Radiation spectrum from relativistic slim accretion discs: an effect of photon trapping

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ABSTRACT

The vertical structure and the emission spectrum of slim accretion discs around massive black holes are investigated for super-critical accretion states. A key process is the effect of photon trapping, which is included in the radiative transfer equation. It is found that the local radiation spectrum emitted from the inner region has a typical Comptonized bremsstrahlung form, and deviates significantly from the blackbody and the modified blackbody spectrum. The spectral hardening factor of the local spectrum has a large radial dependence: it takes a value of $\sim 10^4$ (inner region) and $\sim 3$ (outer region). Owing to the effect of photon trapping, the emergent luminosity becomes small compared with the gravitational energy released through viscosity. The effect is remarkable for a large accretion rate and a spin parameter close to unity. In particular, for an extreme Kerr case with an accretion rate of $100L_{\text{Edd}}/c^2$, where $L_{\text{Edd}}$ is the Eddington luminosity, the emergent luminosity is approximately 2 per cent of the released energy. A comparison with the standard model is also discussed for an accretion rate under the Eddington limit.

Key words: accretion, accretion discs – radiative transfer.

1 INTRODUCTION

Recently, great attention has been paid to advection-dominated accretion flow (ADAF) in investigating various characteristics of accretion-powered astronomical sources such as active galactic nuclei. Historically, the concept of ADAF already appeared in the model to explain the spectral transition of galactic black hole candidates (Ichimaru 1977). However, the importance of the ADAF model has only recently been recognized owing to extensive theoretical efforts (Narayan & Yi 1994, 1995; Abramowicz et al. 1995). One of the most important processes in ADAF is energy transport by advective flows of the accreting gas and the trapped photons. This process gives rise to a disc structure that is distinct from the Shakura–Sunyaev-type (Shakura & Sunyaev 1973) standard disc model. To apply the theoretical models to observations and extract the physical information on the sources, a detailed examination of a radiation spectrum from ADAF is indispensable. The radiation spectrum from optically thin ADAF, including a detailed radiation mechanism, has been investigated and applied to the observations (Manmoto, Mineshige & Kusunose 1997; Manmoto 2000). Optimally thick ADAF was first introduced by Abramowicz et al. (1988) and dubbed ‘slim disc’. Many authors tried to investigate the spectrum from slim discs. In those studies, the emission spectrum was obtained using approximations in calculating the local emission spectrum radiated from the surface of slim discs, such as a blackbody approximation (Mineshige et al. 2000), a modified blackbody approximation (Suszczewicz, Malkan & Abramowicz 1996), a disc atmosphere approach introducing the vertical structure (Wang et al. 1999; Zampieri, Turolla & Szuszkiewicz 2001) and so on. However, the local emission spectra from slim discs are subject to the effect of ‘photon trapping’ within the flow, because the time-scale for photon diffusion from the interior of the disc to its surface exceeds the inflow time-scale near the central black holes, and as a result, it is possible that a substantial amount of the radiation generated in the flow is trapped in accreting gas, and the radiative flux emitted from the disc surface becomes fairly small compared with the dissipative flux generated through viscous heating. Thus in this sense, the emission spectrum from the slim disc has not been fully investigated. Beloborodov (1998, hereafter B98) suggested that local radiation in slim discs is concentrated in the Wien peak of electron temperature owing to a strong deviation from thermodynamic equilibrium between the electron and radiation when an accretion rate approaches a critical accretion rate defined as the rate corresponding to the Eddington luminosity in the standard model. Although this is true within the standard model (Czerny & Elvis 1987; Ross, Fabian & Mineshige 1992; Shimura & Takahara 1993, ST93 hereafter), a radiative transfer equation for vertical direction must be solved to clarify this problem for slim discs. Since a substantial amount of radiation is expected to be emitted from the inner region, the
detailed investigation of the emission spectrum from there is of
great importance.

Thus we are motivated to study the radiation spectrum from the
relativistic slim discs in this paper. To perform this investigation,
first, we solve the full set of relativistic disc equations and obtain
the global structure. Next, using the global structure, we solve the
vertical structure and radiative transfer at each radius and obtain
the local spectrum. In the present study, we treat slim discs around
massive black holes focusing on the active galactic nuclei. We pay
particular attention to the inner region of discs with super-critical
(sup-Eddington) accretion states because the situation in which
the advective effect becomes important is in question.

Throughout the present study, we treat a black hole of mass
$M = 10^8 M_{\odot}$, which is a typical value for active galactic nuclei, and
use the following normalization for the radial coordinate $r$ and the
accretion rate $M : r_e = r / R_S$ with $R_S = 2GM / c^2$. $M_e \equiv M / M_{\ast}$
with $M_{\ast} = M_{\ast} / c^2$ and $L_{\ast} = 4\pi G M_{\ast} / c \gamma_1$ being the Edding-
ton luminosity, where $\gamma_1$ and $\rho_p$ are the Thomson cross-section and
the mass of a proton, respectively.

2 BASIC EQUATIONS

The purpose of this study is to investigate the vertical structure
and the local radiation spectrum at each radius of the slim disc.
The method for obtaining the local emission spectrum at the given
radius is composed of two steps. In the first step, the global structure
(the radial distribution of the vertically integrated or the vertically
averaged structure) of the slim disc is calculated by solving the
full set of relativistic disc equations under the condition of steady
axisymmetric accretion flow to obtain the approximate radial profile
of the physical quantities required in the calculation of the vertical
structure. In the second step, using the quantities obtained from the
calculation of the first step, the vertical structure and the emission
spectrum are solved at the given radius. A cylindrical coordinate system ($r, \phi, z$) is used to describe steady axisymmetric ($\partial / \partial t = \partial / \partial \phi = 0$) accretion flows.

2.1 The first step: solution for a global structure

The method of calculation for the global structure is basically the
same as that of Mamamoto (2000). In the present paper, we have
reduced the formulation to that for the one-temperature and optically
thick state. The given parameters are a mass of the central black hole
($M$), the accretion rate ($\dot{M}$), the normalized spin parameter ($a_*$)
and the viscous parameter ($\alpha$; $\alpha = 0.1$ is adopted through this paper).
The Kerr metric in the equatorial plane formulated by Novikov &
Thorne (1973) is used. The basic equation comprises five height-
integrated equations.

The continuity equation is

$$ \dot{M} = -2\pi \Delta^{1/2} \Sigma \gamma_1 V_r, $$

(1)

with

$$ \Delta = r^2 - R_S r + R_S^2 \gamma_1^2 / 4 $$

$$ \gamma_1 = \left(1 - \beta_e^2\right)^{-1/2}, \quad \beta_e = V_e / c, $$

(2)

where $\Sigma$ is the mass column density, $V_r$ is the radial inflow
velocity of the gas measured in the corotating frame, which is orbiting
with the same angular velocity as the accreted gas. The momentum
equation for a radial ($r$) motion is

$$ \gamma_1^2 V_r \frac{dV_r}{dr} = - \frac{1}{\Sigma} \frac{dW}{dr} - \frac{\gamma_1^2 A G M}{r^3 \Delta} \left(\frac{\Omega - \Omega_+}{\Omega_+ \Omega_-}\right) $$

+ 2 \frac{\gamma_1^2 A G M}{r^3 \Delta} \left(\frac{\Omega - \Omega_-}{\Omega_+ \Omega_-}\right), $$

(3)

with

$$ A = r^4 + R_S^2 \gamma_1^2 r^2 / 4 + \frac{R_S^2 \gamma_1^2}{4}, \quad \gamma_6 = \left(1 - \beta_6^2\right)^{-1/2} $$

$$ \beta_6 = \frac{A}{c r^2 \Delta^{1/2}} \left(\frac{\Omega_+ - \Omega_-}{2 A}\right), $$

(4)

where $W$ is the vertically integrated pressure, $\Omega_\pm$ is the angular velocity
of the accreted gas, $\Omega_\pm$ is the angular velocity for the corotating
($+$) and counterrotating ($-$) Keplerian motions. The momentum equation
for an azimuthal ($\phi$) motion is

$$ \frac{\dot{M}}{2\pi} (J - J_{\ast}) = \alpha A \Delta^{1/2} \gamma_2 \frac{r^3}{r^5} W^{\gamma_1} $$

$$ J = \frac{\gamma_6 A \gamma_2}{r^3}, $$

(5)

where $J$ is the angular momentum, $J_{\ast}$ is the angular momentum
swallowed by the black hole. In equation (5), the angular momentum
extraction caused by the radiation is neglected. The energy equation
is

$$ Q_{\text{vis}} = \frac{M W}{4\pi r} \frac{1}{\Sigma \Gamma_1 - 1} \frac{dW}{dr} $$

$$ \times \left[ \frac{d \ln W}{dr} - \Gamma_1 \frac{d \ln \Sigma}{dr} \right.$$

$$ \left. + \left(\Gamma_1 - 1\right) \frac{d \ln h}{dr} \right] + F^- \right] $$

(6)

with

$$ Q_{\text{vis}} = - \frac{1}{4\pi r} \frac{\gamma_2 A}{\Delta^{1/2} r} M (J - J_{\ast}) \frac{d\Omega}{dr}, $$

(7)

where $Q_{\text{vis}}$ is the viscous heating rate per unit surface for a half-side
of the disc, $h$ is the vertical half thickness of the disc, $\Gamma_1$ and $\Gamma_3$
are the ratios of the generalized specific heat (Kato, Fukue & Mineshige
1998), $F^-$ is the local radiative flux from a half-side of the disc. The
hydrostatic balance in the vertical direction is constructed with a
thin-disc approximation:

$$ W \frac{\rm d}{\rm d} = \frac{GM}{r^2} h^2 g_{\text{var}}, $$

(8)

where $g_{\text{var}}$ is the relativistic correction term given in Petz & Appl
(1997). Although the thin-disc approximation may not hold for ac-
cretion rates considered in this paper, we adopted it to simplify the
problem significantly.

Then, to obtain explicit forms for $\Sigma, W$ and $F^-$, we use the
following simple vertical structure in this step:

$$ \rho = \rho_0 \left[1 - (z / h)^2\right]^3, \quad p = \rho_0 \left[1 - (z / h)^2\right]^4 $$

$$ T = T_0 \left[1 - (z / h)^2\right] \quad (z \leq h), $$

(9)

Note that this vertical structure is used in the first step only. In the
second step described in the next section, this simple structure is
not adopted. Thus, $\Sigma, W$ and $F^-$ are expressed as

$$ \Sigma = \int_{-h}^{h} \rho \ dz = 0.914 \rho_0 h, \quad W = \int_{-h}^{h} p \ dz = 0.813 \rho_0 h $$

$$ p_0 = \frac{1}{3} T_0^4 + \frac{2}{m_p} \rho_0 T_0, $$

(10)

$$ F^- = \frac{8\pi c T_0^4}{3k \rho_0 h}, \quad \kappa = 0.4 + 0.64 \times 10^{23} \rho \tilde{r}^{-7/2} \SI{\text{cm}^2 \text{ g}^{-1}}{.} $$

(11)

where $\rho, p$ and $T$ are the pressure, the mass density, and the tem-
perature, respectively, and the subscript 0 denotes the value at the
equatorial plane. It is assumed that the disc consists of fully
ionized hydrogen. The diffusion approximation is adopted for the
Photon trapping in slim discs. Although $p_0$ in equation (10) and $F^-$ are derived by assuming thermodynamic equilibrium between the electron and the radiation, this assumption is also used in the first step only. The mean density and the mean temperature in $\kappa$ are expressed as $\bar{\rho} = 0.457\rho_0$ and $\bar{T} = 2/3T_0$. The set of equations (1), (3), (5), (6) and (8) is integrated numerically from the outer boundary set at $r_* = 3 \times 10^4$, and unknown variables $V_r$, $\Omega\rho_0$, $T_0$, $h$ are obtained. $J_{in}$ is determined uniquely so that the solution should satisfy the transonic condition. Since the location of the transonic point is not known until we obtain the global solution, we have to adjust $J_{in}$ recursively to obtain a smooth transonic solution. By the calculation of the first step, the radial distribution of the following four physical quantities needed for the calculation of the second step are obtained: (i) $\beta_r$, (ii) $Q_{vis}$, (iii) $g_{pa}$ and (iv) $\tau_0 = 0.4\Sigma_0/2$, where $\tau_0$ is the Thomson thickness from the disc mid-plane to the disc surface.

Fig. 1 shows $-\beta_r$ ($\beta_r \leq 0$) for $(M_*, a_*) = (100, 0.998)$, $(10, 0.998)$ and $(100, 0)$. The position of the marginally stable orbit ($r_{ms} = 0.618$ and $3$ for $a_* = 0.998$ and $0$, respectively) is denoted by an open circle. Fig. 2 shows $Q_{vis} = (\sigma T R S/m c^3)Q_{vis}$. For comparison, we also plot with dotted lines the result from the relativistic standard model (Novikov & Thorne 1973; Page & Thorne 1974):

Figure 1. $-\beta_r$ ($\beta_r \leq 0$) for $(M_*, a_*) = (100, 0.998)$, $(10, 0.998)$ and $(100, 0)$. The position of the marginally stable orbit ($r_{ms} = 0.618$ and $3$ for $a_* = 0.998$ and $0$, respectively) is denoted by an open circle.

Figure 2. The viscous heating rate per unit surface: $Q_{vis} = (\sigma T R S/m c^3)Q_{vis}$. Solid lines show the calculated results while dotted lines show those expected from the relativistic standard model.
\[ Q_{\text{vis}}^{\text{std}} = \frac{3GM M}{8\pi r^2} \frac{C_p}{B C^{1/3}} \]  
\[ B = 1 + a_s \sqrt{1/8r^2} \]  
\[ C = 1 - 1.5/r_* + a_s \sqrt{1/2r_*^2} \]  
\[ \eta = \frac{L_{\text{vis}}}{Mc^2}, \quad L_{\text{vis}} = 2 \int 2\pi r dr Q_{\text{vis}} \]  
\[ \eta_{\text{vis}} = L_{\text{vis}}/Mc^2, \quad L_{\text{vis}} = 2 \int 2\pi r dr Q_{\text{vis}} \]

where \( C_p \) is a function described in Page & Thorne (1974). For \((M_*, a_*) = (100, 0.998)\), \( Q_{\text{vis}}^{\text{std}} \) is approximately 5.6, 2.4 and 1.4 times as large as \( Q_{\text{vis}}^{\text{std}} \) at \( r_* = 1.06, 2.35 \) and 10.5, respectively. For \((M_*, a_*) = (100, 0)\), \( Q_{\text{vis}}^{\text{std}} \) is approximately 1.92 and 1.07 times as large as \( Q_{\text{vis}}^{\text{std}} \) at \( r_* = 4.81 \) and 10.7, respectively.

In the standard model, the energy conversion efficiency of the released gravitational energy \( \eta_{\text{vis}} \)

\[ \eta_{\text{vis}} = \frac{L_{\text{vis}}}{Mc^2}, \quad L_{\text{vis}} = 2 \int 2\pi r dr Q_{\text{vis}} \]

is independent on \( M_* \), where \( L_{\text{vis}} \) is the energy dissipated over a whole disc. It takes a value of \( \eta_{\text{vis}} = 0.32 \) and 0.057 for \( a_* = 0, 0.998 \), respectively. Contrary to the standard disc model, \( \eta_{\text{vis}} \) is not constant over the slim disc. In the case of \( a_* = 0.998, \eta_{\text{vis}} = 1.48 \) and 1.83 for \( M_* = 100 \) and 10, respectively, while \( \eta_{\text{vis}} = 8.46 \times 10^{-2} \) and \( 6.28 \times 10^{-2} \) for \( M_* = 100 \) and 10 in the case of \( a_* = 0 \). The real energy conversion efficiency to radiation, \( \eta_{\text{rad}} = L_{\text{rad}}/Mc^2 \), where \( L_{\text{rad}} \) is the radiative luminosity, is expected to be considerably small as compared with \( \eta_{\text{vis}} \), owing to the effect of photon trapping, which will be discussed later.

Fig. 3 shows the Thomson thickness \( \tau_0 \). It is noted that the behaviour of \( \tau_0 \) in this figure differs from that given by B98. This difference may be caused by the following two reasons. (i) The difference in the way of treating the vertical structure. (ii) The difference of definition of the viscous parameter \( \alpha \). B98 made use of vertically integrated quantities in all the basic equations, while we include the simple vertical structure (equation 9) in obtaining a global structure to treat the value of the disc mid-plane. For the viscous parameter, B98 assumed that the kinematic viscosity \( \nu_{\text{vis}} \) is related to \( \alpha \) as \( \nu_{\text{vis}} = \alpha c_s/h \), where \( c_s \) is the isothermal speed of sound, and treated the derivative of the difference rotation \( \Omega \) without approximation in his angular momentum equation. On the other hand, in deriving the angular momentum equation (equation 5), we used the approximation \( \Omega \sim \Omega_0 \) in addition to \( \nu_{\text{vis}} = \alpha c_s/h \), and neglected the influence of angular velocity, which leads \( \nu_{\text{vis}} \Omega = \alpha c_s/h \Omega_0 \sim -\alpha W/r \).

2.2 The second step: the solution for the vertical structure

The method for solving the vertical structure is based on ST93. In the present work, we modify some of the basic equations to include relativistic effects. Here we repeat the description to show the modification and make the paper more self-contained. It should be noted that the set of equations described in the first step and those described in this section are not self-consistent with each other. Here we stress that the objective of the first step is only to obtain \( \beta, Q_{\text{vis}}, \eta_{\text{vis}}, \) and \( \tau_0 \), which are required for the calculation in this section. Thus, we neglect the inconsistency between two steps.

The hydrostatic equation is expressed as

\[ -\frac{dP}{dz} + \int_{0}^{\infty} F_v \frac{d}{dz} = \frac{m_e GM z}{r^3} s_{pa} \]

\[ l_v = \left( \frac{\lambda^0_{\tau} + \sigma^0_r}{N_e} \right) N_e, \]

where \( z \) is a vertical coordinate, \( P_g \) is the gas pressure, \( F_v \) is the radiative flux in the \( z \)-direction at frequency \( v \), \( l_v \) is the mean free path of a photon, \( N_e \) is the electron number density, \( \lambda^0_{\tau} \) is the transport factor (Grebenev & Sunyaev 1987), \( \sigma^0_r \) is the free–free absorption cross-section. The Gaunt factor for the free–free process is taken from Hummer (1988).

The equation for \( F_v \) in the fluid frame is given by Buchler (1983),

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \beta_v F_v \right) - \frac{1}{5} \left( \frac{\beta_v}{r} + \frac{\partial \beta_v}{\partial r} \right) \frac{\partial}{\partial v} (v F_v) + c \frac{\partial e_v}{\partial z} = -\frac{F_v}{l_v}. \]

where \( e_v \) is the radiation energy density per unit frequency. Here we neglected \( O(\beta_v^2) \) terms. Although these terms are very important in the region near to the black hole where \( \beta_v \sim -1 \), we omitted any influence by these terms for simplicity.
We take the Thomson depth \( \tau \) measured from the disc mid-plane as an independent variable for the vertical coordinate. Thus, \( z \) and \( \tau \) are related by
\[
\frac{d\tau}{dz} = N_e \sigma_T. \tag{16}
\]
The two-temperature state is adopted in the second step. Although this is inconsistent with the assumption of the one-temperature state adopted in the first step, we neglect this inconsistency. The equation of state is given by
\[
P_\epsilon = P_p + P_e \quad \text{with} \quad P_p = N_e k T_p, \quad P_e = N_e k T_e, \tag{17}
\]
where \( P_{\epsilon(p)} \) and \( T_{\epsilon(p)} \) are the proton (electron) pressure and the proton (electron) temperature. Since the bulk of the energy released through viscosity acts to heat the protons, we have the relation
\[
Q^+ = \Gamma_C + q^{p}_{\text{adv}}, \tag{18}
\]
where \( Q^+ \), \( \Gamma_C \) and \( q^{p}_{\text{adv}} \) are the viscous heating rate per unit volume, the energy exchange rate between protons and electrons via Coulomb collisions (Guilbert & Stepney 1985), the rate of advective cooling by protons, respectively. It is assumed that \( Q^+ \) is proportional to the mass density, thus
\[
Q^+ = \frac{N_e \sigma_T}{\tau_0} Q_{\text{vis}}. \tag{19}
\]
The rate of advective cooling by the proton is expressed as
\[
q^{p}_{\text{adv}} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r c \beta_e (\varepsilon_p + P_p) \right] - c \beta_e \frac{\partial P_p}{\partial r}, \tag{20}
\]
where \( \varepsilon_p \) is the internal energy of protons. Since the calculation in the second step cannot explicitly treat the radial gradient of any parameters, we drop the term with respect to \( \varepsilon_p \) from equation (20). This approximation decreases the effect of advective cooling by gas because the value of \( \beta_e \partial \varepsilon_p / \partial r \) is generally expected to be positive in the disc. Adopting a relation for a non-relativistic ideal gas between \( \varepsilon_p \) and \( P_p \) (\( \varepsilon_p = 3/2 P_p \)), the advective cooling rate is reduced to
\[
q^{p}_{\text{adv}} = \frac{5 c}{2} \left( \frac{\beta_e}{r} + \frac{\partial \beta_e}{\partial r} \right) P_e. \tag{21}
\]
Energy transferred from protons to electrons acts to heat the electrons. This energy is partially advected inward by electrons and the rest goes into radiative cooling.
\[
\Gamma_C = q^{p}_{\text{adv}} + \int d\nu \left( -c N_e \sigma_e^e \varepsilon_e + j^{e}_{\nu} + \Lambda_e \right)
\]
\[
q^{e}_{\text{adv}} = \frac{5 c}{2} \left( \frac{\beta_e}{r} + \frac{\partial \beta_e}{\partial r} \right) P_e, \tag{22}
\]
where \( j^{e}_{\nu} \) and \( \Lambda_e \) are the free–free emissivity per unit volume, and the net rate of energy transfer by Compton scattering from electrons to photons. The latter is taken from the Kompaneets model
\[
\Lambda_e = \left( \frac{8 \pi \hbar^3}{c^5} \right) \left[ \frac{\varphi_e}{x^4} \left( T_e \frac{\partial n_e}{\partial x} + n_e + n^+_e \right) \right], \tag{23}
\]
where
\[
\varphi_e = \frac{1}{1 + 4.6 x + 1.1 x^2}, \tag{23}
\]
where \( x = h \nu/m_e c^2 \), \( T_e = k T_e/m_e c^2 \), \( n_e \) is the photon occupation number, \( \varphi_e \) is the relativistic correction factor (Cooper 1971), \( m_e \) is the mass of the electron. Photons are partially advected inwards and the rest is carried to the disc surface. The monochromatic radiation energy equation in the fluid frame is also given by Buchler (1983).
\[
M_e + \frac{\partial F_\nu}{\partial z} = -c N_e \sigma_e^e \varepsilon_e + j^{e}_{\nu} + \Lambda_e, \tag{24}
\]
where \( M_e \) is the advective term that represents the effect of photon trapping. Within the Eddington approximation, \( M_e \) is described as follows:
\[
M_e = \frac{c}{r} \frac{\partial}{\partial r} \left( r \varepsilon_e \beta_e \right) + \frac{c}{3} \left( \frac{\beta_e}{r} + \frac{\partial \beta_e}{\partial r} \right) \left[ \varepsilon_e - \frac{\partial}{\partial r} \left( r \varepsilon_e \right) \right]. \tag{25}
\]
Here we omitted all terms related to a radiative flux in the radial direction and all terms related to \( \beta_e^2 \) from an original equation of Buchler (1983). We show a schematic view concerning an energy conversion process in Fig. 4. A set of equations (14)–(18), (22) and (24) is solved by the complete linearization method (Mihalas 1978) for the variables \( z, N_e, P_p, T_p, T_e, \varepsilon_e, \) and \( F_\nu \).

The calculation for the second step treats the \( z \)-direction only, and we cannot treat the radial gradient of the radiation density \( \partial \varepsilon_e / \partial r \) in equation (25) and \( \partial F_\nu / \partial r \) in equation (15). Thus we roughly estimate the two quantities by assuming that \( \varepsilon_e \) and \( F_\nu \) at each vertical point are proportional to \( \tau_0 Q_{\text{vis}} \). The calculation for the second step treats the \( z \)-direction only, and we cannot treat the radial gradient of the radiation density \( \partial \varepsilon_e / \partial r \) in equation (25) and \( \partial F_\nu / \partial r \) in equation (15). Thus we roughly estimate the two quantities by assuming that \( \varepsilon_e \) and \( F_\nu \) at each vertical point are proportional to \( \tau_0 Q_{\text{vis}} \).
\[
\frac{\partial \varepsilon_e}{\partial r} = \frac{\varepsilon_e}{\tau_0 Q_{\text{vis}}} \frac{\partial}{\partial r} \left( \frac{\tau_0 Q_{\text{vis}}}{\tau_0 Q_{\text{vis}}} \right), \tag{26}
\]
\[
\frac{\partial F_\nu}{\partial r} = \frac{F_\nu}{\tau_0 Q_{\text{vis}}} \frac{\partial}{\partial r} \left( \frac{\tau_0 Q_{\text{vis}}}{\tau_0 Q_{\text{vis}}} \right). \tag{26}
\]
It may not be very intuitive that a series of approximations to the problem (equations 15, 21, 25 and 26) is justified. However, only full two-dimensional calculations will be able to answer the question as to how accurate these calculations are. Although the effects we drop in this paper may be important, we neglect any influence by these effects.

3 RESULTS AND DISCUSSIONS

In the relativistic standard model, a critical accretion rate (the Eddington limit) corresponds to \( M_* = 17.5 \) for \( a_*=0 \) and \( M_* = 3.11 \) for \( a_*=0.998 \). In this paper, we concentrate our discussion on super-critical accretion states for \( a_* = 0 \) and 0.998.
3.1 The case of $a_*=0.998$

First, we show numerical results for $a_*=0.998$ with $M_*=10$ and 100. Fig. 5 shows the vertical temperature profiles at $r_*=1.06, 2.35$ and 10.5 for $M_*=100$. The qualitative behaviour of temperature for $M_*=10$ is the same as that for $M_*=100$. Solid, dash-dotted and dashed lines show the electron ($T_e$), proton ($T_p$) and radiation temperatures, respectively. The radiation temperature $T_e$ is determined from $a_*,T_0^4 = e_0$, where $a_*$ and $e_0$ are the radiation constant and the frequency integrated radiation energy density, respectively. For comparison, the temperature $(T_0$) obtained in the global solution (the first step) is also plotted with dotted lines. From Fig. 5, it is clearly found that the electron temperature ($T_e$) deviates significantly from $T_p$ and $T_0$. The deviation becomes remarkable in the innermost part of the disc. As B98 suggests, this result shows the vertically averaged number density $\bar{N}_* = \tau_0 / z_{\text{max}}$ is used for $\sigma_T^0$ in equation (28). The calculated spectrum is characterized by typical Comptonized bremsstrahlung with a Wien peak, and deviates significantly from the blackbody and the modified blackbody spectrum. The peak frequency and $T_{\text{eff}}$ at each radius are $(X = \log x_T, T_{\text{eff}}) = (-0.5, 4.84 \times 10^{-5})$, $(-1.8, 3.79 \times 10^{-5})$ and $(-3.3, 1.87 \times 10^{-4})$ for $r = 1.06, 2.35, 10.5$, respectively. The temperature $T_{\text{ad}}$, $\bar{N}_*$ and $z_{\text{max}}$ are $(T_{\text{ad}}, \bar{N}_*, z_{\text{max}}) = (5.17 \times 10^{-4}, 3.12 \times 10^7, 2.18)$, $(4.37 \times 10^{-4}, 1.43 \times 10^7, 5.85)$ and $(1.38 \times 10^{-4}, 81.1, 10.1)$ for $r = 1.06, 2.35, 10.5$, respectively. The thin-disc approximation does not hold, which means that an adequate treatment is required for equations (8) and (14).

The emission spectrum has been conventionally approximated by the diluted blackbody $F_v^b$ with $F_v^b = \pi B_v(f_{\text{eff}}) / f_v^4$ with $f_{\text{eff}} = (F_{\text{surf}} / \sigma_{\text{SB}})^{1/4}$. (30) where $B_v$ is the Planck function, $f_v$ is a spectral hardening factor and $\sigma_{\text{SB}}$ is the Stefan–Boltzmann constant. The spectral hardening factor is a useful parameter because we can approximately calculate the emission spectrum from a whole disc if it is known in advance. Here we investigate the behaviour of the spectral hardening factor. The

![Figure 5](https://academic.oup.com/mnras/article-abstract/338/4/1013/1141464/figure5)

**Figure 5.** The vertical temperature profiles at $r_*=1.06, 2.35$ and 10.5 for $(M_*, a_*) = (100, 0.998)$. Solid, dash–dotted and dashed lines show electron, proton and radiation temperature, respectively. For comparison, temperature at the equatorial plane obtained in the first step is also plotted with dotted lines.
Photon trapping in slim discs

best-fitted spectral hardening factor is determined by minimizing the total deviation $I$ defined by

$$ I = \int_{X_1}^{X_2} (\log F_{\nu}^{\text{db}} - \log F_{\nu})^2 \, dX, \quad (31) $$

where $X_1$ and $X_2$ are chosen so as to satisfy $F_{\nu}(X_1) \sim 0.5 F_{\nu}(X_p)$ and $F_{\nu}(X_2) \sim 10^{-3} F_{\nu}(X_p)$, where $X_p$ is the peak frequency. The best-fitted diluted blackbody spectrum is also shown in Fig. 6 with dotted lines. For each local spectrum, $f_s$ amounts to $2.53 \times 10^3, 160, 11.4$ for $r_*=1.06, 2.35, 10.5$. The colour temperature $(f_s T_{\text{eff}})$ at each radius corresponds to an electron temperature of $\tau_T = 3.2, 3.7$ and 16 for $r_*=1.06, 2.35$ and 10.5, where $\tau_T$ is the Thomson depth measured from the disc surface. The relative difference between $F_{\nu}$ and $F_{\nu}^{\text{db}}$ is generally lower than 10 per cent, and it is 50 per cent for the worst point. The spectral hardening factor takes a much larger value at the inner region and depends strongly on $r_*$. To illustrate this dependence, we show in Fig. 7 a radial distribution of the spectral hardening factor. A solid line shows the result for $M_* = 100$ while

Figure 6. The local emission spectrum at $r_* = 1.06, 2.35$ and 10.5 for $(M_*, \alpha_*) = (100, 0.998)$. Solid, dash–dotted and dashed lines show the calculated results, modified blackbody spectrum and the blackbody spectrum at the effective temperature, respectively. The best-fitted dilute blackbody spectrum is also plotted with dotted lines.

Figure 7. The radial distribution of the spectral hardening factor for $\alpha_* = 0.998$. A solid line shows the result for $M_* = 100$ while a dashed line shows that for $M_* = 10$. 

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a dashed line shows that for $M_*=10$. It takes a wide range of the order of $10^{−10}$ for $r_s \lesssim 10$. Even for the outer radius $r_s = 95$, $f_*$ amounts to 2.8. As confirmed by previous research (Ross et al. 1992; Shimura & Takahara 1993, ST95 hereafter) non-relativistic standard discs (Shakura–Sunyaev discs) in active galactic nuclei with an accretion rate under the Eddington limit also have a moderate radial dependence on the spectral hardening factor ($f_s = 2.7−9.2$). However, the dependence is much smaller in comparison with that of slim discs.

For $M_*=100$, the solution of the second step is not found for $r_s \lesssim 0.7$. Thus $f_*$ for $r_s \lesssim 0.7$ is not plotted. This may be caused by the inconsistency between the first step and second step or simple assumptions such as equations (21) and (26).

Fig. 8 shows the radial distribution of the ratio of $F_{\text{surface}}$ to the viscous heating rate per unit surface, namely, $f_* = F_{\text{surface}}/Q_{\text{vis}}$. A solid line shows the result for $M_*=100$ while a dashed line shows that for $M_*=10$. This figure shows how the energy generated through viscosity is converted to the radiative flux. Locally emitted radiation is significantly smaller compared with $Q_{\text{vis}}$ in the inner region. For example, $f_* = 3.79 \times 10^{-3}$, $2.31 \times 10^{-2}$ and $1.41 \times 10^{-1}$ for $r_s = 1.06, 2.35, 10.6$, respectively.

For the whole disc, the emergent luminosity,

$$L_{\text{rad}} = 2 \int 2\pi dr F_{\text{surface}}$$

is $3.36$ and $2.32 L_{\text{Edd}}$ for $M_*=100$ and $10$, where the radial integration is performed for $0.7 \lesssim r_s \lesssim 100$ (50 radial zones) and $0.6 \lesssim r_s \lesssim 100$ (53 zones) for $M_*=100$ and $10$, respectively. The energy generated through viscosity in the region of $0.7 \lesssim r_s \lesssim 100$ amounts to 89 per cent of $L_{\text{vis}}$. Thus, there is approximately a 10 per cent uncertainty in $L_{\text{rad}}$ for $M_*=10$.

The ratio of emergent luminosity to total viscous dissipated energy $L_{\text{rad}}/L_{\text{vis}}$ is $2.27 \times 10^{-3}$ and $1.27 \times 10^{-1}$ for $M_*=100$ and $10$, respectively. The radiative energy conversion efficiency of released gravitational energy $\eta_{\text{rad}} = L_{\text{rad}}/M c^2$ is $3.36 \times 10^{-2}$ and $2.32 \times 10^{-1}$ for $M_*=100$ and $10$. From the above results, it is found that the effect of photon trapping becomes remarkable for a large accretion rate even if we consider the uncertainties. Since $F_{\text{surface}}$ is the quantity in the fluid frame, the effect of bending of light and the gravitational and the Doppler shift of the energy of photons must be considered in obtaining $L_{\text{rad}}$ and $\eta_{\text{rad}}$ in the frame of the observer. Here it is noted that B98 gives $L_{\text{rad}}/L_{\text{vis}} = 0.1$ and 0.5 for $M_*=100$ and 10. This is a gross evaluation compared with our results. However, it is natural that there is some difference in the emergent flux between our model and that of B98 for the same accretion rate because: (i) B98 adopted a one-zone approximation for vertical the direction, and obtained a bulk emergent flux without solving radiative transfer; (ii) the global structure is different for owing to the difference in the way of treating the viscous parameter, etc.

The amount of vertically integrated advective cooling rate caused by gas (protons and electrons) for half-side of the disc at each radius

$$Q_{\text{adv}}^i = \int (q_{\text{adv}} + q_{\text{adv}}^\circ) \, dz$$

is of the order of $10^{-2}−10^{-1} Q_{\text{vis}}$. For example, $Q_{\text{adv}}^i$ is $1.03 \times 10^{-3}$, $7.04 \times 10^{-6}$ and $9.71 \times 10^{-2}$ for $r_s = 1.06, 2.35$ and 10.5, respectively. For the whole disc,

$$L_{\text{adv}}^i = 2 \int 2\pi dr Q_{\text{adv}}^i$$

is $3.57 \times 10^{-3}$ and $7.29 \times 10^{-2} L_{\text{vis}}$ for $M_*=100$ and $10$. Since the amount of energy $(1 - L_{\text{rad}} - L_{\text{adv}}^i) L_{\text{vis}}$ is transported to the central black hole by the trapped radiation, it is confirmed that almost all of the radiation is trapped in the accretion flow. The parameters used and the results obtained at each radius are tabulated in Tables 1 and 2.

3.2 The case of $a_*=0$

Next, we show the results for a non-rotating black hole ($a_*=0$) in Figs 9–12. The results at $r_s = 1.97, 4.81$ and $10.7$ are shown.
The meaning of lines is the same as in the case for $a_*=0.998$. The parameters used and the general features are also shown in Tables 1 and 2. For $M_*=100$, $L_{\text{rad}}/L_{\text{vis}}$ and $\eta_{\text{rad}}$ are 1.97 and 2.34 respectively. Here the solution for $r_* \leq 1.97$ cannot be obtained because of the same reason as in the case for $a_*=0.998$. Since the energy dissipated in the region $1.97 \leq r_* \leq 100$ amounts to 93 per cent of $L_{\text{vis}}$, there are approximately 10 per cent uncertainties in $L_{\text{rad}}$. It is found that the effect of photon trapping is relatively weak compared with the case of $a_*=0.998$.

It is instructive to compare our model with the relativistic standard model. In obtaining the solution of the standard model, we perform the second step only with the following parameters in calculations: (i) $\beta_r=0$, (ii) $Q_{\text{vis}}=Q_{\text{rad}}$ (equation 12), (iii) $g_{\text{pa}} \rightarrow g_{\text{adv}}$: the relativistic correction term $g_{\text{pa}}$ reduces to the formula given by Riffert & Herold (1995), (iv) $\tau_0$ is given by

$$\tau_0 = \frac{8\sqrt{2}}{3} \frac{r_*^{5/2}}{M_* \alpha} \frac{g_{\text{adv}} BC^{1/2}}{D} = \sqrt{3}$$

with

$$D = 1 - \frac{1}{r_*} + a_*^2/4r_*^2.$$  

(35)

The method for obtaining $\tau_0$ is basically the same as that of ST95. However, the resulting presentation of equation (35) is somewhat different because of relativistic effects included in equation (35).

Fig. 13 shows the local emission spectrum for $(M_*, a_*) = (10, 0)$ at $r_* = 4.75$ and 10.4. The solution for $r_* \geq 4.75$ cannot be obtained. Solid lines show the results of our model, while dashed lines show those of the relativistic standard model. For comparison, we also...
plot those of the non-relativistic standard model (Shakura–Sunyaev disc) with dotted lines. The method for non-relativistic discs is quite the same as that of ST93 and ST95. The parameters used and the properties obtained are listed in Table 3. It is found that the spectrum of our model is almost the same as that of the relativistic standard model. This is inferred from the fact that: (i) $\beta_r$ is small ($\beta_r = -1.52 \times 10^{-3}$ at $r_*=4.75$), (ii) $Q_{\text{vis}}$ is nearly the same as $Q_{\text{vis}}^{\text{std}}$. Unlike the non-relativistic standard disc, the spectral hardening factor is almost constant for both our model ($f_s = 2.5–2.7$) and the relativistic standard disc ($f_s = 2.4–3.0$). Although we do not compare with the standard model at a radius within a marginally stable orbit, energy dissipated through viscosity inside the marginally stable orbit is negligibly small. Thus, we expect that there is no prominent difference between the emission spectrum from slim discs with an accretion rate under the Eddington limit and that of the relativistic standard model.
Figure 12. The radial distribution of the ratio of $F_{\text{surface}}$ to the viscous heating rate per unit surface: $f_r = F_{\text{surface}}/Q_{\text{vis}}$ for $a_*=0$. A solid line shows the result for $M_* = 100$ while a dashed line shows that for $M_* = 10$.

Figure 13. The local emission spectrum at $r_* = 4.75$ and 10.4 for $(M_*, a_*) = (10, 0)$. Solid lines show the calculated results of our model while dashed lines show those expected from the relativistic standard model. The case for non-relativistic discs is also plotted with dotted lines.

Table 3. Comparison with the standard model for $(M_*, a_*) = (10, 0)$.

| $r_*$   | $-\beta_*$ | $Q_{\text{vis}}$ | $\tau_0$ | $g_{\text{pl}(r_0)}$ | $T_{\text{eff}}$ | $f_s$  | $f_r$ | $Q_{\text{adv}}/Q_{\text{vis}}$ |
|---------|-------------|------------------|-----------|----------------------|-----------------|--------|-------|--------------------------------|
| 4.75    | $1.52 \times 10^{-3}$ | 7.33             | 7.83 $\times 10^2$ | 1.46                     | $1.88 \times 10^{-5}$ | 2.58  | 9.57 $\times 10^{-1}$ | 2.65 $\times 10^{-6}$ |
| 4.75    | std-R       | 6.30             | 5.02 $\times 10^2$ | 1.46                     | $1.83 \times 10^{-5}$ | 2.65  | –     | –                              |
| 4.75    | std-NR      | 13.2             | 1.88 $\times 10^2$ | –                       | $2.20 \times 10^{-5}$ | 8.27  | –     | –                              |
| 10.4    | $4.28 \times 10^{-4}$ | 2.16             | 1.18 $\times 10^3$ | 1.17                     | $1.33 \times 10^{-5}$ | 2.62  | 8.24 $\times 10^{-1}$ | 4.92 $\times 10^{-6}$ |
| 10.4    | std-R       | 2.00             | 4.26 $\times 10^2$ | 1.17                     | $1.37 \times 10^{-5}$ | 2.87  | –     | –                              |
| 10.4    | std-NR      | 2.83             | 2.71 $\times 10^2$ | –                       | $1.49 \times 10^{-5}$ | 4.54  | –     | –                              |

std-R denotes the relativistic standard model, while std-NR denotes the non-relativistic standard model.
3.3 Concluding remarks

We investigated the vertical structure and the local emission spectrum of slim accretion discs around massive black holes ($M = 10^8 \, M_\odot$) for super-critical accretion states. It is found that the local radiation spectrum emitted from the inner region has the form of a typical Comptonized bremsstrahlung, and deviates significantly from the blackbody of the local effective temperature. The spectral hardening factor has a large radial dependence; it takes a value of $\sim 10^4$ (inner region) and $\sim 3$ (outer region). Owing to the effect of photon trapping, the emergent luminosity becomes considerably smaller compared with the gravitational energy released through viscosity. The effect of photon trapping becomes stronger with an increase of the accretion rate and the spin parameter. For the extreme Kerr case with an accretion rate of $100 L_{\text{Edd}}/c^2$, the emergent luminosity is approximately 2 per cent of the released energy, and the rest is transported into the black hole by the radiation trapped in the accreting gas. For comparison with the standard model, it is found that there is no prominent difference between the emission spectrum from slim discs with an accretion rate under the Eddington limit and that of the relativistic standard model.

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