Abstract

We systematically classify $1/2$, $1/4$ and $1/8$ BPS equations in SUSY gauge theories in $d = 6, 5, 4, 3$ and $2$ with eight supercharges, with gauge groups and matter contents being arbitrary. Instantons (strings) and vortices (3-branes) are only allowed $1/2$ BPS solitons in $d = 6$ with $\mathcal{N} = 1$ SUSY. We find two $1/4$ BPS equations and the unique $1/8$ BPS equation in $d = 6$ by considering configurations made of these field theory branes. All known BPS equations are rederived while many new $1/4$ and $1/8$ BPS equations are found in dimension less than six by dimensional reductions.
1 Introduction

Bogomol’nyi-Prasad-Sommerfield (BPS) states\(^{1}\) are the most important ingredients for recent developments in non-perturbative aspects of supersymmetric (SUSY) field theory, string theory and M-theory. This is because they saturate lower energy bound called the Bogomol’nyi bound, break/preserve half supercharges, belong to short multiplets of supersymmetry, and therefore are non-perturbatively stable\(^{2}\). Their masses (tension) appear as central (tensorial) charges in the SUSY algebras. D-branes are such objects in string theory\(^{3}\). D-brane configurations made of various dimensional D-branes and NS5-branes are very powerful tools to study non-perturbative aspects of SUSY field theory realized on some D-branes\(^{4}\). These brane configurations can be obtained from configuration made of M5- and M2-branes in M-theory through various dualities. Therefore classifying possible configuration of M5- and M2-branes is important. Such classification including possible angles between branes was established in\(^{5}\).

In the field theory side, there exist four kinds of fundamental 1/2 BPS solitons\(^{6,7}\); instantons\(^{8,9}\), monopoles\(^{10,11,12}\), vortices\(^{13–17}\) and domain walls\(^{18–30}\), which are of co-dimension four, three, two and one, respectively.\(^{2}\) These BPS solitons can be regarded as “field theory branes”. In fact, recently, many kinds of 1/4 BPS composite states of several BPS solitons, resembling with D-brane configurations in string theory, have been found in SUSY gauge field theories (or hyper-Kähler sigma models) with eight supercharges in\(^{37–55}\). Further study of similarity between branes in string/M theory and branes in field theory is earnestly desired. However it has not been classified yet what kinds of composite solitons are allowed for BPS states in field theories with eight supercharges.

The purpose of this paper is systematically classifying possible 1/2, 1/4 and 1/8 BPS equations in SUSY gauge theories in\(^{d} = 6, 5, 4, 3, 2\) with eight supercharges, with gauge groups and matter contents being arbitrary. First of all we find that only instantons (strings) and vortices\(^{3–7}\)

\(^{1}\)Instantons are 1/2 BPS states if embedded into four-dimensional Euclidean space in\(^{d} = 5, \mathcal{N} = 2 (\mathcal{N} = 1)\) SUSY gauge theories with sixteen (eight) supercharges.

\(^{2}\)In addition to these fundamental solitons, there exist several associated solitons: they are an instanton and a monopole with electric flux, called a dyonic instanton\(^{31}\) and a dyon\(^{32}\), respectively, and stationary time-dependent lumps and kinks, called Q-lumps\(^{33}\) and Q-kinks\(^{20}\), respectively. Here lumps\(^{34}\) (equivalently sigma-model instantons\(^{35}\) in Euclidean two dimensional space) can be obtained from semilocal vortices\(^{36}\) in the strong gauge coupling limit.
Table 1: 1/4 BPS and 1/8 BPS configurations in $d = 6$. Here “○” denotes the world-volume directions of the solitons, whereas “×” denotes their codimensional directions. Throughout this paper, we use the following Roman uppercases to represent various solitons: I: instanton, M: monopole, H: Hitchin vortex, N: Nahm wall, V: vortex, W: domain wall. The indices on these letters imply that those solitons extend to various directions.

| 1/4 IVV | 0 | 1 | 2 | 3 | 4 | 5 |
|---------|---|---|---|---|---|---|
| Instanton | ○ | × | × | × | × | ○ |
| Vortex | ○ | × | × | ○ | ○ | ○ |
| Vortex | ○ | ○ | ○ | × | × | ○ |

| 1/4 VVV | 0 | 1 | 2 | 3 | 4 | 5 |
|---------|---|---|---|---|---|---|
| Vortex | ○ | ○ | × | × | ○ | ○ |
| Vortex | ○ | × | ○ | × | ○ | ○ |
| Vortex | ○ | × | × | ○ | × | ○ |
| Vortex | ○ | × | ○ | × | ○ | ○ |

| 1/8 IV$^6$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----------|---|---|---|---|---|---|
| Instanton | ○ | × | × | × | × | ○ |
| Vortex | ○ | ○ | × | × | ○ | ○ |
| Vortex | ○ | × | ○ | × | ○ | ○ |
| Vortex | ○ | × | × | ○ | × | ○ |
| Vortex | ○ | ○ | × | × | × | ○ |
| Vortex | ○ | ○ | × | × | ○ | ○ |

Branes) are allowed for 1/2 BPS states in $d = 6$, by classifying projection operators on fermions. Second, intersection rules for orthogonal instanton-strings and vortex-3-branes are obtained. By considering orthogonally intersecting field theory branes, we find that all possible 1/4 and 1/8 BPS field theoretical brane configurations in $d = 6$ are summarized in Table 1. Third, we derive two 1/4 BPS equations and the unique 1/8 BPS equation in $d = 6$ corresponding to these brane configurations. Fourth, all BPS equations in dimensions less than six are derived from them by the Scherk-Schwarz and/or ordinary dimensional reductions.

---

3We first assume minimal kinetic term for hypermultiplets. We can consider hyper-Kähler nonlinear sigma models whose target spaces are hyper-Kähler quotients of the flat spaces, by taking strong gauge coupling limit. BPS equations for these associated hyper-Kähler nonlinear sigma models are also available.

4$d = 6, \mathcal{N} = 1$ SUSY reduces to $d = 5, \mathcal{N} = 1; d = 4, \mathcal{N} = 2; d = 3, \mathcal{N} = 4$ or $d = 2, \mathcal{N} = (4,4)$ SUSY.
known BPS equations are rederived for particular choice of gauge group and representations of matter hypermultiplets, while several new 1/4 and 1/8 BPS equations are found. Some of these equations describe monopoles and/or domain walls also. Therefore all the field theory branes are unified in \( d = 6 \) into instanton-strings and vortex-3-branes, as if D-branes, fundamental string and NS5-branes are unified into M2- and M5-branes in eleven dimensional M-theory.

We also discuss electrically charged solitons. They are new solitons traveling with the speed of light in one direction in \( d = 6 \), which we call wavy solitons. When dimensionally reduced along that direction, they become stationary solitons in dimensions less than six. They are dyonic solitons including dyonic instantons \([31]\) (dyons \([32]\)) in \( d = 5 \) (\( d = 4 \)).

When 1/2 BPS instantons, monopoles and Hitchin-type vortices (in the Hitchin system) are constructed in systems coupled with the Higgs fields, they become 1/4 BPS composite states of instantons accompanied with vortices, those of monopoles with vortices and domain walls, and those of Hitchin vortices with domain walls, respectively. In Abelian gauge theory, they carry negative instanton, monopole and Hitchin charges and these charges contribute negatively to the total energy at intersections of solitons. The first one was found in \([13]\) and was called an “intersecton”. The second one was found in \([41]\) and later was called a “boojum” \([42]\). The last one is a domain wall junction with junction charge (interpreted as the Hitchin charge) \([50]\). In this paper we point out that these 1/4 BPS states are related by dimensional reductions, similarly to the well-known descent relation between instantons, monopoles and Hitchin vortices.

In this paper we discuss only projections orthogonal to each other, implying orthogonal branes (in the absence of fluxes).\(^5\) We do not discuss angles between branes as was done in \([5]\) for M2- and M5-branes in M-theory. We leave it as a future problem. We believe that this paper provides a guide for pursuing similarity between branes in string/M-theory and field theory branes.

This paper is organized as follows. In Sect. 2, we present Lagrangian in \( d = 6 \) and sets of projection operators for 1/2, 1/4 and 1/8 BPS states. Sect. 3, 4 and 5 deal with 1/2, 1/4 and 1/8 BPS states, respectively. Sect. 6 is devoted to conclusion and discussion.

Before closing the introduction we summarize previously known composite BPS solitons in SUSY gauge theories and hyper-Kähler nonlinear sigma models with eight supercharges:

\(^5\)In the presence of magnetic fluxes they can have angle even though projections are orthogonal \([41, 50]\). We do not exclude this possibility of angles between branes in this paper.
Lump-strings ending on a wall was found in a hyper-Kähler sigma model \[37\]. It was promoted to a vortex-string ending on a wall in \(U(1)\) gauge theory \[38\]. These composite states are 1/4 BPS.

A monopole in the Higgs phase is attached by vortices. The total system becomes a composite 1/4 BPS state of a monopole and vortices \[39\].

Even if the above [1/4 WV] and [1/4 VM] are simultaneously considered, a composite state of walls, vortices and monopoles is still 1/4 BPS \[41\]. Moduli were completely determined and all the exact solutions were obtained in the strong gauge coupling limit (reducing to hyper-Kähler sigma model).

A monopole in the Higgs phase is attached by vortices. The total system becomes a composite 1/4 BPS state of a monopole and vortices \[39\].

Even if the above [1/4 WV] and [1/4 VM] are simultaneously considered, a composite state of walls, vortices and monopoles is still 1/4 BPS \[41\]. Moduli were completely determined and all the exact solutions were obtained in the strong gauge coupling limit (reducing to hyper-Kähler sigma model).

Instanton-particles in \(d = 5\) are stabilized in the Higgs phase, if they are supported by a vortex sheet \[43\]. The 1/4 BPS equation for this also contains intersecting vortices with intersecting point carrying an instanton charge. Dimensional reductions of this equation to \(d = 4, 3, 2\) reduce to 1/2, 1/4 BPS equations known so far, except for [1/4 VVV] below.

Walls were found to constitute a junction as 1/4 BPS states in \(d = 4, \mathcal{N} = 1\) SUSY theories \[46\]. A wall junction carries a junction charge in addition to two wall charges. Some exact solutions are known \[47\]. As first example in theories with eight supercharges, intersecting walls were found in a hyper-Kähler nonlinear sigma model \[48\]. A \(Z_3\) symmetric wall junction was constructed in \(d = 4, \mathcal{N} = 2\) \(U(1)\) gauge theory \[49\]. All solutions of wall junctions, called wall webs, have been recently found \[50, 51\] where all the exact solutions have been obtained in the strong gauge coupling limit.

A set of 1/4 BPS equations for triply intersecting lump-strings was found in \(d = 4, \mathcal{N} = 2\) hyper-Kähler nonlinear sigma models \[54\]. It was lifted in \[55\] up to intersecting lump-membranes in \(d = 5, \mathcal{N} = 1\) theory. Promoting them to gauge theories have not been done yet. Only this set of equations cannot be obtained from the above [1/4 IVV] by dimensional reduction.

Only one set of 1/8 BPS equations has been known so far in theories with eight supercharges. It is time dependent extension of the above [1/4 VVV], describing triply intersecting Q-lump-strings (membranes) \[54, 55\].

Note Added. When we were preparing the manuscript of this paper we were informed from Kimyeong Lee and Ho-Ung Yee that there exist some overlap between their paper \[57\] and
this paper. They also derived 1/8 BPS equations. They constructed a perturbative solution for one of 1/8 BPS equations and discussed 1/4 BPS dyonic solitons in \( d = 4 \).

2 Models and SUSY Projection Operators

2.1 Models

We begin with the 6 dimensional supersymmetric gauge theory with eight supercharges (\( \mathcal{N} = 1 \) in 6 dimensions). The SUSY multiplets which we will consider in this paper are vector multiplets and hypermultiplets. The physical fields contained in the vector multiplet are the gauge field \( W_M \) (\( M = 0, 1, 2, 3, 4, 5 \)) and the gaugino \( \lambda^i \). The gaugino \( \lambda^i \) belongs to a doublet of the \( SU(2)_R \) \((i = 1, 2)\) and is the \( SU(2) \)-Majorana Weyl spinor, namely \( \gamma_7 \lambda^i = \lambda^i \) and \( \lambda^i = C\varepsilon^{ij}(\bar{\lambda}_j)^T \). Here \( \gamma_7 \) is defined by \( \gamma_7 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \) and \( C \) is the charge conjugation matrix in 6 dimensions. The physical fields in the hypermultiplets are scalar fields \( H^A_i \) which are \( SU(2)_R \) doublets and the hyperinos \( \psi^A \) (Dirac spinor) whose chirality is \( \gamma_7 \psi^A = -\psi^A \). The indices \( A, B, \cdots \) stand for both the gauge group and the flavor symmetry. Then the bosonic Lagrangian with the minimal kinetic terms in 6 dimensions is given by

\[
\mathcal{L}_6 = -\frac{1}{4g_I^2} F_{IMN}^I F_{LMN}^I + (\mathcal{D}_M H^{iA})^* \mathcal{D}^M H^{iA} - \frac{1}{2g_I^2} (Y^I_a)^2, \tag{2.1}
\]

where we define

\[
Y^I_a = g_I^2 \left[ \zeta_a^I - (H^{iA})^* (\sigma_a)^i_j (T_I)^A_B H^{jB} \right]. \tag{2.2}
\]

We imply the summation over repeated indices. Here \( I \) is the index for generators \( T_I \) of the gauge group, and \( g_I \) denotes the gauge coupling constant taking the same value for each group in the product gauge group. In our conventions, a covariant derivative and a field strength are given by \( \mathcal{D}_M = \partial_M + iW_M \) and \( F_{MN} = -i[\mathcal{D}_M, \mathcal{D}_N] \) (with \( X = X^I T_I \)), respectively. Real constants \( \zeta^I_a \) \((a = 1, 2, 3)\), called the Fayet-Iliopoulos (FI) parameters, can have nonzero values only for \( I \) corresponding to the Abelian subgroups. Masses for hypermultiplets are prohibited in \( d = 6 \). (They can have masses in lower dimensions as shown below.)

\[\text{Our convention for the metric is } \eta_{MN} = \text{diag}(+, -, \cdots, -). \text{ The gamma matrices satisfy the Clifford algebra } \{\gamma_M, \gamma_N\} = \eta_{MN} \text{ and the charge conjugation matrix is defined by } C^{-1} \gamma_M C = \gamma^T_M \text{ and } C^T = -C.\]
The SUSY transformations for the spinor fields in 6 dimensions are given by

\[
\delta \varepsilon^i = \frac{1}{2} \gamma^{\mu \nu} F_{\mu \nu} \varepsilon^i + Y_a (i \sigma_a)^i_j \varepsilon^j, \quad \delta \psi = -\sqrt{2} i \gamma^\mu D_\mu H^i \varepsilon^i. \tag{2.3}
\]

Here, a parameter \( \varepsilon^i \) for the SUSY transformation is an \( SU(2) \) Majorana-Weyl spinor which satisfies \( \varepsilon^i = C \varepsilon^i_j (\bar{\varepsilon}^j) \), \( \gamma_7 \varepsilon^i = + \varepsilon^i \).

We can obtain the \( d(<6) \) dimensional supersymmetric gauge theory with eight supercharges, by performing the Scherk-Schwarz (SS) and/or the trivial dimensional reductions \( (6-d) \)-times from the 6 dimensional theory, after compactifying the \( p \)-th \( (p=5, 4, \cdots, d) \) direction to \( S^1 \) with radius \( R_p \). We consider the twisted boundary condition for the dimensional reduction along the \( x^p \)-direction, given by

\[
H^i A(x^\mu, x^p + 2\pi R_p) = H^i A(x^\mu, x^p) e^{i \alpha_{pA}}, \quad (|\alpha_{pA}| \ll 2\pi), \tag{2.4}
\]

where \( \mu(=0, \cdots, d-1) \) is a spacetime index in \( d \) dimensions. If we consider effective action at sufficiently low energy, we can ignore the infinite towers of the Kaluza-Klein modes and have the lightest fields as functions of the \( d \) dimensional spacetime coordinates

\[
W_\mu(x^\mu, x^p) \to W_\mu(x^\mu), \quad W_p(x^\mu, x^p) \to -\Sigma_p(x^\mu), \tag{2.5}
\]

\[
H^i A(x^\mu, x^p) \to \frac{1}{\prod_p \sqrt{2\pi R_p}} H^i A(x^\mu(x^p)) \exp \left( i \sum_p m_p A x^p \right), \quad m_p \equiv \frac{\alpha_{pA}}{2\pi R_p}. \tag{2.6}
\]

Integrating the 6 dimensional Lagrangian in Eq.\((2.1)\) over the \( x^p \)-coordinates, we obtain the \( d \) dimensional Lagrangian

\[
\mathcal{L}_d = \frac{-1}{4 g_7^2} F_{\mu \nu}^I F_{\mu \nu}^I + \frac{1}{2 g_7^2} \mathcal{D}_\mu \Sigma_p^I \mathcal{D}^\mu \Sigma_p^I + (\mathcal{D}_\mu H^{i A})^* \mathcal{D}^\mu H^{i A} + \frac{1}{4 g_7^2} [\Sigma_p, \Sigma_q]^I \right)^2 - \frac{1}{2 g_7^2} (Y_a^I)^2 - |H^{i A} m_{p A} - \Sigma_p^I (T_I)^A_B H^{i B}|^2, \tag{2.7}
\]

where we have redefined the gauge couplings and the FI parameters in \( d \) dimensions from 6 dimensions as \( g_7^2 \to \left( \prod_p 2\pi R_p \right) g_7^2, \quad \zeta^I_a \to \zeta^I_a \left( \prod_p 2\pi R_p \right) \). By construction one finds that there exist \( (6-d) \) real mass parameters for hypermultiplets in \( d \) dimensions.

In this paper we are mainly interested in composite states of solitons living inside the Higgs branches of gauge theories with eight supercharges, in particular when we consider vortices and/or domain walls. Therefore we prepare sufficiently large number of hypermultiplets. For the massless hypermultiplets, the Higgs branches are hyper-\(K\)ähler quotient \([56]\) given by \( \mathcal{M}_{\text{vac}} = \)
\{H^I_a | Y^I_a = 0\}/G$ where $G$ denotes the gauge group. The real dimension of the Higgs branch is
$$\dim \mathcal{M}_{\text{vac}} = 4(n_H - \dim G)$$
with $n_H$ the number of hypermultiplets. When we introduce masses in lower dimensions by the SS dimensional reductions, the Higgs branch of vacua is lifted by masses except for fixed points of the $U(1)$ Killing vectors \[58\] induced by the $U(1)$ actions in Eq. \[2.4\]. In the strong gauge coupling limit, the model reduces to a hyper-Kähler nonlinear sigma model on the hyper-Kähler manifold $\mathcal{M}_{\text{vac}}$ with a potential given by the square of the Killing vectors \[58\].

A typical model often considered so far is $U(N_C)$ gauge theory coupled to $N_F (\geq N_C)$ hypermultiplets in the fundamental representation with one triplet of FI parameters. There exists the unique Higgs vacuum for $N_F = N_C$, called the color-flavor locking phase, up to gauge transformation. Massless vacua for $N_F > N_C$ are given by the cotangent bundle over the complex Grassmann manifold $G_{N_F,N_C} \simeq SU(N_F)/[SU(N_C) \times SU(N_F-N_C) \times U(1)] \[56\]$. Introducing non-degenerate masses, only $N_F!/[(N_C! \times (N_F-N_C)!)$ discrete points remain as vacua \[59\]. In the strong gauge coupling limit, the model reduces to the massive hyper-Kähler nonlinear sigma model on $T^*G_{N_F,N_C}$. Another model often considered is a hyper-Kähler nonlinear sigma model on the ALE space. It can be obtained in the strong coupling limit from $U(1)^N$ gauge theory coupled to $(N+1)$ hypermultiplets with suitable $U(1)$ charges and $N$ triplets of FI-parameters \[60\]. The $N$ centers in the ALE space are vacua for massive case. In \[28\] we have discussed domain walls in the hyper-Kähler nonlinear sigma model on the cotangent bundle over the Hirzebruch surface $F_n$, which can be obtained by $U(1)^2$ gauge theory coupled to four hypermultiplets with suitable $U(1)^2$ charges and with two parallel triplets of FI-parameters.

### 2.2 Projection operators

There exist several BPS solitons in the supersymmetric gauge theories. The typical property of BPS solitons is that they partially preserve supersymmetry. The most elementary objects are 1/2 BPS solitons preserving a half of eight supercharges. They are characterized by supercharges which solitons preserve. An $SU(2)$ Majorana-Weyl spinor parameter $\varepsilon$ for unbroken SUSY is specified by the equation

$$\gamma^i \varepsilon_i = \pm \varepsilon, \quad \text{Tr}(\gamma) = 0,$$  \hspace{1cm} (2.8)
where \( \Gamma \) is an operator satisfying \( \Gamma^2 = 1 \otimes 1 \). The traceless condition is required for choosing a half of supercharges by operator \( \Gamma \). Since the SUSY transformation parameter \( \epsilon^i \) is the \( SU(2) \) Majorana-Weyl spinor, \( \Gamma \) has to satisfy the following two conditions for consistency,

\[
B^{-1}\Gamma B = \Gamma^*, \quad [\gamma_7, \Gamma] = 0,
\]

with \( B^{ij} \equiv \gamma^0 C \epsilon^{ij} \). If a soliton with codimension \( D \) exists in 6 dimensions, the Lorentz symmetry \( SO(1, 5) \) is broken to \( SO(1, 5-D) \otimes SO(D) \). Therefore the operator \( \Gamma \) has to be invariant under \( SO(1, 5-D) \otimes SO(D) \) transformation. From the second condition in Eq. (2.9) the operator \( \Gamma \) has to consist of a product of even number of the \( \gamma^M \) matrices. Therefore, the codimension \( D \) of the 1/2 BPS solitons must be even in 6 dimensions. Combining this fact with the first condition in Eq. (2.9), we find that possible operators are given by

\[
\text{type IVa : } \Gamma = \gamma^{nm} i \sigma_a, \\
\text{type IVb : } \Gamma = \gamma^{0n},
\]

where \( n, m = 1, 2, \ldots, 5 \) and \( a = 1, 2, 3 \). As we will see in the following section, the first operator which we call type I Va implies existence of 1/2 BPS vortices of codimension two in the \( x^n-x^m \) plane, whose world-volume extends to the time direction and the four space directions except for \( x^n \) and \( x^m \). Vortices are 3-branes in \( d = 6 \). On the other hand the second operator which we call type IV b implies existence of 1/2 BPS instantons of codimension four, whose world-volume extends to the time direction and one space direction \( x^n \). Instantons are strings (1-branes) in \( d = 6 \). Thus we conclude that the elementary 1/2 BPS solitons in 6 dimensions are vortices and instantons, which are 3-branes and strings, respectively.

When there exist two or more 1/2 BPS solitons preserving 1/2 supercharges different from each other, the composite state of these solitons breaks more than four supercharges out of eight. If we impose different 1/2 BPS projection operators for the Killing spinors with general angles, the SUSY will be completely broken. However, composite solitons determined by orthogonal 1/2 BPS projection operators can preserve 1/4 or 1/8 SUSY, like orthogonal D-brane/M-brane configurations in superstring/M theory. In this article we concentrate on composite states of the orthogonal 1/2 BPS solitons. With respect to the projection operator \( \Gamma \) for the supertransformation parameter in Eq. (2.8), there exist two sets of \( \Gamma \)’s for 1/4 BPS states.

One type which we call type IIa is a set of three projections

\[
\text{type IIa : } \Gamma = \{ \gamma^{12} i \sigma_3, \gamma^{23} i \sigma_1, \gamma^{31} i \sigma_2 \}
\]
implying three orthogonal vortex-3-branes. The projections in (2.12) correspond to vortex-3-branes in the $x^1$-$x^2$ plane whose world-volume extends to the $x^0$, $x^3$, $x^4$ and $x^5$-directions, vortex-3-branes in the $x^2$-$x^3$ plane extending to the $x^0$, $x^1$, $x^4$ and $x^5$-directions, and vortex-3-branes in the $x^3$-$x^4$ plane extending to the $x^0$, $x^2$, $x^4$ and $x^5$-directions, respectively (see the second table in p.3). Notice that all projections in (2.12) commute each other (for example $[\gamma^{12} i\sigma_3, \gamma^{23} i\sigma_1] = 0$) and that the product of all of them is proportional to identity ($\gamma^{23} i\sigma_1 \cdot \gamma^{31} i\sigma_2 \cdot \gamma^{12} i\sigma_3 = -1$). These facts imply that the set in Eq. (2.12) project out 3/4 of supercharges, so that composite states of the orthogonal three vortices are 1/4 BPS states in 6 dimensions.

The other set of operators $\Gamma$’s which we call type IIb is given by

$$\text{type IIb : } \Gamma = \{\gamma^{05}, \gamma^{12} i\sigma_3, \gamma^{34} i\sigma_3\}.$$  \hspace{1cm} (2.13)$$

This set of operators leads to 1/4 BPS composite states of 1/2 BPS instanton-strings in the $x^1$-$x^2$-$x^3$-$x^4$ space extending to the $x^0$ and $x^5$-directions, vortex-3-branes in the $x^1$-$x^2$ plane extending to $x^0$, $x^3$, $x^4$, $x^5$-directions and vortex-3-branes in the $x^3$-$x^4$ plane extending to the $x^0$, $x^1$, $x^2$, $x^5$-directions (see the first table in p.3).

Finally, we obtain the unique set of operators $\Gamma$’s for 1/8 BPS composite states, which we call the type I:

$$\text{type I : } \Gamma = \{\gamma^{05}, \gamma^{12} i\sigma_3, \gamma^{34} i\sigma_3, \gamma^{23} i\sigma_1, \gamma^{14} i\sigma_1, \gamma^{31} i\sigma_2, \gamma^{24} i\sigma_2\}.$$ \hspace{1cm} (2.14)$$

These imply instanton-strings and vortex-3-branes in various directions (see the third table in p.3). We can easily show that all of these commute each other and that only three of them are independent operators; 1/8 BPS is realized by taking arbitrary set of three projections except for the sets in type IIa/IIb, for example $\gamma^{05}, \gamma^{12} i\sigma_3$ and $\gamma^{23} i\sigma_1$.

To close this section, it is important to study properties of surviving supercharges under the above projections. It determines SUSY in the effective theory on the world-volume of solitons, and which geometry (Riemann, Kähler or hyper-Kähler) is realized as the moduli space of solitons. In two-dimensional space-time $x^0$-$x^5$, chirality of supercharges can be determined by $\gamma^{05}$ which is in a reducible representation. Thus it is convenient to classify the above projections by eigenvalues of $\gamma^{05}$ on the projected spaces, even in cases that world-volume of solitons is not two dimensional. In the cases of type IVb and type IIb, these sets of projection operators contain just $\gamma^{05}$, and thus, supercharges projected by these operators are all chiral (or anti-chiral). On the other hand, in
the case of type I Va and type I Ib, numbers of surviving chiral and anti-chiral supercharges under these projections are the same. This can be proved by taking a trace of $\gamma^5$ on a projected space of the Killing spinor, for instance in type I Va,

$$\text{Tr} \left( \gamma^5 P \right) = 0, \quad \text{with} \quad P = \frac{1 \pm \gamma^1 \sigma_3}{2} + \gamma_7.$$  \hspace{1cm} (2.15)

Therefore, if we consider solitons with two-dimensional world volume $x^0 - x^5$, a two-dimensional effective action on such solitons has the following supercharges

- type I Va: $N = (2, 2)$
- type I Vb: $N = (4, 0)$
- type IIa: $N = (1, 1)$
- type IIb: $N = (2, 0)$
- type I: $N = (1, 0)$  \hspace{1cm} (2.16)

respectively.

### 3 1/2 BPS Systems

In this section we will derive diverse 1/2 BPS equations of solitons (instantons, monopoles, vortices and domain walls and so on) in $d = 6, 5, 4, 3, 2, 1$ dimensions. Since there are two kinds of 1/2 BPS projections, the type I Va (vortices) and the type I Vb (instantons) in 6 dimensions as we explained in Eqs. (2.10) and (2.11) in the previous section, 1/2 BPS systems in all dimensions will be classified into two series.

In subsections 3.1 and 3.2 we will first begin with the 1/2 BPS equations for instantons and vortices in 6 dimensions, respectively, and then derive the 1/2 BPS equations for other solitons via the trivial dimensional reductions and/or the Scherk-Schwarz dimensional reductions.

#### 3.1 Series of the type IVb

Let us start with the 1/2 BPS equations which are derived by a half of supercharges specified by the type IVb operator (2.11) in 6 dimensions

$$\gamma^5 \varepsilon^i = \gamma^{1234} \varepsilon^i = \eta \varepsilon^i, \quad \eta = \pm 1.$$  \hspace{1cm} (3.1)

On the space projected by these Killing conditions, the representation of gamma matrices is reducible and the matrices $\gamma^5$ and $\gamma^4$ can be rewritten as $\eta \gamma^0$ and $\eta \gamma^{123}$, respectively. The
corresponding 1/2 (anti-) BPS equations can be derived by imposing the 1/2 SUSY condition \((3.1)\) to the supersymmetric transformation law \((2.3)\). The SUSY transformation laws \((2.3)\) for the gaugino \(\lambda^i\) and hyperino \(\psi^A\) can be rewritten by using the Killing conditions \((3.1)\) as

\[
\delta \lambda^i = \left[ \frac{1}{2} \sum_{n,m,l=1}^3 (F_{nm} - \eta \epsilon_{nmkl} F_{kl}) \gamma^{nm} + \sum_{n=1}^4 (F_{0n} + \eta F_{5n}) \gamma^0 \eta + \eta F_{05} \right] \varepsilon^i + Y_a (i \sigma_a)^i_j \varepsilon^j, \quad (3.2)
\]

\[
\delta \psi^A = -\sqrt{2} i \left[ \gamma^0 (D_0 + \eta D_5) H^{iA} + \gamma^5 D_n H^{iA} \right] \epsilon_{ij} \epsilon^j, \quad (3.3)
\]

respectively. By imposing \(\delta \lambda^i = 0\) and \(\delta \psi^A = 0\), we find 1/2 BPS equations

\[
F_{nm} = \frac{\eta}{2} \epsilon_{nmkl} F_{kl}, \quad Y_a = 0, \quad D_n H^i = 0, \quad (3.4)
\]

\[
F_{0n} + \eta F_{5n} = 0, \quad F_{05} = 0, \quad (D_0 + \eta D_5) H^i = 0, \quad (3.5)
\]

with \(n,m,k,l = 1, \cdots, 4\). As discussed in subsection \(3.1.2\) when we consider solitons with non-trivial configurations of \(W_0^I\) or time dependent solutions (instanton-string with traveling wave), we need to take into account the Gauss law

\[
\frac{1}{g_I^2} D_n F_{0m}^I = 2 i (T_I)^A_B H^{iB} \leftrightarrow D_0 (H^{iA})^*, \quad (3.6)
\]

in addition to the above BPS equations.

In the following, subsection \(3.1.1\) is devoted to a review whereas subsection \(3.1.2\) is new.

### 3.1.1 Instantons and their descendants

Let us review well-known BPS equations for instantons and their descendants obtained by dimensional reductions. Topological charges for these solitons emerge in also 1/4, 1/8 BPS states as explained in the following sections. Especially, these charges exist even in Abelian gauge theories and have negative contributions to the energy of those systems, where those negative charges should be interpreted as binding energy of composite solitons.

**Instantons:** Let us first consider time-independent (static) BPS solutions in six dimensions, by ignoring the \(x^5\)-dependence and by setting \(W_0 = W_5 = 0\) in Eq.\((3.3)\), namely by considering only Eq.\((3.4)\). The third equation in Eq.\((3.4)\) implies \([D_n, D_m] H^{iA} = i (F_{nm})^A_B H^{iB} = 0\), that is, instanton configurations require that charged matter should vanish. This is consistent with the extended Derrick’s theorem \([6]\) which insists that there are no finite energy solutions whose codimension is 4 when the scalar fields have nonzero vacuum expectation value. Notice that the
FI parameters $\zeta^I_a$ should be turned off for consistency between $Y^a = 0$ and $H^{iA} = 0$. Instantons become essentially the same with those in the SUSY gauge theory with sixteen supercharges by adding adjoint hypermultiplets implicitly. Thus Eqs. (3.4) reduce to the ordinary (anti) self-dual (SDYM) equations of 1/2 BPS instantons

$$F_{nm} = \frac{\eta}{2} \epsilon_{nmkl} F_{kl}. \quad (3.7)$$

The solutions of the (anti) SDYM equations saturate the BPS energy bound

$$E_m = \eta \mathcal{I}_{1234},$$
$$\mathcal{I}_{1234} = \int d^4 x \frac{1}{4 g_I^2} \epsilon_{nmkl} F^I_{nm} F^I_{kl} = \frac{8 \pi^2}{g_I^2} k^I, \quad (3.8)$$

with $m, n, k, l = 1, 2, 3, 4$. Here $E_m$ is energy for static solutions of instanton (we ignore the volume of instanton-string along the $x^5$-direction), and $\mathcal{I}_{pqr}$ denotes the charge of the instanton localized in the $x^p-x^q-x^r-x^s$ space. As we will see below soon, the instanton charge $\mathcal{I}_{mnkl}$ reduces to the charges of other solitons when we perform dimensional reduction several times. We denote the instanton charge after the dimensional reduction along the $x^k$-direction as $\mathcal{I}_{mn\check{k}}$, with the check "\check{}" on the index implying to omit that index.

**Monopoles**: By performing trivial dimensional reduction along the $x^4$-coordinate, the (anti) SDYM equations for the 1/2 BPS instantons reduce to the 1/2 BPS equations of the monopoles

$$\frac{1}{2} \epsilon_{nml} F_{ml} = - \eta \mathcal{D}_n \Sigma_4, \quad (n, m, l = 1, 2, 3) \quad (3.9)$$

This is well-known Bogomol’nyi equation for BPS monopole, whose solutions saturate the BPS energy bound

$$E_m = \eta \mathcal{M}_{123} \geq 0,$$
$$\mathcal{M}_{123} = \mathcal{I}_{1234} = - \int d^3 x \frac{1}{g_I^2} \epsilon_{nmk} F^I_{nm} \mathcal{D}_k \Sigma_4^I = - \int d^3 x \frac{1}{g_I^2} \epsilon_{nmk} \partial_k (F^I_{nm} \Sigma^I_4), \quad (3.10)$$

with $n, m, k = 1, 2, 3$.

**The Hitchin system**: By performing trivial dimensional reduction further along the $x^2$-direction, the 1/2 BPS equations (3.9) of the monopoles reduce to the 1/2 BPS equations for the Hitchin system, given by

$$F_{13} = - \eta [\Sigma_2, \Sigma_4], \quad \mathcal{D}_1 \Sigma_2 = \eta \mathcal{D}_3 \Sigma_4, \quad \mathcal{D}_1 \Sigma_4 = - \eta \mathcal{D}_3 \Sigma_2. \quad (3.11)$$
Solutions saturate the BPS bound

\[ E_m = \eta H_{13} \geq 0, \]

\[ H_{13} \equiv \mathcal{I}_{1234} = \int d^2x \frac{4}{g_f^2} \partial_1 ((D_3 \Sigma_4^I) \Sigma_2^I). \]  

(3.12)

We call solitons carrying this topological charge with codimension two as “Hitchin vortices”. It is known that finite energy solutions for the Hitchin system exist in compact spaces but not in non-compact spaces, because of divergence of energy \[61\]. However finite energy solutions (as domain wall junctions) exist in the Higgs phase even in a non-compact space \[50, 51\].

The Nahm equation (dual monopole): The 1/2 BPS equations for the Hitchin system reduce to

\[ D_1 \Sigma_p = -\frac{\eta}{2} \epsilon_{pqr} \Sigma_q \Sigma_r, \quad p, q, r = 2, 3, 4 \]  

(3.13)

by trivial dimensional reduction along the \( x^3 \)-coordinate. This is called the (1/2 BPS) Nahm equation. The BPS bound is obtained as

\[ E_m = \eta N_1 \geq 0, \]

\[ N_1 \equiv \mathcal{I}_{1234} = \int dx^1 \frac{1}{3g_f^2} \epsilon_{pqr} \partial_1 \left( \Sigma_p^I \Sigma_q \Sigma_r^I \right), \]  

(3.14)

with \( p, q, r = 1, 2, 3 \). We call solitons carrying this topological charge with codimension one as “Nahm walls”. The energy of solutions diverge again in non-compact space, but we do not know at this stage if there exist finite energy solutions as the Hitchin system in the Higgs phase.

Equations in zero dimension: If we fully reduce the codimension for instantons, we obtain reduced ADHM-like equations

\[ [\Sigma_p, \Sigma_q] = \frac{\eta}{2} \epsilon_{pqr} \Sigma_r, \quad (p, q, r, s = 1, \ldots, 4). \]  

(3.15)

One might consider that if this equation would admit a solution it described a space-time filling brane in \( d = 1 \) or \( 2 \) and gave a model for partial breaking of SUSY \[62\] in that dimension. Unfortunately, this equation, however, has no non-trivial solution, \( [\Sigma_p, \Sigma_q] \neq 0 \), because the Bogomol’nyi bound vanishes in this case. This can be confirmed directly by

\[ E_m \propto \epsilon_{pqr} \text{Tr} \{ [\Sigma_p, \Sigma_q] [\Sigma_r, \Sigma_s] \} = -\epsilon_{pqr} \text{Tr} \{ \Sigma_q [\Sigma_p, [\Sigma_r, \Sigma_s]] \} = 0. \]  

(3.16)

In this paper, we ignore the infinite tower of Kaluza-Klein modes as explained before. However if we would consider their contributions, we could obtain the ADHM equations as dual of four-dimensional SDYM equations.
3.1.2 Solitons with electric flux

*Wavy solitons*: Let us next consider the solitons with the electric flux, by taking Eq. (3.5) into account in addition to Eq. (3.4). Since $H^{iA}$ has to vanish, Eq. (3.5) reduces under dimensional reductions, taken $(6 - d)$-times except for $x^5$, to

$$ F_{0n} + \eta F_{5n} = 0, \quad D_0 \Sigma_p + \eta D_5 \Sigma_p = 0, \quad F_{05} = 0, \quad (3.17) $$

with $p = d - 1, \cdots, 4$ and $n = 1, \cdots, d - 2$. For any static instanton (monopole, Hitchin vortex or Nahm wall) background configurations $W_{\text{static}}^n$ and $\Sigma_{\text{static}}^p$, these equations reduce with an appropriate gauge choice to

$$ \partial_0 + \eta \partial_5 = 0, \quad W_0 + \eta W_5 = 0. \quad (3.18) $$

General solutions for the first equation can be obtained by promoting arbitrary moduli parameters $\phi^i$ of solutions for the static BPS equations to arbitrary functions of $x^0 - \eta x^5$:

$$ \phi^i \to \phi^i(x^0 - \eta x^5). \quad (3.19) $$

This solution indicates that waves on BPS solitons propagate with the speed of light along the $x^5$-direction, to which those solitons extend, and oscillation of magnetic objects induce oscillation of electro-magnetic waves. In the case of time-dependent solitons, one should solve the Gauss law

$$ D_n F_{0n} + i \sum_{p=d-1}^4 [\Sigma_p, D_0 \Sigma_p] = 0, \quad (3.20) $$

in addition to the BPS equations. This equation should be solved with respect to $W_0$ ($W_5$). Let us consider the case that the moduli parameters in (3.19) are taken to be the center of mass (translational zero modes) $\phi^m$ ($m = 1, \cdots, d - 1$). We thus have

$$ W_n(x^0, x^5, x^m) = W_{\text{static}}^n(x^m - \phi^m(x^0 - \eta x^5)), \quad (3.21) $$

$$ \Sigma_p(x^0, x^5, x^m) = \Sigma_{\text{static}}^p(x^m - \phi^m(x^0 - \eta x^5)), \quad (3.22) $$

with $n, m = 1, \cdots, d - 2$ and $p, q = d - 1, \cdots, 4$, where the second equation is not needed for instantons in $d = 6$. In this case, we find that the Gauss law (3.20) is automatically satisfied by setting

$$ W_0(-\eta W_5) = -\frac{d\phi^m}{dx^0} W_m. \quad (3.23) $$
The energy of wavy configurations is given by
\[ E = -\eta S_5 + \int dx^5 E_m (\geq 0), \tag{3.24} \]
where the fifth component of the Poynting vector \( S_5 \) defined by
\[ S_5 = \frac{1}{g_f^2} \int d^{d-1}x \left( F_{0n}^I F_{5n}^I + \sum_{p=d-1}^4 D_0 \Sigma_p^I D_5 \Sigma_p^I \right) \]
\[ = -\frac{\eta}{2g_f^2} \int d^{d-1}x \left[ (F_{0n}^I)^2 + (F_{5n}^I)^2 + \sum_{p=d-1}^4 (D_0 \Sigma_p^I)^2 + (D_5 \Sigma_p^I)^2 \right] \tag{3.25} \]
represents the energy excited by waves on solitons.

**Dyonic solitons:** When we perform trivial dimensional reduction from equations for wavy solitons along the \( x^5 \)-direction, the equations for the dyonic solitons\(^7\) in \((d-1)\)-dimensions are obtained. Eq. (3.17) reduces to
\[ F_{0n} + \eta D_n \Sigma_5 = 0, \quad D_0 \Sigma_p - i\eta [\Sigma_5, \Sigma_p] = 0, \quad D_0 \Sigma_5 = 0. \tag{3.26} \]
For any instanton background configuration \( W_n \), these equations can be solved by setting
\[ \partial_0 = 0, \quad W_0 - \eta \Sigma_5 = 0. \tag{3.27} \]
The configuration of the adjoint scalar \( \Sigma_5 \) is determined by the Gauss law as
\[ D_n F_{0n} + i \sum_{p=d-1}^4 [\Sigma_p, D_0 \Sigma_p] = 0, \quad \rightarrow \quad D_n D_n \Sigma_5 - \sum_{p=d-1}^4 [\Sigma_p, [\Sigma_p, \Sigma_5]] = 0. \tag{3.28} \]
It is known that there exists the unique solution to this equation once an instanton configuration and an asymptotic value of \( \Sigma_5 \) are given \([31]\). The Poynting vector given in Eq. (3.25) reduces to the electric charge of the dyonic instanton as
\[ S_5 \rightarrow Q_e = \frac{1}{g_f^2} \int d^{d-1}x \left( F_{0n}^I D_n \Sigma_5^I + i \sum_{p=d-1}^4 D_0 \Sigma_p^I [\Sigma_p, \Sigma_5] \right) = \frac{1}{g_f^2} \int d^{d-1}x \partial_n (F_{0n}^I \Sigma_5^I) , \tag{3.29} \]
and the energy density becomes
\[ E = -\eta Q_e + E_m (\geq 0). \tag{3.30} \]
\(^7\)Electrically charged instanton in \( d = 5 \) is called dyonic instanton \([31]\) whereas dyon is electrically charged monopole in \( d = 4 \) \([32]\). The dyonic instanton (field theory supertube) in 6 dimensions with 16 supercharges was obtained by boosting the \( x^5 \)-direction with the speed of light and reducing that direction \([63]\).
3.2 Series of the type IVA: vortices and domain walls

This subsection is devoted to a review. Let us next investigate 1/2 BPS solitons which preserve four supercharges specified by the type IVA projection operator $\gamma^{nm}i\sigma_a$ on the Killing spinors, given in Eq. (2.10). Especially, we deal with the projection

$$\gamma^{12} (i\sigma_3\varepsilon)^i = \xi \varepsilon^i, \quad \xi = \pm 1. \tag{3.31}$$

By using these Killing conditions, the conditions $\delta\lambda^i = 0$ and $\delta\psi^A = 0$ in the SUSY transformation laws (2.3) lead to the 1/2 BPS equations

$$F_{\alpha\beta} = F_{\alpha 1} = F_{\alpha 2} = 0, \quad D_\alpha H^i = 0, \tag{3.32}$$

$$F_{12} = \xi Y_3, \quad Y_1 = Y_2 = 0, \quad D_1 H^i - \xi i(\sigma_3)^i_j D_2 H^j = 0. \tag{3.33}$$

with $\alpha, \beta = 0, 3, 4, 5$. Eq. (3.32) is satisfied by simply turning off $W_\alpha$ and dependence of $x^\alpha$. In this section, we omit the directions $x^0, x^3, x^4$ and $x^5$.

**Vortices:** The remaining equation (3.33) are

$$F^I_{12} = \xi g_I^2 [\zeta^I_3 - (H^{I^A})^*(\sigma_3)^i_j (T_I)^A B H^{ JB}], \quad Y_1 = Y_2 = 0, \tag{3.34}$$

$$0 = D_1 H^i - \xi i(\sigma_3)^i_j D_2 H^j. \tag{3.35}$$

These equations are called vortex equations. They reduce to the BPS equations for the Abrikosov-Nielsen-Olesen (ANO) vortices [13] in the case of $U(1)$ gauge theory with one charged hypermultiplet, whereas to those for non-Abelian vortices to which much attention has been recently paid in [14] [15] [16] [17] in the case $U(N_C)$ gauge theory coupled to $N_F (\geq N_C)$ hypermultiplets in the fundamental representation with equal $U(1)$ charges. In order to obtain finite energy vortex solutions, we need to turn on nonzero FI parameters $\zeta_\alpha^I$. We consider the case that all the FI parameters take values in the third direction as $\zeta_\alpha^I = (0, 0, \zeta_3^I)$. Then the condition $Y_1 = Y_2 = 0$ requires that a half of the Higgs fields must vanish. The energy of the BPS vortices is given by the topological charge as

$$E = \xi V_{12}^{(3)}. \tag{3.36}$$

Here we have defined

$$V_{nm}^{(a)} \equiv \int d x^n d x^m [\zeta_\alpha^I F_{nm}^{IL} + 2\partial_n J_m, a] = 2\pi k I_\alpha^I \zeta_\alpha^I, \quad J_{n, a} \equiv i(H^{I^A})^* \sigma_a D_n H^{I^A}. \tag{3.37}$$
with $k_I$ denoting the vorticity and $X \overset{\leftrightarrow}{\partial} Y \equiv (-\partial XY + X\partial Y)/2$.

**Domain walls:** By performing the SS dimensional reduction along one of codimensions on the 1/2 BPS equations for vortices, we can derive 1/2 BPS equations for domain walls. For example, let us perform SS dimensional reduction along the $x^2$-direction. Then we get the following first order equations

\[
\mathcal{D}_1 \Sigma_2 = -\xi Y_3, \quad Y_1 = Y_2 = 0, \quad (3.38)
\]

\[
\mathcal{D}_1 H^{iA} = \xi (\sigma_3)^i_j \left( \Sigma_2 H^{jA} - m_{2,A} H^{jA} \right). \quad (3.39)
\]

We call these equations domain wall equations. They reduce to BPS equations discussed in [20]–[24] in the case of $U(1)$ gauge theory coupled to $N_F$ hypermultiplets with equal $U(1)$ charge, whereas to those in [25]–[27] in the case $U(N_C)$ gauge theory with $N_F(>N_C)$ hypermultiplets in the fundamental representation. The topological charge of the 1/2 BPS vortices reduces to that of the 1/2 BPS walls by the SS dimensional reduction

\[
\mathcal{W}_{1,2}^{(3)} = \mathcal{V}_{12}^{(3)} = \int dx^1 \partial_1 \left[ -\zeta^I \Sigma^I_2 - i(H^{iA})^*(\sigma_3 \Sigma_2 H)^{iA} + i(H^{iA})^*(\sigma_3 H)^{iA} m_{2,A} \right] \quad (3.40)
\]

where we have defined

\[
\mathcal{W}^{(a)}_{m,n} \equiv \mathcal{V}^{(a)}_{mn}. \quad (3.41)
\]

Then the energy of the 1/2 BPS wall is given by this charge:

\[
E = \xi \mathcal{W}_{1,2}^{(3)}. \quad (3.42)
\]

**Equations in zero dimension:** Finally, if we fully reduce the codimensions of solitons, the Bogomol’nyi bound vanishes in this case of the type IVa as in the type IVb. Therefore, spacetime-filling brane (partially breaking SUSY vacua) cannot be obtained as a result of the dimensional reductions.

There exist no wavy or dyonic extensions of these solitons with keeping the 1/2 BPS conditions. Instead they become 1/4 BPS as discussed in section 4.1.5.

### 3.3 Summary of 1/2 BPS systems

We have seen that all 1/2 BPS states are classified into two classes, the types IVa and IVb, with respect to unbroken SUSY. Unbroken SUSY in the case of two dimensional world-volume is
\( \mathcal{N} = (2, 2) \) or \( (4, 0) \) for the types IVa or IVb, respectively. The fundamental solitons in the type IVa are vortices whereas those in the type IVb are instantons in \( d = 6 \). All other solitons have been obtained by (SS) dimensional reductions. Wavy and dyonic solitons exist for the type IVb. We summarize all descendants in Table 2.

Table 2: 1/2 BPS states: Here, \( \tilde{d} \) denotes spacetime dimensions of a soliton world-volume (\( \tilde{d} = 1 \) denotes a particle, \( \tilde{d} = 2 \) a string, \( \tilde{d} = 3 \) a membrane, and \( \tilde{d} = 3 \) a 3-brane). See the caption of Table 1 for notations of the Roman uppercase letters. More notations are: w-: wavy object (or with electric flux), Q-: electric-charged object [that is (Q-)M means dyon].

| \( \tilde{d} \) | \( \mathcal{N} \) | \( d = 6 \) | \( d = 5 \) | \( d = 4 \) | \( d = 3 \) | \( d = 2 \) |
|---|---|---|---|---|---|---|
| 2 | (4,0) | (w-)I | (w-)M | (w-)H | (w-)N | wave |
| 1 | 4 | - | (Q-)I | (Q-)M | (Q-)H | (Q-)N |
| 3 | 2 | V | W | - | - | - |
| 2 | (2,2) | - | V | W | - | - |
| 1 | 4 | - | - | V | W | - |

The types IVa and IVb exhibit very different features. Since BPS solitons in the types IVb and IVa preserve \( \mathcal{N} = (4, 0) \) and \( (2, 2) \) SUSY, respectively, in the case of two dimensional world volume, the moduli space for these solitons are described by \( \mathcal{N} = (4, 0) \) nonlinear sigma models on hyper-Kähler manifolds (with torsion) \[^{67}\] and \( \mathcal{N} = (2, 2) \) nonlinear sigma models on Kähler manifolds \[^{68}\], respectively. Second, BPS solitons in the type IVb naturally exist in theories with sixteen supercharges (typically in \( d = 4, \mathcal{N} = 4 \) SUSY Yang-Mills theory) whereas BPS solitons in the type IVa can exist in theories with four supercharges (\( d = 4, \mathcal{N} = 1 \) SUSY theory).

We would like to note that there exist the two kinds of objects in co-dimensions two and one. The Hitchin vortices (type IVb) and the ordinary vortices of the ANO type (type IVa) are with codimension two, whereas the Nahm walls (type IVb) and the ordinary walls (type IVa) are with codimension one. The Hitchin vortices and the Nahm walls live inside the unbroken (non-Abelian) phase whereas ordinary vortices and walls live inside the Higgs phase. This distinction becomes very important when we consider 1/4 and 1/8 BPS composite states in the following sections. As denoted in the text there exist no finite energy solutions for Hitchin vortices or
Before closing this section we make comments on solutions known so far. The method to construct multi-instantons and multi-BPS monopoles were established by Atiyah, Hitchin, Drinfeld and Manin (ADHM) [9] and Nahm [11], respectively. Their moduli spaces are both hyper-Kähler manifolds. The $\mathcal{N} = (4,0)$ nonlinear sigma model on the ADHM moduli space was considered in [64]. These BPS instantons can be realized by D$p$-branes on D$(p + 4)$-branes [65] whereas BPS monopoles by D$p$-branes ending on (stretched between) D$(p + 2)$-branes [66] in type IIA/IIB superstring theories. These brane configurations give clear physical interpretations of the ADHM/Nahm constraints as the F- and D- flatness conditions in the SUSY gauge theory on D$p$-brane world-volume. On the other hand 1/2 BPS equations in the type IVa can be (partially) solved by the method of the moduli matrix (walls in [25, 28] and vortices in [15, 16, 44]). Fully exact solutions can be obtained in the strong gauge coupling limit in which the model reduces to a hyper-Kähler nonlinear sigma model. Some brane configurations are also available; vortices by Hanany and Tong [14] and walls in [21, 27, 30].

4 1/4 BPS Systems

In the previous section we have discussed different 1/2 BPS solitons; instanton, monopole, vortex and domain wall including their wavy and dyonic extension. In this section we will present several 1/4 BPS equations for composite states of those 1/2 BPS solitons. As we have mentioned in Sec.2 there exist two sets of 1/4 SUSY projection operators, the type IIa in Eq.(2.12) and the type IIb in Eq.(2.13). Similarly to the method in Sec.3 we will start with the 1/4 BPS equations in 6 dimensions and derive diverse 1/4 BPS equations in $d = 5, 4, 3$ by the SS dimensional reductions. These 1/4 BPS states certainly involve the 1/2 BPS vortices and/or 1/2 BPS domain walls which are typical solitons in the Higgs phase of supersymmetric gauge theories. Therefore, the Higgs fields play important roles in the following.

4.1 Series of the type IIb

In this section, subsections 4.1.1, 4.1.2 and 4.1.3 are devoted to a review whereas subsections 4.1.4 and 4.1.5 are new contributions.
4.1.1 Instanton - Vortex

We begin with the type IIb 1/4 SUSY projections in 6 dimensions

\[ \gamma^{05} \varepsilon^i = \eta \varepsilon^i, \quad \gamma^{12} (i \sigma_3 \varepsilon)^i = \xi \varepsilon^i, \quad \gamma^{34} (i \sigma_3 \varepsilon)^i = \xi' \varepsilon^i \]  

(4.1)

where \( \eta, \xi, \xi' \) are \( \pm 1 \) satisfying \( \eta \xi \xi' = -1 \). Imposing invariance to these 1/4 SUSY as \( \delta \lambda^i = 0 \) and \( \delta \psi^A = 0 \) given in Eq.(2.3), we can derive a set of 1/4 BPS equations

\[ F_{13} + \eta F_{24} = 0, \quad F_{14} - \eta F_{23} = 0, \quad \xi F_{12} + \xi' F_{34} = Y_3, \quad Y_1 = Y_2 = 0, \]  

(4.2)

\[ D_1 H^i - \xi i (\sigma_3)^i_j D_2 H^j = 0, \quad D_3 H^i - \xi' i (\sigma_3)^i_j D_4 H^j = 0. \]  

(4.3)

We may call these equations as the SDYM-Higgs equations. They comprise the (anti) SDYM equations for instantons localized in the \( x^1-x^2-x^3-x^4 \) space, and the BPS equations for the vortices living in the \( x^1-x^2 \) plane and in the \( x^3-x^4 \) plane. The world-volume of these solitons can be summarized as follows.

| IVV | 0 | 1 | 2 | 3 | 4 (5) | sign |
|-----|---|---|---|---|-----|------|
| Instanton | \( \bigcirc \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \bigcirc \) | \( \eta \) |
| Vortex | \( \bigcirc \) | \( \times \) | \( \times \) | \( \bigcirc \) | \( \bigcirc \) | \( \bigcirc \) | \( i \sigma_3 \) | \( \xi \) |
| Vortex | \( \bigcirc \) | \( \bigcirc \) | \( \bigcirc \) | \( \times \) | \( \times \) | \( \bigcirc \) | \( i \sigma_3 \) | \( \xi' \) |

Here “\( \bigcirc \)” denotes the world-volume directions of the solitons, whereas “\( \times \)” denotes their codimensional directions. From the above table we find that this composite soliton trivially extends to the \( x^5 \)-direction and therefore that the dimensionally reduced configuration exists in 5 dimensions also. The energy of this composite solitons can be written by the sum of the three topological charges as

\[ E_m = \eta I_{1234} + \left( \xi V_{12}^{(3)} \right) v_{34} + \left( \xi' V_{34}^{(3)} \right) v_{12}, \]  

(4.4)

where we have ignored the sixth direction \( x^5 \). Here \( I_{1234} \) is the instanton charge \( (3.38) \), and \( V_{mn}^{(3)} \)'s are the vortex charge \( (3.37) \), and the quantities \( v_{34} \) and \( v_{12} \) denote the world-volumes of vortices along \( x^3-x^4 \) and \( x^1-x^2 \), respectively.

The equations \( (4.2) \) and \( (4.3) \) were firstly found in general Kähler manifold as a base space by mathematicians \( (53) \), without the BPS property or SUSY. There they are simply called vortex equations although they contain instantons also. These equations are rediscovered by Hanany and
Tong in [39] in the case of $U(N_C)$ gauge theory coupled to $N_F$ hypermultiplets in the fundamental representation with equal $U(1)$ charges. We showed in [43] that these equations are 1/4 BPS and then studied this system with $N_F = N_C$ in 5 dimensions in detail. In the paper [43] we further found that the 1/4 BPS equations (4.2) and (4.3) describe two different kinds of composite states, 1) instantons in the Higgs phase and 2) intersections of vortices in the $x^1\cdot x^2$ and the $x^3\cdot x^4$ planes.

1) As the former concerns, when we put ordinary instantons into the Higgs phase by turning on the FI-parameters, they can be stabilized if the magnetic flux is squeezed to form a vortex similarly to the Meissner effect of the monopoles in the Higgs phase. The total system becomes instantons lying inside vortex-3-branes (membranes) in $d = 6$ ($d = 5$). The instanton in the Higgs phase has the same mass $8\pi^2/g^2$ with that of the ordinary instanton.$^8$

2) In the latter case, the 1/4 BPS equations (4.2) and (4.3) describe a very different situation from the instantons in the Higgs phase [43]. It can exist even for the Abelian gauge theory, contrary to the ordinary instantons which do not exist due to the trivial instanton charge $\pi_3 = 0$. Let us consider the intersection of vortices with vorticity $k$ in the $x^1\cdot x^2$ plane and vortices with the vorticity $k'$ in the $x^3\cdot x^4$ plane. Interestingly, the contribution of the instanton charge to the energy is nonzero for intersecting vortices. It can be easily shown as follows. First we recall

$$\xi \int dx^1 dx^2 F_{12} = 2\pi \xi k > 0 \quad \text{and} \quad \xi' \int dx^3 dx^4 F_{34} = 2\pi \xi' k' > 0.$$ 

Then the instanton contribution can be calculated as $E_m = \eta \mathcal{L}_{1234} = (2\eta/g^2) \int dx^4 F_{12} F_{34} = (\eta kk')8\pi^2/g^2$. Due to the relation $\eta \xi \xi' = -1$, the contribution is always negative and is proportional to the product of the vorticities $E_m = -|kk'|8\pi^2/g^2$. Such objects with the negative energy should not be regarded as solitons, but they are rather binding energy for the intersections of the solitons.

Thus the 1/4 BPS equations (4.2) and (4.3) describe both instantons in the Higgs phase and intersections of the vortices. As we will see in the following subsections, the monopole charge and the Hitchin charge, derived from the instanton charge by dimensional reductions, also can become either positive energies of ordinary solitons or negative binding energies localized on intersections of solitons, depending on configurations.

$^8$These instantons can be regarded as lumps in the $\mathbb{C}P^{N-1}$ model as the effective field theory on the vortex in the case of $U(N)$ gauge theory with $N$ hypermultiplets in the fundamental representation [43].
4.1.2 Monopole - Vortex - Wall

As was shown in the Sec.3, the 1/2 BPS equations for instantons and vortices reduce to those for monopoles and domain walls, respectively, by performing the SS dimensional reduction along one of their codimensions. When we perform the SS dimensional reduction (along, for example, the $x^4$-direction) in the above 1/4 BPS eqs. (4.2) and (4.3) for composite states of instantons, vortices in the $x^1$-$x^2$ plane and vortices in the $x^3$-$x^4$ plane, we derive another set of 1/4 BPS equations for composite states of monopoles in the $x^1$-$x^2$-$x^3$ space, vortices in the $x^1$-$x^2$ plane and domain walls perpendicular to the $x^3$-coordinate, given by

\[
F_{13} - \eta D_2 \Sigma_4 = 0, \quad D_1 \Sigma_4 + \eta F_{23} = 0, \quad \xi F_{12} - \xi' D_3 \Sigma_4 = Y_3, \quad Y_1 = Y_2 = 0, \quad (4.5)
\]
\[
D_1 H^{iA} - \xi i(\sigma_3)^i_j D_2 H^{jA} = 0, \quad D_3 H^{iA} - \xi'(\sigma_3)^i_j \left[ (\Sigma_4)^A_B H^{jB} - m_{4,A} H^{jA} \right] = 0. \quad (4.6)
\]

The codimensions and the world-volume directions of solitons in this 1/4 BPS system is summarized as follows.

| MVW | 0 | 1 | 2 | 3 (5) | sign |
|-----|---|---|---|------|------|
| Monopole | ○ | × | × | × | ○ | $\eta$ |
| Vortex | ○ | × | × | ○ | ○ | $i\sigma_3$ | $\xi$ |
| Wall | ○ | ○ | ○ | × | ○ | $i\sigma_3$ | $\xi'$ |

This 1/4 BPS system trivially depends on the $x^5$-direction, so that these composite solitons exist in four dimensional spacetime. The energy of this composite solitons is three sum of the topological charges of the monopoles, vortices and walls

\[
E_m = \eta M_{123} + \left( \xi \mathcal{V}^{(3)}_{12} \right) v_3 + \left( \xi' \mathcal{W}^{(3)}_{3,4} \right) v_{12}, \quad (4.7)
\]

with the monopole charge (3.10), the vortex charge (3.37), and the domain wall charge (3.41).

The 1/4 BPS equations (4.5) and (4.6) (without walls) were firstly found by Tong in [39] in the case of $d = 4$, $\mathcal{N} = 2$ SUSY $U(N_C)$ gauge theory with $N_F = N_C$ hypermultiplets in the fundamental representation. Similarly to the 1/4 BPS states of the instantons, vortices and vortices discussed in the last subsection, the monopole charge in this 1/4 BPS system also exhibits two different features. One is a monopole in the Higgs phase [39]. If an ordinary monopole of the non-Abelian gauge theory in the Coulomb phase is put into the Higgs phase, magnetic fluxes from it are squeezed due to the Meissner effect to form vortices, making the total system a composite
state of a monopole attached by vortices.\textsuperscript{9} In that situation the mass of the monopole is the same with the mass $4\pi |\Delta \langle \Sigma_4 \rangle|/g^2$ of an ordinary monopole in the Coulomb phase. On the other hand, if we consider $N_F (N_C)$ hypermultiplets domain walls are also allowed \cite{41}. In the Abelian gauge theory the 1/4 BPS equation \cite{45} and \cite{40} describe the composite state of the vortices ending on the walls rather than the monopoles in the Higgs phase. Interestingly, the monopole charge takes a nonzero value even for the Abelian gauge theory. In fact it gives the negative energy contribution $-4k\pi |\Delta \langle \Sigma_4 \rangle|/g^2$ when $k$ ANO vortices penetrate a wall. A vortex is divided by the wall into two vortices attached to one and the other sides of the wall, and end points of these two vortices can be separated along the wall world-volume.\textsuperscript{10} Each has the half $-2\pi |\Delta \langle \Sigma_4 \rangle|/g^2$ of the (negative) unit monopole charge. Although these negative energies are localized at the ending points of the vortices on the wall, they should not be regarded as solitons. Rather they are binding energy of the junction called boojums \cite{41,42}. We constructed implicit solutions for this system in $d = 5$, $\mathcal{N} = 1$ gauge theory. Moreover we have obtained all the exact solutions in the strong gauge coupling limit \cite{41}. Now we stand at the position where the negative binding energy of the boojum can be understood from the viewpoint of the SS dimensional reduction. First recall that the instanton charge $\mathcal{I}_{mnkl}$ reduces to the monopole charge $\mathcal{M}_{mnk} = \mathcal{I}_{mnk}$. This relation is kept even when instantons and monopoles are put into the Higgs phase. Furthermore, this relation is also correct for the charges as the binding energy for intersections of vortices and vortices ending on walls. When we push to perform the SS dimensional reduction once more as we will see below soon, very similar discussion will be able to be done again.

We do not know if there exist composite states made of only monopoles and walls without vortices, but if they exist they are 1/4 BPS. However they are likely impossible because of the Meissner effect.

\textsuperscript{9}These monopoles can be regarded as kinks in the $\mathbb{C}P^{N-1}$ model with a potential as the effective field theory on a vortex in the case of $U(N)$ gauge theory with massive $N$ hypermultiplets in the fundamental representation, as found by Tong \cite{39}.

\textsuperscript{10}This is a peculiar situation for this system. In the case of vortex intersections discussed in the previous section one vortex cannot be divided by the other vortex from their dimensionality.
4.1.3 Hitchin vortex - Wall (Domain wall junction)

As we have seen in Sec 3, the 1/2 BPS equations of monopoles reduce to those for the Hitchin system by one dimensional reduction. Vortices reduce to walls by the SS dimensional reduction. Therefore, performing the SS dimensional reduction in Eqs. (4.5) and (4.6) along $x^2$, we obtain the following 1/4 BPS equations for composite states of Hitchin vortices in the $x^3$-$x^1$ plane, walls perpendicular to the $x^1$-direction and walls perpendicular to the $x^3$-direction, given by

\begin{align}
F_{13} + \eta i [\Sigma_2, \Sigma_4] &= 0, \quad \mathcal{D}_1 \Sigma_4 + \eta \mathcal{D}_3 \Sigma_2 = 0, \quad (4.8) \\
\xi \mathcal{D}_1 \Sigma_2 + \xi' \mathcal{D}_3 \Sigma_4 &= -Y_3, \quad Y_1 = Y_2 = 0, \quad (4.9) \\
\mathcal{D}_1 H^{iA} - \xi (\sigma_3)^i_j [ (\Sigma_2)^A B H^{jB} - m_{2,A} H^{jA} ] &= 0, \quad (4.10) \\
\mathcal{D}_3 H^{iA} - \xi' (\sigma_3)^i_j [ (\Sigma_4)^A B H^{jB} - m_{4,A} H^{jA} ] &= 0. \quad (4.11)
\end{align}

The configuration can be summarized as follows.

|        | 0 | 1 | 3 | (5) | sign  |
|--------|---|---|---|-----|-------|
| Hitchin vortex | × | × | × |     | 1_2 \ η |
| Wall     | × | × | × |     | i \sigma_3 \ \xi |
| Wall     | × | × | × |     | i \sigma_3 \ \xi' |

The energy of this 1/4 BPS states is given by

\[ E_m = \eta \mathcal{H}_{13} + \left( \xi \mathcal{W}_{1,2}^{(3)} \right) v_3 + \left( \xi' \mathcal{W}_{3,4}^{(3)} \right) v_1. \quad (4.12) \]

Recently, the 1/4 BPS equations (4.8), (4.9), (4.10) and (4.11) have been found [50] in the case of $d = 4, \mathcal{N} = 2 U(N_C)$ gauge theory coupled to $N_F$ hypermultiplets with complex masses in the fundamental representation with equal $U(1)$ charges. All possible moduli have been determined, and all the exact solutions have been obtained in the strong gauge coupling limit. Especially, the 1/4 BPS states in the Abelian gauge theory have been extensively studied, and it has been found that several walls can constitute webs of walls, like $(p,q)$ string/5-brane webs in superstring theory. In the Abelian gauge theory, the charge for the Hitchin vortices does not give positive contributions to the energy. Instead, it gives negative binding energy localized at the junction lines (points) of the walls in $d = 3 \ (d = 4)$. There were many works on the 1/4 BPS domain wall junctions in theories with both eight supercharges and four supercharges. All exact solutions of the wall junctions known so far carry negative junction charges as binding energy [47, 50].
The negativeness of the Hitchin charge as binding energy of the domain wall junctions can be understood by remembering that it is obtained by the SS dimensional reduction(s) of the monopole charge for the boojums at the end points of vortices on the wall (or of the negative instanton charge localized at the vortex intersections).

On the other hand, we have seen in the last two subsections that there exist 1/4 BPS states with positive energy of instantons or monopoles, namely instantons or monopoles in the Higgs phase, respectively. This observation strongly suggests that there should exist the 1/4 BPS composite states of the Hitchin vortices with positive energy, which are put into the Higgs phase by the FI-term. In fact we found in [51] that the Hitchin vortices with positive energy live on the junction lines (points) of walls in the non-Abelian gauge theories. Generally, walls in the non-Abelian gauge theories can constitute very ample webs, containing both negative and positive Hitchin charges. Therefore wall webs in the non-Abelian gauge theory have a rich structure which does not exist in the Abelian gauge theory [51].

4.1.4 Hitchin vortex - Vortex, and Nahm wall - Wall

If we perform the SS dimensional reduction in Eqs. (4.5) and (4.6) along the $x^3$-direction instead of the $x^2$-direction, we obtain a set of 1/4 BPS equations which include the 1/2 BPS equations for the Hitchin vortices and for ordinary vortices, where both objects would lie perpendicular to the $x^1$-$x^2$ plane. By taking a further SS dimensional reduction along the $x^2$-direction after the above operation, we obtain the equations containing the 1/2 BPS equations for the Nahm walls and for ordinary walls, where both objects would be perpendicular to the $x^1$-direction. These objects could lie as the following.

| HV | 0 | 1 | 2 (5) |
|-----|---|---|-------|
| Hitchin vortex | O × × O | vortext | O × × O |

| NW | 0 | 1 (5) |
|-----|---|-------|
| Nahm wall | O × O | wall | O × O |

It seems, however, quite difficult for these objects, a Hitchin vortex and an ordinary vortex, or a Nahm wall and an ordinary wall to coexist. This is because the Hitchin vortices and the Nahm walls exist in an unbroken phase of non-Abelian gauge group, whereas ordinary walls and vortices exist in the Higgs phase and their tension is proportional to the FI-term of the Abelian gauge.
group. As a further negative evidence, there exist the condition,

$$\Phi H^1 = \Phi^\dagger H^2 = 0, \quad (\Phi)^A_B = (\Sigma_3 - i\xi'\Sigma_4)^A_B - (m_{3,A} - i\xi'm_{4,A})\delta^A_B$$

(4.13)
as a part of the reduced 1/4 BPS equations, implying that the existence of one excludes the other. Therefore, it is likely the case that this 1/4 BPS system contains only one kind of these BPS solitons and then it reduces to a 1/2 BPS state.

### 4.1.5 Wavy and dyonic extension

Similarly to 1/2 BPS equations of the type IVb, the 1/4 BPS equations of the type IIb can be extended to wavy or dyonic objects by turning on the $x^0$ and $x^5$ dependence, since both projection operators have $\gamma^{05}$.

**Wavy solitons:** In addition to the 1/4 BPS equations (4.2) and (4.3) and their descendants given by the SS dimensional reductions, taken $(6-d)$-times along the $x^p$-directions $(p = d − 1, \cdots, 4)$, 1/4 BPS equations for wavy extensions are obtained, to yield

$$F_{0n} + \eta F_{5n} = 0, \quad D_0 \Sigma_p + \eta D_5 \Sigma_p = 0, \quad F_{05} = 0, \quad (D_0 + \eta D_5) H^I = 0,$$

(4.14)

with $n = 1, \cdots, d − 2$. These reduce to the same simple equations with Eq. (3.27) with appropriate gauge fixing under a given static background. We obtain their wavy extension by promoting any moduli parameters $\phi^i$ of that background to arbitrary functions $\phi^i(x^0 − \eta x^5)$. The solutions have to satisfy the Gauss law

$$\frac{1}{g_1^2} D_0 F_{0n}^I + i \sum_{p=d-1}^{4} [\Sigma_p, D_0 \Sigma_p]^I - 2i(T^I)^A_B H^{iB} \mathcal{D}_0 (H^{iA})^* = 0$$

(4.15)
in addition to the BPS equations. The energy is given by sum of the energies of the solitons and the fifth component of the Poynting vector, $E = E_m − \eta S'_5$ with

$$S'_5 = \frac{1}{g_1^2} \int d^{d-1} x \left( F_{0n}^I F_{5n}^I + \sum_{p=d-1}^{4} D_0 \Sigma_p^I D_5 \Sigma_p^I + (D_0 H^{iA})^* (D_5 H^{iA}) + (D_5 H^{iA})^* (D_0 H^{iA}) \right)$$

(4.16)

**Dyonic solitons:** When we perform the trivial dimensional reduction along the $x^5$-direction, 1/4 BPS wavy solitons reduce to 1/4 BPS dyonic solitons. By this dimensional reduction, Eqs. (4.14) reduce to

$$F_{0n} + \eta D_0 \Sigma_5 = 0, \quad D_0 \Sigma_p + i\eta [\Sigma_5, \Sigma_p] = 0, \quad D_0 \Sigma_5 = 0, \quad D_0 H^{iA} - i\eta (\Sigma_5)^A_B H^{iB} = 0.$$
These can be solved by the solution given in Eq. (3.27). Combining Eqs. (4.17), the Gauss law
\[ \frac{1}{g_i^2} \left( D_n F_{0n}^I + i \sum_{p=d-1}^4 [\Sigma_p, D_0 \Sigma_p]^I \right) - 2i (T^I)^A_B H^{iB} \hat{D}_0 (H^{IA})^* = 0 \] (4.18)
determines the configuration of the $\Sigma_5$. The fifth component of the Poynting vector given in Eq. (4.16) reduces to the electric charge
\[ S_5' \to Q_e = \frac{1}{g_i^2} \int d^{d-1}x \left( F_{0n}^I D_n \Sigma_5^I + i \sum_{p=d-1}^4 D_0 \Sigma_p [\Sigma_p, \Sigma_5]^I \right. \\ \left. - i (D_0 H^{iA})^* (\Sigma_5 H^{iA}) + i (\Sigma_5 H^{iA})^* (D_0 H^{iA}) \right) \]
\[ = \frac{1}{g_i^2} \int d^{d-1}x \partial_n (\Sigma_5^I F_{0n}^I) \] (4.19)
and the energy is given by
\[ E = -\eta Q_e + E_m (\geq 0). \]

### 4.2 Series of the type IIa

While the type IIb series of the 1/4 BPS equations in the previous subsection has been found recently and studied so far, the most of 1/4 BPS equations of the type IIa given in this subsection are unknown and new. We do not make an effort to solve the BPS equations in this paper. Instead we concentrate on finding new 1/4 BPS equations, classifying them and discussing their physical properties. We leave detailed analysis of these 1/4 BPS equations as future problems.

#### 4.2.1 Vortices

Let us turn our attention to the other 1/4 SUSY projection in 6 dimensions. We consider the following type IIa operators
\[ \gamma^{23} (i \sigma_1 \varepsilon)^i = \xi_1 \varepsilon^i, \quad \gamma^{31} (i \sigma_2 \varepsilon)^i = \xi_2 \varepsilon^i, \quad \gamma^{12} (i \sigma_3 \varepsilon)^i = \xi_3 \varepsilon^i, \quad \xi_1 \xi_2 \xi_3 = -1. \] (4.20)

By imposing SUSY preservation $\delta \lambda^i = 0$ and $\delta \psi^A = 0$ in the SUSY transformation laws with the Killing conditions (4.20), we find new 1/4 BPS equations
\[ F_{\alpha\beta} = F_{\alpha n} = 0, \quad D_\alpha H^i = 0, \] (4.21)
\[ \sum_{n=1}^3 (\tau_n)^i_j D_n H^j = 0, \quad \frac{1}{2} \epsilon_{nmli} F_{ml} = \xi_n Y_n, \] (4.22)
where we have defined \( \tau_n \equiv \xi_n i \sigma_n \) and \( \alpha, \beta = 0, 4, 5 \) and \( n, m, l = 1, 2, 3 \). Equation (4.21) is satisfied by simply ignoring the \( x^\alpha \) dependence and by ignoring \( W_\alpha \). Equation (4.22) consists of three 1/2 BPS equations describing vortices in the \( x^1-x^2 \) plane, those in the \( x^2-x^3 \) plane and those in the \( x^3-x^1 \) plane. The configuration can be summarized as follows.

| VVV | 0 | 1 | 2 | 3 | (4) | (5) | FI sign |
|-----|---|---|---|---|-----|-----|--------|
| Vortex | \( \bigcirc \) | \( \bigcirc \) | \( \times \) | \( \times \) | \( \bigcirc \) | \( \bigcirc \) | \( i \sigma_1 \) | \( \xi_1 \) |
| Vortex | \( \bigcirc \) | \( \times \) | \( \bigcirc \) | \( \times \) | \( \bigcirc \) | \( \bigcirc \) | \( i \sigma_2 \) | \( \xi_2 \) |
| Vortex | \( \bigcirc \) | \( \times \) | \( \times \) | \( \bigcirc \) | \( \bigcirc \) | \( \bigcirc \) | \( i \sigma_3 \) | \( \xi_3 \) |

Here \( \zeta^I_1, \zeta^J_2, \zeta^K_3 \) (\( I \neq J \neq K \neq I \)) denote first, second, third components of three different triplets of the FI-parameters, respectively, as explained below. The energy of these composite states is the sum of the energy of vortices in the three directions, given by

\[
E_m = \left( \xi_3 \nu_{12}^{(3)} \right) v_3 + \left( \xi_1 \nu_{23}^{(1)} \right) v_1 + \left( \xi_2 \nu_{31}^{(2)} \right) v_2. \tag{4.23}
\]

In the case that the moduli space of vacua is a non-trivial Higgs branch \( M_{\text{vac}} \), the model reduces to the hyper-K"ahler nonlinear sigma model on \( M_{\text{vac}} \) in the strong gauge coupling limit. Vortices reduce to lumps and the equations (4.21) and (4.22) reduce to those for lump intersections found in [54, 55]. (The Killing conditions (4.20) were already found in [54].) The energy bound (4.23) reduces to the sum of the pull-back of three complex structures \( \mathbf{I}, \mathbf{J} \) and \( \mathbf{K} \) on the hyper-K"ahler manifold \( M_{\text{vac}} \) to the \( x^1-x^2, x^2-x^3 \) and \( x^3-x^1 \) planes, respectively.

We need to turn on an FI parameter to obtain a vortex solution. The direction of the FI parameters \( (\zeta^I_1, \zeta^J_2, \zeta^K_3) \) in the three dimensional space has to be parallel to the direction of the unit vector \( (n_1, n_2, n_3) \) in the 1/2 SUSY projection operator \( \gamma^{nm}(\vec{n} \cdot i \vec{\sigma}) \) for the vortices in the \( x^n-x^m \) plane [54]. Therefore we need to introduce several (at least three) FI parameters in different directions with gauge group containing several \( U(1) \) factors, to obtain intersecting vortices. Such FI-parameters were summarized in the table above.

We do not know if there exist nontrivial junctions of vortices, like domain wall webs [50] in subsection 4.1.3 (or \( (p, q) \) string webs). One negative circumstantial evidence is the fact that this system does not contain junction charge unlike domain wall junction with the Hitchin charge or intersecting vortices with the instanton charge. However this may not be crucial because wall junctions and intersecting vortices of the type IIb exist in the strong gauge coupling limit in which these junction charges vanish.
4.2.2 Vortex - Wall

By performing the SS dimensional reduction along the $x^3$-direction in the above 1/4 BPS system of three orthogonal vortices, we can find another set of new 1/4 BPS equations for vortices and walls, given by

$$F_{12} = \xi_3 Y_3, \quad D_1 \Sigma_3 = \xi_2 Y_2, \quad D_2 \Sigma_3 = -\xi_1 Y_1,$$

$$\sum_n (\tau_n)^{ij} D_n H_{jA} = i \tau_3 \left[ (\Sigma_3)^A B H^{jB} - m_{3,A} H^A \right],$$

with $n = 1, 2$. The configuration can be summarized as follows

| VWW | 0 | 1 | 2 | (4) | (5) | FI | sign |
|-----|---|---|---|-----|-----|----|------|
| Wall | 〇 | 〇 | × | 〇 | 〇 | $i \sigma_1$ | $\xi_1$ |
| Wall | 〇 | × | 〇 | 〇 | 〇 | $i \sigma_2$ | $\xi_2$ |
| Vortex | 〇 | × | × | 〇 | 〇 | $i \sigma_3$ | $\xi_3$ |

The energy of these composite states can be calculated as

$$E_m = \xi_3 V_{12}^{(3)} + \left( \xi_1 W_{2,3}^{(1)} \right) v_1 + \left( -\xi_2 W_{1,3}^{(2)} \right) v_2.$$  (4.26)

Vortex-strings in this system are parallel to both walls perpendicular to $x^1$ and those to $x^2$. This system is very similar to the system admitting wall junctions discussed in subsection 4.1.3.

In the previous case, codimension two objects are the Hitchin vortices, which can live only in the unbroken phase. They are able to be localized at junction lines of walls, because gauge group is almost recovered inside walls. They can play a role of binding energy of walls. However codimension two objects in the current case are ordinary vortices which live in the Higgs phase. Therefore they cannot come inside walls but rather live outside walls. Since they cannot become binding energy, we do not know if there can exist nontrivial wall junctions in this case; this system may admit only trivially (or at most asymptotically) orthogonally intersecting walls. However at least in the strong gauge coupling limit, the junction charge vanishes in the previous case. This implies that junction charge is not needed to make wall junctions in general. Therefore some nonlinear sigma models may admit wall junctions with vortex(lump)-strings outside walls.
4.3 Summary of 1/4 BPS systems

In this section we have shown that all 1/4 BPS states are classified into two classes, the types IIa and IIb, which preserve $\mathcal{N} = (1, 1)$ and $(2, 0)$, respectively, in the case of two dimensional world-volume. According to this classification, there exist two sets of 1/4 BPS equations in $d = 6$. We have obtained diverse 1/4 BPS equations in dimensions less than six. They admit various composite solitons as summarized in Table 3. Most of the 1/4 BPS equations in the type IIb have been known already, whereas those in the type IIa have not been found yet and are new.

|   | $d = 6$ | $d = 5$ | $d = 4$ | $d = 3$ | $d = 2$ |
|---|---------|---------|---------|---------|---------|
| $\mathcal{N} = 1$ | (w-)IVV | (w-)MVW | (w-)HWW | (w-)NW | -       |
| $\mathcal{N} = 2$ | VVV | VWW | WW | - | -       |

Table 3: 1/4 BPS states. Definitions of all symbols here are given in the caption of Table 1.

Like the type IVa and IVb in 1/2 BPS states, the type IIa and type IIb exhibit very different features. Since composite states contain solitons with different dimensions, it depends if there exist normalizable zero modes or not. If they exist dynamics of these composite states can be described by nonlinear sigma models on the target space as the moduli space. According to the unbroken SUSY which they preserve, their target spaces as moduli spaces are Riemann manifolds or Kähler manifolds (with torsion) [69] for the type IIa or IIb, respectively.

By the group including the present authors, the implicit solutions for the most of the 1/4 BPS equations of the type IIb were obtained except for HV and NW in the table. For instance MVW was solved in [41], IVV in [43] and HWW (wall webs) in [50, 51]. All the exact solutions are available in the strong gauge coupling limit. For some cases, the moduli spaces were determined and explicit relations between moduli parameters and actual soliton configurations were examined. On the other hand, no solutions are found yet for the 1/4 BPS equations of the type
5 1/8 BPS Systems: the type I

The results in this section are new.

5.1 Instanton - Vortex

As we mentioned in Sec.3, the fundamental 1/2 BPS solitons in the 6 dimensions with \( \mathcal{N} = 1 \) SUSY are the vortex (3-brane) and the instanton (1-brane) like M2 and M5-brane which are fundamental 1/2 BPS objects in M-theory. In the previous section, we have found that two composite states of the vortices (V) and the instantons (I), namely the type IIa (VVV) and the type IIb (IVV), are possible as the 1/4 BPS states. Furthermore, there exists the unique 1/8 BPS composite state of the vortices and the instantons (the type I). A set of the projection operators on the Killing spinor, \( \Gamma_a \varepsilon = \lambda_a \varepsilon \), is given as follows

\[
\Gamma_a = \left\{ \gamma^{05}, \gamma^{12} i\sigma_3, \gamma^{34} i\sigma_3, \gamma^{23} i\sigma_1, \gamma^{14} i\sigma_1, \gamma^{31} i\sigma_2, \gamma^{24} i\sigma_2 \right\},
\]

(5.1)

where we assign signs \( \lambda_a = (\eta, \xi_3, -\eta \xi_3, \xi_1, -\eta \xi_1, \xi_2, -\eta \xi_2) \) with \( \xi_1 \xi_2 \xi_3 = -1 \). Requiring the invariance \( \delta \lambda^i = 0 \) and \( \delta \psi^A = 0 \) in Eq. (2.3) for the 1/8 SUSY projected by the above operators, we get the following 1/8 BPS equations

\[
\frac{1}{2} \epsilon_{nml} F_{ml} - \eta F_{n4} = \xi_n Y_n, \quad (m,n,l = 1,2,3),
\]

(5.2)

\[
\sum_{m=1}^{4} (q_m)^i j D_m H^j = 0,
\]

(5.3)

\[
q_m \equiv (\xi_1 i \sigma_1, \xi_2 i \sigma_2, \xi_3 i \sigma_3, -\eta 1_2).
\]

(5.4)
It may be interesting to note the quaternionic structure $q_m$ in these equations. We summarize this 1/8 BPS configuration below.

| IV² | 0 | 1 | 2 | 3 | 4 | 5 | sign |
|-----|---|---|---|---|---|---|------|
| Instanton | ○ | × | × | × | × | ○ | $1_2$ $\eta$ |
| Vortex | ○ | ○ | × | × | ○ | ○ | $i\sigma_1$ $\xi_1$ |
| Vortex | ○ | × | ○ | × | ○ | ○ | $i\sigma_2$ $\xi_2$ |
| Vortex | ○ | × | × | ○ | ○ | ○ | $i\sigma_3$ $\xi_3$ |
| Vortex | ○ | × | ○ | ○ | × | ○ | $i\sigma_1$ $-\eta\xi_1$ |
| Vortex | ○ | ○ | × | ○ | × | ○ | $i\sigma_2$ $-\eta\xi_2$ |
| Vortex | ○ | ○ | ○ | × | × | ○ | $i\sigma_3$ $-\eta\xi_3$ |

This 1/8 BPS configurations of the instantons and the vortices are independent on $x^5$, so these also exist in 5 dimensions. This configuration without instantons was expected in [55] without any equations in the context of hyper-Kähler nonlinear sigma models in $d = 5$.

Since the set of the projection operators in Eq. (5.1) contains $\gamma^{05}$, we can add Eq. (3.5) to the above 1/8 BPS equations still preserving the same 1/8 SUSY. Similarly to the type IVb and the type IIb, Eq. (3.5) gives wavy extensions, and the dyonic extension when we perform the trivial dimensional reduction along the $x^5$-direction. Then the energy of this composite state is the sum of energies of instantons and vortices with energy coming from the electric flux

$$E = -\eta X + \eta L_{1234} + \left(\xi_3 \psi_{12}^{(3)}\right) v_{34} + \left(\xi_1 \psi_{23}^{(1)}\right) v_{14} + \left(\xi_2 \psi_{31}^{(2)}\right) v_{24} - \left(\eta \xi_3 \psi_{34}^{(3)}\right) v_{12} - \left(\eta \xi_1 \psi_{14}^{(1)}\right) v_{23} - \left(\eta \xi_2 \psi_{24}^{(2)}\right) v_{13},$$

(5.5)

where $X$ denotes the 5th component of the Poynting vector (4.16) or the electric charge (4.19) of the dyonic solitons.

### 5.2 Descendants of the type I

#### 5.2.1 Monopole - Vortex - Wall

By performing the SS dimensional reduction along the $x^4$-direction, the type I 1/8 BPS states in 6 dimensions given in the previous subsection reduce to other 1/8 BPS composite states which
contain monopoles, vortices and walls. The corresponding 1/8 BPS equations are of the form

\begin{align}
\frac{1}{2} \epsilon_{nml} F_{ml} + \eta D_n \Sigma_4 &= \xi_n Y_n, \\
\sum_{n=1}^3 (\tau_n)^i_j D_n H^j &= -i \eta \left[ (\Sigma_4)^A B H^{iB} - m_{4,A} H^{iA} \right].
\end{align}

(5.6) \hspace{1cm} (5.7)

The configurations are summarized in the following table.

| MV$^3$W$^3$ | 0 | 1 | 2 | 3 (5) | sign |
|-------------|---|---|---|--------|------|
| Monopole    | o | x | x | x | o | $1_2$ | $\eta$ |
| Vortex      | o | o | x | x | o | $i\sigma_1$ | $\xi_1$ |
| Vortex      | o | x | o | x | o | $i\sigma_2$ | $\xi_2$ |
| Vortex      | o | x | x | o | o | $i\sigma_3$ | $\xi_3$ |
| Wall        | o | x | o | o | o | $i\sigma_1$ | $-\eta \xi_1$ |
| Wall        | o | o | x | o | o | $i\sigma_2$ | $-\eta \xi_2$ |
| Wall        | o | o | o | x | o | $i\sigma_3$ | $-\eta \xi_3$ |

Their energy is given by the sum of contributions from monopoles, vortices and walls, and energy coming from the electric flux or the electric charge:

\begin{align}
E &= -\eta X + \eta M_{123} + \left( \xi_3 Y_{12}^{(3)} \right) v_3 + \left( \xi_1 Y_{23}^{(1)} \right) v_1 + \left( \xi_2 Y_{31}^{(2)} \right) v_2 \\
&- \left( \eta \xi_3 W_{3,4}^{(3)} \right) v_{12} - \left( \eta \xi_1 W_{1,4}^{(1)} \right) v_{23} - \left( \eta \xi_2 W_{2,4}^{(2)} \right) v_{31}
\end{align}

(5.8)

with $X$ denoting the 5th component of the Poynting vector (4.16) or the electric charge (4.19) of the dyonic solitons.

At this stage we cannot expect what kinds of configuration are actually possible as solutions of the 1/8 BPS equations (5.6) and (5.7). As one of interesting possibilities, we expect that there may exist a 1/8 BPS state of a vortex-string network. Such expectation seems to be natural because it has been well established that two vortices extending to opposite directions can be connected at a monopole junction point, as a 1/4 BPS state of a monopole in the Higgs phase [39]. Since this 1/8 BPS system contains monopoles and vortices extending to different three directions, two or more vortices may meet at a monopole junction point, and junctions may constitute a three-dimensional vortex-string network, not like two-dimensional $(p,q)$ string webs.
5.2.2 Hitchin vortex - Vortex - Wall

When we further dimensionally reduce the $x^3$-direction, we obtain a system of vortices and Hitchin vortices both living in the $x^1$-$x^2$ plane and walls perpendicular to the $x^2$ direction and walls to the $x^1$ direction in $d = 4$, $\mathcal{N} = 2$ gauge theory. The BPS equations for this system can be obtained as the dimensional reduction of the BPS equations (5.6) and (5.7), to give

$$D_1 \Sigma_3 + \eta D_2 \Sigma_4 = \xi_2 Y_2, \quad -D_2 \Sigma_3 + \eta D_1 \Sigma_4 = \xi_1 Y_1, \quad F_{12} - i \eta [\Sigma_3, \Sigma_4] = \xi_3 Y_3,$$  
(5.9)

$$\sum_{n=1}^2 (\tau_n)^i_j D_n H^j = -\xi_3 (\sigma_3)^i_j [ (\Sigma_3)^A B H^{jB} - m_{3,A} H^{jA} ] - i\eta [ (\Sigma_4)^A B H^{iB} - m_{4,A} H^{iA} ].$$
(5.10)

The energy of these 1/8 BPS states is given by

$$E = -\eta X + \eta \mathcal{H}_{12} + \xi_3 \mathcal{W}^{(3)}_{12}$$
$$+ \left( \xi_1 \mathcal{W}^{(1)}_{2,3} - \xi_2 \mathcal{W}^{(2)}_{2,4} \right) v_1 - \left( \xi_2 \mathcal{W}^{(2)}_{1,3} + \eta \xi_1 \mathcal{W}^{(1)}_{1,4} \right) v_2$$
(5.11)

with $X$ denoting the 5th component of the Poynting vector (4.16) or the electric charge (4.19) of the dyonic solitons. The configurations are summarized in the following table.

| HVW^2W^2 | 0 | 1 | 2 | (5) | sign |
|-----------|---|---|---|-----|-----|
| Hitchin vortex | × | × | × | 1_2 | $\eta$ |
| Wall | × | × | × | $i\sigma_1$ | $\xi_1$ |
| Wall | × | × | × | $i\sigma_2$ | $\xi_2$ |
| Vortex | × | × | × | $i\sigma_3$ | $\xi_3$ |
| Wall | × | × | × | $i\sigma_1$ | $-\eta \xi_1$ |
| Wall | × | × | × | $i\sigma_2$ | $-\eta \xi_2$ |

The 1/8 BPS equations (5.9) and (5.10) were eventually obtained in [49] in the case of $U(1)$ gauge theory coupled to two hypermultiplets with equal $U(1)$ charges, although the authors in [49] did not realize that they are 1/8 BPS and just analyzed 1/4 BPS wall junction (with Hitchin vortices) without ordinary vortices at that time. More general solutions would become wall junctions with vortices outside walls, but it is difficult to construct them at this stage.

Another interesting possibility of solitons is the Hitchin vortex inside vortices, both extending to fifth direction. For concreteness let us consider $U(2)$ gauge theory with two hypermultiplets in the fundamental representation with equal $U(1)$ charges. Since the Hitchin vortex live in
unbroken phase (of non-Abelian gauge group), it cannot live inside a single vortex, where only
U(1) gauge group is recovered. If positions of two vortices eventually coincide but their orienta-
tions are in the upper left and the lower right elements, U(2) gauge group is recovered at that coincident point (line). Then we expect that a Hitchin vortex lives inside those two coincident
vortices with gluing them together, to become a 1/8 BPS composite state of the Hitchin vortex
and two vortices.

When we perform the further SS dimensional reduction along the $x^2$-direction, we get another
set of 1/8 BPS equations in $d = 3$ (2) which contains the Nahm walls and walls (NW$^3$).

Before closing this section let us give a comment about supersymmetry in $d = 4$. After the SS
dimensional reductions twice along the $x^4$ and $x^3$-directions from $d = 6$, the projection operator
$\gamma^{34i}\sigma_3$ does not give any restriction for the BPS equations and just separate eight supercharges
to 2 sets of four supercharges. This means that the 1/8 BPS equations (5.9) and (5.10) can be
embedded into the model with $\mathcal{N} = 1$ in 4 dimensions as a 1/4 BPS equations.

## 5.3 Summary of 1/8 BPS systems

Since 1/8 BPS states preserve just one SUSY, a set of 1/8 BPS equations is the unique in the
maximal dimension $d = 6$. All 1/8 BPS equations in dimension less than six can be obtained
from it by dimensional reductions. We have written down BPS equations in $d = 5$ and $d = 4$,
but we have stopped it there because it is a simple task. All 1/8 BPS system in gauge theory
with eight supercharges are summarized in Table 4. One can find the rests of 1/8 BPS systems
in lower dimensions.

| $d = 6$ | $d = 5$ | $d = 4$ | $d = 3$ | $d = 2$ |
| --- | --- | --- | --- | --- |
| $\mathcal{N} = 1$ | $\mathcal{N} = 1$ | $\mathcal{N} = 2$ | $\mathcal{N} = 4$ | $\mathcal{N} = (4,4)$ |
| $\tilde{d} = 2$ | (w-)IV$^6$ | (w-)MV$^3$W$^3$ | (w-)HVW$^2$W$^2$ | (w-)NW$^3$ |
| $\tilde{d} = 1$ | - | (Q-)IV$^6$ | (Q-)MV$^3$W$^3$ | (Q-)HVW$^2$W$^2$ |
| | | | | (Q-)NW$^3$ |

Since all 1/2 or 1/4 BPS equations can be obtained by simply ignoring some projections,
their solutions automatically satisfy the unique 1/8 BPS equations (5.2) and (5.3). Therefore
1/8 BPS equations contain all possible BPS states in theories with eight supercharges. However, unlike the cases of 1/2 or 1/4 BPS states, solving 1/8 BPS equations is very difficult. It is still difficult to find even a special 1/8 BPS solution. That remains as a future problem.

6 Conclusion and Discussion

We have systematically derived 1/4 BPS and 1/8 BPS equations describing composite states made of various 1/2 BPS solitons in SUSY gauge theories in $d = 6, 5, 4, 3, 2$ with eight supercharges. First of all we have derived, in 6 dimensions, all the 1/2 BPS, 1/4 BPS and 1/8 BPS equations by clarifying the projection operators giving the Killing conditions on fermions of unbroken supersymmetry transformation. Then we have found their descendant BPS equations in lower dimensions by performing the SS and/or the trivial dimensional reductions. While we have found lots of new BPS equations, we have also rederived known BPS equations. Let us briefly summarize which are new contribution found in this paper and which are not.

We have found that the known 1/2 BPS objects in SUSY gauge theories with eight supercharges can be classified into two classes, according to the chirality of unbroken supercharges: the type IVa defined by the projections (2.10) and the type IVb defined by the projections (2.11), which preserve $\mathcal{N} = (2, 2)$ (non-chiral) SUSY and $\mathcal{N} = (4, 0)$ (chiral) SUSY, respectively, in the case of two dimensional world-volume. The type IVa contains the vortex equations (3.34) and (3.35) studied in Refs. [13]–[17] and domain-wall equations (3.38) and (3.39) studied in Refs. [20]–[30]. The type IVb contains the well established soliton equations: the instanton equations (3.7), the monopole equations (3.9), the Hitchin equation (3.11) and the Nahm equation (3.13). As a new result, we have found an extension of solitons in the type IVb to wavy solitons (3.17), along the $x^5$-direction of whose world-volume electric waves travel. We have constructed concrete solutions (3.23) of these wavy solitons. They are reduced to known dyonic solitons (3.26) by the dimensional reduction along the $x^5$-direction. The dyonic instantons [31] and dyons [32] are such solitons.

As to 1/4 BPS objects, we have classified them into the type IIa defined by the projections (2.12) and the type IIb defined by the projections (2.13) which preserve $\mathcal{N} = (1, 1)$ (non-chiral) SUSY and $\mathcal{N} = (2, 0)$ (chiral) SUSY, respectively, in the case of two dimensional world-volume.
Recently much attention have been paid to the type IIB solitons: the vortex-instanton system described by Eqs. (4.2) and (4.3) was analyzed in [43], the monopole-vortex-domain-wall system described by Eqs. (4.5) and (4.6) was analyzed in Refs. [37]–[42] and the system of domain wall junctions (4.8)-(4.11) was analyzed in Refs. [49]–[52]. We also have discussed in Sec. 4.1.4 possibility of new composite states made of Hitchin vortices and ordinary vortices, and of Nahm walls and domain walls. Besides the descent relations between these 1/4 BPS objects, we have pointed out that topological charges in the type IVb, namely the instanton charge, the monopole charge and the Hitchin charge, arise at junction points of different solitons. In the Abelian gauge theory the junction charges contribute negative masses to the total energy while the topological charges with positive masses arise in the non-Abelian gauge theory as usual. As a new result, we have found that the type IIB has the wavy and the dyonic extensions (4.14) with keeping the 1/4 BPS condition, much like solitons of the type IVb.

On the other hand, unlike the type IIB, the type IIA includes many new interesting equations which have not been studied yet. The new equations (4.21) and (4.22) describe intersecting vortex-strings extending to three different directions. These equations reduce in the strong gauge coupling limit to those for intersecting lump-strings in a hyper-Kähler sigma model [54, 55]. The other new equations (4.24) and (4.25) describe composite states of intersecting domain walls and vortex-strings which sit outside walls and are parallel to the intersection line. For both sets of equations, constructing solutions is not succeeded yet and remains as a future problem.

Finally, we have explored the unique set of the 1/8 BPS equations (5.2) and (5.3) in six dimensions from the type I projections (2.14). These equation describe instanton-strings and vortex-3-branes in six different directions. These equations (5.2) and (5.3) are completely new and are nicely written in terms of the quaternion. We then have obtained the 1/8 BPS descendants of Eqs. (5.2) and (5.3): the equations (5.6) and (5.7) describing monopoles with triply intersecting vortices and triply intersecting domain walls, and the equations (5.9) and (5.10) describing two kinds of intersecting domain walls with the Hitchin charge and vortices outside walls. We have pointed out a three-dimensional network of vortex-strings connected by monopoles as a possible solution of Eqs. (5.6) and (5.7). We have shown that all of these 1/8 BPS equations have the wavy and the dyonic extensions. We can call the equations (5.2) and (5.3) as the mother equations of the Yang-Mills-Higgs system in the sense that all the other BPS equations including 1/2 BPS and 1/4 BPS equations can be derived from these equations by the SS and/or the trivial dimensional
reductions and/or by truncating appropriate coupling constants and/or fields. Solving the mother equations (5.2) and (5.3) or determining their moduli space is one of final goal of this subject.

We present several discussion and future directions here.

**D-brane configurations.** In this paper we have considered general gauge group with general matter contents. The mostly discussed model is the $U(N_C)$ gauge theory coupled to hypermultiplets in the fundamental representation with equal $U(1)$ charge. This model can be realized as the effective gauge theory on $N_C$ D$p$-branes under the background of $N_F$ D$(p+4)$-branes separated to one, two, three and four directions, corresponding to real, complex, triplet and quartet masses for $p = 1, 2, 3, 4$, respectively. In that model, 1/2 BPS walls are realized as $N_C$ kinky D$p$-branes traveling between $N_F$ separated D$(p+4)$-branes [21, 27]. In the case of 1/4 BPS wall junctions [50] in Sect.4.1.4, two of transversal directions of D3-branes, in a plane orthogonal to D7-branes with complex positions, depend on two world-volume directions D3-branes [52]. Brane configurations for vortices were given by Hanany and Tong [14]. Duality between vortices and domain walls is understood as T-duality between corresponding brane configurations [10]. Brane configuration for a monopole in the Higgs was also obtained by Hanany and Tong [39]. Diverse BPS equations found in this paper will provide more variations of ample D-brane configurations.

**The Nahm transformation between solutions of vortices and walls and between composite states.** The type IVb equations are the members of the well-established series of instantons (SDYM), BPS monopoles, the Hitchin vortices, the Nahm walls (the Nahm data), if we take the Kaluza-Klein modes into account. The Nahm transformation maps between the ADHM/Nahm data and instanton/monopole solutions. Therefore we expect that there should exist a Nahm-like transformation among the type IVa equations for vortices and domain walls. We also suspect similar relation for the type Ib equations describing, for instance, instantons/monopoles in the Higgs phase.

**Higher co-dimensions/dimensions and higher SUSY.** The same analysis should be applied to theories with sixteen supercharges. The maximal dimension is ten in this case. Therefore BPS solitons with higher codimensions and/or in higher dimensions are possible. For instance a BPS soliton of co-dimension five was found in [70]. The Donaldson-Uhlenbeck-Yau instantons [71] are of co-dimension six and are 1/4 BPS in $d = 7$ [72, 63].

\[ \text{The octernionic instantons} \]

\[ \text{It has been recently pointed out in [73] that the SDYM-Higgs equations (4.2) and (4.3) in } d = 4 + 1 \text{ can} \]

39
are of co-dimension eight \([74, 75]\). Classification of these BPS solitons of higher codimensions is interesting extension.

*Non-commutative solitons.* We have not considered solitons in non-commutative space. Extensions of BPS equations to non-commutative solitons should be possible.

**Acknowledgements**

We are pleased to express much appreciation to Norisuke Sakai for fruitful discussions and continuous encouragements. We would like to thank Kimyeong Lee and Ho-Ung Yee for kindly informing us their results. We are also grateful to Kazutoshi Ohta and Nobuyoshi Ohta for useful comments. The work of K. O. and M. N. (M. E. and Y. I.) is supported by Japan Society for the Promotion of Science under the Post-doctoral (Pre-doctoral) Research Program. M. N. wishes to thank KIAS for their hospitality.

**References**

[1] E. B. Bogomolny, “Stability Of Classical Solutions,” Sov. J. Nucl. Phys. 24, 449 (1976) [Yad. Fiz. 24, 861 (1976)]; M. K. Prasad and C. M. Sommerfield, “An Exact Classical Solution For The 'T Hooft Monopole And The Julia-Zee Dyon,” Phys. Rev. Lett. 35, 760 (1975).

[2] E. Witten and D. I. Olive, “Supersymmetry Algebras That Include Topological Charges,” Phys. Lett. B 78, 97 (1978).

[3] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string,”; “String theory. Vol. 2: Superstring theory and beyond,” Cambridge, UK: Univ. Pr. (1998).

[4] A. Hanany and E. Witten, “Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics,” Nucl. Phys. B 492, 152 (1997) [arXiv:hep-th/9611230]; E. Witten, “Solutions of four-dimensional field theories via M-theory,” Nucl. Phys. B 500, 3 (1997) be derived from the Donaldson-Uhlenbeck-Yau equations in \(d = 6 + 1\) by the \(SU(2)\) equivariant dimensional reduction on \(S^2\), at least in the case of \(U(1)\) gauge group.
[arXiv:hep-th/9703166]; A. Giveon and D. Kutasov, “Brane dynamics and gauge theory,” Rev. Mod. Phys. 71, 983 (1999) [arXiv:hep-th/9802067].

[5] N. Ohta and P. K. Townsend, “Supersymmetry of M-branes at angles,” Phys. Lett. B 418, 77 (1998) arXiv:hep-th/9710129.

[6] N. Manton and P. Sutcliffe, “Topological solitons”, Cambridge Univ. Press, (2004).

[7] R. Rajaraman, “Solitons And Instantons. An Introduction To Solitons And Instantons In Quantum Field Theory,” Amsterdam, Netherlands: North-holland (1982).

[8] A. A. Belavin, A. M. Polyakov, A. S. Schwarz and Y. S. Tyupkin, “Pseudoparticle Solutions Of The Yang-Mills Equations,” Phys. Lett. B 59 (1975) 85.

[9] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, “Construction Of Instantons,” Phys. Lett. A 65, 185 (1978).

[10] P. Goddard, J. Nuyts and D. I. Olive, “Gauge Theories And Magnetic Charge,” Nucl. Phys. B 125, 1 (1977).

[11] W. Nahm, “A Simple Formalism For The BPS Monopole,” Phys. Lett. B 90, 413 (1980).

[12] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and H. Murayama, “NonAbelian monopoles,” Nucl. Phys. B 701, 207 (2004) arXiv:hep-th/0405070.

[13] A. A. Abrikosov, “On The Magnetic Properties Of Superconductors Of The Second Group,” Sov. Phys. JETP 5, 1174 (1957) [Zh. Eksp. Teor. Fiz. 32, 1442 (1957)]; H. B. Nielsen and P. Olesen, “Vortex-Line Models For Dual Strings,” Nucl. Phys. B61, 45 (1973).

[14] A. Hanany and D. Tong, “Vortices, instantons and branes,” JHEP 0307, 037 (2003) arXiv:hep-th/0306150; R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, “Nonabelian superconductors: Vortices and confinement in N = 2 SQCD,” Nucl. Phys. B 673, 187 (2003) arXiv:hep-th/0307287; M. Eto, M. Nitta and N. Sakai, “Effective theory on non-Abelian vortices in six dimensions,” Nucl. Phys. B 701, 247 (2004) arXiv:hep-th/0405161; V. Markov, A. Marshakov and A. Yung, “Non-Abelian vortices in N = 1* gauge theory,” Nucl. Phys. B 709, 267 (2005) arXiv:hep-th/0408239; S. Bolognesi, “The holomorphic tension of vortices,” JHEP 0501, 044 (2005) arXiv:hep-th/0411075; S. Bolognesi, “The holomorphic tension of nonabelian vortices and the quark = dual-quark condensate,” Nucl. Phys. B 719, 67 (2005) arXiv:hep-th/0412241; A. Gorsky, M. Shifman and A. Yung, “Non-Abelian Meissner effect in Yang-Mills theories at weak
coupling,” Phys. Rev. D 71, 045010 (2005) arXiv:hep-th/0412082; M. Shifman and A. Yung, “Non-abelian flux tubes in SQCD: Supersizing world-sheet supersymmetry,” arXiv:hep-th/0501211. S. Bolognesi and J. Evslin, “Stable vs unstable vortices in SQCD,” arXiv:hep-th/0506174.

[15] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Moduli space of non-Abelian vortices,” Phys. Rev. Lett. 96, 161601 (2006) arXiv:hep-th/0511088.

[16] M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, “Non-Abelian vortices on cylinder: Duality between vortices and walls,” Phys. Rev. D 73, 085008 (2006) arXiv:hep-th/0601181.

[17] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Manifestly supersymmetric effective Lagrangians on BPS solitons,” Phys. Rev. D 73, 125008 (2006) arXiv:hep-th/0602289.

[18] M. Cvetic, F. Quevedo and S. J. Rey, “Stringy domain walls and target space modular invariance,” Phys. Rev. Lett. 67, 1836 (1991); M. Cvetic, S. Griffies and S. J. Rey, “Static domain walls in N=1 supergravity,” Nucl. Phys. B 381, 301 (1992) arXiv:hep-th/9201007.

[19] G. R. Dvali and M. A. Shifman, “Dynamical compactification as a mechanism of spontaneous supersymmetry breaking,” Nucl. Phys. B 504, 127 (1997) arXiv:hep-th/9611123.

[20] E. R. C. Abraham and P. K. Townsend, “Q kinks,” Phys. Lett. B 291, 85 (1992); “More on Q kinks: A (1+1)-dimensional analog of dyons,” Phys. Lett. B 295, 225 (1992).

[21] N. D. Lambert and D. Tong, “Kinky D-strings,” Nucl. Phys. B 569, 606 (2000) arXiv:hep-th/9907098.

[22] J. P. Gauntlett, D. Tong and P. K. Townsend, “Multi-domain walls in massive supersymmetric sigma-models,” Phys. Rev. D 64, 025010 (2001) arXiv:hep-th/0012178; D. Tong, “The moduli space of BPS domain walls,” Phys. Rev. D 66, 025013 (2002) arXiv:hep-th/0202012; “Mirror mirror on the wall: On two-dimensional black holes and Liouville JHEP 0304, 031 (2003) arXiv:hep-th/0303151; K. S. M. Lee, “An index theorem for domain walls in supersymmetric gauge theories,” Phys. Rev. D 67, 045009 (2003) arXiv:hep-th/0211058; M. Shifman and A. Yung, “Localization of non-Abelian gauge fields on domain walls at weak coupling (D-brane prototypes II),” Phys. Rev. D 70, 025013 (2004) arXiv:hep-th/0312257;
[23] M. Arai, M. Naganuma, M. Nitta, and N. Sakai, “Manifest supersymmetry for BPS walls in $N = 2$ nonlinear sigma models,” Nucl. Phys. B 652, 35 (2003) [arXiv:hep-th/0211103]; “BPS Wall in N=2 SUSY Nonlinear Sigma Model with Eguchi-Hanson Manifold” in Garden of Quanta - In honor of Hiroshi Ezawa, Eds. by J. Arafune et al. (World Scientific Publishing Co. Pte. Ltd. Singapore, 2003) pp 299-325, [arXiv:hep-th/0302028]; M. Arai, E. Ivanov and J. Niederle, Nucl. Phys. B 680, 23 (2004) [arXiv:hep-th/0312037].

[24] Y. Isozumi, K. Ohashi, and N. Sakai, “Exact wall solutions in 5-dimensional SUSY QED at finite coupling,” JHEP 0311, 060 (2003) [arXiv:hep-th/0310189]; “Massless localized vector field on a wall in $D = 5$ SQED with tensor multiplets,” JHEP 0311, 061 (2003) [arXiv:hep-th/0310130].

[25] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Construction of non-Abelian walls and their complete moduli space,” Phys. Rev. Lett. 93, 161601 (2004) [arXiv:hep-th/0404198]; “Non-Abelian walls in supersymmetric gauge theories,” Phys. Rev. D 70, 125014 (2004) [arXiv:hep-th/0405194];

[26] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Non-Abelian walls and vortices in supersymmetric theories,” in the Proceedings of 12th International Conference on in the Supersymmetry and Unification of Fundamental Interactions (SUSY 04), Tsukuba, Japan, 17-23 Jun 2004, edited by K. Hagiwara et al. (KEK, 2004) p.1 - p.16 [arXiv:hep-th/0409110]; “Walls and vortices in supersymmetric non-Abelian gauge theories,” in the Proceedings of “NathFest” at PASCOS conference, Northeastern University, Boston, Ma, August 2004, edited by G. Alverston, M.T. Vaughn. Hackensack, (World Scientific, 2005) p.229 - p.238 [arXiv:hep-th/0410150]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Solitons in supersymmetric gauge theories,” AIP Conf. Proc. 805, 266 (2005) [arXiv:hep-th/0508017].

[27] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, “D-brane construction for non-Abelian walls,” Phys. Rev. D 71, 125006 (2005) [arXiv:hep-th/0412024].

[28] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai and Y. Tachikawa, “Global structure of moduli space for BPS walls,” Phys. Rev. D 71, 105009 (2005) [arXiv:hep-th/0503033].

[29] N. Sakai and Y. Yang, “Moduli space of BPS walls in supersymmetric gauge theories”, Comm. Math. Phys. [arXiv: hep-th/0505136].
[30] A. Hanany and D. Tong, “On monopoles and domain walls,” arXiv:hep-th/0507140.

[31] N. D. Lambert and D. Tong, “Dyonic instantons in five-dimensional gauge theories,” Phys. Lett. B 462, 89 (1999) arXiv:hep-th/9907014.

[32] B. Julia and A. Zee, “Poles With Both Magnetic And Electric Charges In Nonabelian Gauge Theory,” Phys. Rev. D 11, 2227 (1975).

[33] R. A. Leese, “Q lumps and their interactions,” Nucl. Phys. B 366, 283 (1991); E. Abraham, “Nonlinear sigma models and their Q lump solutions,” Phys. Lett. B 278, 291 (1992).

[34] R. S. Ward, “Slowly Moving Lumps In The Cp**1 Model In (2+1)-Dimensions,” Phys. Lett. B 158, 424 (1985); I. Stokoe and W. J. Zakrzewski, “Dynamics Of Solutions Of The Cp**1 And Cp**2 Models In (2+1)-Dimensions,” Z. Phys. C 34, 491 (1987).

[35] A. M. Polyakov and A. A. Belavin, “Metastable States Of Two-Dimensional Isotropic Ferromagnets,” JETP Lett. 22, 245 (1975) [Pisma Zh. Eksp. Teor. Fiz. 22, 503 (1975)]; A. M. Perelomov, “Chiral Models: Geometrical Aspects,” Phys. Rept. 146, 135 (1987).

[36] T. Vachaspati and A. Achucarro, “Semilocal cosmic strings,” Phys. Rev. D 44, 3067 (1991); “Semilocal and electroweak strings,” Phys. Rept. 327, 347 (2000) [Phys. Rept. 327, 427 (2000)] arXiv:hep-ph/9904229.

[37] J. P. Gauntlett, R. Portugues, D. Tong and P. K. Townsend, “D-brane solitons in supersymmetric sigma-models,” Phys. Rev. D 63, 085002 (2001) arXiv:hep-th/0008221.

[38] M. Shifman and A. Yung, “Domain walls and flux tubes in N = 2 SQCD: D-brane prototypes,” Phys. Rev. D 67, 125007 (2003) arXiv:hep-th/0212293.

[39] D. Tong, “Monopoles in the Higgs phase,” Phys. Rev. D 69, 065003 (2004) arXiv:hep-th/0307302; R. Auzzi, S. Bolognesi, J. Evslin and K. Konishi, “Non-abelian monopoles and the vortices that confine them,” Nucl. Phys. B 686, 119 (2004) arXiv:hep-th/0312233; A. Hanany and D. Tong, “Vortex strings and four-dimensional gauge dynamics,” JHEP 0404, 066 (2004) arXiv:hep-th/0403158; M. Shifman and A. Yung, “Non-Abelian string junctions as confined monopoles,” Phys. Rev. D 70, 045004 (2004) arXiv:hep-th/0403149; R. Auzzi, S. Bolognesi and J. Evslin, “Monopoles can be confined by 0, 1 or 2 vortices,” JHEP 0502, 046 (2005) arXiv:hep-th/0411074.

[40] M. A. C. Kneipp and P. Brockill, “BPS string solutions in non-Abelian Yang-Mills theories,” Phys. Rev. D 64, 125012 (2001) arXiv:hep-th/0104171; M. A. C. Kneipp, “Z(k)
string fluxes and monopole confinement in non-Abelian theories,” Phys. Rev. D 68, 045009 (2003) [arXiv:hep-th/0211049]; “Color superconductivity, Z(N) flux tubes and monopole confinement in deformed N = 2* super Yang-Mills theories,” Phys. Rev. D 69, 045007 (2004) [arXiv:hep-th/0308086]; “Color superconductivity, BPS strings and monopole confinement in N = 2 and N = 4 super Yang-Mills theories,” Braz. J. Phys. 34, 1335 (2004) [arXiv:hep-th/0401234].

[41] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “All exact solutions of a 1/4 Bogomol’nyi-Prasad-Sommerfield equation,” Phys. Rev. D 71, 065018 (2005) [arXiv:hep-th/0405129].

[42] N. Sakai and D. Tong, “Monopoles, vortices, domain walls and D-branes: The rules of interaction,” JHEP 0503, 019 (2005) [arXiv:hep-th/0501207]; R. Auzzi, M. Shifman and A. Yung, “Studying boojums in N = 2 theory with walls and vortices,” Phys. Rev. D 72, 025002 (2005) [arXiv:hep-th/0504148].

[43] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Instantons in the Higgs phase,” Phys. Rev. D 72, 025011 (2005) [arXiv:hep-th/0412048].

[44] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Solitons in the Higgs phase: The moduli matrix approach,” J. Phys. A 39, R315 (2006) [arXiv:hep-th/0602170].

[45] E. R. C. Abraham and P. K. Townsend, “Intersecting Extended Objects In Supersymmetric Field Theories,” Nucl. Phys. B 351, 313 (1991).

[46] G. W. Gibbons and P. K. Townsend, “A Bogomolnyi equation for intersecting domain walls,” Phys. Rev. Lett. 83, 1727 (1999) [arXiv:hep-th/9905196]; S. M. Carroll, S. Hellerman and M. Trodden, “Domain wall junctions are 1/4-BPS states,” Phys. Rev. D 61, 065001 (2000) [arXiv:hep-th/9905217].

[47] H. Oda, K. Ito, M. Naganuma and N. Sakai, “An exact solution of BPS domain wall junction,” Phys. Lett. B 471, 140 (1999) [arXiv:hep-th/9910095]; K. Ito, M. Naganuma, H. Oda and N. Sakai, “Nonnormalizable zero modes on BPS junctions,” Nucl. Phys. B 586, 231 (2000) [arXiv:hep-th/0004188]; “An exact solution of BPS junctions and its properties,” Nucl. Phys. Proc. Suppl. 101, 304 (2001) [arXiv:hep-th/0012182]; M. Naganuma, M. Nitta and N. Sakai, “BPS walls and junctions in SUSY nonlinear sigma models,” Phys. Rev. D 65, 045016 (2002) [arXiv:hep-th/0108179]; “BPS walls and junctions in N = 1 SUSY nonlinear
sigma models,” Proceedings of 3rd International Sakharov Conference On Physics, edited by A. Semikhatov et al. (Scientific World Pub., 2003) p.537 - p.549, [arXiv:hep-th/0210205].

[48] J. P. Gauntlett, D. Tong and P. K. Townsend, “Supersymmetric intersecting domain walls in massive hyper-Kaehler sigma models,” Phys. Rev. D 63, 085001 (2001) [arXiv:hep-th/0007124].

[49] K. Kakimoto and N. Sakai, “Domain wall junction in N = 2 supersymmetric QED in four dimensions,” Phys. Rev. D 68, 065005 (2003) [arXiv:hep-th/0306077].

[50] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Webs of walls,” Phys. Rev. D 72, 085004 (2005) [arXiv:hep-th/0506135].

[51] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Non-abelian webs of walls,” Phys. Lett. B 632, 384 (2006) [arXiv:hep-th/0508241].

[52] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, “D-brane configurations for domain walls and their webs,” AIP Conf. Proc. 805, 354 (2006) [arXiv:hep-th/0509127].

[53] S. B. Bradlow, “Vortices In Holomorphic Line Bundles Over Closed Kahler Manifolds,” Commun. Math. Phys. 135, 1 (1990); I. Mundet i Riera, “Yang-Mills-Higgs theory for symplectic fibrations,” arXiv:math.sg/9912150; “A Hitchin-Kobayashi correspondence for Kahler fibrations,” J. Reine Angew. Math. 528, 41 (2000); K. Cieliebak, A. Rita Gaio, D. A. Salamon, “J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions,” Internat. Math. Res. Notices, 831 (2000) arXiv:math.SG/9909122; J. M. Baptista, “Vortex equations in abelian gauged sigma-models,” Commun. Math. Phys. 261, 161 (2006) arXiv:math.dg/0411517.

[54] M. Naganuma, M. Nitta and N. Sakai, “BPS lumps and their intersections in N = 2 SUSY nonlinear sigma models,” Grav. Cosmol. 8, 129 (2002) [arXiv:hep-th/0108133].

[55] R. Portugues and P. K. Townsend, “Sigma-model soliton intersections from exceptional calibrations,” JHEP 0204, 039 (2002) [arXiv:hep-th/0203181].

[56] U. Lindström and M. Roček, “Scalar Tensor Duality And N=1, N=2 Nonlinear Sigma Models,” Nucl. Phys. B 222, 285 (1983); N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, “Hyperkahler Metrics And Supersymmetry,” Commun. Math. Phys. 108, 535 (1987).
[57] K. M. Lee and H. U. Yee, “New BPS Objects in N=2 Supersymmetric Gauge Theories,” arXiv:hep-th/0506256.

[58] L. Alvarez-Gaume and D. Z. Freedman, “Potentials For The Supersymmetric Nonlinear Sigma Model,” Commun. Math. Phys. 91, 87 (1983).

[59] M. Arai, M. Nitta and N. Sakai, “Vacua of massive hyper-Kaehler sigma models of non-Abelian quotient,” Prog. Theor. Phys. 113, 657 (2005) arXiv:hep-th/0307274; to appear in the Proceedings of the 3rd International Symposium on Quantum Theory and Symmetries (QTS3), September 10-14, 2003, arXiv:hep-th/0401084; to appear in the Proceedings of the International Conference on “Symmetry Methods in Physics (SYM-PHYS10)” held at Yerevan, Armenia, 13-19 Aug. 2003 arXiv:hep-th/0401102; to appear in the Proceedings of SUSY 2003 held at the University of Arizona, Tucson, AZ, June 5-10, 2003 arXiv:hep-th/0402065.

[60] G. W. Gibbons, P. Rychenkova and R. Goto, “HyperKaehler quotient construction of BPS monopole moduli spaces,” Commun. Math. Phys. 186, 585 (1997) arXiv:hep-th/9608085.

[61] S. A. Cherkis and A. Kapustin, “Nahm transform for periodic monopoles and N = 2 super Yang-Mills theory,” Commun. Math. Phys. 218, 333 (2001) arXiv:hep-th/0006050.

[62] I. Antoniadis, H. Partouche and T. R. Taylor, “Spontaneous Breaking of N=2 Global Supersymmetry,” Phys. Lett. B 372, 83 (1996) arXiv:hep-th/9512006; K. Fujiwara, H. Itoyama and M. Sakaguchi, “Supersymmetric U(N) gauge model and partial breaking of N = 2 supersymmetry,” Prog. Theor. Phys. 113, 429 (2005) arXiv:hep-th/0409060; “Partial breaking of N = 2 supersymmetry and of gauge symmetry in the U(N) gauge model,” Nucl. Phys. B 723, 33 (2005) arXiv:hep-th/0503113.

[63] P. K. Townsend, “Field theory supertubes,” Comptes Rendus Physique 6, 271 (2005) arXiv:hep-th/0411206.

[64] E. Witten, “Sigma models and the ADHM construction of instantons,” J. Geom. Phys. 15, 215 (1995) arXiv:hep-th/9410052.

[65] E. Witten, “Small Instantons in String Theory,” Nucl. Phys. B460, 541 (1996) arXiv:hep-th/9511030; M. R. Douglas, “Branes within branes,” arXiv:hep-th/9512077
[66] M. B. Green and M. Gutperle, “Comments on Three-Branes,” Phys. Lett. B 377, 28 (1996) [arXiv:hep-th/9602077]; D. E. Diaconescu, “D-branes, Monopoles and Nahm Equations,” Nucl. Phys. B 503, 220 (1997) [arXiv:hep-th/9608163].

[67] P. S. Howe and G. Papadopoulos, “Ultraviolet Behavior Of Two-Dimensional Supersymmetric Nonlinear Sigma Models,” Nucl. Phys. B 289, 264 (1987); “Further Remarks On The Geometry Of Two-Dimensional Nonlinear Sigma Models,” Class. Quant. Grav. 5, 1647 (1988); P. S. Howe, A. Opfermann and G. Papadopoulos, “Twistor spaces for QKT manifolds,” Commun. Math. Phys. 197, 713 (1998) [arXiv:hep-th/9710072].

[68] B. Zumino, “Supersymmetry And Kahler Manifolds,” Phys. Lett. B 87, 203 (1979).

[69] C. M. Hull and E. Witten, “Supersymmetric Sigma Models And The Heterotic String,” Phys. Lett. B 160, 398 (1985); E. Witten, “Two-dimensional models with (0,2) supersymmetry: Perturbative aspects,” [arXiv:hep-th/0504078]

[70] H. Kihara, Y. Hosotani and M. Nitta, “Generalized monopoles in six-dimensional non-Abelian gauge theory,” Phys. Rev. D 71, 041701 (2005) [arXiv:hep-th/0408068]; E. Radu and D. H. Tchrakian, “Static BPS ‘monopoles’ in all even spacetime dimensions,” Phys. Rev. D 71, 125013 (2005) [arXiv:hep-th/0502025].

[71] S. K. Donaldson, “Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles,” Proc. Lond. Math. Soc. 50, 1 (1985); “Infinite determinants, stable bundles, and curvature,” Duke Math. J. 54, 231 (1987); K. Uhlenbeck and S. T. Yau, “On the existence of Hermitian Yang-Mills connections in stable bundles,” Commun. Pure Appl. Math. 39, 257 (1986).

[72] D. S. Bak, K. M. Lee and J. H. Park, “BPS equations in six and eight dimensions,” Phys. Rev. D 66, 025021 (2002) [arXiv:hep-th/0204221].

[73] A. D. Popov and R. J. Szabo, “Quiver gauge theory of nonabelian vortices and noncommutative instantons in higher dimensions,” J. Math. Phys. 47, 012306 (2006) [arXiv:hep-th/0504025]; O. Lechtenfeld, A. D. Popov and R. J. Szabo, “Rank two quiver gauge theory, graded connections and noncommutative vortices,” [arXiv:hep-th/0603232]

[74] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, “First Order Equations For Gauge Fields In Spaces Of Dimension Greater Than Four,” Nucl. Phys. B 214, 452 (1983).
[75] B. Grossman, T. W. Kephart and J. D. Stasheff, “Solutions To Yang-Mills Field Equations In Eight-Dimensions And The Last Hopf Map,” Commun. Math. Phys. 96, 431 (1984) [Erratum-ibid. 100, 311 (1985)]; D. H. Tchrakian, “Spherically Symmetric Gauge Field Configurations With Finite Action In 4 P-Dimensions (P = Integer),” Phys. Lett. B 150, 360 (1985).