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\textbf{VALUE DISTRIBUTION AND SPECTRAL THEORY OF SCHRÖDINGER OPERATORS WITH $L^2$-SPARSE POTENTIALS}

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\textbf{Abstract.} We apply the methods of value distribution theory to the spectral asymptotics of Schrödinger operators with $L^2$-sparse potentials.

1. Introduction

A real valued, locally integrable function $V$, defined on the half-line $0 \leq x < \infty$, is said to be a $L^2$-sparse potential if, given arbitrary $\delta, N > 0$, there exists a subinterval $(a, b)$ of $[0, \infty)$ such that $b - a = N$ and $\int_a^b (V(x))^2 \, dx < \delta$. In other words, if $V$ is $L^2$-sparse then one can find arbitrarily long intervals on which the $L^2$ norm of $V$ is arbitrarily small. Given an $L^2$-sparse potential, we can define a Schrödinger operator $T = -\frac{d^2}{dx^2} + V(x)$ acting in $L^2(0, \infty)$ and subject to Dirichlet boundary condition at $x = 0$. By considering an appropriate sequence of approximate eigenfunctions (see for example [G], Theorem 22) one may verify that the Weyl spectrum of $T$ contains the whole of $\mathbb{R}^+$. It follows that we have the limit point case at infinity, so that $T$ can be uniquely defined as a self-adjoint operator, subject to the single boundary condition at $x = 0$.

Any $L^2$-sparse potential is a sum $V_1 + V_2$, where $V_1$ is a sparse potential and $V_2 \in L^2(0, \infty)$; here a potential $V$ is said to be sparse if arbitrarily long intervals exist on which $V$ is identically zero. There is a considerable literature on sparse potentials and their perturbations, in particular establishing conditions for the existence of absolutely continuous and singular continuous spectra.

For recent results in this field, see [KLS, R, SS] and references therein.

Spectral theory for the Schrödinger operator $T$ can be closely linked to the theory of value distribution for real-valued functions, and in particular value distribution for functions which are defined as boundary values of Herglotz functions. (A Herglotz or Nevanlinna function is a function of a complex variable, analytic in the upper half-plane with positive imaginary part.)

For a measurable function $F_+ : \mathbb{R} \to \mathbb{R}$, the value distribution may be described by means of a map $\mathcal{M} : (A, S) \mapsto \mathcal{M}(A, S) \in \mathbb{R} \cup \{ \infty \}$, called the value distribution function of $F_+$, and defined for Borel subsets $A, S$ of $\mathbb{R}$ by

$$\mathcal{M}(A, S) = |A \cap F_+^{-1}(S)|. \quad (1)$$

Here $| \cdot |$ stands for Lebesgue measure. Thus $\mathcal{M}(A, S)$ is the Lebesgue measure of the set of $\lambda \in A$ for which $F_+(\lambda) \in S$. In the particular case that $F_+$ is the almost everywhere boundary value of a Herglotz function, i.e.

$$F_+(\lambda) = \lim_{d \to 0^+} F(\lambda + id), \quad \text{almost all } \lambda \in \mathbb{R},$$

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we can write (see [BP1])
\[ \mathcal{M}(A, S) = \lim_{d \to 0^+} \frac{1}{\pi} \int_A \theta(F(\lambda + id), S) \, d\lambda, \]  
(2)
where \( \theta(z, S) \) denotes the angle subtended at a point \( z \) by the Borel subset \( S \) of the real line. (For \( \lambda \in \mathbb{R} \), we define \( \theta(\lambda, S) \) to be \( \pi \chi_S(\lambda) \), where \( \chi_S \) is the characteristic function of the set \( S \).) In fact, given \( A \) and \( S \) with \( |A| < \infty \), the limit in (2) will exist for any Herglotz function \( F \) (whether or not \( F \) has real boundary values a.e.) and may be used to define the value distribution function \( \mathcal{M} \) associated with an arbitrary Herglotz function. In general \( \mathcal{M} \) may not describe the value distribution of any single real-valued function \( F_+(\lambda) \), but there will always be sequences \( \{F^{(n)}\} \) of real-valued functions for which \( \mathcal{M} \) describes the limiting value distribution.

Value distribution for boundary values of Herglotz functions is also closely connected with the geometric properties of the upper half-plane, regarded as a hyperbolic space [BP1, BP2]. Given two points \( z_1, z_2 \in \mathbb{C}^+ \), we define a measure of separation
\[ \gamma(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{\text{Im} z_1 \sqrt{\text{Im} z_2}}}, \]  
(3)
which is related to the hyperbolic distance \( D(z_1, z_2) \) by the equation
\[ \gamma(z_1, z_2) = 2 \sinh \left( \frac{1}{2} D(z_1, z_2) \right). \]
The relevance of hyperbolic distance to estimates of value distribution comes from the fact that if \( F_1 \) and \( F_2 \) are two Herglotz functions satisfying the estimate
\[ \gamma(F_1(z), F_2(z)) < \epsilon, \]
for all \( z \) such that \( \text{Im} z = d \) and \( \text{Re} z \in A \), then the value distribution \( \mathcal{M}_2(A, S) \) associated with \( F_2 \) is a good approximation to the value distribution \( \mathcal{M}_1(A, S) \) associated with \( F_1 \), in the sense that
\[ |\mathcal{M}_1(A, S) - \mathcal{M}_2(A, S)| \leq \epsilon |A| + 2E_A(d). \]  
(4)
Here \( E_A(d) \) is an error estimate which is an increasing function of \( d \), and which converges to zero in the limit \( d \to 0 \), for fixed Borel set \( A \). For details of this and related results, see [BP1, BP2]. Estimates such as (4) imply that if \( \{F^{(n)}\}_{n=1,2,3,...} \) is a sequence of Herglotz functions converging uniformly to \( F(z) \) for \( z \) lying in any fixed compact subset of \( \mathbb{C}^+ \), then the value distribution associated with \( F^{(n)} \) will converge in the limit \( n \to \infty \) to the value distribution associated with \( F \).

The main purpose of this paper is to apply the above analysis to the spectral asymptotics of the Schrödinger operator with \( L^2 \)-sparse potential, as described by solutions \( f(x, \lambda) \) of the Schrödinger equation at real spectral parameter \( \lambda \). Herglotz functions of particular interest in this context are the Weyl \( m \)-function \( m(z) \) for the operator \(-\frac{d^2}{dx^2} + V \) in \( L^2(0, \infty) \) with Dirichlet boundary condition at \( x = 0 \), and the Weyl \( m \)-function \( m^N(z) \) for \(-\frac{d^2}{dz^2} + V \) regarded as operating in \( L^2(N, \infty) \) for some fixed \( N > 0 \), with Dirichlet boundary condition at \( x = N \). Estimates of both of these \( m \)-functions may be carried out, for \( z \) in some fixed compact subset of \( \mathbb{C}^+ \), in terms of the logarithmic derivative \( f'(x, z)/f(x, z) \), for asymptotically large \( x \), of particular solutions \( f(\cdot, z) \) of the Schrödinger equation at complex spectral parameter \( z \). The main general results of the paper are presented in Theorems 1 and 2. Theorem 1 provides an estimate of the large \( x \)
asymptotics of \( f'/f \), based on an \( L^2 \) bound for the potential across a finite interval. Theorem 2 is an analysis of asymptotic value distribution in the case of \( L^2 \)-sparse potential, linking this to the asymptotics of \( m^N \).

Finally, we indicate some consequences of the analysis for spectral theory of \( L^2 \)-sparse potentials, implying in particular the absence of absolutely continuous spectrum at negative \( \lambda \).

2. Asymptotics of \( v'/v \)

We consider the differential expression \( \tau = -\frac{d^2}{dx^2} + V(x) \) on the half-line \( 0 \leq x < \infty \), where the potential function \( V \) is assumed to be real valued and integrable over any finite subinterval of \([0, \infty)\). Assume limit-point case at infinity, implying that a self-adjoint operator \( T = -\frac{d^2}{dx^2} + V(x) \) can be defined, acting in \( L^2(0, \infty) \) and subject to a Dirichlet boundary condition at \( x = 0 \).

The Weyl \( m \)-function \( m(z; V) \) may be defined in terms of solutions \( f(\cdot, z) \) of the Schrödinger equation at complex spectral parameter \( z \), namely

\[
-\frac{d^2}{dx^2}f(x, z) + V(x)f(x, z) = zf(x, z) \quad (\text{Im } z > 0, 0 \leq x < \infty).
\]

First define two solutions \( u(x, z), v(x, z) \) of (5), subject respectively to initial conditions

\[
\begin{align*}
  u(0, z) &= 1, & u'(0, z) &= 0, \\
  v(0, z) &= 0, & v'(0, z) &= 1,
\end{align*}
\]

where prime denotes differentiation with respect to \( x \). (Solutions of (5) and (6) with \( z \) replaced by a real spectral parameter \( \lambda \) will be denoted by \( u(x, \lambda), v(x, \lambda) \) respectively and, for fixed \( x \), are the boundary values of \( u(x, z), v(x, z) \) as \( z \) approaches the real axis.)

Then (in the limit-point case at infinity) we define \( m(z; V) \) uniquely by the condition that

\[
u(\cdot, z) + m(z; V)v(\cdot, z) \in L^2(0, \infty).
\]

An alternative characterisation of the Weyl function is that if \( f(\cdot, z) \) is any (non-trivial) \( L^2(0, \infty) \) solution of (5), then

\[
m(z; V) = \frac{f'(0, z)}{f(0, z)}.
\]

It follows from the limit point/limit circle theory [CL] that \( m(z; V) \) is an analytic function of \( z \) for \( \text{Im } z > 0 \). In addition \( \text{Im } m(z; V) > 0 \) for \( \text{Im } z > 0 \), so that \( m(z; V) \) is a Herglotz function (analytic in the upper half-plane with positive imaginary part).

Given any \( N > 0 \), we can also define the Dirichlet \( m \)-function \( m^N(z; V) \) for the truncated problem on the interval \( N \leq x < \infty \), and an analysis of the large \( N \) asymptotics of \( m^N \) will play an important role in this paper. Here we are strongly motivated by the recent results of Deift and Killip [DK] for \( L^2 \) potentials.

Since, according to equation (5), the \( m \)-function is dependent on the logarithmic derivative of a solution \( f(\cdot, z) \) of equation (5), a first step in our analysis will be to carry out a comparison between logarithmic derivatives of solutions of equation (5) as the potential is varied. We begin with the logarithmic derivative of the solution \( v(\cdot, z) \) subject to the initial conditions (6). Here it is \(-v'/v\) rather than \( v'/v \) that is a Herglotz function for \( x > 0 \). The following elementary estimate provides a bound for the \( \gamma \)-separation of the logarithmic derivative as the potential is varied.
Lemma 1. Let \( v(x, z), \tilde{v}(x, z) \) be solutions of equation (3) with potentials \( V(x), \tilde{V}(x) \) respectively, and subject to initial conditions
\[
v(0, z) = \tilde{v}(0, z) = 0, \quad v'(0, z) = \tilde{v}'(0, z) = 1.
\]
Then, for any \( x > 0 \),
\[
\gamma \left( -\frac{v'(x, z)}{v(x, z)}, -\frac{\tilde{v}'(x, z)}{\tilde{v}(x, z)} \right) \leq (\text{Im } z)^{-1} \left( \frac{\int_0^x \left( V(t) - \tilde{V}(t) \right)^2 |\tilde{v}(t, z)|^2 \, dt}{\left( \int_0^x |\tilde{v}(t, z)|^2 \, dt \right)^{1/2}} \right)^{1/2}
\]
(8)

Proof. Abbreviating the notation for simplicity, we have
\[
\gamma \left( -\frac{v'}{v}, -\frac{\tilde{v}'}{\tilde{v}} \right) = \frac{|v' - \tilde{v}'|}{\sqrt{\text{Im } (-\frac{v}{v}) \text{ Im } (-\frac{\tilde{v}}{\tilde{v}})}},
\]
where
\[
|v' - \tilde{v}'| = \frac{|\tilde{v}v' - v\tilde{v}'|}{|\tilde{v}v|}.
\]
(10)

Using the Schrödinger equation \(-v'' + Vv = zv\), and similarly for \( \tilde{v} \), we have
\[
\frac{d}{dx}(\tilde{v}v' - v\tilde{v}') = \tilde{v}v'' - v\tilde{v}'' = \left( V - \tilde{V} \right) v\tilde{v},
\]
which, with the initial conditions, gives
\[
\tilde{v}v' - v\tilde{v}' = \int_0^x \left( V(t) - \tilde{V}(t) \right) v(t)\tilde{v}(t) \, dt.
\]

We also have
\[
\text{Im } \left( \frac{v'}{v} \right) = \frac{1}{2i} \left( \frac{\overline{v}}{v} \frac{v'}{v} \right) = \frac{1}{2i|v|^2} (v\overline{v}' - \overline{v}v'),
\]
which again on considering \( \frac{d}{dx} (v\overline{v}' - \overline{v}v') \) gives
\[
\text{Im } \left( -\frac{v'}{v} \right) = \frac{\text{Im } z}{|v|^2} \int_0^x |v(t)|^2 \, dt,
\]
with a similar equation for \( \tilde{v} \). Using (3) and (10), and substituting for \( \text{Im } (-v'/v) \), \( \text{Im } (-\tilde{v}'/\tilde{v}) \) and \( (\tilde{v}' - \tilde{v}') \) results in the bound
\[
\gamma \left( -\frac{v'}{v}, -\frac{\tilde{v}'}{\tilde{v}} \right) = \frac{\left| \int_0^x \left( V(t) - \tilde{V}(t) \right) v(t)\tilde{v}(t) \, dt \right|}{(\text{Im } z) \left( \int_0^x |v(t)|^2 \, dt \int_0^x |\tilde{v}(t)|^2 \, dt \right)^{1/2}},
\]
from which (8) follows on applying Schwarz’s inequality to the integral in the numerator.

If both potentials \( V, \tilde{V} \) are bounded, we can use the result of Lemma 1 to derive simple bounds for the separation \( \gamma \) between the two logarithmic derivatives. For example we have, from (8), for any \( L > 0 \),
\[
\gamma \left( -\frac{v'}{v}, -\frac{\tilde{v}'}{\tilde{v}} \right) \bigg|_{x=L} \leq \frac{1}{\text{Im } z} \sup_{t \in [0, L]} |V(t) - \tilde{V}(t)|.
\]
In particular, we see that any uniformly convergent sequence \( V_n \) of potentials will result in a corresponding sequence \( -v'_n/v_n \) which will converge uniformly in \( \gamma \)-separation (and hence also uniformly in the hyperbolic metric).
SCHRÖDINGER OPERATORS WITH $L^2$-SPARSE POTENTIALS

We turn now to the case of a potential subject to an $L^2$-type condition, for which we take in the first instance the comparison potential to be $\tilde{V}(x) = 0$. Let $v(x, z)$ be defined as before to be the solution of equation \( (5) \) with potential $V(x)$ and subject to \( v(0, z) = 0, v'(0, z) = 1 \), and let $v_0(x, z)$ satisfy the equation

\[
- \frac{d^2 v_0(x, z)}{dx^2} = z v_0(x, z)
\]

with the same initial conditions. Again we take $\text{Im} z > 0$, and write $\sqrt{z} = a + ib$ with $a, b$ real and $a, b > 0$. An explicit expression for $v_0$ is then

\[
v_0(x, z) = \frac{1}{2} \left( e^{ix\sqrt{z}} - e^{-ix\sqrt{z}} \right) = (2 (b-i) a) \left( e^{-iax} e^{bx} - e^{iax} e^{-bx} \right),
\]

so that

\[
|v_0(x, z)|^2 = \left( 2 \left( a^2 + b^2 \right) \right)^{-1} \left( \cosh 2bx - \cos 2ax \right),
\]

and, from \( (5) \), we have

\[
\gamma \left( - \frac{v'}{v} - \frac{v_0'}{v_0} \right) \bigg|_{x=L} \leq \left( \frac{\int_0^L V(t)^2 \left( \cosh 2bt - \cos 2at \right) dt}{\text{Im} z} \left( \int_0^L \left( \cosh 2bt - \cos 2at \right) dt \right)^{1/2} \right)^{1/2}. \tag{11}
\]

Here the integral in the numerator may be written

\[
- \int_0^L \left\{ (\cosh 2bt - \cos 2at) \frac{d}{dt} \int_t^L V(s)^2 \, ds \right\} \, dt = \int_0^L \left\{ \left( 2b \sinh 2bt + 2a \sin 2at \right) \int_t^L V(s)^2 \, ds \right\} \, dt
\]

\[
\leq \int_0^L \left\{ (2b \sinh 2bt + 2a) \int_0^L V(s)^2 \, ds \right\} \, dt
\]

\[
= (2aL + \cosh 2bL - 1) \int_0^L V(s)^2 \, ds \tag{12}
\]

To complete the estimate of \( (11) \), we need a lower bound for the denominator integral, which comes to

\[
\frac{\sinh 2bL}{2b} - \frac{\sin 2aL}{2a}.
\]

We shall make the assumption $L \geq 1/\sqrt{|z|}$. Such a condition, with $\sqrt{z} = a + ib$, implies that either $L \geq 1/ (\sqrt{2}a)$ or $L \geq 1/ (\sqrt{2}b)$. (If $L < 1/ (\sqrt{2}a)$ and $L < 1/ (\sqrt{2}b)$ then $|z| = a^2 + b^2 < 1/\sqrt{2}a + 1/\sqrt{2}b = 1/\sqrt{2}$, which contradicts the assumption.)

We consider the two possibilities in turn:

**Case 1 :** $L \geq 1/ \sqrt{2}a$

From the bound $\sinh x/x > 1$ for $x > 0$, we have

\[
\frac{\sinh 2bL}{2b} > L,
\]

whereas

\[
\left| \frac{\sin 2aL}{2a} \right| \leq \frac{1}{2a} \leq \frac{L}{\sqrt{2}},
\]

so that

\[
\left| \frac{\sin 2aL}{2a} \right| < \frac{1}{\sqrt{2}} \frac{\sinh 2bL}{2b},
\]
and it follows that
\[
\frac{\sinh 2bL}{2b} - \frac{\sin 2aL}{2a} > \left(1 - \frac{1}{\sqrt{2}}\right) \frac{\sinh 2bL}{2b}.
\] (13)

**Case 2 :** \(L \geq \frac{1}{\sqrt{2}b}\)

Since the function \(\sinh x/x\) is increasing for \(x \geq 0\), we then have
\[
\frac{\sinh 2bL}{2b} \geq \frac{L \sinh \sqrt{2}}{\sqrt{2}},
\]
whereas
\[
\left|\frac{\sin 2aL}{2a}\right| < L.
\]
Hence in this case we find
\[
\left|\frac{\sin 2aL}{2a}\right| < \sqrt{2} \sinh \sqrt{2} \left(\frac{\sinh 2bL}{2b}\right),
\]
so that
\[
\frac{\sinh 2bL}{2b} - \frac{\sin 2aL}{2a} > \left(1 - \frac{\sqrt{2}}{\sinh \sqrt{2}}\right) \frac{\sinh 2bL}{2b}.
\] (14)

Noting that \(\sinh \sqrt{2} < 2\), we see that the bound (14) holds both in case 1 and in case 2.

Using (12) and (14) as upper and lower bounds for the numerator and denominator respectively of (11), we have, now, for \(L \geq 1/\sqrt{|z|}\), the estimate
\[
\gamma \left(-\frac{\nu'(L, z)}{\nu(L, z)}, -\frac{\nu_0'(L, z)}{\nu_0(L, z)}\right) \leq \frac{1}{\Im z} \left(\frac{(2aL + \cosh 2bL - 1) \int_0^L V(s)^2 \, ds}{\left(1 - \frac{\sqrt{2}}{\sinh \sqrt{2}}\right) \frac{\sinh 2bL}{2b}}\right)^{1/2}
\]
\[
= \frac{1}{\Im z} \left(1 - \frac{\sqrt{2}}{\sinh \sqrt{2}}\right)^{-1/2} \left(2a \left(\frac{2bL}{\sinh 2bL}\right) + 2b \left(\frac{\cosh 2bL - 1}{\sinh 2bL}\right)\right)^{1/2}
\]
\[
\times \left(\int_0^L V(s)^2 \, ds\right)^{1/2}
\]
Noting that
\[
\frac{2bL}{\sinh 2bL} < 1
\]
and that
\[
\frac{\cosh 2bL - 1}{\sinh 2bL} = \tanh bL < 1,
\]
we can use the estimate \((a + b)^{1/2} \leq (2 \left(|a^2 + b^2|\right))^{1/4} = (2|z|)^{1/4}\) to obtain the following result.

**Lemma 2.** Define \(v(x, z)\) as in Lemma 1, and let \(v_0(x, z)\) be the corresponding solution of (5) with zero potential. Then, for any \(L \geq 1/\sqrt{|z|}\), we have the bound
\[
\gamma \left(-\frac{\nu'(L, z)}{\nu(L, z)}, -\frac{\nu_0'(L, z)}{\nu_0(L, z)}\right) \leq \frac{C|z|^{1/4}}{\Im z} \left(\int_0^L V(s)^2 \, ds\right)^{1/2},
\] (15)
where $C$ is a positive constant. (In fact we can take $C = \sqrt{2} \left( \frac{1}{\sqrt{2}} - \frac{1}{\sinh \sqrt{2}} \right)^{-1/2}$ in which case $C < 3.3$.)

Notice that Lemma 2 provides a simple bound for the hyperbolic distance between $-v'/v$ and $-v_0'/v_0$ at $x = L$, in terms of the $L^2$ norm of the potential $V$ across the interval $[0, L]$.

Since, as is easily verified, we have
\[
\lim_{L \to \infty} -\frac{v_0'(L, z)}{v_0(L, z)} = i\sqrt{z},
\]
we can make a comparison, for large $L$, of $-v_0'/v_0$ with its asymptotic limit, leading to the following result.

**Lemma 3.** With $v_0(x, z)$ defined as in Lemma 2, for any $L \geq \frac{1}{\sqrt{|z|}}$ we have the bound
\[
\gamma \left( -\frac{v_0'(L, z)}{v_0(L, z)}, i\sqrt{z} \right) \leq C' \left( 1 + \left( \frac{b}{a} \right)^2 \right)^{1/2} \sqrt{e^{4bL} - 1}^{1/2},
\]
where $C'$ is a positive constant. (In fact we can take $C' = 2^{1/4}C$, where $C$ is the constant defined in Lemma 2 in which case $C' < 3.9$.)

**Proof.** Explicitly, we have
\[
\frac{v_0'}{v_0} = \frac{(ia - b)(e^{-iax}e^{bx} + e^{iax}e^{-bx})}{e^{-iax}e^{bx} - e^{iax}e^{-bx}},
\]
and multiplying numerator and denominator by the complex conjugate of the denominator gives
\[
\text{Im} \left( -\frac{v_0'}{v_0} \right) = \frac{2a \sinh 2bx - 2b \sin 2ax}{|e^{-iax}e^{bx} - e^{iax}e^{-bx}|^2},
\]
Moreover,
\[
\left| -\frac{v_0'}{v_0} - i\sqrt{z} \right| = \left| -\frac{v_0'}{v_0} + b - ia \right| = \frac{2\sqrt{a^2 + b^2}e^{-bx}}{|e^{-iax}e^{bx} - e^{iax}e^{-bx}|}.
\]
Putting these results together we find, at $x = L$,
\[
\gamma \left( -\frac{v_0'}{v_0}, i\sqrt{z} \right) \bigg|_{x=L} = \left( \frac{2 \left( a^2 + b^2 \right)}{a} \right)^{1/2} \frac{e^{-bL}}{(a \sinh 2bL - b \sin 2aL)^{1/2}}.
\]
Substituting in the denominator the lower bound obtained previously in (14) and simplifying, we arrive at (16).

In using (16) to make precise estimates of the convergence to $i\sqrt{z}$ of $-v_0'/v_0$, it is useful to note the inequalities:

(i) if $\Re z \geq 0$ then $b/a \leq 1$;
(ii) $1 + \left( \frac{b}{a} \right)^2 \leq 4 \left( 1 + \left( \frac{\Re z}{\Im z} \right)^2 \right)$;
(iii) $b > \frac{\Im z}{2\sqrt{|z|}}$.

These inequalities imply, in particular, that $-v_0'/v_0$ converges uniformly in hyperbolic norm to $i\sqrt{z}$, for $z$ in any fixed compact subset of the upper half-plane.
3. Estimates of \( u \) and \( v \) for \( L^1 \)-bounded potentials

We consider solutions \( u(x, z), v(x, z) \) of equation (5) on a fixed interval \( 0 \leq x \leq N \), subject to initial conditions (6) at \( x = 0 \). We compare these solutions with the corresponding solutions \( u_0(x, z), v_0(x, z) \) with zero potential, and subject to the same initial conditions as for \( u \) and \( v \).

Lemma 4. Let \( K \) be a fixed compact subset of \( \mathbb{C}^+ \), and let \( N > 0 \) be fixed. Then, given any \( \epsilon > 0 \), there exists \( \delta_0 > 0 \) (\( \delta_0 \) depending on \( \epsilon, N \) and \( K \)) such that, for any potential function \( V \) satisfying

\[
\int_0^N |V(t)| \, dt < \delta_0,
\]

we have, for all \( z \in K \) and for all \( x \in [0, N] \),

\[
|u(x, z) - u_0(x, z)| < \epsilon, \quad |v(x, z) - v_0(x, z)| < \epsilon.
\]

Proof. The proof is a standard perturbation argument using the Gronwall inequality.

Let \( M \) be the \( 2 \times 2 \) transfer matrix given by

\[
M = M(x, z) = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix},
\]

and let

\[
M_0 = \begin{pmatrix} u_0 & v_0 \\ u'_0 & v'_0 \end{pmatrix}.
\]

Then

\[
\frac{dM}{dx} = \begin{pmatrix} 0 & 1 \\ V - z & 0 \end{pmatrix} M, \quad \frac{dM_0}{dx} = \begin{pmatrix} 0 & 1 \\ -z & 0 \end{pmatrix} M_0,
\]

and we have

\[
\frac{d}{dx} (M_0^{-1}M) = VA (M_0^{-1}M),
\]

where

\[
A = A(x, z) = (-v_0, u_0)^T (u_0, v_0)
\]

and \( V = V(x) \). Hence

\[
(M_0^{-1}M) (x) = I + \int_0^x V(t) A(t) (M_0^{-1}M) (t) \, dt,
\]

where \( I \) is the \( 2 \times 2 \) identity matrix and, for notational convenience, we have suppressed the dependence on \( z \). If \( \|A\| \) denotes operator norm of the matrix \( A \) in the two-dimensional space \( l_2 \), we have, for \( x \geq 0 \),

\[
\| (M_0^{-1}M) (x) - I \| \leq \int_0^x |V(t)| \|A(t)\| \, dt + \int_0^x |V(t)| \|A(t)\| \| (M_0^{-1}M) (t) - I \| \, dt.
\]

An application of the Gronwall inequality now leads to the bound, valid for all \( x \in [0, N] \),

\[
\| M(x) - M_0(x) \| \leq \| M_0(x) \| \| (M_0^{-1}M) (x) - I \|
\leq \| M_0(x) \| \left\{ \exp \left( \int_0^N |V(t)| \|A(t)\| \, dt \right) - 1 \right\}.
\] (17)
Noting that
\[ \| M_0(x) \| \leq (|u_0|^2 + |v_0|^2 + |u'_0|^2 + |v'_0|^2)^{1/2}, \]
and
\[ \| A \| = |u_0|^2 + |v_0|^2, \]
we see that both \( \| M_0(x, z) \| \) and \( \| A(t, z) \| \) are bounded for \( x, t \in [0, N] \) and \( z \in K \).

The result of the Lemma now follows from (17) and the observation that
\[ |u - u_0| \leq \| M - M_0 \|, \quad |v - v_0| \leq \| M - M_0 \|. \]

The following Corollary is a straightforward consequence of the Lemma.

**Corollary 1.** Let \( K \) be a fixed compact subset of \( \mathbb{C}^+ \), and let \( N > 0 \) be fixed. Define \( u, v, u_0, v_0 \) as in Lemma 4. Then given any \( \epsilon > 0 \), there exists \( \delta_0 > 0 \) (\( \delta_0 \) depending on \( \epsilon, N \) and \( K \)) such that, for all potential functions \( V \) satisfying the \( L^2 \) bound
\[ \int_{0}^{N} |V(t)|^2 \, dt < \delta, \]
the estimate
\[ \gamma \left( \frac{-f'(L, z)}{f(L, z)}, i \sqrt{z} \right) < \epsilon \] (19)
holds for all \( z \in K \).

**Proof.** In using the \( \gamma \) measure of separation to carry out the estimate (19), it should be noted that, unlike the hyperbolic metric which is a function of \( \gamma \), the separation \( \gamma(z_1, z_2) \) between two points \( z_1, z_2 \in \mathbb{C}^+ \) does not satisfy the triangle inequality. However, the following result can be useful as a substitute for the triangle inequality:

If \( z_1, z_2, z_3 \in \mathbb{C}^+ \) and it is given that
\[ \gamma(z_1, z_2) < \alpha, \quad \gamma(z_2, z_3) < \beta, \quad \text{with} \quad 0 < \alpha, \beta \leq 2, \]
then it follows that \( \gamma(z_1, z_3) < \sqrt{2}(\alpha + \beta) \). (To verify this result, note that if \( 0 < \alpha, \beta \leq 2 \) and
\[
\gamma(z_1, z_2) = 2 \sinh \left( \frac{D(z_1, z_2)}{2} \right) < \alpha, \quad \gamma(z_2, z_3) = 2 \sinh \left( \frac{D(z_2, z_3)}{2} \right) < \beta,
\]
then
\[
\gamma(z_1, z_3) = 2 \sinh \left( \frac{D(z_1, z_3)}{2} \right) \\
\leq 2 \sinh \left( \frac{D(z_1, z_2) + D(z_2, z_3)}{2} \right) \\
= 2 \sinh \left( \frac{D(z_1, z_2)}{2} \right) \cosh \left( \frac{D(z_2, z_3)}{2} \right) + 2 \sinh \left( \frac{D(z_2, z_3)}{2} \right) \cosh \left( \frac{D(z_1, z_2)}{2} \right) \\
\leq \alpha \sqrt{1 + \frac{\beta^2}{4}} + \beta \sqrt{1 + \frac{\alpha^2}{4}} \\
\leq (\alpha + \beta) \sqrt{2}
\]
as required.) As a simple consequence of this result, the three inequalities \( \gamma(z_1, z_2) < \frac{\alpha}{6}, \gamma(z_2, z_3) < \frac{\beta}{6}, \gamma(z_3, z_1) < \frac{\gamma}{6} \), with \( 0 < \epsilon < 1 \), together imply that \( \gamma(z_1, z_4) < \epsilon \).

If, then, we define \( u, v, u_0, v_0 \) as in the proofs of the previous Lemmas, it will be sufficient, to verify (19), to show that if \( z \in K \) then we have the three inequalities, at \( x = L \),
\[
\gamma \left( -\frac{f'}{f} - \frac{v'}{v} \right) < \frac{\epsilon}{6}, \quad \gamma \left( -\frac{v'}{v} - \frac{v_0}{v_0} \right) < \frac{\epsilon}{6}, \quad \gamma \left( -\frac{v_0}{v_0} i \sqrt{z} \right) < \frac{\epsilon}{6}. \tag{20}
\]
We begin by fixing the value of \( N \). Given \( \epsilon > 0 \) and a compact subset \( K \) of \( \mathbb{C}^+ \), we take \( N = N(\epsilon, K) \) to satisfy, for all \( z \in K \), the three inequalities
\[
\int_0^N \text{Im} \left( \overline{\pi_0 v_0} \right) \, dt > \frac{12}{\epsilon \text{Im} z}, \tag{21a}
\]
\[
C' \left( 1 + \left( \frac{1}{2} \right)^2 \right)^{1/2} < \frac{\epsilon}{6}, \tag{21b}
\]
\[
N > \frac{1}{\sqrt{|z|}}. \tag{21c}
\]
That \( N \) may be chosen to satisfy the first of these inequalities for \( z \in K \) follows from the fact that \( \int_0^\infty \text{Im} \left( \overline{\pi_0 v_0} \right) \, dt = \infty \) and that, for fixed \( N \), the integral \( \int_0^N \text{Im} \left( \overline{\pi_0 v_0} \right) \, dt \) depends continuously on \( z \) for \( \text{Im} z > 0 \). In the second inequality we have \( \sqrt{z} = a + ib \), where both \( b \) and \( b/a \) are bounded for \( z \in K \); the constant \( C' \) is defined in the proof of Lemma 3. Note also that \( 1/\sqrt{|z|} \) is bounded for \( z \in K \) in the third inequality.

From the Corollary to Lemma 3 we know that, for \( z \in K \), the integral \( \int_0^N \text{Im} \left( \overline{\pi v} \right) \, dt \) is close to \( \int_0^N \text{Im} \left( \overline{\pi_0 v_0} \right) \, dt \) provided that \( \int_0^N |V(t)| \, dt \) is sufficiently small. In particular, the inequality (21a) implies that there exists \( \delta_0 = \delta_0(\epsilon, K) > 0 \) such that, for all \( z \in K \), we have
\[
\int_0^N |V(t)| \, dt < \delta_0 \Rightarrow \int_0^N \text{Im} \left( \overline{\pi v} \right) \, dt > \frac{6}{\epsilon \text{Im} z}. \tag{22}
\]
Having fixed the values of \( N \) and \( \delta_0 \), now define \( \delta = \delta(\epsilon, K) \) to satisfy the two inequalities
\[
(i) \quad N \delta < \delta_0^2, \\
(ii) \quad \frac{C |z|^{1/4}}{\text{Im} z} \sqrt{\delta} < \frac{\epsilon}{6} \text{ for all } z \in K.
\]
Here the constant $C$ has been defined in the statement of Lemma 2. Now suppose that $L \geq N$ and $\int_0^L |V(t)|^2 \, dt < \delta$. By the Schwarz inequality we then have
\[
\int_0^N |V(t)| \, dt \leq \left( N \int_0^N |V(t)|^2 \, dt \right)^{1/2} < (\delta N)^{1/2} < \delta_0,
\]
by inequality (i). Hence, (22) implies that
\[
\int_0^N \text{Im} (uv) \, dt > \frac{6}{\epsilon} \text{Im} z.
\]
By Lemma 3 of [BP1] (see also Lemma 2 of [BP2]) we have, for any solution $f$ of (5) satisfying $\text{Im} (-f'(0, z)/f(0, z)) > 0$,
\[
\gamma \left(-\frac{f'}{f}, -\frac{\text{Im} z}{\epsilon} \right)_{x=L} \leq \frac{1}{\text{Im} z} \int_0^L \text{Im} (\text{Im} v) \, dt < \frac{\epsilon}{6}.
\]
Thus we have derived the first inequality in (20). The second inequality in (20) follows from (15) and (ii) above, using $\int_0^L |V(t)|^2 \, dt < \delta$. We can also use Lemma 3 with the inequality (21b) to complete the proof of (20), which also completes the proof of the Theorem.

We now explore some consequences of Theorem 1 in the case of $L^2$-sparse potentials. Let $V$ be an $L^2$-sparse potential. Then a sequence of subintervals $\{(a_k, b_k)\} (k = 1, 2, 3, \ldots)$ of $\mathbb{R}^+$ can be found such that, with $L_k = b_k - a_k$,
\[
\lim_{k \to \infty} L_k = \infty \quad \text{and} \quad \lim_{k \to \infty} \int_{a_k}^{b_k} (V(t))^2 \, dt = 0.
\]
Given a fixed, bounded, measurable subset $A$ of $\mathbb{R}$, having closure $\overline{A}$, and given any $\epsilon > 0$, we first of all find $d > 0$ ($d$ depending on $\epsilon$ and $A$) such that $E_A(d) < \epsilon |A|/2$. Here $E_A(\cdot)$ is the error estimate on the right hand side of (4), and from (4) we deduce that
\[
|\mathcal{M}_1(A, S) - \mathcal{M}_2(A, S)| < 2\epsilon |A|,
\]
provided $\gamma (F_1(z), F_2(z)) < \epsilon$ for all $z \in K$, where $K$ is the compact subset of $\mathbb{C}^+$ defined by the conditions $\text{Im} z = d$, $\text{Re} z \in \overline{A}$.

Now use Theorem 1 to define $\delta$ and $N$ such that, for all $L \geq N$ and for all potentials $V$ satisfying the bound $\int_0^L |V(t)|^2 \, dt < \delta$ we have
\[
\gamma \left(-\frac{f'(L, z)}{f(L, z)}, i\sqrt{z} \right) < \epsilon.
\]
(24)
Here $f(\cdot, z)$ is a solution of the Schrödinger equation (4) for which
\[
\text{Im} \left(-\frac{f'(0, z)}{f(0, z)} \right) > 0.
\]
We take $k$ sufficiently large (say $k > k_0$) so that $L_k \geq N$ and such that the bound $\int_{a_k}^{b_k} (V(t))^2 \, dt < \delta$ is satisfied by our sparse potential $V$.

We can now apply (24) with $L = L_k$, where $f$ is a suitably chosen solution of the Schrödinger equation (4), but with potential modified by an appropriate change of $x$-coordinate. There are two separate cases to be considered:
Firstly, define \( f(x, z) = v(x + a_k, z) \) (for \( 0 \leq x \leq L_k = b_k - a_k \)). Then, for \( x \in [0, L_k] \), \( f(\cdot, z) \) satisfies the Schrödinger equation \((\text{2})\) with potential \( V(x + a_k) \). Moreover, we have
\[
\int_0^{L_k} (V(t + a_k))^2 \, dt = \int_{a_k}^{b_k} (V(t))^2 \, dt < \delta.
\]
Hence \((\text{2})\) is satisfied in this case, and we have
\[
\gamma \left( \frac{v'(b_k, z)}{v(b_k, z)}, i\sqrt{z} \right) < \epsilon.
\]
From \((\text{2})\) we now deduce that the respective value distributions for the Herglotz functions \(-v'(b_k, z)/v(b_k, z)\) and \(i\sqrt{z}\) differ by at most \(2\epsilon |A|\), for all \( k > k_0 \).

Secondly, let \( F(\cdot, z) \) be a (non-trivial) solution in \( L^2(0, \infty) \) of the Schrödinger equation \((\text{2})\), with sparse potential \( V \). The \( m \)-function \( m^{a_k}(z) \) for the Schrödinger operator \(-\frac{d^2}{dx^2} + V\) acting in \( L^2(a_k, \infty) \) is then given by
\[
m^{a_k}(z) = \frac{F'(a_k, z)}{F(a_k, z)}.
\]
We can now define \( f(\cdot, z) \) by
\[
f(x, z) = F(b_k - x, z) \quad (0 \leq x \leq L_k)
\]
so that \( f(\cdot, z) \) satisfies the Schrödinger equation with potential \( V(b_k - x) \). Since \( F'(b_k, z)/F(b_k, z) \) has positive imaginary part, we also have \( \text{Im} \left( -f'(0, z)/f(0, z) \right) > 0 \). In this case, an application of \((\text{2})\) with \( L = L_k \) results in the estimate
\[
\gamma \left( m^{a_k}(z), i\sqrt{z} \right) < \epsilon,
\]
and it follows as before that the respective value distributions for the Herglotz functions \( m^{a_k} \) and \( i\sqrt{z} \) differ by at most \(2\epsilon |A|\), for all \( k > k_0 \).

The following Theorem summarises the situation regarding asymptotic value distribution in the case of \( L^2 \)-sparse potentials\(^1\). The Theorem implies in particular, for the special case of \( L^2 \) potentials, that the value distribution of \( v'(N, \lambda)/v(N, \lambda) \) approaches an asymptotic limit as \( N \to \infty \).

**Theorem 2.** Let \( v(\cdot, \lambda) \) be the solution of the Schrödinger equation at real spectral parameter \( \lambda \), subject to initial conditions \( v(0, \lambda) = 0, v'(0, \lambda) = 1 \), in the case of an \( L^2 \)-sparse potential \( V \).

Let \( \{ (a_k, b_k) \} \) be a sequence of subintervals of \( \mathbb{R}^+ \), for which \( \lim_{k \to \infty} (b_k - a_k) = \infty \) and \( \lim_{k \to \infty} \int_{a_k}^{b_k} |V(t)|^2 \, dt = 0 \).

Then for Borel subsets \( A, S \) of \( \mathbb{R} \), with \( |A| < \infty \), we have
\[
\lim_{k \to \infty} \frac{1}{\pi} \int_A \theta \left( m^{a_k}_+(\lambda), S \right) \, d\lambda = \frac{1}{\pi} \int_A \theta \left( i\sqrt{\lambda}, S \right) \, d\lambda,
\]
\[
\lim_{k \to \infty} \left| \left\{ \lambda \in A : \frac{v'(b_k, \lambda)}{v(b_k, \lambda)} \in S \right\} \right| = \frac{1}{\pi} \int_A \theta \left( i\sqrt{\lambda}, -S \right) \, d\lambda.
\]

The conclusion of the Theorem, which applies in the first instance in the case that \( A \) is bounded and of finite measure, may be extended to the more general case in which \( A \) is not necessarily bounded. (Let \( A \) have finite measure. Given \( \epsilon > 0 \), fix \( N \) sufficiently large that the complement of \([-N, N] \cap A \) has measure less than \( \epsilon \). Denoting by \( A_N \) this truncated set, the theorem may be

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\(^1\)We are indebted to A. Pushnitski for pointing out the close connection between estimates of \( m \)-functions at complex \( z \) and asymptotic resolvent estimates in the case of potentials with an \( L^2 \) condition.
applied first of all to $A_N$, which is bounded. Since the integrals to be estimated are then within $\epsilon$ of the corresponding integrals for the set $A$, the more general conclusion follows on letting $\epsilon$ approach zero.)

5. Spectral analysis

Here we present some consequences of Theorem 2 for the spectral theory of Schrödinger operators with $L^2$-sparse potentials. The first result implies that absolutely continuous spectrum can occur only for $\lambda > 0$.

**Corollary 2.** Suppose $V$ is $L^2$-sparse. Then the support of the a.c. measure $\mu_{ac}$ of $T = -\frac{d^2}{dx^2} + V$ is contained in $\mathbb{R}^+$.

**Proof.** Suppose the contrary. Then if $\mu_{ac}$ is the a.c. part of the spectral measure, we can find a subset $A$ of $\mathbb{R}^-$ having finite Lebesgue measure for which $\mu_{ac}(A) > 0$. Then $|A| > 0$, and we may also suppose that $A$ is a subset of an essential support of $\mu_{ac}$.

Now define intervals $(a_k, b_k)$ as in Theorem 2, and set $N_k = (a_k + b_k)/2$. Then $N_k$ may be regarded either as the left hand endpoint of an interval $(N_k, b_k)$, or as the right hand endpoint of an interval $(a_k, N_k)$. An application of Theorem 2 then implies that

$$\lim_{k \to \infty} \frac{1}{\pi} \int_A \theta \left( m_{+}^{N_k}(\lambda), S \right) d\lambda = \frac{1}{\pi} \int_A \theta \left( i\sqrt{\lambda}, S \right) d\lambda,$$

(25)

whereas

$$\lim_{k \to \infty} \left| \left\{ \lambda \in A : \frac{v'(N_k, \lambda)}{v(N_k, \lambda)} \in S \right\} \right| = \frac{1}{\pi} \int_A \theta \left( i\sqrt{\lambda}, -S \right) d\lambda.$$

(26)

Since $A$ is a subset of an essential support of $\mu_{ac}$, we also have

$$\lim_{k \to \infty} \left| \left\{ \lambda \in A : \frac{v'(N_k, \lambda)}{v(N_k, \lambda)} \in S \right\} \right| - \frac{1}{\pi} \int_A \theta \left( m_{+}^{N_k}(\lambda), S \right) d\lambda = 0.$$

(27)

(For a proof of this result, which holds for any sequence $N_k$ with $N_k \to \infty$, and for arbitrary locally $L^1$ potentials, see [BP1].) Equations (24), (25) and (27) now imply that

$$\int_A \theta \left( i\sqrt{\lambda}, S \right) d\lambda = \int_A \theta \left( i\sqrt{\lambda}, -S \right) d\lambda.$$

(28)

However $i\sqrt{\lambda} \in \mathbb{R}^-$ for $\lambda \in A$, and taking $S = \mathbb{R}^-$ we see that the left-hand-side of (28) is strictly positive, whereas the right-hand-side is zero.

Hence we have a contradiction, and the Corollary is proved.

There are interesting applications of Corollary 2 to $L^2$ perturbations of slowly oscillating potentials such as $\cos \sqrt{x}$. For example, if $V(x) = \cos \sqrt{x} + V_0$ with $V_0 \in L^2(\mathbb{R}^+)$, then $V(x) - 1$ is an $L^2$-sparse potential, and it follows from Corollary 2 that $T = -\frac{d^2}{dx^2} + V$ has no a.c. measure for $\lambda < 1$. (In fact, $[-1, 1]$ is contained in the singular spectrum of $T$; for related results on spectral theory with slowly oscillating potentials see [S].)

We can also consider various perturbations of $L^2$-sparse potentials. A typical result is the following:
Corollary 3. Let $V$ be a $L^2$-sparse potential. Define intervals $\{(a_k, b_k)\}$, with $N_k = (a_k + b_k) / 2$, as in the proof of Corollary 3. Then the Schrödinger operator
\[ -\frac{d^2}{dx^2} + V(x) + \sum_{k=1}^{\infty} \delta(x - N_k) \]
has purely singular spectral measure.

Proof. The proof follows from Theorem 2, using similar arguments to those applied in [BP1, BP2] to the special case in which $V$ is a sparse rather than $L^2$-sparse potential.

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