Instabilities in Anisotropic Chiral Plasmas

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Using the Berry-curvature modified kinetic equation we study instabilities in anisotropic chiral plasmas. It is demonstrated that even for a very small value of the anisotropic parameter the chiral-imbalance instability is strongly modified. The instability is enhanced when the modes propagates in the direction parallel to the anisotropy vector and it is strongly suppressed when the modes propagate in the perpendicular direction. Further the instabilities in the jet-plasma system is also investigated. For the case when the modes are propagating in direction parallel to the stream velocity we find that there exist a new branch of the dispersion relation arising due to the parity odd effects. We also show that the parity-odd interaction can enhance the streaming instability.

I. INTRODUCTION

The scope of applying kinetic theory to understand variety of many-body problems arising in various branches of physics is truly enormous [1]. The conventional Boltzmann or Vlasov equations imply that the vector current associated with the gauge charges is conserved. But till recently a very important class of physical phenomena associated with the CP-violation or the triangle-anomaly were left out of the purview of a kinetic theory. In such a phenomenon the axial current is not conserved. It should be noted here that there exists a several models of hydrodynamics which incorporates the effect of CP-violation [2–5]. But a hydrodynamical approach requires that the system under consideration remains in a thermal and chemical equilibrium. However, many applications of the chiral (CP-violating) physics may involve a non-equilibrium situation e.g. during the early stages of relativistic heavy-ion collisions. Therefore it is highly desirable to have a proper kinetic theory framework to tackle the CP-violating effect. Recently there has been a lot of progress in developing such a kinetic theory. In Ref. [6–11] it was shown that if the Berry curvature [12] has nonzero flux across the Fermi-surface then the particles on the surface can exhibit a chiral anomaly in presence of an external electromagnetic field. In this formalism chiral-current \( j^\mu \) is not conserved and it can be attributed to Adler-Bell-Jackiw anomaly [13–15]. It can be shown that if a system of charged fermions does not conserve parity, it can develop an equilibrium electric current along an applied external magnetic field [16]. This is so called chiral-magnetic effect (CME). It has been suggested that a strong magnetic field created in relativistic-heavy-ion experiments can lead to CME in the quark-gluon plasma [17–19]. Indeed the recent experiments with STAR detector at Relativistic Heavy Ion Collider (RHIC) qualitatively agree with a local parity violation. However, more investigations are required to attribute this charge asymmetry with the CME [20, 21]. The idea that a Berry-phase can influence
the electronic properties [e.g. \cite{22} and references cited therein] is well-known in condensed matter literature and it can have applications in Weyl’s semimetal \cite{23}, graphene \cite{24} etc. There exists a deep connection between a CP-violating quantum field theory and the kinetic theory with the Berry curvature corrections. In Ref. \cite{25} it was shown that the parity-odd and parity-even correlations calculated using the modified kinetic theory are identical with the perturbative results obtained in next-to-leading order hard dense loop approximation.

In this work we aim to apply the kinetic theory with the Berry curvature curvature corrections to some non-equilibrium situations. We first note that the results obtained in Refs. \cite{6,25} are limited to low temperature regime $T \ll \mu$, where $\mu$ is chiral chemical potential, when the Fermi surface is well-defined. Recently Ref. \cite{26} argues that the domain of validity of the modified kinetic theory can be extended beyond the Fermi surface to include the effect of finite temperature. As expected from the considerations of quantum-field theoretic approach \cite{27-29} the parity-odd contribution remains temperature independent. Recently using the modified-kinetic theory \cite{25} in presence of the chiral imbalance the collective modes in electromagnetic or quark-gluon plasmas were analyzed \cite{30}. In such a system CP-violating effect can split transverse waves into two branches \cite{31}. It was found in Ref. \cite{30} that in the quasi-static limit i.e. for $\omega \ll k$, where $\omega$ and $k$ respectively denote frequency and wave-number of the transverse wave, there exists an unstable mode. The instability can lead to the growth of Chern-Simons number (or magnetic-helicity in plasma physics parlance) at expense of the chiral imbalance. Similar kind of instabilities were found in Refs. \cite{32-36} in different context.

In the present work we study collective modes in anisotropic chiral plasmas. In many realistic situations in condensed matter physics (see for example \cite{37,38}) and in plasma physics \cite{39} it is important to consider initial distribution function $n_0^p$ to be anisotropic in the momentum space. It is well-known that momentum anisotropy can lead to so called Weibel instability of transverse waves in plasma which can generate a magnetic field in the plasma \cite{40,41}. The Weibel instability is closely related with the streaming instabilities in plasma. Such instabilities may play an important role in thermalization of the quark-gluon plasma created in relativistic heavy-ion collision experiments \cite{12-46}. In this work we generalize the modified kinetic theory to consider anisotropic chiral plasma. In particular we consider two important cases: (i) when the distribution function $n_0^p$ has a preferred direction (an anisotropy) in the momentum space. (ii) when a stream of charged particles travel in a thermally equilibrated chiral-plasma. We believe that the results presented here will be useful in studying Weyl metals and quark-gluon plasmas created in relativistic heavy-ion collisions. The paper is organized as follows: In section II we give a brief introduction of the basic equations of the Berry-curve modified kinetic and the Linear response theories. Section III contains the case of Weibel-instability in anisotropic chiral-plasma. Section IV deals with instability related with the jet-plasma interaction. Section VI contains summary and conclusions.

**II. BASIC EQUATIONS**

The Berry curvature modified collisionless kinetic (Vlasov) equation for distribution function $n_p$ \cite{25} can be written as:

\[
\dot{n}_p + \frac{1}{1 + eB \cdot \Omega_p} \left[ \left( e\tilde{E} + e\tilde{v} \times B + (e^2\tilde{E} \cdot B)\Omega_p \right) \cdot \frac{\partial n_p}{\partial p} + \left( \tilde{v} + e\tilde{E} \times \Omega_p + e(\tilde{v} \cdot \Omega_p)B \right) \cdot \frac{\partial n_p}{\partial x} \right] = 0,
\]  

(1)
where \( \mathbf{v} = \partial \mathbf{p} / \partial \mathbf{x}, \) \( e \mathbf{E} = e \mathbf{E} - \partial \mathbf{p} / \partial \mathbf{x}, \) \( \mathbf{p} = p(1 - e \mathbf{B} \cdot \mathbf{p}) \) and \( \mathbf{p} = \pm \mathbf{p}/2p^3. \) Here \( \pm \) sign corresponds to right and left handed fermions respectively. In absence of the Berry curvature term (i.e. \( \Omega_p = 0 \)) \( \mathbf{p} \) is independent of \( x, \) Eq. (1) reduces to the standard Vlasov equation. Particle density \( n \) can be defined as

\[
n = \int \frac{d^3p}{(2\pi)^3}(1 + e \mathbf{B} \cdot \mathbf{p})n_p,
\]

whereas the current density \( \mathbf{j} \) can be defined as:

\[
\mathbf{j} = -\int \frac{d^3p}{(2\pi)^3} [\epsilon_p \partial_p n_p + e (\mathbf{\Omega}_p \cdot \partial_p n_p) \epsilon_p \mathbf{B} + \epsilon_p \mathbf{\Omega}_p \times \partial_x n_p] + e \mathbf{E} \times \sigma,
\]

where \( \partial_p = \partial / \partial \mathbf{p} \) and \( \partial_x = \partial / \partial x. \) The last term on the right hand side of the above equation represents the anomalous Hall current with \( \sigma \) given as follows:

\[
\sigma = \int \frac{d^3p}{(2\pi)^3} \mathbf{\Omega}_p n_p.
\]

**A. Maxwell Equation, Propagator and Dispersion relation**

Using the above expression for the number and current densities one can write the Maxwell equation as,

\[
\partial_t F^{\mu \nu} = j^{\mu}_{\text{ind}} + j^{\mu}_{\text{ext}}.
\]

Here \( j^{\mu}_{\text{ext}} \) is an external current. The induced current \( j^{\mu}_{\text{ind}} \) can be expressed in terms of gauge field \( A_\mu(k) \) via linear response theory in Fourier space as,

\[
j^{\mu}_{\text{ind}} = \Pi^{\mu \nu}(K) A_\nu(K),
\]

where \( \Pi^{\mu \nu}(K) \) is the retarded self energy in Fourier space. Here we have denoted a Fourier transform any quantity \( F(x,t) \) by \( F(K) = \int d^4xe^{-i(\omega t - k \cdot x)}F(x,t). \) Now one can write Eq. (5) in the Fourier space as

\[
[K^2 g^{\mu \nu} - K^\mu K^\nu + \Pi^{\mu \nu}(K)] = -j^{\mu}_{\text{ext}}(K).
\]

By choosing temporal gauge \( A_0 = 0 \) we can write the above equation as,

\[
[\Delta^{-1}(K)]^{ij} E^j = [(k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K)]E^j = i\omega j^{\mu}_{\text{ext}}(k).
\]

From this one can define

\[
[\Delta^{-1}(K)]^{ij} = (k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K).
\]

By finding inverse of the object \( [\Delta^{-1}(K)]^{ij} \) one can obtain the expression for the propagator \( [\Delta(K)]^{ij} \) whose poles can give the dispersion relation. In a linear response theory we are interested in the induced current by a linear-order deviation in the gauge field. We follow the power counting scheme of Ref. [25]: gauge field \( A_\mu = O(\epsilon) \) and derivatives \( O(\delta), \) where \( \epsilon \) and \( \delta \) are small and independent parameters. In this scheme one considers deviations in the current and the distribution function up to \( O(\epsilon \delta). \) Under this counting scheme one can write the kinetic equation as:

\[
(\partial_t + \mathbf{v} \cdot \partial_x) n_p + (e \mathbf{E} + e \mathbf{v} \times \mathbf{B} - \partial_x \mathbf{e}_p) \cdot \partial_p n_p = 0
\]

where \( \mathbf{v} = \mathbf{p}/p. \)
III. COLLECTIVE MODES IN ANISOTROPIC CHIRAL PLASMA

We consider the equilibrium distribution of the form \( n_p^0 = 1/[e^{(p - \mu)/T} + 1] \). Following the power counting scheme that we have introduced above one can write:

\[
r_p^0 = n_p^{0(0)} + e n_p^{0(\delta)},
\]

where, \( n_p^{0(0)} = \frac{1}{[e^{(p - \mu)/T} + 1]} \) and \( n_p^{0(\delta)} = \left( \frac{B_v}{2\pi T} \right)^2 \frac{e^{(p - \mu)/T}}{[e^{(p - \mu)/T} + 1]^2} \). In order to bring in effect of anisotropy we follow the arguments of Ref. \([46]\). It is assumed that the anisotropic equilibrium distribution function can be obtained from a spherically symmetric distribution function by rescaling of one direction in the momentum space. Thus we assume that there is a momentum anisotropy in direction of a unit vector \( \hat{n} \). Noting that \( p = |p| \), we replace \( p \to \sqrt{p^2 + \xi (p \cdot \hat{n})^2} \) in Eq.\( (11) \) to get the anisotropic distribution function. Here \( \xi \) is an adjustable anisotropy parameter satisfying a condition \( \xi > -1 \). It is convenient to define a new variable \( \tilde{p} \) such that \( \tilde{p} = p\sqrt{1 + \xi (v \cdot \hat{n})^2} \). Using this new variable one can write

\[
n_p^{0(0)} = \frac{1}{[e^{(\tilde{p} - \mu)/T} + 1]} \quad \text{and} \quad n_p^{0(\delta)} = \left( \frac{B_v}{2\pi T} \right)^2 \frac{e^{(\tilde{p} - \mu)/T}}{[e^{(\tilde{p} - \mu)/T} + 1]^2}.
\]

The anomalous Hall current term in Eq.\( (3) \) can vanish if the distribution function is spherically symmetric in the momentum space. However, for an anisotropic distribution function this may not be true in general. Since the Hall-current term depends on electric field, it can be of order \( O(\delta) \) or higher. As we are interested in finding deviations in current and distribution function up to order \( O(\delta) \), only \( n_p^{0(0)} \) would contribute to the Hall current term. Next, we consider \( \sigma \) from Eq.\( (4) \) which can be written as

\[
\sigma = \frac{1}{2} \int d\Omega d\tilde{p} \frac{v}{[1 + \xi (v \cdot \hat{n})]^2} \frac{1}{1 + e^{(\tilde{p} - \mu)/T}}.
\]

Since \( v \) is a unit vector one can express \( v = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) in spherical coordinates. By choosing \( \hat{n} \) in \( z \)-direction, without any loss of generality, one can have \( v \cdot \hat{n} = \cos \theta \). Thus the angular integral in the above equation becomes \( \int d(\cos \theta) d\phi \frac{v}{(1 + \xi \cos^2 \phi)^{1/2}} \). Therefore \( \sigma_x \) and \( \sigma_y \) components of Eq.\( (12) \) will vanish as \( \int_0^{2\pi} \sin \phi \cos \phi d\phi = 0 \) and \( \int_0^{2\pi} \cos \phi d\phi = 0 \). While \( \sigma_z \) will vanish because integration with respect to \( \cos \theta \) variable will yield it (\( \sigma_z \)) to be zero. Thus the anomalous Hall current term will not contribute for the problem at the hand. Now one can write current \( j \) as follows,

\[
\dot{j} = -e \int \frac{d^3p}{(2\pi)^3} [\epsilon_p \partial_p n_p + e(\Omega_p \cdot \partial_p n_p) \epsilon_p \Omega_p \times \partial_x n_p].
\]

The distribution function can be decomposed into separate scales as follows,

\[
n_p = n_p^0 + e(n_p^{(\epsilon)} + n_p^{(\delta)}).
\]

Now the kinetic equation \( (10) \) can be split into two equations valid at \( O(\epsilon) \) and \( O(\epsilon \delta) \) scales as written below,

\[
(\partial_t + v \cdot \partial_x) n_p^{(\epsilon)} = -(E + v \times B) \cdot \partial_p n_p^{0(0)}
\]

\[
(\partial_t + v \cdot \partial_x) (n_p^{0(\delta)} + n_p^{(\delta)}) = -\frac{1}{e} \partial_x \epsilon_p \cdot \partial_p n_p^{0(0)}
\]
Equation for the current defined in Eq. (13) can also split into $O(\epsilon)$ and $O(\epsilon\delta)$ scales as given below,

\begin{equation}
\mathbf{j}^{(\epsilon)} = e^2 \int \frac{d^3p}{(2\pi)^3} v^{\mu} \eta^{(\epsilon)}_p
\end{equation}

\begin{equation}
\mathbf{j}^{(\epsilon\delta)} = e^2 \int \frac{d^3p}{(2\pi)^3} \left[ \nu^i \eta^{(\epsilon\delta)}_p - \left( \frac{\nu^i \partial n^{(0)}_{p}}{2p} \partial \nu^j \right) B^i - \epsilon^{ijk} v^j \partial n^{(\epsilon)}_p \right]
\end{equation}

The self-energy or polarization tensor in Eq. (6) contains parity-even $\Pi^+_{ij}$ and parity-odd $\Pi^-_{ij}$ parts and thus one can write $\Pi = \Pi^+_{ij} + \Pi^-_{ij}$. Using Eqs. (6, 15, 16, 17, 18) one can obtain the expression for $\Pi^+_{ij}$ and $\Pi^-_{ij}$ as:

\begin{equation}
\Pi^+_{ij}(K) = m_D^2 \int \frac{d\Omega}{4\pi} \frac{v^i (v^j + \xi (\mathbf{v} \cdot \hat{n}) \hat{n}^j)}{(v.k)^2} \left( \delta^{ij} + \frac{v^j k^i}{v.k + i\epsilon} \right),
\end{equation}

\begin{equation}
\Pi^-_{im}(K) = C_E \int \frac{d\Omega}{4\pi} \left[ i e^{imk} v^j (\omega + \xi (\mathbf{v} \cdot \hat{n})(\mathbf{k} \cdot \hat{n})) \left( \frac{v^j + \xi (\mathbf{v} \cdot \hat{n}) \hat{n}^j}{(1 + \xi (\mathbf{v} \cdot \hat{n})^2)^{3/2}} \right) \right.
\end{equation}

We would like to mention that one can write the total current $\mathbf{j} = \mathbf{j}^\epsilon + \mathbf{j}^{\epsilon\delta}$ where $\mathbf{j}^\epsilon$ and $\mathbf{j}^{\epsilon\delta}$ respectively denote the vector and axial currents. $\mathbf{j}^\epsilon$ gives contribution of order of the square of plasma frequency or $m_D^2$. The plasma frequency contains additive contribution from the densities of all species i.e. right-handed particle/antiparticles and left-handed particles/antiparticles. The axial current arises due to chiral imbalance its contribution from each plasma specie, depends upon $\epsilon \Omega_p$. Since $\epsilon \Omega_p$ can change sign depending on the plasma specie therefore definition of $C_E$ contains both positive and negative signs. Consequently a relative signs of fermion and anti-fermion are different in $m_D^2$ and $C_E$. After performing above integrations one can get $m_D^2 = \frac{e^2}{2\pi^2} \int_0^\infty d\tilde{p} \tilde{p}^2 \left[ \frac{\partial n^{(0)}_{\tilde{p}}(\tilde{p} - \mu)}{\partial \tilde{p}} + \frac{\partial n^{(0)}_{\tilde{p}}(\tilde{p} + \mu)}{\partial \tilde{p}} \right]$ and $C_E = \frac{e^2 \mu}{4\pi^2}$. Where $\mu$ is the chemical potential for chiral fermions. It is to be noted that $C_E = 0$ when there is no chiral-chemical potential where as $m_D^2 \neq 0$. It can also be noticed that the terms with anisotropy parameter $\xi$ are contributing in parity-odd part of the self-energy. Further we would like to note that we have used expression for right-handed current by adding particle and anti-particle contributions in obtaining Eqs. (19, 21). Contributions from the left-handed particles can be just be added very easily and Eq. (21) will exactly match with the one given in Ref. [26]. Introduction of chemical potential $\mu$ for chiral fermions requires some qualification. Physically the chiral chemical potential imply an imbalance between the right handed and left handed fermion. This in turn related to the topological charge [17, 32]. It should be noted here that due to the axial anomaly chiral chemical potential is not associated with any conserved charge. It can still be regarded as ‘chemical potential’ if its variation is sufficiently slow [30].
A. Finding the Poles of $[\Delta(K)]^{ij}$ or Dispersion relation

In order to get the expression for the propagator $\Delta^{ij}$ it is necessary to write $\Pi^{ij}$ in a tensor decomposition. For the present problem we need six independent projectors. For an isotropic parity-even plasma one may need the transverse $P_T^{ij}$ and the longitudinal $P_L^{ij}$ tensor projectors. Due to anisotropy coming due to the presence direction $\mathbf{n}$ one needs two more projectors $P_n^{ij}$ and $P_{kn}^{ij}$ [17]. To account for parity odd effect we have to include two anti-symmetric operators $P_A^{ij}$ and $P_{An}^{ij}$. Thus we write $\Pi^{ij}$ into the basis spanned by the above six operators as:

$$\Pi^{ij} = \alpha P_T^{ij} + \beta P_L^{ij} + \gamma P_n^{ij} + \delta P_{kn}^{ij} + \lambda P_A^{ij} + \chi P_{An}^{ij}. \quad (22)$$

where, $P_T^{ij} = \delta^{ij} - k^i k^j / k^2$, $P_L^{ij} = k^i k^j / k^2$, $P_n^{ij} = n^i n^j / n^2$, $P_{kn}^{ij} = k^i n^j + k^j n^i$, $P_A^{ij} = i \epsilon^{ijk} n^k$ and $P_{An}^{ij} = i \epsilon^{ijk} n^k$. $\alpha, \beta, \gamma, \delta, \lambda$ and $\chi$ are some scalar functions of $k$ and $\omega$ which are yet to be determined.

Similarly we can write $[\Delta^{-1}(k)]^{ij}$ appearing in Eq.(8) as

$$[\Delta^{-1}(K)]^{ij} = C_T P_T^{ij} + C_L P_L^{ij} + C_n P_n^{ij} + C_{kn} P_{kn}^{ij} + C_A P_A^{ij} + C_{An} P_{An}^{ij}. \quad (23)$$

Using Eqs.(9, 22, 23) one can express relationship between $C$’s and the scalar functions defined in Eq.(22) as:

$$C_T = k^2 - \omega^2 + \alpha, C_L = -\omega^2 + \beta, C_n = \gamma, C_{kn} = \delta, C_A = \lambda, C_{An} = \chi. \quad (24)$$

It should be noted that $\alpha = (P_T^{ij} - P_n^{ij}) \Pi^{ij}$, $\beta = P_L^{ij} \Pi^{ij}$, $\gamma = (2P_n^{ij} - P_T^{ij}) \Pi^{ij}$, $\delta = \frac{1}{2k^2n^2} P_{kn}^{ij} \Pi^{ij}$, $\lambda = -\frac{1}{2} P_A^{ij} \Pi^{ij}$ and $\chi = -\frac{1}{2k^2} P_{An}^{ij} \Pi^{ij}$. In the limit $\xi \to 0$, using Eqs.(19, 20) one calculate $\alpha|_{\xi=0} = \Pi_T$, $\beta|_{\xi=0} = \frac{\omega^2}{k^2} \Pi_L$, $\gamma|_{\xi=0} = 0$, $\delta|_{\xi=0} = 0$, $\lambda|_{\xi=0} = -\frac{\Pi_A}{2}$ and $\chi|_{\xi=0} = 0$ where,

$$\Pi_T = m_D^2 \frac{\omega^2}{2k^2} \left[ 1 + \frac{k^2 - \omega^2}{2\omega k} \ln \frac{\omega + k}{\omega - k} \right],$$

$$\Pi_L = m_D^2 \frac{\omega^2}{2k^2} \left[ \frac{\omega}{2k} \ln \frac{\omega + k}{\omega - k} - 1 \right],$$

$$\Pi_A = -2kC_E \left[ 1 - \frac{\omega^2}{k^2} \right] \left[ 1 - \frac{\omega}{2k} \ln \frac{\omega + k}{\omega - k} \right]. \quad (25)$$

Scalar functions $\Pi_T$, $\Pi_L$ and $\Pi_A$ respectively representing the transverse, longitudinal and the axial part of the self-energy decomposition in the tensorial basis in the $\xi = 0$ limit as defined in Ref. [30].

Next, we expand $[\Delta(K)]^{ij}$ in the tensor projector basis as:

$$[\Delta(K)]^{ij} = a P_L^{ij} + b P_T^{ij} + c P_n^{ij} + d P_{kn}^{ij} + e P_A^{ij} + f P_{An}^{ij}. \quad (26)$$

It is rather easy but rather cumbersome to express the coefficients $a$, $b$, $c$, $d$, $e$ and $f$ in terms of the coefficients $C$’s appearing in Eq. [23] using the relation $[\Delta^{-1}(K)]^{ij} [\Delta(K)]^{ij} = \delta^{ij}$. The dispersion relation can be obtained by equating denominators of the expressions for $a$, $b$, $c$, $d$, $e$ and $f$ with zero. In the present case denominator for $a$, $b$, $c$, $d$, $e$ and $f$ is same therefore the dispersion relation can be written as:

$$2k n^2 C_A C_{An} C_{kn} + C_A^2 C_L + n^2 C_{An}^2 (C_n + C_T) - C_T (-k^2 n^2 C_{kn}^2 + C_L (C_n + C_T)) = 0. \quad (27)$$
This general form of the dispersion relation is quite complicated. The expression for \( \alpha, \beta, \gamma \) and \( \delta \) exactly match with those given in Ref. \[46\]. The new contribution comes in terms of the coefficients \( \lambda \) and \( \chi \) which contains the effect of parity violation. But note that the standard criteria for the plasma instability (Weibel) \[44\] are not applicable here because of the parity violation. For small anisotropy parameter \( \xi \) it is possible to evaluate all the integrals in the dispersion relation analytically.

**B. Analysis of the collective modes in small \( \xi \) limit**

Using \( \hat{k} \cdot \hat{n} = \cos(\theta_n) \) we can express \( \alpha, \beta, \gamma, \delta, \lambda \) and \( \chi \) up to linear order in \( \xi \) as follows,

\[
\alpha = \Pi_T + \xi \left[ \frac{z^2}{12} (3 + 5 \cos 2\theta_n) m_D^2 - \frac{1}{6} (1 + \cos 2\theta_n) m_D^2 + \frac{1}{4} \Pi_T \left( (1 + 3 \cos 2\theta_n) - z^2 (3 + 5 \cos 2\theta_n) \right) \right]; \\
z^{-2} \beta = \Pi_L + \xi \left[ \frac{1}{6} (1 + 3 \cos 2\theta_n) m_D^2 + \Pi_L \left( \cos 2\theta_n - \frac{z^2}{2} (1 + 3 \cos 2\theta_n) \right) \right]; \\
\gamma = \frac{\xi}{3} (3 \Pi_T - m_D^2) (z^2 - 1) \sin^2 \theta_n; \\
\delta = \frac{\xi}{3k} (4z^2 m_D^2 + 3\Pi_T (1 - 4z^2)) \cos \theta_n; \\
\lambda = -\frac{\mu k e^2}{4\pi^2} \left[ (1 - z^2) \frac{\Pi_L}{m_D^2} \right] - \xi \frac{\mu k e^2}{32\pi^2} \left[ (1 - z^2) \frac{\Pi_L}{m_D^2} \left( (1 + 7 \cos 2\theta_n) - 3z^2 (1 + 3 \cos 2\theta_n) \right) \right] \\
+ \frac{1}{3} (11 \cos 2\theta_n) - z^2 (3 + 5 \cos 2\theta_n) ; \\
\chi = \xi [f(\omega, k)],
\]

where \( z = \frac{\omega}{k} \) and \( f(\omega, k) \) is some function \( k \) and \( \omega \). But in the present analysis its exact form of \( f(\omega, k) \) form may not be required. Using the above equations and Eqs. (25, 24) one can finally express Eq.(27) in terms of \( k \) and \( \omega \). One can notice from Eq.(28) that the most significant contribution for \( \gamma, \delta, \lambda \) and \( \chi \) is \( O(\xi) \). Thus in the present scheme of approximation one can write Eq.(27) up to \( O(\xi) \) as:

\[
C_A^2 C_L - C_T C_L (C_n + C_T)) = 0,
\]

which in turn can give following two branches of the dispersion relation,

\[
C_A^2 - C_T^2 - C_n C_T = 0, \\
C_L = 0.
\]

First, we would like to note that when \( C_A = 0 \), Eqs.(30, 31) reduces to exactly the same dispersion relation discussed in Ref.[46] for an anisotropic plasma where there is no parity violating effect. Let us consider Eq.(30), it can be written as:

\[
(k^2 - \omega^2)^2 + (k^2 - \omega^2)(2\alpha + \gamma) + \alpha^2 + \alpha\gamma - \lambda^2 = 0.
\]

This equation is a quadratic equation in \( (k^2 - \omega^2) \) and it’s solutions can be written as,

\[
(k^2 - \omega^2) = \frac{-(2\alpha + \gamma) \pm 2\lambda}{2}.
\]
It is of particular interest to consider the quasi-static limit \(|\omega| \ll k\), in this limit expressions for \(\alpha \sim \Pi_T\) and \(\beta \sim \omega^2 k^2 \Pi_L\) and \(\lambda \sim -\frac{\Pi_L}{2}\). Now \(\Pi_L\), \(\Pi_T\) and \(\Pi_A\) can be obtained by expanding Eq. (25) in the quasi static limit as:

\[
\Pi_T|_{|\omega| < k} = \left( \mp \frac{\pi}{4} \langle k \rangle \right) m_D^2;
\]
\[
\Pi_L|_{|\omega| < k} = m_D^2 \left[ \mp \frac{\pi}{2} \frac{\omega}{k} - 1 \right]
\]
\[
\Pi_A|_{|\omega| < k} = -\frac{\mu k e^2}{2\pi}\left( \frac{\Pi_L|_{|\omega| < k}}{m_D^2} \right)
\]

(34)

Therefore in quasi-stationary limit one can write positive branch of Eq. (33) as \(\omega = i\rho(k)\) where, \(\rho(k)\) is given by the following expression,

\[
\rho(k) = \pm \frac{\sqrt{4\alpha_0^3 A^3}}{\pi^2 m_D^2} k_N^2 \left[ \frac{1}{2} - \frac{\xi}{12} \left( 1 + 5 \cos 2\theta_n + \frac{\xi}{12} \left( 1 + 3 \cos 2\theta_n \right) \frac{\pi^2 m_D^2}{\mu^2 \alpha_0^2 k_N} \right) \right]
\]

(35)

Thus \(\omega\) is purely an imaginary number and its real-part is zero i.e. \(Re(\omega) = 0\). Eq. (35) have positive and negative signs. The negative sign Eq. (35) is unphysical. This can be seen from the fact when \(\Pi_A = 0\) and \(\xi = 0\) the negative branch of Eq. (35) gives an instability. But in this case there is no source of free energy either in terms chiral imbalance or in terms of anisotropy in momentum space. Henceforth we ignore the negative sign. Further we have defined \(\alpha_e = \frac{e^2}{4\pi}\) as the electromagnetic coupling and \(k_N = \frac{\pi}{\mu k}\) normalized wave number. Positive \(\rho(k) > 0\) implies an instability as \(e^{-i(i\rho(k))t} \sim e^{i\rho(k)t}\).

In the denominator of Eq. (35) the terms containing \(\alpha\) can dropped as compared to unity for \(k_N \sim O(1)\) because \(\frac{\mu^2 \alpha_0^2 k_N}{\pi^2 m_D^2} \ll \frac{\alpha_0}{\pi^2} k_N \ll 1\). The denominator now can be written as \((1 + \xi \cos 2\theta_n)\). One can expand the denominator in powers of \(\xi\) and keep only linear term in Eq. (35). Next one can notice that among all \(\xi\)-dependent terms in the numerator the term with \(\alpha_e^2\) will dominate. Thus one can write,

\[
\rho(k) = \left( \frac{4\alpha_0^3 A^3}{\pi^2 m_D^2} \right) k_N^2 \left[ 1 - k_N + \frac{\xi}{12} \left( 1 + 3 \cos 2\theta_n \right) \frac{\pi^2 m_D^2}{\mu^2 \alpha_0^2 k_N} \right]
\]

(36)

One can get an upper bound on \(\xi\) by substituting \(|\omega| = \rho(k) = k\) in above equation. For \(\theta_n = 0\) and \(k_N = 1\) upper bound is \(\xi = \frac{3\pi}{4}\). Before we analyze the interplay between the chiral-imbalance and the Weibel instabilities, it is instructive to qualitatively understand their origin. First consider the chiral-imbalance instability. For a such a plasma ‘chiral-charge’ density \(n\) is given by \(\partial_t n + \nabla \cdot j = \frac{2\alpha}{\pi} E \cdot B\). From this one can estimate the axial charge density \(n \sim \alpha k A^2\) where \(A\) is the gauge-field. The number and energy densities of the plasma respectively given by \(\mu T^2\) and \(\mu^2 T^2\). The typical energy for the gauge field \(\epsilon_A \sim k^2 A^2\). From the above value of the wave-vector it can be seen that \(\epsilon_A = \mu^2 T^2 \frac{T^2}{\alpha^2 A^2}\). Thus for \(\frac{T_e^2}{\alpha^2} < A\), the energy in the gauge field is lower than the energy of the particle. This leads to the chiral-imbalance instability. The Weibel instability arises when the equilibrium distribution function of the plasma has anisotropy in the momentum space. The anisotropy in the momentum space can be regarded as anisotropy in temperature. Suppose there is plasma which is hotter in \(y\)-direction than \(x\) or \(z\) direction. If in this situation a disturbance with a magnetic-field \(B = B_0 \cos(kx)\) which arises say from noise, then the
Lorentz-force can produce current-sheets where the magnetic field changes its sign. The current-sheet in turn enhancing the original magnetic field [10, 41].

The unity term in the square bracket of Eq. (36) is due to the chiral imbalance, while $\xi$ dependent term is due to momentum anisotropy. First consider the case when $\xi = 0$, the above equation reduces to the dispersion-relation in Ref. [30] describing the instability due to the chiral imbalance in the range $0 < k_N < 1$. Next, we consider the case when there is no chiral imbalance, in this case Eq. (36) can be written as,

$$\rho(k) = \left( \frac{4\alpha_2^3\mu^3}{\pi^4m_D^2} \right) k_N^2 \left[ -k_N + \frac{\xi}{12} (1 + 3\cos 2\theta_n) \frac{\pi^2m_D^2}{\mu^2\alpha_e^2k_N} \right].$$  \hspace{1cm} (37)

This gives unstable modes for Weibel instability, in the quasi-static limit $|\omega| << k$, when the following condition on $k_N$ is satisfied,

$$0 < k_N < \sqrt{\frac{\xi}{12} (1 + 3\cos 2\theta_n) \frac{\pi m_D}{\mu\alpha_e}}.$$  \hspace{1cm} (38)

Thus the chiral imbalance and Weibel instabilities have overlapping ranges. Maximum growth rates for the chiral instability is $\Gamma_{ch} \approx \frac{4}{27} \left( \frac{4\alpha_2^3\mu^3}{\pi^4m_D^2} \right)$ and for Weibel instability $\Gamma_w = 2 \left( \frac{4\alpha_2^3\mu^3}{\pi^4m_D^2} \right) \left( \frac{\pi^2m_D^2}{9\mu^2\alpha_e^2} \right)^{3/2}$ at $\theta_n = 0$. For $\mu \sim T$, $m_D^2/\mu^2 = \frac{2\alpha_e}{3\pi}(3 + \pi^2)$ and the ratio $\Gamma_w/\Gamma_{ch} \sim \frac{1}{2} \left[ \frac{2\alpha_e}{3\pi}(3 + \pi^2) \right]^{3/2}$. Thus both the instabilities will have comparable growth rate when $\xi_c \sim 2^{3/2} \left[ \frac{2\alpha_e}{3\pi(3 + \pi^2)} \right]$. For $\xi < \xi_c$, chiral instability will dominate else the Weibel instability will dominate. In figure (1) we plot the dispersion relation given by Eq. (35) as function of $k_N$ for various values of $\xi$ and propagation angle $\theta_n$. y-axis shows the $Re[\omega]$ and $Im[\omega]/\left( \frac{4\alpha_2^3\mu^3}{\pi^4m_D^2} \right)$ . Note that $Im[\omega] = \rho(k)$ and $Re[\omega] = 0$. First note that when $\xi = 0$ the unstable modes could only be due to the chiral-imbalance. The blue curves in fig.(1a,1b, 1e) depict this case. For the sake of comparison we have also plotted the pure Weibel modes by dropping the unity from Eq. (36). The green curves in fig(1a,1b,1e) represent this case. When $\xi \neq 0$ there is a contribution from both the instabilities and the condition for the instability can be written as

$$\frac{\xi}{12} (1 + 3\cos 2\theta_n) \left( \frac{\pi^2m_D^2}{\mu^2\alpha_e^2} \right) + k_N - k_N^2 > 0. \hspace{1cm} (39)$$

Thus for a sufficiently large values of $k_N$ there is always a damping and this is consistent with the findings of Weibel instability [10, 41]. The above inequality can be solved rather easily and since $\frac{\pi^2m_D^2}{4\mu^2\alpha_e^2} \gg 1$ one can write the condition for the instability as

$$0 < k_N < 1 + \frac{\xi}{3} (1 + 3\cos 2\theta_n) \frac{\pi^2m_D^2}{4\mu^2\alpha_e^2}. \hspace{1cm} (40)$$

First one can notice from condition (40) that by increasing $\xi$ the contribution of Weibel instability increases significantly when $\theta_n = 0$. This is because $\frac{\pi^2m_D^2}{4\mu^2\alpha_e^2} \gg 1$ and as $\xi$ becomes sufficiently large the Weibel instability terms start dominating over the terms due to the chiral-imbalance. Further, we have already noted that the chiral instability occurs within the range $0 < k_N < 1$. From condition (40) one can see that for small values of $\theta_n$, the range
of the instability can go beyond $k_N = 1$. For example when $\theta_n = 0$, the condition for the instability is $k_N < 1 + \frac{4}{3} \xi \left( \frac{\pi^2 m_0^2}{4\mu^2\alpha^2} \right)$.

\[ \frac{\pi^2 m_0^2}{4\mu^2\alpha^2} \]

![Graphs](image)

**FIG. 1:** Shows plots of real and imaginary part of the dispersion relation. Here $\theta_n$ is the angle between the wave vector $k$ and the anisotropy vector. Real part of dispersion relation is zero. Fig. (1a-1b) show plots for three cases: (i) Pure chiral (no anisotropy), (ii) Pure Weibel (chiral chemical potential=0) and (iii) When both chiral and Weibel instabilities are present. Fig. (1c-1d) represent the case when both the instabilities are present but the anisotropy parameter varies at different values of $\theta_n$. Fig. (1e) represents the case when for a particular value of $\theta_n \sim \theta_c$ both the instabilities have equal growth rates. Here frequency is normalized in unit of $\omega/\left( \frac{4\mu^3 m_0^3}{\pi^4 m_D^2} \right)$ and wave-number $k$ by $k_N = \frac{\pi}{\mu \alpha} k$.

For instance when $\theta_n = \pi/2$, range of the instability reduces from $k_N = 1$ and it is given by $k_N < 1 - \frac{2}{3} \xi \left( \frac{\pi^2 m_0^2}{4\mu^2\alpha^2} \right)$. This can be seen in the behavior of the plots of the dispersion relation shown in Figs.(1a-1b). It is interesting to note that when the anisotropic parameter $\xi$, satisfies the condition $-1 < \xi < 0$, the conclusions about the range of the instabilities can be altered and now the range of instability would reduce from $k_N = 1$ for $\theta_n = 0$ case and it increases for $\theta_n = \pi/2$ case. The negative value of $\xi$ signifies that the distribution function in the momentum space is stretched in the direction of the anisotropy vector $n$. In
fig.(1a) and fig (1b) the red curve respectively show the instance when both the instability occurs simultaneously for $\theta_n = 0$ and $\theta_n = \pi/2$ cases for $\xi = 0$. Figure(1a) shows that for $\theta_n = 0$ the combined growth rate of the instability is significantly higher than the pure chiral and the pure Weibel cases. In this case the range of the instability also increases in comparison with the pure cases. However for $\theta_n = \pi/2$, the Weibel instability is absent and the combined mode also is damped as can be seen from fig.(1b). Figures (1c and 1d) show that range and the growth rate of the combined instability sensitively depends on $\xi$. As $\xi$ value increases the Weibel instability dominates over the chiral instability. However there exists a some critical value for $\theta_c = \frac{1}{2} \cos^{-1} \left( \frac{2}{\sqrt{3}} \frac{2/3}{\hat{\xi} \pi^2 m_D^2} - \frac{1}{3} \right)$ for a given $\xi$, where both the instabilities contribute equally. This case is depicted in fig.(1e).

IV. STREAM PASSING THROUGH CHIRAL PLASMA

Another important class of the problem that deals with anisotropic situation is the case when a stream of particles moving in a thermalized back-ground plasma [39]. The stream can loose its energy and momentum by interacting with the plasma. This kind of problem have applications in variety of fields including quark-gluon plasma [48–52]. In the present work for background plasma we consider the background distribution function in momentum space $n_p^0$ to be isotropic. We write $n_p^0 = n_p^{0(0)} + e n_p^{0(\delta)}$ where, $n_p^{0(0)} = \frac{1}{(2\pi)^3 2p^0 T}$ and $n_p^{0(\delta)} = \left( \frac{B \cdot v}{2p^0 T} \right) \frac{e^{p/v/T} - 1}{e^{p/v/T} + 1}$ which is same as considered in Ref. [30]. Note that we can obtain this form of the distribution function from the equilibrium distribution function considered in the previous section by setting the anisotropy parameter $\xi = 0$. Next consider a jet or stream of chiral fermions traveling in the background plasma with the spatial component of the four-velocity $u = u^0 v_{st}$, where $u^0$ is temporal component of the four velocity of the stream and $v_{st}$ is the three vector of the stream velocity. $v_{st}$ is same and constant for all the particles in the jet. In this case we consider the following equilibrium distribution function for the jet:

$$n_p^0 = (2\pi)^3 \bar{n} v_{st}^3 (p - \Lambda u) (1 + e B \Omega_p)$$

where, $\bar{n}$ describes the density of the jet particles and it is considered to be constant. Here $\Lambda$ is the scale of energy of the jet. The term with $e B \cdot \Omega_p$ is Berry curvature correction to the distribution function of the stream and it is $O(\epsilon \delta)$.

Now one can consider the perturbations in the distribution function of the background plasma and the jet. The self-energy expression for the background plasma can be simply obtained by taking $\xi \rightarrow 0$ limit from Eqs.(19) as,

$$\Pi^i_+(K) = \alpha P_L^{ij} + \beta P_T^{ij}$$

$$\Pi^i_-(K) = \lambda P_A^{ij}$$

where $\alpha = \Pi_T$, $\beta = \omega_k^2 \Pi_L$, $\lambda = -\frac{\Pi_A}{2}$. $\Pi_T$, $\Pi_L$ and $\Pi_A$ are given by Eq.(25).

For the case of the jet anomalous Hall-current term in general can be non-zero. Due to the presence the delta function in the distribution function in Eq. (41) it is rather easy to calculate the expressions for parity-even $\Pi_{ij}^{st}$ and parity-odd $\Pi_{ij}^{st}$ parts of the self-energy tensor associated with the jet and they are given below:

$$\Pi_{ij}^{st}(K) = \omega_{st}^2 \left[ \delta^{ij} + \frac{k^i v_{st}^j + k^j v_{st}^i}{\omega - k \cdot v_{st}} - \frac{(\omega - k^2) v_{st}^i v_{st}^j}{(\omega - k \cdot v_{st})^2} \right]$$
In what follows we choose the streaming velocity \( v \). Next, we use Eqs. (9, 46) to analyze the modes in a jet-plasma system. The total self-energy of the system can be obtained by adding the contributions from the background plasma and the jet:

\[
\Pi^{ij}(K) = \Pi_{+st}^{ij}(K) + \Pi_{-st}^{ij}(K) + \Pi_{stst}^{ij}(K)
\]

Next, we use Eqs. (9, 46) to analyze the modes in a jet-plasma system.

A. Study of collective modes of the system of chiral plasma with a stream

In order to analyze the collective mode one can evaluate determinant of \( [\Delta^{-1}(K)]^{ij} \):

\[
det([\Delta^{-1}(K)]^{ij}) = det((k^2 - \omega^2)\delta^{ij} - k^i k_j + \Pi^{ij}(K)) = 0.
\]

In what follows we choose the streaming velocity \( v_{st} \) in \( z \)-direction only and the wave propagation vector \( \mathbf{k} \) has a component in a direction parallel to \( v_{st} \), i.e. \( k_z \).

1. When \( \mathbf{k} \) parallel to \( v_{st} \)

In this case \( \mathbf{k} \cdot v_{st} = kv_{st} \), solution of equation (47) gives the following dispersion relation:

\[
(4\bar{\Lambda}^2(k^2 - \omega^2 + \alpha + \omega_{st}^2) - (2\bar{\Lambda}\lambda + (3k - v_{st}(\omega + 2kv_{st}))\omega_{st}^2)^2) \left( \beta + \omega^2 \left( -1 - \frac{(-1 + v_{st}^2)}{\omega - kv_{st}} \right) \right) = 0,
\]

where \( \bar{\Lambda} = \Lambda/(1 - v_{st}^2)^{1/2} \). Thus there exists two separate branches for the mode of propagation,

\[
\left( \beta + \omega^2 \left( -1 - \frac{(-1 + v_{st}^2)}{\omega - kv_{st}} \right) \right) = 0,
\]

\[
(4\bar{\Lambda}^2(k^2 - \omega^2 + \alpha + \omega_{st}^2)^2 - (2\bar{\Lambda}\lambda + (3k - v_{st}(\omega + 2kv_{st}))\omega_{st}^2)^2) = 0.
\]

Eq. (49) is exactly same as discussed in Ref. [50] and it solutions will not be discussed here. However, interestingly this branch does not get any correction due to parity-odd effect considered in this work. Eq. (50) is a new branch of the dispersion relation arising entirely due to the parity odd effect. Next, we analyze this new branch in the quasi static limit \( |\omega| << k \), one can write

\[
\alpha_{|\omega|<<k} \approx -i\frac{\pi \omega}{4k}m_D^2,
\]

\[
\beta_{|\omega|<<k} \approx -m_D^2\frac{\omega^2}{k^2},
\]

\[
\lambda_{|\omega|<<k} \approx \frac{\mu ke^2}{4\pi^2}.
\]
From Eq.\textsuperscript{[50],[51]} and using $\omega = A + iB$ where $A$ and $B$ are real and imaginary part of $\omega$ one can obtain:

$$B(k) = \frac{4\alpha\mu k^2}{\pi^2 m_D^2} \left[ 1 - \frac{\pi k}{\alpha \mu} - \frac{\pi \omega_D^2}{\alpha \mu k} \left( 1 - \frac{3(1-v_{3st}^2)}{2} \frac{k_1}{\Lambda} + v_{3st}^2 \left( 1 - v_{3st}^2 \right)^{1/2} \frac{k_1}{\Lambda_1} \right) \right] \left( 1 + \left( \frac{2kv_{3st} \left( 1 - v_{3st}^2 \right)^{1/2} \omega_D^2}{\pi m_D^2 \Lambda} \right)^2 \right)^{1/2}. \quad (52)$$

In the above equation first term inside the square bracket is arising due to the chiral-imbalance in the plasma. The third term $\frac{\pi \omega_D^2}{\alpha \mu k} \left( 1 - \frac{3(1-v_{3st}^2)}{2} \frac{k_1}{\Lambda} + v_{3st}^2 \left( 1 - v_{3st}^2 \right)^{1/2} \frac{k_1}{\Lambda_1} \right)$ is the effect of streaming. Note that the terms with $\frac{k}{\Lambda}$ are the parity violation or chiral imbalance contribution to the stream. For the case when $\omega_{st} = 0$, one recovers the chiral-imbalance instability discussed in Ref.\textsuperscript{[30]}. For $\omega_{st} \neq 0$ one can see three terms in small bracket of numerator in Eq.\textsuperscript{[52]} competing with each other may give overall positive or negative contribution to the instability depending on the values $k$, $v_{3st}$ and $\Lambda$. It is convenient to define the total plasma frequency $\omega_t$ using $\omega_t^2 = \omega_p^2 + \omega_{st}^2$, where $\omega_p$ is the plasma frequency and $(\omega_p^2 = \frac{m_D^2}{2})$. Using this we introduce normalize frequency $\omega_1 = \omega/\omega_t$ and wave-number $k_1 = k/\omega_t$. Further we have the following parameters: $b = \omega_{st}/\omega_t^2$, $\mu_1 = \mu/\omega_t$, $\Lambda_1 = \Lambda/\omega_t$. It should be noted that parameters $\mu_1$ arises due to the parity-odd effect and it was not there in Ref.\textsuperscript{[50]}. For a finite temperature plasma when $\mu \sim T$, one can have $\mu_1 = 3\pi \left( \frac{2(1-b)}{3+\pi^2} \right)^{1/2}$.

For Heavy ion collisions typical value of $\Lambda$ can be taken 4 – 100 GeV/c\textsuperscript{[53],[54]} and $\Lambda_1 \approx 30$ or greater depending upon $\Lambda$. However for Weyl metals values of $\Lambda$ can be much lower and it may have different values. One can now rewrite Eq.\textsuperscript{[52]} as follows:

$$B_1(k) = \frac{4\alpha\mu k^2}{3\pi^2(1-b)} \left[ 1 - \frac{\pi k_1}{\mu_1 k_1} - \frac{\pi b}{\alpha \mu_1 k_1} \left( 1 - \frac{3(1-v_{3st}^2)}{2} \frac{k_1}{\Lambda_1} + v_{3st}^2 \left( 1 - v_{3st}^2 \right)^{1/2} \frac{k_1}{\Lambda_1} \right) \right] \left( 1 + \left( \frac{2kv_{3st} \left( 1 - v_{3st}^2 \right)^{1/2} \omega_D^2}{3\pi(1-b)\Lambda_1} \right)^2 \right)^{1/2}. \quad (53)$$

In Fig.(2) we have shown the plots of $Im[\omega_1]$ i.e. $B_1(k) = B(k)/\omega_t$ versus $k_1$ for various values of parameters $b$, stream-velocity $v_{3st}$ and $\Lambda_1$. Fig.2(a) shows the plots of $B_1(k)$ as a function of $k_1$ for different values of $b$ while $v_{3st}$ and $\Lambda_1$ are kept fixed at $v_{3st} = 0.9$ and $\Lambda_1 = 30$. The case with $b = 0$ corresponds to the case when there is no stream and the dispersion relation gives the same instability for the plasma background considered in Ref.\textsuperscript{[30]}. But by increasing $b$ the background plasma instability is reduced because the term with factor $-\frac{\pi b}{\alpha \mu_1 k_1}$ in Eq.\textsuperscript{[53]} gives a strong negative contribution to the instability. Keeping $b = 0.01$ and $\Lambda_1$ in the similar ballpark as in Ref.\textsuperscript{[50]} (relevant for a QGP) and $v_{3st} = 0.9$ can strongly suppress the background instability. Fig.2(b) shows how the plots varies with different values of $\Lambda_1$ while we have kept parameters $b$ and $v_{3st}$ fixed at $b = 0.01$ and $v_{3st} = 0.9$. Note that case with $b = 0$ is shown for just making a comparison with the background plasma instability. In this case one can see that when $\Lambda_1 \ll 1$ is the instability is enhanced compared to the background plasma case with $b = 0$. This is arising because of the parity-odd contribution to the self-energy coming from the jet. The reason for this is, the term with coefficient $1/\Lambda_1$ dominates in Eq.\textsuperscript{[53]} and make a strong positive contribution to the instability. As we increase the value of $\Lambda_1$ the instability is strongly suppressed. Fig.2(c) shows the case when $b$ and $\Lambda_1$ are kept fixed at $b = 0.03$ and $\Lambda_1 = 0.1$ while parameter $v_{3st}$ varies. One can see here that parity-odd terms in jet can enhance the instability around
$v_{3st} = 0.3$. But the contribution from the parity-odd terms in jet reduces significantly as $v_{3st} \to 1$.

FIG. 2: show plots of dispersion relation of the instability in a chiral-plasma background with a stream for the situation when the wave-vector $k_1$ propagating in the direction parallel to the stream velocity $v_{st}$. The $b = 0$ corresponds to the situation when there is no stream in the background plasma. Fig.(2a) shows how the instability varies for different values of $b$ while the stream velocity $v_{3st} = 0.9$ and $\Lambda_1 = 30$. Fig.(2b) shows that for a given values of $b$ and $v_{3st}$, the parity-odd terms in the jet self-energy can enhance the instability. Fig.(2c) shows the dependence of the instability on the stream velocity for given values of $b$ and $\Lambda_1$. Inset Figures in Fig. (2b,2c) shows the instability for $b=0$ case with better resolution.
2. When \( \mathbf{k} \) perpendicular to \( \mathbf{v}_{st} \)

In this case \( \mathbf{k} \cdot \mathbf{v}_{st} = 0 \). By choosing \( \mathbf{k} \) to be in x-direction and \( \mathbf{v} \) in z-direction, Eq. (47) gives following dispersion relation in limit \( |\omega| \ll k \):

\[
(\lambda_1^2 - (k_1^2 + \alpha_1 + b)^2) (-\omega_1^2 + \beta_1 + b) - (k_1^2 + \alpha_1 + b) \left( -1 + \frac{\beta_1}{\omega_1^2} \right) bk_1^2 v_3^2 \right)
\]

\[
+ 6 \left( \frac{k_1(1-v_3^2)^{1/2}}{2\Lambda_1} \right) \lambda_1 b \left( \omega_1^2 - \beta_1 \right) \left( 1 - \frac{k_1^2 v_3^2}{6\omega_1^2} \right) + \left( 1 - \frac{v_3^2}{3} \right) b
\]

\[
+ \left( \frac{k_1(1-v_3^2)^{1/2}}{2\Lambda_1} \right)^2 b^2 \left[ 9 \left( \omega_1^2 \left( 1 - \frac{v_3^2}{2} \right) - \beta_1 \left( 1 - \frac{k_1^2 v_3^2}{3\omega_1^2} \left( 1 - \frac{v_3^2}{3} \right) \right) + \frac{k_1^2 v_3^4}{9} - b \right) \right]
\]

\[
+ v_3^2 (-2k_1^2 + \alpha_1 + 7b) = 0 \quad (54)
\]

where, \( \lambda_1 = \frac{\mu_1 k_1 \alpha_1}{\pi} \left( 1 + i \frac{\pi \omega_1}{2 k_1} \right) \), \( \alpha_1 = -i \frac{3\pi (1-b) \omega_1}{4 k_1^2} \), \( \beta_1 = -3(1-b) \frac{\omega_1^2}{k_1^2} \). Note that here we have introduced same normalized variable and the parameters defined for \( \mathbf{k} \parallel \mathbf{v}_{st} \) case. It is very clear from the above equation that for \( b = 0 \) one gets the dispersion relation for the background-plasma (30). If the terms with parameter \( b \) are kept and the terms with \( \mu_1 \) and \( \frac{1}{\Lambda_1} \) are dropped, one can obtain the standard dispersion relation obtained in [e.g. see Ref.50] for jet-plasma system for a parity-even case.

In Fig. (3) we plot the \( \omega_1 \) calculated using Eq. (54) as function of \( k_1 \) for different values of parameters \( b, \Lambda_1 \) and \( v_{3st} \). In Fig. 3(a) we have plotted the positive roots of imaginary part of \( \omega_1 \) with respect to \( k_1 \). The comparison of the root with \( b = 0.01, \Lambda_1 = 30 \) and \( v_{3st} = 0.06 \) is made with the parity-even plasma[50] and the no-jet case \( b = 0 \) with [30]. Thus from the Fig. 3(a) it is clear that the presence of parity-violation effect in the stream enhances the instability. Next, Fig 3(b) we have increased the stream velocity to \( v_{3st} = 0.065 \) from its value \( v_{3st} = 0.06 \) and all other condition remains same between the two figures. In this case also the instability is enhanced due to increased velocity of jet and the parity-odd effect. Note that in comparison with the instability in no-jet case the finite jet has a much stronger instability. In Fig. 3(c) we have shown how the streaming instability will change by changing parameter \( \Lambda_1 \). For \( \Lambda_1 < 1 \), the parity-odd terms can enhance the instability provided stream-velocity remains sufficiently small. Next, in Fig. 3(d) we have shown the variation of the instability by changing the parameter \( b \) while parameters \( \Lambda_1 \) and \( v_{3st} \) are kept fixed. The instability decreases as we decreases the value of \( b \) but the three different curve covers the different \( k_1 \) values. One can see that when \( b = 0.001 \) the system can be unstable for larger \( k_1 \) value as compared to cases when \( b = 0.02 \) and 0.1. Further, we would like to note that when the stream velocity increases and approach unity, the parity-odd contribution from the jet becomes negligible and the contribution to the instability from the parity-odd background remains much weaker. In this limit the parity-even contribution in Ref.[50] can remain unaltered.
FIG. 3: show plots of dispersion relation of instabilities in a chiral plasma with a stream passing through it when \( k_1 \) perpendicular to \( v_{3st} \). Fig. 3(a) shows a comparison in the instabilities when a stream with parameters \( b = 0.01 \) and \( v_{3st} = 0.06 \), \( \Lambda_1 = 30 \) passing through chiral plasma [red (solid) curve] to the cases, when there is no streaming i.e. \( b = 0 \) [blue (dotted) curve] and when there is stream with \( b = 0.01, v_{3st} = 0.06 \) passing through parity even plasma [green (dashed) curve]. Fig. 3(b) shows the same comparison at higher stream velocity keeping other parameter same as in case of Fig. 3(a). Fig. 3(c) shows the effect on instability by changing the parameter \( \Lambda_1 \) keeping parameters \( b = 0.02 \) and \( v_{3st} = 0.1 \) fixed. Fig. 3(d) shows the effect on instability by changing parameter \( b \) keeping parameters \( v_{3st} = 0.4 \) and \( \Lambda_1 = 30 \) fixed.
V. SUMMARY AND CONCLUSIONS

We have studied collective modes in anisotropic chiral plasmas. In particular we have considered two cases of the instabilities in anisotropic plasma namely Weibel instability and jet-plasma interaction. We have shown that even for small values of the anisotropy parameter $\xi \ll 1$, the range and the magnitude of the chiral-imbalance instability is strongly modified. For $\xi > 0$, the growth rate and the range increases significantly when the wave-vector $k$ is in the direction parallel to the anisotropy vector $n$. The instability can become weaker when $k$ is in the direction perpendicular to $n$. In this case modes are strongly damped when one increases value of $\xi$.

We have also studied the dispersion relation of a jet-plasma system with parity-odd effect. We have shown that for the case when the wave-vector is in direction parallel to the stream velocity of the jet there can be two branches of the dispersion relation. The standard branch that could arise in a parity-even plasma and the new branch that is arising solely due to the parity odd effects of the chiral plasma. The standard branch does not have any correction due to the Berry-curvature terms. For the new branch we have shown that the chiral-imbalance instability is suppressed when the stream frequency $\omega_{st}$ or parameter $b$ increases. Further, if the jet energy scale $\Lambda$ is much lower than that found in the heavy ion-collision, the parity-odd effect in the stream can enhance the chiral-imbalance instability. Such lower values of $\Lambda$ may be relevant for Weyl metals. However for the parameters of the jets in the ballpark of relativistic heavy-ion collisions the chiral-imbalance instability is strongly suppressed. For the case when the wave-vector is in direction perpendicular to the stream-velocity we have shown that the parity-odd effect can strongly enhance the streaming-instability when the stream velocity is small. However, when the stream velocity become large the enhancement to the instability due to the parity-odd effect become very small. We hope that the results presented here can be applicable to the relativistic heavy-ion collisions and Weyl metals.

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