Unified treatment and classification of superintegrable systems with integrals quadratic in momenta on a two dimensional manifold *

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Abstract

In this paper we prove that the two dimensional superintegrable systems with quadratic integrals of motion on a manifold can be classified by using the Poisson algebra of the integrals of motion. There are six general fundamental classes of superintegrable systems. Analytic formulas for the involved integrals are calculated in all the cases. All the known superintegrable systems are classified as special cases of these six general classes.

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I Introduction

In classical mechanics, an integrable system on a manifold of $N$ dimensions, is a system which has $N$ integrals of motion in evolution. Superintegrable (or maximally integrable) system is a system possessing the maximum number of constants of motion, i.e. $2N - 1$ integrals of motion.

The simplest case is the two dimensional superintegrable problems with integrals of motion, which are linear and quadratic functions of momenta. The investigation of such superintegrable systems on a two dimensional manifold is a quite old problem, dated on 19th century. Initially the problem was formulated as a geometry problem. The challenge was to find two dimensional manifolds whose the geodesics are curves which possess additional integrals than the free Hamiltonian. This problem was studied in the four volume treatise of Darboux Leçons sur la Théorie Générale des Surfaces [1]. The main result of this study is that, there are five classes of general forms of metrics, whose the geodesics have three integrals of motion (the Hamiltonian and two additional functionally independent integrals). These metrics are called ”formes essentielles” and they depend on four parameters. All the metrics having more than two integrals of motion can be obtained as partial cases of these ”formes essentielles” by choosing appropriate values of the four parameters. The five classes of metrics are tabulated in ”Tableau VII” by Koenigs[1] vol IV p.385

In Classical Mechanics language, Darboux and Koenigs results can be translated as searching manifolds, where the free Hamiltonian accepts quadratic integrals of motion. The evolution of this problem is to find superintegrable systems, whose Hamiltonian is the free Hamiltonian plus a potential and these systems possess additional quadratic integrals of motion.

The simplest integrable and superintegrable systems are the systems defined on the real plane. A comprehensive review of the real two-dimensional integrable classical systems on the plane is given in Ref [2, 3]. The complex superintegrable systems with quadratic moments on a flat space were recently catalogued by Kalnins, Miller, Pogosyan et al [4, 5, 6, 7]. The flat space is a two dimensional manifold with curvature zero. The Drach potentials are also systems defined on a manifold with curvature zero. The Drach systems with quadratic integrals of motion were investigated by Rañada [8]. The superintegrable systems on the hyperbolic plane were studied in [9, 10]. These systems were studied separately, while they are connected by obvious coordinate transformations. Therefore they are naturally connected, i.e.
there is a common classification scheme of all these systems \[9, 10\].

The case of non flat space is under current intensive investigation. The superintegrable systems on the sphere were studied in \[8, 9, 11\] and they were classified in \[6\]. The sphere is a special case of manifold with constant curvature. In refs \[9, 10\] the superintegrable systems on the sphere and on the hyperbolic plane were studied using an unified formulation.

In the case of manifolds of non-constant curvature, the known examples of superintegrable systems are those which are defined on manifolds which are surfaces of revolution. The corresponding problem of the geodesics with three quadratic integrals of motion on surfaces of revolution was treated by Koenigs in [1] n° 5, vol IV, p. 377]. Recently Kalnins and collaborators classified the superintegrable systems on a surface of revolution \[12, 13\] using the manifolds which were provided by Koenigs.

In two recent papers Kalnins, Kress and Miller \[14, 15\] give a comprehensive study of the two-dimensional superintegrable systems. In \[15\], they prove that the general Koenigs essential forms of metrics correspond to the most general forms of superintegrable systems. Also they have shown that every two-dimensional superintegrable system is Stäckel equivalent to a two-dimensional non degenerate superintegrable system on a constant curvature space.

Kress \[16\] in collaboration with Kalnins and Miller studied the Stäckel equivalence classes of the superintegrable systems on the spaces of constant curvature, and they have shown that there are six equivalence classes. The general classes of Koenigs classification given by the Table VII in \[1\] are five. That means that there is a sixth class which should be added in the Koenigs classification scheme. In this paper we investigate this sixth class, which completes the Koenigs classification. This class is the nondegenerate superintegrable system generated by the case VI of Koenigs.

An interesting question is, whether there could be a general classification scheme of superintegrable systems with quadratic integrals of motion, which contains all the equivalence classes of superintegrable systems on a manifold with constant curvature and the general classes of manifolds, which were introduced by Koenigs. The classification schemes are based on the Darboux relations derived for the invariants, which are defined on a specific manifold. In this paper we propose a classification scheme based on the properties of the Poisson algebra of the integrals of motion. Then we show that there is indeed such a classification scheme, which determines the supporting manifold metric. We must notice that the Kress \[16\] equivalence classes are derived by
classifying the Poisson algebra of integrals of motion. The proposed classes of superintegrable systems in this paper correspond to the equivalent classes studied in ref [16]. Analytic formulas for the metrics of the permitted manifolds, the potentials and the integrals of motion are calculated.

This paper is organized as follows: In section II the general form of the integrable two dimensional system with one quadratic integral of motion is derived. The results of this section correspond to the Darboux treatise paragraphs [1] n° 593, vol. III, p.30], [1] n° 593, vol. III, p.31], but we give a brief modern derivation of the formulas including the potentials in our discussion. These formulas will be used in the following sections. The carrying manifold is a Liouville or a Lie surface. Therefore there are two classes of integrable systems. In a specific coordinate system, which is called Liouville (or Lie) system, the analytic expressions of the potential and the integrals of motion are given and the action is calculated. In Section III the Poisson algebra of the integrals of a superintegrable two dimensional system is discussed. This algebra is a quadratic algebra, the coefficients of the quadratic terms are characteristic of the carrying manifold. In Section IV we prove that the coefficients of the Poisson algebra impose the classification of the superintegrable systems with two quadratic integrals in six fundamental classes. The method of analytic calculation of form of the permitted carrying manifolds, the potentials and the integrals of motion are discussed. In this section we prove that the general form of the superintegrable potential can be written as a fraction \( V = \frac{w(x, y)}{g(x, y)} \) and the two functions \( w(x, y) \) and the metric \( g(x, y) \) are two solutions of the same partial differential equation. The existence of the Poisson algebra was assumed as obvious by several authors [4]–[7], [17]–[29]. In Appendix A we give a proof of the existence of the Poisson algebra, for two dimensional superintegrable systems with quadratic integrals of motion. In Section V the analytic formulas of the manifolds and integrals are given for all the six fundamental classes of superintegrable systems. From these analytic formulas we can show that there are new superintegrable systems, because they are defined on manifolds which have not constant curvature and are not surfaces of revolution. In Section VI the superintegrable systems corresponding to the Koenigs essential forms of Table VII are given. In Section VII the superintegrable systems on a surface of revolution are studied. We find that there is a new system which was not revealed by the other classification schemes. In Section VIII the superintegrable systems on a manifold with curvature zero are studied and in Section IX the systems on a manifold with constant curvature are listed. In Section X the systems with a linear
and a quadratic integral of motion are discussed. Finally, in Section X the results of the paper are summarized.

II Integrable systems on a two dimensional manifold

Let us consider an integrable system defined on a two dimensional manifold with metric:

\[ ds^2 = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2 \]

There is a conformal coordinate system where the metric can be written as:

\[ ds^2 = g(x, y)dxdy \]  

(1)

The passage from the original coordinate system \((u, v)\) to the conformal one \((x, y)\) can be realized by using the Beltrami partial differential equation. We must notice that the choice of the conformal coordinate system is not unique, i.e. there are several conformal coordinate systems for a given metric, these systems are conformally equivalent.

In a conformal coordinate system the general form of the Hamiltonian is:

\[ H = \frac{p_x p_y}{g(x, y)} + V(x, y) \]  

(2)

where the Hamiltonian is a quadratic form of the momenta.

Let us consider an integral of motion which is quadratic in momenta. The most general form can be written as:

\[ I = A(x, y)p_x^2 + B(x, y)p_y^2 - 2p_x p_y \frac{\beta(x, y)}{g(x, y)} + Q(x, y) \]  

(3)

By definition the Poisson bracket between the Hamiltonian and the integral of motion is zero:

\[ \{ I, H \}_P = \frac{\partial I}{\partial x} \frac{\partial H}{\partial p_x} - \frac{\partial I}{\partial p_x} \frac{\partial H}{\partial x} + \frac{\partial I}{\partial y} \frac{\partial H}{\partial p_y} - \frac{\partial I}{\partial p_y} \frac{\partial H}{\partial y} = 0 \]  

(4)

The above equality implies restrictions on the functions involved in equations (2) and (3). The left hand side of the above equation is an odd function of
cubic order in the momenta. The coefficients of \( p^3_x \) and \( p^3_y \) must be zero:

\[
\frac{\partial A}{\partial y} = 0 \quad \Rightarrow \quad A = A(x) \\
\frac{\partial B}{\partial x} = 0 \quad \Rightarrow \quad B = B(y)
\]

(5)

The coefficients of \( p^2_x p_y \) and \( p^2_y p_x \) in (4) must be indeed zero:

\[
\frac{\partial \beta}{\partial y} = A(x) \frac{\partial g}{\partial x} + \frac{g}{2} A'(x) = a(x) \frac{\partial (a(x)g(x, y))}{\partial x} \\
\frac{\partial \beta}{\partial x} = B(y) \frac{\partial g}{\partial y} + \frac{g}{2} B'(y) = b(y) \frac{\partial (b(y)g(x, y))}{\partial y}
\]

(6)

(7)

where

\[
A(x) = a^2(x), \quad B(y) = b^2(y)
\]

The partial \( x \)-derivative of the right hand side in (6) is equal to the \( y \)-derivative of the right hand side in equation (7), therefore:

\[
(A''(x) - B''(y))g(x, y) + 3A'(x) \frac{\partial g}{\partial x} - 3B'(y) \frac{\partial g}{\partial y} + 2A(x) \frac{\partial^2 g}{\partial x^2} - 2B(y) \frac{\partial^2 g}{\partial y^2} = 0
\]

(8)

or

\[
\frac{\partial}{\partial x} \left( a(x) \frac{\partial}{\partial x} (a(x)g(x, y)) \right) = \frac{\partial}{\partial y} \left( b(y) \frac{\partial}{\partial y} (b(y)g(x, y)) \right)
\]

(9)

The coefficients of \( p_x \) and \( p_y \) in (4) must be zero:

\[
\frac{\partial Q}{\partial y} = 2A(x) \frac{\partial V}{\partial x} + 2\beta(x, y) \frac{\partial V}{\partial y} \\
\frac{\partial Q}{\partial x} = 2B(y) \frac{\partial V}{\partial y} + 2\beta(x, y) \frac{\partial V}{\partial y}
\]

(10)

The above relations imply:

\[
g(x, y) \left( 2A(x) \frac{\partial^2 V}{\partial x^2} - 2B(y) \frac{\partial^2 V}{\partial y^2} + 3A'(x) \frac{\partial V}{\partial x} - 3B'(y) \frac{\partial V}{\partial y} \right) + \\
+ 4 \left( A(x) \frac{\partial g}{\partial x} \frac{\partial V}{\partial x} - B(y) \frac{\partial g}{\partial y} \frac{\partial V}{\partial y} \right) = 0
\]

(11)
At this point we must to distinguish two cases. In the first case $A(x)$ and $B(y)$ are both different from zero, whereas in the second case $B(y)$ is assumed to be zero.

**Class I:** $A(x)B(y) \neq 0$

Following the method given in Koenigs original paper, we can choose a new coordinate system

$$\xi = \int \frac{dx}{\sqrt{A(x)}} \quad \text{and} \quad \eta = \int \frac{dy}{\sqrt{B(y)}}$$

where the associated momenta are:

$$p_\xi = \sqrt{A(x)} p_x, \quad \text{and} \quad p_\eta = \sqrt{B(y)} p_y$$

In this case the metric is written

$$ds^2 = \hat{g}(\xi, \eta) d\xi d\eta, \quad \text{where} \quad \hat{g}(\xi, \eta) = g(x, y) \sqrt{A(x) B(y)}$$

In these new coordinate system $(\xi, \eta)$ all the above equations are considerably simplified. We can easily show that the formulas are the same by replacing $x \to \xi, \ y \to \eta$ and fixing $A(x) = 1$ and $B(y) = 1$. For simplicity reasons we omit the hat on the metric $\hat{g}(\xi, \eta) \to g(\xi, \eta)$. Equations (2) and (3) are written:

$$H = \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta)$$

$$I = p_\xi^2 + p_\eta^2 - 2 p_\xi p_\eta \frac{\beta(\xi, \eta)}{g(\xi, \eta)} + Q(\xi, \eta)$$

We call these specific coordinates $(\xi, \eta)$ **Liouville coordinates**. In Liouville coordinates the Hamiltonian $H$ and the integral $I$ are written as it was given in the above equation (12).

In the Liouville coordinates, equation (9) is considerably simplified:

$$\frac{\partial^2 g}{\partial \xi^2} - \frac{\partial^2 g}{\partial \eta^2} = 0$$

the general solution is:

$$g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta)$$

where $F(u)$ and $G(v)$ are arbitrary functions. The above metric characterizes a surface which is called Liouville surface in the geometry textbooks.
Therefore we have shown that a system is integrable in Class I only when the surface is a Liouville surface. The surfaces of constant curvature or the rotation surfaces are Liouville surfaces, but there are Liouville surfaces which have not constant curvature and they are not rotation surfaces.

The function $\beta(\xi, \eta)$ can be calculated from equations (6)

$$\frac{\partial \beta}{\partial \xi} = \frac{\partial g}{\partial \eta}, \quad \frac{\partial \beta}{\partial \eta} = \frac{\partial g}{\partial \xi}$$

then

$$\beta(\xi, \eta) = F(\xi + \eta) - G(\xi - \eta)$$  (13)

The potential $V(\xi, \eta)$ in Liouville coordinates is the solution of equation (11)

$$(F(\xi + \eta) + G(\xi - \eta)) \left( \frac{\partial^2 V}{\partial \xi^2} - \frac{\partial^2 V}{\partial \eta^2} \right) + 2F'(\xi + \eta) \left( \frac{\partial V}{\partial \xi} - \frac{\partial V}{\partial \eta} \right) + 2G'(\xi + \eta) \left( \frac{\partial V}{\partial \xi} + \frac{\partial V}{\partial \eta} \right) = 0$$

The general solution of the above equation in Liouville coordinates is

$$V(\xi, \eta) = \frac{f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$  (14)

The functions $f(u)$ and $g(v)$ are arbitrary functions. The function $Q(\xi, \eta)$ is determined from equations (10) and the solution is easily calculated:

$$Q(\xi, \eta) = 4 \frac{f(\xi + \eta) G(\xi - \eta) - g(\xi - \eta) F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$  (15)

Usually is more convenient the use of the coordinates $u, v$, which are defined by:

$$\xi = u + iv, \quad \eta = u - iv, \quad p_\xi = \frac{p_u - ip_v}{2}, \quad \text{and} \quad p_\eta = \frac{p_u + ip_v}{2}$$

The Hamiltonian $H$ and the integral can be written as:

$$H = \frac{p_u^2 + p_v^2 + 4(f(u) + g(v))}{4(F(u) + G(v))}$$

$$I = \frac{p_u^2 G(v) - p_v^2 F(u)}{F(u) + G(v)} + 4 \frac{f(u)G(v) - g(v)F(u)}{F(u) + G(v)}$$
The above formula has been investigated in a different context in [30]. In these coordinates the action \( S(u, v) \) satisfy the following equations:

\[
E = H(u, v, p_u, p_v), \quad J = I(u, v, p_u, p_v)
\]

\[
p_u = \frac{\partial S}{\partial u}, \quad p_v = \frac{\partial S}{\partial v}
\]

and we can find the action \( S(u, v) \) by separation of variables.

\[
S = -Et + \int \sqrt{4E F(u) + J - 4f(u)} \, du + \int \sqrt{4E G(v) - J - 4g(v)} \, dv
\]

**CLASS II:** \( B(y) = 0 \)

We can choose a new coordinate system

\[
\xi = \int \frac{dx}{\sqrt{A(x)}} \quad \text{and} \quad \eta = y
\]

the associated momenta are:

\[
p_\xi = \sqrt{A(x)} \, p_x, \quad \text{and} \quad p_\eta = p_y
\]

In this case the metric is written

\[
ds^2 = \tilde{g}(\xi, \eta) \, d\xi \, d\eta, \quad \text{where} \quad \tilde{g}(\xi, \eta) = g(x, y) \, \sqrt{A(x)}
\]

We can easily show that the formulas are the same by replacing \( x \to \xi, y \to \eta \) and fixing \( A(x) = 1 \) and \( B(y) = 0 \). For simplicity reasons we omit the hat on the metric \( \tilde{g}(\xi, \eta) \to g(\xi, \eta) \). Equations (2) and (3) are written:

\[
H = \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta)
\]

\[
I = p_\xi^2 - 2p_\xi p_\eta \frac{\beta(\xi, \eta)}{g(\xi, \eta)} + Q(\xi, \eta)
\]

We call these specific coordinates \((\xi, \eta)\) **Lie coordinates**. In Lie coordinates the Hamiltonian \( H \) and the integral \( I \) are written as it is given in the above equation (17).

In Lie coordinates equation (9) is written:

\[
\frac{\partial^2 g}{\partial \xi^2} = 0
\]
The general solution is:

\[ g(\xi, \eta) = F(\eta)\xi + G(\eta) \]

where \( F(\eta) \) and \( G(\eta) \) are arbitrary functions. The above metric characterizes a surface which will be called Lie surface. Therefore we have shown that a system is integrable in Case 2 only when the surface is a Lie surface.

Equations (6) imply:

\[ \frac{\partial \beta}{\partial \eta} = \frac{\partial g}{\partial \xi} \]

\[ \frac{\partial \beta}{\partial \xi} = 0 \]

\[ \Rightarrow \begin{cases} 
  g(\xi, \eta) = F(\eta)\xi + G(\eta) \\
  \beta(\xi, \eta) = \int F(\eta) \, d\eta 
\end{cases} \tag{18} \]

Equation (11) is written:

\[ (F(\eta)\xi + G(\eta)) \frac{\partial^2 V}{\partial \xi^2} + 2F(\eta) \frac{\partial V}{\partial \xi} = 0 \]

and the general solution of the above equation is:

\[ V(\xi, \eta) = \frac{f(\eta)\xi + g(\eta)}{F(\eta)\xi + G(\eta)} \tag{19} \]

The functions \( F(\eta), \ G(\eta), \ f(\eta) \) and \( g(\eta) \) are arbitrary functions. In this case the solution of the system of equations (10) is given by:

\[ Q(\xi, \eta) = -2\frac{f(\eta)\xi + g(\eta)\int F(\eta) \, d\eta}{F(\eta)\xi + G(\eta)} + 2\int f(\eta) \, d\eta \tag{20} \]

The action integral in this case can be easily calculated:

\[ S = \xi \sqrt{J - 2\left( \int f(\eta) \, d\eta - E \int F(\eta) \, d\eta \right) - \int d\eta \frac{g(\eta) - EG(\eta)}{\sqrt{J - 2\left( \int f(\eta) \, d\eta - E \int F(\eta) \, d\eta \right)}}} \]

where

\[ E = H(\xi, \eta, p_\xi, p_\eta), \quad J = I(\xi, \eta, p_\xi, p_\eta) \]

\[ p_\xi = \frac{\partial S}{\partial \xi}, \quad p_\eta = \frac{\partial S}{\partial \eta} \]

The above findings are summarized in the following Proposition.
Proposition 1 A Hamiltonian $H$ quadratic in momenta, which is defined on a two dimensional manifold possesses an integral of motion $I$ quadratic in momenta, only in two cases:

**Class I:**
The manifold is a Liouville surface, i.e. there is a coordinate system $(\xi, \eta)$ where the metric can be written:

$$ds^2 = g(\xi, \eta)d\xi d\eta \quad \text{and} \quad g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta)$$

$$H = \frac{p_\xi p_\eta}{F(\xi + \eta) + G(\xi - \eta)} + \frac{f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

and simultaneously the integral of motion is written

$$I = p_\xi^2 + p_\eta^2 - 2p_\xi p_\eta \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} + 4 \frac{f(\xi + \eta) G(\xi - \eta) - g(\xi - \eta) F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

**Class II:**
The manifold is a Lie surface, i.e. there is a coordinate system $(\xi, \eta)$ where the metric can be written:

$$ds^2 = g(\xi, \eta)d\xi d\eta \quad \text{and} \quad g(\xi, \eta) = F(\eta) \xi + G(\eta)$$

$$H = \frac{p_\xi p_\eta}{F(\eta) \xi + G(\eta)} + \frac{f(\eta) \xi + g(\eta)}{F(\eta) \xi + G(\eta)}$$

and simultaneously

$$I = p_\xi^2 - 2p_\xi p_\eta \frac{F(\eta) d\eta}{F(\eta) \xi + G(\eta)} - 2 \left( f(\eta) \xi + g(\eta) \right) \frac{F(\eta) d\eta}{F(\eta) \xi + G(\eta)} + 2 \int f(\eta) d\eta$$

The above specific choice of coordinate system $(\xi, \eta)$ will be called Lie coordinate system.

The integrable systems which belong in Class I and II are well known integrable systems see Ref. [12], the supporting manifold is a Liouville or a Lie surface.

We must notice that the Hamiltonian and the integral of an integrable system determine uniquely the Liouville (or Lie) coordinate system. Therefore the use of this privileged system is imposed by the notion of integrability. In the next sections we shall work in this special coordinate system, which is denoted exclusively by the coordinates $\xi$ and $\eta$. In Class I integrable systems the system is separable, while in Class II systems there is no separation of variables in general.
III Poisson algebra of superintegrable systems with two quadratic integrals of motion

If a system is superintegrable on a two dimensional manifold, that means that there are three functionally independent integrals of motion $H$, $A$ and $B$. In this section, we assume that these integrals of motion are quadratic functions of the momenta and there are no other integrals of motion, which are linear functions of the momenta. Regarding the Hamiltonian $H$ and the first integral $A$, we can choose the Liouville coordinate system and in this system:

$$H = \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta)$$

As we have shown in the previous section the system is integrable with a square integral of motion in two cases. The integral of motion $A$ in the Liouville coordinate system is written:

$$A = p_\xi^2 + b p_\eta^2 - 2p_\xi p_\eta \frac{\sigma(\xi, \eta)}{g(\xi, \eta)} + \Theta(\xi, \eta)$$

where

$$b = \begin{cases} 1 \text{ in Class I, where } g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta) & \text{(Liouville system)} \\ 0 \text{ in Class II, where } g(\xi, \eta) = F(\eta) \xi + G(\eta) & \text{(Lie system)} \end{cases}$$

The integral of motion $B$ is assumed to be indeed a quadratic function of momenta, thus the general form in Liouville coordinates is

$$B = A(\xi)p_\xi^2 + B(\eta)p_\eta^2 - 2p_\xi p_\eta \frac{\beta(\xi, \eta)}{g(\xi, \eta)} + Q(\xi, \eta)$$

By definition the following relations are satisfied:

$$\{H, A\}_P = \{H, B\}_P = 0$$  \hspace{1cm} (21)

From the integrals of motion $A$ and $B$, we can construct the integral of motion:

$$C = \{A, B\}_P$$  \hspace{1cm} (22)

The integral of motion $C$ is not a new independent integral of motion, which is a cubic function of the momenta. The integral $C$ is not functionally independent from the integrals $H$, $A$ and $B$ as it will be shown later. The fact
that, the integral $C$ is a cubic function of momenta, implies the impossibility of expressing $C$ as a polynomial function of the other integrals, which are quadratic functions of momenta. We shall prove that the square of this cubic polynomial is indeed a cubic combination of the integrals. Starting from the integral of motion $C$, we can construct the (non independent) integrals \{A, C\} and \{B, C\}. These integrals are quartic functions of the momenta, i.e. functions of fourth order. In Appendix [A] we show that these integrals can be expressed as quadratic combinations of the integrals $H$, $A$, and $B$. Therefore the following relations are valid:

\begin{align}
\{A, C\}_P &= \alpha A^2 + \beta B^2 + 2\gamma AB + \delta A + \epsilon B + \zeta \tag{23} \\
\{B, C\}_P &= aA^2 + bB^2 + 2cAB + dA + eB + z \tag{24}
\end{align}

By taking an appropriate rotation of the integrals $A$ and $B$ we can always consider the case $\beta = 0$. 

The existence of the Poisson algebra \textcolor{red}{(23)}-\textcolor{red}{(24)} is not evident. The above form was assumed as obvious by several authors [4]–[7], [17]–[29]. In Classical mechanics the Poisson algebra was not considered as an important point, but in Quantum Mechanics the existence of Poisson algebra permits the algebraic treatment of the superintegrable system \textcolor{red}{[20]}. In this paper we prove that the superintegrable systems can be classified using the properties of the Poisson algebra. The superintegrability is a global property of the system, and this fact is reflected in the Poisson algebra structure, which is indeed a global property. The mathematical proof of the existence of the algebra \textcolor{red}{(23)}-\textcolor{red}{(24)} for the two dimensional superintegrable systems can be found in the Appendix \textcolor{red}{[A]}.

The Jacobi equality for the Poisson brackets induces the relation

\[ \{A, \{B, C\}_P\}_P = \{B, \{A, C\}_P\}_P \]

The following relations

\[ b = -\gamma, \quad c = -\alpha \quad \text{and} \quad e = -\delta \]

must be satisfied.

The integrals $A$, $B$ and $C$ satisfy the quadratic Poisson algebra:

\begin{align}
\{A, B\}_P &= C \\
\{A, C\}_P &= \alpha A^2 + 2\gamma AB + \delta A + \epsilon B + \zeta \tag{25} \\
\{B, C\}_P &= aA^2 - \gamma B^2 - 2\alpha AB + dA - \delta B + z
\end{align}
where $\alpha, \gamma, a$ are constants and

\[
\begin{align*}
\delta &= \delta(H) = \delta_0 + \delta_1 H \\
\epsilon &= \epsilon(H) = \epsilon_0 + \epsilon_1 H \\
\zeta &= \zeta(H) = \zeta_0 + \zeta_1 H + \zeta_2 H^2 \\
d &= d(H) = d_0 + d_1 H \\
z &= z(H) = z_0 + z_1 H + z_2 H^2
\end{align*}
\]

where $\delta_i, \epsilon_i, \zeta_i, d_i$ and $z_i$ are constants. The associative algebra, whose the generators satisfy equations (25), is the general form of the closed Poisson algebra of the integrals of superintegrable systems with integrals quadratic in momenta.

The quadratic Poisson algebra (25) possess a Casimir which is a function of momenta of degree 6 and it is given by:

\[
K = C^2 - 2\alpha A^2 B - 2\gamma AB^2 - 2\delta AB - \epsilon B^2 - 2\zeta B + \frac{2}{7}a A^3 + dA^2 + 2z A = k_0 + k_1 H + k_2 H^2 + k_3 H^3
\]

(26)

Obviously

\[
\{K, A\}_P = \{K, B\}_P = \{K, C\}_P = 0
\]

Therefore the integrals of motion of a superintegrable two dimensional system with quadratic integrals of motion close a constrained classical quadratic Poisson algebra (25), corresponding to a Casimir equal at most to a cubic function of the Hamiltonian (26).

In the general case of a superintegrable system the integrals are not necessarily quadratic functions of the momenta, but rather polynomial functions of the momenta. The case of the systems with a quadratic and a cubic integral of motion are studied by Tsiganov [31, 32].

IV Classes of superintegrable systems on a two dimensional manifold with quadratic integrals of motion

The main result of the previous section is that the definition of the Casimir of the Poisson algebra, given by equation (26) determines uniquely the Poisson algebra. This Poisson algebra is specific for each superintegrable system,
therefore it can be used for the classification of the possible superintegrable systems. Usually the proposed classifications of superintegrable systems assumed the definition of the manifold metric and the superintegrable systems were fixed for the given metric. In this paper we propose a classification which is based on the Poisson algebra. Let us consider a superintegrable system, which is described by a Hamiltonian $H$ and two integrals of motion $A$ and $B$. The integrability of the system imposes several choices which are determined two by Classes I and II of integrable systems, as it has been shown in section III These classes of super integrable systems are:

**Class I** This class contains superintegrable systems, whose manifold metric and integrals of motion are written in a specific coordinate:

$$ds^2 = g(\xi, \eta) d\xi d\eta, \quad g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta) \quad (27)$$

$$H = \frac{p_\xi p_\eta}{F(\xi + \eta) + G(\xi - \eta)} + \frac{f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} \quad (28)$$

and

$$A = \frac{p_\xi^2 + p_\eta^2 - 2p_\xi p_\eta \int F(\eta) d\eta}{F(\xi + \eta) + G(\xi - \eta)} + \frac{\beta(\xi, \eta)}{F(\xi + \eta) + G(\xi - \eta)} + Q(\xi, \eta) \quad (29)$$

The second integral of motion has the general form:

$$B = A(\xi) p_\xi^2 + B(\eta) p_\eta^2 - 2p_\xi p_\eta \int \frac{f(\eta)}{F(\xi + \eta) + G(\xi - \eta)} F(\eta) d\eta + Q(\xi, \eta) \quad (30)$$

where $A(\xi)$ and $B(\eta)$ are not zero.

**Class II** This class contains superintegrable systems, whose manifold metric and integrals of motion are written in a specific coordinate:

$$ds^2 = g(\xi, \eta) d\xi d\eta, \quad g(\xi, \eta) = F(\eta)\xi + G(\eta) \quad (31)$$

$$H = \frac{p_\xi p_\eta}{F(\eta)\xi + G(\eta)} + \frac{f(\eta)\xi + g(\eta)}{F(\eta)\xi + G(\eta)} \quad (32)$$

and

$$A = \frac{p_\xi^2 - 2p_\xi p_\eta \int \frac{F(\eta) d\eta}{F(\eta)\xi + G(\eta)} - 2\left(\int f(\eta)\xi + g(\eta)\right)\int \frac{F(\eta) d\eta}{F(\eta)\xi + G(\eta)} + 2\int f(\eta) d\eta}{F(\eta)\xi + G(\eta)} \quad (33)$$
The second integral of motion has the general form:

\[ B = A(\xi)p_\xi^2 + B(\eta)p_\eta^2 - 2p_\xi p_\eta \frac{\beta(\xi, \eta)}{F(\eta)\xi + G(\eta)} + Q(\xi, \eta) \]  

(34)

where \( A(\xi) \) and \( B(\eta) \) are not zero.

The existence of a second integral of motion \( B \) implies that there is another Liouville coordinate system \((X, Y)\) corresponding to the pair of integrals \( H \) and \( B \). In this system there are analytic formulas for the second integral of motion by using the results of Proposition 1. The analytic calculation of these formulas will be given in the following subsections.

The Class I superintegrable system is described by the integrals of motion which are given by equations (27–30). Correspondingly, the Class II superintegrable system is described by the integrals of motion which are given by equations (31–34). These integrals of motion should satisfy the relation (26), which is an identity relation between polynomials of sixth degree for the momenta. The coefficients of \( p_\xi^6 \) and \( p_\eta^6 \) in (26) vanish and the following identities are true:

\[
6 \left( A'(\xi) \right)^2 = 3\gamma A^2(\xi) + 3\alpha A(\xi) - a \\
6 \left( B'(\eta) \right)^2 = 3\gamma B^2(\eta) + 3\alpha B(\eta) - a
\]  

(35)

In equation (30), the coefficients of \( p_\xi^2 \) and \( p_\eta^2 \) in the integral \( B \) are determined by the solution of the above equations (35).

The superintegrable systems on a manifold can be classified by the possible solutions of equations (35). The integral of motion \( A \) has a standard form, given by equation (29), this form is the Liouville form and the coefficients of \( p_x^2 \) and \( p_y^2 \) are equal to 1. The second integral of motion \( B \) can be replaced by any combination of the form:

\[ B \rightarrow qB + rH + sA \]

where \( q, r, s \) are arbitrary constants. From this fact we can show that there are six subclasses of possible solutions of equation (35).

- **Subclass I** This class corresponds to

  \[ \gamma = 0, \quad \alpha = 0, \quad a \neq 0 \]

  if we choose \( a = -6 \), then

  \[ A(\xi) = \xi, \quad B(\eta) = \eta \]
• **Subclass I**₂ This class corresponds to
  \[ \gamma = 0, \quad \alpha \neq 0, \quad a = 0 \]
  if we choose \( \alpha = 8 \), then
  \[ A(\xi) = \xi^2, \quad B(\eta) = \eta^2 \]

• **Subclass I**₃ This class corresponds to
  \[ \gamma \neq 0, \quad \alpha = 0, \quad a \neq 0 \]
  if we choose \( \gamma = 2 \), then we can show that all cases are equivalent to
  the choice
  \[ A(\xi) = \left( e^\xi + e^{-\xi} \right)^2, \quad B(\eta) = \left( e^\eta + e^{-\eta} \right)^2 \]
  and

• **Subclass II**₁ This class corresponds to
  \[ \gamma = 0, \quad \alpha = 0, \quad a = 0 \]
  then
  \[ A(\xi) = 1, \quad B(\eta) = 1 \]

• **Subclass II**₂ This class corresponds to
  \[ \gamma = 0, \quad \alpha = 0, \quad a \neq 0 \]
  if we choose \( a = -6 \), then
  \[ A(\xi) = \xi, \quad B(\eta) = \eta \]

• **Subclass II**₃ This class corresponds to
  \[ \gamma = 0, \quad \alpha \neq 0, \quad a = 0 \]
  if we choose \( \alpha = 8 \), then we can show that
  \[ A(\xi) = \xi^2, \quad B(\eta) = \eta^2 \]

These classes of solutions will be studied in detail in Section V.
IV-a  Class I superintegrable systems

Starting from the definition of functions $A(\xi)$ and $B(\eta)$ we can solve equations (35) and the superintegrable system is fully determined. This procedure will be sketched in detail in the next paragraphs.

Let us start by the known solutions $A(\xi)$, $B(\eta)$ of equations (35).

Equation (35) is written

\begin{align*}
(A''(\xi) - B''(\eta)) (F(\xi + \eta) + G(\xi - \eta)) + \\
+ 3A'(\xi) (F'(\xi + \eta) + G'(\xi - \eta)) - 3B'(\eta) (F'(\xi + \eta) - G'(\xi - \eta)) + \\
+ 2 (A(\xi) - B(\eta)) (F''(\xi + \eta) + G''(\xi - \eta)) &= 0
\end{align*}

(36)

In the next paragraphs we will show that, the above equation can be separated in two second order differential equations for the involved functions $F(u)$ and $G(v)$. The general solution of these equations are given by:

\begin{align*}
F(u) &= \lambda_1 F_1(u) + \lambda_2 F_2(u) + \lambda_3 F_3(u) + \lambda_4 F_4(u) \\
G(v) &= \ell_1 G_1(v) + \ell_2 G_2(v) + \ell_3 G_3(v) + \ell_4 G_4(v)
\end{align*}

(37)

where $F_k(u)$ and $G_k(u)$ are functions which are not generally partial independent solutions of two second order differential equation (36). Between the eight parameters $\lambda_k$ and $\ell_k$ for $k = 1, 2, 3, 4$ only four among of them are independent.

After the calculation of the functions $F(u)$ and $G(v)$ we can calculate the function $\beta(x, y)$ from equation (35). The general form of the potential $V(\xi, \eta)$ is given by equation (14).

After some elementary (but rather lengthy) algebraic calculation we can show that equation (11) leads to a differential equation for the functions $f(u)$ and $g(v)$ which are involved in the definition (14) of the potential:

\begin{align*}
&f(\xi + \eta) \{ -3 B'(\eta) (F'(\xi + \eta) - G'(\xi - \eta)) + 3 A'(\xi) (F'(\xi + \eta) + G'(\xi - \eta)) + \\
&+ 2 (A(\xi) - B(\eta)) (F''(\xi + \eta) + G''(\xi - \eta)) \} + \\
&+ g(\xi - \eta) \{ -3 B'(\eta) (F'(\xi + \eta) - G'(\xi - \eta)) + 3 A'(\xi) (F'(\xi + \eta) + G'(\xi - \eta)) + \\
&+ 2 (A(\xi) - B(\eta)) (F''(\xi + \eta) + G''(\xi - \eta)) \} - \\
&- (F(\xi + \eta) + G(\xi - \eta)) \{ -3 B'(\eta) (f'(\xi + \eta) - g'(\xi - \eta)) + \\
&+ 3 A'(\xi) (f'(\xi + \eta) + g'(\xi - \eta)) + 2 (A(\xi) - B(\eta)) (f''(\xi + \eta) + g''(\xi - \eta)) \} = 0
\end{align*}

We can eliminate in the above equation the functions $F(u)$ and $G(v)$, which satisfy equation (36) and finally the functions involved in the definition of
the potential satisfy the following equation:

\[
(A''(\xi) - B''(\eta)) (f(\xi + \eta) + g(\xi - \eta)) + \\
+ 3A'(\xi) (g'(\xi + \eta) + g'(\xi - \eta)) - 3B'(\eta) (g'(\xi + \eta) - g'(\xi - \eta)) + \\
+ 2 (A(\xi) - B(\eta)) (f''(\xi + \eta) + g''(\xi - \eta)) = 0
\]  

(38)

This equation is indeed the same as (36) and the general solution has been given by equation (37). Then we have proved the following general proposition:

**Proposition 2** The general form of the potential \(V(\xi, \eta)\) in Liouville coordinates for a superintegrable system of Class I is given by the general formula:

\[
V(\xi, \eta) = \frac{f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)}
\]

where both the pairs of functions \(f(u), g(v)\) and \(F(u), G(v)\) satisfy the same differential equation (36) and (38).

Therefore the solutions of equation (38) are given by:

\[
f(u) = \rho_1 F_1(u) + \rho_2 F_2(u) + \rho_3 F_3(u) + \rho_4 F_4(u)
\]

\[
g(v) = r_1 G_1(v) + r_2 G_2(v) + r_3 G_3(v) + r_4 G_4(v)
\]

(39)

The equations of motion don’t depend on the shifts of the potential by one constant, this fact explains physically the identity of differential equations (36) and (38).

The calculation of the second integral of motion \(B\) is now straightforward. If the functions \(A(x)\) and \(B(y)\) are fixed as solutions of the characteristic equations (35) then we can introduce the new coordinates:

\[
X = \int \frac{dx}{\sqrt{A(x)}} \quad \text{and} \quad Y = \int \frac{dy}{\sqrt{B(y)}}
\]

(40)

In these coordinates the metric of the manifold can be written

\[
ds^2 = g(x, y) \, dx \, dy = \tilde{g}(X, Y) \, dX \, dY
\]

(41)

where

\[
\tilde{g}(X, Y) = g(x, y) \sqrt{A(x) \, B(y)}
\]
The integrability of the above Hamiltonian implies:
\[ H = \frac{p_x p_y}{g(x, y)} + V(x, y) = \frac{p_x p_y}{\tilde{g}(X, Y)} + \tilde{V}(X, Y) \]

The integrability of the above Hamiltonian implies:
\[ \tilde{g}(X, Y) = \tilde{F}(X + Y) + \tilde{G}(X - Y) \] (42)

The above relation implies:
\[
\begin{align*}
\tilde{g} \left( \frac{U + V}{2}, \frac{U - V}{2} \right) &= \tilde{F}(U) + \tilde{G}(V) \\
\tilde{g} \left( \frac{U + d}{2}, \frac{U - d}{2} \right) &= \tilde{F}(U) + \tilde{G}(d) \\
\tilde{g} \left( \frac{c + V}{2}, \frac{c - V}{2} \right) &= \tilde{F}(c) + \tilde{G}(V) \\
\tilde{g} \left( \frac{c + d}{2}, \frac{c - d}{2} \right) &= \tilde{F}(c) + \tilde{G}(d)
\end{align*}
\]

where \( c \) and \( d \) are two arbitrary constants. Therefore the functions \( \tilde{F}(U) \) and \( \tilde{G}(V) \) are calculated up to one constant by:
\[
\begin{align*}
\tilde{F}(U) &= \tilde{g} \left( \frac{U + d}{2}, \frac{U - d}{2} \right) - \frac{1}{2} \tilde{g} \left( \frac{c + d}{2}, \frac{c - d}{2} \right) + \mu \\
\tilde{G}(V) &= \tilde{g} \left( \frac{c + V}{2}, \frac{c - V}{2} \right) - \frac{1}{2} \tilde{g} \left( \frac{c + d}{2}, \frac{c - d}{2} \right) - \mu
\end{align*}
\] (43)

where \( \mu \) can be an arbitrary chosen constant. Starting from the well known form of the potential:
\[ V(x, y) = \frac{w(x, y)}{g(x, y)}, \quad w(x, y) = f(x + y) + g(x - y) \]

we can show that
\[ V(x, y) = \tilde{V}(X, Y) = \frac{\tilde{w}(X, Y)}{\tilde{g}(X, Y)}, \quad \tilde{w}(X, Y) = \tilde{f}(X + Y) + \tilde{g}(X - Y) = w(x, y) \sqrt{A(x) B(y)} \] (44)

The functions \( \tilde{f}(U) \) and \( \tilde{g}(V) \) are calculated by:
\[
\begin{align*}
\tilde{f}(U) &= \tilde{w} \left( \frac{U + d}{2}, \frac{U - d}{2} \right) - \frac{1}{2} \tilde{w} \left( \frac{c + d}{2}, \frac{c - d}{2} \right) \\
\tilde{g}(V) &= \tilde{w} \left( \frac{c + V}{2}, \frac{c - V}{2} \right) - \frac{1}{2} \tilde{w} \left( \frac{c + d}{2}, \frac{c - d}{2} \right)
\end{align*}
\]

Then the second integral of motion in the coordinates \( X, Y \) is:
\[
B = \frac{p_x^2 + p_y^2}{\tilde{F}(X+Y)+\tilde{G}(X-Y)} + \frac{4\tilde{f}(X+Y)\tilde{g}(X-Y)-\tilde{g}(X-Y)\tilde{f}(X+Y)}{\tilde{F}(X+Y)+\tilde{G}(X-Y)}
\] (45)

After calculating the above integral in the coordinates \( X, Y \), we can compute analytically the second integral \( B \) in the original coordinates \( x, y \).
IV-b  Class II superintegrable systems

Equation (8) is written

\[
(A''(x) - B''(y)) (F(y)x + G(y)) + \\
+3A'(x)F(y) - 3B'(y) (F'(y)x + G'(y)) + \\
+2 (A(x) - B(y)) (F''(y)x + G''(y)) = 0
\]  

(46)

The general solutions of equation (46) are given by:

\[
F(y) = \lambda_1 F_1(y) + \lambda_2 F_2(y) + \lambda_3 F_3(y) + \lambda_4 F_4(y) \\
G(y) = \ell_1 G_1(y) + \ell_2 G_2(y) + \ell_3 G_3(y) + \ell_4 G_4(y)
\]

(47)

where \( F_k(y) \) and \( G_k(y) \) are partial independent solutions of two second order differential equations with several constant parameters. Between the eight parameters \( \lambda_k \) and \( \ell_k \) for \( k = 1, 2, 3, 4 \) only four among them are linearly independent.

After the calculation of the functions \( F(y) \) and \( G(y) \) we can calculate the function \( \beta(x, y) \) from equation (18). The general form of the potential \( V(x, y) \) is given by equation (19). After some elementary (but rather complicated) algebraic calculation we can show that, equation (20) leads to a differential equation for the functions \( f(y) \) and \( g(y) \), which are involved in the definition (14) of the potential:

\[
(A''(x) - B''(y)) (f(y)x + g(y)) + \\
+3A'(x)f(y) - 3B'(y) (f'(y)x + g'(y)) + \\
+2 (A(x) - B(y)) (f''(y)x + g''(y)) = 0
\]

(48)

Then we have proved the following general proposition:

**Proposition 3** The general form of the potential \( V(\xi, \eta) \) in Liouville coordinates for a superintegrable system of Class II is given by the general formula:

\[
V(\xi, \eta) = \frac{f(\eta)\xi + g(\eta)}{F(\eta)\xi + G(\eta)}
\]

where both the pairs of functions \( f(u), g(v) \) and \( F(u), G(v) \) satisfy the same differential equation (46) and (48).

Therefore the solutions of equation (48) are given by:

\[
f(y) = \rho_1 F_1(y) + \rho_2 F_2(y) + \rho_3 F_3(y) + \rho_4 F_4(y) \\
g(y) = r_1 G_1(y) + r_2 G_2(y) + r_3 G_3(y) + r_4 G_4(y)
\]

(49)
The equations of motion don’t depend on the shifts of the potential by one constant, this fact explains physically the identity of differential equations (36) (or (16) ) and (38) (or (18)).

The calculation of the second integral of motion \( B \) is now straightforward, we can use the same procedure of solution as it has been described by equations (40–45).

V Classification of two dimensional superintegrable systems with two quadratic integrals of motion

In this section we give the analytical solutions for the different classes of superintegrable systems. As we have shown there are two general classes of superintegrable systems, each class has 3 subclasses.

V-a Class I superintegrable systems

V-a.1 Subclass I of superintegrable systems

\[
A(\xi) = \xi, \quad B(\eta) = \eta \\
F(u) = 4\lambda u^2 + \kappa u + \nu/2, \quad G(v) = -\lambda v^2 + \mu/v^2 + \nu/2 \\
f(u) = 4\ell u^2 + ku + n/2, \quad g(v) = -\ell v^2 + m/v^2 + n/2
\]  

(50)

\[
ds^2 = g(\xi,\eta) \, d\xi \, d\eta, \quad g(\xi,\eta) = F(\xi + \eta) + G(\xi - \eta) \\
H = \frac{p_\xi p_\eta}{g(\xi,\eta)} + V(\xi,\eta) \quad V(\xi,\eta) = \frac{w(\xi,\eta)}{g(\xi,\eta)}, \quad w(\xi,\eta) = f(\xi + \eta) + g(\xi - \eta)
\]

The other integral of motion is:

\[
A = p_\xi^2 + p_\eta^2 - 2p_\xi p_\eta \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} + 4\frac{f(\xi + \eta) G(\xi - \eta) - g(\xi - \eta) F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}
\]
We introduce the functions:

\[
\begin{align*}
\tilde{F}(u) &= \frac{\lambda u^6}{256} + \frac{\kappa u^4}{128} + \frac{\nu u^2}{16} - \frac{\mu}{u^2} \\
\tilde{G}(v) &= -\frac{\lambda v^6}{256} - \frac{\kappa v^4}{128} - \frac{\nu v^2}{16} + \frac{\mu}{v^2} \\
\tilde{f}(u) &= \frac{\ell u^6}{256} + \frac{k u^4}{128} + \frac{n u^2}{16} - \frac{m}{u^2} \\
\tilde{g}(v) &= -\frac{\ell v^6}{256} - \frac{k v^4}{128} - \frac{n v^2}{16} + \frac{m}{v^2}
\end{align*}
\]

The second integral of motion is:

\[
B = p_X^2 + p_Y^2 - 2p_X p_Y \frac{\tilde{F}(X + Y) - \tilde{G}(X - Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)} + 4 \frac{\tilde{f}(X + Y)\tilde{G}(X - Y) - \tilde{g}(X - Y)\tilde{F}(X + Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)}
\]

where

\[
X = 2\sqrt{\xi}, \quad p_X = \sqrt{\xi}p_\xi, \quad Y = 2\sqrt{\eta}, \quad p_Y = \sqrt{\eta}p_\eta
\]

The constants of the Poisson algebra are:

\[
\begin{align*}
\alpha &= 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 16(\kappa H - k), \quad \epsilon = 256(\lambda H - \ell) \\
\zeta &= -32(\kappa H - k)(\nu H - n), \quad a = -6, \quad d = 8(\nu H - n) \\
z &= 8(\nu H - n)^2 - 128(\lambda H - \ell)(\mu H - m)
\end{align*}
\]

\[
K = 32(\nu H - n)^3 + 512(\lambda H - \ell)(\mu H - m)(\nu H - n) - 64(\kappa H - k)^2(\mu H - m)
\]

V-a.2 Subclass I_2 of superintegrable systems

\[
\begin{align*}
A(\xi) &= \xi^2, \quad B(\eta) = \eta^2 \\
F(u) &= \lambda u^2 + \frac{k}{u^2} + \frac{\nu}{2}, \quad G(v) = -\lambda v^2 + \frac{\mu}{v^2} + \frac{\nu}{2} \\
f(u) &= \ell u^2 + \frac{k}{u^2} + \frac{n}{2}, \quad g(v) = -\ell v^2 + \frac{m}{v^2} + \frac{n}{2}
\end{align*}
\]
\[ds^2 = g(\xi, \eta) \, d\xi \, d\eta, \quad g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta)\]

\[H = \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta) \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = f(\xi + \eta) + g(\xi - \eta)\]

The other integral of motion is:

\[A = p_\xi^2 + p_\eta^2 - 2p_\xi p_\eta \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} + 4 \frac{f(\xi + \eta) G(\xi - \eta) - g(\xi - \eta) F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}\]

We introduce the functions:

\[\tilde{F}(u) = 4 \lambda e^{2u} + \nu e^u, \quad \tilde{G}(v) = \frac{\kappa e^v}{(1 + e^v)^2} + \frac{\mu e^v}{(-1 + e^v)^2}\]

\[\tilde{f}(u) = 4 \ell e^{2u} + n e^u, \quad \tilde{g}(v) = \frac{k e^v}{(1 + e^v)^2} + \frac{m e^v}{(-1 + e^v)^2}\] (53)

The second integral of motion is:

\[B = p_X^2 + p_Y^2 - 2p_X p_Y \frac{\tilde{F}(X + Y) - \tilde{G}(X - Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)} + 4 \frac{\tilde{f}(X + Y) \tilde{G}(X - Y) - \tilde{g}(X - Y) \tilde{F}(X + Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)}\]

where

\[X = \ln \xi, \quad p_X = \xi p_\xi, \quad Y = \ln \eta, \quad p_Y = \eta p_\eta\]

The constants of the Poisson algebra are:

\[\alpha = 8, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0, \quad \epsilon = 256(\lambda H - \ell)\]
\[\zeta = -32(\nu H - n)^2 + 256(\lambda H - \ell) ((\mu - \kappa) H - (m - k))\]
\[a = 0, \quad d = 0, \quad z = 32 \left((\kappa + \mu) H - (k + m)\right) (\nu H - n)\]

\[K = 256 (\lambda H - \ell) ((\kappa + \mu) H - (k + m))^2 + 128 ((\kappa - \mu) H - (k - m)) (\nu H - n)^2\]

V-a.3 Subclass I$_3$ of superintegrable systems

\[A(\xi) = (e^\xi + e^{-\xi})^2, \quad B(\eta) = (e^n + e^{-n})^2\]
\[
F(u) = \frac{\kappa e^{2u}}{(-1 + e^{2u})^2} + \frac{\lambda e^u (1 + e^{2u})}{(-1 + e^{2u})^2}, \quad G(v) = \frac{\mu e^{2v}}{(-1 + e^{2v})^2} + \frac{\nu e^v (1 + e^{2v})}{(-1 + e^{2v})^2}
\]

\[
f(u) = \frac{k e^{2u}}{(-1 + e^{2u})^2} + \frac{\ell e^u (1 + e^{2u})}{(-1 + e^{2u})^2}, \quad g(v) = \frac{m e^{2v}}{(-1 + e^{2v})^2} + \frac{n e^v (1 + e^{2v})}{(-1 + e^{2v})^2}
\]

\[
ds^2 = g(\xi, \eta) \, d\xi \, d\eta, \quad g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta)
\]

\[
H = \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta) \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = f(\xi + \eta) + g(\xi - \eta)
\]

The other integral of motion is:

\[
A = p_\xi^2 + p_\eta^2 - 2 p_\xi p_\eta \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} + 4 \frac{\tilde{F}(\xi + \eta) G(\xi - \eta) - g(\xi - \eta) F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}
\]

We introduce the functions:

\[
\tilde{F}(u) = \frac{(\kappa + 2 \lambda)}{4} \tan^2 u + \frac{2 \nu - \mu}{4} \cot^2 u + \frac{\lambda + \nu}{2}
\]

\[
\tilde{G}(v) = \frac{(2 \lambda - \kappa)}{4} \tan^2 v + \frac{\mu + 2 \nu}{4} \cot^2 v + \frac{\lambda + \nu}{2}
\]

\[
\tilde{f}(u) = \frac{(k + 2 \ell)}{4} \tan^2 u + \frac{2 n - m}{4} \cot^2 u + \frac{\ell + n}{2}
\]

\[
\tilde{g}(v) = \frac{(2 \ell - k)}{4} \tan^2 v + \frac{m + 2 n}{4} \cot^2 v + \frac{\ell + n}{2}
\]

The second integral of motion is:

\[
B = p_X^2 + p_Y^2 - 2 p_X p_Y \frac{\tilde{F}(X + Y) - \tilde{G}(X - Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)} + 4 \frac{\tilde{f}(X + Y) \tilde{G}(X - Y) - \tilde{g}(X - Y) \tilde{F}(X + Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)}
\]

where

\[
X = \arctan(e^\xi), \quad p_X = (e^\xi + e^{-\xi}) p_\xi, \quad Y = \arctan(e^\eta), \quad p_Y = (e^\eta + e^{-\eta}) p_\eta
\]
The constants of the Poisson algebra are:

\[ \begin{align*}
\alpha &= -32 \quad \beta = 0 \quad \gamma = 8 \quad \delta = 0 \quad \epsilon = 0 \quad \zeta = -32 (\lambda H - \ell) (\nu H - n) \\
\alpha &= 0 \quad \delta = 64 (k - m) - 64 (\kappa - \mu) H \\
\gamma &= 32 ((\lambda - \nu) H - (\ell - n))^2 - 32 ((\kappa H - k) (\mu H - m)) \\
K &= 64 (\kappa H - k) (\nu H - n)^2 - 64 (\lambda H - \ell)^2 (\mu H - m)
\end{align*} \]

V-b Class II superintegrable systems

V-b.1 Subclass II of superintegrable systems

\[ \begin{align*}
A(\xi) &= 1, \quad B(\eta) = 1 \\
F(\eta) &= \kappa \eta + \lambda, \quad G(\eta) = \mu \eta + \nu \\
f(\eta) &= k \eta + \ell, \quad g(\eta) = m \eta + n \\
ds^2 &= g(\xi, \eta) \, d\xi \, d\eta, \quad g(\xi, \eta) = \xi F(\eta) + G(\eta) \\
H &= \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta), \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = \xi f(\eta) + g(\eta)
\end{align*} \]

The other integral of motion is:

\[ A = p_\xi^2 - 2 \frac{p_\xi p_\eta \int F(\eta) \, d\eta}{g(\xi, \eta)} - 2 \frac{(\xi f(\eta) + g(\eta)) \int F(\eta) \, d\eta}{g(\xi, \eta)} + 2 \int f(\eta) \, d\eta \]

We introduce the functions:

\[ \begin{align*}
\tilde{F}(u) &= \frac{\kappa u^2}{4} + \frac{(\lambda + \mu) u}{2} + \frac{\nu}{2} \\
\tilde{G}(v) &= -\frac{\kappa v^2}{4} + \frac{(\lambda - \mu) v}{2} + \frac{\nu}{2} \\
\tilde{f}(u) &= \frac{k u^2}{4} + \frac{(\ell + m) u}{2} + \frac{n}{2} \\
\tilde{g}(v) &= -\frac{k v^2}{4} + \frac{(\ell - m) v}{2} + \frac{n}{2}
\end{align*} \]
The second integral of motion is:

\[
B = p_\xi^2 + p_\eta^2 - 2 p_\xi p_\eta \frac{\bar{F}(\xi + \eta) - \bar{G}(\xi - \eta)}{\bar{F}(\xi + \eta) + \bar{G}(\xi - \eta)} + 4 \frac{\bar{f}(\xi + \eta) \bar{G}(\xi - \eta) - \bar{g}(\xi - \eta) \bar{F}(\xi + \eta)}{\bar{F}(\xi + \eta) + \bar{G}(\xi - \eta)}
\]

The constants of the Poisson algebra are:

\[
\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 8(k - \kappa H), \quad \epsilon = 0, \quad \zeta = 8(\lambda H - \ell)^2, \\
a = 0, \quad d = 16(k - \kappa H), \quad z = 8(\lambda H - \ell)^2 - (\mu H - m)^2, \\
K = 16(\nu H - n)^2(kH - k) - 32(\lambda H - \ell)(\mu H - m)(\nu H - n)
\]

V-b.2 Subclass II_2 of superintegrable systems

\[
A(\xi) = \xi, \quad B(\eta) = \eta \\
F(\eta) = \frac{\kappa}{\sqrt{\eta}} + \lambda, \quad G(\eta) = 3\kappa \sqrt{\eta} + \lambda \eta + \frac{\mu}{\sqrt{\eta}} + \nu \\
f(\eta) = \frac{k}{\sqrt{\eta}} + \ell, \quad g(\eta) = 3k \sqrt{\eta} + \ell \eta + \frac{m}{\sqrt{\eta}} + n
\]

\[
ds^2 = g(\xi, \eta) \, d\xi \, d\eta, \quad g(\xi, \eta) = \xi F(\eta) + G(\eta)
\]

\[
H = \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta), \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = \xi f(\eta) + g(\eta)
\]

The other integral of motion is:

\[
A = p_\xi^2 - 2 \frac{p_\xi p_\eta}{g(\xi, \eta)} \int F(\eta) \, d\eta - 2 \frac{(\xi f(\eta) + g(\eta))}{g(\xi, \eta)} \int F(\eta) \, d\eta + 2 \int f(\eta) \, d\eta
\]
We introduce the functions:

\[
\tilde{F}(u) = \frac{\lambda u^4}{128} + \frac{\kappa u^3}{16} + \frac{\nu u^2}{16} + \frac{\mu u}{4}
\]

\[
\tilde{G}(v) = -\frac{\lambda v^4}{128} + \frac{\kappa v^3}{16} + \frac{\mu v}{4} - \frac{\nu v^2}{16}
\]

\[
\tilde{f}(u) = \frac{\ell u^4}{128} + \frac{k u^3}{16} + \frac{n u^2}{16} + \frac{m u}{4}
\]

\[
\tilde{g}(v) = -\frac{\ell v^4}{128} + \frac{k v^3}{16} + \frac{m v}{4} - \frac{n v^2}{16}
\]

The second integral of motion is:

\[
B = p_R^2 + p_T^2 - 2p_R p_T \frac{\tilde{F}(X + Y) - \tilde{G}(X - Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)} + 4 \frac{\tilde{f}(X + Y)\tilde{G}(X - Y) - \tilde{g}(X - Y)\tilde{F}(X + Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)}
\]

where \( X = 2\sqrt{\xi}, \quad p_X = \sqrt{\xi} p_\xi, \quad Y = 2\sqrt{\eta}, \quad p_Y = \sqrt{\eta} p_\eta \)

The constants of the Poisson algebra are:

\[
\begin{align*}
\alpha &= 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 4 (\ell - \lambda H), \quad \epsilon = 0, \quad \zeta = 8(\kappa H - k)^2 \\
a &= -6, \quad d = 8 (\nu H - n), \quad z = -8 (\kappa H - k)(\mu H - m) - 2(\nu H - n)^2 \\
K &= 8 (\lambda H - \ell)(\mu H - m)^2 - 16 (\kappa H - k)(\mu H - m)(\nu H - n)
\end{align*}
\]

V-b.3 Subclass II_3 of superintegrable systems

\[
\begin{align*}
A(\xi) &= \xi^2, \quad B(\eta) = \eta^2 \\
F(\eta) &= \lambda \eta + \frac{\kappa}{\eta^3}, \quad G(\eta) = \nu + \frac{\mu}{\eta^2} \\
f(\eta) &= \ell \eta + \frac{k}{\eta^3}, \quad g(\eta) = n + \frac{m}{\eta^2}
\end{align*}
\]
\[ ds^2 = g(\xi, \eta) \, d\xi \, d\eta, \quad g(\xi, \eta) = \xi \, F(\eta) + G(\eta) \]

\[ H = \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta), \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = \xi \, f(\eta) + g(\eta) \]

The other integral of motion is:

\[ A = p_\xi^2 - \frac{2 \, p_\xi p_\eta \int F(\eta) \, d\eta}{g(\xi, \eta)} - \frac{2 \, (\xi \, f(\eta) + g(\eta)) \int F(\eta) \, d\eta}{g(\xi, \eta)} + 2 \int f(\eta) \, d\eta \]

We introduce the functions:

\[ \tilde{F}(u) = \lambda \, e^{2u} + \nu \, e^u, \quad \tilde{G}(v) = \kappa \, e^{2v} + \mu \, e^v \]

\[ \tilde{f}(u) = \ell \, e^{2u} + n \, e^u, \quad \tilde{g}(v) = k \, e^{2v} + m \, e^v \]  \hspace{1cm} (61)

The second integral of motion is:

\[ B = p_X^2 + p_Y^2 - 2 \, p_X p_Y \, \frac{\tilde{F}(X + Y) - \tilde{G}(X - Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)} + \]

\[ + 4 \, \frac{\tilde{f}(X + Y)\tilde{G}(X - Y) - \tilde{g}(X - Y)\tilde{F}(X + Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)} \]

where

\[ X = \ln \xi, \quad p_X = \xi \, p_\xi, \quad Y = \ln \eta, \quad p_Y = \eta \, p_\eta \]

The constants of the Poisson algebra are:

\[ \alpha = 8, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0, \quad \epsilon = 0, \quad \zeta = 32 \left( \kappa \, H - k \right) \left( \lambda \, H - \ell \right), \]

\[ a = 0, \quad d = 0, \quad z = 32 \left( \mu \, H - m \right) \left( \nu \, H - n \right) \]

\[ K = 64 \left( \lambda \, H - \ell \right) \left( \mu \, H - m \right)^2 - 64 \left( \kappa \, H - k \right) \left( \nu \, H - n \right)^2 \]

All the above superintegrable systems generally are defined on manifolds which have neither constant curvature nor are they surfaces of revolution. All the known superintegrable systems are defined on manifolds of constant curvature or on surfaces of revolution. Therefore we have proved that there are new superintegrable systems, which have not yet been studied.
Superintegrable systems corresponding to Koenigs essential forms

Using the coordinate transformation

\[ \xi = \frac{1}{2} x^2, \quad p_\xi = \frac{p_x}{x}, \quad \eta = \frac{1}{2} y^2, \quad p_\eta = \frac{p_y}{y} \]

the metric of the Class I superintegrable systems is reduced to the metric of the essential form VII.4 [1, vol IV, p.385], if

\[ \kappa = 16A_2, \quad \lambda = 16A_3, \quad \mu = -A_0, \quad \nu = 4A_1 \]

The corresponding superintegrable system (using the coordinates of Ref. [1]) is given by the Hamiltonian:

\[ H = \frac{p_x p_y + w(x, y)}{g(x, y)} \]

\[ g(x, y) = A_0 \left[ \frac{1}{(x+y)^2} - \frac{1}{(x-y)^2} \right] + A_1 [(x+y)^2 - (x-y)^2] + A_2 [(x+y)^4 - (x-y)^4] + A_3 [(x+y)^6 - (x-y)^6] \]

\[ w(x, y) = a_0 \left[ \frac{1}{(x+y)^2} - \frac{1}{(x-y)^2} \right] + a_1 [(x+y)^2 - (x-y)^2] + a_2 [(x+y)^4 - (x-y)^4] + a_3 [(x+y)^6 - (x-y)^6] \]

where only three of the constants \( a_0, a_1, a_2, a_3 \) are independent, i.e. we can put one among them equal to zero. Using relations (51) we have that in \( x, y \) coordinates the other integral of motion is:

\[ A(x, y) = \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 - p_x p_y \frac{\Phi(x, y) - \Psi(x, y)}{\Phi(x, y) + \Psi(x, y)} + \frac{2}{\Phi(x, y) + \Psi(x, y)} \]

where

\[ \Phi(x, y) = A_0 \frac{1}{(x+y)^2} + A_1 (x+y)^2 + A_2 (x+y)^4 + A_3 (x+y)^6 \]

\[ \Psi(x, y) = -A_0 \frac{1}{(x-y)^2} - A_1 (x-y)^2 - A_2 (x-y)^4 - A_3 (x-y)^6 \]

\[ \Phi(x, y) = a_0 \frac{1}{(x+y)^2} + a_1 (x+y)^2 + a_2 (x+y)^4 + a_3 (x+y)^6 \]

\[ \Psi(x, y) = -a_0 \frac{1}{(x-y)^2} - a_1 (x-y)^2 - a_2 (x-y)^4 - a_3 (x-y)^6 \]
while using relations (50) we have that in \(x, y\) coordinates the second integral of motion is:

\[
B(x, y) = \frac{1}{x^2}p_x^2 + \frac{1}{y^2}p_y^2 - 2\frac{p_x p_y}{xy} \frac{\Phi(x, y) - \Psi(x, y)}{\Phi(x, y) + \Psi(x, y)} + 4\frac{\phi(x, y)\Psi(x, y) - \psi(x, y)\Phi(x, y)}{xy \Phi(x, y) + \Psi(x, y)}
\]

where

\[
\Phi(x, y) = \frac{1}{2}A_1 \left[ (x + y)^2 - (x - y)^2 \right] + A_2 \left[ (x + y)^4 - (x - y)^4 \right] + A_3 \left[ (x + y)^6 - (x - y)^6 \right] + (x + y)^4(x - y)^2 - (x + y)^2(x - y)^4
\]

\[
\Psi(x, y) = A_0 \left[ \frac{1}{(x+y)^2} - \frac{1}{(x-y)^2} \right] + \frac{1}{2}A_1 \left[ (x + y)^2 - (x - y)^2 \right] - A_3 \left[ (x + y)^4(x - y)^2 - (x + y)^2(x - y)^4 \right]
\]

\[
\phi(x, y) = \frac{1}{2}a_1 \left[ (x + y)^2 - (x - y)^2 \right] + a_2 \left[ (x + y)^4 - (x - y)^4 \right] + a_3 \left[ (x + y)^6 - (x - y)^6 \right] + (x + y)^4(x - y)^2 - (x + y)^2(x - y)^4
\]

\[
\psi(x, y) = a_0 \left[ \frac{1}{(x+y)^2} - \frac{1}{(x-y)^2} \right] + \frac{1}{2}a_1 \left[ (x + y)^2 - (x - y)^2 \right] - a_3 \left[ (x + y)^4(x - y)^2 - (x + y)^2(x - y)^4 \right]
\]

**Class I_2** Using the coordinate transformation

\[
\xi = -\frac{1}{2}\cos(2x), \quad p_\xi = \frac{p_x}{\sin(2x)}, \quad \eta = -\frac{1}{2}\cos(2y), \quad p_\eta = \frac{p_y}{\sin(2y)};
\]

the metric of the Class I_2 superintegrable systems is reduced to the metric of the essential form VII.4 \[^1\] vol IV, p.385, if

\[
\kappa = A_1, \quad \lambda = -8A_3, \quad \mu = -A_0, \quad \nu = -2A_2
\]

The corresponding superintegrable system (using the coordinates of ref \[^1\]) is given by the Hamiltonian:

\[
H = \frac{p_x p_y + w(x, y)}{g(x, y)}
\]
that in can put one among them equal to zero. Using relations (53) we have
\[ w(x, y) = a_0 \left[ \frac{1}{\sin^2(x + y)} - \frac{1}{\sin^2(x - y)} \right] + a_1 \left[ \frac{1}{\cos^2(x + y)} - \frac{1}{\cos^2(x - y)} \right] + a_2 \left[ \cos 2(x + y) - \cos 2(x - y) \right] + a_3 \left[ \cos 4(x + y) - \cos 4(x - y) \right] \]

where only three of the constants \( a_0, a_1, a_2, a_3 \) are independent, i.e. we can put one among them equal to zero. Using relations (53) we have that in \( x, y \) coordinates the other integral of motion is:
\[
A(x, y) = \frac{1}{8} \cot^2(2x) p_x^2 + \frac{1}{8} \cot^2(2y) p_y^2 - \frac{p_x p_y}{2 \tan(2x) \tan(2y)} \frac{\Phi(x, y) - \Psi(x, y)}{\Phi(x, y) + \Psi(x, y)} + \frac{1}{\tan(2x) \tan(2y)} \frac{\tilde{\Phi}(x, y) - \tilde{\Psi}(x, y)}{\Phi(x, y) + \Psi(x, y)} \]

where
\[
\tilde{\Phi}(x, y) = A_2[\cos 2(x + y) - \cos 2(x - y)] + A_3[\cos 4(x + y) - \cos 4(x - y)]
\]
\[
\tilde{\Psi}(x, y) = A_0 \left[ \frac{1}{\sin^2(x + y)} - \frac{1}{\sin^2(x - y)} \right] + A_1 \left[ \frac{1}{\cos^2(x + y)} - \frac{1}{\cos^2(x - y)} \right]
\]
\[
\tilde{\phi}(x, y) = a_2[\cos 2(x + y) - \cos 2(x - y)] + a_3[\cos 4(x + y) - \cos 4(x - y)]
\]
\[
\tilde{\psi}(x, y) = a_0 \left[ \frac{1}{\sin^2(x + y)} - \frac{1}{\sin^2(x - y)} \right] + a_1 \left[ \frac{1}{\cos^2(x + y)} - \frac{1}{\cos^2(x - y)} \right]
\]

while using relations (52) we have that in \( x, y \) coordinates the second integral of motion is:
\[
B(x, y) = \frac{1}{\sin^2(2x) p_x^2 + \sin^2(2y) p_y^2 - 2 \frac{p_x p_y}{\sin(2x) \sin(2y)} \frac{\Phi(x, y) - \Psi(x, y)}{\Phi(x, y) + \Psi(x, y)} + 4 \frac{1}{\sin(2x) \sin(2y)} \frac{\tilde{\phi}(x, y) - \tilde{\psi}(x, y)}{\tilde{\phi}(x, y) + \tilde{\psi}(x, y)} \]

where
\[
\Phi(x, y) = A_1 \left[ \frac{1}{\cos^2(x + y)} - \frac{1}{\cos^2(x - y)} \right] + \frac{1}{2} A_2 [\cos 2(x + y) - \cos 2(x - y)] + 4 A_3 \cos^2(x + y) \cos^2(x - y) [\cos 2(x + y) - \cos 2(x - y)]
\]
\[
\Psi(x, y) = A_0 \left[ \frac{1}{\sin^2(x + y)} - \frac{1}{\sin^2(x - y)} \right] + \frac{1}{2} A_2 [\cos 2(x + y) - \cos 2(x - y)] - 4 A_3 \sin^2(x + y) \sin^2(x - y) [\cos 2(x + y) - \cos 2(x - y)]
\]
Using the coordinate transformation

\[
\phi(x, y) = a_1 \left[ \frac{1}{\cos^2(x+y)} - \frac{1}{\cos^2(x-y)} \right] + \frac{1}{2} a_2 \left[ \cos(2x+y) - \cos(2x-y) \right] + 4 a_3 \cos^3(x+y) \cos^2(x-y) \left[ \cos(2x+y) - \cos(2x-y) \right]
\]

\[
\psi(x, y) = a_0 \left[ \frac{1}{\sin^2(x+y)} - \frac{1}{\sin^2(x-y)} \right] + \frac{1}{2} a_2 \left[ \cos(2x+y) - \cos(2x-y) \right] - 4 a_3 \sin^2(x+y) \sin^2(x-y) \left[ \cos(2x+y) - \cos(2x-y) \right]
\]

**Class I**

Using the coordinate transformation

\[
\xi = \ln \left( \frac{\varphi(x) - \varphi(x_1)}{\Delta} \right) + \frac{1}{2} \ln \left( \frac{\varphi(x_1) - \varphi(x_2)}{\Delta} \right), \quad p_\xi = \frac{\varphi(2x) - \varphi(x_1)}{2\Delta} p_x
\]

\[
\eta = \ln \left( \frac{\varphi(x) - \varphi(x_1)}{\Delta} \right) + \frac{1}{2} \ln \left( \frac{\varphi(x_1) - \varphi(x_2)}{\Delta} \right), \quad p_\eta = \frac{\varphi(2y) - \varphi(x_1)}{2\Delta} p_y
\]

where

\[
\Delta^2 = (\varphi(x_1) - \varphi(x_2))(\varphi(x_1) - \varphi(x_3))
\]

the metric of the Class I superintegrable systems is reduced to the metric of the essential form VII.1 [1] vol IV, p.385, if

\[
\kappa = 2(A_2 + A_3), \quad \lambda = A_2 - A_3, \quad \mu = -2(A_0 + A_1), \quad \nu = -A_0 + A_1
\]

The corresponding superintegrable system (using the coordinates of ref [1]) is given by the Hamiltonian:

\[
H = \frac{p_x p_y + w(x, y)}{g(x, y)}
\]

where

\[
g(x, y) = A_0 (\varphi(x+y) - \varphi(x-y)) + A_1 (\varphi(x+y+w) - \varphi(x-y+w)) + A_2 (\varphi(x+y+w) - \varphi(x-y+w)) + A_3 (\varphi(x+y+w) - \varphi(x-y+w))
\]

\[
w(x, y) = a_0 (\varphi(x+y) - \varphi(x-y)) + a_1 (\varphi(x+y+w) - \varphi(x-y+w)) + a_2 (\varphi(x+y+w) - \varphi(x-y+w)) + a_3 (\varphi(x+y+w) - \varphi(x-y+w))
\]

where only three of the constants \(a_0, a_1, a_2, a_3\) are independent, i.e. we can put one among them equal to zero. Using relations (54) we have
that in \( x, y \) coordinates the other integral of motion is:

\[
A(x, y) = \frac{1}{4\Delta x} (\wp(2x) - \wp'(\omega_1)) p_x^2 + \frac{1}{4\Delta y} (\wp(2y) - \wp'(\omega_1)) p_y^2 - 2 \frac{\wp(2x) - \wp(\omega_1)}{\wp'(\omega_1)} \sqrt{\wp(2y) - \wp'(\omega_1)} p_x p_y \Phi(x, y) \Psi(x, y) \phi(x, y) \psi(x, y) \Phi(x, y) + \frac{\wp(2x) - \wp(\omega_1)}{\wp'(\omega_1)} \sqrt{\wp(2y) - \wp'(\omega_1)} \phi(x, y) \psi(x, y) \Phi(x, y) + \Psi(x, y) \\
\]

where

\[
\Phi(x, y) = A_2 (\wp(x + y + \omega_2) - \wp(x - y + \omega_2)) + A_3 (\wp(x + y + \omega_3) - \wp(x - y + \omega_3)) \\
\Psi(x, y) = A_0 (\wp(x + y) - \wp(x - y)) + A_1 (\wp(x + y + \omega_1) - \wp(x - y + \omega_1)) \\
\phi(x, y) = a_2 (\wp(x + y + \omega_2) - \wp(x - y + \omega_2)) + a_3 (\wp(x + y + \omega_3) - \wp(x - y + \omega_3)) \\
\psi(x, y) = a_0 (\wp(x + y) - \wp(x - y)) + a_1 (\wp(x + y + \omega_1) - \wp(x - y + \omega_1))
\]

while using relations \([55]\) we have that in \( x, y \) coordinates the second integral of motion is:

\[
B(x, y) = \frac{1}{(\wp(\omega_3) - \wp(\omega_1)) (\wp(\omega_2) - \wp(\omega_3))} (\wp(2x) - \wp'(\omega_3)) p_x^2 + \frac{1}{(\wp(\omega_3) - \wp(\omega_1)) (\wp(\omega_2) - \wp(\omega_3))} (\wp(2y) - \wp'(\omega_3)) p_y^2 - 2 \frac{\wp(2x) - \wp(\omega_3)}{\wp'(\omega_1)} \sqrt{\wp(2y) - \wp'(\omega_3)} p_x p_y \Phi(x, y) \Psi(x, y) \phi(x, y) \psi(x, y) \Phi(x, y) + \Psi(x, y) \\
\]

where

\[
\tilde{\Phi}(x, y) = A_1 (\wp(x + y + \omega_1) - \wp(x - y + \omega_1)) + A_2 (\wp(x + y + \omega_2) - \wp(x - y + \omega_2)) - \frac{1}{2} (A_0 + A_1 + A_2 + A_3) \sqrt{\wp(2x) - \wp(\omega_1)} \sqrt{\wp(2y) - \wp(\omega_1)} \\
\tilde{\Psi}(x, y) = A_0 (\wp(x + y) - \wp(x - y)) + A_3 (\wp(x + y + \omega_3) - \wp(x - y + \omega_3)) + \frac{1}{2} (A_0 + A_1 + A_2 + A_3) \sqrt{\wp(2x) - \wp(\omega_1)} \sqrt{\wp(2y) - \wp(\omega_1)}
\]
\[ \tilde{\phi}(x, y) = a_1 (\varphi(x + y + \omega_1) - \varphi(x - y + \omega_1)) + \\
+ a_2 (\varphi(x + y + \omega_2) - \varphi(x - y + \omega_2)) - \\
- \frac{1}{2} (a_0 + a_1 + a_2 + a_3) \frac{\sqrt{\varphi(2x) - \varphi(\omega_1)} \sqrt{\varphi(2y) - \varphi(\omega_1)}}{(\varphi(\omega_2) - \varphi(\omega_1)) (\varphi(\omega_3) - \varphi(\omega_1))} \]

\[ \tilde{\psi}(x, y) = a_0 (\varphi(x + y) - \varphi(x - y)) + \\
+ a_3 (\varphi(x + y + \omega_3) - \varphi(x - y + \omega_3)) + \\
+ \frac{1}{2} (a_0 + a_1 + a_2 + a_3) \frac{\sqrt{\varphi(2x) - \varphi(\omega_3)} \sqrt{\varphi(2y) - \varphi(\omega_3)}}{(\varphi(\omega_1) - \varphi(\omega_3)) (\varphi(\omega_2) - \varphi(\omega_1))} \]

**Class II_1** This case is not covered by Table VII of Koenigs [1] Vol. IV, p. 385. This class corresponds to the Kress[16] equivalence class [0, 11] of the nondegenerate superintegrable systems \( E_{11}, E_{20} \) of ref [6].

The Hamiltonian is

\[ H = \frac{p_\xi \eta + k\xi \eta + \ell \xi + m\eta + \nu}{\kappa \xi \eta + \lambda \xi + \mu \eta + \nu} \]

the integrals of motion are

\[ A = p_\xi^2 - 2p_\xi p_\eta - \frac{\xi^2 + \eta^2}{\kappa \xi \eta + \lambda \xi + \mu \eta + \nu} + \frac{2 \left( k \eta^2 + \ell \eta \right)}{\kappa \xi \eta + \lambda \xi + \mu \eta + \nu} - \\
\frac{\xi \eta^2 + \lambda \eta}{\kappa \xi \eta + \lambda \xi + \mu \eta + \nu} (k \xi \eta + \ell \xi + m \eta + n) \]

\[ B = p_\xi^2 + p_\eta^2 - 2p_\xi p_\eta - \frac{\xi^2 + \eta^2}{\kappa \xi \eta + \lambda \xi + \mu \eta + \nu} + 2 \left( \frac{k}{2} \left( \xi^2 + \eta^2 \right) + \ell \eta + m \xi \right) - \\
\frac{\xi \eta^2 + \lambda \eta}{\kappa \xi \eta + \lambda \xi + \mu \eta + \nu} (k \xi \eta + \ell \xi + m \eta + n) \]

The case VI_6 in Table VI [1] Vol. IV, p.384] Koenigs studied separately the cases where \( \kappa = 0 \) and \( \kappa \neq 0, \lambda = \mu = 0. \)

**Class II_2** Using the coordinate transformation

\[ \xi = \frac{1}{2} x^2, \quad p_\xi = \frac{p_x}{x}, \quad \eta = \frac{1}{2} y^2, \quad p_\eta = \frac{p_y}{y} \]

the metric of the Class II_2 superintegrable systems is reduced to the metric of the essential form VII.5 [1] Vol IV, p.385], if

\[ \kappa = 2\sqrt{2}A_1, \quad \lambda = 16A_0, \quad \mu = \sqrt{2}A_3, \quad \nu = 4A_2 \]
The corresponding superintegrable system (using the coordinates of ref [1]) is given by the Hamiltonian:

\[ H = \frac{p_x p_y}{g(x, y)} \]

where

\[ g(x, y) = A_0 [(x + y)^4 - (x - y)^4] + A_1 [(x + y)^3 - (x - y)^3] + A_2 [(x + y)^2 - (x - y)^2] + A_3 [(x + y) - (x - y)] \]

\[ w(x, y) = a_0 [(x + y)^4 - (x - y)^4] + a_1 [(x + y)^3 - (x - y)^3] + a_2 [(x + y)^2 - (x - y)^2] + a_3 [(x + y) - (x - y)] \]

where only three of the constants \( a_0, a_1, a_2, a_3 \) are independent, i.e. we can put one among them equal to zero. Using relations (59) we have that in \( x, y \) coordinates the other integral of motion is:

\[ A(x, y) = \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 + p_x p_y \frac{\Phi(x, y) - \Psi(x, y)}{\Phi(x, y) + \Psi(x, y)} + 2 \frac{\tilde{\phi}(x, y) \tilde{\psi}(x, y) - \tilde{\psi}(x, y)}{\Phi(x, y) + \Psi(x, y)} \]

where

\[ \Phi(x, y) = 16A_0 x + 4A_1 \]
\[ \phi(x) = 16a_0 x + 4a_1 \]

while using relations (58) we have that in \( x, y \) coordinates the second integral of motion is:

\[ B(x, y) = \frac{1}{y^2} p_y^2 - 2 \frac{p_x p_y}{g(x, y)} \int \Phi(x) dx - \frac{w(x, y)}{g(x, y)} \int \Phi(x) dx + \int \phi(x) dx \]

where

\[ \Phi(x) = 16A_0 x + 4A_1 \]
\[ \phi(x) = 16a_0 x + 4a_1 \]
The coordinate transformation

\[
\xi = -\frac{1}{2} \cos(2y), \quad p_\xi = \frac{1}{\sin(2y)} p_y, \quad \eta = e^{2i x}, \quad p_\eta = -\frac{i e^{-2i x}}{2} p_x
\]

the metric of the Class II₃ superintegrable systems is reduced to the metric of the essential form VII.3 [11] vol IV, p.385, if

\[
\kappa = 2(A_1+i A_0), \quad \lambda = 2(i A_0-A_1), \quad \mu = -\frac{1}{2}(A_3+i A_2), \quad \nu = \frac{1}{2}(A_3-i A_2)
\]

where

\[
g(x, y) = A_0 [\sin(4(x + y) - \sin(4(x - y))] + A_1 [\cos(4(x + y) - \cos(4(x - y))] + A_2 [\sin(2(x + y) - \sin(2(x - y))] + A_3 [\cos(2(x + y) - \sin(2(x - y))]
\]

\[
w(x, y) = a_0 [\sin(4(x + y) - \sin(4(x - y))] + a_1 [\cos(4(x + y) - \cos(4(x - y))] + a_2 [\sin(2(x + y) - \sin(2(x - y))] + a_3 [\cos(2(x + y) - \sin(2(x - y))]
\]

where only three of the constants \(a_0, a_1, a_2, a_3\) are independent, i.e. we can put one among them equal to zero. Using relations (61) we have that in \(x, y\) coordinates the other integral of motion is:

\[
A(x, y) = -\frac{1}{4} p_x^2 + \frac{1}{4} \cot^2(2y) p_y^2 + \frac{p_x p_y}{2 \tan(2y)} \frac{\tilde{\Phi}(x, y) - \tilde{\Psi}(x, y)}{\Phi(x, y) + \tilde{\Psi}(x, y)}
\]

where

\[
\tilde{\Phi}(x, y) = (A_0 + i A_1) e^{4i x} \sin(4y) + (A_2 + i A_3) e^{2i x} \sin(2y)
\]

\[
\tilde{\Psi}(x, y) = (A_0 - i A_1) e^{-4i x} \sin(4y) + (A_2 - i A_3) e^{-2i x} \sin(2y)
\]

\[
\tilde{\phi}(x, y) = (a_0 + i a_1) e^{4i x} \sin(4y) + (a_2 + i a_3) e^{2i x} \sin(2y)
\]

\[
\tilde{\psi}(x, y) = (a_0 - i a_1) e^{-4i x} \sin(4y) + (a_2 - i a_3) e^{-2i x} \sin(2y)
\]

while using relations (61) we have that in \(x, y\) coordinates the second integral of motion is:

\[
B(x, y) = \frac{1}{\sin^2(2y)} p_y^2 - 2 \frac{p_x p_y}{g(x, y)} \int \Phi(x) dx - 2 \int \frac{\omega(x, y)}{g(x, y)} \Phi(x) dx + 2 \int \phi(x) dx
\]

where

\[
\Phi(x, y) = -8A_0 \cos(4x) + 8A_1 \sin(4x)
\]

\[
\phi(x, y) = -8a_0 \cos(4x) + 8a_1 \sin(4x)
\]

38
VII Superintegrable potentials on a surface of revolution with two quadratic integrals of motion

A manifold which is described by a metric of the form
\[ ds^2 = g(x + y)dxdy \quad \text{or} \quad ds^2 = g(x - y)dxdy \]
is called a surface of revolution.

The above condition is possible only for a specific choice of the parameters \( \kappa, \lambda, \mu \) and \( \nu \). In many cases the superintegrable systems can be calculated by using the general forms which are studied in Section [V]. The general forms of these systems by revolution in many instances are given by the formulas:

\[
H = \frac{p_\xi p_\eta + f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta)} \quad \text{or} \quad H = \frac{p_\xi p_\eta + f(\xi + \eta) + g(\xi - \eta)}{G(\xi - \eta)}
\]

and

\[
H = \frac{p_X p_Y + \tilde{f}(X + Y) + \tilde{g}(X - Y)}{F(X + Y)} \quad \text{or} \quad H = \frac{p_X p_Y + \tilde{f}(X + Y) + \tilde{g}(X - Y)}{G(X - Y)}
\]

But we must notice that the Liouville or the Lie coordinates are not always the appropriate ones for concluding whether a surface is a surface by revolution. Among the parameters \( \kappa, \lambda, \mu \) and \( \nu \) the surfaces of revolution are determined by two independent parameters. In Table 3 these special values of the parameters \( \kappa, \lambda, \mu \) and \( \nu \) are shown. We notice the corresponding potentials given in references [13] and [12]. This classification scheme shows that there is the case \( R_{11} \) which is not given in the above references and it is a new not known superintegrable system. A detailed description of the superintegrable systems on surfaces of revolution are given further in the present section.

[R1: Class I \( \kappa = 0, \lambda = 0 \)] The form of the Hamiltonian in Liouville coordinates is given by

\[
H = \frac{p_\eta p_\xi}{\nu + \frac{\mu}{(\xi - \eta)^2}} + \frac{k (\eta + \xi) + 4 \ell (\eta + \xi)^2 - \ell (\xi - \eta)^2 + \frac{m}{(\xi - \eta)^2} + n}{\nu + \frac{\mu}{(\xi - \eta)^2}}
\]
| Class | $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | Essential from [1] | Classes from [16] |
|-------|--------|--------|------|------|------------------|------------------|
| $I_1$ | $16A_2$ | $16A_3$ | $-A_0$ | $4A_1$ | VII.4            | 3, 2             |
| $I_2$ | $A_1$   | $-8A_3$ | $-A_0$ | $-2A_2$| VII.2            | [21, 2]          |
| $I_3$ | $(A_2 + A_3)$ | $(A_2 - A_3)$ | $-2(A_0 + A_1)$ | $-A_0 + A_1$ | VII.1            | [111, 1]         |
| $II_1$ |               |             |       |       |                  | 0, 11            |
| $II_2$ | $2\sqrt{2}A_1$ | $16A_0$ | $\sqrt{2}A_3$ | $4A_2$ | VII.5            | [3, 11]          |
| $II_3$ | $2(A_1 + iA_0)$ | $2(iA_0 - A_1)$ | $-\frac{1}{2}(A_3 + iA_2)$ | $\frac{1}{2}(A_3 - iA_2)$ | VII.3            | [21, 0]          |

Table 1: Essential forms of Table VII in Ref. [1] and equivalence classes of ref. [16]

| Class | $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | Potentials by revolution from ref [13] | Potentials by revolution from ref [12] |
|-------|--------|--------|------|------|----------------------------------------|----------------------------------------|
| $R_1$ | $I_1$  | 0      | 0    | 0    | 2$[A]$                                 |                                        |
| $R_2$ |        | -      | 0    | 0    |                                        | (1)                                    |
| $R_3$ | $I_2$  | 0      | 0    | 0    | 2$[B]$                                 |                                        |
| $R_4$ |        | 0      | 0    | 0    | 3$[B]$                                 |                                        |
| $R_5$ |        | 0      | 0    | 0    | 4$[A]$                                 |                                        |
| $R_6$ |        | $-\mu$ | 0    | 0    | 2$[C]$                                 |                                        |
| $R_7$ | $I_3$  | 0      | 0    | 0    | 4$[B]$                                 |                                        |
| $R_8$ |        | $-\mu$ | $\nu$ | 0    | 4$[C]$                                 |                                        |
| $R_9$ | $II_1$ | 0      | $\mu$ | 0    |                                        | (2)                                    |
| $R_{10}$ |        | 0      | 0    | 0    | 3$[A]$                                 |                                        |
| $R_{11}$ | $II_2$ | 0      | 0    | 0    |                                        | new                                    |
| $R_{12}$ | $II_3$ | 0      | $-\kappa$ | $-\nu$ | 3$[D]$                                 |                                        |
| $R_{13}$ |        | $-\mu$ | 0    | 0    | 3$[C]$                                 |                                        |

Table 2: Potentials by revolution with two quadratic integrals of motion
By the coordinate transformation
\[
\xi = \frac{v + i u}{2}, \quad \eta = \frac{v - i u}{2}, \quad p_\xi = p_v - i p_u, \quad p_\eta = p_v + i p_u
\]
and putting \( \mu = -1, \nu = 1 \), the Hamiltonian 2\( [A] \) of reference [13] is obtained
\[
H = \frac{u^2}{u^2 + 1} \left( p^2_u + p^2_v + k v + 4 \ell \left( \frac{1}{4} u^2 + v^2 \right) - \frac{m + n}{u^2} \right) + n
\]

**R2: Class I \( \lambda = 0, \mu = 0 \)** The form of the Hamiltonian in Liouville coordinates is given by
\[
H = \frac{p_\eta p_\xi}{\kappa (\eta + \xi) + \nu} + \frac{k (\eta + \xi) + 4 \ell (\eta + \xi) - \ell (\xi - \eta)^2 + \frac{m}{(\xi - \eta)^2} + n}{\kappa (\eta + \xi) + \nu}
\]
By the coordinate transformation
\[
\xi = u + i v, \quad \eta = u - i v, \quad p_\xi = \frac{1}{2} (p_u - i p_v), \quad p_\eta = \frac{1}{2} (p_u + i p_v)
\]
and putting \( \kappa = 1/2, \nu = 0 \) the Hamiltonian (1) of reference [12] is obtained
\[
H = \frac{p^2_u + p^2_v}{4 u} + \frac{16 \ell (4 u^2 + v^2)}{4 u} + \frac{n}{u} - \frac{m}{4 u v^2} + 2 k
\]

**R3: Class I \( \lambda = 0, \kappa = 0 \)** The form of the Hamiltonian in Liouville coordinates is given by
\[
H = \frac{p_\eta p_\xi}{\nu + \frac{\mu}{(\eta - \xi)^2}} + \frac{k (\eta + \xi) + \ell (\eta + \xi)^2 - \ell (\xi - \eta)^2 + \frac{m}{(\xi - \eta)^2} + n}{\nu + \frac{\mu}{(\xi - \eta)^2}}
\]
By the coordinate transformation
\[
\xi = \frac{v + i u}{2}, \quad \eta = \frac{v - i u}{2}, \quad p_\xi = p_v - i p_u, \quad p_\eta = p_v + i p_u
\]
and putting \( \mu = -1, \nu = 1 \), the Hamiltonian 2\( [B] \) of reference [13] is obtained
\[
H = \frac{u^2}{u^2 + 1} \left( p^2_u + p^2_v + \ell (u^2 + v^2) - \frac{m + n}{u^2} + \frac{k}{v^2} \right) + n
\]
The form of the Hamiltonian in coordinates $(X,Y)$ is given by

$$H = p_X p_Y + 4\ell e^{2(X+Y)} + ne^{(X+Y)} + k \frac{e^{X+Y}}{(1+e^{X+Y})^2} + m \frac{e^{X+Y}}{(-1+e^{X+Y})^2}.$$  

By the coordinate transformation

$$X = -\frac{1}{2} \ln \left( \frac{4}{(iv+u)^2} \right), \quad Y = -\frac{1}{2} \ln \left( \frac{4}{(v+iu)^2} \right),$$

$$p_X = \frac{1}{2} ((iv+u)p_u + (v-iv)p_v), \quad p_Y = -\frac{1}{2} ((iv-u)p_u - (v+iu)p_v)$$

and putting $\lambda = 1, \nu = 4$ the Hamiltonian $3[B]$ of reference [13] is obtained

$$H = \frac{p_u^2 + p_v^2 + \frac{k}{u^2} - \frac{m}{v^2} + n - 4l}{4 + u^2 + v^2} + l$$

The form of the Hamiltonian in coordinates $(X,Y)$ is given by

$$H = p_X p_Y + 4\ell e^{2(X+Y)} + ne^{(X+Y)} + k \frac{e^{X+Y}}{(1+e^{X+Y})^2} + m \frac{e^{X+Y}}{(-1+e^{X+Y})^2}.$$  

By the coordinate transformation

$$X = \ln \left( \frac{1}{2}(x - iy) \right), \quad Y = \ln \left( \frac{1}{2}(x + iy) \right)$$

$$p_X = \frac{1}{2} ((x - iy)p_x + (ix + y)p_y), \quad p_Y = \frac{1}{2} ((x + iy)p_x + (y - ix)p_y)$$

Putting $\kappa = \frac{2-\alpha}{4}, \mu = \frac{2+\alpha}{4}$, the Hamiltonian $4[A]$ of reference [13] is obtained

$$H = -\frac{4x^2y^2 \left( p_x^2 + p_y^2 + n + \frac{1}{4} ((2 + \alpha)k + (\alpha - 2)m) \left( \frac{1}{x^2} + \frac{1}{y^2} \right) + \ell \left( x^2 + y^2 \right) \right)}{y^2(\alpha - 2) + x^2(2 + \alpha)} + k + m$$
The form of the Hamiltonian in Liouville coordinates is given by

\[
H = \frac{p_\eta p_\xi}{(\eta+\xi)^2} + \lambda (\eta + \xi)^2 - \lambda (\xi - \eta)^2 + \frac{\mu}{(\xi - \eta)^2} + \frac{\eta + \xi}{(\eta+\xi)^2} - \ell (\eta - \xi)^2 + \frac{m}{(\xi - \eta)^2} + n
\]

By the coordinate transformation

\[
\xi = u + i v, \eta = u - i v, p_\xi = \frac{1}{2}(p_u - i p_v), p_\eta = \frac{1}{2}(p_u + i p_v)
\]

and putting \(\lambda = \frac{1}{16}, \mu = -1\), the Hamiltonian 2[C] of reference [13] is obtained

\[
H = \frac{p_u^2 + p_v^2 + 4 n + \frac{k-16 \ell}{u^2} - \frac{16 \ell + m}{v^2}}{u^2 + v^2 + \frac{1}{u^2} + \frac{1}{v^2}} + 16 \ell
\]

The form of the Hamiltonian in Liouville coordinates is given by

\[
H = \frac{(e^{\eta+\xi}-e^{\eta-\xi})^2}{(e^{\eta+\xi}+e^{\eta-\xi})^2} + \frac{\mu (e^{\eta+\xi}-e^{\eta-\xi})^2}{\nu (e^{\eta+\xi}+e^{\eta-\xi})^2} + \frac{\nu (e^{2(\eta+\xi)})}{\mu} + \frac{\nu (1+e^{2(\eta+\xi)})}{\mu} + \frac{m (e^{2(\xi-\eta)})}{\nu} + \frac{n (e^{2(\xi-\eta)})}{\nu}
\]

By the coordinate transformation

\[
\xi = v + i u, \eta = v - i u, p_\xi = \frac{1}{2}(p_v - i p_u), p_\eta = \frac{1}{2}(p_v + i p_u)
\]

and putting \(\mu = \alpha, \nu = 1\), the Hamiltonian 4[B] of reference [13] is obtained

\[
H = -\frac{\sin^2(2u)(p_u^2 + p_v^2 + \frac{k+2 \ell}{\sinh^2(v)} + \frac{2 \ell - k}{\cosh^2(v)}) + n \alpha - m}{2 \cos(2u) + \alpha} + n
\]
$$\textbf{R_3: Class I_3}} \; \kappa = -\mu, \; \lambda = \nu$$ The form of the Hamiltonian in Liouville coordinates is given by

$$H = \frac{\mu + \nu}{\eta + \xi} \frac{p_\eta p_\xi}{(\xi - \eta - e^{-\xi - \eta})^2} + \frac{\mu + \nu}{\eta + \xi} \frac{p_\eta p_\xi}{(\xi - \eta - e^{\xi + \eta - (\xi + \eta)})^2} + \frac{k}{\eta + \xi} \frac{e^{\xi + \eta - \xi}}{(\eta + \xi)^2} \frac{e^{\eta + \xi} (1 + e^{2(\eta + \xi)})}{(\xi - \eta - e^{-\xi - \eta})^2} + m \frac{e^{\eta + \xi} (1 + e^{2(\eta + \xi)})}{(\xi - \eta - e^{-\xi - \eta})^2}$$

By the coordinate transformation

$$\xi = \arcsinh(\tan(\phi - i \omega)), \; \eta = \arcsinh(\tan(\phi + i \omega))$$

$$p_\xi = \frac{1}{2} \cos^2(\phi - i \omega) \sqrt{1 + \tan^2(\phi - i \omega)} \left( p_\phi + i p_\omega \right)$$

$$p_\eta = \frac{1}{2} \cos^2(\phi + i \omega) \sqrt{1 + \tan^2(\phi + i \omega)} \left( p_\phi - i p_\omega \right)$$

and putting $\mu = \alpha, \; \nu = 1$, the Hamiltonian $4[C]$ of reference [13] is obtained

$$H = -\frac{\frac{1}{\sinh^2(2\omega)}}{\frac{1}{\sinh^2(2\omega)}} - \frac{e^{1/2} \cos(\phi) + \cos^2(\omega)}{\alpha^{1/2}} + \frac{e^{1/2} \sin(\phi) \sinh(\omega)}{\alpha^{1/2}}$$

where $c_1 = -\frac{m(2 + \alpha) + (k + 2 \ell) \cdot (2 - 2 \alpha)}{8 \alpha}$, $c_2 = -\frac{k(-2 + \alpha) + (m + 2 \alpha) \cdot (2 - 2 \alpha)}{8 \alpha}$

and $c_3 = \frac{(m + 2 \alpha) \cdot (2 - 2 \alpha)}{8 \alpha}$

$$\textbf{R_4: Class II_1}} \; \kappa = 0, \; \lambda = \mu$$ The form of the Hamiltonian in Liouville coordinates is given by

$$H = \frac{p_\eta p_\xi}{\mu (\eta + \xi)} + \frac{k \eta \xi + \ell \xi + m \eta + n}{\mu (\eta + \xi)}$$

By the coordinate transformation

$$\xi = u + i v, \; \eta = u - i v, \; p_\xi = \frac{1}{2} (p_u - i p_v), \; p_\eta = \frac{1}{2} (p_u + i p_v)$$

and putting $\mu = 1/2, \; \nu = 0$, the Hamiltonian (2) of reference [12] is obtained

$$H = \frac{p_u^2 + p_v^2}{4 u} + \frac{n}{u} + \frac{i (\ell - m) v}{u} + \frac{k (u^2 + v^2)}{u} + \ell + m$$
The form of the Hamiltonian in Liouville coordinates is given by

\[ H = \frac{p_\eta p_\xi}{\nu + \eta \kappa \xi} + \frac{n + m \eta + \ell \xi + k \eta \xi}{\nu + \eta \kappa \xi} \]

by the coordinate transformation

\[ \xi = u + i v, \eta = u - i v, p_\xi = \frac{1}{2} (p_u - i p_v), p_\eta = \frac{1}{2} (p_u + i p_v) \]

and putting \( \kappa = \frac{1}{4}, \nu = 1 \), the Hamiltonian 3\([A]\) of reference [13] is obtained

\[ H = \frac{p_u^2 + p_v^2 + 4 (\ell + m) u + 4 i (\ell - m) + 4 (n - 4 k)}{4 + u^2 + v^2} + 4 k \]

The form of the Hamiltonian in Liouville coordinates is given by

\[ H = \frac{p_\eta p_\xi}{\lambda (\eta + \xi) + \nu} + \frac{n + m + \ell \sqrt{(\eta + \xi) + k (3 \eta + \xi)}}{\lambda (\eta + \xi) + \nu} \]

by the coordinate transformation

\[ \xi = u + i v, \eta = u - i v, p_\xi = \frac{1}{2} (p_u - i p_v), p_\eta = \frac{1}{2} (p_u + i p_v) \]

and putting \( \lambda = \frac{1}{2}, \nu = 0 \), we have

\[ H = \frac{p_u^2 + p_v^2}{4 u} + \frac{m + (n + 2 \ell u) \sqrt{u - i v} + k (4 u - 2 i v)}{u \sqrt{u - i v}} \]

For this Hamiltonian the additional integrals of motion have the form

\[ A = -\frac{i}{2} X_1 - \frac{K^2}{2} - \frac{2 \sqrt{u - i v}}{u} \left( \frac{m + \frac{n \sqrt{u - i v}}{2} - i k v}{u} \right) \]

\[ B = X_2 - \frac{2 v}{u} \left( \frac{m (i u + \frac{v}{2})}{u \sqrt{u - i v}} + \frac{\left( \frac{n \sqrt{u - i v}}{2} - i k v \right)}{u \sqrt{u - i v}} \right) \]
where $K$, $X_1$, $X_2$ are the three integrals of the free motion of reference [12].

$$K = p_v, \quad X_1 = p_u p_v - \frac{v}{2u} \left( p_u^2 + p_v^2 \right) \quad \text{and} \quad X_2 = p_v (v p_u - u p_v) - \frac{v^2}{4u} \left( p_u^2 + p_v^2 \right)$$

This system was not included in reference [12]. Therefore is a new superintegrable system, which is studied here for the first time as far as we known.

**$R_{12}$: Class II$_3$ $\kappa = -\lambda, \mu = -\nu$**

The form of the Hamiltonian in $(\xi, \eta)$ coordinates is given by

$$H = \frac{p\xi \rho_{\eta} + \ell \xi \eta + \frac{k \xi}{\eta^2} + \frac{m}{\eta^2} + n}{\lambda \xi \eta + \frac{k \xi}{\eta^2} + \frac{m}{\eta^2} + \nu}$$

By the coordinate transformation

$$\xi = 2 \sqrt{u v}, \quad \eta = i \sqrt{\frac{u}{v}}, \quad p\xi = \frac{1}{2} \left( \sqrt{\frac{u}{v}} p_u + \sqrt{\frac{v}{u}} p_v \right), \quad p\eta = -i \sqrt{u v} p_u + i \left( \sqrt{\frac{v}{u}} p_v \right)$$

and putting $\kappa = \frac{1}{4}, \lambda = -\frac{1}{4}, \mu = i, \nu = -i$ the Hamiltonian 3[D] of reference [13] is obtained

$$H = \frac{u^2 p_u^2 - v^2 p_v^2 + 2 (2 (\ell - k) + i n) u + 2 (2 (\ell - k) - i m) v - 2 (k + \ell)}{(u + v)(2 + u - v)} + 2 (k - \ell)$$

**$R_{13}$: Class II$_3$ $\kappa = -\mu, \lambda = 0, \nu = 0$**

The form of the Hamiltonian in $(\xi, \eta)$ coordinates is given by

$$H = \frac{p\xi \rho_{\eta} + \ell \xi \eta + \frac{k \xi}{\eta^2} + \frac{m}{\eta^2} + n}{\frac{k \xi}{\eta^2} + \frac{m}{\eta^2}}$$

By the coordinate transformation

$$\xi = \frac{v - u}{\sqrt{u v}}, \quad \eta = \frac{2}{\sqrt{u v}}$$

$$p\xi = -\frac{u \sqrt{u v}}{u + v} p_u + \frac{v \sqrt{u v}}{u + v} p_v, \quad p\eta = -\frac{1}{2} \sqrt{u v} (u p_u + v p_v)$$

and putting $\kappa = -4, \mu = 4$, the Hamiltonian 3[C] of reference [13] is obtained

$$H = \frac{u^2 p_u^2 - v^2 p_v^2 + \frac{k + m}{2} (u + v) + 2 n \frac{u + v}{u v} - 4 \ell \frac{u^2 - v^2}{u v} - \frac{k}{4}}{(u + v)(2 + u - v)}$$

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**VIII Superintegrable potentials on a manifold with curvature zero**

Let us consider the manifold corresponding to the metric in the Liouville coordinates:

\[ ds^2 = g(\xi, \eta) \, d\xi \, d\eta \]

the curvature is defined by:

\[ K = -\frac{1}{2g} \frac{\partial^2}{\partial \xi \partial \eta} \ln g = 0 \quad (62) \]

The above constraint imposes restrictions on the choice of the parameters \( \kappa, \lambda, \mu \) and \( \nu \). In Table 3 we can see the possible choices of the values of the above parameters.

In this category there are three families of potentials:

- The potentials on the complex \( E \) plane corresponding to the Hamiltonian:
  \[ H = p_x^2 + p_y^2 + V(x, y) \]
  these potentials are classified in reference [5] and finally the final list of potentials are given in reference [6]. In this section we follow the enumeration of the potentials given by this exhaustive list [6].

- The Drach potentials corresponding to the Hamiltonian
  \[ H = p_x p_y + V(x, y) \]
  these potentials are classified in reference [8]. In this section we follow the enumeration of this reference for the Drach potentials with quadratic integrals of motion.

- The potentials on the real hyperbolic plane \( H_2 \) corresponding to the Hamiltonian:
  \[ H = p_x^2 - p_y^2 + V(x, y) \]
  these potentials are classified in reference [9].

Generally the condition (62) restricts the choices of the constants \( \kappa, \lambda, \mu \) and \( \nu \) in one independent parameter. These values characterize the permitted metrics in the classification given in Section [V]. The permitted choices are given by the following list.
The form of the Hamiltonian in Liouville coordinates is given by

\[ H = \frac{p_\eta p_\xi}{\nu} + \frac{k (\xi + \eta) + 4 \ell (\xi + \eta)^2 - \ell (\xi - \eta)^2 + \frac{m}{(\xi - \eta)^2} + n}{\nu} \]

By the coordinate transformation:

\[ \xi = x + i\ y, \ \eta = x - i\ y, \ p_\xi = \frac{1}{2} (p_x - i\ p_y), \ p_\eta = \frac{1}{2} (p_x + i\ p_y) \]

and putting \( \nu = \frac{1}{4} \), the complex plane Hamiltonian \( E_2 \) of reference [6] is obtained

\[ H = p_x^2 + p_y^2 + 16 \ell \left( 4 x^2 + y^2 \right) + 8 k x - \frac{m}{y^2} + 4 n \]

Also the potential \( V^\alpha \) of reference [9] is generated.

By the coordinate transformation:

\[ \xi = \frac{x}{r}, \ \eta = y, \ p_\xi = r p_x, \ p_\eta = p_y \] where \( r \) is a constant

and putting \( \nu = r \), the Drach Hamiltonian \((a)(r \neq 0)\) of reference [8] is obtained:

\[ H = p_x p_y + \frac{k (x + r y)}{r^2} + \frac{m r}{(x - r y)^2} + \frac{3 \ell (x^2 + r^2 y^2)}{r^3} + \frac{10 \ell x y}{r^2} + \frac{n}{r} \]

This system is generated by the two systems of revolution \( R_1 \) and \( R_2 \) see Table 6. In these cases two parameters among the \( \kappa, \lambda, \mu \) and \( \nu \) are zero. When a third parameter is zero these systems are degenerated to the system \( F_1 \) in Table 6.

The form of the Hamiltonian in Liouville coordinates is given by

\[ H = \frac{p_\eta p_\xi}{\nu} + \frac{k (\xi + \eta)^2 + \ell (\xi + \eta)^2 - \ell (\xi - \eta)^2 + \frac{m}{(\xi - \eta)^2} + n}{\nu} \]

By the coordinate transformation:

\[ \xi = x + i\ y, \ \eta = x - i\ y, \ p_\xi = \frac{1}{2} (p_x - i\ p_y), \ p_\eta = \frac{1}{2} (p_x + i\ p_y) \]
and putting $\nu = \frac{1}{4}$, the complex plane Hamiltonian $E_1$ of reference [6] and the Hamiltonian $V^b$ of reference [9] are obtained

$$H = p_x^2 + p_y^2 + 16 \ell \left( x^2 + y^2 \right) + \frac{k}{x^2} - \frac{m}{y^2} + 4n$$

By the coordinate transformation:

$$\xi = \frac{x}{r}, \eta = y, p_\xi = rp_x, p_\eta = p_y \text{ where } r \text{ is a constant}$$

and putting $\nu = r$, the Drach Hamiltonian $(b) (r \neq 0)$ of reference [8] is obtained:

$$H = p_\xi p_\eta + \frac{4 \ell xy}{r^2} + \frac{m r}{(x - r y)^2} + \frac{k r}{(x + r y)^2} + \frac{n}{r}$$

$F_3$: **Class I** $\kappa = 0, \mu = 0, \nu = 0$ The form of the Hamiltonian in Liouville coordinates is given by

$$H = \frac{p_\eta p_\xi}{4 \lambda \xi \eta} + \frac{k}{(\eta + \xi)^2} - \frac{\ell (\eta + \xi)^2}{(\eta - \xi)^2} + \frac{m}{(\xi - \eta)^2} + n$$

By the coordinate transformation:

$$\xi = \sqrt{\frac{x + iy}{2}}, \eta = \sqrt{\frac{x - iy}{2}}$$

$$p_\xi = \sqrt{2(x + iy)} (p_x - ip_y), p_\eta = \sqrt{2(x - iy)} (p_x + ip_y)$$

and putting $\lambda = 1$, the complex plane Hamiltonian $E_{16}$ of reference [6]

$$H = p_x^2 + p_y^2 + \frac{1}{\sqrt{x^2 + y^2}} \left( \frac{n}{2} + \frac{k}{x + \sqrt{x^2 + y^2}} + \frac{m}{x - \sqrt{x^2 + y^2}} \right) + \ell$$

and the Hamiltonian $V^c$ of reference [9]

$$H = p_x^2 + p_y^2 + \frac{n}{\sqrt{x^2 + y^2}} + \frac{-k + m}{y^2 \sqrt{x^2 + y^2}} + \frac{k - m}{y^2}$$

are obtained
By the coordinate transformation:

\[ \xi = \sqrt{\frac{x}{r}}, \quad \eta = \sqrt{y}, \quad p_\xi = 2\sqrt{r}x \quad p_x, \quad p_\eta = 2\sqrt{y} \quad p_y \]

and putting \( \lambda = r \), the Drach Hamiltonian \((g)(r \neq 0)\) of reference \([8]\) is obtained:

\[ H = p_x \quad p_y + \frac{n}{\sqrt{xy}} + \frac{(m-k)}{2} \frac{r}{(x-ry)^2} + \frac{k+m}{4} \sqrt{\frac{x}{y}} \frac{r}{(x-ry)^2} + \ell \frac{r}{r} \]

The form of the Hamiltonian in Liouville coordinates is given by

\[ H = \frac{p_\eta \quad p_\xi}{\kappa \eta \xi} + \frac{n + m \cdot \eta + (\ell + k \cdot \eta)}{\kappa \eta \xi} \]

By the coordinate transformation:

\[ \xi = \sqrt{\frac{x}{x^2+y^2} + \frac{x}{\sqrt{x^2+y^2}}}, \quad \eta = \sqrt{\frac{x}{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2}}} \]

\[ p_\xi = \sqrt{\frac{x}{x^2+y^2} + \frac{x}{\sqrt{x^2+y^2}}} p_x + \frac{x}{\sqrt{y}} \left( \frac{x}{x^2+y^2} + \frac{x}{\sqrt{x^2+y^2}} \right) p_y \]

\[ p_\eta = \sqrt{\frac{x}{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2}}} p_x + \frac{i}{\sqrt{2}} \frac{x}{x^2+y^2} p_y \]

and putting \( \kappa = 1 \), the complex plane Hamiltonian \( E_{20} \) of reference \([6]\) and the Hamiltonian \( V_d \) of reference \([9]\) are obtained

\[ H = p_x^2 + p_y^2 + \frac{1}{x^2+y^2} \left( n + \frac{\ell + m}{\sqrt{2}} \right) \frac{x}{x^2+y^2} + \right( \frac{\ell - m}{\sqrt{2}} \frac{x}{x^2+y^2} \right) + k \]

By the coordinate transformation:

\[ \xi = 2\sqrt{x}, \quad \eta = 2\sqrt{y}, \quad p_\xi = \sqrt{x} \quad p_x, \quad p_\eta = \sqrt{y} \quad p_y \]

and putting \( \kappa = \frac{1}{4} \), the Drach Hamiltonian \((c)\) of reference \([8]\) is obtained:

\[ H = p_x \quad p_y + \frac{n}{\sqrt{xy}} + \frac{2m}{\sqrt{x}} + \frac{2\ell}{\sqrt{y}} + 4k \]
\[ \textbf{F}_5: \text{ Class II}_1 \quad \kappa = 0, \lambda = 0, \nu = 0 \]

The form of the Hamiltonian in Liouville coordinates is given by

\[
H = \frac{p_\eta p_\xi + k \eta \xi + \ell \xi + m \eta + n}{\eta \mu}
\]

By the coordinate transformation:

\[
\xi = x + i \, y = z, \quad \eta = 2 \sqrt{x - i \, y} = 2 \sqrt{z}
\]

\[
p_\xi = \frac{1}{2} (p_x - i \, p_y), \quad p_\eta = \frac{1}{2} \sqrt{x - i \, y} (p_x + i \, p_y)
\]

and putting \( \mu = \frac{1}{8} \), the complex plane Hamiltonian \( E_{11} \) of reference [6] is obtained

\[
H = p_x^2 + p_y^2 + 8 k \, z + 4 \ell \frac{z}{\sqrt{z}} + \frac{4n}{\sqrt{z}} + 8m
\]

Using the coordinate transformation:

\[
\xi = x, \quad \eta = 2 \sqrt{y}, \quad p_\xi = p_x, \quad p_\eta = \sqrt{y} \, p_y
\]

and putting \( \mu = \frac{1}{2} \), the Drach Hamiltonian \((e)(r = 0)\) of reference [8] is obtained:

\[
H = p_x \, p_y + \frac{n}{\sqrt{y}} + 2 k \, x + \ell \frac{x}{\sqrt{y}} + 2m
\]

\[ \textbf{F}_6: \text{ Class II}_2 \quad \kappa = 0, \lambda = 0, \mu = 0 \]

The form of the Hamiltonian in Liouville coordinates is given by

\[
H = \frac{p_\eta p_\xi + k \xi \eta + \ell \xi + m \eta + n}{\eta \nu}
\]

By the coordinate transformation:

\[
\xi = x + i \, y, \quad \eta = x - i \, y, \quad p_\xi = \frac{1}{2} (p_x - i \, p_y), \quad p_\eta = \frac{1}{2} (p_x + i \, p_y)
\]

and putting \( \nu = \frac{1}{4} \), the complex plane Hamiltonian \( E_9 \) of reference [6] is obtained

\[
H = p_x^2 + p_y^2 + \frac{8k \, (2 \, x - i \, y)}{\sqrt{x - i \, y}} + 8 \ell \, x + \frac{4m}{\sqrt{x - i \, y}} + 4n
\]
By the coordinate transformation:

\[ \xi = \frac{x}{r}, \quad \eta = y, \quad p_\xi = r \, p_x, \quad p_\eta = p_y \]

and putting \( \nu = r \), the Drach Hamiltonian \((e)(r \neq 0)\) of reference [8] is obtained:

\[
H = p_x \, p_y + \frac{m}{\sqrt{y}} + \frac{\ell}{r^2} (x + r \, y) + \frac{k}{r^2} \left(x + 3 \, r \, y\right) + \frac{n}{r}
\]

**F_7: Class II_2 \( \kappa = 0, \lambda = 0, \nu = 0 \)** The form of the Hamiltonian in Liouville coordinates is given by

\[
H = \frac{p_\eta \, p_\xi}{\mu} \sqrt{\eta} + \frac{k \, \xi}{\mu} + \frac{\ell}{\mu} \sqrt{\eta} \xi + \frac{3}{2} \frac{k \, \eta}{\mu} + \frac{\ell}{\mu} \left(\frac{\eta}{\mu}\right)^2 + \frac{m}{\mu} + \frac{n \, \sqrt{\eta}}{\mu}
\]

Using the coordinate transformation:

\[
\xi = x + i \, y = z, \quad \eta = -\frac{(x - i \, y)^2}{2} = -\frac{z^2}{2}
\]

\[
p_\xi = \frac{1}{2} \left(p_x - i \, p_y\right), \quad p_\eta = -\frac{1}{2(x - i \, y)} \left(p_x - i \, p_y\right)
\]

and putting \( \mu = -\frac{1}{4 \sqrt{2}} \), the complex plane Hamiltonian \( E_{10} \) of reference [6] is obtained

\[
H = p_x^2 + p_y^2 - 4 \, n \, z + 4 \sqrt{2} \, i \, k \left(z - \frac{3}{2} \, z^2\right) - 4 \, \ell \left(z \, z - \frac{1}{2} \, z^3\right) + 4 \, i \, \sqrt{2} \, m
\]

By the coordinate transformation:

\[
\xi = x, \quad \eta = y^2, \quad p_\xi = p_x, \quad p_\eta = \frac{1}{2y} \, p_y
\]

and putting \( \mu = \frac{1}{2} \), the corresponding Drach Hamiltonian is obtained:

\[
H = p_x \, p_y + 2 \, k \left(x + 3 \, y^2\right) + 2 \, l \, y \left(x + y^2\right) + 2 \, n \, y + 2 \, m
\]

This potential was not included in the list of reference [8].
The form of the Hamiltonian in Liouville coordinates is given by

\[ H = \frac{p_\eta p_\xi}{\nu} + \frac{k \xi}{\eta^3 \nu} + \frac{\ell \eta \xi}{\nu} + \frac{m}{\eta^2 \nu} + \frac{n}{\nu} \]

Using the coordinate transformation:

\[ \xi = x + i \ y = z, \ \eta = x - i \ y = \bar{z}, \ \rho_\xi = \frac{1}{2} (p_x - i \ p_y), \ \rho_\eta = \frac{1}{2} (p_x + i \ p_y) \]

and putting \( \mu = 0, \ \nu = \frac{1}{4} \), the complex plane Hamiltonian \( E_8 \) of reference [6] is obtained

\[ H = p_x^2 + p_y^2 + \frac{4 \ k \ z}{z^3} + \frac{4 \ m \ y^2}{x^2} + 4 \ell \ z \bar{z} + 4 \ n \]

By the coordinate transformation:

\[ \xi = x, \ \eta = y, \ \rho_\xi = p_x, \ \rho_\eta = p_y \]

and putting \( \nu = 1 \), the Drach Hamiltonians \((f)\) of reference [8] is obtained:

\[ H = p_x p_y + \ell \ x \ y + \frac{m}{y^2} + \frac{k \ x}{y^3} + n \]

The form of the Hamiltonian in \((\xi, \eta)\) coordinates is given by

\[ H = \frac{p_\xi p_\eta}{\frac{1}{\eta^3} + \nu} + \frac{m}{\eta^2} + \frac{k \ |\xi|^2 + \ell \ |\eta|^2}{\frac{1}{\eta^3} + \nu} \xi \]

By the coordinate transformation

\[ \xi = \frac{1}{2} (x + i \ y), \ \eta = \frac{x - i \ y - \sqrt{(x - i \ y)^2 + 4 \mu \ \nu}}{4 \ \nu} \]

\[ p_\xi = p_x - i \ p_y, \ p_\eta = -\frac{2 \nu (p_x + i \ p_y) \sqrt{(x - i \ y)^2 + 4 \mu \ \nu}}{x - i \ y - \sqrt{(x - i \ y)^2 + 4 \mu \ \nu}} \]
and putting $\mu = -\frac{c}{4}, \nu = \frac{c}{4}$ the complex plane Hamiltonian $E_7$ of reference [6] is obtained

$$H = p_x^2 + p_y^2 + \frac{-2(m+n)}{\sqrt{\pi^2-c^2}} + \frac{-c^2(k+\ell)}{\sqrt{\pi^2-c^2} (\pi+\sqrt{\pi^2-c^2})} - \frac{4k}{c^2} w \overline{w} + \frac{2(m-n)}{c}$$

where $w = x + i y, \overline{w} = x - i y$

Using the coordinate transformation:

$$\xi = y, \eta = \frac{x + \sqrt{x^2 + 4 \mu \nu}}{2 \nu}, p_\xi = p_y, p_\eta = \frac{2 \nu \sqrt{x^2 + 4 \mu \nu}}{x + \sqrt{x^2 + 4 \mu \nu}} p_x$$

and putting $\nu = \frac{\sqrt{r}}{2}, \mu = \frac{\sqrt{r}}{2}$ the Drach Hamiltonian $(i) \ (r \neq 0)$ of reference [8] is obtained

$$H = p_x p_y + \frac{\ell - k}{r} x y + \frac{n-m}{\sqrt{x^2 + r}} + \frac{k+\ell (2x^2 + r) y}{\sqrt{x^2 + r}}$$

\textbf{F_{10} Class II_3, $\mu = 0, \nu = 0$} The form of the Hamiltonian in Liouville coordinates is given by

$$H = \frac{p_\eta p_\xi}{\eta \lambda \xi} + \frac{k}{\eta^4 \lambda} + \frac{\ell}{\eta^3 \lambda \xi} + \frac{m}{\eta \lambda \xi} + \frac{n}{\eta \lambda \xi}$$

Using the coordinate transformation:

$$\xi = 2 \sqrt{x - i y} = 2 \sqrt{z}, \quad \eta = 2 \sqrt{x + i y} = 2 \sqrt{z}$$

$$p_\xi = \frac{1}{2} \sqrt{x - i y} (p_x + i p_y), \quad p_\eta = \frac{1}{2} \sqrt{x + i y} (p_x - i p_y)$$

and putting $\kappa = 0, \lambda = \frac{1}{16}$, the complex plane Hamiltonian $E_{17}$ of reference [6] is obtained

$$H = p_x^2 + p_y^2 + \frac{k}{z^2} + 16 \ell + \frac{m}{z \sqrt{z^2}} + \frac{4n}{\sqrt{z^2}}$$

Using the coordinate transformation:

$$\xi = 2 \sqrt{y}, \eta = 2 \sqrt{x}, p_\xi = \sqrt{y} p_y, p_\eta = \sqrt{x} p_x$$

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and putting $\kappa = 0, \lambda = \frac{1}{4}$, the Drach Hamiltonians $(d)$ of reference [8] is obtained:

$$H = p_x p_y + \frac{k}{4 x^2} + 4 \ell + \frac{m}{4 \sqrt{y} x^2} + \frac{n}{\sqrt{y} x}$$

**F11: Class II$_3 \mu = 0, \nu = 0$** The form of the Hamiltonian in $(\xi, \eta)$ coordinates is given by

$$H = \frac{p_\xi p_\eta}{(\frac{k}{\eta^2} + \lambda \eta) \xi} + \frac{(\frac{k}{\eta^2} + l \eta) \xi + m \eta + n}{(\frac{m}{\eta^2} + \lambda \eta) \xi}$$

By the coordinate transformation

$$\xi = 2 \sqrt{x + i y}, \eta = \sqrt{x - i y + \sqrt{(x - i y)^2 - 4}}$$

and putting $\kappa = -\frac{1}{2}, \lambda = \frac{1}{8}$ we have the Hamiltonian in the complex plane $E_{19}$ of reference [6]

$$H = p_x^2 + p_y^2 + \frac{(k + 4) \sqrt{w}}{\sqrt{w^2 - 4}} + \frac{-m \sqrt{2}}{\sqrt{w(w + 2)}} + \frac{m \sqrt{2}}{\sqrt{w(w - 2)}} + 4 \ell - k$$

By the coordinate transformation:

$$\xi = 2 \sqrt{y}, \eta = 2 \sqrt{x + \sqrt{x^2 - r^2}}, p_\xi = \sqrt{y} p_y, p_\eta = \frac{\sqrt{x^2 - r^2}}{\sqrt{x + \sqrt{x^2 - r^2}}} p_x$$

and putting $\kappa = -2r^2, \lambda = \frac{1}{8}$ the Drach Hamiltonian $(h)(r \neq 0)$ of reference [8] is obtained

$$H = p_x p_y + \frac{4 n x + m}{4 \sqrt{r^2 x} \sqrt{y(x - r)}} + \frac{4 n x - m}{4 \sqrt{r^2 x} \sqrt{y(x + r)}} + \frac{16 \ell r^2 + k}{4 r^2 \sqrt{x^2 - r^2}} + \frac{16 \ell r^2 - k}{4 r^2}$$
IX Superintegrable potentials on a manifold with constant curvature

Let us consider the manifold corresponding to the metric in the Liouville coordinates:

\[ ds^2 = g(\xi, \eta) \, d\xi \, d\eta \]

the curvature is defined by:

\[ K = -\frac{1}{2g} \frac{\partial^2}{\partial\xi\partial\eta} \ln g = \text{Constant} \quad (63) \]

The above constraint imposes restrictions on the choice of the parameters \( \kappa, \lambda, \mu \) and \( \nu \). In Table 4 we can see the possible choices of the values of the above parameters.

**C\textsubscript{1}: Class I\textsubscript{1}** \( \kappa = 0, \lambda = 0, \mu = 1/K, \nu = 0 \) The form of the Hamiltonian in Liouville coordinates is given by

\[
H = K p_\eta p_\xi (\xi - \eta)^2 + K (\xi - \eta)^2 \left( k (\eta + \xi) + 4 \ell (\eta + \xi)^2 - \ell (\xi - \eta)^2 + \frac{m}{(\xi - \eta)^2} + n \right)
\]

For \( K = 1 \) and using the coordinate transformation:

\[
\xi = e^{i} \phi \tan(\frac{\theta}{2}) , \quad p_\xi = \frac{\cot(\frac{\theta}{2})}{2 e^{i} \phi} \left( -i p_\phi + p_\theta \sin\theta \right) \\
\eta = -e^{i} \phi \cot(\frac{\theta}{2}) , \quad p_\eta = \frac{\tan(\frac{\theta}{2})}{2 e^{i} \phi} \left( i p_\phi + p_\theta \sin\theta \right)
\]

the Hamiltonian \( S_1 \) of reference [6] in spherical coordinates \( \theta, \phi \) is obtained:

\[
H = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} - 8k e^{3i} \phi \frac{\cos\theta}{\sin^4\theta} + 16 \ell e^{4i} \phi \left( 1 + 2 \cos(2\theta) \right) \frac{1}{\sin^4\theta} + 4n e^{2i} \phi \frac{1}{\sin^2\theta} + m
\]

**C\textsubscript{2}: Class I\textsubscript{2}** \( \kappa = -1/K, \lambda = 0, \mu = 0, \nu = 0 \) The form of the Hamiltonian in Liouville coordinates is given by

\[
H = -K p_\eta p_\xi (\eta + \xi)^2 - K (\eta + \xi)^2 \left( \frac{k}{(\eta + \xi)^2} + \ell (\eta + \xi)^2 - \ell (\xi - \eta)^2 + \frac{m}{(\xi - \eta)^2} + n \right)
\]
| Class | I₁ | I₂ | I₃ | Plane potentials from ref [6] | Drach potentials from ref [8] | Drach potentials from ref [9] |
|-------|----|----|----|-----------------------------|-----------------------------|-----------------------------|
| F₁    | 0  | 0  | 0  | E₂                         | (a) (r ≠ 0)                 | Vₐ             |
| F₂    | 0  | 0  | 0  | E₁                         | (b) (r ≠ 0)                 | V₆             |
| F₃    | 0  | 0  | 0  | E₁₆                        | (g) (r ≠ 0)                 | V₉             |
| F₄    | 0  | 0  | 0  | E₂₀                        | (c)                         | V₄             |
| F₅    | 0  | 0  | 0  | E₁₁                        | (e) (r = 0)                 | V₉             |
| F₆    | 0  | 0  | 0  | E₁₀                        | (e) (r ≠ 0)                 | V₉             |
| F₇    | 0  | 0  | 0  | E₈                         | (f)                         | V₉             |
| F₈    | 0  | 0  | 0  | E₇                         | (i) (r ≠ 0)                 | V₉             |
| F₉    | 0  | 0  | 0  | E₁₇                        | (d)                         | V₉             |
| F₁₀   | 0  | 0  | 0  | E₁₉                        | (h) (r ≠ 0)                 | V₉             |

Table 3: Potentials with curvature zero and two quadratic integrals of motion

| Class | κ  | λ   | μ   | ν   | Potentials on the sphere (K=1) from Ref [6] | Potentials from Ref [9] |
|-------|----|-----|-----|-----|--------------------------------------------|------------------------|
| C₁    | 0  | 0   | 1/K | 0   | S₁                                         |                        |
| C₂    | −1/K | 0   | 0   | 0   | S₂                                         |                        |
| C₃    | −1/K | 0   | 1/K | 0   | S₄                                         |                        |
| C₄    | 0   | 0   | 4/K | 0   |                                            | U₉, U₆                |
| C₅    | 0   | 2/K | 1/K | 0   | S₉                                         |                        |
| C₆    | −2/K | −1/K | 2/K | −1/K | S₇                                         |                        |
| C₇    | −4/K | 0   | 4/K | 0   | S₈                                         |                        |

Table 4: Potentials with constant curvature and two quadratic integrals of motion
For $K = 1$ and using the coordinate transformation:

$$
\begin{align*}
\xi &= -\frac{1}{2} i e^i \phi \cot(\frac{\theta}{2}) , \quad p_\xi = \frac{\tan(\frac{\theta}{2}) (p_\phi - i p_\theta \sin \theta)}{e^i \phi} \\
\eta &= -\frac{1}{2} i e^i \phi \tan(\frac{\theta}{2}) , \quad p_\eta = \frac{\cot(\frac{\theta}{2}) (p_\phi + i p_\theta \sin \theta)}{e^i \phi}
\end{align*}
$$

the Hamiltonian $S_2$ of reference [6] in spherical coordinates $\theta, \phi$ is obtained:

$$
H = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} - k - \ell e^4 i \phi \csc^2 \theta - m \sec^2 \theta + n e^2 i \phi \csc^2 \theta
$$

**C$_3$: Class I$_2$** $\kappa = -1/K$, $\lambda = 0$, $\mu = 1/K$, $\nu = 0$  The form of the Hamiltonian in Liouville coordinates is given by

$$
H = K \frac{p_\eta p_\xi}{(\xi-\eta)^2} + K \frac{k}{(\eta+\xi)^2} + \ell (\eta + \xi)^2 - \ell (\xi - \eta)^2 + \frac{m}{(\xi-\eta)^2} + n
$$

For $K = 1$ and using the coordinate transformation:

$$
\begin{align*}
\xi &= 2 \sqrt{1 - e^i \phi \cot(\frac{\theta}{2})} , \quad p_\xi = \frac{-i p_\phi - p_\theta \sin \theta}{2 \sqrt{1 - e^i \phi \cot(\frac{\theta}{2})}} \\
\eta &= 2 \sqrt{1 + e^i \phi \tan(\frac{\theta}{2})} , \quad p_\eta = \frac{-i p_\phi + p_\theta \sin \theta}{2 \sqrt{1 + e^i \phi \tan(\frac{\theta}{2})}}
\end{align*}
$$

the Hamiltonian $S_4$ of reference [6] in spherical coordinates $\theta, \phi$ is obtained:

$$
H = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} + (m - k) - i (k + m) \cot \theta - 64 \ell e^2 i \phi \csc^2 \theta - 4 n e^i \phi
$$

**C$_4$: Class I$_3$** $\kappa = 0$, $\lambda = 0$, $\mu = 4/K$, $\nu = 0$  The form of the Hamiltonian in Liouville coordinates is given by

$$
H = \frac{(-1 + e^2 (-\eta + \xi))^2 K p_\eta p_\xi}{4 e^2 (-\eta + \xi)} + \frac{(-1 + e^2 (-\eta + \xi))^2 K}{4 e^2 (-\eta + \xi)} \left( \frac{k e^2 (\eta + \xi) + \ell e^4 i \phi (1 + e^2 (\eta + \xi))}{(-1 + e^2 (\xi + \eta))^2} + \frac{m e^2 (-\eta + \xi) + n e^4 i \phi (1 + e^2 (-\eta + \xi))}{(-1 + e^2 (-\eta + \xi))^2} \right)
$$

For $K = 1$ and using the coordinate transformation

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\[ \xi = \frac{1}{2} \ln(-i e^{i \phi} \cot(\theta)), \quad p_\xi = -i p_\phi - p_\theta \sin \theta \]
\[ \eta = \frac{1}{2} \ln(i e^{i \phi} \tan(\theta)), \quad p_\eta = -i p_\phi + p_\theta \sin \theta \]

the above Hamiltonian in spherical coordinates \( \theta, \phi \) has the form

\[ H = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} + \frac{k}{4 \sin^2 \theta \sin^2 \phi} + \frac{\ell \cos \phi}{2 \sin^2 \theta \sin^2 \phi} + \frac{m}{4} - \frac{n}{2} \cot \theta \]

This is the Hamiltonian \( U^c \) of Ref. [9] in spherical coordinates. This potential is missing in the classification scheme of Ref. [6].

\[ C_5: \text{Class I}_3 \kappa = 0, \lambda = 0, \mu = 2/K, \nu = 1/K \]

The form of the Hamiltonian in Liouville coordinates is given by

\[ H = K \left[ \frac{p_\theta p_\xi}{2 e^{2(\xi-\eta)}} + \frac{e^\xi e^{-\eta} (1+e^{2(\xi-\eta)})}{(-1+e^{2(\xi-\eta)})^2} \right] + \]
\[ + K \left[ \frac{e^\eta e^{-\xi} (1+e^{2(\eta+\xi)})}{(-1+e^{2(\eta+\xi)})^2} \right] + \frac{e^\xi e^{-\eta} (1+e^{2(\eta+\xi)})}{(-1+e^{2(\eta+\xi)})^2} \]
\[ + K \left[ \frac{e^\eta e^{-\xi} (1+e^{2(\xi-\eta)})}{(-1+e^{2(\xi-\eta)})^2} \right] + \frac{e^\xi e^{-\eta} (1+e^{2(\xi-\eta)})}{(-1+e^{2(\xi-\eta)})^2} \]

For \( K = 1 \) and using the coordinate transformation:

\[ \xi = \ln(e^{i \phi} \tan(\theta)), \quad p_\xi = -i p_\phi + p_\theta \sin \theta \]
\[ \eta = \ln(-e^{i \phi} \cot(\theta)), \quad p_\eta = -i p_\phi + p_\theta \sin \theta \]

the Hamiltonian \( S_9 \) of reference [6] and the Hamiltonian \( U^b \) of ref. [9] in spherical coordinates \( \theta, \phi \) are obtained:

\[ H = \frac{p_\theta^2}{\sin^2 \theta} + \]
\[ + \frac{m+2n}{4} + (k + 2 \ell \cos(2 \varphi)) \csc^2 \theta \csc^2(2 \phi) - \frac{(m-2n)}{4} \sec^2 \theta \]

while using the coordinate transformation:

\[ \xi = \frac{1}{4} \pi + \ln\left(\frac{1+i e^{i \phi} \tan(\theta)}{1-i e^{i \phi} \tan(\theta)}\right), \quad \eta = \frac{1}{4} \pi + \ln\left(\frac{1-i e^{i \phi} \cot(\theta)}{1+i e^{i \phi} \cot(\theta)}\right), \]

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The form of the Hamiltonian in Liouville coordinates is given by
\[
p_{\xi} = -\frac{(p_{\phi} \left( \cot\left(\frac{\phi}{2}\right) + 2i^{2} \phi \tan\left(\frac{\phi}{2}\right) \right) + 2i^{2} p_{\phi} \cos\left(\frac{\phi}{2}\right) + 2i^{2} \phi \tan\left(\frac{\phi}{2}\right) \right)^{2}}{4e^{i\phi}} \]
\[
p_{\eta} = \frac{1}{4} p_{\phi} \left( 1 + 2i^{2} \phi + \left( -1 + 2i^{2} \phi \right) \cos(\theta) \right) + \frac{p_{\phi} \left( 2i^{2} \phi \cot\left(\frac{\phi}{2}\right) + \tan(\phi) \right)}{4e^{i\phi}}
\]

we have the Hamiltonian $U^{\alpha}$ of ref. [9] in spherical coordinates.

\[
H = p_{\phi}^{2} + \frac{p_{\phi}^{2}}{\sin^{2}\theta} + \frac{m+2n}{4} + \frac{(2n-m) \csc^{2}\theta \sec^{2}\phi}{4} + \frac{k-2l}{2 (\cos \theta - \cos \phi \sin \theta)^{2}} + \frac{k+2l}{2 (\cos \theta + \cos \phi \sin \theta)^{2}}
\]

**C6:** Class I$_{3}$ $\kappa = -2/K$, $\lambda = -1/K$, $\mu = 2/K$, $\nu = -1/K$ The form of the Hamiltonian in Liouville coordinates is given by

\[
H = 
\frac{\sqrt{1 - e^{2i} \phi \cot\left(\frac{\phi}{2}\right)}}{21 e^{i\phi} \cot\left(\frac{\phi}{2}\right)} + \frac{p_{\phi}^{2}}{\sin^{2}\theta} + \frac{\left( -1 + e^{2i} \phi \cot\left(\frac{\phi}{2}\right) \right) \left( 1 + e^{2i} \phi \tan\left(\frac{\phi}{2}\right) \right)^{2}}{4e^{i\phi}}
\]

For $K = 1$ and using the coordinate transformation

$\xi = \text{arcsinh}(-i e^{i\phi} \cot\left(\frac{\phi}{2}\right))$, $p_{\xi} = \frac{\sqrt{1 - e^{2i} \phi \cot\left(\frac{\phi}{2}\right)}}{21 e^{i\phi} \cot\left(\frac{\phi}{2}\right)} + \frac{\left( -1 + e^{2i} \phi \cot\left(\frac{\phi}{2}\right) \right) \left( 1 + e^{2i} \phi \tan\left(\frac{\phi}{2}\right) \right)^{2}}{4e^{i\phi}}$

$\eta = \text{arcsinh}(-i e^{i\phi} \tan\left(\frac{\phi}{2}\right))$, $p_{\eta} = \frac{\sqrt{1 - e^{2i} \phi \tan\left(\frac{\phi}{2}\right)}}{21 e^{i\phi} \tan\left(\frac{\phi}{2}\right)} + \frac{\left( -1 + e^{2i} \phi \tan\left(\frac{\phi}{2}\right) \right) \left( 1 + e^{2i} \phi \cot\left(\frac{\phi}{2}\right) \right)^{2}}{4e^{i\phi}}$

the Hamiltonian $S_{7}$ of reference [6] and the Hamiltonian $U^{\alpha}$ of ref. [9] in spherical coordinates $\theta$, $\phi$ are obtained:

\[
H = p_{\phi}^{2} + \frac{p_{\phi}^{2}}{\sin^{2}\theta} + \frac{\left( -1 + e^{2i} \phi \cot\left(\frac{\phi}{2}\right) \right) \left( 1 + e^{2i} \phi \tan\left(\frac{\phi}{2}\right) \right)^{2}}{4e^{i\phi}} + \frac{\left( k-2l-m-2n \right) \sec^{2}\theta}{8} + \frac{\left( k-2l-m+2n \right) \sec^{2}\theta \sin(\phi) \tan(\theta)}{8 \sqrt{\cos^{2}\theta + \sin^{2}\theta}}
\]

**C7:** Class I$_{3}$ $\kappa = -4/K$, $\lambda = 0$, $\mu = 4/K$, $\nu = 0$ The form of the Hamiltonian in Liouville coordinates is given by

\[
H = 
\frac{\sqrt{1 - e^{2i} \phi \cot\left(\frac{\phi}{2}\right)}}{21 e^{i\phi} \cot\left(\frac{\phi}{2}\right)} + \frac{p_{\phi}^{2}}{\sin^{2}\theta} + \frac{\left( -1 + e^{2i} \phi \cot\left(\frac{\phi}{2}\right) \right) \left( 1 + e^{2i} \phi \tan\left(\frac{\phi}{2}\right) \right)^{2}}{4e^{i\phi}}
\]

For $K = 1$ and using the coordinate transformation

$\xi = \text{arcsinh}(-i e^{i\phi} \cot\left(\frac{\phi}{2}\right))$, $p_{\xi} = \frac{\sqrt{1 - e^{2i} \phi \cot\left(\frac{\phi}{2}\right)}}{21 e^{i\phi} \cot\left(\frac{\phi}{2}\right)} + \frac{\left( -1 + e^{2i} \phi \cot\left(\frac{\phi}{2}\right) \right) \left( 1 + e^{2i} \phi \tan\left(\frac{\phi}{2}\right) \right)^{2}}{4e^{i\phi}}$

$\eta = \text{arcsinh}(-i e^{i\phi} \tan\left(\frac{\phi}{2}\right))$, $p_{\eta} = \frac{\sqrt{1 - e^{2i} \phi \tan\left(\frac{\phi}{2}\right)}}{21 e^{i\phi} \tan\left(\frac{\phi}{2}\right)} + \frac{\left( -1 + e^{2i} \phi \tan\left(\frac{\phi}{2}\right) \right) \left( 1 + e^{2i} \phi \cot\left(\frac{\phi}{2}\right) \right)^{2}}{4e^{i\phi}}$
For $K = 1$ and using the coordinate transformation

$$\xi = \frac{1}{2} \ln \left( \frac{i (-1 + \sqrt{1 + (\sigma + i \tau)^2})}{\sigma + i \tau} \right), \quad \eta = \frac{1}{2} \ln \left( \frac{i (1 + \sqrt{1 + (\sigma - i \tau)^2})}{\sigma - i \tau} \right)$$

$\ p_\xi = \sqrt{1 + (\sigma + i \tau)^2} \ ((\sigma + i \tau) p_\sigma + (-i \sigma + \tau) p_\tau)$

$\ p_\eta = i \sqrt{1 + (\sigma - i \tau)^2} \ ((i \sigma + \tau) p_\sigma - (\sigma - i \tau) p_\tau)$

the Hamiltonian $S_8$ of reference [6] in horospherical coordinates $\sigma, \tau$ is obtained.

$$H = -(p\sigma^2 + p\tau^2) \sigma^2 - \frac{(k+m)(-1+\sigma^2+\tau^2)}{8 \sqrt{\sigma^4+(-1+\tau^2)^2+2\sigma^2(1+\tau^2)}} + \frac{(n-l)(\sigma^2-\tau^2)}{4 \sqrt{(\sigma^2+\tau^2)(\sigma^2+(-1+\tau)^2)}} + \frac{m-k}{8}$$

We use the inverse transformation

$$\sigma \rightarrow \frac{i e^{2\eta}}{1+e^{4\eta}}, \quad \frac{i e^{2\xi}}{1+e^{4\xi}}, \quad \tau \rightarrow -\frac{e^{2\eta}}{1+e^{4\eta}} - \frac{e^{2\xi}}{1+e^{4\xi}}$$

for the verification of Poisson brackets. The above transformations are more appropriate for the corresponding calculations.

**X Superintegrable systems with a linear and a quadratic integral**

In the case of a linear integral of motion and a quadratic integral of motion, there is a Liouville coordinate system where the Hamiltonian and the linear integral of motion are written as:

$$I = (p_\xi + p_\eta)^2 \quad \text{or} \quad A = (p_\xi - p_\eta)^2$$

and

$$H = \frac{p_\xi p_\eta}{G(\xi - \eta)} + \frac{g(\xi - \eta)}{G(\xi - \eta)} \quad \text{or} \quad H = \frac{p_\xi p_\eta}{F(\xi + \eta)} + \frac{f(\xi + \eta)}{F(\xi + \eta)}$$

From the forms of the integral of motion, which are given in Section $\text{V}$ we can find all the possible subclasses corresponding to a linear and quadratic
integral of motion in Liouville coordinates. In all the above cases we remark that the potential depends on two parameters among the $k$, $\ell$, $m$ and $n$. In Table 5 we give the possible cases of superintegrable systems with a linear and a quadratic integral of motion:

In Table 6 the possible superintegrable systems which are defined on a surface of revolution with a linear and a quadratic integral of motion are listed.

In Table 7 the possible superintegrable systems which are defined on a surface of zero curvature with a linear and a quadratic integral of motion are listed.

In Table 8 the possible superintegrable systems which are defined on a surface of constant curvature with a linear and a quadratic integral of motion are listed.

**XI Discussion**

The findings of this paper are summarized as follows:

1. The superintegrable systems with quadratic integrals of motion can be classified in six subclasses. Each subclass depends on seven parameters. Four among these parameters ($\kappa$, $\lambda$, $\mu$ and $\nu$) determine the metric of the manifold, on which the system is determined. These parameters are characteristic of the system’s “kinetic” energy. The remaining three parameters define the potential (The potential depend on four parameters $k$, $\ell$, $m$ and $n$ but only three among them are independent). For each subclass, the analytic explicit forms of the manifold metric, the potential and the integrals of motion are calculated. Also the constants of the quadratic Poisson algebra of integrals of motion are given as functions of the energy and the eight parameters are given.

2. All the known two dimensional superintegrable systems are systems defined on surfaces of constant curvature or on surfaces of revolution. All these systems are classified in these six classes. Each class is characterized by the values four parameters $\kappa$, $\lambda$, $\mu$ and $\nu$, which are determined by the form of the assumed manifold. If we fix the manifold, let us suppose that the manifold is a manifold with negative constant curvature, the possible values of the of the parameters $\kappa$, $\lambda$, $\mu$ and $\nu$ are calculated for each subclass. Therefore we can ’’guess” the existence
Table 5: General superintegrable integrable systems with a linear and a quadratic integral of motion

| Class | \( \kappa \) | \( \lambda \) | \( \mu \) | \( \nu \) | \( k \) | \( \ell \) | \( m \) | \( n \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( GL_1 \) | \( I_1 \) | 0 | 0 | \( \cdot \) | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) |
| \( GL_2 \) | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) |
| \( \approx GL_1 \) | \( I_2 \) | 0 | 0 | \( \cdot \) | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) |
| \( GL_3 \) | \( \cdot \) | 0 | \( \cdot \) | 0 | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) |
| \( GL_4 \) | \( \cdot \) | 0 | \( \cdot \) | \( \cdot \) | 0 | 0 | 0 | \( \cdot \) | \( \cdot \) |
| \( GL_5 \) | \( I_3 \) | 0 | 0 | \( \cdot \) | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) |
| \( \approx GL_1 \) | \( \kappa = \mp 2\lambda \) | \( \mu = \pm 2\nu \) | \( k = \mp 2\ell \) | \( \cdot \) | 0 | \( m = \pm 2n \) | \( \cdot \) | \( \cdot \) |
| \( GL_6 \) | \( II_1 \) | 0 | \( \lambda = \pm \mu \) | \( \cdot \) | \( \cdot \) | 0 | \( \ell = \pm m \) | \( \cdot \) | \( \cdot \) |
| \( GL_7 \) | \( II_2 \) | 0 | \( \cdot \) | 0 | \( \cdot \) | 0 | \( \cdot \) | \( \cdot \) | \( \cdot \) |
| \( \approx GL_3 \) | \( II_3 \) | 0 | \( \cdot \) | 0 | \( \cdot \) | 0 | \( \cdot \) | \( \cdot \) | \( \cdot \) |

Table 6: Potentials by revolution with a linear and a quadratic integral of motion

| Class | \( \kappa \) | \( \lambda \) | \( \mu \) | \( \nu \) | \( k \) | \( l \) | \( m \) | \( n \) | Potentials by revolution from ref \([13]\) | Potentials by revolution from ref \([12]\) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|------------------|------------------|
| \( RL_1 \) | \( I_1 \) | 0 | 0 | \( \cdot \) | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) | 2.2\([D]\) |
| \( RL_2 \) | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) | \( (3) \) |
| \( RL_3 \) | \( I_2 \) | \( \cdot \) | 0 | 0 | \( \cdot \) | 0 | 0 | \( \cdot \) | \( \cdot \) | 4.2\([D]\) |
| \( RL_4 \) | \( II_1 \) | 0 | \( \cdot \) | 0 | \( \cdot \) | 0 | \( \cdot \) | \( \cdot \) | \( \cdot \) | 3.2\([E]\) |

Table 6: Potentials by revolution with a linear and a quadratic integral of motion
### Table 7: Potentials with curvature zero and with a linear and a quadratic integral of motion

| Class | | | | | | | Potentials from ref 6 |
|-------|---|---|---|---|---|---|-------------------|
| FL₁   | I₁ | 0 | 0 | 0 | 0 | 0 | E₆ |
| FL₂   |     | 0 | 0 | 0 | 0 | 0 | E₅ |
| FL₃   | I₂ | 0 | 0 | 0 | 0 | 0 | E₃ |
| FL₄   |     | 0 | 0 | 0 | 0 | 0 | E₁₈ |
| FL₅   | I₁ | 0 | 0 | 0 | 0 | 0 | E₄ |
| FL₆   | I₂ | 0 | 0 | 0 | 0 | 0 | E₁₃ |
| FL₇   | I₃ | 0 | 0 | 0 | 0 | 0 | E₁₄ |
| FL₈   |     | 0 | 0 | 0 | 0 | 0 | E₁₂ |

### Table 8: Potentials with constant curvature and with a linear and a quadratic integral of motion

| Class | | | | | | | Sphere potentials (K=1) from ref 6 |
|-------|---|---|---|---|---|---|-------------------|
| I₁    | 0 | 0 | 1/K | 0 | 0 | 0 | S₅ |
| I₂    | -1/K | 0 | 0 | 0 | 0 | 0 | S₃ |
| -1/K | 0 | 1/K | 0 | 0 | 0 | 0 | S₆ |

Table 7: Potentials with curvature zero and with a linear and a quadratic integral of motion

Table 8: Potentials with constant curvature and with a linear and a quadratic integral of motion
of the permitted superintegrable systems and to classify these systems in tables. Using this technique we can classify all the possible known superintegrable systems and to investigate the possible missing potentials. With this method a new superintegrable system was found for the class I superintegrable systems by revolution, given in ref [5].

3. Generally for any values of the parameters \( \kappa, \lambda, \mu \) and \( \nu \) the associated manifolds are neither surfaces of constant curvature nor surfaces of revolution. Therefore we have investigated superintegrable systems, which are not yet known. We believe that all the possible superintegrable systems with two quadratic integrals of motion are investigated.

4. The two dimensional ‘non degenerate’ [5] superintegrable systems are classified by the values of the constants of the Darboux equations [9, 5] and the constants of the system. The relation of these constants with the constants of the quadratic Poisson algebra is explained.

5. The six classes of superintegrable systems are the equivalence classes of Stäckel equivalent systems.

From the above discussion several open problems arise:

- The superintegrable systems for the case of cubic integrals of motion are under investigation [8, 31, 32, 33, 34, 35, 36]. The general structure of these systems is recently investigated [14] but the general form and their classification is not yet known for manifolds which carry integrable systems with one third order integral of motion.

- The quantum counterparts of the general six subclasses of superintegrable systems with quadratic integrals of motion are not yet known. In Section II, the separation of variables of these systems has been explicitly written. The form of the separation of Schroedinger equation can be written in a Liouville coordinate system. The solutions of the quantum Schroedinger equation can be calculated. This work is under current investigation. The form of the Poisson algebra can be generalized in a quadratic associative algebra, whose energy eigenvalues are generally calculated by using deformed oscillator techniques [18, 19, 20, 21, 22]. From the form of the Poisson algebra, one can be guess that the energy eigenvalues of these quantum systems are roots of cubic polynomials.
The general form of the three dimensional superintegrable systems with quadratic integrals of motion is not yet known. The Poisson algebras and the associated quantum counterparts for the three dimensional superintegrable systems are not yet fully studied. Recently [37, 38] the quantum quadratic algebras have been written down, which are not generally closed as polynomial algebras. A systematic calculation of energy eigenvalues with algebraic methods has not been performed yet.
Appendix

A Polynomial combinations of integrals

Let consider a superintegrable system with two quadratic integrals of motion. The general forms of the Hamiltonian and the integrals of motion in Liouville coordinates are:

\[
H = \frac{p_\xi p_\eta}{g(\xi, \eta)} + V(\xi, \eta)
\]

\[
A = p_\xi^2 + k p_\eta^2 - 2c(\xi, \eta)\frac{p_\xi p_\eta}{g(\xi, \eta)} + q(\xi, \eta), \quad k = 0 \text{ or } 1
\]

\[
B = a^2(\xi)p_\xi^2 + b^2(\eta)p_\eta^2 - 2\beta(\xi, \eta)\frac{p_\xi p_\eta}{g(\xi, \eta)} + Q(\xi, \eta)
\]

In this Appendix we consider that the system has quadratic integrals of motion. We assume that the system has not any linear integral of motion. That assumption excludes the system \(H = p_x^2 + p_y^2\), because it possesses two linear integrals of motion. In this Appendix, we shall prove the following proposition:

**Proposition 4** Let \(M\) be an integral of motion, which is a polynomial function of the momenta of even order. We assume that this integral contains only monomials of momenta of even order, i.e.

\[
M = \sum_{k+\ell=\text{even}}^{2n} \alpha_{k, \ell}(\xi, \eta) p_\xi^k p_\eta^\ell
\]

Then \(M\) is a polynomial of order \(n\) of the integrals \(H, A, B\).

The existence of three integrals of motion implies that the integral \(M\) is functionally dependent integral, i.e. there is some smooth function \(\Phi\) (non generally a polynomial one) such that:

\[
\Phi(M, A, B, H) = 0 \quad \text{or} \quad M = f(A, B, H)
\]
but the function $f(A, B, H)$ is not evident that it is a polynomial one. From (64) we can see that

$$p_\xi^2 = \begin{bmatrix} A + 2cH - q - 2cV & k \\ B + 2\beta H - Q - 2\beta V & b^2(\xi, \eta) \end{bmatrix}$$

$$p_\eta^2 = \begin{bmatrix} 1 \\ a^2(\xi, \eta) & b^2(\xi, \eta) \end{bmatrix}$$

$$p_\xi p_\eta = gH - V$$

The above equations imply that the momentum monomials $p_\xi^k p_\eta^l$ inside the sum sign in (65) can be written as polynomials of the integrals $A, B, H$ with coefficients, which depend on $\xi$ and $\eta$. Therefore the integral of motion $M$ is written:

$$M = \sum_{0 \leq i+j+k \leq n} c_{ijk}(\xi, \eta) A^i B^j H^k$$

Generally the coefficients $c_{ijk}$ should be constants not depending on the variables $\xi, \eta$. If these functions are non constant functions $c_{ijk}(\xi, \eta)$ then we choose a fixed value of these parameters $\xi_0, \eta_0$. In general there is an infinity of trajectories passing through these position values $\xi_0, \eta_0$. Each trajectory is characterized by a special value of the integrals $H, A, B$, therefore $M$ is a polynomial of fixed constants for an infinity of trajectories. For another pair of parameters $\xi_1, \eta_1$ we have another choice of the coefficients in (65) therefore there is a relation of the form:

$$\sum_{0 \leq i+j+k \leq n} (c_{ijk}(\xi_1, \eta_1) - c_{ijk}(\xi_0, \eta_0)) A^i B^j H^k = 0$$

The above relations means that the integrals of motion $H, A, B$ are not functionally independent functions, that is a contradiction to assumption initial regarding the independence of the integrals. Therefore we have proved that the polynomial expansion (65) of the integral $M$ is indeed unique, when $M$ is a even polynomial of the momenta.

A direct application of Proposition 4 is the following Proposition:

**Proposition 5** Let a superintegrable two dimensional system have even quadratic integrals of motion $H, A, B$. If we put $C = \{A, B\}$. The integrals
\{A, C\} and \{B, C\} can written as quadratic polynomials of the integrals. The integral \(C^2\) is a cubic polynomial of the integrals \(H, A, B\).

The above proposition was taken as a conjecture in the previous work \[18\]–\[22\] and \[23\]–\[25\]. Here we prove that this assumption is indeed true. A generalization of Proposition 5 is indeed true for superintegrable two dimensional systems with an integral, which is an odd cubic polynomial in momenta \[32 \, [31\]. This Proposition means that the superintegrable two dimensional systems with even quadratic integrals correspond to a quadratic Poisson algebra, which is characteristic for the superintegrable system. The situation in three dimensional superintegrable systems is not clear \[37, 38\].

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References

[1] G. Darboux, *Leçons sur la Théorie Générale des Surfaces* (1898). The online edition of this treatise can be found in the University of Michigan Historical Mathematics Collection site \http://www.hti.umich.edu\n
[2] Friš J., Smorodinsky Ya. A., Uhlir M. and Winternitz P., On Higher Symmetries in Quantum Mechanics, Phys. Lett. 16 354 (1965); Friš J., Smorodinsky Ya. A., Uhlir M. and Winternitz P., Symmetry Groups in Classical and Quantum Mechanics, Sov. J. Nucl. Physics 4 444 (1967); Makarov A. A., Smorodinsky Ya. A., Valiev Kh. Winternitz P. A systematic Search for Nonrelativistic Systems with Dynamical Symmetries, Nuovo Cimento A 52 1061 (1967)

[3] J. Hietarinta, Direct Methods for the Search of the 2nd Invariant, Phys. Rep C147, 87 (1987)

[4] Kalnins E.G., Miller W., Pogosyan G.S., Superintegrability and Associated Polynomial Solutions - Euclidean-Space and the Sphere in 2 Dimensions. J. Math. Phys.37 6439 (1996)
[5] Kalnins E.G., Miller W., Pogosyan G.S., *Completeness of multiseparable superintegrability in $E_{2,C}$*, J. Phys. A: Math. Gen. **33** 4105 (2000)

[6] Kalnins E.G., Kress J.M., Pogosyan G.S. and Miller W., *Completeness of superintegrability in two-dimensional constant-curvature spaces* J. Phys. A-Math. Gen. **34** 4705 (2001)

[7] Kalnins E.G., Pogosyan G.S. and Miller W. *Completeness of multiseparable superintegrability in two dimensions*, Phys. Atom. Nucl. **65** 1047 (2002)

[8] Rañada M. F., *Superintegrable N=2 Systems, Quadratic Constants of Motion, and Potentials of Drach*. J. Math Phys. **38** 4165 (1997)

[9] Rañada M. F. and Santander M., *Superintegrable systems on the two-dimensional sphere $S^2$ and the hyperbolic plane $H^2$*, J. Math. Phys. **40** 5026 (1999)

[10] Rañada M. F. and Santander M., *On harmonic oscillators on the two-dimensional sphere $S^2$ and the hyperbolic plane $H^2$, Part II*, J. Math. Phys. **43** 2479 (2003)

[11] Kalnins E.G., Miller W., Pogosyan G.S., *Completeness of multiseparable superintegrability on the complex 2-sphere* Preprint of the Waikato ISSN 1174-1570 (2000)

[12] Kalnins E. G., Kress J. M., Winternitz P. *Superintegrability in a two-dimensional space of nonconstant curvature*, J. Math. Phys. **43** 970 (2002)

[13] Kalnins E. G., Kress J. M., Miller W., Winternitz P. *Superintegrable systems in Darboux spaces*, J. Math. Phys. **44** 5811 (2003)

[14] Kalnins E. G., Kress J. M., Miller W., *Second-order superintegrable systems in conformally flat spaces. I. Two-dimensional classical structure theory*, J. Math. Phys. **46** 053509 (2005)

[15] Kalnins E. G., Kress J. M., Miller W., *Second-order superintegrable systems in conformally flat spaces. II. Two-dimensional Stäckel transform*, J. Math. Phys. **46** 053510 (2005)
[16] Kress J. M. *Some brief notes on the equivalence of superintegrable potentials in two dimensions*, contribution in the 2nd Intern. Workshop on Superintegrable Systems, Dubna Russia June 27-July 2005

[17] Létourneau P. and Vinet L., *Superintegrable systems: Polynomial Algebras and Quasi-Exactly Solvable Hamiltonians*, Ann. Phys. (NY) **243**, 144 (1995)

[18] Bonatsos D., Daskaloyannis C., Kokkotas K., *Quantum-Algebraic Description of Quantum Superintegrable Systems in 2 Dimensions*, Phys Rev A **48** R3407 (1993)

[19] Bonatsos D., Daskaloyannis C., Kokkotas K., *Deformed Oscillator Algebras for 2-Dimensional Quantum Superintegrable Systems*, Phys. Rev A **50** 3700 (1994)

[20] Daskaloyannis C., *Polynomial Poisson algebras for two-dimensional classical superintegrable systems and polynomial associative algebras for quantum superintegrable systems* Czech. J. Phys. **50** 1209 (2000)

[21] Daskaloyannis C., *Quadratic Poisson algebras of two-dimensional classical superintegrable systems and quadratic associative algebras of quantum superintegrable systems*, J. Math. Phys. **42** 1100 (2001)

[22] Daskaloyannis C., *Polynomial associative algebras of quantum superintegrable systems*, Phys. Atom. Nuclei **65** 1008 (2002)

[23] Higgs, P. W., *Dynamical symmetries in a spherical geometry I*, J. Phys. A: Math. Gen. **12**, 309 (1979)

[24] Gal’bert O. F., Granovskii Ya. I. and Zhedanov A. S., *Dynamical symmetry of anisotropic singular oscillator*, Phys. Lett. A**153**, 177 (1991)

[25] Zhedanov A. S., *The ”Higgs algebra” as a ”quantum” deformation of su(2)*, Mod. Phys. Lett. A**7**, 507 (1992)

[26] Ya. I. Granovskii, I. M. Lutzenko and A. S. Zhedanov, *Mutual Integrability, Quadratic Algebras and Dynamical Symmetry*, Ann. Phys. NY **217**, 1 (1992)
[27] Granovskii Ya. I., Zhedanov A. S. and Lutzenko I. M., Quadratic Algebras and Dynamics into the Curved Space, I. the oscillator, Teor. Mat. Fiz. 91, 207 (1992) (in russian)

[28] Granovskii Ya. I., Zhedanov A. S. and Lutzenko I. M., Quadratic Algebras and Dynamics into the Curved Space, I. the Kepler Problem, Teor. Mat. Fiz. 91, 396 (1992) (in russian)

[29] Granovskii Ya. I., Zhedanov A. S. and Lutzenko I. M., Quadratic Algebra as a Hidden Symmetry of the Hartmann Potential, J. Phys. A: Math-Gen 24 3887 (1991)

[30] Kalnins E. G., Kress J. M., Miller W., Pogosyan G. Complete sets of invariants for dynamical systems that admit a separation of variables J. Math. Phys. 43 3592 (2002)

[31] Tsyganov A.V., Degenerate integrable systems on the plane with a cubic integral of motion, Theor. Math. Phys. 124 1217 (2000)

[32] Tsyganov A.V., The Drach superintegrable systems, J. Phys. A-Math. Gen. 33 7407 (2000)

[33] Karlovini M. and Rosquist K., A unified treatment of cubic invariants at fixed and arbitrary energy J. Math. Phys. 41 370 (2000)

[34] Gravel S. and Winternitz P., Superintegrability with third-order integrals in quantum and classical mechanics J. Math. Phys. 43 5902 (2002)

[35] Gravel S., Superintegrable systems with third-order integrals in classical and quantum mechanics Theor. Math. Phys. 137 1439 (2003)

[36] Gravel S., Hamiltonians separable in Cartesian coordinates and third-order integrals of motion, J. Math. Phys. 45 1003 (2004)

[37] Kalnins E. G., Williams G. C., Miller W. Jr. and Pogosyan G. S., Superintegrability in three-dimensional Euclidean space, J. Math. Phys. 40, 708 (1999)

[38] Kress J.M. and Kalnins E.G., Multiseparability and superintegrability in three dimensions, Phys. Atom. Nuclei 65 1047 (2002)