A Riemann-Hilbert approach to the Harry-Dym equation on the line

Yu Xiao\textsuperscript{1}, Engui Fan\textsuperscript{1*}

\textsuperscript{1} School of Mathematical Science, Fudan University, Shanghai 200433, P.R. China

Abstract

In this paper, we consider the Harry-Dym equation on the line with decaying initial value. The Fokas unified method is used to construct the solution of the Harry-Dym equation via a $2 \times 2$ matrix Riemann Hilbert problem in the complex plane. Further, one-cups solution solution is expressed in terms of solutions of the Riemann Hilbert problem.

\textbf{Keywords:} Harry-Dym equation, Riemann-Hilbert problem, Initial-value problem, One-cups solution solution

1 Introduction

The following nonlinear partial differential equation
\begin{equation}
q_t - 2\left(\frac{1}{\sqrt{1 + q}}\right)_{xxx} = 0
\end{equation}
is known as the Harry-Dym equation \cite{1}. This equation was obtained by Harry-Dym and Martin Kruskal as an evolution equation solvable by a spectral problem based on the string equation instead of the Schrodinger equation. The Harry-Dym equation has interest in the study of the Saffman-Taylor problem which describes the motion of a two-dimensional interface between a viscous and a nonviscous fluid \cite{2}. The Harry-Dym equation

*Corresponding author and e-mail address: faneg@fudan.edu.cn
shares many of the properties typical of the soliton equations. It is a completely integrable equation which can be solved by the inverse scattering transform\cite{3}. It has a bi-Hamiltonian structure \cite{4}, an infinite number of conservation laws and infinitely many symmetries \cite{5}, and has reciprocal Backlund transformations to the KdV equation \cite{6}. The Harry-Dym equation has been solved in different methods such as the inverse scattering method \cite{3}, the Bäcklund transformation technique \cite{7}, the straightforward method \cite{8}. Especially, the Wadati obtained the one-cups soliton solution \cite{3}

\[ q(x, t) = \tanh^{-4}(\kappa x - 4\kappa^3 t + \kappa x_0 + \varepsilon_+) - 1, \]

\[ \varepsilon_+ = \frac{1}{\kappa}[1 + \tanh(\kappa x - 4\kappa^3 t + \kappa x_0 + \varepsilon_+)]. \]

by using inverse scattering transformation.

The main aim of this paper is to develop the inverse scattering method, based on an Riemann-Hilbert problem for solving nonlinear integrable systems called unified method \cite{9} which has been further developed and applied in different equations with initial value problems on the line \cite{10, 11, 12, 13, 14} and initial boundary value problem on half line \cite{15, 16, 17}. In this paper, we consider the initial value problem of the Harry-Dym equation

\[ q_t - 2\left(-\frac{1}{\sqrt{1+q}}\right)_{xxx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.2) \]

\[ q(x, 0) = q_0(x), \]

where the \( q_0(x) \) is a smoothly real-valued function and decay as \( |x| \to \infty \).

The organization of the paper is as follows. In the following section 2, we perform the spectral analysis of the associated Lax pair for the Harry-Dym equation. In section 3, we formulate the main Riemann-Hilbert problem associated with the initial value problem \eqref{eq:1.2}. In section 4, we obtain one-cups solition solution in the terms of Riemann-Hilbert problem, which has a similar, but not the same form constructed by the inverse scattering method \cite{3}.
2 Spectral analysis

2.1 A Lax pair

In general, the matrix Riemann Hilbert problem is defined in the $\lambda$ plane and has explicit $(x,t)$ dependence, while for the Harry-Dym equation (1.2), we need to construct a new matrix Riemann Hilbert problem with explicit $(y,t)$ dependence, where $y(x,t)$ is a function which is an unknown from the initial value condition. For this purpose, we make a transformation

$$\rho = \sqrt{1 + q},$$

the equation (1.2) can be expressed by

$$(\rho^2)_t - 2(\frac{1}{\rho})_{xxx} = 0.$$

Then the initial value problem (1.2) is transformed into

$$(\rho^2)_t - 2(\frac{1}{\rho})_{xxx} = 0, x \in \mathbb{R}, t > 0,$$

$$\rho(x,0) = \rho_0(x) = \sqrt{1 + q_0(x)},$$

$$\rho_0(x) \to 1, |x| \to \infty.$$

It was shown that the equation (1.2) admits the following Lax pair [3]

$$\begin{cases}
\psi_{xx} = -\lambda^2 (1 + q) \psi, \\
\psi_t = 2\lambda^2 \left[ \frac{2}{\sqrt{1+q}} \psi_x - \left( \frac{1}{\sqrt{1+q}} \right)_x \psi \right]. 
\end{cases}$$

(2.2)

Making a transformation

$$\rho = \sqrt{1 + q}, \ \varphi = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix},$$

then the Lax pair (2.2) can be written in matrix form

$$\begin{cases}
\varphi_x = M \varphi, \\
\varphi_t = N \varphi,
\end{cases}$$

(2.3)
where
\[ M = \begin{pmatrix} 0 & 1 \\ -\lambda^2 \rho^2 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -2\lambda^2 \left( \frac{1}{\rho} \right)_x & 4\lambda^2 \frac{1}{\rho} \\ -4\lambda^4 \rho - 2\lambda^2 \left( \frac{1}{\rho} \right)_{xx} & 2\lambda^2 \left( \frac{1}{\rho} \right)_x \end{pmatrix}. \]

Further by the gauge transformations
\[ \phi = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda \rho} & 0 \\ 0 & \sqrt{\lambda \rho} \end{pmatrix} \phi. \]

we have
\[
\begin{cases} 
\phi_x + i\lambda \rho \sigma_3 \phi = U \phi, \\
\phi_t + i(\lambda \frac{1}{\rho} x + 4\lambda^3) \sigma_3 \phi = V \phi,
\end{cases}
\tag{2.4}
\]

where
\[
U(x,t) = \frac{1}{2} \rho \sigma_2, \quad V(x,t,\lambda) = -\lambda \frac{1}{\rho} \left( \frac{1}{\rho} \right)_{xx} \sigma_1 - 2\lambda^2 \left( \frac{1}{\rho} \right)_x \sigma_2.
\]

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

It is clear that as \(|x| \to \infty\), \(U(x,t) \to 0, V(x,t,\lambda) \to 0\). We define a real-valued function \(y(x,t)\) by
\[ y(x,t) = x + \int_x^\infty (1 - \rho(\xi,t)) d\xi. \]

It is obvious that
\[ y_x = \rho(x,t), \quad y_t = -\int_x^\infty \rho_t(\xi,t) d\xi. \]

The conservation law
\[ \rho_t - \left( -\frac{1}{2} \left( \frac{1}{\rho} \right)_x \right)^2 + \frac{1}{\rho} \left( \frac{1}{\rho} \right)_{xx} = 0 \]
implies that
\[ y_t = -\frac{1}{2} \left( \frac{1}{\rho} \right)_x^2 + \frac{1}{\rho} \left( \frac{1}{\rho} \right)_{xx}. \]

Extending the column vector \(\phi\) to be a \(2 \times 2\) matrix and letting
\[ \mu = \phi \exp(i\lambda y(x,t) \sigma_3 + 4i\lambda^3 t \sigma_3), \]

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then $\mu$ solves
\begin{equation}
\begin{aligned}
\mu_x + i\lambda y_x [\sigma_3, \mu] &= \widetilde{U}\mu, \\
\mu_t + i(\lambda y_t + 4\lambda^3)[\sigma_3, \mu] &= \widetilde{V}\mu,
\end{aligned}
\end{equation}
which can be written in full derivative form
\[ d(e^{i(y(x,t)x+4\lambda^3t)}\sigma_3\mu) = e^{i(y(x,t)x+4\lambda^3t)}(\widetilde{U}dx + \widetilde{V}dt)\mu, \]
where
\[ \widetilde{U} = U, \]
\[ \widetilde{V} = -\frac{1}{2}i\lambda((\frac{1}{\rho})_x)^2\sigma_3 - \lambda\frac{1}{\rho}(\frac{1}{\rho})_{xx}\sigma_1 - 2\lambda^2(\frac{1}{\rho})_x\sigma_2, \]
$[\sigma_3, \mu] = \sigma_3\mu - \mu\sigma_3$. As $|x| \to \infty$, $\widetilde{V} \to 0$. The lax pair in the \[(2.5)\] is very convenient for dedicated solutions via integral Volterra equation, which is also what we study in the following paper.

**Remark 2.1** By the representation of $M, N$ and $U, V$ in \[(2.3)\] and \[(2.4)\] respectively, we find that $\psi_x, \psi_t$ and $\phi_x, \phi_t$ have no singularity in $\lambda = 0$. Therefore, $\phi$ has no real singularity in $\lambda = 0$.

**2.2 Eigenfunctions** We define two eigenfunctions $\mu_{\pm}$ of equation \[(2.5)\] as the solutions of the following two Volterra integral equation in the $(x,t)$ plane
\begin{equation}
\mu(x,t,\lambda) = I + \int_{(x',t')}^{(x,t)} e^{-i\lambda(y(x,t) - y(x',t)) + 4i\lambda^3(t-t')}\sigma_3 (\widetilde{U}(x',t)\mu(x',t,\lambda)dx' + \widetilde{V}(x',t,\lambda)\mu(x',t,\lambda))d\tau
\end{equation}
where $I$ is the $2 \times 2$ identity matrix, $\sigma_3$ acts on a $2 \times 2$ matrix $A$ by $\sigma_3 A = A \sigma_3$. Since the integrated expression is independent of the path of integration, we choose the particular initial points of integration to be parallel to the $x$-axis and obtain that $\mu_+$ and $\mu_-$
\[ \mu_+(x,t,\lambda) = I - \int_{x}^{\infty} e^{-i\lambda(y(x,t) - y(x',t))}\sigma_3 \widetilde{U}(x',t)\mu_+(x',t,\lambda)dx', \]
\[ \mu_-(x, t, \lambda) = I + \int_{-\infty}^{x} e^{-i\lambda(y(x,t)-y(x',t))}\sigma_3 \tilde{U}(x', t)\mu_-(x', t, \lambda)dx'. \] \tag{2.7}

Define the following sets

\[ D_1 = \{ \lambda \in \mathbb{C} | \text{Im} \lambda > 0 \}, \]
\[ D_2 = \{ \lambda \in \mathbb{C} | \text{Im} \lambda < 0 \}. \]

Since any fixed \( t \), \( y_x = \rho(x, t) > 0 \), \( y(x, t) \) is increasing function of \( x \) for fixed \( t \). as \( x - x' < 0 \), \( y(x, t) - y(x', t) < 0 \); as \( x - x' > 0 \), \( y(x, t) - y(x', t) > 0 \). We can deduce that the second column vectors of \( \mu_+, \mu_- \) are bounded and analytic for \( \lambda \in \mathbb{C} \) provided that \( \lambda \) belongs to \( D_1, D_2 \), respectively. We denote these vectors with superscripts (1),(2) to indicate the domains of their boundedness. Then

\[ \mu_+ = (\mu_+^{(2)}, \mu_+^{(1)}), \mu_- = (\mu_-^{(1)}, \mu_-^{(2)}). \]

For any \( x, t \), the following conditions are satisfied

\[ (\mu_-^{(1)}, \mu_+^{(1)}) = I + O(1/\lambda), \lambda \to \infty, \lambda \in D_1, \]
\[ (\mu_+^{(2)}, \mu_-^{(2)}) = I + O(1/\lambda), \lambda \to \infty, \lambda \in D_2, \]
\[ \mu_\pm = I + O(1/\lambda), \lambda \to \infty. \]

2.3 Spectral functions For \( \lambda \in \mathbb{R} \), the eigenfunction \( \mu_+, \mu_- \) being the solution of the system of differential equation (2.5) are related by a matrix independent of \( (x, t) \). We define the spectral function by

\[ \mu_+(x, t, \lambda) = \mu_-(x, t, \lambda)e^{-i(\lambda y(x,t)+4\lambda^3 t)\sigma_3} s(\lambda). \] \tag{2.8}

From (2.5), we get

\[ \text{det}(\mu_\pm(x, t, \lambda)) = 1. \] \tag{2.9}

Since \( \overline{U}(x, t) = -\tilde{U}(x, t) \), the \( \mu_\pm(x, t, \lambda) \) have the relations

\[ \begin{aligned}
\mu_{\pm 11}(x, t, \lambda) &= \mu_{\pm 22}(x, t, \lambda), & \mu_{\pm 21}(x, t, \lambda) &= \mu_{\pm 12}(x, t, \lambda), \\
\mu_{\pm 11}(x, t, -\lambda) &= \mu_{\pm 22}(x, t, \lambda), & \mu_{\pm 12}(x, t, -\lambda) &= \mu_{\pm 21}(x, t, \lambda). 
\end{aligned} \] \tag{2.10}
The spectral function $s(\lambda)$ can be written as
\[
s(\lambda) = \begin{pmatrix} \overline{a(\lambda)} & b(\lambda) \\ b(\lambda) & a(\lambda) \end{pmatrix},
\]
(2.11)

\[
s(\lambda) = I - \int_{-\infty}^{+\infty} e^{i\lambda y(x,0)\sigma_3} \tilde{U}(x',0) \mu_+(x',0,\lambda) dx', \quad \text{Im}\lambda = 0.
\]
(2.12)

From the (2.9), $\det(s(\lambda)) = 1$. Equation (2.8) and (2.9) imply $a(\lambda)$ and $b(\lambda)$ have the following properties:

- $a(\lambda)$ is analytic in $D_1$ and continuous for $\lambda \in \bar{D}_1$.
- $b(\lambda)$ is continuous for $\lambda \in R$.
- $a(\lambda)a(\bar{\lambda}) - b(\lambda)b(\bar{\lambda}) = 1, \quad \lambda \in R$.
- $a(\lambda) = 1 + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty, \quad \lambda \in D_1$.
- $b(\lambda) = O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty, \quad \lambda \in R$.

2.4 Residue conditions We assume that $a(\lambda)$ has $N$ simple zeros $\{\lambda_j\}_{j=1}^N$ in the upper half plane. These eigenvalues are purely imaginary. The second column of equation (2.8) is
\[
\mu_+^{(1)} = b(\lambda)\mu_-^{(1)} e^{-2i(\lambda y(x,t)+4\lambda^3 t)} + \mu_-^{(2)} a(\lambda).
\]
(2.13)

For the (2.9) and equation (2.13), it yields
\[
a(\lambda) = \det(\mu_-^{(1)}, \mu_+^{(1)})
\]
where we have used that both sides are well defined and analytic in $D_1$ to extend the above relation to $\bar{D}_1$. Hence if $a(\lambda_j) = 0$, the $\mu_-^{(1)}, \mu_+^{(1)}$ are linearly dependent vectors for each $x$ and $t$, i.e. there exist constants $b_j \neq 0$ such that
\[
\mu_-^{(1)} = b_j e^{2i(\lambda_j y(x,t)+4\lambda_j^3 t)} \mu_+^{(1)}, \quad x \in R, \ t > 0.
\]
Recalling the symmetries in the (2.10), we find
\[ \mu_\pm^{(2)} = \bar{b}_j e^{-2i(\bar{\lambda}_j y(x,t) + 4\bar{\lambda}_j^3 t)} \mu_\pm^{(2)}, \quad x \in \mathbb{R}, t > 0. \]

Consequently, the residues of \( \mu_\pm^{(1)}/a \) and \( \mu_\pm^{(2)}/a(\lambda) \) at \( \lambda_j \) and \( \bar{\lambda}_j \) are
\[
\text{Res}_{\lambda=\lambda_j} \frac{\mu_\pm^{(1)}(x,t,\lambda)}{a(\lambda)} = C_j e^{2i(\lambda_j y(x,t) + 4\lambda_j^3 t)} \mu_\pm^{(2)}(x,t,\lambda_j), \quad j = 1, \ldots, N,
\]
\[
\text{Res}_{k=\lambda_j} \frac{\mu_\pm^{(2)}(x,t,\lambda)}{a(\lambda)} = \bar{C}_j e^{-2i(\bar{\lambda}_j y(x,t) + 4\bar{\lambda}_j^3 t)} \mu_\pm^{(1)}(x,t,\bar{\lambda}_j), \quad j = 1, \ldots, N,
\]
where \( C_j = \frac{b_j}{a(k_j)}, \quad \dot{a}(k) = \frac{da}{dk}. \)

**Remark 2.2** There is the relation of \( \mu_\pm \) that the \( s(\lambda) \) is the scattering matrix for the one dimensional Schrödinger equation

\[
W_{yy} + \lambda^2 W = f(y)W
\]

via the Liouville transformation:
\[
y = x + \int_x^\infty (1 - \rho(\xi,0)) d\xi, \quad W(y,\lambda) = \psi(y,\lambda)\rho_0(y)
\]
\[
\rho_0(y) = \rho_0(x), \quad f(y) = \frac{1}{2}(\rho_{0yy}\rho_0^{-1} - \frac{1}{2}\rho_{0y}^2\rho_0^{-2}).
\]

Therefore, in terms of spectral problem of Schrödinger equation, we deduce that \( a(\lambda) \) only has pure imaginary of simple poles in the upper plane.

### 3 The Riemann-Hilbert Problem

**3.1 A Riemann-Hilbert problem for \((x,t)\)** We now apply uniform method to solve the initial value problem for equation (2.1) on the line, and the solution can be expressed in terms of a \( 2 \times 2 \) matrix Riemann-Hilbert problem. Let \( M(x,t,\lambda) \) be defined by
\[
M_+ = \begin{pmatrix} \mu_\pm^{(1)}/a(\lambda) & \mu_\pm^{(1)} \\ \mu_\pm^{(2)}/a(\lambda) & \mu_\pm^{(2)} \end{pmatrix}, \quad \lambda \in D_1;
M_- = \begin{pmatrix} \mu_\pm^{(2)} & \mu_\pm^{(2)} \\ \mu_\pm^{(1)}/a(\lambda) & \mu_\pm^{(1)} \end{pmatrix}, \quad \lambda \in D_2 \quad (3.1)
\]
and the $M$ satisfied the jump condition:

$$M_+(x, t, \lambda) = M_-(x, t, \lambda)J(x, t, \lambda), \quad Im\lambda = 0,$$

where

$$J(x, t, \lambda) = \begin{pmatrix}
\frac{1}{a(\lambda)a(\lambda)} & \frac{b(\lambda)}{a(\lambda)}e^{-2i(\lambda y(x, t)+4\lambda^3 t)} \\
-\frac{b(\lambda)}{a(\lambda)}e^{2i(\lambda y(x, t)+4\lambda^3 t)} & 1
\end{pmatrix}, \quad Im\lambda = 0. \tag{3.2}
$$

These definitions imply

$$detM(x, t, \lambda) = 1 \tag{3.3}$$

and

$$M(x, t, \lambda) = I + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty. \tag{3.4}$$

This contour of RH problem is the real axis.

The jump matrix $J(x, t, \lambda)$, the spectral $a(\lambda)$ and $b(\lambda)$ are dependent on the $y(x, t)$, while $y(x, t)$ is not involve initial data. Therefore, this RH problem can’t be formulated in terms of initial alone. In order to overcome this problem, we will reconstruct a new jump matrix by changing

$$(x, t) \to (y, t), \quad y = y(x, t),$$

$y$ is a new scale. Then we can transform this RH problem into the RH problem parametrized by $(y, t)$.

3.2 A Riemann-Hilbert problem for $(y, t)$

3.1 The theorem Let $q_0(x), x \in R$ be a smooth function and decay as $|x| \to \infty$. Moreover $1 + q_0(x) > 0$. Define the $\bar{U}_0, \rho_0$ and $y_0(x)$ as follows:

$$\bar{U}_0(x) = \frac{1}{2} \frac{\rho_{0x}(x)}{\rho_0(x)} \sigma_2, \quad \rho_0(x) = \sqrt{1 + q_0(x)},$$

$$y_0(x) = x + \int_x^\infty (1 - \rho_0(\xi))d\xi.$$

Let $\mu_+(x, 0, \lambda)$ and $\mu_-(x, 0, \lambda)$ be the unique solution of the Volterra linear integral equation $[2.5]$ evaluated at $t = 0$ with $\bar{U}_0(x, 0) = \bar{U}_0(x), \rho_0(x) = \cdots$
\( \rho(x, 0) \) and \( y_0(x) = y(x, 0) \). Define \( a(\lambda), b(\lambda), C_j \) by

\[
\begin{pmatrix} b(\lambda) \\ a(\lambda) \end{pmatrix} = [s(\lambda)]_2, \quad s(\lambda) = I - \int_{-\infty}^{+\infty} e^{i\lambda y_0(x)} \hat{U}_0(x') \mu_+(x', 0, \lambda) dx', \quad Im\lambda = 0
\]

and

\[
[\mu_-(x, 0, \lambda_j)]_1 = \hat{a}(\lambda_j) C_j e^{2i\lambda_j y_0(x)} [\mu_+(x, 0, \lambda_j)]_2, \quad j = 1, \ldots, N, \quad (3.5)
\]

where \([A]_1 [A]_2\) denotes the first (second) column of a \(2 \times 2\) matrix \( A \). We assume that \( a(\lambda) \) has \( N \) simple zeros \( \{\lambda_j\}_{j=1}^N \) in the upper half plane and are pure imaginary. Then

- \( a(\lambda) \) is defined for \( k \in \bar{D}_1 \) and analytic in \( D_1 \).
- \( b(\lambda) \) is defined for \( \lambda \in \mathbb{R} \).
- \( a(\lambda)a(\bar{\lambda}) - b(\lambda)b(\bar{\lambda}) = 1, \quad \lambda \in \mathbb{R} \).
- \( a(\lambda) = 1 + O\left(\frac{1}{\lambda}\right), \lambda \to \infty, \lambda \in D_1 \).
- \( b(\lambda) = O\left(\frac{1}{\lambda}\right), \lambda \to \infty, \lambda \in \mathbb{R} \).

Suppose there exists a uniquely solution \( q(x, t) \) of equation (1.2) with initial data \( q_0(x) \) such that \( \rho_0(x) = \sqrt{1 + q_0(x)} \) has sufficient smoothness and decay for \( t > 0 \). Then \( q(x, t) \) is given in parametric form by

\[
q(x(y, t), t) = e^{\int_{\gamma}^{x(y, t)} m(y', t) dy'} - 1
\]

and the function \( x(y, t) \) is defined by

\[
x(y, t) = y + \int_{-\infty}^{y} (e^{-4 \int_{\xi}^{x} m(\xi, t) d\xi} - 1) dy', \quad (3.8)
\]

where \( m(y, t) = -i \lim_{\lambda \to \infty} (\lambda M(y, t, \lambda))_{12} \), and \( M(y, t, \lambda) \) is the uniquely solution of the following RH problem

- \( M(y, t, \lambda) = \begin{cases} M_-(y, t, \lambda), & \lambda \in \mathbb{D}_2, \\
M_+(y, t, \lambda), & \lambda \in \mathbb{D}_1. \end{cases} \)

is a sectionally meromorphic function.
- \( M_+(y, t, \lambda) = M_-(y, t, \lambda) J^y(y, t, \lambda) \), \( \Im \lambda = 0 \),

where \( J^y(y, t, \lambda) \) is defined by

\[
J^y(y, t, \lambda) = \begin{pmatrix}
\frac{1}{a(\lambda)} & \frac{b(\lambda)}{a(\lambda)} e^{-2i(\lambda y + 4\lambda^3 t)} \\
\frac{b(\lambda)}{a(\lambda)} e^{2i(\lambda y + 4\lambda^3 t)} & 1
\end{pmatrix}, \quad \Im \lambda = 0.
\] (3.9)

- \( M(y, t, \lambda) = I + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty. \) (3.10)

- The possible simple poles of the first column of \( M_+(y, t, \lambda) \) occur at \( \lambda = \lambda_j, \ j = 1, \ldots, N \), and the possible simple poles of the second column of \( M_-(y, t, \lambda) \) occur at \( \lambda = \bar{\lambda}_j, \ j = 1, \ldots, N \). The associated residues are given by

\[
\text{Res}_{\lambda = \lambda_j} [M(y, t, \lambda)]_1 = C_j e^{2i(\lambda_j y + 4\lambda_j^3 t)} [M(y, t, \lambda_j)]_2, \quad j = 1, \ldots, N,
\] (3.11)

\[
\text{Res}_{\lambda = \lambda_j} [M(y, t, \lambda)]_2 = \bar{C}_j e^{-2i(\bar{\lambda}_j y + 4\bar{\lambda}_j^3 t)} [M(y, t, \bar{\lambda}_j)]_1, \quad j = 1, \ldots, N.
\] (3.12)

Proof: Assume that \( \mu(x, t) \) is the solution of equation (2.5), the asymptotic expansion of it

\[
\mu(x, t, \lambda) = I + \frac{\mu^{(1)}(x, t)}{\lambda} + \frac{\mu^{(2)}(x, t)}{\lambda^2} + \frac{\mu^{(3)}(x, t)}{\lambda^3} + O\left(\frac{1}{\lambda^4}\right), \lambda \to \infty
\]

into the \( x \)-part of equation (2.5), where \( \mu^{(1)}(x, t), \mu^{(2)}(x, t) \) and \( \mu^{(3)}(x, t) \) are \( 2 \times 2 \) matrices, dependent on \( x, t \). by considering the terms of \( O(1) \), We get

\[
4\mu^{(1)}_{12}(x, t) = -\frac{\rho_x(x, t)}{\rho(x, t)}.
\] (3.13)

By construction of the new RH problem about \( (y, t, \lambda) \), we can deduce that

\[
\mu^{(1)}_{12}(x, t) = -i \lim_{\lambda \to \infty} \lambda [M(y, t, \lambda)]_{12} = m(y, t).
\] (3.14)
Then
\[ -\frac{1}{4} \rho_x(x,t) = m(y,t). \]  
Equation (3.13) can be expressed in terms of \( y = y(x,t) \). Indeed, using \( \frac{dy}{dx} = \rho \), then (3.15) becomes
\[ -\frac{1}{4} \rho y \rho = m(y,t). \]  
As \( |y| \to \infty \), \( \rho(y,t) \to 1 \), by the evaluation of (3.16), we get
\[ \rho(y,t) = e^{4 \int_{y}^{\infty} m(y',t) dy'}. \]
Therefore,
\[ q(x,t) = e^{8 \int_{y}^{\infty} m(y',t) dy'} - 1 \]
As \( |x| \to \infty \), \( |y| \to \infty \) and \( \frac{dy}{dx} = \rho > 0 \), so
\[ x = y + \int_{-\infty}^{y} (e^{-4 \int_{\xi}^{\infty} m(\xi',t) d\xi} - 1) dy'. \]

Remark 3.1 It follows from the symmetries (2.10) that the solution \( M(y,t,\lambda) \) of Riemann Hilbert problem in the 3.1 theorem has the symmetries
\[ \{ M_{11}(y,t,\lambda) = M_{22}(y,t,\lambda), \quad M_{12}(y,t,\lambda) = M_{21}(y,t,\lambda), \quad M_{11}(y,t,-\lambda) = M_{22}(y,t,\lambda), \quad M_{12}(y,t,-\lambda) = M_{21}(y,t,\lambda). \]  

4 Soliton solution

The solitons correspond to spectral data \( \{a(\lambda), b(\lambda), C_j\} \) for which \( b(\lambda) \) vanishes identically. In this case the jump matrix \( J_{(y)}(y,t,\lambda) \) in the (3.9) is the identity matrix and the RH problem of 3.1 theorem consists of finding a meromorphic function \( M(y,t,\lambda) \) satisfying (3.10) and the residue conditions (3.11) and (3.12). From (3.10) and (3.11), we get
\[ [M(y,t,\lambda)]_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{C_j}{\lambda - \lambda_j} e^{2i(\lambda_j y + 4\lambda_j^3 t)} [M(y,t,\lambda_j)]_2. \]  

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for the symmetries (3.17), equation (4.1) can be written as

\[
\left( \frac{M_{22}(y,t,\lambda)}{M_{12}(y,t,\lambda)} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{C_j}{\lambda_j - \lambda} e^{2i(\lambda_j y + 4\lambda_j^3 t)} \left( \frac{M_{12}(y,t,\lambda_j)}{M_{22}(y,t,\lambda_j)} \right).
\]

(4.2)

Evaluation at \( \bar{\lambda}_n \), equation (4.2) becomes

\[
\left( \frac{M_{22}(y,t,\bar{\lambda}_n)}{M_{12}(y,t,\lambda)} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{C_j}{\lambda_n - \lambda_j} e^{2i(\lambda_j y + 4\lambda_j^3 t)} \left( \frac{M_{12}(y,t,\lambda_j)}{M_{22}(y,t,\lambda_j)} \right), \quad n = 1, \ldots, N.
\]

(4.3)

Solving this algebraic system for \( M_{12}(y,t,\lambda_j), M_{22}(y,t,\lambda_j) \), \( n = 1, \ldots, N \), and substituting them into (4.1) provides an explicit expression for the \([M(y,t,\lambda)]_1\).

In terms of the symmetries (3.17), we can get that \( M_{12}(y,t,\lambda_j) \), which solves the Riemann Hilbert problem. Then

\[
-i \lim_{\lambda \to \infty} (\lambda M(y,t,\lambda))_{12} = m(y,t) = -i \sum_{j=1}^{N} C_j e^{2i(\lambda_j y + 4\lambda_j^3 t)} M_{12}(y,t,\lambda_j).
\]

Therefore, the \( N \) soliton solution \( q(x,t) \) is expressed by the (3.7).

### 4.1 One-soliton solution

In this section we derive an explicit formulation for the one-soliton solution, which arise when \( a(\lambda) \) has a pure imaginary \( \lambda_1 \) of simple zero. Letting \( N = 1 \) in (4.3), from the the symmetries of (2.10), we can deduce that \( a(\lambda_1) = a(-\lambda_1) = 0 \), then \( \lambda_1 = -\bar{\lambda}_1 \) and \( \dot{a}(\lambda_1) = \dot{a}(-\lambda_1) \). Since the \( b_1 \) is a real constant, we find that \( C_1 = -\bar{C}_1 \), thus \( C_1 \) is a pure imaginary. Making use of the symmetries of (3.17), we can obtain

\[
M_{22}(y,t,\lambda_1) = 1 + \frac{C_1}{\lambda_1 - \lambda} e^{2i(\lambda_1 y + 4\lambda_1^3 t)} M_{12}(y,t,\lambda_1),
\]

\[
M_{12}(y,t,\lambda_1) = \frac{C_1}{\lambda_1 - \lambda} e^{2i(\lambda_1 y + 4\lambda_1^3 t)} M_{22}(y,t,\lambda_1).
\]

Then,

\[
M_{22}(y,t,\lambda_1) = \frac{(\bar{\lambda}_1 - \lambda_1)^2}{(\bar{\lambda}_1 - \lambda_1)^2 + |C_1|^2 e^{2i(\lambda_1 y + 4\lambda_1^3 t)} e^{-2i(\lambda_1 y + 4\lambda_1^3 t)}}.
\]

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Substituting this result into the (4.3), we get

\[ M_{12}(y, t, \lambda) = \frac{C_1(\bar{\lambda} - \lambda)^2}{(\lambda - \bar{\lambda})[\bar{\lambda}(\lambda + 4\lambda^3)] + |C_1|^2e^{2i(\lambda_1y + 4\lambda^3)}]. \] (4.4)

Let \( \lambda_1 = i\varepsilon, \varepsilon > 0 \) and, In order to conveniently study the properties of the one soliton solution, we choose \( C_1 = \pm 2i\varepsilon \). When \( C_1 = -2i\varepsilon \), substituting both parameters into the (4.4), it comes into being

\[ M_{12}(y, t, \lambda) = 2i\varepsilon e^{-2(\varepsilon y - 4\varepsilon^3 t)} \] (4.5)

Then,

\[-i \lim_{\lambda \to \infty} (\lambda M(y, t, \lambda))_{12} = -(\arctanh e^{-2(\varepsilon y - 4\varepsilon^3 t)})_{y} \]

where the \( \arctanh x \) is the inverse function of \( \tanh x \). Furthermore,

\[
\int_{y}^{\infty} m(y', t) dy' = -i \int_{y}^{\infty} \lim_{\lambda \to \infty} (\lambda M(y', t, \lambda))_{12} dy'
\]

\[
= - \int_{y}^{\infty} (\arctanh e^{-2(\varepsilon y - 4\varepsilon^3 t)})_{y'} dy'
\]

\[
= \arctanh e^{-2(\varepsilon y - 4\varepsilon^3 t)}. \] (4.6)

The solution \( q(x, t) \) in (3.7) can transforms into

\[ q(x, t) = e^{8\arctanh e^{-2(\varepsilon y - 4\varepsilon^3 t)}} - 1. \] (4.7)

Let \( \alpha(y, t) = e^{\arctanh e^{-2(\varepsilon y - 4\varepsilon^3 t)}} \), we find that \( L\alpha(y, t) = \arctanh e^{-2(\varepsilon y - 4\varepsilon^3 t)} \), then

\[ \tanh(L\alpha(y, t)) = e^{-2(\varepsilon y - 4\varepsilon^3 t)} \]

i.e.

\[
\frac{e^{L\alpha(y, t)} - e^{-L\alpha(y, t)}}{e^{L\alpha(y, t)} + e^{-L\alpha(y, t)}} = e^{-2(\varepsilon y - 4\varepsilon^3 t)},
\]

we deduce

\[ \alpha^2(y, t) = - \tanh^{-1}(-\varepsilon y + 4\varepsilon^3 t). \]
Equation (4.7) can be written as

\[ q(x,t) = (e^{\arctan h e^{-2(\varepsilon y - 4\varepsilon^3 t)}})^8 - 1 = \tanh^{-4}(-\varepsilon y + 4\varepsilon^3 t) - 1. \]  
(4.8)

Substituting \( y \) with \( x \), (4.8) becomes

\[ q(x,t) = \tanh^{-4}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon \gamma(x,t)) - 1 \]  
(4.9)

where \( \gamma(x,t) = \int_x^\infty (1 - \rho(\xi,t))d\xi, \rho(x,t) = \tanh^{-2}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon \gamma(x,t)) \). Then (4.9) can be varied as \((1 + q(x,t))^\frac{1}{2} - 1 = \cosh^2(-\varepsilon x + 4\varepsilon^3 t - \varepsilon \gamma(x,t))\), hence the one soliton solution \( q(x,t) \) has a singularity at the peak of the soliton so called cusp soliton.

When \( \lambda_1 = i\varepsilon \) and \( C_1 = 2i\varepsilon \), the corresponding one soliton solution \( q(x,t) \) of (1.2) can be expressed

\[ q(x,t) = \tanh^{-4}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon \gamma(x,t)) - 1 \]  
(4.10)

where \( \gamma(x,t) = \int_x^\infty (1 - \rho(\xi,t))d\xi, \rho(x,t) = \tanh^{-2}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon \gamma(x,t)) \).

**4.2 Remark** In this paper, we use the uniform method to obtain the solution \( q(x,t) \) of equation (1.2) expressed by the (4.9) and (4.10). While the [3] applies the inverse scattering method to get the solution \( q(x,t) \). If \( \varepsilon = \kappa(\kappa \text{ in the [3]} \), To the one soliton solution, when \( C_1 = -2i\varepsilon \), expression of the solution in both paper is similar, identically with \(-\varepsilon x + 4\varepsilon^3 t \) in the \( \tanh^{-4}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon \gamma(x,t)) \)

and \( \kappa x - 4\kappa^3 t \) in the \( \tanh^{-4}(\kappa x - 4\kappa^3 t - \kappa x_0 + \varepsilon_+) \) in [3]. There is different point that the expression of one soliton solution in the two papers, one is dependent of the \(-\varepsilon \gamma(x,t) \) of \( x \), the other is \(-\kappa x_0 + \varepsilon_+ \) of \( x \).
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