Non Asymptotic Bounds for Optimization via Online Multiplicative
Stochastic Gradient Descent

Riddhiman Bhattacharya
University of Minnesota

Abstract
The gradient noise of Stochastic Gradient Descent (SGD) is considered to play a key role in its properties
(e.g. escaping low potential points and regularization). Past research has indicated that the covariance
of the SGD error done via minibatching plays a critical role in determining its regularization and
escape from low potential points. It is however not much explored how much the distribution of the
error influences the behavior of the algorithm. Motivated by some new research in this area, we prove
universality results by showing that noise classes that have the same mean and covariance structure
of SGD via minibatching have similar properties. We mainly consider the Multiplicative Stochastic
Gradient Descent (M-SGD) algorithm as introduced by Wu et al. [28], which has a much more general
noise class than the SGD algorithm done via minibatching. We establish non asymptotic bounds for the
M-SGD algorithm mainly with respect to the Stochastic Differential Equation corresponding to SGD
via minibatching. We also show that the M-SGD error is approximately a scaled Gaussian distribution
with mean 0 at any fixed point of the M-SGD algorithm. We also establish bounds for the convergence
of the M-SGD algorithm in the strongly convex regime.

Keywords: Multiplicative Stochastic Gradient Descent, Stochastic Gradient Descent, diffusion, Wasser-
stein distance, optimization, Central Limit Theorem

1 Introduction
Stochastic Gradient Descent (SGD) is traditionally focused on finding the minimum value of a function
(also called objective function in optimization literature) taken into consideration due to our given scientific
problem. See, for example, Mertikopoulos et al. [22]; Zhou et al. [31]; Jin et al. [18]; Ghadimi and Lan [13];
Robbins and Munroe [25]. It has proven to be an extremely effective method in tackling hard problems in
mutiple fields such as machine learning [2], statistics [26], electrical engineering [14], etc.

The SGD algorithm and some of its properties are now well known especially in the context of machine
learning. The iterative version of the SGD algorithm is given as

\[ x_{k+1} = x_k - \gamma_k \nabla g(x_k) - \gamma_k \xi(x_k)_{k+1}, \text{ for } k = 0, 1, 2, \ldots \] (1.1)

Here \( \gamma_k \) is called the step size (it is possible to choose \( \gamma_k = \gamma \) for all \( k \)), \( g(\cdot) \) is our objective function
(function for which we want the optimum) and \( \xi(x_k)_{k+1} \) is the error term that may or may not depend
on the current point \( x_k \). The SGD algorithm can be thought of as a stochastic generalization of the Gradient
Descent (GD) algorithm which is one of the oldest algorithms for optimization along with Newton’s

\[^{1}\text{School of Statistics, University of Minnesota, 224 Church Street, S. E., MN55455, USA, bhatt240@umn.edu.}
\]
Method. One of the most popular methods for performing SGD in practice is to perform SGD via mini-batching (see Battou et al. [3]). This algorithm is a subcase of equation 1.1 and is given as

\textbf{Algorithm 1:} Stochastic Gradient Descent via Minibatching

\begin{verbatim}
1 Input $x_0$ as the starting point and the sequence $\eta_k > 0$ as the step size at iteration $k$, training data $x_1, x_2, \cdots, x_l$, the loss function $l(\cdot, x_i)$ and a batch size $b$.
2 for $k = 0, 1, 2, \cdots, K - 1$ do
3 Calculate $y_{k+1} = y_k - \gamma_k \frac{1}{b} \sum_{i \in B} \nabla l(y_k, x_i)$, where $B$ is a subset from $1, 2, \cdots, l$ such that $|B| = b$.
4 end for
5 Output $y_K$ or $\bar{y} = \frac{1}{K} \sum_{i=0}^{K} y_i$ as per the problem requirement
\end{verbatim}

From this point onwards, we refer to this algorithm as minibatch SGD.

Though initially proposed to remedy the computational problem of Gradient Descent, recent studies have shown that minibatch SGD has the property of inducing an implicit regularization which prevents the overparametrized models from converging to the minima (see Zhang et al. [30]; Hoffer et al. [16]; Keskar et al. [19]). This implies that according to empirical findings, smaller batch size minibatch SGD improves model accuracy for problems with flat minima while large size minibatch SGD improves model accuracy for problems with sharper minima. This phenomenon lead to an investigation by Wu et al. [28] which serves as the inspiration for our research. In this paper the authors introduced the Multiplicative Stochastic Gradient Descent (M-SGD) algorithm which is stated below (Algorithm 2).

The term $\mathcal{V}$ as defined in Algorithm 2 is the same as $\xi(x_k)_{k+1}$ in equation 1.1. The $\frac{1}{n}$ term is added to $\mathcal{V}$ in Algorithm 2 to give us the gradient term in the SGD equation. This algorithm has been extensively applied by Wu et al. [28] in the context of deep learning. In their paper, Wu et al. exhibit that for certain problems the M-SGD algorithm gives good performance and provide simulation results using CIFAR data. The authors mainly consider the problem of online linear regression and exhibit the convergence of the M-SGD algorithm on average to the optimum point (see Theorem 1 in Wu et al.). The authors also consider certain noise classes for $\mathcal{V}$ and establish some theoretical results for the same.

\textbf{Algorithm 2:} Multiplicative Stochastic Gradient Descent (M-SGD)

\begin{verbatim}
2 Input: Initial parameter $\theta_0 \in \mathbb{R}^p$, training data $\{(x_i, y_i)\}^n_{i=1}$, loss function $l_i(\theta) = l((x_i, y_i), \theta) \in \mathbb{R}$ and step size (also called learning rate) as $\gamma > 0$.
3 for $k = 0, 1, 2 \cdots, K - 1$ do
4 Generate the sampling noise $\mathcal{V} \in \mathbb{R}^n$.
5 Compute the sampling vector $W = \frac{1}{n} l + \mathcal{V}$, where $l$ is the vector with all entries 1.
6 Update the randomized loss as $L(\theta_k) = L(\theta_k)W$, where $L(\theta_k) = [l_1(\theta_k), l_2(\theta_k), \cdots, l_n(\theta_k)]$.
7 Update the parameter as $\theta_{k+1} = \theta_k - \gamma \nabla L(\theta_k)$.
8 end for
9 Output $y_K$
\end{verbatim}

Motivated by the work of Wu et al. [28] and the applications of the M-SGD algorithm in deep learning, we study the properties of the online version of the M-SGD algorithm. Another motivation is that, the general SGD algorithm has shown to perform with considerable success for convex optimization problems and with some success for nonconvex optimization problems (Polyak and Juditsky [24]; Bottou et al. [2]; Ghadimi and Lan [13]). Due to the stochastic nature of the algorithm it is believed that the SGD escapes
low potential points (see Hu et al. [17]) and thus has the scope to visit multiple of them. This is extremely similar to sampling and is more observed in SGD with fixed stepsize i.e. $\gamma_k = \gamma$ for all $k$. There has been enormous amount of work for SGD with both fixed step size and variable step size, with different error structures, for example Dieuleveut et al. [11]; Leluc and Portier [20]; Yu et al. [29] and many more.

Taking all this as our inspiration (mainly, Wu et al.), we study an online version of the M-SGD algorithm in which at each stage of the iteration we generate data $u_1, u_2, \ldots, u_n \overset{iid}{\sim} Q$, where $Q$ is some probability measure such that loss function defined as $l(\theta, u_i)$ is an unbiased estimator of our objective function $g(\theta)$.

Also, at each stage, we generate the weight vector $W$ as expressed in Algorithm 2. Our training data are random vectors in $\mathbb{R}^p$ ($p$ being the dimension of the problem). One can think of $u_1, u_2, \ldots, u_n$ as being generated from some “true” parameter in the parameter space or if we already have fixed training data $x_1, x_2, \ldots, x_t$ then we may think of $u_i$ as being one of the training data values with uniform probability i.e. $u_i = x_j$ with probability $\frac{1}{t}$ for all $i, j$. Thus our set up helps us address a large class of problems and permits a level of flexibility. Our iterative algorithm is as follows

$$y_{k+1} = y_k - \gamma \left( \sum_{i=1}^{n} w_{i,k} \nabla l(y_k, u_{i,k}) \right), \text{ for } k = 0, 1, 2, \ldots$$

(1.2)

where $\gamma$ is the step size and $l$ is the loss function. Note that we have $w_{i,k}, u_{i,k}$ instead of $w_i$ and $u_i$. This is because we refresh both the data and the “weights” at each step of the iteration. Hence this is an online version of the M-SGD algorithm as in we use new data at each iteration.

**Main results and organization of the paper**

In the first part of our paper we show that at each point the $\theta$ the scaled online M-SGD error term, defined as $\sum_{i=1}^{n} w_{i,k} (\nabla l(\theta, u_{i,k}) - \nabla g(\theta))$, is approximately Gaussian for large $m, n$ values. To do this we make certain assumptions on $W$ which are further discussed in Section 2. In the subsequent part of the paper, we derive non-asymptotic bounds for the algorithm defined by equation (1.2) with respect to the SGD diffusion which is again defined in Section 2. Further on, we provide non-asymptotic rates for the algorithmic guarantees in the case of strongly convex functions which still serves as the standard to analyse any optimization algorithm. In the end we show that adding Gaussian noise with the mixing rate of $\sqrt{2\gamma}$ makes this a valid sampling algorithm in the case where the target is strongly log-concave. We do this by deriving non-asymptotic bounds for the current point with the stationary distribution.

The rest of the paper is organised as follows- in Section 2 we introduce our problem with assumptions and some of the definitions that we shall be using throughout this paper. In Section 3 we state our main theorems. In Section 4 we exhibit our simulation results for certain key examples. Section 5 (Appendix) contains all the proofs of the main results.

## 2 Preliminaries

We consider an objective function $g(\cdot)$ and a loss function $l(\cdot, \cdot)$. In some cases we assume $g(\cdot)$ to be strongly convex, but in general we do not impose that condition on $g$. We have sample data(also called training data) as $u_1, u_2, \ldots, u_n$ as iid data from some distribution $Q$. Our choice of $Q$ should be such that $\mathbb{E}(l(\theta, u_i)) = g(\theta)$ for all $\theta \in \mathbb{R}^p$. We refer to $l$ as the “loss function”. We consider $|\cdot|$ to be the Euclidean norm and denote the transpose of any matrix $A$ as $A^\prime$. 

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We analyse the following algorithms

\[
\begin{align*}
\tilde{x}_{k+1} &= \tilde{x}_k - \gamma \nabla g(\tilde{x}_k), \quad (2.1) \\
x_{k+1} &= x_k - \gamma \nabla g(x_k) + \frac{\gamma}{\sqrt{m}} \sigma(x_k) \xi_{k+1}, \quad (2.2) \\
x_{n,k+1} &= x_{n,k} - \gamma \sum_{i=1}^{n} w_{i,k} \nabla l(x_{n,k}, u_{i,k}), \quad (2.3)
\end{align*}
\]

which are respectively the gradient descent algorithm (2.1), the SGD algorithm with scaled Gaussian error (2.2) and the M-SGD algorithm (2.3). Here \(k = 0, 1, 2, \ldots, K\) and \(\gamma \) is the learning rate or the step size. We also consider the following stochastic processes

\[
\begin{align*}
D_t &= D_{k\gamma} - (t - k\gamma) \nabla g(D_{k\gamma}) + \sqrt{\frac{\gamma}{m}} \sigma(D_{k\gamma}) (B_t - B_{k\gamma}), \quad (2.4) \\
dX_t &= -\nabla g(X_t) + \sqrt{\frac{\gamma}{m}} \sigma(X_t) dB_t, \quad (2.5) \\
d\tilde{X}_t &= -\nabla g(\tilde{X}_t) dt, \quad (2.6) \\
Y_{n,t} &= Y_{n,k\gamma} - \gamma (t - k\gamma) \sum_{i=1}^{n} w_{i,k} \nabla l(Y_{n,k\gamma}, u_{i,k}). \quad (2.7)
\end{align*}
\]

In all these equations \(t \in (k\gamma, (k + 1)\gamma]\) with \(k \leq K\) and \(K\gamma = T\), where \(T\) is the time horizon of all the 4 stochastic processes as defined in equations (2.4)-(2.7). All the initial points are fixed at a single point denoted by \(x_0\). The equations (2.6), (2.4) and (2.7) are continuous versions of (2.1), (2.2) and (2.3) respectively. Equation (2.5) is the diffusion corresponding to minibatch sgd (Hu et al. [17]). Our claim is that M-SGD is close to the minibatch SGD in its properties irrespective of the distribution of the weights. By “close to”, we imply the Wasserstein-2 distance function between two probability measures denoted as \(W_2(\cdot, \cdot)\). The definition of the Wasserstein-2 distance is as such

\[
W_2(\mu, \nu) = \inf_{Q \in \Gamma(\mu, \nu)} \left( \int_{X \times Y} |x - y|^2 Q(dx, dy) \right)^{1/2}. \quad (2.8)
\]

Here \(\Gamma(\mu, \nu)\) denotes the set of all couplings of the probability measures \(\mu, \nu\). Note that since random variables generate a probability measure on the real line or \(\mathbb{R}^p\) for any \(p\), we can also define the Wasserstein-2 distance wrt random variables in analogous fashion.

We now, state our assumptions.

**Assumption 1.** All starting points of equations (2.1)-(2.7) are the same fixed quantity and are denoted by \(x_0\).

**Assumption 2.** The loss function \(l(\theta, u)\) is continuously twice differentiable in \(\theta\). Also \(\forall \ u \in \mathbb{R}^p, \exists \ h_1(u) > 0 \) such that \(\forall \ \theta_1, \theta_2 \in \mathbb{R}^p\), we have

\[
|\nabla l(\theta_1, u) - \nabla l(\theta_2, u)| \leq h_1(u) |\theta_1 - \theta_2|. \quad (2.9)
\]

There exists \(\theta_0\) such that \(\nabla l(\theta_0, u)\) is \(L^3(\Omega, P)\); \(h_1(u)\) has finite moment generating function with \(\mathbb{E} [\exp(\tilde{\gamma} h_1(U))] < \infty\) for some \(\tilde{\gamma} > 0\).

**Remark 2.1.** Note that the first part of Assumption 2 implies that \(\mathbb{E} |\nabla l(\theta, u)|^3 < \infty\) for all \(\theta\). This is easy to see as \(\mathbb{E} |\nabla l(\theta, u)|^3 \leq 4(|\nabla l(\theta_0, u)|^3 + h_1(u)|\theta - \theta_0|^3)\).
**Remark 2.2.** Note that Assumptions 2 implies that for all $\theta$, we have

$$E(\nabla l(\theta, u)) = \nabla g(\theta).$$

Indeed one can see this by using the dominated convergence theorem (DCT). A detailed explanation of this is provided in Lemma 5.1 in the Appendix.

**Remark 2.3.** Note that Assumptions 2 also implies

$$|\nabla g(\theta_1) - \nabla g(\theta_2)| \leq L |\theta_1 - \theta_2|,$$

for all $\theta$, where $L = Eh(u)$.

The remark 2.3 is a standard assumption in optimization literature.

**Assumption 3.** There exists $L_1 > 0$ such that

$$||\sigma(\theta_1) - \sigma(\theta_2)||_2 \leq L_1 |\theta_1 - \theta_2|,$$

where $|| \cdot ||_2$ is the spectral norm of the operator and $\sigma(\cdot) = \text{Var}(\nabla l(\cdot, u))$

**Remark 2.4.** Note that Assumption 3 also implies

$$||\sigma(\theta_1) - \sigma(\theta_2)||_F \leq \sqrt{p}L_1 |\theta_1 - \theta_2|,$$

where $|| \cdot ||_F$ denotes the Frobenius norm of a matrix.

This is easy to see as $||A||_F^2 = \sum_{i=1}^{p} e_i^t A' A e_i \leq \sum_{i=1}^{p} |A e_i|^2 \leq \sum_{i=1}^{p} ||A||^2_2 = p \ ||A||^2_2$. This assumption implies that the covariance structure for the randomness in the training data has some level of linear control.

**Remark 2.5.** All our results are valid for any $\sigma(\theta)$ such that $\sigma(\theta)\sigma(\theta)' = \sigma^2(\theta)$ with $\sigma(\theta)$ as Lipschitz in the spectral norm. For the sake of convenience we assume $\sigma(\theta) = \sigma^2(\theta)^{1/2}$ throughout our work.

This implies that finding any such Lipschitz matrix suffices.

**Assumption 4.** Given any $n$, the weight vectors at each iteration in equation (2.3) are iid $W = (w_1, w_2, w_3, \cdots, w_n)'$ where $W$ is any random vector with $E(W) = \frac{1}{n}(1, 1, \cdots, 1)'$ and the variance covariance matrix is $\Sigma$ i.e.

$$W \sim \left(\frac{1}{n}(1, 1, \cdots, 1)', \Sigma\right),$$

where $(\Sigma)_{i,i} = \sigma_{ii} = \frac{n-m}{mn}$ and $(\Sigma)_{i,j} = \sigma_{ij} = -\frac{n-m}{mn(n-1)}$.

An immediate example of such a case is Algorithm 1 which is the most widely used SGD algorithm in practice. In this case the $w_{i,k} = 1/m$ is it is included in the sample and is 0 otherwise. Here $m$ denotes the “minibatch size”. This is the hypergeometric setup and it is not hard to show that the mean and the variance of the weights are as advertised above in Assumption 4. Indeed it is easy to see that $E(w_i) = \frac{1}{n}$,
\[ Cov(w_i, w_j) = E(w_i w_j) - \frac{1}{n^2} \]
\[ = \frac{1}{m^2} P(w_i = \frac{1}{m}, w_j = \frac{1}{m}) - \frac{1}{n^2} \]
\[ = \frac{1}{m^2} \left( \frac{n-2}{m-2} \right) - \frac{1}{n^2} \]
\[ = \frac{1}{m^2} \frac{m(m-1)}{n(n-1)} - \frac{1}{n^2} \]
\[ = - \frac{n-m}{mn^2(n-1)} \]

and \( Var(w_i) = \frac{n-m}{mn^2} \). This gives an intuition for the mean vector and the covariance matrix in the general case. One point to note is that the variance matrix for \( W \) is not strictly positive which implies that \( W \) lies in a lower dimensional space. Indeed, this is true as one has \( \sum_{i=1}^{n} w_{i,k} = 1 \) almost surely for all \( k \) which is evident from the definition of \( w_{i,k} \).

**Assumption 5.** At each iteration step \((k)\) in equation (2.3), \( u_{i,k} \) are generated iid from the same \( Q \) i.e.

\[ u_{i,k} \overset{iid}{\sim} Q, \quad \text{for all } i = 0, 1, 2, \cdots, n \text{ and } k = 0, 1, 2, \cdots, K \]

where \( Q \) denotes the probability measure such that \( E(l(\theta, u_{i,k})) = g(\theta) \) for all \( \theta \) and \( i, k \).

This assumption on the training data is natural. However, it is our strong belief that most of our results still hold even when we do not refresh our data at each iteration step.

Next we make further assumptions on \( W \) which enable us in proving the Gaussian nature of the M-SGD error. Define \( l = (1, 1, \cdots, 1)' \), i.e. the vector of all entries 1. Note that

\[ \sqrt{\frac{n-m}{mn(n-1)}} \left( I - \frac{1}{n} l l' \right) = \Sigma^{1/2}, \]

where \( \Sigma \) is defined in Assumption 4. Also, note that if we scale the weights by \( m \), we have

\[ Y = (Y_1, Y_2, \cdots, Y_n)' \overset{\Delta}{=} m(w_1, w_2, \cdots, w_n)', \quad (2.10) \]

In this case, \( E(Y) = \frac{m}{n}(1, 1, \cdots, 1)' \) and

\[ Var(Y) = \frac{m}{n} \left( 1 - \frac{m}{n} \right) \begin{bmatrix} 1 & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ -\frac{1}{n-1} & 1 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{n-1} & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & 1 \end{bmatrix}_{n \times n} \]

**Assumption 6.** For \( W \) as defined in Assumption 4 we further have

\[ W = \sqrt{\frac{n-m}{mn(n-1)}} \left( I - \frac{1}{n} l l' \right) X + \frac{1}{n} l, \quad (2.11) \]

where \( X = (X_1, X_2, \cdots, X_n) \) and \( X_1, X_2, \cdots, X_n \) are iid sub-Gaussian with mean \( \mu \) and variance 1.
This assumption enables us to consider a large class of distributions for $W$. Note that we would require higher moment conditions to prove CLT in any case due to the dependent structure of $W$. This assumption addressed this and covers a large class of distributions.

Next we state a second assumption on the $W$ vector which includes the case of the weights being non-negative.

**Assumption 7.** For the random variables $Y_i$ as defined in (2.10), we have $Y_i$ are exchangeable and $Y_i \geq 0$ with
\[
E(Y_i^3) \leq \frac{o(m^{3/2})}{n}, \quad E(Y_i^4) \leq \frac{o(m^2)}{n} \quad \text{and} \quad E(Y_i^2 Y_j^2) = \left(\frac{m}{n}\right)^2 + o\left(\left(\frac{m}{n}\right)^2\right).
\]

Note that we need two separate assumptions as in the first assumption the structure of $W$ forces $W$ to have negative values. This is addressed in the second assumption.

In the later sections, we use the assumption of strong convexity on $g$. The assumption is as such

**Assumption 8.** A function $g$ is $\lambda$-strongly convex if
\[
\lambda I \leq \nabla^2 g(\theta) \quad \text{for some } \lambda > 0 \quad \text{and all } \theta \in \mathbb{R}^p.
\]

**Remark 2.6.** Note that this also implies $g(x) \geq g(y) + (x - y)'\nabla g(y) + \frac{\lambda}{2}|x - y|^2$ for all $x, y \in \mathbb{R}^p$.

Note that the assumption of strong convexity indeed forces $g$ to have a minima. In fact, we can say that there exits a unique $x^*$ such that $\inf_x g(x) = g(x^*)$.

## 3 Main Results

Throughout this paper, we denote $g$ to be the objective function and $l(\theta, u)$ as the loss function. We consider the problem in the regime where $T = \gamma K$, where $T$ is fixed. As mentioned in Algorithm 2, $\gamma$ is the step size and $K$ is the number of iterations. We continue the same notation in Algorithm 3 below. We also have $n \to \infty$, $m \to \infty$ with $m/n \to \gamma^*$ and $0 \leq \gamma^* \leq 1$. The following is the algorithmic representation of the online M-SGD algorithm

**Algorithm 3:** Online Multiplicative Stochastic Gradient Descent (Online M-SGD)

1. **Input:** Starting point $x_0 \in \mathbb{R}^p$, loss function $l((x, u) \in \mathbb{R}$ and step size (also called learning rate) as $\gamma > 0$.
2. **for** $k = 0, 1, 2 \cdots, K - 1$ **do**
3. Generate data for the $k$-th iteration $u_{1,k}, u_{2,k}, \ldots, u_{n,k}$.
4. Generate the weight vector $W_k \in \mathbb{R}^n$, for the $k$-th iteration; where $W_k = (w_{1,k}, w_{2,k}, \ldots, w_{n,k})'$, as per Assumption 4.
5. Update to the next step as $x_{k+1} = x_k - \gamma \sum_{i=1}^n w_{i,k} \nabla l(x_k, u_{i,k})$.
6. **end for**
7. **Output** $x_K$

Note that the update step in Algorithm 3 can also be expressed as
\[
x_{k+1} = x_k - \gamma \nabla g(x_k) - \gamma \sum_{i=1}^n w_{i,k} \left(\nabla l(x_k, u_{i,k}) - \nabla g(x_k)\right)
\]
owing to the fact \( \sum_{i=1}^{n} w_{i,k} = 1 \), which follows from Assumption 4. The term \( \sum_{i=1}^{n} w_{i,k} (\nabla l(x_{k},u_{i,k}) - \nabla g(x_{k})) \) is called the M-SGD error. We first exhibit that for any \( \theta \) a scaled version of this error is approximately Gaussian or for all \( \theta \in \mathbb{R}^{p} \), we have

\[
\sqrt{m} \sum_{i=1}^{n} w_{i,k} (\nabla l(\theta, u_{i,k}) - \nabla g(\theta)) \overset{d}{\approx} N(0, \sigma^{2}(\theta)).
\]

The symbol \( \overset{d}{\approx} \) implies has approximate distribution. This approximation holds when \( n \) is large. Though this problem is well known for the minibatch case, for general \( W_{k} = (w_{1,k}, w_{2,k}, \cdots, w_{n,k})' \), this problem is not much explored. The gradient term in Algorithm 3 is given as \( \sum_{i=1}^{n} w_{i,k} \nabla l(x_{k}, u_{i,k}) \). In the minibatch case, \( w_{i,k} = \frac{1}{m} \) if the \( i \)-th training sample is selected else \( w_{i,k} = 0 \). Also \( \sum_{i=1}^{n} w_{i,k} = 1 \) for all \( k \).

We claim that if \( W_{k} \) has mean and covariance structure as given in Assumption 4 some key properties of the minibatch SGD are retained. This in some way is a universality of the weights. Based on the relation between \( m, n \) we divide the first problem into cases (\( \gamma^{*} = 1 \) and \( \gamma^{*} < 1 \)) and analyse the Gaussianity of the error term in Algorithm 3. For ease of notation, we shall write \( u_{i,k} = u_{i} \) for this section as the Gaussianity property is independent of \( k \).

Given the above setup, we have the following result.

**Theorem 1.** Under the regime \( m/n \to 1 \), with Assumptions 4 we have

\[
\lim_{n \to \infty} \mathbb{E} \left( \sqrt{m} \left( \sum_{i=1}^{n} w_{i} \nabla l(\theta, u_{i}) - \nabla g(\theta) \right) - \sqrt{n} \left( \sum_{i=1}^{n} \frac{1}{n} \nabla l(\theta, u_{i}) - \nabla g(\theta) \right) \right)^{2} = 0
\]

and

\[
\lim_{n \to \infty} \mathbb{E} \left( \sqrt{n} \left( \sum_{i=1}^{n} w_{i} \nabla l(\theta, u_{i}) - \nabla g(\theta) \right) - \sum_{i=1}^{n} \frac{1}{n} \nabla l(\theta, u_{i}) \right)^{2} = 0.
\]

The proof is provided in the Appendix. Using the above result, we instantly get the following CLT

**Corollary 1.** In the regime \( m/n \to 1 \) as \( n \to \infty \), we have

\[
\sqrt{m} \sum_{i=1}^{n} w_{i} (\nabla l(\theta, u_{i}) - \nabla g(\theta)) \overset{d}{\to} N(0, \sigma^{2}(\theta))
\]

as \( n \to \infty \) where the weights \( w_{i} \) are as in Assumption 4.

In the regime, \( m/n \to \gamma^{*} \) where \( 0 \leq \gamma^{*} < 1 \), the question becomes a little complicated. In this particular scenario we need to make use of the additional assumptions as mentioned in the preliminaries (Section 2) in order to guarantee us a central limit theorem.

We invoke Assumptions 4 and 6 or 7 to help us in obtaining the following CLT.

**Theorem 2.** We consider the regime \( m/n \to \gamma^{*} \); \( 0 \leq \gamma^{*} < 1 \) as \( n \to \infty \), \( m \to \infty \), with \( w = (w_{1}, w_{2}, \ldots, w_{n}) \) as defined in Assumption 4. We also take the Assumptions 4 and 6 or 7 to be true and \( u_{i} \sim Q \) iid, where the probability measure \( Q \) is such that \( \mathbb{E}(\nabla l(\theta, u_{i})) = \nabla g(\theta) \). In such a setting, we have

\[
\sqrt{m} \left( \sum_{i=1}^{n} w_{i} \nabla l(\theta, u_{i}) - \nabla g(\theta) \right) \overset{d}{\to} N(0, \sigma^{2}(\theta))
\]

as \( n \to \infty \).
The proof of this theorem is given in the Appendix.

**Remark 3.1.** The CLT still holds if we consider iid mean zero random variables with finite third moment instead of $(\nabla l(\theta, u_i) - \nabla g(\theta))$. The asymptotic variance is dependent on the distribution of the iid random variables.

The proof of Theorem 2 provides evidence for the remark.

**Example for Assumption 6**

An example of weights where this structure is observed is when $X \sim N(0, I)$. We can easily check that the assumptions for Assumption 6 are satisfied here.

**Example for Assumption 7**

The simplest example for the positive case is the minibatch/hypergeometric random variables where $w_i = \frac{1}{m}$ if the $i$-th element is selected and $w_i = 0$ otherwise. Also $\sum_{i=1}^{n} w_i = 1$ which implies exactly $m$ indices are selected out of $n$. In this case it is easy to verify that the Assumptions 4 and Assumption 7 hold. However, we provide a more non-trivial example which is the Dirichlet Distribution.

Consider a vector $w \sim \text{Dir}((\frac{m-1}{n-m}, \frac{m-1}{n-m}, \ldots, \frac{m-1}{n-m}))$, where the parameter vector is of dimension $n$.

**Lemma 3.1.** If $X \sim \text{Dir}(\alpha)$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then

$$E\left(\prod_{i=1}^{n} X_i^{\beta_i}\right) = \frac{\Gamma(n)}{\Gamma(n)} \frac{\Gamma\left(\sum_{i=1}^{n} \alpha_i + \beta_i\right)}{\Gamma\left(\sum_{i=1}^{n} \alpha_i\right)} \prod_{i=1}^{n} \frac{\Gamma\left(\alpha_i + \beta_i\right)}{\Gamma\left(\alpha_i\right)}.$$

The proof of the above lemma is simple hence we skip it. Note that a $w$ which follows the given Dirichlet distribution has the property that $w_i$ are exchangeable, $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. Also, some minor calculations will show that

$$w = (w_1, w_2, \ldots, w_n) \sim \left(\frac{1}{n}(1, 1, 1, \ldots, 1), \Sigma\right).$$

Here $\Sigma$ is defined as previously. We define $Y_i = m \cdot w_i$. We show that $Y_i$ satisfy the assumptions that we have mentioned before. Note that $Y = (Y_1, Y_2, \ldots, Y_n)$ has the same mean and variance covariance matrix as desired.

We already have $Y_i \geq 0$ and $\{Y_1, Y_2, \ldots, Y_n\}$ exchangeable. The following result is required to establish Assumption 7 for this scenario.

**Lemma 3.2.** If $w \sim \text{Dir}((\frac{m-1}{n-m}, \frac{m-1}{n-m}, \ldots, \frac{m-1}{n-m}))$, then it satisfies the conditions of Assumption 7.

**Proof.** We start with the third moment

$$E(Y_i^3) = m^3 E(w_i^3) = m^3 E(w_1^3)$$

$$= m^3 \frac{\Gamma\left(\sum_{i=1}^{n} \frac{m-1}{n-m}\right)}{\Gamma\left(\sum_{i=1}^{n} \frac{m-1}{n-m} + 3\right)} \frac{\Gamma\left(\frac{m-1}{n-m} + 3\right)}{\Gamma\left(\frac{m-1}{n-m}\right)}$$

$$= m^3 \frac{\left(\frac{m-1}{n-m} + 2\right)\left(\frac{m-1}{n-m} + 1\right)}{n} \frac{\left(\frac{n(m-1)}{n-m} + 2\right)\left(\frac{n(m-1)}{n-m} + 1\right)}{\left(\frac{n(m-1)}{n-m}\right)}.$$
This implies,

\[
\mathbb{E}(Y_i^3) \leq \frac{m^3}{n} \left( \frac{1}{n} + 2 \frac{n - m}{n(m-1)} \right) \left( \frac{1}{n} + \frac{n - m}{n(m-1)} \right) \\
\leq \frac{m}{n} \left( \frac{m}{n} + 2 \frac{m(n - m)}{n(m-1)} \right) \left( \frac{m}{n} + m(n - m) \right) \\
= O\left( \frac{m}{n} \right),
\]

where the second step follows from the Lemma 3.1. Hence we establish \( \mathbb{E}(Y_i^3) \leq \frac{o(m^2)}{n} \). We can similarly argue for \( \mathbb{E}(Y_i^4) \). In fact,

\[
\mathbb{E}(Y_i^4) = m^4 \mathbb{E}(w_i^4) = m^4 \mathbb{E}(w_1^4) \\
= m^4 \frac{\Gamma \left( \sum_{i=1}^{n} \frac{m - 1}{n - m} \right)}{\Gamma \left( \sum_{i=1}^{n} \frac{m - 1}{n - m} + 4 \right)} \cdot \frac{\Gamma \left( \frac{m - 1}{n - m} + 4 \right)}{\Gamma \left( \frac{m - 1}{n - m} \right)}.
\]

This implies that,

\[
\mathbb{E}(Y_i^4) = \frac{m^4}{n} \frac{\left( \frac{m - 1}{n - m} + 3 \right) \left( \frac{m - 1}{n - m} + 2 \right) \left( \frac{m - 1}{n - m} + 1 \right)}{\left( \frac{m - 1}{n - m} + 3 \right) \left( \frac{m - 1}{n - m} + 2 \right) \left( \frac{m - 1}{n - m} + 1 \right)} \\
\leq \frac{m^4}{n} \frac{\left( \frac{1}{n} + 3 \frac{n - m}{n(m - 1)} \right) \left( \frac{1}{n} + 2 \frac{n - m}{n(m - 1)} \right) \left( \frac{1}{n} + \frac{n - m}{n(m - 1)} \right)}{\left( \frac{1}{n} + 3 \frac{n - m}{n(m - 1)} \right) \left( \frac{1}{n} + 2 \frac{n - m}{n(m - 1)} \right) \left( \frac{1}{n} + \frac{n - m}{n(m - 1)} \right)} \\
= \frac{m}{n} \left( \frac{m}{n} + 3 \frac{m(n - m)}{n(m - 1)} \right) \left( \frac{m}{n} + 2 \frac{m(n - m)}{n(m - 1)} \right) \left( \frac{m}{n} + \frac{m(n - m)}{n(m - 1)} \right) \\
= O\left( \frac{m}{n} \right).
\]

Hence we establish the required condition for the fourth power as \( \mathbb{E}(Y_i^4) = \frac{o(m^2)}{n} \). The last step is to show that \( \mathbb{E}(Y_i^2 Y_j^2) \) satisfies the assumption.

\[
\mathbb{E}(Y_i^2 Y_j^2) = m^4 \mathbb{E}(w_i^2 w_j^2) = m^4 \mathbb{E}(w_1^2 w_2^2) \\
= m^4 \frac{\Gamma \left( \sum_{i=1}^{n} \frac{m - 1}{n - m} \right)}{\Gamma \left( \sum_{i=1}^{n} \frac{m - 1}{n - m} + 4 \right)} \cdot \left[ \frac{\Gamma \left( \frac{m - 1}{n - m} + 2 \right)}{\Gamma \left( \frac{m - 1}{n - m} \right)} \right]^2.
\]

This leads to

\[
\mathbb{E}(Y_i^2 Y_j^2) = \frac{m^4}{n^2} \left( \frac{m - 1}{n - m} + 3 \right) \left( \frac{m - 1}{n - m} + 2 \right) \left( \frac{m - 1}{n - m} + 1 \right) \cdot \left( \frac{m - 1}{n - m} \right)^2 \\
\leq \frac{m^4}{n^2} \left( \frac{1}{n} + \frac{n - m}{n(m - 1)} \right)^2 \\
= \frac{m^2}{n^2} \left( \frac{m}{n} + \frac{m(n - m)}{n(m - 1)} \right)^2 - 1 \right) + \frac{m^2}{n^2} \\
= o\left( \frac{m^2}{n^2} \right) + \frac{m^2}{n^2}.
\]

Hence Assumption 7 is satisfied. This implies that the CLT holds for Dirichlet weights with the parameter vector \((\frac{m - 1}{n - m}, \frac{m - 1}{n - m}, \ldots, \frac{m - 1}{n - m})'\).
Note that if we rewrite the iteration step in Algorithm 3 as
\[ x_{k+1} = x_k - \nabla g(x_k) + \frac{\gamma}{\sqrt{m}} \left( \sum_{i=1}^{n} \sqrt{m} w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x_k)) \right), \]
where the term \( \sum_{i=1}^{n} \sqrt{m} w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x_k)) \) according to Theorem 2 is approximately normal given \( x_k \). Hence, we might intuitively consider this algorithm to be equivalent to that given by
\[ x_{k+1} \approx x_k - \nabla g(x_k) + \frac{\gamma}{\sqrt{m}} \sigma(x_k) Z_{k+1} \]
where \( Z_{k+1} \) is a standard Gaussian random variable in \( p \) dimensions. This provides motivation for the next result where we show that equations (2.2) and (2.3) are close in the squared mean sense.

**Theorem 3.** Suppose Assumptions 1-5 hold. Recall \( x_{n,k} \) and \( x_k \) from equations (2.3) and (2.2). Then for any \( k \leq K = \left[ \frac{T}{\gamma} \right] \) with \( m = 4^{K-1} \), we have
\[ \mathbb{E} |x_{n,k} - x_k|^2 \leq K_1 \cdot \gamma, \]
where \( K_1 \) is a constant dependent only on \( T, L, L_1 \) and \( p \).

The proof of Theorem 3 is provided in the Appendix where more information on the constant \( K_1 \) can be found.

The above theorem shows that M-SGD and SGD with scaled normal errors are indeed close in terms of the step size and the minibatch size. This is because the rest of terms can be bounded by realizing that \( \gamma = T/K \). Note that in all our work we take \( t \) to be in the compact interval \([0,T]\). Also note that the minibatch size increases exponentially with the maximum number of iterations in order to get reasonable bounds. This implies that in general we need exponentially large number samples for these two algorithms to be close in the most general case. However, given that we are not using any specific property of the distribution of the weights \( w_{i,k} \), this is a universal rate for all classes of weights with specified mean and variance covariance. Hence this is a guarantee we have even in the worst case scenarios for any weights with such mean and covariance structure. In fact, this is the first result of its kind in literature and we hypothesize that may be obtained when considering a specific class of weight distributions.

Next we establish a nonasymptotic bound between (2.4) and (2.5). There have been previous works attempting to address this problem. The main work is by Wu et al. [28] in which the authors derive bounds in their setting which assumes the boundedness of the loss function. Here we do this in our set up which assumes the loss function and the covariance function are Lipschitz in the parameter. Recall that here \( W_2 \) is the 2-Wasserstein distance which has been defined in the preliminaries section in (2.8). Our main aim here is to show that the M-SGD algorithm is close to the SGD diffusion in the step size. The way we go about this is to construct a linear version of the algorithm (2.7) and then show that this process is close to the SGD diffusion (2.5). We first show that the M-SGD process defined in (2.7) is close to the linearization of the discrete SGD algorithm (2.4) in the Wasserstein distance. Then we show that (2.4) is close to (2.5) in the Wasserstein distance. This ultimately helps us in showing that (2.7) and (2.5) are close in the Wasserstein distance.

In the following theorem we show that the Wasserstein distance between (2.7) and (2.4) is in the order of the step-size.

**Theorem 4.** Suppose Assumptions 4-5 hold. Recall \( D_t \) and \( X_t \) as the stochastic processes defined in equations (2.4) and (2.5) respectively. Then for any \( t \in (0, T] \) and any \( m \geq 1 \), we have
\[ W_2^2(D_t, X_t) \leq C_{11} \gamma^2 + C_{12} \gamma, \]
where \(C_{11}, C_{12}\) are constants dependent only on \(T, L, L_1, p\).

The proof of Theorem 4 is furnished in the Appendix where more information on the constants \(C_{11}, C_{12}\) is also provided.

We have shown that \(Y_{n,t}\) and \(D_t\) as defined by equations (2.7) and (2.4) are close in the Wasserstein distance in the Appendix. This brings us to one of our main results.

**THEOREM 5.** Suppose Assumptions [1-5 hold. Recall \(Y_{n,t}\) and \(X_t\) as the stochastic processes defined by (2.7) and (2.5). Then for any \(t \in (0, T]\) and \(m = 4K\) we have

\[
W_2^2(Y_{n,t}, X_t) \leq C_{21}\gamma^2 + C_{22}\gamma,
\]

where \(C_{21}, C_{22}\) are constants dependent only on \(T, L, L_1, p\).

The proof of Theorem 5 is furnished in the Appendix where more information on the constants \(C_{21}, C_{22}\) is also provided.

In essence we observe that the M-SGD algorithm is close in distribution to the diffusion for the SGD algorithm at each point in order of the square root of the step size if \(m = 4K\). Note that the bounds obtained in Theorems 3 and 5 are similar functions of the step size \(\gamma\). However, we also note that in Theorem 3 the dependence on \(m\) is much weaker as in \(m\) can take any value greater than 1. In Theorem 5 the “minibatch” size needs to be exponentially large in terms of the maximum iteration number. This is due to the fact that the distribution of the weights in Theorem 5 is unknown and hence needs a large sample and minibatch size to establish the same rate. For specific problems, we should be able to relax the condition on \(m\).

Next we invoke the assumption of strong convexity of \(g\) and derive bounds for the convergence of the M-SGD algorithm in squared mean to the optimal point. We also show that the expected value of the function also converges to the optimum value. Let \(x^* = \arg\min_x g(x)\).

**THEOREM 6.** Under Assumptions [1-5] and 8 under the regime \(0 < \gamma < \frac{2}{L}\) and \(m > \frac{2pL L_1^2 \gamma}{\lambda^2 (2 - L \gamma)}\), Algorithm 2 exhibits the following

\[
\mathbb{E}(g(x_{k+1}) - g(x^*)) \leq \left[ 1 - \lambda \gamma (2 - L \gamma) + \frac{2pL L_1^2 \gamma^2}{m \lambda} \right]^{k+1} (g(x_0) - g(x^*))
\]

\[
+ \frac{L \gamma}{m \left( \lambda (2 - L \gamma) - \frac{2pL L_1^2 \gamma}{m \lambda} \right)} \|\sigma(x^*)\|_F^2,
\]

\[
\mathbb{E}|x_{k+1} - x^*|^2 \leq \frac{2}{\lambda} \left[ 1 - \lambda \gamma (2 - L \gamma) + \frac{2pL L_1^2 \gamma^2}{m \lambda} \right]^{k+1} (g(x_0) - g(x^*))
\]

\[
+ \frac{2}{\lambda} \left[ \frac{L \gamma}{m \left( \lambda (2 - L \gamma) - \frac{2pL L_1^2 \gamma}{m \lambda} \right)} \right] \|\sigma(x^*)\|_F^2.
\]

The proof of Theorem 6 is given in the Appendix. Note that the existence of an optima follows from strong convexity. Also note that since \(x^*\) is the optimum value, we have \(g(x_0) - g(x^*) > 0\). This is nonrandom as both \(x_0\) and \(x^*\) are fixed points.
4 Simulations

4.1 First Example Central Limit Theorem: Dirichlet weights

This example exhibits the CLT for Dirichlet weights as in Theorem 2. Note that instead of $\nabla l(\theta, u) - \nabla g(\theta)$, we can take any sequence of iid random variables with mean zero and finite third moment by Remark 3.1. In this example we consider $U_i \sim Unif(-1,1)$ for all $i$. Note that we take dimension of this problem as $1$ i.e. $p = 1$ with $n = 10^4$, $m = 2000$. We generate $10^4$ samples of $\sqrt{m} \sum_{i=1}^{n} w_i U_i$. In this example the weight vector $W = (w_1, w_2, \ldots, w_n)$ is simulated from $Dir((\frac{1999}{8000}, \frac{1999}{8000}, \ldots, \frac{1999}{8000}))$ which is the Dirichlet distribution with parameter vector of length $10^4$, given as $(\frac{1999}{8000}, \frac{1999}{8000}, \ldots, \frac{1999}{8000})'$. The results is exhibited in Figure 1. From the plot it seems that the samples are distributed as per the normal distribution.

![Histogram](image)

Figure 1: Histogram of the 10000 samples of $\sqrt{m} \sum_{i=1}^{n} w_{i,k} U_i$. Here we have $p = 1$ with $n = 10^4$, $m = 2000$. The weight vector $W = (w_1, w_2, \ldots, w_n)$ is simulated from $Dir((\frac{1999}{8000}, \frac{1999}{8000}, \ldots, \frac{1999}{8000}))$. The plot indicates the Gaussian nature of the samples.

4.2 Second Example Central Limit Theorem: Normal weights

We consider Gaussian weights instead of Dirichlet and verify the CLT as given in Theorem 2. Similar to the last example we consider $U_i = (U_{i1}, U_{i2}, \ldots, U_{i5})$ where each $U_{ij} \sim Unif(-1,1)$ iid as our data. We have $W \sim N(\mu, \Sigma)$ where $\mu$ and $\Sigma$ are as defined in Assumption 4. The dimension of the problem is taken to be $5$ i.e. $p = 5$ with $n = 10^4$, $m = 2000$ and $10^3$ samples of $\sqrt{m} \sum_{i=1}^{n} w_{i,k} U_i$ are generated. We observe the distribution of the resultant data. The histogram of the one dimensional projection of the data along the standard basis is presented in Figure 2.

4.3 Fourth Example (Convergence in Case of Convexity)

In this example we examine our algorithm in the context of the Logistic regression problem. Consider $t \in \mathbb{N}$ and data given to us in the form $(y_i, x_i)_{i=1}^{t} \in \mathbb{R}^{p+1}$, where $y_i \in \{0,1\}$ and $x_i \in \mathbb{R}^{p}$. Our objective function is given as the negative log-likelihood plus an $L_2$-regularization penalty. The objective function is

$$g(\beta) = \frac{1}{t} \left[ -\sum_{i=1}^{t} y_i x_i^\prime \beta + \sum_{i=1}^{t} \log \left(1 + e^{x_i^\prime \beta}\right) \right] + \kappa |\beta|^2,$$  \hspace{1cm} (4.1)

where $\kappa > 0$ is some constant.
Figure 2: Histogram of the 10000 samples of $\sqrt{m}\sum_{i=1}^{n} w_{i,k} U_i$. Here we have $p = 6$ with $n = 10^4$, $m = 2000$. The weight vector $W = (w_1, w_2, \cdots, w_n)$ is distributed as per $N(\mu, \Sigma)$ where $\mu$ and $\Sigma$ are as specified in Assumption 4.

We choose our training data to be the random samples of $(y_i, x_i)$ done with replacement. That is for each $u_i \in (u_1, u_2, \cdots, u_n)$, we have $u_i = (y_j, x_j)$ with probability $\frac{1}{t}$ for all $i, j$. The parameter for the problem is $\beta \in \mathbb{R}^p$. Note that this objective function as defined in (4.1), is strongly convex with Lipschitz gradients. Indeed this is easy to see as

$$\nabla^2 g(\beta) = \frac{1}{t} \left[ \sum_{i=1}^{t} \frac{e^{x_i'\beta}}{(1 + e^{x_i'\beta})^2} x_i x_i' \right] + 2\kappa I.$$  

It is immediate that the above matrix is positive definite with $||\nabla^2 g(\beta)||_2 \geq 2\kappa$. Also, as $x_i$ are fixed data points, $||\nabla^2 g(\beta)||_2 \leq \frac{1}{t} \lambda_{max}(XX') + 2\kappa$, where $X = [x_1, x_2, \cdots, x_t]$ and $\lambda_{max}(XX')$ denotes the largest eigenvalue of $XX'$. Hence $\nabla g(\beta)$ is Lipschitz.

Define $u_i = (v_i, u_{1,i}, u_{2,i}, \cdots, u_{p,i})$ and $\tilde{u}_i = (u_{1,i}, u_{2,i}, \cdots, u_{p,i})$. Also define the loss function as

$$l(\beta, u) = -v_i \tilde{u}_i' \beta + \log \left(1 + e^{\tilde{u}_i' \beta}\right) + \kappa ||\beta||^2.$$  

(4.2)

Note that the loss function is also strongly convex and Lipschitz in $\beta$. It can also be easily seen that the loss function is unbiased for the objective function. We need to find a matrix $\sigma(\beta)$ such that $\sigma(\beta)\sigma(\beta)' = Var(\nabla l(\beta, u))$ and $\sigma(\beta)$ is Lipschitz in $\beta$ in the $||\cdot||_2$ norm. Now,

$$\nabla l(\beta, u) = -v_i \tilde{u}_i + \frac{e^{\tilde{u}_i' \beta}}{1 + e^{\tilde{u}_i' \beta}} u_i + 2\kappa \beta.$$  

Define $z_i = (y_i, x_i)$. We have

$$Var(\nabla l(\beta, u)) = \frac{1}{t} \sum_{i=1}^{t} (\nabla l(\beta, z_i) - \nabla g(\beta)) (\nabla l(\beta, z_i) - \nabla g(\beta))'.$$

Define

$$A = \frac{1}{\sqrt{t}} \left[ (\nabla l(\beta, z_1) - \nabla g(\beta)), (\nabla l(\beta, z_2) - \nabla g(\beta)), \cdots, (\nabla l(\beta, z_t) - \nabla g(\beta)) \right].$$

Note that

$$Var(\nabla l(\beta, u)) = AA'.$$
Hence, for this problem, we may take

\[ \sigma(\beta) = \frac{1}{\sqrt{t}} \left[ (\nabla l(\beta, z_1) - \nabla g(\beta)), (\nabla l(\beta, z_2) - \nabla g(\beta)), \cdots, (\nabla l(\beta, z_t) - \nabla g(\beta)) \right]. \]

It can easily be seen now that \( \sigma(\beta) \) is Lipschitz in the Frobenius norm. We have

\[ \|\sigma(\beta_1) - \sigma(\beta_2)\|_F \leq \frac{1}{\sqrt{t}} \sum_{i=1}^{t} |\nabla l(\beta_1, z_i) - \nabla l(\beta_2, z_i)| + \sqrt{t} |\nabla g(\beta_1) - \nabla g(\beta_2)|. \]

As \( \nabla l \) and \( \nabla g \) are both Lipschitz in \( \beta \), we have \( \sigma(\beta) \) as a Lipschitz function in \( \beta \).

Note that the above argument for \( \sigma(\beta) \) being Lipschitz can be applied to large class of problems with such variance covariance matrix. The only two conditions necessary to establish this is that both \( \nabla l(\cdot) \) and \( \nabla g(\cdot) \) is Lipschitz. Also an implicit assumption in this case is that the data is fixed.

We provide simulation example to exhibit convergence for the algorithm. Consider \( p = 6 \) with the number of data points as \( t = 10^4 \). The number of samples we choose randomly with replacement is \( n = 10^3 \) and the “minibatch” size is \( m = 10 \). We consider 5 values of \( \kappa \) as \((.5, .1, .05, .01, .001)\). The data is generated as \( y_i \sim Ber(1/2) \) iid and \( x_i \) are random standard Gaussian. The weights at each step of the iteration are generated as \( W \sim N(\mu, \Sigma) \) where \( \mu \) and \( \Sigma \) are provided in Assumption 4. Note that this gives the true \( \beta = 0 \). After each iteration is complete we replicate it and take the norm of all the replicated \( \hat{\beta} \) and take their average. This gives us an approximation of \( E|\beta|^2 \). We plot this and show that it converges to 0 at different rates which depend on \( \kappa \).

![MSE vs Iterations Plot for Regularized Logistic Regression problem. Each coloured line is the MSE corresponding to a different value of \( \kappa \). For each experiment \( t = 10^4, n = 10^3, m = 10 \). The dimension is \( p = 6 \) and \( m = 10 \).](image)

**References**

[1] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, 2013.

[2] Léon Bottou, Frank E Curtis, and Jorge Nocedal. Optimization methods for large-scale machine learning. *Siam Review*, 60(2):223–311, 2018.

[3] Léon Bottou et al. Stochastic gradient learning in neural networks. *Proceedings of Neuro-Nımes*, 91(8):12, 1991.

[4] John Charles Butcher. *Numerical methods for ordinary differential equations*. John Wiley & Sons, 2016.
[5] Yuan Cao and Quanquan Gu. Generalization bounds of stochastic gradient descent for wide and deep neural networks. *Advances in Neural Information Processing Systems*, 32:10836–10846, 2019.

[6] Pratik Chaudhari and Stefano Soatto. Stochastic gradient descent performs variational inference, converges to limit cycles for deep networks. In *2018 Information Theory and Applications Workshop (ITA)*, pages 1–10. IEEE, 2018.

[7] Arnak Dalalyan. Further and stronger analogy between sampling and optimization: Langevin monte carlo and gradient descent. In *Conference on Learning Theory*, pages 678–689. PMLR, 2017.

[8] Arnak S Dalalyan. Theoretical guarantees for approximate sampling from smooth and log-concave densities. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 79(3):651–676, 2017.

[9] Hadi Daneshmand, Jonas Kohler, Aurelien Lucchi, and Thomas Hofmann. Escaping saddles with stochastic gradients. In *International Conference on Machine Learning*, pages 1155–1164. PMLR, 2018.

[10] Alexandre Défossez and Francis Bach. Averaged least-mean-squares: Bias-variance trade-offs and optimal sampling distributions. In *Artificial Intelligence and Statistics*, pages 205–213. PMLR, 2015.

[11] Aymeric Dieuleveut, Alain Durmus, and Francis Bach. Bridging the gap between constant step size stochastic gradient descent and markov chains. *arXiv preprint arXiv:1707.06386*, 2017.

[12] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.

[13] Saeed Ghadimi and Guanghui Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013.

[14] Noah Golmant, Nikita Vemuri, Zhewei Yao, Vladimir Feinberg, Amir Gholami, Kai Rothauge, Michael W Mahoney, and Joseph Gonzalez. On the computational inefficiency of large batch sizes for stochastic gradient descent. *arXiv preprint arXiv:1811.12941*, 2018.

[15] Jeff Heaton. Ian goodfellow, yoshua bengio, and aaron courville: Deep learning, 2018.

[16] Elad Hoffer, Itay Hubara, and Daniel Soudry. Train longer, generalize better: closing the generalization gap in large batch training of neural networks. *arXiv preprint arXiv:1705.08741*, 2017.

[17] Wenzhe Hu, Chris Junchi Li, Lei Li, and Jian-Guo Liu. On the diffusion approximation of nonconvex stochastic gradient descent, 2018.

[18] Chi Jin, Praneeth Netrapalli, Rong Ge, Sham M. Kakade, and Michael I. Jordan. On nonconvex optimization for machine learning: Gradients, stochasticity, and saddle points, 2019.

[19] Nitish Shirish Keskar, Dheevatsa Mudigere, Jorge Nocedal, Mikhail Smelyanskiy, and Ping Tak Peter Tang. On large-batch training for deep learning: Generalization gap and sharp minima. *arXiv preprint arXiv:1609.04836*, 2016.

[20] Rémi Leluc and François Portier. Towards asymptotic optimality with conditioned stochastic gradient descent. *arXiv preprint arXiv:2006.02745*, 2020.
5 Appendix

We divide this section into three parts. The first part consists the proofs of the CLT results in Theorems 1 and 2. The next part consists of the proofs of Theorems 3, 4 and 5. The last part of the section consists the proof of Theorem 6.
Proof of Theorems 1 and 2

We present the following lemma which shall be used throughout our work.

**Lemma 5.1.** Under Assumption 2, we have

\[ \mathbb{E} (\nabla l(\theta, u)) = \nabla g(\theta). \]

**Proof.** We prove this for 1 dimension as that suffices for the general case as expectation distributes over all the components of the vector. Here \( \theta \) is a fixed point at which we differentiate. Note that

\[ \frac{l(\theta_1, u) - l(\theta, u)}{\theta_1 - \theta} = \nabla l(\xi, u) \]

for some \( \xi \) using the mean value theorem. Note that, since differentiation is a local property, we can force \( \theta_1 \in B(\theta, 1) \), where \( B(\theta, 1) \) denotes the ball centred at \( \theta \) with radius 1. This also forces \( \xi \in B(\theta, 1) \). Also note,

\[ |\nabla l(\xi, u)| \leq |\nabla l(\theta, u)| + h(u) |\xi - \theta| \]

\[ \leq |\nabla l(\theta, u)| + h(u). \]

The last line follows as \( \xi \in B(\theta, 1) \).

This implies

\[ \frac{l(\theta_1, u) - l(\theta, u)}{\theta_1 - \theta} \leq \frac{l(\theta_1, u) - l(\theta, u)}{\theta_1 - \theta} \leq |\nabla l(\theta, u)| + h(u). \]

The last term is independent of \( \theta_1 \) and is integrable. Hence we can use DCT and we are done. \( \square \)

**Proof of Theorem 1.** We begin by noting the fact,

\[ \mathbb{E} \left( \sqrt{m} \left( \sum_{i=1}^{n} w_i \nabla l(\theta, u_i) - \nabla g(\theta) \right) - \sqrt{n} \left( \sum_{i=1}^{n} \frac{1}{n} \nabla l(\theta, u_i) - \nabla g(\theta) \right) \right)^2 \]

\[ = \mathbb{E} \left| \sum_{i=1}^{n} \left( \sqrt{m} w_i - \frac{1}{\sqrt{n}} \right) \nabla l(\theta, u_i) + \left( \sqrt{n} - \sqrt{m} \right) \nabla g(\theta) \right|^2. \]

Now, the last term is equal to,

\[ \mathbb{E} \left( \sum_{i=1}^{n} \left( \sqrt{m} w_i - \frac{1}{\sqrt{n}} \right) \nabla l(\theta, u_i), \sum_{i=1}^{n} \left( \sqrt{m} w_i - \frac{1}{\sqrt{n}} \right) \nabla l(\theta, u_i) \right) \]

\[ + 2 \mathbb{E} \left( \sum_{i=1}^{n} \left( \sqrt{m} w_i - \frac{1}{\sqrt{n}} \right) \nabla l(\theta, u_i), \left( \sqrt{n} - \sqrt{m} \right) \nabla g(\theta) \right) + \left( \sqrt{n} - \sqrt{m} \right)^2 |\nabla g(\theta)|^2. \]

We condition on \( w = (w_1, w_2, \ldots, w_n) \) and get the above expression equal to

\[ \mathbb{E} \left[ \sum_{1 \leq i, j \leq n} \left( \sqrt{m} w_i - \frac{1}{\sqrt{n}} \right) \left( \sqrt{m} w_j - \frac{1}{\sqrt{n}} \right) \mathbb{E}_w \langle \nabla l(\theta, u_i), \nabla l(\theta, u_j) \rangle \right] \]

\[ + 2 \mathbb{E} \left[ \sum_{i=1}^{n} \left( \sqrt{m} w_i - \frac{1}{\sqrt{n}} \right) \left( \sqrt{n} - \sqrt{m} \right) |\nabla g(\theta)|^2 \right] \]

\[ + \left( \sqrt{n} - \sqrt{m} \right)^2 |\nabla g(\theta)|^2. \]
Here $E_w$ denotes the conditional expectation with respect to the weights. Using the fact $E(w_i) = \frac{1}{n}$ and some minor manipulation, the second term is $-2(\sqrt{n} - \sqrt{m})^2 |\nabla g(\theta)|^2$. Hence, we have

$$
E \left[ \sum_{1 \leq i, j \leq n} \left( \sqrt{mw_i} - \frac{1}{\sqrt{n}} \right) \left( \sqrt{mw_j} - \frac{1}{\sqrt{n}} \right) E_w \langle \nabla l(\theta, u_i), \nabla l(\theta, u_j) \rangle \right] 
- (\sqrt{n} - \sqrt{m})^2 |\nabla g(\theta)|^2
$$

$$
= E \left[ \sum_{i=1}^n \left( \sqrt{mw_i} - \frac{1}{\sqrt{n}} \right)^2 E_w |\nabla l(\theta, u_i)|^2 \right] 
+ \sum_{1 \leq i, j \leq n, i \neq j} \left( \sqrt{mw_i} - \frac{1}{\sqrt{n}} \right) \left( \sqrt{mw_j} - \frac{1}{\sqrt{n}} \right) E_w \langle \nabla l(\theta, u_i), \nabla l(\theta, u_j) \rangle 
- (\sqrt{n} - \sqrt{m})^2 |\nabla g(\theta)|^2
$$

$$
= E \left[ \sum_{i=1}^n \left( \sqrt{mw_i} - \frac{1}{\sqrt{n}} \right)^2 (\text{Tr} \sigma^2(\theta) + |\nabla g(\theta)|^2) \right] 
+ \sum_{1 \leq i, j \leq n} \left( \sqrt{mw_i} - \frac{1}{\sqrt{n}} \right) \left( \sqrt{mw_j} - \frac{1}{\sqrt{n}} \right) |\nabla g(\theta)|^2
- (\sqrt{n} - \sqrt{m})^2 |\nabla g(\theta)|^2
$$

Using the covariance structure of the weights, the last expression reduces to

$$
2 \left[ \text{Tr} \sigma^2(\theta) + |\nabla g(\theta)|^2 \right] \left( 1 - \frac{m}{n} \right) + |\nabla g(\theta)|^2 \left[ (\sqrt{m} - \sqrt{n})^2 - 2 + 2 \frac{m}{n} \right] - (\sqrt{n} - \sqrt{m})^2 |\nabla g(\theta)|^2
$$

$$
= 2 \left( 1 - \frac{m}{n} \right) \text{Tr} \sigma^2(\theta).
$$

Using this, in the regime $m \rightarrow 1$ as $n \rightarrow \infty$, we get the first conclusion for Theorem 1. The second conclusion to Theorem 1 can be derived similarly.

**Lemma 5.2.** Under the Assumptions 3, 4 and Assumption 6 or 7, we have

$$m \sum_{i=1}^n w_i^2 \overset{a.s.}{\rightarrow} 1,$$

when $\frac{m}{n} \rightarrow \gamma^*$ and $0 \leq \gamma^* \leq 1$.

**Proof.** We shall prove the result for Assumption 6 and Assumption 7 separately.

**Proof in the case of Assumption 6.**

Noticing the fact that $w = \Sigma^{1/2} X + \frac{1}{n} l$, we have

$$m \sum_{i=1}^n w_i^2 = m \left( X' \Sigma X + \frac{1}{n} \right).$$

We also note that $\Sigma = \frac{n-m}{mn(n-1)} (I - \frac{1}{n} l' l)$. Using this,

$$m \sum_{i=1}^n w_i^2 = \frac{n-m}{n(n-1)} X' \left( I - \frac{1}{n} l' l \right) X + \frac{m}{n}$$

$$= \frac{n-m}{n-1} \left( \frac{1}{n} X' X - \bar{X}^2 \right) + \frac{m}{n}.$$
Now, the above expression converges to 1 both almost surely and in $L^1$-norm, whatever $\gamma^*$ is. If $\gamma^* = 0$, the second term converges to 0 and the first term converges to 1 almost surely using Law of Large Numbers (also $L^1$ as $X_1$ is sub-Gaussian). If $\gamma^* = 1$, the second term converges to 1 and the first term converges to 0. If $0 < \gamma^* < 1$, we have the second term converging to $\gamma^*$ and the first converging to $1 - \gamma^*$. Hence we conclude the proof for Assumption 6.

**Proof in the case of Assumption 7**

Recall that $Y_i = m w_i$. Using this and the fact that $w_i$ are exchangeable, we have

$$E \left( \frac{1}{m} \sum_{i=1}^{n} Y_i^2 \right) = \frac{n}{m} E(Y_1^2) = 1.$$  

And

$$Var \left( \frac{1}{m} \sum_{i=1}^{n} Y_i^2 \right) = E \left( \frac{1}{m} \sum_{i=1}^{n} Y_i^2 \right)^2 - \left[ E \left( \frac{1}{m} \sum_{i=1}^{n} Y_i^2 \right) \right]^2 = \frac{1}{m^2} E \left( \sum_{i=1}^{n} Y_i^4 \right) + \frac{1}{m^2} E \left( 2 \sum_{1 \leq i < j \leq n} Y_i Y_j^2 \right) - 1 = \frac{n}{m^2} E(Y_1^4) + \frac{n(n-1)}{m^2} E(Y_1^2 Y_2^2) - 1 \leq o\left( \frac{m^2}{m^2} \right) + \frac{n(n-1)}{m^2} \left( o\left( \frac{m}{n} \right)^2 + \frac{(m)}{n} \right)^2 - 1.$$

The last two lines follow from the assumptions for positive weights. This shows that variance converges to 0. Hence we conclude the proof.

Note that the matrix $\Sigma = U \Lambda U'$ where

$$\Lambda = \frac{n - m}{mn(n-1)} \begin{bmatrix} I_{n-1} & 0_{n-1} \\ 0'_{n-1} & 0 \end{bmatrix},$$  

and we can choose $U = [x_1, x_2, \ldots, x_n]$, such that $x_i = \sqrt{\frac{n-i}{n-i+1}} \cdot (0, 0, \ldots, 1, -\frac{1}{n-1}, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1})'$ i.e. a scalar times first $i-1$ entries 0, 1 in the $i^{th}$ entry and the rest $\frac{1}{n-1}$, for $1 \leq i \leq n-1$ and $x_n = l/\sqrt{n}$. Also note that $\Sigma^{1/2} = \sqrt{\frac{n-m}{mn(n-1)}} \cdot (I_n - \frac{1}{n} ll')$.

**Lemma 5.3.** Under Assumptions 3, 4 and Assumption 6 or 7 we have

$$m^{3/2} \sum_{i=1}^{n} |w_i|^3 \overset{L^1}{\longrightarrow} 0$$

as $n \to \infty$ with $\frac{m}{n} \to \gamma^*$ and $0 \leq \gamma^* \leq 1$.

**Proof.** We prove the result for Assumption 6 and Assumption 7 separately.

**Proof in the case of Assumption 6**

We make two observations- $w_i$ all have the same distribution (this is easy to see, using the fact the $X_i$ are iid and $w_i = \sum_{j=1}^{n-1} \sqrt{\frac{n-m}{mn(n-1)}} \left(x'_j X \right) x_{j,i} + \frac{1}{n}$ since $W = \Sigma^{1/2} X + \frac{1}{n} l$, where $x_j$ are as defined previously.
Also note that we can take \( X_i \) to have 0 mean as \( \Sigma^{1/2} x = \Sigma^{1/2} (X - \mu \cdot 1) + \Sigma^{1/2} \mu \cdot 1 = \Sigma^{1/2} (X - \mu \cdot 1) \).

The last step follows as \( \mu \) is a scalar and \( \Sigma^{1/2} 1 = 0 \). With this we have

\[
m^{3/2} \sum_{i=1}^{n} |w_i|^3 = m^{3/2} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sqrt{\frac{n - m}{mn(n - 1)}} (x_j'X) x_{j,i} + \frac{1}{n} \leq 4 \left( \frac{n - m}{n(n - 1)} \right)^{3/2} \sum_{i=1}^{n} \sum_{j=1}^{n-1} (x_j'X) x_{j,i} + 4 \frac{m^{3/2}}{n^2}.
\]

It is easy to see that the second term, as for \( 0 \leq \gamma^* \leq 1 \), converges to 0.

Define

\[
T_i = \sum_{j=1}^{n-1} (x_j'X) x_{j,i}.
\]

We need to check if

\[
4 \left( \frac{n - m}{n(n - 1)} \right)^{3/2} \sum_{i=1}^{n} \mathbb{E} |T_i|^3 \to 0 \text{ as } n \to \infty.
\]

We show \( \frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E} |T_i|^3 \to 0 \) as \( n \to \infty \).

Now

\[
\frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E} |T_i|^3 = \frac{1}{\sqrt{n}} \mathbb{E} |T_1|^3,
\]

which follows from the fact that \( T_i \) are just centered and scaled \( w_i \) and hence have the same distribution.

Also,

\[
T_i = \sum_{j=1}^{n-1} (x_j'X) x_{j,i} = \sum_{j=1}^{n-1} \left( \sum_{k=1}^{n} x_{j,k} X_k \right) x_{j,i} = \sum_{k=1}^{n} \left( \sum_{j=1}^{n-1} x_{j,i} x_{j,k} \right) X_k.
\]

Using this, by Hoeffding inequality we get,

\[
\mathbb{P} (|T_1| > t) \leq 2 \exp \left\{-\frac{ct^2}{K^2 \sum_{k=1}^{n} (\sum_{j=1}^{n-1} x_{j,i} x_{j,k})^2} \right\},
\]

where \( K \) is a positive constant which depends on the distribution of \( X_1 \) and \( c > 0 \) is another such positive constant.

Note that our choice of \( x_i \) ensure that \( x_{j,i} = 0 \) when \( j > i \); so, in this case, \( x_{1,1} \) is the only non-zero value. Also note that by our construction, \( x_{1,1} = \sqrt{\frac{n-1}{n}} \). Thus

\[
\sum_{k=1}^{n} \left( \sum_{j=1}^{n-1} x_{j,1} x_{j,k} \right) = \left( 1 - \frac{1}{n} \right) \sum_{k=1}^{n} x_{j,k}^2 = 1 - \frac{1}{n}.
\]
This implies the fact
\[ P \left( |T_1| > t \right) \leq 2 \exp \left\{ -\frac{ct^2}{K} \right\} \]
for some constants \( c, K \). Hence, there exists some constant \( C \in \mathbb{R}_+ \) such that
\[ \mathbb{E}|T_1|^3 \leq C. \]
Hence \( \frac{1}{\sqrt{n}} \mathbb{E}|T_1|^3 \to 0 \) as \( n \to \infty \). Hence we conclude the proof for Assumption 6.

**Proof in the case of Assumption 7**

Recall \( Y_i = m w_i \). Proof in the case of Assumption 6 follows from the inequality
\[ \mathbb{E} \left( \frac{1}{m^{3/2}} \sum_{i=1}^{n} Y_i^3 \right) \leq \frac{1}{m^{3/2}} o(m^{3/2}). \]
Hence, we conclude the proof for Assumption 7. \( \square \)

**Lemma 5.4.** Under Assumptions 3 and Assumption 6 or 7, in the regime \( \frac{m}{n} \to \gamma^* \) with \( 0 \leq \gamma^* \leq 1 \), we have
\[ \mathbb{E} \left[ \frac{\sum_{i=1}^{n} |w_i|^3}{(\sum_{i=1}^{n} w_i^2)^{3/2}} \right] \to 0 \quad \text{as} \quad n \to \infty. \]

*Proof.* We know the inequality
\[ \sum_{i=1}^{n} a^p \leq \left( \sum_{i=1}^{n} a_i \right)^p, \]
where \( a_i \geq 0 \) and \( p > 1 \). Using this, we know that
\[ \frac{\sum_{i=1}^{n} |w_i|^3}{(\sum_{i=1}^{n} w_i^2)^{3/2}} \leq 1. \]
We also know from Lemma 5.2 and Lemma 5.3: \( m^{3/2} \sum_{i=1}^{n} |w_i|^3 \to 0 \) as \( n \to \infty \) and \( m \sum_{i=1}^{n} w_i^2 \xrightarrow{a.s.} 1 \) as \( n \to \infty \). Hence,
\[ \frac{\sum_{i=1}^{n} |w_i|^3}{(\sum_{i=1}^{n} w_i^2)^{3/2}} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty. \]
Using DCT, we are done. Note that this proof works for both Assumption 6 and Assumption 7. \( \square \)

*Proof of Theorem 2.* Let us consider \( p = 1 \), where \( p \) is the dimension.
Consider,
\[ \left| \mathbb{P} \left( \sqrt{m} \sum_{i=1}^{n} w_i (\nabla l(\theta, u_i) - \nabla g(\theta)) \leq x \right) - \Phi_{\sigma}(x) \right|, \]
where \( \Phi_{\sigma}(x) \) is the cdf of \( N(0, \sigma^2(\theta)) \).
Define \( (\nabla l(\theta, u_i) - \nabla g(\theta)) = X_i \). Thus \( X_i \) are iid with mean 0.
For this particular case, we take w.l.o.g., \( \sigma^2(\theta) = \sigma^2 = 1 \).
Hence, now the problem reduces to proving
\[ \left| \mathbb{P} \left( \sqrt{m} \sum_{i=1}^{n} w_i X_i \leq x \right) - \Phi(x) \right|. \]
goes to zero where $\Phi$ is the cdf of standard normal.

Now,
\[
\left| E \left( \sqrt{m} \sum_{i=1}^{n} w_i X_i \mid w \right) - E \left( \Phi(x) \right) \right| \leq E \left| P \left( \sqrt{m} \sum_{i=1}^{n} w_i X_i \leq x \mid w \right) - \Phi(x) \right|
\]
\[
\leq E \left[ \sum_{i=1}^{n} E \left[ |w_i X_i|^3 \mid w \right] \right] + E \left| \Phi \left( \frac{x}{m \sum_{i=1}^{n} w_i^2} \right) - \Phi(x) \right|
\]
\[
= \left| X_1 \right|^3 E \left[ \sum_{i=1}^{n} \frac{|w_i|^3}{\left( \sum_{i=1}^{n} w_i^2 \right)^{3/2}} \right] + E \left| \Phi \left( \frac{x}{m \sum_{i=1}^{n} w_i^2} \right) - \Phi(x) \right|.
\]

Hence the proof is completed.

Note that all the above steps work for both Assumption 6 and Assumption 7. Now, it has been proved in Lemma 5.4 that the first term goes to 0. Using $m \sum_{i=1}^{n} w_i^2 \overset{a.s.}{\longrightarrow} 1$ as $n \to \infty$ and the fact that $||\Phi||_{\infty} < 1$, we get that the second term converges to zero as well using DCT. We can extend to any dimension using Cramer-Wold device.

\[\square\]

**Proofs of Theorem 3, 4 and 5**

Now we take a close look at the M-SGD algorithm. We start by showing a result about gradient descent.

**Lemma 5.5.** Under Assumption [1] for the gradient descent algorithm $\tilde{x}_k$ defined by (2.1) and its continuous version $\tilde{X}_t$ defined by (2.6), we have
\[
\left| \tilde{x}_k - \tilde{X}_k \right| \leq C_1 k \gamma (1 + L \gamma)^k.
\]

**Proof.** Note that $\tilde{X}_0 = \tilde{x}_0$ by Assumption [1]. Let $t \in [0, \gamma]$. Note that
\[
\tilde{X}_t = \tilde{x}_0 - \int_0^t \nabla g(\tilde{X}_s) ds;
\]
\[
\tilde{x}_1 = \tilde{x}_0 - \gamma \nabla g(\tilde{x}_0).
\]

Hence, we have
\[
\left| \tilde{X}_t - \tilde{x}_1 \right| \leq \int_0^t \left| \nabla g(\tilde{X}_s) - \nabla g(\tilde{x}_0) \right| + (\gamma - t) \left| \nabla g(\tilde{x}_0) \right|
\]
\[
\leq L \int_0^t \left| \tilde{X}_s - \tilde{x}_0 \right| + \gamma \left| \nabla g(\tilde{x}_0) \right|
\]
\[
\leq L \int_0^t \left| \tilde{X}_s - \tilde{x}_1 \right| + \gamma |\tilde{x}_1 - \tilde{x}_0| + \gamma \left| \nabla g(\tilde{x}_0) \right|
\]
\[
\leq L \int_0^t \left| \tilde{X}_s - \tilde{x}_1 \right| + L \gamma \left| \nabla g(\tilde{x}_0) \right| + \gamma \left| \nabla g(\tilde{x}_0) \right|
\]
\[
\leq \left| \nabla g(\tilde{x}_0) \right| \gamma (1 + L \gamma) + L \int_0^t \left| \tilde{X}_s - \tilde{x}_1 \right|.
\]

Using Gronwall Lemma, we get
\[
\left| \tilde{X}_t - \tilde{x}_1 \right| \leq \left| \nabla g(\tilde{x}_0) \right| \gamma (1 + L \gamma) e^{L \gamma}.
\]
Note that as we are in the regime $0 \leq k \leq [T/\gamma]$, we can bound $e^{L\gamma}$ by $e^T$ where as stated before $T$ is fixed. Let the induction step in $t \in [(k-1)\gamma, k\gamma]$ hold as

$$\left| \tilde{X}_t - \tilde{x}_k \right| \leq |\nabla g(\tilde{x}_0)| k\gamma e^{kL\gamma} (1 + \gamma L)^k.$$

Let $t \in [k\gamma, (k+1)\gamma]$. In this case,

$$\left| \tilde{X}_t - \tilde{x}_{k+1} \right| \leq \left| \tilde{X}_t - \tilde{x}_k \right| + L \int_{k\gamma}^t \left| \tilde{X}_s - \tilde{x}_k \right| ds + ((k+1)\gamma - t) |\nabla g(\tilde{x}_k)|$$

$$\leq |\nabla g(\tilde{x}_0)| k\gamma e^{kL\gamma} (1 + \gamma L)^k + L \int_{k\gamma}^t \left| \tilde{X}_s - \tilde{x}_{k+1} \right| + (1 + L\gamma) \gamma |\nabla g(\tilde{x}_k)|.$$

Noting that $|\nabla g(\tilde{x}_k)| \leq (1 + \gamma L)^k |\nabla g(\tilde{x}_0)|$ and adding some terms we can see that

$$\left| \tilde{X}_t - \tilde{x}_{k+1} \right| \leq |\nabla g(\tilde{x}_0)| (k + 1) \gamma e^{kL\gamma} (1 + \gamma L)^{k+1} + L \int_{k\gamma}^t \left| \tilde{X}_s - \tilde{x}_{k+1} \right|.$$

Using Gronwall the induction step is completed. This completes the proof.

**Lemma 5.6.** For any $a, b > 0, x \geq 1$, we have

$$\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^x \leq \exp(a + b + 1)^2.$$

Proof. The proof is not hard to see as

$$\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^x = \exp \left[ x \log \left(1 + \frac{a}{x} + \frac{b}{x^2}\right) \right]$$

$$\leq \exp \left\{ x \left[ 0 + \left( \frac{a}{x} + \frac{b}{x^2} \right) + \left( \frac{a}{x} + \frac{b}{x^2} \right)^2 \right] \right\}.$$

The last line follows using Taylor expansion. Hence the proof follows.

**Lemma 5.7.** Under Assumptions $\mathbb{A}, \mathbb{B}$, we have for the algorithms described by (2.2) and (2.1)

$$\mathbb{E} |x_{k+1} - \tilde{x}_{k+1}|^2 \leq \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m},$$

where

$$\tilde{C}_1 = e^{\pi^2/6} \exp \left(1 + 2TL + T^2L^2 + 2TL_1 + 2T^2L_1 + T^2L_1^2 + T^2p^2L_1^2 \right)^2 \|\sigma(x_0)\|_F^2$$

and

$$\tilde{C}_2 = e^{\pi^2/6} (T^2 + 1) \left( C_1^2 T^2 e^{2LT} + 2 \sup_{0 \leq t \leq T} \left\| \sigma(\tilde{X}_t) \right\|_F^2 \right).$$

Proof of Lemma 5.7

$$x_{k+1} - \tilde{x}_{k+1} = (x_k - \tilde{x}_k) - \gamma (\nabla g(x_k) - \nabla g(\tilde{x}_k)) + \sqrt{\frac{\gamma}{m}} \sigma(x_k) \sqrt{\gamma} \xi_{k+1}.$$

This implies that

$$|x_{k+1} - \tilde{x}_{k+1}| \leq |x_k - \tilde{x}_k| + \gamma |\nabla g(x_k) - \nabla g(\tilde{x}_k)| + \sqrt{\frac{\gamma}{m}} |\sigma(x_k)| \sqrt{\gamma} \xi_{k+1}.$$
Using the fact that $\nabla g$ is Lipschitz, we have

$$ |x_{k+1} - \bar{x}_{k+1}| \leq |x_k - \bar{x}_k| + \gamma L |x_k - \bar{x}_k| + \frac{\gamma}{\sqrt{m}} |\sigma(x_k)\xi_{k+1}| $$

$$ |x_{k+1} - \bar{x}_{k+1}| \leq (1 + \gamma L) |x_k - \bar{x}_k| + \frac{\gamma}{\sqrt{m}} |(\sigma(x_k) - \sigma(\bar{x}_k))\xi_{k+1}| + \frac{\gamma}{\sqrt{m}} |\sigma(\bar{x}_k)\xi_{k+1}|. $$

Using the fact $|\sigma(x_k) - \sigma(\bar{x}_k)\xi_{k+1}| \leq |\sigma(x_k) - \sigma(\bar{x}_k)|_2 |\xi_{k+1}|$ and Assumption 3 above, we have

$$ |x_{k+1} - \bar{x}_{k+1}| \leq (1 + \gamma L) |x_k - \bar{x}_k| + \frac{\gamma}{\sqrt{m}} L |x_k - \bar{x}_k| |\xi_{k+1}| + \frac{\gamma}{\sqrt{m}} |\sigma(\bar{x}_k)\xi_{k+1}| $$

$$ \leq \left[(1 + \gamma L) + \frac{\gamma}{\sqrt{m}} L |\xi_{k+1}|\right] |x_k - \bar{x}_k| + \frac{\gamma}{\sqrt{m}} |\sigma(\bar{x}_k)\xi_{k+1}|. $$

Using this and squaring and then using a version of Jensen inequality, we get,

$$ |x_{k+1} - \bar{x}_{k+1}|^2 \leq \left(1 + \frac{1}{k^2}\right) \left[(1 + \gamma L) + \frac{\gamma}{\sqrt{m}} L |\xi_{k+1}|\right]^2 |x_k - \bar{x}_k|^2 + \frac{1 + k^2}{m} \frac{\gamma^2}{|\sigma(\bar{x}_k)\xi_{k+1}|^2}. $$

Taking expectation, we get,

$$ \mathbb{E}|x_{k+1} - \bar{x}_{k+1}|^2 \leq \left(1 + \frac{1}{k^2}\right) \mathbb{E}\left[(1 + \gamma L) + \frac{\gamma}{\sqrt{m}} L |\xi_{k+1}|\right]^2 \mathbb{E}|x_k - \bar{x}_k|^2 + \frac{1 + k^2}{m} \frac{\gamma^2}{\mathbb{E}|\sigma(\bar{x}_k)\xi_{k+1}|^2} $$

$$ \leq \prod_{j=1}^k \left(1 + \frac{1}{j^2}\right) \left[(1 + \gamma L)^2 + \frac{2\gamma}{\sqrt{m}} L_1 p (1 + \gamma L) + \frac{\gamma^2}{m} p^2 L_1^2\right]^k \mathbb{E}|x_1 - \bar{x}_1|^2 $$

$$ + \sum_{j=1}^{k-1} \prod_{i=1}^j \left(1 + \frac{1}{(k - j + 1)^2}\right) \frac{\gamma^2}{m} \frac{(j^2 + 1)}{|\sigma(\bar{x}_{k-j})|_F^2} + \frac{\gamma^2}{m} \frac{(k^2 + 1)}{|\sigma(\bar{x}_k)|_F^2}. $$

Now we know from AM-GM inequality, that, for any $k \in \mathbb{N},$

$$ \prod_{j=1}^k \left(1 + \frac{1}{j^2}\right) \leq \left(1 + \frac{1}{k} \sum_{j=1}^k \frac{1}{j^2}\right)^k. $$

Using the fact $\sum_{j=1}^k \frac{1}{j^2} \leq \sum_{j=1}^\infty \frac{1}{j^2} = \frac{\pi^2}{6},$ we have

$$ \prod_{j=1}^k \left(1 + \frac{1}{j^2}\right) \leq \left(1 + \frac{\pi^2}{6k}\right)^k $$

$$ \leq e^{\pi^2/6}. $$

Also, note that

$$ \mathbb{E}|x_1 - \bar{x}_1|^2 = \frac{\gamma^2}{m} ||\sigma(x_0)||_F^2. $$

Therefore, using Lemma 5.6 the first term is less than

$$ e^{\pi^2/6} \exp \left(1 + 2TL + T^2 L^2 + 2TL_1 p + 2T^2 LL_1 p + T^2 p^2 L_1^2\right)^2 \frac{\gamma^2}{m} \mathbb{E}||\sigma(x_0)||_F^2 = C_1 \frac{\gamma^2}{m}. $$

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For the second term, we have
\[
\sum_{j=1}^{k-1} \prod_{i=1}^{j} \left( 1 + \frac{1}{(k-j+1)^2} \right)^2 \frac{\gamma^2 (j^2 + 1)}{m} \left\| \sigma(\tilde{x}_{k-j}) \right\|_F^2 + \frac{\gamma^2 (k^2 + 1)}{m} \left\| \sigma(\tilde{x}_k) \right\|_F^2
\]
\[
\leq \frac{T^2 + 1}{m} \sup_{0 \leq k \leq T/\gamma} \left\| \sigma(\tilde{x}_k) \right\|_F^2 \cdot k \prod_{j=1}^{k} \left( 1 + \frac{1}{j^2} \right)
\]
\[
\leq e^{\pi^2/6} \frac{(T^2 + 1) k}{m} \sup_{0 \leq k \leq T/\gamma} \left\| \sigma(\tilde{x}_k) \right\|_F^2.
\]

Also, we have
\[
(\text{Tr} \sigma^2(\tilde{x}_k))^{1/2} = \left\| \sigma(\tilde{x}_k) \right\|_F
\]
\[
\leq \left\| \sigma(\tilde{x}_k) - \sigma(\tilde{x}_{k\gamma}) \right\|_F + \left\| \sigma(\tilde{x}_{k\gamma}) \right\|_F
\]
\[
\leq L_1 \sqrt{p} |\tilde{x}_k - \tilde{x}_{k\gamma}| + \sup_{0 \leq t \leq T} \left\| \sigma(\tilde{x}_t) \right\|_F
\]
\[
\leq L_1 \sqrt{p} C_1 k \gamma (1 + L \gamma)^k + \sup_{0 \leq t \leq T} \left\| \sigma(\tilde{x}_t) \right\|_F.
\]

The second inequality follows from the relation between spectral norm and Frobenius norm and the last one follows from the fact that the discretized version of gradient descent is only an order of the stepsize away from its continuous counterpart. Hence,
\[
\sup_{0 \leq k \leq T/\gamma} \left\| \sigma(\tilde{x}_k) \right\|_F^2 \leq 2 L_1^2 \gamma^2 C_1^2 T^2 e^{2LT} + 2 \sup_{0 \leq t \leq T} \left\| \sigma(\tilde{x}_t) \right\|_F^2.
\]

Hence, the second terms is less than
\[
e^{\pi^2/6} \frac{k(T^2 + 1)}{m} \left( 2 C_1^2 T^2 e^{2LT} + 2 \sup_{0 \leq t \leq T} \left\| \sigma(\tilde{x}_t) \right\|_F^2 \right) = \tilde{C}_2 \frac{k}{m}.
\]

Hence, we have
\[
E \left| x_{k+1} - \tilde{x}_{k+1} \right|^2 \leq \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m}.
\]

\[\square\]

**Note:**

Using Lemma 5.6, we have the rate in the previous lemma as \( \frac{k}{m} \). This is because all the other terms are bounded in the regime \( \gamma K = T \).

Next we show \((2.3)\) and \((2.1)\) are close in the Wasserstein distance. To show the above statement we need one more lemma.

**Lemma 5.8.** Under Assumptions 4, 5, we have
\[
\mathbb{E} \left\| \sum_{i=1}^{n} w_{i,k} (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\tilde{x}_k, u_{i,k})) \right\|^2 \leq 2 \left( \frac{n - m}{mn} + 1 \right) \mathbb{E} \left( h^2(u) \right) \mathbb{E} \left| x_{k,n} - \tilde{x}_k \right|^2.
\]
Proof. We start with bounding the $L_2$ norm.

$$
\mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})) \right|^2 \leq \mathbb{E} \left( \sum_{i=1}^{n} \left( w_{i,k} - \frac{1}{n} \right) (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})) \right)^2 
+ \sum_{i=1}^{n} \frac{1}{n} (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k}))^2.
$$

Hence

$$
\mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})) \right|^2 \leq 2 \mathbb{E} \left( \sum_{i=1}^{n} \left( w_{i,k} - \frac{1}{n} \right) (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})) \right)^2
+ 2 \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} h(u_i) |x_{n,k} - \bar{x}_k| \right)^2.
$$

Hence

$$
\mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})) \right|^2 = 2 \mathbb{E} \left( \sum_{i=1}^{n} \left( w_{i,k} - \frac{1}{n} \right)^2 |\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})| \right)^2
+ \sum_{i \neq j} \left( w_{i,k} - \frac{1}{n} \right) \left( w_{j,k} - \frac{1}{n} \right) (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k}))'(\nabla l(x_{n,k}, u_{j,k}) - \nabla l(\bar{x}_k, u_{j,k}))
+ 2 \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} h(u_i) \right)^2 \mathbb{E} |x_{n,k} - \bar{x}_k|^2.
$$

Hence

$$
\mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})) \right|^2 \leq 2 \left( \sum_{i=1}^{n} \frac{n-m}{mn^2} \mathbb{E} |\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})| \right)^2
- \sum_{i \neq j} \frac{n-m}{mn^2(n-1)} \mathbb{E} |\nabla g(x_{n,k}) - \nabla g(\bar{x}_k)|^2
+ \left( \frac{1}{n} \text{Var}(h(u)) + \mathbb{E}(h(u))^2 \right) \mathbb{E} |x_{k,n} - \bar{x}_k|^2.
$$

From this, we get

$$
\mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})) \right|^2 \leq 2 \left( \sum_{i=1}^{n} \frac{n-m}{mn^2} \mathbb{E} |\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})| \right)^2
+ 2 \left( \frac{1}{n} \text{Var}(h(u)) + \mathbb{E}(h(u))^2 \right) \mathbb{E} |x_{k,n} - \bar{x}_k|^2.
$$

This implies that

$$
\mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} (\nabla l(x_{n,k}, u_{i,k}) - \nabla l(\bar{x}_k, u_{i,k})) \right|^2 \leq 2 \left( \sum_{i=1}^{n} \frac{n-m}{mn^2} \mathbb{E} (h^2(u)) \mathbb{E} |x_{n,k} - \bar{x}_k|^2 \right)
+ 2 \left( \frac{1}{n} \text{Var}(h(u)) + \mathbb{E}(h(u))^2 \right) \mathbb{E} |x_{k,n} - \bar{x}_k|^2
\leq 2 \left( \frac{n-m}{mn} + 1 \right) \mathbb{E} (h^2(u)) \mathbb{E} |x_{k,n} - \bar{x}_k|^2.
$$

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Lemma 5.9. With $x_{n,k}$ and $\tilde{x}_k$ as defined in (2.3) and (2.1), under Assumptions 7, we have

$$\mathbb{E} |x_{n,k+1} - \tilde{x}_{k+1}|^2 \leq K_1^* \frac{4k^2}{m},$$

where

$$K_1^* = T \cdot \exp \left( 1 + 2T^2L^2 + 6T^2 \mathbb{E} (h^2(u)) \right)^2 \cdot \left( 2p^2L^2T^2 e^{2TL} + 2 \sup_{0 \leq t \leq T} ||\sigma(\hat{X}_t)||_F^2 \right).$$

Proof. Using the definition of (2.3) and (2.1) and defining

$$R_w (x_{n,k}, U_k) = \sqrt{m} \sum_{i=1}^n w_{i,k} (\nabla l(x_{n,k}, u_{i,k}) - \nabla g(x_{n,k})),$$

we have

$$x_{n,k+1} - \tilde{x}_{k+1} = x_{n,k} - \tilde{x}_k - \gamma (\nabla g(x_{n,k}) - g(\tilde{x}_k)) - \gamma \sum_{i=1}^n w_{i,k} (\nabla l(x_{n,k}, u_{i,k}) - \nabla g(x_{n,k}))$$

$$= x_{n,k} - \tilde{x}_k - \gamma (\nabla g(x_{n,k}) - g(\tilde{x}_k)) - \frac{\gamma}{\sqrt{m}} R_w (x_{n,k}, U_k)$$

$$= x_{n,k} - \tilde{x}_k - \gamma (\nabla g(x_{n,k}) - g(\tilde{x}_k)) - \frac{\gamma}{\sqrt{m}} (R_w (x_{n,k}, U_k) - R_w (\tilde{x}_k, U_k)) - \frac{\gamma}{\sqrt{m}} R_w (\tilde{x}_k, U_k).$$

This implies that

$$|x_{n,k+1} - \tilde{x}_{k+1}| \leq |x_{n,k} - \tilde{x}_k| + 2\gamma |\nabla g(x_{n,k}) - \nabla g(\tilde{x}_k)|$$

$$+ \gamma \left| \sum_{i=1}^n w_{i,k} (\nabla l(x_{n,k}, u_i) - \nabla l(\tilde{x}_k, u_i)) \right| + \gamma \left| \sum_{i=1}^n w_{i,k} (\nabla l(\tilde{x}_k, u_i) - \nabla g(\tilde{x}_k)) \right|.$$

From this, squaring both sides of the inequality and using Jensen’s inequality, we get

$$|x_{n,k+1} - \tilde{x}_{k+1}|^2 \leq 4 |x_{n,k} - \tilde{x}_k|^2 + 8\gamma^2 L^2 |x_{n,k} - \tilde{x}_k|^2$$

$$+ 4\gamma^2 \left| \sum_{i=1}^n w_{i,k} (\nabla l(x_{n,k}, u_i) - \nabla l(\tilde{x}_k, u_i)) \right|^2 + 4\gamma^2 \left| \sum_{i=1}^n w_{i,k} (\nabla l(\tilde{x}_k, u_i) - \nabla g(\tilde{x}_k)) \right|^2.$$

Thus, taking expectation, we get

$$\mathbb{E} |x_{n,k+1} - \tilde{x}_{k+1}|^2 \leq 4 \mathbb{E} |x_{n,k} - \tilde{x}_k|^2 + 8\gamma^2 L^2 \mathbb{E} |x_{n,k} - \tilde{x}_k|^2$$

$$+ 4\gamma^2 \mathbb{E} \left| \sum_{i=1}^n w_{i,k} (\nabla l(x_{n,k}, u_i) - \nabla l(\tilde{x}_k, u_i)) \right|^2 + 4\gamma^2 \mathbb{E} \left| \sum_{i=1}^n w_{i,k} (\nabla l(\tilde{x}_k, u_i) - \nabla g(\tilde{x}_k)) \right|^2.$$

This implies that

$$\mathbb{E} |x_{n,k+1} - \tilde{x}_{k+1}|^2 \leq 4 \mathbb{E} |x_{n,k} - \tilde{x}_k|^2 + 8\gamma^2 L^2 \mathbb{E} |x_{n,k} - \tilde{x}_k|^2$$

$$+ 4\gamma^2 \left( \frac{n - m}{mn} + 1 \right) \mathbb{E} (h^2(u)) \mathbb{E} |x_{n,k} - \tilde{x}_k|^2 + \frac{4\gamma^2}{m} ||\sigma(\tilde{x}_k)||_F^2.$$

$$= 4 \left[ 1 + 2\gamma^2 L^2 + 2\gamma^2 \left( \frac{n - m}{mn} + 1 \right) \mathbb{E} (h^2(u)) \right] \mathbb{E} |x_{n,k} - \tilde{x}_k|^2 + \frac{4\gamma^2}{m} ||\sigma(\tilde{x}_k)||_F^2.$$

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Hence, we can say that,

\[ \mathbb{E} \left| x_{n,k+1} - \tilde{x}_{k+1} \right|^2 \leq \sum_{j=0}^{k} 4^{j+1} \left[ 1 + 2\gamma^2 L^2 + 2\gamma^2 \left( \frac{n-m}{mn} + 1 \right) \mathbb{E} \left( h^2(u) \right) \right]^j \frac{\gamma^2}{m} \left\| \sigma(\tilde{x}_{k-j}) \right\|^2_F \]

\[ \leq \frac{\gamma^2}{m} (k+1) 4^{k+1} \left[ 1 + 2\gamma^2 L^2 + 2\gamma^2 \left( \frac{n-m}{mn} + 1 \right) \mathbb{E} \left( h^2(u) \right) \right]^k \sup_{0 \leq k \leq \left\lfloor \frac{T}{4} \right\rfloor} \left\| \sigma(\tilde{x}_k) \right\|^2_F \]

\[ \leq \frac{\gamma^2}{m} (k+1) 4^{k+1} \left[ 1 + 2\gamma^2 L^2 + 2\gamma^2 \left( \frac{n-m}{mn} + 1 \right) \mathbb{E} \left( h^2(u) \right) \right]^k \left[ 2p^2 L_1^2 C_2^2 k^2 \gamma^2 (1 + L\gamma)^{2k} + 2 \sup_{0 \leq t \leq T} \left\| \sigma(\tilde{X}_t) \right\|^2_F \right]. \]

From Lemma \[5.6\] we can see that

\[ \left[ 1 + 2\gamma^2 L^2 + 2\gamma^2 \left( \frac{n-m}{mn} + 1 \right) \mathbb{E} \left( h^2(u) \right) \right]^k \leq \exp \left( 1 + 2T^2 L^2 + 6T^2 \mathbb{E} \left( h^2(u) \right) \right)^2. \]

We use the fact \( (\frac{n-m}{mn} + 1) \leq 3 \) in the above bound. Also, using the fact \( K\gamma = T \), we have

\[ \left[ 2p^2 L_1^2 C_2^2 k^2 \gamma^2 (1 + L\gamma)^{2k} + 2 \sup_{0 \leq t \leq T} \left\| \sigma(\tilde{X}_t) \right\|^2_F \right] \leq 2p^2 L_1^2 C_1^2 T^2 e^{2TL} + 2 \sup_{0 \leq t \leq T} \left\| \sigma(\tilde{X}_t) \right\|^2_F. \]

Hence, we can say that,

\[ \mathbb{E} \left| x_{n,k+1} - \tilde{x}_{k+1} \right|^2 \leq K_1^* \frac{4^k \gamma}{m}, \]

where

\[ K_1^* = T \cdot \exp \left( 1 + 2T^2 L^2 + 6T^2 \mathbb{E} h^2(u) \right)^2 \left( 2p^2 L_1^2 C_1^2 T^2 e^{2TL} + 2 \sup_{0 \leq t \leq T} \left\| \sigma(\tilde{X}_t) \right\|^2_F \right). \]

Thus we conclude the proof. \(\square\)

**Note:**
As we can see that the rate here is \( 4^k \gamma/m \) with \( K_1^* \) as a constant dependent on \( T, L, L_1, p \)

Using the above results, we get,

**Proof of Theorem** Combining Lemma \[5.7\] and Lemma \[5.9\] and using Cauchy Schwarz, we have

\[ \mathbb{E} \left| x_{n,k+1} - x_{k+1} \right|^2 \leq 2 \mathbb{E} \left| x_{n,k+1} - \tilde{x}_{k+1} \right|^2 + 2 \mathbb{E} \left| x_{k+1} - \tilde{x}_{k+1} \right|^2 \]

\[ \leq 2\tilde{C}_1 \frac{\gamma^2}{m} + 2\tilde{C}_2 \frac{k}{m} + 2K_1^* \frac{4^k \gamma}{m}. \]

Which gives us

\[ \mathbb{E} \left| x_{n,k+1} - x_{k+1} \right|^2 \leq K_1 \frac{4^k \gamma}{m}. \]

Where \( K_1 = 2\tilde{C}_1 \frac{\gamma}{T} + 2\tilde{C}_2 \frac{k}{4^k \gamma} + 2K_1^* \). Note that \( \gamma/4^k < T \) and \( \frac{k}{4^k \gamma} = \frac{k^2}{4^k T} < \frac{1}{7} \) since \( k^2 < 4^k \) for all \( k \geq 1 \). \(\square\)
Now we present few lemmas which shall be very important for our next steps.

**Lemma 5.10.** Under Assumptions \footnote{[15]} we have the following inequalities

\[
\mathbb{E} |\nabla g(D_{n,k\gamma})|^2 \leq \tilde{C}_3 + \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m},
\]

where \( \tilde{C}_3 = 4L^2C_1^2T^2e^{2LT} + 4 \sup_{0 \leq t \leq T} |g(\tilde{X}_t)|^2 \) and \( \tilde{C}_1, \tilde{C}_2 \) are defined in Lemma 5.7.

Also,

\[
\mathbb{E} (\text{Tr} \sigma^2(D_{n,k\gamma})) \leq \tilde{C}_4 + \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m},
\]

where \( \tilde{C}_4 = 4L^2C_1^2T^2e^{2LT} + 4 \sup_{0 \leq t \leq T} \left| \sigma(\tilde{X}_t) \right|^2 \).

*Proof.*

\[
\mathbb{E} |\nabla g(D_{n,k\gamma})|^2 = \mathbb{E} |\nabla g(x_k)|^2 
\leq 2L^2 \mathbb{E} |x_k - \tilde{x}_k|^2 + 4L^2 \left| \tilde{x}_k - \tilde{X}_k \right|^2 + 4 \left| \nabla g(\tilde{X}_k) \right|^2.
\]

Using Lemma 5.7 and Lemma 5.5 we find that

\[
\mathbb{E} |\nabla g(D_{n,k\gamma})|^2 \leq 4L^2C_1^2k^2\gamma^2 (1 + L\gamma)^{2k} + 4 \sup_{0 \leq t \leq T} |g(\tilde{X}_t)|^2 
+ \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m}
\leq 4L^2C_1^2T^2e^{2LT} + 4 \sup_{0 \leq t \leq T} |g(\tilde{X}_t)|^2 
+ \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m}
= \tilde{C}_3 + \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m},
\]

where \( \tilde{C}_3 = 4L^2C_1^2T^2e^{2LT} + 4 \sup_{0 \leq t \leq T} |g(\tilde{X}_t)|^2 \) and \( \tilde{C}_1, \tilde{C}_2 \) are defined in Lemma 5.7.

We can do the same with \( \sigma \)

\[
\mathbb{E} (\text{Tr} \sigma^2(D_{n,k\gamma})) = \mathbb{E} \left| \sigma(x_k) \right|^2_F 
\leq 2L^2 \mathbb{E} |x_k - \tilde{x}_k|^2 + 2 \left| \sigma(\tilde{x}_k) \right|^2_F 
\leq 2L^2 \mathbb{E} |x_k - \tilde{x}_k|^2 + 4pL^2C_1^2k^2\gamma^2 (1 + L\gamma)^{2k} + 4 \sup_{0 \leq t \leq T} \left| \sigma(\tilde{X}_t) \right|^2_F.
\]

Using this fact, and the bounds from Lemma 5.7 and Lemma 5.5, we have

\[
\mathbb{E} (\text{Tr} \sigma^2(D_{n,k\gamma})) \leq 4pL^2C_1^2k^2\gamma^2 (1 + L\gamma)^{2k} + 4 \sup_{0 \leq t \leq T} \left| \sigma(\tilde{X}_t) \right|^2_F 
+ \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m}
\leq 4pL^2C_1^2T^2e^{2LT} + 4 \sup_{0 \leq t \leq T} \left| \sigma(\tilde{X}_t) \right|^2_F 
+ \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m}
\leq \tilde{C}_4 + \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m},
\]

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where $\hat{C}_4 = 4pL_2^1C_1^2T^2e^{2LT} + 4\sup_{0 \leq s \leq T}\|\sigma(\tilde{X}_t)\|_F^2$ and $\hat{C}_1, \hat{C}_2$ are as defined in Lemma 5.7.

**Lemma 5.11.** Under Assumptions 1-5, for $t \in (k\gamma, (k+1)\gamma]$, we have

$$
\int_{k\gamma}^{t} \mathbb{E} |D_{n,s} - D_{n,k\gamma}|^2 \leq K_{11}\gamma^3 + K_{12}\frac{\gamma^2}{m}.
$$

where $K_{11}, K_{12}$ are functions dependent on $L, L_1, T, p$.

**Proof of Lemma 5.11.** Using the definition of $D_{n,s}$, we have

$$
\mathbb{E} |D_{n,s} - D_{n,k\gamma}|^2 = \mathbb{E} \left[ -\nabla g(D_{n,k\gamma})(s - k\gamma) + \sqrt{\frac{\gamma}{m}}\sigma(D_{n,k\gamma})(B_s - B_{k\gamma}) \right]^2
$$

$$
\leq 2\mathbb{E} |\nabla g(D_{n,k\gamma})(s - k\gamma)|^2 + 2\frac{\gamma}{m} \mathbb{E} \left| \int_{s}^{k\gamma} \sigma(D_{n,k\gamma}) dB_t \right|^2
$$

$$
= 2(s - k\gamma)^2 \mathbb{E} |\nabla g(D_{n,k\gamma})|^2 + 2\frac{\gamma}{m} \mathbb{E} (\text{Tr}\sigma^2(D_{n,k\gamma}))(s - k\gamma).
$$

Now, hence, for $t \in [k\gamma, (k+1)\gamma]$, we have

$$
\int_{k\gamma}^{t} \mathbb{E} |D_{n,s} - D_{n,k\gamma}| \leq 2\int_{k\gamma}^{t} (s - k\gamma)^2 \mathbb{E} |\nabla g(D_{n,k\gamma})|^2 + 2\frac{\gamma}{m} \int_{k\gamma}^{t} \mathbb{E} |\sigma(D_{n,k\gamma})|^2_{F}(s - k\gamma).
$$

Now, using Lemma 5.10, the first term is bounded by

$$
2\int_{k\gamma}^{t} (s - k\gamma)^2 \mathbb{E} |\nabla g(D_{n,k\gamma})|^2 \leq 2\int_{k\gamma}^{t} (s - k\gamma)^2 \left( \hat{C}_3 + \hat{C}_1\frac{\gamma^2}{m} + \hat{C}_2\frac{k}{m} \right)
$$

$$
\leq \frac{2\gamma^3}{3} \hat{C}_3 + \frac{2}{3} \hat{C}_1\frac{\gamma^5}{m} + \frac{2}{3} \hat{C}_2\frac{\gamma^2 T}{m}.
$$

Now, for the second term we can do the exact same thing.

$$
2\frac{\gamma}{m} \int_{k\gamma}^{t} \mathbb{E} |\sigma(D_{n,k\gamma})|^2_{F}(s - k\gamma) \leq 2\int_{k\gamma}^{t} \frac{\gamma}{m} (s - k\gamma) \left( \hat{C}_4 + \hat{C}_1\frac{\gamma^2}{m} + \hat{C}_2\frac{k}{m} \right)
$$

$$
\leq \frac{\gamma^3}{m} \hat{C}_4 + \hat{C}_1\frac{\gamma^5}{m^2} + \hat{C}_2\frac{\gamma^2 T}{m^2}.
$$

Which gives us,

$$
2\frac{\gamma}{m} \int_{k\gamma}^{t} \mathbb{E} |\sigma(D_{n,k\gamma})|^2_{F}(s - k\gamma) \leq \gamma^3 \left( \frac{2}{3} \hat{C}_3 + \frac{1}{m}\hat{C}_4 \right) + \frac{\gamma^5}{m} \left( \frac{2}{3} + \frac{1}{m} \right) \hat{C}_1 + \frac{\gamma^2 T}{m} \left( \frac{2}{3} + \frac{1}{m} \right) \hat{C}_2.
$$

Combining the above terms we get

$$
\int_{k\gamma}^{t} \mathbb{E} |D_{n,s} - D_{n,k\gamma}|^2 \leq \hat{K}_{11}\gamma^3 + \hat{K}_{12}\frac{\gamma^5}{m} + \hat{K}_{13}\frac{\gamma^2}{m},
$$

where

$$
\hat{K}_{11} = \left( \frac{2}{3} \hat{C}_3 + \frac{1}{m}\hat{C}_4 \right),
$$

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Proof of Theorem 4.

We focus on the interval $(k \gamma, (k+1) \gamma]$, where $K_{12}$ can be taken as

$$K_{12} = \frac{5}{3} \tilde{C}_1,$$

and

$$K_{13} = \frac{5T}{3} \tilde{C}_1.$$

Hence, we can write

$$\int_k^t E |D_{n,s} - D_{n,k\gamma}|^2 \leq K_{11} \gamma^2 + K_{12} \frac{\gamma^2}{m}.$$

where $K_{11} = \tilde{K}_{11}$, $K_{12} = \tilde{K}_{12} + \tilde{K}_{13}$. This completes the proof. \[\square\]

Next we prove Theorem 4.

Proof of Theorem 4. We focus on the interval $(k \gamma, (k+1) \gamma]$, i.e. $t \in (k \gamma, (k+1) \gamma]$. With some abuse of notation we define $j \gamma$ for $j = 0, 1, 2, \ldots, k-1$; with $jk = t$. We use this abuse of notation as this helps with otherwise cumbersome notation. Using the definition of (2.4) and (2.5), we have

$$|X_t - D_t| \leq \left| \int_0^t \nabla g(X_s) - \int_0^t \sum_{j=0}^{k-1} \nabla g(D_{j\gamma}) I_{[j\gamma,(j+1)\gamma]} \right| ds + \sqrt{\frac{\gamma}{m}} \left| \int_0^t \sum_{j=0}^{k-1} (\sigma(X_s) - \sigma(D_{j\gamma})) dB_s \right|.$$

This gives us

$$|X_t - D_t| \leq \frac{k-1}{j\gamma} \left| \int_0^t \nabla g(X_s) - \nabla g(D_{n,j\gamma}) \right| ds + \sqrt{\frac{\gamma}{m}} \left| \int_0^t \sum_{j=0}^{k-1} (\sigma(X_s) - \sigma(D_{n,j\gamma})) dB_s \right|.$$

Hence, using triangle inequality, we have

$$|X_t - D_t| \leq L \frac{k-1}{j\gamma} \left| X_s - D_{n,s} \right| + L \frac{k-1}{j\gamma} \left| D_{n,s} - D_{n,j\gamma} \right|$$

$$+ \frac{\gamma}{m} \sum_{j=0}^{k-1} \left| \int_0^t (\sigma(X_s) - \sigma(D_{n,s})) dB_s \right| + \frac{\gamma}{m} \sum_{j=0}^{k-1} \left| \int_0^t (\sigma(D_{n,s}) - \sigma(D_{n,j\gamma})) dB_s \right|.$$

$$\leq L \int_0^t \left| X_s - D_{n,s} \right| + L \sum_{j=0}^{k-1} \left| D_{n,s} - D_{n,j\gamma} \right| + \frac{\gamma}{m} \int_0^t \left| \sigma(X_s) - \sigma(D_{n,s}) \right| dB_s$$

$$+ \sqrt{\frac{\gamma}{m}} \sum_{j=0}^{k-1} \left| \int_0^t (\sigma(X_s) - \sigma(D_{n,j\gamma})) dB_s \right|. $$
Now squaring both sides and applying Cauchy-Schwarz inequality first on all the terms and then the first two integrals (on the second term we apply CS twice once for the sum and then for the integral), we get

\[ |X_t - D_t|^2 \leq 4L^2 t \int_0^t |X_s - D_{n,s}|^2 + 4L^2 k\gamma \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} |D_{n,s} - D_{n,j\gamma}|^2 \]

\[ + 4 \frac{\gamma}{m} \int_0^t (|\sigma(X_s) - \sigma(D_{n,s})| dB_s)^2 \]

\[ + 4 \frac{\gamma}{m} \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} (|\sigma(D_{n,s}) - \sigma(D_{n,j\gamma})| dB_s)^2 \].

Taking expectation, we get,

\[ \mathbb{E} |X_t - D_t|^2 \leq 4L^2 t \int_0^t \mathbb{E} |X_s - D_{n,s}|^2 + 4L^2 k\gamma \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} \mathbb{E} |D_{n,s} - D_{n,j\gamma}|^2 \]

\[ + 4 \frac{\gamma}{m} \int_0^t \mathbb{E} \left| |\sigma(X_s) - \sigma(D_{n,s})| \right|_F^2 ds + 4 \frac{\gamma}{m} \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} \mathbb{E} \left| |\sigma(D_{n,s}) - \sigma(D_{n,j\gamma})| \right|_F^2 ds. \]

Using Ito Isometry on the last two expressions, we get,

\[ \mathbb{E} |X_t - D_t|^2 \leq 4L^2 t \int_0^t \mathbb{E} |X_s - D_{n,s}|^2 + 4L^2 k\gamma \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} \mathbb{E} |D_{n,s} - D_{n,j\gamma}|^2 \]

\[ + 4 \frac{\gamma}{m} \int_0^t \mathbb{E} |\sigma(X_s) - \sigma(D_{n,s})|_F^2 ds + 4 \frac{\gamma}{m} \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} \mathbb{E} |\sigma(D_{n,s}) - \sigma(D_{n,j\gamma})|_F^2 ds. \]

Using the fact \( \sigma \) is Lipschitz, we get,

\[ \mathbb{E} |X_t - D_t|^2 \leq 4L^2 t \int_0^t \mathbb{E} |X_s - D_{n,s}|^2 + 4L^2 k\gamma \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} \mathbb{E} |D_{n,s} - D_{n,j\gamma}|^2 \]

\[ + 4 \frac{\gamma}{m} \frac{p L_1^2}{k} \int_0^t \mathbb{E} |X_s - D_{n,s}|^2 ds + 4 \frac{k\gamma}{m} \frac{p L_1^2}{k} \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} \mathbb{E} |D_{n,s} - D_{n,k\gamma}|^2 ds \]

\[ = \left(4L^2 t + 4 \frac{\gamma}{m} \frac{p L_1^2}{k} \right) \int_0^t \mathbb{E} |X_s - D_{n,s}|^2 + \left(4L^2 k\gamma + 4 \frac{k\gamma}{m} \frac{p L_1^2}{k} \right) \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} \mathbb{E} |D_{n,s} - D_{n,j\gamma}|^2. \]

Now from the last step, we can apply Gronwall inequality and get,

\[ \mathbb{E} |X_t - D_t|^2 \leq \left(4L^2 k\gamma + 4 \frac{k\gamma}{m} \frac{p L_1^2}{k} \right) \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} \mathbb{E} |D_{n,s} - D_{n,j\gamma}|^2 \cdot \exp \left(4L^2 t + 4 \frac{\gamma}{m} \frac{p L_1^2}{k} t \right). \]

Using \( \gamma K = T \), we get,

\[ \mathbb{E} |X_t - D_t|^2 \leq \left(4L^2 T + 4 \frac{T}{m} \frac{p L_1^2}{k} \right) \sum_{j=0}^{k-1} \int_{j\gamma}^{(j+1)\gamma} \mathbb{E} |D_{n,s} - D_{n,j\gamma}|^2 \cdot \exp \left(4L^2 T^2 + 4 \frac{\gamma}{m} \frac{p L_1^2}{k} T \right). \]
Using Lemma 5.11, we have

\[
\mathbb{E} |X_t - D_t|^2 \leq \left[ \left( 4L^2T + 4\frac{T}{m}pL_1^2 \right) \sum_{j=0}^{k-1} \left( K_{11}\gamma^3 + K_{12}\frac{\gamma^2}{m} \right) \right] \cdot \exp \left( 4L^2T^2 + 4\frac{\gamma}{m}pL_1^2T \right)
\]

\[
\leq \left( 4L^2T + 4\frac{T}{m}pL_1^2 \right) \left( K_{11}T\gamma^2 + K_{12}\frac{T\gamma}{m} \right) \cdot \exp \left( 4L^2T^2 + 4\frac{\gamma}{m}pL_1^2T \right)
\]

\[
= C_{11}\gamma^2 + C_{12}\frac{T\gamma}{m},
\]

where

\[
C_{11} = \left( 4L^2T + 4\frac{T}{m}pL_1^2 \right) \exp \left( 4L^2T^2 + 4\frac{\gamma}{m}pL_1^2T \right) T K_{11},
\]

and

\[
C_{12} = \left( 4L^2T + 4\frac{T}{m}pL_1^2 \right) \exp \left( 4L^2T^2 + 4\frac{\gamma}{m}pL_1^2T \right) T K_{12}.
\]

Hence

\[
W_2^2(X_t, D_t) \leq C_{11}\gamma^2 + C_{12}\frac{T\gamma}{m}.
\]

Thus we are done. \(\square\)

We prove another lemma before we proceed to one of the main theorems.

**Lemma 5.12.** Under the Assumptions 1-5,

\[
\mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} \nabla l(Y_{n,k\gamma}, u_{i,k}) - \sum_{i=1}^{n} w_{i,k} \nabla l(D_{n,k\gamma}, u_{i,k}) \right|^2 \leq \frac{1}{m} \left( \mathbb{E} \left( h^2(U) \right) - L^2 \right) \mathbb{E} |Y_{n,k\gamma} - D_{n,k\gamma}|^2.
\]

**Proof of Lemma 5.12** Again we start by bounding the \(L_2\) norm

\[
\mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} \nabla l(Y_{n,k\gamma}, u_{i,k}) - \sum_{i=1}^{n} w_{i,k} \nabla l(D_{n,k\gamma}, u_{i,k}) \right|^2
\]

\[
= \mathbb{E} \sum_{i=1}^{n} w_{i,k}^2 \left| \nabla l(Y_{n,k\gamma}, u_{i,k}) - l(D_{n,k\gamma}, u_{i,k}) \right|^2
\]

\[
+ \mathbb{E} \sum_{i,j,i \neq j} w_{i,k} w_{j,k} \left( \nabla l(Y_{n,k\gamma}, u_{i,k}) - \nabla l(D_{n,k\gamma}, u_{i,k}) \right)' \left( \nabla l(Y_{n,k\gamma}, u_{j,k}) - \nabla l(D_{n,k\gamma}, u_{j,k}) \right).
\]
Hence
\[
\mathbb{E} \left[ \sum_{i=1}^{n} w_{i,k} \nabla l(Y_{n,k \gamma}, u_{i,k}) - \sum_{i=1}^{n} w_{i,k} \nabla l(D_{n,k \gamma}, u_{i,k}) \right]^2 \\
\leq \frac{1}{m} \mathbb{E} \left( h^2(u) \right) \mathbb{E} |Y_{n,k \gamma} - D_{n,k \gamma}|^2 \\
+ \sum_{i,j,i \neq j} \frac{m-1}{mn(n-1)} \mathbb{E} \left| \nabla g(Y_{n,k \gamma}) - \nabla g(D_{n,k \gamma}) \right|^2 \\
\leq \frac{1}{m} \mathbb{E} \left( h^2(u) \right) \mathbb{E} |Y_{n,k \gamma} - D_{n,k \gamma}|^2 \\
+ L^2 \left( 1 - \frac{1}{m} \right) L^2 \mathbb{E} |Y_{n,k \gamma} - D_{n,k \gamma}|^2.
\]

This completes the proof. \(\square\)

Next we exhibit that the interpolated M-SGD process and the interpolated SGD with scaled normal error are close. Here \(t \in (k \gamma, (k + 1) \gamma]\).

**THEOREM 7.** Under the stated Assumptions [1][3] for \(t \in (k \gamma, (k + 1) \gamma]\), we have
\[
W^2_t(Y_{n,t}, D_t) \leq \tilde{J}_1 \frac{4k \gamma}{m},
\]
where \(\tilde{J}_1\) is a constant dependent on \(T, L, L_1, p, \mathbb{E} \left( h^2(u) \right)\).

**Proof of Theorem 7** Using the definitions of (2.4) and (2.7), we have
\[
|Y_{n,t} - D_t| \leq |Y_{n,k \gamma} - D_{k \gamma}| + \gamma (t - k \gamma) \left| \sum_{i=1}^{n} w_{i,k} \nabla l(Y_{n,k \gamma}, u_{i,k}) - \nabla g(D_{k \gamma}) \right| \\
+ \sqrt{\frac{\gamma}{m}} |\sigma(D_{k \gamma}) (B_t - B_{k \gamma})| \\
\leq |Y_{n,k \gamma} - D_{k \gamma}| + \gamma (t - k \gamma) \left| \sum_{i=1}^{n} w_{i,k} \nabla l(Y_{n,k \gamma}, u_{i,k}) - \nabla l(D_{k \gamma}, u_{i,k}) \right| \\
+ \gamma (t - k \gamma) \left| \sum_{i=1}^{n} w_{i,k} \nabla l(D_{k \gamma}, u_{i,k}) - \nabla g(D_{k \gamma}) \right| + \sqrt{\frac{\gamma}{m}} |\sigma(D_{k \gamma}) (B_t - B_{k \gamma})|.
\]

Thus, we get
\[
|Y_{n,t} - D_t|^2 \leq 4 |Y_{n,k \gamma} - D_{k \gamma}|^2 + 4 \gamma^2 (t - k \gamma)^2 \left| \sum_{i=1}^{n} w_{i,k} \nabla l(Y_{n,k \gamma}, u_{i,k}) - \nabla l(D_{k \gamma}, u_{i,k}) \right|^2 \\
+ 4 \gamma^2 (t - k \gamma)^2 \left| \sum_{i=1}^{n} w_{i,k} \nabla l(D_{k \gamma}, u_{i,k}) - \nabla g(D_{k \gamma}) \right|^2 + 4 \frac{\gamma}{m} |\sigma(D_{k \gamma}) (B_t - B_{k \gamma})|^2.
\]

Taking expectation, we get,
\[
\mathbb{E} |Y_{n,t} - D_t|^2 \leq 4 \mathbb{E} |Y_{n,k \gamma} - D_{k \gamma}|^2 + 4 \gamma^2 (t - k \gamma)^2 \mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} \nabla l(Y_{n,k \gamma}, u_{i,k}) - \nabla l(D_{k \gamma}, u_{i,k}) \right|^2 \\
+ 4 \gamma^2 (t - k \gamma)^2 \mathbb{E} \left| \sum_{i=1}^{n} w_{i,k} \nabla l(D_{k \gamma}, u_{i,k}) - \nabla g(D_{k \gamma}) \right|^2 + 4 \frac{\gamma}{m} \mathbb{E} |\sigma(D_{k \gamma}) (B_t - B_{k \gamma})|^2.
\]

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This implies
\[ \mathbb{E} |Y_{n,t} - D_t|^2 \leq 4 \mathbb{E} |Y_{n,k\gamma} - D_{k\gamma}|^2 + 4 \gamma^2 (t - k\gamma)^2 \left( \frac{1}{m} \mathbb{E} \left( h^2(u) \right) - L^2 \right) + L^2 \mathbb{E} |Y_{n,k\gamma} - D_{k\gamma}|^2 \]
\[ + 4 \gamma^2 (t - k\gamma)^2 \frac{1}{m} \mathbb{E} ||\sigma(D_{k\gamma})||^2_F + 4 \frac{\gamma^2}{m} (t - k\gamma) \mathbb{E} ||\sigma(D_{k\gamma})||^2_F. \]

The last line follows using Lemma 5.12 and Ito isometry. Therefore, we get
\[ \mathbb{E} |Y_{n,t} - D_t|^2 \leq \left[ 4 + 4 \gamma^2 (t - k\gamma) \left( \frac{1}{m} \mathbb{E} \left( h^2(u) \right) - L^2 \right) + L^2 \right] \mathbb{E} |x_{n,k} - x_k|^2 \]
\[ + 2 \left[ \frac{1}{m} 4 \gamma^2 (t - k\gamma)^2 + 4 \frac{\gamma}{m} (t - k\gamma) \right] \left( pL^2 \mathbb{E} |D_{k\gamma} - \tilde{x}_k|^2 + ||\sigma(\tilde{x}_k)||^2_F \right). \]

Here we use the fact that \( \mathbb{E} |Y_{n,k\gamma} - D_{k\gamma}|^2 = \mathbb{E} |x_{n,k} - x_k|^2 \) and we also use the fact \( \mathbb{E} |D_{k\gamma} - \tilde{x}_k|^2 = \mathbb{E} |x_k - \tilde{x}_k|^2 \). We have also used the fact that \( \sigma \) is Lipschitz. Using this fact along with Theorem 3 and Lemma 5.7 we get,
\[ \mathbb{E} |Y_{n,t} - D_t|^2 \leq J_1 + J_2 \]
where
\[ J_1 = \left\{ 4 + 4 \gamma^2 (t - k\gamma) \left( \frac{1}{m} \mathbb{E} \left( h^2(u) \right) - L^2 \right) \right\} \cdot K_1^{4\frac{k\gamma}{m}}, \]
and
\[ J_2 = 2 \left[ \frac{1}{m} 4 \gamma^2 (t - k\gamma)^2 + 4 \frac{\gamma}{m} (t - k\gamma) \right] \]
\[ \cdot \left\{ \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m} + \left[ 2C_1^2 k^2 \gamma^2 L_1^2 p (1 + L\gamma)^{2k} + 2 \sup_{0 \leq t \leq T} ||\sigma(\tilde{X}_t)||^2_F \right] \right\} \]
\[ \leq 2 \left[ \frac{1}{m} 4 \gamma^2 (t - k\gamma)^2 + 4 \frac{\gamma}{m} (t - k\gamma) \right] \]
\[ \cdot \left\{ \tilde{C}_1 \frac{\gamma^2}{m} + \tilde{C}_2 \frac{k}{m} + \left[ 2C_1^2 T^2 L_1^2 pe^{2TL} + 2 \sup_{0 \leq t \leq T} ||\sigma(\tilde{X}_t)||^2_F \right] \right\}. \]

Note that, \( J_1 \leq \frac{4^k \gamma}{m} J_{11} \), where
\[ J_{11} = K_1 \left[ 4 + 4 \gamma^3 \left( \frac{1}{m} \left( \mathbb{E} \left( h^2(u) \right) - L^2 \right) \right) \right]. \]

Also, we have
\[ J_2 = J_{21} \frac{\gamma^4}{m^2} + J_{22} \frac{\gamma}{m^2} + J_{23} \frac{\gamma^2}{m} \]
where \( J_{21}, J_{22}, J_{23} \) can be chosen as
\[ J_{21} = 8 \tilde{C}_1, \]
\[ J_{22} = 8 \tilde{C}_2 T, \]
\[ J_{23} = 8 \left[ 2C_1^2 T^2 L_1^2 pe^{2TL} + 2 \sup_{0 \leq t \leq T} ||\sigma(\tilde{X}_t)||^2_F \right]. \]

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Hence, we have

\[ E |Y_{n,t} - D_t|^2 \leq \frac{4^k \gamma J_{11} + J_{21} \gamma^4}{m^2} + \frac{J_{22} \gamma^2}{m^2} + \frac{J_{23} \gamma^2}{m} \]

\[ \leq \tilde{J}_1 \frac{4^k \gamma}{m}, \]

where we can take \( \tilde{J}_1 \) as

\[ \tilde{J}_1 = J_{11} + J_{21} + J_{22} + J_{23}. \]

Hence

\[ W_2^2(Y_{n,t}, D_t) \leq \tilde{J}_1 \frac{4^k \gamma}{m}. \]

We are done.

Next we prove one of our main theorems.

Proof of Theorem 5. Let \( t \in [k \gamma, (k + 1) \gamma) \) Hence, using the fact that the Wasserstein distance exhibits the inequality \( W_2^2(\mu, \nu) \leq 2W_2^2(\mu, P) + 2W_2^2(P, \nu) \) (where \( \mu, \nu, P \) are probability measures), we have

\[ W_2^2(Y_{n,t}, X_t) \leq 2W_2^2(Y_{n,t}, D_t) + 2W_2^2(D_t, X_t) \]

\[ \leq 2\tilde{J}_1 \frac{4^k \gamma}{m} + 2C_{11} \gamma^2 + 2C_{12} \frac{\gamma}{m} \]

\[ \leq C_{21} \gamma^2 + C_{22} \frac{4^k \gamma}{m}, \]

where \( C_{21} = 2C_{11} \) and \( C_{22} = 2\tilde{J}_1 + C_{12} \). Hence we conclude the proof.

Proof of Theorem 6

In the case of the objective function \( g \) being strongly convex, we derive bounds for the M-SGD algorithm with the structure as mentioned previously.

We consider the algorithm. Under the Assumptions 1-5 and 8 we exhibit that the algorithm converges to the global optimum on average.

Proof of Theorem 4. We know

\[ x_{k+1} = x_k - \gamma \sum_{i=1}^{n} w_{i,k} \nabla l(x_k, u_{i,k}) \]

\[ = x_k - \gamma \nabla g(x_k) + \gamma \sum_{i=1}^{n} w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x_k)). \]
The last line uses the fact that $\sigma$. Replacing it is $L \gamma$. Hence from the above, taking expectation, we get

$$
\nabla g(x_k) + \sum_{i=1}^n w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x))
$$

Hence

$$
g(x_{k+1}) = g(x_k - \gamma \nabla g(x_k) + \gamma \sum_{i=1}^n w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x)))
$$

$$
= g(x_k) - \gamma \nabla g(x_k)' \left( \nabla g(x_k) + \sum_{i=1}^n w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x)) \right)
$$

$$
+ \frac{\gamma^2}{2} \left( \nabla g(x_k) + \sum_{i=1}^n w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x)) \right)' \nabla^2 g(\hat{x}_k)
$$

$$
\left( \nabla g(x_k) + \sum_{i=1}^n w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x)) \right).
$$

The last line follows from the fact that $\nabla g$ is Lipschitz.

Hence from the above, taking expectation, we get

$$
\mathbb{E}(g(x_{k+1})) \leq \mathbb{E}(g(x_k)) - \gamma |\nabla g(x_k)|^2 - \gamma \nabla g(x_k)' \left( \sum_{i=1}^n w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x)) \right)
$$

$$
+ \frac{L\gamma^2}{2} \left| \nabla g(x_k) + \sum_{i=1}^n w_{i,k} (\nabla l(x_k, u_{i,k}) - \nabla g(x)) \right|^2.
$$

The second line follows as this is the online version of the algorithm i.e. we refresh the $u_i$ at each iteration and $\nabla l(\cdot, u)$ is unbiased for $g(\cdot).$ Also we use the definition that $\sigma(\cdot)$ is the variance of the same.

From Boyd and Vandenberghe, 2004, we have

$$
|\nabla g(x)|^2 \geq 2\lambda (g(x) - g(x^*)) , \forall x.
$$

Thus, we have from the previous inequality

$$
\mathbb{E}(g(x_{k+1})) \leq \mathbb{E}(g(x_k)) - \gamma \lambda (2 - L\gamma) \mathbb{E}(g(x_k) - g(x^*)) + \frac{L\gamma^2}{2m} \mathbb{E} \| \sigma(x_k) \|^2_F
$$

$$
\leq \mathbb{E}(g(x_k)) - \gamma \lambda (2 - L\gamma) \mathbb{E}(g(x_k) - g(x^*)) + \frac{L\gamma^2}{2m} \left( 2 \mathbb{E} \| \sigma(x^*) \|^2_F + 2pL_1^2 \mathbb{E} |x_k - x^*|^2 \right).
$$

The last line uses the fact that $\sigma$ is $\sqrt{pL_1}$ Lipschitz in the Frobenius norm which follows from the fact that it is $L_1^2$ lipschitz in the spectral norm. Using the fact that $g$ is $\lambda$-strongly convex, one has

$$
g(y) - g(x) \geq \nabla g(x)'(y - x) + \frac{\lambda}{2} |y - x|^2.
$$

Replacing $y = x_k$ and $x = x^*$ and using the fact that $\nabla g(x^*) = 0$, we have $g(x_k) - g(x^*) \geq \frac{\lambda}{2} |x_k - x^*|^2.$
Thus from the final line, substracting $g(x^*)$ to both sides, we get,
\[
\mathbb{E}(g(x_{k+1}) - g(x^*)) \leq \mathbb{E}(g(x_k) - g(x^*)) - \lambda \gamma (2 - L \gamma) \mathbb{E}(g(x_k) - g(x^*)) + \frac{2pLL^2\gamma^2}{m\lambda} \mathbb{E}(g(x_k) - g(x^*)) + \frac{L\gamma^2}{m} ||\sigma(x^*)||_F^2
\]
\[
= \left[1 - \lambda \gamma (2 - L \gamma) + \frac{2pLL^2\gamma^2}{m\lambda}\right] \mathbb{E}(g(x_k) - g(x^*)) + \frac{L\gamma^2}{m} ||\sigma(x^*)||_F^2.
\]

Note that due to our final assumption on $\gamma$, $m$, \[1 - \lambda \gamma (2 - L \gamma) + \frac{2pLL^2\gamma^2}{m\lambda}\] < 1. Calling this quantity as $r$, $\mathbb{E}(g(x_k) - g(x^*)) = a_k$ and $||\sigma(x^*)||_F^2 = B$, we have the last line as
\[
a_{k+1} \leq ra_k + \frac{L\gamma^2}{m} B
\]
\[
\leq r^{k+1} a_0 + \frac{L\gamma^2}{m} B \left(1 + r + r^2 + \cdots + r^k\right)
\]
\[
\leq r^{k+1} a_0 + \frac{L\gamma^2}{m(1 - r)} B.
\]
And the first result follows from the last line. The second result follows using strong convexity of $g$ which implies that $g(y) - g(x^*) \geq \frac{1}{2} ||y - x^*||^2$. \qed