The explicit formula for Gauss-Jordan elimination and error analysis✩

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Abstract

The explicit formula for the elements of the successive intermediate matrices of the Gauss-Jordan elimination procedure is used for error analysis in the case that the procedure is applied to systems of linear equations. Stability conditions in terms of relative precision and size of determinants are given, such that the Gauss-Jordan procedure leads to a solution respecting the original imprecisions in the right-hand member. The solution is the same as given by Cramer’s Rule. We model imprecisions with the help of non-standard analysis. A direct proof by induction is given of the explicit formula for the intermediate matrices.

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1. Introduction

Gantmacher’s book [9] on linear algebra contains an explicit formula for the elements $a_{ij}^{(k)}$ of the matrix obtained after $k$ Gaussian operations applied

✩The explicit formula for Gauss-Jordan elimination and error analysis

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to a matrix \( A = [a_{ij}] \), below and to the right of the \( k^{th} \) pivot \( a_{kk}^{(k)} \). The formula is given in terms of quotients of minors, and follows when applying Gaussian elimination to the two minors, which happen to have common factors all wiping themselves out, except for \( a_{ij}^{(k)} \). We extend this method to all elements of the matrix \( A^{(k)} = [a_{ij}^{(k)}] \), and derive that above the pivot a similar formula holds, with alternating sign.

We apply this explicit formula to the error analysis of the solution of systems of linear equations with coefficient matrix \( A \) and right-hand member \( b \), in the particular case where \( A \) is square, non-singular, reduced and properly arranged, meaning that the pivots are maximal and are located on the principal diagonal. We formulate stability conditions, the most important expressing that the relative imprecisions of elements of \( A \), when compared to \( \det(A) \), should be at most of the same order as the relative imprecision of the right-hand member. Then if \( \det(A) \) is not too small, the Gauss-Jordan procedure solves the system within the bounds given by the imprecisions in the right-hand member, i.e., there is no significant blow-up of errors. We also show that Cramer’s Rule applied to stable systems leads to the same outcome. We keep track of the order of magnitude of the imprecisions at each step of the Gauss-Jordan procedure. We distinguish between division by the pivot and the operations with the rows, since they have different impact on the imprecisions; thus the pivots will now be of the form \( a_{k+1k+1}^{(2k)} \). We show that each step transforms a stably system into a new stable system. The main tools are lower and upper bounds for the pivots, which are quotients of successive principal minors.

The above considerations on imprecisions and orders of magnitude figuring in error propagation are formalized in Nonstandard Analysis. Notions as ”big” and ”small” are not modelled functionally as limit behaviour, but by numbers, which can be ”unlimited” or ”infinitesimal”. The imprecisions and orders of magnitude are modelled by convex subgroups of the nonstandard reals, called \((scalar) neutrices\), the vagueness being reflected by the invariance under some additions, a formalization of the Sorites property \([8, 25]\); we were inspired by the functional neutrices of Van der Corput’s Theory of Neglecting \([2]\). The representation of imprecisions occurring in the coefficients and the right-hand member of a system of equations by scalar neutrices facilitates individual treatment and direct calculations, due to the absence of functional dependence.

The status of the explicit formula for the Gauss-Jordan elimination pro-
procedure seems to be uncertain. As regards to the formula for the lower part of the intermediate matrices Gantmacher refers to \[11\]. \[19\] contains some historical observations and presents a proof using identities of determinants. The formula including also the alternating behaviour at the upper part of the intermediate matrices is contained in \[18\]. The proof given uses also identities of determinants. We give a direct proof by induction, based on the simplification of quotients of minors indicated in Gantmacher’s book, in combination with a formula for the pivots in terms of quotients of principal minors.

There exist an extensive literature on error analysis for the Gauss-Jordan elimination procedure, see e.g. \[26\], \[20\] \[10\], \[12\] and \[19\]. Often the elimination concerns only the lower part of the matrix, after which solutions are found by successive substitutions. Some key-notions are the growth factor

\[
\rho \equiv \frac{\max_{i,j,k} |a_{i,j}^{(k)}|}{\max_{i,j} |a_{i,j}|}
\]

and the condition number in the form of the product of norms

\[
\text{cond}(A) \equiv \|A\| \|A^{-1}\|,
\]

and they are often used to obtain bounds for the errors as a function of \(k\). The principal tools in our setting are estimates of determinants and its principal minors. This seems somewhat natural, since by Cramer’s Rule the solution of linear systems is stated in the form of quotients of determinants, and the Gauss-Jordan operations are carried out with quotients of minors; we point out that there exists a relationship between the orders of magnitude of determinants and its principal minors. Our stability conditions imply invariance of the order of magnitude in the right-hand member, and allow for a moderate increase in the imprecisions of the coefficients of the matrices, which at any stage should remain small with respect to the determinants.

This article has the following structure. In section 2 we state the explicit formula for all the entries of the intermediate matrices of the Gauss-Jordan procedure and prove it by induction. We outline our approach to error analysis in Section 3 this section also contains some background on nonstandard analysis, the neutrices and external numbers (sums of a nonstandard real number and a neutrix), and calculus with matrices of external numbers, obtained earlier in \[23\]. In Section 4 we recall the notion of flexible systems of \[13\], which are the systems of linear equations with external numbers we use to model error propagation. We define the notion of stability, and state the main theorems, which indicate the solution sets of stable systems, and
state that they may be obtained both by the Gauss-Jordan procedure and
by Cramer’s Rule. The validity of Cramer’s Rule was earlier proved in [13]
for a class of non-homogeneous systems. In Section 5 we show the impact
of each step of the Gauss-Jordan procedure in our model of error propagation
and derive that a stable system is always transformed into a stable system.
These results and the generalization of Cramer’s Rule proved in Section 6
allow us to prove the main theorems in Section 7. Concrete examples of flex-
able systems of equations are contained in Section 8. Some results suggest
the possibility to obtain simplifications, by neglecting terms in the coefficients
of the equations which may be too specified, compared with the imprecisions
in the right-hand member; we illustrate this numerically.

2. Gauss-Jordan operations

We start with some definitions and notations related to the Gauss-Jordan
operations, where we use the common representation by matrix multiplica-
tions. Explicit expressions for these matrices are given in Theorem 2.10, and
explicit formulas for the intermediate matrices are given in Theorem 2.9.

We will always consider $m \times n$ matrices with $m, n \in \mathbb{N}$, $m, n \geq 1$.

Notation 2.1. We denote by $\mathcal{M}_{m,n}(\mathbb{R})$ the set of all $m \times n$ matrices over
the field $\mathbb{R}$.

Notation 2.2. Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$. For each $k \in \mathbb{N}$ such that $1 \leq k \leq \min\{m, n\}$, let $1 \leq i_1 < \cdots < i_k \leq m$ and $1 \leq j_1 < \cdots < j_k \leq n$.

1. We denote the $k \times k$ submatrix consisting of the rows with indices
   $\{i_1, \ldots, i_k\}$ and columns with indices $\{j_1, \ldots, j_k\}$ by $A_{i_1 \ldots i_k}^{j_1 \ldots j_k}$.

2. We denote the corresponding $k \times k$ minor by $m_{i_1 \ldots i_k}^{j_1 \ldots j_k} = \det (A_{i_1 \ldots i_k}^{j_1 \ldots j_k})$.

3. For $1 \leq k \leq \min\{m, n\}$ we may denote the principal minor of order $k$
   by $m_k = m_{1 \ldots k}^{1 \ldots k}$. We define formally $m_0 = 1$.

By appropriately changing rows and columns we may always assume that
the absolute value of the principal minor $m_{k+1}$ is larger than or equal to the
absolute value of the minors of order $k + 1$ which share the the first $k$ rows
and columns. This is stated in the next proposition, which is well-known.
Proposition 2.3. Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$. By if necessary changing rows and columns of $A$, we may obtain that for every $k \in \mathbb{N}$ such that $1 \leq k + 1 \leq \min\{m, n\}$

$$|m^{1..k_i}_{1..k_j}| \leq |m_{k+1}|$$

for $k + 1 \leq i \leq m$ and $k + 1 \leq j \leq n$.

As a consequence, if one of the minors $m^{1..k_i}_{1..k_j}$ is non-zero, the principal minor $m_{k+1}$ is non-zero. So if $r$ is the rank of $A$, the principal minors $m_1, \ldots, m_r$ are all non-zero.

Definition 2.4. Assume $A \in \mathcal{M}_{m,n}(\mathbb{R})$ has rank $r \geq 1$. Then $A$ is called properly arranged, if for $1 \leq k \leq r$ formula (1) holds whenever $k+1 \leq i \leq m$ and $k+1 \leq j \leq n$, and diagonally eliminable up to $r$ if $m_k \neq 0$ for $1 \leq k \leq r$; if $r = n = m$ we say that $A$ is diagonally eliminable.

Convention 2.5. To avoid some complications of notation, In the remaining part of this section we assume that the matrices $A = [a_{ij}]_{m \times n}$ are diagonally eliminable up to $r$, more precisely, if $A$ has rank $r$, the square submatrix $[a_{ij}]_{1 \leq i \leq r, 1 \leq j \leq r}$ is non-singular, with all its principal minors non-zero. This can be assumed without loss of generality because of Proposition 2.3.

Definition 2.6. Let $A = [a_{ij}]_{m \times n} \in \mathcal{M}_{m,n}(\mathbb{R})$ be of rank $r \geq 1$ and diagonally eliminable up to $r$. For every $q$ with $1 \leq q \leq 2r - 1$ the Gauss-Jordan operation matrix $G_q$ and the intermediate matrix $A^{(q)}$ are defined as follows. We let $G_1 = \left[ g_{ij}^{(1)} \right]_{m \times m}$ be the matrix which corresponds to the multiplication of the entries of the first line of $A$ by $1/a_{11}$, such that the first pivot of $A^{(1)} = G_1 A$ becomes $a^{(1)}_{11} = 1$. This means that

$$G_1 = \left[ g_{ij}^{(1)} \right]_{m \times m} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$
column of $\mathcal{A}^{(1)}$, except for $a_{11}^{(1)}$, and let $\mathcal{A}^{(2)} = G_2\mathcal{A}^{(1)}$, i.e.

$$G_2 = \begin{bmatrix} g_{ij}^{(2)} \end{bmatrix}_{m \times m} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m1} & 0 & \cdots & 1 \end{bmatrix},$$

and

$$\mathcal{A}^{(2)} = \begin{bmatrix} a_{ij}^{(2)} \end{bmatrix}_{n \times p} = \begin{bmatrix} 1 & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(2)} & \cdots & a_{mn}^{(2)} \end{bmatrix}.$$

Assuming that $G_{2k}$ and $\mathcal{A}^{(2k)}$ are defined for $k < r$, the matrix $G_{2k+1}$ corresponds to the multiplication of row $k + 1$ of $\mathcal{A}^{(2k)}$ by $1/a_{k+1,k+1}^{(2k)}$, leading to $\mathcal{A}^{(2k+1)} \equiv G_{2k+1}\mathcal{A}^{(2k)}$, and the matrix $G_{2k+2}$ corresponds to transforming the entries of column $k$ of $\mathcal{A}^{(2k+1)}$ into zero, except for the entry $a_{k+1,k+1}^{(2k+1)}(= 1)$, resulting in $\mathcal{A}^{(2k+2)} \equiv G_{2k+2}\mathcal{A}^{(2k+1)}$. So we have $G_{2k+1} = \begin{bmatrix} g_{ij}^{(2k+1)} \end{bmatrix}_{n \times n}$, where

$$g_{ij}^{(2k+1)} = \begin{cases} 1 & \text{if } i = j \neq k + 1 \\ 0 & \text{if } i \neq j \\ \frac{1}{a_{k+1,k+1}^{(2k)}} & \text{if } i = j = k + 1 \end{cases},$$

and $G_{2k+2} = \begin{bmatrix} g_{ij}^{(2k+2)} \end{bmatrix}_{n \times n}$, where

$$g_{ij}^{(2k+2)} = \begin{cases} 0 & \text{if } j \notin \{i, k + 1\} \\ 1 & \text{if } i = j \\ -a_{ik+1}^{(2k+1)} & \text{if } i \neq k + 1, j = k + 1 \end{cases}.$$

Because the product $G_{2k+1}\mathcal{A}^{(2k)}$ corresponds to the Gaussian operation of multiplying the $(k + 1)^{th}$ row of the matrix $\mathcal{A}^{(2k)}$ by the non-zero scalar $\frac{1}{a_{k+1,k+1}^{(2k)}}$, one has

$$\det(G_{2k+1}) = \frac{1}{a_{k+1,k+1}^{(2k)}}.$$
On the other hand, the product $G_{2k}A^{(2k-1)}$ corresponds to the repeated Gauss-Jordan operation of adding a scalar multiple of a row to some other row of $A^{(2k-1)}$, implying that

$$\det(G_{2k}) = 1.$$  

We justify the above definition by showing that $a_{kk}^{(2k-2)} \neq 0$ for $1 \leq k \leq r$. We prove first a lemma.

**Lemma 2.7.** Let $A = [a_{ij}]_{m \times n} \in M_{m,n}(\mathbb{R})$ be of rank $r \geq 1$ and diagonally eliminable up to $r$. Then $m_{11} = a_{11} \neq 0$. If $r > 1$, assuming that $a_{11}, a_{22}, \ldots, a_{kk}^{(2k-2)} \neq 0$ for $1 \leq k < r$, it holds that

$$m_{k+1} = a_{11}a_{22}^{(2k-2)} \cdots a_{kk}^{(2k-2)} a_{k+1k+1}^{(2k)}$$

and $a_{k+1k+1}^{(2k)} \neq 0$.

**Proof.** Obviously $m_{11} = a_{11} \neq 0$. Assume that $r > 1$, $1 \leq k < r$ and also $a_{22}, \ldots, a_{kk}^{(2k-2)} \neq 0$. Then we may apply the Gauss-Jordan operations up to $2k$ and obtain

$$m_{k+1} = \det\begin{bmatrix}
  a_{11} & \cdots & a_{1k} & a_{1k+1} \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{k1} & \cdots & a_{kk} & a_{kk+1} \\
  a_{k+11} & \cdots & a_{k+1k} & a_{k+1k+1}
\end{bmatrix}$$

$$= a_{11} \cdots a_{kk}^{(2k-2)} \det\begin{bmatrix}
  1 & \cdots & 0 & a_{1k+1}^{(2k-1)} \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & 1 & a_{kk+1}^{(2k-1)} \\
  0 & \cdots & 0 & a_{k+1k+1}^{(2k)}
\end{bmatrix}.$$  

The Laplace-expansion applied to the last row yields

$$m_{k+1} = a_{11} \cdots a_{kk}^{(2k-1)} a_{k+1k+1}^{(2k)} \det\begin{bmatrix}
  1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & 1
\end{bmatrix} = a_{11} \cdots a_{kk}^{(2k-1)} a_{k+1k+1}^{(2k)}.$$  

Because $m_{k+1} \neq 0$ we derive that $a_{k+1k+1}^{(2k)} \neq 0$. 

$\square$
Proposition 2.8. Let \( A = [a_{ij}]_{m \times n} \in \mathcal{M}_{m,n}(\mathbb{R}) \) be of rank \( r > 1 \) and diagonally eliminable up to \( r \). Then for \( 1 \leq k < r \) it holds that \( a_{k+1k+1}^{(2k)} \neq 0 \) and
\[
a_{k+1k+1}^{(2k)} = \frac{m_{k+1}}{m_k}.
\] (3)

Proof. Applying Lemma 2.7 and using induction, we derive that \( a_{11}^{(2)}, \ldots, a_{k+1k+1}^{(2k)} \neq 0 \). Then (3) follows from (2) applied to \( k+1 \) and \( k \). \( \square \)

Using similar methods for calculating minors, we derive expressions for the entries \( a_{ik+1}^{(2k+2)} \) of the matrices of even order \( G_{2k+2} \), also in terms of quotients of minors.

Theorem 2.9 (Explicit expressions for Gauss-Jordan elimination). Let \( A = [a_{ij}]_{m \times n} \in \mathcal{M}_{m,n}(\mathbb{R}) \) be of rank \( r > 1 \) and diagonally eliminable up to \( r \). Let \( k < r \). Then

\[
A^{(2k)} = \begin{bmatrix}
1 & \cdots & 0 & a_{1k+1}^{(2k)} & \cdots & a_{1n}^{(2k)} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & a_{kk}^{(2k)} & \cdots & a_{kn}^{(2k)} \\
0 & \cdots & 0 & a_{k+1k+1}^{(2k)} & \cdots & a_{k+1n}^{(2k)} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & a_{mk+1}^{(2k)} & \cdots & a_{mn}^{(2k)}
\end{bmatrix},
\]

where
\[
a_{ij}^{(2k)} = \begin{cases}
(-1)^{k+i} \frac{m_{1\ldots i-1i+1\ldots k\ldots j}}{m_k} & \text{if } 1 \leq i \leq k, j \geq k+1 \\
\frac{m_{1\ldots i\ldots k\ldots j}}{m_{k}} & \text{if } i \geq k+1, j \geq k+1
\end{cases}.
\] (4)

Proof. Firstly, let \( k+1 \leq i \leq m \) and \( k+1 \leq j \leq n \). Let
\[
U_{ij} = \begin{bmatrix}
a_{11} & \cdots & a_{1k} & a_{1j} \\
\vdots & \ddots & \vdots & \vdots \\
a_{k1} & \cdots & a_{kk} & a_{kj} \\
a_{i1} & \cdots & a_{ik} & a_{ij}
\end{bmatrix}
\]
Then \( \det(\mathcal{U}_{ij}) = m_{1\ldots kj}^{1\ldots ki} \). If we apply the first \( 2k \) Gauss-Jordan operations to \( \mathcal{U}_{ij} \), we obtain

\[
\det(\mathcal{U}_{ij}) = a_{11} \cdots a_{kk}^{(2k-2)} \det \begin{bmatrix}
1 & \cdots & 0 & a_{1j}^{(2k-1)} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & a_{kj}^{(2k-1)} \\
0 & \cdots & 0 & a_{ij}
\end{bmatrix} = m_k a_{ij}^{(2k)}.
\]

Hence \( a_{ij}^{(2k)} = \frac{m_{1\ldots kj}^{1\ldots ki}}{m_k} \).

Secondly, we let \( 1 \leq i < k + 1 \) and \( k + 1 \leq j \leq n \). Let

\[
\mathcal{V}_{ij} = \begin{bmatrix}
a_{11} & \cdots & a_{1i-1} & a_{1i+1} & \cdots & a_{1k} & a_{1j} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
a_{i-11} & \cdots & a_{i-1i-1} & a_{i-1i+1} & \cdots & a_{i-1k} & a_{i-1j} \\
a_{i1} & \cdots & a_{ii-1} & a_{ii+1} & \cdots & a_{ik} & a_{ij} \\
a_{i+11} & \cdots & a_{i+1i-1} & a_{i+1i+1} & \cdots & a_{i+1k} & a_{i+1j} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
a_{k1} & \cdots & a_{ki-1} & a_{ki+1} & \cdots & a_{kk} & a_{kj}
\end{bmatrix}.
\]

Then

\[
\det(\mathcal{V}_{ij}) = m_{1\ldots i-1\ldots i+1\ldots kj}^{1\ldots k}.
\]

Let \( \mathcal{V}'_{ij} \) be the matrix obtained by applying the first \( 2k \) Gauss-Jordan operations to \( \mathcal{V}_{ij} \). Then, using (2)

\[
\det(\mathcal{V}'_{ij}) = a_{11} \cdots a_{kk}^{(2k-2)} \det \begin{bmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 & a_{1j}^{(2k)} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & \cdots & 0 & a_{ij}^{(2k)} \\
0 & \cdots & 0 & 0 & \cdots & 0 & a_{ij}^{(2k)} \\
0 & \cdots & 0 & 1 & \cdots & 0 & a_{ij}^{(2k)} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1 & a_{kj}^{(2k)}
\end{bmatrix}
\]

\[
= m_k \det(\mathcal{V}'_{ij}).
\]

Expanding \( \det(\mathcal{V}'_{ij}) \) along the \( i \)th row, we derive that

\[
\det(\mathcal{V}'_{ij}) = (-1)^{i+k} a_{ij}^{(2k)}.
\]
Combining, we conclude that
\[ a_{ij}^{(2k)} = (-1)^{i+k} \frac{m_1^{1...i-1}m_{1...i}^{i+1...k}m_k}{m_k}. \]

The next theorem gives explicit formulas for the matrices \( G_p \) associated to the Gauss-Jordan operations. At odd order \( q = 2k + 1 \) we have to divide the row \( k + 1 \) by \( a_{k+1,k+1}^{(2k)} \) as given by (3), and at even order \( q = 2k + 2 \), in the column \( j = k + 1 \) we have to subtract by \( a_{k+1,j}^{(2k)} \) as given by (4).

**Theorem 2.10** (Explicit expressions for Gauss-Jordan operations). Let \( A = [a_{ij}]_{m \times n} \in \mathcal{M}_{m,n}(\mathbb{R}) \) be of rank \( r > 1 \) and diagonally eliminable up to \( r \). For \( k < r \) the Gaussian elimination matrix of odd order \( G_{2k+1} = [g_{ij}^{(2k+1)}]_{m \times m} \) satisfies
\[
g_{ij}^{(2k+1)} = \begin{cases} 
1 & \text{if } i = j \neq k+1 \\
0 & \text{if } i \neq j \\
\frac{m_k}{m_{k+1}} & \text{if } i = j = k+1
\end{cases}.
\]
and the Gaussian elimination matrix of even order \( G_{2k+2} = [g_{ij}^{(2k+2)}]_{m \times m} \)
satisfies
\[
g_{ij}^{(2k+2)} = \begin{cases} 
0 & \text{if } j \notin \{i, k+1\} \\
1 & \text{if } i = j \\
(-1)^{k+i+1} \frac{m_1^{1...k}m_{1...i}^{i+1...k+1}}{m_k} & \text{if } 1 \leq i \leq k, j = k+1 \\
-\frac{m_1^{1...k}m_{1...k}^{k+1}}{m_k} & \text{if } i > k+1, j = k+1
\end{cases}.
\]

**Definition 2.11.** Let \( A = [a_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{R}) \) be a diagonally eliminable matrix. The Gauss-Jordan procedure for \( A \), denoted by \( G(A) \), is given by the successive multiplications of matrices \( G(A) = G_{2n}(G_{2n-1}(\cdots (G_1(A)) \cdots )) \).

Observe that for \( k \leq n \):
\[
\det(G_1) \det(G_2) \cdots \det(G_{2k+1}) = \frac{1}{m_1} \frac{m_2}{m_2} \cdots \frac{m_k}{m_{k+1}} = \frac{1}{m_{k+1}},
\]
which results from (5).
For $1 \leq k \leq n$, the inverses of the matrices of the Gauss-Jordan procedure $G$ are well-defined, as follows. For odd indices we have $G_{2k+1}^{-1} = \left( (g_{ij}^{-1})^{(2k+1)} \right)$, with

$$
(g_{ij}^{-1})^{(2k+1)} = \begin{cases}
1 & \text{if } i = j \neq k + 1 \\
0 & \text{if } i \neq j \\
\frac{m_{k+1}}{m_k} & \text{if } i = j = k + 1
\end{cases},
$$

and for even indices $G_{2k+2}^{-1} = \left( (g_{ij}^{-1})^{(2k+2)} \right)_{n \times n}$, with

$$
(g_{ij}^{-1})^{(2k+2)} = \begin{cases}
0 & \text{if } j \notin \{i, k + 1\} \\
1 & \text{if } i = j \\
(-1)^{k+i} \frac{m_1 \cdots i_{-1} \cdots k_{-1} k+1}{m_k} & \text{if } 1 \leq i \leq k, j \geq k + 1. \\
\frac{m_1 \cdots k_i}{m_k} & \text{if } i > k + 1, j \geq k + 1
\end{cases}
$$

The product $G_1^{-1} \equiv G_{2k}^{-1}(G_2^{-1}(\cdots (G_{2n}^{-1}(A)) \cdots ))$ will be called the inverse Gauss-Jordan procedure.

3. On error analysis of systems of linear equations

In [13] and [22] neutrices and external numbers have been applied to study error propagation in the solution of systems of linear equations, called ”flexible systems”, and in linear optimization. [22] contains a general ”parameter method” to determine the shape of the error sets around a vector solution obtained by exact Gauss-elimination of a real system, but does not study the propagation of errors resulting from its operations. In [13] conditions were given for the validity of the Cramer formula as a solution set for non-singular flexible systems; systems satisfying these conditions will be called stable. In the remaining sections we study error propagation by applying Gauss-Jordan elimination directly to flexible systems. For stable systems with a reduced properly arranged coefficient matrix we use the expressions for the entries of the intermediate matrices of Section 2 to show that the Gauss-Jordan procedure respects stability, does not augment the neutrices in the right-hand member, and gives rise to the same solution as Cramer’s Rule. The method may be generalized to singular systems, which is the intended subject of a second article [24].
In Section 4 we recall the definition of flexible systems, formulate the stability conditions and state the main theorems, which give solution formulas in terms of Cramer’s rule and Gauss-Jordan elimination, and indicate their relationship. The theorems are proved in Section 7 using properties of error propagation in the Gauss-Jordan procedure established in Section 5 and some complements to the results on Cramer’s Rule of [13], presented in Section 6. The final Section 8 contains some examples. Below we recall basic properties of the calculus of external numbers and matrices with external numbers.

We adopt the axiomatic form of nonstandard analysis Internal Set Theory IST of [17]; an important feature is that, next to the standard numbers, infinitesimals and infinitely large numbers are already present within the ordinary set of real numbers \( \mathbb{R} \). We use only bounded formulas, and then neutrices and external numbers are well-defined external sets in the extension \( HST \) of a bounded form of IST given by Kanovei and Reeken in [14]. For introductions to IST we refer to e.g. [4], [3] or [16] and for introductions to external numbers and illustrative examples we refer to [15], [6] or [7]; the latter contains an introduction to a weak form of nonstandard analysis sufficient for a practical understanding of our approach. We mention one important tool: External induction permits induction for all IST-formulas over the standard natural numbers. An overview of the calculus of matrices with external numbers and its determinants is contained in [23].

Below we recall briefly some definitions and useful properties in relation to neutrices and external numbers, and the matrix calculus built upon them.

**Remark 3.1.** Throughout this article we use the symbol \( \subseteq \) for inclusion and \( \subset \) for strict inclusion.

A real number is **limited** if it is bounded in absolute value by a standard natural number, and real numbers larger in absolute value than all limited numbers are called **unlimited**. Its reciprocals, together with 0, are called **infinitesimal**. **Appreciable** numbers are limited, but not infinitesimal. The set of limited numbers is denoted by \( \mathcal{L} \), the set of infinitesimals by \( \mathcal{O} \), the set of positive unlimited numbers by \( \mathcal{\mathcal{F}} \) and the set of positive appreciable numbers by \( \mathcal{A} \); these sets are all external.

The calculation rules which are a consequence of IST imply that the sum of two limited numbers is limited, and the sum of two infinitesimals is infinitesimal. It follows that \( \mathcal{L} \) and \( \mathcal{O} \) have the group property. A (**scalar**) neutrix \( N \) is an additive convex subgroup of \( \mathbb{R} \). So \( \mathcal{L} \) and \( \mathcal{O} \) are neutrices, and except for \( \{0\} \) and \( \mathbb{R} \), all neutrices are external sets. Let \( \varepsilon \in \mathbb{R} \) be
a positive infinitesimal. Other examples of neutrices are \( \varepsilon \mathcal{L} \), \( \varepsilon \varnothing \), \( M_\varepsilon \equiv \bigcap_{n \in \mathbb{N}} [-\varepsilon^n, \varepsilon^n] = \mathcal{L} \varepsilon^\infty \) and \( \mu_\varepsilon \equiv \bigcup_{n \in \mathbb{N}} [-e^{-1/(n\varepsilon)}, e^{-1/(n\varepsilon)}] = L e^{-\alpha/\varepsilon} \); as groups they are not isomorphic. Let \( \mathcal{N} \) be a neutrix. Clearly \( \mathcal{N} \mathcal{L} = \mathcal{N} \).

An absorber of \( \mathcal{N} \) is a real number \( a \) such that \( a \mathcal{N} \subseteq \mathcal{N} \). No appreciable number is an absorber of any neutrix, and in the examples above the infinitesimal number \( \varepsilon \) is an absorber of \( \mathcal{L} \) and \( \varnothing \), but not of \( M_\varepsilon \) and \( \mu_\varepsilon \).

Neutrices are ordered by inclusion, and if the neutrix \( \mathcal{A} \) is contained in the neutrix \( \mathcal{B} \), we may write \( \mathcal{B} = \max\{\mathcal{A}, \mathcal{B}\} \).

An external number is the Minkowski-sum of a real number and a neutrix. So each external number has the form \( \alpha = a + \mathcal{A} = \{a + x | x \in \mathcal{A}\} \), where \( A \) is called the neutrix part of \( \alpha \), denoted by \( N(\alpha) \), and \( a \in \mathbb{R} \) is called a representative of \( \alpha \). Identifying \( \{a\} \) and \( a \), the real numbers are external numbers with \( N(\alpha) = \{0\} \). We call \( \alpha \) zeroless if \( 0 \notin \alpha \), and neutrical if \( \alpha = N(\alpha) \). Notions as limited, infinitesimal and absorber may be extended in a natural way to external numbers.

The collection of all neutrices is not an external set in the sense of [14], but a definable class, denoted by \( \mathcal{N} \). Also the external numbers form a class, denoted by \( \mathcal{E} \).

The rules for addition, subtraction, multiplication and division of external numbers of Definition 3.2 below are in line with the rules of informal error analysis [21]. Here they are defined formally as Minkowski operations on sets of real numbers.

**Definition 3.2.** Let \( a, b \in \mathbb{R}, A, B \) be neutrices and \( \alpha = a + A, \beta = b + B \) be external numbers.

1. \( \alpha \pm \beta = a \pm b + A + B = a + b + \max\{A, B\} \).

2. \( \alpha \beta = ab + Ab + Ba + AB = ab + \max\{ab, bA, AB\} \).

3. If \( \alpha \) is zeroless, \( \frac{1}{\alpha} = \frac{1}{a} + \frac{A}{a^2} \).

If \( \alpha \) or \( \beta \) are zeroless, in Definition 3.2 we may neglect the neutrix product \( AB \). Definition 3.2 does not permit to divide by neutrices. However we will allow for division of neutrices in terms of division of groups.

**Definition 3.3.** Let \( A, B \in \mathcal{N} \). Then we define

\[ A : B = \{c \in \mathbb{R} | cB \subseteq A\} \]
In practice, the application of the pointwise operations defined above is rather straightforward. In some cases more care is needed. This is true in particular for the distributive law, which takes the following form.

**Theorem 3.4.** *(Distributivity with correction term)* Let $\alpha, \beta, \gamma = c + C$ be external numbers. Then

$$\alpha \gamma + \beta \gamma = (\alpha + \beta)\gamma + C\alpha + C\beta.$$  \hfill (6)

Because a neutrix term is added in the right-hand side of (6), we always have the following form of subdistributivity.

**Corollary 3.5.** *(Subdistributivity)* Let $\alpha, \beta, \gamma$ be external numbers. Then

$$(\alpha + \beta)\gamma \subseteq \alpha \gamma + \beta \gamma.$$ 

Theorem 3.7 below gives conditions such that the common distributive law holds, i.e. the correction terms figuring in (6) may be neglected. To this end we recall the notions of relative uncertainty and oppositeness.

**Definition 3.6.** *[15, 3]* Let $\alpha = a + A$ and $\beta = b + B$ be external numbers and $C$ be a neutrix.

1. The relative uncertainty $R(\alpha)$ of $\alpha$ is defined by $A/\alpha$ if $\alpha$ is zeroless, otherwise $R(\alpha) = \mathbb{R}$.

2. $\alpha$ and $\beta$ are opposite with respect to $C$ if $(\alpha + \beta)C \subset \max(\alpha C, \beta C)$.

**Theorem 3.7.** Let $\alpha, \beta, \gamma = c + C$ be external numbers. Then $\alpha \gamma + \beta \gamma = (\alpha + \beta)\gamma$ if and only if $R(\gamma) \subseteq \max(R(\alpha), R(\beta))$, or $\alpha$ and $\beta$ are not opposite with respect to $C$.

Simple and important special cases are given by

$$(x + N)\beta = x\beta + N\beta$$

and

$$x(\alpha + \beta) = x\alpha + x\beta,$$

whenever $x \in \mathbb{R}$, $N \in \mathcal{N}$ and $\alpha, \beta \in \mathbb{E}$.

An order relation is given as follows.
Definition 3.8. Let $\alpha, \beta \in \mathbb{E}$. We define

$$\alpha \leq \beta \iff \forall a \in \alpha \exists b \in \beta (a \leq b).$$

If $\alpha \cap \beta = \emptyset$ and $\alpha \leq \beta$, then $\forall a \in \alpha \forall b \in \beta (a < b)$ and we write $\alpha < \beta$.

The relation $\leq$ is an order relation indeed, and compatible with the operations, with some small adaptations [15, 7]. The inverse order relation is given by

$$\alpha \geq \beta \iff \forall a \in \alpha \exists b \in \beta (a \geq b),$$

and $\alpha > \beta$ if $\forall a \in \alpha \forall b \in \beta (a > b)$. Clearly $\alpha < \beta$ implies $\beta < \alpha$. However, both $\emptyset \leq \mathbb{E}$ and $\emptyset \geq \mathbb{E}$ hold. External numbers $\alpha$ such that $0 \leq \alpha$ are called non-negative.

The absolute value of an external number $\alpha = a + A$ is defined by $|\alpha| = |a| + A$. Notice that this definition does not depend on the choice of the representative of $\alpha$.

Next proposition lists some useful general properties of external numbers.

**Proposition 3.9.** [15, 13] Let $\alpha = a + A$ be a zeroless external number, $B$ be a neutrix and $n \in \mathbb{N}$ be standard. Then

1. $\alpha B = aB$ and $\frac{B}{\alpha} = \frac{B}{a}$.
2. $R(\alpha), R(1/\alpha) \subseteq \emptyset$.
3. $\alpha \cap \emptyset = \emptyset$.
4. $N((a + A)^n) = a^{n-1}A$.
5. If $\alpha$ is limited and is not an absorber of $B$, then

$$\alpha B = \frac{B}{\alpha} = B.$$

We turn now to matrices with entries in the form of external numbers. We give only a brief account and refer to [23] for more details, proofs and examples.

We denote by $\mathcal{M}_{m,n}(\mathbb{E})$ the class of all $m \times n$ matrices

$$\mathcal{A} = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn}
\end{bmatrix},$$
where $\alpha_{ij} \in \mathbb{E}$ for $1 \leq i \leq m, 1 \leq j \leq n$. We use the common notation $\mathcal{A} = [\alpha_{ij}]_{m \times n}$. When $m = n$ we simply write $\mathcal{M}_n(\mathbb{E})$. For $\mathcal{A}, \mathcal{B} \in \mathcal{M}_{m \times n}(\mathbb{E})$ we write $\mathcal{A} \subseteq \mathcal{B}$ if $\alpha_{ij} \subseteq \beta_{ij}$ for all $i, j$ such that $1 \leq i \leq m, 1 \leq j \leq n$. We always suppose that $m, n$ are standard.

**Notation 3.10.** For matrices $\mathcal{A} = [\alpha_{ij}]_{m \times n} \equiv [a_{ij} + A_{ij}]_{m \times n} \in \mathcal{M}_{m,n}(\mathbb{E})$ we define

$$|\alpha| = \max_{1 \leq i \leq m, 1 \leq j \leq n} |\alpha_{ij}|, \overline{A} = \max_{1 \leq i \leq m, 1 \leq j \leq n} A_{ij}.$$ 

**Definition 3.11.** A matrix of external numbers $\mathcal{A}$ is said to be **limited** if $\alpha \subset \mathcal{L}$ and **reduced**, if $\alpha = \alpha_{11}$ and $\alpha_{11} = 1 + A_{11}$, with $A_{11} \subseteq \emptyset$, while all other entries have representatives which in absolute value are at most 1.

**Definition 3.12.** A matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ is said to be **neutrical** if all of its entries are neutrices. A matrix $I_n \in \mathcal{M}_n(\mathbb{E})$ such that $I_n = I_n + \mathcal{A}$, with $\mathcal{A} \subseteq [\emptyset]_{n \times n}$, is called a near-identity matrix.

Below we give some properties of matrix operations, most are proved in [23]. Due to the subdistributivity property of external numbers, the multiplication of matrices with external numbers is not distributive, but it is also not associative.

The following general property of inclusion is an immediate consequence of the fact that, given external numbers $\alpha, \beta, \gamma$ such that $\alpha \subseteq \beta$, one has $\gamma \alpha \subseteq \gamma \beta$.

**Proposition 3.13.** Let $\mathcal{A} \in \mathcal{M}_{m,n}(\mathbb{R})$ and $\mathcal{B}, \mathcal{C} \in \mathcal{M}_{n,p}(\mathbb{E})$. If $\mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{A} \mathcal{B} \subseteq \mathcal{A} \mathcal{C}$.

Because subdistributivity holds for external numbers, it holds also for the calculus of matrices of external numbers. Next proposition gives a condition for distributivity.

**Proposition 3.14.** [23] Let $\mathcal{A} = [\alpha_{ij}]_{m \times n} \in \mathcal{M}_{m,n}(\mathbb{E})$ and $\mathcal{B} = [\beta_{ij}]_{n \times p}, \mathcal{C} = [\gamma_{ij}]_{n \times p} \in \mathcal{M}_{n,p}(\mathbb{E})$. If $\max_{1 \leq i \leq m, 1 \leq j \leq n} R(\alpha_{ij}) \leq \min_{1 \leq i \leq m, 1 \leq j \leq n} \max\{R(\beta_{ij}), R(\gamma_{ij})\}$, then $\mathcal{A}(\mathcal{B} + \mathcal{C}) = \mathcal{A} \mathcal{B} + \mathcal{A} \mathcal{C}$.

For subassociativity to hold conditions are needed, and associativity holds under stronger conditions.
Proposition 3.15. [23] Let $A \in \mathcal{M}_{m,n}(E), B \in \mathcal{M}_{n,p}(E)$ and $C \in \mathcal{M}_{p,q}(E)$. Then

1. $(AB)C \subseteq A(BC)$ if $A$ is a real matrix or $B, C$ are both non-negative.
2. $A(BC) \subseteq (AB)C$ if $C$ is a real matrix or $A, B$ are both non-negative.

Proposition 3.16. [23] Let $A \in \mathcal{M}_{m,n}(E), B \in \mathcal{M}_{n,p}(E)$ and $C \in \mathcal{M}_{p,q}(E)$. Then $A(BC) = (AB)C$ if one of the following conditions is satisfied:

1. $A$ and $C$ are both real matrices.
2. $B$ is a neutrical matrix.
3. $A, B, C$ are all non-negative matrices.

If $A$ is square, the determinant $\Delta \equiv \det(A) \equiv d + D$ is defined in the usual way through sums of signed products [22]; if $\Delta$ is zeroless, the matrix $A$ is called non-singular.

It is easily proved that the determinant of a limited matrix is limited, as are its minors. The neutrix of these determinants does not exceed the biggest neutrix of the entries.

Proposition 3.17. [23] Let $n \in \mathbb{N}$ be standard and $A \in \mathcal{M}_n(E)$ be limited. Then there exists a limited number $L > 0$ such that whenever $k \in \{1, \ldots, n\}$ and $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_k \leq n$

$$|m_{i_1 \ldots i_k}^{j_1 \ldots j_k}| \leq L.$$ 

In particular $|\Delta| \leq L$. Moreover $N(\Delta) \subseteq \overline{A}$.

The Laplace expansion happens to hold with inclusion [23], and this implies that the minors $\Delta_{i,j}$, obtained by eliminating row $i$ and column $j$ from the matrix $A$ for some $i, j$ with $1 \leq i, j \leq n$, satisfy the following property of order of magnitude.

Proposition 3.18. [23] Let $A \in \mathcal{M}_n(E)$ be a reduced square matrix of order $n$. Suppose that $\Delta$ is zeroless. Then for each $j \in \{1, \ldots, n\}$ there exists $i \in \{1, \ldots, n\}$ such that $|\Delta_{i,j}| > \Theta \Delta$. 

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4. The Gauss-Jordan procedure and its solution sets

We recall the definition of flexible systems of linear equations of \([13]\) in slightly modified form, and show equivalence with the earlier definition. We present simple examples illustrating the difference with ordinary systems of linear equations. For the particular case of square non-singular systems we define a notion of stability, meaning that the Gauss-Jordan operations give rise to at most a moderate increase of imprecisions. At the end we state solution formulas for stable systems.

**Definition 4.1.** Let \(n \in \mathbb{N}\) be standard and \(\xi_1, \ldots, \xi_n\) be external numbers. Then \(\xi \equiv (\xi_1, \ldots, \xi_n)^T\) is called an external vector. For \(1 \leq i \leq n\), let \(\xi_i = x_i + X_i\). Then \(x \equiv (x_1, \ldots, x_n)^T\) is called a representative vector and \(X \equiv (X_1, \ldots, X_n)^T\) is called the associated neutrical vector, and we may write \(\xi = x + X\).

**Definition 4.2.** A flexible system is a system of inclusions

\[
\begin{align*}
\alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n & \subseteq \beta_1 \\
\vdots & \vdots \\
\alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n & \subseteq \beta_m
\end{align*}
\]

where \(m, n\) are standard natural numbers, \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(\alpha_{ij} \equiv a_{ij} + A_{ij}\) and \(\beta_i \equiv b_i + B_i\) are external numbers for \(1 \leq i \leq m\) and \(1 \leq j \leq n\). We denote the matrix \([\alpha_{ij}]_{m \times n}\) by \(A\) and the external vector \((\beta_1, \ldots, \beta_n)^T\) by \(B\). The matrix \(P \equiv [a_{ij}]_{m \times n}\) will be called a representative matrix and the matrix \(A \equiv [A_{ij}]_{m \times n}\) will be called the associated neutrical matrix. We define \(|B| = \max_{1 \leq i \leq n} |\beta_i|\) and \(\underline{B} = \min_{1 \leq i \leq n} B_i\). We write \(\Delta = \det(A)\).

The system (7) is equivalent with the inclusion \(Ax \subseteq B\), and usually is written in the matrix form

\[A|B\].

**Definition 4.3 (\([13]\)).** A vector \(x\) is called an admissible solution of the flexible system \(A|B\) if it satisfies the system. The solution \(S\) of \(A|B\) is the (external) set of all admissible solutions. If the Minkowski product \(AS\) satisfies \(AS = B\) we call \(S\) exact.
Remark 4.4. In [13] and [22] flexible systems with variables in the form of external numbers have been considered, i.e. systems of the form

\[
\begin{cases}
\alpha_{11}\xi_1 + \alpha_{12}\xi_2 + \cdots + \alpha_{1n}\xi_n \subseteq \beta_1 \\
\vdots \\
\alpha_{m1}\xi_1 + \alpha_{m2}\xi_2 + \cdots + \alpha_{mn}\xi_n \subseteq \beta_m
\end{cases}
\]  

(8)

an admissible solution being an external vector \( \xi \) satisfying the system.

In the present article we study only the system (7) with real variables, because it facilitates some calculations. In a sense the systems (8) and (7) are equivalent, because of Proposition 4.5.

Proposition 4.5. An external vector \( \xi \) is an admissible solution of (8) if and only if every representative \( x \) of \( \xi \) is an admissible solution of (7).

Proof. Let \( \xi = (\xi_1, \ldots, \xi_n) \). It is obvious that if the inclusions of (8) are satisfied by \( (\xi_1, \ldots, \xi_n) \), they are also satisfied by any representative vector \( x = (x_1, \ldots, x_n) \).

Conversely, let \( 1 \leq i \leq m \), and assume that \( \alpha_{11}x_1 + \cdots + \alpha_{in}x_n \subseteq \beta_i \) whenever \( x_1 \in \xi_1, \ldots, x_n \in \xi_n \). Let \( t \in \tau \equiv \alpha_{11}\xi_1 + \cdots + \alpha_{in}\xi_n \). It follows from the definition of the Minkowski operations that for all \( j \) with \( 1 \leq j \leq n \) there exist \( a_{ij} \in \alpha_{ij} \) and \( x_j \in \xi_j \) such that \( t = a_{i1}x_1 + \cdots + a_{in}x_n \). Then \( t \in \beta_i \). Hence \( \tau \subseteq \beta_i \).

We conclude that the external vector \( \xi \) is an admissible solution of (8) if and only if all its representative vectors are. \( \square \)

Not always it is possible to obtain exact solutions. A simple example is given by the equation \( \otimes \xi \subseteq \mathcal{L} \). The associated equation \( \otimes x \subseteq \mathcal{L} \) has only limited solutions and \( \xi = \mathcal{L} = \{ x \in \mathbb{R} | x \text{ limited} \} \) satisfies the original equation with the strict inclusion \( \otimes \mathcal{L} = \otimes \subseteq \mathcal{L} \). Due to the appearance of imprecisions, systems of full rank may have no solution at all, for example the equation \( (1 + \otimes)x \subseteq 1 + \varepsilon \mathcal{L} \), with \( \varepsilon \simeq 0 \). Also the solution in the sense of Definition 7 is not always an external vector in the sense of Definition 4.1. This is illustrated by the next example.

Example 4.6. Consider the flexible system

\[
\begin{cases}
(1 + \otimes)x + (1 + \varepsilon \otimes)y \subseteq \otimes \\
(1 + \varepsilon \mathcal{L})x - (1 + \varepsilon \mathcal{L})y \subseteq \varepsilon \mathcal{L}.
\end{cases}
\]  

(9)
As shown in [22] the solution is given by the neutrix

$$N = \bigotimes \left( \frac{1}{2} \right) + \varepsilon \mathcal{L} \left( \frac{1}{2} \right),$$

(10)

which is not a neutricing vector.

Classically the solution of a system of linear equations is the sum of a vector and a linear space. Next theorem shows that the solution of the flexible system (7) is the sum of a real vector and a neutrix.

**Theorem 4.7.** Let $S$ be the solution of the system $\mathcal{A} | \mathcal{B}$, assumed to be non-empty. Then there is a real vector $x$ and a neutrix $N \subseteq \mathbb{R}^n$ such that $S = x + N$.

**Proof.** Let $x, y \in S$ be two real admissible solutions, and $z = x - y$. Using subdistributivity we obtain

$$\mathcal{A}z = \mathcal{A}(x - y) \subseteq \mathcal{A}x - \mathcal{A}y \subseteq \mathcal{B} - \mathcal{B} = N(\mathcal{B}).$$

(11)

If $z' \in \mathbb{R}^n$ satisfies (11), again by subdistributivity we have $\mathcal{A}(z + z') \subseteq \mathcal{A}z + \mathcal{A}z' \subseteq N(\mathcal{B}) + N(\mathcal{B}) = N(\mathcal{B})$, and if $k$ is limited, also $kz$ satisfies (11), for $kz = k\mathcal{A}z \subseteq kN(\mathcal{B}) = N(\mathcal{B})$. Hence $N \equiv \{ z \in \mathbb{R}^n | \mathcal{A}z \subseteq N(\mathcal{B}) \}$ is a neutrix. From this we derive that $S = x + N$. \(

The neutrix $N$ of Theorem 4.7 is the direct sum, with respect to an orthonormal basis, of (at most) $n$ scalar neutrices, as shown in [1]. This means that an appropriate rotation turns the solution into an external vector; for the neutrix (10) we may take a rotation over $\pi/4$.

In the present article we study systems with square reduced non-singular matrices. We introduce some useful notions for these matrices.

**Definition 4.8.** Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{E})$ be reduced and non-singular. We say that $\mathcal{A}$ is properly arranged if it has a properly arranged matrix $P$ of representatives. We then say that $\mathcal{A}$ is properly arranged with respect to $P$.

It follows from Proposition 2.3 that up to changing rows and columns, one may suppose that the non-singular matrix $\mathcal{A}$ has a properly arranged diagonally eliminable representative matrix $P$. The following example shows that $P$ does not need to be reduced. Let $\varepsilon \simeq 0, \varepsilon > 0$. The reduced matrix of external numbers

$$\mathcal{A} \equiv \begin{bmatrix} 1 + \bigotimes & 1 + \varepsilon \\ 0 & 1 \end{bmatrix}$$
is "properly arranged" in the sense of Definition 2.4 but a properly arranged
matrix of representatives \( P = [a_{ij}] \) cannot be reduced. We may remediate to
this, by taking a representative \( \alpha \in \alpha_{11} \) such that \( \alpha \geq |a_{ij}| \) for \( 1 \leq i, j \leq n \),
and transform \( \mathcal{A} \) to the reduced matrix \( \mathcal{A}' \equiv \mathcal{A}/\alpha \) and \( \mathcal{B} \) to \( \mathcal{B}' \equiv \mathcal{B}/\alpha \).
Notice that necessarily \( \alpha \simeq 1 \), so the neutrices of the corresponding system
are unchanged. As a consequence, without restriction of generality we may
assume that the properly arranged representative matrix \( P \) of Definition 4.8
is reduced. The matrix \( P \) is non-singular, for it is a representative matrix
of the nonsingular matrix \( \mathcal{A} \). Hence \( P \) is diagonally eliminable. For \( \mathcal{A} \) to be
diagonally eliminable itself, we will ask that the pivots are zeroless. This is
done Definition 4.8 below. First we introduce some notation.

**Notation 4.9.** Let \( \mathcal{A} = [\alpha_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{E}) \) be reduced, non-singular and
properly arranged with respect to some matrix of representatives \( P \). We write
\( \mathcal{A}^{(0)} = [\alpha_{ij}^{(0)}]_{n \times n} = \mathcal{A}, \mathcal{A}^{(0)} = [A_{ij}^{(0)}]_{n \times n} = A, B^{(0)} = B \) and \( [B]^{(0)} = [B^{-1}]^{(0)} = [B] \). For \( 1 \leq q \leq 2n \) we write

\[
\mathcal{A}^{(q)} = \mathcal{G}_q^\mathcal{P} (\mathcal{G}_{q-1}^\mathcal{P} (\cdots (\mathcal{G}_1^\mathcal{P} (\mathcal{A})) \cdots )) \equiv [\alpha_{ij}^{(q)}]_{n \times n} \\
\equiv [a_{ij}^{(q)} + A_{ij}^{(q)}]_{n \times n} = [a_{ij}^{(q)}]_{n \times n} + A^{(q)}
\]

\[
\overline{A^{(q)}} = \max_{1 \leq i \leq j \leq n} |a_{ij}^{(q)}|
\]

\[
\mathcal{B}^{(q)} = \mathcal{G}_q^\mathcal{P} (\mathcal{G}_{q-1}^\mathcal{P} (\cdots (\mathcal{G}_1^\mathcal{P} (\mathcal{B})) \cdots )) \equiv [\beta_1^{(q)}, \cdots , \beta_n^{(q)}]^T
\]

\[
|\overline{\beta^{(q)}}| = \max_{1 \leq i \leq n} |\beta_i^{(q)}|
\]

\[
[B]^{(q)} = \mathcal{G}_q^\mathcal{P} (\mathcal{G}_{q-1}^\mathcal{P} (\cdots (\mathcal{G}_1^\mathcal{P} [B]) \cdots ))
\]

\[
[B^{-1}]^{(q)} = \left( (\mathcal{G}_q^\mathcal{P})^{-1} ((\mathcal{G}_{q-1}^\mathcal{P})^{-1} (\cdots (\mathcal{G}_1^\mathcal{P} [B])^{-1} \cdots )) \right),
\]

where \( [B] = [B, B, \ldots , B]^T \).

**Definition 4.10.** Let \( \mathcal{A} \in \mathcal{M}_n(\mathbb{E}) \) be reduced, non-singular and properly
arranged with respect to some matrix of representatives \( P \). We say that \( \mathcal{A} \) is
diagonally eliminable (with respect to \( P \)) if \( \alpha_{k+1k+1}^{(2k)} \) is zeroless for \( 1 \leq k < n \).

We recall the notion of relative uncertainty from [13], and use it to define
stable matrices. Proposition 4.12, to be proved in Section 5, states that if a
reduced stable matrix has a diagonally eliminable matrix of representatives,
it is itself diagonally eliminable.
Definition 4.11. Let $A = [\alpha_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{E})$ be a limited non-singular matrix. The relative uncertainty $R(A)$ is defined by $R(A) = \overline{A}/\Delta$. The matrix $A$ is called stable if $R(A) \subseteq \varnothing$.

The biggest neutrix $\overline{A}$ occurring in a limited matrix $A$ is always contained in $\varnothing$, but if $\Delta$ is infinitesimal, for the matrix to be stable, the entries need to be sharper.

Proposition 4.12. Let $A \in \mathcal{M}_n(\mathbb{E})$ be reduced, non-singular, properly arranged and stable. Then $A$ is diagonally eliminable.

We now carry over some of the above notions to systems of equations, in particular we define a notion of stability for systems. We first introduce some additional notions and recall some results on solution by Cramer’s rule of [13].

Definition 4.13. The system $A|B$ is called reduced if $A$ is reduced, homogeneous if $B$ is a neutrix vector, upper homogeneous if $\overline{B}$ is a neutrix and uniform if the neutrices of the right-hand side $B_i \equiv B$ are all the same. When $m = n$, the system is called non-singular if $A$ is non-singular, properly arranged respectively diagonally eliminable (with respect to a matrix of representatives $P$) if $A$ is properly arranged respectively diagonally eliminable with respect to $P$.

Definition 4.14. Let $B = (\beta_1, \ldots, \beta_n)^T$ be an external vector. If $\overline{\beta}$ is zero-less, its relative precision $R(B)$ is defined by$$R(B) = \frac{B}{\overline{\beta}}.$$In the special case that $\overline{\beta} = B$ for some neutrix $B$, we define$$R(B) = \frac{B}{B} : B.$$Observe that, whenever $1 \leq i \leq n$, it holds that$$R(B)\beta_i \subseteq B,$$and that $R(B) \subseteq \varnothing$ if the system is not upper homogeneous.

Definition 4.15. Consider the system $A|B$ with $A \in \mathcal{M}_n(\mathbb{E})$ non-singular. Let $\Delta = \det(A) \equiv d + D$. For $1 \leq i \leq n$, let $M_i$ be the matrix obtained
from $\mathcal{A}$ by the substitution of the $i^{th}$ column by the right-hand member $\mathcal{B}$ and $M_i(b)$ the matrix obtained from $\mathcal{A}$ by the substitution of the $i^{th}$ column by a representative vector $b$ of $\mathcal{B}$. We write

$$\xi(b, d)^T = \left(\frac{\det(M_1(b))}{d}, \ldots, \frac{\det(M_n(b))}{d}\right)^T,$$

$$\xi(b)^T = \left(\frac{\det(M_1(b))}{\Delta}, \ldots, \frac{\det(M_n(b))}{\Delta}\right)^T,$$

and

$$\xi^T = \left(\frac{\det(M_1)}{\Delta}, \ldots, \frac{\det(M_n)}{\Delta}\right)^T. \quad (13)$$

The external vector $\xi$ is called the Cramer-solution if every representative vector $x$ satisfies $\mathcal{A}|\mathcal{B}$.

We recall a theorem on the solution of non upper homogeneous systems by Cramer’s rule from [13], here stated for limited coefficient matrices.

**Theorem 4.16.** Assume $\mathcal{A} \in M_n(\mathbb{E})$ is limited and non-singular and the system $\mathcal{A}|\mathcal{B}$ is not upper homogeneous.

1. If $R(\mathcal{A}) \subseteq R(\mathcal{B})$, for every $d \in D$ and representative vector $b$ of $\mathcal{B}$ the external vector $\xi(b, d)^T$ is an admissible solution.

2. If $R(\mathcal{A}) \subseteq R(\mathcal{B})$ and $\Delta$ is not an absorber of $\mathcal{B}$, the Cramer-solution is given by the external vector $\xi(b)^T$, whenever $b$ is a representative vector of $\mathcal{B}$.

3. If the system $\mathcal{A}|\mathcal{B}$ is uniform, $R(\mathcal{A}) \subseteq R(\mathcal{B})$, and $\Delta$ is not an absorber of $\mathcal{B}$, the external vector $\xi^T$ is its Cramer-solution, and as such it is maximal.

The condition $R(\mathcal{A}) \subseteq R(\mathcal{B})$ expresses that in a sense the coefficient matrix is more precise than the right-hand member, and then by Theorem [13.1611] we get a non-empty part of the solution; for example, if we consider the equation $\alpha x \subseteq \beta$ with $\alpha, \beta \in \mathbb{E}$ zeroless and $R(\alpha) \subseteq R(\beta)$, we will get $b/a$ for some $a \in \alpha$ and $b \in \beta$ instead of the whole solution $\beta/\alpha$. The condition cannot be substituted by $\overrightarrow{A} \subseteq B$. Indeed, the equation

$$(1 + \circ)x \subseteq \omega + \mathcal{E}$$
has no admissible solution for unlimited \( \omega \in \mathbb{R} \).

In line with the conditions of Theorem 4.16, we introduce the notion of stable systems. In case the system is uniform, we show that formula (13) is valid also in the homogeneous case, and that we find the same solution when applying the Gauss-Jordan operations in the form of the matrices \( G_h \) of Definition 2.6. The fact that the operations are carried out with precise, real numbers may be seen to correspond to the numerical practice of choosing relatively simple coefficients to effectuate these operations. It makes the Gauss-Jordan procedure also more operational for, as suggested by Propositions 3.15 and 3.16, in the context of the systems with real variables (7) we have more chance that matrix multiplication respects (sub)associativity. If the coefficient matrix is properly arranged, the notion of stability is respected by the steps of the Gauss-Jordan procedure, which do not lead to a significative increase of imprecisions in the coefficient matrix and leave the imprecision of the right-hand member invariant, finally resulting in the same imprecision for the solution. Observe that the successive choice of the biggest minors reflects the numerical strategy of choosing pivots as large as possible.

**Definition 4.17.** Let \( A \in \mathcal{M}_n(\mathbb{E}) \) be limited and non-singular. The system \( A|B \) is said to be stable if

1. \( A \) is stable.
2. \( R(A) \subseteq R(B) \).
3. \( \Delta \) is not an absorber of \( B \).

Condition (3) expresses that the determinant \( \Delta \), which of course must be non-zero, should not be too small.

**Convention 4.18.** From now on we always suppose that the system \( A|B \) is square, i.e. \( A \in \mathcal{M}_n(\mathbb{E}) \), and that the system is non-singular, reduced, properly arranged with respect to a reduced matrix of representatives and uniform.

As for non-singular systems, only the condition of uniformity is restrictive. It is an essential condition, as illustrated by the following simple example.

**Example 4.19.** Consider the system (9) of Example 4.6. Assume we subtract the first equation from the second. Then we get
\[
\begin{aligned}
(1 + \otimes)x + (1 + \varepsilon\otimes)y & \subseteq \otimes \\
\otimes x - 2(1 + \varepsilon\ell)y & \subseteq \otimes.
\end{aligned}
\]

The obvious solution is the neutrical vector \( K \equiv (\otimes, \otimes) \). However the representative vector \((0, \sqrt{\varepsilon})\) does not satisfy the second equation of (9), and \( N \subseteq K \), with \( N \) the exact solution given by (10).

Observe that if the neutrices \( B_1, \ldots, B_n \) of the right-hand member of the system \( A|B \) are not the same, one could substitute them by \( B \); its solutions automatically satisfy the inclusions of the original system. The exact approach of the already mentioned article [22] allows for different neutrices in the right-hand side, as in equation (9). Notice that an upper homogeneous uniform system is homogeneous.

**Definition 4.20.** Assume the system \( A|B \) is properly arranged with respect to a representative matrix \( P \) of \( A \). The Gauss-Jordan solution \( G^P \) of \( A|B \) with respect to \( P \) is defined by

\[
G^P = \{ x \in \mathbb{R}^n | (G^P(A))x \subseteq G^P(B) \}.
\]

If \( G^P \) does not depend on the choice of \( P \), we simply speak of the Gauss-Jordan solution, denoted by \( G \).

Still assuming Convention 4.18 we now state the principal results for stable systems, which we will summarize into one main theorem.

The first theorem states that the Gauss-Jordan operation does not alter the solution set of a stable system.

**Theorem 4.21.** Suppose that the system \( A|B \) is stable. Then the Gauss-Jordan solution \( G \) is well-defined, and a real vector \( x \) is an admissible solution if and only if \( x \in G \).

The second theorem extends Theorem 4.16.3 on Cramer’s Rule to homogeneous systems.

**Theorem 4.22.** If the system \( A|B \) is stable, its solution is given by the external vector (13).

The third theorem states that the solutions given by Theorem 4.21 and Theorem 4.22 are equal.
Theorem 4.23. Assume that the system $\mathcal{A}|\mathcal{B}$ is stable. Then the Gauss-Jordan solution is equal to the Cramer-solution.

The fourth theorem gives again an explicit expression for the solution, now as a result of the Gauss-Jordan procedure applied to the right-hand member. To this end we may choose any representative matrix $P$ of the coefficient matrix, provided it is reduced and properly arranged.

Theorem 4.24. Suppose that the system $\mathcal{A}|\mathcal{B}$ is stable, and properly arranged with respect to a representative matrix $P$ of $\mathcal{A}$. Then $\mathcal{G}^P(\mathcal{B})$ is the Gauss-Jordan solution of $\mathcal{A}|\mathcal{B}$.

Finally we resume the above results in the form of one main theorem.

Theorem 4.25 (Main Theorem). Assume the system $\mathcal{A}|\mathcal{B}$ is stable, and properly arranged with respect to a representative matrix $P$ of $\mathcal{A}$. Let $S$ be its solution. Then

$$S = G = \mathcal{G}^P(\mathcal{B}) = \begin{pmatrix} \frac{\det(M_1)}{\Delta}, & \ldots, & \frac{\det(M_n)}{\Delta} \end{pmatrix}^T. \quad (15)$$

5. Stability and Gauss-Jordan operations

We show that the Gauss-Jordan operations transform a stable system into a stable system, while the neutrix part of the right-hand member remains unchanged. Our principal tools are properties of the order of magnitude of entries, minors and neutrix parts of the intermediate matrices.

Remark 5.1. We recall from the previous section that a reduced matrix $\mathcal{A}$ has always a reduced representative matrix, and from now on we always suppose that a representative matrix is reduced.

The first property is valid without assuming stability.

Proposition 5.2. Let $\mathcal{A} = [a_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{E})$ be a reduced non-singular matrix, such that it admits a properly arranged representative matrix $P$. Then $a_{ij}^{(q)}$ is limited whenever $1 \leq q \leq 2n$ and $1 \leq i, j \leq n$.

Proof. We apply external induction. Because $P$ is reduced, it holds that $|a_{ij}| \leq 1$ for $1 \leq i, j \leq n$, and since $a_{ij}^{(1)} = a_{ij}$ the same is true for $|a_{ij}^{(1)}|$. 26
It follows that $|a_{ij}^{(2)}| = |a_{ij}| \leq 1$ for $1 \leq j \leq n$ and $|a_{ij}^{(2)}| = |a_{ij} - a_{i1}a_{1j}| \leq |a_{ij}| + |a_{i1}| |a_{1j}| \leq 2$ for $2 \leq i \leq n, 1 \leq j \leq n$. Hence $a_{ij}^{(2)}$ is limited for $1 \leq i, j \leq n$. As for the induction step, let $k \leq n - 1$ and suppose that $a_{ij}^{(q)}$ is limited for $q \leq 2k$ and $1 \leq i, j \leq n$. Because the $j^{th}$ column of $P^{(2k+1)}$ is a unit vector for $1 \leq j \leq k$, the entries of these columns are limited. For $1 \leq i \leq n, k+1 \leq j \leq n$ one has

$$a_{ij}^{(2k+1)} = \begin{cases} a_{ij}^{(2k)} & \text{if } i \neq k+1 \\ \frac{a_{ij}^{(2k)}}{m_{k+1}^{1...k_i}} & \text{if } i = k+1. \end{cases}$$

So $a_{ij}^{(2k+1)} = a_{ij}^{(2k)}$ is limited for $i \neq k+1$ and $k+1 \leq j \leq n$ by the induction hypothesis, and because $P$ is properly arranged, also for $i = k+1$ and $k+1 \leq j \leq n$, since $|a_{ij}^{(2k+1)}| \leq \left| \frac{m_{k+1}^{1...k_i}}{m_{k+1}} \right| \leq 1$. Combining, we see that $a_{ij}^{(2k+1)}$ is limited for $1 \leq i, j \leq n$.

As for $P^{(2k+2)}$, in addition to the first $k$ columns, also the $(k+1)^{th}$ column is a unit vector, i.e. has limited components. Because the elements of $P^{(2k+1)}$ are limited we derive that $a_{ij}^{(2k+2)} = a_{ij}^{(2k+1)}$ is limited for $k+2 \leq j \leq n$, and $a_{ij}^{(2k+2)} = a_{ij}^{(2k+1)} - a_{ik+1}^{(2k+1)} a_{k+1j}^{(2k+1)}$ is limited for $1 \leq i \leq n, i \neq k+1$ and $k+1 \leq j \leq n$. Hence $a_{ij}^{(2k+2)}$ is limited for $1 \leq i, j \leq n$.

The next proposition implies that the pivots in the Gauss-Jordan procedure with respect to a reduced properly arranged matrix of representatives remain limited, and the neutrices of all entries below and to the right of the pivot $a_{k+1k+1}^{(2k)}$ of the intermediate matrices $A^{(2k)}$ do not exceed the biggest neutrix of the original matrix. The latter bound is a consequence of formula (1). This formula does not hold everywhere, and in the remaining part of the matrices we have larger bounds for the neutrices, as will be shown in Proposition 5.9.

**Proposition 5.3.** Let $A = [a_{ij}]_{n \times n} \in M_n(\mathbb{E})$ be a reduced non-singular matrix, such that it admits a properly arranged representative matrix $P$.

1. $m_{k+1}/m_k$ is limited whenever $1 \leq k \leq n - 1$.

2. For $1 \leq k < n$ and $k+1 \leq i, j \leq n$

$$N\left( \alpha_{ij}^{(2k)} \right) \subseteq \overline{A}. \quad (16)$$

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Proof. By Proposition 2.8 we have \( m_{k+1}/m_k = a_{k+1,k+1}^{(2k+2)} \), and the latter is limited by Proposition 5.2.

First, observe that because \( a_{11} = 1 \) for \( 2 \leq i, j \leq n \),

\[
A^{(2)}_{ij} = A_{ij} + a_{i1}A_{1j}.
\]

(17)

Also, in case \( k \geq 2 \), one has \( A_{kj}^{(2k-1)} = A_{ij}^{(2k-2)} \) for \( k+1 \leq i, j \leq n \). It follows that for \( k+1 \leq i, j \leq n \) one has

\[
A_{ij}^{(2k)} = A_{ij}^{(2k-1)} + a_{ik}A_{kj}^{(2k-1)} = A_{ij}^{(2k-2)} + \frac{a_{ij}^{(2k-2)}}{a_{kk}^{(2k-2)}},
\]

We derive from formula (4) that

\[
A_{ij}^{(2k)} = A_{ij}^{(2k-2)} + \frac{m_{1...k-1i}^{1...k-1k}/m_{k-1}}{m_k} = \frac{m_{1...k-1i}^{1...k-1k}}{m_k}.
\]

Hence

\[
A_{ij}^{(2k)} = A_{ij}^{(2k-2)} + \frac{m_{1...k-1i}^{1...k-1k}}{m_k}A_{k,j}^{(2k-2)}
\]

(18)

for \( k+1 \leq i, j \leq n \).

We will now prove (16) by external induction. For \( 2 \leq i, j \leq n \) we have by (17)

\[
A_{ij}^{(2)} = A_{ij} + a_{i1}A_{1j} \subseteq \overline{A} + \overline{A} = \overline{A}.
\]

Suppose that \( A_{ij}^{(2k-2)} \subseteq \overline{A} \) for all \( i, j \) such that \( k \leq i, j \leq n \). Because \( A \) is properly arranged, for \( i, j \geq k+1 \) it holds that

\[
\left| \frac{m_{1...k-1i}^{1...k-1k}}{m_k} \right| \leq 1,
\]

hence by (18)

\[
A_{ij}^{(2k)} = A_{ij}^{(2k-2)} + \frac{m_{1...k-1i}^{1...k-1k}}{m_k}A_{k,j}^{(2k-2)} \subseteq \overline{A} + \overline{A} = \overline{A}.
\]

\[\square\]

To prove Proposition 4.12 saying that a stable reduced non-singular properly arranged matrix is diagonally eliminable, we present first some notation and some auxiliary results on the order of magnitude of minors.

Notation 5.4. Let \( A = [a_{ij}]_{n \times n} \in \mathcal{M}_n(E) \) be a reduced non-singular matrix, such that it admits a properly arranged representative matrix \( P = [a_{ij}]_{n \times n} \), and let \( 1 \leq q \leq 2n \). We write \( d = \det(P) \), \( d^{(q)} = \det(P^{(q)}) \) and \( \Delta^{(q)} = \det(A^{(q)}) = d^{(q)} + D^{(q)} \).
Lemma 5.5. Let $A = [\alpha_{ij}]_{n \times n} \in M_n(\mathbb{E})$ be a reduced non-singular matrix, such that it admits a properly arranged representative matrix $P$. Let $1 \leq q \leq 2n$. Let $k$ be such that $q = 2k - 1$ or $q = 2k$. Then $|d^{(q)}| = \left| \frac{d}{m_k} \right| > \varnothing \Delta$.

Proof. Let $1 \leq k \leq n$ and $q = 2k - 1$ or $q = 2k$. In both cases

$$|d^{(q)}| = |\det(G_q) \det(G_{q-1}) \cdots \det(G_1)d|$$

(19)

Suppose $|d^{(q)}| \subseteq \varnothing \Delta$. Then $d \in m_k \varnothing \Delta$ by (19). By Proposition 3.17 it holds that $d \in \varnothing \Delta$. Hence $d \in \varnothing \Delta \cap \Delta$. Because $\Delta$ is zeroless, this contradicts Proposition 3.9.3. Hence $|d^{(q)}| > \varnothing \Delta$. 

Proposition 5.6. Let $A = [\alpha_{ij}]_{n \times n} \in M_n(\mathbb{E})$ be a reduced non-singular matrix, such that it admits a reduced properly arranged representative matrix $P$.

1. Let $1 \leq k \leq n - 1$. If $\alpha_{11}, \alpha_{22}, \ldots, \alpha_{kk}^{(2k)}$ is zeroless, then $|\frac{m_{k+1}}{m_k}| > \varnothing \Delta$.

2. If $A$ is stable, then $|\frac{m_{k+1}}{m_k}| > \varnothing \Delta$ and $\alpha_{kk}^{(2k)}$ is zeroless for all $k$ such that $1 \leq k \leq n - 1$.

Proof. For $1 \leq k \leq n - 1$ we have

$$A^{(2k)} = \begin{bmatrix}
1 + A_{11} & A_{12}^{(2k)} & \cdots & A_{1k}^{(2k)} & \alpha_{1(2k)}^{(2k)} & \cdots & \alpha_{1n}^{(2k)} \\
A_{21}^{(2k)} & 1 + A_{22}^{(2k)} & \cdots & \alpha_{2k}^{(2k)} & \alpha_{2(2k)}^{(2k)} & \cdots & \alpha_{2n}^{(2k)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{k1}^{(2k)} & A_{k2}^{(2k)} & \cdots & 1 + A_{kk}^{(2k)} & \alpha_{k(k+1)}^{(2k)} & \cdots & \alpha_{kn}^{(2k)} \\
A_{(k+1)1}^{(2k)} & A_{(k+1)2}^{(2k)} & \cdots & A_{(k+1)k}^{(2k)} & \alpha_{(k+1)(k+1)}^{(2k)} & \cdots & \alpha_{(k+1)n}^{(2k)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{n1}^{(2k)} & A_{n2}^{(2k)} & \cdots & A_{nk}^{(2k)} & \alpha_{n(k+1)}^{(2k)} & \cdots & \alpha_{nn}^{(2k)}
\end{bmatrix}$$
Suppose on contrary that \(\frac{m_{k+1}}{m_k} = a_{k+1,k+1}^{(2k)} = \frac{m_{k+1}}{m_k} \in \emptyset \Delta\). From \(|a_{ij}^{(2k)}| = \frac{|m_{i...k_j}^{1...k_i}|}{m_k} \leq |a_{k+1,k+1}^{(2k)}|\) for \(k+1 \leq i, j \leq n\) one derives that \(a_{ij}^{(2k)} \in \emptyset \Delta\) for \(k+1 \leq i, j \leq n\). Let \(S_{n-k}\) be the set of all permutations of \(\{k+1, \ldots, n\}\). Then because \(\Delta\) is limited, \(d^{(2k)} = \sum_{\sigma \in S_{n-k}} \text{sgn}(\sigma)a_{k+1,k+1}^{(2k)} \cdot \ldots \cdot a_{n \sigma(n)}^{(2k)} \in (\emptyset \Delta)^{n-k} \subseteq \emptyset \Delta\), in contradiction to Lemma 5.5. Hence \(\frac{|m_{k+1}}{m_k}| > \emptyset \Delta\).

2. By \(16\) and stability, for \(1 \leq r \leq n-1\)

\[
N(\alpha_{r+1r+1}^{(2r)}) \subseteq \emptyset \Delta. \tag{20}
\]

We use now external induction. Because \(A\) is reduced, the element \(\alpha_{11}\) is zeroless. Then \(\frac{|m_{2}}{m_{1}}| > \emptyset \Delta\) by Part 1. Then \(\alpha_{22}^{(2)}\) is zeroless by \(20\).

Assume that \(\frac{|m_{r+1}}{m_{r}}| > \emptyset \Delta\) for \(1 \leq r \leq k-1\), where \(k \leq n\). Then by \(20\) it holds that \(\alpha_{r+1r+1}^{(2r)} = \frac{m_{r+1}}{m_{r}} + N\left(\alpha_{r+1r+1}^{(2r)}\right)\) is zeroless for \(1 \leq r \leq k-1\).

Then Part 1 implies that \(\frac{|m_{k+1}}{m_{k}}| > \emptyset \Delta\). Also \(N(\alpha_{k+1k+1}^{(2k)}) \subseteq \emptyset \Delta\) by \(20\), hence \(\alpha_{k+1k+1}^{(2k)} = \frac{m_{k+1}}{m_{k}} + N\left(\alpha_{k+1k+1}^{(2k)}\right)\) is zeroless.

**Proof of Proposition 4.12.** By Proposition 5.6.2 it holds that \(\alpha_{k+1k+1}^{(2k)}\) is zeroless for \(1 \leq k \leq n-1\). Hence \(A\) is diagonally eliminable with respect to \(P\).

Theorem 5.7 respectively Theorem 5.8 give bounds on the pivots and the entries of the intermediate matrices of the Gauss-Jordan procedure, and the inverse procedure.

**Theorem 5.7.** Let \(A = [\alpha_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{E})\) be a reduced, non-singular stable matrix, which is properly arranged with respect to a matrix of representatives \(P\). Then for \(1 \leq k < n\)

\[
\emptyset \Delta < \left| \frac{m_{k+1}}{m_{k}} \right| \in \mathcal{L}. \tag{21}
\]
and

\[ \Theta < \left| \frac{m_k}{m_{k+1}} \right| \in \frac{\mathcal{E}}{\Delta}, \quad (22) \]

**Proof.** Formula (21) follows from Part 1 of Proposition 5.3 and Part 2 of Proposition 5.6. Taking multiplicative inverses, we derive (22). \( \square \)

**Theorem 5.8.** Let \( \mathcal{A} = [\alpha_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{E}) \) be a reduced, non-singular stable matrix, which is properly arranged with respect to a matrix of representatives \( P \).

1. Let \( 1 \leq k < n \). Then the \( k^{th} \) diagonal element of \( \mathcal{G}_P^{2k+1} \) satisfies \( g_{kk}^{(2k+1)} \in \frac{\mathcal{E}}{\Delta} \), and the elements of \( \mathcal{G}_P^{2k+2} \) are all limited.

2. All elements of the matrices \( (\mathcal{G}^{-1})_q^P, 1 \leq q \leq 2n \) of the inverse Gauss-Jordan procedure are limited.

**Proof.**

1. The property is a direct consequence of Theorem 5.7 and Proposition 5.2.

2. For the intermediate matrices of odd index the property follows from (21), and for the intermediate matrices of even index \( q = 2k, k < n \) the property follows from the fact that \( \left| (g^{-1})_{ik+1}^{(2k+2)} \right| = \left| g_{ik+1}^{(2k+2)} \right| \) for \( 1 \leq i \leq n, 1 \leq k \leq n - 1 \), and Part 1. \( \square \)

With the help of Theorem 5.8 we derive for each Gauss-Jordan operation an overall bound for the increase of the neutrix parts of the coefficient matrix of a stable system. The bound is bigger than the bound valid for general systems of Proposition 5.3, but the latter bound only holds for the submatrix below to the right of the pivot; this is due to the fact that, the matrix being properly arranged, only fractions of \( \mathcal{A} \) have been added to the original neutrices. The overall bound will be used to derive bounds for the neutrices of the determinants of the coefficient matrices occurring in the Gauss-Jordan procedure, implying that the matrices remain non-singular.

**Proposition 5.9.** Let \( \mathcal{A} \in \mathcal{M}_n(\mathbb{E}) \) be a reduced, non-singular stable matrix, which is properly arranged with respect to a matrix of representatives \( P \). Then for all \( k \) such that \( 1 \leq k \leq n \),

\[ \overline{A^{(2k)}} = \overline{A^{(2k-1)}} \subseteq \frac{\overline{A}}{m_k}, \]

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Proof. We will apply external induction. For \( k = 1 \), because \( m_1 = a_{11} = 1 \), one has \( A^{(2k-1)} = A^{(1)} = \overline{A} = \overline{A}/m_1 \). By Part I of Theorem 5.8 it holds that \( g_{ij}^{(2k)} = g_{ij}^{(2)} \) is limited for \( 1 \leq i, j \leq n \), hence \( A^{(2)} = A^{(1)} = \overline{A}/m_1 \).

As for the induction step, let \( k < n \) and suppose that \( A^{(2k-1)} = A^{(2k)} \subseteq \overline{A}/m_k \). Then \( A^{(2k+1)} \subseteq \frac{m_k}{m_{k+1}} A^{(2k)} \subseteq \frac{m_k}{m_{k+1}} \overline{A} = \overline{A}/m_{k+1} \). Again, by Part I of Theorem 5.8 one has \( A^{(2k+2)} = A^{(2k+1)} = \overline{A}/m_{k+1} \). \( \square \)

Proposition 5.10. Let \( A = [\alpha_{ij}]_{n \times n} \in M_n(\mathbb{E}) \) be a reduced, non-singular stable matrix, which is properly arranged with respect to a matrix of representatives \( P \). Let \( 1 \leq q \leq 2n \). Then

1. \( \Delta^{(q)} \) is zeroless.
2. \( \varnothing \Delta < \Delta^{(q)} \subseteq \mathcal{L} \).
3. \( \overline{A^{(q)}} \subseteq \varnothing \Delta^{(q)} \subseteq \varnothing \).
4. \( A^{(q)} \) is limited, non-singular and stable.

Proof. 1. Let \( q = 2k \) or \( q = 2k - 1 \) with \( 1 \leq k \leq n \). By Lemma 5.5, one has \( |d^{(q)}| = \left| \frac{d}{m_k} \right| \). Because the matrix is non-singular and stable, it holds that \( \overline{A} \subseteq \varnothing \Delta < |d| \), and because it is also reduced, it follows from Proposition 5.9 that \( \overline{A^{(q)}} \subseteq \overline{A}/m_k \). Hence \( |d^{(q)}| > A^{(q)} \). Also \( D^{(q)} \subseteq \overline{A^{(q)}} \) by Proposition 5.2 and Proposition 3.17. Hence \( \Delta^{(q)} \) is zeroless.

2. We show first that \( D^{(q)} \subseteq \varnothing \). Indeed, suppose \( \varnothing \subseteq D^{(q)} \). Then \( \mathcal{L} \subseteq D^{(q)} \). By Proposition 3.17 and Proposition 5.2 it holds that \( d^{(q)} \) is limited. This implies that \( \Delta^{(q)} \) is a neutrix, in contradiction to Part I. Hence \( D^{(q)} \subseteq \varnothing \), which implies that \( \Delta^{(q)} = d^{(q)} + D^{(q)} \subseteq \mathcal{L} \). It also follows from Part I that \( \Delta^{(q)} \subseteq (1 + \varnothing)d^{(q)} \). Now \( d^{(q)} > \varnothing \Delta \) by Lemma 5.5, hence also \( \Delta^{(q)} > \varnothing \Delta \).

3. Let \( 1 \leq q \leq 2n \). Then \( q = 2k \) or \( q = 2k - 1 \) with \( 1 \leq k \leq n \). By Proposition 5.9 the stability of the matrix \( A \), and Lemma 5.5, one has
\[
\overline{A^{(q)}} \subseteq \overline{A}/m_k \subseteq \varnothing d/m_k = \varnothing d^{(q)} = \varnothing \Delta^{(q)}.
\]
Then Part 2 implies that $A^{(q)} \subseteq \emptyset$.

By Proposition 5.2 the matrix $A^{(q)}$ is limited. By Part 1 the matrix $A^{(q)}$ is non-singular. Then $A^{(q)}$ is stable by Part 3. 

**Theorem 5.11.** Let $A \in \mathcal{M}_n(\mathbb{E})$ be a reduced, non-singular stable matrix, which is properly arranged with respect to a matrix of representatives $P$. Then $G^P(A)$ is a near-identity matrix.

**Proof.** Let $A$ be the associated neutrical matrix of $A$. By Proposition 3.14 we have $G^P(A) = G^P(P) + G^P(A) = I + A'$, where $A' = [A'_{ij}]_{n \times n}$ is a neutrical matrix. By Part 3 of Proposition 5.10 one has $A' \subseteq [\emptyset]_{n \times n}$. Hence $G^P(A)$ is a near-identity matrix. 

We turn now to the effects of the Gauss-Jordan procedure to the right-hand member of the system $A|B$. We always assume that the system satisfies Convention 1.

In contrast to the possible increase of the neutrix parts of the coefficient matrix of a stable system, the pivots of the Gauss-Jordan procedure do not give rise to changes in its neutrix part, and the same is true for the inverse procedure. The invariance of the neutrix part of the right-hand member will be a consequence of the next proposition.

**Proposition 5.12.** Suppose that the flexible system $A|B$ is properly arranged with respect to a representative matrix $P$ and stable. Then for all $k$ such that $1 \leq k \leq n-1$

$$\frac{m_{k+1}}{m_k}B = \frac{m_k}{m_{k+1}}B = B.$$ 

**Proof.** Let $1 \leq k \leq n-1$. By formula (21) it holds that $\emptyset \Delta \triangleleft \frac{m_{k+1}}{m_k} \in \mathcal{L}$. The fact that $\Delta$ is not an absorber of $B$ and Proposition 3.9 imply that $\frac{m_{k+1}}{m_k}B = B$. It follows that $\frac{m_k}{m_{k+1}}B = B$ for $1 \leq k \leq n-1$. 

**Theorem 5.13.** Suppose that the flexible system $A|B$ is properly arranged with respect to a representative matrix $P$ and stable. Then for all $q$ such that $1 \leq q \leq 2n$ one has $[B]^{(q)} = [B]$ and $[B^{-1}]^{(q)} = [B]$. In particular $G^P[B] = [B]$ and $(G^P)^{-1}[B] = [B]$.

**Proof.** We will apply External Induction. Because $a_{11} = 1$, we have $[B]^{(1)} = G_1^P[B] = I[B] = [B]$. 

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As for the induction step, let \( q < 2n \) and suppose that \([B]^{(q)} = [B]\). We consider two cases.

Case 1: \( q + 1 = 2k + 1 \) for some \( k \in \{1, \ldots, n-1\} \). By the induction hypothesis and Proposition 5.12 we have

\[
[B]^{(q+1)} = G_{q+1}^P [B]^{(q)} = G_{2k+1}^P [B]
\]

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{m_k}{m_{k+1}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1
\end{bmatrix} \cdot \begin{bmatrix} B \\ \vdots \end{bmatrix} = \begin{bmatrix} B \\ \vdots \end{bmatrix}.
\]

Case 2: \( q + 1 = 2k + 2 \) for some \( k \in \{1, \ldots, n-1\} \). By Theorem 5.8(1) all entries of the matrix \( G_{2k+2}^P \) are limited, and the elements of its diagonal are equal to 1. Then it follows from Case 1 that

\[
[B]^{(q+1)} = G_{q+1}^P [B]^{(q)} = G_{2k+2}^P [B]^{(2k+1)} = G_{2k+2}^P [B] = [B].
\]

In particular \( G^P[B] = [B]^{(2n)} \). This proves the theorem for the Gaussian procedure \( G^P \). The proof for the inverse procedure is similar.

\[\square\]

**Proposition 5.14.** Suppose that the flexible system \( A|B \) is properly arranged with respect to a representative matrix \( P \) and stable. Then

\[
(G^P)^{-1} (G^P(B)) = B.
\]

**Proof.** Let \( B = b + B \). By Proposition 3.14 and Theorem 5.13

\[
(G^P)^{-1} (G^P(B)) = (G^P)^{-1} (G^P(b + B)) = (G^P)^{-1} (G^P(b) + G^P(B))
\]

\[
= (G^P)^{-1} (G^P(b) + B) = (G^P)^{-1} (G^P(b)) + (G^P)^{-1} (B)
\]

\[
= \left((G^P)^{-1} G^P\right)(b) + B = b + B = B.
\]

\[\square\]

Proposition 5.15 shows that the neutrix of the right-hand member is invariant by multiplying and dividing by the determinants \( \Delta \) and \( \Delta^{(q)} \).
Proposition 5.15. For stable systems \( A|B \) it holds that

\[
\Delta B = \frac{B}{\Delta} = B. \tag{23}
\]

Moreover, for \( 1 \leq q \leq 2n \) the determinant \( \Delta^{(q)} \) is not an absorber of \( B \), and

\[
\Delta^{(q)} B = \frac{B}{\Delta^{(q)}} = B. \tag{24}
\]

Proof. It follows from the fact that \( \Delta \) is zeroless and Proposition 3.17 that \( \varnothing \Delta < \Delta \subset \mathcal{L} \). Also \( \Delta \) is not an absorber of \( B \). Then (23) follows from Proposition 3.9.5. By Proposition 5.10.2 we have \( \varnothing \Delta < \Delta^{(q)} \subset \mathcal{L} \). Then also \( \Delta^{(q)} \) is not an absorber of \( B \), hence (24) holds by Proposition 3.9.5.

Proposition 5.16 shows that the neutrical part of the matrices \( A^{(q)} \) do not blow up the neutrix part of the right-hand member \( B \). The proposition is an easy consequence of Proposition 5.10.3.

Proposition 5.16. Suppose that the flexible system \( A|B \) is properly arranged with respect to a representative matrix \( P \) and stable. Let \( A \) be the associated neutrical matrix of \( A \). Then \( A^{(q)}[B] \subseteq [B] \) for all \( q \) such that \( 1 \leq q \leq 2n \).

Proof. Let \( 1 \leq k \leq 2n \). By Part 3 of Proposition 5.10 we have \( A^{(q)} \subseteq [\varnothing]_{n \times n} \). Clearly \( [\varnothing]_{n \times n}[B] \subseteq [B] \).

Theorem 5.17 shows that the Gauss-Jordan operations respect the relation between the relative imprecisions of the coefficient matrix and the right-hand member given by Definition 4.17.2.

Theorem 5.17. Suppose that the flexible system \( A|B \) is properly arranged with respect to a representative matrix \( P \) and stable. Then for \( 0 \leq q \leq 2n \)

\[
R(A^{(q)}) \subseteq R(B^{(q)}). \tag{25}
\]

As a consequence,

\[
R(\mathcal{G}^P A) \subseteq R(\mathcal{G}^P B). \tag{26}
\]

Proof. Because the system is stable, formula (25) holds for \( q = 0 \). Let \( 1 \leq q \leq 2n \). We show by external induction that for \( 0 \leq q \leq 2n \) and \( 1 \leq i, j \leq n \)

\[
A^{(q)}_{ij} B^{(q)} \subseteq B. \tag{27}
\]
For $q = 0$ we have by stability and (12)

$$A_{ij}^{(0)} \beta^{(0)} \subseteq \overline{A \beta} \subseteq \Delta R(A) \beta \subseteq \Delta R(B) \beta \subseteq \Delta B = B.$$ 

Assuming that the property (27) holds for $q$, we will prove it for $q + 1$. Because $\beta^{(q+1)} = \beta_p^{(q+1)}$ for some $p \in \{1, \ldots, n\}$,

$$|\beta^{(q+1)}(1)| = |\beta_p^{(q+1)}| = \left| \sum_{j=1}^{n} g_{pj}^{(q+1)} \beta_j^{(q)} \right| \leq \sum_{j=1}^{n} |g_{pj}^{(q+1)}| \left| \beta_j^{(q)} \right| \leq \sum_{j=1}^{n} |g_{pj}^{(q+1)}| |\beta^{(q)}|.$$ 

Also

$$A_{ij}^{(q+1)} = g_{i1}^{(q+1)} A_{ij}^{(q)} + \cdots + g_{in}^{(q+1)} A_{nj}^{(q)}.$$ (28)

If $q + 1 = 2k + 2$ for some $k \in \{1, \ldots, n - 1\}$, by the induction hypothesis and Theorem 5.8 one has

$$A_{ij}^{(q+1)} \beta^{(q+1)} \subseteq \left( g_{i1}^{(q+1)} A_{ij}^{(q)} + \cdots + g_{in}^{(q+1)} A_{nj}^{(q)} \right) \left( \sum_{j=1}^{n} |g_{ij}^{(q+1)}| |\beta^{(q)}| \right)$$

$$= \left( g_{i1}^{(q+1)} A_{ij}^{(q)} \beta^{(q)} + \cdots + g_{in}^{(q+1)} A_{nj}^{(q)} \beta^{(q)} \right) \left( \sum_{j=1}^{n} |g_{ij}^{(q+1)}| \right)$$

$$\subseteq (LB + \cdots + LB) \beta \subseteq B.$$ 

If $q + 1 = 2k + 1$ for some $k \in \{1, \ldots, n - 1\}$, we consider separately the cases $i \neq k + 1$ and $i = k + 1$.

Case 1: For $i \neq k + 1$ and $1 \leq i \leq n$, the row $g_{i}^{(q+1)}$ is a unit vector, so the neutrices of the $i^{th}$ row of $A^{(q+1)}$ satisfy $A_{ij}^{(q+1)} = A_{ij}^{(q)}$ for $1 \leq j \leq n$. Also

$$\beta^{(q+1)} = \left( \beta_1^{(q)}, \ldots, \beta_k^{(q)} m_k^{(q)}, \beta_{k+1}^{(q)}, \beta_{k+2}^{(q)}, \ldots, \beta_n^{(q)} \right).$$

If $\beta^{(q+1)} = \beta_s^{(q)}$ for some $s \in \{1, \ldots, n\} \setminus \{k + 1\}$, and for $i \neq k + 1, 1 \leq i \leq n$ and $1 \leq j \leq n$ one has by the induction hypothesis

$$A_{ij}^{(q+1)} \beta^{(q+1)} = A_{ij}^{(q)} \beta_s^{(q)} \subseteq A_{ij}^{(q)} \beta^{(q)} \subseteq B.$$ 

If $\beta^{(q+1)} = \frac{m_k}{m_{k+1}} \beta_{k+1}^{(q)}$, then for $i \neq k + 1, 1 \leq i \leq n$ and $1 \leq j \leq n$ it follows from the induction hypothesis and Proposition 5.12 that

$$A_{ij}^{(q+1)} \beta^{(q+1)} = A_{ij}^{(q)} \frac{m_k}{m_{k+1}} \beta_{k+1}^{(q)} \subseteq \frac{m_k}{m_{k+1}} B = B.$$
Case 2: For \( i = k + 1 \), by formula (28) one has for \( 1 \leq j \leq n \)

\[
A_{k+1j}^{(q+1)} = A_{k+1j}^{(q)} \frac{m_k}{m_{k+1}}.
\]

If \( \overline{\beta^{(q+1)}} = \beta_s^{(q)} \) for some \( s \in \{1, \ldots, n\} \setminus \{k + 1\} \), due to Proposition 5.12 one has for \( 1 \leq j \leq n \)

\[
A_{k+1j}^{(q+1)} \beta_{k+1j}^{(q+1)} = \frac{m_k}{m_{k+1}} A_{k+1j}^{(q)} \beta_{k+1j}^{(q)} \subseteq \frac{m_k}{m_{k+1}} B = B.
\]

If \( \overline{\beta^{(q+1)}} = \frac{m_k}{m_{k+1}} \beta_{k+1j}^{(q)} \), again using Proposition 5.12 we find for \( 1 \leq j \leq n \)

\[
A_{k+1j}^{(q+1)} \beta_{k+1j}^{(q+1)} = \frac{m_k}{m_{k+1}} A_{k+1j}^{(q)} \frac{m_k}{m_{k+1}} \beta_{k+1j}^{(q)} \subseteq \left( \frac{m_k}{m_{k+1}} \right)^2 B = B.
\]

Combining, we see that property (27) holds for all \( q \) such that \( 1 \leq q \leq 2n \).

It follows directly from (27) that

\[
\overline{A^{(q)} \beta^{(q)}} \subseteq B. \tag{29}
\]

To finish the proof, we observe first that \( \Delta^{(q)} \) is zeroless by Proposition 5.10, hence \( R(A^{(q)}) \) is well-defined. Also \( B/\Delta^{(q)} = B \) by Proposition 5.15.

We consider separately the cases that \( \overline{\beta^{(q)}} \) is zeroless and that \( \overline{\beta^{(q)}} = B \) is neutricial.

If \( \overline{\beta^{(q)}} \) is zeroless, by Proposition 5.2, Proposition 3.17 and Proposition 5.10 \( \Delta^{(q)} = \det A^{(q)} \) is limited and zeroless. By Lemma 5.5 and formula (29) one has for \( 1 \leq i, j \leq n \)

\[
R(A^{(q)}) = \frac{A^{(q)}}{\Delta^{(q)}} \subseteq \frac{1}{\Delta^{(q)}} \frac{B}{\beta^{(q)}} = \frac{B}{\beta^{(q)}} = R(B^{(q)}).
\]

If \( \overline{\beta^{(q)}} = B \) is neutricial, formula (29) takes the form \( \overline{A^{(q)} B} \subseteq B \). Then

\[
R(A^{(q)}) B = \frac{A^{(q)}}{\Delta^{(q)}} \frac{B}{\beta^{(q)}} = \overline{A^{(q)} B} \subseteq B.
\]

We conclude from Theorem 5.13 that \( R(A^{(q)}) \subseteq B : B = R([B^{(q)}]) = R(B^{(q)}) \).

Formula (26) is obtained by putting \( q = 2n \). \( \square \)
Corollary 5.18. Each Gauss-Jordan operation transforms a stable system into a stable system.

Proof. Let $1 \leq k \leq 2n$. By Proposition 5.10.3 the matrix $A^{(q)}$ is limited, non-singular and stable. By Proposition 5.15 it holds that $\Delta^{(q)}$ is not an absorber of $B$. Then Theorem 5.17 implies that the system $A^{(q)}|B^{(q)}$ is stable. \hfill $\square$

6. Stability and Cramer’s rule

Theorem 4.22 concerning the solution of stable systems by Cramer’s rule was essentially shown in [13] for non-homogeneous systems.

The proof uses estimates for the size of the determinants of the matrices $M_i$ and the neutrix parts of the formulas given by Cramer’s rule. We reprove some of these estimates, now also for homogeneous systems. As before, we suppose that the system $A|B$ satisfies Convention 4.18.

Lemma 6.1. Assume the system $A|B$ is stable. Then for $1 \leq j \leq n$

1. $|\det(M_j)| \leq 2n!|\beta|.$

2. $N(\det(M_j)) \subseteq \beta \cdot A + B.$

Proof. Let $S_n$ be the set of all permutations of $\{1, \ldots, n\}$ and $\sigma \in S_n$. Put

$$\gamma_\sigma = \alpha_{\sigma(1)} \cdots \alpha_{\sigma(j-1)} \alpha_{\sigma(j-1)} \cdots \alpha_{\sigma(n)}.$$

Because the system is reduced,

$$|\gamma_\sigma| \leq \alpha^{n-1} \leq (1 + \odot)^{n-1} = 1 + \odot,$$ (30)

and, as a consequence of Proposition 3.9.4

$$N(\gamma_\sigma) = N \left( \prod_{1 \leq k \leq n, k \neq j} (a_{\sigma(k)}k + A_{\sigma(k)k}) \right) \subseteq N(1 + A)^{n-1} = A.$$ (31)

It follows from (30) that

$$|\det(M_j)| \leq \sum_{\sigma \in S_n} |\gamma_\sigma \beta_\sigma(j)| \leq \sum_{\sigma \in S_n} |(1 + \odot)|\beta|| = n!(1 + \odot)|\beta| = 2n!|\beta|.$$
2. It follows from (31) and (30) that

\[ N\left(\det(M_j)\right) = N\left(\sum_{\sigma \in S_n} \text{sgn}(\sigma) \gamma_{\sigma} \beta_{\sigma(j)} \right) = \sum_{\sigma \in S_n} N(\gamma_{\sigma} \beta_{\sigma(j)}) \]

\[ = \sum_{\sigma \in S_n} (\beta_{\sigma(j)} N(\gamma_{\sigma}) + \gamma_{\sigma} N(\beta_{\sigma(j)})) \]

\[ \subseteq \sum_{\sigma \in S_n} (\beta A + (1 + \varnothing) B) = \beta A + B. \]

\[ \square \]

Proposition 6.2. Assume the system \( A|B \) is stable. Then for \( 1 \leq j \leq n \)

\[ N\left(\frac{\det(M_j)}{\Delta}\right) = B. \]

As a consequence, if the system is homogeneous, for \( 1 \leq j \leq n \)

\[ \frac{\det(M_j)}{\Delta} = B. \]

Proof. Let \( D = N(\Delta) \). By Proposition 3.9.2, Lemma 6.1 and Proposition 3.17, we have for \( 1 \leq j \leq n \)

\[ N\left(\frac{\det(M_j)}{\Delta}\right) = \frac{1}{\Delta} N(\det(M_j)) + \det(M_j) N\left(\frac{1}{\Delta}\right) \]

\[ = \frac{1}{\Delta} N(\det(M_j)) + \det(M_j) \frac{D}{\Delta^2} \]

\[ \subseteq \frac{1}{\Delta}(\beta A + B) + 2n!\frac{D}{\Delta^2} \]

\[ \subseteq \frac{\beta A}{\Delta} + \frac{B}{\Delta} + \frac{A}{\Delta^2}. \]

From the stability condition \( R(A) \subseteq R(B) \) we derive both in the homogeneous and inhomogeneous case that \( \frac{\beta A}{\Delta} \subseteq B \). Then we obtain from (32) and Proposition 5.15 that

\[ N\left(\frac{\det(M_j)}{\Delta}\right) \subseteq \frac{\beta A}{\Delta} + \frac{B}{\Delta} + \frac{1}{\Delta} \left(\frac{\beta A}{\Delta}\right) \subseteq B + B/B = B. \]
It follows from Proposition 3.18 that $|\Delta_{ij}| > 0\Delta$ for some $i \in \{1, \ldots, n\}$. Because $\Delta$ is not an absorber of $B$, also $\Delta_{ij}$ is not an absorber of $B$. Hence $B \subseteq B\Delta_{ij}$. Using the fact that products containing a neutrix have always the same sign and subdistibutivity, we derive that

$$B \subseteq B\Delta_{1j} + \cdots + B\Delta_{nj} \subseteq \det \begin{bmatrix} 1 + A_{11} & \cdots & \alpha_1(j-1) & B & \alpha_1(j+1) & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{n(j-1)} & B & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix}$$

$$= N\left(\det(M_j)\right).$$

Then by Proposition 5.15

$$B = \frac{B}{\Delta} = \frac{N\left(\det(M_j)\right)}{\Delta} \subseteq \frac{N\left(\det(M_j)\right)}{\Delta} + \det(M_j)N\left(\frac{1}{\Delta}\right) = N\left(\frac{\det(M_j)}{\Delta}\right).$$

Combining (33) and (34), we conclude that $B = N\left(\frac{\det(M_j)}{\Delta}\right)$ for $1 \leq j \leq n$.

As a consequence, if the system is homogeneous, it holds that $\frac{\det(M_j)}{\Delta} = N\left(\frac{\det(M_j)}{\Delta}\right) = B$ for $1 \leq j \leq n$. $\square$

7. Proofs of the main theorems

The proof of Theorem 4.21 uses properties of (sub)associativity of the matrix product, Proposition 5.14 based on a distributivity property, and the invariance of the neutrix part of the right-hand member under the Gauss-Jordan procedure of Theorem 5.13.

Proof of Theorem 4.21. Let $S$ be the solution of $\mathcal{A}|\mathcal{B}$, and $P$ be a properly arranged matrix of representatives of $\mathcal{A}$.

Assume first that $x$ is an admissible solution, i.e. $x \in S$. Then $\mathcal{A}x \subseteq \mathcal{B}$. By Proposition 3.16 and Proposition 3.13

$$(G^P(\mathcal{A})) x = G^P(\mathcal{A}x) \subseteq G^P(\mathcal{B}).$$
Hence $x \in G^P$. Conversely, assume that $x \in G^P$. Then $(G^P(A)x) \subseteq G^P(B)$. Using Proposition 3.15, Proposition 3.16 and Proposition 5.14 we derive that

$$
Ax = I(Ax) = ((G^P)^{-1}G^P)(Ax)
\subseteq (G^P)^{-1}(G^P(Ax)) = (G^P)^{-1}((G^P(A))x)
\subseteq (G^P)^{-1}(G^P(B)) = B.
$$

Hence $x \in S$. Combining, we see that $S = G^P$. Consequently $G^P$ does not depend on the choice of $P$, hence $G \equiv G^P$ is well-defined. We conclude that $S = G$.

**Proof of Theorem 4.22.** Let $x = (x_1, x_2, \ldots, x_n)^T \in \xi$. In order to show that $x$ satisfies the system $A|B$, assume first that the system is not homogeneous. By Theorem 4.16 the external vector $\xi$ given by (13) is the solution of the system $A|B$, hence $x$ satisfies $A|B$. Secondly, assume that the system $A|B$ is homogeneous. Then $\xi = (B, B, \ldots, B)^T$ by Proposition 6.2. By direct verification we see that $\xi$ satisfies (8). Again $x$ is a solution of the system $A|B$ by Proposition 4.5.

Suppose now that $x$ is an admissible solution of system $A|B$. Let $P = [a_{ij}]_{n \times n}$ be a representative matrix for $A$. Then for $1 \leq i \leq n$ there exists $b_i \in \beta_i$ such that

$$
\begin{cases}
a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
\vdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n = b_n
\end{cases}
$$

Let $b = (b_1, \ldots, b_n)^T$. For $1 \leq i \leq n$, let $M_i^P(b)$ be the determinant of the matrix obtained from $P$ by substituting the $i^{th}$ column by $b$. By Cramer’s rule, one has $x_i = \frac{M_i^P(b)}{d} \in \frac{\det(M_i)}{\Delta}$ for $1 \leq i \leq n$. Hence $x \in \xi$.

**Proof of Theorem 4.23.** The theorem follows from Theorem 4.21 and Theorem 4.22.

The solution of the system whose coefficient matrix is the identity matrix is of course the right-hand member. We use Theorem 4.23 to show that this property remains valid if the coefficient matrix is a near-identity matrix, provided the system is stable.
Theorem 7.1. Let $\mathcal{A}$ be a near-identity matrix and $\mathcal{B} = b + \mathcal{B}$. Suppose that the system $\mathcal{A}|\mathcal{B}$ is stable. Then $\mathcal{B}$ is the solution of the system.

Proof. Put $\xi = (\xi_1, \ldots, \xi_n)^T$ with $\xi_i = \det(M_i)/\Delta$ for $1 \leq i \leq n$. We have $\mathcal{A} = I_n + A$ with $A \subseteq [\emptyset]_{n \times n}$, so $I_n$ is a representative matrix of $\mathcal{A}$, and $b_i$ is a representative of $M_i$ for $1 \leq i \leq n$. It follows from the stability that $\Delta = 1 + D$ with $D \subseteq \overline{\mathcal{A}} \subseteq \emptyset$. In addition, by Proposition 6.2 it holds that $N \left( \frac{\det(M_i)}{\Delta} \right) = B$ for $1 \leq i \leq n$. Then

$$\xi_i = b_i + N \left( \frac{\det(M_i)}{\Delta} \right) = b_i + B = \beta_i$$

for $1 \leq i \leq n$, i.e. $\xi = \mathcal{B}$. By Theorem 4.22 the vector $\mathcal{B}$ is the Cramer-solution of the system. Because of Theorem 4.23 the vector $\mathcal{B}$ is also the Gauss-Jordan solution of the system. Then it is the solution by Theorem 4.21. $\square$

Theorem 4.24 says that, as in the real case, the solution of $\mathcal{A}|\mathcal{B}$ is given by the Gauss-Jordan solution. The result follows from the fact that the Gauss-Jordan procedure, which due to Corollary 5.18 does not affect the stability of the system, leads to a stable system whose matrix of coefficients is a near identity matrix, with solution equal to right-hand member by Theorem 7.1.

Proof of Theorem 4.24. By Theorem 5.11 it holds that $\mathcal{G}^P(\mathcal{A})$ is a near-identity matrix. By Corollary 5.18 the system $x \subseteq \mathcal{G}^P(\mathcal{B})$ is stable. Then $\mathcal{G}^P(\mathcal{B})$ is the solution of the system $\mathcal{G}^P(\mathcal{A})x \subseteq \mathcal{G}^P(\mathcal{B})$ by Theorem 7.1. Hence $\mathcal{G}^P(\mathcal{B})$ satisfies (14), so it is the Gauss-Jordan solution of the system $\mathcal{A}|\mathcal{B}$. $\square$

Proof of Theorem 4.25. The solution $S$ is equal to the Gauss-Jordan solution $G$ by Theorem 4.21 which also says that the application of the Gauss-Jordan procedure does not depend on the choice of matrix of representatives $P$. Then $G = \mathcal{G}^P(\mathcal{B})$ by Theorem 4.24. Also $G$ is equal to the Cramer-solution by Theorem 4.23 which takes the form (13) by Theorem 4.22. $\square$
8. Examples

We give three straightforward examples of the Gauss-Jordan procedure with imprecisions in the coefficient matrices and the right-hand member. We will see that systems based on first order approximations sometimes generate expansions for the solutions. We also wish to illustrate how, when a precision in the right-hand member is prescribed, the stability conditions suggest the necessary precision in the coefficients to match this requirement. This may lead to neglecting terms and simplifications.

We start with a $3 \times 3$ system. We verify that the system is stable, and show the Gauss-Jordan procedure in some detail.

Example 8.1. We consider the system

$$
\begin{cases}
(1 + \varepsilon^2 \varnothing) x_1 + x_2 + (1 + \varepsilon^3 \varnothing) x_3 \subseteq 1 + \varepsilon \varnothing \\
(1 + \varepsilon^3 \varnothing) x_1 + (\frac{-1}{2} + \varepsilon^2 \varnothing) x_2 - \frac{1}{2} x_3 \subseteq -2 + \varepsilon \varnothing \\
(\frac{1}{2} \varepsilon + \varepsilon^3 \varnothing) x_1 + \frac{1}{2} x_2 + (1 + \varepsilon^2 \varnothing) x_3 \subseteq \varepsilon + \varepsilon \varnothing
\end{cases}
$$

(35)

where $\varepsilon$ is a positive infinitesimal. Let $A$ be its matrix of coefficients and $B$ be the right-hand member. The matrix is reduced and non-singular, with $\Delta = \det A = -\frac{3}{4} + \varepsilon^2 \varnothing$ zeroless. Let

$$
P = \begin{bmatrix}
1 & 1 & 1 \\
\frac{1}{2} \varepsilon & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} \varepsilon & \frac{1}{2} & 1
\end{bmatrix}.
$$

(36)

Then $P$ is a representative matrix of $A$, and is properly arranged. Indeed, formula (1) is obvious for $k = 1$, and is also satisfied for $k = 2$ with $m_2 = m_{12}^2 = -\frac{3}{4}$, $m_{13}^2 = -\frac{3}{4}$, $m_{12}^3 = \frac{1}{2} - \frac{1}{2} \varepsilon$, $m_{13}^3 = 1 - \frac{1}{2} \varepsilon$. As a consequence, $A$ is properly arranged. Because $R(A) = \varepsilon^2 \varnothing \subseteq R(B) = \varepsilon \varnothing$ and $\Delta B = (-\frac{3}{4} + \varepsilon^2 \varnothing) \varepsilon \varnothing = \varepsilon \varnothing = B$, the system is stable. By Theorem 4.23 the solution may be obtained by the Gauss-Jordan procedure. It is given by

$$
S \equiv \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix} = \begin{bmatrix}
-1 + \varepsilon \varnothing \\
4 - 3 \varepsilon + \varepsilon \varnothing \\
-2 + 3 \varepsilon + \varepsilon \varnothing
\end{bmatrix},
$$

(37)

which we verify in detail. Observe that the second and third coordinate of $S$ have the form of a truncated expansion. Also some expansions appear in the
coefficients of the intermediate matrices. We get the following succession of
stable systems:

\[
\begin{align*}
\mathcal{A}|\mathcal{B} &= \begin{bmatrix}
1 + \varepsilon^2 \theta & 1 + \varepsilon^3 \mathcal{L} & 1 + \varepsilon \mathcal{O} \\
1 + \varepsilon^3 \mathcal{L} & -\frac{1}{2} + \varepsilon^2 \theta & -\frac{1}{2} & -2 + \varepsilon \mathcal{O} \\
\frac{1}{2} \varepsilon + \varepsilon^3 \mathcal{O} & \frac{1}{2} & 1 + \varepsilon^2 \theta & \varepsilon + \varepsilon \mathcal{O}
\end{bmatrix} \\
\rightarrow L_2 - L_1 & \begin{bmatrix}
1 + \varepsilon^2 \theta & 1 & 1 + \varepsilon^3 \mathcal{L} & 1 + \varepsilon \mathcal{O} \\
\varepsilon^2 \theta & 1 + \varepsilon^2 \theta & 1 + \varepsilon^3 \mathcal{L} & 2 + \varepsilon \mathcal{O} \\
\varepsilon^3 \mathcal{O} & \frac{1}{2} - \frac{1}{2} \varepsilon & 1 - \frac{1}{2} \varepsilon + \varepsilon^2 \theta & \frac{1}{2} \varepsilon + \varepsilon \mathcal{O}
\end{bmatrix} \\
\rightarrow -\frac{2}{3} L_2 & \begin{bmatrix}
1 + \varepsilon^2 \theta & 1 + \varepsilon^2 \theta & 1 + \varepsilon^3 \mathcal{L} & 1 + \varepsilon \mathcal{O} \\
\varepsilon^2 \theta & 1 + \varepsilon^2 \theta & 1 + \varepsilon^3 \mathcal{L} & 2 + \varepsilon \mathcal{O} \\
\frac{1}{2} \varepsilon + \varepsilon^2 \theta & \frac{1}{2} + \varepsilon^2 \theta & -1 - \frac{3}{2} \varepsilon + \varepsilon \mathcal{O}
\end{bmatrix} \\
\rightarrow L_3 - \left( \frac{1}{2} \varepsilon \right) L_2 & \begin{bmatrix}
1 + \varepsilon^2 \theta & \varepsilon^2 \theta & \varepsilon^3 \mathcal{L} & -1 + \varepsilon \mathcal{O} \\
\varepsilon^2 \theta & 1 + \varepsilon^2 \theta & 1 + \varepsilon^3 \mathcal{L} & 2 + \varepsilon \mathcal{O} \\
\varepsilon^2 \theta & \varepsilon^2 \theta & 1 + \varepsilon^2 \theta & -2 + 3 \varepsilon + \varepsilon \mathcal{O}
\end{bmatrix} \\
\rightarrow 2 L_3 & \begin{bmatrix}
1 + \varepsilon^2 \theta & \varepsilon^2 \theta & \varepsilon^3 \mathcal{L} & -1 + \varepsilon \mathcal{O} \\
\varepsilon^2 \theta & 1 + \varepsilon^2 \theta & \varepsilon^2 \theta & 4 - 3 \varepsilon + \varepsilon \mathcal{O} \\
\varepsilon^2 \theta & \varepsilon^2 \theta & 1 + \varepsilon^2 \theta & -2 + 3 \varepsilon + \varepsilon \mathcal{O}
\end{bmatrix} \\
\equiv \mathcal{T} | S,
\end{align*}
\]

where \( \mathcal{T} \subseteq I_3 + [\theta]_{3 \times 3} \) is a near-identity matrix. The system \( \mathcal{T} | S \) being stable, by Theorem 7.1 the external vector \( S \) solves the latter system, and it is the Gauss-Jordan solution of the system \( (35) \) by definition. By Theorem 4.21 it is its solution, and by Theorem 4.23 it is equal to the Cramer solution, which we may verify directly.

The next Example shows a 4 \( \times \) 4 system, with similar approach.

**Example 8.2.** Let \( \varepsilon > 0 \) be infinitesimal. Consider the reduced flexible system

\[
\begin{align*}
(1 + \varepsilon^2 \theta) x_1 + x_2 + & \quad \frac{1}{2} x_3 + \frac{1}{2} x_4 \leq -1 + \varepsilon \mathcal{L} \\
- x_1 + x_2 + (\frac{1}{2} + \varepsilon \mathcal{O}) x_3 + & \quad \frac{1}{2} x_4 \leq \varepsilon \mathcal{L} \\
x_2 - & \quad \frac{1}{2} x_3 + (1 + \varepsilon \mathcal{O}) x_4 \leq -\frac{1}{2} + \varepsilon \mathcal{L} \\
(\frac{1}{2} + \varepsilon \mathcal{L}) x_1 & \quad + \quad x_3 + \quad x_4 \leq 2 + \varepsilon \mathcal{L}.
\end{align*}
\]

(38)
Let
\[
A = \begin{bmatrix}
1 + \varepsilon^2 \otimes & 1 & \frac{1}{2} & \frac{1}{2} \\
-1 & 1 \varepsilon \otimes & \frac{1}{2} \\
0 & 1 & -\frac{1}{2} & 1 + \varepsilon \otimes \\
\frac{1}{2} + \varepsilon \mathcal{L} & 0 & 1 & 1 \\
\end{bmatrix},
\]
\[
X = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
-1 + \varepsilon \mathcal{L} \\
\varepsilon \mathcal{L} \\
-\frac{1}{2} + \varepsilon \mathcal{L} \\
2 + \varepsilon \mathcal{L} \\
\end{bmatrix}.
\]

Then \( \Delta = \det A = -3 + \varepsilon \mathcal{L} \) is zeroless, \( R(A) = \overline{A}/\overline{\Delta} = \varepsilon \mathcal{L} \), \( R(B) = B/\overline{\beta} = \varepsilon \mathcal{L} \) and \( \Delta B = \varepsilon \mathcal{L} = B \). Hence \( \overline{A} = \varepsilon \mathcal{L} \subset \otimes = \otimes \Delta \), \( R(A) \subseteq R(B) \) and \( \Delta \) is not an absorber of \( B \), so the system is stable.

Let
\[
P = \begin{bmatrix}
1 & 1 & 1/2 & 1/2 \\
-1 & 1 & 1/2 & 1/2 \\
0 & 1 & -1/2 & 1 \\
1/2 & 0 & 1 & 1 \\
\end{bmatrix}.
\]

Then \( P \) is a reduced representative matrix of \( A \), with \( d \equiv \det(P) = -3 \).
A short calculation shows that \( P \) is properly arranged, with \( m_2 = 2 \) and \( m_3 = -2 \).

We may proceed in a similar way as in Example 8.1 and derive that
\[
X = \mathcal{G}P\mathcal{B} = \begin{bmatrix}
-1/2 + \varepsilon \mathcal{L} \\
-13/8 + \varepsilon \mathcal{L} \\
3/4 + \varepsilon \mathcal{L} \\
3/2 + \varepsilon \mathcal{L} \\
\end{bmatrix}.
\]

is the Gauss-Jordan solution of the system. Then \( X \) is also the Cramer-solution by Theorem 4.25.

The next example deals with a system having a coefficient matrix with an infinitesimal determinant. Yet it is not an absorber of the neutrix occurring in the right-hand member. Also the remaining conditions for stability hold, so the Gauss-Jordan procedure still works. The solution will be an external vector with coordinates in the form of an expansion starting with a "singular", i.e. unlimited term.

**Example 8.3.** Let \( \varepsilon \) be a positive infinitesimal. We will use the microhalos \( M_\varepsilon = \mathcal{L} \varepsilon^\mathcal{K} \) and \( M_{\varepsilon 1} = \mathcal{L} \varepsilon_1^\mathcal{K} \), where \( \varepsilon_1 = \varepsilon^\omega \) with \( \omega \in \mathbb{N} \) unlimited. Consider the system
\[
\begin{cases}
(1 + \mathcal{L} \varepsilon_1^\mathcal{K}) x + y \subseteq 1 + \mathcal{L} \varepsilon^\mathcal{K} \\
x + (1 - \varepsilon + \mathcal{L} \varepsilon^\mathcal{K}) y \subseteq 2 + \mathcal{L} \varepsilon^\mathcal{K}.
\end{cases}
\]
Let \( \mathcal{A} = \begin{bmatrix} 1 + \mathcal{L} \varepsilon^\infty & 1 \\ \varepsilon & 1 - \varepsilon + \mathcal{L} \varepsilon^\infty \end{bmatrix} \). Then \( \Delta = \det \mathcal{A} = -\varepsilon + \mathcal{L} \varepsilon^\infty \) is zeroless.

One easily verifies that the system is stable. Applying the Gauss-Jordan procedure we obtain

\[
A|B = \begin{bmatrix} 1 + \mathcal{L} \varepsilon^\infty & 1 \\ \varepsilon & 1 - \varepsilon + \mathcal{L} \varepsilon^\infty \end{bmatrix} \rightarrow L_2 - L_1 \begin{bmatrix} 1 + \mathcal{L} \varepsilon^\infty & 1 \\ \mathcal{L} \varepsilon^\infty & -\varepsilon + \mathcal{L} \varepsilon^\infty \end{bmatrix} \rightarrow \frac{-1}{\varepsilon} L_2 \begin{bmatrix} 1 + \mathcal{L} \varepsilon^\infty & 1 \\ \mathcal{L} \varepsilon^\infty & 1 + \mathcal{L} \varepsilon^\infty \end{bmatrix} \rightarrow L_1 - L_2 \begin{bmatrix} 1 + \mathcal{L} \varepsilon^\infty & \mathcal{L} \varepsilon^\infty \\ \mathcal{L} \varepsilon^\infty & 1 + \mathcal{L} \varepsilon^\infty \end{bmatrix}
\]

By Theorem 4.23 the vector \( \xi = \left( \frac{1}{\varepsilon} + 1 + \mathcal{L} \varepsilon^\infty, -\frac{1}{\varepsilon} + \mathcal{L} \varepsilon^\infty \right)^T \) is the solution of the system.

Consider a system \( \mathcal{A}|\mathcal{B} \) such that some of the entries of \( \mathcal{A} \) are given in the form of expansions. We show that if the terms \( t \) of the expansions satisfy \( t/\Delta \subseteq R(\mathcal{B}) \), they may be neglected, and we may solve as well the system \( \mathcal{A}'|\mathcal{B} \), with \( \mathcal{A}' \) obtained from \( \mathcal{A} \) by neglecting these terms. We may roughly interpret this by the possibility to neglect decimals in a coefficient matrix, if compared with the determinant they are small with respect to the imprecisions of the right-hand member. We will illustrate this with the help of the examples 8.1 and 8.2.

We prove first some general properties.

**Proposition 8.4.** Let \( \mathcal{A} \in M_n(\mathbb{E}) = [a_{ij}]_{n \times n} = [a_{ij} + A_{ij}]_{n \times n} \) be a non-singular stable matrix, properly arranged with respect to a reduced representative matrix \( P = [a_{ij}]_{n \times n} \). Let \( \mathcal{A}' \equiv [\alpha'_{ij}]_{n \times n} \) be defined by

\[
\alpha'_{ij} = a_{ij} + A'_{ij},
\]

with \( A'_{ij} \subseteq \overline{A} \) for \( 1 \leq i, j \leq n \). Then the matrix \( \mathcal{A}' \) is non-singular and stable.

**Proof.** Because \( \mathcal{A} \) is limited, the matrix \( \mathcal{A}' \) is also limited. Let \( d = \det(P) \) and \( \Delta' = \det(\mathcal{A}') \). Because \( A'_{ij} \subseteq \overline{A} \) for \( 1 \leq i, j \leq n \), the matrix \( \mathcal{A} \) is non-singular and stable and the matrix \( P \) is reduced, it holds that \( \Delta' \subseteq d + \overline{A} \subseteq (1 + \ominus)d \). So \( \Delta' \) is zeroless, hence \( \mathcal{A}' \) non-singular. In addition
\[
\begin{align*}
\overline{A}' & \subseteq \overline{A} \\
\frac{\Delta'}{(1 + \odot)d} & = \overline{A} \subseteq \odot.
\end{align*}
\]

Hence \(A'\) is stable. \(\square\)

The proposition below considers systems \(A|B\) in the sense of Convention 4.18.

**Proposition 8.5.** Let \(A|B\) be a stable system with solution \(S = G^P(B)\), where \(P = [a_{ij}]_{n \times n}\) is a reduced properly arranged representative matrix. Let \(Q = [q_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{R})\) be a reduced properly arranged matrix such that for \(1 \leq i, j \leq n\)

\[q_{ij} - a_{ij} \in A.\] (41)

Let \(A' \equiv [\alpha'_{ij}]\) with \(\alpha'_{ij} = q_{ij} + A'_{ij}\) and \(A'_{ij} \subseteq A\) for \(1 \leq i, j \leq n\). Then \(A'|B\) is a stable system, with solution \(G^Q(B) = S\).

**Proof.** Put \(A'' = Q + (\overline{A})_{n \times n}\); note that the system \(A''|B\) satisfies Convention 4.18. It follows from (41) that \(A'' = P + (\overline{A})_{n \times n}\), so by Proposition 8.4 the system \(A''|B\) is stable and has the same solution as \(A|B\), i.e. \(G^P(B)\). By Theorem 4.21 we have \(G^P(B) = G^Q(B)\). Again applying Proposition 8.4 and Theorem 4.21 we see that the system \(A'|B\) is stable, and that \(G^Q(B)\) is its solution. \(\square\)

We will apply Proposition 8.5 to the systems of Example 8.6 and Example 8.7 below.

**Example 8.6.** (Continuation of Example 8.1) Put

\[
\begin{align*}
(1 + \varepsilon^2 \odot) x_1 + (1 + \varepsilon^2 \odot) x_2 + (1 + \varepsilon^2 \odot) x_3 & \subseteq 1 + \varepsilon \\
(1 + \varepsilon^2 \odot) x_1 + (-\frac{1}{2} + \varepsilon^2 \odot) x_2 + (-\frac{1}{2} + \varepsilon^2 \odot) x_3 & \subseteq -2 + \varepsilon \\
(\frac{1}{2} + \varepsilon^2 \odot) x_1 + (\frac{1}{2} + \varepsilon^2 \odot) x_2 + (1 + \varepsilon^2 \odot) x_3 & \subseteq \varepsilon + \varepsilon \odot.
\end{align*}
\]

Let \(A'\) be the coefficient matrix of 8.6. Note that the matrix \(P\) given by 39 is a representative matrix of both \(A\) and \(A'\), and that the associated neutricial matrices satisfy \(A \subseteq A'\) with \(\overline{A'} = \overline{A}\), and that the vectors in the right-hand side of both systems are the same. Then by Proposition 8.5 the solution of the system 8.6 is also given by 37.

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Example 8.7. (Continuation of Example 8.2.) Consider the system
\[
\begin{cases}
(1 + \varepsilon) x_1 + (1 - \varepsilon) x_2 + (1/2 + 2\varepsilon^2) x_3 + 1/2 x_4 \leq -1 + \varepsilon L \\
(-1 + 3\varepsilon) x_1 + x_2 + (1/2 + \varepsilon^2 + \varepsilon^2 \odot) x_3 + 1/2 x_4 \leq -1 + \varepsilon L \\
x_2 - 1/2 x_3 + (1 - 3\varepsilon^2 + \varepsilon^2 \odot) x_4 \leq -1 + \varepsilon L \\
(1/2 + \varepsilon + \varepsilon \odot) x_1 + (1 + \varepsilon L) x_3 + (1 + \varepsilon \odot) x_4 \leq 2 + \varepsilon L.
\end{cases}
\]

Let \( A' \) be the coefficient matrix of the system. It has the representative matrix
\[
Q = \begin{bmatrix}
1 & 1 - \varepsilon & 1/2 + 2\varepsilon^2 & 1/2 \\
-1 + 3\varepsilon & 1 & 1/2 + \varepsilon^2 & 1/2 \\
0 & 1 & -1/2 & 1 - 3\varepsilon^2 \\
1/2 + \varepsilon & 0 & 1 & 1
\end{bmatrix},
\]
which differs from the matrix \( P \) given by (39), which is the representative matrix of the matrix \( A \) of Example 8.2. One verifies that \( Q \) is non-singular, reduced and properly arranged, with \( \text{det}(Q) \in 3 + \varepsilon \) zeroless, \( m_1 = 1 \), \( m_2 \in 2 - 2\varepsilon + \odot \varepsilon \) and \( m_3 \in 2 - 5/2\varepsilon + \odot \varepsilon \). The entries of \( Q \) and \( P \) differ for at most a limited multiple of \( \varepsilon \), which is contained in \( \overline{A} = \varepsilon \). Also \( \overline{A'} = \varepsilon = \overline{A} \). Then by Proposition 8.5 one may as well solve the system using the simpler matrix \( P \), with solution given by (40).

We illustrate Example 8.1/8.6 and Example 8.2/8.7 numerically. We assume that \( \varepsilon = 0.01 \), and represent \( \odot \) by \([-0.1, 0.1]\) and \( \varepsilon \) by the interval \([-2, 2]\). We will not apply interval calculus, but use the extreme values of the numerical intervals, with the help of a binomially distributed random variable.

Working with the matrix (36) and the right-hand member \((1, -2, 1/100)^T\), we find the exact solution
\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-1 \\
3.97 \\
107 \\
100
\end{bmatrix} = \begin{bmatrix}
-1 \\
3.97 \\
-1.97
\end{bmatrix}.
\]

To represent the coefficient matrix of Example 8.1 we may consider, say,
\[
A' = \begin{bmatrix}
1.00001 & 0.99999 & 1.000002 \\
0.999998 & -0.50001 & -0.5 \\
0.0049999 & 0.5 & 1.00001
\end{bmatrix}.
\]
Rounded off at 7 significative digits, we find the solution

\[ x' = \begin{bmatrix} -0.999970 \\ 3.969929 \\ -1.969945 \end{bmatrix}. \]

The largest deviation with respect to the exact solution is about 0.000071 in the second coordinate, which is significantly smaller than 0.001, i.e. the absolute value of the bounds of the interval representing \( \Theta \varepsilon \).

In Example 8.6 all entries of the coefficient matrix are imprecise. In order to compare with the numerical matrix \( A' \), we choose a matrix \( A'' \) using a randomization which is the same for the imprecise coefficients of \( A' \) and put

\[ A'' = \begin{bmatrix} 1.00001 & 0.99999 & 1.00001 \\ 0.99999 & -0.50001 & -0.49999 \\ 0.004999 & 0.49999 & 1.00001 \end{bmatrix}. \]

Rounded off at 7 significative digits, we find the solution

\[ x'' = \begin{bmatrix} -0.999943 \\ 3.969928 \\ -1.969915 \end{bmatrix}. \]

As expected the result is not as good as \( x' \), still the largest deviation of about 0.000085 for the third coordinate lies well within the interval \([-0.001, 0.001]\) representing \( \Theta \varepsilon \).

Finally we illustrate Proposition 8.5 by comparing the solution of the system (38) when using the representative matrices \( P \) given by (39) and \( Q \) given by (42). With the matrix \( P \) and the right-hand member \( b^T = (-1, 0, -1/2, 2)^T \) we find the exact solution

\[ x = G^P b = \begin{bmatrix} -1/2 \\ -13/8 \\ 3/4 \\ 3/2 \end{bmatrix} \begin{bmatrix} -0.5 \\ 1.625 \\ 0.75 \\ 1.5 \end{bmatrix}. \]

The solution \( x' \) for the matrix

\[ Q' = \begin{bmatrix} 1 & 0.99 & 0.5002 & 0.5 \\ -0.97 & 1 & 0.5001 & 0.5 \\ 0 & 1 & -0.5 & 0.9997 \\ 0.51 & 0 & 1 & 1 \end{bmatrix} \]

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is, rounding off at 7 significative digits,

\[ x' = G^\ast b = \begin{bmatrix} -0.5159373 \\ -1.632099 \\ 0.7537178 \\ 1.509410 \end{bmatrix}. \]

We observe the largest deviation with respect to the exact solution in the second coordinate, with a value of about 0.007. This is 0.7 times the value 0.01 chosen for \( \varepsilon \), so it can be considered to lie within \( \mathcal{E} \varepsilon \).

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