Some definite integrals of Srinivasa Ramanujan and its consequences

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Abstract
In this paper, we obtain analytical solutions of some definite integrals of Srinivasa Ramanujan [Mess. Math., XLIV, 75-86, 1915] in terms of Meijer’s G-function by using Laplace transforms of \( \sin(\beta x^2) \), \( \cos(\beta x^2) \), \( x\sin(\beta x^2) \) and \( x\cos(\beta x^2) \). Further, we obtain some infinite summation formulas connected with Meijer’s G-function and numeric values of some infinite series.

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1. Introduction and Preliminaries

For the sake of conciseness of this paper, we use the following notations [2, p.33]

\[ \mathbb{N} := \{1, 2, \ldots\}; \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \quad \mathbb{Z}^- := \mathbb{Z} \cup \{0\}, \]

and

\[ \mathbb{R}^+ := \{a \in \mathbb{R} : a > 0\} \quad ; \quad \mathbb{C}^+ := \{p \in \mathbb{C} : \Re(p) > 0\} \]

where the symbols \( \mathbb{N} \) and \( \mathbb{Z} \) denote the set of natural number and integers; as usual, the symbols \( \mathbb{R} \) and \( \mathbb{C} \) denote the set of real and complex numbers.

The following three theorems of Srinivasa Ramanujan are available in the collected papers [5] edited by Hardy- Aiyar- Wilson, without giving any analytical proofs. In our recent communication [8] we have verified numerically the following three theorems of Ramanujan, with the help of Wolfram Mathematica software[see [8] in Tables 6.1, 6.2; 7.1,7.2;8.1,8.2].

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**Theorem I**: The first theorem of Ramanujan edited by G.H. Hardy et.al [5, p.59, eq.(3) and (3')] and [9, p.75, eq.(3 and 3')] is given below.

If
\[ \Phi_1(n) = \int_0^\infty \frac{\cos(\pi nx^2)}{\cosh(\pi x)} \, dx, \quad (1.2) \]

and
\[ \Psi_1(n) = \int_0^\infty \frac{\sin(\pi nx^2)}{\cosh(\pi x)} \, dx, \quad (1.3) \]

then
\[ \Phi_1(n) = \sqrt{\frac{2}{n}} \, \Psi_1 \left( \frac{1}{n} \right) + \Psi_1(n), \quad (1.4) \]

and
\[ \Psi_1(n) = \sqrt{\frac{2}{n}} \, \Phi_1 \left( \frac{1}{n} \right) - \Phi_1(n), \quad (1.5) \]

where \( n \in \mathbb{R}_> \).

**Theorem II**: The second theorem of Ramanujan edited by G.H. Hardy et.al [5, p.60, eq.(6) and (6')] and [9, p.76, eq.(6 and 6')] is given below.

If
\[ \Phi_2(n) = \int_0^\infty \frac{\cos(\pi nx^2)}{\{1 + 2\cosh(2\pi x/\sqrt{3})\}} \, dx, \quad (1.6) \]

and
\[ \Psi_2(n) = \int_0^\infty \frac{\sin(\pi nx^2)}{\{1 + 2\cosh(2\pi x/\sqrt{3})\}} \, dx, \quad (1.7) \]

then
\[ \Phi_2(n) = \sqrt{\frac{2}{n}} \, \Psi_2 \left( \frac{1}{n} \right) + \Psi_2(n), \quad (1.8) \]

and
\[ \Psi_2(n) = \sqrt{\frac{2}{n}} \, \Phi_2 \left( \frac{1}{n} \right) - \Phi_2(n), \quad (1.9) \]

where \( n \in \mathbb{R}_> \).

**Theorem III**: The third theorem of Ramanujan edited by G.H. Hardy et.al [5, p.60, eq.(10) and (10')]; p.66, after eq.(44); see also [11] and [9, p.76-77, eq.(10 and 10')] is given below.

If
\[ \Phi_3(n) = \int_0^\infty \frac{\cos(\pi nx)}{\{-1 + \exp(2\pi \sqrt{x})\}} \, dx, \quad (1.10) \]

and
\[ \Psi_3(n) = \frac{1}{2\pi n} + \int_0^\infty \frac{\sin(\pi nx)}{\{-1 + \exp(2\pi \sqrt{x})\}} \, dx = \frac{1}{2\pi n} + \Psi_3(n), \quad (1.11) \]

then
\[ \Phi_3(n) = \frac{1}{n} \sqrt{\frac{2}{n}} \, \Psi_3 \left( \frac{1}{n} \right) - \Psi_3(n), \quad (1.12) \]
are available.

Again for particular values of \( n \), we get above numeric values of the integrals (1.14)-(1.17).

**Setting** \( n = 1 \), in the Ramanujan’s integrals (1.2),(1.3),(1.6) and (1.7), we have

\[
\Phi_1(1) = \int_0^\infty \frac{\cos(x^2)}{\cosh(x)} \, dx = \frac{1}{2\sqrt{2}},
\]

(1.14)

\[
\Psi_1(1) = \int_0^\infty \frac{\sin(x^2)}{\cosh(x)} \, dx = -1 + \sqrt{2},
\]

(1.15)

\[
\Phi_2(1) = \int_0^\infty \frac{\cos(x^2)}{\{1 + 2 \cosh(2\pi x/\sqrt{3})\}} \, dx = \frac{2 - \sqrt{6} + \sqrt{2}}{8},
\]

(1.16)

\[
\Psi_2(1) = \int_0^\infty \frac{\sin(x^2)}{\{1 + 2 \cosh(2\pi x/\sqrt{3})\}} \, dx = -\sqrt{12} + \sqrt{2} + \sqrt{6}/8.
\]

(1.17)

When we put \( t = 0 \) in the results [9, p.79, eq.23; p.80, eq.24 and p.84, eq.41], after simplification we get above numeric values of the integrals (1.14)-(1.17).

Again for particular values of \( n \), some values of Ramanujan’s integral \( \Phi_3(n) \) [9, p.85 (eq. 48)] are available.

\[
\Phi_3(1) = \int_0^\infty \frac{\cos(\pi x)}{\{-1 + \exp(2\pi\sqrt{x})\}} \, dx = \frac{2 - \sqrt{2}}{8},
\]

(1.18)

\[
\Phi_3(2) = \int_0^\infty \frac{\cos(2\pi x)}{\{-1 + \exp(2\pi\sqrt{x})\}} \, dx = \frac{1}{16};
\]

(1.19)

\[
\Phi_3(4) = \int_0^\infty \frac{\cos(4\pi x)}{\{-1 + \exp(2\pi\sqrt{x})\}} \, dx = \frac{3 - \sqrt{2}}{32},
\]

(1.20)

\[
\Phi_3(6) = \int_0^\infty \frac{\cos(6\pi x)}{\{-1 + \exp(2\pi\sqrt{x})\}} \, dx = \frac{13 - 4\sqrt{3}}{144},
\]

(1.21)

\[
\Phi_3\left(\frac{1}{2}\right) = \int_0^\infty \frac{\cos\left(\frac{x}{2}\right)}{\{-1 + \exp(2\pi\sqrt{x})\}} \, dx = \frac{1}{4\pi},
\]

(1.22)

\[
\Phi_3\left(\frac{2}{5}\right) = \int_0^\infty \frac{\cos\left(\frac{2x}{5}\right)}{\{-1 + \exp(2\pi\sqrt{x})\}} \, dx = \frac{8 - 3\sqrt{5}}{16}.
\]

(1.23)

From eqns (1.11) and (1.13), we obtain

\[
\Psi_3'(n) = \int_0^\infty \frac{\sin(\pi nx)}{\{-1 + \exp(2\pi\sqrt{x})\}} \, dx = \frac{1}{n} \sqrt{\left(\frac{2}{n}\right)} \Phi_3\left(\frac{1}{n}\right) + \Phi_3(n) - \frac{1}{2\pi n}.
\]

(1.24)

Setting \( n = 1, 2, \frac{1}{2} \) in the above eq. (1.24), using values of \( \Phi_3(1), \Phi_3(2) \) and \( \Phi_3\left(\frac{1}{2}\right) \) from the eqns (1.18),(1.19) and (1.22), after simplification we get the following three results:

\[
\Psi_3'(1) = \int_0^\infty \frac{\sin(\pi x)}{\{-1 + \exp(2\pi\sqrt{x})\}} \, dx = \frac{\pi\sqrt{2} - 4}{8\pi},
\]

(1.25)
\[
\Psi^r_1(2) = \int_0^\infty \frac{\sin(2\pi x)}{\{-1 + \exp(2\pi \sqrt{x})\}} \, dx = \frac{\pi - 2}{16\pi},
\]
(1.26)  \\
\[
\Psi^r_2(1/2) = \int_0^\infty \frac{\sin(\pi \sqrt{x})}{\{-1 + \exp(2\pi \sqrt{x})\}} \, dx = \frac{\pi - 3}{4\pi}.
\]
(1.27)

Binomial function is given by

\[
(1 - z)^{-a} = 1_F_0 \begin{pmatrix} a; \, z \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n,
\]
(1.28)

where \(|z| < 1, \ a \in \mathbb{C}.

The Meijer’s \( G \) function is defined by means of the Mellin-Barnes type contour integral [10, Sec.(1.5), eq. (1)]. When \( k = 1, 2, \ldots, n \) and \( \ell = 1, 2, \ldots, m \), and \( \alpha_k - \beta_{\ell} \neq 0 \)

then

\[
G_{m,n}^{p,q} \begin{pmatrix} | \alpha_1, \ldots, \alpha_n; \alpha_{n+1}, \ldots, \alpha_p \rangle \\ \beta_1, \ldots, \beta_m; \beta_{m+1}, \ldots, \beta_q \end{pmatrix} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \prod_{j=1}^{m} \Gamma(\beta_j - \zeta) \prod_{j=1}^{n} \Gamma(1 - \alpha_j + \zeta) - \prod_{j=m+1}^{p} \Gamma(\alpha_j - \zeta) \prod_{j=n+1}^{q} \Gamma(1 - \beta_j + \zeta) (z)^{\zeta} \, d\zeta,
\]
(1.29)

where \( z \neq 0 \), and \( m, n, p, q \) are non negative integers such that \( 1 \leq m \leq q; 0 \leq n \leq p \).

Suppose:

\[
\triangle = m + n - \frac{1}{2}(p + q),
\]
(1.30)

and

\[
\omega = (\beta_1 + \ldots + \beta_m + \beta_{m+1} + \ldots + \beta_q) - (\alpha_1 + \ldots + \alpha_n + \alpha_{n+1} + \ldots + \alpha_p).
\]
(1.31)

(i) If \(|\arg(z)| < \triangle \pi \) and \( \triangle > 0 \), then the integral (1.29) is converges.
(ii) If \(|\arg(z)| = \triangle \pi \) and \( \triangle \geq 0 \), then the integral (1.29) converges absolutely when \( p = q \) and \( \Re(\omega) < -1 \).
(iii) If \(|\arg(z)| = \triangle \pi \) and \( \triangle \geq 0 \), then the integral (1.29) also converges absolutely when \( p \neq q \) and

\[
(q - p) \xi > 1 - \left( \frac{q - p}{2} \right) + \Re(\omega),
\]
(1.32)

where \( \zeta = \xi + i\eta; \xi \) is so chosen that, for \( \eta \to +\infty; \xi \) and \( \eta \) are real.

Elementary property of G-function is given by [10, p.46, eq. (6)]

\[
G_{m,n}^{p,q} \begin{pmatrix} | \alpha_1, \ldots, \alpha_n; \alpha_{n+1}, \ldots, \alpha_p \rangle \\ \beta_1, \ldots, \beta_m; \beta_{m+1}, \ldots, \beta_q \end{pmatrix} = G_{q,p}^{m,n} \begin{pmatrix} \frac{1}{z} | 1 - \beta_1, \ldots, 1 - \beta_m; 1 - \beta_{m+1}, \ldots, 1 - \beta_q \end{pmatrix},
\]
(1.33)
when \( p > q \), then we can apply above property \((1.33)\) through out the paper.

The representation of sine function in terms of G-Function [3, 6, 7, 11], is given by

\[
\sin(z) = \sqrt{\pi} \, G_{0.2}^{1,0} \left( \frac{z^2}{4} ; \frac{1}{2} , 0 \right), \tag{1.34}
\]

\[
= \sqrt{\pi} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma\left(1 - \zeta\right)}{\Gamma\left(1/2 + \zeta\right)} \left(\frac{z^2}{4}\right)^{\zeta} d\zeta. \tag{1.35}
\]

The above Mellin-Barnes type contour integral representation \((1.35)\) of sine function is obtained by using the definition \((1.29)\) in the equation \((1.34)\).

The representation of cosine function in terms of G-Function [3, 6, 7, 11] is given by

\[
\cos(z) = \sqrt{\pi} \, G_{0.2}^{1,0} \left( \frac{z^2}{4} ; 0, \frac{1}{2} \right), \tag{1.36}
\]

\[
= \sqrt{\pi} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma\left(-\zeta\right)}{\Gamma\left(1/2 + \zeta\right)} \left(\frac{z^2}{4}\right)^{\zeta} d\zeta. \tag{1.37}
\]

The above Mellin-Barnes type contour integral representation \((1.37)\) of cosine function is obtained by using the definition \((1.29)\) in the equation \((1.36)\).

For every positive integer \( m \) \([10, \text{p.22, Eq.(26)}]\), we have

\[
\Gamma(mz) = (2\pi)^{(1-m)/2} \, m^{mz-1/2} \prod_{j=1}^{m} \Gamma \left( z + \frac{j-1}{m} \right), \quad mz \in \mathbb{C} \setminus \mathbb{Z}_0. \tag{1.38}
\]

The equation \((1.38)\) is known as Gauss-Legendre multiplication theorem for Gamma function.

Laplace transform of \( t^{z-1} \) \([3, \text{p.12, eq.(33)}]\) is given by

\[
\mathcal{L}[t^{z-1}, S] = \int_0^\infty e^{-St} t^{z-1} dt = \frac{\Gamma(z)}{S^z}, \tag{1.39}
\]

where \( \Re(S) > 0 \) and \( 0 < \Re(z) < \infty \).

The Laplace transforms of sine and cosine functions associated with Meijer’s G-function are given by

\[
\mathcal{L}[\sin(\beta x^2)] = \int_0^\infty e^{-\alpha x} \sin(\beta x^2) dx = \frac{1}{\alpha \pi \sqrt{2}} G_{3,1}^{1,3,1} \left( \frac{64\beta^2}{\alpha^4} \bigg| \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, 0 \right), \tag{1.40}
\]

\[
\mathcal{L}[\cos(\beta x^2)] = \int_0^\infty e^{-\alpha x} \cos(\beta x^2) dx = \frac{1}{\alpha \pi \sqrt{2}} G_{3,1}^{1,3,1} \left( \frac{64\beta^2}{\alpha^4} \bigg| \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, 0 \right), \tag{1.41}
\]

\[
\mathcal{L}[x \sin(\beta x^2)] = \int_0^\infty e^{-\alpha x} x \sin(\beta x^2) dx = \frac{2\sqrt{2}}{\alpha^2 \pi} G_{3,1}^{1,3,1} \left( \frac{64\beta^2}{\alpha^4} \bigg| \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2} \right), \tag{1.42}
\]

\[
\mathcal{L}[x \cos(\beta x^2)] = \int_0^\infty e^{-\alpha x} x \cos(\beta x^2) dx = \frac{\sqrt{2}}{\alpha \pi} G_{3,1}^{1,3,1} \left( \frac{64\beta^2}{\alpha^4} \bigg| \frac{1}{2}, \frac{1}{4}, \frac{1}{2} \right), \tag{1.43}
\]
I. The first Ramanujan integrals holds true

\[ \mathcal{L}[x \cos(\beta x^2)] = \int_0^\infty e^{-ax} \cos(\beta x^2) \, dx = \frac{2\sqrt{2}}{\alpha^2 \pi} G_{3,1}^{1,3} \left( \frac{64\beta^2}{\alpha^4} \bigg| \begin{array}{c} \frac{1}{4}, \frac{1}{2}, 0 \end{array} \right), \quad (1.43) \]

where \( \Re(\alpha) > 0 \).

**Proof:** The formulas (1.40)-(1.43) are obtained by using the Mellin-Barnes type contour integrals (1.35) and (1.37) of sine and cosine functions. Then change the order of integration in double integrals, take Laplace transform with the help of eq. (1.39) and apply Gauss-Legendre multiplication theorem (1.38) for the factors of \( \Gamma(4\zeta + 1), \Gamma(4\zeta + 2) \) and finally use the definition of Meijer’s G-function (1.29), we get above results (1.40)-(1.43); see also [4].

In this paper our integrals (1.40)-(1.43) play an important role in our investigation. Using Laplace transforms formulas (1.40)-(1.43), we solved all Ramanujan’s integrals involved in three theorems. Further, we obtain six infinite summation formulas involving Meijer’s G-function. Numeric values of thirteen infinite series are also discussed.

2. Analytical solutions of Ramanujan’s integrals

Each of the following Ramanujan’s integrals involving infinite series of Meijer’s G-function holds true:

I. The first Ramanujan integrals holds true

\[ \Psi_1(b) = \int_0^\infty \frac{\sin(b \pi x^2)}{\cosh(\pi x)} \, dx = \frac{\sqrt{2}}{\pi^2} \sum_{r=0}^\infty \left[ \frac{(-1)^r}{(1 + 2r)} G_{3,1}^{1,3} \left( \frac{64b^2}{\pi^2(1 + 2r)^4} \bigg| \begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \end{array} \right) \right], \quad (2.1) \]

\[ \Phi_1(b) = \int_0^\infty \frac{\cos(b \pi x^2)}{\cosh(\pi x)} \, dx = \frac{\sqrt{2}}{\pi^2} \sum_{r=0}^\infty \left[ \frac{(-1)^r}{(1 + 2r)} G_{3,1}^{1,3} \left( \frac{64b^2}{\pi^2(1 + 2r)^4} \bigg| \begin{array}{c} \frac{1}{4}, \frac{3}{4}, 0 \end{array} \right) \right]. \quad (2.2) \]

II. The second Ramanujan integrals holds true

\[ \Psi_2(b) = \int_0^\infty \frac{\sin(b \pi x^2)}{1 + 2 \cosh(\frac{2\pi x}{\sqrt{3}})} \, dx \]

\[ = \frac{\sqrt{3}}{\pi^2 2 \sqrt{2}} \sum_{p,q=0}^\infty \left[ \frac{(1)_{p+q}(1)_{2q+p}(-1)^{p+q}}{(2)_{2q+p} p! q!} G_{3,1}^{1,3} \left( \frac{36b^2}{\pi^2} \bigg| \begin{array}{c} \frac{(1)_{2q+p}}{(2)_{2q+p}} \end{array} \right) \right] \], \quad (2.3) \]

\[ \Phi_2(b) = \int_0^\infty \frac{\cos(b \pi x^2)}{1 + 2 \cosh(\frac{2\pi x}{\sqrt{3}})} \, dx \]

\[ = \frac{\sqrt{3}}{\pi^2 2 \sqrt{2}} \sum_{p,q=0}^\infty \left[ \frac{(1)_{p+q}(1)_{2q+p}(-1)^{p+q}}{(2)_{2q+p} p! q!} G_{3,1}^{1,3} \left( \frac{36b^2}{\pi^2} \bigg| \begin{array}{c} \frac{(1)_{2q+p}}{(2)_{2q+p}} \end{array} \right) \right]. \quad (2.4) \]

III. The third Ramanujan integrals holds true

\[ \Psi_3^r(b) = \int_0^\infty \frac{\sin(b \pi x)}{-1 + \exp(2\pi \sqrt{3})} \, dx = \frac{\sqrt{2}}{\pi^2} \sum_{r=0}^\infty \left[ \frac{(1)_r}{(2)_r} \right]^2 G_{3,1}^{1,3} \left( \frac{4b^2}{\pi^2} \bigg| \begin{array}{c} \frac{(1)_r}{(2)_r} \end{array} \right) \right] \]
\[ \Phi_3(b) = \int_{0}^{\infty} \frac{\cos(b\pi x)}{\{-1 + \exp(2\pi \sqrt{x})\}} \, dx = \frac{\sqrt{2}}{\pi} \sum_{r=0}^{\infty} \left\{ \frac{(1)_r}{(2)_r} \right\}^2 \frac{4b^2}{\pi} \left\{ \frac{(1)_r}{(2)_r} \right\} \left[ -\frac{1}{4}, \frac{1}{4}, 0 \right] \right\] \quad (2.6)

**Proof:** Suppose left hand side of the eq.(2.1) is denoted by \( \Psi_1(b) \) upon using the well known results of hyperbolic functions and Binomial function (1.28), we obtain

\[ \Psi_1(b) = 2 \int_{0}^{\infty} e^{-\pi x} (1 + e^{-2\pi x})^{-1} \sin(b\pi x^2) \, dx, \quad (2.7) \]

\[ = 2 \int_{0}^{\infty} e^{-\pi x} F_0 \left( \begin{array}{c} 1 ; \\ -1 \end{array} ; -e^{-2\pi x} \right) \sin(b\pi x^2) \, dx. \quad (2.8) \]

Change the order of integration and summation in above equation, which yield

\[ \Psi_1(b) = 2 \sum_{r=0}^{\infty} (-1)^r \int_{0}^{\infty} e^{-(\pi+2\pi r)x} \sin(b\pi x^2) \, dx. \quad (2.9) \]

Use Laplace formula (1.40) in the eq.(2.9). Then we get the right hand side stated in (2.1).

Similarly, proof of the integral (2.2) by using the Laplace formula (1.41) is much akin as result (2.1), which we have already discussed in a detailed manner.

Again suppose left hand side of the eq.(2.3) is denoted by \( \Psi_2(b) \), we have

\[ \Psi_2(b) = \int_{0}^{\infty} \frac{\sin(b\pi x^2)}{\left(1 + e^{2\pi x/\sqrt{3}}\right)} \, dx, \quad (2.10) \]

\[ = \int_{0}^{\infty} \frac{1}{\left(1 + e^{2\pi x/\sqrt{3}}\right)} \left\{ 1 + \frac{e^{-2\pi x/\sqrt{3}}}{1 + e^{2\pi x/\sqrt{3}}} \right\}^{-1} \sin(b\pi x^2) \, dx. \quad (2.11) \]

Change the order of integration and summation in above equation, which yield

\[ \Psi_2(b) = \sum_{q=0}^{\infty} (-1)^q \int_{0}^{\infty} \frac{e^{-2\pi q\sqrt{3}}}{\left(1 + e^{2\pi x/\sqrt{3}}\right)^{q+1}} \sin(b\pi x^2) \, dx, \quad (2.12) \]

\[ = \sum_{q=0}^{\infty} (-1)^q \int_{0}^{\infty} F_0 \left( \begin{array}{c} q+1 ; \\ -1 \end{array} ; -e^{-2\pi x/\sqrt{3}} \right) e^{-\left(4\pi q + 2\pi\sqrt{3}\right)x/\sqrt{3}} \sin(b\pi x^2) \, dx, \quad (2.13) \]

\[ = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{p+q}(1)_{p+q}}{p!q!} \int_{0}^{\infty} e^{-\left(4\pi q + 2\pi\sqrt{3}\right)x/\sqrt{3}} \sin(b\pi x^2) \, dx. \quad (2.14) \]

Use Laplace formula (1.41) in the eq.(2.14). Then we get the right hand side stated in (2.3).

Similarly, proof of the integral (2.4) by using the Laplace formulas (1.41) is much akin as result (2.3), which we have already discussed in a detailed manner.

Suppose left hand side of the eq.(2.5) is denoted by \( \Psi_3(b) \), we have

\[ \Psi_3(b) = \int_{0}^{\infty} e^{-2\pi \sqrt{3}} (1 - e^{-2\pi \sqrt{3}})^{-1} \sin(b\pi x) \, dx, \quad (2.15) \]
Setting $\sqrt{x} = t$, when $x = 0, t = 0$, when $x \to \infty, t \to \infty$, in the L.H.S. of eq.(2.17), we obtain

$$
\Psi_3^*(b) = 2 \sum_{r=0}^{\infty} \int_{0}^{\infty} e^{-(2\pi+2\pi r)^{t/2}} \sin(b \pi t^2) dt.
$$  (2.18)

We change the order of integration and summation in above equation, which yield

$$
\Psi_3^*(b) = \sum_{r=0}^{\infty} \int_{0}^{\infty} e^{-(2\pi+2\pi r)^{t/2}} \sin(b \pi t^2) dt.
$$  (2.17)

Similarly, proof of the integral (2.6) by using the Laplace formula (1.43) is much akin as result (2.5), which we have already discussed in a detailed manner.

3. Application of Ramanujan’s integrals in three theorems

The following infinite summation formulas involving Meijer’s G-function hold true.

$$
\sum_{r=0}^{\infty} \left[ \frac{(-1)^r}{(1+2r)} G_{3,1}^{1,3} \left( \frac{64n^2}{\pi^2(1+2r)^4} \right| \begin{array}{c} 1, \frac{3}{4}, 0 \\ \frac{1}{2} \end{array} \right) \right] = \sum_{r=0}^{\infty} \frac{(-1)^r}{(1+2r)} \left[ \frac{\sqrt{2}}{n} G_{3,1}^{1,3} \left( \frac{64}{(n \pi)^2(1+2r)^4} \right| \begin{array}{c} 1, \frac{1}{2}, \frac{3}{4} \\ \frac{1}{2} \end{array} \right) + G_{3,1}^{1,3} \left( \frac{64n^2}{\pi^2(1+2r)^4} \right| \begin{array}{c} 1, \frac{1}{2}, \frac{3}{4} \\ \frac{1}{2} \end{array} \right) \right],
$$  (3.1)

$$
\sum_{r=0}^{\infty} \left[ \frac{(-1)^r}{(1+2r)} G_{3,1}^{1,3} \left( \frac{64n^2}{\pi^2(1+2r)^4} \right| \begin{array}{c} 1, \frac{1}{2}, \frac{3}{4} \\ \frac{1}{2} \end{array} \right) \right] = \sum_{r=0}^{\infty} \frac{(-1)^r}{(1+2r)} \left[ \frac{\sqrt{2}}{n} G_{3,1}^{1,3} \left( \frac{64}{(n \pi)^2(1+2r)^4} \right| \begin{array}{c} 1, \frac{3}{4}, 0 \\ 0 \end{array} \right) - G_{3,1}^{1,3} \left( \frac{64n^2}{\pi^2(1+2r)^4} \right| \begin{array}{c} 1, \frac{3}{4}, 0 \\ 0 \end{array} \right) \right],
$$  (3.2)

$$
\sum_{p,q=0}^{\infty} \frac{(1+p)(1+q)(-1)^{p+q}}{(2)_{2q+p} \ p! q!} G_{3,1}^{1,3} \left( \frac{36n^2}{\pi^2} \left( \frac{(1)_{2q+p}}{(2)_{2q+p}} \right)^4 \right| \begin{array}{c} 1, \frac{3}{4}, 0 \\ 0 \end{array} \right) = \sum_{p,q=0}^{\infty} \frac{(1+p)(1+q)(-1)^{p+q}}{(2)_{2q+p} \ p! q!} \left[ \frac{\sqrt{2}}{n} G_{3,1}^{1,3} \left( \frac{36}{(n \pi)^2} \left( \frac{(1)_{2q+p}}{(2)_{2q+p}} \right)^4 \right| \begin{array}{c} 1, \frac{1}{2}, \frac{3}{4} \\ \frac{1}{2} \end{array} \right) + G_{3,1}^{1,3} \left( \frac{36n^2}{\pi^2} \left( \frac{(1)_{2q+p}}{(2)_{2q+p}} \right)^4 \right| \begin{array}{c} 1, \frac{1}{2}, \frac{3}{4} \\ \frac{1}{2} \end{array} \right) \right],
$$  (3.3)
The numerical values of thirteen infinite series associated with Meijer’s $G$-function hold true

$$
\sum_{p,q=0}^{\infty} \frac{(1)_{p+q}(1)_{2q+p}(-1)^{p+q}}{(2)^{2q+p} p!q!} \left( \frac{36n^2}{\pi^2} \left\{ \frac{(1)_{2q+p}}{(2)^{2q+p}} \right\}^4 \left| \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, 0 \right| \right) = \sum_{p,q=0}^{\infty} \frac{(1)_{p+q}(1)_{2q+p}(-1)^{p+q}}{(2)^{2q+p} p!q!} \left[ \sqrt{\frac{2}{n}} \left( \frac{36}{(n\pi)^2} \left\{ \frac{(1)_{2q+p}}{(2)^{2q+p}} \right\}^4 \left| \frac{1}{4}, \frac{3}{4}, 0 \right| \right) - \left( \frac{36n^2}{\pi^2} \left\{ \frac{(1)_{2q+p}}{(2)^{2q+p}} \right\}^4 \left| \frac{1}{4}, \frac{3}{4}, 0 \right| \right) \right], \quad (3.4)
$$

$$
\sum_{r=0}^{\infty} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^2 \left( \frac{4n^2}{\pi^2} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^4 \left| -\frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right| \right) = -\frac{\pi^2}{2n\sqrt{2}} + \sum_{r=0}^{\infty} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^2 \left[ \frac{\sqrt{2}}{n\sqrt{n}} \left( \frac{4}{(n\pi)^2} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^4 \left| -\frac{1}{4}, \frac{1}{4}, 0 \right| \right) + \left( \frac{4n^2}{\pi^2} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^4 \left| -\frac{1}{4}, \frac{1}{4}, 0 \right| \right) \right], \quad (3.5)
$$

$$
\sum_{r=0}^{\infty} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^2 \left( \frac{4n^2}{\pi^2} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^4 \left| -\frac{1}{4}, \frac{1}{4}, 0 \right| \right) = \frac{\pi^2}{2} \left( \frac{1}{\sqrt{n}} - \frac{1}{n\sqrt{2}} \right) + \sum_{r=0}^{\infty} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^2 \left[ \frac{\sqrt{2}}{n\sqrt{n}} \left( \frac{4}{(n\pi)^2} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^4 \left| -\frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right| \right) - \left( \frac{4n^2}{\pi^2} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^4 \left| -\frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right| \right) \right]. \quad (3.6)
$$

**Proof:** Put the values of $\Phi_1(n), \Phi_1(1/n); \Psi_1(n), \Psi_1(1/n); \Phi_2(n), \Phi_2(1/n); \Psi_2(n), \Psi_2(1/n); \Phi_3(n), \Phi_3(1/n); \Psi_3(n), \Psi_3(1/n); \Phi_3^*(n), \Psi_3^*(1/n)$ with the help of Ramanujan’s integrals (2.1)-(2.6) in the equations (1.4), (1.5), (1.8), (1.9), (1.12), (1.13), after simplification. Then we get six infinite summation formulas (3.1)-(3.6).

4. **Numerical values of some infinite series containing Meijer’s $G$-function**

The numerical values of thirteen infinite series associated with Meijer’s $G$-function hold true

$$
\sum_{r=0}^{\infty} \left[ \frac{(-1)^r}{(1+2r)} \right] G_{3,1}^{1,3} \left( \frac{64}{\pi^2(1+2r)^4} \left| \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 0 \right| \right) = \frac{\pi^2}{4}(-1+\sqrt{2}), \quad (4.1)
$$

$$
\sum_{r=0}^{\infty} \left[ \frac{(-1)^r}{(1+2r)} \right] G_{3,1}^{1,3} \left( \frac{64}{\pi^2(1+2r)^4} \left| \frac{1}{4}, \frac{3}{4}, 0 \right| \right) = \frac{\pi^2}{4}. \quad (4.2)
$$
\[
\sum_{p,q=0}^{\infty} \left[ \frac{(1)_{p+q}(1)_{2q+p}(-1)^{p+q}}{(2)_{2q+p} p! q!} G_{3,1}^{1,3} \left( \frac{36}{\pi^2} \left\{ \frac{(1)_{2q+p}}{(2)_{2q+p}} \right\}^4 \mid \frac{1}{3}, \frac{3}{4}, 0 \right) \right] = \frac{\pi^2}{2\sqrt{3}} (-\sqrt{6} + \sqrt{3} + 4), \quad (4.3)
\]

\[
\sum_{p,q=0}^{\infty} \left[ \frac{(1)_{p+q}(1)_{2q+p}(-1)^{p+q}}{(2)_{2q+p} p! q!} G_{3,1}^{1,3} \left( \frac{36}{\pi^2} \left\{ \frac{(1)_{2q+p}}{(2)_{2q+p}} \right\}^4 \mid \frac{1}{4}, \frac{3}{4}, 0 \right) \right] = \frac{\pi^3}{2\sqrt{3}} (\sqrt{2} - \sqrt{3} + 1), \quad (4.4)
\]

\[
\sum_{r=0}^{\infty} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^2 G_{3,1}^{1,3} \left( \frac{4}{\pi^2} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^4 \mid -\frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right) = \frac{\pi^2}{8} (\pi - 2\sqrt{2}), \quad (4.5)
\]

\[
\sum_{r=0}^{\infty} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^2 G_{3,1}^{1,3} \left( \frac{16}{\pi^2} \left\{ \frac{(1)_{r}}{(2)_{r}} \right\}^4 \mid -\frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right) = \frac{\pi^2 \sqrt{2}}{32} (\pi - 2), \quad (4.6)
\]

Proof: Taking a particular value \( b = 1 \) in the Ramanujan integrals (2.1)–(2.4) and comparing with the corresponding values of the integrals (1.14)–(1.17), then we get some numeric values of infinite series (4.1)–(4.4). Again, taking some particular values of \( b = 1, 2, 1/2 \) in the Ramanujan integral (2.5) and comparing with values of the integrals (1.25)–(1.27), then we get (4.5)–(4.7). Similarly, taking some particular values of \( b \) that is \( b = 1, 2, 4, 6, 1/2, 2/5 \) in the Ramanujan integral (2.6) and comparing with the values of integrals (1.18)–(1.23), which gives some numeric values of infinite series (4.8)–(4.13).
Concluding remarks

We have given the analytical solutions of Srinivasa Ramanujan integrals in terms of Meijer’s $G$-function. Further, we have described some infinite summation formulas connected with Meijer’s $G$-function and numeric values of some infinite series are also discussed. The numerical values of involved Ramanujan integrals are mentioned in our communicated paper [8], that supports to the corresponding our results of this paper. We conclude our present investigations by observing that several other similar integrals (2.1)-(2.6) and consequences of infinite summation formulas of section-3 and its applications in numeric values of infinite series in section-4, can also be deduced in an analogous manner.

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