SUBSPACE-HYPERCYCLIC CONDITIONAL TYPE OPERATORS
ON Lp-SPACES

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ABSTRACT. A conditional weighted composition operator $T_u : L^p(\Sigma) \to L^p(A)$
($1 \leq p < \infty$), is defined by $T_u(f) := E_A(uf \circ \varphi)$, where $\varphi : X \to X$ is a
measurable transformation, $u$ is a weight function on $X$ and $E_A$ is the condi-
tional expectation operator with respect to $A$. In this paper, we study the
subspace-hypercyclicity of $T_u$ with respect to $L^p(A)$. First, we show that if
$\varphi$ is a periodic nonsingular transformation, then $T_u$ is not $L^p(A)$-hypercyclic.
The necessary conditions for the subspace-hypercyclicity of $T_u$ are obtained
when $\varphi$ is non-singular and finitely non-mixing. For the sufficient conditions,
the normality of $\varphi$ is required. The subspace-weakly mixing and subspace-
topologically mixing concepts are also studied for $T_u$. Finally, we give an
example which is subspace-hypercyclic while is not hypercyclic.

1. Introduction and Preliminaries

Suppose that $T$ is a bounded linear operator on a topological vector space $X$. If
there is a vector $x \in X$ such that the orbit $\text{orb}(T, x) := \{T^nx : n = 0, 1, 2, ...\}$ is
dense in $X$, then $T$ will be hypercyclic and $x$ is called a hypercyclic vector. Here,
$T^n$ stands for the $n$-th iterate of $T$ and $T^0$ is the identity map $I$. Let $M$ be a
closed and non-trivial subspace of $X$. An operator $T$ is subspace-hypercyclic with
respect to $M$ ($M$-hypercyclic), if there is a a vector $x \in X$ such that $\text{orb}(T, x) \cap M$
is dense in $M$. Also an operator $T$ is subspace-transitive with respect to $M$, if for
any non-empty open set $U, V \subseteq M$, there exists an $n \in \mathbb{N}$ such that $T^{-n}(U) \cap V$
contains an open non-empty subset of $M$. An operator $T$ is subspace-topologically
mixing with respect to $M$, if for any non-empty open set $U, V \subseteq M$, there exists
an $N \in \mathbb{N}$ such that $T^{-n}(U) \cap V$ contains an open non-empty subset of $M$ for each
$n \geq N$. It is called subspace-weakly mixing if $T \oplus T$ is subspace-hypercyclic with
respect to $M \oplus M$.

The study of subspace-hypercyclic linear operators was initiated by B. F. Madore
and R. A. Martínez-Avendaño [24]. They found out that subspace-hypercyclicity
like as hypercyclicity, can occur only on infinite-dimensional spaces and even sub-
spaces. Also, they proved an interesting Kitai’s type subspace-hypercyclicity crite-
ron on a topological vector space as follows.

Assume that there exist $D_1$ and $D_2$, dense subsets of $M$, and an increasing sequence
of positive integers $(n_k)$ such that

- $T^{n_k}x \to 0$ for all $x \in D_1$;

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topologically mixing, Measurable transformation, Normal, Radon-Nikodym derivative, Conditional expectation, aperiodic.
Let \( 1 \leq \sigma \) and denote by \( \chi_{\sigma M} \). Note that always \( h \) class of all \( A \in \mathcal{M} \) will be used frequently. Then \( T \) will be used frequently. 

For the dynamics of linear operators the survey articles [29], [8], [27], [31], [1], [24] and the books [6], [16] are useful.

Let \((X, \Sigma, \mu)\) be a complete \( \sigma \)-finite measure space and \( A \) is a \( \sigma \)-finite subalgebra of \( \Sigma \). For each \( 1 \leq p < \infty \), the Banach space \( L^p(X, A, \mu|_A) \) is denoted by \( L^p(A) \) simply. All comparisons between two functions or two sets are to be interpreted as holding up to a \( \mu \)-null set. The support of any \( \Sigma \)-measurable function \( f \) is defined by \( \sigma(f) = \{ x \in X : f(x) \neq 0 \} \). The characteristic function of any set \( A \) and the class of all \( A \)-measurable and simple functions on \( X \) with finite supports will be denoted by \( \chi_A \) and \( \mathcal{S}^A(X) \), respectively.

A \( \Sigma \)-measurable transformation \( \varphi : X \to X \) is called non-singular whenever \( \mu \circ \varphi^{-1} \) is absolutely continuous with respect to \( \mu \), which is symbolically shown by \( \mu \circ \varphi^{-1} \ll \mu \). In this case, Radon-Nikodym property is denoted by \( h := \frac{d\mu \circ \varphi^{-1}}{d\mu} \).

A \( \Sigma \)-measurable transformation \( \varphi : X \to X \) is called periodic if \( \varphi^n = I \) for some \( n \in \mathbb{N} \). It is called aperiodic, if it is not periodic. Also, if for each subset \( F \in \Sigma \) with finite measure, there exists an \( N \in \mathbb{N} \) such that \( F \cap \varphi^n(F) = \emptyset \) for every \( n > N \), then \( \varphi \) is called finitely non-mixing.

Set \( \Sigma_\infty = \bigcap_{n=1}^{\infty} \varphi^{-n}(\Sigma) \) and suppose that \( h \) is \( \Sigma_\infty \)-measurable. The assumption \( \mu \circ \varphi^{-1} \ll \mu \) implies that \( \mu \circ \varphi^{-n} \ll \mu \) for all \( n \in \mathbb{N} \) and then

\[
\begin{align*}
h_n & := \frac{d\mu \circ \varphi^{-n}}{d\mu} = \frac{d\mu \circ \varphi^{-n}}{d\mu} \cdots \frac{d\mu \circ \varphi^{-1}}{d\mu} \\
& = (h \circ \varphi^{-(n-1)}) \cdots (h \circ \varphi^0) = \prod_{i=0}^{n-1} h \circ \varphi^{-i}.
\end{align*}
\]

Note that always \( h \circ \varphi > 0 \) and \( h_n = h^n \) whenever \( h \circ \varphi = h \). When it is restricted to a \( \sigma \)-subalgebra \( A \), is denoted by \( h_n^A = \frac{d\mu \circ \varphi^{-n}}{d\mu|_A} \).

The change of variable formula

\[
\int \varphi^{-n}(A) f \circ \varphi^n d\mu = \int_A h_n f d\mu, \quad A \in \Sigma, \; f \in L^1(\Sigma),
\]

will be used frequently.

When \( \varphi(\Sigma) \subseteq \Sigma \) and \( \mu \circ \varphi \ll \mu \), then a measure \( \mu \) is called normal with respect to \( \varphi \) and in this case \( h^2 = \frac{d\mu \circ \varphi}{d\mu} \) is defined. Now, consider that

\[
h^2 = \left( \frac{d\mu}{d\mu \circ \varphi} \right)^{-1} = \left( \frac{d\mu \circ \varphi^{-1}}{d\mu} \circ \varphi \right)^{-1} = \frac{1}{h \circ \varphi}
\]

and

\[
h_n^2 := \frac{d\mu \circ \varphi^n}{d\mu} = (h^2 \circ \varphi^{(n-1)}) \cdots (h^2 \circ \varphi^0) = \prod_{i=0}^{n-1} h^2 \circ \varphi^i = \prod_{i=1}^{n} (h \circ \varphi^i)^{-1},
\]

\( h_n^2 \circ \varphi > 0, \; h_{n+1}^2 = h^2 h_n^2 \circ \varphi. \)

Let \( 1 \leq p \leq \infty \). For any non-negative \( \Sigma \)-measurable functions \( f \) or for any \( f \in \mathcal{S}^A(X) \)
For the fundamental properties of the conditional type operators, the reader is referred to different settings has been studied in [29, 11, 7, 4, 5, 3, 1, 8, 31].

Translations, conditional weighted translations and weighted composition operators have examples given.

The hypercyclicity of the well-known operators such as weighted shifts, weighted translations, conditional weighted translations and weighted composition operators in different settings has been studied in [29, 25, 18, 22].

A weighted composition operator $uC_\varphi : L^p(\Sigma) \to L^p(\Sigma)$ defined by $f \mapsto uf \circ \varphi$ is bounded if and only if $J \in L^\infty(\Sigma)$, where $J := hE^A(|u|^p) \circ \varphi^{-1}$, and in this case $\|uC_\varphi\|^p = \|J\|_\infty$ (see [19, 30, 20]).

Subspace-hypercyclicity of the well-known operators such as weighted shifts, weighted translations, conditional weighted translations and weighted composition operators in different settings has been studied in [29, 11, 7, 4, 5, 3, 1, 8, 31].

Separability and infinite-dimension are two essential objects for the underlying space to admit a hypercyclic vector [6, 16]. To that end, it is important to know that $L^p(X, \Sigma, \mu)$ is separable if and only if $(X, \Sigma, \mu)$ is separable, i.e., there exists a countable $\sigma$-subalgebra $\mathcal{F} \subseteq \Sigma$ such that for each $\epsilon > 0$ and $A \in \Sigma$ we have $\mu(\Delta B) < \epsilon$ for some $B \in \mathcal{F}$. For more details consult [24].

In this paper, we will survey the dynamics of a conditional weighted composition operator $T_u = E^A(uf \circ \varphi)$ on $L^p(\Sigma)$ spaces. First, we prove that $T_u$ cannot be $L^p(\mathcal{A})$-hypercyclic if $\varphi$ is a periodic non-singular transformation. In addition, the necessary conditions for the subspace-hypercyclicity of $T_u$ are then given provided that $\varphi$ is non-singular and finitely non-mixing. For the sufficient conditions, we also require that $\varphi$ is normal. The subspace-weakly mixing and subspace-topologically mixing concepts are also studied for $T_u$. At the end, about what we argued, an examples is given.

2. Subspace-hypercyclicity of $T_u$ On $L^p(\Sigma)$

In this section, the $L^p(\mathcal{A})$-hypercyclicity of a conditional weighted composition operator $T_u$ is studied. When $\varphi$ is periodic transformation, it is seen that $T_u$ is not $L^p(\mathcal{A})$-hypercyclic. But, when it is aperiodic, by Kitai’s subspace-hypercyclicity
Theorem 2.1. Let \( \varphi \) be a periodic non-singular transformation and \( \varphi^{-1}A \subseteq A \). Then a conditional weighted composition operator \( T_u : L^p(\Sigma) \to L^p(A) \) is not subspace-hypercyclic with respect to \( L^p(A) \), for each \( 1 \leq p < \infty \).

Proof. Suppose that there exists an \( m \in \mathbb{N} \) such that \( \varphi^m = I \). Since \( \varphi^{-1}A \subseteq A \), the orbit of \( T_u \) at each \( f \in L^p(\Sigma) \) is written as follows:

\[
\text{orb}(T_u, f) = \{ f, T_u f, \ldots, T_u^{m-1} f \} \cup \{ T_u^{m+1} f, T_u^{m+2} f, \ldots, T_u^{2m} f \} \cup \ldots
\]

\[
\cup \{ T_u^{km+1} f, T_u^{km+2} f, \ldots, T_u^{(k+1)m} f \} \cup \ldots
\]

\[
= \{ f, E^A(uf \circ \varphi), E^A(uf \circ \varphi) \circ \varphi, \ldots, \prod_{i=0}^{m-2} E^A(uf \circ \varphi) \circ \varphi \}
\cup \{ \prod_{i=0}^{m-1} E^A(u) \circ \varphi \cdot \prod_{i=0}^{m-2} E^A(u) \circ \varphi \cdot \prod_{i=0}^{m-1} E^A(u) \circ \varphi \}
\cup \{ (\prod_{i=0}^{m-1} E^A(u) \circ \varphi) \cdot (\prod_{i=0}^{m-2} E^A(u) \circ \varphi) \cdot (\prod_{i=0}^{m-1} E^A(u) \circ \varphi) \}
\cup \ldots
\]

Now we consider that \( \| \prod_{i=0}^{m-1} E^A(u) \circ \varphi \|_\infty \leq 1 \). Since \( \| T_u \| \leq \| J \|^{1/p}_{\infty}, \| T_u^n \| \leq \| T_u \|^n \leq \| J \|^n_{1/p}, \) and for each \( n \in \mathbb{N} \) we have

\[
\| T_u^n f \|_p \leq \max\{ \| f \|_p, \| E^A(uf \circ \varphi) \|_p, \| E^A(u) E^A(uf \circ \varphi) \|_p, \ldots, \}
\| \prod_{i=0}^{m-2} E^A(u) \circ \varphi \cdot \prod_{i=0}^{m-1} E^A(u) \circ \varphi \cdot \prod_{i=0}^{m-2} E^A(u) \circ \varphi \}
\leq \| f \|_p \max\{ 1, \| J \|^{1/\infty}_{\infty}, \| J \|^{2/\infty}_{\infty}, \ldots, \| J \|^{m-1/\infty}_{\infty} \}.
\]

Therefore, \( \text{orb}(T_u, f) \) is a bounded subset and cannot be dense in \( L^p(A) \).

In the second case \( \| \prod_{i=0}^{m-1} E^A(u) \circ \varphi \|_\infty > 1 \), assume that \( T_u \) is subspace-hypercyclic with respect to \( L^p(A) \). Then there exists a subset \( F \in A \) with \( 0 < \mu(F) < \infty \) for each \( \varepsilon > 0 \), such that \( \| \prod_{i=0}^{m-1} E^A(u) \circ \varphi \| > 1 \). Then there is a subspace-hypercyclic vector \( f \in L^p(A) \) and \( n \in \mathbb{N} \) such that

\[
\| f - 2 \chi_F \|_p < \varepsilon \quad \text{and} \quad \| (T_u^{m+1})^n f \|_p < \varepsilon.
\]
We set \( S = \{ t \in F : |f(t)| < 1 \} \) and note that \( \chi_S \leq \chi_S |f - 2| \leq \chi_S |f - 2\chi_F| \). Thus, \( \mu(S) < \varepsilon^p \). On the other hand,

\[
\varepsilon^p > \| (T_u^n)^m f \|_p^p = \int_X \left| \prod_{i=0}^{m-1} E^A(u) \circ \varphi^i f \circ \varphi^m \right| d\mu
\]

\[
= \int_X \left| \prod_{i=0}^{m-1} E^A(u) \circ \varphi^i |n_p| f \right| d\mu \geq \int_{F-S} \left| f \right| d\mu \geq \mu(\chi_{F-S}).
\]

Therefore, \( \mu(F) = \mu(S) + \mu(F-S) < 2\varepsilon^p \), which is a contradiction. \( \square \)

**Remark 2.2.** If \( \varphi \) is a periodic non-singular transformation, \( \varphi^{-1}A \subseteq A \) and \( u = 1 \), then a conditional composition operator \( T_u f = E^A(f \circ \varphi) \) is not subspace-hypercyclic with respect to \( L^p(A) \) either. Since its orbit at \( f \in L^p(\Sigma) \) i.e., \( \text{orb}(T_u, f) = \{ f, E^A(f \circ \varphi), E^A(f \circ \varphi) \circ \varphi, E^A(f \circ \varphi) \circ \varphi^2, \ldots, E^A(f \circ \varphi) \circ \varphi^m \} \) is a bounded subset. Indeed,

\[
\| T_u^n f \|_p \leq \| f \|_p \max \{ 1, \| h \|^\frac{2}{3}, \| h \|_{3}, \cdots, \| h \|^\frac{m-1}{2} \}.
\]

**Corollary 2.3.** Suppose that \( A = \varphi^{-1} \Sigma \) and \( \varphi \) is a periodic non-singular transformation. Then

\[
\text{orb}(T_u, f) = \left\{ f, E^{\varphi^{-1} \Sigma}(u) f \circ \varphi, E^{\varphi^{-1} \Sigma}(u) E^{\varphi^{-1} \Sigma}(u) \circ \varphi \circ \varphi^2, \ldots, \prod_{i=0}^{m-1} E^{\varphi^{-1} \Sigma}(u) \circ \varphi^i f \right\}
\]

and hence \( T_u \) is not subspace-hypercyclic with respect to \( L^p(\varphi^{-1} \Sigma) \), for each \( 1 \leq p < \infty \).

**Theorem 2.4.** Let \( \varphi : X \rightarrow X \) be a non-singular and finitely non-mixing transformation and \( \varphi^{-1}A \subseteq A \). Suppose that \( T_u : L^p(\Sigma) \rightarrow L^p(A) \) is subspace-hypercyclic with respect to \( L^p(A) \). Then for each subset \( F \in A \) with \( 0 < \mu(F) < \infty \), there exists a sequence of \( A \)-measurable sets \( \{ V_n \} \subseteq F \) such that \( \mu(V_n) \rightarrow \mu(F) \) as \( k \rightarrow \infty \), and there is a sequence of integers \( (n_k) \) such that

\[
\lim_{k \rightarrow \infty} \| \left( \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi^i \right)^{-1} \|_{\infty} = 0
\]

and

\[
\lim_{k \rightarrow \infty} \| \left( \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi^i \right) \circ \varphi^{-n_k} \|_{\infty} = 0.
\]

**Proof.** Let \( F \in A \) be an arbitrary set with \( 0 < \mu(F) < \infty \) and let \( \varepsilon > 0 \) be an arbitrary. A transformation \( \varphi \) is finitely non-mixing and hence, there is an \( N \in \mathbb{N} \) such that \( F \cap \varphi^n(F) = \emptyset \) for each \( n > N \). Choose \( \varepsilon_1 \) such that \( 0 < \varepsilon_1 < \frac{\varepsilon}{1 + \varepsilon} \). Since the set of all subspace-hypercyclic vectors for \( T_u \), is dense in \( L^p(A) \), there exist a subspace-hypercyclic vector \( f \in L^p(A) \) and \( m \in \mathbb{N} \) with \( m > N \) such that

\[
\| f - \chi_F \|_p < \varepsilon_1^2 \quad \text{and} \quad \| T_u^m f - \chi_F \|_p < \varepsilon_1^2.
\]
Put \( P_{\varepsilon_1} = \{ t \in F : |f(t) - 1| \geq \varepsilon_1 \} \) and \( R_{\varepsilon_1} = \{ t \in X - F : |f(t)| \geq \varepsilon_1 \} \). Then we have

\[
\varepsilon_1^{2p} > \| f - \chi_F \|^p_p = \int_X |f - \chi_F|^p d\mu \\
\geq \int_{P_{\varepsilon_1}} |f(x) - 1|^p d\mu(x) + \int_{R_{\varepsilon_1}} |f(x)|^p d\mu(x) \\
\geq \varepsilon_1^p (\mu(P_{\varepsilon_1}) + \mu(R_{\varepsilon_1})).
\]

Then, \( \max \{ \mu(P_{\varepsilon_1}), \mu(R_{\varepsilon_1}) \} < \varepsilon_1^p \). Set \( S_{m,\varepsilon_1} = \{ t \in F : \prod_{i=0}^{m-1} E^A(u) \circ \varphi^i f \circ \varphi^{-m}(t) \geq \varepsilon_1 \} \) and now consider the following relationships:

\[
\varepsilon_1^{2p} > \| T^{m}_u f - \chi_F \|^p_p \\
= \int_X \left| \prod_{i=0}^{m-2} E^A(u) \circ \varphi^i E^A(uf \circ \varphi) \circ \varphi^{-m-1} - \chi_F \right|^p d\mu \\
\geq \int_{S_{m,\varepsilon_1}} \left| \prod_{i=0}^{m-2} E^A(u) \circ \varphi^i E^A(uf \circ \varphi) \circ \varphi^{-m-1}(t) - 1 \right|^p d\mu(t) \\
\geq \int_{S_{m,\varepsilon_1}} \left| \prod_{i=0}^{m-1} E^A(u) \circ \varphi^i f \circ \varphi^{-m}(t) - 1 \right|^p d\mu(t) \\
\geq \varepsilon_1^p \mu(S_{m,\varepsilon_1})
\]

to deduce that \( \mu(S_{m,\varepsilon_1}) < \varepsilon_1^p \). But for an arbitrary \( t \in F \), it is readily seen that \( \varphi^{-m}(t) \notin F \) because of \( F \cap \varphi^{-m}(F) = \emptyset \). Hence, for each \( t \in F - (S_{m,\varepsilon_1} \cup \varphi^{-m}(R_{\varepsilon_1})) \), we have

\[
\left| \prod_{i=0}^{m-1} E^A(u) \circ \varphi^{-m-1}(t) \right| < \frac{|f \circ \varphi^{-m}(t)|}{1 - \varepsilon_1} < \frac{\varepsilon_1}{1 - \varepsilon_1} < \varepsilon.
\]

Now, let \( U_{m,\varepsilon_1} = \varphi^{-m}(\{ t \in F : \sqrt{\int_{h^A_m(t)} |E^A(u) \circ \varphi^{-m}(t) f(t)|} \geq \varepsilon_1 \}) \). Here, we remind that \( \prod_{i=0}^{m-1} E^A(u) \circ \varphi^i = \prod_{i=1}^{m} E^A(u) \circ \varphi^i \) on \( \sigma(h^A_m) \). Use the change of variable formula to obtain that

\[
\varepsilon_1^{2p} > \| T^{m}_u f - \chi_F \|^p_p \\
= \int_X \left| \prod_{i=0}^{m-1} E^A(u) \circ \varphi^i f \circ \varphi^{-m} - \chi_F \right|^p d\mu \\
\geq \int_X \left| E^{\varphi^{-m}(A)}(\prod_{i=0}^{m-1} E^A(u) \circ \varphi^i) f \circ \varphi^{-m} - E^{\varphi^{-m}(A)}(\chi_F) \right|^p d\mu \\
\geq \int_{U_{m,\varepsilon_1}} \left| E^{\varphi^{-m}(A)}(\prod_{i=0}^{m-1} E^A(u) \circ \varphi^i) f \circ \varphi^{-m} \right|^p d\mu \\
\geq \int_{U_{m,\varepsilon_1}} \left| E^{\varphi^{-m}(A)}(\prod_{i=0}^{m-1} E^A(u) \circ \varphi^i) f \circ \varphi^{-m} \right|^p h^A_m d\mu \\
\geq \varepsilon_1^p \mu(U_{m,\varepsilon_1}),
\]

which implies in turn that \( \mu(U_{m,\varepsilon_1}) < \varepsilon_1^p \). That \( E^{\varphi^{-m}(A)}(\chi_F) = 0 \) is concluded of the fact that \( F \cap \varphi^{-m}(F) = \emptyset \). Note that for each \( t \in F - (\varphi^{-m}(U_{m,\varepsilon_1}) \cup P_{\varepsilon_1}) \),
we have
\[ \sqrt[p]{h_m(t)} \left| E^{-m}(A) \left( \prod_{i=0}^{m-1} E^A(u) \circ \varphi^i \right) \circ \varphi^{-m}(t) f(t) \right| < \frac{\varepsilon_1}{1 - \varepsilon_1} < \varepsilon. \]

Finally, put \( V_{m, \varepsilon_1} := F - \left( P_{\varepsilon_1} \cup \varphi^{-m}(R_{m, \varepsilon_1}) \cup S_{m, \varepsilon_1} \cup \varphi^m(U_{m, \varepsilon_1}) \right) \). Then, clearly \( \mu(F - V_{m, \varepsilon_1}) < 4 \varepsilon_1^p, \| \prod_{i=0}^{m-1} E^A(u) \circ \varphi^i \| \infty < \varepsilon \) and
\[ \| \sqrt[p]{h_m} \left( \prod_{i=0}^{m-1} E^A(u) \circ \varphi^i \right) \circ \varphi^{-m} \| V_{m, \varepsilon_1} \| \infty < \varepsilon. \]

By induction, for each \( k \in \mathbb{N} \) we get a measurable subset \( V_k \subseteq F \) and an increasing subsequence \( (n_k) \) such that \( \mu(F - V_k) < 4(\frac{\varepsilon}{p})^p, \| \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi^i \| V_k \| \infty < \varepsilon \) and \( \| \sqrt[p]{h_{n_k}} \left( \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi^i \right) \circ \varphi^{-n_k} \| V_k \| \infty < \varepsilon \).

**Theorem 2.5.** Let \( T_u : L^p(\Sigma) \to L^p(A) \) be bounded with \( \sigma(u) = X \), and let \( \varphi \) be a normal and finitely non-mixing transformation provided that \( \varphi^{-1} A \subseteq A \subseteq \Sigma_{\infty} \) and \( \sup_n \| h_n^{-1} \| \infty < \infty \). If for each subset \( F \subseteq A \) with \( 0 < \mu(F) < \infty \), there exists a sequence of \( A \)-measurable sets \( \{V_k\} \subseteq F \) such that \( \mu(V_k) \to \mu(F) \) as \( k \to \infty \), and there is a sequence of integers \( (n_k) \) such that
\[ \lim_{k \to \infty} \| \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi^i \| V_k \| \infty = 0 \]
and
\[ \lim_{k \to \infty} \| \sqrt[p]{h_{n_k}} \left( \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi^i \right) \circ \varphi^{-n_k} \| V_k \| \infty = 0, \]
then \( T_u \) is subspace-hypercyclic with respect to \( L^p(A) \).

**Proof.** Since, \( S^A(X) \) is dense in \( L^p(A) \), we may take \( D_1 = D_2 = S^A(X) \) in the subspace-hypercyclicity’s criterion. For an arbitrary \( f \in S^A(X) \), one can easily find \( \{V_k\} \subseteq \sigma(f) \) such that \( \mu(V_k) \to \mu(\sigma(f)) \) and find an \( N_1 \) such that \( \varphi^n(\sigma(f)) = \emptyset \) for each \( n > N_1 \). Now, for each \( n_k > N_1 \) define the vector \( f_{n_k} = \frac{f \circ \varphi^{-n_k}}{\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi^i} \). Since \( \varphi^{-1} A \subseteq A \subseteq \Sigma_{\infty} \), then \( f_k \in L^p(A) \) and the simple computations show that \( T_u f_{n_k} \to f \). Now, we will show that \( \| T_u f_{n_k} \|_p \to 0 \) and \( \| f_{n_k} \|_p \to 0 \) as \( k \to \infty \). For an arbitrary \( \varepsilon > 0 \), there exist \( M, N_1 \in \mathbb{N} \), sufficiently large such that \( V_{N_1} \subseteq \sigma(f) \) and
\[ \mu(\sigma(f) - V_{N_1}) < \frac{\varepsilon}{M \| f \|_p^p}. \]
By Egoroff’s theorem, there exists an \( N_2 \) such that for each \( n_k > N_2 \),
\[ \| \sqrt[p]{h_{n_k}} \left( \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi^i \right) \circ \varphi^{-n_k} \| V_{N_1} \| \infty < \frac{\varepsilon}{M \| f \|_p^p} \] on \( V_{N_1} \). So, there exists a non-negative real number \( M \) such that \( \| \sqrt[p]{h_{n_k}} \left( \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi^i \right) \circ \varphi^{-n_k} \| \infty \leq M < \infty \) on \( \sigma(f) \). Now, by the change of variable formula, for each \( n_k > N = \max\{N_1, N_2\} \)
we have
\[
\|T_{u}^{n_{k}} f\|_{p} = \int_{X} \prod_{i=0}^{n_{k}-2} E^{A}(u) \circ \varphi^{i} E^{A}(u f \circ \varphi) \circ \varphi^{n_{k}-1} |f| d\mu
\]
\[
= \int_{X} \prod_{i=0}^{n_{k}-1} E^{A}(u) \circ \varphi^{i} f \circ \varphi^{n_{k}} |f| d\mu
\]
\[
= \int_{\sigma(f)} \prod_{i=0}^{n_{k}-1} E^{A}(u) \circ \varphi^{i} \circ \varphi^{-n_{k}} f |f|^{p} h_{n_{k}} d\mu
\]
\[
= \int_{\sigma(f)-V_{N}} \prod_{i=0}^{n_{k}-1} E^{A}(u) \circ \varphi^{i} \circ \varphi^{-n_{k}} f |f|^{p} h_{n_{k}} d\mu
\]
\[
+ \int_{V_{N}} \prod_{i=0}^{n_{k}-1} E^{A}(u) \circ \varphi^{i} \circ \varphi^{-n_{k}} f |f|^{p} h_{n_{k}} d\mu
\]
\[
< \| \sqrt{h_{n_{k}}} \prod_{i=0}^{n_{k}-1} E^{A}(u) \circ \varphi^{i} \circ \varphi^{-n_{k}} \|_{p_{\infty}} \| f \|_{p_{\infty}} \mu(\sigma(f)-V_{N})
\]
\[
+ \frac{\varepsilon}{\| f \|_{p_{\infty}}^{p}} < 2\varepsilon.
\]
By taking into account that \( \sup_{n} \| h_{n}^{A} \|_{\infty} < \infty \), we have
\[
\lim_{k \to \infty} \| f_{k} \|_{p} = \lim_{k \to \infty} \int_{X} \frac{f \circ \varphi^{-n_{k}}}{\prod_{i=0}^{n_{k}-1} E^{A}(u) \circ \varphi^{i} \circ \varphi^{-n_{k}}} |f| d\mu
\]
\[
= \lim_{k \to \infty} \int_{\sigma(f)} \frac{f}{\prod_{i=0}^{n_{k}-1} E^{A}(u) \circ \varphi^{i}} |f|^{p} h_{n_{k}} d\mu
\]
\[
\leq \sup_{k} \| h_{n_{k}}^{A} \|_{\infty} \left( \lim_{k \to \infty} \int_{\sigma(f)-V_{N}} \frac{f}{\prod_{i=0}^{n_{k}-1} E^{A}(u) \circ \varphi^{i}} |f| d\mu
\]
\[
+ \lim_{k \to \infty} \int_{V_{N}} \frac{f}{\prod_{i=0}^{n_{k}-1} E^{A}(u) \circ \varphi^{i}} |f| d\mu
\]
\[
= 0.
\]
Finally, it is clear that \( T_{u}^{n_{k}} L^{p}(A) \subseteq L^{p}(A) \) for all \( k \in \mathbb{N} \), because of \( \varphi^{-1}A \subseteq A \) and hence \( T_{u} \) satisfies in the subspace-hypercyclicity criterion and is subspace-hypercyclic.

**Proposition 2.6.** Suppose that \( \varphi : X \to X \) is a normal and finitely non-mixing transformation with \( \varphi^{-1}(A) \subseteq A \subseteq \Sigma_{\infty} \). Let \( \sup_{n} \| h_{n}^{A} \|_{\infty} < \infty \) and \( \sigma(u) = X \).

Then the following conditions are equivalent:

(i) \( T_{u} \) satisfies the subspace-hypercyclic criterion.

(ii) \( T_{u} \) is subspace-hypercyclic with respect to \( L^{p}(A) \).

(iii) \( T_{u} \oplus T_{u} \) is subspace-hypercyclic with respect to \( L^{p}(A) \oplus L^{p}(A) \).

(iv) \( T_{u} \) is subspace-weakly mixing.

**Proof.** (i) \( \Rightarrow \) (ii). Note that if an operator satisfies the subspace-hypercyclic criterion, then it is subspace-transitive and hence is subspace-hypercyclic [24 Theorem 3.5]. For the implication (ii) \( \Rightarrow \) (iii), we show that \( T_{u} \oplus T_{u} \) is subspace-topologically transitive, according [24 Theorem 3.3]. To begin, pick two pairs
of non-empty open sets \((A_1, B_1)\) and \((A_2, B_2)\) in \(L^p(A) \oplus L^p(A)\) arbitrarily. For \(j = 1, 2\), choose the functions \(f_j, g_j \in S^A(X)\) with \(f_j \in A_j\) and \(g_j \in B_j\). Let \(F = \sigma(f_1) \cup \sigma(f_2) \cup \sigma(g_1) \cup \sigma(g_2)\). Then \(\mu(F) < \infty\). Assume that \(\{V_k\} \subseteq F\), \((\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i^{-1})\) and \(\{\sqrt{h_{n_k} A} E^{\varphi^{-n_k}(A)}(\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i) \circ \varphi^{-n_k}\}\) are as provided by Theorem 2.3. There is an \(N_1 \in \mathbb{N}\) such that for all \(n > N_1\), \(F \cap \varphi^n(F) = \emptyset\). Moreover, for each \(\varepsilon > 0\) there exists \(N_2 \in \mathbb{N}\) such that for each \(k > N_2\) and \(n_k > N_1\), \(\|\sqrt{h_{n_k} A} E^{\varphi^{-n_k}(A)}(\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i) \circ \varphi^{-n_k}\|_p < \varepsilon\) on \(V_k\). Hence, for \(k > N_2\), we get that

\[
\|T_u^{n_k} (f_j \chi_{V_k})\|_p^p = \int_X |T_u^{n_k} (f_j \chi_{V_k})|^p d\mu
= \int_X \left| \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i(f_j \chi_{V_k}) \circ \varphi^{-n_k}\right|^p d\mu
= \int_{V_k} \left| \prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i\right|^p h_{n_k}^p d\mu < \varepsilon.
\]

Now, define a map \(D_\varphi(f) = \frac{f \varphi^{-1}}{E^A(u) \circ \varphi^{-1}}\) on the subspace \(S^A(X)\). Then for each \(f \in S^A(X), T_u^{n_k} D_\varphi^{n_k}(f) = f\). Again, we may find an \(N_3 \in \mathbb{N}\) such that for each \(k > N_3\) and \(n_k > N_1\), \(\|\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i\|_\infty < \frac{\varepsilon}{\max\{\|f\|_\infty, \|g\|_\infty\}}\) on \(V_k\), where \(M = \sup_n \|h_{n_k}^2\|_\infty < \infty\). On the other hand, for each \(k > N_3\) note that

\[
\|D_\varphi^{n_k}(g_j \chi_{V_k})\|_p^p = \int_{V_k} \frac{|g_j \circ \varphi^{-n_k}|^p}{|\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i|} \prod_{i=0}^{n_k} E^A(u) \circ \varphi_i h_{n_k}^p d\mu
= \int_{V_k} \frac{|g_j|^p}{\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i} h_{n_k}^p d\mu < \varepsilon.
\]

For each \(k \in \mathbb{N}\), let \(f_{j,k} = f_j \chi_{V_k} + D_\varphi^{n_k}(g_j \chi_{V_k})\). Then we have \(f_{j,k} \in L^p(A)\),

\[
\|f_{j,k}^p - f_j^p\|_p \leq \|f_j\|_\infty \mu(F - V_k) + \|D_\varphi^{n_k}(g_j \chi_{V_k})\|_p^p
\]

and

\[
\|T_u^{n_k} f_{j,k}^p - g_j^p\|_p \leq \|g_j\|_\infty \mu(F - V_k) + \|T_u^{n_k} (f_j \chi_{V_k})\|_p^p.
\]

Hence, \(\lim_{k \to \infty} f_{j,k} = f_j\), \(\lim_{k \to \infty} T_u^{n_k} f_{j,k} = g_j\) and \(T_u^{n_k}(A_j) \cap B_j \neq \emptyset\) for some \(k \in \mathbb{N}\). Moreover, since \(\varphi^{-1}(A) \subseteq A\) then \(T_u^{n_k}(L^p(A)) \subseteq L^p(A)\). So \(T_u \oplus T_u\) is subspace-hypercyclic on \(L^p(A) \oplus L^p(A)\).

To prove the implication \((iv) \Rightarrow (i)\), we use Bés-Peris’s approach stated in [10, Theorem 4.2]. Assume that \(T_u \oplus T_u\) is subspace-hypercyclic on \(L^p(A) \oplus L^p(A)\) with subspace-hypercyclic vector \(f \oplus g\). Note that for each \(n \in \mathbb{N}\), the operator \(I \oplus T_u^n\) has dense range and commutes with \(T_u \oplus T_u\), therefore \(\text{orb}(I \oplus T_u^n, f \oplus g) = (I \oplus T_u^n) \text{orb}(T_u \oplus T_u, f \oplus g)\). Eventually \(f \oplus T_u^n g\) is subspace-hypercyclic vector as well. We show that the subspace-hypercyclic criterion is satisfied by \(D_1 = D_2 = \text{orb}(T_u \oplus T_u, f \oplus g)\). Let \(U\) be an arbitrary open neighborhood of 0 in \(L^p(A)\). Hence, one can find a sequence \((g_k) \subseteq U\) and an increasing sequence of integers \((n_k)\) such that \(T_u^{n_k} f \oplus T_u^{n_k} g_k \to 0 \oplus g\) and \(g_k \to 0\). Clearly, \(T_u^{n_k}(L^p(A)) \subseteq L^p(A)\).

**Corollary 2.7.** Under the assumptions of Proposition 2.6, the following conditions are equivalent:
(i) $T_u$ is subspace-topologically mixing on $L^p(A)$.
(ii) For each $A$-measurable subset $F \subseteq X$ with $0 < \mu(F) < \infty$, there exists a sequence of $A$-measurable sets $\{V_n\} \subseteq F$ such that $\mu(V_n) \to \mu(F)$ as $n \to \infty$ and $\lim_{n \to \infty} \|\left(\prod_{i=0}^{n-1} E^A(u) \circ \varphi_i \right)^{-1}v_n\| = \lim_{n \to \infty} \|\sqrt[n]{\prod_{i=0}^{n-1} E^A(u) \circ \varphi_i \circ \varphi^{-n}}v_n\| = 0$.

Proof. By Theorem 2.5 and Proposition 2.6, the implication (ii) implies (i) is established, just use the full sequences instead of subsequences. For the implication (i) ⇒ (ii), let $\varepsilon > 0$ and $F \in A$ with $0 < \mu(F) < \infty$ be arbitrary. Consider a non-empty and open subset $U = \{f \in L^p(A) : \|f - \chi_F\|_p < \varepsilon\}$. Since $T_u$ is subspace-topologically mixing and $\varphi$ is finitely non-mixing, one may find $N \in \mathbb{N}$ such that for all $n > N$, $T_u^n(U) \cap U \neq \emptyset$ and $F \cap \varphi^n(F) = \emptyset$. Hence, for each $n > N$, we can choose a function $f_n \in U$ such that $T_u^n f_n \in U$. Then $\|f_n - \chi_F\|_p < \varepsilon$ and $\|T_u^n f_n - \chi_F\|_p < \varepsilon$. The rest of the proof can be proceed like as Theorem 2.3.

Example 2.8. Let $X = \mathbb{R}$ be the real line with Lebesgue measure $\mu$ on the algebra $\Sigma$ of all Lebesgue measurable subsets of $\mathbb{R}$. Let $A$ be the $\sigma$-subalgebra generated by the symmetric intervals about the origin. For a positive real number $t$ define the transformation $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = x + t$, $x \in \mathbb{R}$. Clearly, $\varphi^{-1} A \subseteq A \subseteq \Sigma$ and in this setting, $E^A(f) = \frac{f(x) + f(-x)}{2}$, which is the even part of $f \in L^p(\Sigma)$. Fix $r > 1$ and define the weight function $u$ on $\mathbb{R}$ by

$$u(x) = \begin{cases} 2x + r, & 1 \leq x, \\ -x^2 + \frac{4}{r} + 2, & -1 < x < 1, \\ x^3 + \frac{1}{r}, & x \leq -1. \end{cases}$$

Then, we have

$$E^A(u)(x) = \begin{cases} r, & 1 \leq x, \\ -x^2 + 2, & -1 < x < 1, \\ \frac{1}{r}, & x \leq -1. \end{cases}$$

For an arbitrary $F = [-a, a]$, take $V_k = (-a + \frac{1}{k}, a - \frac{1}{k})$. In this case, one may easily find a sequence $(n_k)$ such that both quantities $\|\left(\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i \right)^{-1}v_k\|_\infty$ and $\|\sqrt[n_k]{\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i \circ \varphi^{-n_k}}v_k\|_\infty$ tend zero as $k \to \infty$. Because, $h_{n_k}^A = h_{n_k} = 1$ and $\|\prod_{i=0}^{n_k-1} E^A(u) \circ \varphi_i \circ \varphi^{-n_k}\| = \prod_{i=1}^{n_k} E^A(u) \circ \varphi^{-i}$, since $\varphi$ is onto (or $\sigma(h_{n_k}^A) = \mathbb{R}$). Therefore, by Theorem 2.3, $T_u$ is subspace-hypercyclic with respect to $L^p(A)$ while it is not hypercyclic on $L^p(\Sigma)$ [3, Theorem 2.3]. For this, just consider that $\|\sqrt[n_k]{\prod_{i=0}^{n_k-1} u \circ \varphi^{-i}}v_k\|_\infty = \|\prod_{i=1}^{n_k} u \circ \varphi^{-i}\|_\infty \to 0$.

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