Abstract. We prove that there is no class-dual for almost all sublinear models on graphs.

1 Introduction

We prove the following result:

Theorem 1. There is no class-dual of a sublinear function with probability one.

The proof is based on material presented in

[1] B. Jain. Flip-Flop Sublinear Models for Graphs, S+SSPR 2014.

The content of [1] is not included in this contribution. Section 1 introduces the formalism necessary to derive the proof. Section 2 proves auxiliary results. Finally, the proof of Theorem 1 is presented in Section 3.

2 Preliminaries

Let $G$ be a permutation group and $X_G$ be the quotient of the group action $G$ on the matrix space $X = \mathbb{R}^{n \times n}$, where $n$ is some number not less than the number of vertices of the largest graphs. By $\pi$ we denote the natural projection from $X$ to $X_G$. For details we refer to [12]. The stabilizer of $w \in X$ is a subgroup defined by

$$G_w = \{ \gamma \in G : \gamma w = w \}.$$ 

If $G_w$ is the trivial group, then each element of the orbit $[w]$ has a trivial stabilizer. A graph $X$ is said to be regular, if there is a representation $x \in X$ with trivial stabilizer $G_x$.

Suppose that $f(X) = W \cdot X + b$ is a sublinear function. A cross section with basepoint $u \in X$ is an injective map $\phi : X_G \to X$ satisfying

1. $W \cdot X = \phi(W)^T \phi(X)$
2. $\pi \circ \phi(X) = X$
for all $X \in \mathcal{X}_G$. We may regard the map $\phi$ as an isometric embedding of the graph space into some Euclidean space with respect to $w = \phi(W)$. Note that $\phi$ is not uniquely determined, even for fixed $w = \phi(W)$. The closure of the image $\phi(X)$ is the Dirichlet (fundamental) domain with basepoint (center) $w$ defined by

$$D_w = \left\{ x \in \mathcal{X} : w^T x \geq \tilde{w}^T x, \tilde{w} \in W \right\}.$$ 

A Dirichlet domain is a convex polyhedral cone of dimension $\dim(\mathcal{X})$ with the following properties: (1) $D_w$ is well-defined, (2) $x \in D_w$ iff $(x, w)$ is an optimal alignment, (3) $\pi(D_w) = X_G$, and (4) $\pi$ is injective on the interior of $D_w$ [2].

3 Auxiliary Results

**Lemma 1.** Let $\phi : \mathcal{X}_G \to \mathcal{X}$ be a fundamental cross section with basepoint $w$. Then $\gamma \circ \phi$ is a fundamental cross section with basepoint $\gamma w$ for all $\gamma \in G$.

**Proof.** From the axioms of a group action follows that $\gamma$ is bijective. A cross section $\phi$ is injective by definition. Then the composition $\phi' = \gamma \circ \phi$ is also injective. We show that $W \cdot X = \phi'(W)^T \phi'(X)$. We have

$$\phi'(W)^T \phi'(X) = \gamma w^T \gamma x,$$

where $w = \phi(W)$ and $x = \phi(X)$. Since $G$ is a permutation group, the mapping $\gamma$ is orthogonal. As an orthogonal mapping, $\gamma$ preserves the inner product, that is $\gamma w^T \gamma x = w^T x$. Since $\phi$ is a fundamental cross section with basepoint $w$, we have

$$W \cdot X = \phi(W)^T \phi(X) = w^T x = \gamma w^T \gamma x = \phi'(W)^T \phi'(X).$$

Note that this part also shows that the $\phi'$ is a cross section with basepoint $\gamma w$.

Finally, we show that $\pi \circ \phi'(X) = X$ for all $X \in \mathcal{X}_G$. Since $\phi$ is a fundamental cross section, the vector $x = \phi(X)$ projects to $X$ via $\pi$. The graph $X$ can be regarded as the orbit $[x]$ of $x$. As an element of $[x]$ the vector $\gamma x$ also projects to $X$ via $\pi$. This shows $\pi \circ \phi'(X) = X$. \qed

**Lemma 2.** Let $\phi : \mathcal{X}_G \to \mathcal{X}$ be a fundamental cross section with basepoint $w$ and Dirichlet domain $D_w$. Then $w$ is an interior point of $D_w$ if and only if $W = \pi(w)$ is regular.

**Proof.** Let $w$ be an interior point of $D_w$. We assume that $W$ is not regular. Then there is a $\gamma \in G \setminus \{\text{id}\}$ such that $w = \gamma w$. According to Lemma 1, the composition $\phi' = \gamma \circ \phi$ is a fundamental cross section with basepoint $\gamma w$ and Dirichlet domain $\gamma D_w = D_w$. From $w = \gamma w$ follows that

$$D = \text{int} D_w \cap \gamma D_w \neq \emptyset,$$

where $\text{int} S$ denotes the interior of a set $S \subseteq \mathcal{X}$. As an intersection of open convex sets, the set $D$ is also open and convex. Then there are $n+1$ points $x_0, \ldots, x_n \in D$.
in general position, where \( n = \dim(X) \). By definition of a Dirichlet domain, we have
\[
w^T x_i = w^T \gamma x_i
\]
for all \( i \in \{0, \ldots, n\} \). This implies \( x_i = \gamma x_i \), because cross sections are injective. Observe that \( G \) acts isometrically on \( X \). Since two isometries are the same if they coincide on \( n + 1 \) points in general position, we obtain \( \gamma = \text{id} \). This contradicts our assumption that there is a \( \gamma \neq \text{id} \) such that \( w = \gamma w \). Hence, \( W \) is regular.

Now we assume that \( W \) is regular and show that there is a representation \( w \) of \( W \) such that \( w \in \text{int} D_w \). The boundary of \( D_w \) is of the form
\[
\text{bd } D_w = \bigcup_{\gamma \in G \setminus \{\text{id}\}} D_w \cap \gamma D_w.
\]
Since \( W \) is regular, \( \gamma w \neq w \) for all \( \gamma \in G \setminus \{\text{id}\} \).

Suppose that \( \tilde{w} \neq w \) is another representation of \( W \) with Dirichlet domain \( D_{\tilde{w}} \). The bisection of \( D_w \) and \( D_{\tilde{w}} \) is defined by the set
\[
\mathcal{H}(w, \tilde{w}) = \{ x \in D : w^T x = \tilde{w}^T x \},
\]
where \( D = D_w \cap D_{\tilde{w}} \). Since \( w \) and \( \tilde{w} \) are unequal, \( \mathcal{H}(w, \tilde{w}) \) is a subset of a hyperplane \( \mathcal{H} \) defined by the equation \( h(x) = (w - \tilde{w})^T x \). The hyperplane \( \mathcal{H} \) is perpendicular to the vector \( v = w - \tilde{w} \) and passes through the midpoint of the connecting line between \( w \) and \( \tilde{w} \). This shows that \( w \) is not a point on \( \mathcal{H}(w, \tilde{w}) \). Since \( \tilde{w} \) was chosen arbitrarily, \( w \) is in the interior of \( D_w \). \( \square \)

**Lemma 3.** Let \( W \in X_G \) be regular. Suppose that \( \phi \) is a fundamental cross section with basepoint \( w \) and Dirichlet domain \( D_w \). Then \( -w \notin D_w \).

**Proof.** Observe that \( w \neq 0 \) and \( w \neq -w \) for all representations \( w \) of a regular graph \( W \). In addition, with \( W \) the graph \( W' = \pi(-w) \) is also regular. As shown in [1], we have
\[
W \cdot W = \max_{\gamma \in G} \gamma w^T w = w^T w. \tag{1}
\]
Let \( W' \) be the graph represented by \( -w \). From eq. (1) follows
\[
-w^T w = -\min_{\gamma} \gamma w^T w < \max_{\gamma} -w^T w = W' \cdot W \tag{2}
\]
Strict inequality in eq. (2) follows from regularity of \( W \) and \( W' \) together with \( w \neq -w \). Thus, \( -w \) is not an element of the Dirichlet domain \( D_w \) with basepoint \( w \). \( \square \)
Proof of Theorem 1

Suppose that each graph \( X \in \mathcal{X}_G \) has a class label \( y \in \mathcal{Y} = \{\pm 1\} \). By \( \mathbb{C}[f] \) we denote the expected misclassification error of the sublinear function \( f \). Consider a sublinear function of the form \( f(X) = W \cdot X + b \), where \( W \) is regular and \( b \neq 0 \). The equation \( f(X) = 0 \) defines a decision surface \( \mathcal{H}_f \) that separates the graph space \( \mathcal{X}_G \) into two disjoint regions \( \mathcal{R}_+(f) \) and \( \mathcal{R}_-(f) \). By construction we have \( \mathbb{C}[f] = 0 \).

Let \( \phi \) be a fundamental cross section with basepoint \( w = \phi(W) \) and Dirichlet domain \( \mathcal{D}_w \). By \( \mathcal{D}_+(f) = \phi(\mathcal{R}_+(f)) \) and \( \mathcal{D}_-(f) = \phi(\mathcal{R}_-(f)) \) we denote the images of both class regions \( \mathcal{R}_+(f) \) and \( \mathcal{R}_-(f) \) in \( \mathcal{D}_w \). The hyperplane separating \( \mathcal{D}_+(f) \) and \( \mathcal{D}_-(f) \) is defined by the equation \( h(x) = w^T x + b = 0 \). By construction, the expected misclassification error of \( h(x) \) is \( \mathbb{C}_X[h] = 0 \). In addition, \( h(x) \) is the unique global minimum of \( \mathbb{C}_X[\cdot] \) over all linear functions.

Now we relabel both class regions \( \mathcal{R}_+(f) \) and \( \mathcal{R}_-(f) \) such that all graphs from the positive class region \( \mathcal{R}_+(f) \) have negative labels and all graphs from the negative class region \( \mathcal{R}_-(f) \) have positive labels. We denote the relabeled regions in \( \mathcal{X}_G \) by \( \overline{\mathcal{R}}_+(f) \) and \( \overline{\mathcal{R}}_-(f) \), resp., and similarly the relabeled regions in \( \mathcal{D}_w \) by \( \overline{\mathcal{D}}_+(f) \) and \( \overline{\mathcal{D}}_-(f) \). For the relabeled variant, the linear classifier in \( \mathcal{D}_w \) determined by \( h'(x) = -w'^T x + b \) has also minimal misclassification error \( \mathbb{C}_X[h'] = 0 \) and is the unique minimizer of \( \mathbb{C}_X[\cdot] \). By Lemma 3 the opposite direction \(-w\) of basepoint \( w \) is not an element of the Dirichlet domain \( \mathcal{D}_w \).

Suppose that \( f'(X) = W' \cdot X + b' \) is a sublinear function such that

\[
\mathbb{C}[f'] = \min_g \mathbb{C}[g]
\]

under the relabeled setting with class regions \( \overline{\mathcal{R}}_+(f) \) and \( \overline{\mathcal{R}}_-(f) \). Let \( w' = \phi(W') \) be a representation of \( W' \) in \( \mathcal{D}_w \) and let \( \phi' \) be a fundamental cross section with basepoint \( w' = \phi'(W') \) and Dirichlet domain \( \mathcal{D}_{w'} \). We consider the intersection \( \mathcal{D} = \mathcal{D}_w \cap \mathcal{D}_{w'} \). According to [3], Theorem 12, the intersection of two convex polyhedral cones is again a convex polyhedral cone. We show that \( \mathcal{D} \) has co-dimension 0. For this, we first show that the relative interior of the intersection \( \mathcal{D} \) contains an inner point \( z \) from \( \mathcal{D}_w \). Since \( W \) is regular, \( w \) is an inner point of \( \mathcal{D}_w \) lying in \( \mathcal{D} \). Two cases can occur: (i) \( w \) is in the relative interior of \( \mathcal{D} \); and (ii) \( w \) lies on the boundary of \( \mathcal{D} \). If (i) holds, we set \( z = w \). Suppose that (ii) holds. Since \( w \) and \( w' \) are distinct elements of the intersection \( \mathcal{D} \), any point \( z = \lambda w + (1-\lambda) w' \) with \( 0 < \lambda < 1 \) lies in the relative interior of \( \mathcal{D} \) by convexity.

Next we show that there is an \( \varepsilon > 0 \) such that the open ball \( B(z, \varepsilon) \) with center \( z \) and radius \( \varepsilon \) is contained in \( \mathcal{D} \). We choose \( \varepsilon < \min(\lambda, 1-\lambda) \). Then \( B(z, \varepsilon) \subset \mathcal{D}_w \), because \( z \) is an interior point of \( \mathcal{D}_w \). Suppose that \( B(z, \varepsilon) \) is not contained in \( \mathcal{D} \). Since \( z \) lies in the relative interior of \( \mathcal{D} \), the co-dimension of \( \mathcal{D} \) is positive. In addition, \( z \) lies on the boundary of \( \mathcal{D}_{w'} \). This implies that \( z \) is also a boundary point of \( \mathcal{D}_{w'} \), which is a contradiction of our construction. Hence, the co-dimension of \( \mathcal{D} \) is zero.

Since \( b \neq 0 \), the hyperplane defined by equation \( h(x) = w^T x + b = 0 \) does not pass through the origin \( 0 \). Boundaries of any Dirichlet domain are
supported by hyperplanes passing through 0. Thus the intersection $\mathcal{D}$ contains an open set $\mathcal{U}$ separated by the hyperplane segment $\mathcal{H}_f$. Then there are $n + 1$ points $x_0, \ldots, x_n \in \mathcal{D}$ in general position that fix the unique hyperplane. Since $-w \notin \mathcal{D}_w$, we have $\mathbb{C}[f'] > 0$ implying that $f'$ is not a dual of $f$.

The probabilistic statement follows from the fact that interior points are regular and therefore have Lebesgue measure one. With respect to the bias $b$, the set $\mathbb{R} \setminus \{0\}$ also has Lebesgue measure one. Combining all parts we arrive at the assertion.

\[ \Box \]

References

1. B. Jain and K. Obermayer. Structure Spaces. The Journal of Machine Learning Research, 10:2667–2714, 2009.
2. B. Jain and K. Obermayer. Learning in Riemannian Orbifolds. arXiv preprint arXiv:1204.4294, 2012.
3. M. Gerstenhaber. Theory of convex polyhedral cones. Activity analysis of production and allocation, 298–316, 1951.