Lepton Mixing and Cancellation of the Dirac Mass Hierarchy in SO(10) GUTs with Flavor Symmetries $T_7$ and $\Sigma(81)$

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Abstract

In SO(10) grand unified theories (GUTs) the hierarchy which is present in the Dirac mass term of the neutrinos is generically as strong as the one in the up-type quark mass term. We propose a mechanism to partially or completely cancel this hierarchy in the light neutrino mass matrix in the seesaw context. The two main ingredients of the cancellation mechanism are the existence of three fermionic gauge singlets and of a discrete flavor symmetry $G_f$ which is broken at a higher scale than SO(10). Two realizations of the cancellation mechanism are presented. The realization based on the Frobenius group $T_7 \simeq \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ leads to a partial cancellation of the hierarchy and relates maximal $2 - 3$ lepton mixing with the geometric hierarchy of the up-quark masses. In the realization with the group $\Sigma(81)$ the cancellation is complete and tri-bimaximal lepton mixing is reproduced at the lowest order. In both cases, to fully accommodate the leptonic data we take into account additional effects such as effects of higher-dimensional operators involving more than one flavon. The heavy neutral fermion mass spectra are considered. For both realizations we analyze the flavon potential at the renormalizable level as well as ways to generate the Cabibbo angle.
1 Introduction

The electric charge quantization as well as the possible gauge coupling unification at high energies are strong hints for a GUT [1]. Especially an SO(10) GUT [2] looks very appealing, since it allows one to unify all fermions of one generation including the right-handed neutrino, \( N \equiv (\nu_R)^c \). However, it is difficult to reconcile this unification with the observation of a strong hierarchy among the charged fermion masses, but only a mild one among the neutrino masses. Indeed, a salient feature of the simplest versions of an SO(10) GUT is that the Dirac mass matrix \( m_D \) of the neutrinos has the same structure as the mass matrix of the up-quarks, i.e. it is strongly hierarchical. In the type-I seesaw mechanism [3] this matrix appears twice and thereby, in general, leads to an even stronger hierarchy among the light neutrino masses contradicting observations. Furthermore, the diverse mixing patterns of quarks and leptons have to be explained. It seems that the lepton sector reveals special features in the mixings, such as \( \mu - \tau \) symmetry [4] or tri-bimaximal mixing (TBM) [5]. TBM, for example, can be understood in non-unified models with the help of discrete or continuous flavor symmetries, such as \( A_4 \) [6] and \( \Delta(27) \) [7] or SO(3) [8] and SU(3) [9]. In general this requires, however, that the fermions residing in different representations of the Standard Model (SM) gauge group, such as left- and right-handed components, transform differently under \( G_f \), so that an extension of these models to a GUT is not straightforward. There are several recent attempts to resolve this problem [10–13].

In this paper we propose a mechanism to break the strong correlation of the up-quark and the neutrino mass matrix in SO(10)\(^1\). For this purpose, we assume the existence of additional fermionic GUT singlets \( S_i \) and a discrete group \( G_f \) to constrain and correlate different couplings. The fields \( S_i \) can mix with neutrinos only and thereby lead to different properties of quark and lepton mixings as well as to the smallness of neutrino masses [17–21]. Each of the fermion generations is accompanied by one single \( S_i \). In our context, the mass matrix of the neutral fermions in the basis \( (\nu_L, N, S) \), is of the following form

\[
\begin{pmatrix}
0 & m_D & m_{\nu S} \\
m_D^T & 0 & M_{NS} \\
m_{\nu S}^T & M_{NS}^T & M_{SS}
\end{pmatrix}
\begin{pmatrix}
\nu_L \\
N \\
S
\end{pmatrix}.
\]  

(1)

Block diagonalization of the matrix in Eq.(1) yields the effective light neutrino mass matrix

\[
m_\nu \approx m_{\nu}^{DS} + m_{\nu}^{LS},
\]

(2)

where

\[
m_{\nu}^{DS} = m_D \left( M_{NS}^{-1T} M_{SS} M_{NS}^{-1} \right) m_D^T
\]

is the double seesaw (DS) contribution [22], and

\[
m_{\nu}^{LS} = - \left[ m_D \left( m_{\nu S} M_{NS}^{-1} \right)^T + \left( m_{\nu S} M_{NS}^{-1} \right) m_D^T \right]
\]

(4)

is the so-called linear seesaw (LS) contribution to the mass matrix [19]. If \( m_{\nu}^{LS} \ll m_{\nu}^{DS} \), the

\(^1\)Other approaches to this problem can be found in [14–16].

\(^2\)This hierarchy arises, for instance, if \( m_D, m_{\nu S} \) are of the order of the weak scale, \( M_{NS} \) is of the order of the GUT scale and the masses of the SO(10) singlets are around the Planck scale.
main contribution can be written as

\[ m_\nu \approx m_\nu^{DS} = F M_{SS} F^T, \]

where

\[ F \equiv m_D M_{NS}^{-1} T. \]

If the hierarchy present in the Dirac mass matrix \( m_D \) is cancelled by the same or a similar hierarchy in \( M_{NS} \), \( F \) may turn out to be proportional to a matrix with \( \mathcal{O}(1) \) entries. Then the structure of the light neutrino mass matrix \( m_\nu \) is non-hierarchical, provided that there is no hierarchy in the Majorana mass matrix \( M_{SS} \) of the singlets \( S_i \). We call this complete cancellation of the (Dirac mass) hierarchy. We refer to partial cancellation, if \( F \) still contains some hierarchy. These possibilities are actually a generalization of the requirement \( F \propto 1 \) which arises if \( M_{NS}^T \) is proportional to \( m_D \) \cite{17,18,20}. In this case, called the Dirac screening mechanism \cite{20}, the light neutrino mass matrix is proportional to \( M_{SS} \) and therefore neutrino masses and lepton mixings are completely decoupled from charged fermion masses and quark mixings. In the context of the type-I seesaw mechanism a similar cancellation has been presented in \cite{23}.

In this paper we show how the cancellation mechanism can be realized in models with the discrete flavor symmetries \( T_7 \) \cite{24} and \( \Sigma(81) \) \cite{25,26}. The \( T_7 \) realization leads to maximal atmospheric mixing and very small \( \theta_{13} \) through a partial cancellation of the up-quark mass hierarchy in the neutrino sector. The large value of the solar mixing angle \( \theta_{12} \) however cannot be explained in this way and also not through the effects of higher-dimensional operators involving more than one flavon. To generate large \( \theta_{12} \) we have to introduce an additional Higgs field. It only contributes to the LS term, while not disturbing the partial cancellation arising in the DS contribution. In the \( \Sigma(81) \) setup the up-quark mass hierarchy is completely cancelled and the resulting neutrino mass matrix is compatible with TBM. However, the atmospheric mass squared difference vanishes. This problem is resolved, if higher-dimensional operators are included into the analysis. In the quark sector we maintain diagonal mass matrices at leading order with the mass hierarchy of the up-quarks in both models. To generate the Cabibbo angle we consider higher-dimensional operators with additional \( 16_H \) fields. Furthermore, we calculate the mass spectrum of the heavy neutral fermions. Finally, the flavon potential is analyzed at the renormalizable level.

The paper is structured as follows: in Section 2 we describe our GUT context and show the prerequisites which have to be fulfilled to setup the cancellation mechanism. In Section 3 we present the \( T_7 \) realization, and in Section 4 the \( \Sigma(81) \) realization of this mechanism. In both setups the corrections arising from higher-dimensional operators with products of more than one flavon are calculated, the generation of the Cabibbo angle is discussed as well as the mass spectrum of the heavy neutral fermions. Additionally, the flavon potentials are presented. We summarize our results in Section 5. Details of the group theory of \( T_7 \) and \( \Sigma(81) \) as well as of the study of the higher-dimensional operators in both realizations are given in the Appendices.

## 2 Cancellation of the Dirac Mass Hierarchy

We consider an SO(10) model in which the SM fermions and the right-handed neutrinos are unified into three \( 16_i, i = 1, 2, 3 \). In addition, we introduce three fermionic SO(10)
singlets $S_i$. In order to guarantee that the gauge coupling is perturbative also above the GUT scale we assume the existence of only low-dimensional SO(10) representations for the Higgs fields \cite{27,28}: $H \sim 10$, $16_H$, $\Delta \sim \overline{16}$ and $45_H$. Thus, no 126-dimensional scalar representation is introduced. The non-existence of the latter causes the zeros in the mass matrix in Eq.(11).

The 10-dimensional representation $H$ is responsible for the Dirac masses of the fermions, i.e. $m_D \propto \langle H \rangle$. $\langle H \rangle$ stands for the weak scale vacuum expectation value (VEV) for up quarks and neutrinos or for down quarks and charged leptons depending on the context. The 16-plet scalar $\Delta$ connects the fermions in $16_i$ and the singlets $S_j$. It therefore gives rise to $m_{\nu S} \propto \langle \Delta \rangle_\nu$ and $M_{NS} \propto \langle \Delta \rangle_N$ in Eq.(1), where $\langle \Delta \rangle_\nu$ and $\langle \Delta \rangle_N$ denote the weak and the GUT scale VEVs, respectively. Note that if there is only one multiplet $\Delta$, the matrices $m_{\nu S}$ and $M_{NS}$ stem from the same coupling and are hence proportional to each other up to renormalization group (RG) corrections. According to Eq.(4) this leads to the proportionality $m_{\nu S}^L \propto m_D + m_D^T$ \cite{19}.

To explain the existence of three generations we unify $16_i$ into a $3$ under $G_f$. By choosing the representation of $G_f$ to be complex we prevent the existence of an invariant coupling $16_i16_i H$ (if $H$ transforms trivially under $G_f$) which leads to degenerate mass spectra for the fermions. Exactly for this reason the group $A_4$ is not applicable. In contrast to this, the transformation properties of the fermionic singlets $S_i$ are determined by the requirement to obtain a phenomenologically viable model. It turns out that in our two realizations it is favorable to choose the three singlets $S_i$ to transform as three inequivalent one-dimensional representations of the flavor group $G_f$, instead of unifying them into a three-dimensional representation. Obviously, a successful model should be able to describe more features of the fermions than just the existence of three generations. In this paper, we concentrate on models which explain the different hierarchies of the charged fermions and the neutrinos through the cancellation mechanism and (some of) the prominent features of the lepton mixings, while simultaneously resulting in vanishing quark mixing at leading order. Nevertheless, we ensure that the Cabibbo angle can be generated at subleading order.

The scalars in our model are separated into two groups, the GUT Higgs and flavon fields, in order to disentangle the GUT and the flavor breaking scales. The GUT Higgs multiplets, $H$, $16_H$, $\Delta$ and $45_H$, do not transform under $G_f$, while the flavon fields $\chi_i$ are gauge singlets carrying flavor indices. We are discussing realizations with a minimal number of flavon fields, which is three in our case.

As mentioned before, in order to achieve a cancellation of the mass hierarchy encoded in $m_D$ the elements of $M_{NS}$ should have a similar hierarchy. One possibility to relate $m_D$ and $M_{NS}$ is to further unify the left-handed neutrinos (and therefore $16_i$) and the fermionic singlets $S_j$, e.g. into an $E_6$ representation \cite{20}. Another one is to assume that a flavor symmetry dictates the relation between $m_D$ and $M_{NS}$. However, when applying a flavor symmetry, in general, more than one Higgs field (which form non-trivial multiplets of the symmetry group) contribute to $m_D$ and $M_{NS}$, respectively. Since these fields are in 10- and 16-dimensional representations of SO(10), respectively, it is not obvious how to properly relate their VEVs through the Higgs potential in order to ensure that $m_D$ and $M_{NS}$ have a similar hierarchy. Therefore, we consider the possibility to have additional fields in the theory, the flavons, which

\footnote{This is true, if we only consider the part relevant for neutrino masses. The operators required for generating the Cabibbo angle also involve GUT Higgs fields in non-trivial one-dimensional representations of the flavor group. However, this does not alter the statement that the GUT and the flavor symmetry breaking are disentangled in both realizations.}
are necessary to build invariants under $G_f$

$$\frac{\alpha}{\Lambda} \mathbf{16} \mathbf{16} H \chi + \frac{\beta}{\Lambda} \mathbf{16} S \Delta \chi' + M_{SS} S S,$$

(7)

with either $\chi' = \chi$ or $\chi' = \chi^\ast$. Here $\alpha$ and $\beta$ are complex three-by-three matrices. The fact that the same flavon field (or its conjugate) enters both interaction terms in Eq. (7) leads to the required correlation of the mass matrices $m_D$ and $M_{NS}$. The couplings involving $\mathbf{16}$ in Eq. (7) are non-renormalizable and suppressed by the cutoff scale $\Lambda$. This scale is not fixed a priori, but a natural choice would be the Planck scale $M_P$. Since also the mass of the top quark stems from such a coupling, the ratio $\langle \chi \rangle / \Lambda$ cannot be small for all fields $\chi_i$. Thus, a careful study of higher-dimensional operators arising from multi-flavon insertions is mandatory. We assume, for simplicity, that the singlets $S_i$ acquire a direct Majorana mass $M_{SS}$ at the lowest order. However, in general, this mass term is also corrected by operators involving the flavon fields $\chi$ or could be even generated solely through these operators.

The idea to erase the Dirac mass hierarchy in the light neutrino mass matrix by introducing flavon fields has been previously addressed in [23]. There, the hierarchy of $m_D$ is cancelled in the context of the type-I seesaw mechanism by a quadratic hierarchy in the Majorana mass matrix $M_{RR}$ of the right-handed neutrinos. This cancellation is complete. Since $M_{RR}$ is strongly hierarchical, sequential dominance is realized which leads (with additional constraints on the vacuum alignment) to TBM. The gauge group is the Pati-Salam group, and either SO(3) or $A_4$ have been employed as $G_f$. Compared to this model our approach has the advantage, that it can be reconciled with an embedding into SO(10) without introducing extra dimensions.

3 $T_7$ Realization

The group $T_7$ is of order 21 and contains five irreducible representations which are denoted by $\mathbf{1}_1$, $\mathbf{1}_2$, $\mathbf{1}_3$ and $\mathbf{3}$, $\mathbf{3}^\ast$. The representations $\mathbf{1}_2$ and $\mathbf{1}_3$ as well as $\mathbf{3}$ and $\mathbf{3}^\ast$ are complex conjugated to each other. $T_7$ is a subgroup of SU(3) [24]. This group has properties similar to those of the well-known group $A_4$ except for the crucial difference that its three-dimensional representation is complex. Due to this difference the product $\mathbf{3} \times \mathbf{3}$ does not contain the invariant $\mathbf{1}_1$, and therefore the renormalizable coupling $\mathbf{16}, \mathbf{16}, H$ (for $H \sim \mathbf{1}_1$ under $T_7$) is forbidden. It is interesting to note that $T_7$ is the smallest discrete group with a complex irreducible three-dimensional representation. In the following model we assume the existence of low-scale supersymmetry.

3.1 Masses and Mixing at the Lowest Order

To explain the three generations of SM fermions we assign $\mathbf{16}$ to $\mathbf{3}$. For $H \sim \mathbf{1}_1$ and $\chi_i \sim \mathbf{3}^\ast$ the Dirac mass matrix which results from the first term of Eq. (7) is diagonal and the VEVs of $\chi_i$ determine the charged fermion mass hierarchy. In order to achieve a partial cancellation of this hierarchy in $m_\nu$ we assign the three fermionic SO(10) singlets to the three

$^4$By introducing additional symmetries, such as a U(1) symmetry, one might be able to forbid all operators with more than one flavon. However, in this paper we would like to concentrate on the simplest models with the least number of additional symmetries.
one-dimensional representations, $S_i \sim \mathbf{1}_i$. The Higgs multiplet $\Delta$ connecting the 16-plets $\mathbf{16}_i$ and $S_j$ is invariant under $T_7$. Thus, also in this case we generate terms of the form as in Eq. (7). These assignments are collected in the Table 1. The Yukawa couplings can be constructed using the Clebsch-Gordan coefficients given in the Appendix A

$$L_Y = \alpha \langle \mathbf{16}_3 H \mathbf{16}_1 \chi_1 + \mathbf{16}_1 H \mathbf{16}_2 \chi_2 + \mathbf{16}_2 H \mathbf{16}_2 \chi_3 \rangle / \Lambda$$
$$+ \beta_1 \langle \mathbf{16}_1 \chi_1 + \mathbf{16}_2 \chi_2 + \mathbf{16}_3 \chi_3 \rangle \Delta S_1 / \Lambda$$
$$+ \beta_2 \langle \mathbf{16}_1 \chi_1 + \omega \mathbf{16}_2 \chi_2 + \omega^2 \mathbf{16}_3 \chi_3 \rangle \Delta S_2 / \Lambda$$
$$+ \beta_3 \langle \mathbf{16}_1 \chi_1 + \omega^2 \mathbf{16}_2 \chi_2 + \omega^2 \mathbf{16}_3 \chi_3 \rangle \Delta S_3 / \Lambda$$
$$+ A S_1 S_1 + B (S_2 S_3 + S_3 S_2) + \text{h.c.} \tag{8}$$

They generate matrices $m_D$, $M_{NS}$ and $M_{SS}$ of the form $^5$

$$m_D = \frac{\alpha \langle H \rangle}{\Lambda} \left( \begin{array}{ccc} \langle \chi_2 \rangle & 0 & 0 \\ 0 & \langle \chi_3 \rangle & 0 \\ 0 & 0 & \langle \chi_1 \rangle \end{array} \right),$$

$$M_{NS} = \frac{\langle \Delta \rangle_N}{\Lambda} \left( \begin{array}{ccc} \beta_1 \langle \chi_1 \rangle & \beta_2 \langle \chi_1 \rangle & \beta_3 \langle \chi_1 \rangle \\ \beta_1 \langle \chi_2 \rangle & \omega \beta_2 \langle \chi_2 \rangle & \omega^2 \beta_3 \langle \chi_2 \rangle \\ \beta_1 \langle \chi_3 \rangle & \omega^2 \beta_2 \langle \chi_3 \rangle & \omega^3 \beta_3 \langle \chi_3 \rangle \end{array} \right)$$

$$= \frac{\langle \Delta \rangle_N}{\Lambda} \left( \begin{array}{ccc} \langle \chi_1 \rangle & 0 & 0 \\ 0 & \langle \chi_2 \rangle & 0 \\ 0 & 0 & \langle \chi_3 \rangle \end{array} \right) \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{array} \right) \left( \begin{array}{ccc} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{array} \right),$$

$$M_{SS} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{pmatrix},$$

with $\omega \equiv e^{2\pi i / 3}$. Assuming the dominance of the DS contribution we obtain

$$m_{\nu} \approx \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N} \right)^2 D_X \left( \begin{array}{ccc} \tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} & \tilde{A} - \tilde{B} \\ \tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} & \tilde{A} - \tilde{B} \\ \tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} & \tilde{A} - \tilde{B} \end{array} \right) D_X,$$  \tag{10}

where

$$D_X \equiv \text{diag} \left( \frac{\langle \chi_2 \rangle}{\langle \chi_1 \rangle}, \frac{\langle \chi_3 \rangle}{\langle \chi_2 \rangle}, \frac{\langle \chi_1 \rangle}{\langle \chi_3 \rangle} \right), \quad \tilde{A} \equiv \frac{A}{9\beta_1^2}, \quad \tilde{B} \equiv \frac{B}{9\beta_2\beta_3} \tag{11}$$

$^5$Throughout this work we assume that $\alpha$, $\langle H \rangle$, $\langle \Delta \rangle_N$ and $\epsilon$ are real and positive.
For simplicity, we will only consider the case in which the VEVs $\langle \chi_i \rangle$ are real and positive. To produce the hierarchy of the up-quark masses, these VEVs have to be chosen as

$$\frac{\langle \chi_2 \rangle}{\langle \chi_1 \rangle} \approx \epsilon^4, \quad \frac{\langle \chi_3 \rangle}{\langle \chi_1 \rangle} \approx \epsilon^2 \quad \text{with} \quad \epsilon \approx 0.05.$$  \hspace{1cm} (12)

The corresponding flavon potential is discussed in Section 3.5. The ratio $\langle \chi_1 \rangle/\Lambda$ cannot be small, i.e.

$$\eta \equiv \frac{\langle \chi_1 \rangle}{\Lambda} \sim \mathcal{O}(1) ,$$  \hspace{1cm} (13)

to guarantee the large mass of the top quark. Using Eq. (12) we obtain from Eq. (10)

$$m_\nu \approx \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N \epsilon^2} \right)^2 \begin{pmatrix} (\tilde{A} + 2\tilde{B})\epsilon^{12} & (\tilde{A} - \tilde{B})\epsilon^{6} & (\tilde{A} - \tilde{B})\epsilon^{6} \\ A + 2\tilde{B} & A - \tilde{B} & A - \tilde{B} \\ \tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} & \tilde{A} - \tilde{B} \end{pmatrix}. \hspace{1cm} (14)$$

This matrix has a dominant, $\mu - \tau$ symmetric $2 - 3$ block, while the elements in the first row and column are strongly suppressed. Therefore, the $2 - 3$ mixing is maximal, $\theta_{23} = \pi/4$, and the two other mixing angles are very small, especially the solar mixing angle has to be generated by additional contributions. The mass spectrum is normally ordered with $m_1 \ll m_2, m_3$. For $|2\tilde{A} + \tilde{B}| < 3|\tilde{B}|$ we find

$$m_2 = \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N \epsilon^2} \right)^2 |2\tilde{A} + \tilde{B}| , \quad m_3 = 3 \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N \epsilon^2} \right)^2 |\tilde{B}|$$  \hspace{1cm} (15)

and therefore

$$r \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{31}} = \frac{m^2_2 - m^2_1}{m^2_3 - m^2_1} \approx \frac{|2\tilde{A} + \tilde{B}|^2}{9|\tilde{B}|^2}. \hspace{1cm} (16)$$

The smallness of the ratio $r$ is achieved, for example, if $\tilde{A} \approx -\frac{1}{4}\tilde{B}$. According to Eq. (14) the DS enhances the neutrino mass terms in the $2 - 3$ block, and consequently, the absolute scale of the neutrino masses by a factor of $\epsilon^{-4} \approx 1.6 \cdot 10^5$. This originates from the fact that in $m_D$ and in the diagonal factor of $M_{\nu}$ the VEVs of the flavons do not follow the same ordering. For this reason, the cancellation of the hierarchy is only partial. Furthermore, the condition $m_3 \lesssim 1$ eV implies (for $\langle H \rangle = 174$ GeV)

$$\frac{\langle \tilde{A}, \tilde{B} \rangle}{10^{16} \text{GeV}} \left( \frac{10^{16} \text{GeV}}{\langle \Delta \rangle_N} \right)^2 \lesssim 2 \cdot 10^{-3}. \hspace{1cm} (17)$$

This is not satisfied for natural values of parameters $\tilde{A}, \tilde{B} \sim M_{Pl} = 1.22 \cdot 10^{19}$ GeV and $\langle \Delta \rangle_N \sim M_{GUT} = 2 \cdot 10^{16}$ GeV. We can fulfill Eq. (17) by either lowering the scale of the parameters $\tilde{A}$ and $\tilde{B}$ (and thus $\tilde{A}$ and $\tilde{B}$) down to $10^{13}$ GeV or by increasing the scale of the VEV $\langle \Delta \rangle_N$ from $10^{16}$ GeV up to $10^{19}$ GeV. The first possibility turns out to be favorable, since we can lower the scale of $\tilde{A}$ and $\tilde{B}$ by several orders of magnitude by introducing an additional symmetry which forbids the direct mass term. Then we have to check whether this modification of scales alters our assumption that $m^L_\nu \ll m^DS_\nu$. The LS term is of the form

$$m^L_\nu = -2 \frac{\langle \Delta \rangle_\nu}{\langle \Delta \rangle_N} m_D = -2 \alpha \eta \frac{\langle \Delta \rangle_\nu}{\langle \Delta \rangle_N} \langle H \rangle \begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (18)

It does not have a $1/\epsilon$ enhancement like the DS contribution and is still of the generic size $10^{-3}$ eV. Therefore it does not change the maximal atmospheric mixing originating from Eq. (14). At the same
time, being diagonal, this term does not generate a sizable $1 - 2$ mixing angle. A non-diagonal LS contribution can originate from the introduction of a second 16-plet, $\Delta' \sim \overline{16}$. This possibility is compatible with lowering the scale of $A$ and $B$. This issue is discussed in Section 3.3.

One might raise the question whether lowering the mass scale of the singlets invalidates the DS formula shown in Eq. (3). This is not the case, as has been discussed in [29]. For $M_{SS} \to 0$ total lepton number conservation is restored and the light neutrinos will be massless. The right-handed neutrinos and singlets will then combine into three heavy Dirac fermions. We will discuss the mass spectrum of the right-handed neutrinos and the singlets in more detail in Section 3.4.

### 3.2 Effects of Higher-Dimensional Operators

As the large top quark mass requires $\eta \sim O(1)$, a careful study of the higher-dimensional operators of the form

$$\hat{O} \left( \frac{\chi_i}{\Lambda} \right)^n$$

(19)

$n = 2, 3, \ldots$ is mandatory. Here $\hat{O}$ denotes $16 \ 16 \ H$, $16 \ S \ \Delta$ or $\Lambda SS$\footnote{Note that there is no renormalizable coupling between the singlets $S_i$ and the flavons, since the singlets are in one-dimensional representations of $T_7$, whereas the fields $\chi_i$ form a triplet.}. For $\langle \chi_i \rangle$ as given in Eq. (12), the products of $\chi_i$ which contribute down to $O(\epsilon^6)$ are collected in the Table 2. We list all contributions up to the $6^{th}$ power of $\epsilon$, $O(\epsilon^6)$, since we have seen in the previous section that entries up to the $4^{th}$ power, $O(\epsilon^4)$, are generated in the mass matrices $m_D$ and $M_{NS}$ by the insertion of one flavon. It should be noted that the $n^{th}$ order in the Table 2 has to be multiplied by $\eta^n < 1$. Thus, contributions from such operators get more and more suppressed as $n$ increases. Choosing, for instance, $\eta \approx 0.47$, equivalent to $\eta \approx \epsilon^{1/4}$, makes it sufficient to consider operators up to $n = 17$. In the Table 10 presented in the Appendix C we show as which component of a $T_7$ covariant the monomials given in the Table 2 actually transform for arbitrary $n$. Using these results one finds that all matrix elements of $m_D$, $M_{NS}$ and $M_{SS}$ get corrections at all orders in $\epsilon^2$, i.e. $O(1)$, $O(\epsilon^2)$, $\ldots$ accompanied by appropriate suppression factors $\eta^n$. Despite the suppression factor $\eta^n$ these operators destroy the

| Structure | Transformation Properties under Generator A | Order in $\epsilon$ |
|-----------|---------------------------------------------|---------------------|
| $\chi^n_1$ | $e^{\mp \frac{2\pi \epsilon}{7} n} \chi^n_1$ | $O(1)$ |
| $\chi_1^{-1} \chi_2$ | $e^{-\frac{2\pi \epsilon}{7} (n+1)} \chi_1^{-1} \chi_2$ | $O(\epsilon)$ |
| $\chi_1^{-1} \chi_3$ | $e^{-\frac{2\pi \epsilon}{7} (n+3)} \chi_1^{-1} \chi_3$ | $O(\epsilon^2)$ |
| $\chi_1^{-2} \chi_2 \chi_3$ | $e^{-\frac{2\pi \epsilon}{7} (n+4)} \chi_1^{-2} \chi_2 \chi_3$ | $O(\epsilon^3)$ |
| $\chi_1^{-2} \chi_3$ | $e^{-\frac{2\pi \epsilon}{7} (n+1)} \chi_1^{-2} \chi_3$ | $O(\epsilon^4)$ |
| $\chi_1^{-3} \chi_3^3$ | $e^{\frac{2\pi \epsilon}{7} (n+2)} \chi_1^{-3} \chi_3^3$ | $O(\epsilon^6)$ |

Table 2: List of products of $\chi_i$ which lead to contributions down to $O(\epsilon^n)$ for $\langle \chi_1 \rangle / \Lambda = \eta \sim O(1)$, $\langle \chi_2 \rangle / \langle \chi_1 \rangle \approx \epsilon^4$ and $\langle \chi_3 \rangle / \langle \chi_1 \rangle \approx \epsilon^2$. Note that for the order $n$ the factor $\eta^n$ has to be included. In the second column we show the behavior of the monomials under the generator $A$ (see the Appendix A) which uniquely determines their $T_7$ transformation properties (apart from the fact that one cannot specify as which component of a $T_7$ transforms with this information).
lowest order result for the fermion mass matrices. In the following we will discuss how to solve this problem by adding another symmetry to the model.

### 3.2.1 Majorana Masses of Singlets

A special problem arises for the masses of the singlets $S_i$. As we have seen in the previous section, most probably the scale of the singlet mass terms has to be (much) below the Planck scale to accommodate the mass scale of the light neutrinos. However, corrections stemming from the insertion of $n$ flavons to the singlet masses can be of the order $\Lambda n$ with $\Lambda$ around the Planck scale. Such contributions can strongly affect the absolute neutrino mass scale as well as the lepton mixing angles, since they are in general not of the same form as the leading order structure. In order to avoid this we invoke an additional symmetry which constrains all higher-dimensional operators and forbids the direct mass term of the singlets. In a minimal setup, no additional fields are introduced, but $M_{SS}$ is generated by operators of the structure $S_i S_j \chi^n/\Lambda^{n-1}$. As $S_i S_j$ transforms as singlet, also the covariants of the type $\chi^n$ have to transform as singlets. In general, it is possible to construct all three different singlet representations at a given order $n$ in $\chi^n/\Lambda^n$ which generates all matrix elements of $M_{SS}$ at the same order. However, at the third order there is only one covariant and it transforms as the trivial singlet $1$ with respect to $T_7$. Therefore the operators of the form $S_i S_j \chi^3/\Lambda^2$ explicitly read

$$a S_1 S_1 \chi^1 \chi^2 \chi^3/\Lambda^2 + b (S_2 S_3 + S_3 S_2) \chi_1 \chi_2 \chi_3/\Lambda^2 + \text{h.c.}$$

and lead to $M_{SS}$ as displayed in Eq.(9) with $A = a \eta^6 \Lambda$ and $B = b \eta^3 e^6 \Lambda$. They give exactly the same result for $m_\nu$ as the matrix $M_{SS}$ stemming from a direct mass term. However, the parameters $A$ and $B$ are of the order $e^6 \Lambda \approx 10^{11}$ GeV. So, they even overcompensate the factor $1/e^4$ appearing in $m_\nu$ in Eq.(14). To correctly adjust the light neutrino mass scale one also has to assume that the VEV of $\Delta$ is smaller than the GUT scale, $\langle \Delta \rangle \approx 10^{15}$ GeV.

As the structure of the covariants given in the Table 2 and the Table 10 for $n+7$ equals the one for $n$, we choose a $Z_7$ symmetry to suppress all higher-dimensional operators relative to the leading order. Then, corrections to the terms of Eq.(20) are suppressed with a relative factor $\eta^7$ and entries which vanish at leading order are generated by operators of the form $S_i S_j \chi^10/\Lambda^9$ giving contributions of order $e^6 \eta^{10} \Lambda$. Thus, all corrections to $M_{SS}$ are well under control. A viable charge assignment which forbids the direct mass term of the singlets and allows operators of type $16 \times 16 \times H \chi/\Lambda$, $16 \times S \Delta \chi/\Lambda$ as well as $S S \chi^3/\Lambda^2$ is presented in the Table 3.

| Field | $T_7$ | $Z_7$ |
|-------|-------|-------|
| $16$  | $3$   | $3$   |
| $S_1$ | $\frac{1}{1}$ | $2$   |
| $S_2$ | $\frac{1}{1}$ | $2$   |
| $S_3$ | $\frac{1}{1}$ | $2$   |
| $H$   | $1$   | $1$   |
| $\Delta$ | $1$ | $1$ |
| $\chi_i$ | $1$ | $1$ |

Table 3: The $Z_7$ charge assignment of all fields. A field $\phi$ with charge $q$ transforms as $e^{\frac{2\pi i q}{7}} \phi$ under $Z_7$.
3.2.2 Cabibbo Angle

The additional $Z_7$ symmetry also constrains the higher-dimensional operators contributing to $m_D$ and $M_{NS}$ to operators with $7k + 1$ flavon insertions ($k = 0, 1, 2, ...$). These contributions do not spoil the leading order result. The matrix elements which are non-vanishing at leading order are corrected by contributions which arise in the same order of $\epsilon$ as the leading order, but with an additional suppression factor $\eta^7 \approx \epsilon^2$. This is due to the above mentioned periodicity in 7 of the structure of the covariants given in the Table 2 and the Table 10. Higher-dimensional operators lead to non-vanishing off-diagonal elements of $m_D$ which are at most of the order of $O(\epsilon^6\eta^8)$. Thus we cannot generate the Cabibbo angle through them by simply introducing a second 10-dimensional Higgs field, $H' \sim 10$. One possibility to obtain the Cabibbo angle is to consider operators of the form [27, 28]

$$\frac{1}{M} \left\langle 16' 16' 16_H 16'_H \right\rangle \left(\frac{\chi}{\Lambda}\right)^n$$

(21)

with $M$ being the mass of the particles mediating this interaction. Thereby, new Higgs fields $16_H$ and $16'_H$ have to be introduced. When $16_H$ acquires a weak scale VEV $\left\langle 16_H \right\rangle$ and $16'_H$ a GUT scale VEV $\left\langle 16'_H \right\rangle$, the operators contribute to the down quark and the charged lepton mass matrix and the size of the contributions is $\left\langle 16_H \right\rangle \left\langle 16'_H \right\rangle N \langle \chi \rangle^n / \Lambda^n$. If the Higgs fields $16_H, 16'_H$ do not transform under $T_7$ and $Z_7$, these operators cannot lead to a sizable contribution to the Cabibbo angle. Therefore, we assume that $16_H$ and $16'_H$ have non-trivial $Z_7$ charges, $Q(16_H) = Q(16'_H) = 6$. Then the lowest dimensional operator which is invariant under $T_7$ and $Z_7$ is of the form

$$\frac{1}{M} \left\langle 16' 16' 16_H 16'_H \right\rangle \left(\frac{\chi}{\Lambda}\right)^3.$$ 

(22)

Its contributions to the mass matrix of the down quarks and charged leptons read $^9$

$$\langle 16_H \rangle \left(\frac{\langle 16'_H \rangle N}{M}\right) \begin{pmatrix} O(\epsilon^4\eta^3) & O(\eta^3) & O(\epsilon^6\eta^3) \\ O(\epsilon^4\eta^3) & O(\epsilon^2\eta^3) & 0 \end{pmatrix}.$$ 

(23)

Assuming $\langle 16_H \rangle \approx 100$ GeV and $\langle 16'_H \rangle N / M \approx \epsilon^2$ (e.g. for $\langle 16'_H \rangle N \approx 2 \cdot 10^{16}$ GeV the mediator scale is $M \approx 8 \cdot 10^{18}$ GeV) we find that the $1 \rightarrow 2$ quark mixing angle $\vartheta_{12} \approx \eta^2 \approx 0.22$, thus reproducing the Cabibbo mixing. The observed values of the quark mixing angles $\vartheta_{13}$ and $\vartheta_{23}$ cannot be generated in this way, but by introducing operators with a different structure. The contributions to the diagonal elements of the down quark and the charged lepton mass matrix are suppressed by at least $\epsilon^2$ (stemming from $\langle 16'_H \rangle N / M$) compared to $m_D$. The mass hierarchy generated in the mass matrix $m_D$ is slightly changed, i.e. the mass of the first generation is now of order $\epsilon^3\eta$ instead of $\epsilon^4\eta$. This enhancement is welcome, as the mass hierarchy in the down quark and the charged lepton sector is milder than in the up-quark masses. However, this is still not sufficient, since the masses of the first two generations turn out to be too suppressed compared to the mass of the third generation. The subleading contributions to operators of structure Eq. (22) are suppressed by a factor $\eta^7$ and carry the same suppression factors in $\epsilon$ as the leading order so that they can be safely neglected. Since the operators of structure Eq. (22) also contribute to the charged lepton mass matrix, they induce a charged lepton mixing angle of the size of the Cabibbo angle in the $1 \rightarrow 2$ sector. This then corrects the results for the lepton mixing angles as well. However, this contribution alone is too small to generate a large $1 \rightarrow 2$ leptonic mixing. Thus, we still have to consider another mechanism to generate a sizable $1 \rightarrow 2$ mixing angle in the neutrino sector. This is discussed in Section 3.3.

$^9$Note that these contributions are not necessarily symmetric in flavor space. However, the elements $(ij)$ and $(ji)$ have the same order in the parameters $\epsilon$ and $\eta$. 

9
3.2.3 Light Neutrino Mass Matrix

Calculating finally the light effective neutrino mass matrix $m_\nu$ with the corrected matrices $m_D$, $M_{NS}$ and $M_{SS}$ we find that the first row and column of the neutrino mass matrix receive small corrections from higher-dimensional ($Z_7$ invariant) operators. The $1 - 2$ and $1 - 3$ entries are corrected by contributions of $O(e^4 \eta^7)$ relative to the $2 - 3$ sub-block. $\nu_1$ remains approximately massless and $m_2^2$ and $m_3^2$ receive corrections of $O(\eta^7)$ relative to the leading order result. The corrections are negligible for all mixing angles. Also RG corrections cannot generate a sizable $\theta_{12}$, since the neutrino masses have a strong normal hierarchy and the value of $\theta_{12}$ at leading order is small. Therefore, we discuss in the following how a sizable $1 - 2$ mixing angle can be generated in the neutrino sector through an additional Higgs field contributing only to the LS term.

3.3 Contribution from the Linear Seesaw Term

In the previous sections, the LS term has been neglected. Since the LS contribution coming from $\Delta$ alone is diagonal in flavor space, see Eq.(18), it cannot lead to a large solar mixing angle anyway. We now extend our setup by a second Higgs 16-plet which we denote as $\Delta'$. In the simplest case it has the same transformation properties under $T_7$ and $Z_7$ as the Higgs field $\Delta$, i.e. it is a $T_7$ singlet and has a charge $Q(\Delta') = 1$ under $Z_7$. The additional couplings are given by

$$L_{\Delta'} = \beta'_1 (16, \chi_1 + 16, \chi_2 + 16, \chi_3) \Delta' S_1/\Lambda$$
$$+ \beta'_2 (16, \chi_1 + \omega 16, \chi_2 + \omega^2 16, \chi_3) \Delta' S_2/\Lambda$$
$$+ \beta'_3 (16, \chi_1 + \omega^2 16, \chi_2 + \omega 16, \chi_3) \Delta' S_3/\Lambda.$$

(24)

Note, that it is always possible to find a linear combination of $\Delta$ and $\Delta'$ with a vanishing GUT scale VEV. Therefore we assume $\langle \Delta' \rangle_N = 0$, so that the cancellation mechanism in the DS contribution is not affected. In the presence of $\Delta'$ the proportionality $m_{\nu S} \propto M_{NS}$ is not maintained anymore and therefore non-diagonal elements in $m_\nu$ are generated. At the leading order the LS contribution equals

$$m_{\nu}^{LS} = \frac{-\alpha \langle H \rangle \langle \Delta \rangle_\nu \eta}{3e^2 \langle \Delta \rangle_N} \left(2 \left(3 \langle \Delta \rangle_\nu + \sum_{i=1}^3 \gamma_i \right) e^6 \sum_{i=1}^3 \frac{\gamma_i \omega^{-i-1}}{\mathcal{O}(\epsilon^4)} \sum_{i=1}^3 \frac{\gamma_i \omega^{i-1}}{\mathcal{O}(\epsilon^2)} \right),$$

(25)

where $\gamma_i \equiv \beta'_i/\beta_i$ and we assume $\langle \Delta \rangle_\nu \lesssim \langle \Delta' \rangle_\nu$ such that the main contribution is due to $\Delta'$. In order to produce a large angle $\theta_{12}$, the LS contribution has to be comparable to the DS contribution. The dominant terms of the neutrino mass matrix are

$$m_\nu \approx \left(\frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N e^2} \right)^2 \left( -2X \left(3 \langle \Delta \rangle_\nu + \sum_{i=1}^3 \gamma_i \right) e^6 -X \sum_{i=1}^3 \frac{\gamma_i \omega^{-i-1}}{\tilde{A}+2\tilde{B}} -X \sum_{i=1}^3 \frac{\gamma_i \omega^{i-1}}{\tilde{A}-\tilde{B}} \right).$$

(26)

The SO(10) Higgs VEVs can be adjusted such that $X = \frac{\langle \Delta \rangle_N \langle \Delta' \rangle e^2 \eta}{3\alpha(H)}$ leads to the correct hierarchy between the elements of the first row/column and the $2 - 3$ sub-block: the mass parameters of the singlets, encoded in $\tilde{A}$ and $\tilde{B}$, see Eq.(11), have to be smaller than $\langle \Delta \rangle_N$. The resulting mixing angles equal

$$\tan 2\theta_{12} \approx \frac{\sqrt{2}(2\gamma_1 - \gamma_2 - \gamma_3) X}{\sqrt{6} |\tilde{B}|}, \quad \sin \theta_{13} \approx \frac{(\gamma_2 - \gamma_3) X}{\sqrt{6} |\tilde{B}|}, \quad \theta_{23} \approx \frac{\pi}{4}.$$

(27)
under the assumptions $|2\tilde{A} + \tilde{B}| < 3|\tilde{B}|$ and $|\langle\gamma_2 - \gamma_3\rangle X| \ll |\tilde{B}|$. Hence a large $\theta_{12}$ and small $\theta_{13}$ can be accommodated. Contributions coming from the diagonalization of the charged lepton mass matrix, may lead to a Cabibbo angle-size contribution to $\theta_{12}$, together with smaller corrections to $\theta_{13}$ and $\theta_{23}$. For the angle $\theta_{12}$ such a contribution however can be compensated by an appropriate choice of the parameter combination $|2\gamma_1 - \gamma_2 - \gamma_3|$ in Eq. (27) to match the experimental value. The light neutrino masses are also corrected by the LS contribution, especially $m_1$ and $m_2$

$$
m_1 \approx \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N \epsilon^2} \right)^2 |2\tilde{A} + \tilde{B}| \frac{\sin^2 \theta_{12}}{\cos 2\theta_{12}}, \quad m_2 \approx \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N \epsilon^2} \right)^2 |2\tilde{A} + \tilde{B}| \frac{\cos^2 \theta_{12}}{\cos 2\theta_{12}} \quad (28)
$$

$$
m_3 \approx 3 \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N \epsilon^2} \right)^2 |\tilde{B}|.
$$

As one can see, the LS term leads to large changes in the $1 - 2$ sector, but mainly preserves the $2 - 3$ sector. Thus maximal atmospheric mixing is still a prediction of the $T_7$ realization and the other mixing angles and masses can be fitted to the experimental data. For instance, from the best fit values of the mass squared differences and of $\theta_{12}$ [30] we deduce $|2\tilde{A} + \tilde{B}| \approx 1.13 \cdot 10^3 \text{GeV}$, $|\tilde{B}| \approx 3.38 \cdot 10^6 \text{GeV}$ and $|2\gamma_1 - \gamma_2 - \gamma_3| \approx 0.0833$ for $X \approx 2.25 \cdot 10^{10} \text{GeV}$ and $\langle \Delta \rangle_N = 10 \text{GeV}$. Note that we already used here that $\alpha = 1$. There is no particular prediction for the angle $\theta_{13}$ and it vanishes for $\gamma_2 = \gamma_3$. The neutrinos obey a normal hierarchy with the lightest neutrino mass being $m_1 \approx 0.00424 \text{eV}$. Eq. (28) shows that our model allows for a non-trivial relation between the ratio $m_1/m_2$ and $\theta_{12}$

$$
\frac{m_1}{m_2} \approx \tan^2 \theta_{12} \quad (29)
$$

which leads to $m_1/m_2 \approx 0.437$ for the best fit value of $\theta_{12}$ [30]. The corrections coming from the RG running below the mass scale of the lightest heavy neutral fermion are small due to the normal hierarchy in the light neutrino masses. However, the effects can be larger above this scale. We briefly comment on these effects in the next section. Finally, note that contributions from higher-dimensional operators to the LS term are also controlled by the $Z_7$ symmetry, as it was discussed in Section 3.2 for the DS term.

### 3.4 Masses of Right-handed Neutrinos and Singlets

In this section we consider the mass spectrum of the heavy neutral leptons, the right-handed neutrinos $N_i$ and the singlets $S_i$. Neglecting the mixing with light neutrinos we can write the corresponding six-by-six mass matrix as

$$
(N, S) \left( \begin{array}{cc} 0 & M_{NS} \\ M_{NS}^T & M_{SS} \end{array} \right) \left( \begin{array}{c} N \\ S \end{array} \right).
$$

The matrices $M_{NS}$ and $M_{SS}$ are given in Eq. (9). As we have established in the previous subsections to accommodate the light neutrino mass scale and to protect the mass matrix of the singlets against too large contributions from higher-dimensional operators, the VEV $\langle \Delta \rangle_N$ has to be $10^{15} \text{GeV}$ and the parameters $A, B \approx e^0 \Lambda \approx 10^{11} \text{GeV}$. We find for the mass
spectrum of these heavy states (HS) \(^{10}\)

\[
M_{HS1} \approx \eta^2 \frac{\langle \Delta \rangle_N}{|A + 2B|} e^8, \quad M_{HS2} \approx 9 \frac{|\beta_1 \beta_2 \beta_3|^2}{|\beta_1^2 \beta_2^2 + \beta_1^2 \beta_3^2 + \beta_2^2 \beta_3^2|} |\tilde{A} + 2\tilde{B}|, \tag{31}
\]

\[
M_{HS3,4} \approx \sqrt{3} \eta \sqrt{\frac{\beta_1^2 \beta_2^2 + \beta_1^2 \beta_3^2 + \beta_2^2 \beta_3^2}{\beta_1^2 + \beta_2^2 + \beta_3^2}} \langle \Delta \rangle_N e^2 \quad \text{and} \quad M_{HS5,6} \approx \eta \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} \langle \Delta \rangle_N.
\]

As one can see, the four heaviest states form two pseudo-Dirac pairs, while the two lowest lying states have a certain mass splitting. A rough estimate of the size of the masses leads to \(M_{HS1} \approx 10^8\) GeV, \(M_{HS2} \approx 10^{11}\) GeV, \(M_{HS3,4} \approx 10^{12}\) GeV and \(M_{HS5,6} \approx 10^{15}\) GeV. For our numerical example which we discuss at the end of Section 3.3 we can choose the following values of \(A\) and \(\beta_i\) to arrive at the numbers given above \(^{11}\)

\[
\tilde{A} \approx -1.12 \cdot 10^9\text{ GeV}, \quad \beta_1 \approx 3.34, \quad \beta_2 \approx 1.34 \quad \text{and} \quad \beta_3 \approx 2.15. \tag{32}
\]

The masses of the heavy states are then \(M_{HS1} \approx 1.49 \cdot 10^9\) GeV, \(M_{HS2} \approx 6.06 \cdot 10^{10}\) GeV, \(M_{HS3,4} \approx 4.35 \cdot 10^{12}\) GeV and \(M_{HS5,6} \approx 1.97 \cdot 10^{15}\) GeV. Analyzing the decomposition of the mass eigenstates we find that the state with the smallest mass \(M_{HS1}\) mainly consists of the second right-handed neutrino \(N_2\), the one with \(M_{HS2}\) is a mixture of the three singlets, the states with \(M_{HS3,4}\) decompose into the third right-handed neutrino and the singlets, while the states with the largest masses mainly involve the first right-handed neutrino \(N_1\) and the three singlets. Due to this fact we expect that above \(M_{HS2}\) larger RG corrections to the light neutrino masses and mixings originate from the \(\mathcal{O}(1)\) couplings of the singlets to the first left-handed neutrino and above \(M_{HS3,4}\) additionally from the \(\mathcal{O}(1)\) coupling of the third right-handed neutrino to the left-handed neutrinos.

### 3.5 Flavon Superpotential

Here we briefly comment on the flavon superpotential and on a possibility to achieve the vacuum structure shown in Eq.\((12)\). The potential can be minimized in the supersymmetric limit, since effects from soft breaking terms are expected to be negligible. The renormalizable part of the superpotential without the \(Z_3\) symmetry, which is necessary to control the higher order corrections to the fermion mass matrices, has the form

\[
W_\chi = \kappa \chi_1 \chi_2 \chi_3 \tag{33}
\]

where \(\kappa\) is a dimensionless coupling constant. As one can see, the F-terms, \(\kappa \chi_i \chi_j, \ i < j, \ i, j \in \{1, 2, 3\}\) are all zero, if (at least) two of the three VEVs \(\langle \chi_i \rangle\) vanish. Therefore, we can choose the minimum in which only \(\langle \chi_1 \rangle\) is non-vanishing which is a good lowest order approximation to the vacuum structure shown in Eq.\((12)\). Note that the value of \(\langle \chi_1 \rangle\) is not fixed by the potential.

\(^{10}\)For this calculation we assumed that all parameters are real.

\(^{11}\)The choice of the parameters \(\beta_i\) is more or less free (they should remain in the perturbative regime), since we can always use the couplings \(\beta_i^\prime\) to appropriately adjust the values of \(\gamma_i\) which play a role in the calculation of the mixing angles above. For \(\tilde{B}\) we simply took \(\tilde{B} \approx 3.38 \cdot 10^9\) GeV.
If we impose the $Z_7$ symmetry on the flavon potential, the term in Eq.(33) is forbidden. The lowest order terms of the flavon potential are then

$$W_\chi = \frac{a_1}{\Lambda^4} (\chi_1^7 + \chi_2^7 + \chi_3^7) + \frac{a_2}{\Lambda^4} (\chi_1\chi_2\chi_3)(\chi_1^3\chi_3 + \chi_1\chi_2^3 + \chi_2\chi_3^3).$$

However, the F-terms derived from this superpotential have no configuration $\langle \chi_1 \rangle \neq 0$ and $\langle \chi_{2,3} \rangle = 0$ as solution. Thus, $Z_7$ should be explicitly broken in the flavon superpotential or other fields should exist apart from the flavons $\chi_i$. One possibility to reconcile the VEV structure and the $Z_7$ symmetry is to introduce a new field $\varphi$ with $T_7$ and $Z_7$ properties $(3^*,5)$.[12] The renormalizable flavon superpotential,

$$W_\chi = \kappa (\varphi_1\chi_2\chi_3 + \varphi_2\chi_1\chi_3 + \varphi_3\chi_1\chi_2),$$  

(34)

allows the configuration $\langle \chi_1 \rangle \neq 0$ and $\langle \chi_{2,3} \rangle = 0$. In order to study the issue whether the hierarchical structure of the VEVs $\langle \chi_i \rangle$ can be achieved with such a potential we have to discuss the contributions to the potential arising from higher-dimensional operators consisting of more than two flavons and the field $\varphi$. This can be done in a similar fashion as done for the contributions to the fermion mass matrices. However, such a study is beyond the scope of this paper. Also the fact that terms involving flavon fields might arise as corrections to the GUT Higgs potential is not discussed here.

4 $\Sigma(81)$ Realization

The $\Sigma(81)$ realization of the cancellation mechanism differs from the $T_7$ one in two aspects: (i) the cancellation of the up-quark mass hierarchy in $m_\nu$ is complete and (ii) it requires a non-supersymmetric framework, since the fields $\chi_i$ are involved in the coupling of $16$ to $S_j$, whereas their complex conjugates $\chi_i^*$ appear in the Yukawa couplings $16_1^*16_j^*H$. The group $\Sigma(81)$, previously discussed in [25,26], has one- and three-dimensional representations. Its order is 81 and therefore it has nine one-dimensional, $1_i$, $i = 1,\ldots,9$, and eight three-dimensional $3_i$, $i = 1,\ldots,8$, representations. Apart from $1_1$, all of them are complex. Contrary to $T_7$, $\Sigma(81)$ is a subgroup of U(3), but not of SU(3). Further group theoretical aspects of $\Sigma(81)$ are summarized in the Appendix B.

4.1 Masses and Mixing at the Lowest Order

The three generations of fermions $16_i$ are assigned to one of the six faithful[13] three-dimensional representations, $3_i$, $i = 1,\ldots,6$. Without loss of generality we choose $3_1$. To arrive at a diagonal $m_D$ we assign $H$ to $1_1$ and $\chi_i$ to $3_2$, so that $\chi_i^*$ contributes to $(m_D)_{ii}$ only. The SO(10) singlets, $S_i$, transform as $1_1,1_2$ and $1_3$, and the Higgs field $\Delta$ as $1_1$. These properties are summarized in the Table 4. As a result the matrix structure of $M_{NS}$ and $M_{SS}$ is the same as in the case

[12]Here we additionally have to assume the existence of a $U(1)_R$ symmetry under which the fermions of the model transform with charge +1, the Higgs fields and flavons acquiring VEVs with 0 and the field $\varphi$ has charge +2. The flavon superpotential then has to be linear in the field $\varphi$. This is done along the lines of [31]. In the simplest case we can assume that the $U(1)_R$ is explicitly broken in the GUT Higgs potential.

[13]A faithful representation of a group has as many distinct representation matrices as elements exist in the group.
Table 4: Particle assignment in the Σ(81) realization. $^{16}_i$ and $S_i$ are fermions, $H$ and $∆$ are Higgs fields and $\chi_i$ are flavons. Note that $^3_2$ is equivalent to $^2_3$.

of $T_7$, see Eq.(9), while the matrix $m_D$ is of the form

$$m_D = \frac{\alpha \langle H \rangle}{\Lambda} \begin{pmatrix}
\langle \chi_1 \rangle^* & 0 & 0 \\
0 & \langle \chi_2 \rangle^* & 0 \\
0 & 0 & \langle \chi_3 \rangle^*
\end{pmatrix}.$$  \hspace{1cm} (35)

As in the $T_7$ realization we assume that the VEVs $\langle \chi_i \rangle$ can be chosen as real and positive. To reproduce the up-quark masses we take

$$\frac{\langle \chi_1 \rangle}{\langle \chi_3 \rangle} \approx \epsilon^4, \quad \frac{\langle \chi_2 \rangle}{\langle \chi_3 \rangle} \approx \epsilon^2 \quad \text{and} \quad \eta = \frac{\langle \chi_3 \rangle}{\Lambda} \sim \mathcal{O}(1) \quad \text{with} \quad \epsilon \approx 0.05.$$  \hspace{1cm} (36)

We assume again the dominance of the DS contribution in the light neutrino mass matrix, which is given by

$$m_\nu \approx \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N} \right)^2 \begin{pmatrix}
\tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} & \tilde{A} - \tilde{B} \\
\tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} & \tilde{A} - \tilde{B} \\
\tilde{A} + 2\tilde{B} & \tilde{A} - \tilde{B} & \tilde{A} - \tilde{B}
\end{pmatrix}.$$  \hspace{1cm} (37)

with $\tilde{A}$ and $\tilde{B}$ being defined in Eq.(11). The hierarchy of the up-quark masses encoded in the VEVs of $\chi_i$ is completely erased in $m_\nu$ without any further assumptions on $\langle \chi_i \rangle$ and/or the couplings. Thus, $\langle \Delta \rangle_N$ and $A$ and $B$ ($\tilde{A}$ and $\tilde{B}$) are of their generic size, $\langle \Delta \rangle_N = 2 \cdot 10^{16}$ GeV and $A, B \sim M_{Pl} = 1.22 \cdot 10^{19}$ GeV, to produce a light neutrino mass scale of 1 eV. As always $\langle H \rangle$ is fixed to 174 GeV. Note that the form of $m_\nu$ in Eq.(37) is the most general one invariant under $S_3$ [32] permuting the three neutrino generations. The matrix $m_\nu$ is diagonalized by the TBM matrix, however the mass spectrum contains two equal masses

$$m_2 = 3 \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N} \right)^2 |\tilde{A}| \quad \text{and} \quad m_1 = m_3 = 3 \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N} \right)^2 |\tilde{B}|.$$  \hspace{1cm} (38)

Here $m_2$ corresponds to the state with tri-maximal mixing and $m_3$ is the mass of the state with bimaximal mixing which is degenerate with the state with mass $m_1$. Thus, the atmospheric mass squared difference $\Delta m_{31}^2$ vanishes. The solar mass squared difference is given by

$$\Delta m_{21}^2 = 9 \left( \frac{\alpha \langle H \rangle}{\langle \Delta \rangle_N} \right)^4 (|\tilde{A}|^2 - |\tilde{B}|^2).$$  \hspace{1cm} (39)
As one can see, we have to assume that $|\tilde{A}| \approx |\tilde{B}|$ for $\Delta m_{21}^2$ being small. This corresponds to a quasi-degenerate neutrino mass spectrum [32]. $\Delta m_{31}^2$ can be generated by higher-dimensional corrections as will be discussed in the next section.

The Majorana mass matrix $M_{NN}$ of the right-handed neutrinos can be obtained by the seesaw formula due to the hierarchy $M_{NS} \ll M_{SS}$

$$M_{NN} \approx -M_{NS} M_{SS}^{-1} M_{NS}^T. \quad (40)$$

Then using the matrices $M_{NS}$ and $M_{SS}$ of Eq.(9) we obtain the analytic formulae for the right-handed neutrino masses (for real parameters)

$$M_{NN1} \approx \epsilon^8 \left\langle \Delta \right\rangle_N^2 \frac{\eta^2}{|A + 2B|}, \quad M_{NN2} \approx \epsilon^4 \left\langle \Delta \right\rangle_N^2 \frac{\tilde{A} + 2\tilde{B}}{3B (2A + B)}, \quad M_{NN3} \approx \left\langle \Delta \right\rangle_N^2 \frac{2\tilde{A} + \tilde{B}}{9A\tilde{B}} \quad (41)$$

with $\tilde{A}$ and $\tilde{B}$ according to Eq.(11). Hence, the masses of the right-handed neutrinos are expected to be strongly hierarchical

$$\{\epsilon^8, \epsilon^4, 1\} \cdot \eta^2 \cdot \left\langle \Delta \right\rangle_N^2 / M_{Pl} \approx \{283 \text{ GeV}, 4.53 \cdot 10^7 \text{ GeV}, 7.24 \cdot 10^{12} \text{ GeV} \} \quad (42)$$

for $\eta \approx 0.47$. This strong hierarchy is due to the hierarchy in $M_{NS}$ and we find one very low lying state in the spectrum of the right-handed neutrinos. Its phenomenology, detection and observable consequences are determined by the mixing with active neutrinos. The mixing angle with the flavor state $\nu_f$ is described by

$$\tan \theta_f \approx \frac{\langle m_D U_{NN} \rangle_{fI}}{M_{NN1}} \quad (43)$$

where $U_{NN}$ is the mixing matrix which diagonalizes the Majorana mass matrix of the right-handed neutrinos Eq.(10), $M_{NN} = U_{NS}^* M_{NN}^{\text{diag}} U_{NN}^T$. If there is no strong hierarchy of the elements of $m_\nu$, i.e. $\tilde{A} + 2\tilde{B}$ and $\tilde{A} - \tilde{B}$ are of the same order, an estimate of the angle is straightforward

$$\tan \theta_f \approx \frac{\alpha \left\langle H \right\rangle |\tilde{A} + 2\tilde{B}|}{\left\langle \Delta \right\rangle_N^2 \eta \epsilon^4} \sim \frac{\left\langle H \right\rangle M_{Pl}}{\left\langle \Delta \right\rangle_N^2 \eta \epsilon^4} \approx 2 \cdot 10^{-6}. \quad (44)$$

It is much below the sensitivity of present and future experiments which is $\sim 10^{-2} - 10^{-1}$ [33]. In particular, the rate of production of $N_1$ at LHC is negligible.

The masses of the singlets are to a good approximation given by $|A|$, $|B|$ and $|\tilde{B}|$, so that two of them are nearly degenerate.

### 4.2 Effects of Higher-Dimensional Operators

The number of the higher-dimensional operators containing more than one flavon $\chi_i$ increases in comparison to the $T_7$ realization, since in a non-supersymmetric theory also the complex
| Order in $\epsilon$ | Operator Structure | No. of Operators at Order $n$ |
|---------------------|--------------------|-------------------------------|
| $O(\epsilon^0)$    | $\chi_3^m (\chi_3^*)^{n-m}$ ($m = 0, ..., n$) | $n + 1$ |
| $O(\epsilon^4)$    | $\chi_3^m (\chi_3^*)^{n-1-m} \chi_1^{(*)}$ ($m = 0, ..., n - 1$) | $2n$ |
| $O(\epsilon^2)$    | $\chi_3^m (\chi_3^*)^{n-1-m} \chi_2^2$ ($m = 0, ..., n - 1$) | $2n$ |
| $O(\epsilon^6)$    | $\chi_3^m (\chi_3^*)^{n-2-m} \chi_1 \chi_2$ ($m = 0, ..., n - 2$) | $4(n - 1)$ |
|                     | $\chi_3^m (\chi_3^*)^{n-2-m} \chi_1 \chi_2$ ($m = 0, ..., n - 2$) | |
|                     | $\chi_3^m (\chi_3^*)^{n-2-m} \chi_1 \chi_2^2$ ($m = 0, ..., n - 2$) | $3(n - 1)$ |
| $O(\epsilon^4)$    | $\chi_3^m (\chi_3^*)^{n-3-m} \chi_2^2$ ($m = 0, ..., n - 3$) | $4(n - 2)$ |

Table 5: List of products of $\chi_i$ and $\chi_i^*$ which lead to contributions down to $O(\epsilon^6)$ to the entries of the mass matrices $m_D$, $M_{NS}$ and $M_{SS}$ under the assumption that $\langle \chi_1 \rangle / \langle \chi_3 \rangle \approx \epsilon^4$, $\langle \chi_2 \rangle / \langle \chi_3 \rangle \approx \epsilon^2$ and $\langle \chi_3 \rangle / \Lambda = \eta \sim O(1)$. In the third column we list the number of operators with a certain structure. As one can see, this number depends linearly on $n$. Note that also here a monomial with $n$ fields leads to an additional suppression factor $\eta^n$.

conjugated fields $\chi_i^*$ have to be taken into account. The general structure of the operators in the Lagrangian is

$$O^\frac{\chi_{i_1}^{n_1} (\chi_{i_2}^*)^{n_2}}{\Lambda^n}$$  \hspace{1cm} (45)$$

with $n_1 + n_2 \equiv n \geq 2$, $n_{1,2} = 0, 1, 2, ...$. Again, $O$ denotes $16 \ 16 \ H$, $16 \ S \Delta$ or $\Lambda SS$. To study the contributions from these higher-dimensional operators we determine the leading order terms in $\epsilon^2$ up to $O(\epsilon^6)$ for the VEVs of the form as given in Eq. (36). This is done analogously to the $T_7$ case. As above, we assume $\eta \approx \epsilon^{1/4} \approx 0.47$. The relevant monomials in the fields $\chi_i$ and $\chi_i^*$ are displayed in the Table 5. Notice that the number of relevant operators increases with the number of flavons $n$. This could compensate the suppression factor $\eta^n$ which arises for any operator containing $n$ flavon fields. The transformation properties of the monomials under $\Sigma(81)$ are given in the Table 11 of the Appendix C. Using these properties we find that for the elements of the matrices $m_D$, $M_{NS}$ and $M_{SS}$, which do not vanish at the leading order, the corrections are of the same order in $\epsilon^2$ as the leading contribution, and/or have a higher power of $\epsilon^2$. Therefore, the hierarchy of the elements in the small parameter $\epsilon$ is not destroyed by the corrections. These elements will be of the form $(m_D)_{ij} = (m_D)^{LO}_{ij} (1 + O(\eta^k))$ and $(M_{NS})_{ij} = (M_{NS})^{LO}_{ij} (1 + O(\eta^l))$ with $k, l \geq 1$, respectively, if the corrections are taken.
into account. The off-diagonal elements of $m_D$ no longer vanish and the full matrix has the form

$$m_D \sim \alpha \langle H \rangle \begin{pmatrix} e^4 \eta (1 + \mathcal{O}(\eta^2)) & \mathcal{O}(e^6 \eta^2) & \mathcal{O}(e^4 \eta^2) \\ . & e^2 \eta (1 + \mathcal{O}(\eta^2)) & \mathcal{O}(e^2 \eta^2) \\ . & . & \eta (1 + \mathcal{O}(\eta)) \end{pmatrix}.$$  

(46)

As one can see, the off-diagonal elements are smaller than the corresponding (in the same row or column) diagonal elements, at least by a factor $\eta$. Consequently, the hierarchy of the eigenvalues of $m_D$ is not changed by these elements. Corrections to the diagonal elements, which are of the same order as the leading order term, will require a re-adjustment of the parameters in a numerical fit to the experimental data. The contributions to the Majorana mass matrix $M_{SS}$ of the singlets arise from insertions of at least two flavons, since the singlets $S_i$ transform as one-dimensional representations under $\Sigma(81)$, whereas the flavons form a triplet. At the level of two flavon insertions all elements of $M_{SS}$ receive contributions of the order $\eta^2 \Lambda$. Assuming that $A$ and $B$ as well as the cutoff scale $\Lambda$ are of order $M_{Pl}$, we conclude that the corrections to $M_{SS}$ can amount to $\eta^2 \approx 0.22$ of the leading order terms. Since the matrix structure of these subleading corrections differs from the one of the leading order, these corrections could be important.

4.2.1 Cabibbo Angle

The simplest possibility to generate the quark mixings would be to assume the existence of a second Higgs $H'$ transforming as $10$ under $\text{SO}(10)$ and trivially under $\Sigma(81)$. Considering this possibility we see from Eq.(46) that the induced mixing angles are of order

$$\langle \vartheta_{12}, \vartheta_{13}, \vartheta_{23} \rangle \sim \left(\mathcal{O}(e^4 \eta), \mathcal{O}(e^4 \eta), \mathcal{O}(e^2 \eta)\right).$$

(47)

These are, however, much smaller than the observed mixing angles. Similarly to the $T_7$ realization, it is possible to generate the Cabibbo angle by introducing operators of the form

$$\frac{1}{M} \langle 16 \rangle^* \langle 16' \rangle \frac{\chi^n (\chi^*)^m}{\Lambda^{n+m}}$$

with at least one of the Higgs 16-plets transforming as a non-trivial one-dimensional representation under $\Sigma(81)$, $\mathbf{1}_i$, $i = 2, \ldots, 9$. For instance, we can assume that $\mathbf{16}_H$ (which obtains a weak scale VEV) transforms as $\mathbf{14}$, while $\mathbf{16}'_H$ (having a GUT scale VEV) is invariant. Through this we ensure that these Higgs fields only break $\Sigma(81)$ at the weak and not at the GUT scale. Therefore the flavor and the GUT symmetry breaking are still mainly induced by different sets of fields. The operators of structure Eq.(48) yield contributions to the down quark and the charged lepton mass matrix of the form

$$\langle \mathbf{16}'_H \rangle^\nu \left(\frac{\langle \mathbf{16}'_H \rangle^N}{M}\right) \begin{pmatrix} \mathcal{O}(e^6 \eta^3) & \mathcal{O}(\eta) & \mathcal{O}(e^2 \eta) \\ . & \mathcal{O}(e^6 \eta^3) & \mathcal{O}(e^4 \eta) \\ . & . & \mathcal{O}(e^6 \eta^3) \end{pmatrix}.$$  

(49)

The contributions to the diagonal elements are in general very small and therefore always subleading. Using Eq.(46) and Eq.(49) we find that a correct value of the $1 - 2$ mixing,\footnote{See footnote 9}
can be obtained if $\langle 16_H \rangle_\nu \approx 100 \text{ GeV}$ and $\langle 16_H' \rangle_N/M \approx \epsilon^{5/2}$. The latter indicates that the mediator scale $M$ is of the order of the Planck scale. The mass hierarchy of the charged leptons and down quarks is expected to be \{\epsilon^3\eta, \epsilon^2\eta, \eta\}. This is similar to the $T_7$ case. The corrections to the matrix elements shown in Eq. (49) due to multi-flavon insertions are at most of the same order in $\epsilon$ as the leading contribution. In addition they are suppressed by some higher power of $\eta$. This is the same as for the corrections to the matrices $m_D$ and $M_{NS}$. As in the $T_7$ model, the 1 − 3 and 2 − 3 quark mixing angles are too small. However, we can expect that they are generated by operators with a structure different from the ones discussed here. Again, we expect that the contributions from the operators of structure Eq. (48) to the charged lepton mass matrix result in a Cabibbo angle-size mixing in the 1 − 2 sector, which can have a certain impact on the lepton mixings.

### 4.2.2 Light Neutrino Mass Matrix

Finally, we discuss the corrections to the effective light neutrino mass matrix coming from higher-dimensional operators. There are no simple analytic formulae for neutrino masses and leptonic mixing angles, because the elements of $m_\nu$ are all of the same order, if there are no further restrictions on the couplings. We only indicate some features which can be directly deduced from the general expression of the mass matrix and show with a numerical example that a viable neutrino mass matrix can be obtained. One can see that (i) $\Delta m^2_{31}$ is proportional to $\eta$ and it does not depend on corrections coming from $M_{SS}$ at leading order, since these are proportional to $\eta^2$ and (ii) contributions of higher-dimensional operators yield differences in the diagonal elements of the mass matrix Eq. (37) which are of relative order $O(\eta)$. Therefore, to reproduce a large atmospheric mixing angle one should require $|\tilde{A} - \tilde{B}| > \eta |\tilde{A}|, \eta |\tilde{B}|$. Taking into account the fact that the smallness of the solar mass squared difference $\Delta m^2_{21}$ enforces $|\tilde{A}| \approx |\tilde{B}|$, as explained above, we find that the relative phase of $\tilde{A}$ and $\tilde{B}$ has to be around $\pi$ such that $\tilde{B} \approx -\tilde{A}$. This is used as a restriction in our numerical search whose results are detailed in the next section.

### 4.3 Phenomenology of Neutral Fermions

We perform a numerical search to show that through the inclusion of higher-dimensional operators the model can accommodate neutrino masses and lepton mixings. In our search we consider a certain (small) set of these operators which contribute to $m_D$ and $M_{NS}$. Their coefficients are real random numbers whose absolute value lies in the interval $[0.1, \eta^{-1}] \approx [0.1, 2.1]$. One example is

\[
m_D = \begin{pmatrix}
1.74664 \cdot 10^{-6} & 0 & 7.38353 \cdot 10^{-7} \\
6.99757 \cdot 10^{-4} & 2.98637 \cdot 10^{-5} & 0.454331 \\
. & . & .
\end{pmatrix} \langle H \rangle, \tag{50a}
\]

\[16\] The up-quark masses $m_u$ and $m_c$ are a factor of two too small at the GUT scale for $m_t(M_{GUT})$ of about 79 GeV [34]. Contributions either from certain multi-flavon insertions not taken into account here or from operators with an SO(10) structure different from $16_1 16_2 H$ may lift the masses of the up and the charm quark.
Using $\langle H \rangle = 174$ GeV, $\langle \Delta \rangle_N = 2 \cdot 10^{16}$ GeV and $M_{Pl} = 1.22 \cdot 10^{19}$ GeV we find for the effective light neutrino mass matrix $m_{\nu}$

$$m_{\nu} \approx \begin{pmatrix} -2.4129 + i \cdot 7.2799 \cdot 10^{-2} & 2.4662 + i \cdot 1.0216 \cdot 10^{-4} & 2.7217 - i \cdot 1.7759 \cdot 10^{-2} \\ 0 & -2.4181 - i \cdot 7.3181 \cdot 10^{-2} & 4.5802 + i \cdot 4.5367 \cdot 10^{-2} \\ 0 & 0 & -1.7962 - i \cdot 2.3591 \cdot 10^{-2} \end{pmatrix} \cdot 10^{-2} \text{eV}. \quad (51)$$

Apparently, some corrections to the lowest order matrix from the higher-dimensional operators are sizable, since the diagonal elements are no longer equal, but $|\langle m_{\nu} \rangle_{22}| \approx 1.347 |\langle m_{\nu} \rangle_{33}|$, and also the off-diagonal elements differ, e.g. $|\langle m_{\nu} \rangle_{12}| \approx 0.538 |\langle m_{\nu} \rangle_{23}|$. This is a result of the interplay between different relatively small corrections in the matrices $m_D$ and $M_{NS}$.

The matrix $m_{\nu}$ in Eq. (51) yields the following mass squared differences and mixing parameters

$$\Delta m_{21}^2 = 7.58 \cdot 10^{-5} \text{eV}^2, \quad \Delta m_{33}^2 = 2.58 \cdot 10^{-3} \text{eV}^2, \quad r = 0.0294,$$

$$\theta_{12} = 31.4^\circ, \quad \theta_{13} = 1.14^\circ, \quad \theta_{23} = 46.5^\circ,$$

$$\delta = 323.2^\circ, \quad \varphi_1 = 299.3^\circ, \quad \varphi_2 = 160.8^\circ,$$

with the Dirac and Majorana phases defined according to [35]. These results are within the 2σ bounds of [30]. Notice that the deviation of the angle $\theta_{12}$ from its TBM value is significant.

The masses of the light neutrinos are $m_1 = 0.0437$ eV, $m_2 = 0.0446$ eV and $m_3 = 0.0670$ eV. Thus, the light neutrinos are normally ordered and have only a mild hierarchy. The sum of the masses, $\sum m_i = 0.155$ eV, is below the cosmological bound [36]. In the numerical analysis, we did not consider RG corrections. Although they might lead to sizable corrections due to the large hierarchy in the right-handed neutrino masses [20], these corrections can be included in a redefinition of the coefficients of the higher-dimensional operators. Similarly, we neglect possible corrections to the mixing angles coming from the non-diagonality of the charged lepton mass matrix. These corrections to the $1-2$ mixing angle are of the order of the Cabibbo angle, whereas they are smaller for $\theta_{13}$ and $\theta_{23}$.

In this numerical example we find for the masses of the right-handed neutrinos $\approx \{3.83 \text{ TeV}, \quad 1.47 \cdot 10^{10} \text{ GeV}, \quad 1.16 \cdot 10^{12} \text{ GeV}\}$. This mass spectrum differs from the estimate in the lowest order presented in Section 4.1. Especially the two lighter states are heavier than before. The difference can be partly attributed to the actual value of $1/|A + 2\tilde{B}|$ and $|A + 2\tilde{B}|/|3\tilde{B}(2A + \tilde{B})|$ and partly to the fact that we included some next-to-leading order corrections to the matrix $M_{NS}$. 

\[ M_{NS} = \begin{pmatrix} 5.57371 \cdot 10^{-6} & 4.80734 \cdot 10^{-6} & 4.00556 \cdot 10^{-6} \\ 2.23041 \cdot 10^{-3} & 1.92369 \cdot 10^{-3} \omega & 1.60262 \cdot 10^{-3} \omega^2 \\ 1.03422 & 0.886329 \omega^2 & 0.702275 \omega \end{pmatrix} \langle \Delta \rangle_N, \quad (50b) \]

\[ M_{SS} = \begin{pmatrix} 0.884095 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (50c) \]
4.4 Flavon Potential

The renormalizable part of the flavon potential is given by [13]

\[ V_\chi(\chi_i) = M_\chi^2 \sum_j |\chi_j|^2 + \left[ \kappa e^{i\sigma} \sum_j \chi_j^3 + \text{h.c.} \right] + \lambda_1 \sum_j |\chi_j|^4 + \lambda_2 \sum_{j<k} |\chi_j|^2 |\chi_k|^2, \] (53)

where \( M_\chi \) is the mass parameter, \( \lambda_{1,2} \) and \( \kappa \) are real constants, and the phase \( \sigma \) lies in the interval \([0, 2\pi)\). In order to analyze the flavon potential, we parameterize the fields \( \chi_i \) as \( \chi_i \equiv X_i e^{i\xi_i} \) with \( X_i \geq 0 \) and \( \xi_i \in [0, 2\pi) \). Then

\[ V_\chi(X_i, \xi_j) = M_\chi^2 \sum_k X_k^2 + \lambda_1 \sum_k X_k^4 + \lambda_2 \sum_{k<l} X_k^2 X_l^2 + 2\kappa \sum_k X_k^3 \cos(\sigma + 3\xi_k). \] (54)

The extremization conditions for the VEVs \( \langle X_1 \rangle \) and \( \langle \xi_1 \rangle \) read

\[ \frac{\partial V_\chi}{\partial X_1} = 2X_1 (M_\chi^2 + 2\lambda_1 X_1^2 + \lambda_2 X_1^2 + 2\kappa X_1 \cos(\sigma + 3\xi_1)) = 0 \] (55a)

\[ \frac{\partial V_\chi}{\partial \xi_1} = -6\kappa X_1^2 \sin(\sigma + 3\xi_1) = 0. \] (55b)

The corresponding equations for \( \langle X_{2,3} \rangle \) and \( \langle \xi_{2,3} \rangle \) are obtained by cyclic permutation in the index \( i = 1, 2, 3 \). Eq. (55b) is solved by either a vanishing VEV \( \langle X_i \rangle \) or by the relation \( 3 \langle \xi_i \rangle + \sigma = n_i \pi \) (\( n_i = 0, \pm 1, \pm 2, \ldots \)). Note that \( n_i \) can be chosen independently for the three \( \xi_i \). This allows one to choose a vacuum configuration in which two of the three fields \( X_i \) (equivalent to \( \chi_i \)) have a vanishing VEV. Such a configuration is a good approximation to the vacuum structure used to accommodate the charged fermion mass hierarchy, see Eq. (36).

If we set \( \langle X_1 \rangle = \langle X_2 \rangle = 0, \langle X_3 \rangle \neq 0 \) and require that the extremum is a minimum of the potential, we obtain the following

\[ \langle X_3 \rangle = \frac{3\kappa + \sqrt{9\kappa^2 - 8M_\chi^2 \lambda_1}}{4\lambda_1}, \quad \langle \xi_3 \rangle = -\frac{\sigma \pm \pi}{3} \] (56)

together with the condition \( 9\kappa^2 \geq 8M_\chi^2 \lambda_1 \). The real value of the VEVs of \( \chi_i \) assumed in our previous discussion can be achieved by a suitable choice of the phase \( \sigma \). The questions how to generate the VEVs \( \langle \chi_{1,2} \rangle \) and how to ensure the hierarchy of these VEVs are not answered in this paper, since this requires a careful study of the higher-dimensional operators with more than four flavons contributing to the scalar potential.

5 Summary

In the context of SO(10) grand unified models we have presented a mechanism which cancels partially or completely the hierarchy of the Dirac mass matrix in the light neutrino mass matrix \( m_\nu \). The ingredients of the mechanism are: (i) the existence of three fermionic SO(10)

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13Possible terms containing GUT Higgs fields and the flavons \( \chi \) and/or \( \chi^* \), which are invariant under all symmetries of the model, are assumed to be fine-tuned in order to keep the flavon and the GUT Higgs potential appropriately separated.
singlets, which realize the double seesaw mechanism and (ii) a discrete non-Abelian flavor symmetry which restricts the form of the relevant matrices $m_D$, $M_{NS}$ and $M_{SS}$, so that the structures of $m_D$ and $M_{NS}$ are correlated. The charged fermion mass matrices are diagonal at leading order and the hierarchy of the up-quarks can be reproduced. The framework of such a model can be supersymmetric or not, depending on the choice of the flavor group.

We discuss the two discrete groups $T_7$ and $\Sigma(81)$, which lead to such a cancellation. In the first case the hierarchy is partially cancelled, and the existence of the dominant $2 - 3$ block in the light neutrino mass matrix, responsible for maximal atmospheric mixing, is related to the geometric hierarchy of the up-quarks, $m_u : m_c : m_t = \epsilon^4 : \epsilon^2 : 1$ for $\epsilon \approx 0.05$. In the $\Sigma(81)$ model we can achieve a complete cancellation of the up-quark hierarchy in $m_\nu$. The resulting $m_\nu$ is compatible with TBM, however the atmospheric mass squared difference vanishes at leading order. In none of the two models we can accommodate all features of the neutrino masses and lepton mixings by considering only the leading order. Thus, further effects have to be taken into account. It turns out that in the $T_7$ model the inclusion of a second Higgs in the SO(10) representation $\mathbf{16}$ can produce a linear seesaw contribution leading to a large $1 - 2$ mixing angle without disturbing the cancellation mechanism from which $\theta_{23} \approx \frac{\pi}{4}$ originates. In the $\Sigma(81)$ model the situation is even simpler, since the introduction of operators involving more than one flavon field already lifts the degeneracy in the light neutrino mass spectrum and allows to reproduce the experimental data.

Since the top quark mass is also generated by non-renormalizable operators, at least one of the expansion parameters $\langle \chi_i \rangle / \Lambda$ has to be of order one. As a consequence, contributions to the fermion masses from operators with more than one flavon are in general not small and have to be carefully studied. For the $T_7$ model we showed that an additional $Z_7$ symmetry can forbid all operators which invalidate the predictions made at lowest order. Additionally, the introduction of the $Z_7$ symmetry helps to solve the problem of a too large light neutrino mass scale by allowing the singlets to acquire masses only through non-renormalizable operators of the form $SS\chi^n / \Lambda^{n-1}$. In the $\Sigma(81)$ realization we observe that the higher-dimensional operators have the same structure in the small expansion parameter $\epsilon \approx 0.05$ as the leading order terms. These operators are therefore much less dangerous than in the $T_7$ model and even help to produce the correct atmospheric mass squared difference.

The Cabibbo angle in the quark sector can be accommodated in both models, if additional operators of the structure as found in Eq. (21) and Eq. (48) are introduced which only contribute to the down quark and the charged lepton mass matrix. The key feature is that the Higgs fields $\mathbf{16}_H$, $\mathbf{16}_H'$ have to transform as non-trivial one-dimensional representations of the flavor group.

Additionally, we study in both models the mass spectrum of the heavy neutral fermions. In case of the $T_7$ model we find that these particles have masses between $10^8$ GeV and $10^{15}$ GeV. In contrast to that the right-handed neutrinos are strongly hierarchical in the $\Sigma(81)$ case with the lightest right-handed neutrino being as light as $\sim 400$ GeV to $\sim 4$ TeV. The singlets, however, are much heavier with masses around the Planck scale. Although one right-handed neutrino is very light, its effects are practically unobservable due to the tiny coupling to the active neutrinos.

Finally, we compute the flavon (super-)potential of the models. In both cases the vacuum structure can be well approximated by a configuration in which only one of the three flavons has a non-vanishing VEV. Such a configuration can be a minimum in both potentials. To study
further whether the hierarchy among the other VEVs, which are zero at leading order, can be appropriately produced we however would need to consider non-renormalizable operators with several flavons also in the (super-)potential.

In conclusion, we consider SO(10) GUTs with three additional fermionic singlets which can reconcile the different mass hierarchies in the charged fermion and the neutrino sector and explain the peculiar mixing pattern among the leptons with the help of discrete non-Abelian flavor symmetries. Furthermore, we study several aspects of these models such as the effects of higher-dimensional operators, the generation of the Cabibbo angle, the mass spectrum of the heavy neutral fermions and the flavon potential at leading order.

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Table 6: Character table of $T_7$. $\omega \equiv e^{\frac{2\pi i}{7}}$ and $\xi \equiv \frac{1}{2}(-1 + i\sqrt{7})$, so that $\xi = \rho + \rho^2 + \rho^4$ where $\rho \equiv e^{\frac{2\pi i}{7}}$. $C_i$ are the classes of the group, $^\circ C_i$ is the order of the $i^{th}$ class, i.e. the number of distinct elements contained in this class, $^\circ h_{C_i}$ is the order of the elements $S$ in the class $C_i$, i.e. the smallest integer ($> 0$) for which the equation $S^{'h_{C_i}} = 1$ holds. Furthermore the table contains one representative for each class $C_i$ given as product of the generators $A$ and $B$ of the group. Finally, $\psi^{(\mu)}$ denotes the so-called $\psi$-value of a representation $\mu$ which indicates whether $\mu$ is real ($\psi^{(\mu)} = 1$), pseudo-real ($\psi^{(\mu)} = -1$) or complex ($\psi^{(\mu)} = 0$).

A Group Theory of $T_7$

In this appendix we show the character table, Kronecker products, one set of generators and the resulting Clebsch-Gordan coefficients for the group $T_7$. As already mentioned above, this group is very similar to $A_4$ with the crucial difference that the two three-dimensional representations of $T_7$ are complex and not real like the one of $A_4$.

The character table is presented in the Table 6. The used generators for the three-dimensional representations are [37]

$$3 : A = \begin{pmatrix} e^{\frac{2\pi i}{7}} & 0 & 0 \\ 0 & e^{\frac{4\pi i}{7}} & 0 \\ 0 & 0 & e^{\frac{6\pi i}{7}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$3^* : A = \begin{pmatrix} e^{-\frac{2\pi i}{7}} & 0 & 0 \\ 0 & e^{-\frac{4\pi i}{7}} & 0 \\ 0 & 0 & e^{-\frac{6\pi i}{7}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

They fulfill the relations [37]

$$A^7 = 1, \quad B^3 = 1, \quad AB = BA^4.$$
symmetric part. The non-trivial Clebsch-Gordan coefficient \( s \) for \((a_1, a_2, a_3)\) pairs are found in the Table 7. Six of the eight three-dimensional representations are faithful.

The irreducible representations are \( \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8} \) and for the triplets we have \( \tilde{3}_A \equiv \tilde{3}_2, \tilde{3}_B \equiv \tilde{3}_3, \tilde{3}_B \equiv \tilde{3}_4, \tilde{3}_C \equiv \tilde{3}_5, \tilde{3}_C \equiv \tilde{3}_6, \tilde{3}_D \equiv \tilde{3}_7, \) and \( \tilde{3}_D \equiv \tilde{3}_8. \)

### Table 7: The representations of the group \( \Sigma(81) \) and their complex conjugates.

| rep. | \( \tilde{1} \) | \( \tilde{2} \) | \( \tilde{3} \) | \( \tilde{4} \) | \( \tilde{5} \) | \( \tilde{6} \) | \( \tilde{7} \) | \( \tilde{8} \) |
|------|--------|--------|--------|--------|--------|--------|--------|--------|
| rep. \* | \( \tilde{1} \) | \( \tilde{3} \) | \( \tilde{1} \) | \( \tilde{4} \) | \( \tilde{1} \) | \( \tilde{6} \) | \( \tilde{1} \) | \( \tilde{8} \) |
(a) Kronecker products with one-dimensional representations

| rep. | \( \mathbf{1}_1 \) | \( \mathbf{1}_2 \) | \( \mathbf{1}_3 \) |
|------|-----------------|-----------------|-----------------|
| \( \mathbf{1}_1 \) | \( \mathbf{1}_1 \) | \( \mathbf{1}_2 \) | \( \mathbf{1}_3 \) |
| \( \mathbf{1}_2 \) | \( \mathbf{1}_2 \) | \( \mathbf{1}_3 \) | \( \mathbf{1}_1 \) |
| \( \mathbf{1}_3 \) | \( \mathbf{1}_3 \) | \( \mathbf{1}_1 \) | \( \mathbf{1}_2 \) |
| \( \mathbf{2}_1 \) | \( \mathbf{2}_1 \) | \( \mathbf{2}_1 \) | \( \mathbf{2}_1 \) |
| \( \mathbf{2}_2 \) | \( \mathbf{2}_2 \) | \( \mathbf{2}_2 \) | \( \mathbf{2}_2 \) |

Table 8: Kronecker products of \( \Sigma(81) \) relevant for the discussion of the leading order of the fermion masses. Note that \([\mu \times \mu]\) denotes the symmetric and \{\(\mu \times \mu\)\} the anti-symmetric part of the product \(\mu \times \mu\).

The generators of all representations are presented in the Table 9.

Some of the Kronecker products are already shown in [25, 26]. In the Table 8 we show the products which we need to discuss the lowest order.

The non-trivial Clebsch-Gordan coefficients are for \((a_1, a_2, a_3)^T \sim \mathbf{3}_1\) \((i = 1, 2)\), \(c \sim \mathbf{1}_1\), \(c' \sim \mathbf{1}_2\) and \(c'' \sim \mathbf{1}_3\),

\[
\begin{align*}
\mathbf{3}_i \times \mathbf{1}_1 & : (a_1 c, a_2 c, a_3 c)^T \sim \mathbf{3}_i \\
\mathbf{3}_i \times \mathbf{1}_2 & : (a_1 c', \omega a_2 c', \omega^2 a_3 c')^T \sim \mathbf{3}_i \\
\mathbf{3}_i \times \mathbf{1}_3 & : (a_1 c'', \omega^2 a_2 c'', \omega a_3 c'')^T \sim \mathbf{3}_i 
\end{align*}
\]

For \((a_1, a_2, a_3)^T, (a'_1, a'_2, a'_3)^T \sim \mathbf{3}_1\) the structure of the Clebsch-Gordan coefficients is

\[
(a_1 a'_1, a_2 a'_2, a_3 a'_3)^T \sim \mathbf{3}_3, \quad (a_2 a'_3, a_3 a'_1, a_1 a'_2)^T \sim \mathbf{3}_4, \quad (a_3 a'_2, a_1 a'_3, a_2 a'_1)^T \sim \mathbf{3}_4.
\]

Similarly, for \((b_1, b_2, b_3)^T, (b'_1, b'_2, b'_3)^T \sim \mathbf{3}_2\) we find

\[
(b_1 b'_1, b_2 b'_2, b_3 b'_3)^T \sim \mathbf{3}_1, \quad (b_2 b'_3, b_3 b'_1, b_1 b'_2)^T \sim \mathbf{3}_3, \quad (b_3 b'_2, b_1 b'_3, b_2 b'_1)^T \sim \mathbf{3}_3.
\]

For \((a_1, a_2, a_3)^T \sim \mathbf{3}_1\) and \((b_1, b_2, b_3)^T \sim \mathbf{3}_2\) the covariant combinations read

\[
\begin{align*}
a_1 b_1 + a_2 b_2 + a_3 b_3 & \sim \mathbf{1}_1, \quad a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \sim \mathbf{1}_2, \quad a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \sim \mathbf{1}_3, \\
(a_3 b_2, a_2 b_1, a_1 b_3)^T & \sim \mathbf{3}_7, \quad (a_2 b_3, a_1 b_2, a_3 b_1)^T \sim \mathbf{3}_8.
\end{align*}
\]

\(^{18}\)The remaining ones can be obtained via the formulae given in [38].
Table 9: Generators of $\Sigma(81)$. We show three generators A, B and C for each representation, although it is enough to take the generators A and C in order to reproduce the whole group. Note that $\omega \equiv e^{2\pi i \nu}$.

C Higher-Dimensional Operators
| Order | $O(1)$ | $O(e^*)$ | $O(e^2)$ | $O(e^*)$ | $O(e^2)$ | $O(e^3)$ |
|-------|--------|--------|--------|--------|--------|--------|
| $n = 1$ | $\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^2 \\ \chi_2^2 \\ \chi_3^2 \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^3 \\ \chi_2^3 \\ \chi_3^3 \end{pmatrix} \sim 2^*$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ |
| $n = 2$ | $\begin{pmatrix} \chi_1^3 \\ \chi_2^3 \\ \chi_3^3 \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^4 \\ \chi_2^4 \\ \chi_3^4 \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^5 \\ \chi_2^5 \\ \chi_3^5 \end{pmatrix} \sim 2^*$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ |
| $n = 3$ | $\begin{pmatrix} \chi_1^6 \\ \chi_2^6 \\ \chi_3^6 \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^7 \\ \chi_2^7 \\ \chi_3^7 \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^8 \\ \chi_2^8 \\ \chi_3^8 \end{pmatrix} \sim 2^*$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ |
| $n = 4$ | $\begin{pmatrix} \chi_1^9 \\ \chi_2^9 \\ \chi_3^9 \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^{10} \\ \chi_2^{10} \\ \chi_3^{10} \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^{11} \\ \chi_2^{11} \\ \chi_3^{11} \end{pmatrix} \sim 2^*$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ |
| $n = 5$ | $\begin{pmatrix} \chi_1^1 \\ \chi_2^1 \\ \chi_3^1 \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^2 \\ \chi_2^2 \\ \chi_3^2 \end{pmatrix} \sim 2^*$ | $\begin{pmatrix} \chi_1^3 \\ \chi_2^3 \\ \chi_3^3 \end{pmatrix} \sim 2^*$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ | $\chi_1 \chi_2 \chi_3 \sim \mathbf{1}_1$ |

Table 10: Higher-dimensional operators of $T_7$. We assume that the VEVs of the fields $\chi_i$ are of the form $\langle \chi_1 \rangle = \eta \Lambda$, $\langle \chi_2 \rangle / \langle \chi_1 \rangle \approx e^2$ with $\eta \sim O(1)$ and $e \approx 0.05$. Note that in the cases in which the one-dimensional representation $\mathbf{1}_1$ is given as a polynomial, i.e. for $n > 3$, also similar polynomials forming the representations $\mathbf{1}_2, \mathbf{3}$ exist leading to contributions of the same order in $e$ as the one transforming as $\mathbf{1}_1$. Note further the periodicity of the covariants in $n$, e.g. for $n = 4$ one finds that the combination $\chi_1^4 \chi_2 \chi_3$ is the first component of a triplet $2^*$ and similarly for $n = 4 + 7 = 11$ the monomial $\chi_1^{12} \chi_2 \chi_3 = \chi_1^9 \chi_2 \chi_3$ belongs to the first component of a $2^*$. When using this table to compute the contributions to the fermion masses, one has to take into account a factor $\eta^n < 1$ for each covariant of order $n$. 
| Order in $\epsilon$ | Operator Structure | Transformation Property |
|---------------------|--------------------|-------------------------|
| $\mathcal{O}(1)$    | $\chi_3^m (\chi_3^*)^{n-m} \ (m = 0, ..., n)$ | $1_{\mathcal{O}(1)}$ for $(2m - n) \ mod \ 3 = 0$
|                     |                    | $2^{\text{nd}} \ \text{comp. of } \chi_{3,3} \ (2m - n) \ mod \ 3 = 2$
|                     |                    | $3^{\text{rd}} \ \text{comp. of } \chi_{3,3} \ (2m - n) \ mod \ 3 = 1$
|                     |                    | $3^{\text{rd}} \ \text{comp. of } \chi_{3,1} \ (2m - n) \ mod \ 3 = 1$
| $\mathcal{O}(\epsilon^2)$ | $\chi_3^m (\chi_3^*)^{n-1-m} \chi_1 \ (m = 0, ..., n - 1)$ | $1^{\text{st}} \ \text{comp. of } \chi_{2,2} \ (2m - n + 1) \ mod \ 3 = 0$
|                     |                    | $2^{\text{nd}} \ \text{comp. of } \chi_{3,2} \ (2m - n + 1) \ mod \ 3 = 1$
|                     |                    | $3^{\text{rd}} \ \text{comp. of } \chi_{3,2} \ (2m - n + 1) \ mod \ 3 = 2$
|                     |                    | $1^{\text{st}} \ \text{comp. of } \chi_{2,3} \ (2m - n + 1) \ mod \ 3 = 0$
|                     |                    | $2^{\text{nd}} \ \text{comp. of } \chi_{3,3} \ (2m - n + 1) \ mod \ 3 = 2$
| $\mathcal{O}(\epsilon^3)$ | $\chi_3^m (\chi_3^*)^{n-1-m} \chi_1^* \ (m = 0, ..., n - 1)$ | $1^{\text{st}} \ \text{comp. of } \chi_{3,3} \ (2m - n + 1) \ mod \ 3 = 0$
|                     |                    | $3^{\text{rd}} \ \text{comp. of } \chi_{3,3} \ (2m - n + 1) \ mod \ 3 = 2$
| $\mathcal{O}(\epsilon^4)$ | $\chi_3^m (\chi_3^*)^{n-2-m} \chi_1 \ (m = 0, ..., n - 2)$ | $1^{\text{st}} \ \text{comp. of } \chi_{3,4} \ (2m - n + 2) \ mod \ 3 = 2$
|                     |                    | $2^{\text{nd}} \ \text{comp. of } \chi_{3,4} \ (2m - n + 2) \ mod \ 3 = 2$
|                     |                    | $3^{\text{rd}} \ \text{comp. of } \chi_{3,4} \ (2m - n + 2) \ mod \ 3 = 1$
| $\mathcal{O}(\epsilon^5)$ | $\chi_3^m (\chi_3^*)^{n-2-m} \chi_2 \ (m = 0, ..., n - 2)$ | $1^{\text{st}} \ \text{comp. of } \chi_{3,5} \ (2m - n + 2) \ mod \ 3 = 0$
|                     |                    | $2^{\text{nd}} \ \text{comp. of } \chi_{3,5} \ (2m - n + 2) \ mod \ 3 = 2$
|                     |                    | $3^{\text{rd}} \ \text{comp. of } \chi_{3,5} \ (2m - n + 2) \ mod \ 3 = 1$
| $\mathcal{O}(\epsilon^6)$ | $\chi_3^m (\chi_3^*)^{n-3-m} \chi_3 \ (m = 0, ..., n - 3)$ | $1^{\text{st}} \ \text{comp. of } \chi_{3,6} \ (2m - n) \ mod \ 3 = 0$
|                     |                    | $2^{\text{nd}} \ \text{comp. of } \chi_{3,6} \ (2m - n) \ mod \ 3 = 2$
|                     |                    | $3^{\text{rd}} \ \text{comp. of } \chi_{3,6} \ (2m - n) \ mod \ 3 = 1$

Table 11: Higher-dimensional operators of $\Sigma(81)$. We assume the vacuum structure $\langle \chi_1 \rangle / \langle \chi_3 \rangle \approx \epsilon^4$, $\langle \chi_2 \rangle / \langle \chi_3 \rangle \approx \epsilon^2$, $\langle \chi_3 \rangle = \eta \Lambda$ with $\eta \sim \mathcal{O}(1)$ and $\epsilon \approx 0.05$. Analogous to $T_7$, we can uniquely identify as which component of a three-dimensional representation a certain monomial in the fields $\chi_1$ and $\chi_3^*$ transforms by using the three elements $S_1 = C^2$, $S_2 = A^2 C^2 A$ and $S_3 = A B^2 C A^2$ of the group which are products of the generators $A$, $B$ and $C$, see the Table. The resulting transformation properties are shown in the third column. The number of operators with a certain transformation property is approximately $|\mathcal{O}|$ for larger values of $n$. 


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