GEOMETRIC AND SPECTRAL PROPERTIES OF DIRECTED GRAPHS UNDER A LOWER RICCI CURVATURE BOUND

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Abstract. For undirected graphs, the Ricci curvature introduced by Lin-Lu-Yau has been widely studied from various perspectives, especially geometric analysis. In the present paper, we discuss generalization problem of their Ricci curvature for directed graphs. We introduce a new generalization by using the mean transition probability kernel which appears in the formulation of the Chung Laplacian. We conclude several geometric and spectral properties of directed graphs under a lower Ricci curvature bound extending previous results in the undirected case.

1. Introduction

Ricci curvature is one of the most fundamental objects in Riemannian geometry. Based on a geometric observation on (smooth) Riemannian manifolds, Ollivier [30] has introduced the coarse Ricci curvature for (non-smooth) metric spaces by means of the Wasserstein distance which is an essential tool in optimal transport theory. Modifying the formulation in [30], Lin-Lu-Yau [23] have defined the Ricci curvature for undirected graphs. It is well-known that a lower Ricci curvature bound of Lin-Lu-Yau [23] implies various geometric and analytic properties (see e.g., [7], [9], [19], [23], [27], [31], and so on).

There have been some attempts to generalize the Ricci curvature of Lin-Lu-Yau [23] for directed graphs. The third author [38] has firstly proposed a generalization of their Ricci curvature (see Remark 3.7 for its precise definition). He computed it for some concrete examples, and given several estimates. Eidi-Jost [14] have recently introduced another formulation (see Remark 3.7). They have applied it to the study of directed hypergraphs.

We are now concerned with the following question: What is the suitable generalization of the Ricci curvature of Lin-Lu-Yau [23] for directed graphs? In this paper, we provide a new Ricci curvature for directed graphs, examine its basic properties, and conclude several geometric and analytic properties under a lower Ricci curvature bound. Our formulation is as follows (more precisely, see Section 2 and Subsection 3.1): Let \((V, \mu)\) denote a simple, strongly connected, finite weighted directed graph, where \(V\) is the vertex set, and \(\mu : V \times V \to [0, \infty)\) is the (non-symmetric) edge weight. For the transition probability kernel \(P : V \times V \to [0, 1]\), we consider the mean transition probability kernel \(\mathcal{P} : V \times V \to [0, 1]\) defined as

\[
\mathcal{P}(x, y) := \frac{1}{2} \left( P(x, y) + \frac{m(y)P(y, x)}{m(x)} \right),
\]

where \(m : V \to (0, 1]\) is the so-called Perron measure on \(V\). We denote by \(d : V \times V \to [0, \infty)\) the (non-symmetric) distance function on \(V\), and by \(W\) the associated Wasserstein distance. For \(x, y \in V\) with \(x \neq y\), we define the Ricci curvature \(\kappa(x, y)\) by

\[
\kappa(x, y) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( 1 - \frac{W(\nu^x_{\varepsilon}, \nu^y_{\varepsilon})}{d(x, y)} \right),
\]

where

\[
\nu^x_{\varepsilon} = \frac{m(x)}{\varepsilon} \mathcal{P}(x, y) + \frac{\varepsilon}{2} \int_0^\varepsilon \frac{m(y)P(y, x)}{m(x)} \mathcal{P}(x, y) \, ds.
\]
where $\nu^x_\varepsilon : V \to [0, 1]$ is a probability measure on $V$ defined as
\[
\nu^x_\varepsilon(z) := \begin{cases} 
1 - \varepsilon & \text{if } z = x, \\
\varepsilon \mathcal{P}(x, z) & \text{if } z \neq x.
\end{cases}
\]
In the undirected case (i.e., the weight $\mu$ is symmetric), our Ricci curvature coincides with that of Lin-Lu-Yau [23]. We also note that the third author [38] and Eidi-Jost [14] have used different probability measures from $\nu^x_\varepsilon, \nu^y_\varepsilon$ to define their Ricci curvatures (see Remark 3.7).

One of remarkable features of our Ricci curvature is that it controls the behavior of the symmetric Laplacian introduced by Chung [11], [12]. Here we recall that the Chung Laplacian $L$ is defined as
\[
L f(x) := f(x) - \sum_{y \in V} \mathcal{P}(x, y) f(y)
\]
for a function $f : V \to \mathbb{R}$. For instance, we will derive lower bounds of the spectrum of the Chung Laplacian $L$ under a lower Ricci curvature bound (see Theorems 1.2 and 5.3).

1.1. Main results and organization. In Section 2, we prepare some notations, recall basic facts on the theory of directed graphs (see Subsections 2.1 and 2.2), and optimal transport theory (see Subsection 2.3). In Section 3, we define our Ricci curvature (see Subsection 3.1), give its estimates (see Subsection 3.2), and calculate it for some concrete examples (see Subsection 3.3). In Section 4, we mention the relation between our Ricci curvature and the curvature-dimension inequality of Bakry-Émery type.

In Section 5, we prove several comparison geometric results under a lower Ricci curvature bound. First, we will extend the diameter comparison of Bonnet-Myers type, and the eigenvalue comparison of Lichnerowicz type that have been obtained by Lin-Lu-Yau [23] in the undirected case to our directed case (see Subsection 5.1). Next, we will generalize the volume comparison of Bishop type that has been established by Paeng [31] in the undirected case (see Subsection 5.2). We further formulate a Laplacian comparison for the distance function from a single point (see Subsection 5.3). To formulate such Laplacian comparison, we introduce a notion of the asymptotic mean curvature around each vertex as follows: For $x \in V$, we define the asymptotic mean curvature $\mathcal{H}_x$ around $x$ by
\[
\mathcal{H}_x := L \rho_x(x),
\]
where $\rho_x : V \to \mathbb{R}$ is the distance function from $x$ defined as $\rho_x(y) := d(x, y)$ (see Remark 3.3 for the reason why we call it the asymptotic mean curvature). It holds that $\mathcal{H}_x \leq -1$. Furthermore, in the undirected case, we have $\mathcal{H}_x = -1$; in particular, this notion plays an essential role in the case where $(V, \mu)$ is not undirected.

On Riemannian manifolds with a lower Ricci curvature bound, it is well-known that several comparison geometric results hold for hypersurfaces with a lower mean curvature bound (see the pioneering work of Heintze-Karcher [17], and e.g., [8], [25], [26]). In a spirit of Heintze-Karcher comparison, under a lower Ricci curvature bound, and a lower asymptotic mean curvature bound, we formulate our Laplacian comparison as follows:

**Theorem 1.1.** Let $x \in V$. For $K \in \mathbb{R}$ we assume $\kappa(x, y) \geq K$ at $y \in V \setminus \{x\}$. For $\Lambda \in (-\infty, -1]$ we further assume $\mathcal{H}_x \geq \Lambda$. Then we have
\[
(1.1) \quad L \rho_x(y) \geq K \rho_x(y) + \Lambda.
\]

Münch-Wojciechowski [27] have established Theorem 1.1 in the undirected case (see Theorem 4.1 in [27]). Here we emphasize that in the undirected case, the lower asymptotic mean curvature bound has not been supposed since $\mathcal{H}_x = -1$ in that case. We compare Theorem 1.1 with a similar comparison result on Riemannian manifolds (see Subsection 5.4).
In Section 6, we study the Dirichlet eigenvalues of the $p$-Laplacian $\mathcal{L}_p$ defined by
\[
\mathcal{L}_p f(x) := \sum_{y \in V} |f(x) - f(y)|^{p-2}(f(x) - f(y))\mathcal{P}(x, y)
\]
for $p \in (1, \infty)$, where we notice that $\mathcal{L}_2 = \mathcal{L}$. For a non-empty subset $V$ of $V$ with $V \neq V$, let $\lambda_2^D(V)$ stand for the smallest Dirichlet eigenvalue over $V$ (more precisely, see Subsection 6.1). We first prove an inequality of Cheeger type for $\lambda_2^D(V)$ (see Subsection 6.2). For $x \in V$, the \textit{inscribed radius} $\text{InRad}_x V$ of $V$ at $x$ is defined by
\[
\text{InRad}_x V := \sup_{y \in V} \rho_x(y).
\]
Combining the Cheeger inequality and Theorem 1.1, we obtain the following lower bound of the Dirichlet eigenvalue over the outside of a metric ball under our lower curvature bounds, and an upper inscribed radius bound:

\textbf{Theorem 1.2.} \textit{Let $x \in V$ and $p \in (1, \infty)$. For $K \in \mathbb{R}$ we assume $\inf_{y \in V \setminus \{x\}} \kappa(x, y) \geq K$. For $\Lambda \in (-\infty, -1]$ we also assume $\mathcal{H}_x \geq \Lambda$. For $D > 0$ we further assume $\text{InRad}_x V \leq D$. Then for every $R \geq 1$ with $KR + \Lambda > 0$, we have}

\[
\lambda_2^D(E_R(x)) \geq \frac{2^{p-1}}{p^p} \left( \frac{KR + \Lambda}{D} \right)^p,
\]

\textit{where $E_R(x) := \{ y \in V \mid \rho_x(y) \geq R \}$.

Theorem 1.2 is new even in the undirected case.}

\section{Preliminaries}

In this section, we review basics of directed graphs. We refer to [16] for the notation and basics of the theory of undirected graph.

\subsection{Directed graphs}

Let $(G, \mu)$ be a finite weighted directed graph, namely, $G = (V, E)$ is a finite directed graph, and $\mu : V \times V \to [0, \infty)$ is a function such that $\mu(x, y) > 0$ if and only if $x \to y$, where $x \to y$ means that $(x, y) \in E$. We will denote by $n$ the cardinality of $V$. The function $\mu$ is called the \textit{edge weight}, and we write $\mu(x, y)$ by $\mu_{xy}$. We notice that $(G, \mu)$ is undirected if and only if $\mu_{xy} = \mu_{yx}$ for all $x, y \in V$, and simple if and only if $\mu_{xx} = 0$ for all $x \in V$. For $x \in V$ and $\Omega \subset V$ we set

\[
\mu(x) := \sum_{y \in V} \mu_{yx}, \quad \mu(\Omega) := \sum_{x \in \Omega} \mu(x).
\]

We also note that $(G, \mu)$ has no isolated points if and only if $\mu(x) > 0$ for all $x \in V$. The weighted directed graph can be denoted by $(V, \mu)$ since $\mu$ contains full information of $E$. Thus in this paper, we use $(V, \mu)$ instead of $(G, \mu)$.

For $x \in V$, its \textit{outer neighborhood} $N_x$, \textit{inner one} $\overline{N}_x$, and \textit{neighborhood} $N_x$ are defined as
\[
N_x := \{ y \in V \mid x \to y \}, \quad \overline{N}_x := \{ y \in V \mid y \to x \}, \quad N_x := N_x \cup \overline{N}_x,
\]

respectively. Its \textit{outer degree} $\deg(x)$ and \textit{inner degree} $\overline{\deg}(x)$ are defined as the cardinality of $N_x$ and $\overline{N}_x$, respectively. We say that $(V, \mu)$ is \textit{unweighted} if $\mu_{xy} = 1$ whenever $x \to y$, and then $\mu(x) = \deg(x)$ for all $x \in V$. In the unweighted case, $(V, \mu)$ is said to be \textit{Eulerian} if $\deg(x) = \overline{\deg}(x)$ for all $x \in V$. An Eulerian graph is called \textit{regular} if $\deg(x)$ (or equivalently, $\overline{\deg}(x)$) does not depend on $x$. Furthermore, for $r \geq 1$, a regular graph is called \textit{$r$-regular} if we possess $\deg(x) = r$ (or equivalently, $\overline{\deg}(x) = r$) for all $x \in V$. 
For \(x, y \in V\), a sequence \(\{x_i\}_{i=0}^l\) of vertexes is called a \textit{directed path} from \(x\) to \(y\) if \(x_i \rightarrow x_{i+1}\) for all \(i = 0, \ldots, l-1\). The number \(l\) is called its length. Furthermore, \((V, \mu)\) is called \textit{strongly connected} if for all \(x, y \in V\), there exists a directed path from \(x\) to \(y\). Notice that if \((V, \mu)\) is strongly connected, then it has no isolated points. For strongly connected \((V, \mu)\), the (non-symmetric) distance function \(d : V \times V \rightarrow [0, \infty)\) is defined as follows: \(d(x, y)\) is defined to be the minimum of the length of directed paths from \(x\) to \(y\). For a fixed \(x \in V\), the \textit{distance function} \(\rho_x : V \rightarrow \mathbb{R}\), and the \textit{reverse distance function} \(\overrightarrow{\rho}_x : V \rightarrow \mathbb{R}\) from \(x\) are defined as

\[
(2.2) \quad \rho_x(y) := d(x, y), \quad \overrightarrow{\rho}_x(y) := d(y, x).
\]

We further define the \textit{inscribed radius} \(\text{InRad}_x V\) of \(V\) at \(x\) by

\[
(2.3) \quad \text{InRad}_x V := \sup_{y \in V} \rho_x(y).
\]

For \(L > 0\), a function \(f : V \rightarrow \mathbb{R}\) is said to be \(L\)-Lipschitz if

\[
f(y) - f(x) \leq L d(x, y)
\]

for all \(x, y \in V\). We remark that \(\rho_x\) is 1-Lipschitz, but \(\overrightarrow{\rho}_x\) is not always 1-Lipschitz. Let \(\text{Lip}_L(V)\) stand for the set of all \(L\)-Lipschitz functions on \(V\).

\textbf{Remark 2.1.} The non-symmetric distance function also appears in the Finsler geometry (see e.g., [5], [34]). We refer to [28], [29] for the notation and terminology concerning the distance.

\textbf{2.2. Laplacians.} Let \((V, \mu)\) be a strongly connected, finite weighted directed graph. We recall the formulation of the Laplacian on \((V, \mu)\) introduced by Chung [11], [12]. The \textit{transition probability kernel} \(P : V \times V \rightarrow [0, 1]\) is defined as

\[
(2.4) \quad P(x, y) := \frac{\mu_{xy}}{\mu(x)},
\]

which is well-defined since \((V, \mu)\) has no isolated points. Since \((V, \mu)\) is finite and strongly connected, the Perron-Frobenius theorem implies that there exists a unique (up to scaling) positive function \(m : V \rightarrow (0, \infty)\) such that

\[
(2.5) \quad m(x) = \sum_{y \in V} m(y) P(y, x).
\]

A probability measure \(m : V \rightarrow (0, 1]\) on \(V\) satisfying (2.5) is called the \textit{Perron measure}. For a non-empty subset \(\Omega \subset V\), its \textit{measure} is defined as

\[
(2.6) \quad m(\Omega) := \sum_{x \in \Omega} m(x).
\]

\textbf{Remark 2.2.} When \((V, \mu)\) is undirected or Eulerian, the Perron measure \(m\) is given by

\[
m(x) = \frac{\mu(x)}{\mu(V)};
\]

in particular, if \((V, \mu)\) is a regular graph, then \(m(x) = 1/n\) for all \(x \in V\) (see Examples 1, 2, 3 in [11]). Here we recall that \(n\) is the cardinality of \(V\).

We denote by \(m\) the Perron measure. We define the \textit{reverse transition probability kernel} \(\overrightarrow{P} : V \times V \rightarrow [0, 1]\), and the \textit{mean transition probability kernel} \(P : V \times V \rightarrow [0, 1]\) by

\[
(2.7) \quad \overrightarrow{P}(x, y) := \frac{m(y)}{m(x)} P(y, x), \quad P := \frac{1}{2}(P + \overrightarrow{P}).
\]
Let $\mathcal{F}$ stand for the set of all functions on $V$. Chung \cite{11}, \cite{12} has introduced the following (positive) Laplacian $\mathcal{L} : \mathcal{F} \to \mathcal{F}$ on $(V, \mu)$:

\begin{equation}
\mathcal{L} f(x) := f(x) - \sum_{y \in V} \mathcal{P}(x, y) f(y).
\end{equation}

We will also use the negative Laplacian $\Delta : \mathcal{F} \to \mathcal{F}$ defined by

\begin{equation}
\Delta := -\mathcal{L}.
\end{equation}

The inner product and the norm on $\mathcal{F}$ are defined by

\[(f_0, f_1) := \sum_{x \in V} f_0(x) f_1(x) m(x), \quad \|f\| := (f, f)^{1/2};\]

respectively. We define a function $m : V \times V \to [0, \infty)$ by

\begin{equation}
m(x, y) := \frac{1}{2} (m(x) P(x, y) + m(y) P(y, x)) = m(x) P(x, y).
\end{equation}

We write $m(x, y)$ by $m_{xy}$. The following basic properties hold: (1) $m_{xy} = m_{yx}$; (2) $m_{xy} > 0$ if and only if $y \in \mathcal{N}_x$ (or equivalently, $x \in \mathcal{N}_y$); (3) $P(x, y) = m_{xy}/m(x)$.

We also have the following integration by parts formula, which can be proved by the same calculation as in the proof of Theorem 2.1 in \cite{16}:

**Proposition 2.3.** Let $\Omega \subset V$ be a non-empty subset. Then for all $f_0, f_1 : V \to \mathbb{R}$,

\[\sum_{x \in \Omega} \mathcal{L} f_0(x) f_1(x) m(x) = \frac{1}{2} \sum_{x, y \in \Omega} (f_0(y) - f_0(x))(f_1(y) - f_1(x)) m_{xy} \]

\[- \sum_{x \in \Omega} \sum_{y \in V \setminus \Omega} (f_0(y) - f_0(x)) f_1(x) m_{xy}.
\]

In particular,

\[(\mathcal{L} f_0, f_1) = \frac{1}{2} \sum_{x, y \in V} (f_0(y) - f_0(x))(f_1(y) - f_1(x)) m_{xy} = (f_0, \mathcal{L} f_1).
\]

In virtue of Proposition 2.3, $\mathcal{L}$ is symmetric with respect to the inner product.

**Remark 2.4.** Besides the Chung Laplacian $\mathcal{L}$, there are several generalizations of the undirected graph Laplacian for directed graphs. For instance, Bauer \cite{6} has studied spectral properties of the (non-symmetric) Laplace operator $\mathcal{L}_0 : \mathcal{F} \to \mathcal{F}$ defined as

\[\mathcal{L}_0 f(x) := f(x) - \frac{1}{\sum_{y \in V} \mu_{yx}} \sum_{y \in V} \mu_{yx} f(y),\]

which is equivalent to the operator $\mathcal{L}_1 : \mathcal{F} \to \mathcal{F}$ defined as

\[\mathcal{L}_1 f(x) := f(x) - \sum_{y \in V} P(x, y) f(y)\]

in the sense that the spectrum of $\mathcal{L}_0$ on $(V, \mu)$ coincides with that of $\mathcal{L}_1$ on the directed graph that is obtained from $(V, \mu)$ by reversing all edges (see Definition 2.1 in \cite{6}). On the other hand, Yoshida \cite{40} has recently introduced the (non-linear) submodular Laplace operator in the context of discrete convex analysis, which can be applied to the study of directed graphs (see Example 1.5 in \cite{40}). He formulated an inequality of Cheeger type for the eigenvalues of the submodular Laplace operator. We stress that $(V, \mu)$ does not need to be strongly connected when we define the Laplace operators in \cite{6}, \cite{40}, unlike the Chung Laplacian.
2.3. Optimal transport theory. We recall the basic facts on the optimal transport theory, and refer to [36], [37]. Let \((V, \mu)\) denote a strongly connected, finite weighted directed graph. For two probability measures \(\nu_0, \nu_1\) on \(V\), a probability measure \(\pi: V \times V \to [0, \infty)\) is called a coupling of \((\nu_0, \nu_1)\) if
\[
\sum_{y \in V} \pi(x, y) = \nu_0(x), \quad \sum_{x \in V} \pi(x, y) = \nu_1(y).
\]
Let \(\Pi(\nu_0, \nu_1)\) denote the set of all couplings of \((\nu_0, \nu_1)\). The \((L^1-)\) Wasserstein distance from \(\nu_0\) to \(\nu_1\) is defined as
\[
W(\nu_0, \nu_1) := \inf_{\pi \in \Pi(\nu_0, \nu_1)} \sum_{x, y \in V} d(x, y) \pi(x, y).
\]
This is known to be a (non-symmetric) distance function on the set of all probability measures on \(V\). We also note that \(W(\delta_x, \delta_y) = d(x, y)\) for all \(x, y \in V\), where \(\delta_x: V \to \{0, 1\}\) denotes the Dirac measure at \(x\) defined as
\[
\delta_x(z) := \begin{cases} 
1 & \text{if } z = x, \\
0 & \text{otherwise}. 
\end{cases}
\]
A coupling \(\pi\) is called optimal if it attains the infimum of (2.11). It is well-known that for any \(\nu_0, \nu_1\), there exists an optimal coupling (cf. Theorem 4.1 in [37]).

The distance \(W\) enjoys the following jointly convexity property (cf. Section 7.4 in [36]):

**Proposition 2.5.** Let \(t \in [0, 1]\). For any four probability measures \(\nu_0, \nu_1, \sigma_0, \sigma_1\) on \(V\),
\[
W((1-t)\nu_0 + t\nu_1, (1-t)\sigma_0 + t\sigma_1) \leq (1-t)W(\nu_0, \sigma_0) + tW(\nu_1, \sigma_1).
\]

We also recall the following Kantorovich-Rubinstein duality formula (cf. Theorem 5.10 and Particular Cases 5.4 and 5.16 in [37], and see also Subsection 2.2 in [29]):

**Proposition 2.6.** For any two probability measures \(\nu_0, \nu_1\) on \(V\), we have
\[
W(\nu_0, \nu_1) = \sup_{f \in Lip_1(V)} \sum_{x \in V} f(x) (\nu_1(x) - \nu_0(x)).
\]

3. Ricci curvature

In this section, we propose a generalization of the Ricci curvature of Lin-Lu-Yau [23] for directed graphs, and investigate its basic properties. In what follows, we denote by \((V, \mu)\) a simple, strongly connected, finite weighted directed graph.

3.1. Definition of Ricci curvature. Let us introduce our Ricci curvature. For \(\varepsilon \in [0, 1]\) and \(x \in V\), we define a probability measure \(\nu^\varepsilon_x: V \to [0, 1]\) by
\[
\nu^\varepsilon_x(z) := \begin{cases} 
1 - \varepsilon & \text{if } z = x, \\
\varepsilon & \text{if } z \neq x,
\end{cases}
\]
where \(\mathcal{P}\) is defined as (2.7). Note that \(\nu^\varepsilon_x\) is a probability measure since \((V, \mu)\) is simple, and it is supported on \(\{x\} \cup \mathcal{N}_x\), where \(\mathcal{N}_x\) is defined as (2.1).

We also notice the following useful property:

**Lemma 3.1.** For every \(f: V \to \mathbb{R}\) it holds that
\[
\sum_{z \in V} f(z) \nu^\varepsilon_x(z) = (f + \varepsilon \Delta f)(x),
\]
where \(\Delta\) is defined as (2.9).
Proof. From straightforward computations we deduce
\[ \sum_{z \in V} f(z) \nu^x_z(z) = (1 - \varepsilon)f(x) + \varepsilon \sum_{z \in V \setminus \{x\}} \mathcal{P}(x, z)f(z) = (f + \varepsilon \Delta f)(x). \]
Here we used the simpleness of \((V, \mu)\) in the second equality. This proves (3.2). \(\square\)

For \(x, y \in V\) with \(x \neq y\), we set
\[ \kappa_{\varepsilon}(x, y) := 1 - \frac{W(\nu^x_x, \nu^y_y)}{d(x, y)}, \]
where \(W\) is defined as (2.11). We will define our Ricci curvature as the limit of \(\kappa_{\varepsilon}(x, y)/\varepsilon\) as \(\varepsilon \to 0\). To do so, we first verify the following (cf. Lemma 2.1 in [23], and see also [10], [27]):

**Lemma 3.2.** Let \(x, y \in V\) with \(x \neq y\). Then \(\kappa_{\varepsilon}(x, y)\) is concave in \(\varepsilon \in [0, 1]\). In particular, \(\kappa_{\varepsilon}(x, y)/\varepsilon\) is non-increasing in \(\varepsilon \in (0, 1]\).

**Proof.** Fix \(\varepsilon_0, \varepsilon_1, t \in [0, 1]\) with \(\varepsilon_0 \leq \varepsilon_1\), and set \(\varepsilon_t := (1 - t)\varepsilon_0 + t\varepsilon_1\). We can check that
\[ \nu^x_{\varepsilon_t} = (1 - t)\nu^x_{\varepsilon_0} + t\nu^x_{\varepsilon_1}, \quad \nu^y_{\varepsilon_t} = (1 - t)\nu^y_{\varepsilon_0} + t\nu^y_{\varepsilon_1}. \]
Proposition 2.5 tells us that
\[ \kappa_{\varepsilon_t}(x, y) = 1 - \frac{W(\nu^x_{\varepsilon_t}, \nu^y_{\varepsilon_t})}{d(x, y)} = 1 - \frac{W((1 - t)\nu^x_{\varepsilon_0} + t\nu^x_{\varepsilon_1}, (1 - t)\nu^y_{\varepsilon_0} + t\nu^y_{\varepsilon_1})}{d(x, y)} \geq 1 - \frac{1}{d(x, y)}(1 - t)W(\nu^x_{\varepsilon_0}, \nu^y_{\varepsilon_0}) + tW(\nu^x_{\varepsilon_1}, \nu^y_{\varepsilon_1}) = (1 - t)\kappa_{\varepsilon_0}(x, y) + t\kappa_{\varepsilon_1}(x, y). \]
Therefore, we arrive at the concavity.

By letting \(\varepsilon_0 \to 0\) in (3.4), and by \(\kappa_0(x, y) = 0\),
\[ \frac{\kappa_{\varepsilon_t}(x, y)}{\varepsilon_t} \geq \frac{(1 - t)\kappa_{\varepsilon_0}(x, y) + t\kappa_{\varepsilon_1}(x, y)}{(1 - t)\varepsilon_0 + t\varepsilon_1} = \frac{\kappa_{\varepsilon_1}(x, y)}{\varepsilon_1} \]
for \(t \in (0, 1]\), and hence \(\kappa_{\varepsilon}(x, y)/\varepsilon\) is non-increasing in \(\varepsilon \in (0, 1]\). We conclude the lemma. \(\square\)

Lin-Lu-Yau [23] have shown Lemma 3.2 in the undirected case (see Lemma 2.1 in [23]).

In view of Lemma 3.2, it suffices to show that \(\kappa_{\varepsilon}(x, y)/\varepsilon\) is bounded from above by a constant which does not depend on \(\varepsilon\). In order to derive the boundedness, we consider the **asymptotic mean curvature** \(\mathcal{H}_x\) around \(x\) that is already introduced in Subsection 1.1, and the **reverse asymptotic mean curvature** \(\mathcal{H}^{-}_x\) defined as
\[ \mathcal{H}_x := \mathcal{L}\rho_x(x), \quad \mathcal{H}^{-}_x := \mathcal{L}^{-}\rho_x(x), \]
where \(\mathcal{L}\) is defined as (2.8), and \(\rho_x\) and \(\mathcal{L}^{-}\rho_x\) are done as (2.2). More explicitly,
\[ \mathcal{H}_x = -\sum_{y \in V} \mathcal{P}(x, y)d(x, y) = -\frac{1}{2} - \frac{1}{2} \sum_{y \in V} \mathcal{P}(x, y)d(x, y), \]
\[ \mathcal{H}^{-}_x = -\sum_{y \in V} \mathcal{P}(x, y)d(x, y) = -\frac{1}{2} \sum_{y \in V} \mathcal{P}(x, y)d(x, y) - \frac{1}{2} \]
for \(\mathcal{P}, \mathcal{P}^{-}\) defined as (2.4), (2.7). We have \(\mathcal{H}_x \leq -1\) and \(\mathcal{H}^{-}_x \leq -1\) since \(\min\{d(x, y), d(y, x)\} = 1\) for \(y \in \mathcal{N}_x\). Furthermore, we see \(\mathcal{H}_x = \mathcal{H}^{-}_x = -1\) in the undirected case (see Remark 2.2).

**Remark 3.3.** The formulation of asymptotic mean curvature is based on the following observation concerning Riemannian geometry: Let \((M, g)\) be a Riemannian manifold (without boundary). We denote by \(d_g\) the Riemannian distance, and by \(\mathcal{L}_g\) the Laplacian defined as the minus of the trace of Hessian. For a fixed \(x \in M\), let \(\rho_{g, x}\) stand for the distance function...
from \(x\) defined as \(\rho_{g,x} := d_{g}(x, \cdot)\). For a sufficiently small \(R > 0\), we consider the metric sphere \(S_{g,R}(x)\) with radius \(R\) centered at \(x\). Then the (inward) mean curvature of \(S_{g,R}(x)\) at \(y \in S_{g,R}(x)\) is equal to \(\mathcal{L}_{g} \rho_{x}(y)\). We notice that in the manifold case, the mean curvature tends to \(-\infty\) as \(R \to 0\), unlike the graph case.

For \(x, y \in V\), we define the mixed asymptotic mean curvature \(\mathcal{H}(x, y)\) by

\[
\mathcal{H}(x, y) := -(\mathcal{H}_{x} + \mathcal{H}_{y}).
\]

We have \(\mathcal{H}(x, y) \geq 2\); moreover, the equality holds in the undirected case.

We now present the following upper estimate of \(\kappa_{\varepsilon}(x, y)/\varepsilon\) in terms of the mixed asymptotic mean curvature (cf. Lemma 2.2 in [23]):

**Lemma 3.4.** For all \(\varepsilon \in [0, 1]\) and \(x, y \in V\) with \(x \neq y\), we have

\[
\frac{\kappa_{\varepsilon}(x, y)}{\varepsilon} \leq \frac{\mathcal{H}(x, y)}{d(x, y)}.
\]

**Proof.** By the triangle inequality, we have

\[
W(\nu_{x}^{\varepsilon}, \nu_{y}^{\varepsilon}) \geq W(\delta_{x}, \delta_{y}) - W(\delta_{x}, \nu_{x}^{\varepsilon}) - W(\nu_{y}^{\varepsilon}, \delta_{y}) = d(x, y) - W(\delta_{x}, \nu_{x}^{\varepsilon}) - W(\nu_{y}^{\varepsilon}, \delta_{y}).
\]

From Lemma 3.1, it follows that

\[
W(\nu_{x}^{\varepsilon}, \nu_{y}^{\varepsilon}) = \sum_{z \in V} d(x, z)\nu_{x}^{\varepsilon}(z) = \varepsilon \Delta \rho_{x}(x) = -\varepsilon \mathcal{H}_{x},
\]

\[
W(\nu_{y}^{\varepsilon}, \delta_{y}) = \sum_{z \in V} d(z, y)\nu_{y}^{\varepsilon}(z) = \varepsilon \Delta \rho_{y}(y) = -\varepsilon \mathcal{H}_{y}.
\]

This yields

\[
W(\nu_{x}^{\varepsilon}, \nu_{y}^{\varepsilon}) \geq d(x, y) + \varepsilon(\mathcal{H}_{x} + \mathcal{H}_{y}) = d(x, y) - \varepsilon \mathcal{H}(x, y).
\]

We obtain

\[
\frac{\kappa_{\varepsilon}(x, y)}{\varepsilon} = \frac{1}{\varepsilon} \left( 1 - \frac{W(\nu_{x}^{\varepsilon}, \nu_{y}^{\varepsilon})}{d(x, y)} \right) \leq \frac{\mathcal{H}(x, y)}{d(x, y)}.
\]

We complete the proof. \(\square\)

**Remark 3.5.** Lin-Lu-Yau [23] proved Lemma 3.2 in the undirected case (see Lemma 2.2 in [23]). We emphasize that in the undirected case, \(\mathcal{H}(x, y)\) has not appeared in the right hand side of (3.9). Actually, its right hand side is equal to \(2/d(x, y)\) in that case.

In virtue of Lemmas 3.2, 3.4, we can define our Ricci curvature as follows:

**Definition 3.6.** For \(x, y \in V\) with \(x \neq y\), we define the Ricci curvature by

\[
\kappa(x, y) := \lim_{\varepsilon \to 0} \frac{\kappa_{\varepsilon}(x, y)}{\varepsilon}.
\]

In undirected case, this is nothing but the Ricci curvature introduced by Lin-Lu-Yau [23].

**Remark 3.7.** Similarly to the Laplacian, besides our Ricci curvature \(\kappa(x, y)\), there might be some generalizations of the undirected Ricci curvature of Lin-Lu-Yau [23] for directed graphs (cf. Remark 2.4). The third author [38] firstly proposed the following generalization:

\[
\overrightarrow{\kappa}(x, y) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( 1 - \frac{W(\overrightarrow{\nu}_{x}^{\varepsilon}, \overrightarrow{\nu}_{y}^{\varepsilon})}{d(x, y)} \right),
\]

where

\[
\overrightarrow{\nu}_{z}^{\varepsilon}(z) := \begin{cases} 1 - \varepsilon & \text{if } z = x, \\ \varepsilon P(x, z) & \text{if } z \neq x. \end{cases}
\]
This can be called the **out-out type Ricci curvature** since we consider the Wasserstein distance from the outer probability measure $\overrightarrow{\nu}^\varepsilon_x$ to the outer one $\overrightarrow{\nu}^\varepsilon_y$. On the other hand, Eidi-Jost [14] considered the **in-out type Ricci curvature**, and used them for the study of directed hypergraphs (see Definition 3.2 in [14]). In our setting, their in-out type Ricci curvature can be formulated as follows:

$$\overleftarrow{\kappa}(x, y) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(1 - \frac{W(\overrightarrow{\nu}^\varepsilon_x, \overrightarrow{\nu}^\varepsilon_y)}{d(x, y)}\right),$$

where

$$\overrightarrow{\nu}^\varepsilon_x(z) := \begin{cases} 1 - \varepsilon & \text{if } z = x, \\ \varepsilon \frac{\mu_{zx}}{\sum_{y \in V} \mu_{yx}} & \text{if } z \neq x. \end{cases}$$

It seems that we can also consider the following **out-in type Ricci curvature** $\overrightarrow{\kappa}(x, y)$, and the **in-in type Ricci curvature** $\overleftarrow{\kappa}(x, y)$ defined as follows (cf. Section 8 in [14]):

$$\overrightarrow{\kappa}(x, y) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(1 - \frac{W(\overrightarrow{\nu}^\varepsilon_x, \overrightarrow{\nu}^\varepsilon_y)}{d(x, y)}\right), \quad \overleftarrow{\kappa}(x, y) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(1 - \frac{W(\overleftarrow{\nu}^\varepsilon_x, \overleftarrow{\nu}^\varepsilon_y)}{d(x, y)}\right).$$

Our Ricci curvature satisfies the following property (see Lemma 2.3 in [23]):

**Proposition 3.8** ([23]). Let $K \in \mathbb{R}$. If $\kappa(z, z') \geq K$ for all edges $(z, z') \in E$, then $\kappa(x, y) \geq K$ for any two distinct vertices $x, y \in V$.

Lin-Lu-Yau [23] obtained Proposition 3.8 in the undirected case (see Lemma 2.3 in [23]). We can prove Proposition 3.8 by the same argument as in the undirected case. We omit it.

### 3.2. Estimates of Ricci curvature

In the present subsection, we discuss several upper and lower bounds of our Ricci curvature.

For $x, y \in V$ with $x \neq y$, we set

$$\mathcal{D}(x, y) := \max \{d(x, y), d(y, x)\}. $$

We first study a lower bound of our Ricci curvature (cf. Theorems 2 and 5 in [19]).

**Proposition 3.9.** For $x, y \in V$ with $x \neq y$, we have

$$\kappa(x, y) \geq -2\frac{\mathcal{D}(x, y)}{d(x, y)} \left(1 - \mathcal{P}(x, y) - \mathcal{P}(y, x)\right)_+ + \frac{1}{d(x, y)} (d(x, y) + \mathcal{D}(x, y) - \mathcal{H}(y, x))$$

$$- \frac{\mathcal{D}(x, y) - d(y, x)}{d(x, y)} (\mathcal{P}(x, y) + \mathcal{P}(y, x)), $$

where $(\cdot)_+$ denotes its positive part. Moreover, if $(x, y) \in E$, then we have

$$\kappa(x, y) \geq -2d(y, x) \left(1 - \mathcal{P}(x, y) - \mathcal{P}(y, x)\right)_+ + (1 + d(y, x) - \mathcal{H}(y, x)). $$

**Proof.** Jost-Liu [19] shown (3.11) in the undirected case, whose primitive version has been established by Lin-Yau [24] (see Theorem 5 in [19], and also Proposition 1.5 in [24], Theorem 2 in [19]). We will calculate along the line of the proof of Theorem 2 in [19].

In view of Lemma 3.2, it suffices to prove that $\kappa_1(x, y)$ is bounded from below by the right hand side of (3.11). To do so, let us estimate $W(\nu_1^\varepsilon, \nu_2^\varepsilon)$ from above. Note that $\nu_1^\varepsilon(z) = \mathcal{P}(x, z)$
and \( \nu_y^1(z) = \mathcal{P}(y, z) \) for all \( z \in V \). From Proposition 2.6 we deduce
\[
W(\nu_x^1, \nu_y^1) = \sup_{f \in \text{Lip}_1(V)} \sum_{z \in V} f(z)(\nu_y^1(z) - \nu_x^1(z))
= \sup_{f \in \text{Lip}_1(V)} \left\{ \left( \sum_{z \in V \setminus \{x\}} (f(z) - f(y)) \mathcal{P}(y, z) \right) - \left( \sum_{z \in V \setminus \{y\}} (f(z) - f(x)) \mathcal{P}(x, z) \right) + (f(y) - f(x)) (1 - \mathcal{P}(x, y) - \mathcal{P}(y, x)) \right\}.
\]

The triangle inequality yields
\[
f(z) - f(y) \leq d(y, z), \quad f(z) - f(x) \geq -d(z, x), \quad |f(y) - f(x)| \leq D(x, y)
\]
for any \( f \in \text{Lip}_1(V) \), and hence
\[
W(\nu_x^1, \nu_y^1) \leq \sum_{z \in V \setminus \{x\}} d(y, z)\mathcal{P}(y, z) + \sum_{z \in V \setminus \{y\}} d(z, x)\mathcal{P}(x, z)
+ D(x, y) |1 - \mathcal{P}(x, y) - \mathcal{P}(y, x)|
= (-\mathcal{H}_y - d(y, x)\mathcal{P}(y, x)) + (-\mathcal{H}_z - d(y, x)\mathcal{P}(y, x))
+ D(x, y) \left( 2 (1 - \mathcal{P}(x, y) - \mathcal{P}(y, x)) + (1 - \mathcal{P}(x, y) - \mathcal{P}(y, x)) \right)
= \mathcal{H}(y, x) - d(y, x) (\mathcal{P}(x, y) + \mathcal{P}(y, x))
+ D(x, y) \left( 2 (1 - \mathcal{P}(x, y) - \mathcal{P}(y, x)) + (1 - \mathcal{P}(x, y) - \mathcal{P}(y, x)) \right)
= 2D(x, y) \left( 1 - \mathcal{P}(x, y) - \mathcal{P}(y, x) \right) - (D(x, y) - \mathcal{H}(y, x))
+ (D(x, y) - d(y, x)) (\mathcal{P}(x, y) + \mathcal{P}(y, x)),
\]
here we used (3.6), (3.7). This proves (3.10).

When \((x, y) \in E\), we have \( d(x, y) = 1 \). Furthermore,
\[
D(x, y) = \max \{d(x, y), d(y, x)\} = \max \{1, d(y, x)\} = d(y, x)
\]
since \( d(y, x) \geq 1 \). Substituting these equalities into (3.10), we see that the first term in the right hand of (3.10) becomes that of (3.11), the second term also does, and the third term vanishes. Thus we obtain (3.11).

\[\square\]

Remark 3.10. In the undirected case, under the same setting as in Proposition 3.9, we have \( d(y, x) = 1 \) and \( \mathcal{H}(x, y) = 2 \). In particular, the second term of the right hand side of (3.11) vanishes, and hence its right hand side coincides with that of Theorem 5 in \cite{19}.

We possess a refined lower bound for regular graphs (cf. Theorem 3 in \cite{19}).

Proposition 3.11. For \( r \geq 1 \), let \((V, \mu)\) be an \( r\)-regular graph. Then for all edge \((x, y) \in E\),
\[
\kappa(x, y) \geq \frac{1 - r}{2r} - \sum_{z \in N_x \setminus \N_y} \frac{\ln \text{Rad}_z V - 1}{2r},
\]
where \( N_x \) and \( \N_y \) are defined as (2.1), and \( \ln \text{Rad}_z V \) is done as (2.3).

Proof. We take a coupling between \( \nu_x^\varepsilon \) and \( \nu_y^\varepsilon \). Our transfer plan moving \( \nu_x^\varepsilon \) to \( \nu_y^\varepsilon \) should be as follows:
(1) Move the mass of \( 1 - \varepsilon \) from \( x \) to \( y \). The distance is 1;
(2) Move the mass of \( \varepsilon/2r \) from \( y \) to a fixed \( y_0 \in N_y \), and move the mass of \( \varepsilon/2r \) from a fixed \( x_0 \in \N_x \) to \( x \). These distances are 1;
and hence
\[ \sum_{z \in N_x \setminus \overline{N_y}} \] where we set \( N \)

Further, for every \( z \in N_x \setminus \overline{N_y} \) to a vertex in \( N_y \setminus \{y_0\} \). The distance is 3;

(5) Move the mass of \( \varepsilon/2r \) from a vertex \( z \in N_x \setminus \overline{N_y} \) to a vertex in \( \overline{N_y} \setminus N_x \). The distance is at most \( \text{InRad}_z V - 1 \).

By this transfer plan, calculating the Wasserstein distance from \( \nu_x^\varepsilon \) to \( \nu_y^\varepsilon \), we have
\[
W(\nu_x^\varepsilon, \nu_y^\varepsilon) \leq (1 - \varepsilon) \times 1 + \frac{\varepsilon}{2r} \times 1 + \frac{\varepsilon}{2r} \times 1 + (r - 1) \frac{\varepsilon}{2r} \times 3
+ \sum_{z \in N_x \setminus \overline{N_y}} \frac{\varepsilon}{2r} \times (\text{InRad}_z V - 1)
= 1 - \varepsilon \left( \frac{1 - r}{2r} - \sum_{z \in N_x \setminus \overline{N_y}} \frac{\text{InRad}_z V - 1}{2r} \right).
\]

This completes the proof. \( \square \)

We next examine an upper bound (cf. Theorems 4 and 7 in [19]).

**Proposition 3.12.** For every edge \((x, y) \in E\) we have

\[
(3.12) \quad \kappa(x, y) \leq \mathcal{P}(x, y) + \mathcal{P}(y, x) + \left( \sum_{z \in N_x \cap N_y} \mathcal{P}(x, z) \right) \land \left( \sum_{z \in N_x \cap N_y} \mathcal{P}(y, z) \right),
\]

where \( s \land t := \min\{s, t\} \).

**Proof.** We show the desired inequality by modifying the proof of Theorem 4 in [19]. Take a coupling \( \pi \) of \((\nu_x^\varepsilon, \nu_y^\varepsilon)\). Note that \( \nu_x^\varepsilon \) is supported on \( \{x\} \cup N_x \). Hence \( \pi \) is supported on

\[
N_\pi := (N_{xy} \times N_{xy}) \setminus ((N_y \setminus \{(x) \cup N_x\}) \times N_{xy}) \cup (N_{xy} \times (N_x \setminus \{(y) \cup N_y\})),
\]

where we set \( N_{xy} := N_x \cup N_y \). Let us define a subset \( N_{\pi, 0} \) of \( N_\pi \) as

\[
N_{\pi, 0} := N_x \setminus \{(x, x), (y, y)\} \cup ((N_x \cap N_y) \times (N_x \cap N_y)).
\]

Notice that \( d(z, z') \geq 1 \) for all \((z, z') \in N_{\pi, 0}\). It follows that

\[
(3.13) \quad \sum_{z, z' \in V} d(z, z') \pi(z, z') \geq \sum_{(z, z') \in N_{\pi, 0}} \pi(z, z')
= 1 - \left( \pi(x, x) + \pi(y, y) + \sum_{(z, z') \in (N_x \cap N_y) \times (N_x \cap N_y)} \pi(z, z') \right).
\]

By \( \pi \in \Pi(\nu_x^\varepsilon, \nu_y^\varepsilon) \), we have

\[
(3.14) \quad \pi(x, x) \leq \sum_{z \in V} \pi(z, x) = \nu_x^\varepsilon(x) = \varepsilon \mathcal{P}(y, x), \quad \pi(y, y) \leq \sum_{z' \in V} \pi(y, z') = \nu_y^\varepsilon(y) = \varepsilon \mathcal{P}(y, y).
\]

Further, for every \( z \in N_x \cap N_y \) we see

\[
\sum_{z' \in N_x \cap N_y} \pi(z, z') \leq \sum_{z' \in V} \pi(z, z') = \nu_x^\varepsilon(z) = \varepsilon \mathcal{P}(x, z),
\]

and hence

\[
(3.15) \quad \sum_{(z, z') \in (N_x \cap N_y) \times (N_x \cap N_y)} \pi(z, z') \leq \varepsilon \sum_{z \in N_x \cap N_y} \mathcal{P}(x, z).
\]
Similarly, for every $z' \in \mathcal{N}_x \cap \mathcal{N}_y$,
\[
\sum_{z \in \mathcal{N}_x \cap \mathcal{N}_y} \pi(z, z') \leq \sum_{z \in V} \pi(z, z') = \nu^*_{\varepsilon}(z') = \varepsilon \mathcal{P}(y, z'),
\]
and thus
\[
(3.16) \quad \sum_{(z, z') \in (\mathcal{N}_x \cap \mathcal{N}_y) \times (\mathcal{N}_y \cap \mathcal{N}_y)} \pi(z, z') \leq \varepsilon \sum_{z' \in \mathcal{N}_y \cap \mathcal{N}_y} \mathcal{P}(y, z').
\]
We now combine (3.13), (3.14), (3.15), (3.16). Since $\pi$ is arbitrary, we conclude
\[
W(\nu^*_{\varepsilon}, \nu^*_{\varepsilon}) \geq 1 - \varepsilon \left( \mathcal{P}(x, y) + \mathcal{P}(y, x) + \left( \sum_{z \in \mathcal{N}_x \cap \mathcal{N}_y} \mathcal{P}(x, z) \right) \wedge \left( \sum_{z \in \mathcal{N}_y \cap \mathcal{N}_y} \mathcal{P}(y, z) \right) \right).
\]
By $d(x, y) = 1$ and by the definition of $\kappa(x, y)$, we arrive at the desired one. \qed

Remark 3.13. We observe that the right hand side of (3.12) is at most
\[
\mathcal{P}(x, y) + \mathcal{P}(y, x) + \sum_{z \in \mathcal{N}_x \cap \mathcal{N}_y} \mathcal{P}(x, z).
\]
that is smaller than or equal to $1 + \mathcal{P}(y, x)$. Therefore we can conclude a simpler estimate $\kappa(x, y) \leq 1 + \mathcal{P}(y, x)$.

3.3. Examples. In this subsection, we consider some examples, and calculate their Ricci curvature. For $K \in \mathbb{R}$, we say that $(V, \mu)$ has constant Ricci curvature $K$ if $\kappa(x, y) = K$ for all edges $(x, y) \in E$. In this case we write $\kappa(V, \mu) = K$.

We first present a directed graph of positive Ricci curvature.

Example 3.14. We consider the directed unweighted complete graph with $n$ vertices, denoted by $K_n$ (see Figure 1). Let $V = \{x_1, \cdots, x_n\}$ be a set of all vertices of $K_n$. For a vertex $x_1$, the probability measure $\nu^*_{\varepsilon}$ satisfies
\[
\nu^*_{\varepsilon}(x_i) = \begin{cases} 
1 - \varepsilon & \text{if } i = 1, \\
\varepsilon & \text{if } i \in \{2, n - 1\}, \\
\frac{\varepsilon}{n - 2} & \text{if } i \in \{3, \cdots, n - 2\}, \\
0 & \text{otherwise}.
\end{cases}
\]
We see $\kappa(K_3) = 3/2$. When $n \geq 4$, we have
\[
\kappa(x_1, x_i) = \begin{cases} 
1 & \text{if } i = 2, \\
\frac{2(n - 2) + 1}{2(n - 2)} & \text{if } i \in \{3, \cdots, n - 1\}.
\end{cases}
\]

We next present a flat directed graph.

Example 3.15. We consider the directed unweighted cycle with $n$ vertices, denoted by $C_n$ (see Figure 2). Let $V = \{x_1, \cdots, x_n\}$ be a set of all vertices on $C_n$. For a vertex $x_1$ we have
\[
\nu^*_{\varepsilon}(x_i) = \begin{cases} 
1 - \varepsilon & \text{if } i = 1, \\
\frac{\varepsilon}{2} & \text{if } i \in \{2, n\}, \\
0 & \text{otherwise}.
\end{cases}
\]
For $n \geq 4$ we see $\kappa(C_n) = 0$. 
We also provide a directed graph with negatively curved edges.

**Example 3.16.** We consider the directed graph shown in Figure 3. Since this graph is an Eulerian graph and the probability measure $\nu^x$ is

$$
\nu^x(z) = \begin{cases} 
1 - \varepsilon & \text{if } z = x, \\
\frac{\varepsilon}{4} & \text{if } z \in \mathcal{N}_x, \\
0 & \text{otherwise},
\end{cases}
$$

it is easy to show that $\kappa(x, y) = -2$. 

**Figure 1.** Directed complete graphs

**Figure 2.** Directed cycles

**Figure 3.** Directed graph with negatively curved edges
4. Curvature-dimension conditions

The aim of this section is to study the relation between our Ricci curvature and the curvature-dimension inequalities of Bakry-Émery type.

4.1. Curvature-dimension inequalities. Let us recall the notion of \( \Gamma \) operator (or \( \text{carré du champ} \)), and the \( \Gamma_2 \) operator (or \( \text{carré du champ itéré} \)) of Bakry-Émery [4] to formulate the curvature-dimension inequality. The \( \Gamma \) operator, and the \( \Gamma_2 \) operator for the (negative) Chung Laplacian \( \Delta \) are defined as follows (see [4], and also Chapter 14 in [37]):

\[
\Gamma(f_0, f_1) := \frac{1}{2} (\Delta(f_0 f_1) - f_0 \Delta f_1 - f_1 \Delta f_0),
\]

\[
\Gamma_2(f_0, f_1) := \frac{1}{2} (\Delta \Gamma(f_0, f_1) - \Gamma(f_0, \Delta f_1) - \Gamma(f_1, \Delta f_0))
\]

for functions \( f_0, f_1 : V \to \mathbb{R} \).

For a function \( f : V \to \mathbb{R} \), we define a function \( Gf : V \to \mathbb{R} \) by

\[
Gf(x) := \frac{1}{4} \sum_{y,z \in V} (f(x) - 2f(y) + f(z))^2 P(x, y)P(y, z).
\]

We begin with the following formulas:

**Proposition 4.1.** For all \( f : V \to \mathbb{R} \) we have

\[
\Gamma(f, f)(x) = \frac{1}{2} \sum_{y \in V} (f(x) - f(y))^2 P(x, y),
\]

\[
\Delta \Gamma(f, f)(x) = \frac{1}{2} \sum_{y,z \in V} (f(x) - 2f(y) + f(z))^2 P(x, y)P(y, z)
\]

\[
- \left( \sum_{y \in V} (f(x) - f(y)) P(x, y) \right) \left( \sum_{z \in V} (f(z) - f(y)) P(y, z) \right),
\]

\[
2\Gamma(f, \Delta f)(x) = - (\Delta f(x))^2 - \left( \sum_{y \in V} (f(x) - f(y)) P(x, y) \right) \left( \sum_{z \in V} (f(z) - f(y)) P(y, z) \right).
\]

In particular,

\[
\Gamma_2(f, f) = Gf - \Gamma(f, f) + \frac{1}{2}(\Delta f)^2.
\]

**Proof.** We recall that the Perron measure \( m \) and the value \( m_{xy} \) are defined as (2.5) and (2.10), respectively. Keeping in mind \( P(x, y) = m_{xy}/m(x) \), we can show the desired formulas from the same calculation as that done by Lin-Yau [24] (see Lemmas 1.4, 2.1 and (2.2) in [24], and also Subsection 2.2 in [19]). The calculation is left to the readers. \( \square \)

We define the **triangle function** \( \mathcal{T} : V \to \mathbb{R} \) as follows (cf. Subsection 3.1 in [19]):

\[
\mathcal{T}(x) := \inf_{y \in N_x} |N_x \cap N_y|,
\]

where \( |\cdot| \) denotes its cardinality.

Based on Proposition 4.1, we formulate the following curvature-dimension inequality:

**Theorem 4.2.** For all \( f : V \to \mathbb{R} \), we have

\[
\Gamma_2(f, f) \geq \frac{1}{2}(\Delta f)^2 + K \Gamma(f, f),
\]
where a function $\mathcal{K} : V \to \mathbb{R}$ is defined as

$$\mathcal{K}(x) := \left( \inf_{y \in \mathcal{N}_x} \mathcal{P}(y, x) \right) \left\{ 2 + \frac{T(x)}{2} \left( \inf_{y \in \mathcal{N}_z} \inf_{z \in \mathcal{N}_y} \frac{\mathcal{P}(y, z)}{\mathcal{P}(y, x)} \right) \right\} - 1.$$

Proof. Jost-Liu [19] have proved a similar curvature-dimension inequality in the undirected case (cf. Theorems 9 and 10 in [19]). We show the desired inequality along the line of the proof of Theorem 9 in [19]. In view of (4.2), it suffices to show that $\mathcal{G} f/\Gamma(f, f)$ is bounded from below by $\mathcal{K} + 1$. From (4.1) we deduce

$$\mathcal{G} f(x) = \frac{1}{4} \sum_{y, z \in V} (f(x) - 2f(y) + f(z))^2 \mathcal{P}(x, y) \mathcal{P}(y, z)$$

$$= \sum_{y \in V} (f(x) - f(y))^2 \mathcal{P}(x, y) \mathcal{P}(y, x)$$

$$+ \frac{1}{4} \sum_{y \in V, z \in V \setminus \{x\}} (f(x) - 2f(y) + f(z))^2 \mathcal{P}(x, y) \mathcal{P}(y, z)$$

$$\geq 2\mathcal{K}_0(x) \Gamma(f, f)(x) + \frac{1}{4} \sum_{y \in \mathcal{N}_x} \sum_{z \in \mathcal{N}_y \setminus \{x\}} (f(x) - 2f(y) + f(z))^2 \mathcal{P}(x, y) \mathcal{P}(y, z)$$

$$\geq 2\mathcal{K}_0(x) \Gamma(f, f)(x) + \frac{1}{4} \sum_{y \in \mathcal{N}_x} \sum_{z \in \mathcal{N}_y \cap \mathcal{N}_y} (f(x) - 2f(y) + f(z))^2 \mathcal{P}(x, y) \mathcal{P}(y, z),$$

where a function $\mathcal{K}_0 : V \to \mathbb{R}$ is defined as

$$\mathcal{K}_0(x) := \inf_{y \in \mathcal{N}_x} \mathcal{P}(y, x).$$

We estimate the second term of the right hand side of (4.3). Define $\mathcal{G}_0 f : V \to \mathbb{R}$ by

$$\mathcal{G}_0 f(x) := \sum_{y \in \mathcal{N}_x} \sum_{z \in \mathcal{N}_x \cap \mathcal{N}_y} (f(x) - 2f(y) + f(z))^2 \mathcal{P}(x, y) \mathcal{P}(y, z).$$

By using $\mathcal{P}(x, y) = m_{xy}/m(x)$, we rewrite $\mathcal{G}_0 f$ as

$$\mathcal{G}_0 f(x) = \frac{1}{m(x)} \sum_{y \in \mathcal{N}_x} \sum_{z \in \mathcal{N}_x \cap \mathcal{N}_y} \frac{m_{xy}}{m(y)} (f(x) - 2f(y) + f(z))^2 m_{yz}$$

$$= \frac{1}{m(x)} \sum_{y \in \mathcal{N}_x} \mathcal{P}(y, x) \sum_{z \in \mathcal{N}_x \cap \mathcal{N}_y} (f(x) - 2f(y) + f(z))^2 m_{yz}.$$

In particular,

$$\mathcal{G}_0 f(x) \geq \frac{\mathcal{K}_0(x)}{m(x)} \sum_{y \in \mathcal{N}_x} \sum_{z \in \mathcal{N}_x \cap \mathcal{N}_y} (f(x) - 2f(y) + f(z))^2 m_{yz}.$$

We now observe that for $y \in \mathcal{N}_x$ and $z \in \mathcal{N}_x \cap \mathcal{N}_y$,

$$(f(x) - 2f(y) + f(z))^2 m_{yz} + (f(x) - 2f(z) + f(y))^2 m_{zy}$$

$$\geq ((f(x) - 2f(y) + f(z))^2 + (f(x) - 2f(z) + f(y))^2) m_{yz}$$

$$= ((f(x) - f(y))^2 + 4(f(y) - f(z))^2 + (f(x) - f(z))^2) m_{yz}$$

$$\geq \mathcal{K}_1(x) ((f(x) - f(y))^2 m_{xy} + (f(x) - f(z))^2 m_{xz}),$$
where a function $K_1 : V \to \mathbb{R}$ is defined as

$$K_1(x) := \inf_{y \in N_x} \inf_{z \in N_y \cap N_z} \frac{m_{yz}}{m_{yx} m_{yx}} = \inf_{y \in N_x} \inf_{z \in N_y \cap N_z} \frac{\mathcal{P}(y, z)}{\mathcal{P}(y, x)}.$$  

From (4.5), and the triangle argument which is the main idea of the proof of Theorem 9 in [19], it follows that the right hand side of (4.4) is greater than or equal to

$$\frac{K_0(x) K_1(x)}{m(x)} \sum_{y \in N_x} |N_x \cap N_y| (f(x) - f(y))^2 m_{xy}.$$  

Hence, (4.1) leads us to

$$\mathcal{G}_0 f(x) \geq \mathcal{T}(x) K_0(x) K_1(x) \sum_{y \in N_x} (f(x) - f(y))^2 \mathcal{P}(x, y) = 2\mathcal{T}(x) K_0(x) K_1(x) \Gamma(f, f)(x).$$  

Therefore,

$$\mathcal{G} f(x) \geq 2K_0(x) \Gamma(f, f)(x) + \frac{1}{4} \mathcal{G}_0 f(x) \geq K_0(x) \left(2 + \frac{1}{2} \mathcal{T}(x) K_1(x)\right) \Gamma(f, f)(x) = (K(x) + 1) \Gamma(f, f)(x).$$  

This completes the proof. \qed

We can immediately derive the following simple one from Theorem 4.2:

**Corollary 4.3.** For all $f : V \to \mathbb{R}$, we have

$$\Gamma_2(f, f) \geq \frac{1}{2} (\Delta f)^2 + \tilde{K} \Gamma(f, f)$$

on $V$, where a function $\tilde{K} : V \to \mathbb{R}$ is defined as

$$\tilde{K}(x) := 2 \inf_{y \in N_x} \mathcal{P}(y, x) - 1.$$  

Lin-Yau [24] have established Corollary 4.3 in the undirected case (see Theorems 1.2 and 1.3 in [24], and also Theorem 8 in [19]).

**Remark 4.4.** In the unweighted case, the third author [39] has shown

$$\Gamma_2(f, f) \geq \frac{1}{2} (\Delta f)^2 + K' \Gamma(f, f)$$

for a function $K' : V \to \mathbb{R}$ defined as

$$K'(x) := \min \left\{ \inf_{y \in N_x} \mathcal{P}(y, x), \inf_{y \in N_x} \frac{\mathcal{P}(y, x)}{\mathcal{P}(y, x)} \right\} - 1,$$  

which is slightly weaker than Corollary 4.3.

4.2. **Ricci curvatures and curvature-dimension inequalities.** By Proposition 3.12 we obtain the following relation between our Ricci curvature and curvature-dimension inequality:

**Corollary 4.5.** For $K \in \mathbb{R}$, we assume $\inf_{x,y} \kappa(x, y) \geq K$, where the infimum is taken over all $x, y \in V$ with $x \neq y$. Then for every $f : V \to \mathbb{R}$ we have

$$\Gamma_2(f, f) \geq \frac{1}{2} (\Delta f)^2 + \tilde{K} \Gamma(f, f)$$

on $V$, where a function $\tilde{K} : V \to \mathbb{R}$ is defined as

$$\tilde{K}(x) := 2K - 3 + \frac{K - 1}{2} \mathcal{T}(x) \left( \inf_{y \in N_x} \inf_{z \in N_y \cap N_z} \frac{\mathcal{P}(y, z)}{\mathcal{P}(y, x)} \right).$$
Proof. Let us fix \( x \in V \). For every \( y \in N_x \), Proposition 3.12 implies \( \kappa(x, y) \leq 1 + P(y, x) \), and hence \( P(y, x) \geq K - 1 \) (see Remark 3.13). Similarly, in virtue of Proposition 3.12, we also see \( P(y, x) \geq K - 1 \) for all \( y \in \overline{N}_x \). Thus we obtain \( \inf_{y \in N_x} P(y, x) \geq K - 1 \). Due to Theorem 4.2, we complete the proof. \( \square \)

Also, combining Proposition 3.12 with Corollary 4.3 implies:

**Corollary 4.6.** For \( K \in \mathbb{R} \), we assume \( \inf_{x, y} \kappa(x, y) \geq K \), where the infimum is taken over all \( x, y \in V \) with \( x \neq y \). Then for every \( f : V \to \mathbb{R} \) we have

\[
\Gamma_2(f, f) \geq \frac{1}{2}(\Delta f)^2 + (2K - 3)\Gamma(f, f).
\]

5. Comparison geometric results

In the present section, we study various comparison geometric results.

5.1. Diameter and eigenvalue comparisons. In this subsection, we formulate a diameter comparison of Bonnet-Myer type, and an eigenvalue comparison of Lichnerowicz type, which have been obtained by Lin-Lu-Yau [23] in the undirected case.

We first state the following diameter comparison:

**Theorem 5.1.** Let \( x, y \in V \) with \( x \neq y \). If \( \kappa(x, y) > 0 \), then

\[
d(x, y) \leq H(x, y) \kappa(x, y),
\]

where \( H(x, y) \) is defined as (3.8).

**Proof.** We complete the proof by letting \( \varepsilon \to 0 \) in (3.9). \( \square \)

Lin-Lu-Yau [23] have proved Theorem 5.1 in the undirected case (see Theorem 4.1 in [23]).

We next produce an eigenvalue comparison. We denote by \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \) the eigenvalues of \( L \). We here notice that for any non-zero function \( f : V \to \mathbb{R} \), its associated Rayleigh quotient is given by

\[
\mathcal{R}(f) := \frac{1}{2} \sum_{x, y \in V} (f(y) - f(x))^2 m_{xy} / (f, f)
\]

in view of Proposition 2.3. To derive an eigenvalue comparison, for \( \varepsilon > 0 \), we consider the \( \varepsilon \)-averaging operator \( \mathcal{A}_\varepsilon : \mathcal{F} \to \mathcal{F} \) defined as

\[
\mathcal{A}_\varepsilon f(x) := \sum_{z \in V} f(z) \nu_\varepsilon(z),
\]

where \( \nu_\varepsilon \) is defined as (3.1). Let us verify the following:

**Lemma 5.2.** For \( L > 0 \), let \( f : V \to \mathbb{R} \) be an \( L \)-Lipschitz function. For \( K > 0 \), we assume \( \inf_{x, y \in V} \kappa_\varepsilon(x, y) \geq K \), where \( \kappa_\varepsilon \) is defined as (3.3), and the infimum is taken over all \( x, y \in V \) with \( x \neq y \). Then \( \mathcal{A}_\varepsilon f \) is \( (1 - \varepsilon K)L \)-Lipschitz.

**Proof.** Let \( x, y \in V \). Using Proposition 2.6, we have

\[
\mathcal{A}_\varepsilon f(y) - \mathcal{A}_\varepsilon f(x) = \sum_{z \in V} f(z)(\nu_\varepsilon(z) - \nu_\varepsilon(z)) \\
\leq LW(\nu_\varepsilon, \nu_\varepsilon) = (1 - \varepsilon \kappa_\varepsilon(x, y)) L d(x, y) \leq (1 - \varepsilon K) L d(x, y).
\]

This proves the lemma. \( \square \)
Based on Lemma 5.2, we show the following:

**Theorem 5.3.** For $K > 0$, we assume $\inf_{x,y} \kappa(x,y) \geq K$, where the infimum is taken over all $x, y \in V$ with $x \neq y$. Then we have

$$\lambda_1 \geq K.$$

**Proof.** This estimate has been obtained by Lin-Lu-Yau [23] in the undirected case (see Theorem 4.2 in [23], and cf. Proposition 30 in [30] and Theorem 4 in [7]). One can show the desired inequality by the same argument as in the proof of Theorem 4.2 in [23], or Proposition 30 in [30]. We only give an outline of the proof.

Let $F_c \subset F$ denote the orthogonal complement of the set of all constant functions on $V$.

For sufficiently small $\varepsilon > 0$ the spectral radius $\text{SpecRad}(A^\varepsilon)$ of $A^\varepsilon$ over $F_c$ is equal to $1 - \varepsilon \lambda_1$. On the other hand, the spectral radius is known to be characterized as

$$\text{SpecRad}(A^\varepsilon) = \lim_{k \to \infty} \| (A^\varepsilon)^k \|_{op}^{1/k},$$

here $\| \cdot \|_{op}$ is the operator norm induced from the norm

$$\| f \|_{\text{Var}}^2 := \frac{1}{2} \sum_{x, y \in V} (f(y) - f(x))^2 m_{xy}$$
on $F_c$. In view of Lemma 5.2, the same argument as in [23], [30] leads to

$$1 - \varepsilon \lambda_1 = \text{SpecRad}(A^\varepsilon) = \lim_{k \to \infty} \| (A^\varepsilon)^k \|_{op}^{1/k} \leq 1 - \varepsilon K.$$

This completes the proof. \(\square\)

In the undirected case, there are some further works concerning diameter comparisons and eigenvalue comparisons (see e.g., [7], [13]).

### 5.2. Volume comparisons.

In this subsection, for $x \in V$, we study volume comparisons for the (forward) metric sphere and metric ball defined as

$$S_R(x) := \{ y \in V \mid \rho_x(y) = R \}, \quad B_R(x) := \{ y \in V \mid \rho_x(y) \leq R \}.$$

To formulate our volume comparison, we prepare the following quantities:

$$D_x := \sup_{y \in N_x} \rho_x(y), \quad \widehat{D}_x := \sup_{y \in N_x} \overleftarrow{\rho}_x(y), \quad \widehat{D} := \sup_{x \in V} \widehat{D}_x.$$

In the undirected case, these values are equal to 1. We also define

$$\mathfrak{M} := \inf_{x \in V} \inf_{y \in N_x} m_{xy}.$$

In the undirected and unweighted case, we see $\mathfrak{M} = 1$.

We show the following volume comparison result under the assumption that not $\kappa(x,y)$ but $\kappa_1(x,y)$ is bounded from below by constant (cf. Theorem 1 and Corollary 3 in [31]):

**Theorem 5.4.** Let $x \in V$. For $K \in \mathbb{R}$ we assume $\inf_{y \in V \setminus \{x\}} \kappa_1(x,y) \geq K$. Then

$$\frac{m(S_{R+1}(x))}{m(S_R(x))} \leq \frac{D_x + \frac{1}{2} \widehat{D} - KR}{(1 + \widehat{D})\mathfrak{M}},$$

where $m(S_R(x))$ is defined as (2.6).

**Proof.** Paeng [31] has obtained a similar result in the undirected and unweighted case (see Theorem 1 and Corollary 3 in [31]). We will prove the desired inequality along the line of the proof of Theorem 1 in [31]. Since $m$ is a probability measure, we have

$$m(S_{R+1}(x)) \leq \sum_{y \in S_R(x)} m(N_y \cap S_{R+1}(x)) \leq \sum_{y \in S_R(x)} |N_y \cap S_{R+1}(x)|,$$
where $| \cdot |$ denotes its cardinality. Here we used

$$S_{R+1}(x) = \bigcup_{y \in S_R(x)} (N_y \cap S_{R+1}(x)).$$

Let us fix $y \in S_R(x)$, and take an optimal coupling $\pi$ of $(\nu_x^1, \nu_y^1)$. It holds that

$$(1 - \kappa_1(x, y))R = (1 - \kappa_1(x, y))d(x, y) = W(\nu_x^1, \nu_y^1) = \sum_{z \in N_x} \sum_{z' \in N_y} d(z, z')\pi(z, z')$$

$$= \sum_{z \in N_x} \sum_{z' \in N_y \cap S_R(x)} d(z, z')\pi(z, z') + \sum_{z \in N_x} \sum_{z' \in N_y \setminus S_{R+1}(x)} d(z, z')\pi(z, z').$$

For all $z \in N_x$ and $z' \in N_y \cap S_{R+1}(x)$ we see

$$d(z, z') \geq d(x, z') - d(x, z) \geq (R + 1) - D_x = R - D_x + 1.$$

On the other hand, for all $z \in N_x$ and $z' \in N_y \setminus S_{R+1}(x)$,

$$d(z, z') \geq d(x, y) - d(x, z) - d(z', y) \geq R - D_x - \frac{\kappa}{R} \geq R - D_x - \frac{\kappa}{R}.$$ 

Therefore from $\nu_x^1(z') = \mathcal{P}(x, z')$ and $\nu_y^1(z') = \mathcal{P}(y, z')$, we conclude

$$(1 - K)R \geq (1 - \kappa_1(x, y))R$$

$$\geq (R - D_x + 1) \sum_{z \in N_x} \sum_{z' \in N_y \cap S_R(x)} \pi(z, z') + (R - D_x - \frac{\kappa}{R}) \sum_{z \in N_x} \sum_{z' \in N_y \setminus S_{R+1}(x)} \pi(z, z')$$

$$= (R - D_x + 1) \sum_{z' \in N_y \cap S_{R+1}(x)} \mathcal{P}(y, z') + (R - D_x - \frac{\kappa}{R}) \sum_{z' \in N_y \setminus S_{R+1}(x)} \mathcal{P}(y, z')$$

$$= (1 + \frac{\kappa}{R}) \sum_{z' \in N_y \cap S_{R+1}(x)} \mathcal{P}(y, z') + (R - D_x - \frac{\kappa}{R}).$$

By using $\mathcal{P}(y, z) = m_{yz}/m(y)$, we obtain

$$\mathcal{M}|N_y \cap S_{R+1}(x)| \leq \sum_{z \in N_y \cap S_{R+1}(x)} m_{yz} \leq \frac{D_x + \frac{\kappa}{R} - KR}{1 + \frac{\kappa}{R}} m(y).$$

This yields

$$(5.2) \sum_{y \in S_R(x)} |N_y \cap S_{R+1}(x)| \leq \frac{D_x + \frac{\kappa}{R} - KR}{(1 + \frac{\kappa}{R})\mathcal{M}} m(S_R(x)).$$

Combining (5.1) and (5.2), we arrive at the desired inequality. \hfill \Box

One can also conclude the following results by using Theorem 5.4 along the line of the proof of Theorem 1 in [31].

**Corollary 5.5.** Let $x \in V$. For $K \in \mathbb{R}$ we assume $\inf_{y \in V \setminus \{x\}} \kappa_1(x, y) \geq K$. Then we have

$$m(S_R(x)) \leq m(x) \prod_{i=0}^{R-1} \left( \frac{D_x + \frac{\kappa}{R} - i K}{(1 + \frac{\kappa}{R})\mathcal{M}} \right).$$

**Corollary 5.6.** Let $x \in V$. For $K \in \mathbb{R}$ we assume $\inf_{y \in V \setminus \{x\}} \kappa_1(x, y) \geq K$. Then we have

$$m(B_R(x)) \leq m(x) \left( 1 + \sum_{j=1}^{R} \prod_{i=0}^{j-1} \left( \frac{D_x + \frac{\kappa}{R} - i K}{(1 + \frac{\kappa}{R})\mathcal{M}} \right) \right).$$
Corollary 5.6 can be viewed as an analogue of Bishop (or rather Heintze-Karcher) volume comparison theorem on Riemannian manifold under a lower Ricci curvature bound.

In the undirected case, there is a further work on volume comparisons (see [9]).

5.3. Laplacian comparisons. In this subsection, we prove Theorem 1.1.

Let \( x, y \in V \) with \( x \neq y \). We define the gradient operator by

\[
\nabla_{xy} f := \frac{f(y) - f(x)}{d(x, y)}
\]

for \( f : V \to \mathbb{R} \). Notice that if \( f \) is \( L \)-Lipschitz, then \( \nabla_{xy} f \leq L \). We first show the following:

**Lemma 5.7.** Let \( x, y \in V \) with \( x \neq y \). We have

\[
(5.3) \quad \frac{\kappa_\varepsilon(x, y)}{\varepsilon} = \inf_{f \in \text{Lip}_1(V)} \left( \frac{1}{\varepsilon} (1 - \nabla_{xy} f) + \nabla_{xy} f \right).
\]

**Proof.** Using Proposition 2.6 and Lemma 3.1, we have

\[
W(\nu_x^\varepsilon, \nu_y^\varepsilon) = \sup_{f \in \text{Lip}_1(V)} \sum_{z \in V} f(z) \left( \nu_x^\varepsilon(z) - \nu_y^\varepsilon(z) \right)
= \sup_{f \in \text{Lip}_1(V)} \left( (f(y) + \varepsilon \Delta f(y)) - (f(x) + \varepsilon \Delta f(x)) \right)
= d(x, y) \sup_{f \in \text{Lip}_1(V)} \nabla_{xy}(f + \varepsilon \Delta f).
\]

This leads us that

\[
\frac{\kappa_\varepsilon(x, y)}{\varepsilon} = \inf_{f \in \text{Lip}_1(V)} \left( \frac{1}{\varepsilon} (1 - \nabla_{xy} (f + \varepsilon \Delta f)) \right) = \inf_{f \in \text{Lip}_1(V)} \left( \frac{1}{\varepsilon} (1 - \nabla_{xy} f) + \nabla_{xy} \mathcal{L} f \right).
\]

We complete the proof. \( \Box \)

We set

\[
\mathcal{F}_{xy} := \{ f \in \text{Lip}_1(V) \mid \nabla_{xy} f = 1 \}.
\]

Note that \( \rho_x \) belongs to \( \mathcal{F}_{xy} \). Let us prove the following formula, which was obtained by Münch-Wojciechowski [27] in the undirected case (see Theorem 2.1 in [27]):

**Proposition 5.8.** Let \( x, y \in V \) with \( x \neq y \). Then we have

\[
\kappa(x, y) = \inf_{f \in \mathcal{F}_{xy}} \nabla_{xy} \mathcal{L} f.
\]

**Proof.** From Lemma 5.7 we deduce

\[
\frac{\kappa_\varepsilon(x, y)}{\varepsilon} = \inf_{f \in \text{Lip}_1(V)} \left( \frac{1}{\varepsilon} (1 - \nabla_{xy} f) + \nabla_{xy} \mathcal{L} f \right) \leq \inf_{f \in \mathcal{F}_{xy}} \nabla_{xy} \mathcal{L} f.
\]

Letting \( \varepsilon \to 0 \), we obtain \( \kappa(x, y) \leq \inf_{f \in \mathcal{F}_{xy}} \nabla_{xy} \mathcal{L} f \).

Let us show the opposite inequality. From the finiteness of \((V, \mu)\), it follows that \( \text{Lip}_1(V) \) is compact with respect to the standard topology on \( \mathbb{R}^n \). For each \( \varepsilon > 0 \), by the compactness of \( \text{Lip}_1(V) \), and the continuity of the function

\[
f \mapsto \frac{1}{\varepsilon} (1 - \nabla_{xy} f) + \nabla_{xy} \mathcal{L} f,
\]

there exists a function \( f_\varepsilon \in \text{Lip}_1(V) \) which attains the infimum in the right hand side of (5.3). Using the compactness of \( \text{Lip}_1(V) \) again, we can find a sequence \( \{ \varepsilon_k \}_{k=1}^\infty \) of positive numbers with \( \varepsilon_k \to 0 \) as \( k \to \infty \) such that \( f_{\varepsilon_k} \) converges to some \( f_0 \in \text{Lip}_1(V) \). The limit of \( \kappa_\varepsilon(x, y)/\varepsilon \) exists due to Lemmas 3.2 and 3.4, and hence Lemma 5.7 yields

\[
\nabla_{xy} f_0 = \lim_{k \to \infty} \nabla_{xy} f_{\varepsilon_k} = 1.
\]
This means $f_0 \in F_{xy}$. Thus we conclude
\[
\kappa(x, y) = \lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon} (1 - \nabla_{xy} f_\varepsilon) + \nabla_{xy} L f_\varepsilon \right) \geq \lim_{k \to \infty} \nabla_{xy} L f_\varepsilon_k = \nabla_{xy} L f_0 \geq \inf_{f \in F_{xy}} \nabla_{xy} L f,
\]
where the first inequality follows from $\nabla_{xy} f_\varepsilon \leq 1$. This completes the proof. □

We are now in a position to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let $x \in V$. For $K \in \mathbb{R}$ we assume $\kappa(x, y) \geq K$ at $y \in V \setminus \{x\}$. For $\Lambda \in (-\infty, -1]$ we further assume $H_{x} \geq \Lambda$. Since $\rho_x$ belongs to $F_{xy}$, Proposition 5.8 yields
\[
K \leq \kappa(x, y) \leq \nabla_{xy} L f = L \rho_x(y) - L \rho_x(x) \frac{d(x, y)}{d(x, y)} \leq L \rho_x(y) - \Lambda \frac{d(x, y)}{d(x, y)},
\]
and hence (1.1). Thus we complete the proof of Theorem 1.1. □

5.4. Note on Laplacian comparisons. As already mentioned in Subsection 1.1, on Riemannian manifolds with a lower Ricci curvature bound, it is well-known that several comparison geometric results hold for hypersurfaces with a lower mean curvature bound. We now compare Theorem 1.1 with a Laplacian comparison on weighted manifolds with boundary under a lower Ricci curvature bound, and a lower mean curvature bound for the boundary obtained in [33]. We will find a similarity between them.

Let $(M, g, \phi)$ be a weighted Riemannian manifold with boundary with weighted measure
\[
m_\phi := e^{-\phi} v_g,
\]
where $v_g$ is the Riemannian volume measure. The weighted Laplacian is defined as
\[
L_\phi := L_g + g(\nabla \phi, \nabla \cdot),
\]
here $\nabla$ is the gradient. The weighted Ricci curvature is defined as follows ([4], [22]):
\[
\text{Ric}_\phi := \text{Ric}_g + \text{Hess} \phi,
\]
where $\text{Ric}_g$ is the Ricci curvature determined by $g$, and Hess is the Hessian. Let $\text{Ric}_{\phi, M}$ be its infimum over the unit tangent bundle. Let $\text{Int} M$ and $\partial M$ stand for the interior and boundary of $M$, respectively. Let $\rho_{\partial M} : M \to \mathbb{R}$ denote the distance function from $\partial M$ defined as $\rho_{\partial M} := d_g(\partial M, \cdot)$, which is smooth on $\text{Int} M \setminus \text{Cut} \partial M$. Here $\text{Cut} \partial M$ is the cut locus for the boundary (for its precise definition, see e.g., Subsection 2.3 in [32]). For $z \in \partial M$, the weighted mean curvature of $\partial M$ at $z$ is defined as
\[
\mathcal{H}_{\phi, z} := H_{g, z} + g(\nabla \phi, u_z),
\]
where $H_{g, z}$ is the (inward) mean curvature induced from $g$, and $u_z$ is the unit inner normal vector on $\partial M$ at $z$. Set $\mathcal{H}_{\phi, \partial M} := \inf_{z \in \partial M} \mathcal{H}_{\phi, z}$.

The second author [33] has shown the following Laplacian comparison inequality under a similar lower curvature bound to that of Theorem 1.1 (see Lemma 6.1 in [33]):

Lemma 5.9 ([33]). For $K \in \mathbb{R}$ we assume $\text{Ric}_{\phi, M} \geq K$. For $\Lambda \in \mathbb{R}$ we further assume $\mathcal{H}_{\phi, \partial M} \geq \Lambda$. Then on $\text{Int} M \setminus \text{Cut} \partial M$, we have
\[
(5.4) \quad L_\phi \rho_{\partial M} \geq K \rho_{\partial M} + \Lambda.
\]

One can observe that the form of our Laplacian comparison inequality (1.1) is same as that of (5.4). The second author [33] derived a relative volume comparison of Heintze-Karcher type from Lemma 5.9 (see Theorem 6.3 in [33], and cf. [17], Theorem 2 in [26]).
6. Dirichlet eigenvalues of $p$-Laplacian

Let $\mathcal{V}$ denote a non-empty subset of $V$ with $\mathcal{V} \neq V$. The purpose of this last section is to establish a lower bound of the Dirichlet eigenvalues of the $p$-Laplacian on $\mathcal{V}$ under our lower curvature bounds.

6.1. Dirichlet $p$-Poincaré constants. Let $p \in (1, \infty)$. For a non-zero function $f : V \to \mathbb{R}$, its $p$-Rayleigh quotient is defined by

$$R_p(f) := \frac{1}{2} \sum_{x,y \in V} |f(y) - f(x)|^p m_{xy} \sum_{x \in V} |f(x)|^p m(x).$$

We define the Dirichlet $p$-Poincaré constant over $\mathcal{V}$ by

$$\lambda^D_p(\mathcal{V}) := \inf_{f \in \mathcal{F}_\mathcal{V} \setminus \{0\}} R_p(f),$$

where $\mathcal{F}_\mathcal{V}$ denotes the set of all function $f : V \to \mathbb{R}$ with $f|_{V \setminus \mathcal{V}} = 0$.

We briefly mention the relation between the Dirichlet $p$-Poincaré constant and the Dirichlet eigenvalues of $p$-Laplacian (cf. [15], [18]). The $p$-Laplacian $L_p : \mathcal{F} \to \mathcal{F}$ is defined by

$$L_p f(x) := \sum_{y \in V} |f(x) - f(y)|^{p-2}(f(x) - f(y))P(x, y).$$

The $2$-Laplacian $L_2$ coincides with the Chung Laplacian $\mathcal{L}$. A real number $\lambda$ is said to be a Dirichlet eigenvalue of $L_p$ on $V$ if there is a non-zero function $f \in \mathcal{F}_V$ such that

$$L_p f = \lambda |f|^{p-2} |f|.$$

The smallest Dirichlet eigenvalue of the $p$-Laplacian $L_p$ on $V$ can be variationally characterized as $\lambda^D_p(V)$.

6.2. Cheeger inequalities. We first formulate an inequality of Cheeger type in our setting to derive a lower bound of the Dirichlet $p$-Poincaré constant. We will refer to the argument of the proof of Theorem 4.8 in [16], and Theorem 3.5 in [21]. We introduce the Dirichlet isoperimetric constant for $\mathcal{V}$. For a non-empty $\Omega \subset V$, its boundary measure is defined as

$$m(\partial \Omega) := \sum_{y \in \Omega} m_{yz}.$$

We define the Dirichlet isoperimetric constant on $\mathcal{V}$ by

$$I^D_V := \inf_{\Omega} \frac{m(\partial \Omega)}{m(\Omega)} ,$$

where $m(\Omega)$ is defined as (2.6), and the infimum is taken over all non-empty subsets $\Omega \subset V$.

For $f : V \to \mathbb{R}$ and $t \in \mathbb{R}$, we set

$$\Omega_{f,t} := \{x \in V \mid f(x) > t\}.$$

We present the following co-area formula (cf. Lemma 3.4 in [16]):

**Lemma 6.1.** For every $f : V \to \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} m(\partial \Omega_{f,t}) \, dt = \frac{1}{2} \sum_{y,z \in V} |f(y) - f(z)| m_{yz}.$$

*Proof.* For an interval $I \subset \mathbb{R}$, let $1_I$ denote its indicator function. For each $t \in \mathbb{R}$ we see that $m(\partial \Omega_{f,t})$ is equal to

$$\sum_{y \in \Omega_{f,t}} \sum_{z \in V \setminus \Omega_{f,t}} m_{yz} = \sum_{f(z) < f(y)} 1_{\{f(z), f(y)\}}(t) m_{yz},$$
where the summation in the right hand is taken over all ordered pairs \((y, z) \in V \times V\) with \(f(z) < f(y)\). Integrating the above equality with respect to \(t\) over \((-\infty, \infty)\), we deduce
\[
\int_{-\infty}^{\infty} m(\partial \Omega_{f,t}) \, dt = \sum_{f(z) < f(y)} \int_{-\infty}^{\infty} 1_{[f(z), f(y))}(t) \, m_{yz}
= \sum_{f(z) < f(y)} (f(y) - f(z)) \, m_{yz} = \frac{1}{2} \sum_{y,z \in V} |f(y) - f(z)| \, m_{yz}.
\]
We obtain the desired equality.

Lemma 6.1 leads us to the following (cf. Lemma 4.9 in [16]):

**Lemma 6.2.** For every non-negative function \(f \in \mathcal{F}_V\),
\[
\frac{1}{2} \sum_{y,z \in V} |f(y) - f(z)| \, m_{yz} \geq \mathcal{I}^D_V \sum_{x \in V} f(x) \, m(x).
\]

**Proof.** Since \(f \in \mathcal{F}_V\), the set \(\Omega_{f,t}\) is contained in \(V\) for every \(t \geq 0\), and hence \(m(\partial \Omega_{f,t}) \geq \mathcal{I}^D_V \, m(\Omega_{f,t})\). Lemma 6.1 implies
\[
\frac{1}{2} \sum_{y,z \in V} |f(y) - f(x)| \, m_{yz} = \int_{0}^{\infty} m(\partial \Omega_{f,t}) \, dt \geq \mathcal{I}^D_V \int_{0}^{\infty} m(\Omega_{f,t}) \, dt
= \mathcal{I}^D_V \sum_{x \in V} \int_{0}^{\infty} 1_{[0,f(x))}(t) \, dt = \mathcal{I}^D_V \sum_{x \in V} f(x) \, m(x).
\]
This proves the lemma. \(\square\)

We recall the following inequality that has been obtained by Amghibech [3] (see [3], and see also Lemma 3.8 in [21]):

**Lemma 6.3 ([3], [21]).** Let \(p \in (1, \infty)\). Then for every non-negative function \(f : V \to \mathbb{R}\), and for all \(x, y \in V\) we have
\[
|f(y)^p - f(x)^p| \leq p |f(y) - f(x)| \left( \frac{f(y)^p + f(x)^p}{2} \right)^{1/q},
\]
where \(q\) is determined by \(p^{-1} + q^{-1} = 1\).

Summarizing the above lemmas, we conclude the following inequality of Cheeger type (cf. Theorem 4.8 in [16]):

**Proposition 6.4.** For \(p \in (1, \infty)\) we have
\[
\chi^D_p(V) \geq \frac{2^{p-1}}{p^p} (\mathcal{I}^D_V)^p.
\]

**Proof.** We fix \(f \in \mathcal{F}_V \setminus \{0\}\). We apply Lemma 6.2 to a non-negative function \(|f|^p\) which belongs to \(\mathcal{F}_V\). Using Lemma 6.3 and the triangle inequality, we obtain
\[
\mathcal{I}^D_V \sum_{x \in V} |f(x)|^p \, m(x) \leq \frac{1}{2} \sum_{x,y \in V} ||f(y)|^p - |f(x)|^p| \, m_{xy}
\leq \frac{p}{2} \sum_{x,y \in V} ||f(y)| - |f(x)|| \left( \frac{|f(y)|^p + |f(x)|^p}{2} \right)^{1/q} \, m_{xy}
\leq \frac{p}{2} \sum_{x,y \in V} |f(y) - f(x)| \left( \frac{|f(y)|^p + |f(x)|^p}{2} \right)^{1/q} \, m_{xy},
\]
where \( q \) is determined by \( p^{-1} + q^{-1} = 1 \). The Hölder inequality yields
\[
\mathcal{I}_V^D \sum_{x \in V} |f(x)|^p \mathbf{m}(x) \leq \frac{p}{2} \sum_{x,y \in V} |f(y) - f(x)| m_{xy}^{1/p} \left( \frac{|f(y)|^p + |f(x)|^p}{2} \mathbf{m}_{xy} \right)^{1/q}
\]
\[
= \frac{p}{2} \left( \sum_{x,y \in V} |f(y) - f(x)|^p \mathbf{m}_{xy} \right)^{1/p} \left( \frac{1}{2} \sum_{x,y \in V} (|f(y)|^p + |f(x)|^p) \mathbf{m}_{xy} \right)^{1/q}.
\]

We possess
\[
(\mathcal{I}_V^D)^p \leq \left( \frac{p}{2} \right)^p \frac{\sum_{x,y \in V} |f(y) - f(x)|^p \mathbf{m}_{xy}}{\sum_{x \in V} |f(x)|^p \mathbf{m}(x)} = \frac{p^p}{2^{p-1}} \mathcal{R}_p(f).
\]

Thus we arrive at the desired inequality. \( \square \)

**Remark 6.5.** We provide a brief historical remark on the Cheeger inequality (without boundary condition) for graphs (for more details, cf. [35] and the references therein). Alon-Milman [2], Alon [1] established the Cheeger inequality for undirected graphs, and for the graph Laplacian. Chung [11] extended it to the directed case. Amghibech [3] generalized it for the graph \( p \)-Laplacian in the undirected case.

### 6.3. Dirichlet eigenvalue estimates.

For \( x \in V \) and \( R \geq 1 \) we set
\[
E_R(x) := \{ y \in V \mid \rho_x(y) \geq R \}.
\]

We obtain the following isoperimetric inequality for \( E_R(x) \):

**Proposition 6.6.** Let \( x \in V \). For \( K \in \mathbb{R} \) we assume \( \inf_{y \in V \setminus \{x\}} \kappa(x,y) \geq K \). For \( \Lambda \in (-\infty, -1] \) we also assume \( \mathcal{H}_x \geq \Lambda \). For \( D > 0 \) we further assume \( \text{InRad}_x V \leq D \). Then for every \( R \geq 1 \) with \( KR + \Lambda > 0 \), we have
\[
\mathcal{I}_{E_R(x)}^D \geq \frac{KR + \Lambda}{D}.
\]

**Proof.** Fix a non-empty \( \Omega \subset E_R(x) \). From Proposition 2.3 we derive
\[
- \sum_{y \in \Omega} \mathcal{L} \rho_x(y) \mathbf{m}(y) = \sum_{y \in \Omega} \sum_{z \in V \setminus \Omega} (\rho_x(z) - \rho_x(y)) \mathbf{m}_{yz} \geq - \sum_{y \in \Omega} \sum_{z \in V \setminus \Omega} \rho_x(y) \mathbf{m}_{yz} \geq -D \mathbf{m}(\partial \Omega).
\]

On the other hand, Theorem 1.1 leads us that for all \( y \in \Omega \)
\[
\mathcal{L} \rho_x(y) \geq K \rho_x(y) + \Lambda \geq KR + \Lambda.
\]

Therefore,
\[
(KR + \Lambda) \mathbf{m}(\Omega) \leq \sum_{y \in \Omega} \mathcal{L} \rho_x(y) \mathbf{m}(y) \leq D \mathbf{m}(\partial \Omega).
\]

We complete the proof. \( \square \)

**Remark 6.7.** On Riemannian manifolds with boundary with a lower Ricci curvature bound and a lower mean curvature bound for the boundary, it is well-known that one can derive a lower bound of its Dirichlet isoperimetric constant from a Laplacian comparison theorem for the distance function from the boundary, and integration parts formula (see Proposition 4.1 in [20], Lemma 8.9 in [32], and cf. Theorem 15.3.5 in [34]). Proposition 6.6 can be viewed as an analogue of such a result on manifolds with boundary (cf. Subsection 5.4).
We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $x \in V$ and $p \in (1, \infty)$. For $K \in \mathbb{R}$ we assume $\inf_{y \in V \setminus \{x\}} \kappa(x, y) \geq K$. For $\Lambda \in (-\infty, -1]$ we also assume $\mathcal{H}_x \geq \Lambda$. For $D > 0$ we further assume $\text{InRad}_x V \leq D$.

Combining Propositions 6.4 and 6.6, we have

$$\lambda_p^D (E_R(x)) \geq \frac{2^{p-1}}{p^p} \left( \frac{\mathcal{X}^D (\mathcal{X}^D (x))}{p^p} - \frac{K + \Lambda}{D} \right)^p.$$ 

We arrive at the desired inequality (1.2). Thus we complete the proof of Theorem 1.2. 

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