Sheafifying Consistent Histories

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Abstract

Isham’s topos-theoretic perspective on the logic of the consistent-histories theory [34] is extended in two ways. First, the presheaves of consistent sets of history propositions in their corresponding topos originally proposed in [34] are endowed with a Vietoris-type of topology and subsequently they are sheafified with respect to it. The category resulting from this sheafification procedure is the topos of sheaves of sets varying continuously over the Vietoris-topologized base poset category of Boolean subalgebras of the universal orthoalgebra $\mathcal{UP}$ of quantum history propositions. The second extension of the topos in [34] consists in endowing the stalks of the aforementioned sheaves, which were originally inhabited by structureless sets, with further algebraic structure that also enjoys a quantum causal interpretation à la [51, 42, 53, 55, 56] so as to arrive at the topos of consistent-histories of quantum causal sets. Not being able to resist the temptation, we speculate on a possible application of such topos-theoretic models to the problem of quantum gravity (i.e., when spacetime structure, causality and its dynamics are supposed to be treated quantum mechanically)—an application that has been anticipated on general grounds by Butterfield and Isham [14] and partly worked out in a special finitary algebraic, sheaf-theoretic and categorical setting by this author [51, 42, 53, 54]. In particular, rather general, but striking, similarities between the topos of consistent-histories of quantum causal sets that arises from our second extension of [34], the topos of finitary spacetime sheaves of non-abelian incidence algebras modeling a dynamical and locally finite quantum causality and its associated non-commutative topology in [42, 53, 54], as well as the recently proposed quantum spacetime scenario based on the so-called quantum causal histories of Markopoulou [43], are exposed. The paper closes with this author’s personal views, anticipations and speculations about the future of the general research program of applying sheaf and topos-theoretic ideas primarily to quantum gravity and then to quantum logic.

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1 Introduction cum Motivation

The consistent-histories (CH) approach to quantum theory \cite{28, 49, 23} presents a sound alternative interpretation of quantum mechanics—one that is less instrumentalist or operational and philosophically more realist than the standard ‘Copenhagen’ one, for it purports to avoid altogether the notorious Heisenberg \textit{schnitt} of the usual theory. Perhaps, such an approach is more fit to support a quantum theory of the universe as a whole (\textit{i.e.}, quantum cosmology) \cite{24, 29} where an observer-system split appears to be highly inappropriate and \textit{prima facie} meaningless. Certainly, one expects that ideas that were born out of the CH version of quantum theory should apply straightforwardly to our quite general endeavor of applying quantum mechanical concepts, results and techniques to the structure and dynamics of spacetime \cite{34, 14}, as well as to the associated problem of quantizing causality \cite{43}; altogether, to the general and by now quite broad and diverse quantum gravity research program.

The quantum sort of logic that underlies the CH theory\footnote{Hereafter we will refer to this logic as ‘quantal logic’ in order to distinguish it from the quantum logic proper that underlies the standard quantum mechanics \cite{5}.} has been beautifully exposed in \cite{33}. Subsequently, this quantal logic was subjected to a topos-theoretic analysis \cite{34} which revealed the theory’s strong ‘neorealist’ undertones in the following sense: the universal orthoalgebra $\mathcal{UP}$ of history propositions admits non-trivial localizations or ‘contextualizations’ (of truth) over its classical Boolean subalgebras. More technically speaking, it was shown that one cannot meaningfully assign truth or semantic values to propositions about histories globally in $\mathcal{UP}$, but that one can only do so locally, that is to say, when the propositions live in certain Boolean sublattices of $\mathcal{UP}$—the classical sites, or windows \cite{12, 13, 15}, or even points \cite{53, 46, 47} within the quantum lattice $\mathcal{UP}$\footnote{Here we will use the names ‘orthoalgebra’ and ‘lattice’ interchangeably for $\mathcal{UP}$, although strictly speaking the latter is a stronger algebraic structure than the former \cite{22}.}. Moreover, the simultaneous consideration of all such Boolean subalgebras and all consistent sets of history propositions\footnote{Isham’s assumption of all consistent sets of history propositions may be called “the principle of histories’ democracy”. See next paragraph.} leads one to realize that the ‘internal logic’ of the CH theory is neither classical (Boolean) nor quantum proper, but intuitionistic\footnote{As mentioned earlier, Isham in \cite{34} uses the epithet ‘neorealist’ for the quantal logic of the CH theory in its topos-theoretic guise. Quite reasonably, we feel, one could also coin this logic ‘neoclassical’ \cite{53}—this name referring to the departure of the Brouwerian logic of the topos of consistent-histories in \cite{24} from the two-valued Boolean lattice calculus obeyed by the states of a classical mechanical system which are modeled after point subsets of its phase space. See also the next section for more about this significant departure of the quantal logic of the CH theory from classical Boolean logic.}. This result befits the fact that the relevant mathematical
structure involved in [34], namely, the collection of presheaves of sets varying over the poset category of Boolean sublattices of $\mathcal{UP}$, is an example of an abstract mathematical structure known as a topos [38, 3, 39, 1], for it is a general result in category theory that every topos has an internal logic that is strongly typed and intuitionistic [24, 37, 21, 39]. This result also seems to suit the primitive intuition that some kind of ‘many-world-views’ [34], or to the same effect, ‘all-consistent-histories-view’ of the logic of the CH scheme will point to the appropriate semantics of the theory, since a modal Kripke-type of ‘possible-worlds’ semantics [36] has been found to apply to a very similar topos of presheaves of variable sets structure underlying the nondistributive quantum logic proper [57] as it too was nicely revealed by Isham et al. in the trilogy [12, 13, 15].

The reader must have noticed by now that no allusion to the measure or probability-theoretic attributes of the CH theory has been made so far. From a physical point of view, one of the most tantalizing questions one can raise about the CH theory is what singles out or ‘realizes’ a complete set of history propositions as the ‘actual’ or ‘real’ one from all other such sets. In the first place, it was the consideration of all the sets of history propositions that are consistent relative to a $\mathcal{C}$-valued measure $d$—the so-called ‘decoherence functional’—that led Isham [34] to question whether one should restrict probability assignments solely on propositions about $d$-consistent sets rather than, say, consider the larger ensemble of history propositions that are complete, but not necessarily consistent relative to a decoherence functional. Isham’s challenging of $d$-consistency paid off since he postulated the aforementioned principle of histories’ democracy which guided him rather straightforwardly to the notion of sieves of $d$-consistent coarse-grainings of complete sets of history propositions, then to their associated valuation presheaves and ultimately to the neorealist topos thereof.

Similarly, in the present paper we are not going to occupy ourselves with prob-

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5As it is also emphasized in [34], the (locally) intuitionistic quantal logic of the CH theory, while nonclassical (ie, non-Boolean) in the sense that the law of excluded middle (tertium non datur) does not hold in it, it still is distributive (at least locally; see below), in contradistinction to quantum logic proper which at a ‘global’ level (ie, when the focus is not restricted solely on propositions dwelling in the logic’s Boolean sublattices or classical points) appears to obey a prominently non-distributive lattice calculus [1], although it too may be shown to be locally intuitionistic and neorealist when viewed from a topos-theoretic perspective [12, 13, 15, 57].

6We will return to discuss this trilogy in more detail in the last section.

7For the notion of completeness of a set of history propositions, see [34].

8Again, see [34] for a brief discussion about this object and the $d$-consistent sets of history propositions associated with it.

9That is to say, to assume ab initio all $d$-consistent sets, rather than single out by hand a preferred or actual one.
abilistic features of the CH theory, for Isham’s topos-theoretic scheme seems to us to be convincing, rich and compelling enough. Rather, we are going to attempt to extend his work [34] in two fronts, which may be coined (i) ‘the base front’, and (ii) ‘the stalk front’, for reasons to be explained below:

- (i) **The base front**: The first extension of Isham’s paper [34] consists in an attempt to endow the presheaves of sets of consistent-history propositions with a suitable topology. This ‘topologization of histories’ will lead us effortlessly to sheafifying their respective presheaves, thus convert the topos organization of the latter to the topos of sheaves of continuously variable sets over the base poset category of Boolean subalgebras of $\mathcal{UP}$ now regarded as a background topological substratum proper. Such a move, apart from its mathematical naturalness,$^{10}$ is expected to unveil otherwise concealed (by the conventional non-topos-theoretic histories formalism) topological features of the CH theory. In any case, from a purely topos-theoretic perspective on the CH theory, such a move appears to be all the more legitimate, because the collection of sheaves (of algebraic structures of any kind) over a locale—the most general sort of a topological space [39]—appears to be the most canonical paradigm of a topos $\mathbf{Sh}(X)$. At least, having topologized the base space over which the overlying objects$^{11}$ are varying, we are able to qualify Lawvere’s adverb ‘continuously’ in [38]. In any case, it would be nice to have a consistent-histories analogue of the topos $\mathbf{Sh}(X)$ of sheaves of sets varying over a continuous spacetime manifold $X$—the mathematical universe in which arguably all quantum, albeit flat, field theories have been hitherto formulated $^{12}$.

- (ii) **The stalk front**: The topologization and concomitant sheafification of the presheaves in (i) will be followed by an algebraization of the stalks of the resulting sheaves. That is to say, instead of considering only structureless sets as inhabiting the stalks of the resulting sheaves as in [34], we will assume that the latter are occupied by so-called quantum causal sets which are finitary algebraic structures $^{11}$. The ultimate hope of such a move is to be able to view the resulting topos of continuously variable quantum causal sets

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$^{10}$That is to say, it is customary in mathematics when a presheaf-like structure appears during the development of a theory, that the next rather natural question that one may ask is whether the base space admits a topology; hence, whether the presheaf can be converted to a sheaf $^{1}$.

$^{11}$In [34] these objects are structureless sets.

$^{12}$This will also prompt us to look for a ‘curvaceous’ topos, since the CH theory purports to address the problem of quantum gravity [34, 14, 53, 54]. See also (ii) next as well as section 4.
as the proper mathematical universe in which to study the dynamical variations of quantum causality—a dynamics that is expected to be at the heart of yet another algebraic approach to quantum gravity \[51, 52, 42, 53, 54, 55, 56\]. We will also see that this second stalk-wise extension of Isham’s topos-theoretic perspective on the CH theory, together with its quantum causal interpretation, is very similar to Markopoulou’s recent quantum causal histories scenario for quantum spacetime structure and gravity \[43\] which purports to be a successful fusion of general ideas from the CH theory with ideas from another quite promising finitistic-causal approach to quantum gravity coined ‘causal set theory’ which has been around for more than a decade now \[6, 63, 64, 66, 67\]. Certainly, a bonus from soldering algebraic quantum structures of significant operational character, like the quantum causal sets in \[51, 42, 53, 55, 56\], on consistent-histories is that it enables us to reinstate to a certain extent some sort of operationality (if not strict instrumentalism!) in the CH theory—a theory whose interpretational philosophy at first sight appears to have a purely realist flavor \[34\].

The present paper is organized as follows: in the next section we give a brief review of the neorealist consistent-histories topos constructed in \[34\] and we highlight its features that are of relevance to our labors in the subsequent sections. In section 3 we topologize the base poset category \(B\) of Boolean subalgebras of \(\mathcal{U}\mathcal{P}\) over which sets are assumed to vary in \[34\] by endowing history propositions with a Vietoris-type of topology, and then we sheafify the associated presheaves of sets in their topos organization relative to the locale of open subsets of history propositions in the Vietoris-like topological space that they constitute. In section 4 we algebraize the stalks of sheaves that resulted from the topologization-sheafification procedure of the previous section by assuming that these fibers are inhabited by finitary algebraic quantum causal sets \[51, 42, 53, 54, 56\], rather than merely by structureless sets as in \[34\]. Thus we arrive at the **topos of consistent-histories of quantum causal sets** (QCHT)\[14\] and compare it with the quantum causal histories scenario proposed by Markopoulou in \[43\]. We also compare the QCHT with certain sheaf-theoretic models, of a finitary, causal and quantal flavor, for the kinematics of Lorentzian quantum gravity suggested in \[42\], as well as with some related algebraic attempts of this author to arrive at a cogent

\[\text{So that, for instance, even probabilities are interpreted as propensities in the CH theory, and in a strong sense history propositions are about the universe ‘as such’ or ‘in itself’ \[34\]. This seems to tie well with the aforementioned existence of an ‘internal logic’ for every topos, hence it further justifies Isham’s fundamental insight of assuming a topos perspective on the CH theory.}\]

\[\text{Initials for ‘Quantum Causal Histories Topos’}.\]
non-commutative or ‘quantum’ topology, of a strong finitistic and causal flavor, for quantum gravity in [53, 54]. In the concluding section we discuss various formal, but rather impressive, similarities between our QCHT and another topos-like structure that has recently appeared in connection with the famous Kochen-Specker ‘no-go’ theorem (or paradox!) of quantum logic proper [12, 13, 15]. The paper closes with some of this author’s personal and undoubtedly subjective views about the future course of development of the sheaf and topos-theoretic approach to the CH theory in particular and to the broader quantum gravity research program in general. More specifically, the possibility of infusing some differential geometric ideas and constructions to the CH theory by sheaf and topos-theoretic means, ultimately with an eye towards applying the resulting structures to quantum gravity, is briefly entertained at the end.

2 Isham’s neorealist consistent-history topos revisited

In this section we recall briefly concepts and results from the topos-theoretic perspective on the logic of the CH theory assumed in [34] that are going to be of relevance to the rest of the paper. The reader should refer to Isham’s original paper for a more thorough analysis of these elements.

The first element of structure of the quantal logic of the CH theory is that its propositions (about histories) form an orthoalgebra \( \mathcal{UP} \). A representation of \( \mathcal{UP} \) by projection operators in a suitable tensor product Hilbert space \( \mathcal{H} \) instantly reveals its ‘inherently quantum’ nature in the sense that the resulting projection lattice \( \mathcal{L}(\mathcal{H}) \) is characteristically non-distributive [33]—the quintessential feature of quantum logic proper [5]. A question that might occur to a quantum logician who is familiar with the logic of the CH theory and who is of a strong philosophical or ‘toposophical’ bent [15] is whether, apart from the unified formal mathematical or ‘syntactic’ structure that underlies both the quantal logic of the CH theory and the usual quantum logic of the conventional quantum mechanics (\( \text{ie} \), the non-distributive orthomodular lattice calculus [34]), there are other deeper similarities, or perhaps more importantly, differences between the two logics. For instance, one may enquire whether the valuation or ‘truth-theoretic’ and other associated ‘semantic’ aspects of the two schemes are also analogous to or fundamentally different from each other, and whether a topos-

\[ \text{It is not uncommon in mathematics’ social jargon for a categorist or topos-theorist who is interested in wider applications or philosophical extensions of topos theory to be called a ‘toposopher’}. \]
theoretic stance against these theories will shed light on such a comparison. After all, in spite of their impressive formal mathematical analogies at the syntactic proposition lattice level, the physical semantics or philosophical interpretation of the two theories are significantly different, as it was briefly mentioned in the introduction.

It is fair to say that Isham’s assumption of a topos-theoretic perspective on the CH theory was predominantly motivated by an interest to explore more ‘qualitative’ truth-theoretic, semantic or ‘valuational’ aspects of the quantal logic of consistent-histories, although there were also other ‘quantitative’ probabilistic or measure-theoretic aspects of the CH theory that appealed to him originally. The latter, however, we are only loosely going to address here. Below we summarize the basic ideas and results from by itemizing them:

- **Presheaves:** Presheaves arise from considering the notion of varying sets in the context of the CH theory. In particular, of central importance in is the notion of presheaves of sets over the poset category of Boolean subalgebras of , denoted by . provides an abstract ‘temporal’ background (base) domain (space) over which sets (or algebraic structures of any kind) are supposed to vary—its abstract character consisting in our identifying the notion of ‘temporal order’ or ‘succession’ with the process of coarse-graining of consistent-histories. The structures inhabiting the presheaves are seen as generalized ‘truth spaces’—realms in which truth assignments or ‘valuation functions’ on the history propositions in take their values. As a result, and from a geometrical perspective, the Boolean sub-

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16 The reader must await section for a brief comparison between the quantal and quantum logics from a topos-theoretic viewpoint.

17 The reader is assured that sophisticated topos-theoretic jargon and highly technical concepts or intricate results from topos theory will be seldom used in this physically, rather than mathematically, oriented paper. When a technical concept is mentioned, or when a theorem and result is quoted, references to the relevant mathematics literature, rather than an analytical discussion, will be given.

18 For the technical notion of presheaves, consult . We will discuss them more analytically in section .

19 is the collection of Boolean sublattices of ordered by set-theoretic inclusion which may be interpreted as ‘coarse-graining of histories’ in the sense that reads ‘’ or equivalently that ‘’. For presheaves of algebraic structures more elaborate than sets, see section .

20 The use of a poset as a ‘temporal support’ or as a ‘general domain of variation of a causal nature’ is also used in . In connection with presheaves of sets varying in time, the coarse-graining relation defining the base poset category had an analogous connotation for Isham in . Again, we will encounter such structures in section .

21 For the technical notion of valuations, refer to .
algebras $W$ of $\mathcal{UP}$ in $\mathcal{B}$ may be viewed as the ‘classical localization sites’ or ‘points of contextualization of truth’, or even as ‘classical windows of access to the quantum system’s states’ [34, 12, 13, 57, 33], within the quantum space $\mathcal{UP}$. The interpretation for them that we favor here is a more ‘temporal’ one as ‘local stages of truth’ [34]—frozen instances or ‘snapshots’ of truth value assignments on compatible history propositions living in each $W$—in the ‘global flow of truth over the partially ordered support $\mathcal{B}$’. Let us call the presheaves of the form $\text{Set}^\mathcal{B}$ ‘the valuation presheaves associated with $\mathcal{UP}$’. 

- **(b) Sieves:** Sieves arise [34] in close connection with the presheaves $\text{Set}^\mathcal{B}$. Isham naturally arrived at sieves by questioning whether only so-called second-level propositions about histories relative to a decoherence functional $d$ should be considered in probabilistic predictions about histories in the CH theory. In effect, he noticed that by coarse-graining a complete set $C$ of history propositions that is not $d$-consistent one could obtain a set $C'$ that is; moreover, any further coarsenings of $C'$ still yield $d$-consistent sets—the two defining properties of a sieve structure on a poset such as $\mathcal{B}$. We may call these sieves ‘the coarse-graining sieves on $\mathcal{B}$’. An even more suggestive result from [34] is that for every object $W_0$ in the poset category $\mathcal{B}$ (i.e., at every stage during the ‘unfolding of truth’ in $\mathcal{UP}$) the collection of all coarse-graining sieves based or soldered at $W_0$ form a logico-algebraic structure isomorphic to a Heyting algebra which is supposed to encode the lattice calculus of intuitionistic logic [26, 37, 4, 39]. Let us symbolize this object by $\Omega(W_0)$. This discussion brings us to the crucial fact about the collection of all presheaves $\text{Set}^\mathcal{B}$ of sets varying over $\mathcal{B}$.

- **(c) The topos of presheaves:** The category of all objects of the form $\text{Set}^\mathcal{B}$ and presheaf morphisms between them is an example of a topos [26, 37, 1, 39].

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23: The classical Boolean windows $W$ in $\mathcal{UP}$ are generated by propositions whose corresponding projection operators on the (closed) subspaces of $\mathcal{L}(\mathcal{H})$ commute with each other, hence the epithet ‘compatible’ [33, 14, 57].

24: For the technical definition of sieves, consult [39].

25: Roughly, a second-level proposition about a history is of the form ‘history $a$ is realized with probability $p$ relative to a chosen decoherence functional $d$’; hence, second-level history propositions are about $d$-consistent sets of histories in $\mathcal{UP}$ where the usual Kolmogorov axioms of probability theory appear to apply rather naturally [34].

26: Thus effectively the consideration of all complete sets of history propositions in $\mathcal{UP}$ (histories’ democracy).

27: See [26, 37, 1, 39] for a definition of this lattice structure.

28: For the notion of (pre)sheaf morphisms, refer to [3, 51, 40].
which we may symbolized as $\mathcal{T}_{CH}$. The aforementioned Heyting algebra object $\Omega$ in $\mathcal{T}_{CH}$ is known as the topos’ subobject classifier. By interpreting the latter as a generalized truth or semantic value space, Isham interpreted the lattice morphisms object-wise (or window-wise) in $\mathcal{T}_{CH}$ of the form

$$\mathfrak{U} : W \to \Omega(W), (\forall W \in B)$$  \hspace{1cm} (1)

as localized (or contextualized) valuations or truth value assignments to (second-level) history propositions in $\mathcal{UP}$. In fact, this is a ‘corollary’ of the following theorem: the presheaves in the consistent-histories topos $\mathcal{T}_{CH}$ admit no global sections over $\mathcal{UP}$; they only do so locally, that is, when restricted over the Boolean sublattices of $\mathcal{UP}$. We may resume this by saying that ‘in the quantal logic of the CH theory truth is localized or contextualized on the classical Boolean subalgebras of the universal quantum proposition lattice $\mathcal{UP}$’—which discussion brings us to an even more ‘universal’ result in topos theory.

- (d) **The internal language of the topos $\mathcal{T}_{CH}$**: The internal language or logic of the consistent-histories topos $\mathcal{T}_{CH}$ is intuitionistic type theory. This is effectively encoded in the subobject classifier $\Omega$ of $\mathcal{T}_{CH}$ which, as noted above, is a Heyting algebra—the logic algebra of intuitionism. This is in striking asymphony with the logic of the topos $\mathcal{Set}$ of sets whose subobject classifier is the Boolean binary alternative, hence whose internal logic is inherently Boolean. Due to this difference in logic, $\mathcal{Set}$ is thought of as a universe of constant or ‘frozen’ (perhaps in time) sets, while $\mathcal{T}_{CH}$ may be thought of as a realm of variable sets. Furthermore, it is precisely due to this generalization of the Boolean binary alternative of the classical logic of constant sets in $\mathcal{Set}$ to the Heyting algebra subobject classifier of the intuitionistic logic of variable sets in $\mathcal{T}_{CH}$ that Isham coined the latter topos ‘neorealist’ in

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29 The consistent-histories topos.
30 See [26, 37, 1, 39, 34] for a description of this object.
31 The Boolean windows in $B$ providing the localization sites or contexts.
32 For the technical notion of sections of (pre)sheaves, see [39, 40].
33 That local sections of the presheaves give rise to their subobjects in $\mathcal{T}_{CH}$, hence to presheaf morphisms (valuations) à la [40], is a well known fact in topos theory [1, 39, 43].
34 As noted in the introduction, the main characteristic of the Heyting calculus of intuitionistic logic is that double negation of a proposition is not its assertion, which reflects a violation of the law of excluded middle of the two-valued Boolean logic which has a unipotent negation unary operation.
35 The trivial Boolean algebra $\{0, 1\}$ consisting of the truth values 0 (false) and 1 (true).
and, in a slightly different context, this author ‘neoclassical’ in [53]. Neorealism then pertains precisely to the localization or contextualization of truth value assignments in the ‘globally’ non-distributive quantal logic of the CH theory over its Boolean subalgebras in that mutually compatible history propositions dwelling in the latter take truth values in a Heyting algebra truth space which is ‘larger’ (although still distributive!) than the Boolean binary alternative 2 of classical (hence realist!) set-logic.

We may summarize (c) and (d) in a geometrical sense by saying that the non-distributive quantal logic of the CH theory is ‘warped’ or ‘curved’ relative to its classical (Boolean) sublogics [27, 53] and, internally in its topos $T_{CH}$, it is locally intuitionistic—certainly not two-valued, but still distributive. This intimate logico-geometric interplay is allowed by the very essence of topos theory which is widely known by now to unify logic and geometry at a deep level [35, 39, 53].

In the next section we impart a topology of a special kind to the presheaf objects in $T_{CH}$ and subsequently we sheafify them. The resulting category is the topos of sheaves of sets over consistent-histories, thus we can expose topological traits of the CH theory as well as qualify the variable sets in $T_{CH}$ to ones being continuously variable in the sense of Lawvere [38].

3 Topologizing and sheafifying consistent-histories

In this section we get our hands dirty and become a bit more technical than before, although we still present everything at a ‘physical level of rigor’ always referring to the relevant mathematics literature for technical intricacies and results, as well as to the pivotal paper [34] for more analytically discussion of various constructions and facts.

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36 Only this observation could prompt one to look for some sort of ‘quantum set theory’—a quantal extension of the Boolean calculus of classical sets in the topos $\mathsf{Set}$ of constant sets so as to account for the way quanta (represented by some kind of ‘quantum sets’) actually combine with each other. The upshot of such an endeavor would be the development of a corresponding ‘quantum topology’ for small-scale spacetime structure [31, 32, 21]—to which we will return, in a bit more detail, in section 4. The general quantum set theory project was originally conceived by von Neumann [7] and has been significantly developed over the years along Grassmann and Clifford algebraic lines by Finkelstein and coworkers. For the latest word from that research front, refer to [20, 62]. Of course, another possibility would be to formulate directly a ‘quantum topos’—a universe which would be a quantum version of $\mathsf{Set}$ thus it would provide a natural habitat for quantum sets. The search for such a quantum topos structure has been very broad and diverse [7, 8, 13, 14, 17, 51, 54, 55, 56, 57, 58], thus we will touch it only peripherally, and from a CH-theoretic point of view, in sections 4 and 5.
about history propositions that are going to be quoted and used below. We extend the topos $\mathcal{T}_{CH}$ in [34] by providing a Vietoris-type of topology to the second-level history propositions dwelling in its presheaf objects $\text{Set}^B$ by using the notion of sets ‘trapped’ in $B$. Subsequently, this ‘topologization of histories’ will enable us to sheafify the objects of $\mathcal{T}_{CH}$.

### 3.1 The abstract Vietoris topology

First, we give a short and watered down exposition of the abstract Vietoris topology that one can give to the collection $\mathcal{C}(X)$ of closed subsets $C$ of a topological space $X$. More details may be found in [44, 3]. So, let $X$ be a topological space. With respect to any open subset $U$ of $X$ one can define:

- (i) **The ‘nerve’ of $U$ in $\mathcal{C}(X)$:**

$$U_{\mathcal{C}(X)}^\cap := \{ C \in \mathcal{C}(X) : C \cap U \neq \emptyset \}$$  \hspace{1cm} (2)

- (ii) **The ‘member’ of $U$ in $\mathcal{C}(X)$:**

$$U_{\mathcal{C}(X)}^\subseteq := \{ C \in \mathcal{C}(X) : C \subseteq U \}$$  \hspace{1cm} (3)

With these definitions of nerve (2) and member (3), the Vietoris topology on $\mathcal{C}(X)$ is defined as the one generated by all basic sets of the form $U_{\mathcal{C}(X)}^\cap$ and $U_{\mathcal{C}(X)}^\subseteq$. We readily apply this abstract definition to consistent-histories next.

### 3.2 A Vietoris-type of topology for consistent-histories

The seed for the idea to endow consistent-histories with a Vietoris-like topology can again be found in [34]. As the principal motivation for considering the Vietoris topologization of consistent-histories one may regard Isham’s observation that the collection of second-level semantic values $\mathcal{V}<a,p>$, relative to a chosen decoherence functional $d$, at an object (stage of truth) $W_0$ in $B$ of the form

37Strictly speaking, the collection of all nerves and members provides a *sub-basis* for the Vietoris topology on $\mathcal{C}(X)$, not a generating set (ie, a basis) proper [44, 3].
\[ \mathfrak{M}_{W_0}(<a,p>) := \begin{cases} \{W \subseteq W_0 : W \in \mathcal{B}^d \text{ and } a \in W\} & \text{if } d(a,a) = p \\ \emptyset & \text{otherwise} \end{cases} \] (4)

do not form a logic algebra since the right hand side of the defining equation (4) is not a sieve\(^{38}\). The result then is that one cannot identify the subobject classifier \( \Omega \) in the topos \( \mathcal{T}_{CH} \) of presheaves of varying sets over \( \mathcal{B}^d \) with the Heyting logic algebra of the collection of all sieves localized at the truth stage \( W_0 \), as we mentioned in (b)-(d) of the previous section. In fact, at first sight one feels that one cannot apply at all the theory of varying sets \(^{38,34}\), thus \textit{a fortiori} one cannot view \( \mathcal{T}_{CH} \) as a topos of presheaves of sets varying over \( \mathcal{B}^d \), once the coarse-graining sieve structure and the intuitionistic logic calculus of all such sieves breaks down object-wise (\( \text{ie, locally} \) in the base poset category \( \mathcal{B}^d \). On the other hand, Isham points out that “\( \ldots \text{in itself this does not rule out the use of (4), but} \) it implies that any logical structure on the set of semantic values must be obtained in a way that is different from our anticipated use of the topos of varying sets \( \textbf{Set}^\mathcal{B} \). One possibility is...” to exploit the notion of ‘trapped sets’. We do this now.

One may observe, as Isham did in \(^{34}\), that second-level semantic values of the sort defined by expression (4) do not form a logic algebra, because they do not close algebraically under set-theoretic union\(^{40}\). To actually close them one may define finite sets \( F \) of history propositions that are ‘trapped’ in Boolean subalgebras \( W \) of \( \mathcal{U}\mathcal{P} \) in \( \mathcal{B}^d \) which coarse-grain a particular stage of truth \( W_0 \), as follows:

\[ \mathfrak{T}_F^{d}(W_0) := \{W \subseteq W_0 : W \in \mathcal{B}^d \text{ and } F \cap W \neq \emptyset\} \] (5)

where, plainly from (4), \( \mathfrak{M}_a(W_0) = \mathfrak{T}_F^{d}(W_0)|_{F = \{a\}} \).

The interesting feature of such trapped sets is that although they close under set-theoretic union, they do not under intersection. To establish \( \cap \)-closure one can consider the nerves of a collection \( F \) of finite sets \( F \) of history propositions in \( \mathcal{U}\mathcal{P} \) relative to the coarse-grainings of \( W_0 \) in \( \mathcal{B}^d \), as defined below:

\[ \mathfrak{T}_{F = \{F_1:F_2: \ldots :F_n\}}^{d}(W_0) := \{W \subseteq W_0 : W \in \mathcal{B}^d \text{ and } (F_1 \cap W \neq \emptyset \& F_2 \cap W \neq \emptyset \& \ldots \& F_n \cap W \neq \emptyset)\} \] (6)

\(^{38}\)To convince oneself of this fact, refer to \(^{34}\). We also note that \( \mathcal{B}^d \) in \(^{3}\) is the poset category on non-trivial Boolean subalgebras of \( \mathcal{U}\mathcal{P} \) metrized by the \( \mathbb{C} \)-valued probability measure \( d \).

\(^{39}\)That is, losing the sieve structure object-wise in \( \mathcal{B}^d \).

\(^{40}\)See (A.2) in appendix A of \(^{34}\).
The alert reader may have directly noticed in connection with (6) that some kind of Vietoris topology could be imposed on consistent-histories in view of this expression’s striking formal similarity with the nerve (3) and member (6) expressions defining the abstract Vietoris topology on $\mathcal{C}(X)$. This is indeed so: one may define a Vietoris-type of topology on $\mathcal{B}^d$ in $\mathcal{U} \mathcal{P}$ by taking as sub-basis the collection of all sets of the form $\mathcal{T}_F(W)$ (\(\forall W \in \mathcal{B}^d\)) as $F$ ranges over all finite subsets of $\mathcal{U} \mathcal{P}$. Let us symbolize this topology by $\mathcal{V}^d$. We have thus effectively topologized the base poset category $\mathcal{B}^d$ and, as Isham remarks in appendix A of [34], the topological space $(\mathcal{B}^d, \mathcal{V}^d)$ may be regarded as the truth or semantic value space for consistent-history propositions—the range of valuations localized or contextualized on the Boolean windows $W$ of $\mathcal{U} \mathcal{P}$.

Of course, this Vietoris-type of topology $\mathcal{V}^d$ assigned on $\mathcal{B}^d$, although it is not the same Heyting logic algebra as in the case of the collection of all coarse-graining sieves on $W_0$ which characterizes the neorealist topos $\mathcal{T}_{CH}$ of varying sets proper in [34], it still qualifies as a perfectly legitimate example of an abstract (open set) topology—a complete distributive lattice (of open subsets of the topological space $(\mathcal{B}^d, \mathcal{V}^d)$) commonly known as a locale [39, 41]. Thus, by distorting a bit one’s point of view, one can still think (perhaps in an oblique sense) of $\mathcal{T}_{CH} \equiv \text{Set}^{\mathcal{B}^d}$ as the topos of presheaves of sets varying over the poset category $\mathcal{B}^d$—with this base space now having been suitably topologized by $\mathcal{V}^d$. However, for the sake of consistency, accuracy and clarity we must define presheaves of sets over trapped sets of history propositions

$$\text{T}_{\text{re}} : (\mathcal{B}^d, \mathcal{V}^d) \equiv \mathcal{L}_{CH} \longrightarrow \text{Set}$$  \hspace{1cm} (7)

in complete analogy to the presheaf objects in $\mathcal{T}_{CH}$

$$\text{Pre} : (\mathcal{B}^d) \longrightarrow \text{Set}$$  \hspace{1cm} (8)

where in (7) the base topological space $(\mathcal{B}^d, \mathcal{V}^d)$ is identified with the aforementioned locale $\mathcal{L}_{CH}$ of its open subsets [4]. The collection of the presheaves $\text{T}_{\text{re}}$ over the locale $\mathcal{L}_{CH}$ is another example of an abstract topos structure [39], which we may symbolize by $\mathcal{T}_{\text{re}}$. \mathcal{T}_{\text{re}}, in complete analogy with $\mathcal{T}_{CH}$, may be thought of as universe of

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[41] Interestingly enough, and from a logic perspective, a complete distributive lattice is also known as a complete Heyting algebra [39]. The consistent-history topos $\mathcal{T}_{CH}$ is, topologically speaking, ‘locally localic’; while, logically speaking, its internal logic is ‘locally Heyting’ (i.e., neorealist) [39, 42, 43].

[42] Note also in connection with (6) that a nickname for ‘presheaves over trapped sets’ can be ‘tresheaves’, hence the symbol $\text{T}_{\text{re}}$.

[43] Quite reasonably, we think, this ‘topos of tresheaves’ may be coined ‘tropos’ (gk. for ‘manner’ or ‘idiosyncracy’). This seems to be a suitable name for the topos in focus in view of the peculiar character of the unusual Vietoris topology carried by trapped sets in its tresheaf objects.
sets varying over the locale $\mathcal{L}_{CH}$, hence its internal logic too is neorealist in the sense of Isham [34].

The bonus from working in $\mathcal{F}_{CH}$ rather than in $\mathcal{T}_{CH}$ is that having a sound background topological space $\mathcal{L}_{CH}$ over which sets vary in the tropos, we can sheafify its presheaf objects relative to the Vietoris topology $V^d$, that is to say, we can promote the contravariant functors (i.e., presheaves [4, 33, 40, 52]) in (7) to 'local homeomorphisms' (i.e., sheaves [4, 33, 40, 52]) between the base topological space $\mathcal{L}_{CH}$ and the fiber or stalk space $\text{Set}$. We do this in the next subsection.

### 3.3 Sheafifying consistent-histories

Our sheafification of the presheaves in (7) will be rather swift. The procedure is quite a standard one and can be found in more detail in [4, 33, 10, 33]. First we present the general case of presheaves of functions over a topological space $X$, then we particularize it to our case of tresheaves of sets over the locale $\mathcal{L}_{CH}$ in $\mathcal{T}_{CH}$.

Initially, we note that presheaf maps such as the ones in (7) and (8) are assignments to each open subset $U$ in a topological space $X$ of function-like objects of the form $S : U \to S(U)$, and to each pair $(U, V)$ of open subsets in $X$ nested by (strict) inclusion (i.e., $U \subset V$) of a so-called restriction map $\rho_{UV} : S(V) \to S(U)$, subject to the following conditions:

- (a) Identity: $\rho_{UU} = \text{Id}$.
- (b) Composition: $\rho_{UV} \circ \rho_{VW} = \rho_{UW}$ ($U \subset V \subset W$).

Thus, as it was mentioned above, this general definition of presheaves prompts one to think of them as a collection of functions on the open subsets of a topological space equipped with restriction maps between them when their open set domains in $X$ are nested by inclusion. In fact, one can construct a topological space $S$—the so-called

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44 In other words, the non-existence of global sections of valuations (i.e., the localization of semantic values or truth) in $\mathcal{T}_{CH}$ carries through to $\mathcal{F}_{CH}$ so that localization maps à la (1) occur in the latter’s tresheaves, although, as noted above, in our tropos the subobject classifier is a complete Heyting algebra or locale different from the $\Omega$ of the coarse-graining sieves in (1).

45 For the Tres and Pres in (7) and (8) the objects assigned are sets in the $\subseteq$-poset category $\text{Set}$.

46 From this standard definition of presheaves one can see clearly why they are called contravariant functors: the direction of $\subseteq$-arrows in the base or source poset category $X$ (i.e., when the topological space $X$ is regarded as the locale of its open subsets) is reversed in the target poset category by the presheaf maps.

47 As it was mentioned before, for the tresheaves in $\mathcal{F}_{CH}$ these functions-like objects are just structureless sets in $\text{Set}$, but let us present the general case first.
sheaf space \[\text{sheaf space}\]
—starting from a presheaf of functions on a given topological base space \(X\). Let us recall briefly this well known construction which is commonly known as sheafification.

First, for every open subset \(U\) in the background topological space \(X\) define the so-called “sections’ selection map” \(\sigma_U\) from the set of presheaf functions \(S(U)\) to a family \(\Gamma\) of continuous functions on \(U\): \(\sigma_U : S(U) \rightarrow \Gamma(U,S)\).

The elements of \(\Gamma(U,S)\) are called the continuous sections of \(S\) over \(U\).

Second, define point-wise in \(X\) (ie, for all \(x \in X\)) the stalks (or fibers) \(S_x\) of the sheaf space \(S\) as direct or inductive limits of the \(S(U)\) presheaf maps above in the following way:

\[
S_x := \lim_{U \in B(x)} \{S(U) : x \in U\} \equiv \bigcup\{S(U) : x \in U\}/\tilde{x}
\]

where \(\tilde{x}\) is the following equivalence relation between the functions in the \(S(U)\)s:

\[
f \sim g \iff \rho_{W,U \cap V}(f) = \rho_{W,U \cap V}(g), \ (f \in S(U), g \in S(V))
\]

for some open neighborhood \(W\) of \(x\) in the ‘nerve’ of \(U\) and \(V\) (ie, for \(W \subset U \cap V\)).

As a non-topologized set, the sheaf space is the disjoint union or direct sum of its stalks: \(S = \bigcup_x S_x\).

Third, we topologize \(S\) as follows: define the germ of \(f\) at \(x\), with \(f \in S(U)\) and \(x \in U\), to be the \(\tilde{x}\)-equivalence class of \(f\), and symbolize it by \([f]_x\). Then, as a basis for the topology on \(S\) we take the following family of open subsets:

\[
\mathfrak{B}[S(X)] := \{(x, [f]_x) : x \in U\}
\]

A continuous section in \(\Gamma(U,S)\) can then be defined relative to this basis as:

\[
\sigma_U(f)(x) = [f]_x \ (x \in U)
\]

and it is plain to see that the germs of \(S\)’s continuous sections dwell in its stalks, that is to say, \([f]_x \in S_x\). In fact, in this construction of the topology on the sheaf space

\[\sigma_U\] is assumed to commute with the \(\rho\)s in (a) and (b) above.

Refer to \[\] for a definition of inductive systems of maps and their direct limits.

In (9), it is supposed that \(U\) varies over a basis \(\mathfrak{B}(x)\) of open neighborhoods of \(x\), while the maps in \(S(U)\) constitute an inductive system of maps.
relative to the base topological space \( X \), one can easily verify that the function \( \pi : S \rightarrow X \)—called ‘the projection of the sheaf space on the base space’—is a \textit{local homeomorphism} \([9, 39, 40, 52]\) acting on the basic open sets of \( B \) in \((11)\) as:

\[
\pi(x, [f]_x) = x\tag{13}
\]

By a \textit{sheaf} one understands in general such a \textit{local homeomorphism} \(51\).

The stalks \( S_x \) of the sheaf carry the discrete topology \(52\), but as a topological space proper it is generated by the germs of its continuous sections \([f]_x\) inhabiting these very stalks. This is a well known \textit{cliché} in sheaf theory, namely, that a \textit{sheaf} is (generated by the germs of) \textit{its continuous sections} \([9, 39, 40, 41]\).

So this is how a sheaf \( S \) arises from or is generated by a presheaf \( S \) on a topological space \( X \). In fact, one can go the other way around and note that the maps \( U \rightarrow \Gamma(U, S) \) constitute a presheaf that satisfies the following ‘collation’ properties:

- (a) If \( U \) is covered by a family \( \{U_i\} \) of open subsets (\textit{ie}, \( U = \bigcup U_i \)) and \( s_1, s_2 \) are sections in \( \Gamma(U, S) \) such that \( s_1|_{U_i} = s_2|_{U_i} \) (\( \forall i \)), then \( s_1 = s_2 \).
- (b) Let \( \{U_i\} \) be as above. If \( s_i \in \Gamma(U_i, S) \) satisfy \( s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \) (\( \forall i, j \)), then there is an element \( s \in \Gamma(U, S) \) such that \( s|_{U_i} = s_i \) for each \( i \).

Supposing that the presheaf \( S : U \rightarrow S(U) \), subject to the usual \( \rho \)-restrictions as before, satisfies these two glueing properties, one can show that the selection maps \( \sigma_U \) are in fact isomorphisms. That is to say, any presheaf satisfying (a) and (b) above can be obtained as the presheaf of continuous sections of a sheaf. Thus a sheaf may be reconstructed from its presheaf of sections—the aforementioned \textit{cliché} vindicated.

As a matter of fact, these two procedures opposite to each other, namely, sheafification of a (complete) presheaf \(53\) and ‘pre-sheafification’ from the continuous sections of a (spatial) sheaf \(54\) are functors adjoint to each other \(55\) denoted by \( S \) and \( \Gamma \), respectively \([40, 41]\).

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51 Another, perhaps physically more intuitive, way to say this is that the base topological space \( X \) and the overlying sheaf space \( S \) are locally (\textit{ie}, \( U \)-wise in \( X \)) topologically equivalent or indistinguishable and the sheaf \( \pi \) implements this equivalence or indistinguishability \([4, 15]\). I wish to thank Tasos Mallios for bringing to my attention this (very physically-minded) definition of a sheaf originally due to Lazard \([4, 11]\).

52 This is another way of saying, as above, that as a non-topologized set \( S = \bigoplus_{x \in X} S_x \).

53 See \([33, 40, 41]\) for a definition of complete presheaves.

54 Again, see \([39, 40, 41]\) for a definition of spatial sheaves.

55 See \([33]\) for a definition of adjoint functors.
This exposition of the general sheafification procedure suffices for our intention to sheafify the particular tresheaves of sets in (\ref{34}). To this end, we make the following identifications:

- (i) The base topological space in our case is the locale $L_{CH}$ of open subsets of the Vietoris topological space $(B^d, V^d)$ of trapped sets.

- (ii) The target category (ie, the range of the presheaf maps $\text{Tre}(U); U \in L_{CH}$) is Set—the poset category of structureless sets ordered by set-theoretic inclusion.

- (iii) The sheaf space we will call $S_{CH}$, while the sheaves resulting from the sheafification functor $S$ on the $\text{Tres}, S_{CH}(L_{CH})$.

- (iv) The basic open sets generating the topology on $S_{CH}$ are of the form $(a, [s]_a)$, where $a$ is just a singleton trapped set in $L_{CH}$ (arguably, of point-like or ‘atomic’ character!) and $[s]_a$ the fiber over it consisting of the $\tilde{a}$-equivalence classes of sets in Set relative to the atomic history proposition $a$, much like (\ref{9}) expressed in the general case of functions over the point-sets of $X$ rather than structureless sets. In connection with (\ref{9}), we also note that the basic sets covering $a$ are taken, of course, from nerves and members in the sub-basis of $(B^d, V^d)$ that trap $a$ (ie, $T^d_{\mathcal{F}={\{a\}}}$).

Applying the aforementioned sheafification functor $S$, we have thus effectively obtained sheaves of sets varying continuously over the Vietoris-topologized poset category $B^d$ without making use of the latter’s coarse-graining poset structure and its associated local sieve-valued logical semantics. As a result, the collection $\mathcal{T}'_{CH} := \{S_{CH}(L_{CH})\}_{\sigma}$ of sheaves of sets over $L_{CH}$ and sheaf morphisms between them is a topos whose internal logic is inevitably intuitionistic, but not identical to the neorealist internal logic proper of $\mathcal{T}_{CH}$ in (\ref{34}), as we contended earlier.

This concludes our presentation of sheafifying consistent-history propositions in $\mathcal{UP}$, thus extending (\ref{34}) at the base front. The next ‘reasonable’ thing that one could do is to endow the stalks of the $S_{CH}(L_{CH})$s in $\mathcal{T}'_{CH}$ with more algebraic structure, thus further extend Isham’s topos $\mathcal{T}_{CH}$ in (\ref{34}) even at the stalk front. This is what we do in the next section. Before we do that, in the next subsection we discuss the physicality of the Vietoris topology on trapped sets and the sheafification process associated with it.

\footnote{Not necessarily an atomic proposition in the universal ortholattice $\mathcal{UP}$.}
\footnote{The sets belonging to the equivalence class $[s]_a$ are ‘extensionally equal’ (ie, with respect to set-theoretic equality).}
\footnote{The superscript ‘$\sigma$’ over $\mathcal{T}_{CH}$ indicating ‘sheafification’ of the latter’s tresheaf objects to sheaves.
4 Algebraizing the stalks: the quantum causal histories topos

In the present section we extend Isham’s work \[34\] at the stalk front as mentioned earlier by assigning more algebraic structure to the stalks of the sheaf objects $\mathcal{G}_{CH}(\mathcal{L}_{CH})$ of the tropos $\mathcal{T}_{CH}$, which stalks have so far been assumed to be occupied by structureless sets. Again, the procedure is quite a standard one: all that one has to make sure is, loosely speaking, that the additional algebraic structure employed in the fibers is compatible with or respects, locally at least, the ‘horizontal’ continuity of the base topological space—its local topology so to speak—as it may, to preserve the sheaf structure.\(^{59}\) Again, as we did for the sheafification of presheaves in the previous section, first we describe briefly the general procedure, which one may call sheaf-algebraization and can be found in more detail in \[3, 33, 39, 40\], then we specify the algebraic structures added to be the finite dimensional incidence Rota algebras modeling quantum causal sets (qausets) in \[12, 13, 24\]. Subsequently, we define the aforementioned Quantum Causal Histories Topos (QCHT) to be $\mathcal{T}^\alpha_{CH}$ and we discuss briefly its affinities with Markopoulou’s quantum causal histories scenario for quantum spacetime structure and gravity advocated in \[13\]. We also find some suggestive similarities with the curved finitary spacetime sheaves of qausets proposed in \[12\] as a locally finite, causal and quantal model of (the kinematics of) the ever elusive Lorentzian quantum gravity, as well as with this model’s non-commutative or quantum topological traits detected in \[53\].

4.1 Rota-algebraizing the $\mathcal{G}_{CH}$

So, first we present the general sheaf-algebraization procedure à la Mallios \[10, 11\]: the additional algebraic structures most commonly given to the stalks of a sheaf of structureless sets are $\mathbb{C}$-algebras or modules over such algebras. The most elementary example is that of a sheaf of (abelian) groups (i.e., a group sheaf) $\mathcal{G}$ on a topological space $X$ whose stalks $\mathcal{G}_x$ are groups so that the (commutative) group operation, usually denoted by ‘$+$’, is continuous in the following sense: defining the ‘fiber product’ $\circ$ to be

$$\mathcal{G} \times_X \mathcal{G} := \{(g, g') \in \mathcal{G} \times \mathcal{G} : \pi(g) = \pi(g')\} \equiv \mathcal{G} \circ \mathcal{G}$$  \hspace{1cm} (14)$$

Another way to say this is that the extra algebraic operations defined stalk-wise in the sheaves under focus should be continuous.\(^{59}\)

The superscript ‘$\alpha$’ added to $\mathcal{T}^\alpha_{CH}$ indicating now the Rota-algebraization of the set-inhabited stalks of the tropos’ sheaf objects $\mathcal{G}_{CH}(\mathcal{L}_{CH})$. For the latter we also write $\mathcal{G}^\alpha(\mathcal{L}_{CH})$.\(^{60}\)
the map

\[ \mathcal{G} \circ \mathcal{G} \ni (g, g') \mapsto g + g' \in \mathcal{G}_x \subset \mathcal{G} \ (\pi(g) = \pi(g') = x \in X) \quad (15) \]

is continuous. Moreover, one can prove that the unary operation of inverting the group elements stalk-wise (i.e., \( g \mapsto g^{-1} \equiv -g \), \( g \in \mathcal{G}_x \)), hence of subtracting elements fiber-wise (i.e., \( g_x' + g_x^{-1} \equiv g_x' - g_x \), are also continuous in the manner above. Concomitantly, the group’s neutral element 0 (i.e., \( 0_x = g_x + g_x^{-1} \), \( \forall g_x \in \mathcal{G}_x, \ \forall x \in X \)) is defined to be a global continuous section of \( \mathcal{G} \).

Similarly to the definition of abelian group sheaves \( \mathcal{G} \), one can define (unital) ring sheaves \( \mathcal{R} \), \( k \)-algebra sheaves \( \mathcal{A} \) (\( k = \mathbb{R} \) or \( \mathbb{C} \)), as well as sheaves \( \mathcal{M} \) of modules over such \( k \)-algebras (\( k \)-module sheaves) by appropriately making sure that the extra structures imposed are continuous stalk-wise in the respective sheaves.

Particularizing the general sheaf-algebraization technique above to our case of interest, we assume that the stalks of the sheaf objects \( \mathcal{G}(\mathcal{L}_{CH}) \) in the topos \( \mathcal{T}_{CH}^{\varphi_0} \) are occupied not by sets, but by finite dimensional non-abelian incidence Rota \( \mathcal{C} \)-algebras \( \vec{\Omega} \) representing quasets. The resulting structures are sheaves \( \mathcal{G}_{CH}(\mathcal{L}_{CH}) \) of quasets over the Vietoris-topologized trapped sets of consistent-history propositions—in brief, ‘sheaves of consistent-histories of quasets.’ The topos structure having as objects these sheaves and as arrows sheaf morphisms between them is called the ‘Quantum Causal Histories Topos’ and is abbreviated by the name’s initials (QCHT). Like the general paradigm of a topos of sheaves of rings or algebras over a topological space \( X \) or a locale \( \mathcal{L} \) can be interpreted as a mathematical universe of continuously variable rings or algebras varying with respect to the background ‘parameter space’ \( X \) or \( \mathcal{L} \), so the quasets inhabiting the stalks of the sheaves in the QCHT may be viewed as variable objects varying (continuously) relative to the

\[ \text{Not insisting that the algebraic product is necessarily commutative. In fact, we will see shortly that the particular sheaf-algebraization of interest to us here will employ non-commutative rings and non-abelian \( \mathcal{C} \)-algebras.} \]

\[ \text{For instance, the rings’ multiplication unit 1 defines a continuous global section of } \mathcal{R}, \text{ while } k\text{-scalar multiplication is continuous in } \mathcal{A}. \]

\[ \text{In fact, since the incidence Rota algebras } \vec{\Omega} \text{ modeling quasets are } \mathbb{Z}\text{-graded } \mathbb{C}\text{-modules of ‘discrete differentials’ over their commutative subalgebras } \vec{\Omega}^0 \text{ of point-like ‘stationaries’ } [55, 51, 12, 60], \text{ their sheaves are } \mathcal{M}\text{-sheaves in the sense above. This observation, namely, that the } \mathcal{M}\text{-sheaves of } \vec{\Omega} \text{ support discrete differential calculi and a discrete Riemannian geometry } \text{à la } \text{Dimakis et al.} [18, 17] \text{ over consistent-histories, could prompt one to apply in this direction some very general, but powerful, concepts, results and techniques from Mallios’ Abstract Differential Geometry on Vector and Algebra Sheaves [10, 11] to the more conrete task of applying CH-theoretic ideas to quantum gravity. Such a possibility is roughly sketched in the subsection 4.3 below, as well as in the concluding section.} \]
Vietoris topology carried by the base poset category $B^d$ in the universal consistent-histories orthoalgebra $\mathcal{UP}$.

We strongly feel that the ultimate challenge for physics is to find a plausible dynamics that ‘quantifies’ this topos-variability of qausets, so as to qualify the topos-theoretic perspective to some kind of algebraic quantum gravity model proper \cite{42, 53}. From this perspective one may perhaps get a clearer view of the supposedly central role that the CH theory plays in our quest for a cogent quantum theory of gravity. Unfortunately, in this paper we will not go as far as to give an explicit dynamics for qausets in their QCHT. Rather, we are going to content ourselves with drawing close connections between the QCHT and two recently proposed models of the kinematics of a finitary (ie, locally finite), causal and quantal version of (Lorentzian) gravity in \cite{43} and \cite{42}. If anything, these connections will give us hints of how to develop further, and hopefully in the immediate future, the research program of applying ideas from the CH theory to the problem of quantum gravity by sheaf and topos-theoretic means.

### 4.2 Affinities between the QCHT and quantum causal histories

Our brief comparison of the QCHT and Markopoulou’s quantum causal histories scenario in \cite{43} is centered around the observation that in the latter the base poset category on whose objects (vertices) finite dimensional Hilbert spaces $\mathcal{H}$ (of the same dimensionality)—realms in which states of a quantum system of a finite number of ‘degrees of freedom’ (presumably, spacetime) are supposed to live—are localized, is taken to be a causal set (causet) $\vec{P}$ in the sense of Sorkin et al. \cite{6, 63, 64, 66, 67, 68}. From a finite number of Hilbert spaces soldered on a finite number of a causally (or ‘space-like’) separated event-vertices in $\vec{P}$, tensor-product ‘compound’ Hilbert spaces were then formed as befits the $\mathcal{H}$-representation theory of the CH formalism \cite{33}; moreover, unitary maps (modeling transitions) between such tensor product spaces were defined in a way that respects the reflexivity, antisymmetry and transitivity properties of the base poset $\vec{P}$. Thus, quantum causal histories and their unitary transformation theory were born and were held as sound unifications of the basic ideas of causet theory and the CH theory. The important thing to notice in this scenario is that the abstract temporal support on which consistent-histories are localized and relative to which they are supposed to vary, is provided by the finitary poset $\vec{P}$—

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\textsuperscript{64} In a nutshell, a causet is a locally finite poset.

\textsuperscript{65} That is to say, in a way that respects the causal topology of the causet base space $\vec{P}$ \cite{2, 23}. 
needless to point out, as Markopoulou already did, that the collection of all ‘causal future sieves’ based on any vertex of \( \vec{P} \) form a Heyting algebra or locale.

Similarly, in our QCHT the abstract temporal background on which quasets are localized and with respect to which they are supposed to vary, is also a poset, namely, the complete distributive lattice \( \mathcal{L}_{CH} \), so that the QCHT itself is ‘locally localic’ (or logically speaking, ‘neorealist’). The similarity becomes even more prominent if one decides to consider finite dimensional Hilbert space representations associated with the finite dimensional incidence Rota algebras dwelling in the stalks of the sheaf objects \( \mathcal{S}^\alpha_{CH}(\mathcal{L}_{CH}) \) in the QCHT. The bundles associated with the \( \mathcal{S}^\alpha_{CH}(\mathcal{L}_{CH}) \)'s are then such finite dimensional \( \mathcal{H} \)-vector sheaves in the sense of Mallios. Unitary-like transitions between the \( \mathcal{H} \)-stalks of these associated \( \mathcal{H} \)-sheaves are then induced by geometric morphisms on the QCHT.

On the other hand, however, there is prima facie a significant obstacle in carrying further this analogy between the QCHT and Markopoulou’s quantum causal histories. If one decides to make use of the whole tensor product panoply of the CH theory underlying (ie, providing the base space for) the sheaf objects \( \mathcal{S}^\alpha_{CH}(\mathcal{L}_{CH}) \) of the QCHT, one is bound to encounter the following rather subtle technical difficulty. With the vector \( \mathcal{H} \)-sheaves associated to the \( \mathcal{S}^\alpha_{CH}(\mathcal{L}_{CH}) \)'s a rather undesirable ‘stalk-collapse phenomenon’ is observed whereby sections of two distinct \( \mathcal{H} \)-stalks over two distinct ‘atomic’ history propositions (ie, \( \mathcal{H}_{a_1} \) and \( \mathcal{H}_{a_2} \)) ‘merge’ or ‘collapse’ into a single stalk (now in the tensor product sheaf \( \bigotimes \mathcal{H}_i \) over \( \mathcal{L}_{CH} \)) when the underlying propositions tensor combine with each other as \( a_1 \otimes a_2 \). Since in the quantal logic of the CH theory \( \otimes \) represents the phenomenon of quantum entanglement or quantum coherence, one may infer from the aforementioned stalk-collapse phenomenon that the usual ‘classical’ tensor product structure is rather inadequate for representing the purely quantum behavior of entanglement, at least in a sheaf-theoretic context. Indeed, one may get a stronger feeling for the pathological character of this stalk-collapse if one assumes that the associated consistent-histories \( \mathcal{H} \)-sheaves are soldered on the points

\[ \text{Such finite dimensional Hilbert space } \mathcal{H} \text{ matrix representations were studied in [73].} \]

\[ \text{See [40] for a general definition of associated sheaves to vector, algebra and, more importantly, principal } \mathcal{G} \text{-sheaves. In the case of the } \mathcal{H} \text{-vector sheaves associated with the } \mathcal{S}^\alpha_{CH}(\mathcal{L}_{CH}) \text{’s in the QCHT, their sections represent generalized quantum states.} \]

\[ \text{One may recall that, in general, with any bijective lattice morphism } f : \mathcal{L} \rightarrow \mathcal{L}' \text{ between two locales (which } f \text{ is, in effect, a homeomorphism between these two abstract pointless topological spaces), there is associated a pair of adjoint functors—the so-called ‘pushout’ } f_* : \text{Sh}(\mathcal{L}) \rightarrow \text{Sh}(\mathcal{L}') \text{ and ‘pullback’ } f^* : \text{Sh}(\mathcal{L}') \rightarrow \text{Sh}(\mathcal{L}) \text{—between the respective categories or topos of sheaves (of any algebraic structures) over them [39]. Within the particular QCHT, such functor pairs } (f_*, f^*) \text{—commonly known as geometric morphisms—are induced by elements } f \text{ of the group Aut}(\mathcal{L}_{CH}) \text{ of automorphisms of } \mathcal{L}_{CH}. \]
of a classical continuous spacetime manifold $M$ (or even on the pointless locale $\mathcal{L}_M$ of open subsets about them, but this example is not as clear as in the case of sheaves on the pointed $M$).

In such a hypothetical model of consistent-histories of qausets varying continuously over a background (possibly curved) spacetime continuum $M$, it is easy to see (at least from a more heuristic and physical point of view) why the tensor product $\otimes$ and the ‘classical’ definition of a sheaf do not seem to go hand in hand: when one considers the tensor product of two distinct stalks in a vector sheaf like the associated $\mathcal{H}(M)$, as when one combines two distinct quanta in the usual quantum theory, the two stalks ‘collapse’ to a tensor product stalk over a single spacetime point-event of the classical base spacetime manifold $M$. This phenomenon is characteristic in both classical and quantum field theories where, when we coherently combine or entangle systems by tensor multiplication, their spacetime coordinates combine by identification. “This mathematical practice expresses a certain physical practice: to learn the time, we do not look at the system but at the sun (or nowadays) at the laboratory clock, both prominent parts of the episystem” [20], and it should be emphasized that the episystem is always regarded as being classical in the sense of Bohr. All in all, this stalk-collapse pathology is begging for a radical revision of ‘classical’ tensor product $\mathcal{H}$-sheaves over classical topological spaces in the sense that we should search for a new ‘quantum tensor product’ structure $\otimes_q$ that soundly represents quantum entanglement and at the same time it evades the stalk-collapse observed in classical tensor product sheaves over classical pointed topological spaces (or even over their pointless locales).

Interestingly enough, and closely related to the quest for the $\otimes_q$ above, current researchers in quantum logic proper as well as in non-commutative or quantal generalizations of classical topological spaces (locales), are also looking for a similar quantal tensor product-like structure, which is non-commutative but distributes over the ‘$qor$’

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69 Such an assumption to use the spacetime manifold $M$ as a base space would suit, for instance, a possible continuous spacetime consistent-histories theory with an eye towards applying ideas from the CH theory to the classical and, hopefully, to the ever elusive quantum theory of gravity.

70 Now, generalized quantum states of qausets are represented by $\mathcal{H}(M)$’s continuous sections.

71 Here, the classical base spacetime manifold $M$.

72 Or even sheaves of tensor products of general Banach spaces.

73 This symbol should not be confused with the one commonly used for the $q$-deformed group product of Hopf algebras and related quantum groups.

74 Inevitably, this quantal version of the classical tensor product structure is expected to be accompanied by a quantum revision of ‘classical’ sheaves and of the classical spacetime manifold topology on which these are defined (see the following paragraph and subsection). I wish to thank Chris Isham for a timely exchange on precisely such a possibility of a new ‘quantum tensor product’ structure.
connective of the usual quantum logic \( \otimes \), as it may, to replace the commutative, but not distributive over \( qor \), ‘\( q\text{and} \)’ operation of quantum logic\(^{75} \). People have looked for such a \( \otimes_q \) structure in Girard’s linear logic\(^{76} \) and in Yetter’s non-commutative version of it\(^{77} \), as well as in Mulvey’s quantale ‘\( \text{and} \)’ or non-commutative \( \cap \)-like operation ‘\&’\(^{45, 46} \). This author too has entertained the possibility for an analogous \( \otimes_q \)-connective in his endeavor to ‘localize non-commutatively’ in the context of quantum gravity, that is to say, to develop a non-commutative topology and its associated sheaf or scheme theory\(^{77} \) for the small-scale structure of spacetime\(^{53, 54} \) mainly motivated by the curved finitary spacetime sheaves (finsheaves) of qausets theme in\(^{42} \). The search for the ‘right’ \( \otimes_q \) structure continues to stimulate quantum logicians and non-commutative topologists alike.

### 4.3 Affinities between the QCHT and curved finsheaves of qausets

The concluding words in the last subsection prompt us to present briefly some analogies between the QCHT, the curved finsheaves of qausets in\(^{42} \), the latter’s non-commutative topology suggested in\(^{53} \) and the related idea of a quantum topos for quantum gravity entertained in\(^{54} \). More details about all these ideas can be found in the corresponding citations and the references therein.

In\(^{42} \), curved principal \( G \)-finsheaves\(^{52} \) of qausets\(^{51} \) were proposed as reticular, causal and quantal replacements of the curved Lorentzian spacetime manifold \( M \) of general relativity—classical gravity’s kinematical structure. As base spaces for these finsheaves, causets \( \vec{P} \) were assumed in a manner similar to the \( H \)-localization spaces in the quantum causal histories scenario of Markopoulou that we briefly encountered above. These finsheaves were subsequently subjected to a ‘classicalization coarse-graining’ procedure in the sense that an inverse system or net (\( ie \), coarse-graining poset category \( \otimes \) in\(^{34} \)) consisting of finer-and-finer such finsheaves possessed at the limit of infinite localization or resolution or refinement (of spacetime into its point-events\(^{43, 22} \)) a limit \( G \)-sheaf isomorphic to the spin-Lorentzian principal fiber bundle classical gravity\(^{77} \). In turn, this inverse limit localization procedure was physically interpreted as Bohr’s correspondence principle in a way originally proposed in the

\(^{75}\)Jim Lambek in private communication.

\(^{76}\)I wish to thank Steve Selesnick for bringing to my attention Girard’s work and Mulvey’s original quantale paper \( 15 \).

\(^{77}\)This is just a \( G \)-bundle with structure group \( SL(2, \mathbb{C}) \)—the double cover of the orthochronous Lorentz local gauge group of general relativity.
case of discrete quantum spacetime topologies modeled after finite dimensional Rota incidence algebras by Zapatrin and this author in [55].

Since, as it was also mentioned earlier, qausets are discrete differential manifolds à la Dimakis et al. [17, 18], discrete \( sl(2, \mathbb{C}) \)-valued connections \( \mathcal{D} \) were defined as \( \mathcal{G} \)-finsheaf morphisms by following closely Mallios’ Abstract Differential Geometry on Vector Sheaves theory in [10, 11]. The central point made in [12] is that these \( \mathcal{G} \)-finsheaves admit no global \( \mathcal{D} \)-sections, so that they qualify as being ‘curved’. Subsequently, and in [53, 54], the idea was pitched to organize these \( \mathcal{G} \)-finsheaves into a topos-like structure \( \mathbf{Sh}_{f eq} \) which may be viewed as a locally finite, causal and quantal substitute for the ‘classical’ topos \( \mathbf{Sh}(M) \) of sheaves of sets over the spacetime continuum \( M \). The aforementioned non-existence of global \( \mathcal{D} \)-sections of the reticular \( \mathcal{G} \)-sheaf objects of \( \mathbf{Sh}_{f eq} \) is completely analogous to the non-existence of global valuations in the QCHT or in the presheaf topos \( \mathcal{T}_{CH} \) of [34]. The former entails a non-trivial curvature form on the \( \mathcal{G} \)-finsheaves. A reasonable question one might ask is whether there is an analogous ‘quantum logical curvature form’ on the \( \mathcal{G}_{CH}^{CH}(\mathcal{L}_{CH}) \) objects in the QCHT.\(^{79}\). What is worth stressing at this point is that only topos theory allows for such a close logico-geometric interplay, that is to say, to speak to speak of a geometric spacetime curvature (gravity) in the \( \mathcal{G} \)-finsheaves of [42] and of a sort of quantum logical or semantic curvature in the warped sheaves of the QCHT.\(^{80}\).

As we mentioned in the previous subsection, the problematic tensor product stalk collapse in \( \mathcal{H} \)-vector sheaves, here too we mention a problem that may arise with the curved \( \mathcal{G} \)-finsheaves of qausets in [27]. The finite dimensional vector \( \mathcal{H} \)-sheaves associated with them, whose sections represent quantum states of qausets, are supposed to carry a representation of the reticular spin-Lorentz structure group of the principal \( \mathcal{G} \)-finsheaves. If on top we would like to emulate the situation in the quantum causal histories approach discussed above thus wish to implement unitary transitions between the stalks of the associated \( \mathcal{H} \)-sheaves, we would soon run into significant problems, because there are no finite dimensional unitary representations of the Lorentz group since it is non-compact. Of course, one could resort to an ‘easy-way-out’ by saying on the one hand that the reticular and quantal version of the spin-Lorentz structure group of the \( \mathcal{G} \)-finsheaves of qausets neither a continuous (Lie) nor even a ‘classical’ group any more, and on the other that the continuous spacetime manifold together with the continuous group of its symmetries somehow ‘emerges’ (as a macroscopic ef-

\(^{78}\)The topos of (f)initary, (c)ausal and (q)uantal sheaves of qausets.

\(^{79}\)This \( \mathbf{Sh}_{f eq} \) is one candidate for the quantum topos structure briefly alluded to in footnote 36.

\(^{80}\)We will return to this question in the next section where we entertain the possibility of a cohomological classification of the \( \mathcal{A} \)-sheaf objects of the QCHT à la Mallios [10, 11].
fect) from such quantal $G$-finsheaf substrata and it does not have to be accounted for at quantum scales\textsuperscript{81}. On the other hand, one could ultimately question the validity of unitarity in the quantum deep, since the latter is a non-local conception (\textit{eg}, in non-relativistic quantum mechanics unitarity involves an integral over all space, while in quantum field theory, over all spacetime), but that would appear to kill quantum causal histories altogether.

We conclude this subsection by remarking on a possible ‘consistent-histories of non-commutative or quantum spacetime topologies’ scenario in the QCHT. In \cite{53} it was argued that the non-abelian Rota incidence algebras modeling qca sets are also finitistic non-commutative topologies suitable for a quantum theoreisis of spacetime topology\textsuperscript{82}. Very recently, this gave birth to the related idea that spacetime topology can be regarded as a quantum observable of a foam-like nature \cite{56}. However, much earlier, and in the context of Rota-algebras and their $\mathcal{H}$ representations, not only spacetime topology had been conceived as being subject to some sort of quantum measurements and dynamical fluctuations \cite{47}, but also that one could even formulate a histories theory for such quantum spacetime topology measurements and variations \cite{10}. Even more suggestive is the observation that for the formulation of a theory of \textit{quantum topology} on the lattice of all topologies on a set $X$ of fixed finite cardinality (and canonical (\textit{ie}, Hamiltonian) dynamical transitions between them) \cite{31}, Vietoris-type of topologies like the one we used here for topologizing and concomitantly sheafifying the CH theory may play a crucial role \cite{3, 32}. Thus the question arises: can we marry all these diverse ideas under the single QCHT roof?—a question generating a quest that is certainly worth pursuing further in the future.

5 Brief comparison with the Kochen-Specker topos and a future outlook

To the future project that closed the last section we would like to add and discuss briefly a couple more below.

The first project for the immediate future that we would like to suggest is to try to relate the two strikingly similar topoi of presheaves of sets that arise in connection

\textsuperscript{81}See \cite{12} for more arguments about this.

\textsuperscript{82}Furthermore, in \cite{53} and subsequently in \cite{54} it was conjectured that the topos $\mathbb{Sh}_{f_{eq}}$ of the ‘non-commutative sheaves’ mentioned in the previous paragraph may be the canonical example of a \textit{quantum topos} structure in the same way that the collection of (commutative) sheaves over a locale is the canonical paradigm of a ‘classical’ topos. See also \cite{4} for a similar, but technically much more sophisticated, conception of quantum toposi and the non-commutative topology/sheaf theory that they encode.
with the semantic analysis of the quantal logic of the CH theory in [34] (ie, the $T_{CH}$ above) on the one hand, and on the other from considering similar valuation-localizations over the Boolean sublattices of a quantum lattice that result from viewing the famous Kochen-Specker (KS) theorem of quantum logic proper from a topos-theoretic perspective [12, 13, 15, 57].

It is immediately transparent from comparing the $T_{CH}$ and $T_{KS}$ topos that both use a poset category of Boolean sublattices of the proposition ortholattices underlying their quantal and quantum logics respectively as base spaces for semantic localizations, both toposes possess presheaf objects that do not admit global valuation-sections thus, as a result, both are ‘warped’ relative to their classical Boolean subalgebras or sublogics and have a neorealist (ie, Heyting) logic calculus as their internal contextualized logic. Indeed, these remarkable similarities call for a closer comparison between the quantal logic of the CH theory and the usual quantum logic in the illuminating light of sheaf and topos theory. At the same time, this unified topos perspective is even more formidable if one considers the significant differences on the interpretational side between the logic of the CH theory, whose propositions are in a strong sense ‘diachronic’ (ie, about entire histories of quantum systems), and quantum logic proper whose propositions are well known to be ‘synchronic’ or instantaneous (ie, at a single moment of time about the observable properties of quantum systems)—the dramatic differences between the (neo)realist philosophy supporting the CH theory and the operationalist one supporting the usual ‘Copenhagen’ quantum theory aside.

The second project that we would like to bring forth is a possible infusion of differential geometric ideas into the CH theory with an eye towards applying the resulting structures to quantum gravity—a problem that consistent histories are expected to address sooner or later. As we mentioned in the previous section, we hope to apply quite straightforwardly concepts and results from Mallios’ general and abstract (ie, axiomatic) treatment of the usual differential calculus on manifolds by means of vector and algebra sheaves to the sheaves $\mathcal{G}^0_{CH}(\mathcal{L}_{CH})$ of consistent-histories of qausets and their associated $H$-state sheaves in their QCHT. In particular, since it has been well

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83Let us call this topos ‘$T_{KS}$’—‘the Kochen-Specker topos’.

84This ‘warped’ or, geometrically speaking, curved character of the quantal and quantum logics of the CH theory and the usual quantum theory respectively is to be contrasted against the ‘intrinsically flat’ character of classical Boolean logic. From a sheaf-theoretic perspective, for example, it is well known that any Boolean algebra may be equivalently cast as the algebra of global sections of a sheaf of $2$s over its Stone space, hence it is flat [20, 4, 57]. In this way one may also understand the fact that in the intuitionistic (or neorealist!) topos $\text{Sh}(X)$ of sheaves of continuously variable sets over a topological manifold $X$, $\text{Set}$—the archetypal Boolean topos of constant sets whose subobject classifier is $2$—arises as an instantaneous ‘snapshot’ or localization of a geometric morphism kind of $\text{Sh}(X)$ on $X$’s points or ‘instances’ [11].
established that the finite dimensional incidence Rota algebras modeling qausets are discrete differential manifolds supporting discrete differential calculi in the sense that a reticular version of the nilpotent Kähler-Cartan differential $d$ has been shown to be operative on these algebra sheaves; this $d$ may be able to trigger the start of a de Rham-like cohomology theory on the $\mathfrak{S}^\alpha_{CH}(\mathfrak{L}_{CH})$ objects in the QCHT—arguably the focal point in Mallios’ aufbau of the usual $C^\infty$-differential geometric constructions in a more abstract, but powerful, sheaf-theoretic setting. Related to the above, and as it was also briefly mentioned earlier, since the $G$-finsheaves of qausets are curved, one could also entertain the possibility of cohomologically classifying à la Mallios the similarly warped algebra and vector sheaves of histories of qausets in their QCHT by means of a *characteristic curvature class* $\mathfrak{F}$, in a manner analogous to how it is usually done in the case of the locally trivial fiber bundles that are of interest to Yang-Mills theories and, possibly, to (quantum) gravity.

The possibility of bringing together such fiber bundle techniques and sheaf cohomology ideas from Mallios’ work, and apply them to the quantum causal history sheaves $\mathfrak{S}^\alpha_{CH}$ of interest here, rests on the observation that the sheaf $\mathcal{S}$ of germs of continuous sections of a locally trivial $k$-vector bundle $\mathbf{B}$ ($k = \mathbb{C}, \mathbb{R}, \cdots$) has the property that there is a an open cover $\mathcal{U} = \{U_i\}$ of the base topological space $X$ such that

$$\Gamma(U_i, \mathcal{S}) \cong \bigoplus_n C^0(U_i, k)$$

where $C^0(U_i, k)$ denotes the collection of continuous functions from $U_i$ into $k$. It is well known that this sheaf preserves the action of the bundle’s Čech cocycles so that $\mathbf{B}$ may be reconstructed from its sheaf of germs of sections, as well as from the algebraic structure of the set of sections which, in turn, inherits the algebraic structure of the objects that one may assume to inhabit the stalks of the sheaf $\mathfrak{S}^\alpha_{CH}$—these two constructions being essentially equivalent. So, in view of the (quantum) logical character of the algebra sheaves $\mathfrak{S}^\alpha_{CH}(\mathfrak{L}_{CH})$ and the sheaf cohomology project anticipated above, one may reasonably ask: is there some kind of ‘quantum logical curvature’ characteristic class $\mathfrak{F}$ classifying the $\mathfrak{S}^\alpha_{CH}(\mathfrak{L}_{CH})$ objects in the QCHT?—certainly a question worth pondering on. However, the possibility of such a geometrical characterization of ‘quantum logical’ sheaves, such as our $\mathfrak{S}^\alpha_{CH}(\mathfrak{L}_{CH})$s, essentially by algebraic means (ie, via sheaf cohomology), hence also the possibility of bringing together, quite unexpectedly, quantum logic and quantum gravity concepts.

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85 See section 4 and [40, 41].
and constructions, we have only lately begun to fathom—being surely guided in our quests by the illuminating light of topos theory. Perhaps, such a potential conceptual unity between quantum spacetime structure and dynamics on the one hand, and quantum logic on the other, achieved by applying sheaf and topos-theoretic ideas to the CH theory and quantum logic in particular, as well as to the general quantum gravity program [14, 42, 53], will further vindicate Finkelstein’s deep insight in [20] that “logics come from dynamics” [54].

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