A Decision Procedure for String Logic with Equations, Regular Membership and Length Constraints

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Abstract. In this paper, we consider the satisfiability problem for string logic with equations, regular membership and Presburger constraints over length functions. The difficulty comes from multiple occurrences of string variables making state-of-the-art algorithms non-terminating. Our main contribution is to show that the satisfiability problem in a fragment where no string variable occurs more than twice in an equation is decidable. In particular, we propose a semi-decision procedure for arbitrary string formulae with word equations, regular membership and length functions. The essence of our procedure is an algorithm to enumerate an equivalent set of solvable disjuncts for the formula. We further show that the algorithm always terminates for the aforementioned decidable fragment. Finally, we provide a complexity analysis of our decision procedure to prove that it runs, in the worst case, in factorial time.

Keywords: String logic · Satisfiability · Decision Procedure · Inductive Predicates

1 Introduction

There has been significant recent interest in reasoning about web and database programs for bug finding [3] and vulnerability verification [17] due to a huge number of security threats over the Internet. In these reasoning systems, solvers for constraint languages over strings (a.k.a. string solvers) plays a central role. The problem of solving word equations had been established. In 1977, Makanin notably proved that the satisfiability problem of word equations is decidable [22]. Following up the great Makanin's seminal paper, many studies either improved complexity for this algorithm [18,11,23] or search for a minimal and complete set of solutions [15,24]. However, reasoning about web applications and database programs typically requires a constraint language including word equations, regular membership and arithmetic on length functions. As an example, a function which generates new user accounts is often required to validate validity of user-name (whether it contains some special characters i.e., '@') and password (whether its length is longer than a certain number i.e., 8). Since the length constraints implied by a word equation is not always represented with finitely many equations in numeric form described by Plandowski [24], developing a decision procedure for the combined theories is not straightforward.

There has been a few studies on foundations for string formulas which combine word equations, regular membership and length constraints. Ganesh et. al. presented decidability result for the combination of word equations and linear arithmetic [10]. The
formulas in this fragment are restricted such that no string variable occurs twice in an equation. Abdulla et. al. further extended the result with regular membership to acyclic fragment [1]. Liang et. al. formalized the acyclic fragment without word equations using the calculus in [21]. Finally, Ganesh et. al. have recently shown the undecidability of the satisfiability problem for the theories over string equations, length function, and string-number conversion predicate [9]. So far, there is no decision procedure supporting for a fragment of word equations and length functions beyond the acyclic fragment discussed above.

Practical approaches to solving constraints of string logic have been developed dramatically. Initial approaches [13,14,28,29] which are based on automata have difficulties in handling string constraints related to length functions. To overcome this problem, bounded approaches - automata-based [17,12] as well as bit vector-based [6,25] - support those queries whose string variables have bounded lengths. These approaches could efficiently support for satisfiability (SAT). However, they may not be sound for unsatisfiability (UNSAT). Recently, unbounded approaches [31,20,26,27,30] support words as primitive type and are successfully integrated into Satisfiability Modulo Theories framework. The main technique used in these solvers is “Unfold-and-Match” which is to incrementally reduce the size of the input, via splitting and/or unfolding process. Although this technique is effective and efficient for a large number of queries over the combined theories of string and arithmetic, it does not work for those queries which have more than one occurrence of every string variables. For instance, the solvers [31,19,20,26,2] did not terminate when deciding satisfiability for the following formula which has two occurrences of the string variable $s$:

\[ \pi \equiv a \cdot b \cdot s = s \cdot b \cdot a \]

For efficiency, new heuristics has recently introduced in [30] and [27] to avoid such non-termination. However, these approaches are not complete. Our main contribution is a decision procedure for the constraint language including the formula $\pi$ above.

In this work, we present a new semi-decision procedure, called $S_{21SEA}$, for a fragment of string logic, called SEA, which includes word equations, regular membership and arithmetical constraints over length functions. The proposed procedure provides an answer, which is either SAT (with a model, a valuation assignment to variables of the input) or UNSAT, for the satisfiability problem. Different to the existing approaches, we propose inductive predicate to model string variable together with length function. The core idea of $S_{21SEA}$ is an algorithm to enumerate the complete set of solutions for a given SEA formula. Each solution is solvable i.e., is defined in a sound and complete base logic, called 0SEA fragment.

$S_{21SEA}$ takes a formula in SEA logic as input. It iteratively constructs a series of unfolding trees for the input by unfolding inductive predicates in a complete manner until either a SAT leaf or a proof of UNSAT is identified. In each iteration, it examines every leaves of the tree (the disjunction of which is equivalent to the input formula) with under-approximation, over-approximation and back-link construction for cyclic proofs. In particular, $S_{21SEA}$ first checks satisfiability for leaves which are in the base logic. These leaves are under-approximation of the input and are precisely decided. Second, $S_{21SEA}$ over-approximates open (non-unsatisfiable) leaves prior to checking their unsatisfiability. Next, remaining open leaves are either linked back to an interior nodes (to
form a partial cyclic proof). Leaves which are either unsatisfiable, or linked are marked closed. Otherwise, they are open. Finally, if all leaves are closed then $S_2^{1SEA}$ returns \texttt{UNSAT}. Otherwise, it chooses an open leaf in a depth-first manner for unfolding inductive predicates, matching and moving to the next iteration. For unfolding, $S_2^{1SEA}$ applies an Unfold-and-Match strategy on the leading terms (either string variables or constant characters) of the left-hand-side (LHS) and right-hand-side (RHS) of a word equation.

Our main contribution is a decidable subfragment, called $1SEA$, so as the proposed procedure always terminates. There are two restrictions on $1SEA$ formulas. The first restriction is that either (i) no string variable occurs twice in an equation or (ii) no string variable occurs more than twice in an equation with some additional restrictions in arithmetic. The second restriction applied on formulas with multiple word equations is that every formulas deduced by $S_2^{1SEA}$ satisfy the first restriction. Our Unfold-and-Match strategy ensures that notational length of the equation decreases at least one for type (i) formulas and does not increase for type (ii) formulas. This makes $S_2^{1SEA}$ solver terminating for formulas in the $1SEA$ fragment. We undertake a complexity analysis of our decision procedure which shows that, in the worst case, it runs in linear time for type (i) and in factorial time for type (ii) of $1SEA$.

Contributions. We make the following primary contributions.

- We propose semi-decision procedure $S_2^{1SEA}$ for word equations, regular expression and arithmetic constraints on length functions.
- We present a subfragment where $S_2^{1SEA}$ always terminates and thus becomes a decision procedure.
- We provide computational complexity results for the satisfiability on the decidable fragments.

2 Preliminaries

In this section, we present the string logic $SEA$. We also describe a normalized form which our solver is built upon.

2.1 $SEA$ String Logic

Concrete string models assume a finite alphabet $\Sigma$, set of finite words over $\Sigma^*$, and a set of integer numbers $\mathbb{Z}$. We work with a set $U$ of string variables denoting words in $\Sigma^*$, and a set $I$ of arithmetical variables.

Syntax The syntax of quantifier-free string formulas in $SEA$ is presented in Fig. 1. Regular expressions $R$ does not contains any string variables. We use $E$ to denote a word equation and $Es$ a conjunctive sequence of word equations. $Es_i$ to denote the $i^{th}$ word equation in the sequence. We use $w_{|k|}$ for an arbitrary word in $\Sigma^*$ with length $k$, and $w^n$ to denote the word which is a concatenation of $n$ word $w$, i.e. $w^n \equiv w \cdots w$ ($n$ copies). We use $\pi[t_1/t_2]$ for a substitution of all occurrences of $t_2$ in $\pi$ to $t_1$. We define inductive predicate $\text{STR}$ to encode string variables as follows.
### Fig. 1. Syntax

**Definition 1 (STR Predicate)** A string variable is defined via the inductive predicate \( \text{STR} \) as:

\[
\text{STR}(u, n) \equiv u = \epsilon \land n = 0 \lor \text{STR}(u_1, n_1) \land u = c \cdot u_1 \land n_1 = n - 1 \land n > 0,
\]

where \( u \) and \( n \) are parameters: \( n \) is the length of string variable \( u \) and \( c \in \Sigma \).

This predicate has the invariant \( n \geq 0 \). In the inductive rule, \( u_1 \) is a subterm of \( u \) and \( u = c \cdot u_1 \) is a subterm constraint. This subterm is important for cyclic proof to detect isomorphic word equations. A string variable may be in bare form (without a \( \text{STR} \) predicate) or \( \text{STR} \) predicate instance. We emphasize that \( \text{STR} \) instances are generated and used by our solver. They do not appear in the user-provided formulas. We inductively define length function of a string term \( tr \), denoted as \( |tr| \), as follows.

\[
|\epsilon| = 0 \quad |c| = 1 \quad |w| = k \quad |\text{STR}(u, n)| = n \quad |tr_1 \cdot tr_2| = |tr_1| + |tr_2|
\]

**Definition 2 (Equation Size)** Size of a word equation \( tr_1 = tr_2 \) is the sum of the notational length of \( tr_1 \) and \( tr_2 \).

We use \( E(n) \) to denote a word equation with size \( n \). For example, size of the word equation \( a \cdot b \cdot s = s \cdot b \cdot a \) is 6.

**Semantics** The semantics in this logic is mostly standard. Every regular expression \( R \) is evaluated to the language \( L(R) \). We define

\[
SStacks \overset{\text{def}}{=} U \to \Sigma^* \quad ZStacks \overset{\text{def}}{=} I \to \mathbb{Z}
\]

The semantics is given by a forcing relation: \( \eta, \beta_\eta \models \pi \) that forces the interpretation on both string \( \eta \) and arithmetic \( \beta_\eta \) to satisfy the constraint \( \pi \) where \( \eta \in SStacks \), \( \beta_\eta \in ZStacks \), and \( \pi \) is a formula.

The semantics of our language is formalized as in Figure 2. We use \texttt{true} (\texttt{false}) to syntactically denote a valid (unsatisfiable, respectively) formula. If \( \eta, \beta_\eta \models \pi \), we use the pair \( \langle \eta, \beta_\eta \rangle \) to denote a solution of the formula \( \pi \).

### 2.2 Normalized Form

We would like to remark that word disequalities can be eliminated using the approach in [1]. Thus, we only consider formulas which contain only one word equation in the normalized form.
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\( \pi \equiv \Sigma \cap I \cap \Lambda \) where

(i) \( \Sigma \) is a conjunction of word equations, (ii) \( I \) a conjunction of regular expressions, (iii) \( I \) is a conjunction of arithmetic constraints, (iv) and finally \( \Lambda \) is a conjunction of subterm relations obtained from unfolding inductive string predicates. We notice that if it is unambiguous, we sometimes use \( \Sigma, I, \) and \( \Lambda \) as sets instead of conjunctions. And while string variables in \( \Sigma \) may be encoded with the inductive predicates, those in \( \Lambda \) are not. For every string inductive predicate \( \text{STR}(u,n) \), its invariant \( n \geq 0 \) must be implied by \( I \). Each \( \Lambda \) is of the form either \( s_1 = c \cdot s_2 \) or \( s_1 = s_2 \cdot s_3 \), They are deduced during solving a formula and dedicated for constructing a model to witness \( \text{SAT} \).

3 Illustrative Example

We illustrate how \( S2_{\text{SEA}} \) solver solves satisfiability through the following example:

\[
\pi \equiv a \cdot b \cdot s = s \cdot b \cdot a \land s \in ((ab)^* \cdot a) \land |s| \cdot 2 = 0
\]

Initially, function \( \text{init}_{\text{SEA}} \) pairs the string variable \( s \) in the word equation with a fresh inductive predicate \( \text{STR}(u,n) \) and transforms the constraint \( |s| \) into a fresh integer variable i.e., \( n \). Let \( \pi_0 = \text{init}_{\text{SEA}}(\pi) \), \( \pi_0 \) is as follows.

\[
\pi_0 \equiv a \cdot b \cdot \text{STR}(u,n) = \text{STR}(u,n) \cdot b \cdot a \land s \in ((ab)^* \cdot a) \land n \cdot 2 = 0 \land s = u
\]

To decide satisfiability, \( S2_{\text{SEA}} \) solver systematically constructs unfolding trees for the input \( \pi_0 \). Starting from the unfolding tree \( T_0 \) with one node \( \pi_0 \), \( S2_{\text{SEA}} \) derives unfolding trees for \( \pi_0 \) as in Figure 3. In this figure, underlined leaves are closed, \( \text{star} \) leaves are linked and \( T_2 \) is a cyclic proof. As the word equation in \( \pi_0 \) contains inductive predicates, \( \pi_0 \) is not considered for under-approximation. For over-approximation, \( S2_{\text{SEA}} \) replaces every word equations \( tr_1 = tr_2 \) by their corresponding length constraints \( |tr_1| = |tr_2| \). As so, the over-approximation of \( \pi_0 \) is: \( \text{inv}_0 \equiv 2 + n = n \land s \in ((ab)^* \cdot a) \land n \cdot 2 = 0 \). Since \( \text{inv}_0 \) is not unsatisfiable, \( S2_{\text{SEA}} \) unfolds the predicate instance \( u \) in \( \pi_0 \) to obtain the tree \( T_1 \) with two leaves \( \pi_{11} \) and \( \pi_{12} \) as follows.

\[
\begin{align*}
\pi_{11} & \equiv a \cdot b \cdot b \cdot a \land s \in ((ab)^* \cdot a) \land n \cdot 2 = 0 \land n = 0 \land s = u \wedge u = \epsilon \\
\pi_{12} & \equiv b \cdot a \cdot \text{STR}(u,n_1) = \text{STR}(u,n_1) \cdot b \cdot a \land s \in ((ab)^* \cdot a) \land n \cdot 2 = 0 \land n > 0 \land n_1 = n - 1 \land s = u_1 \wedge u_1 = a \cdot u
\end{align*}
\]

Fig. 2. Semantics

Fig. 3. Tree \( T_2 \).
In the 2\textsuperscript{nd} iteration, while $\pi_{11}$ is classified as unsatisfiable (unsat cores are underlined), $\pi_{12}$ is kept open as $\pi_{12}$ is not unsatisfiable. $S2\text{SEA}$ unfolds $\pi_{12}$ to obtain $T_2$ with two leaves as follows.

$\pi_{21} \equiv b \cdot a = b \cdot a \land s \in ((ab)^* \cdot a) \land n \cdot 2 = 0 \land n > 0 \land n_1 = n - 1 \land n_1 = 0 \land s = u_1 \land u_1 = a \cdot u \land u = \epsilon$

$\pi_{22} \equiv a \cdot b \cdot \text{STR}(u, l_2, r) = \text{STR}(u, n_2) \cdot b \cdot a \land s \in ((ab)^* \cdot a) \land n \cdot 2 = 0 \land n > 0 \land n_1 = n - 1 \land n_1 > 0 \land n_2 = n - 1$.

In the 3\textsuperscript{rd} iteration, while $\pi_{21}$ is marked closed through under-approximation checking, $\pi_{22}$ is linked back to $\pi_0$ by function $fp_{\text{SEA}}$. $fp_{\text{SEA}}$ links $\pi_{22}$ back to $\pi_0$ through the following steps.

1. First, it discards subterm constraints of $\pi_0$ and $\pi_{22}$ as these constraints are for counter-model construction and not for UNSAT checking. Let the remaining formula of $\pi_0$ and $\pi_{22}$ be $\pi'_0$ and $\pi'_{22}$, respectively.
2. Secondly, it substitutes the remaining of $\pi_{22}$ with the substitution $\theta$ where $\theta = [n'/n, n/n_2]$ and $\pi'_{22} \equiv a \cdot b \cdot \text{STR}(u, n) = \text{STR}(u, n_2) \cdot b \cdot a \land s \in ((ab)^* \cdot a) \land n' \cdot 2 = 0 \land n' > 0 \land n' = n_1 - 1 \land n_1 > 0 \land n_2 = n - 1$.
3. Finally, it checks whether the string-related part of $\pi'_{22}$ is identical to its counterpart in $\pi'_0$ and arithmetic of $\pi'_{22}$ implies the arithmetic of $\pi'_0$ i.e.,

$$n' \cdot 2 = 0 \land n' > 0 \land n' = n_1 - 1 \land n_1 > 0 \land n_2 = n - 1 \models n' \cdot 2 = 0$$

4 S2\text{SEA} Solver

In this section, we present the semi-decision procedure S2\text{SEA}. We first describe an overview of S2\text{SEA}.

4.1 Overview

The proposed satisfiability solvers S2\text{SEA} is an instantiation of the general satisfiability procedure S2SAT presented in [19]. S2SAT supports for a sound and complete base theory (logic) $L$ augmented with inductive predicates. The base theory $L$ must satisfy the following properties: (i) $L$ is closed under propositional combination and supports boolean variables; (ii) there exists a complete decision procedure for $L$. We use $\pi^i$ to denote a formula in $L$ and $\pi$ to denote a formula in the extended theory. Semantically, $\pi \equiv \bigvee_{i=0}^{n} \pi^i$, $n \geq 0$. We remark that in this work the base logic is $0\text{SEA}$ and the extended logic is $\text{SEA}$ which augmented the base logic with the inductive predicate $\text{STR}$. More inductive predicates to represent recursive functions (i.e., $\text{replaceAll}$) might be investigated in future work.

The instantiated satisfiability procedure S2\text{SEA} is presented in Algorithm 1. Intuitively, to decide satisfiability for a formula, e.g. $\pi$, S2\text{SEA} systematically enumerates an equivalent set of base formulas for $\pi$. Particularly, starting from $T_0$ which has one initialized node $\pi_0$, S2\text{SEA} iteratively constructs series of unfolding trees $T_i$ for $\pi$. An iteration of the algorithm is described in lines 3-13. Function $\text{UA}_{\text{SEA}}$ at line 3 checks whether there exists a leaf is in base logic and satisfiable. Function $\text{OA}_{\text{SEA}}$ at line 6 overapproximates a leaf (into the base logic) prior to checking its unsatisfiability. Function
Algorithm 1: S2\textsubscript{SEA} Solver

\begin{verbatim}
input : \pi
output: SAT or UNSAT
1 \texttt{i} \leftarrow 0; \pi_0 \leftarrow \text{init}\textsubscript{SEA}(\pi); T_0 \leftarrow \{\pi_0\}; /* initialize */
2 \textbf{while} true \textbf{do}
3 \hspace{1em} (is\_sat, T_i) \leftarrow \text{UA}\textsubscript{SEA}(T_i); /* check SAT */
4 \hspace{1em} \textbf{if} is\_sat \textbf{then} \textbf{return} SAT; /* SAT */
5 \hspace{1em} \textbf{else}
6 \hspace{2.5em} T_i \leftarrow \text{OA}\textsubscript{SEA}(T_i); /* prune UNSAT */
7 \hspace{2.5em} \textbf{if} is\_closed(T_i) \textbf{then} \textbf{return} UNSAT; /* UNSAT */
8 \hspace{2.5em} \textbf{else}
9 \hspace{4em} \pi_i \leftarrow \text{dfs}(T_i); i \leftarrow i+1;
10 \hspace{4em} T_i \leftarrow \text{unfold}\textsubscript{SEA}(\pi_i);
11 \hspace{2.5em} \textbf{end}
12 \hspace{1em} \textbf{end}
13 \textbf{end}
\end{verbatim}

\texttt{fp}\textsubscript{SEA} at line 7 links a leaf to an interior node to form a (partial) cyclic proof. Otherwise, it is marked open. At line 8, if all leaf nodes are closed, S2\textsubscript{SEA} returns UNSAT. Otherwise, at line 10 function \texttt{dfs} chooses an open leaf in a breadth-first manner and function \texttt{unfold}\textsubscript{SEA} unfolds the selected leaf at line 11.

The construction of cyclic proofs is the most interesting feature of the S2\textsubscript{SAT} framework. Intuitively, a cyclic proof is an unfolding tree whose some leaves are marked closed and remaining leaves are linked back to interior nodes. Function \texttt{fp}\textsubscript{SEA} is based on some weakening and substitution principles [19]. The soundness of cyclic proof is as follows.

\textbf{Theorem 4.1} ([19]) \textit{If there is a cyclic proof of \pi, \pi is UNSAT.}

As an instantiation of S2\textsubscript{SAT} framework, S2\textsubscript{SEA} is sound for both SAT and UNSAT. Its soundness is ensured under the following assumptions: the base logic 0SEA is both sound and complete, functions \texttt{UA}\textsubscript{SEA}, \texttt{OA}\textsubscript{SEA} and \texttt{fp}\textsubscript{SEA} are sound, and function \texttt{unfold}\textsubscript{SEA} has complete property (i.e. let \texttt{unfold}\textsubscript{SEA}(\pi) \equiv \pi_1 \lor ... \lor \pi_k \text{ then } \pi \models \pi_1 \lor ... \lor \pi_k). S2\textsubscript{SEA} always terminates for SAT. However, it may, in general, not terminate for UNSAT.

In the rest of this section, we define 0SEA formulas which is the foundation of the base logic of S2\textsubscript{SEA} (subsection 4.2). Next, in subsection 4.3 we present in details functions of S2\textsubscript{SEA}: \texttt{init}\textsubscript{SEA}, \texttt{UA}\textsubscript{SEA} (for under-approximation), \texttt{OA}\textsubscript{SEA} (for over-approximation), \texttt{fp}\textsubscript{SEA} (for cyclic proofs) and \texttt{unfold}\textsubscript{SEA} (for tree expansion). We discuss correctness, termination and computational complexity results in the next section.

\subsection*{4.2 0SEA Fragment}

In this paragraph, we define 0SEA formulae which are based on linear formulas and dependency directed graph.

\textbf{Definition 3 (Linear Formulas)} A formula in SEA is said to be linear if it contains no equality or disequality where a string-typed variable appears more than once.
Algorithm 2: Dependency Graph Construction

input : (s, Es)
output : G
1 G←vertex(s); WL←{s};
2 while WL≠∅ do
3   s_i←head(WL); WL←tail(WL);
4   is_exist, tr_i, tr_d, Es←choose_intersect(s_i, Es);
5   if is_exist then
6     if FV(tr_d) == ∅ then
7       foreach s_j ∈ FV(tr_i) do
8         G←vertex(s_j); /* mark s_j as leaf */
9       end
10   else
11     foreach s_j ∈ FV(tr_d) do
12       G←vertex(s_j); G←edge(s_i, s_j); WL←WL∪{s_j};
13     end
14   else /* mark s_i as leaf */
15 end

In the following, we present an algorithm to construct a dependency directed graph for a conjunction of word equations.

Let Es≡∧(tr_i=tr_r_i | i∈1...n} be a conjunctive set of word equations. For each string variable in Es, we construct its dependency graph as in Algorithm 2. This algorithm takes inputs as a pair of variable s and a set of equations Es. It initially generates a graph with one node s and a waiting list WL with one variable s. Function vertex create a new node if the node does not exist. In each iteration, it looks for dependent variables of a variable s_i in the head of WL. In particular, it uses function choose_intersect at line 4 to extract from Es a word equation, e.g. tr_i=tr_d, such that s_i∈FV(tr_i) (FV(π) returns free variables in π). In this case, it returns all variables in tr_d as dependent variables of s_i. In lines 6-9, for each word equation of the form s_1·s_2···s_k=w where w is a word in Σ*, we mark s_1, s_2, ..., s_k as leaves. We remark that when a node is marked as leaf, its out-going edges are removed and it is never added into the waiting list. Otherwise, it adds a directed edge from s_i to a dependent node s_j using function edge. We notice that there may be more than one edge between two nodes.

Definition 4 (0SEA Formulas) A formula π is said to be in 0SEA fragment if π is linear and for all dependency graphs G built for each string variable in π, G does not contain any cycle.

We find that 0SEA fragment is equivalent to the acyclic form presented in [1], and thus satisfiability problem for 0SEA formulas is decidable. We explicitly state this decidability as follows.

Theorem 4.2 (0SEA Decidability [1]) The satisfiability problem for 0SEA is decidable.
4.3 S2\textsubscript{1SEA} Instantiation

The satisfiability procedure S2\textsubscript{1SEA} is an instantiation of the generic framework S2SAT presented in Algorithm 1. S2\textsubscript{1SEA} takes a formula \( \pi \) as input, initially pairs each bare string variable in word equations with a fresh string inductive predicate. (using function \texttt{init\_1SEA}), and then systematically enumerates disjuncts \( \pi^b \). S2\textsubscript{1SEA} can produce two possible outcomes: SAT with a model obtained from a satisfiable formula \( \pi^b \), or UNSAT with a proof; non-termination is classified as UNKNOWN. We recap that while our discussion focuses on formulas with only string equalities, a string disequality can be reduced to a finite set of equalities. An implementation for such reduction can be found in [1].

In the rest of this subsection, we present the base logic and instantiation of functions \texttt{init\_1SEA}, \texttt{UA\_1SEA}, \texttt{OA\_1SEA}, \texttt{fp\_1SEA}, and \texttt{unfold\_1SEA}.

**Base Logic** The base formula of S2\textsubscript{1SEA} is defined as follows.

**Definition 5 (Base Formula)** Let \( \pi \equiv E\setminus T \setminus I \setminus A \). \( \pi \) is a base formula of solver S2\textsubscript{1SEA} if it is in fragment OSEA and Es does not contain any inductive predicate instance.

We use \( \texttt{sat}^b(\pi^b) \) to denote the satisfiability checking for base formula \( \pi^b \). Both function \texttt{UA\_1SEA} and \texttt{OA\_1SEA} invoke \( \texttt{sat}^b(...) \) to discharge base formulas.

**Initializing** Let \( \pi \equiv E\setminus T \setminus I \setminus A \) be the input. Function \texttt{init\_1SEA} pairs each string variable in Es with a predicate instance STR. In particular, for each variable \( s_i \), we generate a new inductive predicate \( \texttt{STR}(u_i, n_i) \) where \( u_i \) and \( n_i \) are fresh variables, conjoins the constraint \( s_i = u_i \) into \( A \), and conjoins a conjunction of invariant of each length function \( \land \{ n_i \geq 0 \} \) into \( I \). After that, we replace all length function of \( s_i \), i.e., exhaustively reduce all expression \( |tr_i| \) and then substitute each \( |s_i| \) expression in \( I \) by the corresponding variable \( n_i \).

**Approximating** For soundness of \( \texttt{SAT} \), under-approximation function \texttt{UA\_1SEA} only considers base leaves, those leaves which are in the base logic. Over-approximation function \texttt{OA\_1SEA} reduces each leaf with inductive predicates to a base formula by replacing each word equation \( tr_1 = tr_2 \) with the corresponding length constraint \( |tr_1| = |tr_2| \). For example, the following formula

\[
\pi \equiv \texttt{STR}(u,n_u) \cdot \texttt{STR}(v,n_v) \cdot \texttt{STR}(u,n_u) \cdot a \cdot \texttt{STR}(u,n_u) \cdot \texttt{STR}(t,n_t) \land n_u \geq 0 \land n_v \geq 0 \land n_t \geq 0
\]

is over approximated into \( \pi' \equiv n_u = n_v + n_u + 1 + n_u + n_t \land n_u \geq 0 \land n_v \geq 0 \land n_t \geq 0 \). \( \pi' \) is passed to \( \texttt{sat}^b(...) \) to check its satisfiability. As \( \pi' \) is unsatisfiable, so is \( \pi \).

**Expanding** S2\textsubscript{1SEA} chooses an open leaf, e.g. node \( i \), in a depth-first manner (at line 10 of Algorithm 1) and unfolds it using function \texttt{unfold\_1SEA}. The function \texttt{unfold\_1SEA} chooses one word equation of the node \( i \), e.g. \( tr_i = tr_r \), and examines two leading terms at the head of \( tr_i \) and \( tr_r \). After that, it unfolds a predicate instance \( \texttt{STR} \) accordingly, matches/consumes and returns a set \( L \) of formulas. If this set is empty, the algorithm marks the node \( i \) closed. Otherwise, for each formula in \( L \) it creates a new node \( j \) and new edge from \( i \) to \( j \).
Function `unfoldSEA` is the core of our algorithm. It aims to reduce word equations to base disjuncts. Intuitively, it applies Unfold-and-Match on the leading (first) term (string variable or character constant) of both sides of an equation. In particular, this function examines the following three cases.

**Case 1.** In this case, the leading terms at LHS and RHS are characters in the alphabet. It then matches these two characters, reduces the size of the word equation and thus makes progressing. Two subcases are formalized as follows.

```
[UNF–1SEA–CONST–SUCCE]
unfold(tr1 = tr2 ∧ π) → L
```
```
[UNF–1SEA–CONST–FAIL]
c1 ≠ c2
```

In the first sub-case (rule `[UNF–1SEA–CONST–SUCCE]`), these two terms are identical; function `unfoldSEA` consumes them and makes progressing. In the second sub-case (rule `[UNF–1SEA–CONST–FAIL]`), these two terms are not identical; function `unfoldSEA` returns an empty set and classifies this leaf unsatisfiable.

**Case 2.** In the second case, one leading term is a character `c` and another is a predicate instance `STR(u, n)`. This case is formalized by the following two rules corresponding to two cases where the inductive predicate is in LHS ([`UNF–1SEA–SMALL–L`]) or RHS ([`UNF–1SEA–SMALL–R`]).

```
[UNF–1SEA–SMALL–L]
E1 = (tr1 = tr2 ∧ ES)[ε/STR(u, n)] I1 = I ∧ n = 0 A1 = A ∧ u = ε
E2 = (STR(u, n1) ∨ tr1 = tr2 ∧ ES)[c · STR(u, n1)/STR(u, n)]
fresh u1, n1 I2 = I ∧ n1 = n − 1 ∧ n > 0 A2 = A[u1/u] ∧ u1 = c · u
```
```
[UNF–1SEA–SMALL–R]
E1 = (c · tr1 = tr2 ∧ ES)[ε/STR(u, n)] I1 = I ∧ n = 0 A1 = A ∧ u = ε
E2 = (tr1 = STR(u, n) ∨ tr2 ∧ ES)[c · STR(u, n1)/STR(u, n)]
L1 = I ∧ n1 = n − 1 ∧ n > 0 A2 = A[u1/u] ∧ u1 = c · u
```

In these rules, function `unfoldSEA` does case split by unfolding the predicate to consider two cases: `u` is an empty word or it is a word whose the first character is `c`. In the latter case, our system substitutes `STR(u, n)` by the concatenation `c · STR(u, n1)` where `n1 = n − 1`. The reuse of variable `u` is critical to identify back-links in the unfolding trees. After this selectively unfolding, `unfoldSEA` matches the character in both sides and makes progressing (i.e., reducing the size of the word equations).

**Case 3.** In the last case, the leading terms on both LHS and RHS are inductive predicate instances, e.g. `STR(s1, n1)` and `STR(s2, n2)`.

```
[UNF–1SEA–BIG]
E1 = (tr1 = tr2 ∧ ES)[u1/u2] I1 = I ∧ n1 = n2 A1 = A ∧ u1 = u2 L1 = {E1 ∧ I ∧ A1}
E2 = (STR(u1, n3) ∨ tr1 = tr2 ∧ ES)[u2/n3] · STR(u2, n2)/STR(u1, n3)
L2 = {E2 ∧ I ∧ A2}
E3 = (tr1 = STR(u2, n3) ∨ tr2 ∧ ES)[STR(u1, n1) ∧ STR(u2, n3)/STR(u2, n3)]
L3 = {E3 ∧ I ∧ A3}
```
```
unfoldSEA(STR(u1, n1) ∨ tr1 = tr2 ∧ ES)[u2/n3] ∧ I → L1 ∪ L2 ∪ L3
```

10
Function $\text{unfold}_{1\text{SEA}}$ expands the tree through a big-step unfolding. As shown in rule $[\text{UNF} - 1\text{SEA} - \text{BIG}]$, it considers the following three subcases: (i) two string variables are identical (i.e., $u_1 = u_2$ in the first line); (ii) $u_2$ is a substring of $u_1$ (i.e., $u_1$ is substituted by $u_2 \cdot u_1$ in the second and third lines); and (iii) $u_1$ is a substring of $u_2$ (i.e., $u_2$ is substituted by $u_1 \cdot u_2$ in the fourth and fifth lines). We notice that while the first subcase make progressing (i.e., reducing the size of the word equations), the remaining two cases do not.

**Linking Back** Function $\text{fp}_{1\text{SEA}}$ attempts to link remaining open leaves back to interior nodes so as to form a fixpoint (i.e., a pre-proof for induction proving) [19]. This function is implemented through some weakening and substitution principles. In particular, function $\text{fp}_{1\text{SEA}}$ links a leaf to an interior node if after some substitution, (i) the leaf has isomorphic word equations and regular membership to the inter node; and (ii) its arithmetical part implies the arithmetical part of the inter node. We notice that the subterm constraints in each leaf are for counter-model construction and are discarded during this linking. The substitutions are identified based on isomorphic string terms and well-founded ordering relations $R$ over arithmetical variables. In the following, we define isomorphic relation between word equations. The isomorphic relation between regular expression is similar.

**Definition 6 (isomorphic equations)** The equations $E_1$ and $E_2$ are isomorphic if $E_1$ and $E_2$ become identical when we replace all string variables $u$ in $E_1$ by $\text{permute}(u)$ and all characters $c$ in $E_1$ by $\text{permute}(c)$, where $\text{permute}(u)$ is a permutation function on $U$, and $\text{permute}(c)$ is a permutation function on the alphabet $\Sigma$.

In the next section, we will describe a decidable subfragment which includes arithmetic based on classes of well-founded ordering relations.

## 5 Correctness

In this section, we discuss the soundness and termination of our solver. We also provide a complexity analysis of our decision procedure to show that it runs, in the worst case, in linear time for $0\text{SEA}$ and factorial time for $1\text{SEA}$.

### 5.1 Soundness

The soundness of our $S2_{1\text{SEA}}$ algorithm relies on the correctness of functions $\text{UA}_{1\text{SEA}}$, $\text{OA}_{1\text{SEA}}$ and $\text{unfold}_{1\text{SEA}}$. The soundness of functions $\text{UA}_{1\text{SEA}}$ and $\text{OA}_{1\text{SEA}}$ is straightforward. Additionally, it is easy to verify that our unfolding rules have the complete property. We state the correctness of the proposed $S2_{1\text{SEA}}$ algorithm as follows.

**Theorem 5.1 (Soundness)** Let $T_{i+1}$ be the unfolding tree obtained after expanding the tree $T_i$ using function $\text{unfold}_{1\text{SEA}}$. Then

- $T_i$ has a SAT leaf with a solution $⟨\eta, \beta_\eta⟩$ implies that there exists $\eta' \subseteq \eta$ and $\beta_\eta' \subseteq \beta_\eta$ such that $T_{i+1}$ has a SAT leaf with solution $⟨\eta', \beta_\eta'⟩$.
- $T_{i+1}$ has a SAT leaf with a solution $⟨\eta', \beta_\eta'⟩$ implies that $T_i$ has a SAT leaf with a solution $⟨\eta, \beta_\eta⟩$ where $\eta' \subseteq \eta$ and $\beta_\eta' \subseteq \beta_\eta$.
5.2 Decidable Fragment

In this section, we show that our solver terminates for the subfragment 1SEA which is defined as follows.

1SEA Formulae The arithmetical constraints over length functions of 1SEA formulas are restricted on periodic relations \( R \) which is defined as follows. For each string variable \( \text{STR}(u_i,x_i) \), let \( u'_i (\text{STR}(u'_i,x'_i)) \) be subterm of \( u_i \) where \( x'_i > x_i \). Finally, let \( \vec{x} \) and \( \vec{x}' \) be sequences of \( k \) such \( x_i \) variables. \( R \in \mathbb{Z}^k \times \mathbb{Z}^k \) is an integer relation over variables \( \vec{x} \) and \( \vec{x}' \), its transitive closure \( R^+ = \bigcup_{i=1}^{\infty} R_i \) where \( R^0 = R \) and \( R^{i+1} = R^i \circ R \) for all \( i \geq 1 \). Relation \( R \) is defined as one of the two following form.

- Octagonal relation. An octagonal relation is a finite conjunction of constraints of the form \( R(x_1,x_2) \equiv x_1 + x_2 \leq k \) where \( k \) is an integer constraint, \( x_1,x_2 \in \vec{x} \cup \vec{x}' \).
- Finite linear affine relation. A linear affine relation is a finite conjunction of constraints of the form \( R(\vec{x},\vec{x}') \equiv C \vec{x} \geq D \land \vec{x}' = A \vec{x} + B \), where \( A \in \mathbb{Z}^{k \times k} \), \( C \in \mathbb{Z}^{p \times k} \) are matrices, and \( B \in \mathbb{Z}^k \), \( D \in \mathbb{Z}^p \). A linear affine relation is finite if the set \( \{ A^i | i \geq 0 \} \) is finite.

For example \( R(x_1,x_2) \equiv x_1 - x_2 = 5 \) is an octagonal relation as it is equivalent to \( R(x_1,x_2) \equiv x_1 - x_2 \leq 5 \lor x_2 - x_1 \leq -5 \). Especially, the authors in [7] show that the transitive closure of these periodic relations is Presburger-definable and effectively computable. In other words, these relations are ultimately periodic. The set of periodic is defined as follows.

**Definition 7** A set \( S \) of integers is defined to be ultimately periodic if there are some \( M \geq 0 \), \( p > 0 \) such that \( n \in S \) iff \( n + p \in S \) for all \( n \geq M \). Then we call the set \((M,p)\)-periodic.

The set \((M,p)\)-periodic is important for the complexity analysis.

**Definition 8 (1SEA Formulae)** A formula \( \pi \) is said to be in 1SEA fragment if either it is in 0SEA subfragment or it satisfies the two following restrictions

1. For all dependency graph \( G \) built for each string variable in \( \pi \), \( G \) contains at most one cycle, including self-cycle.
2. zero or more arithmetical periodic constraints [7] (as defined above) on the length functions of string variables.

**Termination and Complexity** Function \( \text{unfold}_{1SEA} \) produces a set of new formulas whose either i) size are decreased or ii) all variables in the chosen word equation are suffix of the corresponding in the input and there is at least one variable is strict suffix. Hence, \( S_{21SEA} \) procedure always terminates for \( \text{SAT} \). The substitution in the rules \( \text{UNF}_{-1SEA-\text{SMALL-}} \) and \( \text{UNF}_{-1SEA-\text{BIG}} \) may infinitely increase the sizes of word equations when these equations include multiple occurrences of one variable. Thus, in general, \( S_{21SEA} \) algorithm may not terminate. In the following, we show that \( S_{21SEA} \) indeed terminates for 0SEA and 1SEA formulas. We also provide computational complexity analyses.
Theorem 5.2  Let \( \pi \equiv E_1 \land \ldots \land E_M \land Y \land A \) be in the GSEA fragment. \( S_2 \) terminates for \( \pi \). If \( M \) word equations are of the form \( \text{tr}_i = \text{tr}_j \) where \( i \in \{1, \ldots, M\} \), and \( N \) is the longest notational length of these word equations, then the length of every path of the derived unfolding trees for \( \pi \) is \( O(2^M N) \).

Proof  As \( \pi \) is in GSEA, it is linear as well as there no cycle in dependency graphs derived for its every string variables. As \( \pi \) is linear, the size of the word equation obtained from unfolding the word equation \( E \) is less than the size of \( E \). Furthermore, as there is no cycle in any dependency graph, the formulas after the substitution while unfolding using either rule \([\text{UNF} - \text{SEA} - \text{SMALL} - \pi] \) or rule \([\text{UNF} - \text{SEA} - \text{BIG}] \), are still linear. Thus, \( \pi \) is reduced to a set of base formulas in finite steps.

We remark that after each unfolding on the word equation \( E \), while the size of result decreases at least one, the size of the each remaining word equation in \( E \)s increases at most one. Thus, whenever reducing one word equation to size 0, size of each remaining word equations in \( E \)s increases \( O(N) \). Based on this fact, the complexity is \( O((1+2^1+2^2+\ldots+2^{(M-1)})N) \). Indeed, we can prove the computational complexity above by induction on \( M \).

This theorem implies that \( S_2 \) solves a GSEA with one word equation, in the worst case, in linear time. In the next theorem, we show that \( \text{SEA} \) indeed terminates for a formula with multiple word equation.

Theorem 5.3 (SEA Termination)  Let \( \pi \equiv E_1 \land \ldots \land E_M \land Y \land A \) be in the 1SEA fragment. \( S_2 \) terminates for \( \pi \).

Proof  The proof for the formula in GSEA is given in Theorem 5.2. In the following, we consider the formula which is in another case. We remark that unfolding rules of function \( \text{unfold} \) decrease the size of on-processing (the first one in these rules) word equation at least one and increases the size of each remaining equation in \( E \)s at most one during the substitution. As the input formula \( \pi \) is in the 1SEA fragment, neither (i) this on-processing equation includes any string variable which occurs more than twice nor (ii) any dependency graphs derived for variables of \( \pi \) contains more than one loop. (i) guarantees that size of the on-processing word equation after unfolded is never longer than the size of original equation. (ii) ensures that \( \pi \) is still in the 1SEA fragment after the substitution. As a permutation of a word equation with a given length is finite, these equations are isomorphic to an inner node after a finite number of unfoldings. We notice that, in these rules \([\text{UNF} - \text{SEA} - \text{SMALL} - \pi] \) and \([\text{UNF} - \text{SEA} - \text{BIG}] \), the new subterm constraints are generated on length functions and they are \( R \) periodic relations which are Presburger definable. This means they can be reduced to an equivalent Presburger constraints in finite time. Hence, function \( f_{\pi, 1SEA} \) can always link back every leaves after a finite number of unfoldings. Thus, \( S_2 \) can always link back every leaves after a finite number of unfoldings. Thus, \( S_2 \) terminates for a 1SEA formula.

Finally, we state the computational complexity of the satisfiability problem for 1SEA. For simplicity, we only discuss the case where \( \pi \) contains one word equation. The proof for the complexity relies on the following lemma which states that given a periodic relation corresponding a set \( S \), any formula derived from the unfolding of this relation corresponds to a set \( S' \) and \( S' \subseteq S \).

Lemma 1. If \( S \neq \emptyset \) is \((M, p)\)-periodic and \( S' = \{ y \mid y = kpx, \ x \in S \} \), then \( S' \) is \((M, kp)\)-periodic and \( S' \subseteq S \) for \( k \) is an integer and \( k > 0 \).
It is easy to show that if \( x \in S \) and \( x \geq M \), then \( kpx \geq M \), \( kpx \in S' \) and \( kpx + kp \in S' \).

**Theorem 5.4 (1SEA Complexity)** Let \( \pi \equiv tr_1 = tr_r \land \overline{Y} \land I \land \Lambda \) be in the 1SEA fragment. The length of every path in the derived unfolding trees for \( \pi \) is \( O(N^2(N!)) \) where \( N \) is the size of the equation \( tr_1 = tr_r \).

**Proof** This complexity result is based on the following four facts.

1. Size of a word equation of any node in the derived unfolding trees for \( \pi \) is less than or equal \( N \); Hence, there are \( O(N) \) possibilities for the length.
2. There are \( O(N!) \) possibilities to arrange a sequence of \( N \) symbols of the respective either string variables or characters.
3. For every arrangement, i.e. a word equation, there are \( O(N) \) possibilities to distinguish two sides (LHS and RHS) of the equation.
4. In a path, arithmetical part of a \( S_2\text{SEA} \) formula is a disjunct of the unfolding from its descendant. From lemma \( \square \) the set \( S' \) of this disjunct is a subset of set corresponding its descendant. Thus, the function \( f_{\text{SEA}} \) can always link the arithmetical part of such above leaf to any its descendant nodes.

\( \square \)

**6 Related Work**

Makanin notably provided a mathematical proof for the satisfiability problem of word equation \([22]\). In the sequence of papers, Plandowski et al. showed that the complexity of this problem is PSPACE \([15,11,23,24]\). Beside the development of the foundation for the acyclic form \([1,21]\) as discussed in section \( \square \) Ganesh et al. presented undecidability result for quantified string-based formulas \([10]\). In the rest of this section, we summarize the development of related works on practical string solvers.

**Automata-based Solvers.** Finite automata provides a natural encoding for string with regular membership constraints. Rex \([28]\) encodes strings as symbolic finite automata (SFA). Each SFA transition is transformed into SMT constraints. Java String Analyzer (JSA) \([8]\) is specialized for Java string constraints. JSA approximates string constraints into multi-level automaton. \([13,14]\) provides a reasoning over string with priori length bounds. Recent work in \([5]\) provides a length-bound approach for solving string constraints and further counting the number of solution to such constraints. Recently, authors in \([12]\) proposes a DPLL(T)-based approach to unbounded string constraints with regular expressions and length function. \([?]\) described a new method based on a scalable logic circuit representation to support various string and automata manipulation operations and counter-example generation. In our view, inductive predicate could represent automaton. Thus, it is interesting to investigate how we could adapt the proposed algorithm \( S_2\text{SEA} \) for the prolems based on automata.
Bit-vector-based Solvers. Hampi solver [17] reduces fixed-sized string constraints to bit-vector problem and then satisfiability. The Kazula solver [25] extends Hampi with concatenation operation. It first solves arithmetical constraints and then enumerates possible fixed-length versions of an input formula using Hampi. In [6], strings are represented as arrays. Discharging string with length constraints are performed through two phases. First an integer-based over-approximation of the string constraint is solved and then fixed-length string constraints are then decided in a second phase.

Word-based Solvers. Z3str [31] implements string theory as an extension of Z3 SMT solver through string plug-in. It supports unbounded string constraints with a wide range of string operations. Intuitively, it solves string constraints and generates string lemmas to control with z3’s congruence closure core. Z3str2 [30] improves Z3str by proposing a detection of those constraints beyond the tractable fragment, i.e. overlapping arrangement, and pruning the search space for efficiency. Similar to Z3str, CVC4-based string solver [20] communicates with CVC4’s equality solver to exchange information over string. S3 [26] enhances Z3str to incrementally interchange information between string and arithmetic constraints. S3P [27] further extends S3 to detect and prune non-minimal subproblems while searching for a proof. While the technique in S3P aims for satisfiable formulae, it may returns unknown for unsatisfiable formulas due to absence of multiple occurrences of each string variable. Our solver can support well for both classes of queries in case of less than or equal to two occurrences of each string variable.

7 Conclusion and Future Work

We have presented the semi-decision procedures S2SEA for the problem of solving satisfiability of a SEA formula with word equations, regular membership and length functions. We have shown that the proposed procedure terminates for the subfragment 0SEA and provided its computational complexity.

For future work, we would like to implement the proposed decision procedure S2SEA based on the generic S2SAT framework [19]. As the S2SAT framework naturally supports arbitrary user-defined predicates, we might extend the proposed decision procedure with inductive predicates encoding recursive string functions (i.e., function replace) [27]. We were hoping that such extension helps enhance the completeness of the string logic augmented with these recursive functions.

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