alpha(\xi) \right)^2 \in [0, \infty),$$

is convergent.
for \( n = 1, 2, 3, \ldots \), converges to the continuous convex function

\[
\frac{1}{2} (\kappa f_\epsilon + \frac{1}{\kappa} \alpha)^2 : [W, \xi] \in \mathbb{R}^{MN} \times \mathbb{R} \mapsto \frac{1}{2} \left( \kappa f_\epsilon(W) + \frac{1}{\kappa} \alpha(\xi) \right)^2 \in [0, \infty),
\]

on \( \mathbb{R}^{MN} \times \mathbb{R} \), in the sense of Mosco, as \( n \to \infty \);

\( \# 2 \) the sequence \( \{ \frac{1}{4\delta_n} \| \cdot \|_4^{N+1} \} \) of continuous convex functions

\[
\frac{1}{4\delta_n} \| \cdot \|_4^{N+1} : \mathbf{v} \in \mathbb{R}^M \mapsto \frac{1}{4\delta_n} \| \mathbf{v} \|_4^{N+1} \in [0, \infty),
\]

converges to the convex function

\[
\frac{1}{4\delta} \| \cdot \|_4 : \mathbf{v} \in \mathbb{R}^M \mapsto \frac{1}{4\delta} \| \mathbf{v} \|_4 \in [0, \infty),
\]

on \( \mathbb{R}^M \), in the sense of Mosco, as \( n \to \infty \);

\( \# 3 \) the sequence \( \{ \frac{1}{N+1} \| v_n (\cdot) \|_4^{N+1} \} \) of continuous convex functions

\[
\frac{1}{N+1} \| v_n (\cdot) \|_4^{N+1} : \mathbf{v} \in \mathbb{R}^M \mapsto \frac{1}{N+1} \| \mathbf{v} \|_4^{N+1} \in [0, \infty),
\]

converges to the convex function

\[
\frac{1}{N+1} \| v (\cdot) \|_4^{N+1} : \mathbf{v} \in \mathbb{R}^M \mapsto \frac{1}{N+1} \| \mathbf{v} \|_4^{N+1} \in [0, \infty),
\]

on \( \mathbb{R}^M \), in the sense of Mosco, as \( n \to \infty \).
for all \([\eta, u] \in \mathcal{X}, \{[\eta_n, u_n]\}_{n=1}^{\infty} \subset \mathcal{X}\), such that \([\eta_n, u_n] \to [\eta, u]\) weakly in \(\mathcal{X}\) as \(n \to \infty\).

Next, given \(u \in D(\Phi_{\kappa, \nu, \delta}^{\kappa})\) by means of the standard regularization method of Sobolev functions and a diagonal argument, we can find a sequence of smooth functions \(\{\tilde{u}_n\}_{n=1}^{\infty} \subset C^\infty(\Omega; \mathbb{R}^M)\), such that as \(n \to \infty\),

\[
\begin{cases}
\tilde{u}_n \rightarrow u \text{ in } \mathbb{W}, \\
v_n \tilde{u}_n \rightarrow \nu u \text{ in } W^{1,N+1}(\Omega; \mathbb{R}^M).
\end{cases}
\]

(4.8)

Note that (4.5), (4.8), (A1), and (A4) lead to as \(n \to \infty\),

\[
\begin{cases}
\kappa f_{\kappa_n}^\prime(\nabla \tilde{u}_n) \rightarrow \kappa f_{\kappa}^\prime(\nabla u) \text{ in } H, \\
\delta_n \frac{1}{4} \tilde{u}_n \rightarrow \delta^{-\frac{1}{4}} u \text{ in } L^4(\Omega; \mathbb{R}^M).
\end{cases}
\]

Therefore, the optimality condition for the Mosco convergence of \(\{\Phi_{\kappa_n, \nu_n, \delta_n}^{\kappa}\}_{n=1}^{\infty}\) is obtained via the following computation:

\[
\begin{align*}
&\Phi_{\kappa_n, \nu_n, \delta_n}^{\kappa}(\eta, \tilde{u}_n) - \Phi_{\kappa, \nu, \delta}^{\kappa}(\eta, u) \\
\leq & \left(\|\kappa f_{\kappa_n}^\prime(\nabla \tilde{u}_n) - \kappa f_{\kappa}^\prime(\nabla u)\|_H + \|\kappa_n^{-1} - \kappa^{-1}\|_{L^0(H)}\right) \\
& \cdot \sup_{n \in \mathbb{N}} \left|\kappa f_{\kappa_n}^\prime(\nabla \tilde{u}_n) + \kappa f_{\kappa}^\prime(\nabla u) + \left(\kappa_n^{-1} + \kappa^{-1}\right)\alpha(\eta)\right|_H \\
+ & \frac{1}{N+1} \left|\nabla(\nu_n \tilde{u}_n)\right|_{L^{N+1}(\Omega; \mathbb{R}^{MN})}^{N+1} - \left|\nabla(\nu u)\right|_{L^{N+1}(\Omega; \mathbb{R}^{MN})}^{N+1} \\
+ & \frac{1}{4} \left|\delta_n \frac{1}{4} \tilde{u}_n\right|_{L^4(\Omega; \mathbb{R}^M)}^4 - \left|\delta^{-\frac{1}{4}} u\right|_{L^4(\Omega; \mathbb{R}^M)}^4 \\
\to 0, & \text{ as } n \to \infty.
\end{align*}
\]

\(\Box\)

**Remark 4** Let \(\{\varepsilon_n\}_{n=1}^{\infty} \subset [0, \infty), \{\delta_n\}_{n=1}^{\infty} \subset (0, \infty), \{\nu_n\}_{n=1}^{\infty} \subset [0, \infty), \) and \(\{\kappa_n\}_{n=1}^{\infty} \subset (0, \infty)\) be the sequences as in (4.5). Additionally, let \(\eta \in H\) be a fixed function, and let \([\eta_n]\) be a sequence such that

\(\eta_n \to \eta \text{ in } V, \text{ as } n \to \infty\).

(4.9)

Then, as a corollary of Lemma 1, we can show that

\(\Phi_{\kappa_n, \nu_n, \delta_n}^{\kappa}(\eta_n, \cdot) \to \Phi_{\kappa, \nu, \delta}^{\kappa}(\eta, \cdot)\) on \(\mathcal{X}\), in the sense of Mosco, as \(n \to \infty\).

(4.10)

In fact, the lower-bound condition for the Mosco convergence (4.10) is a straightforward consequence of Lemma 1. Also, we can verify the condition of optimality by using the sequence \(\{\tilde{u}_n\} \subset C^\infty(\Omega; \mathbb{R}^M)\) as in (4.8).
The following result is a direct consequence of the standard variational method (cf. Brézis 1973; Barbu 2010). Therefore, we omit its proof.

**Lemma 2** For any \( u \in W \), the functional \( \Phi^\kappa_{\epsilon, \nu, \delta} : \eta \in H \rightarrow \Phi^\kappa_{\epsilon, \nu, \delta}(\eta, u) \in [0, \infty) \) is proper l.s.c. and convex on \( H \). The subdifferential \( \partial_\eta \Phi^\kappa_{\epsilon, \nu, \delta}(\cdot, u) \subset H \times H \) is a single-valued operator, with domain

\[
D(\partial_\eta \Phi^\kappa_{\epsilon, \nu, \delta}(\cdot, u)) = D(\partial \Psi_0) = \{ \tilde{\eta} \in H^2(\Omega) | \nabla \eta \cdot n_\Gamma = 0 \text{ on } \Gamma \},
\]

and

\[
\partial_\eta \Phi^\kappa_{\epsilon, \nu, \delta}(\eta, u) = -\Delta \eta + \alpha'(\eta) f_\epsilon(\nabla u) + \frac{1}{\kappa^2} \alpha(\eta) \alpha'(\eta), \quad \forall \eta \in D(\Phi^\kappa_{\epsilon, \nu, \delta}).
\]

**Lemma 3** For any \( \eta \in V \), the functional \( \Phi^\kappa_{\epsilon, \nu, \delta}(\eta, \cdot) : u \in X \mapsto \Phi^\kappa_{\epsilon, \nu, \delta}(\eta, u) \in [0, \infty] \) is proper l.s.c. and convex on \( X \). Moreover, the following two items are equivalent:

(\( O \)) \([ u, u^*, u] \in \partial_\eta \Phi^\kappa_{\epsilon, \nu, \delta}(\eta; \cdot) \) in \( X \times X \);

(\( I \)) \([ u, u^*] \in X \times X, u \in D(\Phi^\kappa_{\epsilon, \nu, \delta}(\eta; \cdot)) \subset \{ u \in W \cap L^4(\Omega; \mathbb{R}^M) | v u \in W^{1, N+1}(\Omega; \mathbb{R}^M) \}, \)

and there exists \( W_u \in L^\infty(\Omega; \mathbb{R}^{MN}) \) such that

(i-a) \( W_u \in \partial f_\epsilon(\nabla u) \) a.e. in \( \Omega \),

(i-b) \( \text{div}(\alpha(\eta) W_u + \kappa^2 \nabla u + v |v \nabla u|^{N-1} v \nabla u) + \frac{1}{\delta} |u|^2 u \in X \),

(i-c) \( \alpha(\tilde{\eta}) W_u + \kappa^2 \nabla u + v |v \nabla u|^{N-1} v \nabla u) n_\Gamma = 0 \text{ in } L^2(\Gamma; \mathbb{R}^M), \)

(i-d) \( u^* = -\text{div}(\alpha(\eta) W_u + \kappa^2 \nabla u + v |v \nabla u|^{N-1} v \nabla u) + \frac{1}{\delta} |u|^2 u \) in \( X \).

**Proof** When \( \epsilon > 0 \), this result is an immediate consequence of the theory of elliptic variational inequalities (cf. Brézis 1973; Barbu 1976, 2010). Moreover, the operator \( \partial_\eta \Phi^\kappa_{\epsilon, \nu, \delta}(\eta; \cdot) : X \times X \) is single valued, and

\[
\begin{align*}
\partial_\eta \Phi^\kappa_{\epsilon, \nu, \delta}(\tilde{\eta}, u) &= -\text{div}(\alpha(\tilde{\eta}) [\nabla f_\epsilon](\nabla u) + \kappa^2 \nabla u + v |v \nabla u|^{N-1} v \nabla u) \\
&\quad + \frac{1}{\delta} |u|^2 u \in X,
\end{align*}
\]

subject to \( \nabla u n_\Gamma = 0 \) in \( L^2(\Gamma; \mathbb{R}^N) \).

Hence, we only give the proof in the case that \( \epsilon = 0 \).

We define a set-valued map \( A^\kappa_{\nu, \delta} : X \rightarrow 2^X \), with domain

\[
D(A^\kappa_{\nu, \delta}) := \left\{ u \in D(\Phi^\kappa_{0, \nu, \delta}(\eta; \cdot)) \bigg| \text{there exists } W_u \in L^\infty(\Omega; \mathbb{R}^{MN}) \text{ such that (i-a)–(i-c) hold.} \right\},
\]

as

\[
u, \delta
\]

\[
\begin{align*}
\nu, \delta
\end{align*}
\]

\[
\begin{align*}
\nu, \delta
\end{align*}
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\[
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\nu, \delta
\end{align*}
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\[
\begin{align*}
\nu, \delta
\end{align*}
\]
Next, we prove the following equality, which leads to the conclusion:

\[ \partial_u \Phi^\kappa_{0,v,\delta}(\eta, \cdot) = A^\kappa_{v,\delta} \text{ in } X \times X. \] (4.14)

**Claim #1:** \( A^\kappa_{v,\delta} \subset \partial_u \Phi^\kappa_{0,v,\delta} \text{ in } X \times X. \)

Suppose that \( u \in D(A^\kappa_{v,\delta}) \) and \( u^* \in A^\kappa_{v,\delta} u \) in \( X \). Then, from (A4) and (i-a)–(i-d), we get

\[
(u^*, z - u)_X = \langle -\text{div}(\alpha(\eta) W_u + \kappa^2 \nabla u + \nu|\nu \nabla u|^{N-1} \nabla u), z - u \rangle_{W} \\
+ \frac{1}{\delta} (|u|^2_{W, \Omega} - |u|^2_{W, \Omega}) \leq \int_{\Omega} \alpha(\eta) |\nabla(z - u)|^2 dx + \frac{1}{\delta} \int_{\Omega} |u|^2 \cdot (z - u) dx \\
\leq \int_{\Omega} \alpha(\eta) (|\nabla z| - |\nabla u|)^2 dx + \frac{\kappa^2}{2} \int_{\Omega} (|\nabla z|^2 - |\nabla u|^2) dx \\
+ \frac{1}{N + 1} \int_{\Omega} (|\nu \nabla z|^{N+1} - |\nu \nabla u|^{N+1}) dx + \frac{1}{4\delta} \int_{\Omega} (|z|^4 - |u|^4) dx \\
= \Phi^\kappa_{0,v,\delta}(\eta, z) - \Phi^\kappa_{0,v,\delta}(\eta, u), \text{ for any } z \in D(\Phi(\eta; \cdot)).
\]

Thus,

\[ u \in D(\partial_u \Phi^\kappa_{0,v,\delta}(\eta, \cdot)) \text{ and } u^* \in \partial_u \Phi^\kappa_{0,v,\delta}(\eta, u) \text{ in } X, \]

which implies

\[ A^\kappa_{v,\delta} \subset \partial_u \Phi^\kappa_{0,v,\delta}(\eta, \cdot) \text{ in } X \times X. \]

**Claim #2:** \( (A^\kappa_{v,\delta} + I_X)X = X \).

Since \( (A^\kappa_{v,\delta} + I_X)X \subset X \) is trivial, it suffices to prove the converse inclusion. Let us take any \( w \in X \). By applying Minty’s theorem, we can find a net of functions \( \{ u_\varepsilon \mid 0 < \varepsilon \leq 1 \} \subset W \)

\[
\{ u_\varepsilon := (\partial_u \Phi^\kappa_{\varepsilon,v,\delta}(\bar{\eta}, \cdot) + I_X)^{-1}w \mid 0 < \varepsilon \leq 1 \} \text{ in } X,
\]

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i.e.,

$$w - u_\varepsilon \in \partial_u \Phi^{\kappa}_{\varepsilon, v, \delta}(\tilde{\eta}, u_\varepsilon) \text{ in } \mathbb{X}, \text{ for any } 0 < \varepsilon \leq 1. \tag{4.15}$$

Therefore, by (4.11),

$$\int_{\Omega} (\alpha(\tilde{\eta})|\nabla f_\varepsilon| |\nabla u_\varepsilon| + \kappa^2 \nabla u_\varepsilon + \nu |\nabla u_\varepsilon|^2) : \nabla z dx + \frac{1}{\delta} \int_{\Omega} |u_\varepsilon|^2 u_\varepsilon \cdot z dx = \int_{\Omega} (w - u_\varepsilon) \cdot z dx, \quad \text{for all } z \in D(\Phi^{\kappa}_{\varepsilon, v, \delta}(\eta, \cdot)) \subset D(\Phi^{\kappa}_{\varepsilon, v, \delta}(\tilde{\eta}, \cdot)), \quad \varepsilon > 0. \tag{4.16}$$

We take $z = u_\varepsilon \in D(\Phi^{\kappa}_{\varepsilon, v, \delta}(\tilde{\eta}, \cdot))$ in (4.16). Then, with (A1), (A4)–(A5) and Young’s inequality in mind, we deduce that

$$\frac{1}{2} \|u_\varepsilon\|_\mathbb{X}^2 + \frac{\kappa^2}{2} \|\nabla u_\varepsilon\|_{L^2(\Omega; \mathbb{R}^{MN})}^2 + \nu \|\nabla u_\varepsilon\|_{L^{N+1}(\Omega; \mathbb{R}^{MN})}^{N+1} \leq \frac{1}{2} \|w\|_\mathbb{X}^2, \quad \text{for any } 0 < \varepsilon \leq 1. \tag{4.17}$$

Therefore, we can find a function $u \in D(\Phi^{\kappa}_{\varepsilon, v, \delta}(\tilde{\eta}, \cdot))$ and a sequence $1 > \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > \varepsilon_n \downarrow 0$ as $n \to \infty$, such that as $n \to \infty$,

$$\begin{cases} u_n := u_{\varepsilon_n} \to u \text{ in } \mathbb{X}, \text{ weakly in } \mathbb{W}, \text{ weakly in } L^4(\Omega; \mathbb{R}^M), \text{ a.e. in } \Omega, \\
\nu \nabla u_n \to \nu \nabla u \text{ weakly in } L^{N+1}(\Omega; \mathbb{R}^{MN}). \end{cases} \tag{4.18}$$

By (4.18), (4.15) and Remark 4, we can apply Remark 5 (Fact 1) to see that

$$w - u \in \partial_u \Phi^{\kappa}_{0, v, \delta}(\tilde{\eta}, u) \text{ in } \mathbb{X},$$

and

$$\Phi^{\kappa}_{\varepsilon_n, v, \delta}(\tilde{\eta}, u_n) \to \Phi^{\kappa}_{0, v, \delta}(\tilde{\eta}, u) \text{ as } n \to \infty. \tag{4.19}$$

Thanks to (4.2), (4.16)–(4.19), (4.1)–(3), and Remark 5 (Fact 2), we obtain
\[
\leq \Phi_0^\kappa (\tilde{\eta}, \mathbf{u}) - \int_{\Omega} \left( \kappa |\nabla \mathbf{u}| + \frac{1}{\kappa} \alpha (\tilde{\eta}) \right)^2 dx \\
- \frac{1}{\delta} \int_{\Omega} |\mathbf{u}|^4 dx - \frac{1}{N + 1} \int_{\Omega} |\nabla \mathbf{u}|^{N+1} dx = \frac{\kappa^2}{2} \int_{\Omega} |\nabla \mathbf{u}|^2 dx.
\] (4.20)

Having in mind (4.18), (4.20) and the uniform convexity of \( L^2 \)-based topologies, we can get as \( n \to \infty \),
\[
\begin{align*}
\{ & u_n \to u \text{ in } W, \text{ a.e. in } \Omega, \\
& \nabla u_n \to \nabla u \text{ in } L^2(\Omega; \mathbb{R}^{MN}), \text{ a.e. in } \Omega.
\end{align*}
\] (4.21)

Additionally, by the boundedness of \( \Omega \), and Lions’ Lemma Lions (1969, Lemma 1.3, in page 12), one can show from (4.18) and (4.21) that as \( n \to \infty \),
\[
\begin{align*}
|u_n|^2 u_n & \to |u|^2 u \text{ weakly in } L^4(\Omega; \mathbb{R}^M), \\
|\nabla u_n|^{N-1} \nabla u_n & \to |\nabla u|^{N-1} \nabla u \text{ weakly in } L^{N/(N-1)}(\Omega; \mathbb{R}^{MN}),
\end{align*}
\]
i.e.,
\[
\begin{align*}
& \int_{\Omega} |v|\nabla u_n|^{N-1} \nabla u_n : \nabla z dx + \frac{1}{\delta} \int_{\Omega} |u_n|^2 u_n \cdot z dx \\
& \to \int_{\Omega} |v|\nabla u|^{N-1} \nabla u : \nabla z dx + \frac{1}{\delta} \int_{\Omega} |u|^2 u \cdot z dx,
\end{align*}
\] (4.22)

for any \( z \in D(\Phi_0^\kappa, \cdot, \cdot) \), as \( n \to \infty \).

On the other hand, from (A4), one can see that
\[
\| [\nabla f_{\epsilon_n}] (\nabla u_n) \| (= \| \frac{\nabla u_n}{\sqrt{\epsilon_n^2 + |\nabla u_n|^2}} \|) \leq 1, \text{ a.e. in } \Omega, \text{ for any } n \in \mathbb{N},
\]
\[
\begin{align*}
\{ & W \in L^2(\Omega; \mathbb{R}^{MN}) \mapsto \| f_{\epsilon_n} (W) \|_{L^1(\Omega)} \in \mathbb{R} \}_{n=1}^\infty \text{ converges to the convex function } \\
& W \in L^2(\Omega; \mathbb{R}^{MN}) \mapsto \| f_0 (W) \|_{L^1(\Omega)} (= \| W \|_{L^1(\Omega; \mathbb{R}^{MN})}),
\end{align*}
\]
on \( L^2(\Omega; \mathbb{R}^{MN}) \), in the sense of Mosco, as \( n \to \infty \).
\( (4.21) \) and \( \#4 \) enable us to say that
\[
[\nabla f_{\epsilon_n}] (\nabla u_n) \to M_u \text{ weakly- } \ast \text{ in } L^\infty(\Omega; \mathbb{R}^{MN}) \text{ as } n \to \infty,
\] (4.23)
for some \( M_u \in L^\infty(\Omega; \mathbb{R}^{MN}) \), by taking a subsequence if necessary. In view of
\( (4.21), (4.23), \#5 \), Remark 5 (Fact 1), and Brézis (1973, Proposition 2.16), one can see that
In particular, this identity leads to

\[ M_u \in \partial f_0(\nabla u) \text{ a.e. in } \Omega. \quad (4.24) \]

With (4.18), (4.22), and (4.23) in mind, letting \( n \to \infty \) in (4.16) with \( \varepsilon = \varepsilon_n \) yields

\[
\int_{\Omega} (\alpha(\eta) M_u + \kappa^2 \nabla u + v|v \nabla u|^{N-1} v \nabla u) : \nabla z \, dx + \frac{1}{\delta} \int_{\Omega} |u|^2 u \cdot z \, dx
= \int_{\Omega} (w - u) \cdot z \, dx, \quad \text{for all } z \in D(\Phi_{0,v,\delta}^\kappa(\tilde{\eta}, \cdot)).
\]

(4.25)

In particular,

\[
(w - u, \varphi_0)_{\mathcal{X}} = \int_{\Omega} (\alpha(\eta) M_u + \kappa^2 \nabla u + v|v \nabla u|^{N-1} v \nabla u) : \nabla \varphi_0 \, dx
+ \frac{1}{\delta} \int_{\Omega} |u|^2 u \cdot \varphi_0 \, dx,
\]

for any \( \varphi_0 \in C^\infty_c(\Omega; \mathbb{R}^M) \). This implies

\[
\begin{align*}
-\text{div}(\alpha(\eta) M_u + \kappa^2 \nabla u + v|v \nabla u|^{N-1} v \nabla u) + \frac{1}{\delta} |u|^2 u &= w - u \in \mathcal{X}, \\
-\text{div}(\alpha(\eta) M_u + \kappa^2 \nabla u + v|v \nabla u|^{N-1} v \nabla u) - w - u - \frac{1}{\delta} |u|^2 u &\in L^\frac{4}{3}(\Omega; \mathbb{R}^M),
\end{align*}
\]

in the distributional sense in \( \Omega \). Furthermore, applying Green’s formula, one can observe that

\[
\int_{\Gamma} \left[ (\alpha(\eta) M_u + \kappa^2 \nabla u + v|v \nabla u|^{N-1} v \nabla u) \mathbf{n}_\Gamma \right] \cdot \varphi \, d\Gamma
= \int_{\Omega} (\alpha(\eta) M_u + \kappa^2 \nabla u + v|v \nabla u|^{N-1} v \nabla u) : \nabla \varphi \, dx
- \int_{\Omega} (w - u - \frac{1}{\delta} |u|^2 u) \cdot \varphi \, dx
= 0, \quad \text{for any } \varphi \in C^\infty_c(\overline{\Omega}; \mathbb{R}^M).
\]

This identity leads to

\[
\left[ (\alpha(\eta) M_u + \kappa^2 \nabla u + v|v \nabla u|^{N-1} v \nabla u) \mathbf{n}_\Gamma \right] = 0 \text{ in } L^2(\Gamma; \mathbb{R}^M). \quad (4.26)
\]

As a consequence of (4.25) and (4.26), we obtain Claim #2.

On account of Claims #1–#2 and the maximality of \( \partial_u \Phi_{0,v,\delta}^\kappa(\tilde{\eta}, \cdot) \) in \( \mathcal{X} \times \mathcal{X} \), we deduce coincidence (4.14), and we conclude the proof. \( \square \)

Now, we denote by \( [\partial_{\eta} \Phi_{\varepsilon,v,\delta}^\kappa \times \partial_u \Phi_{\varepsilon,v,\delta}^\kappa] \subset \mathcal{X} \times \mathcal{X} \) a (possibly) set-valued operator, such that

\[
[U, U^*] = [[\eta, u], [\eta^*, u^*]] \in [\partial_{\eta} \Phi_{\varepsilon,v,\delta}^\kappa \times \partial_u \Phi_{\varepsilon,v,\delta}^\kappa] \text{ in } \mathcal{X} \times \mathcal{X}
\]

and

\[
\begin{align*}
\bullet \ & \eta \in D(\partial_{\eta} \Phi_{\varepsilon,v,\delta}^\kappa(\cdot, u)), \text{ and } u \in D(\partial_u \Phi_{\varepsilon,v,\delta}^\kappa(\cdot, \cdot)), \\
\bullet \ & [\eta^*, u^*] \in \partial_{\eta} \Phi_{\varepsilon,v,\delta}^\kappa(\eta, u) \times \partial_u \Phi_{\varepsilon,v,\delta}^\kappa(\eta, u) \text{ in } \mathcal{X}.
\end{align*}
\]
Note that the inclusion $\partial \Phi^x_{\varepsilon,\nu,\delta} \subset \left[ \partial \eta \Phi^x_{\varepsilon,\nu,\delta} \times \partial u \Phi^x_{\varepsilon,\nu,\delta} \right]$ is easily seen to hold. We show that, in fact, equality holds.

**Lemma 4** The following two items hold:

(I) $\left[ \partial \eta \Phi^x_{\varepsilon,\nu,\delta} \times \partial u \Phi^x_{\varepsilon,\nu,\delta} \right]$ is semi-monotone in $\mathcal{X} \times \mathcal{X}$, i.e., there exists a positive constant $R_0 > 0$, such that $\left[ \partial \eta \Phi^x_{\varepsilon,\nu,\delta} \times \partial u \Phi^x_{\varepsilon,\nu,\delta} \right] + R_0 \mathcal{I}$ is monotone in $\mathcal{X} \times \mathcal{X}$, where $\mathcal{I}$ is the identity on $\mathcal{X} \times \mathcal{X}$.

(II) $\partial \Phi^x_{\varepsilon,\nu,\delta} = \left[ \partial \eta \Phi^x_{\varepsilon,\nu,\delta} \times \partial u \Phi^x_{\varepsilon,\nu,\delta} \right]$ in $\mathcal{X} \times \mathcal{X}$.

**Proof** We set

$$R_0 := 1 + \frac{2}{\kappa^2} \| \alpha' \|_{L^\infty(\mathbb{R})}^2,$$  

(4.27)

and we will show that (I) is satisfied with this constant $R_0$.

Let us assume that in $\mathcal{X} \times \mathcal{X}$,

$$\left[ \left[ \eta, u \right], \left[ \eta^*, u^* \right] \right] \in \left[ \partial \eta \Phi^x_{\varepsilon,\nu,\delta} \times \partial u \Phi^x_{\varepsilon,\nu,\delta} \right] + R_0 \mathcal{I},$$

$$\left[ \left[ \tilde{\eta}, \tilde{u} \right], \left[ \tilde{\eta}^*, \tilde{u}^* \right] \right] \in \left[ \partial \eta \Phi^x_{\varepsilon,\nu,\delta} \times \partial u \Phi^x_{\varepsilon,\nu,\delta} \right] + R_0 \mathcal{I}.$$

Therefore,

$$\left( \left[ \eta^*, u^* \right] - \left[ \tilde{\eta}^*, \tilde{u}^* \right], \left[ \eta, u \right] - \left[ \tilde{\eta}, \tilde{u} \right] \right)_{\mathcal{X}}$$

$$= (\eta^* - \tilde{\eta}^*, \eta - \tilde{\eta})_H + (u^* - \tilde{u}^*, u - \tilde{u})_H$$

$$= I_1 + I_2 + I_3,$$  

(4.28a)

with

$$I_1 := \| \nabla (\eta - \tilde{\eta}) \|^2_{H^1(\Omega)} + \kappa^2 \| \nabla (u - \tilde{u}) \|^2_{H^1(\Omega)} + \frac{1}{\delta} \int_{\Omega} \left( |u|^2 - |\tilde{u}|^2 \right) \cdot (u - \tilde{u}) \, dx$$

$$+ \int_{\Omega} \nu^2 \left( |\nabla u|^{N-1} \nabla u - |\nabla \tilde{u}|^{N-1} \nabla \tilde{u} \right) \cdot (\nabla u - \nabla \tilde{u}) \, dx$$

$$+ R_0 \left( \| \eta - \tilde{\eta} \|^2_H + \| u - \tilde{u} \|^2_\mathcal{X} \right)$$

$$\geq \| \nabla (\eta - \tilde{\eta}) \|^2_{H^1(\Omega)} + \kappa^2 \| \nabla (u - \tilde{u}) \|^2_{H^1(\Omega)} + R_0 \left( \| \eta - \tilde{\eta} \|^2_H + \| u - \tilde{u} \|^2_\mathcal{X} \right),$$  

(4.28b)

$$I_2 := \left( \alpha'(\eta) f_\epsilon(\nabla u) - \alpha'(\tilde{\eta}) f_\epsilon(\nabla \tilde{u}), \eta - \tilde{\eta} \right)_H$$

$$+ \kappa^2 \left( \alpha(\eta) \alpha'(\eta) - \alpha(\tilde{\eta}) \alpha'(\tilde{\eta}), \eta - \tilde{\eta} \right)_H$$

$$= \int_{\Omega} f_\epsilon(\nabla u) \left( \alpha'(\eta) - \alpha'(\tilde{\eta}) \right) (\eta - \tilde{\eta}) \, dx$$

$$+ \int_{\Omega} \alpha'(\tilde{\eta}) \left( f_\epsilon(\nabla u) - f_\epsilon(\nabla \tilde{u}) \right) (\eta - \tilde{\eta}) \, dx$$
by (A3) and Hölder’s inequality. Finally,

\[ I_3 := \int_\Omega (\alpha(\eta)[(\nabla f_\varepsilon)](\nabla u) - \alpha(\tilde{\eta})[\nabla f_\varepsilon](\nabla \tilde{u})) \cdot (\nabla u - \nabla \tilde{u}) \, dx \]

\[ = \int_\Omega \alpha(\eta) \cdot [(\nabla f_\varepsilon)(\nabla u) - [\nabla f_\varepsilon](\nabla \tilde{u})) \cdot (\nabla u - \nabla \tilde{u}) \, dx \]

\[ + \int_\Omega (\alpha(\eta) - \alpha(\tilde{\eta}))[\nabla f_\varepsilon](\nabla \tilde{u})) \cdot (\nabla u - \nabla \tilde{u}) \, dx \]

\[ \geq -\|\alpha'\|_{L^\infty(\mathbb{R})}\|\eta - \tilde{\eta}\| H \|\nabla (u - \tilde{u})\|_{[\mathcal{X}]^N} \] \hspace{1cm} (4.28d)

by (A3), (A4), and Hölder’s inequality. Due to (4.27), using Young’s inequality, the inequalities in (4.28) lead to

\[
\left(\left[\eta^*, u^*\right] - [\tilde{\eta}^*, \tilde{u}^*], [\eta, u] - [\tilde{\eta}, \tilde{u}]\right)_{\mathcal{X}} \geq \|\nabla(\eta - \tilde{\eta})\|_{[H]^N}^2 + \kappa^2 \|\nabla (u - \tilde{u})\|_{[\mathcal{X}]^N}^2
\]

\[
+ R_0\left\|\eta - \tilde{\eta}\right\|_{H}^2 + \|u - \tilde{u}\|_{\mathcal{X}}^2 - 2\|\alpha'\|_{L^\infty(\mathbb{R})}\|\eta - \tilde{\eta}\| \|\nabla (u - \tilde{u})\|_{[\mathcal{X}]^N}^2
\]

\[
\geq \|\nabla(\eta - \tilde{\eta})\|_{[H]^N}^2 + \kappa^2 \|\nabla (u - \tilde{u})\|_{[\mathcal{X}]^N}^2 + R_0\left\|\eta - \tilde{\eta}\right\|_{H}^2 + \|u - \tilde{u}\|_{\mathcal{X}}^2
\]

\[
- 2\frac{\|\alpha'\|_{L^\infty(\mathbb{R})}^2}{\kappa^2}\|\eta - \tilde{\eta}\|_{H}^2 - \frac{\kappa^2}{2} \|\nabla (u - \tilde{u})\|_{[\mathcal{X}]^N}^2
\]

\[
\geq \|\eta - \tilde{\eta}\|_{H}^2 + \|u - \tilde{u}\|_{\mathcal{X}}^2 + \frac{\kappa^2}{2} \|\nabla (u - \tilde{u})\|_{[\mathcal{X}]^N}^2 \geq 0,
\]

which implies the (strict) monotonicity of the operator \([\partial_\eta \Phi_{\varepsilon,v,\delta}^\kappa \times \partial_u \Phi_{\varepsilon,v,\delta}^\kappa] + R_0 \mathcal{I}\) in \(\mathcal{X} \times \mathcal{X}\).

Next, we prove that (II) holds. Since \(\partial \Phi_{\varepsilon,v,\delta}^\kappa \subseteq \left[\partial_\eta \Phi_{\varepsilon,v,\delta}^\kappa \times \partial_u \Phi_{\varepsilon,v,\delta}^\kappa\right]\) in \(\mathcal{X} \times \mathcal{X}\), we can see that

\[\varepsilon 6) \partial \Phi_{\varepsilon,v,\delta}^\kappa + R_0 \mathcal{I} \subseteq \left[\partial_\eta \Phi_{\varepsilon,v,\delta}^\kappa \times \partial_u \Phi_{\varepsilon,v,\delta}^\kappa\right] + R_0 \mathcal{I} \text{ in } \mathcal{X} \times \mathcal{X}, \text{ i.e., } \partial \Phi_{\varepsilon,v,\delta}^\kappa \text{ is contained in a monotone graph } \left[\partial_\eta \Phi_{\varepsilon,v,\delta}^\kappa \times \partial_u \Phi_{\varepsilon,v,\delta}^\kappa\right] + R_0 \mathcal{I} \text{ in } \mathcal{X} \times \mathcal{X}.
\]

Moreover, by Barbu (2010, Theorem 2.10) and Brézis (1973, Corollary 2.11), we also have

\[\varepsilon 7) \partial \left(\Phi_{\varepsilon,v,\delta}^\kappa + \frac{R_0}{\kappa^2} \cdot \|\|_{\mathcal{X}}^2\right) = \partial \Phi_{\varepsilon,v,\delta}^\kappa + R_0 \mathcal{I} \text{ in } \mathcal{X} \times \mathcal{X}, \text{ and hence, } \left[\partial_\eta \Phi_{\varepsilon,v,\delta}^\kappa \times \partial_u \Phi_{\varepsilon,v,\delta}^\kappa\right] + R_0 \mathcal{I} \text{ coincides with the maximal monotone graph } \partial \left(\Phi_{\varepsilon,v,\delta}^\kappa + \frac{R_0}{\kappa^2} \cdot \|\|_{\mathcal{X}}^2\right) \text{ in } \mathcal{X} \times \mathcal{X}.
\]

Thus, item (II) is a direct consequence of \(\varepsilon 6)\) and \(\varepsilon 7)\). \(\square\)

We can now give the proof of Theorems 1 and 2.

**Proof of Theorem 1** On account of Lemmas 2 and 3, one can observe that the system \((P)_{\varepsilon,v,\delta}^\kappa\) is equivalent to the following Cauchy problem, denoted by \((\mathcal{C}P)_{\varepsilon,v,\delta}^\kappa\).
Cauchy problem $(\tilde{CP})_{\varepsilon, \nu, \delta}^\kappa$:

\[
\begin{align*}
U'(t) + \left[ \partial_\eta \Phi_{\varepsilon, \nu, \delta}^\kappa \times \partial_u \Phi_{\varepsilon, \nu, \delta}^\kappa \right](U(t)) + \mathcal{G}_\delta^\kappa(U(t)) & \geq 0 \text{ in } \mathcal{X}, \ t \in (0, T), \\
U(0) & = U_0 \text{ in } \mathcal{X}.
\end{align*}
\]

Additionally, by Lemma 4, the above Cauchy problem $(\tilde{CP})_{\varepsilon, \nu, \delta}^\kappa$ is equivalent to the Cauchy problem $(CP)_{\varepsilon, \nu, \delta}^\kappa$.

Thus, $(P)_{\varepsilon, \nu, \delta}^\kappa$, $(CP)_{\varepsilon, \nu, \delta}^\kappa$, and $(\tilde{CP})_{\varepsilon, \nu, \delta}^\kappa$ are equivalent to each other, and this finishes the proof of Theorem 1. \(\square\)

**Proof of Theorem 2** First, we show that item (I) holds. From hypothesis (A6) with \(u_0 \in L^4(\Omega; \mathbb{R}^M)\) and \(v_0 \in W^{1,N+1}(\Omega; \mathbb{R}^M)\), we get \(U_0 = [\eta_0, u_0] \in D(\Phi_{\varepsilon, \nu, \delta}^\kappa)\). Hence, by applying general theories in evolution equations, e.g., Barbu (2010, Theorem 4.1, p. 158), Brézis (1973, Theorem 3.6 and Proposition 3.2), Ito et al. (1998, Section 2) and Kenmochi (1981, Theorem 1.1.2), we immediately obtain the existence and uniqueness of a solution \(U = [\eta, u] \in L^2(0, T; \mathcal{X}) \cap L^\infty(0, T; \mathcal{M})\), and \(\Phi_{\varepsilon, \nu, \delta}^\kappa(U) \in L^\infty(0, T)\). (4.29)

Observe that (4.2) and (4.29) imply the regularity stated in (4.4). Additionally, for any \(0 < T < \infty\), there exists a positive constant \(C_1\), independent of \(U_0\), such that

\[
\|U'\|_{L^2(0,T;\mathcal{X})}^2 + \sup_{t \in (0,T)} \Phi_{\varepsilon, \nu, \delta}^\kappa(U(t)) \leq C_1 \left( 1 + \|U_0\|_{\mathcal{X}}^2 + \Phi_{\varepsilon, \nu, \delta}^\kappa(U_0) \right). (4.30)
\]

Concerning (4.3), we take \(h(s) := s - \max\{0, \min\{s, 1\}\}\) and consider \(\varphi = \varphi_t := h(\eta(t)) \in V\) in Definition 2. Thus, integrating in \((0, \tau)\) for any \(\tau \in (0, T)\), and having (A3) in mind, we get

\[
\frac{1}{2} \int_0^\tau \int_\Omega \partial_t(h^2(\eta)) \, dx \, dt = -\int_0^\tau \int_\Omega g(\eta)h(\eta) \, dx \, dt - \int_0^\tau \int_\Omega \alpha'(\eta)h(\eta) f_\varepsilon(\nabla u) \, dx \, dt - \int_0^\tau \int_\Omega |\nabla h(\eta)|^2 \, dx \, dt \leq -\int_0^\tau \int_\Omega g(\eta)h(\eta) \, dx \, dt.
\]

Moreover, by (A2),

\[
g(\eta)h(\eta) = \begin{cases} 
    g(\eta)\eta \geq C|\eta| + g(0)\eta \geq -C\eta^2 & \text{if } \eta \leq 0, \\
    0 & \text{if } \eta \in [0, 1], \\
    g(\eta)(\eta - 1) \geq -C|\eta - 1|^2 & \text{if } \eta > 1
\end{cases}
\]

where \(C\) is the Lipschitz constant of \(g\). Therefore,

\[
\frac{1}{2} \int_0^\tau \int_\Omega \partial_t(h^2(\eta)) \, dx \, dt \leq C \int_0^\tau \int_\Omega h^2(\eta) \, dx \, dt,
\]

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and (4.3) follows from Gronwall inequality.

Next, we verify item (II). We fix a bounded open interval \( I \subset (0, T) \). Taking into account (4.5)–(4.7), (4.2), and (4.30), one can observe that \( \{ \Phi_{\kappa_n, \nu_n, \delta_n}^{\infty}(U_0, n) \}_{n=1}^\infty \) is bounded, and furthermore,

\[
\begin{align*}
\{ U_n \}_{n=1}^\infty &= \{ [\eta_n, u_n] \}_{n=1}^\infty \text{ is bounded in } W^{1,2}(I; \mathcal{X}) \cap L^\infty(I; \mathcal{W}), \\
\{ \nu_n \nabla u_n \}_{n=1}^\infty &= \{ \nu_n \nabla u_n \}_{n=1}^\infty \text{ is bounded in } L^\infty(I; L^{N+1}(\Omega; \mathbb{R}^M)).
\end{align*}
\]

Therefore, applying the Poincaré–Wirtinger inequality and general compactness theories, such as Ascoli’s theorem (cf. Simon 1987, Corollary 4), we find subsequences of \( \{ \kappa_n, \nu_n, \delta_n \}_{n=1}^\infty \) and \( \{ U_n \}_{n=1}^\infty \) (not relabeled), such that

\[
\begin{align*}
U_n &\rightarrow U \text{ in } C(\bar{I}; \mathcal{X}), \text{ weakly in } W^{1,2}(I; \mathcal{X}), \text{ and weakly-* in } L^\infty(I; \mathcal{W}), \\
\alpha(\eta_n) &\rightarrow \alpha(\eta) \text{ in } L^2(I; H), \\
\nu_n u_n &\rightharpoonup \nu u \text{ weakly in } L^{N+1}(I; W^{1,N+1}(\Omega; \mathbb{R}^M)), \text{ as } n \rightarrow \infty.
\end{align*}
\]

(4.31a)

Also, by Lemma 1 and (Fact 2), we have that \( \{ \hat{\Phi}_n \}_{n=1}^\infty \), a sequence of proper, l.s.c., and convex functions on \( L^2(I; \mathcal{X}) \), defined as

\[
\begin{align*}
\hat{\Phi}_n(\tilde{U}) &= \begin{cases} 
\int_I \Phi_{\kappa_n, \nu_n, \delta_n}^{\infty}(\tilde{U}(t)) \, dt, & \text{if } \Phi_{\kappa_n, \nu_n, \delta_n}^{\infty}(\tilde{U}) \in L^1(I), \\
\infty, & \text{otherwise},
\end{cases}
\end{align*}
\]

for \( n = 1, 2, 3, \ldots \), converges to a proper, l.s.c., and convex function \( \hat{\Phi} \) on \( L^2(I; \mathcal{X}) \), defined as

\[
\begin{align*}
\hat{\Phi}(\tilde{U}) &= \begin{cases} 
\int_I \Phi_{\kappa_n, \nu_n, \delta_n}^{\infty}(\tilde{U}(t)) \, dt, & \text{if } \Phi_{\kappa_n, \nu_n, \delta_n}^{\infty}(\tilde{U}) \in L^1(I), \\
\infty, & \text{otherwise};
\end{cases}
\end{align*}
\]

on \( L^2(I; \mathcal{X}) \), in the sense of Mosco, as \( n \rightarrow \infty \).

From (4.31) and (Fact 1), it is seen that

\[
[ U, -U' - G_\delta(U) ] \in \partial \hat{\Phi}(U) \text{ in } L^2(I; \mathcal{X}) \times L^2(I; \mathcal{X}),
\]

(4.32)

and

\[
\hat{\Phi}_n(U_n) \rightarrow \hat{\Phi}(U) \text{ as } n \rightarrow \infty.
\]

(4.33)
Note that (4.6), (4.32), and (Fact 1) enable us to say that $U = [\eta, u]$ is a solution to the Cauchy problem (CP)$_{\eta, \nu, \delta}^\kappa$. So, due to uniqueness of solutions, the convergences stated in (4.31) hold for the whole sequence and not only for subsequences.

On the other hand, from (A3), (A4), and (4.31), it is inferred that

$$\lim_{n \to \infty} \frac{1}{\kappa_n^2} \int_I \int_{\Omega} \alpha(\eta_n)^2 \, dxdt = \frac{1}{\kappa^2} \int_I \int_{\Omega} \alpha(\eta)^2 \, dxdt,$$

(4.34a)

$$\lim_{n \to \infty} \frac{1}{2} \int_I \int_{\Omega} |\nabla \eta_n|^2 \, dxdt \geq \frac{1}{2} \int_I \int_{\Omega} |\nabla \eta|^2 \, dxdt,$$

(4.34b)

$$\lim_{n \to \infty} \frac{1}{\delta_n^2} \int_I \int_{\Omega} |u_n|^4 \, dxdt \geq \frac{1}{\delta} \int_I \int_{\Omega} |u|^4 \, dxdt,$$

(4.34c)

$$\lim_{n \to \infty} \frac{N+1}{N+1} \int_I \int_{\Omega} |\nu_n \nabla u_n|^{N+1} \, dxdt \geq \frac{1}{N+1} \int_I \int_{\Omega} |\nu \nabla u|^{N+1} \, dxdt,$$

(4.34d)

$$\lim_{n \to \infty} \frac{\kappa_n^2}{2} \int_I \int_{\Omega} f_{\epsilon_n}(\nabla u_n)^2 \, dxdt \geq \frac{\kappa^2}{2} \int_I \int_{\Omega} f_{\epsilon}(\nabla u)^2 \, dxdt,$$

(4.34e)

and

$$\lim_{n \to \infty} \int_I \int_{\Omega} \alpha(\eta_n) f_{\epsilon_n}(\nabla u_n) \, dxdt \geq \lim_{n \to \infty} \int_I \int_{\Omega} \alpha(\eta) f_{\epsilon}(\nabla u) \, dxdt$$

$$- \lim_{n \to \infty} \|\alpha(\eta_n) - \alpha(\eta)\|_{L^2(I; H)} \sup_{n \in \mathbb{N}} \left( \int_I \int_{\Omega} (\epsilon_n^2 + |\nabla u_n|^2) \, dxdt \right)^{\frac{1}{2}}$$

$$- \|\alpha(\eta)\|_{H} \lim_{n \to \infty} \left( \int_I \int_{\Omega} |\epsilon_n - \epsilon|^2 \, dxdt \right)^{\frac{1}{2}}$$

$$\geq \int_I \int_{\Omega} \alpha(\eta) f_{\epsilon}(\nabla u) \, dxdt.$$  

(4.34f)

From (4.33) and (4.34), it follows that

$$\begin{align*}
\lim_{n \to \infty} \|
abla \eta_n\|_{L^2(I; [H]^N)}^2 &= \|
abla \eta\|_{L^2(I; [H]^N)}^2, \\
\lim_{n \to \infty} \|
abla u_n\|_{L^2(I; L^2(\Omega; \mathbb{R}^MN))}^2 &= \|
abla u\|_{L^2(I; L^2(\Omega; \mathbb{R}^MN))}^2, \\
\lim_{n \to \infty} \|
abla \nu \nabla u_n\|_{L^{N+1}(I; L^{N+1}(\Omega; \mathbb{R}^MN))}^{N+1} &= \|
abla \nu \nabla u\|_{L^{N+1}(I; L^{N+1}(\Omega; \mathbb{R}^MN))}^{N+1}.
\end{align*}$$

(4.35)

Now, taking into account (4.31) and (4.35) and applying the uniform convexity of $L^2$-based topologies, one can observe that as $n \to \infty$,

$$\begin{align*}
\eta_n &\to \eta \text{ in } L^2(I; V), \\
u_n \nabla u_n &\to \nu \nabla u \text{ in } L^{N+1}(I; L^{N+1}(\Omega; \mathbb{R}^MN)).
\end{align*}$$

(4.36)

Thus, (6) and (7) are direct consequences of (4.31) and (4.36).
5 Proof of Main Theorem

In this section, we show the proof of Main Theorem. To see this, we take limits in \((P)_{\kappa, \nu, \delta}^\varepsilon\) as \(\delta, \nu, \varepsilon \to 0^+\), respectively.

First of all, we derive an energy inequality for \(F_{\kappa, \nu, \delta}^\varepsilon\) and a priori estimates for the approximate solutions.

5.1 Energy Inequality and A Priori Estimates

Lemma 5 Let \(U_0 = [\eta_0, u_0] \in \mathcal{W}\), with \(u_0 \in L^4(\Omega; \mathbb{R}^M), \nu u_0 \in W^{1,N+1}(\Omega; \mathbb{R}^M)\), and let \(U_{\varepsilon, \nu, \delta} = [\eta_{\varepsilon, \nu, \delta}, u_{\varepsilon, \nu, \delta}]\) be a solution to \((P)_{\kappa, \nu, \delta}^\varepsilon\).

Then, for any \(T > 0\), \(U_{\varepsilon, \nu, \delta}\) satisfies the following energy inequality

\[
F_{\varepsilon, \nu, \delta}(U_{\varepsilon, \nu, \delta}(s)) + \int_0^s \|\partial_t U_{\varepsilon, \nu, \delta}(t)\|^2_X \, dt \leq F_{\varepsilon, \nu, \delta}(U_0) \quad \text{for all } s \in [0, T]. \tag{5.1}
\]

Moreover, it follows that

\[
\begin{align*}
\{ & U_{\varepsilon, \nu, \delta} \in W^{1,2}(0, T; \mathcal{X}) \cap L^\infty(0, T; \mathcal{W}), \\
& \kappa^2 \nabla u_{\varepsilon, \nu, \delta} \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^{MN})), \nu \nabla u_{\varepsilon, \nu, \delta} \in L^\infty(0, T; L^{N+1}(\Omega; \mathbb{R}^{MN})),
\end{align*}
\tag{5.2}
\]

and

\[
\begin{align*}
\delta^{-1}\|u_{\varepsilon, \nu, \delta}\|^2 - 1& \|L^\infty(0,T;H) \leq F_{\varepsilon, \nu, \delta}(U_0), \\
\|\alpha'(\eta_{\varepsilon, \nu, \delta})[\nabla f_{\varepsilon}(\nabla u_{\varepsilon, \nu, \delta})]\|_{L^\infty(\Omega_T; \mathbb{R}^{MN})} & \leq C,
\end{align*}
\tag{5.3}
\]

for some constant \(C > 0\), independent of \(\varepsilon, \nu, \) and \(\delta\).

Proof Multiplying both sides of the equation of \(u_{\varepsilon, \nu, \delta}\) in \((P)_{\kappa, \nu, \delta}^\varepsilon\) by \(\partial_t u_{\varepsilon, \nu, \delta}\), we have

\[
\|\partial_t u_{\varepsilon, \nu, \delta}\|^2_X + \frac{d}{dt}\Psi^\kappa_{\varepsilon, \nu, \delta}(\eta_{\varepsilon, \varepsilon, \delta}, u_{\varepsilon, \nu, \delta}) - \int_\Omega \alpha'(\eta_{\varepsilon, \nu, \delta})\partial_t \eta_{\varepsilon, \nu, \delta} f_{\varepsilon}(\nabla u_{\varepsilon, \nu, \delta}) dx = 0. \tag{5.4}
\]

Also, multiplying both sides of the equation of \(\eta_{\varepsilon, \nu, \delta}\) in \((P)_{\kappa, \nu, \delta}^\varepsilon\) by \(\partial_t \eta_{\varepsilon, \nu, \delta}\), it follows that

\[
\|\partial_t \eta_{\varepsilon, \nu, \delta}\|^2_H + \frac{d}{dt}\Psi_0(\eta_{\varepsilon, \nu, \delta}) + \int_\Omega \alpha'(\eta_{\varepsilon, \nu, \delta})\partial_t \eta_{\varepsilon, \nu, \delta} f_{\varepsilon}(\nabla u_{\varepsilon, \nu, \delta}) dx = 0. \tag{5.5}
\]

Adding up (5.4)–(5.5) and integrating from 0 to \(s\), we get (5.1).

The regularity and the a priori estimates are immediately obtained from Theorem 2 and (5.1).

Lemma 6 If \(|u_0| \leq 1 \text{ a.e. in } \Omega\), then solutions \(u_{\varepsilon, \nu, \delta}\) to \((P)_{\kappa, \nu, \delta}^\varepsilon\) satisfy \(|u_{\varepsilon, \nu, \delta}| \leq 1 \text{ a.e. in } \Omega\).
Proof We define the following nonnegative monotone increasing function on $[0, \infty)$:

$$
\chi(z) := \frac{(z - 1)_+}{z} = \begin{cases} 
0 & \text{for } 0 \leq z \leq 1, \\
\frac{z - 1}{z} & \text{for } z > 1.
\end{cases}
$$

Multiplying both sides of the equation of $u_{\varepsilon,v,\delta}$ in $(P^\varepsilon)_{\varepsilon,v,\delta}$ by $u_{\varepsilon,v,\delta} \chi(|u_{\varepsilon,v,\delta}|)$, and using $(A5)$, we have

$$
\frac{1}{2} \frac{d}{dt} \int_{|u_{\varepsilon,v,\delta}| > 1} (|u_{\varepsilon,v,\delta}| - 1)^2 dx + \int_{|u_{\varepsilon,v,\delta}| > 1} \alpha(\eta_{\varepsilon,v,\delta}) (\nabla f_{\varepsilon}) (\nabla u_{\varepsilon,v,\delta}) \\
+ \kappa^2 \nabla u_{\varepsilon,v,\delta} + \nu |\nabla u_{\varepsilon,v,\delta}|^{N-1} |\nabla u_{\varepsilon,v,\delta}| : \nabla \left( u_{\varepsilon,v,\delta} \chi(|u_{\varepsilon,v,\delta}|) \right) dx
$$

$$
= - \int_{|u_{\varepsilon,v,\delta}| > 1} (\alpha(\eta_{\varepsilon,v,\delta}) (\nabla f_{\varepsilon}) (\nabla u_{\varepsilon,v,\delta}) \\
+ \kappa^2 |\nabla u_{\varepsilon,v,\delta}|^2 + |\nabla u_{\varepsilon,v,\delta}|^{N+1}) \chi(|u_{\varepsilon,v,\delta}|) dx
$$

$$
- \int_{|u_{\varepsilon,v,\delta}| > 1} \left( \frac{\alpha(\eta_{\varepsilon,v,\delta})}{\sqrt{\varepsilon^2 + |\nabla u_{\varepsilon,v,\delta}|^2}} + \kappa^2 + \nu |\nabla u_{\varepsilon,v,\delta}|^{N-1} \right) \frac{|\nabla u_{\varepsilon,v,\delta}|^2 |\nabla u_{\varepsilon,v,\delta}|}{4|u_{\varepsilon,v,\delta}|^3} dx \leq 0.
$$

Here,

$$
\nabla u_{\varepsilon,v,\delta} : \frac{u_{\varepsilon,v,\delta} \nabla |u_{\varepsilon,v,\delta}|}{|u_{\varepsilon,v,\delta}|^2} = \frac{\nabla u_{\varepsilon,v,\delta} : (\nabla u_{\varepsilon,v,\delta}) u_{\varepsilon,v,\delta}}{|u_{\varepsilon,v,\delta}|^3} = \frac{|(\nabla u_{\varepsilon,v,\delta}) u_{\varepsilon,v,\delta}|^2}{|u_{\varepsilon,v,\delta}|^3} = \frac{|\nabla u_{\varepsilon,v,\delta}|^2}{4|u_{\varepsilon,v,\delta}|^3}.
$$

Therefore,

$$
\int_{|u_{\varepsilon,v,\delta}(t)| > 1} (|u_{\varepsilon,v,\delta}(t)| - 1)^2 dx \leq \int_{|u_{\varepsilon,v,\delta}(0)| > 1} (|u_{\varepsilon,v,\delta}(0)| - 1)^2 dx = 0.
$$

Hence, $|u_{\varepsilon,v,\delta}| \leq 1$ a.e. in $\Omega$. \hfill \Box

At the end of this subsection, we introduce some compactness results which can be proved as Chen et al. (1994, Theorem 2.1) and Misawa (2001, Lemma 9) by using the fact that the operator $\text{div} \left( \alpha(\eta_{\varepsilon,v,\delta}) (\nabla f_{\varepsilon}) (\nabla \cdot) + \kappa^2 (\nabla \cdot) + \nu |\nabla \cdot|^{N+1} (\nabla \cdot) \right)$ is uniformly elliptic.

**Lemma 7** (cf. Barrett et al. 2008, Lemma 2.2) Let $\varepsilon > 0$, $\nu > 0$ and $\kappa > 0$ be fixed. Let $\{w_{\varepsilon,v,\delta}\}_{\delta > 0}$ be bounded in $W^{1,2}(0, T; \mathbb{X}) \cap L^\infty(0, T; \mathbb{W})$, $\{\eta_{\varepsilon,v,\delta}\}_{\delta > 0}$ be bounded in $W^{1,2}(0, T; H) \cap L^\infty(0, T; V)$, and $\{f_{\varepsilon,v,\delta}\}_{\delta > 0}$ be bounded in $L^1(0, T; L^1(\Omega; \mathbb{R}^M))$ uniformly in $\delta$, respectively. Moreover, we assume that $0 \leq \eta_{\varepsilon,v,\delta} \leq 1$ a.e. in $\Omega_T$ and
Lemma 8 (cf. Barrett et al. 2008, Lemma 2.2) Let \( \varepsilon > 0 \) and \( \kappa > 0 \) be fixed. Let \( \{w_{\varepsilon, v}\}_{v>0} \) be bounded in \( W^{1,2}(0, T; \mathbb{X}) \cap L^\infty(0, T; \mathbb{W}) \), \( \{\eta_{\varepsilon, v}\}_{v>0} \) be bounded in \( W^{1,2}(0, T; H) \cap L^\infty(0, T; V) \), and \( \{f_{\varepsilon, v}\}_{v>0} \) be bounded in \( L^1(0, T; L^1(\Omega; \mathbb{R}^M)) \) uniformly in \( v \), respectively. Moreover, we assume that \( 0 \leq \eta_{\varepsilon, v} \leq 1 \) a.e. in \( \Omega_T \) and \( w_{\varepsilon, v} \) satisfies the equation, for \( \varepsilon > 0 \),

\[
\partial_t w_{\varepsilon, v} - \text{div}(\alpha(\eta_{\varepsilon, v})[\nabla f_{\varepsilon}](\nabla w_{\varepsilon, v}) + \kappa^2 \nabla w_{\varepsilon, v} + v|\nabla w_{\varepsilon, v}|^{N-1} v \nabla w_{\varepsilon, v}) = f_{\varepsilon, v} \quad \text{in } \Omega_T,
\]

in the sense of distributions. Then, \( \{w_{\varepsilon, v}\}_{\varepsilon > 0} \) is precompact in \( L^q(0, T; W^{1,q}(\Omega; \mathbb{R}^M)) \) for all \( 1 \leq q < 2 \).

Lemma 9 (cf. Barrett et al. 2008, Lemma 2.3) Let \( \kappa > 0 \) be fixed. Let \( \{w_{\varepsilon}\}_{\varepsilon > 0} \) be bounded in \( W^{1,2}(0, T; \mathbb{X}) \cap L^\infty(0, T; \mathbb{W}) \), \( \{\eta_{\varepsilon}\}_{\varepsilon > 0} \) be bounded in \( W^{1,2}(0, T; H) \cap L^\infty(0, T; V) \), and \( \{f_{\varepsilon}\}_{\varepsilon > 0} \) be bounded in \( L^1(0, T; L^1(\Omega; \mathbb{R}^M)) \) uniformly in \( \varepsilon \), respectively. Moreover, we assume that \( 0 \leq \eta_{\varepsilon} \leq 1 \) a.e. in \( \Omega_T \) and \( w_{\varepsilon} \) satisfies the equation, for \( \varepsilon > 0 \),

\[
\partial_t w_{\varepsilon} - \text{div}(\alpha(\eta_{\varepsilon})[\nabla f_{\varepsilon}](\nabla w_{\varepsilon}) + \kappa^2 \nabla w_{\varepsilon}) = f_{\varepsilon} \quad \text{in } \Omega_T,
\]

in the sense of distributions. Then, \( \{w_{\varepsilon}\}_{\varepsilon > 0} \) is precompact in \( L^1(0, T; W^{1,1}(\Omega; \mathbb{R}^M)) \).

5.2 The Limit as \( \varepsilon \to 0 \)

In this subsection, we study the \( \varepsilon \to 0 \) limit problem in \( (P)_{\varepsilon, v, \delta} \), assuming \( 0 < \varepsilon < \varepsilon_0 \); i.e., we solve the following problem.

Problem \((P)_{\varepsilon, v, \delta} : \)

\[
\begin{align*}
\partial_t \eta_{\varepsilon, v} - \Delta \eta_{\varepsilon, v} + g(\eta_{\varepsilon, v}) + \alpha(\eta_{\varepsilon, v}) f_{\varepsilon}(\nabla u_{\varepsilon, v}) &= 0 \quad \text{in } \Omega_T, \\
\partial_t u_{\varepsilon, v} - \text{div}(\alpha(\eta_{\varepsilon, v})[\nabla f_{\varepsilon}](\nabla u_{\varepsilon, v}) + \kappa^2 \nabla u_{\varepsilon, v} + v|\nabla u_{\varepsilon, v}|^{N-1} v \nabla u_{\varepsilon, v}) &= \mu_{\varepsilon, v} u_{\varepsilon, v} \quad \text{in } \Omega_T,
\end{align*}
\]

subject to the initial and boundary conditions

\[
\begin{align*}
\nabla \eta_{\varepsilon, v} \cdot \mathbf{n} &= 0, \quad (\alpha(\eta_{\varepsilon, v})[\nabla f_{\varepsilon}](\nabla u_{\varepsilon, v}) + \kappa^2 \nabla u_{\varepsilon, v} + v|\nabla u_{\varepsilon, v}|^{N-1} v \nabla u_{\varepsilon, v}) \mathbf{n} = 0 \quad \text{on } \Gamma_T, \\
\eta_{\varepsilon, v}(0, x) &= \eta_0(x), \quad u_{\varepsilon, v}(0, x) = u_0(x), \quad x \in \Omega,
\end{align*}
\]

where

\[
\mu_{\varepsilon, v} := (\alpha(\eta_{\varepsilon, v})[\nabla f_{\varepsilon}](\nabla u_{\varepsilon, v}) + \kappa^2 \nabla u_{\varepsilon, v} + v|\nabla u_{\varepsilon, v}|^{N-1} v \nabla u_{\varepsilon, v}) : \nabla u_{\varepsilon, v}, \quad \text{a.e. in } \Omega_T.
\]
Note that the dependence on $\kappa$ is not anymore needed or used. Therefore, we have removed it.

**Theorem 3** Let $U_0 = [\eta_0, u_0] \in \mathcal{W}$ with $|u_0| = 1$ in $\Omega$ and $v u_0 \in W^{1,N+1}(\Omega; \mathbb{R}^M)$. Then, there exists $U_{\varepsilon,v} = [\eta_{\varepsilon,v}, u_{\varepsilon,v}] \in C([0, T]; \mathcal{X})$ such that $U_{\varepsilon,v}$ satisfies $(P)_{\varepsilon,v}$ in the sense of distributions,

\[
\begin{align*}
U_{\varepsilon,v} \in W^{1,2}(0, T; \mathcal{X}) \cap L^\infty(0, T; \mathcal{W}), \\
\kappa^2 \nabla u_{\varepsilon,v} \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^{MN})), \\
v \nabla u_{\varepsilon,v} \in L^\infty(0, T; L^{N+1}(\Omega; \mathbb{R}^{MN})), \\
0 \leq \eta_{\varepsilon,v} \leq 1 \quad \text{a.e. in } \Omega_T, \quad |u_{\varepsilon,v}| = 1 \quad \text{a.e. in } \Omega_T,
\end{align*}
\]

and

\[
\mathcal{F}_{\varepsilon,v}(U_{\varepsilon,v}(s)) + \int_0^s \| \partial_t U_{\varepsilon,v}(t) \|^2 d\bar{\chi} + \varepsilon, \nu
\leq \mathcal{F}_{\varepsilon,v}(U_0) \leq \mathcal{F}_{\varepsilon,0,v}(U_0) \quad \text{for all } s \in [0, T],
\]

\[
\left\| \alpha(\eta_{\varepsilon,v})|\nabla f_\varepsilon|(|\nabla u_{\varepsilon,v}| + \kappa^2 \nabla u_{\varepsilon,v}) \right\|_{L^\infty(\Omega_T; \mathbb{R}^{MN})} \leq C,
\]

\[
\left\| \text{div} \left( \alpha(\eta_{\varepsilon,v})|\nabla f_\varepsilon|(|\nabla u_{\varepsilon,v}| + \kappa^2 \nabla u_{\varepsilon,v}) \right) \right\|_{L^2(0,T; L^1(\Omega; \mathbb{R}^M))} < C,
\]

\[
\left\| \text{div} \left( \left( \alpha(\eta_{\varepsilon,v})|\nabla f_\varepsilon|(|\nabla u_{\varepsilon,v}| + \kappa^2 \nabla u_{\varepsilon,v}) \right) \wedge u_{\varepsilon,v} \right) \right\|_{L^2(\Omega_T; \Lambda_2(\mathbb{R}^M))} < C,
\]

where the constant $C > 0$ is independent of $\varepsilon$ and $v$.

**Proof** We observe from (A5) that since $|u_0| = 1$, $\Pi_\delta(u_0) = 0$. Hence,

\[
\mathcal{F}_{\varepsilon,v,\delta}(U_0) = \mathcal{F}_{\varepsilon,v}(U_0) < \mathcal{F}_{\varepsilon,0,v}(U_0) =: C < +\infty.
\]

Therefore, recalling the energy inequality (5.1), the uniform estimates (5.2)–(5.3) and the invariance principle $|U_{\varepsilon,v,\delta}| \leq 1$ on $\Omega_T$ (Lemma 6) and $0 \leq \eta_{\varepsilon,v,\delta} \leq 1$ a.e. in $\Omega_T$, we obtain that there exist a subsequence $\{U_{\varepsilon,v,\delta_n}\}_n$ and a function $U_{\varepsilon,v} \in C([0, T]; \mathcal{X}) \cap W^{1,2}(0, T; \mathcal{X}) \cap L^\infty(0, T; \mathcal{W})$ such that

\[
\begin{align*}
U_{\varepsilon,v,\delta_n} & \to U_{\varepsilon,v} \quad \text{in } C([0, T]; \mathcal{X}), \text{ weakly in } W^{1,2}(0, T; \mathcal{X}), \\
\text{and weakly-}^* \text{ in } L^\infty(0, T; \mathcal{W}), \\
v u_{\varepsilon,v,\delta_n} & \to v u_{\varepsilon,v} \quad \text{weakly-}^* \text{ in } L^\infty(0, T; W^{1,N+1}(\Omega; \mathbb{R}^M)), \\
|u_{\varepsilon,v,\delta_n}| & \to 1 \quad \text{strongly in } L^2(0, T; H),
\end{align*}
\]

as $n \to \infty$, by an Aubin-type compactness theorem Simon (1987, Corollary 4). Thus, we also see that

\[
|u_{\varepsilon,v}| = 1 \quad \text{a.e. in } \Omega_T.
\]
By $|u_{e,v,\delta}| \leq 1$ in $\Omega_T$, the Lebesgue dominated convergence theorem implies that
\[ u_{e,v,\delta_n} \to u_{e,v} \quad \text{strongly in } L'(0, T; L'(\Omega; \mathbb{R}^M)) \quad \text{as } n \to \infty, \quad (5.13) \]
for all $r \in [1, \infty)$.

By $|u_{e,v,\delta}| \leq 1$ in $\Omega_T$ again, we can show that
\[
\int_0^T \int_\Omega |\sigma_\delta(u_{e,v,\delta})| \, dxdt \leq \frac{1}{\delta} \int_0^T \int_\Omega (1 - |u_{e,v,\delta}|^2)^2 \, dxdt \\
+ \frac{1}{\delta} \int_0^T \int_\Omega (1 - |u_{e,v,\delta}|^2)^2 |u_{e,v,\delta}|^2 \, dxdt.
\]

By (5.3), the first term of the right-hand side in the above inequality is uniformly bounded in $\delta$. Multiplying both sides of the equation of $u_{e,v,\delta}$ by $u_{e,v,\delta}$, integrating by parts, applying H"{o}lder inequality, and (5.7)–(5.10), we see that
\[
\text{By (2.15). Setting } \delta = \delta_n \text{ and letting } n \to \infty, \text{ we get}
\]
\[
\int_0^T \int_\Omega \left\{ (\partial_t u_{e,v} \wedge u_{e,v}, \omega) + \sum_{i=1}^N ((\alpha(\eta_{e,v})[\nabla f_e](\nabla u_{e,v,\delta})) \wedge u_{e,v,\delta}) \right\} \, dxdt = 0, \quad (5.15)
\]
for \( \omega \in L^\infty(0, T; W^{1, N+1}(\Omega_T; \mathbb{R}^M)) \) by (5.11), (5.13), and (5.14).

Taking \( \omega = (u_{e,v} \wedge \psi) \) in (5.15) for \( \psi \in C^1(\overline{\Omega_T}; \mathbb{R}^M) \) and using (2.9) and (2.15), we have

\[
\int_0^T \int_{\Omega_T} \left\{ (\partial_t u_{e,v} \wedge u_{e,v} \wedge (u_{e,v} \wedge \psi)) + \sum_{i=1}^N (\alpha(\eta_{e,v})[\nabla f_e](\partial_{x_i} u_{e,v})
+ \kappa^2 \partial_{x_i} u_{e,v} + v|\nabla u_{e,v}|N-1 v \partial_{x_i} u_{e,v} \wedge \partial_{x_i} (u_{e,v} \wedge (u_{e,v} \wedge \psi))) \right\} dx dt = 0.
\]

(5.16)

Noting (5.12), we see that for a.e. in \( \Omega_T \),

\[
\left\{ \begin{array}{l}
\partial_t u_{e,v} \cdot u_{e,v} = 0, \\
(\alpha(\eta_{e,v})[\nabla f_e](\nabla u_{e,v}) + \kappa^2 \nabla u_{e,v} + v|\nabla u_{e,v}|N-1 v \nabla u_{e,v})u_{e,v} = 0.
\end{array} \right.
\]

(5.17)

According to (2.6), we obtain

\[
u_{e,v} \wedge (u_{e,v} \wedge \psi) = (u_{e,v} \cdot \psi) u_{e,v} - \psi.
\]

(5.18)

Having (5.16)–(5.18) in mind, we can see that

\[
\int_0^T \int_{\Omega_T} \left\{ \partial_t u_{e,v} \cdot \psi + (\alpha(\eta_{e,v})[\nabla f_e](\nabla u_{e,v}) + \kappa^2 \nabla u_{e,v} + v|\nabla u_{e,v}|N-1 v \nabla u_{e,v}) : \nabla \psi \right\} dx dt
= \int_0^T \int_{\Omega_T} \mu_{e,v}(u_{e,v} \cdot \psi) dx dt,
\]

(5.19)

for \( \psi \in C^1(\overline{\Omega_T}; \mathbb{R}^M) \) by (2.2), (2.8), (2.9), (2.16).

On the other hand, it follows that

\[
\| \alpha'(\eta_{e,v,\delta_n})[\nabla f_e](\nabla u_{e,v,\delta_n}) \|_{L^\infty(\Omega_T; \mathbb{R}^{MN})} \leq \frac{\| \alpha' \|_{C([0,1])}}{\alpha^*} \| \alpha(\eta_{e,v,\delta_n})[\nabla f_e](\nabla u_{e,v,\delta_n}) \|_{L^\infty(\Omega_T; \mathbb{R}^{MN})} < \infty.
\]

Hence, we have

\[
\alpha'(\eta_{e,v,\delta_n})[\nabla f_e](\nabla u_{e,v,\delta_n}) \rightharpoonup \alpha'(\eta_{e,v})[\nabla f_e](\nabla u_{e,v}) \quad \text{weakly-}^* \quad \text{in} \quad L^\infty(\Omega_T; \mathbb{R}^{MN}),
\]

(5.20)

as \( n \to \infty \). By (5.11) and (5.20), it follows that

\[
\int_0^T (\partial_t \eta_{e,v}(t) + g(\eta_{e,v}(t)) + \alpha'(\eta_{e,v}(t)) f_e(\nabla u_{e,v}(t)), \varphi)_H dt
+ \int_0^T (\nabla \eta_{e,v}(t), \nabla \varphi)_H dt = 0,
\]

for any \( \varphi \in L^2(0, T; V) \cap L^\infty(\Omega_T) \).
Finally, by (5.1) and (5.11), we see that the energy inequality (5.7) immediately holds. Moreover, estimates (5.8)–(5.10) also follow by (5.7), (5.19), and (5.15), respectively. □

5.3 The Limit as \( \nu \to 0 \)

Now, the aim is to solve the limit problem in (P)\(_{\varepsilon,\nu}\), as \( \nu \to 0 \) assuming \( 0 < \varepsilon < \varepsilon_0 \); i.e., we solve Problem (P)\(_{\varepsilon}\):

\[
\begin{cases}
\partial_t \eta_\varepsilon - \Delta \eta_\varepsilon + g(\eta_\varepsilon) + \alpha' (\eta_\varepsilon) f_\varepsilon (\nabla u_\varepsilon) = 0 \text{ in } \Omega_T, \\
\partial_t u_\varepsilon - \text{div} (\alpha (\eta_\varepsilon) |\nabla f_\varepsilon| (\nabla u_\varepsilon) + \kappa^2 \nabla u_\varepsilon) = \mu_\varepsilon u_\varepsilon \text{ in } \Omega_T,
\end{cases}
\]

subject to the initial and boundary conditions

\[
\begin{aligned}
\nabla \eta_\varepsilon \cdot n = 0, & \quad (\alpha (\eta_\varepsilon) |\nabla f_\varepsilon| (\nabla u_\varepsilon) + \kappa^2 \nabla u_\varepsilon) n = 0 \text{ on } \Gamma_T, \\
\eta(0, x) = \eta_0(x), & \quad u_\varepsilon(0, x) = u_0(x), \quad x \in \Omega,
\end{aligned}
\]

where

\[ \mu_\varepsilon := (\alpha (\eta_\varepsilon) |\nabla f_\varepsilon| (\nabla u_\varepsilon) + \kappa^2 \nabla u_\varepsilon) : \nabla u_\varepsilon, \text{ a.e. in } \Omega_T. \]

**Theorem 4** Let \( U_0 = [\eta_0, u_0] \in \mathcal{W} \) with \( u_0 \in S^{M-1} \) in \( \Omega \). Then, there exists \( U_\varepsilon = [\eta_\varepsilon, u_\varepsilon] \in C ([0, T]; \mathcal{X}) \) such that \( U_\varepsilon \) satisfies (P)\(_{\varepsilon}\) in the sense of distributions,

\[
\begin{cases}
U_\varepsilon \in W^{1,2}(0, T; \mathcal{X}) \cap L^\infty (0, T; \mathcal{W}), \\
\kappa^2 \nabla u_\varepsilon \in L^\infty (0, T; L^2 (\Omega; \mathbb{R}^{MN})), \\
0 \leq \eta_\varepsilon \leq 1 \quad \text{a.e. } \Omega_T, \quad u_\varepsilon \in S^{M-1} \text{ a.e. in } \Omega_T,
\end{cases}
\]

and

\[
\begin{align*}
F_\varepsilon (U_\varepsilon (s)) + \int_0^s \| \partial_t U_\varepsilon (t) \|_X^2 \, dt & \leq F_\varepsilon (U_0) \quad \text{for all } s \in [0, T], \\
\| \alpha (\eta_\varepsilon) |\nabla f_\varepsilon| (\nabla u_\varepsilon) \|_{L^\infty (\Omega_T; \mathbb{R}^{MN})} & \leq C, \\
\| \text{div} (\alpha (\eta_\varepsilon) |\nabla f_\varepsilon| (\nabla u_\varepsilon) + \kappa^2 \nabla u_\varepsilon) \|_{L^2 (0, T; L^1 (\Omega; \mathbb{R}^{MN}))} & < C, \\
\| \text{div} (\alpha (\eta_\varepsilon) |\nabla f_\varepsilon| (\nabla u_\varepsilon) + \kappa^2 \nabla u_\varepsilon) \wedge u_\varepsilon \|_{L^2 (\Omega_T; \Lambda^2 (\mathbb{R}^{M}))} & < C,
\end{align*}
\]

where \( C \) does not depend on \( \varepsilon \). Moreover, if \( u_0 \in S^{M-1}_{+, r} \) in \( \Omega \) for \( r \in (0, 1) \), then \( u_\varepsilon \in S^{M-1}_{+, r} \) a.e. in \( \Omega_T \).
Proof Let \( \{U_{0,n}\}_{n=1}^{\infty} = \{[\eta_{0,n}, u_{0,n}]\}_{n=1}^{\infty} \subset \mathcal{U} \) with \( |u_{0,n}| = 1 \) a.e. in \( \Omega \), and \( \{v_{n} u_{0,n}\}_{n=1}^{\infty} \subset W^{1,N+1}(\Omega; \mathbb{R}^{M}) \) be a sequence of initial data satisfying (4.6). By the same argument as that in the proofs of Theorems 2 and 3, we obtain the existence of a subsequence \( \{U_{\varepsilon,v_{n}}\}_{n} \) and a function \( U_{\varepsilon} \in C([0, T]; \mathbb{R}) \cap W^{1,2}(0, T; \mathbb{R}) \cap L^{\infty}(0, T; \mathcal{W}) \) such that

\[
\begin{aligned}
U_{\varepsilon,v_{n}} &\to U_{\varepsilon} \quad \text{in } C([0, T]; \mathbb{R}), \text{ weakly in } W^{1,2}(0, T; \mathbb{R}), \\
v_{n} (|v_{n} \nabla u_{\varepsilon,v_{n}}|^{N-1} v_{n} \nabla u_{\varepsilon,v_{n}}) &\to 0 \quad \text{in } L^{2}(0, T; L^{2}(\Omega; \mathbb{R}^{MN})), \\
|u_{\varepsilon,v_{n}}| &\to 1 \quad \text{strongly in } L^{2}(0, T; H),
\end{aligned}
\]

as \( n \to \infty \). Therefore, we also see that

\[
u_{n} (|\nabla u_{\varepsilon,v_{n}}|^{N-1} \nabla u_{\varepsilon,v_{n}}) \to 0 \quad \text{in } L^{2}(0, T; L^{2}(\Omega; \mathbb{R}^{MN})),
\]

(5.25)

(5.26)

The rest of the proof follows the same way as Theorem 3 using Lemma 8 instead of Lemma 7. In fact, we have

\[
\int_{0}^{T} \int_{\Omega} \left\{ (\partial_{t} u_{\varepsilon} \wedge u_{\varepsilon}), \omega \right\} + \sum_{i=1}^{N} \left\{ (\alpha(\eta_{\varepsilon})|\nabla f_{\varepsilon}|(\partial_{x_{i}} u_{\varepsilon}) + \kappa^{2} \partial_{x_{i}} u_{\varepsilon} \wedge u_{\varepsilon}, \partial_{x_{i}} \omega) \right\} \, dx \, dt = 0,
\]

for \( \omega \in L^{\infty}(0, T; H^{1}(\Omega; \Lambda_{2}(\mathbb{R}^{M}))) \), and

\[
\int_{0}^{T} \int_{\Omega} \left\{ \partial_{t} u_{\varepsilon} \cdot \psi + (\alpha(\eta_{\varepsilon})|\nabla f_{\varepsilon}| \nabla u_{\varepsilon}) + \kappa^{2} \nabla u_{\varepsilon} : \nabla \psi \right\} dx \, dt = \int_{0}^{T} \int_{\Omega} \mu_{\varepsilon}(u_{\varepsilon} \cdot \psi) dx \, dt,
\]

for \( \psi \in C^{1}(\overline{\Omega_{T}}; \mathbb{R}^{M}) \).

The rest of the proof is the same as that of Theorem 3, and we omit the details.

Finally, let \( u_{0} \in S^{M-1}_{+,r} \) for \( r \in (0, 1) \). Therefore, Theorem 6 (invariance principle) implies that

\[
u_{n} (|\nabla u_{\varepsilon,v_{n}}|^{N-1} \nabla u_{\varepsilon,v_{n}}) \to 0 \quad \text{in } L^{2}(0, T; L^{2}(\Omega; \mathbb{R}^{MN})),
\]

which, by (5.25) implies that \( u_{\varepsilon} \in S^{M-1}_{+,r} \) a.e. in \( \Omega_{T} \).

5.4 The Limit as \( \varepsilon \to 0 \)

Finally, we solve the initial system (P) by letting \( \varepsilon \to 0 \). As \( \varepsilon \to 0 \), the limit problem is formulated as follows.

Problem (P):

\[
\begin{aligned}
\partial_{t} \eta - \Delta \eta + g(\eta) + \omega'(\eta)|\nabla u| &\equiv 0 \quad \text{in } \Omega_{T}, \\
\partial_{t} u - \text{div}(\alpha(\eta)B + \kappa^{2} \nabla u) &\equiv \mu u \quad \text{in } \Omega_{T},
\end{aligned}
\]
subject to the initial and boundary conditions
\[
\begin{align*}
\nabla \eta \cdot n &= 0, \quad (\alpha(\eta) B + \kappa^2 \nabla u) n = 0 \text{ on } \Gamma_T, \\
\eta(0, x) &= \eta_0(x), \quad u(0, x) = u_0(x), \quad x \in \Omega,
\end{align*}
\]
where
\[
B \in \text{Sgn}^{M,N}(\nabla u), \quad \text{and } \mu := (\alpha(\eta) B + \kappa^2 \nabla u) : \nabla u, \text{ a.e. in } \Omega_T.
\]

Now, to prove the Main Theorem, it suffices to show that the next result holds.

**Theorem 5** Let \( U_0 = [\eta_0, u_0] \in \mathcal{W} \) with \( u_0 \in S_{M-1} \) in \( \Omega \). Then, there exist \( U = [\eta, u] \in L^2(0, T; X) \) and \( B \in L^\infty(\Omega_T; \mathbb{R}^{MN}) \) such that
\[
\begin{align*}
U \in W^{1,2}(0, T; X) \cap L^\infty(0, T; \mathcal{W}), \\
\kappa^2 \nabla u \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^{MN})), \\
0 \leq \eta \leq 1 \text{ a.e. in } \Omega_T, \quad u \in S_{M-1} \text{ a.e. in } \Omega_T,
\end{align*}
\]
and
\[
\mathcal{F}(U(s)) + \int_0^s \| \partial_t U(t) \|^2_X dt \leq \mathcal{F}(U_0) \quad \text{for all } s \in [0, T].
\]

Also, there exists a constant \( C > 0 \) such that
\[
\begin{align*}
\| \alpha(\eta) B \|_{L^\infty(\Omega_T; \mathbb{R}^{MN})} &\leq C, \\
\| \text{div}(\alpha(\eta) B + \kappa^2 \nabla u) \|_{L^2(0, T; L^1(\Omega; \mathbb{R}^M))} &< C, \\
\| \text{div}(\alpha(\eta) B + \kappa^2 \nabla u) \wedge u \|_{L^2(\Omega_T; \Lambda^2(\mathbb{R}^M))} &< C.
\end{align*}
\]

Moreover, if \( u_0 \in S_{M-1} \) in \( \Omega \) for \( r \in (0, 1) \), then \( u \in S_{M-1} \) a.e. in \( \Omega_T \).

**Proof** Let \( U_\varepsilon \) be the solution to \((P)_\varepsilon\) constructed in Sect. 5.3. Due to (5.21), (5.22)–(5.24), (5.26), we can find a subsequence \( \{ U_{\varepsilon_n} \}_n \) and a function \( U \in C([0, T]; \mathcal{X}) \cap W^{1,2}(0, T; \mathcal{X}) \cap L^\infty(0, T; \mathcal{W}) \) such that
\[
\begin{align*}
U_{\varepsilon_n} &\rightharpoonup U \text{ in } C([0, T]; \mathcal{X}), \text{ weakly in } W^{1,2}(0, T; \mathcal{X}), \\
&\quad \text{weakly-} \ast \quad \text{in } L^\infty(0, T; \mathcal{W}), \\
|U_{\varepsilon_n}| &\rightarrow 1 \text{ strongly in } L^2(0, T; H),
\end{align*}
\]
as \( n \rightarrow \infty \), by an Aubin-type compactness theorem Simon (1987, Corollary 4). Therefore, we also see that
\[
\begin{align*}
\quad u \in S_{M-1} \quad \text{a.e. in } \Omega_T.
\end{align*}
\]
Since $|u| = 1$ in $\Omega_T$, the Lebesgue dominated convergence theorem implies that

$$u_{\varepsilon_n} \rightarrow u \text{ strongly in } L^r(0, T; L^r(\Omega; \mathbb{R}^M)) \text{ as } n \rightarrow \infty,$$

for all $r \in [1, \infty)$.

Next, we set $f_\varepsilon := \mu_\varepsilon u_\varepsilon$ and we note that $f_\varepsilon$ is uniformly bounded in $L^1(0, T; L^1(\Omega; \mathbb{R}^M))$ in $\varepsilon$. By Lemma 9, it follows that

$$\nabla u_{\varepsilon_n} \rightarrow \nabla u \text{ strongly in } L^1(0, T; L^1(\Omega; \mathbb{R}^{MN})) \text{ as } n \rightarrow \infty.$$

The above convergence and (5.22) imply that there exists $B \in L^\infty(\Omega_T; \mathbb{R}^{MN})$ such that

$$\alpha(\eta_{\varepsilon_n})[\nabla f_{\varepsilon_n}](\nabla u_{\varepsilon_n}) \rightarrow \alpha(\eta)B \text{ weakly-}* \text{ in } L^\infty(\Omega_T; \mathbb{R}^{MN}),$$

$$\alpha(\eta_{\varepsilon_n})[\nabla f_{\varepsilon_n}](\nabla u_{\varepsilon_n}) + \kappa^2 \nabla u_{\varepsilon_n} \rightarrow \alpha(\eta)B + \kappa^2 \nabla u \text{ weakly in } L^2(\Omega_T; \mathbb{R}^{MN}),$$

as $n \rightarrow \infty$. In addition, we see that

$$\alpha(\eta)B : \nabla u = \alpha(\eta)|\nabla u| \text{ a.e. in } \Omega_T.$$

Letting $n \rightarrow \infty$ in (5.27), we get

$$\int_0^T \int_\Omega \{((\partial_t u \wedge u), \omega)_2 + \sum_{i=1}^N (((\alpha(\eta)B_i + \kappa^2 \partial_{x_i} u) \wedge u), \partial_{x_i} \omega)_2\}dxdt = 0,$$

(5.29)

for $\omega \in C^1(\overline{\Omega_T}; \Lambda_2(\mathbb{R}^M))$ (here, $B = (B_i)$, $B_i \in L^\infty(\Omega_T; \mathbb{R}^M)$). Since $B \in L^\infty(\Omega_T; \mathbb{R}^{MN})$ and $\kappa^2 \nabla u \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^{MN}))$, a standard density argument implies that (5.29) holds for any $\omega \in L^\infty(0, T; W^{1,2}(\Omega; \Lambda_2(\mathbb{R}^M)))$. Taking $\omega = (u \wedge \psi)$ for $\psi \in C^1(\overline{\Omega_T}; \mathbb{R}^M)$ in (5.29), we have

$$\int_0^T \int_\Omega \left\{((\partial_t u \wedge u) \wedge (u \wedge \psi) + \sum_{i=1}^N (((\alpha(\eta)B_i + \kappa^2 \partial_{x_i} u) \wedge u) \wedge \partial_{x_i} (u \wedge \psi)) \right\}dxdt = 0.$$

In a similar way to Sect. 5.2, we obtain that $\mu = (\alpha(\eta)B + \kappa^2 \nabla u) : \nabla u \in L^1(0, T; L^1(\Omega))$ and the following equation is satisfied

$$\int_0^T \int_\Omega \{\partial_t u \cdot \psi + (\alpha(\eta)B + \kappa^2 \nabla u) : \nabla \psi\}dxdt = \int_0^T \int_\Omega \mu u \cdot \psi dxdt.$$

The rest of the proof is the same as that of Theorem 3 and we omit it. Finally, in the case that $u_0 \in S_0^{M-1}$ a.e. in $\Omega_T$, by Theorem 5 and (5.28) we easily get the conclusion.
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Appendix A

In this appendix, we recall the notion of Mosco convergence and some results about it that we use in the paper.

Definition 3 (Mosco convergence: cf. Mosco 1969) Let $X$ be an abstract Hilbert space. Let $\Psi : X \to (-\infty, \infty]$ be a proper, l.s.c., and convex function, and let $\{\Psi_n\}_{n=1}^{\infty}$ be a sequence of proper, l.s.c., and convex functions $\Psi_n : X \to (-\infty, \infty), n = 1, 2, 3, \ldots$. Then, it is said that $\Psi_n \to \Psi$ on $X$, in the sense of Mosco, as $n \to \infty$, iff. the following two conditions are fulfilled:

- **(M1) Lower-bound condition:**
  \[
  \lim_{n \to \infty} \Psi_n(\hat{w}_n) \geq \Psi(\hat{w}), \text{ if } \hat{w} \in X, \{\hat{w}_n\}_{n=1}^{\infty} \subset X, \text{ and } \hat{w}_n \rightharpoonup \hat{w} \text{ weakly in } X, \text{ as } n \to \infty.
  \]

- **(M2) Optimality:** for any $\hat{w} \in D(\Psi)$, there exists a sequence $\{\hat{w}_n\}_{n=1}^{\infty} \subset X$ such that $\hat{w}_n \rightharpoonup \hat{w}$ in $X$ and $\Psi_n(\hat{w}_n) \to \Psi(\hat{w})$, as $n \to \infty$.

If a net of convex functions $\{\hat{\Psi}_\varepsilon\}_{\varepsilon \in \Xi}$ is labeled by a continuous argument $\varepsilon \in \Xi$ with a range $\Xi \subset \mathbb{R}$, then for any $\varepsilon_0 \in \Xi$, we will say that $\{\hat{\Psi}_\varepsilon\}_{\varepsilon \in \Xi}$ Mosco converges to $\hat{\Psi}_{\varepsilon_0}$, as $\varepsilon \to \varepsilon_0$, if all subsequences $\{\hat{\Psi}_{\varepsilon_{n}}\}_{n=1}^{\infty}$ Mosco converge to $\hat{\Psi}_{\varepsilon_{0}}$, for all sequences $\{\varepsilon_{n}\}_{n=1}^{\infty} \subset \Xi$, satisfying $\varepsilon_{n} \to \varepsilon_{0}$ as $n \to \infty$.

Remark 5 Let $X$, $\Psi$, and $\{\Psi_n\}_{n=1}^{\infty}$ be as in Definition 3. Then, the following hold.

**Fact 1** (cf. Attouch 1984, Theorem 3.66 and Kenmochi 1981, Chapter 2) Assume that

\[
\Psi_n \to \Psi \text{ on } X, \text{ in the sense of Mosco, as } n \to \infty, \quad (5.30)
\]

and

\[
\left\{
\begin{array}{ll}
[w, w^*] \in X \times X, & [w_n, w_n^*] \in \partial \Psi_n \text{ in } X \times X, n \in \mathbb{N}, \\
w_n \to w \text{ in } X \text{ and } w_n^* \rightharpoonup w^* \text{ weakly in } X, \text{ as } n \to \infty.
\end{array}
\right.
\]
Then, it holds that

\[ [w, w^n] \in \partial \Psi \text{ in } X \times X, \text{ and } \Psi_n(w_n) \to \Psi(w), \text{ as } n \to \infty. \]

**Fact 2** (cf. Colli et al. 2017, Lemma 4.1 and Giga et al. 2004, Appendix) Let \( d \in \mathbb{N} \), and let \( S \subset \mathbb{R}^d \) be a bounded open set. Then, under the Mosco convergence as in (5.30), a sequence \( \{ \hat{\Psi}_n^S \}_{n=1}^\infty \) of proper, l.s.c., and convex functions on \( L^2(S; X) \), defined as

\[
\begin{align*}
\hat{\Psi}_n^S(w) &= \int_S \Psi_n(w(t)) \, dt, & \text{if } \Psi_n(w) \in L^1(S), \\
&= \infty, & \text{otherwise,}
\end{align*}
\]

converges to a proper, l.s.c., and convex function \( \hat{\Psi}^S \) on \( L^2(S; X) \), defined as

\[
\begin{align*}
\hat{\Psi}(z) &= \int_S \Psi(z(t)) \, dt, & \text{if } \Psi(z) \in L^1(S), \\
&= \infty, & \text{otherwise;}
\end{align*}
\]

on \( L^2(S; X) \), in the sense of Mosco, as \( n \to \infty \).

**Example 1** (Example of Mosco convergence) Let \( \{ f_\varepsilon \}_{\varepsilon \geq 0} \subset C(\mathbb{R}^d) \) be the sequence of non-expansive convex functions defined in (A4). Then, for any \( \varepsilon, \tilde{\varepsilon} \geq 0 \), the uniform estimate

\[
|f_\varepsilon(W) - f_{\tilde{\varepsilon}}(\tilde{W})| \leq \|\varepsilon, W - [\tilde{\varepsilon}, \tilde{W}]\|_{\mathbb{R}^{1+MN}} \leq |\varepsilon - \tilde{\varepsilon}| + \|W - \tilde{W}\|_{\mathbb{R}^{MN}},
\]

for all \( \varepsilon, \tilde{\varepsilon} \geq 0 \) and \( y, \tilde{y} \in \mathbb{R}^d \), immediately implies

\[
f_\varepsilon \to f_{\varepsilon_0} \text{ uniformly on } \mathbb{R}^{MN}, \text{ as } \varepsilon \to \varepsilon_0.
\]

This uniform convergence leads to

\[
f_\varepsilon \to f_{\varepsilon_0} \text{ on } \mathbb{R}^{MN}, \text{ in the sense of Mosco, as } \varepsilon \to \varepsilon_0.
\]

**Remark 6** We recall that for a sequence of proper l.s.c. and convex functions on a Hilbert space, the uniform convergence implies the convergence in the sense of Mosco.

**Appendix B**

In this appendix, we give some insight and results related to the group of rotations \( SO(3) \) as a Riemannian manifold that we use throughout the paper.

First of all, according to Euler’s theorem, any rotation can be characterized by an angle of rotation \( w \in [0, 2\pi] \) and an axis of rotation \( n \in S^2 \). Moreover, the range
of the angle can be taken into \([0, \pi]\) (note that \((w, n)\) and \((-w, -n)\) correspond to the same rotation) and in case \(w \neq \pi\), this representation is unique. It is well known that \(SO(3)\) is diffeomorphic to the real projective space \(\mathbb{P}^3(\mathbb{R})\). However, we will use instead that \(SO(3)\) is diffeomorphic to a quotient group in the unit hypersphere \(S^3\) in \(\mathbb{R}^4\).

We construct the following surjective homeomorphism between the group of unit quaternions (which is isomorphic to the unit special group \(SU(2)\)) and \(SO(3)\):

\[
q = (q^0, q^1, q^2, q^3) \mapsto q(\bullet)q^{-1}.
\]

It is easy to see that this is a two-to-one homeomorphism and that for any rotation there are two unit quaternions associated to it: \(q\) and \(-q\). Moreover, the following relation between quaternions and axis-angle parametrization is obtained:

\[
q^0 = \pm \left| \cos \left( \frac{w}{2} \right) \right|, \quad q^i = \pm \left| \sin \left( \frac{w}{2} \right) \right| n_i, \quad i = 1, \ldots, 3.
\]

Therefore, since the group of unit quaternions is isomorphic to \(S^3\), we conclude that \(SO(3)\) is diffeomorphic to the quotient group \(S^3/ \sim\) where \(\sim\) is the equivalence class of antipodal points in the hypersphere; i.e., \(p \sim q\) iff \(p + q = 0\). Furthermore, we see that points in the equator (i.e., of the form \((0, p_1, p_2, p_3)\)) correspond to the angle of rotation \(w = \pi\). Therefore, it is obtained that rotations with angle of rotation \(w \in [0, \pi]\) can be uniquely identified as points in the open upper hemisphere \(S^3_+\); i.e., \((p_0, p_1, p_2, p_3)\) with \(0 < p_0 \leq 1\). We also denote by \(p_0\) the north pole on the sphere.

In the paper, we restrict ourselves to initial rotations \(P_0\) such that the angle of rotation \(wp_0\) satisfies \(0 \leq wp_0 < \pi\). Therefore, we can equivalently work with initial data \(u_0 \in S^3_+\) and with the geometry of \(S^3_+\), which is much simpler and it has been used much wider in variational constraint problems than that of \(SO(3)\). However, in order that the solution to the flow still belongs to \(S^3_+\) and therefore it can be identified with a rotation, we need the following result:

**Theorem 6** (Invariance principle) Suppose that \(u_0 \in B_g(p_0; R)\), with \(R < \frac{\pi}{2}\) (equivalently \(wp_0 \in [0, r]\) with \(r < \pi\)). Then, the solution to \((P)_{\varepsilon, \nu}\) satisfies

\[
u_{\varepsilon, \nu} \in B_g(p_0; R), \quad \text{a.e. in } \Omega.
\]

**Proof** The proof follows the proof of Giacomelli et al. (2019, Lemma 4). We proceed by contradiction. Let \(T^* = \inf\{t \in [0, T]: u(t; \Omega) \not\in B_g(p_0; R)\}\). Due to continuity of \(u\), there is a \(\delta > 0\) such that \(u(t; \Omega) \subset B_g(p_0; \frac{\pi}{2})\) for \(t \in [0, T^* + \delta]\).

We now take the equation for \(u_{\varepsilon, \nu}\) in \((P)_{\varepsilon, \nu}\) and we take the projection \(\pi_{u_{\varepsilon, \nu}}\) from \(\mathbb{R}^M\) to \(T_{u_{\varepsilon, \nu}}S^{M-1}\). Noting that \(\pi_{u_{\varepsilon, \nu}}(\mu_{\varepsilon, \nu}, u_{\varepsilon, \nu}) = 0\), we get

\[
\partial_t u = \pi_u \text{div}(Z),
\]

with

\[
u_{\varepsilon, \nu} = u_{\varepsilon, \nu}, \quad Z := \alpha(\eta_{\varepsilon, \nu}) (\nabla f_{\varepsilon})(\nabla u_{\varepsilon, \nu}) + \kappa^2 \nabla u_{\varepsilon, \nu} + \nu|\nabla u_{\varepsilon, \nu}|^{N-1} \nu \nabla u_{\varepsilon, \nu}.
\]

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We choose on $B_g(p_0; \frac{\pi}{2})$ a polar coordinate system $p \mapsto (p^r; p^{\theta_1}, \ldots, p^{\theta_{M-2}})$ centered at $p_0$. Next, we compute the second equation in $(P)_{\epsilon, \nu}$ for the radial coordinate.

The metric in the polar coordinates around the north pole is the following one:

$$g = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \sin^2(r) & 0 & \cdots & 0 \\
0 & 0 & \sin^2(r) \sin^2(\theta_1) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sin^2(r) \sin^2(\theta_1) \cdots \sin^2(\theta_{M-3})
\end{pmatrix}.$$

Therefore, the Christoffel symbols of the second kind for the variable $r$ are

$$\Gamma^r = -\frac{\sin(2r)}{2} \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \sin^2(\theta_1) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sin^2(\theta_1) \cdots \sin^2(\theta_{M-3})
\end{pmatrix}.$$

Next, we note that (see Eells and Sampson 1964)

$$\pi_{u}^i (\text{div} Z)^i = \text{div} Z^i + \sum_{j,k,l} \Gamma^i_{j,k}(u) u^{j}_x Z^k_{x} \quad i = r, \theta_1, \ldots, \theta_{M-2},$$

for any $Z \in W^{1,1}(\Omega; \mathbb{R}^{MN})$ such that $Z \in T_u(S^{M-1})$.

Thus, we get that the equation for the radial coordinate in (5.31) is the following one:

$$u^r_t = \text{div} Z^r - \frac{\sin 2u^r}{2} \left( \frac{\alpha(\eta)}{\sqrt{\varepsilon^2 + |\nabla u|^2}} + \kappa^2 + \nu^{N+1} |\nabla u|^{N-1} \right)$$

$$\times \left( |\nabla u^{\theta_1}|^2 + \sum_{i=1}^{M-3} \sin^2(u^{\theta_1}) \cdots \sin^2(u^{\theta_i}) |\nabla u^{\theta_{i+1}}|^2 \right).$$

Therefore, since $u^r \in [0, \frac{\pi}{2}]$,

$$u^r_t \leq \text{div} Z^r.$$

Thus,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u^r - R)^2_+ = \int_{\Omega} u^r_t (u^r - R)_+ \leq \int_{\Omega} \text{div} Z^r (u^r - R)_+$$

$$= -\int_{\Omega \cap \{u^r > R\}} Z^r : \nabla u^r$$

$$= -\int_{\Omega \cap \{u^r > R\}} |\nabla u^r|^2 \left( \frac{\alpha(\eta)}{\sqrt{\varepsilon^2 + |\nabla u|^2}} + \kappa^2 + \nu^{N+1} |\nabla u|^{N-1} \right)$$

$$\leq 0.$$

This finishes the proof.
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