Rigorous proof of attractive nature for Casimir force of $p$-odd hypercube

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Abstract

The Casimir effect giving rise to an attractive force between the configuration boundaries that confine the massless scalar field is rigorously proven for odd dimensional hypercube with the Dirichlet boundary conditions and different spacetime dimensions $D$ by the Epstein zeta function regularization.

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In 1948 Casimir calculated an extraordinary property that two uncharged metallic plates would have an attractive force in vacuum[1]. This force is the strong function of \( a \) and is measurable only for \( a < 1\mu m[2] \). Boyer[3] numerically calculated the Casimir force of a thin spherical shell, who found that the sphere tends to be expanded. By using the mean approximation, the stresses are directed outwards for a cubic cavity, which tend to expand the cavity[4]. Therefore, one may imagine the spherical shell to be deformed into a cubic shell, and expect that this deformation does not change the resulting stresses. It is interesting that the Casimir force of massless scalar field may be repulsive for \( p \)-odd cavity with unequal edges[5]. On the other hand, few physicists would nowadays argue against the statement that the zeta function regularization procedure has proven to be a very powerful and elegant technique. Rigorous extension of the proof of Epstein zeta function regularization has been obtained[6]. The generalized \( \zeta \)-function has many interesting applications, e.g., in the piecewise string[7] and branes[8]. In this paper, we rigorously proof that the Casimir force is attractive for \( p \)-odd hypercube with the Dirichlet boundary conditions and spacetime dimension \( D \) less than critical value \( D_c \).

A Hermitian massless scalar field \( \phi(t, x^a, x^T) \) is confined in the interior of \((D-1)\)-dimensional rectangular cavity \( \Omega \) with \( p \) edges of finite lengths \( L_1, L_2, \cdots, L_p \) and \( D-1-p \) edges with characteristic lengths of order \( \lambda \gg L_a \) where \( i = 1, \cdots, D-1; a = 1, \cdots, p; T = p+1, \cdots, D-1 \). We consider the case of Dirichlet boundary conditions, i.e., \( \phi(t, x^a, x^T)|_{\partial \Omega} = 0 \). The Casimir energy density is [5]

\[
\varepsilon_p^D(L_1, L_2, \cdots, L_p) = -\frac{\pi^{(D-p)/2}}{2^{D-p+1} \Gamma(D-p/2)} \frac{1}{L_1^2} \frac{1}{L_2^2} \cdots \frac{1}{L_p^2} \frac{1}{2} \zeta(D-p)
\]

(1)

where the Epstein \( \zeta \) function \( \zeta_p(a_1, a_2, \cdots, a_p; s) \) is defined as

\[
\zeta_p(a_1, a_2, \cdots, a_p; s) = \frac{\sum_{\{n\}} \left( \sum_{j=1}^{\infty} a_j n_j^2 \right)^{-s}}{\sum_{\{n\}}}
\]

(2)

Assuming \( L_1 = L_2 = \cdots = L_p = L \), Eq.(1) can be reduced to

\[
\varepsilon_p^D(L) = \frac{L^{p-D} \sum_{q=0}^{p-1} (-1)^{q+1} C_p^q \left( \frac{D-q}{2} \right) A(1, \cdots, 1; \frac{D-q}{2})}{2^{D+1}}
\]

(3)

where the Epstein zeta function \( A(a_1, a_2, \cdots, a_p; s) \) is defined as

\[
A(a_1, a_2, \cdots, a_p; s) = \sum_{\{n\}}' \left( \sum_{j=1}^{\infty} a_j n_j^2 \right)^{-s}
\]

(4)

where the prime means that the term \( n_1 = n_2 = \cdots = n_p = 0 \) has to be excluded.

Using Eqs.(2)-(4) and the Mellin transformation of \( \exp(-b\tau) \)

\[
\int_0^\infty \tau^{s-1} e^{-b\tau} d\tau = b^{-s} \Gamma(s)
\]

(5)
we have

\[
\begin{align*}
\varepsilon^D_p (L_1, L_2, \ldots, L_p) &= L_p \varepsilon^D_{p-1} (L_1, L_2, \ldots, L_{p-1}) \\
&+ \frac{\pi^{(D-p)/2}}{2^{D-p+2}} \Gamma \left( -\frac{D-p}{2} \right) E_{p-1} \left( \frac{1}{L_1^2}, \ldots, \frac{1}{L_{p-1}^2}; -\frac{D-p}{2} \right) \\
&- \frac{1}{2^{D-p+1}} L_p \sum_{k=0}^{\infty} \frac{(16\pi)^{-k}}{k!} \prod_{j=1}^{k} (D-p+1)^2 - (2j-1)^2 \\
&\times \sum_{\{n\}} n_p^{(p-D-2-2k)/2} \left( \frac{n_1^2}{L_1^2} + \cdots + \frac{n_{p-1}^2}{L_{p-1}^2} \right) \frac{D-p-2k}{4} \exp \left[ -2\pi L_p n_p \left( \frac{n_1^2}{L_1^2} + \cdots + \frac{n_{p-1}^2}{L_{p-1}^2} \right)^{\frac{1}{4}} \right]
\end{align*}
\]

and

\[
\varepsilon^D_p (L) = \frac{L_p^{p-D}}{2^{D+1}} \pi^{-D/2} \int_0^\infty d\tau (\sqrt{\tau})^{D-2} \left\{ \left[ 1 - \left( \frac{\pi}{\tau} \right)^{\frac{2}{D}} \right]^p - \left[ \theta_3 (0, e^{-\tau}) - \left( \frac{\pi}{\tau} \right)^{\frac{2}{D}} \right]^p \right\}
\]

where the elliptic \( \theta \) function

\[
\theta_3 (0, q) = \sum_{m=-\infty}^{\infty} q^{m^2}
\]

when \( p = 2j + 1 \) (\( j \) is a positive integer), it is obvious that \( \varepsilon^D_p (L) < 0 \) for any \( D \), since \( \theta_3 (0, e^{-\tau}) > 1 \) and integrand is always negative between the integration limits. On the other hand, numerical calculations show that the energy density is positive for \( p \) is even and \( D \leq 6 \) in the \( p = 2j \) case [9]. Noticed that the terminology "Casimir force" for \( L_i \) direction is in proportion to the derivative of Casimir energy with respective to \( L_i \), we need to study firstly the behaviour of Casimir energy so as to discuss the nature of Casimir force. We can prove the following lemmata for the Casimir energy.

**Lemma 1**    For two spacetime dimensions \( D_1 \) and \( D_2 \), if \( D_2 > D_1 \), and \( \varepsilon^{D_2}_{2j} (L) \leq 0 \), then \( \varepsilon^{D_2}_{2j} < 0 \).

**Proof**    Defining

\[
k(\tau) \equiv \left[ 1 - \left( \frac{\pi}{\tau} \right)^{\frac{2}{D}} \right]^{2j} - \left[ \theta_3 (0, e^{-\tau}) - \left( \frac{\pi}{\tau} \right)^{\frac{2}{D}} \right]^{2j}
\]

which has only one real root \( 0 < \tau_0 < \pi \) for any \( j \). Since \( k(\tau) > 0 \) for \( 0 < \tau < \tau_0 \) and \( k(\tau) < 0 \) for \( \tau_0 < \tau < \infty \), we have

\[
\varepsilon^{D_2}_{2j} (L) = \frac{L_p^{D_2-D_1}}{2^{D_2+1}} \pi^{-D_2/2} \left( \int_{0}^{\tau_0} + \int_{\tau_0}^{\infty} \right) d\tau (\sqrt{\tau})^{D_1+(D_2-D_1)-2} k(\tau)
\]

\[
< \left( \frac{1}{2L} \right)^{D_2-D_1} \varepsilon^{D_1}_{2j} (L) \leq 0
\]
Lemma 2 There exists a particular of spacetime dimension $D_c = 6$, such that for $D \leq D_c, \varepsilon_D^2(L) > 0$ and for $D > D_c, \varepsilon_D^2(L) < 0$.

Proof In the $p = 2$ case, Eq.(7) can be reduced to

$$
\varepsilon_2^D(L) = \frac{L^{2-D}}{2^{D-1} \pi^{D/2}} \left[ 2\sqrt{\pi} \Gamma \left( \frac{D-1}{2} \right) \zeta \left( \frac{D-1}{2} \right) \beta \left( \frac{D-1}{2} \right) - \Gamma \left( \frac{D}{2} \right) \zeta \left( \frac{D}{2} \right) \beta \left( \frac{D}{2} \right) \right]
$$

(11)

Where $\zeta(r)$ is the Riemann $\zeta$ function and $\beta(r)$ is the Dirichlet series $\beta(r) \equiv \sum_{j=0}^{\infty} \left( \frac{-1}{j+1} \right)^r$.

Using these known one-dimensional series, we have $\varepsilon_2^D(L) = 0.04104, \varepsilon_2^4(L) = 0.00483, \varepsilon_2^5(L) = 0.00081, \varepsilon_2^6(L) = 0.00011$ and $\varepsilon_2^7(L) = -1.9 \times 10^{-5}$ if $L$ is the chosen unit length. thus, we show Lemma 2 from Lemma 1.

Lemma 3 If $j$ is large enough, then $\varepsilon_{2j}^D(L) < 0$.

Proof For $\tau > 4\pi > \tau_0$, one can easily find

$$
|k(\tau)| > \frac{pe^{-\tau}}{2^{2j-2}}
$$

(13)

From Eq.(7), we have

$$
\varepsilon_{2j}^D(L) < \frac{L^{2j-D}}{2^{D+1} \pi^{D/2}} \left( 2\pi^j \tau_0^{\frac{1}{2}} - \int_{64\pi}^{\infty} d\tau (p^{-1})^{1/2} |k(\tau)| \right)
$$

$$
< 2^D \left( 4\sqrt{\pi} L \right)^{2j-D} \left[ 4^{-2j} \tau_0^{\frac{1}{2}} - p \left( 4\sqrt{\pi} \right)^{-1} e^{-64\pi} \right]
$$

(14)

then, for $j$ large enough, $\varepsilon_{2j}^D(L) < 0$.

Lemma 4 If $D$ is large enough, then $\varepsilon_{2j}^D(L) < 0$ for any $j$.

Proof From Eq.(7), we have

$$
\varepsilon_{2j}^D(L) < \frac{L^{2j-D}}{2^{D+1} \pi^{D/2}} \left( \int_0^{\tau_0} + \int_{\tau_0}^{\infty} \right) d\tau (\sqrt{\tau})^{D-2} k(\tau)
$$

$$
< \frac{L^{2j-D} \tau_0^{D-1}}{2^{D+2} \tau^{D/2}} \left[ \frac{1}{2^{D-1}} \int_0^{\tau_0} d\tau k(\tau) + \int_{\tau_0}^{\infty} d\tau k(\tau) \right]
$$

(15)

Since the first term is positive and the second term is negative in the brackets of Eq.(15), when $D$ is large enough, then $\varepsilon_{2j}^D(L) < 0$ for any $j$.

Lemma 5 The Casimir force of a $p$-dimensional rectangular cavity with the Dirichlet boundary conditions can be written as terms of $(p-1)$-dimensional Casimir energy density and multiseries with exponential factors.

Proof From the recursion relation of the Casimir energy density Eq.(6), per unit area
This completes the proof of Lemma 5.

Since the second and the third terms are obviously negative in the right hand of Eq.(16), so that the pressure is always negative(directed inwards) if the first term is also negative. From Lemmata 1-4, we show that there exist a critical value of the spacetime dimension $D_c$ for which $\epsilon_D > 0$ if $D \leq D_c$ and $\epsilon_D < 0$ if $D > D_c$. However, $2j + 1 > D_c$ if $j > 14$. In this case there is no critical $D_c$ since all $\epsilon_D$ for $D = 2j + 1, 2j + 2, \cdots$ are negative. We summarize the above in the following.

**Theorem** The Casimir effect gives rise to an attractive force between the configuration boundaries that confine the massless scalar field for $p$-odd hypercube with the Dirichlet boundary conditions and $D \leq D_c, p \leq 29$.

Furthermore, we show the dependence of the critical value $D_c$ on $p$ in Table 1 using numerical calculation for possible physical application.

**Table 1** The critical value $D_c$ for massless scalar fields satisfying Dirichlet boundary conditions inside a hypercube with $p$-odd unit sides in a $D$-dimensional spacetime.

| $p$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 | 29 |
|-----|---|---|---|---|----|----|----|----|----|----|----|----|----|----|
| $D_c$ | 7 | 9 | 11 | 12 | 14 | 16 | 17 | 19 | 21 | 23 | 24 | 26 | 28 | 30 |

Finally, we shall give a brief discussion on our result. In spite of an impressive literature on the Casimir effect[10], the query whether its attractive or repulsive character changes by going to higher dimensions had never been elucidated for Dirichlet boundary conditions. We analytically show that Casimir force of $p$-odd hypercube is attractive, in contrast with the result of Ref.[11]. Our result is consistent with numerical calculation[9]. It may be worth emphasizing that Epstein zeta function is the fundamental zeta function associated with higher dimensions.
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