Zero-Rate Thresholds and New Capacity Bounds for List-Decoding and List-Recovery

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Abstract—In this work we consider the list-decodability and list-recoverability of arbitrary $q$-ary codes, for all integer values of $q \geq 2$. A code is called $(p, L)_q$-list-decodable if for every radius $pn$ Hamming ball contains less than $L$ codewords; $(p, \ell, L)_q$-list-recoverability is a generalization where we place radius $pn$ Hamming balls on every point of a combinatorial rectangle with side length $\ell$ and again stipulate that there be less than $L$ codewords. Our main contribution is to precisely calculate the maximum value of $p$ for which there exist infinite families of positive rate $(p, \ell, L)_q$-list-recoverable codes, the quantity we call the zero-rate threshold. Denoting this value by $p_*$, we in fact show that codes correcting a $p_* - \varepsilon$ fraction of errors must have size $O(1)$, i.e., independent of $n$. Such a result is typically referred to as a “Plotkin bound.” To complement this, a standard random code with expurgation construction shows that there exist positive rate codes correcting a $p_* - \varepsilon$ fraction of errors. We also follow a classical proof template (typically attributed to Elias and Bassalygo) to derive from the zero-rate threshold other tradeoffs between rate and decoding radius for list-decoding and list-recovery. Technically, proving the Plotkin bound boils down to demonstrating the Schur convexity of a certain function defined on the $q$-simplex as well as the convexity of a univariate function derived from it. We remark that an earlier argument claimed similar results for $q$-ary list-decoding; however, we point out that this earlier proof is flawed.

Index Terms—Combinatorial mathematics, error correction codes.

I. INTRODUCTION

Given a code $C \subseteq [q]^n$, a fundamental problem of coding-theory is to determine how “well-spread” $C$ can be if we also insist that $C$ have large rate $R = \log_2 |C|$. The most basic way of quantifying “well-spread” is by insisting that all pairs of codewords are far apart. That is, we hope that the minimum distance $d := \min\{d_H(c, c') : c \neq c' \in C\}$ is large, where $d_H(\cdot, \cdot)$ denotes Hamming distance, i.e., the number of coordinates on which the two strings differ. Equivalently, given any word $y \in [q]^n$, we have that $|B_H(y, r) \cap C| \leq 1$, where $r = \lfloor d/2 \rfloor$ and $B_H(y, r) = \{x \in [q]^n : d_H(x, y) \leq r\}$ denotes the Hamming ball of radius $r$ centered at $y$.

One can naturally relax this requirement to the notion of list-decodability: instead of upper-bounding $|B_H(y, r) \cap C|$ by 1, we upper bound it by a larger integer $L - 1$.1 Equivalently, if we place Hamming balls of radius $r$ on each codeword of $C$, no vector in $[q]^n$ is covered by $L$ or more balls. If $C$ satisfies this property we call it $(p, L)_q$-list-decodable. Initially introduced by Elias and Wozencraft in the 1950’s [16], [17], [55], this relaxed notion of decoding has been intensively studied in recent years, in part motivated by purely coding-theoretic concerns, but also due to its connections with theoretical computer science more broadly [5], [25], [35], [38], [41], [52].

A further generalization of list-decoding is provided by list-recoverability. In this case, one considers tuples of input lists $\mathbf{Y} = (Y_1, \ldots, Y_k)$ where each $Y_i \subset [q]$ is of size at most $\ell$, and the requirement is that the number of codewords $c$ satisfying $|\{i \in [n] : c_i \neq Y_i\}| \leq pn$ is at most $L - 1$. Such a code is deemed $(p, \ell, L)_q$-list-recoverable. Note that $(p, 1, L)_q$-list-recoverability is the same as $(p, L)_q$-list-decoding, demonstrating that list-recoverability is a more general notion. While it was originally defined as an abstraction required for the task of uniquely-list-decoding concatenated codes [20], [21], [22], [23], it has since found myriad further applications in computer science more broadly, e.g., in cryptography [32], [33], randomness extraction [30], hardness amplification [14], group testing [34], [45], streaming algorithms [15], and beyond. When it comes to list-decoding and list-recovery, the optimal tradeoff between decoding-radius $p$ and rate $R$ is well-understood if one is satisfied with list-sizes $L = O(1)$. That is, there exist $(p, \ell, O(\ell/\varepsilon))_q$-list-recoverable codes of rate $1 - H_q(\varepsilon) - \varepsilon$ where

$$H_q(\varepsilon) := p \log_q \left( \frac{q - \ell}{p} \right) + (1 - p) \log_q \left( \frac{\ell}{1 - p} \right);$$

considerly, if the rate is at least $1 - H_q(\varepsilon) + \varepsilon$ then it will not be list-recoverable for any $L = o(q^{n})$ [49, Theorem 2.4.12]. (Note that setting $\ell = 1$ recovers the more well-known list-decoding capacity theorem.) While this already provides some

1We find it most convenient to let $L$ denote 1 more than the list-size, which is admittedly nonstandard, but will make our computations much cleaner.

2Or indeed, if we insist on $L$ just being subexponential.

3For $\ell = 1$, $H_{q,1}$ reduces to the $q$-ary entropy function denoted by $H_q$. 

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“coarse-grained” information concerning the list-decodability/list-recoverability of codes, it leaves many questions unanswered. 

For example, one can ask about the maximum rate of a $(p,3)_{q}$-list-decodable code. That is, what is the maximum rate of a code that never contains more than 2 points from a Hamming ball of radius $pn$? However, this question as stated appears to be quite difficult to solve: any improvement for the special case of $L = 2$ and $q = 2$ would require improving either on the Gilbert-Varshamov bound [24], [53] (on the “possibility” side) or the linear programming bounds [12], [42], [54] (on the “impossibility” side). Unfortunately, despite decades of interest in this basic question hardly any asymptotic improvements on these bounds have been provided in the past fifty years.

Zero-Rate Thresholds for List-Decoding and -Recovery: We therefore begin by targeting a more modest question: what is the maximum $p_{\ell}$ such that for any $p < p_{\ell}$ there exist infinite families of $q$-ary $(p,\ell,L)_{q}$-list-recoverable codes of positive rate? That is, imagining the curve describing the achievable tradeoffs with the rate $R$ on the $y$-axis and decoding radius $p$ on the $x$-axis, instead of asking to describe this entire curve, we simply seek to determine the point where this curve crosses the $x$-axis (clearly, this curve is monotonically decreasing).

Over the binary alphabet, setting $\ell = 1$ and $L = 2$ in this question we recover a famous result of Plotkin [46]: the maximum fraction of errors that can be uniquely-decoded by a general $q$-ary codes is $1/4$. Over general $q$-ary alphabets, this value is similarly known to be $\frac{q-1}{2q}$ (folklore; see, e.g., [29, Theorem 4.4.1]). The value of $p_{\ell}(2,1,L)$ has been computed by Blinovsky [6] for all $L$, and is known to be

$$p_{\ell}(2,1,L) = \frac{1}{2} - \frac{(2k)(k)}{22k+1}$$

if $L = 2k$ or $L = 2k + 1$.

While this expression is quite impenetrable at first glance, here is a natural probabilistic interpretation: given $x_{1},\ldots,x_{L} \in \{0,1\}$, let $pl(x_{1},\ldots,x_{L})$ denote the number of times the more popular bit appears.\(^4\) We then have (see Remark 4 for details)

$$p_{\ell}(2,1,L) = \frac{1}{L} \sum_{(x_{1},\ldots,x_{L}) \sim Bern(1/2)^{\otimes L}} pl(X_{1},\ldots,X_{L})$$

where the notation $(X_{1},\ldots,X_{L}) \sim Bern(1/2)^{\otimes L}$ denotes that $L$ independent unbiased bits are sampled.

It is then not difficult to conjecture the value for $p_{\ell}(q,\ell,L)$: if $pl_{i}(x_{1},\ldots,x_{L})$ denotes the top-$\ell$-plurality value of $x_{1},\ldots,x_{L} \in \{q\}$, i.e., $pl_{i}(x_{1},\ldots,x_{L}) = \max_{\sum \in [q]:|\Sigma| = \ell} \left\{ i \in [L] : x_{i} \in \Sigma \right\}$, then it should be that

$$p_{\ell}(q,\ell,L) = \frac{1}{L} \sum_{(X_{1},\ldots,X_{L}) \sim Unif([q])^{\otimes L}} pl_{i}(X_{1},\ldots,X_{L}).$$

This quantity is fairly natural: one can interpret it as the minimum radius of a list-recovery ball (i.e., a set of the form

$$\{v \in [q]^{n} : v_{i} \in Y_{i} \text{ for at least } (1-p)n \ i \in [n]\}$$

that will contain $L$ codewords in the “typical” case. For the case of $\ell = 1$, i.e., $q$-ary list-decoding, a proof is claimed in [8] and [9]; however, as we outline in Section A this proof is flawed. In this work we provide a rigorous derivation of Equation (1.1) for all values of $\ell$, $L$ and $q$ with $1 \leq \ell \leq q$.

More precisely, we obtain the following results:

- A proof that $(p,\ell,L)_{q}$-list-recoverable $q$-ary codes with $p > p_{\ell}(q,\ell,L)$ have constant-size, i.e., independent of $n$. This should be interpreted as a generalization of the Plotkin bound [46], which states that binary codes uniquely-decodable from a $1/4 + \epsilon$ fraction of errors have size at most $O(1/\epsilon)$. For this reason we call our result a “Plotkin bound for list-recovery.”
- Adapting the Elias-Bassalygo argument [3], we subsequently derive upper bounds on the rate of $(p,\ell,L)_{q}$-list-recoverable $q$-ary codes whenever $p < p_{\ell}(q,\ell,L)$.
- To complement this, we show that there exist infinite families of positive rate $q$-ary codes that are $(p,\ell,L)_{q}$-list-recoverable whenever $p < p_{\ell}(q,\ell,L)$. We are therefore justified in calling $p_{\ell}(q,\ell,L)$ the zero-rate threshold for list-recovery.

Before this work, for a generic tuple of constants $q,p,\ell,L$, the problem of identifying the zero-rate threshold was not considered and our work closes this problem in full generality. Moreover, to our knowledge, there does not appear to be any effective lower or upper bounds in the literature that hold for generic parameters. For sporadic configurations of parameters corresponding to settings including binary list-decoding with even $L$ and certain hash codes (see Section I-B), there exist better bounds using ad-hoc techniques that seem difficult to generalize. Therefore, in a sense for “most” parameters, our bounds are the best known.

As mentioned, our upper and lower bounds both vanish at the zero-rate threshold $p = p_{\ell}(q,\ell,L)$. Moreover, as $L$ increases, they both converge to the list-decoding/-recovery capacity. In particular $p_{\ell}(q,\ell,L)$ converges to $1 - \ell/q$. Our bounds, together with several related expressions from prior works (see Section I-B), are plotted in Figure 1 as functions of $p$ for various configurations of parameters $q,\ell,L$.

We now describe our techniques in more detail.

A. Our Techniques

Sudan Convexity of the Function $f_{q,L,\ell}$: Following prior work [8],\(^5\) our task requires us to answer the following question. Consider the function on distributions $P$ over the alphabet $[q]$ defined as

$$f_{q,L,\ell}(P) := \sum_{(X_{1},\ldots,X_{L}) \sim P^{\otimes L}} pl_{i}(X_{1},\ldots,X_{L}).$$

Analogously to before, the notation $(X_{1},\ldots,X_{L}) \sim P^{\otimes L}$ means that $L$ independent samples are taken from the distribution $P$. A crucial ingredient for deriving the Plotkin bound is a demonstration that this function is minimized by the uniform distribution.

\(^4\)We use $pl$ to stand for “plurality”. However, we caution that this function does not output the more popular symbol (as is perhaps more in line with the standard meaning of plurality), but the number of $i \in [L]$ for which $x_{i}$ equals the most popular symbol.

\(^5\)In fact, [8] only considers list-decoding, so a slight adaptation of this argument is required for list-recovery.
There is a well-studied class of functions on finite distributions with the property that they are minimized by the uniform distribution: Schur convex functions. These are the functions that are monotonically-increasing with respect to the majorization-ordering, which compares vectors of real numbers by first sorting the vectors in descending order and then checking to see if all the prefix sums of one vector is greater than or equal to the prefix sums of the other. The important detail for us is that the uniform vector \((1/q, \ldots, 1/q) \in \mathbb{R}^q\), corresponding to the uniform distribution, is majorized by every other vector corresponding to a distribution over \([q]\).

To demonstrate the Schur convexity of this function, we use the Schur-Ostrowski criterion, which states that Schur-convexity is equivalent to the non-negativity of a certain expression involving partial derivatives. Showing that this expression is non-negative boils down to a combinatorial accounting game, where we can show that the positive contributions arising from certain terms exceed the negative contributions arising from others.

### Convexity of the Univariate Function

**Convexity of the Univariate Function** \(g_{q,\ell,L}(w)\): Another important technical ingredient that we need for the proof of the Plotkin bound is the convexity of the univariate function

\[
g_{q,\ell,L}(w) := f_{q,\ell,L}(P_{q,\ell,w}),
\]

where the distribution \(P_{q,\ell,w} = (p_1, \ldots, p_q)\) is defined as

\[
p_i = \begin{cases} \frac{w}{\ell} & \text{if } i \leq q - \ell \\ \frac{1-w}{L-q+1} & \text{if } i \geq q - \ell + 1
\end{cases}
\]

In order to show the function is convex, we prove the second derivative is non-negative. In differentiating, we use the expression for \(g_{q,\ell,L}\) in terms of \(f_{q,\ell,L}\) and apply the chain rule. Showing the resulting expression is positive is again a sort of combinatorial accounting game: we can show the positive terms contribute more than the negative terms.

Quite interestingly, for \(\ell = 1\) (i.e., the case relevant for list-decoding) we only prove the convexity of the function \(f_{q,1,L}\) on the interval \([0, (q-1)/q]\). Fortunately, as we can also easily show that \(g_{q,1,L}\) decreases on the interval \([0, (q-1)/q]\) and then increases on the interval \([(q-1)/q, 1]\).\(^6\) Convexity of \(f_{q,1,L}\) on \([0, (q-1)/q]\) suffices for our purposes. And indeed, this is not an artifact of the proof: Blinovsky had already observed that convexity of \(f_{q,1,L}\) does not hold on the entire interval \([0, 1]\).\(^8\) However, for \(\ell \geq 2\) we obtain that convexity of \(f_{q,\ell,L}\) does indeed hold on the entire interval \([0, 1]\). We note that the second derivative does behave qualitatively differently, so this is perhaps not too surprising in hindsight; we comment on this further in Remark 5.

### Plotkin Bound

**Plotkin Bound**: Armed with these (Schur-)convexity results, we aim to prove a Plotkin bound for list-decoding/recovery. That is, if a \(q\)-ary code is \((p, \ell, L)_q\)-list-recoverable with \(p \geq p_a(q, \ell, L) + \varepsilon\), how large can the code be? Following the template of the standard argument (although certain subtleties arise when generalizing to list-recovery), we can show that such a code must be of constant size, i.e., independent of \(n\).

Informally, the argument begins with a "preprocessing step" that prunes away some (but, crucially, not too many) code-words and yields a more structured subcode that we can subsequently analyze. The codewords of this subcode are very "balanced" in the sense that all patterns of symbols appear with roughly the same frequency. In particular, every pattern of length \(t\) should appear roughly a \(1/q^t\) fraction of the time (or the code is very "biased," in which case a separate argument bounds its size).

To analyze this subcode \(C^\ell\) we apply a double-counting argument to the average radius to cover \(L\)-subsets (where for list-recoverability, this radius is measured via the distance to a tuple of input lists). The lower bound on this quantity follows quite naturally from the list-decodability/recoverability of the code, together with the "balancedness" of the subcode. For the upper bound, we compute the radius of an \(L\)-subset in terms of the empirical distribution of a coordinate \(k \in [n]\), i.e., each \(x \in [q]\) is assigned probability mass \(P_k(x) = \frac{1}{M} \sum_{x \in \mathcal{C}} \mathbb{1}\{x_k = x\}\). By the Schur convexity of the function \(f_{q,\ell,L}\) and the convexity of the univariate function \(g_{q,\ell,L}\), we can bound this in terms of a distribution placing total mass \(w \leq \frac{q-\ell}{q}\) on the last \(\ell\) elements of \([q]\) and mass \(\frac{1-w}{L}\) on each of the others. The result then follows.

We remark that, due to our use of Ramsey-theoretic arguments, the precise bound we obtain on the code size is quite poor. We have made no effort to optimize this constant. However, we do believe it would be interesting to improve this bound; we discuss this further in Section XIII.

**Elias-Bassalygo-Style Bound**: After deriving this Plotkin bound, a well-known argument template (typically attributed to Elias and Bassalygo [3]) allows one to derive more general tradeoffs between the rate \(R\) and the noise-resilience parameters \((p, \ell, L)_q\). Informally, this proceeds by covering the space \([q]^n\) by a bounded number of list-recovery balls. The radius of these balls is carefully chosen to allow one to apply the Plotkin bound to the subcodes obtained by taking the intersection of the code with these balls. On the other hand, the number of list-recovery balls needed to cover \([q]^n\), known as the covering number, can be sharply estimated. From the above two bounds (the Plotkin bound and the covering number), a bound on the size of the whole code can be derived.

**Possibility Result: Random Code With Expurgation**: To complement the Plotkin bound, we show that if the decoding radius \(p\) is less than \(p_a(q, \ell, L)\) then there exist infinite families of \((p, \ell, L)_q\)-list-recoverable \(q\)-ary codes. This justifies our "zero-rate threshold" terminology for \(p_a(q, \ell, L)\). The argument is completely standard, obtained by sampling a random code and subsequently expurgating codewords to destroy all \(size-L\) lists that can fit into Hamming balls of radius \(np\). In fact, the lower bound on achievable rate is derived from the exact large deviation exponent of a certain quantity known as the average radius (cf. Definition 12) of a tuple of random vectors. Therefore the bound holds under a stronger notion called average-radius list-recovery: namely, for any subset of \(L\) codewords \(x_1, \ldots, x_j\) and any tuple of input lists \((Y_1, \ldots, Y_n)\), we have

\[
\sum_{j=1}^{L} \{i \in [n] : x_{j,i} \neq Y_i\} > Lpn.
\]
Comparison to Claimed Proofs in [8] and [9]: A proof the Plotkin bound for q-ary list-decoding was previously claimed. The crux again boils down to understanding the functions \( f_{q,d} \) and \( g_{q,d} \) which was done in [8] and [9] respectively. As detailed in Section A, both proofs are flawed and counterexamples can be constructed for certain technical steps therein. We do not see an easy redemption of these proofs and instead follow a different route based on Schur convexity. Our argument leads to a stronger statement and is, in our modest opinion, cleaner and easier to verify. More importantly, it admits seamless extension to the list-recovery setting for which the problem has never been studied to our knowledge.

B. Discussion on Related Work

Lower Bounds for Small q and/or L: For the case of \((p, 3)\) list-decoding, it was shown in [27, Theorem 6.1] that the threshold rate\(^7\) of random binary linear code equals
\[
\frac{1}{2} (2 - H_2(3p) - 3p \log_2(3)).
\] (I.2)

The term threshold refers to the critical rate below which a random binary linear code is \((p, 3)\)-list-decodable with high probability and above which it is not with high probability. This result was recently extended to the following two cases [51]. For \((p, 4)\)-list-decoding, the threshold rate of random binary linear code is lower bounded by [51, Theorem 1.3]
\[
\frac{1}{3} \min_{x_1, x_2 > 0 : x_1 + 2x_2 \leq 4p} 3 - \eta_2(x_1, x_2) - 2x_1 - x_2 \log_2(3). \tag{I.3}
\]

Here we use the notation
\[
\eta_q(x_1, \ldots, x_t) := \sum_{i=1}^t x_i \log_q x_i + \left(1 - \sum_{i=1}^t x_i\right) \log_q 1 - \frac{1}{1 - \sum_{i=1}^t x_i}
\]

for a partial probability vector \((x_1, \ldots, x_t) \in \mathbb{R}_{\geq 0}^t\) satisfying \( t \leq q \) and \( x_1 + \cdots + x_t \leq 1 \). Note that \( \eta_2(x) = H_2(x) \), however, this is no longer the case for \( q > 2 \). Moreover, for \((p, 3)\)-q-list-decoding, [51, Theorem 1.5] showed that the threshold rate of random code is at least
\[
\frac{1}{2} \min_{x_1, x_2 > 0 : x_1 + 2x_2 \leq 3p} \left\{ 2 - \eta_q(x_1, x_2) - x_1 \log_q(3(q - 1)) \right. \\
\left. - x_2 \log_q(1)(q - 2) \right\}. \tag{I.4}
\]

Our general lower bound (cf. Theorem 16) for list-recovery (numerically) matches Equations (I.2) to (I.4) upon particularizing the parameters \( q, \ell, L \) suitably. See Figures 1a to 1c. It is possible to analytically prove this observation, though we do not pursue it in the current paper. The rationale underlying this phenomenon is that the threshold rate of random linear codes for list-recovery is expected to match the rate achieved by random codes with expurgation (with the notable exception of zero-error list-recovery [26]). This conjecture, in its full generality, remains unproved, although it is partially justified in several recent works [26], [27], [43], [51].

Hash Codes: One may note that for \( \ell \geq 2 \), our upper and lower bounds typically exhibit a large gap even at \( p = 0 \). See Figures 1e to 1h. We provide evidence below indicating that closing this gap is in general a rather challenging task and necessarily requires significantly new ideas. Let us focus on the vertical axis \( p = 0 \), known as zero-error list-recovery. We observe that some configurations of \( q, \ell, L \) in this regime encode several longstanding open questions in combinatorics. Indeed, consider \( \ell = q - 1, L = q \). The \((0, q - 1, q)\) list recoverability condition can then be written as: for any \( \mathcal{Y}_1, \ldots, \mathcal{Y}_n \in \mathcal{G}_{q-1}^q \),
\[
|\{x \in \mathcal{C} : |\{j \in [n] : x_j \notin \mathcal{Y}_j\} = 0\}| \leq L - 1,
\]

i.e.,
\[
|\{x \in \mathcal{C} : \forall j \in [n], x_j \in \mathcal{Y}_j\}| \leq L - 1.
\]

Taking the contrapositive, we note that this condition is further equivalent to: for any \( \{x_1, \ldots, x_L\} \in \mathcal{Y}_L^q \), there exists \( j \in [n] \) such that \( \{x_{1,j}, \ldots, x_{L,j}\} = q \). In words, for any q-tuple of codewords in a \((0, q - 1, q)\)-q-list-recoverable code, there must exist one coordinate such that the corresponding q-ary symbols in the tuple are all distinct. Such a code is also known as a q-hashing in combinatorics. It is well-known [19], [40] that a probabilistic construction yields such codes of rate\(^8\) at least
\[
C_{(q-1,q)q} \geq \frac{1}{q - 1} \log_q \frac{1}{1 - \frac{q - 1}{q}}. \tag{I.5}
\]

In the same paper [19] also proved an upper bound
\[
C_{(q-1,q)q} \leq \frac{q^4}{q^q - 1} \log_q(2). \tag{I.6}
\]

Another upper bound
\[
C_{(q-1,q)q} \leq \log_q \frac{q}{q - 1}. \tag{I.7}
\]

can be proved using either a double-counting argument (a.k.a. first moment method), or (hyper)graph entropy [37], [39], [40]. Equation (I.6) is much better than Equation (I.7) for \( q \geq 4 \). However, the latter bound \( \log_q \frac{q}{q} \) remains the best known for \( q = 3 \) (called the trieffference problem by Körner). For larger \( q \), both lower [56] and upper bounds [2], [10], [11], [13], [28] can be improved. However, improving the bound for \( q = 3 \) is recognized as a formidable challenge. We will show in Remark 9 in Section XII that our lower bound for list-recovery (cf. Theorem 16) recovers Equation (I.5) for q-hashing upon setting \( \ell = q - 1, L = q \). Furthermore, our upper bound Theorem 10 recovers Equation (I.7) for q-hashing (cf. Remark 7).

A generalization of q-hashing known as \((q, L)\)-hashing \((q \geq L)\) can also be cast as zero-error list-recoverable codes with more general values of \( \ell, L \). Indeed, taking \( L = \ell + 1 \) and \( \ell \leq q - 1 \), we can write \((0, \ell, \ell + 1)\)-list-recoverability

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\(^7\)We warn the reader not to confuse this concept with that of the zero-rate threshold.

\(^8\)The bounds in [19] and [37] are slightly adjusted so that they are consistent with our definition of code rate which adopts a \( \log_q \) normalization (cf. Definition 6).
alternatively as: for any \( \{x_1, \ldots, x_{\ell+1}\} \in \binom{\mathcal{C}}{\ell+1} \), there exists \( j \in [n] \) such that \( \{x_{1j}, \ldots, x_{\ell+1j}\} = \ell + 1 \). This is in turn the precise definition of \( (q, \ell + 1)\)-hashing. It can be immediately seen that \( (q, q)\)-hashing is nothing but \( q\)-hashing. The upper and lower bounds in \([19]\) also extend to \( (q, \ell + 1)\) -hashing and read as follows:

\[
\frac{1}{\ell} \log \frac{1}{1 - \left(\frac{q}{q+1}\right)^{\ell+1}!} \leq C_{(0,\ell,\ell+1)} \leq \frac{\ell!}{q^\ell} \log(q - \ell + 1).
\]  

(1.8)

Our lower bound for list-recovery in Theorem 16 also recovers the above lower bound for \( (q, \ell + 1)\)-hashing by \([19]\) upon setting \( L = \ell + 1 \) (see Remark 8). The upper bound was later improved in \([37]\) for \( q > L \) using the notion of hypergraph entropy:

\[
C_{(0,\ell,\ell+1)\geq} \leq \min_{0 \leq j \leq \ell - 1} \frac{\left(\frac{q}{q+1}\right)(j + 1)!}{q^{\ell+1}} \log q - j \ , q - j - 1, \]  

(1.9)

though it coincides with Equation (1.6) when \( \ell = q - 1 \). Some improved upper bounds in \([13]\) and \([28]\) apply to \( (q, \ell + 1)\)-hashing as well. To the best of our knowledge, no improvement on lower bounds is known for \( \ell < q - 1 \).

Zero-Rate Thresholds for General Adversarial Channels: The problem of locating the zero-rate threshold has been addressed in a much more general context \([57]\). The results in \([57]\) on general adversarial channel model can be specialized to the list-recovery setting and read as follows. Given \( q, p, \ell, L \), define the confusability set \( \mathcal{K}(p, \ell, L) \), as the set of \( \ell \)-partitions of an integer \( q \) (cf. Definition 15) of all “confusable” \( \ell \)-tuple of codewords in the sense that they can fit into a certain list-recovery ball (cf. Definition 3) of radius \( np \). Specifically,

\[
\mathcal{K}(p, \ell, L) : = \left\{ \sum_{y \in \binom{\mathcal{G}}{\ell}} P_{X_1, \ldots, X_L, Y = y} \in \Delta([q]^L) : \right. \\
\quad \forall i \in [L], \sum_{(x,y)\in\Delta([q]^L)} P_{X_1,Y}(x,y) \leq p, \right\},
\]


In the above definition, we use \( \Delta(\Sigma) \) to denote the probability simplex over a finite set \( \Sigma \), whose elements are typically denoted by \( P \), perhaps with subscripts. In other cases, vectors are denoted by \textit{boldface} letters, e.g., \( \alpha \), and the \( i \)-th entry of \( \alpha \) is denoted by \( a_i \). The symmetric group of order \( n \) is denoted by \( S_n \). For a finite set \( \Sigma \) and an integer \( 0 \leq k \leq |\Sigma| \), denote by

\[
\binom{\Sigma}{k} := \{ \Pi \subset \Sigma : |\Pi| = k \}
\]

the collection of all size-\( k \) subsets of \( \Sigma \). We use \( \exp(x) \) and \( \exp_p(x) \) (where \( b \in \mathbb{R} \)) to denote \( e^x \) and \( b^x \), respectively. For a positive integer \( k \), we use \( \lfloor k \rfloor \) to denote the set \( \{ 1, 2, \ldots, k \} \). We use the following notation for \( q \)-partitions of an integer \( L \). For integers \( q \geq 1, L \geq 0 \),

\[
\mathcal{A}_{q,L} := \left\{ (a_1, \ldots, a_q) \in \mathbb{Z}_{\geq 0}^q : \sum_{i=1}^q a_i = L \right\}.
\]

For \( \alpha = (a_1, \ldots, a_q) \in \mathcal{A}_{q,L} \), we often simplify notation for the multinomial coefficient and write

\[
\binom{L}{\alpha} := \binom{L}{a_1, \ldots, a_q} = \frac{L!}{\prod_{i=1}^q (a_i!)}. \]

It is proved in \([57]\) that the zero-rate threshold \( p_\emptyset(q, \ell, L) \) can be expressed as the smallest \( p \) such that all completely positive distributions are confusable:

\[
p_\emptyset(q, \ell, L) = \inf \left\{ p \in [0,1] : CP_q^\otimes \cap \Delta([q]^L) \subset \mathcal{K}(p, \ell, L) \right\}.
\]  

(1.10)

The above characterization is \textit{single}-letter in the sense that it is independent of the blocklength \( n \). For \( q, \ell, L \) independent of \( n \) (which is assumed to be the case in the current paper), the optimization problem on the RHS of Equation (1.10) can be solved in constant time. However, it does not immediately provide an explicit formula of \( p_\emptyset(q, \ell, L) \) and analytically solving the optimization problem does not appear easy to the authors. On the other hand, the characterization \( p_\emptyset(q, \ell, L) = 1 - \frac{1}{L} \mathbb{E} \left[ \rho_p(X_1, \ldots, X_L) \right] \) (where the expectation is over \( (X_1, \ldots, X_L) \sim \text{Unif}([q]^L) \), cf. Equation (1.1)) in this paper can be seen as the explicit solution to the optimization problem, though the way it is obtained is \textit{not} by solving the latter problem per se. Instead, we prove the characterization from the first principle by leveraging specific structures of list-recovery. We hope that our characterization can shed light on the geometry of the high-dimensional polytopes – the confusability set and the set of CP distributions – involved in the characterization in Equation (1.10).

Organization: We begin with the necessary preliminaries in Section II, and subsequently state our main results in Section III. Sections IV to XI contain proofs of our main technical results. Section XII derives a lower bound on list-recovery capacity via a random coding argument. We summarize our results and state open problems in Section XIII and provide acknowledgements in Section D.
Fig. 1. Plots of upper and lower bounds in [6], [8], [27], and [51] and this work for various values of $q$.

For any $a \in \mathbb{R}^q$, let

$$\max_{\ell} \{a\} := \max_{Q \in \{q\}} \sum_{i \in Q} a_i,$$

denote the largest partial sum of $\ell$ coordinates from $a$.

**B. Basic Definitions**

Throughout the paper, $q \in \mathbb{Z}_{\geq 2}$ denotes the alphabet size, $L - 1 \in \mathbb{Z}_{\geq 1}$ denotes the list size, $n \in \mathbb{Z}_{\geq 1}$ denotes the blocklength. The parameter $\ell$ used in list-recovery takes integer values between 1 and $q - 1$. The $\ell = 1$ case reduces to list-decoding and the $\ell = q$ case is trivial (cf. Definition 4).

1) **List-Decodability and List-Recoverability:**

**Definition 1 (Hamming Metric):** We equip the space $[q]^n$ with the Hamming metric:

$$d_H(x, y) := \sum_{i=1}^{n} \mathbb{1}\{x_i \neq y_i\}$$
for \( x, y \in [q]^n \). The Hamming weight of a vector \( x \in [q]^n \) is defined as
\[
\omega_H(x) := d_H(x, q),
\]
where \( q = (q, q, \ldots, q) \in [q]^n \) is the all-\( q \) vector.

A Hamming ball or Hamming sphere centered around \( y \in [q]^n \) of radius \( 0 \leq r \leq n \) is defined as
\[
B_H(y, r) := \{ x \in [q]^n : d_H(x, y) \leq r \}.
\]

or
\[
S_H(y, r) := \{ x \in [q]^n : d_H(x, y) = r \},
\]
respectively.

**Remark 1**: Our definition of Hamming weight is nonstandard; typically, the weight is defined to be the distance from the all-0 string. However, as our arguments regularly work with probability distributions over the alphabet, and in said arguments it is convenient for us to view these probability distributions as vectors in real-space, it is notionally simplest for us to assume the alphabet is \( q = \{1, 2, \ldots, q\} \). We therefore have chosen to make \( q \) the “distinguished” symbol from the alphabet, i.e., it takes the role that 0 typically plays. (Note further that as we are not concerned with linear codes, there is no mathematical reason that we should have 0 playing this distinguished role.)

**Definition 2 (List-Decodability):** A code \( C \subset [q]^n \) is said to be \((p, L)_q\)-list-decodable if for every \( y \in [q]^n \),
\[
|C \cap B_H(y, np)| < L.
\]

**Remark 2**: When we talk about a code \( C \subset [q]^n \), we always mean a sequence of codes \( \{C_n\}_n \) for an infinite increasing sequence of \( n \in \mathbb{Z}_{\geq 1} \).

Analogous definitions can be made for list-recovery.

**Definition 3**: Given a vector \( x \in [q]^n \) and a tuple of sets \( \mathcal{Y} \in \binom{[q]}{\ell}^n \) for \( 1 \leq \ell \leq q - 1 \), we define
\[
d_{LR}(x, \mathcal{Y}) := \sum_{i=1}^n \mathbf{1}\{x_i \notin \mathcal{Y}_i\}, \quad d_{LR}(x) := d_{LR}(x, \mathcal{Y}_\ell),
\]
where \( \mathcal{Y}_\ell := \{q-\ell+1, \ldots, q\}^{\binom{[q]}{\ell}^n} \). We also define the list-recovery ball and sphere centered at \( \mathcal{Y} \in \binom{[q]}{\ell}^n \) of radius \( 0 \leq r \leq n \) as
\[
B_{LR}(\mathcal{Y}, r) := \{ x \in [q]^n : d_{LR}(x, \mathcal{Y}) \leq r \},
\]
\[
S_{LR}(\mathcal{Y}, r) := \{ x \in [q]^n : d_{LR}(x, \mathcal{Y}) = r \},
\]
respectively.

**Definition 4 (List-Recoverability):** A code \( C \subset [q]^n \) is said to be \((p, \ell, L)_q\)-list-recoverable if for every \( \mathcal{Y} \in \binom{[q]}{\ell}^n \),
\[
|C \cap B_{LR}(\mathcal{Y}, np)| < L.
\]

2) **Radius, Capacity and Zero-Rate Threshold:**

**Definition 5 (Radius):** The radius of an \( L \)-set of vectors \( x_1, \ldots, x_L \in [q]^n \) is defined as the radius of the smallest Hamming ball containing the set \( \{x_1, \ldots, x_L\} \):
\[
\text{rad}_L(C) := \min_{y \in [q]^n} \max_{i \in [L]} \omega_H(x_i, y).
\]

The \( L \)-radius of a code \( C \subset [q]^n \) of size at least \( L \) is defined as the maximum radius of \( L \)-lists of codewords in \( C \):
\[
\text{rad}_L(C) := \max_{\mathcal{L} \in \binom{[q]}{\ell}^n} \text{rad}_L(C).
\]

In fact, the radius of a code characterizes its list-decodability. Indeed, inspecting Definition 2 and Equation (II.2), one observes that a code \( C \subset [q]^n \) is \((p, L)_q\)-list-decodable if and only if \( \text{rad}_L(C) > n p \).

A quantity of central interest in the study of combinatorial list-decoding is the list-decoding capacity defined as follows.

**Definition 6 (Rate, List-Decoding Capacity and Zero-Rate Threshold):** The rate of a code \( C \subset [q]^n \) is defined as
\[
R(C) := \frac{1}{n} \log_q |C|.
\]

The \((p, L)_q\)-list-decoding capacity is defined as the maximum rate of \((p, L)_q\)-list-decodable codes:
\[
C_{(p, L)_q} := \limsup_{n \to \infty} \max_{C \subset [q]^n, |C| = n} R(C_n).
\]

Since \( C_{(p, L)_q} \) is a decreasing function in \( p \), we define the zero-rate threshold for \((p, L)_q\)-list-decodling as the maximum \( p \) such that the capacity is strictly positive:
\[
p_*(q, L) := \sup \{ p > 0 : C_{(p, L)_q} > 0 \}.
\]

Again, we extend the above definitions to list-recovery. Let us start with \( \ell \)-radius for list-recovery.

**Definition 7 (\( \ell \)-Radius):** The \( \ell \)-radius of an \( L \)-set of vectors \( x_1, \ldots, x_L \in [q]^n \) is defined as the radius of the smallest list-recovery ball containing the set \( \{x_1, \ldots, x_L\} \):
\[
\text{rad}_{\ell}(x_1, \ldots, x_L) := \min_{\mathcal{Y}} \max_{i \in [L]} d_{LR}(x_i, \mathcal{Y}).
\]

The \( (\ell, L) \)-radius of a code \( C \subset [q]^n \) of size at least \( L \) is defined as the maximum \( \ell \)-radius of \( L \)-lists of codewords in \( C \):
\[
\text{rad}_{\ell,L}(C) := \max \text{rad}_{\ell}(L).\]

Capacity and zero-rate threshold for list-recovery are similarly defined by noting that \((p, \ell, L)_q\)-list-recoverability of any \( C \subset [q]^n \) is equivalent to \( \text{rad}_{\ell,L}(C) > n p \).

**Definition 8 (Capacity and Zero-Rate Threshold for List-Recovery):** The \((p, \ell, L)_q\)-list recovery capacity is defined as the largest rate of any \((p, \ell, L)_q\)-list-recoverable code:
\[
C_{(p, \ell, L)_q} := \limsup_{n \to \infty} \max_{C \subset [q]^n, |C| = n} R(C_n),
\]
and the zero-rate threshold is defined to be the smallest value of \( p \) at which \( C_{(p, \ell, L)_q} \) first hits zero:
\[
p_*(q, \ell, L) := \sup \{ p > 0 : C_{(p, \ell, L)_q} > 0 \}.
\]

3) **Average Radius:** Tightly related to the above definition of radius is another widely-studied notion called average radius. To introduce this concept, let us first define the plurality of a sequence of symbols.

**Definition 9 (Plurality):** For \( L \in \mathbb{Z}_{\geq 2} \) and \( x_1, \ldots, x_L \in [q] \), the plurality of \( x_1, \ldots, x_L \) is defined to be the number of times the most frequent symbol appears:
\[
\mathbf{pl}(x_1, \ldots, x_L) := \max_{a \in [a]} \sum_{i=1}^L \mathbf{1}\{x_i = a\}.
\]

The maximizer is denoted by \( \text{argpl}(x_1, \ldots, x_L) \in [q] \).

The notion of average radius can then be defined using plurality.
Definition 10 (Average Radius): The average radius of an $L$-set of vectors $x_1, \ldots, x_L \in [q]^n$ is defined as

$$\overline{\operatorname{rad}}(x_1, \ldots, x_L) := \frac{1}{n} \sum_{j=1}^{n} \left(1 - \frac{1}{L} \mathbb{P}(x_{1j}, \ldots, x_{Lj})\right).$$

We pause to explain the motivation for defining average radius in the above particular way. It is well-known in the literature [31, Lemma 2.3] and not hard to check that the average radius defined in Equation (II.4) is the solution to the following optimization problem:

$$\min_{y \in [q]^n} \mathbb{E}_{t \sim \text{Unif}(\{L\})} \left[d_H(x_I, y)\right],$$

and the minimizer is given by $y^* \in [q]^n$ defined as $y^* = \arg\max_{y \in [q]^n} \mathbb{E}_{t \sim \text{Unif}(\{L\})} \left[d_H(x_I, y)\right]$.

Definition 11 ($\ell$-Plurality): Let $L \in \mathbb{Z}_{\geq 2}$ and $1 \leq \ell \leq q-1$ be an integer. The $\ell$-plurality of an $L$-tuple of symbols $x_1, \ldots, x_L \in [q]$ is defined as the total number of times that the top $\ell$ most frequent symbols appear:

$$\operatorname{pl}_\ell(x_1, \ldots, x_L) := \max_{\Sigma \subseteq \{q\}^\ell} \sum_{i=1}^{\ell} \mathbb{I}\{x_i \in \Sigma\}.$$  

The maximizer is denoted by $\arg\operatorname{pl}_\ell(x_1, \ldots, x_L) \subseteq \{q\}^\ell$.

Remark 3: It is easy to see that $\operatorname{pl}_1(\cdot) = \operatorname{pl}(\cdot)$.

Definition 12 ($\ell$-Average Radius): The $\ell$-average radius of an $L$-set of vectors $x_1, \ldots, x_L \in [q]^n$ is defined as

$$\overline{\operatorname{rad}}_\ell(x_1, \ldots, x_L) := \sum_{j=1}^{L} \left(1 - \frac{1}{L} \operatorname{pl}_\ell(x_{1j}, \ldots, x_{Lj})\right).$$

Similarly, Equation (II.6) is the solution to the following relaxation of Equation (II.3):

$$\min_{\mathcal{Y} \subseteq \{q\}^n} \mathbb{E}_{t \sim \text{Unif}(\{L\})} \left[d_{LR}(x_I, \mathcal{Y})\right],$$

with minimizer $\mathcal{Y}^* \subseteq \{q\}^n$ given by $\mathcal{Y}^* = \arg\max_{\mathcal{Y} \subseteq \{q\}^n} \mathbb{E}_{t \sim \text{Unif}(\{L\})} \left[d_{LR}(x_I, \mathcal{Y})\right]$ for every $i \in [n]$.

III. MAIN RESULTS

A. $q$-Ary List-Decoding

Define $f_{q, L} : \Delta([q]) \to \mathbb{R}_{\geq 0}$ as

$$f_{q, L}(P) := \frac{1}{L} \mathbb{E}_{(X_1, \ldots, X_L) \sim P \otimes [L]} \left[\mathbb{P}(X_1, \ldots, X_L)\right]$$

for $P \in \Delta([q])$.

For $w \in [0, 1]$, let $P_{q, w} \in \Delta([q])$ denote the following probability vector:

$$P_{q, w} := \left(\frac{w}{q-1}, \ldots, \frac{w}{q-1}, 1-w\right).$$

Define $g_{q, L} : [0, 1] \to \mathbb{R}_{\geq 0}$ as

$$g_{q, L}(w) := f_{q, L}(P_{q, w}).$$

Definition 13 (Majorization): Let $a, b \in \mathbb{R}^d$. Let $a^i, b^i \in \mathbb{R}^d$ denote the vectors obtained by sorting the elements in $a$ and $b$ in descending order, respectively. We say that $a$ majorizes $b$, written as $a \geq_{\text{m}} b$, if

$$\sum_{i=1}^{k} a^i \geq \sum_{i=1}^{k} b^i$$

for every $k \in [d]$, and

$$\sum_{i=1}^{d} a_i = \sum_{i=1}^{d} b_i.$$

Definition 14 (Schur Convexity): A function $f : \mathbb{R}^d \to \mathbb{R}$ is called Schur-convex if $f(x) \geq f(y)$ for every $x, y \in \mathbb{R}^d$ such that $x \geq_{\text{m}} y$ (in the sense of Definition 13).

Theorem 1 (Schur Convexity of $f_{q, L}$): For any $q \in \mathbb{Z}_{\geq 2}$ and $L \in \mathbb{Z}_{\geq 2}$, the function $f_{q, L} : \Delta([q]) \to \mathbb{R}_{\geq 0}$ defined in Equation (III.1) is Schur convex.

Proof: See Section IV.

Theorem 2 (Convexity of $g_{q, L}$): For any $q \in \mathbb{Z}_{\geq 2}$ and $L \in \mathbb{Z}_{\geq 2}$, the function $g_{q, L} : [0, 1] \to \mathbb{R}_{\geq 0}$ defined in Equation (III.3) is convex in the interval $[0, (q-1)/q]$.

Proof: See Section V.

Remark 4: In the binary case (i.e., $q = 2$), understanding the functions $f_{2, L}$ and $g_{2, L}$ is an easier task. In fact, $f_{2, L}$ collapses to a univariate function and coincides with $g_{2, L}$. It can be computed [7, Eqn. (2.15) and (2.16)] that for $L = 2k, 2k+1$,

$$p_s(2, L; w) := 1 - \frac{1}{L} g_{2, L}(w) = \sum_{i=1}^{k} \frac{(2k-1)}{i} (w(1-w))^i,$$

and

$$\frac{q^2}{c w^2} p_s(2, L; w) = -k \left(\frac{2k}{k}\right)^i (w(1-w))^k.$$

The concavity (see also [47, Lemma 8]) and monotonicity of $p_s(2, L; w)$ immediately follow. Such explicit computation cannot be performed in the $q > 2$ case (and for list-recovery) and we have to work with summations like in Lemma 2.

Other approaches to arguing monotonicity such as induction [1, Lemma 8(d)] do not seem to work well either for larger $q$.

As convexity only holds in the interval $[0, (q-1)/q)$, we will also require the following monotonicity properties, which follow easily from the Schur convexity of $f_{q, L}$.

Lemma 1: For any $q \in \mathbb{Z}_{>2}$ and $L \in \mathbb{Z}_{>2}$, the function $g_{q, L} : [0, 1] \to \mathbb{R}_{\geq 0}$ defined in Equation (III.3) is non-increasing on $[0, (q-1)/q]$ and non-decreasing on $[(q-1)/q, 1]$.

Proof: See Section B.

Define $p_s(q, L; w) := 1 - \frac{1}{L} g_{q, L}(w).$

Theorem 3 (Plotkin Bound for $q$-ary List-Decoding): Fix any $q \in \mathbb{Z}_{\geq 2}$ and $L \in \mathbb{Z}_{\geq 2}$. Let $C \subseteq \mathbb{Y}^n$ be an arbitrary $(p, L, \mathcal{L})$-list-decodable code with $p = p_s(q, L; \frac{1}{q-1}) + \tau$
for any constant $\tau \in (0, 1)$. Then there exists a constant $M_a = M_a(q, L, \tau)$ independent of $n$ such that $|C| \leq M_a$. As a consequence, in particular we have
\[
p_a(q, L) \leq p_a \left( q, L; \frac{q - 1}{q} \right) = 1 - \frac{1}{L} g_{q, L} \left( \frac{q - 1}{q} \right). \]

The proof of this theorem can be found in Section VI. Specifically, a theorem (cf. Theorem 12) of the above kind will be first proved for approximately constant-weight codes in which all codewords have approximately the same Hamming weight. This theorem can then be used to prove Theorem 3 above (see Corollary 4 for a more quantitative version) by partitioning a general (weight-unconstrained) code into a constant number of almost constant-weight subcodes.

The upper bound on the zero-rate threshold in Theorem 3 is in fact sharp. It turns out that positive rate $(p, L)_q$-list-decodable codes exist for any $p$ strictly smaller than the bound $1 - \frac{1}{L} g_{q, L} \left( \frac{q - 1}{q} \right)$ in Theorem 3. Indeed, Blinovsky [8] proved the following lower bound on the $(p, L)_q$-list-decoding capacity which remains the best known to date. It can also be implied by our lower bound (Theorem 9 below) for list-recovery upon setting $\ell = 1$.

**Theorem 4 ([8, Sec. 2]):** For any $q \in \mathbb{Z}_{\geq 2}$, $L \in \mathbb{Z}_{\geq 2}$ and $0 \leq p < p_a \left( q, L; \frac{q - 1}{q} \right)$, the following lower bound on the $(p, L)_q$-list-decoding capacity holds:
\[
C_{(p, L)_q} \geq \frac{L}{L - 1} - \frac{1}{L - 1} \left( \lambda_a + \log_q \left( \sum_{a \in A_{q, L}} \left( \frac{L}{a} \right)^{\frac{1}{L} \max \{a\}} \right) \right) \times \exp_q \left( -\lambda_a \left( 1 - \frac{1}{L} \max \{a\} \right) \right),
\]
where $\lambda_a = \lambda_a(q, L, p)$ is the solution to the following equation
\[
p = \frac{\sum_{a \in A_{q, L}} \left( \frac{L}{a} \right)^{q - \lambda_a \left( 1 - \frac{1}{L} \max \{a\} \right)}}{\sum_{a \in A_{q, L}} \left( \frac{L}{a} \right)^{q - \lambda_a \left( 1 - \frac{1}{L} \max \{a\} \right)}}.
\]

Blinovskiy’s lower bound is plotted in Figure 1d. It is not hard to verify [8, Eqn. (19)] that the lower bound above vanishes at
\[
p = q^{-L} \sum_{a \in A_{q, L}} \left( \frac{L}{a} \right)^{\frac{1}{L} \max \{a\}} \left( 1 - \frac{1}{L} \max \{a\} \right),
\]
and the corresponding $\lambda_a$ equals 0.

Theorems 3 and 4 together pin down the exact value of $p_a(q, L)$ shown in the following corollary.

**Corollary 1:** For any $q \in \mathbb{Z}_{\geq 2}$ and $L \in \mathbb{Z}_{\geq 2}$, the zero-rate threshold $p_a(q, L)$ for $(p, L)_q$-list-decoding is given by
\[
p_a(q, L) = p_a \left( q, L; \frac{q - 1}{q} \right) = 1 - \frac{1}{L} g_{q, L} \left( \frac{q - 1}{q} \right) = q^{-L} \sum_{a \in A_{q, L}} \left( \frac{L}{a} \right)^{\frac{1}{L} \max \{a\}}. \tag{III.5}
\]

The last equality above follows from the definition of $g_{q, L}$ in Equation (III.3) and some simple algebra:
\[
1 - \frac{1}{L} g_{q, L} \left( \frac{q - 1}{q} \right) = 1 - \frac{1}{L} f_{q, L} (\text{Unif}([q])) = 1 - \frac{1}{L} \sum_{a \in A_{q, L}} \left( \frac{L}{a} \right) \max \{a\} q^{-L} = q^{-L} \sum_{a \in A_{q, L}} \left( \frac{L}{a} \right) \left( 1 - \frac{1}{L} \max \{a\} \right),
\]
where the second equality uses the expression of $f_{q, L}$ in Lemma 2, and the last equality uses the multinomial theorem
\[
f_{q, L} = \sum_{a \in A_{q, L}} \left( \frac{L}{a} \right)^{\frac{1}{L} \max \{a\}}.
\]
From now on, we will use $p_a(q, L)$ to denote the RHS of Equation (III.5).

**Theorem 5 (Elias-Bassalygo Bound for q-ary List-Decoding):** Fix any $q \in \mathbb{Z}_{\geq 2}$, $L \in \mathbb{Z}_{\geq 2}$ and $0 \leq p < p_a(q, L)$. Then the $(p, L)_q$-list-decoding capacity can be upper bounded as $C_{(p, L)_q} \leq 1 - H_q(w_{q, L})$ where $w_{q, L}$ is the solution to the equation $p_a(q, L; w) = p$ in $w \in [0, (q - 1)/q]$.

**Proof:** The above theorem is implied by Theorem 13 proved in Section VII. The latter theorem shows that for any $(p, L)_q$-list-decodable code $C \subset [q]^n$ with $p < p_a(q, L)$ and sufficiently small constant $\tau > 0$, $|C|$ is at most $B \cdot n^{1.5-qn(1-H_q(w_{q, L, \tau}))}$, where $B = B(q, L, \tau)$ is a constant and $w_{q, L, \tau}$ is the solution to $p_a(q, L; w) = p - \tau$. Taking $\tau \to 0$ and neglecting polynomial factors, we obtain the upper bound on the list-decoding capacity.

The above upper bound is plotted in Figure 1d.

**B. List-Recovery**

Define $f_{q, L, \ell, w}: \Delta([q]) \to \mathbb{R}_{\geq 0}$ as
\[
f_{q, L, \ell, w}(P) := \mathbb{E}_{(X_1, \ldots, X_L) \sim P^{\otimes L}} [p_f(X_1, \ldots, X_L)] \tag{III.6}
\]
for $P \in \Delta([q])$. Define $g_{q, L, \ell, w}: [0, 1] \to \mathbb{R}_{\geq 0}$ as
\[
g_{q, L, \ell, w}(w) := f_{q, L, \ell, w}(P_{q, L, \ell, w}), \tag{III.7}
\]
where the distribution $P_{q, L, \ell, w} \in \Delta([q])$ is defined as
\[
P_{q, L, \ell, w}(i) = \begin{cases} \frac{w}{q - \ell}, & 1 \leq i \leq q - \ell, \\ \frac{1 - w}{\ell - w}, & q - \ell + 1 \leq i \leq q. \end{cases} \tag{III.8}
\]

**Theorem 6 (Schur Convexity of $f_{q, L, \ell, w}$):** For any $q \in \mathbb{Z}_{\geq 2}$, $L \in \mathbb{Z}_{\geq 2}$ and integer $1 \leq \ell \leq q - 1$, the function $f_{q, L, \ell, w}: \Delta([q]) \to \mathbb{R}_{\geq 0}$ defined in Equation (III.6) is Schur convex.

**Proof:** See Section VIII.

**Theorem 7 (Convexity of $g_{q, L, \ell, w}$):** For any $q \in \mathbb{Z}_{\geq 2}$, $L \in \mathbb{Z}_{\geq 2}$ and integer $2 \leq \ell \leq q - 1$, the function $g_{q, L, \ell, w}: \Delta([q]) \to \mathbb{R}_{\geq 0}$ defined in Equation (III.7) is convex in the interval $w \in [0, 1]$.

**Proof:** See Section IX.

Define
\[
p_a(q, \ell, L; w) := 1 - \frac{1}{L} g_{q, L, \ell, w}. \tag{III.9}
\]

**Theorem 8 (Plotkin Bound for List-Recovery):** Fix any $q \in \mathbb{Z}_{\geq 2}$, $L \in \mathbb{Z}_{\geq 2}$ and integer $2 \leq \ell \leq q - 1$. Let
\(C \subset [q]^n\) be an arbitrary \((p, \ell, L)\)-q-list-recoverable code with 
\(p = p_{\text{s}}(q, \ell, L, \frac{q-\ell}{q}) + \tau\) for any constant \(\tau \in (0, 1)\). Then there exists a constant 
\(M_{\text{s}} = M_{\text{s}}(q, \ell, \tau)\) independent of \(n\) such that 
\(|C| \leq M_{\text{s}}\). This implies, in particular, 
\[p_{\text{s}}(q, \ell, L) \leq p_{\text{s}}\left(q, \ell, L, \frac{q-\ell}{q}\right) = 1 - \frac{1}{L} g_{q,\ell,L}(q - \ell) q\].

The proof structure is similar to that of Theorem 3. We first prove the analogous statement for almost constant-weight codes (in which all codewords have approximately the same list-recovery weight) in Theorem 14 and then pass to general codes by weight partitioning (cf. Corollary 5). Since the technical proofs bear many similarities to those in the list-decoding case, we only present proof sketches in Section X.

To complement Theorem 8, we prove in Section XII the following lower bound on the \((p, \ell, L)\)-q-list-recovery capacity. To the best of our knowledge, this is the first lower bound for list-recovery with \(q, \ell, L\) all being constants (independent of \(p\) and \(n\)). We believe that improving it likely requires novel techniques beyond expurgation.

**Theorem 9:** For any \(q \in \mathbb{Z}_{\geq 2}\), \(L \in \mathbb{Z}_{\geq 2}\), integer \(2 \leq \ell \leq q - 1\) and \(0 \leq p < p_{\text{s}}(q, \ell, L)\), the following lower bound on the \((p, \ell, L)\)-q-list-recovery capacity holds:

\[
\begin{align*}
C_{(p, \ell, L)_{q}} & \geq \frac{L}{L-1} \left( - \frac{1}{L} \sum_{a \in A_{q,L}} L a - \log_q \left( \sum_{a \in A_{q,L}} L a \right) \right) - \frac{1}{L} \log_q \left( \sum_{a \in A_{q,L}} L a \right) \\
& \times \exp_q \left( - \lambda_{\text{s}} \left( 1 - \frac{1}{L} \max_{a} \{ a \} \right) \right)
\end{align*}
\]

where \(\lambda_{\text{s}} = \lambda_{\text{s}}(q, \ell, L, p)\) is the solution to the following equation

\[
p = \sum_{a \in A_{q,L}} \left( L a \right) q^{-\lambda_{\text{s}} \left( 1 - \frac{1}{L} \max_{a} \{ a \} \right)} \left( 1 - \frac{1}{L} \max_{a} \{ a \} \right).
\]

Similar to the list-decoding case (Theorem 4), the above lower bound vanishes at

\[p = q^{-L} \sum_{a \in A_{q,L}} \left( L a \right) \left( 1 - \frac{1}{L} \max_{a} \{ a \} \right),\]

and the corresponding \(\lambda_{\text{s}}\) equals 0.

Theorems 8 and 9 jointly determine the value of \(p_{\text{s}}(q, \ell, L)\) shown in the corollary below.

**Corollary 2:** For any \(q \in \mathbb{Z}_{\geq 2}\), \(L \in \mathbb{Z}_{\geq 2}\) and integer \(2 \leq \ell \leq q - 1\), the zero-rate threshold \(p_{\text{s}}(q, \ell, L)\) for \((p, \ell, L)\)-q-list-recovery is given by

\[
\begin{align*}
p_{\text{s}}(q, \ell, L) &= p_{\text{s}}\left(q, \ell, L, \frac{q-\ell}{q}\right) = 1 - \frac{1}{L} g_{q,\ell,L}(q - \ell) q \\
&= q^{-L} \sum_{a \in A_{q,L}} \left( L a \right) \left( 1 - \frac{1}{L} \max_{a} \{ a \} \right).
\end{align*}
\]

The above corollary follows in a similar manner to Corollary 1 using the explicit expression of \(f_{q,\ell,L}\) in Lemma 2.

From now on, we use \(p_{\text{s}}(q, \ell, L)\) to refer to the same quantity as the RHS of Equation (III.10).

**Theorem 10 (Elias–Bassalygo Bound for List-Recovery):** Fix any \(q \in \mathbb{Z}_{\geq 2}\), \(L \in \mathbb{Z}_{\geq 2}\), integer \(2 \leq \ell \leq q - 1\) and \(0 \leq p < p_{\text{s}}(q, \ell, L)\). Then the \((p, \ell, L)\)-q-list-recovery capacity can be upper bounded as

\[
C_{(p, \ell, L)_{q}} \leq 1 - H_{q,L}(w_{q,\ell,L})\] where \(w_{q,\ell,L}\) is the solution to the equation

\[p_{\text{s}}(q, \ell, L, w) = p\]

in \(w \in [0, (q-\ell)/q]\).

**Proof:** Parallel to Theorem 5, the above theorem is immediately implied by a finite-blocklength version Theorem 15 (analogous to Theorem 13) whose full proof is presented in Section XI.

C. Expressions for \(f_{q,\ell,L}, g_{q,\ell,L}\) and Their Derivatives

Before concluding this section, we collect representations for \(f_{q,\ell,L}, g_{q,\ell,L}\) and their derivatives which will be useful later.

For \(\ell \in [q]\), \(e_i\) denotes the length-\(q\) vector with a 1 in the \(i\)-th coordinate and 0’s elsewhere.

**Lemma 2:** We have, for all \(1 \leq \ell \leq q - 1\), \(P = (p_1, \ldots, p_q) \in \Delta([q])\) and \(j, k \in [q]\),

\[
f_{q,\ell,L}(P) = \sum_{a \in A_{q,L}} \left( L a \right) \max_{a} \{ a \} \left( \prod_{i=1}^{q} p_i^{a_i} \right) ;
\]

\[
\frac{\partial}{\partial p_j} f_{q,\ell,L}(P) = L \sum_{a \in A_{q,L-1}} \left( L - 1 \right) \max_{a} \{ a + e_j \} \left( \prod_{i=1}^{q} p_i^{a_i} \right) ;
\]

\[
\frac{\partial^2}{\partial p_i \partial p_j} f_{q,\ell,L}(P_1, \ldots, p_q) = L(L-1) \sum_{a \in A_{q,L-2}} \left( L - 2 \right) \max_{a} \{ a + e_j + e_k \} \left( \prod_{i=1}^{q} p_i^{a_i} \right).
\]

Furthermore, defining

\[
G_{\ell}(a) := \frac{1}{(q-\ell)^2} \sum_{1 \leq i,j \leq q-\ell} \max_{a} \{ a + e_i + e_j \}
\]

\[
- \frac{2}{(q-\ell)\ell} \sum_{1 \leq i,j \leq q-\ell+1} \max_{a} \{ a + e_i + e_j \}
\]

\[
+ \frac{1}{\ell^2} \sum_{q-\ell+1 \leq i,j \leq q} \max_{a} \{ a + e_i + e_j \},
\]

we have

\[
g''_{q,\ell,L}(w) = L(L-1) \frac{w}{q-\ell} \left( \frac{w}{q-\ell} \right)^{L-2} \sum_{a \in A_{q,L-2}} \left( L - 2 \right) a \left( \prod_{i=1}^{q} p_i^{a_i} \right) \frac{(q-\ell)(1-w)}{\ell w} G_{\ell}(a).
\]

**Proof:** See Section C.

IV. SCHUR CONVEXITY OF \(f_{q,L}\): PROOF OF THEOREM 1

First, we provide the criterion for Schur-convexity that we use when proving \(f_{q,L}\) (and later, \(f_{q,\ell,L}\)) is Schur-convex.

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Theorem 11 (Schur–Ostrowski Criterion, [50]): Let \( f: \mathbb{R}^d \to \mathbb{R} \) be a symmetric function, i.e.,

\[
f(x_1, \ldots, x_d) = f(x_{\pi(1)}, \ldots, x_{\pi(d)})
\]

for every \( \pi \in S_d \). Suppose all first partial derivatives of \( f \) exist. Then \( f \) is Schur-convex if and only if

\[
(x_i - x_j) \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) f(x_1, \ldots, x_d) \geq 0
\]

for all \( x \in \mathbb{R}^d \) and \( 1 \leq i \neq j \leq d \).

**Proof of Theorem 1:** By Equation (III.12) of Lemma 2, we have

\[
\frac{\partial}{\partial p_j} f_{q,L}(P) = L \sum_{a \in A_{q,L-1}} \left( \frac{L-1}{a} \right) \max\{a + e_j\} \left( \prod_{i=1}^{q} p_i^{a_i} \right).
\]

By symmetry, it is without loss of generality

\[
\left( \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_j} \right) f_{q,L}(p_1, \ldots, p_q) \geq 0.
\]

This shows that \( \left( \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_j} \right) f_{q,L}(p_1, \ldots, p_q) > 0 \) and the proof of Theorem 1 is finished.

\[\square\]

**V. Convexity of \( g_{q,L} \): Proof of Theorem 2**

**Proof of Theorem 2:** We will show that \( g_{q,L}'' \geq 0 \) on the interval \( [0, (q-1)/q] \). By Equation (III.14) of Lemma 2, we have

\[
g_{q,L}(w) = L(L-1) \left( \frac{w}{q-1} \right)^{L-2} \sum_{a \in A_{q,L-2}} \left( \frac{L-2}{a} \right) \left( \frac{1}{(q-1) - \frac{w}{q}} \right)^{a_q} G(a)
\]

where

\[
G(a) = \frac{1}{(q-1)^2} \sum_{i,j \in [q-1]} \max\{a + e_i + e_j\}
\]

\[
- \frac{2}{q-1} \sum_{i \in [q-1]} \max\{a + e_i + e_q\}
\]

\[
+ \max\{a + 2e_q\}.
\]

We now seek to lower bound \( G(a) \) for a given \( a \in A_{q,L-2} \). Let \( r \) denote the number of distinct \( i \in [q] \) for which \( a_i = \max\{a\} \). By symmetry, it is without loss of generality to assume \( a_1 \geq a_2 \geq \cdots \geq a_{q-1} \). Further, observe that

\[
\max\{a\} \left( \sum_{i,j \in [q-1]} \frac{1}{(q-1)^2} - \sum_{i \in [q-1]} \frac{2}{q-1} + 1 \right) = 0.
\]

Thus, we have

\[
G(a) = \frac{1}{(q-1)^2} \sum_{i,j \in [q-1]} (\max\{a + e_i + e_j\} - \max\{a\})
\]

\[
- \frac{2}{q-1} \sum_{i \in [q-1]} (\max\{a + e_i + e_q\} - \max\{a\})
\]

\[
+ (\max\{a + 2e_q\} - \max\{a\}).
\]

We will lower bound each of the three terms separately. We consider two cases.
Case \( a_q < \max\{a\} \): In this case, we claim \( G(a) \geq \frac{r(r-1)}{q(q-1)} \). Observe our assumption implies \( a_1 = \cdots = a_r = \max\{a\} \). Firstly,

\[
\sum_{i,j \in [q-1]} (\max\{a + e_i + e_j\} - \max\{a\}) \\
\geq 2r + r(q-2) + r(q-1 - r) \\
= 2r(q-1) - r(r-1).
\]

Indeed, for \( i, j \in [q-1] \) we note that if \( i = j \leq r \) then \( \max\{a + e_i + e_j\} - \max\{a\} = 2; i \leq r \) and \( i \neq j \) then \( \max\{a + e_i + e_j\} - \max\{a\} = 1 \); and if \( i > r \) and \( j \leq r \) then again \( \max\{a + e_i + e_j\} - \max\{a\} = 1 \). Note further that these conditions on \( (i, j) \in [q-1] \) define disjoint sets, and that for all other choices of \( i, j \in [q-1] \) we have \( \max\{a + e_i + e_j\} - \max\{a\} \geq 0 \).

Next,

\[
\sum_{i \in [q-1]} (\max\{a + e_i + e_q\} - \max\{a\}) = r
\]

as if \( i \leq r \) then \( \max\{a + e_i + e_q\} - \max\{a\} = 1 \) and otherwise \( \max\{a + e_i + e_q\} - \max\{a\} = 0 \).

Finally, it’s clear \( \max\{a + 2e_q\} - \max\{a\} \geq 0 \). Combining these inequalities and plugging them into (V.3) conclude \( G(a) \geq \frac{r(r-1)}{q(q-1)} \), as claimed.

Case \( a_q = \max\{a\} \): In this case, we claim \( G(a) \geq \frac{(q-1)^2}{r(r-1)} \). Observe our assumptions yield \( a_i = \max\{a\} \) if and only if \( i \leq r-1 \) or \( i = q \). Firstly, we have

\[
\sum_{i,j \in [q-1]} (\max\{a + e_i + e_j\} - \max\{a\}) \\
\geq 2(r-1) + (r-1)(q-2) + (q-1 - (r-1)) \\
= 2(r-1)(q-1) - (r-1)(r-2).
\]

The argument is completely analogous to the previous case, upon replacing \( r \) by \( r-1 \).

Next, note that for all \( i \leq q-1 \) we have \( \max\{a + e_i + e_q\} - \max\{a\} = 1 \) so

\[
\sum_{i \in [q-1]} (\max\{a + e_i + e_q\} - \max\{a\}) = q-1,
\]

and furthermore \( \max\{a + 2e_q\} - \max\{a\} = 2 \). Plugging these into Equation (V.3) we find \( G(a) \geq \frac{(q-1)^2}{r(r-1)-(r-1)(r-2)} \).

We now combine these bounds as follows to lower bound \( g''_{q,L}(w) \). Consider the equivalence relation defined on \( A_{q,L-2} \) obtained by identifying tuples that can be obtained from one another by coordinate permutation. Stated differently, let the symmetric group \( S_q \) act on \( A_{q,L-2} \) by coordinate permutation, and identify tuples lying in the same orbit. Let \( R \) be a collection of representatives for this equivalence relation, and given \( a^* \in R \) let \( O(a^*) \) denote the equivalence class to which \( a^* \) belongs. As equivalence classes form a partition, we have

\[
g''_{q,L}(w) = L(L-1) \left( \frac{w}{q-1} \right)^{L-2} \sum_{a \in A_{q,L-2}} \left( \frac{L-2}{a} \right) G(a) \\
\times \left( (q-1) \frac{1-w}{w} \right)^{a_q} G(a) \\
\times \sum_{a^* \in R} \left( \frac{L-2}{a^*} \right) \left( (q-1) \frac{1-w}{w} \right)^{a_q} G(a).
\]

It therefore suffices to fix \( a^* \in R \) and show

\[
\sum_{a^* \in R} \left( (q-1) \frac{1-w}{w} \right)^{a_q} G(a) \geq 0. \tag{V.4}
\]

Towards this end, let \( s := \max\{a^*\} \) and let \( r \) denote the number of \( i \in [q] \) for which \( a^*_i = s \). Define further \( O_1(a^*) := \{ a \in O(a^*) : a_q < s \} \) and \( O_2(a^*) := O(a^*) \setminus O_1(a^*) = \{ a \in O(a^*) : a_q = s \} \). Thus, for \( a \in O_1(a^*) \) we have \( G(a) \geq \frac{(q-1)^2}{r(r-1)} \) and for \( a \in O_2(a^*) \), \( G(a) \geq \frac{(q-1)^2}{r(r-1)-(r-1)(r-2)} \). Further, note that for any \( a \in O(a^*) \) we have that if we sample a random permutation \( \Pi \in S_q \) then the probability that the \( a^*_{\Pi} \in O_2(a^*) \) is \( \frac{1}{q} \), since \( a^*_{\Pi} \) is vector obtained after permuting \( a^* \)’s coordinates according to \( \Pi \). Therefore \( \frac{|O_2(a^*)|}{|O(a^*)|} = \frac{1}{q} \) and so \( \frac{|O_2(a^*)|}{|O(a^*)|} = \frac{q-1}{q} \).

Finally, the assumption \( w \in \left[ 0, \frac{q-1}{q} \right] \) implies that \( \left( (q-1)\frac{1-w}{w} \right)^s \geq 1 \), which guarantees that for \( t \geq w \), we have \( \left( (q-1)\frac{1-w}{w} \right)^s \geq \left( (q-1)\frac{1-w}{w} \right)^{\frac{q}{q-1}} \). This implies that for any \( a \in O_1(a^*) \), we have \( a_q < s \) and thus \( \left( (q-1)\frac{1-w}{w} \right)^s \geq \left( (q-1)\frac{1-w}{w} \right)^{\frac{q}{q-1}} \).

Therefore,

\[
\sum_{a^* \in O(a^*)} \left( (q-1) \frac{1-w}{w} \right)^{a_q} G(a) \\
= \sum_{a^* \in O_1(a^*)} \left( (q-1) \frac{1-w}{w} \right)^{a_q} G(a) \\
+ \sum_{a^* \in O_2(a^*)} \left( (q-1) \frac{1-w}{w} \right)^{a_q} G(a) \\
\geq \sum_{a^* \in O_1(a^*)} \left( (q-1) \frac{1-w}{w} \right)^{a_q} \frac{r(r-1)}{(q-1)^2} \\
+ \sum_{a^* \in O_2(a^*)} \left( (q-1) \frac{1-w}{w} \right)^{a_q} \frac{2r(r-1)(q-1) - (r-1)(r-2)}{(q-1)^2} \\
= \left( (q-1) \frac{1-w}{w} \right)^s \cdot |O(a^*)| \cdot \frac{r}{q} \cdot \frac{r(r-1)}{(q-1)^2} \\
+ \frac{r}{q} \cdot \frac{2r(r-1)(q-1) - (r-1)(r-2)}{(q-1)^2} \\
= \left( (q-1) \frac{1-w}{w} \right)^s \cdot |O(a^*)| \cdot \frac{r-1}{q(q-1)^2} \\
\cdot \frac{2r(q-1) - r(q-r) - (r-2)}{(q-1)^2}.
\]
As $1 \leq r \leq q$ and $q \geq 2$, it is immediate that the above expression is non-negative. This establishes Equation (4.4) and therefore completes the proof of Theorem 2.

VI. Plotkin Bound for $q$-ary List-Decoding: Proof of Theorem 3

Before diving into the proof, let us introduce the concept of type which will be used in the current section and Section X.

Definition 15 (Type): Let $X_1, \ldots, X_k$ be $k$ arbitrary finite sets. Then the type $T_{x_1, \ldots, x_k} \in \Delta \left( \prod_{i=1}^{k} X_i \right)$ of a $k$-tuple of vectors $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ is defined as its empirical distribution, i.e.,

$$T_{x_1, \ldots, x_k}(z_1, \ldots, z_k) := \frac{1}{n} \sum_{j=1}^{n} \prod_{i=1}^{k} \mathbb{1}\{x_{i,j} = z_i\}$$

for any $(z_1, \ldots, z_k) \in X_1 \times \cdots \times X_k$.

Suppose that we are given a code $C \subseteq [q]^n$ of size $M$ and therefore rate $R = \frac{1}{n} \log_q M$. Suppose that $C$ is guaranteed to be $(p, L_q)$-list-decodable for a given $L \in \mathbb{Z}_{\geq 2}$. The goal is to upper bound $M$ and therefore $R$.

Theorem 12 (Plotkin Bound, $q$-ary List-Decoding, Approximately Constant Weight): Let $C \subseteq [q]^n$ be a $(p, L_q)$-list-decodable code with $p = p_q(q, L; w) + \tau$ for $w \in [0, (q-1)/q]$ and some small constant $\tau > 0$. Let

$$0 < \varepsilon_1 \leq \frac{L \tau}{8 \text{Lip}(g_{q,L})},$$

where $\text{Lip}(g_{q,L})$ depends only on $q, L$, and will be defined in Equation (VI.26). Assume that every codeword in $C$ has Hamming weight between $\max\{0, n(w-\varepsilon_1)\}$ and $\min\{n(w+\varepsilon_1), n(q-1)/q\}$. Then

$$|C| < N(c, L, M_0), \quad (VI.1)$$

where

$$c := \left( \frac{802^L q^{4L^2-2}}{2^{2\tau}} + 1 \right)^q,$$

$$M_0 := \max \left\{ \frac{2^{11} L^2 q^{2L}}{\tau^2} + L - 2, \quad (L - 1) L \left( \frac{p_q(q, L; w)}{L \text{Lip}(g_{q,L})} + 2 + \varepsilon_1 \right) + 1 \right\}, \quad (VI.2)$$

and the function $N(\cdot, \cdot, \cdot)$ denotes the Ramsey number given by Theorem 18.

Proof: Suppose $p = p_q(q, L; w) + \tau$ and we are given a $(p, L_q)$-list-decodable code $C_1 \subseteq [q]^n$ all of whose codewords have Hamming weight between $n(w-\varepsilon_1)$ and $n(w+\varepsilon_1)$. The goal is to show that $|C_1| \leq M_0$ for some $M_0$ that depends on $q, L, w, \tau$, but not on $n$. Suppose towards a contradiction that Equation (VI.1) holds in the reverse direction,

$$|C_1| \geq N(c, L, M_0). \quad (VI.3)$$

The first step is to pass to an almost “equi-coupled” subcode $C_2 \subset C_1$. This follows from a standard Ramsey argument. Let $\varepsilon_2$ be a small positive constant defined as

$$\varepsilon_2 := \tau^2 \frac{802^L q^{4L^2-2}}{3}. \quad (VI.4)$$

We build an $L$-uniform complete hypergraph $\mathcal{H}_{L,[C_1]}$ in which each codeword in $C_1$ is a vertex and every $L$-list of codewords is connected by a hyperedge. We then assign colours to the hyperedges of $\mathcal{H}_{L,[C_1]}$. To this end, we quantize the probability simplex $\Delta([q]^L)$. By a standard covering argument, there exists an $\varepsilon_2$-net $N \subseteq \Delta([q]^L)$ w.r.t. $\|\cdot\|_\infty$ of size

$$|N| \leq \left( \frac{q^L}{2\varepsilon_2} \right)^q \left( \frac{q^L}{2\varepsilon_2} + 1 \right)^q := c.$$

The covering property of $N$ guarantees that for any $P_{X_1, \ldots, X_L} \in \Delta([q]^L)$, there exists $Q_{X_1, \ldots, X_L} \in N$ such that $\|P_{X_1, \ldots, X_L} - Q_{X_1, \ldots, X_L}\|_\infty \leq \varepsilon_2$. Now, viewing distributions in $N$ as colours, we assign a colour $Q_{X_1, \ldots, X_L}$ to a hyperedge $\{x_1, \ldots, x_L\} \in \binom{[q]}{L}$ iff $\|T_{x_1, \ldots, x_L} - Q_{X_1, \ldots, X_L}\|_\infty \leq \varepsilon_2$. Note that such $Q_{X_1, \ldots, X_L}$ must exist by the construction of $N$. In case it is not unique, pick an arbitrary one. In this way, we get a $c$-colouring of all hyperedges of $\mathcal{H}_{L,[C_1]}$. By Theorem 18, as long as $|C_1| \geq N(c, L, M_1)$, $\mathcal{H}_{L,[C_1]}$ must contain as subgraph a complete $L$-uniform hypergraph of size at least $M_2$. The hypothesis Equation (VI.3) guarantees that $M$ is sufficiently large:

$$M \geq M_0 = \max \left\{ \frac{2^{11} L^2 q^{2L}}{\tau^2} + L - 2, \quad (L - 1) L \left( \frac{p_q(q, L; w)}{L \text{Lip}(g_{q,L})} + 2 + \varepsilon_1 \right) + 1 \right\}. \quad (VI.5)$$

Furthermore, all hyperedges in the subgraph get the same colour. Let us collect vertices (i.e., a subset of codewords in $C_1$) in this subgraph into a set $C_2$. The monochromatic property of $C_2$ implies that there exists $Q_{X_1, \ldots, X_L} \in N$,\n
$$\|T_{x_1, \ldots, x_L} - Q_{X_1, \ldots, X_L}\|_\infty \leq \varepsilon_2 \quad (VI.6)$$

for every $\{x_1, \ldots, x_L\} \in \binom{[q]}{L}$. This $C_2$ is our desired subcode in which all lists have roughly the same type.

The second step is to argue that $Q_{X_1, \ldots, X_L}$ in fact must be approximately symmetric, where symmetric means $Q_{X_1, \ldots, X_L} = Q_{X_{\pi(1)}, \ldots, X_{\pi(L)}}$ for every $\pi \in S_L$, otherwise $|C_2|$ must be small. Let $T_{x_1, \ldots, x_M} \in \Delta[q]^M$ denote the joint type of all codewords in $C_2$. According to the first step of the proof, $T_{x_1, \ldots, x_M}$ clearly meets the assumptions of Equation (19) with distribution $Q_{X_1, \ldots, X_L}$ and constant $\varepsilon_2$. Therefore Equation (19) guarantees that

$$\max_{\pi \in S_L} \|Q_{X_1, \ldots, X_L} - Q_{X_{\pi(1)}, \ldots, X_{\pi(L)}}\|_\infty \leq 2L^3 \sqrt{\frac{2L}{M - (L - 2)}} + 4L^3 \sqrt{q^{L-2} \varepsilon_2} + L^2 \varepsilon_2 \leq 2L^3 \sqrt{\frac{2L}{M - (L - 2)}} + 5L^3 \sqrt{q^{L-2} \varepsilon_2}. \quad (VI.7)$$

10For instance, one can take $N = \varepsilon_2 + 2\varepsilon_2 Z$ to be the coordinate-wise quantization.

11We enumerate the codewords in $C_1$ according to an arbitrary fixed order. Any subset of codewords in $C_1$ is by default listed according to this order.

12For convenience we assume that the size of this subgraph (later defined to be $C_2$) is exactly $M$. Otherwise, we throw away the last few vertices (i.e., codewords).
In words, as long as $M$ is sufficiently large, $Q_{x_1,\ldots,x_L}$ must be approximately symmetric. More specifically, since Equation (VI.5) guarantees
\[
M \geq \frac{211L^7q^{2L}}{\tau^2} + L - 2,
\]
by the choice of $\varepsilon_2$ (cf. Equation (VI.4)), Equation (VI.7) implies that
\[
\max_{\pi \in S_L} \|Q_{x_1,\ldots,x_L} - Q_{x_{\pi(1)},\ldots,x_{\pi(L)}}\|_\infty \\
\leq 2L^3\sqrt{\frac{2L}{211L^7q^{2L}/\tau^2}} + 5L^3\cdot \frac{\tau}{80L^3q^L} = \frac{\tau}{8q^L}.
\]
We define the RHS to be $\varepsilon_3$,
\[
\varepsilon_3 := \frac{\tau}{8q^L}.
\]
At this point, we have got a subcode $C_2 \subset C_1$ with the following properties.
1) $|C_2| = M$;
2) every codeword in $C_2$ has Hamming weight between $n(w - \varepsilon_1)$ and $n(w + \varepsilon_1)$;
3) there exists a distribution $Q_{x_1,\ldots,x_L} \in \Delta([q]^L)$ satisfying Equation (VI.8) such that Equation (VI.6) holds for every $L$-list in $C_2$;
4) $C_2$ is $(p, L, q)$-list-decodable.

The third step is to apply a well-known double counting trick to $C_2$. Let us bound from both sides the following quantity
\[
\frac{1}{ML} \sum_{(i_1,\ldots,i_L) \in [M]^L} \text{rad}(x_{i_1},\ldots,x_{i_L}),
\]
(10.1)
that is, the average radius averaged over (potentially repeated) lists in $C_2$.

A lower bound on Equation (VI.10) follows from list-decodability of $C_2$. Indeed, suppose towards a contradiction that $\text{rad}(x_1,\ldots,x_L) \leq n(p_a(q, L; w) + \varepsilon_4)$ for some list $\{x_1,\ldots,x_L\} \in (C_2)_L$. Here we choose
\[
\varepsilon_4 := \frac{2}{L} \cdot \text{Lip}(q_{g,L}) \cdot \varepsilon_1,
\]
(11.1)
and $\text{Lip}(q_{g,L})$ will be defined in Equation (VI.26). Then by the definition of average radius,
\[
\frac{1}{L} \sum_{i=1}^{L} d_H(x_i, y) \leq n(p_a(q, L; w) + \varepsilon_4),
\]
(12.1)
where $y$ is the centroid of the list whose coordinates attain the plurality of the corresponding coordinates of the list: $y_i = \arg\max\{x_{i_1,\ldots,i_L}\}$ for every $i \in [n]$. If $\arg\max\{x_{i_1,\ldots,i_L}\}$ is not unique, take an arbitrary one. Observe that for any $1 \leq i \neq j \leq L$, $d_H(x_i, y)$ and $d_H(x_j, y)$ have approximately the same value which does not depend on the choice of $i$ and $j$.
\[
\frac{1}{n} |d_H(x_i, y) - d_H(x_j, y)| = \left| \sum_{(x_1,\ldots,x_L,y) \in [q]^{L+1}} T_{x_1,\ldots,x_L,y}(x_1,\ldots,x_L,y) \right| \cdot \left( \prod_{x \neq y} 1 - \prod_{x \neq y} \right)
\]
(13.1)
Equation (VI.13) holds since
\[
\frac{1}{n} \sum_{i=1}^{L} d_H(x_i, y) \geq d_H(x_j, y) - nq^L(2\varepsilon_2 + \varepsilon_3).
\]
Therefore, by Definition 5,
\[
\text{rad}(x_1,\ldots,x_L)
\]
\leq \max_{i \in [L]} d_H(x_i, y)
\leq \frac{1}{L} \sum_{i=1}^{L} d_H(x_i, y) + nq^L(2\varepsilon_2 + \varepsilon_3)
\leq n(p_a(q, L; w) + \varepsilon_4 + q^L(2\varepsilon_2 + \varepsilon_3))
\leq n(p_a(q, L; w) + \tau/4 + \varepsilon_3)
\leq n(p_a(q, L; w) + \tau/4)
\leq n(p_a(q, L; w) + \tau).
(18.1)
Equation (VI.16) is due to Equation (VI.12), Equation (VI.17) follows from the choices of $\varepsilon_2$ (cf. Equation (VI.4)), $\varepsilon_3$ (cf. Equation (VI.11)) and that
\[
\varepsilon_2 = \left( \frac{\tau}{80L^3q^L} \right)^2, \frac{1}{q^{L-2}} \leq \left( \frac{\tau}{16q^L} \right)^2 \leq \frac{\tau}{16q^L}.
\]
That is, we have found a list $\{x_1,\ldots,x_L\} \in (C_2)_L$ whose radius is at most the RHS of Equation (VI.18) which is less
than $np$. This contradicts $(p, L)_i$-list-decodability of $C_2$ (cf. Property 4). Hence we can conclude that
\[
\bar{\text{rad}}(x_{i_1}, \ldots, x_{i_L}) \geq n(p_*(q, L; w) + \varepsilon_4)
\] (VI.19)
for every $\{i_1, \ldots, i_L\} \in [M]^L$. For those lists $(i_1, \ldots, i_L) \in [M]^L$ of indices which are not all distinct, we simply lower bound $\bar{\text{rad}}(x_{i_1}, \ldots, x_{i_L}) \geq 0$. Therefore, we have the following lower bound\(^\dagger\) on Equation (VI.10):
\[
\frac{1}{ML} \sum_{(i_1, \ldots, i_L) \in [M]^L} \bar{\text{rad}}(x_{i_1}, \ldots, x_{i_L}) \\
\geq \frac{(M - L + 1)^L}{ML} n(p_*(q, L; w) + \varepsilon_4) \\
= \left(1 - \frac{L - 1}{M}\right)^L n(p_*(q, L; w) + \varepsilon_4) \\
\geq \left(1 - \frac{(L - 1)L}{M}\right) n(p_*(q, L; w) + \varepsilon_4).
\] (VI.20)
Next, we upper bound Equation (VI.10).

\[
\frac{1}{ML} \sum_{(i_1, \ldots, i_L) \in [M]^L} \bar{\text{rad}}(x_{i_1}, \ldots, x_{i_L}) \\
= \frac{1}{ML} \sum_{(i_1, \ldots, i_L) \in [M]^L} \sum_{k \in [n]} \left(1 - \frac{1}{L} \text{pl}(x_{i_1}, \ldots, x_{i_L}, k)\right) \\
= \frac{1}{ML} \sum_{(i_1, \ldots, i_L) \in [M]^L} \sum_{\{1, \ldots, n\} \subseteq [q]} \sum_{g \in [q]^L} \left(\prod_{j=1}^L \left(\frac{1}{M} \sum_{i \in [M]} 1\{x_{i_1,j}, \ldots, x_{i_L,j} = x_j\}\right)\right) \\
\times \left(1 - \frac{1}{L} \text{pl}(x_{i_1}, \ldots, x_{i_L})\right) \\
= \sum_{k \in [n]} \sum_{(x_1, \ldots, x_L) \in [q]^L} \left(\prod_{j=1}^L P_k(x_j)\right) \left(1 - \frac{1}{L} \text{pl}(x_{i_1}, \ldots, x_{i_L})\right) \\
= \sum_{k \in [n]} \sum_{a \in \Delta_{q,L}} \left(\frac{L}{a}\right) \left(\sum_{i=1}^q P_k(i)^a_i\right) \left(1 - \frac{1}{L} \max\{a\}\right) \\
= \sum_{k \in [n]} \left(1 - \frac{1}{L} f_{q,L}(P_k)\right).
\] (VI.21)

In Equation (VI.21), we define the empirical distribution $P_k \in \Delta([q])$ of the $k$-th ($k \in [n]$) column of $C_2$ as
\[
P_k(x) = \frac{1}{M} \sum_{x \in C_2} 1\{x_k = x\}
\]
for any $x \in [q]$. In Equation (VI.22), we use the multinomial theorem and the fact that $P_k$ is a probability vector:
\[
\sum_{a \in \Delta_{q,L}} \left(\frac{L}{a}\right) \left(\sum_{i=1}^q P_k(i)^a_i\right) = 1.
\]
Let $w_k$ ($1 \leq k \leq n$) denote the (fractional) weight of the $k$-th column of $C_2$, i.e., $w_k := 1 - P_k(q)$. By Schur convexity of $f_{q,L}$ (cf. Theorem 1), $f_{q,L}(P_k) \geq f_{q,L}(P_{0_w})$ for every $1 \leq k \leq n$. Therefore,
\[
\frac{1}{n} \sum_{k \in [n]} \left(1 - \frac{1}{L} f_{q,L}(P_k)\right) \leq \frac{1}{n} \sum_{k \in [n]} \left(1 - \frac{1}{L} f_{q,L}(P_{0_w})\right) \\
= \frac{1}{n} \sum_{k \in [n]} \left(1 - \frac{1}{L} g_{q,L}(w_k)\right).
\] (VI.23)

Denote by
\[
\overline{w} := \frac{1}{n} \sum_{k \in [n]} w_k
\] (VI.24)
the average (fractional) weight of $C_2$. At this point, we would like to use convexity of $g_{q,L}$ (cf. Theorem 2) to deduce
\[
\frac{1}{n} \sum_{k \in [n]} g_{q,L}(w_k) \geq g_{q,L}(\overline{w}).
\] (VI.25)

However, convexity of $g_{q,L}$ only holds on the interval $[0, (q - 1)/q]$ whereas it is not guaranteed that $w_k \in [0, (q - 1)/q]$ for every $k \in [n]$. Fortunately, this is not going to cause an issue due to the monotonicity property of $g_{q,L}$ (cf. Lemma 1). We can perform the following surgery on $(w_k)_{k \in [n]}$ after which the value of $\sum_{k \in [n]} g_{q,L}(w_k)$ will increase and the value of $\sum_{k \in [n]} w_k$ will not change. Suppose that not all $w_k$ lie in the interval $[0, (q-1)/q]$. Then we must be able to find a pair of weights $w_i$, $w_j$ ($1 \leq i \neq j \leq n$) such that $w_i < (q-1)/q$, $w_j > (q-1)/q$. We then replace $w_i$, $w_j$ with $w_i + \varepsilon$, $w_j - \varepsilon$, respectively, where $\varepsilon := \min\{w_i, w_j - (q-1)/q\}$. One of the new $w_i$, $w_j$ becomes equal to $(q-1)/q$ and the other one may not be equal. We then repeat such operations until no $w_k > (q-1)/q$ can be found. This process will terminate since each codeword has Hamming weight at most $n(q-1)/q$, so $\frac{1}{n} \sum_{k \in [n]} w_k \leq (q - 1)/q$. With slight abuse of notation, we denote the weights after the surgery still by $(w_k)_{k \in [n]}$. With the new weights, convexity of $g_{q,L}$ can be applied and Equation (VI.25) is valid.

Note that by Property 2 of $C_2$,
\[
\overline{w} = \frac{1}{M} \sum_{x \in C_2} \frac{1}{n} w_{f_{q,L}(x)}(x) \\
eq \frac{1}{M} \sum_{x \in C_2} \frac{1}{n} w_{f_{q,L}(x)}(x) \\
\in \left[\max\{0, w - \varepsilon_1\}, \min\{w + \varepsilon_1, (q - 1)/q\}\right].
\]
Let
\[
\text{Lip}(g_{q,L}) := \max_{w \in [0,(q-1)/q]} |g_{q,L}(w)|
\] (VI.26)
be the Lipschitz constant of $g_{q,L}$. Then
\[
g_{q,L}(\overline{w}) \geq g_{q,L}(w) - \text{Lip}(g_{q,L}) \cdot \varepsilon_1.
\]
Therefore, continuing with Equation (VI.23), we have shown that Equation (VI.10) is upper bounded by
\[
\frac{1}{n} \sum_{k \in [n]} \left( 1 - \frac{1}{L} g_q, L(w_k) \right)
\leq 1 - \frac{1}{L} g_q, L(w)
\leq 1 - \frac{1}{L} g_q, L(w) + \frac{1}{L} \cdot \text{Lip}(g_q, L) \cdot \varepsilon_1
= p_*(q, L; w) + \frac{1}{L} \cdot \text{Lip}(g_q, L) \cdot \varepsilon_1.
\] (VI.27)

Finally, combining the upper (Equation (VI.27)) and lower (Equation (VI.20)) bounds on Equation (VI.10), we get
\[
\left( 1 - \frac{(L - 1)L}{M} \right) (p_*(q, L; w) + \varepsilon_4)
\leq p_*(q, L; w) + \frac{1}{L} \cdot \text{Lip}(g_q, L) \cdot \varepsilon_1.
\] (VI.28)

where
\[
M > \frac{(L - 1) L (p_*(q, L; w) + \varepsilon_4)}{\varepsilon_4 - \frac{1}{L} \cdot \text{Lip}(g_q, L) \cdot \varepsilon_1}
= (L - 1) L \left( \frac{p_*(q, L; w)}{\frac{1}{L} \cdot \text{Lip}(g_q, L) \cdot \varepsilon_1} + 2 \right)
\] (VI.29)

which holds by the hypothesis Equation (VI.5). We have reached a contradiction which implies that Equation (VI.3) must not hold and therefore the conclusion of Theorem 12 is true.

By tuning the slack factors in Theorem 12, one can easily obtain an analogous bound for exactly constant-weight list-decodable codes.

**Corollary 3 (Plotkin Bound, q-ary List-Decoding, Constant Weight):** Let \( C \subset [q]^n \) be a \((p, L)\)-list-decodable code with \( p = p_*(q, L; w) + \tau \) for \( w \in [0, (q - 1)/q] \) and some small constant \( \tau > 0 \). Assume that every codeword in \( C \) has Hamming weight \( nw \). Then
\[
|C| < N(c, L, M_0),
\]
where
\[
c := \left( \frac{80^2 L^6 q^{4L - 2}}{2\tau^2} + 1 \right)^{q^L},
M_0 := \max \left\{ \frac{2^{11} L^7 q^{2L}}{\tau^2} + L - 2, \left( \frac{4 p_*(q, L; w)}{\tau} + 1 \right) \right\},
\]
and the function \( N(\cdot, \cdot, \cdot) \) denotes the Ramsey number given by Theorem 18.

**Proof:** The corollary follows by setting \( \varepsilon_1 = 0 \) and \( \varepsilon_4 = \tau/4 \) in Equation (VI.28) in the proof of Theorem 12.

By partitioning the code into approximately constant-weight subcodes, one can obtain a bound analogous to that in Theorem 12 for codes without weight constraints.

**Corollary 4 (Plotkin Bound, q-ary List-Decoding, Weight Unconstrained):** Let \( C \subset [q]^n \) be a \((p, L)\)-list-decodable code with \( p = p_*(q, L) + \tau \) for some small constant \( \tau > 0 \). Then
\[
|C| < q \left( \frac{4 \text{Lip}(g_q, L)}{L\tau} + 1 \right) N(c, L, M_0),
\]
where \( \text{Lip}(g_q, L) \) depends only on \( q, L \) and has been defined in Equation (VI.26),
\[
c := \left( \frac{80^2 L^6 q^{4L - 2}}{2\tau^2} + 1 \right)^{q^L},
M_0 := \max \left\{ \frac{2^{11} L^7 q^{2L}}{\tau^2} + L - 2, \left( \frac{8 p_*(q, L)}{\tau} + 1 \right) \right\},
\]
and the function \( N(\cdot, \cdot, \cdot) \) denotes the Ramsey number given by Theorem 18.

**Proof:** We slice \( C \) into a sequence of subcodes each of which is almost constant-weight for some weight at most \( n (q - 1)/q \), so that Theorem 12 can be applied. Let \( \varepsilon_1 \) be a small positive constant defined as
\[
\varepsilon_1 := \frac{L\tau}{8 \text{Lip}(g_q, L)},
\]
where \( \text{Lip}(g_q, L) \) only depends on \( q, L \) and has been defined in Equation (VI.26). Let \( \mathcal{W} := 2\varepsilon_1 Z \cap [0, (q - 1)/q] \) be an \( \varepsilon_1 \)-net of the interval \([0, (q - 1)/q]\). For \( w \in \mathcal{W} \) and \( u \in [q] \) define
\[
\mathcal{X}_{w,u} := \{ x \in [q]^n : \max \{0, (w - \varepsilon_1) n\} \leq d_H(x, u) \leq \min \{ (w + \varepsilon_1) n, n(q - 1)/q \} \},
\]
where \( u = (u, u, \ldots, u) \) denotes the all-\( u \) vector. First, we note that
\[
[q]^n \subseteq \bigcup_{w \in \mathcal{W}} \bigcup_{u \in [q]} \mathcal{X}_{w,u}.
\]
Indeed, given any \( x \in [q]^n \) we have that for some \( u \in [q] \),
\[
d_H(x, u) \leq n \frac{2 - q}{q}, \quad \text{and thus for some } w \in \mathcal{W} \text{ we have}
\max \{0, (w - \varepsilon_1) n\} \leq d_H(x, u) \leq \min \{ (w + \varepsilon_1) n, n(q - 1)/q \}. \] (VI.31)

Define \( \mathcal{C}_{w,u} := \mathcal{C} \cap \mathcal{X}_{w,u} \). For each subcode \( \mathcal{C}_{w,u} \) with \( u \neq q \), we interchange \( u \) and \( q \) in each codeword and denote, with slight abuse of notation, the resulting subcode again by \( \mathcal{C}_{w,u} \). This does not affect the \((p, L)\)-\( q \)-list-decodability of \( \mathcal{C}_{w,u} \) since list-decodability is preserved under permutation on the alphabet. However, Equation (VI.31) then becomes a guarantee on the Hamming weight of each codeword in the subcode:
\[
\max \{0, (w - \varepsilon_1) n\} \leq d_H(x) \leq \min \{ (w + \varepsilon_1) n, n(q - 1)/q \}. \] (VI.32)

To summarize, at this point, we have obtained \( K \) subcodes \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_K \subset \mathcal{C} \) such that
1) \( \{C_i\}_{i=1}^K \) forms a covering of \( C \), i.e., \( \bigcup_{i=1}^K C_i \supseteq C \);
2) for each \( i \in [K] \), the codewords in \( C_i \) all have Hamming weight between \( \max \{0, n(w_i - \varepsilon_1) \} \) and \( \min \{ n(w_i + \varepsilon_1), n(q - 1)/q \} \) for some \( w_i \in [0, (q - 1)/q] \);
3) for each \( i \in [K] \), \( C_i \) is \((p, L)\)-\( q \)-list-decodable;
For each $i \in [K]$, define a subcode
\[ C_i := C \cap B_H(c_i, n w_{q,L,\tau}). \]
that is, the subcode obtained by restricting $C$ to the Hamming ball centered around $c_i$ of radius $n w_{q,L,\tau}$. Note that different subcodes may not be disjoint. Let \( \varepsilon_1 := \frac{4L\text{lip}(g_{q,L})}{L} \). Take an \( \varepsilon_1 \)-net \( W := 2\varepsilon_1 \mathbb{Z} \cap [0, w_{q,L,\tau}] \) of the interval \([0, w_{q,L,\tau}]\).

Note that \( |W| \leq \frac{1}{2\varepsilon_1} + 1 = \frac{4L\text{lip}(g_{q,L})}{L} + 1 \). For each $i \in [K]$, further define
\[ C_{i,w} := C_i \cap \left( B_H(c_i, \min\{n(w + \varepsilon_1), n w_{q,L,\tau}\}) \setminus B_H(c_i, \max\{0, n(w - \varepsilon_1)\}) \right), \]
for each $w \in W$, that is, the subcode obtained by collecting codewords in $C$ whose Hamming distance to $c_i$ is within $n(w \pm \varepsilon_1)$. We then re-center each $C_{i,w}$ by permuting the symbols in $C_{i,w}$ so that $C_{i,w} \subset B_H(q, \min\{n(w + \varepsilon_1), n w_{q,L,\tau}\}) \setminus B_H(q, \max\{0, n(w - \varepsilon_1)\})$. Denote, with slight abuse of notation, the resulting subcode again by $C_{i,w}$.

Now each codeword in $C_{i,w}$ has Hamming weight between $\max\{0, n(w - \varepsilon_1)\}$ and $\min\{n(w + \varepsilon_1), n w_{q,L,\tau}\}$. This operation does not affect list-decodability of $C_{i,w}$. For any $w \in W$, \( p_*(q, L; w) + \tau < p_*(q, L; w_{q,L,\tau}) + \tau = p \).

The inequality follows since (i) by Lemma 1, \( p_*(q, L; w) \) is non-decreasing on \([0, (q - 1)/q]\) and non-increasing on \([(q - 1)/q, 1]\), and (ii) \( w \leq w_{q,L,\tau} \leq (q - 1)/q \). The equality is by the choice of $w_{q,L,\tau}$ (cf. Equation (VII.1)).

Therefore, each $C_{i,w}$ satisfies the assumptions of Theorem 12 and has size at most
\[ |C_{i,w}| \leq q \left( \frac{4L\text{lip}(g_{q,L})}{L\tau} + 1 \right) N(c, L, M_0) =: M_w, \]
where $c, M_0$ are defined in Equation (VI.29) and we define $M_w$ as the constant on the RHS. The covering property Equation (VII.2) implies
\[ \bigcup_{i=1}^{K} \bigcup_{w \in W} C_{i,w} = C, \]
which finishes the proof.

VIII. SCHUR CONVEXITY OF $f_{q,L,\tau}$:
PROOF OF THEOREM 6

The proof of Theorem 6 follows similar ideas to those used in Theorem 1 with suitable adjustments.

Proof of Theorem 6: The proof of Theorem 6 again follows from verifying the Schur-Ostrowski criterion (Theorem 11).

First, by Equation (III.12) of Lemma 2, we have
\[ \frac{\partial}{\partial p_j} f_{q,L,\ell}(p_1, \cdots, p_q) = L \sum_{a \in A_q, \ell - 1} \left( \frac{L - 1}{a} \right) \max_{\{a + e_j\}} \left( \prod_{i=1}^{q} p_i^{a_i} \right), \]
for all $j \in [q]$. 

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Let now \( 1 \leq k \neq j \leq q \) and assume \( p_k > p_j \). The goal is to show \( \left( \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_j} \right) f_{q,q,L,\ell} \geq 0 \). We have
\[
\left( \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_j} \right) f_{q,q,L,\ell}(p_1, \ldots, p_q) = L \sum_{a \in A_{q,q,L-1}} \left( L - 1 \right) \left( \prod_{i=1}^{q} p_i^{q_i} \right) \left( \max_{\ell} \{ a + e_k \} - \max_{\ell} \{ a + e_j \} \right).
\]

Now observe that
\[
\max_{\ell} \{ a + e_k \} - \max_{\ell} \{ a + e_j \} = \begin{cases} 1, & k \in \arg\max_{\ell} \{ a \}, j \notin \arg\max_{\ell} \{ a \} \\ -1, & k \notin \arg\max_{\ell} \{ a \}, j \in \arg\max_{\ell} \{ a \} \\ 0, & \text{otherwise} \end{cases}
\]

where we have defined
\[
\arg\max_{\ell} \{ a \} := \arg\max_{\ell} \sum_{i \in Q} a_i 2^i
\]
to be the \( \ell \)-subset that achieves \( \max_{\ell} \{ a \} \).

It follows, via similar manipulations to Equation (IV.2), that
\[
\frac{1}{L} \left( \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_j} \right) f_{q,q,L,\ell}(p_1, \ldots, p_q) = \sum_{a \in A_{q,q,L-1}} \left( L - 1 \right) \left( \prod_{i \in [q] \setminus \{k,j\}} p_i^{q_i} \right) \left( p_k^{q_k} p_j^{q_j} - p_k^{q_j} p_j^{q_k} \right).
\]

Finally, the above expression can be seen positive due to the assumption \( p_k > p_j \) and the condition \( a_k \geq a_j \). \( \Box \)

IX. CONVEXITY OF \( g_{q,q,L,\ell} \): PROOF OF THEOREM 7

Theorem 7 can be proved in an analogous way to Theorem 2 by showing \( g_{q,q,L,\ell} \geq 0 \) with the adjustment of replacing \( \max \) with \( \max_{\ell} \). And indeed, quite pleasantly, the argument actually becomes simpler when \( \ell \geq 2 \).

Proof of Theorem 7: By Equation (III.14) from Lemma 2, we have
\[
g_{q,q,L,\ell}(w) = L(L-1) \sum_{a \in A_{q,q,L-2}} \left( \frac{w}{q-\ell} \right)^{L-2} a_{q-\ell+1} \ldots a_q \frac{w}{\ell w} G_{\ell}(a)
\]
where
\[
G_{\ell}(a) = \frac{1}{(q-\ell)^2} \sum_{1 \leq i,j \leq q-\ell} \max_{\ell} \{ a + e_i + e_j \} - \frac{2}{(q-\ell)\ell} \sum_{i,j=q-\ell+1}^{q} \max_{\ell} \{ a + e_i + e_j \}
\]
\[
+ \frac{1}{\ell^2} \sum_{q-\ell+1 \leq i,j \leq q} \max_{\ell} \{ a + e_i + e_j \}.
\]

In this case, it turns out that \( G_{\ell}(a) \geq 0 \) for all \( a \in A_{q,q,L-2} \); this clearly suffices to derive the non-negativity of \( g_{q,q,L,\ell}(w) \) for all \( w \in [0,1] \).

We first observe that
\[
\sum_{1 \leq i,j \leq q-\ell} \frac{1}{(q-\ell)^2} - \sum_{i=1}^{q-\ell} \sum_{j=q-\ell+1}^{q} \frac{2}{(q-\ell)\ell} + \frac{1}{\ell^2} \sum_{q-\ell+1 \leq i,j \leq q} 1 = 0.
\]

Therefore
\[
G_{\ell}(a) = \frac{1}{(q-\ell)^2} \sum_{i,j \in [q-\ell]} \left( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} \right)
\]
\[
- \frac{2}{(q-\ell)\ell} \sum_{i=1}^{q-\ell} \sum_{j=q-\ell+1}^{q} \left( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} \right)
\]
\[
+ \frac{1}{\ell^2} \sum_{q-\ell+1 \leq i,j \leq q} \left( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} \right);
\]

as before, we lower bound each of the three terms in the above expression.

Fix \( j \in [q] \) such that \( a_j \) is an \( \ell \)-th largest element in \( a \).

Let \( S = \{ i \in [q] : a_i \geq a_j, a_i \notin a \} \). By the definition of \( a_j \), we have \( |S| \geq \ell \). Let \( S_1 = S \cap [q-\ell], S_2 = S - S_1 \). Assume that the size of \( S_1, S_2 \) are \( r_1, r_2 \) respectively.

First, we claim
\[
\sum_{1 \leq i,j \leq q-\ell} \left( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} \right)
\]
\[
\geq 2r_1^2 + 2r_1(q-\ell - r_1) = 2r_1(q-\ell).
\]

Indeed, if \( i,j \in S_1 \) then \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} = 2 \); if \( i \in S_1 \) and \( j \notin S_1 \) then \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} = 1 \); and if \( i \notin S_1 \) and \( j \in S_1 \) then \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} = 1 \)

As we have \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} \geq 0 \) for all \( i,j \), the claimed lower bound follows.

Next,
\[
\sum_{i=1}^{q-\ell} \sum_{j=q-\ell+1}^{q} \left( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} \right) = 2r_1r_2 + r_1(\ell - r_2) + r_2(q-\ell - r_1) = r_1 \ell + r_2(q-\ell).
\]

as \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} = 2 \) if \( i \in S_1, j \in S_2 \), \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} = 1 \) if \( i \in S_1, j \notin S_2 \) or \( i \notin S_1, j \in S_2 \); and otherwise \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} = 0 \). Finally,
\[
\sum_{q-\ell+1 \leq i,j \leq q} \left( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} \right)
\]
\[
\geq 2r_2^2 + 2r_2(\ell - r_2) = 2r_2 \ell.
\]

This is because if \( i,j \in S_2 \) then \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} = 2 \); if \( i \in S_2 \) and \( j \notin S_2 \) then \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} = 1 \); and if \( i \notin S_2 \) and \( j \in S_2 \) then \( \max_{\ell} \{ a + e_i + e_j \} - \max_{\ell} \{ a \} = 1 \). Thus, \( G_{\ell}(a) \geq \frac{1}{(q-\ell)^2} 2r_1(q-\ell) - \frac{2r_1(\ell - 2r_2(q-\ell))}{(q-\ell)\ell} + \frac{1}{\ell^2} 2r_2 \ell = 0 \), as claimed. \( \Box \)
Assume that every codeword in $w$ seems to be a natural analogue to the condition in Theorem 12. This makes sense as $a + e_i + e_j$ increments the count of up to 2 alphabet symbols, so (if necessary) both can be included in a set achieving $\arg\max_{\ell} \{ a + e_i + e_j \}$. This is responsible for the simplification in the argument, and more importantly yields convexity on the entire interval $[0, 1]$ (which we recall is false in general for the $\ell = 1$ case).

X. Plotkin Bound for List-Recovery: Proof of Theorem 8

**Theorem 14 (Plotkin Bound, List-Recovery):** Assume $2 \leq \ell \leq q - 1$ is an integer. Let $C \subseteq \mathbb{F}_q^n$ be a $(p, \ell, L)_q$-list-recoverable code with $p = p_x(q, \ell, L; w) + \tau$ for $w \in [0, 1]$ and some small constant $\tau \in (0, 1)$. Let

$$0 < \varepsilon_1 < \frac{L\tau}{8\text{Lip}(g_{q,L,\ell})},$$

where $\text{Lip}(g_{q,L,\ell})$ denotes the Lipschitz constant of $g_{q,L,\ell}$:

$$\text{Lip}(g_{q,L,\ell}) := \max_{w \in [0,1]} |g_{q,L,\ell}(w)|.$$  \hspace{1cm} (X.1)

Assume that every codeword in $C$ has list-recovery weight between $n(w - \varepsilon_1)$ and $n(w + \varepsilon_1)$. Then

$$|C| < N(c, L, M_0)$$

where

$$c := \frac{(80L^6 q^{4L-2} + 1)^{qL}}{2^{qL-2}},$$

$$M_0 := \max \left\{ \frac{2^{11}L^7 q^{2L}}{\tau^2} + 2 + 1, (L - 1)L \left( \frac{p_x(q, \ell, L; w)}{q \cdot \text{Lip}(g_{q,L,\ell})} \cdot \varepsilon_1 + 2 \right) + 1 \right\},$$

and the function $N(\cdot, \cdot, \cdot)$ denotes the Ramsey number given by Theorem 18.

**Remark 6:** Unlike the list-decoding version (cf. Theorem 12), $w$ here is not required to lie in $[0, (q - \ell)/q]$ which seems to be a natural analogue to the condition in Theorem 12. Instead, the proof below works for any $w \in [0, 1]$ thanks to Theorem 7 which guarantees convexity of $g_{q,L,\ell}$ on the whole $[0, 1]$ (see Equation (X.12) in the proof for more details). Note that the latter statement is not true for list-decoding (cf. Theorem 2).

**Proof of Theorem 8:** The proof largely follows the structure of the proof of Theorem 12. Suppose $p = p_x(q, \ell, L; w) + \tau$ and we are given a $(p, \ell, L)_q$-list-recoverable code $C_1 \subseteq \mathbb{F}_q^n$ all of whose codewords have list-recovery weight between $n(w - \varepsilon_1)$ and $n(w + \varepsilon_1)$. The goal is to show that $|C_1| \leq M_0$ for some $M_0$ depending only on $q, \ell, L, w, \tau$, but not on $n$. Suppose towards a contradiction that Equation (X.2) holds in the reverse direction. Then according to the first and second steps in the proof of Theorem 12, we can extract a subcode $C_2 \subseteq C_1$ satisfying the following properties

1) $|C_2| = M \geq M_0$ where $M_0$ is defined in Equation (X.3).

2) every codeword in $C_2$ has list-recovery weight between $n(w - \varepsilon_1)$ and $n(w + \varepsilon_1)$;

3) there exists a distribution $Q_{X_1, \ldots, X_L} \in \Delta([q]^L)$ satisfying Equations (VI.7) to (VI.9) such that Equation (VI.16) holds for every $L$-list in $C_2$ with $\varepsilon_2, \varepsilon_3$ to be determined in Equation (X.14);

4) $C_2$ is $(p, \ell, L)_q$-list-recoverable.

The rest of the proof is devoted to a pair of upper and lower bounds on

$$\frac{1}{M^L} \sum_{(i_1, \ldots, i_L) \in [M]^L} \overline{\text{rad}}\left(x_{i_1}, \ldots, x_{i_L}\right),$$

which in turn implies an upper bound on $M$.

A lower bound on Equation (X.4) follows from the list-recoverability guarantee of $C_2$. Indeed, suppose towards a contradiction that $\overline{\text{rad}}(x_1, \ldots, x_L) \leq n(p_x(q, \ell, L; w) + \varepsilon_4)$ for some list $\{x_1, \ldots, x_L\} \in (C_2)_L$ and $\varepsilon_4$ will be determined in Equation (X.14). Then by the definition of $\ell$-average radius,

$$\frac{1}{L} \sum_{i=1}^{L} d_{LR}(x_i, Y) \leq \frac{n(p_x(q, \ell, L; w) + \varepsilon_4)}{L}$$

where $Y \in ([q]^n)_L$ is defined as $Y_i := \arg\max_{j \in [L]} \{x_{i,j} \}$ for every $i \in [n]$. Due to the equicoupledness property (Equation (VI.6)) and the approximate symmetry (Equation (VI.8)) of $Q_{X_1, \ldots, X_L}$, the distance from any codeword $x_i$ to $Y$ is approximately the same.

$$\frac{1}{n} |d_{LR}(x_i, Y) - d_{LR}(x_j, Y)| = \left| \sum_{(x_1, \ldots, x_L) \in [q]^L} \sum_{y \in ([q]^L)} T_{x_1, \ldots, x_L, y}(x_1, \ldots, x_L, Y) \times (1 \{x_i \neq Y\} - 1 \{x_j \neq Y\}) \right| = \left| \sum_{(x_1, \ldots, x_L) \in [q]^L} \sum_{y \in ([q]^L)} \left( T_{x_1, \ldots, x_L, y}(x_1, \ldots, x_L, Y) \right) - T_{\sigma_{i,j}(x_1, \ldots, x_L, y)}(x_1, \ldots, x_L, Y) \right| \leq \left| \sum_{(x_1, \ldots, x_L) \in [q]^L} \sum_{y \in ([q]^L)} \left( T_{x_1, \ldots, x_L, y}(x_1, \ldots, x_L, Y) \right) - T_{\sigma_{i,j}(x_1, \ldots, x_L, y)}(x_1, \ldots, x_L, Y) \right| \\ \leq \left\| T_{x_1, \ldots, x_L} - Q_{X_1, \ldots, X_L} \right\|_1 + \left\| Q_{X_1, \ldots, X_L} - Q_{\sigma_{i,j}(X_1, \ldots, X_L)} \right\|_1 + \left\| T_{\sigma_{i,j}(x_1, \ldots, x_L)} - Q_{\sigma_{i,j}(x_1, \ldots, x_L)} \right\|_1 \leq q^L(2\varepsilon_2 + \varepsilon_3),$$

where the first equality follows from a similar reasoning for Equation (VI.13). We can then deduce an upper bound on the
maximum radius of \( \{ x_1, \cdots, x_L \} \).

\[
\text{rad}(x_1, \cdots, x_L) \\
\leq \max_{i \in [L]} d_{LR}(x_i, \mathcal{Y}) \\
\leq \frac{1}{L} \sum_{i=1}^{L} d_{LR}(x_i, \mathcal{Y}) + n q L (2 \varepsilon_2 + \varepsilon_3) \\
\leq n (p_\ast (q, \ell, L; w) + \varepsilon_4 + q L (2 \varepsilon_2 + \varepsilon_3)) \\
< n (p_\ast (q, \ell, L; w) + \tau). 
\]

Equations (X.7) and (X.8) are due to Equation (X.6) and Equation (X.5), respectively. Equation (X.9) follows from the definitions of \( \varepsilon_2, \varepsilon_3, \varepsilon_4 \) in Equation (X.14). However, the existence of a list with radius strictly smaller than \( np \) violates the list-recoverability guarantee of \( C_2 \). We therefore have

\[
\text{rad}(x_{i_1}, \cdots, x_{i_L}) \geq n (p_\ast (q, \ell, L; w) + \varepsilon_4)
\]

for every \( \{ i_1, \cdots, i_L \} \in [M]^L \). The desired lower bound on Equation (X.4) follows from the same calculations leading to Equation (VI.20):

\[
\frac{1}{M^L} \sum_{(i_1, \cdots, i_L) \in [M]^L} \text{rad}(x_{i_1}, \cdots, x_{i_L}) \\
\geq \left( 1 - \frac{(L - 1) L}{M} \right) n (p_\ast (q, \ell, L; w) + \varepsilon_4). 
\]

We then turn to deriving an upper bound on Equation (X.4). Repeating the calculations leading to Equation (VI.22) with \( p_l \) and \( \max_x \) replaced by \( p_l \) and \( \max_x \), respectively, one has

\[
\frac{1}{M^L} \sum_{(i_1, \cdots, i_L) \in [M]^L} \frac{1}{n} \cdot \text{rad}(x_{i_1}, \cdots, x_{i_L}) \\
= \frac{1}{n} \sum_{k \in [n]} \sum_{x_1, \cdots, x_L} \left( \prod_{j=1}^{L} P_k(x_j) \right) \\
\times \left( 1 - \frac{1}{L} p_l(x_1, \cdots, x_L) \right) \\
= \frac{1}{n} \sum_{k \in [n]} \left( \frac{q}{L} \right)^{\alpha_k} \left( \prod_{j=1}^{q} P_k(i)^{a_i} \right) \left( 1 - \frac{1}{L} \max_x \{ a \} \right) \\
= \frac{1}{n} \sum_{k \in [n]} \left( 1 - \frac{1}{L} f_{q, \ell, L}(P_k) \right) \\
\leq \frac{1}{n} \sum_{k \in [n]} \left( 1 - \frac{1}{L} f_{q, \ell, L}(P_{k, \ell, w_k}) \right) \\
= \frac{1}{n} \sum_{w \in \mathcal{W}} \left( 1 - \frac{1}{L} g_{q, \ell, L}(w_k) \right) \\
\leq 1 - \frac{1}{L} g_{q, \ell, L}(\mathcal{W}) \\
\leq p_\ast (q, \ell, L; w) + \frac{1}{L} \cdot \text{Lip}(g_{q, \ell, L}) \cdot \varepsilon_1.
\]

where \( P_k \in \Delta([q]) \) denotes the empirical distribution (a.k.a. type) of the \( k \)-th column of \( C_2 \),

\[
w_k := \sum_{x=1}^{q-\ell} P_k(x)
\]
denotes the (relative) list-recovery weight of the \( k \)-th column of \( C_2 \) and \( \mathcal{W} \) is defined in Equation (VI.24). In Equations (X.11) and (X.12) we use the Schur convexity of \( f_{q, \ell, L} \) (guaranteed by Theorem 6) and convexity of \( g_{q, \ell, L} \) (guaranteed by Theorem 7), respectively. (Here it is helpful to recall the definitions of \( P_{q, \ell, w} \) and \( g_{q, \ell, L} \), see Equations III.8 and (III.7)). Note that here we do not need to further massage the weights \( (w_k)_{k \in [n]} \) (see the paragraph following Equation (VI.25)) thanks to the fact that convexity of \( g_{q, \ell, L} \) holds on the whole interval \([0, 1]\) for \( \ell \geq 2 \) (cf. Theorem 7).

Finally, choosing

\[
\varepsilon_2 = \frac{\tau^2}{80^2 L^6 q L^2}, \\
\varepsilon_3 = \frac{\tau}{8 q L}, \\
\varepsilon_4 = \frac{2}{L} \text{Lip}(g_{q, \ell, L}) \varepsilon_1 
\]

allows us to conclude from Equations (X.10) and (X.13) that

\[
M \leq (L - 1) L \left( \frac{p_\ast (q, \ell, L; w)}{\varepsilon_1} - 2 \right)
\]

which contradicts our assumption that Equation (X.2) holds in the reverse direction. This finishes the proof. \( \square \)

**Corollary 5** (Plotkin Bound, List-Recovery, Weight Unconstrained): Assume \( 2 \leq \ell \leq q - 1 \) is an integer. Let \( C \subset [q]^n \) be a \( (p, \ell, L) \)-list-recoverable code with \( p = p_\ast (q, \ell, L) + \tau \) for some small constant \( \tau > 0 \). Then

\[
|C| < \left( \frac{4 \text{Lip}(g_{q, \ell, L}) + 1}{L \tau} \right) N(c, L, M_0)
\]

where \( \text{Lip}(g_{q, \ell, L}) \) is defined in Equation (X.1),

\[
c := \left( \frac{80^2 L^6 q^4 L^2 - 2}{2 \tau^2} \right)^{\frac{q L}{L^2}} + 1,
\]

\[
M_0 := \max \left\{ \frac{211 L^7 q^2 L}{\tau^2} + L - 2, \\
(L - 1) L \left( \frac{8 p_\ast (q, \ell, L; w) + 2}{\tau} + 1 \right) \right\},
\]

and the function \( N(\cdot, \cdot, \cdot, \cdot) \) denotes the Ramsey number given by Theorem 18.

**Proof:** The proof follows the same idea used in the proof of Corollary 4 with minor modifications. We partition the code according to list-recovery weight and apply Theorem 14 to each subcode. A proof sketch is presented below.

Let \( \varepsilon_1 = \frac{\tau}{8 \text{Lip}(g_{q, \ell, L})} \) and \( \mathcal{W} := 2 \varepsilon_1 Z \cap [0, 1] \). Define for each \( w \in \mathcal{W} \),

\[
C_w := \{ x \in C : \max\{0, (w - \varepsilon_1) n\} \leq w_{LR}(x) \leq \min\{(w + \varepsilon_1) n, n\} \}.
\]
Each $C_w$ satisfies the condition in Theorem 14 and therefore has size at most $|C_w| \leq N(c, L, M_w)$ where $M_w$ equals

$$M_w := \max \left\{ \frac{21 L^{\ell+2} + 2 L}{\varepsilon^2} + 2, \left( L - 1 \right) L \left( \frac{p_0(q, \ell, L; w)}{2} \cdot \text{rad}_q(g_{q, L, \ell}) \cdot \varepsilon + 2 \right) + 1 \right\},$$

We conclude that

$$|C| \leq \sum_{w \in W} |C_w| \leq |W| \cdot N(c, L, M_0) \leq \left( \frac{1}{2\varepsilon} + 1 \right) N(c, L, M_0),$$

where the middle inequality follows from the monotonicity of $g_{q, L, \ell}(w)$. Specifically, $g_{q, L, \ell}(w)$ is non-increasing on $[0, (q-\ell)/q]$ and non-decreasing on $[(q-\ell)/q, 1]$. This follows from the Schur convexity of $f_{q, L, \ell}$. The argument is analogous to the proof of Lemma 1 and is therefore omitted. The proof of Corollary 5 is then finished. □

XI. ELIAS–BASSALYGO BOUND FOR LIST-RECOVERY: PROOF OF THEOREM 9

**Theorem 15 (Elias–Bassalygo Bound, List-Recovery):** Let $C \subset [q]^n$ be a $(p, \ell, L)_q$-list-recoverable code with $p < p_0(q, \ell, L)$. For any sufficiently small constant $\tau \in (0, 1)$, let $w_{q, \ell, L, \tau}$ be the unique solution to the following equation in $w \in [0, (q-\ell)/q]$,

$$p_\tau(q, \ell, L; w) = p - \tau.$$  \hspace{1cm} (XI.1)

Then

$$|C| \leq \left( \frac{4 \text{Lip}(g_{q, L, \ell})}{L \tau} \right)^2 N(c, L, M_0) \cdot n \ln(q) \cdot \sqrt{8 n w_{q, \ell, L, \tau}(1 - w_{q, \ell, L, \tau})} \cdot q^{n(1 - H_q(\ell(w_{q, \ell, L, \tau})) + 1)},$$

where $c, M_0$ are defined in Corollary 5.

The proof follows verbatim that of Theorem 13 with the application of Lemma 5 and Theorem 12 replaced with Lemma 6 (proved in Section D) and Theorem 14, respectively. The details are omitted.

**Remark 7:** We specialize the bound in Theorem 10 to the case $p = 0$, $\ell \leq q - 1$, $L = \ell + 1$ and $\lambda = (q, \ell + 1)$-hashing. The fixed point equation $p_\tau(q, \ell, L; w) = p$ becomes

$$0 = 1 - \frac{1}{\ell + 1} \sum_{x_1, \ldots, x_{\ell+1} \sim P_{q, L, w}} [p_\tau(q, x_1, \ldots, x_{\ell+1})],$$

or

$$\sum_{x_1, \ldots, x_{\ell+1} \sim P_{q, L, w}} [p_\tau(q, x_1, \ldots, x_{\ell+1})] = \ell + 1,$$  \hspace{1cm} (XII.1)

where $P_{q, L, w} \in \Delta^q([q])$ (recall Equation III.8) is a probability vector whose first $q - \ell$ entries are all equal to $\frac{w}{L - \ell}$ and rest entries are all equal to $\frac{L - 1}{L - \ell}$. We see that the only way to ensure the expected $\ell$-plurality of $\ell + 1$ (random) symbols to be $\ell + 1$ is to set $w = 0$ so that each $X_i$ can only take $\ell$ possible values (specifically, $q - \ell + 1, \ldots, q$) with positive probability. Therefore, the unique solution to Equation (XII.1) is $w_{q, \ell, \ell+1} = 0$. The capacity upper bound in Theorem 10 then becomes

$$1 - H_{q, \ell}(0) = 1 - \log_q \ell = \log_q \frac{q}{\ell}.$$ Further setting $\ell = q - 1$ yields $\log_q \frac{q}{q - 1}$ which recovers the upper bound due to Körner and Marton for $q$-hashing [37] (see also Equation (I.7)).

**XII. LOWER BOUND ON LIST-RECOVERY CAPACITY:**

**Proof of Theorem 9**

We prove a slightly rephrased version of Theorem 9 below.

**Theorem 16 (Random Coding With Expurgation Lower Bound):** Let $q, \ell, L$ be positive integers such that $q, L \geq 2$, $\ell \leq q$. Suppose $p < p_0(q, \ell, L)$. Let $(X_1, \ldots, X_L) \sim \text{Unif}([q])^\otimes L$. Then

$$C_{(p, \ell, L)_q} > \frac{1}{L - 1} \left( - \lambda_q p - \log_q \mathbb{E} \left[ \exp_q \left( - \lambda_q \cdot \text{rad}_q(X_1, \ldots, X_L) \right) \right] \right),$$  \hspace{1cm} (XII.1)

where $\lambda_q = \lambda_q(q, \ell, L, p)$ is the solution to the following equation

$$p = \frac{\mathbb{E} \left[ q^{-\lambda_q \cdot \text{rad}_q(X_1, \ldots, X_L)} \cdot \text{rad}_q(X_1, \ldots, X_L) \right]}{\mathbb{E} \left[ q^{-\lambda_q \cdot \text{rad}_q(X_1, \ldots, X_L)} \right]},$$  \hspace{1cm} (XII.2)

and

$$\text{rad}_q(x_1, \ldots, x_L) := 1 - \frac{1}{L} \sum_{i=1}^L p_q(x_i, \ldots, x_L)$$

for any $(x_1, \ldots, x_L) \in [q]^L$.

**Remark 8:** Expanding out the expectation, we can write the above bound in a more explicit (though less informative) way.

$$C_{(p, \ell, L)_q} > \frac{L}{L - 1} \left\{ \lambda_q p + \log_q \left( \sum_{a \in A_{q, L}} \left( \frac{L}{a} \right) \right) \lambda_q \left( 1 - \frac{1}{L} \max_{a} \{a\} \right) \right\},$$

where $\lambda_q = \lambda_q(q, \ell, L, p)$ is the solution to the following equation

$$p = \frac{\sum_{a \in A_{q, L}} \left( \frac{L}{a} \right) \exp_q \left( - \lambda_q \left( 1 - \frac{1}{L} \max_{a} \{a\} \right) \right) \left( 1 - \frac{1}{L} \max_{a} \{a\} \right) }{\sum_{a \in A_{q, L}} \left( \frac{L}{a} \right) \exp_q \left( - \lambda_q \left( 1 - \frac{1}{L} \max_{a} \{a\} \right) \right) }.$$  \hspace{1cm} (XII.3)

**Remark 9:** We specialize the bound Equation (XII.1) to the case $p = 0$, $L = \ell + 1$, $\ell \leq q - 1$ corresponding to $(q, \ell + 1)$-hashing.

$$- \frac{1}{L} \log_q \left[ q^{-(\ell + 1)} \sum_{a \in A_{q, \ell + 1}} \left( \frac{\ell + 1}{a} \right) \exp_q \left( - \lambda_q \left( 1 - \frac{1}{L} \max_{a} \{a\} \right) \right) \right] \hspace{1cm} (XII.3)$$

$$= - \frac{1}{L} \log_q \left[ q^{-(\ell + 1)} \sum_{a \in A_{q, \ell + 1}} \left( \frac{\ell + 1}{a} \right) \right] q^{-\lambda_q}.$$
occurs. For the former case, there are Equation (XII.2) with $p$ zeros. Otherwise, the latter case (11) occurs if and only if $\ell$. Therefore, Equation (XII.1) evaluates to $\lambda$ yields $\log q$.

Equation (XII.3) is obtained by expanding the expectation in Equation (XII.1). Equation (XII.4) follows since for $a \in A_q, \ell + 1$, $\max x \{ a \}$ can be either $\ell$ or $\ell + 1$. We then use the multinomial theorem $\sum_{a \in A_{\ell + 1}} q^n = q^{\ell + 1}$ to get Equation (XII.5). Observe that the former case (max $x \{ a \} = \ell$) occurs if and only if $a$ is a length-$q$ vector with $\ell + 1$ ones and $q - \ell - 1$ zeros. Otherwise, the latter case (max $x \{ a \} = \ell + 1$) occurs. For the former case, there are $(\ell + 1)$ ways to choose the locations of ones, which leads to Equation (XII.6) where we use $1_{\ell + 1}$ to denote the all-one vector of length $\ell + 1$.

By similar manipulations, the fixed point equation Equation (XII.2) with $p = 0$, $L = \ell + 1$ becomes:

$$0 = \frac{(\ell + 1)(\ell + 1)q^{-\lambda_p}}{(\ell + 1)\ell q^{-\lambda_p} + q^{\ell + 1} - (\ell + 1)!} \cdot \frac{1}{1 - (\ell + 1)!}.$$

It is obvious that the unique solution to the above equation is $\lambda_p = q$. Therefore, Equation (XII.2) evaluates to

$$\frac{1}{\ell} \log q \frac{1}{1 - (\ell + 1)!},$$

which recovers the lower bound due to Fredman and Komlós [19] (see also Equation (I.8)). Further setting $\ell = q - 1$ yields

$$\frac{1}{q - 1} \log q \frac{1}{1 - \frac{\ell}{q^{\ell + 1}}},$$

which recovers the Fredman–Komlós lower bound for $q$-hashing [37] (see also Equation (I.5)).

**Proof of Theorem 16:** Let $C \in [q]^{M \times n}$ be a codebook consisting $M$ codewords from $[q]^n$. Each entry of each codeword is sampled independently and uniformly from $[q]$. Denote $C = \{X_1, \ldots, X_M\}$.

For an arbitrary $L$-list $\{i_1, \ldots, i_L\} \in [M]_L$ of indices, we have

$$\mathbb{E} \left[ \frac{1}{n} \text{rad}_\ell (X_{i_1}, \ldots, X_{i_L}) \right] = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ 1 - \frac{1}{L} \text{pl}_\ell (X_{i_1,j}, \ldots, X_{i_L,j}) \right] = \mathbb{E} \left[ (X_1, \ldots, X_L) \sim \text{Unif}([q])^L \right] \left[ 1 - \frac{1}{L} \text{pl}_\ell (X_1, \ldots, X_L) \right] = \sum_{(x_1, \ldots, x_L) \in [q]^L} q^{-L} \left( \frac{L}{a} \right) \left( 1 - \frac{1}{L} \max \{ a \} \right) > p.$$

Therefore, by Cramér’s large deviation theorem (Theorem 17), we have

$$E := \lim_{n \to \infty} -\frac{1}{n} \log q \text{Pr} \left[ \text{rad}_\ell (X_{i_1}, \ldots, X_{i_L}) \leq np \right] = \lim_{n \to \infty} -\frac{1}{n} \log q \text{Pr} \left[ \sum_{j=1}^{n} \left( 1 - \text{pl}_\ell (X_{i_1,j}, \ldots, X_{i_L,j}) \right) \leq np \right] = \sup_{\lambda > 0} \left\{ -\lambda p - \log q \mathbb{E} \left[ q^{-\lambda} \left( 1 - \frac{1}{L} \text{pl}_\ell (X_1, \ldots, X_L) \right) \right] \right\},$$

where the expectation is taken over $(X_1, \ldots, X_L) \sim \text{Unif}([q])^L$. For any $\lambda \in \mathbb{R}$, we then compute the moment generating function $\mathbb{E} \left[ \exp \left( -\lambda \left( 1 - \frac{1}{L} \text{pl}_\ell (X_1, \ldots, X_L) \right) \right) \right]$ in a similar way.

$$\mathbb{E} \left[ (X_1, \ldots, X_L) \sim \text{Unif}([q])^L \right] \left[ q^{-\lambda} \left( 1 - \frac{1}{L} \text{pl}_\ell (X_1, \ldots, X_L) \right) \right] = q^{-L} \sum_{a \in A_{q,L}} \left( \frac{L}{a} \right) \exp \left( \lambda \left( 1 - \frac{1}{L} \max \{ a \} \right) \right) = M(\lambda).$$

Note that

$$M'(\lambda) = q^{-L} \sum_{a \in A_{q,L}} \left( \frac{L}{a} \right) \exp \left( \lambda \left( 1 - \frac{1}{L} \max \{ a \} \right) \right) \times \left( 1 - \frac{1}{L} \max \{ a \} \right).$$

Then

$$E = \sup_{\lambda > 0} \left\{ -\lambda p - \log q M(-\lambda) \right\}.$$
Or more explicitly,
\[
P = \frac{\sum_{a \in \mathcal{A}_{q,L}} \binom{L}{a} q^{-\lambda_a \left(1 - \frac{1}{L} \max_{f} \{a\}\right)} \left(1 - \frac{1}{L} \max_{f} \{a\}\right)}{\sum_{a \in \mathcal{A}_{q,L}} \binom{L}{a} q^{-\lambda_a \left(1 - \frac{1}{L} \max_{f} \{a\}\right)}}.
\]

Finally, by a standard random coding with expurgation argument, we expunge one codeword from every \( L \)-list for which \( \text{rad}_{L} \) is at most \( np \). The expected total number of expurgated codewords is at most \( \binom{M}{L} q^{-nE + o(n)} \leq q^{o(RL - E + o(n))} \). If this number is less than \( M/2 \), then we end up with a \( (p, \ell, L) \)-list-recoverable code of rate at least \( M/2 \). Therefore the following rate \( R \) can be achieved
\[
R = \frac{E}{L - 1} \tag{XII.7}
\]
\[
= \frac{1}{L - 1} \left(\lambda - p - \log_q M(\lambda)\right)
\]
\[
= \frac{1}{L - 1} \left\{ - \lambda - p - \log_q \sum_{a \in \mathcal{A}_{q,L}} \binom{L}{a} \times \exp \left( \frac{-\lambda}{1 - \frac{1}{L} \max_{\mathcal{A}_{q}} \{a\}} \right) \right\}
\]
\[
= \frac{L}{L - 1} \left( - \lambda - p + \log_q \left( \sum_{a \in \mathcal{A}_{q,L}} \binom{L}{a} \times \exp \left( \frac{-\lambda}{1 - \frac{1}{L} \max_{\mathcal{A}_{q}} \{a\}} \right) \right) \right)
\]

Indeed, we discard one codeword per bad list, i.e., a list \( \mathcal{L} \in \binom{n}{L} \) satisfying \( \text{rad}_{L}(\mathcal{L}) \leq np \). Since there are \( \binom{M}{L} \) \( n \)-list recoverable lists, by linearity of expectation, the number of codewords to be discarded is expected to be \( \exp_q(nRL - nE) \). As long as \( \exp_q(nRL - nE) \ll q^{o(R)} \), we are left with a \( (p, \ell, L) \)-list-recoverable code of rate essentially \( R \). The proof is then finished by noting that the requirement is met by the choice of \( R \) in Equation (XII.7).

\[\square\]

### Appendix A

#### Discussion of Blinovskys Results [8], [9]

As mentioned in Section I, part of the motivation of this work is to fill in the gaps in the proofs in [8] and [9] for \( q \)-ary list-decoding. We discuss in detail below the issues therein. The main result in [8] is a Plotkin bound (as our Theorem 3) for an arbitrary \( q \)-ary list-decodable code \( C \subseteq [q]^n \). For the sake of brevity, we assume in the proceeding discussion that \( C \) is \( w \)-constant weight. Additional bookkeeping is needed to handle small deviations in the weight, as we did in the proof of Theorem 12. The skeleton of the proof in [8] follows Blinovskys proof in the binary case [6] which we adopt here as well: (i) pass to an (approximately) equi-coupled subcode \( C' = \{x_1, \cdots, x_M\} \subseteq C \) using a Ramsey reduction; (ii) handle asymmetric coupling using Komlos argument (and its order-\( L \) generalization [4]); (iii) prove an upper bound on the size \( M \) of the subcode \( C' \) using a double-counting argument. In completing the double-counting argument, one is required to upper bound the average radius (averaged over all \( L \)-lists in the subcode) by the zero-rate threshold
\[
p_{\text{zr}}(q, L; w) = 1 - \frac{1}{L} \text{rad}_{q,L}(w);
\]

\[
\frac{1}{M} \sum_{(i_1, \cdots, i_L) \in [M]^L} \text{rad}((x_{i_1}, \cdots, x_{i_L}))
\]
\[
= \sum_{k=1}^{n} \left(1 - \frac{1}{L} f_{q,L}(P_k)\right) \leq n \left(1 - \frac{1}{L} \text{rad}_{q,L}(w)\right), \tag{A.1}
\]
where \( P_k \in \Delta([q]) \) is the empirical distribution of the \( k \)-th column of \( C' \in [q]^{M \times n} \). The equality in Equation (A.1) is by elementary algebraic manipulations (see Equation (VI.22) for details). To show the inequality in Equation (A.1), we need the following properties of the functions \( f_{q,L} \) and \( g_{q,L} \):

1. For any \( P = (p_1, \cdots, p_q) \in \Delta([q]) \), we have \( f_{q,L}(P) \geq g_{q,L}(1 - p_q) \). In words, uniformizing \( P \) except one entry will only make \( f_{q,L} \) no larger.

2. \( g_{q,L} \) is convex as a univariate real-valued function on \( [0, (q - 1)/q] \).

If these properties hold, one can deduce Equations (VI.23) and (VI.25) from which Equation (A.1) follows. However, we observe that the proofs in [8] and [9] for both properties above are problematic.

To show Item 1 above, the idea in [8] is to show instead monotonicity of \( f_{q,L} \) under the so-called Robin Hood operation which averages two distinct entries of \( P \). Specifically, [8] attempts to show
\[
f_{q,L}(p_1, \cdots, p_i, \cdots, p_j, \cdots, p_q) \geq f_{q,L}\left(p_1, \cdots, \frac{P_i + P_j}{2}, \cdots, \frac{p_i + p_j}{2}, \cdots, p_q\right), \tag{A.2}
\]
for any \( 1 \leq i < j \leq q \). This suffices since a sequence of Robin Hood operations can turn \( P \) into \( P_{r>1} = p_q \) (defined in Equation (III.2)). [8] then proceeds to show Equation (A.2) by checking the derivative of a certain function related to the Robin Hood operation. Specifically, fix \( \{p_k\}_{k \in [q] \setminus \{i,j\}} \) and assume \( p_i + p_j = c \) (or equivalently \( \sum_{k \in [q] \setminus \{i,j\}} p_i = 1 - c \)) for some constant \( 0 < c < 1 \). Consider the function

---

**XIII. Conclusion**

In this work, we addressed the basic question of determining the maximum achievable decoding radius for positive rate list-recoverable codes, i.e., we pinned down the list-recovery zero-rate threshold. We then adapted known techniques to show that codes correcting more errors must in fact have constant size. Subsequently, we transferred this bound to give upper bounds on the rate of list-recoverable codes for all values of decoding radius.

As we apply general Ramsey-theoretic tools in bounding the size of list-recoverable codes in the zero-rate regime, our dependence on the corresponding parameters is quite poor, and indeed, we made no efforts to optimize these constants. However, for list-decodable binary codes in the zero-rate, a recent work of Alon et al. [1] derived new (and, in some cases, tight) upper bounds on their size. Obtaining similarly improved size upper bounds for \( q \)-ary list-decodable/-recoverable codes in the zero-rate regime therefore appears to be a natural next step.
\[ F_{q,L}(p) = \frac{1}{L} g_{q,L}(p, \ldots, p, c-p, \ldots, p), \]
i.e., \( f_{q,L}(p) \) evaluated at \( P \) with \( p_i = p, p_j = c-p \). The proof of Equation (4.2) is reduced to proving \( F_{q,L}(p) \leq 0 \) for \( p \in [0, c/2] \) and \( F_{q,L}(p) \geq 0 \) for \( p \in [c/2, c] \). If true, it implies that \( F_{q,L}(P) \) is minimized at \( p_i = p_j = c/2 \) with fixed \( (p_k)_{k \in [q], \langle i,j \rangle} \). However, we note that the expression of \( F_{q,L}(p) \) (see the second displayed equation on page 27 of [8]) is incorrect. Upon correcting it, we do not see an easy way to argue its non-positivity/negativity. In particular, the claim in [8] that \( F_{q,L}(p) \), as a sum of multiple terms, is term-wise non-positive/negative can be in general falsified by counterexamples.

The proof (attempt) of Item 2 is deferred to a subsequent paper [9]. The methodology thereof is similar to ours, i.e., verifying \( g_{q,L}'' \geq 0 \). However, the expression of \( g_{q,L}'' \) in [9] is not exactly correct (see the first displayed equation on page 36 of [9] and compare it with ours in Equation (V.1)\(^{14}\) and we have trouble verifying the case analysis of the second derivative of \( G(\cdot) \) (see Equation (V.2) in our notation, denoted by \( \gamma(\cdot) \) in [9]) following that expression.

In contrast to Blinovsky’s approach [8], [9], we deduce the monotonicity property of \( f_{q,L} \) (cf. Item 1 above) from a stronger property: Schur convexity (cf. Theorem 1). Also, we believe that our proof of the convexity of \( g_{q,L} \) (cf. Item 2 above) is cleaner, more transparent and easier to verify. Both results can be extended to list-recovery setting. Another advantage is that the monotonicity property of \( g_{q,L} \) (specifically, \( g_{q,L}'' \) is non-increasing in \([0, (q-1)/q]\) and non-decreasing in \([(q-1)/q, 1]\) which is needed in the proof of the Plotkin bound appears to be a simple consequence of the Schur convexity of \( f_{q,L} \) (see Lemma 1). In [9], this is proved by checking the first derivative of \( g_{q,L} \) which involves somewhat cumbersome calculations and case analysis.

**Appendix B**

**Monotonicity Properties of \( g_{q,L} \): Proof of Lemma 1**

In this section we prove Lemma 1, which we recall here for convenience.

**Lemma 1:** For any \( q \in \mathbb{Z}_{\geq 2} \) and \( L \in \mathbb{Z}_{\geq 2} \), the function \( g_{q,L} : [0, 1] \to \mathbb{R} \geq 0 \) defined in Equation (III.3) is non-increasing on \([0, (q-1)/q]\) and non-decreasing on \([(q-1)/q, 1]\).

**Proof:** As \( g_{q,L}(w) = f_{q,L}(P_{q,w}) \) and \( f_{q,L} \) is Schur convex, the proposition follows from the following pair of facts:

- if \( w \leq u \leq \frac{q-1}{q} \) then \( P_{q,w} \geq P_{q,u} \);
- if \( w \geq u \geq \frac{q-1}{q} \) then \( P_{q,w} \geq P_{q,u} \).

Looking at the first item, if \( w \leq u \leq \frac{q-1}{q} \) then \( P_{q,w} = (1-w, \frac{w}{q-1}, \ldots, \frac{w}{q-1}) \) and \( P_{q,u} = (1-u, \frac{u}{q-1}, \ldots, \frac{u}{q-1}) \). It therefore suffices to show that \( \forall k \in \{0, 1, \ldots, q-1\}, \quad 1-w+k: \frac{w}{q-1} \geq 1-u+k: \frac{u}{q-1}, \)

which is rearranges to

\( \forall k \in \{0, \ldots, q-1\}, \quad u-w \geq (u-w) \frac{k}{q-1}. \)

As \( u-w \geq 0 \), the above is indeed true for all \( 0 \leq k \leq q-1 \).

Considering now the second item, if \( w \geq u \geq \frac{q-1}{q} \) then \( P_{q,w} = (\frac{w}{q-1}, \ldots, \frac{w}{q-1}, 1-w) \) and \( P_{q,u} = (\frac{u}{q-1}, \ldots, \frac{u}{q-1}, 1-u) \). Thus, the fact that \( P_{q,w} \geq P_{q,u} \) follows immediately from the fact that

\( \forall k \in \{1, \ldots, q-1\}, \quad k: \frac{w}{q-1} \geq k: \frac{u}{q-1}. \)

\[ \square \]

**Appendix C**

**Expressions for \( f_{q,L} \), \( g_{q,L} \), \( \ell \) and Their Derivatives:**

**Proof of Lemma 2**

First, by applying the multinomial theorem, we derive that

\[
\begin{align*}
\frac{\partial}{\partial p_j} f_{q,L}(P_1, \ldots, P_q) &= \sum_{(x_1, \ldots, x_L) \sim P \otimes L} \left[ \prod_{i=1}^q p_{x_i} \right] \\
&= \sum_{(x_1, \ldots, x_L) \sim P \otimes L} \left( \frac{L}{a_1, \ldots, a_q} \right) \max \{a_1, \ldots, a_q\} \left( \prod_{i=1}^q p_i^{a_i} \right).
\end{align*}
\]

Next, for any \( j \in [q] \), one can then compute \( \frac{\partial}{\partial p_j} f_{q,L} \) as follows.

\[
\begin{align*}
\frac{\partial}{\partial p_j} f_{q,L}(P_1, \ldots, P_q) &= \sum_{(a_1, \ldots, a_q) \in \mathcal{A}_{q,L}} \left( \frac{L}{a_1, \ldots, a_q} \right) \prod_{i=1}^q p_i^{a_i} \\
&= \sum_{(a_1, \ldots, a_q) \in \mathcal{A}_{q,L}} a_1! \cdots (a_j-1)! \cdots a_q! \left( \prod_{i=1}^q p_i^{a_i} \right) p_j^{a_j-1} \\
&= \sum_{a_1+\cdots+a_q=L} \left( \frac{L-1}{a_1! \cdots a_j! \cdots a_q!} \right) \left( \prod_{i=1}^q p_i^{a_i} \right) p_j^{a_j-1} \\
&= L \sum_{a_1+\cdots+a_q=L-1} \left( \frac{L-1}{a_1! \cdots a_j! \cdots a_q!} \right) \left( \prod_{i=1}^q p_i^{a_i} \right) p_j^{a_j-1} \\
&\quad \times \max \{a_1, \ldots, a_j+1, \ldots, a_q\} \left( \prod_{i=1}^q p_i^{a_i} \right) (C.1)
\end{align*}
\]
In Equation (C.1), we apply the change of variable.

Now, for $j, k \in [q]$, similar reasoning leads to

$$
\frac{\partial}{\partial \sqrt{p} \sqrt{q}} f_{q,L}(p_1, \ldots, p_q)
= L \sum_{(a_1, \ldots, a_k) \in A_{q,L-1}} \left( \frac{L-1}{a} \right) \frac{\partial}{\partial \sqrt{a}} \left( \prod_{i=1}^{q} p_i^{a_i} \right) a_k p_k^{a_k-1}
\times \max \{a + e_j\} \left( \prod_{i \in [q] \setminus \{k\}} p_i^{a_i} \right) a_k p_k^{a_k-1}
= L(L-1) \sum_{(a_1, \ldots, a_k) \in A_{q,L-2}} \left( \frac{L-2}{a} \right) \frac{\partial}{\partial \sqrt{a}} \left( \prod_{i=1}^{q} p_i^{a_i} \right)
\times \max \{a + e_j + e_k\} \left( \prod_{i \in [q]} p_i^{a_i} \right)
= L(L-1) \sum_{(a_1, \ldots, a_k) \in A_{q,L-2}} \left( \frac{L-2}{a} \right) \frac{\partial}{\partial \sqrt{a}} \left( \prod_{i=1}^{q} p_i^{a_i} \right)
\times \max \{a + e_j + e_k\} \left( \prod_{i=1}^{q} p_i^{a_i} \right).
$$

Next, note that for any $a \in A_{q,L-2}$, if $P_{q,L,w} = (p_1, \ldots, p_q)$,

$$
\prod_{i=1}^{q} p_i^{a_i} = \left( \frac{w}{q - \ell} \right)^{a_1 + \cdots + a_{q-\ell}} \left( \frac{1 - w}{\ell w} \right)^{a_{q-\ell+1} + \cdots + a_q}.
$$

Thus, we conclude

$$
g_{q,L,w}(w)
= \sum_{1 \leq i \leq j \leq q-\ell} L(L-1) \sum_{a \in A_{q,L-2}} \left( \frac{L-2}{a} \right) \max \{a + e_i + e_j\}
+ \sum_{q-\ell+1 \leq i \leq j \leq q} \left( \frac{w}{q - \ell} \right)^{a_{q-\ell+1} + \cdots + a_q} \left( \frac{1}{q - \ell} \right)^{a_{q-\ell} + \cdots + a_q}.
$$

Recalling the definition of $G_{\ell}(a)$, the result follows.

**Appendix D**

**Auxiliary Lemmas**

**Theorem 17 (Cramér [36, Theorem 23.3]):** Let $X_1, \cdots, X_n$ be a sequence of $n$ i.i.d. real-valued random variables. Define $S_n := \frac{1}{n} \sum_{i=1}^{n} X_i$. Fix any $q > 1$. Suppose $\log_q E[\exp_q (\lambda X_1)] < \infty$ for every $\lambda \in \mathbb{R}$. Then for any $t > E[X_1]$,

$$
\limsup_{n \to \infty} \frac{- \lambda}{n} \log_q \Pr[S_n \geq t] \leq \sup_{\lambda \in \mathbb{R}} \left\{ \lambda \log_q \mathbb{E}[\exp_q (\lambda X_1)] \right\}.
$$

**Theorem 18 (Hypergraph Ramsey [18], [48]):** Let $c, k, M \in \mathbb{Z}_{\geq 2}$. Then there exists a positive integer $N = N(c, k, M)$ such that if the hyperedges of a complete $k$-uniform hypergraph $H_{n,N}$ (i.e., a hypergraph in which every subset of $k$ vertices is connected by a hyperedge) of size $n$, number of
vertices) $N$ are coloured with $c$ different colours, then $\mathcal{H}_{k,N}$ must contain a subgraph which (i) is a complete $k$-uniform hypergraph of size at least $M$, and (ii) has the same colour on all of its hyperedges.

**Theorem 19** ([4], [8]): Let $M \geq 2L - 2$. Let $X_1, \cdots, X_M$ be a sequence of $[q]$-valued random variables. Let $P_{X_1, \cdots, X_M}$ denote their joint distribution. Suppose there exists a distribution $Q_{X_1, \cdots, X_L} \in \Delta([q]^L)$ and a constant $\varepsilon > 0$ such that

$$\left\| P_{X_1, \cdots, X_L} - Q_{X_1, \cdots, X_L} \right\|_\infty \leq \varepsilon$$

for every $1 \leq i_1 < \cdots < i_L \leq M$. Then

$$\max_{\pi \in S_L} \left\| Q_{X_1, \cdots, X_L} - Q_{X_{\pi(1)}, \cdots, X_{\pi(L)}} \right\|_\infty \leq 2L^3 \frac{2L}{M - (L - 2)} + 4L^3 \sqrt{q^{L-2} \varepsilon + L^2 \varepsilon}.$$

**Lemma 3** ([44, pp. 308-310]): Suppose $w \in [0,1]$ satisfies $1 \leq nw \leq n - 1$. Then

$$\frac{2^n H(w)}{\sqrt{8nw(1-w)}} \leq \left( \frac{n}{nw} \right) \leq 2^n H(w).$$

Bounding the volume of a ball by the outermost sphere and applying Lemma 3, we immediately obtain the following lemma.

**Lemma 4**: For any $q \in \mathbb{Z}_{\geq 2}$ and $w \in [0,1]$ such that $1 \leq nw \leq n - 1$,

$$\frac{1}{\sqrt{8nw(1-w)}} \cdot q^{n H_q(w)} \leq |B_H(q,nw)| \leq nw \cdot q^{n H_q(w)}.$$

Furthermore, suppose $1 \leq \ell \leq q$ is an integer and let $\mathcal{Y}_\ell := \{q - \ell + 1, \ldots, q\} \in ([q]^n)$. Then

$$\frac{1}{\sqrt{8nw(1-w)}} \cdot q^{n H_q(\ell, w)} \leq |B_{LR}(\mathcal{Y}_\ell, nw)| \leq nw \cdot q^{n H_q(\ell, w)}.$$

**Lemma 5** (Covering): There exists a set $\{c_1, \cdots, c_K\} \subset [q]^n$ of vectors such that

1) $\bigcup_{i=1}^K B_{LR}(c_i, nw) = [q]^n$;

2) $K \leq n \ln(q) \cdot \sqrt{8nw(1-w)} \cdot q^{n(1 - H_q(\ell, w))} + 1$.

**Proof**: The lemma follows from a simple random construction. Let $C_1, \cdots, C_K$ be points independent and uniformly distributed in $[q]^n$, for $K$ satisfying

$$K \geq n \ln(q) \cdot \sqrt{8nw(1-w)} \cdot q^{n(1 - H_q(\ell, w))} + 1.$$ (D.1)

Let us upper bound the probability that the union of the list-recovery balls centered around $C_1, \cdots, C_K$ does not cover $[q]^n$.

$$\Pr[\exists x \in [q]^n, \forall i \in [K], d_{LR}(x, C_i) > nw] \leq q^n \left( 1 - \frac{\left| B_H(q, nw) \right|}{q^n} \right)^K \leq q^n \left( 1 - \frac{q^{n H_q(w)}}{\sqrt{8nw(1-w)}} \cdot \frac{1}{q^n} \right)^K.$$ (D.2)

**Lemma 6** (Covering, List-Recovery): There exists a set $\{c_1, \cdots, c_K\} \subset ([q]^n)$ of vectors such that

1) $\bigcup_{i=1}^K B_{LR}(c_i, nw) = [q]^n$;

2) $K \leq n \ln(q) \cdot \sqrt{8nw(1-w)} \cdot q^{n(1 - H_q(\ell, w))} + 1$.

**Proof**: The lemma follows from a simple random construction. Let $C_1, \cdots, C_K$ be points independent and uniformly distributed in $[q]^n$, for $K$ satisfying

$$K \geq n \ln(q) \cdot \sqrt{8nw(1-w)} \cdot q^{n(1 - H_q(\ell, w))} + 1.$$ (D.5)

Let us upper bound the probability that the union of the list-recovery balls centered around $C_1, \cdots, C_K$ does not cover $[q]^n$.

$$\Pr[\exists x \in [q]^n, \forall i \in [K], d_{LR}(x, C_i) > nw] \leq q^n \left( 1 - \Pr_{c \sim ([q]^n)}[d_{LR}(q, c) \leq nw] \right)^K \leq q^n \left( 1 - \frac{\Pr_{c \sim ([q]^n)}[d_{LR}(q, c) \leq nw]}{q^n} \right)^K.$$ (D.6)
Bounding Equation (D.6) in a similar way to the proof of Lemma 5, we have
\[
\Pr \left[ 3x \in [q]^n, \forall i \in [K], \, d_{L_1}(x, C_{i}) > nw \right] \
\leq q^n \exp \left( \frac{-n(1 - H_{q,\ell}(w))}{8nw(1 - w)K} \right) < 1,
\]
where the last strict inequality follows from the choice of \( K \) (cf. Equation (D.5)). According to the probabilistic method, the lemma follows.

Remark 10: Though not needed in this paper, a converse statement can be shown that the exponents \( 1 - H_{q,\ell}(w) \) in Lemma 5 and \( 1 - H_{q,\ell}(w) \) in Lemma 6 cannot be further reduced. That is, any covering (not necessarily the random ones used in Lemmas 5 and 6) of \([q]^n\) must have size at least \( q^{n(1 - H_{q,\ell}(w)) - \varepsilon} \) w.r.t. the Hamming metric or at least \( q^{n(1 - H_{q,\ell}(w)) - \varepsilon} \) w.r.t. the list-recovery metric, for any constant \( \varepsilon > 0 \).

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