IMPROVEMENTS OF SOME OPERATOR INEQUALITIES INVOLVING POSITIVE LINEAR MAPS VIA THE KANTOROVICH CONSTANT

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Abstract. We present some operator inequalities for positive linear maps that generalize and improve the derived results in some recent years. For instant, if $A$ and $B$ are positive operators and $m, m', M, M'$ are positive real numbers satisfying either one of the condition $0 < m \leq B \leq m' < M' \leq A \leq M$ or $0 < m \leq A \leq m' < M' < B \leq M$, then

$$\Phi^p(A \nabla_v B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1}v B^{-1})) \leq \left( \frac{K(h)}{4^{1/2}K^{r_1}(\sqrt{h'})} \right)^{p} \Phi^p(A^{\#}v B)$$

and

$$\Phi^p(A \nabla_v B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1}v B^{-1})) \leq \left( \frac{K(h)}{4^{1/2}K^{r_1}(\sqrt{h'})} \right)^{p} (\Phi(A)^{\#}v \Phi(B))^p,$$

where $\Phi$ is a positive unital linear map, $0 \leq \nu \leq 1$, $p \geq 2$, $r = \min\{\nu, 1 - \nu\}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$, $K(h) = \frac{(1+h)^2}{4h}$ and $r_1 = \min\{2r, 1-2r\}$. We also obtain a reverse of the Ando inequality for positive linear maps via the Kantorovich constant.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ whose identity is denoted by $I$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and in this case we write $A \geq 0$. We write $A > 0$ if $A$ is a positive invertible operator. The absolute value of $A$ is denoted by $|A|$, that is $|A| = (A^*A)^{1/2}$. For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$, we say $A \leq B$ if $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between

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the $C^*$-algebra $C(sp(A))$ of continuous functions on the spectrum $sp(A)$ of a self-adjoint operator $A$ and the $C^*$-algebra generated by $A$ and $I$. If $f, g \in C(sp(A))$, then $f(t) \geq g(t)$ \hspace{1em} (t \in sp(A)) \hspace{1em}$ implies that $f(A) \geq g(A)$. A linear map $\Phi$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. If $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible, then the $\nu-$weighted arithmetic mean and geometric mean of $A$ and $B$ denoted by $A \nabla_\nu B$ and $A_\# B$, respectively, which are defined by

$$A \nabla_\nu B = \nu A + (1 - \nu)B$$

$$A_\# B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right)^\nu A^{\frac{1}{2}},$$

respectively, where $0 \leq \nu \leq 1$. In case of $\nu = \frac{1}{2}$, we write $A \nabla B$ and the $A_\# B$ for the arithmetic mean and the geometric mean, respectively. The well-known $\nu-$weighted arithmetic-geometric (AM-GM) operator inequality says that if $A, B \in \mathcal{B}(\mathcal{H})$ are positive and $0 \leq \nu \leq 1$, then $A_\# B \leq A \nabla_\nu B$; see [7]. For $\nu = \frac{1}{2}$, we obtain the AM-GM operator inequality

$$A_\# B \leq \frac{A + B}{2}.$$  \hspace{1em} (1.1)

For further information about the AM-GM operator inequality and positive linear maps inequalities we refer the reader to [1, 2, 3, 8, 14] and references therein. Lin [11] presented a reverse of inequality (1.1) for a positive linear map $\Phi$ and positive operators $A, B \in \mathcal{B}(\mathcal{H})$ such that $m \leq A, B \leq M$ as follows:

$$\Phi \left( \frac{A + B}{2} \right) \leq K(h)\Phi(A_\# B),$$  \hspace{1em} (1.2)

where $K(h) = \frac{(1+h)^2}{4h}$ and $h = \frac{M}{m}$. The constant $K(t) = \frac{(t+1)^2}{4t} (t > 0)$ is called the Kantorovich constant which satisfies the following properties:

(i) $K(1, 2) = 1$;
(ii) $K(t, 2) = K(\frac{1}{t}, 2) \geq 1 \hspace{1em} (t > 0)$;
(iii) $K(t, 2)$ is monotone increasing on the interval $[1, \infty)$ and monotone decreasing on the interval $(0, 1]$.

The Lowner-Heinz theorem [9] says that if $A, B \in \mathcal{B}(\mathcal{H})$ are positive, then for $0 \leq p \leq 1$,

$$A \leq B \hspace{1em} \text{implies} \hspace{1em} A^p \leq B^p.$$  \hspace{1em} (1.3)

In general (1.3) is not true for $p > 1$. In [11], the author showed that inequality (1.2) can be squared that is,

$$\Phi^2 \left( \frac{A + B}{2} \right) \leq K^2(h)\Phi^2(A_\# B)$$  \hspace{1em} (1.4)
and
\[ \Phi^2 \left( \frac{A+B}{2} \right) \leq K^2(h)(\Phi(A)\Phi(B))^2. \] (1.5)

It follows (1.3), (1.4) and (1.5) that for \(0 < p \leq 2\) we have
\[ \Phi^p \left( \frac{A+B}{2} \right) \leq K^p(h)\Phi^p(A\Phi(B)) \] (1.6)

and
\[ \Phi^p \left( \frac{A+B}{2} \right) \leq K^p(h)(\Phi(A)\Phi(B))^p. \] (1.7)

It is natural to ask whether inequalities (1.6) and (1.7) are true for \(p > 2\). In [6], the authors gave a positive answer to this question and proved the following theorem:

**Theorem 1.1.** Let \(0 < m \leq A, B \leq M\). Then for every positive unital linear map \(\Phi\) and for every \(p \geq 2\)
\[ \Phi^p \left( \frac{A+B}{2} \right) \leq \left( \frac{(M+m)^2}{4^p Mm} \right)^p \Phi^p(A\Phi(B)) \] (1.8)

and
\[ \Phi^p \left( \frac{A+B}{2} \right) \leq \left( \frac{(M+m)^2}{4^p Mm} \right)^p (\Phi(A)\Phi(B))^p. \] (1.9)

The next result is a further generalization [2]:

**Theorem 1.2.** [2] Let \(0 < m \leq A, B \leq M\). Then for every positive unital linear map \(\Phi\), \(0 \leq \nu \leq 1\) and for every \(p > 0\)
\[ \Phi^p \left( A\nabla_\nu B + 2rMm \left( A^{-1}\nabla B^{-1} - A^{-1}\Phi(B)^{-1} \right) \right) \leq \alpha^p \Phi^p(A\Phi(B)) \]
\[ \Phi^p \left( A\nabla_\nu B + 2rMm \left( A^{-1}\nabla B^{-1} - A^{-1}\Phi(B)^{-1} \right) \right) \leq \alpha^p (\Phi(A)\Phi(B))^p, \]
where \(r = \min\{\nu, 1 - \nu\}\) and \(\alpha = \max\left\{ \frac{(M+m)^2}{4^p Mm}, \frac{(M+m)^2}{4^p Mm} \right\} \).

The authors of [17] proved the following theorem, which is another improvement of inequalities (1.8) and (1.9).

**Theorem 1.3.** [17] Let \(0 < m \leq A \leq m' < M' \leq B \leq M\). Then for every positive unital linear map \(\Phi\), \(0 \leq \nu \leq 1\) and for every \(p \geq 2\)
\[ \Phi^p(A\nabla_\nu B) \leq \left( \frac{K(h)}{4^p K(\nu h')} \right)^p \Phi^p(A\Phi(B)) \]
\[ \Phi^p(A\nabla_\nu B) \leq \left( \frac{K(h)}{4^p K(\nu h')} \right)^p (\Phi(A)\Phi(B))^p, \]
where \(r = \min\{\nu, 1 - \nu\}\), \(h = \frac{M}{m}\), \(h' = \frac{M'}{m'}\) and \(K(h) = \frac{(1+h)^2}{4h} \).
In this article, we give some operator inequalities involving positive linear maps that generalize inequalities (1.8), (1.9) and refine some results in [2, 17]. Moreover, we obtain a reverse of Ando’s inequality.

2. SOME OPERATOR INEQUALITIES INVOLVING POSITIVE LINEAR MAPS

We begin this section with several essential lemmas.

Lemma 2.1. [4] (Choi’s inequality) Let \( A \in \mathbb{B}(\mathcal{H}) \) be positive and \( \Phi \) be a positive unital linear map. Then

\[
\Phi(A^{-1}) \leq \Phi(A^{-1}).
\] (2.1)

The next lemma, part (i) is proved for matrices but a careful investigation shows that it is true for operators on an arbitrary Hilbert space; see [13, page 79].

Lemma 2.2. [5, 9, 2] Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be positive and \( \alpha > 0 \). Then

(i) \( ||AB|| \leq \frac{1}{4}||A + B||^2 \).

(ii) \( ||A^\alpha + B^\alpha|| \leq ||(A + B)^\alpha|| \).

(iii) \( A \leq \alpha B \) if and only if \( ||A^{1/2}B^{-1/2}|| \leq \alpha^{1/2} \).

To obtain our results, we need to prove the following lemma. Its proof is similar to that of [16, Theorem 3.1].

Lemma 2.3. Suppose that \( A, B \in \mathbb{B}(\mathcal{H}) \) are positive and \( m, m', M, M' \) are positive real numbers satisfying either one of the following conditions:

1. \( 0 < m \leq B \leq m' < M' \leq A \leq M \);
2. \( 0 < m \leq A \leq m' < M' \leq B \leq M \).

Then for every \( 0 \leq \nu \leq 1 \),

\[
2r \left( A^{-1} \nabla B^{-1} - A^{-1} \nabla B^{-1} \right) + K_r \left( \sqrt{h'} \right) \left( A^{-1} \nabla \nu B^{-1} \right) \leq \left( A^{-1} \nabla \nu B^{-1} \right),
\] (2.2)

where \( r = \min\{\nu, 1 - \nu\} \), \( r_1 = \min\{2r, 1 - 2r\} \), \( K(h) = \frac{(1 + h)^2}{4h} \), \( h = \frac{M}{m} \) and \( h' = \frac{M'}{m'} \).

Proof. It follows from [16, Lemma 2.3] that

\[
2r \left( \frac{1 + x}{2} - \sqrt{x} \right) + K_r(\sqrt{x})x^\nu \leq (1 - \nu) + \nu x
\]

for any \( x > 0 \). The first condition, that is, \( 0 < m \leq B \leq m' < M' \leq A \leq M \) ensures that \( 1 < h' I = \frac{M'}{m} I \leq A^{1/2} B^{-1} A^{1/2} \leq \frac{M}{m} I = h I \). By setting \( X = A^{1/2} B^{-1} A^{1/2} \), we see
The last above inequality follows by the increasing property of the function $K(t)$ on the interval $(1, +\infty)$; see [7]. Finally, multiplying the both sides of inequality (2.3) by $A^{-\frac{1}{2}}$, we obtain the desired result. The inequality can be proved under the second condition (2) in a similar way. □

Our first main result is the following:

**Theorem 2.4.** Suppose that $A, B \in \mathbb{B}(\mathcal{H})$ are positive and $m, m', M, M'$ are positive real numbers satisfying either one of the following conditions:

1. $0 < m \leq B \leq m' \leq A \leq M$;
2. $0 < m \leq A \leq m' \leq M' \leq B \leq M$.

Then for every positive unital linear map $\Phi$ and every $0 \leq \nu \leq 1$

$$
\Phi^2(A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \\
\leq \left( \frac{K(h)}{K_1(\sqrt{h'})} \right)^2 \Phi^2(A_{\sharp \nu}^\sharp B) 
$$

and

$$
\Phi^2(A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \\
\leq \left( \frac{K(h)}{K_1(\sqrt{h'})} \right)^2 (\Phi(A)_{\sharp \nu}^\sharp \Phi(B))^2, 
$$

where $r = \min\{\nu, 1 - \nu\}$, $K(h) = \frac{(1+h)^2}{4h}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r_1 = \min\{2r, 1 - 2r\}$.

**Proof.** We shall prove inequality (2.4), and leave inequality (2.5) to the reader because the proof is similar. By Lemma 2.3, inequality (2.4) is equivalent to

$$
\left\| \Phi \left( A\nabla_{\nu}B + 2rMm \left( A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1} \right) \right) MmK_1 \left( \sqrt{h'} \right) \Phi^{-1}(A_{\sharp \nu}^\sharp B) \right\| \\
\leq \frac{(M + m)^2}{4}.
$$
Using Lemma 2.2, inequalities (2.1), (2.2) and the linear property of $\Phi$, we obtain

$$
\left\| \Phi \left( A\nabla_v B + 2r M M \left( A^{-1}\nabla B^{-1} - A^{-1} B^{-1} \right) \right)MM K^{r_1} \left( \sqrt{h'} \right) \Phi^{-1} (A_\nu^\sharp B) \right\|
$$

$$
\leq \frac{1}{4} \left\| \Phi \left( A\nabla_v B + 2r M M \left( A^{-1}\nabla B^{-1} - A^{-1} B^{-1} \right) \right) \right\|
$$

$$
+ MM K^{r_1} \left( \sqrt{h'} \right) \Phi^{-1} (A_\nu^\sharp B^{-1}) \right\|^2
$$

$$
= \frac{1}{4} \left\| \Phi (A\nabla_v B) + M M \left( \Phi \left( 2r \left( A^{-1}\nabla B^{-1} - A^{-1} B^{-1} \right) \right) \right) \right\|
$$

$$
+ K^{r_1} \left( \sqrt{h'} \right) \left( A_\nu^\sharp B^{-1} \right) \right\|^2
$$

$$
\leq \frac{1}{4} \left\| \Phi (A\nabla_v B) + M M \Phi \left( A^{-1}\nabla_v B^{-1} \right) \right\|^2
$$

$$
\leq \frac{(M + m)^2}{4}.
$$

The last above inequality holds since by our assumptions,

$$
A + M M A^{-1} \leq M M \quad \text{and} \quad B + M M B^{-1} \leq M M.
$$

By multiplying the inequalities above by $(1 - \nu)$ and $\nu$, respectively, and then summing up the derived inequalities, we get

$$
A\nabla_v B + M M \left( A^{-1}\nabla_v B^{-1} \right) \leq M + m.
$$

Since $\Phi$ is a positive linear map, we obtain

$$
\Phi \left( A\nabla_v B \right) + M M \Phi \left( A^{-1}\nabla_v B^{-1} \right) \leq M + m.
$$

So, inequality (2.4) holds.

\[ \square \]

**Corollary 2.5.** Suppose that $A, B \in \mathbb{B}(\mathcal{H})$ are positive and $m, m', M, M'$ are positive real numbers satisfying either one of the following conditions:

1. $0 < m \leq B \leq m' < M' \leq A \leq M$;

2. $0 < m \leq A \leq m' < M' \leq B \leq M$.

Then for every positive unital linear map $\Phi$, $0 \leq \nu \leq 1$ and for every $0 < p \leq 2$

$$
\Phi^p \left( A\nabla_v B + 2r M M \left( A^{-1}\nabla B^{-1} - A^{-1} B^{-1} \right) \right)
$$

$$
\leq \left( \frac{K(h)}{K^{r_1} \left( \sqrt{h'} \right)} \right)^p \Phi^p (A_\nu^\sharp B)
$$
and

\[
\Phi^p(A \nabla v B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1}B^{-1})) \\
\leq \left( \frac{K(h)}{4^{\frac{p}{2} - 1} K r_1 \left( \sqrt{h'} \right)} \right)^p (\Phi(A)^{\sharp v} \Phi(B))^p,
\]

where \( r = \min\{\nu, 1 - \nu\}, \quad K(h) = \frac{(1 + h)^2}{4h}, \quad h = \frac{M}{m}, \quad h' = \frac{M'}{m} \) and \( r_1 = \min\{2r, 1 - 2r\} \).

**Proof.** If \( 0 < p \leq 2 \), then \( 0 < \frac{p}{2} \leq 1 \). Using inequalities (1.3), (2.4) and (2.5), we get the desired results. \( \Box \)

**Theorem 2.6.** Suppose that \( A, B \in \mathbb{B}(\mathcal{K}) \) are positive and \( m, m', M, M' \) are positive real numbers satisfying either one of the following conditions:

1. \( 0 < m \leq B \leq m' < M' \leq A \leq M \);
2. \( 0 < m \leq A \leq m' < M' \leq B \leq M \).

Then for every positive unital linear map \( \Phi \), \( 0 \leq \nu \leq 1 \) and for every \( p \geq 2 \), we have

\[
\Phi^p(A \nabla v B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1}B^{-1})) \\
\leq \left( \frac{K(h)}{4^{\frac{p}{2} - 1} K r_1 \left( \sqrt{h'} \right)} \right)^p (\Phi(A)^{\sharp v} \Phi(B))^p \quad (2.6)
\]

and

\[
\Phi^p(A \nabla v B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1}B^{-1})) \\
\leq \left( \frac{K(h')}{4^{\frac{p}{2} - 1} K r_1 \left( \sqrt{h'} \right)} \right)^p (\Phi(A)^{\sharp v} \Phi(B))^p, \quad (2.7)
\]

where \( r = \min\{\nu, 1 - \nu\}, \quad K(h) = \frac{(1 + h)^2}{4h}, \quad K(h') = \frac{(1 + h')^2}{4h'}, \quad h = \frac{M}{m}, \quad h' = \frac{M'}{m} \) and \( r_1 = \min\{2r, 1 - 2r\} \).

**Proof.** Since the proof of inequality (2.7) is similar to the proof of inequality (2.6), we only prove inequality (2.6). By Lemma 2.2, inequality (2.6) is equivalent to

\[
\left\| \Phi^\frac{p}{2} \left( A \nabla v B + 2rMm \left( A^{-1} \nabla B^{-1} - A^{-1}B^{-1} \right) \right) \right\|^{\frac{p}{2}} \leq \left( \frac{K(h)}{4^{\frac{p}{2} - 1} K r_1 \left( \sqrt{h'} \right)} \right)^{\frac{p}{2}}.
\]
Using Lemma 2.2, inequalities (2.1), (2.2), and applying the same reasoning as in the last inequality of Theorem 2.4, we have

\[
M^{\frac{m}{2m}} \| \Phi^\frac{m}{2m} (A\nabla_\nu B + 2rMm (A^{-1}\nabla B^{-1} - A^{-1,2}B^{-1})) K^{-\frac{m}{2m}} (\sqrt{h}) \Phi^{-\frac{m}{2m}} (A^\sharp_\nu B) \|
\]

\[
= \| \Phi^\frac{m}{2m} (A\nabla_\nu B + 2rMm (A^{-1}\nabla B^{-1} - A^{-1,2}B^{-1})) M^{\frac{m}{2m}} K^{\frac{m}{2m}} (\sqrt{h}) \Phi^{-\frac{m}{2m}} (A^\sharp_\nu B) \|
\]

\[
\leq \frac{1}{4} \| \Phi^\frac{m}{2m} (A\nabla_\nu B + 2rMm (A^{-1}\nabla B^{-1} - A^{-1,2}B^{-1}))
\]

\[
+ M^{\frac{m}{2m}} K^{1} (\sqrt{h}) \Phi^{-1} (A^\sharp_\nu B) \|^{2}
\]

\[
= \frac{1}{4} \| \Phi^\frac{m}{2m} (A\nabla_\nu B + 2rMm (A^{-1}\nabla B^{-1} - A^{-1,2}B^{-1})) + MmK^{1} (\sqrt{h}) \Phi^{-1} (A^\sharp_\nu B) \|^{p}
\]

\[
\leq \frac{1}{4} \| \Phi^\frac{m}{2m} (A\nabla_\nu B + 2rMm (A^{-1}\nabla B^{-1} - A^{-1,2}B^{-1})) + MmK^{1} (\sqrt{h}) \Phi (A^{-1,2}B^{-1}) \|^{p}
\]

\[
= \frac{1}{4} \| \Phi (A\nabla_\nu B) + Mm (2r (A^{-1}\nabla B^{-1} - A^{-1,2}B^{-1}) + K^{1} (\sqrt{h}) (A^{-1,2}B^{-1})) \|^{p}
\]

\[
\leq \frac{1}{4} \| \Phi (A\nabla_\nu B) + Mm (A^{-1}\nabla B^{-1}) \|^{p}
\]

\[
\leq \frac{1}{4} (M + m)^{p}.
\]

Thus we get the desired result. \(\square\)

**Remark 2.7.** For \(p \geq 1\), we have

\[
\Phi^p (A\nabla_\nu B) \leq \Phi^p (A\nabla_\nu B) + (2rMm)^p \Phi^p \left(A^{-1}\nabla B^{-1} - A^{-1,2}B^{-1}\right).
\]

On the other hand, Lemma 2.2 yields that

\[
\| \Phi^p (A\nabla_\nu B) \| \leq \| \Phi^p (A\nabla_\nu B) + (2rMm)^p \Phi^p \left(A^{-1}\nabla B^{-1} - A^{-1,2}B^{-1}\right) \|
\]

\[
\leq \| \Phi^p \left(A\nabla_\nu B + 2rMm \left(A^{-1}\nabla B^{-1} - A^{-1,2}B^{-1}\right)\right) \|.
\]

Therefore, Theorem 2.6 is a refinement of Theorem 1.3 for the operator norm and \(p \geq 2\).

**Remark 2.8.** Since the Kantorovich constant \(K(h)\) is an increasing function on the interval \((1, +\infty)\) and also \(K(h) \geq 1\) for every \(h > 0\), so Theorem 2.6 is a refinement of Theorem 1.2; see [7].
Zhang [18] obtained the following inequalities for $p \geq 4$:

$$\Phi^p(A \nabla B) \leq \left( \frac{K(h) (M^2 + m^2)}{4^p Mm} \right)^p \Phi^p(A^\sharp B);$$

$$\Phi^p(A \nabla B) \leq \left( \frac{K(h) (M^2 + m^2)}{4^p Mm} \right)^p (\Phi(A)^\sharp \Phi(B))^p.$$

Recently, the authors of [17] improved the above inequalities as follows:

$$\Phi^p(A \nabla_B B) \leq \left( \frac{K(h) (M^2 + m^2)}{4^p Mm K^r(h')} \right)^p \Phi^p(A^\sharp \nu B); \quad (2.8)$$

$$\Phi^p(A \nabla_B B) \leq \left( \frac{K(h) (M^2 + m^2)}{4^p Mm K^r(h')} \right)^p (\Phi(A)^\sharp \nu \Phi(B))^p. \quad (2.9)$$

In the following theorem, we show some refinements of inequalities (2.8) and (2.9).

**Theorem 2.9.** Let $A, B \in \mathcal{B}(\mathcal{H})$ are positive and $m, m', M, M'$ are positive real numbers satisfying either one of the following conditions:

1. $0 < m \leq B \leq m' < M' \leq A \leq M$;
2. $0 < m \leq A \leq m' < M' \leq B \leq M$.

Then for every positive unital linear map $\Phi$, $0 \leq \nu \leq 1$ and for every $p \geq 4$

$$\Phi^p(A \nabla_B (B + 2r Mm(A^{-1} \nabla B^{-1} - A^{-1} B^{-1}))) \leq \left( \frac{K(h) (M^2 + m^2)}{4^p Mm K^r(h')} \right)^p \Phi^p(A^\sharp \nu B) \quad (2.10)$$

and

$$\Phi^p(A \nabla_B (B + 2r Mm(A^{-1} \nabla B^{-1} - A^{-1} B^{-1}))) \leq \left( \frac{K(h) (M^2 + m^2)}{4^p Mm K^r(h')} \right)^p (\Phi(A)^\sharp \nu \Phi(B))^p, \quad (2.11)$$

where $r = \min\{\nu, 1 - \nu\}$, $K(h) = \frac{(1 + h)^p}{4^p h}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r_1 = \min\{2r, 1 - 2r\}$. 
Proof. It follows from Lemma 2.2 and Theorem 2.4 that

$$M^2 m^2 \Phi \left( A \nabla_v B + 2rm (A^{-1} \nabla B^{-1} - A^{-1}B^{-1}) \right) \Phi^{-\frac{2}{3}}(A^*_{\nu \nu} B)$$

$$= \left\| \Phi^{\frac{2}{3}}(A \nabla_v B + 2rm (A^{-1} \nabla B^{-1} - A^{-1}B^{-1})) \right\| \left\| M^2 m^2 \Phi^{-\frac{2}{3}}(A^*_{\nu \nu} B) \right\|$$

$$\leq \frac{1}{4} \left\| \frac{K^{r_1}}{K} \Phi^2 \left( A \nabla_v B + 2rm (A^{-1} \nabla B^{-1} - A^{-1}B^{-1}) \right) + \frac{M^2 m^2 K(h)}{K^{r_1}} \Phi^{-2}(A^*_{\nu \nu} B) \right\|^2$$

$$= \frac{1}{4} \left\| \frac{K^{r_1}}{K} \Phi^2 \left( A \nabla_v B + 2rm (A^{-1} \nabla B^{-1} - A^{-1}B^{-1}) \right) + \frac{M^2 m^2 K(h)}{K^{r_1}} \Phi^{-2}(A^*_{\nu \nu} B) \right\|^2$$

$$\leq \frac{1}{4} \left\| \frac{K(h)(M^2 + m^2)}{K^{r_1}} \left( \Phi^2 \left( A^*_{\nu \nu} B \right) + M^2 m^2 \Phi^{-2} \left( A^*_{\nu \nu} B \right) \right) \right\|^2$$

$$\leq \frac{1}{4} \left( \frac{K(h)(M^2 + m^2)}{K^{r_1}} \right)^{\frac{2}{3}}.$$

It follows from $0 < m \leq A^*_{\nu \nu} B \leq M$ and the linearity $\Phi$ that $0 < m \leq \Phi \left( A^*_{\nu \nu} B \right) \leq M$. In addition, for every $T \in \mathcal{B}(\mathcal{H})$ such that $0 < m \leq T \leq M$, we have

$$M^2 m^2 T^{-2} + T^2 \leq M^2 + m^2.$$

Now, by putting $\Phi \left( A^*_{\nu \nu} B \right)$ in the latter inequality, we obtain the last inequality. Hence,

$$M^2 m^2 \Phi \left( A \nabla_v B + 2rm (A^{-1} \nabla B^{-1} - A^{-1}B^{-1}) \right) \Phi^{-\frac{2}{3}}(A^*_{\nu \nu} B)$$

$$\leq \left( \frac{K(h)(M^2 + m^2)}{4mK^{r_1}(h')} \right)^{\frac{2}{3}}.$$
By Lemma 2.2, the last inequality implies inequality (2.10). Analogously, we can prove inequality (2.11). □

Remark 2.10. Note that inequalities (2.10) and (2.11) are refinements of (2.8) and (2.9) for the operator norm, respectively.

Theorem 2.11. Let $A, B \in \mathcal{B}(\mathcal{H})$ are positive and $m, m', M, M'$ positive real numbers satisfying either one of the following conditions:

1. $0 < m \leq B \leq m' < M' \leq A \leq M$.
2. $0 < m \leq A \leq m' < M' \leq B \leq M$.

Then for every positive unital linear map $\Phi$ and $0 \leq \nu \leq 1$

$$
\Phi^p (A \nabla_{\nu} B + 2r Mm (A^{-1} \nabla B^{-1} - A^{-1} \nabla B^{-1} )) \leq \left( K^{-\frac{r_\alpha}{2}} (\sqrt{h}) K_{\frac{\alpha}{2}} (h) (M^\alpha + m^\alpha) \right)^{\frac{2\mu}{\alpha}} \Phi^p (A^{\nu}_{\nu} B), \quad (2.12)
$$

$$
\Phi^p (A \nabla_{\nu} B + 2r Mm (A^{-1} \nabla B^{-1} - A^{-1} \nabla B^{-1} )) \leq \left( K^{-\frac{r_\alpha}{2}} (\sqrt{h}) K_{\frac{\alpha}{2}} (h) (M^\alpha + m^\alpha) \right)^{\frac{2p}{\alpha}} \Phi (A^{\nu}_{\nu} \Phi (B))^p, \quad (2.13)
$$

where $1 \leq \alpha \leq 2$, $K(h) = \frac{(1+h)^2}{4h}$, and $p \geq 2\alpha$.

Proof. By Lemma 2.2, inequality (2.12) is equivalent to the following inequality

$$
\left\| \Phi^\frac{\alpha}{2} (A \nabla_{\nu} B + 2r Mm (A^{-1} \nabla B^{-1} - A^{-1} \nabla B^{-1} )) \Phi^{-\frac{\alpha}{2}} (A^{\nu}_{\nu} B) \right\| \leq \left( K^{-\frac{r_\alpha}{2}} (\sqrt{h}) K_{\frac{\alpha}{2}} (h) (M^\alpha + m^\alpha) \right)^{\frac{\alpha}{\mu}} \frac{4M_{\frac{1}{2}}^\frac{\alpha}{2} m_{\frac{1}{2}}^\frac{\alpha}{2}}{4M_{\frac{1}{2}}^\frac{\alpha}{2} m_{\frac{1}{2}}^\frac{\alpha}{2}}.
$$
By using Lemma 2.2 and Theorem 2.4, one can obtain

\[
M^{\frac{\alpha}{2}} m^{\frac{\alpha}{2}} \left\| \Phi^{\frac{\alpha}{2}} \left( A\nabla_v B + 2r M m \left( A^{-1} \nabla B^{-1} - A^{-1} B^{-1} \right) \right) \Phi^{-\frac{\alpha}{2}} (A^{\dagger}_v B) \right\|
\]

\[
= \left\| \Phi^{\frac{\alpha}{2}} \left( A\nabla_v B + 2r M m \left( A^{-1} \nabla B^{-1} - A^{-1} B^{-1} \right) \right) M^{\frac{\alpha}{2}} m^{\frac{\alpha}{2}} \Phi^{-\frac{\alpha}{2}} (A^{\dagger}_v B) \right\|
\]

\[
\leq \frac{1}{4} \left\| \frac{K^{\frac{\alpha}{2}}}{K^{\frac{\alpha}{2}}(h)} \Phi^{\frac{\alpha}{2}} \left( A\nabla_v B + 2r M m \left( A^{-1} \nabla B^{-1} - A^{-1} B^{-1} \right) \right) \right\|^{2}
\]

\[
= \frac{1}{4} \left\| \frac{K^{\frac{\alpha}{2}}}{K^{\frac{\alpha}{2}}(h)} \Phi^{\frac{\alpha}{2}} \left( A\nabla_v B + 2r M m \left( A^{-1} \nabla B^{-1} - A^{-1} B^{-1} \right) \right) \right\|^{2}
\]

\[
\leq \frac{1}{4} \left( \frac{K^{\frac{\alpha}{2}}}{K^{\frac{\alpha}{2}}(h)} \right)^{\frac{\alpha}{2}} \left( \Phi^{-\alpha} (A^{\dagger}_v B) \right) \left\| \frac{K^{\frac{\alpha}{2}}}{K^{\frac{\alpha}{2}}(h)} \Phi^{-\alpha} (A^{\dagger}_v B) \right\|^{2}
\]

\[
\leq \frac{1}{4} \left( \frac{K^{\frac{\alpha}{2}}(h)(M^{\alpha} + m^{\alpha})}{K^{\frac{\alpha}{2}}(\sqrt{h})} \right)^{\frac{\alpha}{2}}
\]
Now, by setting $\Phi(A^\sharp \nu B)$ in the latter inequality we obtain the last inequality. This proves inequality (2.12). By utilizing the same ideas as in the proof of inequality (2.12), we can reach inequality (2.13).

**Remark 2.12.** If we take $\alpha = 1, 2$, then Theorem 2.11 reduces to Theorem 2.9 and Theorem 2.6, respectively.

### 3. Reverse of Ando’s inequality

For positive operators $A, B \in \mathbb{B}(\mathcal{H})$, we know [4] that for every positive unital linear map $\Phi$

$$\Phi(A^\sharp B) \leq \Phi(A)^\sharp \Phi(B).$$  \hspace{1cm} (3.1)

Ando’s inequality says that if $A, B$ be positive operators and $\Phi$ be a positive unital linear map, then

$$\Phi(A^\sharp \nu B) \leq \Phi(A)^\sharp \nu \Phi(B).$$ \hspace{1cm} (3.2)

The author [10] presented the following theorem that can be viewed as a reversed version of (3.1).

**Theorem 3.1.** If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$, then for every positive linear map $\Phi$ and some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$

$$\Phi(A)^\sharp B \leq \frac{\sqrt{M} + \sqrt{m}}{2\sqrt{Mm}} \Phi(A^\sharp B),$$ \hspace{1cm} (3.3)

where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$.

Seo [15] improved inequality above and obtained the following inequality:

**Theorem 3.2.** Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive such that $0 < m_1^2 \leq A \leq M_1^2$, $m_2^2 \leq B \leq M_2^2$, $m = \left(\frac{m_2}{M_1}\right)^2$ and $M = \left(\frac{M_2}{m_1}\right)^2$. Then for every positive unital linear map $\Phi$ and $0 \leq \nu \leq 1$

$$\Phi(A)^\sharp \nu B \leq K(m, M, \nu)^{-1} \Phi(A^\sharp \nu B),$$ \hspace{1cm} (3.4)

where $K(m, M, \nu) = \frac{mM^\nu - Mm^\nu}{(\nu-1)(M-m)} \left(\frac{\nu-1}{\nu} \frac{M^\nu - m^\nu}{mM^\nu - Mm^\nu}\right)^\nu$.

In this section, we give a refinement of inequality (3.4). To achieve this, we need the following theorem:

**Theorem 3.3.** [19] Suppose that $A, B \in \mathbb{B}(\mathcal{H})$ are positive and $m, m', M, M'$ are positive real numbers satisfying either one of the following conditions:
Then for $0 \leq \nu \leq 1$

\[ A \nabla_v B \geq K^r(h)(A^*_{\nu}B), \quad (3.5) \]

where $r = \min\{\nu, 1 - \nu\}$, $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$.

**Theorem 3.4.** Let $A, B \in \mathbb{B}({\mathcal H})$ such that $0 < m_1^2 \leq A \leq M_1^2$, $m_2^2 \leq B \leq M_2^2$, $m = \left(\frac{m_2}{M_1}\right)^2$ and $M = \left(\frac{M_2}{m_1}\right)^2$. If $M_1 < m_2$, then for every positive unital linear map $\Phi$ and $0 \leq \nu \leq 1$,

\[ \Phi(A)^{\#}_{\nu}\Phi(B) \leq K(m, M, \nu)^{-1}K(h)^{-r}\Phi(A^*_{\nu}B), \quad (3.6) \]

where $K(m, M, \nu) = \frac{m^\nu M^\nu - m^\nu M^\nu}{(\nu - 1)(M - m)} \left(\frac{M^\nu - m^\nu}{M^\nu - m^\nu}\right)^\nu$, $r = \min\{\nu, 1 - \nu\}$, $K(h) = \frac{(1 + h)^2}{4\nu}$ and $h = \frac{m_2^2}{M_1^2}$. Similarly, one can prove the inequality for $M_2 < m_1$ and $h = \frac{M_2^2}{m_1^2}$.

**Proof.** For $t \in [m, M]$. We put $F(t) = \nu t^{1-\nu} + (1 - \nu)\lambda_0 t^{-\nu}$, where

\[ \mu_0 = \frac{\nu(M - m)}{M^\nu - m^\nu}, \quad \lambda_0 = \frac{\nu}{1 - \nu}\frac{M^{1-\nu} - m^{1-\nu}}{m^{1-\nu} - M^{1-\nu}}. \]

Easy computation shows that $\max_{t \in [m, M]} F(t) = F(M) = F(m)$ and $F(M) = F(m) = \mu_0$. Hence

\[ \nu t^{1-\nu} + (1 - \nu)\lambda_0 t^{-\nu} \leq \mu_0. \quad (3.7) \]

Using the fact that $0 < m_1^2 \leq A \leq M_1^2$ and $m_2^2 \leq B \leq M_2^2$, we get $mI \leq C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq MI$. Considering inequality (3.7) with $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we obtain

\[ \nu C + (1 - \nu)\lambda_0 I \leq \mu_0 C^\nu. \]

Multiplying both sides of the latter inequality by $A^\frac{1}{2}$, we have

\[ \nu \Phi(B) + (1 - \nu)\lambda_0 \Phi(A) \leq \mu_0 \Phi(A^*_{\nu}B). \quad (3.8) \]

Using (3.5) for two operators $\lambda_0 \Phi(A)$ and $\Phi(B)$ yields that

\[ \lambda_0^{1-\nu}\Phi(A)^{\#}_{\nu}\Phi(B) \leq K(h, 2)^{-r}(\nu \Phi(B) + (1 - \nu)\lambda_0 \Phi(A)). \quad (3.9) \]

From (3.8) and (3.9), we obtain inequality (3.6). $\square$

**Remark 3.5.** Note that the right side of inequality (3.6) is a better bound than inequality (3.4), since the Kantorovich constant $K(h)$ is increasing on the interval $(1, +\infty)$. 
Remark 3.6. If we put $\nu = \frac{1}{2}$ in Theorem 3.4, then we obtain a refinement of (3.3), since the Kantorovich constant $K(h)$ is increasing on the interval $(1, +\infty)$.

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