DE DONDER FORM FOR SECOND ORDER GRAVITY

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Abstract. We show that the De Donder form for second order gravity, defined in terms of Ostrogradski’s version of the Legendre transformation applied to all independent variables, is globally defined by its local coordinate descriptions. It is a natural differential operator applied to the diffeomorphism invariant Lagrangian of the theory.

1. Introduction. In 1929, De Donder formulated an approach to study first order variational problems for several independent variables in terms of a differential form obtained by the Legendre transformation in each independent variable [9] and [10]. The De Donder form is a field theory analogue of the Poincaré-Cartan form, which was introduced for a single independent variable. It is a basis of the multisymplectic formulation of field theory, which is called also a polysymplectic theory or De Donder-Weyl theory. The first application of the De Donder form to general relativity in the Palatini formulation [19] was given in [20]. For further developments see [3], [13], [14], [23] and references quoted there.

In 1936, Lepage [15] constructed a family of forms, each of which can be used in the same way as the De Donder form to reduce the original variational problem to a system of equations in exterior differential forms. In 1977, Aldaya and Azcárraga [1] studied generalizations of the Lepage construction to higher order Lagrangians, for which they used the term Poincaré-Cartan forms. Here, we use the term De Donder form for the Poincaré-Cartan form of Aldaya and Azcárraga which is obtained from the Lagrangian by Ostrogradski’s generalization of the Legendre transformation [18] in all independent variables.

The usual expression for a De Donder form is given in terms of coordinates on an appropriate jet bundle induced by a coordinate patch on the space of variables. For a generic Lagrangian, if the number of independent and the number of dependent variables are greater than 1, this expression depends on the choice of coordinates. Therefore, it does not define a global form. This leads to a search for additional geometric structures, which would ensure global existence of such forms, see [4], [11] and references cited there.

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The aim of this paper is to show that, for second order general relativity with a diffeomorphism invariant Lagrangian $L$, the coordinate expression for the De Donder form is independent of the choice of coordinates. This implies that the De Donder construction for second order gravity yields a unique form $\Theta$, which is given by a natural differential operator applied to the invariant Lagrangian $L$. Therefore, we can use $\Theta$ to obtain an invariant multisymplectic formulation of second order gravity for any choice of invariant Lagrangian. For the first order Lagrangians, our results do not produce new results. However, for genuinely second order invariant Lagrangians our result is not only new but also unexpected (at least to some experts in the field).

We are grateful to the referee, who gave us a list of recent papers dealing with a multisymplectic covariant description of gravitation [5], [6], [8], [12] and [24], which were unknown to us, and asked us to relate our results to the results contained in these papers. These papers deal mainly with refinements of the Palatini formulation of the variational problem for the Hilbert Lagrangian in general relativity. In this formulation, components of metric tensor and connection coefficients are considered to be independent variables. This leads to a variational problem with first order Lagrangian. Gaset and Roman-Roy, [12], present a multisymplectic formulation of the Einstein-Hilbert model of General Relativity. Even though they are concerned with the Hilbert Lagrangian, they write formulae, which are valid for a general second order theory. At this stage, we can compare this paper with ours. Our results ensure covariance of their multisymplectic formulation.

The paper is organized as follows. In Section 2, we present a brief review of some fundamentals of jet bundles. We exhibit the results obtained in this work in Section 3, where we also discuss multisymplectic formulation of the second order gravity. Since our proofs are mainly computational and require a lot of attention to details, they are presented in Section 4.

2. Geometric background.

2.1. Jets. Let $M$ be a 4-dimensional manifold representing the space-time of general relativity, and $N \subset \otimes_2^{\text{sym}} T^* M$ be the bundle of Lorentzian frames on $M$. We denote the canonical projection by $\pi : N \to M$. A Lorentzian metric on $M$ is a section $\sigma : M \to N$ of $\pi : N \to M$. If $(x^\mu) = (x^1, ..., x^4)$ are local coordinates on $M$ with domain $U$, we denote by $(x^\mu, y_{\mu \nu})$ the induced coordinates on $\pi^{-1}(U) \subset N$. In these coordinates, a section $\sigma$ restricted to $U$ is given by

$$\sigma_U : U \to \pi^{-1}(U) : x \mapsto (y_{\mu \nu}) = g_{\mu \nu}(x^\lambda), \quad (1)$$

where $g_{\mu \nu}(x^\lambda) = g_{\mu \nu}(x^1, ..., x^4)$ are smooth functions of the coordinates $(x^\lambda)$.\footnote{Here, we work in the smooth category. In application to concrete cases, one has to choose a suitable function space.}

The first derivatives of sections form the first jet bundle $J^1(M, N)$ with the source projection $\pi^1 : J^1(M, N) \to M$, the target projection $\pi^1_0 : J^1(M, N) \to N$ and the induced coordinates $(x^\mu, y_{\mu \nu}, z_{\mu \nu \lambda})$ such that

$$\pi^1 : J^1(M, N) \to M : (x^\mu, y_{\mu \nu}, z_{\mu \nu \lambda}) \mapsto (x^\mu), \quad (2)$$

$$\pi^1_0 : J^1(M, N) \to N : (x^\mu, y_{\mu \nu}, z_{\mu \nu \lambda}) \mapsto (x^\mu, y_{\mu \nu}).$$
The first jet extension of a section \( \sigma_U : U \to \pi^{-1}(U) \subseteq N \) is
\[
j^1\sigma_U : U \to (\pi^1)^{-1}(U) \subseteq J^1(M, N) : x \mapsto (x^\mu, y_{\mu\nu}, z_{\mu\nu\lambda}) = (x^\mu, g_{\mu\nu}(x), g_{\mu\nu,\lambda}(x)),
\]
where
\[
g_{\mu\nu,\lambda} = \frac{\partial}{\partial x^\lambda} g_{\mu\nu} = \partial_\lambda g_{\mu\nu}.
\]
Similarly, for \( k = 2, 3, \ldots \), we have the \( k \)-jet bundle \( J^k(M, N) \) with local coordinates \((x^\mu, y_{\mu\nu}, z_{\mu\nu,\lambda_1}, \ldots, z_{\mu\nu,\lambda_1,\ldots,\lambda_k})\), the source map \( \pi^k : J^k(M, N) \to M \), the target map \( \pi^0_0 : J^k(M, N) \to N \) and the forgetful maps \( \pi^k_i : J^k(M, N) \to J^i(M, N) \) defined for \( k > l \geq 0 \). The \( k \)-jet extension of a section \( \sigma_U : U \to \pi^{-1}(U) \subseteq N \) is
\[
j^k\sigma_U : U \to (\pi^k)^{-1}(U) \subseteq J^k(M, N)
\]
\[x \mapsto (x^\mu, y_{\mu\nu}, z_{\mu\nu,\lambda_1}, \ldots, z_{\mu\nu,\lambda_1,\ldots,\lambda_k})\]
\[= (x^\mu, g_{\mu\nu}(x), g_{\mu\nu,\lambda}(x), \ldots, g_{\mu\nu,\lambda_1,\ldots,\lambda_k}(x)).\]

Local contact forms are
\[
dy_{\mu\nu} - z_{\mu\nu,\lambda}dx^\lambda, \; dz_{\mu\nu,\lambda} - z_{\mu\nu,\lambda\rho}dx^\rho, \; \ldots, \; dz_{\mu\nu,\lambda_1,\ldots,\lambda_k} - z_{\mu\nu,\lambda_1,\ldots,\lambda_k}dx^{\lambda_k},\]
where summation over repeated indices is assumed. A section \( \sigma : M \to J^k(M, N) \) of the source projection \( \pi^k \) is said to be holonomic if it is the \( k \)-jet extension of \( \sigma = \pi^0_0 \circ \rho : M \to N \). The importance of the local contact forms stems from the fact that a section \( \rho : M \to J^k(M, N) \) is holonomic if and only if the pull-back of every local contact form by \( \rho \) vanishes.

Let \( Y \) be a vector field on \( N \). For every \( k \geq 0 \), the local 1-parameter group \( e^{Yt} \) of local diffeomorphisms of \( N \) preserving the projection map \( \pi : N \to M \) gives rise to a local 1-parameter group \( e^{Y^k_{\mu\nu}t} \) of local diffeomorphisms of \( J^k(M, N) \), which preserve the ideal generated by contact forms, and intertwine forgetful maps. In other words, the following diagram
\[
\begin{array}{ccc}
J^k(M, N) & \xrightarrow{e^{Y^k_{\mu\nu}t}} & J^k(M, N) \\
\downarrow{\pi^k_i} & & \downarrow{\pi^k_i} \\
J^i(M, N) & \xrightarrow{e^{Y^i_{\mu\nu}t}} & J^i(M, N)
\end{array}
\]
commutes for \( k > l \). The vector field \( X^k \) on \( J^k(M, N) \) is called the prolongation of \( X \) to \( J^k(M, N) \). For more details on jet bundles see [17].

2.2. Variational problem. Let \( \Lambda \) be the Lagrange form of the second order gravity. This means that \( \Lambda \) is a semi-basic 4-form on \( J^2(M, N) \). In local coordinates,
\[
\Lambda = L(x^\mu, y_{\mu\nu}, z_{\mu\nu,\lambda}, z_{\mu\nu,\lambda\rho})dx^1 \wedge \cdots \wedge dx^4,
\]
where \( L(x^\mu, y_{\mu\nu}, z_{\mu\nu,\lambda}, z_{\mu\nu,\lambda\rho}) \) is a scalar density with respect to the transformations of \( J^2(M, N) \) induced by coordinate transformations in \( M \). For the sake of simplicity, we set
\[
L(x^\mu, y_{\mu\nu}, z_{\mu\nu,\lambda}, z_{\mu\nu,\lambda\rho}) = L(x, y, z) \quad \text{and} \quad dx^1 \wedge \cdots \wedge dx^4 = d_4x,
\]
so that equation (6) reads \( \Lambda = L(x, y, z)d_4x \).
Let $\sigma : M \to N$ be a section of $\pi : N \to M$. If $U \subseteq M$ has compact closure $\overline{U}$, the action $A_U$ on $\sigma$ is the integral

$$A_U[\sigma] = \int_U (j^2\sigma)^* \Lambda.$$  \hfill (8)

A section $\sigma$ is a critical point of the action $A_U$ if, for every vertical vector field $Y$ on $N$ that vanishes on the boundary of $\pi^{-1}(U)$ up to second order,

$$\int_U (j^2\sigma)^* \mathcal{L}_Y \Lambda = 0,$$  \hfill (9)

where $Y^2$ is the prolongation of $Y$ to $J^2(M, N)$ and $\mathcal{L}_Y \Lambda$ is the Lie derivative of $\Lambda$ with respect to $Y^2$. The condition that $Y^2$ is the prolongation of $Y$ is equivalent to the classical condition that variations and derivatives commute.

3. **De Donder form.** Following references [21] and [22], we present here the geometric description of the De Donder construction adapted to the second order gravity.

**Definition 3.1.** De Donder form corresponding to a Lagrangian $\Lambda = Ld_4x$ on $J^3(M, N)$ is a form $\Theta$ on $J^3(M, N)$ such that, in local coordinates $(x^\mu)$ on $M$,

$$\Theta = \pi_2^3 Ld_4x + p^{\mu \alpha \beta}(dz_{\mu \alpha \gamma} - z_{\mu \alpha \gamma} dx^\gamma) \wedge (\partial_{\beta} J^3 dx) +$$

$$+ p^{\mu \alpha \beta} (dy_{\mu \beta} - z_{\mu \beta} dx^\beta) \wedge (\partial_{\alpha} J^3 dx),$$  \hfill (10)

where $p^{\mu \alpha \beta}$ and $p_{\mu \alpha \beta}$ are functions on $J^3(M, N)$ such that, for every local section $\sigma : M \to N : x \mapsto (x, g_{\mu \nu}(x))$ of $\pi$,

$$(j^3 \sigma)^* p^{\mu \alpha \beta} = (j^3 \sigma)^* \frac{\partial L}{\partial z_{\mu \alpha \beta}},$$  \hfill (11)

$$(j^3 \sigma)^* p_{\mu \alpha \beta} = (j^3 \sigma)^* \frac{\partial L}{\partial x^\alpha} - \frac{\partial}{\partial x^\beta} \left( (j^3 \sigma)^* \frac{\partial L}{\partial z_{\mu \alpha \beta}} \right).$$

Equations (10) and (11) define a $4$-form $\Theta$ on the domain of coordinates on $J^3(M, N)$ defined by local coordinates on $M$. For a generic Lagrangian, these local forms do not define a global form on $J^3(M, N)$. The main result of our paper is the following theorem.

**Theorem 3.2.** For second order gravity with an invariant Lagrangian form $Ld_4x$ on $J^3(M, N)$, equations (10) and (11) define a global De Donder form $\Theta$ on $J^3(M, N)$, given by a natural differential operator applied to $Ld_4x$.

**Proof** of Theorem 3.2 is given in Section 4.

3.1. **Field equations.** Since $\Theta$ differs from $\pi_2^3 Ld_4x$ by terms proportional to contact forms, for every section $\sigma$ of $\pi : N \to M$,

$$A_U[\sigma] = \int_U (j^2\sigma)^* \Lambda = \int_U (j^2\sigma)^* Ld_4x = \int_U (j^3\sigma)^* (\pi_2^3 Ld_4x) = \int_U (j^3\sigma)^* \Theta.$$  \hfill (12)

Thus, replacing in the variational principle the Lagrangian form $\Lambda = Ld_4x$ by the corresponding De Donder form does not change the value of the action $A_U[\sigma]$.

**Proposition 3.3.** For every vector field $X$ on $J^3(M, N)$ tangent to fibres of the target map $\pi_3^0 : J^3(M, N) \to N$ and every section $\sigma$ of $\pi : N \to M$,

$$(j^3\sigma)^* (X \mathcal{J} d\Theta) = 0.$$  \hfill (13)
Proof. Equation (10) yields
\[d\Theta = \pi_2^3 dL \wedge d_{4x} - p^\mu_{\nu\alpha\beta} d z_{\nu\alpha\gamma} \wedge dx_\gamma \wedge (\partial_\beta J_{4d4x})\]
and compute the following
\[\frac{\partial}{\partial z_{\mu\nu\alpha\beta}} (j^3 \sigma)^*(X_{\mu\nu\alpha\beta} d4x) = \frac{\partial}{\partial z_{\mu\nu\alpha\beta}} (j^3 \sigma)^* X_{\mu\nu\alpha\beta} d4x\]
and compute the following
\[(j^3 \sigma)^*(X_{\mu\nu\alpha\beta} d4x) = \partial_\beta ((j^3 \sigma)^* p^\mu_{\nu\alpha\beta})(j^3 \sigma)^* X_{\mu\nu\alpha\beta} d4x\]

Consider a vector field in form
\[X = X_{\mu\nu\alpha\beta} \frac{\partial}{\partial z_{\mu\nu\alpha\beta}} + X_{\mu\nu\alpha} \frac{\partial}{\partial z_{\mu\nu\alpha}},\]

and compute the following
\[(j^3 \sigma)^*(X_{\mu\nu\alpha\beta} d4x) = \partial_\beta ((j^3 \sigma)^* p^\mu_{\nu\alpha\beta})(j^3 \sigma)^* X_{\mu\nu\alpha\beta} d4x\]

Here, we used the fact that
\[(j^3 \sigma)^* [\pi_2^3 dL] = (j^2 \sigma)^* dL.\]

Observe that one has
\[(j^3 \sigma)^* d p^\mu_{\nu\alpha\beta} \wedge (j^3 \sigma)^* X_{\mu\nu\alpha} (\partial_\beta J_{4d4x}) = \partial_\beta ((j^3 \sigma)^* p^\mu_{\nu\alpha\beta})(j^3 \sigma)^* X_{\mu\nu\alpha} d4x\]
so that
\[(j^3 \sigma)^*(X_{\mu\nu\alpha\beta} d4x) = \frac{\partial}{\partial z_{\mu\nu\alpha\beta}} (j^3 \sigma)^* X_{\mu\nu\alpha\beta} d4x\]
and compute the following
\[(j^3 \sigma)^*(X_{\mu\nu\alpha\beta} d4x) = \partial_\beta ((j^3 \sigma)^* p^\mu_{\nu\alpha\beta})(j^3 \sigma)^* X_{\mu\nu\alpha\beta} d4x\]

by equation (11). \qed

Lemma 3.4. For each vector field \(Y\) on \(N\), which projects to a vector field on \(M\), and every section \(\sigma\) of \(\pi: M \to N\),
\[(j^3 \sigma)^*(L_{Y^3} [\Theta - \pi_2^3 J_{4d4x}]) = 0,\]
where \(Y_3\) is the prolongation of \(Y\) to \(J^3(M,N)\).

Proof. See Lemma 3 in reference [22]. \qed
Taking these results into account and using Stokes’ Theorem, we can rewrite equation (9) in the form
\begin{align}
\int_U (j^2 \sigma)^* \mathcal{L}_{Y^2} \Lambda &= \int_U (j^2 \sigma)^* \mathcal{L}_{Y^2} (Ld_4 x)) = \int_U (j^2 \sigma)^* \mathcal{L}_{Y^2} (\pi_3^2 * Ld_4 x)) = \\
&= \int_U (j^3 \sigma)^* \mathcal{L}_{Y^3} \Theta = \int_U (j^3 \sigma)^* [Y^3 \mathcal{L} d \Theta + d (Y^3 \mathcal{J} d \Theta)] = \\
&= \int_U (j^3 \sigma)^* [Y^3 \mathcal{L} d \Theta] + \int_{\partial \mathcal{U}} (j^3 \sigma)^* [(Y^3 \mathcal{J} d \Theta)] = \\
&= \int_U (j^3 \sigma)^* Y^3 \mathcal{J} d \Theta = 0,
\end{align}
where \( \partial \mathcal{U} \) is the boundary of \( \mathcal{U} \), and the integral over the boundary vanishes because we assume that \( Y \) vanishes on \( \partial \mathcal{U} \) to second order. Proposition 3.3 implies that in equation (16) we may replace the prolongation \( Y^3 \) of a vector field \( Y \) on \( N \) tangent to fibres of \( \pi: M \to N \) by arbitrary vector field \( X \) on \( J^3 (M, N) \) tangent to fibres of the source map \( \pi^3: J^3 (M, N) \to M \) and vanishes on \( (\pi^3)^{-1} (\partial \mathcal{U}) \). Therefore, the variational principle (9) is equivalent to
\begin{align}
\int_U (j^3 \sigma)^* [X \mathcal{J} d \Theta] = 0,
\end{align}
where \( X \) is an arbitrary vector field on \( J^3 (M, N) \) tangent to fibres of the source map \( \pi^3: J^3 (M, N) \to M \). The Fundamental Theorem in the Calculus of Variations ensures that the variational principle (17) is equivalent to
\begin{align}
(j^3 \sigma|_{\mathcal{U}})^* [X \mathcal{J} d \Theta] = 0
\end{align}
for every vector field \( X \) on \( J^3 (M, N) \) tangent to fibres of the source map \( \pi^3: J^3 (M, N) \to M \). Equation (18) is the De Donder equation for the second order gravity with invariant Lagrangian \( L \).

We can show directly that equation (18) is a system of equations in differential forms equivalent to the Euler-Lagrange equations corresponding to \( L \). Let
\begin{align}
X = X_{\mu \nu \alpha \beta \gamma} \frac{\partial}{\partial z_{\mu \nu \alpha \beta \gamma}} + X_{\mu \nu \alpha \beta} \frac{\partial}{\partial z_{\mu \nu \alpha \beta}} + X_{\mu \nu \alpha} \frac{\partial}{\partial z_{\mu \nu \alpha}} + X_{\mu \nu} \frac{\partial}{\partial y_{\mu \nu}}
\end{align}
be a vector field tangent to fibres of the source map, and let \( \sigma: (x^\lambda) \to g_{\mu \nu} (x^\lambda) \) be a section of \( \pi: N \to M \). Introducing the notation
\begin{align}
P_{\mu \nu \alpha \beta} = (j^3 \sigma)^* p_{\mu \nu \alpha \beta} \quad \text{and} \quad P_{\mu \nu} = (j^3 \sigma)^* p_{\mu \nu}
\end{align}
we can write the left hand side of equation (13) in the from
\begin{align}
(j^3 \sigma)^* [X \mathcal{J} d \Theta]
&= \left[ X_{\mu \nu \beta} \left( (j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu \nu \beta}} - P_{\mu \nu \alpha \beta} \right) + X_{\mu \nu \alpha} \left( (j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu \nu \alpha}} - P_{\mu \nu \alpha} \right) \right] d_4 x \\
&+ \left[ X_{\mu \nu} (j^2 \sigma)^* \frac{\partial L}{\partial y_{\mu \nu}} \right] d_4 x - \left[ X_{\mu \nu \alpha} P_{\beta}^{\mu \nu \alpha \beta} + X_{\mu \nu} P_{\alpha}^{\mu \nu \alpha} \right] d_4 x.
\end{align}
Since components of $X$ are arbitrary, equation (13) reads
\[
(j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu \nu \alpha \beta}} - P_{\mu \nu \alpha \beta} = 0,
\]
(20)
\[
(j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu \nu \alpha}} - P_{\mu \nu \alpha} = 0,
\]
(21)
\[
(j^2 \sigma)^* \frac{\partial L}{\partial y_{\mu \nu}} - P_{\mu \nu} = 0.
\]
(22)

Equation (20) is the definition of $P_{\mu \nu \alpha \beta}$, equation (21) is the definition of $P_{\mu \nu \alpha}$, while equation (22) is equivalent to the Euler-Lagrange equations
\[
(j^2 \sigma)^* \frac{\partial L}{\partial y_{\mu \nu}} - \frac{\partial}{\partial x^{\alpha}} \left( (j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu \nu \alpha}} \right) + \frac{\partial^2}{\partial x^{\alpha} \partial x^{\beta}} \left( (j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu \nu \alpha \beta}} \right) = 0.
\]
(23)

3.2. **Example: Hilbert Lagrangian.** Hilbert Lagrangian of general relativity is
\[
L_{\text{Hilbert}} = R[g] \sqrt{-\det g},
\]
(24)
where $R[g]$ is the scalar curvature of the Lorentzian metric $g$. Since $L_{\text{Hilbert}}$ depends linearly on the second derivatives of the metric, the corresponding Euler-Lagrange equations are of the second order. The Arnowitt, Deser and Misner Hamiltonian formalism for general relativity, [2], see also [16], is based on the Palatini formalism, [19], in which metric and connection are independent dynamical variables. The De Donder form for the Palatini formulation of general relativity was given in [20].

**Proposition 3.5.** The De Donder form for the second order Hilbert Lagrangian $L_{\text{Hilbert}}$, expressed in local coordinates, is
\[
\Theta_{\text{Hilbert}} = \Gamma_{\lambda \mu \nu} \Gamma_{\alpha \beta \gamma} \left( g^{\alpha \gamma} g^{\mu \nu} g^{\beta \lambda} - g^{\beta \lambda} g^{\alpha \mu} g^{\gamma \nu} \right) \sqrt{-\det g} d_4 x
\]
(25)
\[
+ \frac{1}{2} \left( -g^{\alpha \mu} \Gamma^\beta_\mu - g^{\beta \lambda} \Gamma^\alpha_\lambda + \Gamma^{\alpha \beta \mu} \right) \sqrt{-\det g} d_4 x
\]
\[
+ \frac{1}{2} \left( g^{\alpha \mu} g^{\beta \nu} + g^{\alpha \nu} g^{\beta \mu} - 2 g^{\alpha \beta} g^{\mu \nu} \right) \sqrt{-\det g} d_4 x
\]
\[
+ \partial \Gamma^{\alpha \beta \nu} \cdot \partial \left( \partial x^{\mu} d_4 x \right),
\]
where $\Gamma^\alpha = g^{\mu \nu} \Gamma^\alpha_{\mu \nu}$.

**Proof** of Proposition 3.5 is given in Section 4.

3.3. **Example: Matter and gravitation.** In the study of second order gravity, we cannot ignore the interaction of gravity with matter. If $L$ is the Lagrangian for the gravity alone and $L_{\text{matter}}$ is the Lagrangian for the matter such that the total Lagrangian $L_{\text{total}} = L + L_{\text{matter}}$ is invariant under the group of diffeomorphisms of $M$, we conjecture that the statement of Theorem 3.2 also holds for the De Donder form $\Theta_{\text{total}}$ associated to the total Lagrangian $L_{\text{total}}$. Here, we illustrate it with the case of second order gravity interacting with a scalar field $\phi$ given by the Lagrangian form
\[
L_{\text{matter}}[g, \phi] d_4 x = \left[ \frac{1}{2} g^{\mu \nu} \partial \phi \partial x^\mu \partial x^\nu + V(\phi) \right] \sqrt{-\det g} d_4 x.
\]
(26)
where $V(\phi)$ is a known function. We may consider matter field to be a section $\phi$ of the trivial bundle
\[
\kappa : \mathbb{R} \times M \to M : (t, x) \mapsto x.
\]
(27)
We understand, the Lagrangian $L_{\text{matter}}$, exhibited in (26), as of the first order in $\phi$ and depending parametrically on the Lorentzian metric $g$. Introducing local coordinates $(x^\mu, t, z_\mu)$ on the first jet bundle $J^1(M, \mathbb{R} \times M)$, we write the contact form as $dt - z_\mu dx^\mu$, and $g$-dependent function $L_{\text{matter}}$ as

$$L_{\text{matter}}^g(x, t, z_\mu) = \frac{1}{2} g^{\mu\nu} z_\mu z_\nu + V(z) \sqrt{-\det g}.$$  \hfill (28)

In the coordinate representation, the De Donder form for the present case is defined as

$$\Theta^\mu_{\text{matter}} = L_{\text{matter}}^g dx + q^\mu (dt - z_\mu dx^\nu) \wedge (\partial_\mu \mathbf{d}x),$$  \hfill (29)

where $q^\mu$ is a function on $J^1(M, K)$ such that, for every local section $\phi$ of the trivial fibration $\kappa$,

$$(j^1 \phi)^* q^\mu = (j^1 \phi)^* \frac{\partial}{\partial z_\mu} L_{\text{matter}}^g = g^{\mu\nu} z_\nu \sqrt{-\det g}.$$  \hfill (30)

The total space is the fibre product $J^3(M, N) \times_M J^1(M, K)$ of the fibrations $\pi^3 : J^3(M, N) \to M$ and $\kappa^1 : J^1(M, K) \to M$. The total De Donder form $\Theta_{\text{total}}$ is the pull back of the sum of the De Donder form $\Theta$ for the gravity and the De Donder form $\Theta_{\text{matter}}$ of the matter to the this Whitney product. In coordinates, the total De Donder form is given by

$$\Theta_{\text{total}} = L_{\text{matter}}^g dx + p^{\mu\nu\alpha\beta} (dz_{\mu\nu\alpha\gamma} d\gamma) \wedge (\partial_\beta \mathbf{d}x) + p^{\mu\nu} (dy_{\mu\nu} - z_{\mu\nu\alpha\beta} dx^\alpha) \wedge (\partial_3 \mathbf{d}x) + L_{\text{matter}}^g dx + q^\mu (dt - z_\mu dx^\nu) \wedge (\partial_\mu \mathbf{d}x),$$  \hfill (31)

where, for every local section

$$\sigma \times \phi : M \to N \times_M (\mathbb{R} \times M) : x \mapsto (x, g_{\mu\nu}(x), \phi(x))$$

of the fibration $\pi \times_M \kappa$ from the product manifold $N \times_M (\mathbb{R} \times M)$ to the base manifold $M$, the coefficient functions are

$$(j^3 \sigma)^* p^{\mu\nu\alpha\beta} = (j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu\nu\alpha\beta}},$$  \hfill (32)

$$(j^3 \sigma)^* p^{\mu\nu} = (j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu\nu}} - \frac{\partial}{\partial x^\nu} \left( (j^2 \sigma)^* \frac{\partial L}{\partial x^\nu} \right),$$  \hfill (33)

$$(j^1 \phi)^* q^\mu = g^{\mu\nu} \phi_\nu \sqrt{-\det g}.$$  \hfill (34)

In equation (31) we did not put any pull-back signs to make the coordinate picture more transparent. Moreover, we replaced the subscript $g$ over $L_{\text{matter}}^g$ in equation (3.1) by $y$ in order to indicate dependence of $L_{\text{matter}}$ on the variable $y_{\mu\nu} = g_{\mu\nu}(x)$.

**Proposition 3.6.** For second order gravity with invariant Lagrangian $L$ on $J^2(M, N)$, the expressions in equations (31) and (32-34) are independent of the choice of coordinates $(x^\mu)$ on $M$. Hence, they define a global 4-form $\Theta_{\text{total}}$ on the fibre product $J^3(M, N) \times_M J^1(M, K)$.

**Proof.** of Proposition 3.6 is given in Section 4.

As in preceding section, the Euler-Lagrange equations for the total Lagrangian $L_{\text{total}}$ are equivalent to the De Donder equations for the total De Donder form $\Theta_{\text{total}}$.
4. Proofs.

4.1. Proof of Theorem 3.2. Recall that the De Donder form corresponding to a Lagrangian form \( \Lambda = L d_4 x \) on \( J^2(M, N) \) is a form \( \Theta \) on \( J^3(M, N) \), defined by

\[
\Theta = \pi_3^* L d_4 x + p^{\mu\nu\alpha\beta} (dz_{\mu\nu} - z_{\mu\nu\alpha\gamma} dx^\gamma) \wedge (\partial_\beta L d_4 x) + p^{\mu\alpha} (dy_{\mu\nu} - z_{\mu\nu\beta} dx^\beta) \wedge (\partial_\alpha L d_4 x),
\]

in the domain of the coordinate chart on \( J^3(M, N) \) induced by a coordinate chart \( (x^\mu) \) on \( M \), where \( \pi_3^* L d_4 x \) is the pull-back of the Lagrangian form by the forgetful map \( \pi_3^* : J^3(M, N) \to J^2(M, N) \), while \( p^{\mu\nu\alpha\beta} \) and \( p^{\mu\alpha} \) are Ostrogradski’s momenta. In other words, \( p^{\mu\nu\alpha\beta} \) and \( p^{\mu\alpha} \) are functions on \( J^3(M, N) \) such that

\[
(j^3 \sigma)^* p^{\mu\nu\alpha\beta} = (j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu\nu\alpha\beta}},
\]

\[
(j^3 \sigma)^* p^{\mu\alpha} = (j^2 \sigma)^* \frac{\partial L}{\partial x^\alpha} - (j^2 \sigma)^* \frac{\partial L}{\partial x^\mu} \left( (j^2 \sigma)^* \frac{\partial L}{\partial z_{\mu\nu\alpha\beta}} \right),
\]

for every local section \( \sigma \) of \( \pi \).

Our aim in this section is to study transformation laws of components of \( \Theta \) with respect to coordinate transformations in \( J^3(M, N) \) induced by an orientation preserving coordinate transformation

\[
(x^{\mu'}) \to (x^\mu) = (x^\mu (x'^\mu)).
\]

on \( M \). It induces a local coordinate transformation on \( N \) given by

\[
(x^{\mu'}, y_{\mu'\nu'}) \to (x^\mu, y_{\mu\nu} = y_{\mu'\nu'} x^{\mu'}_{\mu\nu'}).
\]

Further, we have the local expressions for transformations for the jet coordinates

\[
z_{\mu'\nu'\alpha'} = z_{\mu\nu\alpha} x^{\mu'}_{\mu\nu} x^{\nu'}_{\nu\alpha} + y_{\mu\nu} x^{\mu'}_{\mu\nu} x^{\nu'}_{\nu'\alpha'} + y_{\mu'\nu'} x^{\mu'}_{\mu'\nu'} x^{\nu'}_{\nu'\alpha'}
\]

\[
z_{\alpha'\beta'\gamma'\lambda'} = z_{\alpha\beta\gamma\lambda} x^{\alpha'}_{\alpha\beta} x^{\beta'}_{\beta\gamma} x^{\gamma'}_{\gamma\lambda} + z_{\alpha\beta\gamma} x^{\alpha'}_{\alpha\beta} x^{\beta'}_{\beta\gamma} x^{\gamma'}_{\gamma\lambda} + z_{\alpha\beta\gamma} x^{\alpha'}_{\alpha\beta} x^{\beta'}_{\beta\gamma} x^{\gamma'}_{\gamma\lambda} + z_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'} + z_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'} + z_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'}
\]

\[
+ z_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'} + z_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'} + z_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'} + y_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'} + y_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'} + y_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'}
\]

\[
+ y_{\alpha'\beta'\gamma'} x_{\alpha'\beta'} x^{\gamma'}_{\gamma'\lambda'}.
\]

By assumption the Lagrangian form \( \Lambda = L d_4 x \) is invariant under the transformations (37) through (40). This implies that under these transformations, \( L \) transforms as a scalar density. In other words,

\[
L \left( x^{\mu'}, y_{\mu'\nu'\alpha'}, z_{\mu'\nu'\alpha'\beta'} \right) \det(x^{\lambda}_{\lambda}) = L \left( x^\mu, y_{\mu\nu}, z_{\mu\nu\alpha}, z_{\mu\nu\alpha\beta} \right).
\]

We cite [22] for the global character of the boundary form \( \Xi \) presented in the following Lemma.

**Lemma 4.1.** Since the boundary form

\[
\Xi = p^{\mu\alpha\beta} (dz_{\mu\alpha} - z_{\mu\alpha\gamma} dx^\gamma) \wedge (\partial_\beta L d_4 x) + p^{\mu\alpha} (dy_{\mu\nu} - z_{\mu\nu\beta} dx^\beta) \wedge (\partial_\alpha L d_4 x)
\]

is a form of \( J^3(M, N) \) on \( J^3(M, N) \), we have

\[
\Xi = p^{\mu\alpha\beta} (dz_{\mu\alpha} - z_{\mu\alpha\gamma} dx^\gamma) \wedge (\partial_\beta L d_4 x) + p^{\mu\alpha} (dy_{\mu\nu} - z_{\mu\nu\beta} dx^\beta) \wedge (\partial_\alpha L d_4 x)
\]

for every local section \( \sigma \) of \( \pi \).
is defined to be a four-form on $J^3(M,N)$, under the change of coordinates (37) through (40), the coefficients $p^{\mu\nu\alpha}$ and $p^{\mu\nu\alpha\beta}$ transform as follows:

$$
p^{\mu\nu\alpha} = \left( p^{\mu'\nu'\alpha'} x_{\mu'} x_{\nu'} x_{\alpha'} + p^{\mu'\nu'\alpha/\beta'} x_{\beta'} x_{\alpha'} \right) \left( x^{\mu}\partial_{\mu'} x^{\nu} x_{\alpha'} + x^{\mu'} x_{\nu'} x_{\alpha'} \right) \det(x^\chi_{\alpha'}),$$

$$
p^{\mu\nu\alpha\beta} = p^{\mu'\nu'\alpha'/\beta'} x_{\mu'} x_{\nu'} x_{\alpha'} x_{\beta'}, \det(x^\chi_{\alpha'}).$$

(43) (44)

**Proof.** Notice that, the boundary term $\Xi$ in (42) is the sum of two four-forms, label them as $\Xi_1$ and $\Xi_2$, respectively. In order to deduce the transformation properties of the coefficients $p^{\mu\nu\alpha}$ and $p^{\mu\nu\alpha\beta}$, we express $\Xi_1$ and $\Xi_2$ in primed coordinates using the transformations (37) through (40) under the assumption that $\Xi_1$ and $\Xi_2$ are 4-forms on $J^3(M,N)$.

Consider first the term $\Xi_1 = p^{\mu\nu\alpha\beta}(dz_{\mu\nu\alpha\beta} - z_{\mu\nu\alpha\beta} dx^\gamma) \wedge (\partial_{\beta'} d\alpha x)$.

Taking exterior differential of the coordinate transformation (39), we get

$$
dz_{\mu'\nu'\alpha'} = x^{\mu}_{\mu'} x^{\nu}_{\nu'} x^{\alpha}_{\alpha'} d\mu_{\mu'} + z_{\mu\nu\alpha} x_{\mu'} x_{\nu'} x_{\alpha'} d\mu_{\mu'} + z_{\mu\nu\alpha} x_{\mu'} x_{\nu'} d\alpha_{\alpha'},$$

and using the transformation rule (40) of $z_{\mu'\nu'\alpha'}$, one computes

$$
z_{\mu'\nu'\alpha'\gamma} dx^\gamma = z_{\mu'\nu'\alpha'} x_{\mu'} x_{\nu'} x_{\alpha'} dx^\gamma + z_{\mu\nu\alpha} x_{\mu'} x_{\nu'} dx^\gamma + z_{\mu\nu\alpha} x_{\mu'} dx_{\alpha'},$$

$$+ z_{\mu\nu\alpha} x_{\mu'} dx_{\alpha'} + z_{\mu\nu\alpha} dx_{\alpha'}. \quad (45)$$

We subtract the one-form $z_{\mu'\nu'\alpha'\gamma} dx^\gamma$, exhibited in (46), from the one-from $dz_{\mu'\nu'\alpha'}$ in (45). While taking the difference, see that the second line of (45) cancels with the second line of (46). The second and the third terms both in the third and the forth lines of (45) cancel with the second and the third terms of the third and the forth lines of (46), respectively. Eventually, we arrive at the following expression

$$
dz_{\mu'\nu'\alpha'} - z_{\mu'\nu'\alpha'\gamma} dx^\gamma = x^{\mu}_{\mu'} x^{\nu}_{\nu'} x^{\alpha}_{\alpha'} \left( d\mu_{\mu'} - z_{\mu\nu\alpha} dx^\gamma \right) + \left( x^{\mu}_{\mu'} x^{\nu}_{\nu'} + x^{\mu'} x_{\nu'} x_{\alpha'} \right) \left( dy_{\nu'} - z_{\mu\nu\alpha} dx^\gamma \right).$$

(47)

On the other hand, it is immediate to see that

$$\partial_{\beta'} dx^\gamma = x^\gamma_{\alpha'} \partial dx^\gamma_{\alpha'} \partial_{\beta'} dx^\gamma.$$

Hence the first term in the boundary form can be obtained by first taking the exterior product of the one form $dz_{\mu'\nu'\alpha'} - z_{\mu'\nu'\alpha'\gamma} dx^\gamma$ and the third form $\partial_{\beta'} dx^\gamma$ then by multiplying the product with $p^{\mu'\nu'\alpha'\beta'}$. This shows that $\Xi_1$ expressed in

---

2Since $\Xi$ depends on the third jet variables only through $p^{\mu\nu\alpha}$, we need not write explicit transformation rules for the third jets.
terms of the primed coordinates is
\[
\Xi_1 = p^{\mu\nu\alpha\beta}(dz_{\mu\nu} - z_{\mu\nu\alpha\beta}dx^\gamma) \land (\partial_\beta J_{d4}x^\gamma)
\]
\[
= p^{\mu\nu\alpha\beta} x^\mu_{\mu\nu} x^\nu_{\mu\alpha} x^\alpha_{\beta} z_{\mu\nu\alpha\gamma} \det(x^\lambda_\alpha) \left( dz_{\mu\nu} - z_{\mu\nu\alpha\gamma}dx^\gamma \right) \land \partial_\beta J_{d4}x
\]
\[
+ p^{\mu\nu\alpha\beta} \left( x^\mu_{\mu\nu} x^\nu_{\mu\alpha} + x^\mu_{\mu\nu} x^\nu_{\mu\alpha} \right) x^\alpha_{\beta} \det(x^\lambda_\alpha) \left( dy_{\mu\nu} - z_{\mu\nu\gamma}dx^\gamma \right) \land \partial_\beta J_{d4}x.
\] (48)

So that we have derived the first term in the boundary form (42) in terms of the primed coordinates.

As a second step, we write the one-form \( dy_{\mu\nu} - z_{\mu\nu\beta}dx^\beta \) in terms of the primed coordinates. By substituting the transformations of \( y_{\mu\nu} \) in (38) and \( z_{\mu\nu\beta} \) in (39) into this one-form, we have
\[
\Xi_2 = p^{\mu\nu\beta}(dy_{\mu\nu} - z_{\mu\nu\beta}dx^\beta) \land (\partial_\beta J_{d4}x)
\]
\[
= p^{\mu\nu\beta} x^\mu_{\mu\nu} x^\nu_{\mu\beta} \left( dy_{\mu\nu} - z_{\mu\nu\gamma}dx^\gamma \right) x^\beta_{\beta} \det(x^\lambda_\alpha) \land \partial_\beta J_{d4}x. \quad (49)
\]

The sum of the four-forms in (48) and (49) is the boundary form in primed coordinates. Explicitly we have that
\[
\Xi' = \Xi_1 + \Xi_2
\]
\[
= p^{\mu\nu\alpha\beta} x^\mu_{\mu\nu} x^\nu_{\mu\alpha} x^\alpha_{\beta} \det(x^\lambda_\alpha) \left( dz_{\mu\nu} - z_{\mu\nu\gamma}dx^\gamma \right) \land \partial_\beta J_{d4}x
\]
\[
+ p^{\mu\nu\alpha\beta} x^\mu_{\mu\nu} x^\nu_{\mu\alpha} x^\alpha_{\beta} \left( dy_{\mu\nu} - z_{\mu\nu\gamma}dx^\gamma \right) \land \partial_\beta J_{d4}x
\]
\[
+ p^{\mu\nu\alpha\beta} \left( x^\mu_{\mu\nu} x^\nu_{\mu\alpha} + x^\mu_{\mu\nu} x^\nu_{\mu\alpha} \right) x^\alpha_{\beta} \left( dz_{\mu\nu} - z_{\mu\nu\gamma}dx^\gamma \right) \land \partial_\beta J_{d4}x.
\]

Since \( \Xi \) is a 4-form, its expression \( \Xi' \) in primed coordinates gives the same form as the expression in the original coordinates. That is \( \Xi = \Xi' \), which implies that
\[
p^{\mu\nu\alpha}\ = \ \left( p^{\mu\nu\alpha} x^\mu_{\mu\nu} y^\nu_{\mu\alpha} + p^{\mu\nu\alpha} x^\mu_{\mu\nu} x^\alpha_{\beta} \right) \det(x^\lambda_\alpha),
\]
\[
p^{\mu\nu\alpha}\ = \ p^{\mu\nu\alpha} x^\mu_{\mu\nu} x^\nu_{\mu\alpha} x^\alpha_{\beta} \det(x^\lambda_\alpha).
\]

This completes the proof of Lemma 4.1.

In Lemma 4.1 we showed that arbitrary smooth functions \( p^{\mu\nu\alpha\beta} \) and \( p^{\mu\nu\alpha} \) on \( J^3(M, N) \) define a 4-form
\[
p^{\mu\nu\alpha\beta}(dz_{\mu\nu} - z_{\mu\nu\gamma}dx^\gamma) \land (\partial_\beta J_{d4}x) + p^{\mu\nu\alpha}(dy_{\mu\nu} - z_{\mu\nu\beta}dx^\beta) \land (\partial_\beta J_{d4}x)
\]
on $J^3(M,N)$ provided that, under coordinate transformations (37) through (40), they transform according to equations (43) and (44). In the present case, the coefficients $p^{\mu\nu\alpha\beta}$ and $p^{\mu\nu\alpha}$ are defined as the Ostrogradski’s momenta, which implies equations (36) for every section $\sigma$ of $\pi: N \rightarrow M$. Note that any function $f$ on $J^3(M,N)$ is uniquely determined by its pull-backs $(j^3\sigma)^*f$ for all sections $\sigma$ of $\pi: N \rightarrow M$. Therefore we may write

$$p^{\mu\nu\alpha\beta} = \frac{\partial L}{\partial z^{\mu\nu\alpha\beta}}(x^\lambda, y_{\mu\nu}, z_{\mu\nu\alpha}, z_{\mu\nu\alpha\beta}),$$  \hspace{1cm} (50)$$

$$p^{\mu\nu\alpha} = \frac{\partial L}{\partial z^{\mu\nu\alpha}}(x^\lambda, y_{\mu\nu}, z_{\mu\nu\alpha}, z_{\mu\nu\alpha\beta}) - D_\beta p^{\mu\nu\alpha\beta},$$

where $D_\beta p^{\mu\nu\alpha\beta}$ is the total divergence given by

$$D_\beta p^{\mu\nu\alpha\beta} = D_\beta \left[ \frac{\partial L}{\partial z^{\mu\nu\alpha\beta}}(x^\lambda, y_{\mu\nu}, z_{\mu\nu\alpha}, z_{\mu\nu\alpha\beta}) \right]$$

$$= \frac{\partial}{\partial x^\beta} \frac{\partial L}{\partial z^{\mu\nu\alpha\beta}}(x^\lambda, y_{\mu\nu}, z_{\mu\nu\alpha}, z_{\mu\nu\alpha\beta})$$

$$+ z_{\sigma\tau\beta} \frac{\partial}{\partial y^{\sigma\tau}} \frac{\partial L}{\partial z^{\mu\nu\alpha\beta}}(x^\lambda, y_{\mu\nu}, z_{\mu\nu\alpha}, z_{\mu\nu\alpha\beta})$$

$$+ z_{\rho\sigma\tau\beta} \frac{\partial}{\partial z^{\rho\sigma\tau}} \frac{\partial L}{\partial z^{\mu\nu\alpha\beta}}(x^\lambda, y_{\mu\nu}, z_{\mu\nu\alpha}, z_{\mu\nu\alpha\beta})$$

In order to simplify computations, we use the following notations $P^{\mu\nu\alpha\beta}(x^\gamma) = (j^3\sigma)^*p^{\mu\nu\alpha\beta}(x^\gamma)$ and $P^{\mu\nu\alpha}(x^\gamma) = (j^3\sigma)^*p^{\mu\nu\alpha}(x^\gamma)$ introduced in (19). With these notations,

$$P^{\mu\nu\alpha}(x^\lambda) = (j^2\sigma)^* \left( \frac{\partial L}{\partial z^{\mu\nu\alpha}}(x^\lambda) - p^{\mu\nu\alpha\beta}(x^\lambda) \right),$$  \hspace{1cm} (52)$$

where $p^{\mu\nu\alpha\beta}(x^\lambda) = \frac{\partial}{\partial x^\beta} P^{\mu\nu\alpha\beta}(x^\lambda)$.

**Lemma 4.2.** If $L dx$ is a second order Lagrangian form on $J^2(M,N)$, invariant under the coordinate transformations (37) through (40), Ostrogradski’s momenta given by equation (50) satisfy the transformation rules (43) and (44).

**Proof.** We start with the second momentum $p^{\mu\nu\alpha\beta}$ and obtain the transformation law (44) as follows,

$$p^{\mu\nu\alpha\beta} = \frac{\partial}{\partial z^{\mu\nu\alpha\beta}} L(x^\mu, y_{\mu\nu}, z_{\mu\nu\alpha}, z_{\mu\nu\alpha\beta})$$

$$= \frac{\partial}{\partial z^{\mu'\nu'\alpha'\beta'}} L(x^{\mu'}, y_{\mu'\nu'}, z_{\mu'\nu'\alpha'}, z_{\mu'\nu'\alpha'\beta'}) \det(x^\lambda) \frac{\partial}{\partial z^{\mu'\nu'\alpha'\beta'}}$$

$$= p^{\mu'\nu'\alpha'\beta'} \det(x^\lambda) x^{\mu'} x^{\nu'} x^{\alpha'} x^{\beta'},$$

where we have used the chain rule, the prolonged coordinate transformation for $z_{\mu'\nu'\alpha'\beta'}$ given in (40), and equation (41).
In order to show that the first momentum $P^{\mu \nu \alpha}$ satisfies transformation law (43), start first with the term $\partial L/\partial z_{\mu \nu \alpha}$. As in equation (53),

\[
\frac{\partial L}{\partial z_{\mu \nu \alpha}} = \frac{\partial z_{\mu' \nu' \alpha'}}{\partial z_{\mu \nu \alpha}} \frac{\partial L}{\partial z_{\mu' \nu' \alpha'}} \det(x^X_{\lambda}) + \frac{\partial z_{\mu' \nu' \alpha'}}{\partial z_{\mu \nu \alpha}} \frac{\partial L}{\partial z_{\mu' \nu' \alpha'}} \det(x^X_{\lambda})
\]

\[
= x^\mu_{\mu'} x^\nu_{\nu'} x^\alpha_{\alpha'} \frac{\partial L}{\partial z_{\mu' \nu' \alpha'}} \det(x^X_{\lambda}) + \left( x^\mu_{\mu'} x^\nu_{\nu'} x^\alpha_{\alpha'} + x^\nu_{\nu'} x^\mu_{\mu'} x^\alpha_{\alpha'} + x^\alpha_{\alpha'} x^\mu_{\mu'} x^\nu_{\nu'} \right) \frac{\partial L}{\partial z_{\mu' \nu' \alpha'}} \det(x^X_{\lambda}).
\]

(54)

To compute the divergence term, we work with pull-backs $P^{\mu \nu \alpha} = (j^3 \sigma)^* P^{\mu \nu \alpha}$ and $P^{\mu \nu \alpha \beta} = (j^3 \sigma)^* P^{\mu \nu \alpha \beta}$, which allows replacing total derivative by partial derivative, see equation (52). We obtain

\[
(j^3 \sigma)^* (D_{\beta} P^{\mu \nu \alpha \beta}) = (j^3 \sigma)^* P^{\mu \nu \alpha \beta},
\]

\[
= \left( P^{\mu' \nu' \alpha' \beta'} \det(x^X_{\lambda}) x^\mu_{\mu'} x^\nu_{\nu'} x^\alpha_{\alpha'} x^{\beta'}_{\beta} \right) \frac{\partial L}{\partial z_{\mu' \nu' \alpha' \beta'}} \det(x^X_{\lambda}) + \left( P^{\mu' \nu' \alpha' \beta'} \det(x^X_{\lambda}) x^\mu_{\mu'} x^\nu_{\nu'} x^\alpha_{\alpha'} \right) \frac{\partial L}{\partial z_{\mu' \nu' \alpha' \beta'}} x^{\gamma}_{\gamma}
\]

\[
= P^{\mu' \nu' \alpha' \beta'} \det(x^X_{\lambda}) x^\mu_{\mu'} x^\nu_{\nu'} x^\alpha_{\alpha'} + P^{\nu' \mu' \alpha' \beta'} \det(x^X_{\lambda}) (x^\mu_{\mu'} x^\nu_{\nu'} x^\alpha_{\alpha'}) x^{\beta}_{\beta}
\]

\[
+ P^{\mu' \nu' \alpha' \beta'} \det(x^X_{\lambda}) (x^\mu_{\mu'} x^\nu_{\nu'} x^\alpha_{\alpha'}) x^{\beta}_{\beta} x^{\gamma}_{\gamma}.
\]

(55)

Note that

\[
\det(x^X_{\lambda})_{\beta'} = \frac{\partial \det(x^X_{\lambda})_{\gamma}}{\partial x^\gamma_{\beta}} \frac{\partial x^{\gamma}_{\beta}}{\partial x^{\gamma}_{\beta'}} = \det(x^X_{\lambda}) x^{\gamma}_{\gamma} \frac{\partial x^{\gamma}_{\beta}}{\partial x^{\gamma}_{\beta'}}.
\]

(56)

and

\[
\frac{\partial}{\partial x^{\beta'}} \left( x^\gamma_{\beta} x^\beta_{\gamma} \right) = \frac{\partial}{\partial x^{\beta'}} \delta^\gamma_{\beta} = 0,
\]

which implies

\[
x^{\beta}_{\gamma} \frac{\partial x^{\gamma}_{\beta}}{\partial x^{\beta'}} = -x^\gamma_{\beta} \frac{\partial x^{\beta}_{\gamma}}{\partial x^{\beta'}}.
\]

(57)

Therefore, the second term on the right hand side of equation (55) reads

\[
P^{\mu' \nu' \alpha' \beta'} \det(x^X_{\lambda})_{\beta'} (x^\mu_{\mu'} x^\nu_{\nu'} x^\alpha_{\alpha'}) = P^{\nu' \mu' \alpha' \beta'} \det(x^X_{\lambda}) x^{\gamma}_{\gamma} \frac{\partial x^{\gamma}_{\beta}}{\partial x^{\gamma}_{\beta'}} x^\mu_{\mu'} x^\nu_{\nu'} x^\alpha_{\alpha'}
\]

\[
= -P^{\mu' \nu' \alpha' \beta'} \det(x^X_{\lambda}) x^{\gamma}_{\gamma} x^{\mu}_{\mu'} x^{\nu}_{\nu'} x^\alpha_{\alpha'}.
\]
In the light of this, we can rewrite $F_{\mu\nu}^{\alpha\beta}$ in (55) as
\[ P_{\mu\nu}^{\alpha\beta} = P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma})(j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}}) (x_{\alpha'}^{\gamma}) + P_{\mu\nu}^{\alpha\beta} \det(x_{\alpha'}^{\gamma}) \]
\[ + (x_{\alpha'}^{\gamma}x_{\mu'}^{\mu}x_{\nu'}^{\nu}x_{\alpha'}^{\alpha}) (j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}}) \]
\[ - P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma})(x_{\mu'}^{\mu}x_{\nu'}^{\nu}) - P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma})(x_{\mu'}^{\mu}x_{\nu'}^{\nu}) \]
\[ = P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma})(x_{\mu'}^{\mu}x_{\nu'}^{\nu}x_{\alpha'}^{\alpha}) (j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}}) \]
\[ + P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma})(x_{\mu'}^{\mu}x_{\nu'}^{\nu}x_{\alpha'}^{\alpha}) (j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}}) \]
\[ = P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma})(x_{\mu'}^{\mu}x_{\nu'}^{\nu}x_{\alpha'}^{\alpha}) \]
\[ + (\alpha_{\beta'}^{\mu'}x_{\mu'}^{\mu}x_{\nu'}^{\nu}) P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma}). \] 
(59)

In order to arrive at the coordinate transformation for Ostrogradski’s momentum $P_{\mu\nu}^{\alpha\beta}$, we take the difference of $(j^2\sigma)^*(\partial L/\partial z_{\mu\nu\alpha\beta})$ in (54) and $P_{\mu\nu}^{\alpha\beta}$ in (58). So that,
\[ P_{\mu\nu}^{\alpha\beta} = (j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu\nu\alpha\beta}})(x_{\mu}, y_{\mu\nu}, z_{\mu\nu\alpha\beta}, z_{\mu\nu\alpha\beta}) - P_{\mu\nu}^{\alpha\beta} \]
\[ = x_{\mu'}^{\mu}x_{\nu'}^{\nu}x_{\alpha'}^{\alpha}(j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}}) \]
\[ + (x_{\mu'}^{\mu}x_{\nu'}^{\nu}x_{\alpha'}^{\alpha}) (j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}}) \]
\[ - x_{\mu'}^{\mu}x_{\nu'}^{\nu}x_{\alpha'}^{\alpha} - P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma})(x_{\mu'}^{\mu}x_{\nu'}^{\nu}) \]
\[ = (j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}})(x_{\mu'}, y_{\mu'\nu'}, z_{\mu'\nu'\alpha\beta}, z_{\mu'\nu'\alpha\beta}) \]
\[ + (\alpha_{\beta'}^{\mu'}x_{\mu'}^{\mu}x_{\nu'}^{\nu}) (j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}}) - P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma})(x_{\mu'}^{\mu}x_{\nu'}^{\nu}) \]
\[ = P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma})(x_{\mu'}^{\mu}x_{\nu'}^{\nu}x_{\alpha'}^{\alpha}) \]
\[ + (\alpha_{\beta'}^{\mu'}x_{\mu'}^{\mu}x_{\nu'}^{\nu}) (j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}}) - P_{\mu\nu}^{\alpha'\beta'} \det(x_{\alpha'}^{\gamma}) \]
\[ = (j^2\sigma)^*(\frac{\partial L}{\partial z_{\mu'\nu'}}) - P_{\mu\nu}^{\alpha'\beta'}. \] 

Here, we used notation (19) in the primed coordinates, which yields
\[ P_{\mu\nu}^{\alpha'\beta'} = (j^3\sigma)^* p_{\mu'\nu'\beta'} = (j^2\sigma)^* (\frac{\partial L}{\partial z_{\mu'\nu'\alpha'\beta'}}). \]
\[ P_{\mu\nu}^{\alpha'\beta'} = (j^3\sigma)^* p_{\mu'\nu'\alpha'} = (j^2\sigma)^* (\frac{\partial L}{\partial z_{\mu'\nu'\alpha'}}) \]
\[ - (j^3\sigma)^* (\frac{\partial L}{\partial z_{\mu'\nu'\alpha'}}). \]
Equation (59) may be rewritten as
\[ p^{\mu\nu\alpha} = p^{\mu\nu\alpha'} x^\mu x^\nu x^\alpha \text{det}(x^\lambda) \]
\[ + \left( x^{\alpha'}_{\beta\gamma} x^\mu x^\nu x^\alpha + x^{\alpha'}_{\beta\gamma} x^\mu x^\nu x^\alpha \right) p^{\mu\nu\alpha'\beta'} \text{det}(x^\lambda), \]
Since this equation is valid for every section \( \sigma \) of \( \pi : N \to M \), it follows that
\[ p^{\mu\nu\alpha} = p^{\mu\nu\alpha'} x^\mu x^\nu x^\alpha \text{det}(x^\lambda) \]
\[ + \left( x^{\alpha'}_{\beta\gamma} x^\mu x^\nu x^\alpha + x^{\alpha'}_{\beta\gamma} x^\mu x^\nu x^\alpha \right) p^{\mu\nu\alpha'\beta'} \text{det}(x^\lambda), \]
where \( p^{\mu\nu\alpha} \) and \( p^{\mu\nu\alpha'\beta'} \) are Ostrogradski’s momenta corresponding to the Lagrangian form \( L_d x \). (50). This completes proof of Lemma 4.2.

It follows from Lemma 4.1 and Lemma 4.2 that for an invariant Lagrangian form \( L_d x \), the corresponding boundary form has the same expression in the class of coordinate system on \( M \), which differ by orientation preserving transformations. Therefore, the boundary form \( \Xi \) is globally defined and is given by a natural differential operator applied to the to the Lagrangian form \( L_d x \). Since \( \Theta = \pi^*_N L_d x + \Xi \), it follows that the De Donder form \( \Theta \) is globally defined and is also given by a natural differential operator applied to the to the Lagrangian form \( L_d x \). This completes proof of Theorem 3.2.

4.2. Proof of Proposition 3.5. The outline of the proof is as follows. First, we will write the Hilbert Lagrangian (24) in terms of the metric tensor and its partial derivatives. Such kind of a local presentation of the Hilbert Lagrangian will enable us to prove the Lemma 4.3 where we shall exhibit the induced Ostrogradski’s momenta. Then, we will be ready for the calculation of the De Donder form (25) in an explicit form.

Recall that the Christoffel symbols of the first kind \( \Gamma_{\mu\nu} \) and the Christoffel symbols of the second kind \( \Gamma^\rho_{\mu\nu} \) are defined and related as
\[ \Gamma^\rho_{\mu\nu} = g^{\rho\lambda} \Gamma_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\nu\mu,\lambda}), \]
where \( g^{\rho\lambda} \) is the dual of the metric tensor \( g_{\rho\lambda} \) whereas \( g_{\mu\lambda,\nu} \) denotes the partial derivative of \( g_{\mu\lambda} \) with respect to \( x^\nu \). It is possible to write the Christoffel symbols in a pure contravariant form
\[ \Gamma_{\mu\nu} = g^{\lambda\alpha} g^{\beta\gamma} \Gamma_{\alpha\beta\gamma}, \]
For future reference, we define here some symbols by contacting the Christoffel symbols
\[ \Gamma^\Lambda = g^{\mu\nu} \Gamma_{\mu\nu} = g_{\mu\nu} \Gamma^\Lambda, \]
\[ \Delta^\nu = g^{\mu\nu} \Delta_{\mu} = g_{\nu\lambda} \Gamma^\Lambda_{\mu\nu}, \]
\[ \Gamma_\rho = g_{\rho\lambda} \Gamma^\Lambda = g^{\mu\nu} \Gamma_\rho_{\mu\nu}, \]
\[ \Delta_{\mu} = \Gamma_\mu^\Lambda = g^{\mu\nu} \Gamma_{\mu\nu}. \]
Taking the derivative of the identity \( g^{\rho\lambda} g_{\mu\nu} = \delta^\rho_\gamma \), one arrives at relation between \( g^{\rho\lambda}_{\gamma\lambda} \) and \( \Gamma_{\mu\nu} \)
\[ g^{\rho\lambda}_{\gamma\lambda} = -g^{\rho\mu} g^{\delta\nu} g_{\mu\nu,\gamma} = -g^{\rho\mu} g^{\delta\nu} \left( \Gamma_{\mu\nu\gamma} + \Gamma_{\nu\mu\gamma} \right), \]
whereas the contraction of this yields
\[ g^{\alpha \delta} = -g^{\alpha \mu} g^{\delta \nu} (\Gamma_{\mu \nu \alpha} + \Gamma_{\nu \mu \alpha}) \]  
(67)

Recall also that, the Riemann and the Ricci tensors are
\[ R_{\beta \gamma} = \Gamma_{\beta \delta, \gamma} - \Gamma_{\beta \gamma, \delta} + \Gamma_{\mu \gamma} \Gamma_{\beta \delta} - \Gamma_{\mu \delta} \Gamma_{\beta \gamma}, \]  
(68)
\[ R_{\beta \delta} = R_{\gamma \delta \alpha \beta} + \Gamma_{\beta \delta, \alpha} - \Gamma_{\beta \alpha, \delta} + \Gamma_{\mu \alpha} \Gamma_{\gamma \delta} - \Gamma_{\mu \delta} \Gamma_{\gamma \alpha}, \]  
(69)
respectively. Here, \( \Gamma_{\beta \delta} \) denotes the partial derivative of \( \Gamma_{\beta \delta} \) with respect to \( x^\gamma \).

In this local representation, the scalar curvature is defined to be
\[ R = g^{\beta \gamma} R_{\beta \gamma} = g^{\beta \gamma} \left( \Gamma_{\beta \gamma, \alpha} - \Gamma_{\beta \alpha, \gamma} + \Gamma_{\mu \alpha} \Gamma_{\beta \gamma} - \Gamma_{\mu \gamma} \Gamma_{\beta \alpha} \right). \]  
(70)

Note that the presentation (70) is in terms of the Christoffel symbols of the second kind. It is possible to write \( R \) in terms of the Christoffel symbols of the first kind and its partial derivative as well. Simply, by substituting the definition in (60), we compute
\[ R = g^{\beta \gamma} \left( g^{\alpha \delta} \Gamma_{\delta \beta \gamma} \right)_{, \alpha} - g^{\beta \gamma} \left( g^{\alpha \delta} \Gamma_{\delta \beta \alpha} \right)_{, \gamma} + g^{\beta \gamma} g^{\alpha \rho} g^{\mu \sigma} (\Gamma_{\rho \mu \alpha} \Gamma_{\sigma \beta \gamma} - \Gamma_{\rho \mu \gamma} \Gamma_{\sigma \beta \alpha}) 
+ g^{\beta \gamma} g^{\alpha \delta} (\Gamma_{\delta \beta \gamma, \alpha} - \Gamma_{\delta \alpha, \gamma} + \Gamma_{\delta \beta \gamma} g^{\alpha \rho} g^{\mu \sigma} (\Gamma_{\rho \mu \alpha} \Gamma_{\gamma \delta \alpha} + \Gamma_{\rho \mu \gamma} \Gamma_{\beta \delta \alpha})). \]

Notice that, the symbols \( \Gamma_{\rho \mu \alpha} \) contain the first derivative \( g_{\mu \nu, \lambda} \) of the metric \( g_{\mu \nu} \). So that the partial derivative \( \Gamma_{\beta \gamma, \alpha} \) of the symbols are containing the second partial derivative \( g_{\mu \nu, \lambda} \) of \( g_{\mu \nu} \). In accordance with this, we understand the scalar curvature \( R \) as the sum of two terms, say \( R_1 \) and \( R_2 \) by putting all the first order terms involving \( \Gamma_{\rho \mu \alpha} \) into \( R_1 \), and by putting all the second order terms involving \( \Gamma_{\delta \beta \gamma, \alpha} \) into \( R_2 \), that is \( R = R_1 + R_2 \) and
\[ R_1 = g^{\beta \gamma} \left( g^{\alpha \delta} \Gamma_{\delta \beta \gamma} - g^{\alpha \delta} \Gamma_{\delta \beta \alpha} \right) + g^{\beta \gamma} g^{\alpha \rho} g^{\mu \sigma} (\Gamma_{\rho \mu \alpha} \Gamma_{\gamma \beta \gamma} - \Gamma_{\rho \mu \gamma} \Gamma_{\beta \delta \alpha}) \]
\[ R_2 = g^{\beta \gamma} g^{\alpha \delta} (\Gamma_{\delta \beta \gamma, \alpha} - \Gamma_{\delta \alpha, \gamma} + \Gamma_{\delta \beta \gamma} g^{\alpha \rho} g^{\mu \sigma} (\Gamma_{\rho \mu \alpha} \Gamma_{\gamma \delta \alpha} + \Gamma_{\rho \mu \gamma} \Gamma_{\beta \delta \alpha})). \]

Therefore, we write the Lagrangian as
\[ L = R_1 \sqrt{-\det g} + R_2 \sqrt{-\det g}, \]  
(71)
where \( R_1 \) depends on \( g_{\mu \nu, \alpha} \) linearly while \( R_2 \) depends quadratically on \( g_{\mu \nu, \alpha \beta} \). A simplification is possible for \( R_1 \). See that,
\[ R_1 = g^{\beta \gamma} \left( -g^{\alpha \mu} g^{\delta \nu} (\Gamma_{\mu \nu \alpha} + \Gamma_{\nu \mu \alpha}) \Gamma_{\delta \beta \gamma} + g^{\alpha \mu} g^{\beta \nu} (\Gamma_{\nu \gamma \alpha} + \Gamma_{\gamma \nu \alpha}) \Gamma_{\delta \beta \gamma} \right) 
+ g^{\beta \gamma} g^{\alpha \rho} g^{\mu \sigma} (\Gamma_{\rho \mu \alpha} \Gamma_{\sigma \beta \gamma} - \Gamma_{\rho \mu \gamma} \Gamma_{\sigma \beta \alpha}) \]
\[ = \Gamma_{\mu \nu \lambda \alpha \beta \gamma} (\Gamma_{\mu \nu \alpha \beta \gamma} - \Gamma_{\mu \nu \alpha \beta \gamma}) + \Gamma_{\mu \nu \lambda \alpha \beta \gamma} (\Gamma_{\mu \nu \alpha \beta \gamma} - \Gamma_{\mu \nu \alpha \beta \gamma}) \]
\[ = \Gamma_{\mu \nu \lambda \alpha \beta \gamma} (\Gamma_{\mu \nu \alpha \beta \gamma} - \Gamma_{\mu \nu \alpha \beta \gamma}), \]  
(72)
where we have employed the identities in (66) and (67) in the first line. In the third line, the first term in the parentheses is canceling with the fifth term, and the third term in the parentheses is canceling with the sixth term. We write \( R_2 \) in terms of
the metric tensor

\[ R_2 = g^\beta_\gamma g^\alpha_\delta (\Gamma_{\delta_\gamma,\alpha} - \Gamma_{\delta_\alpha,\gamma}) \]
\[ = \frac{1}{2} g^\beta_\gamma g^\alpha_\delta (g_{\beta_\delta,\gamma_\alpha} + g_{\gamma_\delta,\alpha_\gamma} - g_{\beta_\delta,\gamma_\alpha} - g_{\beta_\delta,\gamma_\alpha} + g_{\gamma_\delta,\alpha_\gamma} + g_{\beta_\delta,\gamma_\alpha}) \]
\[ = \frac{1}{2} g_{\mu_\nu,\alpha_\beta} (g^{\mu_\alpha} g^{\nu_\beta} + g^{\mu_\beta} g^{\nu_\alpha} - 2 g^{\mu_\nu} g^{\alpha_\beta}). \quad (73) \]

In the following Lemma, we are stating the conjugate momenta induced by the Hilbert Lagrangian.

**Lemma 4.3.** The Ostrogradski’s momenta induced by the Hilbert Lagrangian (71) are

\[ P^{\mu_\nu_\alpha} = \frac{1}{2} (-g^{\mu_\alpha} \Gamma^\nu - g^{\nu_\alpha} \Gamma^\mu + \Gamma^{\mu_\nu_\alpha}) \sqrt{-\det g} \]  
\[ P^{\mu_\nu_\alpha_\beta} = \frac{1}{2} (g^{\mu_\alpha} g^{\nu_\beta} + g^{\mu_\beta} g^{\nu_\alpha} - 2 g^{\mu_\nu} g^{\alpha_\beta}) \sqrt{-\det g}, \]  
\[ \quad (74, 75) \]

where the symbol \( \Gamma^\nu \) is the one defined in (62).

**Proof.** First recall the definition of the Ostrogradski’s momenta

\[ P^{\mu_\nu_\alpha} = \frac{\partial L}{\partial g_{\mu_\nu,\alpha}} - P^{\mu_\nu_\alpha_\beta}, \quad P^{\mu_\nu_\alpha_\beta} = \frac{\partial L}{\partial g_{\mu_\nu,\alpha_\beta}}. \]

By substituting the exhibition of the Hilbert Lagrangian given in (71), we can rewrite the momenta as

\[ P^{\mu_\nu_\alpha} = \frac{\partial R_1}{\partial g_{\mu_\nu,\alpha}} \sqrt{-\det g} - P^{\mu_\nu_\alpha_\beta}, \quad P^{\mu_\nu_\alpha_\beta} = \frac{\partial R_2}{\partial g_{\mu_\nu,\alpha_\beta}} \sqrt{-\det g}, \quad (76) \]

where \( R_1 \) and \( R_2 \) as the ones in (72) and (73), respectively. It is immediate to observe that the second momenta is

\[ P^{\mu_\nu_\alpha_\beta} = \frac{1}{2} \left( g^{\mu_\alpha} g^{\nu_\beta} + g^{\mu_\beta} g^{\nu_\alpha} - 2 g^{\mu_\nu} g^{\alpha_\beta} \right) \sqrt{-\det g}. \quad (77) \]

Notice from (76) that, in order to determine the first momenta \( P^{\mu_\nu_\alpha} \), we need to take the divergence of the second momenta \( P^{\mu_\nu_\alpha_\beta} \), given in (77), with respect to \( x^\beta \). For this, we start with taking the partial derivative of \( \sqrt{-\det g} \) with respect to \( x^\beta \) as follows

\[ \left( \sqrt{-\det g} \right)_{,\beta} = \frac{1}{2 \sqrt{-\det g}} (-1) \frac{\partial}{\partial x^\beta} (\det g) = \frac{-1}{2 \sqrt{-\det g}} (\det g) g^{\sigma_\tau, \beta} g_{\sigma_\tau, \beta} \]
\[ = \frac{1}{2} \sqrt{-\det g} g_{\sigma_\tau, \beta} (\Gamma_{\sigma_\tau, \beta} + \Gamma_{\tau_\sigma, \beta}) \]
\[ = \sqrt{-\det g} g_{\sigma_\tau, \beta} \Gamma_{\sigma_\tau, \beta} = \sqrt{-\det g} \Delta_\beta, \quad (78) \]
where the symbol $\Delta_\beta$, in (65), has been substituted in the last line of the calculation. On the other hand, we take the divergence
\[
\left( g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - 2 g^{\mu\nu} g^{\alpha\beta} \right)_{,\beta}
\]
\[
= g_{\beta\gamma}^{\mu} g^{\nu\beta} + g^{\mu\alpha} g_{\beta\gamma}^{\nu} + g^{\mu\beta} g_{\beta\gamma}^{\nu} + g^{\mu\beta} g_{\beta\gamma}^{\nu} - 2 g^{\mu\nu} g_{\beta\gamma}^{\alpha} - 2 g^{\mu\nu} g_{\beta\gamma}^{\alpha} 
\]
\[
= -g_{\beta\gamma}^{\nu} g_{\beta\gamma}^{\mu} (\Gamma_{\mu\alpha\rho} + \Gamma_{\rho\alpha\mu}) - g^{\mu\alpha} g_{\beta\gamma}^{\mu} (\Gamma_{\gamma\beta\rho} + \Gamma_{\beta\gamma\rho}) - g^{\mu\beta} g_{\beta\gamma}^{\mu} (\Gamma_{\gamma\alpha\rho} + \Gamma_{\alpha\gamma\rho}) + 2 g^{\mu\nu} g_{\beta\gamma}^{\alpha} (\Gamma_{\gamma\beta\rho} + \Gamma_{\beta\gamma\rho}) + 2 g^{\alpha\beta} g_{\beta\gamma}^{\gamma} (\Gamma_{\gamma\alpha\rho} + \Gamma_{\alpha\gamma\rho}) 
\]
\[
= -\Gamma_{\gamma\beta\alpha} - \Gamma_{\alpha\beta\gamma} - g^{\mu\alpha} (\Gamma_{\nu} + \Delta_{\nu}) - g^{\nu\alpha} (\Gamma_{\mu} + \Delta_{\mu}) - 2 g^{\mu\nu} (\Gamma_{\alpha} + \Delta_{\alpha}) + 2(\Gamma_{\alpha\mu} + \Gamma_{\nu\mu}) + 2 \Gamma_{\gamma\beta\alpha} + \Gamma_{\alpha\beta\gamma} - (\Gamma_{\alpha\mu} + \Gamma_{\nu\mu}) - (\Gamma_{\alpha\mu} + \Gamma_{\nu\mu}),
\]
(80)
where, the identities (66) and (67) have been used in the second line, and the symbols $\Gamma_{\alpha}$ and $\Delta_{\alpha}$, defined in (62) and (63) have been substituted in the sixth and the seventh lines. In the light of the calculations in (78) and (80), the divergence of the second momenta $P^{\mu\nu\alpha\beta}$ turns out to be
\[
P_{\alpha\beta}^{\mu\nu\alpha\beta} = \frac{1}{2} (g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - 2 g^{\mu\nu} g^{\alpha\beta}) \left( \sqrt{-\det g} \right)_{,\beta}
\]
\[
= \frac{1}{2} (g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - 2 g^{\mu\nu} g^{\alpha\beta}) \sqrt{-\det g} \Delta_{\beta}
\]
\[
= \frac{1}{2} (g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - 2 g^{\mu\nu} g^{\alpha\beta}) \sqrt{-\det g} \Delta_{\beta}
\]
\[
= \frac{1}{2} \left( 2 g^{\mu\nu} (\Gamma_{\alpha} + \Delta_{\alpha}) - g^{\mu\alpha} (\Gamma_{\nu} + \Delta_{\nu}) - g^{\nu\alpha} (\Gamma_{\mu} + \Delta_{\mu}) \right) \sqrt{-\det g}
\]
\[
= \frac{1}{2} \left( 2 g^{\mu\nu} (\Gamma_{\alpha} + \Delta_{\alpha}) - g^{\mu\alpha} (\Gamma_{\nu} + \Delta_{\nu}) - g^{\nu\alpha} (\Gamma_{\mu} + \Delta_{\mu}) \right) \sqrt{-\det g}
\]
(81)
where the identity $g^{\nu\beta} \Delta_{\beta} = \Delta_{\nu}$ has been used. Notice that all the terms involving $\Delta_{\nu}$ canceling each other in the calculation. Let us now concentrate on the first term $\partial R_1 / \partial g_{\mu\nu,\alpha} \sqrt{-\det g}$ in the momenta $P^{\mu\nu\alpha}$, applying the chain rule, we have that
\[
\frac{\partial R_1}{\partial g_{\mu\nu,\alpha}} \sqrt{-\det g} = \frac{\partial R_1}{\partial \Gamma_{\lambda\mu\nu}} \frac{\partial \Gamma_{\lambda\mu\nu}}{\partial g_{\alpha\beta,\gamma}} \sqrt{-\det g}.
\]
Notice that, the partial derivative of $R_1$ with respect to the Christoffel symbol of the first kind $\Gamma_{\lambda\mu\nu}$ is computed to be
\[
\frac{\partial R_1}{\partial \Gamma_{\lambda\mu\nu}} = 2 \Gamma_{\alpha\beta\gamma} \left( -g^{\lambda\alpha} g^{\mu\nu} g^{\beta\gamma} + g^{\lambda\nu} g^{\mu\alpha} g^{\beta\gamma} \right),
\]
(82)
whereas the partial derivative of $\Gamma_{\lambda\mu\nu}$ with respect to $g_{\alpha\beta,\gamma}$ is
\[
\frac{\partial \Gamma_{\lambda\mu\nu}}{\partial g_{\alpha\beta,\gamma}} = \frac{1}{4} \left( \left( \delta^\beta_\mu \delta^\gamma_\alpha \delta^\gamma_\delta + \delta^\beta_\mu \delta^\gamma_\alpha \delta^\gamma_\delta - \delta^\beta_\mu \delta^\gamma_\delta \delta^\beta_\alpha \right) + \left( \delta^\beta_\mu \delta^\delta_\alpha \delta^\gamma_\nu + \delta^\beta_\mu \delta^\delta_\alpha \delta^\nu_\delta - \delta^\beta_\mu \delta^\nu_\delta \delta^\beta_\alpha \right) \right).
\]
(83)
Here, the factor $1/4$ is the manifestation of the symmetry of the metric tensor. We multiply the expressions (82) and (83) and arrange the terms, so that we arrive at

$$\frac{\partial R_2}{\partial g_{\mu\nu,\alpha}} \sqrt{-\det g} = g^{\mu\nu} \Gamma^\alpha - g^{\mu\alpha} \Gamma^\nu - g^{\nu\alpha} \Gamma^\mu - \Gamma^{\alpha\mu\nu} + \Gamma^{\nu\mu\alpha} + \Gamma^{\mu\nu\alpha} \sqrt{-\det g}.$$  \hspace{1cm} (84)

Now we are ready to write the first momenta $P^{\mu\nu\alpha}$, for this simply take the difference of (84) and (81), this gives

$$P^{\mu\nu\alpha} = \frac{\partial R_2}{\partial g_{\mu\nu,\alpha}} \sqrt{-\det g} - P_{\beta}^{\mu\nu\alpha\beta}$$

$$= (g^{\mu\nu} \Gamma^\alpha - g^{\mu\alpha} \Gamma^\nu - g^{\nu\alpha} \Gamma^\mu - \Gamma^{\alpha\mu\nu} + \Gamma^{\nu\mu\alpha} + \Gamma^{\mu\nu\alpha}) \sqrt{-\det g}$$

$$- \frac{1}{2} (2g^{\mu\nu} \Gamma^\alpha - g^{\mu\alpha} \Gamma^\nu - g^{\nu\alpha} \Gamma^\mu - 2\Gamma^{\alpha\mu\nu} + \Gamma^{\nu\mu\alpha} + \Gamma^{\mu\nu\alpha}) \sqrt{-\det g}$$

$$= \frac{1}{2} (-g^{\mu\nu} \Gamma^\alpha - g^{\nu\alpha} \Gamma^\mu + \Gamma^{\nu\mu\alpha} + \Gamma^{\mu\nu\alpha}) \sqrt{-\det g}$$  \hspace{1cm} (85)

where the first and fourth terms in the second and the third lines are canceling each other, respectively.

We are now ready to prove Proposition 3.5. In the present framework, the De Donder form turns out to be

$$\Theta_{\text{Hilbert}} = R_1 \sqrt{-\det g} d_4 x + R_2 \sqrt{-\det g} d_4 x + \Xi_{\text{Hilbert}},$$

where $\Xi_{\text{Hilbert}}$ is the boundary form induced by the Hilbert Lagrangian. Explicitly, the boundary form is

$$\Xi_{\text{Hilbert}} = (P^{\alpha\beta\mu} g_{\alpha\beta} + P^{\alpha\beta\mu\nu} g_{\alpha\beta,\nu\mu} \partial x) \sqrt{-\det g} d_4 x - (P^{\alpha\beta\mu} g_{\alpha\beta,\mu} + P^{\alpha\beta\mu\nu} g_{\alpha\beta,\nu\mu}) d_4 x.$$

By substituting the conjugate momenta $P^{\mu\nu\alpha}$ and $P^{\mu\nu\alpha\beta}$, respectively given in (74) and (75), one has

$$\Xi_{\text{Hilbert}} = \frac{1}{2} \left( -g^{\alpha\mu} \Gamma^\beta - g^{\beta\mu} \Gamma^\alpha + \Gamma^{\beta\mu\alpha} + \Gamma^{\mu\alpha\beta} \right) \sqrt{-\det g} g_{\alpha\beta} \partial x \wedge (\frac{\partial}{\partial x} d_4 x)$$

$$+ \frac{1}{2} \left( g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - 2g^{\alpha\beta} g^{\mu\nu} \right) \sqrt{-\det g} g_{\alpha\beta,\nu \mu} \partial x \wedge (\frac{\partial}{\partial x} d_4 x)$$

$$- \frac{1}{2} \left( g^{\alpha\mu} \Gamma^\beta - g^{\beta\mu} \Gamma^\alpha + \Gamma^{\beta\mu\alpha} + \Gamma^{\mu\beta\alpha} \right) \sqrt{-\det g} g_{\alpha\beta,\mu} \partial x \wedge (\frac{\partial}{\partial x} d_4 x)$$

$$- \frac{1}{2} \left( g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - 2g^{\alpha\beta} g^{\mu\nu} \right) \sqrt{-\det g} g_{\alpha\beta,\nu \mu} \partial x.$$

Substitution of the boundary form $\Xi_{\text{Hilbert}}$ and the terms $R_1$ and $R_2$ in (72) and (73) leads to the following expression of the De Donder form

$$\Theta_{\text{Hilbert}} = \frac{1}{2} g_{\mu\nu,\alpha\beta} \left( g^{\mu\alpha} g^{\nu\beta} + g^{\nu\beta} g^{\mu\alpha} - 2g^{\mu\nu} g^{\alpha\beta} \right) \sqrt{-\det g} d_4 x$$

$$+ \Gamma_{\mu\nu\alpha\beta} \left( -g^{\alpha\mu} g^{\beta\nu} g^{\gamma\lambda} + g^{\alpha\lambda} g^{\beta\nu} g^{\gamma\mu} \right) \sqrt{-\det g} d_4 x$$

$$+ \frac{1}{2} \left( -g^{\alpha\mu} \Gamma^\beta - g^{\beta\mu} \Gamma^\alpha + \Gamma^{\beta\mu\alpha} + \Gamma^{\mu\beta\alpha} \right) \sqrt{-\det g} g_{\alpha\beta} \partial x \wedge (\frac{\partial}{\partial x} d_4 x)$$

$$+ \frac{1}{2} \left( g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - 2g^{\alpha\beta} g^{\mu\nu} \right) \sqrt{-\det g} g_{\alpha\beta,\nu \mu} \partial x \wedge (\frac{\partial}{\partial x} d_4 x)$$

$$- \frac{1}{2} \left( g^{\alpha\mu} \Gamma^\beta - g^{\beta\mu} \Gamma^\alpha + \Gamma^{\beta\mu\alpha} + \Gamma^{\mu\beta\alpha} \right) g_{\alpha\beta,\mu} d_4 x$$

$$- \frac{1}{2} g_{\alpha\beta,\nu \mu} \left( g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - 2g^{\alpha\beta} g^{\mu\nu} \right) \sqrt{-\det g} d_4 x.$$
Notice that, the first and the last terms are canceling since they are the negative of each other. So that there remain

\[
\Theta_{\text{Hilbert}} = (\Gamma_{\mu\nu} \Gamma_{\alpha\beta\gamma} \left( -g^{\lambda\alpha} g^{\mu\nu} g^{\beta\gamma} + g^{\lambda\alpha} g^{\mu\gamma} g^{\nu\beta} \right) \sqrt{-\det gd_4 x} \\
- \frac{1}{2} \left( -g^{\alpha\mu} \Gamma^\beta - g^{\beta\mu} \Gamma^\alpha + \Gamma^{\beta\alpha\mu} + \Gamma^{\alpha\beta\mu} \right) g_{\alpha\beta,\mu} \sqrt{-\det gd_4 x} \\
+ \frac{1}{2} \left( -g^{\alpha\mu} \Gamma^\beta - g^{\beta\mu} \Gamma^\alpha + \Gamma^{\beta\alpha\mu} + \Gamma^{\alpha\beta\mu} \right) \sqrt{-\det gd_4 x} \cdot \left( \frac{\partial}{\partial x^\mu} J_{d_4} x \right) \\
+ \frac{1}{2} \left( g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - 2g^{\alpha\beta} g^{\mu\nu} \right) \sqrt{-\det gd_4 x} \cdot \left( \frac{\partial}{\partial x^\nu} J_{d_4} x \right).
\]

Let us simply concentrate on the second line of this expression. A simple calculation gives

\[
\frac{1}{2} \left( -g^{\alpha\mu} \Gamma^\beta - g^{\beta\mu} \Gamma^\alpha + \Gamma^{\beta\alpha\mu} + \Gamma^{\alpha\beta\mu} \right) g_{\alpha\beta,\mu}
= \frac{1}{2} \left( -g^{\alpha\mu} \Gamma^\beta - g^{\beta\mu} \Gamma^\alpha + \Gamma^{\beta\alpha\mu} + \Gamma^{\alpha\beta\mu} \right) \left( \Gamma_{\alpha\beta\mu} + \Gamma_{\beta\alpha\mu} \right)
= \frac{1}{2} \Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} (-g^{\gamma\alpha} g^{\mu\nu} g^{\beta\lambda} - g^{\beta\gamma} g^{\mu\nu} g^{\alpha\lambda} - g^{\beta\gamma} g^{\mu\nu} g^{\alpha\lambda} - g^{\gamma\alpha} g^{\mu\nu} g^{\beta\lambda})
+ \frac{1}{2} \Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} \left( g^{\beta\lambda} g^{\alpha\mu} g^{\gamma\nu} + g^{\alpha\lambda} g^{\beta\mu} g^{\gamma\nu} + g^{\alpha\lambda} g^{\beta\mu} g^{\gamma\nu} + g^{\beta\lambda} g^{\alpha\mu} g^{\gamma\nu} \right)
= \Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} (g^{\beta\lambda} g^{\mu\nu} g^{\alpha\gamma} + g^{\alpha\lambda} g^{\mu\nu} g^{\beta\gamma} - g^{\alpha\lambda} g^{\mu\nu} g^{\beta\gamma} - g^{\alpha\lambda} g^{\mu\nu} g^{\beta\gamma}),
\]

where we used the identity (66) in the first line, and the identity (61) in the third line, and in the fourth line we sum up the similar terms. This simplification means that the coefficient of the basis \( \sqrt{-\det gd_4 x} \) can be written as

\[
\Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} \left( -g^{\lambda\alpha} g^{\mu\nu} g^{\beta\gamma} + g^{\alpha\lambda} g^{\mu\nu} g^{\beta\gamma} \right)
- \Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} \left( g^{\beta\gamma} g^{\mu\nu} g^{\alpha\lambda} + g^{\alpha\lambda} g^{\mu\nu} g^{\beta\gamma} \right)
= \Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} \left( g^{\lambda\alpha} g^{\mu\nu} g^{\beta\gamma} + \Gamma_{\mu\lambda\nu} \Gamma_{\alpha\beta\gamma} g^{\alpha\gamma} g^{\mu\nu} g^{\beta\lambda} - \Gamma_{\mu\lambda\nu} \Gamma_{\alpha\beta\gamma} g^{\beta\lambda} g^{\alpha\mu} g^{\gamma\nu} \right)
- \Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} \left( g^{\beta\gamma} g^{\mu\nu} g^{\alpha\lambda} + \Gamma_{\mu\lambda\nu} \Gamma_{\alpha\beta\gamma} g^{\alpha\gamma} g^{\mu\nu} g^{\beta\lambda} - \Gamma_{\mu\lambda\nu} \Gamma_{\alpha\beta\gamma} g^{\beta\lambda} g^{\alpha\mu} g^{\gamma\nu} \right)
= \Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} \left( g^{\beta\gamma} g^{\mu\nu} g^{\alpha\lambda} + \Gamma_{\mu\lambda\nu} \Gamma_{\alpha\beta\gamma} g^{\alpha\gamma} g^{\mu\nu} g^{\beta\lambda} - \Gamma_{\mu\lambda\nu} \Gamma_{\alpha\beta\gamma} g^{\beta\lambda} g^{\alpha\mu} g^{\gamma\nu} \right)
- \Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} \left( g^{\beta\gamma} g^{\mu\nu} g^{\alpha\lambda} + \Gamma_{\mu\lambda\nu} \Gamma_{\alpha\beta\gamma} g^{\alpha\gamma} g^{\mu\nu} g^{\beta\lambda} - \Gamma_{\mu\lambda\nu} \Gamma_{\alpha\beta\gamma} g^{\beta\lambda} g^{\alpha\mu} g^{\gamma\nu} \right),
\]

where we have cancelled the first and last terms in the first line, and used the symmetry of the Christoffel symbol in the third line. Eventually, the De Donder form for the Hilbert Lagrangian becomes

\[
\Theta_{\text{Hilbert}} = \Gamma_{\lambda\mu\nu} \Gamma_{\alpha\beta\gamma} \left( g^{\alpha\gamma} g^{\mu\nu} g^{\beta\lambda} - g^{\beta\lambda} g^{\alpha\nu} g^{\gamma\mu} \right) \sqrt{-\det gd_4 x} \\
+ \frac{1}{2} \left( -g^{\alpha\mu} \Gamma^\beta - g^{\beta\mu} \Gamma^\alpha + \Gamma^{\beta\alpha\mu} + \Gamma^{\alpha\beta\mu} \right) \sqrt{-\det gd_4 x} \cdot \left( \frac{\partial}{\partial x^\mu} J_{d_4} x \right) \\
+ \frac{1}{2} \left( g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - 2g^{\alpha\beta} g^{\mu\nu} \right) \sqrt{-\det gd_4 x} \cdot \left( \frac{\partial}{\partial x^\nu} J_{d_4} x \right).
\]
A comparison of this form and the one in (25) shows that the proof of Proposition 5 in Section 3.2 is achieved.

4.3. Proof of Proposition 3.6. Referring to Proposition 3.5, to prove Proposition 3.6 we need to just focus on the De Donder form (29) for the matter. See that it is composed of a Lagrangian term and the boundary term. We label the boundary term as

$$\Xi_3 = q^\mu (dt - z_\nu dx^\nu) \wedge (\partial_\mu \int_4 dx),$$

where the coefficient function $q^\mu$ reads (37) for a local section. Let us first show that, $\Xi_3$ is invariant under a coordinate transformation on the base manifold $M$ given in (37). See that

$$q^\mu = g^\mu\nu \phi_\nu \sqrt{- \det (g_{\mu\nu})} = g^\mu\nu x^\mu_{\beta\delta} \phi_\alpha \partial_\nu \sqrt{- \det (g_{\mu\nu} x^\mu_{\beta\delta})}$$

$$= \det (x^\mu_{\lambda}) x^\mu_{\mu\nu} g^\mu\nu \phi_\nu \sqrt{- \det (g_{\mu\nu})}$$

$$= q^\mu \det (x^\mu_{\lambda}) x^\mu_{\mu\nu}.$$

For the basis we recall the transformation in (47), and compute

$$(dt - z_\nu dx^\nu) = (dt - z_\nu x_{\nu\alpha} dx^\alpha) = dt - z_\nu dx^\nu.$$

Collecting all these, one sees that formulation of the boundary form remains the same under coordinate transformation

$$q^\mu \sqrt{- \det g(dt - z_\nu dx^\nu)} \wedge (\partial_\mu \int_4 dx)$$

$$= q^\mu \det (x^\mu_{\lambda}) x^\mu_{\mu\nu} (dt - z_\nu dx^\nu) \wedge (\det (x^\mu_{\lambda}) x^\mu_{\mu\nu} \partial_\nu \int_4 dx)$$

$$= q^\mu (dt - z_\nu dx^\nu) \wedge (\partial_\mu \int_4 dx).$$

On the other hand, the Lagrangian term is

$$L_{\text{matter}}^g \int_4 dx = \frac{1}{2} g^\mu\nu z_\mu z_\nu + V(z) \sqrt{- \det g} \int_4 dx$$

$$= \frac{1}{2} g^\mu\nu x^\mu_{\alpha\beta} x^\nu_{\gamma\delta} z_\alpha z_\beta x^\beta_{\mu\nu} x^\delta_{\mu\nu} + V(z') \sqrt{- \det g} \int_4 dx'$$

$$= \frac{1}{2} g^\mu\nu z_\mu z_\nu + V(z') \sqrt{- \det g} \int_4 dx',$$

where we have employed the fact that $\sqrt{- \det g} \int_4 dx$ is invariant under the coordinate transformation. Notice that, assumption of the invariance of the function $V$ leads to the invariance of $L_{\text{matter}}^g \int_4 dx$.

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