Spontaneous symmetry-breaking in optomechanical cavity

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Abstract

A theoretical consideration of the so-called "membrane-in-the-middle" optomechanical cavity revealed that it undergoes a spontaneous symmetry breaking as a function of the transparency of the membrane. Such typical features of this phenomenon as a square-root development of the order parameter and divergency of the critical susceptibility were identified. In the contract to classical spontaneous-symmetry-breaking systems of ferromagnets and ferroelectrics, in the considered system, this divergency remains, due to interference effects, an "internal" property of the system, which does not reveal itself in any divergency of its observables. A spontaneous symmetry breaking in an optomechanical cavity might pave a new way to enhanced optomechanical interactions.

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Spontaneous symmetry breaking is a process, by which a physical system in a symmetric state ends up in an asymmetric state. Such an evolution is characterized by the so-called order parameter, which is zero in the symmetric state acquiring non-zero values in the asymmetric state. One also speaks about the appearance of non-zero order parameter as a phase transition. This is a general phenomenon, the manifestations of which span from the particle physics and cosmology to the condensed matter physics, where ferromagnetism and ferroelectricity are the classical examples. An important variable used for the description of this phenomenon is the so-call conjugated field. The conjugated field is a perturbation, which induces a non-zero order parameter in the symmetric state. A characteristic feature of spontaneous symmetry breaking is a divergence (or a strong increase in the case of a discontinuous transition) of the susceptibility of the order parameter to the conjugated field at the breaking point, which is called critical susceptibility. It is this feature of spontaneous symmetry breaking that is behind the most of applications of ferroelectrics, profiting from highly enhanced dielectric constant, which plays the role of the critical susceptibility.

The parity-time symmetry breaking is currently a hot topic in optics. In this paper, we theoretically analyse the performance of a simple optomechanical system to identify a spontaneous symmetry breaking in the spatial field distribution in the cavity. Here, the difference of the decay rates of two optical modes play the role of the order parameter while the mechanical displacement plays the role of the conjugated force. As a result, the divergency of the critical susceptibility translates into anomalously large dissipative optomechanical coupling constants of the optical modes. Such a situation, being similar to that in ferroelectrics, however, is yet essentially different: due to interference effects, the divergencies of the coupling constants of the individual modes do not translate into any singularity of the total optomechanical response.

The system addressed is the so-called "membrane-in-the-middle" optomechanical cavity, which, during the past decade, has been attracting appreciable attention of theorists and experimentalists. It is schematically depicted in Fig.1. For the case where the membrane is set exactly half-way between the end mirrors, we are interested in the resonance frequencies, decay rates, constants of optomechanical coupling of the optical modes as well as in the optomechanical signal in the light back-scattered from the cavity.
FIG. 1. Schematic of a membrane-in-the-middle optomechanical cavity. The left mirror is semi-transparent, the right mirror is perfectly reflecting. The membrane is shown shifted from the middle of the cavity by distance $x$. Running electromagnetic waves are schematically shown with arrows and labeled with their complex amplitudes.

To be specific, we set the following scattering matrixes:

$$
\begin{pmatrix}
  i\tau & -\rho \\
  -\rho & i\tau
\end{pmatrix},
\begin{pmatrix}
  0 & -1 \\
  -1 & 0
\end{pmatrix}, \text{ and } \begin{pmatrix}
  it & -r \\
  -r & it
\end{pmatrix}
$$

(1)

for the the left mirror, right mirror, and membrane, respectively, where the amplitude transmission coefficients are on the diagonals. We set $\rho$, $\tau$, $r$, and $t$ as real and positive ($r^2 + t^2 = 1$ and $\rho^2 + \tau^2 = 1$).

We consider the eigen modes (with complex eigen values, in general) of the system. Thus, setting the amplitudes of the input field $G_0 = 0$ (see Fig.1), the complex field amplitudes $G_1$, $G_2$, $U_1$, and $U_2$ (values at the membrane) are linked with the following relations

$$
G_1 = itU_2 - rU_1 \\
G_2 = -rU_2 + itU_1 \\
G_1e^{ik(l-x)} = -U_1e^{-ik(l-x)} \\
\rho G_2e^{ik(l+x)} = -U_2e^{-ik(l+x)}
$$

(2)

where for the definition of $l$ and $x$ see Fig.1. These relations imply the following equation
for the resonance $k$-vector (complex, in general)

$$
(e^{-2ikl} - re^{-2ikx})(\rho^{-1}e^{-2ikl} - re^{2ikx}) + t^2 = 0.
$$

(3)

In the absence of dissipation (energy loss from the cavity), i.e. at $\rho = 1$, Eq. (3) determines a well-known relation$^6$, $\cos 2kl = r \cos 2kx$, for a real resonance wavevector. The complex wave vector, $k$, satisfying (3) defines the resonance frequencies, decay rates, and optomechanical coupling constants (dispersive and dissipative$^{15–17}$) as follows

$$
\omega_c = c \text{Re}[k], \quad \gamma = -2c \text{Im}[k]
$$

(4)

$$
\frac{d\omega_c}{dx} = c \text{Re} \left[ \frac{dk}{dx} \right], \quad \text{and} \quad \frac{d\gamma}{dx} = -2c \text{Im} \left[ \frac{dk}{dx} \right],
$$

(5)

respectively. In what follows we study these parameters as function of $t$ and $\tau$ for the central position of the mirror, i.e. at $x = 0$. We address the case of practical interest where $\tau \ll 1$, in what follows, usually keeping only the lowest order terms in this parameter.

At $x = 0$, the solutions to (3) reads

$$
e^{-2ikl} = \frac{r(1 + \rho) \pm \sqrt{(1 - \rho)^2 - (1 + \rho)^2t^2}}{2}.
$$

(6)

where $\pm$ corresponds to the two modes of the doublets merging in the limit of non-transparent membrane, i.e. at $t = 0$. A remarkable feature of this solution is that at $t > t_0 \equiv (1 - \rho)/(1 + \rho) \approx \tau^2/4 \ll 1$, the square root in (6) is imaginary, implying the same damping rate for the modes of the doublets while, in the opposite case, it is real, implying, in turn, the degeneracy of the doublet frequencies. These results are exact.

For the decay rates, using (6) and (4), straightforward calculations yield, at $t > t_0$,

$$
\gamma_0 = \frac{c\tau^2}{4l}
$$

(7)

while, at $t < t_0$,

$$
\gamma_{+, -} = \gamma_0[1 \mp \sqrt{1 - (t/t_0)^2}].
$$

(8)

The $t$-dependence of the cavity decay rates according to (7) and (8) is shown in Fig.2 The spontaneous symmetry breaking behaviour with the typical square-root development, c.f. Ref. 18, of the order parameter, $\gamma - \gamma_0$, in the ”asymmetric” state, i.e. at $t < t_0$, is clearly seen here.

Such a symmetry breaking can also be identified in the field profiles of the modes. For the system addressed, such profiles calculated neglecting the energy loss from the input mirror
Typical square-root development of the order parameter, $\gamma - \gamma_0$, is seen.

(i.e. at $\tau = 0$) are known\textsuperscript{6,8} to be symmetric or antisymmetric with respect to the middle of the cavity. For non-zero $\tau$, this, however, is not always the case. To characterize the field profiles, we pass from the amplitudes at the membrane, $G_1$ and $G_2$, to those to at the end mirrors: $G_L = G_2 e^{ikl}$ and $G_R = G_1 e^{ikl}$. From Eqs.\textsuperscript{(2)} one readily finds\textsuperscript{10}

\[
\frac{G_L}{G_R} = \frac{r(1 - \rho) \pm \sqrt{(1 - \rho)^2 - (1 + \rho)^2 t^2} 2it\rho}{2it\rho}.
\]

(9)

In the "symmetric" state, i.e. at $t > t_0$, Eq.\textsuperscript{(9)} yields

\[
\left| \frac{G_L}{G_R} \right|^2 = 1/\rho \approx 1
\]

(10)

for both modes of the doublets, reproducing, to within the accepted accuracy, the results of the dissipation-free calculations. Note that (10), being valid for the both modes, actually means that the energy fluxes in the modes towards the input mirror are the same if the energy stored in the modes are the same either. This implies the same decay rates of the modes. At the same time, in the "asymmetric" state, i.e. at $t < t_0$, Eq.\textsuperscript{(9)} results in an
asymmetric field profiles. Now to within the accepted accuracy of calculations one finds
\[
\left| \frac{G_L}{G_R} \right|^2 = \left( \frac{t_0}{t} \right)^2 \left[ 1 \mp \sqrt{1 - (t/t_0)^2} \right]. \quad (11)
\]
Notably, for the modes corresponding to \( \mp \) in this equation, it implies
\[
G_R^+ \Rightarrow 0 \quad \text{and} \quad G_L^- \Rightarrow 0 \quad \text{at} \quad t \Rightarrow 0, \quad (12)
\]
respectively. The above relations clearly suggest that, below the critical value of \( t \), there appears an onset of localization the modes into the two halves of the cavity. It is also evident that these relations are qualitatively consistent with (8).

The spontaneous symmetry breaking revealed above in the mode decay rates and field profiles also manifests itself in the frequencies of the optical modes. Since the spontaneous symmetry breaking occurs at \( t = t_0 \ll 1 \), the further analysis is done in the approximation of small \( t \). Let us conceder the modes of a doublet, which at \( t \to 0 \) merge into a single mode with with frequency \( \omega_0 \) (evidently \( e^{-2i\omega_0 t/c} = 1 \)). In the absence of dissipation one readily finds [8] for the frequencies of these modes
\[
\omega_c - \omega_0 = \mp \Omega_0 \quad \Omega_0 = \frac{ct}{2l} \quad (13)
\]
However, the incorporation of the dissipation dramatically affects this result: Eqs.(6) and (4) imply that, at \( t < t_0 \), as was mentioned above, the doublet stays degenerate at finite \( t \), i.e.
\[
\omega_c = \omega_0, \quad (14)
\]
while at \( t > t_0 \),
\[
\omega_c - \omega_0 = \pm \frac{c \sqrt{t^2 - t_0^2}}{2l}. \quad (15)
\]
The cavity mode splitting for small \( t \) is schematically illustrated in Fig.3.

To address the the critical anomaly associated with the spontaneous symmetry breaking, the conjugated force is to be identified. For the order parameter \( \gamma - \gamma_0 \), a displacement of the membrane from the middle of the cavity, \( x \), (see Fig.1) can be taken as the conjugated force such that the dissipative coupling constant \( \frac{d\gamma}{dx} \) appears to play the role of the critical susceptibility.

Using (3) and (6), one finds
\[
\frac{dk}{dx} = \pm \frac{r}{l} \frac{k}{\sqrt{1 - t^2 (1+\rho)^2/(1-\rho)^2}}. \quad (16)
\]
Next, to within the accepted accuracy, using (5), Eq. (16) implies: At $t > t_0$,

$$
\frac{d\gamma}{dx} = \pm \frac{2ck_{00}}{l} \frac{1}{\sqrt{(t/t_0)^2 - 1}}.
$$

(17)

while, at $t < \tau^2/4$,

$$
\frac{d\gamma}{dx} = \mp \frac{\gamma_{+,-}}{l} \frac{1}{\sqrt{1 - (t/t_0)^2}}.
$$

(18)

The dependence of the absolute value of dissipative coupling constants of the doublet modes given by Eqs. (17) and Eqs. (18) is shown in Fig. 3 where, at $t < t_0$, relatively small values of $d\gamma/dx$ are shown as zero. Here, another characteristic feature of the spontaneous symmetry breaking - divergence of the critical susceptibility - is seen, c.f. Ref. 18.

One sees that, in the symmetric state, this value may readily exceed the typical value of the dispersive coupling constant for an optomechanical Fabry-Perot cavity of the same length. Note that even, not very close to the transition, the absolute values of the dissipative optomechanical coupling constant are much larger than those discussed in the literature.16,20

As it is clear from (16), the dispersive coupling constants of the doublet modes also
exhibit an anomaly, implying via (5) to within the accepted accuracy: at $t < t_0$,

$$\frac{d\omega_c}{dx} = \pm \frac{ck_{00}}{t} \frac{1}{\sqrt{1 - (t/t_0)^2}}$$

(19)

while, at $t > t_0$,

$$\frac{d\omega_c}{dx} = \pm \frac{\gamma_+ - \gamma_-}{2l} \frac{1}{\sqrt{(t/t_0)^2 - 1}}.$$

(20)

The dependence of the dispersive optomechanical coupling constants given by Eqs. (19) and (20) is schematically shown Fig. 4 where, at $t > \tau^2/4$, relatively small values of $d\omega_c/dx$ are shown as zero.

In view of the opposite sign of the coupling constants of modes of the doublet, the divergences identified above does not necessarily mean that of the optomechanical signal in the light reflected from the cavity. To check this, such a signal was calculated using the method...
known in the gravitational wave community as the input-output relations approach\textsuperscript{21, 22}, which for linear optomechanical problems of this kind provides an exact solution.

Using this approach, we considered the system, schematically shown depicted in Fig. 1 to be pumped from the left mirror with a strong monochromatic light of frequency $\omega_L$ and amplitude $G_0$. We are interested in modifications of the light scattered back from the cavity, which are caused by small and slow displacements $x(t)$ of the membrane from its central position, assuming $x(\Omega)k_L \ll 1$ and $\Omega \ll \omega_L$, where $k_L = \omega_L/c$ and $x(\Omega)$ is the Fourier transform of $x(t)$ at the frequency $\Omega$. The Fourier transform at the frequency $\Omega$ of the $x$-modulated part of amplitude of the backscattered light reads (see SI)

$$u_0^{(x)}(\Omega) = -i \frac{8G_0k_Lx(\Omega)}{\tau^2} B(k, k_L)$$

$$B(k, k_L) = \tau^4 \frac{C(k, k_L)}{2D(k)D(k_L)}$$

where $k = (\omega_L + \Omega)/c$, $D(z) = r - e^{-2izl} + \rho(r - e^{2izl})$, and $C(k, k_L) = \cos[(k + k_L)l] - r \cos[(k - k_L)l]$. Here $|B(k, k_L)|$ has the meaning of the absolute value of optomechanical signal normalized the maximal absolute value of that for the considered system with the perfectly reflecting membrane.

It is seen from Eqs. (21) and (22) that divergency of the optomechanical signal at $t \approx t_0$ may occur only if $D(k)D(k_L)$ tends here to zero. One readily check that it is not the case, moreover, even no cusp at $t \approx t_0$ in the $u_0^{(x)}$ vs. $t$ dependence is present. Thus, in the case of optomechanical cavity, such a characteristic feature of spontaneous symmetry breaking as the divergency of the critical susceptibly remains an "internal" property of the system, which does not reveal itself in any divergency of its observables. This makes a big contrast with the manifestation of the spontaneous symmetry breaking in ferroelectrics.

The fact that divergencies of the optomechanical coupling constants of the individual modes of the doublet disappear from the output signal can be easily rationalized. Evidently, if the frequencies and dampings of the doublet modes were equal, the coupling constants of the modes that differ only in the sign would result the total cancelation of their contributions to the optomechanical signal. In our system the modes differ either in frequency or in damping such that the full cancelation does not take place. Instead, at $t > t_0$ where the dampings of the modes are equal while the frequencies, according to (15), are split by $\delta \omega_c = c\sqrt{t^2 - t_0^2}/l$, one expects the signal to be proportional $|\delta \omega_c d\gamma/dx|$. In view of (17),
being equal to $\omega_0^2 \tau^2/(2l^2)$, this product does not contain any singularity. On the same lines one can show that, at $t < t_0$, the divergency of the dispersive coupling constant $d\omega_c/dx$ is washed out from the optomechanical output signal either.

Though such a fingerprint of the spontaneous symmetry breaking as the divergence of the critical susceptibility does not reveal itself in the optomechanical output of the system, the spectrum of the optomechanical signal is affected by an essential modification of the frequency and damping of the modes of the doublet caused by the symmetry breaking. As an example we give an expression for $|B(k, k_L)|$ calculated at $k_L = \omega_0/c$ and keeping the lowest term in $\Omega, \tau$, and $t$, which reads (see SI)

$$|B(\Omega/c + \omega_0/c, \omega_0/c)| = \frac{\gamma_0^2}{\sqrt{(\Omega^2 - \Omega_0^2)^2 + \gamma_0^2 \Omega^2}}.$$  

(23)

We compare this expression with that where $(\Omega^2 - \Omega_0^2)^2 + \gamma_0^2 \Omega^2$ is replaced with $|(\Omega - \Omega_0 + i\gamma_0/2)(\Omega + \Omega_0 + i\gamma_0/2)|^2 = (\Omega^2 - \Omega_0^2)^2 + \gamma_0^2 (\Omega^2 + \Omega_0^2)/2 + \gamma_0^4/16$, the latter corresponding to the frequencies and dampings of the doublet modes calculated neglecting the spontaneous symmetry breaking. Such a comparison is presented in Fig.5. It is cleanly seen that, far from the summery breaking point, at $t = 4t_0$, (a), the optomechanical signal is hardly affected by the symmetry breaking phenomenon while approaching the transition, at $t/\tau^2 = 1.4t_0$, (b), and further on, at $t/\tau^2 = 0.65t_0$, (c), the impact is appreciable.

To summarize. A theoretical consideration of the so-called "membrane-in-the-middle" optomechanical cavity, which is a quite popular optomechanical system, revealed that it undergoes a spontaneous symmetry breaking as a function of the transparency of the membrane. Such typical features of this phenomenon as a square-root development of the order parameter and divergency of the critical susceptibility were identified. In the contract to classical spontaneous-symmetry-breaking systems of ferromagnets and ferroelectrics, in the considered system, this divergency remains an "internal" property of the system, which, due to interference effects, does not reveal itself in any divergency of its observables. At the same time, the spectrum of the optomechanical signal is affected by the phenomenon. There is no reason to consider the suppression of the divergency of the critical susceptibility in to optomechanical output signal to be a general phenomenon. Thus, a spontaneous symmetry breaking in an optomechanical cavity might pave a new way to enhanced optomechanical interactions.
FIG. 5. Spectre of the absolute value of optomechanical signal normalized to the maximal absolute
value of that for the considered system with the perfectly reflecting membrane (i.e. at $t = 0$),
denoted as $|B|$: a) $t = 0.4 t_0$, b) $t = 1.4 t_0$, c) $t = 0.65 t_0$. The spontaneous symmetry
breaking takes place at $t = t_0$. Curved dashed lines show the spectrum if the frequency and
damping of the modes were calculated neglecting the spontaneous symmetry breaking. Vertical dashed
lines show the frequencies $\Omega/\gamma_0$ where $2\Omega_0$ is frequency splitting in the doublet
calculated neglecting damping, i.e. at $\tau = 0$. The pumping light frequency equals to the resonance
frequency at $t = 0$, i.e. $\omega_L = \omega_0$. 
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I. APPENDIX

We are interested in an optomechanical signal of the ”membrane-in-the-middle” optomechanical cavity schematically depicted in Fig[1]. Specifically, we consider small deviations $x$ of the membrane from its central position and calculate the $x$-dependent component of the backscattered light when the cavity is exited with a strong coherent light of frequency $\omega_L$. To be specific, we set the following scattering matrixes:

\[
\begin{pmatrix}
i\tau & -\rho \\
-\rho & i\tau
\end{pmatrix}
\]  
(24)

for the the left mirror,

\[
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}
\]  
(25)

for the the right mirror, and

\[
\begin{pmatrix}
it & -r \\
r & it
\end{pmatrix}
\]  
(26)

for the membrane.

A. General

A theory of the system in question was already offered in a number of papers[6–8], a comprehensive treatment being given using a perturbation approach[5,8]. At the same time, the linear optomechanical problem we are interested in can also be treated practically exactly by using the so-called input-output relations approach[21,23] popular in the gravitational wave community. Below we implement such an approach, as yielding a result, which is free from the limitations of the customary Langevin-equation formalism.
Following this approach, in the frame rotating with the frequency $\omega_L$, we present all amplitudes of the fields (see Fig.1) as a sum of a large constant part and a small fluctuating part, e.g.

\begin{align*}
G_0(t) &= G_{00} + g_0(t) & G_2(t) &= G_{20} + g_2(t) \\
U_0(t) &= U_{00} + u_0(t) & U_2(t) &= U_{20} + u_2(t) \tag{27}
\end{align*}

etc.

For our system, in view of (24), (25), and (26), the following equations

\begin{align*}
U_{00} &= i\tau G_{20} e^{ik_Ll} - \rho G_{00} \\
U_{20} e^{-ik_Ll} &= -\rho G_{20} e^{ik_Ll} + i\tau G_{00} \\
G_{20} &= itU_{10} - rU_{20} \\
G_{10} &= -rU_{10} + itU_{20} \\
G_{10} &= -U_{10} e^{-2ik_Ll}. \tag{28}
\end{align*}

are satisfied for the constant parts, where $k_L = \omega_L/c$. The solution to this set of equation reads

\begin{align*}
U_{10} &= \frac{t\tau e^{ik_Ll}}{D(k_L)} G_{00} \tag{29}
\end{align*}

and

\begin{align*}
U_{20} &= \frac{i\tau e^{ik_Ll}(r - e^{-2ik_Ll})}{D(k_L)} G_{00} \tag{30}
\end{align*}

where

\begin{align*}
D(z) &= r - e^{-2izl} + \rho(r - e^{2izl}). \tag{31}
\end{align*}

The Fourier transform of fluctuating parts of the amplitudes (denoted as $g_0(\Omega)$, $u_0(\Omega)$, etc) meet the relations:

\begin{align*}
u_0(\Omega) &= i\tau g_2(\Omega) e^{ikl} - \rho g_0(\Omega) \\
u_2(\Omega) e^{-ikl} &= -\rho g_2(\Omega) e^{ikl} + i\tau g_0(\Omega) \\
g_2(\Omega) &= itu_1(\Omega) - ru_2 + 2irU_{20}k_Lx(\Omega) \\
g_1(\Omega) &= -ru_1(\Omega) + itu_2(\Omega) - 2irU_{10}k_Lx(\Omega) \\
g_1(\Omega) &= -u_1(\Omega) e^{-2ikl}. \tag{32}
\end{align*}
where $k = k_L + \Omega/c$ and $x(\Omega)$ is the Fourier transform of $x(t)$. Here it is assumed that $x(\Omega)k_L \ll 1$ and $\Omega \ll \omega_L$.

Starting from (28) and (32), the Fourier transform of complex amplitude of backscattered light, $u_0(\Omega)$, reads

$$u_0(\Omega) = -\frac{D(k)^{*}}{D(k)}g_0 - iu_m B(k, k_L)$$  \hspace{1cm} (33)

$$u_m = \frac{8G_{00}k_L x(\Omega)}{\tau^2}$$  \hspace{1cm} (34)

$$B(k, k_L) = \tau^4 \frac{C(k, k_L)}{2D(k)D(k_L)}$$  \hspace{1cm} (35)

$$C(k, k_L) = \cos[(k + k_L)l] - r \cos[(k - k_L)l].$$  \hspace{1cm} (36)

Equations (33)-(36) brings us to Eqs.(21) and (22) of the main text.

**B. Nearly resonance excitation**

Consider the case of excitation with $\omega_L$ not far from $\omega_0$, which is the resonance frequency of the half-cavity, neglecting the dissipation. We are infested in a narrow spectral range about $\omega_L$. To characterize the the detuning with respect to $\omega_0$ and the frequency range of interest, which we keep in mind to be about the mechanical frequency, we introduce dimensionless parameters

$$Q = |\omega_L - \omega_0|l \ll 1$$

and

$$q = |k - \omega_L/c|l = |\Omega l/c| \ll 1.$$  

Next we expand $B(k, k_L)$ with respect to small parameters of the problem $t$, $\tau$, $Q$, and $q$, keeping the lowest order terms. We readily find

$$D(k_L) = -t^2 + 4Q^2 + i\tau^2Q,$$

$$D(k) = -t^2 + 4(Q + q)^2 + i\tau^2(Q + q),$$

$$C(k, k_L) = \frac{t^2 - 4Q^2 - 4Qq}{2},$$

and

$$B(k, k_L) = \frac{\tau^4}{4} \frac{t^2 - 4Q^2 - 4Qq}{[t^2 - 4Q^2 - i\tau^2Q][t^2 - 4(Q + q)^2 - i\tau^2(Q + q)]}. \hspace{1cm} (37)$$
In the case of the "resonance" excitation, i.e. at $\omega_L = ck_{00}$ i.e. $Q = 0$, Eq. (37) boils down to the from

$$B(k, k_L) = \frac{\tau^4}{4} \frac{1}{t^2 - 4q^2 - i\tau^2q}.$$  

(38)

which, using the definitions $\Omega_0 \equiv \frac{\gamma_0}{2\Omega}$ and $\gamma_0 \equiv \frac{c\tau^2}{4t}$, can be rewritten as

$$B(k, k_L) = \frac{\gamma_0^2}{\Omega_0^2 - \Omega^2 - i\gamma_0\Omega_0}.$$  

(39)

This brings us to Eq.(23) of the main text.

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In the absence of the dissipation, i.e. at $\rho = 1$, this way we can find the profiles of the true physical fields in the system. However, as far as the dissipation is involved this is not exactly the case since, in view of complex $k$-vectors, our approach is manipulating with the slightly increasing waves between the mirrors and membrane instead of perfectly sinusoidal. Nevertheless, such a disparity will introduce a relative small mistake about $k_{il} \approx \tau^2 \ll 1$ such that one can trust the aforementioned calculations to within $\tau^2$, which is actually the precision mostly accepted in the work.