Thermoelectric response and entropy of fractional quantum Hall systems

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We study thermoelectric transport properties of fractional quantum Hall systems based on exact diagonalization calculation. Based on the relation between thermoelectric response and thermal entropy, we demonstrate that thermoelectric Hall conductivity $\alpha_{xy}$ has powerlaw scaling $\alpha_{xy} \propto T^\eta$ for gapless composite Fermi-liquid states at filling number $\nu = 1/2$ and $1/4$ at low temperature ($T$), with exponent $\eta \sim 0.5$ distinctly different from Fermi liquids. The powerlaw scaling remains unchanged for different forms of interaction including Coulomb and short-range ones, demonstrating the robustness of non-Fermi-liquid behavior at low $T$. In contrast, for $1/3$ fractional quantum Hall state, $\alpha_{xy}$ vanishes at low $T$ with an activation gap associated with neutral collective modes rather than charged quasiparticles. Our results establish a new manifestation of the non-Fermi-liquid nature of quantum Hall fluids at finite temperature.

Introduction.— Thermoelectric phenomena that provide the direct conversion between heat and electricity are interesting and useful. Decades of research has been devoted to finding materials and methods to increase thermoelectric energy conversion efficiency [1, 2]. Recent theoretical works suggested the possibility of record-high thermoelectric conversion efficiency in semiconductors and semimetals under a quantizing magnetic field [3, 4], where thermoelectric response is directly related to entropy [5–8]. Based on this relation, it is found that thermopower of 3D Dirac/Weyl materials in the quantum limit increases unboundedly with the magnetic field [3, 9–12].

Very recently, it is shown that two-dimensional quantum Hall systems can reach thermoelectric figure of merit on the order of unity down to low temperature ($T$), as a consequence of the thermal entropy from the massive Landau level (LL) degeneracy [4].

The degeneracy of a partially filled Landau level is lifted by disorder and electron-electron interaction. Therefore, thermoelectric response of quantum Hall systems is expected to depart from the noninteracting and clean limit when thermal energy $k_B T$ is smaller than disorder-induced Landau level broadening $\Gamma$ or a characteristic energy scale proportional to electron interaction strength. Previous works [6, 13] have shown that disorder leads to a $T$-linear thermoelectric Hall conductivity $\alpha_{xy} \propto T$ for $k_B T \ll \Gamma$, in accordance with thermal entropy of disorder broadened Landau levels.

On the other hand, in clean systems electron-electron interaction lifts the massive Landau level degeneracy and forms the many-body ground state at fractional filling. These include gapped fractional quantum Hall (FQH) states [14] and gapless composite Fermi liquids [15, 16], which provide a fertile ground for exotic quantum states of matter. After nearly four decades of theoretical and experimental studies, ground state properties and low-energy excitations at various fractional fillings are largely understood. In contrast, much less is known about FQH liquids at finite temperature. Based on the relation between the entropy and the thermoelectric transport coefficient, a linear $T$ scaling behavior for thermopower $S_{xx}$ has been conjectured [17] for composite Fermi liquid states. While there are a few measurements [19–25], to our knowledge no theoretical or numerical calculations on finite $T$ thermoelectric transport coefficients for fractional quantum Hall states exists so far.

In this work, we investigate thermoelectric Hall response for interacting quantum Hall systems through exact diagonalization calculations of entropy of finite size systems at finite temperature. For even denominator filling numbers $\nu = 1/2$ and $1/4$, we identify robust power law scaling behavior of the thermoelectric Hall conductivity $\alpha_{xy} \propto T^\eta$ in a wide temperature range, with the exponent $\eta \sim 0.5$ distinctly different from the linear $T$ behavior of Fermi liquids. We further show that the scaling behavior is robust against weak disorder, and the exponent $\eta$ gradually increases with disorder strength. In contrast, for $1/3$ FQH system, we observe a vanishing $\alpha_{xy}$ at low $T$ below the excitation gap. Our prediction of anomalous power-law temperature dependence of thermoelectric response establishes a new fundamental property of $\nu = 1/2$ and $\nu = 1/4$ quantum Hall fluids, which can be measured in future experiments.

Model and Method.— We consider a two dimensional (2D) electron system subject to a perpendicular strong magnetic field, whose energy spectrum is composed of discrete LLs. Throughout this work we assume the cyclotron energy is much larger than other energy scales set by interaction, disorder scattering or temperature, so that it suffices to work with the restricted Hilbert space of the partially filled LL.

The many-body Hamiltonian can be written as

\begin{equation}
H = \sum_{i<j} \sum_q e^{-q^2/2} V(q)e^{iq(R_i - R_j)} + \sum_i \sum_q e^{-q^2/4} U_q e^{iq R_i},
\end{equation}

where $R_i$ is the guiding center coordinate of the $i$-th electron, $V(q) = 2\pi e^2/\epsilon q$ is the Coulomb potential, and $U_q$ is the impurity potential with the wave vector $q$. We set the magnetic length $\ell = 1$ and $e^2/\epsilon \ell = 1$ for convenience. The Gaussian white noise potential we use is generated according to the correlation relation in $q$-space $\langle U_q U'_{q'} \rangle = (W^2/A) \delta_{q-q'}$, which corresponds to $\langle U(r) U(r') \rangle = W^2 \delta(r-r')[26]$ in real space, where $W$ is the strength of the disorder and $A$ is the area of the system. The filling fraction is then defined as $\nu = N_e/N_s$, where $N_e$ and $N_s$ are the number of electrons in the partially filled LL and the number of the flux quanta, respectively.
Thermoelectric conductivity $\alpha_{ij}$ is defined by the electrical current generated by a temperature gradient in the absence of any voltage (short-circuit condition), or via Onsager relation, by the heat current generated by a voltage difference at a uniform temperature. Since heat current is carried by thermal excitations, thermoelectric conductivity is purely a property of the partially occupied LL. In our case, its value depends on temperature $k_B T$ and disorder strength $W$ (both in units of $e^2/\ell\epsilon$). While thermoelectric conductivity is conceptually convenient for theoretical analysis, experiments usually measure thermopower $S_{xx}$ and Nernst signal $S_{xy}$ directly. These are given by the product of $\alpha_{ij}$ and resistivity $\rho_{jk}$: $S_{ik} = \alpha_{ij}\rho_{jk}$.

We first consider the clean limit $W = 0$. As shown explicitly for both non-interacting systems [4, 6] and generic interacting electron fluid[17, 18], in the absence of disorder, thermoelectric Hall conductivity $\alpha_{xy}$ is directly proportional to entropy density $s$: $\alpha_{xy} = s/B$. This remarkable formula enables us to obtain $\alpha_{xy}$ by numerically calculating entropy—a thermodynamic property—without invoking the Kubo formula for transport coefficients.

We perform thermal entropy calculations based on exact calculation of energy spectrum of the Hamiltonian, and obtain $\alpha_{xy} = S/N_s$ (in units of $k_B e/h$), where $S = sA$ is the thermal entropy of the system. We consider systems with sub-dimension of Hilbert space up to $N_s = 102348$ (2119036) for full (partial) diagonalizations, which is slightly smaller than the sizes used in ground state simulations and gives reliable results for all temperature regime we considered.

**Thermoelectric Hall response of composite Fermion liquids**—We first consider interacting quantum Hall systems without disorder scattering. Two even denominator filling numbers $\nu = 1/2$ and 1/4 will be considered first, where low temperature behavior of such systems is controlled by the physics of the composite Fermi-liquid[16]. By exact diagonalization, we can study systems with up to the number of flux quanta $N_q = 28$ for electrons at 1/2 filling using magnetic translational symmetry. By obtaining all energy eigenvalues
of the system, we determine $\alpha_{xy}$ from the entropy per flux. As shown in Fig. 1(a-b), we show $\alpha_{xy}$ for different system sizes with $N_s = 12$ to 28 for both $n = 0$ and $1$ LLs. The $\alpha_{xy}$ grows with $T$ monotonically, and saturates towards a universal value $\ln(2) + k_BT/h$ determined by the entropy per flux for LL at half-filling at high $T$ regime with a small correction due to the constraint of using canonical ensemble with fixed particle number for finite size systems. As shown in insets of Fig. 1(a-b), we identify a powerlaw behavior at low temperature $\alpha_{xy} \propto T^n$ as a straight line fitting to the data in the algorithmic plots, and find the exponent $\eta \sim 0.54 \pm 0.03$ and $\sim 0.44 \pm 0.03$ for $n = 0$ and $1$, respectively. We remark that although the nature of groundstates for the $n = 0$ and $n = 1$ are different at $T = 0$ limit corresponding to the composite-Fermi liquid and Moore-Read non-Abelian FQH state[27], respectively, $\alpha_{xy}$ of both systems in the small to intermediate $T$ can demonstrate similar scaling behavior. This is because $\alpha_{xy}$ is controlled by composite-Fermi liquid behavior once $k_BT$ is comparable to the excitation gap of the FQH of $n = 1$ LL. The exponent $\eta$ appears to be slightly different for these two cases. Furthermore, we compare $\alpha_{xy}$ of systems with different type of electron-electron interactions including the Haldane short-range pseudopotential, and find quantitative similar results. Importantly, we have demonstrated that $\alpha_{xy}$ at low $T$ behaves distinctly different from the linear $T$ scaling of the Fermi liquid behavior. The finite size effect only occurs for very low $T \lesssim 0.01$, which is expected due to finite energy level spacing for such systems.

Now we move on to $\nu = 1/4$ filling number, where larger systems can be accessed with $N_s$ up to 32 (40) for full (partial) diagonalization. As shown in Fig. 2(a-b) for systems with $N_s = 16$ to 40 for both LLs $n = 0$ and $n = 1$, $\alpha_{xy}$ increases with $T$ rapidly and it saturates to a universal value ($0.562$) determined by the entropy per flux for the $1/4$ partially filled LL at high $T$ limit. As shown in the insets of Fig. 2(a-b), we demonstrate a powerlaw behavior at low temperature $\alpha_{xy} \propto T^n$, where almost all data points from low $T$ regime can be well fitted by such a scaling behavior up to $T \sim 0.07$, beyond which $\alpha_{xy}$ starts to saturate. Clearly, the finite size effect is reduced comparing to $\nu = 1/2$ case as we can access larger systems with larger $N_s$ at $\nu = 1/4$ filling. The exponent is identified to be $\eta \sim 0.45 \pm 0.03$ and $\sim 0.53 \pm 0.03$ for $n = 0$ and $1$, respectively. The average exponent for the power-law behavior is consistent with $\eta \sim 0.50$ for both $\nu = 1/4$ and $1/2$ filling numbers, indicating a possible universal scaling behavior for $\alpha_{xy}$ at low $T$.

**Thermoelectric Hall response for different electron filling numbers—** At low temperature, correlated states emerge for interacting quantum Hall systems. The nature of the states dependent on the electron filling number. As we have shown before, the even denominator state is either gapless (for $n = 0$ case) or having a small excitation gap for QHE state ($1/2$ filling of $n = 1$ LL), which exhibits powerlaw scaling for $\alpha_{xy}$ down to very low temperature. To explore possible distinct physics from thermoelectric Hall effect of a FQH state with a robust gap, we present results of $1/3$ FQH with Coulomb interaction for pure system at $W = 0$. As shown in Fig. 3(a) with flux quanta $N_s = 12 − 30$, we find that $\alpha_{xy}$ at low $T < 0.01$ decreases with the increase of $N_s$, while $N_s \alpha_{xy} = ln3/N_s$ in accordance with the three fold degeneracy of the system. The data for the largest system $N_s = 30$ is obtained through Lanczos for lowest 600 states in each momentum sector, which allows us to obtain lower temperature $\alpha_{xy}$ accurately. The temperature regime to see such a vanishing $\alpha_{xy}$ behavior is $T \lesssim 0.01$, indicating a finite gap for such a fractionalized topological state. Beyond $T > 0.01$, $\alpha_{xy}$ increases with $T$ very sharply, and saturates to a universal value at larger $T$ side. To compare these data with the activation behavior of a gapped system, we use the following form $\alpha_{xy} - ln3/N_s \propto exp(-E_g/k_BT)$ to fit the low $T$ data. The constant term $ln3/N_s$ is the size dependent contribution from topological degeneracy. As shown in the inset of Fig. 3(a), we identify the collective excitation gap $E_g = 0.06$, which should be distinguished from the quasiparticle charge gap. Therefore thermoelectric response provides a new experimental way to detect collective excitations.

We compare $\alpha_{xy}$ at other filling numbers with the behavior of composite Fermion liquids systems at $\nu = 1/2$ and $1/4$. In Fig. 3(b), we show $\alpha_{xy}$ vs. $\nu$ for filling numbers $\nu = 1/8−1/2$ at fixed $N_s = 24$ (the result is symmetric about $\nu = 1/2$ due to particle-hole symmetry). At low temperature, the gapped $1/3$ FQH state has suppressed $\alpha_{xy}$ as discussed above. Other systems are either gapless, or having tiny gaps
all the data collapse into one universal curve as a function of $N$. In this case, the Hilbert space has a dimension $N_s$ to include both terms in the Hamiltonian to model realistic fractional Fermi-liquid states, or thermally excited QHE states. Where these interacting systems are either in gapless composite Fermi-liquid states, or thermally excited QHE states. The noninteracting systems with strong magnetic field as also obtained for graphene systems based on Kubo formula. The $\alpha_{xy}$ saturates to an universal value $\ln 2 * k_B e/h$ as $T$ approaches the order of $1$. Now we turn to the effect of weak random disorder for interacting quantum Hall systems. In this case, the magnetic translational symmetry is broken due to momentum nonconserving scattering present in the Hamiltonian. So we have to diagonalize the whole Hilbert space for $N_e$ electrons occupying $N_s$ orbitals. This limits us to smaller systems with $N_s = 16$ and $18$. As shown in Fig. 4(b) for $N_s = 16$ at filling number $\nu = 1/2$, thermal entropy is always an increasing function of $T$ for different $W$, which saturates to the universal value determined by the maximum entropy per orbital in such systems. Entropy decreases monotonically at small $T$ regime with the increase of $W$. In the smaller and intermediate $T$ regime, entropy appears to follow the powerlaw scaling behavior $s \propto T^\eta$, with the exponent $\eta$ increases from around 0.62 for $W = 0.02$, to $\eta \sim 0.90$ for $W = 0.1$. Very similar results are obtained for $N_s = 16$ and 18, indicating these behaviors are robust as they are associated with the thermal entropy. Quantitatively, the exponents determined here are less reliable due to limited system sizes we can access.

**Summary and discussion.**—We now discuss the importance of our work for understanding the thermoelectric Hall effect of different quantum Hall systems at low temperature. Since most of quantum Hall systems realized in experiments have high mobility, the interaction effect plays essential role in lifting LL degeneracy. Our work focuses on finite temperature, where these interacting systems are either in gapless composite Fermi-liquid states, or thermally excited QHE states. The nonlinear power law scaling behavior we established for $\nu = 1/2$ and 1/4 filling numbers strongly proves that these systems at finite temperature are strongly correlated electron fluids, distinctly different from Fermi liquids for which the linear scaling law of $\alpha_{xy}$ is expected from the low energy excitations around the Fermi level. The scaling behavior of $\alpha_{xy} \propto T^\eta$ (with $\eta \sim 0.5$) appears to be very general for quantum Hall systems at different filling numbers. In particular, the $\nu = 1/2$ and 1/4 quantum Hall states have much enhanced $\alpha_{xy}$ at the lowest temperature, and therefore are promising candidates for thermoelectric energy conversion with high efficiency.

Our calculations can be naturally extended to other quantum Hall systems with different LL degeneracy or multi-component interactions such as graphene. The intriguing strange metal transport we discovered calls for a theory of fractional quantum Hall liquids at finite temperature, which we will provide in forthcoming works.

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