CHIRAL LAGRANGIANS
AND
KAON CP-VIOLATION

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Abstract

These lectures are an introduction to the subject of chiral effective Lagrangians of the Standard Model and their applications, mostly in the sector of non–leptonic kaon decays, with special emphasis on CP–violation. The first lecture gives an introduction to the phenomenological description of the $K^0 - \bar{K}^0$ system and $K \rightarrow \pi\pi$ decays. In the second lecture I give an overview of the basic ideas behind the chiral perturbation theory ($\chi$PT) –approach to hadron dynamics at low energies. The study of the weak interactions of $K$–particles within the framework of $\chi$PT is the subject of the third lecture. The fourth lecture is an overview of various models of the QCD low–energy effective action which have been developed during the last few years. The fifth lecture is dedicated to a discussion of the CP–violation $\epsilon$ and $\epsilon'$ parameters, and to the study of the decay mode: $K_L \rightarrow \pi^0 e^+ e^-$. 

Key-Words : CP–Violation, Kaon Decays, Chiral Perturbation Theory.

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1 Phenomenology of $K^0 - \bar{K}^0$ Mixing and CP–Violation in $K \to 2\pi$ Decays

The purpose of this first lecture is to introduce the basics of the phenomenological description of the $K^0 - \bar{K}^0$ system. There is practically no theory behind this description. It is only based on first principles: the superposition principle, Lorentz invariance, and general invariance properties under the P, C and T symmetries. The basic idea is to reduce the description of this system to a minimum of phenomenological parameters which, eventually, an underlying theory –like the Standard Model– should be able to predict. There are many reviews on this subject. I recommend the reader to consult Ref. 1 for an excellent historical overview and complementary information.

1.1 Phenomenology of $K^0 - \bar{K}^0$ Mixing

In the absence of the weak interactions, the $K^0$ and $\bar{K}^0$ particles produced by the strong interactions are stable eigenstates of strangeness with eigenvalues $\pm 1$. In the presence of the weak interaction they become unstable. The states with an exponential time dependence law ($\tau$ is the proper time)

$$|K_L\rangle \to e^{-iM_L\tau} |K_L\rangle \quad \text{and} \quad |K_S\rangle \to e^{-iM_S\tau} |K_S\rangle,$$

are linear superpositions of the eigenstates of strangeness:

$$|K_L\rangle = \frac{1}{\sqrt{|p|^2 + |q|^2}} (p |K^0\rangle + q |\bar{K}^0\rangle),$$

$$|K_S\rangle = \frac{1}{\sqrt{|p|^2 + |q|^2}} (p |K^0\rangle - q |\bar{K}^0\rangle),$$

where $p$ and $q$ are complex numbers and CPT–invariance, which is a property of the Standard Model in any case, has been assumed. The parameters $M_{L,S}$ in Eq. (1) are also complex

$$M_{L,S} = m_{L,S} - \frac{i}{2} \Gamma_{L,S},$$

with $m_{L,S}$ the masses and $\Gamma_{L,S}$ the decay widths of the long–lived and short–lived neutral kaon states.

As we shall see, experimentally, the $|K_S\rangle$ and $|K_L\rangle$ states are very close to the CP-eigenstates

$$|K^0_1\rangle = \frac{1}{\sqrt{2}} (|K^0\rangle - |\bar{K}^0\rangle) \quad \text{and} \quad |K^0_2\rangle = \frac{1}{\sqrt{2}} (|K^0\rangle + |\bar{K}^0\rangle)$$

with

$$\text{CP} \ |K^0_1\rangle = + |K^0_1\rangle \quad \text{and} \quad \text{CP} \ |K^0_2\rangle = - |K^0_2\rangle.$$
This is characterized by the small complex parameter $\tilde{\epsilon}$,

$$\tilde{\epsilon} = \frac{p - q}{p + q};$$

(7)
in terms of which,

$$|K_{L,S}\rangle = \frac{1}{\sqrt{1 + |\tilde{\epsilon}|^2}} (|K^0_{2,1}\rangle + \tilde{\epsilon} |K^0_{1,2}\rangle).$$

(8)

According to Eqs. (1) and (2), a state initially pure $|K^0\rangle$ evolves, in a period of time $\tau$ to a state which is a superposition of $|K^0\rangle$ and $|\bar{K}^0\rangle$:

$$|K^0\rangle \rightarrow \frac{1}{2} [e^{-iM_L\tau} + e^{-iM_S\tau}] |K^0\rangle + \frac{1}{2} \frac{p}{q} [e^{-iM_L\tau} - e^{-iM_S\tau}] |\bar{K}^0\rangle;$$

(9)

and, likewise

$$|\bar{K}^0\rangle \rightarrow \frac{1}{2} [e^{-iM_L\tau} + e^{-iM_S\tau}] |\bar{K}^0\rangle + \frac{1}{2} \frac{q}{p} [e^{-iM_L\tau} - e^{-iM_S\tau}] |K^0\rangle.$$

(10)

For a small period of time $\delta\tau$ we then have

$$|K^0\rangle \rightarrow |K^0\rangle - i\delta\tau (\mathcal{M}_{11} |K^0\rangle + \mathcal{M}_{12} |\bar{K}^0\rangle);$$

(11)

$$|\bar{K}^0\rangle \rightarrow |\bar{K}^0\rangle - i\delta\tau (\mathcal{M}_{21} |K^0\rangle + \mathcal{M}_{22} |\bar{K}^0\rangle),$$

(12)

where

$$\mathcal{M}_{ij} = \frac{1}{2} \left( \begin{array}{cc} M_L + M_S & \frac{2}{q}(M_L - M_S) \\ \frac{2}{p}(M_L - M_S) & M_L + M_S \end{array} \right).$$

(13)

This is the complex mass–matrix of the $K^0 - \bar{K}^0$ system.

In full generality, the mass–matrix $\mathcal{M}_{ij}$ admits a decomposition, similar to the one of the complex parameters $M_{L,S}$ in Eq. (4), in terms of an absorptive part $\Gamma_{ij}$ and a dispersive part $M_{ij}$:

$$\mathcal{M}_{ij} = M_{ij} - i \frac{1}{2} \Gamma_{ij}.$$  

(14)

In a given quantum field theory, like e.g., the Standard Electroweak Model, the complex $K^0 - \bar{K}^0$ mass–matrix is defined via the transition matrix $T$ which characterizes $S$–matrix elements. More precisely, the off–diagonal absorptive matrix element $\Gamma_{12}$ for example, is given by the sum of products of on–shell matrix elements:

$$\Gamma_{12} = \sum_{\Gamma} \int d\Gamma (\langle \Gamma | T | K^0 \rangle)^* \langle \Gamma | T | K^0 \rangle,$$

(15)

where the sum is extended to all possible states $|\Gamma\rangle$ to which the states $|K^0\rangle$ and $|\bar{K}^0\rangle$ can decay. The symbol $d\Gamma$ denotes the phase space measure appropriate to
the particle content of the state $\Gamma$. The corresponding matrix element $M_{12}$ is defined by the dispersive principal part integral

$$M_{12} = \frac{1}{\pi} \phi \int ds \frac{1}{m^2_K - s} \Gamma_{12}(s) + \text{"local – terms"}. \quad (16)$$

The fact that $M_{11} = M_{22}$ in Eq. (13) is a consequence of CPT–invariance. In general, if we have a transition between an initial state $|IN\rangle$ and a final state $|FN\rangle$, CPT–invariance relates the matrix elements of this transition to the one between the corresponding CPT–transformed states $|F\bar{N}\rangle$ and $|ar{T\bar{N}}\rangle$, where $|ar{T\bar{N}}\rangle$ denotes the state obtained from $|IN\rangle$ by interchanging all particles into antiparticles (this is the meaning of the bar symbol in $|\bar{\text{IN}}\rangle$), and taking the mirror image of the kinematic variables: $[(E, \vec{p}) \rightarrow (E, -\vec{p}); (\sigma^0, \vec{\sigma}) \rightarrow (-\sigma^0, \vec{\sigma})]$, as well as their motion reversal image: $[(E, \vec{p}) \rightarrow (E, -\vec{p}); (\sigma^0, \vec{\sigma}) \rightarrow (\sigma^0, -\vec{\sigma})]$. (These kinematic changes are the meaning of the prime symbol in $|\bar{\text{IN}}\rangle$.) Altogether, CPT–invariance implies then:

$$\langle FN | T | IN \rangle = \langle \bar{IN} | T | \bar{FN} \rangle.$$

Since, for the $K^0$-states: $| (K^0) \rangle = | K^0 \rangle$, the CPT–invariance relation implies

$$\mathcal{M}_{11} = \mathcal{M}_{22}. \quad (18)$$

The off–diagonal matrix elements in Eq. (13) are also related by CPT–invariance, plus the hermiticity property of the $T$-matrix in the absence of strong final state interactions; certainly the case when the $|IN\rangle$ and $|FN\rangle$ states are $|K^0\rangle$ and $|\bar{K}^0\rangle$. In general, in the absence of strong final state interactions, we have:

$$\langle \bar{IN} | T | \bar{FN} \rangle = \left( \langle FN | T | \bar{IN} \rangle \right)^*.$$

This relation, together with the CPT–invariance relation in (17) implies then:

$$\mathcal{M}_{12} = (\mathcal{M}_{21})^*. \quad (20)$$

There are a number of interesting constraints between the various phenomenological parameters we have introduced. With $M_{12}$ and $\Gamma_{12}$ defined in Eqs. (16) and (15) and using Eqs. (13, 14, and 17), we have

$$\frac{q}{p} = \frac{1 - \bar{\epsilon}}{1 + \bar{\epsilon}} = \frac{1}{2} \frac{\Delta m + i \frac{1}{2} \Delta \Gamma}{M_{12} - \frac{i}{2} \Gamma_{12}} = \frac{M_{21} - \frac{i}{2} \Gamma_{21}}{\frac{i}{2} (\Delta m + \frac{i}{2} \Delta \Gamma)}, \quad (21)$$

where

$$\Delta m \equiv m_L - m_S \quad \text{and} \quad \Delta \Gamma \equiv \Gamma_S - \Gamma_L. \quad (22)$$

As already discussed, CPT–invariance implies

$$M_{21} = (M_{12})^* \quad \text{and} \quad \Gamma_{21} = (\Gamma_{12})^*. \quad (23)$$
Experimentally, the masses $m_{L,S}$ and widths $\Gamma_{L,S}$ are well measured, and in what follows they will be used as known parameters. (There is no way for theory at present to do better than experiments in the determination of these parameters...) The precise values for the masses and widths can be found in the Particle Data Booklet (PDB). Nevertheless, it is important to keep in mind some orders of magnitude:

\begin{align}
\Gamma_{S}^{-1} & \simeq 0.9 \times 10^{-10}\text{sec.}; \\
\Gamma_{L} & \simeq 1.7 \times 10^{-3}\Gamma_{S};  \\
\Delta m & \simeq 0.5\Gamma_{S}.
\end{align}

### 1.2 The Bell–Steinberger Unitarity Constraint

Let us consider a state $|\Psi\rangle$ to be an arbitrary superposition of the short–lived and long–lived kaon states:

$$|\Psi\rangle = \alpha |K_{S}\rangle + \beta |K_{L}\rangle.$$  

(27)

The total decay rate of this state must be compensated by a decrease of its norm:

$$\sum_{\Gamma} |\langle \Gamma | T | \Psi \rangle|^2 = -\frac{d}{d\tau} |\Psi|^2.$$  

(28)

The change in rate is governed by the mass matrix defined by Eq. (11). Equating terms proportional to $|\alpha|^2$ and $|\beta|^2$ in both sides of Eq. (28) results in the trivial relations:

$$\Gamma_{L} = \sum_{\Gamma} \int d\Gamma |\langle \Gamma | T | K_{L} \rangle|^2;$$  

(29)

$$\Gamma_{S} = \sum_{\Gamma} \int d\Gamma |\langle \Gamma | T | K_{S} \rangle|^2.$$  

(30)

The mixed terms, proportional to $\alpha\beta^*$ and $\alpha^*\beta$, lead however to a highly non–trivial relation, first derived by Bell and Steinberger:

$$-i(M_{L}^* - M_{S}) \langle K_{L} | K_{S} \rangle = \sum_{\Gamma} \int d\Gamma ((\langle \Gamma | T | K_{L} \rangle)^*)^* \langle \Gamma | T | K_{S} \rangle.$$  

(31)

Notice that

$$\langle K_{L} | K_{S} \rangle = \frac{|p|^2 - |q|^2}{|p|^2 + |q|^2} = \frac{2\text{Re} \tilde{\epsilon}}{1 + |\tilde{\epsilon}|^2}.$$  

(32)

The l.h.s. of Eq. (31) can be expressed in terms of measurable physical parameters with the result

$$\left(\frac{\Gamma_{S} + \Gamma_{L}}{2} - i\Delta m\right) \frac{2\text{Re} \tilde{\epsilon}}{1 + |\tilde{\epsilon}|^2} = \sum_{\Gamma} \int d\Gamma ((\langle \Gamma | T | K_{L} \rangle)^*)^* \langle \Gamma | T | K_{S} \rangle.$$  

(33)
The r.h.s. of this equation can be bounded, using the Schwartz inequality, with the result
\[ \left| \Gamma_S + \Gamma_L - i\Delta m \right| \frac{2\text{Re} \tilde{\epsilon}}{1 + |\tilde{\epsilon}|^2} \leq \sqrt{\Gamma_L \Gamma_S}. \] (34)
Inserting the experimental values for \( \Gamma_S, \Gamma_L \) and \( \Delta m \), results in an interesting bound for the non-orthogonality of the \( K_L \) and \( K_S \) states [see eq (32)]:
\[ \frac{2\text{Re} \tilde{\epsilon}}{1 + |\tilde{\epsilon}|^2} \leq 2.9 \times 10^{-2}, \] (35)
indicating also that the admixture of \( K_0^0(K_2^0) \) in \( K_L(K_S) \) has to be rather small.

It is possible to obtain further information from the unitarity constraint in (33), if one uses the experimental fact that the \( 2\pi \) states are by far the dominant terms in the sum over hadronic states \( \Gamma \). One can then write the r.h.s. of Eq. (33) in the form
\[ \sum_{\pi\pi} \int d(\pi\pi) \langle \langle \pi\pi \mid T \mid K_L \rangle \rangle^* \langle \pi\pi \mid T \mid K_S \rangle + \gamma \Gamma_S. \] (36)
It is possible to obtain a bound for \( \gamma \), by considering other states than \( 2\pi \) in the sum of the r.h.s. in Eq. (33) and applying the Schwartz inequality to individual sets of states separated by selection rules. The contribution from the various semileptonic modes, for example, is known to be smaller than
\[ \left| \sum_{\text{lep. modes}} \int \cdots \right| \ll 10^{-3} \Gamma_S; \] (37)
and the contribution from the \( 3\pi \)–states
\[ \left| \sum_{3\pi} \int \cdots \right| \ll 10^{-3} \Gamma_S. \] (38)
We conclude that to a good approximation we can restrict the Bell–Steinberger relation to \( 2\pi \)–states. We shall later come back to this inequality, but first we have to discuss the phenomenology of the dominant \( K \to \pi\pi \) transitions.

### 1.3 \( K \to \pi\pi \) Amplitudes

In the limit where CP is conserved the states \( K_S(K_L) \) become eigenstates of CP; i.e., the states \( K_1^0(K_2^0) \) introduced in (3) with eigenvalues CP = +1(CP = -1). On the other hand a state of two–pions with total angular momentum \( J = 0 \) has CP = +1. Therefore, the observation of a transition from the long–lived component of the neutral kaon system to a two–pion final state is evidence for CP–violation. The first observation of such a transition to the \( \pi^+\pi^- \) mode was made by Christenson, Cronin, Fitch, and Turlay \cite{4} in 1964, with the result
\[ \frac{\Gamma_L(\pi^+\pi^-)}{\Gamma_L(\text{all})} = (2 \pm 0.4) \times 10^{-3}. \] (39)
Since then the transition to the $\pi^0\pi^0$ mode has also been observed, as well as the phases of the amplitude ratios

$$\eta_{+-} = \frac{\langle \pi^+\pi^- | T | K_L \rangle}{\langle \pi^+\pi^- | T | K_S \rangle} \quad \text{and} \quad \eta_{00} = \frac{\langle \pi^0\pi^0 | T | K_L \rangle}{\langle \pi^0\pi^0 | T | K_S \rangle},$$

(40)

with the results:

$$\eta_{+-} = (2.269 \pm 0.023) \times 10^{-3} e^{i(44.3 \pm 0.8)^\circ};$$

(41)

$$\eta_{00} = (2.259 \pm 0.023) \times 10^{-3} e^{i(43.3 \pm 1.3)^\circ}. $$

(42)

In order to make a phenomenological analysis of $K \to \pi\pi$ transitions, it is convenient to express the states $|\pi^+\pi^-\rangle$ and $|\pi^0\pi^0\rangle$ in terms of well defined isospin $I = 0$, and $I = 2$ states. (The $I = 1$ state in this case is forbidden by Bose statistics.):

$$|++\rangle = \sqrt{\frac{2}{3}} |0\rangle + \sqrt{\frac{1}{3}} |2\rangle;$$

(43)

$$|00\rangle = \sqrt{\frac{2}{3}} |2\rangle - \sqrt{\frac{1}{3}} |0\rangle.$$

(44)

The reason for introducing pure isospin states, is that the matrix elements of transitions from $K^0$ and the $\bar{K}^0$ states to the same $(\pi\pi)_I$ state can be related by CPT–invariance plus Watson’s theorem on final state interactions. The relation in question is the following

$$e^{-2i\delta_I} \langle \text{I} | T | K^0 \rangle = (\langle \text{I} | T | \bar{K}^0 \rangle)^*,$$

(45)

where $\delta_I$ denotes the appropriate $J = 0$, isospin $I \pi\pi$ phase–shift at the energy of the neutral kaon mass.

The proof of this relation is rather simple. With $S = 1 + iT$, the unitarity of the $S$–matrix, $SS\dagger = 1$, implies

$$T\dagger T = i(T\dagger - T).$$

(46)

If one takes matrix elements of this operator relation between an initial state $K^0$, and a final $2\pi$–state with isospin $I$, we then have

$$\sum_F \langle \text{I} | T\dagger \langle F | T | K^0 \rangle = i\langle \text{I} | T\dagger | K^0 \rangle - i\langle \text{I} | T | K^0 \rangle,$$

(47)

where we have inserted a complete set of states $\sum | F\rangle\langle F \rangle = 1$ between $T$ and $T\dagger$. The crucial observation is that, in the strong interaction sector of the $S$–matrix, only the state $F = I$ can contribute to the $T\dagger$–matrix element. All the other states are suppressed by selection rules; e.g., the $3\pi$–states have opposite $G$–parity than
the $2\pi$–states; the $\pi l\nu$–states are not related to $2\pi$–states by the strong interactions alone; etc. Then, introducing the $\pi\pi$ phase–shift definition:

$$\langle I | S | I \rangle = e^{2i\delta_I},$$  \hspace{1cm} (48)

results in the relation

$$i(e^{-2i\delta_I} - 1)\langle I | T | K^0 \rangle = i\langle I | T | K^0 \rangle - i\langle I | T | K^0 \rangle^*, \hspace{1cm} \text{(49)}$$

We can next use CPT–invariance [recall Eq. (17), which in our case implies the relation: $\langle K^0 | T | I \rangle^* = (\langle I | T | \bar{K}^0 \rangle)^*$.] The result in Eq. (45) then follows.

As a consequence of the relation we have proved, we can use in full generality the following parametrization for $K^0(\bar{K}^0) \rightarrow (\pi\pi)_I$ amplitudes:

$$\langle I | T | K^0 \rangle = iA_I e^{i\delta_I}; \hspace{1cm} \text{(50)}$$

$$\langle I | T | \bar{K}^0 \rangle = -iA_I^* e^{i\delta_I}. \hspace{1cm} \text{(51)}$$

One possible quantity we can introduce to characterize the amount of CP–violation in $K \rightarrow 2\pi$ transitions is the parameter

$$\epsilon = \frac{A[K_L \rightarrow (\pi\pi)_{I=0}]}{A[K_S \rightarrow (\pi\pi)_{I=0}]}.$$  \hspace{1cm} (52)

This parameter is related to the $\tilde{\epsilon}$–parameter introduced in Eq. (6); as well as to the complex $A_0$-amplitude defined in (50) and (51), in the following way

$$\epsilon = \frac{(1 + \tilde{\epsilon})A_0 - (1 - \tilde{\epsilon})A_0^*}{(1 + \tilde{\epsilon})A_0 + (1 - \tilde{\epsilon})A_0^*}. \hspace{1cm} (53)$$

i.e.,

$$\epsilon = \frac{\tilde{\epsilon} + i\frac{\text{Im}A_0}{\text{Re}A_0}}{1 + i\frac{\text{Im}A_0}{\text{Re}A_0}}. \hspace{1cm} (54)$$

This is a good place to comment on the history of phase conventions in neutral $K$–decays. In their pioneering paper on the phenomenology of the $K - \bar{K}$ system, Wu and Yang \cite{wu-yang} chose to freeze the arbitrary relative phase between the $K^0$ and $\bar{K}^0$ states, with the choice $\text{Im}A_0 = 0$. With this convention, $\epsilon = \tilde{\epsilon}$. In fact, the parameter $\epsilon$ is phase–convention independent; while neither $\tilde{\epsilon}$, nor $A_I$ are. Indeed, under a small arbitrary phase change of the $K^0$–state:

$$| K^0 \rangle \rightarrow e^{-i\varphi} | K^0 \rangle,$$  \hspace{1cm} (55)

the parameters $A_I$, $M_{12}$, and $\tilde{\epsilon}$ change as follows:

$$\text{Im}A_I \rightarrow \text{Im}A_I - \varphi \text{Re}A_I; \hspace{1cm} (56)$$

$$\text{Im}M_{12} \rightarrow \text{Im}M_{12} + \varphi \Delta m; \hspace{1cm} (57)$$

$$\tilde{\epsilon} \rightarrow \tilde{\epsilon} + i\varphi; \hspace{1cm} (58)$$

$$-i\varphi,$$  \hspace{1cm} (58)
while $\epsilon$ remains invariant. The Wu–Yang phase convention was made prior to the development of the electroweak theory. In the standard model, the conventional way by which the freedom in the choice of relative phases of the quark–fields has been frozen, is not compatible with the Wu–Yang convention. Since $\epsilon$ is convention independent, we shall keep it as one of the fundamental parameters. Then, however, we need a second parameter which characterizes the amount of intrinsic CP–violation specific to the $K \to 2\pi$ decay, by contrast to the CP–violation in the $K^0 - \bar{K}^0$ mass–matrix. The parameter we are looking for has to be sensitive then to the lack of relative reality of the the two isospin amplitudes $A_0$ and $A_2$. This is the origin of the famous $\epsilon'$–parameter, which we shall next discuss.

In general, we can define three independent ratios of the $K_{L,S} \to (2\pi)_{I=0,2}$ transition amplitudes. One is the $\epsilon$–parameter in (52). Two other natural ratios are

$$\frac{A[K_L \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]} \quad \text{and} \quad \omega \equiv \frac{A[K_S \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]}.$$ (59)

Both ratios can be expressed in terms of the $\tilde{\epsilon}$–parameter introduced in Eq. (7), and the complex $A_I$–amplitudes defined in (50) and (51):

$$\frac{A[K_L \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]} = \frac{(1 + \tilde{\epsilon})A_2 - (1 - \tilde{\epsilon})A_2^* e^{i(\delta_2 - \delta_0)}}{(1 + \tilde{\epsilon})A_0 + (1 - \tilde{\epsilon})A_0^*} = \frac{\text{Re} A_2 + \tilde{\epsilon} \text{Im} A_2}{1 + i\tilde{\epsilon} \frac{\text{Im} A_2}{\text{Re} A_0}} e^{i(\delta_2 - \delta_0)}. \quad (60)$$

and

$$\omega \equiv \frac{A[K_S \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]} = \frac{(1 + \tilde{\epsilon})A_2 + (1 - \tilde{\epsilon})A_2^* e^{i(\delta_2 - \delta_0)}}{(1 + \tilde{\epsilon})A_0 + (1 - \tilde{\epsilon})A_0^*} = \frac{\text{Re} A_2 + \tilde{\epsilon} \text{Im} A_2}{1 + i\tilde{\epsilon} \frac{\text{Im} A_2}{\text{Re} A_0}} e^{i(\delta_2 - \delta_0)}. \quad (61)$$

The $\epsilon'$–parameter is then defined as the following combination of these ratios:

$$\epsilon' = \frac{1}{\sqrt{2}} \left( \frac{A[K_L \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]} - \epsilon \times \omega \right). \quad (62)$$

From these results, and using the expression for $\epsilon$ we obtained in Eq. (54), we finally get

$$\epsilon' = \frac{i}{\sqrt{2}} \frac{(1 - \tilde{\epsilon}^2)e^{i(\delta_2 - \delta_0)}}{(\text{Re} A_0 + i\tilde{\epsilon} \text{Im} A_0)^2} (\text{Im} A_2 \text{Re} A_0 - \text{Im} A_0 \text{Re} A_2), \quad (63)$$

an expression which clearly shows the proportionality to the lack of relative reality between the $A_0$ and $A_2$ amplitudes.
We shall next establish contact with the parameters \( \eta_{+-} \) and \( \eta_{00} \), which were introduced in Eq. (40, and which are directly accessible to experiment. Using Eqs. (43), (44), as well as the definitions of \( \epsilon, \epsilon' \), and \( \omega \) above, one finds

\[
\eta_{+-} = \epsilon + \epsilon' \frac{1}{1 + \frac{1}{\sqrt{2}} \omega}; \tag{64}
\]

\[
\eta_{00} = \epsilon - 2\epsilon' \frac{1}{1 - \sqrt{2}\omega}. \tag{65}
\]

So far, we have made no approximations in our phenomenological analysis of the \( K^0 - \bar{K}^0 \) mass–matrix and \( K \to 2\pi \) decays. It is however useful to try to thin down in some way the exact expressions we have derived, by taking into account the relative size of the various phenomenological parameters which appear in the expressions above. The strategy will be to neglect first, terms which are products of CP–violation parameters. For example, in Eq. (61), we have introduced the parameter \( \omega \), which a priori we can reasonably expect to be dominated by the term

\[
\omega \simeq \frac{\text{Re}A_2}{\text{Re}A_0} e^{i(\delta_2 - \delta_0)}. \tag{66}
\]

We can justify this approximation by the fact that non–leptonic \( \Delta I = \frac{3}{2} \) transitions, although suppressed with respect to the \( \Delta I = \frac{1}{2} \) transitions, are nevertheless larger than the observed CP–violation effects. Notice that the amplitude \( A_2 \) is responsible for the deviation from an exact \( \Delta I = \frac{1}{2} \) rule. The ratio \( \frac{\text{Re}A_2}{\text{Re}A_0} \) can be obtained from the experimentally known branching ratios \( \Gamma(K_S \to \pi^+\pi^-) \) and \( \Gamma(K_S \to \pi^0\pi^0) \). More precisely, correcting for the phase–space effects, one must compare the normalized decay rates:

\[
\gamma(1, 2) = \frac{\Gamma(K \to \pi_1\pi_2)}{16\pi M \sqrt{1 - \frac{(m_1 + m_2)^2}{M^2}} \sqrt{1 - \frac{(m_1 - m_2)^2}{M^2}}}, \tag{67}
\]

where the denominator here is the two–body phase space factor for the mode \( K \to \pi_1\pi_2 \), (\( M \) is the mass of the \( K \)-particle and \( m_{1,2} \) the pion masses.) Then, we have

\[
\frac{\gamma_S(+-)}{2\gamma_S(00)} = 1 + 3\sqrt{2} \frac{\text{Re}A_2}{\text{Re}A_0} \cos(\delta_2 - \delta_1) + \mathcal{O}(\frac{\alpha}{\pi}). \tag{68}
\]

Experimentally, from the PDB \[3\], one finds

\[
\frac{\gamma_S(+-)}{2\gamma_S(00)} = 1.109 \pm 0.012, \tag{69}
\]

and using the present experimental information on \( (\delta_2 - \delta_1) \), [see the discussion in Sec. 5.,] we find, with neglect of radiative corrections

\[
\frac{\text{Re}A_2}{\text{Re}A_0} = (+22.2)^{-1}. \tag{70}
\]
I shall later discuss some of the qualitative dynamical explanations, within the standard model, of how this small number appears. It is fair to say however, that a reliable calculation of this ratio is still lacking at present.

Using the approximations

$$\tilde{\epsilon} \mathrm{Im}A_0 \ll \mathrm{Re}A_0 \quad \text{and} \quad \tilde{\epsilon}^2 \ll 1,$$

we can rewrite $\epsilon'$ in a simpler form

$$\epsilon' \simeq \frac{1}{\sqrt{2}} e^{i(\delta_2 - \delta_0 + \frac{\pi}{2})} \frac{\mathrm{Re}A_2}{\mathrm{Re}A_0} \left( \frac{\mathrm{Im}A_2}{\mathrm{Re}A_0} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} \right),$$

(72)

clearly showing the fact that $\epsilon'$ is proportional to direct CP-violation in $K \rightarrow 2\pi$ transitions and is also suppressed by the $\Delta I = \frac{1}{2}$ selection rule.

The same approximations in Eq. (71), when applied to $\epsilon$, lead to

$$\epsilon \simeq \tilde{\epsilon} + i \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}. $$

(73)

Let us next go back to the mass matrix equations in (21) which, expanding in powers of $\tilde{\epsilon}$, we can rewrite as follows

$$1 - 2\tilde{\epsilon} \simeq \frac{\mathrm{Re}M_{12} - \frac{i}{2}\mathrm{Re}\Gamma_{12}}{\frac{1}{2}(\Delta m + \frac{i}{2}\Delta \Gamma)} - i \frac{\mathrm{Im}M_{12} - \frac{i}{2}\mathrm{Im}\Gamma_{12}}{\frac{1}{2}(\Delta m + \frac{i}{2}\Delta \Gamma)}. $$

(74)

To a first approximation, neglecting CP-violation effects altogether, we find that

$$\mathrm{Re}M_{12} \simeq \frac{\Delta m}{2} \quad \text{and} \quad \mathrm{Re}\Gamma_{12} \simeq -\frac{\Delta \Gamma}{2}. $$

(75)

If furthermore, we restrict the sum over intermediate states in $\Gamma_{12}$ [see Eq. (15)] to $2\pi$-states, an approximation which we have already seen to be rather good [see Eqs. (37) and (38)] we can write

$$\Gamma_{12} \simeq (-iA_0^*e^{i\delta_0})^*iA_0e^{i\delta_0} = -(\mathrm{Re}A_0 + i\mathrm{Im}A_0)^2, $$

(76)

from where it follows that

$$\frac{\mathrm{Im}\Gamma_{12}}{\mathrm{Re}\Gamma_{12}} \simeq \frac{2\mathrm{Re}A_0\mathrm{Im}A_0}{\mathrm{Re}A_0^2 + \mathrm{Im}A_0^2} \simeq \frac{2\mathrm{Im}A_0}{\mathrm{Re}A_0}. $$

(77)

Then, using the empirical fact that $\Delta m \simeq \frac{\Delta \Gamma}{2}$, and $\Gamma_L \ll \Gamma_S$, we finally arrive at the simplified expression

$$\tilde{\epsilon} \simeq \frac{1}{1 + i \left( \frac{\mathrm{Im}M_{12}}{\Delta m} + \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} \right)}. $$

(78)
and, using Eq. (73),
\[ \epsilon \simeq \frac{1}{\sqrt{2}} e^{i \frac{\pi}{4}} \left( \frac{\text{Im} M_{12}}{\Delta m} + \frac{\text{Im} A_0}{\text{Re} A_0} \right). \] (79)

This is as much as one can do, within a strict phenomenological analysis of the CP–violation in K–decays. We have reduced the problem to the knowledge of two parameters: \( \epsilon \) in Eq. (73), and \( \epsilon' \) in Eq. (72). We shall come back to these parameters in Sec. 5. There, we shall discuss what predictions for these fundamental parameters can be made at present within the framework of the Standard Model. As we shall see, the main difficulty comes from the lack of quantitative understanding of the low–energy sector of the strong interactions. In terms of QCD, the sector in question is the one of the interactions between the states with lowest masses: the octet of the pseudoscalar particles (\( \pi, K, \eta \)). It seems therefore appropriate to examine the possibility of describing the interactions of these particles within the framework of an effective Lagrangian of the Standard Model at very low–energies. This will be the subject of the next three lectures.

2 Introduction to Chiral Perturbation Theory

Chiral perturbation theory (\( \chi \)PT) is the effective field theory of quantum chromodynamics (QCD) at low energies. In this lecture I shall give an overview of the basic ideas behind the \( \chi \)PT–approach to hadron dynamics at low energies. This will give us the basis for a study of possible applications of the \( \chi \)PT–approach to the non–leptonic weak interactions of \( K \)–particles, which will then be the subject of the next lecture.

There have already been several sets of TASI–lectures in previous years dedicated to \( \chi \)PT. The reader should consult them for complementary information\[6,7,8,9\]. Some recent review articles can be found in\[10,11\].

2.1 An Overview of the Basic Ideas in the \( \chi \)PT-Approach.

In the limit where the heavy quark fields \( t, b, \) and \( c \) are integrated out, and the masses of the light quarks \( u, d \) and \( s \) are set to zero, the QCD Lagrangian is invariant under global \( SU(3)_L \times SU(3)_R \) rotations \( (V_L, V_R) \) of the left–handed and right–handed quark triplets
\[ q_L \equiv \frac{1-\gamma_5}{2} q \quad \text{and} \quad q_R \equiv \frac{1+\gamma_5}{2} q ; \quad q = u, d, s. \] (80)
\[ q_L \to V_L q_L \quad \text{and} \quad q_R \to V_R q_R. \] (81)

Formally, the global symmetry of the QCD–Lagrangian is in fact larger. The full symmetry group is \( SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A \). The \( U(1)_A \) symmetry is broken however at the quantum level by the abelian axial anomaly; the Adler–Bell–Jackiw\[12\] anomaly. The \( U(1)_V \) quark–number symmetry is trivially realized
in the mesonic sector. At the level of the hadronic spectrum, this chiral–$SU(3)$ symmetry of the QCD–Lagrangian is however not apparent. Although the low–lying hadronic states can indeed be neatly classified in irreducible representations of the famous $SU(3)$ symmetry of the Eightfold Way, there do not appear degenerate multiplets with opposite parity. In QCD it is therefore expected, (and there are some good theoretical reasons for it), as well as numerical evidence from lattice QCD simulations, that the chiral–$SU(3)$ global symmetry is spontaneously broken down to the diagonal $SU(3)_{V=L+R}$ group of the Eightfold Way. This pattern of spontaneously broken symmetry implies specific constraints on the dynamics of the strong interactions between the low–lying pseudoscalar states ($\pi, K, \eta$), which are the massless Nambu–Goldstone bosons associated to the “broken” chiral generators. As a result of the spontaneous symmetry breaking, there appears a mass–gap in the hadronic spectrum between the ground state of the octet of $0^−$–pseudoscalars and the lowest hadronic states which become massive in the chiral limit $m_u = m_d = m_s = 0$; i.e., the octet of $1^−$ vector–meson states and the octet of $1^+$ axial–vector–meson states.

The basic idea of the $\chi$PT–approach is that in order to describe the physics at energies within this gap region, it may be more useful to formulate the strong interactions of the low–lying pseudoscalar particles in terms of an effective low–energy Lagrangian of QCD, with the octet of Nambu–Goldstone fields ($\chi$ are the eight $3 \times 3$ Gell–Mann matrices, with $\text{tr} \lambda^a \lambda^b = 2 \delta_{ab}$)

$$\Phi(x) = \frac{\chi}{\sqrt{2}} \cdot \chi(x) = \begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^- & \overline{K^0} & -2\eta/\sqrt{6} \end{pmatrix},$$ (82)

as explicit degrees of freedom, rather than in terms of the quark and gluon fields of the usual QCD Lagrangian. In the conventional formulation, the Nambu–Goldstone fields are collected in a unitary $3 \times 3$ matrix $U(x)$ with $\text{det} U = 1$, which under chiral–$SU(3)$ transformations ($V_L, V_R$) is chosen to transform linearly:

$$U \rightarrow V_R U V_L^\dagger.$$ (83)

The effective Lagrangian we look for has to be then a sum of chirally invariant terms with increasing number of derivatives of $U$. For example, to lowest order in the number of derivatives, only one independent term can be constructed which is invariant under ($V_L, V_R$) transformations:

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{4} f_\pi^2 \text{tr} \partial_\mu U(x) \partial^\mu U^\dagger(x),$$ (84)

where, as I shall soon explain in more detail, the normalization $f_\pi$ is fixed in such a way that the corresponding axial–current with the quantum numbers of the pion, obtained from this Lagrangian, is the one which induces the $\pi \rightarrow \mu \nu$ transition. An
explicit representation of $U$ is

$$U(x) = \exp \left( -i \frac{1}{f_\pi} \vec{\lambda} \cdot \vec{\varphi}(x) \right);$$  \hspace{1cm} (85)$$

and, from experiment,

$$f_\pi \simeq 92.4 \text{MeV}. \hspace{1cm} (86)$$

Because of the non-linearity in $\varphi$, processes with different numbers of pseudoscalar mesons are then related. These are the successful current–algebra relations of the 60’s which the effective Lagrangian formulation above [14] incorporates in a compact way.

The low–energy effective Lagrangian $\mathcal{L}_{\text{eff}}^{(2)}$ describes physical amplitudes by means of the lowest order term in a Taylor expansion in powers of momenta:

$$\mathcal{A}(p_1, p_2, \ldots) = \sum a_{ij}^{(2)} p_i \cdot p_j + \mathcal{O}(p^4).$$  \hspace{1cm} (87)$$

The expansion has no constant $\mathcal{O}(p^0)$ term, due to the fact that the lowest order Lagrangian in (84) has two derivatives. As an exercise you should check some examples, like elastic $\pi^+\pi^0$ scattering, where the induced amplitude one finds is:

$$\mathcal{A}(p_+, p_0, p'_+, p'_0) = \frac{(p'_+ - p_+)^2}{f_\pi^2}.$$  \hspace{1cm} (88)$$

We expect this effective Lagrangian description to be useful for values of invariants of momenta sufficiently small as compared to the scale $\Lambda_\chi$ where spontaneous chiral symmetry breaking (S$\chi$SB) occurs in QCD:

$$p^2/\Lambda^2_\chi \ll 1.$$  \hspace{1cm} (89)$$

It seems also reasonable to expect $\Lambda_\chi$ to be of the same order of magnitude as the masses of the lowest states which become massive due to S$\chi$SB, i.e.,

$$M_\rho(770\text{MeV}) \leq \Lambda_\chi \leq M_A(1260\text{MeV}).$$  \hspace{1cm} (90)$$

As we shall see, these expected features are justified phenomenologically.

It is convenient to promote the global chiral–$SU(3)$ symmetry to a local $SU(3)_L \times SU(3)_R$ gauge symmetry. This can be accomplished by adding appropriate quark bilinear couplings with external field sources to $\mathcal{L}^{(0)}_{QCD}$, the usual QCD Lagrangian with massless quarks:

$$\mathcal{L}_{QCD}(x) = \mathcal{L}^{(0)}_{QCD}(x) + \bar{\psi}(x)\gamma^\mu[v_\mu(x) + \gamma_5 a_\mu(x)]q(x) - \bar{\psi}(x)[s(x) - i\gamma_5 p(x)]q(x).$$  \hspace{1cm} (91)$$

The external field sources $v_\mu$, $a_\mu$, $s$ and $p$ are Hermitian $3 \times 3$ matrices in flavour, and colour singlets. Under chiral–$SU(3)$ gauge transformations ($V_L(x)$, $V_R(x)$), they are required to transform as follows:

$$l_\mu \equiv v_\mu - a_\mu \rightarrow V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger,$$  \hspace{1cm} (92)$$

$$r_\mu \equiv v_\mu + a_\mu \rightarrow V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger.$$  \hspace{1cm} (93)$$

13
and

\[ s + ip \rightarrow V_R(s + ip)V_L^\dagger. \]  (94)

In the presence of these external field sources, the possible terms in \( \mathcal{L}_{\text{eff}}^{(2)} \) with the lowest chiral dimension, i.e., \( \mathcal{O}(p^2) \) are

\[ \mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{4} f_\pi^2 \left\{ \text{tr} D_\mu U D^\mu U^\dagger + \text{tr} \left( \chi U^\dagger + U \chi^\dagger \right) \right\}, \]  (95)

where \( D_\mu \) denotes the covariant derivative

\[ D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu, \]  (96)

and

\[ \chi = 2B[s(x) + ip(x)], \]  (97)

with \( B \) a constant, which like \( f_\pi \), is not fixed by symmetry requirements alone. Once special directions in flavour space (like the ones selected by the electroweak Standard Model couplings) are fixed for the external fields, the chiral symmetry is then explicitly broken. In particular, the choice

\[ s + ip = \mathcal{M} = \text{diag} (m_u, m_d, m_s) \]  (98)

takes into account the explicit breaking due to the quark masses in the underlying QCD Lagrangian.

Notice that the quark bilinear \( \bar{q}_L^j(x) q_R^i(x) \) has the same transformation properties, under chiral–SU(3), as the \( U(x) \)–matrix field. In fact, \( U(x) \) can be viewed as the low–energy parametrization of the Nambu–Goldstone field excitations of the vacuum. To lowest order in the chiral expansion, the constant \( B \) appears then as the parameter which fixes the relative normalization to the light quark condensate:

\[ \langle 0 | \bar{q}^i q^j | 0 \rangle = -f_\pi^2 B \delta_{ij}. \]  (99)

The relation between the physical pseudoscalar masses and the quark masses, to lowest order in the chiral expansion, is fixed by identifying quadratic terms in \( \varphi \) in the expansion of the second term in (95), with the result

\[ \chi = \begin{pmatrix} m_{\pi^0}^2 + M_{K^+}^2 - M_{K^0}^2 & 0 & 0 \\ 0 & m_{\pi^0}^2 - M_{K^+}^2 + M_{K^0}^2 & 0 \\ 0 & 0 & -m_{\pi^0}^2 + M_{K^+}^2 + M_{K^0}^2 \end{pmatrix}. \]  (100)

The mass matrix in the \( \pi^0, \eta \) basis is not diagonal, because of a small admixture between these two fields. The appropriate diagonalization leads to the mass eigenvalues:

\[ M_{\pi^0}^2 = 2\hat{m} B - \epsilon + \mathcal{O}(\epsilon^2), \]  (101)

\[ M_\eta^2 = \frac{2}{3}(\hat{m} + 2m_s) B + \epsilon + \mathcal{O}(\epsilon^2), \]  (102)
where
\[ \epsilon = \frac{B(m_u - m_d)^2}{4(m_s - m)} , \]
and
\[ \hat{m} \equiv \frac{1}{2}(m_u + m_d) . \]

There are a number of important relations which follow from these lowest order results:

- The Gell-Mann–Oakes–Renner relation\(^ {19}\)
  \[ f_\pi^2 M_\pi^2 = -\hat{m} < 0|\bar{u}u + \bar{d}d|0> . \] \hspace{1cm} (105)

- The Current Algebra mass ratios\(^ {19, 20}\)
  \[ \frac{B}{2\hat{m}} = \frac{M_{K^+}}{(m_u + m_d)} = \frac{M_K^0}{(m_d + m_s)} \approx \frac{3 M_{\eta_s}^2}{2\hat{m} + 4m_s} . \] \hspace{1cm} (106)

- The Gell-Mann–Okubo mass formula\(^ {21, 22}\)
  \[ 3 M_\eta_s^2 = 4 M_K^2 - M_\pi^2 . \] \hspace{1cm} (107)

Barring the possibility that the parameter \( B \) may be unexpectedly small (see however\(^ {23}\), and references thereof,) and neglecting the small \( O(\epsilon) \) effects, we are led to two quark mass ratios estimates\(^ {23}\):

\[ \frac{m_d - m_u}{m_d + m_u} = \frac{M_{K^0}^2 - M_{K^+}^2 - (M_{\pi^0}^2 - M_{\pi^+}^2)_{EM}}{M_{\pi^0}^2} = 0.29 ; \] \hspace{1cm} (108)

and

\[ \frac{m_s - \hat{m}}{2\hat{m}} = \frac{M_{K^0}^2 - M_{\pi^0}^2}{M_{\pi^0}^2} = 12.6 , \] \hspace{1cm} (109)

where we have subtracted the pion squared mass difference, to take into account the electromagnetic contribution to the charged pseudoscalar mesons self–energies. This electromagnetic contribution, in the chiral limit \((m_{u,d,s} = 0,)\) gives a common mass to the charged \( K^+ \) and \( \pi^+ \) mesons.

As already pointed out, in the case of the Standard Model, not all the external gauge field sources \( l_\mu \) and \( r_\mu \) correspond to physical gauge fields. In the same way that the scalar and pseudoscalar sources are frozen to the quark mass matrix, as indicated in\(^ {98}\), the explicit chiral symmetry breaking induced by the electroweak currents of the Standard Model corresponds to the following choice:

\[ r_\mu = eQ_R[A_\mu - \tan \vartheta_W Z_\mu]; \]

\[ l_\mu = eQ_R[A_\mu - \tan \vartheta_W Z_\mu] + \frac{e}{\sin \vartheta_W} Q_L^{(3)} Z_\mu \]
\[ + \frac{e}{\sqrt{2} \sin \vartheta_W} [Q_L^{(+)} W_\mu^{(+)} + Q_L(-) W_\mu^{(-)}] . \] \hspace{1cm} (111)
Here the $Q$’s are the electroweak matrices:

$$Q_L = Q_R = Q = \frac{1}{3} \text{diag}(2, -1, -1), \quad Q_L^{(3)} = \text{diag}(1, -1, -1); \quad (112)$$

and

$$Q_L^{(+)} = (Q_L^{(-)})^\dagger = \begin{pmatrix}
0 & V_{ud} & V_{us} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}. \quad (113)$$

We can now compute, in a straightforward way, the chiral realization—to lowest order in the chiral expansion—of the electroweak currents of the Standard Model. They follow by taking appropriate variations of the lowest order effective action $\Gamma^{(2)} = \int d^4x L^{(2)}(x)$, with respect to the external field sources:

$$J_L^\mu \equiv \bar{q}_L \gamma^\mu q_L \equiv \frac{\delta L^{(2)}}{\delta l^\mu} = \frac{i}{2} f_{\pi}^2 (D^\mu U) U^\dagger, \quad (114)$$

$$J_R^\mu \equiv \bar{q}_R \gamma^\mu q_R = \frac{\delta L^{(2)}}{\delta r^\mu} = \frac{i}{2} f_{\pi}^2 (D^\mu U) U^\dagger. \quad (115)$$

Expanding $U$ in powers of $\Phi$-matrix fields [see Eqs.(82) and (85)], we have:

$$J_L^\mu = f_\pi \frac{1}{\sqrt{2}} D^\mu \Phi - \frac{i}{2} [\Phi (D^\mu \Phi) - (D^\mu \Phi) \Phi] + \mathcal{O}(\Phi^3), \quad (116)$$

$$J_R^\mu = - f_\pi \frac{1}{\sqrt{2}} D^\mu \Phi - \frac{i}{2} [\Phi (D^\mu \Phi) - (D^\mu \Phi) \Phi] + \mathcal{O}(\Phi^3). \quad (117)$$

Let us now consider the decay $\pi^+ \rightarrow \mu^+ \nu_\mu$ we mentioned earlier as an example. The hadronic matrix element can be readily computed using the form of the axial current deduced from the expressions just above, with the result

$$< 0|(J_A^\mu)_{12}|\pi^+ > = i \sqrt{2} f_\pi p^\mu, \quad (118)$$

explicitly showing the physical meaning of the $f_\pi$ constant in the chiral Lagrangian.

You are now in the position of being able to calculate any observable you wish to lowest order in the chiral expansion; but I am sure you are more ambitious. How does one calculate higher order chiral corrections? This is what we shall now discuss in the next subsection.

### 2.2 Chiral Perturbation Theory to $\mathcal{O}(p^4)$

In QCD, the generating functional $Z[v, a, s, p]$ of the Green’s functions of colour singlet quark currents, is defined via the path–integral formula

$$\exp \{ iZ[v, a, s, p] \} = \int Dq D\bar{q} D\mathcal{G} \exp \left\{ i \int d^4x L_{QCD} \right\}, \quad (119)$$
with \( \mathcal{L}_{QCD} \) defined as in eq.(91). The physical Green’s functions of a specific flavour are then obtained by functional derivatives with respect to the appropriate external field sources \( v, a, s, \) and \( p \). The chiral symmetry properties which we have discussed imply that, at sufficiently small energies, there exists an effective Lagrangian \( \mathcal{L}_{\text{eff}} \) of the Nambu–Goldstone field modes alone, in the presence of external field sources, such that

\[
\exp \left\{ iZ[v, a, s, p] \right\} = \int \mathcal{D}U[\Phi(x)] \exp \left\{ i \int d^4x \mathcal{L}_{\text{eff}}[U; v, a, s, p] \right\}.
\]

(120)

To lowest order in the chiral expansion, \( \mathcal{O}(p^2) \) as we have seen, the generating functional reduces to the classical action

\[
Z^{(2)}[v, a, s, p] = \int d^4x \mathcal{L}_{\text{eff}}^{(2)}(U; v, a, s, p),
\]

(121)

with \( \mathcal{L}_{\text{eff}}^{(2)} \) as defined in eq.(95).

To next–to–leading order in the chiral expansion, i.e.\( \mathcal{O}(p^4) \), the generating functional \( Z[v, a, s, p] \) gets contributions from three different sources:

- The most general local effective chiral Lagrangian of \( \mathcal{O}(p^4) \).
- The one–loop functional generated from the lowest–order \( \mathcal{L}_{\text{eff}}^{(2)} \) Lagrangian.
- The well known Wess–Zumino–Witten functional \( ^{[24,25]} \) induced by the non–abelian chiral anomaly \( ^{[26]} \).

2.2.1 *The Chiral Lagrangian of \( \mathcal{O}(p^4) \)*

The ingredients we have at our disposal to build this Lagrangian, their chiral transformation properties, as well as their chiral power counting are as follows:

\[
\begin{align*}
U, & \quad V_R U V_L^\dagger, & \quad \mathcal{O}(p^0); \\
D_\mu U, & \quad V_R D_\mu U V_L^\dagger, & \quad \mathcal{O}(p); \\
\chi, & \quad V_R \chi V_L^\dagger, & \quad \mathcal{O}(p^2); \\
F_{\mu\nu}^L, & \quad V_L F_{\mu\nu}^L V_L^\dagger, & \quad \mathcal{O}(p^2); \\
F_{\mu\nu}^R, & \quad V_R F_{\mu\nu}^R V_R^\dagger, & \quad \mathcal{O}(p^2).
\end{align*}
\]

Here, \( F_{\mu\nu}^L \) and \( F_{\mu\nu}^R \) are the strength field tensors associated to the external \( l_\mu \) and \( r_\mu \) sources:

\[
F_{\mu\nu}^L = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu], \quad \text{and} \quad F_{\mu\nu}^R = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu].
\]

(122)

Furthermore, since we shall only use \( \mathcal{L}_{\text{eff}}^{(4)} \) at tree level, we can eliminate possible terms using the \( \mathcal{O}(p^2) \) equations of motion obeyed by the \( U \)-functional:

\[
(\Box U)U^\dagger - U(\Box U^\dagger) = \chi U^\dagger - U\chi^\dagger - \frac{1}{3} \text{tr}(\chi U^\dagger - U\chi^\dagger).
\]

(123)
Remember also that traceless $3 \times 3$ matrices $A_i$ obey the constraint
\[
\text{tr}A_1A_2A_3A_4 + \text{tr}A_1A_3A_2A_4 + \text{tr}A_1A_4A_3A_2 \\
+ \text{tr}A_1A_2A_4A_3 + \text{tr}A_1A_3A_4A_2 + \text{tr}A_1A_4A_2A_3 + \\
- \text{tr}A_1A_2\text{tr}A_3A_4 - \text{tr}A_1A_3\text{tr}A_2A_4 - \text{tr}A_1A_4\text{tr}A_2A_3 = 0. \quad (124)
\]

Other constraints which we must also impose are that only terms which are invariant under parity and charge conjugation should be allowed. The most general Lagrangian which satisfies all these conditions is the following:
\[
L_{\text{eff}}^{(4)} = L_1 \text{tr} \left( D_\mu U^\dagger D^\nu U \right)^2 + L_2 \text{tr} D_\mu U^\dagger D_\nu U \text{tr} D^\mu U^\dagger D^\nu U + \\
L_3 \text{tr} D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U + \\
L_4 \text{tr} D_\mu U^\dagger D^\mu U \text{tr} \left( \chi U + U^\dagger \chi \right) + L_5 \text{tr} D_\mu U^\dagger D^\mu U \left( \chi U + U^\dagger \chi \right) + \\
L_6 \left[ \text{tr} \left( \chi U + U^\dagger \chi \right) \right]^2 + L_7 \left[ \text{tr} \left( \chi U - U^\dagger \chi \right) \right]^2 + \\
L_8 \left( U^\dagger \chi U \chi^\dagger + U^\dagger \chi U^\dagger \chi \right) + \\
iL_9 \text{tr} \left( F^\mu_\nu D_\mu U D_\nu U^\dagger + F^\mu_\nu D_\mu U^\dagger D_\nu U \right) + L_{10} \text{tr} U^\dagger F^\mu_\nu U F_{\mu\nu} + \\
H_1 \text{tr} \left( F^\mu_\nu F_{\mu\nu} + F^\mu_\nu F_{\mu\nu} \right) + H_2 \text{tr} \chi \chi^\dagger. \quad (125)
\]

In this Lagrangian the parameters $L_i$, $i = 1, 2, 3, \ldots, 10$ are dimensionless coupling constants, which like $f_\pi$ and $B$ in the lowest order effective Lagrangian, are not fixed by chiral symmetry requirements alone. The terms proportional to the coupling constants $H_1$ and $H_2$ involve only the external fields. As a result these coupling constants cannot be fixed from low–energy observables alone. Only if we had a detailed knowledge of the dynamics of how the effective chiral Lagrangian emerges from the underlying QCD Lagrangian could we give an unambiguous definition of $H_1$ and $H_2$. By contrast, as we shall later discuss, most of the other couplings can be fixed from low energy observables. Of course, in QCD, the $L_i$ constants, much the same as $f_\pi$ and $B$, are in principle calculable parameters in terms of the intrinsic $\Lambda_{QCD}$ scale only. We shall come back to this important question later in Sec. 4. For the time being we shall only be interested in the phenomenological determination of these constants. However, for that, we need to discuss first the possible contributions to low–energy observables from chiral loops.

2.2.2 Chiral Loops

Simple power counting tells us that loops generated by the lowest order Lagrangian are badly divergent. This is not a surprise: the non–linear sigma model in 4–
dimensions is not renormalizable i.e., an infinite number of local counter–terms are required. Here, however, we are considering the chiral Lagrangian as an effective field theory for low energies. Order by order in the momentum expansion of the effective theory, it is possible to specify a renormalizable framework. Of course, as we go to higher and higher powers of momentum, more and more local counter–terms will appear, with couplings which are not fixed by chiral symmetry properties alone, and eventually the predictive power of the effective field theory formulation will therefore disappear. To define the loop integrals it is necessary to fix a regularization which preserves the symmetries of the Lagrangian. The well known dimensional regularization technique does the job beautifully. Since by construction, the $\mathcal{O}(p^4)$ Lagrangian $\mathcal{L}_{\text{eff}}^{(4)}$ contains all possible terms which are allowed by chiral invariance, all the one loop divergences –which by power counting can only give rise to local $\mathcal{O}(p^4)$ terms– can all be absorbed by suitable renormalizations of the $L_i$ and $H_{1,2}$ constants. This program has been explicitly realized in the papers of Gasser and Leutwyler 28, 27, 29. In fact, these authors have explicitly constructed the one–loop functional $Z_{\text{loop}}^{(4)}[v, a, s, p]$ at the required level of non–locality needed for all practical calculations. They do the path integral around the classical functional $U(\Phi_{\text{cl}}[v, a, s, p])$, solution of the equations of motion in (123) with the boundary condition $U(0) = 1$. The method consists in expanding $\mathcal{L}_{\text{eff}}^{(2)}(U; v, a, s, p)$ around $\Phi = \Phi_{\text{cl}}$ and then doing the functional integral over the fluctuations $\xi = \Phi - \Phi_{\text{cl}}$. In order to obtain the one–loop effective action it is sufficient to expand $\mathcal{L}_{\text{eff}}^{(2)}$ to $O[(\xi)^2]$:

$$
\Gamma_{\text{eff}}^{(2)} = \int d^4x \mathcal{L}_{\text{eff}}^{(2)}[\Phi_{\text{cl}}] - \frac{f_2^2}{2}(\xi, D\xi) + O(\xi^3),
$$

where $D$ is a known, but rather complicated, differential operator acting on the fluctuating $\xi$–variables. The one–loop functional is then embodied in the master formula:

$$
Z_{\text{loop}}^{(4)}[v, a, s, p] = \frac{i}{2} \text{tr log} \, D. \quad (127)
$$

It is relatively easy to extract the singular part of $Z_{\text{loop}}^{(4)}[v, a, s, p]$. It corresponds to a local effective Lagrangian, exactly like the one in eq.(125), with couplings $L_i$ and $H_j$:

$$
L_i^{\text{loop}} = \Gamma_i \Lambda^{\text{loop}}, \quad i = 1, 2, 3, ... 10; \quad H_i^{\text{loop}} = \tilde{\Gamma}_j \Lambda^{\text{loop}}, \quad j = 1, 2, \quad (128)
$$

where

$$
\Lambda^{\text{loop}} = \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left[ \log(4\pi) + \Gamma'(1) + 1 \right] \right\}, \quad j = 1, 2; \quad (129)
$$

and $\Gamma_i$, $\tilde{\Gamma}_j$ have the following rational values:

$$
\Gamma_1 = \frac{3}{32}, \quad \Gamma_2 = \frac{3}{16}, \quad \Gamma_3 = 0, \quad \Gamma_4 = \frac{1}{8}.
$$
The renormalized couplings $L_i^r(\mu)$ and $H_j^r(\mu)$, are then defined by subtracting from the tree level $L_i$ and $H_j$ the one–loop singular contributions:

$$L_i = L_i^r(\mu) + \Gamma_i \Lambda_{\text{loop}}, \quad \text{and} \quad M_j = H_j^r(\mu) + \tilde{\Gamma}_j \Lambda_{\text{loop}}.$$  

The renormalized coupling constants depend of course on the scale $\mu$ introduced by the dimensional regularization. The running in $\mu$ is governed by the coefficients $\Gamma_i$ (and $\tilde{\Gamma}_j$), which play the rôle of one–loop $\beta$–functions:

$$L_i^r(\mu) = L_i^r(\mu') + \frac{\Gamma_i}{16\pi^2} \log \frac{\mu'}{\mu}. \quad (132)$$

The $\mu$–scale dependence cancels however in the full $O(p^4)$ calculation of a given physical observable. The non-polynomial contribution to a specific physical process will in general have a logarithmic $\mu$–scale dependence – the so called chiral logarithms – which cancels with the $\mu$–dependence of the tree level contribution modulated by the $L_i(\mu)$–constants. Let us consider a typical example to illustrate this feature: the electromagnetic mean squared radius of the pion. The pion electromagnetic form factor, when expanded in a Taylor series in powers of the momentum transfer, has the following structure:

$$F^{\pm}(q^2) = 1 + \frac{1}{6} < r^2 >^{\pm} q^2 + \cdots. \quad (133)$$

The result of the calculation of $< r^2 >^{\pm}$, to lowest non–trivial $O(p^4)$ in $\chi$PT is

$$< r^2 >^{\pm} = \frac{12}{f_\pi^2} L_9^r(\mu) - \frac{1}{16\pi^2 f_\pi^2} \left[ \log \frac{M_{\pi}^2}{\mu^2} + \frac{1}{2} \log \frac{M_K^2}{\mu^2} + \frac{3}{2} \right]. \quad (134)$$

The first term in the r.h.s. shows the tree level contribution from the renormalized $\mathcal{L}_{\text{eff}}^{(4)}$ Lagrangian; the rest of the terms are generated by the –finite though $\mu$–dependent– contribution from the Feynman diagram loops (one–pion loop and one–kaon loop,) generated by the lowest $\mathcal{L}_{\text{eff}}^{(2)}$ Lagrangian. The scaling law of the $L_9^r(\mu)$–constant, as shown in (132), is the same as the one of the chiral logarithm, and therefore the whole contribution is scale invariant, as it should be.

There are a number of interesting generic features which emerge from this example, and which we next wish to point out:

- The factor $\frac{q^2}{16\pi^2 f_\pi^2}$ is a characteristic factor of the loop–expansion. More precisely, this factor appears modulated by the number of active flavour loops $n_f$. Therefore, we expect chiral loops to contribute $O[n_f \frac{q^2}{16\pi^2 f_\pi^2} \times \log^s s]$ to physical processes in general.
• Experimentally \( \langle r^2 \rangle_{\pi^\pm} = (0.439\pm0.008)\text{fm}^2 \), showing that the contribution of the \( L_9 \)-constant, for any reasonable value of \( \mu \), dominates the electromagnetic mean squared radius of the pion. In other words, in this example, the tree level \( \mathcal{O}(p^4) \) contribution largely dominates the “chiral logs” induced from the lowest order Lagrangian. This is a fact which can be easily understood within the framework of the \( 1/N_c \)-expansion\(^*\): in the large–\( N_c \)-limit, \( L_9 \) and \( f_\pi^2 \) are \( \mathcal{O}(N_c) \); therefore the chiral loop contribution is \( 1/N_c \)-suppressed with respect to the tree level contribution.

• The phenomenological value we get, from this observable, for the \( L_9 \) coupling constant at the scale of the \( \rho \)-meson mass: \( L_9^{(\rho)}(770\text{MeV}) = (6.9\pm0.7) \times 10^{-3} \) is in the range of the expected order of magnitude if, as we have assumed a priori, the chiral Lagrangian is a good effective theory for energies below the \( \chi \)SB scale \( \Lambda_\chi \). With a factor \( \frac{1}{4}f_\pi^2 \) pulled out from \( L^{(4)}_{\text{eff}} \), (the same \( \frac{1}{4}f_\pi^2 \)-factor which modulates the lowest order \( L^{(2)}_{\text{eff}} \) Lagrangian,) the dimensionless couplings \( L_i \), can then be traded by \( \frac{1}{\Lambda_i} \) couplings, such that \( L_i^\prime \equiv \frac{1}{\Lambda_i} \). It seems reasonable to expect the \( \Lambda_i \)-constants to be of the same order of magnitude as \( \Lambda_\chi \). Then, from (90), we expect \( M_\rho(770\text{MeV}) \leq \Lambda_i \leq M_{A_1}(1260\text{MeV}) \); i.e., \( L_i^\prime \sim 10^{-3} \) to \( 5 \times 10^{-5} \). We see that \( L_9 \) falls in this expected range; and in fact, as we shall soon see, so do all the other \( L_i \)-couplings.

There is a wealth of experimental information on low energy hadron physics, which has permitted the phenomenological determination of the \( L_i \)-constants. The power of the chiral approach is that once these constants have been fixed from some experiment, or spectral function sum rules, there are of course predictions –and therefore tests– for other observables. In Table 1, I have collected the most recent compilation of the \( L_i \)-’s. One can also read in the same Table 1, the experimental source which has been used for the determination of the appropriate constant. It would be nice to improve the accuracy of some of the low–energy experiments. Hopefully, the DAΦNE–project at Frascati will eventually provide some of this improvement. A lot of theoretical work has recently been made in view of this project. This work has been published as a reference guide\(^2\). One can find there an update of many recent phenomenological applications of \( \chi \)PT.

2.2.3 The Non-Abelian Chiral Anomaly

Although the QCD Lagrangian with external sources is formally invariant under local chiral transformations, this is no longer true for the associated generating functional. The anomalies of the fermionic determinant break chiral symmetry at the quantum level. The anomalous change of the generating functional under an

\(^*\) This is the approximation proposed by 't Hooft\(^1\), where the number of colours \( N_c \) in QCD is large with the product \( \alpha_s N_c \) kept fixed.
Table 1: Phenomenological values of the renormalized couplings $L^*_i(M_\rho)$.

| $i$ | $L^*_i(M_\rho) \times 10^3$ | Source |
|-----|----------------------------|--------|
| 1   | $0.7 \pm 0.5$              | $K_{e4}$, $\pi\pi \to \pi\pi$ |
| 2   | $1.2 \pm 0.4$              | $K_{e4}$, $\pi\pi \to \pi\pi$ |
| 3   | $-3.6 \pm 1.3$             | $K_{e4}$, $\pi\pi \to \pi\pi$ |
| 4   | $-0.3 \pm 0.5$             | Zweig rule |
| 5   | $1.4 \pm 0.5$              | $F_K : F_\pi$ |
| 6   | $-0.2 \pm 0.3$             | Zweig rule |
| 7   | $-0.4 \pm 0.2$             | Gell-Mann–Okubo, $L_5$, $L_8$, Sum Rules |
| 8   | $0.9 \pm 0.3$              | $M_{K^0} - M_{K^+}$, $L_5$, $(m_s - \hat{m}) : (m_d - m_u)$ |
| 9   | $6.9 \pm 0.7$              | $\langle r^2 \rangle_{\pi\pi}$ |
| 10  | $-5.5 \pm 0.7$             | $\pi \to e\nu\gamma$ |

infinitesimal chiral transformation

$$V_{L,R} = 1 + i\alpha \mp i\beta + \ldots$$  \hspace{0.75in} (135)

is given by \cite{26}:

$$\delta Z[v, a, s, p] = -\frac{N_c}{16\pi^2} \int d^4x \text{tr}\beta(x) \Omega(x),$$  \hspace{0.75in} (136)

where

$$\Omega(x) = \varepsilon^{\mu\nu\sigma\rho} [v_{\mu\nu}v_{\sigma\rho} + \frac{4}{3} \nabla_\mu a_\nu \nabla_\sigma a_\rho + \frac{2}{3} i \{v_{\mu\nu}, a_\sigma a_\rho\} + \frac{8}{3} i a_\sigma v_{\mu\nu} a_\rho + \frac{4}{3} a_\mu a_\nu a_\sigma a_\rho], \quad \varepsilon_{0123} = 1;$$  \hspace{0.75in} (137)

and

$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i [v_\mu, v_\nu], \quad \nabla_\mu a_\nu = \partial_\mu a_\nu - i [v_\mu, a_\nu].$$  \hspace{0.75in} (138)

This anomalous variation of $Z$ is an $O(p^4)$ effect in the chiral counting.

Chiral Symmetry is the basic requirement to construct the effective $\chi$PT Lagrangian. Since chiral symmetry is explicitly violated by the anomaly at the fundamental QCD level, one is forced to add an effective functional with the property that its change under chiral gauge transformations reproduces (136). Such a functional was first constructed by Wess and Zumino \cite{24}. An interesting topological interpretation was later found by Witten \cite{25}. The functional in question, has the following explicit form:

$$\Gamma[U, \ell, r]_{ZW} = -\frac{iN_c}{240\pi^2} \int d\sigma^{ijklm} \text{tr}\left\{\Sigma^L_i \Sigma^L_j \Sigma^L_k \Sigma^L_l \Sigma^L_m \right\}$$

$$-\frac{iN_c}{48\pi^2} \int d^4x \varepsilon_{\mu\nu\alpha\beta} \left(W(U, \ell, r)^{\mu\nu\alpha\beta} - W(1, \ell, r)^{\mu\nu\alpha\beta}\right),$$  \hspace{0.75in} (139)
with

\[ W(U, \ell, r)_{\mu \alpha \beta} = \text{tr} \left\{ u_{\mu} \ell_{\nu} \epsilon_{\alpha \beta} U^\dagger r_{\beta} + \frac{1}{4} u_{\mu} u_{\nu} r_{\rho} U^\dagger \ell_{\alpha} U^\dagger r_{\beta} + i u_{\mu} \ell_{\nu} \epsilon_{\alpha \beta} U^\dagger r_{\beta} 
\]
\[ + i \partial_{\mu} r_{\nu} U_{\alpha} U^\dagger r_{\beta} - i \Sigma_{\mu} U_{\nu} U^\dagger \ell_{\alpha} U^\dagger r_{\beta} + \Sigma_{\mu} U_{\nu} \partial_{\alpha} U^\dagger r_{\beta} - \Sigma_{\mu} U_{\nu} \partial_{\alpha} U^\dagger r_{\beta} - i \Sigma_{\mu} U_{\nu} \Sigma_{\alpha} \epsilon_{\beta} \right\} + \text{h.c.}, \]
\[ (L \leftrightarrow R), \tag{140} \]

where

\[ \Sigma_{\mu}^L = U^\dagger \partial_{\mu} U, \quad \Sigma_{\mu}^R = U \partial_{\mu} U^\dagger, \tag{141} \]

and \((L \leftrightarrow R)\) stands for the interchanges \( U \leftrightarrow U^\dagger, \ell_{\mu} \leftrightarrow r_{\mu} \) and \( \Sigma_{\mu}^L \leftrightarrow \Sigma_{\mu}^R \). The integration in the first term of Eq. (139) is over a five–dimensional manifold whose boundary is four–dimensional Minkowski space. The integrand is a surface term; therefore both the first and the second terms of \( \Gamma_{WZW} \) are \( O(p^4) \) according to the chiral counting rules.

Since the effect of anomalies is completely calculable, their translation from the fundamental quark–gluon level to the effective chiral level is unaffected by hadronization problems. The anomalous action (139) has no free parameters. It is responsible for the \( \pi^0 \rightarrow 2\gamma, \eta \rightarrow 2\gamma \) decays, and the \( \gamma 3\pi, \gamma \pi^+\pi^-\eta \) interactions among others. The five–dimensional surface term generates interactions among five or more Goldstone bosons.

The variation of the anomalous action \( \Gamma[U, \ell, r]_{WZW} \) with respect to appropriate external field sources, generates the chiral realization of the anomalous electroweak currents of the Standard Model, much the same as we calculated the electroweak currents in eqs. (114), (115) induced from the lowest order chiral Lagrangian. With

\[ L_{\mu} \equiv i U^\dagger D_{\mu} U, \tag{142} \]

and \( F_{\mu \nu}^L, F_{\mu \nu}^R \) the strength field tensors (122) associated to \( l_{\mu} \) and \( r_{\mu} \), we have

\[ \frac{\delta \Gamma_{WZW}}{\delta (l_{\mu})_{ji}} = \frac{N_c}{48\pi^2} \varepsilon^{\mu \nu \rho \sigma} \left[ i L_{\nu} L_{\rho} L_{\sigma} + \left\{ F_{\nu \rho}^L + \frac{1}{2} U^\dagger F_{\nu \rho}^R U, L_{\sigma} \right\} \right]_{ij}. \tag{143} \]

This anomalous current is in fact defined, up to a chirally non–covariant polynomial in the external fields \( l, r \). Only the covariant anomalous current above is measurable. Of particular interest for semileptonic decays of \( K \)–mesons, is the strangeness changing current which, in the presence of external electromagnetic interactions, couples to the charged \( W^\pm \) in the Standard Model:

\[ J_{\mu}^{\text{anom.}} = -\frac{e}{\sqrt{2} \sin \theta_W} \frac{N_c}{48\pi^2} \times \]
\[ \varepsilon^{\mu \nu \rho \sigma} \text{tr} Q_{\pm} \left[ D_{\nu} U^\dagger D_{\rho} U D_{\sigma} U^\dagger U + i e F_{\nu \rho} \left\{ U^\dagger D_{\sigma} U, Q + \frac{1}{2} U^\dagger Q U \right\} \right] + \text{h.c.}, \tag{144} \]
where, here,
\[ D_\mu = \partial_\mu U + ieA_\mu[Q,U] \]  \tag{145}

is the covariant derivative with respect to electromagnetism only, \((Q\) is the charge matrix in \((112)\)) and \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) the electromagnetic field strength tensor. The first term in the r.h.s. contributes to \(K_{l4}\)-decays such as
\[ K^+(p) \rightarrow \pi^+(p_+)\pi^-(p_-)e^+\nu_e, \quad q = p - p_+ - p_-, \]  \tag{146}

via the vector current, \((H\) denotes the conventional phenomenological parametrization of the vector form factor,)
\[ V^\mu = -H(q^2)\frac{1}{M_K^3} z^{\mu\nu\rho\sigma} q^\nu(p_++p_-)^\rho(p_--p_-)^\sigma, \]  \tag{147}

with \[ H(0) = -\frac{N_c}{16\pi^2 f_\pi^2} \frac{4 M_K^3}{3 \sqrt{2} f_\pi} = -2.7. \]  \tag{148}

Experimentally \[ H_{\text{threshold}} = -2.68 \pm 0.68. \]  \tag{149}

The second term in \((144)\) contributes to radiative semileptonic \(K\)-decays. An update of the rich phenomenology of these processes can be found in the DA\(\Phi\)NE physics handbook \[ 30. \]

3 Weak Interactions and Chiral Perturbation Theory

In this lecture we shall study the weak interactions of \(K\)-particles within the framework of \(\chi PT\). We shall be particularly concerned with processes induced by virtual \(W^\pm\)-exchange between quark currents, in the presence of the strong and electromagnetic interaction. A central issue in the study of the non-leptonic weak decays of \(K\)-mesons, is the question of the origin of the \(\Delta I = 1/2\) selection rule. Let me start by explaining what the problem is.

There is experimental evidence, both from \(K\)-decays and hyperon decays, that the rates of non-leptonic strangeness changing transitions \(\Delta S = 1\) with isospin change \(\Delta I = 1/2\) are particularly enhanced. This phenomenological fact is referred to as “the \(\Delta I = 1/2\) rule”. The explanation of this selection rule has been a continuous challenge to theorists for the last three decades! There are good qualitative indications that in the Standard Model, the observed \(\Delta I = 1/2\) rule has a dynamical origin; but no clear quantitative description of the enhancement has yet been exhibited.

To illustrate the problem let us try to make a simple guess of the strength of the transitions expected for \(K \rightarrow \pi\pi\) decays in the Standard Model. For such
processes the $W$–mass can be considered as infinitely heavy, and the exchange of
the $W^\pm$–field between two charged quark currents, ignoring for the moment
gluonic interactions between the two quark currents, is described by a simple
effective four–quark Hamiltonian:
\[ H_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* 4(\bar{s}_L \gamma^\mu u_L)(\bar{u}_L \gamma_\mu d_L) + \text{h.c.} \]  
with e.g.,

\[ (\bar{s}_L \gamma^\mu u_L) \equiv \sum_\alpha \bar{s}^{(\alpha)}(x) \gamma^\mu \left( \frac{1-\gamma^5}{2} \right) u^{(\alpha)}(x), \]

and where $\alpha$ denotes colour indices. Here $G_F$ is the Fermi coupling constant
$(1.166 \times 10^{-5}\text{GeV}^{-2})$ and $V_{ud}$, $V_{us}$ are matrix elements of the Cabibbo–Kobayashi–
Maskawa flavour mixing matrix, (see the lectures of Roberto Peccei\cite{37}.) At this
level of approximation, the effective Hamiltonian appears as the factorized product
of two quark currents. In the previous section, and to lowest order in the chiral
expansion, we have explicitly worked out the realization of these currents in terms
of pseudoscalar fields, [see Eqs.(114) and (115),] with the result:

\[ (\bar{s}_L \gamma^\mu u_L) \Rightarrow -\frac{1}{\sqrt{2}} f_\pi \partial_\mu K^+ \]

\[ -\frac{i}{2\sqrt{2}} \left[ \left( \pi^0 \partial_\mu K^+ \right) + \sqrt{3} \left( \eta \partial_\mu K^+ \right) + \sqrt{2} \left( \pi^+ \partial_\mu K^0 \right) \right] + \cdots ; \quad (152) \]

\[ (\bar{u}_\gamma^\mu d_L) \Rightarrow -\frac{1}{\sqrt{2}} f_\pi \partial_\mu \pi^- \]

\[ -\frac{i}{2} \left[ \sqrt{2} \left( \pi^- \partial_\mu \pi^0 \right) + \left( K^0 \partial_\mu K^- \right) \right] + \cdots . \quad (153) \]

We have all the ingredients now to calculate the $K \rightarrow \pi \pi$ isospin amplitudes $A_I$;
$I = 0, 2$ which we introduced in the Sec.1., [see Eqs.(50) and (51),] with the result:

\[ A_0 = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{2}{3} \sqrt{2} f_\pi \left( M_K^2 - M_\pi^2 \right), \quad (154) \]

\[ A_2 = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{1}{3} 2 f_\pi \left( M_K^2 - M_\pi^2 \right). \quad (155) \]

As already mentioned in Sec.1 as well, experimentally, from the ratio of the decay
rates $\Gamma(K_\text{S} \rightarrow \pi^+\pi^-)$ and $\Gamma(K_\text{S} \rightarrow \pi^0\pi^0)$, and with neglect of radiative corrections,
one finds

\[ \text{Re} \left( \frac{A_0}{A_2} \right) \bigg|_{\text{exp}} \simeq 22.2, \quad (156) \]
i.e., a factor of sixteen larger than our factorization estimate! More precisely, taking also into account the experimental rate for \( K^+ \to \pi^+ \pi^0 \), we are led to the conclusion that our first guess underestimates the \( \Delta I = 1/2 \) amplitude by a factor of eight, and overestimates the \( \Delta I = 3/2 \) amplitude by a factor of two.

One may be tempted to conclude that the factorization assumption we have used to make our estimate must indeed be a very naive picture. It turns out however, that this is precisely the result predicted by the leading behaviour of the \( 1/N_c \)-expansion in QCD; i.e., the limit where the number of colours \( N_c \) is taken to be large with \( \alpha_s N_c \) fixed. We shall see later why the large–\( N_c \) estimate fails in this case to give the right order of magnitude.

### 3.1 Short–Distance Reduction to an Effective Four–Quark Hamiltonian

In QCD, the composite operator

\[
Q_2 = 4 (\bar{s}_L \gamma^\mu u_L)(\bar{u}_L \gamma_\mu d_L),
\]

is not multiplicatively renormalizable. As first pointed out by Gaillard and Lee and by Altarelli and Maiani, the effect of gluon exchanges generates a new \( \Delta S = 1 \) operator (\( \alpha \) and \( \beta \) are quark colour indices; summation over repeated Greek letters is understood)

\[
Q_1 = 4 s^\alpha_L \gamma^\mu u^\beta_L \bar{u}^\beta_L \gamma_\mu d^\alpha_L,
\]

which mixes with the standard \( Q_2 \) operator in (157) under renormalization. It was later noticed by Vainshtein, Zakharov and Shifman, that the effect of the so-called “penguin” diagrams where one light quark line is attached by gluon exchange to a weak \( s - d \) self–energy like diagram, brings further \( \Delta S = 1 \) operators \( [q_R = \frac{1}{2} (1 + \gamma_5)q(x)]:\)

\[
Q_3 = 4 (\bar{s}_L \gamma^\mu d_L) \sum_{q=u,d,s} (\bar{q}_L \gamma_\mu q_L),
\]

\[
Q_4 = 4 s^\alpha_L \gamma^\mu d^\beta_L \sum_{q=u,d,s} \bar{q}^\beta_L \gamma_\mu q^\alpha_L,
\]

\[
Q_5 = 4 (\bar{s}_L \gamma_\mu d_L) \sum_{q=u,d,s} (\bar{q}_R \gamma_\mu q_R),
\]

\[
Q_6 = 4 s^\alpha_L \gamma^\mu d^\beta_L \sum_{q=u,d,s} \bar{q}^\beta_R \gamma_\mu q^\alpha_R,
\]

which also mix with \( Q_2 \) and among themselves under renormalization. The operators

\[
Q_- = Q_2 - Q_1
\]

and \( Q_i, i = 3, 4, 5, 6 \), induce pure \( \Delta I = 1/2 \) transitions, but only four of these operators are independent, since

\[
Q_- + Q_3 - Q_4 = 0.
\]
Under $SU(3)_L \times SU(3)_R$ transformations, $Q_-$ and the operators $Q_i$, $i = 3, 4, 5, 6$, transform like $(8_L, 1_R)$ operators, while the combination

$$Q^{(27)} = 2Q_2 + 3Q_1 - Q_3$$

(165)

transforms like a $(27_L, 1_R)$ operator which induces both $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ transitions via its components

$$Q^{(27)} = \frac{4}{3} \left( Q_{1/2}^{(27)} + 5Q_{3/2}^{(27)} \right),$$

(166)

where

$$Q_{1/2}^{(27)} = (\bar{s}_L \gamma^\mu d_L) (\bar{u}_L \gamma_\mu u_L) + (\bar{s}_L \gamma^\mu u_L) (\bar{u}_L \gamma_\mu d_L) + 2 (\bar{s}_L \gamma^\mu d_L) (\bar{d}_L \gamma_\mu d_L) - 3 (\bar{s}_L \gamma^\mu d_L) (\bar{s}_L \gamma_\mu s_L),$$

(167)

and

$$Q_{3/2}^{(27)} = (\bar{s}_L \gamma^\mu d_L) (\bar{u}_L \gamma_\mu u_L) + (\bar{s}_L \gamma^\mu u_L) (\bar{u}_L \gamma_\mu d_L) - (\bar{s}_L \gamma^\mu d_L) (\bar{d}_L \gamma_\mu d_L).$$

(168)

The operator $Q^{(27)}$ is multiplicatively renormalizable and does not mix with the others.

The procedure to go from the Lagrangian of the Standard Model to an effective electroweak Hamiltonian, where only the degrees of freedom of the light quark fields $u, d, s$ appear, consists in using the asymptotic freedom property of QCD to successively integrate out the fields with heavy masses down to scales $\mu^2 < m_c^2$, i.e. below the charm quark mass. The appropriate technique is the operator product expansion and the use of renormalization group equations to compute the various Wilson coefficient functions of the four–quark operators $Q_i$, $i = 1, \ldots, 6$. The inclusion of virtual electromagnetic interactions in the process of integrating out the fields with heavy masses brings in four new four–quark operators to the effective electroweak Hamiltonian. With $e_q$ denoting the corresponding quark charges in units of the electric charge, the new operators are:

$$Q_7 = 6 (\bar{s}_L \gamma^\mu d_L) \sum_{q=u,d,s} e_q (\bar{q}_R \gamma_\mu q_R),$$

(169)

$$Q_8 = 6 \bar{s}_L^\alpha \gamma^\mu d_L^\beta \sum_{q=u,d,s} e_q \bar{q}_R^\beta \gamma_\mu q_R^\alpha,$$

(170)

$$Q_9 = 6 (\bar{s}_L \gamma^\mu d_L) \sum_{q=u,d,s} e_q (\bar{q}_L \gamma_\mu q_L),$$

(171)

$$Q_{10} = 6 \bar{s}_L^\alpha \gamma^\mu d_L^\beta \sum_{q=u,d,s} e_q \bar{q}_L^\beta \gamma_\mu q_L^\alpha.$$
Under the action of the chiral group $SU(3)_L \times SU(3)_R$ the operators $Q_7$ and $Q_8$ transform like combinations of $(8_L, 1_R)$ and $(8_L, 8_R)$ operators, while $Q_9$ and $Q_{10}$ transform like combinations of $(8_L, 1_R)$ and $(27_L, 1_R)$.

Owing to the big value of the top quark mass, higher–order electroweak contributions turn out to be relevant when analyzing some CP–violation effects. These additional corrections do not generate new operators, but contribute to the Wilson coefficient functions of the operators $Q_i$, $i = 1, \ldots, 10$.

In processes with leptons in the final state, three more operators need still to be considered:

$$Q_{11} = 4 (\bar{s}_L \gamma^\mu d_L) \sum_{l=\mu,e} (\bar{l}_L \gamma_\mu l_L),$$

$$Q_{12} = 4 (\bar{s}_L \gamma^\mu d_L) \sum_{l=\mu,e} (\bar{l}_R \gamma_\mu l_R),$$

$$Q_{13} = 4 (\bar{s}_L \gamma^\mu d_L) (\bar{\nu}_L \gamma^\mu \nu_L).$$

Since these operators are bilinear in the quark–fields, they have to be considered separately whenever they may be relevant. The operators $Q_{11}$ and $Q_{12}$ play an important rôle in the transition $K_L \to \pi^0 e^+ e^-$ which we shall study in Sec.5.

In the Standard Model, the $\Delta S = 1$ non–leptonic interactions can then be described by an effective Hamiltonian

$$\mathcal{H}_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} C_i(\mu) Q_i + \text{h.c.},$$

where $Q_i$ are the ten local four–quark operators introduced above, and $C_i(\mu)$ the modulating Wilson coefficients which are functions of the masses of the fields which have been integrated out, i.e. $t$, $Z$, $W$, $b$, and $c$, as well as of the overall renormalization scale $\mu$. The Wilson coefficients of the ten four–quark operators take into account the effect of the strong interactions down to scales of $O(\mu)$, in the presence of virtual electroweak interactions, which in practice are kept to $O(\alpha)$.

The Wilson coefficients $C_i(\mu)$ obey the renormalization group equation ($\frac{\alpha_s^2}{4\pi} = \alpha_s$)

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right] \tilde{C} \left( \frac{M_W^2}{\mu^2}, \alpha_s, \alpha \right) = \gamma^T(\alpha_s, \alpha) \tilde{C} \left( \frac{M_W^2}{\mu^2}, \alpha_s, \alpha \right),$$

where $\beta(\alpha_s)$ denotes the QCD beta function

$$\beta(\alpha_s) = \beta_1 \frac{\alpha_s}{\pi} + \beta_2 \left( \frac{\alpha_s}{\pi} \right)^2 + \cdots,$$

$$\beta_1 = \frac{1}{6} (-11N_c + 2n_f),$$

$$\beta_2 = \frac{-17}{3} \left( \frac{N_c}{2} \right)^2 + \frac{1}{2} \frac{N_c^2 - 1}{2N_c} + \frac{5}{12} N_c n_f.$$
To be extracted from the calculation of the full Feynman diagrams at the $\alpha$ scale. In the limit the $C$ to be scaled from the large $t$ quark operators. The Wilson coefficients of the full set of four–quark operators has of gluon exchanges between quark current bilinears brings in, by mixing, new four–quark–mass and $W$–mass scales down to a renormalization mass scale $\Lambda_{\overline{\text{MS}}}$, below which the QCD perturbative evaluation is no longer trustworthy. This long running of scales brings in large logarithms, the effect of the long evolution from short–distances to long–distances is incorporated. To see this more explicitly, it suffices to compute the lowest order Wilson coefficients in the sector of the $Q_1$ and $Q_2$ four–quark operators. In the basis of the $Q_1 = Q_1 - Q_2$ and $Q_2 = Q_1 + Q_2$ operators, you will find

$$C^\pm(\mu^2) = \left( \frac{\alpha_s(M_W^2)}{\alpha_s(\mu^2)} \right) \left[ \frac{\pm 3/2(1+1/N_c)}{-\beta_0} \right].$$

The net effect is that transitions induced by the $Q_-$–operator are enhanced by a factor $\left( \log \frac{M_W^2}{\Lambda_{\overline{\text{MS}}}^2} / \log \frac{\mu^2}{\Lambda_{\overline{\text{MS}}}^2} \right)^{4/7}$; while the transitions induced by the $Q_+$–operator are

\[\begin{align*}
\mu^2 \frac{d}{d\mu^2} (Q_i) = & -\frac{1}{2} \gamma_{ij}(\alpha_s, \alpha) \langle Q_j \rangle. 
\end{align*}\]

In perturbation theory, the matrix elements of $\gamma$ which are needed for a calculation of the Wilson coefficients at the leading and next–to–leading logarithmic approximation, are the coefficients of the first four terms in the expansion

$$\gamma(\alpha_s, \alpha) = \frac{\alpha_s N_c}{\pi} \frac{1}{2} \gamma_s^{(0)} + \frac{\alpha}{\pi} \frac{1}{2} \gamma_s^{(0)} + \left( \frac{\alpha_s N_c}{\pi} \right)^2 \frac{1}{4} \gamma_s^{(1)} + \frac{\alpha_s N_c}{\pi} \alpha \frac{1}{4} \gamma_s^{(1)} + \ldots,$$

The most recent calculations of the full $10 \times 10$ anomalous dimension matrix corresponding to this expansion, can be found in Refs. [23].

The general solution of the renormalization group equation above is given by

$$\vec{C}(\frac{M_W^2}{\mu^2}, \alpha_s(\mu), \alpha) = \left\{ T_{\alpha_s} \exp \int_{\alpha_s(M_W)}^{\alpha_s(\mu)} dz \frac{\gamma^T(z, \alpha)}{z^\beta(z)} \right\} \vec{C}(1, \alpha_s(M_W), \alpha),$$

where $T_{\alpha_s}$ denotes ordering in the QCD coupling constant increasing from right to left. The vector $\vec{C}(1, \alpha_s(M_W), \alpha)$, which defines the initial boundary conditions, has to be extracted from the calculation of the full Feynman diagrams at the $M_W$ mass scale. In the limit $\alpha_s = \alpha = 0$: $C_2(M_W) = 1$ and all the other $C_i$ vanish. To $O(\alpha)$, the $C_i$ with $i = 1, 2, 3, 7$ and $9$ get contributions, while $C_8 = C_{10} = 0$. The two–loop final expressions for the Wilson coefficient functions are rather elaborate. I shall not reproduce them here. They can be found in the references quoted earlier [23].

We can now understand why our factorization estimate, earlier, failed. The effect of gluon exchanges between quark current bilinears brings in, by mixing, new four–quark operators. The Wilson coefficients of the full set of four–quark operators has to be scaled from the large $t$–mass and $W$–mass scales down to a renormalization mass scale $\mu \simeq 1$GeV, below which the QCD perturbative evaluation is no longer trustworthy. This long running of scales brings in large logarithms, the effect of their size being governed by the anomalous dimension matrix of the four–quark operators. It is only when terms of at least $O(N_c)$ are taken into account in the weak amplitudes, that the effect of the long evolution from short–distances to long–distances is incorporated. To see this more explicitly, it suffices to compute the lowest order Wilson coefficients in the sector of the $Q_1$ and $Q_2$ four–quark operators.
depressed by a factor \( \left( \log \frac{\mu^2}{\Lambda^2} / \log \frac{\Lambda^2}{\Lambda_{\text{MS}}^2} \right)^{2/7} \). Although this does not explain the observed \( \Delta I = 1/2 \) enhancement, it goes well in the right direction, and points towards a solution of the problem: the possibility that the enhancement due to the anomalous dimensions continues in the evolution of the hadronic matrix elements at low energies.

Another interesting issue, which appears at the two loop level of the perturbative calculations, is the question of renormalization scheme dependence. The Wilson coefficient functions have been calculated in two schemes: the ’t Hooft–Veltman renormalization scheme \( 51 \), and dimensional regularization with an anticommuting \( \gamma_5 \). The scheme dependence can only be removed by matching the calculation of the \( C_i \)’s with a similar calculation of the matrix elements of the \( Q_i(\mu) \) operators in the same scheme. Unfortunately, the technology to calculate low energy matrix elements of light four–quark operators is not yet developed to the degree of sophistication of perturbative QCD. Only approximate methods have been developed so far: lattice models of QCD (see the lectures of Steven Sharpe \( 16 \)); various versions of QCD sum rules \( 52, 53, 54, 55 \); the \( 1/N_c \) approach developed by Bardeen, Buras and Gérard \( 56, 57, 58 \); and more recently, the QCD low–energy effective action approach \( 54 \). We postpone the discussion of some of these methods to the following sections, where we shall review estimates of various \( \chi \)PT coupling constants relevant to \( K \)–decays.

### 3.2 The Effective Four–Quark Hamiltonian in \( \chi \)PT

The effective four–quark Hamiltonian that we have derived in the previous subsection contains only light–quark fields degrees of freedom. The light quark fields still have strong interactions mediated by the gluon fields of the QCD–Lagrangian. Now we want to find the effective chiral realization of this Hamiltonian in terms of (pseudo) Nambu–Goldstone degrees of freedom only; i.e., we want a description in terms of an effective chiral Lagrangian –with the same chiral transformation properties as the four–quark Hamiltonian– which will be incorporated as a weak perturbation to the strong chiral effective Lagrangian we discussed in Sec. 2. We also want to construct the weak effective Lagrangian in terms of a chiral expansion in powers of derivatives and quark masses, much the same as we did for the sector of the strong interactions.

As we saw in the last subsection, there are essentially\( ^\dagger \) two types of four–quark operators, according to their transformation properties under chiral–\( SU(3) \): those which transform as \( (8_L, 1_R) \) and those which transform as \( (27_L, 1_R) \). To lowest order in powers of derivatives, all the possible operators one can construct with these transformation properties are the following: With \( L_\mu(x) \) the \( 3 \times 3 \) flavour

\( ^\dagger \) There is also the combination of the operators \( Q_7 \) and \( Q_8 \) which transforms like \( (8_L, 8_R) \) which has to be considered separately \( 45 \).
matrix field
\[ \mathcal{L}_\mu(x) \equiv -i \frac{f_\pi^2}{2} U(x)^\dagger D_\mu U(x), \]
which by itself transforms as \( \mathcal{L}_\mu \to V_L \mathcal{L}_\mu V_L^\dagger \), we can construct the two operators
\[ \mathcal{L}_8(x) = \sum_i (\mathcal{L}_\mu)_{2i} (\mathcal{L}_\mu)_{i3}; \]
\[ \mathcal{L}_{27}(x) = \frac{2}{3} (\mathcal{L}_\mu)_{21} (\mathcal{L}_\mu)_{13} + (\mathcal{L}_\mu)_{23} (\mathcal{L}_\mu)_{11}. \]

The second operator induces both \( \Delta I = 1/2 \) and \( \Delta I = 3/2 \) transitions via its components:
\[ \mathcal{L}_{27}(x) = \frac{1}{9} \mathcal{L}_{27}^{(1/2)}(x) + \frac{5}{9} \mathcal{L}_{27}^{(3/2)}(x), \]
where
\[ \mathcal{L}_{27}^{(1/2)}(x) = (\mathcal{L}_\mu)_{21} (\mathcal{L}_\mu)_{13} + (\mathcal{L}_\mu)_{13} \left[ 4 (\mathcal{L}_\mu)_{11} + 5 (\mathcal{L}_\mu)_{22} \right], \]
\[ \mathcal{L}_{27}^{(3/2)}(x) = (\mathcal{L}_\mu)_{21} (\mathcal{L}_\mu)_{13} + (\mathcal{L}_\mu)_{23} \left[ (\mathcal{L}_\mu)_{11} - (\mathcal{L}_\mu)_{22} \right]. \]

The most general effective Lagrangian we are looking for, to lowest order in the chiral expansion, is then\(^\dagger\)
\[ \mathcal{L}_{\text{eff}}^{\Delta S=1}(x) = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^\dagger 4 \left[ g_8 \mathcal{L}_8 + g_{27} \mathcal{L}_{27}^{(3/2)} + \frac{1}{5} g_{27} \mathcal{L}_{27}^{(1/2)} \right] + \text{h.c.} \]

The two constants \( g_8 \) and \( g_{27} \) are dimensionless constants, real constants in as far as CP-violation effects are neglected, which like \( f_\pi \), \( B \), and the \( L_i \) coupling constants of the strong Lagrangian, cannot be determined from symmetry arguments alone. With the factors of \( f_\pi \) included in the definition of \( \mathcal{L}_\mu \) in (185), the constants \( g_8, g_{27} \) are of \( \mathcal{O}(1) \) in the large-\( N_c \) expansion. We can get their phenomenological values from a comparison between the expressions for the \( K \to \pi \pi \) isospin amplitudes \( A_I \) calculated with the Lagrangian above:
\[ A_0 = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^\dagger (g_8 + \frac{1}{5} g_{27}) \sqrt{2} f_\pi \left( M_K^2 - M_\pi^2 \right), \]
\[ A_2 = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^\dagger g_{27} 2 f_\pi \left( M_K^2 - M_\pi^2 \right), \]
and experiment; with the result\(^\dagger\):
\[ |g_8 + \frac{1}{5} g_{27}|_{\text{exp.}} \approx 5.1, \quad |g_{27}|_{\text{exp.}} \approx 0.16. \]

\(^\dagger\) A coupling \( (\chi U^\dagger + U \chi^\dagger)_{23} \) is also possible a priori. However, for on-shell processes, this term can be rotated away so as to maintain the normalization condition \( < U > = 1 \). The physical effect is then of higher order. This term has still physical relevance, even to lowest order, for off-shell non-leptonic Green’s functions, and therefore brings in an extra coupling constant which is not fixed by symmetry requirements alone.
From a comparison between the $A_I$-amplitudes calculated above, and the results of our factorization estimate in Eqs. (154) and (155), we can also read the values of the coupling constants $g_{8,27}$ corresponding to this approximation:

$$g_8|_{\text{fact.}} = \frac{3}{5}, \quad g_{27}|_{\text{fact.}} = \frac{1}{3},$$

rather far away, as already discussed, from the experimental values.

Once the couplings $g_8$ and $g_{27}$ have been fixed phenomenologically, there follow a wealth of non-trivial predictions for $K \to \pi\pi\pi$ decays and some radiative $K$–decays as well. Concerning the latter, there appear some interesting features, which I think are worth pointing out, because they reveal the power and the simplicity of the chiral approach:

- **$K$-decay amplitudes with any number of real or virtual photons and at most one pion in the final state vanish to lowest $O(p^2)$ in the chiral expansion**.

The reason for it is due to the fact that electromagnetic gauge invariance requires physical amplitudes to have a number of chiral powers higher than just the two powers allowed by the lowest order effective Lagrangian. Processes like

$$K_S \to \gamma\gamma, \quad K_L \to \pi^0\gamma\gamma;$$

$$K^+ \to \pi^+\gamma\gamma, \quad K_S \to \pi^0\gamma\gamma;$$

$$K^+ \to \pi^+e^+e^-, \quad K_L \to \pi^0e^+e^-, \quad (196)$$

are all of this type. They are at least $O(p^4)$ in the chiral expansion.

- **$K$-decay amplitudes with two pions and any number of real or virtual photons in the final state factorize, at $O(p^2)$, into the corresponding $K \to \pi\pi$ on–shell amplitude times a universal bremsstrahlung–like amplitude.**

The proof consists in a simple adaptation of a well known theorem due to F. Low $^{[61]}$, to lowest order in $\chi$PT. [A sketch of the proof for one photon is given in Ref. $^{[62]}$. Applications to $K \to \pi\pi\gamma$ can be found in Ref. $^{[31]}$]  

### 3.2.1 Weak Amplitudes to $O(p^4)$ in $\chi$PT

The full analysis of the one loop divergences generated by the lowest order weak Lagrangian, in the presence of the strong effective chiral Lagrangian; as well as the classification of the possible local terms of $O(p^4)$ was first made in $^{[64]}$; and, using different techniques in $^{[65],[66]}$ as well. The number of terms is too large to do a phenomenological determination of the couplings, as it has been done in the strong interaction sector. Even restricting the attention to the effective realization of the four–quark operators which transform like $\left(8_L, 1_R\right)$, still leaves twenty two possible terms that contribute to non–leptonic $K$–decays with possible external photons or virtual $Z$–bosons. To determine phenomenologically these couplings is not a
question of calculation complexity; it is simply that there is not enough available experimental information to do the job! The only possible way to do $\chi$PT usefully in the sector of the non–leptonic weak interactions is to combine the chiral expansion with other approximation methods, like e.g., the $1/N_c$–expansion; or to resort to models of the low–energy effective action of QCD that one can first test in the strong interaction sector.

The art of the game in making clean $\chi$PT predictions for non–leptonic weak processes, consists in finding subsets of observables which to $O(p^4)$ are fully given by a chiral loop only, like e.g., $K_S \to \gamma\gamma$ \cite{foot:1}, and $K_L \to \pi^0\gamma\gamma$ \cite{foot:2}, or which involve a small number of unknown $O(p^4)$ local couplings, like $K \to \pi l^+ l^-$ decays \cite{foot:3,foot:4}. For a recent review on the the state of the art see Ref. \cite{foot:5}. I shall discuss some of these processes in Sec. 5.

Another sector where it has been possible to test the validity of the chiral expansion in non–leptonic weak interactions is in $K \to 2\pi$ and $K \to 3\pi$ decays. The description of these decays to $O(p^4)$ in $\chi$PT involves seven linear combinations of coupling constants. Altogether, imposing isospin and Bose symmetries, these decays can be parametrized in terms of twelve observables. Five of these parameters, the quadratic slopes in the Dalitz plot for the various $K \to 3\pi$ modes, vanish to lowest order in the chiral expansion. The resulting five constraints can be formulated in terms of neat predictions \cite{foot:6} for the slopes. The five predictions are compatible with experiment within errors. Another important result which also emerges from the $K \to 2\pi, 3\pi$ amplitude analysis \cite{foot:7}, and which is relevant for the understanding of the underlying dynamics of the $\Delta I = 1/2$ rule, is the fact that, in the presence of the $O(p^4)$ corrections, the fitted value for $|g_8|$ is $\simeq 30\%$ smaller than the lowest order determination in Eq. (194). This is, again, another little factor which, like the short–distance enhancement that we discussed earlier in subsec. 3.1, helps towards explaining the $\Delta I = 1/2$ rule, but the bulk of the enhancement remains still unrav–elled. Let me note that, at the same order of approximation, the coupling $g_{27}$ has no large corrections.

3.2.2 Penguins in the Large–$N_c$ Limit

It has often been speculated that the bulk of the origin of the $\Delta I = 1/2$ enhancement comes from the large matrix elements of the Penguin–generated $Q_6$ operator in (162). By Fierz reordering, this operator can also be written as

$$Q_6 = -8 \sum_{q=u,d,s} (\bar{s}_L q_R) (\bar{q}_R d_L).$$

I find it very instructive to discuss the chiral realization of this operator \cite{foot:8}, because it reveals a number of interesting features; and also because it provides an excellent example of the way that, when systematically combined, the chiral expansion and the large–$N_c$ expansion may eventually help us to make substantial progress in the understanding of low–energy QCD.
Using Eqs. (91) and (97), the term which couples quark bilinears to external scalar and pseudoscalar sources can be written as follows

\[-\bar{q}(s - i\gamma_5 p)q = -\frac{1}{2B}(\bar{q}_R \chi q_L + \bar{q}_L \chi^\dagger q_R).\]  

(200)

In the large–$N_c$ limit, the operator $Q_6$ in (199) factorizes into the product of two $\bar{q}q$–densities. The chiral realization of these densities to $O(p^2)$ in the chiral expansion and keeping only those terms which may eventually contribute to the lowest order $g_8 L_8$–piece in the Lagrangian in (191), proceeds as follows:

\[(\bar{s}_L q_R) \equiv 2B \frac{\delta L_{\text{eff}}}{\delta \chi_{3i}} = 2B \left\{ \frac{1}{4f_\pi^2} U_{13} + L_5 (UD_\mu U^\dagger D^\mu U)_{13} + \cdots \right\},\]  

(201)

\[(\bar{q}_R i d_L) \equiv 2B \frac{\delta L_{\text{eff}}}{\delta \chi_{i2}} = 2B \left\{ \frac{1}{4f_\pi^2} U^\dagger_{2i} + L_5 (D_\mu U^\dagger D^\mu U U^\dagger)_{2i} + \cdots \right\}.\]  

(202)

Since $UU^\dagger = U^\dagger U = 1$, there is no contribution to $Q_6$ of $O(p^0)$, and to leading order, both in the chiral expansion and in the large–$N_c$ expansion, we find

\[Q_6 \Rightarrow -8 \times 4B^2 \frac{1}{4f_\pi^2} 2L_5 (D^\mu U^\dagger D_\mu U)_{23}.\]  

(203)

We can cast this result in terms of the contribution to the $g_8$–coupling, induced by the term in the effective four–quark weak Hamiltonian proportional to the $Q_6$–operator:

\[\langle g_8 \rangle_{6^{\text{th}}\text{–Penguin}} = -16L_5 C_6 (\mu^2) \left[ \frac{<\bar{\psi}\psi>}{f_\pi^3} \right]^2.\]  

(204)

As I said before, there are a number of interesting features emerging from this result:

- Remember that in the large–$N_c$ limit $f_\pi^2 L_5$ and $<\bar{\psi}\psi>$ are all $O(N_c)$. Because of the fact that $C_6 \sim \alpha_s$, $\langle g_8 \rangle_{6^{\text{th}}\text{–Penguin}}$ is next–to–leading in the large–$N_c$ limit. It is precisely this next–to–leading contribution that we have succeeded in calculating explicitly.

- The Wilson coefficient $C_6$ has an imaginary part induced by the CP–violation phase in the Cabibbo–Kobayashi–Maskawa matrix. It is precisely this contribution that, in the Standard Model, induces the $\epsilon'$–amplitude we discussed in Sec. 1. The fact that we can say something quantitative about the size of the modulating coupling $\langle g_8 \rangle_{6^{\text{th}}\text{–Penguin}}$ is of course welcome for phenomenology. We shall come back to this in Sec. 5.

- In QCD the matrix element $<\bar{\psi}\psi>$ is $\mu$–scale dependent. The scale dependence is given by the anomalous dimension of the $\bar{\psi}\psi$–operator. On the other hand, the coupling constant $g_8$ is a scale independent quantity. This example exhibits explicitly the cancellation between the $\mu$–scale dependence of the
short–distance Wilson coefficient $C_6(\mu^2)$ and the $\mu$–scale dependence of the long–distance four–quark operator bosonization, which appears via the constant $(\langle \bar{\psi} \psi \rangle)^2$. In the large–$N_c$ limit the operator $Q_6$ does not mix with the others because, in that limit, the anomalous dimension matrix in Eq.(182): 
\[ [\gamma_s^{(0)}]_{ij} \rightarrow -\frac{3}{2}N_c \delta_{66}. \] In that limit, $C_6$ depends on $\mu$ via $C_6 \sim \alpha(\mu^2)^{9/11}$, which exactly cancels the $\mu$–depedance of $(\langle \bar{\psi} \psi \rangle)^2$ evaluated in the same large–$N_c$ limit.

- Of the parameters which appear in the r.h.s. of Eq.(204), the one which has the largest uncertainty is $\langle \bar{\psi} \psi \rangle$. The most recent determination from an update of QCD sum rules methods gives the value 
\[ \langle \bar{\psi} \psi \rangle (1\text{GeV}^2) = -(0.013 \pm 0.003)\text{GeV}^3. \] (205)

It is also questionable which value of $f_\pi$ one should use in the r.h.s. of Eq.(204). At the level of approximation that the calculation has been made, it seems appropriate to take the value of $f_\pi$ in the chiral limit; i.e., $f_\pi^{(0)} \simeq 86\text{MeV}$. Then, using this value for $f_\pi$, the value $L_5 \simeq 1.4 \times 10^{-3}$ which is in Table 1, and the result of the $C_6$–calculation in 49, I find

\[ [\mathbf{gs}]^{6\text{th–Penguin}} \simeq 0.9. \] (206)

From this result, one is led to conclude that, unless the next–to–leading order $1/N_c$ corrections to this calculation are huge, the bulk of the $\Delta I = 1/2$ enhancement does not come from the Penguin $Q_6$–operator. Barring this possibility, the long–distance contribution from the $Q_6$–operator remains then the most likely candidate to provide the bulk of the enhancement.

### 3.3 $\Delta S = 2$ Non-Leptonic Weak Interactions

The Standard Model predicts strangeness changing transitions with $\Delta S = 2$ via two virtual $W$–exchanges between quark lines, the so–called box diagrams. The reduction via the operator product expansion results in an effective Hamiltonian which is proportional to the local four–quark operator

\[ Q_{\Delta S=2} \equiv (\bar{s}_L \gamma^\mu d_L)(\bar{s}_\gamma \gamma^\mu d_L), \] (207)

modulated by products of the flavour mixing matrix elements

\[ \lambda_q = V_{qd}^* V_{qs}, \quad q = u, c, t; \] (208)

times functions $F_{1,2,3}$ of the heavy masses $m_t^2, M_W^2, m_b^2, m_c^2$ of the fields which have been integrated out:

\[ \mathcal{H}_{\text{eff}}^{\Delta S=2} = \frac{G_F^2 M_W^2}{4\pi^2} \left[ \lambda_c^2 F_1 + \lambda_t^2 F_2 + 2\lambda_c \lambda_t F_3 \right] C_{\Delta S=2}(\mu^2) Q_{\Delta S=2}. \] (209)
The operator $Q_{\Delta S=2}$ is multiplicatively renormalizable and has an anomalous dimension $\gamma(\alpha_s)$ defined by the equation

$$\mu^2 \frac{d}{d\mu^2} < Q_{\Delta S=2} > = -\frac{1}{2} \gamma(\alpha_s) < Q_{\Delta S=2} >$$  \hspace{1cm} (210)

At the one–loop level

$$\gamma(\alpha_s) = \frac{\alpha}{\pi} \gamma_1 + O\left(\frac{\alpha}{\pi}\right)^2, \quad \gamma_1 = \frac{3}{2} (1 - 1/N_c);$$  \hspace{1cm} (211)

and

$$C_{\Delta S=2}(\mu^2) = \alpha_s(\mu^2) \left[ -\frac{\gamma_1}{-\pi^2} = -2/9 \right].$$  \hspace{1cm} (212)

The matrix element

$$< K^0 | Q_{\Delta S=2}(0) | K^0 > \equiv \frac{4}{3} f_K^2 M_K^2 B_K(\mu^2);$$  \hspace{1cm} (213)

defines the so–called $B_K$–parameter, which governs $K^0 - \bar{K}^0$ mixing at short distances, and is one of the crucial unknown parameters in the phenomenological studies of CP–violation in the Standard Model. The definition above is such that in the so–called vacuum saturation approximation

$$(B_K)_{VS} = 1.$$  \hspace{1cm} (214)

In the large–$N_c$ limit, the four quark operator $Q_{\Delta S=2}$ factorizes into a product of the two left–handed current operators: $(\bar{s}_L \gamma^\mu d_L)$ times $(\bar{s}_L \gamma_\mu d_L)$. Each of these currents, to lowest order in $\chi$PT, has a bosonic realization as indicated in Eq.(114); i.e.,

$$(\bar{s}_L \gamma^\mu d_L) \Rightarrow -\frac{1}{\sqrt{2}} f_\pi \partial_\mu K^0 + \cdots.$$  \hspace{1cm} (215)

This approximation results then in a value: $B_K = \frac{3}{4} f_K^2$. There are however chiral corrections of $O(\mu^4)$ to this determination which are still leading in the large–$N_c$ limit. The leading $1/N_c$–corrections come from the $L_5$ term in $L_{\text{eff}}^{(4)}$; as well as from the $K^0$ wave–function renormalization, with the result

$$(\bar{s}_L \gamma^\mu d_L) \Rightarrow -\frac{1}{\sqrt{2}} f_\pi \left( 1 + \frac{8M_K^0}{f_\pi^2} L_5 \right) \left( 1 - \frac{4M_K^0}{f_\pi^2} L_5 \right) \partial_\mu K^0 + \cdots.$$  \hspace{1cm} (216)

where the first factor in the r.h.s. is the one coming from $L_{\text{eff}}^{(4)}$. (Notice that the contribution from chiral logarithms is $1/N_c$ suppressed.) The net result is a renormalization of $f_\pi$. This renormalized $f_\pi$–coupling is a first approximation to the physical $f_K$–coupling constant which is measured in $K^+ \to \mu^+ \nu_\mu$ decays: $f_K \simeq 113$MeV. To the extent that the two are identified, one then obtains the result which is often quoted in the literature as the large–$N_c$ prediction:

$$B_K |_{N_c \to \infty} = \frac{3}{4}.$$  \hspace{1cm} (217)
4 Models of the QCD Low–Energy Effective Action

The purpose of this lecture is to give an overview on models of the low–energy hadronic interactions, which have been developed during the last few years, and which try to focus on general properties that the QCD low–energy effective action is expected to have. The aim here is to find generic features of these models which, eventually, one may be able to promote to the rank of a low–energy effective action derived from QCD, hopefully within a well defined set of approximations.

Let me first formulate the problem which one would like to solve. For this purpose, it is convenient to promote the global chiral–SU(3) symmetry to a local gauge symmetry, in the same way that has already been discussed in Sec. 2. It is also convenient to use a path integral representation for the generating functional \( \Gamma(v,a,s,p) \) of the Green’s functions of quark currents:

\[
e^{i\Gamma(v,a,s,p)} = \frac{1}{Z} \int \mathcal{D}G_\mu \det \mathcal{D} \exp\left(-i \int d^4x \frac{1}{4} \tilde{G}^{\mu\nu} \tilde{G}^{\mu\nu}\right),
\]

(218)

with \( \mathcal{D} \) the Dirac operator

\[
\mathcal{D} = \gamma^\mu (\partial_\mu + ig_5 G_\mu) - i\gamma^\mu (v_\mu + \gamma_5 a_\mu) + i(s - i\gamma_5 p);
\]

(219)

where \( G_\mu \) is the gluon field, \( \tilde{G}^{\mu\nu} \) the gluon field strength tensor, and \( v_\mu, a_\mu, s, p \) external field sources. The normalization factor \( Z \) is such that \( \Gamma(0,0) = 1 \).

The chiral symmetry of the underlying QCD theory implies that \( \Gamma(v,a,s,p) \) admits a low–energy representation

\[
e^{i\Gamma(v,a,s,p)} = \frac{1}{Z} \int \mathcal{D}U \exp\left[i \int d^4x \mathcal{L}_{\text{eff}}(U;v,a,s,p)\right],
\]

(220)

in terms of an effective Lagrangian \( \mathcal{L}_{\text{eff}}(U;v,a,s,p) \) with \( U(x) \) a 3\( \times \)3 unitary matrix, with \( \det U = 1 \), which collects the octet of pseudoscalar fields \( (\pi,K,\eta) \).

There is only one term in \( \mathcal{L}_{\text{eff}} \) which is known from first principles. It is the term associated with the existence of anomalies in the fermionic determinant \( D \). The corresponding effective action is the Wess and Zumino functional that we have discussed in Sect. 2. All possible other terms in \( \mathcal{L}_{\text{eff}} \), are not fixed by symmetry requirements alone. It would be nice to find, at least approximatively, a dynamical scheme rooted in QCD, with as few free parameters as possible (ideally \( \Lambda_{\text{QCD}} \) only!), which allows for a derivation of the coupling constants in the effective chiral Lagrangian. As we have seen in Sec. 3, the need for such an approximate dynamical scheme is urgently needed at present to make progress in the phenomenology of non–leptonic flavour dynamics.

The various types of models which have been discussed in the literature can be classified, roughly speaking, in one of the following entries:

- QCD in the Large–\( N_c \) Limit.
• Low–Lying Resonances Dominance Models.
• The Constituent Chiral Quark Model.
• Effective Action Approach Models.
• The Extended Nambu and Jona-Lasinio Model (ENJL-model.)

4.1 QCD in the Large–$N_c$ limit

It would be a major breakthrough, if one could derive the low–energy effective Lagrangian of the interactions between Nambu–Goldstone modes in the large–$N_c$ limit of QCD. So far, it has only been possible to obtain constraints among various coupling constants in this limit; but not their values in terms, say, of $\Lambda_{QCD}$. A typical example is the relation: $2L_1 = L_2$, which, as first noticed by Gasser and Leutwyler, follow in the large–$N_c$ limit of QCD. Unfortunately, nobody can claim as yet to be able to compute, say $L_2$, in that limit. Often in the literature, there appear statements about “large–$N_c$ predictions” but, in fact, they have been all derived with some extra ad hoc assumptions.

An interesting approach to do approximate calculations within the framework of the $1/N_c$–expansion is the one proposed by Bardeen, Buras and Gérard, which they have applied extensively to the calculation of non–leptonic weak matrix elements. The basic idea is to start with the factorized form of the four–quark operators in the effective weak Hamiltonian, and to do one–loop chiral perturbation theory, keeping track of the quadratic divergences which appear. If one was able to work with the full hadronic low–energy effective Lagrangian, it would be possible to obtain a smooth matching between the scale dependence of the Wilson coefficients, calculated at short–distances, and the hadronic matrix elements calculated with the full hadronic low–energy effective Lagrangian. The hope with the approach proposed by Bardeen, Buras and Gérard is that the numerical matching of the quadratic long–distance scale with the logarithmic short–distance scale, may turn out to be already a good first approximation to the problem one would like to solve. The technology of their approach is explained with detail in their papers.

4.2 Low–Lying Resonances Dominance Models

There has been quite a lot of progress during the last few years in understanding the rôle of resonances in $\chi$PT. At the phenomenological level, it turns out that the observed values of the $L_i$–constants are practically saturated by the contribution from the lowest resonance exchanges between the pseudoscalar particles; and particularly by vector–exchange, whenever vector mesons can contribute. The specific form of an effective chiral invariant Lagrangian describing the couplings of vector and axial–vector particles to the (pseudo) Nambu–Goldstone modes is not uniquely fixed by chiral symmetry requirements alone. When the vector fields describing heavy vector particles are integrated out, different field theory descriptions may
lead to different predictions for the $L_i$–couplings. It has been shown however that if a few QCD short–distance constraints are imposed, the ambiguities of different formulations are then removed. The most compact effective Lagrangian formulation, compatible with the short–distance constraints, has two free parameters: $f_\pi$ and $M_V$. When the vector and axial–vector fields are integrated out, it leads to specific predictions for five of the $L_i$ constants:

$$
L_1^{(V)} = L_2^{(V)} / 2 = -L_3^{(V)} / 6 = L_9^{(V)} / 8 = -L_{10}^{(V+A)} / 6 = \frac{f_\pi^2}{16M_V^2} \simeq 0.6 \times 10^{-3}, \tag{221}
$$

in good agreement, within errors, with experiment. [See Table 1.]

It is fair to conclude that the old phenomenological concept of vector meson dominance (VMD) can now be formulated in a way compatible with the chiral symmetry properties and the short–distance behaviour of QCD.

In view of this success, there have been various suggestions of extensions of VMD–models in the literature. In particular, one would like to extend the idea of VMD to the non–leptonic sector of the weak interactions, in order to have a useful low–energy chiral effective Lagrangian formulation. (Remember the problem of the proliferation of couplings we have discussed in Sec. 3.) At the strict phenomenological level, one is forced to introduce weak couplings between vector (axial–vector) fields and/or the (pseudo) Nambu–Goldstone fields. The fact that these weak coupling constants are experimentally unknown, forces one to resort to models if the idea of VMD is to be pursued.

Models like the Geometric Model and the Quark Resonance Model are attempts to reduce the number of free coupling parameters which are allowed in principle by chiral symmetry requirements alone. However, the precise relation of these models to the specific assumptions which one is making within the underlying QCD–theory to justify them, remains unclear.

### 4.3 The Constituent Chiral Quark Model

This model was introduced by Georgi and Manohar, in an attempt to reconcile the successful features of the Constituent Quark Model, with the chiral symmetry requirements of QCD. The basic assumption of the model is the idea that between the scale of chiral symmetry breaking $\Lambda_\chi$ and the confinement scale $\sim \Lambda_{QCD}$ the underlying QCD–theory, may admit a useful effective Lagrangian realization in terms of constituent quark fields $Q$; pseudoscalar particles; and, perhaps, “gluons”. The Lagrangian in question has the form

$$
\mathcal{L}_{\text{eff}}^{\text{GM}} = i\bar{Q}\gamma_\mu (\partial_\mu + ig_\ast G_\mu + \Gamma_\mu)Q + \frac{i}{2}g_A\bar{Q}\gamma_5 \gamma^\mu \xi_\mu Q - M_Q\bar{Q}Q + \frac{1}{4}f_\pi^2 tr D_\mu U D^\mu U^\dagger - \frac{1}{4} G_{\mu\nu} \bar{G}^{\mu\nu}. \tag{222}
$$
Some explanations about the notation here are in order. Remember that under chiral rotations \((V_L, V_R)\), \(U\) transforms like: \(U \rightarrow V_R UV_L\). The unitary matrix \(U\) is the product of the so-called left and right coset representatives: \(U = \xi_R \xi_L^\dagger\) and, without lost of generality, one can always choose the gauge where \(\xi_L^\dagger = \xi_R \equiv \xi\). The coset representative \(\xi\), \((U = \xi \xi)\) transforms like:

\[
\xi \rightarrow V_R \xi h^\dagger = h \xi V_L^\dagger \quad h \in SU(3)_V,
\]

where \(h\) denotes the rotation induced by the chiral transformation \((V_L, V_R)\) in the diagonal \(SU(3)_V\). In Eq. (222) the constituent quark fields \(Q\) transform like

\[
Q \rightarrow hQ, \quad h \in SU(3)_V.
\]

In the presence of external sources\(^\S\),

\[
\Gamma_\mu = \frac{1}{2} \{ \xi^\dagger [\partial_\mu - i(v_\mu + a_\mu)] \xi + \xi [\partial_\mu - i(v_\mu - a_\mu)] \xi^\dagger \} \quad (225)
\]

and

\[
\xi_\mu = i \xi^\dagger D_\mu U \xi^\dagger. \quad (226)
\]

The free parameters of the theory are \(f_\pi\), \(M_Q\), and \(g_A\). The QCD coupling constant is assumed to have entered a regime (below \(\Lambda_{\chi}\)) where its running is frozen and is taken to be constant.

The merit of this model is that it automatically digests the phenomenological successes of the constituent quark model, in a way compatible with chiral symmetry. We shall in fact see, that effective Lagrangians of the Georgi–Manohar type, do indeed appear in practically all QCD low–energy models where quarks are not confined. The weak point of the model is its “vagueness” about the gluonic sector. In the absence of a dynamical justification for the “freezing” of the QCD running coupling constant, it is very unclear what the “left out” gluonic interactions mean; and in fact, in most applications they are simply ignored.

### 4.4 Effective Action Approach Models

The basic idea in this class of models is to make some kind of drastic approximation to compute the non–anomalous part of the QCD–fermionic determinant in the presence of external \(v_\mu\) and \(a_\mu\) fields, but with the external \(s\) and \(p\) fields frozen to the quark matrix

\[
s + ip = M = \text{diag}(m_u, m_d, m_s).
\]

For this purpose, it is convenient to perform a chiral rotation of the quark fields \(q_{L,R}(x) \equiv \frac{1}{2} (1 \pm \gamma_5) q(x)\) in the initial QCD–Lagrangian:

\[
q_L(x) \Rightarrow Q_L(x) = \xi(x) q_L(x), \quad q_R \Rightarrow Q_R(x) = \xi^\dagger(x) q_R(x), \quad (227)
\]

\(^\S\) The original formulation of the model of Georgi and Manohar\(^4\) was in fact made without external fields.
with $\xi$ chosen so that $\xi \xi = U$. Under chiral rotations $(V_L, V_R)$, the quark fields of the rotated basis transform like the constituent chiral quarks of the Georgi–Manohar Lagrangian:

$$Q_{L,R} \rightarrow h(x) Q_{L,R},$$

with $h(\phi(x))$ the rotation in $SU(3)_V$ induced by the chiral transformation $(V_L, V_R)$; $h$ is the same object which appears in Eq. (224). The QCD–Lagrangian in the rotated field basis, and in Euclidean space has then the following form:

$$L_{\text{QCD}}(E) = -\frac{1}{4} G^{(a)}_{\mu\nu} G_{\mu\nu}^{(a)} + \bar{Q} D_E Q,$$

with $D_E$ the Euclidean Dirac operator ($\gamma_{\mu} \equiv -i\gamma_{\mu}$ are Hermitian Dirac matrices with positive metric)

$$D_E = \tilde{\gamma}_{\mu}(\partial_{\mu} + ig_{\mu} G_{\mu} + \Gamma_{\mu} - \frac{i}{2} \gamma_5 \xi_{\mu}) - \frac{1}{2}(\Sigma - \gamma_5 \Delta),$$

where $\Gamma_{\mu}$ and $\xi_{\mu}$ are the same as in Eqs. (225) and (226); and

$$\Sigma = \xi^\dagger M \xi^\dagger + \xi M \xi, \quad \Delta = \xi^\dagger M \xi^\dagger - \xi M \xi.$$  

The $\Sigma$ and $\Delta$ terms break explicitly the chiral symmetry.

The Euclidean effective action $\Gamma_E(U, v, a, M)$ associated to $L_{\text{QCD}}^{(E)}$ is then given by

$$\exp \Gamma_E(U, v, a, M) = \frac{1}{Z} \int \mathcal{D}G_{\mu} \exp \left[ - \int d^4z \frac{1}{4} G^{(a)}_{\mu\nu} G^{(a)}_{\mu\nu} \right] \exp \Gamma_E(G_{\mu}; U, v, a, M) \equiv \frac{1}{Z} \int [\mathcal{D}G_{\mu}] \exp \Gamma_E(G_{\mu}; U, v, a, M),$$

with

$$\exp \Gamma_E(G_{\mu}; U, v, a, M) = \int \mathcal{D}Q \mathcal{D}Q \exp \left( \int d^4z \bar{Q} D_E Q \right) = \det D_E.$$  

In fact, for the non–anomalous part of the effective action, it is sufficient to consider the modulus of the determinant. Then,

$$\Gamma_E(G_{\mu}; U, v, a, M) = \frac{1}{2} \log \det D_E.$$  

It is in trying to evaluate the modulus of the fermionic determinant that one encounters the first problem. There is need of a regularization; and if possible, a regularization which shows the explicit cut–off dependence. The simplest regularization in that respect is the proper–time cut–off method, where

$$\Gamma_E = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Tr} \exp(-\tau D_E^\dagger D_E),$$
where Tr stands for trace over Dirac $\gamma$–matrices, $SU(3)$–colour matrices, Gell-Mann’s flavour–$SU(3)$ matrices, and Euclidean space.

Since $D_E^\dagger D_E$ is a second–order elliptic operator, the proper–time integrand in (235) has a well defined power series expansion in powers of $\tau$; this is the so called heat-kernel expansion:

$$ (x \mid e^{-\tau D_E^\dagger D_E} \mid y) = \frac{1}{16\pi^2\tau^2} e^{-\frac{(x-y)^2}{4\tau}} \sum_0^\infty H_n(x, y) \tau^n. \quad (236) $$

The functions $H_n(x, y)$ are known in the literature as Seeley-DeWitt coefficients. Here we encounter the second problem: for $n \geq 2$, the proper–time integrals are infrared divergent. We need an infrared regulator to integrate the large–$\tau$ behaviour.

Now, it just happens that, with neglect of the gluonic coupling in the Dirac operator in (230), and for some of the $\mathcal{O}(p^4)$ terms in the effective action, the proper time integral in (235) is convergent both in the ultraviolet and infrared domains. Therefore, the values of the corresponding $L_i$–coupling constants in this brutal approximation turn out to be finite quantities. The results in question are:

$$ 8L_1 = 4L_2 = L_9 = \frac{N_c}{48\pi^2}; \quad L_3 = L_{10} = -\frac{N_c}{96\pi^2}. \quad (237) $$

Numerically, these results, when compared to the phenomenological determinations listed in Table 1, are surprisingly good. Any attempt however, to improve on these results, and/or to compute other couplings of the low–energy effective Lagrangian, necessarily brings in the question of the infrared behaviour of the underlying theory.

A simple suggestion which has been proposed, in connection with the infrared behaviour, is to parametrize the effect of spontaneous symmetry breaking, by adding to the QCD–Lagrangian a phenomenological order parameter like–term:

$$ \Delta L_{QCD} \equiv -M_Q(\bar{q}_R U q_L + \bar{q}_L U^\dagger q_R), \quad (238) $$

which introduces at the same time the $U$–field in a way non–invariant under $U \rightarrow -U$, and the mass parameter $M_Q$ which provides the infrared regulator needed in the evaluation of the low energy effective action. It is easy to see that in the presence of this term, and with $M_Q > 0$, there is quark condensation: $<\bar{\psi}\psi> \neq 0$ and negative. Furthermore, in the presence of this term, it is also possible to evaluate the effect of large–$N_c$ gluonic interactions which appear as inverse powers of $M_Q$ times the appropriate vacuum expectation values of gauge invariant gluonic operators. For example, in the chiral limit, and including the leading gluonic contributions, one gets:

$$ f_\pi^2 = \frac{N_c}{16\pi^2} 4M_Q^2 \left[ \log \frac{\Lambda^2}{M_Q^2} + \frac{\pi^2}{6N_c} <\frac{\alpha_s}{\pi}GG> + \frac{1}{360N_c} \frac{g^3GGG}{M_Q^4} + \cdots \right]. \quad (239) $$

There also appear gluonic corrections to some of the previous results for the $L_i$–couplings:

$$ L_3 = L_{10} = -\frac{N_c}{96\pi^2} \left[ 1 + \frac{\pi^2}{5N_c} <\frac{\alpha_s}{\pi}GG> + \mathcal{O}(\frac{1}{M_Q^4}) \right]. \quad (240) $$
The positive sign of this correction helps towards the agreement with the experimental values. Notice however that the gluon condensate which appears here, is an average over configurations from $1/\Lambda^2$ to roughly $1/M_Q^2$ distances; and therefore cannot be related easily to the usual gluon condensate which appears in the phenomenology of QCD–sum rules. The predictions for the other $L_i$–coupling constants to $\mathcal{O}(M_Q^2)$. remain the same as those in (237).

Applications of this approach to the non–leptonic weak interactions have also been made. The problem here is to find the effective action of a given four–quark operator. For example, we have seen in Sec. 3, that the $\Delta S = 2$ transitions, after the heavy degrees of freedom have been integrated out, are governed by the four–quark operator

$$Q_{\Delta S=2} \equiv (\bar{s}_L \gamma^\mu d_L)(\bar{s}_L \gamma^\mu d_L),$$

modulated by products of flavour–mixing matrix elements, times Wilson coefficient functions resulting from the short–distance integration. The Euclidean effective action of $Q_{\Delta S=2}$ in an external gluonic background, is given by the expression (see Ref. 54 for technical details):

$$<Q_{\Delta S=2}> \bigg|_{G_{\mu,U,v,a,M,M_Q}} = -\text{Tr} \left( D_E^{-1} \frac{\delta \mu D_E}{\delta l_\mu(x)_{32}} \right) \text{Tr} \left( D_E^{-1} \frac{\delta \mu D_E}{\delta l_\mu(x)_{32}} \right) + \text{Tr} \left( D_E^{-1} \frac{\delta \mu D_E}{\delta l_\mu(x)_{32}} D_E^{-1} \frac{\delta \mu D_E}{\delta l_\mu(x)_{32}} \right).$$

(241)

The first term in the r.h.s. is the one induced by the configuration where the two currents in $Q_{\Delta S=2}$ are factorized, the term leading in the $1/N_c$–expansion; the second term contains the non–factorizable contributions, which are next–to–leading in the $1/N_c$–expansion. These contributions have been evaluated in Ref. 54 including, in addition to the well known $\mathcal{O}(N_c^2)$ factorizable contributions, the subleading $\mathcal{O}(N_c)$ and $\mathcal{O}(\alpha_s N_c)$ terms. The corresponding result for the $B_K$–parameter defined in it subsec. 3.3 reads as follows:

$$B_K = \frac{3}{4} \left[ 1 + \frac{1}{N_c} \left( 1 - \frac{N_c}{2} \frac{\langle \frac{\alpha_s}{\pi} GG >}{16\pi^2 f_\pi^4} + \mathcal{O}(\alpha_s N_c)^2 \right) \right].$$

(242)

The corresponding results for the other coupling constants of the $\mathcal{O}(p^2)$ non–leptonic effective Lagrangian discussed in Sect. 3, evaluated at the same approximation 54 as the $B_K$–parameter above, are the following:

$$g_2^{(3/2)} = C_+(\mu^2) \frac{2}{3} \left[ 1 + \frac{1}{N_c} \left( 1 - \frac{N_c}{2} \frac{\langle \frac{\alpha_s}{\pi} GG >}{16\pi^2 f_\pi^4} + \mathcal{O}(\alpha_s N_c)^2 \right) \right].$$

(243)

and

$$g_8 = C_-(\mu^2) \left[ 1 - \frac{1}{N_c} \left( 1 - \frac{N_c}{2} \frac{\langle \frac{\alpha_s}{\pi} GG >}{16\pi^2 f_\pi^4} + \mathcal{O}(\alpha_s N_c)^2 \right) \right].$$
\[ + C_+ (\mu^2) \frac{1}{5} \left[ 1 + \frac{1}{N_c} \left( 1 - \frac{N_c < \frac{\alpha_s}{\pi} G G >}{2 \frac{16\pi^2 f_\pi^4}{1}} + \mathcal{O}(\alpha_s N_c)^2 \right) \right] \]
\[ + C_4 (\mu^2) - C_6 (\mu^2) 16 L_5 \left[ \frac{< \bar{\psi} \psi >}{f_\pi^3} \right]^2 + \mathcal{O}(\alpha_s N_c). \]  

(244)

There are a number of interesting features which emerge from these results, worth commenting upon.

- The results of the so called vacuum saturation approximation (VSA), often used in the literature, are obtained from those above when the terms \( \mathcal{O}(\alpha_s N_c) \) are dropped. This model calculation shows however that, numerically, the neglected VSA–terms are as important as those retained.

- The results in eqs. (242) and (243) satisfy indeed the chiral limit symmetry relation between \( \Delta S = 2 \) and \( \Delta I = 3/2 \), \( \Delta S = 1 \) transitions first observed by Donoghue, Golowich and Holstein \( 86 \).

- Equations (242), (243), and (244) show an interesting correlation between the \( 1/N_c \)–corrections to the different couplings. The same correction which decreases the \( C_+ \)–modulated terms, and hence the \( B_K \)–parameter as well, increases the \( C_- \)–modulated term; and hence the \( \Delta I = 1/2 \)–transitions. It can be shown, in full generality, that this is in fact a general property of the full \( \mathcal{O}(N_c) \)–corrections in the chiral limit \( 87 \).

- The term proportional to \( C_6 (\mu^2) \) in Eq.(244) is the result of the next–to–leading \( 1/N_c \) calculation we have already discussed in the subsubsec. 3.2.2.

- The cancellation of the \( \mu^2 \)–dependence of the other Wilson coefficients with the bosonization of the corresponding four–quark operators, is more involved. To exhibit this cancellation explicitly, requires the knowledge of the full dynamics of next–to–leading order in the \( 1/N_c \)–expansion. It is clear that, in as far as one doesn't know the origin of the phenomenological term in eq.(238), this cannot be shown. It is possible however to show that the logarithmic \( \alpha_s \)–corrections to the bosonized operators are weighted by the same anomalous dimension factors as those of the original four–quark operators; i.e.,

\[ < Q_i > \sim - \frac{1}{2} \gamma_{ij}^{(1)} \frac{\alpha_s}{\pi} \log(\frac{\mu^2}{M_Q^2}) < Q_j >. \]

Further calculations within the framework of the Effective Action Approach Model discussed here can be found in references \( 88 \), \( 89 \), and \( 90 \). A possible generalization of the constituent mass ansatz term in \( 238 \) to a non–local form has also been suggested \( 91 \). Phenomenological applications using a non–local constituent mass term can be found in Ref. \( 92 \).
4.5 The Extended Nambu and Jona-Lasinio Model (ENJL–model)

Since the early work of Nambu and Jona-Lasinio, there have been many suggestions in the literature proposing models of the type first discussed by these authors, as relevant models for low–energy hadron dynamics. [For a recent review where earlier references can be found see.] The scenario suggested in Refs., which I shall follow here, assumes that at intermediate energies below or of the order of the spontaneous chiral symmetry breaking scale \( \Lambda_\chi \), the leading operators of higher dimension which, after integration of the high frequency modes of the quark and gluon fields down to the scale \( \Lambda_\chi \), become relevant in the QCD–Lagrangian, are those which can be cast in the form of four–fermion operators, i.e.,

\[
\mathcal{L}_{QCD} \Rightarrow \mathcal{L}_{QCD}^\chi + \mathcal{L}_{S,P} + \mathcal{L}_{V,A} + \cdots,
\]

where

\[
\mathcal{L}_{S,P} = \frac{1}{N_c} \frac{8\pi^2}{\Lambda_\chi^2} G_S \sum_{i,j} (\bar{q}_R^i q_L^j)(\bar{q}_L^i q_R^j),
\]

and

\[
\mathcal{L}_{V,A} = -\frac{1}{N_c} \frac{8\pi^2}{\Lambda_\chi^2} G_V \sum_{i,j} [(\bar{q}_L^i \gamma^\mu q_L^j)(\bar{q}_L^j \gamma^\mu q_L^i) + L \leftrightarrow R].
\]

Here \( i,j \) denote \( u, d, \) and \( s \) flavour indices and summation over colour degrees of freedom within each bracket is understood; \( q_{L,R} \equiv \frac{1}{2}(1 \pm \gamma_5)q \). The couplings \( G_{S,V} \) are dimensionless functions of the ultraviolet integration cut–off \( \Lambda \). They are expected to grow as \( \Lambda \) approaches the critical value \( \Lambda_\chi \), where spontaneous chiral symmetry breaking occurs. (This is the reason why the operators \( \mathcal{L}_{S,P} \) and \( \mathcal{L}_{V,A} \) become relevant.) In QCD, and with the factor \( N_c^{-1} \) pulled out, both couplings \( G_S \) and \( G_V \) are \( \mathcal{O}(1) \) in the large–\( N_c \) limit. These constants are in principle calculable functions of the ratio \( \Lambda / \Lambda_{QCD} \). In practice however, the calculation requires non–perturbative knowledge of QCD in the region where \( \Lambda \simeq \Lambda_\chi \), and we shall take \( G_S \) and \( G_V \), as well as \( \Lambda_\chi \), as independent unknown parameters. The \( \chi \) index in \( \mathcal{L}_{QCD}^\chi \) means that only the low–frequency modes \( \Lambda \leq \Lambda_\chi \) of the quark and gluon fields are to be considered from now onwards.

Notice that in QCD, couplings of the type \( \mathcal{L}_{S,P} \) and \( \mathcal{L}_{V,A} \) appear naturally from gluon exchange between two QCD colour currents. Using Fierz rearrangement, one has in the large–\( N_c \) limit:

\[
g_s^2 \sum_a \bar{q}^a \gamma^\mu \frac{\Lambda_\chi}{2} q)(\bar{q}^a \gamma^\mu \frac{\Lambda_\chi}{2} q) \Rightarrow \frac{1}{N_c} \frac{8\pi^2}{\Lambda_\chi^2} \frac{4\alpha_s N_c}{\pi} \sum_{i,j} (\bar{q}_R^i q_L^j)(\bar{q}_L^i q_R^j) + \frac{1}{N_c} \frac{8\pi^2}{\Lambda_\chi^2} \frac{4\alpha_s N_c}{\pi} \sum_{i,j} [(\bar{q}_L^i \gamma^\mu q_L^j)(\bar{q}_L^j \gamma^\mu q_L^i) + L \leftrightarrow R];
\]

i.e.; \( G_V = G_S/4 = \frac{\alpha_s N_c}{\pi} \) in this case. The two operators \( \mathcal{L}_{S,P} \) and \( \mathcal{L}_{V,A} \) have however different anomalous dimensions, and it is therefore not surprising that \( G_S \neq 4G_V \) for the corresponding physical values.
If furthermore, one assumes that the relevant gluonic effects for low–energy physics are those already absorbed in the new couplings $G_S$ and $G_V$, then

$$\mathcal{L}^\chi_{QCD} \Rightarrow i\bar{q} Dq$$

in Eq. (247) with $D$ the Dirac operator given in Eq. (249), where now the gluon field $G_\mu$ plays the rôle of an external colour field source. There is no gluonic kinetic term any longer.

As is well known from the early work of Nambu and Jona-Lasinio, the operator $L_{S,P}$, for values of $G_S > 1$, is at the origin of the spontaneous chiral symmetry breaking. This can best be seen following the standard procedure of introducing auxiliary field variables to convert the four–fermion coupling operators into bilinear quark operators. For this purpose, one introduces a $3 \times 3$ auxiliary field matrix $M(x)$ in flavour space; the so called collective field variables, which under chiral–$SU(3)$ transform as

$$M \rightarrow V_R M V_L^\dagger;$$

and uses the functional integral identity:

$$\exp \left[ i \int d^4x \frac{1}{N_c} \frac{8\pi^2}{\Lambda^2} G_S \sum_{i,j} (\bar{q}_R^i q_L^j)(\bar{q}_L^j q_R^i) \right] = \int \mathcal{D}M \exp \left[ i \int d^4x \{ - (\bar{q}_L M^\dagger q_R + h.c.) - N_c \frac{\Lambda^2}{8\pi^2} G_S \text{tr}MM^\dagger \} \right]. \quad (248)$$

By polar decomposition

$$M = \xi H \xi,$$

with $\xi \xi = U$ unitary and $H$ hermitian.

Next, we look for translational–invariant solutions, which minimize the effective action;

$$\frac{\partial \Gamma_{\text{eff}}}{\partial M} \big|_{H=\langle H \rangle=M_Q, \xi=1, v=a=s=p=0} = 0.$$  

The minimum is reached when all the eigenvalues of $\langle H \rangle$ are equal, i.e., $\langle H \rangle = M_Q 1$; and the minimum condition leads to

$$\text{Tr}(x \mid \frac{1}{\not{D}} \mid x) = -2M_Q N_c \frac{\Lambda^2}{8\pi^2} G_S \int d^4x. \quad (249)$$

The trace in the l.h.s. of this equation is formally proportional to $\langle \bar{\psi}\psi \rangle$. The calculation however requires a regularization, with $\Lambda^2$ the ultraviolet cut–off. We choose the proper time regularization. [See e.g., Ref. 46 for technical details.] Then

$$\langle \bar{\psi}\psi \rangle = -\frac{N_c}{16\pi^2} 4 M_Q^3 \Gamma(-1, \frac{M_Q^2}{\Lambda^2}) \quad (250)$$
and the minimum condition in Eq. (249) leads to the so-called *gap equation*:

\[
\frac{M_Q}{G_S} = M_Q \left\{ \exp \left( -\frac{M_Q^2}{\Lambda^2} \right) - \frac{M_Q^2}{\Lambda^2} \Gamma(0, \frac{M_Q^2}{\Lambda^2}) \right\}.
\]  

(251)

The functions

\[
\Gamma(n - 2, x \equiv \frac{M_Q^2}{\Lambda^2}) = \int_x^\infty \frac{dz}{z} e^{-z} z^{n-2}, \quad n = 1, 2, 3, \ldots,
\]

are incomplete gamma functions. Equations (250) and (251) show the existence of two phases with regards to chiral symmetry. The unbroken phase corresponds to the trivial solution \(M_Q = 0\), which implies \(<\bar{\psi}\psi> = 0\). The broken phase corresponds to the possibility that the coupling \(G_S\) increases as we decrease the ultraviolet cut–off \(\Lambda\) down to \(\Lambda_\chi\), allowing for solutions to Eq. (251) with \(M_Q > 0\) and therefore \(<\bar{\psi}\psi> \neq 0\) and negative. In this phase the Hermitian auxiliary field \(H(x)\) develops a non–vanishing vacuum expectation value, which is at the origin of a constituent chiral quark mass term [see the r.h.s. of Eq.(248)]:

\[-M_Q(q_L^U\bar{q}_R + q_R^U\bar{q}_L) = -M_Q\bar{Q}Q,\]

like the one which appears in the Georgi–Manohar model; and like the one proposed in the effective action approach of Ref. [85].

In the presence of the operator \(L_{V,A}\), we need two more auxiliary \(3 \times 3\) complex field matrices \(L_\mu(x)\) and \(R_\mu(x)\) to rearrange the Lagrangian in (245) into an equivalent Lagrangian which is only quadratic in the quark fields. Under chiral \((V_L, V_R)\) transformations these collective field variables are chosen to transform as follows:

\[L_\mu \to V_L L_\mu V_L^\dagger, \quad R_\mu \to V_R R_\mu V_R^\dagger.\]

Then, the following functional identity follows:

\[
\exp(-i \int d^4x \frac{1}{N_c} \frac{8\pi^2}{\Lambda^2} G_V \sum_{i,j} [(\bar{q}_L^i \gamma^\mu q_L^j)(\bar{q}_L^j \gamma_\mu q_L^i) + L \leftrightarrow R]) =
\]

\[
\int \mathcal{D}L_\mu \mathcal{D}R_\mu \exp[i \int d^4x \left\{ \bar{q}_L^i \gamma^\mu L_\mu q_L^j + N_c \frac{\Lambda^2}{8\pi^2} G_V \frac{1}{4} tr L_\mu L_\mu + L \leftrightarrow R \right\}] .
\]  

(252)

It is convenient to trade the auxiliary field matrices \(L_\mu(x)\) and \(R_\mu(x)\) by new vector field matrices

\[W_\mu^{(\pm)} = \xi L_\mu \xi + \xi R_\mu \xi,\]

which transform homogeneously under chiral transformations \((V_L, V_R)\); i.e.,

\[W_\mu^{(\pm)} \to h W_\mu^{(\pm)} h^\dagger,\]

47
with $h$ the $SU(3)_V$ rotation induced by $(V_L, V_R)$. The fermionic determinant can then be obtained using standard techniques, like for example the heat kernel expansion we described earlier. When computing the resulting effective action, there appears a mixing term between the fields $W_{\mu}^{(-)}$ and $\xi_{\mu}$. One needs a new redefinition of the auxiliary field $W_{\mu}^{(-)}$:

$$W_{\mu}^{(-)} \rightarrow \hat{W}_{\mu}^{(-)} + (1 - g_A)\xi_{\mu},$$

in order to diagonalize the quadratic form in the variables $W_{\mu}^{(-)}$ and $\xi_{\mu}$. It is this mixing which is at the origin of an effective axial coupling of the constituent quarks with the Nambu–Goldstone modes:

$$\frac{1}{2}ig_A\vec{Q}\gamma^\mu\gamma_5\xi_{\mu}\vec{Q},$$

a term like the axial coupling which appears in the Georgi–Manohar model, but with a specific form for the axial coupling constant $g_A$:

$$g_A = \frac{1}{1 + G_V \frac{4M_Q}{\Lambda_X} \left(\Gamma(0, \frac{M_Q}{\Lambda_X})\right)}.$$ (253)

In terms of Feynman diagrams this result can be understood as an infinite sum of constituent quark bubbles, with a coupling at the end to the pion field. These are the diagrams generated by the $G_V$ four-fermion coupling to leading order in the $1/N_c$–expansion. The quark propagators in these diagrams are constituent quark propagators, solution of the Schwinger-Dyson which is at the origin of the gap equation in (251). In the limit where $G_V = 0, g_A = 1$; but in general $g_A \neq 1$ to leading order in the $1/N_c$–expansion.

Kinetic terms for the auxiliary field variables are also generated by the functional integral over the quark fields $Q$ and $\bar{Q}$. The resulting Lagrangian, after wave–function rescaling of the auxiliary fields, has the form of a constituent chiral quark model, with scalar $S(x)$, vector $V(x)$, and axial–vector $A(x)$ field couplings:

$$\mathcal{L}^{ENJL}_{\text{eff}} = \frac{i}{\sqrt{2}f_V}V_\mu Q - M_Q\bar{Q}Q$$

$$+ \frac{i}{2}g_A\bar{Q}\gamma_5\gamma^\mu(\xi_{\mu} - \sqrt{2}\frac{f_A}{f_A}A_{\mu})Q - \frac{1}{\lambda_S}\bar{Q}S(x)Q$$

$$+ \frac{1}{2}tr[\partial_{\mu}S\partial^{\mu}S - M^2_SS]$$

$$- \frac{1}{4}tr[(\partial_{\mu}V_\nu - \partial_{\nu}V_\mu)(\partial^{\mu}V^{\nu} - \partial^{\nu}V^{\mu}) - 2MV_\mu V^{\mu}]$$

$$- \frac{1}{4}tr[(\partial_{\mu}A_\nu - \partial_{\nu}A_\mu)(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) - 2MA_\mu A^{\mu}]$$

$$+ \frac{1}{4}f^2\pi trD_\mu UD^{\mu}U^\dagger + \mathcal{O}(p^4)\text{terms},$$ (254)
where \( \Gamma_\mu \) and \( \xi_\mu \) are the same as those defined in Eqs.(225) and (226), and the coupling constants and masses are now expressed in terms of only three input parameters. As input parameters, we can either fix: \( G_S, G_V, \) and \( \Lambda_\chi \); or the more physical parameters: \( M_Q, \Lambda_\chi, g_A \).

The coupling constants are then:

\[
f_\pi^2 = \frac{N_c}{16\pi^2} 4M_Q^2g_A\Gamma(0, M_Q^2/\Lambda_\chi^2),
\]

\[
f_V^2 = \frac{N_c}{16\pi^2} \frac{2}{3} \Gamma(0, M_Q^2/\Lambda_\chi^2),
\]

\[
f_A^2 = \frac{N_c}{16\pi^2} \frac{2}{3} g_A^2 [\Gamma(0, M_Q^2/\Lambda_\chi^2) - \Gamma(1, M_Q^2/\Lambda_\chi^2)],
\]

\[
\lambda_S^2 = \frac{N_c}{16\pi^2} \frac{2}{3} [3\Gamma(0, M_Q^2/\Lambda_\chi^2) - 2\Gamma(1, M_Q^2/\Lambda_\chi^2)];
\]

and the masses:

\[
M_V^2 = 6M_Q^2 \frac{g_A}{1 - g_A},
\]

\[
M_A^2 = 6M_Q^2 \frac{1}{1 - g_A} \frac{1}{\Gamma(0, M_Q^2/\Lambda_\chi^2)} \frac{1}{\Gamma(1, M_Q^2/\Lambda_\chi^2)};
\]

and

\[
M_S^2 = 4M_Q^2 \frac{1}{1 - \frac{2}{3} \frac{\Gamma(1, M_Q^2/\Lambda_\chi^2)}{\Gamma(0, M_Q^2/\Lambda_\chi^2)}}.
\]

In the absence of the vector and axial–vector four–fermion like coupling i.e., when \( G_V = 0 \): \( g_A = 1, M_V \to \infty \) and \( M_A \to \infty \). Then the vector and axial–vector interactions decouple, and the model becomes equivalent to the Constitutuent Chiral Quark Model of Georgi and Manohar, with \( g_A = 1 \) and a non–trivial coupling to a scalar field.

The functional integration over the quark fields and the auxiliary \( S(x), V(x), \) and \( A(x) \) fields results in an effective action among the Nambu–Goldstone boson particles, with all the couplings fixed by the three parameters \( M_Q, \Lambda_\chi, \) and \( g_A \). The explicit results one gets for the \( L_i \) constants which appear in the large–\( N_c \) limit at \( \mathcal{O}(p^4) \) in the chiral expansion are shown in Table 2. The reason why the constant \( L_7 \) does not appear in this Table is that, phenomenologically, this constant gets a large contribution from the integration of the heavy singlet \( \eta' \) particle. However, in the chiral limit, the mass of the \( \eta' \) is induced by the axial–\( U(1) \) anomaly, which only appears to next–to–leading order in the \( 1/N_c \)–expansion. By definition, the ENJL–model, as formulated here, ignores this effect. In order to take these next–to–leading effects in \( 1/N_c \) systematically, together with the chiral expansion, one has to resort to a \( U(3) \times U(3) \) formulation of the effective theory \[99]. The constants
and \( L_6 \) are of next–to–leading order in the \( 1/N_c \)-expansion; this is the reason why they do not appear in Table 2 either. We also show in Table 2 the numerical results of the fit 1 discussed in Ref. [2]. These results correspond to the set of input parameter values:

\[
M_Q = 265 \text{ MeV}, \quad \Lambda_X = 1165 \text{ MeV}, \quad g_A = 0.61.
\]  

(262)

Table 2: The \( L_i \)-coupling constants in the ENJL–model of Ref. [2], with \( g_A \) defined in Eq. (253), and \( \Gamma_n \equiv \Gamma(n, M_Q^2/\Lambda_X^2) \). The second column gives the results corresponding to the input parameter values in (262). The third column gives the experimental values of Table 1.

| The \( L_i \) couplings of \( \mathcal{O}(p^4) \) in the ENJL–model | Fit 1 | Experiment |
|---------------------------------------------------------------|------|------------|
| \( L_1 = \frac{N_c}{16\pi^2} \frac{1}{48}[(1 - g_A^2)\Gamma_0 + 4g_A^2(1 - g_A^2)\Gamma_1 + 2g_A^4\Gamma_2] \) | 0.85 | 0.7 ± 0.5 |
| \( L_2 = 2L_1 \) | | 1.7 | 1.2 ± 0.4 |
| \( L_3 = -\frac{N_c}{16\pi^2} \frac{1}{8} \left[(1 - g_A^2)^2\Gamma_0 + 4g_A^2(1 - g_A^2)\Gamma_1 + \frac{5}{3}g_A^4[2\Gamma_1 - 4\Gamma_2 + 3\frac{1}{\Gamma_0}(\Gamma_0 - \Gamma_1)^2] \right] \) | | -4.2 | -3.6 ± 1.3 |
| \( L_5 = \frac{N_c}{16\pi^2} \frac{1}{4}g_A^3[\Gamma_0 - \Gamma_1] \) | | 1.6 | 1.4 ± 0.5 |
| \( L_8 = \frac{N_c}{16\pi^2} \frac{1}{16}g_A^3[\Gamma_0 - \frac{2}{3}\Gamma_1] \) | | 0.8 | 0.9 ± 0.3 |
| \( L_9 = \frac{N_c}{16\pi^2} \frac{1}{6}[(1 - g_A^2)\Gamma_0 + 2g_A^2\Gamma_1] \) | | 7.1 | 6.9 ± 0.7 |
| \( L_{10} = -\frac{N_c}{16\pi^2} \frac{1}{6}[(1 - g_A^2)\Gamma_0 + g_A^2\Gamma_1] \) | | -5.9 | -5.5 ± 0.7 |

The overall picture which emerges from this simple model is quite remarkable. The main improvement with respect to the results obtained in the effective action approach model discussed in subsec. 4.4 is on the constants \( L_5 \) and \( L_8 \), where the combined effect of the vector and scalar degrees of freedom leads to rather simple results modulated by powers of the \( g_A \)-constant, which agree very well with the phenomenological determinations. One of the characteristic features of the ENJL–model, is that it interpolates successfully between pure VMD–type predictions and those of the constituent chiral quark model. A nice illustration is the result for \( L_9 \) in Table 2, where the first term is the one coming from vector–exchange, while the second one comes from the chiral quark loop integral.
There is no difficulty to reproduce the anomalous Wess–Zumino–Witten functional within the ENJL–model [97].

QCD two–point functions, beyond the low–energy expansion, have also been evaluated in the ENJL–model [98]. This involves calculations to leading order in the $1/N_c$–expansion (i.e., an infinite number of chains of fermion bubbles; but no loops of chains,) and to all orders in powers of momenta $Q^2/\Lambda^2$. As a result, vector and axial–vector correlation functions have a VMD–like form, but with slowly varying couplings and masses. For the transverse invariant functions for example, the results are

$$\Pi^{(1)}_V(Q^2) = \frac{2f^2_V(Q^2)M^2_V(Q^2)}{M^2_V(Q^2) - Q^2},$$

and

$$\Pi^{(1)}_A(Q^2) = \frac{2f^2_A(Q^2) + 2f^2_A(Q^2)M^2_A(Q^2)}{M^2_A(Q^2) - Q^2},$$

where

$$f^2_V(Q^2) = 4 \frac{N_c}{16\pi^2} \int_0^1 dx x(1-x)\Gamma(0,xQ \equiv [M_{Q^2} + x(1-x)Q^2]/\Lambda^2)).$$

The product

$$2f^2_V(Q^2)M^2_V(Q^2) = N_c \frac{\Lambda^2}{8\pi^2} \frac{1}{G_V}$$

is scale invariant.

With

$$g_A(Q^2) = \frac{1}{1 + G_V \frac{4M_{Q^2}}{\Lambda^2} \int_0^1 dx \Gamma(0,xQ)},$$

the other couplings are fixed by

$$f^2_A(Q^2) = g^2_A(Q^2)f^2_V(Q^2),$$

and the relations

$$f^2_V(Q^2)M^2_V(Q^2) = f^2_A(Q^2)M^2_A(Q^2) + f^2_\pi(Q^2),$$

and

$$f^2_V(Q^2)M^4_V(Q^2) = f^2_A(Q^2)M^4_A(Q^2).$$

The last two equations are the $Q^2$–dependent version of the $1^{\text{st}}$– and $2^{\text{nd}}$–Weinberg sum rules [100].

In the case of the scalar two–point function there appears a pole in the $Q^2$–summed expression at $M_S = 2M_Q$. The case of the other two–point functions is somewhat more involved because they mix through the four–fermion interaction terms. The corresponding results can be found in Ref.[98]. Calculations of the low–energy behaviour of some three–point functions in the ENJL–model have also been
made \cite{10}. Corrections due to possible four–quark operators of higher dimension involving derivative terms, have also been studied recently \cite{102}.

In principle, the ENJL–model can be applied to obtain a systematic calculation of the low–energy constants of the weak non–leptonic Lagrangian, like the $B_K$–parameter, $g_8$ and $g_{27}$. The difficulties are mainly technical; but progress is underway.

5 Rare Kaon Decays and CP–Violation

This last lecture is dedicated to a discussion of the theoretical status, within the Standard Model, of the CP–violation parameters $\epsilon$ and $\epsilon'$ which we have introduced in Sec. 1. and also to the study of the decay $K_L \rightarrow \pi^0 e^+ e^-$. Other than the $K \rightarrow \pi \pi$ decays which we have discussed, this rare decay mode seems to be the most promising candidate to observe direct CP–violation in $K$–physics in the near future.

5.1 The Parameters $\epsilon$ and $\epsilon'$ Revisited

In Sec. 1. we have seen that CP–violation in $K \rightarrow \pi \pi$ decays is governed by the parameters $\epsilon$ and $\epsilon'$, [cf. Eqs.(52) and (53).] We have also seen that, to a good approximation, these parameters can be written as follows:

$$\epsilon \simeq \frac{1}{\sqrt{2}} e^{i \pi/4} \left( \frac{\text{Im} M_{12}}{\Delta m} + \frac{\text{Im} A_0}{\text{Re} A_0} \right),$$

(267)

and

$$\epsilon' \simeq \frac{1}{\sqrt{2}} e^{i(\delta_2 - \delta_0 + \pi/2)} \frac{\text{Re} A_2}{\text{Re} A_0} \left( \frac{\text{Im} A_2}{\text{Re} A_2} - \frac{\text{Im} A_0}{\text{Re} A_0} \right).$$

(268)

The question we want to discuss here, is the present status in the determination of the various components which appear in these expressions.

5.1.1 The mass difference $\Delta m$.

The $K_L – K_S$ mass difference $\Delta m$ is well determined experimentally \cite{3}:

$$\Delta m \equiv m_{K_L} - m_{K_S} = (3.522 \pm 0.016) \times 10^{-12} \text{MeV}.$$  

(269)

As we have shown in Sec. 1., $\Delta m$ is related to the real part of the off–diagonal $K^0 - \bar{K}^0$ mass matrix, [cf. Eq.(52)]:

$$\Delta m \simeq 2 \text{Re} M_{12}.$$

In the Standard Model, there are both short–distance and long–distance contributions to $\text{Re} M_{12}$. The short–distance contributions, from the effective $\Delta S = 2$
Hamiltonian described in subsec. 3.3, are modulated by the same unknown four–quark matrix element as the contribution to \( \text{Im} M_{12} \); i.e., the so called \( B_K \)–parameter defined in Eq.(213). If one was able to calculate the long–distance component to \( \text{Re} M_{12} \), it would be possible to fix the value of the \( B_K \)–factor from the measured value of \( \Delta m \). Unfortunately, as I shall next explain, this is not the case.

In principle, the long–distance contribution to \( \text{Re} M_{12} \), and hence to \( \Delta m \), can be evaluated in \( \chi PT \). It appears at the one–loop level, from diagrams with two vertex insertions of the effective \( \Delta S = 1 \) chiral Lagrangian of \( O(p^2) \) described in subsec. 3.2, in the presence of the strong interactions. These chiral loops are however divergent, and therefore they necessarily bring in new \( O(p^4) \), \( \Delta S = 2 \) local couplings which are not fixed by chiral symmetry requirements alone. Here again, one has to resort to models of the low–energy effective action to make further progress. Estimates of the long–distance contributions to \( \Delta m \), show that they are important \( 10^3 \). This is not surprising: the fact that nominally they are of higher order, \( O(p^4) \) in the chiral expansion, is here largely compensated by the relative \( \Delta I = \frac{1}{2} \) enhancement of the two lowest \( O(p^2) \) vertices. In the absence of a reliable way to calculate the needed couplings of the local \( O(p^4) \) \( \Delta S = 2 \) effective Lagrangian, there appears no way to get rid of the unknown \( B_K \)–factor in the ratio \( \frac{\text{Im} M_{12}}{\text{Re} M_{12}} \); and this is the reason why, for phenomenological purposes, one is forced to use the experimental value of \( \Delta m \) in Eq.(267).

5.1.2 The phase–shift difference \( \delta_2 - \delta_0 \).

Contrary to the situation concerning \( \Delta m \), the \( \pi - \pi \) phase shift difference \( \delta_2 - \delta_0 \) is poorly known experimentally. (It is not easy to extract \( \pi - \pi \) elastic scattering amplitudes from physical observables.) Here, \( \chi PT \) does better! One can extract this phase shift difference from the calculation of the elastic \( \pi - \pi \) amplitudes to \( O(p^4) \), without introducing new unknown parameters, with the result \( 10^4 \):

\[
\delta_{I=2}^J(M_K^2) - \delta_{I=0}^J(M_K^2) = -45^\circ \pm 6^\circ. \quad (270)
\]

5.1.3 \( \text{Re} A_0 \) and \( \text{Re} A_2 \).

There is very little we can do as yet, theoretically, concerning \( \text{Re} A_0 \) and \( \text{Re} A_2 \). As we have seen in Sec. 3., their calculation to lowest order in \( \chi PT \), involves the couplings \( g_8 \) and \( g_{27} \), which are not fixed by symmetry requirements alone. The models discussed in Sec. 4. are not yet developed to the required degree of precision for theory to be useful here. With neglect of electromagnetic corrections, the ratio \( \frac{\text{Re} A_2}{\text{Re} A_0} \), can be extracted from the physical branching ratio \( \frac{\Gamma(K_S \rightarrow \pi^+ \pi^-)}{\Gamma(K_S \rightarrow \pi^0 \pi^0)} \), as already discussed in Sec. 1., [cf. Eqs. (77) to (79)], with the result

\[
\frac{\text{Re} A_0}{\text{Re} A_2} = +22.2. \quad (271)
\]
Since \( \text{Im} A_I \ll \text{Re} A_I \), we can also extract \( \text{Re} A_0 \) from the experimental \( \Gamma(K \to \pi\pi) \) rates with the result
\[
|\text{Re} A_0| = 3.3 \times 10^{-7} \text{GeV}. \tag{272}
\]

To make predictions about \( \epsilon \) and \( \epsilon' \) we still need to know \( \text{Im} M_{12} \) and \( \text{Im} A_I \) for \( I = 0, 2 \).

### 5.1.4 The \( \epsilon \)-parameter.

It is presently known that, experimentally\footnote{See the lectures of Sunil Somalwar\cite{105} for a description of the relevant experiments.}: \( \epsilon' \ll \epsilon \). On the other hand, the Bell-Steinberger inequality we have discussed in subsec. 1.2, when restricted to the \( 2\pi \) intermediate states, can be written as follows:
\[
(\frac{\Gamma_S + \Gamma_L}{2} + i\Delta m) 2\text{Re} \bar{\epsilon} = \epsilon \Gamma_S + \epsilon'[\Gamma_S(+-) - 2\Gamma_S(00)]. \tag{273}
\]

The second term in the r.h.s. is suppressed in two ways: i) because \( \epsilon' \ll \epsilon \); ii) because of the \( \Delta I = 1/2 \)–rule. Furthermore, using the empirical facts that \( \Gamma_L \ll \Gamma_S \) and \( \Delta m \simeq \Gamma_S/2 \), we arrive at the constraint
\[
(1 + i)\text{Re} \bar{\epsilon} \simeq \epsilon. \tag{274}
\]

With \( \bar{\epsilon} \) given by Eq.\((78)\) and \( \epsilon \) by Eq.\((267)\), this constraint implies the neglect of \( \frac{\text{Im} A_0}{\text{Re} A_0} \) versus \( \frac{\text{Im} M_{12}}{\Delta m} \). We conclude that, to a good approximation, \( \epsilon \) is completely governed by the size of \( M_{12} \), evaluated in the usual phase convention of flavour mixing in the Standard Model. With \( \frac{\text{Im} A_0}{\text{Re} A_0} \) versus \( \frac{\text{Im} M_{12}}{\Delta m} \) neglected, it is reasonable to neglect as well the long distance effects contributing to \( \text{Im} M_{12} \). In fact, it has been argued that the two effects largely cancel each other\cite{103}. To this approximation, \( \epsilon \) is then entirely given by the local structure of the \( \Delta S = 2 \) box diagrams which generate the four–quark operator in \((207)\).

It has become conventional in the phenomenology of flavour dynamics, to use an approximate form of the flavour mixing matrix; the Cabibbo–Kobayashi–Maskawa matrix\footnote{See the lectures of Roberto Peccei\cite{37} for further details on this parametrization. For a recent update of the Unitarity Triangle see Ref.\cite{106}.}:
\[
V = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3\sigma e^{-i\delta} \\
-\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
A\lambda^3(1 - \sigma e^{i\delta}) & -A\lambda^2 & 1
\end{pmatrix} + \mathcal{O}(\lambda^4), \tag{275}
\]
where \( \lambda \) denotes the Cabibbo angle,
\[
\lambda \simeq |V_{us}| = 0.2205 \pm 0.0018; \tag{276}
\]
and \( A \) and \( \sigma \) are parameters of order one. The phase \( \delta \) is at the origin of the CP–violation in the Standard Model. The merit of this parametrization is that it clearly...
shows the orders of magnitude of the various mixing matrix elements. Often, the notation
\[ \sigma e^{-is} = \rho - i\eta, \] (277)
is also used. In this parametrization, the Jarlskog invariant which governs all the observables which violate CP–invariance \[^{107}\] is given by the product:
\[ |J_{\text{CP}}| = 2\eta \lambda^6 A^2. \] (278)
The \(A\)-parameter is fixed from \(B\)-decays into states containing charm:
\[ A \simeq \lambda^{-2} |V_{cb}| = 0.82 \pm 0.12; \] (279)
and \(\sigma\) is fixed, once \(|V_{ub}|\) is measured:
\[ \lambda \sigma \simeq \frac{|V_{ub}|}{|V_{cb}|} = 0.08 \pm 0.02. \] (280)

In terms of this parametrization of the mixing matrix, and to the approximation that we have finally adopted, the \(\epsilon\)-parameter can then be written as follows:
\[ \epsilon = \frac{1}{\sqrt{2}} e^{i\pi/4} \frac{4}{\Delta m} \frac{1}{3} \frac{G_F^2}{2M_K} \frac{1}{2M_K} \frac{1}{2\pi^2} M_W^2 \times \hat{B}_K \times \eta \lambda^6 A^2 \left[ \eta_3 S(r_c, r_t) - \eta_1 S(r_c) + A^2 \lambda^4 (1 - \rho) \eta_2 S(r_t) \right]. \] (281)

In this expression, \(\hat{B}_K\) denotes the scale invariant \(B_K\)-parameter:
\[ \hat{B}_K \equiv \alpha_s(\mu^2)^{-2/9} B_K(\mu^2); \] (282)
and \(S(r_c, r_t)\) and \(S(r_t)\) are the box–diagram functions of the ratios \(r_q \equiv \frac{m_q^2}{M_V^2}\), with \(m_q\) the corresponding quark mass running at the pole value:
\[ S(r_t) = \frac{r_t}{4} \left[ 1 + \frac{9}{1 - r_t} - \frac{6}{(1 - r_t)^2} - \frac{6 r_t^2 \log r_t}{(1 - r_t)^3} \right]; \] (283)
and
\[ S(r_c, r_t) = r_c \left[ \log \frac{r_t}{r_c} - \frac{3 r_t}{4(1 - r_t)} \left( 1 + \frac{r_t}{1 - r_t} \log r_t \right) \right]. \] (284)

The \(\eta_i\) parameters correct for short–distance gluon exchanges in the \(\Delta S = 2\) box–diagrams. \[^{108}\] With \(\hat{B}\) factored out, they are renormalization scale invariant, with the values** :
\[ \eta_1 = 1.10, \quad \eta_2 = 0.57, \quad \eta_3 = 0.36. \] (285)
** The \(\eta_1\)-parameter is rather sensitive to \(m_c\) and \(\Lambda_{QCD}\), which explains the different values found in the literature. Here we use \[^{109}\] \(\Lambda_{MS}^{(5)} = (240 \pm 90) \text{MeV}\), which corresponds to \(\Lambda_{MS}^{(3)} = (350 \pm 150) \text{MeV}\); for the charm quark pole mass we take \[^{110}\] \(m_c = (1.47 \pm 0.05) \text{MeV}\).
Equation (281), with the value of $\epsilon$ fixed from experiment $|\epsilon| = (2.26 \pm 0.02) \times 10^{-3}$, and for a given value of $\hat{B}_K$, determines then a hyperbola in the $(\rho, \eta)$ plane. A useful approximation to Eq. (281), for quick numerical estimates is

$$|\epsilon| = (3.4 \times 10^{-3}) \eta A^2 \hat{B}_K \left[ 1 + 1.3 A^2 (1 - \rho) \left( \frac{m_\pi}{M_W} \right)^{1.6} \right].$$

(286)

### 5.1.5 Calculations of the $\hat{B}_K$-parameter.

There exist several calculations of this parameter using very different techniques. The problem is that, although the errors from some of the calculations are rather small, the central values are still too dispersed. Two simple calculations are the ones we have already discussed in subsec. 3.3: the vacuum saturation approximation which gives

$$(B_K)_{\text{VS}} = 1;$$

(287)

and the large–$N_c$ prediction, which corresponds to

$$B_K|_{N_c \to \infty} = 3/4.$$ (288)

None of these two calculations can distinguish between $\hat{B}_K$ and $B_K(\mu^2)$.

We also pointed out in subsec. 4.4 that, as first observed by Donoghue, Golowich and Holstein, there is a symmetry relation in the chiral limit between $\Delta S = 2$ and $\Delta I = 3/2$, $\Delta S = 1$ transitions. Using the known $K^+ \to \pi^+\pi^0$ decay rate, it is then possible to fix $B_K$ in that limit, with the result

$$\hat{B}_K|_{\chi}\text{PT to } O(p^2) = 0.37.$$ (289)

The calculation of $B_K$ within the effective action approach model we have discussed in subsec. 4.4 shows how to reconcile this result with the large–$N_c$ result above. The next–to–leading corrections in the $1/N_c$–expansion appear to be large and negative. The same pattern appears in the $1/N_c$–approach developped by Bardeen, Buras and Gérard. From their matching of short–distances to long–distances, these authors give the estimate:

$$\hat{B}_K = 0.70 \pm 0.10.$$ (290)

This result includes the effect of the one loop $O(p^4)$ chiral corrections as well, which are large and positive. Higher $O(p^4)$ chiral corrections, including the effect of the local $O(p^4)$ terms, evaluated within the effective action approach model, have also been recently estimated, with the result

$$\hat{B}_K = 0.42 \pm 0.06.$$ (291)

There are a variety of QCD Sum Rules that have been used to estimate $B_K$ as well. The description of the techniques involved in these calculations goes beyond the scope of these lectures. I shall however give some results. The most recent
estimate of $B_K$ using two-point function sum rules of the $\Delta S = 2$ four-quark local operator $\Delta S = 2$ gives

$$\hat{B}_K \bigg|_{2\text{p.f. sum rules}} = 0.39 \pm 0.10.$$ (292)

The same approach reproduces rather well the observed $K^+ \to \pi^+\pi^0$ decay rate. (It overestimates by less than 15% the coupling constant $g^{(3/2)}_{27}$.)

The calculations based on QCD Sum Rules for three-point functions give a large variety of outputs. I shall only quote the result of Ref. [112], wherefrom one can trace earlier results:

$$\hat{B}_K \bigg|_{3\text{p.f. sum rules}} = 0.5 \pm 0.1 \pm 0.2$$ (293)

Lattice–QCD simulations of the $B_K$–parameter tend to find results in the higher range of the estimates we have presented here. They are discussed by Steve Sharpe in his lectures at this TASI school.

In my opinion, the only safe claim that theorists can make at present is that the value of $\hat{B}_K$ is likely to lie within the range:

$$0.35 \leq \hat{B}_K \leq 0.80.$$ (294)

5.1.6 The ratio $\epsilon'/\epsilon$.

As discussed in the lectures of Sunil Somalwar [105], there is experimental information on this ratio of parameters from the measurement of the ratio of branching ratios [cf. Eqs. (40), (64), and (65)]:

$$\left| \frac{\eta_{\pi \pi}}{\eta_{\pi \pi}} \right| \simeq 1 + 6 \text{Re}(\frac{\epsilon'}{\epsilon}) ,$$ (295)

with the results:

$$\text{Re}(\frac{\epsilon'}{\epsilon}) = \begin{cases} 
(23 \pm 6.5) \times 10^{-4} & \text{CERN – NA31}, \\
(7.4 \pm 6.0) \times 10^{-4} & \text{FERMI lab – E731}.
\end{cases} (296)$$

The phases in the expressions of $\epsilon$ and $\epsilon'$ being practically the same, the theoretical prediction for the ratio $\frac{\epsilon'}{\epsilon}$ turns out to be a real number. If furthermore, we use the experimental input in Eq. (271) as well as the experimental determination of $|\epsilon| \simeq 2.3 \times 10^{-3}$, we get

$$\frac{\epsilon'}{\epsilon} = -\frac{1}{\sqrt{2}} \left( \frac{\text{Re}A_0}{\text{Re}A_2} \simeq 22.2 \right) \left( |\epsilon| \simeq 2.3 \times 10^{-3} \right) \frac{1}{\text{Im}A_0} \left( 1 - \frac{22.2 \text{Im}A_2}{\text{Im}A_0} \right).$$ (297)

The dominant contribution to $\text{Im}A_0$ originates in the diagrams which give rise to the Penguin-operator $Q_6$. They bring the factors $V_{td}V_{us}^*, V_{td}V_{cs}^*$, and hence the CP-violation phase which contributes to $\text{Im}A_0$. We have seen in subsec. 3.2, that to lowest order in $\chi$PT, the amplitude $A_0$ takes the form,

$$A_0 = -\frac{G_F}{\sqrt{2}} V_{td}V_{us}^* (g_8 + g_{27}^{(1/2)}) \sqrt{2} f_\pi (M_K^2 - m_\pi^2).$$

57
The operator $Q_6$ transforms like an $(8_L, 1_R)$ operator and contributes only to $g_8$. In the large--$N_c$ expansion, its contribution to $g_8$ is *next-to-leading*. As we discussed in 3.2.2, it is quite remarkable that, at this order of approximation, this contribution can be calculated exactly in terms of known parameters!. Remember the result [cf. Eq. (204)]:

$$\text{Im} g_8 = -\text{Im} C_6 (\mu^2) 16L_5 \left[ \frac{\langle \bar{\psi}\psi \rangle}{f^3_\pi} \right]^2 \{1 + \mathcal{O}(1/N_c) + \mathcal{O}(p^2) \}. \quad (298)$$

With this result in hand, we can rewrite Eq. (297) in the following way:

$$\frac{\epsilon'}{\epsilon} = \frac{1}{\sqrt{2}} \frac{1}{22.2} \frac{\eta \lambda^4 A^2}{(2.3 \times 10^{-3})} \left( \frac{0.09 \pm 0.01}{\text{Re}[g_8 + g_{27}^{(1/2)}] \approx 5.1} \right) 16L_5 \left[ \frac{\langle \bar{\psi}\psi \rangle}{f^3_\pi} \right]^2$$

$$\times \left( 1 - 22.2 \frac{\text{Im} A_2}{\text{Im} A_0} \right) \{1 + \mathcal{O}(1/N_c) + \mathcal{O}(p^2) \}. \quad (299)$$

Several comments concerning this result are in order:

- The factor $(0.09 \pm 0.01)$ in the r.h.s. is the value of the calculated short–distance Wilson coefficient $\text{Im} C_6$, once the overall flavour factor $\eta \lambda^4 A^2$ has been pulled out. The error here comes from the uncertainties in the top–quark mass and in $\Lambda_{\overline{MS}}$.

- As we already mentioned in subsec. 3.2, the largest uncertainty from the calculated matrix element of the “penguin” $Q_6$–operator is due to the poorly known value of the quark condensate $\langle \bar{\psi}\psi \rangle$. [Cf. Eq. (205).] It is customary to trade this factor by the value of the strange–quark mass, using the Gell-Mann–Oakes–Renner relation in Eq. (105) of lowest order $\chi$PT. I prefer not to do that, because then one should also discuss corrections to the Gell-Mann–Oakes–Renner relation, which complicates things even further.

- Because of the large coefficient $-22.2$–modulating $\text{Im} A_2/\text{Im} A_0$ in the r.h.s. of the final expression for $\epsilon'/\epsilon$ above, a reliable estimate of the ratio $\text{Im} A_2/\text{Im} A_0$ is clearly needed. There are two sources of contributions to this term:

  i) Isospin breaking effects,

  ii) Electroweak “penguin” effects, where the gluon exchange of the ordinary “penguin”–like diagram is replaced by a photon or a $Z$–vector boson. Because of the isovector component in the coupling of the photon and the $Z$ to the quark–currents, the electroweak “penguin” diagrams lead to effective operators which can induce $\Delta I = 3/2$–transitions.

Both effects i) and ii) have been shown to go in the same direction, and they decrease the leading effect of the $Q_6$–operator that we have calculated above. The effect ii) has become particularly important because, as the top–quark
mass \( m_t \) increases, for \( m_t > M_W \), it grows roughly as \( m_t^2 \). [See Refs. \textsuperscript{10,11} for a detailed discussion.] Isospin breaking effects due to \( \pi^0 - \eta - \eta' \) mixing have been estimated \textsuperscript{11} to contribute

\[
\left( \frac{\text{Im} A_2}{\text{Im} A_0} \right)_{\eta+\eta'} \simeq 1.4 \times 10^{-2}.
\]

- Some attempts to estimate the other \( \mathcal{O}(\frac{1}{\Lambda}) \) and the other \( \mathcal{O}(g^2) \) corrections in the r.h.s. of Eq. (293) have also been made. The results however are controversial. New efforts in that direction are urgently needed, if we want to compare usefully with the next generation of \( \epsilon'/\epsilon \) experiments.

Due to the various uncertainties we have discussed, it is difficult to claim, at present, a theoretical prediction for \( \epsilon'/\epsilon \) better than the limits:

\[
10^{-4} \leq \frac{\epsilon'}{\epsilon} \leq 2 \times 10^{-3}.
\]

5.2 The Decay \( K_L \rightarrow \pi^0 e^+ e^- \) and CP–Violation

Transitions like \( K \rightarrow \pi \gamma \), with \( \gamma \) a real photon, are forbidden by gauge invariance. They are allowed, however, for virtual photons \( \gamma^* \), which can produce real lepton pairs. Because of the absence of strangenes changing neutral currents in the Standard Model, transitions like \( K \rightarrow \pi l^+ l^- \), with \( l = e, \mu \), are then governed by the interplay of weak non–leptonic and electromagnetic interactions. To lowest order in the electromagnetic coupling constant they are expected to proceed, in principle dominantly, via one–photon exchange. This is certainly the case for the \( K^\pm \rightarrow \pi^\pm l^+ l^- \) and \( K_S \rightarrow \pi^0 l^+ l^- \) decays. The transition \( K^0_2 \rightarrow \pi^0 \gamma^* \rightarrow \pi^0 l^+ l^- \), via one virtual photon, is however forbidden by CP–invariance. It is then not obvious any longer whether the physical decay \( K_L \rightarrow \pi^0 l^+ l^- \) will be still dominated by the CP–suppressed \( \gamma^* \)–virtual transition; or whether a transition via two virtual photons, which is of a higher order in the electromagnetic coupling but CP–allowed, may dominate. With the possibility of reaching branching ratios for the mode \( K_L \rightarrow \pi^0 e^+ e^- \) as small as \( 10^{-12} \) in the near future dedicated experiments, the theoretical study of this mode has become of major importance.

5.2.1 The one–photon exchange amplitude

The interesting thing about this transition amplitude is that it gets a direct contribution coming from the terms in the \( \Delta S = 1 \) short–distance Hamiltonian proportional to the operators \( Q_{11} \) and \( Q_{12} \) in subsec. 3.1. These are the operators induced by “penguin”–like diagrams, where the gluon exchange of the ordinary “penguin”–like diagram is replaced by a photon or a \( Z \)–vector boson. Much the same as in the case of the \( Q_6 \)–operator which we have already discussed, these diagrams bring in
the factors $V_{td}V_{ts}^*, V_{cd}V_{cs}^*$, and hence the CP–violation phase. The most recent calculation of the corresponding Wilson coefficients, can be found in Ref. [115]. Another interesting issue concerning this transition, is that the hadronic matrix element here is the one of a quark bilinear operator; and by isospin symmetry is related to the well known hadronic matrix element of charged $K_{13}$–decays. The main sources of errors in the calculation of this direct transition amplitude are then the mass of the top quark, and the CP–violation $\eta$–parameter [see Eq.(281)], which modulates the overall amplitude. Letting the $\hat{B}_K$–parameter be within the range in Eq.(294); and for a top mass $150\text{GeV} \leq m_t \leq 200\text{GeV}$, the expected branching ratio from this direct source of CP–violation is

$$1.3 \times 10^{-12} \leq \text{Br}(K_L \to \pi^0 e^+ e^-)|_{\gamma\text{dir.}} \leq 5.8 \times 10^{-12}. \quad (302)$$

Let me remind you that the present experimental upper limit is [116] (90% C.L.)

$$\text{Br}(K_L \to \pi^0 e^+ e^-)|_{\exp.} < 5.5 \times 10^{-9}, \quad (303)$$

still quite far away.

The other source of CP–violation, usually called “indirect”, is the one induced by the $K_0^0$–component of the $K_L$ state, which brings in the CP–violation admixture parameter $\tilde{\epsilon}$, i.e., the parameter $\epsilon$ to a very good approximation. The problem here is then reduced to the evaluation of the CP–conserving transition $K_S \to \pi^0 e^+ e^-$. The analysis of $K^+ \to \pi^+ l^+ l^−$ decays in general, within the framework of $\chi$PT has been made in Refs. [59, 60]. To $O(p^4)$ in the chiral expansion, the corresponding decay amplitudes get contributions both from one chiral loop graphs, and from tree level contributions from local operators of $O(p^4)$. In fact, only two local operators of the $O(p^4)$ non–leptonic effective Lagrangian contribute to the CP–conserving amplitudes:

$$\mathcal{L}^\Delta_{\text{eff}}^S=1(x) \doteq -ieG_8 \frac{2}{f^2} F^{\mu\nu} \left\{ w_1 \text{tr}(Q \lambda_6-i\lambda_7 \mathcal{L}_\mu \mathcal{L}_\nu) + w_2 \text{tr}(Q \mathcal{L}_\mu \lambda_6-i\lambda_7 \mathcal{L}_\nu) \right\}, \quad (304)$$

where $\mathcal{L}_\mu(x)$ is the $3 \times 3$ flavour matrix field we introduced in Eq.(185); $Q$ the electric charge matrix: diag$(2/3, -1/3, -1/3)$; $F^{\mu\nu}$ the electromagnetic field strength tensor; $\lambda_{6-i7}$ the $SU(3)$ Gell-Mann matrix: $\lambda_6-i\lambda_7$; and $w_{1,2}$ are two dimensionless coupling constants not fixed by chiral symmetry requirements alone. The overall constant $G_8$ denotes the combination of couplings

$$G_8 = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* g_8, \quad |G_8| \approx 9 \times 10^{-6} \text{GeV}^{-2}. \quad (305)$$

The mode $K^+ \to \pi^+ e^+ e^-$ turns out to be particularly interesting because both its branching ratio and the pion energy spectrum have now been measured in the BNL–E777 experiment [117]. In full generality, one can predict the $K^+ \to \pi^+ l^+ l^−$
decay rates as a function of the unknown scale invariant combination of coupling constants

\[
\mathbf{w}_+ = -\frac{1}{3}(4\pi^2)[\mathbf{w}_1 - \mathbf{w}_2 + 3(\mathbf{w}_2^r - 4L_9^r)] - \frac{1}{6} \log \frac{M_K^2 m_\pi^2}{\nu^4}.
\]

(306)

where \( \mathbf{w}_1, \mathbf{w}_2 \) and \( L_9 \) are renormalized couplings at the scale \( \nu \); \( L_9 \) is the same strong coupling constant which governs the mean squared radius of the pion \[\text{see the discussion in section 2.2.2}]. Viceversa, the measured branching ratio \[\text{determines two possible solutions for the combination of constants } \mathbf{w}_+. \text{ The degeneracy between the two possible solutions can be lifted from the measurement of the invariant mass distribution of the final lepton pair, (the } q^2 \text{–dependence; or the } \pi^+ \text{ energy–spectrum in the } K^+ \text{ rest–system). A fit to the high– } q^2 \text{ spectrum from the latest BNL-E777 data \[\text{favours the positive solution, with the result:}}\]

\[
\mathbf{w}_+ = 0.89 + 0.24 - 0.14 ;
\]

(308)

while the positive solution, extracted from the measured decay rate in the same experiment corresponds to the value:

\[
\mathbf{w}_+^{BR} = 1.2 + 0.4 - 0.5 .
\]

(309)

The two numbers are consistent with each other within less than two standard deviations; a fact which provides an independent check of the level of accuracy of \( \mathcal{O}(p^4) - \chi \)PT to describe this process.

Unfortunately, the determination of the combination of constants \( \mathbf{w}_+ \) is not enough to predict the \( K^0 \to \pi^0 e^+ e^- \) decay rate. To the same order in the chiral expansion, the corresponding transition amplitude brings in the combination of constants:

\[
\mathbf{w}_s = -\frac{1}{3}(4\pi^2)[\mathbf{w}_1 - \mathbf{w}_2] - \frac{1}{3} \log \frac{M_K^2}{\nu^2} .
\]

(310)

Clearly, one has to resort to models to go any further in making a prediction.

In Ref.\[\text{59}\] it was pointed out that the relation \( \mathbf{w}_2 - 4L_9 = 0 \) holds for the divergent parts of the corresponding regularized coupling constants. It is also valid for the full couplings if, as it happens in many models, the mesonic effective strangeness–changing current which couples to the virtual photon is required to transform as a pure \( SU(3) \) octet. In general however, this current is allowed to have terms which transform as 10 and \( \bar{10} \) \( SU(3) \)–representations as well. Therefore, as emphasized in \[\text{59}\], the constraint: \( \mathbf{w}_2 = 4L_9 \), is not required by general chiral invariance arguments alone. It is nevertheless tempting to see what the prediction for the \( K_S \)–mode is in the class of models which satisfy this relation. Using the center value
for \( w_+ = 1.2 \) from the measurement of the \( K^+ \)-mode branching ratio quoted above, one gets

\[
\text{Br}(K_S \to \pi^0 e^+ e^-) \big|_{w_2 = 4L_9'} \simeq 5.4 \times 10^{-10},
\]

(311)

which in turn, corresponds to an “indirect” CP-violation branching ratio

\[
\text{Br}(K_L \to \pi^0 e^+ e^-) \big|_{1\gamma \text{ind.}(w_2 = 4L_9')} \simeq 1.6 \times 10^{-12},
\]

(312)
on the lower range of the “direct” CP-violation prediction.

It is important to analyze the sensitivity of this result to models which do not satisfy the constraint \( w_2 = 4L_9' \). This has been done in Ref. [89] and more recently in Ref. [118]. The outcome is that the \( K_S \)-branching ratio is rather sensitive to small variations of the octet–dominance constraint. As a result, “indirect” CP-violation branching ratios comparable, if not bigger, than the “direct” prediction cannot be excluded for the time being. Once more, we find the need to develop good models of the low–energy effective action, if one wants to make further progress. There is not much else that \( \chiPT \) can do here, except wait for the experimentalists to measure the rate of \( K_S \to \pi^0 e^+ e^- \); not an easy task!

5.2.2 The two-photon exchange amplitude

The transition \( K_2 \to \pi^0 \gamma^* \gamma^* \to \pi^0 e^+ e^- \) is CP-allowed. The lowest non-trivial order calculation of this process in \( \chiPT \) involves at least two loops. It is possible, however, to obtain a lower bound to the \( 2\gamma \)-exchange rate from the calculation of the contribution to the absorptive part of the amplitude \( A(K_1^0 \to \pi^0 e^+ e^-) \) coming from the \( \gamma \gamma \)-discontinuity.

The general kinematics decomposition of the subprocess \( K_2^0 \to \pi^0 \gamma \gamma \) involves two invariant amplitudes. With

\[
K_2(p) \to \pi^0(p') + \gamma(q_1) + \gamma(q_2), \quad p^2 = M^2, \quad p'^2 = M'^2, \quad q_1^2 = q_2^2 = 0,
\]
gauge invariance and Lorentz invariance restrict the form of the transition amplitude to

\[
\mathcal{M}[K_2(p) \to \pi^0(p') \gamma(q_1) \gamma(q_2)] = G_\text{em}^\alpha \epsilon_{\mu
u}(q_1) \epsilon_{\rho\sigma}(q_2) \left[ A(y, z) (q_2^\mu q_1^\nu - q_1 \cdot q_2 g^{\mu\nu}) + 2B(y, z) \left( p \cdot q_1 q_2^\mu p'^\nu + p \cdot q_2 q_1^\nu p'^\mu - p'^\mu p'^\nu q_1 \cdot q_2 - p \cdot q_1 p \cdot q_2 g^{\mu\nu} \right) \right],
\]

(313)

where

\[
y = \frac{p \cdot (q_1 - q_2)}{M^2}, \quad \text{and} \quad z = \frac{(q_1 + q_2)^2}{M^2}.
\]

(314)

Bose symmetry requires the amplitudes \( A \) and \( B \) to be symmetric functions of \( q_1 \) and \( q_2 \), i.e.

\[
A(-y, z) = A(y, z), \quad B(-y, z) = B(y, z).
\]

(315)
To lowest non–trivial order in $\chi$PT, only the amplitude proportional to $A$ contributes; and in fact, to that order, $A$ depends only on $z$ i.e., the invariant mass squared of the $\gamma\gamma$–pair. None of the local terms of $O(p^4)$ in the effective $\Delta S = 1$ non–leptonic Lagrangian can contribute to this decay. As a result, the full contribution at lowest non–trivial order in the chiral expansion, comes only from the finite chiral one loop amplitude. The predicted $z$–distribution has a very characteristic shape; being highly peacked at the higher end of the spectrum. This prediction has been subsequently confirmed experimentally by the CERN-NA31 collaboration. However the $O(p^4)$ predicted branching ratio:

$$\text{Br}(K_L \to \pi^0 e^+ e^-) = 6.8 \times 10^{-7},$$

turns out to be smaller than the observed rates:

$$\text{Br}(K_L \to \pi^0 e^+ e^-) = \begin{cases} (1.7 \pm 0.2 \pm 0.2) \times 10^{-6} & \text{NA31}^{120}, \\ (1.86 \pm 0.60 \pm 0.60) \times 10^{-6} & \text{E731}^{121}. \end{cases}$$

It has been claimed, however, that phenomenologically expected higher order effects can explain this discrepancy.

The term proportional to the $A$–amplitude in Eq.(313) has a tensor structure which, when inserted in the two–body $\gamma\gamma$ phase space integral to get the corresponding $K^0_2 \to \pi^0 e^+ e^-$ amplitude, requires a flip in the helicity of the electron line for the transition to be allowed; a fact which brings in the electron mass $m_e$ as an overall suppression factor in amplitude. This amplitude structure leads to a branching ratio:

$$\text{Br}(K^0_2 \to \pi^0 \gamma\gamma \to \pi^0 e^+ e^-) \bigg|_{A(\gamma\gamma)} \simeq 5 \times 10^{-15},$$

too small, by several orders of magnitude to be competitive with the one–photon exchange, CP–violating amplitude, contributing to $K_L \to \pi^0 e^+ e^-$. The interesting issue here is that, contrary to what happens with the contribution from the $A(\gamma\gamma)$–amplitude which we have just discussed, the tensor structure proportional to the $B$–amplitude in Eq.(313), when inserted in the two–photon phase space integral of the process $K^0_2 \to \pi^0 \gamma^*\gamma^* \to \pi^0 e^+ e^-$ has contributions which are allowed without requiring an electron helicity flip; i.e., they are not $m_e$–suppressed. Although the $B(\gamma\gamma)$–amplitude first appears at $O(p^6)$ in $\chi$PT; the fact that it can induce a helicity allowed transition to $K^0_2 \to \pi^0 \gamma^*\gamma^* \to \pi^0 e^+ e^-$, makes it “the dominant amplitude” for this process. There have been several estimates in the literature of the $B$–amplitude in Eq.(313), based on vector meson dominance ($VMD$) models. Here, however, one has to be careful not to overestimate its size and therefore spoil the observed shape of the invariant $\gamma\gamma$–mass distribution in the $K_L \to \pi^0 e^+ e^-$ decay. Many of the early models are in fact now ruled out by the NA31 experiment; but they have been very useful to sharpen our views on the rôle of vector mesons in $\chi$PT in general, with the results we already discussed in subsec. 4.2.
Within the framework of \( \chi \)PT, one can parametrize the local \( \mathcal{O}(p^6) \) contributions to \( K_L \to \pi^0\gamma\gamma \) induced by vector–exchanges, (including vector–exchanges in direct weak transitions,) by an effective vector coupling \( a_V \), such that \( B \):

\[
B = -2a_V G_8 M_K^2 \frac{\alpha}{\pi}.
\]  

In terms of this parametrization, the contribution to the \( K_0^0 \to \pi^0 e^+ e^- \) amplitude is as follows:

\[
A(K_0^0 \to \pi^0 e^+ e^-) |_{B(\gamma\gamma)} = iG_8 \frac{\alpha^2}{3\pi} a_V \bar{u}(k)\gamma \cdot p \frac{p \cdot (k-k')}{M_K^2} v(k'),
\]  

which leads to a branching ratio \( \text{Br}(K_0^0 \to \pi^0 e^+ e^-) |_{B(\gamma\gamma)} = 4.4 \times 10^{-12} a_V^2 \) .

The NA31–collaboration \(^{120}\) has performed an analysis of \( K_L \to \pi^0 e^+ e^- \) events with an invariant \( \gamma\gamma \)–mass: \( M_{\gamma\gamma} < 240 \text{MeV} \), using this parametrization, and obtained the following limits:

\[
-0.32 < a_V < 0.19 \quad (90\% \text{ C.L.}).
\]  

The rate in Eq.(321), and hence the result in Eq.(322) does not take into account, however, the non–polynomial structure of the \( B \)–amplitude in Eq.(313) due to chiral loops. A recent analysis \(^{122}\) which tries to fold the phenomenology of both local and non–polynomial effects in the \( B \)–amplitude with the observed data from the NA31–experiment, results in a branching ratio for \( K_L \to \pi^0 e^+ e^- \) from the \( \gamma\gamma \)–discontinuity:

\[
\text{Br}(K_L \to \pi^0 e^+ e^-) \bigg|_{\gamma\gamma \text{Abs.}} = \begin{cases} 0.13 \times 10^{-12}, & a_V = 0, \\ 1.8 \times 10^{-12}, & a_V = -0.9. \end{cases}
\]

The value \( a_V = -0.9 \) is the one which reproduces the observed \( z \)–spectrum in the decay rate of \( K_L \to \pi^0 \gamma\gamma \), in the presence of the non–polynomial ansatz for the \( B \)–amplitude which has been used in Ref. \(^{122}\). The two effects combined also raise the predicted \( K_L \to \pi^0 \gamma\gamma \) branching ratio to \( 1.6 \times 10^{-6} \), in good agreement with the experimental values in Eq.(317).

As a summary, the theoretical prospects for more refined predictions on the decay \( K_L \to \pi^0 e^+ e^- \) are the following:

- The branching ratio expected from “direct” CP–violation is rather well known. The uncertainty in Eq.(322) will be reduced once the top quark mass is better determined; and the \( B_K \)–factor is pinned down more accurately – either by theoretical improved calculations, or phenomenologically –.
• The error in the branching ratio expected from the CP–conserving transition induced via two intermediate photons, can be reduced with a combined effort of more accurate measurements of the mode $K_L \rightarrow \pi^0\gamma\gamma$ on the one hand, and an improvement in the phenomenological ansatz of the $B$–amplitude in the theoretical analysis on the other.

• The largest uncertainty, at the moment, comes from the “indirect” CP–violation transition. Here, the only way I can see to make progress is via the development of good models of the QCD low–energy effective action; e.g., the extension of the succesful ENJL–model in the strong sector to non–leptonic weak transitions. Parallel to this theoretical effort, there should be, of course, some progress as well in obtaining more experimental results in rare $K$–decays, so as to have enough observables to test the models.

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THERE ARE A LOT OF INTERESTING THINGS TO BE DONE BOTH FOR THEORISTS AND EXPERIMENTALISTS!!

Thank you.

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