Tangential interpolatory projections for structure-preserving model order reduction of large-scale sparse second-order index 1 differential-algebraic equations

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Abstract

This paper studies model order reduction of second-order index-1 descriptor systems using tangential interpolation projection method based on Iterative Rational Krylov Algorithm (IRKA). Our primary focus is to reduce the system into second-order form so that the structure of the original system can be preserved. For this purpose, the IRKA based tangential interpolatory method is modified to deal with the second-order structure of the underlying descriptor system efficiently in an implicit way. The paper also shows that by exploiting the symmetric properties of the system the implementing computational costs can be reduced significantly. Theoretical results are verified for the model reduction of piezo actuator based adaptive spindle support which is second-order index-1 differential algebraic form. The efficiency and accuracy of the method is demonstrated by analyzing the numerical results.

\textbf{keywords:} Interpolatory projections, Iterative Rational Krylov Algorithm, structure-preserving model order reduction, second-order index-1 systems, piezo actuator based adaptive spindle support

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1 Introduction

We discuss the Iterative Rational Krylov Algorithm (IRKA) based tangential interpolation projection technique for the model reduction of second-order differential algebraic equations (DAEs) together with output equation which are given by

\begin{align}
\dot{v}(t) + L_{11}v(t) + K_{11}v(t) + K_{12}\eta(t) &= F_1u(t), \\
K_{21}v(t) + K_{22}\eta(t) &= F_2u(t), \\
H_1v(t) + H_2\eta(t) + D_0\eta(t) &= y(t),
\end{align}

where \( v(t) \in \mathbb{R}^{n_1}, \eta(t) \in \mathbb{R}^{n_2} \) are the states, \( u(t) \in \mathbb{R}^m \) are the inputs and \( y(t) \in \mathbb{R}^p \) are the outputs, and matrices \( M_{11}, L_{11}, K_{11}, K_{12}, K_{21}, \) and \( K_{22} \) are sparse. The matrix \( D_0 \in \mathbb{R}^{p \times m} \) represents the direct feed-through from the input to the output. We consider that number of inputs and outputs is greater than one i.e., the system is multi-inputs and multi-outputs (MIMO). We also assume that the block matrix \( K_{22} \) is non-singular. In the previous literature see, e.g., \cite{1} such system was defined as index-1 system. This system is called symmetric if the matrices \( M_{11}, L_{11}, K_{11}, K_{22} \) and \( D_0 \) are symmetric, and \( K_{21} = K_{12}^T, H_1 = F_1^T \) and \( H_2 = F_2^T \).

Such structure systems arise in many applications, for examples in the modeling of the mechanical and electrical networks (see e.g., \cite{2}) where the constraints are imposed to control the dynamic behavior of the systems, or mechatronics \cite{3} in which mechanical and electrical components are coupled with each other. In the specific case of the model example which is used for our numerical experiments, the index-1 character results from the certain machine tools; adaptive spindle support (ASS) \cite{4,5} based on piezo actuators. See, a bit details in Section 4.1.

If the model is very large, performing the simulation with it has prohibitively expensive computational effort, or is simply impossible due to the limited computer memory. Therefore, we want to approximate a large-scale system by a substantially small-scale system which approximates the main features of the original system but is much faster to evaluate.

Model Order Reduction (MOR) of the index-1 descriptor system \cite{1} was studied in several literature, see, e.g., \cite{6,1,7,8}. All these literature focused onto the system theoretic method Balanced Truncation (BT) considering either second-order-to-first-order or second-order-to-second-order reduction techniques. To implement the method one has to compute and store the Gramian factors of the system. Computing the Gramian factors by solving continuous-time algebraic Lyapunov equations is a huge computational task and often considered as a drawback of the method.

On the other hand Iterative Rational Krylov Algorithm (IRKA) based interpolatory methods as introduced in \cite{9} is computationally efficient. Therefore, recently this method is applied frequently for the model reduction of large-scale dynamical systems. The method was generalized for first-order descriptor system \cite{10}. The idea was also extended in \cite{11,12} for the second-order-to-second-order reduction of second-order standard systems. Authors in \cite{13} discussed this method to obtain reduced first-order state space model from the second-order index-1 system in \cite{1}. Until now there is no investigation of this method for the second-order-to-second-order model reduction of second-order index-1 systems.
This paper is mainly devoted to close this gap.

In this paper we will discuss the Structure-Preserving Model Order Reduction (SPMOR) i.e., second-order-to-second-order model order reduction of the second-order index-1 descriptor systems applying tangential interpolation projection based on IRKA. Generally, second-order index-1 system \([1]\) can be converted into a second-order standard system. Then the proposed method can be applied to the converted system following the procedure as in \([12]\). However, such conversion will destroy the sparsity pattern and turn the system into dense form. The dense system not only consumes a large-scale computer memory but also leads to additional computational complexities. For a large-scale system, like the Adaptive Spindle Support (ASS) model considered in this paper, converting into dense form is forbidden. We develop SPMOR algorithm for the system \([1]\) without converting the system dense form explicitly. For this purpose, the standard IRKA based interpolatory methods as in \([11]\) would be modified to deal with the second-order structure. Many cases, in the real-life applications, see, e.g. \([14, 15]\), the model we use in the numerical experiments, systems are in symmetric form. This paper also shows how to accelerate the computation by exploiting the symmetric properties of the system. The proposed techniques are applied to the large-scale real-life model, piezo actuator based adaptive spindle support. The efficiency of the method is discussed by the numerical results. The results are also compared with that of the Balanced Truncation.

2 IRKA based tangential interpolatory methods

The goal of this section is to review the basic idea of the tangential interpolation techniques based on IRKA from the previous literature. At first we introduce the method for the first-order generalized systems. Then the idea would be generalized for the second-order standard systems. This section also recalls some important definitions and essential notations, theorems etc., that will be used in the next sections.

2.1 Tangential interpolation for first-order systems

We briefly discuss the IRKA based tangential interpolation method for the MIMO generalized state space system

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Da_u(t),
\end{align*}
\]

where \(E \in \mathbb{R}^{k \times k}\) is non-singular, and \(A \in \mathbb{R}^{k \times k}, B \in \mathbb{R}^{k \times p}, C \in \mathbb{R}^{m \times k}\) and \(Da \in \mathbb{R}^{m \times p}\). The transfer-function matrix of this system is defined by \(G(s) = C(sE - A)^{-1}B + Da\), where \(s \in \mathbb{C}\). Applying the tangential interpolatory framework we want to construct a \(r\) dimensional \((r \ll k)\) reduced-order model

\[
\begin{align*}
\dot{\hat{E}} \dot{\hat{x}}(t) &= \hat{A} \hat{x}(t) + \hat{B}u(t), \\
\hat{y}(t) &= \hat{C} \hat{x}(t) + \hat{Da}_u(t),
\end{align*}
\]

such that its transfer function-matrix \(\hat{G}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{Da}\) interpolates the original one, \(G(s)\), at selected points in the complex plane along with selected
directions. The points are called interpolation points and the directions are called tangential directions. We use the procedure illustrated in [11] to make this problem more precisely as follows. Selecting interpolation points \( \{ \alpha_i \}_{i=1}^r \), right tangential directions \( \{ b_i \}_{i=1}^r \) and left tangential directions \( \{ c_i \}_{i=1}^r \) construct an \( n \times r \) projection matrices

\[
V = [(\alpha_1 E - A)^{-1} B b_1, \cdots, (\alpha_r E - A)^{-1} B b_r],
\]

\[
W = [(\alpha_1 E - A)^{-T} C^T c_1, \cdots, (\alpha_r E - A)^{-T} C^T c_r].
\] (4)

The approximating \( x(t) \) by \( V \hat{x}(t) \) and enforcing the Petrov-Galerkin condition provided as

\[
W^T (EV \hat{x}(t) - AV \hat{x}(t) - Bu(t)) = 0, \quad \hat{y}(t) = CV \hat{x}(t) + Du(t),
\]
construct the reduced matrices in [3] as

\[
\hat{E} := W^T EV, \quad \hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad \hat{D}_a := D_a.
\] (5)

The reduced model obtained by this procedure satisfies

\[
G(\alpha_i) b_i = \hat{G}(\alpha_i) b_i, \quad c_i^T G(\alpha_i) b_i = c_i^T \hat{G}(\alpha_i) b_i \quad \text{and} \quad c_i^T G'(\alpha_i) b_i = c_i^T \hat{G}'(\alpha_i) b_i,
\] (6)

for \( i = 1, 2, \ldots, r \), which is known as Hermite bi-tangential interpolation conditions.

The quality of the reduced order model (ROM) can be measured by \( |y - \hat{y}| \), which can, in frequency domain, also be expressed in terms of the transfer function error

\[
\| G(\cdot) - \hat{G}(\cdot) \|.
\] (7)

A common choices for the error norm are the \( \mathcal{H}_\infty \) or \( \mathcal{H}_2 \)-norms (see, e.g. [10]). To minimize the error, choice of interpolation points and tangential directions are crucial tasks. They depend on the reduced-order model; hence are not known priorly. The Iterative Rational Krylov Algorithm (IRKA) introduced in [17] resolves the problem by iteratively correcting the interpolation points and the directions as summarized in Algorithm [11]

### 2.2 Tangential interpolation for standard second-order systems

Let us move to review of second-order linear time invariant (LTI) system

\[
M \ddot{z}(t) + L \dot{z}(t) + K z(t) = Fu(t), \quad y(t) = Hz(t) + D_u u(t),
\] (8)

where \( M, L \) and \( K \) are non-singular, and \( z(t) \) is the \( n \) dimensional state vector. Consider that the system is MIMO, and its transfer function matrix can be defined as

\[
\hat{G}(s) = H(s^2 M + s L + K)^{-1} F + D_a; \quad s \in \mathbb{C}.
\] (9)

Transform the system into a first order form in [12], in which \( x(t) = \begin{bmatrix} \dot{z}(t)^T \ z(t)^T \end{bmatrix}^T \)

and the coefficient matrices are replaced by

\[
\hat{E} := \begin{bmatrix} 0 & M \\ M & L \end{bmatrix}, \quad \hat{A} := \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix}, \quad \hat{B} := \begin{bmatrix} 0 \\ F \end{bmatrix}, \quad \hat{C} := \begin{bmatrix} 0 \\ H \end{bmatrix} \quad \text{and} \quad \hat{D}_a = D_a.
\] (10)
Algorithm 1: IRKA for First-Order MIMO Systems [17].

| Algorithm 1: IRKA for First-Order MIMO Systems [17]. |
|-------------------------------------------------------|
| **Input**: $E, A, B, C, D_a$.                           |
| **Output**: $E, A, B, C, D_a := D_a$.                  |
| 1 Make the initial selection of the interpolation points $\{\alpha_i\}_{i=1}^r$ and the tangential directions $\{b_i\}_{i=1}^r$ and $\{c_i\}_{i=1}^r$. |
| 2 $V = \left[ (\alpha_1 E - A)^{-1} B b_1, \ldots, (\alpha_r E - A)^{-1} B b_r \right]$, & $W = \left[ (\alpha_1 E^T - A^T)^{-1} C^T c_1, \ldots, (\alpha_r E^T - A^T)^{-1} C^T c_r \right]$. |
| 3 **while** (not converged) **do**                      |
| 4     Compute $\hat{E} = W^T EV$, $\hat{A} = W^T AV$, $\hat{B} = W^T B$ and $\hat{C} = CV$. |
| 5     for $i = 1, \ldots, r$. **do**                    |
| 6         Compute $A z_i = \lambda_i \hat{E} z_i$ and $y_i^* \hat{\alpha} = \lambda_i y_i^* \hat{E}$ for $\alpha_i \leftarrow -\lambda_i$, |
| 7     $b_i^* \leftarrow -y_i^* \hat{B}$ and $c_i^* \leftarrow \hat{C} z_i^*$. |
| 8     **end for**                                      |
| 9     **Repeat** Step 2.                              |
| 10   **end while**                                    |
| 11 Construct the reduced-order matrices $\hat{E} = W^T EV$, $\hat{A} = W^T AV$, $\hat{B} = W^T B$ and $\hat{C} = CV$. |

Although there are several first-order representations of the second-order system as shown in [18], we particularly interested to this form (10) since this representation yields first-order symmetric system if $M$, $L$, $K$ and $D_a$ are symmetric, and $F$ and $H$ are transposes of each other. Once system in (8) is converted into the system in (10), Algorithm 1 can be applied to obtain a reduced model. However, the reduced model is first order form and one can not go back to second-order representation since the second-order structure is already disintegrated. Therefore we aim to obtain a $r$ dimensional ($r \ll n$) second-order reduced model

$$\hat{M} \ddot{z}(t) + \hat{L} \dot{z}(t) + \hat{K} z(t) = \hat{F} u(t), \quad \hat{y}(t) = \hat{H} \dot{z}(t) + \hat{D}_a u(t).$$

(11)

where using the projection matrices $V_s, W_s \in \mathbb{R}^{n \times r}$, the coefficient matrices are obtained as follows

$$\hat{M} = W_s^T M V_s, \quad \hat{L} = W_s^T L V_s, \quad \hat{K} = W_s^T K V_s,$n

$$\hat{F} = W_s^T F, \quad \hat{H} = H V_s \quad \text{and} \quad \hat{D}_a := D_a.$$ (12)

We want to achieve this by applying tangential interpolatory techniques. It can be shown that the transfer-function matrix of the second-order system (8) is coincided with the transfer-function matrix of its first-order representation in (10), i.e.,

$$\hat{G}(s) = H(s^2 M + sL + K)^{-1} F + D_a = \hat{C}(s \hat{E} - \hat{A})^{-1} \hat{B} + \hat{D}_a.$$

Therefore, based on the discussion above the interpolatory projection method can directly be applied to (8) for the reduced model in (11). Considering interpolation points $\{\alpha_i\}_{i=1}^r$, right tangential directions $\{b_i\}_{i=1}^r$ and left tangential directions $\{c_i\}_{i=1}^r$ construct $V_s$ and $W_s$ as follows

$$V_s = \left[ (\alpha_1^2 M + \alpha_1 L + K)^{-1} F b_1, \ldots, (\alpha_r^2 M + \alpha_r L + K)^{-1} F b_r \right].$$

$$W_s = \left[ (\alpha_1^2 M + \alpha_1 L + K)^{-T} H^T c_1, \ldots, (\alpha_r^2 M + \alpha_r L + K)^{-T} H^T c_r \right].$$ (13)
If the reduced-order model \(11\) is constructed by \(V_s\) and \(W_s\), the reduced transfer-function matrix \(\hat{\tilde{G}}(s) = \tilde{H}(s^2\tilde{M} + s\tilde{L} + \tilde{K})^{-1}\tilde{F} + \tilde{D}_a\) tangentially interpolates \(\tilde{G}(s)\) satisfying the interpolation conditions as in (6).

In some articles, see, e.g., [19, 12] the SPMOR of second-order system via tangential interpolations was discussed from the first-order representations as in (10). There the authors discussed that due to the structure of the system the projectors \(V, W \in \mathbb{R}^{2n \times r}\) as computed in Algorithm 1 can be intersected into two equal parts. Then the reduced order model \(11\) can be constructed by using those partitions as left and right projection matrices. See, for examples [12] details.

Note that if the second-order system \(8\) is symmetric the projection matrices \(V_s\) and \(W_s\) are coincided. In that case we can reduced the computational cost to construct the reduced models. Another important issue for the SPMOR is to update the interpolation points and tangential directions. We leave this to discuss in the next section.

3 SPMOR for second-order index-1 descriptor systems

In this section our goal is to develop interpolatory projections for SPMOR of second-order index-1 DAEs \(1\). In Section 1 we already have mentioned that second-order index-1 DAEs can be converted into second-order standard system. In a large-scale system, this conversion is however infeasible. This section mainly devoted without such converting how to apply the tangential interpolatory methods for the SPMOR of second-order DAEs.

IRKA based sparse tangential interpolation. Recall the second-order index-1 system \(1\), and rewrite the system in Matrix-vector form:

\[
\begin{bmatrix}
M_{11} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{v}(t) \\
\dot{\eta}(t)
\end{bmatrix} +
\begin{bmatrix}
L_{11} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\bar{v}}(t) \\
\dot{\bar{\eta}}(t)
\end{bmatrix} +
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
v(t) \\
\eta(t)
\end{bmatrix} =
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\begin{bmatrix}
u(t) \\
u(t)
\end{bmatrix},
\tag{14a}
\]

\[
y(t) = \begin{bmatrix}
H_1 & H_2
\end{bmatrix}
\begin{bmatrix}
v(t) \\
\eta(t)
\end{bmatrix} + D_a u(t).
\tag{14b}
\]

The transfer-function matrix of the system is defined by

\[
\tilde{G}(s) = \tilde{H}(s^2\tilde{M} + s\tilde{L} + \tilde{K})^{-1}\tilde{F} + \tilde{D}_a.
\tag{15}
\]

From second equation of (14a) we obtain

\[
\eta(t) = -K_{22}^{-1}K_{21}v(t) + K_{22}^{-1}F_2 u(t).
\]

Inserting this identity into the first equation of (14a) and equation (14b), and some algebraic manipulations yields

\[
\tilde{M}\ddot{v}(t) + \tilde{L}\dot{v}(t) + \tilde{K}v(t) = \tilde{F} u(t), \quad \text{and} \quad \tilde{y}(t) = \tilde{H}v(t) + \tilde{D}_a u(t),
\tag{16}
\]
respectively, where

\[
\begin{align*}
\mathcal{M} & := M_{11}, \quad \mathcal{L} := L_{11} \\
\mathcal{K} & := K_{11} - K_{12}K_{22}^{-1}K_{21}, \quad \mathcal{F} := F_1 - K_{12}K_{22}^{-1}F_2, \\
\mathcal{H} & := H_1 - H_2K_{22}^{-1}K_{21}, \quad \mathcal{D}_a := D_a + H_2K_{22}^{-1}F_2.
\end{align*}
\] (17)

This system is continuous-time LTI system and can be compared with the standard second-order system as in [8]. The transfer-function matrix of the system (16) is given by

\[
\mathcal{G}(s) = \mathcal{H}(s^2\mathcal{M} + s\mathcal{L} + \mathcal{K})^{-1}\mathcal{F} + \mathcal{D}_a.
\] (18)

The following observation shows that systems (14) and (16) are equivalent.

**Theorem 1.** The transfer-function matrices \(\bar{\mathcal{G}}(s)\) and \(\mathcal{G}(s)\) as defined in (15) and (18), respectively are equal.

**Proof.** Plugging \(\bar{\mathcal{H}}, \, \bar{\mathcal{M}}, \, \bar{\mathcal{D}}, \, \bar{\mathcal{K}}\) and \(\bar{\mathcal{L}}\) from (14) into (15) we obtain

\[
\bar{\mathcal{G}}(s) = [H_1 \quad H_2] \left( s^2 \begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix} + s \begin{bmatrix} L_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \mathcal{D}_a
\]

\[
= [H_1 \quad H_2] \left[ s^2M_{11} + sL_{11} + K_{11} \begin{bmatrix} K_{12} \\ K_{22} \end{bmatrix} \right]^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \mathcal{D}_a.
\] (19)

Consider that

\[
\begin{bmatrix} s^2M_{11} + sL_{11} + K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},
\]

which leads

\[
\begin{bmatrix} s^2M_{11} + sL_{11} + K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.
\]

This implies

\[
(s^2M_{11} + sL_{11} + K_{11})x_1 + K_{12}x_2 = F_1, \quad K_{21}x_1 + K_{22}x_2 = F_2.
\] (20)  

Equation (21) gives

\[
x_2 = -K_{22}^{-1}K_{21}x_1 + K_{22}^{-1}F_2.
\]

Inserting this identity into Equation (20) we have

\[
x_1 = (s^2M_{11} + sL_{11} + K_{11} - K_{12}K_{22}^{-1}K_{21})^{-1}(F_1 - K_{12}K_{22}^{-1}F_2).
\] (22)

Equation (19) implies

\[
\bar{\mathcal{G}}(s) = [H_1 \quad H_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathcal{D}_a = H_1x_1 + H_2x_2 + \mathcal{D}_a.
\]
Using $x_1$ and $x_2$, and some algebraic manipulations leads the above equation to
\[
\tilde{G}(s) = (H_1 - H_2K_{22}^{-1}K_{21})(s^2M_{11} + sL_{11} + K_{11} - K_{12}K_{22}^{-1}K_{21})^{-1} \\
(F_1 - K_{12}K_{22}^{-1}F_2) + (D_a + H_2K_{22}^{-1}F_2).
\]

Now following (17) we obtain
\[
\tilde{G}(s) = \mathcal{H}(s^2M + sL + K)^{-1}F + D_a,
\]
which leads to the desired conclusion.

We are now ready to discuss the interpolatory methods for second-order descriptor system [1]. In context of Theorem [3] dynamical systems [1], [14] and (16) are equivalent. Therefore, instead of applying the proposed model reduction method onto the descriptor system (1) we can apply the equivalent form (15).

**Theorem 2.** Let $G(s) = G_1(s) + G_2(s)$, where $G_1(s)$ and $G_2(s)$ are the strictly proper and polynomial parts, respectively, be the transfer function matrix of the original system and $\tilde{G}(s) = \hat{G}_1(s) + \hat{G}_2(s)$, where $\hat{G}_1(s)$ and $\hat{G}_2(s)$ are strictly proper and polynomial parts, respectively, be the transfer function matrix of its reduced system. If $\tilde{G}(s)$ minimizes the overall error $\|G - \tilde{G}\|$, then $G_2(s) = \hat{G}_2(s)$ and $G_1(s)$ minimizes the error $\|G_1 - \hat{G}_1\|$.

**Proof.** For a proof see, e.g., [10] Algorithm 4.1.

As a consequence of this theorem, to apply interpolatory tangential methods via IRKA onto (19), the interpolation points and tangential directions are computed based on the strictly proper part of the transfer-function matrix. One has to make sure that in the final reduced-order model has the same polynomial part as the original one. Therefore, we will modify the standard IRKA discussed in Section 2 as follows to meet these changes.

Select a set of interpolation points $\{\alpha_i\}_{i=1}^r$ and left tangential directions $\{c_i\}_{i=1}^r$ and construct $V_s$ and $W_s$ as follows
\[
V_s = [(\alpha_1^2M + \alpha_1L + K)^{-1}FH_1, \ldots, (\alpha_r^2M + \alpha_rL + K)^{-1}FH_r], \\
W_s = [(\alpha_1^2M + \alpha_1L + K)^{-T}HR_1, \ldots, (\alpha_r^2M + \alpha_rL + K)^{-T}HR_r].
\]

Applying $V_s$ and $W_s$ onto the system (16) the following reduced-order model is constructed
\[
\tilde{M}\tilde{v}(t) + \tilde{L}\dot{\tilde{v}}(t) + \tilde{K}\tilde{v}(t) = \tilde{F}u(t), \quad \text{and} \quad \tilde{y}(t) = \tilde{H}\tilde{v}(t) + \tilde{D}_au(t),
\]
(24)

where the reduced matrices are formed as follows
\[
\tilde{M} = W_s^T\mathcal{M}V_s, \quad \tilde{L} = W_s^T\mathcal{L}V_s, \quad \tilde{K} = W_s^T\mathcal{K}V_s, \quad \tilde{F} = W_s^T\mathcal{F}, \quad \tilde{H} = \mathcal{H}V_s \quad \text{and} \quad \tilde{D}_a := D_a.
\]
(25)

These reduced matrices can also be formed using the block matrices from the descriptor system (1) as
\[
\tilde{K}_{11} = W_s^TK_{11}V_s, \quad \tilde{K}_{12} = W_s^TK_{12}, \quad \tilde{K}_{21} = K_{21}V_s, \quad \tilde{F}_1 = W_s^TF_1, \quad \tilde{H}_1 = H_1V_s \\
\tilde{M} := W_s^TM_{11}V_s, \quad \tilde{L} := W_s^TL_{11}V_s, \quad \tilde{K} := \tilde{K}_{11} - \tilde{K}_{12}\tilde{K}_{22}^{-1}\tilde{K}_{21}, \\
\tilde{F} := \tilde{F}_1 - \tilde{K}_{12}\tilde{K}_{22}^{-1}\tilde{F}_2, \quad \tilde{H} := \tilde{H}_1 - H_2\tilde{K}_{22}^{-1}\tilde{K}_{21}, \\
\tilde{D}_a := D_a + H_2\tilde{K}_{22}^{-1}\tilde{F}_2,
\]
(26)
which however show that the reduced-matrices can be constructed without forming the dense system \([16]\).

Now the important issue is that how to construct the transformation matrices \(V_s\) and \(W_s\) from the sparse system.

To construct \(V_s\) in \((23)\) at \(i\)-th iteration the vector \(v_i = (\alpha_i^2 M + \alpha_i L + K)^{-1} F b_1\) is obtained by solving the linear system
\[
(\alpha_i^2 M + \alpha_i L + K)v_i = F b_1. \tag{27}
\]

Plugging \(M\), \(L\), \(K\) and \(F\) from \((17)\) we obtain
\[
(\alpha_i^2 M_{11} + \alpha_i L_{11} + K_{11} - K_{12} K_{22}^{-1} K_{21}) v_i = (F_1 - W_s^T K_{12} K_{22}^{-1} F_2) b_i,
\]
which implies to
\[
\begin{bmatrix}
\alpha_i^2 M_{11} + \alpha_i L_{11} + K_{11} \\
K_{21}
\end{bmatrix}
\begin{bmatrix}
v_i^* \\
1
\end{bmatrix}
= 
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
b_i, \tag{28}
\]
for \(v_i\). Although the dimension of this linear system is higher than that of \((27)\), it is sparse and therefore it can be treated using a sparse direct solver \([20, \text{Ch. 5}]\), or any suitable iterative solver \([21]\) efficiently. Similarly each vector \(w_i = (\alpha_i^2 M + \alpha_i L + K)^{-T} H c_i\) in \(W_s\) of \((23)\) can be formed by solving the sparse linear system. Which again implies to
\[
\begin{bmatrix}
\alpha_i^2 M_{11}^T + \alpha_i L_{11}^T + K_{11}^T \\
K_{21}^T
\end{bmatrix}
\begin{bmatrix}
w_i^* \\
1
\end{bmatrix}
= 
\begin{bmatrix}
H_1^T \\
H_2^T
\end{bmatrix}
c_i. \tag{29}
\]

In this way \(V_s\) and \(W_s\) can be constructed without forming the dense system \([17]\) explicitly.

**Update interpolation points and tangential directions.** We have mentioned earlier that in the tangential interpolatory methods, selection of tangential points and tangential directions are important task. Since they depend on the reduced-order model, they are not known \textit{a priori}. From Section 2 we already have known that an iterative algorithm IRKA has overcome this problem. Here we also follow Step 7 in Algorithm 1 to select \(r\) interpolation points along with left and right tangential directions. In our case we construct
\[
\begin{align*}
\hat{E} := \begin{bmatrix} 0 & \hat{M} \end{bmatrix}, \quad \hat{A} := \begin{bmatrix} \hat{M} & 0 \\ \hat{L} & -K \end{bmatrix}, \quad \hat{B} := \begin{bmatrix} 0 \\ \hat{F} \end{bmatrix} \quad \text{and} \quad \hat{C} := \begin{bmatrix} 0 \\ \hat{H} \end{bmatrix}. \tag{30}
\end{align*}
\]

Then apply Algorithm 1 using the inputs: \(\hat{E}, \hat{A}, \hat{B}\) and \(\hat{C}\) to find \(r \times r\) matrices \(\hat{A}\) and \(\hat{E}\). The interpolation points are updated by choosing the mirror images of the eigenvalues of the pair \((\hat{A}, \hat{E})\) as the next interpolations points. The tangential directions are updated in the similar way as Algorithm 1.

The whole procedure discussed above to construct a structure-preserving reduced-order model for the second-order index-1 DAEs \([1]\) that is summarized in Algorithm 2.
Algorithm 2: IRKA for Second-Order Index-1 Descriptor Systems.

Input: \( M_{11}, L_{11}, K_{11}, K_{12}, K_{21}, K_{22}, F_1, F_2, L_1, L_2 \) and \( D_a \) from (1).

Output: \( \hat{M}, \hat{L}, \hat{K}, \hat{F}, \hat{H} \) and \( \hat{D}_a \) as in (24).

1. Make the initial selection of the interpolation points \( \{\alpha_i\}_{i=1}^r \) and the tangential directions \( \{b_i\}_{i=1}^r \) and \( \{c_i\}_{i=1}^r \).

2. Construct the projection matrices

\[
V_s = [v_1, v_2, \ldots, v_r] \quad \text{and} \quad W_s = [w_1, w_2, \ldots, w_r],
\]

where \( v_i \) & \( w_i \); \( i = 1, \ldots, r \) are the solutions of the linear systems (28) and (29), respectively.

3. while (not converged) do

4. Compute \( \hat{M}, \hat{L}, \hat{K}, \hat{F} \) and \( \hat{H} \) by (26).

5. Construct \( \hat{E}, \hat{A}, \hat{B} \) and \( \hat{C} \), then using Algorithm 1 compute \( \hat{A}, \hat{E} \in \mathbb{R}^{r \times r} \).

6. Compute \( \hat{A}z_i = \lambda_i \hat{E}z_i \) and \( y_i^* \hat{A} = \lambda_i y_i^* \hat{E} \) for \( \alpha_i \leftarrow -\lambda_i, b_i^* \leftarrow -y_i^* \hat{B} \) and \( c_i^* \leftarrow \hat{C}z_i^* \).

7. Repeat Step 2.

8. \( i = i + 1 \).

9. end while

10. Construct the reduced-order matrices \( \hat{M}, \hat{L}, \hat{K}, \hat{F} \) and \( \hat{H} \) as in (26).

Back to index 1 form. Algorithm 2 yields standard reduced-order model (24) form second-order index-1 DAEs (1). A little algebraic manipulation again turns (24) into a index-1 form

\[
\begin{bmatrix}
\hat{M} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{v}(t) \\
\dot{\eta}(t)
\end{bmatrix}
+ \begin{bmatrix}
\hat{L} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{v}(t) \\
\dot{\eta}(t)
\end{bmatrix}
+ \begin{bmatrix}
\hat{K}_{11} & \hat{K}_{12} \\
\hat{K}_{21} & \hat{K}_{22}
\end{bmatrix}
\begin{bmatrix}
v(t) \\
\eta(t)
\end{bmatrix}
= \begin{bmatrix}
\hat{F}_1 \\
\hat{F}_2
\end{bmatrix}
u(t),
\]

\[
y(t) = \begin{bmatrix}
\hat{H}_1 & \hat{H}_2
\end{bmatrix}
\begin{bmatrix}
v(t) \\
\eta(t)
\end{bmatrix}
+ Dau(t),
\]

where all the block matrices are defined in (26). Note that this turnover, however is not too much beneficiary if the algebraic part of the system is still large.

Setting with symmetric system. When system (1) as defined in Section 1 is symmetric, then the computed \( V_s \) and \( W_s \) in Algorithm 2 are coincided. Therefore, we can compute only \( V_s \) and reduced-order model in (24) can be constructed by forming the reduced matrices in (26) by using \( W_s = V_s \). In this way the constructed reduced-order model is also symmetric. Moreover, the ROM preserves the definiteness of the original system. Therefore, the stability is also preserved.

4 Numerical results

We investigate the performance of the proposed techniques, Algorithm 2 by applying to a set of data for the FE discretized model of the adaptive spindle support (ASS). This section discusses the numerical results which are obtained
by using MATLAB 8.5.0 (R2015a) on a Windows machine having INTEL XEON SILVER 4114 CPU with a 2.20-GHz clock speed, 2 Cores each and 64-GB of total RAM. In the following first we briefly introduced the molder example. Then different dimensions reduced models are compared with the original models using frequency domain analysis. The results obtained by the proposed algorithm are also compared with that of the balanced truncation.

4.1 Model example

Piezo-actuator based adaptive spindle support as shown in Figure 1(a) is a machine tool which is mounted in a parallel kinematic machine shown in Figure 1(b), in order to gain additional positioning freedom during machining operations. A detail of such complex mecharonic model can be found, for example in [4, 22, 23] for more details.

Figure 1: (a) Piezo-actuator based adaptive spindle support (ASS) and (b) real component mounted on the test bench 3pod (Source [4]).

The purpose of the piezo-sensor and piezo-actuator are to control active vibration or shunt damping so that a high quality product can be ensured.

For analyzing the mechanical design and performance of the ASS, using the finite element method a mathematical model as shown in (1) was formed, where time dependent state vector \( v(t) \) consists of the components of mechanical displacements and \( \eta(t) \) are the electrical charges and in which \( M_{11}, L_{11} \) and \( K_{11} \) are the mass, damping and stiffness, respectively. Moreover, the block \( K_{22} \) is electrical and \( K_{21} = K^{T}_{12} \) is coupling terms, the general force quantities (mechanical forces and electrical charges) are chosen as the input quantities \( u \), and the corresponding general displacements (mechanical displacements and electrical potential) are the output quantities \( y \). In experimental data the block matrices \( M_{11}, L_{11}, K_{11} \) and \( K_{22} \) are symmetric, and also the output matrix \( H \) is equal to the transpose of the input matrix \( F \). Therefore, the the system is symmetric. The dimension of the original model is \( n = 290 \, 137 \), which consists of \( n_1 = 282 \, 699 \) differential equations and \( n_2 = 7 \, 438 \) algebraic equations and number of Inputs\((m)\)/outputs\((p)\) is 9.
Table 1: Speed-up comparisons for ROMs against full model by IRKA

| Model          | Time per cycle (sec) | Speed-up |
|----------------|----------------------|----------|
| full model     | 100.949071           | 1        |
| 50 dim ROM     | 0.011203             | 9011     |
| 40 dim ROM     | 0.010584             | 9538     |
| 30 dim ROM     | 0.009338             | 10811    |
| 20 dim ROM     | 0.008960             | 11267    |
| 10 dim ROM     | 0.007948             | 12701    |

4.2 Frequency domain analysis

We compute the ROMs of different dimensions by applying Algorithm 2. The algorithm is stopped by the maximum number of iteration 50 steps. The frequency domain comparisons of the full model and different dimensional ROMs are investigated in Figure 2 on the range $[10^{-1}, 10^4]$ [rad/s]. At each iteration of Algorithm 2, to update the interpolation points and tangential directions we have used Algorithm 1 with the tolerance $10^{-10}$ and maximum 30 iterations. Figure 2a shows that transfer functions of all the ROMs with different dimensions obtained by IRKA acceptably match to the transfer function of the full model. Figure 2b and figure 2c represent the absolute errors and the relative errors of the ROMs respectively from which we observe that the errors of the ROMs are reasonably acceptable. It is evident that the ROMs of different dimensions preserves the fundamental and vital attributes of the full model. Because of that, the achieved ROMs can be implemented instead the full model to perform the necessary activities of the real controller.

Table 1 represents the speed-up of the frequency responses of ROMs obtained by IRKA against the full model. For the convenient comparison, we have counted the execution time for a single cycle of the frequency responses of the full model and the ROMs of different dimensions. It has been observed that the obtained ROMs generated by IRKA can accelerate the system speed many times.

4.3 Comparison with the balanced truncation

To compare the ROMs obtained by IRKA and Balanced Truncation (BT) of the ASS model, we have considered Algorithm 2 of this work with the Algorithm 2 in [13] for the balancing based method. We have considered the 20 dimensional ROM obtained by the IRKA based interpolatory method and that of the balancing based method. Figure 3 describes the comparison of the relative errors of the ROMs of these two methods. It has been observed that the 20 dimensional ROM achieved by IRKA gives better approximation except for some points than the 20 dimensional ROM achieved by BT method. On the other hand if we compare the execution time then IRKA is faster than the BT which is observed in Table 2. In this table the CPU time is computed on the machine mentioned above. Table 2 illustrates that the interpolatory based method IRKA can perform 4 times faster than the Balanced Truncation (BT) method to find the ROMs on the basis of time. Note that balanced truncation takes more time to compute the low-rank Gramian factors.
Figure 2: Comparison of original and different dimensional reduced systems (dimensions indicated in the legend) computed by Algorithm 2.
Figure 3: Relative errors of 20 dimensional reduced models by the IRKA and BT

Table 2: Speed-up comparisons for 20 dim ROMs by IRKA and BT

| Method | Time (day) | Speed-up |
|--------|------------|----------|
| BT     | 12         | 1        |
| IRKA   | 3          | 4        |
4.4 Stability

Stability is one of the pivot features of a real-world system. For engineering applications, system stability is one of the fundamental requirements. A model order reduction (MOR) technique is said to be well structure-preserving if it can provide stable ROMs. Figure (4) shows the stability of the target model through eigenvalue comparisons.

Figure 4 depicts that the eigenvalues corresponding to all of the ROMs lie on the left-half plane of the complex domain. So, it can be said that, the ROMs found by the Algorithm 2 can be efficiently applied to acquire the stable ROMs of different dimensions.

5 Conclusions

This paper is devoted to develop the interpolatory tangential method via IRKA for SPMOR of large-scale sparse second-order index-1 DAEs without computing the ODE system (index-0) explicitly. In this context to modify the classical IRKA, we have discussed the techniques to construct the reduced matrices in sparse form by implicitly producing the two transformation matrices. For this intention, the selection of interpolation points and tangential directions is a very crucial task which has been determined. We have also examined that the computational complexity can be drastically reduced for the symmetric system by constructing only one projection matrix with preserving the stability and the symmetry of the system. The performance of the proposed technique has been applied to a very large model of an ASS employing Piezo actuators with 29017
DoF, that manifest the applicability of the proposed method in the real-world engineering applications.

From the numerical computations, it has been investigated that even very lower dimensional ROMs found by the proposed method preserve the system attributes and input-output behaviors in the acceptable level. The transfer functions of the full model and the achieved ROMs are very identical in the frequency domain. The comparison of the proposed method with the BT method indicates the similarity by the transfer functions and behaviors of the ROMs, whereas IRKA provides the ROMs many times faster than that of BT method in the simulations. The display of the eigenvalues of the various ROMs attained by the IRKA shows the stability preservation of the proposed method.

6 Acknowledgment

This work is partially funded by Bangladesh Bureau of Educational Information and Statistics (BANBEIS) under the project, ID MS20191055. First author is also a fellow of University Grant Commission (UGC), Bangladesh.

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