QUASICIRCLES AND HYPERBOLIC ZERO PACKING

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Abstract. We look at the work of Oleg Ivrii connected with the dimension of quasicircles for asymptotically small quasiconformality parameter $k$. We intend to make this work more easily accessible. Our main focus is the integral means spectrum associated with normalized conformal mappings of the exterior disk which have quasiconformal extensions to the whole plane with small dilatation parameter $k$. Moreover, we address the estimates from above only, not the sharpness from below.

1. Introduction

1.1. Basic notation. We use the same basic notation as in [2]. So, we write $\mathbb{R}$ for the real line, $\mathbb{C}$ for the complex plane, $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ for the extended complex plane (the Riemann sphere). We use $ds$ for normalized arc length measure, and $dA$ for normalized area measure, while $\partial_z$, $\bar{\partial}_z$ are the standard Wirtinger derivatives, and $\Delta_z = \partial_z \bar{\partial}_z$. We let $D$ denote the open unit disk, $T := \partial D$ the unit circle, and $D_e$ the exterior disk. We use the sesquilinear forms $\langle \cdot, \cdot \rangle_T$ and $\langle \cdot, \cdot \rangle_D$ where the measure is $ds$ and $dA$, respectively. We write $1_E$ for the indicator function of a subset $E$.

1.2. The Bloch space and the Bloch seminorm. The Bloch space consists of those holomorphic functions $g : D \to \mathbb{C}$ that are subject to the seminorm boundedness condition

$$\|g\|_{B(D)} := \sup_{z \in D} (1 - |z|^2) |g'(z)| < +\infty.$$ 

Let $\text{aut}(D)$ denote the group of sense-preserving Möbius automorphism of $D$. By direct calculation,

$$\|g \circ \gamma\|_{B(D)} = \|g\|_{B(D)}, \quad \gamma \in \text{aut}(D),$$

which says that the Bloch seminorm is invariant under all Möbius automorphisms of $D$. An immediate observation we can make at this point is that

Date: July 29, 2022.

2020 Mathematics Subject Classification. Primary 46E20, 30E05, 46E22.

Key words and phrases. Quasicircles, fractal dimension, integral means spectrum, asymptotic variance, geometric zero packing.

This research was supported by Vetenskapsrådet (VR grant 2020-03733), by the Leverhulme trust (grant VP1-2020-007), and by grant 075-15-2021-602 of the Government of the Russian Federation for the state support of scientific research, carried out under the supervision of leading scientists.
provided that \( g(0) = 0 \), we have the estimate
\[
|g(z)| \leq \|g\|_{B(D)} \int_0^{|z|} \frac{dt}{1 - t^2} = \frac{1}{2} \|g\|_{B(D)} \log \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D},
\]
which is sharp pointwise. This estimate is related with the interpretation of \( f \in B(D) \) as the hyperbolically Lipschitz continuous functions.

1.3. The Bergman projection of bounded functions. For \( \mu \in L^1(\mathbb{D}) \), let
\[
\mathbf{P}\mu(z) := \int_{\mathbb{D}} \frac{\mu(w)}{(1 - zw)^2} dA(w), \quad z \in \mathbb{D},
\]
be its Bergman projection. Restricted to \( L^2(\mathbb{D}) \), it is the orthogonal projection onto the subspace of holomorphic functions. In addition, it acts boundedly on \( L^p(\mathbb{D}) \) for each \( p \) in the interval \( 1 < p < +\infty \) (see, e.g., [?]). The derivative of \( \mathbf{P}\mu \) is given by
\[
(\mathbf{P}\mu)'(z) := 2 \int_{\mathbb{D}} \frac{\bar{w}\mu(w)}{(1 - zw)^3} dA(w), \quad z \in \mathbb{D},
\]
so that if \( \mu \in L^\infty(\mathbb{D}) \) with \( \|\mu\|_{L^\infty(\mathbb{D})} \leq 1 \), we get that (see Perälä [6])
\[
(1 - |z|^2) |(\mathbf{P}\mu)'(z)| \leq 2 \int_{\mathbb{D}} \frac{|w|}{|1 - zw|^2} dA(w)
\]
\[
= 2 \sum_{k=0}^{+\infty} \left[ \frac{(3/2)^k}{(3/2 + k)(k!)^2} \right] |z|^{2k} \leq 2 \Gamma(\frac{3}{2})^{-2} (1 - |z|^2)^{-1} = \frac{8\pi^{-1}}{1 - |z|^2}.
\]

But what happens if \( m \) vanishes on a big enough disk? Can we then get a better estimate? By the change-of-variables formula,
\[
(\mathbf{P}\mu)'(z) = 2 \int_{\mathbb{D}} \frac{\tilde{\phi}(\xi) \mu \circ \tilde{\phi}(\xi)}{(1 - z\tilde{\phi}(\xi))^3} |\tilde{\phi}'(\xi)|^2 dA(\xi),
\]
so that with
\[
\phi(\xi) = \frac{z - \xi}{1 - \bar{z}\xi},
\]
we find that
\[
(\mathbf{P}\mu)'(z) = \frac{2}{1 - |z|^2} \int_{\mathbb{D}} \frac{\bar{z} - \bar{\xi}}{(1 - \bar{z}\xi)^2} \mu \circ \phi(\xi) dA(\xi).
\]

Lemma 1.1. If \( \mu \circ \phi = 0 \) holds on \( \mathbb{D}(0, \varrho) \) for some \( \varrho \) with \( 0 < \varrho < 1 \), then
\[
(1 - |z|^2) |(\mathbf{P}\mu)'(z)| \leq 2 \int_{\mathbb{D} \setminus \mathbb{D}(0, \varrho)} \frac{1}{|1 - \bar{z}\xi|} dA(\xi) = O \left( (1 - \varrho) \log \frac{1}{1 - \varrho} \right).
\]

The details of the estimate of the integral are left to the interested reader.
1.4. **Exponential integrability.** Suppose $g = P\mu$, where $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_{L^\infty(\mathbb{D})} \leq 1$. Let $g_r$ denotes the dilate $g_r(\zeta) := g(r\zeta)$. It is of importance to control the integrals

$$\int_T |e^{tg_r}| ds$$

for $t \in \mathbb{C}$ which are close to 0. The following estimate was offered by Ivrii [4].

**Theorem 1.2.** (Ivrii) In the above setting, we have the estimate

$$\int_T |e^{tg_r}| ds = O\left(\frac{1}{(1 - r^2)^{(2 + o(1))|t|^2/4}}\right)$$

uniformly as $|t| \to 0$. Here, $\Sigma^2$ is an absolute constant with $0 < \Sigma^2 \leq 1$.

The constant $\Sigma^2$ was introduced by Astala, Ivrii, Perälä, and Prause [1]. Later, Hedenmalm [2] found another interpretation of the constant in terms of a zero packing problem.

**Theorem 1.3.** (Hedenmalm) It holds that

$$\Sigma^2 = 1 - \rho_H < 1,$$

where

$$\rho_H := \lim \inf_{r \to 1^-} \frac{\int_{\mathbb{D}(0,r)} \Phi_f(z) \frac{dA(z)}{1 - |z|^2}}{\int_{\mathbb{D}(0,r)} \Phi_f(z) \frac{dA(z)}{1 - |z|^2 \log^2 1 - |z|^2}} = \lim \inf_{r \to 1^-} \frac{\int_{\mathbb{D}(0,r)} \Phi_f(z) \frac{dA(z)}{1 - |z|^2}}{\log^2 1 - |z|^2}.$$

Here, the infimum runs over all polynomials $f$, and we use the notation

$$\Phi_f(z) := \left((1 - |z|^2)|f(z)| - 1\right)^2, \quad z \in \mathbb{D}.$$

Ivrii’s arXiv preprint [4] has unfortunately not been published. However, this does not mean that Ivrii’s contribution is flawed. Here we shed light the various aspects of Ivrii’s argument which we believe are correct or at least may be fixed so that they are correct.

2. **Exponential integrability**

2.1. **The Littlewood-Paley identity.** We use a variant of the Littlewood-Paley identity, for $f \in H^2$, where $H^2$ is the usual Hardy space of Taylor series with square summable coefficients:

$$\int_T |f|^2 ds = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|^2} dA(z)$$

$$= \int_{\mathbb{D}} |f(z)|^2 dA(z) + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) dA(z).$$
We apply this identity to \( f(z) = \exp(\frac{1}{2} g_r(z)) \), where \( g_r(z) = g(r z) \) is shorthand for the dilated function, with \( 0 < r < 1 \).

\[
(2.2) \quad \int_\mathbb{T} |e^{t g_r}| ds = \int_\mathbb{D} |e^{t g_r}| dA + \frac{|t|^2}{4} \int_\mathbb{D} (1 - |z|^2)|g_r)'|^2 |e^{t g_r}| dA(z)
\]

\[
\leq r^{-2} \int_{D(0,r)} |e^{t g}| dA + \frac{|t|^2}{4} \int_{D(0,r)} (1 - |z|^2)^2 |g|'^2 |e^{t g}| \frac{dA(z)}{1 - |z|^2}.
\]

We may estimate the first integral on the right-hand side uniformly:

\[
r^{-2} \int_{D(0,r)} |e^{t g}| dA \leq \int_\mathbb{D} |e^{t g}| dA \leq \int_\mathbb{D} \left( \frac{1 + |z|}{1 - |z|} \right)^{8\pi^{-1}|t|} dA(z)
\]

\[
= \int_0^1 \left( \frac{1 + r}{1 - r} \right)^{8\pi^{-1}|t|} 2r dr \leq \frac{4^{8\pi^{-1}|t|}}{1 - 8\pi^{-1}|t|},
\]

provided that \(|t| < \pi/8\). In particular, we find that

\[
r^{-2} \int_{D(0,r)} |e^{t g}| dA \leq \int_\mathbb{D} |e^{t g}| dA \leq 4, \quad |t| \leq \frac{\pi}{16}.
\]

We implement this estimate into (2.2), and obtain that

\[
(2.3) \quad \int_\mathbb{T} |e^{t g_r}| ds \leq 4 + \frac{|t|^2}{4} \int_{D(0,r)} (1 - |z|^2)^2 |g|'^2 |e^{t g}| \frac{dA(z)}{1 - |z|^2}, \quad |t| \leq \frac{\pi}{16}.
\]

By Perälä’s estimate (1.1), the function

\[
(2.4) \quad N_g(z) := (1 - |z|^2)^2 |g(z)|^2
\]

is uniformly bounded,

\[
(2.5) \quad 0 \leq N_g \leq \frac{64}{\pi^2},
\]

and the estimate (2.3) simplifies:

\[
(2.6) \quad \int_\mathbb{T} |e^{t g_r}| ds \leq 4 + \frac{|t|^2}{4} \int_{D(0,r)} N_g |e^{t g}| \frac{dA(z)}{1 - |z|^2}, \quad |t| \leq \frac{\pi}{16}.
\]

We proceed to analyze this inequality.

2.2. A toy calculation. Let us suppose for the moment that \( N_g \leq \eta \) holds for some positive \( \eta \). This is of course true for \( \eta = 64\pi^{-2} \), but we are now interested in smaller values of \( \eta \). Under this hypothesis, we see from (2.6) that

\[
(2.7) \quad \int_\mathbb{T} |e^{t g_r}| ds \leq 4 + \eta \frac{|t|^2}{4} \int_{D(0,r)} |e^{t g}| \frac{dA(z)}{1 - |z|^2}, \quad |t| \leq \frac{\pi}{16}.
\]

Let us write

\[
I(r, t) := \langle |e^{t g}| \rangle_{\mathbb{T}(0,r)} := \int_\mathbb{T} |e^{t g_r}| ds,
\]

...
and observe that
\[ \int_{B(0,r)} |e^{tg}| \frac{dA(z)}{1-|z|^2} = \int_0^r 2udu \int_0^r |e^{tgu}| ds = \int_0^r I(u,t) \frac{2udu}{1-u^2}, \]
so that (2.7) would assert that for $|t| \leq \frac{\pi}{16},$
\[ I(r,t) \leq 4 + \eta \frac{|t|^2}{4} \int_0^r I(u,t) \frac{2udu}{1-u^2}. \]

(2.8) It is possible to argue (by iteration, or in terms of differential inequalities) that
the estimate (2.8) leads to the upper bound
\[ I(r,t) \leq 4 (1 - r^2)^{-\eta |t|^2/4}. \]
If this is true with some fixed $\eta < 1$, say $\eta = \frac{1}{2}$, then we would get the desired growth estimate of $I(r,t)$ automatically. To be able to defeat the estimate $N_g \leq \eta$ the function $g'$ should exhibit some growth. If we instead want $N_g \geq \eta$
to hold on a reasonably fat set, the natural way to achieve the growth is by
having zeros in $g'$. We now discuss the rather opposite behavior of the functions
$e^{tg}$ and $N_g$, respectively. Here, $e^{tg}$ is basically hyperbolically locally constant,
while $N_g$ is instead oscillatory.

2.3. The scale associated with $e^{tg}$. The function $g$ is in the Bloch space, and
hence Lipschitz continuous in the hyperbolic metric. Then $tg$ has a Lipschitz
constant which is $O(|t|)$, which makes it asymptotically constant on hyperbolic
disks of any fixed radius (in fact, it is basically constant over hyperbolic distances
that are $o(|t|^{-1})$). On the other hand, the function $N_g$ can be much more oscillatory.
Let us discretize the argument in the toy calculation, but without making
any fixed assertion regarding the size of $N_g$. We let $L$ be a large positive real,
which we think of as fixed, and consider successive radii
\[ r_k = \frac{1-e^{-Lk}}{1+e^{-Lk}}, \quad k = 0, 1, 2, \ldots, \]
which tend to 1 quickly. We consider the annuli
\[ A_k := \mathbb{D}(0, r_k) \setminus \bar{\mathbb{D}}(0, r_{k-1}), \quad k = 1, 2, 3, \ldots. \]
The hyperbolic width of the annulus $A_k$ equals
\[ \int_{r_{k-1}}^{r_k} 2du \frac{1}{1-u^2} = \log \frac{1+r_k}{1-r_k} - \log \frac{1+r_{k-1}}{1-r_{k-1}} = L, \]
while the weighted area is
\[ |A_k|_{A_{k-1}} := \int_{A_k} \frac{dA(z)}{1-|z|^2} = \log \frac{1-r_k^2}{1-r_k^2} = L + O(e^{-L(k-1)}). \]
We introduce the notation
\[ \langle h \rangle_{A_k} = \frac{\int_{A_k} h \frac{dA}{1-|z|^2}}{\int_{A_k} \frac{dA}{1-|z|^2}} = \frac{1}{|A_k|} \int_{A_k} h(z) \frac{dA(z)}{1-|z|^2} \]
for the weighted average of \( h \) on the annulus \( A_k \). The object that we will focus on are the annular averages
\[ \langle |e^{tg}| \rangle_{A_k} \]
for \( t \) close to 0. This quantity models \( I(r_k, t) \). Indeed, since \( e^{tg} \) is essentially constant in the radial direction inside \( A_k \) for small \( |t| \), the integral is mainly over the angular direction, which gives that
\[ \langle |e^{tg}| \rangle_{A_k} = e^{O(L|t|)} I(r_k, t) = e^{O(L|t|)} \langle |e^{tg}| \rangle_{\pi(0, r_k)}. \]

2.4. The scale associated with \( N_g \). The function \( N_g \) is bounded, but it can be highly oscillatory. It varies at much shorter distances. We are interested in maximizing the average quantity
\[ \langle N_g |e^{tg}| \rangle_{A_k}. \]
Since \( N_g \) and \( |e^{tg}| \) vary at different scales, it is tempting to think like a probabilist and suggest that we have independent stochastic variables, so that
\[ \langle N_g |e^{tg}| \rangle_{A_k} \sim \langle N_g \rangle_{A_k} \langle |e^{tg}| \rangle_{A_k}. \]
Let us carry on and see where this line of thinking would lead us. We obtain from (2.6) and the assumed independence that
\[ e^{O(L|t|)} \langle |e^{tg}| \rangle_{A_k} \leq 4 + \frac{|t|^2}{4} \int_{D(0, r_k)} N_g |e^{tg}| \frac{dA}{1-|z|^2} \]
\[ \sim 4 + \frac{|t|^2}{4} \sum_{j=1}^{k} |A_j| \langle N_g \rangle_{A_j} \langle |e^{tg}| \rangle_{A_j}. \]
Moreover, let us suppose that for big \( k \),
\[ \langle N_g \rangle_{A_j} \leq \sigma_g(L)^2, \]
where \( \sigma_g(L)^2 = \sigma_g^2 + o(1) \) as \( L \to +\infty \). The nonnegative parameter \( \sigma_g^2 \) would express how big \( N_g \) can be, on average, in a rather precise sense. Then we would have that
\[ e^{O(L|t|)} \langle |e^{tg}| \rangle_{A_k} \leq 4 + \frac{|t|^2}{4} \sum_{j=1}^{k} |A_j| \langle N_g \rangle_{A_j} \langle |e^{tg}| \rangle_{A_j} \]
\[ \leq M + \frac{|t|^2}{4} \sigma_g(L)^2 \sum_{j=1}^{k} |A_j| \langle |e^{tg}| \rangle_{A_j}. \]
Moreover, iteration would give an estimate of the type

\[
\langle |e^{tg}| \rangle_{\mathbb{A}_k} = O\left( \exp \left( e^{O(|t|)} \frac{|t|^2 \sigma_g(L)^2 Lk}{4} \right) \right)
= O\left( \frac{1}{(1 - \frac{r_k^2}{L})^2 e^{O(|t|)} |t|^2 \sigma_g(L)^2 / 4} \right).
\]

This estimate along the radii \( r_k \) may be extended (with minor losses) to all radii \( r \) if we take into account the hyperbolic Lipschitz property with small Lipschitz constant.

3. Grid of boxes

3.1. Boxes of hyperbolic size \( \asymp L \). It is actually not so difficult to make the above reasoning more rigorous. We split the annulus \( \mathbb{A}_k \) into into equal “boxes”, by cutting in the radial direction. Each box \( Q_{k,l} \) is of the type \( \theta_{l-1} \leq \theta \leq \theta_l \) and \( r_{k-1} \leq r \leq r_k \), so that \( \theta_l - \theta_{l-1} \) is independent of \( l \). We will need each box to be of the angular width \( O\left( L e^{-Lk} \right) \), so there should be on the order of magnitude \( \asymp L^{-1} e^{Lk} \) boxes. In each box \( Q = Q_{k,l} \) we would try to maximize the average \( \langle N_g \rangle_Q \). In each such box \( Q \), \( |e^{tg}| \) is roughly constant, so that

\[
\langle N_g |e^{tg}| \rangle_Q \sim \langle N_g \rangle_Q \langle |e^{tg}| \rangle_Q
\]
holds. All we need now is to have an estimate of the form

\[
\langle N_g \rangle_Q \leq \sigma_g(L)^2,
\]
with \( \sigma_g(L)^2 \leq \Sigma^2 + o(1) \), where \( \Sigma^2 < 1 \) is independent of \( g \). Indeed, then we obtain from (3.1) that

\[
\langle N_g |e^{tg}| \rangle_{\mathbb{A}_k} = \frac{1}{\#(Q)} \sum_Q \langle N_g |e^{tg}| \rangle_Q
\leq e^{O(|t|)} \sigma_g(L)^2 \sum_Q \langle N_g |e^{tg}| \rangle_Q = e^{O(|t|)} \sigma_g(L)^2 \langle |e^{tg}| \rangle_{\mathbb{A}_k}.
\]

This would permit us to apply the iteration argument which leads to (2.9), as desired. But what kind of argument would be able to give us (3.2)?

3.2. Localization of \( \mu \) to boxes. Here, we recall the formula (2.4), and split \( \mu = \mu_Q + \mu_{Q^c} \), where \( \mu_Q = 1_Q \mu \) and \( \mu_{Q^c} = 1_{\mathbb{D} \setminus Q} \mu \). Then \( g = P \mu = g_Q + g_{Q^c} \), where \( g_Q = P \mu_Q \) and \( g_{Q^c} = P \mu_{Q^c} \). We estimate

\[
(1 - |z|^2)(|P \mu_{Q^c})'(z)| \leq 2(1 - |z|^2) \int_{\mathbb{D} \setminus Q} \frac{|w|}{|1 - z \bar{w}|^3} dA(w) = O(L e^{-L/2})
\]
for \( z \in \frac{1}{2} Q \), if we invoke Lemma 1.1, assuming that the corresponding hyperbolic distance between \( \frac{1}{2} Q \) and \( \mathbb{D} \setminus Q \) is at least \( L/2 \). This means that for the size of \( N_g \) in \( \frac{1}{2} Q \), only \( \mu_Q \) matters. Here, it is actually better for us to work with bigger subboxes \( (1 - \epsilon)Q \) in place of \( \frac{1}{2} Q \), which we can if \( L \) is assumed big enough so
that $\epsilon L$ is also big. Here, we think of $(1 - \epsilon)Q$ as a correspondingly smaller box with the same mass center point and all other sizes shrunk by the factor $1 - \epsilon$. This procedure of going from the box $Q$ to the slightly smaller box $(1 - \epsilon)Q$ is explained in some detail in the subsection below. The relevance of $\epsilon L$ is that it should measure the hyperbolic distance between $(1 - \epsilon)Q$ and $\mathbb{D} \setminus Q$, and we need $\epsilon L$ big to be able to ensure localization in terms of Lemma 1.1.

We aim to show that $\langle N_g \rangle_{(1-\epsilon)Q} \leq \sigma_g(L)^2$ holds, with $\sigma_g(L)^2 = \sigma_g^2 + o(1)$, and $\sigma_g^2 \leq \Sigma^2$. As a first step, we find the box $Q = Q_{k,i}$ for fixed $k$ such that $\langle N_g \rangle_{(1-\epsilon)Q}$ is maximal (we maximize over $l$), say $\langle N_g \rangle_{(1-\epsilon)Q} = \alpha$. Then for that $Q = Q_{\max}$, $\mu_Q$ must a rather optimal, and we may use a periodization procedure to replace the original $\mu$ in the entire annulus by copies of $\mu_{Q_{\max}}$ so as to make $\langle N_g \rangle_{(1-\epsilon)Q} \approx \alpha$ for all the other boxes in the annulus. This would make $\langle N_g \rangle_{\Lambda_k} \approx \alpha + O(\epsilon)$. We would now like to work with optimization along all the annuli at the same time. In fact, is better to have scale invariance as well, which is only approximately true for the disk. The point is that if we have found an optimal box $Q = Q_{\max}$, we would like to use $\mu_Q$ to replace $\mu$ along all the other annuli as well, by just scaling the optimal $\mu_Q$ (like we did for rotations before). Neglecting the technicalities, we should then have $\langle N_g \rangle_{\Lambda_k} \approx \alpha + O(\epsilon)$ for all big $j$. We should think of the correct value of $\alpha$ to appear in the limit as we zoom in at the circle $\mathbb{T}$, when the circle begins to look like a straight line.

3.3. Summing over boxes (or annuli). It remains to find out how big can be this optimal value $\alpha$. By summing over successive annuli we obtain

$$\int_{\mathbb{D}(0,r_k)} |g'(z)|^2(1 - |z|^2)dA = \sum_{j=1}^k |A_j|_{A-1} \langle N_g \rangle_{\Lambda_j} \approx (\alpha + O(\epsilon))(Lk + O(1)),$$

so that

$$\frac{\int_{\mathbb{D}(0,r_k)} |g'(z)|^2(1 - |z|^2)dA}{\int_{\mathbb{D}(0,r_k)} (1 - |z|^2)^{-1}dA(z)} \approx \alpha + O(\epsilon).$$

The left-hand side expression equals the average of $N_g$ on the disk $\mathbb{D}(0,r_k)$. That average would be trivially dominated by the maximal local average $\langle N_g \rangle_Q$, but our argument (which needs to be made more technically sophisticated) gives that the local maximal average over big boxes is the same as the maximal average over the disks $\mathbb{D}(0,r_k)$. By the Littlewood-Paley identity (2.1),

$$\int_{\mathbb{D}(0,r)} |g'(z)|^2(1 - |z|^2)dA = r \int_{\mathbb{T}(0,r)} |g|^2ds - \int_{\mathbb{D}(0,r)} |g|^2dA$$

it follows that the optimal value of $\alpha$ equals McMullen’s asymptotic variance:

$$\sigma_g^2 := \limsup_{r \to 1^-} \int_{\mathbb{T}} |g(r)|^2ds \frac{1}{\log(\frac{1}{1-r^2})}. $$

Moreover, by Theorem 1.3,

$$\sup_g \sigma_g^2 = \Sigma^2 = 1 - \rho_{\mathbb{D}} < 1,$$
where the supremum runs over all \( g = \mathcal{P}_\mu \) with \( \|\mu\|_{L^\infty(D)} \).

3.4. Wrapping up the proof. We return to the various boxes \( Q \). We need to keep \( L \) big, \( \epsilon \) small, but \( \epsilon L \) big too. Then the averages \( \langle N_g \rangle_{(1-\epsilon)Q} \) approximate well \( \langle N_g \rangle_Q \), given that \( N_g \) enjoys the bound (2.5).

The careful analysis of the boxes would require a better way to represent the disk which is stable under dilatation, not just rotation. We may model \( \mathbb{D} \) in terms of logarithmic coordinates. We write \( z = e^{i\pi \zeta} \), where \( \text{Im} \zeta > 0 \) and \( \text{Re} \zeta \in \mathbb{R}/2\mathbb{Z} \). The boxes are now just rectangles that form a grid. The Bergman projection may be expressed in these coordinates, if \( D \) is the fundamental domain in the upper half-plane with real part between \(-1 \) and \( 1 \):

\[
\mathcal{P}_\mu(e^{i\pi \zeta}) = \int_D \frac{1}{(1 - e^{i\pi(\zeta - \bar{\xi})})^2} \mu(e^{i\pi \xi}) e^{-2\pi \text{Im} \xi} dA(\xi).
\]

Similarly, we find that

\[
(3.3) \quad (1 - |z|^2)(\mathcal{P}_\mu)'(z) \bigg|_{z = e^{i\pi \zeta}} = 2(1 - e^{-2\pi \text{Im} \zeta}) \int_D \frac{e^{-i\pi \bar{\xi}}}{(1 - e^{i\pi(\zeta - \bar{\xi})})^3} \mu(e^{i\pi \xi}) e^{-2\pi \text{Im} \xi} dA(\xi),
\]

and we observe that for \( \zeta, \xi \in D \), the formula

\[
(3.4) \quad \frac{1}{(e^{i\pi(\zeta - \bar{\xi})} - 1)^3} = \sum_{j=-1}^{1} \left\{ (i\pi(\zeta - \bar{\xi} + 2j))^{-3} - \frac{3}{2}((i\pi(\zeta - \bar{\xi} + 2j))^{-2} + (i\pi(\zeta - \bar{\xi} + 2j))^{-1} \right\} + O(1)
\]

allows us to control the singularity. The sum over \( j \in \{-1, 0, 1\} \) is due to the fact that the left-hand side expression is 2-periodic in the variable \( \zeta - \bar{\xi} \). But we may as well focus our attention to the contribution with \( j = 0 \), and observe that the contributions associated with the second and third term on the right-hand side of (3.4) are negligible, i.e. correspond to asymptotically vanishing contributions in (3.3) as \( \text{Im} \zeta \to 0 \) (i.e., as \( |z| \to 1 \)). Similarly,

\[
e^{-i\pi \bar{\xi}} e^{-2\pi \text{Im} \xi} = 1 - i\pi \bar{\xi} - 2\pi \text{Im} \xi + O(|\xi|^2),
\]

where only the constant 1 makes a significant contribution. For \( \zeta \in D \) with \( |\zeta| \leq \frac{1}{2} \), we see from (3.3) that

\[
(3.5) \quad (1 - |z|^2)(\mathcal{P}_\mu)'(z) \bigg|_{z = e^{i\pi \zeta}} = -4\pi(\text{Im} \zeta + O(\text{Im} \zeta)^2) \int_{D \cap \mathbb{H}} \frac{1}{(i\pi(\zeta - \bar{\xi}))^3} \mu(e^{i\pi \xi}) dA(\xi) + o(1),
\]

as \( \text{Im} \zeta \to 0 \). Effectively this reduces to the study of the the derivative of the Bergman projection on the upper half-plane \( \mathbb{H} \), since the main kernel in (3.5) appears this way. Moreover, in \( \mathbb{H} \), we have access to both translation and dilatation invariance. This is key to making the argument with boxes effective.
Some clarifying words should perhaps be added regarding the boxes \((1 - \epsilon)Q\) compared with \(Q\). More precisely, how should the size reduction be made? It is easier to explain these matters in the context of the upper half-plane. We work with the hyperbolic metric \(d_{\mathbb{H}}(\zeta) = (\text{Im}\,\zeta)^{-1}|d\zeta|\) and weighted area element \(dA_{-1}(\zeta) = (\text{Im}\,\zeta)^{-1}dA(\zeta)\). A prototypical box \(Q\) of size \(L\) would look like this:
\[
Q = \{ \zeta \in \mathbb{H} : 0 < \text{Re} \,\zeta < L, \ e^{-L} < \text{Im} \,\zeta < 1 \},
\]
Other boxes could be a horizontal translate of this one, or a dilate of such a translate. The weighted area of \(Q\) is
\[
|Q|_{A_{-1}} = \int_{Q} \frac{dA(\zeta)}{\text{Im}\,\zeta} = \pi^{-1}L^2.
\]
We choose \((1 - \epsilon)Q\) to be the smaller box
\[
(1 - \epsilon)Q = \{ \zeta \in \mathbb{H} : \epsilon L < \text{Re} \,\zeta < (1 - \epsilon)L, \ e^{-(1-\epsilon)L} < \text{Im} \,\zeta < e^{-\epsilon L} \}.
\]
Then the weighted area of the box \((1 - \epsilon)Q\) is given by
\[
|(1 - \epsilon)Q|_{A_{-1}} = \int_{(1-\epsilon)Q} \frac{dA(\zeta)}{\text{Im}\,\zeta} = \pi^{-1}(1 - 2\epsilon)^2L^2.
\]
Moreover, the hyperbolic distance from any point of \((1 - \epsilon)Q\) to any point of the complement \(\mathbb{H} \setminus Q\) is at least \(\geq \epsilon L\).

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