Dynamical system approach of non-minimal coupling in AdS/CFT cosmology

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Our purpose is to develop a non minimal coupling scalar field in the context of the anti de Sitter/conformal field theory (AdS/CFT) correspondence. The dynamics of this model are studied by rewriting the cosmological field equations in the form of a system of autonomous differential equations. In particular, the analysis is considered by investigating a minimal coupling, a conformal coupling and a non minimal coupling scalar field to the curvature. We present and discuss issues of stability and viability of this model for different behaviors of our universe.

Keywords: Dynamical system, braneworld model, AdS/CFT correspondence, non-minimal coupling

I. INTRODUCTION

In recent years, extra dimensional space-time has taken considerable research interest such as braneworld models in which ordinary matter fields are considered to live on the boundary of a high-dimensional bulk space-time, the brane. In particular, Randall and Sundrum [1] proposed a model (RSII) in which a brane with a positive tension is embedded in five dimensional anti de Sitter space. The cosmology of this model shows that at low energies general relativity on the brane can be recovered, while in high energy limit gravity becomes five dimensional. The RSII model can also be considered as an example of the holographic principle [2]−[5] that has emerged in M theory. Indeed, holography suggests that higher-dimensional gravitational dynamics may be determined from knowledge of the fields on a lower-dimensional boundary. A concrete illustration of this holographic principle is The AdS/CFT correspondence. This kind of correspondence asserts that there is an equivalence between a gravitational theory in d-dimensional anti de Sitter space-time and a conformal field theory living in a (d−1)-dimensional boundary space-time [6]. This equivalence or duality is best understood in the context of string theory with d=5, where the duality relates type IIB superstring theory on $AdS_{5} \times S^{5}$ and $N = 4$ supersymmetric Yang Mills theory with gauge group SU(N) in four dimensions [7, 8]. The RSII model with its $AdS_{5}$ metric satisfies this correspondence to lowest perturbative order [9]. In this paper, the AdS/CFT correspondence is the subject of our framework in order to derive the modified equations.

On the other hand, Dvali, Gabadadze and Porrati [10] suggested a model with a bulk as a flat Minkowski space-time, but a reduced gravity term appears on the brane without tension. This setup is based on a modification of the gravitational theory in an induced gravity perspective [11]−[14]. Generally, induced gravity effect can be viewed as a quantum correction in any braneworld model for instance the Randall-Sundrum model.

In the present paper, we study the effect of an induced gravity term which is a function of a scalar field on the brane within the AdS/CFT correspondence using the dynamical system analysis.

In this framework, we consider a scenario where a scalar field is coupled to gravity non-minimally. The motivation for including a non-minimal coupling term arises at the quantum level when quantum corrections to the scalar field theory are considered. This kind of non-minimal coupling to gravity has been discussed enough in four dimensions [15]−[19] and also in extra dimensions [20]−[33]. It turns out that in general relativity, the coupling constant is valued to 1/6 [34]. We analyze this coupling value as a particular case.

Since explicit solutions of the evolution equations cannot be obtained in this setup, the theory of dynamical systems has proven to be a very powerful scheme to obtain exact solutions and a qualitative description of the global dynamics. Dynamical systems methods are widely used in cosmology. Indeed, these approaches have been applied to study the dynamics of extended theories of gravity [35]−[40].

The paper is organized as follows. In Sec. II, we establish our basic non-minimal coupling approach of the scalar field to gravity in the context of the AdS/CFT correspondence. In Sec. III, we present the basic cosmological equations to describe the evolution of a non minimal coupled scalar field such as the modified Friedmann equation, in which we show the existence of two branches of solution as a function of scalar field, and the equation of motion. In Sec. IV, a dynamical systems analysis is developed by fixing the choice of an exponential potential and a monomial form of the coupling function. Three cases are of interest, the minimal coupling, the conformal coupling and the non minimal coupling. Finally, we present our summary and conclusions in Sec. V.
II. THE SETUP

In this work, we will analyse the model described by the action \[ S = \int_{\text{bulk}} d^5x \sqrt{-g} \left( \frac{1}{2\kappa_5^2} R_5 - \Lambda_5 \right) + \int_{\text{brane}} d^4x \sqrt{-g} (-\Lambda_4 + \Sigma_\phi), \] (2.1)
where $\kappa_5^2$ is the 5D gravitational constant, $R_5$ is the Ricci scalar of the five-dimensional metric $g^{(5)}$, and $\Lambda_5$ is the bulk cosmological constant. $g$ is the induced metric on the brane, $\Lambda_4$ is the brane tension and $\Sigma_\phi$, the Lagrangian density of the non-minimal scalar field localized on the brane, is defined as

\[ \Sigma_\phi = f(\phi)R + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi). \] (2.2)

where $\nabla_\mu$ is the covariant derivative associated with the induced metric on the brane, $V(\phi)$ is the scalar field potential, and $f(\phi) \equiv \frac{1}{2} \left( \frac{\Lambda_4}{\kappa_5^2} - \alpha(\phi) \right)$ is a coupling between the scalar field $\phi$ and the induced gravity $R$.

The gravitational field equations through the AdS/CFT correspondence are obtained by extremizing the variation of the dual action with respect to the metric tensor \[ \kappa_4 \bar{G}_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{CFT} - \lambda g_{\mu\nu}, \] (2.3)
where $\bar{G}_{\mu\nu}$ denotes the Newton’s constant ($8\pi G_N = \kappa_4^2$), $T_{\mu\nu}$ is the total energy-momentum tensor and $T_{\mu\nu}^{CFT}$ denotes the CFT energy momentum tensor.

The total energy-momentum tensor $T_{\mu\nu}$ and the effective cosmological constant on the brane $\lambda$ are given by \[ T_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(f)} - 2f(\phi)G_{\mu\nu}, \] (2.4)
\[ \lambda = \frac{\kappa_5^2}{2} \Lambda_5 + \frac{\kappa_4^4}{12} \Lambda_4^2. \] (2.5)

The energy-momentum tensor of the conformal field theory, $T_{\mu\nu}^{CFT}$, cannot be written in the local covariant form, however its trace writes \[ T^{CFT}_{\mu\nu} = c \left( R^\alpha_\beta R^\beta_\alpha - \frac{1}{3} R^2 \right), \] (2.6)
where $c$ is the conformal anomaly related to the AdS/CFT length.

The total energy-momentum tensor $T_{\mu\nu}$ Eq. (2.4) has been split of into a scalar field energy-momentum tensor,

\[ T_{\mu\nu}^{(\phi)} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 - g_{\mu\nu} V(\phi), \] (2.7)

and into a non-minimal coupling energy momentum tensor

\[ T_{\mu\nu}^{(f)} = 2 \nabla_\mu \nabla_\nu f(\phi) - 2 \Box f(\phi) g_{\mu\nu}. \] (2.8)

For a spatially flat Friedmann-Robertson-Walker universe (FRW), we may define the conformal field energy momentum tensor as \[ T^{CFT}_{\mu\nu} = \left( \begin{array}{cc} -\sigma & \sigma \\ 0 & \sigma \delta_{ij} \end{array} \right), \] (2.9)

The Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, and the equation of conservation of the energy-momentum, $\nabla^\mu T_{\mu\nu} = 0$, implies that $\nabla^\mu T_{\mu\nu}^{CFT} = 0$, which amounts to

\[ \dot{\sigma} + 3H(\sigma + \sigma_p) = 0, \] (2.10)
where $H$ is the Hubble parameter.

Furthermore, the trace of the conformal anomaly equation (2.6) simplifies to

\[ \sigma - 3\sigma_p = 24cH^2(\dot{H} + H^2), \] (2.11)
and Eq. (2.10) becomes

\[ \dot{\sigma} + 4H\sigma - 24cH^3(H^2 + \dot{H}) = 0, \] (2.12)
whose solution reads

\[ \sigma = \chi_{\text{rad}} + 6cH^4, \] (2.13)
where $\chi_{\text{rad}}$ is an effective radiation term. During inflation, this term is rapidly redshifted as $a^{-4}$ away and its contribution can be neglected \[ \text{[13].} \]

III. BASIC COSMOLOGICAL EQUATIONS

In this section we will consider the following spatially flat isotropic and homogeneous FRW brane

\[ ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \] (3.1)
where $a(t)$ is the scale factor, $\delta_{ij}$ is a symmetric 3-dimensional metric and $x^i$, $i = 1, 2, 3$ are the comoving spatial coordinates.

From the (00)-component of the field equations together with the equations (2.4) and (2.13), the modified Friedmann equation on this spatially flat brane can be obtained as

\[ H^2 = \frac{\kappa_5^2}{3} \left( \rho + \lambda + 6cH^4 \right), \] (3.2)
where $\rho$ is the total energy density and the effective gravitational coupling, $\kappa_{eff}^2$, is given by

\[ \kappa_{eff}^2 = \frac{\kappa_4^2}{2} \left( \frac{\chi_{\text{rad}}}{\lambda} \right). \] (3.3)
Following the notation introduced in [30], we can write the total energy density and the pressure of the universe respectively as

\[ \rho = \rho^{(\phi)} + \rho^{(\alpha)}, \quad p = p^{(\phi)} + p^{(\alpha)}, \]  
\[ (3.4) \]

where

\[ \rho^{(\phi)} = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p^{(\phi)} = \frac{1}{2} \dot{\phi}^2 - V(\phi), \]  
\[ (3.5) \]

\[ \rho^{(\alpha)} = 3H \frac{d\alpha(\phi)}{dt} \quad \text{and} \quad p^{(\alpha)} = -2H \frac{d\alpha(\phi)}{dt} - \frac{d^2\alpha(\phi)}{dt^2}. \]  
\[ (3.6) \]

The modified Friedmann equation Eq. (3.2) can be rewritten as

\[ H^2 = \frac{1}{4\kappa^2_{eff}} \left[ 1 \pm \frac{\rho + \lambda}{\rho_{max}} \right], \]  
\[ (3.7) \]

where \( \rho_{max} = \frac{3}{4\kappa^2_{eff}}. \)

In the limit \( \alpha(\phi) \to 0 \) we recover the modified Friedmann equation of the Randall-Sundrum cosmology in the context of the AdS/CFT correspondence [41, 43] with a minimally coupled scalar field.

Furthermore, the modified Raychaudhuri equation can be deduced from Eq. (2.3) as

\[ \frac{\ddot{a}}{a} = -\frac{\kappa^2_{eff}}{6} \left( \rho + 3p - 2\lambda - 12cH^2(2\frac{\ddot{a}}{a} - H^2) \right). \]  
\[ (3.8) \]

Finally, minimising the action (2.1) with respect to variation of the scalar field, \( \phi \), we obtain the equation of motion in the FRW geometry as

\[ \ddot{\phi} + 3H\dot{\phi} + \frac{1}{2} \alpha'(\phi)R + V'(\phi) = 0, \]  
\[ (3.9) \]

where the prime denotes the derivative with respect to the scalar field \( \phi \).

The intrinsic Ricci scalar for a flat FRW brane is

\[ R = 6(\dot{H} + 2H^2). \]  
\[ (3.10) \]

IV. A DYNAMICAL SYSTEMS APPROACH

In order to simplify the analysis of Eqs (3.2), (3.8) and (3.9), the method taken up is the dynamical systems study. In this section, we present the phase space of the non-minimally coupled scalar field in detail, exact solutions and their stability.

The first step in the implementation of the Dynamical System Approach (DSA) is the introduction of the general dimensionless variables

\[ x_1 \equiv \frac{1}{\sqrt{6\kappa_4 H}}, \quad x_2 \equiv \sqrt{\frac{\alpha(\phi)}{2cH}}, \quad x_3 \equiv \frac{\dot{\alpha}(\phi)}{2cH^3}, \]

\[ y \equiv \frac{\phi}{\sqrt{12cH^2}}, \quad z \equiv \frac{V(\phi)}{\sqrt{6cH^2}}. \]  
\[ (4.1) \]

The Friedmann constraint Eq. (3.2) with respect to the dimensionless variables (4.1) becomes

\[ 1 = x_1^2(1 - Ax_1^2) - x_2^2 - y^2 - z^2 - x_3. \]  
\[ (4.2) \]

The dynamical variables Eq. (4.1) are non-compact, i.e. their values do not have finite bounds as in [35–37, 45]. We will come back to this point in the conclusion. The cosmological equations become equivalent to the following autonomous system:

\[ \frac{dx_1}{dN} = -x_1 \frac{\dot{H}}{H^2}, \]  
\[ (4.3a) \]

\[ \frac{dx_2}{dN} = x_1 - 2x_2 \frac{\dot{H}}{H^2}, \]  
\[ (4.3b) \]

\[ \frac{dx_3}{dN} = 6\Gamma y^2 - 6qx_2(3y - 6qx_2 + r) - 3(x_3 + 6q^2x_2^2) \frac{\dot{H}}{H^2}, \]  
\[ (4.3c) \]

\[ \frac{dy}{dN} = -3y - 6qx_2 - rz - 2x_2 \frac{\dot{H}}{H^2}(2y + 3qx_2), \]  
\[ (4.3d) \]

\[ \frac{dz}{dN} = ry - 2z \frac{\dot{H}}{H^2}, \]  
\[ (4.3e) \]

where

\[ \Gamma \equiv \alpha''(\phi), \quad r \equiv \frac{V'(\phi)}{\sqrt{2V(\phi)}H}, \]

\[ q \equiv \frac{\alpha'(\phi)}{\sqrt{6\alpha(\phi)}} \quad \text{and} \quad A \equiv \frac{\kappa^2_4}{6\lambda}. \]  
\[ (4.5) \]
The prime denotes the derivative with respect to the scalar field $\phi$. 

From the first equation of the dynamical system (1.3a)-(1.3e), one can notice that the system has two invariant manifolds $x_1 = 0$ and $\dot{H}/H^2 = 0$. The most interesting, from a physical point of view, is the last one $\dot{H}/H^2 = 0$. 

The critical points of any dynamical system can be extracted by setting $dx_i/dN = 0$ ($i = 1, 2, ... n$), while their properties are determined by the eigenvalues $\mu_i$ of its Jacobian matrix, $J$, which is also called the stability matrix

$$J = \left(\begin{array}{ccc}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{array}\right),$$

(4.6)

where $f_i \equiv (dx_i/dN)$. 

The critical points are classified according to the sign of their eigenvalues by using linear stability method as:

- **Attractor critical point**, If all eigenvalues have negative real parts. In this case the point would attract all nearby trajectories and is viewed as stable.

- **Repeller critical point**, If all eigenvalues have positive real parts where trajectories are repelled from the fixed point and we speak in this situation of an unstable point.

- If there is mixture of both positive and negative real parts of eigenvalues, then the corresponding critical point is called a saddle. This point will attract nearby trajectories in some directions but repels them along others.

However, If at least one of the eigenvalues is zero, the linear stability theory fails to describe the stability of the critical point which is called non-hyperbolic. In this case other techniques have to be employed to study the stability properties, such as the Centre Manifold Theory (CMT) [46–49], the Lyapunov function method [50–52] and Kosambi-Cartan-Chern theory [53].

### A. Example of $\alpha(\phi)$ and $V(\phi)$

Since the dynamical system Eqs. (1.3a)-(1.3e) is complicated to analyze in its full generality, we consider a particular case in order to illustrate our purpose. Concerning the scalar field potential, we choose an exponential function which has many implications in cosmological inflation [54] [55]

$$V = V_0 e^{-\kappa \phi},$$

(4.7)

where $V_0$ corresponds to the maximum value of the potential and $\kappa$ is a constant and we choose the following form of the coupling $\alpha(\phi)$ as

$$\alpha(\phi) = \alpha_0 \phi^2,$$

(4.8)

where $\alpha_0$ is a constant parameter. If one chooses this monomial form of $\alpha(\phi)$, the set of phase space variables reduces to a four dimensional by writing the variable $x_3$ as

$$x_3 = 2\sqrt{6\alpha_0} y x_2.$$  

(4.9)

In the next subsections we will consider first two special values of $\alpha_0$, namely the minimal coupling for $\alpha_0 = 0$ and the conformal coupling for $\alpha_0 = 1/6$.

#### 1. Minimal coupling

To illustrate our purpose, we begin by the simple case, namely the minimal one where $\alpha_0 = 0$. In that case the variable $x_2$ is equal to zero. Using the Friedmann constraint Eq. (4.2) and Eq.(4.7), the system (1.3a)-(1.3e) reduces to the following autonomous two-dimensional system in terms of the dynamical variables

$$\frac{dx_1}{dN} = -\frac{3x_1 y^2}{1 - x_1^2};$$

(4.10a)

$$\frac{dy}{dN} = -3y(1 + \frac{2y^2}{1 - x_1^2}) + \sqrt{3}b \frac{1 - y^2 + x_1^2 (1 - Ax_1^2)}{x_1}.$$  

(4.10b)

This nonlinear autonomous system has four critical points $A_{\pm}$ and $B_{\pm}$. Their properties are given in Table I and are summarized below.

- **Critical points $A_{\pm}$** exist for $A \leq 1/4$ (see Eq. (4.5)) i.e. the effective cosmological constant on the brane satisfy $\lambda \leq \lambda_{\text{max}}$ where, $\lambda_{\text{max}} = 3/2\kappa^2$. These points correspond to the case where the kinetic energy density of the scalar field and its potential energy density $V$ vanish (as the Hubble rate remains finite). This means that there is no dynamical motion of the the scalar field.

The Friedmann equation of these fixed points writes

$$H = \pm \sqrt{\frac{\Lambda_+}{3}},$$

(4.11)

where $\Lambda_+ = \frac{\lambda \kappa^2}{1 - \sqrt{1 - \frac{\lambda}{\lambda_{\text{max}}}}}$.

Therefore, we conclude that the dynamic of the universe is governed by the effective cosmological constant. we notice also that the critical point $A_+$ corresponds to an expanding de Sitter universe while $A_-$ represents a contracting one.
**Stability**

The stability properties of these critical points, \( \mathcal{A}_\pm \) and \( \mathcal{B}_\pm \), are obtained by evaluating the eigenvalues of the Jacobian matrix \( J \) in the neighborhood of these critical points. We obtain the following eigenvalues \( \mu_1 = 0 \) and \( \mu_2 = -3 \) for each critical point. We notice that these points are nonhyperbolic.

The stability properties of these points are obtained by applying the CMT to the 2D-system Eqs. (4.10a) and (4.10b) (the analysis details are given in the Appendix).

Around the critical point \( \mathcal{A}_+ \), the stability depends on the value of the constant \( A \).

For \( A < 0 \) (i.e. \( \lambda < 0 \)), the critical point \( \mathcal{A}_+ \) is saddle, whereas for \( 0 < A \leq \frac{1}{4} \) both critical points \( \mathcal{A}_+ \) and \( \mathcal{B}_+ \) are unstable.

Fig. 1 shows the phase space and the position of the critical points of the system (4.10a)-(4.10b).

We conclude that in the case of minimal coupling, the resulting Hubble rate, can be considered as solutions at early times. This means that in the past, each trajectory begins in a de Sitter state as the solution behaves like a cosmological constant of an arbitrary value for \( A > 0 \).

### 2. Conformal coupling

We now consider the case of a conformally coupled scalar field on the brane [30, 56], with conformal coupling \( \alpha_0 = \frac{1}{6} \), and a vanishing potential [30]. In what follows, we present the results of our dynamical system (4.3a)-(4.3e) for the conformal coupling. The system reduces to

\[
\begin{align*}
\frac{dx_1}{dN} &= -x_1 \ g(x_1, x_2, y), \tag{4.13a} \\
\frac{dx_2}{dN} &= y - x_2 \ g(x_1, x_2, y), \tag{4.13b} \\
\frac{dy}{dN} &= -3y - 2x_2 - (2y + x_2) \ g(x_1, x_2, y), \tag{4.13c}
\end{align*}
\]

where

\[
g(x_1, x_2, y) = \frac{3x_1^2(1 - Ax_1^2) + x_1^2 + y^2 + 2x_2y - 3}{2(1 - x_1^2)}. \tag{4.14}
\]

In Table I, we present the coordinates of each critical point and the results of their stability analysis by means of the signs of the real parts of the eigenvalues of the Jacobian matrix.

- **Critical points** \( \mathcal{C}_\pm \) exist for \( A \leq 1/4 \) and amount to assume that the solution is a de Sitter universe

\[
(H, \phi) = (\pm \sqrt{\frac{A-3}{3}}, 0). \tag{4.15}
\]

The stability of these two points depend on the value of the constant \( A \) given by (4.5).

For \( A < 0 \) (i.e. \( \lambda < 0 \)), the critical points \( \mathcal{C}_\pm \) are stable since all eigenvalues are negative, whereas for \( 0 < A \leq \frac{1}{4} \), the two critical points \( \mathcal{C}_\pm \) are saddle since one of the eigenvalues is positive while the others are negative.

- **Critical points** \( \mathcal{D}_\pm \) exist for \( 0 < A \leq 1/4 \) and represent also a de Sitter universe

\[
(H, \phi) = (\pm \sqrt{\frac{A_+}{3}}, 0). \tag{4.16}
\]

Finally, accordingly to their eigenvalues, the critical points \( \mathcal{D}_\pm \) are saddle points.

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2 We restrict our analysis to the critical points \( \mathcal{A}_+ \) and \( \mathcal{B}_+ \) since we are assuming an expanding universe, i.e. \( H > 0 \).

3 The Klein-Gordon equation (3.9) is conformally invariant if \( V = 0 \) or \( V = \lambda \phi^4 \) for the conformal coupling [20, 56, 58].
(a) We have taken $A = -1$ and $b = -2$.

(b) We have taken $A = 1/8$ and $b = 1$.

FIG. 1. Phase plot (blue arrows) and critical points (colored dots) of the system (4.10a)-(4.10b), for a minimal coupled scalar field with an exponential potential (4.7). It seems that point $\mathcal{B}_{\pm}$ is saddle, but in fact it is not, it is unstable from CMT point of view, see Appendix A.

| Point | $x_1$ | $x_2$ | $y$ | Existence | Eigenvalues | Stability | Physical State |
|-------|-------|-------|-----|-----------|-------------|-----------|----------------|
| $\mathcal{C}_{\pm}$ | $\pm \sqrt{\frac{1 - \sqrt{1 - 4A}}{2A}}$ | 0 | 0 | $A \leq \frac{1}{4}$ | $\mu_1 = \frac{3(1 - 4A + \sqrt{1 - 4A})}{2A}$, $\mu_2 = -2$, $\mu_3 = -1$ | Stable for $A < 0$ Saddle for $0 < A \leq \frac{1}{4}$ | de Sitter universe |
| $\mathcal{D}_{\pm}$ | $\pm \sqrt{\frac{1 + \sqrt{1 - 4A}}{2A}}$ | 0 | 0 | $0 < A \leq \frac{1}{4}$ | $\mu_1 = \frac{3(1 - 4A + \sqrt{1 - 4A})}{2A}$, $\mu_2 = -2$, $\mu_3 = -1$ | Saddle | de Sitter universe |

TABLE II. Coordinates of the critical points of the system (4.13a)-(4.13c), and their properties.

(a) Projection of perturbations along $x_1$-axis.  
(b) Projection of perturbations along $x_2$-axis.  
(c) Projection of perturbations along $y$-axis.  
(d) Slow roll parameter, $\epsilon$.

FIG. 2. Projection of perturbations of $\mathcal{C}_+$ along $x_1$, $x_2$, $y$ axis and slow roll parameter, $\epsilon$, vs the e-fold number, $N$, for $A = -1/4$.  

We notice that in the conformal coupling case, the critical point \( C_+ \) is the future attractor if and only if \( \lambda \) is negative while the critical point \( D_+ \) is always a stable saddle point meaning that it cannot be the past attractor.

To confirm the stability of the critical point \( C_+ \), we perturb the solutions around this point in order to analyse numerically this property. In Figs. 2a, 2b and 2c we plot the projection of perturbations of the system (4.13a)-(4.13c) along \( x_1 \)-axis, \( x_2 \)-axis and \( y \)-axis respectively with respect to \( N \).

From these figures we notice that trajectories of the perturbed solutions approach the coordinates of \( C_+ \) from both \( x_1 \) and \( x_2 \), respectively as \( N \to \infty \). From these behaviours, we can conclude that the critical point \( C_+ \) is a saddle solution which is in agreement with our analytical result. Furthermore, this point corresponds to a slow roll parameter \( \epsilon \equiv -\frac{\dot{H}}{H^2} = 0 \) which means that this point may sustain inflation. As we can notice from Fig. 2d the universe remains eternally in the inflation era even though we perturb it around this attractor solution.

### 3. Non minimal coupling

In what follows we will assume a positif non-minimal coupling constant \( \alpha_0 \) non equal to 0 and 1/6.

In this subsection and due to the complexity of our system, we shall restrict our analysis to the case of \( \lambda = 0 \), by choosing \( \Lambda_4 = \sqrt{-6\Lambda_5/k^2} \) for \( \Lambda_5 < 0 \) in Eq. (2.5). The system of the differentiable Eqs. (4.3a)-(4.3c) reduces by considering the constraint Eq. (4.2) to the following autonomous system

\[
\begin{align*}
\frac{dx_1}{dN} &= -x_1 g(x_1, x_2, y), \\
\frac{dx_2}{dN} &= \sqrt{6\alpha_0} y - x_2 g(x_1, x_2, y), \\
\frac{dy}{dN} &= -y(3 + \frac{3b}{x_1}(y + 2\sqrt{6\alpha_0}x_2)) - x_2(2\sqrt{6\alpha_0} + \frac{3b}{x_1}x_2) + \sqrt{6\alpha_0}y - \frac{3b}{x_1}x_2 - (2y + \sqrt{6\alpha_0}x_2) g(x_1, x_2, y),
\end{align*}
\]  

where

\[
g(x_1, x_2, y) = \frac{3y^2 (x_1(1 - 2\alpha_0) + \sqrt{6\alpha_0}bx_2) + 3x_2 \sqrt{6\alpha_0}b (x_1^2 - x_2^2) + 2\sqrt{6\alpha_0}bx_2 y(2x_1 + 3\sqrt{6\alpha_0}bx_2) + 3x_2 (4\alpha_0 x_1 x_2 + \sqrt{2\alpha_0}b)}{x_1(1 - x_1^2 + x_2^2(1 - 6\alpha_0))},
\]  

\[
\text{The fixed points of the system (4.17)-(4.19) are illustrated in table III}
\]

- The critical point \( E \) is formed of a continuous line of critical points, called a critical line or line of non-isolated equilibrium points \[29\]. This critical line exists for an infinite number of critical points for all values of \( x_1 \) that verify the condition of existence \( x_1 > 1 \) and \( b > 0 \). The dynamic of the universe for this critical line is dominated by the potential energy density, i.e. \( V \not= 0 \) and \( \phi = 0 \) such that \( \phi = \phi_E \). The Friedmann equation and the equation of motion of the scalar field of this critical line write respectively as

\[
H^2 = \frac{1}{4\epsilon\kappa_{eff,c}^2} \left( 1 \pm \sqrt{1 - \frac{8\epsilon\kappa_{eff,c}^2}{3} V(\phi_E)} \right),
\]  

\[
V'(\phi_E) = -6\alpha'(\phi_E)H^2,
\]

where \( V(\phi_E) < \frac{3}{8\epsilon\kappa_{eff,c}^2} \).

The slow-roll parameter \( \epsilon \) is equal to zero (\( \epsilon = 0 \)), which means that this critical line corresponds to inflation.

- The critical line \( F \) exists for \( x_1 < -1 \). We restrict our analysis to the critical line \( E \) since we assume an expanding universe (\( H > 0 \)). Indeed, the critical line \( F \) does not correspond to an expanding universe due to the condition of existence for \( x_1 \propto H^{-1} \) (see table III).

In order to discuss the stability analytically, we use the linear theory. The stability of these lines is shown in Fig. 3. From both figures 3a and 3b we conclude that the critical lines \( E \) and \( F \) are either stable or saddle. Consequently, the critical line \( E \) corresponds to a non-minimally coupled inflation attractor solution for a specific values of

\[5\] We ignore \( C_- \) since we are assuming only expanding universe.
Table III. Critical lines, Stability, and the existence of the system (4.17)-(4.19) for an exponential potential (4.7) and a non-minimal function (4.8) with $\lambda = 0$.

| Point $x_1$ | $x_2$ | $y$ | $z$ | $\epsilon$ | Existence | Stability | Description |
|------------|------|----|----|---------|---------|---------|-----------|
| $E$ | $-\sqrt{2a_0 x_1 + \sqrt{2a_0 x_1^2 + b^2(-1 + x_1^2)}}$ | 0 | $\sqrt{2x_1 \sqrt{2a_0 (x_1^2 - 1) + 2a_0 x_1^2} - 4a_0 x_1^2}$ | 0 | $x_1 > 1$ & $b > 0$ | Stable/ Saddle | Potential domination |
| $F$ | $-\sqrt{2a_0 x_1 - \sqrt{2a_0 x_1^2 + b^2(-1 + x_1^2)}}$ | 0 | $-2x_1 \sqrt{2a_0 (x_1^2 - 1) + 2a_0 x_1^2} - 4a_0 x_1^2$ | 0 | $x_1 < -1$ & $b < 0$ | Stable/ Saddle | Potential domination |

Our model parameters $a_0$ and $b$ in addition to the choice of the value of the dimensionless variable $x_1$.

However the stability can also be found numerically by perturbing the system around the critical line. We plot in Fig. 4 the projection plots on $x_1$, $x_2$, $y$ and $z$ separately for $a_0 = 0.2$ and $b = 1$.

From Fig. 4a, it seems that the trajectories are parallel to an horizontal axis, and that any perturbation of the system near $x_1$ makes it an arbitrary constant as $N \to \infty$.

We can also see from Fig. 4b and 4c that for each value of $x_1$, the corresponding trajectories of $x_2$ and $z$ also approach the value $\left(\sqrt{b^2 (x_1^2 - 1) + 2a_0 x_1^2} - \sqrt{2a_0 x_1^2} / b\right)$ and $\sqrt{2x_1 \sqrt{2a_0 (x_1^2 - 1) + 2a_0 x_1^2} - 4a_0 x_1^2} / b$ respectively as $N \to \infty$. Some numerical values of any perturbation near $x_1$, $x_2$ and $z$ are also shown in Fig. 4. For example for $x_1 = 2.20$ the corresponding critical point coordinates $x_2$ and $z$ are 1.01 and 1.67 respectively as $N \to \infty$.

From Fig. 4e, we notice that trajectories of the perturbed solutions approach $y = 0$ as $N \to \infty$.

From these behaviours it is evident that the system comes back to the critical point following the perturbation, which means that the critical line $E$ is an attractor line for $a_0 = 0.2$ and $b = 1$. These plots support strongly our analytical findings.

In order to obtain a complete information about the structure of the phase space of the dynamical system (4.3a)-(4.3c) it is necessary to investigate the dynamical behavior for $a_0 < 0$. To this aim, we extend the previous study by including negative values of $a_0$ in Eq. (4.8) to search for any possible attractor solutions.

To keep the definition of the dimensionless variables as in (4.1), one has to consider a non-minimal function as $\zeta(\phi) \equiv -\alpha(\phi)$, where $a_0 \equiv -\zeta_0$ and $\zeta_0$ is a positif constant. It deserves to be mentioned that the constraint equation Eq. (4.2) reads in this case

$$z^2 = -1 + x_1^2 + x_2^2 + 2\sqrt{6a_0 y x_2} - y^2.$$  \hfill (4.23)

The set of differential Eqs. (4.3a)-(4.3c) reduces to the following dynamical system

FIG. 3. Blue region corresponds to the stable region of the critical lines $E$ and $F$, while it is saddle otherwise.
(a) Projection of perturbations along $x_1$-axis.  
(b) Projection of perturbations along $x_2$-axis.  
(c) Projection of perturbations along $y$-axis.  
(d) Projection of perturbations along $z$-axis.  

**FIG. 4.** Projection of perturbations along $x_1$, $x_2$, $y$ axis for $\alpha_0 = 0.2$ and $b = 1$. 

\[
\frac{dx_1}{dN} = -x_1 \, g(x_1, x_2, y), 
\]
\[
\frac{dx_2}{dN} = \sqrt{6} \zeta_0 y - x_2 \, g(x_1, x_2, y), 
\]
\[
\frac{dy}{dN} = -y(3 + \sqrt{3}b \frac{x_1}{x_1} (y - 2\sqrt{6} \zeta_0 x_2)) + x_2(2\sqrt{6} \zeta_0 + \sqrt{3}b x_2) + \sqrt{3}b x_1 \, (\frac{\sqrt{3}b}{x_1} + (-2y + \sqrt{6} \zeta_0 x_2) \, g(x_1, x_2, y)), 
\]

where

\[
g(x_1, x_2, y) = \frac{3y^2 (x_1(1 + 2\zeta_0) - \sqrt{2} \zeta_0 b x_3) + 3x_2 \sqrt{2} \zeta_0 b (x_2^2 + x_1^2) + \sqrt{6} \zeta_0 x_2 y(-4x_1 + 6\sqrt{6} \zeta_0 b x_2) + 3x_2(4\zeta_0 x_1 x_2 - \sqrt{2} \zeta_0 b))}{x_1(1 - x_1^2 - x_2^2(1 + 6\zeta_0))}. 
\]

The system formed by the equations (4.24)–(4.26) has two critical lines. The coordinates of these critical lines with their qualitative behaviour are given in table IV.

- For both critical lines $\mathcal{G}$ and $\mathcal{H}$ the dynamic of the universe is dominated by the potential energy density as $\dot{\phi}$ vanishes while $V(\phi) \neq 0$, with the solution of the Hubble parameter writes as Eq. (4.21). The parameter $\epsilon$ evaluated at these critical lines is also equal to 0 which means that these lines correspond to inflation.

Examination of the stability conditions Fig. 5 indicates that the state can be stable (or saddle) during inflation. The critical line $\mathcal{G}$ is always saddle in the region of existence ($b < 0$), while $\mathcal{H}$ is an attractor solution for $x_1 > 1$ and $-\sqrt{\frac{2x_1^2}{x_1^2-1}} < b < 0$. To check the stability of the critical line $\mathcal{H}$ numerically, we perturb the solutions around the critical point. We again plot the projections plots on $x_1$, $x_2$, $y$ and $z$ separately for $\zeta_0 = 0.2$ and $b = -0.1$. Like previous case, from Figs. 6a-6d, it is clear that the


| Point | $x_1$ | $x_2$ | $y$ | $z$ | $\epsilon$ | Existence | Stability | Description |
|-------|-------|-------|-----|-----|-------------|-----------|-----------|-------------|
| $\mathcal{G}$ | $x_1$ | $\sqrt{-x_1^2 + b^2 + 2\zeta_0 x_1^2 - \sqrt{2\zeta_0 x_1}}$ | 0 | $\frac{4\zeta_0 x_1^2 - 2\sqrt{2\zeta_0}}{b^2}$ | $0$ | $x_1 > 1 \& -\sqrt{\frac{2x_1^2}{\epsilon^2 - 1}} < b < 0$ | Stable | Potential d. |
| $\mathcal{H}$ | $x_1$ | $\sqrt{-x_1^2 + b^2 + 2\zeta_0 x_1^2 - \sqrt{2\zeta_0 x_1}}$ | 0 | $\frac{4\zeta_0 x_1^2 + 2\sqrt{2\zeta_0 x_1} + 1}{b^2}$ | $0$ | $b < 0$ | Saddle | Potential d. |

TABLE IV. Critical lines, Stability, and the existence of the system Eqs. (4.24)-(4.26).

(a) Line $\mathcal{G}$.

(b) Line $\mathcal{H}$.

FIG. 5. Blue (Green) region corresponds to the stable (saddle) region of the critical lines $\mathcal{G}$ and $\mathcal{H}$.

critical line $\mathcal{H}$ is an attractor for $\zeta_0 = 0.2$ and $b = -0.1$.

(a) Projection along $x_1$-axis.

(b) Projection along $x_2$-axis.

(c) Projection along $y$-axis.

(d) Projection along $z$-axis.

FIG. 6. Projection of perturbations for $\zeta_0 = 0.2$ and $b = -0.1$. 
V. SUMMARY AND CONCLUSIONS

The present study is devoted to the dynamical system analysis, in order to investigate the impact of non-minimal coupling of the scalar field to the Ricci curvature within the AdS/CFT correspondence.

Exact analytic solutions cannot be obtained for the modified Friedmann equation due to the complicated form of the evolution equation. The application of the basic tools of the dynamical systems theory helps us to deeply understand the dynamics of this cosmological model and to determine analytically the global behaviour of the system. The stability of the critical points of the dynamical system is studied by means of linear theory, centre manifold theory in the case of a non hyperbolic critical points and numerically to support our results.

We have started by considering a minimal case where we have shown that the system admits two unstable de Sitter state critical points with no dynamical motion of the scalar field. We have also considered the special case of a conformally coupled scalar field for which we have obtained a future attractor point; this solution corresponds to an inflation era with no exit. For both cases, minimal and conformal coupling, the dynamic of the universe is governed by the effective cosmological constant.

For a positive non-minimal coupling constant, we have found one critical line that corresponds to a future attractor de Sitter inflationary era for specific values of our model parameters $\alpha_0$ and $b$. For a negative non-minimal coupling constant, we have found two critical lines. One of them is saddle while the second one is always a future attractor solution describing a de Sitter inflation scenario.

Finally, one of the interesting results of including non-minimal coupling of the scalar field to the intrinsic curvature on the brane is that we obtain a future attractor solution which corresponds to a scenario where the content of the universe is completely dominated by the exponential potential and a de Sitter inflationary era. Even though the success of this study of non-minimal gravity within the AdS/CFT correspondence, the non compactness of our dynamical variables makes the analysis incomplete due to lack of the dynamical analysis at infinity of the phase space. Consequently, there could be missed critical points. This issue will be the subject of the next forthcoming paper.

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Appendix A: Centre Manifold Theory

In Sec. [IV] we have mentioned that if the eigenvalues of the Jacobian matrix [[4.6]] has one eigenvalue with zero real part while the other one is negative, the critical point is called non-hyperbolic and the linear approach fails to determine the stability properties. Different methods can be employed to study the stability properties in this situation such as the Lyapunov stability [40,52], centre manifold theory (CMT) [46,49] and Kosambi-Cartan-Chern theory [53]. This Appendix is devoted to show how we get the stable conditions of the non-hyperbolic critical points $A_+$ and $B_+$ using the CMT.

In what follows, we present the detailed calculus to find the stable conditions of the critical point $A_+$. To this purpose and in order to simplify the dynamical system Eqs. (4.10a)-(4.10b), we define a new variable $\tau$ as $d/d\tau = x_1(1-x_1^2) d/dN$.

We recall that for any dynamical system $\dot{x} = f(x)$, the new dynamical system $\dot{\chi} = \chi(x)f(x)$, where $\chi(x)$ is a positive function, has the same critical points with the same stability properties.

For the critical point $A_+$, the function

\[
\begin{align*}
\chi(x) &= x_1(1-x_1^2) & \text{is positive for } A < 0 \\
\chi(x) &= x_1(x_1^2 - 1) & \text{is positive for } A > 0
\end{align*}
\]

Our dynamical system (4.10a)-(4.10b) becomes for $A < 0$

\[
\begin{align*}
\frac{dx_1}{d\tau} &= -3x_1^2 y^2, \\
\frac{dy}{d\tau} &= -3y (1-x_1^2 + 2y^2) + \sqrt{3}b(-1-y^2 + x_1^2(1-Ax_1^2))(1-x_1^2).
\end{align*}
\]

The first step is to consider a specific transformation: $X = x_1 - k$ and $Y = y$ in order to move the critical point $A_+(k,0)$ to the origin of the phase space $(0,0)$, where $k = \sqrt{(1-\sqrt{1-4A})/2A}$. We obtain the new dynamical system

\[
\begin{align*}
\frac{dX}{d\tau} &= -3Y^2 (k + X)^2, \\
\frac{dY}{d\tau} &= 3Y(k + X) (-1 + (k + X)^2 - 2Y^2) + \sqrt{3}b(k + X - 1)(k + X + 1) \\
& \quad \times ((k + X)^2 (A(k + X)^2 - 1) + Y^2 + 1).
\end{align*}
\]

Our dynamical system has the required form, i.e. the fixed point sits at the origin $(0,0)$ and the system does not contain any linear term of $X$ in the first equation. We rewrite the above system as

\[
\begin{align*}
\frac{d}{d\tau} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} F(X,Y) \\ G(X,Y) \end{pmatrix},
\end{align*}
\]
and using the Leibnitz rule
\[
\frac{dY}{d\tau} = \frac{d}{d\tau} \left( Y \right),
\]
In this coordinate, the dynamic of the CM, for \( X \) sufficiently small, can be written as
\[
\frac{dX}{d\tau} = F(X, H(X)),
\]
Assuming that \( H(X) \) is of the form
\[
H(X) = a_1X^2 + b_1X^3 + O(X^4),
\]
and using the Leibnitz rule \( dY/d\tau = (dH/dX)(dX/d\tau) \), one obtains by combining the second row of Eq. \( (A3) \)
where \( \mu_1 \) is the eigenvalue equal to zero, \( \mu_2 \) is a non-zero eigenvalue and, from Eqs. \( (A2a)-(A2b) \), the two functions \( F \) and \( G \) are

\[
F(X, Y) = -3Y^2(k + X)^2,
\]
\[
G(X, Y) = 3Y(k + X)(-2Y^2 + (k + X)^2 - 1) + Y^2(k + X - 1)(k + X + 1) + 3\sqrt{3}hX^2k^2 (A(5X^2 - 2) - 2)
\]
\[
+ \sqrt{3}hX^2(15Ak^4 + 20Ak^3X + 6AkX^3 - 4(A + 1)kX + AX^4 - (A + 1)X^2 + 2).
\]
and satisfy
\[
F(0, 0) = 0, \quad \nabla F(0, 0) = 0,
\]
\[
G(0, 0) = 0, \quad \nabla G(0, 0) = 0.
\]
The centre manifold (CM) suggests that its geometrical space is tangent at \((0,0)\) to the eigenspace of the non-zero eigenvalue \( \mu_2 \). We may assume from the definition of the CM that \( Y = H(X) \) with the following conditions:
\[
H(0) = 0, \quad \nabla H(0) = 0.
\]
Comparing coefficients of equal order in \( X \) of \( (A10) \), we find the coefficients \( a_1 \) and \( b_1 \)
\[
a_1 = \frac{b(-15Ak^4 + 6(A + 1)k^2 - 2)}{\sqrt{3}h(k^2 - 1)},
\]
\[
b_1 = \frac{b(25Ak^6 - (9A + 14)k^4 + 2(A + 4)k^2 - 2)}{\sqrt{3}h(k^2 - 1)^2}.
\]
Since the system \( (A2a)-(A2b) \) has a CM, the evolution of this system is given by
\[
\frac{dX}{d\tau} = \frac{b^2(15Ak^4 - 6(A + 1)k^2 + 2)^2}{(k^2 - 1)^2}X^4 + O(X^5)
\]
We notice that the coefficient of the fourth order of \( X \) is negative for \( A < 0 \) and consequently, around the fixed point \( A_+ \) the system is saddle in the centre manifold. We repeat the calculation in the case of \( A > 0 \) for the fixed point \( A_+ \) and for the fixed point \( A_- \) (since its existence is for \( A > 0 \)) we conclude that these points are unstable.
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