Counting false entries in truth tables of bracketed formulae connected by implication

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July 14, 2010

Abstract

In this paper we count the number of rows $f_n$ with the value “false” in the truth tables of all bracketed formulae with $n$ distinct variables connected by the binary connective of implication. We find a recurrence and an asymptotic formulae for $f_n$. We also show that the ratio of $f_n$ to the total number of rows converges to $(3 - \sqrt{3})/6$.

Keywords: Propositional logic, implication, Catalan numbers, asymptotics
AMS classification: 05A15, 05A16, 03B05
1 Introduction

In this paper we study enumerative and asymptotic questions on formulae of propositional calculus which are correctly bracketed chains of implications.

For brevity, we represent truth values of propositional variables and formulae by 1 for “true” and 0 for “false”.

We begin by stating some important notions of propositional logic. The propositional language consists of propositional variables \( p_1, p_2, \ldots, p_n \) and symbols called connectives. The well known connectives are ‘not’, ‘and’, ‘or’, ‘implies’, and ‘if and only if’, which we write as \( \neg, \land, \lor, \rightarrow, \) and \( \leftrightarrow \), respectively. The formulae of propositional logic, are expressions that can be obtained recursively from propositional variables by applying connectives. More precisely:

1. A propositional variable is a formula.
2. If \( \phi \) and \( \psi \) are formulae, then so are \( \neg \phi, \phi \rightarrow \psi, \phi \leftrightarrow \psi, \phi \land \psi, \phi \lor \psi \).

For unambiguity, brackets are also used in formulae. For example, we need to be able to distinguish \( p_1 \rightarrow (p_2 \rightarrow p_3) \) from \( (p_1 \rightarrow p_2) \rightarrow p_3 \). Note that, in U.K., left and right brackets are denoted by the symbols ‘[’ and ‘]’, respectively, whereas in U.S., they are denoted by the symbols ‘[’ and ‘]’.

Any formula, \( \phi \), which involves the propositional variables \( p_1, \ldots, p_n \) can be used to define a function of \( n \) variables, called ‘a truth function’ or ‘a propositional function’, that is, a function from \( \{0,1\}^n \) to \( \{0,1\} \). Since, \( |\{0,1\}^n| = |\{0,1\}|^n = 2^n \), the \( n \)-ary Cartesian product \( \{0,1\}^n \) has \( 2^n \) elements. Which is the number of rows of a truth table with \( n \) variables. As is well known, there are \( 2^{2^n} \) propositional functions, each of which can be represented by a formula involving the connectives \( \neg, \lor \) and \( \land \).

The function represented by a formula is conveniently calculated using a truth table. Where each row of the truth table corresponds to a valuation. A valuation is a function \( \nu \) from the set of propositions \( \{p_1, \ldots, p_n\} \) to the set \( \{1,0\} \). Thus a valuation is an assignment of values to the variables \( p_1, \ldots, p_n \), with consequent assignment of values to formulae.

For more information on standard propositional logic the reader can refer to the following books, [3] and [4].

We are interested in \textit{bracketed implications}, which are formulae obtained from \( p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_n \) by inserting brackets so that the result is well-formed, where \( p_1, \ldots, p_n \) are distinct propositions.
The binary connective $\rightarrow$ “implies” is defined as usual by the rule that, for any valuation $\nu$,

$$\nu(\phi \rightarrow \psi) = \begin{cases} 0 & \text{if } \nu(\phi) = 1 \text{ and } \nu(\psi) = 0, \\ 1 & \text{otherwise}. \end{cases}$$

**Example 1.1** Here are the truth tables, (merged into one), for the two bracketed implications in $n = 3$ variables. Where the corresponding rows with the value false are in blue:

| $p_1$ | $p_2$ | $p_3$ | $p_1 \rightarrow (p_2 \rightarrow p_3)$ | $(p_1 \rightarrow p_2) \rightarrow p_3$ |
|-------|-------|-------|---------------------------------------|--------------------------------------|
| 1     | 1     | 1     | 1                                     | 1                                    |
| 1     | 1     | 0     | 0                                     | 0                                    |
| 1     | 0     | 1     | 1                                     | 1                                    |
| 1     | 0     | 0     | 1                                     | 1                                    |
| 0     | 1     | 1     | 1                                     | 1                                    |
| 0     | 1     | 0     | 1                                     | 0                                    |
| 0     | 0     | 1     | 1                                     | 1                                    |
| 0     | 0     | 0     | 1                                     | 0                                    |

It is well known that two formulae are logically equivalent if they define the same propositional function. Consequently they must have the same truth table. Our concern is with the set of propositional functions defined by bracketed implications. The following uniqueness lemma shows that it suffices to work with the formulae.

**Lemma 1.2** Two bracketed implications are logically equivalent if and only if they are equal.

**Proof** We show how to recover the bracketing from the propositional function defined by such a formula. Our proof is by induction on $n$, the result is trivial for $n \leq 2$. Suppose that the proposition function defined by a formula on $t$ distinct variables $p_1, \ldots, p_t$, where $1 \leq t < n$, recovers the bracketing.

Let $\phi$ be a bracketed implication. Let valuations $\nu_i$ and $\nu_{i,j}$ be defined by

$$\nu_i(p_j) = \begin{cases} 0 & \text{if } j = i, \\ 1 & \text{otherwise}. \end{cases} \quad \nu_{i,j}(p_k) = \begin{cases} 0 & \text{if } k = i \text{ or } k = j, \\ 1 & \text{otherwise}. \end{cases}$$
Now it is straightforward to check that $\nu_n(\phi) = 0$, while $\nu_i(\phi) = 1$ for $i \neq n$.

Suppose that $\phi$ has the form $\psi \to \chi$, where $\psi$ and $\chi$ are bracketed implications involving $p_1, \ldots, p_r$ and $p_{r+1}, \ldots, p_n$ respectively. Then, for $i \leq r$, we have $\nu_{i,n}(\chi) = 0$, while $\nu_{i,n}(\psi) = 1$ if $i < r$, $\nu_{r,n}(\psi) = 0$. We conclude that $\nu_{i,n}(\phi) = 0$ if $i < r$ while $\nu_{r,n}(\phi) = 1$. Hence we can determine the value of $r$. By the induction hypothesis, the bracketings of $\psi$ and $\chi$ are determined by the propositional function, and hence the bracketing of $\phi$ is determined.

□

We could also consider permuted bracketed implications, which are formulae obtained from $p_1 \to p_2 \to \cdots \to p_n$ by permuting the propositions and then inserting brackets, where $p_1, \ldots, p_n$ are distinct propositions. More precisely: these are well-formed bracketings of $p_{\iota_1} \to p_{\iota_2} \to \cdots \to p_{\iota_n}$, where $(\iota_1, \ldots, \iota_n)$ is a permutation of $(1, \ldots, n)$. Here the situation is less satisfactory; we can count formulae, but the analogue of our uniqueness lemma does not hold (for example, $p_1 \to (p_2 \to p_3)$ and $p_2 \to (p_1 \to p_3)$ define the same propositional function), and we do not know how to count propositional functions represented by permuted bracketed implications, or the rows with value “false” in the corresponding truth tables.

2 The number of false rows

It is well known that the number of bracketings of a product of $n$ terms is the Catalan number

$$C_n = \frac{1}{n} \binom{2n-2}{n-1}, \text{ with } C_0 = 0$$

whose generating function is

$$\sum_{n \geq 1} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2}$$

(see [2 page 61]). Then $C_n$ is the number of bracketed implications in $n$ propositional variables, and by the uniqueness lemma of the preceding section, it is also the number of propositional functions or truth tables defined by such formulae.
Proposition 2.1 Let \( f_n \) be number of rows with the value “false” in the truth tables of all bracketed implications with \( n \) distinct variables. Then

\[
f_n = \sum_{i=1}^{n-1} (2^i C_i - f_i) f_{n-i}, \quad \text{with } f_0 = 0, \ f_1 = 1.
\]

Proof A row with the value false comes from an expression \( \psi \rightarrow \chi \) where \( \nu(\psi) = 1 \) and \( \nu(\chi) = 0 \). If \( \psi \) contains \( i \) variables, then \( \chi \) contains \( n - i \), and the number of choices is given by the summand in the proposition. \( \square \)

Example 2.2

\[
f_1 = 1, \ f_2 = (2^1 C_1 - f_1) f_1 = 1,
\]

and

\[
f_3 = (2^1 C_1 - f_1) f_2 + (2^2 C_2 - f_2) f_1 = 1 + 3 = 4
\]

which coincides with the result we had from Example 1.1.

Using this Proposition, it is straightforward to calculate the values of \( f_n \) for small \( n \). The first 22 values are

\[
\{f_n\}_{n \geq 1} = 1, 1, 4, 19, 104, 614, 3816, 24595, 162896, 1101922, 7580904, \\
52878654, 373100272, 2658188524, 19096607120, 138182654595, \\
1006202473888, 7367648586954, 54214472633064, \\
400698865376842, 2973344993337520, 22142778865313364, \ldots
\]

Let \( g_n \) be the total number of rows in all truth tables for bracketed implications with \( n \) variables. It is clear that \( g_n = 2^n C_n \), with \( g_0 = 0 \). Let \( F(x) \) and \( G(x) \) be the generating functions for \( f_n \), and \( g_n \), respectively. That is, \( F(x) = \sum_{n \geq 1} f_n x^n \), and \( G(x) = \sum_{n \geq 1} g_n x^n \). Then Proposition 2.1 gives

\[
F(x) = x + F(x)(G(x) - F(x)) \tag{1}
\]

where \( G(x) \) can be obtained from the generating function of \( C_n \) by replacing \( x \) by \( 2x \): that is,

\[
G(x) = (1 - \sqrt{1 - 8x})/2. \tag{2}
\]

Substituting the equation (2) into the equation (1) gives the following quadratic equation:

\[
2F(x)^2 + F(x) \left(1 + \sqrt{1 - 8x} \right) - 2x = 0. \tag{3}
\]

Solving equation (3) gives the following proposition:
Proposition 2.3 The generating function for the sequence \( \{f_n\}_{n \geq 1} \) is given by

\[
F(x) = -1 - \sqrt{1 - 8x} + \frac{\sqrt{2 + 2\sqrt{1 - 8x} + 8x}}{4}.
\]

(As with the Catalan numbers, the choice of sign in the square root is made to ensure that \( F(0) = 0 \).) With the help of Maple we can obtain the first 22 terms of the above series, and hence give the first 22 values of \( f_n \); these agree with the values found from the recurrence relation.

3 Asymptotic analysis

In this section we want to get an asymptotic formula for the coefficients of the generating function \( F(x) \) from Proposition [2.3] We use the following result [1, page 389]:

Proposition 3.1 Let \( a_n \) be a sequence whose terms are positive for sufficiently large \( n \). Suppose that \( A(x) = \sum_{n \geq 0} a_n x^n \) converges for some value of \( x > 0 \). Let \( f(x) = (-\ln(1 - x/r))^b(1 - x/r)^c \), where \( c \) is not a positive integer, and we do not have \( b = 0 \) and \( c = 0 \). Suppose that \( A(x) \) and \( f(x) \) each have a singularity at \( x = r \) and that \( A(x) \) has no singularities in the interval \((-r, r)\). Suppose further that \( \lim_{x \to r} \frac{A(x)}{f(x)} \) exists and has nonzero value \( \gamma \). Then

\[
a_n \sim \begin{cases} 
\gamma \left( \frac{n-c-1}{n} \right)^{b-n} (\ln n)^{b-n}, & \text{if } c \neq 0, \\
\gamma b(\ln n)^{b-1}, & \text{if } c = 0.
\end{cases}
\]

Note 3.2 We also have

\[
\binom{n-c-1}{n} \sim \frac{n^{-c-1}}{\Gamma(-c)},
\]

where the standard gamma-function

\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt
\]

satisfies \( \Gamma(x + 1) = x\Gamma(x) \) and \( \Gamma(1/2) = \sqrt{\pi} \). It follows that \( \Gamma(-1/2) = -\sqrt{\pi}/2 \).
Recall that \( G(x) = (1 - \sqrt{1-8x})/2 \), therefore
\[
F(x) = \frac{(G(x) - 1) + \sqrt{(1 - G(x))^2 + 4x}}{2}.
\]

Before studying \( F(x) \), we first study \( G(x) \). This \( G(x) \) could easily be studied by using the explicit formula for its coefficients, which is \( 2^n \binom{3n-2}{n-1}/n \). But our aim is to understand how to handle the square root singularity. A square root singularity occurs while attempting to raise zero to a power which is not a positive integer. Clearly the square root, \( \sqrt{1-8x} \), has a singularity at 1/8. Therefore by Proposition 3.1 \( r = 1/8 \). We have \( G(1/8) = 1/2 \), so we would not be able to divide \( G(x) \) by a suitable \( f(x) \) as required in Proposition 3.1. To create a function which vanishes at \( 1/8 \), we simply look at \( A(x) = G(x) - 1/2 \) instead. That is, let
\[
f(x) = (1 - x/r)^{1/2} = (1 - 8x)^{1/2}.
\]
Then
\[
\gamma = \lim_{x \to 1/8} \frac{A(x)}{\sqrt{1 - 8x}} = -\frac{1}{2}.
\]

Now by using Proposition 3.1 and Note 3.2,
\[
g_n \sim -\frac{1}{2} \left( \frac{n - 3}{n} \right) \left( \frac{1}{8} \right)^n \sim -\frac{1}{2} \frac{8^n n^{-3/2}}{\Gamma(-1/2)} = \frac{2^{3n-2}}{\sqrt{\pi n^3}}.
\]

We are now ready to tackle \( F(x) \), and state the main theorem of the paper.

**Theorem 3.3** Let \( f_n \) be number of rows with the value false in the truth tables of all the bracketed implications with \( n \) distinct variables. Then
\[
f_n \sim \left( \frac{3 - \sqrt{3}}{6} \right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}.
\]

**Proof** We have
\[
F(x) = \frac{-1 - \sqrt{1 - 8x} + \sqrt{2 + 2\sqrt{1 - 8x} + 8x}}{4}.
\]

We find that \( r = 1/8 \), and \( f(x) = \sqrt{1 - 8x} \). Since \( F(1/8) = (-1 + \sqrt{3})/4 \neq 0 \), we need a function which vanishes at \( F(1/8) \), thus we let \( A(x) = F(x) - F(1/8) \).
\[
\lim_{x \to 1/8} \frac{A(x)}{f(x)} = \lim_{x \to 1/8} \frac{-\sqrt{1-8x} + \sqrt{2+2\sqrt{1-8x}+8x} - \sqrt{3}}{4\sqrt{1-8x}}.
\]

Let \( v = \sqrt{1-8x} \). Then

\[
\gamma = \lim_{v \to 0} \frac{-v + \sqrt{(1+v)(3-v)} - \sqrt{3}}{4v} = \lim_{v \to 0} \frac{-v + \sqrt{3+2v-v^2} - \sqrt{3}}{4v} = \lim_{v \to 0} \frac{-1 + (1-v)(3+2v-v^2)^{-1/2}}{4} = \frac{3 - \sqrt{3}}{12},
\]

where we have used l'Hôpital's Rule in the penultimate line.

Finally,

\[
f_n \sim -\frac{3 - \sqrt{3}}{12} \left( \frac{n - \frac{3}{2}}{n} \right) \left( \frac{1}{8} \right)^{-n} \sim \left( \frac{3 - \sqrt{3}}{6} \right) \frac{2^{3n-2}}{\sqrt{\pi n^3}},
\]

and the proof is finished. □

The importance of the constant \((3 - \sqrt{3})/6 = 0.2113248654\) lies in the following fact:

**Corollary 3.4** Let \( g_n \) be the total number of rows in all truth tables for bracketed implications with \( n \) variables, and \( f_n \) the number of rows with the value “false”. Then \( \lim_{n \to \infty} f_n / g_n = (3 - \sqrt{3})/6 \).

The table below illustrates the convergence.
Corollary 3.5 Let $t_n$ be the number of rows with the value “true” in the truth tables of all bracketed formulae with \( n \) distinct variables connected by the binary connective of implication. Then

\[
  t_n = g_n - f_n, \quad \text{with } t_0 = 0,
\]

and for large \( n \),

\[
  t_n \sim \left( \frac{3 + \sqrt{3}}{6} \right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}.
\]

Using this Corollary 3.5, it is straightforward to calculate the values of \( t_n \). The table below illustrates this up to \( n = 10 \).

| \( n \) | \( f_n \) | \( g_n \) | \( f_n/g_n \) |
|--------|--------|--------|----------|
| 1      | 1      | 2      | 0.5      |
| 2      | 1      | 4      | 0.25     |
| 3      | 4      | 16     | 0.25     |
| 4      | 19     | 80     | 0.2375   |
| 5      | 104    | 448    | 0.2321428571 |
| 6      | 614    | 2688   | 0.228422619 |
| 7      | 3816   | 16896  | 0.2258522727 |
| 8      | 24595  | 109824 | 0.2239492279 |
| 9      | 162896 | 732160 | 0.2224868881 |
| 10     | 1101922| 4978688| 0.2213277876 |

For \( n = 100 \) the ratio is 0.2122908650, and for \( n = 1000 \) it is 0.2114211279.

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