1. Definitions and statement of the main theorem

The purpose of this note is to prove the following theorem:

**Theorem 4.1.** Let $S$ be a closed orientable surface with genus $g(S) \geq 3$. Then the pants complex of $S$ has only one end. In fact, there are constants $K = K(S)$ and $M_3 = M_3(S)$ so that, if $R > M_3$, any pair of pants decompositions $P$ and $Q$, at distance greater than $KR$ from a basepoint, can be connected by a path which remains at least distance $R$ from the basepoint.

Recall that a pants decomposition of $S$ consists of $3g(S) - 3$ disjoint essential non-parallel simple closed curves on $S$. Each component of the complement of the curves is a sphere with 3 holes or pair of pants. Then the pants complex $C_P(S)$ is the metric graph whose vertices are pants decompositions of $S$, up to isotopy. Two vertices $P, P'$ are connected by an edge if $P, P'$ differ by an elementary move. In an elementary move all curves in the pants are fixed except for one curve $\alpha$. Removing $\alpha$, the component of the complement of the remaining curves that contains $\alpha$ is either a punctured torus or a sphere with 4 holes. Then $\alpha$ is replaced by a curve $\beta$ contained in this domain that intersects $\alpha$ minimally; in the punctured torus case once, and in the sphere case twice. All edges of $C_P(S)$ are assigned length 1. We let $d(\cdot, \cdot)$ be the distance function in $C_P(S)$. The pants complex $C_P(S)$ is known to be connected [HT].

Recall that a metric space $(X, d)$ has one end if for any basepoint $O \in X$ and any radius $R$ the complement of $B_R = B_R(O)$, the ball of radius $R$ centered at $O$, has only one unbounded component. It is easy to see that the definition does not depend on the choice of the point $O$. Clearly having one end is a quasi-isometry invariant. So, following Brock [B], our theorem implies:

**Corollary 1.1.** Teichmüller space, equipped with the Weil-Peterson metric, has only one end.

Finally, recall that the curve complex $C(S)$ is the complex whose $k$-simplices consist of $k + 1$ distinct isotopy classes of essential simple closed curves on $S$ that have disjoint representatives on $S$. Or, in the case of a once-punctured torus and four-punctured sphere, $C(S)$ is the Farey graph. From the metric point of view we will only interested in the 1-skeleton of $C(S)$. Each edge is assigned length 1. We let $d_S(\cdot, \cdot)$ denote the distance function in $C(S)$.
2. The set of handle curves is connected

In this section we prove two combinatorial facts. First, the set of handle curves in the curve complex is connected and second, any pants decomposition is a bounded distance (in the pants complex) from a decomposition containing a handle curve.

Again assume $S$ is a closed orientable surface with genus three or greater. We will call $\alpha$ a handle curve in $S$ if $\alpha$ separates $S$ into two surfaces: the handle $S(\alpha) \cong T^2 \setminus \{\text{pt}\}$ and the rest of the surface.

We will need the following result. It was first proved by Farb and Ivanov \cite{FI} by different methods. Another proof was given by McCarthy and Vautaw \cite{MV} by methods similar to ours. We include a proof for completeness.

**Proposition 2.1.** If $g(s) \geq 3$, the subcomplex $h(S) \subset C(S)$ of handle curves is connected.

**Remark 2.2.** Note that the hypothesis $g(S) \geq 3$ cannot be removed; it is easy to check that the handle curves for a closed surface of genus 2 do not form a connected set.

**Remark 2.3.** Note that Proposition 2.1 immediately implies that the set of separating curves in $C(S)$ is also connected.

**Remark 2.4.** We note that Proposition 2.1 may be generalized to the case $\partial S \neq \emptyset$. A still open question is the higher connectivity of $h(S)$.

Before we begin the proof we will require a bit of terminology. Let $i(\cdot, \cdot)$ denote the geometric intersection number of two essential simple closed curves in $S$. Also, if $\delta$ is a dividing curve in $S$ we say that an arc $\beta'$ is a wave for $\delta$ if $\beta' \cap \delta = \partial \beta'$ and $\beta'$ is essential as a properly embedded arc in $S \setminus \delta$. We say that two waves $\beta'$ and $\beta''$ for $\delta$ link if $\beta' \cap \beta'' = \emptyset$, both $\beta'$ and $\beta''$ meet the same side of $\delta$, and $\partial \beta'$ separates $\partial \beta''$ inside $\delta$.

Finally we define double surgery as follows. Suppose we are given a linking pair of waves $\beta'$ and $\beta''$ for an essential dividing curve $\delta_0$. Form the closed regular neighborhood $U = \text{neigh}(\delta_0 \cup \beta' \cup \beta'')$. Let $\delta_1$ be the component of $\partial U$ which is not homotopic in $S$ to $\delta_0$. We say that $\delta_1$ is obtained from $\delta_0$ via double surgery along $\beta'$ and $\beta''$. Note that $\delta_1$ is necessarily a dividing curve and is disjoint from $\delta_0$. If the component of $S \setminus \delta_0$ containing $\beta' \cup \beta''$ is not a handle then $\delta_1$ is also essential.

We are now equipped to prove the proposition:

**Proof of Proposition 2.1.** Let $\alpha, \beta \in h(S)$ be handle curves and $S$ a closed orientable surface of genus at least three. Suppose that $\alpha$ and $\beta$ are tight: $\alpha$ has been isotoped to make $|\alpha \cap \beta| = i(\alpha, \beta)$. If $i(\alpha, \beta) = 0$ then there is nothing to prove. If $i(\alpha, \beta) > 0$ we will find a curve $\gamma \in h(S)$ with $i(\gamma, \alpha) = 0$ and $i(\gamma, \beta) < i(\alpha, \beta)$. By induction, $\gamma$ will be connected to $\beta$ in $h(S)$, proving the proposition.

We find $\gamma$ via the following inductive procedure. Recall that $S(\alpha)$ is the handle which $\alpha$ bounds. To begin, we define $\delta_0 \subset S \setminus S(\alpha)$ to be a parallel copy of $\alpha$, still intersecting $\beta$ tightly. At stage $k$ by induction we will be given an essential dividing curve $\delta_k$ where

- $i(\alpha, \delta_k) = 0$,
- $\delta_k$ is tight with respect to $\beta$, and
- $i(\delta_k, \beta) < i(\delta_{k-1}, \beta)$, if $k > 0$.

Let $T_k$ be the component of $S \setminus \delta_k$ which does not contain $\alpha$. If $T_k$ is a handle, then we take $\gamma = \delta_k$ and we are done with the inductive procedure. If $i(\delta_k, \beta) = 0$ then we may take $\gamma$
to be any handle curve inside $T_k$. As this $\gamma$ satisfies $i(\alpha, \gamma) = i(\beta, \gamma) = 0$ this would finish the proposition. From now on we assume that $T_k$ is not a handle and that $i(\delta_k, \beta) > 0$. 

We now attempt to do a double surgery of $\delta_k$ into $T_k$. Either we will find $\gamma$ directly or the curve resulting from double surgery, $\delta_{k+1}$, will satisfy the induction hypothesis. As the geometric intersection with $\beta$ is always decreasing, this procedure will stop after finitely many steps yielding the desired handle curve.

So all that remains is to do the double surgery. Recall that we are given $\alpha, \beta$ tightly intersecting handle curves and we are also given $\delta_k$ satisfying the induction hypotheses. Recall also that $T_k$ is the component of $S \setminus \delta_k$ which does not contain $\alpha$. Recall $T_k$ is not a handle and that $i(\delta_k, \beta) > 0$.

Suppose further that $\beta', \beta'' \subset \beta \cap T_k$ are linking waves for $\delta_k$. As described above we may form $\delta_{k+1}$ via a double surgery along $\beta'$ and $\beta''$. Isotope $\delta_{k+1}$, in the complement of $\delta_k$, to be tight with respect to $\beta$. As noted in the definition of double surgery, $\delta_{k+1}$ is an essential dividing curve which is disjoint from $\alpha$. Finally note that $i(\delta_{k+1}, \beta) \leq i(\delta_k, \beta) - 4$. Thus all of the induction hypotheses are satisfied.

Suppose now that we cannot find linking waves among the arcs $\beta \cap T_k$. Choose instead an outermost wave $\beta' \subset \beta \cap T_k$: that is, there exists an arc $\delta_k' \subset \delta_k$ such that $\delta_k' \cap \beta = \partial \delta_k' = \partial \beta'$.

Here there are two remaining cases. If $\delta_k' \cup \beta'$ is a separating curve take $\delta_{k+1} = \delta_k' \cup \beta'$ and note that the induction hypotheses are easily verified. The final possibility is that $\delta_k' \cup \beta'$ is not separating. In this case choose a properly embedded essential arc $\beta'' \subset T_k$ such that $\beta'' \cap \beta = \emptyset$ and $|\beta'' \cap \delta_k'| = 1$. Then $\beta'$ and $\beta''$ link. Do a double surgery along these waves to obtain $\delta_{k+1}$. Isotope $\delta_{k+1}$, in the complement of $\delta_k$, to be tight with respect to $\beta$. Again, all of the induction hypotheses are easily verified, as we have $i(\delta_{k+1}, \beta) \leq i(\delta_k, \beta) - 2$. This completes the second induction step and hence completes the proof of Proposition 2.1.

We also require

**Lemma 2.5.** There is a constant $M_3 = M_3(S)$ such that the pants decompositions containing a handle curve are $M_3$-dense in the space of all pants decompositions.

**Proof.** The mapping class group acts co-compactly on the space of pants decompositions. □

### 3. Subsurface projections and distances

Here we give two lemmas studying the pants complex. The first gives a condition for a pants decomposition to lie outside of a large ball about the origin in $C_P(S)$ while the second provides us with useful paths laying outside of such a ball. This uses techniques developed by Masur and Minsky [MM].

Given a subsurface $W \subset S$ and a curve $\gamma$ that intersects $W$, we may define a projection of $\gamma$ to $C(W)$ which associates to $\gamma$ a collection of curves in $C(W)$. Namely, the intersections of $\gamma$ with $W$ fall into finitely many homotopy classes of disjoint arcs (and curves) relative to $\partial W$. Let $\alpha$ be any such arc or curve. Let $U$ be a regular neighborhood of $\alpha \cup \partial W$. Then $\partial U \setminus \partial W$ is a curve in $C(W)$ if $\alpha$ is a curve or is an arc connecting distinct components of $\partial W$. Otherwise $\partial U \setminus \partial W$ is a pair of curves in $C(W)$. Clearly this curve or pair of curves in $C(W)$ depend only on the homotopy class of $\alpha$. If $\alpha, \beta$ are disjoint homotopy classes of arcs, then any pair of curves $\alpha', \beta'$ built out of this surgery satisfy $d_W(\alpha', \beta') \leq 2$ (MM)
Lemma 2.3). Thus we can define $\pi_W(\gamma)$ to be the corresponding subset of diameter at most 2 in $C(W)$.

Similarly, given a pants decomposition $P$ we may project each curve of $P$ that intersects $W$ into $W$. We denote the resulting image set in $C(W)$, which has diameter at most 4, by $\pi_W(P)$. By $d_W(P, P')$ we mean the distance in the curve complex of $W$ between the sets $\pi_W(P)$ and $\pi_W(P')$.

Let $[x]_C$ be the function on $N$ giving zero if $x < C$ and giving $x$ if $x \geq C$. We will need the following result from [MM] (see Theorem 6.12 and Section 8 of that paper):

There is a constant $C_0 = C_0(S) \geq 1$ such that for any $C \geq C_0$ there are constants $M_1 = M_1(C) \geq C$ and $M_2 = M_2(C) \geq 0$ with the following property: for any pants decompositions $P, P'$ we have

$$
\frac{1}{M_1} \sum_V [d_V(P, P')]_C - M_2 \leq d(P, P') \leq M_1 \sum_V [d_V(P, P')]_C + M_2,
$$

(1)

where the sums range over subsurfaces $V \subset S$ with essential boundary and where $V$ is not an annulus nor is it a thrice-punctured sphere.

Fix now such a $C > 2$. It follows from equation (1) that the projections that are at the critical values $C, C + 1$ cannot account for the entire distance. Namely there are constants $c = c(C) > 1$ and $M_4 = M_4(C) > 0$ such that

$$
\sum_V [d_V(P, P')]_C \leq c \cdot \sum_V [d_V(P, P')]_{C + 2} + M_4.
$$

(2)

Choose $K = K(C) > 0$ so that for all $R \geq 1$,

$$
\frac{1}{2cM_1} ((K - 1)R - M_2 - M_1M_4 - cM_1^2(R + M_2)) > M_1(R + M_2)
$$

(3)

Also, choose a basepoint $O \in C_P(S)$ and let $B_R = B_R(O)$ be the ball of radius $R$ centered at $O$.

Lemma 3.1. Fix a handle curve $\alpha$ and some curve $\alpha'' \subset S(\alpha)$ satisfying $d_{S(\alpha)}(O, \alpha'') > M_1(R + M_2)$. For any pants decomposition $P$ containing $\alpha''$ we have $P \notin B_R$.

Proof. Note that $d_{S(\alpha)}(O, \alpha'') \geq C$. So, by the left inequality of (1) any pants $P$ containing $\alpha''$ has

$$
d(P, O) \geq \frac{1}{M_1} [d_{S(\alpha)}(P, O)]_C - M_2 \geq \frac{1}{M_1} d_{S(\alpha)}(\alpha'', O) - M_2 > R.
$$

As the Farey graph for $S(\alpha)$ has infinite diameter, and as the diameter of $\pi_{S(\alpha)}(O)$ is bounded, such curves $\alpha''$ exist in abundance. We now turn to the existence of paths lying outside of the $R$-ball about the basepoint.

Lemma 3.2. Suppose $P_0$ is a pants decomposition of $S$ such that $P_0 \notin B_{(K-1)R}$ and $P_0$ contains a curve $\alpha$ which bounds a handle $S(\alpha)$. Then there is a path $P_t$ starting at $P_0$ such that

- for all $t$, $P_t|(S \setminus S(\alpha)) = P_0|(S \setminus S(\alpha))$.
- for all $t$, $P_t \notin B_R$
• The endpoint of the path, $P_1$, contains a curve $\alpha'' \subset S(\alpha)$ which does not appear in any pants decomposition in $B_R$.

Proof. Let $\alpha' \in P_0$ be the curve strictly contained in $S(\alpha)$. Consider a geodesic segment in the Farey graph connecting $\alpha'$ to $\beta \in \pi_{S(\alpha)}(O)$, where $\beta$ is chosen as close as possible to $\alpha'$. Extend this segment through $\alpha'$ to a geodesic ray $L$ in the direction opposite $\beta$. The ray $L$ meets the segment only at $\alpha'$. Move along $L$ distance more than $M_1 (R + M_2)$ from $\alpha'$ to a point $\alpha''$. Let $P_i$ be the path obtained by making elementary moves along the curves in $L$ and fixing the pants in $S \setminus S(\alpha)$.

Suppose first that $d_{S(\alpha)}(\beta, \alpha') > M_1 (R + M_2)$. Then by Lemma 3.1, any pants $\hat{P}$ containing any $\alpha_t \in L$ has $d(\hat{P}, O) > R$. So Lemma 3.2 holds in this case.

Next suppose that $d_{S(\alpha)}(\beta, \alpha') \leq M_1 (R + M_2)$. Then by (1) and (2)

$$(K - 1) R \leq d(P_0, O) \leq M_1 \sum V \sum [d_V(P_0, 0)]_C + M_2 \leq$$

$$\leq c M_1 \sum V \sum [d_V(P_0, 0)]_{C + 2} + M_1 M_4 + M_2 \leq$$

$$\leq c M_1 \sum V \sum [d_V(P_0, 0)]_{C + 2} + c M_1^2 (R + M_2) + M_1 M_4 + M_2.$$

Let $V$ be any subsurface disjoint from $S(\alpha)$. Since $P_i$ is constant in $V$, the projection $\pi_V(P_i)$ is constant. Now let $V$ be a subsurface that intersects $S(\alpha)$ or strictly contains $S(\alpha)$. Since $\alpha \in P_i$, it follows that $\pi_V(P_i)$ contains $\pi_V(\alpha)$. Since each $\pi_V(P_i)$ has diameter at most 2, $d_V(P_i, O) \geq d_V(P_0, O) - 2$. Thus for any subsurface $V$ not isotopic to $S(\alpha)$, as $C \geq 2$, we have $[d_V(P_i, O)]_C \geq \frac{1}{2} [d_V(P_0, O)]_{C + 2}$. Thus,

$$\sum V \sum [d_V(P_i, O)]_C \geq \frac{1}{2} \sum V \sum [d_V(P_0, 0)]_{C + 2} \geq$$

$$\geq \frac{1}{2c M_1} ((K - 1) R - M_2 - M_1 M_4 - c M_1^2 (R + M_2)) > M_1 (R + M_2),$$

the last inequality following from (3). So, by (1),

$$d(P_i, O) > R.$$

Finally, as the pants decomposition $P_1$ contains $\alpha''$ and

$$d_{S(\alpha)}(O, \alpha'') \geq d_{S(\alpha)}(\alpha', \alpha'') > M_1 (R + M_2)$$

we have $P_1 \notin B_R$, by Lemma 3.1. \hfill $\square$

4. Proof of the theorem

Recall the statement:

Theorem 4.1. Let $S$ be a closed orientable surface with genus $g(S) \geq 3$. Then the pants complex of $S$ has only one end. In fact, there are constants $K = K(S)$ and $M_3 = M_3(S)$ so that, if $R > M_3$, any pair of pants decompositions $P$ and $Q$, at distance greater than $KR$ from a basepoint, can be connected by a path which remains at least distance $R$ from the basepoint.
Proof. We take $M_3$ as defined in Section 2 and $K$ as defined in Section 3.

First move $P$ and $Q$ a distance at most $M_3 < R$ to obtain pants decompositions $P_0$ and $Q_0$ which contain handle curves $\alpha_P \in P_0$, $\alpha_Q \in Q_0$ and such that $P_0, Q_0 \notin B_{(K-1)R}$.

Apply Lemma 3.2 twice in order to connect $P_0$ and $Q_0$ to pants decompositions $P_1$ and $Q_1$ satisfying all of the conclusions of the lemma. Let $\alpha''_P$ and $\alpha''_Q$ be the curves lying in the handles $S(\alpha_P) \subset P_1$ and $S(\alpha_Q) \subset Q_1$ respectively.

We must now construct a path from $P_1$ to $Q_1$. Consider first the case where $\alpha_P \neq \alpha_Q$.

Applying Proposition 2.1 we connect $\alpha_P \in P_1$ and $\alpha_Q \in Q_1$ by a path $\{\alpha_i\}_{i=1}^n$ of handle curves in $\mathcal{C}(S)$, the curve complex of $S$. Here we have $\alpha_1 = \alpha_P$, $\alpha_n = \alpha_Q$, and $n > 1$. Note that in this step the hypothesis $g(S) > 2$ is used. Choose, for $i \in \{2, 3, \ldots, n-1\}$, any curve $\alpha''_i \subset S(\alpha_i)$ such that $d_S(\alpha_i)(O, \alpha''_i) > M_1(R + M_2)$. Set $\alpha''_1 = \alpha''_P$ and $\alpha''_n = \alpha''_Q$. Let $P_n = Q_1$.

Inductively, we connect $P_i$ by a path to $P_{i+1}$ where, first, every pants decomposition in the path contains $\alpha_i$ and $\alpha''_i$ and, second, $P_{i+1}$ also contains $\alpha_{i+1}$ and $\alpha''_{i+1}$. (This is possible because $C_P(S \setminus S(\alpha_i))$ is connected.) By Lemma 3.1 this path lies outside of the ball of radius $R$ and we are done.

In the case which remains we have $\alpha_P = \alpha_Q$. So there is no need for Proposition 2.1. Instead we choose any handle curve $\beta$ which is disjoint from $\alpha_P$. Note that $\beta$ exists as $g(S) > 2$. Choose also any $\beta''$ satisfying the hypothesis of Lemma 3.1. We now consider the sequence $\alpha_P, \beta, \alpha_Q$ as a path of length two in the curve complex and connect $P_1$ to $Q_1$ as in the previous paragraph. This completes the proof. \qed

References

[FI] B. Farb, N. Ivanov Torelli geometry and commensurations of the Torelli group, preprint

[B] J. Brock The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores J. Amer. Math. Soc. 16 (2003) 495-535.

[HT] A. Hatcher, W. Thurston A presentation for the mapping class group Topology 19 (1980) 221-237

[MM] H. Masur, Y. Minsky Geometry of the complex of curves II: Hierarchical structure GAFA 10 (2000) 902-974