A New Type Coefficient Conjugate on the Gradient Methods for Impulse Noise Removal in Images

Basim A. Hassan\textsuperscript{1,*}, Haneen A. Alashoor\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, College of Computers Sciences and Mathematics, University of Mosul, Iraq

\textbf{Abstract.} The conjugate gradient methods are outstanding by choosing a suitable for coefficient conjugate. In this paper, a modified version of the conjugate gradient algorithm suggested by Hideaki and Yasushi [4] is proposed in order to show that the new method is globally convergent, under standard assumptions. To exemplify the efficiency of the new method, its performance is examined for impulse noise removal in images.

\textbf{2020 Mathematics Subject Classifications}: 90C30, 65K05, 49M37

\textbf{Key Words and Phrases}: Derivative coefficient conjugate gradient, Convergence property, Image restoration problems

\section{1. Introduction}

Image declaring [16] from noisy data is a fundamental problem in image processing. Two-phase approach, consists of two phases. Firstly, accurate detection of the location of impulse noise using a median-type filter (AMF) [17]. Let $x$ be the true image with M-by-N pixels, and $x_{i,j}$ denote the gray level of $x$, $y$ signifying the observed noisy image of $x$ corrupted by the salt-and-pepper noise, $\bar{y}$ is defined by the image obtained by applying the adaptive median filter method to the noisy image $y$ in the first phase. Secondly, recovering the noise pixels by minimizing the following functional:

$$f_{\alpha}(u) = \sum_{(i,j) \in N} \left[ |u_{i,j} - y_{i,j}| + \frac{\beta}{2}(S_{i,j}^1 + S_{i,j}^2) \right]$$ (1)

Where $u_{i,j} = \lfloor u_{i,j} \rfloor_{(i,j) \in N}$ is a column vector of length $|N|$, $\beta$ is the regularization parameter and:

$$S_{i,j}^1 = 2 \sum_{(m,n) \in P_{i,j} \cap N} \varphi_\alpha(u_{i,j} - y_{m,n}), \quad S_{i,j}^2 = \sum_{(m,n) \in P_{i,j} \cap N} \varphi_\alpha(u_{i,j} - y_{m,n})$$ (2)

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i4.4579

Email addresses: basimah@uomosul.edu.iq (B. A. Hassan), haneen19921006@gmail.com (H. A. Alashoor)
The noise candidate indices set $N^c = \{(i, j) \in A/\bar{y}_{i,j} \neq y_{i,j} \text{ and } y_{i,j} = s_{\text{max}} \text{ or } s_{\text{min}}\}$, $s_{\text{max}}$ is the maximum of the noisy pixel and $s_{\text{min}}$ denotes the minimum of the noisy pixel. $A = 1, 2, 3, ..., M \times 1, 2, 3, ..., N$, $V_{i,j} = (V_{i,j} \cap N^c) \cup (V_{i,j} \cap N)$ is the neighborhood of $(i, j)$, and $\varphi_\alpha$ is an edge preserving potential function having the parameter $\alpha$, example of such $\varphi_\alpha = \sqrt{\alpha + x^2}$, $\alpha > 0$. Similar optimization problems arise with non-smooth regularizations, where $F_\alpha(u)$ is of the form (1) with $S_{i,j}^1 + S_{i,j}^2$ smooth and $|u_{i,j} - y_{i,j}|$ non-smooth at zero. In formula, the function which is minimized is a half-quadratic smooth approximation of $F_\alpha(u)$ as:

$$f_\alpha(u) = \sum_{(i,j) \in \mathcal{N}} [(2 \times S_{i,j}^1 + S_{i,j}^2)]$$

(3)

More details can be found in [16],[17].

The minimization of formula (3) has been accomplished by the method of conjugated gradients such as:

$$f(x^*) = \min_{x \in \mathbb{R}^N} f(u).$$

(4)

In formula [15], the k-th iteration, a step-length $\alpha_k$ is obtained by a line search technique and the next iterate is set to:

$$u_{k+1} = u_k + \alpha_k d_k.$$  

(5)

If $f$ is a convex quadratic, its one-dimensional minimizer along the ray $u_k + \alpha_k d_k$ can be computed analytically, and is given by:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k}.$$ 

(6)

For general non-linear functions, it is necessary to use an iterative procedure. More details can be found in formula [14].

In the convergence analysis and implementations of conjugate gradient methods, the Wolfe condition is often used to find the step length $\alpha_k$ satisfying:

$$f(u_k + \alpha_k d_k) \leq f(u_k) + \delta \alpha_k g_k^T d_k$$

$$d_k^T g(u_k + \alpha_k d_k) \geq \sigma d_k^T g_k$$

(7)

(8)

where $0 < \delta < \sigma < 1$. More details can be found in [13].

The conjugate gradient direction for the next iteration has the following form:

$$d_{k+1} = -g_{k+1} + \beta_k d_k$$

(9)

where $\beta_k$ is a scalar. Two famous ways of choosing $\beta_k$ are:

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \quad \beta_k^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k}$$

(10)
These were given by Fletcher-Reeves (FR) method [4], and the Dai - Yuan (DY) method [2] independently. Its algorithms have important properties including the globally convergent property.

Recently, Hideaki and Yasushi [11] and Basim [8] proposed conjugate gradient methods which significantly differ in using both their gradient and function values with higher accuracy in the approximation of the curvature, in which the parameters are specified as follows:

$$\beta_{HY}^{k} = \frac{\|g_{k+1}\|^2}{2(f_k - f_{k+1})/a_k} \quad \beta_{B}^{k} = \frac{\|g_{k+1}\|^2}{(f_k - f_{k+1})/a_k - g_k^T d_k/2}$$

These algorithms are significantly efficient in their computational efficiency also. The reported improved performance for this modification for unconstrained problems is a quadratic model in order to optimize the benefits of the original conjugate gradient methods.

The main goal of this study is derivation of the new coefficient conjugate and to think differently of the denominator $d_k^T G v_k$ based on the quadratic model. Our method is found to be numerically coherent and also efficient in image restoration.

2. Deriving new coefficient conjugate for conjugate gradient methods

We will assume that $f$ is quadratic and exact line search are being used. This is a studies on the behaviors of one of the family of conjugate gradient optimization methods, which was introduced by Hideaki and Yasushi in 2011. Hideaki and Yasushi choice of $\beta_k$ is:

$$\beta_k = \frac{g_{k+1}^T Q s_k}{d_k^T Q s_k}$$

where $Q$ is Hessian matrix and where $\beta_k$ is satisfies the conjugacy condition:

$$d_{k+1}^T Q d_k = 0$$

To modify this method we will introduce a good approximation to the $d_k^T Q s_k$. So by Taylor formula we have:

$$f(u) = f(u_{k+1}) + g_{k+1}^T (u - u_{k+1}) + \frac{1}{2}(u - u_{k+1})^T Q(u_k)(u - u_{k+1})$$

Define the gradient step:

$$g_{k+1} = g_k + Q(u_k)s_k$$

Note that equation (14) and equation (15) result that:

$$1/2 s_k^T Q(u_k)s_k = (f_{k+1} - f_k - g_k^T s_k)$$
Now, by equation (16) and equation (6) we can see that:

$$d^T_k Q(u_k) s_k = (f_{k+1} - f_k)/a_k - 3/2d^T_k g_k$$  \hspace{1cm} (17)

Using (12) and (17), we obtain the following formula for computing $\beta_k$:

$$\beta_k = \frac{g^T_{k+1} y_k}{(f_{k+1} - f_k)/\alpha_k - 3/2d^T_k g_k}$$  \hspace{1cm} (18)

On the other hand, using (6) in (17) and putting in (12), then got another formula:

$$\beta_k = \frac{g^T_{k+1} y_k}{(f_{k+1} - f_k)/\alpha_k + 3/2d^T_k y_k}$$  \hspace{1cm} (19)

Since $f$ is quadratic function and exact line search is employed, then:

$$\beta_k = \frac{\|g_{k+1}\|^2}{(f_{k+1} - f_k)/\alpha_k - 3/2d^T_k g_k}$$  \hspace{1cm} (20)

and

$$\beta_k = \frac{\|g_{k+1}\|^2}{(f_{k+1} - f_k)/\alpha_k + 3/2d^T_k y_k}$$  \hspace{1cm} (21)

Given our formula for modified HY formula, so-called BNC and BTC.

At this point, we describe a new algorithm, called BNC and BTC.

**Algorithms BNC and BTC.**

**Stage 1.** An initial point $u_1$. Set $d_1 = -g_1$. If $\|g_1\| = 10^{-6}$, then stop.

**Stage 2.** Determine the $\alpha_k > 0$ satisfying the Wolfe conditions (7) and (8).

**Stage 3.** Let $x_{k+1} = x_k + \alpha_k d_k$ and $g_{k+1} = g(x_{k+1})$. If $\|g_{k+1}\| = 10^{-6}$, then stop.

**Stage 4.** Compute $\beta_k$ by the formulae (20-21), then generate $d_{k+1}$ by (9).

**Stage 5.** Set $k = k + 1$ and continue with step 2.

The nice property for any a good algorithm should accomplish the sufficient descent condition. Specially, in this study will provide additional property are very vital in the convergence proving. Summarizing the above discussion, the following theorem is obtained.

**Theorem 1.** Let $x_k$ and $d_k$ be generated by BNC method, then we obtain:

$$d^T_{k+1} g_{k+1} < 0 \text{ and } d^T_{k+1} g_{k+1} = \beta_k d^T_k g_k$$  \hspace{1cm} (22)

**Proof.** It is visible that $d_k = -g_k$ satisfies $d^T_k g_k < 0$. Suppose that $d^T_k g_k < 0$ for any $k$. Directly from (11) and (21) we conclude that:

$$d^T_{k+1} g_{k+1} = -g^T_{k+1} g_{k+1} + \beta_k d^T_k g_{k+1}$$

$$= -\beta_k ((f_{k+1} - f_k)/a_k - 3/2d^T_k g_k) + \beta_k d^T_k g_{k+1}$$  \hspace{1cm} (23)
Since the gradient of the function (14) is (15), we can conclude that:

\[ d^T_{k+1} g_{k+1} = \beta_k d^T_k g_k \]  

(24)

Because \( d^T_k g_k < 0 \), then we obtain:

\[ d^T_{k+1} g_{k+1} < 0 \]  

(25)

so the proof is completed. A similar result holds for the BTC formula.

3. Convergence Analysis

In order to analyze the global convergence of the BDC and BTC methods, we begin by impose the following assumptions:

1. The level set “\( D = u/f(u) \leq f(u_0) \)” is bounded, where \( u_0 \) is the starting point.

2. The gradient is Lipschitz continuous in some neighborhood \( D \) which contains \( L_0 \); i.e. there exist \( L \), such that:

\[ \|g(\upsilon) - g(\omega)\| \leq L\|\upsilon - \omega\|, \forall \upsilon, \omega \in D. \]  

(26)

For more details see [2],[5].

In [18], Zoutendijk introduced the general result are important feature in proving the convergence results.

**Lemma 1.** Let the Assumptions (1) and (2) holds true. If the direction \( d_k \) is descent and the sequence generated by (5) where \( \alpha_k \) satisfies (7) and (8), then:

\[ \sum_{k=1}^{\infty} \frac{(g^T_k d_k)^2}{\|d_k\|^2} < \infty. \]  

(27)

**Theorem 2.** Suppose that assumptions (1) and (2) hold. If formula \( \beta_k \) satisfies (21), then Then we have:

\[ \lim_{k \to \infty} \inf |g_k| = 0. \]  

(28)

**Proof.** The prove is by induction \( \|g_k\| \neq 0 \) for all \( k \in n \). Rewriting (9) as \( d_{k+1} + g_{k+1} = \beta_k d_k \). Accordingly, squaring both sides, we have:

\[ |d_{k+1}|^2 + |g_{k+1}|^2 + 2d^T_{k+1} g_{k+1} = (\beta_k)^2 |d_k|^2 \]  

(29)

Using (23) to (27) implies:

\[ |d_{k+1}|^2 = \frac{(d^T_{k+1} g_{k+1})^2}{(d^T_k g_k)^2} |d_k|^2 - 2d^T_{k+1} g_{k+1} - |g_{k+1}|^2 \]  

(30)
Dividing both sides of (28) by \((d_{k+1}^T g_k + 1)^2\), we get:

\[
\frac{|d_{k+1}|^2}{(d_{k+1}^T g_k + 1)^2} = \frac{|d_k|^2}{(d_k^T g_k + 1)^2} - \frac{|g_{k+1}|^2}{(d_{k+1}^T g_{k+1})^2} - \frac{2}{d_{k+1}^T g_{k+1}}
\leq \frac{|d_k|^2}{(d_k^T g_k + 1)^2} - \left( \frac{|g_{k+1}|}{(d_{k+1}^T g_{k+1})^2} + \frac{1}{|g_{k+1}|^2} \right) + \frac{1}{|g_{k+1}|^2}
\]

Hence, we obtained:

\[
\frac{|d_{k+1}|^2}{(d_{k+1}^T g_k + 1)^2} \leq \sum_{i=1}^{k+1} \frac{1}{|g_i|^2}
\]

Assume that there exists \(c_1 > 0\) such that \(|g_k| \geq c_1\) for all \(k \in \mathbb{N}\). Then:

\[
\frac{|d_{k+1}|^2}{(d_{k+1}^T g_{k+1})^2} < \frac{k+1}{c_1^2}
\]

Equation (31) shows that:

\[
\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{|d_k|^2} = \infty
\]

Based on Lemma 1, we get \(\lim_{k \to \infty} \inf |g_k| = 0\) holds. A similar result holds for the BTC formula.

4. Numerical results

The main goal we carry out some numerical experiments for minimization of 4 test images taken from [3]. To test and compare the computation effect of the proposed BNC, BTC in this paper with FR method whose results be given by [4]. There are different methods than this methods that we can see in [6, 7, 9, 10, 12]. In the numerical experiments, BNC, BTC and FR methods uses the following parameters: \(\delta = 0.0001\) and \(\sigma = 0.5\).

Numerical results of the BNC and FR are listed in Table 1. Here denotes the NI/NF/PSNR denote the number of iteration, function evaluations and PSNR (peak signal to noise ratio), which is defined as:

\[
PSNR = 10 \log_{10} \frac{255^2}{\frac{1}{MN} \sum_{i,j} (u_{i,j}^r - u_{i,j}^*)^2}
\]

where \(u_{i,j}^r\) and \(u_{i,j}^*\) denote the pixel values of the restored image and the original image, respectively. We stop the iterations, if:
\[ \frac{|f(u_k) - f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \text{ and } |f(u_k)| \leq 10^{-4}(1 + |f(u_k)|) \]  

(36)

For more details see [1],[3].

| Image | Noise level r (%) | FR-Method | NI | NF | PSNR (dB) | BNC-Method | NI | NF | PSNR (dB) | BTC-Method | NI | NF | PSNR (dB) |
|-------|------------------|------------|----|----|-----------|------------|----|----|-----------|------------|----|----|-----------|
| Le    | 50               | 82         | 153| 30.5529 | 45         | 93         | 30.5049 | 15 | 33         | 35.7945    |
|       | 70               | 81         | 155| 27.4824 | 46         | 91         | 27.4686 | 15 | 35         | 34.3739    |
|       | 90               | 108        | 211| 22.8583 | 53         | 100        | 22.8008 | 15 | 33         | 31.6911    |
| ho    | 50               | 52         | 53 | 30.6845 | 28         | 56         | 34.7098 | 15 | 32         | 39.1522    |
|       | 70               | 63         | 116| 31.2564 | 37         | 73         | 31.3444 | 15 | 31         | 36.3978    |
|       | 90               | 111        | 214| 25.287  | 49         | 96         | 24.927  | 14 | 30         | 34.52      |
| El    | 50               | 35         | 36 | 33.9129 | 26         | 47         | 33.8859 | 16 | 32         | 35.0033    |
|       | 70               | 38         | 39 | 31.864  | 28         | 49         | 31.8548 | 16 | 31         | 33.456     |
|       | 90               | 65         | 114| 28.2019 | 39         | 70         | 28.2239 | 14 | 28         | 32.6548    |
| c512  | 50               | 59         | 87 | 35.5359 | 30         | 65         | 35.4408 | 12 | 26         | 42.0829    |
|       | 70               | 78         | 142| 30.6259 | 35         | 75         | 30.7049 | 12 | 27         | 40.3123    |
|       | 90               | 121        | 236| 24.3962 | 45         | 96         | 24.935  | 13 | 31         | 36.9659    |

Table 1: Numerical results for the NI, NF and PSNR.

As we can see from Table (1), our proposed algorithm is competitive and promising has a very great advantage over the FR method in term of the the number of iterations and the number of function evaluations.

Figure 1: From left to right: 50, 70, 90% noise, FR method, BNC and BTC method 256 × 256 Lena image.
Figure 2: From left to right: 50, 70, 90% noise, FR method, BNC and BTC method 256 x 256 House image.

Figure 3: From left to right: 50, 70, 90% noise, FR method, BNC and BTC method 512 x 512 Elaine image.
5. Conclusions

We presented a powerful conjugate gradient technique. In addition to meeting the adequate descent criterion, the proposed approach is globally convergent. According to numerical findings, the approach operates well in practice and is superior than the widely used FR method. We also looked at our approach’s aptitude for resolving several practical problems. In this manner, a typical issue from applications for image processing was taken into account. We demonstrated the acceptability of the image that our approach restored.

References

[1] Jian-Feng Cai, Raymond Chan, and Benedetta Morini. Minimization of an edge-preserving regularization functional by conjugate gradient type methods. In Image Processing Based on Partial Differential Equations, pages 109–122. Springer, 2007.

[2] Yu-Hong Dai and Yaxiang Yuan. A nonlinear conjugate gradient method with a strong global convergence property. SIAM Journal on optimization, 10(1):177–182, 1999.

[3] Yuhong Dai, Jiye Han, Guanghui Liu, Defeng Sun, Hongxia Yin, and Ya-xiang Yuan. Convergence properties of nonlinear conjugate gradient methods. SIAM Journal on Optimization, 10(2):345–358, 2000.
[4] Reeves Fletcher and Colin M Reeves. Function minimization by conjugate gradients. *The computer journal*, 7(2):149–154, 1964.

[5] William W Hager and Hongchao Zhang. A new conjugate gradient method with guaranteed descent and an efficient line search. *SIAM Journal on optimization*, 16(1):170–192, 2005.

[6] BA Hassan and RM Sulaiman. A new class of self-scaling for quasi-newton method based on the quadratic model. *Indonesian Journal of Electrical Engineering and Computer Science*, 21(3):1830–1836, 2021.

[7] Basim A Hassan. A modified quasi-newton methods for unconstrained optimization. *Pure Appl. Math*, 42(2019):504–511, 2019.

[8] Basim A Hassan. A new formula for conjugate parameter computation based on the quadratic model. *Indonesian Journal of Electrical Engineering and Computer Science*, 3(3):954–961, 2019.

[9] Basim A Hassan. A new type of quasi-newton updating formulas based on the new quasi-newton equation. *Numerical Algebra, Control & Optimization*, 10(2):227, 2020.

[10] Basim A Hassan, Kanikar Muangchoo, Fadhil Alfarag, Abdulkarim Hassan Ibrahim, and Auwal Bala Abubakar. An improved quasi-newton equation on the quasi-newton methods for unconstrained optimizations. *Indonesian Journal of Electrical Engineering and Computer Science*, 22(2):389–397, 2020.

[11] Hideaki Iiduka and Yasushi Narushima. Conjugate gradient methods using value of objective function for unconstrained optimization. *Optimization Letters*, 6(5):941–955, 2012.

[12] Hawraz N Jabbar and Basim A Hassan. Two-versions of descent conjugate gradient methods for large-scale unconstrained optimization. *Indonesian Journal of Electrical Engineering and Computer Science*, 22(3):1643, 2021.

[13] Xian-zhen Jiang and Jin-bao Jian. A sufficient descent dai–yuan type nonlinear conjugate gradient method for unconstrained optimization problems. *Nonlinear Dynamics*, 72(1):101–112, 2013.

[14] Jorge Nocedal and Stephen J Wright. Numerical optimization, springer series in operations research. *Siam J Optimization*, 2006.

[15] Elijah Polak and Gerard Ribiere. Note sur la convergence de méthodes de directions conjuguées. *Revue française d’informatique et de recherche opérationnelle. Série rouge*, 3(16):35–43, 1969.

[16] Wei Xue, Junhong Ren, Xiao Zheng, Zhi Liu, and Yueyong Liang. A new dy conjugate gradient method and applications to image denoising. *IEICE TRANSACTIONS on Information and Systems*, 101(12):2984–2990, 2018.
[17] Gaohang Yu, Jinhong Huang, and Yi Zhou. A descent spectral conjugate gradient method for impulse noise removal. *Applied mathematics letters*, 23(5):555–560, 2010.

[18] G Zoutendijk. Nonlinear programming, computational methods. *Integer and nonlinear programming*, pages 37–86, 1970.