A note on the Schur multiplier of a nilpotent Lie algebra

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Abstract

For a nilpotent Lie algebra $L$ of dimension $n$ and $\dim(L^2) = m$ ($m \geq 1$), we find the upper bound $\dim(M(L)) \leq \frac{1}{2}(n+m-2)(n-m-1)+1$, where $M(L)$ denotes the Schur multiplier of $L$. In case $m = 1$ the equality holds if and only if $L \cong H(1) \oplus A$, where $A$ is an abelian Lie algebra of dimension $n-3$ and $H(1)$ is the Heisenberg algebra of dimension 3.

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1 Introduction

It is well known that restrictions on the Schur multiplier of finite $p$–groups ($p$ a prime) are related to significant information on the Schur multiplier $M(L)$ of a nilpotent Lie algebra $L$ of dimension $n$. Many times it is possible to get structural results on $L$ only looking at the size of $M(L)$. This fact was already noted by Batten and others in [1, Theorem 5]. Given $t(L) = \frac{1}{2}n(n-1) - \dim(M(L))$, they prove in [1, Theorem 3] that $t(L) = 1$ if and only if $L \cong H(1)$, where $H(1)$ is the Heisenberg algebra of dimension 3. Moreover [1, Theorem 5] shows that $t(L) = 2$ if and only if $L \cong H(1) \oplus A$, where $A$ is an abelian algebra of $\dim(A) = 1$.

To convenience of the reader, we recall that a finite dimensional Lie algebra $L$ is called Heisenberg provided that $L^2 = Z(L)$ and $\dim(L^2) = 1$. Such algebras are odd dimensional with basis $v_1, \ldots, v_{2m}, v$ and the only nonzero multiplication between basis elements is $[v_{2i-1}, v_{2i}] = -[v_{2i}, v_{2i-1}] = v$. 

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for \( i = 1, \ldots, m \). The symbol \( H(m) \) denotes the Heisenberg algebra of dimension \( 2m + 1 \).

There are successive contributions on the same line of investigation, once we prescribe a value for \( t(L) \). In \([3]\) the cases \( t(L) = 3, 4, 5, 6 \) are studied and weaker characterizations are obtained (see \([3]\) Theorems 2, 3, 4)). Under the same prospective we should read \([2, 5]\), where homological machineries are involved.

It is instructive to note that in \([1]\) Section 1 Batten and others declare explicitly that their contributions originated from the classification of Zhou in \([9]\), where a corresponding situation for \( p \)-groups was analyzed. In a certain sense the same motivation allows us to write the present paper.

A classic restriction of Jones \([4]\) Theorem 3.1.4] on the Schur multiplier of a non-abelian \( p \)-group has been recently improved by the first author. More precisely it is proved in \([6]\) that a non-abelian \( p \)-group \( G \) of order \( p^n \) with derived subgroup of order \( p^k \) has

\[
|M(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.
\]

In particular,

\[
|M(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1}
\]

and the equality holds in this last bound if and only if \( G = H \times Z \), where \( H \) is extra special of order \( p^3 \) and exponent \( p \), and \( Z \) is an elementary abelian \( p \)-group.

The present paper is devoted to obtain similar results for Lie algebras.

2 Preliminaries

The present section illustrates how the ideas of Zhou \([9]\) have been adapted in \([3]\) to the context of Lie algebras. This is an important feedback for our main theorems. For instance, the following is analogous to \([4]\) Theorem 2.5.2).

**Proposition 2.1.** (See \([1]\) Lemma 4). Let \( L \) be a finite dimensional Lie algebra, \( K \) an ideal of \( L \) and \( H = L/K \). Then there exists a finite dimensional Lie algebra \( J \) and an ideal \( M \) of \( J \) such that

(i) \( L^2 \cap K \cong J/M \);

(ii) \( M(L) \cong M \);

(iii) \( M(H) \) is an epimorphic image of \( J \).
The Schur multiplier of the direct product of two finite groups is equal to the direct product of the Schur multipliers of the two factors plus the tensor product of the abelianization of the two groups (see [4, Theorem 2.2.10]). This is a general fact of homology, which is known as the Künneth Formula (see [7]), and it is true also for two finite dimensional Lie algebras $H$ and $K$. The Künneth Formula was originally obtained by Schur in 1904 (see [5]). We recall that the symbol $\otimes$ denotes the usual tensor product of abelian Lie algebras. Then we have

$$M(H \oplus K) = M(H) \oplus M(K) \oplus (H/H^2 \otimes K/K^2).$$

At this point we may give a short proof of [1, Theorem 1] as follows.

**Theorem 2.2.** Let $A$ and $B$ be finite dimensional Lie algebras. Then

$$\dim(M(A \oplus B)) = \dim(M(A)) + \dim(M(B)) + \dim(A/A^2 \otimes B/B^2).$$

**Proof.** Use the Künneth Formula above mentioned. \hfill $\square$

The following result is proved in [4, Theorem 2.5.5 (ii)] for groups.

**Corollary 2.3.** Let $L$ be a finite dimensional Lie algebra, $K$ an ideal of $L$ and $H = L/K$. Then

$$\dim(M(L)) + \dim(L^2 \cap K) \leq \dim(M(H)) + \dim(M(K)) + \dim(H/H^2 \otimes K/K^2).$$

**Proof.** From Theorem 2.2 it is enough to prove that $\dim(M(L)) + \dim(L^2 \cap K) \leq \dim(M(H \oplus K))$. From Proposition 2.1

$$\dim(M(L)) + \dim(L^2 \cap K) = \dim(M) + \dim(J/M) = \dim(J) = \dim(\epsilon(M(H)))$$

for a suitable epimorphism $\epsilon$ of Lie algebras. Since $\epsilon$ sends systems of generators in systems of generators, $\dim(\epsilon(M(H))) \leq \dim(M(H \otimes K))$, as claimed. \hfill $\square$

Now we mention three results from [1, 2, 3, 5] to convenience of the reader.

**Lemma 2.4.** (See [2, Example 3] and [5, Theorem 24]).

(i) $\dim(M(H(1))) = 2$.

(ii) $\dim(M(H(m))) = 2m^2 - m - 1$ for all $m \geq 2$. 

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Lemma 2.5. (See [1] Lemma 3].) A Lie algebra $L$ of dimension $n$ is abelian if and only if $\dim(M(L)) = \frac{1}{2}n(n - 1)$.

Proposition 2.6. (See [3 Proposition 1]). A nilpotent Lie algebra $L$ of dimension $n$ has $\dim(L^2) + \dim(M(L)) \leq \frac{1}{2}n(n - 1)$.

Hardy and others declare in [3] that they were working on higher values of $t(L)$ (namely, the case $t(L) \geq 9$ is still open). Their investigations come from the bound in Proposition 2.6.

3 Main Theorem

The following result provides a bound which is less than the bound in Proposition 2.6 except for the case $L \cong H(1)$.

Theorem 3.1. Let $L$ be a nilpotent Lie algebra of $\dim(L) = n$ and $\dim(L^2) = m$ $(m \geq 1)$. Then

$$\dim M(L) \leq \frac{1}{2}(n + m - 2)(n - m - 1) + 1.$$  

Moreover, if $m = 1$, then the equality holds if and only if $L \cong H(1) \oplus A$, where $A$ is an abelian Lie algebra of $\dim(A) = n - 3$.

Proof. Assume $\dim(L^2) = 1$. Then $L/L^2$ is an abelian Lie algebra of $\dim(L/L^2) = n - 1$. Since $L^2 \subseteq Z(L)$, we may consider a complement $H/L^2$ of $Z(L)/L^2$ in $L/L^2$. So we have $L = H + Z(L)$ and $L^2 = H^2$. On the other hand, $H \cap Z(L) \subseteq H$ and so $Z(H) = L^2$. Since $L^2 \subseteq Z(L)$, $L^2$ must have a complement $K$ in $Z(L)$. Let $L^2 \oplus K = Z(L)$ so we have $L \cong K \oplus H$. By Theorem 2.2

$$\dim(M(K \oplus H)) = \dim(M(K)) + \dim(M(H)) + \dim(K/K^2 \otimes H/H^2).$$

Since $H$ is a Heisenberg algebra and $K$ is abelian, two cases should be considered.

Case 1. Assume $\dim(H) = 2m + 1$ for $m \geq 2$. By Lemmas 2.4 and 2.5

$$\dim(M(H)) = 2m^2 - m - 1,$$

$$\dim(M(K)) = \frac{1}{2}(n - 2m - 1)(n - 2m - 2),$$

$$\dim(K \otimes H/H^2) = \dim(K) \cdot \dim(H/H^2) = (n - 2m - 1)(2m).$$
and we deduce that
\[
\dim(M(K\oplus H)) = \frac{1}{2}(n-2m-1)(n-2m-2) + (2m^2-m-1) + (n-2m-1)(2m) \\
= \frac{1}{2}n(n-3) < \frac{1}{2}(n-1)(n-2) + 1.
\]

Case 2. Assume \(m = 1\) and \(H \cong H(1)\). By Lemmas [2.4] and [2.5],
\[
\dim(M(H)) = 2,
\]
\[
\dim(M(K)) = \frac{1}{2}(n-3)(n-4),
\]
\[
\dim(K \otimes H/H^2) = \dim(K) \cdot \dim(H/H^2) = (n-3) \cdot (2)
\]
and we deduce that
\[
\dim(M(K \oplus H)) = \frac{1}{2}(n-3)(n-4) + 2 + 2(n-3) \\
= \frac{1}{2}n(n-3) + 2.
\]

Now we proceed by induction on \(m\).

Let \(m > 1\). Since \(0 \neq L^2 \cap Z(L)\), there exists an ideal \(K\) of dimension 1 contained in \(L^2 \cap Z(L)\). By induction hypothesis and Lemma [2.3], we have
\[
1 + \dim(M(L)) \leq \dim(M(L/K)) + \dim(M(K)) + \dim(L/L^2 \otimes K)
\]
and so
\[
\dim(M(L)) \leq \frac{1}{2}(n+m-4)(n-m-1) + 1 + n - m - 1 \\
= \frac{1}{2}(n+m-2)(n-m-1) + 1,
\]
as claimed.

\[\square\]

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