On tight 4-designs in Hamming association schemes

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Abstract

We use triple intersection numbers of association schemes to show non-existence of tight 4-designs in Hamming association schemes $H(n, 6)$. Combining with a result by Noda (1979), this completes the classification of tight 4-designs in $H(n, q)$.

1 Introduction

Design theory is devoted to finding subsets that globally approximate the whole spaces such as Hamming association schemes $H(n, q)$, Johnson association schemes $J(v, k)$, or the real unit sphere $S^{d-1}$. The design theories for these spaces have been studied separately as orthogonal arrays, as block designs and as spherical designs, respectively. A unifying framework to study designs are $Q$-polynomial association schemes developed by Delsarte [7].

A fundamental research on designs is to determine the minimum cardinality of a design with given strength $t$, i.e., a $t$-design. The lower bounds of $t$-designs were provided for $H(n, q)$ by Rao [18], for $J(v, k)$ by Ray-Chaudhuri and Wilson [19] (see also [25]), and for $S^{d-1}$ by Delsarte, Goethals and Seidel [8]. We call a design tight if it achieves the corresponding lower bound.

An important necessary condition for the existence of tight designs in the Johnson association schemes was established by Wilson (according to [7, Page 6], see also [19]), and the result was extended to tight designs in $Q$-polynomial association schemes [7] including the Johnson association schemes and the Hamming association schemes (see Theorem 2.3), and in the real unit sphere [8].

In 1979, Noda [17] used Theorem 2.3 to show the following.

Theorem 1.1. Let $C$ be a tight 4-design in the Hamming association scheme $H(n, q)$. Then one of the following holds:

1. $|C| = (16, 5, 2)$,
2. $|C| = (243, 11, 3)$,
3. $|C| = (9a^2(9a^2 - 1)/2, (9a^2 + 1)/5, 6)$, where $a$ is a positive integer such that $a \equiv 0 \pmod{3}$, $a \equiv \pm 1 \pmod{5}$ and $a \equiv 5 \pmod{16}$.

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The existence and uniqueness for (1) or (2) had been shown (see Examples 3.4 and 3.5 in Section 3). It remains open to determine whether the third case exists or not.

In this paper, we show that there is no tight 4-design as in Theorem 1.1(3). We briefly outline how to prove the non-existence result. Due to Delsarte’s work, a tight 4-design $C$ in $H(n,6)$ yields an association scheme of 2 classes. Decompose the vertex set $C$ into disjoint 6 subsets which can be identified with orthogonal arrays of strength 3 in $H(n-1,6)$. We then apply an analogue of the result in [21] to these subsets in the Hamming association schemes to construct another association scheme $S$, which is shown to be $Q$-antipodal. This property allows us to calculate the triple intersection numbers with respect to some triples of vertices of $S$. In the case when $S$ corresponds to a tight 4-design as in Theorem 1.1(3), certain triple intersection numbers turn out to be non-integral, which leads to a contradiction. This completes the classification of tight 4-designs in $H(n,q)$.

The existence and classification problems of tight 2e-designs in $H(n,q)$ have been extensively studied. Together with our result (see Corollary 4.2), its current state is as follows.

**Theorem 1.2.** The following hold.

1. [10, Theorem 7.5] For $e = 1, q = 2$, a tight 2-design in $H(n,2)$ is equivalent to a Hadamard matrix of order $n + 1$.
2. [10, Theorem 3.1] For $e = 1, q \geq 3$, there exists a tight 2-design in $H(q^2, q)$ for any prime power $q$.
3. [11] For $e \geq 3, q \geq 3$, there is no tight 2e-design in $H(n,q)$.
4. [17] For $e = 2$, if there exists a tight 4-design $C$ in $H(n,q)$, then one of the following occur:
   (a) $(|C|, n, q) = (16, 5, 2)$,
   (b) $(|C|, n, q) = (243, 11, 3)$.
5. [16] For $e = 3, q = 2$, if there exists a tight 6-design in $H(n, 2)$, then $n = 7, 23$.

The organization of the paper is as follows. In Section 2, we prepare basic notions for association schemes and orthogonal arrays. In Section 3, we show that tight 4-designs in $H(n,q)$ yield $Q$-antipodal $Q$-polynomial association scheme of 4 classes. Finally, in Section 4, we analyze triple intersection numbers with respect to some triples of vertices of the scheme obtained in Section 3 to conclude that there are no tight 4-designs in $H(n,6)$.

## 2 Preliminaries

In this section we prepare the notions needed in subsequent sections.

### 2.1 Association schemes

Let $X$ be a finite set of vertices and $\{R_0, R_1, \ldots, R_D\}$ be a set of non-empty subsets of $X \times X$. Let $A_i$ denote the adjacency matrix of the graph $(X, R_i)$ ($0 \leq i \leq D$). The pair $(X, \{R_i\}_{i=0}^D)$ is called a (symmetric) association scheme of $D$ classes if the following conditions hold:

1. $A_0 = I_{|X|}$, which is the identity matrix of size $|X|$,
2. $\sum_{i=0}^D A_i = J_{|X|}$, which is the square all-one matrix of size $|X|$,
3. $A_i^T = A_i$ ($1 \leq i \leq D$),
4. $A_i A_j = \sum_{k=0}^D p_{ij}^k A_k$, where $p_{ij}^k$ are nonnegative integers ($0 \leq i, j \leq D$).

The nonnegative integers $p_{ij}^k$ are called intersection numbers. The vector space $A$ over $\mathbb{R}$ spanned by the matrices $A_i$ forms an algebra. Since $A$ is commutative and semisimple, there exists a unique basis of $A$ consisting of primitive idempotents $E_0 = \frac{1}{|X|} J_{|X|}, E_1, \ldots, E_D$. Since the algebra $A$ is closed under the entry-wise multiplication denoted by $\circ$, we define the Krein parameters...
$q^k_{ij}$ ($0 \leq i, j, k \leq D$) by $E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{D} q^k_{ij} E_k$. It is known that the Krein parameters are non-negative real numbers (see [7, Lemma 2.4]). Since both $\{A_0, A_1, \ldots, A_D\}$ and $\{E_0, E_1, \ldots, E_D\}$ form bases of $\mathcal{A}$, there exists a matrix $Q = (Q_{ij})_{i,j=0}^{D}$ with $E_i = \frac{1}{|X|} \sum_{j=0}^{D} Q_{ji} A_j$. The matrix $Q$ is called the second eigenmatrix of $(X, \{R_i\}_{i=0}^{D})$. An association scheme $(X, \{R_i\}_{i=0}^{D})$ is said to be $Q$-polynomial if, for some ordering of $E_1, \ldots, E_D$ and for each $i$ ($0 \leq i \leq D$), there exists a polynomial $v^*_i(x)$ of degree $i$ such that $Q_{ji} = v^*_i(Q_{j1})$ ($0 \leq j \leq D$). It is also known that an association scheme is $Q$-polynomial if and only if the matrix of Krein parameters $L^*_1 := (q^k_{ij})_{i,j=0}^{D}$ is a tridiagonal matrix with nonzero superdiagonal and subdiagonal [1, p. 193] and $q^k_{ij} = 0$ whenever the triple $(i, j, k)$ does not satisfy the triangle inequality (i.e., when $|i - j| < k$ or $i + j > k$). For a $Q$-polynomial association scheme, set $a^*_i = q^1_{1,i}$, $b^*_i = q^1_{1,i+1}$, and $c^*_i = q^1_{1,i-1}$. These Krein parameters are usually gathered in the Krein array\[ \{b^*_0, b^*_1, \ldots, b^*_{D-1}; c^*_1, c^*_2, \ldots, c^*_D\}, \]as the remaining Krein parameters of a $Q$-polynomial association scheme can be computed from them. We say that a $Q$-polynomial association scheme is $Q$-antipodal if $b^*_i = c^*_{D-i}$ except possibly for $i = \lfloor D/2 \rfloor$. We simply say $Q$-antipodal association schemes for $Q$-antipodal $Q$-polynomial association schemes. In a $Q$-antipodal association scheme, we have $q^k_{ij} = 0$ whenever $i + j + k > 2D$ and the triple $(D - i, D - j, D - k)$ does not satisfy the triangle inequality. See [6] and [15] for more results on $Q$-antipodal association schemes.

There exists a matrix $G = (G_0, G_1, \ldots, G_D)$ whose rows and columns are indexed by $X$, satisfying that $GG^T = |X|I_{|X|}$ and $G$ diagonalizes the adjacency matrices, where $E_i = \frac{1}{|X|} \sum_{j=0}^{D} G_j G_i^T$ ($0 \leq i \leq D$) [7, p. 11]. We then define the $i$-th characteristic matrix $H_i$ of a non-empty subset $C$ of $X$ as the submatrix of $G_i$ that lies in the rows indexed by $C$. (Throughout this paper, a subset $C$ of $X$ is always non-empty.) A subset $C$ of $X$ for a $Q$-polynomial association scheme $(X, \{R_i\}_{i=0}^{D})$ is a $t$-design if its characteristic vector $\chi_C$ satisfies that $\chi_C^T E_i \chi = 0$ ($1 \leq i \leq t$).

For a triple of vertices $u, v, w \in X$ and integers $i, j, k$ ($0 \leq i, j, k \leq D$) we denote by \[ [u \ v \ w \ i \ j \ k] \] (or simply $[i \ j \ k]$) when it is clear which triple $(u, v, w)$ we have in mind) the number of vertices $x \in X$ such that $(u, x) \in R_i$, $(v, x) \in R_j$ and $(w, x) \in R_k$. We call these numbers triple intersection numbers. They have first been studied in the case of strongly regular graphs [4], and later also for distance-regular graphs, see for example [5, 9, 12, 13, 14, 22].

Unlike the intersection numbers, the triple intersection numbers depend, in general, on the particular choice of $(u, v, w)$. Nevertheless, for a fixed triple $(u, v, w)$, we may write down a system of $3D^2$ linear Diophantine equations with $D^3$ triple intersection numbers as variables, thus relating them to the intersection numbers, cf. [13]:

\[
\sum_{\ell=0}^{D} [\ell \ j \ k] = p^U_{jk}, \quad \sum_{\ell=0}^{D} [i \ \ell \ k] = p^V_{ik}, \quad \sum_{\ell=0}^{D} [i \ j \ \ell] = p^W_{ij}, \quad (2.1)
\]

where $(v, w) \in R_U$, $(u, w) \in R_V$, $(u, v) \in R_W$, and

\[ [0 \ j \ k] = \delta_{jW} \delta_{kV}, \quad [i \ 0 \ k] = \delta_{iW} \delta_{kU}, \quad [i \ j \ 0] = \delta_{iV} \delta_{jU}. \]

Moreover, the following theorem sometimes gives additional equations.

**Theorem 2.1.** ([5, Theorem 3], cf. [3, Theorem 2.3.2]) Let $(X, \{R_i\}_{i=0}^{D})$ be an association scheme of $D$ classes with second eigenmatrix $Q$ and Krein parameters $q^k_{ij}$ ($0 \leq i, j, k \leq D$). Then,

\[ q^k_{ij} = 0 \iff \sum_{r,s,t=0}^{D} Q_{ri} Q_{sj} Q_{tk} [u \ v \ w] = 0 \quad \text{for all } u, v, w \in X. \]
2.2 Hamming association schemes and orthogonal arrays

Let \( V = \{1, 2, \ldots, q\} \) (\( q \geq 2 \)) and \( X = V^n \). For \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X \), define the **Hamming distance** \( d(x, y) \) to be the number of indices \( i \) with \( x_i \neq y_i \). Suppose that \( R_i = \{(x, y) \mid x, y \in X, d(x, y) = i\} \) for \( i = 0, 1, \ldots, n \). Then the pair \((X, \{R_i\}_i=0^n)\) is an association scheme, which is called the **Hamming association scheme** \( H(n, q) \). The Hamming association scheme has the second eigenmatrix \( Q = (K_{n,q,i}(j))_{i,j=0}^n \) and is a \( Q \)-polynomial association scheme with the polynomials \( v_i^*(x) = K_{n,q,i}((q-1)n-x)/q \), where \( K_{n,q,i}(x) \) is the Krawtchouk polynomial of degree \( i \) defined as \( K_{n,q,i}(x) = \sum_{j=0}^i (-1)^j (q-1)^{i-j} \binom{n-x}{x-j} \).

An orthogonal array \( OA(N, n, q, t) \) is an \( N \times n \) matrix \( M \) with entries the numbers \( 1, 2, \ldots, q \) such that in any \( N \times t \) submatrix of \( M \) all possible row vectors of length \( t \) occur equally often \([10]\). Let \( C \) be the set of row vectors of \( M \). We identify the orthogonal array \( M \) with the subset \( C \subset X \). It is known from \([7, \text{Theorem } 4.4]\) that an orthogonal array \( OA(N, n, q, t) \) is equivalent to a \( t \)-design \( C \) with \(|C| = N \) in the Hamming association scheme \( H(n, q) \).

For \( t = 2e \), the lower bound on \( N \) was given by Rao \([18]\) as follows:

\[
N \geq \sum_{k=0}^e \binom{n}{k} (q-1)^k. \tag{2.2}
\]

An orthogonal array is said to be **complete** or **tight** if it achieves (2.2).

The **degree set** of an orthogonal array \( C \) is the set |\(S(C)|\) of Hamming distances of \( x, y \) among distinct \( x, y \in C \), and the **degree** \( s \) of \( C \) is defined as \( s = |S(C)| \). It is known that a tight \( 2e \)-design has degree \( e \) \([7, \text{Theorem } 5.12]\). The following lemma characterizes designs in terms of their characteristic matrices. The subsequent lemma and theorems are valid for any \( Q \)-polynomial association scheme, but we state these only for \( H(n, q) \).

**Lemma 2.2.** \([7, \text{Theorem } 3.15]\) Let \( C \) be a subset in \( H(n, q) \). The following conditions are equivalent:

1. \( C \) is a \( t \)-design,
2. \( H_i^\top H_t = \delta_{\ell i} |C| I \) for \( 0 \leq k + \ell \leq t \).

Then the following theorems are crucial.

**Theorem 2.3.** Let \( C \) be a tight \( 2e \)-design in \( H(n, q) \) with degree set \( S = \{\alpha_1, \ldots, \alpha_s\} \). Then \(|C| \prod_{i=1}^e (1 - x/\alpha_i) = \sum_{j=0}^e K_{n,q,j}(x) \) holds. In particular, \( \sum_{j=0}^e K_{n,q,j}(x) \) has exactly \( e \) distinct integral zeros in the interval \([1, n]\).

**Proof.** See \([7, \text{Theorem } 5.21]\). \(\square\)

Let \( C \) be a subset in \( H(n, q) \) with degree set \( S(C) = \{\alpha_1, \ldots, \alpha_s\} \). Set \( \alpha_0 = 0 \). Define \( S_i = \{(x, y) \in C \times C \mid d(x, y) = \alpha_i\} \) \((0 \leq i \leq s)\).

**Theorem 2.4.** Let \( C \) be a \( t \)-design in \( H(n, q) \) with degree \( s \). If \( t \geq 2s - 2 \), then the pair \((C, \{S_i\}_{i=0}^s)\) is a \( Q \)-polynomial association scheme of \( s \) classes.

**Proof.** Let \( A_i \) be the adjacency matrix of the graph \((C, S_i)\) for each \( i \), and \( A \) the vector space spanned by \( A_0, A_1, \ldots, A_s \).

Let \( H_i \) be the \( i \)-th characteristic matrix of \( C \) \((0 \leq i \leq s - 1)\). Define \( F_i = \frac{1}{|C|} H_i H_i^\top \) \((0 \leq i \leq s - 1)\). Set \( F_0 = I - \sum_{i=0}^{s-1} F_i \) Then \( F_0 = \frac{1}{|C|} \sum_{j=0}^s K_{n,q,j}(\alpha_i) A_j \) \((0 \leq i \leq s - 1)\) by \([7, \text{Theorem } 3.13]\) and \( F_0 = \frac{1}{|C|} \sum_{j=0}^s f(\alpha_j) A_j \) where \( f(z) = |C| \prod_{i=1}^e (1 - z/\alpha_i) - \sum_{j=0}^s K_{n,q,j}(z) \). Then \( F_i \neq O \) and \( F_i \in A \) for each \( i \).

By Lemma 2.2, we have \( F_i F_j = \delta_{ij} F_i \) \((0 \leq i, j \leq s - 1)\), from which it follows that \( F_i F_s = F_i F_{s-1} = O \) \((0 \leq i \leq s - 1)\) and \( F_i^2 = F_s \). These show that \( \{F_0, F_1, \ldots, F_s\} \) form a set of mutually orthogonal idempotents of \( A \). Therefore \( A \) is closed under matrix multiplication and the pair
(C, \{S_i\}_{i=0}^{4}) is an association scheme. Note that \( F_i \) is written as a polynomial of degree \( i \) in \( F_1 \) with respect to the entrywise product. Therefore the scheme is \( Q \)-polynomial. 

3 Tight 4-designs in \( H(n, q) \) and \( Q \)-antipodal association schemes of 4 classes

Let \( C \) be a tight 4-design in \( H(n, q) \) with degree set \( S(C) = \{\alpha_1, \alpha_2\} \) where \( \alpha_1, \alpha_2 \) (\( \alpha_1 < \alpha_2 \)) are the zeros of \( \sum_{j=0}^{1} K_{n,q,j}(x) = 0. \) Set \( \alpha_0 = 0, \) and define \( S_1 = \{(x, y) \in C \times C \mid d(x, y) = \alpha_i \} \) for each \( i. \) By Theorem 2.4, the pair \((C, \{S_i\}_{i=0}^{4})\) is an association scheme of 2 classes. In this section, we decompose \( S_1 \) into two subsets so that a tight 4-design in \( H(n, q) \) yields a \( Q \)-antipodal association scheme of 4 classes.

Define \( C_i \) to be

\[
C_i = \{(x_1, \ldots , x_n) \mid (i, x_2, \ldots , x_n) \in C \} \quad (1 \leq i \leq q).
\]

Then \( C = \bigcup_{i=1}^{q} \{(i) \times C_i \} \) holds. Note that \( C_i \) is obtained from \( C \) by deleting the first coordinate of the vectors with \( x_1 = i \) in \( C \) and \( |C_i| = |C|/q \) for each \( i. \) Setting \( \tilde{C} = \bigcup_{i=1}^{q} C_i, \) we will consider further combinatorial structure on \( \tilde{C} \) based on its partition \( \tilde{C} = \bigcup_{i=1}^{q} C_i. \)

Denote by \( H_k^{(i)} \) the \( k \)-th characteristic matrix of \( C_i \) in \( H(n-1, q), \) and observe that \( C_i \) is a 3-design with degree 2 in \( H(n-1, q). \) First we claim the following lemma, which is crucial to construct an association scheme on \( \tilde{C}. \)

**Lemma 3.1.** Let \( C \) be a tight 4-design in \( H(n, q). \) Define \( F_{\ell}^{(i,j)} \) to be

\[
F_{\ell}^{(i,j)} = \frac{1}{\sqrt{|C_i||C_j|}} H_\ell^{(i)}(H^{(j)}_\ell)^\top \quad (1 \leq i, j \leq q, \ \ell \in \{0, 1\})
\]

and

\[
F_{\ell}^{(i,i)} = I - F_{\ell}^{(i,i)} - F_{\ell}^{(i,i)} \quad (1 \leq i \leq q).
\]

Then \( F_{\ell}^{(i,j)} F_{\ell}^{(j,k)} = \delta_{\ell\ell'} F_{\ell}^{(i,k)} \) holds for \( 1 \leq i, j, k \leq q \) and \( \ell, \ell' \in \{0, 1\}, \) and \( F_{\ell}^{(i,i)} F_{\ell}^{(i,i)} = F_{\ell}^{(i,j)} F_{\ell}^{(j,j)} = O \) holds for \( 1 \leq i, j \leq q \) and \( \ell \in \{0, 1\}. \)

**Proof.** By Lemma 2.2. ∎

Recall \( \tilde{C} = \bigcup_{i=1}^{q} C_i. \) Then \( \tilde{C} \) is a subset in \( H(n-1, q) \) and \( S(\tilde{C}) = \{\alpha_1, \alpha_2, \alpha_1 - 1, \alpha_2 - 1\}. \)

Define \( S_0, S_1, \ldots , S_4 \) by \( S_0 = \{(x, y) \in \tilde{C} \times \tilde{C} \mid d(x, y) = 0 \} \) and

\[
\begin{align*}
\tilde{S}_{2i+1} &= \{(x, y) \in \tilde{C} \times \tilde{C} \mid d(x, y) = \alpha_i \} , \\
\tilde{S}_{2i} &= \{(x, y) \in \tilde{C} \times \tilde{C} \mid d(x, y) = \alpha_i \} ,
\end{align*}
\]

for \( i \in \{1, 2\}. \) The following theorem is the main theorem in this section.

**Theorem 3.2.** Let \( C \) be a tight 4-design in \( H(n, q). \) Then \((\tilde{C}, \{\tilde{S}_i\}_{i=0}^{4})\) is a \( Q \)-antipodal association scheme of 4 classes with Klein array

\[
\{(n-1)(q-1), (n-2)(q-1), 2(q-1), 1; 1, 2, (n-2)(q-1), (n-1)(q-1)\}.
\]

**Proof.** Let \( A_i \) be the adjacency matrix of the graph \((\tilde{C}, \tilde{S}_i) \) (\( 0 \leq i \leq 4 \)), and let \( A_i \) be the vector space spanned by \( A_0, A_1, \ldots , A_4 \) over \( \mathbb{R}. \)

Since each \( C_i \) is a 3-design with degree 2 in \( H(n-1, q), \) \( C_i \) provides an association scheme of 2 classes by Theorem 2.4. It follows from the proof of Theorem 2.4 that the primitive idempotents of \( C_i \) are \( F_0^{(i,i)}, F_1^{(i,i)}, F_2^{(i,i)} := I - F_0^{(i,i)} - F_1^{(i,i)} \), where \( F_\ell^{(i,i)} = \frac{1}{|S_i|} H_\ell^{(i)}(H^{(i)}_\ell)^\top \) for \( \ell \in \{0, 1\}. \)
Now we define $E_0, E_1, \ldots, E_4$ as

\[
E_i = \frac{1}{q} \begin{pmatrix}
F_i^{(1,1)} & F_i^{(1,2)} & \ldots & F_i^{(1,q)} \\
F_i^{(2,1)} & F_i^{(2,2)} & \ldots & F_i^{(2,q)} \\
\vdots & \vdots & \ddots & \vdots \\
F_i^{(q,1)} & F_i^{(q,2)} & \ldots & F_i^{(q,q)}
\end{pmatrix}
\]

for $i \in \{0, 1\}$,

\[
E_2 = \begin{pmatrix}
F_2^{(1,1)} & O & \ldots & O \\
O & F_2^{(2,2)} & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & F_2^{(q,q)}
\end{pmatrix},
\]

\[
E_{4-i} = \frac{1}{q} \begin{pmatrix}
(q-1)F_i^{(1,1)} & -F_i^{(1,2)} & \ldots & -F_i^{(1,q)} \\
-F_i^{(2,1)} & (q-1)F_i^{(2,2)} & \ldots & -F_i^{(2,q)} \\
\vdots & \vdots & \ddots & \vdots \\
-F_i^{(q,1)} & -F_i^{(q,2)} & \ldots & (q-1)F_i^{(q,q)}
\end{pmatrix}
\]

for $i \in \{0, 1\}$.

Note that each $E_i$ is a non-zero matrix. Since the matrices

\[
\begin{pmatrix}
F_i^{(1,1)} & O & \ldots & O \\
O & F_i^{(2,2)} & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & F_i^{(q,q)}
\end{pmatrix},
\]

\[
\begin{pmatrix}
O & F_i^{(1,2)} & \ldots & F_i^{(1,q)} \\
F_i^{(2,1)} & O & \ldots & F_i^{(2,q)} \\
\vdots & \vdots & \ddots & \vdots \\
F_i^{(q,1)} & F_i^{(q,2)} & \ldots & O
\end{pmatrix}
\]

are written as a linear combinations of $A_0, A_1, \ldots, A_4$, so are the matrices $E_0, E_1, \ldots, E_4$. From Lemma 3.1, it follows that $E_0, E_1, \ldots, E_4$ are mutually orthogonal idempotents. Thus $\mathcal{A}$ is closed under the matrix multiplication, and $(\tilde{C}, \{\tilde{S}_i\}_{i=0}^4)$ is an association scheme of $4$ classes with the primitive idempotents $E_0, E_1, \ldots, E_4$. The second eigenmatrix $Q$ is given as:

\[
Q = \begin{pmatrix}
1 & (n-1)(q-1) & \frac{1}{2}(n^2 - 3n + 2)(q-1)^2 & (n-1)(q-1)^2 & q-1 \\
1 & \frac{1}{2}(q-2+d) & 0 & \frac{1}{2}(q-2+d) & -1 \\
1 & \frac{1}{2}(q-2-d) & \frac{1}{2}q(q-d) & \frac{1}{2}(q-1)(-q-2+d) & q-1 \\
1 & \frac{1}{2}(q-2-d) & \frac{1}{2}q(q+d) & \frac{1}{2}(q-1)(-q-2-d) & q-1
\end{pmatrix},
\]

where $d = \sqrt{q^2 + 4(n-2)(q-1)}$. Then the matrix $L^*_i$ is

\[
L^*_i = \begin{pmatrix}
0 & (n-1)(q-1) & 0 & 0 & 0 \\
1 & q-2 & (n-1)(q-1) & 0 & 0 \\
0 & 2 & n(q-1) - 3q + 1 & 2(q-1) & 0 \\
0 & 0 & (n-2)(q-1) & q-2 & 1 \\
0 & 0 & 0 & (n-1)(q-1) & 0
\end{pmatrix}.
\]

Therefore the scheme is a $Q$-antipodal scheme with the given Krein array. \qed

**Remark 3.3.** The association scheme $(\tilde{C}, \{\tilde{S}_i\}_{i=0}^4)$ is a fission scheme of $(C, \{S_i\}_{i=0}^4)$ in the following way. Let $\phi$ be a mapping from $C$ to $\tilde{C}$ defined by $\phi(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n)$ and extended from $C \times C$ to $\tilde{C} \times \tilde{C}$ with respect to entrywise. Then $\phi(S_0) = \tilde{S}_0$ and $\phi(S_i) = \tilde{S}_{2i-1} \cup \tilde{S}_{2i}$ for $i = 1, 2$ hold.

**Example 3.4.** There exists a unique tight $4$-design in $H(5,2)$. It is the dual code of the repetition code of length $5$. By Theorem 3.2, it yields a $Q$-antipodal association scheme of $4$ classes with Krein array $\{4, 3, 2, 1; 1, 2, 3, 4\}$ (i.e., the Hamming association scheme $H(4,2)$).
Example 3.5. There exists a unique tight 4-design in $H(11,3)$, namely the dual code of ternary Golay code. By Theorem 3.2, it yields a $Q$-antipodal association scheme of 4 classes with Krein array $\{20, 18, 4, 1; 1, 2, 18, 20\}$.

4 Triple intersection numbers of a $Q$-antipodal association scheme of 4 classes

In this section we calculate triple intersection numbers of a $Q$-antipodal association scheme of 4 classes obtained from a tight 4-design in $H((9a^2 + 1)/5, 6)$ where $a$ is a positive integer such that $a \equiv 0 \pmod{3}$, $a \equiv \pm 1 \pmod{5}$ and $a \equiv 5 \pmod{16}$.

Let $C$ be a tight 4-design in $H((9a^2 + 1)/5, 6)$. The corresponding association scheme $(\tilde{C}, \{\tilde{S}_i\}_{i=0}^4)$ has Krein array $\{9a^2 - 4, 9a^2 - 9, 10, 1; 1, 2, 9a^2 - 9, 9a^2 - 4\}$. By substituting $3a = r$, we get the Krein array $\{r^2 - 4, r^2 - 9, 10, 1; 1, 2, r^2 - 9, r^2 - 4\}$. This parameter set is feasible for all odd $r \geq 5$ (i.e., the intersection numbers and multiplicities are nonnegative integers, and the Krein parameters are nonnegative real numbers).

An association scheme with such parameters has $r^2(r^2 - 1)/2$ vertices and is $Q$-antipodal, so many of its Krein parameters are zero. For a chosen triple of vertices of the association scheme, this allows us to augment the system of equations (2.1) with new equations derived from Theorem 2.1. We used the sage-drg package [23] (see also [24]) for the SageMath computer algebra system [20] to derive the following result.

Theorem 4.1. Let $(X, \{R_i\}_{i=0}^4)$ be a $Q$-polynomial association scheme with Krein array $\{r^2 - 4, r^2 - 9, 10, 1; 1, 2, r^2 - 9, r^2 - 4\}$. Then $r = 9$.

Proof. Since the Krein array above is obtained from the Krein array in Theorem 3.2 by setting $n = (r^2 + 1)/5, q = 6$, we may write the corresponding second eigenmatrix as

$$Q = \begin{pmatrix}
1 & r^2 - 4 & \frac{1}{2}(r^2 - 4)(r^2 - 9) & 5(r^2 - 4) & 5 \\
1 & r + 2 & 0 & -r - 2 & -1 \\
1 & r - 4 & -6(r - 3) & 5(r - 4) & 5 \\
1 & -r + 2 & 0 & r - 2 & -1 \\
1 & -r - 4 & 6(r + 3) & -5(r + 4) & 5
\end{pmatrix}.$$

As noted above, $r$ must be odd and at least 5 for the intersection numbers $p^k_{ij}$ ($0 \leq i, j, k \leq 4$) to be all nonnegative and integral. In particular, we have $p^k_{11} = (r^2 - 3r + 6)(r^2 - 1)/12 > 0$ for all such $r$, so we can choose $u, v, w \in X$ such that $(u, v), (u, w), (v, w) \in R_1$.

Solving the system of equations (2.1) for the triple $(u, v, w)$ augmented by equations derived from Theorem 2.1 for each zero Krein parameter yields a one-parametrical solution (see the notebook QPoly-d4-tight4design.ipynb on the sage-drg package repository for computation details). Let $\alpha = \{1 2 3\}$, and write $r = 2t + 1$. Then we may express

$$[1 1 1] = t^4 + 2t^3 + 2t^2 - 3\alpha - \frac{5r + 4 - 9/r}{8}.$$

Clearly, this expression can only be integral when $r$ divides 9. Since we must have $r \geq 5$, this leaves $r = 9$ as the only feasible solution.

Corollary 4.2. A tight 4-design as in Theorem 1.1(3) does not exist.

Proof. Let $(\tilde{C}, \{\tilde{S}_i\}_{i=0}^4)$ be the association scheme corresponding to a tight 4-design in $H((9a^2 + 1)/5, 6)$. By Theorem 3.2, its Krein array matches that of Theorem 4.1 with $r = 3a$, from which $a = 3$ follows. But this fails the condition $a \equiv \pm 1 \pmod{5}$, so such a design cannot exist.
Theorem 4.1 allows for the existence of a \( Q \)-polynomial association scheme with Krein array \( \{77, 72, 10, 1, 2, 72, 77\} \). No such scheme is known, however such a scheme would have as a subscheme a strongly regular graph (i.e., an association scheme of 2 classes) with parameters \((v, k, \lambda, \mu) = (540, 154, 28, 50)\). This parameter set is also feasible, but no example is known, see [2].

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