Effective Dimension of Exp-concave Optimization

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Abstract
We investigate the role of the effective dimension \(d_\lambda\) in determining both the statistical and the computational costs associated with exp-concave stochastic minimization. Our main statistical result is a nearly tight bound of order \(d_\lambda/\varepsilon\) on the sample complexity of any algorithm that approximately minimizes the empirical risk. Our main algorithmic contribution is a fast preconditioned method that solves the ERM problem in time \(\tilde{O} \left( \min \left\{ \frac{1}{\lambda} \left( \text{nnz}(A) + d_\lambda^2 \right) : \lambda' \geq \lambda \right\} \right)\), where \(\text{nnz}(A)\) is the number of nonzero entries in the data.

Our results shed a light on two central sketching approaches named “sketch-and-solve” and “sketch-to-preconditioning”. Our statistical result render the first approach redundant (in the context of bounded exp-concave minimization). On the contrary, our computation results highlight the efficacy of the latter approach. Our analysis emphasizes interesting connections between leverage scores, algorithmic stability and regularization, which might be of independent interest.

1 Introduction

Exp-concave stochastic optimization underlies many important machine learning problems such as linear regression, logistic regression and portfolio selection. While the worst-case complexity of exp-concave stochastic optimization is fairly understood ([23, 29, 19, 16]), a promising avenue is to investigate these complexities under distributional assumptions. A common distributional condition which can be exploited potentially is fast eigen-decay (measured quantitatively by the notion of effective dimension (see Equation (3))) ([13, 5, 26, 1]). Namely, in many machine learning problems, the eigenvalues associated with the population covariance matrix exhibit a
fast decay, where the tail of the eigenvalues are significantly smaller than the
desired precision. Naturally, this phenomenon suggests a sketch-and-solve
approach, where a sufficiently accurate solution is obtained by projecting
the data onto a low-dimensional space and solving the smaller problem. In-
deed, many algorithmic ideas in this spirit have been suggested in the recent
years (e.g. [3, 26]).

A more sophisticated approach, which we name sketch-to-precondition
([2, 9]), is to enhance the performance of first-order optimization methods
via preconditioning, where the preconditioner is based on a coarse low-rank
approximation to the data matrix. The main message of our paper is as
follows:

**Main message:** The sample complexity of any algorithm minimizing
an exp-concave empirical risk scales optimally with the effective dimension,
rendering the sketch-and-solve approach useless in the statistical setting.
On the other hand, the sketch-to-precondition approach is effective for op-
timization and can be accelerated via model selection.

To illustrate this message, we next describe our results in the context of
both linear and Kernelized $\ell_2$-regression.

### 1.1 Results for Linear and Kernel $\ell_2$-regression

Consider the task of minimizing

\[
F(w) = \frac{1}{2} \mathbb{E}_{(x,y) \sim \mathcal{D}}[(w^\top x - y)^2],
\]

over a compact set $\mathcal{W} \subseteq \mathbb{R}^d$. Here, $\mathcal{D}$ is a distribution over $\mathbb{R}^d \times [-1, 1]$ which
satisfies $\mathbb{P}_{x \sim \mathcal{D}} (\forall w \in \mathcal{W}: |\langle w, x \rangle| \leq 1) = 1$. We denote the minimizer by $w^*$. As usual, the input to the learning algorithm consists of an i.i.d. sample
$S = ((x_i, y_i))_{i=1}^n \sim \mathcal{D}^n$. Our focus is on algorithms that minimize the em-
pirical risk over $\mathcal{W}$. Although regularization is not needed for generalization
purposes (as shown by [16]), for reasons that will become apparent soon, we
introduce a ridge parameter

\[
\lambda \equiv \frac{\epsilon}{B^2}, \quad B \equiv \text{diam}(\mathcal{W}),
\]

and consider the minimization problem:

\[
\hat{w}_\lambda \equiv \arg\min \left\{ \hat{F}_\lambda(w) \equiv \frac{1}{2n} \sum_{i=1}^n (w^\top x_i - y_i)^2 + \frac{\lambda}{2} \|w\|^2 : w \in \mathcal{W} \right\}.
\]


Tight sample complexity bound in terms of the effective dimension: we define the sample complexity \( n : \mathbb{R}_{>0} \rightarrow \mathbb{N} \) as the minimal number of samples required for ensuring that \( F(\hat{w}) - F(w^*) \leq \epsilon \). As we mentioned above, sample complexity bounds for this formulation are well-understood. Namely, results from \([19, 29, 16]\) imply that \( n(\epsilon) = \Theta \left( \min \left\{ \frac{d}{\epsilon}, \frac{\|\epsilon\|_2^2}{\epsilon} \right\} \right) \).

We refer to the leftmost term as a dimension-dependent fast rate (i.e., it scales with \( 1/\epsilon \) rather than with \( 1/\epsilon^2 \)), whereas the right term is a dimension-independent slow rate. While the above bound is tight, it can be significantly improved if the spectrum of the covariance of the underlying data decays fast. A common measure used to capture this decay is the effective dimension, defined by

\[
d_{\lambda} = d_{\lambda} \left( \mathbb{E}_{x \in \mathcal{D}_x} [xx^\top] \right) = \sum_{i=1}^{d} \frac{\lambda_i}{\lambda_i + \lambda},
\]

where \( \lambda_1 \geq \ldots \geq \lambda_d \) are the eigenvalues of the population covariance matrix \( C = \mathbb{E}_{x \in \mathcal{D}_x} [xx^\top] \). Clearly, \( d_{\lambda} \leq d \). However, it is very typical that most of the eigenvalues are dominated by \( \lambda \), and consequently \( d_{\lambda} \ll d \). For example, if the spectrum decays exponentially, the effective dimension is polylogarithmic in \( B \) ([13]). Our sample complexity bound in this setting is as follows.

**Theorem 1.** The sample complexity of linear regression satisfies

\[
n(\epsilon) = O \left( \frac{d_{\lambda}}{\epsilon} \right).
\]

where \( \lambda = \frac{\epsilon}{B^2} \).

We also prove a nearly matching lower bound (Theorem 6) in a high accuracy regime and specify our bounds for several regimes of interest corresponding to eigendecay patterns.

Essentially, we enjoy the best of the two worlds, as our bound is is both fast (in terms of \( \epsilon \)) and dimension-independent. Also note that the bound is independent of the \( \ell_2 \) diameter, \( B \). The only dependence on \( B \) is implicit through the definition of \( \lambda \). Indeed, while \( B \) can be trivially used to bound the magnitude of the prediction, such a bound is often loose due to a failure of the \( \ell_2 \)-metric to capture the geometry of the problem (e.g., due to sparsity).

**Redundancy of Sketch-and-Solve:** It is instructive to examine the sketch-and-solve approach ([26]), whereby one uses leverage score sampling to find
a small $(1 \pm \epsilon)$-spectral approximation to the empirical covariance (respectively, the kernel) matrix using a subsample of size $\tilde{O}\left(\frac{d}{\epsilon}\right)$, and then solves the corresponding smaller problem (see Section C of [26] for more details).

While there are relatively efficient methods for approximating the leverage scores, their computation is clearly more involved than sampling uniformly at random. In some sense, our sample complexity result shows that the sketch-and-solve is redundant.\(^1\) Namely, our bound implies that the same (additive) accuracy we can attain the same (additive) accuracy by sampling a training sub-sequence of the same size uniformly at random.

**Efficacy of Sketch-to-Precondition in Optimization:** A different approach is to use ridge leverage score sampling in order to improve the condition number of the optimization problem. Instead of aiming at $(1 \pm \epsilon)$-spectral approximation, we draw only $\tilde{O}(d\lambda)$ samples to compute a constant spectral approximation to the empirical covariance matrix. This approximation is used to reduce the condition number to a constant order (see Section 5). Notably, maintaining this preconditioner (i.e., computing it and multiplying any $d$-dimensional vector by its inverse) can be done in time $O(d^2\lambda d)$. By endowing Gradient Descent (GD) with this preconditioner, we can find an $\epsilon$-approximate ERM in time $\tilde{O}(\text{nnz}(A) + d^2\lambda d)$.

As we discussed above, the regularization parameter used in practice is often chosen via model selection. Both our sample and computational complexity bounds shed light on the bias-complexity trade-off reflected by the choice of $\lambda$. Namely, as we increase $\lambda$, the effective dimension (and hence the complexity) become smaller, whereas the bias increases. In Section 6 we show that even if we have already chosen a desired regularization parameter $\lambda$ (e.g., $\lambda = \epsilon/B^2$), as we describe next, we may still achieve a significant gain by performing optimization with $\lambda' > \lambda$. Namely, the effective dimension associated with $\lambda' > \lambda$ might be much smaller, and we can compensate for using a larger ridge parameter by repeating the optimization process $O(\lambda'/\lambda)$ times. The main challenge we need to tackle is that the cost of computing the effective dimension associated with each candidate parameter $\lambda'$ dominates the entire optimization process. The main contribution described in Section 6 is a new algorithm which finds the best ridge candidate by iteratively sharpening its estimates to the corresponding effective dimensions.

**Theorem 2.** There exists an algorithm that finds an $\epsilon$-approximate mini-

\(^1\)Note that the boundedness assumption is crucial here. Sketch-and-solve can be very helpful when instances are not bounded (and instead of additive accuracy we aim at multiplicative accuracy).
mizer to (2) in time

\[ \tilde{O} \left( \min_{N \geq \lambda} \left\{ \text{nnz}(A) + \frac{N^2}{\lambda} d^2 \right\} \right). \]

In Appendix D we explain how the above results extend to the kernel setting.

2 Related work

2.1 Sample complexity bounds

To the best of our knowledge, the first bounds for empirical risk minimization for kernel ridge regression in terms of the effective dimension have been proved by [31]. By analyzing the Local Rademacher complexity ([6]), they proved an upper bound of \( O(d_\lambda B^2/\epsilon) \) on the sample complexity. On the contrary, our bound has no explicit dependence on \( B \). More recently, [13] used compression schemes ([21]) together with results on leverage score sampling from [26] in order to derive a bound in terms of the effective dimension with no explicit dependence on \( B \). However, their rate is slow in terms of \( \epsilon \).

Aside from improving the above aspects in terms of accuracy, rate and explicit dependence on \( B \), our analysis is arguably simple and underscores nice connections between algorithmic stability and ridge leverage scores.

2.1.1 Online Newton Sketch

The Online Newton Step (ONS) algorithm due to [17] is a well-established method for minimizing exp-concave loss functions both in the stochastic and the online settings. As hinted by its name, each step of the algorithm involves a conditioning step that resembles a Newton step. Recent papers reanalyzed ONS and proved upper bounds on the regret (and consequently on the sample complexity) in terms of the effective dimension ([22, 8]). We note that using a standard online to batch reduction, the regret bound of [22] implies the same (albeit a little weaker in terms of constants) sample complexity bounds as this paper. While ONS is certainly appealing in the context of regret minimization, in the statistical setting, our paper establishes the sample complexity bound \textit{irrespective} of the optimization algorithm used for the intermediate ERM step, thereby establishing that the computational overhead resulted by conditioning in ONS is not required.\footnote{We also do not advocate ONS for offline optimization, as it does not yield linear rate (i.e., \( \log(1/\epsilon) \) iterations).}
2.2 Sketch-and-Solve vs. Sketch-to-Precondition

As we discussed above, the sketch-and-solve approach (e.g. see the nice survey by [30]) has gained considerable attention recently in the context of enhancing both discrete and continuous optimization ([22, 15, 14, 9]). As we briefly mentioned above, a recent paper by [26] suggested to combine ridge leverage score sampling with the Nyström method to compute a spectral approximation of the Kernel matrix. As an application, they consider the problem of Kernel ridge regression and describe how this spectral approximation facilitates the task of finding $\epsilon$-approximate minimizer in time $O(ns^2)$, where $s = \tilde{O}(d\lambda/\epsilon)$. Based on Corollary 4 (with $n = O(d\lambda/\epsilon)$), our complexity is better by factor of $\Omega(\min\{1/\epsilon^2, d\lambda/\epsilon\})$.

We would like to stress that our results only obviate the necessity of the sketch-and-solve approach in the statistical setting, where we assume boundedness and aim at additive error bounds. On the other hand, most of sketch-and-solve results (e.g., [26]) are multiplicative and do not require boundedness.

The Sketch-to-precondition approach is more appealing in scenarios where machine precision accuracy is required ([30][Section 2.6]). In Appendix 5 we review this approach in detail and describe a corresponding preconditioned GD that solves the empirical risk in time $O(\text{nnz}(A) + d^2\lambda)$ (or $O(n^2 + d^2\lambda n)$ respectively in the Kernel setting). A different application of the sketch-to-precondition approach, due to [2], focuses on polynomial Kernels and yields an algorithm whose runtime resembles our running time but also scales exponentially with the polynomial degree.

3 Preliminaries

3.1 Problem Setting

We consider the problem of minimizing the expected risk

$$F(w) = \mathbb{E}_{(x,y) \sim \mathcal{D}}[\phi_y(w^\top x)],$$

over a compact and convex set $W \subseteq \mathbb{R}^d$ whose diameter is denoted by $B$. Following [16], we assume that for all $y \in \mathcal{Y}$, $\phi_y$ is twice-continuously differentiable and satisfies the following assumptions:

1. **Lipschitzness**: for all $(w, x) \in W \times \mathcal{X}$, $|\phi'_y(w^\top x)| \leq \rho$.

2. **Strong convexity**: for all $(w, x) \in W \times \mathcal{X}$, $\phi''_y(w^\top x) \geq \alpha$. 


3. Smoothness: for all \((w, x) \in \mathcal{W} \times \mathcal{X}\), \(\phi''(w^\top x) \leq \beta\).

As noted in [16], our framework includes all known \(\alpha/\rho^2\)-exp-concave functions. A prominent example illustrated below is bounded \(\ell_2\)-regression. Further examples include logistic regression and log-loss ([17]).

**Example 1.** Bounded \(\ell_2\)-regression: let \(Y \in \{-1, 1\}\) and let \(W, X\) be two compact sets in \(\mathbb{R}^d\) such that \(\forall w \in W, x \in X, |w^\top x| \leq 1\). The loss is defined by \(\phi_y(z) = \frac{1}{2}(z - y)^2\). It is easily verified that \(\alpha = \beta = 1\) and \(\rho = 2\).

The input to the learning algorithm consists of an i.i.d. sample \(S = ((x_1, y_1), \ldots, (x_n, y_n)) \sim \mathcal{D}^n\). A popular practice is regularized loss minimization (RLM) which, given a regularization parameter \(\lambda\), is defined as

\[
\hat{w}_\lambda = \arg\min_{w \in W} \hat{F}_\lambda(w) = \arg\min_{w \in W} \left( \frac{1}{n} \sum_{i=1}^{n} \phi_{y_i} (w^\top x_i) + \frac{\lambda}{2} \|w\|^2 \right).
\]

We also define the unregularized empirical loss as

\[
\hat{F}(w) = \sum_{i=1}^{n} \phi_{y_i} (w^\top x_i).
\]

The strong convexity of \(\phi\) implies the following property of the empirical loss (e.g. see Lemma 2.8 of [28]).

**Lemma 1.** Given a sample \(S\), let \(\hat{w}_\lambda\) be as defined in Equation (5). Then for all \(w \in W\),

\[
\hat{F}_\lambda(w) - \hat{F}_\lambda(\hat{w}_\lambda) \geq \frac{\alpha}{2} (w - \hat{w}_\lambda)^\top \left( \frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top + \frac{\lambda}{\alpha} I \right) (w - \hat{w}_\lambda).
\]

### 3.2 Sketching via leverage-score sampling

In this section we define the notion of ridge leverage scores, relate it to the effective dimension and explain how sampling according to these scores facilitates the task of spectral approximation.

Given a sample \((x_1, \ldots, x_n)\), we define the data matrix by

\[
A = [a_1, \ldots, a_n] = n^{-1/2} [x_1; \ldots, x_n] \in \mathbb{R}^{n \times d}
\]

\(^3\)This assumption is only required for our optimization results.
Given a ridge parameter \( \lambda > 0 \), we define the \( i \)-th leverage score by

\[
\tau_{\lambda,i} = a_i^\top (A^\top A + \lambda I)^{-1} a_i.
\]

It’s easily seen that \( d_\lambda = \sum \tau_{\lambda,i} \). The following lemma intuitively says that the (ridge) leverage score captures the importance of the \( i \)-th example in composing the column space of the covariance matrix. The proof is detailed in Appendix F.

**Lemma 2.** For a ridge parameter \( \lambda > 0 \) and for any \( i \in [n] \), \( \tau_{\lambda,i} \) is the minimal scalar \( t \geq 0 \) such that \( a_i a_i^\top \leq t(A^\top A + \lambda I) \).

The notion of leverage scores give rise to a natural algorithm for spectral approximation by sampling rows with probability proportional to the corresponding ridge leverage scores. Before describing the sampling procedure, we define the goal of spectral approximation.

**Definition 1. (Spectral approximation)** We say that a matrix \( \tilde{A} \) is a \((\lambda, \epsilon)\)-spectral approximation to \( A \) if

\[
1 - \epsilon \leq \frac{1}{1 + \epsilon}(A^\top A + \lambda I) \leq \tilde{A}^\top \tilde{A} + \lambda I \leq A^\top A + \lambda I.
\]

**Definition 2. (Ridge Leverage Score Sampling)** Let \((u_i)_{i=1}^n \) be a sequence of ridge leverage score overestimates, i.e., \( u_i \geq \tau_{\lambda,i} \) for all \( i \). For a fixed positive constant \( c > 0 \) and accuracy parameter \( \epsilon \), define \( p_i = \min\{1, ce^{-u_i} \log d\} \) for each \( i \in [n] \). Let \( \text{Sample}(u, \epsilon) \) denote a function which returns a diagonal matrix \( S \in \mathbb{R}^{n \times n} \), where \( S_{i,i} = ((1 + \epsilon)p_i)^{-1/2} \) with probability \( p_i \) and 0 otherwise.

**Theorem 3.** \([24, 26]\) Let \((u_i)_{i=1}^n \) be ridge leverage score overestimates, and let \( S = \text{Sample}(u, \epsilon) \).

1. With high probability, \( SA \) is a \((\lambda, \epsilon)\)-spectral approximation to \( A \).
2. With high probability, \( S \) has at most \( \tilde{O}(\epsilon^{-2}\|u\|_1) \) nonzero entries. In particular, if \( \tau_{\lambda,i} \leq u_i \leq C \tau_{\lambda,i} \) for some constant \( C > 1 \), then \( S \) has at most \( \tilde{O}(\epsilon^{-2}d_\lambda) \) nonzero entries.
3. There exists an algorithm which computes \((u_i)_{i=1}^n \) with \( \frac{1}{2} \tau_{\lambda,i} \leq u_i \leq \tau_{\lambda,i} \) for all \( i \) in time \( \tilde{O}(\text{nnz}(A) + d_\lambda^2 d) \)

\(^4\)We use the symbols \( c \in (0, 1), C \geq 1 \) to denote global constants.
3.3 Stability

In this section we define the notion of algorithmic stability, a common tool to bound the generalization error of a given algorithm. Analogously to the definition of $\hat{w}_\lambda$ in (2), for each $i \in [n]$, we define $\hat{w}_{\lambda,i}$ to be the predictor produced by the algorithm on the sample $S^{(i)}$, obtained from $S$ by replacing the $i^{th}$ example with a fresh i.i.d. pair $(x'_i, y'_i)$. We can now define the stability terms

$$\Delta_i = f_i(\hat{w}_{\lambda,i}) - f_i(\hat{w}_\lambda) \text{ and } \Delta'_i = f_{i'}(\hat{w}_\lambda) - f_{i'}(\hat{w}_{\lambda,i}).$$

The following theorem relates the expected generalization error to the expected average stability.

**Theorem 4** ([7]). We have

$$\mathbb{E}_{S \sim \mathcal{D}^n}[F(\hat{w}_\lambda) - \hat{F}(\hat{w}_\lambda)] = \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \Delta_i\right].$$

4 Sample Complexity Bounds for Exp-Concave Minimization

In this section we show nearly tight sample complexity bounds for exp-concave minimization based on the effective dimension. Let $\hat{C} = \frac{1}{n} \sum_{i=1}^n x_ix_i^\top$.

**Theorem 5.** For any $\lambda > 0$ the excess risk of RLM is bounded as follows:

$$\mathbb{E}_{S \sim \mathcal{D}^n}[F(\hat{w}_\lambda) - F(w^*)] \leq \frac{8\rho^2 d_{\alpha}(C)}{\alpha n} + \frac{\lambda}{2}B^2$$

Choosing $\lambda = \epsilon/\alpha B^2$ gives us the following corollary.

**Corollary 1.** The sample complexity is bounded as $n(\epsilon) \leq \frac{16\rho^2 d_{\alpha}(C)}{\alpha \epsilon}$

**Remark 1.** To obtain high-probability bounds (rather than in expectation) we can employ the validation process suggested in [25].

**Proof of Theorem 5.** For a given sample $S = (x_i, y_i)_{i=1}^n$, define $\hat{\tau}$ to be the associated leverage scores with ridge parameter $\lambda/\alpha$. We first use Theorem 4 to relate the excess risk to the average stability:

$$\mathbb{E}[F(\hat{w}_\lambda) - F(w^*)] = \mathbb{E}[F(\hat{w}_\lambda) - \hat{F}(\hat{w}_\lambda)]$$
$$+ \mathbb{E}[\hat{F}(\hat{w}_\lambda) - F(w^*)] \leq \mathbb{E}[F(\hat{w}_\lambda) - \hat{F}(\hat{w}_\lambda)]$$
$$+ \mathbb{E}[\hat{F}_\lambda(\hat{w}_\lambda) - \hat{F}_\lambda(w^*)] + \frac{\lambda}{2}B^2 \leq \mathbb{E}\left[n^{-1} \sum \Delta_i\right]$$
$$+ \frac{\lambda}{2}B^2$$
It is left to bound the average stability. Towards this end we fix some $i \in [n]$. By the mean value theorem there exists $z = \alpha \hat{w}_\lambda + (1 - \alpha)\tilde{w}_\lambda$ with $\alpha \in [0,1]$ such that $\Delta_i = \phi'_y(x_i^T z) x_i^T (\hat{w}_\lambda - \tilde{w}_\lambda)$. We now have that

$$\Delta_i \leq \rho \cdot |x_i^T (\hat{w}_\lambda - \tilde{w}_\lambda)| = \rho \cdot \sqrt{(\hat{w}_{\lambda,i} - \tilde{w}_\lambda)^T x_i x_i^T (\hat{w}_\lambda - \tilde{w}_\lambda)}$$

(Lemma 2)

$$\leq \rho \cdot \sqrt{\frac{2n \cdot \hat{\tau}}{\alpha} \cdot \left(\hat{C} + \frac{\Delta_i}{\alpha}\right)} (\hat{w}_{\lambda,i} - \tilde{w}_\lambda)$$

(Lemma 1)

$$\leq \rho \cdot \sqrt{\frac{2n \cdot \hat{\tau}}{\alpha} \cdot \frac{\Delta_i + \Delta'_i}{n}} ,$$

where the last inequality uses the fact that $\hat{F}_\lambda(\hat{w}_{\lambda,i}) - \tilde{F}_\lambda(\hat{w}_\lambda) \leq \frac{\Delta_i + \Delta'_i}{n}$. Similarly, $\Delta'_i \leq \rho \cdot \sqrt{\frac{2n \cdot \hat{\tau}}{\alpha} \cdot \Delta_i + \Delta'_i}$, where $\hat{\tau}p$ is the $i$-th ridge leverage score corresponding to $S^{(i)}$.

Combining the above and using the inequality $(a + b)^2 \leq 2a^2 + 2b^2$, we obtain that

$$\Delta_i + \Delta'_i \leq \frac{4\rho^2 (\hat{\tau} + \hat{\tau}p)}{\alpha} \Rightarrow \frac{1}{n} \sum_{i=1}^{n} (\Delta_i + \Delta'_i) \leq \frac{4\rho^2}{\alpha n} \sum_{i=1}^{n} (\hat{\tau} + \hat{\tau}p) .$$

Since $\Delta_i$ and $\Delta'_i$ (similarly, $\hat{\tau}p$) are distributed identically, the result now follows from the following lemma whose proof is given in Appendix A.

**Lemma 3.** Let $x$ be a random variable supported in a bounded set of $\mathbb{R}^d$ with $\mathbb{E}[x x^T] = C$. Let $\hat{C} = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T$ where $x_i$ are i.i.d copies of $x$. Then we have that for any fixed $\lambda > 0$

$$\mathbb{E}[d_\lambda(\hat{C})] \leq 2d_\lambda(C)$$

We now state a nearly matching lower bound on the sample complexity. To exhibit a lower bound we consider the special case of linear regression. Notably, our lower bound holds for any spectrum specification. The proof appears in Appendix C

**Theorem 6.** Given numbers $B > 0$ and $\lambda_1 \geq \ldots \geq \lambda_d \geq 0$, define $d_\lambda = \sum_{i=1}^{d} \frac{\lambda_i}{\alpha + \lambda}$ and $\Lambda = \text{diag}(\lambda_1 \ldots \lambda_d)^5$. Then for any algorithm there exist a

\footnote{For any $x \in \mathbb{R}^d$, $\text{diag}(x) \in \mathbb{R}^{d \times d}$ is a diagonal matrix with $i^{th}$ entry $x_i$}
distribution $\mathcal{D}$ over $\mathbb{R}^d \times \mathbb{R}$ such that for any algorithm that returns a linear predictor $\hat{w}$, given $n \geq 2d/3$ independent samples from $\mathcal{D}$, satisfies

$$E_{S \sim \mathcal{D}^n} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}} \left[ \frac{1}{2} (\hat{w}^T x - y)^2 \right] \right] - \min_{w: \|w\| \leq B} \mathbb{E} \left[ \frac{1}{2} (\hat{w}^T x - y)^2 \right] \geq \frac{d_{\gamma/(n-B^2)}}{n}$$

for any $\gamma$ satisfying

$$d_{\gamma/(n-B^2)} - \sum_{i=1}^{d} \left( \frac{\lambda_i}{\gamma_i + \frac{1}{(n-B^2)}} \right)^2 \leq \gamma$$

(7)

To put the bound achieved by Theorem 6 into perspective we specialize the bound achieved for two popular cases for eigenvalue profiles defined in [13]. We say that a given eigenvalue profile $\lambda_1 \ldots \lambda_d \geq 0$ satisfies (C,p) Polynomial Decay if there exists numbers $C, p > 0$ such that $\lambda_i \leq Ci^{-p}$. Similarly it satisfies C-Exponential Decay if there exists a number $C > 0$ such that $\lambda_i \leq Ce^{-i}$. The following table specifies nearly matching upper and lower bounds for polynomial and exponential decays (see exact statements in Appendix E).

| Decay                      | Upper Bound | Lower Bound |
|----------------------------|-------------|-------------|
| Polynomial Decay (degree p)| $O \left( \frac{d \gamma^p}{\epsilon^2} \right)$ | $\Omega \left( \frac{d \gamma^p}{\epsilon^2} \right)$ |
| Exponential Decay          | $O \left( \frac{d \gamma^p}{\epsilon} \right)$ | $\tilde{\Omega} \left( \frac{d \gamma^p}{\epsilon} \right)$ |

5 Sketch-to-precondition: an overview

In this section we describe in more detail the sketch-to-precondition approach and specify it to exp-concave stochastic optimization. This scheme will serve as a basis for the acceleration technique presented in the next section.

For concreteness, suppose we apply Gradient Descent (GD) to minimize the regularized risk (5). Denote by $\hat{C}$ the empirical covariance matrix $\frac{1}{n} \sum_{i=1}^{n} x_i x_i^T$. As we assume that $\phi_y$ is $\beta$-smooth and $\alpha$-strongly convex, it can be easily verified that the entire regularized risk is $(\alpha \lambda_d \hat{C} + \lambda)$-strongly convex and $(\beta \lambda_1 \hat{C} + \lambda)$-smooth. Denote $\hat{F}$'s strong convexity and smoothness parameters by $\hat{\alpha}$ and $\hat{\beta}$, respectively. The quantity $\hat{\kappa} = \hat{\beta}/\hat{\alpha}$ is referred
to as the condition number of the regularized risk. It is well known (e.g., see [27]) that GD converges after $O(\hat{\beta}/\hat{\alpha})$ iterations. Note that if the eigen-decay is fast, the condition number may be much larger that the so-called functional condition number

$$\tilde{\kappa} \equiv \frac{\beta + \lambda}{\alpha + \lambda}.$$ 

Preconditioning can be seen as a change of variable, where instead of optimizing $\hat{F}(w)$ over $W$, we optimize $\hat{F}(P^{-1/2}w)$ over $P^{1/2}W$, where $P > 0$ is called a preconditioner. It can be easily verified (e.g. see [14]) that this operation amounts to replacing each instance $x_i$ with $P^{-1/2}x_i$ (after decomposing the regularization into a suitable form). Straightforward calculations show that The Hessian of $\hat{F}$ at any point $w$ becomes

$$P^{-1/2} \left( \frac{1}{n} \sum_{i=1}^{n} \phi''_{y_i}(w^\top x_i)x_i x_i^\top + \lambda I \right) P^{-1/2}.$$ 

Therefore, if $P$ satisfies $P \leq \hat{C} + \lambda I \leq 3P$, the smoothness and strong convexity of $\phi$ imply that the resulted condition number is $O(\beta/\alpha) = O(\kappa)$. Using Theorem 3, we can compute a $(1/2, \lambda)$-spectral approximation to the data matrix in time $\tilde{O}(\text{nnz}(A) + d_\lambda^2 d)$. Furthermore, multiplying any $d$-dimensional vector with the inverse of this approximation can be done in time $O(d_\lambda^2 d)$. Note that the gradient at some point $w'_t \in P^{1/2}W$ is $P^{-1/2} \nabla \hat{F}(P^{-1/2}w'_t)$. By maintaining both $w'_t$ and $w_t$ and assuming that $\nabla \hat{F}(w)$ can be computed in time $O(\text{nnz}(X))$, we are able to perform a single step of preconditioned GD in time $\tilde{O}(\text{nnz}(A) + d_\lambda^2 d)$. Overall, we obtain the following result.

**Theorem 7.** There exists an algorithm that finds an $\epsilon$-approximate minimizer to (5) in time $\tilde{O}(\kappa(\text{nnz}(A) + d_\lambda^2 d))$.

In the Kernel setting we need to make some modifications to this scheme. First, we need to form the Gram matrix in time $O(n^2)$. Furthermore, as the number of samples replaces the intrinsic dimension, maintaining the preconditioner costs $O(d_\lambda^2 n)$ rather than $O(d_\lambda^2 d)$.

Finally, we remark that by using more advanced first-order methods such as Accelerated SVRG ([18, 20]), we can obtain a better dependence on $\kappa$. However, to keep our presentation simple we stick to GD.
6 Optimizing the Tradeoff between Oracle Complexity and Effective Dimensionality

As explained in the introduction, given a ridge parameter $\lambda$, we may prefer to perform optimization with a different ridge parameter $\lambda' > \lambda$ in order to accelerate the optimization process.

6.1 The Proximal Point algorithm: overview

Before quantifying the tradeoff reflected by the choice of $\lambda'$, we need to explain how to reduce minimization w.r.t. $\hat{F}_{\lambda}$ to minimization w.r.t. $\hat{F}_{\lambda'}$. The basic idea is to repeat the minimization process for $\lambda'$ epochs. We demonstrate this idea using the Proximal Point algorithm (PPA) due to [12]. For a fixed $\bar{w} \in \mathcal{W}$, define $\hat{F}_{\lambda, \bar{w}}(w) = \hat{F}(w) + \frac{\lambda}{2}\|w - \bar{w}\|^2$. Suppose we start from $w_0 = 0$. At time $t$, we find a point $w_t$ satisfying

$$
\hat{F}_{\lambda, w_{t-1}}(w_t) - \min_{w \in \mathcal{W}} \hat{F}_{\lambda, w_{t-1}}(w) \leq \frac{c\lambda}{\lambda'} \left( \hat{F}_{\lambda, w_{t-1}}(w_{t-1}) - \min_{w \in \mathcal{W}} \hat{F}_{\lambda, w_{t-1}}(w) \right).
$$

Lemma 4. [12] Applying PPA with $\lambda' \geq \lambda$ yields $\epsilon$-approximate minimizer to $\hat{F}_{\lambda}$ after $t = O(\lambda'/\lambda)$ epochs, i.e., $\hat{F}_{\lambda}(w_t) - \min_{w \in \mathcal{W}} \hat{F}_{\lambda}(w) \leq \epsilon$.

6.2 Quantifying the tradeoff

Applying PPA while using sketch-to-preconditioning as described in Section 5 yields the following complexity bound:

$$
\tilde{O} \left( \min \left\{ \frac{\lambda' \beta}{\lambda \alpha} \cdot (\text{nnz}(A) + d^2_{\lambda'} d) : \lambda' \geq \lambda \right\} \right).
$$

Focusing on the (reasonable) regime where $d^2_{\lambda'} d \geq \text{nnz}(X)$, we note that $d_{\lambda}/d_{\lambda'}$ may be large as $\lambda'/\lambda$. Notably, while the deterioration in runtime scales linearly on $\lambda'/\lambda$, the improvement in terms of $d_{\lambda'}/d_{\lambda}$ is quadratic. For instance, if $\frac{d_{\lambda}}{d_{\lambda'}} \approx \sqrt{\lambda} \approx \sqrt{\lambda}$, the computational gain is $\Omega(\sqrt{\lambda})$.

Therefore, we wish to minimize the complexity term

$$
\psi(\lambda') := \frac{\lambda'}{\lambda} d^2_{\lambda'}
$$

over all possible $\lambda' \geq \lambda$. To this end, suppose that we had an access to an oracle that computes $d_{\lambda'}$ for a given parameter $\lambda' > 0$. Using the continuity

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6The term involving $\text{nnz}(A)$ can be easily optimized w.r.t. the ridge parameter $\lambda'$.  

of the effective dimension, we could optimize the above quantity over a
discrete set of the form \{\lambda, 2\lambda, \ldots, 2^{C \log d} \lambda \}.\(^7\)

The main difficulty stems from the fact that the cost of implementing this oracle already scales with
\(d^2 \lambda d\).

### 6.3 Efficient tuning using undersampling

Our second main contribution is a novel approach for minimizing (8) in
negligible amount of time.

**Theorem 8.** There exists an algorithm which receives a data matrix \(A \in \mathbb{R}^{n \times d}\) and a regularization parameter \(\lambda > 0\), and with high probability outputs
a regularization parameter \(\bar{\lambda}\) satisfying

\[
\frac{\bar{\lambda}}{\lambda} d^2 = O(\psi^*) \pm O \left( \min_{\lambda' \geq \lambda} \left\{ \frac{\lambda'}{\lambda} d^2 \right\} \right).
\]

The runtime of the algorithm is \(\tilde{O} \left( \text{nnz}(X) + \min_{\lambda' \geq \lambda} \left\{ \frac{\lambda'}{\lambda} d^2 \right\} \right)\).

**Corollary 2.** There exists an algorithm that finds an \(\epsilon\)-approximate solution
to (5) in time

\[
\tilde{O} \left( \min \left\{ \frac{\lambda'}{\lambda} \cdot \frac{\beta}{\alpha} \cdot (\text{nnz}(A) + d^2) : \lambda' \geq \lambda \right\} \right).
\]

The main idea behind Theorem 8 is that instead of (approximately) computing the effective dimension for each candidate \(\lambda'\), we guess the optimal complexity \(\psi^*\) and employ undersampling to test whether a given candidate
\(\lambda'\) attains the desired complexity. The key ingredient to this approach is
described in the following theorem.

**Theorem 9.** Let \(A \in \mathbb{R}^{n \times d}\), \(\lambda' > 0\) and \(m \in \mathbb{R}_{> 0}\). There exists an algorithm
that verifies whether \(d_{\lambda'}(A^\top A) = O(m)\) in time \(\tilde{O}(\text{nnz}(A) + dm^2)\).

**Proof.** (of Theorem 8) Starting from a small constant \(C > 0\) as our “guess”
for \(\psi^*\), we double our guess until finding a candidate \(\lambda'\) which satisfies
the desired bound. According to Theorem 9, for each guess \(\tilde{\psi} \in \mathbb{R}_{> 0}\) and
candidate \(\lambda' > 0\), the complexity of verifying whether \(d_{\lambda'}^2 \leq \frac{1}{N} \tilde{\psi}\) is at most
\(O \left( \frac{\lambda'}{\lambda} d_{\tilde{\psi}} \right) = O(d \psi^*)\). The number of such tests is logarithmic, hence the
theorem follows. \(\square\)

\(^7\)Clearly, the optimal ridge parameter can not be larger than \(d^3 \lambda\).
6.3.1 Proof of Theorem 9

Inspired by [10, 11], our strategy is to use undersampling to obtain sharper estimates to the ridge leverage scores. We start by incorporating an undersampling parameter $\alpha \in (0, 1)$ into Definition 2.

**Definition 3. (Ridge Leverage Score Undersampling)** Let $(u_i)_{i=1}^n$ be a sequence of ridge leverage score overestimates, i.e., $u_i \geq \tau_{\lambda,i}$ for all $i$. For some fixed positive constant $c > 0$, accuracy parameter $\epsilon$, define $p_i = \min\{1, ce^{-2\alpha u_i \log d}\}$ for each $i \in [n]$. Let $\text{Sample}(u, \epsilon, \alpha)$ denote a function which returns a diagonal matrix $S \in \mathbb{R}_{\geq 0}^{n \times n}$, where $S_{i,i} = ((1 + \epsilon)\frac{p_i}{\alpha})^{-1/2}$ with probability $p_i$ and 0 otherwise.

Note that while we reduce each probability $p_i$ by factor $\alpha$, the definition of $S_{i,i}$ neglects this modification. Hence, our undersampling is equivalent to sampling according to Definition 2 and preserving each row with probability $1 - \alpha$. By employing undersampling we cannot hope to obtain a constant approximation to the true ridge leverage scores. However, as we describe in the following theorem, this strategy still helps us to sharpen our estimates to the ridge leverage scores.

**Theorem 10.** Let $u_i \geq \tau_{\lambda,i}$ for all $i$ and let $\alpha \in (0, 1)$ be an undersampling parameter. Given $S = \text{Sample}(u, 1/2, \alpha)$, we form new estimates $(u^{(\text{new})}_i)_{i=1}^n$ by

$$u^{(\text{new})}_i := \min\{a_i(A^\top S^\top S A + \lambda I)^{-1} a_i, u_i\}. \tag{9}$$

Then with high probability, each $u^{(\text{new})}_i$ is an overestimate of $\tau_{\lambda,i}$ and $\|u^{(\text{new})}\|_1 \leq 3d_\lambda/\alpha$.

The proof of the theorem (which is similar to Theorem 3 of [10] and Lemma 13 of [11]) is provided in Appendix B. Equipped with this result, we employ the following strategy in order to verify whether $d_\lambda = O(m)$. Applying the lemma with $\alpha = 6m/\|u\|_1$, we have that if $d_\lambda \leq m$ then $\|\tilde{\tau}\|_1 \leq n/2$. This gives rise to the following test:

1. If $\|u^{(\text{new})}\|_1 \leq m$, accept the hypothesis that $d_\lambda \leq m$.
2. If $\|u^{(\text{new})}\|_1 \geq \|u\|_1/2$, reject the hypothesis that $d_\lambda \leq m$.
3. Otherwise, apply Theorem 10 to obtain a new vector of overestimates, $(u^{(\text{new})}_i)_{i=1}^n$. 

15
Proof. (of Theorem 9) Note that the rank of the matrix $SA$ is $\tilde{O}(m)$ with high probability. Hence, each step of the testing procedure costs $\tilde{O}(\text{nnz}(A) + m^2 d)$\footnote{Namely, we can compute $(\tilde{A}^T \tilde{A} + \lambda I)^{-1}$ in time $O(m^2 d)$. Thereafter, computing $(\tilde{r}_i)_{i-1}$ can be done in time $O(\text{nnz}(A))$.}. Since our range of candidate ridge parameters is of logarithmic size and each test consists of logarithmic number of steps, the theorem follows using the union bound.

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A Concentration of The Effective Dimension

Proof of Lemma 3. Let $\lambda > 0$ and denote the spectral decomposition of $C$ by $\sum_{i=1}^{d} \lambda_i u_i u_i^T$. Let $k = \arg\max\{j \in [d] : \lambda_j \geq \lambda\}$. Note that

$$\forall i \in [k] \quad \frac{\lambda_i}{\lambda_i + \lambda} \geq \frac{1}{2}, \quad \forall i > k \quad \frac{\lambda_i}{\lambda_i + \lambda} \geq \frac{\lambda_i}{2\lambda}.$$ 

Therefore,

$$d_\lambda(C) = \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_i + \lambda} + \sum_{i>k} \frac{\lambda_i}{\lambda_i + \lambda} \geq \frac{1}{2} \left( k + \sum_{i=k}^{\lambda_i} \frac{\lambda_i}{\lambda} \right) \quad (10)$$

Denote the eigenvalues of $\hat{C} = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T$ by $\hat{\lambda}_1, \ldots, \hat{\lambda}_d$. Since for any $i \in [d]$, $\frac{\lambda_i}{\lambda_i + \lambda} \leq 1$, we have that

$$d_\lambda(\hat{C}) = \sum_{i=1}^{k} \frac{\hat{\lambda}_i}{\hat{\lambda}_i + \lambda} + \sum_{i>k} \frac{\hat{\lambda}_i}{\hat{\lambda}_i + \lambda} \leq k + \sum_{i>k} \frac{\hat{\lambda}_i}{\lambda} \leq k + \frac{\sum_{i>k} \hat{\lambda}_i}{\lambda}. \quad (11)$$

We now consider the random variable $\sum_{i>k} \hat{\lambda}_i$. To argue about this random variable consider the following identity which follows from the Courant-Fisher min-max principle for real symmetric matrices:

$$\sum_{i>k} \hat{\lambda}_i = \min \left\{ \text{tr}(V^T \hat{C} V) : V \in \mathbb{R}^{d \times k}, V^T V = I \right\}.$$ 

Let $U_{i \geq k}$ be the $d \times d - k$ matrix with the columns $u_{k+1} \ldots u_d$. We now have that

$$\mathbb{E}[\sum_{i>k} \hat{\lambda}_i] = \min \left\{ \text{tr}(V^T \hat{C} V) : V \in \mathbb{R}^{d \times k}, V^T V = I \right\}$$

$$\leq \min \left\{ \mathbb{E} \left[ \text{tr}(V^T \hat{C} V) \right] : V \in \mathbb{R}^{d \times k}, V^T V = I \right\}$$

$$\leq \mathbb{E} \left[ \text{tr}(U_{i \geq k}^T \hat{C} U_{i \geq k}) \right] = \text{tr}(U_{i \geq k}^T \hat{C} U_{i \geq k})$$

$$= \sum_{i>k} \lambda_i \quad (12)$$

Combining Equation (10), Equation (11) and Equation (12) and taking expectations we get that

$$\mathbb{E}[d_\lambda(\hat{C})] \leq 2d_\lambda(C) \quad \square$$

20
B Ridge Leverage Score Undersampling

In this section we prove Theorem 10. The next lemma intuitively says only a small fraction of $A$’s rows might have a high leverage score.

**Lemma 5.** Let $A \in \mathbb{R}^{n \times d}$, $\lambda > 0$ and denote by $d_\lambda$ the effective dimension of $A$. For any $u \in \mathbb{R}_n$ there exists a diagonal rescaling matrix $W \in [0, 1]^{n \times n}$ such that for all $i \in [n]$, $\tau_{\lambda, i}(WA) \leq u_i$ and $\sum_{i: W_{ii} \neq 1} u_i \leq d_\lambda$.

**Proof.** We prove the lemma by considering a hypothetical algorithm which constructs a sequence $(W^{(1)}, W^{(2)}, \ldots)$ of $n \times n$ diagonal matrices s.t. $W^{(t)}$ converges to some $W$ which possesses the desired properties. Initially, the algorithm sets $W^{(1)} = I$. At each time $t > 1$ it modifies a single entry $W_{ii}$ corresponding to (any) index $i \in [n]$ for which $\tau_{\lambda, i}(WA) > u_i$; namely, it chooses $W^{(t+1)}_{ii} \in (0, 1)$ such that $\tau_{\lambda, i}(W^{(t+1)}A) = u_i$. It is not hard to verify the following (e.g., see Lemma 5 and 6 of [10]):

- We can always find $W^{(t+1)}_{ii}$ such that $\tau_{\lambda, i}(W^{(t+1)}A) = u_i$.
- For any $j \neq i$, $\tau_{\lambda, j}(W^{(t+1)}A) \geq \tau_{\lambda, j}(W^{(t)}A)$.
- Since the entries of $W^{(t)}$ are bounded and monotonically decreasing, $W^{(t)} \rightarrow W \in [0, 1]^{n \times n}$ satisfying $\tau_{\lambda, i}(WX) \leq u_i$.

It is left to show that indeed, $\sum_{i: W_{ii} \neq 1} u_i \leq d_\lambda$. Let $k$ be the first iteration such that $W_{ii}^{(k)} \neq 0$ for all $i \in \{j : W_{jj}^{(k)} \neq 1\}$. For each such $i$, consider the last iteration $k_i \leq k$ where we reduced $W_{ii}$ such that $\tau_{\lambda, i}(W^{(k_i)}) = u_i$. As was mentioned above, in any intermediate iteration $t \in \{k_i + 1, \ldots, k\}$, we could only increase the $i$-th leverage score. Therefore, $\tau_{\lambda, i}(W^{(k)}A) \geq u_i$. Since $W \leq I$, it follows that

$$\sum_{i: W_{ii} \neq 1} u_i \leq \sum_{i: W_{ii} \neq 1} \tau_{\lambda, i}(W^{(k)}A) \leq \sum_{i=1}^n \tau_{\lambda, i}(W^{(k)}A).$$

$$\leq tr \left( X^T \left( W^{(k)} \right)^2 X \left( X^T \left( W^{(k)} \right)^2 A + \lambda I \right)^{-1} \right)$$

$$\leq \sum_{i=1}^d \frac{\lambda_i (A^T W^{(k)} A)^2}{\lambda_i (A^T A) + \lambda}$$

$$\leq \sum_{i=1}^d \frac{\lambda_i (A^T A)}{\lambda_i (A^T A) + \lambda} = d_\lambda.$$
Proof. (of Theorem 10) By Lemma 5, there exists a diagonal matrix $W \in [0,1]^{n \times d}$ satisfying

$$\forall i \quad \tau_{\lambda,i}(WA) \leq \alpha u_i, \quad \alpha \sum_{i: W_{i,i} \neq 1} u_i \leq d_\lambda.$$ 

Therefore,

$$\sum_{i=1}^{n} u_i^{\text{(new)}} \leq \sum_{i: W_{i,i} = 1} a_i^T (A^T S^T SA + \lambda I)^{-1} a_i + \sum_{W_{i,i} \neq 1} \tau_{\lambda,i} \leq \sum_{i: W_{i,i} = 1} a_i^T (A^T S^T SA + \lambda I)^{-1} a_i + \frac{d_\lambda}{\alpha}.$$

We would like to upper bound the first term in the RHS. Since $W \leq I$, for all $i \in [n]$,

$$a_i^T (A^T S^T SA + \lambda I)^{-1} a_i \leq a_i^T (A^T W S^T SWA + \lambda I)^{-1} a_i$$

Also, it is clear that for any $i$ for which $W_{i,i} = 1$,

$$a_i^T (A^T W S^T SWA + \lambda I)^{-1} a_i = (WA)_{i,i}^T (A^T W S^T SWA + \lambda I)^{-1} (WA)_{i,i}.$$  

Finally, since the sampling matrix $S$ was chosen according to $(\alpha u_i)_{i=1}^n$, which form valid overestimates of $(\tau_{\lambda,i}(WA))_{i=1}^n$, Theorem 3 implies that with high probability,

$$\frac{1}{2} A^T W^2 A \leq AW S^T SWA \leq A^T W^2 A.$$

$$\Rightarrow (\forall i \in [n]) \quad (WA)_{i,i}^T (A^T W S^T SWA + \lambda I)^{-1} (WA)_{i,i} \leq 2 \tau_{\lambda,i}(WA)$$

We deduce that

$$\sum_{i: W_{i,i} = 1} u_i^{\text{(new)}} \leq 2 \sum_{i: W_{i,i} = 1} \tau_{\lambda,i}(WA) \leq 2 \sum_{i=1}^{n} \tau_{\lambda,i}(WA) \leq 2d_\lambda.$$ 

All in all, 

$$\sum_{i=1}^{n} \tau'_{\lambda,i} \leq \frac{3d_\lambda}{\alpha}.$$

\[\square\]
C Proof of Our Lower Bound

Proof of Theorem 6. Owing to Yao’s minimax principle, it is sufficient to exhibit a randomized choice of data distributions against which a deterministic algorithm achieves an excess risk lower bounded as above. To this end consider \( \mathcal{X} = \{ \sqrt{d\lambda_1} e_1, \ldots, \sqrt{d\lambda_d} e_d \} \) and \( \mathcal{Y} = \{-1, 1\} \). Define the randomized choice of data distribution by selecting a vector \( \sigma_i \sim \{-1, 1\}^d \) uniformly randomly. The randomized distribution is now defined as first defining the marginal distribution over \( x \) as

\[
\Pr(x = e_i) = 1/d \quad i = 1 \ldots d
\]

Further given \( \sigma \) the conditional distribution of \( y \) is defined as

\[
\Pr\left[ y = \pm 1 | X = \sqrt{d\lambda_i} e_i \right] = \frac{1}{2} (1 \pm \sigma_i b),
\]

where \( b = \sqrt{d/6m} \). Note that \( \mathbb{E}[xx^T] = \Lambda \). Consider the following definitions

\[
F(w) \triangleq \mathbb{E}\left[ \frac{1}{2} (w^T x - y)^2 \right]
\]

Further for any \( \lambda > 0 \) define

\[
F_\lambda(w) \triangleq F(w) + \frac{\lambda}{2} \|w\|^2 \quad w^*_\lambda \triangleq \arg \min_{w \in \mathbb{R}^d} F_\lambda(w)
\]

A straightforward calculation shows that

\[
(w^*_\lambda)_i = \frac{b \sqrt{\lambda_i}}{\sqrt{d} \lambda_i + \lambda} \sigma_i.
\]

Further via complementary slackness we have that there exists some \( \lambda^* \) for which \( \|w^*_\lambda\| = B \). First note that since \( \|\hat{w}\| \leq B = \|w^*_\lambda\| \), we have that

\[
F(\hat{w}) - F(w^*_\lambda) \geq F_\lambda(\hat{w}) - F_\lambda(w^*_\lambda).
\]

Therefore it is sufficient to bound the quantity on the RHS which the following claim shows.

Claim 1. There exists a constant \( c \) such that for any \( \lambda > 0 \),

\[
\mathbb{E}[F_\lambda(\hat{w})] - F_\lambda(w^*_\lambda) = \Omega\left( \frac{d\lambda}{n} \right)
\]

Further we have that
Claim 2. For any $\gamma > 0$ define $\lambda(\gamma) = \frac{\gamma}{n-B^2}$. We have that $\lambda(\gamma) \geq \lambda^*$ if the following holds

$$d(\gamma) - \sum_{i=1}^{d} \left( \frac{\lambda_i}{\lambda_i + \lambda(\gamma)} \right)^2 \leq \gamma$$

Putting together Equation (13) and Claims 1 and 2 gives us that if $\gamma$ satisfies Equation (7) we have that

$$E[r_F p - F(w^*_*) - F(\hat{w})] \geq \Omega \left( \frac{\lambda^*}{n} \right) \geq \Omega \left( \frac{d(\gamma)}{n} \right)$$

which finishes the proof.

Proof of Claim 2. Note that for any $\gamma$ to ensure that $\lambda(\gamma) \geq \lambda^*$ if and only if $\|w^*_\lambda(\gamma)\| \leq B$. To check the latter consider the following

$$\|w^*_\lambda(\gamma)\|^2 = \sum_{i=1}^{d} \frac{\lambda_i}{(\lambda_i + \lambda(\gamma))^2} = \frac{1}{n \cdot \lambda(\gamma)} \sum_{i=1}^{d} \left( \frac{\lambda_i}{\lambda_i + \lambda(\gamma)} - \frac{\lambda^2_i}{(\lambda_i + \lambda(\gamma))^2} \right) \leq B^2$$

where the inequality follows from the definition of $\lambda(\gamma)$ and the condition on $\gamma$.

Proof of Claim 1. Consider the following equations

$$E[F(\hat{w}) - F(w^*_\lambda)] = E[\|\hat{w} - w^*_\lambda\|_{\lambda + \lambda}^2]$$

$$\geq E \left[ \sum_{i=1}^{d} (\lambda_i + \lambda) (w^*_\lambda)^2 \mathbf{1}_{\hat{w}_i(w^*_\lambda) 
\leq 0} \right]$$

$$= E \left[ \frac{b^2}{d} \sum_{i=1}^{d} \left( \frac{\lambda_i}{\lambda_i + \lambda} \right) \cdot \mathbf{1}_{\hat{w}_i(w^*_\lambda) 
\leq 0} \right]$$

$$= \frac{b^2}{d} \sum_{i=1}^{d} \frac{\lambda_i}{\lambda_i + \lambda} \cdot \Pr[\hat{w}_i \cdot (w^*_\lambda)_i \leq 0] . \quad (14)$$

We will now consider the term $\Pr[\hat{w}_i \cdot (w^*_\lambda)_i \leq 0]$. Note that since $\sigma_i$ has the same sign as $(w^*_\lambda)_i$, we have that

$$\Pr[\hat{w}_i \cdot (w^*_\lambda)_i \leq 0] = \Pr[w_i \geq 0|\sigma_i \leq 0] + \Pr[w_i \leq 0|\sigma_i \geq 0]$$

$$= 1 - \Pr[w_i \leq 0|\sigma_i \leq 0] + \Pr[w_i \leq 0|\sigma_i \geq 0]$$

$$\geq 1 - [\Pr[w_i \leq 0|\sigma_i \leq 0] - \Pr[w_i \leq 0|\sigma_i \geq 0]]$$

$$\geq 1 - \frac{1}{2} \sqrt{D_{KL}(p(S|\sigma_i \leq 0) \| p(S|\sigma_i \geq 0))} \quad (15)$$
Further we have that

\[ D_{KL} \left( p (S|\sigma_i \leq 0) \mid \mid p (S|\sigma_i \geq 0) \right) = m \cdot D_{KL} \left( p ((x,y)|\sigma_i \leq 0) \mid \mid p ((x,y)|\sigma_i \geq 0) \right) \]

(16)

Further we have that

\[ p ((x,y)|\sigma_i) = \frac{1}{d} \cdot p ((x,y)|\sigma_i, x = e_i) + \left( 1 - \frac{1}{d} \right) p ((x,y)|\sigma_i, x \neq e_i) \]

We can now use the joint convexity of KL divergence to obtain that

\[ D_{KL} \left( p ((x,y)|\sigma_i \leq 0) \mid \mid p ((x,y)|\sigma_i \geq 0) \right) \leq \]

\[ \frac{1}{d} \cdot D_{KL} \left( p ((x,y)|\sigma_i \leq 0, x_i = e_i) \mid \mid p ((x,y)|\sigma_i \geq 0, x_i = e_i) \right) \]

\[ + \left( 1 - \frac{1}{d} \right) D_{KL} \left( p ((x,y)|\sigma_i \leq 0, x_i \neq e_i) \mid \mid p ((x,y)|\sigma_i \geq 0, x_i \neq e_i) \right) \]

Note that the distribution on \((x,y)\) is independent of \(\sigma_i\) conditioned on \(x_i \neq e_i\) and therefore the second term above is zero and therefore we have that

\[ D_{KL} \left( p ((x,y)|\sigma_i \leq 0) \mid \mid p ((x,y)|\sigma_i \geq 0) \right) \leq \frac{1}{d} D_{KL} \left( p (y|\sigma_i \leq 0, x_i = e_i) \mid \mid p (y|\sigma_i \geq 0, x_i = e_i) \right) \]

The RHS now is the KL divergence between two Bernoulli random variables with parameters \(\frac{1}{d} (1 + b)\) and \(\frac{1}{d} (1 - b)\) respectively. Following arguments similar to [29](Lemma 4) this can be seen to be bounded by \(6b^2\) when \(b \leq 1/2\).

Therefore we have that

\[ D_{KL} \left( p ((x,y)|\sigma_i \leq 0) \mid \mid p ((x,y)|\sigma_i \geq 0) \right) \leq \frac{6b^2}{d} \]

Putting the above together with Equation (14), Equation (15) and Equation (16) we get that

\[ \mathbb{E}[F_\lambda(\hat{w}) - F_\lambda(w_\lambda^*)] \geq \frac{b^2}{d} \sum_{i=1}^{d} \frac{\lambda_i}{\lambda_i + \lambda} \left( 1 - \sqrt{\frac{3nb^2}{d}} \right) \geq 0.28 \cdot \frac{b^2 \cdot d \lambda}{d} = \Omega \left( \frac{d \lambda}{n} \right) \]

Proof of Corollary 5. The proof in both cases follows by choosing \(\gamma\) to ensuring that \(d \lambda nB^2 \leq \gamma\) and using Theorem 6. For the case of \((C, p)\)-polynomial decay it can be seen using Theorem 11 that the condition is satisfied by choosing \(\gamma = \left( \frac{C}{p-1} \right)^{1/(p+1)} \left( \frac{1}{nB^2} \right)^{p+1} + 2\) and in the case of \(C\)-exponential decay it can be obtained by setting \(\gamma = O(\log(nB^2) \log(\log(nB^2)))\).
D Kernel $\ell_2$-regression

Let $\phi : \mathbb{R}^d \to \mathcal{H}$ be a feature mapping into a (possibly infinite-dimensional) Hilbert space $\mathcal{H}$. Similarly to (1), we consider the minimization of $F(w) = \frac{1}{2} \mathbb{E}_{(x,y) \sim \mathcal{D}}[(\langle w, x \rangle - y)^2]$ over a compact and convex subset $\mathcal{W} \subseteq \mathcal{H}$. We may also assume that predictions are bounded by 1 and denote the diameter of $\mathcal{W}$ by $B$. Since Theorem 1 does not depend on the intrinsic dimension, we conclude the following.

**Corollary 3.** Theorem 1 holds also in the kernel setting.

To extend our algorithmic ideas, we consider the minimization of the dual of (2):

$$\hat{F}_\lambda(\alpha) = \frac{1}{2n} \|K\alpha - y\|^2,$$

where $K_{i,j} = \langle x_i, x_j \rangle$ is the Gram matrix. Since the eigenvalues of the Gram matrix coincide with those of the empirical covariance matrix, the effective dimension associated with $\frac{1}{n}K$ coincides with the effective dimension of the primal problem. Consequently, applying preconditioned GD to the dual problem yields the same convergence rate, albeit forming the Gram matrix yields an additional cost of order $n^2$. However, due to our sample complexity bounds, we can bound $n$ by $O(d_\lambda/\epsilon)$. Overall, we obtain the following result.

**Corollary 4.** There exists an algorithm that finds $\epsilon$-approximate minimizer w.r.t. Kernel $\ell_2$-regression in time

$$O \left( \min_{\lambda \geq \lambda} \frac{n'}{\lambda} \left( \frac{d_\lambda^2}{\epsilon^2} + \frac{d_\lambda^3}{\epsilon} \right) \right)$$

E Sample Complexity Bounds for Common Decay Patterns

**Theorem 11.** ([13]) Given an eigenvalue profile $\Lambda = \lambda_1, \ldots, \lambda_d \geq 0$, define $d_\lambda = \sum_{i=1}^d \frac{\lambda}{\lambda_i + \lambda}$. We have that

- If $\Lambda$ satisfies $(C, p)$-Polynomial Decay, then $d_\lambda \leq \left( \frac{C}{(p-1)\lambda} \right)^{1/p}$.

---

9While the proof of the theorem refers to the empirical covariance matrix (which requires more careful treatment in the infinite dimensional case), we can always use random features as in [4] to enforce finite dimensionality without critically modifying any quantity appearing in our analysis.
If $\Lambda$ satisfies $C$-Exponential Decay, then $d_\lambda \leq \log \left( \frac{C}{(e-1)\lambda} \right)$.

Combining Theorem 11 with Theorem 6 we get the following corollary.

**Corollary 5.** Given numbers $B > 0$ and an eigenvalue profile $\Lambda = \lambda_1 \geq \ldots \geq \lambda_d \geq 0$. Then there exists sets $\mathcal{X}, \mathcal{Y}$ and a distribution over $\mathcal{X}, \mathcal{Y}$ such that for any algorithm that returns a linear predictor $\hat{w}$ given $m \geq 2d/3$ independent samples from $D$ satisfies

- If $\Lambda$ has $(C,p)$-polynomial decay then
  \[
  \mathbb{E}_{S \sim D^m} \left[ \mathbb{E}_{(x,y) \sim D} \left[ \frac{1}{2} (\hat{w}^T x - y)^2 \right] \right] - \min_{w : \|w\| \leq B} \mathbb{E} \left[ \frac{1}{2} (\hat{w}^T x - y)^2 \right] \geq \Omega \left( \frac{d \left( \frac{1}{nB^2} \right)^p (p+1)}{n} \right)
  \]
- If $\Lambda$ has $C$-exponential decay then
  \[
  \mathbb{E}_{S \sim D^m} \left[ \mathbb{E}_{(x,y) \sim D} \left[ \frac{1}{2} (\hat{w}^T x - y)^2 \right] \right] - \min_{w : \|w\| \leq B} \mathbb{E} \left[ \frac{1}{2} (\hat{w}^T x - y)^2 \right] \geq \Omega \left( \frac{d \log(\log B) \log(\log(\log B))}{nB^2} \right)
  \]

**F Technical Lemmas**

**Proof.** (of Lemma 2) Fix a scalar $t > 0$. By definition, $a_i a_i^\top \leq t(A^\top A + \lambda I)$ if and only if for every vector $v \in \mathbb{R}^d$,

\[
v^\top a_i a_i^\top v_i \leq tv^\top (A^\top A + \lambda I)v
\]

Substituting $v = (A^\top A + \lambda I)^{1/2}u$, we get that $a_i a_i^\top \leq t(A^\top A + \lambda I)$ if and only if for every $u \in \mathbb{R}^d$,

\[
u^\top (A^\top A + \lambda I)^{1/2}(A^\top A + \lambda I)(A^\top A + \lambda I)^{1/2}u \leq t
\]

Using that $(A^\top A + \lambda I)^{1/2}a_i a_i^\top (A^\top A + \lambda I)^{1/2}$ is a rank-1 matrix, the above equivalence can be rewritten as follows:

\[
a_i a_i^\top \leq t(A^\top A + \lambda I) \iff (A^\top A + \lambda I)^{1/2}a_i a_i^\top (A^\top A + \lambda I)^{1/2} \leq tI
\]

\[
\iff \lambda_1((A^\top A + \lambda I)^{1/2}a_i a_i^\top (A^\top A + \lambda I)^{1/2}) \leq t
\]

\[
\iff \text{tr}((A^\top A + \lambda I)^{1/2}a_i a_i^\top (A^\top A + \lambda I)^{1/2}) \leq t
\]

\[
\iff a_i^\top (A^\top A + \lambda I)^{1/2}(A^\top A + \lambda I)^{1/2}a_i \leq t,
\]

\[
\tau_{\lambda_i}(A)
\]
where the last equivalence stems from the cyclic invariance of the trace. The chain of equivalences implies that \( \tau_{\lambda,i} \) is the minimal scalar for which \( a_i a_i^T \preceq t(A^T A + \lambda I) \). \qed