A CAMERON–STORVICK THEOREM FOR THE ANALYTIC FEYNMAN INTEGRAL ASSOCIATED WITH GAUSSIAN PATHS ON A WIENER SPACE AND APPLICATIONS

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Abstract. The purpose of this paper is to establish a Cameron–Storvick theorem for the analytic Feynman integral of functionals in non-stationary Gaussian processes on Wiener space. As interesting applications, we apply this theorem to evaluate the generalized analytic Feynman integral of certain polynomials in terms of Paley–Wiener–Zygmund stochastic integrals.

1. Introduction and preliminaries. Let $C_0[0, T]$ denote the one-parameter Wiener space, that is, the space of all real-valued continuous functions $x$ on $[0, T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0, T]$ and let $m$ denote the Wiener measure. Then, as it is well-known, $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space.

In [1] Cameron derived an integration by parts formula for the Wiener measure $m$. This is the first infinite dimensional integration by parts formula. In [11] Donsker also established this formula using a different method, and applied it to study Fréchet–Volterra differential equations. In [16, 17] Kuo and Lee developed the parts formula to abstract Wiener spaces and applied their formula to evaluate some functional integrals. The integration by parts formula on $C_0[0, T]$ introduced in [1] was improved in [7, 21] to study the parts formulas involving the analytic Feynman integral and the analytic Fourier–Feynman transform. Since then the parts formula for the analytic Feynman integral is called the Cameron–Storvick theorem by many mathematicians.

The Wiener process used in [1, 7, 21] is a stationary process. The purpose of this paper is to establish a Cameron–Storvick theorem for the analytic Feynman integral of functionals in non-stationary Gaussian processes defined on the Wiener space $C_0[0, T]$. As interesting applications, we use our parts formula to evaluate the generalized analytic Feynman integral of certain polynomials in terms of Paley–Wiener–Zygmund stochastic integrals.

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2. Preliminaries. In this section we first present a brief background and some well-known results about the Wiener space $C_0[0,T]$.

A subset $B$ of $C_0[0,T]$ is said to be scale-invariant measurable provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set $N$ is said to be scale-invariant null provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional $F$ is said to be scale-invariant measurable provided $F$ is defined on a scale-invariant measurable set and $F(\rho \cdot)$ is Wiener-measurable for every $\rho > 0$.

The Paley–Wiener–Zygmund (PWZ) stochastic integral [18] plays a key role throughout this paper. Let $\{\phi_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $L_2[0,T]$, each of whose elements is of bounded variation on $[0,T]$. Then for each $v \in L_2[0,T]$, the PWZ stochastic integral $\langle v, x \rangle$ is defined by the formula

$$
\langle v, x \rangle = \lim_{n \to \infty} \int_0^T \sum_{j=1}^n (v, \phi_j) (t) dx(t)
$$

for all $x \in C_0[0,T]$ for which the limit exists, where $(\cdot, \cdot)_2$ denotes the $L_2$-inner product. For each $v \in L_2[0,T]$, the limit defining the PWZ stochastic integral $\langle v, x \rangle$ is essentially independent of the choice of the complete orthonormal set $\{\phi_n\}_{n=1}^{\infty}$ and it exists for s-a.e. $x \in C_0[0,T]$. If $v$ is of bounded variation on $[0,T]$ then $\langle v, x \rangle$ equals the Riemann–Stieltjes integral $\int_0^T v(t) dx(t)$ for s-a.e. $x \in C_0[0,T]$, and for all $v \in L_2[0,T]$, $\langle v, x \rangle$ is a Gaussian random variable on $C_0[0,T]$ with mean zero and variance $\|v\|_2^2$. For a more detailed study of the PWZ stochastic integral, see [14, 19].

Throughout this paper we let

$$
\text{Supp}_2[0,T] = \{h \in L_2[0,T] : m_L(\text{supp}(h)) = T\}
$$

$$
= \{h \in L_2[0,T] : h \neq 0 \text{ m}_L\text{-a.e. on } [0,T]\}
$$

where $m_L$ denotes Lebesgue measure on $[0,T]$.

For any $h \in \text{Supp}_2[0,T]$, let $Z_h : C_0[0,T] \times [0,T] \to \mathbb{R}$ be the stochastic process given by

$$
Z_h(x,t) = (h \chi_{[0,t]}, x),
$$

where $\chi_{[0,t]}$ denotes the indicator function of the set $[0,t]$. Next, let $\beta_h(t) = \int_0^t h^2(u) du$. The stochastic process $Z_h$ on $C_0[0,T] \times [0,T]$ is a Gaussian process with mean zero and covariance function

$$
\int_{C_0[0,T]} Z_h(x,s) Z_h(x,t) m(dx) = \beta_h(\min\{s,t\}).
$$

In addition, by [23, Theorem 21.1], $Z_h(\cdot,t)$ is stochastically continuous in $t$ on $[0,T]$. If $h \in \text{Supp}_2[0,T]$ is of bounded variation on $[0,T]$, then for all $x \in C_0[0,T]$, $Z_h(x,t)$ is continuous in $t$. Also, for any $h_1, h_2 \in \text{Supp}_2[0,T]$,

$$
\int_{C_0[0,T]} Z_{h_1}(x,s) Z_{h_2}(x,t) m(dx) = \int_0^{\min\{s,t\}} h_1(u) h_2(u) du.
$$

Of course if $h(t) \equiv 1$ on $[0,T]$, then the process $W$ on $C_0[0,T] \times [0,T]$ given by $(w,t) \mapsto W_t(x) = Z_1(x,t) = x(t)$ is a Wiener process (standard Brownian motion). We note that the coordinate process $Z_1$ is stationary in time, whereas the stochastic process $Z_h$ generally is not. For more detailed studies on the stochastic process $Z_h$, see [9, 10, 20].
Let $\mathbb{C}, \mathbb{C}_+$ and $\mathbb{C}_+$ denote the set of complex numbers, complex numbers with positive real part and non-zero complex numbers with nonnegative real part, respectively. For each $\lambda \in \mathbb{C}$, $\lambda^{1/2}$ denotes the principal square root of $\lambda$; i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$ for all $\lambda \in \mathbb{C}_+$.

**Definition 2.1.** Let $h$ be a function in $\text{Supp}_2[0,T]$ and let $F$ be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_0[0,T]$ such that

$$\int_{C_0[0,T]} F(\lambda^{-1/2} Z_h(x, \cdot)) \text{d}m(dx) = J(h; \lambda)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(h; \lambda)$ analytic on $\mathbb{C}_+$ such that $J^*(h; \lambda) = J(h; \lambda)$ for all $\lambda > 0$, then $J^*(h; \lambda)$ is defined to be the generalized analytic Wiener integral (associated with the Gaussian paths $Z_h(x, \cdot)$) of $F$ over $C_0[0,T]$ with parameter $\lambda$, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0[0,T]} \text{anw}_\lambda F(Z_h(x, \cdot)) \text{d}m(dx) = J^*(h; \lambda).$$

Let $q \neq 0$ be a real number and let $F$ be a functional such that $\int_{C_0[0,T]} \text{anw}_\lambda F(Z_h(x, \cdot)) \text{d}m(dx)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral (associated with the Gaussian paths $Z_h(x, \cdot)$) of $F$ with parameter $q$ and we write

$$\int_{C_0[0,T]} \text{anf}_q F(Z_h(x, \cdot)) \text{d}m(x) = \lim_{\lambda \to q^+} \int_{C_0[0,T]} \text{anw}_\lambda F(Z_h(x, \cdot)) \text{d}m(dx).$$

**Remark 2.2.** Note that if $h(t) \equiv 1$ on $[0,T]$, then this definition agrees with the previous definition of the analytic Feynman integral [6, 7, 21].

Next we give the definition of the first variation $\delta F$ of a functional $F$. The following definition is due to by Chang, Cho, Kim, Song and Yoo [8].

**Definition 2.3.** Let $F$ be a Wiener measurable functional on $C_0[0,T]$ and let $w \in C_0[0,T]$. Then

$$\delta_{h_1,h_2} F(x|w) \equiv \delta F(Z_{h_1}(x, \cdot) | Z_{h_2}(w, \cdot))$$

$$= \frac{\partial}{\partial \mu} F(Z_{h_1}(x, \cdot) + \mu Z_{h_2}(w, \cdot)) \bigg|_{\mu=0} \quad (2.1)$$

(if it exists) is called the first variation of $F$ in the direction $w$.

Throughout this paper we shall always choose $w$ to be an element of $C_0'[0,T]$ where

$$C_0'[0,T] = \{ w \in C_0[0,T] : w \text{ is absolutely continuous on } [0,T] \text{ with } w' \in L_2[0,T] \}$$

$$= \left\{ w \in C_0[0,T] : w(t) = \int_0^t z(s) ds \text{ for some } z \in L_2[0,T] \right\}.$$

Let $BV[0,T]$ denote the space of the functions of bounded variation on $[0,T]$. Then we obtain the following formula: for each $\varphi \in BV[0,T]$ and each $h \in \text{Supp}_2[0,T]$,

$$\langle \varphi, Z_h(x, \cdot) \rangle = \langle \varphi h, x \rangle \quad (2.2)$$
for s.a.e. \( x \in C_0[0,T] \). Furthermore if \( x \in C_0'[0,T] \) with \( x(t) = \int_0^t \alpha(s)ds \), then
\[
\langle \varphi, Z_h(x,\cdot) \rangle = (\varphi h, \alpha)_2.
\] (2.3)

We finish this section by stating the following well-known translation theorem using the above notation [2, 3, 15].

**Theorem 2.4** (Translation Theorem). Let \( F \) be Wiener integrable over \( C_0[0,T] \) and let \( x_0 \in C_0'[0,T] \). Then
\[
\int_{C_0[0,T]} F(x + x_0)dm(x)
= \exp \left\{ -\frac{1}{2} \|x_0\|_2^2 \right\} \int_{C_0[0,T]} F(x) \exp \{ \langle x_0, x \rangle \} m(dx).
\] (2.4)

### 3. A translation theorem.

It is well-known that there is no quasi-invariant measure on infinite dimensional linear spaces (see for instance [22]). Thus, there is no quasi-invariant probability measure on the Wiener space \((C_0[0,T], M, m)\). Based on such circumstance, numerous constructions and applications of the translation theorem (Cameron–Martin theorem) for integrals on infinite-dimensional spaces have been studied in various research fields in Mathematics and Physics. Most of the results in the literature are concentrated on Wiener spaces.

The Cameron–Martin translation theorem on classical Wiener space was introduced in [2, 3]. On the other hand, Cameron and Storvick [5, 6] presented a translation theorem for the analytic Feynman integral of functionals on the Wiener space \( C_0[0,T] \).

In our next theorem, we obtain a translation theorem for the generalized Wiener integral on \( C_0[0,T] \).

**Theorem 3.1.** Let \( h_1 \) be a function in \( \text{Supp}_2[0,T] \) and let \( F \) be a functional on \( C_0[0,T] \) such that \( F(Z_{h_1}(x,\cdot)) \) is Wiener integrable over \( C_0[0,T] \). Given \( \varphi \in BV[0,T] \), let \( x_1 \in C_0'[0,T] \) be defined by
\[
x_1(t) = \int_0^t \varphi(s)h_1(s)ds.
\]
Then for each \( h_2 \in \text{Supp}_2[0,T] \),
\[
\int_{C_0[0,T]} F(Z_{h_1}(x,\cdot) + Z_{h_2}(x_1,\cdot))m(dx)
= \exp \left\{ -\frac{1}{2} \|\varphi h_2\|_2^2 \right\} \int_{C_0[0,T]} F(Z_{h_1}(x,\cdot)) \exp \{ \langle \varphi, Z_{h_2}(x,\cdot) \rangle \} m(dx).
\] (3.1)

**Proof.** We first note that
\[
Z_{h_2}(x_1,t) = \int_0^t h_2(s) \left[ \int_0^s \varphi(u)h_1(u)du \right] ds
= \int_0^t h_1(s)\varphi(s)h_2(s)ds
= \int_0^t h_1(s) \left[ \int_0^s \varphi(u)h_2(u)du \right] = Z_{h_1}(x_2,t)
\]
where
\[
x_2(t) = \int_0^t \varphi(s)h_2(s)ds.
\] (3.2)
Using this we see that

\[ Z_{h_1}(x, t) + Z_{h_2}(x_1, t) = Z_{h_2}(x + x_2, t). \]  

(3.3)

Also, using equation (2.2) it follows that

\[ \langle \varphi h_2, x \rangle = \langle \varphi, Z_{h_2}(x, \cdot) \rangle. \]  

(3.4)

Hence letting \( G(x) = F(Z_{h_1}(x, \cdot)) \) and using equations (3.3), (2.4) with \( F \) and \( x_0 \) replaced with \( G \) and \( x_2 \), (3.2), and (3.4), it follows that

\[
\int_{C_0[0,T]} F(Z_{h_1}(x, \cdot) + Z_{h_2}(x_1, \cdot)) m(dx) \\
= \int_{C_0[0,T]} F(Z_{h_1}(x + x_2, \cdot)) m(dx) \\
= \int_{C_0[0,T]} G(x + x_2) m(dx) \\
= \exp \left\{ - \frac{1}{2} \| x_2 \|^2 \right\} \int_{C_0[0,T]} G(x) \exp \{ \langle x_2, x \rangle \} m(dx) \\
= \exp \left\{ - \frac{1}{2} \| \varphi h_2 \|^2 \right\} \int_{C_0[0,T]} F(Z_{h_1}(x, \cdot)) \exp \{ \langle \varphi, Z_{h_2}(x, \cdot) \rangle \} m(dx)
\]

as desired. \( \Box \)

4. A Cameron–Storvick theorem. In [1] Cameron introduced the first variation (a kind of Gâteaux derivative) of functionals on \( C_0[0,T] \) and obtained a formula involving the Wiener integral of the first variation. In [7] Cameron and Storvick established a similar result for the analytic Feynman integral of functionals on \( C_0[0,T] \). They also applied their celebrated result to establish the existence of the analytic Feynman integral of unbounded functionals on \( C_0[0,T] \). In this section we establish the Cameron–Storvick theorem for the analytic Feynman integral of functionals in Gaussian paths on \( C_0[0,T] \).

Now we state the integration by parts formula for the generalized Wiener integral.

**Theorem 4.1.** Let \( h_1 \) and \( h_2 \) be functions in \( \text{Supp}^2[0,T] \) and given \( \varphi \in BV[0,T] \), let \( w_{\varphi h_1} \in C_0'[0,T] \) be defined by

\[ w_{\varphi h_1}(t) = \int_0^t \varphi(s) h_1(s) ds. \]

Let \( F \) be a functional on \( C_0[0,T] \) such that \( F(Z_{h_1}(x, \cdot)) \) is Wiener integrable over \( C_0[0,T] \). Furthermore assume that

\[
\int_{C_0[0,T]} \left| \delta F(Z_{h_1}(x, \cdot)|Z_{h_2}(w_{\varphi h_1}, \cdot)) \right| m(dx) < +\infty.
\]  

(4.1)

Then

\[
\int_{C_0[0,T]} \delta F(Z_{h_1}(x, \cdot)|Z_{h_2}(w_{\varphi h_1}, \cdot)) m(dx) \\
= \int_{C_0[0,T]} \langle \varphi, Z_{h_2}(x, \cdot) \rangle F(Z_{h_1}(x, \cdot)) m(dx).
\]  

(4.2)
that for each \( \rho > 0 \), assume that Lemma 4.2.

**Proof.** By using (2.1) and (3.1) with \( x_1 \) replaced with \( w_{\varphi h_1} \), it follows that

\[
\int_{C_0[0,T]} \delta F(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{\varphi h_1},\cdot)) m(dx)
\]

\[
= \int_{C_0[0,T]} \frac{\partial}{\partial \mu} F(Z_{h_1}(x,\cdot) + \mu Z_{h_2}(w_{\varphi h_1},\cdot)) \bigg|_{\mu=0} m(dx)
\]

\[
= \frac{\partial}{\partial h} \left( \int_{C_0[0,T]} F(Z_{h_1}(x,\cdot) + \mu Z_{h_2}(w_{\varphi h_1},\cdot)) m(dx) \right) \bigg|_{\mu=0}
\]

\[
= \frac{\partial}{\partial \mu} \left( \exp \left\{ -\frac{\mu^2 \| \varphi h_2 \|^2}{2} \right\} \right)
\]

\[
\times \int_{C_0[0,T]} F(Z_{h_1}(x,\cdot)) \exp\{\mu \langle \varphi, Z_{h_2}(x,\cdot) \rangle\} m(dx) \bigg|_{\mu=0}
\]

\[
= \int_{C_0[0,T]} \langle \varphi, Z_{h_2}(x,\cdot) \rangle F(Z_{h_1}(x,\cdot)) m(dx).
\]

The second equality of (4.3) follows from (4.1) and Theorem 2.27 in [12]. \( \square \)

**Lemma 4.2.** Let \( h_1, h_2, \varphi, w_{\varphi h_1} \), and \( F \) be as in Theorem 4.1. For each \( \rho > 0 \), assume that \( F(\rho Z_{h_1}(x,\cdot)) \) is Wiener integrable over \( C_0[0,T] \). Furthermore assume that for each \( \rho > 0 \),

\[
\int_{C_0[0,T]} |\delta F(\rho Z_{h_1}(x,\cdot)|\rho Z_{h_2}(w_{\varphi h_1},\cdot))| m(dx) < +\infty.
\]

Then

\[
\int_{C_0[0,T]} \delta F(\rho Z_{h_1}(x,\cdot)|\rho Z_{h_2}(w_{\varphi h_1},\cdot)) m(dx)
\]

\[
= \int_{C_0[0,T]} \langle \varphi, Z_{h_2}(x,\cdot) \rangle F(\rho Z_{h_1}(x,\cdot)) m(dx).
\]

**Proof.** Let \( G(x) = F(\rho x) \). Then

\[
G(Z_{h_1}(x,\cdot) + \mu Z_{h_2}(w_{\varphi h_1},\cdot)) = F(\rho Z_{h_1}(x,\cdot) + \mu \rho Z_{h_2}(w_{\varphi h_1},\cdot))
\]

and

\[
\frac{\partial}{\partial \mu} G(Z_{h_1}(x,\cdot) + \mu Z_{h_2}(w_{\varphi h_1},\cdot)) \bigg|_{\mu=0} = \frac{\partial}{\partial \mu} F(\rho Z_{h_1}(x,\cdot) + \rho \mu Z_{h_2}(w_{\varphi h_1},\cdot)) \bigg|_{\mu=0}.
\]

Thus \( \delta G(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{\varphi h_1},\cdot)) = \delta F(\rho Z_{h_1}(x,\cdot)|\rho Z_{h_2}(w_{\varphi h_1},\cdot)) \). Hence by equation (4.2) with \( F \) replaced with \( G \), we have

\[
\int_{C_0[0,T]} \delta F(\rho Z_{h_1}(x,\cdot)|\rho Z_{h_2}(w_{\varphi h_1},\cdot)) m(dx)
\]

\[
= \int_{C_0[0,T]} \delta G(Z_{h_1}(x,\cdot)|Z_{h_2}(w_{\varphi h_1},\cdot)) m(dx)
\]

\[
= \int_{C_0[0,T]} \langle \varphi, Z_{h_2}(x,\cdot) \rangle G(Z_{h_1}(x,\cdot)) m(dx)
\]

\[
= \int_{C_0[0,T]} \langle \varphi, Z_{h_2}(x,\cdot) \rangle F(\rho Z_{h_1}(x,\cdot)) m(dx)
\]
which establishes (4.4).

Next we present a Cameron–Storvick theorem for the generalized analytic Feynman integral.

**Theorem 4.3.** Let \( h_1, h_2, \varphi, w_{\varphi h_1}, \) and \( F \) be as in Lemma 4.2. Then if either member of the following equation exists, both generalized analytic Feynman integrals below exist, and for each \( q \in \mathbb{R} \setminus \{0\}, \)

\[
\int_{C_0[0,T]}^{\text{and}\, q} \delta F(Z_{h_1}(x, \cdot)|Z_{h_2}(w_{\varphi h_1}, \cdot)) \, m(dx)
= -i q \int_{C_0[0,T]}^{\text{and}\, q} \langle \varphi, Z_{h_2}(x, \cdot) \rangle F(Z_{h_1}(x, \cdot)) \, m(dx).
\]

**Proof.** Let \( \rho > 0 \) be given. Since \( h_1 \in \text{Supp}_2[0,T], (1/\rho) h_1 \in \text{Supp}_2[0,T] \) and

\[
w_{\varphi h_1}(t) = \int_0^t \varphi(s) h_1(s) \, ds = \rho \int_0^t \frac{1}{\rho} \varphi(s) h_1(s) \, ds.
\]

Let \( y_{(\varphi/\rho) h_1}(t) = \int_0^t (1/\rho) \varphi(s) h_1(s) \, ds \). By equation (4.4) with \( w_{\varphi h_1} \) and \( \varphi \) replaced with \( y_{(\varphi/\rho) h_1} \) and \( \rho^{-1} \varphi \) respectively,

\[
\int_{C_0[0,T]}^{\rho} \delta F(\rho Z_{h_1}(x, \cdot)|Z_{h_2}(w_{\varphi h_1}, \cdot)) \, m(dx)
= \int_{C_0[0,T]}^{\rho} \delta F(\rho Z_{h_1}(x, \cdot)|\rho Z_{h_2}(y_{(\varphi/\rho) h_1}, \cdot)) \, m(dx)
= \int_{C_0[0,T]}^{\rho} \langle \rho^{-1} \varphi, Z_{h_2}(x, \cdot) \rangle F(\rho Z_{h_1}(x, \cdot)) \, m(dx)
= \rho^{-2} \int_{C_0[0,T]}^{\rho} \langle \varphi, \rho Z_{h_2}(x, \cdot) \rangle F(\rho Z_{h_1}(x, \cdot)) \, m(dx).
\] (4.5)

Now let \( \rho = \lambda^{-1/2} \). Then equation (4.5) becomes

\[
\int_{C_0[0,T]}^{\lambda^{-1/2}} \delta F(\lambda^{-1/2} Z_{h_1}(x, \cdot)|Z_{h_2}(w_{\varphi h_1}, \cdot)) \, m(dx)
= \lambda \int_{C_0[0,T]}^{\lambda^{-1/2}} \langle \varphi, \lambda^{-1/2} Z_{h_2}(x, \cdot) \rangle F(\lambda^{-1/2} Z_{h_1}(x, \cdot)) \, m(dx).
\] (4.6)

Since \( \rho > 0 \) was arbitrary, we conclude that equation (4.6) holds for all \( \lambda > 0 \). We now use Definition 2.1 to obtain our desired conclusions. \( \square \)

**Corollary 4.4.** Under the assumptions as given in Theorem 4.3, if either member of the following equation exists, both generalized analytic Feynman integrals below exist, and for each \( q \in \mathbb{R} \setminus \{0\}, \)

\[
\int_{C_0[0,T]}^{\text{and}\, q} \langle \varphi, Z_{h_2}(x, \cdot) \rangle F(Z_{h_1}(x, \cdot)) \, m(dx)
= \frac{i}{q} \int_{C_0[0,T]}^{\text{and}\, q} \delta F(Z_{h_1}(x, \cdot)|Z_{h_2}(w_{\varphi h_1}, \cdot)) \, m(dx).
\] (4.7)
5. Generalized analytic Feynman integral of polynomials in terms of PWZ stochastic integrals. In this section we present interesting examples to which equation (4.7) can be applied.

**Example 5.1.** Let $h_1$ and $h_2$ be elements of $\text{Supp}_1[0, T]$ and given a non-zero function $g$ in $BV[0, T]$, set $F(x) = \langle g, x \rangle$. Then using equations (2.1), (2.2) and (2.3) it follows that for any $w_z$ in $C^2_0[0, T]$ with $w_z(t) = \int_0^t z(s)ds$ for $t \in [0, T]$,\[
\delta F(Z_{h_1}(x, \cdot)|Z_{h_2}(w_z, \cdot)) = \frac{\partial}{\partial \mu} F(Z_{h_1}(x, \cdot) + \mu Z_{h_2}(w_z, \cdot)) \bigg|_{\mu=0}
= \frac{\partial}{\partial \mu} \{ \langle g, Z_{h_1}(x, \cdot) \rangle + \mu \langle g, Z_{h_2}(w_z, \cdot) \rangle \} \bigg|_{\mu=0}
= \langle g, Z_{h_2}(w_z, \cdot) \rangle
= \langle gh_2, w_z \rangle
= \langle gh_2, z \rangle_2.
\]
In particular, setting $w_{gh_1}(t) = \int_0^t g(s)h_1(s)ds$ for $t \in [0, T]$, we obtain that
\[
\delta F(Z_{h_1}(x, \cdot)|Z_{h_2}(w_{gh_1}, \cdot)) = \langle gh_2, w_{gh_1} \rangle = \langle gh_2, gh_1 \rangle_2 = \langle gh_1, gh_2 \rangle_2. \tag{5.2}
\]
Next using equation (4.7) with $\varphi$ and $w_{\varphi h_1}$ replaced with $g$ and $w_{gh_1}$, respectively, and with $F(x) = \langle g, x \rangle$, and using equation (5.2), we obtain the formula
\[
\int_{C^0[0, T]} \langle g, Z_{h_2}(x, \cdot) \rangle \langle g, Z_{h_1}(x, \cdot) \rangle m(dx)
= \int_{C^0[0, T]} \langle g, Z_{h_2}(x, \cdot) \rangle F(Z_{h_1}(x, \cdot)) m(dx)
= \frac{i}{q} \int_{C^0[0, T]} \delta F(Z_{h_1}(x, \cdot)|Z_{h_2}(w_{gh_1}, \cdot)) m(dx)
= \frac{i}{q} \langle gh_1, gh_2 \rangle_2. \tag{5.3}
\]
**Remark 5.2.** Using (2.2), it follows that
\[
\langle g, Z_{h_2}(x, \cdot) \rangle \langle g, Z_{h_2}(x, \cdot) \rangle = \langle gh_1, x \rangle \langle gh_2, x \rangle.
\]
In this case, the two Gaussian random variables $\langle gh_1, x \rangle$ and $\langle gh_2, x \rangle$ have different Gaussian distributions. Thus in order to calculate the analytic Feynman integral (see Remark 2.2 above)
\[
\int_{C^0[0, T]} \langle gh_1, x \rangle \langle gh_2, x \rangle m(dx),
\]
we might apply the Gram–Schmidt process to the subset $\{ gh_1, gh_2 \}$ of $L^2[0, T]$ and use a well-known Wiener integration theorem. The Wiener integration theorem is as follows: for a Wiener integrable functional $F : C^0[0, T] \rightarrow \mathbb{C}$ of the form $F(x) = f(\langle \alpha_1, x \rangle, \ldots, \langle \alpha_n, x \rangle)$, where $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is a Lebesgue measurable function
and \( \{\alpha_1, \ldots, \alpha_n\} \) is an orthonormal set of functions in \( L_2[0, T] \),
\[
\int_{C_0[0,T]} F(x) m(dx) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} u_j^2 \right\} du_1 \cdots du_n. \tag{5.4}
\]

In our next example, for any positive integer \( m \in \{3, 4, \ldots\} \), we obtain a recurrence relation for the generalized analytic Feynman integral of the monomials \( \prod_{j=1}^{m} \langle g, Z_{h_j}(x, \cdot) \rangle \) of the PWZ stochastic integrals.

**Example 5.3.** Let \( m \geq 3 \) be a positive integer and let \( g \) be as in Example 5.1 above. Let \( \{h_1, \ldots, h_{m-1}, h_m\} \) be a finite sequence of functions in \( \text{Supp}_2[0, T] \) and set
\[
F(x) = \prod_{j=1}^{m-1} \langle g, Z_{h_j}(x, \cdot) \rangle = \prod_{j=1}^{m-1} \langle gh_j, x \rangle.
\]

First, using equation (2.1) it follows that for all \( w \in C_0'[0, T] \),
\[
\delta F(x|Z_{h_m}(w, \cdot)) = \delta F(Z_1(x, \cdot)|Z_{h_m}(w, \cdot))
\]
\[
= \left. \frac{\partial}{\partial \mu} F(Z_1(x, \cdot) + \mu Z_{h_m}(w, \cdot)) \right|_{\mu=0}
\]
\[
= \left. \frac{\partial}{\partial \mu} \prod_{j=1}^{m-1} \langle gh_j, Z_1(x, \cdot) + \mu Z_{h_m}(w, \cdot) \rangle \right|_{\mu=0}
\]
\[
= \left. \frac{\partial}{\partial \mu} \sum_{l=1}^{m-1} \prod_{\substack{j=1 \atop j \neq l}}^{m-1} \langle gh_j, Z_1(x, \cdot) \rangle \langle gh_l, Z_{h_m}(w, \cdot) \rangle \right|_{\mu=0}
\]
Then in particular, setting \( w_g(t) = \int_0^t g(s) ds \) and using (2.2) and (2.3), it follows that
\[
\delta F(Z_1(x, \cdot)|Z_{h_m}(w_g, \cdot)) = \sum_{l=1}^{m-1} \prod_{\substack{j=1 \atop j \neq l}}^{m-1} \langle gh_j, Z_1(x, \cdot) \rangle \langle gh_l, Z_{h_m}(w_g, \cdot) \rangle
\]
\[
= \sum_{l=1}^{m-1} \langle gh_l, gh_m \rangle_2 \prod_{\substack{j=1 \atop j \neq l}}^{m-1} \langle gh_j, Z_1(x, \cdot) \rangle. \tag{5.5}
\]

Hence, using equation (4.7) with \( \varphi \equiv g \), \( h_1 \equiv 1 \) and \( h_2 \) replaced with \( h_m \) and with \( F(x) = \prod_{j=1}^{m} \langle gh_j, x \rangle \) and using equation (5.5), we obtain the formula
\[
I_m = \int_{C_0[0,T]} \prod_{j=1}^{m} \langle g, Z_{h_j}(x, \cdot) \rangle m(dx)
\]
\[
= \int_{C_0[0,T]} \langle g, Z_{h_m}(x, \cdot) \rangle \prod_{j=1}^{m-1} \langle gh_j, x \rangle m(dx) \tag{5.6}
\]
and completely evaluate the generalized analytic Feynman integral

\[
\int_{C_0[0,T]} I_{\text{anf}_q} (g, Z_{h_1}(x, \cdot)) \langle g, Z_{h_2}(x, \cdot) \rangle \langle g, Z_{h_3}(x, \cdot) \rangle m(dx)
\]

where \(n\) is a finite family of subsets, \(S\), of \(L_2[0,T]\) with \(|S| = l\), \(c_S \in \mathbb{C}\) for each \(S \in \mathcal{S}_l\) and \(c_0 \in \mathbb{C}\).

Remark 5.4. Applying equations (5.6) and (5.3) and the linearity of the analytic Feynman integral, we can calculate the analytic Feynman integral of the polynomials \(P\) having the form

\[
P(x) = \sum_{l=1}^{n} \sum_{S \in \mathcal{S}_l} c_S \prod_{h \in S} \langle g, Z_h(x, \cdot) \rangle + c_0,
\]

where \(n \in \mathbb{N}\), \(\mathcal{S}_l\) is a finite family of subsets, \(S\), of \(L_2[0,T]\) with \(|S| = l\), \(c_S \in \mathbb{C}\) for each \(S \in \mathcal{S}_l\) and \(c_0 \in \mathbb{C}\).
Next we introduce the Fourier–Hermite polynomials on the Wiener space $C_0[0, T]$. For each $m = 0, 1, 2, \ldots$, let $H_m(u)$ denote the Hermite polynomial of degree $m$ given by

$$H_m(u) = (-1)^m (m!)^{-1/2} e^{u^2/2} \frac{d^m}{du^m}(e^{-u^2/2}). \quad (5.7)$$

Then, as is well-known, the set $\{(2\pi)^{-1/4} H_m(u)e^{-u^2/4} : m = 0, 1, 2, \ldots\}$ is a complete orthonormal set in $L_2(\mathbb{R})$. Next let $\{\alpha_j\}_{j=1}^\infty$ be a complete orthonormal set of functions of bounded variation on $[0, T]$. For each positive integer $j$ and nonnegative integer $m$, let $\phi_{m,j}$ be the functional on $C_0[0, T]$ given by

$$\phi_{m,j}(x) = H_m \left( \int_0^T \alpha_j(t)dx(t) \right) = H_m((\alpha_j, x))$$

with $H_m$ given by $(5.7)$. In addition, for each $j = 1, 2, \ldots, k$, let $\gamma_j$ be a nonnegative integer, and let $\Psi_{m_1, \ldots, m_k}(x)$ be the functional on $C_0[0, T]$ defined by

$$\Psi_{m_1, \ldots, m_k}(x) = \phi_{m_1, 1}(x) \cdots \phi_{m_k, k}(x) = \prod_{j=1}^k H_{\gamma_j}((\alpha_j, x)). \quad (5.8)$$

For example, we observe that $H_0(u) = 1$, $H_1(u) = u$, $H_2(u) = (2!)^{-1/2}(u^2 - 1)$, and $H_3(u) = (3!)^{-1/2}(u^3 - 3u)$. Hence

$$\Psi_{1,3,2}(x) = \phi_{1,1}(x)\phi_{3,2}(x)\phi_{2,3}(x)
= H_1((\alpha_1, x))H_3((\alpha_2, x))H_2((\alpha_3, x))
= (2!)^{-1/2}(3!)^{-1/2}(\alpha_1, x)((\alpha_2, x)^3 - 3(\alpha_2, x)^2 - 1)
= (2!)^{-1/2}(3!)^{-1/2} \left\{ (\alpha_1, x)(\alpha_2, x)^3(\alpha_3, x)^2 - 3(\alpha_1, x)(\alpha_2, x)^3(\alpha_3, x)^2 - 3(\alpha_1, x)(\alpha_2, x)^2(\alpha_3, x)^2 
- (\alpha_1, x)(\alpha_2, x)^3 + 3(\alpha_1, x)(\alpha_2, x)^2 \right\}.$$

We note that $\Psi_{m_1, \ldots, m_k}(x) = \Psi_{m_1, \ldots, m_k, 0, \ldots, 0}(x)$ for all positive integers $k$.

The functionals given by $(5.8)$ are called the Fourier–Hermite polynomials (or functionals). It was shown by Cameron and Martin [4] that the Fourier–Hermite polynomials form a complete orthonormal set in $L_2(C_0[0, T])$. To prove the completeness of the class of the Fourier–Hermite polynomials, Cameron and Martin used the Wiener integration formula $(5.4)$ above. The use is based on the followings: given a complete orthonormal set $\{\alpha_j\}_{j=1}^\infty$, each of whose elements is of bounded variation on $[0, T]$, let $\gamma_j(x) = (\alpha_j, x)$ for each $j \in \mathbb{N}$. Then the class $\{\gamma_j(x)\}_{j=1}^\infty$ is a sequence of i.i.d standard Gaussian random variables. Thus, using equation $(5.4)$, one can calculate the Wiener integrals, $\int_{C_0[0, T]} \Psi_{m_1, \ldots, m_k}(x)\mathcal{m}(dx)$ of the Fourier–Hermite polynomials $\Psi_{m_1, \ldots, m_k}$.

However, we will pose the following question.

**Question 5.5.** How to calculate the generalized analytic Feynman integral

$$\int_{C_0[0, T]}^{\text{anf}} \prod_{j=1}^n H_{\gamma_j}((\alpha_j, Z_{h_j}(x), \cdot))\mathcal{m}(dx) \quad (5.9)$$

where $\{h_1, \ldots, h_n\}$ is a finite sequence in $\text{Supp}_2[0, T]$ ?.
When we evaluate the generalized analytic Feynman integral given by (5.9) we might not use the Wiener integration formula (5.4) because the set of Gaussian random variables \( \{ (\alpha_j, \mathcal{Z}_{h_j}(x, \cdot)) \}_{j=1}^n = \{ (\alpha_j h_j, x) \}_{j=1}^n \) is generally not independent. However, we will calculate the generalized analytic Feynman integral (5.9) via the following examples.

**Example 5.6.** Let \( h_1 \) and \( h_2 \) be elements of \( \text{Supp}_2 [0, T] \) and set \( F(x) = x(T) = \langle 1, x \rangle \). Then using the techniques as those used in (5.1) we observe that for \( w_{\alpha_1, h_1} \) in \( C_0'[0, T] \) with \( w_{\alpha_1, h_1}(t) = \int_0^t \alpha_1(s) h_1(s) ds \),

\[
\delta F(\mathcal{Z}_{\alpha_1, h_1}(x, \cdot) | \mathcal{Z}_{\alpha_2, h_2}(w_{\alpha_1, h_1}, \cdot)) = \langle \alpha_2 h_2, w_{\alpha_1, h_1} \rangle = \langle \alpha_1 h_1, \alpha_2 h_2 \rangle_2. \tag{5.10}
\]

Next using equation (4.7) with \( \varphi \equiv 1 \), with \( h_2 \) and \( h_1 \) replaced with \( \alpha_2 h_2 \) and \( \alpha_1 h_1 \), respectively, and with \( F(x) = \langle 1, x \rangle \), and using equation (5.10), we obtain the formula

\[
\int_{C_0'[0, T]} \langle \alpha_2, \mathcal{Z}_{h_2}(x, \cdot) \rangle \langle \alpha_1, \mathcal{Z}_{h_1}(x, \cdot) \rangle m(dx) = \int_{C_0'[0, T]} \langle 1, \mathcal{Z}_{\alpha_2, h_2}(x, \cdot) \rangle \langle 1, \mathcal{Z}_{\alpha_1, h_1}(x, \cdot) \rangle m(dx)
\]

\[
= \int_{C_0'[0, T]} \langle 1, \mathcal{Z}_{\alpha_2, h_2}(x, \cdot) \rangle F(\mathcal{Z}_{\alpha_1, h_1}(x, \cdot)) m(dx)
\]

\[
= i \int_{C_0'[0, T]} \delta F(\mathcal{Z}_{\alpha_1, h_1}(x, \cdot) | \mathcal{Z}_{\alpha_2, h_2}(w_{\alpha_1, h_1}, \cdot)) m(dx)
\]

\[
= i \langle \alpha_1 h_1, \alpha_2 h_2 \rangle_2.
\]

Let \( m \geq 3 \) be a positive integer and \( \{ h_1, \ldots, h_{m-1}, h_m \} \) be a finite sequence of functions in \( \text{Supp}_2 [0, T] \) and set

\[
F(x) = \prod_{j=1}^{m-1} \langle \alpha_j, \mathcal{Z}_{h_j}(x, \cdot) \rangle = \prod_{j=1}^{m-1} \langle \alpha_j h_j, x \rangle.
\]

Using equation (2.1) it follows that for all \( w \in C_0'[0, T] \),

\[
\delta F(x | \mathcal{Z}_{\alpha_m, h_m}(w, \cdot)) = \delta F(\mathcal{Z}_1(x, \cdot) | \mathcal{Z}_{\alpha_m, h_m}(w, \cdot))
\]

\[
= \frac{\partial}{\partial \mu} F(\mathcal{Z}_1(x, \cdot) + \mu \mathcal{Z}_{\alpha_m, h_m}(w, \cdot)) \bigg|_{\mu=0}
\]

\[
= \frac{\partial}{\partial \mu} \prod_{j=1}^{m-1} \langle \alpha_j h_j, \mathcal{Z}_1(x, \cdot) + \mu \mathcal{Z}_{\alpha_m, h_m}(w, \cdot) \rangle \bigg|_{\mu=0}
\]

\[
= \frac{\partial}{\partial \mu} \prod_{j=1}^{m-1} \left\{ \langle \alpha_j h_j, \mathcal{Z}_1(x, \cdot) \rangle + \mu \mathcal{Z}_{\alpha_m, h_m}(w, \cdot) \right\} \bigg|_{\mu=0}
\]

\[
= \sum_{l=1}^{m-1} \prod_{j=1}^{m-1} \langle \alpha_j h_j, \mathcal{Z}_1(x, \cdot) \rangle \langle \alpha_l h_l, \mathcal{Z}_{\alpha_m, h_m}(w, \cdot) \rangle.
\]

Then in particular, setting

\[
w_t(t) = t = \int_0^t ds
\]
and

\[ w_{\alpha_m}(t) = \int_0^t \alpha_m(s) ds, \]

it follows that

\[
\delta F(Z_1(x, \cdot)|Z_{h_m}(w_{\alpha_m}, \cdot)) = \delta F(Z_1(x, \cdot)|Z_{\alpha_m h_m}(w_1, \cdot))
\]

\[
= \sum_{l=1}^{m-1} \prod_{j=1}^{m-1} \langle \alpha_j h_j, Z_1(x, \cdot) \rangle \langle \alpha_l h_l, Z_{\alpha_m h_m}(w_1, \cdot) \rangle
\]

\[
= \sum_{l=1}^{m-1} (\alpha_l h_l, \alpha_m h_m) z \prod_{j=1}^{m-1} \langle \alpha_j h_j, Z_1(x, \cdot) \rangle.
\]

Hence, using equation (4.7) with \( \varphi \equiv \alpha_m, h_1 \equiv 1, h_2 \) and \( w_{\varphi,1} \) replaced with \( h_m \) and \( w_{\alpha_m} \) and with \( F(x) = \prod_{j=1}^{m-1} \langle \alpha_j h_j, x \rangle \), and using equation (5.12), we obtain the formula

\[
\int_{C_0[0,T]} \bigl( \langle \alpha_j, Z_{h_j}(x, \cdot) \rangle m(\cdot) \bigr) dx
\]

\[
= \int_{C_0[0,T]} \langle \alpha_m, Z_{h_m}(x, \cdot) \rangle \prod_{j=1}^{m-1} \langle \alpha_j h_j, Z_1(x, \cdot) \rangle m(\cdot) dx
\]

\[
= \int_{C_0[0,T]} \langle \alpha_m, Z_{h_m}(x, \cdot) \rangle F(Z_1(x, \cdot)) m(\cdot) dx
\]

\[
= \frac{i}{q} \int_{C_0[0,T]} \delta F(Z_1(x, \cdot)|Z_{h_m}(w_{\alpha_m}, \cdot)) m(\cdot) dx
\]

\[
= \frac{i}{q} \sum_{l=1}^{m-1} (\alpha_l h_l, \alpha_m h_m) z \int_{C_0[0,T]} \prod_{j=1}^{m-1} \langle \alpha_j h_j, Z_1(x, \cdot) \rangle m(\cdot) dx
\]

\[
= \frac{i}{q} \sum_{l=1}^{m-1} (\alpha_l h_l, \alpha_m h_m) z \prod_{j=1}^{m-1} \langle \alpha_j, Z_{h_j}(x, \cdot) \rangle m(\cdot) dx.
\]

Using (5.13) inductively, and (5.11) we can obtain a positive answer of the Question 5.5 above. More precisely, using the recursive formula (5.13) and a tedious calculation, we conclude that

\[
\int_{C_0[0,T]} \prod_{j=1}^{m} \langle \alpha_j, Z_{h_j}(x, \cdot) \rangle m(\cdot) dx
\]

\[
= \begin{cases} 
0, & \text{if } m \text{ is odd} \\
\left( \frac{i}{q} \right)^{m/2} \sum_{k} \prod_{i} \langle \alpha_i h_i, \alpha_{j_k} h_{j_k} \rangle z \text{, if } m \text{ is even}
\end{cases}
\]

where the sum is over all partitions of \( \{1, 2, \ldots, m\} \) into disjoint pairs \( \{i_k, j_k\} \). This result subsumes Wick’s theorem (see [13, Theorem 1.28]) in quantum field theory.

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