THE NAVIER–STOKES EQUATIONS WITH BODY FORCES DECAYING COHERENTLY IN TIME

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Abstract. The long-time behavior of solutions of the three-dimensional Navier–Stokes equations in a periodic domain is studied. The time-dependent body force decays, as time $t$ tends to infinity, in a coherent manner. In fact, it is assumed to have a general and complicated asymptotic expansion which involves complex powers of $e^t$, $t$, $\ln t$, or other iterated logarithmic functions of $t$. We prove that all Leray–Hopf weak solutions admit an asymptotic expansion which is independent of the solutions and is uniquely determined by the asymptotic expansion of the body force. The proof makes use of the complexifications of the Gevrey–Sobolev spaces together with those of the Stokes operator and the bilinear form of the Navier–Stokes equations.

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1. Introduction

Let \( \mathbf{x} \in \mathbb{R}^3 \) and \( t \in \mathbb{R} \) denote the space and time variables, respectively. Let the (kinematic) viscosity be denoted by \( \nu > 0 \), the velocity vector field by \( \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3 \), the pressure by \( p(\mathbf{x}, t) \in \mathbb{R} \), and the body force by \( \mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^3 \). The Navier–Stokes equations (NSE) are

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} = -\nabla p + \mathbf{f} \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty),
\]

\[
\text{div} \mathbf{u} = 0 \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty).
\]

The initial condition is

\[
\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}),
\]

where \( \mathbf{u}^0(\mathbf{x}) \) is a given divergence-free vector field.

Throughout the paper, we use the following notation

\[
\mathbf{u}(t) = \mathbf{u}(\cdot, t), \quad \mathbf{f}(t) = \mathbf{f}(\cdot, t), \quad \mathbf{u}^0 = \mathbf{u}^0(\cdot).
\]

In the case of a potential force, that is, \( \mathbf{f}(\mathbf{x}, t) = -\nabla \phi(\mathbf{x}, t) \) for some scalar function \( \phi \), Foias and Saut prove in [22] that any non-trivial, regular solution \( \mathbf{u}(t) \), in a bounded or periodic domain \( \Omega \), admits an asymptotic expansion (as \( t \to \infty \))

\[
\mathbf{u}(t) \sim \sum_{n=1}^{\infty} q_n(t)e^{-\mu_n t},
\]

in Sobolev spaces \( H^m(\Omega)^3 \), for all \( m \geq 0 \). Above, \( q_n(t) \)'s are polynomials in \( t \) with values in functional spaces. Also, \( (\mu_n)_{n=1}^{\infty} \) is a divergent, strictly increasing sequence of positive numbers. The interested reader is referred to [11, 20] for early results on the solutions’ asymptotic behavior, [19, 20, 21, 22, 23] for associated normalization map and invariant nonlinear manifolds, [14, 15, 16] for the corresponding Poincaré-Dulac normal form, [12, 13] for their applications to analysis of helicity, statistical solutions, and decaying turbulence, and [17] for a survey on the subject.

In the case of periodic domains, it is then improved in [28] that the expansion (1.3) holds in Gevrey–Sobolev spaces \( G_{\alpha,\sigma} \) for any \( \alpha, \sigma > 0 \), see definition (2.6) in Section 2 below, which have much stronger norms than those in \( H^m(\Omega)^3 \). When the force \( f \) is not potential, the asymptotic expansion of Leray–Hopf weak solutions is established in [29] for an exponentially decaying force. Namely, if the force has an asymptotic expansion

\[
f(t) \sim \sum_{n=1}^{\infty} p_n(t)e^{-\gamma_n t},
\]

where \( p_n(t) \)'s are function-valued polynomials in \( t \), then \( \mathbf{u}(t) \) has an asymptotic expansion of type (1.3). The case of power-decaying forces is treated in [7]. Roughly speaking, if

\[
f(t) \sim \sum_{n=1}^{\infty} \phi_n t^{-\gamma_n},
\]

then all Leray–Hopf weak solutions \( \mathbf{u}(t) \) admit a similar asymptotic expansion

\[
\mathbf{u}(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-\mu_n}.
\]

Above \( \phi_n \)'s and \( \xi_n \)'s belong to some Gevrey–Sobolev space \( G_{\alpha,\sigma} \).
More general asymptotic expansion results for the NSE with other types of decaying forces in time are obtained in [6]. They include the asymptotic expansions in terms of logarithmic and iterated logarithmic functions. Furthermore, the asymptotic expansions of type (1.3) are derived in [27] for the Lagrangian trajectories associated with the solutions of the NSE. The Foias–Saut expansion is also established for dissipative wave equations in [33]. It is developed further for general nonlinear ordinary differential equations (ODEs). Specifically, it is investigated first in [32] for autonomous analytic systems, and recently in [9] for systems with non-smooth nonlinearity, and in [8, 26] for nonautonomous systems. In particular, the forcing functions in [26] are much more general and complex than (1.4) and (1.5) above. For other complicated asymptotic expansions for higher order ODEs with a different approach, see [2, 1, 3, 4, 5] and references therein.

The current paper aims to develop the Foias–Saut asymptotic expansion theory for the NSE to cover a very large class of forces. Its direct motivation is our previous work [26] for nonlinear ODE systems. We will obtain a very general result for the NSE in which the forces and solutions have complicated asymptotic expansions. We describe the ideas briefly below.

Let \( \psi(t) \) be \( t, \ln t, \) or some iterated logarithmic function of \( t \). Suppose the body force \( f(t) \) has an asymptotic expansion, as \( t \to \infty \),

\[
f(t) \sim \sum_{n=1}^{\infty} \phi_n(t) \psi(t)^{-\mu_n} \tag{1.7}
\]

in some Gevrey–Sobolev space \( G_{\alpha,\sigma} \). The \( \phi_n(t) \)'s are \( G_{\alpha,\sigma} \)-valued functions and are combinations of complex powers of \( e^t, t, \ln t, \ln \ln t, \) etc. such that the dominant decaying modes in (1.7) are still \( \psi(t)^{-\mu_n} \)'s. We will prove that any Leray–Hopf weak solution \( u(t) \) admits an asymptotic expansion

\[
u(t) \sim \sum_{n=1}^{\infty} \xi_n(t) \psi(t)^{-\mu_n} \tag{1.8}
\]

in \( G_{\alpha,\sigma} \), where each \( \xi_n(t) \) is uniquely determined by \( \phi_1(t), \phi_2(t), \ldots, \phi_n(t) \). The asymptotic expansions (1.7) and (1.8) are much more general and sophisticated than (1.3), (1.4), (1.5), (1.6).

For example, thanks to the complex powers, \( \phi_n(t) \)'s and \( \xi_n(t) \)'s may contain sine or cosine of \( t, \ln t, \ln \ln t, \ln \ln \ln t, \) etc. In order to deal with the complex powers, we utilize the complexifications of the Gevrey–Sobolev spaces, and the complexifications of the Stokes operator \( A \) and the bilinear form \( B \).

The paper is organized as follows. In section 2 we review the standard functional setting for the NSE. We recall in Theorem 2.4 the main asymptotic estimate, as \( t \to \infty \), for any Leray–Hopf weak solution \( u(t) \). Theorem 2.7 is the asymptotic expansion of \( u(t) \) whenever \( f(t) \) has an asymptotic expansion in terms of exponential decaying functions (in time), see Definition 2.6. The proof of Theorem 2.7 is now standard and omitted. However, it serves as the starting point for developing a new asymptotic theory in this paper. The definition of our asymptotic expansions is in section 3, see Definition 3.4. Because of its complicated nature, more involved functions are introduced before that. The main idea in dealing with complicated functions in this paper is the complexification. In section 4, we review the basic facts about general complexification and also present the specific ones for the NSE. The complexified Stokes operator \( A_C \) and complexified bilinear form \( B_C \) are introduced in Definition 4.3. One of the most crucial linear transformations in this paper is \( Z_{AC} \), which is defined in Definition 4.5. It is used in approximating, for large time, the solutions of
the linearized NSE. That result is Theorem 5.1 in section 5. This theorem is the building block in our later construction of the asymptotic approximation of solutions of the NSE. The main results are stated and proved in section 6. Theorem 6.7 deals with the case when \( f(t) \) has coherent power decay, while Theorem 6.11 deals with the case when \( f(t) \) has coherent logarithmic or iterated logarithmic decay. Their formulations and proofs rely on the complexified Gevrey–Sobolev spaces \( G_{\alpha,\sigma,C} \), and complex linear and bilinear mappings \( A_C \) and \( B_C \) mentioned above. The last section – section 7 – recasts Theorems 6.7 and 6.11 into a form that only uses the standard (real) Gevrey–Sobolev spaces \( G_{\alpha,\sigma} \), see Theorem 7.3.

Finally, it is noteworthy that we do not complexify the NSE, only use complexified spaces and mappings as technical tools.

2. Preliminaries on the NSE

We use the following notation throughout the paper.

- \( N = \{1, 2, 3, \ldots\} \) denotes the set of natural numbers, and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \).
- Number \( i = \sqrt{-1} \).
- For any vector \( x \in \mathbb{C}^n \), the real part, respectively, imaginary part, Euclidean norm, of \( x \) is denoted by \( \text{Re}\, x \), respectively, \( \text{Im}\, x, |x| \).
- Let \( f \) and \( h \) be non-negative functions on \([T_0, \infty)\) for some \( T_0 \in \mathbb{R} \). We write \( f(t) = O(h(t)) \), implicitly meaning as \( t \to \infty \), if there exist numbers \( T \geq T_0 \) and \( C > 0 \) such that \( f(t) \leq Ch(t) \) for all \( t \geq T \).

**Definition 2.1.** Let \( S \) be a subset of \( \mathbb{C} \).

(i) We say \( S \) preserves the addition if \( x + y \in S \) for all \( x, y \in S \).

(ii) We say \( S \) preserves the unit increment if \( x + 1 \in S \) for all \( x \in S \).

(iii) The additive semigroup generated by \( S \) is defined by

\[
\langle S \rangle = \left\{ \sum_{j=1}^{N} z_j : N \in \mathbb{N}, z_j \in S \text{ for } 1 \leq j \leq N \right\}.
\]

(iv) The real part of \( S \) is \( \text{Re}\, S = \{\text{Re}\, z : z \in S\} \).

Regarding Definition 2.1 it is obvious that \( \langle S \rangle \) preserves the addition, and \( \text{Re}\, \langle S \rangle = \langle \text{Re}\, S \rangle \).

2.1. Backgrounds. We recall the standard functional setting for the NSE in periodic domains, see e.g. [10, 35, 34, 18].

Let \( \ell_1, \ell_2, \ell_3 \) be fixed positive numbers. Denote \( \mathbf{L} = (\ell_1, \ell_2, \ell_3) \), \( \ell_* = \max\{\ell_1, \ell_2, \ell_3\} \), the domain \( \Omega = (0, \ell_1) \times (0, \ell_2) \times (0, \ell_3) \), and its volume \( |\Omega| = \ell_1 \ell_2 \ell_3 \).

Let \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) be the standard basis of \( \mathbb{R}^3 \). A function \( g(\mathbf{x}) \) is said to be \( \Omega \)-periodic if

\[
g(\mathbf{x} + \ell_j \mathbf{e}_j) = g(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^3, \ j = 1, 2, 3,
\]

and is said to have zero average over \( \Omega \) if

\[
\int_{\Omega} g(\mathbf{x}) d\mathbf{x} = 0.
\]

In this paper, we focus on the case when the force \( f(\mathbf{x}, t) \) and solutions \( (u(\mathbf{x}, t), p(\mathbf{x}, t)) \) are \( \Omega \)-periodic. By rescaling the variables \( \mathbf{x} \) and \( t \), we assume throughout, without loss of generality, that \( \ell_* = 2\pi \) and \( \nu = 1 \).
Let $L^2(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$, for integers $m \geq 0$, denote the standard Lebesgue and Sobolev spaces on $\Omega$. The standard inner product and norm in $L^2(\Omega)^3$ are denoted by $\langle \cdot, \cdot \rangle$ and $| \cdot |$, respectively. (We warn that this notation $| \cdot |$ also denotes the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{C}^n$, for any $n \in \mathbb{N}$, but its meaning will be clear based on the context.)

Let $\mathcal{V}$ be the set of all $\Omega$-periodic trigonometric polynomial vector fields which are divergence-free and have zero average over $\Omega$. Define

$$H, \text{ resp. } V = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3, \text{ resp. } H^1(\Omega)^3.$$ 

On $V$, we use the following inner product

$$\langle \langle \mathbf{u}, \mathbf{v} \rangle \rangle = \sum_{j=1}^{3} \frac{\partial \mathbf{u}}{\partial x_j} \cdot \frac{\partial \mathbf{v}}{\partial x_j} \quad \text{for all } \mathbf{u}, \mathbf{v} \in V, \quad (2.1)$$

and denote its corresponding norm by $\| \cdot \|$.

We use the following embeddings and identification

$$V \subset H = H' \subset V',$$

where each space is dense in the next one, and the embeddings are compact. Let $\mathcal{P}$ denote the orthogonal (Leray) projection in $L^2(\Omega)^3$ onto $H$.

The Stokes operator $A$ is a bounded linear mapping from $V$ to its dual space $V'$ defined by

$$\langle A \mathbf{u}, \mathbf{v} \rangle_{V', V} = \langle \mathbf{u}, \mathbf{v} \rangle \quad \text{for all } \mathbf{u}, \mathbf{v} \in V.$$

As an unbounded operator on $H$, the operator $A$ has the domain $\mathcal{D}(A) = V \cap H^2(\Omega)^3$, and, under the current consideration of periodicity condition,

$$A \mathbf{u} = -\mathcal{P} \Delta \mathbf{u} = -\Delta \mathbf{u} \in H \quad \text{for all } \mathbf{u} \in \mathcal{D}(A). \quad (2.2)$$

There exist a complete orthonormal basis $(\mathbf{w}_n)_{n=1}^{\infty}$ of $H$, and a sequence of positive numbers $(\lambda_n)_{n=1}^{\infty}$ so that

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots, \quad \lim_{n \to \infty} \lambda_n = \infty, \quad (2.3)$$

$$A \mathbf{w}_n = \lambda_n \mathbf{w}_n \quad \forall n \in \mathbb{N}. \quad (2.4)$$

Note that each $\lambda_n$ is an eigenvector of $A$. Also, each $\lambda_n$ has finite multiplicity.

We denote by $(\Lambda_n)_{n=1}^{\infty}$ the strictly increasing sequence of the above eigenvalues $\lambda_n$’s. Then we still have $\Lambda_n \to \infty$ as $n \to \infty$.

Denote $\mathcal{S}(A) = \{\lambda_n : n \in \mathbb{N}\} = \{\Lambda_n : n \in \mathbb{N}\}$, which is the spectrum of $A$.

For $\Lambda \in \mathcal{S}(A)$, we denote by $R_\Lambda$ the orthogonal projection from $H$ to the eigenspace of $A$ corresponding to $\Lambda$, and set

$$P_\Lambda = \sum_{\lambda \in \mathcal{S}(A), \Lambda \leq \lambda} R_\lambda.$$ 

Note that each linear space $P_\Lambda H$ is finite dimensional.

For $\alpha, s, \sigma \in \mathbb{R}$, define, for $\mathbf{u} = \sum_{n=1}^{\infty} c_n \mathbf{w}_n \in H$ with $c_n = \langle \mathbf{u}, \mathbf{w}_n \rangle$,

$$A^\alpha \mathbf{u} = \sum_{n=1}^{\infty} c_n \lambda_n^\alpha \mathbf{w}_n, \quad e^{sA} \mathbf{u} = \sum_{n=1}^{\infty} c_n e^{s\lambda_n} \mathbf{w}_n, \quad e^{\sigma A^{1/2}} \mathbf{u} = \sum_{n=1}^{\infty} c_n e^{\sigma \sqrt{\lambda_n}} \mathbf{w}_n \quad (2.5)$$

whenever the defined element belongs to $H$. More precisely, the formulas in (2.5) are respectively defined on the domains $\mathcal{D}(L) = \{\mathbf{u} \in H : L \mathbf{u} \in H\}$ for $L = A^\alpha, e^{sA}, e^{\sigma A^{1/2}}$. 


In the current case of a periodic domain, it is convenient to formulate the NSE using the Fourier series. For \( k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \), denote

\[
\textbf{k}_L = 2\pi \left( \frac{k_1}{\ell_1}, \frac{k_2}{\ell_2}, \frac{k_3}{\ell_3} \right).
\]

It is known that \( \mathcal{G}(A) = \{ |\textbf{k}_L|^2 : k \in \mathbb{Z}^3, k \neq 0 \} \). Note that the minimum of \( \mathcal{G}(A) \) is 1.

For \( \alpha, s, \sigma \in \mathbb{R} \) and \( u(x) = \sum_{k \neq 0} \hat{u}_k e^{i\textbf{k}_L \cdot x} \in H \), we have

\[
A^\alpha u = \sum_{k \neq 0} |\textbf{k}_L|^2 |\hat{u}_k|^2 \hat{u}_k e^{i\textbf{k}_L \cdot x}, \quad e^s A u = \sum_{k \neq 0} e^{|\textbf{k}_L|^2} |\hat{u}_k|^2 \hat{u}_k e^{i\textbf{k}_L \cdot x}, \quad e^\sigma A^{1/2} u = \sum_{k \neq 0} e^{\sigma |\textbf{k}_L|} |\hat{u}_k|^2 \hat{u}_k e^{i\textbf{k}_L \cdot x}.
\]

Let \( \alpha, \sigma \geq 0 \). The Gevrey–Sobolev spaces are defined by

\[
G_{\alpha,\sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) = \{ u \in H : |u|_{\alpha,\sigma} = |A^\alpha e^{\sigma A^{1/2}} u| < \infty \}.
\]

Each \( G_{\alpha,\sigma} \) is a real Hilbert space with the inner product

\[
\langle u, v \rangle_{G_{\alpha,\sigma}} = \langle A^\alpha e^{\sigma A^{1/2}} u, A^\alpha e^{\sigma A^{1/2}} v \rangle \quad \text{for } u, v \in G_{\alpha,\sigma}.
\]

Note that \( G_{0,0} = \mathcal{D}(A^0) = H, G_{1/2,0} = \mathcal{D}(A^{1/2}) = V, G_{1,0} = \mathcal{D}(A) \). The inner product in (2.7) when \( \alpha = \sigma = 0 \), respectively, \( \alpha = 1/2, \sigma = 0 \), agrees with \( \langle \cdot, \cdot \rangle \) on \( H \), respectively, \( \langle \cdot, \cdot \rangle \) on \( V \) indicated at the beginning of this subsection and (2.1). Subsequently, \( \|u\| = |\nabla u| = |A^{1/2} u| \) for \( u \in V \). Also, the norms \( |\cdot|_{\alpha,\sigma} \) are increasing in \( \alpha, \sigma \), hence, the spaces \( G_{\alpha,\sigma} \) are decreasing in \( \alpha, \sigma \).

Denote, for \( \sigma \geq 0 \), the space \( E^{\infty,\sigma} = \bigcap_{\alpha > 0} G_{\alpha,\sigma} \).

Regarding the nonlinear term in the NSE, a bounded bilinear mapping \( B : V \times V \to V' \) is defined by

\[
\langle B(u, v), w \rangle_{V',V} = b(u, v, w) \overset{\text{def}}{=} \int_\Omega ((u \cdot \nabla)v) \cdot w \, dx, \quad \text{for all } u, v, w \in V.
\]

In particular,

\[
B(u, v) = \mathcal{P}((u \cdot \nabla)v), \quad \text{for all } u, v \in \mathcal{D}(A).
\]

The problems (1.1) and (1.2) can now be rewritten in the functional form as

\[
du(t) + A u(t) + B(u(t), u(t)) = f(t) \quad \text{in } V' \text{ on } (0, \infty),
\]

\[
u(0) = u^0 \in H.
\]

(We refer the reader to the books [33, 10, 35, 34] for more details.)

The next definition makes precise the meaning of weak solutions of (2.9).

**Definition 2.2.** Let \( f \in L^2_{\text{loc}}([0, \infty), H) \). A Leray–Hopf weak solution \( u(t) \) of (2.9) is a mapping from \([0, \infty)\) to \( H \) such that

\[
u \in C([0, \infty), H_w) \cap L^2_{\text{loc}}([0, \infty), V), \quad \nu' \in L^{4/3}_{\text{loc}}([0, \infty), V'),
\]

and satisfies

\[
\frac{d}{dt} \langle u(t), v \rangle + \langle u(t), v \rangle + b(u(t), u(t), v) = \langle f(t), v \rangle
\]

in the distribution sense in \((0, \infty)\), for all \( v \in V \), and the energy inequality

\[
\frac{1}{2} |u(t)|^2 + \int_0^t \|u(\tau)\|^2 \, d\tau \leq \frac{1}{2} |u(t_0)|^2 + \int_{t_0}^t \langle f(\tau), u(\tau) \rangle \, d\tau
\]
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holds for \( t_0 = 0 \) and almost all \( t_0 \in (0, \infty) \), and all \( t \geq t_0 \). Here, \( H_w \) denotes the topological vector space \( H \) with the weak topology.

A regular solution is a Leray–Hopf weak solution that belongs to \( C([0, \infty), \mathcal{V}) \).

If \( t \mapsto u(T + t) \) is a Leray–Hopf weak solution, respectively, regular solution, then we say \( u \) is a Leray–Hopf weak solution, respectively, regular solution on \( [T, \infty) \).

It is well-known that a regular solution is unique among all Leray–Hopf weak solutions.

We assume throughout the paper the following.

Assumption 2.3. The function \( f \) in equation (2.9) belongs to \( L^\infty_{\text{loc}}([0, \infty), H) \).

Under Assumption 2.3, for any \( u^0 \in H \), there exists a Leray–Hopf weak solution \( u(t) \) of (2.9) and (2.10), see e.g. [18]. We will study the large-time behavior of \( u(t) \) in details. Of course, it will depend on the large-time behavior of the force \( f(t) \). Hence, we will specify more conditions on \( f(t) \) later.

We recall well-known inequalities that will be used throughout. Let \( \alpha \geq 0 \) and \( \sigma > 0 \).

Denote

\[
d_0(\alpha, \sigma) = \begin{cases} 
e^{-\sigma}, & \text{if } \alpha = 0, \\ \left( \frac{\alpha}{\sigma} \right)^{\alpha} \max_{x \geq 0} (x^\alpha e^{-\sigma x}), & \text{if } \alpha > 0. \end{cases}
\]

Then

\[
|A^\alpha e^{-\sigma A} v| \leq d_0(\alpha, \sigma) |v| \quad \forall v \in H,
\]

(2.12)

\[
|A^\alpha e^{-\sigma A^{1/2}} v| \leq d_0(2\alpha, \sigma) |v| \quad \forall v \in H,
\]

(2.13)

and, by writing \( A^\alpha v = (A^\alpha e^{-\sigma A^{1/2}}) e^{\sigma A^{1/2}} v \) and applying (2.13),

\[
|A^\alpha v| \leq d_0(2\alpha, \sigma) |e^{\sigma A^{1/2}} v| \quad \forall v \in G_{0, \sigma}.
\]

(2.14)

For the bilinear mapping \( B(u, v) \), it follows from its boundedness that there exists a constant \( K_\ast > 0 \) such that

\[
\|B(u, v)\|_{\mathcal{V}'} \leq K_\ast \|u\| \|v\| \quad \forall u, v \in \mathcal{V}.
\]

(2.15)

The estimate of the Gevrey norms \( |B(u, v)|_{\alpha, \sigma} \), for \( \alpha = 0 \), was established by Foias and Temam in [24]. For other values of \( \alpha \), we recall here a useful inequality from [28, Lemma 2.1].

There exists a constant \( K > 1 \) such that if \( \sigma \geq 0 \) and \( \alpha \geq 1/2 \), then

\[
|B(u, v)|_{\alpha, \sigma} \leq K^\alpha |u|_{\alpha + 1/2, \sigma} |v|_{\alpha + 1/2, \sigma} \quad \forall u, v \in G_{\alpha + 1/2, \sigma}.
\]

(2.16)

2.2. Decaying force and asymptotic estimates. We recall a previous result on the eventual regularity and asymptotic estimates, as time tends to infinity, for the Leray–Hopf weak solutions of the NSE.

Theorem 2.4 ([6, Theorem 3.4]). Let \( F \) be a continuous, decreasing, non-negative function on \([0, \infty)\) that satisfies

\[
\lim_{t \to \infty} F(t) = 0.
\]

(2.17)

Suppose there exist \( \sigma \geq 0 \) and \( \alpha \geq 1/2 \) such that

\[
|f(t)|_{\alpha, \sigma} = O(F(t)).
\]

(2.18)
Let $u(t)$ be a Leray–Hopf weak solution of (2.9). Then there exists $\hat{T} > 0$ such that $u(t)$ is a regular solution of (2.9) on $[\hat{T}, \infty)$, and for any $\varepsilon, \lambda \in (0, 1)$, and $a_0, a, \theta_0, \theta \in (0, 1)$ with $a_0 + a < 1, \theta_0 + \theta < 1$, there exists $C > 0$ such that

$$|u(\hat{T} + t)|_{\alpha+1-\varepsilon, \sigma} \leq C\left(e^{-a_0 t} + e^{-2\theta_0 t} + F^{2\lambda}(\theta a t) + F(at)\right) \quad \forall t \geq 0.$$  

If, in addition, $F$ satisfies

(i) there exist $k_0 > 0$ and $D_1 > 0$ such that

$$e^{-k_0 t} \leq D_1 F(t) \quad \forall t \geq 0,$$  

(ii) for any $a \in (0, 1)$, there exists $D_2 = D_2,a > 0$ such that

$$F(at) \leq D_2 F(t) \quad \forall t \geq 0,$$  

then

$$|u(\hat{T} + t)|_{\alpha+1-\varepsilon, \sigma} \leq CF(t) \quad \forall t \geq 0.$$  

Theorem 2.4 will be used with different specific choices of the function $F(t)$.

### 2.3. Asymptotic expansions in the case of exponentially decaying force.

First, we recall the definition of the asymptotic expansions studied in [22] originally, and then in [33, 32, 28, 29, 30].

#### Definition 2.5.

Let $X$ be a linear space over $\mathbb{R}$ or $\mathbb{C}$.

(i) A function $g : \mathbb{R} \to X$ is an $X$-valued S-polynomial if it is a finite sum of the functions in the set

$$\left\{ t^m \cos(\omega t) Z, t^m \sin(\omega t) Z : m \in \mathbb{Z}, \omega \in \mathbb{R}, Z \in X \right\}.$$

(ii) Denote by $\mathcal{F}_0(X)$, respectively, $\mathcal{F}_1(X)$ the set of all $X$-valued polynomials, respectively, $S$-polynomials.

#### Definition 2.6.

Let $(X, \| \cdot \|_X)$ be a normed space over $\mathbb{R}$ or $\mathbb{C}$, and $(\gamma_k)_{k=1}^{\infty}$ be a divergent sequence of strictly increasing nonnegative real numbers. Let $\mathcal{F} = \mathcal{F}_0$ or $\mathcal{F}_1$. A function $g : (T, \infty) \to X$, for some $T \in \mathbb{R}$, is said to have an asymptotic expansion

$$g(t) \sim \sum_{k=1}^{\infty} p_k(t)e^{-\gamma_k t} \quad \text{in } X,$$

where each $p_k$ belongs to $\mathcal{F}(X)$ for $k \in \mathbb{N}$, if one has, for any $N \geq 1$, there is a number $\mu > \gamma_N$ such that

$$\|g(t) - \sum_{k=1}^{N} p_k(t)e^{-\gamma_k t}\|_X = O(e^{-\mu t}).$$

Below, the asymptotic expansions (2.22) and (2.23), and equation (2.24) are said to hold in $E^{\infty, \sigma}$, which means that they hold in $G_{\alpha, \sigma}$ for all $\alpha \geq 0$.

#### Theorem 2.7.

Let $(\mu_n)_{n=1}^{\infty}$ being a divergent, strictly increasing sequence of positive numbers. Moreover, the set $S \overset{\text{def}}{=} \{ \mu_n : n \in \mathbb{N} \}$ preserves the addition and contains $\mathcal{S}(A)$. 
Let $\mathcal{F} = \mathcal{F}_0$ or $\mathcal{F}_1$. Assume that there exist a number $\sigma \geq 0$ and functions $p_n \in \mathcal{F}(E^{\infty,\sigma})$, for all $n \in \mathbb{N}$, such that $f(t)$ has the asymptotic expansion

$$f(t) \sim \sum_{n=1}^{\infty} p_n(t)e^{-\mu_n t} \text{ in } E^{\infty,\sigma}. \quad (2.22)$$

Let $u(t)$ be a Leray–Hopf weak solution of (2.9). Then there exist functions $q_n \in \mathcal{F}(E^{\infty,\sigma})$, for all $n \in \mathbb{N}$, such that $u(t)$ has the asymptotic expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t)e^{-\mu_n t} \text{ in } E^{\infty,\sigma}. \quad (2.23)$$

Moreover, the mappings

$$u_n(t) \overset{\text{def}}{=} q_n(t)e^{-\mu_n t} \text{ and } f_n(t) \overset{\text{def}}{=} p_n(t)e^{-\mu_n t}$$

satisfy the following ordinary differential equations in the space $E^{\infty,\sigma}$

$$\frac{d}{dt}u_n(t) + Au_n(t) + \sum_{m,j \geq 1, \mu_m + \mu_j = \mu_n} B(u_m(t), u_j(t)) = f_n(t), \quad t \in \mathbb{R}, \quad (2.24)$$

for all $n \in \mathbb{N}$.

Theorem 2.7 can be proved by combining the proof of [29, Theorem 2.2] with the general treatment of $\mathcal{S}(A)$ and the class $\mathcal{F}_1$ as in [30]. We omit its proof here.

Our goal is to establish similar results to Theorem 2.7 when the force $f(t)$ belongs to a very large class of decaying, but not exponentially decaying, functions. We describe them in the next section.

3. The asymptotic expansions of interest

We describe the asymptotic expansions that will be studied in details in this paper. They are new to the NSE but were already used in our previous work [26] for systems of ODEs in the Euclidean spaces.

3.1. Basic functions and their properties. In this paper, we make use of only single-valued complex functions. To avoid any ambiguity we recall basic definitions and properties of elementary complex functions.

For $z \in \mathbb{C}$ and $t > 0$, the exponential and power functions are defined by

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \text{ and } t^z = \exp(z \ln t). \quad (3.1)$$

When $t = e = \exp(1)$ in (3.1), one has the usual identity $e^z = \exp(z)$.

If $z = a + ib$ with $a, b \in \mathbb{R}$, then

$$t^z = t^a \left( \cos(b \ln t) + i \sin(b \ln t) \right) \text{ and } |t^z| = t^a.$$

The standard properties of the power functions still hold, namely,

$$t^{z_1}t^{z_2} = t^{z_1+z_2}, \quad (t_1t_2)^z = t_1^z t_2^z, \quad (t^z)^m = t^{mz} = (t^m)^z, \quad \frac{d}{dt}(t^z) = zt^{z-1},$$

for any $t, t_1, t_2 > 0$, $z, z_1, z_2 \in \mathbb{C}$, and $m \in \mathbb{Z}_+$. We will also deal with the following iterated exponential and logarithmic functions.
Definition 3.1. Define the iterated exponential and logarithmic functions as follows:

\[
E_0(t) = t \text{ for } t \in \mathbb{R}, \quad \text{and } E_{m+1}(t) = e^{E_m(t)} \text{ for } m \in \mathbb{Z}^+, \quad t \in \mathbb{R},
\]

\[
L_{-1}(t) = e^t, \quad L_0(t) = t \text{ for } t \in \mathbb{R}, \quad \text{and}
\]

\[
L_{m+1}(t) = \ln(L_m(t)) \text{ for } m \in \mathbb{Z}^+, \quad t > E_m(0).
\]

For \( k \in \mathbb{Z}^+ \), define

\[
\mathcal{L}_k = (L_1, L_2, \ldots, L_k) \quad \text{and} \quad \widehat{\mathcal{L}}_k = (L_{-1}, L_0, L_1, \ldots, L_k).
\]

Explicitly,

\[
L_1(t) = \ln t, \quad L_2(t) = \ln \ln t, \quad L_k(t) = (\ln t, \ln \ln t, \ldots, L_k(t)) \quad \text{and} \quad \widehat{\mathcal{L}}_k(t) = (e^t, t, \ln t, \ln \ln t, \ldots, L_k(t)).
\]

Note that \( \widehat{\mathcal{L}}_k(t) \) belongs to \( \mathbb{R}^{k+2} \) and extends \( \mathcal{L}_k(t) \in \mathbb{R}^k \) to include two more coordinates \( e^t \) and \( t \). Same as in [26], we continue to use \( \widehat{\mathcal{L}}_k \) to formulate the results in this paper. They are more general than the previous results using \( \mathcal{L}_k \) obtained in [8].

It is clear, for \( m \in \mathbb{Z}^+ \), that \( L_m(t) \) is positive and increasing for \( t > E_m(0) \),

\[
L_m(E_{m+1}(0)) = 1, \quad \lim_{t \to \infty} L_m(t) = \infty. \tag{3.2}
\]

Also,

\[
\lim_{t \to \infty} \frac{L_k(t)^\lambda}{L_m(t)} = 0 \text{ for all } k > m \geq -1 \text{ and } \lambda \in \mathbb{R}. \tag{3.3}
\]

For \( m \in \mathbb{N} \), the derivative of \( L_m(t) \) is

\[
L'_m(t) = \frac{1}{t \prod_{k=1}^{m-1} L_k(t)} = \frac{1}{\prod_{k=0}^{m-1} L_k(t)}. \tag{3.4}
\]

With the use of the L’Hospital rule and (3.5), one can prove, by induction, that it holds, for any \( T \in \mathbb{R} \) and \( c > 0 \),

\[
\lim_{t \to \infty} \frac{L_m(T + ct)}{L_m(t)} = \begin{cases} c, & \text{for } m = 0, \\ 1, & \text{for } m \geq 1. \end{cases} \tag{3.6}
\]

Consequently, if \( T, T' > E_m(0) \) and \( c, c' > 0 \), then there are numbers \( C, C' > 0 \) such that

\[
C' \leq \frac{L_m(T + ct)}{L_m(T' + c't)} \leq C \text{ for all } t \geq 0. \tag{3.7}
\]

We recall a fundamental integral estimate that will be used throughout.

Lemma 3.2 ([8, Lemma 2.5]). Let \( m \in \mathbb{Z}^+ \) and \( \lambda > 0, \gamma > 0 \) be given. For any number \( T_* > E_m(0) \), there exists a number \( C > 0 \) such that

\[
\int_0^t e^{-\gamma(t-\tau)} L_m(T_* + \tau)^{-\lambda} d\tau \leq C L_m(T_* + t)^{-\lambda} \quad \text{for all } t \geq 0. \tag{3.8}
\]
3.2. Definition of the asymptotic expansions. We will study a large class of asymptotic expansions which involve the following types of power functions of several variables and complex exponents. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

For $z = (z_{-1}, z_0, z_1, \ldots, z_k) \in (0, \infty)^{k+2}$ and $\alpha = (\alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathbb{R}^{k+2}$, define

$$z^{\alpha} = \prod_{j=-1}^{k} z_j^{\alpha_j}.$$  

For $\mu \in \mathbb{R}$, $m, k \in \mathbb{Z}$ with $k \geq m \geq -1$, denote by $\mathcal{E}_{\mathbb{K}}(m, k, \mu)$ the set of vectors $\alpha$ in (3.9) such that

$$\Re(\alpha_j) = 0 \text{ for } -1 \leq j < m \text{ and } \Re(\alpha_m) = \mu.$$  

Particularly, $\mathcal{E}_{\mathbb{R}}(m, k, \mu)$ is the set of vectors $\alpha = (\alpha_{-1}, \alpha_0, \ldots, \alpha_k) \in \mathbb{R}^{k+2}$ such that

$$\alpha_{-1} = \ldots = \alpha_{m-1} = 0 \text{ and } \alpha_m = \mu.$$  

For example, when $m = -1$, $k \geq -1$, $\mu = 0$, the set

$$\mathcal{E}_{\mathbb{R}}(-1, k, 0)$$  

is the collection of vectors $\alpha$’s in (3.9) with $\Re(\alpha_{-1}) = 0$. Let $k \geq m \geq -1$, $\mu \in \mathbb{R}$, and $\alpha \in \mathcal{E}_{\mathbb{K}}(m, k, \mu)$. Using (3.4), one can verify that, see, e.g., equation (3.14) in [26],

$$\lim_{t \to \infty} \frac{\hat{L}_k(t)^\alpha}{L_m(t)^{\mu+\delta}} = 0 \quad \text{for any } \delta > 0.$$  

Definition 3.3. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and $X$ be a linear space over $\mathbb{K}$.

(i) For $k \geq -1$, define $\mathcal{P}(k, X)$ to be the set of functions of the form

$$p(z) = \sum_{\alpha \in S} z^{\alpha} \xi_{\alpha} \text{ for } z \in (0, \infty)^{k+2},$$  

where $S$ is some finite subset of $\mathbb{R}^{k+2}$, and each $\xi_{\alpha}$ belongs to $X$.

(ii) Let $k \geq m \geq -1$ and $\mu \in \mathbb{R}$. Define $\mathcal{P}_m(k, \mu, X)$ to be the set of functions of the form (3.12), where $S$ is a finite subset of $\mathcal{E}_{\mathbb{K}}(m, k, \mu)$ and each $\xi_{\alpha}$ belongs to $X$.

Below are immediate observations about Definition 3.3:

(a) $\mathcal{P}(k, X)$ contains all polynomials from $\mathbb{R}^{k+2}$ to $X$, in the sense that, if $p : \mathbb{R}^{k+2} \to X$ is a polynomial, then its restriction on $(0, \infty)^{k+2}$ belongs to $\mathcal{P}(k, X)$.

(b) Each $\mathcal{P}(k, X)$ is a linear space over $\mathbb{K}$.

(c) If $m > k \geq -1$, then, by the standard embedding

$$\mathbb{K}^{k+2} = \mathbb{K}^{k+2} \times \{0\}^{m-k} \subset \mathbb{R}^{m+2},$$

one can embed $\mathcal{P}(k, X)$ into $\mathcal{P}(m, X)$. See Remark (c) after Definition 2.7 in [8].

(d) One has

$$q \in \mathcal{P}_m(k, \mu, X) \text{ if and only if } \exists p \in \mathcal{P}_m(k, 0, X), \forall z = (z_{-1}, z_0, \ldots, z_k) \in (0, \infty)^{k+2} : q(z) = p(z)z_m^{\mu}.$$  

(e) For any $k \geq m \geq 0$ and $\mu \in \mathbb{R}$, one has

$$\mathcal{P}_m(k, \mu, X) \subset \mathcal{P}_m(k, 0, X).$$

Now, we define the asymptotic expansions in which the power or logarithmic or iterated logarithmic functions are the main decaying modes.
Definition 3.4. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and $(X, \| \cdot \|_X)$ be a normed space over $\mathbb{K}$. Suppose $g$ is a function from $(T, \infty)$ to $X$ for some $T \in \mathbb{R}$, and $m_* \in \mathbb{Z}_+$. Let $(\gamma_k)_{k=1}^{\infty}$ be a divergent, strictly increasing sequence of positive numbers, and $(n_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{N} \cap [m_*, \infty)$. We say
\[
g(t) \sim \sum_{k=1}^{\infty} p_k(\widehat{L}_k(t)), \text{ where } p_k \in \mathcal{P}_{m_*}(n_k, -\gamma_k, X) \text{ for } k \in \mathbb{N},
\]
if, for each $N \in \mathbb{N}$, there is some $\mu > \gamma_N$ such that
\[
\left\| g(t) - \sum_{k=1}^{N} p_k(\widehat{L}_k(t)) \right\|_X = \mathcal{O}(L_{m_*}(t)^{-\mu}).
\]

By using the equivalence (3.13), we have the following equivalent form of (3.15)
\[
g(t) \sim \sum_{k=1}^{\infty} \widehat{g}_k(\widehat{L}_{n_k}(t)) L_{m_*}(t)^{-\gamma_k}, \text{ where } \widehat{g}_k \in \mathcal{P}_{m_*}(n_k, 0, X) \text{ for } k \in \mathbb{N}.
\]
(3.16)

Note that the function $\widehat{g}_k(\widehat{L}_{n_k}(t))$ in (3.16) does not contribute any extra $L_{m_*}(t)^r$, with some $r \in \mathbb{R}$, to the decaying mode $L_{m_*}(t)^{-\gamma_k}$. For example, when $m_* = 0$ the asymptotic expansion (3.16) reads as
\[
g(t) \sim \sum_{k=1}^{\infty} \widehat{g}_k(\widehat{L}_{n_k}(t)) t^{-\gamma_k}, \text{ where } \widehat{g}_k \in \mathcal{P}_0(n_k, 0, X) \text{ for } k \in \mathbb{N}.
\]

When $m_* = 0$, respectively, $m = 1$, $m \geq 2$, we say the function $g(t)$ in (3.15) has coherent power, respectively, logarithmic, iterated logarithmic, decay (as $t \to \infty$).

4. Complexification

We will use the idea of complexification, which we recall below in a brief and convenient form. For more details, see, e.g., [25, section 77].

4.1. Complexification of real linear spaces. Let $X$ be a linear space over $\mathbb{R}$. Its complexification is $X_C = X + iX$ with the following natural addition and scalar multiplication. For any $z = x + iy$ and $z' = x' + iy'$ in $X_C$ with $x, x', y, y' \in X$, and any $c = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$, define
\[
z + z' = (x + x') + i(y + y'),
\]
\[
cz = (ax - by) + i(bx + ay).
\]

Then $X_C$ is a linear space over $\mathbb{C}$ and $X \subset X_C$.

For $z = x + iy \in X_C$, with $x, y \in X$, its conjugate is defined by $\overline{z} = x - iy = x + i(-y)$. When more explicit notation is needed, we denote this $\overline{z}$ by $\text{conj}_{X_C}(z)$. Obviously, $z + \overline{z} = 2x \in X$. One can also verify that
\[
\overline{cz} = c\overline{z} \text{ for all } c \in \mathbb{C}, z \in X_C.
\]
Suppose $(X, \langle \cdot, \cdot \rangle_X)$ is an inner product space over $\mathbb{R}$. Then the complexification $X_C$ is an inner product space over $\mathbb{C}$ with the corresponding inner product $\langle \cdot, \cdot \rangle_{X_C}$ defined by
\[
\langle x + iy, x' + iy' \rangle_{X_C} = \langle x, x' \rangle_X + \langle y, y' \rangle_X + i(\langle y, x' \rangle_X - \langle x, y' \rangle_X) \text{ for } x, x', y, y' \in X.
\]
Denote by $\| \cdot \|_X$ and $\| \cdot \|_{XC}$ the norms on $X$ and $X_C$ induced from their respective inner products. Then
\[
\|x + iy\|_{XC} = \left(\|x\|^2 + \|y\|^2\right)^{1/2} \quad \text{and} \quad \|\overline{z}\|_{XC} = |z|_{XC} \quad \text{for all } x, y \in X \text{ and } z \in X_C.
\]

For any $k \in \mathbb{N}$, the complexification of $X = \mathbb{R}^k$ is $\mathbb{C}^k$. If $z = (z_1, \ldots, z_k) \in \mathbb{C}^k = X_C$, then $\text{conj}_{X_C}(z)$ is the standard conjugate vector $\overline{z} = (\overline{z_1}, \ldots, \overline{z_k})$ and $\|z\|_{XC}$ is the standard Euclidean norm $|z| = (|z_1|^2 + \ldots + |z_k|^2)^{1/2}$.

Let $S$ be a subset of $\mathbb{C}^k$ with $k \in \mathbb{N}$. We say $S$ preserves the conjugation if the conjugate $\overline{z}$ of any $z \in S$ also belongs to $S$.

### 4.2. Complexification of real linear operators

Let $X$ and $Y$ be real linear spaces, and $X_C$ and $Y_C$ be their complexifications. Let $L$ be a linear mapping from $X$ to $Y$. The complexification $L_C$ is the mapping $L_C : X_C \to Y_C$ defined by
\[
L_C(x_1 + ix_2) = Lx_1 + iLx_2 \quad \text{for all } x_1, x_2 \in X.
\]

Clearly, $L_C$ is the unique linear extension of $L$ from $X$ to $X_C$.

**Lemma 4.1.** Assume $X$, $Y$ and $Z$ are real inner product spaces, and $L : X \to Y$ and $T : X \to Z$ are linear mappings that satisfy
\[
\|Lx\|_Y \leq C\|Tx\|_Z \quad \text{for all } x \in X, \quad \text{for some constant } C \geq 0.
\]

Let $L_C : X_C \to Y_C$ and $T_C : X_C \to Z_C$ be the complexifications of $L$ and $T$. Then
\[
\|L_Cx\|_{Y_C} \leq C\|T_Cx\|_{Z_C} \quad \text{for all } x \in X_C.
\]
(4.1)

**Proof.** Let $x = x_1 + ix_2 \in X_C$, then
\[
\|L_Cx\|^2_{Y_C} = \|Lx_1\|^2_X + \|Lx_2\|^2_X \leq C^2(\|Tx_1\|^2_Z + \|Tx_2\|^2_Z) = C^2\|T_Cx\|^2_{Z_C}.
\]
Therefore, we obtain (4.1). □

**Corollary 4.2.** If $L : X \to Y$ is a bounded linear mapping between two real inner product spaces, then $L_C$ is also a bounded linear mapping, and
\[
\|L_C\|_{B(X_C, Y_C)} = \|L\|_{B(X, Y)}.
\]
(4.2)

Here, $\| \cdot \|_{B(\cdot, \cdot)}$ denotes the norm of a bounded linear mapping.

**Proof.** Applying Lemma 4.1 to $Z = X$, $T = \text{Id}_H$ and $C = \|L\|_{B(X,Y)}$ yields $\|L\|_{B(X,Y)}$ is an upper bound of the set $S = \{\|Lx\|_{Y_C} : x \in X_C, \|x\|_{X_C} = 1\}$.

Suppose $(x_n)_{n=1}^\infty$ is a sequence in $X$ with $\|x_n\|_{X} = 1$ and $\|Lx_n\|_Y \to \|L\|_{B(X,Y)}$ as $n \to \infty$. Because $x_n \in X$, we have $\|x_n\|_{X_C} = \|x_n\|_{X} = 1$ and $L_Cx_n = Lx_n \in Y$. Thus, $\|L_Cx_n\|_{Y_C} = \|Lx_n\|_Y \to \|L\|_{B(X,Y)}$ as $n \to \infty$. Therefore, $\|L\|_{B(X,Y)}$ equals the supremum of the set $S$, hence, it is the norm of $L_C$ and we obtain (4.2). □

### 4.3. Specific complexification for the NSE

For $\alpha, \sigma \geq 0$, denote the complexification $(G_{\alpha,\sigma})_C$ of $G_{\alpha,\sigma};$ it is a complex Hilbert space and we abbreviate its norm $\| \cdot \|_{G_{\alpha,\sigma}}$ by $| \cdot |_{\alpha,\sigma}.$ In particular, $\| \cdot \|_{H_C}$ is denoted by $| \cdot |$, and $\| \cdot \|_{V_C}$ by $\| \cdot \|.$

**Definition 4.3.** Considering the Stokes operator $A$ given by (2.2), let $A_C$ denote its complexification. Specifically, $A_C : G_{1,0;C} \to H_C$ is defined by
\[
A_C(u + iv) = Au + iAv \quad \text{for } u, v \in G_{1,0}.
\]
(4.3)
Consider the bilinear form $B$ given by (2.3), its complexification is $B_C : G_{1,0,C} \times G_{1,0,C} \to H_C$ defined by
\[
B_C(u_1 + iv_1, u_2 + iv_2) = B(u_1, u_2) - B(v_1, v_2) + i(B(u_1, v_2) + B(v_1, u_2)) \quad (4.4)
\]
for $u_1, u_2, v_1, v_2 \in G_{1,0}$.

Then $A_C$ is the unique linear mapping that extends $A$ from $G_{1,0}$ to $G_{1,0,C}$. Thanks to Corollary 4.2, $A_C$ is a bounded linear mapping from $G_{1,0,C}$ to $H_C$. Similarly, $B_C$ is the unique bilinear mapping that extends $B$ from $G_{1,0} \times G_{1,0}$ to $G_{1,0,C} \times G_{1,0,C}$. Moreover, $B_C$ is a bounded bilinear mapping from $G_{1,0,C} \times G_{1,0,C}$ to $H_C$.

One can verify from (4.3) and (4.4) that
\[
\overline{A_Cw} = A_C\overline{w} \quad \text{for all} \quad w \in G_{1,0,C},
\]
\[
B_C(\overline{w_1}, \overline{w_2}) = \overline{B_C(w_1, w_2)} \quad \text{for all} \quad w_1, w_2 \in G_{1,0,C}. \quad (4.5)
\]

Let $(\lambda_n)_{n=1}^\infty$ and $(w_n)_{n=1}^\infty$ be as in subsection 2.1, see (2.3) and (2.4). It is clear that $(w_n)_{n=1}^\infty$ is a complete orthonormal basis of $H_C$ and
\[
A_Cw_n = \lambda_n w_n \quad \text{for all} \quad n \in \mathbb{N}. \quad (4.6)
\]

We make the following two remarks.

(a) The eigenvalues of $A_C$ are exactly $\mathcal{S}(A)$. Indeed, thanks to (4.6), $\lambda_n$’s already are eigenvalues of $A_C$. Suppose $A_Cw = \lambda w$ for some $\lambda \in \mathbb{C}$, with $w \in H_C$, $w \neq 0$. Suppose $w = \sum_{n=1}^\infty c_n w_n \in H_C$, with $c_{n_0} \neq 0$ for some $n_0 \in \mathbb{N}$. We then have
\[
\sum_{n=1}^\infty \lambda_n c_n w_n = \sum_{n=1}^\infty \lambda c_n w_n.
\]
It implies $\lambda_{n_0} c_{n_0} = \lambda c_{n_0}$, hence, $\lambda = \lambda_{n_0} \in \mathcal{S}(A)$.

(b) For any $\Lambda \in \mathcal{S}(A)$, the eigenspace of $A_C$ corresponding to $\Lambda$ is $(R_\Lambda H)_C$. We quickly verify this fact here. We temporarily denote the eigenspace of $A_C$ corresponding to $\Lambda$ by $S_\Lambda$. It is clear that $(R_\Lambda H)_C \subset S_\Lambda$. Now, suppose $A_C(x + iy) = \Lambda(x + iy)$ for $x, y \in H$ with $x + iy \neq 0$. Since $\Lambda \in \mathbb{R}$, one has $Ax = \Lambda x$ and $Ay = \Lambda y$. Thus $x, y \in R_\Lambda H$, and, hence, $x + iy \in (_\Lambda H)_C$. We then have $S_\Lambda \subset (R_\Lambda H)_C$, and, consequently, $S_\Lambda = (R_\Lambda H)_C$.

For $\Lambda \in \mathcal{S}(A)$, define $R_{\Lambda,C}$ to be the orthogonal projection from $H_C$ to the space $(R_\Lambda H)_C$. For $\Lambda \in \mathcal{S}(A)$, define
\[
P_{\Lambda,C} = \sum_{\lambda \in \mathcal{S}(A), \lambda \leq \Lambda} R_{\lambda,C}.
\]

In the similar way to (2.3), we define for $\alpha, s, \sigma \in \mathbb{R}$ and $w = \sum_{n=1}^\infty c_n w_n \in H_C$ with $c_n = \langle w, w_n \rangle_{H_C} \in \mathbb{C},$
\[
A_C^2w = \sum_{n=1}^\infty \lambda_n^2 c_n w_n, \quad e^{sA_C}w = \sum_{n=1}^\infty e^{s \lambda_n} c_n w_n, \quad e^{\sigma A_C^{1/2}}w = \sum_{n=1}^\infty e^{\sigma \sqrt{\lambda_n}} c_n w_n, \quad (4.7)
\]
whenever the defined element belongs to $H_C$.

Let $w \in H_C$. Then $w = u + iv$ for $u, v \in H$. Assume $u$ and $v$ have the Fourier series
\[
u = \sum_{k \neq 0} \tilde{u}_k e^{ik\cdot x} \quad \text{and} \quad \nu = \sum_{k \neq 0} \tilde{v}_k e^{ik\cdot x}. \quad (4.8)
\]
(We do not combine the Fourier coefficients of $u$ and $iv$, and, hence, do not use the formal addition $\sum_{k\neq 0}(\hat{u}_k + i\hat{v}_k)e^{ikL\cdot x}$ for $w$.)

For $A \in \mathcal{G}(A)$, we have

$$R_{A,C}w = \sum_{|k_L|^2 = \Lambda} \hat{u}_k e^{ikL \cdot x} + i \sum_{|k_L|^2 = \Lambda} \hat{v}_k e^{ikL \cdot x},$$

$$P_{A,C}w = \sum_{0 < |k_L|^2 \leq \Lambda} \hat{u}_k e^{ikL \cdot x} + i \sum_{0 < |k_L|^2 \leq \Lambda} \hat{v}_k e^{ikL \cdot x} .$$

Clearly, one has $R_{A,C} = (R_A)_C$ and $P_{A,C} = (P_A)_C$.

From (4.7), we explicitly have

$$A^\alpha_Cw = \sum_{k \neq 0} |k_L|^{2\alpha} \hat{u}_ke^{ikL \cdot x} + i \sum_{k \neq 0} |k_L|^{2\alpha} \hat{v}_ke^{ikL \cdot x},$$

$$e^{sA_C}w = \sum_{k \neq 0} e^{s|k_L|^2} \hat{u}_k e^{ikL \cdot x} + i \sum_{k \neq 0} e^{s|k_L|^2} \hat{v}_k e^{ikL \cdot x},$$

$$e^{\sigma A^{1/2}_C} w = \sum_{k \neq 0} e^{\sigma|k_L|^2} \hat{u}_k e^{ikL \cdot x} + i \sum_{k \neq 0} e^{\sigma|k_L|^2} \hat{v}_k e^{ikL \cdot x} .$$

It follows that

$$A^\alpha_C = (A^\alpha)_C, \quad e^{sA_C} = (e^{sA})_C, \quad e^{\sigma A^{1/2}_C} = (e^{\sigma A^{1/2}})_C. \quad (4.9)$$

For $\alpha, \sigma \geq 0$ and $w \in G_{a,\sigma,\mathbb{C}}$, we have $|w|_{a,\sigma} = |w|_{a,\sigma,\mathbb{C}} = |A^\alpha_C e^{\alpha A^{1/2}_C} w|$.

Thanks to the relations in (4.9) and Lemma 4.1 and Corollary 4.2, inequalities (2.12), (2.13) and (2.14) are still valid for complexified spaces and operators, namely,

$$|A^\alpha_C e^{-\sigma A_C}v| \leq d_0(\alpha, \sigma)|v| \quad \forall v \in H_C, \quad (4.10)$$

$$|A^\alpha_C e^{-\alpha A^{1/2}_C}v| \leq d_0(2\alpha, \sigma)|v| \quad \forall v \in H_C, \quad (4.11)$$

$$|A^\alpha_C v| \leq d_0(2\alpha, \sigma)|e^{\sigma A^{1/2}_C}v| \quad \forall v \in G_{0,\sigma,\mathbb{C}}. \quad (4.12)$$

For $c \in \mathbb{C}$, we naturally define the linear mapping $A_C + c : G_{1,0,\mathbb{C}} \rightarrow H_C$ by

$$(A_C + c)w = A_Cw + cw \text{ for } w \in G_{1,0,\mathbb{C}}.$$ 

Let $\omega \in \mathbb{R}$, we explicitly have

$$(A_C + i\omega)(u + iv) = (Au - \omega v) + i(\omega u + Av), \quad \text{for } u, v \in G_{1,0}. \quad (4.11)$$

Let $\alpha, \sigma \geq 0$. Assume $w = u + iv \in G_{a+1,\sigma,\mathbb{C}}$, with $u, v \in G_{a+1,\sigma}$. Elementary calculations based on (4.11) give

$$|(A_C + i\omega)w|_{a,\sigma}^2 = |Au - \omega v|_{a,\sigma}^2 + |Av + \omega u|_{a,\sigma}^2 = |Au|_{a,\sigma}^2 + |Av|_{a,\sigma}^2 + \omega^2(|u|_{a,\sigma}^2 + |v|_{a,\sigma}^2).$$

Thus,

$$|(A_C + i\omega)w|_{a,\sigma}^2 = |A_Cw|_{a,\sigma}^2 + \omega^2 |w|_{a,\sigma}^2 \text{ for } w \in G_{a+1,\sigma,\mathbb{C}}. \quad (4.12)$$

Consequently, $(A_C + i\omega)w \in G_{a,\sigma,\mathbb{C}}$ and

$$|(A_C + i\omega)w|_{a,\sigma}^2 \leq (1 + \omega^2)|w|_{a+1,\sigma}^2, \text{ for } w \in G_{a+1,\sigma,\mathbb{C}}.$$ 

Moreover, it follows (4.12) that the restriction of $A_C + i\omega$ on $G_{a+1,\sigma,\mathbb{C}}$ is one-to-one. Assume $u$ and $v$ have the Fourier series as in (1.8). We have from (1.8) and (4.11) that

$$(A_C + i\omega)w = \left(\sum (|k_L|^2\hat{u}_k - \omega\hat{v}_k)e^{ikL \cdot x}\right) + i \left(\sum (\omega\hat{u}_k + |k_L|^2\hat{v}_k)e^{ikL \cdot x}\right). \quad (4.13)$$
To calculate \((A_c + i\omega)^{-1}w\), we formally compute
\[
\begin{pmatrix}
|k_L|^2 & -\omega \\
\omega & |k_L|^2
\end{pmatrix}^{-1}
\begin{pmatrix}
\hat{u}_k \\
\hat{v}_k
\end{pmatrix} =
\frac{1}{|k_L|^4 + \omega^2}
\begin{pmatrix}
|k_L|^2 & \omega \\
-\omega & |k_L|^2
\end{pmatrix}
\begin{pmatrix}
\hat{u}_k \\
\hat{v}_k
\end{pmatrix}.
\]
Thus, formally
\[
(A_c + i\omega)^{-1}w = \left(\sum \frac{|k_L|^2 \hat{u}_k + \omega \hat{v}_k e^{ik_L x}}{|k_L|^4 + \omega^2} w \right) + i \left(\sum \frac{|k_L|^2 \hat{v}_k - \omega \hat{u}_k e^{ik_L x}}{|k_L|^4 + \omega^2} \right),
\]
and, hence,
\[
|(A_c + i\omega)^{-1}w|^2_{a+1,\sigma} = |\Omega| \sum \frac{|k_L|^2 |\hat{u}_k + \omega \hat{v}_k|^2 + |k_L|^2 |\hat{v}_k - \omega \hat{u}_k|^2}{(|k_L|^4 + \omega^2)^2} |k_L|^{4(\alpha+1)e^{2\sigma}|k_L|}
= |\Omega| \sum \frac{(|k_L|^4 + \omega^2)(|\hat{u}_k|^2 + |\hat{v}_k|^2)}{|k_L|^4 + \omega^2} |k_L|^{4(\alpha+1)e^{2\sigma}|k_L|}
= |\Omega| \sum \frac{|\hat{u}_k|^2 + |\hat{v}_k|^2}{|k_L|^4 + \omega^2} |k_L|^{4(\alpha+1)e^{2\sigma}|k_L|}.
\]

Now, assume \(w \in G_{\alpha,\sigma,C}\). Then
\[
|(A_c + i\omega)^{-1}w|^2_{a+1,\sigma} \leq |\Omega| \sum (|\hat{u}_k|^2 + |\hat{v}_k|^2) |k_L|^{4\alpha e^{2\sigma}|k_L|} = |u|^2_{a,\sigma} + |v|^2_{a,\sigma} = |w|^2_{a,\sigma} < \infty.
\]

Therefore, \((A_c + i\omega)^{-1}w\) exists in \(G_{\alpha+1,\sigma,C}\) and is given by (4.14).

We have proved the following facts.

Lemma 4.4. For any numbers \(\alpha, \sigma \geq 0\) and \(\omega \in \mathbb{R}\), one has \(A_c + i\omega\) is a bijective, bounded linear mapping from \(G_{\alpha+1,\sigma,C}\) to \(G_{\alpha,\sigma,C}\) with
\[
|(A_c + i\omega)^{-1}w|^2_{a+1,\sigma} = |w|^2_{a,\sigma} \text{ for all } w \in G_{\alpha,\sigma,C}.
\]

Lemma 4.4 particularly asserts, when \(\alpha = \sigma = 0\), that \(A_c + i\omega\) is a bijective, bounded linear mapping from \(G_{1,0,C}\) to \(H_C\).

It follows (4.14), recalling \(\omega \in \mathbb{R}\), \(w = u + iv\) and \(\bar{w} = u - iv\), that
\[
(A_c + i\omega)^{-1}w = (A_c - i\omega)^{-1}\bar{w} = (A_c + i\omega)^{-1}\bar{w}.
\]

Next, inequality (2.10) for \(B\) can be extended for \(B_C\) as follows.

For \(\alpha \geq 1/2\), \(\sigma \geq 0\), one has
\[
|B_C(w_1, w_2)|_{a,\sigma} \leq \sqrt{2K^\alpha}|w_1|_{a+1/2,\sigma}|w_2|_{a+1/2,\sigma} \forall w_1, w_2 \in G_{a+1/2,\sigma,C}.
\]

Indeed, if \(w_j = u_j + iv_j \in G_{a+1/2,\sigma,C}\), with \(u_j, v_j \in G_{a+1/2,\sigma}\) for \(j = 1, 2\), then
\[
|B_C(w_1, w_2)|^2_{a,\sigma} = |B(u_1, u_2) - B(v_1, v_2)|^2_{a,\sigma} + |B(u_1, v_2) + B(v_1, u_2)|^2_{a,\sigma}
\leq 2 \left( |B(u_1, u_2)|^2_{a,\sigma} + |B(v_1, v_2)|^2_{a,\sigma} + |B(u_1, v_2)|^2_{a,\sigma} + |B(v_1, u_2)|^2_{a,\sigma} \right).
\]

Applying inequality (2.10) gives
\[
|B_C(w_1, w_2)|^2_{a,\sigma} \leq 2 \left( |u_1|^2_{a+1/2,\sigma}|u_2|^2_{a+1/2,\sigma} + |v_1|^2_{a+1/2,\sigma}|v_2|^2_{a+1/2,\sigma}
+ |u_1|^2_{a+1/2,\sigma}|v_2|^2_{a+1/2,\sigma} + |v_1|^2_{a+1/2,\sigma}|u_2|^2_{a+1/2,\sigma} \right)
\leq 2K^\alpha (|u_1|^2_{a+1/2,\sigma} + |v_1|^2_{a+1/2,\sigma})(|u_2|^2_{a+1/2,\sigma} + |v_2|^2_{a+1/2,\sigma}).
\]

Hence, we obtain (4.17).
The following linear transformation $\mathcal{Z}_{\mathcal{A}C}$ will play a crucial role in our presentation.

**Definition 4.5.** Given an integer $k \geq -1$.

1. Let $p \in \mathcal{P}_-(k,0,H_0)$ be given by (3.12) with $z \in (0,\infty)^{k+2}$ and $\alpha \in \mathbb{C}^{k+2}$ as in (3.9). Define the function $\mathcal{Z}_{\mathcal{A}C}p : (0,\infty)^{k+2} \rightarrow G_{1,0,C}$ by

$$
(\mathcal{Z}_{\mathcal{A}C}p)(z) = \sum_{\alpha \in \mathcal{S}} z^\alpha (A_\alpha + \alpha_{-1})^{-1} \xi_\alpha.
$$

2. By mapping $p \mapsto \mathcal{Z}_{\mathcal{A}C}p$, one defines the linear transformation $\mathcal{Z}_{\mathcal{A}C}$ on $\mathcal{P}_-(k,0,H_0)$ for $k \geq -1$.

Note that each $\alpha = (\alpha_{-1}, \alpha_0, \ldots, \alpha_k)$ in (4.18) belongs to $\mathcal{E}_k(-1,k,0)$, which, by (3.10), yields $\text{Re}(\alpha_{-1}) = 0$. Therefore, $(A_\alpha + \alpha_{-1})^{-1} \xi_\alpha$ exists thanks to Lemma 4.4.

If $\alpha_{-1} = 0$ for all $\alpha \in S$ in (4.18), then $\mathcal{Z}_{\mathcal{A}C}p = A_\mathcal{C}^{-1}p$. Moreover, in the case $p \in \mathcal{P}_-(k,0,H)$, which corresponds to $K = \mathbb{R}$, then $\alpha_{-1} = 0$, $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ and $\xi_\alpha \in H$ for all $\alpha \in S$ in (4.18), hence, $\mathcal{Z}_{\mathcal{A}C}p = A^{-1}p$.

The following properties are direct consequences of Lemma 4.4.

**Lemma 4.6.** Given numbers $\alpha, \sigma \geq 0$, the following statements hold true.

1. For $k \geq -1$, $\mathcal{Z}_{\mathcal{A}C}$ maps $\mathcal{P}_-(k,0,G_{\alpha,\sigma,C})$ into $\mathcal{P}_-(k,0,G_{\alpha+1,\sigma,C})$.

2. For any integers $k \geq m \geq 0$ and number $\mu \in \mathbb{R}$, $\mathcal{Z}_{\mathcal{A}C}$ maps $\mathcal{P}_m(k,\mu,G_{\alpha,\sigma,C})$ into $\mathcal{P}_m(k,\mu,G_{\alpha+1,\sigma,C})$.

**Proof.** Part (i) follows Lemma 4.4 directly. Consider Part (ii). Let integers $k, m$ satisfy $k \geq m \geq 0$ and $\mu$ be a real number. Let $p \in \mathcal{P}_m(k,\mu,G_{\alpha,\sigma,C})$. Thanks to relation (3.14), $p \in \mathcal{P}_-(k,0,G_{\alpha,\sigma,C})$, thus $\mathcal{Z}_{\mathcal{A}C}p$ is well-defined. Note that the powers $\alpha$'s in (4.18) for $(\mathcal{Z}_{\mathcal{A}C}p)(z)$ are the same as those appearing in (3.12) for $p(z)$. Also, each $\xi_\alpha$ in (4.18) belongs to $G_{\alpha,\sigma,C}$. Therefore, $\mathcal{Z}_{\mathcal{A}C}p$ belongs to $\mathcal{P}_m(k,\mu,G_{\alpha+1,\sigma,C})$. \(\square\)

5. **Asymptotic approximations for solutions of the linearized NSE**

The main asymptotic approximation for solutions of the linearized NSE with a decaying force is the following Theorem 5.1. It is an extension of the asymptotic approximation result [26, Theorem 5.5] from finite dimensional spaces to infinite dimensional spaces with particular Stokes operator. It also generalizes [7, Lemma 2.3] and [6, Theorem 3.2].

**Theorem 5.1.** Given numbers $\alpha, \sigma \geq 0$, $\mu > 0$, integers $m \in \mathbb{Z}_+$ and $k \geq m$, and a real number $T_*$ such that $T_* > E_k(0)$ and $T_* \geq E_{m+1}(0)$. Let $p$ be in $\mathcal{P}_m(k,-\mu,G_{\alpha,\sigma,C})$ and satisfy

$$
p(\hat{L}_k(t)) \in G_{\alpha,\sigma} \text{ for all } t \in [T_*, \infty).
$$

Let $g$ be a function from $[T_*, \infty)$ to $G_{\alpha,\sigma}$ that satisfies

$$
|g(t)|_{\alpha,\sigma} \leq M L_m(t)^{-\mu-\delta_0} \text{ a.e. in } (T_*, \infty),
$$

for some positive numbers $\delta_0$ and $M$.

Suppose $w \in C([T_*, \infty),H_w) \cap L^1_{\text{loc}}([T_*, \infty),V)$, with $w' \in L^1_{\text{loc}}([T_*, \infty),V')$, is a weak solution of

$$
w' = -Aw + p(\hat{L}_k(t)) + g(t) \text{ in } V' \text{ on } (T_*, \infty),
$$

(5.3)
i.e., it holds, for all \( v \in V \), that
\[
\frac{d}{dt} \langle w, v \rangle = -\langle w, v \rangle + \langle p(\tilde{L}_k(t)) + g(t), v \rangle \text{ in the distribution sense on } (T_*, \infty).
\]
Assume, in addition, \( w(T_*) \in \mathcal{G}_{\alpha, \sigma} \). Then the following statements hold true.

(i) \( w(t) \in \mathcal{G}_{\alpha+1-\varepsilon, \sigma} \) for all \( \varepsilon \in (0, 1) \) and \( t > T_* \).

(ii) Let \( \varepsilon \in (0, 1) \) and
\[
\delta_* = \begin{cases} 
\text{any number in } (0, 1) \cap (0, \delta_0], & \text{for } m = 0, \\
\delta_0, & \text{for } m \geq 1.
\end{cases}
\]

Then there exists a constant \( C > 0 \) such that
\[
\left| w(t) - (\mathcal{Z}_{A_c} p)(\tilde{L}_k(t)) \right|_{\alpha+1-\varepsilon, \sigma} \leq C L_m(t)^{-\mu-\delta_*} \quad \forall t \geq T_* + 1. 
\] (5.4)

We prepare for the proof of Theorem 5.1 with the following calculations and estimates.

Consider \( \beta = (\beta_1, \beta_0, \ldots, \beta_k) \in \mathcal{E}_k(m, k, -\mu) \) with \( m \in \mathbb{Z}_+, k \geq m \) and \( \mu > 0 \). Given \( T_* > E_k(0) \), set
\[
h(t) = \prod_{j=0}^k L_j(T_* + t)^{\beta_j} \text{ for } t \geq 0.
\]
By (3.11), one has
\[
|h(t)| = \mathcal{O}(L_m(T_* + t)^{-\mu+s}) \text{ for all } s > 0.
\] (5.5)

The derivative of \( h(t) \) can be computed, by the product rule and (3.5), as
\[
h'(t) = (T_* + t)^{-1} \left\{ \beta_0 + \sum_{j=1}^k \left[ \beta_j \prod_{\ell=1}^j L_\ell(T_* + t)^{-1} \right] \right\} h(t).
\] (5.6)

By (5.5) and (5.6), it holds, for any \( s > 0 \), that
\[
|h'(t)| = \mathcal{O}((T_* + t)^{-1} L_m(T_* + t)^{-\mu+s}).
\] (5.7)

Consider \( m = 0 \). Let \( \gamma \) be an arbitrary number in \((0, 1)\). Taking \( s = 1 - \gamma > 0 \) in (5.7) and using the continuity of \( h'(t) \) and \((T_* + t)^{-\mu-\gamma}\) on \([0, \infty)\), we obtain
\[
|h'(t)| \leq C(T_* + t)^{-\mu-\gamma} \text{ for all } t \geq 0,
\] (5.8)
for some positive constant \( C \).

Consider \( m \geq 1 \). Taking \( s = \mu > 0 \) in (5.7), we infer, similar to (5.5), that
\[
|h'(t)| \leq C'(T_* + t)^{-1} \text{ for all } t \geq 0,
\] (5.9)
for some positive constant \( C' \).

**Proof of Theorem 5.1.** Part (i) is from Lemma 2.4(i) of [7]. Below, we prove part (ii).

Thanks to (3.2) and (3.3), the function \( t \in [0, \infty) \mapsto L_m(T_* + t) \) is increasing and maps \([0, \infty)\) into \([1, \infty)\).

Let \( N \in \mathbb{N} \), denote \( \Lambda = \Lambda_N \), let \( A_A = A|_{P_A H} \) and \( A_{C,A} = A_C|_{P_{A_c} H_C} \). Then \( A_A \) is a linear operator on \( P_A H \), and, in fact, \( A_{C,A} : P_{A_c} H_C \rightarrow P_{A_c} H_C \) is the complexification \((A_A)_C\) of \( A_A \).
By applying projection $P_\Lambda$ to equation \[5.3\] and using the variation of constants formula, one obtains, for $t \geq 0$,

$$P_\Lambda w(T_* + t) = e^{-tA_\Lambda}P_\Lambda w(T_*) + \int_0^t e^{-(t-\tau)A_\Lambda}P_\Lambda p(\mathcal{L}_k(T_* + \tau))d\tau$$

$$+ \int_0^t e^{-tA_\Lambda}P_\Lambda g(T_* + \tau)d\tau. \tag{5.10}$$

(For the validity of the variation of constants formula \[5.10\], see the arguments between (2.17) and (2.19) of \cite{7}, and also \cite[Lemma 4.2]{28}.)

Assume

$$p(z) = \sum_{\beta \in S} z^\beta \xi_\beta = \sum_{\beta \in S} p_\beta(z), \text{ where } p_\beta(z) = z^\beta \xi_\beta, \; \xi_\beta \in G_{\alpha,\sigma,\mathcal{C}},$$

with $S$ being a finite subset of $E_C(m, N, -\mu).

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in S$. Since $\beta \in E_C(m, N, -\mu)$ and $m \geq 0$, we have $\text{Re}(\beta_{-1}) = 0$. Hence,

$$\beta_{-1} = i\omega_\beta \text{ for some } \omega_\beta \in \mathbb{R}. \tag{5.11}$$

By defining $h_\beta(t) = \prod_{j=0}^k L_j(T_* + t)^{\beta_j}$, we can write

$$p_\beta(\mathcal{L}_k(T_* + t)) = e^{\beta_{-1}(T_* + t)}h_\beta(t)\xi_\beta.$$

In the calculations below, we use $A_{\beta,\Lambda}$ to denote

$$(A_C + \beta_{-1})|_{P_{\Lambda,c}H_C} = A_{\Lambda,\Lambda} + \beta_{-1}I_{P_{\Lambda,c}H_C}$$

Then $A_{\beta,\Lambda}$ is an invertible linear transformation from $P_{\Lambda,c}H_C$ to itself, see formulas \[4.13\] and \[4.14\] when $w \in P_{\Lambda,c}H_C$.

The second term on the right-hand side of \[5.10\] can be computed as the following

$$\int_0^t e^{-tA_\Lambda}P_\Lambda p(\mathcal{L}_k(T_* + \tau))d\tau = \int_0^t e^{-(t-\tau)A_{\Lambda,C}}P_{\Lambda,C}p(\mathcal{L}_N(T_* + \tau))d\tau$$

$$= \sum_{\beta \in S} \int_0^t e^{-(t-\tau)A_{\Lambda,C}}(e^{(t-1)(T_* + t)}h_\beta(\tau)P_{\Lambda,C}\xi_\beta)d\tau = \sum_{\beta \in S} H_\beta(t), \tag{5.12}$$

where

$$H_\beta(t) = e^{\beta_{-1}(T_* + t)} \int_0^t h_\beta(\tau)e^{-(t-\tau)A_{\beta,\Lambda}}P_{\Lambda,C}\xi_\beta d\tau.$$

Applying integration by parts gives

$$H_\beta(t) = e^{\beta_{-1}(T_* + t)} \left\{ h_\beta(\tau)A_{\beta,\Lambda}^{-1}e^{-(t-\tau)A_{\beta,\Lambda}}P_{\Lambda,C}\xi_\beta \bigg|_{\tau=t}^{\tau=0} - J_\beta(t) \right\}$$

$$= e^{\beta_{-1}(T_* + t)}h_\beta(t)A_{\beta,\Lambda}^{-1}P_{\Lambda,C}\xi_\beta - e^{\beta_{-1}T_*}h_\beta(0)A_{\beta,\Lambda}^{-1}e^{-tA_{\Lambda,C}}P_{\Lambda,C}\xi_\beta - e^{\beta_{-1}(T_* + t)}J_\beta(t),$$

where

$$J_\beta(t) = \int_0^t h_\beta'(\tau)A_{\beta,\Lambda}^{-1}e^{-(t-\tau)A_{\beta,\Lambda}}P_{\Lambda,C}\xi_\beta d\tau. \tag{5.13}$$

Note that

$$e^{\beta_{-1}(T_* + t)}h_\beta(t)A_{\beta,\Lambda}^{-1}P_{\Lambda,C}\xi_\beta = (A_{\Lambda,\Lambda} + \beta_{-1}I_{P_{\Lambda,c}H_C})^{-1}P_{\Lambda,C}p_\beta(\mathcal{L}_k(T_* + t))$$

$$= P_{\Lambda,C}(Z_{A_\Lambda} p_\beta)(\mathcal{L}_k(T_* + t)),$$
and, similarly,
\[ e^{-T_x h_x(0)} A_{\beta, \Lambda}^{-1} e^{-tA_{\Lambda}} P_{\Lambda, C} \xi_\beta = e^{-tA_{\Lambda}} P_{\Lambda, C} (\mathcal{Z}_{\Lambda} p_\beta)(\mathcal{L}_k(T_*)). \]
Summing in \( \beta \) gives
\[ \sum_{\beta \in S} H_{\beta}(t) = P_{\Lambda, C} (\mathcal{Z}_{\Lambda} p)(\mathcal{L}_k(T_* + t)) - e^{-tA_{\Lambda}} P_{\Lambda, C} (\mathcal{Z}_{\Lambda} p)(\mathcal{L}_k(T_*)) \]
\[ - \sum_{\beta \in S} e^{-t A_{\Lambda}} J_{\beta}(t). \]
Combining (5.10), (5.12) and (5.14), we obtain
\[ P_{\Lambda, C} w(T_* + t) = P_{\Lambda, C} (\mathcal{Z}_{\Lambda} p)(\mathcal{L}_k(T_* + t)) + e^{-tA_{\Lambda}} P_{\Lambda, C} W_* \]
\[ - \sum_{\beta \in S} e^{-t A_{\Lambda}} J_{\beta}(t) + \int_0^t e^{-(t-\tau)A_{\Lambda}} P_{\Lambda} g(T_* + \tau) d\tau. \]
where \( W_* = w(T_*) - (\mathcal{Z}_{\Lambda} p)(\mathcal{L}_k(T_*)). \)
By the theorem’s own hypothesis, \( w(T_*) \in G_{\alpha, \sigma}. \) By the assumption \( p \in \mathcal{P}_m(k, -\mu, G_{\alpha, \sigma, C}) \) and Lemma 4(ii), one has \((\mathcal{Z}_{\Lambda} p)(\mathcal{L}_k(T_*)) \in G_{\alpha, \sigma, C}. \) Therefore, \( W_* \in G_{\alpha, \sigma, C} \) and \( |W_*|_{\alpha, \sigma} \) is a number in \([0, \infty].\)
It follows (5.15) and, thanks to (5.11), the fact \(|e^{-t A_{\Lambda}}| = 1\) that
\[ \left| P_{\Lambda, C} \left( w(T_* + t) - (\mathcal{Z}_{\Lambda} p)(\mathcal{L}_k(T_* + t)) \right) \right|_{\alpha + 1, -\epsilon, \sigma} \leq |e^{-tA_{\Lambda}} P_{\Lambda, C} W_*|_{\alpha + 1, \sigma} \]
\[ + \sum_{\beta \in S} |J_{\beta}(t)|_{\alpha + 1, \sigma} + \left| \int_0^t e^{-(t-\tau)A_{\Lambda}} P_{\Lambda} g(T_* + \tau) d\tau \right|_{\alpha + 1, -\epsilon, \sigma}. \]
We estimate each term on the right-hand side of (5.16) separately. Let \( \delta, \theta \in (0, 1) \) and \( t \geq 1.\)
For the first term on the right-hand side of (5.16),
\[ |e^{-tA_{\Lambda}} P_{\Lambda, C} W_*|_{\alpha + 1, \sigma} \leq |e^{-tA_{\Lambda}} W_*|_{\alpha + 1, \sigma} = |A_{\Lambda} e^{-tA_{\Lambda}} W_*|_{\alpha, \sigma} = |A_{\Lambda} e^{-(1-\delta)tA_{\Lambda}} (e^{-\delta tA_{\Lambda}} W_*)|_{\alpha, \sigma}. \]
We estimate the last norm by applying inequality (4.10) to \( \alpha = 1, \sigma = (1-\delta)t \) and \( v = A_{\Lambda} e^{\sigma A_{\Lambda} t/2} (e^{-tA_{\Lambda}} W_*), \) and using the fact \( t \geq 1.\) It yields
\[ |e^{-tA_{\Lambda}} P_{\Lambda, C} W_*|_{\alpha + 1, \sigma} \leq \frac{1}{e(1-\delta)} |e^{-\delta tA_{\Lambda}} W_*|_{\alpha, \sigma} \leq \frac{e^{-\delta t}}{e(1-\delta)} |W_*|_{\alpha, \sigma}. \]
For the second term on the right-hand side of (5.16), we rewrite \( J_{\beta}(t) \) in (5.13) as
\[ J_{\beta}(t) = \int_0^t e^{-\beta_1(t-\tau)h_\beta(\tau)} (A_{\Lambda} + \beta_{-1} I_{D_{\Lambda, C}}) (e^{-(t-\tau)A_{\Lambda}} P_{\Lambda, C} \xi_\beta) d\tau \]
\[ = \int_0^t e^{-\beta_1(t-\tau)h_\beta(\tau)} (A_{\Lambda} + \beta_{-1})^{-1} e^{-(t-\tau)A_{\Lambda}} P_{\Lambda, C} \xi_\beta d\tau. \]
Note, again, from (5.11) that \(|e^{-\beta_1(t-\tau)}| = 1.\) Therefore,
\[ |J_{\beta}(t)|_{\alpha + 1, \sigma} \leq \int_0^t |h_\beta(\tau)| |(A_{\Lambda} + \beta_{-1})^{-1} e^{-(t-\tau)A_{\Lambda}} P_{\Lambda, C} \xi_\beta|_{\alpha + 1, \sigma} d\tau. \]
By inequality (3.15),
\[ |J_\beta(t)|_{0+1,\sigma} \leq \int_0^t |h_\beta' (\tau)| e^{-(t-\tau)\Lambda} P_\Lambda C_\xi |\beta|_{0+1,\sigma} d\tau \leq \int_0^t |h_\beta' (\tau)| e^{-(t-\tau)\Lambda} |\beta|_{0+1,\sigma} d\tau. \]

In the remainder of this proof, C_1, C_2, \ldots, C_{10} are some positive constants which are independent of N and t.

Case m = 0. Using estimate (5.8) for \( \gamma = \delta_0 \in (0, 1) \) and applying inequality (3.8) give
\[ |J_\beta(t)|_{0+1,\sigma} \leq C_1 \int_0^t e^{-(t-\tau)(T_\sigma + \tau)} - \delta_0 \sum_{\beta \varepsilon S} |\beta|_{0+1,\sigma} d\tau \leq C_2 (T_\sigma + t)^{-(1-\delta_0)} |\beta|_{0+1,\sigma}. \quad (5.18) \]

Case m \geq 1. Using estimate (5.9) and applying inequality (3.8) give
\[ |J_\beta(t)|_{0+1,\sigma} \leq C_3 \int_0^t e^{-(t-\tau)(T_\sigma + \tau)} - 1 |\beta|_{0+1,\sigma} d\tau \leq C_4 (T_\sigma + t)^{-(1-\delta_0)} |\beta|_{0+1,\sigma}. \]

Consequently,
\[ |J_\beta(t)|_{0+1,\sigma} \leq C_5 L_m (T_\sigma + t)^{-(1-\delta_0)} |\beta|_{0+1,\sigma}. \quad (5.19) \]

Summing up in \( \beta \) the inequalities (5.18) and (5.19), one obtains, for both cases m = 0 and m \geq 1,
\[ \sum_{\beta \varepsilon S} |J_\beta(t)|_{0+1,\sigma} \leq C_6 L_m (T_\sigma + t)^{-(1-\delta_0)} \sum_{\beta \varepsilon S} |\beta|_{0+1,\sigma}. \quad (5.20) \]

Consider the last term on the right-hand side of (5.16). We have
\[
\left| \int_0^t e^{-(t-\tau)\Lambda} P_\Lambda g(T_\sigma + \tau) d\tau \right|_{0+1,\sigma} \leq \int_0^t \left| e^{-(t-\tau)\Lambda} A^{1-\varepsilon} g(\tau) \right|_{0+1,\sigma} d\tau
\]
\[ = \int_0^\theta \left| e^{-(t-\tau)\Lambda} A^{1-\varepsilon} g(\tau) \right|_{0+1,\sigma} d\tau + \int_\theta^t \left| e^{-(t-\tau)\Lambda} A^{1-\varepsilon} g(\tau) \right|_{0+1,\sigma} d\tau = I_1 + I_2, \]

where \( I_1 \) is the integral from 0 to \( \theta t \), and \( I_2 \) is the integral from \( \theta t \) to \( t \).

The integral \( I_1 \) is bounded by
\[ I_1 \leq \int_0^\theta \left| e^{-(t-\tau)\Lambda} A g(\tau) \right|_{0+1,\sigma} d\tau. \]

Note, in the last integral, that \( t - \tau \geq (1-\theta)t \). Then
\[ |e^{-(t-\tau)\Lambda} A g(\tau)|_{0+1,\sigma} = |A e^{-(t-\tau)(1-\delta)\Lambda} e^{-(t-\tau)\delta A} g(\tau)|_{0+1,\sigma} \leq \left| \left( A e^{-(1-\theta)t(1-\delta)\Lambda} \right) e^{-(t-\tau)\delta A} g(\tau) \right|_{0+1,\sigma}. \]

To estimate the last norm, we apply inequality (2.12) to \( \alpha = 1, \sigma = (1-\theta)t(1-\delta) \) and \( v = A^\alpha e^{\delta A} e^{-(t-\tau)\delta A} g(\tau) \), and also use the fact \( t \geq 1 \). It results in
\[ |e^{-(t-\tau)\Lambda} A g(\tau)|_{0+1,\sigma} \leq \frac{e^{-(t-\delta)\delta A} g(\tau)|_{0+1,\sigma}}{e(1-\theta)t(1-\delta)} \leq \frac{e^{-(t-\tau)\delta} g(\tau)|_{0+1,\sigma}}{e(1-\theta)(1-\delta)}. \]

We bound \( e^{-(t-\tau)\delta} \) from above by \( e^{-(t-\theta)\delta} \), and bound \( |g(\tau)|_{0+1,\sigma} \) by (5.2). Combining these with inequality (3.8), we obtain
\[
I_1 \leq \frac{M}{e(1-\theta)(1-\delta)} \int_0^\theta e^{-(t-\theta)\delta} L_m (T_\sigma + \tau)^{-(1-\delta_0)} d\tau \leq \frac{MC_7 L_m (T_\sigma + \theta t)^{-(1-\delta_0)}}{e(1-\theta)(1-\delta)}. \quad (5.22)
\]
The integral $I_2$ can be estimated in the same way as in part (c) of the proof of Theorem 3.2(ii) in [6] replacing $F(t)$ by $L_m(T_\ast + t)^{-\mu - \delta_0}$. It results in, for $t \geq 1$,

$$I_2 \leq \left[ \frac{1 - \varepsilon}{\varepsilon (1 - \delta)} \right]^{1 - \varepsilon} ML_m(T_\ast + \theta t)^{-\mu - \delta_0} \left\{ (1 - \theta)^{\varepsilon \delta} + \frac{(1 - \theta)^{\varepsilon - 1}}{\delta} e^{-\delta (1 - \theta)} \right\} \leq (1 - \delta)^{\varepsilon - 1} ML_m(T_\ast + \theta t)^{-\mu - \delta_0} (1 - \theta)^{\varepsilon - 1} \left\{ \frac{1}{\varepsilon} + \frac{1}{\delta} \right\}. \quad (5.23)$$

Combining (5.21) with estimates (5.22) and (5.23), and using (3.7) to compare $L_m(T_\ast + \theta t)$ with $L_m(T_\ast + t)$, we obtain

$$\left| \int_0^t e^{-(t-\tau)A} P_A g(T_\ast + \tau) d\tau \right|_{\alpha + 1 - \varepsilon, \sigma} \leq C_8 L_m(T_\ast + t)^{-\mu - \delta_0}. \quad (5.24)$$

We take $\delta = \theta = 1/2$ now. Combining the above (5.10), (5.17), (5.20) and (5.24), and noticing that $\delta_0 \leq \delta_0$, we obtain

$$|P_{A,c}(w(T_\ast + t) - (Z_{A,c} p)(\hat{L}_k(T_\ast + t)))|_{\alpha + 1 - \varepsilon, \sigma} \leq C_9 \left\{ e^{-t/2}|W_s|_{\alpha, \sigma} + L_m(T_\ast + t)^{-\mu - \delta_\ast} \left( 1 + \sum_{\beta \in S} |\xi_\beta|_{\alpha, \sigma} \right) \right\} \forall t \geq 1.$$

By comparing $e^{-t/2}$ with $L_m(T_\ast + t)^{-\mu - \delta_\ast}$, we deduce

$$|P_{A,c}(w(T_\ast + t) - (Z_{A,c} p)(\hat{L}_k(T_\ast + t)))|_{\alpha + 1 - \varepsilon, \sigma} \leq C_{10} L_m(T_\ast + t)^{-\mu - \delta_\ast} \forall t \geq 1. \quad (5.25)$$

By passing $N \to \infty$ in (5.25), and the fact $(Z_{A,c} p)(\hat{L}_k(T_\ast + t)) \in G_{\alpha + 1, \sigma}$ we obtain, for $t \geq 1$, $w(T_\ast + t) \in G_{\alpha + 1 - \varepsilon, \sigma}$ and

$$|w(T_\ast + t) - (Z_{A,c} p)(\hat{L}_k(T_\ast + t))|_{\alpha + 1 - \varepsilon, \sigma} \leq C_{10} L_m(T_\ast + t)^{-\mu - \delta_\ast}. \quad (5.26)$$

By replacing $T_\ast + t$ with $t$ in (5.26), we obtain (5.4). The proof is complete. \qed

**Remark 5.2.** The following remarks on Theorem 5.1 are in order.

(a) According to Theorem 5.1, the solution $w(t)$ can be approximated, as $t \to \infty$, by function $(Z_{A,c} p)(\hat{L}_k(t))$, which is specifically determined by the the linear operator $A$ and the dominant coherent decay $p(\hat{L}_k(t))$ in equation (5.3) of $w$. The difference between $w(t)$ and $(Z_{A,c} p)(\hat{L}_k(t))$ decays faster than the decaying rate $L_m(t)^{-\mu}$ of $p(\hat{L}_k(t))$.

(b) The function $(Z_{A,c} p)(\hat{L}_k(t))$ itself satisfies an ODE, see Lemma 6.2 below.

(c) Thanks to Lemma 6.3 $(Z_{A,c} p)(\hat{L}_k(t))$ belongs to the complex linear space $G_{\alpha + 1, \sigma, c}$. We will investigate in more details in section 6 whether it can belong to the real linear space $G_{\alpha + 1, \sigma}$, together with condition (5.1) for $p(\hat{L}_k(t))$.

(d) For the purpose of this paper, we only consider functions $p, g$ and $w$ with values in real linear spaces. It, at least, allows us to avoid repeating the proof of part (1). In the case equation (5.3) is already set up with complex linear spaces, then the complexification is not needed. The result and proof are similar, and, in fact, simpler. See [26] Theorem 5.5] for such a result in finite dimensional spaces.
6. Main results and proofs

In order to state the main results – Theorems 6.7 and 6.11 below – we introduce the following important linear transformations \( M_j \) and \( R \), in addition to \( Z_{A_c} \) in Definition 4.5.

**Definition 6.1.** Let \( X \) be a linear space over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). Given an integer \( k \geq -1 \), let \( p \in \mathcal{P}(k,X) \) be given by (3.12) with \( z \in (0, \infty)^{k+2} \) and \( \alpha \in \mathbb{K}^{k+2} \) as in (3.9).

(i) Define, for \( j = -1, 0, \ldots, k \), the function \( M_j p : (0, \infty)^{k+2} \to X \) by

\[
(M_j p)(z) = \sum_{\alpha \in S} \alpha_j z^\alpha \xi_\alpha. \tag{6.1}
\]

(ii) In the case \( k \geq 0 \), define the function \( R p : (0, \infty)^{k+2} \to X \) by

\[
(R p)(z) = \sum_{j=0}^{k} z_0^{-1} z_1^{-1} \cdots z_j^{-1} (M_j p)(z). \tag{6.2}
\]

(iii) By mapping \( p \mapsto M_j p \) and, respectively, \( p \mapsto R p \), one defines linear transformation \( \mathcal{M}_j \) on \( \mathcal{P}(k,X) \) for \( -1 \leq j \leq k \), and, respectively, linear transformation \( \mathcal{R} \) on \( \mathcal{P}(k,X) \) for \( k \geq 0 \).

In particular, when \( j = -1, 0 \), formula (6.1) reads as

\[
\mathcal{M}_{-1} p(z) = \sum_{\alpha \in S} \alpha_{-1} z^\alpha \xi_\alpha \quad \text{and} \quad \mathcal{M}_0 p(z) = \sum_{\alpha \in S} \alpha_0 z^\alpha \xi_\alpha.
\]

An equivalent definition of \((R p)(z)\) in (6.2) is

\[
(R p)(z) = \frac{\partial p(z)}{\partial z_0} + \sum_{j=1}^{k} z_1^{-1} z_2^{-1} \cdots z_j^{-1} \frac{\partial p(z)}{\partial z_j}.
\]

The powers \( \alpha \)'s in (6.1) for \( M_j p(z) \) are the same as those that appear in (3.12) for \( p(z) \). Consequently, we have the following properties.

(a) For \( k \geq m \geq 0 \) and \( \mu \in \mathbb{R} \), if \( p \) is in \( \mathcal{P}_m(k,\mu,X) \), then so are all \( M_j p \)'s, for \( -1 \leq j \leq k \).

(b) \( R p(z) \) has the same powers of \( z_{-1} \) as \( p(z) \).

(c) If \( p \in \mathcal{P}_0(k,\mu,X) \), then \( R p \in \mathcal{P}_0(k,\mu-1,X) \).

**Lemma 6.2** (\cite{26} Lemma 5.6). Let \( (X, \| \cdot \|_X) \) be a normed space over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). If \( k \in \mathbb{Z}_+ \) and \( q \in \mathcal{P}(k,X) \), then

\[
\frac{d}{dt}(\mathcal{L}_k(t)) = \mathcal{M}_{-1} q(\mathcal{L}_k(t)) + \mathcal{R} q(\mathcal{L}_k(t)) \quad \text{for} \quad t > E_k(0). \tag{6.3}
\]

In particular, when \( k \geq m \geq 1 \), \( \mu \in \mathbb{R} \), and \( q \in \mathcal{P}_m(k,\mu,X) \), one has

\[
\left\| \frac{d}{dt}(\mathcal{L}_k(t)) - \mathcal{M}_{-1} q(\mathcal{L}_k(t)) \right\|_X = \left\| \mathcal{R} q(\mathcal{L}_k(t)) \right\|_X = O(t^{-\gamma}) \quad \text{for all} \quad \gamma \in (0, 1). \tag{6.4}
\]

In fact, the statements and proofs of \cite{26} Lemma 5.6 are for the space \( X = \mathbb{C}^n \). However, they equally hold true for any normed space \( X \) as stated in Lemma 6.2 above.

Considering \( \mathcal{M}_{-1} \) in relation with \( Z_{A_c} \) defined in Definition 4.5, we clearly have

\[
(A_c + \mathcal{M}_{-1})(Z_{A_c} p) = p \quad \forall p \in \mathcal{P}_{-1}(k,0,H_C), \tag{6.5}
\]

\[
Z_{A_c}((A_c + \mathcal{M}_{-1})p) = p \quad \forall p \in \mathcal{P}_{-1}(k,0,G_{1,0,C}). \tag{6.6}
\]
It is clear in (6.5) and (6.6) that $\mathcal{M}_{-1}$ is defined for $K = \mathbb{C}$ and $X = G_{1,0,\mathbb{C}}$.

In dealing with power, logarithmic and iterated logarithmic functions valued in real linear spaces, we have the following counterpart of Definition 3.3.

**Definition 6.3.** Let $X$ be a linear space over $\mathbb{R}$, and $X_\mathbb{C}$ be its complexification.

(i) Define $\mathcal{P}(k, X_\mathbb{C}, X)$ to be set of functions of the form

\[ p(z) = \sum_{\alpha \in S} z^\alpha \xi_\alpha \text{ for } z \in (0, \infty)^{k+2}, \tag{6.7} \]

where $S$ is a finite subset of $\mathbb{C}^{k+2}$ that preserves the conjugation, and each $\xi_\alpha$ belongs to $X_\mathbb{C}$, with

\[ \xi_\alpha = \overline{\xi_\alpha} \forall \alpha \in S. \tag{6.8} \]

(ii) Define $\mathcal{P}_m(k, \mu, X_\mathbb{C}, X)$ to be set of functions in $\mathcal{P}(k, X_\mathbb{C}, X)$ with the restriction that the set $S$ in (6.7) is also a subset of $\mathcal{E}_\mathbb{C}(m, k, \mu)$.

We have $z^\alpha = \overline{z^\alpha}$ for all $z \in (0, \infty)^{k+2}$ and $\alpha \in \mathbb{C}^{k+2}$. Because of this fact and the conjugation condition (6.8), each function $p$ in the class $\mathcal{P}(k, X_\mathbb{C}, X)$ is, in fact, $X$-valued.

We remark that the classes $\mathcal{P}(k, X_\mathbb{C}, X)$ and $\mathcal{P}_m(k, \mu, X_\mathbb{C}, X)$ are (additive) subgroups of $\mathcal{P}(k, X_\mathbb{C})$, but not linear spaces over $\mathbb{C}$.

We examine the restrictions of the mappings $M_j$'s, $R$ and $Z_{A_\mathbb{C}}$ on the new classes in Definition 6.3.

**Lemma 6.4.** Let $X$ be a real linear space and $X_\mathbb{C}$ be its complexification. The following statements hold true.

(i) Each $M_j$, for $-1 \leq j \leq k$, maps $\mathcal{P}(k, X_\mathbb{C}, X)$ into itself, $R$ maps $\mathcal{P}(k, X_\mathbb{C}, X)$, for $k \geq 0$, into itself.

(ii) All $M_j$'s, for $-1 \leq j \leq k$, map $\mathcal{P}_m(k, \mu, X_\mathbb{C}, X)$ into itself for any integers $k \geq m \geq 0$ and real number $\mu$.

(iii) $R$ maps $\mathcal{P}_0(k, \mu, X_\mathbb{C}, X)$ into $\mathcal{P}_0(k, \mu - 1, X_\mathbb{C}, X)$ for any $k \in \mathbb{Z}_+$ and $\mu \in \mathbb{R}$.

Lemma 6.4 is proved in [26] Lemma 10.7 for $X = \mathbb{R}^n$ and $X_\mathbb{C} = \mathbb{C}^n$, but the proofs work also for general spaces $X$ and $X_\mathbb{C}$.

**Lemma 6.5.** Let $\alpha, \sigma \geq 0$. The following statements hold true.

(i) For $k \geq 1$, $Z_{A_\mathbb{C}}$ maps $\mathcal{P}_{-1}(k,0,H_\mathbb{C},H)$ into $\mathcal{P}_{-1}(k,0,G_{1,0,\mathbb{C}},G_{1,0})$.

(ii) For any integers $k \geq m \geq 0$ and number $\mu \in \mathbb{R}$, $Z_{A_\mathbb{C}}$ maps $\mathcal{P}_m(k, \mu, G_{\alpha,\sigma,\mathbb{C}}, G_{\alpha,\sigma})$ into $\mathcal{P}_m(k, \mu, G_{\alpha+1,\sigma,\mathbb{C}}, G_{\alpha+1,\sigma})$.

**Proof.** We prove part (i) first. Let $p \in \mathcal{P}_{-1}(k,0,H_\mathbb{C},H)$. Then $Z_{A_\mathbb{C}}p \in \mathcal{P}_{-1}(k,0,G_{1,0,\mathbb{C}})$ thanks to Lemma 4.4(ii) applied to $\alpha = \sigma = 0$.

Let $p$ be as in (6.4). Consider formula (4.18) of $Z_{A_\mathbb{C}}p$. We write

\[ Z_{A_\mathbb{C}}p(z) = \sum_{\alpha \in S} z^\alpha \eta_\alpha, \text{ where } \eta_\alpha = (A_\mathbb{C} + \alpha_\mathbb{C})^{-1}\xi_\alpha. \]

Using (6.8) and (4.16), one has

\[ \eta_\alpha = (A_\mathbb{C} + \alpha_\mathbb{C})^{-1}\xi_\alpha = (A_\mathbb{C} + \alpha_\mathbb{C})^{-1}\xi_\alpha = (A_\mathbb{C} + \alpha_\mathbb{C})^{-1}\xi_\alpha = \eta_\alpha. \]

This implies $Z_{A_\mathbb{C}}p \in \mathcal{P}_{-1}(k,0,G_{\alpha,\sigma,\mathbb{C}},G_{\alpha,\sigma})$.

The proof of part (ii) is similar by using Lemma 4.4(ii) and property (6.9). □
6.1. **Case 1: the force has coherent power decay.** This subsection is focused on the case when the force \( f(t) \) in (2.9) has coherent power decay. More specifically, we assume the following.

**Assumption 6.6.** Suppose there exist real numbers \( \sigma \geq 0, \alpha \geq 1/2, \) a strictly increasing, divergent sequence of positive numbers \( (\mu_n)_{n=1}^{\infty} \) that preserve the addition and unit increment, an increasing sequence \( (M_n)_{n=1}^{\infty} \) in \( \mathbb{Z}_+ \), and functions \( p_n \in \mathcal{P}_0(M_n, -\mu_n, G_{\alpha,\sigma}, G_{\alpha,\sigma}) \) for all \( n \in \mathbb{N} \) such that, in the sense of Definition 3.4 with \( m_* = 0 \),

\[
f(t) \sim \sum_{n=1}^{\infty} p_n(\mathcal{L}_{M_n}(t)) \text{ in } G_{\alpha,\sigma}.
\tag{6.10}
\]

Our first main result on the asymptotic expansion of the Leray–Hopf weak solutions is the next theorem.

**Theorem 6.7.** Under Assumption 6.6 let \( q_n \), for \( n \in \mathbb{N} \), be defined recursively by

\[
q_n = \mathcal{Z}_{A_C} \left( p_n - \sum_{1 \leq m,j \leq n-1, \mu_m + \mu_j = \mu_n} B_C(q_m, q_j) - \chi_n \right) \text{ for } n \geq 1,
\tag{6.11}
\]

with

\[
\chi_n = \begin{cases} 
\mathcal{R}q_\lambda, & \text{if } \exists \lambda \in [1, n-1] \text{ such that } \mu_\lambda + 1 = \mu_n, \\
0, & \text{otherwise}.
\end{cases}
\tag{6.12}
\]

Let \( u(t) \) be any Leray–Hopf weak solution of (2.9). For any \( N \in \mathbb{N} \), there exists a number \( \delta_N > 0 \) such that it holds, for any \( \rho \in (0,1) \),

\[
\left| u(t) - \sum_{n=1}^{N} q_n(\mathcal{L}_{M_n}(t)) \right|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_N-\delta_N}).
\tag{6.13}
\]

Consequently, any Leray–Hopf weak solution \( u(t) \) of (2.9) has the asymptotic expansion

\[
u(t) \sim \sum_{n=1}^{\infty} q_n(\mathcal{L}_{M_n}(t)) \text{ in } G_{\alpha+1,\rho,\sigma} \text{ for any } \rho \in (0,1).
\tag{6.14}
\]

Certainly, \( \chi_1 = 0 \) in (6.12) and, hence,

\[
q_1 = \mathcal{Z}_{A_C} p_1.
\tag{6.15}
\]

For \( n \in \mathbb{N} \), the index \( \lambda \) in (6.12), if exists, is unique, and \( \lambda < n \). Thus, equation (6.11) is, indeed, a recursive formula in \( n \).

**Lemma 6.8.** For any \( n \in \mathbb{N} \), one has \( q_n \in \mathcal{P}_0(M_n, -\mu_n, G_{\alpha+1,\sigma}, G_{\alpha,\sigma}) \).

**Proof.** We prove by induction. By (6.15), we have \( q_1 = \mathcal{Z}_{A_C} p_1 \). Because \( p_1 \in \mathcal{P}_0(M_1, -\mu_1, G_{\alpha,\sigma}, G_{\alpha,\sigma}) \), then by the virtue of Lemma 6.4 it follows that \( q_1 \in \mathcal{P}_0(M_1, -\mu_1, G_{\alpha+1,\sigma}, G_{\alpha+1,\sigma}) \).

Therefore, the statement is true for \( n = 1 \).

Let \( n \geq 2 \). Suppose \( q_j \in \mathcal{P}_0(M_j, -\mu_j, G_{\alpha,\sigma+1}, G_{\alpha+1,\sigma}) \) for \( 1 \leq j \leq n-1 \).
As a consequence of (6.16) and Lemma 6.2(iii), we have
\[ R q_j \in \mathcal{P}_0(M_j, -\mu_j - 1, G_{\alpha, \sigma}, G_{\alpha, \sigma}) \text{ for } 1 \leq j \leq n - 1. \] (6.17)

For \( 1 \leq j \leq n - 1 \), we have \( M_j \leq M_n \) and can use the embedding in property (c) after Definition 3.3 to have
\[ q_j \in \mathcal{P}_0(M_n, -\mu_j, G_{\alpha, \sigma + 1, \xi}, G_{\alpha + 1, \sigma}). \] (6.18)

Consider formula (6.11) for \( q_n \). By our assumption on the asymptotic expansion of \( f(t) \), we already know
\[ p_n \in \mathcal{P}_0(M_n, -\mu_n, G_{\alpha, \sigma}, G_{\alpha, \sigma}). \] (6.19)

Suppose \( \chi_n = R q_\lambda \) as in (6.12). We infer \( \lambda < n \), and by applying (6.17) for \( j = \lambda \), we have
\[ R q_\lambda \in \mathcal{P}_0(M_\lambda, -\mu_\lambda - 1, G_{\alpha, \sigma}, G_{\alpha, \sigma}). \] (6.20)

Because \( M_\lambda \leq M_n \) and \( \lambda + 1 = \mu_n \), it follows (6.20), the embedding in property (c) after Definition 3.3 that \( \chi_n = R q_\lambda \) satisfies
\[ \chi_n \in \mathcal{P}_0(M_n, -\mu_n, G_{\alpha, \sigma}, G_{\alpha, \sigma}). \] (6.21)

In the case \( \chi_n = 0 \) in (6.12) then, surely, (6.21) holds true.
Consider \( B_C(q_m, q_j) \) with \( \mu_m + \mu_j = \mu_n \). By (6.18), we can write
\[ q_m = \sum_{\alpha \in S_m} z^\alpha \xi_\alpha \text{ and } q_j = \sum_{\beta \in S_j} z^\beta \eta_\beta, \]
where \( S_m \) and \( S_j \) are finite subsets of \( \mathcal{E}_C(0, M_n, -\mu_m) \) and, respectively, \( \mathcal{E}_C(0, M_n, -\mu_j) \) that preserve the conjugation, all \( \xi_\alpha \)'s and \( \eta_\beta \)'s belong to \( G_{\alpha + 1, \sigma, \xi} \) with
\[ (\xi_\alpha = \xi_\alpha \forall \alpha \in S_m) \text{ and } (\eta_\beta = \eta_\beta \forall \beta \in S_j). \] (6.22)

We have
\[ B_C(q_m, q_j) = B_C \left( \sum_{\alpha \in S_m} z^\alpha \xi_\alpha, \sum_{\beta \in S_j} z^\beta \eta_\beta \right) = \sum_{(\alpha, \beta) \in S_m \times S_j} z^{\alpha + \beta} B_C(\xi_\alpha, \eta_\beta) \]
\[ = \frac{1}{2} \sum_{(\alpha, \beta) \in S_m \times S_j} \left( z^{\alpha + \beta} B_C(\xi_\alpha, \eta_\beta) + z^{\alpha + \beta} B_C(\xi_\alpha, \eta_\beta) \right). \]

Consider each summand in the last sum. Clearly, the powers \( \alpha + \beta \) and \( \alpha + \beta \) are conjugates of each other, and belong to \( \mathcal{E}_C(0, M_n, -\mu_m - \mu_j) = \mathcal{E}_C(0, M_n, -\mu_n) \).

Since \( \xi_\alpha, \eta_\beta \in G_{\alpha + 1, \sigma, \xi} \), we have \( B_C(\xi_\alpha, \eta_\beta) \in G_{\alpha + 1/2, \sigma, \xi} \). Similarly, \( B_C(\xi_\alpha, \eta_\beta) \in G_{\alpha + 1/2, \sigma, \xi} \). By properties (6.22) and (4.5),
\[ B_C(\xi_\alpha, \eta_\beta) = B_C(\xi_\alpha, \eta_\beta) = \overline{B_C(\xi_\alpha, \eta_\beta)}. \]

Hence, each \( z^{\alpha + \beta} B_C(\xi_\alpha, \eta_\beta) + z^{\alpha + \beta} B_C(\xi_\alpha, \eta_\beta) \) belongs to \( \mathcal{P}_0(M_n, -\mu_n, G_{\alpha + 1/2, \sigma, \xi}, G_{\alpha + 1/2, \sigma}) \).
So does their finite sum over \( \alpha \)'s and \( \beta \)'s. Thus,
\[ B_C(q_m, q_j) \in \mathcal{P}_0(M_n, -\mu_n, G_{\alpha + 1/2, \sigma, \xi}, G_{\alpha + 1/2, \sigma}). \] (6.23)

Summing up in \( m, j \) and combining with the facts (6.19) and (6.21), we obtain
\[ \sum_{\mu_m + \mu_j = \mu_n} B_C(q_m, q_j) + p_n - \chi_n \in \mathcal{P}_0(M_n, -\mu_n, G_{\alpha, \sigma}, G_{\alpha, \sigma}). \]
Applying $Z_{Ac}$ to this element and using Lemma 6.8(ii), we have
\[ q_n \in \mathcal{P}_0(M_n, -\mu_n, G_{\alpha+1,\sigma}, G_{\alpha+1}). \]

By the induction principle, $q_n \in \mathcal{P}_0(M_n, -\mu_n, G_{\alpha+1,\sigma}, G_{\alpha+1})$ for all $n \in \mathbb{N}$. \hfill \Box

As a consequence of the proof of Lemma 6.8, see (6.17), (6.21) and (6.23), we have
\[ B_C(q_m, q_j) \in \mathcal{P}_0(M_n, -\mu_n, G_{\alpha,\sigma}, G_{\alpha}) \text{ for any } m, j, n \in \mathbb{N} \text{ with } \mu_n = \mu_m + \mu_j. \] (6.26)

We are now ready to prove Theorem 6.7.

**Proof of Theorem 6.7.** The expansion (6.14) clearly comes from (6.13). Hence, we focus on proving (6.13). Let $m_1 = 0$ and $\psi(t) = L_{m_1}(t) = t$ for $t > 0$. For $n \in \mathbb{N}$, denote
\[ u_n(t) = q_n(\hat{\mathcal{L}}_{M_n}(t)), \quad U_n(t) = \sum_{j=1}^{n} u_j(t) \text{ and } v_n = u(t) - U_n(t), \]
\[ f_n(t) = p_n(\hat{\mathcal{L}}_{M_n}(t)), \quad F_n(t) = \sum_{j=1}^{n} f_j(t) \text{ and } g_n(t) = f(t) - F_n(t). \]

According to the expansion (6.10), we can assume that
\[ |g_n(t)|_{\alpha,\sigma} = O(\psi(t)^{-\mu_n - \varepsilon_n}), \] (6.27)
for any $n \in \mathbb{N}$, with some $\varepsilon_n > 0$.

By Lemma 6.8 and property (3.11), we have, for any $n \in \mathbb{N}$ and $\delta > 0$,
\[ |f_n(t)|_{\alpha+1,\sigma} = O(\psi(t)^{-\mu_n + \delta}), \] (6.28)
\[ |u_n(t)|_{\alpha+1,\sigma} = O(\psi(t)^{-\mu_n + \delta}), \] (6.29)
\[ |U_n(t)|_{\alpha+1,\sigma} = O(\psi(t)^{-\mu_1 + \delta}). \] (6.30)

As a preparation, we need to establish the large time decay for $u(t)$ first.

Fix a real number $T_*$ such that $T_* > E_k(0)$ and $T_* \geq E_{m_1+1}(0)$. Note that $\psi(t + T_*) \geq 1$ for $t \geq 0$. By (6.27) and (6.28),
\[ |f(t)|_{\alpha,\sigma} \leq |f_1(t)|_{\alpha,\sigma} + |g_1(t)|_{\alpha,\sigma} = O(\psi(t)^{-\mu_1 + \delta}) + O(\psi(t)^{-\mu_1 - \varepsilon_1}) = O(\psi(t + T_*)^{-\mu_1 + \delta}) \] (6.31)
for any $\delta > 0$. The last relation is due to (3.6).

Let $\delta \in (0, \mu_1)$ and define $F(t) = \psi(t + T_*)^{-\mu_1 + \delta}$ for $t \geq 0$. Then $F(t)$ is positive, continuous, decreasing on $[0, \infty)$ and satisfies (2.17). By (6.31), $f$ satisfies (2.18).

Clearly, $F$ also satisfies (2.19) and (2.20).

We now apply Theorem 2.4 with $\varepsilon = 1/2$. Then there exists time $\hat{T} > 0$ and a constant $C > 0$ such that $u(t)$ is a regular solution of (2.9) on $[\hat{T}, \infty)$, and, by estimate (2.21),
\[ |u(\hat{T} + t)|_{\alpha+1/2,\sigma} \leq C\psi(t + T_*)^{-\mu_1 + \delta} \quad \forall t \geq 0. \] (6.32)

It follows (2.16) and (5.32) that
\[ |B(u(\hat{T} + t), u(\hat{T} + t))|_{\alpha,\sigma} \leq K^2|u(\hat{T} + t)|_{\alpha+1/2,\sigma} \leq CK^2\psi(t + T_*)^{-2\mu_1 + 2\delta} \quad \forall t \geq 0. \] (6.33)

By estimates (6.33) and the relations in (3.6), we have
\[ |B(u(t), u(t))|_{\alpha,\sigma} = O(\psi(t)^{-2\mu_1 + 2\delta}). \] (6.34)
For our convenience, we rewrite the desired statement (6.13) as follows: There exists $\delta_N > 0$ such that

$$
\left| u(t) - \sum_{n=1}^{N} u_n(t) \right|_{\alpha+1-\rho,\sigma} = O(\psi(t)^{-\mu N - \delta_N}) \quad \text{for all } \rho \in (0, 1). \tag{6.35}
$$

We will prove (6.35) by induction in $N$. In calculations below, all differential equations hold in $V'$-valued distribution sense on $(T, \infty)$ for any $T > 0$, which is similar to (2.11). One can easily verify them by using (2.15), and the facts $u \in L^2_{\text{loc}}([0, \infty), V')$ and $u' \in L^1_{\text{loc}}([0, \infty), V')$ in Definition 2.2.

The first case $N = 1$. Rewrite the NSE (2.9) as

$$
u' + Au = -B(u, u) + f_1 + (f - f_1) = f_1 + H_1(t), \tag{6.36}
$$

where

$$H_1(t) = -B(u, u) + g_1(t).$$

By (6.37) and taking $\delta = \mu_1/4$ in (6.34), we obtain

$$|H_1(t)|_{\alpha,\sigma} = O(\psi(t)^{-\mu_1-\delta_1}), \quad \text{where } \delta_1 = \min\{\varepsilon_1, \mu_1/2\} > 0. \tag{6.37}
$$

Note that $g_1 = Z_{Ap1}$ and $u_1(t) = q_1(\hat{L}_{M_1}(t))$. By equation (6.36) and estimate (6.37), we can apply Theorem 5.1 to $w = u, p = p_1, k = M_1, \mu = \mu_1$ and $g = H_1$. Then there exists $\delta_1 > 0$ such that

$$|u(t) - u_1(t)|_{\alpha+1-\rho,\sigma} = O(\psi(t)^{-\mu_1-\delta_1}) \quad \text{for all } \rho \in (0, 1).$$

Thus, (6.35) is true for $N = 1$.

The induction step. Let $N \geq 1$ be an integer and assume there exists $\delta_N > 0$ such that

$$|v_N(t)|_{\alpha+1-\rho,\sigma} = O(\psi(t)^{-\mu N - \delta_N}) \quad \text{for all } \rho \in (0, 1). \tag{6.38}
$$

We will find a differential equation for $v_N$. Taking derivative of $v_N(t)$ gives

$$v'_N = u' - \sum_{k=1}^{N} u'_k = -Au - B(u, u) + f(t) - \sum_{k=1}^{N} u'_k. \tag{6.39}
$$

By writing

$$Au = \sum_{k=1}^{N} Au_k + Av_N \quad \text{and} \quad f(t) = \sum_{k=1}^{N} f_k(t) + f_{N+1}(t) + g_{N+1}(t),$$

we have

$$v'_N = -Av_N - B(u, u) + f_{N+1}(t) - \sum_{k=1}^{N} (Au_k + u'_k - f_k(t)) + g_{N+1}(t). \tag{6.40}
$$

We have

$$B(u, u) = B(U_N + v_N, U_N + v_N) = B(U_N, U_N) + h_{N+1,1}, \tag{6.41}
$$

where

$$h_{N+1,1} = B(U_N, v_N) + B(v_N, U_N) + B(v_N, v_N).$$
Write
\[ B(U_N, U_N) = \sum_{m,j=1}^{N} B(u_m, u_j). \] (6.42)

Define the set \( \mathcal{S} = \{ \mu_n : n \in \mathbb{N} \} \). Thanks to Assumption 6.6, the set \( \mathcal{S} \) preserves the addition. Hence, the sum \( \mu_m + \mu_j \) belongs to \( \mathcal{S} \), and, thus, it must be \( \mu_k \) for some \( k \geq 1 \). Therefore, we can split the sum in (6.42) into two parts: \( \mu_m + \mu_j = \mu_k \) for \( k \leq N + 1 \) and for \( k \geq N + 2 \). Hence,
\[ B(U_N, U_N) = \sum_{k=1}^{N+1} \left( \sum_{1 \leq m,j \leq N, \mu_m + \mu_j = \mu_k} B(u_m, u_j) \right) + h_{N+1,2}, \] (6.43)

where
\[ h_{N+1,2} = \sum_{1 \leq m,j \leq N, \mu_m + \mu_j \geq \mu_{N+2}} B(u_m, u_j). \] (6.44)

For \( 1 \leq k \leq N + 1 \), define
\[ J_k(t) = \sum_{\mu_m + \mu_j = \mu_k} B(u_m(t), u_j(t)). \] (6.45)

Note in (6.45) that \( m, j < k \leq N + 1 \), hence, \( m, j \leq N \). It follows (6.40) and (6.43) that
\[ B(u(t), u(t)) = \sum_{k=1}^{N+1} J_k(t) + h_{N+1,1}(t) + h_{N+1,2}(t). \] (6.46)

By formula (6.3), it holds, for \( k \in \mathbb{N} \), that
\[ u'_k = (\mathcal{M}_{-1}q_k + \mathcal{R}q_k) \circ \hat{\mathcal{L}}_{M_k} \text{ on } (E_{M_k}(0), \infty). \] (6.47)

Summing up (6.47) in \( k \) gives
\[ \sum_{k=1}^{N} u'_k = \sum_{k=1}^{N} \mathcal{M}_{-1}q_k \circ \hat{\mathcal{L}}_{M_k} + \sum_{\lambda=1}^{N} \mathcal{R}q_{\lambda} \circ \hat{\mathcal{L}}_{M_{\lambda}} \text{ on } (E_{M_n}(0), \infty). \] (6.48)

Note that we already changed the index \( k \) to \( \lambda \) in the last sum of (6.48).

Regarding the last sum in (6.48), we have from (6.21) that
\[ \mathcal{R}q_{\lambda} \in \mathcal{P}_0(M_{\lambda}, -\mu_{\lambda} - 1, G_{\alpha,\sigma}, G_{\alpha,\sigma}). \] (6.49)

Thanks to Assumption 6.6, \( \mu_{\lambda} + 1 \in \mathcal{S} \). Hence, there exists a unique number \( k \in \mathbb{N} \) such that \( \mu_k = \mu_{\lambda} + 1 \). Because \( \mu_k > \mu_{\lambda} \), we have \( \lambda \leq k - 1 \). Thus, \( \mathcal{R}q_{\lambda} = \chi_k \). Considering two possibilities \( k \leq N + 1 \) and \( k \geq N + 2 \), we rewrite, similar to (6.46),
\[ \sum_{\lambda=1}^{N} \mathcal{R}q_{\lambda} \circ \hat{\mathcal{L}}_{M_{\lambda}} = \sum_{k=1}^{N+1} \chi_k \circ \hat{\mathcal{L}}_{M_k} + h_{N+1,3}(t), \] (6.50)

where
\[ h_{N+1,3} = \sum_{1 \leq \lambda \leq N, \mu_{\lambda} + 1 \geq \mu_{N+2}} \mathcal{R}q_{\lambda} \circ \hat{\mathcal{L}}_{M_{\lambda}}. \] (6.51)
Combining (6.39), (6.46) and (6.50) yields

\[ v'_N + A v_N = f_{N+1}(t) - \sum_{k=1}^{N} X_k(t) - \chi_{N+1} \circ \hat{L}_{MN+1}(t) - J_{N+1}(t) + H_{N+1}(t), \]

where

\[ X_k(t) = (Aq_k + \mathcal{M}_{-1} q_k + \chi_k - p_k) \circ \hat{L}_{M_k}(t) + J_k(t), \]

\[ H_{N+1}(t) = g_{N+1}(t) - h_{N+1,1}(t) - h_{N+1,2}(t) - h_{N+1,3}(t). \]

For \( k \in \mathbb{N} \) and \( z \in (0, \infty)^{M_k+2} \), let

\[ Q_k(z) = \sum_{\mu m + \mu j = \mu k} B(q_m(z), q_j(z)) = \sum_{\mu m + \mu j = \mu k} B_C(q_m(z), q_j(z)). \]

The last relation comes from the fact that \( q_m(z), q_j(z) \in G_{\alpha,\sigma} \).

Obviously, \( Q_k(\hat{L}_{M_k}(t)) = J_k(t) \) for all \( k \in \mathbb{N} \). In (6.53), we write, thanks to (6.55),

\[ Aq_k + \mathcal{M}_{-1} q_k = (A_C + \mathcal{M}_{-1}) q_k. \]

Also, by identity (6.55), we can write

\[ \chi_k - p_k = (A_C + \mathcal{M}_{-1}) \mathcal{Z}_{A_C}(\chi_k - p_k), \quad J_k = ((A_C + \mathcal{M}_{-1}) \mathcal{Z}_{A_C} Q_k) \circ \hat{L}_{M_k}. \]

Therefore,

\[ X_k(t) = \left( (A_C + \mathcal{M}_{-1}) \left( q_k + \mathcal{Z}_{A_C}(\chi_k - p_k + Q_k) \right) \right) \circ \hat{L}_{M_k}(t). \]

For \( 1 \leq k \leq N \), one has from (6.11) that \( q_k = \mathcal{Z}_{A_C}(p_k - Q_k - \chi_k) \), hence, \( X_k = 0 \). It follows from this fact and equation (6.52) that

\[ v'_N(t) + A v_N(t) = (p_{N+1} - Q_{N+1} - \chi_{N+1}) \circ \hat{L}_{MN+1}(t) + H_{N+1}(t). \]

By the assumption on \( p_{N+1} \) and properties (6.25), (6.26), we have

\[ p_{N+1} - Q_{N+1} - \chi_{N+1} \in \mathcal{P}_0(M_{N+1}, -\mu_{N+1}, G_{\alpha,\sigma,C}, G_{\alpha,\sigma}) \]

We estimate \( |H_{N+1}(t)|_{\alpha,\sigma} \) now. For the first term on the right-hand side of (6.54), thanks to (6.27),

\[ |g_{N+1}(t)|_{\alpha,\sigma} = \mathcal{O}(\psi(t)^{-\mu_{N+1} - \varepsilon_{N+1}}). \]

To estimate \( h_{N+1,1}(t) \), given by (6.41), we use property (6.30), estimate (6.38) with \( \rho = 1/2 \), and inequality (2.16) to obtain, for any \( \delta > 0 \),

\[ |h_{N+1,1}(t)|_{\alpha,\sigma} = \mathcal{O}(\psi(t)^{-\mu_1 + \delta - \mu_N - \delta N}) + \mathcal{O}(\psi(t)^{-2\mu_N - 2\delta N}). \]

Because \( 2\mu_N, \mu_1 + \mu_N \in \mathcal{S} \) and \( 2\mu_N, \mu_1 + \mu_N > \mu_N \), we have \( 2\mu_N, \mu_1 + \mu_N \geq \mu_{N+1} \). Then taking \( \delta = \delta_N/2 \) in (6.59) gives

\[ |h_{N+1,1}(t)|_{\alpha,\sigma} = \mathcal{O}(\psi(t)^{-\mu_{N+1} - \delta N/2}). \]

Considering the summand of \( h_{N+1,2}(t) \) in (6.44), applying inequalities (2.16) and (6.29), and taking into account the fact \( \mu_m + \mu_j = \mu_k \geq \mu_{N+2} \), we have

\[ |B(u_m(t), u_j(t))|_{\alpha,\sigma} = \mathcal{O}(\psi(t)^{-\mu_m - \mu_j + \delta}) = \mathcal{O}(\psi(t)^{-\mu_{N+2} + \delta}) \quad \forall \delta > 0. \]
By taking $\delta = \delta_{N+1}'$ with $\delta_{N+1}' = (\mu_{N+2} - \mu_{N+1})/2$, and summing up in $m$ and $j$, we obtain
\[ |h_{N+1,2}(t)|_{\alpha,\sigma} = O(\psi(t)^{-\mu_{N+1}-\delta_{N+1}}), \tag{6.61} \]

Concerning $h_{N+1,3}(t)$, we similarly have, thanks to property (6.49) and the fact $\mu_k + 1 = \mu_{n+1} \geq \mu_{n+2}$ for each summand of $h_{N+1,3}$ in (6.51),
\[ |h_{N+1,3}(t)|_{\alpha,\sigma} = O(\psi(t)^{-\mu_{N+1}-\delta_{N+1}}). \tag{6.62} \]

Therefore, combining (6.54), (6.58), (6.60), (6.61) and (6.62) gives
\[ |H_{N+1}(t)|_{\alpha,\sigma} = O(\psi(t)^{-\mu_{N+1}-\delta_{N+1}}), \tag{6.63} \]
where $\delta_{N+1}' = \min\{\varepsilon_{N+1} + \delta_2/2, \delta_{N+1}'\} > 0$.

From equation (6.56), property (6.57) and estimate (6.63), we can apply Theorem 5.1 to $w = v_N, p = p_{N+1} - Q_{N+1} - \chi_{N+1}$, $k = M_{N+1}, \mu = \mu_{N+1}$ and $g = H_{N+1}$. Then there is a number $\delta_{N+1} > 0$ such that
\[ |v_N(t) - (Z_{\mathcal{A}_c}(p_{N+1} - Q_{N+1} - \chi_{N+1})) \circ \hat{L}_{M_{N+1}}(t)|_{\alpha_1-\rho,\sigma} = O(\psi(t)^{-\mu_{N+1}-\delta_{N+1}}) \]
for all $\rho \in (0,1)$. Note that
\[ (Z_{\mathcal{A}_c}(p_{N+1} - Q_{N+1} - \chi_{N+1})) \circ \hat{L}_{M_{N+1}} = q_{N+1} \circ \hat{L}_{M_{N+1}} = u_{N+1}. \]

Therefore,
\[ |v_N(t) - u_{N+1}(t)|_{\alpha_1-\rho,\sigma} = O(\psi(t)^{-\mu_{N+1}-\delta_{N+1}}) \]
for all $\rho \in (0,1)$.

Because $v_N - u_{N+1} = u - \sum_{n=1}^{N+1} u_n$, the preceding estimate implies that (6.35) is true for $N := N + 1$.

**Conclusion.** By the induction principle, we have (6.35) holds true for all $N \in \mathbb{N}$. The proof is complete. \qed

In Theorem 6.7 both the force $f(t)$ and the solution $u(t)$ have infinite series expansions. The case of finite sum approximations can be treated similarly, see [7, Theorem 4.1] and [6, Theorem 5.6] for details.

**Remark 6.9.** Consider the case when the $p_n$’s in Assumption 6.6 belong to $\mathcal{P}_0(M_n, -\mu_n, G_{\alpha,\sigma})$ corresponding to $\mathcal{K} = \mathbb{R}$ for all $n \in \mathbb{N}$. Then there is no need for the complexification and the proof is much simpler. All $q_n$’s belong to $\mathcal{P}_0(M_n, -\mu_n, G_{\alpha+1,\sigma})$, the bilinear form $B_{\mathcal{A}_c}$ in (6.11) is simply $B$, and, thanks to the second remark after Definition 1.15 the operator $Z_{\mathcal{A}_c}$ in (6.11) is simply $A^{-1}$. (See [3] for results of this type for general nonlinear ODE systems.)

6.2. **Case 2: the force has coherent logarithmic or iterated logarithmic decay.** We deal with the force $f(t)$ having different types of coherent decay as $t \to \infty$. The assumption and result are similar to those in subsection 6.1.

**Assumption 6.10.** Suppose there exist real numbers $\sigma \geq 0$, $\alpha \geq 1/2, m_\ast \in \mathbb{N}$, a strictly increasing, divergent sequence of positive numbers $(\mu_n)_{n=1}^{\infty}$ that preserve the addition, an increasing sequence $(M_n)_{n=1}^{\infty}$ in $\mathbb{N} \cap [m_\ast, \infty)$, and functions $p_n \in \mathcal{P}_{m_\ast}(M_n, -\mu_n, G_{\alpha,\sigma,\mathcal{C}}; G_{\alpha,\sigma})$ for all $n \in \mathbb{N}$ such that $f(t)$ admits the asymptotic expansion (6.10) in the sense of Definition 3.4.
Theorem 6.11. Under Assumption 6.10, let $q_n$, for $n \in \mathbb{N}$, be defined recursively by

$$q_n = \mathcal{Z}_{Ac} \left( p_n - \sum_{1 \leq k, m \leq n-1, \mu_k + \mu_m = \mu_n} B_C(q_k, q_m) \right). \quad (6.64)$$

Then any Leray–Hopf weak solution $u(t)$ of (2.9) has the asymptotic expansion (6.14).

Proof. The proof is the same as that of Theorem 6.7 with the following adjustments.

Similar to Lemma 6.8 by replacing $\mathcal{P}_0$ with $\mathcal{P}_{m_0}$, replacing $\mathcal{E}_C(0, \cdot, \cdot)$ with $\mathcal{E}_C(m_+, \cdot, \cdot)$, and neglecting all the terms $\chi_n$’s in its proof, one obtains

$$q_n \in \mathcal{P}_{m_+}(M_n, -\mu_n, G_{\alpha+1, \sigma}, G_{\alpha+1, \sigma}) \text{ for any } n \in \mathbb{N}.$$  

Set $\psi(t) = L_{m_+}(t)$. In (6.48), by using (6.44) instead of (6.3), the sum $\sum_{k=1}^N R_q \circ \hat{L}_{M_\lambda}$ satisfies

$$\left| \sum_{k=1}^N (R_q \circ \hat{L}_{M_\lambda})(t) \right|_{\alpha, \sigma} = O(t^{-\gamma}), \text{ for any } \gamma \in (0, 1),$$

which is of $O(\psi(t)^{-\mu_{N+1} - \delta})$ for any $\delta > 0$. Then we can neglect (6.50), and take $\chi_k = 0$ for $1 \leq k \leq N + 1$ in all calculations thereafter. The proof results in the asymptotic expansion (6.14) with formula (6.11) of $q_n$ containing $\chi_n = 0$, which yields (6.64). \hfill \Box

Remark 6.12. We have the same observation as Remark 6.9. Namely, if $p_n$’s in Assumption 6.10 belong to $\mathcal{P}_{m_+}(M_n, -\mu_n, G_{\alpha, \sigma})$ corresponding to $K = \mathbb{R}$ for all $n \in \mathbb{N}$, then all $q_n$’s belong to $\mathcal{P}_{m_+}(M_n, -\mu_n, G_{\alpha+1, \sigma})$ for all $n \in \mathbb{N}$. Also, $B_C$ is replaced with $B$ and $\mathcal{Z}_{Ac}$ is replaced with $\mathcal{Z}_A$ in (6.64).

7. Alternative statements

In Theorems 6.7 and 6.11 above, the functions in the asymptotic expansions are still expressed with the use of complex linear spaces $G_{\alpha, \sigma, \mathbb{C}}$. Below, we will remove such expressions and write the results in terms of functions only having values in real linear spaces $G_{\alpha, \sigma}$. The presentation of this section is parallel to [26, Definition 10.11–Theorem 10.12].

Definition 7.1. Let $X$ be a real linear space. Given integers $k \geq m \geq 0$. Define the class $\mathcal{P}_m^1(k, X)$ to be the collection of functions which are the finite sums of the following functions

$$z = (z_1, z_2, \ldots, z_k) \in (0, \infty)^{k+2} \mapsto z\alpha \prod_{j=0}^k \sigma_j(\omega_j z_j)\xi,$$  \quad (7.1)

where $\xi \in X$, $\alpha \in \mathcal{E}_\mathbb{R}(m, k, 0)$, $\omega_j$’s are real numbers, and, for each $j$, either $\sigma_j = \cos$ or $\sigma_j = \sin$.

Define the class $\mathcal{P}_m^0(k, X)$ to be the subset of $\mathcal{P}_m^1(k, X)$ when all $\omega_j$’s in (7.1) are zero.

Note that vector $\alpha = (\alpha_{-1}, \alpha_0, \ldots, \alpha_k)$ in (7.1) satisfies

$$\alpha_{-1} = \alpha_0 = \ldots = \alpha_m = 0, \quad \alpha_{m+1}, \ldots, \alpha_k \in \mathbb{R}.$$  

Let $m \in \mathbb{Z}_+$, $k \geq m$, $-1 \leq j \leq k$ and $\omega \in \mathbb{R}$. For $\xi = x + iy \in X_\mathbb{C}$ with $x, y \in X$, one has

$$L_j(t)^i \omega \xi + L_j(t)^{-i} \omega \xi = 2(\cos(\omega L_{j+1}(t))x - \sin(\omega L_{j+1}(t))y).$$

Consequently, one can prove, by induction, the following.
If \( p \in \mathcal{P}_m(k, 0, X_C, X) \) then
\[
 p \circ \hat{L}_k = q \circ \hat{L}_{k+1} \text{ for some } q \in \mathcal{P}_m^1(k+1, X). \tag{7.2}
\]
Moreover,
\[
 q \in \mathcal{P}_m^1(k, X) \text{ provided } p(z) = \sum_{\alpha \in S} z^\alpha \xi_\alpha \text{ as in Definition 6.3 }
\]
\[
 \text{where } \alpha = (\alpha_1, \alpha_0, \ldots, \alpha_k) \text{ with } \text{Im}(\alpha_k) = 0. \tag{7.3}
\]

The reasons for (7.3) are twofold: there is no term \( L_{k+1}(t)^\beta \), for a real number \( \beta \neq 0 \), in \( p \circ \hat{L}_k(t) \), and there is no term \( L_k(t)^i\omega \) in \( p \circ \hat{L}_k(t) \) to contribute to \( \cos(\omega L_{k+1}(t)) \) and \( \sin(\omega L_{k+1}(t)) \) in \( q \circ \hat{L}_{k+1}(t) \).

We now observe that
\[
 \cos(\omega L_j(t)) = g(\hat{L}_k(t)) \text{ and } \sin(\omega L_j(t)) = h(\hat{L}_k(t)) \tag{7.4}
\]
where \( g \) and \( h \) are two functions in \( \mathcal{P}_m(k, 0, \mathbb{C}, \mathbb{R}) \) which are given explicitly by
\[
 g(z) = \frac{1}{2}(z_j^{i\omega} + z_j^{-i\omega}) \text{ and } h(z) = \frac{1}{2i}(z_j^{i\omega} - z_j^{-i\omega}). \tag{7.5}
\]
Using properties (7.4) and (7.3), one can verify the following facts.
If \( p \in \mathcal{P}_m^1(k, X) \) then
\[
 p \circ \hat{L}_k = q \circ \hat{L}_k \text{ for some } q \in \mathcal{P}_m(k, 0, X_C, X). \tag{7.6}
\]
More specifically,
\[
 q(z) = \sum_{\alpha \in S} z^\alpha \xi_\alpha \text{ as in Definition 6.3 } \text{ where } \alpha = (\alpha_1, \alpha_0, \ldots, \alpha_k) \text{ with } \text{Im}(\alpha_k) = 0. \tag{7.7}
\]

The last condition is due to the fact that the functions \( \cos(\omega L_k(t)) \) and \( \sin(\omega L_k(t)) \) can be converted via (7.4) and (7.5), when \( j = k \), using the functions of the variable \( z_k \).

**Definition 7.2.** Let \((X, \| \cdot \|_X)\) be a normed space over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). Suppose \( g \) is a function from \((T, \infty)\) to \( X \) for some \( T \in \mathbb{R} \). Given \( m_* \in \mathbb{Z}_+ \), let \((\gamma_k)_{k=1}^\infty \) and \((n_k)_{k=1}^\infty \) be the same as in Definition 5.7. We say
\[
g(t) \sim \sum_{k=1}^\infty \hat{p}_k(\hat{L}_{nk}(t))L_{m_*}(t)^{-\gamma_k} \text{ in } X, \text{ where } \hat{p}_k \in \mathcal{P}_{m_*}^1(n, X) \text{ for } k \in \mathbb{N}, \tag{7.8}
\]
if, for any \( N \in \mathbb{N} \), there is a number \( \mu > \gamma_N \) such that
\[
 \left\| g(t) - \sum_{k=1}^N \hat{p}_k(\hat{L}_{nk}(t))L_{m_*}(t)^{-\gamma_k} \right\|_X = O(L_{m_*}(t)^{-\mu}).
\]

We restate Theorems 6.7 and 6.11 using the class \( \mathcal{P}_m^1(k, X) \) and the asymptotic expansions of type (7.8).

**Theorem 7.3.** Let \( m_* \in \mathbb{Z}_+ \) and let sequences \((\mu_m)_{m=1}^\infty \), \((M_m)_{m=1}^\infty \) be the same as in Assumption 6.6 if \( m_* = 0 \), and be the same as in Assumption 6.10 if \( m_* \geq 1 \). Assume there are
numbers \( \alpha \geq 1/2 \) and \( \sigma \geq 0 \) such that \( f(t) \) admits the asymptotic expansion, in the sense of Definition 7.2

\[
f(t) \sim \sum_{n=1}^{\infty} \hat{p}_n(\mathcal{L}_{M_n}(t))L_{m_*}(t)^{-\mu_n} \text{ in } G_{\alpha,\sigma}, \text{ where } \hat{p}_n \in \mathcal{P}_{m_*}^{1}(M_n, G_{\alpha,\sigma}) \text{ for } n \in \mathbb{N}. \tag{7.9}
\]

Then there exist \( \hat{q}_n \in \mathcal{P}_{m_*}^{1}(M_n, G_{\alpha+1,\sigma}) \), for \( n \in \mathbb{N} \), such that any Leray–Hopf weak solution \( u(t) \) of (2.9) admits the asymptotic expansion

\[
u(t) \sim \sum_{n=1}^{\infty} \hat{q}_n(\mathcal{L}_{M_n}(t))L_{m_*}(t)^{-\mu_n} \text{ in } G_{\alpha+1-\rho,\sigma} \text{ for any } \rho \in (0, 1). \tag{7.10}
\]

**Proof.** Let \( u(t) \) be any Leray–Hopf weak solution of (2.9). For each \( n \in \mathbb{N} \), thanks to (7.6) we have \( \hat{p}_n(\mathcal{L}_{M_n}(t)) = \tilde{p}_n(\mathcal{L}_{M_n}(t)) \) for some \( \tilde{p}_n \in \mathcal{P}_{m_*}(M_n, 0, G_{\alpha,\sigma}, G_{\alpha,\sigma}) \). We rewrite (7.9) as

\[
f(t) \sim \sum_{n=1}^{\infty} \hat{p}_n(\mathcal{L}_{M_n}(t))L_{m_*}(t)^{-\mu_n}.
\]

Applying Theorem 6.7 when \( m_* = 0 \) and Theorem 6.11 when \( m_* \geq 1 \), we obtain the asymptotic expansion

\[
u(t) \sim \sum_{k=1}^{\infty} \tilde{q}_n(\mathcal{L}_{M_n}(t))L_{m_*}(t)^{-\mu_n} \text{ in } G_{\alpha+1-\rho,\sigma} \text{ for any } \rho \in (0, 1), \tag{7.11}
\]

where \( \tilde{q}_n \in \mathcal{P}_{m_*}(M_n, 0, G_{\alpha+1,\sigma}, G_{\alpha+1,\sigma}) \) for all \( n \in \mathbb{N} \).

Thanks to property (7.2), we have

\[
\tilde{q}_n(\mathcal{L}_{M_n}(t)) = \tilde{q}_n(\mathcal{L}_{M_n}(1+t)) \text{ for some } \tilde{q}_n \in \mathcal{P}_{m_*}^{1}(M_n + 1, G_{\alpha+1,\sigma}).
\]

We examine \( \tilde{q}_n \) more closely. In fact, \( \tilde{p}_n \) has the representation as in (7.7) with \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{M_n}) \) satisfying \( \text{Im}(\alpha_{M_n}) = 0 \). By the recursive formula (6.11) for \( m_* = 0 \), or (6.63) for \( m_* \geq 1 \), each \( \tilde{q}_n \) has the same property. By the virtue of (7.3), we have \( \tilde{q}_n \in \mathcal{P}_{m_*}^{1}(M_n, G_{\alpha+1,\sigma}) \), and hence, obtain (7.10) from (7.11). \( \square \)

In particular, if \( \hat{p}_n \in \mathcal{P}_{m_*}^{0}(M_n, G_{\alpha,\sigma}) \) for all \( n \in \mathbb{N} \), then \( \tilde{q}_n \in \mathcal{P}_{m_*}^{0}(M_n, G_{\alpha+1,\sigma}) \) for all \( n \in \mathbb{N} \). Indeed, this statement follows Remarks 0.9 and 0.12 with \( p_n(z) = \hat{p}_n(z)z_{m_*}^{-\mu_n} \) and \( \tilde{q}_n(z) = q_n(z)z_{m_*}^{-\mu_n} \) for \( z = (z_1, z_0, \ldots, z_{M_n}) \).

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