Bulk-edge correspondence and topological phases in periodically driven spin-orbit coupled materials in the low-frequency limit

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Abstract

We study the topological phase transitions induced in spin-orbit coupled materials with buckling like silicene, germanene, stanene, etc, by circularly polarised light, beyond the high-frequency regime, and unearth many additional topological phases. We also study the robustness of these phases in the presence of uniform disorder. These phases are characterised by the spin-resolved topological invariants, $C_0^\uparrow, C_0^\downarrow, C_1^\uparrow$ and $C_1^\downarrow$, which specify the spin-resolved edge states traversing the gaps at zero quasi-energy and the Floquet zone boundaries respectively. We show that for each phase boundary, and independently for each spin sector, the gap closure in the Brillouin zone occurs at a high symmetry point.

1. Introduction

Dynamical control of topological phases is one of the most intensely researched topics in recent times [1–10]. Proposals have involved periodic driving in semiconductor systems [4], cold atom (or optical lattice) systems [5], graphene [1–3, 9] and systems with spin–orbit coupling like silicene [10], with a variety of analytical and numerical methods. Apart from band-structure control of a system by renormalization of its dynamical parameters via a periodic drive, novel non-trivial topological phases, which do not have any analogue in static systems, have been explored, theoretically [11–16] as well as in experimental photonic systems [17, 18].

Silicene [19] and other spin-orbit coupled [20] materials like germanane, stanene, etc are recently synthesized materials which have shot into prominence because their buckled nature allows them to be tuned by an electric field through a transition between a band insulator and a topological insulator [21–23]. This tunability, and in particular, the experimental realisation [24] of silicene-based transistors has led to extensive work [25–30] on the interplay of topology and transport in these materials.

More recently, the question of whether topological phases can be controlled in irradiated silicene and similar materials has been studied. Although these materials are time-reversal invariant, in the presence of circularly polarised light, the system breaks time-reversal symmetry and the Chern number classification is an integer and not $Z_2$ invariant. Ezawa [10] showed that at high frequencies and for small amplitudes of driving, new phases in silicene such as quantum Hall insulator, spin-polarized quantum Hall insulator and spin and spin-valley polarized metals can be realised. Further, it was shown [31] that many more topological phases could be realised by performing a systematic Brillouin–Wigner (BW) expansion of the Hamiltonian to second order in the inverse of the frequency, not only in silicene but in other spin-orbit coupled materials. But, as was discussed also in earlier references [31, 32], the BW expansion breaks down when the frequency $\omega$ becomes smaller than the bandwidth of the effective Hamiltonian. The real constraint on the applicability of the BW expansion is a combined bound on both $\omega$ and the amplitude of driving and in fact, the validity of BW increases, even for lower frequencies when the amplitude increases. However, the physical reason for the breakdown of the earlier studies at low frequencies is because, at frequencies comparable to the bandwidth, it is no longer possible to neglect the topology of the quasienergy space, which forms a periodic structure with the single valuedness of the eigenfunction requiring the quasienergies to be within a 'Floquet zone'. Low-frequency driving can lead to...
crossings between the bottom of one Floquet band and the top of the next Floquet band. These crossings are neglected in the BW expansion and hence, the study of the driving at low frequencies requires a new formalism which goes beyond the effective static approximation of a dynamical Hamiltonian.

To characterize the topological nature of these systems that would satisfy the edge-bulk correspondence, one needs to have access to the full time-dependent bulk evolution operator \( U(t) \), evaluated for all intermediate times within the driving period [11]. The invariants thus computed predict the complete Floquet edge-state spectrum. Similar \( Z_2 \) valued indices for periodically driven time-reversal invariant two-dimensional indices have also been found [14]. For a two-band model with the Fermi energy at zero quasi-energy, it was shown that the gaps at zero quasi-energy and at the zone boundary \( \omega / 2 \) gave rise to winding numbers \( C_0 \) and \( C_n \) whose difference gave the Chern number of the band. This formalism has been used to demonstrate the bulk-boundary correspondence in graphene and other systems [9, 33–39].

However, most of the studies have focused on graphene both in the high and low-frequency regimes and silicene-like materials have not been fully explored when they are exposed to low-frequency radiation. In this paper, we study the photoinduced topological phase diagram in these materials by computing the topological invariant for both up and down spin. We also look for the gap closing points in the Brillouin zone and find that it always occurs at the high symmetry points. Further, we compute \( C_0 \) and \( C_n \) by counting the number of edge states at the right and left edges of the sample and verify the bulk-boundary correspondence as illustrated earlier in graphene [9, 33]. Although the question of whether these topological phases obtained in the low-frequency regime are stable or not, requires detailed analysis to answer definitively, here we just address the behaviour of a few phases against uniform disorder by computing the real space Chern number using the coupling matrix approach prescribed in [40]. A full analysis is left for future studies. We further note that our study can also be extended to two-dimensional optical lattices where one can artificially synthesise an effective spin–orbit coupling by a combination of microwave driving and lattice shaking [41], and also to ultracold atoms [42], in both bosonic [43, 44] and fermionic systems [45, 46] where the Raman coupling has been shown to give rise to effective spin–orbit interaction.

### 2. Computation of the dynamical bandstructure

We start with two dimensional Dirac systems which are buckled due to the large ionic radius of the silicon atoms and consequently have a non-coplanar structure, unlike graphene. These materials can be described by a four-band tight-binding model in a hexagonal lattice given by

\[
H = -t \sum_{\langle i,j \rangle,\sigma} c_{i\sigma}^\dagger c_{j\sigma} + \frac{i\lambda}{\sqrt{\Delta}} \sum_{\langle i,j \rangle,\sigma} \sigma n_{i,j} c_{i\sigma}^\dagger c_{j\sigma} + \frac{\sigma E_T}{2} \sum_{i\sigma} \zeta_i c_{i\sigma}^\dagger c_{i\sigma}. \tag{1}
\]

Here, the first term is the kinetic term where \( t \) is the hopping parameter. The second term represents the spin-orbit coupling term where the value of \( \lambda \) depends on the material and \( n_{i,j} = \pm 1 \) depending on whether the next-nearest neighbour hopping is clock-wise or anti-clock-wise. The last term represents the staggering sub-lattice potential due to the buckling. When a beam of circularly polarised light is incident on the sheet, the corresponding electromagnetic potential \( \mathbf{A} = (A_0 \cos(\omega \tau), A_0 \sin(\omega \tau), 0) \) is introduced into the Hamiltonian using Peierls substitution. \( \omega \) is the frequency of light and \( A_0 \) is its amplitude. In the Fourier transformed space, this is written as

\[
H(\tau) = \begin{pmatrix}
|E_z - \delta_x| & \delta_t & 0 & 0 \\
-\delta^*_t & |E_z + \delta_x| & 0 & 0 \\
0 & 0 & |E_z - \delta_x| & \delta_t \\
0 & 0 & \delta^*_t & -|E_z + \delta_x|
\end{pmatrix}, \tag{2}
\]

where

\[
\delta_x(\tau) = \frac{2\lambda}{\sqrt{3}} \left[ \sqrt{3} a_0 \sin \tilde{k}_x - \sin \left( \frac{\sqrt{3} a_0}{2} \tilde{k}_x + \frac{3a_0}{2} \tilde{k}_y \right) - \sin \left( \frac{3a_0}{2} \tilde{k}_x - \frac{3a_0}{2} \tilde{k}_y \right) \right], \tag{3}
\]

with \( \tilde{k}_x = k_x + A \cos(\omega \tau) \) and \( \tilde{k}_y = k_y + A \sin(\omega \tau) \) and

\[
\delta_t(\tau) = t \left[ \exp(-i\alpha \sin(\omega \tau)) + T_e \exp \left( \frac{i\alpha (\sqrt{3} \cos(\omega \tau) + \sin(\omega \tau))}{2} \right) \right. \\
+ T_e \exp \left( \frac{i\alpha (-\sqrt{3} \cos(\omega \tau) + \sin(\omega \tau))}{2} \right) \left. \right] \tag{4}
\]
with $T_0 = \exp(ia_0(\pm \sqrt{3}k_x + 3k_y/2))$. Here, we have defined $a = Aa_0$, where $a_0$ is the lattice constant. We assume the radiation is weak enough such that other effects of the radiation on the underlying tight-binding model can be neglected. Further, we neglect any effect of the magnetic field of the radiation and we also assume that the radiation does not affect the spin structure of the system.

For the bulk system, the vector potential and hence the Hamiltonian is periodic in both the $x$ and $y$ directions. This implies that we can rewrite the Hamiltonian in terms of a Floquet eigenvalue problem with the Hamiltonian given by

$$H_F = -i\frac{\partial}{\partial \tau} + H(\tau),$$

(5)

the eigenfunctions are given by

$$\psi_{k,\phi}(x, y, \tau) = u_0(k_x, k_y, \tau)e^{i\tau k - i\phi \tau}$$

(6)

with $u_0(k_x, k_y, \tau) = u_0(k_x, k_y, \tau + 2\pi/\omega)$, and where $\phi_0$ are the quasienergies or the eigenvalues of $H_F$. The Hamiltonian can now be solved numerically as a function of the amplitude $A_0$, frequency $\omega$ and the sub-lattice potential $E_0$, both for the quasienergy eigenvalues and for the wave-functions and following earlier work [1, 8, 31, 32], we obtain the Floquet band structure. Here again, we assume that the effect of the driving is weak enough that does not alter the underlying lattice system otherwise.

At high frequencies, $\omega$ constitutes a large gap between unperturbed subspaces, and the extended Floquet Hilbert space splits into decoupled subspaces with different photon numbers. Since the perturbation scale of the Hamiltonian, which is the bandwidth $t$, is much smaller than $\omega$, one can use systematic perturbation theory to include virtual processes of emitting and absorbing photons, and up to a given order in perturbation theory, one can obtain an effectively static Hamiltonian as shown in [31]. The Chern numbers for the model can then be computed by integrating the Berry curvature over the whole Brillouin zone [47] using the eigenvectors of the effective Hamiltonian. However, it is expected that such an expansion in $1/\omega$ would fail to predict the correct Chern numbers once the frequency of the drive, $\omega$, becomes comparable to the bandwidth. This is the part of the phase diagram that we shall complete in this paper. In the paper, we also take $t$ to be our unit of energy, which for the case of silicene is $\approx 1.09\text{eV}$, as can be computed from the values given in [48]. The other parameter values, such as $\lambda$ (also from [48]) and $\lambda E_{\perp}$ (which is varied here) are also scaled with respect to $t$. This essentially implies that our phase diagram depends on the ratios of these energies. Other materials such as stanene, germanene, or even artificial systems such as cold atoms or photonic crystals will have different ratios and different phase diagrams, which can be computed in the same way.

2.1. The phase diagram of the Floquet Hamiltonian

As the frequency of the drive becomes comparable to the effective bandwidth of the system, it is essential to now consider the complete nature of the quasi-energy bands in the computation of the topological invariants of the system. As was mentioned in the introduction, the quasi-energy bands (of the two band system) are now identified with two topological invariants, $C_0$ and $C_2$, and the net Chern number of a band is given by

$$C = C_0 - C_2$$

independently for each of the spins.

The Fourier-transformed time-dependent Hamiltonian (equation 2) is block-diagonal in the spin space. For either the $\uparrow$ or the $\downarrow$ spin, it is a $2 \times 2$ Hermitian matrix which encodes the bulk properties of the system. The time evolution operator at stroboscopic times can then be written as

$$U(k, 2\pi/\omega) = Te^{-i\int_0^{2\pi/\omega} H(k, \tau) d\tau},$$

(7)

and the Floquet states $u_0(k_x, k_y, 0)$ are the eigenstates of this operator. The Chern number of each Floquet band is then defined by integrating the Berry curvature of the Floquet states over the whole Brillouin zone -

$$C = \frac{1}{2\pi} \int_{BZ} dk_x dk_y (\nabla \times A_{\text{lower}}(k)), $$

(8)

where $A_{\text{lower}}$ is the Berry connection in terms of Floquet states of the quasi-energy band with quasienergy lying between $(-\omega/2, 0)$. We numerically compute the Chern numbers of the lower band (of both $\uparrow$ and $\downarrow$ spins) following the work by Fukui et al [47].

When the parameter ranges are such that a high-frequency approximation would be valid, the Chern number computed using the effective static Hamiltonian would exactly match the one obtained by considering the Floquet states. In this sense, the following phase diagram that we present complements what has been obtained earlier in [31], and completely specifies the topological phases of the system for all parameter regimes.

The phase diagrams for both the up spin and the down spin bands are presented in figures 1 and 2. In figure 1, we show the Chern number of the lower quasienergy band as a function of the amplitude of the drive versus the frequency, whereas in figure 2, we show it as a function of the amplitude of the drive versus the sub-lattice potential. For lower frequencies, many different phases appear and appear to follow a fractal structure, as
was seen for graphene in [32]. But as such phases are not expected to be protected by a large enough band-gap, we have only shown phases which are ‘large enough’ (occupy enough area in the phase diagram) and we have ignored tinier phases. As \( \alpha \to 2 \) and \( \omega \to 6 \), these phases smoothly go over to the high-frequency phases in [31]. We have also chosen to name only those phases that are large enough to be possible stable phases in calligraphic letters as \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}, \mathcal{F} \) in figures 1 and 2. Note that there are two phases \( \mathcal{B} \) and \( \mathcal{B}' \) which have identical values of the Chern numbers for both the \( \uparrow \) spin band and the \( \downarrow \) spin band. Nevertheless, they are two distinct phases since they occur for different values of \( \omega \) and \( \alpha \) and are not continuously connected to each other and they could have different edge state structures. Note also the existence of a phase \( \mathcal{A} \) which has zero Chern numbers for both spin \( \uparrow \) and spin \( \downarrow \) electrons. We will see later in the next section, that this is a topological phase and has edge states despite having zero Chern numbers.

The lines that separate the phases are when the gap closes and the gap closing typically occurs at the high symmetry points of the Brillouin zone as shown in figure 3. For the lines \( P_5, P_4 \) and \( P_3 \), the gap closes at the \( \Gamma \) point whereas for the \( P_1 \) and \( P_3 \) lines, it closes at the \( K \) point and for the \( P_6 \) line, the closure happens at the half-way point between the \( \Gamma \) point and the \( K \) point. Note that we have concentrated on the spin \( \uparrow \) bands and hence have lines separating region \( \mathcal{C} \) from \( \mathcal{E} \), which have different Chern numbers for \( \uparrow \) spin, but no line separating...
regions \( C \) from \( B \), which have the same Chern number for \( \uparrow \) spin. A similar analysis can be done for the \( \downarrow \) spin case.

We note that the Chern number changes by \( \pm 2 \) at the \( P_5 \) crossing, which essentially implies a quadratic touching of the bands. This is similar to the transition explained in \([9]\) where the Hamiltonian for the first \( \Gamma \) point transition at the Floquet zone boundary was obtained perturbatively and was shown to lead to a Chern number change of \( \pm 2 \). This can only happen at the spherically symmetric \( \Gamma \) point. Along \( P_{123} \) and \( P_4 \), the change in the Chern number is \( \pm 1 \) and the band touching happens at the \( \Gamma \) or \( K \) points. Along \( P_6 \), however, the change in the Chern number is \( \pm 3 \). This can happen at 3 points in the Brillouin zone, symmetric around the \( \Gamma \) point as shown in figure 3. We have also checked that a change of the chirality of the circularly polarized light, besides changing signs of all the Chern numbers also breaks inversion symmetry with respect to the gap closing diagram in figure 3. The blue points are at \( \bar{K} \) instead of \( K \) points and the green points are placed so as to complete the smaller hexagon.

However, the computation of the Chern number does not specify the \( C_0 \) and \( C_\psi \) invariants individually. As the bulk-boundary correspondence in our system comes from these invariants, to discover these two indices, we need to consider the edge-state structure in a system with edges - e.g., a ribbon geometry. This is what we shall discuss in the following section.

### 2.2. Edge states in a ribbon geometry

In this section, we study the quasi-energy band-structure of the model in an infinite zigzag nanoribbon geometry, with a finite width. We identify the four integers \( C_{\psi}, C_{\psi}', C_0, C_\psi \) (defined later) that characterize Floquet topological insulators in our model, in each of the phases in figure 1 and 2, by choosing appropriate values of \( \omega, \alpha \) and \( lE_z \). A representative diagram for the phase \( \mathcal{A} \) has been shown in figure 4 and the remaining diagrams have been relegated to the appendix (figure A1). The spectrum has been shown slightly beyond the first Floquet-Brillouin zone, \(-\omega/2 < \epsilon_0 < \omega/2\), so that the edge states at the zone boundaries are clearly visible.

The first point that we note is the gaps and the edge states at the zone boundaries (at \( \epsilon_0 = \omega/2 \equiv -\omega/2 \)). In the high-frequency regime studied earlier, we had restricted ourselves to frequencies below the zone boundaries (i.e., at \( \epsilon = \pm \omega/2 \)), and hence the edge states at the zone boundary do not appear. However, in this work, our main focus is on the low-frequency regime, and one of our aims is to explicitly check that the Chern number of the band is given by the difference between the number of chiral edge states above and below the band. How do we count the number of chiral edge states? As shown in \([11]\), the number of edge modes are related to the winding number of the Floquet operator. Unlike the Chern number of a band, which depends only on the stroboscopic dynamics of the Floquet operator, the winding number has information about the circulation direction, which gets related to the direction of propagation of the edge states. In a Floquet system, the chirality at a given edge depends on details of the driving and can be either positive or negative, independent of the

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**Figure 3.** Gap closing points in the Brillouin zone along the \( P_i \) (\( i = 1 \ldots 6 \)) phase boundaries (drawn in figure 1) as described in the caption of figure 1.
The chirality of the driving force only provides the required time-reversal breaking. However, at low frequencies, there is no direct relation between the chirality of the drive and the chirality of the edge states, since the drive can lead to multiple gap closings and openings with multiple edge states. Hence, the edge state chirality needs to be explicitly computed for each phase.

Let us now focus on the Floquet band structure in the various different phases. For illustration, let us confine ourselves to the spin-up band. Let us also confine our attention to the left edge (L). The determination of the chirality of the edge state as shown on the graph is made by actually checking whether the right-moving state (positive slope) is at the left edge or at the right edge and similarly whether the left-moving slope (negative slope) is at the left or right edge. This can be done explicitly since we have numerically obtained all the wave-functions. We can now easily count the number of chiral edge states at the band-gap at zero, and at the band-gap at $\omega/2$, in the various plots in the panels in figure 3 and in the appendix. We choose a convention where a right-moving (positive slope in the energy versus momentum plot) at the left edge state is assigned a winding number or chirality $-1$ and a left moving (negative slope) state is assigned a chirality $+1$. We then compute $C_0^s$ by taking it to be $-1$ for each right-moving/ left-moving state and adding up the values. Similarly, in the band-gap at frequency $\omega/2$, we compute $C_s^p$ by taking $-1/1$ for each right-moving/ left-moving state and adding up the values. For instance, in figure 3, for the spin-up band, at zero frequency, there is a single edge state at the left edge which has a negative slope; thus $C_0^s = +1$. At the frequency $\omega/2$ also, there is a single edge state at the left edge with a negative slope, thus $C_1^s = +1$ as well. The Chern number of the $\uparrow$ band in phase $A$ was computed earlier to be $C_1^\uparrow = 1$ which precisely agrees with $C_0^s = C_1^s$, as expected from ([11]).

Using the same method, $C_0^p$ and $C_s^p$ can be computed for each of the phases in figure 1 and 2 and the results are tabulated in table 1. Note that, as expected, the Chern number of the band, $C^s = C_0^s - C_s^p$ in each case. Note also that the phases $A$, $B$, $C$, $E$, $F$ in the table are present in both figures 1 and 2, whereas $B'$, $D$ and $G$ occur only in figure 1 and $H$, $I$ and $J$ only in figure 2. Further, the intensity of a circularly polarized is $I = \frac{1}{2} \epsilon_0 c^2 \omega^2$, which can be written as $I \approx 10^{-4} \epsilon_0 c^2 (\hbar \omega / t) \ W \ cm^{-2}$, where $\epsilon = eA_0 / h$ is written in terms of the driving amplitude. The various phases, for a given driving frequency, of the table thus can be realized by varying the intensity of the drive.

3. Discussions and conclusions

In comparison with earlier studies of irradiated graphene, the main difference for spin-orbit coupled materials is the fact that the phase boundaries for the spin $\uparrow$ electrons and the spin $\downarrow$ electrons occur at different points in the parameter space. Besides, due to the buckling, an external electric field can be applied which can tune the masses...
at the $K$ and $K'$ points. This external tuning parameter helps in finding new phases as seen in figure 2, which do not exist in graphene.

We have also studied the robustness of each of the phases in the presence of (uniform) disorder. As these phases are topologically distinct and thus the edge-modes have a topological origin, it is expected that these phases and the edge-modes will be robust against weak uncorrelated disorder. Here, we check whether certain phases are more robust against disorder than others. The disorder in the system is modelled as an on-site chemical potential which is taken from a normal distribution of standard deviation $\sigma$, where $\sigma$ serves as the strength of the disorder in terms of the hopping parameter $t$. In figure 5, we have plotted the disorder averaged real-space ‘Chern numbers’ of the various phases in figure 1, computed using the coupling matrix approach following [40]. We note that a number of the topological phases are immune to uniform disorder for a reasonable range of the disorder strength, and starts degrading only for larger values, whereas a few topological phases immediately change their character even for a relatively small disorder. For a few of the phases, the robustness against disorder can be understood in terms of the respective values of the quasi-energy gap of the system, but in certain cases (such as contrasting phase $B$ and $D$, see appendix), the robustness against disorder may not be simply related to the quasi-energy gap of the system for each of the phases, which can be compared with the disorder strength $\sigma$ required to change the topological order. This is a surprising outcome and is expected to be related to the structure of the time-dependent Hamiltonian and is a direction for future study. We also note, in passing, that the phase $A$, characterized by zero value of the topological invariant appears to attain the Floquet topological Anderson insulator phase [49, 50] and exhibits two-lead quantized current at the infinite bias limit [51]. Further, the robustness of a certain phase also implies that any transport phenomena, such as a sum-ruled quantum Hall conductance [52–54], should also be protected and might act as signatures to identify the individual phases. This is of particular importance because the lack of knowledge of the occupation of the bands can be circumvented using signatures of the edge states.

The low-frequency analysis in this manuscript focuses on spin-orbit coupled materials which are silicene, germanene and stanene in condensed matter systems. Although the theoretical study of experimentally attainable parameter values requires detailed study as has been done in graphene [55] we provide the values which are used in this study e.g. the phase $A$ can be realised in silicene with frequency $\omega = 2.55$ $t$ which belongs to near-infrared (NIR) in the electromagnetic spectrum (with a hopping parameter, $t \sim 1$ eV); amplitude, $A = 1.5$ in units of the inverse of lattice constant ($a_0 = 3.84$ $\AA$); external electric field, $E_z = 0.173$ $V/A$ [56, 57]; spin-orbit coupling, $\lambda = 0.05$ $t = 3.5$ meV [58]. We note that it might seem experimentally challenging in condensed matter systems, the range of parameter values required to realise proposed Floquet topological phases are accessible experimentally in a photonic crystal structure [59, 60]. Recently, Quelle et al [61] provided a driving protocol to realize anomalous Floquet-Anderson insulating (AFAI) phase in optical lattices.

### Table 1. Spin-resolved topological quantum numbers and the edge states for phases in figures 1, 2.

| Phases | $(C^0, C^1)$ | $C_0^1$ | $C_p^1$ | $C_q^1$ | $C_2^1$ |
|--------|---------------|----------|----------|----------|----------|
| $A$    | (0,0)         | 1        | 1        | 1        | 1        |
| $B, B'$| (+2, +2)      | 0        | −2       | 0        | −2       |
| $C$    | (+2, +3)      | 0        | −2       | 1        | −2       |
| $D$    | (+1, +2)      | −1       | −2       | 0        | −2       |
| $E$    | (+3, +3)      | 1        | −2       | 1        | −2       |
| $F$    | (+1, +1)      | 1        | 0        | 1        | 0        |
| $G$    | (+1, +1)      | −1       | −2       | −1       | −2       |
| $H$    | (+1, +1)      | 0        | −1       | 0        | −1       |
| $I$    | (−2, −2)      | 0        | 2        | 0        | 2        |
| $J$    | (−1, −1)      | 0        | 1        | 0        | 1        |
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Appendix A. Edge states in the ribbon geometry for different phases

In this appendix, we compute the Floquet band structure in a zigzag nanoribbon in all the different phases which have been shown in figures 1 and 2 in the main text. In the main text, the band diagram for phase $\mathcal{A}$ was already shown; here we show the edge-state spectrum for all the remaining phases. The name of the phase, as well as the
values of $C^0_0$ and $C^0_s$ are given in the figure itself. As described in the main text, $C^0_0$ and $C^0_s$ are computed by taking it to be $-1/1+1$ depending on whether the $L$ state (or states) in the appropriate band-gap is right-moving or left-moving at the left edge of the sample and adding up the values. Note that it is not always to visually
determine whether or not the gap exists and in ambiguous cases, we have explicitly mentioned that it is gapped. Note also that in the diagrams of the phases $G$, $H$ and $T$, the edge states are isolated from the bulk states at zero energy even though the spectrum is not gapped (or has an extremely small gap). Thus the computation of the Chern numbers by counting edge states is more reliable than the bulk computation, which can numerically fail in the absence of a well-defined gap.

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**References**

[1] Oka T and Aoki H 2009 Phys. Rev. B 79 081406
[2] Kitagawa T, Berg E, Rudner M and Demler E 2010 Phys. Rev. B 82 235114
[3] Gu Z, Fertig H A, Arovas D P and Auerbach A 2011 Phys. Rev. Lett. 107 216601
[4] Lindner N H, Refael G and Galitski V 2011 Nature Phys. 7 490
[5] Jiang L, Kitagawa T, Alicea J, Akhmerov A R, Pekker D, Refael G, Cirac I J, Demler E, Lukin M D and Zoller P 2011 PhRvRef. 106 220402
[6] Calvo H L, Pastawski H M, Roche S and Fox Torres L E F 2011 Appl. Phys. Lett. 98 232103
[7] Calvo H L, Perez-Piskunow P M, Roche S and Fox Torres L E F 2012 Appl. Phys. Lett. 98 232103
[8] Calvo H L, Perez-Piskunow P M, Pastawski H M, Roche S and Fox Torress L E F 2013 Infl of Phys; Cond. Matt. 25 144202
[9] Dora B, Cayssol J, Simon F and Moessner R 2012 Phys. Rev. Lett. 108 056602
[10] Kundu A and Seradjeh B 2013 Phys. Rev. Lett. 111 136402
[11] Kundu A, Fertig H A and Seradjeh B 2013 Phys. Rev. Lett. 110 236803
[12] Ezawa M 2013 Phys. Rev. Lett. 110 026603
[13] Rudner M S, Lindner N H, Berg E and Levin M 2013 Phys. Rev. X 3 031005
[14] Delplace P, Gomez-Leon A and Platero G 2013 Phys Rev. B 88 245422
[15] Gallego-Marcos F, Platero G, Nietcner C, Schaller G and Brandes T 2014 Phys. Rev. A 90 033614
[16] Carpentier D, Delplace P, Fruchmart M and Gawedzki K 2015 Phys. Rev. Lett. 114 106806
[17] Senthil M A, Claassen M, Kemper A F, Moritz B, Oka T, Freericks J K and Devereaux T P 2015 Nat. Comm. 6 7047
[18] Fisht M V and Efetov K B 2014 Phys Rev. B 90 125416
[19] Macewsky J L, Zeuner J M, Nolte S and Sramek A 2017 Nat. Comm. 8 13756
[20] Mukherjee S, Spracklen A, Valliente M, Andersson E, Oldberg P, Goldman N and Thomson R R 2017 Nat. Comm. 8 13918
[21] Ezawa M 2013 Phys. Rev. Lett. 110 026603
[22] Konschuh S, Gmitra M and Fabian J 2010 Phys Rev. B 82 245412
[23] Kane C L and Mele E J 2005 Phys Rev. Lett. 95 146802
[24] Hasan M Z and Kane C L 2010 Rev. Mod. Phys. 82 3045
[25] Qi X L and Zhang S C 2011 Rev. Mod. Phys. 83 1057
[26] Yao L, Cinquanta E, Chiappe D, Graziantelli C, Fanciulli M, Dubey M, Molle A and Ankinwande D 2015 Nanotechnology. 108 227
[27] Linder J and Yokoyumata T 2014 Phys. Rev. B 89 020504
[28] Rachel S and Ezawa M 2014 Phys. Rev. B 89 195303
[29] Saxena R, Saha A and Rao S 2015 Phys. Rev. B 92 245412
[30] Paul G C, Sarkar S and Saha A 2016 Phys. Rev. B 94 155453
[31] Li K and Zhang Y Y 2016 Phys. Rev. B 94 165441
[32] Sarkar S, Saha A and Gangadharaiah S 2018 Superlattices and Microstructures 123 436–46
[33] Mohan P, Saxena R, Kundu A and Rao S 2016 Phys. Rev. B 94 235419
[34] Mikami T, Kitamura S, Yasuda K, Tsuji N, Oka T and Aoki H 2016 Phys. Rev. B 93 144307
[35] Xiang Wang Y and Li F 2016 Physica B 492 1–6
[36] Perez-Piskunow P M, Fox Torres L E F and Usaj G 2015 Phys. Rev. A 91 043625
[37] Perez-Piskunow P M, Usaj G, Balseiro C A and Fox Torres L E F 2014 Phys. Rev. B 89 121401
[38] Fox Torres L E F, Perez-Piskunow P M, Balseiro C A and Usaj G 2014 Phys. Rev. Lett. 113 266801
[39] Atteia J, Bardarson J H and Cayssol J 2017 Phys. Rev. B 96 245404
[40] Mukherjee B, Mohan P, Sen D and Sengupta K 2018 Phys. Rev. B 97 205415
[41] Rodriguez-Vega M and Seradjeh B 2018 Phys. Rev. Lett. 121 036402
[42] Yi-Fu Z, Yun-You Y, Yan J, Li S, Rui S, Dong-Ning S and Ding-Yu X 2013 Chin. Phys. B 22 11
[43] Grundt F, Li T, Bloch I and Demler E 2017 Phys. Rev. A 95 063617
[44] Dong L, Zhou L, Wu B, Ramachandran B and Pu H 2014 Phys. Rev. A 89 011602
[45] Lin Y, J, Jimenez-Garcia K and Spielman I B 2011 Nature (London) 471 83
[46] Lin Y J, Compton R L, Jimenez-Garcia K, Phillips W D, Porto J V and Spielman I B 2011 Nature Phys. 7 531
[47] Wang P, Yu Z-Q, Fu Z, Miao J, Huang L, Chai S, Zhai H and Zhang J 2012 Phys. Rev. Lett. 109 095301
[48] Cheuk I W, Sommer A T, Hadzibabic Z, Yefsah T, Bakr W S and Zwierlein M W 2012 Phys. Rev. Lett. 109 095302
[49] Fukui T, Hatsuigui Y and Suzuki H 2005 J. Phys. Soc. Jpn. 74 1674–7
[50] Liu CC, Jiang H and Yao Y G 2011 Phys. Rev. B 84 195430
[51] Kitamura S, Gmitra M and Fabian J 2018 Phys. Rev. X 8 021013
[52] Quelle A, Weitenberg C, Stengsli K and Moraes Smith C 2017 New J. Phys. 19 113010
[53] Kundu A, Rudner M, Berg E and Lindner N H 2020 Phys. Rev. B 101 044103
[54] Aaron Farrell and Perez-Berne E 2015 Phys. Rev. Lett. 115 106403
[55] Aaron Farrell and Perez-Berne E 2016 Phys. Rev. B 93 045121
[56] Yap H H, Zhou L, Wang J S and Gong J 2017 Phys. Rev. B 96 165443
[55] Sente J M A, Claassen M, Kemper A F, Moritz B, Oka T, Freericks J K and Devereaux T P 2015 Nat. Commun. 6 7047
[56] Kane C L and Mele E J 2005 Phys. Rev. Lett 95 226801
[57] Saxena R, Saha A and Rao Sumathi 2015 Phys. Rev. B 92 245412
[58] Hattori A, Tanaya S, Yada K, Araida M, Sato M, Hatsugai Y, Shirai S and Tanaka Y 2017 J. Phys.: Condens. Matter 29 115302
[59] Rechtsman M C, Zeuner J M, Plotnik Y, Lumer Y, Podolsky D, Dreisow F, Nolte S, Segev M and Szameit A 2013 Nature 496 196–200
[60] Titum P, Lindner N H, Rechtsman M C and Refael G 2015 Phys. Rev. Lett. 114 056801
[61] Quelle A, Weitenberg C, Sengstock K and Smith C M 2017 New J. Phys. 19 113010