Nonlinear Description of Quantum Dynamics.  
N-level quantum systems

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Abstract. In this work a nonlinear evolution of pure states of a finite dimensional quantum system is introduced, in particular a Riccati evolution equation. It is shown how this class of dynamics is actually a Hamiltonian dynamics in the complex projective space. In this projective space it is shown that there is a nonlinear superposition rule, consistent with its linear counterpart in the Hilbert space. As an example, the developed nonlinear formalism is applied to the semiclassical Jaynes–Cummings model.

Keyword: Quantum dynamics, Nonlinear evolution, Riccati equation, Nonlinear superposition rule, Hamiltonian evolution, Jaynes–Cummings model.

1. Introduction

The nonlinear description of quantum phenomena has currently gained considerable interest [1, 2, 3, 4], not only because this constitutes an alternative description of quantum theory, but also because this evolution presents interesting properties allowing a better understanding of quantum theory itself. For example for the case of $N$-level systems the nonlinear description, specifically a Riccati evolution, has been proposed and studied in [1] and also taken into account in spin Hamiltonian systems in [5].
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The main purpose of this work is to show that an interesting application of the statistical-probabilistic contents of quantum mechanics is the nonlinear description of an \( N \)-level quantum system by means of the nonlinear Riccati equation, as an alternative description of quantum evolution [1]. This constitutes a transition from linear Schrödinger dynamics to a nonlinear Riccati equation. To introduce such a transition, it is first recalled that a pure state in quantum mechanics is an equivalence class, i.e. a ray in the Hilbert space, where the equivalence class may be described infinitesimally by two vector fields in involution: the dilation vector field \( \Delta \) and the vector field \( \Gamma \) associated with the multiplication by a phase factor. In addition, it is possible to see that these vector fields are symmetries of the Schrödinger dynamics because \( \Delta \) and \( \Gamma \) permute with solutions of the Schrödinger equation [11]. This fact allows to reduce the dynamics to a lower dimensional space, specifically the complex projective space \( \mathbb{C}P \) of the Hilbert space, an example of a Hilbert manifold.

Usually in quantum mechanics, pure states are considered to be rank-one projectors and therefore are studied as elements of the space of Hermitian, non-negative and trace one operators. However, in this paper we stick to the Hilbert manifold, i.e., the complex projective space, to stress its intrinsic “non-linearity”. For the analysis on \( \mathbb{C}P \), it is used that it is possible to give a complete description of the complex projective space by means of complex homogeneous coordinates [13, 10], such that in those coordinates the Schrödinger dynamics is projected onto the Riccati dynamics. Furthermore, it is shown that not only the dynamics is projectable but also important geometrical structures may be projected. In particular, the Khäler structure on the Hilbert space is also presented in the complex projective space [13, 14, 10]. This structure allows to show that the nonlinear Riccati evolution is actually Hamiltonian.

It is noticed that in the nonlinear description the evolution of pure states is described by solutions of a Riccati equation that, although it is nonlinear, originates from the linear Schrödinger equation with linear superposition rule, thus there must be a way to superimpose solutions of the Riccati equation, which leads to a nonlinear superposition rule in \( \mathbb{C}P \) [7, 8, 9].

Finally, as an example of nonlinear evolution of quantum systems, the formalism developed is applied to the semiclassical Jaynes–Cummings model of quantum optics, where this model consists of a single two-level atom irradiated by a laser source, i.e. the radiation is considered as a classical field. Furthermore, it is shown that the solution of the associated Riccati equation is connected with Möbius transformations in \( \mathbb{C}P \).

2. From the Linear Schrödinger Equation to a Nonlinear Riccati Evolution

For simplicity the study starts with 2-level systems, also known as q-bit systems, and later the results will be extended to \( N \)-level systems. The quantum evolution for a 2-level system on the Hilbert space is obtained from the solutions of the Schrödinger
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Equation

$$i\hbar \left( \begin{array}{c} \dot{\psi}_1 \\ \dot{\psi}_2 \end{array} \right) = \left( \begin{array}{cc} H_1 & V \\ \bar{V} & H_2 \end{array} \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right),$$

where $H_1, H_2 \in \mathbb{R}$ and $V \in \mathbb{C}$ are possibly time-dependent functions. Alternatively, defining the Hilbert space $\mathcal{H}_0 = \mathcal{H} - \{0\}$, i.e., the Hilbert space $\mathcal{H}$ with the origin removed, one may introduce homogeneous coordinates

$$\pi : \mathcal{H}_0 \rightarrow \mathbb{C} : (\psi_1, \psi_2) \mapsto z = \frac{\psi_1}{\psi_2},$$

which associates to each point in $\mathcal{H}_0$ a point in the complex plane. It is clear that the transformation $\pi$ is not defined for states with $\psi_2 = 0$; however, in order to take into account those states one simply may consider a different chart, i.e., the transformation

$$\tilde{\pi} : \mathcal{H}_0 \rightarrow \mathbb{C} : (\psi_1, \psi_2) \mapsto \zeta = \frac{\psi_2}{\psi_1}. \tag{3}$$

Then, it can be proven that the dynamics in the Hilbert space $\mathcal{H}_0$, given by the Schrödinger equation (1), induces a nonlinear dynamics in the complex plane given by the Riccati equation

$$\dot{z} = \frac{i}{\hbar} \left[ \bar{V} z^2 - (H_1 - H_2) z - V \right], \tag{4}$$

or equivalently for $\zeta = 1/z$ the Riccati equation

$$\dot{\zeta} = \frac{i}{\hbar} \left[ V \zeta^2 + (H_1 - H_2) \zeta - \bar{V} \right]. \tag{5}$$

These equations constitute nonlinear evolution equations for quantum states.

Before addressing the dynamical properties of the Riccati equations (4) and (5), we will establish the geometrical nature of the transformations (2) and (3). For this purpose it is necessary to introduce some geometrical structures in quantum mechanics [13, 10].

It is well-known that a complete measurement in quantum mechanics does not provide us with a uniquely defined vector in the Hilbert space, but rather with an equivalence class of vectors, obtained by multiplication with a complex number, i.e., on the Hilbert space there is a natural action of the Abelian group $\mathbb{C}_0 = \mathbb{C} - \{0\}$ given by

$$|\psi\rangle \mapsto \lambda |\psi\rangle = \rho e^{i\theta} |\psi\rangle \text{ with } \rho > 0. \tag{6}$$

A pure state is an equivalence class, i.e., a ray in the Hilbert space.

Thus, considering an N-level quantum system with Hilbert space $\mathcal{H}_0$ and selecting an orthonormal basis $\{ |e_k\rangle \}_{k=1,\ldots,n}$ in $\mathcal{H}_0$, one may introduce a Cartesian coordinate system $\{ x^k, y^k \}_{k=1,\ldots,n}$ on $\mathcal{H}_0$, namely for any element $|\psi\rangle \in \mathcal{H}_0$ one has that

$$|\psi\rangle = \psi^k |e_k\rangle = (x^k + i y^k) |e_k\rangle. \tag{7}$$

Then, the group action defining the equivalence class may be described infinitesimally by means of two commuting linear vector fields, given in the cartesian coordinates as

$$\Delta = x^k \frac{\partial}{\partial x^k} + y^k \frac{\partial}{\partial y^k} \text{ and } \Gamma = y^k \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial y^k}. \tag{8}$$

† The overbar is used along this work to denote the complex conjugate of a complex quantity.
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where $\Delta$ is the infinitesimal generator of dilations, while $\Gamma$ is the infinitesimal generator of the multiplication by a global phase factor.

Then, on the space $\mathcal{H}_0$ there are two regular distributions related with $\Delta$ and $\Gamma$. The first one is $D = \{ \Delta \}$, and it is possible to see that the quotient space $\mathcal{H}_0/\Phi^D$ may be represented by the unit sphere in $\mathcal{H}_0$, namely

$$S^{2n-1} := \{ |\psi\rangle \in \mathcal{H}_0 | \langle \psi | \psi \rangle = 1 \}.$$ \hfill (9)

In the following, the projection from $\mathcal{H}_0$ to $S^{2n-1}$ is denoted by $\tau$, while an element of $S^{2n-1}$ will be denoted as $|\psi\rangle$, whereas $|\psi\rangle$ is a vector in $\mathcal{H}_0$.

If one now considers the distribution $D_2 = \{ \Gamma, \Delta \}$, which is involutive, it gives rise to a foliation $\Phi^{D_2}$ which is regular. The quotient space $\mathcal{H}_0/\Phi^{D_2}$ is known as the complex projective space $\mathbb{CP}(\mathcal{H}_0)$ associated with $\mathcal{H}_0$, namely

$$\mathbb{CP}(\mathcal{H}_0) := \{ \lambda |\psi\rangle | \lambda \in \mathbb{C} \}.$$ \hfill (10)

The projection from $\mathcal{H}_0$ to $\mathbb{CP}(\mathcal{H}_0)$ is denoted by $\pi$ and the elements of $\mathbb{CP}(\mathcal{H}_0)$, denoted by $[\psi]$ with $|\psi\rangle \in \mathcal{H}_0$, $[\psi]$ are identified with the pure states of a quantum system. They are in one-to-one correspondence with rank-one projectors, i.e.,

$$[\psi] \mapsto \rho_\psi := \frac{|\psi\rangle \langle \psi|}{\langle \psi| \psi \rangle}.$$ \hfill (11)

Furthermore, because the vector field $\Gamma$ is tangent to $S^{2n-1}$, one may consider its restriction $\Gamma_s$ to $S^{2n-1}$ and build the distribution $D_s$ associated with it. So, the quotient space $S^{2n-1}/\Phi^{D_s}$ is precisely the complex projective space $\mathbb{CP}(\mathcal{H}_0)$, and the canonical projection from $S^{2n-1}$ to $\mathbb{CP}(\mathcal{H}_0)$ is denoted by $\upsilon$, and hence it holds that $\pi = \upsilon \circ \tau$. Then, one arrives at the following diagramme

$$\begin{array}{ccc}
\mathcal{H}_0 & \xrightarrow{\tau} & S^{2n-1} \\
\downarrow \pi & & \downarrow \upsilon \\
\mathbb{CP}(\mathcal{H}_0) & & 
\end{array}$$

Therefore, $\mathbb{CP}(\mathcal{H}_0)$ is the space of physical states where one has normalized and gotten rid of the global phase, i.e., we are aiming at a description on the space of pure states described without redundancies.

2.1. \textit{Q-bit system}

To better visualize the situation, let us again consider a 2-level quantum system. In this case the Hilbert space is isomorphic to $\mathbb{C}^2$ where the vectors are given by

$$|\psi\rangle = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix},$$ \hfill (12)
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with $\psi^k \in \mathbb{C}$. Then, considering the system of coordinates in (7), the infinitesimal generators of dilations and multiplication by a phase factor are given by

$$\Delta = x^1 \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial y^1} + x^2 \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial y^2}$$

(13)

and

$$\Gamma = y^1 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial y^1} + x^2 \frac{\partial}{\partial x^2} - y^2 \frac{\partial}{\partial y^2},$$

(14)

respectively. Notice that the integral curves of the linear vector field $\Delta$ constitute a set of disjoint lines passing through each point of $H_0$; then, the family of these lines is a foliation $\Phi^\Delta$ of the space $H_0 \approx \mathbb{C}^2$. Consequently, the leaves of this foliation introduce equivalence relations, which define the quotient space $H_0/\Phi^\Delta$ isomorphic to the space of normalized states $S^3$.

In this case the unit sphere $S^3 \subset \mathbb{C}^2$ of normalized states is given by the normalized vectors

$$\tau(|\psi\rangle) = |\psi\rangle = \frac{1}{\sqrt{\psi^1\psi^1 + \psi^2\psi^2}} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}. \quad (15)$$

On the other hand, because the linear vector field $\Gamma$ is tangent to $S^3$, the integral curves of $\Gamma$ foliate the sphere $S^3$ introducing an equivalence relation with corresponding quotient space $S^3/\Phi^\Gamma$, i.e., the complex projective space $\mathbb{CP}(H_0)$. As we will see next, the complex projective space may be thought of as the unit sphere $S^2 \subset \mathbb{R}^3$ and one may establish the map $\nu : S^3 \to S^2$, known in mathematics as the Hopf fibration [10].

In summary, we have the covering map $\pi = \nu \circ \tau : H_0 \to S^2$. One associates a pure state

$$\rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} = |\psi\rangle\langle\psi|$$

(16)

with a vector $|\psi\rangle$. In coordinates we have that

$$\pi(|\psi\rangle) = \rho_\psi = \frac{1}{\psi^1\psi^1 + \psi^2\psi^2} \begin{pmatrix} \psi^1 \overline{\psi^1} & \psi^1 \overline{\psi^2} \\ \psi^2 \overline{\psi^1} & \psi^2 \overline{\psi^2} \end{pmatrix}. \quad (17)$$

In general, one may immerse the complex projective space for the $q$-bit in the space of $2 \times 2$ Hermitian matrices where a basis is provided by the Pauli matrices and the identity matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (18)$$

and

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (19)$$

i.e. every density matrix may be expressed as

$$\rho = \frac{1}{2}(\mathbb{I} + x^j\sigma_j). \quad (20)$$
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Figure 1. The stereographic projection from the north pole onto the equatorial plane, which maps every point \( x = (x^1, x^2, x^3) \in S^2 \) into points in the complex plane \( z \in \mathbb{C} \).

This means that every quantum state is represented by the unit vector \( (x^1, x^2, x^3) \in S^2 \), such that \( x^j = \text{Tr}\{\sigma_j \rho\} \), and the purity condition \( \rho^2 = \rho \) identifies the unit sphere
\[
S^2 = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}.
\] (21)

However, the description of the complex projective space is complete by means of complex homogeneous coordinates, this is, let \( U_j \subset \mathbb{C}P \), with \( j = 1, 2 \), denote the coordinate patch where \( \psi^j \neq 0 \) defined by
\[
\phi_1 : [\psi^1, \psi^2] \mapsto z = \frac{\psi^1}{\psi^2} \quad \text{and} \quad \phi_2 : [\psi^1, \psi^2] \mapsto \zeta = \frac{\psi^2}{\psi^1},
\] (22)
such that the set of \( (U_j, \phi_j) \) constitutes an atlas for the complex projective space. Thus, alternative expression of the vector states \( |\psi\rangle \in S^3 \) and its image \( \rho_\psi \in S^2 \) is
\[
|\psi\rangle = \frac{e^{i\varphi}}{\sqrt{1 + z \bar{z}}} \begin{pmatrix} z \\ 1 \end{pmatrix}, \quad \rho_\psi = |\psi\rangle\langle\psi| = \frac{1}{1 + z \bar{z}} \begin{pmatrix} z \bar{z} & z \\ \bar{z} & 1 \end{pmatrix},
\] (23)
where \( \varphi \) is a real and possibly time-dependent function. Comparing this expression of the density matrix with the one given in Eq. (20) one obtains the transformation
\[
x^1 = \frac{2z_R}{1 + z \bar{z}}, \quad x^2 = \frac{2z_i}{1 + z \bar{z}}, \quad x^3 = \frac{-1 + z \bar{z}}{1 + z \bar{z}},
\] (24)
where \( z = z_R + i z_i \).

This transformation is simply the stereographic projection from the “north pole” of the sphere onto the equatorial plane, see Fig. 1. For the map \( \zeta = \frac{z^2}{\bar{z}^2} \), one obtains the stereographic projection from the “south pole” of the sphere onto the equatorial plane, i.e.
\[
x^1 = \frac{2\zeta_R}{1 + \zeta \bar{\zeta}}, \quad x^2 = \frac{-2\zeta_i}{1 + \zeta \bar{\zeta}}, \quad x^3 = \frac{1 - \zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}},
\] (25)
with \( \zeta = \zeta_R + i \zeta_i \). Therefore, we have a complete description of \( S^2 \). The obtained complex plane \( \mathbb{C}P \) is known in quantum physics as the ray space description of the quantum system.
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For the general case, i.e., for an N-level quantum system, it is straightforward to adapt the complex homogeneous coordinates. This is, let $U_j \subset \mathbb{CP}^n$ denote the coordinates patch defined by

$$
\phi_j : U_j \to \mathbb{C}^{n-1} : [\psi_1, \ldots, \psi_n] \mapsto (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n),
$$

with

$$
z^k = \frac{\psi^k}{\psi^j}.
$$

for $\psi^j \neq 0$, then the set of $(U_j, \phi_j)$, with $j = 1, \ldots, n$, constitutes an atlas for the complex projective space. Therefore the projection of $\mathcal{H}_0$ onto the complex projective space may be expressed as

$$
\pi : \mathcal{H}_0 \to \mathbb{CP} : |\psi\rangle \mapsto |\psi\rangle = \frac{1}{\sqrt{1 + z^\dagger z}} \begin{pmatrix} z \\ 1 \end{pmatrix},
$$

where the chart $\phi_n$ has been employed, so $z$ is an $(N-1)$-dimensional complex vector, with components $z^k = \frac{\psi^k}{\psi^n}$ for $k = 1, \ldots, n-1$.

Now that one has established the canonical projection $\pi$ from the Hilbert space $\mathcal{H}_0$ to the complex projective space $\mathbb{CP}$, one may apply this results to the dynamics of the system. The Schrödinger dynamics defines a dynamical vector field $X_H$ in the carrier space $\mathcal{T}_H \approx \mathcal{H} \times \mathcal{H}$, given by

$$
X_H = \frac{i}{\hbar} \psi_j J^i_k \frac{\partial}{\partial \psi_k} - \frac{i}{\hbar} H^i_j \psi^j \frac{\partial}{\partial \psi^k}
$$

with $H^i_j$ being an entry of the Hamiltonian operator. Therefore, a major question is whether there is a vector field which describes the Schrödinger motion on $\mathbb{CP}$, or as is said, whether $X_H$ is projectable. In fact, a sufficient condition for projectability (see reference [11] for the formal proof) is

$$
[X_H, \Delta] = [X_H, \Gamma] = 0.
$$

It is interesting to mention that because the dynamics is projectable, it carries leaves of the foliation $\Phi^{D_2}$, into leaves, i.e. the foliation is invariant under $X_H$ [11]. In this sense the group action is a symmetry for the Schrödinger dynamics.

3. Nonlinear Evolution of N-level Quantum Systems

It has been shown that the complex projective space constitutes the space of quantum states, where one has normalized and gotten rid of the global phase. In particular, for the q-bit system we found that the evolution of the states is, in homogeneous coordinates, given by a nonlinear Riccati equation. In this section, this result is generalized to any N-level system as well as, in some details, the structure of the complex projective space of the Hilbert space is discussed.

It is well-known that the Hilbert space $\mathcal{H}_0$ is a Kähler manifold [13, 10], i.e., there are a symplectic form $\omega_\mathcal{H}$, a Riemannian metric tensor $g_\mathcal{H}$ and a complex structure $J_\mathcal{H}$...
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such that $g_H = J_H \circ \omega_H$. Hence, it is expected that similar structures are defined on the complex projective space $\mathbb{CP}$. In fact, considering the homogeneous coordinates in Eq. (28) let us proceed to introduce the 1-form

$$\theta_{FS} = \frac{\hbar}{i} \left( \frac{1}{1 + \bar{z}z} \right) \left[ 1 + \frac{1}{\sqrt{1 + \bar{z}z}} \begin{pmatrix} dz \\ 0 \end{pmatrix} + \left( \frac{1}{\sqrt{1 + \bar{z}z}} \right) d \left( \frac{1}{\sqrt{1 + \bar{z}z}} \right) \right]$$

Thus, the symplectic form $\omega_{FS}$ on $\mathbb{CP}$ is then defined as

$$\omega_{FS} = \frac{\hbar}{i} (d\psi | \wedge | d\psi)$$

$$= \left[ (1 + \bar{z}z) dz_k \wedge dz^k + \frac{1}{2} (\bar{z}_k dz^k - z^k d\bar{z}_k) \otimes (\bar{z}_l dz^l - z^l d\bar{z}_l) \right].$$

Furthermore, by means of the map $\pi$ in (28), one may look at the pullback of $\omega_H$ to $\mathcal{H}_0$ and see that $\omega_H = \pi^* \omega_{FS}$. The same is also true for the metric, i.e., $g_H = \pi^* g_{FS}$, where $g_{FS}$ is the so-called Fubini–Study metric [13], which in homogeneous coordinates has the form

$$g_{FS} = (d\psi | \otimes | d\psi)$$

$$= \left[ (1 + \bar{z}z) dz_k \otimes dz^k + \frac{1}{2} (\bar{z}_k dz^k - z^k d\bar{z}_k) \otimes (\bar{z}_l dz^l - z^l d\bar{z}_l) \right],$$

where $dz_k \otimes dz^k = d\bar{z}_k dz^k + dz_k d\bar{z}_k$. On the other hand, the complex structure on the complex projective space may be obtained by a $(1,1)$-tensor containing vector fields and forms. While vector fields may be projected (if projectable), forms cannot be projected. Therefore one may speak of “related” $(1,1)$-tensor fields by a related $(1,1)$-tensor field

$$J_{FS} = \frac{1}{i} \left( dz^k \otimes \frac{\partial}{\partial z_k} - d\bar{z}_k \otimes \frac{\partial}{\partial \bar{z}_k} \right).$$

Therefore, the complex projective space has a Kähler structure $(\omega_{FS}, g_{FS}, J_{FS})$ [13, 14].

The symplectic form and the Riemannian metric define a Hamiltonian vector field and a gradient vector field, respectively. This means to any quadratic function $e_A \in \mathfrak{g}(\mathbb{CP})$ a gradient and a Hamiltonian vector field is associated by means of the intrinsic definitions

$$g_{FS}(Y_{e_A}, \cdot) = d e_A, \quad \omega_{FS}(X_{e_A}, \cdot) = d e_A,$$

such that

$$J_{FS}(X_{e_A}) = Y_{e_A},$$

which provides an intrinsic global definition of the complex-structure tensor.
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To introduce the definitions in Eq. (35) in complex homogeneous coordinates on $\mathbb{CP}$, we consider the expectation value of an arbitrary observable $A$ which may be expressed in general as

$$e_A = (\psi|A|\psi)$$

$$= \frac{1}{1 + z^\dagger z} \left( z^\dagger (A_1 V + V^\dagger z + A_2) \right),$$

where the matrix $A_1$ is an $(N - 1) \times (N - 1)$-dimensional matrix, $V$ is an $(N - 1)$-component column vector and $A_2$ a real quantity. In particular we can consider the Hamiltonian as the observable with representative matrix

$$\mathcal{H} = \begin{pmatrix} H_1 & V \\ V^\dagger & H_2 \end{pmatrix}.$$ (38)

Thus, the Hamiltonian vector field may be obtained from the right-hand expression in Eq. (35), taking into account the expression in coordinates of the symplectic form in Eq. (32). Namely, one has that the dynamics induced by the Hamiltonian corresponds to

$$\dot{z}^k = i\hbar \left( z^k \partial_{\bar{z}^l} (z^l - |\mathcal{H}|_{l|k}^{\dagger} z^l + H_2 z^k - V^k) \right).$$ (40)

Therefore, the integral curves of this Hamiltonian vector field are provided by the Hamiltonian equations of motion

$$\dot{z}^k = -i \hbar \left( z^k \partial_{\bar{z}^l} \left( \frac{\partial e_H}{\partial z^k} + z^k \partial_{\bar{z}^l} \right) \right)$$

$$= i \hbar \left( z^k \partial_{\bar{z}^l} (z^l - |\mathcal{H}|_{l|k}^{\dagger} z^l + H_2 z^k - V^k) \right),$$

$$\dot{\bar{z}}^k = i \hbar \left( z^k \partial_{\bar{z}^l} \left( \frac{\partial e_H}{\partial z^k} + \bar{z}^l \partial_{\bar{z}^l} \right) \right)$$

$$= -i \hbar \left( \bar{z}^l \partial_{\bar{z}^l} (z^k - |\mathcal{H}|_{l|k}^{\dagger} z^l + H_2 \bar{z}^k - \bar{V}_k) \right).$$ (41)

Therefore the equation of motion of the $N$-level quantum system is given by the well-known matrix Riccati equation [1]

$$\dot{z}^k = i \hbar \left( z^k \partial_{\bar{z}^l} (z^l - |\mathcal{H}|_{l|k}^{\dagger} z^l + H_2 z^k - V^k) \right).$$ (42)

Hence, the matrix Riccati equation is simply the coordinate expression of the Schrödinger equation on the complex-projective space. Moreover, it has been proven that the nonlinear Riccati evolution is actually a Hamiltonian dynamics.
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The gradient vector field can be obtained similarly, i.e., considering now the left-hand expression in Eq. (35) and the expression in coordinates of the Riemannian metric in Eq. (33). However, because the Hamiltonian vector field and the complex structure have been already determined; then, it is possible to obtain the gradient vector field from the property in Eq. (36). Hence the gradient vector field is simply

\[ Y_{\epsilon H} = J_{FS}(X_{\epsilon H}) \]
\[ = \frac{1}{\hbar} \left[ z^k \bar{V}_l z^l - |H_1|^k z^k + H_2 z^k - V^k \right] \frac{\partial}{\partial z^k} \]
\[ + \frac{1}{\hbar} \left[ \bar{z}_l V^l \bar{z}_k - \bar{z}_l |H_1|^l \bar{z}_k + H_2 \bar{z}_k - \bar{V}_k \right] \frac{\partial}{\partial \bar{z}_k}. \] (43)

In addition, one may introduce the Poisson and Jordan brackets for the complex projective space [10, 13]. So, given the expectation values \( e_A \) and \( e_B \) associated to the observables \( A \) and \( B \)

\[ \{e_A, e_B\}_{\text{WFS}} = \omega_{\text{WFS}}(X_{e_A}, X_{e_B}) \]
\[ \{e_A, e_B\}_{\text{GPS}} = g_{FS}(Y_{e_A}, Y_{e_B}), \] (44)

where in complex homogeneous coordinates these brackets have the form

\[ \{e_A, e_B\}_{\text{WFS}} = -\frac{i}{\hbar} (1 + z^l \bar{z}) \left[ \left( \frac{\partial e_A}{\partial z^k} \frac{\partial e_B}{\partial \bar{z}_k} - \frac{\partial e_A}{\partial \bar{z}_k} \frac{\partial e_B}{\partial z^k} \right) \right. \]
\[ \left. + \left( z^k \frac{\partial e_A}{\partial z^k} \bar{z}_l \frac{\partial e_B}{\partial \bar{z}_l} - \bar{z}_l \frac{\partial e_A}{\partial \bar{z}_l} z^k \frac{\partial e_B}{\partial z^k} \right) \right], \] (45)

and

\[ \{e_A, e_B\}_{\text{GPS}} = -\frac{1}{\hbar} (1 + z^l \bar{z}) \left[ \left( \frac{\partial e_A}{\partial z^k} \frac{\partial e_B}{\partial \bar{z}_k} + \frac{\partial e_A}{\partial \bar{z}_k} \frac{\partial e_B}{\partial z^k} \right) \right. \]
\[ \left. + \left( z^k \frac{\partial e_A}{\partial z^k} \bar{z}_l \frac{\partial e_B}{\partial \bar{z}_l} + \bar{z}_l \frac{\partial e_A}{\partial \bar{z}_l} z^k \frac{\partial e_B}{\partial z^k} \right) \right]. \] (46)

Further, after some calculations it is possible to prove that the Poisson and the Jordan brackets are such that

\[ \{e_A, e_B\}_{\text{WFS}} = e \frac{2}{\hbar} \epsilon_{[A,B]} \]
\[ \{e_A, e_B\}_{\text{GPS}} = -\frac{2}{\hbar} \epsilon_{[A,B]+} - e_A e_B, \] (47)

where \([A,B]_+ = AB - BA\) and \([A,B]_- = AB + BA\). Therefore, one has a clear connection between the Poisson brackets and the quantum commutator. In particular, considering the Hamiltonian of the system, \( H \), with expectation value \( e_H \), then the evolution of the expectation value \( e_A \) of an arbitrary observable \( A \) is given by

\[ \frac{de_A}{dt} = \{e_H, e_A\}_{\text{WFS}} = e \frac{2}{\hbar} \epsilon_{[H,A]}_-. \] (48)

This result implies immediately that for the time-independent case \( e_H \) is a first integral of the flow, i.e., the expectation value of the Hamiltonian is conserved. In addition, the expectation value of any observable commuting with \( H \) is also a first integral.

On the other hand, the Jordan bracket is connected with the dispersion and the correlation of the observables. This means, for every couple of observables \( A \) and \( B \) their uncertainties and correlations are given by

\[ \sigma_A^2 = e_A^2 - e_A^2 = -\frac{\hbar}{2} \{e_A, e_A\}_{\text{GPS}} \] (49)
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and

\[ \sigma_{AB} = \frac{1}{2} e^{[A,B]} - e_A e_B = -\frac{\hbar}{2} \{ e_A, e_B \} \theta_{FS}, \]

respectively. Thus, the Riemannian metric carried by the complex projective space takes into account the probabilistic character of quantum mechanics [15].

3.1. Q-bit system

For the q-bit example the expectation value of the Hamiltonian is expressed in homogeneous coordinates by

\[ e_H = \frac{1}{1 + z \bar{z}} \left( \bar{z}, 1 \right) \left( \begin{array}{c} H_1 \\ V \\ \bar{H}_2 \end{array} \right) \left( \begin{array}{c} z \\ 1 \end{array} \right) = \frac{1}{1 + z \bar{z}} \left( \bar{z}H_1 z + \bar{z}V + \bar{V}z + H_2 \right). \]

The Hamiltonian dynamics intrinsically defined in (35) may be found explicitly employing Hamilton’s equations in Eq. (41). Then, Hamilton’s equations of motion are given by

\[ \dot{z} = -\frac{i}{\hbar} \left( 1 + z \bar{z} \right)^2 \frac{\partial e_H}{\partial \bar{z}} = -\frac{i}{\hbar} \left[ \bar{V}z^2 - (H_1 - H_2)z - V \right] \]

\[ \dot{\bar{z}} = \frac{i}{\hbar} \left( 1 + z \bar{z} \right)^2 \frac{\partial e_H}{\partial z} = \frac{i}{\hbar} \left[ V \bar{z}^2 - (H_1 - H_2)\bar{z} - \bar{V} \right], \]

and are identical with the Riccati equation (4). Therefore the Hamiltonian vector field has the form

\[ X_{eH} = \frac{i}{\hbar} \left[ \bar{V}z^2 - (H_1 - H_2)z - V \right] \frac{\partial}{\partial z} - \frac{i}{\hbar} \left[ \bar{V}z^2 - (H_1 - H_2)\bar{z} - \bar{V} \right] \frac{\partial}{\partial \bar{z}}. \]

Furthermore, with the help of the complex structure one may obtain the gradient vectorfield, i.e.

\[ Y_{eH} = J_{FS}(X_{eH}) = 1 \frac{\partial}{\partial z} + \bar{V} \frac{\partial}{\partial \bar{z}}, \]

which is orthogonal to the Hamiltonian vector field.

3.2. Time-dependent Hamiltonian systems

For the N-level quantum system whose Hamiltonian depends explicitly on time the definition (35) for the Hamiltonian vector field is no longer valid, because the differential of the expectation value of the Hamiltonian depends on time. In order to deal with time-dependent systems, as in classical Hamiltonian theory [16], one usually extends the space with an extra dimension representing time. The extended space \( \mathbb{CP}^{\mathbb{R}} = \mathbb{CP} \times \mathbb{R} \) is a manifold endowed with the 1-form

\[ \theta_{FS}^E = \theta_{FS} + e_H dt, \]
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where \( \theta_{FS} \) is the one-form defined in (31) and the expectation value \( e_H \) may depend explicitly on time. Notice that in this manner one is dealing with a contact Hilbert manifold [17, 18]. Then one proceeds to define a dynamics on \( \mathbb{CP}^E \) that correctly extends the Hamiltonian dynamics. A direct calculation shows that the condition

\[
\text{d}\theta_{E}^F(X_{e_H}^E, \cdot) = 0
\]

is satisfied if and only if the extended vector field \( X_{e_H}^E \) takes the form

\[
X_{e_H}^E = X_{e_H} + \frac{\partial}{\partial t},
\]

where \( X_{e_H} \) is given by Eq. (39). Therefore, one obtains the Hamiltonian equations (41), augmented with the trivial equation \( \dot{t} = 1 \). It follows that the evolution of an arbitrary time-dependent expectation value of \( e_A \) is given by

\[
\frac{\text{d}e_A}{\text{d}t} = X_{e_H}^E [e_A] = \{e_H, e_A\}_\omega_{FS} + \frac{\partial e_A}{\partial t},
\]

with the Poisson bracket \( \{e_H, e_A\}_\omega_{FS} \) defined in Eq. (45). Consequently for time-dependent Hamiltonian systems the expectation value of its Hamiltonian is no longer conserved.

4. Nonlinear Superposition Rule

In the Hilbert space, given two solutions of the Schrödinger equation \( |\psi\rangle \) and \( |\phi\rangle \), the state given by linear superposition

\[
|\Psi\rangle = p_1 |\psi\rangle + p_2 |\phi\rangle
\]

is also a solution of the Schrödinger equation. Physically, quoting Dirac, this means that: “Any state may be considered as the result of a superposition of two or more other states, and indeed in an infinite number of ways. Conversely any two or more states may be superposed to give a new state” [6]. To obtain such a superposition rule in \( \mathbb{CP} \), notice that in the nonlinear description the evolution of pure states is described by solutions of a Riccati equation, that although it is nonlinear, originates from the linear Schrödinger equation with linear superposition rule, thus there must be a way to superimpose its solutions.

Let us first establish the superposition rule for the q-bit system. So, employing the superposition principle on the Hilbert space then in the complex projective space the superimposed solution \( |\Psi\rangle \) in homogeneous coordinates (27) is given by

\[
Z = \frac{p_1 \psi^1 + p_2 \phi^1}{p_1 \psi^2 + p_2 \phi^2},
\]

which by construction is a solution of the Riccati equation (4). It is not difficult to realize that it is impossible to express \( Z \) just in terms of the particular solutions \( z_\psi = \frac{\psi^1}{\psi^2} \) and \( z_\phi = \frac{\phi^1}{\phi^2} \). However, introducing the auxiliarly solution

\[
z_0 = \frac{\psi^1 + \phi^1}{\psi^2 + \phi^2},
\]

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after some algebra the superposed solution can be expressed as

\[
Z = \frac{p_1 z_\psi (z_0 - z_\phi) + p_2 z_\phi (z_\psi - z_0)}{p_1 (z_\phi - z_0) + p_2 (z_\psi - z_0)}.
\]  

(62)

This relation between solutions of the Riccati equation is known in the literature as the nonlinear superposition rule [7, 8, 9]. Note that in order to obtain the expression (62) we have chosen the particular solution (61); however, the nonlinear superposition principle establishes that the general solution of a Riccati equation may be expressed as a function \( Z = \Phi(z_0, z_\psi, z_\phi, \kappa) \) of three particular solutions and an arbitrary constant \( \kappa \), thus any arbitrary solution \( z_0 \) may be employed to obtain the relation (62), for a formal proof see [9].

Therefore, it has been proven that the linear superposition principle on the Hilbert space (59) is translated into the complex projective space as a nonlinear superposition rule (62). This means, given three particular solutions \{\( z_0, z_\psi, z_\phi \)\} of the Riccati equation (4) the general solution can be written as

\[
\frac{(Z - z_\psi)(z_0 - z_\phi)}{(Z - z_\phi)(z_0 - z_\psi)} = \kappa,
\]  

(63)

where \( \kappa \) is a constant determined by the initial conditions [8, 10].

The use of an auxiliary solutions in order to obtain a superposition principle of pure states is not new. In fact, in terms of the density matrix description, the rule to add two pure states, by their density operators \( \rho_\psi \) and \( \rho_\phi \), involves the use of a fiducial projector \( \rho_0 \) such that the superimposed operator is

\[
\rho_\Psi = \frac{1}{N} \left( p_1 \rho_\psi + p_2 \rho_\phi + \frac{\sqrt{p_1 p_2} (\rho_\psi \rho_0 \rho_\phi + \text{h.c.})}{\sqrt{\text{Tr}(\rho_\psi \rho_0 \rho_\phi \rho_0)}} \right),
\]  

(64)

with \( N \) a normalization constant, for details see [12]. From this point of view, one should not be surprised that the nonlinear superposition principle involves an auxiliary solution.

Finally, it is straightforward to generalize the nonlinear superposition rule to \( N \)-level systems. The superimposed state \(|\Psi\rangle = p_1 |\psi\rangle + p_2 |\phi\rangle\) is then represented in the homogeneous coordinates of \( \mathbb{CP}(\mathcal{H}) \) by the vector state \( Z \) with components

\[
Z^k = \frac{p_1 z_\psi^k (z_0^k - z_\phi^k) + p_2 z_\phi^k (z_\psi^k - z_0^k)}{p_1 (z_\phi^k - z_0^k) + p_2 (z_\psi^k - z_0^k)}.
\]  

(65)

with \( k = 1, \ldots, n - 1 \), where \( z_\psi^k = \frac{\psi^k}{\psi^n} \), \( z_\phi^k = \frac{\phi^k}{\phi^n} \) and where \( z_0^k \) is the \( k \)-th component of an arbitrary solution \( z_0 \). For example, one may consider \( z_0 \) with components

\[
z_0^k = \frac{\psi^k + \phi^k}{\psi^n + \phi^n}.
\]  

(66)

5. Semiclassical Jaynes–Cummings model

As an example of nonlinear evolution of quantum systems, let us consider the Jaynes–Cummings model of quantum optics. This model consists of a single two-level atom.
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interacting with a single quantized cavity mode of the electromagnetic field [19, 20]. Employing the rotating wave approximation, this interaction is described by the Hamiltonian
\[
H = H_F + H_X + H_I = \hbar \omega \hat{a}^{\dagger} \hat{a} + \frac{1}{2} \hbar \omega_a \sigma_3 + 2 \hbar g (\sigma_+ \hat{a} + \sigma_- \hat{a}^{\dagger}),
\]
(67)
where \{\hat{a}^{\dagger}, \hat{a}\} represent the photon creation and annihilation operators, and \{\sigma_+, \sigma_-\} are the transition operators acting on the atomic states \{E_1, E_2\}. In terms of the Pauli matrices \{\sigma_1, \sigma_2, \sigma_3\} the transition operators have the form
\[
\sigma_+ = \frac{1}{2}(\sigma_1 + i \sigma_2), \quad \sigma_- = \frac{1}{2}(\sigma_1 - i \sigma_2).
\]
(68)
Besides, \omega represents the frequency of the electromagnetic field, \hbar \omega_a is the difference of energy between the two states of the atom, i.e. \hbar \omega_a = E_2 - E_1, and \(g\) is the atom-field coupling constant.

Then, let us consider an effective description of the 2-level atom (reference system) in interaction with the electromagnetic field (environment). By effective description we mean that given a conservative (reference) system we want to take into account the coupling with the environment without consideration of additional degrees of freedom with respect to those possessed by the system. Hence one may consider the expectation value of the Jaynes–Cumming Hamiltonian (67) with respect to the coherent states of the quantum oscillator system \(H_F\), i.e. with respect to the states \(|\alpha e^{i \omega t}\rangle\) such that it satisfies the eigenvalue equation \(\hat{a}|\alpha e^{i \omega t}\rangle = \alpha e^{i \omega t}|\alpha e^{i \omega t}\rangle\), with \(\alpha \in \mathbb{C}\). It is not difficult to show that these states are solutions of the Schrödinger equation associated with \(H_F\). Hence, taking this expectation value, the reduced Hamiltonian
\[
H_R(t) = \frac{1}{2} \hbar \omega_a \sigma_3 + \hbar [\xi_R(t) \sigma_1 - \xi_I(t) \sigma_2]
\]
(69)
is obtained, where we have defined the complex number \(\xi(t) = g \alpha e^{-i \omega t}\) and ignored constant terms. In physical terms, it has been found the Hamiltonian that describes a 2-level atom irradiated by a laser source, i.e. the radiation is considered as a classical field [20].

The expectation value of the Hamiltonian (69) defined on the complex projective space \(\mathbb{CP}\) is given by
\[
e_H = (\psi|H_R|\psi) = \frac{\hbar}{1 + z \bar{z}} \begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \omega_a & \xi(t) \\ \bar{\xi}(t) & -\frac{1}{2} \omega_a \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{\hbar}{1 + z \bar{z}} \begin{pmatrix} \frac{1}{2} \omega_a \bar{z}z + \xi(t) \bar{z} + \bar{\xi}(t)z - \frac{1}{2} \omega_a \end{pmatrix},
\]
(70)
and hence the Hamiltonian equations of motion (52) lead, as it should be, to the Riccati equation
\[
\dot{z} = i \left[\bar{\xi}(t) z^2 - \omega_a z - \xi(t)\right].
\]
(71)
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Figure 2. Phase portrait of (a) the Hamiltonian vector field and (b) the gradient vector field associated to the time-independent case $\omega = 0$ and parameters: $\omega_a = 1, \alpha = 1 + i$.

For the time-independent case, i.e. $\omega = 0$ it is possible obtain the phase portrait of the Hamiltonian vector field and the gradient vector field. The phase portrait of both vector fields are plotted in Fig. 2a and Fig. 2b, respectively. Here one can see that the solutions of the Hamilton vector field for this case are periodic curves except for the two singular solutions where the vector field vanishes. For the gradient vector fields these singular solutions become an attractive and a repulsive singular point.

For the time-dependent case, i.e. $\omega \neq 0$, the solutions of the Riccati equation are well-known, see for example [7, 8, 2, 3]. These solutions are Möbius transformations in $\mathbb{CP}$, namely

$$\Phi(\Sigma, z_0) \mapsto z(t) = \frac{a(t)z_0 + b(t)}{c(t)z_0 + d(t)} \quad \text{with} \quad \Sigma = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

where $z_0$ is the initial condition and $\Sigma$ is a symplectic matrix. For our case of interest we have the matrix

$$\Sigma(t) = \begin{pmatrix} e^{-i\omega t/2} (\cos \Omega t - \frac{i\Delta}{2\Omega} \sin \Omega t) & -i\frac{\alpha}{\Omega} e^{-i\omega t/2} \sin \Omega t \\ -i\frac{\alpha}{\Omega} e^{i\omega t/2} \left(\Omega - \frac{\Delta^2}{4\Omega^2}\right) \sin \Omega t & e^{i\omega t/2} (\cos \Omega t + \frac{i\Delta}{2\Omega} \sin \Omega t) \end{pmatrix},$$

where $\Delta = \omega_a - \omega$ is the difference between frequencies and $\Omega^2 = \frac{\Delta^2}{4} + \xi \bar{\xi}$ is a time-independent frequency. Important solutions of the Riccati equation are the periodic ones, i.e. the solutions such that $z(t) = z(t + T)$ with the period $T$. This condition of periodicity is reflected by $\Sigma(t) = \Sigma(t + T)$, which is satisfied under the constraint

$$\omega_a = \omega \pm \sqrt{\omega^2 - 4\xi \bar{\xi}},$$

with period $T = \frac{\pi \Omega}{2\sqrt{\omega}}$. From this constraint one sees that not all physical systems have periodic solutions, because it is necessary that $\omega^2 \geq 4 \xi \bar{\xi}$ to obtain a real frequency $\omega_a$. 
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Figure 3. Time evolution of the two-level atom in interaction with a classical source of radiation with initial condition \( z_0 = 0 \) and parameters: (a) \( \omega_a = 1, \omega = 1, g = 1, \beta = 1 + i \) and (b) \( \omega_a = 1, \omega = 1, g = 1 \) and \( \beta = 1 \).

Some examples of the quantum evolution in \( \mathbb{CP} \) and in \( S^2 \) are plotted in Fig. 3. For the case in Fig. 3a, we have a periodic orbit with period \( T = 2\pi \) and initial condition \( z_0 = 0 \). On the other hand, in Fig. 3b we consider the same initial condition but, due to the choice of the parameters, we do not have closed curve, but as the time progresses, the curve becomes dense on the sphere.

6. Conclusions

As a conclusion we would like to give a pictorial summary of what has been done. At the end of 1945 and at the beginning of 1946 Pablo Picasso created the set of eleven lithographs in Fig. 4 known as “The Bull”.

One can observe in Fig. 4 that Picasso started drawing the bull in every detail. The result is a lively and realistic image of the animal. However, after the third lithograph the artist started a process of abstraction of the bull, by progressively simplifying and outlining the anatomy of the animal. Thus, after the simplification and reduction of the image in the final lithograph Picasso has reduced the bull to a few lines obtaining the simplest and most abstract form of the bull, all this without losing the absolute essences of the animal.
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Figure 4. Pablo Picasso, the series “The Bull” (“Le taureau”), 1946, lithographs, each 32.4 × 44.2 cm, bpk-Bildagentur No: 00058489, 00052468, 70390790, 00052548, 00052531, 00058490 70390791, 70390792, 70390793, 00110164, 00058491. RMN-Grand Palais/René-Gabriel Ojeda.

Something similar is what we have done. At the beginning of our analysis we started with the linear Schrödinger dynamics in the $2^n$-dimensional Hilbert space $\mathcal{H}_0$. However, after taking into account the symmetry of the dynamics under the action of the dilation group the Hilbert space has been reduced to the odd-dimensional sphere $S^{2n-1}$ of normalized states. In addition, when also the symmetry under multiplication by a global phase change is taken into account, we get a further reduction to the $(2n - 2)$-dimensional complex projective space $\mathbb{CP}$.

Later, it is not only proven that the Schrödinger dynamics is projectable to $\mathbb{CP}$, but also it has been possible to see that the Kähler structure of the Hilbert space is related to the triple $(\omega_{FS}, g_{FS}, J_{FS})$ in the complex projective space. This structure allows to define a Hamiltonian dynamics on $\mathbb{CP}$ whose integral curves are defined by the solutions of the
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matrix Riccati equation. Furthermore, we were able to introduce the Poisson and Jordan brackets, where the former brackets allow to obtain the evolution of the expectation value of the observables, while the second brackets are connected to the dispersion and the correlation of the observables and thus express the probabilistic uncertainty aspect of the quantum system.

Even though one has a nonlinear dynamics, it has been shown that it is possible to introduce a nonlinear superposition rule on $\mathbb{CP}$, would allow for the description of interference phenomena in this non-linear manifold. It is consistent with the linear superposition rule available in the Hilbert space $\mathcal{H}_0$. This nonlinear superposition rule is also an instance of what happens in the more general case of Lie-Scheffers systems \cite{9,8}. Finally, all this formalism has been applied to a concrete physical problem, the semiclassical Jaynes–Cumming model, where the evolution in the complex projective space is given by a Möbius transformation.

Therefore, like Picasso’s bull abstraction, we have started with a problem that, after the analysis of its symmetries, has been reduced to a more abstract and fundamental description. In this process the fundamental parts have been always preserved, i.e. we preserve the physics behind the problem. This is basically the path that theoretical physicist should follow in his/her daily work in general.

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