Distributed Learning with Regularized Least Squares

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Abstract

We study distributed learning with the least squares regularization scheme in a reproducing kernel Hilbert space (RKHS). By a divide-and-conquer approach, the algorithm partitions a data set into disjoint data subsets, applies the least squares regularization scheme to each data subset to produce an output function, and then takes an average of the individual output functions as a final global estimator or predictor. We show with error bounds in expectation in both the $L^2$-metric and RKHS-metric that the global output function of this distributed learning is a good approximation to the algorithm processing the whole data in one single machine. Our error bounds are sharp and stated in a general setting without any eigenfunction assumption. The analysis is achieved by a novel second order decomposition of operator differences in our integral operator approach. Even for the classical least squares regularization scheme in the RKHS associated with a general kernel, we give the best learning rate in the literature.

Keywords: Distributed learning, divide-and-conquer, error analysis, integral operator, second order decomposition.

1. Introduction and Distributed Learning Algorithms

In the era of big data, the rapid expansion of computing capacities in automatic data generation and acquisition brings data of unprecedented size and complexity, and raises a series of scientific challenges such as storage bottleneck and algorithmic scalability (Zhou et al., 2014). To overcome the difficulty, some approaches for generating scalable approximate algorithms have been introduced in the literature such as low-rank approximations of kernel matrices for kernel principal component analysis (Schölkopf et al., 1998; Bach, 2013), incomplete Cholesky decomposition (Fine, 2002), early-stopping of iterative optimization algorithms for gradient descent methods (Yao et al., 2007; Raskutti et al., 2014), and greedy-type algorithms. Another method proposed recently is distributed learning based on a
divide-and-conquer approach and a particular learning algorithm implemented in individual machines (Zhang et al., 2015; Shamir and Srebro, 2014). This method produces distributed learning algorithms consisting of three steps: partitioning the data into disjoint subsets, applying a particular learning algorithm implemented in an individual machine to each data subset to produce an individual output (function), and synthesizing a global output by utilizing some average of the individual outputs. This method can successfully reduce the time and memory costs, and its learning performance has been observed in many practical applications to be as good as that of a big machine which could process the whole data. Theoretical attempts have been recently made in (Zhang et al., 2013, 2015) to derive learning rates for distributed learning with least squares regularization under certain assumptions.

This paper aims at error analysis of the distributed learning with regularized least squares and its approximation to the algorithm processing the whole data in one single machine. Recall (Cristianini and Shawe-Taylor, 2000; Evgeniou et al., 2000) that in a reproducing kernel Hilbert space (RKHS) \((\mathcal{H}_K, \| \cdot \|_K)\) induced by a Mercer kernel \(K\) on an input metric space \(\mathcal{X}\), with a sample \(D = \{(x_i, y_i)\}_{i=1}^N \subset \mathcal{X} \times \mathcal{Y}\) where \(\mathcal{Y} = \mathbb{R}\) is the output space, the least squares regularization scheme can be stated as

\[
 f_{D,\lambda} = \arg\min_{f \in \mathcal{H}_K} \left\{ \frac{1}{|D|} \sum_{(x,y) \in D} (f(x) - y)^2 + \lambda \| f \|_K^2 \right\}.
\]  

Here \(\lambda > 0\) is a regularization parameter and \(|D| = N\) is the cardinality of \(D\). This learning algorithm is also called kernel ridge regression in statistics and has been well studied in learning theory. See e.g. (De Vito et al., 2005; Caponnetto and De Vito, 2007; Steinwart et al., 2009; Bauer et al., 2007; Smale and Zhou, 2007; Steinwart and Christmann, 2008). The regularization scheme (1) can be explicitly solved by using a standard matrix inversion technique, which requires costs of \(O(N^3)\) in time and \(O(N^2)\) in memory. However, this matrix inversion technique may not conquer challenges on storages or computations arising from big data.

The distributed learning algorithm studied in this paper starts with partitioning the data set \(D\) into \(m\) disjoint subsets \(\{D_j\}_{j=1}^m\). Then it assigns each data subset \(D_j\) to one machine or processor to produce a local estimator \(f_{D_j,\lambda}\) by the least squares regularization scheme (1). Finally, these local estimators are communicated to a central processor, and a global estimator \(\overline{f}_{D,\lambda}\) is synthesized by taking a weighted average

\[
 \overline{f}_{D,\lambda} = \sum_{j=1}^m \frac{|D_j|}{|D|} f_{D_j,\lambda}
\]  

of the local estimators \(\{f_{D_j,\lambda}\}_{j=1}^m\). This algorithm has been studied with a matrix analysis approach in (Zhang et al., 2015) where some error analysis has been conducted under some eigenfunction assumptions for the integral operator associated with the kernel, presenting error bounds in expectation.

In this paper we shall use a novel integral operator approach to prove that \(\overline{f}_{D,\lambda}\) is a good approximation of \(f_{D,\lambda}\). We present a representation of the difference \(\overline{f}_{D,\lambda} - f_{D,\lambda}\) in terms of empirical integral operators, and analyze the error \(\overline{f}_{D,\lambda} - f_{D,\lambda}\) in expectation without
any eigenfunction assumptions. As a by-product, we present the best learning rates for the least squares regularization scheme (1) in a general setting, which surprisingly has not been done for a general kernel in the literature (see detailed comparisons below).

2. Main Results

Our analysis is carried out in the standard least squares regression framework with a Borel probability measure $\rho$ on $\mathcal{Z} := \mathcal{X} \times \mathcal{Y}$, where the input space $\mathcal{X}$ is a compact metric space. The sample $D$ is independently drawn according to $\rho$. The Mercer kernel $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ defines an integral operator $L_K$ on $H_K$ as

$$L_K(f) = \int_{\mathcal{X}} K_x f(x) d\rho_X, \quad f \in H_K,$$

where $K_x$ is the function $K(\cdot, x)$ in $H_K$ and $\rho_X$ is the marginal distribution of $\rho$ on $\mathcal{X}$.

2.1 Error Bounds for the Distributed Learning Algorithm

Our error bounds in expectation for the distributed learning algorithm (2) require the uniform boundedness condition for the output $y$, that is, for some constant $M > 0$, there holds $|y| \leq M$ almost surely. Our bounds are stated in terms of the approximation error $\|f_\lambda - f_\rho\|_\rho$, (4)

where $f_\lambda$ is the data-free limit of (1) defined by

$$f_\lambda = \arg\min_{f \in H_K} \left\{ \int_{\mathcal{Z}} (f(x) - y)^2 d\rho + \lambda \|f\|_K^2 \right\},$$

$\|\cdot\|_\rho$ denotes the norm of $L^2_{\rho_X}$, the Hilbert space of square integrable functions with respect to $\rho_X$, and $f_\rho$ is the regression function (conditional mean) of $\rho$ defined by

$$f_\rho(x) = \int_{\mathcal{Y}} y d\rho(y|x), \quad x \in \mathcal{X},$$

with $\rho(\cdot|x)$ being the conditional distribution of $\rho$ induced at $x \in \mathcal{X}$.

Since $K$ is continuous, symmetric and positive semidefinite, $L_K$ is a compact positive operator of trace class and $L_K + \lambda I$ is invertible. Define a quantity measuring the complexity of $H_K$ with respect to $\rho_X$, the effective dimension (Zhang, 2005), to be the trace of the operator $(L_K + \lambda I)^{-1}L_K$ as

$$\mathcal{N}(\lambda) = \text{Tr} \left( (L_K + \lambda I)^{-1}L_K \right), \quad \lambda > 0.$$
and
\[ E \| \mathbf{f}_{D,\lambda} - f_{D,\lambda} \|_K \leq C_\kappa \left( \frac{m}{(N\lambda)^2} + \frac{N(\lambda)}{N\lambda} \right) \sqrt{m} \left\{ \frac{\| f_\lambda - f_\rho \|_\rho m^2}{\sqrt{N}} + M \left( 1 + \frac{m^2}{(N\lambda)^2} + \frac{mN(\lambda)}{N\lambda} \right) \right\}, \]

where \( C_\kappa \) is a constant depending only on \( \kappa \).

To derive the explicit learning rate of algorithm (2), one needs the following assumption as a characteristic of the complexity of the hypothesis space (Caponnetto and De Vito, 2007; Blanchard and Krämer, 2010),
\[ N(\lambda) \leq c\lambda^{-\beta}, \quad \forall \lambda > 0 \quad (7) \]
for some \( 0 < \beta \leq 1 \) and \( c > 0 \). In particular, let \( \{ (\lambda_l, \phi_l) \}_{l=1}^{\infty} \) be a set of normalized eigenpairs of \( L_K \) on \( H_K \) with \( \{ \phi_l \}_{l=1}^{\infty} \) being an orthonormal basis of \( H_K \) and \( \{ \lambda_l \}_{l=1}^{\infty} \) arranged in a non-increasing order, and let
\[ L_K = \sum_{\ell=1}^{\infty} \lambda_\ell \langle \cdot, \phi_\ell \rangle_K \phi_\ell \]
be the spectral decomposition. Since \( N(\lambda) = \sum_l \frac{\lambda_l}{\lambda + \lambda} \leq \sum_l \frac{\lambda_l}{\lambda} = \text{Tr}(L_K)/\lambda \), the condition (7) with \( \beta = 1 \) always holds true with \( c = \text{Tr}(L_K) \leq \kappa^2 \). For \( 0 < \beta < 1 \), \( \lambda_n \leq c' \lambda_n^{-1/\beta} \) implies (7) (see, e.g. Caponnetto and De Vito (2007)). This condition \( \lambda_n \leq c' \lambda_n^{-1/\beta} \) is satisfied, e.g., by the Sobolev space \( W^{m_*}(B(\mathbb{R}^d)) \), where \( B(\mathbb{R}^d) \) is a ball in \( \mathbb{R}^d \) with the integer \( m_* > d/2 \), \( \rho_X \) being the uniform distribution on \( B(\mathbb{R}^d) \), and \( \beta = \frac{d}{2m_*} \) (Steinwart et al., 2009; Edmunds and Triebel, 1996).

The results in (Caponnetto and De Vito, 2007; Steinwart et al., 2009; Zhang et al., 2015) showed that if \( \frac{N(\lambda)}{N\lambda} = O(1) \), then the optimal learning rates of algorithm (2) with \( m = 1 \) can be obtained in the sense that the upper and lower bounds for \( \max_{\rho \in \mathcal{H}_K} E \| f_{D,\lambda} - f_D \|_\rho \) are asymptotically identical. Thus, to derive learning rates for \( E \| \mathbf{f}_{D,\lambda} - f_{D,\lambda} \|_\rho \), a more general case with an arbitrary \( m \) is covered as follows.

**Corollary 2** Assume \(|y| \leq M \) almost surely. If \(|D_j| = \frac{N}{m} \) for \( j = 1, \ldots, m \), and \( \lambda \) satisfies
\[ 0 < \lambda \leq C_0 \quad \text{and} \quad \frac{mN(\lambda)}{N\lambda} \leq C_0, \quad (8) \]
for some constant \( C_0 > 0 \), then we have
\[ E \| \mathbf{f}_{D,\lambda} - f_{D,\lambda} \|_\rho \leq \tilde{C}_\kappa \frac{mN(\lambda)}{N\lambda} \left( \frac{m\sqrt{m}}{\sqrt{N\lambda}} + \frac{M}{\sqrt{m}} \right) \]
and
\[ E \| \mathbf{f}_{D,\lambda} - f_{D,\lambda} \|_K \leq \tilde{C}_\kappa \frac{mN(\lambda)}{N\lambda} \left( \frac{m\sqrt{m}}{\sqrt{N\lambda}} + \frac{M}{\sqrt{m}} \right), \]
where \( \tilde{C}_\kappa \) is a constant depending only on \( \kappa, C_0, \) and the largest eigenvalue of \( L_K \).
In the special case that \( f_\rho \in \mathcal{H}_K \), the approximation error can be bounded as \( \|f_\lambda - f_\rho\|_\rho \leq \|f_\rho\|_K \sqrt{\lambda} \). A more general condition can be imposed for the regression function as

\[
f_\rho = L_K^r(g_\rho) \quad \text{for some } g_\rho \in L^2_{\rho_X}, \ r > 0,
\]

where the integral operator \( L_K \) is regarded as a compact positive operator on \( L^2_{\rho_X} \) and its \( r \)th power is well defined for any \( r > 0 \). The condition (9) means \( f_\rho \) lies in the range of \( L_K^r \), and the special case \( f_\rho \in \mathcal{H}_K \) corresponds to the choice \( r = 1/2 \). Under condition (9), we can obtain from Corollary 2 the following nice convergence rates for the distributed learning algorithm.

**Corollary 3** Assume (9) for some \( 0 < r \leq 1, \ |y| \leq M \) almost surely and \( N(\lambda) = O(\lambda^{-\frac{1}{2r}}) \) for some \( \alpha > 0 \). If \( |D_j| = \frac{N}{m} \) for \( j = 1, \ldots, m \) with

\[
m \leq N^{\frac{1+2\alpha \max(2r-1,0)+2\alpha(2r-1)}{2\alpha \max(2r,1)-4\alpha+4r}} ,
\]

and \( \lambda = \left( \frac{m}{N} \right)^{\frac{2\alpha}{2\alpha \max(2r,1)+1}} \), then we have

\[
E \left| \mathcal{J}_{D,\lambda} - f_{D,\lambda} \right|_\rho = O \left( N^{-\frac{\alpha}{2\alpha \max(2r,1)+1}} m^{-\frac{\frac{1}{2} - \alpha \max(2r-1,0)}{2\alpha \max(2r,1)+1}} \right)
\]

and

\[
E \left| \mathcal{J}_{D,\lambda} - f_{D,\lambda} \right|_K = O \left( N^{-\frac{2\alpha \max(2r-1,0)}{2\alpha \max(2r,1)+1}} m^{-\frac{\frac{1}{2} + 2\alpha - \alpha \max(2r,1)}{2\alpha \max(2r,1)+1}} \right).
\]

In particular, when \( f_\rho \in \mathcal{H}_K \) and \( m \leq N^{\frac{1}{4+6r}} \), the choice \( \lambda = \left( \frac{m}{N} \right)^{\frac{2\alpha}{2\alpha + 1}} \) yields \( E \left| \mathcal{J}_{D,\lambda} - f_{D,\lambda} \right|_\rho = O \left( N^{-\frac{\alpha}{2\alpha + 1}} m^{-\frac{1}{4+2r}} \right) \) and \( E \left| \mathcal{J}_{D,\lambda} - f_{D,\lambda} \right|_K = O \left( \frac{1}{\sqrt{m}} \right) \).

**Remark 4** In Corollary 3, we present learning rates in both \( \mathcal{H}_K \) and \( L^2_{\rho_X} \) norms. The \( L^2_{\rho_X} \)-norm bound is useful because it equals (subject to a constant) the generalization error \( \int_Z (f(x) - y)^2 \, dp \). The \( \mathcal{H}_K \) norm controls the \( L^2_{\rho_X} \) norm since for any \( f \) in \( \mathcal{H}_K \), \( \|f\|_\rho \leq \|f\|_\infty \leq \kappa \|f\|_K \) \cite{Smale2007}; this inequality also implies the application of the \( \mathcal{H}_K \)-norm bound in the mismatched problem where the generalization power is measured in some \( L^2_\mu \)-norm with \( \mu \) different from \( \rho_X \).

**Remark 5** In Corollary 3, the established error bounds are monotonically decreasing with respect to \( m \), which is different from the error analysis in \cite{Zhang2015}. The reason is that we are concerned with the difference between \( \mathcal{J}_{D,\lambda} \) and \( f_{D,\lambda} \). This difference reflects the variance of the distributed learning algorithm. Concerning the learning rate (as shown in Corollary 3 below), the regularization parameter \( \lambda \) should be smaller, and then the learning rate is independent of \( m \), provided \( m \) is not very large.
2.2 Minimax Rates of Convergence for Least Squares Regularization Scheme

The second main result of this paper is a sharp error bound for the least squares regularization scheme \( \|f\|_p \). We can even relax the uniform boundedness to a moment condition that for some constant \( p \geq 1 \),

\[
\sigma^2_{\rho} \in L^p_{\rho x},
\]

where \( \sigma^2_{\rho} \) is the conditional variance defined by \( \sigma^2_{\rho}(x) = \int_Y (y - f_{\rho}(x))^2 \, d\rho(y|x) \).

The following learning rates for the least squares regularization scheme \( \|f\|_p \) will be proved in Section 5. The existence of \( f_\lambda \) is ensured by \( E[y^2] < \infty \).

**Theorem 6** Assume \( E[y^2] < \infty \) and (11) for some \( 1 \leq p \leq \infty \). Then we have

\[
E \left[ \|f_{D,\lambda} - f_\rho\|_p \right] \leq (1 + 59\kappa^4 + 59\kappa^2) (1 + \kappa) \left( 1 + \frac{1}{(N\lambda)^2} + \frac{\mathcal{N}(\lambda)}{N} \right) \left\{ \left( 1 + \frac{1}{\sqrt{N\lambda}} \right) \|f_\lambda - f_\rho\|_p + \sqrt{\|\sigma^2_{\rho}\|_p} \left( \frac{\mathcal{N}(\lambda)}{N} \right)^{\frac{1}{2}(1 - \frac{1}{p})} \left( \frac{1}{N\lambda} \right)^{\frac{1}{2p}} \right\},
\]

(12)

If the parameters satisfy \( \frac{\mathcal{N}(\lambda)}{N\lambda} = O(1) \), we have the following explicit bound.

**Corollary 7** Assume \( E[y^2] < \infty \) and (11) for some \( 1 \leq p \leq \infty \). If \( \lambda \) satisfies (8) with \( m = 1 \), then we have

\[
E \left[ \|f_{D,\lambda} - f_\rho\|_p \right] = O \left( \|f_\lambda - f_\rho\|_p + \sqrt{\mathcal{N}(\lambda)} \frac{\mathcal{N}(\lambda)}{N} \right).
\]

In particular, if \( p = \infty \), that is, the conditional variances are uniformly bounded, then

\[
E \left[ \|f_{D,\lambda} - f_\rho\|_\infty \right] = O \left( \|f_\lambda - f_\rho\|_\infty + \sqrt{\mathcal{N}(\lambda)} \frac{\mathcal{N}(\lambda)}{N} \right).
\]

In particular, when (1) is satisfied, we have the following learning rates.

**Corollary 8** Assume \( E[y^2] < \infty \), (11) for some \( 1 \leq p \leq \infty \), and (12) for some \( 0 < r \leq 1 \). If \( \mathcal{N}(\lambda) = O(\lambda^{-\frac{1}{2r}}) \) for some \( \alpha > 0 \), then by taking \( \lambda = N^{-\frac{2r\alpha}{2\alpha + 2r_1 + 1}} \), we have

\[
E \left[ \|f_{D,\lambda} - f_\rho\|_r \right] = O \left( N^{-\frac{2r\alpha}{2\alpha + 2r_1 + 1} + \frac{1}{2r} \frac{2\alpha - 1}{2\alpha + 2r_1 + 1}} \right).
\]

In particular, when \( p = \infty \) (the conditional variances are uniformly bounded), we have

\[
E \left[ \|f_{D,\lambda} - f_\rho\|_\infty \right] = O \left( N^{-\frac{2r\alpha}{2\alpha + 2r_1 + 1}} \right).
\]

**Remark 9** For \( r \in \left[ \frac{1}{2}, 1 \right] \), [Caponnetto and De Vito, 2007; Steinwart et al., 2008] give the minimax lower bound \( N^{-\frac{2r\alpha}{2\alpha + 2r_1 + 1}} \) for \( E[\|f_{D,\lambda} - f_\rho\|_p^2] \) as \( p \to \infty \). So the convergence rate we obtain in Corollary 8 is sharp.
Combining bounds for $\|F_{D,\lambda} - f_{D,\lambda}\|_\rho$ and $\|f_{D,\lambda} - f_\rho\|_\rho$, we can derive learning rates for the distributed learning algorithm (2) for regression.

**Corollary 10** Assume $|y| \leq M$ almost surely and (9) for some $\frac{1}{2} < r \leq 1$. If $N(\lambda) = O(\lambda^{-\frac{\alpha}{2}})$ for some $\alpha > 0$, $|D_j| = \frac{N}{m}$ for $j = 1, \ldots, m$, and $m$ satisfies the restriction

$$m \leq N \min \left\{ \frac{6(2r-1)+1}{5(4r+1)}, \frac{2(2r-1)}{4r+1} \right\},$$

(13)

then by taking $\lambda = N^{-\frac{2r}{4r+1}}$, we have

$$E \left[ \|F_{D,\lambda} - f_\rho\|_\rho \right] = O \left( N^{-\frac{2r}{4+2r}} \right).$$

**Remark 11** Corollary 10 shows that distributed learning with least squares regularization scheme (2) can reach the minimax rates in expectation, provided $m$ satisfies (13). It should be pointed out that we consider error analysis under (9) with $1/2 < r \leq 1$ while (Zhang et al., 2015) focused on the case (9) with $r = 1/2$. The main novelty of our analysis is that by using a novel second order decomposition for the difference of operator inverses, we remove the eigenfunction assumptions in (Zhang et al., 2015) and provide error bounds for a larger range of $r$.

**Remark 12** In this paper, we only derive minimax rates for the least squares regularization scheme (1) as well as its distributed version (2) in expectation. We guess it is possible to derive error bounds in probability by combining the proposed second order decomposition approach with the analysis in (Caponnetto and De Vito, 2007; Blanchard and Krämer, 2010). We will study it in a future publication.

**Remark 13** Corollary 10 and Corollary 8 suggest that the optimal choice of the regularization parameter $\lambda$ should be independent of the number $m$ of partitions. In particular, for regularized least squares (1), the distributed scheme shares the optimal $\lambda$ with the batch learning scheme. This observation is consistent with the results in (Zhang et al., 2015). We note that there are several parameter selection approaches in literature including cross-validation (Györfy et al., 2002) and the balancing principle (De Vito et al., 2010). It would be interesting to develop some parameter selection method for distributed learning.

3. Comparisons and Discussion

The least squares regularization scheme (1) is a classical algorithm for regression and has been extensively investigated in statistics and learning theory. There is a vast literature on this topic. Here for a general kernel beyond the Sobolev kernels, we compare our results with the best learning rates in the existing literature. Denote the set of positive eigenvalues of $L_K$ as $\{\lambda_i\}_i$ arranged in a decreasing order, and a set of normalized (in $H_K$) eigenfunctions $\{\varphi_i\}_i$ of $L_K$ corresponding to the eigenvalues $\{\lambda_i\}_i$.

Under the assumption that the orthogonal projection $f_\mathcal{H}$ of $f_\rho$ in $L^2_{\rho_X}$ onto the closure of $\mathcal{H}_K$ satisfies (9) for some $\frac{1}{2} \leq r \leq 1$, and that the eigenvalues $\lambda_i$ satisfy $\lambda_i \approx i^{-2\alpha}$ with some $\alpha > 1/2$, it was proved in (Caponnetto and De Vito, 2007) that

$$\lim_{\tau \to \infty} \limsup_{N \to \infty} \sup_{\rho \in \mathcal{P}(\alpha)} \text{Prob} \left[ \|f_{D,\lambda_N} - f_\mathcal{H}\|_\rho^2 > \tau \lambda_N^{2r} \right] = 0,$$
where
\[
\lambda_N = \begin{cases} 
N^{-\frac{2\alpha}{4\alpha + 7}}, & \text{if } \frac{1}{2} < r \leq 1, \\
\left(\frac{\log N}{N}\right)^{\frac{2\alpha}{2\alpha + 1}}, & \text{if } r = \frac{1}{2}, 
\end{cases}
\]
and \(\mathcal{P}(\alpha)\) denotes a set of probability measures \(\rho\) satisfying some moment decay condition (which is satisfied when \(|y| \leq M\)). This learning rate is suboptimal due to the limitation taken for \(\tau \to \infty\) and the logarithmic factor in the case \(r = \frac{1}{2}\). In particular, to have \(\|f_{D,\lambda_N} - f_H\|_\rho^2 \leq \tau_0 \lambda_N^{2r}\) with confidence \(1 - \eta\), one needs to restrict \(N \geq N_\eta\) to be large enough and has the constant \(\tau_\eta\) depending on \(\eta\) to be large enough. Using similar mathematical tools as that in (Caponnetto and De Vito, 2007) and a novel second order decomposition for the difference of operator inverses, we succeed in deriving learning rates in expectation in Corollary 8 by removing the logarithmic factor in the case \(r = \frac{1}{2}\).

Under the assumption that \(|y| \leq M\) almost surely, the eigenvalues \(\lambda_i\) satisfying \(\lambda_i \leq a_i^{-2\alpha}\) with some \(\alpha > 1/2\) and \(a > 0\), and for some constant \(C > 0\), the pair \((K, \rho_X)\) satisfying
\[
\|f\|_\infty \leq C\|f\|_{K_{\rho}}^{\frac{1}{2}}\|f\|_{\rho}^{1 - \frac{1}{2\alpha}}
\]
for every \(f \in \mathcal{H}_K\), it was proved in (Steinwart et al., 2009) that for some constant \(c_{\alpha,C}\) depending only on \(\alpha\) and \(C\), with confidence \(1 - \eta\), for any \(0 < \lambda \leq 1\),
\[
\|\pi_M (f_{D,\lambda}) - f_\rho\|_\rho^2 \leq 9A_2(\lambda) + c_{\alpha,C} \frac{a^{1/(2\alpha)}M^2 \log(3/\eta)}{\lambda^{1/(2\alpha)}N}.
\]
Here \(\pi_M\) is the projection onto the interval \([-M, M]\) defined (Chen et al., 2004; Wu et al., 2006) by
\[
\pi_M(f)(x) = \begin{cases} 
M, & \text{if } f(x) > M, \\
f(x), & \text{if } |f(x)| \leq M, \\
-M, & \text{if } f(x) < -M,
\end{cases}
\]
and \(A_2(\lambda)\) is the approximation error defined by
\[
A_2(\lambda) = \inf_{f \in \mathcal{H}_K} \left\{\|f - f_\rho\|_\rho^2 + \lambda\|f\|_K^2\right\}.
\]
When \(f_\rho \in \mathcal{H}_K\), \(A_2(\lambda) = O(\lambda)\) and the choice \(\lambda_N = N^{-\frac{2\alpha}{4\alpha + 7}}\) gives a learning rate of order \(\|f_{D,\lambda_N} - f_\rho\|_\rho = O \left( N^{-\frac{\alpha}{4\alpha + 7}} \right) \). But one needs to impose the condition (14) for the functions spaces \(L_{\rho_X}^2\) and \(\mathcal{H}_K\), and to take the projection onto \([-M, M]\), although (14) is more general than the uniform boundedness assumption of the eigenfunctions and holds when \(\mathcal{H}_K\) is the Sobolev space and \(\rho_X\) is the uniform distribution (Steinwart et al., 2009; Mendelson and Neeman, 2010). Our learning rates in Corollary 8 do not require such a condition for the function spaces, nor do we take the projection. Learning rates for the least squares regularization scheme (II) in the \(\mathcal{H}_K\)-metric have been investigated in the literature (Smale and Zhou, 2007).

For the distributed learning algorithm (2) with subsets \(\{D_j\}_j=1^m\) of equal size, under the assumption that for some constants \(k > 2\) and \(A < \infty\), the eigenfunctions \(\{\varphi_i\}_i\) satisfy
\[
\|\varphi_i\|_{L_{\rho_X}^{2k}}^{2k} = E \left[|\varphi_i(x)|^{2k}\right] \leq A^{2k}, \quad i = 1, 2, \ldots,
\]
and
\[
P(\alpha) \text{ denotes a set of probability measures } \rho \text{ satisfying some moment decay condition (which is satisfied when } |y| \leq M)\).
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that \( f_\rho \in \mathcal{H}_K \) and \( \lambda_i \leq a_i^{-2\alpha} \) for some \( \alpha > 1/2 \) and \( a > 0 \), it was proved in (Zhang et al., 2015) that for \( \lambda = N^{2\alpha/(2\alpha+1)} \) and \( m \) satisfying the restriction

\[
m \leq c_\alpha \left( \frac{N^{2(k-4)\alpha/(2\alpha+1)}}{A^{4k} \log^k N} \right)^{1/(2\alpha+1)}
\]

with a constant \( c_\alpha \) depending only on \( \alpha \), there holds \( E \left[ \| \mathbf{f}_{D,\lambda} - f_\rho \|_\rho^2 \right] = O \left( N^{-2\alpha/(2\alpha+1)} \right) \). This interesting result was achieved by a matrix analysis approach for which the eigenfunction assumption (15) played an essential role.

The eigenfunction assumption (15) generalizes the classical case that the eigenfunctions are uniformly bounded: \( \| \varphi_i \|_\infty = O(1) \). An example of a \( C^\infty \) Mercer kernel was presented in (Zhou, 2002, 2003) to show that smoothness of the Mercer kernel does not guarantee the uniform boundedness of the eigenfunctions. Furthermore, (Gittens and Mahoney, 2016) provided a practical reason for avoiding uniform boundedness assumption on the eigenfunctions (or eigenvectors) in terms of localization and sparseness. The condition (15), to the best of our knowledge, only holds when \( \mathcal{H}_K \) is the Sobolev space and \( \rho_X \) is the Lebesgue measure or \( K \) is a periodical kernel. It is a challenge to verify (15) for some widely used kernels including the Gaussian kernel. It would be interesting to find practical instance such that (15) holds. Our learning rates stated in Corollary 3 do not require such an eigenfunction assumption. Also, our restriction (10) for the number \( m \) of local processors is more general when \( \alpha \) is close to 1/2. Notice that the learning rates stated in Corollary 3 are for the difference \( \mathcal{F}_{D,\lambda} - f_{D,\lambda} \) between the output function of the distributed learning algorithm (2) and that of the algorithm (1) using the whole data. In the special case of \( r = \frac{1}{2} \), we can see that \( E \left[ \| \mathcal{F}_{D,\lambda} - f_{D,\lambda} \|_\rho^2 \right] = O \left( N^{-\alpha/(2\alpha+1)} m^{-\frac{1}{2\alpha+2}} \right) \), achieved by choosing \( \lambda = \left( \frac{m}{N} \right)^{2\alpha/(2\alpha+1)} \), is smaller as \( m \) becomes larger. This is natural because the error \( E \left[ \| \mathcal{F}_{D,\lambda} - f_{D,\lambda} \|_\rho \right] \) reflects more the sample error and should become smaller when we use more local processors. On the other hand, as one expects, increasing the number \( m \) of local processors would increase the approximation error for the regression problem, which can also be seen from the bound with \( \lambda = \left( \frac{m}{N} \right)^{2\alpha/(2\alpha+1)} \) stated in Theorem 6. The result in Corollary 10 with \( r > \frac{1}{2} \) compensates and gives the best learning rate \( E \left[ \| \mathcal{F}_{D,\lambda} - f_\rho \|_\rho^2 \right] = O \left( N^{-\frac{2\alpha}{2\alpha+1}} \right) \) by restricting \( m \) as in (13).

Besides the divide-and-conquer technique, there are some other widely-used approaches towards the goal of reducing time complexity. For example, the localized learning (Meister and Steinwart, 2016), Nyström regularization (Bach, 2013) and on-line learning (Dekel et al., 2012), to name but a few. A key advantage of the divide-and-conquer technique is that it also reduces the space complexity without a significant lost (as proved in this paper) of prediction power. Although here we only consider the distributed regularized least squares, it would be important also to develop the theory for the distributed variance of other algorithms such as the spectral algorithms (Bauer et al., 2007), empirical feature-based learning (Guo and Zhou, 2012), error entropy minimization (Hu et al., 2015), randomized Kaczmarz (Lin and Zhou, 2015), and so on. It would be important to consider the strategies of parameter selection and data partition for distributed learning.
In this paper, we consider the regularized least squares with Mercer kernels. It would be interesting to minimize the assumptions on the kernel and the domain to maximize the scope of applications. For example, the domain that does not have a metric (Shen et al., 2014), the kernel that is only bounded and measurable (Steinwart and Scovel, 2012), and so on.

4. Second Order Decomposition of Operator Differences and Norms

To analyze the error \( \mathbf{f}_{D,\lambda} - f_{D,\lambda} \), we need the following representation in terms of the difference of inverse operators denoted by

\[ Q_D(x) = (L_{K,D(x)} + \lambda I)^{-1} - (L_K + \lambda I)^{-1} \]  

(16)

and \( Q_{D_j(x)} \) for the data subset \( D_j \). The empirical integral operator \( L_{K,D_j(x)} \) is defined with \( D \) replaced by the data subset \( D_j \).

Define two random variables \( \xi_\lambda \) and \( \xi_0 \) with values in the Hilbert space \( \mathcal{H}_K \) by

\[ \xi_0(z) = (y - f_0(x)) K_x, \quad \xi_\lambda(z) = (y - f_\lambda(x)) K_x, \quad z = (x,y) \in \mathcal{Z}. \]  

(17)

We can derive a representation for \( \mathbf{f}_{D,\lambda} - f_{D,\lambda} \) in the following lemma.

Lemma 14 Assume \( E[y^2] < \infty \). For \( \lambda > 0 \), we have

\[ \mathbf{f}_{D,\lambda} - f_{D,\lambda} = \sum_{j=1}^{m} \frac{|D_j|}{|D|} \left[ (L_{K,D_j(x)} + \lambda I)^{-1} - (L_K + \lambda I)^{-1} \right] \Delta_j \]

(18)

where

\[ \Delta_j := \frac{1}{|D_j|} \sum_{z \in D_j} \xi_\lambda(z) - E[\xi_\lambda], \quad \Delta_D := \frac{1}{|D|} \sum_{z \in D} \xi_\lambda(z) - E[\xi_\lambda], \]

and

\[ \Delta_j' := \frac{1}{|D_j|} \sum_{z \in D_j} \xi_0(z), \quad \Delta_j'' := \frac{1}{|D_j|} \sum_{z \in D_j} (\xi_\lambda - \xi_0)(z) - E[\xi_\lambda]. \]

Proof A well known formula (see e.g. (Smale and Zhou, 2007)) asserts that

\[ f_{D_j,\lambda} - f_\lambda = \left( L_{K,D_j(x)} + \lambda I \right)^{-1} \Delta_j. \]

So we know that

\[ \mathbf{f}_{D,\lambda} - f_\lambda = \sum_{j=1}^{m} \frac{|D_j|}{|D|} \{ f_{D_j,\lambda} - f_\lambda \} = \sum_{j=1}^{m} \frac{|D_j|}{|D|} \left( L_{K,D_j(x)} + \lambda I \right)^{-1} \Delta_j. \]

Also, with the whole data \( D \), we have

\[ f_{D,\lambda} - f_\lambda = \left( L_{K,D(x)} + \lambda I \right)^{-1} \Delta_D. \]  

(19)
But

\[ \Delta_D = \frac{1}{|D|} \sum_{z \in D} \xi_\lambda(z) - E[\xi_\lambda] = \sum_{j=1}^{m} \frac{|D_j|}{|D|} \left\{ \frac{1}{|D_j|} \sum_{z \in D_j} \xi_\lambda(z) - E[\xi_\lambda] \right\} = \sum_{j=1}^{m} \frac{|D_j|}{|D|} \Delta_j. \]

Hence

\[ f_{D,\lambda} - \xi = \sum_{j=1}^{m} \frac{|D_j|}{|D|} (L_{K,D(x)} + \lambda I)^{-1} \Delta_j. \]

Then the first desired expression for \( \mathcal{J}_{D,\lambda} - f_{D,\lambda} \) follows.

By adding and subtracting the operator \((L_K + \lambda I)^{-1}\), writing \( \Delta_j = \Delta_j' + \Delta_j'' \), and noting \( E[\xi_0] = 0 \), we know that the first expression implies (18). This proves Lemma 14. ■

Our error estimates are achieved by a novel second order decomposition of operator differences in our integral operator approach. We approximate the integral operator \( L_K \) by the empirical integral operator \( L_{K,D(x)} \) on \( \mathcal{H}_K \) defined with the input data set \( D(x) = \{x_i\}_{i=1}^N = \{x : (x, y) \in D \text{ for some } y \in \mathcal{Y}\} \) as

\[ L_{K,D(x)}(f) = \frac{1}{|D|} \sum_{x \in D(x)} f(x)K_x = \frac{1}{|D|} \sum_{x \in D(x)} \langle f, K_x \rangle K_x, \quad f \in \mathcal{H}_K, \quad (20) \]

where the reproducing property \( f(x) = \langle f, K_x \rangle_K \) for \( f \in \mathcal{H}_K \) is used. Since \( K \) is a Mercer kernel, \( L_{K,D_j(x)} \) is a finite-rank positive operator and \( L_{K,D_j(x)} + \lambda I \) is invertible.

The operator difference in our study is \( A^{-1} - B^{-1} \) with \( A = L_{K,D(x)} + \lambda I \) and \( B = L_K + \lambda I \). Our second order decomposition for the difference \( A^{-1} - B^{-1} \) is stated as follows.

**Lemma 15** Let \( A \) and \( B \) be invertible operators on a Banach space. Then we have

\[ A^{-1} - B^{-1} = B^{-1} \{ B - A \} B^{-1} + B^{-1} \{ B - A \} A^{-1} \{ B - A \} B^{-1}. \quad (21) \]

In particular, we have

\[ \begin{align*}
(L_{K,D(x)} + \lambda I)^{-1} - (L_K + \lambda I)^{-1} &= (L_K + \lambda I)^{-1} \{ L_K - L_{K,D(x)} \} (L_K + \lambda I)^{-1} \\
+ (L_K + \lambda I)^{-1} \{ L_K - L_{K,D(x)} \} (L_{K,D(x)} + \lambda I)^{-1} &- (L_{K,D(x)} + \lambda I)^{-1} \{ L_K - L_{K,D(x)} \} (L_K + \lambda I)^{-1}. \quad (22)
\end{align*} \]

**Proof** We can decompose the operator \( A^{-1} - B^{-1} \) as

\[ A^{-1} - B^{-1} = B^{-1} \{ B - A \} A^{-1}. \quad (23) \]

This is the first order decomposition.

Write the last term \( A^{-1} \) as \( B^{-1} + (A^{-1} - B^{-1}) \) and apply another first order decomposition similar to (23) as

\[ A^{-1} - B^{-1} = A^{-1} \{ B - A \} B^{-1}. \]

It follows from (23) that

\[ A^{-1} - B^{-1} = B^{-1} \{ B - A \} \{ B^{-1} + A^{-1} \{ B - A \} B^{-1} \}. \]
Then the desired identity (21) is verified. The lemma is proved.

Note that $L_k^{1/2}$ and the $r$th power of the compact positive operator $L_K + \lambda I$ or $L_{K,D(x)} + \lambda I$ is well defined for any $r \in \mathbb{R}$. The following lemma which will be proved in the Appendix provides estimates for the operator $(L_k + \lambda I)^{-1/2} \{L_K - L_{K,D(x)}\}$ in the second order decomposition (22). As in (Caponnetto and De Vito, 2007), we use effective dimensions defined by (6) to estimate operator norms.

**Lemma 16** Let $D$ be a sample drawn independently according to $\rho$. Then the following estimates for the operator norm $\left\| (L_k + \lambda I)^{-1/2} \{L_K - L_{K,D(x)}\} \right\|$ hold.

(a) $E \left[ \left\| (L_k + \lambda I)^{-1/2} \{L_K - L_{K,D(x)}\} \right\|^2 \right] \leq \frac{\kappa^2 N(\lambda)}{|D|}.$

(b) For any $0 < \delta < 1$, with confidence at least $1 - \delta$, there holds

$$\left\| (L_k + \lambda I)^{-1/2} \{L_K - L_{K,D(x)}\} \right\| \leq B_{|D|,\lambda} \log(2/\delta),$$

where we denote the constant

$$B_{|D|,\lambda} = \frac{2\kappa}{\sqrt{|D|}} \left\{ \frac{\kappa}{\sqrt{|D|N}} + \sqrt{N(\lambda)} \right\}.$$  \hspace{1cm} (25)

(c) For any $d > 1$, there holds

$$\left\{ E \left[ \left\| (L_k + \lambda I)^{-1/2} \{L_K - L_{K,D(x)}\} \right\|^d \right] \right\}^{\frac{1}{d}} \leq (2d\Gamma(d) + 1)^{\frac{1}{2}} \cdot B_{|D|,\lambda},$$

where $\Gamma$ is the Gamma function defined for $d > 0$ by $\Gamma(d) = \int_0^\infty u^{d-1} \exp \{-u\} \, du$.

To apply (18) for our error analysis, we also need to bound norms involving $\Delta_j$, $\Delta_j''$ and $\Delta_D$. We are able to give the following estimates even after multiplying with $(L_k + \lambda I)^{-1/2}$ taken from the operator $Q_{D(x)}$ or $Q_{D_j(x)}$, which will be proved in the Appendix.

**Lemma 17** Let $D$ be a sample drawn independently according to $\rho$ and $g$ be a measurable bounded function on $\mathbb{Z}$ and $\xi_g$ be the random variable with values on $\mathcal{H}_K$ given by $\xi_g(z) = g(z)K_{x}$ for $z = (x,y) \in \mathbb{Z}$. Then the following statements hold.

(a) $E \left[ \left\| (L_k + \lambda I)^{-1/2} (K_x) \right\|_K^2 \right] = N(\lambda).$

(b) For any $0 < \delta < 1$, with confidence at least $1 - \delta$, there holds

$$\left\| (L_k + \lambda I)^{-1/2} \left( \frac{1}{|D|} \sum_{z \in D} \xi_g(z) - E[\xi_g] \right) \right\|_K \leq \frac{2\|g\|_\infty \log(2/\delta)}{\sqrt{|D|}} \left\{ \frac{\kappa}{\sqrt{|D|N}} + \sqrt{N(\lambda)} \right\}.$$
5. Deriving of Error Bounds for Least Squares Regularization Scheme

To illustrate how to apply the second order decomposition (22) for operator differences in our integral operator approach, we prove in this section our main result on error bounds for the least squares regularization scheme (1).

Proposition 18 Assume $E[y^2] < \infty$ and (17) for some $1 \leq p \leq \infty$. Then we have

$$E \left[ \|f_{D,\lambda} - f_\lambda\|_\rho \right] \leq (1 + 59\kappa^4 + 59\kappa^2) \left( 1 + \frac{1}{|D|\lambda^2} + \frac{\mathcal{N}(\lambda)}{|D|\lambda} \right) \left\{ \frac{1}{\kappa} \sqrt{||\sigma_\rho^2||_p} \left( \frac{\mathcal{N}(\lambda)}{|D|}\right)^{\frac{1}{p}} \left( \frac{1}{|D|\lambda} \right)^{\frac{1}{p}} + \kappa \|f_\lambda - f_\rho\|_\rho \right\}.$$  

Proof We recall the expression (19) for $f_{D,\lambda} - f_\lambda$ and the notation $Q_{D(x)}$ defined by (16) for the operator difference $(L_{K,D(x)} + \lambda I)^{-1} - (L_K + \lambda I)^{-1}$. Then we see

$$f_{D,\lambda} - f_\lambda = [Q_{D(x)}] \Delta_D + (L_K + \lambda I)^{-1} \Delta_D.$$  

To estimate the $L^2_{\rho_X}$ norm, we use the identity

$$\|g\|_\rho = \|L^{1/2}_K g\|_K, \quad \forall g \in L^2_{\rho_X},$$  

and get

$$\|f_{D,\lambda} - f_\lambda\|_\rho \leq \left\| L^{1/2}_K [Q_{D(x)}] \Delta_D \right\|_K + \left\| L^{1/2}_K (L_K + \lambda I)^{-1} \Delta_D \right\|_K.$$  

We apply the second order decomposition (22), use the bounds $\left\| L^{1/2}_K (L_K + \lambda I)^{-1/2} \right\| \leq 1$, $\left\| (L_{K,D(x)} + \lambda I)^{-1} \right\| \leq 1/\lambda$, and know that

$$\left\| L^{1/2}_K [Q_{D(x)}] \Delta_D \right\|_K \leq \left\| (L_K + \lambda I)^{-1/2} \left\{ L_K - L_{K,D(x)} \right\} (L_K + \lambda I)^{-1} \Delta_D \right\|_K + \left\| (L_K + \lambda I)^{-1/2} \left\{ L_K - L_{K,D(x)} \right\} \left( L_K + \lambda I \right)^{-1} \left\{ L_K - L_{K,D(x)} \right\} (L_K + \lambda I)^{-1} \Delta_D \right\|_K$$

$$\leq \left\| (L_K + \lambda I)^{-1/2} \left\{ L_K - L_{K,D(x)} \right\} \left( \frac{1}{\sqrt{\lambda}} \right) \left\{ L_K + \lambda I \right\}^{-1/2} \Delta_D \right\|_K + \left\| (L_K + \lambda I)^{-1/2} \left\{ L_K - L_{K,D(x)} \right\} \left( L_K + \lambda I \right)^{-1/2} \left\{ L_K - L_{K,D(x)} \right\} (L_K + \lambda I)^{-1/2} \Delta_D \right\|_K.$$  

For convenience, we introduce the notation

$$\Xi_D = \left\| (L_K + \lambda I)^{-1/2} \left\{ L_K - L_{K,D(x)} \right\} \right\|.$$  

Then the above bound can be restated as

$$\left\| L^{1/2}_K [Q_{D(x)}] \Delta_D \right\|_K \leq \left( \frac{\Xi_D}{\sqrt{\lambda}} + \frac{\Xi_D^2}{\lambda} \right) \left\| (L_K + \lambda I)^{-1/2} \Delta_D \right\|_K.$$  

(28)
Hence
\[ \|f_{D,\lambda} - f_{\lambda}\|_p \leq \left( 1 + \frac{\Xi_D}{\sqrt{\lambda}} + \frac{\Xi_D^2}{\lambda} \right) \| (L_K + \lambda I)^{-1/2} \Delta_D \|_K, \]
and by the Schwarz inequality we have
\[ E \left[ \|f_{D,\lambda} - f_{\lambda}\|_p \right] \leq \left\{ E \left[ \left( 1 + \frac{\Xi_D}{\sqrt{\lambda}} + \frac{\Xi_D^2}{\lambda} \right)^2 \right] \right\}^{1/2} \left\{ E \left[ \| (L_K + \lambda I)^{-1/2} \Delta_D \|_K^2 \right] \right\}^{1/2}. \tag{29} \]

To deal with the expected value in (29), as in Lemma 14, we separate \( \Delta_D \) as
\[ \Delta_D = \Delta_D' + \Delta_D'', \]
where
\[ \Delta_D' := \frac{1}{|D|} \sum_{z \in D} \xi_0(z), \quad \Delta_D'' := \frac{1}{|D|} \sum_{z \in D} (\xi_0 - \xi_0)(z) - E[\xi_0]. \]

Then
\[ \left\{ E \left[ \left\| (L_K + \lambda I)^{-1/2} \Delta_D' \right\|_K^2 \right] \right\}^{1/2} \leq \left\{ E \left[ \left\| (L_K + \lambda I)^{-1/2} \Delta_D' \right\|_K^2 \right] \right\}^{1/2} + \left\{ E \left[ \left\| (L_K + \lambda I)^{-1/2} \Delta_D'' \right\|_K^2 \right] \right\}^{1/2}. \tag{30} \]

Observe that
\[ (L_K + \lambda I)^{-1/2} \Delta_D' = \sum_{z \in D} \frac{1}{|D|} (y - f_\rho(x))(L_K + \lambda I)^{-1/2} (K_x). \]

Each term in this expression is unbiased because \( \int_{\mathbb{R}} y - f_\rho(x) d\rho(y|x) = 0 \). This unbiasedness and the independence tell us that
\[ \left\{ E \left[ \left\| (L_K + \lambda I)^{-1/2} \Delta_D' \right\|_K^2 \right] \right\}^{1/2} = \left\{ \frac{1}{|D|} E \left[ \left\| (y - f_\rho(x))(L_K + \lambda I)^{-1/2} (K_x) \right\|_K^2 \right] \right\}^{1/2} = \left\{ \frac{1}{|D|} E \left[ \sigma^2_\rho(x) \left\| (L_K + \lambda I)^{-1/2} (K_x) \right\|_K^2 \right] \right\}^{1/2}. \tag{31} \]

If \( \sigma^2_\rho \in L^\infty \), then \( \sigma^2_\rho(x) \leq \| \sigma^2_\rho \|_\infty \) and by Lemma 17 we have
\[ \left\{ E \left[ \left\| (L_K + \lambda I)^{-1/2} \Delta_D' \right\|_K^2 \right] \right\}^{1/2} \leq \sqrt{\| \sigma^2_\rho \|_\infty} \sqrt{N(\lambda)/|D|}. \]

If \( \sigma^2_\rho \in L^p_{\rho^\infty} \) with \( 1 \leq p < \infty \), we take \( q = \frac{p}{p - 1} \) (\( q = \infty \) for \( p = 1 \)) satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \) and apply the Hölder inequality \( E[|\xi||\eta|] \leq (E[|\xi|^p])^{1/p} (E[|\eta|^q])^{1/q} \) to find
\[ E \left[ \sigma^2_\rho(x) \left\| (L_K + \lambda I)^{-1/2} (K_x) \right\|_K^2 \right] \leq \| \sigma^2_\rho \|_p \left\{ E \left[ \left\| (L_K + \lambda I)^{-1/2} (K_x) \right\|_K^{2q} \right] \right\}^{1/q}. \]
But

\[ \left\| \left[ (L_K + \lambda I)^{-1/2} \right] (K_x) \right\|_K^{2q-2} \leq \left( \frac{\kappa}{\sqrt{\lambda}} \right)^{2q-2} \]

and \( E \left[ \left\| (L_K + \lambda I)^{-1/2} (K_x) \right\|_K^2 \right] = \mathcal{N}(\lambda) \) by Lemma 17. So we have

\[
\left\{ E \left[ \left\| (L_K + \lambda I)^{-1/2} \Delta_D^t \right\|_K^2 \right] \right\}^{1/2} \leq \left\{ \frac{1}{|D|} \left\| \sigma_p^2 \right\|_p \left\{ \frac{\kappa^{2q-2}}{\lambda q-1} \mathcal{N}(\lambda) \right\}^{1/q} \right\}^{1/2} = \sqrt{\left\| \sigma_p^2 \right\|_p \frac{1}{|D|^{q-1}} \left( \frac{\mathcal{N}(\lambda)}{|D|} \right)^{\frac{1}{q}} } \left( 1 \frac{1}{|D|\lambda} \right)^{\frac{1}{p}}.
\]

Combining the above two cases, we know that for either \( p = \infty \) or \( p < \infty \),

\[
\left\{ E \left[ \left\| (L_K + \lambda I)^{-1/2} \Delta_D^t \right\|_K^2 \right] \right\}^{1/2} \leq \sqrt{\left\| \sigma_p^2 \right\|_p \frac{1}{|D|^{q-1}} \left( \frac{\mathcal{N}(\lambda)}{|D|} \right)^{\frac{1}{q}} } \left( 1 \frac{1}{|D|\lambda} \right)^{\frac{1}{p}}.
\]

The second term of (30) can be bounded easily as

\[
\left\{ E \left[ \left\| (L_K + \lambda I)^{-1/2} \Delta_D'' \right\|_K^2 \right] \right\}^{1/2} \leq \frac{1}{\sqrt{|D|}} \left\{ E \left[ \left( f_{\rho}(x) - f_{\lambda}(x) \right)^2 \left\| (L_K + \lambda I)^{-1/2} (K_x) \right\|_K^2 \right] \right\}^{1/2} \leq \frac{1}{\sqrt{|D|}} \left\{ E \left[ \left( f_{\rho}(x) - f_{\lambda}(x) \right)^2 \frac{\lambda^2}{\lambda} \right] \right\}^{1/2} = \frac{\kappa \left\| f_{\rho} - f_{\lambda} \right\|_\rho}{\sqrt{|D|\lambda}}.
\]

Putting the above estimates for the two terms of (30) into (29) and applying Lemma 16 to get

\[
\left\{ E \left[ \left( 1 + \frac{\Xi_D}{\sqrt{\lambda} \lambda} + \frac{\Xi_D^2}{\lambda^2} \right)^2 \right] \right\}^{1/2} \leq 1 + \left\{ E \left[ \frac{\Xi_D^2}{\lambda^2} \right] \right\}^{1/2} + \left\{ E \left[ \frac{\Xi_D^4}{\lambda^2} \right] \right\}^{1/2} \leq 1 + \left\{ \frac{\kappa^2 \mathcal{N}(\lambda)}{|D|\lambda} \right\}^{1/2} + \left\{ \frac{4\mathcal{N}_1^4}{|D|\lambda^2} \right\}^{1/2} \leq 1 + \frac{59\kappa^4}{|D|\lambda^2} + \frac{59\kappa^2 \mathcal{N}(\lambda)}{|D|\lambda},
\]

we know that \( E \left[ \left\| f_{D,\lambda} - f_{\lambda} \right\|_\rho \right] \) is bounded by

\[
\left( 1 + \frac{59\kappa^4}{|D|\lambda^2} + \frac{59\kappa^2 \mathcal{N}(\lambda)}{|D|\lambda} \right) \left( \sqrt{\left\| \sigma_p^2 \right\|_p \frac{1}{|D|^{q-1}} \left( \frac{\mathcal{N}(\lambda)}{|D|} \right)^{\frac{1}{q}} } \left( 1 \frac{1}{|D|\lambda} \right)^{\frac{1}{p}} + \frac{\kappa \left\| f_{\rho} - f_{\lambda} \right\|_\rho}{\sqrt{|D|\lambda}} \right).
\]

Then our desired error bound follows. The proof of the proposition is complete. ■
Proof of Theorem 6 Combining Proposition 18 with the triangle inequality \( \|f_{D,\lambda} - f_{\rho}\|_\rho \leq \|f_{D,\lambda} - f_{\lambda}\|_\rho + \|f_{\lambda} - f_{\rho}\|_\rho \), we know that

\[
E \left[ \|f_{D,\lambda} - f_{\rho}\|_\rho \right] \leq \|f_{\lambda} - f_{\rho}\|_\rho + (1 + 59\kappa^4 + 59\kappa^2) \left( 1 + \frac{1}{(|D|\lambda)^2} + \frac{\mathcal{N}(\lambda)}{|D|\lambda} \right) \left\{ \frac{1}{\kappa^p \sqrt{\|\sigma^2_f\|_\rho}} \left( \frac{\mathcal{N}(\lambda)}{|D|} \right)^{\frac{1}{2}\left(1 - \frac{2}{p}\right)} \left( \frac{1}{|D|\lambda} \right)^{\frac{1}{2}} + \frac{\kappa}{\sqrt{|D|\lambda}} \|f_{\lambda} - f_{\rho}\|_\rho \right\}.
\]

Then the desired error bound holds true, and the proof of Theorem 6 is complete.

Proof of Corollary 7 By the definition of effective dimension,

\[
\mathcal{N}(\lambda) = \sum_{\ell} \frac{\lambda_\ell}{\lambda_\ell + \lambda} \geq \frac{\lambda_1}{\lambda_1 + \lambda}.
\]

Combining this with the restriction (8) with \( m = 1 \), we find \( \mathcal{N}(\lambda) \geq \frac{\lambda_1}{\lambda_1 + \lambda} C_0 \) and \( N\lambda \geq \lambda \left( \frac{1}{C_0} + \frac{1}{N\lambda} \right) \). Putting these and the restriction (8) with \( m = 1 \) into the error bound (12), we know that

\[
E \left[ \|f_{D,\lambda} - f_{\rho}\|_\rho \right] \leq \left( 1 + 59\kappa^4 + 59\kappa^2 \right) \left( 1 + \kappa \right) \left( 1 + \frac{(\lambda_1 + C_0)^2 C_0^2}{\lambda_1^2} + C_0 \right) \left\{ \left( 1 + \sqrt{(\lambda_1 + C_0)/C_0}\right) \|f_{\lambda} - f_{\rho}\|_\rho + \sqrt{\|\sigma^2_f\|_\rho} \left( \frac{\mathcal{N}(\lambda)}{N} \right)^{\frac{1}{2}\left(1 - \frac{2}{p}\right)} \left( \frac{1}{N\lambda} \right)^{\frac{1}{2}} + \frac{\kappa}{\sqrt{|D|\lambda}} \|f_{\lambda} - f_{\rho}\|_\rho \right\}.
\]

Then the desired bound follows. The proof of Corollary 7 is complete.

To prove Corollary 8 we need the following bounds (Smale and Zhou, 2007) for \( \|f_{\lambda} - f_{\rho}\|_\rho \) and \( \|f_{\lambda} - f_{\rho}\|_K \).

Lemma 19 Assume (9) with \( 0 < r \leq 1 \). There holds

\[
\|f_{\lambda} - f_{\rho}\|_\rho \leq \lambda^r \|h_\rho\|_\rho.
\]  (32)

Furthermore, if \( 1/2 \leq r \leq 1 \), then we have

\[
\|f_{\lambda} - f_{\rho}\|_K \leq \lambda^{r-1/2} \|h_\rho\|_\rho.
\]  (33)

Proof of Corollary 8 It follows from Lemma 19 that the condition (9) with \( 0 < r \leq 1 \) implies

\[
\|f_{\lambda} - f_{\rho}\|_\rho \leq \lambda^r \|g_\rho\|_\rho.
\]

If

\[
\mathcal{N}(\lambda) \leq C_0 \lambda^{-\frac{4\alpha}{2\alpha}} \quad \forall \lambda > 0
\]

for some constant \( C_0 \geq 1 \), then the choice \( \lambda = N^{-\frac{2\alpha}{2\alpha}} \) yields

\[
\frac{\mathcal{N}(\lambda)}{N\lambda} \leq \frac{C_0 \lambda^{-\frac{4\alpha}{2\alpha}}}{N} = C_0 N^{-\frac{2\alpha+1}{2\alpha+1}} \leq C_0.
\]

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So \((8)\) with \(m = 1\) is satisfied. With this choice we also have
\[
\left(\frac{\mathcal{N}(\lambda)}{N}\right)^{\frac{1}{2}(1 - \frac{1}{p})} \left(\frac{1}{N\lambda}\right)^{\frac{1}{2p}} \leq C_0 \left(1 - \frac{1}{p}\right) N^{-2\alpha \max\{2r, 1\} - 1 - 2\alpha + \frac{1}{2p}} \frac{1}{2\alpha \max\{2r, 1\} + 1}
= C_0 \left(1 - \frac{1}{p}\right) N^{-\alpha \max\{2r, 1\} - \frac{2\alpha - 1}{2p} \frac{1}{2\alpha \max\{2r, 1\} + 1}}.
\]
Putting these estimates into Corollary \(\ref{corollary:main_result}\), we know that
\[
E\left[\|f_{D,\lambda} - f_\rho\|_\rho\right] = O\left(N^{-\frac{2\alpha}{2\alpha \max\{2r, 1\} + 1}} + N^{-\frac{\alpha \max\{2r, 1\}}{2\alpha \max\{2r, 1\} + 1} + \frac{1}{2p \frac{1}{2\alpha \max\{2r, 1\} + 1}}\right)
= O\left(N^{-\frac{\alpha \min\{2r, 1\}}{2\alpha \max\{2r, 1\} + 1} + \frac{1}{2p \frac{1}{2\alpha \max\{2r, 1\} + 1}}\right).
\]
But we find
\[
\min\{2r, \max\{2r, 1\}\} = 2r
\]
by discussing the two different cases \(0 < r < \frac{1}{2}\) and \(\frac{1}{2} \leq r \leq 1\). Then our conclusion follows immediately. The proof of Corollary \(\ref{corollary:main_result}\) is complete. \(\blacksquare\)

6. Proof of Error Bounds for the Distributed Learning Algorithm

In this section, we prove our first main result on the error \(\overline{f}_{D,\lambda} - f_{D,\lambda}\) in the \(H_K\) metric and \(L_\rho^2\) metric. The following result is more general, allowing different sizes for data subsets \(\{D_j\}\).

**Theorem 20** Assume that for some constant \(M > 0\), \(|y| \leq M\) almost surely. Then we have
\[
E\left[\|\overline{f}_{D,\lambda} - f_{D,\lambda}\|_\rho\right] \leq C'_\kappa \left(\frac{1}{(N\lambda)^2} + \frac{\mathcal{N}(\lambda)}{N\lambda}\right) \left\{\|f_\lambda - f_\rho\|_\rho \sum_{j=1}^m \left(\frac{|D_j|}{|D|}\right)^{\frac{1}{2}} + M \sqrt{\frac{\mathcal{N}(\lambda)}{N\lambda}}\right\}
+ C'_\kappa M \sqrt{\lambda}\left\{\sum_{j=1}^m \left(\frac{|D_j|}{|D|}\right)^{\frac{1}{2}} \left(\frac{1}{|D_j|^2 \lambda^2} + \frac{\mathcal{N}(\lambda)}{|D_j|\lambda}\right)\left\{1 + \left(\frac{1}{|D_j|^2 \lambda^2} + \frac{\mathcal{N}(\lambda)}{|D_j|\lambda}\right)^2\right\}^{1/2}\right\}
\]
and
\[
E\left[\|\overline{f}_{D,\lambda} - f_{D,\lambda}\|_K\right] \leq C'_\kappa \left(\frac{1}{(N\lambda)^2} + \frac{\mathcal{N}(\lambda)}{N\lambda}\right) \left\{\|f_\lambda - f_\rho\|_\rho \sum_{j=1}^m \left(\frac{|D_j|}{|D|}\right)^{\frac{1}{2}} + M \sqrt{\frac{\mathcal{N}(\lambda)}{N\lambda}}\right\}
+ C'_\kappa M \left\{\sum_{j=1}^m \left(\frac{|D_j|}{|D|}\right)^{\frac{1}{2}} \left(\frac{1}{|D_j|^2 \lambda^2} + \frac{\mathcal{N}(\lambda)}{|D_j|\lambda}\right)\left\{1 + \left(\frac{1}{|D_j|^2 \lambda^2} + \frac{\mathcal{N}(\lambda)}{|D_j|\lambda}\right)^2\right\}^{1/2}\right\},
\]
where \(C'_\kappa\) is a constant depending only on \(\kappa\).
Proof  Recall (18) in Lemma 14. It enables us to express

$$L_{K}^{1/2} \left\{ f_{D, \lambda} - f_{D, \lambda} \right\} = J_{1} + J_{2} + J_{3},$$

where the terms $J_{1}, J_{2}, J_{3}$ are given by

$$J_{1} = \sum_{j=1}^{m} \left\{ \frac{|D_{j}|}{|D|} \left[ L_{K}^{1/2} Q_{D_{j}(x)} \right] \Delta'_{j} \right\}, \quad J_{2} = \sum_{j=1}^{m} \left\{ \frac{|D_{j}|}{|D|} \left[ L_{K}^{1/2} Q_{D_{j}(x)} \right] \Delta''_{j} \right\}, \quad J_{3} = -\left[ L_{K}^{1/2} Q_{D(x)} \right] \Delta_{D}.$$  

These three terms will be dealt with separately in the following.

For the first term $J_{1}$ of (34), each summand with $j \in \{1, \ldots, m\}$ can be expressed as

$$\sum_{z \in D_{j}} \frac{1}{|D|} (y - f_{\rho}(x)) \left[ L_{K}^{1/2} Q_{D_{j}(x)} \right] (K_{z}),$$

and is unbiased because $\int_{D} y - f_{\rho}(x) d\rho(y|x) = 0$. The unbiasedness and the independence tell us that

$$E[||J_{1}||_{K}] \leq \left\{ E \left[ ||J_{1}||_{K}^{2} \right] \right\}^{1/2} \leq \left\{ \sum_{j=1}^{m} \left( \frac{|D_{j}|}{|D|} \right)^{2} E \left[ \left[ L_{K}^{1/2} Q_{D_{j}(x)} \right] \Delta'_{j} \right]^{2} \right\}^{1/2}.$$  

Let $j \in \{1, \ldots, m\}$. The relation (28) derived from the second order decomposition (22) in the proof of Proposition 18 yields

$$\left\| L_{K}^{1/2} Q_{D_{j}(x)} \right\| \Delta'_{j} \leq \left( \frac{\Xi_{D_{j}}}{\lambda} + \frac{\Xi_{D_{j}}^{2}}{\lambda} \right) \left( L_{K} + \lambda I \right)^{-1/2} \Delta'_{j} \leq \left( L_{K} + \lambda I \right)^{-1/2} \Delta'_{j}.$$  

Now we apply the formula

$$E[\xi] = \int_{0}^{\infty} \text{Prob} [\xi > t] dt$$

(36)
to estimate the expected value of (35). By Part (b) of Lemma 16 for $0 < \delta < 1$, there exists a subset $Z_{\delta, 1}^{|D_{j}|}$ of $Z^{|D_{j}|}$ of measure at least $1 - \delta$ such that

$$\Xi_{D_{j}} \leq B_{|D_{j}|, \lambda} \log(2/\delta), \quad \forall D_{j} \in Z_{\delta, 1}^{|D_{j}|}.$$  

(37)

Applying Part (b) of Lemma 17 to $g(z) = y - f_{\rho}(x)$ with $\|g\|_{\infty} \leq 2M$ and the data subset $D_{j}$, we know that there exists another subset $Z_{\delta, 2}^{|D_{j}|}$ of $Z^{|D_{j}|}$ of measure at least $1 - \delta$ such that

$$\left\| L_{K} + \lambda I \right\|^{-1/2} \Delta'_{j} \leq \frac{2M}{\kappa} B_{|D_{j}|, \lambda} \log(2/\delta), \quad \forall D_{j} \in Z_{\delta, 2}^{|D_{j}|}.$$  

Combining this with (37) and (35), we know that for $D_{j} \in Z_{\delta, 1}^{|D_{j}|} \cap Z_{\delta, 2}^{|D_{j}|}$

$$\left\| L_{K}^{1/2} Q_{D_{j}(x)} \right\| \Delta'_{j} \leq \left( \frac{B_{|D_{j}|, \lambda}^{2}}{\lambda} + \frac{B_{|D_{j}|, \lambda}^{4}}{\lambda^{2}} \right) \left( \frac{M}{\kappa} \right)^{2} B_{|D_{j}|, \lambda} (2 \log(2/\delta)) 6.$$  

Since the measure of the set $Z_{\delta, 1}^{|D_{j}|} \cap Z_{\delta, 2}^{|D_{j}|}$ is at least $1 - 2\delta$, by denoting

$$C_{|D_{j}|, \lambda} = 64 \left( \frac{B_{|D_{j}|, \lambda}^{2}}{\lambda} + \frac{B_{|D_{j}|, \lambda}^{4}}{\lambda^{2}} \right) \left( \frac{M}{\kappa} \right)^{2} B_{|D_{j}|, \lambda},$$  

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Applying the Schwarz inequality and Lemmas 16 and 17, we get

\[
\text{Prob} \left[ \left\| L_{K}^{1/2} Q D_{j}(x) \right\|_{K} \Delta_{j}' \right\|_{K} > C_{|D_{j}|,\lambda} (\log(2/\delta))^{6} \right] \leq 2\delta.
\]

For \( 0 < t < \infty \), the equation \( C_{|D_{j}|,\lambda} (\log(2/\delta))^{6} = t \) has the solution

\[
\delta_{t} = 2 \exp \left\{ - \left( t/C_{|D_{j}|,\lambda} \right)^{1/6} \right\}.
\]

When \( \delta_{t} < 1 \), we have

\[
\text{Prob} \left[ \left\| L_{K}^{1/2} Q D_{j}(x) \right\|_{K} \Delta_{j}' \right\|_{K} > t \right] \leq 2\delta_{t} = 4 \exp \left\{ - \left( t/C_{|D_{j}|,\lambda} \right)^{1/6} \right\}.
\]

This inequality holds trivially when \( \delta_{t} \geq 1 \) since the probability is at most 1. Thus we can apply the formula (36) to the nonnegative random variable \( \xi = \left\| L_{K}^{1/2} Q D_{j}(x) \right\|_{K} \Delta_{j}' \right\|_{K} \) and obtain

\[
E \left[ \left\| L_{K}^{1/2} Q D_{j}(x) \right\|_{K} \Delta_{j}' \right\|_{K} = \int_{0}^{\infty} \text{Prob} [\xi > t] dt \leq \int_{0}^{\infty} 4 \exp \left\{ - \left( t/C_{|D_{j}|,\lambda} \right)^{1/6} \right\} dt
\]

which equals \( 24\Gamma(6)C_{|D_{j}|,\lambda} \). Therefore,

\[
E [\| J_{1} \|_{K}] \leq \left\{ \sum_{j=1}^{m} \left( \frac{|D_{j}|}{|D|} \right)^{2} 24\Gamma(6)C_{|D_{j}|,\lambda} \right\}^{1/2}
\]

\[
\leq 1536\sqrt{\pi}M \left\{ \sum_{j=1}^{m} \left( \frac{|D_{j}|}{|D|} \right)^{2} \lambda \left( \frac{\kappa^{2}}{|D_{j}|^{2} \lambda^{2} + N(\lambda)} \right)^{2} \left\{ 1 + 8\kappa^{2} \left( \frac{\kappa^{2}}{|D_{j}|^{2} \lambda^{2} + N(\lambda)} \right)^{2} \right\} \right\}^{1/2}.
\]

For the second term \( J_{2} \) of (34), we use the second order decomposition (22) again and obtain

\[
\left\| L_{K}^{1/2} Q D_{j}(x) \right\|_{K} \leq \left( \frac{\Xi_{D_{j}}}{\sqrt{\lambda}} + \frac{\Xi_{D_{j}}^{2}}{\lambda} \right) \left\| (L_{K} + \lambda I)^{-1/2} \right\|_{K} \Delta_{j}' \right\|_{K}.
\]

Applying the Schwarz inequality and Lemmas 16 and 17, we get

\[
E \left[ \left\| L_{K}^{1/2} Q D_{j}(x) \right\|_{K} \Delta_{j}' \right\|_{K} \leq \left\{ E \left[ \left( \frac{\Xi_{D_{j}}}{\sqrt{\lambda}} + \frac{\Xi_{D_{j}}^{2}}{\lambda} \right)^{2} \right] \right\}^{1/2}
\]

\[
\leq \frac{1}{\sqrt{|D_{j}|}} \left\{ E \left[ (f_{\rho}(x) - f_{\lambda}(x))^{2} \left\| (L_{K} + \lambda I)^{-1/2} \right\|_{K} \right] \right\}^{1/2}
\]

\[
\leq \left\{ \frac{\kappa^{2}N(\lambda)}{|D_{j}| \lambda} \right\}^{1/2} + \left\{ \frac{49B_{|D_{j}|,\lambda}^{4}}{\lambda^{2}} \right\}^{1/2} \frac{\kappa\|f_{\rho} - f_{\lambda}\|_{\rho}}{\sqrt{|D_{j}| \lambda}}
\]

\[
\leq \left( \frac{59\kappa^{4}}{|D_{j}|^{2}} + \frac{59\kappa^{2}N(\lambda)}{|D_{j}| \lambda} \right) \frac{\kappa\|f_{\rho} - f_{\lambda}\|_{\rho}}{\sqrt{|D_{j}| \lambda}}.
\]

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It follows that
\[ E[\|J_2\|_K] \leq \left( \frac{59\kappa^4}{(|D|\lambda)^2} + \frac{59\kappa^2N(\lambda)}{|D|\lambda} \right) \sum_{j=1}^{m} \frac{|D|}{|D_j|} \frac{\kappa\|f_{\rho} - f_{\lambda}\|_\rho}{\sqrt{|D_j|\lambda}}. \]

The last term \(J_3\) of (31) has been handled in the proof of Proposition 15 by ignoring the summand \((L_K + \lambda I)^{-1} \Delta_D\) in the expression for \(f_{D,\lambda} - f_{\lambda}\), and we find from the trivial bound \(\|\sigma^2\|_\infty \leq 4M^2\) with \(p = \infty\) that
\[ E[\|J_3\|_K] \leq \left( \frac{59\kappa^4}{(|D|\lambda)^2} + \frac{59\kappa^2N(\lambda)}{|D|\lambda} \right) \left( 2M \left( \frac{N(\lambda)}{|D|} \right)^{\frac{1}{2}} + \frac{\kappa\|f_{\rho} - f_{\lambda}\|_\rho}{\sqrt{|D|\lambda}} \right). \]

Combining the above estimates for the three terms of (34), we see that the desired error bound in the \(L^2_{\rho,\lambda}\) metric holds true.

The estimate in the \(H_K\) metric follows from the steps in deriving the error bound in the \(L^2_{\rho,\lambda}\) metric except that in the representation (33) the operator \(L^{1/2}_K\) in the front disappears. This change gives an additional factor \(1/\sqrt{\lambda}\), the bound for the operator \((L_K + \lambda I)^{-1/2}\), and proves the desired error bound in the \(H_K\) metric.

**Proof of Theorem** 11. Since \(|D_j| = \frac{N}{m}\) for \(j = 1, \ldots, m\), the bound in Theorem 20 in the \(L^2_{\rho,\lambda}\) metric can be simplified as
\[ E[\|J_{D,\lambda} - f_{D,\lambda}\|_\rho] \leq C'_{\kappa} \left( \frac{1}{(N\lambda)^2} + \frac{N(\lambda)}{N\lambda} \right) \left\{ \frac{\|f_{\lambda} - f_{\rho}\|_\rho}{\sqrt{N\lambda}} m^2 + M\sqrt{\lambda} \frac{N(\lambda)}{N\lambda} \right\} + C'_{\kappa} M \frac{\sqrt{\lambda}}{\sqrt{m}} \left( \frac{m^2}{N^2\lambda^2} + \frac{mN(\lambda)}{N\lambda} \right) \left\{ 1 + \left( \frac{m^2}{N^2\lambda^2} + \frac{mN(\lambda)}{N\lambda} \right) \right\}. \]

Notice that the term \(\sqrt{\frac{N(\lambda)}{N\lambda}}\) can be bounded by \(1 + \frac{mN(\lambda)}{N\lambda}\). Then the desired error bound in the \(L^2_{\rho,\lambda}\) metric with \(C_{\kappa} = 2C'_{\kappa}\) follows. The proof for the error bound in the \(H_K\) metric is similar. The proof of Theorem 11 is complete.

**Proof of Corollary** 2. As in the proof of Corollary 17, the restriction (8) implies \(N(\lambda) \geq \frac{\lambda_1}{\lambda_1 + C_0}\) and \(N\lambda \geq \frac{m\lambda_1}{\lambda_1 + C_0}C_0\). It follows that
\[ \frac{m}{(N\lambda)^2} \leq \frac{(\lambda_1 + C_0)C_0}{\lambda_1} \frac{1}{N\lambda} \leq \frac{(\lambda_1 + C_0)^2C_0^2}{\lambda_1^2} \frac{N(\lambda)}{N\lambda}. \]

Putting these bounds into Theorem 11, we know that the expected value \(E[\|J_{D,\lambda} - f_{D,\lambda}\|_\rho]\) is bounded by
\[ C_{\kappa} \left( \frac{(\lambda_1 + C_0)^2C_0^2}{\lambda_1^2} + 1 \right) \frac{N(\lambda)}{N\lambda} \sqrt{m} \left\{ \frac{\|f_{\lambda} - f_{\rho}\|_\rho}{\sqrt{N\lambda}} m^2 + M\sqrt{\lambda} \left( 1 + \frac{(\lambda_1 + C_0)^2C_0^2}{\lambda_1^2} + C_0 \right) \right\}. \]
Corollary 3. This proves the first desired convergence rate. The second rate follows easily. This proves for some constant $C$

\[ \|f_\lambda - f_\rho\|_\rho \leq C \frac{m \sqrt{N} \lambda}{\sqrt{N M}} + M \frac{\sqrt{\lambda}}{\sqrt{m}}, \]

and

\[ E \|f_{D,\lambda} - f_{D,\lambda}\|_K \leq \tilde{C}_\kappa \frac{mN(\lambda)}{N \lambda} \left( \|f_\lambda - f_\rho\|_\rho \frac{m \sqrt{m}}{\sqrt{N \lambda}} + M \frac{\sqrt{\lambda}}{\sqrt{m}} \right), \]

where

\[ \tilde{C}_\kappa := C_\kappa \left( \frac{(\lambda_1 + C_0)^2 C_0^2}{\lambda_1^2} + 1 \right) \left\{ 1 + \frac{(\lambda_1 + C_0)^2 C_0^2}{\lambda_1^2} + C_0 \right\}. \]

This proves Corollary 2. ■

Proof of Corollary 3 If $N(\lambda) \leq C_0 \lambda^{-\frac{1}{\alpha}}$, $\forall \lambda > 0$

for some constant $C_0 \geq 1$, then the choice $\lambda = \left( \frac{m}{N} \right)^{\frac{2\alpha}{(\max{(2,r)} + 1)}}$ satisfies (8). With this choice we also have

\[ \frac{mN(\lambda)}{N \lambda} \leq C_0 \left( \frac{m}{N} \right)^{\frac{2\alpha}{(max{(2,r)} + 1)}}. \]

Since the condition (9) yields $\|f_\lambda - f_\rho\|_\rho \leq \|g_\rho\|_\rho \lambda^r$, we have by Corollary 2

\[ E \|f_{D,\lambda} - f_{D,\lambda}\|_\rho \leq \tilde{C}_\kappa C_0 \left( \frac{m}{N} \right)^{\frac{2\alpha}{(max{(2,r)} + 1)}} \left( \|g_\rho\|_\rho \frac{m}{N} \right)^{\frac{2\alpha}{(max{(2,r)} + 1)}} \left( r - \frac{1}{2} \right) \frac{m \sqrt{m}}{\sqrt{N}} + MN^{-\frac{\alpha}{2(\max{(2,r)} + 1)}} \frac{2\alpha}{(max{(2,r)} + 1)} \frac{m}{2(4\alpha \max{\{2,r\}} + 1)} \right). \]

The inequality $\left( \frac{m}{N} \right)^{\frac{2\alpha}{(max{(2,r)} + 1)}} \left( r - \frac{1}{2} \right) \frac{m \sqrt{m}}{\sqrt{N}} \leq N^{-\frac{\alpha}{2(\max{(2,r)} + 1)}} \frac{2\alpha}{(max{(2,r)} + 1)} \frac{m}{2(4\alpha \max{\{2,r\}} + 1)}$ is equivalent to

\[ m^{\frac{3}{2}} + 2\alpha m^{(2\max{(2,r)} + 1)} + \frac{2\alpha}{2(\max{(2,r)} + 1)} \left( r - \frac{1}{2} \right) \leq N^{-\frac{\alpha}{2(\max{(2,r)} + 1)}} \frac{2\alpha}{2(4\alpha \max{\{2,r\}} + 1)} \frac{m}{2(4\alpha \max{\{2,r\}} + 1)} \]

and it can be expressed as (11). Since (11) is valid, we have

\[ E \|f_{D,\lambda} - f_{D,\lambda}\|_\rho \leq \tilde{C}_\kappa C_0 \left( \frac{m}{N} \right)^{\frac{2\alpha}{(max{(2,r)} + 1)}} \left( \|g_\rho\|_\rho + M \right) N^{-\frac{\alpha}{2(\max{(2,r)} + 1)}} \frac{2\alpha}{2(4\alpha \max{\{2,r\}} + 1)} \frac{m}{2(4\alpha \max{\{2,r\}} + 1)} \]

This proves the first desired convergence rate. The second rate follows easily. This proves Corollary 3. ■

Proof of Corollary 10 By Corollary 8 with the choice $\lambda = N^{-\frac{2\alpha}{4\alpha + 1}}$, we can immediately bound $\|f_{D,\lambda} - f_\rho\|_\rho$ as

\[ E \left[ \|f_{D,\lambda} - f_\rho\|_\rho \right] = O \left( N^{-\frac{2\alpha}{4\alpha + 1}} \right). \]

The assumption $N(\lambda) = O(\lambda^{-\frac{1}{\alpha}})$ tells us that for some constant $C_0 \geq 1$,

\[ N(\lambda) \leq C_0 \lambda^{-\frac{1}{\alpha}}, \quad \forall \lambda > 0. \]
So the choice $\lambda = N^{-\frac{2\alpha}{4\alpha r + 1}}$ yields
\[
\frac{mN(\lambda)}{N\lambda} \leq C_0 \frac{m \lambda^{\frac{1+2\alpha}{2\alpha}}}{N} = C_0 mN^{\frac{1+2\alpha}{4\alpha r + 1}} = C_0 mN^{\frac{2\alpha(1-2r)}{4\alpha r + 1}}. \tag{38}
\]
If $m$ satisfies
\[
m \leq N^{\frac{2\alpha(2r-1)}{4\alpha r + 1}}, \tag{39}
\]
then (5) is valid, and by Corollary 2,
\[
E \left| f_{D,\lambda} - f_{D,\lambda} \right|_{\rho} \leq \tilde{C}_\kappa \left( \frac{mN(\lambda)}{N\lambda} \right)^{\frac{1+2\alpha}{2\alpha}} \left( \frac{m \lambda^{\frac{1+2\alpha}{2\alpha}}}{N} \right)^{\frac{1+2\alpha}{2\alpha}} + \sqrt{mN^{\frac{2\alpha(1-2r)}{4\alpha r + 1}} \lambda^{\frac{1}{2} - r}}.
\]
Thus, when $m$ satisfies
\[
m \leq N^{\frac{6\alpha(2r-1)+1}{8(4\alpha r + 1)}}, \quad m \leq N^{\frac{2\alpha(2r-1)}{4\alpha r + 1}}, \tag{40}
\]
we have
\[
E \left| f_{D,\lambda} - f_{D,\lambda} \right|_{\rho} \leq 2\tilde{C}_\kappa C_0 \left( \frac{m \lambda^{\frac{1+2\alpha}{2\alpha}}}{N} \right)^{\frac{1+2\alpha}{2\alpha}} \left( \frac{m \lambda^{\frac{1+2\alpha}{2\alpha}}}{N} \right)^{\frac{1+2\alpha}{2\alpha}} + \sqrt{mN^{\frac{2\alpha(1-2r)}{4\alpha r + 1}} \lambda^{\frac{1}{2} - r}},
\]
and thereby
\[
E \left[ \left| f_{D,\lambda} - f_{\rho} \right|_{\rho} \right] = O \left( N^{-\frac{2\alpha}{4\alpha r + 1}} \right).
\]
Finally, we notice that (13) is equivalent to the combination of (39) and (40). So our conclusion follows. This proves Corollary 10. \(\blacksquare\)

**Appendix**

To estimate norms of various operators involving the approximation of $L_K$ by $L_{K,D(x)}$, we need the following probability inequality for vector-valued random variables in [Pinelis, 1994].

**Lemma 21** For a random variable $\xi$ on $(\mathcal{Z}, \rho)$ with values in a Hilbert space $(H, \| \cdot \|)$ satisfying $\| \xi \| \leq \tilde{M} < \infty$ almost surely, and a random sample $\{z_i\}_{i=1}^s$ independent drawn according to $\rho$, there holds with confidence $1 - \tilde{\delta}$,
\[
\left\| \frac{1}{s} \sum_{i=1}^s [\xi(z_i) - E(\xi)] \right\| \leq 2\tilde{M} \log(2/\tilde{\delta}) + \sqrt{\frac{2E(\|\xi\|^2) \log(2/\tilde{\delta})}{s}}. \tag{41}
\]
Proof of Lemma 16 We apply Lemma (21) to the random variable \( \eta_1 \) defined by

\[
\eta_1(x) = (L_K + \lambda I)^{-1/2} \langle \cdot, K_x \rangle_K K_x, \quad x \in \mathcal{X} \tag{42}
\]

It takes values in \( HS(\mathcal{H}_K) \), the Hilbert space of Hilbert-Schmidt operators on \( \mathcal{H}_K \), with inner product \( \langle A, B \rangle_{HS} = \text{Tr}(B^T A) \). Here Tr denotes the trace of a (trace-class) linear operator. The norm is given by \( \|A\|_{HS}^2 = \sum_i \|Ae_i\|_{K}^2 \) where \( \{e_i\} \) is an orthonormal basis of \( \mathcal{H}_K \). The space \( HS(\mathcal{H}_K) \) is a subspace of the space of bounded linear operators on \( \mathcal{H}_K \), denoted as \( (L(\mathcal{H}_K), \| \cdot \|) \), with the norm relations

\[
\|A\| \leq \|A\|_{HS}, \quad \|AB\|_{HS} \leq \|A\|_{HS}\|B\|. \tag{43}
\]

Now we use effective dimensions to estimate norms involving \( \eta_1 \). The random variable \( \eta_1 \) defined by (42) has mean \( E(\eta_1) = (L_K + \lambda I)^{-1/2} L_K \) and sample mean \( (L_K + \lambda I)^{-1/2} L_{K,D(x)} \).

Recall the set of normalized (in \( \mathcal{H}_K \)) eigenfunctions \( \{\varphi_i\}_i \) of \( L_K \). It is an orthonormal basis of \( \mathcal{H}_K \). If we regard \( L_K \) as an operator on \( L^2_{\rho_X} \), the normalized eigenfunctions in \( L^2_{\rho_X} \) are \( \{\frac{1}{\sqrt{\lambda_i}} \varphi_i\}_i \), and they form an orthonormal basis of the orthogonal complement of the eigenspace associated with eigenvalue 0. By the Mercer Theorem, we have the following uniform convergent Mercer expansion

\[
K(x, y) = \sum_i \lambda_i \frac{1}{\sqrt{\lambda_i}} \varphi_i(x) \frac{1}{\sqrt{\lambda_i}} \varphi_i(y) = \sum_i \varphi_i(x) \varphi_i(y). \tag{44}
\]

Take the orthonormal basis \( \{\varphi_i\}_i \) of \( \mathcal{H}_K \). By the definition of the HS norm, we have

\[
\|\eta_1(x)\|^2_{HS} = \sum_i \left\| (L_K + \lambda I)^{-1/2} \langle \cdot, K_x \rangle_K K_x \varphi_i \right\|^2_K.
\]

For a fixed \( i \),

\[
\langle \cdot, K_x \rangle_K K_x \varphi_i = \varphi_i(x) K_x,
\]

and \( K_x \in \mathcal{H}_K \) can be expended by the orthonormal basis \( \{\varphi_{\ell}\}_\ell \) as

\[
K_x = \sum_\ell \langle \varphi_{\ell}, K_x \rangle_K \varphi_{\ell} = \sum_\ell \varphi_{\ell}(x) \varphi_{\ell}. \tag{45}
\]

Hence

\[
\|\eta_1(x)\|^2_{HS} = \sum_i \left\| \varphi_i(x) \sum_\ell \varphi_{\ell}(x) (L_K + \lambda I)^{-1/2} \varphi_{\ell} \right\|^2_K = \sum_i \left( \varphi_i(x) \right)^2 \sum_\ell \left( \frac{\varphi_{\ell}(x)}{\lambda_\ell + \lambda}\right)^2.
\]

Combining this with (44), we see that

\[
\|\eta_1(x)\|^2_{HS} = K(x, x) \sum_\ell \frac{\left( \varphi_{\ell}(x) \right)^2}{\lambda_\ell + \lambda}, \quad \forall x \in \mathcal{X} \tag{46}
\]
and
\[ E \left[ \|\eta_1(x)\|_{HS}^2 \right] \leq \kappa^2 E \left[ \sum_{\ell} \frac{(\varphi_\ell(x))^2}{\lambda_\ell + \lambda} \right] = \kappa^2 \sum_{\ell} \int_{\mathcal{X}} \frac{(\varphi_\ell(x))^2}{\lambda_\ell + \lambda} d\rho_x. \]

But
\[ \int_{\mathcal{X}} (\varphi_\ell(x))^2 d\rho_x = \|\varphi_\ell\|_{L^2_{\rho_x}}^2 = \left\| \sqrt{\frac{1}{\lambda_\ell}} \varphi_\ell \right\|_{L^2_{\rho_x}}^2 = \lambda_\ell. \] (47)

So we have
\[ E \left[ \|\eta_1\|_{HS}^2 \right] \leq \kappa^2 \sum_{\ell} \frac{\lambda_\ell}{\lambda_\ell + \lambda} = \kappa^2 \text{Tr} \left( (L_K + \lambda I)^{-1} L_K \right) = \kappa^2 \mathcal{N}(\lambda) \] (48)

and
\[ E \left[ \frac{1}{|D|} \sum_{x \in D(x)} \eta_1(x) - E[\eta_1] \right]_{HS}^2 = E \left[ \left\| (L_K + \lambda I)^{-1/2} \left\{ L_K - L_{K,D(x)} \right\} \right\|_{HS}^2 \right] \leq \frac{\kappa^2 \mathcal{N}(\lambda)}{|D|}. \]

Then our desired inequality in Part (a) follows from the first inequality of (43).

From (45) and (46), we find a bound for \( B \)
\[ B = \frac{1}{\sqrt{\lambda}} \sqrt{\sum_{\ell} (\varphi_\ell(x))^2} \leq \frac{\kappa}{\sqrt{\lambda}} \sqrt{K(x,x)} \leq \frac{\kappa^2}{\sqrt{\lambda}}, \quad \forall x \in \mathcal{X}. \]

Applying Lemma 21 to the random variable \( \eta_1 \) with \( \tilde{M} = \frac{\kappa^2}{\sqrt{\lambda}} \), we know by (43) that with confidence at least 1 − \( \delta \),
\[ \left\| E[\eta_1] - \frac{1}{|D|} \sum_{x \in D(x)} \eta_1(x) \right\|_{HS} \leq \left\| E[\eta_1] - \frac{1}{|D|} \sum_{x \in D(x)} \eta_1(x) \right\|_{HS} \leq \frac{2\kappa^2 \log(2/\delta)}{|D| \sqrt{\lambda}} + \sqrt{\frac{2\kappa^2 \mathcal{N}(\lambda) \log(2/\delta)}{|D|}}. \]

Writing the above bound by taking a factor \( \frac{2\kappa \log(2/\delta)}{|D| \sqrt{\lambda}} \), we get the desired bound (24).

Recall \( \mathcal{B}_{|D|,\lambda} \) defined by (25). Apply the formula (36) for nonnegative random variables to \( \xi = \left\| (L_K + \lambda I)^{-1/2} \left\{ L_K - L_{K,D(x)} \right\} \right\|_{d} \) and use the bound
\[ \text{Prob} [\xi > t] = \text{Prob} \left[ \xi^{1/2} > t^{1/2} \right] \leq 2 \exp \left\{ -\frac{t^{1/2}}{E_{|D|,\lambda}} \right\} \]
derived from (24) for \( t \geq \log^d 2 \mathcal{B}_{|D|,\lambda} \). We find
\[ E \left[ \left\| (L_K + \lambda I)^{-1/2} \left\{ L_K - L_{K,D(x)} \right\} \right\|_{d}^d \right] \leq \log^d 2 \mathcal{B}_{|D|,\lambda} + \int_0^\infty 2 \exp \left\{ -\frac{t^{1/2}}{E_{|D|,\lambda}} \right\} dt. \]
The second term in the right hand of above equation equals $2d B_{d,|D|,\lambda} \int_0^\infty u^{d-1} \exp \{-u\} \, du$. Then the desired bound in Part (c) follows from $\int_0^\infty u^{d-1} \exp \{-u\} \, du = \Gamma(d)$ and the lemma is proved.

**Proof of Lemma 17** Consider the random variable $\eta_2$ defined by

$$
\eta_2(z) = (L_K + \lambda I)^{-1/2} \left( K_x \right), \quad z = (x, y) \in \mathcal{Z}.
$$

It takes values in $\mathcal{H}_K$. By (45), it satisfies

$$
\|\eta_2(z)\|_K = \left\| (L_K + \lambda I)^{-1/2} \left( \sum_{\ell} \varphi_\ell(x) \varphi_\ell \right) \right\|_K = \left( \sum_{\ell} \frac{(\varphi_\ell(x))^2}{\lambda_\ell + \lambda} \right)^{1/2}.
$$

So

$$
E \left[ \left\| (L_K + \lambda I)^{-1/2} \left( K_x \right) \right\|_K^2 \right] = E \left[ \sum_{\ell} \frac{(\varphi_\ell(x))^2}{\lambda_\ell + \lambda} \right] = \mathcal{N}(\lambda).
$$

This is the statement of Part (a).

For Part (b), we consider another random variable $\eta_3$ defined by

$$
\eta_3(z) = (L_K + \lambda I)^{-1/2} \left( g(z) K_x \right), \quad z = (x, y) \in \mathcal{Z}.
$$

It takes values in $\mathcal{H}_K$ and satisfies

$$
\|\eta_3(z)\|_K = |g(z)| \left\| (L_K + \lambda I)^{-1/2} \left( K_x \right) \right\|_K = |g(z)| \left( \sum_{\ell} \frac{(\varphi_\ell(x))^2}{\lambda_\ell + \lambda} \right)^{1/2}.
$$

So

$$
\|\eta_3(z)\|_K \leq \frac{\kappa \|g\|_\infty}{\sqrt{\lambda}}, \quad z \in \mathcal{Z}
$$

and

$$
E \left[ \|\eta_3\|_K^2 \right] \leq \|g\|_\infty^2 E \left[ \sum_{\ell} \frac{(\varphi_\ell(x))^2}{\lambda_\ell + \lambda} \right] = \|g\|_\infty^2 \mathcal{N}(\lambda).
$$

Applying Lemma 21 proves the statement in Part (b).

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