Second order SUSY transformations with ‘complex energies’

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Abstract

Second order supersymmetry transformations which involve a pair of complex conjugate factorization energies and lead to real non-singular potentials are analyzed. The generation of complex potentials with real spectra is also studied. The theory is applied to the free particle, one-soliton well and one-dimensional harmonic oscillator.

Key words: Second-order supersymmetry, irreducible intertwining operators, complex potentials with real spectra, generation of solvable potentials
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1 Introduction

The \( n \)-th order supersymmetric quantum mechanics (\( n \)-SUSY QM), which involves differential intertwining operators of order \( n \), is a useful tool for generating new solvable potentials [1,2,3,4]. Due to its simplicity, the 1-SUSY QM is the most explored; its nonsingular transformations produce partner potentials whose spectra can differ at most in the ground state energy level [5,6,7]. The
difficulty of ‘modifying’ the excited part of the spectrum has been surpassed through the 2-SUSY QM [8,9,10,11,12,13,14], which allows to ‘create’ two new levels $\epsilon_1$, $\epsilon_2$ between two neighboring energies $E_i$, $E_{i+1}$ of the initial Hamiltonian [15]. A similar treatment, implemented for periodic potentials [16], can be used to embed two bound states in a spectral gap above the lowest energy band [17, 18]. In both situations the corresponding 2-SUSY transformations are irreducible, i.e., when obtained as the iteration of two 1-SUSY procedures they will involve always singular intermediate potentials.

Here we will study a different set of ‘irreducible’ 2-SUSY transformations applied to non-periodic potentials, which employs two complex conjugate factorization energies $\epsilon_1$, $\epsilon_2$, $\bar{\epsilon}_2 = \bar{\epsilon}_1$. Irreducibility means now that the intermediate potential is complex although the final one is real. This problem has been addressed previously [19], but up to our knowledge the conditions granting that the final potential will be regular have not been yet examined. We will show as well that the non-singular case leads to intermediate complex potentials having real spectra. These points constitute the subject of this letter, which has been organized as follows. In section 2 the second order SUSY transformations with $\epsilon_1, \epsilon_2 \in \mathbb{C}$, $\epsilon_2 = \bar{\epsilon}_1$ will be analyzed. A prescription for avoiding the singularities in the new potential will be provided in section 3, while section 4 will be devoted to study the intermediate complex potentials. Section 5 will deal with some particular examples as the free particle, one-soliton well and harmonic oscillator potential.

2 Second order supersymmetric quantum mechanics

The standard supersymmetric (SUSY) algebra with generators $Q_1$, $Q_2$ (supercharges) and $H_{ss}$ (SUSY Hamiltonian) reads:

$$\{Q_j, Q_k\} = \delta_{jk}H_{ss}, \quad [H_{ss}, Q_j] = 0, \quad j, k = 1, 2,$$

where $[\cdot, \cdot]$ denotes a commutator and $\{\cdot, \cdot\}$ an anticommutator. The realization $Q_1 = (Q^\dagger + Q)/\sqrt{2}$, $Q_2 = (Q^\dagger - Q)/(i\sqrt{2})$ with

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix},$$

$$H_{ss} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} (\bar{H} - \epsilon_1)(H - \epsilon_2) & 0 \\ 0 & (H - \epsilon_1)(\bar{H} - \epsilon_2) \end{pmatrix},$$

$$A = \frac{d^2}{dx^2} + \eta(x)\frac{d}{dx} + \gamma(x),$$
is called second order supersymmetric quantum mechanics (2-SUSY QM). In this formalism $\tilde{H}$, $\tilde{H}$ are two intertwined Schrödinger Hamiltonians:

$$\tilde{H}A = AH, \quad (5)$$

$$H = -\frac{d^2}{dx^2} + V(x), \quad \tilde{H} = -\frac{d^2}{dx^2} + \tilde{V}(x), \quad (6)$$

and thus the real functions $\eta(x)$, $\gamma(x)$ are related with $V(x)$, $\tilde{V}(x)$ through:

$$\tilde{V} - V = 2\eta', \quad (7)$$

$$(\tilde{V} - V)\eta = 2V' + 2\gamma' + \eta'', \quad (8)$$

$$(\tilde{V} - V)\gamma = \eta V' + V'' + \gamma''. \quad (9)$$

To decouple this system, substitute (7) in (8) and integrate to obtain

$$\gamma = d - V + \eta^2/2 - \eta'/2, \quad (10)$$

where $d \in \mathbb{R}$ is a constant. By plugging (7,10) into (9), multiplying the result by $\eta$ and performing the next integration one arrives to:

$$\eta\eta'' - (\eta')^2/2 + 2\eta^2\left(\eta^2/4 - \eta' - V + d\right) + 2c = 0, \quad (11)$$

$c \in \mathbb{R}$ being another constant. This formalism is useful for generating new solvable potentials. To illustrate it, suppose that $c,d$ are fixed and $\tilde{V}(x)$ is an initial exactly solvable potential. The complete determination of $\tilde{V}(x)$ in (7) requires to find solutions $\eta(x)$ of the nonlinear second order differential equation (11). Let us find them using the Ansatz [12]:

$$\eta'(x) = \eta^2(x) + 2\beta(x)\eta(x) - 2\xi(x), \quad (12)$$

where $\beta(x)$ and $\xi(x)$ are to be determined. After substituting (12) into (11) we obtain a system of equations from which it follows that $\xi^2 = c$. The essential part of the system is the Riccati equation:

$$\beta' + \beta^2 = V - \epsilon, \quad \epsilon = d + \xi. \quad (13)$$

As there exist two possible values of $\epsilon$, $\epsilon_1 = d + \sqrt{c}$ and $\epsilon_2 = d - \sqrt{c}$ (which coincide with the factorization energies in (3)), we indeed are dealing with two equations (13) whose solutions will be denoted by $\beta_1(x)$, $\beta_2(x)$. This leads to a natural classification of the 2-SUSY transformations based on the sign of $c$. 
which will be discussed elsewhere. Here, we restrict ourselves to the complex case for which $c < 0$ and then $\epsilon_1, \epsilon_2 \in \mathbb{C}$, $\epsilon_2 = \bar{\epsilon}_1$. Since $V$ is real we can take $\beta_2(x) = \bar{\beta}_1(x)$, i.e., the problem reduces to solve the Riccati equation for $\beta_1$. In addition, using (12) we get two equivalent expressions for the real $\eta(x)$:

$$
\eta' = \eta^2 + 2\beta_1 \eta - 2i \text{Im}(\epsilon_1),
$$

$$
\eta' = \eta^2 + 2\beta_1 \eta + 2i \text{Im}(\epsilon_1).
$$

(14)

(15)

By subtracting both equations and solving for $\eta$, we obtain:

$$
\eta = \text{Im}(\epsilon_1)/\text{Im}(\beta_1).
$$

(16)

Once we know $\eta$, the 2-SUSY partner potential $\tilde{V}(x)$ is calculated using

$$
\tilde{V} = V + 2 \frac{\text{Im}(\epsilon_1)/\text{Im}(\beta_1)}{\eta}.
$$

(17)

Let us remark that the case we are dealing with has been previously explored [8]. However, we have not found any previous analysis on how to avoid the singularities in $\tilde{V}(x)$, a phenomenon which seems almost unavoidable in the complex case [19].

### 3 The non-singular 2-SUSY potentials

The simplest algorithms departing from and arriving at the exactly solvable potentials should avoid the singularities which might appear in $\tilde{V}(x)$. Notice that a singular $\tilde{V}(x)$ could be treated as a non-singular partner potential of $V(x)$ in a restricted $x$-domain. However, this would require at the end to solve the initial Schrödinger equation with modified boundary conditions losing, in general, the solvability of $H$ [20].

Let us rewrite first the formulae of section 2 in terms of solutions $u_1(x)$ of the Schrödinger equation arising from (13) by the change $\beta(x) = u'(x)/u(x)$ [14]:

$$
-u''(x) + V(x)u(x) = \epsilon u(x).
$$

(18)

Hence $\eta(x) = -2i \text{Im}(\epsilon_1)|u_1|^2/W(u_1, \bar{u}_1)$, where $W(u_1, u_2) = u_1u'_2 - u_2u'_1$ denotes the Wronskian of $u_1, u_2$. From now on it is convenient to work with the real normalized Wronskian $w(x) \equiv W(u_1, \bar{u}_1)/|2i \text{Im}(\epsilon_1)|$. Therefore:
\[
\begin{align*}
    w'(x) &= |u_1(x)|^2, \quad (19) \\
    \eta(x) &= -w'(x)/w(x), \quad (20) \\
    \tilde{V}(x) &= V(x) - 2[w'(x)/w(x)]'. \quad (21)
\end{align*}
\]

In order that \( \tilde{V}(x) \) be non-singular, \( w(x) \) must be nodeless. Since \( w(x) \) is an increasing monotonic function (see (19)), the arising of zeros is avoided if

\[
\lim_{x \to \infty} w(x) = 0 \text{ or } \lim_{x \to -\infty} w(x) = 0. \quad (22)
\]

To ensure this requirement it is sufficient that either

\[
\lim_{x \to \infty} u_1(x) = 0 \text{ or } \lim_{x \to -\infty} u_1(x) = 0. \quad (23)
\]

Such solutions are appropriate for generating non-singular potentials \( \tilde{V}(x) \). Notice that a similar treatment can be designed for systems defined in a generic interval \( x \in (a,b) \subset \mathbb{R} \) by identifying in (22-23) \(-\infty\) with \( a \) and \( \infty \) with \( b \).

4 Complex potentials with real spectrum

Although in principle irreducible, let us decompose the non-singular 2-SUSY transformations of the previous section into two 1-SUSY steps:

\[
\tilde{H} A_2 = A_2 H_1, \quad H_1 A_1 = A_1 H, \quad (24)
\]

where

\[
H_1 = -\frac{d^2}{dx^2} + V_1(x), \quad A_i = \frac{d}{dx} + \alpha_i(x), \quad i = 1, 2. \quad (25)
\]

The 1-SUSY treatment implies that \( \alpha_1, \alpha_2 \) obey the Riccati equations:

\[
\begin{align*}
    -\alpha_1' + \alpha_1^2 &= V(x) - \epsilon_1, \quad (26) \\
    -\alpha_2' + \alpha_2^2 &= V_1(x) - \tilde{\epsilon}_1, \quad (27)
\end{align*}
\]

where \( V_1(x) = V(x) + 2\alpha_1' \) and thus \( \tilde{V}(x) = V(x) + 2(\alpha_1 + \alpha_2)' \). A simple comparison of (13) with (26) leads to

\[
\alpha_1(x) = -\beta_1(x) = -u_1'(x)/u_1(x), \quad (28)
\]
\[ u_1(x) \text{ being a solution of (18) behaving asymptotically as in (23). Moreover, by expanding } A = A_2A_1 \text{ and comparing the result with (4) we find that:} \]

\[ \alpha_2 = -\alpha_1 + \eta = \beta_1 + (\epsilon_1 - \bar{\epsilon}_1)/(\beta_1 - \bar{\beta}_1). \] \hspace{1cm} (29)

This is a particular case of the finite difference Bäcklund algorithm [3, 13, 14], which algebraically determines a solution to (27) in terms of solutions of (26) for two different factorization energies (here \( \epsilon_1 \) and \( \bar{\epsilon}_1 \)). It is interesting as well to factorize the involved Hamiltonians:

\[ H = A_1^- A_1 + \epsilon_1, \quad H_1 = A_1 A_1^- + \epsilon_1, \] \hspace{1cm} (30)

\[ H_1 = A_2^+ A_2 + \bar{\epsilon}_1, \quad \bar{H} = A_2 A_2^- + \bar{\epsilon}_1, \] \hspace{1cm} (31)

\( A_i^- = -d/dx + \alpha_i(x), \quad i = 1, 2. \) Since \( \alpha_1, \alpha_2 \) and \( \epsilon_1 \) are complex, the \( A_i^- \)'s are not adjoint to the \( A_i \)'s but \( A_i^\dagger = -d/dx + \bar{\alpha}_i(x) \), \( i = 1, 2. \)

It is clear now that the complex intermediate potential \( V_1(x) \) is non-singular:

\[ V_1(x) = V(x) - 2[u_1'(x)/u_1(x)]'. \] \hspace{1cm} (32)

To analyse the normalizability of the corresponding eigenfunction associated to \( E_n \),

\[ \psi_n^i(x) = c_n A_1 \psi_n = c_n [\psi_n'(x) - u_1'(x)\psi_n(x)/u_1(x)], \] \hspace{1cm} (33)

we will employ the operator relationship:

\[ \eta A_1 = H - \epsilon_1 + A. \] \hspace{1cm} (34)

From the validity of (23) and the assumption of \( ||\psi_n|| = 1 \), it turns out that \( \bar{\psi}_n = A\psi_n/|E_n - \epsilon_1| \) is normalized, and therefore the function \( \eta A_1 \psi_n = (E_n - \epsilon_1)\psi_n + |E_n - \epsilon_1|\bar{\psi}_n \) is normalizable as well. Thus, for \( A_1 \psi_n \) to be normalizable it is necessary that \( \eta^{-1} \) does not destroy the normalizability of \( (E_n - \epsilon_1)\psi_n + |E_n - \epsilon_1|\bar{\psi}_n \). If this is the case (and this will happen for the examples we discuss below), we obtain a complex potential \( V_1(x) \) with real eigenvalues \( E_n \) [21,22,23]. Let us remark that complex Hamiltonians with real spectra have been studied recently in the context of PT-symmetry and pseudo-Hermiticity [24, 25].
5 Illustrative examples

We shall show that the previous techniques admit very simple applications.

i) Consider firstly the free particle for which \( V(x) = 0 \). The general solution \( u_1(x) \) of the Schrödinger equation (18) for \( \epsilon_1 \in \mathbb{C} \) is a linear combination of

\[
e^{\pm(k_1+ik_2)x}, \tag{35}
\]

where \( \epsilon_1 = -(k_1+ik_2)^2, \ k_1 > 0, \ k_2 \in \mathbb{R} \). In general, such a \( u_1(x) \) does not tend to zero neither when \( x \to -\infty \) nor when \( x \to +\infty \). However, two particular solutions with the required behavior are precisely those of (35). We use them for obtaining the nodeless \( w(x) \):

\[
w(x) = \pm e^{\pm 2k_1x}/(2k_1). \tag{36}
\]

It turns out that \( \tilde{V}(x) \) becomes again the null potential for both \( w(x) \), \( \tilde{V}(x) = 0 \). The intermediate 1-SUSY complex potentials generated by using the two \( u_1(x) \) of (35) are as well trivial, \( V_1(x) = 0 \). Our conclusion is that the null potential can be non-trivially transformed in frames of our algorithm only at the price of creating singularities (compare with [19]).

ii) Consider now the well known one-soliton potential (Pöschl-Teller) [26]

\[
V(x) = -2k_0^2 \text{sech}^2(k_0x) \tag{37}
\]

which is obtained from the null potential by a 1-SUSY transformation employing \( \cosh(k_0x) \), \( k_0 > 0 \). The spectrum of (37) consists of a continuous part \( E \geq 0 \) and a bound state at \( E_0 = -k_0^2 \) with eigenfunction given by:

\[
\psi_0(x) = \sqrt{k_0/2} \ \text{sech}(k_0x). \tag{38}
\]

The general solution \( u_1(x) \) of (18) for (37) with \( \epsilon_1 = -(k_1+ik_2)^2, \ k_1 > 0, \ k_2 \in \mathbb{R} \) is a linear combination of the 1-SUSY transformed eigenfunctions of (35)

\[
e^{\pm(k_1+ik_2)x}[k_0 \tanh(k_0x) \mp (k_1 + ik_2)]. \tag{39}
\]

The solutions (39) are precisely the required ones: if the upper signs are taken, then \( u_1(x) \to 0 \) for \( x \to -\infty \), while the lower signs ensure \( u_1(x) \to 0 \) when \( x \to +\infty \). An explicit calculation leads to the two nodeless \( w(x) \):
\[ w(x) = \pm \frac{k x e^{\pm 2k_1 x}}{2k_1} \cosh[k_0(x \mp x_0)] \cosh(k_0 x), \quad (40) \]

where \( k_0^2 + k_1^2 + k_2^2 \equiv k \cosh(k_0 x_0), \ 2k_0 k_1 \equiv k \sinh(k_0 x_0), \ k, x_0 \in \mathbb{R} \). The two 2-SUSY partner potentials of (37) read now:

\[ \tilde{V}(x) = -2k_0^2 \text{sech}^2[k_0(x \mp x_0)], \quad (41) \]

obtaining just real \( x_0 \)-displaced copies of (37). This result has to do with the Darboux invariance phenomenon recently discovered for the one-soliton well [17, 18].

On the other hand, the two complex intermediate potentials generated by (39) become:

\[ V_1(x) = -2k_0^2 \text{sech}^2[k_0(x \mp x_1)], \quad (42) \]

where now \( k_1 + ik_2 \equiv \kappa \cosh(k_0 x_1), \ k_0 \equiv \kappa \sinh(k_0 x_1) \) define ‘complex displacements’ \( x_1 \in \mathbb{C}, \ \kappa \in \mathbb{C} \). These potentials have a bound state at \( E_0 = -k_0^2 \) whose normalized ‘ground state’ eigenfunction is obtained from (33) by employing the \( \psi_0(x) \) of (38) and the \( u_1(x) \) of (39):

\[ \psi_0^1(x) = \frac{k_0}{|\kappa|} \left[ \frac{1}{k_2} \arctan \left( \frac{k_0 + k_1}{k_2} \right) + \frac{1}{k_2} \arctan \left( \frac{k_0 - k_1}{k_2} \right) \right]^{-\frac{1}{2}} \text{sech}[k_0(x \mp x_1)] \quad (43) \]

The complex potentials (42) were obtained for the first time in [22].

\textbf{iii) Our final example is the harmonic oscillator:}

\[ V(x) = x^2, \quad (44) \]

which has a purely discrete equidistant spectrum \( \{E_n = 2n + 1, \ n = 0, 1, \ldots \} \).

The general solution of (18) for \( \epsilon_1 \in \mathbb{C} \) is now (see [4] and references therein):

\[ u_1(x) = c_1 e^{-x^2} \left[ _1F_1 \left( \frac{1 - \epsilon_1}{4}, \frac{1}{2}; x^2 \right) + 2\nu x \frac{\Gamma(\frac{3 - \epsilon_1}{4})}{\Gamma(\frac{3 + \epsilon_1}{4})} _1F_1 \left( \frac{3 - \epsilon_1}{4}, \frac{3}{2}; x^2 \right) \right], \quad (45) \]

where \( _1F_1(a, c; z) \) is the Kummer hypergeometric series. In general, such a \( u_1(x) \) does not satisfy neither \( \lim_{x \to -\infty} u_1(x) = 0 \) nor \( \lim_{x \to \infty} u_1(x) = 0 \). However, there are two particular values for \( \nu \ (\nu = \pm 1) \) leading to solutions with the required behavior:
\[ u_1(x) = e^{-\frac{x^2}{2}} \left[ \, _1 F_1 \left( \frac{1 - \epsilon_1}{4}, \frac{1}{2}; x^2 \right) \pm 2x \frac{\Gamma(\frac{3 - \epsilon_1}{4})}{\Gamma(1 - \epsilon_1)} \, _1 F_1 \left( \frac{3 - \epsilon_1}{4}, \frac{3}{2}; x^2 \right) \, \right]. \] (46)

Take, e.g., (46) with the upper sign, which in the negative semiaxis \( x = -|x| < 0 \) reduces to:

\[ u_1(x) = \frac{\Gamma(\frac{3 - \epsilon_1}{4})}{\Gamma(\frac{1 - \epsilon_1}{4})} e^{-\frac{|x|^2}{2}} \Psi \left( \frac{1 - \epsilon_1}{4}, \frac{1}{2}; |x|^2 \right), \] (47)

where the Tricomi function \( \Psi(a, c; z) \) is related with \( _1 F_1(a, c; z) \) through (see, e.g., [27]):

\[
\Psi(a, c; z) = \frac{\Gamma(1 - c)}{\Gamma(a - c + 1)} _1 F_1(a, c; z) + \frac{\Gamma(c - 1)}{\Gamma(a)} z^{1-c} _1 F_1(a - c + 1, 2 - c; z) \] (48)

Since the leading term in the asymptotic expansion for \( \Psi(a, c; z) \) is \( z^{-a} \) [28], one can check that \( \lim_{x \to -\infty} u_1(x) = 0 \). Similarly, if the lower sign is chosen we have \( \lim_{x \to \infty} u_1(x) = 0 \).

**Fig. 1.** The real potential \( \tilde{V}(x) \) (black curve) generated from \( V(x) = x^2 \) (gray curve) by means of a ‘complex’ 2-SUSY transformation with \( \epsilon_1 = 10 + 0.1i \) and the \( u_1(x) \) of (46) with the lower minus sign.

Once we have identified the solutions (46) with the right asymptotic behavior, we evaluated \( w(x) \) and then \( \tilde{V}(x) \). The resulting expressions in this case are too long; instead, we are plotting \( \tilde{V}(x) \) for \( \epsilon_1 = 10 + 0.1i \) using (46) with the lower minus sign (see figure 1). Contrasting with the results for the previous examples, in this case the potentials \( \tilde{V}(x) \) are in general different from \( V(x) \). This means that the transformations involving a pair of complex conjugate factorization energies are effective tools in generating isospectral 2-SUSY partner potentials. As a byproduct, we have obtained in a simple way complex potentials \( V_1(x) \) given by (32) with real energy eigenvalues \( E_n = 2n + 1 \). A plot of the ‘ground state’ probability density \( |\psi_0^1(x)|^2 \), illustrating the existence of these bound states for the complex 1-SUSY partner \( V_1(x) \) of the oscillator, is shown in figure 2.
Fig. 2. The ‘ground state’ probability density $|\psi_1^0(x)|^2$ for the complex 1-SUSY partner potential (32) of the oscillator generated by using $u_1(x)$ of (46) with the lower sign and $\epsilon_1 = 10 + 0.1i$.

6 Conclusions

We have shown that the 2-SUSY transformations involving two complex conjugate factorization energies can produce new non-singular potentials isospectral to a given initial one. This non-singular character is shared as well by the intermediate complex potentials arising when those transformations are factorized.

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