SELF DUAL EINSTEIN ORBIFOLDS WITH FEW SYMMETRIES
AS QUATERNION KÄHLER QUOTIENTS

LUCA BISCONTI AND PAOLO PICCINNI

To the memory of Krzysztof Galicki

ABSTRACT. We construct a new family of compact orbifolds $O^4(\Theta)$ with a positive self dual Einstein metric and a one-dimensional group of isometries. Together with another family, introduced in [6] and here denoted by $O^4(\Omega)$, these examples classify all 4-dimensional orbifolds that are quaternion Kähler quotients by a torus of real Grassmannians.

1. Introduction

A classical theorem of Hitchin states that $S^4$ and $\mathbb{C}P^2$, with their symmetric metrics, are the only compact positive self dual Einstein (SDE) 4-manifolds [10]. A classification of compact positive self dual Einstein 4-orbifolds appears much harder, and at the present not fully understood.

First examples of compact self dual Einstein 4-orbifolds of positive scalar curvature were constructed by Galicki and Lawson via their also now classical quaternion Kähler quotient construction. Presently, known examples of such orbifolds include: (i) the $SO(3)$-invariant, cohomogeneity one orbifold metrics on $S^4$ discovered by Hitchin [11]; (ii) the toric orbifold metrics constructed by Boyer, Galicki, Mann and Rees as quaternion Kähler quotients of quaternionic projective spaces [7]; (iii) the $S^1$-invariant orbifold metrics of Galicki and Nitta [9].

The toric orbifold metrics mentioned in (ii) include as special cases the Galicki-Lawson metrics on weighted complex projective spaces $\mathbb{C}P^2(p, q, q)$. All these toric metrics have been completely classified through quaternion Kähler quotients first by Bielawski [2] in a special case and more generally by Calderbank and Singer [8]. Compact positive SDE orbifolds with a one-dimensional isometry group are known just in a few cases, and the only known examples seem to be the ones mentioned in (iii) and the family constructed in [6] again through a quaternion Kähler quotient.

The present paper is devoted to constructing a new family of positive SDE metrics with a one-dimensional isometry group on compact orbifolds. We show that these new examples, together with the ones constructed in [6], classify all such orbifolds that can be obtained as toric quotients from the quaternion Kähler Grassmannians $Gr_4(\mathbb{R}^{n+1}) \cong \frac{SO(n+1)}{SO(n-3) \times SO(4)}$. In fact, actions by a $k$-dimensional torus sitting inside the maximal torus of $SO(n+1)$ and leading to a 4-dimensional positive SDE quotient orbifold give necessarily $n+1 = 6, 7, 8$ (cf. Section 2). Indeed the first case $n+1 = 6$ gives rise to SDE orbifolds with $T^2$-symmetries [7]. The
remaining cases \( n + 1 = 7, 8 \) give rise to intermediate circle quotients related to the groups \( G_2 \) and \( Spin(7) \), respectively: [12], [14].

Let \( M^{4n} \) be a quaternion Kähler manifold of positive scalar curvature. Recall the diagram of fibrations, all consistent with the respective quotient constructions:

In particular, one can consider weighted action of tori \( T_k^k \subset SO(n+1) \subset Sp(n+1) \) on the 3-Sasakian sphere \( S^{4n+3} \subset H^{n+1}, k = \left[\frac{n+1}{2}\right] - 1 \), where \( \Theta \) is a \((k - 1) \times k\) integral matrix. In Section 2 we describe the details when \( n = 7 \) and the acting torus is 3-dimensional. This gives the following:

**Theorem A** Let \( \Theta \in M_{3 \times 4}(\mathbb{Z}) \) be an integral matrix such that each of its \( 3 \times 3 \) minor determinants \( \Delta_{\alpha\beta\gamma} \) does not vanish. Moreover, assume that their sum is non zero, that none of them is equal to the sum of the other three, and that none of the sums of two of them is equal to the sum of the other two. Then, for each such a matrix \( \Theta \), there exists a compact self dual Einstein 4-dimensional orbifold \( O^4(\Theta) \) with positive scalar curvature and a one-dimensional group of isometries. \( \square \)

In fact, the analysis of the action shows that no choice of matrix \( \Theta \) gives a smooth quaternion Kähler quotient of the corresponding quaternion Kähler base \( \mathbb{HP}^7 \), and this is the case also for 3-Sasakian quotient metric on the mentioned \( SO(3) \)-bundle. The singularities of the quotient \( O^4(\Theta) \) can in fact be more conveniently described through the singular locus on its twistor space \( Z^6(\Theta) \). To give a proper formulation of this, denote by \( \tilde{G} \) the group \( Sp(1) \times T_3^3 \Theta \times U(1) \). It acts in a natural way on the quaternionic vector space \( \mathbb{H}^8 \), and denote by \( u_\alpha = z_\alpha + jw_\alpha (\alpha = 1, ..., 8) \) its coordinates. In Section 3 we prove the following:

**Theorem B** Let \( Z^6(\Theta) \) be the twistor space of the self dual Einstein orbifold \( O^4(\Theta) \). Then the singular locus \( \Sigma(\Theta) \) of \( Z^6(\Theta) \) contains at most the following sets.

(i) Two spheres \( S^2 \), whose isotropy depends only on an algebraic sum of the determinants \( \Delta_{\alpha\beta\gamma} \).

(ii) Further 22 disjoint spheres \( S^2 \), whose isotropy depends on the minor determinants \( \Delta_{\alpha\beta\gamma} \). These 2-spheres are obtained as \( G \)-quotients from strata \( S^2_{\alpha\beta\gamma}, S^2_{\beta\gamma\delta} \) or \( S^2_{\gamma\delta} \) on \( \mathbb{H}^8 \), that are \( \tilde{G} \) orbits of loci where some pairs of...
complex coordinates \((z_{2\alpha-1}, z_{2\alpha})\) or \((w_{2\alpha-1}, w_{2\alpha})\) are zero. For example:

\[
S_{123}^{123} = \tilde{G} \cdot \left\{ \left( \begin{array}{cccccc}
   z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\
   0 & 0 & 0 & 0 & 0 & w_7 \\
   0 & 0 & 0 & 0 & 0 & w_8 \\
\end{array} \right) \right\},
\]

\[
S_{234}^{1} = \tilde{G} \cdot \left\{ \left( \begin{array}{cccccc}
   z_1 & z_2 & 0 & 0 & 0 & 0 \\
   0 & 0 & w_3 & w_4 & w_5 & w_6 \\
   0 & 0 & 0 & 0 & 0 & w_7 \\
   0 & 0 & 0 & 0 & 0 & w_8 \\
\end{array} \right) \right\},
\]

\[
S_{34}^{12} = \tilde{G} \cdot \left\{ \left( \begin{array}{cccccc}
   z_1 & z_2 & z_3 & z_4 & 0 & 0 \\
   0 & 0 & 0 & 0 & w_5 & w_6 \\
   0 & 0 & 0 & 0 & w_7 & w_8 \\
\end{array} \right) \right\}.
\]

One obtains in this way eight strata \(S_{\alpha\beta\gamma}^{\alpha\beta\gamma}\) and \(S_{\alpha\beta\gamma}^{\alpha\beta\gamma}\) each of which intersects the zero set of the moment map in two connected components, and six further strata \(S_{\alpha\beta}^{\alpha\beta}\) having a connected intersection with the zero set. All quotients by \(\tilde{G}\) of these connected components are spheres \(S^2\).

(iii) Three sets of at most four points. The points of each set are joined by one of the \(2^2 = 4\) spheres \(S_{\alpha\beta}^{\alpha\beta} \cap N(\Theta) / \tilde{G}\), where \(N(\Theta) \subset S^{31} \subset H^8\) is the zero set of the moment map associated to the \(Sp(1) \times T^3\) action.

We describe also singularities for quotients appeared in \([6]\). We use here the notation \(\Omega \in M_{2 \times 3}(\mathbb{Z})\) for the matrix of weights, entering in the \(\tilde{G} \cdot \Omega = T^2_\Omega \times Sp(1) \times U(1)\) action on the quaternionic vector space \(H^7\) (cf. Section 4). We denote by \(\Delta_{\alpha\beta}\), \(\alpha, \beta = 1, 2, 3\), the minor determinants of the matrix \(\Omega\). It is proved in \([6]\) that, under some hypotheses on the \(\Delta_{\alpha\beta}\), a positive SDE orbifold \(O^4(\Omega)\) with a one-dimensional group of isometries can be constructed. In Section 4 we describe the singular locus at twistor level by proving the following:

**Theorem C** Let \(Z^6(\Omega)\) be the twistor space of the self dual Einstein orbifold \(O^4(\Omega)\). The singular locus \(\Sigma(\Omega) \subset Z^6(\Omega)\) contains at most the following sets:

(i) one sphere \(S^2\), whose isotropy depends only on one of the possible algebraic sums of the \(\Delta_{\alpha\beta}\);

(ii) 12 points, coming from the following strata of the action of \(\tilde{G} \cdot \Omega\) on \(N(\Omega)\) on \(H^7\):

\[
S_{12}^{12} = \tilde{G} \cdot \left\{ \left( \begin{array}{cccccc}
   0 & z_2 & z_3 & z_4 & 0 & 0 \\
   0 & 0 & 0 & 0 & w_6 & w_7 \\
\end{array} \right) \right\}.
\]

and the similarly defined \(S_{13}^{13}, S_{21}^{21}, S_{31}^{31}, S_{12}^{12}, S_{13}^{13}, S_{21}^{21}\).

Any stratum \(S_{\alpha\beta}^{\alpha\beta}\) or \(S_{\alpha\beta}^{\alpha\beta}\) intersects the zero set of the moment map in two connected components. Each of these connected components gives rise to a singular point at the twistor level. Moreover, for each of these points the isotropy depends only on one of the minor determinants \(\pm \Delta_{\alpha\beta}\).

When some of the minor determinants \(\Delta_{\alpha\beta}\) or of their algebraic sums \(\pm 1\) in Theorem B, or similarly when some of the \(\Delta_{\alpha\beta}\) or their algebraic sums are \(\pm 1\) in Theorem C, then the singular loci \(\Sigma(\Theta)\) and \(\Sigma(\Omega)\) do not contain the corresponding sets.
The comparison between singularities in the two cases shows that Theorem A gives rise to a new family of positive SDE orbifolds with a one-dimensional group of isometries.

Acknowledgement and Provenance. This paper is based on the first author’s doctoral thesis, defended at Roma Tor Vergata in 2007 [3]. Both authors express their gratitude to Krzysztof Galicki, for his decisive encouragement and many helpful suggestions and discussions. This paper is dedicated to his memory.

2. The Quotient Orbifolds $O^4(\Omega)$ and $O^4(\Theta)$

A family of 4−dimensional positive SDE orbifolds with one-dimensional group of isometries has been constructed in [6]. We denote here by $O^4(\Omega)$ these orbifolds, a notation that allows to distinguish them from the new orbifolds $O^4(\Theta)$ that will be introduced in the present paper. We recall that the $O^4(\Omega)$ are quaternion Kähler quotients, via a $Sp(1) \times T^2_{\Omega}$ action with convenient weight matrix $\Omega \in M_{2 \times 3}(\mathbb{Z})$ on the torus factor, of the quaternionic projective space $\mathbb{H}P^6$. An alternative quotient construction of the same $O^4(\Omega)$ is through the action of the weighted 2−torus $T^2_{\Omega}$ on the oriented Grassmannian $Gr_4(\mathbb{H}P^7)$. Under suitable assumptions for the weight matrix $\Omega$, orbifold quotients $O^4(\Omega)$ are obtained.

A similar construction by a torus action can be carried out on any quaternion-Kähler oriented Grassmannian

$$Gr_4(\mathbb{R}^{n+1}) \cong \frac{SO(n+1)}{SO(n-3) \times SO(4)},$$

and if a 4-dimensional quotient is desired, one has to look at actions of $(n - 4)$−dimensional tori. The dimension $\left\lfloor \frac{n+1}{2} \right\rfloor$ of the maximal torus in $SO(n+1)$ shows that the possibility of introducing weights in the torus action yields the inequality $n - 4 < \left\lfloor \frac{n+1}{2} \right\rfloor$. Thus:

$$n < 9 \quad (n + 1 \text{ even}) \quad \text{or} \quad n < 8 \quad (n + 1 \text{ odd}),$$

so that the only Grassmannians that can admit such quotients are:

$$Gr_4(\mathbb{R}^6) \cong Gr_2(\mathbb{C}^4), \quad Gr_4(\mathbb{R}^7), \quad \text{and} \quad Gr_4(\mathbb{R}^8).$$

The first two cases in (3) have been examined in [2] and in [6], respectively. The present paper is devoted to the third case and to its comparison with the second one (circle quotients of $Gr_4(\mathbb{R}^6)$ have a two-dimensional group of isometries, and are therefore a priori distinct from orbifolds in the other two families). Thus our main choice is the Grassmannian $Gr_4(\mathbb{R}^8)$, acted on by a 3−torus $T^3_\Theta \subset T^4 \subset SO(8)$, where $\Theta$ is a $3 \times 4$ integral weight matrix.

The action of $T^3_\Theta$ is conveniently described through $2 \times 2$ block diagonal matrices:
\[
A(\Theta) = \begin{pmatrix}
A(\theta_1) & 0 & 0 & 0 \\
0 & A(\theta_2) & 0 & 0 \\
0 & 0 & A(\theta_3) & 0 \\
0 & 0 & 0 & A(\theta_4)
\end{pmatrix} \in SO(8),
\]

where
\[
A(\theta_a) = \begin{pmatrix}
\cos \theta_a & \sin \theta_a \\
-\sin \theta_a & \cos \theta_a
\end{pmatrix}, \quad \theta_a = p_a t + q_a s + l_a r,
\]

with \(t, s, r \in [0, 2\pi]\), and

\[
\Theta = \begin{pmatrix}
p_1 & p_2 & p_3 & p_4 \\
q_1 & q_2 & q_3 & q_4 \\
l_1 & l_2 & l_3 & l_4
\end{pmatrix}
\]
is the matrix of the integral weights defining the action.

Next, recall that the Hopf fibration \(S^{31} \to \mathbb{H}P^7\), acted on isometrically by \(Sp(1)\) via left multiplication of quaternions, gives as quotient:

\[
S^{31} \xrightarrow{\text{Sp}(1)} SO(8) \xrightarrow{\text{SO}(3)} SO(8) \cong G_4(\mathbb{R}^8),
\]

and we are going now to add to it the \(T^3\) action.

Accordingly, we look at \(G = Sp(1) \times T^3\) as a subgroup of the 3-Sasakian isometries of \(S^{31}\). The moment maps \(\mu : S^{31} \to \mathfrak{sp}(1) \otimes \mathbb{R}^3 \cong \mathbb{R}^8\) associated with the \(Sp(1)\) action and: \(\nu : S^{31} \to \mathfrak{u}(1)^3 \otimes \mathbb{R}^3\) associated with \(T^3 \cong U(1)^3\) read respectively:

\[
\mu(u) = \left( \sum_{\alpha=1}^{8} \mathfrak{u}_\alpha u_\alpha \sum_{\alpha=1}^{8} \mathfrak{u}_\alpha j u_\alpha \sum_{\alpha=1}^{8} \mathfrak{u}_\alpha k u_\alpha \right) \in \mathfrak{sp}(1) \otimes \mathbb{R}^3,
\]

and

\[
\nu(u) = \left( \sum_{\alpha=1}^{4} p_\alpha \begin{pmatrix} u_{2\alpha-1} & u_{2\alpha} - u_{2\alpha-1} \end{pmatrix} \right) \in \mathfrak{u}(1)^3 \otimes \mathbb{R}^3,
\]

where \(u \in S^{31} \subset \mathbb{H}^8\).

The zero set \(\mu^{-1}(0)\) can be easily identified with the Stiefel manifold of oriented orthonormal 4-frames in \(\mathbb{R}^8\), and it is therefore natural to look at elements of \(N(\Theta) = \mu^{-1}(0) \cap \nu^{-1}(0)\) as \(4 \times 8\) real matrices \(u = (u_1, u_2, \ldots, u_7, u_8)\), whose columns \(u_\rho\) are coefficients of a quaternion respect to the base \(\{1, i, j, k\}\). Of course any such matrix \(u\) has rank 4.

**Definition 2.1.** Let \(\alpha = 1, 2, 3, 4\). Any pair \((u_{2\alpha}, u_{2\alpha})\) of quaternionic coordinates of \(u \in S^{31} \subset \mathbb{H}^8\) will be called a quaternionic pair.

**Lemma 2.1.** Suppose that all the minor determinants

\[
\Delta_{\alpha \beta \gamma} = \begin{vmatrix}
p_\alpha & q_\alpha & l_\alpha \\
p_\beta & q_\beta & l_\beta \\
p_\gamma & q_\gamma & l_\gamma
\end{vmatrix} \quad (1 \leq \alpha < \beta < \gamma \leq 4),
\]
of \(\Theta\) do not vanish. Then the zero set \(N(\Theta)\) contains no elements \(u\) having a null quaternionic pair.
Proof. Refer to the choice of \((u_7, u_8)\) as a null quaternionic pair on some point of \(N(\Theta)\). Let \(x_\alpha = \bar{u}_{2\alpha - 1}u_{2\alpha} - \bar{u}_{2\alpha}u_{2\alpha - 1}, \ \alpha = 1, 2, 3, 4,\) and rewrite \(\nu(u)\) as

\[
\nu(u) = \Theta \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ l_1 & l_2 & l_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},
\]

where \(\mathbb{A} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ l_1 & l_2 & l_3 \end{pmatrix}\). Since \(det \mathbb{A} = \Delta_{123}\), the equation \(\nu(u) = 0\) is solved by

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = -\mathbb{A}^{-1} \begin{pmatrix} p_4x_4 \\ q_4x_4 \\ l_4x_4 \end{pmatrix},
\]

and \(N(\Theta) \cap \{u_7 = u_8 = 0\}\) has equations:

\[
\begin{aligned}
\bar{u}_1u_2 &= \bar{u}_2u_1 \\
\bar{u}_3u_4 &= \bar{u}_4u_3 \\
\bar{u}_5u_6 &= \bar{u}_6u_5 \\
\mu(u) &= \sum_{\sigma=1}^{6} \bar{u}_\sigma \sigma u_\sigma |_{\sigma = i,j,k} = 0.
\end{aligned}
\]

From (12) we get \(Im(\bar{u}_{2\alpha - 1}u_{2\alpha}) = 0\), i.e. \(\bar{u}_{2\alpha - 1}u_{2\alpha} \in \mathbb{R}\).

Now observe that in equations (12) we can assume, up to a scale, that \(u_\alpha\) belongs to \(Sp(1)\). Thus maps \((u_{2\alpha - 1}, u_{2\alpha}) \in Sp(1) \times Sp(1) \rightarrow \bar{u}_{2\alpha - 1}u_{2\alpha} \in Sp(1), \ \alpha = 1, 2, 3,\) are consequently defined. It follows \(\bar{u}_{2\alpha - 1}u_{2\alpha} = \pm 1, \ u_{2\alpha - 1} = \pm u_{2\alpha},\) and that \(u \in N(\Theta),\) as a a real 4 \times 8 matrix, cannot satisfy all the equations (12): the first three equations force in fact the columns \(u_{2\alpha - 1} = (u_{2\alpha - 1}^1, u_{2\alpha - 1}^2, u_{2\alpha - 1}^3)\) and \(u_{2\alpha} = (u_{2\alpha}^0, u_{2\alpha}^1, u_{2\alpha}^2, u_{2\alpha}^3)\) of each quaternionic pair to be proportional to each other. Thus the matrix \(u\) has at most rank 3, contradicting the assumption of \(u\) as a 4-frame in \(\mathbb{R}^8\). It follows that \(N(\Theta) \cap \{u_7 = u_8 = 0\}\) is empty. \(\square\)

**Proposition 2.1.** The action of \(G = Sp(1) \times T_\Theta^3\) on \(N(\Theta)\) is locally free if and only if all the following determinants:

\[
\Delta_{\alpha \beta \gamma} = \begin{vmatrix} p_\alpha & q_\alpha & l_\alpha \\ p_\beta & q_\beta & l_\beta \\ p_\gamma & q_\gamma & l_\gamma \end{vmatrix} \quad (1 \leq \alpha < \beta < \gamma \leq 4),
\]

and:

\[
1^{\pm 2} 1^{\pm 3} := \begin{vmatrix} p_1 & p_2 & q_1 & q_2 & l_1 & l_2 \\ p_1 & p_3 & q_1 & q_3 & l_1 & l_3 \\ p_1 & p_4 & q_1 & q_4 & l_1 & l_4 \end{vmatrix}
\]

do not vanish.

**Proof.** By Lemma 2.1 the conditions \(\Delta_{\alpha \beta \gamma} \neq 0\) insure that \(N(\Theta)\) has no points with a null quaternionic pair. Then the fixed point equations can be written as:

\[
A(\theta_\alpha) \begin{pmatrix} u_{2\alpha - 1} \\ u_{2\alpha} \end{pmatrix} = \begin{pmatrix} a_\alpha & b_\alpha \\ -b_\alpha & a_\alpha \end{pmatrix} \begin{pmatrix} u_{2\alpha - 1} \\ u_{2\alpha} \end{pmatrix} = \lambda \begin{pmatrix} u_{2\alpha - 1} \\ u_{2\alpha} \end{pmatrix},
\]

where \(a_\alpha = \cos \theta_\alpha, \ b_\alpha = \sin \theta_\alpha,\) and \(\lambda \in Sp(1).\) It follows:

\[
\begin{aligned}
a_\alpha |u_{2\alpha - 1}|^2 + b_\alpha u_{2\alpha} \bar{u}_{2\alpha - 1} = \lambda |u_{2\alpha - 1}|^2, \\
-b_\alpha u_{2\alpha - 1} \bar{u}_{2\alpha} + a_\alpha |u_{2\alpha}|^2 = \lambda |u_{2\alpha}|^2,
\end{aligned}
\]
and:

\[ a_\alpha (|u_{2\alpha -1}|^2 + |u_{2\alpha}|^2) + b_\alpha (u_{2\alpha} \bar{u}_{2\alpha -1} - u_{2\alpha -1} \bar{u}_{2\alpha}) = \lambda (|u_{2\alpha -1}|^2 + |u_{2\alpha}|^2), \]

where by Lemma 2.1 the term multiplying \( \lambda \) is non-zero. Also:

\[ Re \lambda = a_\alpha, \quad Im \lambda = b_\alpha \]

so that \( a_1 = a_2 = a_3 = a_4 \) and \( b_1 = \pm b_2 = \pm b_3 = \pm b_4 \). Therefore:

\[
\begin{align*}
(p_1 \pm p_2)t + (q_1 \pm q_2)s + (l_1 \pm l_2)r &= 2h_{12}^2 \pi \\
(p_1 \pm p_3)t + (q_1 \pm q_3)s + (l_1 \pm l_3)r &= 2h_{23}^2 \pi \\
(p_1 \pm p_4)t + (q_1 \pm q_4)s + (l_1 \pm l_4)r &= 2h_{13}^2 \pi,
\end{align*}
\]

where \( h_{\alpha\beta} \in \mathbb{Z} \). To have a locally free action, we need that all these eight systems have at most discrete solutions, i.e. that the eight determinants \( 1 \pm 1 \pm 3 \) do not vanish. \( \square \)

**Proposition 2.2.** There is no weight matrix \( \Theta \) such that the action of \( G = Sp(1) \times T^3 \) on \( N(\Theta) \) is free.

**Proof.** From the previous proof we see that there is a unique solution for the fixed point equations [IS] if and only if \( 1 \pm 2 \pm 1 \pm 3 \)v \pm 1 \pm 4 \) = 1. On the other hand, the identities:

\[
\begin{align*}
1 \pm 2 \pm 1 \pm 3 & = \Delta_{123} - \Delta_{124} + \Delta_{134} + \Delta_{234} \\
1 \pm 2 \pm 1 \pm 3 & = -\Delta_{123} - \Delta_{124} - \Delta_{134} - \Delta_{234} \\
1 \pm 2 \pm 1 \pm 3 & = \Delta_{123} + \Delta_{124} - \Delta_{134} - \Delta_{234} \\
1 \pm 2 \pm 1 \pm 3 & = -\Delta_{123} + \Delta_{124} + \Delta_{134} + \Delta_{234} \\
1 \pm 2 \pm 1 \pm 3 & = -\Delta_{123} + \Delta_{124} + \Delta_{134} - \Delta_{234} \\
1 \pm 2 \pm 1 \pm 3 & = \Delta_{123} + \Delta_{124} - \Delta_{134} + \Delta_{234} \\
1 \pm 2 \pm 1 \pm 3 & = -\Delta_{123} - \Delta_{124} - \Delta_{134} + \Delta_{234} \\
1 \pm 2 \pm 1 \pm 3 & = \Delta_{123} - \Delta_{124} + \Delta_{134} - \Delta_{234},
\end{align*}
\]

can be solved with respect to the \( \Delta_{\alpha\beta\gamma} \):

\[
\begin{align*}
\Delta_{123} &= -\frac{Y + W}{2} \\
\Delta_{124} &= -\frac{X + Z}{2} \\
\Delta_{134} &= \frac{X + Y - Z + W}{2} \\
\Delta_{234} &= \frac{Z - Y}{2},
\end{align*}
\]

where \( X = 1 \pm 2 \pm 1 \pm 3 \), \( Y = 1 \pm 2 \pm 1 \pm 4 \), \( Z = 1 \pm 2 \pm 1 \pm 3 \), \( W = 1 \pm 2 \pm 1 \pm 4 \). In particular:

\[
\begin{align*}
\pm 1 &= 1 \pm 2 \pm 1 \pm 4 = -(X + Y + W) \\
\pm 1 &= 1 \pm 2 \pm 1 \pm 4 = Z - X - 2Y - W \\
\pm 1 &= 1 \pm 2 \pm 1 \pm 4 = Y + Z + W \\
\pm 1 &= 1 \pm 2 \pm 1 \pm 4 = X + Y - Z,
\end{align*}
\]

and our assumptions \( \Delta_{\alpha\beta\gamma} \neq 0 \) and \( X, Y, Z, W = \pm 1 \) give \( X = Y = W = -Z \), so that \((X, Y, Z, W)\) is either \((1, 1, -1, 1)\) or \((-1, -1, 1, -1)\) and \((\Delta_{123}, \Delta_{124}, \Delta_{134}, \Delta_{234})\) is either \((1, 1, -2, 1)\) or \((-1, -1, -2, -1)\). Since these choices of \((X, Y, Z, W)\) are not a solution for the above system, no free action can be obtained. \( \square \)
We conclude the paragraph by summarising all of this in a statement. Note that Theorem A of the Introduction then follows.

**Theorem 2.1.** The action of $Sp(1) \times T^3_0$ on $N(\Theta) = \nu^{-1}(0) \cap \mu^{-1}(0)$ is never free, and it is locally free if and only if the following conditions hold:

i) $\Delta_{\alpha\beta\gamma} \neq 0$ for any $(\alpha, \beta, \gamma)$,

ii) all the determinants $\Delta^\pm_1 \Delta^\pm_1 \Delta^\pm_1$ are non zero.

In such a case the quotient

\[(22) \quad \mathcal{M}^7(\Theta) = \frac{N(\Theta)}{Sp(1) \times T^3_0} \]

is a compact 7-dimensional 3-Sasakian orbifold and a principal $SO(3)$-bundle over a 4-dimensional orbifold $\mathcal{O}_4(\Theta)$ with a positive SDE metric and a one-dimensional group of isometries.

The orbifolds $\mathcal{M}^7(\Theta)$ are not toric. To see this, look at the foliation on $N(\Theta)$ that gives any such orbifold as 3-Sasakian quotient. Then observe that $N(\Theta)$ is a compact submanifold of $S^{31} \subset H^8$ as the zero locus of the quadratic functions defined by the moment maps $\mu$ and $\nu$. Thus all the isometries of $N(\Theta)$ come from the restriction of the isometries of $S^{31}$ and, projecting to the 4-dimensional base, the group of isometries associated to $\mathcal{O}_4(\Theta)$ turns out to be one-dimensional.

**Examples 2.2.** There are many matrices which satisfy the assumptions of Theorem 2.1 and hence of Theorem A in the Introduction. For example:

\[(23) \quad \Theta_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 9 & 2 & 7 & 1 \\ 40 & 9 & 31 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \]

have minor determinants $\Delta_{\alpha\beta\gamma} = (-2, -1, 1, -1)$ and $\Delta_{\alpha\beta\gamma} = (1, 72, -32, -63)$, respectively. All conditions listed in Theorem 2.1 are easily verified. □

3. **The Singular Locus of $Z^6(\Theta)$**

Let $Z^6(\Theta)$ be the twistor space of any of the orbifolds $\mathcal{O}_4(\Theta)$ constructed in Section 2 and let $\Sigma(\Theta)$ be its singular locus. The zero set $N(\Theta) = \mu^{-1}(0) \cap \nu^{-1}(0) \subset S^{31}$ is acted on by the group $\tilde{G} = G \times U(1) = T^3_0 \times Sp(1) \times U(1)$ that, up to the central $Z_2$, is a subgroup of $Sp(8) \cdot Sp(1) \subset SO(32)$. Let:

\[(24) \quad \Phi : T^3_0 \times Sp(1) \times U(1) \times N(\Theta) \rightarrow N(\Theta) \]

be the action, where:

\[(25) \quad \Phi((A(\Theta), \lambda, \rho))((z, w)) = A(\Theta)\lambda \left( \begin{array}{c} z \\ w \end{array} \right) \rho, \]

and we have identified $H^8 \cong \mathbb{C}^8 \times \mathbb{C}^8$ by $u_\alpha = z_\alpha + jw_\alpha$. Thus $(z, w) = u = (u_1, u_2, \ldots, u_7, u_8) \in H^8$ and we will use both notations $u$ and $(z, w)$. The twistor
space $Z^6(\Theta)$ is the leaf space of the $\bar{G}$—action on $N(\Theta)$. There is a natural stratification of $Z^6(\Theta)$ and we want to see how any singular stratum in $Z^6(\Theta)$ appears from the action of $\bar{G}$ on $N(\Theta)$.

**Definition 3.1.** We say that two points $(z, w), (z_1, w_1) \in N(\Theta)$ define the same $\bar{G}$—stratum $\bar{S}$ of $Z^6(\Theta)$ if their corresponding isotropy subgroups $\bar{G}(z, w), \bar{G}(z_1, w_1)$ are conjugate with respect to the $\bar{G}$—action.

To get the possible isotropy subgroups, fix a point $(z, w) = u$ and write the fixed point equations in quaternionic pairs $(u_{2\alpha - 1}, u_{2\alpha})$, as follows:

$$A(\theta_{\alpha}) \begin{pmatrix} u_{2\alpha - 1} \\ u_{2\alpha} \end{pmatrix} = \begin{pmatrix} \lambda u_{2\alpha - 1} \rho \\ \lambda u_{2\alpha} \rho \end{pmatrix},$$

where $A(\theta_{\alpha}) = \begin{pmatrix} \cos \theta_{\alpha} & \sin \theta_{\alpha} \\ -\sin \theta_{\alpha} & \cos \theta_{\alpha} \end{pmatrix}$, $\lambda = \epsilon + \sigma \in Sp(1), \rho \in U(1)$. Equivalently:

$$A(\theta_{\alpha}) \begin{pmatrix} z_{2\alpha - 1} \\ z_{2\alpha} \\ w_{2\alpha - 1} \\ w_{2\alpha} \end{pmatrix} = \begin{pmatrix} \epsilon & -\sigma \\ \sigma & \tau \end{pmatrix} \begin{pmatrix} z_{2\alpha - 1} & z_{2\alpha} \\ w_{2\alpha - 1} & w_{2\alpha} \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}^T.$$

The following property is easily verified:

**Lemma 3.1.** Let $(z, w)$ be a point in $N(\Theta)$. Then, up to $\bar{G}$—conjugation, we have $\bar{G}(z, w) \subset T^3_6 \times U(1)^c \times U(1)$, where $U(1)^c = \{ \lambda \in Sp(1) | \sigma = 0 \}$.

Thus, orbits through points $(z, w) \in N(\Theta)$ with non trivial isotropy subgroup $\bar{G}(z, w) \subset T^3_6 \times U(1)^c \times U(1)$ give all the $\bar{G}$—strata of $N(\Theta)$ whose projection gives rise to singular strata of $Z^6(\Theta)$.

Rewrite now equations (27) as follows $(\alpha = 1, 2, 3, 4)$:

$$M_{\alpha} := \begin{pmatrix} 0 & -\sigma \rho \\ -\sigma \rho & 0 \\ -\sin \theta_{\alpha} & \cos \theta_{\alpha} - \epsilon \rho \\ \cos \theta_{\alpha} - \epsilon \rho & \sin \theta_{\alpha} \end{pmatrix} \begin{pmatrix} -\sin \theta_{\alpha} & \cos \theta_{\alpha} - \tau \rho \\ \cos \theta_{\alpha} - \tau \rho & \sin \theta_{\alpha} \end{pmatrix} \begin{pmatrix} z_{2\alpha - 1} \\ z_{2\alpha} \\ w_{2\alpha - 1} \\ w_{2\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and note that none of the $M_{\alpha}$ can have rank 4, since otherwise the correspondent quaternionic pair $(u_{2\alpha - 1}, u_{2\alpha})$ would vanish, a contradiction with Lemma 2.1.

**Proposition 3.1.** Let $M_{\alpha}$ be the matrix in formula (28). Then $\det M_{\alpha} = 0$ if and only if at least one of the following four identities

$$\overline{\rho} e^{\pm i \theta_{\alpha}} = \text{Re} \, \epsilon \pm i \sqrt{(\text{Im} \, \epsilon)^2 + |\sigma|^2}$$

holds.

**Proof.** By using the block notation:

$$M_{\alpha} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

one has, for $\sigma \neq 0$:

$$M_{\alpha} = \begin{pmatrix} B & \text{Id} \\ D & 0 \end{pmatrix} \begin{pmatrix} D^{-1}C & \text{Id} \\ A - BD^{-1}C & 0 \end{pmatrix}.$$
where the matrix \((A - BD^{-1}C)\) is given by:

\[
\sigma \frac{1}{|\sigma|^2} \begin{pmatrix}
\overline{\rho}\sin 2\theta_\alpha - 2\sin \theta_\alpha \text{Re} \epsilon & (2\cos \theta_\alpha \text{Re} \epsilon - \overline{\rho}\cos 2\theta_\alpha) - \rho \\
(2\cos \theta_\alpha \text{Re} \epsilon - \overline{\rho}\cos 2\theta_\alpha) - \rho & -\overline{\rho}\sin 2\theta_\alpha + 2\sin \theta_\alpha \text{Re} \epsilon
\end{pmatrix}.
\]

It follows:

\[
\det M_\alpha = \sigma^2 \rho^2 \det (A - BD^{-1}C) = \rho^2 \left( \rho + \overline{\rho}(e^{-i\theta_\alpha})^2 - 2(\text{Re} \epsilon)(e^{-i\theta_\alpha}) \right) \left( \rho + \overline{\rho}(e^{i\theta_\alpha})^2 - 2(\text{Re} \epsilon)(e^{i\theta_\alpha}) \right),
\]

and this is zero if and only if:

\[
\overline{\rho}(e^{-i\theta_\alpha})^2 - 2(\text{Re} \epsilon) + \rho = 0 \quad \text{or} \quad \overline{\rho}(e^{i\theta_\alpha})^2 - 2(\text{Re} \epsilon) + \rho = 0,
\]

that gives the stated condition.

\[\square\]

Remark 3.1. Note that, when \(\sigma = 0\), the condition \(\det M_\alpha = 0\) is equivalent to

\[
\epsilon \rho = e^{\pm i\theta_\alpha} \quad \text{or} \quad \overline{\epsilon} \rho = e^{\pm i\theta_\alpha},
\]

which are special cases of formula (34).

We can rephrase all of this as follows:

Proposition 3.2. For any \((\overline{z}, w) \in N(\Theta) \subset S^{31}\), to get a non trivial solution for the fixed point equations (26) it is necessary that condition (29) holds for some choices of the signs and for \(\alpha = 1, \ldots, 4\).

Assume now \(\sigma = 0\) in system (28) and use in each block \(M_\alpha\) one or two relations among the four in (34). Then we see that equations (28) admit non null solutions \((\overline{z}, w) \in \mathbb{H}^8\), fixed by a subgroup \(H_{((\rho, \rho \overline{\epsilon}), \alpha)}\) of the group generated by the chosen relations. Thus, Proposition 3.2 gives that for any of these solutions \((\overline{z}, w)\) the isotropy subgroup \(\overline{G}_{((\overline{z}, w), \alpha)}\) is contained in \(H_{((\rho, \rho \overline{\epsilon}), \alpha)}\). Depending on the numbers of the relations (34), the following possibilities for the rank of the blocks \(M_\alpha\) can occur:

1) just one of the relations in (34) holds \(\iff\) \(\text{rank } M_\alpha = 3\),
2) two relations in (34) hold \(\iff\) \(\text{rank } M_\alpha = 2\),
3) three or four relations in (34) are satisfied \(\iff\) \(M_\alpha = 0_{4 \times 4}\).

When three or four relations in (29) hold, for each \(\alpha = 1, 2, 3, 4\), they describe the non effectivity. Thus, up to the non effective subgroup, the third case can be ignored. Accordingly:

Lemma 3.2. Assume \(\sigma = 0\) in system (28). If \(\text{rank } M_\alpha = 3\), its solutions are given by any of the following:

1) \(\pm V_1^\alpha = \{(z_{2\alpha-1}, \pm iz_{2\alpha-1}, 0, 0)\}, \quad \rho e = e^{\pm i\theta_\alpha}\),
2) \(\pm V_2^\alpha = \{(0, 0, w_{2\alpha-1}, \pm iw_{2\alpha-1})\}, \quad \rho \overline{e} = e^{\pm i\theta_\alpha}\).
and when rank $M_\alpha = 2$ by any of :

3) $\pm V_3^\alpha = \{(z_{2\alpha - 1}, \pm i z_{2\alpha - 1}, w_{2\alpha - 1}, \pm i w_{2\alpha - 1})\}, \ \rho e = e^{\pm i \theta_\alpha}, \ \bar{\rho} e = e^{\mp i \theta_\alpha}$,

4) $V_4^\alpha = \{(z_{2\alpha - 1}, z_{2\alpha}, 0, 0)\}, \ \rho e = e^{i \theta_\alpha} = e^{-i \theta_\alpha}$,

5) $V_5^\alpha = \{(0, 0, w_{2\alpha - 1}, w_{2\alpha})\}, \ \rho e = e^{i \theta_\alpha} = e^{-i \theta_\alpha}$.

It follows:

**Corollary 3.1.** Let $(z, w)$ be a point in $N(\Theta)$ with non trivial isotropy subgroup \( \overline{G}_{(z, w)} \). Then each quaternionic pair $(u_{2\alpha - 1}, u_{2\alpha})$ of $(z, w)$ belongs to one of the sets $\overline{G} \cdot \pm V_1^\alpha$, $\overline{G} \cdot \pm V_2^\alpha$, $\overline{G} \cdot (\pm, \pm) V_3^\alpha$, $\overline{G} \cdot V_4^\alpha$ or $\overline{G} \cdot V_5^\alpha$.

**Proposition 3.3.** Let $(z, w)$ be a point on a singular $\overline{G}$--orbit of $N(\Theta)$, and assume that the hypotheses of Theorem 3.1 hold. Then at least one of the blocks $M_\alpha$, $\alpha = 1, 2, 3, 4$, has rank $= 2$.

*Proof.* Assume that rank $M_\alpha = 3$ for $\alpha = 1, 2, 3, 4$. Then, for each $M_\alpha$, just one of relations (34) holds. Note first that there are $\gamma \neq \delta$ such that $M_\delta$ satisfies one of the first two identities in (34) and $M_\delta$ one of the remaining two. In particular, $e^{i \theta_\gamma} \neq e^{\pm i \theta_\delta}$ since otherwise $M_\delta$ and $M_\delta$ would have rank $= 2$. In fact, assuming that such indices $\gamma$ and $\delta$ do not exist, then solutions for equations (28) would have quaternionic pairs either contained in a $\pm V_1^\alpha$ or in a $\pm V_2^\alpha$ for all $\alpha$. Thus, these solutions would be $8 \times 4$ real matrices with rank $< 4$, and as such not points in $N(\Theta)$. Thus, let $M_\gamma$ and $M_\delta$ with the mentioned property, so with spaces of solutions of type $\pm V_1^\alpha$ and $\pm V_2^\alpha$. Then, looking at all the indices $\alpha$ we get solutions in any of the following subspaces:

$$
\Sigma_{23}^{(4)} = \left\{ X \in M_{2 \times 8}(\mathbb{C}) \mid X = \begin{pmatrix}
* & * & * & * & * & 0 & 0 & 0
* & * & 0 & 0 & 0 & * & * & *
\end{pmatrix},
\right\}
$$

$$
\Sigma_{44}^{(4)} = \left\{ X \in M_{2 \times 8}(\mathbb{C}) \mid X = \begin{pmatrix}
* & * & * & * & 0 & 0 & 0 & *
* & * & 0 & 0 & * & * & * & *
\end{pmatrix},
\right\},
$$

or in the similarly defined $\Sigma_{12}^{(4)}, \Sigma_{13}^{(4)}, \Sigma_{24}^{(4)}, \Sigma_{134}^{(4)}, \Sigma_{124}^{(4)}, \Sigma_{123}^{(4)}, \Sigma_{124}^{(4)}$, or in $\Sigma_{24}^{(13)}, \Sigma_{13}^{(14)}, \Sigma_{134}^{(14)}, \Sigma_{124}^{(14)}$. In these matrices, the $2 \times 2$ blocks represent elements either of $\pm V_1^\alpha$ or of $\pm V_2^\alpha$. Look for example at $\Sigma_{123}^{(4)}$ with the choices (following notations in (35)): $(u_{2\alpha - 1}, u_{2\alpha}) \in \pm V_1^\alpha$, $\alpha = 1, 3$, $(u_3, u_4) \in \mp V_2^\alpha$ and $(u_7, u_8) \in \mp V_2^\alpha$.

By reading the $T_3^{(4)}$--moment map equations on this set of solutions, we get:

\[
\begin{align*}
p_1|z_1|^2 - p_2|z_2|^2 + p_3|z_3|^2 + p_4|w_7|^2 &= 0 \\
q_1|z_1|^2 - q_2|z_2|^2 + q_3|z_3|^2 + q_4|w_7|^2 &= 0 \\
l_1|z_1|^2 - l_2|z_2|^2 + l_3|z_3|^2 + l_4|w_7|^2 &= 0.
\end{align*}
\]

Similarly, the $Sp(1)$--moment map equation for this choice gives:

\[
|z_1|^2 + |z_3|^2 + |z_5|^2 - |w_7|^2 = 0,
\]

\[
\frac{\theta_1}{\theta_2} = \frac{\theta_3}{\theta_4} = \frac{\theta_5}{\theta_6} = \frac{\theta_7}{\theta_8}.
\]
and we can rewrite all these equations as follows:

\[
\begin{pmatrix}
  p_1 & -p_2 & p_3 & p_4 \\
  q_1 & -q_2 & q_3 & q_4 \\
  l_1 & -l_2 & l_3 & l_4 \\
  1 & 1 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
  |z_1|^2 \\
  |z_3|^2 \\
  |z_5|^2 \\
  |w_7|^2
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix},
\]

where the determinant is one of the \(1^{1+2}2^{1+3}4^{1+4} \neq 0\). Thus we get only the trivial solution. All the other listed cases can be treated similarly. □

**Proposition 3.4.** Let \((z, w)\) be on a singular \(G\)–orbit of \(N(\Theta)\) and assume that \(M_3\) has rank \(2\) for some \(\beta = 1, \ldots, 4\) with solutions of the corresponding system of type 3) in (35) and (36). Then all the four matrices \(M_\alpha\) have rank \(\leq 2\).

**Proof.** Without loss of generality, we can assume \(\beta = 1\) and that the conditions in (41) giving solutions of type 3) in (35) are:

\[
\begin{cases}
  \epsilon \rho = e^{i\theta_1} \\
  \tau \rho = e^{i\theta_1}.
\end{cases}
\]

Since \(\text{rank } M_\alpha = 3, 4\), at least one of relations (42) holds, so that either \(\epsilon \rho = e^{i\theta_1}\) or \(\tau \rho = e^{i\theta_1}\). If one of the first two identities holds we get either \(\epsilon \rho = \tau \rho = e^{i\theta_1}\) or \(\epsilon \rho = \tau \rho = e^{-i\theta_1}\), and thus we also get one of the further conditions \(\tau \rho = e^{\pm i\theta_2}\), so that \(\text{rank } M_\alpha \leq 2\) for \(\alpha = 2, 3, 4\). □

It follows:

**Corollary 3.2.** Assume the hypotheses of Theorem A, and let \((z, w) \in N(\Theta)\) be such that one of its quaternionic pairs belongs to a \((\pm, \pm)\) \(V^3_3\), and another one either to a \(V^3_3\) or to a \(V^3_3\). Then the isotropy subgroup \(G_{(z, w)}\) is trivial. Then, if \((z, w) \in N(\Theta)\) is a singular point with a quaternionic pair \((w_{2a-1}, u_{2a})\) contained in a \((\pm, \pm)\) \(V^3_3\), then all its quaternionic pairs are contained in one of the spaces \((\pm, \pm)\) \(V^3_3\).

Lemma 3.2 can be viewed as a description of strata on \(Z^6(\Theta)\) coming from the action of \(G\) on \(N(\Theta)\). In fact, quaternionic pairs corresponding to singular points in the quotient are listed in (35) and (36). In particular, by Propositions 3.3, 3.4 and Corollary 3.2 we see that singular strata on \(Z^6(\Theta)\) can be distinguished into the following two different families. The first family of singular strata on \(Z^6(\Theta)\) comes from points \((z, w)\) such that all of their quaternionic pairs are contained in \((\pm, \pm)\) \(V^3_3\), \(\alpha = 1, 2, 3, 4\). In the second family, the \((z, w)\) have no quaternionic pairs contained in \((\pm, \pm)\) \(V^3_3\).

We begin now by studying the first mentioned family of singularities on \(Z^6(\Theta)\). Here all quaternionic pairs of \((z, w)\) are in a \((\pm, \pm)\) \(V^3_3\), so that:

\[
T((z, w)) := \begin{pmatrix}
  z_1 & \pm iz_1 \\
  w_1 & \pm iw_1 \\
  z_3 & \pm iz_3 \\
  w_3 & \pm iw_3 \\
  z_5 & \pm iz_5 \\
  w_5 & \pm iw_5 \\
  z_7 & \pm iz_7 \\
  w_7 & \pm iw_7
\end{pmatrix},
\]

where all the signs can be chosen independently. The fixed point equations are:

\[
\begin{cases}
  e^{i(\theta_1, \pm \theta_\alpha)} = 1 \\
  \epsilon \rho = e^{\pm i\theta_1} \\
  \tau \rho = e^{\pm i\theta_1},
\end{cases}
\]
where $\alpha = 1, 2, 3, 4$. It is convenient to introduce the following notation. Fix a pair of signs $(\pm, \pm)$ in relations $\{ e^\rho = e^{\pm \theta^\rho}, \varpi^\rho = e^{\pm \eta^\rho}, \}$ and a triple of signs $(\pm, \pm, \pm)$, in $e^\theta^\alpha = e^{\pm \theta^\alpha}$, $\alpha = 2, 3, 4$. Any space of solutions of (25) is associated to a 5–tuple of signs $\pm = ((\pm), (\pm, \pm), (\pm, \pm))$. Accordingly, we will denote any such space of solutions by $S^\pm$.

Next, consider the intersections of all spaces $S^\pm$ with $N(\Theta)$. A first observation is the following.

**Proposition 3.5.** Let $\pm = ((+, -), (\pm, \pm, \pm))$ or $\pm = ((-, +), (\pm, \pm, \pm))$. Then $S^\pm$ has an empty intersection with $N(\Theta) = \mu^{-1}(0) \cap \nu^{-1}(0)$.

**Proof.** It is sufficient to look at the intersection with $S^{(+, -)}_{(\pm, \pm, \pm)}$ (the other case is symmetric). By reading the $Sp(1)$-moment map equations on points $(z, w)$ in $S^{(+, -)}_{(\pm, \pm, \pm)}$, we see that

\[
\begin{align*}
\sum_{\alpha=1}^{4} (|z_{2\alpha-1}|^2 - |w_{2\alpha-1}|^2) &= 0, \\
\sum_{\alpha=1}^{4} z_{2\alpha-1}w_{2\alpha-1} &= 0.
\end{align*}
\]

The moment map $\nu$ of the $T^3_\Theta$-action yields:

\[
\begin{align*}
\sum_{\alpha=1}^{4} d_\alpha \mathrm{Im} (z_{2\alpha-1}z_{2\alpha} + w_{2\alpha-1}w_{2\alpha}) &= 0, \\
\sum_{\alpha=1}^{4} d_\alpha (z_{2\alpha-1}w_{2\alpha} - z_{2\alpha}w_{2\alpha-1}) &= 0,
\end{align*}
\]

where $d_\alpha = p_\alpha, q_\alpha, l_\alpha, \alpha = 1, 2, 3, 4$. Thus for points in $S^{(+, -)}_{(\pm, \pm, \pm)}$:

\[
\begin{align*}
\sum_{\alpha=1}^{4} (-1)^{m_\alpha} d_\alpha (|z_{2\alpha-1}|^2 - |w_{2\alpha-1}|^2) &= 0, \\
\sum_{\alpha=1}^{4} (-1)^{m_\alpha} d_\alpha z_{2\alpha-1}w_{2\alpha-1} &= 0,
\end{align*}
\]

where the indices $m_\alpha$ depend on the 5-ples of signs. If $\Gamma_\alpha := |z_{2\alpha-1}|^2 - |w_{2\alpha-1}|^2$, we can rewrite all our equations in (45) and (46) as

\[
\begin{align*}
\sum_{\alpha=1}^{4} (-1)^{m_\alpha} p_\alpha \Gamma_\alpha &= 0, \\
\sum_{\alpha=1}^{4} (-1)^{m_\alpha} q_\alpha \Gamma_\alpha &= 0, \\
\sum_{\alpha=1}^{4} (-1)^{m_\alpha} l_\alpha \Gamma_\alpha &= 0, \\
\sum_{\alpha=1}^{4} (\Gamma_\alpha) &= 0
\end{align*}
\]

and we can observe that the first four equation have the same determinant of coefficients as the last four, namely one of the $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \neq 0$. Then:

\[
|z_{2\alpha-1}|^2 = |w_{2\alpha-1}|^2 > 0 \quad \text{and} \quad z_{2\alpha-1}w_{2\alpha-1} = 0,
\]

so that our system does not admit solutions. □

The following Proposition shows the existence in the singular locus of the two 2-spheres appearing in Theorem B (i) of the Introduction.

**Proposition 3.6.** Just one among the spaces $S^{(+, +)}_{(\pm, \pm, \pm)}$ and just one among the $S^{(-, -)}_{(\pm, \pm, \pm)}$ intersects $N(\Theta)$. 
Proof. We outline the argument for the first set of spaces \( S_{(\pm, \pm, \pm)}^{+,+} \), the other case being very similar. On any of the \( S_{(\pm, \pm, \pm)}^{+,+} \) the \( Sp(1) \)-moment map equation is given by:

\[
\begin{align*}
\sum_{a=1}^{4} \left( |z_{2a-1}|^2 - |w_{2a-1}|^2 \right) &= 0, \\
\sum_{a=1}^{4} w_{2a-1} z_{2a-1} &= 0,
\end{align*}
\]

representing the Stiefel manifold \( S = U(4)/U(2) \). The \( T^3 \) moment map equation depends instead on the chosen \( S_{(\pm, \pm, \pm)}^{+,+} \):

\[
\sum_{a=1}^{4} (-1)^{m_a} d_\alpha (|z_{2a-1}|^2 + |w_{2a-1}|^2) = 0,
\]

with \( d_\alpha = p_\alpha, q_\alpha, l_\alpha \). This can be rewritten in quaternionic coordinates:

\[
\sum_{a=1}^{4} (-1)^{m_a} d_\alpha |u_{2a-1}|^2 = 0,
\]

where \( u_{2a-1} = z_{2a-1} + j w_{2a-1} \), and by intersecting with the sphere \( S^{31} \) we get:

\[
\begin{pmatrix}
(-1)^{m_1} p_1 \\
(-1)^{m_2} p_2 \\
(-1)^{m_3} p_3 \\
(-1)^{m_4} p_4
\end{pmatrix}
\begin{pmatrix}
|u_1|^2 \\
|u_3|^2 \\
|u_5|^2 \\
|u_7|^2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
\frac{1}{2}
\end{pmatrix}.
\]

This is equivalent to

\[
\begin{align*}
2|u_1|^2 &= \frac{\pm \Delta_{+\pm \pm}}{\pm \Delta_{1234}^{+\pm \pm}} > 0, \\
2|u_3|^2 &= \frac{\pm \Delta_{13\pm \pm}}{\pm \Delta_{1234}^{+\pm \pm}} > 0, \\
2|u_5|^2 &= \frac{\pm \Delta_{13\pm \pm}}{\pm \Delta_{1234}^{+\pm \pm}} > 0, \\
2|u_7|^2 &= \frac{\pm \Delta_{13\pm \pm}}{\pm \Delta_{1234}^{+\pm \pm}} > 0,
\end{align*}
\]

admitting a unique solution in \( |u_1|^2, |u_3|^2, |u_5|^2, |u_7|^2 \). Then, by looking at relations \( \|u\| \) we see that \( N(\Theta) \) intersects only one of the eigenspaces \( \tilde{S}_{(\pm, \pm, \pm)}^{+,+} \). In particular, in the non-empty intersection case, \( \dim S_{(\pm, \pm, \pm)}^{+,+} \cap N(\Theta) = 9 \) and \( S_{(\pm, \pm, \pm)}^{+,+} \cap N(\Theta)/\tilde{G} \) is diffeomorphic to a \( S^2 \). □

We consider now the second mentioned family of singularities on \( Z^6(\Theta) \), coming from points \( (\varphi, \mu) \in \mathbb{H}^8 \) having no quaternionic pairs in a \( (\pm, \pm) V^\alpha \). Observe first that the remaining eigenspaces \( \pm V_1^\alpha, \pm V_2^\alpha, V_4^\alpha \) and \( V_5^\alpha \), \( \alpha = 1, 2, 3, 4 \) are not invariant for the action of the group \( \tilde{G} \). Accordingly, the corresponding strata \( S \subset \mathbb{H}^8 \) have to be defined as

\[
\tilde{G} \cdot \left\{ \begin{array}{c|c|c|c|c|c|c}
u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{array} \right\},
\]

where \( V \) has quaternionic pairs in any of the \( \pm V_1^\alpha, \pm V_2^\alpha, V_4^\alpha \) and \( V_5^\alpha \).

By examining all cases we get the following:
Proposition 3.7. Let \((z, w) \in N(\Theta)\) be point in the \(V\) defined by (55). Then \((z, w)\) has at most two quaternionic pairs which are contained either in \(\pm V_1^\alpha\) or in \(\pm V_2^\alpha\).

As a consequence, we can list the singular strata \(S\) that will give the 22 spheres mentioned in Theorem B (ii):

\[
i) \quad S_4^{123} = \tilde{G} \cdot \left\{ \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w_7 & w_8 \end{pmatrix} \right\},
\]
\[
\quad \text{and } S_3^{124}, S_2^{134}, S_1^{134}, S_3^{124}, S_1^{234}, S_4^{123},
\]
\[
\quad \pm S_4^{123} = \tilde{G} \cdot \left\{ \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w_7 & \pm iw_7 \end{pmatrix} \right\},
\]
\[
\quad \text{and } \pm S_3^{124}, \pm S_2^{134}, \pm S_1^{134}, \pm S_3^{124}, \pm S_2^{234}, \pm S_4^{123},
\]
\]
\[
(56) \quad iii) \quad S_4^{12} = \tilde{G} \cdot \left\{ \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_5 & w_6 & w_7 & w_8 \end{pmatrix} \right\},
\]
\[
\quad \text{and } S_3^{13}, S_2^{14}, S_1^{24}, S_3^{12}, S_4^{12},
\]
\[
\quad iv) \quad (\pm, \pm) S_4^{12} = \tilde{G} \cdot \left\{ \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_5 \pm iw_5 & w_7 \pm iw_7 \end{pmatrix} \right\},
\]
\[
\quad \text{and } (\pm, \pm) S_3^{13}, (\pm, \pm) S_2^{14}, (\pm, \pm) S_1^{24}, (\pm, \pm) S_3^{12}, (\pm, \pm) S_4^{12}.
\]

Lemma 3.3. Let \(S = \tilde{G} \cdot V\) be any of the strata \((55)\). Then \(\dim S = \dim V + 2\).

Proof. Look at the subgroup \(\tilde{H} = Sp(1) \times U(1)^{\times} \times U(1)\) of \(\tilde{G}\) fixing \(V\) and let:

\[
(57) \quad \tilde{G} \times \tilde{H} V = \{ [(g, u)] \mid g \in \tilde{G} \text{ and } u \in V \},
\]

where \([(g, u)] = [(g_1, u_1)]\) if \((g_1, u_1) = (g h^{-1}, h \cdot u)\) for some \(h \in \tilde{H}\), by definition a \(V\)-vector bundle over \(\tilde{G}/\tilde{H}\). A \(\tilde{G}\)-action on \(\tilde{G} \times (\tilde{G} \times \tilde{H} V)\) is defined as:

\[
(58) \quad g' \cdot [(g, u)] := [(g' g, u)],
\]

so that \(\tilde{G} \cdot V \cong \tilde{G} \times \tilde{H} V\) through the \(\tilde{G}\)-equivariant diffeomorphism

\[
(59) \quad \Gamma : \tilde{G} \cdot V \longrightarrow \tilde{G} \times \tilde{H} V,
\]

\[
\quad g \cdot u \longrightarrow [(g, u)].
\]

Thus \(\tilde{G} \cdot V\) can be looked at as a \(V\)-vector bundle over \(\tilde{G}/\tilde{H} \cong S^2\). \(\square\)

Next, we have:

Theorem 3.1. The strata \(S_\delta^{0,\beta,\gamma}\) and \(S_\gamma^{0,\beta,\delta}\) are such that

\[
i) \quad S_\delta^{0,\beta,\gamma} \cap N(\Theta) = + S_\delta^{0,\beta,\gamma} \cap N(\Theta) \bigcup - S_\delta^{0,\beta,\gamma} \cap N(\Theta),
\]
\[
ii) \quad S_\alpha^{0,\beta,\gamma} \cap N(\Theta) = + S_\alpha^{0,\beta,\gamma} \cap N(\Theta) \bigcup - S_\alpha^{0,\beta,\gamma} \cap N(\Theta),
\]
\[
iii) \quad S_\gamma^{0,\beta,\delta} \cap N(\Theta) \text{ is connected}.
\]
Moreover, \( \Sigma(\Theta) = \bigcup_{\delta, \gamma} \left( S_{\delta, \gamma} \cap N(\Theta) \right) \cup \left( S_{\gamma, \delta} \cap N(\Theta) \right) \bigg/ \tilde{G} \), and each \( S_{\gamma, \delta}^{\alpha, \beta} \) contains four strata, the points of which listed in the proof.

**Proof.** To fix the argument, consider \( S_{4}^{123} \) in point i). The \( Sp(1) \)-moment map equations and the sphere equation yield a system

\[
\begin{aligned}
2 \cdot \sum_{\alpha=1}^{6} |z_{\alpha}|^{2} &= \frac{1}{2}, \\
\sum_{\alpha=1}^{6} |z_{\alpha}|^{2} &= 0,
\end{aligned}
\]

and by lemma 3.4, it follows that \( \dim S_{4}^{123} = 18 \). This system gives in particular \( w_{8} = \pm i w_{7} \) and \( S_{4}^{123} \cap N(\Theta) \subseteq + S_{4}^{123} \cup - S_{4}^{123} \). Note that equations (61) coincide with the \( Sp(1) \)-moment map equations restricted on \( + S_{4}^{123} \) and \( - S_{4}^{123} \). Also the \( T_{3}^{3} \)-moment map equations on \( S_{4}^{123} \)

\[
\begin{aligned}
p_{1} \text{Im}(z_{1} \overline{z}_{2}) + p_{2} \text{Im}(z_{3} \overline{z}_{4}) + p_{3} \text{Im}(z_{5} \overline{z}_{6}) + p_{4} \text{Im}(w_{7} \overline{w}_{8}) &= 0, \\
q_{1} \text{Im}(z_{1} \overline{z}_{2}) + q_{2} \text{Im}(z_{3} \overline{z}_{4}) + q_{3} \text{Im}(z_{5} \overline{z}_{6}) + q_{4} \text{Im}(w_{7} \overline{w}_{8}) &= 0, \\
l_{1} \text{Im}(z_{1} \overline{z}_{2}) + l_{2} \text{Im}(z_{3} \overline{z}_{4}) + l_{3} \text{Im}(z_{5} \overline{z}_{6}) + l_{4} \text{Im}(w_{7} \overline{w}_{8}) &= 0,
\end{aligned}
\]

coincide with the ones on \( + S_{4}^{123} \) and \( - S_{4}^{123} \). Since \( N(\Theta) \) is \( \tilde{G} \)-invariant, it follows \( S_{4}^{123} \cap N(\Theta) = + S_{4}^{123} \cap N(\Theta) \cup - S_{4}^{123} \cap N(\Theta) \). If \( + S_{4}^{123} \) denotes the intersection \( + S_{4}^{123} \cap N(\Theta) \), then:

\[
+ S_{4}^{123} = \tilde{G} \cdot \{ (z, w) \in \mathbb{H}^{5} \mid \text{Im}(z_{1} \overline{z}_{2} + z_{3} \overline{z}_{4} + z_{5} \overline{z}_{6}) = 0, \sum_{\alpha=1}^{6} |z_{\alpha}|^{2} = 0, \sum_{\alpha=1}^{6} |w_{\alpha}|^{2} = 0, w_{8} = i w_{7}, |w_{7}|^{2} = \frac{1}{4} \}.
\]

Moreover, \( \dim \left( S_{4}^{123} \cap N(\Theta) \right) / \tilde{G} = \dim \left( + S_{4}^{123} \cap N(\Theta) \right) / \tilde{G} = 2 \) and this intersection has two connected components. Namely, it is easy to see that both of \( + S_{4}^{123} \cap N(\Theta) / \tilde{G} \) and \( - S_{4}^{123} \cap N(\Theta) / \tilde{G} \) give twistorial lines \( S^{2} \). A similar argument applies to any \( S_{\delta, \gamma}^{\alpha, \beta} \) or \( S_{\gamma, \delta}^{\alpha, \beta} \), yielding sixteen twistorial \( S^{2} \). The remaining six \( S^{2} \) come from the strata \( S_{34}^{\alpha, \beta} \). Refer in particular to \( S_{34}^{123} \) defined in (50, iii). The \( Sp(1) \)-moment map equations and sphere equation are now

\[
\begin{aligned}
\sum_{\alpha=1}^{4} |z_{\alpha}|^{2} &= \frac{1}{2}, \\
2 \cdot \sum_{\alpha=1}^{4} |z_{\alpha}|^{2} &= 0,
\end{aligned}
\]

and the \( T_{3}^{3} \)-moment map equations

\[
\begin{aligned}
p_{1} \text{Im}(z_{1} \overline{z}_{2}) + p_{2} \text{Im}(z_{3} \overline{z}_{4}) + p_{3} \text{Im}(w_{5} \overline{w}_{6}) + p_{4} \text{Im}(w_{7} \overline{w}_{8}) &= 0, \\
q_{1} \text{Im}(z_{1} \overline{z}_{2}) + q_{2} \text{Im}(z_{3} \overline{z}_{4}) + q_{3} \text{Im}(w_{5} \overline{w}_{6}) + q_{4} \text{Im}(w_{7} \overline{w}_{8}) &= 0, \\
l_{1} \text{Im}(z_{1} \overline{z}_{2}) + l_{2} \text{Im}(z_{3} \overline{z}_{4}) + l_{3} \text{Im}(w_{5} \overline{w}_{6}) + l_{4} \text{Im}(w_{7} \overline{w}_{8}) &= 0,
\end{aligned}
\]

Like in the previous case we obtain \( \left( S_{34}^{123} \cap N(\Theta) \right) / \tilde{G} \cong S^{2} \). Note that the four strata \( \pm \pm S_{34}^{123} \subseteq S_{34}^{123} \) and the mentioned argument also gives \( \dim \left( \pm \pm S_{34}^{123} \cap N(\Theta) \right) = 7 \) and \( \dim \left( \pm \pm S_{34}^{123} \cap N(\Theta) \right) / \tilde{G} = 0 \) and these give the points listed in statement (iii) of Theorem B in the Introduction. \( \square \)

**Remark 3.2.** There is a real structure on the twistor space \( Z^{6}(\Theta) \) coming from the multiplication by the second quaternionic unit \( j \) on vectors \( (z, w) \in \mathbb{H}^{5} \):

\[
J((z, w)) = (-w, z).
\]
Under this \( J - \) map our strata transform according to
\[
J(S^{\alpha \beta \gamma}_{\delta}) = J(S^{\delta \alpha \beta \gamma}),
\]
\[
J(\pm S^{\alpha \beta \gamma}_{\delta}) = J(\pm S^{\delta \alpha \beta \gamma}),
\]
\[
J(S^{\alpha \beta \gamma}) = J(S^{\gamma \delta \alpha \beta}).
\]

4. THE QUOTIENT ORBIFOLDS \( O^4(\Omega) \) AND THEIR TWISTOR SPACE \( Z^6(\Omega) \)

Consider now matrices
\[
A(\omega) = \begin{pmatrix}
\cos \omega & \sin \omega \\
-\sin \omega & \cos \omega
\end{pmatrix},
\]
with \( \omega = p_{\alpha} t + q_{\alpha} s \), where \( \alpha = 1, 2, 3 \) and
\[
\Omega = \begin{pmatrix}
p_1 & p_2 & p_3 \\
q_1 & q_2 & q_3
\end{pmatrix}
\]
is a matrix of integral weights.

The sphere \( S^{27} \subset \mathbb{H}^7 \) is acted on by the group \( G^\Omega = Sp(1) \times T^3_\Omega \subset SO(7) \subset Sp(7) \), whose factor \( Sp(1) \) acts by left quaternionic multiplication and the second factor \( T^3_\Omega \) by matrices:

\[
A(\omega) := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & A(\omega_1) & 0 & 0 \\
0 & 0 & A(\omega_2) & 0 \\
0 & 0 & 0 & A(\omega_3)
\end{pmatrix} \in SO(7),
\]

Accordingly, the zero set of the moment map \( N(\Omega) = \mu^{-1}(0) \cap \nu^{-1}(0) \subset S^{27} \subset \mathbb{H}^7 \) admits the following \( G^\Omega - \)strata [6]:
\[
S_0 = \{ u \in N(\Omega) | u_1 = 0 \},
\]
\[
S_1 = \{ u \in N(\Omega) | u_1 \neq 0 \}.
\]

The following lemma points out a minor correction to statements of corollary 2.3 and lemma 3.2 in [6].

**Lemma 4.1.** The \( G^\Omega \) action on \( S_1 = N(\Omega) \cap \{ u_1 \neq 0 \} \) is

i) locally free if and only if \( \Delta_{\alpha \beta} \neq 0 \), \( \forall 1 \leq \alpha < \beta \leq 3 \),

ii) free if and only if \( \gcd(\Delta_{12}, \Delta_{13}, \Delta_{23}) = \pm 1 \).

(The latter is a weaker condition than the one in Lemma 3.2 (ii) of [6].)

**Proof.** Since \( u_1 \neq 0 \), the \( Sp(1) \) factor acts trivially, and it is sufficient to look at the \( T^3_\Omega \) – action. On the other hand, it is easy to see that, like in Lemma 2.1, no quaternionic pair can vanish on \( N(\Omega) \) provided all the minors \( \Delta_{\alpha \beta} \) of \( \Omega \) are non-zero. It follows that the fixed point equations read:

\[
(\cos \omega_\alpha & \sin \omega_\alpha \\
-\sin \omega_\alpha & \cos \omega_\alpha
\end{pmatrix}
\begin{pmatrix}
u_{2\alpha} \\
u_{2\alpha+1}
\end{pmatrix}
= \begin{pmatrix}
u_{2\alpha} \\
u_{2\alpha+1}
\end{pmatrix}, \quad \alpha = 1, 2, 3,
\]

where \( \omega_{\alpha} = p_{\alpha} t + q_{\alpha} s \), and \( t, s \in [0, 2\pi) \), so that \( A(\omega_\alpha) = id_{2 \times 2} \), and \( e^{i(p_{\alpha} t + q_{\alpha} s)} = 1 \), yielding only the trivial solution if and only if \( \gcd(\Delta_{12}, \Delta_{13}, \Delta_{23}) = \pm 1 \). The
Consider now the action of $\tilde{G}^\Omega = T^2_\Omega \times Sp(1) \times U(1)$ on $N(\Omega)$, similar to the one in [23]. Let $(z, w)$ be a point in $N(\Omega) \subset S^{27} \subset \mathbb{H}^7$

\[
T(z, w) := \begin{pmatrix}
z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 \\
w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 
\end{pmatrix},
\]

where $(z_\beta, w_\beta) \in \mathbb{C} \times \mathbb{C}$, $\beta = 1, ..., 7$. The fixed point equations for each quaternionic pairs $(u_\alpha, u_{\alpha+1})$ read now:

\[
A(\omega_\alpha) \begin{pmatrix} z_{2\alpha} \\ z_{2\alpha+1} \\ w_{2\alpha} \\ w_{2\alpha+1} \end{pmatrix} = \begin{pmatrix} \epsilon & -\sigma & \rho \\ \sigma & \tau & 0 \\ \rho & 0 & 0 \end{pmatrix} \begin{pmatrix} z_{2\alpha} \\ z_{2\alpha+1} \\ w_{2\alpha} \\ w_{2\alpha+1} \end{pmatrix}^T,
\]

where $\lambda = \epsilon + j \sigma \in Sp(1)$, $\rho \in U(1)$. These equations can be rewritten as:

\[
\tilde{M}_\alpha := \begin{pmatrix} 0 & -\sigma \rho & -\sin \omega_\alpha & \cos \omega_\alpha - \tau \rho \\ -\sigma \rho & 0 & \cos \omega_\alpha - \tau \rho & \sin \omega_\alpha \\ -\sin \omega_\alpha & \cos \omega_\alpha - \epsilon \rho & 0 & \tau \rho \\ \cos \omega_\alpha - \epsilon \rho & \sin \omega_\alpha & \tau \rho & 0 \end{pmatrix} \begin{pmatrix} z_{2\alpha} \\ z_{2\alpha+1} \\ w_{2\alpha} \\ w_{2\alpha+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

for each $\alpha = 1, 2, 3$. Then, similarly to Lemma 3.1 and Proposition 3.1, we get:

**Proposition 4.1.** Let $(z, w)$ be a point in $N(\Omega)$. Then, up to $\tilde{G}^\Omega$-conjugation, we have that $\tilde{G}^\Omega(z, w) \subset T^2_\Omega \times \{ \lambda \in Sp(1) \mid \sigma = 0 \} \times U(1)$.

**Proposition 4.2.** For each $\alpha = 1, 2, 3$, $\det \tilde{M}_\alpha = 0$ if and only if:

\[
\bar{\rho}e^{\pm i\omega_\alpha} = \text{Re}(\epsilon) \pm i \sqrt{\text{Im}(\epsilon)^2 + |\sigma|^2}.
\]

Now, singularities of the twistor space $Z^6(\Omega)$ can be described by looking at the two strata $S_0$ and $S_1$.

On $S_1 : \{u_1 \neq 0\}$ we have

**Lemma 4.2.** The fixed point equations on $S_1$ and with respect to $u_1$ give:

\[
\rho = \text{Re}(\epsilon) \pm i \sqrt{\text{Im}(\epsilon)^2 + |\sigma|^2}.
\]

**Proof.** In fact:

\[
u_1 = \lambda u_1 \rho, \iff \begin{pmatrix} 1 - \epsilon \rho & \sigma \rho \\ -\sigma \rho & 1 - \tau \rho \end{pmatrix} \begin{pmatrix} z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

where $\lambda = \epsilon + j \sigma \in Sp(1)$ and $\rho \in U(1)$. There are non trivial solutions if and only if the determinant vanishes, and this gives the stated condition. □

**Remark 4.1.** If $\sigma = 0$ in (74), the fixed point equations with respect to the first quaternionic coordinate become $\bar{\epsilon}u_1 \rho = u_1$, i.e. $\epsilon z_1 \rho + j \tau w_1 \rho = z_1 + j w_1$. Then, when $z_1 \neq 0$ and $w_1 \neq 0$, we get $\epsilon \rho = 1$, and $\epsilon = \rho = \pm 1$, conditions that give the non effective subgroup. Moreover, for any $(z, w) \in S_1$, the orbit contains points with both $z_1 \neq 0$ and $w_1 \neq 0$.

Thus, by Proposition 4.1, Lemma 4.2 and the above remark:
**Proposition 4.3.** The stratum $S_1$ does not give rise to any singular point on $\mathcal{Z}^6(\Omega)$.

Now, we look at $S_0 : \{ u_1 = 0 \}$. Since the elements of $S_0$ have three quaternionic pairs, this situation can be treated like the zero set $N(\Theta)$ studied in the previous section. In fact, by using similar arguments of Proposition 3.2 and Lemma 3.2, we see that the possible solutions for the equations (75) belong to one of the space $\pm V_1^\alpha$, $\pm V_2^\alpha$, $(\pm, \pm)V_3^\alpha$, $V_4^\alpha$ or $V_5^\alpha$, listed in (76).

By using Corollary 3.2, we can distinguish two families of strata defined by the action of $\tilde{G}^\Omega$ on $\mathbb{H}^7$. The first family is given by points $(z, w)$ whose quaternionic pairs are contained in $(\pm, \pm)V_3^\alpha$, $\alpha = 1, 2, 3$. Instead, points in the second family have no quaternionic pairs in $(\pm, \pm)V_3^\alpha$. The first family of strata has points in one of the following:

$$S_{(\pm, \pm, \pm)} := \left\{ \begin{array}{c|c|c|c|c|c|c|c} 0 & z_3 & \pm iz_3 & z_5 & \pm iz_5 & z_7 & \pm iz_7 \\ 0 & w_3 & \pm iw_3 & w_5 & \pm iw_5 & w_7 & \pm iw_7 \end{array} \right\},$$

The second family gives rise to:

$$i) \quad S_{12}^{13} := \tilde{G}^\Omega \cdot \left\{ \begin{array}{c|c|c|c|c|c|c|c} 0 & z_2 & z_3 & z_4 & z_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_6 & w_7 \end{array} \right\},$$

$$ii) \quad \pm S_{13}^{12} := \tilde{G}^\Omega \cdot \left\{ \begin{array}{c|c|c|c|c|c|c|c} 0 & z_2 & z_3 & z_4 & z_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_6 & \pm iw_6 \end{array} \right\},$$

and $S_{13}^{12}, S_{23}^{12}, S_{32}^{13}, S_{31}^{23}, S_{12}^{13}, S_{13}^{12}$, as well as $\pm S_{12}^{13}, \pm S_{13}^{12}, \pm S_{32}^{13}, \pm S_{31}^{23}, \pm S_{12}^{13}, \pm S_{13}^{12}$.

We now see the 2-sphere appearing in Theorem C (i). In fact, through the same argument used in Proposition 3.6 we get:

**Proposition 4.4.** Just one of the strata $S_{(\pm, \pm, \pm)}$ intersects the zero set $N(\Omega)$. This intersection generates a 2-sphere $S^2$ on the twistor space $\mathcal{Z}^6(\Omega)$.

As for the second family of singular strata:

**Theorem 4.1.** The strata listed in (79) have no empty intersection with the submanifold $N(\Omega)$. Moreover:

$$i) \quad S^\alpha_\gamma \cap N(\Omega) = +S^\alpha_\gamma \cap N(\Omega) \bigcup -S^\alpha_\gamma \cap N(\Omega),$$

$$ii) \quad S^\alpha_\beta \gamma \cap N(\Omega) = +S^\alpha_\beta \gamma \cap N(\Omega) \bigcup -S^\alpha_\beta \gamma \cap N(\Omega),$$

and each of the connected components $\pm S^\alpha_\gamma \beta \cap N(\Omega)$ and $\pm S^\alpha_\beta \gamma \cap N(\Omega)$ generate a singular point on the twistor space $\mathcal{Z}^6(\Omega)$.

The proof is a consequence of the same dimensional argument used in Theorem 3.1. The connected components in (81) (i) and (ii), provide the singular points’ description in Theorem C (ii).
References

[1] A. Besse, *Einstein Manifolds*, Springer Verlag, Berlin (1984).
[2] R. Bielawski, *Complete hyperkähler Manifolds with a local tri-Hamiltonian $\mathbb{R}^n$—action*, Math. Ann., vol. 314, 505-528 (1999).
[3] L. Bisconti, *Positive Self Dual Einstein Orbifolds with one-dimensional group of Isometries*, Tesi di Dottorato in Matematica, Roma Tor Vergata (2007), [arXiv:math/0703721](https://arxiv.org/abs/math/0703721).
[4] Ch. P. Boyer and K. Galicki, *The Twistor Space of a 3–Sasakian Manifold*, Internat. J. Math., vol. 8, 595-610 (1997).
[5] Ch. P. Boyer and K. Galicki, *Sasakian Geometry*, Mathematical Monographs, Oxford University Press, (2008).
[6] Ch. P. Boyer, K. Galicki and P. Piccinni, *3–Sasakian geometry, nilpotent orbits, and exceptional quotients*, Ann. Global Anal. Geom., Vol. 21, 85-110 (2002).
[7] Ch. P. Boyer, K. Galicki, B. M. Mann and E. G. Rees, *Compact 3–Sasakian 7–Manifolds with arbitrary second Betti number*, Invent. Math., vol. 131, 321-344 (1998).
[8] D. M. J. Calderbank and M. Singer, *Toric self dual Einstein metrics on compact Manifolds*, Duke Math. J., 237-258 (2006).
[9] K. Galicki, K. and T. Nitta, *Nonzero scalar curvature generalisations of the ALE hyper-Kähler metrics*, J. Math. Phys., Vol. 33, no.5, 1765-1771 (1992).
[10] N. J. Hitchin, *On compact four-dimensional Einstein Manifolds*, J. Diff. Geom., Vol. 9, 435-442 (1974).
[11] N. J. Hitchin, *A new family of Einstein metrics*, *Manifolds and Geometry (Pisa 1993)*, Sympos. Math.,XXXVI, Cambridge Univ. Press, Cambridge, 190-222 (1996).
[12] P. Z. Kobak and A. Swann, *Quaternionic Geometry of a Nilpotent Variety*, Math. Ann., Vol. 29, 747-763 (1993).
[13] C. LeBrun and M. Wang (eds), *Surveys in Differential Geometry: Essays on Einstein Manifolds*, Suppl. IV to J. Diff. Geom., Int. Press (1999).
[14] L. Ornea and P. Piccinni, *Cayley 4–frames and a quaternion Kähler reduction related to Spin(7)*, Contemp. Math, Vol. 288, 401-405 (2001).
[15] S. M. Salamon, *Riemannian Geometry and Holonomy Groups*, Pitman Research Notes in Mathematics, Longman, Harlow Essex (1989).

L. B.: I.N.G.V., Sezione di Pisa, via della Faggiola, 1, 56126 Pisa, Italia.
E-mail address: bisconti@pi.ingv.it

P. P.: DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA ‘LA SAPIENZA’, PIAZZALE ALDO MORO 2, I-00185 Roma, Italia.
E-mail address: piccinni@mat.uniroma1.it