THE $L^2$-ALEXANDER TORSION IS SYMMETRIC

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Abstract. We show that the $L^2$-Alexander torsion of a 3-manifold is symmetric. This can be viewed as a generalization of the symmetry of the Alexander polynomial of a knot.

1. Introduction

An admissible triple $(N, \phi, \gamma)$ consists of an irreducible, orientable, compact 3–manifold $N \neq S^1 \times D^2$ with empty or toroidal boundary, a non-zero class $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$ and a homomorphism $\gamma: \pi_1(N) \to G$ such that $\phi$ factors through $\gamma$.

In [DFL14a, DFL14b] we used the $L^2$–torsion (see e.g. [Lü02]) to associate to an admissible triple $(N, \phi, \gamma)$ the $L^2$–Alexander torsion $\tau(2)(N, \phi, \gamma)$ which is a function $\tau(2)(N, \phi, \gamma): \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ that is well-defined up to multiplication by a function of the type $t \mapsto t^m$ for some $m \in \mathbb{Z}$. We recall the definition in Section 6.1.

The goal of this paper is to show that the $L^2$-Alexander torsion is symmetric. In order to state the symmetry result we need to recall that given a 3–manifold $N$ the Thurston norm $[\text{Th86}]$ of some $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$ is defined as

$$x_N(\phi) := \min \{ \chi_-(S) \mid S \subset N \text{ properly embedded surface dual to } \phi \}.$$ 

Here, given a surface $S$ with connected components $S_1 \cup \cdots \cup S_k$, we define its complexity as

$$\chi_-(S) = \sum_{i=1}^k \max \{ -\chi(S_i), 0 \}.$$ 

Thurston [Th86] showed that $x_N$ is a, possibly degenerate, norm on $H^1(N; \mathbb{Z})$. We can now formulate the main result of this paper.

Theorem 1.1. Let $(N, \phi, \gamma)$ be an admissible triple. Then for any representative $\tau$ of $\tau(2)(N, \phi, \gamma)$ there exists an $n \in \mathbb{Z}$ with $n \equiv x_N(\phi) \mod 2$ such that

$$\tau(t^{-1}) = t^n \cdot \tau(t)$$

for any $t \in \mathbb{R}_{>0}$. 

Date: November 11, 2014.

2010 Mathematics Subject Classification. Primary 57M27; Secondary 57Q10.

Key words and phrases. $L^2$-Alexander torsion, duality, Thurston norm.
It is worth looking at the case that $N = S^3 \setminus \nu K$ is the complement of a tubular neighborhood $\nu K$ of an oriented knot $K \subset S^3$. We denote by $\phi_K : \pi_1(S^3 \setminus \nu K) \to \mathbb{Z}$ the epimorphism sending the oriented meridian to 1. Let $\gamma : \pi_1(N) \to G$ be a homomorphism such that $\phi_K$ factors through $\gamma$. We define

$$\tau^{(2)}(K, \gamma) := \tau^{(2)}(S^3 \setminus \nu K, \phi_K, \gamma).$$

If we take $\gamma = \text{id}$ to be the identity, then we showed in [DFL14b] that

$$\tau^{(2)}(K, \text{id}) = \Delta^{(2)}_K(t) \cdot \max\{1, t\},$$

where $\Delta^{(2)}_K(t) : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ denotes the $L^2$-Alexander invariant which was first introduced by Li–Zhang [LZ06a, LZ06b, LZ08] and which was also studied in [DW10, DW13, BA13a, BA13b].

If we take $\gamma = \phi_K$, then we showed in [DFL14b] that the $L^2$-Alexander torsion $\tau^{(2)}(K, \phi_K)$ is fully determined by the Alexander polynomial $\Delta_K(t)$ of $K$ and that in turn $\tau^{(2)}(K, \phi_K)$ almost determines the Alexander polynomial $\Delta_K(t)$. In this sense the $L^2$-Alexander torsion can be viewed as a ‘twisted’ version of the Alexander polynomial, and at least morally it is related to the twisted Alexander polynomial of Lin [Li01] and Wada [Wa94] and to the higher-order Alexander polynomials of Cochran [Co04] and Harvey [Ha05]. We refer to [DFL14a] for more on the relationship and similarities between the various twisted invariants.

If $K$ is a knot, then any Seifert surface is dual to $\phi_K$ and it immediately follows that $x(\phi_K) \leq \max\{2 \cdot \text{genus}(K) - 1, 0\}$. In fact an elementary argument shows that for any non-trivial knot we have the equality $x(\phi_K) = 2 \cdot \text{genus}(K) - 1$. It follows in particular that the Thurston norm of $\phi_K$ is odd. We thus obtain the following corollary to Theorem 1.1.

**Theorem 1.2.** Let $K \subset S^3$ be an oriented non-trivial knot and let $\gamma : \pi_1(N) \to G$ be a homomorphism such that $\phi_K$ factors through $\gamma$. Then there exists an odd $n$ with

$$\tau^{(2)}(K, \gamma)(t^{-1}) = t^n \cdot \tau^{(2)}(K, \gamma)(t) \text{ for any } t \in \mathbb{R}_{>0}.$$
in Section 4. In Section 5 we relate the $L^2$-torsion of a 3-manifold to a relative $L^2$-torsion. Finally, in Section 6 we introduce the $L^2$-Alexander torsion of an admissible triple and we prove Theorem 1.1.

**Conventions.** All manifolds are assumed to be smooth, connected, orientable and compact. All CW-complexes are assumed to be finite and connected. If $G$ is a group then we equip $\mathbb{C}[G]$ with the usual involution given by complex conjugation and by $\overline{g} := g^{-1}$ for $g \in G$. We extend this involution to matrices over $\mathbb{C}[G]$ by applying the involution to each entry.

Given a ring $R$ we will view all modules as left $R$-modules, unless we say explicitly otherwise. Furthermore, given a matrix $A \in M_{m,n}(R)$, by a slight abuse of notation, we denote by $A: R^m \to R^n$ the $R$-homomorphism of left $R$-modules obtained by right multiplication with $A$ and thinking of elements in $R^m$ as the only row in a $(1,m)$-matrix.

**Acknowledgments.** The second author gratefully acknowledges the support provided by the SFB 1085 ‘Higher Invariants’ at the University of Regensburg, funded by the Deutsche Forschungsgemeinschaft DFG. The paper is also financially supported by the Leibniz-Preis of the third author granted by the DFG.

2. Euler structures

In this section we recall the notion of an Euler structure of a pair of CW-complexes and manifolds which is due to Turaev. We refer to [Tu90, Tu01, FKK12] for full details. Throughout this paper, given a space $X$, we denote by $H_1(X)$ the first integral homology group viewed as a multiplicative group.

2.1. **Euler structures on CW-complexes.** Let $X$ be a finite CW-complex of dimension $m$ and let $Y$ be a proper subcomplex. We denote by $p: \tilde{X} \to X$ the universal covering of $X$ and we write $\tilde{Y} := p^{-1}(Y)$. An Euler lift $c$ is a set of cells in $\tilde{X}$ such that each cell of $X \setminus Y$ is covered by precisely one of the cells in the Euler lift.

Using the canonical left action of $\pi = \pi_1(X)$ on $\tilde{X}$ we obtain a free and transitive action of $\pi$ on the set of cells of $\tilde{X} \setminus \tilde{Y}$ lying over a fixed cell in $X \setminus Y$. If $c$ and $c'$ are two Euler lifts, then we can order the cells such that $c = \{c_{ij}\}$ and $c' = \{c'_{ij}\}$ and such that for each $i$ and $j$ the cells $c_{ij}$ and $c'_{ij}$ lie over the same $i$-cell in $X \setminus Y$. In particular there exist unique $g_{ij} \in \pi$ such that $c'_{ij} = g_{ij} \cdot c_{ij}$. We now write $\mathcal{H} = \mathcal{H}_1(X)$ and we denote the projection map $\pi \to \mathcal{H}$ by $\Psi$. We define

$$c'/c := \prod_{i=0}^{m} \prod_{j} (\Psi(g_{ij}))^{(-1)^i} \in \mathcal{H}.$$ 

We say that $c$ and $c'$ are equivalent if $c'/c \in \mathcal{H}$ is trivial. An equivalence class of Euler lifts will be referred to as an *Euler structure*. We denote by $\text{Eul}(X,Y)$ the set of Euler structures. If $Y = \emptyset$ then we will also write $\text{Eul}(X) = \text{Eul}(X,Y)$.
Given \( g \in H \) and \( e \in \text{Eul}(X, Y) \) we define \( g \cdot e \in \text{Eul}(X, Y) \) as follows: pick a representative \( c \) for \( e \) and pick \( \tilde{g} \in \pi_1(X) \) which represents \( g \), then act on one \( i \)-cell of \( c \) by \( g^{(i-1)} \). The resulting Euler lift represents an element in \( \text{Eul}(X, Y) \) which is independent of the choice of the cell. We denote by \( g \cdot e \) the Euler structure represented by this new Euler lift. This defines a free and transitive \( H \)-action on \( \text{Eul}(X, Y) \), with \( (g \cdot e)/e = g \).

If \((X', Y')\) is a cellular subdivision of \((X, Y)\), then there exists a canonical \( H_1(X) \)-equivariant bijection \( \sigma : \text{Eul}(X, Y) \rightarrow \text{Eul}(X', Y') \) which is defined as follows: Let \( e \in \text{Eul}(X, Y) \) and pick an Euler lift for \((X, Y)\) which represents \( e \). There exists a unique Euler lift for \((X', Y')\) such that the cells in the Euler lift of \((X', Y')\) are contained in the cells of the Euler lift of \((X, Y)\). We then denote by \( \sigma(e) \) the Euler structure represented by this Euler lift. This map agrees with the map defined by Turaev [Tu90, Section 1.2].

2.2. **Euler structures of smooth manifolds.** Now we will quickly recall the definition of Euler structures on smooth manifolds. Let \( N \) be a manifold and let \( \partial_0 N \subset \partial N \) be a union of components of \( \partial N \) such that \( \chi(N, \partial_0 N) = 0 \). We write \( H = H_1(N) \). A triangulation of \( N \) is a pair \((X, t)\) where \( X \) is a simplicial complex and \( t : |X| \rightarrow N \) is a homeomorphism. Note that \( t^{-1}(\partial_0 N) \) is a simplicial subspace of \( X \). Throughout this section we write \( Y := t^{-1}(\partial_0 N) \). For the most part we will suppress \( t \) from the notation. Following [Tu90, Section I.4.1] we consider the projective system of sets \( \{\text{Eul}(X, Y)\}_{(X, t)} \) where \((X, t)\) runs over all \( C^1 \)-triangulations of \( N \) and where the maps are the \( H \)-equivariant bijections between these sets induced either by \( C^1 \)-subdivisions or by smooth isotopies in \( N \).

Now we define \( \text{Eul}(N, \partial_0 N) \) by identifying the sets \( \{\text{Eul}(X, Y)\}_{(X, t)} \) via these bijections. We refer to \( \text{Eul}(N, \partial_0 N) \) as the set of Euler structures on \((N, \partial_0 N)\). Note that for a \( C^1 \)-triangulation \( X \) of \( N \) we get a canonical \( H \)-equivariant bijection \( \text{Eul}(X, Y) \rightarrow \text{Eul}(N, \partial_0 N) \).

3. The \( L^2 \)-torsion of a manifold

3.1. **The Fuglede-Kadison determinant and the \( L^2 \)-torsion of a chain complex.** Before we start with the definition of the \( L^2 \)-Alexander torsion we need to recall some key properties of the Fuglede-Kadison determinant and the definition of the \( L^2 \)-torsion of a chain complex of free based left \( \mathbb{C}[G] \)-modules. Throughout the section we refer to [Lü02] and to [DFL14b] for details and proofs.

We fix a group \( G \). Let \( A \) be a \( k \times l \)-matrix over \( \mathbb{C}[G] \). Then there exists the notion of \( A \) being of ‘determinant class’. (To be slightly more precise, we view the \( k \times l \)-matrix \( A \) as a homomorphism \( \mathcal{N}(G)^l \rightarrow \mathcal{N}(G)^k \), where \( \mathcal{N}(G) \) is the von Neumann algebra of \( G \), and then there is the notion of being of ‘determinant class’.) We treat this entirely as a black box, but we note that if \( G \) is residually amenable, e.g., a 3-manifold group [He87] or solvable, then by [Lü94, Sc01, Ci99, ES05] any matrix
over $\mathbb{Q}[G]$ is of determinant class. If the matrix $A$ is not of determinant class then for the purpose of this paper we define $\det_{\mathcal{N}(G)}(A) = 0$. On the other hand, if $A$ is of determinant class, then we define

$$\det_{\mathcal{N}(G)}(A) := \text{Fuglede-Kadison determinant of } A \in \mathbb{R}_{>0}.$$ 

Note that we do not assume that $A$ is a square matrix. We will not provide a definition of the Fuglede-Kadison determinant but we summarize a few key properties in the following theorem which is basically a consequence of [Lü02, Example 3.12] and [Lü02, Theorem 3.14].

**Theorem 3.1.**

1. If $A$ is a square matrix with complex entries such that the usual determinant $\det(A) \in \mathbb{C}$ is non-zero, then $\det_{\mathcal{N}(G)}(A) = |\det(A)|$.
2. The Fuglede-Kadison determinant does not change if we swap two rows or two columns.
3. Right multiplication of a column by $\pm g$ with $g \in G$ does not change the Fuglede–Kadison determinant.
4. For any matrix $A$ over $\mathbb{C}[G]$ we have

$$\det_{\mathcal{N}(G)}(A) = \det_{\mathcal{N}(G)}(A^t).$$

Note that (2) implies that when we study determinants of homomorphisms we can work with unordered bases.

Now let

$$C_* = \left( 0 \to C_l \xrightarrow{\partial_l} C_{l-1} \xrightarrow{\partial_{l-1}} \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_1} C_0 \to 0 \right)$$

be a chain complex of free left $\mathbb{C}[G]$-modules. We can then consider the corresponding $L^2$-Betti numbers $b^{(2)}_i(C_*) \in \mathbb{R}_{\geq 0}$, as defined in [Lü02].

Now suppose that the chain complex is equipped with bases $B_i \subset C_i$, $i = 0, \ldots, l$. If at least one of the $L^2$-Betti numbers $b^{(2)}_i(C_*)$ is non-zero or if at least one the boundary maps is not of determinant class, then we define the $L^2$-torsion $\tau^{(2)}(C_*, B_*) := 0$. Otherwise we define the $L^2$-torsion of the based chain complex $C_*$ to be

$$\tau^{(2)}(C_*, B_*) := \prod_{i=1}^l \det_{\mathcal{N}(G)}(A_i)^{-1} \in \mathbb{R}_{>0}$$

where the $A_i$ denote the boundary matrices corresponding to the given bases. Note that this definition is the multiplicative inverse of the exponential of the $L^2$-torsion as defined in [Lü02, Definition 3.29].

### 3.2. The twisted $L^2$-torsion of a pair of CW-complexes.

Let $(X, Y)$ be a pair of finite CW-complexes and let $e \in \text{Eul}(X, Y)$. We denote by $p: \tilde{X} \to X$ the universal covering of $X$ and we write $\tilde{Y} := p^{-1}(Y)$. Note that the deck transformation turns $C_*(\tilde{X}, \tilde{Y})$ naturally into a chain complex of left $\mathbb{Z}[\pi_1(X)]$-modules.
Now let $G$ be a group and let $\varphi : \pi(X) \to \GL(d, \mathbb{C}[G])$ be a representation. We view elements of $\mathbb{C}[G]^d$ as row vectors. Right multiplication via $\varphi(g)$ thus turns $\mathbb{C}[G]^d$ into a right $\mathbb{Z}[\pi_1(X)]$-module. We then consider the chain complex

$$C^\varphi_*(X, Y; \mathbb{C}[G]^d) := \mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X}, \tilde{Y})$$

of left $\mathbb{C}[G]$-modules.

Now let $e \in \text{Eul}(X, Y)$. We pick an Euler lift $\{c_{ij}\}$ which represents $e$. Throughout this paper we denote by $v_1, \ldots, v_d$ the standard basis for $\mathbb{C}[G]^d$. We equip the chain complex $C^\varphi_*(X, Y; \mathbb{C}[G]^d)$ with the basis provided by the $v_k \otimes c_{ij}$. Therefore we can define

$$\tau^{(2)}(X, Y, \varphi, e) := \tau^{(2)}(C^\varphi_*(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}) \in \mathbb{R}_{\geq 0}.$$ 

We summarize a few properties of the $L^2$-torsion in the following lemma.

**Lemma 3.2.**

1. The number $\tau^{(2)}(X, Y, \varphi, e)$ is well-defined, i.e., independent of the choice of the Euler lift which represents $e$.
2. If $g \in \mathcal{H}_1(X)$, then

$$\tau^{(2)}(X, Y, \varphi, ge) = \det_{\mathcal{N}(G)}(\varphi(g^{-1})) \cdot \tau^{(2)}(X, Y, \varphi, e).$$

3. If $\delta : \pi_1(X) \to \GL(d, \mathbb{C}[G])$ is conjugate to $\varphi$, i.e., if there exists an $A \in \GL(d, \mathbb{C}[G])$ such that $\delta(g) = A \varphi(g) A^{-1}$ for all $g \in \pi_1(X)$, then

$$\tau^{(2)}(X, Y, \delta, e) = \tau^{(2)}(X, Y, \varphi, e).$$

4. If $(X', Y')$ is a cellular subdivision of $(X, Y)$ and if $e' \in \text{Eul}(X', Y')$ is the Euler structure corresponding to $e$, then

$$\tau^{(2)}(X', Y', \varphi, e') = \tau^{(2)}(X, Y, \varphi, e).$$

The proofs are completely analogous to the proofs for ordinary Reidemeister torsion as given in [Tu86, Tu01, FKK12]. In the interest of time and space we will therefore not provide the proofs.

### 3.3. The $L^2$-Alexander torsion for manifolds

Let $N$ be a manifold and let $\partial_0 N \subset \partial N$ be a union of components of $\partial N$. Let $G$ be a group and let $\varphi : \pi(N) \to \GL(d, \mathbb{C}[G])$ be a representation. Finally let $e \in \text{Eul}(N, \partial_0 N)$.

Recall that for any $C^1$-triangulation $f : X \to N$ we get a canonical bijection $\text{Eul}(X, Y) \xrightarrow{f_*} \text{Eul}(N, \partial_0 N)$. Now we define

$$\tau^{(2)}(N, \partial_0 N, \varphi, e) := \tau^{(2)}(X, Y, \varphi \circ f_*, f_*^{-1}(e)).$$

By Lemma 3.2 (4) and the discussion in [Tu90] the invariant $\tau^{(2)}(N, \partial_0 N, \varphi, e) \in \mathbb{R}_{\geq 0}$ is well-defined, i.e., independent of the choice of the triangulation.
4. Duality for torsion of manifolds equipped with Euler structures

4.1. The algebraic duality theorem for \(L^2\)-torsion. Let \(G\) be a group and let \(V\) be a right \(C[G]\)-module. We denote by \(\overline{V}\) the left \(C[G]\)-module with the same underlying abelian group together with the module structure given by \(v \overline{v} p := \overline{p \cdot v}\) for any \(p \in C[G]\) and \(v \in V\). If \(V\) is a left \(C[G]\)-module then we can consider \(\text{Hom}_{C[G]}(V, C[G])\) the set of all left \(C[G]\)-module homomorphisms. Note that the fact that the range \(C[G]\) is a \(C[G]\)-bimodule implies that \(\text{Hom}_{C[G]}(V, C[G])\) is naturally a right \(C[G]\)-module.

In the following let \(C_*\) be a chain complex of length \(m\) of left \(C[G]\)-modules with boundary operators \(\partial_i\). Suppose that \(C_*\) is equipped with a basis \(V_i\) for each \(C_i\). We denote by \(C^\#\) the dual chain complex whose chain groups are the \(C[G]\)-left modules \(C^\#_i := \text{Hom}_{C[G]}(C_{m-i}, C[G])\) and where the boundary map \(\partial^\#_i : C^\#_{i+1} \to C^\#_i\) is given by \((-1)^{m-i}\partial_{m-i-1}\). This means that for any \(c \in C_{m-i}\) and \(d \in C^\#_{i+1}\) we have \(\partial^\#_i(d)(c) = (-1)^{m-i}d(\partial_{m-i-1}(c))\). We denote by \(B^\#_*\) the bases of \(C^\#\) dual to the bases \(V_*\). We have the following lemma.

**Lemma 4.1.** If \(\tau(2)(C_*, B_*) = 0\), then \(\tau(2)(C^\#, B^\#) = 0\), otherwise we have
\[
\tau(2)(C_*, B_*) = \tau(2)(C^\#, B^\#)(-1)^{m+1}.
\]

**Proof.** We first note that by the proof of [Liu02, Theorem 1.35 (3)] the \(L^2\)-Betti numbers of \(C_*\) vanish if and only if the \(L^2\)-Betti numbers of \(C^\#\) vanish. If either does not vanish, then it thus follows that the other does not vanish, and both torsions are zero by definition.

We now suppose that the \(L^2\)-Betti numbers of \(C_*\) vanish. We denote by \(A_i\) the matrices of the boundary maps of \(C_*\) with respect to the given basis. It follows easily from the definitions that the boundary matrices of the chain complex \(C^\#_*\) with respect to the basis \(B^\#_*\) are given by \((-1)^{m-i}A^\dagger_i\). The lemma is now an immediate consequence of the definitions and of Theorem 3.1 (2).

4.2. The duality theorem for manifolds. Before we state our main technical duality theorem we need to introduce two more definitions.

1. Let \(G\) be a group and let \(\varphi : \pi \to GL(d, \mathbb{C}[G])\) be a representation. We denote by \(\varphi^\dagger\) the representation which is given by \(g \mapsto \overline{\varphi(g^{-1})}\).

2. Let \(N\) be an \(m\)-manifold and let \(e \in \text{Eul}(N, \partial N)\). We pick a triangulation \(X\) for \(N\). We denote by \(Y\) the subcomplex corresponding to \(\partial N\). Let \(X^\dagger\) be the CW-complex which is given by the cellular decomposition of \(N\) dual to \(X\). We pick an Euler lift \(\{c_{ij}\}\) which represents \(e \in \text{Eul}(X, Y) = \text{Eul}(N, \partial N)\). For any \(i\)-cell \(c\) in \(\tilde{X}\) we denote by \(c^\dagger\) the unique oriented \((m - i)\)-cell in \(\tilde{X}^\dagger\) which has intersection number +1 with \(c_{ij}\). The Euler lift \(\{c^\dagger_{ij}\}\) defines an element in \(\text{Eul}(X^\dagger) = \text{Eul}(N)\) that we denote by \(e^\dagger\). We refer to [Liu01, Section 1.4] and [FKK12, Section 4] for details.
In this section we will prove the following duality theorem.

**Theorem 4.2.** Let $N$ be an $m$-manifold. Let $G$ be a group and let $\varphi: \pi(N) \to GL(d, \mathbb{C}[G])$ be a representation. Let $e \in \text{Eul}(N, \partial N)$. Then either both $\tau(2)(N, \partial N, \varphi, e)$ and $\tau(2)(N, \varphi^\dagger, e^\dagger)$ are zero, or the following equality holds:

$$\tau(2)(N, \partial N, \varphi, e) = \tau(2)(N, \varphi^\dagger, e^\dagger)^{(-1)^{m+1}}.$$ 

**Proof.** As above we pick a triangulation $X$ for $N$ and we denote by $Y$ the subcomplex corresponding to $\partial N$. Let $X^\dagger$ be the CW-complex which is given by the cellular decomposition of $N$ dual to $X$. In the following we make the identification $\pi = \pi_1(X) = \pi_1(N) = \pi_1(X^\dagger)$.

For the remainder of this section we pick an Euler lift $\{c_{ij}\}$ which represents $e \in \text{Eul}(N, \partial N) = \text{Eul}(X, Y)$. We denote by $c_{ij}^\dagger$ the corresponding dual cells. Theorem 4.2 follows immediately from the definitions and the following claim.

**Claim.** Either both $\tau(2)(C_\varphi^\dagger(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\})$ and $\tau(2)(C_\varphi(X^\dagger; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}^\dagger\})$ are zero, or the following equality holds:

$$\tau(2)(C_\varphi^\dagger(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}) = \tau(2)(C_\varphi(X^\dagger; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}^\dagger\})^{(-1)^{m+1}}.$$

In order to prove the claim we first note that there is a unique, non-singular intersection $\mathbb{Z}$-linear pairing

$$C_{m-i}(\widetilde{X}, \widetilde{Y}) \times C_i(\widetilde{X}^\dagger) \to \mathbb{Z}$$

with the property that $a \cdot b^\dagger = \delta_{ab}$ for any cell $a$ of $\widetilde{X} \setminus \widetilde{Y}$ and any cell $b$ of $\widetilde{X}$. We then consider the following pairing:

$$C_{m-i}(\widetilde{X}, \widetilde{Y}) \times C_i(\widetilde{X}^\dagger) \to \mathbb{Z}[\pi]$$

$$(a, b) \mapsto \langle a, b \rangle := \sum_{g \in \pi}(a \cdot gb)g^{-1}.$$ 

Note that this pairing is sesquilinear in the sense that for any $a \in C_{m-i}(\widetilde{X}, \widetilde{Y})$, $b \in C_i(\widetilde{X}^\dagger)$ and $p, q \in \mathbb{Z}[\pi]$ we have $\langle pa, qb \rangle = q\langle a, b \rangle p$. It is furthermore straightforward to see that the pairing is non-singular. This pairing has the property (see e.g. [Tu01, Claim 14.4]) that the following diagram commutes:

$$
\begin{array}{ccc}
C_{i+1}(\widetilde{X}, \widetilde{Y}) & \times & C_{m-i-1}(\widetilde{X}^\dagger) \\
\downarrow \partial_i & & \uparrow (-1)^{i+1}\partial_{m-i-1} \\
C_i(\widetilde{X}, \widetilde{Y}) & \times & C_{m-i}(\widetilde{X}^\dagger) \\
\end{array} \to \mathbb{Z}[\pi].$$

Put differently, the maps

$$C_i(\widetilde{X}, \widetilde{Y}) \to \text{Hom}_{\mathbb{Z}[\pi]}(C_{m-i}(\widetilde{X}^\dagger), \mathbb{Z}[\pi])$$

$$a \mapsto (b \mapsto \langle a, b \rangle)$$
define an isomorphism of based chain complexes of right \( \mathbb{Z}[\pi] \)-modules. In fact it follows easily from the definitions that the maps define an isomorphism

\[
(C_\ast(\tilde{X}, \tilde{Y}), \{c_{ij}\}) \to (\text{Hom}_{\mathbb{Z}[\pi]}(C_{m-\ast}(X^\dagger), \mathbb{Z}[\pi]), \{(c_{ij})^\ast\})
\]

of based chain complexes of left \( \mathbb{Z}[\pi] \)-modules. Tensoring these chain complexes with \( C[G]^d \) we obtain an isomorphism

\[
(C[G]^d \otimes_{\mathbb{Z}[\pi]} C_\ast(\tilde{X}, \tilde{Y}), \{v_k \otimes c_{ij}\}) \to (C[G]^d \otimes_{\mathbb{Z}[\pi]} \text{Hom}_{\mathbb{Z}[\pi]}(C_{m-\ast}(X^\dagger)^\ast, \mathbb{Z}[\pi]), \{(c_{ij})^\ast \otimes v_k\})
\]

of based chain complexes of \( \mathbb{Z}[G] \)-modules. Furthermore the maps

\[
\begin{align*}
C[G]^d \otimes_{\mathbb{Z}[\pi]} \text{Hom}_{\mathbb{Z}[\pi]}(C_\ast(\tilde{X}^\dagger), \mathbb{Z}[\pi]) & \to \text{Hom}_{C[G]}(C_{\ast}^\dagger(X^\dagger; \mathbb{C}[G]^d), C[G]) \\
v \otimes f & \mapsto \left( C_{\ast}^\dagger(X^\dagger; \mathbb{C}[G]^d) \to C[G] \right)
\end{align*}
\]

\( w \otimes \sigma \mapsto v \varphi(f(\sigma)) \sigma w \)

induce an isomorphism

\[
(C_\ast^\dagger(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}) \to \left( C_\ast^\dagger(X^\dagger; \mathbb{C}[G]^d)^\#, \{(v_k \otimes c_{ij}^\#)^\#\} \right)
\]

of based chain complexes of left \( \mathbb{C}[G] \)-modules. The claim is now an immediate consequence of Lemma 4.1. \( \square \)

5. Twisted \( L^2 \)-torsion of 3-manifolds

Now we are heading towards the proof of Theorem 1.1. Therefore we are turning towards the study of \( L^2 \)-torsions of 3-manifolds. In order to turn Theorem 4.2 into the desired symmetry result we will need to relate the \( L^2 \)-torsions of a 3-manifold \( N \) and the relative \( L^2 \)-torsions of the pair \( (N, \partial N) \). Henceforth we will restrict ourselves to one-dimensional representations since these are precisely the ones which we will need in the proof of Theorem 1.1.

5.1. Canonical structures on tori. Let \( T \) be a torus. We equip \( T \) with a CW-structure with one 0-cell \( p \), two 1-cells \( x \) and \( y \) and one 2-cell \( s \). We write \( \pi = \pi_1(T, p) \) and by a slight abuse of notation we denote by \( x \) and \( y \) the elements in \( \pi \) represented by \( x \) and \( y \). We denote by \( \tilde{T} \) the universal cover of \( T \). Then there exist lifts of the cells such that the chain complex \( C_\ast(\tilde{T}) \) of left \( \mathbb{Z}[\pi] \)-modules with respect to the bases given by these lifts is of the form

\[
0 \to \mathbb{Z}[\pi] \xrightarrow{(y-1, 1-x)} \mathbb{Z}[\pi]^2 \xrightarrow{(1-x, 1-y)} \mathbb{Z}[\pi] \to 0.
\]
We refer to the corresponding Euler structure of \( T \) as the *canonical Euler structure on \( T \).* This definition is identical to the definition provided by Turaev [Tu02, p. 10].

Given a group \( G \) we say that a representation \( \varphi: \pi \to \text{GL}(1, \mathbb{C}[G]) \) is *monomial* if for any \( x \in \pi \) we have \( \varphi(x) = zg \) for some \( z \in \mathbb{C} \) and \( g \in G \). We now have the following lemma.

**Lemma 5.1.** Let \( T \) be the torus and let \( \varphi: \pi(T) \to \text{GL}(1, \mathbb{C}[G]) \) be a monomial representation such that \( b_1^{(2)}(T; \mathbb{C}[G]^d) = 0 \). Let \( e \) be the canonical Euler structure on \( T \). Then

\[
\tau^{(2)}(T, \varphi, e) = 1.
\]

**Proof.** In [DFL14b] we used the canonical Euler structure (even though we did not call it that way) to compute \( \tau^{(2)}(T, \varphi) = 1 \). \( \square \)

5.2. Chern classes on 3-manifolds with toroidal boundary. Let \( N \) be a compact, orientable 3-manifold with toroidal incompressible boundary and let \( e \in \text{Eul}(N, \partial N) \).

Let \( X \) be a triangulation for \( N \). We denote the subcomplexes corresponding to the boundary components of \( N \) by \( S_1 \cup \cdots \cup S_b \). We denote by \( p: \tilde{X} \rightarrow X \) and \( p_i: \tilde{S}_i \rightarrow S_i, i = 1, \ldots , b \) the universal covering maps of \( X \) and \( S_i \). For each \( i \) we identify a component of \( p^{-1}(S_i) \) with \( \tilde{S}_i \).

We pick an Euler lift \( c \) which represents \( e \). For each boundary torus \( S_i \) we pick an Euler lift \( \tilde{s}_i \) to \( \tilde{S}_i \subset p^{-1}(S_i) \subset \tilde{X} \) which represents the canonical Euler structure. The set of cells \( \{ \tilde{s}_1, \ldots , \tilde{s}_b, e \} \) defines an Euler structure \( K(e) \) for \( N \), which only depends on \( e \). Put differently, we just defined a map \( K: \text{Eul}(N, \partial N) \rightarrow \text{Eul}(N) \) which is easily seen to be \( \mathcal{H}_1(N) \)-equivariant.

Given \( e \in \text{Eul}(N) \) there exists a unique element \( g \in \mathcal{H}_1(N) \) such that \( e = g \cdot K(e^\dagger) \). Following Turaev [Tu02, p. 11] we define \( c_1(e) := g \in H_1(N; \mathbb{Z}) \) and we refer to \( c_1(e) \) as the Chern class of \( e \).

5.3. Torsions of 3-manifolds. Let \( \pi \) and \( G \) be groups and let \( \varphi: \pi \to \text{GL}(1, \mathbb{C}[G]) \) be a monomial representation. It follows from the multiplicativity of the Fuglede-Kadison determinant, see [Lück02, Theorem 3.14], that given \( g \in \pi \) the invariant \( \det_{\mathbb{N}(G)}(\varphi(g)) \) only depends on the homology class of \( g \). Put differently, \( \det_{\mathbb{N}(G)} \circ \varphi: H_1(\pi; \mathbb{Z}) \rightarrow \mathbb{R}_{\geq 0} \) descends to a map \( \det_{\mathbb{N}(G)} \circ \varphi: H_1(\pi; \mathbb{Z}) \rightarrow \mathbb{R}_{\geq 0} \). We can now formulate the following theorem.

**Theorem 5.2.** Let \( N \) be a 3-manifold which is either closed or which has toroidal and incompressible boundary. Let \( G \) be a group and let \( \varphi: \pi(N) \to \text{GL}(1, \mathbb{C}[G]) \) be a monomial representation. Suppose that \( b_1^{(2)}(\partial N; \mathbb{C}[G]) = 0 \). Then for any \( e \in \text{Eul}(N, \partial N) \) we have

\[
\tau^{(2)}(N, \partial N, \varphi, e^\dagger) = \det_{\mathbb{N}(G)}(\varphi(c_1(e))) \cdot \tau^{(2)}(N, \varphi, e).
\]
Proof. The assumption that $b_\tau^s(\partial N; \mathbb{C}[G]) = 0$ together with the proof of Theorem 1.35 (2)] implies that $b_\tau^s(N; \mathbb{C}[G]) = 0$ if and only if $b_\tau^s(N, \partial N; \mathbb{C}[G]) = 0$. If both are non-zero, then both torsions $\tau^s(N, \partial N, \varphi, e^\dagger)$ and $\tau^s(N, \varphi, e)$ are zero. For the remainder of this proof we now assume that $b_\tau^s(N; \mathbb{C}[G]) = 0$.

We pick a triangulation $X$ for $N$. As usual we denote by $Y$ the subcomplex corresponding to $\partial N$. Let $e \in \text{Eul}(N, \partial N) = \text{Eul}(X, Y)$. We pick an Euler lift $c_\ast$ which represents $e^\dagger$. We denote the components of $Y$ by $Y_1 \cup \cdots \cup Y_b$ and we pick $\tilde{s}_1^\ast, \ldots, \tilde{s}_b^\ast$ as in the previous section. We write $\tilde{s}_\ast = \tilde{s}_1^\ast \cup \cdots \cup \tilde{s}_b^\ast$. We denote by $\{\tilde{s}_\ast, c_\ast\}$ the resulting Euler lift for $X$. Recall that this Euler lift represents $K(e)$. We have the following claim.

Claim.

$$\tau^s(N, \partial N, \varphi, e^\dagger) = \tau^s(N, \varphi, e).$$

In order to prove the claim we consider the following short exact sequence of chain complexes

$$0 \to \bigoplus_{i=1}^b C^s_\ast(Y_i; \mathbb{C}[G]) \to C^s_\ast(X; \mathbb{C}[G]) \to C^s_\ast(X, Y; \mathbb{C}[G]) \to 0,$$

with the bases

$$\{s_i^\ast\}_{i=1,\ldots,b}, \{\tilde{s}_\ast \cup c_\ast\} \text{ and } \{c_\ast\}.$$

Note that these bases are in fact compatible, in the sense that the middle basis is the image of the left basis together with a lift of the right basis. By Lemma 5.1 we have $\tau^s(C^s_\ast(Y_i; \mathbb{C}[G]), \{s_i^\ast\}) = 1$ for $i = 1, \ldots, b$. Now it follows from the multiplicativity of torsion, see Theorem 3.35], that

$$\tau^s(C^s_\ast(X, Y; \mathbb{C}[G]), \{c_\ast\}) = \tau^s(C^s_\ast(X; \mathbb{C}[G]), \{c_\ast \cup \tilde{s}_\ast\}).$$

Here we used that the complexes are acyclic. This concludes the proof of the claim.

Finally it follows from this claim, the definitions and Lemma 3.2 that

$$\tau^s(N, \partial N, \varphi, e^\dagger) = \tau^s(C^s_\ast(X, Y; \mathbb{C}[G]), \{c_\ast\}) = \tau^s(C^s_\ast(X; \mathbb{C}[G]), \{\tilde{s}_\ast \cup c_\ast\}) = \tau^s(N, \varphi, K(e^\dagger)) = \tau^s(N, \varphi, c_1(e)^{-1}e) = \det_{N[G]}(\varphi(c_1(e))) \cdot \tau^s(N, \varphi, e).$$

\[\square\]

6. The symmetry of the $L^2$-Alexander torsion

6.1. The $L^2$-Alexander torsion for 3-manifolds. Let $(N, \phi, \gamma: \pi_1(N) \to G)$ be an admissible triple and let $e \in \text{Eul}(N)$. Given $t \in \mathbb{R}_{>0}$ we consider the representation

$$\gamma_t: \pi_1(N) \to \text{GL}(1, \mathbb{C}[G])
\quad g \mapsto (t^{\phi(g)}) \gamma(g).$$
Then we denote by \( \tau^{(2)}(N, \phi, \gamma, e) \) the function
\[
\tau^{(2)}(N, \phi, \gamma, e) : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \\
t \mapsto \tau^{(2)}(N, \gamma_t, e).
\]

Note that for a different Euler class \( e' \) we have \( e' = ge \) for some \( g \in \mathcal{H}_1(N) \) and it follows from Lemma 3.2 that
\[
\tau^{(2)}(N, \phi, \gamma, ge)(t) = t^{-\phi(g)} \tau^{(2)}(N, \phi, g, e)(t) \text{ for all } t \in \mathbb{R}_{>0}.
\]

Put differently, the functions \( \tau^{(2)}(N, \phi, \gamma, e) \) and \( \tau^{(2)}(N, \phi, \gamma, ge) \) are equivalent. We denote by \( \tau^{(2)}(N, \phi, \gamma) \) the equivalence class of the functions \( \tau^{(2)}(N, \phi, \gamma, e) \) and we refer to \( \tau^{(2)}(N, \phi, \gamma) \) as the \( L^2 \)-Alexander torsion of \( (N, \phi, \gamma) \).

6.2. Proof of Theorem 1.1. For the reader’s convenience we recall the statement of the proof of Theorem 1.1.

**Theorem 1.1.** Let \( (N, \phi, \gamma) \) be an admissible triple. Then for any representative \( \tau \) of \( \tau^{(2)}(N, \phi, \gamma) \) there exists an \( n \in \mathbb{Z} \) with \( n \equiv x_N(\phi) \mod 2 \) such that
\[
\tau(t^{-1}) = t^n \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.
\]

**Proof.** Let \( e \in \text{Eul}(N) \). We write \( \tau = \tau^{(2)}(N, \phi, \gamma, e) \). Let \( t \in \mathbb{R}_{>0} \). It follows easily from the definitions that \( (\gamma_t)^\dagger = \gamma_{t^{-1}} \). Using Theorems 4.2 and 5.2 we see that the following equalities hold:
\[
\tau(t) = \tau^{(2)}(N, \phi, \gamma, e) = \tau^{(2)}(N, \gamma_t, e) = \tau^{(2)}(N, \partial N, (\gamma_t)^\dagger, e^\dagger) = \tau^{(2)}(N, \partial N, \gamma_{t^{-1}}, e^\dagger)
\]
\[
= \det_{N(G)}(\gamma_{t^{-1}}(c_1(e))) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) = \det_{N(G)}(t^{-\phi(c_1(e))}c_1(e)) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) = t^{-\phi(c_1(e))} \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) = \tau^{(2)}(N, \gamma_{t^{-1}}, e) = t^{-\phi(c_1(e))} \cdot \tau(t^{-1}).
\]

Now it suffices to prove the following claim:

**Claim.** For any \( \phi \in H^1(N; \mathbb{Z}) \) we have
\[
\phi(c_1(e)) = x_N(\phi) \mod 2.
\]

Let \( S \) be a Thurston norm minimizing surface which is dual to \( \phi \). Since \( N \) is irreducible and since \( N \neq S^1 \times D^2 \) we can arrange that \( S \) has no disk components. Therefore we have
\[
x_N(\phi) \equiv x_-(S) \equiv b_0(\partial S) \mod 2\mathbb{Z}.
\]

On the other hand, by [Tu02, Lemma VI.1.2] and [Tu02, Section XI.1] we have that
\[
b_0(\partial S) \equiv c_1(e) \cdot S \mod 2\mathbb{Z}.
\]
where \( c_1(e) \cdot S \) is the intersection number of \( c_1(e) \in H_1(N) = H_1(N) \) with \( S \). Since \( S \) is dual to \( \phi \), we obtain that
\[
\phi(c_1(e)) \equiv c_1(e) \cdot S \equiv b_0(\partial S) \equiv \chi_-(S) \equiv x_N(\phi) \mod 2\mathbb{Z}.
\]
This concludes the proof of the claim.

\[\square\]

6.3. Extending the main result to real cohomology classes. A real admissible triple \((N, \phi, \gamma)\) consists of an irreducible, orientable, compact 3–manifold \( N \neq S^1 \times D^2 \) with empty or toroidal boundary, a non-zero class \( \phi \in H^1(N; \mathbb{R}) = \text{Hom}(\pi_1(N), \mathbb{R}) \) and a homomorphism \( \gamma: \pi_1(N) \to G \) such that \( \phi \) factors through \( \gamma \). Verbatim the same definition as in Section 6.1 associates to \((N, \phi, e)\) a function \( \tau^{(2)}(N, \phi, e): \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \) that is well-defined up to multiplication by a function of the form \( t \mapsto t^r \) for some \( r \in \mathbb{R} \). Furthermore, verbatim the same argument as in the proof of Theorem 1.1 gives us the following result.

**Theorem 6.1.** Let \((N, \phi, \gamma)\) be a real admissible triple. Then for any representative \( \tau \) of \( \tau^{(2)}(N, \phi, \gamma) \) there exists an \( r \in \mathbb{R} \) such that
\[
\tau(t^{-1}) = t^r \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.
\]

The only difference to Theorem 1.1 is that for real cohomology classes \( \phi \in H^1(N; \mathbb{R}) \) we can not relate the exponent \( r \) to the Thurston norm of \( \phi \).

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