An analytic derivation of clustering coefficients for weighted networks

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Abstract. Clustering coefficients are among the most important parameters characterizing the topology of complex networks and have a significant influence on various dynamical processes occurring on networks. On the other hand, a plethora of real-life networks with diverse links can be described better in terms of weighted networks than in terms of binary networks, where all links are homogeneous. However, analytical research on clustering coefficients in weighted networks is still lacking. In this paper, we apply an extended mean-field approach to investigate clustering coefficients for the typical weighted networks proposed by Barrat, Barthélemy and Vespignani (BBV networks) (2004 \textit{Phys. Rev. Lett.} \textbf{92} 228701). We provide an analytical solution to the model, showing how the local clustering of a node in the BBV networks depends on its degree and strength. Our analysis is in good agreement with the results of numerical simulations.

Keywords: disordered systems (theory), exact results
1. Introduction

In the past decade, studies on relatively simple binary (Boolean) networks where edges (links) are either present or absent [1]–[4] took a successful step in the direction of understanding the structure and function of complex networks. However, the connections in many real networks are not homogeneous, which naturally calls for a more sophisticated form, weighted networks [5]–[8], for better modeling these systems, where each connection is assigned a weight to denote a physical property of interest. For a social network, the link weight indicates the intimacy between two individuals [9,10]; for the Internet, the link weight describes the bandwidth of an optical cable between two routers [11]; for the worldwide aviation network, the link weights reflect the annual volume of passengers traveling between two airports [5]; and for the E. coli metabolic network, the link weight encodes the optimal metabolic fluxes between two metabolites [12]. Consequently, weighted networks encode more information and they are a more realistic representation of real systems where individual links (and nodes) are vastly different.

Previous studies on weighted networks have indicated the importance of the link weights of networks and led to the formulation of a long list of models better mimicking real systems [5]–[7], [13]–[26]. The first evolving weighted network was proposed by Yook et al [13], where the topology and weight are driven simultaneously by the preferential attachment rule [27]. However, to some extent, the evolutionary mechanism of weighting for the model did not match the real processes well. Instead, Barrat, Barthélemy, and Vespignani presented a growing model (BBV) [5]–[7] for weighted networks, where the topology and weight coevolve interdependently. Motivated by BBV’s remarkable work, a variety of models and mechanisms for weighted networks have been proposed. For example in a weight-driven model [14], each link carries a positive weight which is drawn from a certain distribution; in a fitness model [15], the weight and topology evolve independently; in traffic-driven evolution models [16]–[19], the evolution of weight and the topology mutually interact. Besides this, accelerating growth [20], deterministic models [21]–[23], weight-dependent deactivation [24], and spatial constraints [25,26] provide some interesting insight as well.

In the light of the influence on the studies of weighted networks mentioned above, the BBV model has been acknowledged to be a pioneer and classic work in physics.
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communities. The main contribution of the work can be ascribed to two aspects: a
new strength-driven preferential attachment mechanism; and local weight and strength
rearrangements for a new link. The results from this model represent many behaviors
observed empirically in real-world networks, e.g., link weight–degree correlation, node
strength–degree correlation, and scale-free topology. Thus, it is considered as a general
starting point for more realistic models as a representation of specific networks.

As is known to us all, a primary purpose of the current studies of complex networks is
to understand how their dynamical behaviors are influenced by underlying geometrical and
topological properties \[4, 28\]. Among many fundamental structural characteristics \[29\],
the clustering coefficient, also known as the transitivity, plays an important role in
the theory of complex networks and has attracted considerable attention. In the sociological
literature, for example, it is referred to as the ‘network density’. For a node \(i\) with \(k_i\) links,
its clustering coefficient \(C_i\) denotes the fraction of these allowable links that actually exist.
The average clustering coefficient for the whole system is defined as the average of \(C_i\) for
each node \[30\]. This quantity is very important because it is one of the two key criteria
for judging whether a network or a complex system exhibits the small-world property \[30\].
Also, it has a profound influence on a variety of crucial fields, such as those of epidemic
processes \[31\]–\[33\], network resilience \[34, 35\] and community structure detection \[36, 37\],
to name but a few. Very recently, on the basis of certain distributions of edges and
triangles, Newman were the first to provide an exact solution for the parameter \[38\] in an
interesting random graph. This work contributed an invaluable tool for investigating the
coefficients for some random graph models \[39\]–\[41\]. Unfortunately, the recipe cannot be
adopted directly for solving problems in networks where the connections are not randomly
created and the distributions of edges and triangles cannot be acquired easily, for example,
BA networks \[27\] and small-world networks \[30\]. So far, most previous related studies on
these networks have been confined to numerical methods, which is, however, prohibitively
difficult for large networks because of the limits of time and computer memory. Hence, it is
necessary to present an analytical solution for the clustering coefficient, toward providing
some useful insight into topological structures of weighted networks.

Despite its importance, to the best of our knowledge, the rigorous computation for
the clustering coefficient of the BBV model has not been addressed yet, contrasting
sharply with the successes as regards other network properties such as the degree, weight
and strength distribution \[7\]. To fill this gap, in this present paper we investigate this interesting quantity analytically. We derive an implicit formula for the clustering
coefficient characterizing the BBV model. The analytic method adopted in this work is
an extended mean-field approach. The precise result obtained shows that the average local
clustering coefficient of the BBV network decreases as a power law in the network order,
the total number of nodes in the networks. To some extent, our research opens a way to
theoretically studying the clustering coefficient of neutral or nonassortative mixing \[42\]
weighted networks. At the same time, our analytical solution may give insight different
from that afforded by the approximate solution of a numerical simulation.

2. Introduction to the BBV model

The BBV model starts from an initial seed of \(N_0\) vertices fully connected by links with
assigned weight \(w_0\). A new vertex \(n\) is added at each time step. This new site is connected

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to \( m \) previously existing vertices (i.e., each new vertex will have initially exactly \( m \) links, all with equal weight \( w_0 \)), choosing preferentially sites with large strength. In other words, an old node \( i \) is chosen according to the following probability:

\[
\Pi_{n\rightarrow i} = \frac{s_i}{\sum_v s_v} \tag{1}
\]

where \( s_i \) is the strength of node \( i \), defined as \( s_i = \sum_j w_{ij} \). The weight \( w_{in} \) of the new link \( L_{in} \) connecting nodes \( n \) and \( i \) is initially set to a given value \( w_0 \). The creation of this edge will introduce a local variations of the edge weight \( w_{ij} \rightarrow w_{ij} + \Delta w_{ij} \), in which node \( j \) is one of the neighbors of node \( i \) and its node strength \( s_j \rightarrow s_j + \Delta w_{ij} \), where this perturbation is proportionally distributed among the links according to their weights:

\[
\Delta w_{ij} = \frac{\delta_i}{s_i} w_{ij}. \tag{2}
\]

Simultaneously, this rule yields a total strength increase for node \( i \) of \( \delta_i + w_0 \), implying that \( s_i \rightarrow s_i + \delta_i + w_0 \). In this paper, we focus on the case of homogeneous coupling with \( \delta_i = \delta = \text{const} \) and equivalently set \( w_0 = 1 \). After the weights and strengths have been updated, the growth process is iterated by introducing a new vertex, until the desired size of the network is reached.

In the BBV networks, the dynamical process for \( s_i \) and \( k_i \) can thus be described through the following evolution:

\[
\frac{ds_i}{dt} = ms_i(t) \left(1 + \frac{\delta}{\sum_v s_v}\right) + \sum_{j \in \Omega(i)} m s_j(t) \frac{\delta w_{ij}(t)}{\sum_v s_v} \frac{\delta w_{ij}(t)}{s_j(t)}, \tag{3}
\]

\[
\frac{dk_i}{dt} = ms_i(t) \frac{\delta s_i(t)}{\sum_v s_v}. \tag{4}
\]

Note that \( \sum_{j \in \Omega(i)} \) runs over the nearest neighbors of the node \( i \). Solving these two equations with initial conditions \( k_i(t_i) = s_i(t_i) = m \), one has

\[
s_i = m \left(1 - \frac{t}{t_i}\right)^{(2\delta+1)/(2\delta+2)} \tag{5}
\]

(references \[7\]) and

\[
k_i = \frac{s_i + 2m\delta}{2\delta + 1}. \tag{6}
\]

After introducing the BBV model, we will investigate analytically the clustering coefficient for all the nodes in the BBV networks. As will be shown, like for the Barabási–Albert (BA) networks \[27\], the average clustering coefficient of the BBV networks rapidly decreases with increasing network size. In previous work \[7\], researchers have presented numerical calculations reporting this result, while we will offer an exact solution to explain it in what follows.
3. An analytical expression for clustering coefficients of the model

Before calculating the clustering coefficient, we first introduce a basic concept—‘what neutral weighted networks are’. In networks, the degree of a node is defined as the number of links that the node has. A relevant topological property is the degree–degree correlation, or so-called assortative mixing. It is defined as a preference for high-degree vertices to attach to other high-degree vertices and vice versa [42, 43]. In contrast, disassortative mixing is defined as a preference for high-degree vertices to preferentially connect with low-degree nodes and vice versa. However, a network having neutral degree correlation means that it is neither assortative nor disassortative. A random network, where links are randomly connected between nodes, is an example of a neutral network.

We introduce, along with the degree correlations, the weighted degree correlations [5], defined as

$$k_{nni}^w = \frac{1}{s_i} \sum_{j=1}^{N} a_{ij} w_{ij} k_j,$$

which measure the effective affinity for connecting with high-degree or low-degree neighbors according to the magnitude of the actual interactions. Also, the behavior of the function $k_{nni}^w(k)$ marks the weighted assortative or disassortative properties. When $k_{nni}^w(k)$ increases with $k$, it means that nodes have a tendency to connect to nodes with a similar or larger degree. In this case the network is defined as assortative. In contrast, if $k_{nni}^w(k)$ is decreasing with $k$, which implies that nodes having large degree are likely to have near neighbors with small degree, then the network is said to be disassortative. Naturally, if correlations are absent, this can be ascribed to weighted neutral networks.

Of particular interest is the BBV model mentioned in section 2 which can give rise to neutral networks for small $\delta$ [7]. However, for increasing $\delta$, the disassortative character and a power law behavior of the $k_{nn}(k)$ emerge. Fortunately, the weighted assortativity $k_{nn}^w(k)$ is a constant, that is, they belong to weighted neutral networks. In this case, we redefine the linking probability of nodes $i$ and $j$,

$$p_{ij} = \frac{s_j s_i}{\sum_v s_v} = \frac{s_j s_i}{2m(1 + \delta)tw_{ij}},$$

in order to extend the mean-field method to this particular network for large $\delta$.

In a BBV network, consider a certain node $i$ at time step $t$. Note that we only consider the case $m > 1$, in that the BBV network is close to tree-like networks when $m = 1$. As is known to us all, for a tree network, the clustering coefficient in the network is equal to zero. By definition [30], the clustering coefficient $C_i = 2E_i/(k_i(k_i - 1))$ of node $i$ depends on two variables, where $E_i$ is the number of links between the $k_i$ neighbors of $i$.

Since in the BBV model only new nodes may create links, the coefficient $C_i$ changes only when its degree $k_i$ changes, i.e., when new nodes create connections to $i$ and $x \in \langle 0, m - 1 \rangle$ of its nearest neighbors. The appropriate equation for changes of $C_i$ is then

$$\frac{dC_i}{dt} = \sum_{x=0}^{m-1} \hat{p}_{ix} \Delta C_x,$$
where $\Delta C_{ix}$ denotes the change of the clustering coefficient when a new node connects to the node $i$ and to $x$ of its first neighbors as well, whereas $\tilde{p}_{ix}$ describes the probability of this event.

Notice that, in the equation (9), $\Delta C_{ix}$ represents the difference between clustering coefficients of the same node $i$ calculated after and before a new node attachment:

$$\Delta C_{ix} = \frac{2(E_i + x)}{k_i(k_i + 1)} - \frac{2E_i}{k_i(k_i - 1)} = \frac{2x}{k_i(k_i + 1)} - \frac{2C_i}{k_i + 1},$$

where $\Delta C_{ix} > 0$. Also, equation (10) provides the restricted region of our method. The probability $\tilde{p}_{ix}$ is a product of two factors:

$$\tilde{p}_{ix} = \frac{s_i}{\sum_v s_v} \left( \frac{m - 1}{x} \right) P^x(1 - P)^{m-1-x}.$$

The first factor is the probability for a new link to end up in $i$, which is given by equation (1). The second one is the probability that $x$ among the rest of the $(m - 1)$ new links connect to the nearest neighbors of $i$. It is equivalent to the probability of $(m - 1)$ Bernoulli trials with the probability $P$ for $x$ successes. So, the probability $P$ can be expressed as:

$$P = \frac{\sum_{j \in \Omega(i)} s_j}{\sum_v s_v} = \frac{\sum_{j \in \Omega(i)} s_j}{2m(1 + \delta)t}.$$  

Replacing the sum $\sum_{j \in \Omega(i)} s_j$ by an integral, one obtains

$$\sum_{j \in \Omega(i)} s_j \sim \int t^1 s_j p_{ij} dt_j$$

where $p_{ij} = \frac{s_is_j}{\sum_v s_v w_{ij}} = \frac{s_is_j}{2m(1 + \delta)t w_{ij}},$ in which $w_{ij}$ is the weight of the link $L_{ij}$. In the BBV networks, $w_{ij}(t) = \left( \frac{t}{t_{ij}} \right)^{\delta/(\delta + 1)},$

where $t_{ij} = \max(i, j)$. Hence, using equation (13), one can find the probability

$$P = \frac{mt^{(4\delta-1)/(2\delta+2)}}{4(1 + \delta)\delta} \left( \frac{1}{2} - \frac{t^{-(2\delta+1)/(2\delta+2)} - t^{-(6\delta+1)/(2\delta+2)}}{2} - \frac{t^{-(2\delta+1)/(2\delta+2)} - t^{-1/(2\delta+2)}}{2} \right).$$

Now, inserting equations (5), (6), (10) and (11) into equation (9) one can obtain

$$\frac{dC_i}{dt} + \frac{m(2\delta + 1)t^{1/(2\delta+2)}C_i}{(1 + \delta)[mt^{(2\delta+1)/(2\delta+2)} + (2m\delta + 2\delta + 1)t_i^{(2\delta+1)/(2\delta+2)}]}$$

$$= \frac{m(m - 1)(2\delta + 1)}{2\delta(\delta + 1)^2} \frac{mt^{(2\delta+1)/(2\delta+2)} + (2m\delta + 2\delta + 1)t_i^{(2\delta+1)/(2\delta+2)}}{mt^{(2\delta+1)/(2\delta+2)} + (2m\delta + 2\delta + 1)t_i^{(2\delta+1)/(2\delta+2)}},$$

In the derivation process, we simplify

$$\frac{mt^{(2\delta+1)/(2\delta+2)} + (2m\delta + 2\delta + 1)t_i^{(2\delta+1)/(2\delta+2)}}{mt^{(2\delta+1)/(2\delta+2)} + (2m\delta + 2\delta + 1)t_i^{(2\delta+1)/(2\delta+2)}} \approx 1$$

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Figure 1. The initial value of the local clustering coefficient $C_i(t_i)$ for (a) $\delta = 0.1$ and (b) $\delta = 0.2$ (averaged over ten BBV networks). Red circles show the analytical solutions given by equation (20).

to make the analytical integral easily executable. Solving the equation for $C_i$ one gets

$$C_i(t) = \left[ mt^{(2\delta+1)/(2\delta+2)} + (2m\delta + 2\delta + 1)t_i^{(2\delta+1)/(2\delta+2)} \right]^{-2} \left[ B_i + \frac{m^2}{4(1+\delta)\delta^2} \right] \times \left( \frac{2t_i^{3\delta/(\delta+1)}t_i^{-(\delta/(\delta+1))} - t_i^{3\delta/(\delta+1)}t_i^{-(2\delta/(\delta+1))} - 6t_i^{\delta/(\delta+1)}}{6} \right),$$

(19)

where $B_i$ is an integration constant for the node $i$ and determined by the initial condition $C_i(t_i)$ that describes the clustering coefficient of the node $i$ exactly at the moment of its attachment $t_i$. Noting that $\Delta C_i > 0$ in equation (10), we have $1/k_i > C_i$. Thus, in this article, it is worth noticing that $\delta$ has to be confined to the region $[0, 1/4)$. The restriction is given by our approximation in equation (18). We believe that this drawback will be addressed in future work.

To clarify the calculation of $C_i(t_i)$, we show another form of the definition of $C_i$ [28, 30]:

$$C_i = \frac{\text{number of triangles connected to vertex } i}{\text{number of triples centered on vertex } i}.$$

It can be rewritten as

$$C_i(t_i) = \frac{\sum_j \sum_k (m^2 - t_i^{-(\delta/(\delta+1))})(1 - t_i^{-(2\delta/(\delta+1))}t_i^{(\delta-1)/(\delta+1)})}{16(m-1)(\delta+1)\delta^2}.$$

(20)

Figure 1 shows the prediction from equation (20) in comparison with numerical results. For small values of $t_i$, the numerical data differ from the theory significantly. This incongruity originates from the formula for the probability of a connection $p_{ij}$ in
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Figure 2. The local clustering coefficient \( C_i(t) \) as a function \( t_i \) for (a) \( \delta = 0.1 \) and (b) \( \delta = 0.2 \). Results are obtained by numerical simulations on the BBV networks (averaged over ten networks). Red lines show the analytical solutions given by equation (19), and black curves show the smoothed form of the simulation data (gray circles).

Taking into account the initial condition \( C_i(t_i) \) from equation (20), one can obtain \( B_i \) for a given node \( i \):

\[
B_i = \frac{m^2(2m\delta + 2\delta + m + 1)^2(1 - t_i^{-\delta/(\delta + 1)})(1 - t_i^{-2\delta/(\delta + 1)})t_i^{\delta/(\delta + 1)}}{16(m - 1)(\delta + 1)\delta^2}
\]

\[
- \frac{m^2}{4(1 + \delta)\delta^2} \left( \frac{t_i^{2\delta/(\delta + 1)}}{3} - \frac{7t_i^{\delta/(\delta + 1)}}{6} \right).
\]

Inserting \( B_i \) to equation (19), one can easily obtain \( C_i(t) \). Notice that both numerical results and analytical solutions of \( C_i(t) \) are shown in figure 2.

To obtain the average clustering coefficient \( C \) of the whole network, the expression of equation (21) has to be averaged over all nodes within a network. Admittedly, it is difficult to find an analytic result for \( C \) uniformly but we can show numerical solutions of BBV networks using the definition

\[
C = \frac{\sum_{i=1}^{t} C_i(t)}{t}.
\]

In figure 3, one can easily find that both the numerical and theoretical results exhibit a power law decay as the size of the networks grows. Hence, when \( t \) tends to infinity, \( C \) gets close to zero. As shown in figure 3, the power law exponent depends on the tunable parameter \( \delta \). For large \( \delta \), the decreasing velocity is relatively slow.
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Figure 3. The clustering coefficient $C$ of a whole BBV network as a function of the network size $t$ for different values of $\delta$. Results are obtained by numerical simulations on BBV networks (averaged over ten networks). The solid lines are numerical solutions given by equation (22).

Generally, the cluster coefficient plays a crucial role in the topology and dynamical processes on networks. Previous studies showed that for the BA networks [27], both local cluster coefficients $C_i(t)$ and their mean can be obtained using mean-field theory as the networks are uncorrelated [44]. For correlated networks, however, the mean-field method cannot be adopted simply. Hence, for most correlated networks, to present the cluster coefficient in an analytical way remains challenging. Admittedly, using an approach based on generating functions in the seminal work [38], Newman derived rigorous results for the parameter for a random graph. However, the key point of calculating the clustering coefficient lies in acquiring the distributions of these corresponding quantities first. Except for random graphs, for most normal networks (weighted or unweighted) reported, these distributions are uncertain and have to be calculated through numerical methods. Hence, comparatively few analytical investigations have been recorded in the literature relating to the parameter, both for random and deterministic weighted networks, as yet. For the BBV networks, we introduce an extended mean-field method for studying the property, as these networks are fortunately weight uncorrelated. Although specialized to the BBV networks here, this method may open up a new way to analytically investigating dynamics on the networks and their weight uncorrelated counterparts.

4. Conclusion

To sum up, the clustering coefficient is an important topological feature of complex networks. It has a profound impact in a variety of crucial fields, such as those of epidemic processes, network resilience and community detection, and so on. In this paper, we have derived analytically the solution for the clustering coefficient of the BBV model which has been attracting much research interest. We found that the average clustering coefficient decays as a power law function of the network order. Our analytical technique could
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guide and shed light on related studies for weighted networks by providing a paradigm for calculating the clustering coefficient.

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References

[1] Albert R and Barabási A-L, 2002 Rev. Mod. Phys. 74 47
[2] Dorogvtsev S N and Mendes J F F, 2002 Adv. Phys. 51 1079
[3] Newman M E J, 2003 SIAM Rev. 45 167
[4] Boccaletti S, Latora V, Moreno Y, Chavez M and Hwang D-U, 2006 Phys. Rep. 424 175
[5] Barrat A, Barthélémy M, Pastor-Satorras R and Vespignani A, 2004 Proc. Nat. Acad. Sci. 101 3747
[6] Barrat A, Barthélémy M and Vespignani A, 2004 Phys. Rev. Lett. 92 228701
[7] Barrat A, Barthélémy M and Vespignani A, 2004 Phys. Rev. E 70 066149
[8] Newman M E J, 2004 Phys. Rev. E 70 056131
[9] Newman M E J, 2001 Proc. Nat. Acad. Sci. 98 404
[10] Newman M E J, 2001 Phys. Rev. E 64 016133
[11] Albert R, Jeong H and Barabási A-L, 1999 Nature 401 130
[12] Almaas E, Kovács B, Oltvai Z N and Barabási A-L, 2004 Nature 427 839
[13] Yook S H, Jeong H, Barabási A-L and Tu Y, 2001 Phys. Rev. Lett. 86 5835
[14] Antal T and Krapivsky P L, 2005 Phys. Rev. E 71 026103
[15] Bianconi G, 2005 Europhys. Lett. 71 1029
[16] Wang W-X, Wang B-H, Hu B, Yan G and Ou Q, 2005 Phys. Rev. Lett. 94 188702
[17] Goh K I, Kahng B and Kim D, 2005 Phys. Rev. E 72 017103
[18] Wang W X, Hu B, Zhou T, Wang B H and Xie Y B, 2005 Phys. Rev. E 72 046140
[19] Wang W X, Hu B, Wang B H and Yan G, 2005 Phys. Rev. E 73 016133
[20] Zhang Z Z, Fang L J, Zhou S G and Guan J H, 2009 Physica A 388 225
[21] Dorogtsev S N and Mendes J F F, 2005 AIP Conf. Proc. 776 29
[22] Zhang Z Z, Zhou S G, Fang L J, Guan J H and Zhang Y C, 2007 Europhys. Lett. 79 38007
[23] Zhang Z Z, Zhou S G, Chen L C, Guan J H, Fang L J and Zhang Y C, 2007 Eur. Phys. J. B 59 99
[24] Wu Z X, Xu X J and Wang Y H, 2005 Phys. Rev. E 71 066124
[25] Barrat A, Barthélémy M and Vespignani A, 2005 J. Stat. Mech. P05003
[26] Mukherjee G and Manna S S, 2006 Phys. Rev. E 74 036111
[27] Barabási A-L and Albert R, 1999 Science 286 509
[28] Newman M E J, 2003 SIAM Rev. 45 167
[29] Costa L D F, Rodrigues F A, Travieso G and Boas P R V, 2007 Adv. Phys. 56 167
[30] Watts D J and Strogatz S H, 1998 Nature 393 440
[31] Pastor-Satorras R and Vespignani A, 2001 Phys. Rev. Lett. 86 3200
[32] Pastor-Satorras R and Vespignani A, 2001 Phys. Rev. E 63 066117
[33] Liljeros F, Edling C R, Amaral L A N, Stanley H E and Aberg Y, 2001 Nature 411 907
[34] Albert R, Jeong H and Barabási A L, 2000 Nature 406 378
[35] Moreira A A, Andrade J S, Herrmann H J and Ikedaou J O, 2009 Phys. Rev. Lett. 102 018701
[36] Newman M E J and Girvan M, 2004 Phys. Rev. E 69 026113
[37] Girvan M and Newman M E J, 2002 Proc. Nat. Acad. Sci. 99 7821
[38] Newman M E J, 2009 Phys. Rev. Lett. 103 058701
[39] Erdös P and Rényi A, 1959 Publ. Math. 6 290
[40] Newman M E J, Strogatz S H and Watts D J, 2001 Phys. Rev. E 64 026118
[41] Molloy M and Reed B, 1995 Random Struct. Algorithms 6 161
[42] Newman M E J, 2002 Phys. Rev. Lett. 89 208701
[43] Newman M E J, 2003 Phys. Rev. E 67 026126
[44] Fronczak A, Fronczak P and Holyst J A, 2003 Phys. Rev. E 68 046126

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