Maximally inhomogeneous Gödel-Farnsworth-Kerr generalizations

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Abstract. It is pointed out that physically meaningful aligned Petrov type D perfect fluid space-times with constant zero-order Riemann invariants are either the homogeneous solutions found by Gödel (isotropic case) and Farnsworth and Kerr (anisotropic case), or new inhomogeneous generalizations of these with non-constant rotation. We present the construction of the line element and the local geometric properties for the latter.

1. Introduction
The discovery by Milson and Pelavas [1, 2] of a Petrov type N pure radiation space-time family for which the theoretically maximal Karlhede bound $q = 7$ is reached, reopened the question whether the upper bounds for given Weyl-Petrov and/or Ricci-Segre type are sharp as well. In any case, a necessary condition is that the space-time is curvature homogeneous of order zero (further denoted by CH$_0$), i.e., its zero-order Cartan-Riemann invariants are all constant. It is well known that ample families of pure radiation metrics satisfy this property $^1$. Here we revise the situation for electrovac fields and their Einstein space limits, and present a theorem classifying all CH$_0$ genuine (i.e. non-Einstein space) perfect fluids, in the case where the Weyl tensor is of aligned Petrov type D. We will focus on the physically relevant models, which turn out to be exhausted by the celebrated homogeneous Gödel universe, its anisotropic generalizations found by Farnsworth and Kerr, and a new family constituted by generalizations of these known solutions, the members of which have non-constant rotation. We mention the construction of good coordinates, summarize the local properties and end with concrete and more general conclusions of this investigation.

2. Inhomogeneous CH$_0$ perfect fluid solutions of Petrov type D
In general, CH$_0$ Petrov type D space-times are characterized by the existence of a Weyl principal complex null frame $(k^a, l^a, m^a, n^a)$ relative to which

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 = \text{const} \neq 0. \quad (1)$$

$k^a$ and $l^a$ spanning the Weyl principal null directions (PND’s).

All double aligned Petrov type D, non-null Einstein-Maxwell ‘electrovac’ fields have been classified and integrated by Debever, Kamran and McLenaghan [3, 4] and independently by

$^1$ I thank Michael Bradley for reminding me of this after the talk
Garcia [5]. From their results, or from the invariant approaches in [6, 7] (see also [8] for a more recent account in the vacuum case, making use of the GHP formalism) one readily infers that the only CH sub 0 solutions in this class are given by

\[
ds^2 = \frac{dx^2}{P(x)} + P(x)dx^2 + \frac{dy^2}{Q(y)} - Q(y)dy^2,
\]

where

\[
P(x) = 1 - (\Lambda + \Phi_0)x^2,
\]

\[
Q(y) = 1 - (\Lambda - \Phi_0)y^2.
\]

They are gravito-electric (Ψ sub 2 = -\Lambda/3), homogeneous, have a complete group G sub 0 of isometries and are attributed to Levi-Civita [9], Robinson [10] and Bertotti [11], the last author giving the more general case with Λ ≠ 0. Putting the electromagnetic field parameter Φ_0 equal to zero one gets all CH sub 0 Einstein spaces (notice that Λ = 0 gives the Petrov type O Minkowski space-time), having the same isometry group.

A Petrov type D space-time represents an aligned perfect fluid if its Einstein tensor has the structure

\[
G_{a b} = (w + p)u_a u_b + p g_{a b}, \quad w + p \neq 0 \quad u^a = \frac{q k^a + l^a}{\sqrt{2q}}
\]

where q > 0, w = w' + Λ and p = p' + Λ are the (geometric or effective) energy density, resp. pressure of the fluid (in which the cosmological constant Λ has been absorbed) and u^a is the fluid 4-velocity, lying in the PND-plane. Space-times of this nature are CH sub 0 if and only if the conditions (1) hold and w and p are constant. Then, by energy-momentum conservation and w + p ≠ 0, u^a is non-accelerating and non-expanding. The only explicitly known CH sub 0 examples are given by

\[
A^2 ds^2 = -du^2 - 2(dt + e^\tau dy)^2 + \frac{e^\nu}{\cosh \nu} (\cos t dx + \sin t e^\tau dy)^2
\]

\[
+ \frac{e^{-\nu}}{\cosh \nu} (-\sin t dx + \cos t e^\tau dy)^2
\]

where A and v are constant. For v = 0 one obtains the G sub 5 shearfree homogeneous Gödel universe [12], and for v ≠ 0 the anisotropic and shearing, yet still homogeneous generalizations discovered by Farnsworth and Kerr [13]. Let us list the local geometric properties of these space-times:

(A) the vector field \( v \equiv \frac{2k-1}{\sqrt{2q}} = \frac{\partial}{\partial u} \) is covariantly constant;

(B) the vorticity and shear of u^a are given by

\[
\omega^a = \omega^a v^a, \quad \omega = a \cosh v = i\sqrt{2q} \rho = i\sqrt{2/q} \mu,
\]

\[
\sigma_{ab} = V m_{(a} m_{b)}, \quad V = a \sinh v e^{i\nu v} = -\sqrt{2q} \sigma = \sqrt{2/q} \lambda,
\]

where the relation with the NP spin coefficients ρ, μ, σ and λ has been added, and where \( e^{i\nu v} \) is a spin gauge field;

(C) the space-time represents gravito-electric ‘stiff dust’, satisfying

\[
A^2 \equiv p = w = -3\Psi_2 = \omega_a \omega^a - \sigma_{ab} \sigma^{ab} = \omega^2 - V \nabla.
\]

Notice that perfect fluids with p = w satisfy the dominant energy condition only when Λ ≤ 0 and, when w is moreover constant, may be interpreted as e.g. dust space-times (Λ = -w) or stiff fluids (Λ = 0).
The question arises whether the Gödel-Farnsworth-Kerr space-times are the only CH$_0$ aligned Petrov type D perfect fluids. In contrast to the Einstein space case mentioned above, however, homogeneity of the curvature invariants does not imply homogeneity of the space-time here. One can deduce the following classification result [14]:

**Theorem.** Any CH$_0$ aligned Petrov type D perfect fluid satisfies property (C). It is either an unphysical member $w = p = 3\Psi_2 < 0$ of Ellis’ LRS II non-rotating dust family [15] or the Stephani [16]-Barnes [17] rotating dust family, or it satisfies properties (A) and (B) as well.

Let us focus on the physically relevant class of space-times satisfying (A)-(C). When $\omega$ is constant one recovers the homogeneous Gödel-Farnsworth-Kerr solutions; when $\omega$ is non-constant, however, new inhomogeneous solutions arise as follows (see [18] for more details). At each point the variables $\omega$, $V$ and $V$ are constrained by the hyperbolic equation $\omega^2 - V V = A^2$. Thus they can be parametrized as in (4)-(5), but this does not give suitable coordinates. A better choice is the parametrization

$$\omega = A \cosh x, \quad V = A \sin y + i \sinh x.$$  

Also, the Jacobi identities allow to set the NP spin coefficients $\alpha$, $\beta$, $\gamma$ and $\epsilon$ to zero. On rectifying the vector field $u = a \partial_t$, the integration of the Cartan equations is quite straightforward. On using $x$, $y$, $t$ and $u$ as coordinates, where $v = \partial / \partial u$, one obtains

$$A^2 ds^2 = dt^2 - \left(\frac{1}{2} \partial \mathcal{F}(x,y) \partial x - \frac{1}{2} \partial \mathcal{F}(x,y) \partial y\right)^2 
+ e^{\mathcal{F}(x,y)} e^x \left(\cos \left(t - \frac{y}{2}\right) dx + \sin \left(t - \frac{y}{2}\right) dy\right)^2 
+ e^{\mathcal{F}(x,y)} e^{-x} \left(- \sin \left(t + \frac{y}{2}\right) dx + \cos \left(t + \frac{y}{2}\right) dy\right)^2,$$  

(7)

$F(x,y)$ being a solution of

$$\frac{\partial^2 F(x,y)}{\partial x^2} + \frac{\partial^2 F(x,y)}{\partial y^2} = 4 \cosh(x) e^{F(x,y)}.$$  

(8)

We emphasize that this is the general line element for inhomogeneous aligned Petrov type D perfect fluids with positive effective energy density $w$.

Finally, denote $t_k$ for the number of functionally independent components of the Riemann tensor and its first $k$ covariant derivatives, relative to the canonically fixed frame at step $k$ in the Karlhede space-time classification algorithm [19], and let $q$ be the Karlhede bound. As there is at least one Killing vector field $\partial / \partial u$ one has $t_0 \leq 3$. The CH$_0$ assumption means precisely $t_0 = 0$, and by direct calculation one finds that $t_1 = 1$ and $2 \leq t_2$, whence $2 \leq t_2 \leq t_3 \leq 3$. Further investigation then shows that $t_2 = t_3$, such that $q = 3$ for any member of the new family. The generic situation is $t_2 = 3$, in which case $\partial / \partial u$ is the only Killing vector field; only a very specific subfamily satisfies $t_2 = 2$, in which case the complete isometry group is Abelian $G_2$.

3. Conclusions

For physically relevant CH$_0$ aligned Petrov type D perfect fluids the constant energy density $w$ equals the pressure $p$ on the one hand, and the difference between the vorticity and shear
amplitudes on the other. In contrast to the homogeneous Gödel-Farnsworth-Kerr models, these amplitudes can be non-constant, resulting in the non-constancy of higher-order curvature invariants and, correspondingly, a dramatic drop of the isometry group dimension of the relevant 3D part of the metric. A special choice of (non-invariantly defined) coordinates leads to the metric (7)-(8).

Put in a much broader context, the results exemplify a generation technique, valid for any metric-based gravitation theory in any space-time dimension: investigate whether a set of (possibly higher-order) invariant relations, valid for a well-known family of space-times, singles out this family. If not, new interesting families may arise.

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