Quantum mechanical potentials exactly solvable in terms of higher hypergeometric functions. I: The third-order case

S. Trachanas
Foundation of Research and Technology Hellas (FORTH)
and Department of Physics, University of Crete, Crete, Greece

Abstract. We present a new six-parameter family of potentials whose solutions are expressed in terms of the hypergeometric functions \( _3F_2, _2F_2 \) and \( _1F_2 \). Both the scattering data and the bound states of these potentials are explicitly computed and the peculiar properties of discrete spectrum are depicted in a suitable phase diagram. Our starting point is a third-order formal eigenvalue equation of the hypergeometric type (with a suitable solution known) which is transformed to the Schrödinger equation by applying the reduction of order technique as the crucial first step. The general preconditions allowing for the reduction to Schrödinger form of an arbitrary eigenvalue equation of higher order, are discussed at the end of the article, and two universal features of the potentials arising this way are also stated and discussed. In this general scheme the Natanzon potentials are the simplest special case, those presented here the next ones, and so on for potentials arising from equations of fourth or higher order.

I. INTRODUCTION

The search for exactly solvable quantum mechanical potentials has a long history dating back to the early days of quantum mechanics. In 1971, Natanzon\(^1\) appeared to have closed the subject by explicitly constructing the full set of potentials \( V(x) \) whose solutions can be expressed in terms of the hypergeometric functions \( _2F_1 \) and \( _1F_1 \).

The subject received a new impetus by the addition to the available tools for exact solution of the Darboux transformation\(^2\): A mapping of the Schrödinger equation to itself which can be applied to any solvable potential and produce an infinite chain of new ones, usually with predetermined spectral changes with respect to the initial potential (addition or removal of bound states, etc.).

Given though that the solutions of these "derived potentials" are also expressed in terms of the hypergeometric functions \( _2F_1 \) and \( _1F_1 \) – or, to be precise, in terms of linear combinations of these functions and their derivatives\(^3\) – it is reasonable to view these potentials as simple "derivatives" of the initial ones. So, a complete description of the now-known exactly solvable potentials is, we believe, this: Exactly solvable are the Natanzon potentials and their Darboux derivatives.

In view of the above, it is clear that new solvable potentials can be discovered only if we enlarge the set of functions within which their solution is sought for. And the most natural such enlargement is the full set of hypergeometric functions \( _pF_q \). These are certainly equally exact functions as \( _2F_1 \) or \( _1F_1 \), the only difference being that they satisfy linear differential equations of order higher than second. Specifically, the function \( _pF_q \) satisfies an equation of order \( q + 1 \) where for any given \( q \) the index \( p \) takes the values \( 0 \leq p \leq q + 1 \) characterizing the various types of hypergeometric functions of the given order.

In particular, if we restrict ourselves to hypergeometric functions of the third order – e.g., the functions \( _3F_2, _2F_2, _1F_2 \) and \( _0F_2 \) – then it is clear that their “use” for constructing new exactly solvable potentials presupposes, as a necessary first step, the reduction of order of the third-order equation satisfied by these functions. But it
is also necessary that this initial third-order equation has an eigenvalue parameter \( \lambda \) suitably located so that it takes the position of energy in the final Schrödinger equation produced by the reduction. How this can be done will be examined in the following section.

II. THIRD-ORDER EIGENVALUE EQUATIONS REDUCIBLE TO THE SCHröDINGER FORM.

As is well-known, if a particular solution \( y = y_0 \) of a linear differential equation is available, the transformation \( y = y_0 Y \)–that is, the factoring out of the known solution– eliminates the \( Y \) term of the new equation and so the further substitution \( Y' = u \) lowers its order by one. In the case of a linear equation of order three

\[
a(x)y''' + b(x)y'' + c(x)y' + d(x)y = 0
\]

the application of this procedure leads to the second order equation

\[
u'' + \left( \frac{b}{a} + 3 \frac{y_0'}{y_0} \right) u' + \left( \frac{c}{a} + \frac{2}{a} \frac{y_0'}{y_0} + 3 \frac{y_0''}{y_0} \right) u = 0,
\]

where \( u = (y/y_0)' \). With the further substitution

\[
u(x) = y_0^{-3/2} \exp \left( -\frac{1}{2} \int \frac{b}{a} dx \right) U(x)
\]

equation (2) is transformed into the canonical form

\[
U'' + \left( \frac{c}{a} - \frac{1}{2} \left( \frac{b}{a} \right)' - \frac{1}{4} \left( \frac{b}{a} \right)^2 + \frac{1}{2} a \frac{y_0'}{y_0} + \frac{3}{2} \left( \frac{y_0'}{y_0} \right)' + \frac{3}{4} \left( \frac{y_0''}{y_0} \right)^2 \right) U = 0
\]

where the first derivative term is missing. From (4) it is now clear that it can be cast into the so-called Liouville form

\[
U'' + (\lambda w(x) - v(x))U = 0
\]

–which is readily reduced to the Schrödinger equation– if the following conditions are met: i) The coefficients \( a(x), b(x) \) and the special solution \( y = y_0 \) of (1), are independent of the eigenvalue parameter \( \lambda \), ii) the coefficient \( c \) depends linearly on \( \lambda \). That is,

\[
c(x, \lambda) = c_0(x) + \lambda c_1(x).
\]

If these conditions are met then (4) is indeed a Liouville equation with weight function \( w(x) = c_1(x)/a(x) \) and Liouville potential –this is a suitable name for the function \( v(x) \) in (3)– as follows

\[
v(x) = -\frac{c_0}{a} + \frac{1}{2} \left( \frac{b}{a} \right)' + \frac{1}{4} \left( \frac{b}{a} \right)^2 - \frac{1}{2} a \frac{y_0'}{y_0} - \frac{3}{2} \left( \frac{y_0'}{y_0} \right)' - \frac{3}{4} \left( \frac{y_0''}{y_0} \right)^2.
\]
As far as equation (5) is concerned it is well known that it can be reduced to the Schrödinger form

$$\psi''(z) + (\lambda - V(z))\psi(z) = 0$$

(8)

where primes stand now for derivatives with respect to $z$– with the so-called Liouville transformation

$$\psi(z) = \psi(x(z)) = w^{1/4}U(x)|_{x = x(z)}$$

(9)

where the final position variable $z$ is defined by

$$z'(x) = \sqrt{w(x)} \Rightarrow z(x) = \int \sqrt{w(x)} \, dx$$

(10)

and the potential $V(x)$ –from now on we will always assume that $x = x(z)$– is given by

$$V(x) = v(x) + \{z, x\}w(x),$$

(11)

where $\{z, x\}$ is the so-called Schwartzian derivative –or simply the Schwartzian– of the function $z(x)$ defined by

$$\{z, x\} = \frac{1}{2} \left( \frac{z''}{z'} \right)' - \frac{1}{4} \left( \frac{z''}{z'} \right)^2 = \frac{1}{4} \left( \frac{w'}{w} \right)' - \frac{1}{16} \left( \frac{w'}{w} \right)^2.$$  

(12)

It is now clear from (12) that the integration constant in (10) is eliminated and can be ignored. Concerning the requirement that the special solution $y_0$ is independent of $\lambda$ –in spite of the fact that $\lambda$ appears in the equation– this leads also to the condition that the coefficient $d$ in (1) is linearly dependent on $\lambda$ like $c$. That is,

$$d(x, \lambda) = d_0(x) + \lambda d_1(x),$$

(13)

whence equation (1) can finally be written as

$$(L + \lambda M)y = 0,$$

(14)

where the operator $L$ has the full order of the equation –that is, three – while $M$ is of first order. We claim therefore that any formal –i.e., with no boundary conditions specified– eigenvalue equation of the form (14) can be reduced to the Schrödinger equation, provided that the solution $y = y_0(x)$ of the first-order equation $My = 0$ is also a solution of $Ly = 0$, and therefore also of the complete equation (14) for all values of the parameter $\lambda$. A suitable generalization of this proposition for equations of arbitrary order will be given in section VII.

III. EXACTLY SOLVABLE EIGENVALUE EQUATIONS OF THIRD ORDER AND THE ASSOCIATED POTENTIALS

The next step is obvious. Starting with the most general third-order linear equation, solvable in terms of hypergeometric functions, we write it in the form (14) –with its parameters restricted so that the solution of $My = 0$ is also a solution of $Ly = 0$– and then transform it into Schrödinger form producing along the way
the respective solvable potential. The most general third-order equation we need is written as

\[ x^2(1 + \omega x)y''' + x(\alpha + \beta x)y'' + (\gamma + \delta x)y' + \left(\frac{\varepsilon}{x} + \zeta\right)y = 0 \]  

(15)

whose form is readily recognizable if we introduce the concept of dimension \( d = m - n \) of the typical term \( x^m y^{(n)} \) (of an arbitrary linear equation with polynomial coefficients) and say that (15) is a bidimensional equation. That is, an equation whose terms can be grouped in only two different sets, each set having a definite dimension. Bidimensional equations are important because their solution in power series leads to a two-term recursion formula—\( a_{k+\ell} = f(k)a_k \)—that allows us to determine the general series coefficient \( a_k \) in closed form as a function of \( k \).

Thus the series represents an exactly known function and the pertinent equation can be classified as exactly solvable on purely mathematical grounds. Furthermore, since the dimension \( d \) of the typical term \( x^m y^{(n)} \) is the displacement on the exponent of a typical power \( x^k \) brought about by the operator \( L = x^m \partial^n \), it follows that for a bidimensional equation with dimensions \( d_1 \) and \( d_2 \), the integer \( \ell \) in the recursion relation \( a_{k+\ell} = f(k)a_k \) which is clearly equal to the difference of the two displacements experienced by the general power of the series-solution when substituted into the equation—\( \ell = d_2 - d_1 \) (\( d_2 > d_1 \) by convention). The quantity \( \ell \) will be called the step of the equation since it indeed tells us that the series-solution proceeds in steps of size \( \ell \). Based on the preceding discussion, Eq. (15) is uniquely specified by the statement that: It is the most general bidimensional equation of third order and step unity.

Concerning our assertion that Eq. (15) is solvable in terms of hypergeometric functions, this follows directly from the existence of a two-term recursion relation and will be discussed further in Appendix A where we will also provide a simple recipe for arriving at the solution using only elementary algebraic operations. Note also that the exponents of power behaviors at zero and infinity, for any bidimensional equation, are determined by the conditions

\[ L_1 x^\mu = L_1(\mu)x^{\mu+d_1} = 0, \quad L_2 x^\nu = L_2(\nu)x^{\nu+d_2} = 0 \]  

(16)

i.e., from the roots of the characteristic polynomials \( L_1(\mu) \) and \( L_2(\nu) \) of the unidimensional operators \( L_1 \) and \( L_2 \) representing the components with the least and the largest-dimension respectively, of the bidimensional operator \( L = L_1 + L_2 \).

From the preceding discussion it follows that our starting point should be the solvable, third-order eigenvalue equation

\[ x^2(1 + \omega x)y''' + x(\alpha + \beta x)y'' + (\gamma + \delta x)y' + \left(\frac{\varepsilon}{x} + \zeta\right)\lambda y = 0 \]

(17)

\[ \lambda \left( (\gamma + \delta x)y' + \left(\frac{\varepsilon}{x} + \zeta\right) y \right) = 0 \]

which has the same form as eq. (14) with

\[ L = x^2(1 + \omega x)\partial^3 + x(\alpha + \beta x)\partial^2 + (\gamma + \delta x)\partial + \left(\frac{\varepsilon}{x} + \zeta\right), \]  

(18)

\[ M = (\gamma + \delta x)\partial + \left(\frac{\varepsilon}{x} + \zeta\right) \quad (\partial = d/dx). \]  

(19)
where \( \gamma, \delta, \varepsilon \) and \( \zeta \) are new, arbitrary parameters, independent of \( \gamma, \delta, \varepsilon \) and \( \zeta \) of Eq. (18) but with the same role as these, hence the similar notation. With \( M \) given by the expression above, the general solution of \( My = 0 \) takes the form 
\[
y = y_0 = x^\alpha (x + \rho)^s
\]
with suitably chosen \( s, s' \) and \( \rho \). But here we will restrict our discussion to the case \( s' = 1 \) since then \( y_0 \) can certainly be also a solution of \( Ly = 0 \) as it is a terminating series about \( x = 0 \) and the equations of hypergeometric type are especially these admitting such solutions.

By regarding now \( s \) and \( \rho \) as given parameters and requesting that \( y_0 = x^\alpha (x + \rho)^s \) be a simultaneous solution of the equations \( My = 0 \) and \( Ly = 0 \), we find that the operator \( M \) can only have the form
\[
M = \left(1 + \frac{x}{\rho}\right)\partial - \left(\frac{s}{x} + \frac{s + 1}{\rho}\right)
\]
while \( L \) has the form of (18), with the parameters \( \delta, \varepsilon \) and \( \zeta \) no longer independent, but expressed in terms of \( \alpha, \beta, \omega, \rho \) and \( s \) as follows:
\[
\begin{align*}
\delta &= \frac{1}{\rho}\left(3(1 - \omega \rho)s^2 + (2\alpha - 2\beta \rho + 3\omega \rho - 3)s\right) \\
\varepsilon &= -(s^3 + (\alpha - 3)s^2 + (2 - \alpha)s) \\
\zeta &= -\frac{1}{\rho}\left((3 - 2\omega \rho)s^3 + (2\alpha - \beta \rho)s^2 + (2\alpha - \beta \rho + 2\omega \rho - 3)s\right).
\end{align*}
\]
As for the parameter \( \gamma \) it can be set equal to zero since it only appears as an additive constant to the potential and can therefore be ignored.

Our assertion becomes now quite specific: We claim that the formal eigenvalue equation of third order
\[
x^2(1 + \omega x)y''' + x(\alpha + \beta x)y'' + \left(\delta x + \lambda \left(1 + \frac{x}{\rho}\right)\right)y' + \left(\frac{\varepsilon - \lambda s}{x} + \zeta - \lambda \frac{1 + s}{\rho}\right)y = 0
\]
with \( \delta, \varepsilon \) and \( \zeta \) as in (21) is reduced to a Schrödinger equation which must therefore be solvable, since Eq. (22) is solvable. More specifically, the expression
\[
w(x) = c_1(x)/a(x)
\]
implies that the weight function in the relevant Liouville equation (Eq. (19)) will be equal to
\[
w(x) = \frac{1 + (x/\rho)}{x^3(1 + \omega x)} \Rightarrow z(x) = \int \sqrt{w(x)} \, dx = \begin{cases} 
\frac{x + \rho}{1 + \omega x} & \text{for } x \to 0 \\
\frac{1 + \omega x}{\sqrt{\rho}} \ln x \to x = e^{x^2} & \text{for } x \to \infty
\end{cases}
\]
where we have assumed \( \rho \) and \( \omega \) to be positive so that the change of variables \( z = z(x) \) maps the region \( 0 < x < \infty \) —between the singular points \( x = 0 \) and \( x = \infty \) of the initial equation— into the full region \( -\infty < z < \infty \) of the final position variable. As for the complete expression of the function \( z = z(x) \), this is a simple elementary function which, nevertheless, does not seem to be invertible in terms of elementary functions even though the existence of an inverse is guaranteed by the fact that \( z'(x) > 0 \) in the region of positive \( x \). But just as in the case of Natanzon potentials, this poses no problem in solving the Schrödinger equation and calculating its spectral properties, since only the asymptotic forms of \( x(z) \) are involved in these calculations and these are explicitly known (Eq. (23)).
also that the initial variable \( x \) is much more suitable for working out the solution; so we will use this variable henceforth in the understanding that at the end one has to make the substitution \( x \rightarrow x(z) \).

Based on the preceding discussion, we can now readily construct the potentials derived from (22) by simply applying formulas (11) through (12). Let us begin with the Schwartzian term \( V_s(x) = \frac{z}{w(x)} \) which is given by

\[
V_s(x) = \frac{\rho}{16} \frac{4\omega^2 x^4 + (4\omega + 12\omega^2 \rho) x^3 + (3 + 18\omega \rho + 3 \omega^2 \rho^2) x^2 + (12\rho + 4\omega \rho^2) x + 4\rho^2}{(x + \rho)^3(1 + \omega x)}
\]

while for the full potential \( V(x) = \frac{v(x)}{w(x)} + V_s(x) \) we get

\[
V(x) = \frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{(x + \rho)^3(1 + \omega x)}
\]

whereby

\[
A = \left( \frac{9}{4} \rho \omega - 3 \right) \omega^2 s^2 + \left( \frac{3}{2} \beta \rho - 2\alpha - 3\omega \rho + 3 \right) \omega s + \left( \frac{\beta^2 \rho}{4\omega} + \rho \omega - \beta \rho \right) \omega
\]

\[
B = \left( \frac{9}{2} \omega^2 \rho^2 - \frac{9}{2} \rho \omega - 3 \right) s^2 + \\
\quad + \left( \frac{-\frac{9}{2} \omega^2 \rho^2 + 3\omega \rho + 3 \omega^2 \beta + \frac{3}{2} \rho \beta - \frac{9}{2} \alpha \omega \rho - 2\alpha + 3 \right) s
\]

\[
\frac{1}{2} \left( \rho^2 \beta^2 - 3\omega \rho^2 \beta + \rho \alpha \beta - 3\alpha \omega \rho + 3 \omega \rho - \rho \beta \right) + \frac{1}{4} \omega \rho (3\omega \rho + 1)
\]

\[
C = \frac{\rho}{4} \left( 9\omega^2 \rho^2 - 3 \right) s^2 + 6(4 - 3\alpha - 2\alpha \omega + 2\beta \rho + \omega \rho^2 \beta - \omega^2 \rho^2)s
\]

\[
+ \rho^2 \beta^2 - 2\omega \rho^2 \beta + 4\alpha \beta \rho - 2\beta \rho - 10\alpha \omega \rho + \alpha^2 - 4\alpha
\]

\[
+ \frac{3}{4} \omega^2 \rho^2 + \frac{9}{2} \omega \rho + \frac{15}{4}
\]

\[
D = \frac{\rho^2}{4} \left( 6(\omega \rho - 3)s^2 + (6\beta \rho - 2\alpha \omega - 12\alpha + 18)s
\right)
\]

\[
+ (2\alpha \beta \rho - 4\alpha \omega \rho + 2\alpha^2 - 6\alpha + \omega \rho + 3)
\]

\[
E = \left( \frac{3}{4} s^2 - \frac{1}{2} (\alpha - 3)s + \frac{1}{4} (\alpha - 1)^2 \right) \rho^3.
\]

Note that even though the potential is a simple rational function of the initial variable \( x \), it nevertheless depends in a complicated way on the five dimensionless parameters \( \alpha, \beta, \omega, \rho \) and \( s \) of the problem at hand. (There are five and not six parameters as we had initially claimed, since the sixth is a scaling parameter of \( z \) which we implicitly set equal to unity—see, e.g., the first numerical coefficient in (20)—so that, together with the substitutions \( \hbar = 2m = 1 \), we can arrive at a complete system of units for the Schrödinger equation.) The form of \( V(z) \)—which is but the form of \( V(x) \) in the interval \( 0 < x < \infty \) stretched to \( -\infty < z < +\infty \)—
will typically be as shown in Figure 1 with $V_0 (= E/\rho^3)$ and $V_\infty (= A/\omega)$ given by

\[
V_0 \equiv V_\infty = -\frac{3}{4}s^2 - \frac{1}{2}(\alpha - 3)s + \frac{1}{4}(\alpha - 1)^2
\]

\[
V_\infty = \left(\frac{9}{4}\rho \omega - 3\right)s^2 + \left(\frac{3}{2}\beta \rho - 2\alpha - 3\omega \rho + 3\right)s + \frac{\beta^2 \rho}{4\omega} + \rho \omega - \beta \rho.
\]

(27)

Even though the expression for the potential in terms of its parameters is quite complex, the solutions of the respective Schrödinger equation look much simpler! To a large extent, this simplification is due to a different parametrization of the problem which occurs naturally once we realize that Eq. (22) has $x^s$ as a solution provided $\lambda$ takes the value

\[
\lambda_0 = -(3s^2 + (2\alpha - 3)s)
\]

(28)

\[\text{Figure 1: A typical form of the potential } V(z)\]

so that the corresponding eigenfunction $\psi_0(x)$ is

\[
\psi_0(x) = \frac{x^q(1 + \omega x)^{-q + (1/4)}}{(x + \rho)^{1/4}},
\]

(29)

where $q$ and $r$ are given by the expressions

\[
q = \frac{3s + \alpha - 1}{2}, \quad r = \frac{3s + (\beta/\omega) - 2}{2}.
\]

(30)

But since for $x \to 0$ and $x \to \infty$ we have $\psi_0(x) \to x^q$ and $\psi_0(x) \to x^r$ respectively, it follows that (28) will satisfy the boundary conditions at the origin and at infinity – i.e., at $-\infty$ and $+\infty$ of the variable $z$ – only if $q > 0$ and $r < 0$. In this case, $\psi_0(x)$ represents the system’s ground state since it has no nodes in this region. The crucial finding now is this: If we take the value (28) as the reference level for the potential energies and eigenvalues of the problem, using instead of $V$ and $\lambda$ the quantities

\[
U = V - \lambda_0, \quad \epsilon = \lambda - \lambda_0
\]

(31)

then a remarkable simplification ensues. $U$ and $\epsilon$ do not depend separately on the parameters $s$, $\alpha$ and $\beta$, but only on their combinations $q$ and $r$. In other words, when
we make this (parametrically dependent) change of reference level, the number of (dimensionless) parameters of the problem gets reduced from five to four; \( q, r, \rho \) and \( \omega \). Note, for example, that subtracting \( \lambda_0 \) from the asymptotic values (27) gives

\[
U_0 \equiv V_0 - \lambda_0 = q^2, \quad U_\infty \equiv V_\infty - \lambda_0 = \rho \omega r^2,
\]

which are much simpler expressions than those before, with the parameters now being \( q, r, \rho \) and \( \omega \). Let’s call this way of parametrizing the problem invariant parametrization; we shall work with this from now on. The expression for the potential \( U(x) \) now becomes as in (25), i.e.,

\[
U(x) = \frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{(x + \rho)^2(1 + \omega x)}
\]

but with new coefficients \( A, B, C \) etc., which are given by

\[
A = \omega^2 \rho r^2 \quad B = 2\omega^2 \rho^2 r^2 + 2\omega \rho qr - \omega \rho q + \omega^2 \rho^2 r + \frac{1}{4} \omega \rho (1 - \omega) \rho \\
C = \rho \left( q^2 + \omega^2 \rho^2 r^2 + 4\omega \rho qr - (1 + \omega)q + \omega \rho (1 + \omega) r + \frac{3}{16} (1 - \omega^2) \rho \right) \\
D = \rho^2 \left( 2q^2 + 2\omega \rho qr - q + \omega qr + \frac{1}{4} (\omega - 1) \right) \\
E = \rho^3 \omega q^2.
\]

IV. RESULTS: THE FULL HYPERGEOMETRIC CASE

As explained in Appendix A, if we use the above parametrization, the solution \( \psi(x) \) of the Schrödinger equation satisfying the boundary condition \( \psi(x = 0) \equiv \psi(z = -\infty) = 0 \) in the discrete spectrum region –or \( \psi(z \to -\infty) \sim e^{ikz} \) in the case of continuous spectrum– can be written as

\[
\psi(x) = \frac{x \sqrt{q^2 - \epsilon}}{(x + \rho)^{1/4}} \left( x(x + \rho)F' + (ax + \rho(a + 1))F \right)
\]

where \( F \equiv {}_3F_2(a, b, c; d, e; -\omega x) \) and \( a, b, c, d, e \) are given by

\[
a = -q + \sqrt{q^2 - \epsilon} \quad d = 2 - q + \sqrt{q^2 - \epsilon} = a + 2 \\
b = \sqrt{q^2 - \epsilon} + \sqrt{r^2 - \sigma} - \sigma \quad e = 1 + 2\sqrt{q^2 - \epsilon} \\
c = \sqrt{q^2 - \epsilon} - \sqrt{r^2 - \sigma} - \sigma
\]

and

\[
\sigma = q - r - 1, \quad g = \frac{1}{\rho \omega}.
\]
In fact it can be shown that expression (35) can also be written in the equivalent and more explicit form

\[ \psi(x) = \frac{x^{q-1}(1 + \omega x)^{q+1/4}}{(x + \rho)^{1/4}} \times \]

\[ \times \left( (\rho + 1) J_2(a, b, c; a + 1, c; -\omega x) + ax J_2(a + 1, b, c; a; -\omega x) \right) \]  \hspace{1cm} (38)

where no derivatives of \( J_2 \) enter. To the best of my knowledge expressions like (38) –with higher-order hypergeometric functions present in the solution of a Schrödinger equation– appear for the first time in the literature.

To impose the boundary condition at \( x \to +\infty \) (i.e., at \( z \to +\infty \)) we also need the asymptotic relation

\[ F(x) \rightarrow \frac{\Gamma(d)\Gamma(e)}{\Gamma(b)\Gamma(c)} \frac{\Gamma(b-a)\Gamma(c-a)}{\Gamma(d-a)\Gamma(e-a)} (\omega x)^{-a} + \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(c)} \frac{\Gamma(a-b)\Gamma(c-b)}{\Gamma(d-b)\Gamma(e-b)} (\omega x)^{-b} \]

\[ + \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a-c)\Gamma(b-c)}{\Gamma(d-c)\Gamma(e-c)} (\omega x)^{-c} \]  \hspace{1cm} (39)

in conjunction with the fact that the first-order operator \( \mathcal{M} \) acting on \( F \) –Eq. (35)– annihilates the term \( x^{-a} \), for the same reason that \( M \) annihilates also the solution \( y_0 \). Thus, we come to the following conclusions.

**A. Scattering states**

Denoting by \( r_R(k, k') \) and \( r_L(k, k') \) the reflection amplitudes from the right and left respectively, we have

\[ r_R(k, k') = \frac{r + i\sqrt{g}k'}{r - i\sqrt{g}k'} \frac{\Gamma(2i\sqrt{g}k')}{\Gamma(-2i\sqrt{g}k')} \times \]

\[ \times \frac{\Gamma(-\sigma - i(k + \sqrt{g}k'))\Gamma(\sigma + 1 - i(k + \sqrt{g}k'))}{\Gamma(-\sigma - i(k - \sqrt{g}k'))\Gamma(\sigma + 1 - i(k - \sqrt{g}k'))} (\omega x)^{2i\sqrt{g}k'} \]  \hspace{1cm} (40)

\[ r_L(k, k') = \frac{-q + ik}{q + ik} \frac{\Gamma(1 + 2ik)}{\Gamma(1 - 2ik)} \times \]

\[ \times \frac{\Gamma(-\sigma + i(k + \sqrt{g}k'))\Gamma(\sigma + 1 + i(k + \sqrt{g}k'))}{\Gamma(-\sigma + i(k - \sqrt{g}k'))\Gamma(\sigma + 1 + i(k - \sqrt{g}k'))} (\omega x)^{-2ik} \]  \hspace{1cm} (41)

where \( k \) and \( k' \) are the wavenumbers for \( z \to -\infty \) and \( z \to +\infty \) respectively. In other words, we have \( k = \sqrt{\epsilon - q^2} \), \( k' = \sqrt{\epsilon - (r^2/g)} \). For the corresponding reflection probability \( P_r \) –which is independent of the direction of incidence– we get

\[ P_r(k, k') = \frac{\cosh^2 \pi(k - \sqrt{g}k') - \cos^2 \pi\sigma}{\cosh^2 \pi(k + \sqrt{g}k') - \cos^2 \pi\sigma} \]  \hspace{1cm} (42)

**B. Bound states**

In this case, the requirement for vanishing \( \psi(x) \) at infinity leads –due to (39)– to the three conditions i) \( a = -n \), ii) \( b = -n \) and iii) \( e = -c = -n \). From these we obtain (see Appendix A for details) the respective regions, red, green and blue, of
the phase diagram in Figure 2. More specifically, the condition \(a = -n\) is actually restricted to the value \(n = 0\) and has a respective (unique) eigenvalue \(\epsilon = 0\); while \(b = -n\) and \(e - c = -n\) lead to the equations (I) and (II) for the green and blue region respectively.

Note however that in order for solutions of, say, Eq. (I) to exist, it is not sufficient that \(\sigma > 0\), but we must also have \(\sigma > f_{\text{min}}\), where \(f_{\text{min}}\) is the minimum value of the function \(f(\epsilon) = \sqrt{q^2 - \epsilon} + \sqrt{r^2 - g\epsilon}\), in the left hand side of (I), which is monotonically decreasing and will therefore cross the horizontal line at height \(\sigma\) only if the above inequality holds. This directly implies that the \(g\)-interval within which bound states can exist is

\[
\frac{r^2 - \sigma^2}{q^2} < g < \frac{r^2}{q^2 - \sigma^2}
\]  

where it is assumed that the left or the right side of these inequalities will be replaced by zero or infinity, respectively, when the corresponding factor \(r^2 - \sigma^2\) or \(q^2 - \sigma^2\) vanishes or becomes negative. It also follows from (43) that a suitable range of values for \(g\) always exists, so the existence of bound states in the green or blue region of the phase diagram is always guaranteed. Of course, the inequality (43) holds also for \(\sigma \to \sigma - n\) -i.e., for the full right hand side of (I) (or of (II))—in which case the number of bound states will depend on the value of \(g\) and will generally be smaller than predicted by the simple positivity of the right hand sides of (I) or (II). Combining the above reasoning with the fact that it will always be \(q - r - 1 < |q| + |r|\), leads us to conclude also that \(\epsilon = 0\) is not a solution of the eigenvalue equation (I), although it can be a solution of (II) in the middle blue region; but ultimately even that is rejected for reasons explained in Appendix A. Hence the only vanishing eigenvalue is the one located in the red region and it represents the absolute ground state of the system. In the special case whereby the potential’s asymptotic values \(U_0\) and \(U_\infty\) are equal -i.e., when \(q^2 = r^2/g\) - then clearly \(f_{\text{min}} = f(\epsilon)|_{\epsilon=q^2=0} = 0\) and (I) is satisfied for all positive \(g\), a fact that also follows directly from the inequality (43). In that case, the eigenvalues are found easily in closed form and their formula (in the green region, say) becomes

![Figure 2: Phase diagram for bound states](image-url)
in the full hypergeometric case.

- **Red region**: \((q > 0, r < 0)\). The eigenvalue \(\epsilon = 0\) exists and represents the absolute ground state of the system.

- **Green region**: \((q - r > 1)\). Bound states exist and are determined by the condition
  \[ \sqrt{q^2 - \epsilon + \sqrt{r^2 - g\epsilon}} = \sigma - n = q - r - N \quad (N = n + 1) \]  (I)

- **Blue region**: \((q - r < 0\) for \(qr > 0\) or \(q - r < -1\) for \(qr < 0\)). Bound states exist and are determined by the condition
  \[ \sqrt{q^2 - \epsilon + \sqrt{r^2 - g\epsilon}} = -\sigma - 1 - n = r - q - n \]  (II)

which can be also deduced from (I) by the substitution \(q \rightarrow -q, r \rightarrow -r\) and \(N \rightarrow n\). Note, however, that in the middle blue region \((qr < 0)\) the value \(n = 0\) is rejected since then the value \(\epsilon = 0\), which is now a solution of (II), does not correspond to a physically acceptable solution (see Appendix A).

- **White regions**: There are no bound states.

\[ \epsilon_n = q^2 - \frac{(\sigma - n)^2}{(1 + \sqrt{g})^2} = q^2 - \frac{(q - r - N)^2}{(1 + \sqrt{g})^2}, \quad N = n + 1 = 1, 2, \ldots \leq q - r. \]  (44)

This will also hold in the blue region upon substituting \(q \rightarrow -q, r \rightarrow -r\); while the further substitution \(N \rightarrow n\) is also necessary in the outer blue regions.

Let us also mention that the wavefunction of the absolute ground state of the system – we say absolute since there now exist various ground states across the different regions of the phase diagram – is given by

\[ \psi_0(x) = \frac{x^q(1 + \omega x)^{r-q+(1/4)}}{(x + \rho)^{1/4}}. \]  (45)

Other features of the phase diagram worth noting are the following:

a) In contrast to all known examples of solvable potentials, in this case the region of parametric space where bound states exist is _doubly connected_. It consists of the two main regions \(q - r > 1\) and \(q - r < 0\) with a gap in-between (white regions) where no bound states exist. b) Every point \((q, r)\), for example in the green-red region, has a _mirror image_ \((-q, -r)\) in the middle blue region with the same bound states except the zero one \((\epsilon = 0)\) which exists in the green but not the blue region. This follows directly from the eigenvalue equations (I) and (II) and the fact that in the middle blue region the value \(n = 0\) \((\Rightarrow \epsilon = 0)\) is rejected. Thus we have pairs of potentials \(U_+ = U(q, r, x)|_{q-r>1}\) and \(U_- = U(-q, -r, x)\) with the same bound states except for \(\epsilon = 0\) which exists for the former but not the latter potential. Whether there is some symmetry behind this “pairing” of potentials in parametric space is an interesting question that deserves further study.

V. CONFLUENT CASE OF THE FIRST KIND

Let us now study the case where the parameter \(\omega\) vanishes, so the solutions (see Appendix A) are hypergeometric functions of the confluent kind – i.e., \(2F_2, 1F_2\) etc. – and their behavior at infinity will vary accordingly, including now not only powers of \(x\) but also exponentials. We will start with the confluent case of the first kind,
for which $\omega = 0$ but $\beta \neq 0$. The expression of the potential in this case emerges readily from (33) and (34) upon taking the limit $\omega \to 0$, which yields

$$U(x) = \frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{(x + \rho)^3}$$

(46)

with

$$A = \frac{1}{4}\rho\beta^2$$
$$B = \frac{1}{2}\rho^2\beta^2 + pq\beta$$
$$C = \rho \left(q^2 + \frac{1}{4}\rho^2\beta^2 + 2pq\beta - q + \frac{1}{2}\rho\beta + \frac{3}{16}\right)$$
$$D = \rho^2 \left(2q^2 + pq\beta - q + \frac{1}{2}\rho\beta - \frac{1}{4}\right)$$
$$E = \rho^3 q^2$$

(47)

and has a typical plot as in Figure 3.

![Figure 3: Typical potential of the confluent family of the first kind. At $-\infty$ the potential tends to a fixed value and at $+\infty$ it rises as a harmonic oscillator.](image)

Let us note here that some of the results for the confluent case derive directly from the previous ones by setting $\omega = 0$, provided they do not represent a qualitative change in the problem, in which case it is safer to redo the calculation. Let us simply mention these results without elaborating much on them. We begin from the relation $z = z(x)$, for which we have

$$z'(x) = \frac{\sqrt{1 + (x/\rho)}}{x} \Rightarrow x(z) \xrightarrow{z \to -\infty} e^z, \quad x(z) \xrightarrow{z \to +\infty} \frac{\rho}{4} z^2.$$ 

(48)

The general expression of the wavefunctions is now

$$\psi(x) = \frac{x^{\sqrt{q^2 - \epsilon}} e^{\beta x/2}}{(x + \rho)^{1/4}} \left(x(x + \rho)F' + (ax + \rho(a + 1))F\right),$$

(49)

where $F(x) \equiv \, _2F_2\left(a, b; c, d; -\beta x\right)$ and

$$a = -q + \sqrt{q^2 - \epsilon} \quad c = 2 - q + \sqrt{q^2 - \epsilon} \quad (= a + 2)$$
$$b = -q + \sqrt{q^2 - \epsilon} + pe + 1 \quad d = 1 + 2\sqrt{q^2 - \epsilon}$$

(50)
with \( q \) as before –formula (30a)– and

\[
p = \frac{1}{\rho \beta}
\]

(51)

where the new parameter \( p \) takes now the position of \( r \) in the phase diagram that we will present shortly. The search for physically acceptable solutions depends crucially on the sign of \( \beta \), or \( p \), and is based on the asymptotic relation

\[
F(x) \rightarrow \frac{\Gamma(b - a) \Gamma(c) \Gamma(d)}{\Gamma(b) \Gamma(c - a) \Gamma(d - a)} (\beta x)^{-a} + \frac{\Gamma(a - b) \Gamma(c) \Gamma(d)}{\Gamma(a) \Gamma(c - b) \Gamma(d - b)} (\beta x)^{-b} + \frac{\Gamma(c) \Gamma(d)}{\Gamma(a) \Gamma(b)} (-\beta x)^{a+b-c-d} e^{-\beta x}.
\]

(52)

The results we obtain are as follows.

A. Scattering states

Since the only scattering that makes sense now is from the left, we need only calculate the reflection amplitude \( r_L(k) = r(k) \), which becomes

\[
r(k) = \frac{-q + ik \Gamma(1 + 2ik) \Gamma(1 - q + pe - ik)}{q + ik \Gamma(1 - 2ik) \Gamma(1 - q + pe + ik)} (-\beta)^{-2ik} \quad (\beta < 0)
\]

\[
r(k) = \frac{-q + ik \Gamma(1 + 2ik) \Gamma(q - pe - ik)}{q + ik \Gamma(1 - 2ik) \Gamma(q - pe + ik)} \beta^{-2ik} \quad (\beta > 0)
\]

(53)

with \(|r(k)| = 1\) of course, since the form of the potential does not allow for the particle to pass through towards positive infinity.

B. Bound states

A similar calculation as before yields now –apart from the vanishing eigenvalue in the region \( q > 0 \), \( p < 0 \), which is found separately (see Appendix A)– the two complementary conditions

\[
\sqrt{q^2 - \epsilon} = -pe + q - N \quad (p < 0, N \geq 1)
\]

(54)

and

\[
\sqrt{q^2 - \epsilon} = pe - q - n \quad (p > 0, n \geq 0)
\]

(55)

from which we can readily obtain the results summarized in the next phase diagram.

Clearly, the phase diagram has the type of mirror symmetry we mentioned before and this raises similar questions as to its origin and interpretation.

Concerning the absolute ground state of the system –i.e. the one with \( \epsilon = 0 \)– this exists only in the red region and its wavefunction is written as

\[
\psi_0(x) = \frac{x^q e^{\beta x/2}}{(x + p)^{1/4}}.
\]

(56)

Note also that the bound states do not depend on the full set of dimensionless parameters \( \rho, \beta, q \) of the potential \( U(x) \) (in the invariant parametrization as before) but only on their two combinations \( p = 1/\rho \beta \) and \( q \). This is contrary to what happens in the scattering data –formulas (53)– where all three parameters enter. Something similar to this occurs in the full hypergeometric case we examined before. The presence of isospectral orbits in parametric space is thus one more feature of the potentials presented here which has to be explained.
• Red region: \((q > 0, p < 0)\). The eigenvalue \(\epsilon = 0\) exists and represents the absolute ground state of the system.

• Green region: \((p < 0 : p \leq (q - 1)/q^2)\). Bound states exist, and they are given by the formula

\[
\epsilon_N = \frac{(pq - \frac{1}{2}) - pN + \sqrt{(pq - \frac{1}{2})^2 + pN}}{p^2}, \quad N = 1, 2, \ldots, \leq (1 - pq)q
\]  

(I)

• Blue region: \((p > 0 : p \geq 1/q\) for \(pq > 0\) and \(p \geq (q + 1)/q^2\) for \(pq < 0\)). Bound states exist, and they are given by the formula

\[
\epsilon_n = \frac{(pq - \frac{1}{2}) + pn + \sqrt{(pq - \frac{1}{2})^2 - pn}}{p^2}, \quad n = 0, 1, \ldots, \leq (pq - 1)q
\]  

(II)

which can be also derived from (I) by the substitution \(p \rightarrow -p\), \(q \rightarrow -q\) and \(N \rightarrow n\). But in the left blue region \((pq < 0)\) the value \(n = 0\) is rejected for the same reasons as before.

• White regions: No bound states exist.

VI. CONFLUENT CASE OF THE SECOND KIND

The expression for the potential \(U(x)\), in the invariant parametrization, emerges directly from (46), (47) upon setting \(\beta = 0\), so there will be

\[
U(x) = \frac{\left(q^2 - q + \frac{3}{16}\right)\rho x^2 + \left(2q^2 - q - \frac{1}{4}\right)\rho^2 x + q^2 \rho^3}{\rho^2 + (x + \rho)^3}
\]  

(57)

with \(x = x(z)\) as in (48), which implies an exponentially fast approach to the asymptotic value \(U_0 = q^2\) of the potential at \(z \rightarrow -\infty\), but a very slow decrease to zero –of the kind \(U \sim 1/z^2\)– at \(z \rightarrow +\infty\). Depending on the range of values of \(q\) the potential \(U(z)\) can take one of the following forms:
The wave functions will now be written as (Eq. 49 with $\beta = 0$)

$$\psi(x) = \frac{x^{\sqrt{q^2 - \epsilon}}}{(x + \rho)^{1/4}} (x(x + \rho)F + (ax + \rho(a + 1))F)$$

where $F(x) = {}_2F_2(a; b, c; -\epsilon x/\rho)$ and

$$a = -q + \sqrt{q^2 - \epsilon}, \quad b = a + 2, \quad c = 1 + 2\sqrt{q^2 - \epsilon}$$

while the asymptotic behavior of $F(\zeta)$ for large $|\zeta|$ – where $\zeta = -\epsilon x/\rho$ in our case – will be given by

$$F(\zeta) \rightarrow \frac{\Gamma(b)\Gamma(c)}{\Gamma(b - a)\Gamma(c - a)}(-\zeta)^{-a}$$

$$+ \frac{\Gamma(b)\Gamma(c)}{2\sqrt{\pi}\Gamma(a)}(-\zeta)^{a} \left( e^{i(\pi \eta + 2\sqrt{\zeta})} + e^{-i(\pi \eta + 2\sqrt{\zeta})} \right) \left( \eta = \frac{1}{2} \left( a - b - c + \frac{1}{2} \right) \right)$$

(60)
e.g., by a linear combination of the three asymptotic behaviors at infinity, which now have the form of two exponentials and one power of $\zeta$ due to the fact that the maximum dimension component of the operator $\mathcal{L}$ in (22) – with $\omega = \beta = 0$ – is now of first order.

On the basis of the above the results we obtain are as follows.

For $\epsilon > U_0$ there are scattering solutions both from the left and from the right, and the corresponding reflection amplitudes are given by the formulas

$$r_L(k) = \frac{-q + ik}{q + ik} \frac{\Gamma(1 + 2ik)}{\Gamma(1 - 2ik)} \left( \frac{\epsilon}{\rho} \right)^{-2ik} e^{-2k\pi}, \quad r_R(k) = i e^{-2k\pi}$$

(61)

where it is noteworthy that, apart from a constant phase (independent of $k$), the reflection amplitude from the right is real.

As for bound states, there is only the eigenvalue $\epsilon = 0$ with a corresponding eigenfunction

$$\psi_0(x) = \frac{x^q}{(x + \rho)^{1/4}}$$

(62)
which will satisfy the boundary conditions \( \psi(0) = \psi(\infty) = 0 \) only if \( 0 < q < 1/4 \). Nevertheless, (62) is not square integrable as one might have expected, due to
the position of the corresponding eigenvalue at the threshold of the continuous spectrum.

It hardly needs mentioning, that all the above results –irrespective of how
they were produced– can be verified directly with a straight substitution in the
Schrödinger equation. To do that, it is advisable to cast that equation in the equiv-
alent \( x \)-form

\[
\frac{1}{w(x)} \psi''(x) - \frac{w'(x)}{2w^2(x)} \psi'(x) + (\epsilon - U(x)) \psi(x) = 0
\]

(63)

where the derivatives and the expressions of the functions \( w \) and \( U \) are with respect
to the initial variable \( x \).

Finally, we note that in both main classes of potentials (full hypergeometric case
and confluent of the first kind) their graphs do not always assume the typical forms
of Figures 1 and 3 but they show interesting variations similar to those in Figure
5. The question of which of these forms may have some special physical interest
will not be discussed here.

VII. PENDING ISSUES AND GENERALIZATIONS

Let us note first that our preceding discussion on potentials that are solvable via
hypergeometric functions of third order is not exhaustive. The first reason for this
is that the solution \( y_0 = x^s(x + \rho) \), which we have chosen to reduce the order of the
initial equation, is just a special case –for \( s' = 1 \)– of the general expression, \( y_0 = x^s(x + \rho)^{s'} \), for the solutions of the first order equation \( My = 0 \). This special case
is undoubtedly the most important, for reasons that may have become obvious by
now. But there are also other possibilities –e.g., \( s' = 2 \)– belonging to the terminating
series category and which can thus be solutions of \( Ly = 0 \), as required. Given
though that the coefficients of the relevant polynomial are now related to one
another (they are all functions of \( \rho \)), the number of independent parameters of the
operator \( L \), and hence of the potential \( V \), will necessarily be reduced, or such a
solution might not even exist.

But there is another reason why our previous discussion is incomplete. We
assumed that \( \rho \) and \( \omega \) are positive –and hence, they fall outside the region \( 0 < x < \infty \)
between the singular points of the original equation– which means that the poten-
tial \( V \) is also finite in this region, and therefore also in \( -\infty < z < +\infty \) into which
\( 0 < x < \infty \) is mapped via the transformation \( z = z(x) \). But if we assume, for
example, that \( \omega < 0 \) –and substitute \( \omega \) for \( -\omega \)– then \( x \) must lie within the interval
\( 0 < x < \omega^{-1} \), which is mapped via the transformation (23) to the seminfinite
z-interval \( -\infty < z < 0 \); and so the corresponding potential \( V(z) \) will also be a
half-interval potential with repulsive core of the type \( 1/z^2 \), for \( z \to 0 \). In the same
fashion one needs to examine all the remaining choices of signs for \( \rho \) and \( \omega \) together
with the corresponding choices of intervals between singular points with respect to
\( x \) or \( z \). What is certain, is that even in the context of the third-order case, the
set of solvable potentials is much wider than what we presented here and deserves
further study, especially if such a study could provide answers to some interesting
special questions.

A third pending issue pertains to the comparison of the present family of solvable
potentials with that of Natanzon. What are the differences or similarities between
the two families? Can one family be deduced from the other by applying the special
Darboux transformation related to the removal of the ground state? We will leave this discussion for a follow-up publication, once we have constructed a much more general framework of studying solvable potentials and developed a general formalism that is immediately applicable to Natanzon potentials as the simplest special case. What is certain is that—apart from the special case $\rho = \omega^{-1}$ (for which the potentials are elementary functions)—there is no overlap between the two families, as evidenced both from the different scattering data and the different functional forms of their solutions. (Hypergeometric functions of the type $3F_2, 2F_2, 1F_2$ in our case, vs. $2F_1, 1F_1$ in the case of Natanzon.)

Next comes the question whether our methodology can be extended to equations of order higher than third, and produce the corresponding families of solvable potentials. While this will be the topic of an upcoming publication, we find it warranted at this point to present, without proof, some of the pertinent conclusions in order to underscore the existence of a general framework for the systematic investigation and enlisting of, potentially, all solvable potentials. Our most basic results are contained in the following three propositions:

**Proposition 1**: Every formal eigenvalue equation of the general form

$$\left( L + \lambda M \right)y = 0$$

where $L$ is any linear differential operator of order $n$ and $M$ a similar operator of order $n - 2$, can be reduced to an eigenvalue equation of second order—and hence, to a Schrödinger equation—provided that each solution of $My = 0$ is also a solution of $Ly = 0$.

**Proposition 2**: The weight function $w(x)$ of the Liouville equation that results from the reduction of any given bidimensional equation $(L + \lambda M)y = 0$—subject to the constraints of Proposition 1—can always be written in the form

$$w(x) = \frac{1 + (x/\rho)}{x^2(1 + \omega x)}.$$  

Therefore, the transformation function $z(x)$ that renders the Liouville equation into Schrödinger form will also be a “universal function”, i.e., the same function for all solvable potentials.

**Proposition 3**: The solvable potentials arising from a bidimensional equation of an arbitrary order $(L + \lambda M)y = 0$—subject to the constraints of Proposition 1—will always have the functional form

$$V(x) = \frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{(x + \rho)^3(1 + \omega x)}$$

where the coefficients $A, B, C$ etc., are functions of the parameters that remain free in the operators $L$ and $M$ once we demand that every solution of $My = 0$ be also a solution of $Ly = 0$.

In subsequent publications, we will rely on these three propositions—and on some crucial extensions of them—in order to extend this work and develop a systematic theory of exact solvability of the Schrödinger equation.

**VIII. DISCUSSION**
As was noted at the outset, the critical new idea of this work is this: Given that the Natanzon search for exactly solvable potentials is exhaustive, we can only produce new ones if we enlarge the set of functions within which their solutions are sought for. And if this extended set were the hypergeometric functions of all orders, then utilizing this set for solving Schrödinger equation would be in principle possible only in conjunction to the idea of order-reduction. We are thus led naturally to the notion of eigenvalue equations of higher order –of the hypergeometric type if we want them to be solvable— that can be reduced to the Schrödinger equation using the technique of order-reduction. And as we saw earlier, this idea is indeed feasible and has actually delivered as a tangible product a new set of solvable potentials, beyond that of Natanzon. We also saw that this notion –provided that propositions 1 through 3 are valid– is extensible to a very elegant general formulation which encompasses, as special cases, both the one studied here –where $L$ was of third and $M$ of first order– as well as the case of Natanzon, with $L$ being of second and $M$ of zeroth order, i.e., a function of $x$.

It follows that the method is in principle extensible to equations of higher than third order, even though it remains to be seen whether the relevant calculations continue to be manageable. Many further questions arise; here is but a short list of these:

a) Could there exist an infinite hierarchy –an infinite “tower”– of solvable potentials, with those of the Natanzon class occupying its “ground level”, the ones we presented here the “first floor” and so on, for the potentials originating from equations of ever higher order? Or does the process terminate leading to an ultimate closed set of solvable potentials?

b) What is the origin of the peculiar topology of the phase diagram for bound states and its mirror symmetry? Are the potential pairs $U_+$ and $U_-$, connected by this symmetry, supersymmetric partners? And what about the property of shape invariance that was shown not to match the initial expectations pertaining to its range of applicability? In view of the preceding discussion, could it be that an old conjecture, namely that this invariance may require a wider class of potentials than that of Natanzon, ought to be investigated further?

c) What is the origin of the reparametrization that reduces the number of active parameters of the system, or of the isospectral orbits in parametric space? Do these properties hint at some sort of symmetry that allows also a purely algebraic approach to the problem?

At a more mathematical level, the mapping of eigenvalue equations of higher order into the Schrödinger eigenvalue equation —a mapping that can be cast in an elegant compact form– could potentially provide valuable new insights to an old equation which clearly continues to hold an element of surprise.

APPENDIX A

CALCULATION OF EIGENVALUES AND EIGENFUNCTIONS

According to the equations (1) through (10) of section II, we have

$$
\psi(x) = (x + \rho)^{7/4}x^q(1 + \omega x)^{r-q+(1/4)} \left( \frac{y}{x^4(x + \rho)} \right)' \tag{A1}
$$
where \( q \) and \( r \) are defined as in (A10) and \( y(x) \) is the solution of the bidimensional equation (22). The latter can be written in terms of the hypergeometric function \( _3F_2 \) as
\[
y(x) = x^\mu \, _3F_2(a, b, c; d, e; -\omega x) \tag{A2}
\]
where the factor \(-\omega_, \) in the argument \( \zeta = -\omega x , \) serves to transfer the finite singular point \( x_0 = -\omega^{-1} \) of (22) to the standard position \( \zeta_0 = 1 \) of the hypergeometric equation, while the factor \( x^\mu \) represents the physically acceptable power behavior of \( x \to 0. \) The possible values of \( \mu \) (physically acceptable or not) are found as solutions of the unidimensional equation (or Euler-type equation)
\[
\mathcal{L}_1 x^\mu = 0 \tag{A3}
\]
where \( \mathcal{L}_1 \) is the lowest-dimension component of the bidimensional operator \( \mathcal{L} = L + \lambda M \) of (22). For the power behavior \( x^\nu \) at infinity the respective equation is
\[
\mathcal{L}_2 x^\nu = 0 \tag{A4}
\]
where \( \mathcal{L}_2 \) is the largest-dimension component of the operator \( \mathcal{L} \). (A3) and (A4) are of course cubic equations for \( \mu \) and \( \nu \), which are however easily solved since we already know (from the known solution \( y_0 = x^\nu (x + \beta) \)) that one behavior at the origin is \( x^\nu \) and one behavior at infinity is \( x^{\nu + 1} \). The exponents of the remaining two behaviors are easily found to be
\[
\mu, \nu = \frac{3 - \alpha - s}{2} \pm \sqrt{V_0 - \lambda}, \quad \nu, \bar{\nu} = \frac{2 - (\beta/\omega) - s}{2} \pm \frac{1}{\sqrt{\omega^2}} \sqrt{V_\infty - \lambda} \tag{A5}
\]
where the specific choice of signs for \( \mu \) and \( \nu \) (positive for \( \mu \) and negative for \( \nu \)) reflects the fact that the physically desirable behaviors are then \( x^\mu \) and \( x^\nu \) for small and large \( x \) respectively. Now we can readily calculate the hypergeometric parameters \( a, b, c, d \) and \( e \) in (A2), by recalling that the parameters \( a_i \) and \( b_j \) in the general hypergeometric series
\[
pFq(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{A6}
\]
have always the same asymptotic meaning: \( x^{-a_i}, \) \( i = 1, \ldots, p \) are the power behaviors of the general solution of the corresponding hypergeometric equation at infinity and \( x^{b_j}, x^{1-b_j}, \) \( j = 1, \ldots, q \) are the power behaviors at the origin. The calculation of \( a, b, c, d \) and \( e \) in (A2) reduces then to simple asymptotic comparison of the two sides for \( x \to 0 \) and \( x \to \infty \). The result is
\[
a = \mu - s - 1, \quad b = \mu - \nu, \quad c = \mu - \bar{\nu} \\
d = \mu - s + 1, \quad e = \mu - \bar{\nu} + 1 \tag{A7}
\]
from which, upon using (A5) and the definition of the new parameters \( q \) and \( r \) (eq. (A10)), we obtain the expressions (30) and the general formula (34) for the solution \( \psi(x) \) that satisfies the boundary condition \( \psi(0) = 0 \) in the region of bound states.

The boundary condition at infinity \( (\psi(\infty) = 0) \) is imposed via the asymptotic form of \( \psi(x) \) for large \( x \) which is written as
\[
\psi(x) \to (a - b) Bx^{-\sqrt{r^2 - g x}} + (a - c) Cx^{\sqrt{r^2 - g x}} \tag{A8}
\]
and is derived from the asymptotic relation (43) for $F(x)$

$$F(x) \equiv \, _3F_2 \rightarrow Ax^{-a} + Bx^{-b} + Cx^{-c}$$

(A9)

by taking into account what we noted earlier: Namely that the first-order operator $\mathcal{M}$ acting on $F$ (Eq. (35)) annihilates the term $x^{-a}$. It follows then from (A8) that the boundary condition at infinity will only be satisfied if $a - c = 0$ or $C = 0$. But the case $a - c = 0$ is rejected at once due to the factor $\Gamma(a - c)$ in the expression for $C$, so we are left with $C = 0$ which can be realized in each of the following ways

i) $a = -n$  
ii) $b = -n$  
iii) $e - c = -n$

(A10)

since the case $d - c = -n$ is also rejected due to the relation $d = a + 2$ and the presence of the factor $\Gamma(a - c)$ in the numerator of $C$. From $\Gamma(d)/\Gamma(a) = a(a+1)$ it is also clear that for case (i) only the values $a = 0$ and $a = -1$ survive, the former reproducing the already known ground-state solution $\epsilon = 0$ (i.e., $\lambda = \lambda_0$) for $q > 0$, $r < 0$, while the latter gives $\psi \equiv 0$ as expected. In case (ii) the pertinent condition is written as

$$\sqrt{q^2 - \epsilon} + \sqrt{r^2 - ge} = q - r - N \quad (N = 1, 2, \ldots)$$

(A11)

and will be satisfied under the conditions laid out in section IV (Eq. (43)) from where the exact number of bound states for the problem is deduced. Note that $\epsilon = 0$ is no solution of (A11) for $n = 0$, i.e., for $N = 1$, and thus the condition (ii) yields only positive eigenvalues. In both the above cases, the hypergeometric function $F(x)$ has polynomial form because both conditions $a = -n$ and $b = -n$ are termination conditions for the respective hypergeometric series.

However, in case (iii) –contrary to what usually happens for bound states– the hypergeometric series does not terminate since in that case both behaviors $x^{-a}$ and $x^{-b}$ survive at infinity, even though only the latter contributes to the asymptotic limit of $\psi(x)$. The condition for calculating the eigenvalues is now written as

$$\sqrt{q^2 - \epsilon} + \sqrt{r^2 - ge} = r - q - n \quad (n = 0, 1, 2, \ldots, \leq r - q)$$

(A12)

but especially in the middle blue region of the relevant phase diagram (Fig. 2) the value $n = 0$ has to be exempted since then (A12) can have as its solution the vanishing eigenvalue $\epsilon = 0$, which, however, does not lead to a physically acceptable solution $\psi(x)$ and must be thus rejected. The reason for this is nontrivial. For $\epsilon = 0$ a flaw arises in the mechanism that “ejects” the undesirable asymptotic behavior for $x \rightarrow \infty$, since then the hypergeometric parameters $a, b$, etc., no longer depend on $\rho$ (see Eq. (43)), so the corresponding solution $F(x)$ has no way of “knowing” the value of $\rho$ although the relevant ejection operator $\mathcal{M} = x(x+\rho)\partial + ax + \rho(a+1)$ in (35) continues to depend on it.

Using the same techniques as above –but with the asymptotic formula (52) instead of (39)– we can also calculate the bound states in the confluent case, obtaining as a result –except for the zero eigenvalue– the two conditions

$$\sqrt{q^2 - \epsilon} = -p\epsilon + q - N, \quad \sqrt{q^2 - \epsilon} = p\epsilon - q - n$$

(A13)

which lead to formulas (I) and (II) of Figure 4.
As for the scattering solutions (in the full hypergeometric case for example), we need the asymptotic form of $\psi(z)$ for $z \to \pm \infty$ which turns out to be

$$
\psi(z) = \begin{cases} 
\frac{\rho^{1/4}(a + 1)e^{ikz}}{z \to -\infty} \\
\frac{-Be^{-ik'z} + Ce^{ik'z}}{z \to \infty}
\end{cases}
$$

(A14)

and it is clear from (A14) that the scattering solution from the right equals $\psi^*$ –that is, $\psi_R = \psi^*$– and therefore

$$
r_R = \left( \frac{B}{C} \right)^*
$$

(A15)

from where –after a few steps of algebra– we readily obtain the results (10) and (11). Finally, the calculation of scattering from the left is done by forming the combination $c_1\psi + c_2\psi^*$ containing only a traveling wave $\exp(ik'z)$ at $+\infty$, and proceeding as before. The spectral data for the confluent cases of first and second kind are calculated in an analogous manner.

Some closing remarks about the connection of hypergeometric functions and bidimensional equations are in order. The connection originates from the fact that bidimensional equations lead to two-term recursion relations, from which the general coefficient of the series-solution is computed in closed form and it turns out to be exactly the same as the general coefficient of the hypergeometric series (A6). Thus the hypergeometric functions emerge naturally as solutions of the bidimensional equations of step unity. The equation they satisfy –take, for instance, the function $3F_2$– will naturally have the form (15), but with $\omega = -1$ (so that the nonzero singular point lies in the standard position $x_0 = 1$) and $\varepsilon = 0$, so that one exponent of the power behavior at the origin can take the typical value zero as in the general hypergeometric series (A6) that solves the relevant equation. As for the exact form of the remaining numerical coefficients of (15), as functions of the hypergeometric parameters $a, b, c, d$ and $e$, this is never needed since to obtain the solutions it suffices to know the power behaviors of $y(x)$ at zero and infinity, together with the standard asymptotic meaning of the hypergeometric parameters.

As for the hypergeometric functions of the confluent type –i.e, with $p < q + 1$– these emerge naturally from those bidimensional equations whose highest-dimension component $L_2$ has no longer the full order of the equation, but is of lower order by one, two, three, etc., exactly as the value of $p$ compared to $q + 1$. In this case, only $p$ behaviors at infinity –where $p$ now stands also for the order of $L_2$– will be powers of $x$; the rest will be exponentials. This alters the character of the point at infinity that now becomes irregular singular. As for the terminology, we have chosen the terms confluent equation (or function) of the first kind if $p = q$, confluent of the second kind if $p = q - 1$, etc. To complete the picture, we note that not only the bidimensional equations of step one can be solved via hypergeometric functions, but also the bidimensional equations of any step $\ell$. The reason for this is simple. It has to do with the (rather obvious) fact that the transformations $t = x^m$ (change of independent variable) and $y = x^{\mu}Y$ (change of dependent variable) preserve the bidimensional character of an equation and merely cause a change in its step, from $\ell$ to $\ell/m$, and a shift in its starting powers by $\mu$, respectively.
It thus follows that we will always be able to write the solutions of any bidimensional equation in the form

$$y(x) = x^\mu \Phi_\eta(a_1, \ldots, a_p; b_1, \ldots, b_q; kx^\ell)$$  \hspace{1cm} (A16)

where \(\Phi_\eta\) is the general solution of the hypergeometric equation with parameters \(a_i\) and \(b_j\), \(\ell\) is the step of the given equation and \(k\) the numerical coefficient that maps all singular points outside the origin (all of which lie on a circle in the complex plane) to the standard position \(\zeta = 1\) where \(\zeta = kx^\ell\). As for the parameters \(\mu\) and \(a_i, b_j\) in (A16) these are obtained readily from the asymptotic comparison of its sides at zero and at infinity, leading directly to the relations

$$\nu_i = \mu - \ell a_i, \quad \mu_j = \mu + \ell (1 - b_j)$$  \hspace{1cm} (A17)

where \(\mu_j\) and \(\nu_i\) are the power behaviors of the given equation at zero and infinity respectively, while of course \(\mu\) must be equal to one of the \(\mu_j\) (e.g. \(\mu = \mu_1\)).

It is clear from the above discussion that the hypergeometric functions and bidimensional equations emerge as the natural framework for the systematic study of the problem of exact solvability of the Schrödinger equation or any other equation for that matter. Our future work on this problem will lie within this systematic framework.

ACKNOWLEDGMENTS

I am grateful to Nick Papanicolaou for helpful discussions and critical comments and Manolis Antonoyiannakis for helpful discussions and encouragement.

BIBLIOGRAPHY

1. G.A. Natanzon, Vestnik Leningrand Univ. 10, 22 (1971); Teor. Mat. Fiz. 38, 146 (1979)
2. G. Darboux, C.R. Acad. Sci. (Paris) 94, 1456 (1882)
3. F. Cooper, N. Ginocchio and A. Khare, Phys. Rev. D 36, 2458 (1987)
4. F. Cooper, A. Khare, U. Sukhatme, Physics Reports 251, 267 (1995)