The algebro-geometric solutions for the modified Camassa-Holm hierarchy

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Abstract

This paper is dedicated to provide theta function representation of algebro-geometric solutions and related crucial quantities for the modified Camassa-Holm (MCH) hierarchy through studying an algebro-geometric initial value problem. Our main tools include the polynomial recursive formalism to derive the MCH hierarchy, the hyperelliptic curve with finite number of genus, the Baker-Akhiezer functions, the meromorphic function, the Dubrovin-type equations for auxiliary divisors, and the associated trace formulas. With the help of these tools, the explicit representations of the Baker-Akhiezer functions, the meromorphic function, and the algebro-geometric solutions are obtained for the entire MCH hierarchy.

1 Introduction

Algebro-geometric solution, an important feature of integrable system, is a kind of explicit solutions closely related to the inverse spectral theory [1, 2, 6-8]. As a degenerated case of the algebro-geometric solution, the multi-soliton solution and periodic solution in elliptic function type may be obtained [1, 5, 21]. A systematic approach, proposed by Gesztesy and Holden to construct algebro-geometric solutions for integrable equations, has been extended to the whole (1+1) dimensional integrable hierarchy, such as

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the AKNS hierarchy, the Camassa-Holm (CH) hierarchy etc. [9]-[12]. Recently, we investigated the Gerdjikov-Ivanov hierarchy and the Degasperis-Procesi hierarchy and obtained their algebro-geometric solutions [15],[16].

The CH equation

\[ u_t - u_{xxt} + 2\gamma u_x + 3uu_x - 2u_xu_xx - uu_{xxx} = 0 \]  (1.1)

where \( \gamma \) is a real constant, was derived from the two-dimensional Euler equations in a search for integrable shallow water equations, with the \( u(x, t) \) is the fluid velocity in the \( x \) direction and \( \gamma \) is a constant related to the critical shallow water wave speed [2]. The CH equation describes the propagation of two-dimensional shallow water waves over a flat bed, also the propagation of axially symmetric waves in hyperelastic rods [20],[22]. The CH and the Degasperis-Procesi (DP) equations are only two integrable members with \( b = 2 \) and \( b = 3 \) from the following family

\[ u_t - uu_{xx} + \gamma bu_x + (b + 1)uu_x = bu_xu_xx + uu_{xxx}, \]  (1.2)

where \( b \) is a constant.

Due to its many remarkable integrable properties, the CH equation has extensively been studied in the last twenty years. The bi-Hamiltonian structure of the CH equation was implied in the work of Fuchssteiner and Fokas [14]. Complete integrability and infinity of conservation laws have been studied in [2],[3],[28]. The inverse scattering transform was developed by Constantin, McKean, Gerdjikov and Ivanov [23],[24]. Other progress in studying the CH equation includes the existence of peaked solitons and geometry of multi-peakons [2],[29], geometric formulations [26], and waves breaking [25]. The classic papers on the algebro-geometric solutions of the CH equation, or more appropriately, one is due to Qiao, the other is Gesztesy and Holden. Qiao obtained the algebro-geometric solution on a symplectic submainfold [19]. Gesztesy and Holden derived the algebro-geometric solutions for the whole CH hierarchy by using the polynomial recursion method [10]-[12].

This paper is concerned with the following equation, called the modified Camassa-Holm (MCH) equation

\[ u_t - u_{xxt} + 3u^2u_x - u^3_x = (4u - 2u_{xx})u_xu_xx + (u^2 - u_x^2)u_{xxx}, \]  (1.3)

where \( u(x, t) \) is the function of spatial variable \( x \) and time variable \( t \). For our convenience, let us rewrite (1.3) as

\[ q_t + q_x(u^2 - u_x^2) + 2q^2u_x = 0, \]  (1.4)
The MCH equation (1.3) was derived as an integrable system by Fuchssteiner, and Olver and Rosenau by applying the tri-Hamiltonian method to the representation of the modified KdV equation [13], [31]. Later, Qiao recovered the MCH equation [13] from the two-dimensional Euler equations by using the approximation procedure, provided the Lax pair and bi-Hamiltonian structure for the MCH equation, and first time proposed the W/M-shape solitons [17]. Recently, Gui, Liu, Olver and Qu showed the wave-breaking and peaked traveling-wave solutions for the MCH equation [30]. However, within the knowledge of the authors, the algebro-geometric solutions of the entire MCH hierarchy are not studied yet.

The main task of this paper focuses on the algebro-geometric solutions of the whole MCH hierarchy in which (1.3) is just the second member. The outline of the present paper is as follows.

In section 2, based on the polynomial recursion formalism, we derive the MCH hierarchy, the associated sequences, and Lax pairs. A hyperelliptic curve $K_n$ with arithmetic genus $n$ is introduced with the help of the characteristic polynomial of Lax matrix $V_n$ for the stationary MCH hierarchy.

In Section 3, we study a meromorphic function $\phi$ such that $\phi$ satisfies a nonlinear second-order differential equation. Then we study the properties of the Baker-Akhiezer function $\psi$, and furthermore the stationary MCH equations are decomposed into a system of Dubrovin-type equations. The stationary trace formulas are obtained for the MCH hierarchy.

In Section 4, we present the first set of our results, the explicit theta function representations of Baker-Akhiezer function, the meromorphic function and the potentials $u$ for the entire stationary MCH hierarchy. Furthermore, we study the initial value problem on an algebro-geometric curve for the stationary MCH hierarchy.

In Sections 5 and 6, we extend the analysis in Sections 3 and 4 to the time-dependent case. Each equation in the MCH hierarchy is permitted to evolve in terms of an independent time parameter $t_r$. As an initial data we use a stationary solution of the $n$th equation and then construct a time-dependent solution of the $r$th equation in the MCH hierarchy. The Baker-Akhiezer function, the meromorphic function, the analogs of the Dubrovin-type equations, the trace formulas, and the theta function representation in Section 4 are all extended to the time-dependent case.

\[ q = u - u_{xx}. \] (1.5)
2 The MCH hierarchy

In this section, let us derive the MCH hierarchy and the corresponding sequence of zero-curvature pairs by using a polynomial recursion formalism. Moreover, we introduce the hyper-elliptic curve connecting to the stationary MCH hierarchy.

Throughout this section, let us make the following hypothesis.

Hypothesis 2.1 In the stationary case, let us assume

\[ u \in C^\infty(\mathbb{C}), \quad \partial^k_x u \in L^\infty(\mathbb{C}), \quad k \in \mathbb{N}_0. \]  

(2.1)

In the time-dependent case, let us assume

\[ u(\cdot, t) \in C^\infty(\mathbb{C}), \quad \partial^k_x u(\cdot, t) \in L^\infty(\mathbb{C}), \quad k \in \mathbb{N}_0, \quad t \in \mathbb{C}, \]

\[ u(x, \cdot), u_{xx}(x, \cdot) \in C^1(\mathbb{C}). \]  

(2.2)

Let us begin with the polynomial recursion formalism. Define \( \{ f_{2l} \}_{l \in \mathbb{N}_0} \), \( \{ g_{2l} \}_{l \in \mathbb{N}_0} \), and \( \{ h_{2l} \}_{l \in \mathbb{N}_0} \) through the following recursive relations

\[
\begin{align*}
f_{2l+1} &= 0, \quad g_{2l+1} = 0, \quad h_{2l+1} = 0, \quad l \in \mathbb{N}_0, \\
g_0 &= -1, \\
g_{2l} &= -\frac{1}{2} \partial^{-1} (u - u_{xx}) \delta_{2l,x}, \quad l \in \mathbb{N}, \\
f_{2l,x} + f_{2l} &= (u - u_{xx}) g_{2l}, \quad l \in \mathbb{N}_0, \\
h_{2l} &= -2(u - u_{xx})^{-1} g_{2l+2,x} - f_{2l}, \quad l \in \mathbb{N}_0.
\end{align*}
\]

(2.3)

where \( \delta_{2l} \) are determined by the following formulas (for details, see [18])

\[
\begin{align*}
\delta_2 &= 2u, \\
\delta_{2l+2} &= G(\delta_{2l}), \quad l \in \mathbb{N}, \\
G &= -e^{-x} \partial^{-1} e^{2x} \partial^{-1} e^{-x} (u - u_{xx}) \partial^{-1} (u - u_{xx}) \partial.
\end{align*}
\]

(2.4)

Apparently, the first few can be computed as follows:

\[
\begin{align*}
f_0 &= u_x - u, \\
f_2 &= -\frac{1}{2} (u - u_{xx}) (u^2 - u_x^2) + (u - u_{xx}) c_1, \\
g_0 &= -1, \\
g_2 &= -\frac{1}{2} (u^2 - u_x^2) + c_1, \\
h_0 &= u + u_x, \\
h_2 &= \frac{1}{2} (u - u_{xx}) (u^2 - u_x^2) - (u - u_{xx}) c_1, \text{ etc.}
\end{align*}
\]

(2.5)
where \( \{ c_l \}_{l \in \mathbb{N}_0} \subset \mathbb{C} \) are integration constants.

Next, we introduce the corresponding homogeneous coefficients \( \hat{f}_{2l}, \hat{g}_{2l}, \) and \( \hat{h}_{2l} \), defined through taking \( c_k = 0 \) for \( k = 1, \cdots, l, l \in \mathbb{N}_0 \),

\[
\hat{f}_0 = u_x - u, \quad \hat{f}_2 = -\frac{1}{2}(u - u_{xx})(u^2 - u_x^2), \quad \hat{f}_{2l} = f_{2l} \big|_{c_k=0, k=1,\ldots,l},
\]
\[
\hat{g}_0 = -1, \quad \hat{g}_2 = -\frac{1}{2}(u^2 - u_x^2), \quad \hat{g}_{2l} = g_{2l} \big|_{c_k=0, k=1,\ldots,l},
\]
\[
\hat{h}_0 = u_x + u, \quad \hat{h}_2 = \frac{1}{2}(u - u_{xx})(u^2 - u_x^2), \quad \hat{h}_{2l} = h_{2l} \big|_{c_k=0, k=1,\ldots,l},
\]

(2.6)

It is easy to see

\[
f_{2l} = \sum_{k=0}^{l} c_{l-k}\hat{f}_{2k}, \quad g_{2l} = \sum_{k=0}^{l} c_{l-k}\hat{g}_{2k}, \quad h_{2l} = \sum_{k=0}^{l} c_{l-k}\hat{h}_{2k}, \quad l \in \mathbb{N}_0, \quad (2.7)
\]

where

\[
c_0 = 1.
\]

(2.8)

Let us now consider the following \( 2 \times 2 \) matrix iso-spectral problem

\[
\psi_x = U(u, \lambda)\psi = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \lambda^{-1}(u - u_{xx}) \\ \frac{1}{2} \lambda^{-1}(u - u_{xx}) & \frac{1}{2} \end{pmatrix} \psi
\]

(2.9)

and an auxiliary problem

\[
\psi_{tn} = V_n(\lambda)\psi,
\]

(2.10)

where \( V_n(\lambda) \) is defined by

\[
V_n(\lambda) = \begin{pmatrix} -G_n & \lambda^{-1}F_n \\ \lambda^{-1}H_n & G_n \end{pmatrix}, \quad \lambda \in \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N}_0.
\]

(2.11)

Assume that \( F_n, G_n, \) and \( H_n \) are polynomials of degree \( 2n \) with \( C^\infty \) coefficients with respect to \( x \). Let us consider the stationary zero-curvature equation

\[
- \lambda^2 G_{n,x} = -V_{n,x} + [U, V_n] = 0,
\]

that is

\[
F_{n,x} = -F_n + (u - u_{xx})G_n, \quad (2.13)
\]
\[
H_{n,x} = -H_n + (u - u_{xx})G_n, \quad (2.14)
\]
\[
\lambda^2 G_{n,x} = -\frac{1}{2}(u - u_{xx})H_n - \frac{1}{2}(u - u_{xx})F_n. \quad (2.15)
\]
From (2.13)-(2.15), a direct calculation shows
\[
\frac{d}{dx} \det(V_n(\lambda, x)) = -\frac{1}{\lambda^2} \frac{d}{dx} \left( \lambda^2 G_n(\lambda, x)^2 + F_n(\lambda, x) H_n(\lambda, x) \right) = 0, \quad (2.16)
\]
and therefore \( \lambda^2 G_n^2 + F_n H_n \) is \( x \)-independent to imply
\[
\lambda^2 G_n^2 + F_n H_n = R_{4n+2} \quad (2.17)
\]
where the integration constant \( R_{4n+2} \) is a monic polynomial of degree \( 4n+2 \) with respect to \( \lambda \). Let \( \{E_m^2\}_{m=1,\ldots,2n+1} \) denote its zeros, then
\[
R_{4n+2}(\lambda) = \prod_{m=1}^{2n+1} (\lambda^2 - E_m^2), \quad \{E_m^2\}_{m=1,\ldots,2n+1} \in \mathbb{C}. \quad (2.18)
\]
In order to derive the corresponding hyper-elliptic curve, we compute the characteristic polynomial \( \det(yI - \lambda V_n) \) of the Lax matrix \( \lambda V_n \),
\[
\det(yI - \lambda V_n) = y^2 - \lambda^2 G_n(\lambda)^2 - F_n(\lambda) H_n(\lambda) = y^2 - R_{4n+2}(\lambda) = 0. \quad (2.19)
\]
Equation (2.19) naturally leads to the hyper-elliptic curve \( \mathcal{K}_n \), where
\[
\mathcal{K}_n : F_n(\lambda, y) = y^2 - R_{4n+2}(\lambda) = 0,
R_{4n+2}(\lambda) = \prod_{m=1}^{2n+1} (\lambda^2 - E_m^2), \quad \{E_m^2\}_{m=1,\ldots,2n+1} \in \mathbb{C}. \quad (2.20)
\]
Actually, it is more convenient to introduce the notations \( z = \lambda^2, \bar{E}_m = E_m^2 \), so that \( \mathcal{K}_n \) becomes the hyper-elliptic curve of genus \( n \in \mathbb{N}_0 \) (possibly with a singular affine part), namely,
\[
\mathcal{K}_n : F_n(z, y) = y^2 - R_{2n+1}(z) = 0,
R_{2n+1}(z) = \prod_{m=1}^{2n+1} (z - \bar{E}_m), \quad \{\bar{E}_m\}_{m=1,\ldots,2n+1} \in \mathbb{C}. \quad (2.21)
\]
The stationary zero-curvature equation (2.12) implies polynomial recursion relations (2.3). Let us introduce the following polynomials \( F_n(\lambda), G_n(\lambda) \) and \( H_n(\lambda) \) with respect to the spectral parameter \( \lambda \),
\[
F_n(\lambda) = \sum_{l=0}^{n} f_{2l} \lambda^{2(n-l)}, \quad (2.22)
\]
\[ G_n(\lambda) = \sum_{l=0}^{n} g_{2l} \lambda^{2(n-l)}, \]  
(2.23)  
\[ H_n(\lambda) = \sum_{l=0}^{n} h_{2l} \lambda^{2(n-l)}. \]  
(2.24)

Inserting (2.22)-(2.24) into (2.13)-(2.15) yields the recursion relations (2.3) for \( f_{2l} \) and \( g_{2l}, l = 0, \ldots, n. \) By using (2.15), we obtain the recursion formula for \( h_{2l}, l = 0, \ldots, n-1 \) in (2.3) and

\[ h_{2n} = -f_{2n}. \]  
(2.25)

Moreover, from (2.14), we have

\[ h_{2n,x} - h_{2n} - (u - u_{xx})g_{2n} = 0, \quad n \in \mathbb{N}_0. \]  
(2.26)

Hence, inserting the relations (2.25) and

\[ f_{2n,x} + f_{2n} - (u - u_{xx})g_{2n} = 0 \]  
(2.27)

into (2.26), we obtain

\[ s - \text{MCH}_n(u) = -2f_{2n,x} = 0, \quad n \in \mathbb{N}_0. \]  
(2.28)

The stationary MCH hierarchy is defined by (2.28). The first few equations are

\[ s - \text{MCH}_0(u) = 2u_x - 2u_{xx} = 0, \]  
\[ s - \text{MCH}_1(u) = (u_x - u_{xxx})(u^2 - u_x^2) + 2(u - u_{xx})^2u_x + (u_x - u_{xxx})c_1 = 0, \]  
etc.

(2.29)

By definition, the set of solutions of (2.28) represents a class of algebro-geometric MCH solutions with \( n \) ranging in \( \mathbb{N}_0 \) and \( c_l \) in \( \mathbb{C}, \ l \in \mathbb{N}. \) We call the stationary algebro-geometric MCH solutions \( u \) as MCH potentials at times.

**Remark 2.2** Here we emphasize that if \( u \) satisfies one of the stationary MCH equations in (2.28), then it must satisfy infinitely many such equations with order higher than \( n \) for certain choices of integration constants \( c_l. \) This is a common characteristic of the general integrable soliton equations such as the KdV, AKNS and CH hierarchies \([12]\).
Next, we introduce the following homogeneous polynomials $\hat{F}_l, \hat{G}_l, \hat{H}_l$ defined by

$$\hat{F}_l(\lambda) = F_l(\lambda)|_{c_k=0}, \quad k=1,\ldots,l = \sum_{k=0}^{l} \hat{f}_{2k} \lambda^{2(l-k)}, \quad l = 0,\ldots,n,$$  
(2.30)

$$\hat{G}_l(\lambda) = G_l(\lambda)|_{c_k=0}, \quad k=1,\ldots,l = \sum_{k=0}^{l} \hat{g}_{2k} \lambda^{2(l-k)}, \quad l = 0,\ldots,n,$$  
(2.31)

$$\hat{H}_l(\lambda) = H_l(\lambda)|_{c_k=0}, \quad k=1,\ldots,l = \sum_{k=0}^{l} \hat{h}_{2k} \lambda^{2(l-k)}, \quad l = 0,\ldots,n-1,$$  
(2.32)

$$\hat{H}_n(\lambda) = -\hat{f}_{2n} + \sum_{k=0}^{n-1} \hat{h}_{2k} \lambda^{2(n-k)}.$$  
(2.33)

Then, the corresponding homogeneous formalism of (2.28) are given by

$$s - \text{MCH}_n(u) = s - \text{MCH}_n(u)|_{c_l=0}, \quad l=1,\ldots,n = 0, \quad n \in \mathbb{N}_0.$$  
(2.34)

Let us conclude this section with the time-dependent MCH hierarchy. The "time-dependent" means that $u$ is a function of both space and time. Let us introduce a deformation parameter $t_n \in \mathbb{C}$ in $u$, replacing $u(x)$ by $u(x,t_n)$, for each equation in the hierarchy. In addition, the definitions (2.9), (2.11) and (2.22)-(2.24) of $U_n, V_n$ and $F_n, G_n$ and $H_n$ are still available. Then, the compatibility condition yields the zero-curvature equation

$$U_{t_n} - V_{n,x} + [U, V_n] = 0, \quad n \in \mathbb{N}_0,$$  
(2.35)

namely,

$$\frac{1}{2}(u_t - u_{xxx}) - F_{n,x} - F_n + (u - u_{xx})G_n = 0, \quad n \in \mathbb{N}_0,$$  
(2.36)

$$-\frac{1}{2}(u_t - u_{xxx}) - H_{n,x} + H_n + (u - u_{xx})G_n = 0, \quad n \in \mathbb{N}_0,$$  
(2.37)

$$\lambda^2 G_{n,x} = -\frac{1}{2}(u - u_{xx})H_n - \frac{1}{2}(u - u_{xx})F_n.$$  
(2.38)

Inserting the polynomial expressions for $F_n, G_n$ and $H_n$ into (2.36) and (2.38), then we have the relations (2.3) for $f_{2l}$ and $g_{2l}, \quad l = 0,\ldots,n$. By
using (2.38), we obtain the recursion formula (2.3) for \( h_{2l} \), \( l = 0, \ldots, n - 1 \)
and
\[
h_{2n} = -f_{2n}.
\] (2.39)
Moreover, from (2.36) and (2.37), we have
\[
\frac{1}{2}(u_{tn} - u_{xxtn}) - f_{2n,x} - f_{2n} + (u - u_{xx})g_{2n} = 0,
\] (2.40)
\[
-\frac{1}{2}(u_{tn} - u_{xxtn}) - h_{2n,x} + h_{2n} + (u - u_{xx})g_{2n} = 0.
\] (2.41)
Hence, by (2.39) and (2.40), (2.41) can be rewritten as
\[
\text{MCH}_n(u) = ut_n - u_{xxtn} - 2f_{2n,x} = 0, \quad (x, t_n) \in \mathbb{C}^2, \quad n \in \mathbb{N}_0.
\] (2.42)
Varying \( n \in \mathbb{N}_0 \) in (2.42) defines the time-dependent MCH hierarchy. The first few equations are
\[
\text{MCH}_0(u) = ut_0 - u_{xxt0} + 2u_x - 2u_{xx} = 0,
\]
\[
\text{MCH}_1(u) = ut_1 - u_{xxt1} + (u_x - u_{xxx})(u^2 - u_x^2) + 2(u - u_{xx})^2u_x + (u_x - u_{xxx})c_1 = 0,
\] (2.43)
etc.
The second equation \( \text{MCH}_1(u) = 0 \) (with \( c_1 = 0 \)) in the hierarchy represents the Modified Camassa-Holm (MCH) equation as discussed in section 1. Similarly, one can introduce the corresponding homogeneous MCH hierarchy by
\[
\text{MCH}_n(u) = \text{MCH}_n(u)_{|c_l=0, \; l=1,\ldots,n} = 0, \quad n \in \mathbb{N}_0.
\] (2.44)
The MCH hierarchy can also be defined in the form of \( h_{2n} \) for the relation \( h_{2n} = -f_{2n} \). The integration constants \( c_l = 0 \) \( l = 1, \ldots, n \) are taken to derive the Modified Camassa-Holm equation \( \text{MCH}_1(u) = 0 \).
In fact, since the Lenard operator recursion formalism is almost universally adopted in the contemporary literature on the integrable soliton equations, it might be worthwhile to adopt the alternative approach using the polynomial recursion relations.

3 The stationary MCH formalism

In this section, we focus our attention on the stationary case. By using the polynomial recursion formalism described in section 2, we define a fundamental meromorphic function \( \phi(P, x) \) on a hyper-elliptic curve \( K_n \). Moreover, we study the properties of the Baker-Akhiezer function \( \psi(P, x, x_0) \), Dubrovin-type equations, and trace formulas.
We emphasize that the analysis about the stationary case described in section 2 also holds here for the present context.

Let 
\[
z = \lambda^2, \quad \tilde{E}_m = E_m^2, \quad (3.1)
\]
then the hyper-elliptic curve \( K_n \) is given by \((2.21)\). \( K_n \) is compactified by joining a point at infinity, \( P_\infty \), but for our convenience, the compactification is still denoted by \( K_n \). Point \( P \) on 
\[
K_n \setminus \{ P_\infty \},
\]
is referred to a pair type \( P = (z, y(P)) \), where \( y(\cdot) \) is the meromorphic function on \( K_n \) satisfying 
\[
\mathcal{F}_n(z, y(P)) = 0.
\]
The complex structure on \( K_n \) is defined in the usual way by introducing local coordinates 
\[
\zeta_{Q_0} : P \to (z - z_0)
\]
near points 
\[
Q_0 = (z_0, y(Q_0)) \in K_n \setminus P_0, \quad P_0 = (0, y(P_0)),
\]
which are neither branch nor singular points of \( K_n \). Near \( P_0 \), the local coordinates are 
\[
\zeta_{P_0} : P \to z^{1/2};
\]
near the branch point \( P_\infty \in K_n \), the local coordinates are 
\[
\zeta_{P_\infty} : P \to z^{-1},
\]
and also the similar case at branch and singular points of \( K_n \). Thus, \( K_n \) becomes a two-sheeted hyper-elliptic Riemann surface \([27]\) with genus \( n \in \mathbb{N}_0 \) (possibly with a singular affine part) in a standard manner.

We also notice that fixing the zeros \( \tilde{E}_1, \ldots, \tilde{E}_{2n+1} \) of \( R_{2n+1} \) discussed in \((2.21)\) leads to the curve \( K_n \) fixed. Then the integration constants \( c_1, \ldots, c_n \) in \( f_{2n} \) are uniquely determined, which are the symmetric functions of \( \tilde{E}_1, \ldots, \tilde{E}_{2n+1} \).

The holomorphic map \( * \), changing sheets, is defined by 
\[
*: \begin{cases} 
K_n \to K_n, \\
P = (z, y_j(z)) \to P^* = (z, y_{j+1(\text{mod} 2)}(z)), \quad j = 0, 1, \\
P^{**} := (P^*)^*, \quad \text{etc.,}
\end{cases}
\quad (3.2)
\]
where \( y_j(z) \), \( j = 0, 1 \) denote the two branches of \( y(P) \) satisfying \( F_n(z, y) = 0 \). Finally, positive divisors on \( K_n \) of degree \( n \) are denoted by

\[
\mathcal{D}_{P_1, \ldots, P_n} : \begin{cases} \\
K_n \to \mathbb{N}_0, \\
P \to \mathcal{D}_{P_1, \ldots, P_n} = \begin{cases} \\
k & \text{if } P \text{ occurs } k \text{ times in } \{P_1, \ldots, P_n\}, \\
0 & \text{if } P \notin \{P_1, \ldots, P_n\}. 
\end{cases}
\end{cases}
\]  

(3.3)

For the notational simplicity, assume \( n \in \mathbb{N} \). The case \( n = 0 \) is treated in Example 4.10.

Next, we define the stationary Baker-Akhiezer function \( \psi(P, x, x_0) \) on \( K_n \setminus \{P_{\infty}, P_0\} \) as follows

\[
\psi(P, x, x_0) = \begin{pmatrix} \psi_1(P, x, x_0) \\ \psi_2(P, x, x_0) \end{pmatrix},
\]

\[
\psi_x(P, x, x_0) = U(u(x), z(P))\psi(P, x, x_0),
\]

\[
z^{1/2} V_n(u(x), z(P)) \psi(P, x, x_0) = y(P) \psi(P, x, x_0),
\]

\[
\psi_1(P, x_0, x_0) = 1; \quad P = (z, y) \in K_n \setminus \{P_{\infty}, P_0\}, \quad x \in \mathbb{C}.
\]

Closely related to \( \psi(P, x, x_0) \) is the following meromorphic function \( \phi(P, x) \) on \( K_n \) defined by

\[
\phi(P, x) = \frac{2z}{q} \left( \frac{\psi_1(P, x, x_0)}{\psi_1(P, x, x_0)} + \frac{1}{2} \right), \quad P \in K_n, \quad x \in \mathbb{C} \tag{3.5}
\]

with

\[
\psi_1(P, x, x_0) = \exp \left( z^{-1} \int_{x_0}^x \frac{1}{2} q(x') \phi(P, x') dx' - \frac{1}{2} (x - x_0) \right),
\]

\[
P \in K_n \setminus \{P_{\infty}, P_0\},
\]

(3.6)

where \( q = u - u_{xx} \).

Then, based on (3.1) and (3.5), a direct calculation shows that

\[
\phi(P, x) = z^{1/2} y + z^{1/2} G_n(z, x) \overline{F_n(z, x)} = \frac{z^{1/2} H_n(z, x)}{y - z^{1/2} G_n(z, x)} \tag{3.7}
\]

and

\[
\psi_2(P, x, x_0) = \psi_1(P, x_0) \phi(P, x) / z^{1/2}. \tag{3.8}
\]
By inspection of the expression of $F_n$ and $H_n$ in (2.22) and (2.24), we notice that $F_n$ and $H_n$ are even functions with respect to $\lambda$. Thus, if $\lambda$ is the root of $F_n$ or $H_n$, then $-\lambda$ is also their root. In this way, we can write $F_n$ and $H_n$ as the following finite products

$$F_n(\lambda) = f_0 \prod_{j=1}^{n} (\lambda^2 - \bar{\mu}_j^2),$$

$$H_n(\lambda) = h_0 \prod_{j=1}^{n} (\lambda^2 - \bar{\nu}_j^2).$$

(3.9)

For our convenience, let

$$z = \lambda^2, \quad \mu_j = \bar{\mu}_j^2, \quad \nu_j = \bar{\nu}_j^2,$$

(3.10)

then $F_n$ and $H_n$ can be rewritten as the following formalism,

$$F_n(z) = f_0 \prod_{j=1}^{n} (z - \mu_j) = (u_x - u) \prod_{j=1}^{n} (z - \mu_j),$$

(3.11)

$$H_n(z) = h_0 \prod_{j=1}^{n} (z - \nu_j) = (u_x + u) \prod_{j=1}^{n} (z - \nu_j).$$

(3.12)

Moreover, defining

$$\hat{\mu}_j(x) = \left( \mu_j(x), -\sqrt{\mu_j(x)} G_n(\mu_j(x), x) \right) \in \mathbb{K}_n, \quad j = 1, \ldots, n, \quad x \in \mathbb{C},$$

(3.13)

and

$$\hat{\nu}_j(x) = \left( \nu_j(x), \sqrt{\nu_j(x)} G_n(\nu_j(x), x) \right) \in \mathbb{K}_n, \quad j = 1, \ldots, n, \quad x \in \mathbb{C},$$

(3.14)

and taking $z = 0$ in (2.17) yields

$$f_{2n} h_{2n} = - f_{2n}^2 = - \prod_{j=1}^{2n+1} \tilde{E}_m.$$

(3.15)

So, we can choose

$$P_0 = (0, f_{2n}) = (0, \prod_{j=1}^{2n+1} \tilde{E}_m^{1/2}).$$

(3.16)
Due to (2.1), \( u \) is smooth and bounded, and therefore \( F_n(z, x) \) and \( H_n(z, x) \) share the same property. Thus, we have

\[
\mu_j, \nu_j \in C(\mathbb{R}), \quad j = 1, \ldots, n, \tag{3.17}
\]

where \( \mu_j, \nu_j \) may have appropriate multiplicities.

The branch of \( y(\cdot) \) near \( P_{\infty} \) is fixed according to

\[
\lim_{|z(P)| \to \infty} \frac{y(P)}{z^{1/2}G_n(z, x)} = 1. \tag{3.18}
\]

Also by (3.7), the divisor \((\phi(P, x))\) of \( \phi(P, x) \) is given by

\[
(\phi(P, x)) = \mathcal{D}_{P_0, \hat{\nu}_1(x), \ldots, \hat{\nu}_n(x)}(P) - \mathcal{D}_{P_{\infty}, \hat{\mu}_1(x), \ldots, \hat{\mu}_n(x)}(P). \tag{3.19}
\]

That means, \( P_0, \hat{\nu}_1(x), \ldots, \hat{\nu}_n(x) \) are the \( n+1 \) zeros of \( \phi(P, x) \) and \( P_{\infty}, \hat{\mu}_1(x), \ldots, \hat{\mu}_n(x) \) are its \( n+1 \) poles. These zeros and poles can be abbreviated in the following form

\[
\hat{\mu} = \{\hat{\mu}_1, \ldots, \hat{\mu}_n\}, \quad \hat{\nu} = \{\hat{\nu}_1, \ldots, \hat{\nu}_n\} \in \text{Sym}^n(K_n). \tag{3.20}
\]

Let us recall the holomorphic map (3.2),

\[
* : \begin{cases} \mathcal{K}_n \to \mathcal{K}_n, \\ P = (z, y_j(z)) \to P^* = (z, y_{j+1(\text{mod } 2)})(z), & j = 0, 1, \\ P^{**} := (P^*)^*, & \text{etc.}, \end{cases} \tag{3.21}
\]

where \( y_j(z), j = 0, 1 \) satisfy \( \mathcal{F}_n(z, y) = 0 \), namely,

\[
(y - y_0(z))(y - y_1(z)) = y^2 - R_{2n+1}(z) = 0. \tag{3.22}
\]

From (3.22), we can easily get

\[
\begin{aligned}
y_0 + y_1 &= 0, \\
y_0y_1 &= -R_{2n+1}(z), \\
y_0^2 + y_1^2 &= 2R_{2n+1}(z).
\end{aligned} \tag{3.23}
\]

Further properties of \( \phi(P, x) \) are summarized as follows.

**Lemma 3.1** Under the assumption (2.1), let \( P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty}, P_0\} \) and \( (z, x) \in \mathbb{C}^2 \), and \( u \) satisfy the \( n \)th stationary MCH equation (2.28). Then

\[
\phi_x(P) + \frac{1}{2}(u - u_{xx})z^{-1}\phi(P) - \phi(P) = -\frac{1}{2}(u - u_{xx}), \tag{3.24}
\]

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\[ \phi(P)\phi(P^*) = -\frac{zH_n(z)}{F_n(z)}, \quad (3.25) \]
\[ \phi(P) + \phi(P^*) = 2\frac{zG_n(z)}{F_n(z)}, \quad (3.26) \]
\[ \phi(P) - \phi(P^*) = z^{1/2} \frac{2y}{F_n(z)}. \quad (3.27) \]

**Proof.** A direct calculation shows that (3.24) holds. Let us now prove (3.25)-(3.27). Without loss of generality, let \( y_0(P) = y(P) \). From (3.7), (2.17) and (3.23), we arrive at

\[
\phi(P)\phi(P^*) = \sqrt{z}y_0 + \sqrt{z}G_n F_n = \sqrt{z}y_1 + \sqrt{z}G_n F_n \\
= z \frac{y_0y_1 + (y_0 + y_1)\sqrt{z}G_n + zG_n^2}{F_n^2} \\
= z \left( -R_{2n+1} + zG_n^2 \right) \\
= z \frac{-F_nH_n}{F_n^2} \\
= -\frac{zH_n}{F_n}, \quad (3.28)
\]

\[
\phi(P) + \phi(P^*) = \sqrt{z}y_0 + \sqrt{z}G_n F_n + \sqrt{z}y_1 + \sqrt{z}G_n F_n \\
= \sqrt{z}(y_0 + y_1) + 2\sqrt{z}G_n F_n \\
= 2zG_n F_n, \quad (3.29)
\]

\[
\phi(P) - \phi(P^*) = \sqrt{z}y_0 + \sqrt{z}G_n F_n - \sqrt{z}y_1 + \sqrt{z}G_n F_n \\
= \sqrt{z}(y_0 - y_1) \\
= \sqrt{z} \frac{2y_0}{F_n} = \sqrt{z} \frac{2y}{F_n}. \quad (3.30)
\]

Let us discuss properties of \( \psi(P, x, x_0) \) below.
Lemma 3.2 Under the assumption \((\ref{2.7})\), let \(P = (z, y) \in K_n \setminus \{P_\infty, P_0\}\), \((x, x_0) \in \mathbb{C}^2\), \(u\) satisfy the \(n\)th stationary MCH equation \((\ref{2.28})\). Then

\[
\psi_1(P, x, x_0) = \left(\frac{F_n(z, x)}{F_n(z, x_0)}\right)^{1/2} \exp\left(\frac{y}{2\sqrt{z}} \int_{x_0}^{x} q(x') F_n(z, x')^{-1} dx'\right), \tag{3.31}
\]

\[
\psi_1(P, x, x_0)\psi_1(P^*, x, x_0) = \frac{F_n(z, x)}{F_n(z, x_0)}, \tag{3.32}
\]

\[
\psi_2(P, x, x_0)\psi_2(P^*, x, x_0) = -\frac{H_n(z, x)}{F_n(z, x_0)}, \tag{3.33}
\]

\[
\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) + \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = 2\sqrt{z} G_n(z, x), \tag{3.34}
\]

\[
\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) - \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = -\frac{2y}{F_n(z, x_0)}. \tag{3.35}
\]

Proof. Equation \((\ref{3.31})\) can be proven through the following procedure. Using \((\ref{2.13})\), the expression of \(\psi_1\), \((\ref{3.3})\) and \((\ref{3.7})\), we obtain

\[
\psi_1(P, x, x_0) = \exp\left(z^{-1} \int_{x_0}^{x} \frac{1}{2} q(x') \sqrt{z} \left(y + \sqrt{z} G_n\right) dx' - \frac{1}{2} (x - x_0)\right)
\]

\[
= \exp\left(\frac{1}{\sqrt{z}} \int_{x_0}^{x} \frac{1}{2} q(x') \frac{y + \sqrt{z} G_n}{F_n} dx' - \frac{1}{2} (x - x_0)\right)
\]

\[
= \exp\left(\frac{1}{\sqrt{z}} \int_{x_0}^{x} \left[\frac{1}{2} q(x') \frac{y}{F_n} + \frac{1}{2} \frac{F_n x'}{F_n}\right] dx'\right), \tag{3.36}
\]

which implies \((\ref{3.31})\). Moreover, \((\ref{3.6})\) together with \((\ref{2.28})\)-(\ref{3.30}) leads to

\[
\psi_1(P)\psi_1(P^*) = \exp\left(z^{-1} \int_{x_0}^{x} \frac{1}{2} q(x') (\phi(P) + \phi(P^*)) dx' - (x - x_0)\right)
\]

\[
= \exp\left(z^{-1} \int_{x_0}^{x} \frac{1}{2} q(x') (2z G_n) dx' - (x - x_0)\right)
\]

\[
= \exp\left(\int_{x_0}^{x} \frac{F_n x'}{F_n} dx'\right)
\]

\[
= \frac{F_n(z, x)}{F_n(z, x_0)}, \tag{3.37}
\]

\[
\psi_2(P, x, x_0)\psi_2(P^*, x, x_0) = z^{-1} \psi_1(P, x, x_0)\phi(P, x)\psi_1(P^*, x, x_0)\phi(P^*, x)
\]

\[
= z^{-1} \frac{F_n(z, x)}{F_n(z, x_0)} \frac{\phi(P, x)}{F_n(z, x)} \frac{(-z H_n(z, x))}{F_n(z, x)}
\]

\[
= -\frac{H_n(z, x)}{F_n(z, x_0)}, \tag{3.38}
\]
Remark 3.3

The definition of stationary Baker-Akhiezer function $\psi$ of the MCH hierarchy is analogous to that in the context of KdV or AKNS hierarchies. But the crucial difference is that $P_0$ is an essential singularity of $\psi$ in the MCH hierarchy, which is the same as in the CH hierarchy, but different from the KdV or AKNS hierarchy. This fact will be shown in the asymptotic expansions of $\psi$ in next section.

Furthermore, we derive Dubrovin-type equations, which are first-order coupled systems of differential equations and govern the dynamics of the zeros $\mu_j(x)$ and $\nu_j(x)$ of $F_n(z, x)$ and $H_n(z, x)$ with respect to $x$. We recall that the affine part of $K_n$ is nonsingular if

$$\{ \tilde{E}_m \}_{m=1, \ldots, 2n+1} \subset \mathbb{C}, \quad \tilde{E}_m \neq \tilde{E}_{m'} \quad \text{for} \ m \neq m', \ m, m' = 1, \ldots, 2n+1.$$  

(3.42)
Lemma 3.4 Assume that (2.1) holds and \( u \) satisfies the \( n \)th stationary MCH equation (2.28).

(i) If the zeros \( \{ \mu_j(x) \}_{j=1,\ldots,n} \) of \( F_n(z,x) \) remain distinct for \( x \in \Omega_\mu \), where \( \Omega_\mu \subseteq \mathbb{C} \) is open and connected, then \( \{ \mu_j(x) \}_{j=1,\ldots,n} \) satisfy the system of differential equations

\[
\mu_{j,x} = \frac{(u - u_{xx})y(\hat{\mu}_j)}{(u_x - u)\sqrt{\mu_j}} \prod_{k=1, k \neq j}^{n} (\mu_j(x) - \mu_k(x))^{-1}, \quad j = 1, \ldots, n, \tag{3.43}
\]

with initial conditions

\[
\{ \hat{\mu}_j(x_0) \}_{j=1,\ldots,n} \in \mathcal{K}_n, \tag{3.44}
\]

for some fixed \( x_0 \in \Omega_\mu \). The initial value problems (3.43), (3.44) have a unique solution satisfying

\[
\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \ldots, n. \tag{3.45}
\]

(ii) If the zeros \( \{ \nu_j(x) \}_{j=1,\ldots,n} \) of \( H_n(z,x) \) remain distinct for \( x \in \Omega_\nu \), where \( \Omega_\nu \subseteq \mathbb{C} \) is open and connected, then \( \{ \nu_j(x) \}_{j=1,\ldots,n} \) satisfy the system of differential equations

\[
\nu_{j,x} = -\frac{(u - u_{xx})y(\hat{\nu}_j)}{(u_x + u)\sqrt{\nu_j}} \prod_{k=1, k \neq j}^{n} (\nu_j(x) - \nu_k(x))^{-1}, \quad j = 1, \ldots, n, \tag{3.46}
\]

with initial conditions

\[
\{ \hat{\nu}_j(x_0) \}_{j=1,\ldots,n} \in \mathcal{K}_n, \tag{3.47}
\]

for some fixed \( x_0 \in \Omega_\nu \). The initial value problems (3.46), (3.47) have a unique solution satisfying

\[
\hat{\nu}_j \in C^\infty(\Omega_\nu, \mathcal{K}_n), \quad j = 1, \ldots, n. \tag{3.48}
\]

Proof. For our convenience, let us focus on (3.43) and (3.45). The proof of (3.46) and (3.48) follows in an identical manner. The derivatives of (3.11) with respect to \( x \) take on

\[
F_{n,x}(\mu_j) = -(u_x - u)\mu_{j,x} \prod_{k=1, k \neq j}^{n} (\mu_j(x) - \mu_k(x)). \tag{3.49}
\]
On the other hand, inserting $z = \mu_j$ into equation (2.13) leads to

$$F_{n,x}(\mu_j) = (u - u_{xx})G_n(\mu_j) = (u - u_{xx})\frac{y(\mu_j)}{-\sqrt{H_j}},$$ (3.50)

Comparing (3.49) with (3.50) gives (3.43). The smoothness assertion (3.45) is clear as long as $\mu_j$ stays away from the branch points $(E_m, 0)$. In case $\hat{\mu}_j$ hits such a branch point, one can use the local chart around $(E_m, 0)$ ($\zeta = \sigma (z - \tilde{E}_m)^{1/2}, \sigma = \pm 1$) to verify (3.45). □

Let us now turn to the trace formulas of the MCH invariants, which are the expressions of $f_2 l$ and $h_2 l$ in terms of symmetric functions of the zeros $\mu_j$ and $\nu_j$ of $F_n$ and $H_n$. Here, we just consider the simplest case.

**Lemma 3.5** If (2.1) holds and $u$ satisfies the $n$th stationary MCH equation (2.28), then

$$-\frac{1}{2}(u - u_{xx})(u^2 - u_x^2) + \frac{1}{2}(u - u_{xx})\sum_{m=1}^{2n+1} \tilde{E}_m = (u - u_x)\sum_{j=1}^{n} \mu_j,$$ (3.51)

$$\frac{1}{2}(u - u_{xx})(u^2 - u_x^2) - \frac{1}{2}(u - u_{xx})\sum_{m=1}^{2n+1} \tilde{E}_m = -(u + u_x)\sum_{j=1}^{n} \nu_j,$$ (3.52)

**Proof.** By comparison of the coefficients of $z^{n-1}(\lambda^{2n-2})$ in (2.22) and (2.24), taking account into (3.11) and (3.12) yields

$$-\frac{1}{2}(u - u_{xx})(u^2 - u_x^2) + (u - u_{xx})c_1 = (u - u_x)\sum_{j=1}^{n} \mu_j,$$ (3.53)

$$\frac{1}{2}(u - u_{xx})(u^2 - u_x^2) - (u - u_{xx})c_1 = -(u + u_x)\sum_{j=1}^{n} \nu_j,$$ (3.54)

On the other hand, considering the coefficient of $z^{2n}(\lambda^{4n})$ in $\lambda^2 G_n^2 + F_n H_n = R_{2n+1}$ leads to

$$2g_0 g_2 + h_0 f_0 = -\sum_{m=1}^{2n+1} \tilde{E}_m,$$ (3.55)

which implies

$$c_1 = \frac{1}{2} \sum_{m=1}^{2n+1} \tilde{E}_m.$$ (3.56)

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The trace formulas in the MCH hierarchy are implicit for the potential \( u \), which is apposed to the general integrable soliton equations such as KdV, AKNS and CH hierarchies. But this does not affect to obtain the algebro-geometric solutions of the MCH hierarchy, and we can still construct \( u \) from trace formulas.

4 Algebro-geometric solutions of the stationary MCH hierarchy

In this section, we continue our study of the stationary MCH hierarchy, and will obtain explicit Riemann theta function representations for the meromorphic function \( \phi \), and in particular for the potentials \( u \) of the stationary MCH hierarchy.

Let us begin with the asymptotic properties of \( \phi \) and \( \psi_j, j = 1, 2 \).

**Lemma 4.1** Assume that \( (2.1) \) holds and \( u \) satisfies the \( n \)th stationary MCH equation \( (2.28) \). Moreover, let \( P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty, P_0\}, (x, x_0) \in \mathbb{C}^2 \). Then

\[
\phi(P) = \frac{2}{u - u_x} \zeta^{-1} + O(1), \quad P \to P_\infty, \quad \zeta = z^{-1}, \tag{4.1}
\]

\[
\phi(P) = i\zeta + O(\zeta^2), \quad P \to P_0, \quad \zeta = z^{1/2}, \tag{4.2}
\]

\[
\psi_1(P, x, x_0) = \exp\left(\int_{x_0}^{x} \left(\frac{u - u_{xx}}{u - u_x} + O(\zeta)\right) dx' - \frac{1}{2}(x - x_0)\right), \quad P \to P_\infty, \quad \zeta = z^{-1}, \tag{4.3}
\]

\[
\psi_2(P, x, x_0) = \left(\frac{2}{u - u_x} \zeta^{-1/2} + O(\zeta^{1/2})\right) \exp\left(\int_{x_0}^{x} \left(\frac{u - u_{xx}}{u - u_x} + O(\zeta)\right) dx' - \frac{1}{2}(x - x_0)\right), \quad P \to P_\infty, \quad \zeta = z^{-1}, \tag{4.4}
\]

and

\[
\psi_1(P, x, x_0) = \exp\left(\frac{1}{\zeta} \int_{x_0}^{x} \frac{1}{2}(u - u_{xx})(i + O(1)) dx' + O(1)\right), \quad P \to P_0, \quad \zeta = z^{1/2}, \tag{4.5}
\]
\( \psi_2(P, x, x_0) \sim (i + O(\xi)) \exp \left( \frac{1}{\xi} \int_{x_0}^x \frac{1}{2}(u - u_{xx})(i + O(1)) \, dx' + O(1) \right) \),

\[ P \to P_0, \quad \xi = z^{1/2}. \quad (4.6) \]

**Proof.** Under the local coordinates \( \xi = z^{-1} \) near \( P_\infty \) and \( \xi = z^{1/2} \) near \( P_0 \), the existence of the asymptotic expansion of \( \phi \) is clear from its explicit expression in (3.7). Next, we use the Riccati-type equation (3.24) to compute the explicit expansion coefficients. Inserting the ansatz

\[ \phi \sim \phi_{-1} z + \phi_0 + O(z^{-1}) \quad (4.7) \]

into (3.24) and comparing the powers of \( z \), then we have (4.1). Similarly, inserting the ansatz

\[ \phi \sim \phi_1 z^{1/2} + \phi_2 z + O(z^{3/2}) \quad (4.8) \]

into (3.24) and comparing the power of \( z^0 \), we obtain (4.2). Finally, expansions (4.3)-(4.6) follow up by (3.6), (3.8), (4.1) and (4.2). \( \square \)

**Remark 4.2** We notice the unusual fact: \( P_0 \) is the essential singularity of \( \psi_j, \ j = 1, 2 \). From (4.5) and (4.6), the leading-order exponential term \( \psi_j, \ j = 1, 2 \), near \( P_0 \) is \( x \)-dependent, which makes the problem worse. One can obtain the analogous result near \( P_\infty \) as displayed in (4.3) and (4.4). This is in sharp contrast to standard Baker-Akhiezer functions that typically feature a linear behavior with respect to \( x \), such as \( \exp(c(x - x_0)) \) and \( \exp(c(x - x_0)\xi^{-1}) \) near \( P_\infty \) and \( P_0 \), respectively.

Let us now introduce the holomorphic differentials \( \eta_l(P) \) on \( K_n \)

\[ \eta_l(P) = \frac{z^{l-1}}{\sqrt{y(P)}} \, dz, \quad l = 1, \ldots, n, \quad (4.9) \]

and choose a homology basis \( \{a_j, b_j\}_{j=1}^n \) on \( K_n \) in such a way that the intersection matrix of the cycles satisfies

\[ a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \ldots, n. \]

Define an invertible matrix \( E \in GL(n, \mathbb{C}) \) as follows

\[ E = (E_{j,k})_{n \times n}, \quad E_{j,k} = \int_{a_k} \eta_j, \quad (4.10) \]

\[ \xi(k) = (c_1(k), \ldots, c_n(k)), \quad c_j(k) = (E^{-1})_{j,k}, \]

\[ 20 \]
and the normalized holomorphic differentials as follows
\[
\omega_j = \sum_{l=1}^{n} c_j(l) \eta_l, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad \int_{b_k} \omega_j = \tau_{j,k}, \quad j, k = 1, \ldots, n. \quad (4.11)
\]

Apparently, the matrix \( \tau \) is symmetric and has a positive-definite imaginary part.

The symmetric functions \( \Phi^{(j)}_{n-1}(\mu) \) and \( \Psi_n(\mu) \) are defined by
\[
\Phi^{(j)}_{n-1}(\mu) = (-1)^{n-1} \prod_{p=1 \atop p \neq j}^{n} \mu_p, \quad (4.12)
\]
\[
\Psi_n(\mu) = (-1)^n \prod_{p=1}^{n} \mu_p. \quad (4.13)
\]

Let us present our results in the following theorem.

**Theorem 4.3** Assume that (2.1) holds.

(i) Suppose that \( \{\hat{\mu}_j\}_{j=1}^{n} \) satisfy the stationary Dubrovin equations (3.43) on \( \Omega_\mu \) and remain distinct for \( x \in \Omega_\mu \), where \( \Omega_\mu \subseteq \mathbb{C} \) is open and connected. Let the associated divisor be
\[
D_{\hat{\mu}(x)} \in \text{Sym}^n(K_n), \quad \hat{\mu} = \{\hat{\mu}_1, \ldots, \hat{\mu}_n\} \in \text{Sym}^n(K_n). \quad (4.14)
\]

Then
\[
\partial_x \Omega_{Q_0}(D_{\hat{\mu}(x)}) = -\frac{u - u_{xx}}{u_x - u} \frac{1}{\Psi_n(\mu(x))} \mathcal{O}(1), \quad x \in \Omega_\mu. \quad (4.15)
\]

In particular, the Abel map does not linearize the divisor \( D_{\hat{\mu}(x)} \) on \( \Omega_\mu \).

(ii) Suppose that \( \{\hat{\nu}_j\}_{j=1}^{n} \) satisfy the stationary Dubrovin equations (3.46) on \( \Omega_\nu \) and remain distinct for \( x \in \Omega_\nu \), where \( \Omega_\nu \subseteq \mathbb{C} \) is open and connected. Let the associated divisor be
\[
D_{\hat{\nu}(x)} \in \text{Sym}^n(K_n), \quad \hat{\nu} = \{\hat{\nu}_1, \ldots, \hat{\nu}_n\} \in \text{Sym}^n(K_n). \quad (4.16)
\]

Then
\[
\partial_x \Omega_{Q_0}(D_{\hat{\nu}(x)}) = \frac{u - u_{xx}}{u_x + u} \frac{1}{\Psi_n(\nu(x))} \mathcal{O}(1), \quad x \in \Omega_\nu. \quad (4.17)
\]

In particular, the Abel map does not linearize the divisor \( D_{\hat{\nu}(x)} \) on \( \Omega_\nu \).
Proof. It is easy to see that

\[
\frac{1}{\mu_j} = \frac{\prod_{p=1}^{n} \mu_p}{\prod_{p=1}^{n} \mu_p} = -\frac{\phi_{n-1}(\mu)}{\Psi_n(\mu)}, \quad j = 1, \ldots, n. \tag{4.18}
\]

Let

\[
\omega = (\omega_1, \ldots, \omega_n), \tag{4.19}
\]

and choose an appropriate base point \(Q_0\). Then we have

\[
\partial_x Q_0(\mathcal{D}_\hat{\mu}(x)) = \partial_x \left( \sum_{j=1}^{n} \int_{Q_0} \omega \right) = \sum_{j=1}^{n} \mu_{j,x} \sum_{k=1}^{n} c(k) \frac{\mu_{j}^{k-1}}{\sqrt{\mu_j y(\mu_j)}}
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{u - u_{xx}}{u_x - u} \frac{1}{\mu_j} \prod_{l=1}^{n} (\mu_j - \mu_l) c(k)
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{u - u_{xx}}{u_x - u} \frac{\mu_{j}^{k-2}}{\prod_{l=1}^{n} (\mu_j - \mu_l)} c(k)
\]

\[
= -\frac{1}{\Psi_n(\mu)} \frac{u - u_{xx}}{u_x - u} \sum_{j=1}^{n} \sum_{k=1}^{n} c(k) \prod_{l=1}^{n} (\mu_j - \mu_l) \phi_{n-1}(\mu)
\]

\[
= \frac{1}{\Psi_n(\mu)} \frac{u - u_{xx}}{u_x - u} \sum_{j=1}^{n} c(k) \delta_{k,1}
\]

\[
= \frac{1}{\Psi_n(\mu)} \frac{u - u_{xx}}{u_x - u} c(1), \tag{4.20}
\]

where we used (E.25) and (E.26) in [12],

\[
(U_n(\mu)) = \left( \frac{\mu_{j}^{k-1}}{\prod_{l=1}^{n} (\mu_j - \mu_l)} \right)_{j,k=1}^{n}, \quad (U_n(\mu))^{-1} = \left( \phi_{n-1}(\mu) \right)_{j=1}^{n}. \tag{4.21}
\]

The analogous results hold for the corresponding divisor \(\mathcal{D}_{2^2(x)}\), which can be obtained in the same way. \(\square\)

Next, we give the special forms for Theorem 4.3, which can be used to provide the proper change of variables to linearize the divisors \(\mathcal{D}_{2^2(x)}\) and
$D_{\hat{\mu}(x)}$ associated with $\phi(P, x)$. We only consider the case about the divisor $D_{\hat{\mu}(x)}$. One may conclude that the analogous results also hold for the other divisor $D_{\hat{\nu}(x)}$.

Let us introduce

$\hat{B}_{Q_0} : K_n \setminus \{P_{\infty}, P_0\} \to \mathbb{C}^n,$

$P \mapsto \hat{B}_{Q_0}(P) = (\hat{B}_{Q_0,1, \ldots, \hat{B}_{Q_0, n})$

\[
\begin{cases}
\int_{Q_0}^P \omega_{P_{\infty}, P_0}^{(3)}, & n = 1 \\
(\int_{Q_0}^P \eta_2, \ldots, \int_{Q_0}^P \eta_n, \int_{Q_0}^P \tilde{\omega}_{P_{\infty}, P_0}^{(3)}), & n \geq 2,
\end{cases}
\]

(4.22)

where

$\tilde{\omega}_{P_{\infty}, P_0}^{(3)} = \frac{z^n}{\sqrt{z y(P)}} dz.$

and

$\hat{\beta}_{Q_0} : \text{Sym}^n(K_n \setminus \{P_{\infty}, P_0\}) \to \mathbb{C}^n,$

$D_Q \mapsto \hat{\beta}_{Q_0}(D_Q) = \sum_{j=1}^n \hat{B}_{Q_0}(Q_j),$ 

(4.23)

$Q = \{Q_1, \ldots, Q_n\} \in \text{Sym}^n(K_n \setminus \{P_{\infty}, P_0\}).$

**Theorem 4.4** Assume that (2.1) holds, and the statements of $\mu_j$ in Theorem 4.3 are all true. Then

$\partial_x \sum_{j=1}^n \int_{Q_0}^P \mu_j(x) \eta_1 = \frac{1}{\Psi_n(\mu(x))} \frac{u - u_{xx}}{u_x - u}, \quad x \in \Omega_{\mu},$ 

(4.24)

$\partial_x \hat{\beta}(D_{\hat{\mu}(x)}) = \begin{cases}
\frac{u - u_{xx}}{u_x - u}, & n = 1, \\
\frac{u - u_{xx}}{u_x - u}(0, \ldots, 0, 1), & n \geq 2,
\end{cases} \quad x \in \Omega_{\mu}.$ 

(4.25)

**Proof.** Equation (4.24) is a special case of (4.15), and (4.25) follows from (4.20). Alternatively, one can follow the same way as shown in Theorem 4.3 to derive (4.24) and (4.25). \(\square\)

Let $\theta(z)$ denote the Riemann theta function associated with $K_n$ and an appropriately fixed homology basis. We assume $K_n$ to be nonsingular. Next, 1

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1 Here we choose the same path of integration from $Q_0$ and $P$ for all integrals in (4.22) and (4.23).
choosing a convenient base point \( Q_0 \in \mathcal{K}_n \setminus \{ P_\infty, P_0 \} \), the vector of Riemann constants \( \Xi_{Q_0} \) is given by (A.66) [12], and the Abel maps \( \mathcal{A}_{Q_0}(\cdot) \) and \( \alpha_{Q_0}(\cdot) \) are defined by

\[
\mathcal{A}_{Q_0} : \mathcal{K}_n \to J(\mathcal{K}_n) = \mathbb{C}^n/\mathbb{L}_n,
\]

\[
P \mapsto \mathcal{A}_{Q_0}(P) = (\mathcal{A}_{Q_0,1}(P), \ldots, \mathcal{A}_{Q_0,n}(P)) = \left( \int_{Q_0}^P \omega_1, \ldots, \int_{Q_0}^P \omega_n \right) \pmod{\mathbb{L}_n}
\]

and

\[
\alpha_{Q_0} : \text{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n),
\]

\[
\mathcal{D} \mapsto \alpha_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_n} \mathcal{D}(P) \mathcal{A}_{Q_0}(P),
\]

where \( \mathbb{L}_n = \{ \bar{z} \in \mathbb{C}^n | \bar{z} = \bar{N} + \tau \bar{M}, \bar{N}, \bar{M} \in \mathbb{Z}^n \} \).

Let

\[
\omega^{(3)}_{P_\infty P_0} = \frac{1}{\sqrt{zy}} \prod_{j=1}^{n} (z - \lambda_j) \, dz
\]

be the normalized differential of the third kind holomorphic on \( \mathcal{K}_n \setminus \{ P_\infty, P_0 \} \) with simple poles at \( P_\infty \) and \( P_0 \) with residues \( \pm 1 \), that is,

\[
\omega^{(3)}_{P_\infty P_0}(P) \xrightarrow{\zeta \to 0} (\zeta^{-1} + O(1)) \, d\zeta, \quad \text{as } P \to P_\infty,
\]

\[
\omega^{(3)}_{P_\infty P_0}(P) \xrightarrow{\zeta \to 0} (-\zeta^{-1} + O(1)) \, d\zeta, \quad \text{as } P \to P_0,
\]

where the local coordinate are given by

\[
\zeta = z^{-1} \quad \text{for } P \text{ near } P_\infty, \quad \zeta = z^{1/2} \quad \text{for } P \text{ near } P_0,
\]

and the constants \( \{ \lambda_j \}_{j=1}^{n} \) are determined by the normalization condition

\[
\int_{a_k} \omega^{(3)}_{P_\infty P_0} = 0, \quad k = 1, \ldots, n.
\]

Then, we have

\[
\int_{Q_0}^P \omega^{(3)}_{P_\infty P_0}(P) \xrightarrow{\zeta \to 0} \ln \zeta + e_0 + O(\zeta), \quad \text{as } P \to P_\infty,
\]
\[
\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty}P_0}(P) = \lim_{\zeta \to 0} -\ln \zeta + d_0 + O(\zeta), \quad \text{as } P \to P_0,
\]  

(4.32)

where constants \(c_0, d_0 \in \mathbb{C}\) arise from the integrals at their lower limits \(Q_0\).

We also notice

\[
\Delta_{Q_0}(P) - \Delta_{Q_0}(P_\infty) = \lim_{\zeta \to 0} U \zeta + O(\zeta^2), \quad \text{as } P \to P_\infty, \quad U = c(n),
\]  

(4.33)

\[
\Delta_{Q_0}(P) - \Delta_{Q_0}(P_0) = \lim_{\zeta \to 0} -2U \zeta + O(\zeta^2), \quad \text{as } P \to P_0, \quad U = c(n).
\]  

(4.34)

The following abbreviations are used for our convenience:

\[
z(P, Q) = \Xi_{Q_0} - \Delta_{Q_0}(P) + \omega_{Q_0}(D_{\tilde{\mu}}),
\]  

\[
P \in \mathcal{K}_n, \quad Q = (Q_1, \ldots, Q_n) \in \text{Sym}^n(\mathcal{K}_n),
\]  

(4.35)

where \(z(\cdot, Q)\) is independent of the choice of base point \(Q_0\).

Moreover, from Theorems 4.3 and 4.4 we note that the Abel map dose not linearize the divisor \(D_{\tilde{\mu}(x)}\). However, the change of variables

\[
x \mapsto \tilde{x} = \int_{x}^{\tilde{x}} dx' \left( \frac{1}{\Psi_n(\mu(x'))} \frac{u - u_{x'}x'}{u_{x'} - u} \right)
\]  

(4.36)

linearizes the Abel map \(\Delta_{Q_0}(D_{\tilde{\mu}(x)}), \mu_j(\tilde{x}) = \mu_j(x), j = 1, \ldots, n\). The intricate relation between the variable \(x\) and \(\tilde{x}\) is discussed detailedly in Theorem 4.5.

Based on the above all these preparations, let us now give an explicit representation for the meromorphic function \(\phi\) and the solution \(u\) of the stationary MCH equations in terms of the Riemann theta function associated with \(\mathcal{K}_n\). Here we assume the affine part of \(\mathcal{K}_n (n \in \mathbb{N})\) to be nonsingular.

**Theorem 4.5** Assume that the curve \(\mathcal{K}_n\) is nonsingular, \((2.1)\) holds, and \(u\) satisfies the \(n\)th stationary MCH equation \((2.28)\) on \(\Omega\). Let \(P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty, P_0\}\), and \(x \in \Omega\), where \(\Omega \subseteq \mathbb{C}\) is open and connected. In addition, suppose that \(D_{\tilde{\mu}(x)}\), or equivalently \(D_{\tilde{\mu}(x)}\) is nonspecial for \(x \in \Omega\). Then, \(\phi\) and \(u\) have the following representations

\[
\phi(P, x) = i \frac{\theta(z(P, \tilde{\mu}(x)))\theta(z(P_0, \mu(x)))}{\theta(z(P_0, \mu(x)))\theta(z(P, \mu(x)))} \exp \left( -d_0 - \int_{Q_0}^{P} \omega^{(3)}_{P_{\infty}P_0} \right),
\]  

(4.37)

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\[ u(x) = \frac{\theta(z(P_0, \hat{\mu}(x)))\theta(z(P_\infty, \hat{\mu}(x)))}{\theta(z(P_\infty, \hat{\mu}(x)))\theta(z(P_0, \hat{\mu}(x)))} \]  
\[ \times \left( \frac{\sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n} U_j \partial_{\omega_j} (\theta(z(P_\infty, \hat{\mu}(x)) + \omega))}{\sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n} U_j \partial_{\omega_j} (\theta(z(P_\infty, \hat{\mu}(x)) + \omega))} \right) \bigg|_{\omega = 0} + 1 \right) \]  

Moreover, let \( \mu_j, j = 1, \ldots, n \) be not vanishing on \( \Omega \). Then, we have the following constraint
\[ \int_{x_0}^{x} \frac{u - u_{x'}x'}{u_{x'} - u} = - (\tilde{x} - \tilde{x}_0) \sum_{j=1}^{n} \left( \int_{a_j}^{(3)} \tilde{w}_{\mu_0}(x) c_j(1) \right) \]
\[ + \ln \left( \frac{\theta(z(P_\infty, \hat{\mu}(x)))\theta(z(P_\infty, \hat{\mu}(x)))}{\theta(z(P_\infty, \hat{\mu}(x)))\theta(z(P_\infty, \hat{\mu}(x)))} \right), \]  

and
\[ \hat{x}(P_\infty, \hat{\mu}(x)) = \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P_\infty) + \hat{\alpha}_{Q_0}(D_{\hat{\mu}}(x)) \]
\[ = \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P_\infty) + \hat{\alpha}_{Q_0}(D_{\hat{\mu}}(x)) - \int_{x_0}^{x} \frac{u - u_{x'}x'}{u_{x'} - u} \Psi_n(\mu(x')) dx'(1) \]
\[ = \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P_\infty) + \hat{\alpha}_{Q_0}(D_{\hat{\mu}}(x)) - \mathcal{C}(1)(\tilde{x} - \tilde{x}_0), \]  
\[ \hat{x}(P_0, \hat{\mu}(x)) = \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P_0) + \hat{\alpha}_{Q_0}(D_{\hat{\mu}}(x)) \]
\[ = \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P_0) + \hat{\alpha}_{Q_0}(D_{\hat{\mu}}(x)) - \int_{x_0}^{x} \frac{u - u_{x'}x'}{u_{x'} - u} \Psi_n(\mu(x')) dx'(1) \]
\[ = \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P_0) + \hat{\alpha}_{Q_0}(D_{\hat{\mu}}(x)) - \mathcal{C}(1)(\tilde{x} - \tilde{x}_0). \]  

Proof. First, let us assume
\[ \mu_j(x) \neq \mu_{j'}(x), \quad \nu_k(x) \neq \nu_k'(x) \quad \text{for} \; j \neq j', k \neq k' \; \text{and} \; x \in \tilde{\Omega}, \]
where \( \tilde{\Omega} \subseteq \Omega \) is open and connected. From (3.19), \( D_{\hat{\mu}} \sim D_{\hat{\mu}_n} \), and \( P_\infty = (P_\infty)^* \neq \{ \hat{\mu}_1, \ldots, \hat{\mu}_n \} \), based on some hypothesis of Theorem A. 31 [12], one can have \( D_{\hat{\mu}} \in \text{Sym}^n(K_n) \) is nonspecial. This argument is symmetric with respect to \( \hat{\mu} \) and \( \hat{\mu} \). Thus, \( D_{\hat{\mu}} \) is nonspecial if and only if \( D_{\hat{\mu}} \) is.

Next, we derive the representations of \( \phi \) and \( u \) in terms of the Riemann theta functions. A special case of Riemann’s vanishing theorem (Theorem A. 26 [12]) yields
\[ \theta(z_{Q_0} - \hat{A}_{Q_0}(P) + \omega_{Q_0}(D_{\hat{\mu}})) = 0 \quad \text{if and only if} \; P \in \{ Q_1, \ldots, Q_n \}. \]
Therefore, the divisor (3.19) of $\phi(P,x)$ suggests considering expression of the following type

$$C(x) \frac{\theta(E_{Q_0} - A_{Q_0}(P) + \alpha_{Q_0}(D_{P}(x)))}{\theta(E_{Q_0} - A_{Q_0}(P) + \alpha_{Q_0}(D_{P}(x)))} \exp\left(\int_{Q_0}^{P} \omega_{P_{\infty},P_0}^{(3)} \right),$$

(4.44)

where $C(x)$ is independent of $P \in K_n$. So, together with the asymptotic expansion of $\phi(P,x)$ near $P_0$ in (4.2), we are able to obtain (4.37).

Let us now construct $u$ from trace formulas (3.51) and (3.52). By comparing (3.51) with (3.52), we have

$$(u - u_x) \sum_{j=1}^{n} \mu_j = (u + u_x) \sum_{j=1}^{n} \nu_j,$$

(4.45)

namely,

$$u + u_x = \frac{\sum_{j=1}^{n} \mu_j}{\sum_{j=1}^{n} \nu_j} (u - u_x).$$

(4.46)

Therefore,

$$u = \frac{1}{2} (u + u_x) + \frac{1}{2} (u - u_x) = \frac{1}{2} \left( \sum_{j=1}^{n} \mu_j + 1 \right) (u - u_x).$$

(4.47)

On the other hand, the asymptotic expansion of $\phi$ near $P_{\infty}$ with taking account into (4.37) and comparing the coefficient of $\zeta^{-1}$ yields

$$u - u_x = 2i \theta(z(P_0, \tilde{\mu}(x))) \theta(z(P_{\infty}, \hat{\mu}(x)))$$

$$\theta(z(P_{\infty}, \tilde{\mu}(x))) \theta(z(P_0, \hat{\mu}(x))).$$

(4.48)

Inserting the expressions of $\sum_{j=1}^{n} \mu_j$, $\sum_{j=1}^{n} \nu_j$ (F.88 [12]), and (4.48) into (4.47) leads to (4.38).

Moreover, from (4.24) and (4.25), one can readily get

$$\sum_{j=1}^{n} \int_{Q_0}^{P_0} \hat{\nu}_j(x) \omega_{P_{\infty},P_0}^{(3)} - \sum_{j=1}^{n} \int_{Q_0}^{P_0} \hat{\nu}_j(x) \omega_{P_{\infty},P_0}^{(3)}$$

$$= \int_{x_0}^{x} \partial_{x'} \left( \sum_{j=1}^{n} \int_{Q_0}^{P_0} \hat{\nu}_j(x') \omega_{P_{\infty},P_0}^{(3)} \right) dx'$$

$$= \int_{x_0}^{x} dx' \frac{u - u_{x'}x'}{u_{x'} - u}.$$
By (F.88) \([12]\), we may arrive at

\[
\sum_{j=1}^{n} \int_{Q_0}^{\hat{\varphi}_{P_\infty,P_0}} \omega^{(3)}_{j} - \sum_{j=1}^{n} \int_{Q_0}^{\hat{\varphi}_{P_\infty,P_0}} \omega^{(3)}_{j} = \sum_{j=1}^{n} \int_{a_j}^{\tilde{\omega}_{P_\infty,P_0}} \omega^{(3)}_{j} \times \left( \sum_{k=1}^{n} \int_{Q_0}^{\hat{\varphi}_{k}} \omega_{j} - \sum_{k=1}^{n} \int_{Q_0}^{\hat{\varphi}_{k}} \omega_{j} \right) + \ln \left( \frac{\theta(z(P_\infty,\hat{\mu}(x)))}{\theta(z(P_0,\hat{\mu}(x)))} \right)
\]

\[
= \sum_{j=1}^{n} \int_{a_j}^{\tilde{\omega}_{P_\infty,P_0}} \int_{Q_0}^{x} \partial_{x'} \left( \sum_{k=1}^{n} \int_{Q_0}^{\hat{\varphi}_{k}} \omega_{j} \right) dx' + \ln \left( \frac{\theta(z(P_\infty,\hat{\mu}(x)))\theta(z(P_0,\hat{\mu}(x)))}{\theta(z(P_\infty,\hat{\mu}(x)))\theta(z(P_0,\hat{\mu}(x)))} \right)
\]

\[= - \int_{x_0}^{x} \left( u - u_{x'x'} \right) \left( \frac{1}{u_{x'} - u} \right) \sum_{j=1}^{n} \left( \int_{a_j}^{\tilde{\omega}_{P_\infty,P_0}} \omega^{(3)}_{j} \right) c_j(1) + \ln \left( \frac{\theta(z(P_\infty,\hat{\mu}(x)))\theta(z(P_0,\hat{\mu}(x)))}{\theta(z(P_\infty,\hat{\mu}(x)))\theta(z(P_0,\hat{\mu}(x)))} \right). \tag{4.50}\]

Hence, inserting the condition of changing of variables (4.36) into (4.50) and comparing with (4.49) can give the constraint (4.39). Equations (4.40) and (4.41) are clear from (4.15). The extension of all results from \( x \in \Omega \) to \( x \in \bar{\Omega} \) follows by the continuity of \( \tilde{\omega}_{Q_0} \) and the hypothesis of \( D_{\hat{\omega}(x)} \) being nonspecial for \( x \in \Omega \).

**Remark 4.6** The stationary MCH solution \( u \) in (4.38) is a quasi-periodic function with respect to the new variable \( \hat{x} \) in (4.36). The Abel map in (4.40) and (4.41) linearizes the divisor \( D_{\hat{\omega}(x)} \) on \( \Omega \) with respect to \( \hat{x} \).

**Remark 4.7** The similar results to (4.40) and (4.41) (i.e. the Abel map also linearizes the divisor \( D_{\hat{\omega}(x)} \) on \( \Omega \) with respect to \( \hat{x} \)) hold for the divisor \( D_{\hat{\omega}(x)} \) associated with \( \phi(P,x) \). The change of variables is

\[
x \mapsto \hat{x} = \int_{x_0}^{x} dx' \left( \frac{1}{\Psi_0(\bar{\varphi}(x'))} \frac{u - u_{x'x'}}{u_{x'} + u} \right). \tag{4.51}\]

**Remark 4.8** Since \( D_{P_0\hat{\omega}} \) and \( D_{P_\infty\hat{\omega}} \) are linearly equivalent, that is,

\[
A_{Q_0}(P_\infty) + \omega_{Q_0}(D_{\hat{\omega}(x)}) = A_{Q_0}(P_0) + \omega_{Q_0}(D_{\hat{\omega}(x)}), \tag{4.52}\]

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we have
\[ a_0 Q(D^\nu(x)) = \Delta + a_0 Q(D^\mu(x)), \quad \Delta = \Delta_{P_0}(P_\infty). \] (4.53)

Hence
\[ z(P, \hat{\nu}) = z(P, \hat{\mu}) + \Delta, \quad P \in \mathcal{K}_n. \] (4.54)

The representations of \( \phi \) and \( u \) in (4.37) and (4.38) can be rewritten in terms of \( D^\nu(x) \), respectively.

Remark 4.9 We have emphasized in Remark 4.2 that the Baker-Akhiezer functions \( \psi \) in (3.6) and (3.8) for the MCH hierarchy enjoy very different from standard Baker-Akhiezer functions. Hence, one may not expect the usual theta function representations of \( \psi_j, j=1,2 \), in terms of ratios of theta functions times a exponential term including \( (x-x_0) \) multiplying a meromorphic differential with a pole with the essential singularity of \( \psi_j \). However, using the properties of symmetric function and (F.89) [13], we obtain

\[
F_n(z) = (u_x - u)(z^{n+1}) = (u_x - u)(z^n + \sum_{i=1}^{n-1} \Psi_{n-i}(\mu) z^i)
\]

\[
= (u_x - u)\left(z^n + \sum_{k=1}^{n} \left(\Psi_{n+1-k}(\lambda) - \sum_{j=1}^{n} c_j(k) \partial_{\omega_j} \ln \left(\frac{\theta(z(P_\infty, \hat{\mu}) + \omega)}{\theta(z(P_0, \hat{\mu}) + \omega)}\right)\right) \right)
\]

\[
= (u_x - u)\left(\prod_{j=1}^{n} (z - \lambda_j) \right)
- \sum_{j=1}^{n} \sum_{k=1}^{n} c_j(k) \partial_{\omega_j} \ln \left(\frac{\theta(z(P_\infty, \hat{\mu}) + \omega)}{\theta(z(P_0, \hat{\mu}) + \omega)}\right) \right)\right|_{\omega=0}^{z^{-k-1}}.
\] (4.55)

and by inserting (4.55) into (3.31), we obtain the theta function representation of \( \psi_1 \). Then, the corresponding theta function representation of \( \psi_2 \) follows by (3.8) and (4.37).

Let us now discuss the trivial case \( n = 0 \) excluded in Theorem 4.5.

Example 4.10 Let \( n = 0, P = (z, y) \in \mathcal{K}_0 \setminus \{P_\infty, P_0\}, \) and \( (x, x_0) \in \mathbb{C}^2 \).
Then, we have
\[ K_0 : \mathcal{F}_0(z, y) = y^2 - R_1(z) = y^2 - (z - \bar{E}_1) = 0, \quad \bar{E}_1 \in \mathbb{C}, \]
\[ u = \bar{E}_1^{1/2}, \]
\[ s - \text{MCH}_0(u) = 2u_x - 2u_{xx} = 0, \]
\[ F_0(z, x) = f_0 = u_x - u, \quad G_0(z, x) = g_0 = -1, \quad H_0(z, x) = h_0 = u_x + u, \]
\[ \phi(P, x) = \frac{z^{1/2}y - z}{u_x - u} = \frac{u_x + u}{z^{1/2}y + 1}, \quad (4.56) \]
\[ \psi_1(P, x, x_0) = \exp \left( \int_{x_0}^{x} \left( \frac{1}{2} \frac{u - u_{x'}}{u_x - u} \frac{y - z^{1/2}}{z^{1/2}} \right) dx' - \frac{1}{2}(x - x_0) \right), \]
\[ \psi_2(P, x, x_0) = \frac{y - z^{1/2}}{u_x - u} \exp \left( \int_{x_0}^{x} \left( \frac{1}{2} \frac{u - u_{x'}}{u_x - u} \frac{y - z^{1/2}}{z^{1/2}} \right) dx' - \frac{1}{2}(x - x_0) \right). \]

The general solution of \( s - \text{MCH}_0(u) = 2u_x - 2u_{xx} = 0 \) is given by
\[ u(x) = a_1 e^x + \bar{E}_1^{1/2}, \quad a_1 \in \mathbb{C}. \quad (4.57) \]

But, according to the condition \( \partial^k u \in L^\infty(\mathbb{C}), k \in \mathbb{N}_0 \) in (2.1), we conclude \( a_1 = 0 \), and therefore \( u = \bar{E}_1^{1/2} \), which is the same as the expression of \( u \) in (4.56) from trace formula (3.51) or (3.52) in the special case \( n = 0 \).

At the end of this section, we turn to the initial value problem in the stationary case. We show that the solvability of the Dubrovin equations (3.43) on \( \Omega_\mu \subseteq \mathbb{C} \) in fact implies equation (2.28) on \( \Omega_\mu \), which amounts to solving the initial value problem in the stationary case.

**Theorem 4.11** Assume that (2.1) holds and \( \{\mu_j\}_{j=1, \ldots, n} \) satisfies the stationary Dubrovin equations (3.43) on \( \Omega_\mu \) and remain distinct and nonzero for \( x \in \Omega_\mu \), where \( \Omega_\mu \subseteq \mathbb{C} \) is open and connected. Then, if \( u \in C^\infty(\Omega_\mu) \) satisfies
\[ -\frac{1}{2}(u - u_{xx})(u^2 - u_x^2) + \frac{1}{2}(u - u_{xx}) \sum_{m=1}^{2n+1} \bar{E}_m = (u - u_x) \sum_{j=1}^{n} \mu_j, \quad (4.58) \]
\( u \) is a solution of the \( n \)th stationary MCH equation (2.28), that is
\[ s - \text{MCH}_n(u) = 0, \quad \text{on} \quad \Omega_\mu. \quad (4.59) \]
Proof. Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, K_n)$, $j = 1, \ldots, n$ of (3.43), let us introduce

$$F_n(z) = (u_x - u) \prod_{j=1}^n (z - \mu_j) \quad \text{on } \mathbb{C} \times \Omega_\mu,$$  

where $u$ is the solution of (4.58) up to multiplicative constant. Given $F_n$ and $u$, let us denote the polynomial $G_n$ by

$$G_n(z) = (u - u_{xx})^{-1}(F_n(z) + F_{n,x}(z)) \quad \text{on } \mathbb{C} \times \Omega_\mu,$$  

and from (4.60), one can see that the degree of $G_n$ is $n$ with respect to $z$. Taking account into (4.61), the Dubrovin equations (3.43) imply

$$y(\hat{\mu}_j) = \mu_j,x (u_x - u) \sqrt{\mu_j} \prod_{k=1, k \neq j}^n (\mu_j - \mu_k)$$

$$= -\sqrt{\mu_j} F_{n,x}(\mu_j) \quad \text{on } \mathbb{C} \times \Omega_\mu,$$  

Hence

$$R_{2n+1}(\mu_j)^2 - \mu_j G_n(\mu_j)^2 = y(\hat{\mu}_j)^2 - \mu_j G_n(\mu_j)^2 = 0, \quad j = 1, \ldots, n.$$  

Next, let us define a polynomial $H_n$ on $\mathbb{C} \times \Omega_\mu$ such that

$$R_{2n+1}(z) - z G_n(z)^2 = F_n(z) H_n(z)$$  

holds. Such a polynomial $H_n$ exists since the left-hand side of (4.64) vanishes at $z = \mu_j$, $j = 1, \cdots, n$ by (4.63). We need to determine the degree of $H_n$. By (4.61), we compute

$$R_{2n+1}(z) - z G_n(z)^2 \bigg|_{z \to \infty} = (u_x + u)(u_x - u)z^{2n} + O(z^{2n-1}),$$  

with $O(z^{2n-1})$ depending on $x$ by inspection. Therefore, combining (4.60), (4.61) and (4.65), we conclude that $H_n$ has degree $n$ with respect to $z$, with the coefficient $(u_x + u)$ of powers $z^{2n}$. Hence, we may write $H_n$ as

$$H_n(z) = (u_x + u) \prod_{j=1}^n (z - \nu_j), \quad \text{on } \mathbb{C} \times \Omega_\mu,$$  

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where \( u \) is the same as that in (4.60). Next, let us consider the polynomial \( P_{n-1} \) by

\[
P_{n-1}(z) = \frac{1}{2}(u - u_{xx})H_n(z) + \frac{1}{2}(u - u_{xx})F_n(z) + zG_{n,x}(z). \tag{4.67}
\]

Using (4.60), (4.61) and (4.66), we see that \( P_{n-1} \) is a polynomial of degree at most \( n-1 \). Differentiating on both sides of (4.64) with respect to \( x \) yields

\[
2zG_n(z)G_{n,x}(z) + F_{n,x}(z)H_n(z) + F_n(z)H_{n,x}(z) = 0 \quad \text{on } \mathbb{C} \times \Omega_{\mu}. \tag{4.68}
\]

Multiplying (4.67) by \( G_n \) and using (4.68), we have

\[
G_n(z)P_{n-1}(z) = \frac{1}{2}F_n(z)[(u - u_{xx})G_n(z) - H_{n,x}(z)]
+ [(u - u_{xx})G_n(z) - F_{n,x}(z)]\frac{1}{2}H_n(z), \tag{4.69}
\]

and therefore on \( \Omega_{\mu} \), we have

\[
G_n(\mu_j)P_{n-1}(\mu_j) = 0, \quad j = 1, \ldots, n. \tag{4.70}
\]

Next, let \( x \in \tilde{\Omega}_{\mu} \subseteq \Omega_{\mu} \), where \( \tilde{\Omega}_{\mu} \) is given by

\[
\tilde{\Omega}_{\mu} = \{ x \in \Omega_{\mu} \mid G(\mu_j(x), x) = \frac{y(\mu_j(x))}{\sqrt{\mu_j(x)}} \neq 0, j = 1, \ldots, n \}\]

\[
= \{ x \in \Omega_{\mu} \mid \mu_j(x) \notin \{ \hat{E}_m \}_{m=1, \ldots, 2n+1}, j = 1, \ldots, n \}, \tag{4.71}
\]

Thus, we have

\[
P_{n-1}(\mu_j(x), x) = 0, \quad j = 1, \ldots, n, \quad x \in \tilde{\Omega}_{\mu}. \tag{4.72}
\]

Since \( P_{n-1} \) is a polynomial of degree at most \( n-1 \), (4.72) implies

\[
P_{n-1} = 0 \quad \text{on } \mathbb{C} \times \tilde{\Omega}_{\mu}, \tag{4.73}
\]

So, (2.15) holds, that is,

\[
zG_{n,x}(z) = \frac{1}{2}(u - u_{xx})H_n(z) - \frac{1}{2}(u - u_{xx})F_n(z) \quad \text{on } \mathbb{C} \times \tilde{\Omega}_{\mu}. \tag{4.74}
\]

Inserting (4.74) and (4.61) into (4.68) yields

\[
[-(u - u_{xx})G_n(z) - H_n(z) + H_{n,x}(z)]F_n(z) = 0, \tag{4.75}
\]

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namely,

\[ H_{n,x}(z) = (u - u_{xx})G_n(z) + H_n(z), \quad \text{on } \mathbb{C} \times \tilde{\Omega}_\mu. \]  

(4.76)

Thus, we obtain the fundamental equations (2.13)-(2.15), and (2.17) on \( \mathbb{C} \times \tilde{\Omega}_\mu \).

In order to extend these results to all \( x \in \Omega_\mu \), let us consider the case where \( \hat{\mu}_j \) admits one of the branch points \( (\tilde{E}_{m_0}, 0) \). Hence, we suppose

\[ \mu_{j_1}(x) \to \tilde{E}_{m_0} \quad \text{as} \quad x \to x_0 \in \Omega_\mu, \]  

for some \( j_1 \in \{1, \cdots, n\} \), \( m_0 \in \{1, \cdots, 2n + 1\} \). Introducing

\[ \zeta_{j_1}(x) = \sigma(\mu_{j_1}(x) - \tilde{E}_{m_0})^{1/2}, \quad \sigma = \pm 1, \]

\[ \mu_{j_1}(x) = \tilde{E}_{m_0} + \zeta_{j_1}(x)^2 \]  

(4.78)

for \( x \) near \( x_0 \), then the Dubrovin equation (3.43) for \( \mu_{j_1} \) becomes

\[ \zeta_{j_1,x}(x) = \frac{c(\sigma)(u - u_{xx})}{2(u_x - u)\sqrt{\tilde{E}_{m_0}} \left( \prod_{m=1, m\neq m_0}^{2n+1} (\tilde{E}_{m_0} - \tilde{E}_m) \right)^{1/2}} \]

\[ \times \prod_{k=1, k\neq j_1}^{n} (\tilde{E}_{m_0} - \mu_k(x))^{-1} (1 + O(\zeta_{j_1}(x)^2)) \]  

(4.79)

for some \( |c(\sigma)| = 1 \). Hence, (4.73)-(4.76) are extended to \( \Omega_\mu \) by continuity. Consequently, we obtain relations (2.13)-(2.15) on \( \mathbb{C} \times \tilde{\Omega}_\mu \), and can proceed as in Section 2 to see that \( u \) satisfies the stationary MCH hierarchy (4.59).

\( \square \)

**Remark 4.12** The result in Theorem 4.11 is derived in terms of \( u \) and \( \{\mu_j\}_{j=1}^{n} \), but we can prove the analogous result in terms of \( u \) and \( \{\nu_j\}_{j=1}^{n} \).

**Remark 4.13** Theorem 4.11 reveals that given \( K_n \) and the initial condition \( \hat{\mu}(x_0) = (\hat{\mu}_1(x_0), \ldots, \hat{\mu}_n(x_0)) \in \text{Sym}^n(K_n) \), or equivalently, the auxiliary divisor \( D_{\hat{\mu}(x_0)} \in \text{Sym}^n(K_n) \) at \( x = x_0 \), \( u \) is uniquely determined in an open neighborhood \( \Omega \) of \( x_0 \) by (4.58) and satisfies the \( n \)th stationary MCH equation (2.28). Conversely, given \( K_n \) and \( u \) in an open neighborhood \( \Omega \) of \( x_0 \), we can construct the corresponding polynomial \( F_n(z, x) \), \( G_n(z, x) \) and \( H_n(z, x) \) for \( x \in \Omega \), and then obtain the auxiliary divisor \( D_{\hat{\mu}(x)} \) for \( x \in \Omega \) from the zeros of \( F_n(z, x) \) and (3.13). In that sense, once the curve \( K_n \) is fixed, elements of the isospectral class of the MCH potentials \( u \) can be characterized by nonspecial auxiliary divisor \( D_{\hat{\mu}(x)} \).
5 The time-dependent MCH formalism

In this section, let us go back to the recursive approach detailed in Section 2 and extend the algebro-geometric analysis in Section 3 to the time-dependent MCH hierarchy.

Throughout this section, we assume that (2.2) holds.

The time-dependent algebro-geometric initial value problem of the MCH hierarchy is to solve the time-dependent $r$th MCH flow with a stationary solution of the $n$th equation as the initial data in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, based on the solution $u^{(0)}$ of the $n$th stationary MCH equation $s - \text{MCH}_n(u^{(0)}) = 0$ associated with $K_n$ and a set of integration constants $\{c_l\}_{l=1,...,n} \subset \mathbb{C}$, we want to build up a solution $u$ of the $r$th MCH flow $\text{MCH}_r(u) = 0$ such that $u(t_0,r) = u^{(0)}$ for some $t_0,r \in \mathbb{C}$, $r \in \mathbb{N}_0$.

We employ the following notation $\tilde{V}_r, \tilde{F}_r, \tilde{G}_r, \tilde{H}_r, \tilde{f}_{2s}, \tilde{g}_{2s}, \tilde{h}_{2l}$ to stand for the time-dependent quantities, which are obtained in $V_n, F_n, G_n, H_n, f_{2l}, g_{2l}, h_{2l}$ by replacing $\{c_l\}_{l=1,...,n}$ with $\{\tilde{c}_s\}_{s=1,...,r}$, where the integration constants $\{c_l\}_{l=1,...,n} \subset \mathbb{C}$ in the stationary MCH hierarchy and $\{\tilde{c}_s\}_{s=1,...,r} \subset \mathbb{C}$ in the time-dependent MCH hierarchy are independent of each other. In addition, we mark the individual $r$th MCH flow by a separate time variable $t_r \in \mathbb{C}$.

Let us now provide the time-dependent algebro-geometric initial value problem as follows

\begin{align*}
\text{MCH}_r(u) &= u_t - u_{xxt} - 2\tilde{f}_{2r,x} = 0, \\
|u|_{t_r=t_0,r} &= u^{(0)}, \\
s - \text{MCH}_n(u^{(0)}) &= -2f_{2n,x}(u^{(0)}) = 0, 
\end{align*}

where $t_0,r \in \mathbb{C}$, $n,r \in \mathbb{N}_0$, $u = u(x,t_r)$ satisfies the condition (2.2), and the curve $K_n$ is associated with the initial data $u^{(0)}$ in (5.2). Noticing that the MCH flows are isospectral, we are going to a further step and assume that (5.2) holds not only at $t_r = t_0,r$, but also at all $t_r \in \mathbb{C}$.

Let us now start from the zero-curvature equations (2.35)

\begin{align*}
U_{t_r} - \tilde{V}_{r,x} + [U, \tilde{V}_r] &= 0, \\
- V_{n,x} + [U, V_n] &= 0,
\end{align*}

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where

\[
U(z) = \left( -\frac{1}{2}z^{-\frac{1}{2}}(u - u_{xx}), \frac{1}{2}z^{-\frac{1}{2}}(u - u_{xx}) \right)
\]

\[
V_n(z) = \left( -G_n(z) z^{-\frac{1}{2}}F_n(z), z^{-\frac{1}{2}}H_n(z) G_n(z) \right)
\]

\[
\tilde{V}_r(z) = \left( -\tilde{G}_r(z) z^{-\frac{1}{2}}\tilde{F}_r(z), z^{-\frac{1}{2}}\tilde{H}_r(z) \tilde{G}_r(z) \right)
\]

and

\[
F_n(z) = \sum_{l=0}^{n} f_{2l} z^{(n-l)} = f_0 \prod_{j=1}^{n} (z - \mu_j), \quad f_0 = u_x - u,
\]

\[
G_n(z) = \sum_{l=0}^{n} g_{2l} z^{(n-l)},
\]

\[
H_n(z) = \sum_{l=0}^{n} h_{2l} z^{(n-l)} = h_0 \prod_{j=1}^{n} (z - \nu_j), \quad h_0 = u_x + u,
\]

\[
\tilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{2s} z^{(r-s)}, \quad \tilde{f}_0 = u_x - u,
\]

\[
\tilde{G}_r(z) = \sum_{s=0}^{r} \tilde{g}_{2s} z^{(r-s)},
\]

\[
\tilde{H}_r(z) = \sum_{s=0}^{r} \tilde{h}_{2s} z^{(r-s)}, \quad \tilde{h}_0 = u_x + u,
\]

for fixed \( n, r \in \mathbb{N}_0 \). Here \( f_{2l}, g_{2l}, h_{2l}, l = 0, \ldots, n \) and \( \tilde{f}_{2s}, \tilde{g}_{2s}, \tilde{h}_{2s}, s = 0, \ldots, r \), satisfy the relations in (2.3).

Moreover, it is more convenient for us to rewrite the zero-curvature equations (5.3) and (5.4) as the following forms

\[
\frac{1}{2}(u_{tr} - u_{xxt}) - \tilde{F}_{r,x} - \tilde{F}_r + (u - u_{xx})\tilde{G}_r = 0,
\]

\[
-\frac{1}{2}(u_{tr} - u_{xxt}) - \tilde{H}_{r,x} + \tilde{H}_r + (u - u_{xx})\tilde{G}_r = 0,
\]

\[
z\tilde{G}_{r,x} = \frac{1}{2}(u - u_{xx})\tilde{H}_r - \frac{1}{2}(u - u_{xx})\tilde{F}_r.
\]
and
\[
F_{n,x} = -F_n + (u - u_{xx})G_n, \quad (5.15)
\]
\[
H_{n,x} = H_n + (u - u_{xx})G_n, \quad (5.16)
\]
\[
zG_{n,x} = -\frac{1}{2}(u - u_{xx})H_n - \frac{1}{2}(u - u_{xx})F_n. \quad (5.17)
\]

From (5.15)-(5.17), we may compute
\[
\frac{d}{dx} \det(V_n(z)) = -\frac{1}{z} \frac{d}{dz} \left( zG_n(z)^2 + F_n(z)H_n(z) \right) = 0, \quad (5.18)
\]
and meanwhile Lemma 5.2 gives
\[
\frac{d}{dt} \det(V_n(z)) = -\frac{1}{z} \frac{d}{dt} \left( zG_n(z)^2 + F_n(z)H_n(z) \right) = 0. \quad (5.19)
\]
Hence, \(zG_n(z)^2 + F_n(z)H_n(z)\) is independent of variables both \(x\) and \(t\), which implies
\[
zG_n(z)^2 + F_n(z)H_n(z) = R_{2n+1}(z). \quad (5.20)
\]
This reveals that the fundamental identity (2.17) still holds in the time-dependent context. Consequently the hyperelliptic curve \(K_n\) is still available by (2.21).

Next, let us introduce the time-dependent Baker-Akhiezer function \(\psi(P, x, x_0, t_r, t_{0,r})\) on \(K_n \setminus \{P_{\infty}, P_0\}\) by
\[
\psi(P, x, x_0, t_r, t_{0,r}) = \begin{pmatrix}
\psi_1(P, x, x_0, t_r, t_{0,r}) \\
\psi_2(P, x, x_0, t_r, t_{0,r})
\end{pmatrix},
\]
\[
\psi_x(P, x, x_0, t_r, t_{0,r}) = U(u(x, t_r), z(P))\psi(P, x, x_0, t_r, t_{0,r}),
\]
\[
\psi_{t_r}(P, x, x_0, t_r, t_{0,r}) = \tilde{V}_r(u(x, t_r), z(P))\psi(P, x, x_0, t_r, t_{0,r}),
\]
\[
z^\frac{1}{2}V_n(u(x, t_r), z(P))\psi(P, x, x_0, t_r, t_{0,r}) = y(P)\psi(P, x, x_0, t_r, t_{0,r}),
\]
\[
\psi_1(P, x_0, x_0, t_{0,r}, t_{0,r}) = 1;
\]
\[
P = (z, y) \in K_n \setminus \{P_{\infty}, P_0\}, \quad (x, t_r) \in \mathbb{C}^2.
\]

where
\[
\psi_1(P, x, x_0, t_r, t_{0,r}) = \exp \left( \int_{t_{0,r}}^{t_r} ds (z^{-1}\tilde{F}_r(z, x_0, s)\phi(P, x_0, s) - \tilde{G}_r(z, x_0, s)) \right)
\]
\[
+ z^{-1} \int_{x_0}^{x} dx' \left( \frac{1}{2} q(x', t_r)\phi(P, x', t_r) - \frac{1}{2} (x - x_0) \right),
\]
\[
P = (z, y) \in K_n \setminus \{P_{\infty}, P_0\}. \quad (5.22)
\]
Closely related to $\psi(P, x, x_0, t_r, t_{0,r})$ is the following meromorphic function $\phi(P, x, t_r)$ on $\mathcal{K}_n$ defined by

$$\phi(P, x, t_r) = \frac{2z}{q(x, t_r)} \left( \frac{\psi_{1,x}(P, x, x_0, t_r, t_{0,r})}{\psi_1(P, x, x_0, t_r, t_{0,r})} + \frac{1}{2} \right),$$

$$P \in \mathcal{K}_n \setminus \{P_\infty, P_0\}, \ (x, t_r) \in \mathbb{C}^2. \tag{5.23}$$

which implies by (5.21) that

$$\phi(P, x, t_r) = \frac{z^{1/2}y + z^{1/2}G_n(z, x, t_r)}{F_n(z, x, t_r)}$$

$$= \frac{z^{1/2}H_n(z, x, t_r)}{y - z^{1/2}G_n(z, x, t_r)}, \tag{5.24}$$

and

$$\psi_2(P, x, x_0, t_r, t_{0,r}) = \psi_1(P, x, x_0, t_r, t_{0,r})\phi(P, x, t_r)/z^{1/2}. \tag{5.25}$$

In analogy to equations (3.13) and (3.14), we define

$$\hat{\mu}_j(x, t_r) = \left( \mu_j(x, t_r), -\sqrt{\mu_j(x, t_r)G_n(\mu_j(x, t_r), x, t_r)} \right) \in \mathcal{K}_n,$$

$$j = 1, \ldots, n, \ (x, t_r) \in \mathbb{C}^2, \tag{5.26}$$

$$\hat{\nu}_j(x, t_r) = \left( \nu_j(x, t_r), \sqrt{\nu_j(x, t_r)G_n(\nu_j(x, t_r), x, t_r)} \right) \in \mathcal{K}_n,$$

$$j = 1, \ldots, n, \ (x, t_r) \in \mathbb{C}^2. \tag{5.27}$$

The regular properties of $F_n$, $H_n$, $\mu_j$ and $\nu_j$ are analogous to those in Section 3 due to assumptions (2.2).

From (5.24), the divisor $(\phi(P, x, t_r))$ of $\phi(P, x, t_r)$ reads

$$(\phi(P, x, t_r)) = \mathcal{D}_{P_0} \hat{\nu}(x, t_r)(P) - \mathcal{D}_{P_\infty} \hat{\mu}(x, t_r)(P) \tag{5.28}$$

where

$$\hat{\mu} = \{\hat{\mu}_1, \ldots, \hat{\mu}_n\}, \ \hat{\nu} = \{\hat{\nu}_1, \ldots, \hat{\nu}_n\} \in \text{Sym}^n(\mathcal{K}_n). \tag{5.29}$$

That means $P_0, \nu_1(x, t_r), \ldots, \nu_n(x, t_r)$ are the $n + 1$ zeros of $\phi(P, x, t_r)$ and $P_\infty, \mu_1(x, t_r), \ldots, \mu_n(x, t_r)$ its $n + 1$ poles.

Further properties of $\phi(P, x, t_r)$ are summarized as follows.
Lemma 5.1 Assume that (2.2), (5.3) and (5.4) hold. Let \( P = (z, y) \in K_n \setminus \{ P_\infty, P_0 \} \), \((x, t_r) \in \mathbb{C}^2\). Then

\[
\phi_x(P) + \frac{1}{2}(u-u_{xx})z^{-1}\phi(P)^2 - \phi(P) = -\frac{1}{2}(u-u_{xx}),
\]

(5.30)

\[
(q\phi(P))_{t_r} = (-2z\tilde{G}_r(z) + 2\tilde{F}_r(z)\phi(P))_x
\]

(5.31)

\[
\phi_{t_r}(P) = \tilde{H}_r(z) + 2\tilde{G}_r(z)\phi(P) - z^{-1}\tilde{F}_r(z)\phi(P)^2,
\]

(5.32)

\[
\phi(P)\phi(P^*) = -\frac{zH_n(z)}{F_n(z)},
\]

(5.33)

\[
\phi(P) + \phi(P^*) = 2\frac{zG_n(z)}{F_n(z)},
\]

(5.34)

\[
\phi(P) - \phi(P^*) = z^{1/2} \frac{2y}{F_n(z)},
\]

(5.35)

Proof. We just need to prove (5.31) and (5.32). Equations (5.30) and (5.33)-(5.35) can be proved as in Lemma 3.1. By using (5.21) and (5.23), we obtain

\[
\frac{1}{2}z^{-1}(q\phi)_{t_r} = (\ln\psi_1)_{xt_r} = (\ln\psi_1)_{t_r} = (\frac{\psi_{1,t_r}}{\psi_1})_x
\]

\[
= \left( \frac{-\tilde{G}_r\psi_1 + z^{-1/2}\tilde{F}_r\psi_2}{\psi_1} \right)_x
\]

\[
= (-\tilde{G}_r + z^{-1}\tilde{F}_r\phi)_x,
\]

(5.36)

which implies the fist line of (5.31). Inserting (5.14) into (5.36) yields the second line of (5.31). Then by the definition of \( \phi \) (5.23), one may have

\[
\phi_{t_r} = z^{1/2} \left( \frac{\psi_2}{\psi_1} \right)_{t_r}
\]

\[
= z^{1/2} \left( \frac{\psi_{2,t_r}}{\psi_1} - \frac{\psi_2\psi_1,t_r}{\psi_1^2} \right)
\]

\[
= z^{1/2} \left( \frac{z^{-1/2}\tilde{H}_r\psi_1 + \tilde{G}_r\psi_2}{\psi_1} - z^{-1/2}\phi\frac{-\tilde{G}_r\psi_1 + z^{-1/2}\tilde{F}_r\psi_2}{\psi_1} \right)
\]

\[
= \tilde{H}_r + 2\tilde{G}_r\phi - z^{-1}\tilde{F}_r\phi^2,
\]

(5.37)
which is (5.32). Alternatively, one can insert (5.12)-(5.14) into (5.31) to obtain (5.32). □

Next, we study the time evolution of $F_n$, $G_n$ and $H_n$ by using zero-curvature equations (5.12)-(5.14) and (5.15)-(5.17).

**Lemma 5.2** Assume that (2.2), (5.3), and (5.4) hold. Then

\[ F_{n,t} = 2(G_n \tilde{F}_r - \tilde{G}_r F_n), \]  
\[ zG_{n,t} = H_r F_n - H_n \tilde{F}_r, \]  
\[ H_{n,t} = 2(H_n \tilde{G}_r - G_n H_r). \]

Equations (5.38) – (5.40) imply

\[ -V_{n,t} + [\tilde{V}_r, V_n] = 0. \]

**Proof.** Differentiating both sides of (5.35) with respect to $t_r$ leads to

\[ (\phi(P) - \phi(P^*))_{t_r} = -2z^{1/2}yF_{n,t}F_n^{-2}. \]

On the other hand, by (5.32), (5.34), and (5.35), the left-hand side of (5.42) equals to

\[ \phi(P)_{t_r} - \phi(P^*)_{t_r} = 2\tilde{G}_r(\phi(P) - \phi(P^*)) - z^{-1}\tilde{F}_r(\phi(P)^2 - \phi(P^*)^2) \]
\[ = 4z^{1/2}y(\tilde{G}_r F_n - \tilde{F}_r G_n)F_n^{-2}. \]

Combining (5.42) with (5.43) yields (5.38). Similarly, differentiating both sides of (5.34) with respect to $t_r$ gives

\[ (\phi(P) + \phi(P^*))_{t_r} = 2z(G_{n,t} F_n - G_n F_{n,t})F_n^{-2}. \]

Meanwhile, by (5.32), (5.33), and (5.34), the left-hand side of (5.44) equals to

\[ \phi(P)_{t_r} + \phi(P^*)_{t_r} = 2\tilde{G}_r(\phi(P) + \phi(P^*)) - z^{-1}\tilde{F}_r(\phi(P)^2 + \phi(P^*)^2) + 2\tilde{H}_r \]
\[ = -2zG_n F_n^{-2}F_{n,t} + 2F_n^{-1}(\tilde{H}_r F_n - \tilde{F}_r H_n). \]

Thus, (5.39) clearly follows by (5.44) and (5.45). Hence, insertion of (5.38) and (5.39) into the differentiation of $zG_n^2 + F_n H_n = R_{2n+1}(z)$ can derive (5.40). Finally, a direct calculation shows that (5.38)-(5.40) are equivalent to (5.41). □

Further properties of $\psi$ is summarized as follows.
Lemma 5.3 Assume that (2.22), (5.3), and (5.4) hold. Let \( P = (z, y) \in K_n \backslash \{P_\infty, P_0\} \), \( (x, x_0, t_r, t_0, r) \in \mathbb{C}^4 \). Then, we have

\[
\psi_1(P, x, x_0, t_r, t_0, r) = \left( \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_0, r)} \right)^{1/2} \times \exp\left( \frac{y}{\sqrt{z}} \int_{t_0, r}^{t_r} ds F_r(z, x_0, s) F_n(z, x_0, s)^{-1} \right. \\
\left. + \frac{y}{2\sqrt{z}} \int_{x_0}^{x} dx' q(x', t_r) F_n(z, x', t_r)^{-1} \right), \quad (5.46)
\]

\[
\psi_1(P, x, x_0, t_r, t_0, r) \psi_1(P^*, x, x_0, t_r, t_0, r) = \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_0, r)}, \quad (5.47)
\]

\[
\psi_2(P, x, x_0, t_r, t_0, r) \psi_2(P^*, x, x_0, t_r, t_0, r) = -\frac{H_n(z, x, t_r)}{F_n(z, x_0, t_0, r)}, \quad (5.48)
\]

\[
\psi_1(P, x, x_0, t_r, t_0, r) \psi_2(P^*, x, x_0, t_r, t_0, r) + \psi_1(P^*, x, x_0, t_r, t_0, r) \psi_2(P, x, x_0, t_r, t_0, r) = 2\frac{\sqrt{z} G_n(z, x, t_r)}{F_n(z, x_0, t_0, r)}, \quad (5.49)
\]

\[
\psi_1(P, x, x_0, t_r, t_0, r) \psi_2(P^*, x, x_0, t_r, t_0, r) - \psi_1(P^*, x, x_0, t_r, t_0, r) \psi_2(P, x, x_0, t_r, t_0, r) = -\frac{2y}{F_n(z, x_0, t_0, r)} \quad (5.50)
\]

**Proof.** In order to prove (5.46), let us first consider the part of time variable in the definition (5.22), that is,

\[
\exp\left( \int_{t_0, r}^{t_r} ds \left[ z^{-1} \widetilde{F}_r(z, x_0, s) \phi(z, x_0, s) - \widetilde{G}_r(z, x_0, s) \right] \right). \quad (5.51)
\]

The integrand in the above integral equals to

\[
\begin{align*}
& z^{-1} \widetilde{F}_r(z, x_0, s) \phi(z, x_0, s) - \widetilde{G}_r(z, x_0, s) \\
& = z^{-1} \widetilde{F}_r(z, x_0, s) z^{1/2} y + z^{1/2} G_n(z, x_0, s) - \widetilde{G}_r(z, x_0, s) \\
& = \frac{y}{\sqrt{z}} \widetilde{F}_r(z, x_0, s) F_n(z, x_0, s)^{-1} + (\widetilde{G}_r(z, x_0, s) G_n(z, x_0, s) \\
& - \widetilde{G}_r(z, x_0, s) F_n(z, x_0, s)^{-1}) F_n(z, x_0, s)^{-1} \\
& = \frac{y}{\sqrt{z}} \widetilde{F}_r(z, x_0, s) F_n(z, x_0, s)^{-1} + \frac{1}{2} \frac{F_{n,s}(z, x_0, s)}{F_n(z, x_0, s)}, \quad (5.52)
\end{align*}
\]
By (5.24), (5.38), and (5.52), (5.51) reads

\[
\left( \frac{F_n(z, x_0, t_r)}{F_n(z, x_0, t_{0,r})} \right)^{1/2} \exp \left( \frac{y}{\sqrt{z}} \int_{t_{0,r}}^{t_r} ds \tilde{F}_r(z, x_0, s) F_n(z, x_0, s)^{-1} \right). \tag{5.53}
\]

On the other hand, the part of space variable in (5.22) can be written as

\[
\left( \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_r)} \right)^{1/2} \exp \left( \frac{y}{2 \sqrt{z}} \int_{x_0}^{x} dx' q(x', t_r) F_n(z, x', t_r)^{-1} \right), \tag{5.54}
\]

which can be proved using the similar procedure to Lemma 3.2. Combining (5.53) with (5.54) yields (5.46). Evaluating (5.46) at the points \( P \) and \( P^* \) with noticing

\[
y(P) + y(P^*) = 0, \tag{5.55}
\]

leads to (5.47). Hence, we have

\[
\psi_2(P, x, x_0, t_r, t_{0,r}) \psi_2(P^*, x, x_0, t_r, t_{0,r}) = z^{-1} \psi_1(P, x, x_0, t_r, t_{0,r}) \phi(P) \psi_1(P^*, x, x_0, t_r, t_{0,r}) \phi(P^*)
\]

\[
= z^{-1} \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})} \frac{-z H_n(z, x, t_r)}{F_n(z, x, t_r) F_n(z, x, t_r)}
\]

\[
= \frac{H_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})}, \tag{5.56}
\]

\[
\psi_1(P, x, x_0, t_r, t_{0,r}) \psi_2(P^*, x, x_0, t_r, t_{0,r}) + \psi_1(P^*, x, x_0, t_r, t_{0,r}) \psi_2(P, x, x_0, t_r, t_{0,r})
\]

\[
= (\phi(P) + \phi(P^*)) \psi_1(P) \psi_1(P^*) / z^{1/2}
\]

\[
= \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})} \frac{2 \sqrt{z} G_n(z, x, t_r) / z^{1/2}}{F_n(z, x, t_r) / z^{1/2}}
\]

\[
= 2 \sqrt{z} G_n(z, x, t_r) / F_n(z, x_0, t_{0,r}), \tag{5.57}
\]

\[
\psi_1(P, x, x_0, t_r, t_{0,r}) \psi_2(P^*, x, x_0, t_r, t_{0,r})
\]

\[
- \psi_1(P^*, x, x_0, t_r, t_{0,r}) \psi_2(P, x, x_0, t_r, t_{0,r})
\]

\[
= (-\phi(P) + \phi(P^*)) \psi_1(P) \psi_1(P^*) / z^{1/2}
\]

\[
= \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})} \frac{-2 z^{1/2} y}{F_n(z, x_0, t_{0,r})} F_n(z, x, t_r) / z^{1/2}
\]

\[
= - \frac{2 y}{F_n(z, x_0, t_{0,r})}. \tag{5.58}
\]
which are \( (5.48)-(5.50) \). □

In analogy to Lemma 3.4, the dynamics of the zeros \( \{ \mu_j(x,t_r) \}_{j=1,...,n} \) and \( \{ \nu_j(x,t_r) \}_{j=1,...,n} \) of \( F_n(z,x,t_r) \) and \( H_n(z,x,t_r) \) with respect to \( x \) and \( t_r \) are described in terms of Dubrovin-type equations (see the following Lemmas). We assume that the affine part of \( K_n \) is nonsingular

\[
\{ \tilde{E}_m \}_{m=1,...,2n+1} \subset \mathbb{C}, \quad \tilde{E}_m \neq \tilde{E}_{m'} \quad \text{for} \quad m \neq m', \quad m, m' = 1, \ldots, 2n+1. \tag{5.59}
\]

**Lemma 5.4** Assume that \((5.2), (5.3), \) and \((5.4)\) hold.

(i) Suppose that the zeros \( \{ \mu_j(x,t_r) \}_{j=1,...,n} \) of \( F_n(z,x,t_r) \) remain distinct for \( (x,t_r) \in \Omega_\mu \), where \( \Omega_\mu \subseteq \mathbb{C}^2 \) is open and connected. Then, \( \{ \mu_j(x,t_r) \}_{j=1,...,n} \) satisfy the system of differential equations

\[
\mu_{j,x} = \frac{(u - u_{xx})y(\hat{\mu}_j)}{(u_x - u)\sqrt{\mu_j}} \prod_{k=1}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 1, \ldots, n, \tag{5.60}
\]

\[
\mu_{j,t_r} = \frac{2 \tilde{F}_r(\mu_j)y(\hat{\mu}_j)}{(u_x - u)\sqrt{\mu_j}} \prod_{k=1}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 1, \ldots, n, \tag{5.61}
\]

with initial conditions

\[
\{ \hat{\mu}_j(x_0,t_0,r) \}_{j=1,...,n} \in K_n, \tag{5.62}
\]

for some fixed \((x_0,t_0,r) \in \Omega_\mu \). The initial value problem \((5.61), (5.62)\) has a unique solution satisfying

\[
\hat{\mu}_j \in C^\infty(\Omega_\mu,K_n), \quad j = 1, \ldots, n. \tag{5.63}
\]

(ii) Suppose that the zeros \( \{ \nu_j(x,t_r) \}_{j=1,...,n} \) of \( H_n(z,x,t_r) \) remain distinct for \( (x,t_r) \in \Omega_\nu \), where \( \Omega_\nu \subseteq \mathbb{C}^2 \) is open and connected. Then \( \{ \nu_j(x,t_r) \}_{j=1,...,n} \) satisfy the system of differential equations

\[
\nu_{j,x} = -\frac{(u - u_{xx})y(\hat{\nu}_j)}{(u_x + u)\sqrt{\nu_j}} \prod_{k=1}^{n} (\nu_j - \nu_k)^{-1}, \quad j = 1, \ldots, n, \tag{5.64}
\]

\[
\nu_{j,t_r} = \frac{2 \tilde{H}_r(\nu_j)y(\hat{\nu}_j)}{(u_x + u)\sqrt{\nu_j}} \prod_{k=1}^{n} (\nu_j - \nu_k)^{-1}, \quad j = 1, \ldots, n, \tag{5.65}
\]

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with initial conditions
\[
\{ \hat{v}_j(x_0, t_0, r) \}_{j=1, \ldots, n} \in K_n,  \tag{5.66}
\]
for some fixed \((x_0, t_0, r) \in \Omega_{\nu}\). The initial value problem (5.65), (5.66) has a unique solution satisfying
\[
\hat{v}_j \in C^\infty(\Omega_{\nu}, K_n), \quad j = 1, \ldots, n.  \tag{5.67}
\]

**Proof.** It suffices to focus on (5.60), (5.61) and (5.63), since the proof procedure for (5.64), (5.65) and (5.67) is similar.

The proof of (5.60) has been given in Lemma 3.4. We just derive (5.61). Differentiating on both sides of (5.6) with respect to \(t_r\) yields
\[
F_{n,t_r}(\mu_j) = -(u_x - u)\mu_{j,t_r} \prod_{k=1}^{n} (\mu_j - \mu_k).  \tag{5.68}
\]
On the other hand, inserting \(z = \mu_j\) into (5.38) and considering (5.26), we arrive at
\[
F_{n,t_r}(\mu_j) = 2G_n(\mu_j) \hat{F}_r(\mu_j) = 2\frac{y(\hat{\mu}_j)}{-\sqrt{\hat{\mu}_j}} \hat{F}_r(\mu_j).  \tag{5.69}
\]
Combining (5.68) with (5.69) leads to (5.61). The smoothness assertion of (5.63) is clear as long as \(\hat{\mu}_j\) stays away from the branch points \((\tilde{E}_m, 0)\). In case \(\hat{\mu}_j\) admits such a branch point, one can use the local chart around \((\tilde{E}_m, 0)\) \((\zeta = \sigma(z - \tilde{E}_m)^{1/2}, \sigma = \pm 1)\) to verify (5.63). \(\square\)

Let us now present the \(t_r\)-dependent trace formulas of the MCH hierarchy, which are used to construct the algebro-geometric solutions \(u\) in section 6. For simplicity, we just take the simplest case.

**Lemma 5.5** Assume that (2.2), (5.3) and (5.4) hold. Then, we have
\[
-\frac{1}{2}(u - u_{xx})(u^2 - u_x^2) + \frac{1}{2}(u - u_{xx}) \sum_{m=1}^{2n+1} \tilde{E}_m = (u - u_x) \sum_{j=1}^{n} \mu_j,  \tag{5.70}
\]
\[
\frac{1}{2}(u - u_{xx})(u^2 - u_x^2) - \frac{1}{2}(u - u_{xx}) \sum_{m=1}^{2n+1} \tilde{E}_m = -(u + u_x) \sum_{j=1}^{n} \nu_j.  \tag{5.71}
\]

**Proof.** The proof is similar to the corresponding stationary case in Lemma 3.5. \(\square\)
6 Time-dependent algebro-geometric solutions

In this final section, we extend the results in section 4 from the stationary MCH hierarchy to the time-dependent case. In particular, we obtain Riemann theta function representations for the Baker-Akhiezer function, the meromorphic function \( \phi \), and the algebro-geometric solutions for the MCH hierarchy.

Let us first consider the asymptotic properties of \( \phi \) in the time-dependent case.

**Lemma 6.1** Assume (2.2), (5.3), and (5.4) hold. Let \( P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty, P_0\} \), \( (x, t_r) \in \mathbb{C}^2 \). Then, we have

\[
\phi(P) \underset{\zeta \to 0}{=} \frac{2}{u - u_x} \zeta^{-1} + O(1), \quad P \to P_\infty, \quad \zeta = z^{-1}, \quad (6.1)
\]

\[
\phi(P) \underset{\zeta \to 0}{=} i \zeta + O(\zeta^2), \quad P \to P_0, \quad \zeta = z^{1/2}. \quad (6.2)
\]

**Proof.** The proof is analogous to the corresponding stationary case in Lemma 4.1. \( \square \)

Next, we study the properties of Abel map, which does not linearize the divisor \( \mathcal{D}_{\mu(x, t_r)} \) and \( \mathcal{D}_{\hat{\mu}(x, t_r)} \) in the time-dependent MCH hierarchy. This is the remarkable difference between the MCH hierarchy and other integrable systems such as KdV and AKNS hierarchies. For that purpose, we introduce some notations of symmetric functions.

Let us define

\[
\mathcal{S}_k = \{ \underline{l} = (l_1, \ldots, l_n) \in \mathbb{N}^k \mid l_1 < \cdots < l_k \leq n \}, \quad k = 1, \ldots, n,
\]

\[
\mathcal{T}_k^{(j)} = \{ \underline{l} = (l_1, \ldots, l_n) \in \mathcal{S}_k \mid l_m \neq j \}, \quad k = 1, \ldots, n - 1, \quad j = 1, \ldots, n.
\]

(6.3)

The symmetric functions are defined by

\[
\Psi_0(\mu) = 1, \quad \Psi_k(\mu) = (-1)^k \sum_{\underline{l} \in \mathcal{S}_k} \mu_{l_1} \cdots \mu_{l_k}, \quad k = 1, \ldots, n,
\]

(6.4)

and

\[
\Phi_0^{(j)}(\mu) = 1,
\]

\[
\Phi_k^{(j)}(\mu) = \sum_{\underline{l} \in \mathcal{T}_k^{(j)}} \mu_{l_1} \cdots \mu_{l_k}, \quad k = 1, \ldots, n - 1, \quad j = 1, \ldots, n.
\]

(6.5)
where $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$. The properties of $\Psi_k(\mu)$ and $\Phi_k^{(j)}(\mu)$ can be found in Appendix E [12]. Here we freely use those relations.

Moreover, for the MCH hierarchy we have

$$
\tilde{F}_r(\mu_j) = (u_x - u) \sum_{s=(r-n)\vee 0}^{r} \tilde{c}_s(F) \Phi_r^{(j)}(\mu),
$$

$$
\tilde{F}_r(\mu_j) = \sum_{s=0}^{r} \tilde{c}_{r-s} F_r(\mu_j) = (u_x - u) \sum_{k=0}^{r\wedge n} \tilde{d}_{r,k}(E) \Phi_k^{(j)}(\mu), \quad r \in \mathbb{N}_0, \tilde{c}_0 = 1,
$$

where

$$
\tilde{d}_{r,k}(E) = \sum_{s=0}^{r-k} \tilde{c}_{r-k-s} \tilde{c}_s(E) \quad k = 0, \ldots, r \wedge n.
$$

**Theorem 6.2** Assume that $\mathcal{K}_n$ is nonsingular and (2.2) holds.

(i) Suppose $\{\hat{\mu}_j\}_{j=1,\ldots,n}$ satisfies the Dubrovin equations (5.60), (5.61) on $\Omega_\mu$ and remain distinct and $\tilde{F}_r(\mu_j) \neq 0$ for $(x, t_r) \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}^2$ is open and connected, and the associated divisor is defined by

$$
D_{\hat{\mu}(x, t_r)} \in \text{Sym}^n(\mathcal{K}_n), \quad \hat{\mu} = \{\hat{\mu}_1, \ldots, \hat{\mu}_n\} \in \text{Sym}^n(\mathcal{K}_n).
$$

Then, we have

$$
\partial_x \alpha_{Q_0}(D_{\hat{\mu}(x, t_r)}) = -\frac{u - u_x}{u_x - u} \frac{1}{\Psi_n(\mu(x, t_r))} \xi(1), \quad (x, t_r) \in \Omega_\mu, (6.9)
$$

$$
\partial_t \alpha_{Q_0}(D_{\hat{\mu}(x, t_r)}) = -\frac{2}{\Psi_n(\mu(x, t_r))} \left( \sum_{k=0}^{r\wedge n} \tilde{d}_{r,k}(E) \Psi_k(\mu(x, t_r)) \right) \xi(1)
$$

$$
+ 2 \left( \sum_{l=1\vee(n+1-r)}^{n} \tilde{d}_{r,n+1-l}(E) \xi(l) \right), \quad (x, t_r) \in \Omega_\mu.
$$

In particular, the Abel map does not linearize the divisor $D_{\hat{\mu}(x, t_r)}$ on $\Omega_\mu$.

(ii) Suppose that $\{\hat{\nu}_j\}_{j=1,\ldots,n}$ satisfies the Dubrovin equations (5.64), (5.65) on $\Omega_\nu$ and remain distinct and $\tilde{H}_r(\nu_j) \neq 0$ for $(x, t_r) \in \Omega_\nu$, where $\Omega_\nu \subseteq \mathbb{C}^2$ is open and connected, and the associated divisor is defined by

$$
D_{\hat{\nu}(x, t_r)} \in \text{Sym}^n(\mathcal{K}_n), \quad \hat{\nu} = \{\hat{\nu}_1, \ldots, \hat{\nu}_n\} \in \text{Sym}^n(\mathcal{K}_n).
$$

$^2m \wedge n = \min\{m, n\}, \ m \vee n = \max\{m, n\}$
Then, we have
\[
\partial_x \omega_Q_0(\mathcal{D}_{\mathcal{D}(x,t_r)}) = \frac{u - u_{xx}}{u_x + u_x} \frac{1}{\Psi_n(\mathcal{D}(x,t_r))} \xi(1), \quad (x,t_r) \in \Omega_\nu, \quad (6.12)
\]
\[
\partial_u \omega_Q_0(\mathcal{D}_{\mathcal{D}(x,t_r)}) = -\frac{2}{\Psi_n(\mathcal{D}(x,t_r))} \left( \sum_{k=0}^{r \wedge n} \tilde{d}_{r,k}(\tilde{E}) \Psi_k(\mathcal{D}(x,t_r)) \right) \xi(1)
\]
\[
+ 2 \left( \sum_{l=0}^{n} \tilde{d}_{r,n+1-l}(\tilde{E}) \xi(l) \right), \quad (x,t_r) \in \Omega_\nu. \quad (6.13)
\]

In particular, the Abel map does not linearize the divisor \(\mathcal{D}_{\mathcal{D}(x,t_r)}\) on \(\Omega_\nu\).

**Proof.** It suffices to prove (6.10), since the proof procedure is similar for (6.13). The proof for (6.9) and (6.12) has been given in the stationary context of Theorem 4.3. Let us first give a fundamental identity (E.17) [12], that is,
\[
(6.14)
\]
Then, together with (6.6) and (4.18), we have
\[
\frac{\tilde{F}_r(\mu_j)}{(u_x - u)\mu_j} = \mu_j^{-1} \sum_{m=0}^{r \wedge n} \tilde{d}_{r,m}(\tilde{E}) \Phi_m^{(j)}(\mu) \quad (6.15)
\]
\[
= \mu_j^{-1} \sum_{m=0}^{r \wedge n} \tilde{d}_{r,m}(\tilde{E}) \left( \mu_j \Phi_{m-1}^{(j)}(\mu) + \Psi_m(\mu) \right)
\]
\[
= \sum_{m=1}^{r \wedge n} \tilde{d}_{r,m}(\tilde{E}) \Phi_{m-1}^{(j)}(\mu) - \sum_{m=0}^{r \wedge n} \tilde{d}_{r,m}(\tilde{E}) \Psi_m(\mu) \Phi_n^{(j)}(\mu) \Psi_n(\mu). \]

So, using (6.15), (6.61), (E.9), (E.25) and (E.26) [12], we obtain
\[
\partial_{x_r} \left( \sum_{j=1}^{n} \int_{Q_0} \xi_j \right) = \sum_{j=1}^{n} \mu_j \sum_{k=1}^{n} c(k) \frac{\mu_j^{-1}}{\sqrt{\mu_j}(\mu_j)}
\]
\[
= 2 \sum_{j=1}^{n} \sum_{k=1}^{n} c(k) \prod_{l=1}^{r \wedge n} \frac{\mu_j^{-1}}{\mu_j(\mu_j - \mu_l) \mu_j(u_x - u)}
\]
\[
= 2 \sum_{j=1}^{n} \sum_{k=1}^{n} c(k) \prod_{l=1}^{r \wedge n} \frac{\mu_j^{-1}}{\mu_j(\mu_j - \mu_l)} \left( - \sum_{m=0}^{r \wedge n} \tilde{d}_{r,m}(\tilde{E}) \Psi_m(\mu) \Phi_n^{(j)}(\mu) \Psi_n(\mu) \right)
\]
\[
+ \sum_{m=1}^{r \wedge n} \tilde{d}_{r,m}(\tilde{E}) \Phi_n^{(j)}(\mu) \Psi_n(\mu). \]
Therefore, we complete the proof of (6.10). \(\Box\)

The following result is a special form of Theorem 6.2, which provides the constraint condition to linearize the divisor \(D_{\hat{\mu}(x,t_r)}\) and \(D_{\hat{\beta}(x,t_r)}\) associated with \(\phi(P,x,t_r)\). We recall the definitions of \(\hat{\beta}Q_0\) and \(\hat{\beta}Q_0\) in (4.22) and (4.23).

**Theorem 6.3** Assume that (2.2) and the statements of \(\{\mu_j\}_{j=1,\ldots,n}\) and \(\{\nu_j\}_{j=1,\ldots,n}\) in Theorem 6.2 hold. Then,

(i) for \(\{\mu_j\}_{j=1,\ldots,n}\), we have

\[
\partial_x \sum_{j=1}^{n} \int_{Q_0} \tilde{\mu}_j(x,t_r) \eta_1 = - \frac{1}{\Psi_n(\mu(x,t_r))} \frac{u - u_{xx}}{u_x - u}, \quad (x,t_r) \in \Omega_{\mu}, \tag{6.17}
\]

\[
\partial_x \hat{\beta}(D_{\hat{\mu}(x,t_r)}) = \begin{cases} \frac{u - u_{xx}}{u_x - u}, & n = 1, \\ \frac{u - u_{xx}}{u_x - u}, & n \geq 2, \end{cases} \quad (x,t_r) \in \Omega_{\mu}, \tag{6.18}
\]

\[
\partial_t \sum_{j=1}^{n} \int_{Q_0} \tilde{\mu}_j(x,t_r) \eta_1 = - \frac{2}{\Psi_n(\mu(x,t_r))} \sum_{k=0}^{r/n} \tilde{d}_{r,k}(\tilde{E})\Psi_k(\mu(x,t_r)) \\
+ 2 \tilde{d}_{r,n}(\tilde{E})\delta_{n,r \land n}, \quad (x,t_r) \in \Omega_{\mu}, \tag{6.19}
\]

\[
\partial_t \hat{\beta}(D_{\hat{\mu}(x,t_r)}) = 2 \left( \sum_{s=0}^{r} \tilde{c}_{r-s}c_{s+1}(\tilde{E}), \ldots, \sum_{s=0}^{r} \tilde{c}_{r-s}c_{s}(\tilde{E}), \sum_{s=0}^{r} \tilde{c}_{r-s}c_{s}(\tilde{E}) \right), \quad \tilde{c}_{-l}(\tilde{E}) = 0, \quad l \in \mathbb{N}, \quad (x,t_r) \in \Omega_{\mu}. \tag{6.20}
\]
(ii) for \( \{\nu_j\}_{j=1,\ldots,n} \), we have

\[
\partial_x \sum_{j=1}^{n} \int_{Q_0} \hat{\rho}_j(x,t_r) \eta_1 = \frac{1}{\Psi_n(\mu(x,t_r))} \frac{u - u_{xx}}{u_x + u}, \quad (x,t_r) \in \Omega_\nu, \tag{6.21}
\]

\[
\partial_t \sum_{j=1}^{n} \int_{Q_0} \hat{\rho}_j(x,t_r) \eta_1 = \begin{cases} 
\frac{-u + u_{xx}}{u_x + u}, & n = 1, \\
\frac{-u + u_{xx}}{u_x + u} (0,\ldots,0,1), & n \geq 2,
\end{cases} \quad (x,t_r) \in \Omega_\nu, \tag{6.22}
\]

\[
\partial_t \sum_{j=1}^{n} \int_{Q_0} \hat{\rho}_j(x,t_r) \eta_1 = 2 \sum_{s=0}^{r} \tilde{c}_{r-s} \tilde{c}_{s+1-n}(\tilde{E}), \ldots, \sum_{s=0}^{r} \tilde{c}_{r-s} \tilde{c}_{s+1-n}(\tilde{E}), \sum_{s=0}^{r} \tilde{c}_{r-s} \tilde{c}_{s}(\tilde{E}), \tag{6.23}
\]

Motivated by Theorems 6.2 and 6.3, the change of variables

\[
x \mapsto \tilde{x} = \int^{x} dx' \left( \frac{1}{\Psi_n(\mu(x'))} \frac{u - u_{xx'}}{u_{x'} - u} \right) \tag{6.25}
\]

and

\[
t_r \mapsto \tilde{t}_r = \int^{t_r} ds \left[ \frac{2}{\Psi_n(\mu(x,t_r))} \sum_{k=0}^{r\wedge n} \tilde{d}_{r,k}(\tilde{E}) \Psi_k(\mu(x,t_r)) \right] - 2 \sum_{l=1}^{n} \tilde{d}_{r,n-1-l}(\tilde{E}) \xi(l) \tag{6.26}
\]

linearizes the Abel map \( \Delta_{Q_0}(D_{\tilde{\mu}(\tilde{x},\tilde{t}_r)}), \tilde{\mu}_j(\tilde{x},\tilde{t}_r) = \mu_j(x,t_r), \) \( j = 1,\ldots,n. \)

The intricate relation between the variables \((x,t_r)\) and \((\tilde{x},\tilde{t}_r)\) is detailedly studied in Theorem 6.4.

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Next, we shall provide an explicit representation of \( \phi \) and \( u \) in terms of the Riemann theta function associated with \( \mathcal{K}_n \) under the assumption of the affine part of \( \mathcal{K}_n \) being nonsingular. We still use \( n \in \mathbb{N} \) for the remainder of this argument to avoid the trivial case \( n = 0 \). By (4.26)-(4.35), one of the principal results reads as follows.

**Theorem 6.4** Suppose that the curve \( \mathcal{K}_n \) is nonsingular, (2.2), (5.3), and (5.4) hold on \( \Omega \). Let \( P = (z,y) \in \mathcal{K}_n \setminus \{P_\infty, P_0\} \), \((x,t),(x_0,t_0)\) \( \in \Omega \), and \( \mathcal{D}_{\hat{\mu}(x,t)} \) or \( \mathcal{D}_{\hat{\mu}(x,t)} \) is nonspecial for \((x,t) \in \Omega \), where \( \Omega \subseteq \mathbb{C}^2 \) is open and connected. Then, \( \phi \) and \( u \) have the following representations

\[
\begin{align*}
\phi(P,x,t_r) &= i \frac{\theta(z(P,\hat{\mu}(x,t_r)))\theta(z(P_0,\hat{\mu}(x,t_r)))}{\theta(z(P_0,\hat{\mu}(x,t_r)))\theta(z(P,\hat{\mu}(x,t_r)))} \exp\left(d_0 - \int_{Q_0}^{P_\infty} \omega_P^{(3)} \right), \\
u(x,t_r) &= \left( \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n} U_j \partial_{\omega_j} \ln \left( \frac{\theta(z(P_\infty,\hat{\mu}(x,t_r)))}{\theta(z(P,\hat{\mu}(x,t_r)))} \right) \right) + 1.
\end{align*}
\]

Moreover, let \( \mu_j, j = 1, \ldots, n, \) be non-vanishing on \( \Omega \). Then, we have the following constraint

\[
\begin{align*}
\int_{x_0}^{x} dx \frac{u'' - u_{x'}x'}{u_{x'} - u} &+ 2(t_r - t_0) \sum_{s=0}^{r} \delta_{r-s} \delta_s (\bar{E}) \\
&= \left( - \int_{x_0}^{x} dx' \frac{u'' - u_{x'}x'}{u_{x'} - u} \prod_{k=1}^{n} \mu_k(x',t) - 2 \sum_{k=0}^{r^n} \tilde{d}_{r,k}(\bar{E}) \int_{t_0}^{t_r} \Psi_k(\mu(x_0,s)) \Psi_n(\mu(x_0,s)) ds \right) \\
&+ \sum_{j=1}^{n} \left( \int_{a_j} \tilde{\omega}_{P_\infty P_0}^{(3)} \right) c_j(1) \\
&+ 2(t_r - t_0, \bar{E}) \sum_{l=1}^{n} \tilde{d}_{r,n+1-l}(\bar{E}) \sum_{j=1}^{n} \left( \int_{a_j} \tilde{\omega}_{P_\infty P_0}^{(3)} \right) c_j(l) \\
&+ \ln \left( \frac{\theta(z(P_\infty,\hat{\mu}(x,t_r)))\theta(z(P,\hat{\mu}(x,t_0)))}{\theta(z(P_0,\hat{\mu}(x,t_r)))\theta(z(P,\hat{\mu}(x,t_0)))} \right),
\end{align*}
\]

\((x,t_r),(x_0,t_0) \in \Omega\)
with
\[
\dot{\hat{\mathcal{V}}} \bigl( P_\infty, \hat{\mu}(x, t_r) \bigr) = \tilde{\hat{\mathcal{V}}} Q_0 - \hat{\Delta}_Q (P_\infty) + \hat{\Delta}_Q (\mathcal{D}_{\hat{\mu}(x, t_r)}) \\
= \tilde{\hat{\mathcal{V}}} Q_0 - \hat{\Delta}_Q (P_\infty) + \hat{\Delta}_Q (\mathcal{D}_{\hat{\mu}(x_0, t_r)}) - \left( \int_{x_0}^x u - u_{x'} \, dx' \right) \mathcal{C}(1)
\]
(6.30)

and
\[
\dot{\hat{\mathcal{V}}} \bigl( P_0, \hat{\mu}(x, t_r) \bigr) = \tilde{\hat{\mathcal{V}}} Q_0 - \hat{\Delta}_Q (P_0) + \hat{\Delta}_Q (\mathcal{D}_{\hat{\mu}(x, t_r)}) \\
= \tilde{\hat{\mathcal{V}}} Q_0 - \hat{\Delta}_Q (P_0) + \hat{\Delta}_Q (\mathcal{D}_{\hat{\mu}(x_0, t_r)}) - \left( \int_{x_0}^x u - u_{x'} \, dx' \right) \mathcal{C}(1)
\]
(6.32)

\[
\dot{\hat{\mathcal{V}}} \bigl( P_\infty, \hat{\mu}(x, t_r) \bigr) = \tilde{\hat{\mathcal{V}}} Q_0 - \hat{\Delta}_Q (P_\infty) + \hat{\Delta}_Q (\mathcal{D}_{\hat{\mu}(x, t_r)}) \\
= \tilde{\hat{\mathcal{V}}} Q_0 - \hat{\Delta}_Q (P_0) + \hat{\Delta}_Q (\mathcal{D}_{\hat{\mu}(x_0, t_r)}) - \left( \int_{x_0}^x u - u_{x'} \, dx' \right) \mathcal{C}(1)
\]
(6.33)

**Proof.** Let us first assume that \( \mu_j, j = 1, \ldots, n \), are distinct and non-vanishing on \( \hat{\Omega} \) and \( F_{\hat{\mu}_j} \neq 0 \) on \( \hat{\Omega}, j = 1, \ldots, n \), where \( \hat{\Omega} \subseteq \Omega \). Then, the representation (6.27) for \( \phi \) on \( \hat{\Omega} \) follows by combining (6.1) with (6.2). The representation (6.28) for \( u \) on \( \hat{\Omega} \) follows by the trace formulas (5.70), (5.71) and (F.89) [12]. In fact, since the proofs of (6.27) and (6.28) are identical to the corresponding stationary results in Theorem 4.5, which can be extended line by line to the time-dependent setting. Here we skip the details.

Let us now turn to the relation (6.29). We first consider the time variation part of (6.29). From (6.20), it is easy to see that
\[
\sum_{j=1}^n \int_{Q_0} \hat{\mu}_j(x_0, t_r) \ \hat{\omega}^{(3)}_{P_\infty P_0} - \sum_{j=1}^n \int_{Q_0} \hat{\mu}_j(x_0, t_0, r) \ \hat{\omega}^{(3)}_{P_\infty P_0}
\]
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On the other hand, combining (6.10) with (F.88) yields

\[ \int_{t_0}^{t_r} \partial_s \left( \sum_{j=1}^{n} \int_{Q_0} \tilde{\omega}_{P_\infty P_0}^{(3)} \right) ds \]

\[ = \int_{t_0}^{t_r} \left( 2 \sum_{s=0}^{r} \tilde{c}_{r-s} \tilde{c}_s \right) ds = 2(t_r - t_{0,r}) \sum_{s=0}^{r} \tilde{c}_{r-s} \tilde{c}_s. \quad \text{(6.34)} \]

The space variation part of (6.29) has been given in the stationary case in

\[ \sum_{j=1}^{n} \int_{Q_0} \tilde{\omega}_{P_\infty P_0}^{(3)} \left( \sum_{j=1}^{n} \int \hat{\omega}_{P_\infty P_0}^{(3)} \right) \]

\[ + \ln \left( \frac{\theta(z(P_\infty, \hat{\mu}(x_0, t_r)))}{\theta(z(P_0, \hat{\mu}(x_0, t_r)))} \right) \]

\[ = \sum_{j=1}^{n} \int_{a_j}^{t_r} \partial_s \left( \sum_{k=1}^{n} \int_{Q_0} \hat{\omega}_{P_\infty P_0}^{(3)} \right) ds \]

\[ + \ln \left( \frac{\theta(z(P_\infty, \hat{\mu}(x_0, t_r)))}{\theta(z(P_0, \hat{\mu}(x_0, t_r)))} \right) \]

\[ \sum_{j=1}^{n} \int_{a_j}^{t_r} \partial_s \left( \sum_{k=1}^{n} \int_{Q_0} \hat{\omega}_{P_\infty P_0}^{(3)} \right) ds \]

\[ = \sum_{j=1}^{n} \int_{a_j}^{t_r} \hat{\omega}_{P_\infty P_0}^{(3)} \right) \int_{t_0}^{t_r} \left( - \frac{2\Psi_k(\mu(x_0, s))}{\Psi_n(\mu(x_0, s))} \right) \sum_{k=0}^{r/n} \tilde{d}_{r,k}(\vec{E})c_j(1) \]

\[ + 2 \sum_{l=1}^{n} \tilde{d}_{r,n+1-l}(\vec{E})c_j(l) ) ds \]

\[ + \ln \left( \frac{\theta(z(P_\infty, \hat{\mu}(x_0, t_r)))}{\theta(z(P_0, \hat{\mu}(x_0, t_r)))} \right) \]

\[ = -2 \sum_{j=1}^{n} \int_{a_j}^{t_r} \hat{\omega}_{P_\infty P_0}^{(3)} \right) \int_{t_0}^{t_r} \frac{\Psi_k(\mu(x_0, s))}{\Psi_n(\mu(x_0, s))} ds \]

\[ + 2(t_r - t_{0,r}) \sum_{l=1}^{n} \tilde{d}_{r,n+1-l}(\vec{E}) \sum_{j=1}^{r/n} \left( \int_{a_j}^{t_r} \hat{\omega}_{P_\infty P_0}^{(3)} \right) c_j(l) \]

\[ + \ln \left( \frac{\theta(z(P_\infty, \hat{\mu}(x_0, t_r)))}{\theta(z(P_0, \hat{\mu}(x_0, t_r)))} \right) \theta(z(P_\infty, \hat{\mu}(x_0, t_r))) \theta(z(P_0, \hat{\mu}(x_0, t_r))). \quad \text{(6.35)} \]
(4.49) and (4.50), that is,
\[ \int_{x_0}^{x} dx' \frac{u - u_{x'x'}}{u_{x'} - u} = - \int_{x_0}^{x} \left( \frac{1}{\prod_{k=1}^{n} \mu_k(x', t_r)} \right) dx' \]
\[ \times \sum_{j=1}^{n} \left( \int_{a_j}^{(3)} \omega_{x,t_r}^{(3)} \right) c_j(1) \]
\[ + \ln \left( \theta(z(P_\infty, \tilde{\mu}(x, t_r))) \theta(z(P_0, \tilde{\mu}(x_0, t_r))) \right) \cdot (6.36) \]

Hence, combining all of these three (6.34), (6.35) and (6.36) leads to (6.29). Equations (6.30)-(6.33) are valid from (6.9) and (6.10). The extension of all results from \( x \in \tilde{\Omega} \) to \( x \in \Omega \) simply follows by the continuity of \( \alpha_{Q_0} \) and the hypothesis of \( D_{\tilde{\mu}(x,t_r)} \) being nonspecial for \( x \in \Omega \). □

**Remark 6.5** A closer look at Theorem 6.4 shows that (6.30)-(6.33) equal to

\[ \hat{\tilde{\mu}}(P_\infty, \tilde{\mu}(x, t_r)) = \hat{\tilde{\mu}}_{Q_0} - \hat{\tilde{\mu}}_{Q_0}(P_\infty) + \hat{\tilde{\mu}}_{Q_0}(D_{\tilde{\mu}(x,t_r)}) \]

\[ = \hat{\tilde{\mu}}_{Q_0} - \hat{\tilde{\mu}}_{Q_0}(P_\infty) + \hat{\tilde{\mu}}_{Q_0}(D_{\tilde{\mu}(x_0,t_r)}) - \hat{\tilde{\mu}}(1)(\tilde{x} - \tilde{x}_0) \]

\[ = \hat{\tilde{\mu}}_{Q_0} - \hat{\tilde{\mu}}_{Q_0}(P_\infty) + \hat{\tilde{\mu}}_{Q_0}(D_{\tilde{\mu}(x_0,t_r)}) - \hat{\tilde{\mu}}(1)(\tilde{t}_r - \tilde{t}_0,r), \]

and

\[ \hat{\tilde{\mu}}(P_0, \tilde{\mu}(x, t_r)) = \hat{\tilde{\mu}}_{Q_0} - \hat{\tilde{\mu}}_{Q_0}(P_0) + \hat{\tilde{\mu}}_{Q_0}(D_{\tilde{\mu}(x,t_r)}) \]

\[ = \hat{\tilde{\mu}}_{Q_0} - \hat{\tilde{\mu}}_{Q_0}(P_0) + \hat{\tilde{\mu}}_{Q_0}(D_{\tilde{\mu}(x_0,t_r)}) - \hat{\tilde{\mu}}(1)(\tilde{x} - \tilde{x}_0) \]

\[ = \hat{\tilde{\mu}}_{Q_0} - \hat{\tilde{\mu}}_{Q_0}(P_0) + \hat{\tilde{\mu}}_{Q_0}(D_{\tilde{\mu}(x_0,t_r)}) - \hat{\tilde{\mu}}(1)(\tilde{t}_r - \tilde{t}_0,r), \]

based on the changing of variables \( x \mapsto \tilde{x} \) and \( t_r \mapsto \tilde{t}_r \) in (6.25) and (6.26). Hence, the Abel map linearizes the divisor \( D_{\tilde{\mu}(x,t_r)} \) on \( \Omega \) with respect to \( \tilde{x}, \tilde{t}_r \). This fact reveals that the Abel map does not effect the linearization of the divisor \( D_{\tilde{\mu}(x,t_r)} \) in the time-dependent MCH case.

**Remark 6.6** The Abel map linearizes the divisor \( D_{\tilde{\mu}(x,t_r)} \) on \( \Omega \) with respect to \( \tilde{x}, \tilde{t}_r \), and the change of variables is given by (4.51) and

\[ t_r \mapsto \tilde{t}_r = \int_{t_0}^{t_r} ds \left( \frac{2}{\Psi_n(\nu(x, t_r))} \sum_{k=0}^{n} \tilde{\mu}_{r,k}(\tilde{E}) \Psi_k(\nu(x, t_r)) \right) \]

\[ - 2 \sum_{l=1}^{n} \tilde{d}_{r,n+1-l}(\tilde{E}) \frac{c(l)}{\theta(1)} \right). \]
Remark 6.7 Remark 4.8 is applicable to the present time-dependent context. Moreover, in order to obtain the theta function representation of \( \psi_j \), \( j = 1, 2, \ldots \), one can write \( \bar{F}_r \) in terms of \( \Psi_k(\mu) \) in analogy to the stationary case studied in Remark 4.9. Here we skip the corresponding details.

In analogy to Example 4.10, the special case \( n = 0 \) is excluded in Theorem 6.4. Let us summarize it as follows. For simplicity, we just consider the elementary case \( n = 0, r = 0 \).

Example 6.8 Suppose \( n = 0, r = 0 \), \( P = (z, y) \in K_0 \setminus \{P_\infty, P_0\} \), and let \( (x, t_r), (x_0, t_{0r}) \in \mathbb{C}^2 \). Then, we have

\[
K_0 : F_0(z, y) = y^2 - R_1(z) = y^2 - (z - \bar{E}_1) = 0 \quad \bar{E}_1 \in \mathbb{C},
\]

\[
u(x, t_0) = \bar{E}_1^{1/2},
\]

\[
\text{MCH}_0(u) = u_{t_0} - u_{xxt_0} + 2u_x - 2u_{xx} = 0,
\]

\[
F_0(z, x) = \bar{F}_0(z, x) = f_0 = u_x - u,
\]

\[
G_0(z, x) = \bar{G}_0(z, x) = g_0 = -1,
\]

\[
H_0(z, x) = \bar{H}_0(z, x) = h_0 = u_x + u,
\]

\[
\phi(P, x, t_0) = \frac{z^{1/2}y - z}{u_x - u} = \frac{u_x + u}{z^{-1/2}y + 1},
\] (6.40)

\[
\psi_1(P, x, x_0, t_0, t_0) = \exp\left( \int_{x_0}^{x} \left( \frac{1}{2} \frac{u - u_{x'x'}}{u_{xx'} - u} \frac{y - \sqrt{z}}{\sqrt{z}} \right) dx' \right)
- \frac{1}{2} (x - x_0) + \int_{t_{00}}^{t} \frac{y}{\sqrt{z}} ds,
\]

\[
\psi_2(P, x, x_0, t_0, t_0) = \frac{y - \sqrt{z}}{u_x - u} \exp\left( \int_{x_0}^{x} \left( \frac{1}{2} \frac{u - u_{x'x'}}{u_{xx'} - u} \frac{y - \sqrt{z}}{\sqrt{z}} \right) dx' \right)
- \frac{1}{2} (x - x_0) + \int_{t_{00}}^{t} \frac{y}{\sqrt{z}} ds.
\]

The general solution of \( \text{MCH}_0(u) = u_{t_0} - u_{xxt_0} + 2u_x - 2u_{xx} = 0 \) is given by

\[
u(x, t_0) = a_1 e^{x + a_2 t_0} + \bar{E}_1^{1/2}, \quad a_1, a_2 \in \mathbb{C}.
\] (6.41)

But, according to the condition \( \partial^k_x u(x, t) \in L^\infty(\mathbb{C}), k \in \mathbb{N}_0, t \in \mathbb{C} \) in (22), we may conclude \( a_1 = 0 \), and therefore derive \( u(x, t_0) = \bar{E}_1^{1/2} \), which equals to the expression of \( u(x, t_0) \) in (6.40) obtained from trace formula (5.70) or (5.71) in the special case \( n = 0 \).
Let us end this section by providing another principle result about algebro-geometric initial value problem of the MCH hierarchy. We will show that the solvability of the Dubrovin equations (5.60) and (5.61) on $\Omega_\mu \subseteq \mathbb{C}^2$ in fact implies (5.3) and (5.4) on $\Omega_\mu$. As pointed out in Remark 4.13, this amounts to solving the time-dependent algebro-geometric initial value problem (5.1) and (5.2) on $\Omega_\mu$. Recalling definition of $\tilde{F}_r(\mu_j)$ introduced in (6.6), then we may present the following result.

**Theorem 6.9** Assume that (2.22) holds and $\{\tilde{\mu}_j\}_{j=1,\ldots,n}$ satisfies the Dubrovin equations (5.60) and (5.61) on $\Omega_\mu$ and remain distinct and nonzero for $(x,t_r) \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}^2$ is open and connected. If $\tilde{F}_r(\mu_j)$ in (5.61) are expressed in terms of $\mu_k$, $k = 1,\ldots,n$ by (6.6), then $u \in C^\infty(\Omega_\mu)$ satisfying

$$-rac{1}{2}(u-u_{xx})(u^2-u_x^2) + \frac{1}{2}(u-u_{xx}) \sum_{m=1}^{2n+1} E_m = (u-u_x) \sum_{j=1}^{n} \mu_j$$

will also satisfy the $r$th MCH equation (5.1), that is,

$$\text{MCH}_r(u) = 0 \quad \text{on } \Omega_\mu,$$

with initial values satisfying the $n$th stationary MCH equation (5.2).

**Proof.** Given the solutions $\tilde{\mu}_j = (\mu_j, y(\tilde{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1,\ldots,n$ of (5.60) and (5.61), we introduce polynomials $F_n, G_n,$ and $H_n$ on $\Omega_\mu$, which are exactly the same as in Theorem 4.11 in the stationary case

$$F_n(z) = (u_x - u) \prod_{j=1}^{n}(z - \mu_j),$$

$$(u-u_{xx})G_n(z) = F_n(z) + F_{n,x}(z),$$

$$H_n(z) = (u_x + u) \prod_{j=1}^{n}(z - \nu_j),$$

$$zG_{n,x}(z) = -\frac{1}{2}(u-u_{xx})H_n(z) - \frac{1}{2}(u-u_{xx})F_n(z),$$

$$H_{n,x}(z) = (u-u_{xx})G_n(z) + H_n(z),$$

$$R_{2n+1}(z) = zG_n^2(z) + F_n(z)H_n(z),$$

where $t_r$ is treated as a parameter. Hence, let us focus on the proof of (5.1).

Let us denote the polynomial $G_r$ of degree $r$ by

$$\frac{1}{2}(u_{t_r} - u_{xx,t_r}) = \tilde{F}_{r,x}(z) + \tilde{F}_r(z) - (u-u_{xx})\tilde{G}_r(z) \quad \text{on } \mathbb{C} \times \Omega_\mu.$$
Next, we want to establish
\[ F_{n,t_r}(z) = 2(G_n(z)\tilde{F}_r(z) - F_n(z)\tilde{G}_r(z)) \] on \( \mathbb{C} \times \Omega_{\mu} \), \hspace{1cm} (6.51)

where \( \tilde{F}_r(z) \) is defined on \( \mathbb{C} \times \Omega_{\mu} \) by
\[ \tilde{F}_r(z) = \sum_{s=0}^r \tilde{c}_{r-s} \tilde{F}_s(z), \quad \tilde{c}_0 = 1 \] \hspace{1cm} (6.52)

with integration constants \( \{\tilde{c}_1, \ldots, \tilde{c}_r\} \subset \mathbb{C} \) and
\[ \tilde{F}_s(z) = (u_x - u) \sum_{\rho=0}^s \hat{c}_{\rho} (\tilde{E}) \sum_{l=0}^{s-\rho} \Psi_{s-l-\rho}(\mu) z^l. \] \hspace{1cm} (6.53)

To prove (6.51), let
\[ \tilde{F}_n(z) = (u - u_{xx})^{-1} F_n(z), \] \hspace{1cm} (6.54)
\[ \tilde{F}_r(z) = (u - u_{xx})^{-1} \tilde{F}_r(z), \] \hspace{1cm} (6.55)
on \( \mathbb{C} \times \Omega_{\mu} \). A direct calculation shows that (6.51) is equivalent to
\[ F_{n,t_r}(z) = 2\tilde{F}_r(z)\tilde{F}_n(z) - 2\tilde{F}_n(z)\tilde{G}_r(z) \] \hspace{1cm} (6.56)

which is similar to that in the AKNS context \footnote{[12]}. So, (6.56) holds due to (F.112) \footnote{[12]}. This in turn proves (6.51).

Next, we denote the polynomial \( \tilde{H}_r \) of degree \( r \) by
\[ z\tilde{G}_{r,x}(z) = -\frac{1}{2}(u - u_{xx})\tilde{H}_r(z) - \frac{1}{2}(u - u_{xx})\tilde{F}_r(z) \] on \( \mathbb{C} \times \Omega_{\mu} \). \hspace{1cm} (6.57)

Then, differentiating on both sides of (6.45) with respect to \( t_r \) and inserting (6.51) and (6.50), we have
\[ F_{n,xt_r} = -2(G_n\tilde{F}_r - F_n\tilde{G}_r) + (u - u_{xx})t_r G_n + (u - u_{xx})G_{n, t_r} \] \hspace{1cm} (6.58)
\[ = 2F_n\tilde{G}_r + 2\tilde{F}_r G_n - 2(u - u_{xx})\tilde{G}_r G_n + (u - u_{xx})G_{n, t_r}. \]

On the other hand, taking the derivative on both sides of (6.51) with respect to \( x \), and using (6.45), (6.47) and (6.57), we obtain
\[ F_{n,tx}(z) = z^{-1}(u - u_{xx})f_n H_r - z^{-1}(u - u_{xx})H_n \tilde{F}_r. \] \hspace{1cm} (6.59)
Consequently, combining (6.58) with (6.59) we conclude
\[ zG_{n,t_r}(z) = F_n(z)\tilde{H}_r(z) - H_n(z)\tilde{F}_r(z) \text{ on } \mathbb{C} \times \Omega_\mu. \] (6.60)

Next, differentiating both sides of (6.49) with respect to \( t_r \), and inserting the expressions (6.51) and (6.60) for \( F_{n,t_r} \) and \( G_{n,t_r} \), we have
\[ H_{n,t_r}(z) = 2(H_n(z)\tilde{G}_r(z) - G_n(z)\tilde{H}_r(z)) \text{ on } \mathbb{C} \times \Omega_\mu. \] (6.61)

Finally, taking the derivative on both sides of (6.60) with respect to \( x \), and inserting (6.45), (6.48) and (6.50) for \( F_{n,x} \), \( H_{n,x} \) and \( \tilde{F}_r \), we arrive at
\[ zG_{n,t_x} = \left( u - u_{xx} \right) G_n\tilde{H}_r - F_n\tilde{F}_r + F_n\tilde{H}_{r,x} - \left( u - u_{xx} \right) G_n\tilde{F}_r - \frac{1}{2}(u - u_{xx})_t_r H_n - (u - u_{xx})H_n\tilde{G}_r. \] (6.62)

On the other hand, differentiating both sides of (6.47) with respect to \( t_r \), and using (6.51) and (6.61) for \( F_{n,t_r} \) and \( H_{n,t_r} \), we have
\[ zG_{n,xt_r} = -\frac{1}{2}(u - u_{xx})_t_r H_n - \frac{1}{2}(u - u_{xx})_t_r(2H_n\tilde{G}_r - 2G_n\tilde{H}_r) - \frac{1}{2}(u - u_{xx})_t_r F_n - \frac{1}{2}(u - u_{xx})(2G_n\tilde{F}_r - 2F_n\tilde{G}_r). \] (6.63)

Therefore, combining (6.62) with (6.63) yields
\[ -F_n\tilde{H}_r + F_n\tilde{H}_{r,x} = -\frac{1}{2}(u - u_{xx})_t_r F_n + (u - u_{xx})F_n\tilde{G}_r, \] (6.64)
which implies
\[ -\frac{1}{2}(u - u_{xx})_t_r = \tilde{H}_{r,x}(z) - \tilde{H}_r(z) - (u - u_{xx})\tilde{G}_r(z) \text{ on } \mathbb{C} \times \Omega_\mu. \] (6.65)

So, we proved (5.12)-(5.17) and (5.38)-(5.40) on \( \mathbb{C} \times \Omega_\mu \) and thus conclude that \( u \) satisfies the \( r \)th MCH equation (5.1) with initial values satisfying the \( n \)th stationary MCH equation (5.2) on \( \mathbb{C} \times \Omega_\mu \).

**Remark 6.10** The result in Theorem 6.9 is presented in terms of \( u \) and \( \{\mu_j\}_{j=1,\ldots,n} \), but of course one can provide the analogous result in terms of \( u \) and \( \{\nu_j\}_{j=1,\ldots,n} \).

The analog of Remark 4.13 directly extends to the current time-dependent MCH hierarchy.
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