Multiple Query Satisfiability of Constrained Horn Clauses*

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Abstract. We address the problem of checking the satisfiability of a set of constrained Horn clauses (CHCs) possibly including more than one query. We propose a transformation technique that takes as input a set of CHCs, including a set of queries, and returns as output a new set of CHCs, such that the transformed CHCs are satisfiable if and only if so are the original ones, and the transformed CHCs incorporate in each new query suitable information coming from the other ones so that the CHC satisfiability algorithm is able to exploit the relationships among all queries. We show that our proposed technique is effective on a non-trivial benchmark of sets of CHCs that encode many verification problems for programs manipulating algebraic data types such as lists and trees.

1 Introduction

Constrained Horn Clauses (CHCs) have been advocated as a logical formalism very well suited for automatic program verification [3, 6]. Indeed, many verification problems can be reduced to problems of checking satisfiability of CHCs [13], and several effective CHC solvers are currently available as back-end tools for program verification purposes [4, 7, 16, 18, 19].

Following the CHC-based verification approach, a program is translated into a set of definite CHCs (that is, clauses whose head is different from false), which capture the semantics of the program, together with a set of queries (that is, clauses whose head is false), which specify the program properties to be verified. Very often the CHC translation of the verification problem generates several queries. In particular, this is the case when the program includes several functions, each one having its own contract (that is, a pair of a pre-condition and a post-condition).

CHC solvers try to show the satisfiability of a set of CHCs of the form: 
\[ P \cup \{\text{false} \leftarrow G_1, \ldots, \text{false} \leftarrow G_n\}, \] where \( P \) is a set of definite CHCs, and

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false ← \(G_1\), \ldots, false ← \(G_n\) are queries, by trying to show in a separate way the satisfiability of each set \(P \cup \{\text{false} ← G_i\}\), for \(i = 1, \ldots, n\). This approach may not be always quite effective, as the solver may not be able to exploit the, possibly mutual, dependencies among the various queries. There is a simple way of combining all queries into one (as done, for instance, in the CHC solver competition [10]): we introduce a new predicate \(f\), and from the above mentioned set of CHCs we get \(P \cup \{\text{false} ← f, f ← G_1, \ldots, f ← G_n\}\). However, also in this case, existing solvers will handle each query separately and then combine the results.

In this paper we propose, instead, a technique that, given a set \(P \cup \{\text{false} ← G_1, \ldots, false ← G_n\}\) of CHCs, derives an equisatisfiable set \(P' \cup \{\text{false} ← G'_1, \ldots, false ← G'_n\}\), for whose satisfiability proof a CHC solver may exploit the mutual interactions among the \(n\) satisfiability proofs, one for each query.

Our technique builds upon the transformation approach for verifying contracts that we presented in previous work and implemented in the VeriCaT tool [8]. The algorithm of VeriCaT takes as input a set of CHCs that manipulate algebraic data types (ADTs) such as lists and trees, and a set of CHCs defining contracts by means of catamorphisms [20], and returns as output a set of CHCs without ADT variables such that the original set is satisfiable if the new set is satisfiable. For CHCs without ADT variables state-of-the-art solvers are more effective in proving satisfiability, and hence validity of contracts.

The objective of the transformation algorithm presented in this paper, which we call \(T_{mq}\), is not to eliminate ADT variables, rather, it is to incorporate into the clauses relative to a particular query some additional constraints that are derived from other queries. These additional constraints are often very beneficial to the CHC solvers when trying to check the satisfiability of a given set of clauses, thereby enhancing their ability to verify program properties. Algorithm \(T_{mq}\) is both sound and complete, that is, the transformed clauses are satisfiable if and only if so are the original ones. The completeness of \(T_{mq}\) is very important because if a property does not hold, it allows us to infer the unsatisfiability of the original clauses. Thus, whenever the solver shows the unsatisfiability of the transformed clauses, we deduce the invalidity of the property to be verified.

2 Preliminary Notions

We consider constrained Horn clauses that are defined in a many-sorted first order language \(L\) with equality (=) whose constraints are expressed using linear integer arithmetic (LIA) and boolean (Bool) expressions. A constraint is a quantifier-free formula \(c\), where the LIA constraints may occur as subexpressions of boolean constraints, according to the SMT approach [2]:

\[
c ::= d \mid B \mid \text{true} \mid \text{false} \mid \neg c \mid c_1 \land c_2 \mid c_1 \lor c_2 \mid c_1 \Rightarrow c_2 \mid c_1 = c_2 \mid \ite(c, c_1, c_2) \mid t = \ite(c, t_1, t_2)
d ::= t_1 = t_2 \mid t_1 < t_2 \mid t_1 \leq t_2 \mid t_1 \geq t_2 \mid t_1 > t_2
\]

where: (i) \(B\) is a boolean variable, (ii) \(\neg, \land, \lor, \Rightarrow\) denote negation, conjunction, disjunction, and implication, respectively, (iii) the ternary function \(\ite\) denotes the if-then-else operator, and (iv) \(t\) is a LIA term of the form \(a_0 + \ldots + a_n \times t \)
morphisms will be shown in the following section. Defined by instances of schemata (A) and (B), respectively. Examples of catamorphisms in \( \sigma \rightarrow \tau \) are distinguished variables, and \( \cdot \rightarrow \cdot \) is the sort of lists of elements of sort \( \cdot \). The equality symbol will be used both for integers and booleans. We will often write \( B = \text{true} \) (or \( B = \text{false} \)) as \( B \) (or \( \sim B \)). The theory of LIA and boolean constraints will be denoted by \( \text{LIA} \cup \text{Bool} \). The integer and boolean sorts are said to be basic sorts. A recursively defined sort (such as the sort of lists and trees) is said to be an algebraic data type (ADT, for short).

An atom is a formula of the form \( p(t_1, \ldots, t_m) \), where \( p \) is a predicate symbol not occurring in \( \text{LIA} \cup \text{Bool} \), and \( t_1, \ldots, t_m \) are first order terms in \( L \). A constrained Horn clause (CHC), or simply, a clause, is an implication of the form \( H \leftarrow c, G \). The conclusion \( H \), called the head, is either an atom or false, and the premise, called the body, is the conjunction of a constraint \( c \) and a conjunction \( G \) of zero or more atoms. A clause is said to be a query if its head is false, and a definite clause, otherwise. Without loss of generality, we assume that every atom of the body of a clause has distinct variables (of any sort) as arguments. The set of all variables occurring in an expression \( e \) is denoted by \( \text{vars}(e) \). By \( \text{bvars}(e) \) (or \( \text{adt-vars}(e) \)) we denote the set of variables in \( e \) whose sort is a basic sort (or an ADT sort). The universal closure of a formula \( \varphi \) is denoted by \( \forall(\varphi) \).

Let \( D \) be the usual interpretation for the symbols of theory \( \text{LIA} \cup \text{Bool} \). By \( M(P) \) denote the least \( D \)-model of a set \( P \) of definite clauses \([17]\).

Now, in order to characterize the class of queries that can be handled using our transformation technique, we introduce (see Definition \([1]\) below) a class of recursive schemata defined by CHCs \([8]\). That class is related to those of morphisms, catamorphisms, and paramorphisms considered in functional programming \([13, 20]\). We will not introduce a new terminology here and we will refer to our schemata as generalized catamorphisms, or catamorphisms, for short.

Let \( f \) be a predicate symbol with \( m+n \) arguments (for \( m \geq 0 \) and \( n \geq 0 \)) whose sorts are \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \), respectively. We say that \( f \) is a functional predicate from \( \alpha_1 \times \ldots \times \alpha_m \) to \( \beta_1 \times \ldots \times \beta_n \), with respect to a set \( P \) of definite clauses, if \( M(P) \models \forall X, Y, Z. (f(X, Y) \land f(X, Z) \rightarrow (Y = Z)) \), where \( X \) is an \( m \)-tuple of distinct variables, and \( Y \) and \( Z \) are \( n \)-tuples of distinct variables. Given the atom \( f(X, Y) \), we say that \( X \) and \( Y \) are the tuples of the input and output variables of \( f \), respectively. Predicate \( f \) is said to be total if \( M(P) \models \forall X \exists Y. f(X, Y) \). In what follows, a ‘total, functional predicate’ \( f \) from a tuple \( \alpha \) of sorts to a tuple \( \beta \) of sorts will be called a ‘total function’ and denoted by \( f \in [\alpha \rightarrow \beta] \) (the set \( P \) of clauses that define \( f \) will be understood from the context).

**Definition 1 (Generalized Catamorphisms).** A generalized list catamorphism, shown in Figure \([1] (A)\), is a total function \( h \in [\sigma \times \text{list}(\beta) \rightarrow \varrho] \), where: (i) \( \sigma, \beta, \varrho, \) and \( \tau \) are (products of) basic sorts, (ii) \( \text{list}(\beta) \) is the sort of lists of elements of sort \( \beta \), (iii) \( \text{base1} \) is a total function in \([\sigma \rightarrow \varrho] \), (iv) \( f \) is a catamorphism in \([\sigma \times \text{list}(\beta) \rightarrow \tau] \) and (v) \( \text{combine1} \) is a total function in \([\sigma \times \beta \times \varrho \times \tau \rightarrow \varrho] \). Similarly, a generalized tree catamorphism is a total function \( t \in [\sigma \times \text{tree}(\beta) \rightarrow \varrho] \) defined as shown in Figure \([1] (B)\).

Note that the above definition is recursive, that is, the predicates \( f \) and \( g \) are defined by instances of schemata (A) and (B), respectively. Examples of catamorphisms will be shown in the following section.
The predicate `ins` is the result of sorting a list `Ys` by possibly mutually recursive predicates, using a CHC solver. CHC transformations allow us to automatically prove properties expressed by those clauses, which will be called program predicates.

We have that `ins_sort(Xs, Ys, Zs)` holds if `Zs` is the ordered list of integers such that `ins_sort([], Ys, Zs)` holds. The predicate `ins_sort` depends on the predicates `empty_list` and `ord_ins`. We have that `empty_list(L)` holds if the list `L` is empty, and `ord_ins(Y, Xs1, Xs2, Ys, Zs)` holds if: (i) the concatenation of the lists `Xs1` and `Xs2` is ordered, (ii) `Y` is greater than or equal to the last element of `Xs1`, and (iii) `ins_sort(Xs', Ys, Zs)` holds, where `Xs'` is the concatenation of `Xs1` and the ordered list obtained by inserting (`according to the ≤ order`) `Y` into `Xs2`. As usual, `append(Xs, Ys, Zs)` holds if `Zs` is the concatenation of the lists `Xs` and `Ys`, and `snoc(Xs, Y, Zs)` holds if `append(Xs, [Y], Zs)` holds.

It is not immediate to see why the above properties for `ins_sort` and `ord_ins` are valid. This is also due to the fact that the predicates `ins_sort` and `ord_ins` are mutually recursive. The goal of this paper is to present a technique based on CHC transformations that allows us to automatically prove properties expressed by possibly mutually recursive predicates, using a CHC solver.

Let us consider the clauses of Figure 2, which are the result of translating an iterative program for Insertion Sort. (Details on how this translation can be performed are outside the scope of this paper.) The clauses in Figure 2 will be called program clauses and the predicates defined by those clauses will be called program predicates.

1. `ins_sort(Xs, [], Xs).`
2. `ins_sort(Xs, [Y|Ys], S) ←`
   - `empty_list(L),`
   - `ord_ins(Y, L, Xs, Ys, S).`
3. `append([], Ys, Ys).`
4. `append([X|Xs], Ys, [X|Ys]) ← append(Xs, Ys, Zs).`
5. `snoc([], X, [X]).`
6. `snoc([H|T], X, [H|TX]) ← snoc(T, X, TX).`
7. `empty_list([]).`
8. `ord_ins(Y, Xs1, [], Ys, Zs) ←`
   - `snoc(Xs1, Y, Xs1Y),`
   - `ins_sort(Xs1Y, Ys, Zs).`
9. `ord_ins(Y, Xs1, [X|Xs2], Ys, Zs) ←`
   - `Y ≤ X, snoc(Xs1, Y, Xs1Y),`
   - `snoc(Xs1Y, X, Xs1XY),`
   - `append(Xs1XY, Xs2, Xs),`
   - `ins_sort(Xs, Ys, Zs).`
10. `ord_ins(Y, Xs1, [X|Xs2], Ys, Zs) ←`
    - `Y > X, snoc(Xs1, X, Xs1X),`
    - `ord_ins(Y, Xs1X, Xs2, Ys, Zs).`
Now we will present the clauses that formalize the properties we want to show in our Insertion Sort example. This set of clauses is made out of two subsets: (A) a set of definite CHCs that define the catamorphisms used in the queries, and (B) a set of queries that specify the program properties to be shown.

For (A), in Figure 3 we present the definite CHCs defining the three catamorphisms we will use. They are: (i) ordered, which takes as input a list $Ls$ and returns a boolean $B$ such that if $Ls$ is ordered, then $B = true$, otherwise $B = false$; (ii) first, which takes as input a list $Ls$ and returns a boolean $B$ and an element $F$ such that if $Ls$ is empty, then $B = false$ and $F = 0$ (this value for $F$ is an arbitrary integer and will not be used elsewhere), otherwise $B = true$ and $F$ is the head of $Ls$; and (iii) last, which is analogous to first, except that it returns the last element $L$, if any, instead of the first element.

The predicates of these catamorphisms are called property predicates.

11. $ordered([], B) \leftarrow B.$
12. $ordered([H|T], B) \leftarrow B = (B1 \Rightarrow (H \leq F \& B2)).$
13. $first([], B, F) \leftarrow \sim B \& F = 0.$
14. $first([H|T], B, F) \leftarrow B \& F = H.$
15. $last([], B, L) \leftarrow \sim B \& L = 0.$
16. $last([H|T], B, L) \leftarrow B \& L = \text{ite}(B1, L1, H), last(T, B1, L1).$

Fig. 3: Property clauses for Insertion Sort: (A) the catamorphisms.

For (B), we present the set of queries that specify the program properties. We assume that: (i) at most one property is specified for each program predicate, and (ii) the property related to the program predicate $p$ is expressed as an implication of the form: $p(...), cata_1(...), \ldots, cata_n(...) \Rightarrow d$, where the $cata_i(...)$’s are catamorphisms and $d$ is a constraint. This implication can be expressed as a query by adding $\sim d$ to its premise and changing its conclusion to false. For our Insertion Sort example, in Figure 4 we specify four properties by introducing a query for the predicates ins_sort, ord_ins, append, and snoc.

For ins_sort, query $q1$ states that given a list of integers $Ys$, if $Xs$ is ordered and ins_sort($Xs, Ys, Zs$) holds, then $Zs$ is ordered. For ord_ins, query $q2$ states that if the concatenation of the lists $Xs1$ and $Xs2$ is ordered (that is, $Xs1$ and $Xs2$ are ordered, and the last element of $Xs1$ is less than or equal to the first element of $Xs2$), $Y$ is greater than or equal to the last element of $Xs1$, and ord_ins($Y, Xs1, Xs2, Ys, Zs$) holds, then $Zs$ is ordered. For append, query $q3$ states that if $Xs1$ and $Xs2$ are ordered, and the last element of $Xs1$ is less than or equal to the first element of $Xs2$ and append($Xs,Ys,Zs$) holds, then $Zs$ is ordered. For snoc, query $q4$ states that if $Xs$ is ordered, and the last element of $Xs$ is less than or equal to $X$, and snoc($Xs, X, XsX$) holds, then $XsX$ is ordered.

At this point, we can check whether or not the properties expressed by the queries $q1$–$q4$ do hold by checking the satisfiability of the program clauses together with the property clauses. In order to perform that satisfiability check, we have used the state-of-the-art CHC solver SPACER, based on Z3 [15]. SPACER failed to return an answer within five minutes. The weakness of CHC solvers for examples like the one presented here, motivates our technique based on CHC transformation. This technique produces an equisatisfiable set of CHCs whose
(q1) \( \text{false} \leftarrow (B1 \Rightarrow B2), \text{ordered}(Xs, B1), \text{ordered}(Zs, B2), \text{ins}_\text{sort}(Xs, Ys, Zs). \)

(q2) \( \text{false} \leftarrow ((B1 \& B2 \& ((B4 \& B5) \Rightarrow L \leq F) \& (B4 \Rightarrow L \leq Y)) \Rightarrow B3), \text{ordered}(Xs1, B1), \text{ordered}(Xs2, B2), \text{ordered}(Zs, B3), \text{last}(Xs1, B4, L), \text{first}(Xs2, B5, F), \text{ord}_\text{ins}(Y, Xs1, Xs2, Ys, Zs). \)

(q3) \( \text{false} \leftarrow ((B1 \& B2 \& ((B4 \& B5) \Rightarrow L \leq F)) \Rightarrow B3), \text{ordered}(Xs, B1), \text{ordered}(Ys, B2), \text{ordered}(Zs, B3), \text{last}(Xs, B4, L), \text{first}(Ys, B5, F), \text{append}(Xs, Ys, Zs). \)

(q4) \( \text{false} \leftarrow ((B1 \& (B2 \Rightarrow L \leq X)) \Rightarrow B3), \text{ordered}(Xs, B1), \text{last}(Xs, B2, L), \text{ordered}(XsY, B3), \text{snoc}(Xs, X, XsY). \)

Fig. 4: Property clauses for Insertion Sort: (B) the queries.

Satisfiability can be, hopefully, easier to verify. Indeed, in our example, SPACER succeeds to prove the satisfiability of the new set of CHCs produced by our transformation algorithm. In Section 7, we will show that our technique improves the effectiveness of state-of-the-art solvers on a non-trivial benchmark.

4 Catamorphism-based Queries

As already mentioned, the translation of a program verification problem to CHCs usually generates two disjoint sets of clauses: (i) the set of program clauses, and (ii) the set of property clauses, with the associated sets of program predicates, and property predicates. Without loss of generality, we will assume that property predicates may occur in the property clauses only. Moreover, in order to define a class of CHCs where our transformation technique always terminates, we will consider property predicates that are catamorphisms (see Definition 1). In the sequel we need the following definitions. An atom is said to be a program atom (or a catamorphism atom), if its predicate symbol is a program predicate (or a catamorphism, respectively). Recall that when writing a catamorphism atom as \( \text{cata}(X, T, Y) \), we stipulate that \( X \) is the (tuple of its) input basic variable(s), \( T \) is the input ADT variable, and \( Y \) is the (tuple of its) output basic variable(s).

Definition 2. A catamorphism-based query is a query of the form:

\[ \text{false} \leftarrow c, \text{cata}_1(X_1, T_1, Y_1), \ldots, \text{cata}_n(X_n, T_n, Y_n), \text{pred}(Z) \]

where: (i) \( \text{pred} \) is a program predicate and \( Z \) is a tuple of distinct variables, (ii) \( c \) is a constraint such that \( \text{vars}(c) \subseteq \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\} \), (iii) \( \text{cata}_1, \ldots, \text{cata}_n \) are catamorphism atoms, (iv) \( Y_1, \ldots, Y_n \) are pairwise disjoint tuples of distinct variables of basic sort not occurring in \( X_1, \ldots, X_n, Z \), (v) \( T_1, \ldots, T_n \) are ADT variables occurring in \( Z \).

The queries of Figure 4 are examples of catamorphism-based queries. Many interesting program properties can be defined as catamorphism-based queries, although, in general, this might require some ingenuity.

5 Transformation Rules

In this section we present the rules that we use for transforming CHCs. These rules are variants of the usual transformation rules for CHCs (and CLP pro-
A and one-step unfolding produces the following set of CHCs

\[ P \in \{ \text{let } \text{Cls:} \} \]

The goal of the transformation rules is to incorporate catamorphisms into program predicates, that is, to derive for each program predicate \( p \), a new predicate \( newp \) whose definition is given by the conjunction of an atom for \( p \) with the catamorphism atoms needed for showing the satisfiability of the query relative to \( p \). In this section we will indicate how this can be done referring our Insertion Sort example, while in the next section we will present a transformation algorithm to perform this task in an automatic way.

A transformation sequence from \( S_0 \) to \( S_n \) is a sequence \( S_0 \Rightarrow S_1 \Rightarrow \ldots \Rightarrow S_n \) of sets of CHCs such that, for \( i = 0, \ldots, n-1 \), \( S_{i+1} \) is derived from \( S_i \), denoted \( S_i \Rightarrow S_{i+1} \), by performing a transformation step consisting in applying one of the following rules R1–R4.

(R1) **Definition Rule.** Let \( D \) be a clause of the form \( newp(X_1, \ldots, X_k) \leftarrow c, \text{Catas}, A \), where: (1) \( newp \) is a predicate symbol not occurring in the sequence \( S_0 \Rightarrow S_1 \Rightarrow \ldots \Rightarrow S_i \) constructed so far, (2) \( \{X_1, \ldots, X_k\} = \text{vars}(\{\text{Catas}, A\}) \), (3) \( c \) is a constraint such that \( \text{vars}(c) \subseteq \text{vars}(\{\text{Catas}, A\}) \), (4) \( \text{Catas} \) is a conjunction of catamorphism atoms, with \( \text{adt-vars(Catas)} \subseteq \text{adt-vars}(A) \), and (5) \( A \) is a program atom. By **definition introduction** we may add \( D \) to \( S_i \) and get \( S_{i+1} = S_i \cup \{D\} \).

The case where \( A \) is absent is accommodated by considering \( A \) to be \( \text{true}(X) \), which holds for every \( X \) of ADT sort.

For \( j = 0, \ldots, n \), by \( \text{Defs}_j \) we denote the set of clauses, called **definitions**, introduced by rule R1 during the construction of the sequence \( S_0 \Rightarrow S_1 \Rightarrow \ldots \Rightarrow S_j \). Thus, \( \text{Defs}_0 = \emptyset \), and for \( j = 0, \ldots, n \), \( \text{Defs}_j \subseteq \text{Defs}_{j+1} \).

In our Insertion Sort example, the set \( S_0 \) consists of all the clauses shown in Figures 2, 3, and 4, and we start off by introducing the following definition (with constraint \( \text{true} \) ), whose body consists of the atoms in the body of query \( q_1 \):

\[ D_1. \text{new1}(X_s, B_1, Z_s, B_2, Y_s) \leftarrow \text{ordered}(X_s, B_1), \text{ordered}(Z_s, B_2), \text{ins}_{\text{sort}}(X_s, Y_s, Z_s) \]

Thus, \( S_1 = S_0 \cup \{D_1\} \).

The clauses for \( newp \) are obtained by first (i) **unfolding** the definition of \( newp \), then (ii) incorporating some catamorphisms and constraints provided by the queries into the clauses derived by unfolding, and (iii) finally, folding using suitable new definitions. Now, we introduce an unfolding rule (see R2 below), which actually is the composition of the unfolding and the application of the functionality property presented in previous work. Let us first define the notion of the one-step unfolding which is a step of symbolic evaluation performed by applying once the resolution rule.

**Definition 3 (One-step Unfolding).** Let \( C : H \leftarrow c, L, A, R \) be a clause, where \( A \) is an atom, and let \( P \) be a set of definite clauses with \( \text{vars}(C) \cap \text{vars}(P) = \emptyset \). Let \( \text{Cls:} \{K_1 \leftarrow c_1, B_1, \ldots, K_m \leftarrow c_m, B_m\} \), with \( m \geq 0 \), be the set of clauses in \( P \), such that: for \( j = 1, \ldots, m \), (i) there exists a most general unifier \( \vartheta_j \) of \( A \) and \( K_j \), and (ii) the conjunction of constraints \( (c, c_j)\vartheta_j \) is satisfiable. The one-step unfolding produces the following set of CHCs:
For instance, in our Insertion Sort example, by unfolding definition $k$ for $C$ based queries, and let the unfolding

Then, by $(R3)$ in the set of CHCs whose satisfiability is under verification for strengthening $E$

Step 2 of the unfolding rule is not performed in this example. At Step 3, clause $S$ is replaced by:

Step 2. (Unfolding of the formation atoms)

while there exists a clause $E$: $H \leftarrow d, L, C, R$ in $UnfCls$, for some conjunctions $L$ and $R$ of atoms, such that $C$ is a catamorphism atom whose argument of ADT sort is not a variable do

Step 3. (Applying Functionality)

while there exists a clause $E$: $H \leftarrow d, L, cat(a(X, T, Y1), cat(a(X, T, Y2), R$ in $UnfCls$, for some catamorphism $cata$ do

Then, by unfolding $D$ we derive $S_{i+1} = (S_i \setminus \{D\}) \cup UnfCls$.

For instance, in our Insertion Sort example, by unfolding definition $D1$, at Step 1 we replace $D1$ by:

E1. new1($A, B, A, C, []$) \leftarrow ordered($A, B$), ordered($A, C$).

E2. new1($A, B, C, D, [EF]$) \leftarrow ordered($A, B$), ordered($C, D$), empty_list($G$), ord_ins($E, G, A, F, C$).

Step 2 of the unfolding rule is not performed in this example. At Step 3, clause $E1$ is replaced by:

E3. new1($A, B, A, C, []$) \leftarrow $B \equiv C$, ordered($A, B$).

Thus, $S_2 = S_0 \cup \{E2, E3\}$.

The query-based strengthening rule allows us to use the queries occurring in the set of CHCs whose satisfiability is under verification for strengthening the body of the other clauses with the addition of catamorphism atoms and constraints.

(R3) Query-based Strengthening Rule. Let $S_i = P \cup Q$, where $P$ is a set of definite clauses obtained by applying the unfolding rule, and $Q$ is a set of catamorphism-based queries, and let $C: H \leftarrow c, Catas^C, A_1, \ldots, A_m$ be a clause in $P$, being the $A_i$’s program atoms. Let $E$ be the clause derived from $C$ as follows:

for $k = 1, \ldots, m$ do

- consider program atom $A_k$: let Catas$^C_k$ be the conjunction of every catamorphism atom $F$ in Catas$^C$ such that adt-vars($A_k$) \cap adt-vars($F$) \neq \emptyset;

- if in $Q$ there exists a query (modulo variable renaming)

$g_k: false \leftarrow c_k, cata_1(X_1, T_1, Y_1), \ldots, cata_n(X_n, T_n, Y_n), A_k$

where $Y_1, \ldots, Y_n$ do not occur in $C$, and the conjunction $cata_1(X_1, T_1, Y_1), \ldots, cata_n(X_n, T_n, Y_n)$ can be split into two subconjunctions $B_1$ and $B_2$ such that: (i) a variant of $B_1$ is a subconjunction of Catas$^C_k$, and (ii) for every catamorphism atom $cata_i(X_i, T_i, Y_i)$ in $B_2$ there is no catamorphism atom $cata_i(V, T_i, W)$ in Catas$^C_k$

then add the conjunction $\sim c_k, B_2$ to the body of $C$. 

Then, by *query-based strengthening of clause C using queries* $q_1, \ldots, q_m$ (some of these queries may be absent), we get the new set $S_{i+1} = (S_i \setminus \{C\}) \cup \{E\}$.

For instance, from clause $E_2$ by query-based strengthening using $q_2$ (note that in $E_2$ the program atom empty_list has no associated query), we get:

$E_4. \ \text{new1}(A, B, C, D, [E\mid F]) \leftarrow$

\begin{align*}
& (L \& B \& ((J \& H) \Rightarrow K \leq I) \& (J \Rightarrow K \leq E)) \Rightarrow D, \\
& \text{ordered}(A, B), \text{ordered}(C, D), \\
& \text{ordered}(G, L), \text{last}(G, J, K), \text{first}(A, H, I), \\
& \text{empty_list}(G), \text{ord_ins}(E, G, A, F, C),
\end{align*}

where the subconjunction $B_1$ mentioned in Rule R3 is in line $(B_1)$, while the subconjunction $B_2$ is in line $(B_2)$. Thus, $S_3 = S_0 \cup \{E_3, E_4\}$. □

The *folding rule* allows us to replace a conjunction of a program atom and catamorphisms by a single atom, whose predicate has been introduced in a previous application of the Definition Rule.

(RA4) *Folding Rule*. Let $C: H \leftarrow c, \text{Catas}^C, A_1, \ldots, A_m$ be a clause in $S_i$, where either $H$ is false or $C$ has been obtained by the unfolding rule, possibly followed by query-based strengthening. For $k = 1, \ldots, m$,

- let $\text{Catas}^C_k$ be the conjunction of every catamorphism atom $F$ in $\text{Catas}^C$ such that $\text{adt-vars}(A_k) \cap \text{adt-vars}(F) \neq \emptyset$;
- let $D_k: H_k \leftarrow d_k, \text{Catas}^D_k, A_k$ be a clause in $\text{Defs}_i$ (modulo variable renaming)
  such that: (i) $\mathbb{D} \models \forall(c \rightarrow d_k)$, and (ii) $\text{Catas}^C_k$ is a subconjunction of $\text{Catas}^D_k$.

Then, by folding $C$ using $D_1, \ldots, D_m$, we derive clause $E: H \leftarrow c, H_1, \ldots, H_m$, and we get $S_{i+1} = (S_i \setminus \{C\}) \cup \{E\}$.

In order to fold clause $E_4$, we introduce two new definitions, one for each program atom occurring in the body of that clause, as follows (the predicate names are introduced by our tool):

$D_2. \ \text{new2}(A, B, C, D, E, F, G, H, I, J, K, L) \leftarrow \text{ordered}(A, B), \text{ordered}(C, D), \\
\text{ordered}(E, F), \text{last}(A, G, H), \text{first}(C, I, J), \text{ord_ins}(K, A, C, L, E).$

$D_3. \ \text{new19}(A, B, C, D) \leftarrow \text{ordered}(A, B), \text{last}(A, C, D), \text{empty_list}(A).$

Thus, $S_4 = S_0 \cup \{E_3, E_4, D_2, D_3\}$.

Now, we apply Rule R4, and from clause $E_4$ we get:

$E_5. \ \text{new1}(A, B, C, D, [E\mid F]) \leftarrow (G \& B \& ((H \& I) \Rightarrow J \leq K) \& (H \Rightarrow J \leq E)) \Rightarrow D, \\
\text{new2}(L, G, A, B, C, D, H, J, I, K, E, F), \text{new19}(L, G, H, J).$

Then, $S_5 = S_0 \cup \{E_3, E_5, D_2, D_3\}$. Also, query $q_1$ can be folded using definition $D_1$, and we get:

$E_6. \ \text{false} \leftarrow \sim (B_1 \Rightarrow B_2), \text{new1}(Xs, B_1, Zs, B_2, Ys).$

Thus, $S_6 = (S_0 \setminus \{q_1\}) \cup \{E_3, E_5, E_6, D_2, D_3\}$. Then, the transformation will continue by looking for the clauses relative to the newly introduced predicates new1 (see Definition D2) and new19 (see Definition D3).

□

The key for understanding our transformation technique is to observe that in our Insertion Sort example, the query $E_6$ and the clauses $E_3$ and $E_5$ that define the new predicate new1, incorporate the program predicate ins_sort together with the catamorphisms that are used in $q_1$. The advantage of performing this transformation is that the solver can look for a model of new1($Xs, B1, Zs, B2, Ys$).
where the constraint $B1 \Rightarrow B2$ holds, instead of looking in a separate way for models of $\text{ordered}(Xs,B1)$, $\text{ordered}(Zs,B2)$, and $\text{ins\_sort}(Xs,Ys,Zs)$ whose conjunction implies $B1 \Rightarrow B2$. Note also that clause E5 has constraints and catamorphisms that come from query-based strengthening. Indeed, E5 is obtained by folding E4 derived by strengthening $E2$ using query $q2$.

The following Theorem 1 whose proof sketch is given in Appendix 1, states the correctness of the transformation rules.

**Theorem 1 (Soundness and Completeness of the Rules).** Let $S_0 \Rightarrow S_1 \Rightarrow \ldots \Rightarrow S_n$ be a transformation sequence using rules R1–R4. Then, $S_0$ is satisfiable if and only if $S_n$ is satisfiable.

Note that the applicability conditions of R3 disallow the application of the rule to a query. Otherwise, we could easily get a satisfiable clause from an unsatisfiable one. Indeed, we could transform $\text{false} \leftarrow \pi(Y) \triangleright \text{cata}(X,Y), p(X)$ into $\text{false} \leftarrow \sim \pi(Y) \& \pi(Y), \text{cata}(X,Y), p(X)$. Note also that folding a clause using itself is not allowed, thus avoiding the transformation of $H \leftarrow c, \text{Catas}, A$ into the trivially satisfiable clause $H \leftarrow c, H$.

The applicability conditions of the rules force a sequence of the transformation rules which is fixed, if the new definitions to be introduced are known. The algorithm that we will present in the next section shows how these definitions can be introduced in an automatic way.

### 6 Transformation Algorithm

In this section we present an algorithm, called $T_{mq}$, which given a set $P$ of definite clauses and a set $Q$ of queries, introduces a set of new predicates and transforms $P \cup Q$ into a new set $P' \cup Q'$ such that: (i) $P \cup Q$ is satisfiable if and only if $P' \cup Q'$ is so, and (ii) each new predicate defined in $P' \cup Q'$ is equivalent to the conjunction of a program predicate and some catamorphisms needed for checking the satisfiability of the queries in $Q$. As an effect of the application of the query-based strengthening rule, the transformed clauses also exploit the interdependencies among the queries in $Q$. This transformation is effective, in particular, in the presence of mutually recursive predicates, like in our Insertion Sort example, where we are able to get new clauses for checking the satisfiability of the query $q1$ for $\text{ins\_sort}$ that take into account the constraints and catamorphisms of the query $q2$ for $\text{ord\_ins}$, and vice versa.

The set of new definitions needed by $T_{mq}$ is computed as the least fixpoint of an operator $\tau_{P,Q}$ that transforms a set $\Delta$ of definitions into a new set $\Delta'$. For introducing that operator, we need some preliminary definitions and functions.

**Definition 4.** A generalization of a pair $(c_1, c_2)$ of constraints is a constraint, denoted $\alpha(c_1, c_2)$, such that $D \models \forall(c_1 \rightarrow \alpha(c_1, c_2))$ and $D \models \forall(c_2 \rightarrow \alpha(c_1, c_2))$. The projection of a constraint $c$ onto a tuple $V$ of variables is a constraint $\pi(c, V)$ such that: (i) $\text{vars}(\pi(c, V)) \subseteq V$ and (ii) $D \models \forall(c \rightarrow \pi(c, V))$.

A set $\Delta$ of definitions is monovariant if it contains at most one definition for each program predicate.
Definition 5. Let \( D_1: \text{newp}1(U_1) \leftarrow c_1, \text{Catas}_1, p(Z) \) and \( D_2: \text{newp}2(U_2) \leftarrow c_2, \text{Catas}_2, p(Z) \) be two definitions for the same predicate \( p \). We say that \( D_2 \) is an extension of \( D_1 \), written \( D_1 \sqsubseteq D_2 \), if (i) \( \text{Catas}_1 \) is a subconjunction of \( \text{Catas}_2 \), and (ii) \( \mathbb{D} \models \forall(c_1 \rightarrow c_2) \). Let \( \Delta_1 \) and \( \Delta_2 \) be two monovariant sets of definitions. We say that \( \Delta_2 \) is an extension of \( \Delta_1 \), written \( \Delta_1 \sqsubseteq \Delta_2 \), if for each \( D_1 \) in \( \Delta_1 \) there exists \( D_2 \) in \( \Delta_2 \) such that \( D_1 \sqsubseteq D_2 \).

Given a set \( \text{Cls} \) of clauses and a set \( \Delta \) of definitions, the \textit{Define} function (see Figure 5) derives a set \( \Delta' \) of definitions that can be used for folding all clauses in \( \text{Cls} \). If \( \Delta \) is monovariant, then also \( \Delta' \) is monovariant. In particular, due to the (Project) case, \( \Delta' \) contains a definition for each program predicate occurring in the body of clauses in \( \text{Cls} \). Due to the (Extend) case, \( \Delta \sqsubseteq \Delta' \).

**Function** \textit{Define}(\( \text{Cls}, \Delta \)): a set \( \text{Cls} \) of clauses; a monovariant set \( \Delta \) of definitions. \textit{Define}(\( \text{Cls}, \Delta \)) returns a monovariant set \( \Delta' \) of new definitions computed as follows.

\[
\Delta' := \Delta;
\]

\textbf{for each clause} \( C: H \leftarrow c, G \) in \( \text{Cls} \) \textbf{do}

\textbf{for each program atom} \( A \) in \( G \) \textbf{do}

- let \( \text{Catas}_A \) be the conjunction of every catamorphism atom \( F \) in \( G \) such that \( \text{adl-vars}(A) \cap \text{adl-vars}(F) \neq \emptyset \)

- \textbf{if} (\( \text{Skip} \)) \textbf{then}

  - \( (\text{Skip}) \): \text{if} in \( \Delta' \) there is a clause \( \text{newp}(U) \leftarrow d, B, A \), for any conjunction \( B \) of catamorphism atoms, such that: (i) \( \text{Catas}_A \) is a subconjunction of \( B \), and (ii) \( \mathbb{D} \models \forall(c \rightarrow d) \), \textbf{then} \textbf{skip}.

- \textbf{else if} \( (\text{Extend}) \) \textbf{then}

  - \( (\text{Extend}) \): \text{if} the definition for the predicate of \( A \) in \( \Delta' \) is the clause \( D: \text{newp}(U) \leftarrow d, B, A \), where \( B \) is a conjunction of catamorphism atoms, and either (i) \( \text{Catas}_A \) is not a subconjunction of \( B \), or (ii) \( \mathbb{D} \not\models \forall(c \rightarrow d) \), \textbf{then} introduce definition \( \text{Ext}D: \text{extp}(V) \leftarrow \alpha(d, c), A, B' \), where: (i) \( \text{extp} \) is a new predicate symbol, (ii) \( V = \text{vars}(\{\alpha(d, c), B', A\}) \), and \( B' \) is the conjunction of the distinct catamorphism atoms occurring either in \( B \) or in \( \text{Catas}_A \);

  - \( \Delta' := (\Delta' \setminus \{D\}) \cup \{\text{Ext}D\} \);

- \textbf{else if} \( (\text{Project}) \) \textbf{then}

  - \( (\text{Project}) \): \text{if} there is no clause in \( \Delta' \) of the form \( K \leftarrow d, B, A \), for any conjunction \( B \) of catamorphism atoms,

  - \textbf{then} introduce definition \( D: \text{newp}(U) \leftarrow \pi(c, I), A, \text{Catas}_A \), where: (i) \( \text{newp} \) is a new predicate symbol, (ii) \( I \) are the input variables of basic sort in \( \{A, \text{Catas}_A\} \), and (iii) \( U = \text{vars}(\{\pi(c, I), A, \text{Catas}_A\}) \);

  - \( \Delta' := \Delta' \cup \{D\} \);

**Function** \textit{Unfold}(\( \Delta, P \)): a set \( \Delta = \{D_1, \ldots, D_n\} \) of definitions; a set \( P \) of definite clauses. \( \text{Unfold}(\Delta, P) = \bigcup_{i=1}^{n} C_i \), where \( C_i \) is the set of clauses derived by unfolding \( D_i \).

**Function** \textit{Strengthen}(\( \text{Cls}, Q \)): a set \( \text{Cls} = \{C_1, \ldots, C_n\} \) of clauses; a set \( Q \) of catamorphism-based queries, at most one query for each program predicate in \( \text{Cls} \). \( \text{Strengthen}(\text{Cls}, Q) = \{E_i | E_i \text{ is derived from } C_i \text{ by query-based strengthening using } Q\} \)

**Function** \textit{Fold}(\( \text{Cls}, \Delta \)): a set \( \text{Cls} = \{C_1, \ldots, C_n\} \) of clauses; a monovariant set \( \Delta \) of definitions.

\( \text{Fold}(\text{Cls}, \Delta) = \{E_i | E_i \text{ is derived from } C_i \text{ by folding } C_i \text{ using definitions in } \Delta\} \).

**Fig. 5:** The \textit{Define}, \textit{Unfold}, \textit{Strengthen}, and \textit{Fold} functions.
The *Unfold* and *Strengthen* functions (see Figure 5) apply the unfolding and query-based strengthening rules, respectively, to sets of clauses.

Now, we define the operator $\tau_{P,Q}$ as follows:

$$
\tau_{P,Q}(\Delta) = \begin{cases} 
\text{Define}(Q, \emptyset) & \text{if } \Delta = \emptyset \\
\text{Define}(\text{Strengthen}(\text{Unfold}(\Delta, P), Q), \Delta) & \text{otherwise}
\end{cases}
$$

In the case where $\Delta$ is the empty set of definitions, $\tau_{P,Q}(\Delta)$ introduces by the *Define* function (Project case), a new definition for each program predicate occurring in a query in $Q$. In the case where $\Delta$ is not empty, $\tau_{P,Q}(\Delta)$ is an extension of $\Delta$ obtained by first unfolding all clauses in $\Delta$, then applying the query-based strengthening rule to the clauses derived by unfolding, and finally applying the *Define* function.

The *Define* function is parametric with respect to the generalization operator $\alpha$ (see the Extend case). In our implementation we use an operator based on widening that ensures stabilization, that is, for any infinite sequence $c_0, c_1, \ldots$ of constraints and any sequence defined as (i) $d_0 = c_0$, and (ii) $d_{k+1} = \alpha(d_k, c_{k+1})$, there exists $m \geq 0$ such that $d_m = d_{m+1}$. The *Strengthen* function ensures that there is a bound on the number of catamorphisms that can be present in a definition. Since, as already mentioned, $\tau_{P,Q}$ is monotonic with respect to $\subseteq$, its least fixpoint $\mathit{lfp}(\tau_{P,Q})$ is equal to $\tau_{P,Q}^n(\emptyset)$, for some finite number $n$ of iterations of $\tau_{P,Q}$. Note that, by construction, $\mathit{lfp}(\tau_{P,Q})$ is monovariant.

Once we have computed the set $\mathit{lfp}(\tau_{P,Q})$ of definitions, we can use them to fold all clauses derived by unfolding and strengthening by applying the *Fold* function (see Figure 5). Thus, the transformation algorithm is defined as follows:

$$
\mathcal{T}_{mq}(P, Q) = \text{Fold}(\text{Strengthen}(\text{Unfold}(\mathit{lfp}(\tau_{P,Q}), P), Q), \mathit{lfp}(\tau_{P,Q}))
$$

Termination of $\mathcal{T}_{mq}$ follows immediately from the fact that $\mathit{lfp}(\tau_{P,Q})$ is computed in a finite number of steps.

**Theorem 2 (Termination of Algorithm $\mathcal{T}_{mq}$).** Let $P$ be a set of definite clauses and $Q$ a set of catamorphism-based queries. Then, Algorithm $\mathcal{T}_{mq}$ terminates for $P$ and $Q$.

By the soundness and completeness of the transformation rules (see Theorem 1), we also get the following result.

**Theorem 3 (Soundness and Completeness of Algorithm $\mathcal{T}_{mq}$).** For any set $P$ of definite clauses and $Q$ of catamorphism-based queries, $P \cup Q$ is satisfiable if and only if $\mathcal{T}_{mq}(P, Q)$ is satisfiable.

We conclude this section by showing the sequence of definitions for the program predicate *snoc* computed by iterating the applications of $\tau_{P,Q}$ in the Insertion Sort example. The definitions in $\mathit{lfp}(\tau_{P,Q})$ are listed in Appendix 2.

- new4($A, B, C, D, E, F, G$) $\leftarrow \text{ordered}(A, B)$, $\text{last}(A, C, D)$, $\text{ordered}(E, F)$, $\text{snoc}(A, G, E)$.
- new5($A, B, C, D, E, F, G, H, I, J, K$) $\leftarrow \text{ordered}(E, J)$, $\text{last}(E, F, G)$, $\text{ordered}(A, D)$, $\text{first}(E, H, I)$, $\text{first}(A, B, C)$, $\text{snoc}(E, K, A)$.
- new13($A, B, C, D, E, F, G, H, I, J, K, L, M$) $\leftarrow \text{ordered}(A, B)$, $\text{last}(A, C, D)$, $\text{ordered}(E, F)$, $\text{last}(E, G, H)$, $\text{first}(E, J, K)$, $\text{first}(A, L, M)$, $\text{snoc}(A, I, E)$.

Note that these three definitions are in the $\subseteq$ relation.
7 Experimental Evaluation

We have implemented algorithm $T_{mq}$ in a tool, called VeriCaT$_{mq}$, which extends VeriCaT [8] by guaranteeing a sound and complete transformation. VeriCaT$_{mq}$ is based on (i) VeriMAP [7] for transforming CHCs, and (ii) SPACER (with Z3 4.11.2) to check the satisfiability of the transformed CHCs.

We have considered 170 problems, as sets of CHCs, with 470 queries in total, equally divided between the class of satisfiable (sat) problems and unsatisfiable (unsat) ones (85 problems and 235 queries for each class). These problems are related to programs that manipulate: (i) lists of integers by performing concatenation, permutation, reversal and sorting, and (ii) binary search trees, by inserting and deleting elements. For list manipulating programs, we have considered properties such as: list length, minimum and maximum element, sum of elements, list content as sets or multisets of elements, and list sortedness (in ascending or descending order). For trees, we have considered size, height, minimum and maximum element, tree content and the binary search tree property.

The problems considered here are derived from those of the benchmark set of previous work [8] with some important differences. We have considered additional satisfiable problems (for instance, those related to Heapsort). In addition to satisfiable problems, we have also considered unsatisfiable problems that have been obtained from their satisfiable counterparts by introducing bugs in the programs: for instance, by not inserting an element in a list, or adding an extra constraint, or replacing a non-empty tree by an empty one. Note also that the transformed CHCs produced by $T_{mq}$ contain both basic variables and ADT variables, whereas those produced by the method presented in previous work [8] contain basic variables only.

For comparing the effectiveness of our method with that of a state-of-the-art CHC solver, we have also run SPACER (with Z3 4.11.2) on the original, non-transformed CHCs, in SMT-LIB format. In Table 1 we summarize the results of our experiments. The first three columns report the name of the program, the total number of problems and queries for each program. The fourth and fifth columns report the number of satisfiable and unsatisfiable problems proved by SPACER before transformation, whereas the last two columns report the number of satisfiable and unsatisfiable problems proved by VeriCaT$_{mq}$.

In summary, VeriCaT$_{mq}$ was able to prove all the 170 considered problems whereas SPACER was able to prove the properties of 84 ‘unsat’ problems out of 85, and none of the ‘sat’ problems. The total time needed for transforming the CHCs was 275 seconds (1.62 s per problem, on average), and checking the satisfiability of the transformed CHCs took about 174 s in total (about 1 s average time, 0.10 s median time). For comparison, SPACER took 30.27 s for checking the unsatisfiability of 84 problems (0.36 s average time, 0.15 s median time). The benchmark and the tool are available at https://fmlab.unich.it/vericatmq.

---

4 Experiments have been performed on an Intel Xeon CPU E5-2640 2.00GHz with 64GB RAM under CentOS with a time limit of 300 s per problem.
For instance, for all the considered list sorting programs (Bubble sort, Heapsort, Insertionsort, Mergesort, Quicksort, Selectionsort and Treesort), VeriCaT\(_{mq}\) was able to prove properties stating that the output list is sorted and has the same multiset of elements of the input list. Similarly, VeriCaT\(_{mq}\) was able to prove that those properties do not hold, if extra elements are added to the output list, or some elements are not copied from the input list to the output list, or a wrong comparison operator is used.

The results obtained by the VeriCaT\(_{mq}\) prototype implementation of our method are encouraging and show that, when used in combination with state-of-the-art CHC solvers, it can greatly improve their effectiveness to prove satisfiability of sets of CHCs with multiple queries, while it does not inhibit their remarkable ability to prove unsatisfiability, although some extra time due to transformation may be required.

### 8 Conclusions and Related Work

Many program verification problems can be translated into the satisfiability problem for sets of CHCs that include more than one query. A notable example is the case where we want to verify the correctness of programs made out of several functions, each of which has its pre-/postconditions [13]. We have proposed an algorithm, called \(T_{mq}\), for transforming a set of CHCs with multiple queries into a new, equisatisfiable set of CHCs that incorporate suitable information about the set of queries contained in the initial set. The advantage gained is that, in order to prove the satisfiability of the transformed CHCs, the CHC solver may exploit the mutual interactions among the satisfiability proofs of the various queries. We have identified a class of queries that specify program properties using catamorphisms on ADTs, such as lists and trees, for which \(T_{mq}\) terminates.
We have implemented algorithm $T_{mq}$ and shown that it improves the effectiveness of the state-of-the-art CHC solver SPACER [18] on a non-trivial benchmark.

Algorithm $T_{mq}$ improves over the transformation algorithm $T_{cata}$ presented in a previous paper [8], which works by eliminating ADTs from sets of CHCs. Instead of the contracts handled by $T_{cata}$, algorithm $T_{mq}$ considers queries, and thus its input is simply a set of CHCs. More importantly, $T_{mq}$ is sound and complete, in the sense that the initial and transformed CHCs are equisatisfiable sets, whereas $T_{cata}$ is only sound (that is, it can be seen as computing an abstraction of the initial clauses), and thus if the transformed clauses are unsatisfiable we cannot infer anything about the satisfiability of the initial CHCs. Completeness is very important in practice, because proving that a set of clauses is unsatisfiable and finding a counterexample can help identify a program bug. The experimental evaluation reported in Section 7 shows that our transformation-based verification technique is able to dramatically improve the effectiveness of the SPACER solver for satisfiable sets of CHCs (where the results of SPACER are very poor), while retaining the excellent results of the solver for unsatisfiable sets of CHCs (for which SPACER is, at least in principle, complete).

Decision procedures for suitable classes of first order formulas defined on catamorphisms [21, 22] have been used in program verifiers [23]. However, we do not propose here any specific decision procedure for catamorphisms and, instead, we transform a set of CHCs with catamorphisms into a new set of CHCs where catamorphisms are, in a sense, compiled away.

Type-based norms, which are a special kind of integer-valued catamorphisms, were used for proving termination of logic programs [5] and for resource analysis [1] via abstract interpretation. Similar abstract interpretation techniques are also implemented in the CiaoPP preprocessor [14] of the Ciao logic programming system. In our approach we do not need to specify a priori any abstract domain where to perform the analysis, and instead, by transformation, we generate new CHCs which incorporate the relations defined by the constraints on the catamorphisms. The problem of showing the satisfiability of CHCs defined on ADTs is a very hot topic and various approaches have been proposed in recent work, including: (i) a proof system that combines inductive theorem proving with CHC solving [25], (ii) lemma generation based on syntax-guided synthesis from user-specified templates [26], (iii) invariant discovery based on finite tree automata [19], and (iv) use of suitable abstractions [12].

A limitation of our approach is that the effectiveness of the transformation may depend on the set of properties specified through the queries. For instance, it may happen that programmers provide partial program specifications (e.g., for a subset of the program functions), and therefore queries only for some program predicates, such as the main program predicates (e.g., $\text{ins} \_\text{sort}$ and $\text{ord} \_\text{ins}$ of our example). In this case, it is essential to have a mechanism that is able to infer from the queries the unspecified catamorphisms for the remaining program predicates (e.g., $\text{append}$ and $\text{snoc}$ which $\text{ins} \_\text{sort}$ and $\text{ord} \_\text{ins}$ depend on). As future work, we plan to extend $T_{mq}$ to propagate the catamorphisms specified in the queries to those program predicates for which no query has been specified.
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Appendix 1

In this appendix we present a proof sketch of Theorem 1 that states the soundness and completeness of the transformation rules.

Proof sketch. The proof of Theorem 1 is based on the correctness of the transformation rules for (constraint) logic programs [1, 9, 24]. In particular, the addition of catamorphisms performed by the query-based strengthening rule, is sound and complete because the catamorphisms are total, functional relations. The correctness of folding is proved by using a method similar to the one introduced by Tamaki and Sato [24], which relies on two facts: (i) we can associate predicates with levels where program predicates have a higher level than catamorphisms, and (ii) only clauses obtained by unfolding with respect to a program atom are folded.

Appendix 2

In this appendix let us first show the seven definitions that have been introduced during the transformation of the given set of CHCs presented in Figures 2, 3, and 4. Note that in our case, the constraints in these definitions are all true.

D1. new1(A, B, C, D, E) ← ordered(A, B), ordered(C, D), ins_sort(A, E, C).
D2. new2(A, B, C, D, E, F, G, H, I, J, K, L) ← ordered(A, B), ordered(C, D), ordered(E, F), last(A, G, H), first(C, I, J), ord_ins(K, A, C, L, E).
new6(A, B, C, D) ← ordered(A, B), first(A, C, D).
new7(A, B, C, D, E, F, G, H, I, J, K, L, M, N) ← first(A, B, C), ordered(A, D), last(E, F, G), first(H, I, J), ordered(H, K), first(E, L, M), ordered(E, N), append(E, A, H).
new13(A, B, C, D, E, F, G, H, I, J, K, L, M) ← ordered(A, B), last(A, C, D), ordered(E, F), last(E, G, H), first(E, J, K), first(A, L, M), snoc(A, I, E).
new17(A, B) ← ordered(A, B).
D3. new19(A, B, C, D) ← ordered(A, B), last(A, C, D), empty_list(A).

Now we list the final set of CHCs which has been derived by our transformation technique using the above definitions. The first four CHCs are the queries derived from the queries q1–q4 of Figure 4 denoting the four properties for Insertion Sort we wanted to prove. We have that this final set of clauses is satisfiable and thus, the four properties are all valid.

E6. false ← ∼(A ⇒ B), new1(C, A, D, B, E),
false ← ∼((A & B & ((C & D) ⇒ E ≤ F) & (C ⇒ E ≤ G)) ⇒ H),
new2(I, A, J, B, K, H, C, E, D, F, G, L).
false ← ∼(A & B & ((C & D) ⇒ E ≤ F) ⇒ G),
new7(H, D, F, B, I, C, E, J, K, L, G, M, N, A),
false ← ∼((A & (B ⇒ C ≤ D)) ⇒ E), new13(F, A, B, C, G, E, H, I, D, J, K, L, M).
new1(A, B, A, B, [ ]) ← new17(A, B).
E5. new1(A, B, C, D, [E | F]) ← (G & B & ((H & I) ⇒ J ≤ K) & (H ⇒ J ≤ E)) ⇒ D,
new2(L, G, A, B, C, D, H, J, I, K, E, F), new19(L, G, H, J).
new2(A, B, [], C, D, E, F, G, H, I, J, K) ← C & ~H & I = 0 &
((B & (F ⇒ G ≤ J)) ⇒ L) & (L ⇒ E), new1(M, L, D, E, K),
new13(A, B, F, G, M, L, N, O, J, P, Q, R, S).
new2(A, B, [C|D], E, F, G, H, I, J, K, L, M) ← (L ≤ C) &
(E = (N ⇒ (C ≤ O & P))) & K = C & ((B & (H ⇒ I ≤ L)) ⇒ Q) &
((Q & (F ⇒ J ≤ C)) ⇒ T) & ((T & P & ((U & N) ⇒ V ≤ O)) ⇒ W) &
(W ⇒ G). new1(X, W, F, G, M).
new7(D, N, O, P, Y, U, V, X, Z, A1, W, B1, C1, T),
new13(D1, Q, R, S, Y, T, U, V, C, E1, F1, G1, H1),
new13(A, B, H, I, D1, Q, R, S, L, I1, J1, K1, L1).
new2(A, B, [C|D], E, F, G, H, I, J, K, L, M) ← (L ≥ C+1) &
(E = (N ⇒ (C ≤ O & P))) & J & K = C & ((B & (H ⇒ I ≤ C)) ⇒ Q) &
((Q & P & ((R & N) ⇒ S ≤ O)) & (R ⇒ S ≤ L)) ⇒ G),
new2(T, Q, D, P, F, G, R, S, N, O, L, M),
new13(A, B, H, I, D, Q, R, S, C, U, V, W, X).
new6([], A, B, C) ← A & ~B & C = 0.
new6([], [B], C, D, E) ← C = (F ⇒ (A ≤ G & H)) & D & E = A, new6(B, H, F, G).
new7(A, B, C, D, [], E, F, A, B, C, D, G, H, I) ← ~E & F = 0 & ~G & H = 0 & I,
new6(A, D, B, C).
new7(A, B, C, D, [], E, F, [], G, H, I, J, K, L, M, N, O) ← G & H = ite(P, Q, E) &
J & K = E & L = (R ⇒ (E ≤ S & T)) & M & N = E &
O = (U ⇒ (E ≤ V & W)) & (W & D & ((P & B) ⇒ Q ≤ C)) ⇒ T),
new7(A, B, C, D, F, P, Q, I, R, S, T, U, V, W).
new13([], A, B, C, [], D, E, F, G, D, H, I, J, K) ← A & ~B & C = 0 &
E = (L ⇒ (D ≤ M & N)) & F & G = ite(O, P, D) & H & I = D &
J & K = 0 & N & ~L & M = 0 & ~O & P = 0.
new13([], A, B, C, [E], F, [], G, H, I, J, K, L, M, N) ← C = (O ⇒ (A ≤ P & Q)) &
D & E = ite(R, S, A) & G = (T ⇒ (A ≤ U & V)) & H & I = ite(W, X, A) &
K & L = A & M & N = A & ((Q & (R ⇒ S ≤ J)) ⇒ V),
new13(B, Q, R, S, F, V, W, X, J, T, U, O, P).
new17([], A) ← A.
new17([], [B], C) ← C = (D ⇒ (A ≤ E & F)), new6(B, F, D, E).
new19([], A, B, C) ← A & ~B & C = 0.