REMARKS ON THE DEFINITION OF A COURANT ALGEBROID

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Abstract. The notion of Courant algebroid was introduced by Liu, Weinstein and Xu in 1997. Its definition consists of five axioms and an defining relation for a derivation. It is shown that two of the axioms and the relation (assuming only the Leibniz rule) follow from the rest of the axioms.

1. Introduction

The Courant bracket on $TM \oplus T^*M$, where $M$ is a smooth manifold, was defined in 1990 by T.Courant [1]. A similar bracket was defined on the double of any Lie bialgebroid by Liu, Weinstein, and Xu [2]. They show that the double is not a Lie algebroid, but a more complicated object which they call a Courant algebroid.

Definition of a Courant algebroid [2]
A Courant algebroid is a vector bundle $E \rightarrow M$ equipped with a nondegenerate symmetric bilinear form $(\ , \ )$, a skew-symmetric bracket $[\ , \ ]$ on $\Gamma E$ and a bundle map $\rho : E \rightarrow TM$ satisfying the following relations:

[C1] $[[x, y], z] + [[z, x], y] + [[y, z], x] = DT(x, y, z) \forall x, y, z \in \Gamma E$,

[C2] $\rho[x, y] = [\rho(x), \rho(y)] \forall x, y \in \Gamma E$,

[C3] $[x, fy] = f[x, y] + (\rho(x)f)y - (x, y)Df \forall x, y \in \Gamma E, \forall f \in C^\infty(M)$,

[C4] $\rho \circ D = 0$, i.e., $(Df, Dg) = 0 \forall f, g \in C^\infty(M)$,

[C5] $\rho(x)(y, z) = ([x, y] + D(x, y), z) + (y, [x, z] + D(x, z)) \forall x, y, z \in \Gamma E$,

where $T(x, y, z) = \frac{1}{3}(([x, y], z) + ([y, z], x) + ([z, x], y))$, and $D : C^\infty(M) \rightarrow \Gamma E$ is the map given by the composition of the dual map $\rho^*$ of $\rho$ and the usual differential $d$, $D = \frac{1}{2}\beta^{-1}\rho^*d$, $\beta$ being the isomorphism between $E$ and $E^*$ given by the bilinear form. Here $\Gamma E$ denotes the space of all smooth sections of $E$.

Remark 1. $D = \frac{1}{2}\beta^{-1}\rho^*d$ is equivalent to $(Df, x) = \frac{1}{2}\rho(x)f \forall x \in \Gamma E, \forall f \in C^\infty(M)$.

In this paper we show that [C3] and [C4] and the property of $D$, i.e., $D = \frac{1}{2}\beta^{-1}\rho^*d$, follow from [C2], [C5] and the Leibniz rule for $D$, (L) below.
2. Main Results

We assume [C2] and [C5] in the definition above, and only that $D$ is a map from $C^\infty(M)$ to $\Gamma E$ satisfying [C5] and the Leibniz rule:

$$(L) \quad D(fg) = fD(g) + gD(f), \quad \forall f, g \in C^\infty(M).$$

We then have

**Proposition 2.1.**

(i) $[x, fy] = f[x, y] + \rho(x)(f)y - (x, y)Df, \quad \forall x, y \in \Gamma E, \forall f \in C^\infty(M),$

(ii) $\rho \circ D = 0, \forall x, y \in \Gamma E.$

**Proof.** First we show (i). By [C5] we have

$$\rho(x)(fy, z) = ([x, fy] + D(xfy), z) + (fy, [x, z] + D(xz)).$$

On the other hand, the Leibniz rule gives

$$\rho(x)(fy, z) = f\rho(x)(yz) + (yz)\rho(x)(f),$$

$$= (f[x, y] + fD(x, y), z) + (fy, [x, z] + D(xz)) + (yz)\rho(x)(f).$$

Hence we have

$$(f[x, y] + fD(x, y), z) + (y, z)\rho(x)(f) = ([x, fy] + D(x, fy), z).$$

Since $(\ , \ )$ is nondegenerate we obtain

$$f[x, y] + fD(x, y) + \rho(x)(fy) = [x, fy] + D(x, fy).$$

By the Leibniz rule we have $D(x, fy) = fD(x, y) + (x, y)Df$ and then get (i), $[x, fy] = f[x, y] + \rho(x)(fy) - (x, y)Df.$

Next we show (ii). By [C2] we have

$$\rho(x, fy) = [\rho(x), f\rho(y)] = f[\rho(x), \rho(y)] + \rho(x)(f)\rho(y).$$

Then applying $\rho$ to the both sides of (i) gives the desired result. ∎

Next we show a map $D$ satisfying [C5] and the Leibniz rule (L) can be written $D = \frac{1}{2}\beta^{-1} \rho^* d.$ By Remark 1, it suffices to show that

**Proposition 2.2.**

$$(Df, y) = \frac{1}{2}\rho(y)(f), \quad \forall y \in \Gamma E, \forall f \in C^\infty(M).$$

**Proof.** If $y = 0$ the identity is trivial. We assume $y \neq 0.$

$$\forall x, y \in \Gamma E \forall f \in C^\infty(M), [C5] \text{ gives }$$

$$\rho(fx)(y, y) = ([fx, y] + D(fx, y), y) + ([fx, y] + D(fx, y), y).$$

We apply Proposition 1 (i) and the Leibniz rule for $D$ to the line above and get

$$\rho(fx)(y, y) = ([fx, y] - \rho(y)(fx) - (x, y)Df + D(fx, y) + (x, y)Df, y)$$

$$+ (fx, y - \rho(y)(fx) + (x, y)Df + D(fx, y) + (x, y)Df, y),$$

$$= f([x, y] + D(x, y), y) + f([x, y] + D(x, y), y)$$

$$+ (-\rho(y)(fx) + 2(x, y)Df, y) + (-\rho(y)(fx) + 2(x, y)Df, y).$$
Again by [C5]
\[ \rho(fx)(y, y) = f\rho(x)(y, y) + 2(-\rho(y)(f)x + 2(x, y)Df, y). \]
Hence we have
\[ 0 = (-\rho(y)(f)x + 2(x, y)Df, y) = -\rho(y)(f)(x, y) + 2(x, y)(Df, y). \]
Since \((, )\) is nondegenerate we obtain
\[ 0 = -\rho(y)(f)y + 2(Df, y)y, = (2(Df, y) - \rho(y)(f))y, \]
which gives \((Df, y) = \frac{1}{2}\rho(y)(f)\).

**Remark 2.1.** Conversely, the Leibniz rule \((L)\) can be derived from Proposition 1(i).
Namely, if a map \(D : C^\infty(M) \to \Gamma E\) satisfies the identity in Proposition 1(i), then
\(D\) fulfills the Leibniz rule \((L)\).

In fact, by Proposition 1 (i) we have
\[ (x, y)Df = f[x, y] + (\rho(x)f)y - [x, fy]. \]
Since right-hand side is in \(\Gamma E\) and \((x, y)\) in \(C^\infty(M)\), we obtain \(Df \in \Gamma E\). Applying Proposition 1 (i) to \([x, (fg)y]\) and \([x, f(gy)]\), we obtain
\[ D(fg) = fD(g) + gD(f). \]

**Remark 2.2.** For a Lie algebroid, Grabowski and Marmo showed that a homomorphism property of \(\rho\) with respect to \([, ]\), i.e., [C2], follows from the rest of the axioms of a Lie algebroid. It is thus natural to ask if, in a Courant algebroid, one can derive [C2] from the rest of the axioms. At present, this is still uncertain.

We can however show the following.
If we assume only [C5] and \((L)\), then
\[ (A) \quad \rho(x)\rho(y)(f) = \rho(x \circ y)(f) + \rho(y)\rho(x)(f) + 2([x, Df] - D(x, Df), y), \]
where \(x \circ y := [x, y] + D(x, y)\). Indeed by [C5] and Proposition 2, we have
\[ \rho(x)\rho(y)(f) = 2(x \circ y, Df) + 2(y, x \circ Df) + \rho(y)\rho(x)(f) - \rho(y)\rho(x)(f). \]
Again by Proposition 2, we obtain
\[ 2(x \circ y, Df) = \rho(x \circ y)(f) \quad \text{and} \quad \rho(y)\rho(x)(f) = 4(y, D(x, Df)). \]
Then we have \((A)\).

By \((A)\) we see [C2] is equivalent to \([x, Df] = D(x, Df)\) as follows. Since \([x, y] = \frac{1}{2}([x \circ y - y \circ x], \rho(x, y) = \frac{1}{2}(\rho(x \circ y) - \rho(y \circ x))\). If \([x, Df] = D(x, Df)\) then \((A)\) gives \(\rho(x \circ y) = [\rho(x), \rho(y)]\) and then we have [C2]. Conversely, if [C2] is assumed then \(\rho \circ D = 0\) by Proposition 1 (ii), which gives \(\rho(x \circ y) = \rho([x, y] + D(x, y)) = \rho[x, y] = [\rho(x), \rho(y)]\). Hence \([x, Df] = D(x, Df)\).

It is to be noticed that a Courant algebroid has the property \([x, Df] = D(x, Df)\) \((\text{[C1]} \text{[C2]})\).

**Remark 2.3.** It is known that a Courant algebroid is given equivalently by the following axioms.

**Alternative definition of a Courant algebroid**

A Courant algebroid is a vector bundle \(E \to M\) equipped with a nondegenerate symmetric bilinear form \((, )\), a non-skew-symmetric bilinear operation \(\circ\) on \(\Gamma E\) and a bundle map \(\rho : E \to TM\) satisfying the following relations:

\[ [C'1] \quad x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z) \quad \forall x, y, z \in \Gamma E, \]
[C’2] \( \rho(x \circ y) = [\rho(x), \rho(y)] \quad \forall x, y \in \Gamma E, \)

[C’3] \( x \circ fy = f(x \circ y) + (\rho(x)f)y \quad \forall x, y \in \Gamma E, \forall f \in C^\infty(M), \)

[C’4] \( x \circ x = D(x, x) \quad \forall x \in \Gamma E, \)

[C’5] \( \rho(x)(y, z) = (x \circ y, z) + (y, x \circ z) \quad \forall x, y, z \in \Gamma E, \)

where \( D : C^\infty(M) \rightarrow \Gamma E \) is the map defined by \( D = \frac{1}{2} \beta^{-1} \rho^*d, \beta \) being the isomorphism between \( E \) and \( E^* \) given by the bilinear form. Here \( \Gamma E \) denotes the space of all smooth sections of \( E \).

The above axioms [C’2], [C’3] and assumption for the derivation \( D = \frac{1}{2} \beta^{-1} \rho^*d \) are also derived from the rest of the axioms and condition (L) in a similar way.

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