Quantum quenches in extended systems

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Abstract. We study in general the time evolution of correlation functions in an extended quantum system after the quench of a parameter in the Hamiltonian. We show that correlation functions in $d$ dimensions can be extracted using methods of boundary critical phenomena in $d + 1$ dimensions. For $d = 1$ this allows us to use the powerful tools of conformal field theory in the case of critical evolution. Several results are obtained in generic dimension in the Gaussian (mean field) approximation. These predictions are checked against the real time evolution of some solvable models that allow us also to understand which features are valid beyond the critical evolution.

All our findings may be explained in terms of a picture generally valid, whereby quasiparticles, entangled over regions of the order of the correlation length in the initial state, then propagate with a finite speed through the system. Furthermore we show that the long time results can be interpreted in terms of a generalized Gibbs ensemble. We discuss some open questions and possible future developments.

Keywords: correlation functions, conformal field theory (theory), quantum phase transitions (theory)

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# 1. Introduction

Suppose that an extended quantum system in $d$ dimensions (for example a quantum spin system), is prepared at time $t = 0$ in a pure state $|\psi_0\rangle$ which is the ground state of some Hamiltonian $H_0$ (or, more generally, in a thermal state at a temperature less than the mass gap to the first excited state). For times $t > 0$ the system evolves unitarily according to the dynamics given by a different Hamiltonian $H$, which may be related to $H_0$ by varying a parameter such as an external field. This variation, or quench, is supposed to be carried out over a timescale much less than the inverse mass gap. How does the state $|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$ evolve? For a finite number of degrees of freedom, the system generically shows a periodic (or quasiperiodic) behaviour with a period that typically increases when
the number of degrees of freedom grows. This is the well-known phenomenon of quantum recurrence. However, in the thermodynamic limit this is no longer necessarily the case, and the natural question arises as to whether the system (or, rather, a macroscopically large subsystem) reaches a stationary state for very large times. To attack this question we consider the simpler question of how the correlation functions, expectation values of products of local observables, evolve and whether they reach constant values for large times.

This problem has its own theoretical interest, being a first step towards the understanding of equilibration in quantum systems. On the same fundamental level we can also ask whether the approach can provide a new tool for the characterization of the collective excitations in strongly correlated systems. However, until recently it has been considered a largely academic question, because the timescales over which most condensed matter systems can evolve coherently without coupling to the local environment are far too short, and the effects of dissipation and noise are inescapable. Recent developments of experimental tools for studying the behaviour of ultracold atoms have revised completely this negative attitude. In fact, thanks to the phenomenon of Feshbach resonance it is possible to tune the coupling of an interacting system to essentially any value and in extremely short times. In addition, the coupling to dissipative degrees of freedom can be much weaker than in ordinary condensed matter systems. This has allowed the observation of, among other things, the collapse and revival of a Bose–Einstein condensate [1] and of the quenching of a spinor condensate [2]. Furthermore, using highly anisotropic optical lattices, it has been possible to build and study essentially one-dimensional systems [3], and, of direct interest for this paper, the coherent non-equilibrium dynamics of these integrable models has been measured [4]. (Other interesting experiments are considered in [5].) These striking experimental results have largely motivated the development of new numerical methods to study non-equilibrium dynamics, the most successful one being the time dependent density matrix renormalization group (DMRG) [6].

On the purely theoretical side these kind of questions were first considered (as far as we are aware) in the 70s in the context of the quantum Ising XY model in [7, 8] (see also [9]–[14] for recent developments). However, in the very last few years, after the previously mentioned experimental progress, the study of quantum quenches has been pursued in a systematic manner and a large number of results is nowadays available. To quote only a few examples, the models considered include several realizations of one-dimensional (1D) Bose gases [15]–[25], Luttinger liquids [26, 27], coupled 1D condensates [28], strongly correlated 1D fermions [29], and mean field fermion condensates [30, 31]. A closely related topic, that will not be considered here, concerns the formation of defects when crossing a critical point with changing the external parameter at a fixed rate, i.e. the so-called quantum Kibble–Zurek mechanism [32].

Most of these papers concern the exact or approximate solutions of very specific models. These are often strongly relevant, since in many cases they can be directly compared with existing experimental results or give very accurate predictions for future investigations. However, from the study of specific models it is difficult to draw general conclusions on the physics of quantum quenches. There have been at least two notable exceptions. In [15] it was conjectured that the asymptotic state at very large times can be described by a generalized Gibbs ensemble (for more details we refer the reader to section 6 where this conjecture will be explicitly discussed in relation to some of our
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findings). In [33] we studied the time evolution after a quench in general, exploiting the path integral approach and the well-known mapping of the quantum problem to a classical one in \( d + 1 \) dimensions. The translational invariance in the (imaginary) time direction is explicitly broken and thus the initial state plays the role of a boundary condition. This limits the range of applicability of field-theoretical methods, but when the Hamiltonian \( H \) is at or close to a quantum critical point we could use the renormalization group (RG) theory of boundary critical behaviour (see, e.g., [34]). From this point of view, particularly powerful analytic results are available for \( d = 1 \) because then the 1+1-dimensional problem is described asymptotically by a boundary conformal field theory (BCFT) [35, 36].

The aim of this paper is twofold: on the one hand we give a detailed description of the methods and the results reported briefly in [33], and on the other we generalize these results and give some new physical insights. Several results, in fact, appear here for the first time. The paper is organized as follows. In section 2 we introduce the path integral formalism that is applied in the following section 3 to one-dimensional critical systems by means of CFT. In section 4 we consider two simple 1D models whose non-equilibrium dynamics is exactly solvable to check the correctness of the previous results and to extend our findings to gapped and lattice systems. In section 5 we generalize the method to higher dimensions. In section 6 we analyse all our results and explain most of the general findings in quantum quenches by means of simple physical arguments. We broadly discuss some open questions and possible future developments.

2. Path integral formulation and surface criticality

Let us consider a lattice quantum theory in \( d \) space dimensions. The lattice spacing is \( a \), and the lattice variables are labelled by a discrete vector variable \( r \). Time is considered to be continuous. The dynamics of the theory is described by the Hamiltonian \( H \). Suppose we prepare this system in a state \( |\psi_0\rangle \) that is not an eigenstate of \( H \) and unitarily evolve it according to \( H \). The expectation value of a local operator \( \mathcal{O}(\{r_i\}) \) at time \( t \) is

\[
\langle \mathcal{O}(t, \{r_i\}) \rangle = \langle \psi_0 | e^{iHt} \mathcal{O}(\{r_i\}) e^{-iHt} |\psi_0\rangle.
\]

We modify this time dependent expectation value as

\[
\langle \mathcal{O}(t, \{r_i\}) \rangle = Z^{-1} \langle \psi_0 | e^{iHt-\epsilon H} \mathcal{O}(\{r_i\}) e^{-iHt-\epsilon H} |\psi_0\rangle,
\]

where we have included damping factors \( e^{-\epsilon H} \) in such a way as to make the path integral representation of the expectation value absolutely convergent. The normalization factor \( Z = \langle \psi_0 | e^{-2\epsilon H} |\psi_0\rangle \) ensures that the expectation value of the identity is one. At the end of the calculation we shall set \( \epsilon \) to zero.

Equation (2) may be represented by an analytically continued path integral in imaginary time over the field variables \( \phi(\tau, r) \), with initial and final values weighted by the matrix elements with \( |\psi_0\rangle \):

\[
\int [d\phi(\tau, r)] \langle \psi_0 | \phi(\tau_2, r) \rangle \langle \phi(\tau_1, r) |\psi_0\rangle \mathcal{O}(\{r_i\}) e^{-\int_{\tau_1}^{\tau_2} L[\phi] d\tau},
\]

where \( \int_{\tau_1}^{\tau_2} L[\phi] d\tau \) is the (Euclidean) action. The operator \( \mathcal{O} \) is inserted at \( \tau = 0 \), and the width of the slab is \( 2\epsilon \). \( \tau_1 \) and \( \tau_2 \) should be considered as real numbers during the calculation, and only at the end should they be continued to their effective values \( \pm \epsilon - it \).
In this way we have reduced the real time non-equilibrium evolution of a $d$-dimensional systems to the thermodynamics of a $d+1$ field theory in a slab geometry with the initial state $|\psi_0\rangle$ playing the role of boundary condition at both the borders of the slab. The validity of our results relies on the technical assumption that the leading asymptotic behaviour given by field theory, which applies to the Euclidean region (large imaginary times), may simply be analytically continued to find the behaviour at large real time.

Equation (2) can be in principle used to describe the time evolution in any theory and for all possible initial conditions. Unfortunately, in confined geometries only a few field theories with very specific boundary conditions can be solved analytically in such a way to have results that can be continued from real to complex values. Some examples of these will be given in section 5. (An interesting case concerns integrable massive boundary field theories as for example the recent application to the dynamics of coupled 1D condensates in [28].) However, the treatment greatly simplifies if a system is at or close to a (quantum) phase transition. In fact, in this case, we can use the powerful tools of renormalization group (RG) theory of boundary critical phenomena (see, e.g., [34]). For example, in the case of a scalar order parameter (corresponding to the Ising universality class), the slab geometry of above is usually described by the action [34]

$$S(\phi) = \int d^d r \int_{\tau_1}^{\tau_2} d\tau \left( \frac{1}{2} (\partial \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4 \right) + \int d^d r c(\phi(\tau_1, r) + \phi(\tau_2, r)),$$

(4)

where $m^2$ measures the distance from criticality, $g > 0$ ensures the stability in the broken-symmetry phase, and $c$ denotes the surface enhancement, i.e. the difference of the interactions on the surface with respect to the bulk. The value of $c$ depends on the boundary conditions. For example, the choice $c = +\infty$ forces the field to vanish on the boundary, thus corresponding to Dirichlet boundary conditions. On the other hand $c = -\infty$ forces the field to diverge at the boundary. Finite values of $c$ correspond to intermediate boundary conditions. According to the RG theory, the different boundary universality classes are characterized by the fixed point of $c$. A complete RG analysis (see, e.g., [34]) shows that the possible fixed points of $c$ are $c^* = \infty, 0, -\infty$, with $c^* = \pm\infty$ stable and $c^* = 0$ unstable. Any other value of $c$ flows under RG transformations to $\pm\infty$. These universality classes are called ordinary ($c^* = \infty$), special ($c^* = 0$), and extraordinary ($c^* = -\infty$) [34].

Thus, for the purpose of extracting the asymptotic behaviour, as long as $|\psi_0\rangle$ is translationally invariant, we may replace it by the appropriate RG-invariant boundary state $|\psi_0^\ast\rangle$ to which it flows. The difference may be taken into account, to leading order, by assuming that the RG-invariant boundary conditions are not imposed at $\tau = \tau_1$ and $\tau_2$ but at $\tau = \tau_1 - \tau_0$ and $\tau = \tau_2 + \tau_0$. In the language of boundary critical behaviour, $\tau_0$ is called the extrapolation length [34]. It characterizes the RG distance of the actual boundary state from the RG-invariant one. It is always necessary because scale-invariant boundary states are not in fact normalizable [36]. It is expected to be of the order of the typical timescale of the dynamics near the ground state of $H_0$, that is the inverse gap $m_0^{-1}$. (This can be checked explicitly for a free field theory, see later.) The effect of introducing $\tau_0$ is simply to replace $\epsilon$ by $\epsilon + \tau_0$. The limit $\epsilon \rightarrow 0+$ can now safely be taken, so the width of the slab is then taken to be $2\tau_0$. For simplicity in the calculations, in the following we will consider the equivalent slab geometry between $\tau = 0$ and $2\tau_0$ with the

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operator $O$ inserted at $\tau = \tau_0 + it$. This is illustrated in the left part of figure 1. In fact, in this geometry we can consider products of operators at different times $t_j$, by analytically continuing their labels to $\tau_0 + it_j$.

Let us discuss to what initial conditions the fixed points correspond. In the case of free boundary conditions, when the order parameter is unconstrained, the continuum limit forces it to vanish there, so it corresponds to the ordinary transition ($c = \infty$). For a lattice model, this corresponds to a completely disordered initial state (e.g. for the Ising chain in a transverse field corresponds to infinite field). In contrast for non-vanishing fixed boundary conditions (e.g. + or − for Ising-like systems), the continuum limit makes the order parameter to diverge at the boundary, thus corresponding to the extraordinary transition ($c = -\infty$).

3. One space dimension and conformal field theory

In this section we specialize the methods just introduced to the case when $H$ is at a quantum critical point whose long distance behaviour is given by a 1 + 1-dimensional CFT, with dispersion relation $\Omega_k = v|k|$. We set $v = 1$ in the following. RG-invariant boundary conditions then correspond to conformally invariant boundary states. In this case the correlation functions are accessible through the powerful tools of boundary CFT.

The main property we will repeatedly use in the following is the relation of correlation functions of (primary) operators among two geometries connected by a conformal transformation. For example, the slab geometry of above is just a two-dimensional strip whose points are labelled by a complex number $w = r + i\tau$ with $0 < \text{Im } w < 2\tau_0$. The strip can be obtained from the upper half-plane (UHP) $\text{Im } z > 0$ by the conformal mapping

$$w(z) = \frac{2\tau_0}{\pi} \log z,$$

with the images of points at the same imaginary time on the strip lying along $\arg z_i = \theta = \pi\tau/2\tau_0$. In the case where $O$ is a product of local primary scalar operators $\Phi_i(w_i)$, the expectation value in the strip is related to the one in the UHP by the standard transformation

$$\left\langle \prod_i \Phi_i(w_i) \right\rangle_{\text{strip}} = \prod_i |w'(z_i)|^{-x_i} \left\langle \prod_i \Phi_i(z_i) \right\rangle_{\text{UHP}},$$

where $x_i$ is the bulk scaling dimension of $\Phi_i$. Note that the expectation values (if any) of the $\Phi_i$ in the ground state of $H$ are supposed to have been subtracted off. The asymptotic
real time dependence is obtained via the analytic continuation $\tau \to \tau_0 + it$, and taking the limit $t, r_{ij} \gg \tau_0$.

In the following subsections we apply these methods to some specific cases.

3.1. The one-point function

In the UHP, the one-point function of a scalar primary field with bulk scaling dimension $x$ is $\langle \Phi(z) \rangle_{\text{UHP}} = A_\Phi^b [2 \Im(z)]^{-x}$, as a simple consequence of scaling invariance. The normalization factor $A_\Phi^b$ is a non-universal amplitude. In CFT the normalizations are chosen in such a way that the bulk two-point functions have unit amplitude (i.e. $\langle \Phi(z_1)\Phi(z_2) \rangle_{\text{bulk}} = |z_2 - z_1|^{-2x}$). This choice fixes unambiguously the amplitude $A_\Phi^b$ that turns out to depend both on the considered field $\Phi$ and on the boundary condition on the real axis $b$. It vanishes if $\Phi$ corresponds to an operator whose expectation value in $|\psi_0\rangle$ vanishes, and thus $\langle \Phi(t) \rangle = 0$, for all times.

When the primary field is not vanishing on the boundary, performing the conformal mapping \textup{eq} \textit{it} gives

$$\langle \Phi(t) \rangle = A_\Phi^b \left[ \frac{\pi}{4\tau_0 \sin(\pi t / (2\tau_0))} \right] \simeq A_\Phi^b \left( \frac{\pi}{2\tau_0} \right)^x e^{-x\pi t / 2\tau_0}. \quad (8)$$

Thus the order parameter (and any other observables described by a primary field) decays exponentially to the ground state value, with a non-universal relaxation time $t_{\text{rel}} = 2\tau_0 / x \Omega$. The ratio of the relaxation times of two different observables equals the inverse of the ratio of their scaling dimensions and it is then \textit{universal}.

The normalization factor $A_\Phi^b$ is known for the simplest boundary universality classes [37]. In the case of $\Phi$ being the order parameter and the boundary condition is fixed ($\langle \psi_0(x) = \infty \rangle A_\Phi^b$ is 1 for the free boson and $2^{1/4}$ for the Ising model.

An important exception to this law is the local energy density (or any piece thereof). This corresponds to the $tt$ component of the energy–momentum tensor $T_{\mu\nu}$. In CFT this is not a primary operator. Indeed, if it is normalized so that $\langle T_{\mu\nu} \rangle_{\text{UHP}} = 0$, in the strip [38] $\langle T_{tt}(r, \tau) \rangle = \pi c / 24 (2\tau_0)^2$ (where $c$ is the central charge of the CFT) so that it does not decay in time. Of course this is to be expected since the dynamics conserves energy. A similar feature is expected to hold for other local densities corresponding to globally conserved quantities which commute with $H$, for example the total spin in isotropic models.

3.2. The two-point function

In the case of the one-point function, scaling invariance was enough to fix the functional dependence on the position in the UHP. However, the form of the two-point function depends explicitly on the boundary universality class and on the operator considered. In the following subsections we will consider the equal time correlation function for the order parameter in the Gaussian and in the Ising universality classes that are easily treated in full generality. At the quantum level they describe (among the other things)
a chain of harmonic oscillators (explicitly considered in section 4.1) and the Ising model in a transverse field (whose real time evolution has been considered in [7]–[9], [11] and is briefly reviewed in section 4.2). Finally we will discuss the general form of the two-point function for asymptotically large time and distance, that can be obtained from general CFT arguments.

3.2.1. The Gaussian model. The content of this subsection has been already reported in [39], during the study of the time evolution of the entanglement entropy, that transforms like the two-point function of a primary field in a boundary Gaussian theory [40]. We report it here for sake of completeness.

For a free boson the two-point function in the UHP is [35]

\[
\langle \Phi(z_1)\Phi(z_2) \rangle_{UHP} = \left( \frac{z_{12}z_{21}}{z_{12}z_{21}z_{22}} \right)^x,
\]

with \(z_{ij} = |z_i - z_j|\) and \(z_{k} = \overline{z}_k\). Note that \(\Phi\) is not the Gaussian field \(\theta(z)\), but its exponential \(\Phi(z) = e^{i\theta(z)}\) (see section 4.1).

Under the conformal mapping (5) we obtain the two-point function on the strip at imaginary time \(\tau\), at distance \(r\) apart

\[
\langle \Phi(r, \tau)\Phi(0, \tau) \rangle_{\text{strip}} = \left[ \left( \frac{\pi}{2\tau_0} \right)^2 \frac{\cosh(\pi r/2\tau_0) - \cos(\pi \tau/\tau_0)}{8 \sinh^2(\pi r/4\tau_0) \sin^2(\pi \tau/2\tau_0)} \right]^x,
\]

that continued to real time \(\tau = \tau_0 + it\) gives

\[
\langle \Phi(r, t)\Phi(0, t) \rangle = \left[ \left( \frac{\pi}{2\tau_0} \right)^2 \frac{\cosh(\pi r/2\tau_0) + \cosh(\pi t/\tau_0)}{8 \sinh^2(\pi r/4\tau_0) \cosh^2(\pi t/2\tau_0)} \right]^x.
\]

In the case where \(r/\tau_0\) and \(t/\tau_0\) are large this simplifies to

\[
\langle \Phi(r, t)\Phi(0, t) \rangle = \left( \frac{\pi}{2\tau_0} \right)^2 \left( \frac{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0} - e^{\pi t/\tau_0}} \right)^x \propto \begin{cases} e^{-\pi t/\tau_0} & \text{for } t < r/2 \\ e^{-\pi r/2\tau_0} & \text{for } t > r/2 \end{cases},
\]

i.e. the two-point function at fixed \(r\) decays exponentially in time up to \(t^* = r/2\) and then saturates to a value that depends exponentially on the separation.

In the case of fixed initial condition, with one-point function given by equation (8), the connected correlation function is

\[
\langle \Phi(r, t)\Phi(0, t) \rangle_{\text{conn}} = \langle \Phi(r, t)\Phi(0, t) \rangle - \langle \Phi(0, t) \rangle^2
\]

\[
\propto \begin{cases} 0 & \text{for } t < r/2 \\ e^{-\pi r/2\tau_0} - e^{-\pi t/\tau_0} & \text{for } t > r/2 \end{cases},
\]

i.e. correlations start developing at \(t^* = r/2\) and, being \(t \gg \tau_0\), at \(t^*\) the connected two-point function almost immediately jumps to its asymptotic value. In the case of disordered initial conditions (\(\psi_0(r) = 0\)), connected and full correlation functions are equal.
3.2.2. The Ising universality class. For the Ising model the two-point function in the UHP is [35]

\[ \langle \Phi(z_1)\Phi(z_2) \rangle_{\text{UHP}} = \left( \frac{z_{12}z_{21}}{z_{12}z_{12}z_{11}z_{22}} \right)^{1/8} F(\eta), \]  

where \( F(\eta) \) is given by

\[ F(\eta) = \frac{\sqrt{1 + \eta^{1/2}} \pm \sqrt{1 - \eta^{1/2}}}{\sqrt{2}}, \]  

and \( \eta \) is the four-point ratio

\[ \eta = \frac{z_{11}z_{22}}{z_{12}z_{21}}. \]  

The sign \( \pm \) depends on the boundary conditions. + corresponds to fixed boundary conditions and – to disordered ones.

The only difference with respect to the Gaussian case is that we have also to map \( F(\eta) \) according to the conformal transformation (5). After simple algebra we have

\[ \eta = \frac{2 \sin^2(\pi\tau/2\tau_0)}{\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)}, \]  

and so

\[ \langle \Phi(r, \tau)\Phi(0, \tau) \rangle_{\text{strip}} = \left[ \left( \frac{\pi}{2\tau_0} \right)^2 \frac{\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)}{8 \sinh^2(\pi r/4\tau_0) \sin^2(\pi\tau/2\tau_0)} \right]^{1/8} \frac{1}{\sqrt{2}} \times \sqrt{1 + \frac{\sqrt{2} \sin(\pi\tau/2\tau_0)}{\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)}} \pm \sqrt{1 - \frac{\sqrt{2} \sin(\pi\tau/2\tau_0)}{\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)}}. \]  

Analytically continuing to real time \( \tau = \tau_0 + it \) we obtain

\[ \langle \Phi(r, t)\Phi(0, t) \rangle = \left[ \left( \frac{\pi}{2\tau_0} \right)^2 \frac{\cosh(\pi r/2\tau_0) + \cosh(\pi t/\tau_0)}{8 \sinh^2(\pi r/4\tau_0) \cosh^2(\pi t/2\tau_0)} \right]^{1/8} \frac{1}{\sqrt{2}} \times \sqrt{1 + \frac{\sqrt{2} \cosh(\pi t/2\tau_0)}{\cosh(\pi r/2\tau_0) + \cosh(\pi t/\tau_0)}} \pm \sqrt{1 - \frac{\sqrt{2} \cosh(\pi t/2\tau_0)}{\cosh(\pi r/2\tau_0) + \cosh(\pi t/\tau_0)}} \right], \]  

that for \( r/\tau_0 \) and \( t/\tau_0 \) much larger than 1 simplifies to

\[ (\pi/2\tau_0)^{1/4} \frac{1}{\sqrt{2}} \left( e^{\pi r/2\tau_0} + e^{\pi t/\tau_0} \right)^{1/8} \left[ \sqrt{1 + \frac{e^{\pi t/2\tau_0}}{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}} \pm \sqrt{1 - \frac{e^{\pi t/2\tau_0}}{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}} \right]. \]
Note that the exponential terms in the square root are always $\ll 1$. Thus for fixed boundary condition we get the free boson result equation (12) with $x = 1/8$. For the connected part we need to subtract $\langle \Phi(0, t) \rangle^2$ given by equation (8) with $A_+^\Phi = 2^{1/4}$. We finally obtain

$$
\langle \Phi(r, t) \Phi(0, t) \rangle_{\text{conn}} = \langle \Phi(r, t) \Phi(0, t) \rangle - \langle \Phi(0, t) \rangle^2
$$

$$
\propto \begin{cases} 
0 & \text{for } t < r/2, \\
\ e^{-\pi r/16\tau_0} - e^{-\pi t/8\tau_0} & \text{for } t > r/2.
\end{cases}
$$

(21)

Thus, also for the Ising model with fixed boundary conditions, connected correlations start developing at $t = t^* = r/2$.

In the case of disordered initial condition, we have

$$
\langle \Phi(r, t) \Phi(0, t) \rangle \propto \left( e^{\pi r/2\tau_0} + e^{\pi t/\tau_0} \right)^{1/8} e^{\pi t/2\tau_0} \sqrt{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}
$$

$$
\sim \begin{cases} 
\ e^{-\pi (r - 3/2t)/4\tau_0} & \text{for } t < r/2, \\
\ e^{-\pi r/16\tau_0} & \text{for } t > r/2,
\end{cases}
$$

(22)

resulting in an exponential space dependence even for $t < r/2$ (clearly in this case the connected correlation function equals the full one).

3.2.3. The general two-point function. From the results reported for the Gaussian and Ising models, it is now relatively simple to understand the general properties of the time dependence of the two-point function in the very general case. The two-point function in the half-plane has the general form [35]

$$
\langle \Phi(z_1) \Phi(z_2) \rangle_{UHP} = \left( \frac{z_{12} \bar{z}_{21}}{z_{12} \bar{z}_{12} \bar{z}_{11} \bar{z}_{22}} \right)^x F(\eta),
$$

(23)

where the function $F(\eta)$ depends explicitly on the considered model. Under the conformal map to the strip we know that the first part of equation (23) transforms according to equation (10). Thus we need only to map $F(\eta)$ that, in the general case, is an unknown function. However, during the study of the Ising model we showed that the analytical continuation of $\eta$ for $t, r \gg \tau_0$ is

$$
\eta \sim \frac{e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}.
$$

(24)

Thus for $t < r/2$ we have $\eta \sim e^{\pi(t-r/2)/\tau_0} \ll 1$ and in the opposite case $t > r/2$ we have $\eta \sim 1$. As a consequence to have the asymptotic behaviour of the two-point function we only need to know the behaviour close to $\eta \sim 0$ (i.e. the behaviour close to the surface) and for $\eta \sim 1$ (i.e. deep in the bulk). Fortunately they are both exactly known. Indeed when $\eta \sim 1$ the two points are deep in the bulk, meaning $F(1) = 1$. Instead for $\eta \ll 1$, from the short distance expansion, we have

$$
F(\eta) \simeq (A_b^\Phi)^2 \eta^{x_b},
$$

(25)

where $x_b$ is the boundary scaling dimension of the leading boundary operator to which $\Phi$ couples and $A_b^\Phi$ is the bulk boundary operator product expansion coefficient that equals the one introduced in equation (8) (see e.g. [37]).

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All the previous observations and the explicit calculations of the previous sections lead for $t > r/2$ to
\[ \langle \Phi(r, t) \Phi(0, t) \rangle \propto e^{-\pi r/2\tau_0}, \tag{26} \]
while for $t < r/2$ we get
\[ \langle \Phi(r, t) \Phi(0, t) \rangle \propto (A_b^\Phi)^2 e^{-\pi t/\tau_0} \times e^{\pi x_b(t-r/2)/\tau_0}. \tag{27} \]
Note that if $\langle \Phi \rangle \neq 0$, $x_b = 0$ and the last factor is absent. The leading term is then just $\langle \Phi \rangle^2$. Thus the leading term in the connected two-point function vanishes for $t < r/2$, and its first non-vanishing contributions is given by subleading terms either in $F$ or in the bulk boundary short distance expansion.

It is very interesting that we only have to know the behaviour as $\eta \to 0$ and to get the results we need for large $r$ and $t$. However, we stress that only a complete calculation (as those performed in the preceding sections) gives the full analytic structure of the CFT result needed to justify the analytical continuation from imaginary to real time. Moreover, the behaviour within a distance $O(\tau_0)$ of the horizon $r = 2t$ depends on the detailed form of $F$.

### 3.3. Correlations functions at different times

Let us consider the case of the two-point function calculated at different real times $\langle \Phi(r, t) \Phi(0, s) \rangle$. This is again obtained by mapping the imaginary time strip to the UHP, but in this case the two points are $w_1 = r + i\tau_1$ and $w_2 = 0 + i\tau_2$, that, at the end of the calculation, must be analytically continued to $\tau_1 = \tau_0 + it$ and $\tau_2 = \tau_0 + is$. See figure 2 for a pictorial representation of the space–time domain in the strip and the resulting mapping to the UHP.

Let us start the discussion with the free boson. The distances appearing in equation (9) are
\[ z_{12}^2 = 1 + e^{\pi r/\tau_0} - 2e^{\pi r/2\tau_0} \cos(\theta_1 - \theta_2), \quad z_{11} = 2 \sin \theta_1, \quad z_{22} = 2e^{\pi r/2\tau_0} \sin \theta_2, \tag{28} \]
where $\theta_i = \pi \tau_i / 2\tau_0$. Thus the correlation function on the strip is
\[ \langle \Phi(r, \tau_1) \Phi(0, \tau_2) \rangle_{\text{strip}} = \left| w'(z_1) \right|^{-x} \left| w'(z_2) \right|^{-x} \langle \Phi(z_1(w)) \langle \Phi(z_2(w)) \rangle_{\text{UHP}} \right|^{\pi \left( \frac{\pi}{2\tau_0} \right)^2 \cosh(\pi r/2\tau_0) - \cos(\tau_1 + \tau_2)/2\tau_0} / 4 \sin(\pi \tau_1/2\tau_0) \sin(\pi \tau_2/2\tau_0)(\cosh(\pi r/2\tau_0) - \cos(\tau_1 - \tau_2)/2\tau_0)} \right|^{x}, \tag{29} \]

Figure 2. Left: space–time region for the correlation functions at different times. Right: conformal mapping to the upper half-plane.
Quantum quenches in extended systems

that for \( \tau_1 = \tau_2 \) reduces to equation (10) as it should. Continuing to real times and considering \( r, t, s, |t - s| \gg \tau_0 \) we obtain

\[
\left( \frac{\pi}{2\tau_0} \right)^{2\Delta} \frac{e^{\Delta \pi r/\tau_0} + e^{(\Delta \pi/\tau_0)(t+s)}}{e^{(\Delta \pi/\tau_0)(t+s)}(e^{\Delta \pi r/\tau_0} + e^{(\Delta \pi/\tau_0)(t-s)})} = \begin{cases} 
- e^{-\pi x(t+s)/4\tau_0} & \text{for } r > t + s, \\
- e^{-\pi x r/4\tau_0} & \text{for } t - s < r < t + s, \\
- e^{-\pi x(t-s)/4\tau_0} & \text{for } r < |t - s|. 
\end{cases}
\]

(30)

Following the line sketched in the previous subsection, it is easy to generalize this result to the most general CFT. In the case of a theory with fixed initial conditions (i.e., \( \langle \Phi \rangle \neq 0 \)) the asymptotic result is the same as before, with only the crossover points being affected by the precise expression for \( F(\eta) \). Instead, in the case where \( \langle \Phi \rangle = 0 \), the first case gains an additional factor \( e^{-\pi x_0(t+s-r)/4\tau_0} \).

Note that the autocorrelation function (i.e. \( r = 0 \)) has only an exponential dependence on the time separation \( t - s \) and does not exhibit ageing in this regime.

3.4. Evolution with boundaries

We now consider the case of time evolution of a half-chain with some boundary condition at \( r = 0 \). For simplicity we assume that the (conformal) boundary condition is of the same kind of the initial boundary condition (for example we fix all the spins at \( t = 0 \) and at the boundary \( r = 0 \) to point in the same direction). The space–time region we have to consider is depicted in figure 3. If different initial and boundary were considered, one needs to insert boundary conditions changing operators at the corners of the figure.

The \( w \) plane is mapped into the UHP by

\[
z(w) = \sin \frac{\pi w}{2\tau_0},
\]

(31)

with the corners at \( \pm \tau_0 \) mapped to \( \pm 1 \). The mapping of \( w_1 \) is

\[
z_1 \equiv z(w_1) = z(-\tau_0 + \tau_1 + ir) = - \cos(\pi \tau_1/2\tau_0) \cosh(\pi r/2\tau_0) + i \sin(\pi \tau_1/2\tau_0) \sinh(\pi r/2\tau_0).
\]

(32)

In the \( z \) plane the one-point function is

\[
\langle \Phi(z_1) \rangle_{\text{UHP}} \propto |\text{Im } z_1|^{-x} \rightarrow [\sin(\pi \tau_1/2\tau_0) \sinh(\pi r/2\tau_0)]^{-x},
\]

(33)
and

$$|w'(z_1)|^2 = \left(\frac{2\tau_0}{\pi}\right)^2 \frac{1}{|1-z|^2} \propto \frac{1}{\cosh(\pi r/\tau_0) - \cos(\pi \tau_1/\tau_0)}.$$  (34)

Thus on the strip we have

$$\langle \Phi(w_1) \rangle_{\text{strip}} = |w'(z_1)|^{-x} \langle \Phi(w(z_1)) \rangle_{\text{UHP}} \propto \left[ \frac{\sin^2(\pi \tau_1/2\tau_0) \sinh^2(\pi r/2\tau_0)}{\cosh(\pi r/\tau_0) - \cos(\pi \tau_1/\tau_0)} \right]^{-x/2},$$  (35)

that continued to real time $\tau_1 = it$ is

$$\langle \Phi(t, r) \rangle \propto \left[ \frac{\cosh(\pi t/\tau_0) + \cosh(\pi r/\tau_0)}{\cosh(\pi t/2\tau_0)^2 \sinh^2(\pi r/2\tau_0)} \right]^{x/2},$$  (36)

and for $t, r \gg \tau_0$ simplifies to

$$\langle \Phi(t, r) \rangle \propto \left[ \frac{e^{\pi r/\tau_0} + e^{\pi t/\tau_0}}{e^{\pi r/\tau_0} - e^{\pi t/\tau_0}} \right]^{x/2} = \begin{cases} e^{-\pi xt/2\tau_0} & \text{for } t < r, \\ e^{-\pi xr/2\tau_0} & \text{for } t > r. \end{cases}$$  (37)

Note that in this case the characteristic time is $t^* = r$ and not $r/2$. This explains also why the entanglement entropy of a semi-infinite chain with free boundary condition at $x = 0$, has characteristic time $t^* = r$ as firstly noted in [41].

3.5. Discussion and interpretation of the CFT results

All the correlation functions calculated so far display two very general features: first there is a sharp horizon (or light-cone) effect at $t = t^* = r/2$ (or $r$) resulting in a behaviour before and after $t^*$ completely different; second the asymptotic long time correlation functions are the same as those at finite temperature $\beta_{\text{eff}} = 4\tau_0$. The light-cone effect is a very general phenomenon and will be discussed in section 6. In the following sections we will point out that also the ‘effective temperature’ is a general phenomenon, but it has some specific CFT features that are worthy of comment.

In fact, it is very easy to understand the technical reason why we find an effective temperature despite the fact that we are studying a pure state at $T = 0$. The finite temperature correlations can be calculated by studying the field theory on a cylinder of circumference $\beta = 1/T$. In CFT a cylinder can be obtained by mapping the complex plane with the logarithmic transformation $\beta/(2\pi) \log z$. The form for the two-point function in the slab depends in general on the function $F(\eta)$—cf equation (15)—but when we analytically continue and take the limit of large real time, we find that effectively the points are far from the boundary, i.e. at $\eta = 1$. Thus we get the same result as we would get if we conformally transformed from the full plane to a cylinder, and from equation (5) the effective temperature is $\beta_{\text{eff}} = 4\tau_0$. A similar argument can be worked out for the multi-point functions as well.

An additional comment concerns what we expect for correlation functions of general operators, not only primary. We have clearly seen that the local energy density does not relax, of course, and this is consistent with its not being primary. But, we also know that at finite temperature $(T_U)_\beta = \pi c/6\beta^2$, that is perfectly compatible with the previous result with $\beta_{\text{eff}} = 4\tau_0$. Furthermore, for large real times, the two-point function of $T_U - \langle T_U \rangle$ does behave as though it was at finite temperature. This means, in particular, that the
energy fluctuations in a large but finite volume (the specific heat) are the same as those at
finite temperature. Thus one is tempted to extend this finite temperature interpretation
to non-primary operators, and indeed it is the case. In fact, in the argument of above for
the equivalence of the long time correlations and finite $T$, there are essentially three steps.
First, we need to write down the form of the correlation function in the half-plane, but this
only depends on special conformal transformation and so is valid for any quasiprimary
operator like $T_{\mu\nu}$. Second we have to transform from the slab to the half-plane: for
non-primary operators this can have some anomalous term. Finally we need to compare
the large limit of this to what one would get transforming directly from the cylinder to
the full plane. However, the two conformal transformations we are comparing are both
logarithmic (in one case $2\tau_0/\pi \log z$, in the other case $\beta/2\pi \log z$), so the anomalous terms
should be the same. Thus the two correlations functions are the same. This argument
works for all quasiprimary operators. Then, since we can get all the non-quasiprimaries
by considering successive operator product expansions with the stress tensor, it also works
for all operators.

4. Exact real time dynamics in simple integrable chains

The results of the previous section rely on the technical assumption that the leading
asymptotic behaviour given by CFT, which applies to the Euclidean region (large
imaginary times), may simply be analytically continued to find the behaviour at large
real time. While such procedures have been shown to give the correct behaviour for
the time dependent correlations in equilibrium, it is important to check them in specific
solvable cases for non-equilibrium evolution.

Thus, as a complement to the CFT calculations, in this section we consider the real
time evolution of two simple analytically tractable models. We solve the dynamics of a
chain of coupled harmonic oscillators and we review and re-analyse some known results
for the Ising XY chain in a transverse magnetic field. Beyond providing examples of the
CFT results (with central charge $c = 1$ and $1/2$ respectively) in the critical case, these
models allow us to take into account the effects of a finite mass gap and of the lattice.

4.1. The chain of harmonic oscillators

The simplest model with an exactly solvable non-equilibrium dynamics is surely a chain
of coupled harmonic oscillators with Hamiltonian

\[ H = \frac{1}{2} \sum_r \left[ \pi_r^2 + m^2 \varphi_r^2 + \sum_j \omega_j^2 (\varphi_{r+j} - \varphi_r)^2 \right]. \]  

We introduce a coupling more general than simple nearest neighbour hopping so as to
allow for a general dispersion relation below. For simplicity we also assume periodic
boundary conditions. $\varphi_n$ and $\pi_n$ are the position and the momentum operators of the $n$th
oscillator, with equal time commuting relations

\[ [\varphi_m, \pi_n] = i \delta_{nm}, \quad [\varphi_n, \varphi_m] = [\pi_n, \pi_m] = 0. \]
The Hamiltonian can be written in diagonal form \( H(m) = \sum_k \Omega_k A^\dagger_k A_k \) with modes

\[
A_k = \frac{1}{\sqrt{2\Omega_k}}(\Omega_k \varphi_k + i\pi_k),
\]

\[
A^\dagger_k = \frac{1}{\sqrt{2\Omega_k}}(\Omega_k \varphi_{-k} - i\pi_{-k}),
\]

\[
\Omega_k^2 = m^2 + 2 \sum_j \omega_j^2 (1 - \cos(2\pi kj/N)).
\]

Note that we use the same symbols for the operators and their Fourier transforms (\( \varphi_k = 1/\sqrt{N} \sum_{n=0}^{N-1} e^{2\pi ikn/N} \varphi_n \) and analogously for \( \pi_k \)).

We consider the scenario in which the system is prepared in a state \( |\psi_0\rangle \), that is ground state of \( H(m_0) \), and at the time \( t = 0 \) the mass is quenched to a different value \( m \neq m_0 \). We use the notation \( \Omega_{0k} \) for the dispersion relation for \( t < 0 \) and the \( \Omega_k \) for the one for \( t > 0 \).

Since \( \langle \psi_0| \varphi_n |\psi_0\rangle = 0 \), the expectation value of the field \( \varphi_n \) vanishes at any time. This example in fact corresponds to the quench from the disordered phase in the language of the previous section. Thus we concentrate our attention on the two-point function

\[
\langle \psi(t)|\varphi_n \varphi_0 \rangle \psi(t)\rangle = \langle \psi_0| \varphi_n^H(t) \varphi_0^H(t) |\psi_0\rangle,
\]

where we introduced the operator in the Heisenberg picture \( \varphi_n^H(t) \), whose time evolution is given by

\[
\varphi_n^H(t) = \sum_{k=0}^{N-1} \sqrt{\frac{2}{N\Omega_k}} \left( e^{i(p_k n - \Omega_k t)} A_k + e^{-i(p_k n - \Omega_k t)} A^\dagger_k \right),
\]

where \( p_k = 2\pi k/N \). Accordingly, the product of the two fields is

\[
\varphi_n^H(t) \varphi_0^H(t) = \frac{2}{N} \sum_{k,k'} \frac{1}{\Omega_k \Omega_{k'}} \left( e^{i(p_k n - \Omega_k t)} A_k + e^{-i(p_k n - \Omega_k t)} A^\dagger_k \right) \left( e^{-i\Omega_{k'} t} A_{k'} + e^{i\Omega_{k'} t} A^\dagger_{k'} \right).
\]

In order to have the time dependent two-point function we need the expectation values of the bilinear combinations of \( A^\dagger \)'s on the initial state, that is annihilated by the \( A_{0k} \). Thus it is enough to write \( A^\dagger \)'s as functions of \( A_0 \)'s, i.e.

\[
A_k = \frac{1}{2} \left[ A_{0k} \left( \sqrt{\Omega_k \Omega_{0k}} + \sqrt{\Omega_{0k} \Omega_k} \right) + A^\dagger_{0-k} \left( \sqrt{\Omega_k \Omega_{0k}} - \sqrt{\Omega_{0k} \Omega_k} \right) \right] = c_k A_{0k} + d_k A^\dagger_{0-k},
\]

\[
A^\dagger_k = \frac{1}{2} \left[ A^\dagger_{0k} \left( \sqrt{\Omega_k \Omega_{0k}} + \sqrt{\Omega_{0k} \Omega_k} \right) + A_{0-k} \left( \sqrt{\Omega_k \Omega_{0k}} - \sqrt{\Omega_{0k} \Omega_k} \right) \right] = c_k A^\dagger_{0k} + d_k A_{0-k},
\]

leading to (we understand \( \langle \cdot \rangle = \langle \psi_0 | \cdot | \psi_0 \rangle \))

\[
\langle A_k A_{k'} \rangle = \langle (c_k A_{0k} + d_k A^\dagger_{0-k})(c_{k'} A_{0k'} + d_{k'} A^\dagger_{0-k'}) \rangle = c_k d_{k'} \langle A_{0k} A^\dagger_{0-k'} \rangle = c_k d_{k'} \delta_{k,-k'},
\]

\[
\langle A^\dagger_k A^\dagger_{k'} \rangle = \langle (c_k A_{0k} + d_k A^\dagger_{0-k})(c_{k'} A^\dagger_{0k'} + d_{k'} A_{0-k'}) \rangle = c_k c_{k'} \langle A_{0k} A^\dagger_{0k'} \rangle = c_k c_{k'} \delta_{k,k'},
\]

\[
\text{doi:10.1088/1742-5468/2007/06/P06008}
\]
\begin{align}
\langle A_k^\dagger A_{k'} \rangle &= \langle (c_k A_{0k} + d_k A_{-k})(c_{k'} A_{0k'} + d_{k'} A_{-k'}) \rangle = d_k d_{k'} \delta_{k,k'}, \quad (50) \\
\langle A_k^\dagger A_{k'}^\dagger \rangle &= \langle (c_k A_{0k} + d_k A_{-k})(c_{k'} A_{0k'} + d_{k'} A_{-k'}) \rangle = d_k c_{k'} \delta_{k,-k'}. \quad (51)
\end{align}

Finally we arrive at
\begin{equation}
\langle \varphi^H_n(t) \varphi^H_0(t) \rangle = \frac{2}{Na} \sum_k \frac{1}{\Omega_k} [c_k d_k e^{ipkn-2\Omega_k t} + c_k^2 + d_k^2 e^{-ipkn} + d_k d_k e^{-i(p_kn-2\Omega_k t)}], \quad (52)
\end{equation}
which, in the thermodynamic limit \((N \to \infty)\), may be written as
\begin{equation}
\langle \varphi^H_r(t) \varphi^H_0(t) \rangle - \langle \varphi^H_r(0) \varphi^H_0(0) \rangle = \int_{BZ} \frac{dk}{(2\pi)^2} e^{ikr} \frac{(\Omega^2_{0k} - \Omega^2_k)(1 - \cos(2\Omega_k t))}{\Omega^2_k \Omega_{0k}}, \quad (53)
\end{equation}
where the integral is on the first Brillouin zone \(|k| < \pi/a\). Note that for \(t = 0\) and for \(m = m_0\) this two-point function reduces to the static one, as it should. This result can also be found by integrating the Heisenberg equations of motion for each mode. For future reference it is also useful to write down explicitly the Fourier transform known as momentum distribution function
\begin{equation}
\rho(k) = \frac{(\Omega^2_{0k} + \Omega^2_k) - (\Omega^2_{0k} - \Omega^2_k) \cos(2\Omega_k t)}{\Omega^2_k \Omega_{0k}}. \quad (54)
\end{equation}

Note that when considering correlation functions in momentum space, a long time limit does not exist and we need to take the time average, in contrast to what happens in real space.

4.1.1. The continuum limit. In equation \((52)\) everything is completely general and applies to any chain with finite lattice spacing. Let us now discuss the continuum limit that is achieved by sending \(N \to \infty\) in such a way that \((1/N) \sum_k \to \int_{-\infty}^{\infty} dp/(2\pi)\), \(k_n \to p\), and \(\Omega^2_k \to \Omega^2_p = m^2 + p^2\).

In this limit the correlation function becomes \((\Delta m^2 = m_0^2 - m^2)\)
\begin{equation}
G(r,t) \equiv \langle \varphi^H_r(t) \varphi^H_0(t) \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipr} \frac{-\Delta m^2 \cos(2\sqrt{p^2 + m^2} t) + m^2 + m_0^2 + 2p^2}{(m^2 + p^2)(m_0^2 + p^2)}. \quad (55)
\end{equation}

Since a closed form for such integral is quite difficult to write down in the most general case, we will only consider some particular cases.

Let us first consider the conformal evolution \((m = 0)\) from an initial state with a very large mass \(m_0 \to \infty\). This should reproduce the CFT result previous section for all the time \(t\), since the correlations of the initial state shrink to a point. We have
\begin{equation}
G(r,t) = m_0 \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipr} \frac{1 - \cos(2pt)}{p^2} = m_0 \left\{ \begin{array}{ll}
0 & \text{for } t < r/2, \\
1/2(2t - r) & \text{for } t > r/2.
\end{array} \right. \quad (56)
\end{equation}
To compare such result with the conformal result given by equation \((12)\), we have to keep in mind that the primary field is not the Gaussian one \(\varphi^H(r,t)\), but the imaginary exponential. Thus we need \(\langle e^{iq\varphi^H(r,t)} e^{-iq\varphi^H(0,t)} \rangle\), with an arbitrary \(q\). Despite of the apparent complexity
of such correlator, it is very simple to obtain it using the standard property of Gaussian integrals
\[
\langle e^{i q x^H(r,t)} e^{-i q x^H(0,t)} \rangle = e^{-q^2((x^H(r,t) - x^H(0,t))^2)/2} = e^{q^2(G(r,t) - G(0,t))},
\]
leading to
\[
\langle e^{i q x^H(r,t)} e^{-i q x^H(0,t)} \rangle = \begin{cases} 
  e^{-q^2m t} & \text{for } t < r/2, \\
  e^{-q^2m r/2} & \text{for } t > r/2,
\end{cases}
\]
that is exactly the same of equation (12) with \(x_\phi \propto q^2\) and \(\tau_0 \propto m_0^{-1}\), confirming that \(\tau_0\) is just proportional to the correlation in the initial state, as its interpretation in terms of the extrapolation length suggests.

The case \(m = 0\) and \(m_0\) finite corresponds to a conformal evolution from a generic state with correlations proportional to \(m_0^{-1}\). Thus we expect the CFT result equation (12) to be true for asymptotic large times and separations.

From equation (55), the two-point function of the Gaussian field is
\[
G(r,t) = \int_{-\infty}^{\infty} dp \frac{e^{ipr}}{2\pi} \frac{-m_0^2 \cos(2pt) + m_0^2 + 2p^2}{p^2 \sqrt{m_0^2 + p^2}},
\]
that can be written as
\[
2\pi G(r,t) = 2K_0(m_0r) + f(m_0r) - \frac{f(m_0(r-2t)) + f(m_0(r+2t))}{2},
\]
where \(K_0(y)\) is the modified Bessel function and
\[
f(y) = 1 + \frac{1}{2} G_{21}^{13} \left( \frac{y^2}{4} \middle| \begin{array}{ccc} 3/2 \\ 0 \\ 1 \\ 1/2 \end{array} \right),
\]
with \(G_{21}^{13}\) the Meijer G function (see e.g. [42]). Note that \(f(x)\) is characterized by \(f''(x) = -K_0(x)\) and \(f(0) = f'(0) = 0\). In the limit \(m_0 \to \infty\), it is easy to show that \(G(r,t)\) reduces to the previous result. In the left panel of figure 4, we report \(G(r,t)\) as function of \(t\) at fixed \(r = 1\) for \(m_0 = 10, 3, 1\). It is evident that a finite \(m_0\) results in smoothing the curve close to \(t = r/2\) and giving an offset in zero. Both the effects are more pronounced as \(m_0\) decreases. For large \(t\), independently on \(m_0 \neq 0\), we have \(G(r,t) = t + O(t^0)\), confirming that the CFT result is correct for asymptotic large times.

The case with arbitrary \(m\) and \(m_0\) is quite cumbersome to be worked out analytically and not really illuminating. For this reason we concentrate here on the massive evolution from a state with \(m_0 \to \infty\). In this case the correlation function reads
\[
G(r,t) = m_0 \int_{-\infty}^{\infty} dp \frac{e^{ipr}}{2\pi} \frac{1 - \cos(2\sqrt{p^2 + m^2}t)}{(m^2 + p^2)}.
\]
In the limit \(t \to \infty\) the cosine term averages to zero, giving
\[
G(r,t = \infty) = m_0 \int_{-\infty}^{\infty} dp \frac{e^{ipr}}{2\pi (m^2 + p^2)} = m_0 \frac{e^{-mr}}{m}.
\]

To understand the time dependence, let us first note that, despite the presence of a square root, the integrand of equation (62) is analytic in \(p\), since the square root is the argument of the even cosine function. Thus we can make the integral in the complex plane...
and use the Cauchy theorem. As long as \( r > 2t \), the behaviour for \( p \to i\infty \) is dominated by \( e^{ipr} \) and we can safely close the contour path in the upper half-plane where the residue at \( p = ir \) is zero. As a consequence \( G(r, t < r/2) = 0 \).

For \( t > r/2 \), the integral is more difficult, since we cannot close the contour in the upper half-plane, because the cosine is ‘larger’ than \( e^{ipr} \) for \( p \to i\infty \). However, the approach to the asymptotic value is easily worked out. In fact, for \( t \gg r \), the term \( e^{ipr} \) in the integral (62) can be approximated with 1, since it is slowly oscillating. Thus we have

\[
G(r, t \gg r) - G(r, t = \infty) \simeq -m_0 \int_{-\infty}^{\infty} \frac{dp \cos(2\sqrt{p^2 + m^2 t})}{(m^2 + p^2)} = -m_0 f_m(t),
\]

where \( f_m(t) = 1/(2m) - t_1 F_2(1/2; 1, 3/2; -m^2 t^2) \), that satisfies \( f'_m(t) = -J_0(2mt) \) and \( f_m(0) = 1/(2m) \). Note that \( f_m(t) \) is just the result for \( r = 0 \). In the complementary region \( 0 < 2t - r \ll m^{-1} \), the integral in equation (62) is dominated by the modes with \( p \gg m \), and so it can be described by the conformal result.

In the right panel of figure 4, we plot the time dependence of \( G(r, t) \) at fixed \( r \) as obtained by numerically integrating equation (62) for \( m = r = 1 \). The plot shows that the conformal result \( t - r/2 \) describes the behaviour close to \( t = r/2 \), while for larger times the asymptotic expression gives an excellent approximation.

We finally note that \( G(r, t) \) at fixed \( t \) displays spatial oscillations (for \( r < 2t \), else it vanishes), as it can be simply realized by a stationary phase argument.

4.1.2. Lattice effects. Another advantage of this simple model is that the effects of the lattice can be easily understood. In fact, by a stationary phase argument, the dominant contribution to (53) in the limits of large \( t \) and \( r \) comes from where, the group velocity \( v_k \equiv \Omega'_k = r/2t \), independently of the explicit form of \( \Omega_k \). Consequently, the two-point function of a gapless model with dispersion relation \( \Omega_k = 2\sin k/2 \) differs from the continuum limit previously derived for \( t > r/2v_m \), where \( v_m \) is the maximum group velocity. In particular \( G(r, t) \) receives a contribution from the slowest mode (\( k = \pi \) with

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whose effect is to add to equation (56) fast oscillations going as \( \cos(2\Omega \pi t) = \cos 4t \). The resulting \( G(r, t) \) is plotted in the inset of figure 4 in the gapless case.

Thus, lattice effects play a more important role in the cases where the asymptotic result vanishes. A typical example is the energy density that in this model is proportional to \( (\varphi_{r+1} - \varphi_r)^2 \). The continuum calculation would just give zero (the mean energy is conserved), but the approach to this value is governed by lattice effects and is dominated by the smallest group velocity that comes from the zone boundary at \( |k| = \pi \).

Indeed, just with a trivial calculation we have

\[
\langle (\varphi_{r+1} - \varphi_r)^2 \rangle = 2\langle \varphi_0^2 \rangle - 2\langle \varphi_0 \varphi_1 \rangle = \int \frac{dk}{2\pi} \frac{(1 - \cos k)(1 - \cos 2\Omega_k t)}{2 \sin^2 k/2},
\]

that displays the typical \( \cos 4t \) from the zone boundary at \( |k| = \pi \).

The lattice dispersion relations are sensitive to the microscopical details of the model and consequently quantities like the energy density show a dependence on this. For example, a lattice massless fermion has dispersion relation \( \Omega_k = 2 \sin k/2 \) and energy density given by \( \sin^2 k \). In this case the time evolution of the energy density is

\[
\int \frac{dk}{2\pi} \frac{(1 - \cos 2\Omega_k t)}{2 \sin^2 k/2},
\]

that for large times goes like \( t^{-3/2} \cos 4t \), resulting in a different power law compared to before.

If the system is quenched to a gapped \( H \) with \( \Omega_k^2 = m^2 + 2(1 - \cos k) \) the maximum group velocity corresponds to a non-zero wavenumber. This gives rise to spatial oscillations in the correlation function (this is true also in the continuum limit as already discussed in the previous subsection).

### 4.2. The Ising XY chain in a transverse magnetic field

The most studied one-dimensional quantum spin model is the so-called XY chain in a transverse field, defined by the Hamiltonian

\[
H = \sum_{j=1}^{N} \left[ \frac{(1 + \gamma)}{2} \sigma_j^x \sigma_{j+1}^x + \frac{(1 - \gamma)}{2} \sigma_j^y \sigma_{j+1}^y - h \sigma_z \right],
\]

where \( \sigma_i^{x,y,z} \) are the Pauli matrices, \( \gamma \) is called anisotropy parameter, and \( h \) is the applied external transverse field. It is well known that for any \( \gamma \neq 0 \) the model undergoes a phase transition at \( h = 1 \), that is in the universality class of the Ising model (defined by \( \gamma = 1 \)). For simplicity we will just consider the Ising case, other values of \( \gamma \neq 0 \) being equivalent.

We consider the non-equilibrium unitary dynamics that follows from a quench of the magnetic field at \( t = 0 \) from \( h_0 \) to \( h_1 \neq h_0 \). Earlier works on this subject by Barouch et al date back to seventies [7, 8]. Exploiting the mapping of this model onto a free fermion, the time evolution of the transverse magnetization (that is not the order parameter) was
obtained exactly [7]:

\[ m_z(t) = \frac{1}{\pi} \int_0^\pi dk \frac{(h_0-h_1) \sin^2 k \cos(2\epsilon_1 t) - (\cos k - h_1)(\cos k - h_1 + \sin^2 k)}{\epsilon_0 \epsilon_1^2}, \]

(68)

where \( \epsilon_i = \epsilon(h_i) \) with \( \epsilon(h) = \sqrt{\sin^2 k + (h - \cos k)^2} \). In [7] it was shown that (for non-exceptional parameters) the approach to the asymptotic value for \( t \to \infty \) is of the form \( t^{-3/2} \cos 4t \). This is simply shown in the case of \( h_0 = \infty \) and \( h_1 = 1 \), when the integral simplifies and we obtain (see also [9])

\[ m_z(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\pi dk \frac{\sin^2 k \cos(2\epsilon_1 t)}{4 \sin^2 k/2} = \frac{1}{2} + \frac{J_1(4t)}{4t}, \]

(69)

where \( J_1 \) is the Bessel function of the first kind whose asymptotic expansion for large argument is \( J_1(x) \sim \frac{\sqrt{2/\pi x}}{\cos(x + \pi/4)} \).

To understand this result we should keep in mind that the transverse magnetization is not the order parameter: it corresponds to the product of two ‘disorder parameters’ at neighbour sites on the dual lattice. Thus it must have the symmetry of an energy operator, for which CFT just predicts a constant asymptotic result. The power law term \( t^{-3/2} \cos(4t) \) is just the lattice correction to the asymptotic. This correction clearly shows the fermionic nature of the model on the basis of what discussed in the previous section.

The asymptotic result of the two-point function for \( t \to \infty \) has been studied in [11]. Calling \( G_n = \langle \Phi(n, \infty)\Phi(0, \infty) \rangle \), in the case \( h_0 = \infty \) it has been found

\[ G_n = \begin{cases} 
\frac{1}{2\pi h_1^n}, & \text{for } h_1 \geq 1, \\
\frac{1}{2\pi} \cos[n \arccos(h_1)], & \text{for } h_1 \leq 1,
\end{cases} \]

(70)

instead for \( h_0 = 0 \)

\[ G_n = \begin{cases} 
\frac{1}{2^n}, & \text{for } h_1 \geq 1, \\
\frac{h_1^{n+1}}{2^n} \cosh \left( (n+1) \ln \left( \frac{1 + \sqrt{1-h_1^2}}{h_1} \right) \right), & \text{for } h_1 \leq 1,
\end{cases} \]

(71)

that for large \( n \) decay exponentially with \( n \). It has been shown that \( G_n \) is decaying exponentially with \( n \) for general \( h_0 \) and \( h_1 \) [11], although closed forms are not available. The exponential decay is the prediction of CFT that (maybe surprisingly) applies even far from the critical point \( h_1 = 1 \).

The time dependence of two-point function has been studied in [9] by means of exact diagonalization of the model with open boundary condition at the two ends \( r = 0, L \). The results of interest for this paper are:

- The connected two-point function of \( \sigma_z \) in the thermodynamic limit and at the critical point is

\[ \langle \sigma_z^c(t)\sigma_0^c(t) \rangle_c = \left[ \frac{r}{2t} J_{2r}(4t) \right]^2 - \frac{r^2 - 1}{4t^2} J_{2r+1}(4t) J_{2r-1}(4t), \]

(72)

which is valid both for \( h_0 = 0, \infty \). Neglecting fast oscillations, \( \langle \sigma_z^c(t)\sigma_0^c(t) \rangle_c \) increases as \( r^2 \) for \( r < 2t \) and then it drops almost immediately to 0.
The autocorrelation function (not connected) of $\sigma_z$ at different times is

$$\langle \sigma_z(t_1)\sigma_z(t_2) \rangle = J_0^2(2t_2 - 2t_1) - \frac{1}{4}[f(t_2 + t_1) \pm g(t_2 - t_1)],$$

(73)

where $f(x) = J_2(2x) + J_0(2x)$, $g(x) = J_2(2x) - J_0(2x)$ and the sign $+(-)$ refers to $h_0 = 0 (h_0 = \infty)$.

Let us comment on these results in view of the general understanding we found so far. For $t < t^*$ all the connected correlation functions are zero, in agreement with CFT. All the oscillation terms of the asymptotic form $\cos(4t)$ are, as we discussed in the previous section, a lattice effect. Concerning the $\sigma_z$ correlator, the $r^2$ dependence for $t > r/2$ is a consequence of the fact that $\sigma_z$ is not primary. The same is true for the two-time correlations function of $\sigma_z$ that also decays as a power law of $t - s$ for large times, instead of the exponential prediction by CFT for primary field.

5. Higher dimensions

Until now we just considered one-dimensional systems. Despite the fact that in low dimensions the effect of fluctuations is more pronounced making the physics highly non-trivial it is desirable to have results in higher dimensions as well. The method presented in section 2 to obtain the non-equilibrium dynamics of a quantum model close to a critical point from the critical behaviour of a system confined in a slab geometry applies to generic dimension $d$ through the study of the Hamiltonian (4). Its analysis proceeds via field-theoretical RG that may provide the all scaling quantities of the model in an expansion close to the upper critical dimensions (u.c.d.), that is $D = d + 1 = 4$. Above the u.c.d. mean field (or Gaussian) results are exact, with logarithmic correction at the u.c.d.. For dimensions lower than the u.c.d., the scaling quantities are obtained as series in $\epsilon = 4 - (d + 1)$. Thus for the time evolution problem the simple mean field solution represents an exact scaling result (apart log corrections) for the physically relevant three-dimensional case. An alternative method to attack analytically the Hamiltonian (4) is to consider an $N$ component field $\phi$ and taking the limit $N \to \infty$, but this will not be employed here.

5.1. Dirichlet boundary conditions: the two-point function

The $D$-dimensional slab geometry with Dirichlet boundary conditions has been the subject of several investigations. The two-point function has been calculated at the first order in $\epsilon$ expansion in [43]. The Gaussian two-point function, with partial Fourier transform in the parallel directions reads [43]

$$G(p, z_1, z_2) = \frac{1}{2b} \left( e^{-b|z_1 - z_2|} - e^{-b(z_1 + z_2)} + \frac{e^{-b(z_1 - z_2)} + e^{-b(z_2 - z_1)} - e^{-b(z_1 + z_2)} - e^{b(z_1 + z_2)}}{e^{2bL} - 1} \right),$$

(74)

with $b = \sqrt{p^2 + m^2}$. We are interested in the case where $L = 2\tau_0$, $z_1 = z_2 = \tau$ that we will analytically continue to $\tau \to \tau_0 + it$, and for computational simplicity we will restrict
to the massless case $m = 0$. Thus in real space and imaginary time, we have ($p = |\mathbf{p}|$)

$$G(r, \tau) = \int \frac{d^3 p}{(2\pi)^3} e^{-i p \cdot r} \frac{1}{2p} \left(1 - e^{-2pr} + \frac{2(1 - \cosh(2pr))}{e^{4p\tau_0} - 1}\right)$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty p^2 dp \frac{1}{2p} \left(1 - e^{-2pr} + \frac{2(1 - \cosh(2pr))}{e^{4p\tau_0} - 1}\right) \int_{-1}^1 d(cos \theta) e^{ipr \cos \theta}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dp \sin pr \left(1 - e^{-2pr} + \frac{2(1 - \cosh(2pr))}{e^{4p\tau_0} - 1}\right). \quad (75)$$

This integral can be performed by making the sum over all the residues coming from the denominator $e^{4p\tau_0} - 1$. The calculation is rather involved, but the final result is very simple:

$$G(r, \tau) = \frac{1}{\pi r \tau_0} \frac{\coth(\pi r/4\tau_0) \sin^2(\pi \tau/2\tau_0)}{\cosh(\pi r/2\tau_0) - \cos(\pi \tau/\tau_0)}. \quad (76)$$

Continuing to real time $\tau = \tau_0 + it$ we obtain

$$G(r, t) = \frac{1}{\pi r \tau_0} \frac{\coth(\pi r/4\tau_0) \cos^2(\pi t/2\tau_0)}{\cosh(\pi r/2\tau_0) + \cos(\pi t/\tau_0)}, \quad (77)$$

that for $r, t \gg \tau_0$ simplifies to

$$G(r, t) \approx \frac{1}{\tau \tau_0} \frac{e^{\pi t/\tau_0}}{e^{\pi t/\tau_0} + e^{\pi r/2\tau_0}} \left\{ \begin{array}{ll} e^{\pi(t-r)/2\tau_0}/r & \text{for } t < r/2, \\ 1/r & \text{for } t > r/2. \end{array} \right. \quad (78)$$

We recall that this result in $d = 3$ is exact apart log corrections. Thus the basic structure of the two-point function in 3D is the same as in 1D, with a characteristic time $t^* = r/2$. Using the result in [43] it is in principle possible to calculate the correlation functions for $d < 3$ in the $\epsilon$ expansion framework. However, this requires the analytical continuation of complicated functions, resulting in a quite cumbersome algebra, as we shall see in the following for a simpler observable.

Equation (74) can be used in principle to determine the Gaussian behaviour in any $d < 3$. Unfortunately the integral one gets is not analytically tractable for $d \neq 1, 3$. A possible strategy would be to perform the analytic continuation before of the integral and then evaluate it through a saddle point approximation. It is straightforward to show that the result obtained in this way is equivalent to what we discuss in section 5.4 where we postpone the analysis for $1 < d < 3$.

### 5.2. Dirichlet boundary conditions: a non-trivial one-point function

In the case of Dirichlet boundary conditions, the order parameter profile in the slab geometry is trivially vanishing. However not all the one-point expectation values are zero. Let us consider as a typical example the operator $\mathcal{O} = \phi^2$ that has been calculated at the first order in $\epsilon$ expansion in [44]. It has the scaling form

$$\langle \mathcal{O}(z, L) \rangle \sim L^{-d+1/\nu} H(z/L), \quad (79)$$

where $\nu$ is the correlation length exponent. In $D = 4$, the function $H(x)$ is [44]

$$H(x) = \frac{\pi^2}{\sin^2(\pi x)} - \frac{\pi^2}{3}. \quad (80)$$
that, using $L = 2\tau_0$ and continuing to $z = \tau = \tau_0 + it$, leads to the real time evolution

$$\langle O(t) \rangle - \langle O(t = \infty) \rangle \simeq \tau_0^2 \frac{\pi^2}{\cosh^2(\pi t/2\tau_0)} \sim \tau_0^2 e^{-\pi t/2\tau_0},$$  

(81)

i.e. $\langle O(t) \rangle$ approaches its asymptotic value exponentially, as it has been found for all primary operators in $d = 1$.

The one-loop result for the $O(N)$ model in $D = 4 - \epsilon$ dimensions is [44]

$$H(x) = \left[\frac{\pi^2}{\sin^2(\pi x)} - \frac{\pi^2}{3}\right] \left(1 + \epsilon \frac{N + 2}{N + 8} \ln \frac{\pi}{\sin \pi x}\right) - \epsilon [\zeta'(2, x) + \zeta'(2, 1 - x)] + \text{const},$$

(82)

where \text{const} stands for terms do not depend on $x$ and $\zeta'$ is the generalized Riemann function. This example clearly shows that within the $\epsilon$ expansion, the scaling functions contain logarithmic contributions that originate from the expansion of power law as e.g. $\sin^\epsilon \pi x = 1 + \epsilon \ln \sin \pi x$. To have a function with good analytical structure to perform the real time continuation, it is desirable to ‘exponentiate’ such logarithms. In [44] the exponentiation procedure led to

$$H(x) = \left(\frac{\pi}{\sin(\pi x)}\right)^{2-1/\nu} \left[\zeta(D - 2, x) + \zeta(D - 2, 1 - x) - 2\zeta(D - 2)\right]$$

$$- \frac{\pi^2}{6} \left(2 - 1/\nu\right) \left(\frac{\pi}{\sin(\pi x)}\right)^{D-2-1/\nu},$$

(83)

with $\nu = 1/2 + (N + 2)/(N + 8)\epsilon/4$. Performing the analytical continuation we have the sum of two exponentials, and in any dimension $d < 3$ the second one has a largest ‘relaxation time’ that hence is dominating for large $t$. Considering only the second term we have

$$\langle O(t \gg \tau_0) \rangle - \langle O(t = \infty) \rangle \propto e^{-(d-2-1/\nu)\pi t/2\tau_0}.$$  

(84)

Note that the subleading term $e^{-(2-1/\nu)\pi t/2\tau_0}$ is multiplied by a log $t$ term arising from the $\zeta$ function.

This example put forward the idea that (at least for Dirichlet boundary conditions) the exponential relaxation of the one-point functions is not only a property of one-dimensional systems, but holds in any dimension (with possibly log corrections) with relaxation times related to the scaling dimensions of the operator.

5.3. Fixed boundary conditions: the one-point function

The case of the extraordinary transition, that corresponds to fixed boundary conditions, has been considered in [45]. The magnetization profile can be written as [45]

$$\phi(z) = \frac{2K_4}{L} \frac{dn(2K_4 z/L)}{sn(2K_4 z/L)},$$

(85)

where $K_4 = K(k_4)$, where $K(k)$ is the elliptic integral, $k_4$ the elliptic modulus that in terms of the parameter of the model is $\phi^2 L^2 = (2K_4)^2(2k_4^2 - 1)$, and $sn(x)$ and $dn(x)$ and the Jacobi functions. Continuing to real time, and using the properties of the Jacobi
functions we obtain
\[ \phi(t) = \sqrt{1 - k_4 \text{cn}(K_4, t/\tau_0, 1 - k_4)} \]
that, contrarily to all the other cases we have considered, is oscillating and not exponentially decreasing.

To our knowledge there are no results in the literature for the magnetization profile. However, using ‘local functional methods’ [46] it has been obtained an approximate profile in \( D = 3 \) that involves, as in mean field theory, Jacobi elliptic functions. It can be easily continued to real time via \( z \to \tau_0 + it \) and again one finds an oscillating behaviour with time. The method exploited in [46] can be used in any dimensions, obtaining an always an oscillating behaviour with a period that diverges as \( D \) approaches 2, recovering equation (7). All these calculations are essentially mean field and we do not know how the inclusion of fluctuations changes them. It can possibly that for the extraordinary transition (i.e. fixed initial conditions), the exponential decay found at \( D = 2 \) is more an exception rather than a rule, because of the simple analytic structure of the trigonometric functions in the complex plane. Another possibility is that fluctuations destroy these oscillations. Only a complete analytical calculation (e.g. in large \( N \)) can help in understanding this point.

5.4. A real time solvable model

As for the one-dimensional case, it is worth to check the results coming from the analytical continuation of large imaginary time with exactly solvable models. The simplest (and probably one of the few) model solvable in generic dimension is the generalization of the Hamiltonian (38) to a \( d \)-dimensional hypercubic lattice. The solution of such model proceeds via Fourier transform as in one dimension. The final result is simply given by equation (53) with the replacement \( dk \to d^d k \), i.e.
\[ \langle \varphi_r(t) \varphi_0(t) \rangle - \langle \varphi_r(0) \varphi_0(0) \rangle = \int_{BZ} \frac{d^d k}{(2\pi)^d} e^{ik\cdot r} \left( \Omega_0^2(k) - \Omega_0^2 \right) \frac{(1 - \cos(2\Omega_0 t))}{\Omega_0^2(\sin(\Omega_0 t))}. \]  
(87)

First of all let us note that this expression in virtually identical to equation (75) if we take \( \tau_0 \propto m_0^{-1} \to 0 \). In fact, taking \( \tau_0 \to 0 \) in equation (75), only the third term matters, since it is \( O(\tau_0^{-1}) \) relative to the first two. Taking \( z = r + it \) and \( b = \Omega_p \) we get \( G(p) \propto \tau_0^{-1} \int \Omega_p^{-2}(1 - \cos 2\Omega_p t) \) that is equation (87) for \( m_0 \to \infty \).

To understand the general features let us consider in details the conformal evolution \( c(\Omega_k = v|k|) \) from a disordered state \( (\Omega_0k = m_0) \). The derivative of the two-point function is \( (k = |k|) \)
\[ \partial_t \langle \varphi_r(t) \varphi_0(t) \rangle = 2m_0 \text{Im} \int \frac{d^d k}{(2\pi)^d} e^{i(k\cdot r - 2\Omega_0 t)}. \]  
(88)

Except for \( d = 1 \), this can be done analytically only in \( d = 3 \), where we can write it as
\[ \sim \int k dk \int d\theta \sin \theta e^{ik(r\cos \theta - 2vt)} \sim \int dk (\sin(kr)/r)e^{2ikvt} \sim (1/r)\delta(vt - r/2). \]  
(89)

Integrating with respect to \( t \) we get zero for \( t < r/2v \) and \( 1/r \) for \( t > r/2v \).

For general \( d \) we have
\[ \partial_t \langle \varphi_r(t) \varphi_0(t) \rangle \propto 2m_0 \text{Im} \int k^{d-2} \frac{dk}{(2\pi)^d} \int d\theta (\sin \theta)^{d-2} e^{ik(r\cos \theta - 2vt)}. \]  
(90)

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By a saddle point argument, we can assume that the dominant behaviour comes from $\theta$ close to zero, so the $\theta$ integral gives

$$\int d\theta \theta^{d-2} e^{-i k r \theta^2 / 2} \sim (k r)^{-(d-1)/2},$$

leading to

$$\int d k \frac{k^{d-2}}{(k r)^{(d-1)/2}} e^{i(k(r-2vt))} \sim r^{-(d-1)/2}(2vt - r)^{-(d-1)/2}\Theta(vt - r/2).$$

Integrating with respect to $t$, we get zero for $t < r / 2v$, as expected by causality, and

$$r^{-(d-1)/2}(2vt - r)^{(3-d)/2},$$

for $t > r / 2v$. It is interesting that this Gaussian correlation function blows up at large $t$ only for $d < 3$, when we expect the fluctuations to become important. It would be interesting to study this in the $\phi^4$ theory for large $N$, by replacing $\phi^4$ by $3\langle \phi^2 \rangle \phi^2$ in the usual way, where now $\langle \phi^2 \rangle$ depends on $t$ and is calculated self-consistently.

6. Physical interpretation and discussion

In this paper we studied in general the non-equilibrium unitary dynamics that follows a sudden quantum quench. We showed that if the Hamiltonian $H$ governing the time evolution is at a critical point, while $H_0$ (i.e. the one for $t < 0$) is not, the expectation value of a class of operators (primary ones in CFT) relaxes to the ground state value exponentially in time with universal ratio of decaying constants. We also found that connected two-point functions of operators at distance $r$ are vanishing for $t < r / 2v$, while for $t > r / 2v$ reach exponentially fast a value that depends exponentially on the separation, in contrast with the power laws typical of equilibrium configuration.

We also considered the real time dynamics of simple exactly solvable models and we found that several of the typical characteristics of the critical points still hold. Roughly speaking, critical points are not special as far as quenching dynamics is concerned. In fact, also for gapped systems, connected correlation functions vanish (or are strongly suppressed) for $t < r / 2v$ and for asymptotic large times resemble those at finite temperature despite the fact that the whole system is in a pure state. Several other examples in the recent literature (see e.g. [9, 15, 24, 26, 39, 41]) gives further evidence that these two effects are actually true in general, at least in the realm of exactly solvable models considered so far.

In the following we give a simple interpretation of these two features separately trying to understanding their physical origin.

6.1. The horizon effect

The qualitative, and many of the quantitative, features found for the time evolution of correlation functions may be understood physically on the basis of a picture we first introduced in [39] to describe the time evolution of the entanglement entropy. Later we generalized it to correlation functions in [33] and it has been largely adopted thereafter [2, 24, 26, 41].
We emphasize that such scheme is not an *ab initio* calculation but rather a simplified
picture which allows us to explain physically our findings. The initial state $|\psi_0\rangle$ has
an (extensively) high energy relative to the ground state of the Hamiltonian $H$ which
governs the subsequent time evolution, and therefore acts as a source of quasiparticle
excitations. Those quasiparticles originating from closely separated points (roughly within
the correlation length $\xi_0 = m_0^{-1}$ of the ground state of $H_0$) are quantum entangled and
particles emitted from far different points are incoherent. If the quasiparticle dispersion
relation is $E = \Omega_k$, the classical velocity is $v_k = \nabla_k \Omega_k$. We assume that there is a
maximum allowed speed $v_m = \max_k |v_k|$. A quasiparticle of momentum $k$ produced at
$r$ is therefore at $r + v_k t$ at time $t$, ignoring scattering effects. This is the only physical
assumption of the argument. Scattering effects are not present in the theories considered
so far, but as evident from the argument outlined below they can play a role for only $t > r/2v_m$ (allowing, perhaps, for a renormalization of $v_m$ by the interactions).

These *free* quasiparticles have two distinct effects. Firstly, incoherent quasiparticles
arriving a given point $r$ from well-separated sources cause relaxation of (most) local
observables at $r$ towards their ground state expectation values. (An exception is the
local energy density which of course is conserved.) Secondly, entangled quasiparticles
arriving at the same time $t$ at points with separation $|r| \gg \xi_0$ induce quantum correlations
between local observables. In the case where they travel at a unique speed $v$ (as in CFT),
therefore, there is a sharp ‘horizon’ or light-cone effect: the connected correlations
do not change from their initial values until time $t \sim |r|/2v$. In the CFT case this
horizon effect is rounded off in a (calculable) manner over the region $t - |r|/2v \sim \tau_0$,
since quasiparticles remain entangled over this distance scale. After this they rapidly
saturate to *time independent* values. For large separations (but still $\ll 2vt$), these
decay exponentially $\sim \exp(-\pi x|r|/2\tau_0)$. Thus, while the generic one-point functions
relax to their ground state values (we recall in CFT this relaxation is exponential
$\sim \exp(-\pi xvt/\tau_0)$), the correlation functions do not, because, at quantum criticality, these
would have a power law dependence. Of course, this is to be expected since the mean
energy is much higher than that of the ground state, and it does not relax.

This simple argument also explains why for the case of a semi-infinite chain the
relevant timescale is $r/v$ rather than $r/2v$, since one of the two particles arriving in $r$
has been reflected from the end of the chain. This has been also stressed in [41], in the
study of the time dependence of entanglement entropy of finite chains with open boundary
conditions.

All our results are consistent with this picture as long as the quasiparticles are
assumed to all propagate at the same speed, resulting from a ‘conformal’ dispersion
relation $\Omega_k = v|k|$. However, it is very simple to generalize this picture to different
dispersion relations, taking into account that each particle propagates at group velocity
$v_k \equiv \Omega_k$ appropriate to the wavenumber $k$. In this case the horizon effect first occurs
at time $t \sim |r|/2v_m$, where $v_m$ is the maximum group velocity. If $v_m$ occurs at a non-
zero wavenumber, it gives rise to spatial oscillations in the correlation function. Thus
again connected correlation function are expected to be strongly suppressed for $t < t^*$
and start developing only after $t^*$. In the case of a general dispersion relation we do not
have a proof, beyond the stationary phase approximation, that the connected correlation
functions remain constant up to this time, but one would expect it on the grounds of causality. (The proof in the case of a relativistic dispersion relation uses Lorentz invariance.

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in an essential way.) However, because there are also quasiparticles moving at speeds less than $v_m$, the approach to the asymptotic behaviour at late times is less abrupt. In fact, for a lattice dispersion relation where $\Omega'_k$ vanishes at the zone boundary, the approach to the limit is slow, as an inverse power of $t$. A similar result applies to the one-point functions. This is consistent with the exact results obtained here and elsewhere.

6.2. The large time limit and the generalized Gibbs ensemble

The existence and the understanding of the asymptotic state resulting from the evolution from an arbitrary state is one of the most interesting problems in statistical mechanics. A robust theory able to predict this state \textit{ab initio} still does not exist. A currently popular idea is that for late times the system (or rather macroscopically large subsystems) ‘look like’ they are in a thermal state, despite the fact that the actual state of the whole system is pure. A common belief is that a region of dimension $r$ can be thermalized by the infinitely large rest of the system which acts as a bath (see e.g. [24,39,47]). But this intriguing idea is not sufficient to give the value of the resulting effective temperature.

A major step toward the clarification of the properties of the asymptotic state has been made by Rigol \textit{et al} [15]. In fact, it was conjectured that if the asymptotic stationary state exists, it is given by a generalized Gibbs ensemble obtained by maximizing the entropy $S = -\text{Tr} \rho \log \rho$, subject to all the constraints imposed by the dynamics [15]. Consequently, denoting with $I_m$ a maximal set of commuting and linearly independent integrals of motion, the density matrix is

$$\rho = Z^{-1} e^{-\sum \lambda_m I_m}, \quad Z = \text{Tr} e^{-\sum \lambda_m I_m}. \quad (94)$$

We note that such a density matrix describes a pure state only if the model under consideration is integrable, i.e. if the number of integral of motions equals the number degrees of freedom. If there are not enough integrals of motion, $\rho$ corresponds to a mixed state and it is not clear to us to which extent it can describe the pure state resulting from the time evolution. The values of the Lagrange multipliers $\lambda_m$ are fixed by the initial conditions:

$$\text{Tr} I_m \rho = \langle I_m \rangle_{t=0}. \quad (95)$$

In the following we will take this generalized Gibbs ensemble as a postulate and we will show how it nicely and naturally explains the ‘effective temperature’ effect observed for large times. However we stress that there is still no proof for this assumption that, to our opinion, cannot be considered on the same fundamental level as the thermal Gibbs ensemble.

Let us consider the chain of harmonic oscillators of the previous section as a typical example. We will soon see that most of the features are quite general. In this case the natural choice for an infinite set of integral of motion is the number of particles with momentum $k$, i.e. $n_k = A_k^\dagger A_k$. Most other observables can be written in terms of these, i.e. $H = \sum_k \Omega_k n_k$. Consequently the expectation value is given by

$$\langle O \rangle_{t=\infty} = \text{Tr} O \rho = \text{Tr} O Z^{-1} e^{-\sum \lambda_k n_k}, \quad (96)$$

that can be seen as a thermal density matrix with a $k$ dependent effective temperature given by

$$\beta_{\text{eff}}(k) \Omega_k = \lambda_k. \quad (97)$$

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Thus an effective temperature already appeared. Note that this state can still be pure because such a temperature is $k$ dependent.

To fix $\lambda_k$ we need $\langle n_k \rangle_{t=0}$. From equation (50) we get

$$
\langle n_k \rangle_{t=0} = \langle A_k^\dagger A_k \rangle_{t=0} = d_k^2 = \frac{1}{4} \left( \frac{\Omega_k}{\Omega_{0k}} + \frac{\Omega^2_{0k}}{\Omega_k} \right) - \frac{1}{2}.
$$

The calculation then proceeds as for a thermal distribution

$$
\langle n_k \rangle_\rho = \text{Tr} n_k \rho = -\frac{\partial}{\partial \lambda_k} \ln Z,
$$

with $Z = \text{Tr} e^{-\sum_k \lambda_k n_k} = \prod_k \sum_{n_k=0}^\infty e^{-\lambda_k n_k} = \prod_k \frac{1}{1 - e^{-\lambda_k}},$

so that

$$
\langle n_k \rangle_\rho = \frac{\partial}{\partial \lambda_k} \sum_k \ln(1 - e^{-\lambda_k}) = \frac{1}{e^{\lambda_k} - 1}.
$$

From this $e^{\lambda_k} = 1 + d_k^{-2}$ and

$$
\beta_{\text{eff}}(k) = \frac{\ln(1 + d_k^{-2})}{\Omega_k}.
$$

At finite temperature the correlation function in momentum space is

$$
\langle \varphi_k \varphi_{-k} \rangle_\beta = \left\langle \frac{2}{\Omega_k} (A_k + A_k^\dagger)(A_{-k} + A_{-k}^\dagger) \right\rangle_\beta
$$

$$
= \frac{2}{\Omega_k} \left( \langle A_k^\dagger A_{-k} \rangle_\beta + \langle A_k A_{-k}^\dagger \rangle_\beta \right) = \frac{2}{\Omega_k} \frac{1 + e^{-\beta\Omega_k}}{1 - e^{-\beta\Omega_k}},
$$

that substituting the previous result for $\beta_{\text{eff}}$ gives

$$
\rho(k)_{t=\infty} = \langle \varphi_k \varphi_{-k} \rangle_{t=\infty} = \frac{2}{\Omega_k} (1 + 2d_k^2) = \frac{\Omega_k^2 + \Omega_{0k}^2}{\Omega_k^2 \Omega_{0k}},
$$

that is exactly the time average of equation (54) reproducing, after Fourier transforming, the well defined correlation in real space for $t \to \infty$.

So this generalized Gibbs ensemble correctly reproduces the exact diagonalization of the model and gives also few insights more. In fact, in the limit $m_0 \to \infty$ the effective temperature is independent from $k$ and $m$ obtaining $\beta_{\text{eff}} = 4/m_0$, explaining a posteriori the simplicity of the results in this case. Note instead that for arbitrary $m_0$ and $m = 0$, i.e. conformal evolution, $\beta_{\text{eff}}(k)$ is a function of $k$. In this case, the large distance properties of correlation functions are described by the mode with $k = 0$, for which, independently of $m_0$ we get $\beta_{\text{eff}}(k = 0, m = 0) = 4/m_0$, consistently with the previous findings and general expectations. Furthermore we can conclude that the large $r$ asymptotic behaviour is always governed by the effective temperature $\beta_{\text{eff}}(k = 0) = -2(\log(|m_0 - m|/(m_0 + m))/m)$. Another interesting feature is that $\beta_{\text{eff}}(k = 0)$ gives the asymptotic behaviour of the correlation functions of all those observables that are effectively coupled with the zero-mode. In the opposite case the relevant temperature is the largest $\beta_{\text{eff}}(k)$ with the $k$ mode coupled with the observable.
Another interesting feature is that on a very rough basis one can be tempted to assume that $\beta_{\text{eff}}(k)$ is directly related to the excess of energy of the mode $k$ generated by the quench, but this is not the case. In fact $\langle H \rangle_{t=0} = \sum_k \Omega_k d_k^2$ that is different from $\beta_{\text{eff}}^{-1}$, being the same only in the limit $d_k^{-2} \to 0$, i.e. $m_0 \to \infty$.

Clearly the same reasoning of before applies every time we consider a model that can be diagonalized in momentum space with a proper choice of the quasiparticles, i.e. every time that $H = \sum_k \Omega_k A_k^\dagger A_k$ for some $A_k$. This means that the excitations are non-interacting. As far as we are aware all the applications of the generalized Gibbs ensemble to the date only concern this kind of models [15,26], but there are few numerical hints suggesting that can be true more generically [29].

Now we outline how we imagine the generalized Gibbs ensemble given by equation (94) could be used to justify the effective temperature scenario for large time for any integrable system, i.e. with a complete set of integrals of motion. The Hamiltonian can be written as $H = \sum_m a_m I_m$, where some $a_m$ can be zero. Thus one can think to an $m$ dependent effective temperature $\beta_{\text{eff}}(m) = \lambda_m / a_m$, but this temperature does not give directly the behaviour of the correlation function for large distance because in general the integral of motions are not diagonal in $k$ space. Thus we can only conjecture that the correlation functions of a given operator $O(r)$ are governed by the largest $\beta_{\text{eff}}(m)$ with $m$ among the integral of motions to which $O(r)$ effectively couples. We stress that this is only a crude argument and we are still not able to put it on a firmer basis.

6.3. Discussion and open questions

We presented a quite complete picture on the time evolution of a quantum system after a sudden quench of one Hamiltonian parameter. Despite the fact that a lot of work has been done, still more is left for future investigation.

The first problem that must be addressed is the real time dynamics of effectively interacting systems. In this direction Bethe ansatz solvable models like the Lieb–Liniger gas and Heisenberg spin chains are among the best candidates for an analytical approach. This would clarify how a non-trivial two-body scattering matrix can modify the time evolution (only inside the light-cone) of the correlation functions and whether the crude argument we outlined for an ‘effective temperature’ for the long time state is valid. To clarify this point also the study of (boundary) integrable massive field theories can be of some help.

Another approach, currently under investigation [48], is the direct analysis of the perturbative expansion for the correlation functions in a $\lambda \phi^4$ field theory. This has some simplifying features in the large $m_0$ limit which may allow it to be resummed to all orders. It is very important to get non-trivial results for such a non-integrable interacting theory.

Numerical computations for non-integrable systems should also be performed to understand if and eventually to which extent the generalized Gibbs ensemble picture is valid beyond integrability. Some numerics concerning this point are already available [22, 23, 29] but still a clear scenario is not emerged. Among non-integrable models special care must be given to disordered systems because of the non-existence of a speed of sound as a consequence of Anderson localization [49]. Some insights can be obtained from those models whose equilibrium behaviour can be analytically obtained by means of the strong disorder renormalization group [50]. Even in this case time dependent DMRG can...
help a general understanding. The time evolution of the entanglement entropy [41] already revealed that in these systems there is no sign of the light-cone (as easily predictable because of the absence of speed of sound) and the effective motion of ‘quasiparticles’ is more diffusive rather than ballistic. Furthermore the results at the largest available times does not look thermal at all, but this can be also due to the time window accessible with numerics.

Another very interesting question is what happens at finite temperature and under what conditions a system can equilibrate. It should be relatively simple to generalize the results for all the ‘quasifree’ models already considered at $T = 0$. Work in this direction is in progress, but it will be almost impossible to consider the same problem for more complicated models.

Finally it is also important to understand the role played by the initial state. We assumed always a translationally invariant one with short range correlations. Thus one natural modification consists in taking a state that is only locally different from the actual ground state. This problem is known in the literature as a ‘local quench’ and it has already been considered to some extent (see e.g. [51]–[53]), but still a general picture as the one outlined here for global quenches does not exist (this is also interesting for possible connections with quantum impurity problems, see [54] as a review). Another natural modification of the initial state is one with long range correlations, such as a critical one and let it evolve with another critical Hamiltonian. This has been done for the Luttinger liquid [26] and the results show the typical functional dependence of a light-cone scenario (i.e. everything depends only on $x \pm 2vt$) but the long time correlations decays as power laws, with exponents that are different from equilibrium ones and can be predicted by the generalized Gibbs ensemble.

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