Comment on “Traversable Wormholes in General Relativity”

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In the letter titled “Traversable Wormholes in General Relativity” [Phys. Rev. Lett. 128, 091104 (2022)], R. A. Konoplya and A. Zhidenko have constructed an asymmetric wormhole solution, which is not symmetric about the throat and is compounded from smooth gravitational and charged Dirac fields. However, the authors have claimed that a physically relevant condition on the throat is imposed to lead to no gravitational force experienced by a stationary observer at the throat. In this comment, we point out that the above condition is unnecessary.

The authors of Ref. [1] claimed that a physically relevant condition on the throat \( x = 0 \) should be imposed. This condition was chosen as [1]

\[
N'(0) = 0,
\]

where the function \( N(x) \) is related to the metric coefficient \( g_{tt} \). The above condition could lead to no gravitational force experienced by a stationary observer at the throat.

The numerical method used by the authors is the shooting method. According to the asymptotic behavior of the gravitational field equations at the throat, one can know that if one only fixes the values of \( F(0), G(0), N(0) \), and \( B'(0) \), it is not enough to determine the first-order derivatives of these functions, and there are still redundant free parameters, which will lead to failure of the shooting method. So, one needs to fix the value of \( N'(0) \).

However, considering that the traversable wormhole solutions are generally asymmetric at the throat, we should have \( N'(0) \neq 0 \), which means that the wormhole should be nonsymmetric about the throat [4]. Of course, it is also possible that the condition \( N'(0) = 0 \) also leads to an asymmetric solution. But this is just a special case. In addition, it is reasonable for an asymmetric wormhole that there is a gravitational force experienced by a stationary observer at the throat. Based on the above two reasons, we propose that it is necessary to extend the condition in Eq. (1) to

\[
N'(0) = \text{Constant},
\]

where the constant may be limited in a range, while each of them corresponds to a certain wormhole solution.

Besides the shooting method, the Newton-Raphson method is also feasible to solve the coupled system of nonlinear ordinary differential equations. We also numerically solve the equations of motion (30) in Ref. [1]. We first solve the equations of motion in \( 0 \leq x \leq 1 \) with the boundary conditions at infinities:

\[
F(1) = G(1) = 0, \quad B(1) = 1, \quad \text{at} \quad x = 1,
\]

and

\[
F(0) = f_i, \quad B(0) = 0, \quad N(0) = n_i, \quad U(0) = \omega, \quad W(0) = w_i, \quad \text{at} \quad x = 0,
\]

where \( f_i, n_i, \omega, \) and \( w_i \) are constants. After numerically solving the equations of motion with the above boundary conditions, we can obtain the solutions of the functions \( F(x), G(x), N(x), B(x), \) and \( W(x) \) in \( 0 \leq x \leq 1 \). Besides, the value of \( N'(0) \) can also be determined. Then, in the range of \( -1 \leq x \leq 0 \), we could continue to adopt the Newton-Raphson method or use the shooting method to solve the equations of motion, and look for smooth solutions.

In TAB. 1, we list some typical results of the values of the parameters describing the wormhole configurations. From the last column, one can see that each wormhole with a set of parameters has a different value of \( N'(0) \). Except for that \( N'(0) \) is equal to zero for \( \omega r_0/\sigma_+ = -0.105347 \) in the third row of the table, the rest of the wormhole solutions has a deviation from the case of \( N'(0) = 0 \). Comparing with the result of TAB. I given in Ref. [1], we conjecture that when there is a big gap between \( \sigma_- \) and \( \sigma_+ \), the solutions with \( N'(0) \neq 0 \) can exist. Meanwhile, in FIG. 1, we show a typical result of the distribution of the wormhole configuration as a function of \( x \) with the parameters given in the last row of TAB. I. In the bottom left panel, we can find that

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TABLE I: The families of wormhole solutions with the parameters $q r_0 = 0.03$ and $\mu r_0 = 0.2$.

| $b_1$ | $g_1$ | $f_1$ | $n_1/\sigma_+$ | $\sigma_-/\sigma_+$ | $Q_+/r_0$ | $Q_-/r_0$ | $M_+/r_0$ | $M_-/r_0$ | $\omega r_0/\sigma_+$ | $N'(0)$ |
|-------|-------|-------|----------------|-------------------|-----------|-----------|-----------|-----------|-------------------|--------|
| 0.289865 | 0.0049816 | 0.033000 | 0.0521623 | 1.62879 | 0.978917 | 0.977803 | 0.977428 | 0.977239 | -0.120251 | -0.0622616 |
| 0.28043 | 0.0177181 | 0.025000 | 0.0491175 | 1.21799 | 0.976157 | 0.975392 | 0.975476 | 0.976952 | -0.115302 | -0.0263646 |
| 0.288948 | 0.0256779 | 0.016095 | 0.0448804 | 1.00440 | 0.975806 | 0.974565 | 0.974637 | 0.973544 | -0.105347 | 0 |
| 0.289748 | 0.0329166 | 0 | 0.0345718 | 0.72252 | 0.978405 | 0.977286 | 0.977742 | 0.975932 | -0.080254 | 0.0421234 |
| 0.29168 | 0.0310424 | -0.028000 | 0.0173830 | 0.24981 | 0.985702 | 0.984627 | 0.985574 | 0.982341 | -0.034964 | 0.1299027 |

FIG. 1: The distribution of the wormhole configuration as a function of $x$ with the parameters given in the last row of Tab. I. In the top left panel, the purple and red lines denote the functions $F(x)$ and $G(x)$, respectively. In the bottom left panel, the red point corresponds to the minimum value of $N(x)$ with $N'(x = -0.0815) = 0$.

It is worth pointing out that there may exist various possible solutions for a given set of parameters. In fact, these solutions can be considered as the ground or excited states of the wormhole, where the Dirac field possessing zero and $n(>0)$ nodes along the radial coordinate corresponds to the ground state and the $n$-th excited state, respectively.

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[1] R. A. Konoplya and A. Zhidenko, “Traversable Wormholes in General Relativity”, Phys. Rev. Lett. 128, no.9, 091104 (2022), [arXiv:2106.05034 [gr-qc]].
[2] J. L. Blázquez-Salcedo, C. Knoll and E. Radu, “Traversable wormholes in Einstein-Dirac-Maxwell theory”, Phys. Rev. Lett. 126, no.10, 101102 (2021), [arXiv:2010.07317 [gr-qc]].
[3] J. L. Blázquez-Salcedo, C. Knoll and E. Radu, “Einstein-Dirac-Maxwell wormholes: ansatz, construction and properties of symmetric solutions”, [arXiv:2108.12187 [gr-qc]].
[4] S. Bolokhov, K. Bronnikov, S. Krasnikov and M. Skvortsova, “A Note on ‘Traversable Wormholes in Einstein–Dirac–Maxwell Theory’”, Grav. Cosmol. 27, no.4, 401-402 (2021), [arXiv:2104.10933 [gr-qc]].