AN EXPLICIT SATURATING SET LEADS TO APPROXIMATE CONTROLLABILITY FOR NAVIER–STOKES EQUATIONS IN 3D CYLINDERS UNDER LIONS BOUNDARY CONDITIONS.

DUY PHAN

ABSTRACT. A saturating set consisting eigenfunctions of Stokes operator in general 3D Cylinders is proposed. The explicit saturating set leads to the approximate controllability for Navier–Stokes equations in 3D cylinders under Lions boundary conditions.

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1. Introduction

We consider the incompressible 3D Navier–Stokes equation in $(0, T) \times \Omega$, under Lions boundary conditions,

$$\begin{align*}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p + h &= 0, \\
\text{div } u &= 0,
\end{align*}$$

where $\Omega \subset \mathbb{R}^3$ is an arbitrary three-dimensional cylinder

$$\Omega = (0, L_1) \times (0, L_2) \times \left(\frac{2L_3S_1}{2\pi}\right),$$

whose boundary is denoted by $\partial \Omega := \left(\{0, L_1\} \times (0, L_2)\right) \cup \left(0, L_1\right) \times \{0, L_2\} \cup \{(2L_3S_1)/2\pi, 0, L_2\}$. As usual $u = (u_1, u_2, u_3)$ and $p$, defined for $(t, x_1, x_2, x_3) \in I \times \Omega$, are respectively the unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, the operators $\nabla$ and $\Delta$ are respectively the well known gradient and Laplacian in the space variables $(x_1, x_2, x_3)$, $(u \cdot \nabla)v$ stands for $(u \cdot \nabla v_1, u \cdot \nabla v_2, u \cdot \nabla v_3)$, $\text{div } u \equiv \sum_{i=1}^3 \partial_{x_i} u_i$, the vector $n$ stands for the outward unit normal vector to $\partial \Omega$, and $h$ is a fixed function.

Notice that this is equivalent to take appropriate mixed Lions–periodic boundary conditions in the infinite channel $\mathbb{R}_C = (0, L_1) \times (0, L_2) \times \mathbb{R}$:

$$\begin{align*}
\left(\text{curl } u - (n \cdot \text{curl } u)n\right) &= 0, \quad \text{on } \left(\{0, L_1\} \times (0, L_2)\right) \cup \left(0, L_1\right) \times \{0, L_2\} \times \mathbb{R},\n\left(u(x_1, x_2, x_3) = u(x_1, x_2, x_3 + 2L_3), \quad (x_1, x_2, x_3) \in \mathbb{R}_C.\right.
\end{align*}$$

This case can be seen as the case where the fluid is contained in a long (infinite) 3D channel with Lions boundary conditions, and with the periodicity assumption on the long (infinite) direction. Lions boundary conditions are a particular case of Navier boundary conditions. For works and motivations concerning Lions and Navier boundary conditions (in both 2D and 3D cases) we refer to [6, 10, 11, 16, 30, 31] and references therein.

Further $A$ maps $V$ onto $V'$, and the operator $A^{-1} \in \mathcal{L}(H)$ is compact. The spaces $H$, $V$, and $D(A)$ will depend on the boundary conditions the fluid will be subjected to. We

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assume that the inclusion $V \subseteq H$ is dense, continuous, and compact. The eigenvalues of $A$, repeated accordingly with their multiplicity, form an increasing sequence $(\lambda_k)_{k \in \mathbb{N}_0}$,

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \ldots,$$

with $\lambda_k$ going to $+\infty$ with $k$.

We can rewrite system (1) as an evolutionary system

$$\dot{u} + Au + B(u, u) + h = \eta, \quad u(0) = u_0,$$

1.1. Saturating sets and approximate controllability. In the pioneering work [3] the authors introduced a method which led to the controllability of finite-dimensional Galerkin approximations of the 2D and 3D Navier–Stokes system, and to the approximate controllability of the 2D Navier–Stokes system, by means of low modes/degenerate forcing.

Hereafter $U \subseteq H$ will stand for a linear subspace of $H$, and we denote

$$\mathcal{B}(a, b) := B(a, b) + B(b, a), \quad \text{for } (a, b) \in U \times U.$$

**Definition 1.1.** Let $\mathcal{C} = \{W_k \mid k \in \{1, 2, \ldots, M\}\}$ and let $E$ be a finite-dimensional space so that $\mathcal{C} \subset E \subset U$. The finite-dimensional subspace $\mathcal{F}_L(E) \subset U$ is given by

$$\mathcal{F}_L(E) := E + \text{span}\{\mathcal{B}(a, b) \mid a \in \mathcal{C}, b \in E, \text{ and } (B(a, a), B(b, b)) \in H \times H\} \cap U.$$

**Definition 1.2.** A given finite subset $\mathcal{C} = \{W_k \mid k \in \{1, 2, \ldots, M\}\} \subset U$ is said $(L, U)$-saturating if for the following sequence of subspaces $\mathcal{G}^j \subset U$, defined recursively by

$$\mathcal{G}^0 := \text{span} \mathcal{C}, \quad \mathcal{G}^{j+1} := \mathcal{F}_L(\mathcal{G}^j),$$

we have that the union $\bigcup_{j \in \mathbb{N}} \mathcal{G}^j$ is dense in $H$.

In [4], section 4, the authors present an explicit saturating set for the 2D Navier–Stokes system. We would like to refer also to the works [7,9,22], where the notion of saturating set was used to derive ergodicity for the Navier–Stokes system under degenerate stochastic forcing (compare the sequence of subsets $\mathcal{Z}_n$ in [9], section 4 with the sequence of subsets $\mathcal{K}^n$ in [3], section 8).

In the pioneering work [3] the set $U$ in (1.2) is taken to be $D(A)$, the same is done in [4,18,19,24]. Later, in [15,20,21], $U$ is taken as $V$ in order to deal either with Navier-type boundary conditions or with internal controls supported in a small subset.

In [24], the method introduced in [3] is developed in the case where the well-posedness of the Cauchy problem is not known. Though the author focuses on no-slip boundary conditions, i.e. $u|_{\partial Q} = 0$, the results also hold for other boundary conditions. The author considers the case of periodic boundary conditions, and presents an explicit saturating set $\mathcal{C}$ (for the case of $(1,1,1)$-periodic vectors) whose 64 elements are eigenfunctions of the Stokes operator (i.e., the Laplacian). For a general period $q = (q_1, q_2, q_3) \in (\mathbb{R}_0)^3$ the existence of a saturating set is also proven [24], section 2.3, Theorem 2.5], though the form of the saturating set is less explicit.

In [17], the approximate controllability also follows from the existence of a $(L, D(A))$-saturating set. For any given length triplet $L = (L_1, L_2, L_3)$ of a 3D rectangle, we present an explicit $(L, D(A))$-saturating set $\mathcal{C}_R$ for the 3D rectangle $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ (which will be recalled below). The elements of $\mathcal{C}_R$ are 81 eigenfunctions of the Stokes operator, under Lions boundary conditions.

In various works of this topic, to tackle different types of boundary conditions as well as domains, some different definitions of saturating set has been proposed. Here we follow
the definition of saturating set as in the previous work (see [17]) since it leads to some advantages in computation.

For further results concerning the controllability and approximate controllability of Navier–Stokes (and also other) systems by a control with low finite-dimensional range (independent of the viscosity coefficient) in several domains (including the 2D Sphere and Hemisphere) we refer the reader to [2, 4, 5, 12–14, 23, 25–27]. We also mention Problem VII raised by A. Agrachev in [1] where the author inquires about the achievable controllability properties for controls taking values in a saturating set whose elements are localized/supported in a small subset $\omega \subset \Omega$. The existence of such saturating sets is an open question (except for 1D Burgers in [15]). The controllability properties implied by such saturating set is an open question. There are some negative results, as for example in the case we consider the 1D Burgers equations in $\Omega = (0,1)$ and take controls in $L^2(\omega, \mathbb{R})$, $w \subset \Omega$, the approximate controllability fails to hold. Instead, to drive the system from one state $u_0 = u(0)$ at time $t = 0$ to another one $u_T = u(T)$ at time $t = T$, we may need $T$ to be big enough. Though we do not consider localized controls here, we refer the reader to the related results in [8, 28] and references therein.

1.2. The main contribution. The main contribution is that we present a saturating set in 3D cylinder consisting finite number of eigenfunctions of Stoke operator (see Theorem 3.2 hereafter). The saturating set has 354 elements (or a simple version with 259 elements in corollary 4.1). In some particular cases, it may exist a saturating set with less elements. However, we want to notice that our goal is not to find a saturating set with minimal number of elements. In all cases, we emphasize that the existence of a $(L, D(A))$-saturating set is independent of the viscosity coefficient $\nu$. In particular, the linear space $\mathcal{G}^1$, where the control $\eta$ takes its value, does not change with $\nu$.

To construct a saturating set, we firstly introduce a system of eigenfunctions in section 3.1. In this case, we have two types of eigenfunctions $Y^{j(k),k}$ and $Z^{j(k),k}$. The form of $Y^{j(k),k}$ are analogous to the one in 3D Rectangles. However the appearence of another type of eigenfunctions $Z^{j(k),k}$ leads to some difficulties. The construction of all eigenfunctions $Z^{j(k),k}$ is based on the expression of $(Z^{j(k),k} \cdot \nabla) Y^{j(m),m} + (Y^{j(m),m} \cdot \nabla) Z^{j(k),k}$. To construct these eigenfunctions $Z^{j(k),k}$, Lemma 3.3 (an analogous version of Lemma 3.1 in [17]) is not enough to prove the linear independence. Therefore another Lemma 3.4 is introduced and used mostly in the proof. Only Lemma 3.4 can be used to prove the linear independence in some cases (for example in Step 3.5.2). Besides we notice that the procedure can be applied analogously in two first directions because we consider Lions boundary conditions in the directions (see the proofs in the Sections 3.4.2 and 3.4.1). However, the third direction must be addressed separately because we consider the periodicity assumption in the third direction. These are some reasons to convince that the proof in the case of 3D cylinder is inspired from the case of 3D rectangle but it cannot follow line by line.

The rest of the paper is organized as follows. In section 2 we recall some results of the approximate controllability for 3D Navier-Stokes equations under Lions boundary conditions. A saturating set in the case of three-dimensional Rectangle will be revisited in section 2.3. In section 3 we construct a $(L, D(A))$-saturating set in the case of three-dimensional cylinder.

2. Preliminaries
where the control } \eta \text{ is a finite-dimensional subspace. Let us consider the system }
\begin{equation}
\dot{u} + Au + B(u, u) + h = \eta, \quad u(0) = u_0,
\end{equation}
in the subspace } H := \{ u \in L^2(\Omega, \mathbb{R}^3) \mid \text{div } u = 0 \text{ and } (u \cdot n)|_{\partial \Omega} = 0 \} \text{ of divergence free vector fields which are tangent to the boundary. We may suppose that } h \text{ and } \eta \text{ take their values in } H \text{ (otherwise we just take their orthogonal projections onto } H \text{). We consider } H, \text{ endowed with the norm inherited from } L^2(\Omega, \mathbb{R}^3), \text{ as a pivot space, that is, } H = H'.
\]
Further we set the spaces
\[ V := \{ u \in H^1(\Omega, \mathbb{R}^3) \mid u \in H \}, \quad D(A) := \{ u \in H^2(\Omega, \mathbb{R}^3) \mid u \in H, \text{ curl } u - ((\text{curl } u) \cdot n)|_{\partial \Omega} = 0 \} \]
Above, for } u, v, w \in V,
\[ A : V \to V', \quad \langle Au, v \rangle_{V', V} := \nu(\text{curl } u, \text{curl } v)_{L^2(\Omega, \mathbb{R}^3)}, \quad (3) \]
\[ B : V \times V \to V', \quad \langle B(u, v), w \rangle_{V', V} := -\int_{\Omega} ((u \cdot \nabla) w) \cdot v \, d\Omega. \quad (4) \]
It turns out that } D(A) = \{ u \in H \mid Au \in H \} \text{ is the domain of } A. \text{ We will refer to } A \text{ as the Stokes operator, under Lions boundary conditions. Further, we have the continuous, dense, and compact inclusions } D(A) \overset{d,c}{\hookrightarrow} V \overset{d,c}{\hookrightarrow} H.
\]
\textbf{Remark 2.1.} The notation } S \hookrightarrow R \text{ above means that the inclusion } S \subseteq R \text{ is continuous. The letter } \text{“d”} \text{ (respectively “c”) means that, in addition, the inclusion is also dense (respectively compact).}

Denoting by } \Pi \text{ the orthogonal projection in } L^2(\Omega, \mathbb{R}^3) \text{ onto } H, \text{ for } u, v \in D(A) \text{ we may write } Au := \Pi(\nu \Delta u), \text{ and } B(u, v) := \Pi((u \cdot \nabla)v).
\]
\textbf{Remark 2.2.} It is clear that the Stokes operator } (3) \text{ is well defined, mapping } V \text{ into } V'. \text{ We also see that the bilinear operator } (4) \text{ maps } V \times V \text{ into } V', \text{ due to the estimate }
\[ B(u, v), w \rangle_{V', V} \leq C_1|u|_{L^6(\Omega, \mathbb{R}^3)}|\nabla w|_{L^2(\Omega, \mathbb{R}^3)}|v|_{L^3(\Omega, \mathbb{R}^3)} \leq C_2|u|_{H^1(\Omega, \mathbb{R}^3)}|w|_{H^1(\Omega, \mathbb{R}^3)}|v|_{H^1(\Omega, \mathbb{R}^3)}. \]
For further estimations on the bilinear operator we refer to [29, section 2.3].

\textbf{2.2. Approximate controllability.} Hereafter } u_0 \in V, h \in L^2_{\text{loc}}(\mathbb{R}_0, H), \text{ and } E \subset D(A) \text{ is a finite-dimensional subspace. Let us consider the system }
\begin{equation}
\dot{u} + Au + B(u, u) + h = \eta, \quad u(0) = u_0,
\end{equation}
where the control } \eta \text{ takes its values in } E.
For simplicity we will denote
\[ I_T := (0, T), \quad \text{and} \quad \overline{I_T} := [0, T], \quad T > 0. \]
\textbf{Definition 2.1.} Let } T \text{ be a positive constant. System } (5) \text{ is said to be } E\text{-approximately controllable in time } T \text{ if for any } \varepsilon > 0 \text{ and any pair } (u_0, \hat{u}) \in V \times D(A), \text{ there exists a control function } \eta \in L^\infty(I_T, E) \text{ and a corresponding solution } u \in C(\overline{I_T}, V) \bigcap L^2(I_T, D(A)), \text{ such that } |u(T) - \hat{u}|_V < \varepsilon.
\]
We recall the Main Theorem in [17] which shows the approximate controllability of 3D Navier-Stokes system from the existence of a } (L, D(A))-\text{saturating set.}
\textbf{Theorem 2.1.} Let } (u_0, \hat{u}) \in V \times V, \varepsilon > 0, \text{ and } T > 0. \text{ If } C \text{ is a } (L, D(A))-\text{saturating set, then we can find a control } \eta \in L^\infty((0, T), G^1) \text{ so that the solution of system } (2) \text{ satisfies } |u(T) - \hat{u}|_V < \varepsilon.
Remark 2.3. In [24], the author introduced a definition of (B, D(A)) saturating set and proved that the existence of a (B, D(A)) saturating set implies the approximate controllability of the 3D Navier-Stokes systems, at time $T > 0$. In [17] and this work, we are using another definition of saturating set, so-called (L, D(A)) saturating set. The main advantage to use this definition is in the computation below.

2.3. An explicit saturating set in 3D Rectangles. In this section, we recall a (L, D(A))-saturating set containing a finite number of suitable eigenfunctions of the Stokes operator $A$ in the 3D rectangle

$$ R = (0, L_1) \times (0, L_2) \times (0, L_3) $$

under Lions boundary conditions, see (3), where $L_1$, $L_2$, and $L_3$ are positive real numbers.

For a given $k \in \mathbb{N}^3$, let $\#_0(k)$ stand for the number of vanishing components of $k$. A complete system of eigenfunctions $Y_k$ is given by

$$ Y_{R}^{j(k), k} := \begin{pmatrix} u_1^{j(k), k} \\ u_2^{j(k), k} \\ u_3^{j(k), k} \end{pmatrix} = \begin{pmatrix} \sin \left( \frac{k_1 \pi x_1}{L_1} \right) & \cos \left( \frac{k_2 \pi x_2}{L_2} \right) & \cos \left( \frac{k_3 \pi x_3}{L_3} \right) \\ \sin \left( \frac{k_1 \pi x_1}{L_1} \right) & \cos \left( \frac{k_2 \pi x_2}{L_2} \right) & \cos \left( \frac{k_3 \pi x_3}{L_3} \right) \\ \sin \left( \frac{k_1 \pi x_1}{L_1} \right) & \cos \left( \frac{k_2 \pi x_2}{L_2} \right) & \cos \left( \frac{k_3 \pi x_3}{L_3} \right) \end{pmatrix}, \quad \#_0(k) \leq 1, \quad (6a) $$

with

$$ \{ w_j^{(k), k} | j(k) \in \{1, 2 - \#_0(k)\} \} \subseteq \{k\}_0 \uparrow \{L\} \quad (6b) $$

a linearly independent and orthogonal family and where

$$ \{k\}_0 \uparrow \{L\} := \{ z \in \mathbb{R}^3 \setminus \{(0,0,0)\} \mid (z, k)_{\{L\}} = 0, \text{ and } z_i = 0 \text{ if } k_i = 0 \}, \quad (6c) $$

$$ (z, k)_{\{L\}} := \frac{z_1 k_1}{L_1} + \frac{z_2 k_2}{L_2} + \frac{z_3 k_3}{L_3}, \quad (6d) $$

Notice that $2 - \#_0(k)$ is the dimension of the subspace $\{k\}_0 \uparrow \{L\}$ and that the orthogonality of the family $\{ w_j^{(k), k} | j(k) \in \{1, 2 - \#_0(k)\} \}$ implies that the family in (6a) is also orthogonal.

We recall the result in [17] Theorem 3.1]

**Theorem 2.2.** The set $C_R := \left\{ Y_{R}^{j(n), n} \mid n \in \mathbb{N}^3, \quad 0 \leq n_i \leq 3, \quad \#_0(n) \leq 1, \quad j(n) \in \{1, 2 - \#_0(n)\} \right\}$ is $(L, D(A))$-saturating.

3. A saturating set in the 3D-cylinder case

3.1. A system of eigenfunctions. We start by observing that the vector functions

$$ Y^{j(k), k} = \begin{pmatrix} u_1^{j(k), k} \\ u_2^{j(k), k} \\ u_3^{j(k), k} \end{pmatrix} = \begin{pmatrix} \sin \left( \frac{k_1 \pi x_1}{L_1} \right) & \cos \left( \frac{k_2 \pi x_2}{L_2} \right) & \cos \left( \frac{k_3 \pi x_3}{L_3} \right) \\ \sin \left( \frac{k_1 \pi x_1}{L_1} \right) & \cos \left( \frac{k_2 \pi x_2}{L_2} \right) & \cos \left( \frac{k_3 \pi x_3}{L_3} \right) \\ \sin \left( \frac{k_1 \pi x_1}{L_1} \right) & \cos \left( \frac{k_2 \pi x_2}{L_2} \right) & \cos \left( \frac{k_3 \pi x_3}{L_3} \right) \end{pmatrix}, \quad (7a) $$

with $\#_0(k) \leq 1$, and also the functions

$$ Z^{j(k), k} = \begin{pmatrix} u_1^{j(k), k} \\ u_2^{j(k), k} \\ -u_3^{j(k), k} \end{pmatrix} = \begin{pmatrix} \sin \left( \frac{k_1 \pi x_1}{L_1} \right) & \cos \left( \frac{k_2 \pi x_2}{L_2} \right) & \cos \left( \frac{k_3 \pi x_3}{L_3} \right) \\ \sin \left( \frac{k_1 \pi x_1}{L_1} \right) & \cos \left( \frac{k_2 \pi x_2}{L_2} \right) & \cos \left( \frac{k_3 \pi x_3}{L_3} \right) \\ -\sin \left( \frac{k_1 \pi x_1}{L_1} \right) & \cos \left( \frac{k_2 \pi x_2}{L_2} \right) & \cos \left( \frac{k_3 \pi x_3}{L_3} \right) \end{pmatrix}, \quad (7b) $$

either with $\#_0(k) \leq 1$ or with $\#_0(k) = 2$ and $k_3 = 0$.

Furthermore we assume that the vectors $u_j^{j(k), k}$ are chosen satisfying
• if \( \#_0(k) = 2 \) and \( k_3 = 0 \), then \( w^{j(k),k} = w^{1,k} = (0, 0, w_3^{1,k}) \), with \( w_3^{1,k} \neq 0 \),
• if \( \#_0(k) \leq 1 \), then

\[
\{ w^{j(k),k} \mid j(k) \in \{1, 2 - \#_0(k)\} \} \subset \{ k_0^{|L|} \},
\]

(7c)
is a linearly independent and orthogonal family, and where

\[
\left\{ k \right\}_{0^{|L|}} := \{ z \in \mathbb{R}^3 \setminus \{ (0, 0, 0) \} \mid (z, k)_{|L|} = 0, \text{ and } z_i = 0 \text{ if } k_i = 0 \},
\]

(7d)

\[
(z, k)_{|L|} := \frac{z_1 k_1}{L_1} + \frac{z_2 k_2}{L_2} + \frac{z_3 k_3}{L_3}.
\]

(7e)

are eigenfunctions of the shifted Stokes operator \( A \) in \( C \) under Lions boundary conditions. Indeed, it is clear that they are eigenfunctions of the usual Laplacian operator. So it remains to check that they are divergence, satisfy the boundary conditions.

**Remark 3.1.** Notice that the functions in (7a) are similar to those in (6a), though the domain \( \Omega \) is different, namely \( x \in \mathbb{R} \times (0, L_1) \times (0, L_2) \times (0, L_3) \) in (6a), and \( x \in \Omega \sim (0, L_1) \times (0, L_2) \times (0, 2L_3) \) in (7a).

The divergence free condition follows from the way the vectors \( w^{j(k),k} \) are chosen in (7). It is also clear that \( u \cdot n \) vanishes at the boundary \( \partial C \). Finally, we can see that the curl is normal to the boundary, from the expressions

\[
\text{curl } Y^{j(k),k} = -\pi \left( \begin{array}{c} \frac{k_2}{L_2} u_3^{j(k),k} - \frac{k_3}{L_3} u_2^{j(k),k} \\ \frac{k_3}{L_3} u_1^{j(k),k} - \frac{k_1}{L_1} u_3^{j(k),k} \\ \frac{k_1}{L_1} u_2^{j(k),k} - \frac{k_2}{L_2} u_1^{j(k),k} \end{array} \right) \cos \left( \frac{k_1 \pi x_1}{L_1} \right) \sin \left( \frac{k_2 \pi x_2}{L_2} \right) \sin \left( \frac{k_3 \pi x_3}{L_3} \right),
\]

(7f)

\[
\text{curl } Z^{j(k),k} = \pi \left( \begin{array}{c} \frac{k_2}{L_2} u_3^{j(k),k} - \frac{k_3}{L_3} u_2^{j(k),k} \\ \frac{k_3}{L_3} u_1^{j(k),k} - \frac{k_1}{L_1} u_3^{j(k),k} \\ \frac{k_1}{L_1} u_2^{j(k),k} - \frac{k_2}{L_2} u_1^{j(k),k} \end{array} \right) \cos \left( \frac{k_1 \pi x_1}{L_1} \right) \cos \left( \frac{k_2 \pi x_2}{L_2} \right) \cos \left( \frac{k_3 \pi x_3}{L_3} \right),
\]

(7g)

which we can derive by direct computations. For example, at the lateral boundary \( x_1 = L_1 \), that is, for \( x \in \{ L_1 \} \times (0, L_2) \times \mathbb{S}^1 \), we have \( n = (1, 0, 0) \) and

\[
\text{curl } Y^{j(k),k} = -\pi \left( \begin{array}{c} \frac{k_2}{L_2} u_3^{j(k),k} - \frac{k_3}{L_3} u_2^{j(k),k} \\ 0 \\ 0 \end{array} \right) \sin \left( \frac{k_2 \pi x_2}{L_2} \right) \sin \left( \frac{k_3 \pi x_3}{L_3} \right),
\]

(7h)

\[
\text{curl } Z^{j(k),k} = \pi \left( \begin{array}{c} \frac{k_2}{L_2} u_3^{j(k),k} - \frac{k_3}{L_3} u_2^{j(k),k} \\ 0 \\ 0 \end{array} \right) \cos \left( \frac{k_2 \pi x_2}{L_2} \right) \cos \left( \frac{k_3 \pi x_3}{L_3} \right),
\]

(7i)

which show that \( \text{curl } Y^{j(k),k} \) and \( \text{curl } Z^{j(k),k} \) have the same direction as the normal vector \( n \).

**Lemma 3.1.** The system of eigenfunctions

\[
\{ Y^{j(k),k}, Z^{j(k),k} \mid k \in \mathbb{N}^3 \text{ and } \#_0(k) \leq 1 \} \bigcup \{ Z^{1,k} \mid k \in \mathbb{N}^3, \#_0(k) = 2, \text{ and } k_3 = 3 \}
\]

is complete.

**Proof.** Recalling that, for \( r > 0 \),

\[
\{ \sin \left( \frac{k \pi x_1}{r} \right) \mid k \in \mathbb{N}_0 \} \text{ and } \{ \cos \left( \frac{k \pi x_1}{r} \right) \mid k \in \mathbb{N} \}
\]
are two complete systems in $L^2((0, r), \mathbb{R})$. And
\[
\{ \sin\left( \frac{k \pi x}{r} \right) \mid k \in \mathbb{N}_0 \} \cup \{ \cos\left( \frac{k \pi x}{r} \right) \mid k \in \mathbb{N} \}
\]
is a complete system in $L^2((0, 2r), \mathbb{R})$. Then the proof can be done by following the arguments in [16, Section 6.6]. We skip the details. \hfill \square

Now we can present the saturating set.

**Theorem 3.2.** The set of eigenfuntions
\[
\mathcal{C} := \left\{ Y_{j(n), n} \mid n \in \mathbb{N}^3, \ #_0(n) \leq 1, \ n_i \leq 4, \ j(n) \in \{ 1, 2 - #_0(n) \} \right\}
\]
\[
\cup \left\{ Z_{j(n), n} \mid n \in \mathbb{N}^3, \ #_0(n) = 2, \ n_3 = 0 \right\}
\]
\[
\cup \left\{ Z_{j(n), n} \mid n \in \mathbb{N}^3, \ #_0(n) \leq 1, \ n_i \leq 4, \ j(n) \in \{ 1, 2 - #_0(n) \} \right\}
\]
is $(L, D(A))$-saturating.

The proof will be presented in Section 3.3. Next, we will introduce some tools which are fruitful in the proof below.

### 3.2. The expression for $(Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k$

Here we will present the expression for the coordinates of $(Z_{j(k), k} \cdot \nabla) Y_{j(m), m} + (Y_{j(m), m} \cdot \nabla) Z_{j(k), k}$ for given eigenfuntions as in [6a]. In order to shorten the following expressions and simplify the writing, we will write
\[
Y^k = Y_{j(k), k}, \quad Y^m = Y_{j(m), m}, \quad w^k = w_{j(k), k}, \quad \text{and} \quad w^m = w_{j(m), m}
\]
by omitting the indexes $j(k), j(m)$. We will also denote
\[
C_i(k_i) := \cos\left( \frac{k_i \pi x_i}{L_i} \right) \quad \text{and} \quad S_i(k_i) := \sin\left( \frac{k_i \pi x_i}{L_i} \right), \quad i \in \{ 1, 2, 3 \}.
\]

Proceeding as in the case of the rectangle, we obtain the same expressions for $(Y^k \cdot \nabla) Y^m$, and for the coordinates of $(Y^k \cdot \nabla) Y^m + (Y^m \cdot \nabla) Y^k$ as in [17 Section 3.1]. By the same argument, we can obtain

\[
(Y^k \cdot \nabla) Z^m = \begin{bmatrix}
Y^k \cdot w_1^m & m_1 \pi C_1(m_1)C_2(m_2)S_3(m_3) \\
-m_1 \pi C_1(m_1)S_2(m_2)S_3(m_3) & m_1 \pi S_1(m_1)C_2(m_2)C_3(m_3)
\end{bmatrix},
\]

\[
(Z^m \cdot \nabla) Y^k = \begin{bmatrix}
Z^m \cdot w_1^k & k \pi C_1(k_1)C_2(k_2)C_3(k_3) \\
-k \pi C_1(k_1)S_2(k_2)C_3(k_3) & -k \pi S_1(k_1)C_2(k_2)S_3(k_3)
\end{bmatrix},
\]

\[
(Z^m \cdot \nabla) Y^k = \begin{bmatrix}
Z^m \cdot w_1^k & k \pi C_1(k_1)C_2(k_2)C_3(k_3) \\
-k \pi C_1(k_1)S_2(k_2)C_3(k_3) & -k \pi S_1(k_1)C_2(k_2)S_3(k_3)
\end{bmatrix},
\]

\[
Z^m \cdot w_3^k & k \pi C_1(k_1)C_2(k_2)C_3(k_3) \\
-k \pi C_1(k_1)S_2(k_2)C_3(k_3) & -k \pi S_1(k_1)C_2(k_2)S_3(k_3)
\end{bmatrix}
\]

\[
Z^m \cdot w_3^k & k \pi C_1(k_1)C_2(k_2)C_3(k_3) \\
-k \pi C_1(k_1)S_2(k_2)C_3(k_3) & -k \pi S_1(k_1)C_2(k_2)S_3(k_3)
\end{bmatrix}
\]

\[
Z^m \cdot w_3^k & k \pi C_1(k_1)C_2(k_2)C_3(k_3) \\
-k \pi C_1(k_1)S_2(k_2)C_3(k_3) & -k \pi S_1(k_1)C_2(k_2)S_3(k_3)
\end{bmatrix}
\]
and, with $\beta_{w^*,m}^{*1*2*3}$ and $\beta_{w^*,k}^{*1*2*3}$ with $\star_1, \star_2, \star_3 \in \{+, -\}$ defined as below

$$\beta_{w^*,m}^{*1*2*3} := \frac{\pi}{8} \left( \star_1 \frac{w^*_1 m_1}{L_1} \star_2 \frac{w^*_2 m_2}{L_2} \star_3 \frac{w^*_3 m_3}{L_3} \right), \quad \text{for} \quad (\star_1, \star_2, \star_3) \in \{+, -\}^3. \quad (8)$$

For example, we have

$$\beta_{w^*,m}^{+++} := \frac{\pi}{8} \left( \frac{w^*_1 m_1}{L_1} + \frac{w^*_2 m_2}{L_2} + \frac{w^*_3 m_3}{L_3} \right), \quad \beta_{w^*,m}^{--} := \frac{\pi}{8} \left( -\frac{w^*_1 m_1}{L_1} + \frac{w^*_2 m_2}{L_2} - \frac{w^*_3 m_3}{L_3} \right).$$

By straightforward computation, we can find the expression for the coordinates of $(Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k$ as follows

$$\left((Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k\right)_1 = \left(w^*_1 \beta_{w^*,m}^{+++} + w^*_1 \beta_{w^*,m}^{+++} \right) S_1(k_1 + m_1) C_2(k_2 + m_2) S_3(k_3 + m_3) + \left(-w^*_1 \beta_{w^*,m}^{+++} + w^*_1 \beta_{w^*,m}^{+++} \right) S_1(k_1 - m_1) C_2(k_2 - m_2) S_3(k_3 - m_3) + \left(-w^*_1 \beta_{w^*,m}^{+++} - w^*_1 \beta_{w^*,m}^{+++} \right) S_1(k_1 + m_1) C_2(k_2 + m_2) S_3(k_3 - m_3) + \left(w^*_1 \beta_{w^*,m}^{+++} - w^*_1 \beta_{w^*,m}^{+++} \right) S_1(k_1 - m_1) C_2(k_2 - m_2) S_3(k_3 + m_3)$$

(9a)

$$\left((Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k\right)_2 = \left(w^*_2 \beta_{w^*,m}^{+++} + w^*_2 \beta_{w^*,m}^{+++} \right) C_1(k_1 + m_1) S_2(k_2 + m_2) S_3(k_3 + m_3) + \left(-w^*_2 \beta_{w^*,m}^{+++} + w^*_2 \beta_{w^*,m}^{+++} \right) C_1(k_1 - m_1) S_2(k_2 - m_2) S_3(k_3 - m_3) + \left(-w^*_2 \beta_{w^*,m}^{+++} - w^*_2 \beta_{w^*,m}^{+++} \right) C_1(k_1 + m_1) S_2(k_2 + m_2) S_3(k_3 - m_3) + \left(w^*_2 \beta_{w^*,m}^{+++} - w^*_2 \beta_{w^*,m}^{+++} \right) C_1(k_1 - m_1) S_2(k_2 - m_2) S_3(k_3 + m_3)$$

(9b)
Lemma 3.4. Also need the following relaxed version.

\[
\left((Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k\right)_3
\]
\[
= \left(-w^{m}_{3}\beta^{++m} - w^{k}_{3}\beta^{m+,k}\right) C_1(k_1 + m_1)C_2(k_2 + m_2)C_3(k_3 + m_3)
\]
\[
+ \left(w^{m}_{3}\beta^{++m} - w^{k}_{3}\beta^{m+,k}\right) C_1(k_1 - m_1)C_2(k_2 - m_2)C_3(k_3 - m_3)
\]
\[
+ \left(-w^{m}_{3}\beta^{++,m} + w^{k}_{3}\beta^{m+,k}\right) C_1(k_1 + m_1)C_2(k_2 + m_2)C_3(k_3 - m_3)
\]
\[
+ \left(w^{m}_{3}\beta^{++,m} + w^{k}_{3}\beta^{m+,k}\right) C_1(k_1 - m_1)C_2(k_2 - m_2)C_3(k_3 + m_3)
\]
\[
(9c)
\]

Remark 3.2. We do not present the coordinates of the vector \((Z^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Z^k\) because we will not need them in the construction of \(Z^{l(n),n}\). The vectors generate functions of the type \(Y^{j(n),n}\).

3.3. Two fruitful lemmas. Next we introduce two fruitful lemmas which play the main role in the proof below. These lemmas help us to avoid explicit and complicated computations of operator \(B(a, b)\) as some works before in 2D cases (see \([15,18,19,21]\)).

Lemma 3.3. Let us be given \(\alpha, \gamma, \in \mathbb{R}^3\) and \(k \in \mathbb{N}^3\). Then

\[
\text{span}\{\Pi Z^k_{\alpha}, \Pi Z^k_{\gamma}\} = \text{span} Z^{(1,2),k}
\]

if, and only if, the family \(\{\alpha, \gamma, k\}\) is linearly independent.

In Lemma 3.3 we denote

\[
Z^n := \left(\begin{array}{c}
z_1 \sin \left(\frac{k_1 \pi x_1}{L_1}\right) \cos \left(\frac{k_2 \pi x_2}{L_2}\right) \sin \left(\frac{k_3 \pi x_3}{L_3}\right) \\
z_2 \cos \left(\frac{k_1 \pi x_1}{L_1}\right) \sin \left(\frac{k_2 \pi x_2}{L_2}\right) \sin \left(\frac{k_3 \pi x_3}{L_3}\right) \\
z_3 \cos \left(\frac{k_1 \pi x_1}{L_1}\right) \cos \left(\frac{k_2 \pi x_2}{L_2}\right) \cos \left(\frac{k_3 \pi x_3}{L_3}\right)
\end{array}\right)
\]

for given \(z \in \mathbb{R}^3\) and \(n \in \mathbb{N}^3\). The proof is analogous to the Lemma 3.1 in \([17]\). We will also need the following relaxed version.

Lemma 3.4. Let us be given \(\alpha, \gamma, \delta \in \mathbb{R}^3\) and \(k \in \mathbb{N}^3\). Then

\[
\text{span}\{\Pi Z^k_{\alpha}, \Pi Z^k_{\gamma}, \Pi Z^k_{\delta}\} = \text{span} Z^{(1,2),k}
\]

if, and only if, at least one of the families \(\{\alpha, \gamma, k\}\), \(\{\alpha, \delta, k\}\), and \(\{\gamma, \delta, k\}\) is linearly independent.

Proof. Notice that the inclusion \(\text{span}\{\Pi Z^k_{\alpha}, \Pi Z^k_{\gamma}, \Pi Z^k_{\delta}\} \subseteq \text{span} Z^{(1,2),k}\) holds true for all \(\alpha, \gamma, \delta \in \mathbb{R}^3\). Then, the reverse inclusion holds if, and only if, we can set two vectors in \(\{\Pi Z^k_{\alpha}, \Pi Z^k_{\gamma}, \Pi Z^k_{\delta}\}\) which are linearly independent. The Lemma follows straightforwardly from Lemma 3.3. \(\square\)
3.4. Proof of Theorem 3.2. Firstly, let us recall the index subsets $S_R^q, R_m^q, L^q_{m_1,m_2}$ defined in the proof of rectangle case (see [17, Section 3.4])

\[ S_R^q := \{ n \in \mathbb{N}^3 \mid 0 \leq n_i \leq q, \#_o(n) \leq 1 \}, \]
\[ C_R^q := \{ y_{j(n),n} \mid n \in S_R^q, \; j(n) \in \{ 1, 2 - \#_o(n) \} \}, \quad q \in \mathbb{N}, \quad q \geq 4, \]
\[ R_m^q := \{ n \in S_R^q \mid n_m = q, \; 0 \leq n_i \leq q-1 \text{ for } i \neq m \}, \]
\[ L^q_{m_1,m_2} := \{ n \in S_R^q \mid n_{m_1} = q = n_{m_2}, \; m_1 \neq m_2, \; 0 \leq n_i \leq q-1, \; i \notin \{ m_1, m_2 \} \}. \]  

Next, we define some new sets

\[ S_C^q := S_R^q \cup \{(n_1,0,0),(0,n_2,0) \mid 0 < n_1 \leq q, \; 0 < n_2 \leq q \}, \quad q \geq 4, \quad q \in \mathbb{N}, \]
\[ C_C^q := \{ y_{j(n),n}, Z_{j(n),n} \mid n \in S_R^q, \; j(n) \in \{ 1, 2 - \#_o(n) \} \} \cup \{ Z^{j(n),n} \mid n \in S_C^q \setminus S_R^q, \; j(n) = 1 \}. \] 

Notice that

\[ S_C^{q+1} = S_R^q \cup (R_1^{q+1} \cup R_2^{q+1} \cup R_3^{q+1}) \cup \{(q+1,0,0),(0,q+1,0)\} \cup \{ (q+1,q+1,q+1) \}. \]

We can see that Theorem 3.2 is a corollary of the following inclusions

\[ C_C^q \subseteq G^{q-1}, \quad \text{for all} \quad q \in \mathbb{N}, \quad q \geq 4. \]  

Let us decompose $C_C^q = C_C^q \cup C_C^q_{YC}$ with

\[ C_C^q_{YC} := \{ y_{j(n),n} \mid n \in S_R^q, \; j(n) \in \{ 1, 2 - \#_o(n) \} \} \]
\[ C_C^q_{Y2C} := \{ Z_{j(n),n} \mid n \in S_R^q, \; j(n) \in \{ 1, 2 - \#_o(n) \} \} \cup \{ Z^{1,n} \mid n \in S_C^q \setminus S_R^q \}. \]

Inspiring from the proof of Theorem 2.2, we get

\[ C_C^q \subseteq G^{q-1}, \quad \text{for all} \quad q \in \mathbb{N}, \quad q \geq 4. \]

So it remains to prove that

\[ C_C^q_{YC} \subseteq G^{q-1}, \quad \text{for all} \quad q \in \mathbb{N}, \quad q \geq 4, \]

which we will prove by induction.

**Base step** By definition we have that $C = C_C^1 \supset C_C^1$ and span $C = G^0$. Therefore

Inclusion (16) holds for $q = 4$.

**Induction step** The induction hypothesis is

\[ C_C^q_{YC} \subseteq G^0 \text{ and the inclusion } C_C^q_{YC} \subseteq G^{q-1} \text{ holds true for a given } q \in \mathbb{N}, \; q \geq 4. \]

(II.H.C-equation 18)

We want to prove that $C_C^{q+1} \subseteq G^q$.

Based on this decomposition (12), we introduce five Lemmas below. We will prove these lemmas in following Sections 3.4.1, 3.4.2, 3.4.3, 3.4.4, 3.4.5

**Lemma 3.5.** $Z^{j(n),n} \in G^q$ for all $n \in R^{q+1}_3$.

**Lemma 3.6.** $Z^{j(n),n} \in G^q$ for all $n \in (R^{q+1}_1 \cup R^{q+1}_2)$.

**Lemma 3.7.** $Z^{j(n),n} \in G^q$ for all $n \in \{(q+1,0,0),(0,q+1,0)\}$.

**Lemma 3.8.** $Z^{j(n),n} \in G^q$ for all $n \in (\ell^{q+1}_1 \cup \ell^{q+1}_2 \cup \ell^{q+1}_3)$.

**Lemma 3.9.** $Z^{q+1,0,0} \cup (q+1,q+1,q+1) \in G^q$.

Following all results in Lemma 3.5, 3.6, 3.7, 3.8, and 3.9, we can conclude that $C_C^{q+1} \subseteq G^q$. By induction hypothesis and (17), we can conclude that the inclusion (16) hold true, which implies the statement of Theorem 3.2. □
3.4.1. Proof of Lemma 3.5. We divide this proof into three steps

- Step 3.5.1: Generating $Z^{1,n}$, $n = (0, l, q + 1)$ or $n = (l, 0, q + 1)$, with $1 \leq l \leq q$.
- Step 3.5.2: Generating $Z^{(1,2),n}$, $n = (1, l, q + 1)$ or $n = (l, 1, q + 1)$, $1 \leq l \leq q$.
- Step 3.5.3: Generating $Z^{(1,2),n}$ with $n = (n_1, n_2, q + 1)$ where $2 \leq n_1 \leq q$ and $2 \leq n_2 \leq q$.

**Step 3.5.1 Generating** $Z^{1,n}$, $n = (0, l, q + 1)$ or $n = (l, 0, q + 1)$, with $1 \leq l \leq q$.

The case $n = (0, l, q + 1)$. We may follow the result in 2D Cylinder addressed in [15]. Indeed from [9] we find that for $k$ and $m$ such that $k_1 = m_1 = 0$,

$$Y^k = \begin{pmatrix} 0 \\ \hat{Y}^k \end{pmatrix} \quad \text{and} \quad Z^k = \begin{pmatrix} 0 \\ \hat{Z}^k \end{pmatrix}$$

where for suitable constants $C_1$ and $C_2$

$$\hat{Y}^k = C_1 \left( \frac{k_3}{L_3} \sin \left( \frac{k_3 \pi z_3}{L_3} \right) \cos \left( \frac{k_2 \pi z_2}{L_2} \right) \sin \left( \frac{k_1 \pi z_1}{L_1} \right) \right) \quad \text{and} \quad \hat{Z}^k = C_1 \left( \frac{k_3}{L_3} \cos \left( \frac{k_3 \pi z_3}{L_3} \right) \sin \left( \frac{k_2 \pi z_2}{L_2} \right) \cos \left( \frac{k_1 \pi z_1}{L_1} \right) \right),$$

where the functions $\hat{Y}^k$ and $\hat{Z}^k$ are eigenfunctions of the Stokes operator in $C_2 = (0, L_2) \times \frac{L_2}{2 \pi} \mathbb{S}^1$ under Lions boundary conditions. Using an argument that is similar to the one used to derive span $\{Y^n \mid n \in \mathbb{S}^{d+1}_R, \#_0(n) = 1 \} \subset G^q$ as in [17] Section 3.4, we can derive that

$$\mathcal{B}(Y^k)Z^k = \begin{pmatrix} 0 \\ \Pi_2 \left( (\hat{Y}^k \cdot \nabla_2) \hat{Z}^m + (\hat{Z}^m \cdot \nabla_2) \hat{Y}^k \right) \end{pmatrix}$$

with $\Pi_2$ being the orthogonal projection onto $H_2 = \{ u \in L^2(C_2, TC_2) \mid u \cdot n = 0, \text{div}_2 u = 0 \}$ and $\nabla_2$ and $\text{div}_2$ being the gradient and divergence operators in $C_2 \sim (0, L_2) \times (0, 2L_3)$.

Therefore from [15] proof of Theorem 4.1] we know that

$$\{Z^{1,n} \mid n = (0, l, q + 1), 0 < l \leq q \} \subset G^q.$$ (19a)

and a similar argument gives us

$$\{Z^{1,n} \mid n = (l, 0, q + 1), 0 < l \leq q \} \subset G^q.$$ (19b)

**Step 3.5.2 Generating** $Z^{(1,2),n}$, $n = (1, l, q + 1)$ or $n = (l, 1, q + 1)$, $1 \leq l \leq q$.

The case $n = (1, l, q + 1)$. Firstly, we choose

$$k = (1, 0, q), \quad m = (0, 1, 1), \quad w^k = (L_1 q, 0, -L_3), \quad w^m = (0, L_2, -L_3).$$

From [9], this choice gives us

$$(Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k = Z^{(1,1,q+1)}_{z^1} + Z^{(1,1,q-1)}_{z^2},$$

for suitable $z^1, z^2 \in \mathbb{R}^3$. By the induction hypothesis [16], we have $Z^{(1,2), (1,1,q-1)} \in G^{q-1} \subset G^q$. It is equivalent that $Z^{(1,1,q-1)}_{z^2} \in G^q$. Next, from

$$\beta^{1*2+}_{w^k,m} = -\frac{\pi}{8}, \quad \beta^{1*2-}_{w^k,m} = \frac{\pi}{8}, \quad \beta^{1*2+}_{w^m,k} = -\frac{\pi}{8}q, \quad \text{and} \quad \beta^{1*2-}_{w^m,k} = \frac{\pi}{8}q,$$
with \( \{\ast_1, \ast_2\} \subseteq \{+, -\} \), following the expressions in [19], we find

\[
\begin{pmatrix}
0 + L_1 q \left( \beta_{++}^{++} - \beta_{++}^{--} + \beta_{++}^{-+} - \beta_{++}^{+-} \right) \\
L_2 \left( \beta_{++}^{++} + \beta_{++}^{--} \operatorname{sign}(0 - 1) - \beta_{++}^{-+} \operatorname{sign}(0 - 1) - \beta_{++}^{+-} \right) + 0 \\
-L_3 \left( -\beta_{+}^{+} - \beta_{+}^{-} + \beta_{+}^{+} + \beta_{+}^{+} \right) - L_3 \left( -\beta_{+}^{+} - \beta_{+}^{-} + \beta_{+}^{+} + \beta_{+}^{+} \right)
\end{pmatrix}

\]

\[
= -\frac{\pi}{2} \begin{pmatrix}
L_1 q^2 \\
L_2 \\
L_3 (q + 1)
\end{pmatrix}
\]

and we can conclude that \( \Pi Z_{z_{\ast_1}}^{(1,1,q+1)} \in \mathcal{G}' \).

Secondly, we choose

\[
k = (0, 1, q), \quad m = (1, 0, 1), \quad w^m = (L_1, 0, -L_3)
\]

which again gives us \((Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k = Z_{z_{\ast_1}}^{(1,1,q+1)} + Z_{z_{\ast_2}}^{(1,1,q-1)}\) with

\[
z_{\ast_2} = -\frac{\pi}{2} \begin{pmatrix}
L_1 \\
L_2 q^2 \\
L_3 (q + 1)
\end{pmatrix}
\]

Again, we get that \( \Pi Z_{z_{\ast_1}}^{(1,1,q+1)} \in \mathcal{G}' \).

We observe that

\[
\begin{align*}
det(n \ z_{\ast_1} \ z_{\ast_2}) &= \frac{\pi^2}{4} (q + 1) \begin{vmatrix}
L_1 q^2 & L_1 \\
L_2 & L_2 q^2 \\
L_3 & L_3
\end{vmatrix} \\
&= \frac{\pi^2}{4} (q + 1)^2 (q - 1) \left( L_1 L_2 q^2 + L_1 L_2 - L_2 L_3 - L_1 L_3 \right),
\end{align*}
\]

and that, since \( q \geq 4 \),

\[
det(n \ z_{\ast_1} \ z_{\ast_2}) = 0 \quad \iff \quad q^2 = \frac{L_2 L_3 + L_1 L_3 - L_1 L_2}{L_1 L_2}.
\]

Thus from Lemma 3.3 we conclude that \( \Pi Z_{z_{\ast_1}}^{(1,1,q+1)} \) and \( \Pi Z_{z_{\ast_2}}^{(1,1,q+1)} \) are not necessarily linearly independent. So, next we want to use Lemma 3.4. For that, we choose the third quadruple

\[
k = (1, 0, q - 1), \quad m = (0, 1, 2), \quad w^m = (L_1(q - 1), 0, -L_3),
\]

which gives us

\[
(Y^k \cdot \nabla) Z^m + (Y^m \cdot \nabla) Z^k = Z_{z_{\ast_1}}^{(1,1,q+1)} + Z_{z_{\ast_2}}^{(1,1,q-3)},
\]

for suitable \( z_{\ast_1}, z_{\ast_2} \in \mathbb{R}^3 \). Since by [16] \( Z^{(1,2),(1,1,q-3)} \in \mathcal{G}' \subset \mathcal{G}' \), we can conclude that \( \Pi Z_{z_{\ast_1}}^{(1,1,q+1)} \in \mathcal{G}' \). We can also find

\[
z_{\ast_2} = -\frac{\pi}{2} \begin{pmatrix}
L_1(q - 1)^2 \\
4L_2 \\
L_3 (q + 1)
\end{pmatrix}
\]
Now, we compute
\[
\frac{4}{\pi^2(q+1)} \det(n, z_{\alpha_1}, z_{\beta_1}) = \det \begin{pmatrix} 1 & L_1q^2 & L_1(q-1)^2 \\ 1 & L_2 & 4L_2 \\ 1 & L_3 & L_3 \end{pmatrix}
\]
\[
= 3L_1L_2q^2 + 2(L_1L_2 - L_1L_3)q + L_1L_3 - L_1L_2 - 3L_2L_3
\]
\[
= -L_1L_2q^4 + 2L_1L_2q^3 + (L_2L_3 + L_1L_3 - L_1L_2)q^2 - 2L_1L_3q + 4L_1L_2 - 4L_2L_3.
\]
From which we conclude that \( \det(n, z_{\alpha_1}, z_{\gamma_1}) = \det(n, z_{\alpha_1}, z_{\beta_1}) = \det(n, z_{\gamma_1}, z_{\beta_1}) = 0 \) if, and only if,
\[
\begin{align*}
L_1L_2q^2 + L_1L_2 - L_2L_3 - L_1L_3 &= 0 \quad (21a) \\
L_2(L_3 - L_1)(q - 2) &= 0 \quad (21b) \\
L_1(L_2 - L_3)(q - 2) &= 0. \quad (21c)
\end{align*}
\]
Since \( q \geq 4 \) and \( L_1, L_2, L_3 > 0 \), from \( 21b \) and \( 21c \) it arrives that \( L_1 = L_2 = L_3 \). In this case, from \( 21a \) we arrive to the contradiction \( q = 1 \). Hence, at least one of the families \( \{n, z_{\alpha_1}, z_{\gamma_1}\}, \{n, z_{\alpha_1}, z_{\beta_1}\}, \) or \( \{n, z_{\gamma_1}, z_{\beta_1}\} \) is linearly independent. From Lemma 3.4 we can conclude that
\[
Z^{(1,2),(1,1,q+1)} \in \mathcal{G}^q. \quad (22)
\]

The case \( n = (1, l, q) + 1 \) with \( 2 \leq l \leq q \).
Assume that
\[
Z^{(k),k} \subset \mathcal{G}^q \quad \text{with} \quad k = (1, l - 2, q + 1). \quad (\text{IH-C11-eq.23})
\]
We will prove that \( Z^{(1,2),(1,l,q+1)} \).

Firstly, we choose
\[
k = (1, l - 1, q), \quad m = (0, 1, 1),
\]
\[
w^k = (0, L_2q, L_3(1 - l)), \quad w^m = (0, L_2, -L_3),
\]
This choice gives us
\[
(\nabla \cdot \nabla) Z^m + (\nabla \cdot \nabla) Y^k = Z^{(1,l,q+1)}_{\alpha_1} + Z^{(1,l-2,q+1)}_{\alpha_2} + Z^{(1,l,q-1)}_{\alpha_3} + Z^{(1,l-2,q-1)}_{\alpha_4}.
\]
From \( \text{(IH.C-eq.18)} \) and \( \text{(IH-C11-eq.23)} \), we have that \( Z^{(1,l-2,q+1)}_{\alpha_2}, Z^{(1,l,q-1)}_{\alpha_3}, \) and \( Z^{(1,l-2,q-1)}_{\alpha_4} \) belong in \( \mathcal{G}^q \). Therefore, we can conclude that \( \Pi Z^{(1,l,q+1)}_{\alpha_1} \in \mathcal{G}^q \).

Next, from
\[
\beta^{++}_{w^k,m} = \beta^{--}_{w^k,m} = \frac{\pi(q - l + 1)}{8} \quad \text{and} \quad \beta^{++}_{w^m,k} = \beta^{--}_{w^m,k} = \frac{\pi(l - q - 1)}{8},
\]
we obtain
\[

z_{\alpha_1} = \begin{pmatrix}
L_2 \left( \beta^{++}_{w^k,m} + \beta^{--}_{w^k,m} \right) + L_2q \left( \beta^{++}_{w^m,k} + \beta^{--}_{w^m,k} \right) \\
L_3 \left( -\beta^{++}_{w^k,m} - \beta^{--}_{w^k,m} \right) + L_3(1 - l) \left( -\beta^{++}_{w^m,k} - \beta^{--}_{w^m,k} \right)
\end{pmatrix}
\]
\[
= -\frac{\pi}{4} \begin{pmatrix}
L_2(q - l + 1)(q - 1) \\
L_3(q - l + 1)(l - 2)
\end{pmatrix}.
\]
Secondly we choose
\[ k = (1, l - 1, q), \quad m = (1, 1, 1), \]
\[ w^k = (L_1q, 0, -L_3), \quad w^m = (0, L_2, -L_3), \]
which allow us to obtain \( Z_{\gamma_1}^{(1, l, q + 1)} \in G^q \), and from
\[ \beta_{w, m}^{+ + } = \beta_{w, m}^{- + } = -\frac{\pi}{8} \quad \text{and} \quad \beta_{w, k}^{+ + } = \beta_{w, k}^{- + } = \frac{\pi}{8}(l - q - 1), \]
we can find
\[ z_{\gamma_1} = -\frac{\pi}{4} \left[ \begin{array}{c} L_1q(q - l + 1) \\ L_2 \\ L_3(q - l + 2) \end{array} \right]. \]

Thirdly, we choose
\[ k = (1, l - 1, q), \quad m = (1, 1, 1), \]
\[ w^k = (0, L_2q, L_3(1 - l)), \quad w^m = (L_1, -L_2, 0), \]
which gives us that \( Z_{\delta_1}^{(1, l, q + 1)} \in G^q \), with
\[ z_{\delta_1} = -\frac{\pi}{4} \left[ \begin{array}{c} -L_1(q - l + 1) \\ L_2(ql - l + 1) \\ L_3(l - 1)^2 \end{array} \right]. \]

Next we compute
\[ \det(n z_{\alpha_1} z_{\gamma_1}) = \frac{\pi^2}{16} q(q - l + 1)^2 \left( L_1 L_2 q^2 - L_1 L_2 - L_2 L_3 - L_1 L_3(l - 2) \right), \]
\[ \det(n z_{\alpha_1} z_{\delta_1}) = \frac{\pi^2}{16} (q - l + 1)^2 \left( L_1 L_2 L_3 + L_2 L_3 + L_1 L_2 - L_1 L_3(l - 2) \right), \]
and observe that \( \det(n z_{\alpha_1} z_{\gamma_1}) = \det(n z_{\alpha_1} z_{\delta_1}) = 0 \), if, and only if,
\[ L_1 L_2 q^2 - L_1 L_2 - L_2 L_3 - L_1 L_3(l - 2) = L_1 L_2 q^2 + L_2 L_3 - L_1 L_2 - L_1 L_3(l - 2) = 0, \]
because \( 2 \leq l \leq q \), which implies \( 2L_2 L_3 = 0 \). This contradicts the fact that \( L_1, L_2, L_3 > 0 \). Therefore one of the families \( \{ n, z_{\alpha_1}, z_{\gamma_1} \} \) or \( \{ n, z_{\alpha_1}, z_{\delta_1} \} \) is linearly independent and, by Lemma 3.4, it follows that \( Z^{(1, 2), (l, q + 1)} \in G^q \).

By induction, using (19), (22), and the induction hypothesis (IH-CN1-eq.23), it follows that
\[ Z^{(1, l, q + 1)} \in G^q, \quad \text{for all} \quad 0 \leq l \leq q \]
and proceeding similarly we can also derive that
\[ Z^{(l, 1, q + 1)} \in G^q, \quad \text{for all} \quad 0 \leq l \leq q. \]

Step 3.5.3: Generating the family \( Z^{(1, 2), n} \) with \( n = (n_1, n_2, q + 1) \) where \( 2 \leq n_1 \leq q \) and \( 2 \leq n_2 \leq q \).

Firstly, we introduce an induction hypothesis. Assume that
\[ Z^{(\kappa), \kappa} \in G^q, \quad \text{(IH-Cn1n2-eq.25)} \]
for \( \kappa \in \{(n_1 - 2, n_2 - 2, q + 1), (n_1 - 2, n_2, q + 1), (n_1, n_2 - 2, q + 1)\} \).

We will prove that \( Z^{(1, 2), (n_1, n_2, q + 1)} \in G^q \).
By choosing

\[ k = (n_1 - 1, n_2 - 1, q), \quad m = (1, 1, 1), \]
\[ u^k = (0, L_2 q, L_3 (1 - n_2)), \quad w^m = (0, L_2, -L_3), \]

we obtain

\[ (Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k = Z^{(n_1, n_2, q+1)} + \sum_{i=2}^{8} Z^{\kappa_i}, \]

with \( \kappa_i \in \{(n_1 - 2, n_2 - 2, q-1), (n_1, n_2 - 2, q-1), (n_1 - 2, n_2, q), (n_1, n_2, q+1)\}(n_1 - 2, n_2 - 2, q+1), (n_1, n_2 - 2, q+1), (n_1 - 2, n_2, q-1)\). Using the inductive hypothesis (eq. 18), we find that \( Z^{(\kappa, \kappa)} \in G^q \), for \( \kappa \in \{(n_1 - 2, n_2 - 2, q-1), (n_1, n_2 - 2, q-1), (n_1 - 2, n_2, q-1)\}. From the inductive hypothesis (eq. 25) we also have \( Z^{(\kappa, \kappa)} \in G^q \), for \( \kappa \in \{(n_1 - 2, n_2 - 2, q+1), (n_1, n_2 - 2, q+1), (n_1 - 2, n_2, q+1)\}. Thus, we can conclude that \( \Pi Z^{(n_1, n_2, q+1)} \in G^q \).

Next, from

\[ \beta^{+++}_{w^k, m} = \frac{\pi}{8} (q - n_2 + 1) \quad \text{and} \quad \beta^{+++}_{w^m, k} = \frac{\pi}{8} (n_2 - q - 1), \]

we obtain

\[ z_{\alpha^1} = -\frac{\pi}{8} \left( \begin{array}{c} 0 \\ L_2 (q - n_2 + 1)(q - 1) \\ L_3 (q - n_2 + 1)(n_2 - 2) \end{array} \right). \]

A second choice is

\[ k = (n_1 - 1, n_2 - 1, q), \quad m = (1, 1, 1), \]
\[ u^k = (L_1 q, 0, L_3 (1 - n_1)), \quad w^m = (0, L_2, -L_3), \]

which gives us \( \Pi Z^{(n_1, n_2, q+1)} \in G^q \). From

\[ \beta^{+++}_{w^k, m} = \frac{1}{8} (q - n_1 + 1), \quad \text{and} \quad \beta^{+++}_{w^m, k} = \frac{1}{8} (n_2 - q - 1), \]

we obtain

\[ z_{\gamma^1} = \frac{\pi}{8} \left( \begin{array}{c} -L_1 (q - n_2 + 1) \\ -L_2 (q - n_1 + 1) \\ L_3 (1 - n_1)(q - n_2 + 1) + L_3 (q - n_1 + 1) \end{array} \right). \]

Another choice is

\[ k = (n_1, n_2 - 1, q), \quad m = (0, 1, 1), \]
\[ u^k = (L_1 (n_2 - 1), -L_2 n_1, 0), \quad w^m = (0, L_2, -L_3), \]

which gives us \( \Pi Z^{(n_1, n_2, q+1)} \in G^q \), with

\[ z_{\delta^1} = \frac{\pi}{8} \left( \begin{array}{c} -L_1 (q - n_2 + 1)(n_2 - 1) \\ -L_2 n_1 (q - n_2) \\ -L_3 n_1 \end{array} \right). \]
Next, from $2 \leq n_1, n_2 \leq q$ and
\[- \frac{64}{\pi^2(q - n_2 + 1)} \det(n \ z_{\alpha^1} z_{\gamma^1}) = \det \begin{pmatrix} n_1 & 0 & -L_1(q - n_2 + 1)(n_2 - 1) \\ n_2 & L_2(q - 1) & L_2(q - n_1 + 1) \\ q + 1 & L_3(n_2 - 2) & L_3(1 - n_1)(q - n_2 + 1) + L_3(q - n_1 + 1) \end{pmatrix} = q(q - n_2 + 1) \left( L_1 L_2 q^2 - L_1 L_2 - L_3 n_1 (n_1 - 1) - L_1 L_3 n_2 (n_2 - 2) \right),\]
\[- \frac{64}{\pi^2(q - n_2 + 1)} \det(n \ z_{\alpha^1} \ z_{\delta^1}) = \det \begin{pmatrix} n_1 & 0 & -L_1 q(q - n_2 + 1) \\ n_2 & L_2(q - 1) & L_2 n_1 (q - n_2) \\ q + 1 & L_3(n_2 - 2) & -L_3 n_1 \end{pmatrix} = (q - n_2 + 1) \left( L_1 L_2 q^2 - L_1 L_2 - L_2 L_3 n_1^2 - L_1 L_3 n_2 (n_2 - 2) \right),\]
we have that $\det(n \ z_{\alpha^1} \ z_{\gamma^1}) = \det(n \ z_{\alpha^1} \ z_{\delta^1}) = 0$ only if $2 L_2 L_3 n_1 = 0$. This contradicts the fact that $L_2, L_3,$ and $n_1$ are positive. Thus, one of the families $\{n, z_{\alpha^1}, z_{\gamma^1}\}$ or $\{n, z_{\alpha^1}, z_{\delta^1}\}$ is linearly independent. By Lemma 3.4 it follows that $Z^{(1,2), (n_1,n_2,q+1)} \in G^q$.

Using (IH.C-eq.18), (19), (24), and the induction hypothesis (IH.C-niln2-eq.25), we conclude that $Z^{(n), n} \in G^q$ with $n = (n_1, n_2, q + 1)$. Finally, we obtain
\[Z^{(n), n} \in G^q \quad \text{for all } n \in R_3^{q+1}. \quad (26)\]

\[\Box\]

3.4.2. Proof of Lemma 3.6. First of all, notice that the cases $n \in R_1^{q+1}$ and $n \in R_2^{q+1}$ are analogous. On the other hand, since we consider the periodicity assumption in the third direction and Lions boundary conditions in the first two directions, these cases must be addressed separately from the case $n \in R_3^{q+1}$ treated in section 3.4.1. Let us take $n \in R_3^{q+1}$. Again we divide this proof into three main steps

- Step 3.6.1: Generating $Z^{1, n}$ with $n = (q + 1, l, 0)$ or $n = (q + 1, 0, l)$, $1 \leq l \leq q$.
- Step 3.6.2: Generating the famility $Z^{(1,2), n}$ with $n = (q + 1, l, 1)$ or $n = (q + 1, 1, l)$, $1 \leq l \leq q$.
- Step 3.6.3: Generating the family $Z^{(1,2), n}$ with $n = (q + 1, n_1, n_2)$, $2 \leq n_1, n_2 \leq q$.

**Step 3.6.1** Generating $Z^{1, n}$ with $n = (q + 1, l, 0)$ or $n = (q + 1, 0, l)$. The case $n = (q + 1, l, 0)$. We choose
\[k = (q, l, 0), \quad m = (1, 0, 0), \quad w^k = (L_1 l, -L_2 q, 0), \quad w^m = (0, 0, 1),\]
which gives us $\Pi \mathcal{Z}^{(q+1, l, 0)}_{Z_{\alpha^1}} \in G^q$. From $\beta_{w^k m^*} = \frac{\pi}{2} l$, $\beta_{w^m k} = 0$, with $\{*, *\} \subseteq \{+,-\}$, we get
\[z_{\gamma^1} = \frac{\pi}{2} \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix}.\]

Observe that $Z^{1, (q+1, l, 0)}_{Z_{\gamma^1}} = \frac{\pi}{2} l \begin{pmatrix} 0 \\ \cos\left(\frac{(q+1)\pi}{L_1}\right) \\ \cos\left(\frac{\pi}{L_2}\right) \end{pmatrix}$, it means that for a suitable constant $\zeta \neq 0$, $Z^{1, (q+1, l, 0)} = \zeta Z^{1, (q+1, l, 0)}_{Z_{\gamma^1}}$. Hence, we conclude that
\[Z^{1, (q+1, l, 0)} \in G^q, \quad \text{for all } \quad 1 \leq l \leq q. \quad (27)\]
To generate $Z^{1,n}$ with $n = (q + 1, 0, l)$, we can use the result for the 2D cylinder in [15]. Notice that, for some constant $ζ \neq 0$,

$$Z^{1,(q+1,0,l)} = ζ \begin{pmatrix}
\frac{2\pi}{L_3} \sin \left(\frac{(q+1)\pi x_1}{L_1}\right) \sin \left(\frac{\pi x_3}{L_3}\right) \\
0 \\
\frac{(q+1)\pi}{L_1} \cos \left(\frac{(q+1)\pi x_1}{L_1}\right) \cos \left(\frac{\pi x_3}{L_3}\right)
\end{pmatrix},$$

which is an eigenfunction of the Stokes operator in the 2D cylinder $(0, L_1) \times \frac{L_2}{\pi} \cdot S^1$, under Lions boundary conditions. It follows, from [15, Theorem 4.1], that

$$Z^{1,(q+1,0,l)} \in G^q, \quad \text{for all} \quad 1 \leq l \leq q. \tag{28}$$

**Step 3.6.2** Generate the family $Z^{(1,2),n}$ with $n = (q + 1, 1, l)$ or $n = (q + 1, l, 1)$. The case $n = (q + 1, 1, 1)$. We firstly choose

$$k = (q, 0, 1), \quad m = (1, 1, 0),$$

$$w^k = (L_1, 0, -L_3q), \quad w^m = (L_1, -L_2, 0),$$

Then, by changing the roles of $k$ and $m$ in from (9), we obtain

$$(Z^k \cdot \nabla) Y^m + (Y^m \cdot \nabla) Z^k = Z^{(q+1,1,1)}_{z_{\alpha^1}} + Z^{(q-1,1,1)}_{z_{\alpha^2}},$$

for suitable $z_{\alpha^1}, z_{\alpha^2} \in \mathbb{R}^3$. By [15, eq.18], we have $Z^{(1,2),(q-1,1,1)} \in G^q$. Therefore we derive that $\Pi Z^{(q+1,1,1)}_{z_{\alpha^1}} \in G^q$, and we can also find

$$z_{\alpha^1} = \frac{\pi}{2} \begin{pmatrix} L_1(q + 1) \\ -L_2 \end{pmatrix}. \tag{29}$$

Secondly, we compute $(Z^k \cdot \nabla) Y^m + (Y^m \cdot \nabla) Z^m$ with the choice

$$k = (q, 1, 1), \quad m = (1, 0, 0),$$

$$w^k = (0, L_2, -L_3), \quad w^m = (0, 0, L_3),$$

Analogously, we obtain that $\Pi Z^{(q+1,1,1)}_{z_{\gamma^1}} \in G^q$, with

$$z_{\gamma^1} = \frac{\pi}{2} \begin{pmatrix} 0 \\ L_2 \\ L_3 \end{pmatrix}. \tag{29}$$

Thirdly, we compute $(Z^m \cdot \nabla) Y^k + (Y^k \cdot \nabla) Z^m$ with

$$k = (q, 1, 1), \quad m = (1, 0, 0),$$

$$w^k = (L_1, 0, -L_3q), \quad w^m = (0, 0, L_3),$$

which gives us $\Pi Z^{(q+1,1,1)}_{z_{\delta^1}} \in G^q$, with

$$z_{\delta^1} = \frac{\pi}{2} \begin{pmatrix} L_1 \\ 0 \\ L_3(q - 1) \end{pmatrix}. \tag{29}$$

Now, observe that $\det(n, z_{\alpha^1}, z_{\gamma^1}) = \det(n, z_{\gamma^1}, z_{\delta^1}) = 0$ if, and only if,

$$L_2L_3q^2 + L_1L_3 + L_2L_3 - L_1L_2 = L_2L_3q^2 - L_2L_3 + L_1L_3 - L_1L_2 = 0,$$

which implies the contradiction $2L_2L_3 = 0$, because $L_2, L_3 > 0$. Therefore, by Lemma 3.4,

$$Z^{(1,2),(q+1,1,1)} \in G^q. \tag{29}$$
The case $n = (q + 1, 1, l)$. Let us introduce the induction hypothesis

$$Z^{(k),l} \in G^q, \quad \text{if} \quad k = (q + 1, 1, l - 2), \quad \text{for given} \ 2 \leq l \leq q. \quad \text{(IH-Cq11-eq.30)}$$

We prove that $Z^{(k),l} \in G^q$ with $k = (q + 1, 1, l)$.

To generate $n = (q + 1, 1, l)$, firstly we compute $(Z^m \cdot \nabla) Y^k + (Y^k \cdot \nabla) Z^m$ with the choice

$$k = (q, 1, l), \quad m = (1, 0, 0),$$
$$w^k = (L_2 l, L_3), \quad w^m = (0, 0, L_3),$$

which allow us to conclude that $\Pi Z^{(q+1,1,l)}_{z_n^1} \in G^q$ with

$$z_\alpha^1 = \frac{\pi}{2} \left( \begin{array}{c} 0 \\ L_2 l^2 \\ L_3 \end{array} \right).$$

Secondly, we compute $(Z^m \cdot \nabla) Y^k + (Y^k \cdot \nabla) Z^m$ with

$$k = (q, 1, l), \quad m = (1, 0, 0),$$
$$w^k = (L_1 l, 0, L_3), \quad w^m = (0, 0, L_3),$$

which gives us $\Pi Z^{(q+1,1,l)}_{z_n^1} \in G^q$, with

$$z_\beta^1 = \frac{\pi}{2} \left( \begin{array}{c} L_1 l^2 \\ 0 \\ L_3 (q - 1) \end{array} \right).$$

Thirdly we compute $(Z^k \cdot \nabla) Y^m + (Y^m \cdot \nabla) Z^k$ with

$$k = (q, 0, l), \quad m = (1, 1, 0),$$
$$w^k = (L_1 l, 0, L_3), \quad w^m = (L_1, -L_2, 0),$$

which gives us $\Pi Z^{(q+1,1,l)}_{z_n^1} \in G^q$, with

$$z_\delta^1 = \frac{\pi}{2} \left( \begin{array}{c} L_1 l (q + 1) \\ -L_2 l \\ L_3 q^2 \end{array} \right).$$

Next we observe that $\det(n z_{a1}, z_{\alpha1}) = \det(n z_{a1}, z_{\delta1}) = 0$ if, and only if,

$$l^3 (L_2 L_3 q^2 - L_2 L_3 + L_1 L_3 - L_1 L_2) = l^3 (q + 1) (L_2 L_3 q^2 + L_2 L_3 + L_1 L_3 - L_1 L_2) = 0$$

which leads to the contradiction $0 = 2 L_1 L_2 l (l - 1) + 2 L_2 L_3 \geq 2 L_2 (L_1 + L_3) > 0$, since $2 \leq l \leq q$. Then from Lemma 3.3 we conclude that $Z^{(q+1,1,l)}_{z_n^1} \in G^q$. By induction, using (27), (29), and the induction hypothesis (IH-Cq11-eq.30), it follows that

$$Z^{(q+1,1,l)}_{z_n^1} \in G^q, \quad \text{for all} \quad 1 \leq l \leq q. \quad \text{(31)}$$

The case $n = (q + 1, l, 1)$. Let us introduce the induction hypothesis

$$Z^{(k),l} \in G^q, \quad \text{if} \quad k = (q + 1, l, l - 2), \quad l \geq 2. \quad \text{(IH-Cql1-eq.32)}$$

We prove that $Z^{(k),l} \in G^q$ with $k = (q + 1, l, l)$.

To generate $n = (q + 1, l, 1)$, firstly we choose

$$k = (q, l, 1), \quad m = (1, 0, 0),$$
$$w^k = (0, L_2, -L_3), \quad w^m = (0, 0, L_3),$$

which allows us to conclude that $\Pi Z^{(q+1,1,l)}_{z_n^1} \in G^q$. By induction, using (27), (29), and the induction hypothesis (IH-Cq11-eq.30), it follows that

$$Z^{(q+1,1,l)}_{z_n^1} \in G^q, \quad \text{for all} \quad 1 \leq l \leq q. \quad \text{(32)}$$
which allow us to conclude that \( \Pi Z^{(q+1,1,l)}_{z^{\alpha}} \in G^q \) with
\[
z_{\alpha} = \frac{\pi}{2} \begin{pmatrix} 0 \\ L_2 \\ L_3 l \end{pmatrix}.
\]

Secondly, we compute \((Z^m \cdot \nabla)Y^k + (Y^k \cdot \nabla)Z^m\) with
\[
k = (q, l, 1), \quad m = (1, 0, 0),
\]
\[
w^k = (L_1, 0, -L_3q), \quad w^m = (0, 0, L_3),
\]
which gives us \(\Pi Z^{(q+1,1,l)}_{z^{\gamma}} \in G^q\), with
\[
z_{\gamma} = \frac{\pi}{2} \begin{pmatrix} L_1 \\ 0 \\ L_3(q - 1) \end{pmatrix}.
\]

Thirdly we compute \((Z^k \cdot \nabla)Y^m + (Y^m \cdot \nabla)Z^k\) with
\[
k = (q, l, 0), \quad m = (1, 0, 1),
\]
\[
w^k = (L_1 l, -L_2 q, 0), \quad w^m = (L_1, 0, -L_3),
\]
which gives us \(\Pi Z^{(q+1,1,l)}_{z^{\delta}} \in G^q\), with
\[
z_{\delta} = \frac{\pi}{2} \begin{pmatrix} L_1 l(q + 1) \\ -L_2 q^2 \\ L_3 l \end{pmatrix}.
\]

Next we observe that \(\det(n z_{\alpha} z_{\gamma} z_{\delta}) = 0\) if, and only if,
\[
L_2 L_3 l(q - 1) + L_1 (L_3 l^2 - L_2) = L_2 L_3 q(q^2 - 1) + L_1 (L_3 l^2 - L_2)(q + 1) = 0
\]
which leads to the contradiction \(0 = (l - 1)(q^2 - 1) - 2 \geq q^2 - 3 > 0\), since \(2 \leq l \leq q\). Then from Lemma 3.4 we conclude that \(Z^{(q+1,1,l)}_{z^{\delta}} \in G^q\). By induction, using (28), (29), and the induction hypothesis \((\text{IH-Cq1n1eq.32})\), it follows that
\[
Z^{(1,2)(q+1,1,l)} \in G^q \quad \text{for all } 1 \leq l \leq q. \tag{33}
\]

**Step 3.6.3 Generating the family** \(Z^{(n)}_{j(n)}\) **with** \(n = (q + 1, n_1, n_2), 2 \leq n_1, n_2 \leq q\).

Let us introduce the inductive hypothesis
\[
Z^{(n)}_{j(n),\kappa} \in G^n, \tag{IH-Cq1n2eq.34}
\]
for \(\kappa \in \{(q + 1, n_1 - 2, n_2 - 2), (q + 1, n_1, n_2 - 2), (q + 1, n_1 - 2, n_2)\}\).

We will prove that \(Z^{(n)}_{j(n),\kappa} \in G^n\) with \(\kappa = (q + 1, n_1, n_2)\).

To generate \(n = (q + 1, n_1, n_2)\), firstly we compute \((Z^m \cdot \nabla)Y^k + (Y^k \cdot \nabla)Z^m\) with
\[
k = (q, n_1, n_2), \quad m = (1, 0, 0),
\]
\[
w^k = (0, L_2 n_2, -L_3 n_1), \quad w^m = (0, 0, L_3),
\]
which leads to \(\Pi Z^{(q+1,n_1,n_2)}_{z^{\alpha}} \in G^q\), with
\[
z_{\alpha} = \frac{\pi}{2} \begin{pmatrix} 0 \\ L_2 n_2^2 \\ L_3 n_1 n_2 \end{pmatrix}.
\]
Secondly we compute \((Z^m \cdot \nabla) Y^k + (Y^k \cdot \nabla) Z^m\) with
\[
k = (q, n_1, n_2), \quad m = (1, 0, 0),
w^k = (L_1 n_2, 0, -L_3 q), \quad w^m = (0, 0, L_3),
\]
which leads us to \(\Pi Z_{z_1}^{q+1,n_1,n_2} \in G^q\) where
\[
z_{\gamma_1} = \frac{\pi}{2} \left( \begin{array}{c} L_1 n_2^2 \\ 0 \\ L_3 n_2 (q - 1) \end{array} \right).
\]
Thirdly we compute \((Z^k \cdot \nabla) Y^m + (Y^m \cdot \nabla) Z^k\) with
\[
k = (q, n_1 - 1, n_2), \quad m = (1, 1, 0),
w^k = (L_1 (n_1 - 1), -L_2 q, 0), \quad w^m = (L_1, -L_2, 0),
\]
which gives us \(\Pi Z_{z_3}^{q+1,n_1,n_2} \in G^q\) with
\[
z_{\delta_1} = \frac{\pi}{4} \left( \begin{array}{c} L_1 (q - n_1 + 1)(n_1 - 2) \\ L_2 (q - n_1 + 1)(2 - q) \\ 0 \end{array} \right).
\]
Now \(\det(n z_{\alpha_1} z_{\gamma_1}) = \det(n z_{\alpha_1} z_{\delta_1}) = 0\) if, and only if,
\[
(1) n_2^2 (L_2 L_3 q^2 - L_2 L_3 + L_1 L_3 n_2^2 - L_1 L_2 n_2^2) = 0 \tag{35a}
(2) (q - n_1 + 1) n_2 (L_2 L_3 n_2 (q + 1)(q - 2) + (n_1 - 2)(L_1 L_3 n_1^2 - L_1 L_2 n_2^2)) = 0 \tag{35b}
\]
Since \(0 < 2 \leq n_2, n_1 \leq q\), from (35a) we have \(L_1 L_3 n_2^2 - L_1 L_2 n_2^2 = L_2 L_3 (1 - q^2)\). Then, after substitution into (35b) and since \(n_2 (q - n_1 + 1) \geq n_2 > 0\), we arrive to the contradiction
\[
0 = L_2 L_3 (q + 1)(2q - n_1 - 2) > L_2 L_3 (q + 1)(q - 2) > 0, \quad \text{because} \quad q \geq 4 \quad \text{and} \quad L_2, L_3 > 0.
\]
Therefore by Lemma 3.4 it follows that \(Z_{z_3}^{q+1,n_1,n_2} \in G^q\).

By induction, using (27), (28), (31), (33), and the induction hypothesis (IH-Cqn1n2-eq.34), we obtain
\[
Z^{j(n),n} \in G^q \quad \text{for all} \quad n \in R_1^{q+1}, \tag{36a}
\]
and a similar argument gives us
\[
Z^{j(n),n} \in G^q \quad \text{for all} \quad n \in R_2^{q+1}. \tag{36b}
\]

3.4.3. Proof of Lemma 3.7. To generate \(n = (q + 1, 0, 0)\), we choose
\[
k = (q, 0, 1), \quad m = (1, 0, 1),
w^k = (L_1, 0, -L_3 q), \quad w^m = (L_1, 0, -L_3),
\]
which gives us
\[
(\gamma^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k = Z_{z_1}^{q+1,0,0} + Z_{z_1}^{q+1,0,0} + Z_{z_1}^{q-1,0,0} + Z_{z_1}^{q-1,0,0}.
\]
From (IH.C-eq.18) and (36), we can conclude that \(\Pi Z_{z_1}^{q+1,0,0} \in G^q\) where
\[
z_{\alpha_1} = -\frac{\pi}{4} \left( \begin{array}{c} 0 \\ 0 \\ L_3 (q - 1)^2 \end{array} \right).
\]
Now, since \(L_3 (q - 1)^2 \neq 0\) we have that \(Z^{1,(q+1,0,0)} = \zeta Z_{z_1}^{(q+1,0,0)}\). Therefore
\[
Z^{1,(q+1,0,0)} \in G^q, \tag{37a}
\]
and a similar argument gives us
\[ Z^{1,(0,q+1,0)} \in \mathcal{G}^q. \] (37b)

3.4.4. *Proof of Lemma 3.8*. Due to two different types of boundary conditions, we divide the proof into two steps

- **Step 3.8.1**: Generating \( Z^j_{(n)} \) with \( n \in L_{q+1,2}^2 \cup L_{q+1,1}^3 \).
- **Step 3.8.2**: Generating \( Z^j_{(n)} \) with \( n \in L_{q+1,1}^1 \).

**Step 3.8.1**: Generating \( Z^j_{(n)} \) with \( n \in L_{q+1,2}^2 \cup L_{q+1,1}^3 \).

To generate \( n = (l, q + 1, q + 1) \), with \( 1 \leq l \leq q \). We start computing \((Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k\) with
\[
k = (l, q - 1, q), \quad m = (0, 2, 1),
w^k = (0, L_{2q}, L_3(1 - q)), \quad w^m = (0, L_2, -2L_3),
\]
and obtain that \( \Pi Z^j_{z_{\alpha_1}} \in \mathcal{G}^q \) with
\[
z_{\alpha_1} = -\frac{\pi}{4} \begin{pmatrix} 0 \\ \frac{L_2(q^2 - 1)}{L_3(q + 1)(q - 3)} \end{pmatrix}.
\]

Next we compute \((Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k\) with
\[
k = (l, q, q), \quad m = (0, 1, 1),
w^k = (L_1q, -L_2l, 0), \quad w^m = (0, L_2, -L_3),
\]
and obtain \( \Pi Z^j_{z_{\gamma_1}} \in \mathcal{G}^q \) with
\[
z_{\gamma_1} = -\frac{\pi}{4} \begin{pmatrix} 0 \\ L_2l \end{pmatrix}.
\]

Since
\[
\frac{16}{\pi^2} \det(n \cdot z_{\alpha_1}, z_{\gamma_1}) = \det \begin{pmatrix} l & 0 \\ q + 1 & L_2(q^2 - 1) \\ q + 1 & L_3(q + 1)(q - 3) \end{pmatrix} = 2L_2l^2(q + 1) > 0,
\]
from Lemma 3.3 we obtain that
\[ Z^{(1,2),(l,q+1,q+1)} \in \mathcal{G}^q, \quad \text{for} \quad 1 \leq l \leq q, \] (38a)

and a similar argument gives us
\[ Z^{(1,2),(q+1,l,q+1)} \in \mathcal{G}^q, \quad \text{for} \quad 1 \leq l \leq q, \] (38b)

*The case* \( n = (0, q + 1, q + 1) \). We can use again [15, Theorem 4.1] to conclude that
\[ Z^{1,(0,q+1,0)} \in \mathcal{G}^q, \] (39a)

and a similarly
\[ Z^{1,(q+1,0,q+1)} \in \mathcal{G}^q. \] (39b)
To generate \( n = (q + 1, q + 1, l) \) with \( 1 \leq l \leq q \), we first compute \((Y^m \cdot \nabla) Z^k + (Z^k \cdot \nabla) Y^m\) with
\[
k = (q, q, l), \quad m = (1, 1, 0),
\]
\[
w^k = (L_1 l, 0, -L_3 q), \quad w^m = (L_1, -L_2, 0),
\]
which gives us \( \Pi Z_{z_{a1}}^{(q+1,q+1,l)} \in \mathcal{G}^q \), with
\[
z_{a1} = \frac{\pi}{4} \left( \begin{array}{c} L_1 l \\ -L_2 l \\ 0 \end{array} \right).
\]
Next, we choose compute \((Y^m \cdot \nabla) Z^k + (Z^k \cdot \nabla) Y^m\) with
\[
k = (q, q - 1, l), \quad m = (1, 2, 0),
\]
\[
w^k = (L_1(q - 1), -L_2 q, 0), \quad w^m = (2L_1, -L_2, 0),
\]
which gives us \( \Pi Z_{z_{a1}}^{(q+1,q+1,l)} \in \mathcal{G}^q \), with
\[
z_{a1} = \frac{\pi}{4} \left( \begin{array}{c} L_1(q + 1)(q - 3) \\ -L_2(q^2 - 1) \\ 0 \end{array} \right).
\]
From
\[
\frac{16}{\pi^2} \left[ n z_{a1} z_{\gamma 1} \right] = \det \left( \begin{array}{ccc} q + 1 & L_1 l & L_1(q + 1)(q - 3) \\ q + 1 & -L_2 l & -L_2(q^2 - 1) \\ l & 0 & 0 \end{array} \right) = \frac{32}{\pi^2} L_1 L_2 l^2(q + 1) > 0,
\]
and Lemma [3.3] we obtain that
\[
Z^{(1,2),(q+1,q+1,l)} \in \mathcal{G}^q, \quad \text{for} \quad 1 \leq l \leq q. \tag{40}
\]
To generate \( n = (q + 1, q + 1, 0) \) we choose
\[
k = (q, q - 1, 1), \quad m = (1, 2, 1),
\]
\[
w^k = (L_1 0, -L_3 q), \quad w^m = (L_1 0, -L_3),
\]
which gives us
\[
(Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k = Z_{z_{a1}}^{(q+1,q+1,0)} + Z_{z_{a2}}^{(q+1,q+1,2)} + \sum_{i=3}^{8} Z_{z_{a1}}^{k_i}
\]
with \( k_i \in \{(q + 1, q - 3, 0), (q + 1, q - 3, 2), (q - 1, q + 1, 0), (q - 1, q + 1, 2), (q - 1, q - 3, 0), (q - 1, q - 3, 2)\} \). Recalling that \( q \geq 4 \), from [H.C-eq.18] and [36] it follows that \( \sum_{i=3}^{8} \Pi Z_{z_{a1}}^{k_i} \in \mathcal{G}^q \) and, from [40] we have that \( \Pi Z_{z_{a2}}^{(q+1,q+1,2)} \in \mathcal{G}^q \). Therefore we obtain \( \Pi Z_{z_{a1}}^{(q+1,q+1,0)} \in \mathcal{G}^q \), and we can find
\[
z_{a1} = \frac{\pi}{8} \left( \begin{array}{c} -L_1(q + 1) \\ 0 \\ 0 \end{array} \right).
\]
Observe that \( Z_{z_{a1}}^{1,(q+1,q+1,0)} = -\frac{3}{8}(q^2 - 1)L_3 \left( \begin{array}{c} 0 \\ 0 \\ \cos\frac{(q+1)\pi x_1}{L_1} \cos\frac{(q+1)\pi x_2}{L_2} \end{array} \right) \), that is, for a suitable constant \( \zeta \neq 0 \), we have \( Z^{1,(q+1,q+1,0)} = \zeta Z_{z_{a1}}^{1,(q+1,q+1,0)} \). In particular \( \Pi Z_{z_{a1}}^{1,(q+1,q+1,0)} = \zeta Z_{z_{a1}}^{1,(q+1,q+1,0)} \).
and it follows
\[ Z^{1,(q+1,q+1,0)}_{\alpha_1} \in \mathcal{G}^q. \] (41)

To sum up, from (38), (39), (40), and (41), it follows that
\[ Z^{j(n),n} \in \mathcal{G}^q, \quad \text{for all} \quad n \in \mathcal{L}_{1,2}^{q+1} \cup \mathcal{L}_{2,3}^{q+1} \cup \mathcal{L}_{3,1}^{q+1}. \] (42)

3.4.5. Proof of Lemma 3.9. Firstly, we compute
\[ (Y^k \cdot \nabla) Z^m + (Z^m \cdot \nabla) Y^k \]
\[ k = (q, q-1, q), \quad m = (1, 2, 1), \]
\[ w^k = (L_1, 0, -L_3), \quad w^m = (2L_3, -L_2, 0), \]
which gives us \( \Pi Z^{(q+1,q+1,q+1)}_{\alpha_1} \in \mathcal{G}^q \) where
\[ z_{\alpha_1} = \frac{\pi}{8} \left( \begin{array}{c} 2L_1(q+1) \\ -L_2(q+1) \\ 0 \end{array} \right). \]

Secondly, we compute
\[ (Y^m \cdot \nabla) Z^k + (Z^k \cdot \nabla) Y^m \]
\[ k = (q, q-1, q), \quad m = (1, 2, 1), \]
\[ w^k = (L_1, 0, -L_3), \quad w^m = (2L_3, -L_2, 0), \]
which gives us \( \Pi Z^{(q+1,q+1,q+1)}_{\gamma_1} \in \mathcal{G}^q \), with
\[ z_{\gamma_1} = \frac{\pi}{8} \left( \begin{array}{c} -L_1(q+1)(q-3) \\ L_2(q^2-1) \\ 0 \end{array} \right). \]

Since
\[ \frac{64}{\pi^2(q+1)^3} \det(n \ z_{\alpha_1} \ z_{\gamma_1}) = \det \left( \begin{array}{ccc} 1 & 2L_1 & -L_1(q-3) \\ 1 & -L_2 & L_2(q-1) \\ 1 & 0 & 0 \end{array} \right) = L_1L_2(q+1) > 0, \]
by Lemma 3.3, we find
\[ Z^{(1,2),(q+1,q+1,q+1)}_{\alpha_1, \gamma_1} \in \mathcal{G}^q. \] (43)

4. Final remarks

Following the approximate controllability by degenerate low modes forcing proven in [17,24], we present an explicit \( (L, D(A)) \)-saturating set in a general 3D Cylinder. This case is as an extended result in the work of 2D Cylinder (see [15]). However we just get the control \( \eta \in L^\infty((0, T), \mathcal{G}^1) \) instead of \( L^\infty((0, T), \mathcal{G}^0) \) in 2D Cylinder case. The reason is that we do not have the equality \( B(Y^k, Y^k) = 0 \) for all \( k \) as in 2D Cylinder case (see more details in [17, Theorem 3.2]).

We underline that the presented saturating set is (by definition) independent of the viscosity coefficient \( \nu \). That is, approximate controllability holds by means of controls taking values in \( \mathcal{G}^1 = \text{span} \mathcal{C} + \text{span} \mathcal{B}(\mathcal{C}, \text{span} \mathcal{C}) = \text{span} (\mathcal{C} \cup \mathcal{B}(\mathcal{C}, \mathcal{C})) \), for any \( \nu > 0 \). It is plausible that a \( (L, D(A)) \)-saturating set with less elements does exists. One of them is introduced in next corollary.
Corollary 4.1. The set of eigenfunctions
\[ C := \{ Y^{(n),n}_j | n \in \mathbb{N}^3, \#_0(n) \leq 1, n_i \leq 3, j(n) \in \{1, 2 - \#_0(n)\} \} \]
\[ \cup \{ Z^{(n),n}_j | n \in \mathbb{N}^3, \#_0(n) = 2, n_3 = 0 \} \]
\[ \cup \{ Z^{(n),n}_j | n \in \mathbb{N}^3, \#_0(n) \leq 1, n_i \leq 4, j(n) \in \{1, 2 - \#_0(n)\} \} \]
is saturating.

Proof. Its proof is obtained obviously form Theorems 3.2 and 2.2. □

As mentioned in the beginning of this work, it is not our goal to find a saturating set with minimal number of elements.

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