Stringy Domain Walls of $\mathcal{N} = 1$, $D = 4$ Supergravity

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Abstract

We examine domain wall solutions of $\mathcal{N} = 1$, $D = 4$ supergravity which preserve half of the supersymmetry and arise from Euclidean M2-brane instantons on M5-branes wrapping associative 3-cycles of $G_2$-holonomy manifolds. We also investigate composite solutions which break an additional half of the supersymmetry.
1 Introduction

There has been considerable interest shown in recent months in the geometry of seven-dimensional manifolds with holonomy $G_2$. At the level of ten and eleven dimensional supergravity theories, the fact that such manifolds preserve a proportion of supersymmetry and possess parallel spinors means that new solutions corresponding to branes in non-trivial backgrounds may be constructed. In this context, new non-compact $G_2$ metrics are particularly important [1], [2], [3]; and they are related to various types of domain wall geometries [4]. Furthermore, such solutions may be used to probe possible dualities between supergravity theories and field theory [5], [6], [7], [8]. Another important aspect of $G_2$ holonomy manifolds is the fact that eleven dimensional supergravity compactified on a 7-manifold of $G_2$ holonomy reduces to $\mathcal{N} = 1, D = 4$ supergravity, which is of special significance phenomenologically.

In this paper we shall examine some aspects of supersymmetric solutions of a $\mathcal{N} = 1, D = 4$ supergravity theory obtained from such a $G_2$ compactification. This investigation is a continuation of that presented in [9], in which some aspects of the moduli space of $G_2$ structures examined in [10] and [11] were used to simplify the couplings of the four-dimensional theory, and some solutions were presented. The examples we consider are motivated by M-brane configurations in eleven dimensions wrapped in various fashions on a compact $G_2$ holonomy manifold. We shall concentrate on solutions which are associated with the wrapping of $M5$-branes on supersymmetric cycles of the $G_2$ manifold associated with calibrated geometries. For example, the geometry of a $M5$-brane in directions 012389 wrapping an associative supersymmetric 3-cycle of the $G_2$ manifold along 1234567 may be represented schematically as

$$M5 : 0 , 1 , 2 , 3 , * , * , * , * , 8 , 9 , * \quad G_2 : - , X , X , X , X , X , X , X , - , - , - , -$$

and compactifying on the $G_2$ manifold produces a 2-brane solution, i.e. a domain wall. We remark that such solutions do not possess, a priori from these constraints, the physical properties of domain walls summarized in [12], such as disconnected supersymmetric extrema of the potential. This depends on the form of the superpotential used in the theory, and we shall examine this in more detail later.

It is also possible for an $M5$-brane to wrap a co-associative supersymmetric calibrated 4-cycle of the $G_2$ manifold. Using the same notation this is given by

$$M5 : 0 , 1 , * , 3 , 4 , * , 6 , * , 8 , * , * \quad G_2 : - , X , X , X , X , X , X , X , - , - , - , -$$

This solution corresponds to a stringy cosmic string solution in four dimensions [13], [14]. Both (1) and (2) preserve $\frac{1}{16}$ of the $D = 11$ supersymmetry, i.e. $\frac{1}{2}$ of the $D = 4$
supersymmetry. It is also possible to take the orthogonal intersection of two $M5$-branes over a 3-brane, wrapping one of the $M5$-branes on a co-associative cycle, and the other on an associative cycle. This gives

$$M5 : 0, 1, 2, 3, *, *, *, 8, 9, *$$

$$M5 : 0, 1, *, 3, 4, *, 6, *, 8, *, *$$

$$G_2 : -, X, X, X, X, X, X, -, -, -.$$ (3)

This solution breaks another half of the eleven dimensional supersymmetry, and preserves only $\frac{1}{4}$ of the $D = 4$ supersymmetry. It is a composite string and domain wall solution.

The plan of this paper is as follows. In section 2 we present the truncated $\mathcal{N} = 1, D = 4$ supergravity action together with its field equations and supersymmetry constraints. We also summarize how geometric constraints imposed by requiring that the compactifying 7-manifold has $G_2$ holonomy simplify the couplings, and discuss some properties of superpotentials. In section 3 we derive various supersymmetry constraints associated with domain wall and composite string and domain wall solutions preserving $\frac{1}{2}$ and $\frac{1}{4}$ of the $\mathcal{N} = 1, D = 4$ supersymmetry respectively. In section 4 we investigate numerically the properties of domain wall type solutions obtained from various M2-brane instanton superpotentials. In section 5 we present some conclusions.

2 $\mathcal{N} = 1 \text{ } D = 4$ Supergravity

In this section we summarize some important details of $\mathcal{N} = 1, D = 4$ supergravity. We also present the constraints obtained on the various couplings when the four dimensional theory is obtained from compactification of eleven-dimensional supergravity on a manifold of $G_2$ holonomy.

2.1 Supergravity Action and Killing Spinor Equations

The geometric data that determine the various couplings of the $\mathcal{N} = 1$ four-dimensional supergravity theory consists of $n$ vector and $m$ chiral multiplets together with

- (i) A Kähler-Hodge manifold $M$ of complex dimension $m$ with Kähler potential $K$.

- (ii) A vector bundle $E$ over $M$ of rank $n$ for which its complexified symmetric product admits a holomorphic section $h$.

- (iii) A locally defined holomorphic function $f$ on $M$.  

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• (iv) Sigma model maps, \( z \), from the four-dimensional spacetime \( \Sigma \) into the manifold \( M \).

• (v) A principal bundle \( P \) on the four-dimensional spacetime \( \Sigma \) with fibre the abelian group \( U(1)^n \) such that the pull back of \( E \) with respect to \( z \) is isomorphic to \( P \times_{U(1)^n} LU(1)^n \), where \( LU(1)^n \) is the Lie algebra of \( U(1)^n \).

The bosonic part of the \( \mathcal{N} = 1, D = 4 \) supergravity action is \([15], [16], [17]\)

\[
L = \sqrt{-g} \left[ \frac{1}{2} R(g) - \frac{1}{4} \text{Re} h_{ab} F_{MN}^a F^{bMN} + \frac{1}{4} \text{Im} h_{ab} F_{MN}^a * F^{bMN} - \gamma_{ij} \partial_M z^i \partial^M z^j - V \right] \tag{4}
\]

where

\[
V = e^K [\gamma^{ij} D_i f D_j f - 3|f|^2] + \frac{1}{2} D_a D^a, \tag{5}
\]

\[
F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a, \tag{6}
\]

\[
D_i f = \partial_i f + \partial_i K f, \tag{7}
\]

\( A_N^a \) are \( U(1) \) (Maxwell) gauge potentials and the \( D_a \) are constants associated to a Fayet-Iliopoulos term. The gauge indices \( a, b = 1, \ldots, n \) are raised and lowered with \( \text{Re} h_{ab} \); \( i, j = 1, \ldots, m \) and \( M, N = 0, \ldots, 3 \) are holomorphic sigma model manifold and spacetime indices, respectively.

In this paper, we shall consider solutions of \( \mathcal{N} = 1, D = 4 \) supergravity which preserve some proportion of the supersymmetry. The Killing spinor equations of (4) in a bosonic background are most conveniently expressed in terms of a real 4-component Majorana spinor \( \epsilon \) as

\[
2(\partial_M + \frac{1}{4} \omega_{MAB} \Gamma^{AB}) \epsilon - (\text{Im}(K_i \partial_M z^i)) + e^K (\text{Re} f - \text{Im} f \Gamma^5) \Gamma_M \epsilon = 0, \tag{8}
\]

\[
( - \frac{1}{2} F_{MN}^a \Gamma^{MN} + \Gamma^5 D^a ) \epsilon = 0 \tag{9}
\]

and

\[
(\text{Re}(\partial_M z^i) - \Gamma^5 \text{Im}(\partial_M z^i)) \Gamma^M \epsilon - e^K (\text{Re}(\gamma^{ij} D_j f) - \Gamma^5 \text{Im}(\gamma^{ij} D_j f)) \epsilon = 0, \tag{10}
\]

where underlined indices \( \underline{A}, \underline{B} \) denote tangent frame indices and \( \Gamma^5 = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \). For our spinor conventions see the appendix.

The field equations of the supergravity action (4) are the following:

• (1) The Einstein equations are:

\[
G_{MN} - \text{Re} h_{ab} F_{ML}^a F^{bL} - 2 \gamma_{ij} \partial_M z^i \partial_N z^j + g_{MN} \left( \frac{1}{4} \text{Re} h_{ab} F_{LP}^a F^{bLP} + \gamma_{ij} \partial_L z^i \partial^L z^j + V \right) = 0. \tag{11}
\]
• (2) The Maxwell field equations are:

$$\partial_M \left[ \sqrt{-g} \left( \text{Re} h_{ab} F^{bMN} - \text{Im} h_{ab} \ast F^{bMN} \right) \right].$$

(12)

• (3) The scalar equations; varying $z^\ell$ gives the equation

$$-\frac{1}{8} \partial_\ell h_{a b} F^{a}_{MN} F^{bMN} \partial_\ell V - \frac{i}{8} \partial_\ell h_{a b} F^{a}_{MN} \ast F^{bMN} + \gamma_{ij} (\nabla_M \partial^M z^j + \Gamma^j_{ik} \partial_M z^i \partial^M z^k) = 0,$$

(13)

where $\nabla_M$ is the covariant derivative with respect to the Levi-Civita connection of the spacetime metric and

$$\partial_\ell V = \partial_\ell (e^K \gamma^{ij} D_i f) D_j f - 2e^K f D_\ell f + \frac{1}{2} \partial_\ell (D_a D^a).$$

(14)

Taking the conjugate of this equation, one obtains the field equation for $z^{\bar{i}}$.

We remark that stringy cosmic string solutions with non-vanishing Fayet-Iliopoulos terms arising from taking $D^a \neq 0$ have been found in [18]. For the remainder of this paper we shall set $D^a = 0$.

2.2 M-Theory Compactification on $G_2$ Manifolds

The relationship between the supergravity action (4) and the action of eleven dimensional supergravity compactified on manifolds of $G_2$ holonomy has been examined in detail in [19] and [9]. We shall summarize some of the results which are of particular use for our purposes. Suppose that the $G_2$-holonomy manifold is $N$; $\{\phi_i; i = 1, \ldots, m = b_3\}$ is a basis of $H^3(N, \mathbb{R})$ and $\{\omega_a; a = 1, \ldots, n = b_2\}$ is a basis of $H^2(N, \mathbb{R})$. Then the complex sigma model co-ordinates may be written as $z^i = -\frac{1}{2} (s^i + ip^i)$ for $s^i, p^i \in \mathbb{R}$. Setting

$$\phi = s^i \phi_i$$

(15)

the various couplings of the four dimensional theory obtained from the compactification of eleven dimensional supergravity are

$$ds^2 = \gamma_{ij} dz^i dz^j = k_{ij}(s) ds^i ds^j + m_{ij}(s) dp^i dp^j$$

$$m_{ij}(s) = \frac{1}{4} \int_N \sqrt{G} d^7 y \left( \phi_i, \phi_j \right)$$

$$\text{Re} h_{ab}(s) = \frac{1}{2} \int_N \sqrt{G} d^7 y (\omega_a, \omega_b) = \frac{1}{2} \int_N \omega_a \wedge \ast \omega_b = -\frac{1}{2} \int_N \omega_a \wedge \omega_b \wedge \phi$$

$$\text{Im} h_{ab}(p) = -\frac{1}{2} p^i \int_N \omega_a \wedge \omega_b \wedge \phi_i = -\frac{1}{2} p^i C_{iab}.$$
We shall denote the volume of the compact $G_2$ holonomy manifold by $\Theta$. With respect to the $G_2$ moduli co-ordinates described here, $\Theta = \Theta(s^i)$ is homogeneous of degree $\frac{7}{3}$ with respect to the $s^i$, and the Kähler potential is related to $\Theta$ by

$$K = -\frac{3}{7} \log \Theta.$$  \hfill (17)

It remains to consider the role played by the superpotential $f$. Such terms do not arise from direct compactifications of eleven dimensional supergravity using the ansatz presented above to four dimensions. However, a superpotential may originate from some non-vanishing 4-form flux $F_0$ along the compact directions. Such a superpotential has been considered in [20], [21] and [22]. In this case, the potential is specified via

$$\text{Re } f(z) = \int_N \phi \wedge F^0.$$ \hfill (18)

However, it has been argued that obtaining $f$ from the 4-form flux is not consistent with the compactness of the $G_2$ manifold [23]. An alternative mechanism for generating potentials is the wrapping of M2-branes on associative 3-cycles of $G_2$ manifolds. The contribution to the superpotential from such an M2-brane instanton is [24]

$$\Delta f(z) = \mu e^{z^i}$$ \hfill (19)

where $\mu > 0$ is constant and $z = z^i \delta_i$ for real constants $\delta_i$. It has however been argued that there generically exist obstructions to the construction of a locally smooth moduli space of associative cycles [25], although there are special cases when there does exist a smooth moduli space. More recently, it has been proposed in [26] and [24] that the contribution from multiple M2-brane instantons may be obtained by taking the sum

$$f(z) = \mu \sum_{n=1}^{\infty} \frac{e^{nz}}{n^2}.$$ \hfill (20)

We shall concentrate on supergravity solutions corresponding to (19) and (20).

3 Stringy Domain Wall Solutions

In order to investigate domain wall solutions, and composite string and domain wall solutions we shall consider the following ansatz;

$$ds^2 = A^2(w, \bar{w}) ds^2(\mathbb{R}^{1,1}) + ds^2_{(2)}$$

$$z^i = z^i(w, \bar{w})$$

$$A^a = 0$$

(21)
where $ds^2_{(2)}$ is a metric on the manifold spanned by $w, \bar{w}$ where $w = x+iy$ and $\bar{w} = x-iy$ for $x, y \in \mathbb{R}$. Without loss of generality, we shall take

$$ds^2_{(2)} = B^2(x, y)(dx^2 + dy^2)$$

(22)

to be diagonal, using the fact that any metric on a Riemann surface is locally conformally flat.

Substituting this ansatz into the Killing spinor equations, we find that

$$\partial_x A \Gamma_x + \partial_y A \Gamma_y \epsilon + A B e^K(\text{Re} f + \text{Im} f \Gamma^5) \epsilon = 0$$

(23)

together with

$$2 \partial_x \epsilon - \partial_x \log A \epsilon + \partial_y \log B \epsilon A \Gamma_y \epsilon - \Gamma^5 \epsilon = 0$$

(24)

and

$$(\text{Re} \partial_x z^i - \Gamma^5 \text{Im} \partial_x z^i) \Gamma_x \epsilon + (\text{Re} \partial_y z^i - \Gamma^5 \text{Im} \partial_y z^i) \Gamma_y \epsilon$$

$$- B e^K (\text{Re} (\gamma^i \bar{D}_j \bar{f}) - \Gamma^5 \text{Im} (\gamma^i \bar{D}_j \bar{f})) \epsilon = 0$$

(25)

The solutions which we shall concentrate on have $f \neq 0$ and preserve $\frac{1}{4}$ of the supersymmetry. We may begin by examining (25). If we work in a real basis, so that $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ with $\epsilon_1, \epsilon_2$ real; then (25) implies

$$\begin{align*}
(\sigma^1 - 1)[2i \partial_w z^i \bar{\eta} + B e^K \gamma^i \bar{D}_j \bar{f} \bar{\eta}] &= 0 \\
(\sigma^1 + 1)[ -2i \partial_{\bar{w}} z^i \bar{\eta} + B e^K \bar{\gamma}^i \bar{D}_j \bar{f} \bar{\eta}] &= 0
\end{align*}$$

(27)

where $\eta = \epsilon_1 + i \epsilon_2$. Suppose now that there exists $i$ such that $\gamma^i \bar{D}_j \bar{f} = 0$. Then for these $i$, these equations may be solved by taking $z^i$ constant. Alternatively, one may have $z^i = z^i(w)$ non-constant holomorphic with $\Gamma^5 \Gamma_x \Gamma_y \epsilon = -\epsilon$; or $z^i = z^i(\bar{w})$ non-constant anti-holomorphic with $\Gamma^5 \Gamma_x \Gamma_y \epsilon = \epsilon$ (however if there exists more that one value of $i$ such that $\gamma^i \bar{D}_j \bar{f} = 0$ then one cannot have a supersymmetric solution with a mixture of corresponding non-constant holomorphic and anti-holomorphic complex scalars).

Suppose now we consider $i$ for which $\gamma^i \bar{D}_j \bar{f} \neq 0$. Define

$$\psi^i = 2i \partial_w z^i [B e^K \gamma^i \bar{D}_j \bar{f}]^{-1}$$

$$\tau^i = -2i \partial_{\bar{w}} z^i [B e^K \gamma^i \bar{D}_j \bar{f}]^{-1}$$

(28)
Then one requires for these $i$;

$$(1 - \sigma^1)(\psi^i \bar{\eta} + \eta) = 0$$
$$(1 + \sigma^1)(\tau^i \bar{\eta} + \eta) = 0$$

There are several possibilities. Firstly, note that one cannot have a supersymmetric solution with both $\psi^i = \tau^i = 0$. If $\psi^i = 0$ then it turns out that $\Gamma^5 \Gamma_x \Gamma_y \epsilon = -\epsilon$. If $\tau^i = 0$, however, then $\Gamma^5 \Gamma_x \Gamma_y \epsilon = \epsilon$. Alternatively, one may have $\psi^i, \tau^i$ both nonzero. It turns out that if both $|\psi^i| \neq 1$ and $|\tau^i| \neq 1$ then the solution cannot be supersymmetric. If however, $|\psi^i| \neq 1$ but $|\tau^i| = 1$ then one has $\Gamma^5 \Gamma_x \Gamma_y \epsilon = -\epsilon$. Another possibility is to take $|\tau^i| \neq 1$ and $|\psi^i| = 1$; then $\Gamma^5 \Gamma_x \Gamma_y \epsilon = \epsilon$. We shall see that these solutions generically preserve $\frac{1}{4}$ of the supersymmetry. They correspond to a superposition of string and domain wall solutions. Before considering this case, we shall consider the remaining case in which we take $|\tau^i| = |\psi^i| = 1$. These conditions have been examined previously in [9] and they give solutions which preserve half of the supersymmetry. It is useful to recap these results here.

### 3.1 Half Supersymmetric Solutions

Writing $\psi^i = e^{i\theta^i}, \tau^i = e^{i\phi^i}$ for real $\theta^i, \phi^i$, the supersymmetry constraint (25) for $|\tau^i| = 1$ and $|\psi^i| = 1$ is satisfied by taking

$$
\epsilon_1 = \sin \phi^i \left( \chi^i \chi^i \right) + \sin \theta^i \left( -\mu^i \mu^i \right)
$$
$$
\epsilon_2 = -(1 + \cos \phi^i) \left( \chi^i \chi^i \right) - (1 + \cos \theta^i) \left( -\mu^i \mu^i \right)
$$

for real $\mu^i, \chi^i$. Analogous reasoning may be used to consider (23). In particular, (23) may be written as

$$(\sigma^1 - 1)[2i\partial_w A \bar{\eta} - ABE \bar{f} \bar{\eta}] = 0$$
$$(\sigma^1 + 1)[-2i\partial_w A \bar{\eta} - ABE \bar{f} \bar{\eta}] = 0 .$$

Defining

$$
\Omega = -2i\partial_w A(ABe \bar{f})^{-1}
$$
$$
\Lambda = 2i\partial_w A(ABe \bar{f})^{-1}
$$

we note that (23) is equivalent to

$$(\sigma^1 - 1)(\Omega \bar{\eta} + \eta) = 0$$
$$(\sigma^1 + 1)(\Lambda \bar{\eta} + \eta) = 0 .$$

Hence the reasoning used to determine the various possible values of $\psi^i, \tau^i$ also applies to $\Omega$ and $\Lambda$. 

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So, we have shown that (23) and (25) imply that $\Gamma_5 \Gamma_x \Gamma_y \epsilon \neq \pm \epsilon$. Furthermore, if $\gamma^{ij} D_j \bar{f} = 0$ then $z^i$ is constant, and if $\gamma^{ij} D_j \bar{f} \neq 0$ then $\psi^i = \Omega$ and $\tau^i = \Lambda$ with $|\Omega| = |\Lambda| = 1$. $\epsilon$ is given by (30).

It is also necessary to examine (24). This constraint may be rewritten as

$$4 \partial_{\bar{w}} \hat{\epsilon} - 2i \partial_{\bar{w}} \log \frac{B}{A} \Gamma_5 \Gamma_y \hat{\epsilon} + i \Gamma^5 (-\partial_{\bar{w}} K + 2K_i \partial_{\bar{w}} z^i) \hat{\epsilon} = 0,$$  

where $\hat{\epsilon} = A^{-\frac{1}{2}} \epsilon$. In this case $\tau^i = \Lambda$ and $\psi^i = \Omega$ imply (for $f \neq 0$)

$$-\partial_{\bar{w}} z^i = A^{-1} \partial_{\bar{w}} A \bar{f}^{-1} \gamma^{ij} D_j \bar{f}$$
$$-\partial_{\bar{w}} z^i = A^{-1} \partial_{\bar{w}} A \bar{f}^{-1} \gamma^{ij} D_j \bar{f}$$

and we solve the supersymmetry constraints by taking $\Lambda = e^{i \phi}$, $\Omega = e^{i \theta}$, for $\theta, \phi \in \mathbb{R}$ with

$$\hat{\epsilon}_1 = \sin \phi \left( \frac{\hat{\lambda}}{\lambda} \right) + \sin \theta \left( \frac{-\hat{\mu}}{\mu} \right)$$
$$\hat{\epsilon}_2 = -(1 + \cos \phi) \left( \frac{\hat{\lambda}}{\lambda} \right) - (1 + \cos \theta) \left( \frac{-\hat{\mu}}{\mu} \right)$$

where $\hat{\lambda}, \hat{\mu} \in \mathbb{R}$. Then (24) implies that

$$4 \partial_{\bar{w}} (\hat{\lambda} \sin \phi) + i(1 + \cos \phi)(\partial_{\bar{w}} (K + 2 \log \frac{B}{A}) - 2K_i \partial_{\bar{w}} z^i) \hat{\lambda} = 0$$
$$-4 \partial_{\bar{w}} ((1 + \cos \phi) \hat{\lambda}) + i \sin \phi(\partial_{\bar{w}} (K + 2 \log \frac{B}{A}) - 2K_i \partial_{\bar{w}} z^i) \hat{\lambda} = 0$$
$$4 \partial_{\bar{w}} (\hat{\mu} \sin \theta) + i(1 + \cos \theta)(\partial_{\bar{w}} (K - 2 \log \frac{B}{A}) - 2K_i \partial_{\bar{w}} z^i) \hat{\mu} = 0$$
$$-4 \partial_{\bar{w}} ((1 + \cos \theta) \hat{\mu}) + i \sin \theta(\partial_{\bar{w}} (K - 2 \log \frac{B}{A}) - 2K_i \partial_{\bar{w}} z^i) \hat{\mu} = 0.$$

This is solved by taking

$$\hat{\lambda} = \frac{\xi}{\sqrt{1 + \cos \phi}}$$
$$\hat{\mu} = \frac{\zeta}{\sqrt{1 + \cos \theta}}$$

for constant $\xi, \zeta \in \mathbb{R}$ and $B, \phi$ and $\theta$ are determined by

$$\partial_{\bar{w}} (2 \phi + i(K + 2 \log \frac{B}{A})) = 2iK_i \partial_{\bar{w}} z^i$$
$$\partial_{\bar{w}} (-2 \theta - i(K + 2 \log \frac{B}{A})) = -2iK_i \partial_{\bar{w}} z^i.$$

We note that these solutions generically preserve $\frac{1}{2}$ of the supersymmetry.

It is straightforward to check that these conditions ensure that the scalar and Einstein field equations hold.
3.2 Quarter Supersymmetric Solutions

We shall concentrate on the case $\Gamma_5 \Gamma_x \Gamma_y \epsilon = -\epsilon$ for which $|\tau^i| = 1$, as the case $\Gamma_5 \Gamma_x \Gamma_y \epsilon = +\epsilon$ follows by analogous reasoning. When $\Gamma_5 \Gamma_x \Gamma_y \epsilon = -\epsilon$, the Killing spinor can be written as

$$\epsilon_1 = \sin \phi^i \left( \lambda^i \right)$$
$$\epsilon_2 = -(1 + \cos \phi^i) \left( \lambda^i \right) .$$

(40)

In addition, (23) may be written as

$$(\sigma^1 + 1)[ -2i \partial_{\bar{w}} A \bar{\eta} - AB \epsilon \bar{k} \bar{f} \eta] = 0 .$$

(41)

Then defining

$$\Lambda = 2i \partial_{\bar{w}} A (AB \epsilon \bar{k} \bar{f})^{-1}$$

we note that (23) is equivalent to

$$(\sigma^1 + 1)(\Lambda \bar{\eta} + \eta) = 0 .$$

(43)

So, for the case $\Gamma_5 \Gamma_x \Gamma_y \epsilon = -\epsilon$, (23) and (25) imply that $z^i = z^i(w)$ is holomorphic iff $\gamma^{ij} D_j \bar{f} = 0$. If however $\gamma^{ij} D_j \bar{f} \neq 0$ then $\tau^i = \Lambda$ and

$$\Lambda = e^{i \phi}$$

(44)

for $\phi = \phi^i \in \mathbb{R}$.

It is also necessary to examine (24). This constraint may be rewritten as

$$4 \partial_{\bar{w}} \hat{\epsilon} - 2i \partial_{\bar{w}} \log \frac{B}{A} \Gamma_5 \Gamma_x \Gamma_y \hat{\epsilon} + i \Gamma^5 (-\partial_{\bar{w}} K + 2K_i \partial_{\bar{w}} z^i) \hat{\epsilon} = 0 .$$

(45)

where $\hat{\epsilon} = A^{-\frac{1}{2}} \epsilon$. Then $\tau^i = \Lambda$ implies that

$$-\partial_{\bar{w}} z^i = A^{-1} \partial_{\bar{w}} A \bar{f}^{-1} \gamma^{ij} D_j \bar{f} .$$

(46)

The supersymmetry constraints are solved by taking

$$\hat{\epsilon}_1 = \sin \phi \left( \hat{\lambda} \right)$$
$$\hat{\epsilon}_2 = -(1 + \cos \phi) \left( \hat{\lambda} \right)$$

(47)

for $\hat{\lambda} \in \mathbb{R}$. Hence (24) is equivalent to

$$4 \partial_{\bar{w}} (\hat{\lambda} \sin \phi) + i (1 + \cos \phi) \left( \partial_{\bar{w}} (K + 2 \log \frac{B}{A}) - 2 K_i \partial_{\bar{w}} z^i \right) \hat{\lambda} = 0$$

$$-4 \partial_{\bar{w}} ((1 + \cos \phi) \hat{\lambda}) + i \sin \phi \left( \partial_{\bar{w}} (K + 2 \log \frac{B}{A}) - 2 K_i \partial_{\bar{w}} z^i \right) \hat{\lambda} = 0 .$$

(48)
This is solved by taking
\[ \hat{\lambda} = \frac{\xi}{\sqrt{1 + \cos \phi}} \] (49)
for constant \( \xi \in \mathbb{R} \) and \( B \) and \( \phi \) are determined by
\[ \partial_\theta (2\phi + i(K + 2\log \frac{B}{A})) = 2iK_i\partial_\theta z^i \] (50)

From these expressions, it follows that these solutions preserve \( \frac{1}{4} \) of the supersymmetry. Again, it is straightforward to check that the conditions (44), (46) and (50) are sufficient to ensure that the scalar and Einstein field equations hold.

4 Half-Supersymmetric \( G_2 \)-Instanton Solutions

Further simplifications of the supersymmetry constraints may be obtained in the special case when the \( \mathcal{N} = 1, D = 4 \) supergravity theory arises from the compactification of \( M \)-theory on a manifold with holonomy \( G_2 \). In particular, as a consequence of the homogeneity properties of the \( G_2 \) manifold volume, we have
\[ \gamma^{ij} K_j = s^i . \] (51)

The conditions presented in the previous section imply that
\[ \partial_\theta (\theta - \phi) = 0 , \] (52)
so that \( \theta - \phi = \) const. Furthermore, we note that \( \partial_\theta A e^{i\theta} + \partial_\theta A e^{i\phi} = 0 \), so it follows that \( A \) depends only on some linear combination of \( x \) and \( y \). Without loss of generality we shall take \( A = A(x) \). Then (35) implies that \( z^i = z^i(x) \), and hence all fields depend only on \( x \). This then fixes the constant to be
\[ \theta - \phi = (2n + 1)\pi , \] (53)
for \( n \in \mathbb{Z} \). In addition, we require that
\[ \partial_x (2\phi + i(K + 2\log \frac{B}{A})) = 2iK_i \partial_x z^i , \] (54)
\[ \partial_x z^i = -A^{-1} \partial_x A (s^i + \bar{f}^{-1}\gamma^{ij} \partial_j \bar{f}) , \] (55)
and
\[ A B e^{\frac{K}{2}} \bar{f} e^{i\phi} = i \partial_x A . \] (56)

Now, (54) implies that \( B = A \) and
\[ 2\partial_x \phi = K_i \partial_x p^i , \] (57)
where we recall for the $G_2$ ansatz under consideration here, $K_i \in \mathbb{R}$. It is straightforward to see that contracting the imaginary portion of (55) with $K_i$ one obtains the relation
\[ \frac{1}{2} K_i \partial_x p^i - \partial_x (\text{Arg } f) = 0, \] (58)
and so (57) and (56) are consistent, as we expect. We shall set $\phi = \text{Arg } f + \frac{\pi}{2}$ so that
\[ A^{-2} \partial_x A = e^{\frac{\phi}{2}}|f|, \] (59)
and we observe that a discontinuity jump of $\pi$ in $\phi$ induces a sign change in (59).

### 4.1 Instantonic Solutions

We proceed to solve the half-supersymmetric constraints (55) and (56) using the instanton induced superpotentials (19) and (20). Taking first the single instanton cover (19) we note that (55) implies that $\partial_x p^i = 0$. We solve this by setting without loss of generality $p^i = 0$ and as with this choice $f \in \mathbb{R}$ we take $\phi = \frac{\pi}{2}$ ($\theta = -\frac{\pi}{2}$). In addition, (55) may be rewritten as
\[ K_i = A^{-2} \chi_i - \delta_i \] (60)
for real constants $\chi_i$ and (56) is
\[ fe^{\frac{\phi}{2}} = -\partial_x A^{-1}. \] (61)

We note that in the case when $\chi_i = 0$ for $i = 1, \ldots, b_3$, $K_i$, $s^i$, and $\Theta$ are constants and $A = \zeta x^{-1}$ for constant $\zeta$, so the spacetime is simply anti-de-Sitter. For $\chi_i \neq 0$, it is clear from (60) that at supersymmetric extrema for which $D_i f = 0$, we require $A \to \infty$.

A more interesting solution is obtained if we take $\delta_i = \alpha \chi_i \neq 0$ for constant $\alpha$. Then the solution obtained is a scaling solution with
\[ s^i = (A^{-2} - \alpha)^{-1} \chi^i \] (62)
for constants $\chi^i$ satisfying $\chi^i \chi_i = 1$. Then the Kähler potential, and the scalar potential and its derivatives are given in terms of $A$ by
\[ K = \log |A^{-2} - \alpha| + \sigma_1 \]
\[ V = \sigma^2 |A^{-2} - \alpha| e^{-\alpha (A^{-2} - \alpha)^{-1}} \left[ \frac{A^{-4}}{(A^{-2} - \alpha)^2} - 3 \right] \] (63)
\[ \frac{\partial V}{\partial z^\ell} = \alpha^{-1} \sigma^2 A^{-2} |A^{-2} - \alpha| e^{-\alpha (A^{-2} - \alpha)^{-1}} \left[ \frac{\alpha^2}{(A^{-2} - \alpha)^2} - 2 \right] \delta_\ell, \]
for constants $\sigma_1, \sigma = \mu^2 e^{\sigma_1}$. The spacetime dependence is fixed by (61) via

$$\partial_x A = \sigma A^2 \sqrt{|A^{-2} - \alpha|} e^{-\frac{\alpha}{2}(A^{-2} - \alpha)^{-1}}.$$  \hfill (64)

It is not possible to obtain a closed form for this solution, however we may examine its generic properties. We begin with the case $\alpha > 0$. This solution has two branches; one with $x_2 < x < x_1$ and $\alpha^{-1/2} < A < \infty$ and the other with $-\infty < x < \infty$ and $0 < A < \alpha^{-1/2}$.

For the first branch, as $x \to x_2^+$, $A \to \alpha^{-1/2}$ and $V \to \infty$. As $x$ increases $V$ decreases and $A$ increases. There is a global minimum of $V = V_{\text{min}} = -2e^{\sqrt{2} \alpha \sigma^2}$ at $x = x_0$, at which $A = \sqrt{2 + \sqrt{2} \alpha^{-1/2}}$. For $x > x_0$, as $A \to \infty$, $V$ increases with $x$ to attain a (local) maximum value of $V = V_{\text{max}} = -3e\alpha^2$. We observe that as a consequence of (64), $A \to \infty$ at some finite $x = x_1 > x_0$. Furthermore, setting $x_1 = 0$ (after perhaps making some constant translation), we note that $A \sim \frac{A}{2}$ as $x \to 0$ for $\kappa < 0$ constant. The behaviour of $A$ and $V$ is sketched in Figure 1.

![Figure 1: Graphs of $A$ and $V$ (first branch): $\alpha > 0$](image)

It is useful to examine motion of a test particle in this background. If $\rho$ is the affine parameter along the geodesic then

$$\frac{d\rho}{dA} = \frac{1}{\sigma \sqrt{|(\epsilon A^2 + E^2)(A^{-2} - \alpha)|}} e^{\frac{2}{2}(A^{-2} - \alpha)^{-1}},$$ \hfill (65)

where $\epsilon = 0$ or $\epsilon = -1$ according to whether we consider a photon or massive particle and $E$ is the constant energy. From this we observe that the particle reaches both $x = 0$ and $x = x_2$ in a finite affine parameter, say $\rho = 0$ and $\rho = \rho_2$ respectively. As $\rho \to \rho_2$ the scalars $s^i$, and the potential $V$ and all of their derivatives are unbounded, and the $G_2$ manifold has a singularity as the $G_2$ volume $\Theta$ vanishes. As $\rho \to 0$ the geometry...
becomes anti-de-Sitter and all derivatives of $V$ vanish. From this it is apparent that the unique (partial) smooth extension of the solution through $\rho = 0$ gives a solution defined on a finite interval of affine parameter $\rho$ with a reflection symmetric double-well potential (Figure 2).

![Figure 2: Extended Potential](image)

At both of the boundaries of this interval where $V \to \infty$, there is a naked curvature singularity at which $R \to \infty$. Although the potential has two minima, they are not supersymmetric. The only supersymmetric (AdS) extremum is the local maximum at $x = 0$.

For the second branch of the $\alpha > 0$ solution, $A \to 0$ as $x \to -\infty$ and $A \to \alpha^{-\frac{1}{2}}$ as $x \to \infty$. $V$ increases from $-\infty$ to a global maximum of $V_{\text{max}} = 2\sigma^2 \alpha e^{-\sqrt{2}}$ at $A = \alpha^{-\frac{1}{2}} \sqrt{2 - \sqrt{2}}$ and then decreases to 0 as $A \to \alpha^{-\frac{1}{2}}$. The global maximum is not supersymmetric, nor is the Minkowski minimum; indeed at the minimum the scalars become unbounded and the $G_2$ volume tends to infinity. Furthermore, an analysis of the test particle motion shows that $A = 0$ is reached in finite affine parameter. This is a naked singularity. The behaviour of $A$ and $V$ for this branch of the solution is sketched in Figure 3.
For the case of $\alpha < 0$, $A$ is monotonic increasing from 0 as $x \to -\infty$ and $A \to \infty$ as $x \to 0$ as $A \sim \frac{\alpha}{x}$ for $\kappa < 0$ constant. $A = 0$ is a naked curvature and $G_2$ singularity. Just as in the first branch of the $\alpha > 0$ solution, this solution may be smoothly extended through $x = 0$ to give a reflection symmetric potential bounded by two disconnected naked singularities. In this case, however, $V$ has only one extremum, a global supersymmetric maximum at $x = 0$ of $V_{\text{max}} = 2\sigma^2\alpha e$. The behaviour of this solution is sketched in Figure 4.

For more general solutions where $\chi_i$ and $\delta_i$ are linearly independent, the analysis is considerably more complicated due to the non-linear structures on the $G_2$-manifold. However, as a consequence of the supersymmetry constraints it is possible to derive
the following useful identities;

\[ \frac{R}{6} = V + \mu^2 e^{K - s^i \delta_i} \]

\[ \frac{R}{6} = \mu^2 e^{K - s^i \delta_i} \gamma \delta_i \delta_j + 2 s^j \delta_j - 2 \]

(66)

and from these it is apparent that if \( V \to \infty \) then \( R \to \infty \) and there is a \( G_2 \) manifold singularity. It is unclear if this necessarily implies that \( A \to 0 \), which is a \( G_2 \) singularity as the \( K_i \) become unbounded; for linearly independent \( \chi_i \) and \( \delta_i \) it is no longer possible for a \( G_2 \) singularity to arise from all of the \( K_i \) vanishing.

We can also examine the solutions obtained when one takes the sum of multiple M2-brane instanton contributions to the superpotential given by (20). In this case, the equations are considerably more complicated. We define

\[ \mathcal{P}(z) = \sum_{i=1}^{\infty} \frac{e^{nz}}{n^2} \]

so that \( f = \mu \mathcal{P} \). In particular, from this form of \( f \) it is apparent that (55) no longer implies that \( \partial_x \psi^i = 0 \). If, however, we restrict the solution to \( s > 0 \) then we may again set \( \psi^i = 0 \) and \( \phi = \frac{\pi}{2} \). The equations simplify even further if we take a scaling solution ansatz

\[ s^i = s(x) \delta^i \]

\[ K_i = s^{-1}(x) \delta_i \]

(68)

where \( \delta^i \) are real constants such that \( \delta_i \delta^i = 1 \). The Kähler potential is given by

\[ e^{\frac{K}{2}} = \tilde{\sigma} s^{-\frac{1}{2}} \]

(69)

for constant \( \tilde{\sigma} > 0 \), and the supersymmetry constraints are solved by imposing

\[ \partial_x s = 2\sigma A s^{\frac{1}{2}} (\mathcal{P} - s \log(1 - e^{-\frac{2}{s}})) \]

\[ A^{-2} \partial_x A = \sigma \mathcal{P} s^{-\frac{1}{2}} , \]

(70)

where \( \sigma = \mu \sqrt{\tilde{\sigma}} \). The potential may be written as

\[ V = \sigma^2 s^{-1} [s^2 \log(1 - e^{-\frac{2}{s}})^2 - 2s (\mathcal{P} \log(1 - e^{-\frac{2}{s}})) - 2 \mathcal{P}^2] . \]

(71)

A numerical analysis of these equations shows that \( A \) and \( V \) behave just as in the second branch of the \( \alpha > 0 \) solution for the single cover instanton superpotential.

So, for these scaling solutions with \( \text{Im} z^i = 0 \), we have shown that the spacetimes typically possess naked curvature and \( G_2 \) singularities. It is apparent that they do not produce supersymmetric AdS minima of the potential, and they do not as they stand readily have an interpretation as domain wall solutions.
More interesting scaling solutions may be obtained by allowing \( \text{Im} z^i \) to vary. In particular, we shall examine the behaviour of the multiple cover superpotential with the scaling solution

\[
\begin{align*}
    s^i &= s(x) \delta^i \\
p^i &= p(x) \delta^i \\
    K_i &= s^{-1}(x) \delta_i
\end{align*}
\]

(72)

where \( \delta^i \) are real constants such that \( \delta_i \delta^i = 1 \). We allow \( p(x) \) to vary and allow for \( s < 0 \) by taking the appropriate analytic continuation of the superpotential. To be specific, we shall restrict \( \mathcal{P} \) to the principal branch of the polylogarithm function. The discontinuities of this function give curvature discontinuities in the spacetime geometry which in turn give rise to domain wall solutions. The Kähler potential is given by

\[
e^\frac{K}{2} = \tilde{\sigma} |s|^{-\frac{1}{2}}
\]

(73)

for constant \( \tilde{\sigma} > 0 \). Setting

\[
\phi = \text{Arg} \mathcal{P}(x) + \frac{\pi}{2}
\]

(74)

where \( z = -\frac{1}{2}(s + ip) \), (55) and (56) imply

\[
\begin{align*}
    \partial_x s &= 2\sigma A s |\mathcal{P}| |s|^{-\frac{1}{2}} (1 - s \text{Re}(\mathcal{P}^{-1}\log(1 - e^z))) \\
    \partial_x p &= 2\sigma A s^2 |s|^{-\frac{1}{2}} \text{Im}(\mathcal{P}^{-1}\log(1 - e^z)) \\
    A^{-2} \partial_x A &= \sigma |\mathcal{P}| |s|^{-\frac{1}{2}},
\end{align*}
\]

(75)

where \( \sigma = \mu \sqrt{\tilde{\sigma}} \). The potential may be written as

\[
V = \sigma^2 |s|^{-1} [s^2 \log(1 - e^z)^2 - 2s \text{Re}(\mathcal{P} \log(1 - e^z)) - 2|\mathcal{P}|^2].
\]

(76)

We have been unable to find an analytic solution to the equations (75). The problem may be simplified somewhat by considering \( A \) and \( s \) as functions of \( p \). Then (75) implies that

\[
\begin{align*}
    s \frac{d}{dp}(\text{Im}\log \mathcal{P}) &= \frac{1}{2} \\
    A^{-1} \frac{dA}{dp} &= \frac{1}{2s^2 \text{Im}(\mathcal{P}^{-1}\log(1 - e^z))}
\end{align*}
\]

(77)

Although we have been unable to solve these equations analytically, they may be solved numerically. Of particular interest is a solution with \( s_1 < s < s_0 \) for \( s_0, s_1 < 0 \), so unlike the other solutions, there are no \( G_2 \) singularities. Furthermore, \( A > A_0 \) for some \( A_0 > 0 \) constant and when \( A \to \infty \) the divergence is according to \( A \sim \frac{\kappa}{x} \) for finite \( x \) and \( \kappa \) constant, so this portion of the spacetime may be continued through this coordinate singularity into another copy of itself. This divergence of the conformal factor
corresponds to a supersymmetric minimum of the potential and \( A = A_0 \) corresponds to a non-supersymmetric maximum of the potential. Although \( A \) and \( \partial_x A \) are smooth at \( A = A_0 \) there is a discontinuity jump in the curvature and in \( \phi \) corresponding to a branch cut in the polylogarithm function \( \mathcal{P} \). The behaviour of this solution for \(-2\pi < p < 2\pi\) is sketched in Figure 5.

![Figure 5: Graphs of A and V (Multiple Cover)](image)

We remark that restricting the superpotential to one particular branch (and so picking up discontinuities at certain points) is essentially equivalent to truncating the spacetime. Indeed, if we instead continue smoothly through to another branch of the polylogarithm, a numerical analysis indicates that just as for the other cases considered previously, \( s \to 0 \) in a finite affine parameter, and a curvature and \( G_2 \) singularity is encountered.

### 5 Conclusions

In this letter we have investigated a class of supergravity solutions obtained from M2-brane instantons wrapping associative cycles which produce supersymmetric extrema of the potential. For solutions preserving half of the supersymmetry arising from the single cover of M2-brane instantons on an associative 3-cycle, we have demonstrated that there are no solutions with AdS minima. However, one may, for example, construct an array of supersymmetric AdS maxima by truncating by hand the first branch of the \( \alpha > 0 \) solution at the minima of the potential and excising the portions of the spacetime containing the singularities. Gluing on copies of the portion containing the maximum one obtains a solution in which \( V \) is \( C^1 \) smooth but there is an array of curvature discontinuities at the minima. For the multiple cover superpotential, the solution is more promising. In particular, there exists a scaling solution which does
have supersymmetric AdS minima. It describes an array of domain walls interpolating between these minima. However, it is clear that for the solution presented here there is no well-defined finite non-zero charge.

It is apparent that some aspects of solutions to the supersymmetry constraints presented in section 3 remain to be addressed. In particular, we have only been able to construct numerically solutions in which the $G_2$ moduli scale uniformly. More general solutions, even for the simple superpotential (19) satisfying (60) may have more interesting spacetime geometries.

In addition, we might expect there to be a quarter supersymmetric solution which corresponds to a superposition of a stringy cosmic string solution (which has $f = 0$ and breaks half of the supersymmetry) with a domain wall solution which will have a more complicated spacetime geometry than the examples considered here. Solutions to the quarter supersymmetric differential equations are more difficult to find. It is perhaps most convenient to eliminate the fields $B$ and $\phi$; this then gives

\[
-\partial_w z^i = A^{-1} \partial_{\bar{w}} A \bar{f}^{-1} \gamma^{ij} D_j \bar{f}
\]

\[
\partial_w \partial_{\bar{w}} A = -|f|^{-2} |\partial_w A|^2 \gamma^{ij} D_j f D_j \bar{f}.
\]

(78)

In principle, given the appropriate boundary conditions for a string-domain wall superposition, one may solve these equations numerically.

**Acknowledgements**

I thank G. Papadopoulos for useful conversations. J.G. is supported by a EPSRC post-doctoral grant. This work is partially supported by SPG grant PPA/G/S/1998/00613.

**Appendix: Spinor Notation**

It is most convenient to present the supersymmetry transformations in terms of a 4-component Majorana spinor $\epsilon$ with real components. We define $\sigma^M = (\sigma^M_{\alpha \dot{\beta}})$ as;

\[
\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(79)

We set $\eta_{MN} = \text{diag}(-1, 1, 1, 1), \epsilon^{12} = \epsilon_{21} = 1$, and to perform the supersymmetry
calculations we define explicitly
\[ \Gamma_x = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \Gamma_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \quad \Gamma_z = \begin{pmatrix} 0 & -\sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \]
(80)
so that
\[ \Gamma^5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]
(81)
With these definitions, the gamma matrices satisfy the Clifford algebra
\[ \Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2\eta_{MN}. \]
(82)

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