Gröbner bases of nested configurations

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Abstract

In this paper we introduce a new and large family of configurations whose toric ideals possess quadratic Gröbner bases. As an application, a generalization of algebras of Segre–Veronese type will be studied.

1 Introduction

Let \( K[t] = K[t_1, \ldots, t_d] \) be the polynomial ring over a field \( K \). A finite set \( A \) of monomials of \( K[t] \) is called a configuration of \( K[t] \) if there exists a nonnegative vector \( (w_1, \ldots, w_d) \in \mathbb{R}_{\geq 0}^d \) such that \( \sum_{i=1}^d w_i a_i = 1 \) for all \( t_1^{a_1} \cdots t_d^{a_d} \in A \). Let \( A \) be a configuration of \( K[t] \). We associate \( A \) with the homogeneous semigroup ring \( K[A] \) which is the subalgebra of \( K[t] \) generated by the monomials of \( A \). Let \( K[X] = K[\{x_M | M \in A\}] \) denote the polynomial ring over \( K \) in the variables \( x_M \) with \( M \in A \), where each \( \deg(x_M) = 1 \). The toric ideal \( I_A \) of \( A \) is the kernel of the surjective homomorphism \( \pi : K[X] \to K[A] \) defined by setting \( \pi(x_M) = M \) for all \( M \in A \). It is known that the toric ideal \( I_A \) is generated by homogeneous binomials. See, e.g., [Stu, Section 4].

A fundamental question in commutative algebra is to determine whether \( K[A] \) is Koszul. A Gröbner basis \( \mathcal{G} \) is called a quadratic Gröbner basis if \( \mathcal{G} \) consists of quadratic homogeneous polynomials. Even though it is difficult to prove that \( K[A] \) is Koszul, the hierarchy \( (i) \implies (ii) \implies (iii) \) is known among the following properties:

(i) \( I_A \) possesses a quadratic Gröbner basis.

(ii) \( K[A] \) is Koszul;

(iii) \( I_A \) is generated by quadratic binomials.

However both \( (ii) \implies (i) \) and \( (iii) \implies (ii) \) are false in general. One can find counterexamples for them in [OHI, Examples 2.1 and 2.2].

Let \( A \) be a configuration of a polynomial ring \( K[t] = K[t_1, \ldots, t_d] \) with \( d \) variables. For each \( i = 1, 2, \ldots, d \), let \( B_i = \{m_1^{(i)}, \ldots, m_{\lambda}^{(i)}\} \) be a configuration of a polynomial ring

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\( K[u^{(i)}] = K[u_1^{(i)}, \ldots, u_{\mu_i}^{(i)}] \) with \( \mu_i \) variables. The \textit{nested configuration} arising from \( A \) and \( B_1, \ldots, B_d \) is the configuration

\[ A(B_1, \ldots, B_d) := \left\{ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \mid 1 \leq r \in \mathbb{Z}, t_{i_1} \cdots t_{i_r} \in A, \ 1 \leq j_k \leq \lambda_k \right\} \]

of a polynomial ring \( K[u^{(1)}, \ldots, u^{(d)}] \). The main result of the present paper is as follows:

**Theorem 3.6.** Work with the same notation as above. If the toric ideals \( I_A, I_{B_1}, \ldots, I_{B_d} \) possess quadratic Gröbner bases, then the toric ideal \( I_{A(B_1, \ldots, B_d)} \) possesses a quadratic Gröbner basis.

In addition, in Section 4, we study a quadratic Gröbner basis of the toric ideal of \( A(B_1, \ldots, B_d) \) where \( A \) comes from a Segre–Veronese configuration.

### 2 Motivation from statistics

In this section we present a statistical problem, which motivates a generalization of Segre–Veronese configuration considered in our previous paper [AHOT].

We consider nested selection of groups and items from the groups. For example, consider an examination on mathematics which consists of \( J \) groups of problems (e.g. algebra, geometry and statistics) and each group \( j \) consists of \( m_j \) individual problems. For simplicity let \( J = 3 \) and \( m_j = 3 \). Label the nine individual problems as \( A_1, A_2, A_3, G_1, G_2, G_3 \) and \( S_1, S_2, S_3 \). Suppose that each examinee is asked to choose two groups of problems and then \( c_j = 2 \) problems from each chosen group \( j \). Then there are 27 patterns of selections of four problems as \( (A_1, A_2, G_1, G_2), (A_1, A_3, G_1, G_2), (A_2, A_3, G_1, G_2), \ldots, (G_2, G_3, S_2, S_3) \).

Now as a simple statistical model suppose that each problem is chosen according to its own attractiveness, independent of the choices of other problems within the same group as well as the choice of other group. Let \( q_{A_1}, q_{A_2}, \ldots, q_{S_3} \) denote the attractiveness of each problem. Then the probabilities of the selections are expressed as

\[
\begin{align*}
\text{Prob}(A_1, A_2, G_1, G_2) &= c q_{A_1} q_{A_2} q_{G_1} q_{G_2}, \\
\vdots \\
\text{Prob}(G_2, G_3, S_2, S_3) &= c q_{G_2} q_{G_3} q_{S_2} q_{S_3},
\end{align*}
\]

where \( c \) is the normalizing constant so that the 27 probabilities sum to one. Now associate a configuration \( A \) to the semigroup ring

\[ K(q_{A_1} q_{A_2} q_{G_1} q_{G_2}, \ldots, q_{G_2} q_{G_3} q_{S_2} q_{S_3}) \]

which is a subring of the polynomial ring \( K(q_{A_1}, q_{A_2}, \ldots, q_{S_3}) \) in nine variables. A system of generators of the toric ideal for \( I_A \), such as the reduced Gröbner basis, is required for statistical test of this model. This example corresponds to \( A(B_1, B_2, B_3) \) where \( A = \{ t_1 t_2, t_1 t_3, t_2 t_3 \} \) and \( B_1, B_2, B_3 \) are copies of \( A \) with different variables.
Enumeration of different selections becomes somewhat more complicated if the same item can be chosen more than once (“sampling with replacement”). Suppose that a customer is given two (identical) coupons, which allow the customer to go to one of several shops and buy two items at a discount at the shop. Buying the same item twice is allowed. For simplicity suppose that there are only two shops A, B and they sell only two different items \{A1, A2\} and \{B1, B2\}, respectively. A person may buy A1 four times, by going to the shop A twice and buying A1 twice each time. Or a person may buy each of A1, A2, B1, B2 once. Note that in this scheme it is not possible to buy three items from shop A and 1 item from shop B. Again we can think of a statistical model that the relative popularity of selections is explained entirely by the attractiveness of each item. This corresponds to Example 3.4 below, where \(A1 = u_1^{(1)}, A2 = u_2^{(1)}, B1 = u_1^{(2)}, B2 = u_2^{(2)}\).

Note that in the above examples we can also consider recursive nesting of subgroups.

3 Nested configurations

In this section, we introduce an effective method to construct semigroup rings whose toric ideals have quadratic Gröbner bases.

Let \(A\) be a configuration of a polynomial ring \(K[t] = K[t_1, \ldots, t_d]\) with \(d\) variables. For each \(i = 1, 2, \ldots, d\), let \(B_i = \{m_1^{(i)}, \ldots, m_{\lambda_i}^{(i)}\}\) be a configuration of a polynomial ring \(K[u^{(i)}] = K[u_1^{(i)}, \ldots, u_{\lambda_i}^{(i)}]\) with \(\mu_i\) variables. The nested configuration arising from \(A\) and \(B_1, \ldots, B_d\) is the configuration

\[
A(B_1, \ldots, B_d) := \left\{ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \mid 1 \leq r \in \mathbb{Z}, t_{i_1} \cdots t_{i_r} \in A, \ 1 \leq j_k \leq \lambda_k \right\}
\]

of a polynomial ring \(K[u^{(1)}, \ldots, u^{(d)}]\).

**Example 3.1.** If \(B_i = \{m_1^{(i)}\}\) for all \(1 \leq i \leq d\), then we have \(K[A(B_1, \ldots, B_d)] \simeq K[A]\).

**Example 3.2.** Let \(A = \{t_1\}\) and let \(B_1 = \{u_1, \ldots, u_\lambda\}\) be the set of variables. Then \(K[A(B_1)]\) is \(r\)th Veronese subring of the polynomial ring \(K[B_1] = K[u_1, \ldots, u_\lambda]\).

**Example 3.3.** Let \(A = \{t_1t_2\}\) and let \(B_1 = \{u_1^{(1)}, \ldots, u_{\lambda_1}^{(1)}\}\) and \(B_2 = \{u_1^{(2)}, \ldots, u_{\lambda_2}^{(2)}\}\) be the sets of variables. Then \(K[A(B_1, B_2)]\) is the Segre product of the polynomial rings \(K[B_1] = K[u_1^{(1)}, \ldots, u_{\lambda_1}^{(1)}]\) and \(K[B_2] = K[u_1^{(2)}, \ldots, u_{\lambda_2}^{(2)}]\).

Let \(\eta\) be the cardinality of \(A(B_1, \ldots, B_d)\) and set \(A(B_1, \ldots, B_d) = \{M_1, \ldots, M_\eta\}\). Let

\[
K[x] = K[x_{M_1}, \ldots, x_{M_\eta}], \\
K[y] = K[y_{i_1 \cdots i_r} \mid 1 \leq r \in \mathbb{Z}, i_1 \leq \cdots \leq i_r, t_{i_1} \cdots t_{i_r} \in A] \\
K[z^{(i)}] = K[z_{\lambda_1}^{(i)}, \ldots, z_{\lambda_i}^{(i)}] \quad (i = 1, 2, \ldots, d)
\]

be polynomial rings. The toric ideal \(I_A\) is the kernel of the homomorphism \(\pi_0 : K[y] \longrightarrow K[t]\) defined by setting \(\pi_0(y_{i_1 \cdots i_r}) = t_{i_1} \cdots t_{i_r}\). The toric ideal \(I_{B_i}\) is the kernel of the
homomorphism $\pi_i : K[z^{(i)}] \rightarrow K[u^{(i)}]$ defined by setting $\pi_i(z_j^{(i)}) = m_j^{(i)}$. The toric ideal $I_{A(B_1, \ldots, B_d)}$ is the kernel of the homomorphism $\pi : K[x] \rightarrow K[u^{(1)}, \ldots, u^{(d)}]$ defined by setting $\pi(x_M) = M$.

Let $\mathcal{G}_i$ be a Gröbner basis of $I_{B_i}$ with respect to a monomial order $<_i$ for $1 \leq i \leq d$. For each $M \in A(B_1, \ldots, B_d)$, the expression $M = m_1^{(i_1)} \cdots m_r^{(i_r)}$ is called standard if

$$\prod_{i_k=j, \ 1 \leq k \leq r} z_{j_k}^{(i_k)}$$

is a standard monomial with respect to $\mathcal{G}_j$ for all $1 \leq j \leq d$.

**Example 3.4.** Let $A = \{t_1^2, t_1 t_2, t_2^2\}$,

$$B_1 = \left\{ m_1^{(1)} = (u_1^{(1)})^2, m_2^{(1)} = u_1^{(1)} u_2^{(1)}, m_3^{(1)} = (u_2^{(1)})^2 \right\},$$

$$B_2 = \left\{ m_1^{(2)} = (u_1^{(2)})^2, m_2^{(2)} = u_1^{(2)} u_2^{(2)}, m_3^{(2)} = (u_2^{(2)})^2 \right\}.$$

Then $A(B_1, B_2)$ consists of the monomials

$$\left( u_1^{(1)} \right)^4, \left( u_1^{(1)} \right)^3 u_2^{(1)}, \left( u_1^{(1)} \right)^2 \left( u_2^{(1)} \right)^2, u_1^{(1)} \left( u_2^{(1)} \right)^3, \left( u_2^{(1)} \right)^4,$$

$$\left( u_1^{(1)} \right)^2 \left( u_1^{(2)} \right)^2, \left( u_1^{(1)} \right)^2 u_1^{(2)} u_2^{(1)}, \left( u_1^{(1)} \right)^2 \left( u_2^{(2)} \right)^2,$$

$$u_1^{(1)} u_2^{(1)} \left( u_2^{(2)} \right)^2, u_1^{(1)} u_2^{(1)} u_1^{(2)} u_2^{(2)}, u_1^{(1)} u_2^{(2)}, \left( u_2^{(2)} \right)^2,$$

$$\left( u_2^{(1)} \right)^2 \left( u_1^{(2)} \right)^2, \left( u_2^{(1)} \right)^2 u_1^{(2)} u_2^{(2)}, \left( u_2^{(1)} \right)^2 \left( u_2^{(2)} \right)^2,$$

$$\left( u_1^{(2)} \right)^4, \left( u_1^{(2)} \right)^3 u_2^{(2)}, \left( u_1^{(2)} \right)^2 \left( u_2^{(2)} \right)^2, u_1^{(2)} \left( u_2^{(2)} \right)^3, \left( u_2^{(2)} \right)^4$$

and, with respect to any monomial order,

$$\mathcal{G}_0 = \{y_{11} y_{22} - y_{12}^2\},$$

$$\mathcal{G}_1 = \{z_1^{(1)} z_3^{(1)} - (z_2^{(1)})^2\},$$

$$\mathcal{G}_2 = \{z_1^{(2)} z_3^{(2)} - (z_2^{(2)})^2\}$$

are Gröbner bases of

$$I_A = \langle y_{11} y_{22} - y_{12}^2 \rangle,$$

$$I_{B_1} = \langle z_1^{(1)} z_3^{(1)} - (z_2^{(1)})^2 \rangle,$$

$$I_{B_2} = \langle z_1^{(2)} z_3^{(2)} - (z_2^{(2)})^2 \rangle,$$
respectively. Let \( >_0 \) be a lexicographic order induced by \( y_{11} >_0 y_{12} >_0 y_{22} \) and let \( >_1 \) a lexicographic order induced by \( z_1^{(i)} >_1 z_2^{(i)} >_1 z_3^{(i)} \) for \( i = 1, 2 \). For example,

\[
M = (u_1^{(i)})^2 (u_2^{(i)})^2 \in A(B_1, B_2)
\]

has two expressions, that is, \( M = m_1^{(i)} m_3^{(i)} \) and \( M = (m_2^{(i)})^2 \). Since \( z_1^{(1)} z_3^{(1)} \) is not standard and \( (z_2^{(1)})^2 \) is standard with respect to \( G_1 \), \( M = m_1^{(1)} m_3^{(1)} \) is not a standard expression and \( M = (m_2^{(1)})^2 \) is a standard expression.

In order to study the relation among \( I_A, I_B, \) and \( I_{A(B_1, \ldots, B_d)} \), we define homomorphisms

\[
\varphi_0 : K[x] \longrightarrow K[y] , \quad \varphi_0 \left( x_{m_1^{(i)} \cdots m_r^{(i)}} \right) = y_{i_1 \cdots i_r},
\]

\[
\varphi_j : K[x] \longrightarrow K[z^{(j)}] , \quad \varphi_j \left( x_{m_1^{(i)} \cdots m_r^{(i)}} \right) = \prod_{i_k = j} z_{j_k}^{(i)},
\]

where \( m_1^{(i)} \cdots m_r^{(i)} \) is the standard expression defined above. For example,

\[
\varphi_1 \left( (u_1^{(i)})^2 (u_2^{(i)})^2 \right)
\]

is not \( z_1^{(1)} z_3^{(1)} \) but \( (z_2^{(1)})^2 \) in Example 3.4. Throughout this paper, we order the monomials of \( A(B_1, \ldots, B_d) \) as \( A(B_1, \ldots, B_d) = \{ M_1, \ldots, M_n \} \) where

\[
\pi_0 \circ \varphi_0 (x_{M_1}) \geq_{lex} \cdots \geq_{lex} \pi_0 \circ \varphi_0 (x_{M_n}) \tag{\ast}
\]

with respect to the lexicographic order \( <_{lex} \) induced by \( t_1 > \cdots > t_d \).

**Lemma 3.5.** Let \( f \) be a binomial in \( K[x] \). Then the following conditions are equivalent:

1. \( f \in I_{A(B_1, \ldots, B_d)} \);
2. \( \varphi_i(f) \in I_{B_i} \) for all \( 1 \leq i \leq d \).

Moreover, if the above conditions hold, then we have \( \varphi_0(f) \in I_A \).

**Proof.** Let \( f = x^\alpha - x^\beta \). Since \( \pi(x_M) = M = \prod_{j=1}^d \pi_j \circ \varphi_j(x_M) \) and each \( \pi_j \circ \varphi_j(x_M) \) belongs to \( K[u^{(j)}] \), we have

\[
f \in I_{A(B_1, \ldots, B_d)} = \ker(\pi) \iff \pi(x^\alpha) = \pi(x^\beta)
\]

\[
\iff \prod_{j=1}^d \pi_j \circ \varphi_j(x^\alpha) = \prod_{j=1}^d \pi_j \circ \varphi_j(x^\beta)
\]

\[
\iff \pi_j \circ \varphi_j(x^\alpha) = \pi_j \circ \varphi_j(x^\beta) \quad \text{for all } 1 \leq j \leq d
\]

\[
\iff \pi_j \circ \varphi_j(f) = 0 \quad \text{for all } 1 \leq j \leq d
\]

\[
\iff \varphi_j(f) \in \ker(\pi_j) = I_{B_j} \quad \text{for all } 1 \leq j \leq d.
\]
Thus (i) $\iff$ (ii) holds.

Recall that $I_{B_i}$ is homogeneous for all $1 \leq i \leq d$. If $M = m^{(i_1)}_{j_1} \cdots m^{(i_r)}_{j_r}$, then

$$\pi_0 \circ \varphi_0(x_M) = t_{i_1} \cdots t_{i_r} = \prod_{j=1}^{d} t_j^{\deg(\varphi_j(x_M))}.$$  

Hence we have

$$\varphi_j(f) \in I_{B_j} \text{ for all } 1 \leq j \leq d$$

$$\implies \deg(\varphi_j(x^\alpha)) = \deg(\varphi_j(x^\beta)) \text{ for all } 1 \leq j \leq d$$

$$\implies \pi_0 \circ \varphi_0(x^\alpha) = \prod_{j=1}^{d} t_j^{\deg(\varphi_j(x^\alpha))} = \prod_{j=1}^{d} t_j^{\deg(\varphi_j(x^\beta))} = \pi_0 \circ \varphi_0(x^\beta)$$

$$\implies \pi_0 \circ \varphi_0(f) = 0$$

$$\implies \varphi_0(f) \in \ker(\pi_0) = I_A$$

as desired. 

\[\square\]

**Theorem 3.6.** Suppose that the toric ideals $I_A$, $I_{B_1}, \ldots, I_{B_d}$ possess quadratic Gröbner bases $G_0$, $G_1, \ldots, G_d$ with respect to $<_0$, $<_1, \ldots, <_d$ respectively. Then the toric ideal $I_{A(B_1,\ldots,B_d)}$ possesses the reduced Gröbner basis $G$ consisting of all binomials of the form $x_{M_\alpha}x_{M_\beta} - x_{M_\gamma}x_{M_\delta}$ where $M_\alpha = m^{(i_1)}_{j_1} m^{(i_2)}_{j_2} \cdots m^{(i_r)}_{j_r}$, $M_\beta = m^{(k_1)}_{\ell_1} m^{(k_2)}_{\ell_2} \cdots m^{(k_s)}_{\ell_s}$ with $\gamma \leq \delta$ and

$$\varphi_j(x_{M_\alpha}x_{M_\beta}) \xrightarrow{G_j} \varphi_j(x_{M_\alpha}x_{M_\delta}) \quad \text{for each } j = 0, 1, \ldots, d, \quad (1)$$

$$i_\lambda = k_\mu \implies j_\lambda \leq \ell_\mu \quad \text{for } 1 \leq \lambda \leq r, \ 1 \leq \mu \leq s. \quad (2)$$

The initial monomial of $x_{M_\alpha}x_{M_\beta} - x_{M_\gamma}x_{M_\delta}$ is $x_{M_\alpha}x_{M_\beta}$.

**Example 3.4 (continued).** The reduced Gröbner basis $G$ in the statement of Theorem 3.6 consists of 105 binomials. For example,

$$x_{m_1^{(1)}m_2^{(1)}m_3^{(1)}m_4^{(1)}m_5^{(1)}m_6^{(1)}} - x_{m_2^{(1)}m_3^{(1)}}^2$$

$$x_{m_1^{(1)}m_1^{(1)}m_3^{(1)}m_4^{(1)}m_5^{(1)}m_6^{(1)}} - x_{m_1^{(1)}m_2^{(1)}m_3^{(1)}m_4^{(1)}m_5^{(1)}}^2$$

$$x_{m_2^{(1)}m_2^{(1)}m_3^{(1)}m_4^{(1)}m_5^{(1)}m_6^{(1)}} - x_{m_1^{(1)}m_2^{(1)}m_3^{(1)}m_4^{(1)}m_5^{(1)}}^2$$

$$x_{m_1^{(1)}m_2^{(1)}}^2 - x_{m_1^{(1)}m_1^{(1)}m_2^{(1)}m_3^{(1)}m_4^{(1)}m_5^{(1)}}$$

belong to $G$ and the initial monomial is the first monomial for each binomial.

**Proof of Theorem 3.6.** Let $x_{M_\alpha}x_{M_\beta} - x_{M_\gamma}x_{M_\delta} \in G$. Thanks to the condition (1) above, we have $\varphi_j(x_{M_\alpha}x_{M_\beta}) - \varphi_j(x_{M_\gamma}x_{M_\delta}) \in I_{B_j}$ for all $1 \leq j \leq d$. By virtue of Lemma 3.3, we have $x_{M_\alpha}x_{M_\beta} - x_{M_\gamma}x_{M_\delta} \in I_{A(B_1,\ldots,B_d)}$. Thus $G$ is a subset of $I_{A(B_1,\ldots,B_d)}$. 

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Since the reduction relation modulo Gröbner bases are Noetherian, [Stu, Theorem 3.12] guarantees that there exists a monomial order such that the monomial $x_{M_a}x_{M_β}$ is the initial monomial for each $x_{M_a}x_{M_β} - x_{M_a}x_{M_λ} \in \mathcal{G}$.

Suppose that there exists a binomial $0 \neq u - v \in I_{A(B_1, \ldots , B_d)}$ such that neither $u$ nor $v$ is divided by the initial monomial of any binomial in $\mathcal{G}$. By virtue of Lemma 1, we have $\varphi_0(u) - \varphi_0(v) \in I_A$ and $\varphi_i(u) - \varphi_i(v) \in I_{B_i}$ for all $1 \leq i \leq d$. Hence $\varphi_i(u) - \varphi_i(v) \xrightarrow{G_i} 0$ for all $0 \leq i \leq d$. Moreover, since neither $u$ nor $v$ is divided by the initial monomial of any binomial in $\mathcal{G}$, we have $\varphi_i(u) \xrightarrow{G_i} \varphi_i(v)$ and $\varphi_i(v) \xrightarrow{G_i} \varphi_i(v)$. Thus $\varphi_i(u) = \varphi_i(v)$ for all $0 \leq i \leq d$. By virtue of $\varphi_0(u) = \varphi_0(v)$ and our convention $(*)$,

$$u = x_{M_{t_1}}x_{M_{t_2}} \cdots x_{M_{t_p}},$$

$$v = x_{M_{t_1}'}x_{M_{t_2}'} \cdots x_{M_{t_p}'};$$

where $1 \leq t_1 \leq \cdots \leq t_p \leq \eta$, $1 \leq t_1' \leq \cdots \leq t_p' \leq \eta$ and

$$M_{t_1} = m_{(i_{1,1})}m_{(i_{1,2})} \cdots m_{(i_{1,r_{t_1}})},$$

$$M_{t_1'} = m_{(i_{1,1})}m_{(i_{1,2})} \cdots m_{(i_{1,r_{t_1}})}.$$ 

Since $\varphi_j(u) = \varphi_j(v)$ for all $1 \leq j \leq d$, we have

$$\prod_{i_{1,1}=j, 1 \leq \xi \leq p, 1 \leq q \leq r_{\xi}} z_{j_{\xi,q}}^{(i_{\xi,q})} = \prod_{i_{1,1}=j, 1 \leq \xi \leq p, 1 \leq q \leq r_{\xi}} z_{j_{\xi,q}}^{(i_{\xi,q})}.$$ 

Thanks to the condition $(2)$, we have

$$i_{\xi,q} = i_{\xi',q'}, \xi < \xi' \implies j_{\xi,q} \leq j_{\xi',q'},$$

$$i_{\xi,q} = i_{\xi',q'}, \xi < \xi' \implies k_{\xi,q} \leq k_{\xi',q'}.$$ 

Hence $M_{t_1} = M_{t_1'}$ for all $1 \leq \xi \leq p$. Thus we have $u = v$ and this is a contradiction. \qed

**Example 3.7.** In the definition of a nested configuration, we assumed that each $B_t$ and $B_j$ have no common variable. If $B_t$ and $B_j$ have a common variable for some $1 \leq i < j \leq \lambda$, then Theorem [3.6] does not hold in general. For example, if $A = \{t_1t_4, t_2t_5, t_3t_6\}$, $B_1 = \{u_1\}$, $B_2 = \{u_2\}$, $B_3 = \{u_3\}$, $B_4 = \{v_1, v_2\}$, $B_5 = \{v_2, v_3\}$ and $B_6 = \{v_1, v_3\}$, then $I_{A(B_1, \ldots , B_6)}$ is a principal ideal generated by a binomial of degree 3.

### 4 Nested configurations arising from Segre–Veronese configurations

A typical class of semigroup rings whose toric ideal possesses a quadratic initial ideal is *algebras of Segre–Veronese type* defined in [OH2, AHOT]. Fix integers $\tau \geq 2$ and $n$ and sets of integers $b = \{b_1, \ldots , b_n\}$, $c = \{c_1, \ldots , c_n\}$, $p = \{p_1, \ldots , p_n\}$ and $q = \{q_1, \ldots , q_n\}$ such that
(i) \( 0 \leq c_i \leq b_i \) for all \( 1 \leq i \leq n \);

(ii) \( 1 \leq p_i \leq q_i \leq d \) for all \( 1 \leq i \leq n \).

Let \( A_{r,b,c,r,s} \subset K[t_1, \ldots, t_d] \) denote the set of all monomials \( \prod_{j=1}^{d} t_j^{f_j} \) such that

(i) \( \sum_{j=1}^{d} f_j = r \).

(ii) \( c_i \leq \sum_{j=p_i}^{q_i} f_j \leq b_i \) for all \( 1 \leq i \leq n \).

Then the affine semigroup ring \( K[A_{r,b,c,r,s}] \) is called an algebra of Segre–Veronese type.

Several popular classes of semigroup rings are algebras of Segre–Veronese type. If \( n = 2, \tau = 2, b_1 = b_2 = c_1 = c_2 = 1, p_1 = 1, p_2 = q_1 + 1 \) and \( q_2 = d \), then the affine semigroup ring \( K[A_{r,b,c,r,s}] \) is the Segre product of polynomial rings \( K[t_1, \ldots, t_q] \) and \( K[t_{q+1}, \ldots, t_d] \). On the other hand, if \( n = d, p_i = q_i = i, b_i = \tau \) and \( c_i = 0 \) for all \( 1 \leq i \leq n \), then the affine semigroup ring \( K[A_{r,b,c,r,s}] \) is the classical \( \tau \)th Veronese subring of the polynomial ring \( K[t_1, \ldots, t_d] \). Moreover, if \( n = d, p_i = q_i = i, b_i = 1 \) and \( c_i = 0 \) for all \( 1 \leq i \leq n \), then the affine semigroup ring \( K[A_{r,b,c,r,s}] \) is the \( \tau \)th squarefree Veronese subring of the polynomial ring \( K[t_1, \ldots, t_d] \). In addition, algebras of Veronese type (i.e., \( n = d, p_i = q_i = i \) and \( c_i = 0 \) for all \( 1 \leq i \leq n \)) are studied in [DeHi] and [Stu].

Let \( K[X] \) denote the polynomial ring with the set of variables

\[
\left\{ x_{j_1 j_2 \cdots j_\tau} \ \bigg| \ 1 \leq j_1 \leq j_2 \leq \cdots \leq j_\tau \leq d, \ \prod_{k=1}^{\tau} t_{j_k} \in A_{r,b,c,r,s} \right\}.
\]

The toric ideal \( I_{A_{r,b,c,r,s}} \) is the kernel of the surjective homomorphism \( \pi : K[X] \rightarrow K[A_{r,b,c,r,s}] \) defined by \( \pi(x_{j_1 j_2 \cdots j_\tau}) = \prod_{k=1}^{\tau} t_{j_k} \). A monomial \( x_{\ell_1 \ell_2 \cdots \ell_\tau} x_{m_1 m_2 \cdots m_\tau} \cdots x_{n_1 n_2 \cdots n_\tau} \) is called sorted if we have

\[
\ell_1 \leq m_1 \leq \cdots \leq n_1 \leq \ell_2 \leq m_2 \leq \cdots \leq n_2 \leq \cdots \leq \ell_\tau \leq m_\tau \leq \cdots \leq n_\tau.
\]

Let \( \text{sort}(\cdot) \) denote the operator which takes any string over the alphabet \( \{1, 2, \ldots, d\} \) and sorts it into weakly increasing order.

The squarefree quadratic Gröbner basis of the toric ideal \( I_{A_{r,b,c,r,s}} \) is given as follows.

**Theorem 4.1** ([Stu, OH2, AHOT]). Work with the same notation as above. Let \( G \) be the set of all binomials

\[
x_{\ell_1 \ell_2 \cdots \ell_\tau} x_{m_1 m_2 \cdots m_\tau} - x_{n_1 n_2 \cdots n_{2\tau-1}} x_{n_2 n_4 \cdots n_{2\tau}}
\]

where

\[
\text{sort}(\ell_1 m_1 \ell_2 m_2 \cdots \ell_\tau m_\tau) = n_1 n_2 \cdots n_{2\tau}.
\]

Then there exists a monomial order on \( K[X] \) such that \( G \) is the reduced Gröbner basis of the toric ideal \( I_{A_{r,b,c,r,s}} \). The initial ideal is generated by squarefree quadratic (nonsorted) monomials.
By virtue of Theorem 3.6 if all of $A, B_1, \ldots, B_d$ are arising from Segre–Veronese configurations, then the toric ideal of the nested configuration $A(B_1, \ldots, B_d)$ possesses a quadratic Gröbner basis. Although the following Gröbner basis is different from that Theorem 3.6 guarantee, the proof is similar.

**Theorem 4.2.** If $K[A]$ is an algebra of Segre–Veronese type, and if the toric ideals $I_{B_1}, \ldots, I_{B_d}$ possess the reduced Gröbner basis $G_i$, then the toric ideal $I_{A(B_1, \ldots, B_d)}$ possesses a quadratic Gröbner basis $G$ consisting of all binomial of the form

$$x^\alpha - x^\beta = x_{m_{11}x_{m_{12}x_{m_{13}}}}^i x_{m_{14}x_{m_{15}x_{m_{16}}}}^{i_2} \cdots x_{m_{1m_2x_{m_{1m_3}}}}^{i_{2r-1}} x_{m_{12}x_{m_{14}x_{m_{15}}}}^{i_{2r}} - x_{m_{21}x_{m_{22}x_{m_{23}}}}^i x_{m_{24}x_{m_{25}x_{m_{26}}}}^{i_2} \cdots x_{m_{2m_2x_{m_{2m_3}}}}^{i_{2r-1}} x_{m_{22}x_{m_{24}x_{m_{25}}}}^{i_{2r}}$$

where $x^\alpha$ is the initial monomial and

$$k_1k_2 \cdots k_{2r} = \text{sort}(i_1i_2 \cdots i_{2r}), \quad (3)$$

$$\varphi_j(x^\alpha) \xrightarrow{G_j} \varphi_j(x^\beta) \text{ for each } 1 \leq j \leq d, \quad (4)$$

$$k_i = k_{i+1} \implies \ell_i \leq \ell_{i+1} \quad (1 \leq i \leq 2r - 1). \quad (5)$$

Moreover, if the initial ideal of $I_{B_i}$ is squarefree for all $i$, then the initial ideal of $I_{A(B_1, \ldots, B_d)}$ is squarefree.

**Proof.** By virtue of Lemma 3.5 $G$ is a subset of $I_{A(B_1, \ldots, B_d)}$. Since both the sorting operation and the reduction relation modulo Gröbner bases are Noetherian, there exists a monomial order such that the first monomial $x^\alpha$ is the initial monomial.

Suppose that there exists a binomial $0 \neq u - v \in I_{A(B_1, \ldots, B_d)}$ such that neither $u$ nor $v$ is divided by any initial monomial of $G$. Let

$$u = x_{m_{11}x_{m_{12}x_{m_{13}}}}^i x_{m_{14}x_{m_{15}x_{m_{16}}}}^{i_2} \cdots x_{m_{1m_2x_{m_{1m_3}}}}^{i_{2r-1}} x_{m_{12}x_{m_{14}x_{m_{15}}}}^{i_{2r}}, \quad v = x_{m_{21}x_{m_{22}x_{m_{23}}}}^i x_{m_{24}x_{m_{25}x_{m_{26}}}}^{i_2} \cdots x_{m_{2m_2x_{m_{2m_3}}}}^{i_{2r-1}} x_{m_{22}x_{m_{24}x_{m_{25}}}}^{i_{2r}}$$

By virtue of Lemma 3.5 we have

$$\text{sort}(i_1i_2 \cdots i_{2r}) = \text{sort}(k_1k_2 \cdots k_{2r}).$$

Thanks to the condition (3), we have

$$i_1i_2 \cdots i_{2r} = \text{sort}(i_1i_2 \cdots i_{2r}) = \text{sort}(k_1k_2 \cdots k_{2r}) = k_1k_2 \cdots k_{2r}.$$

Hence $i_q = k_q$ for all $1 \leq q \leq 2r$. By virtue of Lemma 3.5 we have $\varphi_j(u) - \varphi_j(v) \in I_{B_i}$ for each $1 \leq i \leq d$. Since each $G_i$ consists of quadratic binomials and thanks to the condition (4), $\varphi_j(u) = \varphi_j(v)$ for each $1 \leq i \leq d$. Hence

$$\prod_{i_q = j, \ 1 \leq q \leq 2r} z_q^{(i_q)} = \prod_{k_q = j, \ 1 \leq q \leq 2r} z_q^{(k_q)}.$$
Thanks to the condition (5) together with (6) above, \( j_q = \ell_q \) for all \( 1 \leq q \leq rp \). Thus we have \( u = v \) and this is a contradiction.

Suppose that the initial ideal of \( I_{B_k} \) is squarefree for all \( i \) and that \( x_{m_j^{(i_1)} m_j^{(i_2)} \cdots m_j^{(i_r)}}^2 \) belongs to the initial ideal of \( I_{A(B_1, \ldots, B_d)} \). Then

\[
g = x_{m_j^{(i_1)} m_j^{(i_2)} \cdots m_j^{(i_r)}}^2 - x_{m_j^{(k_{i_1})} m_j^{(k_{i_2})} \cdots m_j^{(k_{i_r}-1)}} x_{m_j^{(k_{i_2})} m_j^{(k_{i_4})} m_j^{(k_{i_r})}} \]

belongs to \( G \). Thanks to the condition (3), we have

\[
g = x_{m_j^{(i_1)} m_j^{(i_2)} \cdots m_j^{(i_r)}}^2 - x_{m_j^{(i_1)} m_j^{(i_2)} m_j^{(i_r)}} x_{m_j^{(i_2)} m_j^{(i_4)} \cdots m_j^{(i_r)}}.
\]

Since \( g \neq 0 \), there exists \( k \) such that \( j_k \neq \ell_{2k} \). Thanks to the condition (5), we have

\[
\varphi_k \left( x_{m_j^{(i_1)} m_j^{(i_2)} \cdots m_j^{(i_r)}}^2 \right) \neq \varphi_k \left( x_{m_j^{(i_1)} m_j^{(i_2)} m_j^{(i_r)}} x_{m_j^{(i_2)} m_j^{(i_4)} \cdots m_j^{(i_r)}} \right).
\]

Hence by the condition (4),

\[
\varphi_k \left( x_{m_j^{(i_1)} m_j^{(i_2)} \cdots m_j^{(i_r)}}^2 \right) = \varphi_k \left( x_{m_j^{(i_1)} m_j^{(i_2)} m_j^{(i_r)}} \right)^2
\]

belongs to the initial ideal of \( I_{B_k} \). Since the initial ideal of \( I_{B_k} \) is squarefree,

\[
\varphi_k \left( x_{m_j^{(i_1)} m_j^{(i_2)} \cdots m_j^{(i_r)}} \right)
\]

belongs to the initial ideal of \( I_{B_k} \). This contradicts that \( \varphi_k \) is defined with a standard expression. Thus the initial ideal of \( I_{A(B_1, \ldots, B_d)} \) is squarefree. \( \square \)

**Example 4.3.** Let \( A = \{ t_1^2 \} \) and \( B_1 = \{ u_1, u_2, u_3 \} \). Then

\[ A(B_1) = \{ u_1^2, u_1 u_2, u_1 u_3, u_2^2, u_2 u_3, u_3^2 \}. \]

The reduced Gröbner basis in Theorem 3.6 consists of the binomials

\[
x_{u_1 u_2}^2 - x_{u_1} x_{u_2}^2 \\
x_{u_1}^2 - x_{u_1} x_{u_2} \\
x_{u_2 u_3}^2 - x_{u_2} x_{u_3} \\
x_{u_1 u_2} x_{u_1 u_3} - x_{u_2} x_{u_2 u_3} \\
x_{u_1 u_3} x_{u_2 u_3} - x_{u_1 u_2} x_{u_2 u_3} \\
x_{u_1 u_3} x_{u_2 u_3} - x_{u_1 u_2} x_{u_2 u_3} \\
\]
and the reduced Gröbner basis in Theorem 4.2 consists of the binomials

\[ \begin{align*}
&x_{u_1}^2 x_{u_2}^2 = x_{u_1 u_2}^2 \\
&x_{u_1} x_{u_2} x_{u_3} = x_{u_1 u_3}^2 \\
&x_{u_2} x_{u_3} = x_{u_2 u_3}^2 \\
&x_{u_1} x_{u_2} x_{u_3} = x_{u_1 u_2} x_{u_1 u_3} \\
&x_{u_1} x_{u_2} x_{u_3} = x_{u_1 u_2} x_{u_2 u_3} \\
&x_{u_1} x_{u_2} x_{u_3} = x_{u_1} x_{u_2} x_{u_2 u_3}
\end{align*} \]

where the initial monomial of each binomial is the first monomial.

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