Critical Temperature of Chiral Symmetry Restoration for Quark Matter with a Chiral Chemical Potential

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Abstract. In this article we study restoration of chiral symmetry at finite temperature for quark matter with a chiral chemical potential, $\mu_5$, by means of a nonlocal Nambu-Jona-Lasinio model. This model allows to introduce in the simplest way possible a Euclidean momentum, $p_E$, dependent quark mass function which decays (neglecting logarithms) as $1/p_E^2$ for large $p_E$, in agreement with asymptotic behaviour expected in QCD in presence of a nonperturbative quark condensate. We focus on the critical temperature for chiral symmetry restoration in the chiral limit, $T_c$, versus $\mu_5$, as well as on the order of the phase transition. We find that $T_c$ increases with $\mu_5$, and that the transition remains of the second order for the whole range of $\mu_5$ considered.

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1. Introduction

Systems with chirality imbalance, namely with a finite chiral density $n_5 = n_R - n_L$ generated by quantum anomalies, have attracted some interest in recent years. In fact gauge field configurations with a finite winding number, $Q_W$, can change fermions chirality according to the Adler-Bell-Jackiw anomaly [1, 2]. In the context of Quantum Chromodynamics (QCD) such nontrivial gauge field configurations with $Q_W \neq 0$ are instantons and sphalerons, the latter being produced copiously at high
temperature. The large number of sphaleron transitions in high temperature phase of QCD suggested the possibility to measure the Chiral Magnetic Effect (CME) in heavy ion collisions. The interest for mediums with a net chirality has then spread from QCD to hydrodynamics and condensed matter systems.

In order to describe systems with finite chirality in thermodynamical equilibrium, it is customary to introduce the chiral chemical potential, $\mu_5$, which is conjugated to $n_5$, see references therein. Naming $\tau$ the typical time scale in which chirality changing processes take place, it can be assumed that $\mu_5 \neq 0$ describes a system in thermodynamical equilibrium with a fixed value of $n_5$ on a time scale much larger than $\tau$. For example in the quark-gluon plasma phase of QCD chirality changing processes have been studied where it has been found that $\tau \simeq 50 \div 140$ fm/c in the temperature range $T \simeq 225 \div 500$ MeV.

An interesting problem in the context of QCD is the study of chiral symmetry restoration at finite temperature and $\mu_5 \neq 0$. Some previous calculations based on chiral models predicted $T_c$, the critical temperature for chiral symmetry restoration, to decrease with $\mu_5$. On the other hand, lattice simulations have shown that $T_c$ increases with $\mu_5$, in agreement with the results obtained by solving Schwinger-Dyson equations.

In this article we study chiral symmetry restoration at finite temperature with $\mu_5 \neq 0$, within a Nambu-Jona-Lasinio (NJL) model with a nonlocal interaction kernel, which has the advantage to introduce in the simplest way possible a Euclidean momentum dependent quark mass function that agrees with QCD.

The plan of the Article is as follows. In Section 2 we briefly describe the nonlocal NJL model we use in our calculation, presenting the several choices we do for the running dependent mass. In Section 3 we compute the critical temperature for chiral symmetry restoration as a function of $\mu_5$, as well as determine the order of the phase transition. In Section 4 we perform a small $\mu_5$ computation of the second Ginzburg-Landau (GL) coefficient in the free energy. Finally in Section 5 we draw our conclusions.

2. NJL model with momentum dependent quark mass function

In this Section we describe the model we use to compute the critical line for chiral symmetry restoration in the $T - \mu_5$ plane. We use a Nambu-Jona-Lasinio (NJL) model (see reviews) with a nonlocal interaction kernel inspired by the Instanton Liquid picture of the QCD vacuum, see for a review, which has the advantage to introduce in the simplest way possible a Euclidean momentum dependent quark mass function that agrees with QCD.
2.1. Thermodynamic potential

In the nonlocal NJL model we use in this study the lagrangian density is given by

\[ \mathcal{L} = \bar{\psi} \left( i \partial^\mu \gamma_\mu + \mu_5 \bar{\psi} \gamma_5 \psi \right) \psi + \mathcal{L}_4, \]

with \( \psi \) being a quark field with Dirac, color and flavor indices. In this equation \( \mu_5 \) is the chiral chemical potential, and its conjugated quantity is the chiral charge density, \( n_5 \equiv n_R - n_L \): a finite \( \mu_5 \) induces a chiral density in the system, and in general the relation between \( n_5 \) and \( \mu_5 \) has to be computed numerically within some model, see for example [25, 26].

In Eq.\( (1) \) \( \mathcal{L}_4 \) corresponds to the interaction term, namely

\[ \mathcal{L}_4 = G \int d^4yd^4z F^*(y - x)F(z - x)\bar{\psi}(y)\psi(z). \]

The interaction term in Eq. \( (2) \) is formally equivalent to a local NJL interaction,

\[ \mathcal{L}_4 = G\langle \bar{\Psi}(x)\Psi(x) \rangle^2, \]

written in terms of the dressed quark fields

\[ \Psi(x) \equiv \int d^4yF(y - x)\psi(y). \]

Chiral symmetry is spontaneously broken by the interaction in Eq. \( (2) \); this leads to a nonvanishing expectation value of the dressed quark field operator

\[ \sigma \equiv G\langle \bar{\Psi}(x)\Psi(x) \rangle \neq 0. \]

Working at finite temperature \( T \) within the well established imaginary time formalism, the thermodynamic potential per unit volume can be written as

\[ \Omega = \frac{\sigma^2}{G} - N_cN_fT \sum_n \int \frac{d^3p}{(2\pi)^3} \log \beta^4(\omega_n^2 + E^2_\pm)(\omega_n^2 + E^2_\pm), \]

where \( \beta = 1/T \) and we have defined

\[ E^2_\pm = (p \pm \mu_5)^2 + M(\omega_n, p)^2. \]

Here \( \omega_n = \pi T(2n + 1) \) corresponds to the fermionic Matsubara frequency and \( M(\omega_n, p) \) denotes the quark mass function to be specified later.

2.2. Quark mass functions

In Eq. \( (7) \) we have introduced the quark mass function

\[ M(p) \equiv -2\sigma C(p), \]

with \( C(p) \equiv F^2(p) \) and \( F(p) \) corresponding to the Fourier transform of the form factor \( F \) in Eq. \( (1) \). The above equation agrees with the results from one-gluon exchange inspired models [56, 57, 51, 52, 59, 58]. Nonlocal models mimic the constituent quark mass function of QCD in presence of spontaneous chiral symmetry breaking [54, 59] for large \( p_F \). In this work we assume several specific functional forms for \( M(p) \) in Eq. \( (8) \).
A class of form factors that we use have the form
\[ C(p^2_E) = \theta(\Lambda^2 - p^2_E) + \theta(p^2_E - \Lambda^2) \frac{\Lambda^2(\log \frac{\Lambda^2}{\Lambda^2_{QCD}})^\gamma}{p^2_E(\log \frac{\Lambda^2}{\Lambda^2_{QCD}})^\gamma}. \] (9)

For the exponent \( \gamma \) in Eq. (9) we consider here three cases: \( \gamma = 0 \) for its simplicity; \( \gamma = 1 \) following [56, 57]; finally \( \gamma = 1 - d_m \), inspired by the quark mass function derived by Politzer [54], where \( d_m = 12/29 \) corresponds to the anomalous mass dimension for \( N_f = 2 \).

We also consider form factors that connect smoothly the infrared and the ultraviolet \( p_E \) domains. In particular, we consider a Yukawa-type form factor, namely [51, 52]
\[ C(p^2_E) = \frac{\Lambda^2}{p^2_E + \Lambda^2}; \] (10)
then we consider a form factor inspired by the Instanton Liquid Model (ILM) of the QCD vacuum, namely [53]
\[ C(p^2_E) = d^2 p^2_E \left\{ \frac{d}{dx} \left[ I_0(x) K_0(x) - I_1(x) K_1(x) \right] \right\}^2, \] (11)
where \( d \) corresponds to the typical instanton size \( d \approx 0.36 \) fm and \( x = |p_E| d/2 \). Finally we consider a nonlocal kernel used in nonlocal NJL model studies [56, 57], namely
\[ C(p^2_E) = \theta(\Lambda^2 - p^2_E) e^{-p^2_E d^2/2} + \theta(p^2_E - \Lambda^2) \frac{\Lambda^2(\log \frac{\Lambda^2}{\Lambda^2_{QCD}})}{p^2_E(\log \frac{\Lambda^2}{\Lambda^2_{QCD}})} e^{-\Lambda^2 d^2/2}, \] (12)
where \( d \) corresponds to the instanton size used in Eq. (11) and \( \Lambda = O(1) \) GeV. Equation (12) offers a smooth version of the form factor in Eq. (9) with \( \gamma = 1 \).

3. The critical temperature and the order of the phase transition

In this Section we compute the critical line for chiral symmetry restoration as a function of the chiral chemical potential, both within a GL expansion of the thermodynamic potential in Eq. (6) and within numerical calculations using the full potential.

3.1. Ginzburg-Landau expansion

Close to a second order phase transition we can write Eq. (6) as
\[ \Omega - \Omega_0 = \frac{\alpha_2}{2} \sigma^2 + \frac{\alpha_4}{4} \sigma^4 + O(\sigma^6), \] (13)
where we have subtracted the thermodynamic potential at \( \sigma = 0 \), namely \( \Omega_0 \); \( \alpha_2 \) and \( \alpha_4 \) can be computed by taking the derivatives of \( \Omega \) with respect to \( \sigma \) at \( \sigma = 0 \). We find
\[ \alpha_2 = \frac{2}{G} - N_c N_f T \sum_n \int \frac{d^3 p}{(2\pi)^3} C^2(\omega_n, p) \]
\[ \times \frac{16(p^2 + \omega_n^2 + \mu_5^2)}{[\omega_n^2 + (p - \mu_5)^2][\omega_n^2 + (p + \mu_5)^2]}, \] (14)
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and

\[ \alpha_4 = -N_c N_f T \sum_n \int \frac{d^3 p}{(2\pi)^3} C_4(\omega_n, p) \]

\[ = -384 \left[ p^4 + 2p^2(\omega_n^2 + 3\mu_n^2) + (\mu_n^2 + \omega_n^2)^2 \right] \]

\[ \left[ \omega_n^2 + (p - \mu_n)^2 \right]^2 \left[ \omega_n^2 + (p + \mu_n)^2 \right]^2. \]  

(15)

The nontrivial solution of the gap equation, \( \partial \Omega/\partial \sigma = 0 \), for \( T \leq T_c \) is given by

\[ \sigma^2(T, \mu_5) = \frac{\alpha_2(T, \mu_5)}{\alpha_4(T, \mu_5)}, \]  

(16)

and the critical temperature is defined by the condition \( \alpha_2(T, \mu_5) = 0 \).

In Fig. 1a we plot the coefficient \( \alpha_2 \) in units of the parameter \( \Lambda^2 \) as a function of temperature, for three different values of \( \mu_5 \). For each value of \( \mu_5 \) the critical temperature \( T_c \) corresponds to \( \alpha_2(T_c) = 0 \). We show data for the nonlocal model with mass function given by Eqs. (9) and (8) with \( \gamma = 0 \) and \( \Lambda = 900 \text{ MeV} \); for other models we obtain qualitatively the same results. We notice that increasing \( \mu_5 \) results in an increasing critical temperature. We also notice that the slope of \( \alpha_2 \) at \( T = T_c \) increases with \( \mu_5 \).

In Fig. 1b we plot the coefficient \( \alpha_4 \) versus temperature, for the same values of \( \mu_5 \) shown in Fig. 1a of the figure. We notice that for any value of \( \mu_5 \) the coefficient \( \alpha_4 \) decreases in magnitude, but it is always positive at the critical temperature meaning the phase transition is a second order one. We also notice that the magnitude of \( \alpha_4 \) at \( T = T_c(\mu_5) \) decreases compared to the case \( \mu_5 = 0 \), implying that the phase transition becomes sharper with increasing \( \mu_5 \). In fact because of Eq. (16) we can write the solution of the gap equation for \( T \approx T_c \) as

\[ \sigma^2 = -\frac{1}{\alpha_4(T_c)} \left. \frac{d\alpha_2}{dT} \right|_{T=T_c} (T - T_c) + O((T - T_c)^2), \]  

(17)

then the slope of the condensate at the critical temperature is given by

\[ \left. \frac{d\sigma^2}{dT} \right|_{T=T_c} = -\frac{1}{\alpha_4(T_c)} \left. \frac{d\alpha_2}{dT} \right|_{T=T_c}, \]  

(18)

which becomes larger as we increase \( \mu_5 \) because \( \alpha_4 \) decreases and the slope of \( \alpha_2 \) at the critical temperature increases with \( \mu_5 \). Our conclusion is that within the range of \( \mu_5 \) explored in our study, we have a firm signal that the phase transition becomes sharper as \( \mu_5 \) is increased, but there is no critical endpoint because \( \alpha_4 \) does not change sign at the critical temperature. This is in agreement with lattice simulations [30, 31] but it is in disagreement with previous model studies which used a different regularization scheme [24, 26, 25, 27], showing how the existence of the critical point in the phase diagram is very sensitive on the regularization prescription, in fact already noticed in [28]. Finally in Fig. 1c: we plot the coefficient \( \alpha_4 \) at \( T = T_c(\mu_5) \) for several models. We notice that although the numerical value of \( \alpha_4 \) strongly depends on the model, we find that it is always positive at \( T = T_c \).
Figure 1. (a). Coefficient $\alpha_2$ in units of the parameter $\Lambda^2$ as a function of temperature, for three different values of $\mu_5$. For each value of $\mu_5$ the critical temperature $T_c$ corresponds to $\alpha_2(T_c) = 0$. (b). Coefficient $\alpha_4$ versus temperature, for three different values of $\mu_5$. (a) and (b) refer to the nonlocal model with mass function given by Eqs. (9) and (8) with $\gamma = 0$ and $\Lambda = 900$ MeV. (c). Evolution of the coefficient $\alpha_4$ computed at $T = T_c(\mu_5)$ versus $\mu_5$ for several models.

3.2. Beyond Ginzburg-Landau expansion

In the previous subsection we have discussed results obtained within a GL expansion of the thermodynamic potential, see Eq. (13). The GL expansion is a useful tool because it allows to study the analytical behaviour of the thermodynamic potential close to the phase transition. As long as we are interested to the critical temperature at a second order phase transition the GL expansion is equivalent to use the full thermodynamic potential in Eq. (6): as a matter of fact, it requires a straightforward calculation to
verify that the gap equation $\partial \Omega / \partial \sigma = 0$ at $T = T_c$ obtained from Eq. (13) coincides with the GL gap equation at $T = T_c$, that is $\alpha_2 = 0$ where $\alpha_2$ is the second order GL coefficient in Eq. (13). On the other hand, for a first order phase transition the GL expansion is not reliable because the value of the condensate at $T = T_c$ might not be small compared to $T$, and the use of the expansion in Eq. (13) might be doubtful.

A natural question therefore arises, namely if the above results, in particular the order of the phase transition, are a mere consequence of the GL expansion or if they are in agreement with those that would be obtained using the gap equation derived from the full thermodynamic potential. In the previous Section we have first computed the temperature at which $\alpha_2 = 0$, identifying this with $T_c$, then we have computed $\alpha_4$ at $T = T_c$ checking its sign: we have then concluded that being $\alpha_4 > 0$ the phase transition is always of the second order regardless the value of $\mu_5$; within the GL approximation $\alpha_4 < 0$ would have been a signal of a first order phase transition. The purpose of this Section is to check the results of the previous Section going beyond the GL expansion of Eq. (13). To this end we compute the condensate defined in Eq. (5) by solving the gap equation $\partial \Omega / \partial \sigma = 0$ with $\Omega$ defined in Eq. (6).

In Fig. 2 we plot the condensate versus temperature for two of the nonlocal NJL models mentioned in Fig. 3, namely the Gaussian model (a) and the Yukawa model with $\Lambda = 900$ MeV (b). We have checked that for the other models we obtain similar results. In the figure $T_{c0}$ corresponds to the critical temperature at $\mu_5 = 0$, while $\sigma_0$ denotes the condensate at $T = 0$ and $\mu_5 = 0$. For both form factors we plot data for several values of $\mu_5$: in particular black solid lines correspond to $\mu_5 = 0$, maroon dotted lines to $\mu_5 = T_{c0}$, orange dashed lines to $\mu_5 = 2T_{c0}$, finally brown dot-dashed lines to $\mu_5 = 2.5T_{c0}$. For both cases we have zoomed to the temperature range close to the critical temperature.

Figure 2. (a). NJL condensate, defined in Eq. (5), versus temperature for several values of $\mu_5$, for the case of the Gaussian form factor of Fig. 3 (b). Condensate for the case of the Yukawa form factor with $\Lambda = 900$ MeV in Fig. 3. In both panels $T_{c0}$ corresponds to the critical temperature at $\mu_5 = 0$, while $\sigma_0$ denotes the condensate at $T = 0$ and $\mu_5 = 0$. For both form factors we plot data for several values of $\mu_5$: in particular black solid lines correspond to $\mu_5 = 0$, maroon dotted lines to $\mu_5 = T_{c0}$, orange dashed lines to $\mu_5 = 2T_{c0}$, finally brown dot-dashed lines to $\mu_5 = 2.5T_{c0}$. For both cases we have zoomed to the temperature range close to the critical temperature.
which is the one relevant for our study.

Data shown in Fig. 2 confirm the results obtained within the GL expansion and presented in the previous subsection. In fact, the critical temperature is found to increase with $\mu_5$. Moreover the condensate vanishes smoothly with increasing temperature, meaning the phase transition is of the second order (a first order phase transition would appear as a discontinuity in the condensate, which we do not find for all the values of $\mu_5$ explored here). We thus can conclude that the main results of the our study, with particular regard to the absence of a first order phase transition line in the $\mu_5 - T$ plane, are not a mere consequence of the GL expansion Eq. (13). The presence of a first order phase transition line, found in NJL and quark-meson model calculations [24, 25, 26, 27] but not found in the nonlocal NJL model calculations, appears thus to be model dependent, in agreement with what anticipated in [28].

3.3. The critical line in the $\mu_5 - T$ plane

In Fig. 3 we plot the critical temperature versus $\mu_5$ for the nonlocal models described in the text. In the figure $T_{c0}$ denotes the critical temperature at $\mu_5 = 0$. In Fig. 3a we collect the results for the sharp models described in the text. Circles correspond to mass function given by Eqs. (9) and (8) with $\gamma = 0$, for two different values of $\Lambda$; diamonds correspond to the same mass function with $\gamma = 1 - d_m$. In Fig. 3a we have also shown the results for two local NJL models. In particular, we denote by squares the
results for a standard local NJL model with a 4-dimensional sharp cutoff on the vacuum term and no cutoff on the thermal part of the free energy; moreover, empty squares correspond to a model dubbed $\Lambda-$NJL, in which there is a 4D sharp cutoff both on the vacuum and on the thermal contribution to the gap equation. In both cases $\Lambda = 900$ MeV. In Fig. 3, we plot the critical temperature for smooth form factors. In particular data with triangles pointing upwards correspond to a Yukawa-like form factor in Eq. (10) with two values of $\Lambda$. Data denoted by triangles pointing downwards correspond to the Instanton Liquid Model (ILM) form factor in Eq. (11). Finally stars correspond to the nonlocal form factor in Eq. (12). In both panels both temperature and chemical potential are measured in units of the critical temperature at $\mu_5 = 0$. In each calculation we have fixed the value of the parameter $\Lambda$ in the form factor, then we have tuned the NJL coupling constant $G$ in order to obtain $T_{c0} = 170$ MeV for any model.

The results in Fig. 3 show that for all the nonlocal models studied in this article the critical temperature increases with $\mu_5$. For large values of $\mu_5$ the results shown in Fig. 3 should be not considered very reliable because we have neglected a possible backreaction on the nonlocal interaction kernel due to $\mu_5$. For the case of local models, we find that the $\Lambda-$NJL model still predicts $T_c$ increases with $\mu_5$, at least up to values of $\mu_5$ of the order of $\Lambda$. This is in agreement with the previous analysis of [28] where a $\Lambda-$NJL with a 3-dimensional cutoff has been considered. For the NJL model result in Fig. 3 we find that $T_c$ increases with $\mu_5$ for small values of $\mu_5$, in agreement with a small $\mu_5$ analysis presented in the following section.

A detailed comparison with lattice data [30, 31] is premature because those data have not been obtained in the chiral limit; moreover, some data on the lattice correspond to $N_c = 2$ QCD while here we consider $N_c = 3$. However, we can at least compare the magnitude of the increase of the critical temperature obtained within the nonlocal models and within the lattice simulations. In Fig. 3 we show lattice results for $T_c(\mu_5)$ for $N_c = 2$ adapted from Ref. [31] in which the critical temperature at $\mu_5 = 0$ is $T_{c0} = 195.8 \pm 0.4$ MeV. We find that among the models considered here, the ones with Gaussian ILM form factors, respectively Eqs. (12) and (11), better reproduce the magnitude of the variation of the critical temperature with $\mu_5$.

4. Small $\mu_5$ analysis

Since the phase transition is of the second order we can use the GL expansion, see Eq. (13), to investigate in more detail the relation between $\mu_5$ and $T_c$ within the model at hand. In particular, we perform in this section a small $\mu_5$ analysis of the coefficient $\alpha_2$ in Eq. (13) to enlighten the differences between local and nonlocal NJL models at finite $\mu_5$. 
4.1. The coefficient $\alpha_{2,2}$ and $T_c$ versus $\mu_5$ for $\mu_5/T \ll 1$

We expand

$$\alpha_2 = \alpha_{2,0} + \mu_5^2 \alpha_{2,2}. \quad (19)$$

The above equation allows to compute, to the lowest order in $\mu_5/T$, the shift of the critical temperature due to $\mu_5$:

$$\delta T_c = -\frac{\alpha_{2,2}(T^0_c)}{a} \mu_5^2, \quad (20)$$

where $T^0_c$ corresponds to the critical temperature at $\mu_5 = 0$ and $a \equiv d\alpha_{2,0}/dT$ at $T = T^0_c$. The quantity $a$ depends on the specific model used but it is positive by definition because $\alpha_{2,0}$ is negative for $T < T_c$ and positive for $T > T_c$, thus the sign of $\delta T$ in Eq. (20) is determined only by the sign of $\alpha_{2,2}$. A straightforward computation starting from Eq. (6) shows that

$$\alpha_{2,2} = -4N_cN_fT \sum_n \int \frac{d^3p}{(2\pi)^3} C^2(\omega_n, p) \frac{2(3p^2 - \omega_n^2)}{(p^2 + \omega_n^2)^3}, \quad (21)$$

where $C$ is the non-local interaction kernel. Once $\alpha_{2,2}$ is known, the critical temperature versus $\mu_5$ can be computed as

$$T_c(\mu_5) = T^0_c \left[ 1 - \frac{\alpha_{2,2}(T^0_c)}{aT^0_c} \mu_5^2 \right]. \quad (22)$$

In Figure 4, we plot the coefficient $\alpha_{2,2}$ computed by Eq. (21) for the several form factors described in Section 2. For all the models of $p_E$–dependent quark mass functions we find that $\alpha_{2,2} < 0$, and because of Eq. (22) this implies $\mu_5$ tends to increase the critical temperature for chiral symmetry restoration within the model at hand. This is different from what is obtained within local models in which critical temperature has been found to decrease with $\mu_5$, with the exception of [23] where renormalization has been used to treat the divergent vacuum term.
4.2. The modes contributions

It is instructive to present an analysis of the coefficient $\alpha_{2,2}$ defined in Eq. (21), in order to enlighten the difference between the nonlocal and local models for what concerns $T_c(\mu_5)$ for small values of $\mu_5$. This analysis follows a similar one presented in [28] for the case of an NJL model with a local interaction kernel and a 3-dimensional cutoff. For simplicity, we focus on the form factor given by Eq. (25) with $\gamma = 0$ which allows easier manipulations and a clearer mode separation. We split $\alpha_{2,2}$ as

$$\alpha_{2,2} = I_1 + I_2 + J_1 + J_2;$$

here we have introduced several contributions depending on the momentum region of quarks and on temperature. These terms are defined as follows. Firstly we add and subtract the $T = 0$ contribution to Eq. (21), that according to the well known rules of finite temperature field theory in the imaginary time formalism reads

$$\alpha_{2,2}^0 = -4N_c N_f \int \frac{d^4p_E}{(2\pi)^4} C^2(p_E) \frac{2(3p^2 - p_4^2)}{(p^2 + p_4^2)^3};$$

then we define

$$I_1 = -4N_c N_f \int_{p_E^2 \leq \Lambda^2} \frac{d^4p_E}{(2\pi)^4} C^2(p_E) \frac{2(3p^2 - p_4^2)}{(p^2 + p_4^2)^3},$$

$$I_2 = -4N_c N_f \int_{p_E^2 > \Lambda^2} \frac{d^4p_E}{(2\pi)^4} C^2(p_E) \frac{2(3p^2 - p_4^2)}{(p^2 + p_4^2)^3},$$

which correspond to the contributions to $\alpha_{2,2}$ at zero temperature of the modes with $p_E^2 \leq \Lambda^2$ and $p_E^2 > \Lambda^2$ respectively. Moreover we define

$$J_1 = -4N_c N_f T \sum_n \int \frac{d^3p}{(2\pi)^3} C^2(\omega_n, p) \frac{2(3p^2 - \omega_n^2)}{(p^2 + \omega_n^2)^3} \bigg|_{\omega_n^2 + p^2 \leq \Lambda^2}$$

$$I_1 = -2 \int_{p_E^2 \leq \Lambda^2} \frac{d^4p_E}{(2\pi)^4} C^2(p_E) \frac{2(3p^2 - p_4^2)}{(p^2 + p_4^2)^3};$$

$$J_2 = -4N_c N_f \sum \int \frac{d^3p}{(2\pi)^3} C^2(\omega_n, p) \frac{2(3p^2 - \omega_n^2)}{(p^2 + \omega_n^2)^3} \bigg|_{\omega_n^2 + p^2 > \Lambda^2}$$

$$I_2 = -2 \int_{p_E^2 > \Lambda^2} \frac{d^4p_E}{(2\pi)^4} C^2(p_E) \frac{2(3p^2 - p_4^2)}{(p^2 + p_4^2)^3};$$

which correspond to the contributions to $\alpha_{2,2}$ at finite temperature of the modes with $p_E^2 \leq \Lambda^2$ and $p_E^2 > \Lambda^2$ respectively.

Evaluation of integrals and summation over Matsubara frequencies in the above equations lead to the following results:

- modes with $p_E^2 \leq \Lambda^2$ at $T = 0$:
  
  $$I_1 = -a_2 \frac{4N_c N_f}{2\pi^2} \log \frac{\Lambda}{m_0};$$

- modes with $p_E^2 > \Lambda^2$ at $T = 0$:
  
  $$I_2 = -a_1 \frac{4N_c N_f}{2\pi^2};$$
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- modes with $p_E^2 \leq \Lambda^2$ at $T > 0$:
  \[ J_1 = \frac{4N_cN_f}{2\pi^2} \left[ a_2 \log \frac{1}{\beta m_0} + |F(\beta \Lambda)| \right] ; \quad (31) \]

- modes with $p_E^2 > \Lambda^2$ at $T > 0$:
  \[ J_2 = \frac{4N_cN_f}{2\pi^2} |G(\beta \Lambda)| ; \quad (32) \]

in order to obtain the above equations we have done some manipulation on the definitions in Eqs. (26) and (26) which allow to extract the analytical contribution shown in Eqs. (29) - (32). The coefficients $a_1 \approx 0.25$ and $a_2 \approx 0.938$ are the results of numerical integration. Moreover we have introduced an infrared cutoff $m_0$ which appears in the intermediate steps of the computation when the contributions are split; this fictitious cutoff disappears when the sum of the contributions is done, as it is clear from Eqs. (31) and (29). In Fig. 5a we plot the functions $F$, $G$ as well as their sum in order to understand the role of the several terms in Eq. (23). In particular the modes in Eq. (30) come from the high momentum part of the Dirac sea; they are not usually considered in a local model calculation because in that case their contribution is divergent hence it is simply subtracted. We notice that this contribution to $\alpha_{2,2}$ is negative, thus it helps to keep the critical temperature at finite $\mu$ higher than that at $\mu = 0$.

4.3. Comparison with local NJL model

The benefit of expansion in Eq. (23) is that it allows to compare easily nonlocal with local models. To this end we introduce a local $\Lambda$–NJL model in which we remove all the modes with $p_E^2 > \Lambda^2$; the coefficient $\alpha_{2,2}$ will be thus given by the sum of Eqs. (29) and (31) namely

\[ \alpha_{2,2}^{\Lambda-NJL} = -\frac{4N_cN_f}{2\pi^2} \left[ a_2 \log \beta \Lambda - |F(\beta \Lambda)| \right] . \quad (33) \]

We also introduce the standard local NJL model in which we remove the ultraviolet modes $p_E^2 > \Lambda^2$ only at $T = 0$, and integrate over all momenta at finite temperature:

\[ \alpha_{2,2}^{NJL} = -\frac{4N_cN_f}{2\pi^2} \left[ a_2 \log \beta \Lambda - |F(\infty)| \right] , \quad (34) \]

where $F(\infty) \equiv \lim_{x \to \infty} F(x)$. Both these models follow the definitions already introduced in [28].

In Fig. 5b we plot the coefficient $\alpha_{2,2}$ for the the local NJL model (green dot-dashed line), the local $\Lambda$–NJL model (maroon dashed line) and the nonlocal model with mass function given by Eqs. (8) and $\gamma = 0$. For the local models there exists a window of $\beta \Lambda$ in which $\alpha_{2,2} > 0$; on the other hand for the nonlocal model considered here we find $\alpha_{2,2} < 0$ for any value of $\beta \Lambda$. The fact that $\alpha_{2,2}$ can be positive in the local models is in part due to the absence of the vacuum term in Eq. (30) which would give a negative contribution to $\alpha_{2,2}$. Moreover, the main difference between the standard local NJL and the $\Lambda$-NJL models is that in the latter the positive contribution $J_2$ of the modes with $p_E^2 > \Lambda^2$ at finite temperature is missing, while in the former the positive
contribution of these modes is added assuming a constant mass function: this explains why $\alpha_{2,2}$ for the $\Lambda$-NJL model is always smaller than the one of the standard local NJL.

The difference between the nonlocal model on the one hand, and the local models on the other hand, is that for the former we find $\alpha_{2,2}$ is always negative, while for the latters there exist windows of $\beta \Lambda$ in which $\alpha_{2,2}$ is positive. This means that depending on the values of $T_c$ and $\Lambda$, $T_c$ can either increase or decrease with $\mu_5$ in local NJL models, the result depending on model parameters. The parameter window in which $\alpha_{2,2}$ is positive (implying $T_c$ decreasing with $\mu_5$) is very tiny for the $\Lambda$-NJL model, but it is quite wide for the standard local NJL model. Considering that for the 4-dimensional regularization typically $\Lambda \approx 1$ GeV and $T_c$ is in the range 150 – 200 MeV, the value of $\beta \Lambda$ at $T = T_c$ turns out to be approximately in the range 5 – 6.7: in this range we find that $\alpha_{2,2}$ is negative, which explains why in this work we find that $T_c$ increases with $\mu_5$ also in the case of the local models.

In Fig. 6 we plot $\alpha_{2,2}$ computed for a 3-dimensional regulator; the calculation steps are similar to those of the models with 4-dimensional regulator so we do not repeat them. In particular the green dot-dashed line corresponds to the model used in [24, 25, 27]. For 3-dimensional regularizations the value of $\Lambda$ is considerably smaller than the one used in the 4-dimensional case, typically of the order of 600 MeV [43, 44, 45] while the range in which $T_c$ runs is the same found in the 4-dimensional case. This implies that $\beta \Lambda$ at $T = T_c$ for 3-dimensional regularization schemes are in the range 3 – 4: for the case of the local NJL model we find this range to be in the region where $\alpha_{2,2}$ is positive, see green dot-dashed line in Fig. 6 meaning that $T_c$ is lowered by $\mu_5$. Similarly for the case of the $\Lambda$-NJL model we find that $\alpha_{2,2}$ is negative implying that $T_c$ increases with $\mu_5$. Comparing the results for the standard local NJL models with 4-dimensional and 3-dimensional regulators we thus conclude that in previous calculations [24, 25, 27] the critical temperature decreases with $\mu_5$ because of an accident driven by the model.
parameters, suggesting that behaviour of $T_c$ to be an artifact of the 3-dimensional regularization.

5. Conclusions

In this article we have presented a model study of the critical temperature of chiral symmetry restoration, $T_c$, as a function of chiral chemical potential, $\mu_5$. We have used a nonlocal NJL model with several Euclidean interaction kernels, chosen to mimick the constituent quark mass of QCD in the ultraviolet.

We have studied the thermodynamic potential both within a Ginzburg-Landau expansion in the vicinity of the second order critical line, and within calculations using the full potential. The main interest of our study has been the computation of the critical temperature versus $\mu_5$. The results about $T_c(\mu_5)$ are collected in Fig. 3 for the different models. We have found that within the nonlocal models used in our study, $T_c$ increases with $\mu_5$ regardless of the interaction kernel used. We remark that our interaction kernels lack of a backreaction of $\mu_5$, hence our results should be taken with a grain of salt for $\mu_5 = O(1 \text{ GeV})$, while they are reliable for smaller values of $\mu_5$. We have also found that $T_c$ increases with $\mu_5$ for a standard NJL model with a 4-dimensional regulator, at least for small values of $\mu_5$. According to these findings, we have concluded that previous works [24, 25, 26, 27] found $T_c$ a decreasing function of $\mu_5$ as a result of an accident driven by the model parameters, suggesting that behaviour of $T_c$ to be an artifact of the 3-dimensional regularization of the standard local NJL model.

We have then checked the order of the phase transition by computing the coefficient $\alpha_4$ of the GL effective potential: we have found that although $\mu_5$ makes the transition sharper because the magnitude of $\alpha_4$ decreases with $\mu_5$ at $T_c$, the coefficient never vanishes as it should happen at the critical endpoint. We have confirmed the results obtained within the GL expansion by performing a calculation considering the full
thermodynamic potential. Our conclusion is that there is no trace of a critical endpoint in the phase diagram, at least within the range of \( \mu_5 \) we have explored in this article. According to this result we can conclude that the presence of the critical point in the \( \mu_5 - T \) plane advertised before \cite{24, 25, 26, 27} is model dependent: in particular its existence depends on the details of the interaction used in the model calculation. This result, as well as our conclusion about \( T_c(\mu_5) \), agree with \cite{28}.

We would like to close this article by doing few considerations about the implications of our study. The main purpose of our investigation is purely theoretical: the interest of a phase diagram of QCD in the \( \mu_5 - T \) plane was suggested in several references \cite{24, 25, 26, 27}, where it was found that the critical temperature for chiral symmetry restoration and for confinement-deconfinement decrease with \( \mu_5 \), and a critical endpoint appears in the phase diagram. Since both these characteristics belong also the would-be phase diagram of QCD in the \( \mu - T \) plane, and because the \( \mu_5 - T \) plane can be accessed by Lattice QCD calculations while QCD at finite \( \mu \) suffers the sign problem, the idea that might derive from \cite{24, 25, 26, 27} is that Lattice QCD studies at finite \( \mu_5 \) can shed a light on QCD in the \( \mu - T \) plane. Therefore the main purpose of our model study has been to check whether the predictions of \cite{24, 25, 26, 27} are general or specific to the model used in the calculations. What we have found is that the latter scenario is actually verified, since classes of effective models exist in which the phase diagram looks quite different from that advertised previously. The scenario depicted here is in agreement with Lattice QCD calculations \cite{30, 31}, and with results obtained by solving Schwinger-Dyson equations at finite \( \mu_5 \) \cite{34, 35}. Therefore we can conclude by stating that we have now three independent calculation schemes that agree on the fact that \( T_c \) increases with \( \mu_5 \), and that the transition line is of the second order. As a consequence it seems unlikely that further investigations at finite \( \mu_5 \) can teach something about the QCD phase structure in the \( \mu - T \) plane.

Regarding the confinement-deconfinement in the \( \mu_5 - T \) plane, Lattice QCD has found no evidence for a split of this crossover from the chiral crossover \cite{30, 31}. In order to study this problem within the models at hand we should augment the NJL model with some physical quantity that is sensitive to the deconfinement: the best candidate model is the NJL model augmented with a coupling to the Polyakov loop (PNJL) \cite{60, 61}. We expect that the picture drawn in this article does not change drastically by turning to the PNJL model, in particular if a coupling between the NJL interaction and the Polyakov loop is taken into account \cite{62, 63}. A study of the problems studied in our article by means of the PNJL model might be the subject of a future study. Moreover, the absence of a critical endpoint in the \( \mu_5 - T \) plane might limit the inhomogenous condensates that have been predicted to develop in the \( \mu - T \) plane near the critical point, see for example \cite{64, 65, 66, 67}. More study related to this topic might be worth of an investigation.

Finally, we would like to mention that during the very final stage of preparation of the present manuscript, Ref. \cite{68} appeared in which the same problem has been studied and an increasing \( T_c \) versus \( \mu_5 \) has been found, in agreement with the results presented
in this article. Moreover during the revision of the manuscript Ref. [69] appeared, in which similar conclusions to the work presented here as well as to that of [28] have been drawn, regarding $T_c(\mu_5)$ and the order of the phase transition at finite $\mu_5$.

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