MOTT’S LAW FOR THE CRITICAL CONDUCTANCE OF MILLER–ABRAHAMS RANDOM RESISTOR NETWORK

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Abstract. In this short note we derive Mott’s law for the critical conductance of the Miller–Abrahams random resistor network on a Poisson point process on $\mathbb{R}^d$, $d \geq 2$, and we give a percolative characterization of the factor preceding the temperature dependent term $\beta^{\frac{d+1}{d+2}}$. We also give mathematical arguments supporting its universality. This note is a preliminary version of a more extended work, where we also discuss the equality between the effective conductance of the resistor network and the critical conductance.

1. Introduction

Mott’s variable range hopping is a mechanism of phonon–assisted electron transport taking place in amorphous solids (as doped semiconductors) in the regime of strong Anderson localisation. It has been introduced by Mott in order to explain the anomalous non–Arrenhius decay of the conductivity at low temperature [9]. We refer to [10, 12, 11] for a detailed discussion. Keeping the language of doped semiconductors, in the regime of low impurity density some effective models have been proposed. One can approximate the localized electrons by classical non–interacting particles moving according to random walks with jump probability rates given by the electron transition rates multiplied by a suitable factor mimicking the effect of Pauli exclusion principle. The mathematical analysis of this random walk has lead, between other, to the derivation of upper and lower bounds of the diffusion constant in agreement with Mott’s law [4, 6]. Another effective model in the regime of low impurity density is given by the random Miller–Abrahams resistor network [8] (shortly, MA resistor network), on which we concentrate here.

We describe the resistor network. The set of vertexes (nodes) of the resistor network is a so called simple point process, i.e. a random locally finite subset $\xi \subset \mathbb{R}^d$. We suppose the law of $\xi$ to be isotropic. Given a realization of $\xi$, independently from the random mechanism generating $\xi$, one attaches to each vertex $x \in \xi$ a random variable $E_x$ called energy mark, in such a way that the energy marks $(E_x)_{x \in \xi}$ are i.i.d. random variables with common law $\nu$. Physically, $E_x$ would be the ground state energy of the electron eigenfunction localized around the impurity $x$. The physically relevant laws $\nu$ (in inorganic doped semiconductors) are of the form $c|E|^\alpha dE$ with support in some bounded

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interval \([-C_0, C_0]\), \(c\) being the normalizing constant and \(\alpha\) be a nonnegative number:

\[
\nu(dE) = \frac{(\alpha + 1)}{2C_0^{\alpha+1}} |E|^\alpha 1(-C_0 \leq E \leq C_0)dE
\]

Then the MA resistor network is obtained by attaching to any unordered pair of sites \(x \neq y\) in \(\xi\) a filament of conductivity \([1, 11]\):

\[
c_{x,y} := \exp\left\{ -\frac{2}{\gamma} |x - y| - \frac{\beta}{2}(|E_x| + |E_y| + |E_x - E_y|) \right\}.
\]

(1)

Above \(\gamma\) denotes the localization length and \(\beta\) denotes the inverse temperature. Note that the skeleton of the resistor network is the complete graph on \(\xi\).

As explained in \([1]\) one expects that the effective conductance of the MA resistor network is well approximated by the critical conductance \(c_c(\beta)\) as \(\beta \to \infty\). To define it, given a number \(c_* > 0\) we denote by \(G(c_*)\) the graph obtained from the MA resistor network by keeping filaments with conductivity at least \(c_*\). Then the critical conductance \(c_c(\beta)\) is characterized by the following two properties: (i) for any value \(c_* > c_c(\beta)\) a.s. the graph \(G(c_*)\) has no unbounded cluster, (ii) for any value \(c_* < c_c(\beta)\) a.s. the graph \(G(c_*)\) has some unbounded cluster.

We will discuss the validity of the approximation of the effective conductance of the MA resistor network with the critical conductance \(c_c(\beta)\) in a future extension of this note. Our aim here is the derivation of Mott’s law for the critical conductance \(c_c(\beta)\), which reads

\[
c_c(\beta) \approx e^{-\kappa \beta^{\alpha+1+d}}, \quad \beta \gg 1,
\]

(2)

for some \(\beta\)-independent constant \(\kappa > 0\). To this aim, we sample the node set \(\xi\) by a homogeneous Poisson point process (the resulting resistor network will be called Poisson MA resistor network). In this case we can provide a formula for \(c_c(\beta)\) (cf. Corollary \([2,3]\)). We will also give a percolation characterization of the factor \(\kappa\) in \([2]\) for Poisson MA resistor networks and provide arguments supporting the universality of the constant \(\kappa\) outside the class of homogeneous Poisson point processes.

We point out that the analysis of the factor \(\kappa\) has lead to a certain debate in the physical literature with different proposals \([11]\). The analysis carried on below is indeed analytic and mathematically rigorous (hence, without any Ansatz).

2. Poisson Miller–Abrahams resistor network

In this section we suppose that the random set \(\xi\) of the nodes of the MA resistor network is given by a homogeneous Poisson point process with density \(\rho\) in \(\mathbb{R}^d\). We denote by \(\xi(A)\) the number of points in \(\xi \cap A\) for \(A \subset \mathbb{R}^d\). We recall that \(\xi\) is a homogeneous Poisson point process with density \(\rho\) in \(\mathbb{R}^d\) if and only if the following two properties are satisfied:
Lemma 2.3. There exists \( \lambda_c > 0 \) such that if \( \lambda < \lambda_c \) then a.s. the graph \( G^*_{1,1,\lambda} \) does not percolate, while if \( \lambda > \lambda_c \) then a.s. the graph \( G^*_{1,1,\lambda} \) percolates.

**Proof.** Since two homogeneous Poisson point processes with density \( \lambda, \lambda' \) can be coupled in a way that the one with smaller density is contained in the other, we get that the function \( h(\lambda) \), defined as the probability that the graph \( G^*_{1,1,\lambda} \) percolates, is weakly increasing. Hence, to get the thesis it is enough to exhibit two positive constants \( \lambda_1, \lambda_2 \) such that \( h(\lambda_1) = 0 \) and \( h(\lambda_2) > 0 \).

Let us consider the graph \( G^{(1)}_{\lambda} := (\xi, E^{(1)}) \), where \( E^{(1)} \) is the set of pairs \( \{x, y\} \) of vertexes of \( \xi \) that satisfies \( 0 < |x - y| \leq 1 \). Then \( G^{(1)}_{\lambda} \) contains the graph \( G^*_{1,1,\lambda} \) and can be seen as the realization of a Boolean model with deterministic radius 1/2 on the Poisson point process \( \xi \) with density \( \lambda \) [7]. Indeed two points \( x, y \) are connected by an edge in \( G^{(1)}_{\lambda} \) if and only if the closed balls centered at \( x \) and \( y \) with radius 1/2 intersect. It is known [7] that for \( \lambda \) small the graph
The condition $G$ percolates and therefore the same happens for large $\lambda$ small.

We now show that $h(\lambda)$ is positive for $\lambda$ large. To this aim we fix $\delta \in (0, 1/3)$ and we consider the graph $G^{(2)}$ in $\mathbb{R}^d$ obtained by taking as vertex set $V^{(2)} := \{ x \in \xi : E_x \in [0, \delta] \}$ and taking as edge set $E^{(2)} := \{ \{x, y\} : x, y \in V^{(2)} \text{ and } |x - y| \leq 1 - 3\delta \}$. By construction, since $\zeta = \beta = 1$, if $\{x, y\} \in E^{(2)}$ then the inequality (4) is satisfied and therefore $\{x, y\} \in \mathcal{E}$. As a consequence $G^{(2)}$ is a subgraph of $G_{1,1,\lambda}$. On the other hand, $G^{(2)}$ can be thought of as a Boolean model associated to the deterministic radius $(1 - 3\delta)/2$, on the Poisson point process $V^{(2)}$ with density $\lambda\nu([0, \delta])$ (note that $\nu([0, \delta]) > 0$). It is known [7] that when this density is large, i.e. for large $\lambda$, the graph $G^{(2)}$ a.s. percolates and therefore the same happens for $G_{1,1,\lambda}$ since $G^{(2)} \subset G_{1,1,\lambda}$. This proves that $h(\lambda) > 0$ for $\lambda$ large. $\square$

**Theorem 2.4.** Let $\lambda_c$ be the universal constant appearing in Lemma 2.3. Fixed $\beta$ and $\rho$, set

$$\zeta_c(\beta, \rho) := \left( \frac{\lambda_c(\beta)}{\rho} \right)^{\frac{\alpha+1}{\alpha+1-\delta}} (\beta C_0) \frac{\alpha+1}{\alpha+1-\delta}. \quad (5)$$

Then, for $\beta > \lambda_c^2 \rho^{\frac{1}{\alpha+1-\delta}} C_0^{-1}$, the following holds:

(i) if $\zeta < \zeta_c(\beta, \rho)$, then a.s. the graph $G_{\zeta_c, \beta, \rho}$ does not percolate;

(ii) if $\zeta > \zeta_c(\beta, \rho)$, then a.s. the graph $G_{\zeta_c, \beta, \rho}$ percolates.

**Proof.** The condition $\beta > \lambda_c^2 \rho^{\frac{1}{\alpha+1-\delta}} C_0^{-1}$ is equivalent to $\zeta < \zeta_c(\beta, \rho)$. Under this condition it is enough to prove the thesis for $\zeta \leq C_0\beta$. Indeed, if we prove the thesis for $\zeta \leq C_0\beta$, then we can proceed as follows for $\zeta > C_0\beta$. Since $\zeta > \zeta_c(\beta, \rho)$, and we fix $\zeta' \in (\zeta_c(\beta, \rho), C_0\beta)$, we already know that $G_{\zeta', \beta, \rho}$ percolates. Since $G_{\zeta', \beta, \rho} \subset G_{\zeta_c, \beta, \rho}$, we conclude that a.s. the graph $G_{\zeta_c, \beta, \rho}$ percolates.

Due to the above claim from now on we restrict to $\zeta \leq C_0\beta$. It is convenient to rewrite (4) as

$$\frac{|x - y|}{\zeta} + \frac{\beta}{\zeta} (|E_x| + |E_y| + |E_x - E_y|) \leq 1. \quad (6)$$

Hence, if $(\beta/\zeta)|E_x| > 1$, then $x$ is an isolated point in the graph $G_{\zeta, \beta, \rho}$ and therefore it does not give any contribution to the existence of the infinite cluster. So $G_{\zeta, \beta, \rho}$ percolates if and only if the graph $\tilde{G}_{\zeta, \beta, \rho} = (\tilde{\xi}, \tilde{E})$ percolates, where $\tilde{G}_{\zeta, \beta, \rho}$ is defined as follows. Its vertex set is given by

$$\tilde{\xi} := \{ x \in \xi : |E_x| \leq \zeta/\beta \},$$

while its edge set $\tilde{E}$ equals $E_{\zeta, \beta, \rho}$. We analyse in detail the graph $\tilde{G}_{\zeta, \beta, \rho}$.

- The vertex set $\tilde{\xi}$ is obtained by thinning $\xi$. In particular, a point $x$ in $\xi$ survives in $\tilde{\xi}$ independently from the other points with probability

$$\nu([-\zeta/\beta, \zeta/\beta]) = \left( \frac{\zeta}{\beta C_0} \right)^{\alpha+1}.$$
As a consequence, $\hat{\xi}$ is a homogeneous Poisson point process with density $\hat{\rho} := (\cdot/\beta C_0)^{\alpha+1} \rho$.

- Let us set $A_x := (\beta/\zeta)E_x$. We claim that, knowing that $x \in \hat{\xi}$, $A_x$ has distribution $\nu$ defined in Lemma 2.3. Indeed, given $0 \leq u \leq 1$,

$$P\left((\beta/\zeta)E_x \in [0, u] \mid ((\beta/\zeta)E_x) \leq 1\right) = \frac{P((\beta/\zeta)E_x \in [0, u])}{P((\beta/\zeta)E_x) \leq 1} = \frac{\nu([0, (\zeta/\beta)u])}{\nu([-\zeta/\beta, \zeta/\beta])}$$

$$= \left(\frac{\zeta u}{\beta C_0}\right)^{\alpha+1} \left(\frac{\zeta}{\beta C_0}\right)^{\alpha+1} = \frac{1}{2} u^{\alpha+1} = \nu([0, u]).$$

(7)

A similar result holds for what concerns the event $((\beta/\zeta)E_x) \in [-u, 0]$, thus the thesis.

- We observe that we can rewrite (6) in terms of the variables $A_x$'s as

$$\frac{|x - y|}{\zeta} + |A_x| + |A_y| + |A_x - A_y| \leq 1.$$  

(8)

Now we consider a third graph $\tilde{\mathcal{G}} = (\hat{\xi}, \hat{\mathcal{E}})$ defined as follows. We first introduce the map $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ as $\phi(x) = x/\zeta$. Then we define $\hat{\xi} := \phi(\hat{\xi})$ and we define $\hat{\mathcal{E}}$ as the set of edges $\{\phi(x), \phi(y)\}$ with $\{x, y\}$ varying among $\mathcal{E}$. To any point $\phi(x) \in \hat{\xi}$ we associate a new energy mark $B_{\phi(x)} := A_x$. Writing $\phi(x) = v$ and $\phi(y) = w$, we get that two vertexes $v \neq w$ in $\hat{\xi}$ are connected by an edge in $\tilde{\mathcal{G}}$ if and only if

$$|v - w| + |B_v| + |B_w| + |B_v - B_w| \leq 1.$$  

(9)

We know that the r.v.'s $B_v$'s are i.i.d. with law $\nu$. Moreover, since $\hat{\xi}$ is the $\phi$–image of the homogeneous Poisson point process $\hat{\xi}$ with density $(\cdot/\beta C_0)^{\alpha+1} \rho$, we get that $\hat{\xi}$ is a homogeneous Poisson point process with density $\lambda := (\zeta^{\alpha+1+d}/(\beta C_0)\zeta^{\alpha+1}) \rho$. As a consequence of the above observations the random graph $\tilde{\mathcal{G}}$ has the same law of the random graph $\mathcal{G}_{1,1,\lambda}$. Since $\mathcal{G}_{\zeta,\beta,\rho}$ percolates whenever the graph $\hat{\mathcal{G}}$ percolates (by construction), from Lemma 2.3 we deduce that $\mathcal{G}_{\zeta,\beta,\rho}$ a.s. does not percolate if $\lambda < \lambda_c$ and that $\mathcal{G}_{\zeta,\beta,\rho}$ a.s. percolates if $\lambda > \lambda_c$. Trivially, the conditions $\lambda < \lambda_c$ and $\lambda > \lambda_c$ equals the conditions $\zeta < \zeta_c(\beta, \rho)$ and $\zeta > \zeta_c(\beta, \rho)$ respectively, thus concluding the proof. \hfill $\square$

Recalling that we have written $c_\ast$ as $e^{-\zeta}$ we get the following result on the critical conductivity:

**Corollary 2.5.** Let $\lambda_c$ be the universal constant appearing in Lemma 2.3. Then the critical conductance $c_c(\beta)$ is given by

$$c_c(\beta) = \exp \left\{ - \frac{\lambda_c}{\rho} \frac{\lambda + 1 + d}{\beta C_0} \frac{\lambda + 1}{\beta C_0} \right\}.$$
3. Final remarks

Let us suppose now that $\xi$ is an ergodic stationary simple point process on $\mathbb{R}^d$ with density $\rho$, i.e. $\mathbb{E}[\xi(B)] = \rho \ell(B)$ for any bounded Borel subset $B \subset \mathbb{R}^d$. Then one could argue as in the proof of Theorem 2.4 and conclude that the graph $\mathcal{G}$ obtained from the Miller–Abrahams resistor network by keeping filaments of conductivity at least $e^{-\zeta}$ percolates if and only if the graph $\hat{\mathcal{G}}$ percolates, where $\hat{\mathcal{G}}$ is obtained by keeping nodes $x$ with $|E_x| \leq \zeta/\beta$ (the other nodes would be isolated in the Miller–Abrahams resistor network) and afterwards rescaling by $\zeta$. By [2, Prop. 9.3.I] the nodes of the graph $\hat{\mathcal{G}}$ behave asymptotically as a homogeneous Poisson point process, hence we expect that the low temperature asymptotics of the critical conductance $c_c(\beta)$ is the same of the Poisson asymptotics given in Corollary 2.5. A more robust analysis of this universality will be given in a future work [3].

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