In this work we show that the \( N \times N \) Toeplitz determinants with the symbols \( z^\mu \exp(-\frac{1}{2}\sqrt{t}(z + 1/z)) \) and \( (1 + z)^\nu(1 + 1/z)^\nu \exp(tz) \) – known \( \tau \)-functions for the P\(_{III}'\) and P\(_{V}\) systems – are characterised by nonlinear recurrences for the reflection coefficients of the corresponding orthogonal polynomial system on the unit circle. It is shown that these recurrences are entirely equivalent to the discrete Painlevé equations associated with the degenerations of the rational surfaces \( D_6^{(1)} \rightarrow E_7^{(1)} \) (discrete Painlevé II) and \( D_5^{(1)} \rightarrow E_6^{(1)} \) (discrete Painlevé IV) respectively through the algebraic methodology based upon of the affine Weyl group symmetry of the Painlevé system, originally due to Okamoto. In addition it is shown that the difference equations derived by methods based upon the Toeplitz lattice and Virasoro constraints, when reduced in order by exact summation, are equivalent to our recurrences. Expressions in terms of generalised hypergeometric functions \( {}_0F_1^{(1)} \), \( {}_1F_1^{(1)} \) are given for the reflection coefficients respectively.

1. Introduction

There are now at least three approaches to systematically obtain recurrences for random matrix averages corresponding to \( \tau \)-functions for Painlevé systems. One, exploited by the present authors \cite{9}, is to use the theory of Schlesinger transformations within the \( \tau \)-function theory of Painlevé systems. Another, due to Borodin \cite{6}, is based on a discrete analogue of the Riemann-Hilbert problem \cite{5,7}. A third due to Adler and van Moerbeke \cite{4} is based on the theory of the integrable Toeplitz lattice and Virasoro constraints. In the work of the present authors, and of Borodin, there is an explicit connection with the discrete Painlevé equations. Thus in all cases the recurrences for the \( \tau \)-functions involve auxiliary quantities which satisfy discrete Painlevé equations. However the recurrences obtained in the work of Adler and van Moerbeke were not, in general, related to discrete Painlevé equations. This then immediately raises the question as to the relationship between the recurrences obtained by Adler and van Moerbeke, and the discrete Painlevé recurrences. In this work, for the recurrences relating to \( \tau \)-functions for the Painlevé III’ and the Painlevé V systems, we will answer this question by showing that in fact the recurrences obtained in \cite{4} are transformed versions of the discrete Painlevé equations. Moreover, we will show that the recurrences of Adler and van Moerbeke, obtained from their theory of the integrable Toeplitz lattice, follow from an approach based on the theory of orthogonal polynomials on the unit circle with semi-classical weights. In the situation...
of a general weight the recurrence relations for the various coefficients appearing in the orthogonal polynomial system are known as Freud or Laguerre-Freud equations \([12,11]\). This theory then provides a fourth approach to systematically obtain recurrences for random matrix averages corresponding to \(\tau\)-functions for Painlevé systems.

In this work attention will be focused on an orthogonal polynomial approach to \(\tau\)-functions for the Painlevé III′ and Painlevé V systems defined as averages over the eigenvalue probability density function for the unitary group \(U(N)\) with Haar (uniform) measure (see e.g. \([8\), Chapter 2]),

\[
\frac{1}{(2\pi)^N N!} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2, \quad (z_j := e^{i\theta_j}, -\pi < \theta_j \leq \pi, j = 1, \ldots, N)
\]

These \(\tau\)-functions are

\[
\tau^{III'}[N](t; \mu) := t^{-N\mu/2} \left\langle \prod_{l=1}^{N} z_l^{\mu} e^{i\sqrt{\pi} (z_l + z_l^{-1})} \right\rangle_{U(N)}
\]

\[
\tau^{V}[N](t; \mu, \nu) := \left\langle \prod_{l=1}^{N} (1 + z_l)^\mu (1 + 1/z_l)^\nu e^{tz_l} \right\rangle_{U(N)}
\]

where in \((1.2)\) it is assumed \(\mu \in \mathbb{Z}\) while in \((1.3)\) it is assumed \(\mu, \nu \in \mathbb{Z}_{\geq 0}\). By noting the identity

\[
(1 + z)^\mu (1 + 1/z)^\nu = z^{(\mu-\nu)/2} |1 + z|^\mu + \nu
\]

we can rewrite \((1.3)\) to read

\[
\tau^{V}[N](t; \mu, \nu) = \left\langle \prod_{l=1}^{N} z_l^{(\mu-\nu)/2} |1 + z_l|^\mu + \nu e^{tz_l} \right\rangle_{U(N)}
\]

which is well defined for \(\Re(\mu + \nu) > -1\). Associated with \((1.2)\) and \((1.5)\) are the weight functions

\[
z^{\mu} e^{i\sqrt{\pi} (z + z^{-1})}, \quad z^{(\mu-\nu)/2} |1 + z|^\mu + \nu e^{tz},
\]

on the unit circle \(z \in \mathbb{T}\) which have the special property of being semi-classical. On this latter point, analogous to the use of the term classical weight function for orthogonal polynomials on the line (see e.g. \([3]\)) we will call a weight function \(w(z)\) on the unit circle classical if its logarithmic derivative is of the form \(g(z)/f(z)\) with \(g(z)\) a polynomial of degree \(\leq 1\), and \(f(z)\) is a polynomial of degree \(\leq 2\). This gives \(w(z) = z^{(\mu-\nu)/2} |1 + z|^\mu + \nu\) as the only classical weight on the unit circle. The weights closest to classical with respect to the degree of the corresponding polynomials \(g\) and \(f\) are

\[
z^{\mu} e^{i\sqrt{\pi} (z + z^{-1})}, \quad z^{(\mu-\nu)/2} |1 + z|^\mu + \nu e^{tz},
\]

and are to be termed semi-classical. In the case of the first weight in \((1.7)\) with \(\mu = 0\), it is known that the corresponding orthogonal polynomials satisfy special recurrence relations \([13,16]\) which lead to a recurrence for the corresponding \(U(N)\) average. We will show that this is also true of the general form of the first weight in \((1.7)\), as well as the second weight in \((1.7)\), thus giving recurrences for \(\tau^{III'}[N]\) and \(\tau^{V}[N]\). In general these are different from those obtained in the Painlevé systems approach, but rather coincide with recurrences obtained by Adler and van Moerbeke from their theory of the Toeplitz lattice and its
Virasoro algebra \[4\]. As already remarked, we are able to show that after appropriate transformations, the recurrences of the two approaches coincide.

Another theme we wish to develop is the solution of the recurrences associated with (1.2) and (1.5) in terms of generalised hypergeometric functions based on Schur polynomials. To define the latter let \(\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_N)\) denote a partition so that the \(\kappa_i\)'s are non-negative integers with \(\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_N \geq 0\), let \(s_\kappa(t_1, \ldots, t_N)\) denote the Schur symmetric polynomial, and define the generalised Pochhammer symbol

\[
[a]^{(1)}_\kappa := \prod_{j=1}^N (a - j + 1)_{\kappa_j}, \quad (a)_l := a(a+1) \cdots (a+l-1).
\]

Also, with \((i, j) \in \kappa\) referring to a node in the Young diagram of \(\kappa\) and \(a(i, j), l(i, j)\) the corresponding arm and leg lengths respectively (see [15]), define the hook length

\[
h_\kappa = \prod_{(i, j) \in \kappa} [a(i, j) + l(i, j) + 1].
\]

With this notation the generalised hypergeometric series of interest is defined through a series representation \[20, 14\]

\[
\sum_{\kappa} [a_1]^{(1)}_\kappa \cdots [a_p]^{(1)}_\kappa s_\kappa(t_1, \ldots, t_N) h_\kappa
\]

for \(p, q \in \mathbb{Z}_{\geq 0}\). The superscript \(1\) in \(aF_q^{(1)}\) indicates \(1.10\) is a special case of a hypergeometric series based on Jack polynomials and depends on a complex parameter \(d\), the case \(d = 1\) corresponding to \(1.10\).

In Section 2 we present some general formulas from the Szegő theory of orthogonal polynomials on the unit circle [18] required for our study of (1.2) and (1.5), the later being carried out in Sections 3 and 4 respectively. In studying [12] and [13] we first present the recurrence scheme following from the work of Adler and van Moerbeke [4], and then proceed to show how the same schemes, and ones of lower order, can be derived from the theory of orthogonal polynomials on the unit circle. We make note of the solution of these recurrences in terms of the generalised hypergeometric functions, before revising the recurrence schemes following from the Painlevé systems approach. The latter do not coincide with the recurrences resulting from the orthogonal polynomial approach, however we show that after an appropriate transformation of variables the recurrences are in fact equivalent. We label the particular discrete equations that arise according to the unambiguous algebraic-geometric classification of Sakai [17] by association with a degeneration of a particular rational surface into another, rather than the previous names employed.

2. Orthogonal Polynomials on the Unit Circle

We will consider orthogonal polynomials with respect to a complex weight function \(w(z)\), analytic in the cut complex \(z\)-plane. The latter specification means \(w(z)\) possesses a Fourier expansion

\[
w(z) = \sum_{k=-\infty}^\infty w_k z^k, \quad w_k = \int_T \frac{dz}{2\pi i z} w(z) z^{-k},
\]
where $\mathbb{T}$ denotes the unit circle $|z| = 1$, appropriately deformed so not to cross the cut, and $z = e^{i\theta}, \theta \in (-\pi, \pi)$. For $\epsilon = 0, \pm 1$ we define the Toeplitz determinants

\[ I_N[w] := \det \left[ \int_{\mathbb{T}} \frac{dz}{2\pi i z} w(z) z^{j+k-1} \right]_{0 \leq j < k \leq N-1} = \det [w_{-\epsilon+j-k}]_{0 \leq j < k \leq N-1}. \]

In the case $\epsilon = 0$, and $\mathbb{T}$ the unit circle without deformation, by virtue of the identity

\[ \det \left[ \int_{\mathbb{T}} \frac{dz}{2\pi i z} w(z) z^{j+k-1} \right]_{0 \leq j < k \leq N-1} = \langle \prod_{l=1}^{N} z_l^l w(z_l) \rangle_{U(N)} \]

we see that (1.2) and (1.5), in the cases $\mu, \nu \in \mathbb{Z}_{\geq 0}$ at least, can be expressed as $I_N^0[w]$ with $w(z)$ as in (1.7). In certain circumstances the weight is real and positive $w(z) = \bar{w}(z)$ and thus the Toeplitz matrix is Hermitian, $\bar{w}_k = w_{-k}$, but in general this will not be true.

If $I_N^0[w]$ is non-zero for each $N = 1, 2, \ldots$ then there exists a system of orthogonal polynomials \{\phi_n(z), n = 0, 1, \ldots\} with the orthonormality property

\[ \int_{\mathbb{T}} \frac{dz}{2\pi i z} \bar{\phi}(z) \phi_n(z) = \delta_{m,n} \]

where

\[ \bar{w}(z) := \frac{w(z)}{w_0}. \]

We introduce special notation for the various coefficients in $\phi_n(z)$ according to

\[ \phi_n(z) = \kappa_n z^n + l_n z^{n-1} + m_n z^{n-2} + \ldots + \phi_n(0) = \sum_{j=0}^{n} \bar{c}_{n,j} z^j, \]

where without loss of generality $\kappa_n$ is chosen to be real and positive. We also define the reciprocal polynomial by

\[ \phi_n^*(z) := z^n \bar{\phi}(1/z) = \sum_{j=0}^{n} \bar{c}_{n,j} z^{-j}, \]

where $\bar{c}$ denotes the complex conjugate. A fundamental quantity is the ratio $r_n = \phi_n(0)/\kappa_n$, known as reflection coefficients because of their role in the scattering theory formulation of orthogonal polynomial systems on the unit circle.

From the Szeg"{o} theory these coefficients and their complex conjugates are related to the above Toeplitz determinants (2.3) by

\[ r_N = (-1)^N \frac{I_N^0[w]}{I_N^0[\bar{w}]}, \quad \bar{r}_N = (-1)^N \frac{I_N^{-1}[w]}{I_N^{-1}[\bar{w}]} \]

In the case that $w(z)$ is not real, $\bar{r}_N$ (notwithstanding the notation) is not the complex conjugate of $r_N$ but rather an independent variable. Note that the normalisation (2.4) implies that $\kappa_0 = 1$ and thus $r_0 = \bar{r}_0 = 1$. Knowledge of $\{r_N\}_{N=0,1,\ldots}$, $\{\bar{r}_N\}_{N=0,1,\ldots}$ is sufficient to compute $\{I_N^0[w]\}_{N=0,1,\ldots}$. For this one uses the general formula (1.8)

\[ \frac{I_{N+1}^0[w]}{I_{N+1}^0[\bar{w}]} = 1 - r_N \bar{r}_N. \]

Our fundamental task is then to obtain recurrences determining the $r_N$ and $\bar{r}_N$. 

For this purpose we require further formulae from the Szegö theory. First, as a consequence of the orthogonality condition we have the mixed linear recurrence relations for $\phi_n$ and $\phi_n^*$,

\begin{align}
\kappa_n \phi_{n+1} &= \kappa_{n+1} z \phi_n + \phi_{n+1}(0) \phi_n^* \\
\kappa_n \phi_n^* &= \kappa_{n+1} \phi_n^* + \bar{\phi}_{n+1}(0) z \phi_n,
\end{align}

as well as the three-term recurrences

\begin{align}
\kappa_n \phi_n(0) \phi_{n+1}(z) + \kappa_{n-1} \phi_{n+1}(0) z \phi_{n-1}(z) &= (\kappa_n \phi_{n+1}(0) + \kappa_{n+1} \phi_n(0)) z \phi_n(z) \\
\kappa_n \bar{\phi}_n(0) \phi_{n+1}(z) + \kappa_{n-1} \bar{\phi}_{n+1}(0) z \phi_{n-1}(z) &= (\kappa_n \bar{\phi}_{n+1}(0) z + \kappa_{n+1} \bar{\phi}_n(0)) \phi_n^*(z).
\end{align}

From the latter one can derive the analogue of the Christoffel-Darboux summation formula

\begin{align}
\sum_{j=0}^n \phi_j(z) \overline{\phi_j(\zeta)} &= \frac{\phi_n^*(z) \overline{\phi_n(\zeta)} - \bar{z} \overline{\phi_n(z) \overline{\phi_n(\zeta)}}}{1 - \bar{z}} \\
&= \frac{\phi_{n+1}^*(z) \overline{\phi_{n+1}(\zeta)} - \phi_{n+1}(z) \overline{\phi_{n+1}(\zeta)}}{1 - \bar{z}}
\end{align}

for $\bar{z} \neq 1$. Identities from the Szegö theory that relate the leading coefficients back to the reflection coefficients are

\begin{align}
\kappa_n^2 &= \kappa_{n-1}^2 + |\phi_n(0)|^2, \\
\frac{l_n}{\kappa_n} &= \sum_{j=0}^{n-1} r_j r_{j+1}, \\
\frac{m_n}{\kappa_n} &= \sum_{j=0}^{n-1} r_j r_{j+1} + r_j r_{j+1} \frac{l_{j-1}}{k_{j-1}}.
\end{align}

Another relevant formula is

\begin{align}
\mathcal{L}_N^D[w] &= \prod_{j=0}^{N-1} \kappa_j.
\end{align}

Finally, with $\pi_n$ denoting an arbitrary polynomial in the linear space of polynomials with degree at most $n$, we can check from the structure \[2.20\] that

\begin{align}
z \phi_n(z) &= \frac{\kappa_n}{\kappa_{n+1}} \phi_{n+1}(z) + \frac{l_n}{\kappa_n} - \frac{l_{n+1}}{\kappa_{n+1}} \phi_n(z) \\
&\quad + \left\{ \frac{l_n}{\kappa_{n-1}} \left( \frac{l_{n+1}}{\kappa_{n+1}} - \frac{l_{n-1}}{\kappa_n} \right) + \frac{m_n}{\kappa_{n-1}} - \frac{m_{n+1}}{\kappa_{n+1}} \frac{\kappa_n}{\kappa_{n-1}} \right\} \phi_{n-1}(z) + \pi_{n-2} \\
z^2 \phi_n(z) &= \frac{\kappa_n}{\kappa_{n+2}} \phi_{n+2}(z) + \left( \frac{l_n}{\kappa_{n+1}} - \frac{l_{n+2}}{\kappa_{n+2}} \frac{\kappa_n}{\kappa_{n+1}} \right) \phi_{n+1}(z) \\
&\quad + \left\{ \frac{l_{n+1}}{\kappa_{n+1}} \left( \frac{l_{n+2}}{\kappa_{n+2}} - \frac{l_n}{\kappa_n} \right) + \frac{m_n}{\kappa_n} \frac{m_{n+2}}{\kappa_{n+2}} \right\} \phi_n(z) + \pi_{n-1}
\end{align}

\begin{align}
\phi_n'(z) &= n \frac{\kappa_n}{\kappa_{n+1}} \phi_{n-1}(z) + \pi_{n-2} \\
z \phi_n'(z) &= n \phi_n(z) - \frac{l_n}{\kappa_{n-1}} \phi_{n-1}(z) + \pi_{n-2} \\
z^2 \phi_n'(z) &= n \phi_n(z) - \left( \frac{\kappa_n}{\kappa_{n+1}} \right) \phi_{n+1}(z) + \left\{ (n-1) \frac{l_n}{\kappa_n} - \frac{l_{n+1}}{\kappa_{n+1}} \right\} \phi_n(z) + \pi_{n-1}
\end{align}

where ' denotes the derivative with respect to $z$. 
How we use the above formulae to produce recurrences for \( r_N \) and \( \bar{r}_N \) differs for the orthogonal polynomial systems corresponding to the two different weights in \([17]\). Let us then treat the two weights separately. Let us refer to the order of a difference equation in \( r_n, \bar{r}_n \) as \( q/p \) where \( q \in \mathbb{Z}_{\geq 0} \) refers to the order of \( r_n \) while \( p \in \mathbb{Z}_{\geq 0} \) refers to the order of \( \bar{r}_n \).

3. The \( P_{III} \) System

For non-integer values of \( \mu \) the weight function

\[
(3.1) \quad w(z) = z^\mu e^{\frac{1}{\sqrt{z(z+1)}},}
\]

has a branch point at \( z = 0 \). Cutting the complex plane along the negative real axis \((-\infty, 0]\) we deform the contour in \([24]\) to the contour starting at \(-\infty, \infty \) running along the real axis on the negative imaginary side to \( z = -1 \), following the circle \( |z| = 1 \) in the anticlockwise direction to return to \( z = -1 \) on the positive imaginary side, then returning to \(-\infty \) along this side of the negative real axis. This contour is standard in the theory of the Bessel function (see \([19]\), pg. 363). Denoting this contour by \( C \), and noting that the integral representation of the Bessel function of pure imaginary argument gives

\[
(3.2) \quad \int_C \frac{dz}{2\pi i z}w(z) = I_\mu(\sqrt{z}),
\]

we see that

\[
I_\mu[w] = \det[I_{\mu+i+j-k}(\sqrt{t})]_{j,k=1,\ldots,N}.
\]

For general \( \mu \) we then define

\[
(3.3) \quad \tau^{III}[N](t; \mu) = \det[I_{\mu+j-k}(\sqrt{t})]_{j,k=1,\ldots,N},
\]

which is consistent with \([12] \) in the case \( \mu \in \mathbb{Z} \).

Recurrences for the reflection coefficients \( r_N, \bar{r}_N \) in the case of the weight \([31]\) can be deduced from the work of Adler and van Moerbeke \([4]\). In their case 3, one specialises their parameters \( d_1 = d_2 = \gamma_1' = \gamma_2' = \gamma_1'' = \gamma_2'' = 0, \gamma = \mu \) and one sets \( P_1(z) = P_2(z) = 1/2\sqrt{z} \). Then making the identification \( r_N = x_N, \bar{r}_N = y_N \) it follows from \([4]\), eq. (0.0.17) that

\[
(3.5) \quad \frac{1}{2} \sqrt{t}v_N(r_{N+1} + r_{N-1}) + Nr_N = 0,
\]

\[
(3.6) \quad \frac{1}{2} \sqrt{t}v_N(\bar{r}_{N+1} + \bar{r}_{N-1}) + N\bar{r}_N = 0,
\]

where \( v_N := 1 - r_N\bar{r}_N \). After specifying the initial conditions

\[
(3.7) \quad r_0 = \bar{r}_0 = 1, \quad r_1 = -\frac{I_{\mu+1}(\sqrt{t})}{I_\mu(\sqrt{t})}, \quad \bar{r}_1 = -\frac{I_{\mu-1}(\sqrt{t})}{I_\mu(\sqrt{t})},
\]

which follow from \([2.8] \) and \([3.3] \), the recurrences \([3.5] \) and \([3.6] \) uniquely determine \( r_N, \bar{r}_N \) for \( N = 2, 3, \ldots \). We note that the order of \( (3.5) \) and \( (3.6) \) is 2/0 and 0/2 respectively and the parameter \( \mu \) does not appear explicitly. In addition we observe that \([3.5] \) and \([3.6] \) have the familiar form of the discrete Painlevé equation associated with degeneration of the rational surfaces \( D_6^{(1)} \rightarrow E_6^{(1)} \) \([17]\) (discrete Painlevé II). We now seek a derivation of \([3.5] \) and \([3.6] \) using the formulae of Section 2 specialised to the weight \([3.1] \). In addition we will show that the orthogonal polynomial theory can be used to derive a pair of coupled difference equations for \( r_N, \bar{r}_N \), both of order 1/1. It will then be shown how \([3.5] \), \([3.6] \) can be deduced from these equations.
Proposition 3.1. The reflection coefficients \( r_N \) corresponding to the Toeplitz determinants \( \Delta_N \) satisfy the coupled 1/1 order recurrences

\[
\begin{align*}
\frac{1}{2} \sqrt{t} (r_{N+1} \bar{r}_N + r_N \bar{r}_{N-1}) + N \frac{r_N \bar{r}_N}{1 - r_N \bar{r}_N} - \mu = 0, \\
\frac{1}{2} \sqrt{t} (\bar{r}_{N+1} r_N + \bar{r}_N r_{N-1}) + N \frac{\bar{r}_N r_N}{1 - r_N \bar{r}_N} + \mu = 0,
\end{align*}
\]

with the initial conditions \( r_1 = 1 \).

Proof. We adapt a method due to Freud [12], and consider two different ways to evaluate the integral

\[
J_1 := \int_C \frac{dz}{2 \pi i z^2} w'(z) \phi_N(z) \phi_{N+1}(z).
\]

Integrating this by parts, employing (2.20) and the orthogonality conditions shows

\[
J_1 = - (N + 1) \frac{\kappa_N}{\kappa_{N+1}} + (N + 1) \frac{\kappa_{N+1}}{\kappa_N}.
\]

Alternatively we note from (3.1) that

\[
\frac{w'}{w} = \frac{\mu}{z} + \frac{1}{2} \sqrt{t} (1 - \frac{1}{z^2}),
\]

substituting this in (3.10), one can again use (2.20) and the orthogonality conditions to show

\[
J_1 = \mu \frac{\kappa_N}{\kappa_{N+1}} + \frac{1}{2} \sqrt{t} \left( \frac{l_N}{\kappa_{N+1}} - \frac{l_{N+2}}{\kappa_{N+2}} \frac{\kappa_{N+1}}{\kappa_{N+2}} \right).
\]

Equating (3.11) and (3.13) and eliminating \( l_N \) using (2.17) gives (3.8). To deduce (3.9) we apply an analogous strategy to

\[
J_2 := \int_C \frac{dz}{2 \pi i z^2} w'(z) \phi_{N+1}(z) \phi_N(z).
\]

To deduce (3.5), (3.6) from (3.8), (3.9) we first subtract (3.8) from (3.9) to find

\[
\frac{1}{2} \sqrt{t} \left( L_{N+1} + L_N \right) + 2 \mu = 0, \quad L_N := \bar{r}_{N+1} r_N - r_N \bar{r}_{N-1}.
\]

Using (3.7) and a Bessel function identity this equation is to be solved subject to the initial condition

\[
L_1 := - \frac{2 \mu}{\sqrt{t}}.
\]

It follows that the solution of (3.15) is the constant

\[
L_N = - \frac{2 \mu}{\sqrt{t}}, \quad N = 1, 2, 3, \ldots
\]

and thus

\[
\frac{1}{2} \sqrt{t} \left( \bar{r}_{N+1} r_N - r_N \bar{r}_{N-1} \right) + \mu = 0.
\]

Using this to substitute for \( \mu \) in (3.8) and (3.9) gives (3.5) and (3.6) respectively. Furthermore (3.18) can be summed to give

\[
\frac{1}{2} \sqrt{t} \left( \frac{l_N}{\kappa_N} + \frac{l_N}{\kappa_N} \right) + \mu N = 0.
\]
We turn our attention now to formulae for $\tau_{\Pi}^{\Pi}[N]$, $r_N$ and $\tilde{r}_N$ in terms of generalised hypergeometric functions. From earlier work \[14\], \[10\] we know

\begin{equation}
(3.20) \quad \tau_{\Pi}^{\Pi}[N](t; \mu) = \left( \frac{\sqrt{t}}{2} \right)^N \frac{1}{\mu!} \frac{\Gamma(j+\mu)}{(j+\mu)!} g F_1^1 \left( ; N + \mu; t_1, \ldots, t_N \right) |_{t_1 = \ldots = t_N = t/4}.
\end{equation}

It follows from this and \[3.3\], \[3.4\] and \[2.8\] that

\begin{align}
(3.21) \quad r_N &= (-1)^N \left( \frac{\sqrt{t}}{2} \right)^N \frac{1}{(\mu+1)!} \frac{\Gamma(j+1+\mu)}{(j+1+\mu)!} g F_0^1 \left( ; N + 1 + \mu; t_1, \ldots, t_N \right) \bigg|_{t_1 = \ldots = t_N = t/4}, \\
(3.22) \quad \tilde{r}_N &= (-1)^N \left( \frac{2}{\sqrt{t}} \right)^N \frac{1}{(\mu)!} \frac{\Gamma(j+1)}{(j+1)!} g F_0^1 \left( ; N - 1 + \mu; t_1, \ldots, t_N \right) \bigg|_{t_1 = \ldots = t_N = t/4}.
\end{align}

Note that the small-$t$ expansions are more forthcoming from these formulae than from the Toeplitz determinants.

Our final point in relation to $\tau_{\Pi}^{\Pi}[N]$ concerns the known recurrence scheme for $\tau_{\Pi}^{\Pi}[N]$ in terms of the variables $p_N, q_N$ specifying the corresponding Hamiltonian in the Painlevé systems approach to $\Pi_{\Pi}^{\Pi}$. The idea here is that we start from the Hamiltonian for the $\Pi_{\Pi}^{\Pi}$ system

\begin{equation}
(3.23) \quad tH_{\pi n}^{\Pi} = q_n^2 p_n^2 - \left( q_n^2 + v_1 q_n - t \right) p_n + \frac{1}{2} (v_1 + v_2) q_n,
\end{equation}

with special parameters $(v_1, v_2) = (v_1^{(0)}, v_2^{(0)}) = (\mu, -\mu)$ and corresponding special values $p = p_0 = 0, q = q_0$ for some particular $q_0$ (see e.g. \[10\]). A sequence of Hamiltonians is constructed from this seed by application of the Schlesinger transformation $T_1$ with the action on the parameters $T_1 \cdot (v_1, v_2) = (v_1 + 1, v_2 + 1)$ and some explicit actions on $p$ and $q$ involving rational functions of $p|(v_1, v_2)$ and $q|(v_1, v_2)$. Thus we set

\begin{equation}
(3.24) \quad tH_{\pi n}^{\Pi} := tH_{\pi n}^{\Pi} \bigg|_{T_1^n (v_1^{(0)}, v_2^{(0)})},
\end{equation}

and introduce the corresponding $\tau$-function $\tau_{\pi n}^{\Pi}$ by the requirement that

\begin{equation}
(3.25) \quad H_{\pi n}^{\Pi} := \frac{d}{dt} \log \tau_{\pi n}^{\Pi}.
\end{equation}

We know from \[11\] that with $(v_1^{(0)}, v_2^{(0)}) = (\mu, -\mu)$ the sequence $\{\tau_{\pi n}^{\Pi}\}_{n=0, 1, 2, \ldots}$ is realised by

\begin{equation}
(3.26) \quad \tau_{\pi n}^{\Pi} = e^{-n \mu/2} \tau_{\Pi}^{\Pi}[N](t; \mu) \bigg|_{t \to 4t}.
\end{equation}

With $p_n := T_1^n p|(v_1^{(0)}, v_2^{(0)})$ and $q_n := T_1^n q|(v_1^{(0)}, v_2^{(0)})$ further theory associated with $T_1$ led to the following recurrence scheme.

**Proposition 3.2 \[11\].** We have

\begin{align}
(3.27) \quad & \frac{\tau_{\Pi}^{\Pi}[N+1] \tau_{\Pi}^{\Pi}[N-1]}{\left( \tau_{\Pi}^{\Pi}[N] \right)^2} \bigg|_{t \to 4t} = p_N, \quad (N = 1, 2, \ldots) \\
(3.28) \quad p_{N+1} = q_N \left( p_N - 1 \right) - \mu q_N + 1 \quad (N = 0, 1, \ldots) \\
(3.29) \quad q_{N+1} = \frac{t}{q_N} + \frac{(N+1)t}{q_N (q_N (p_N - 1) - \mu) + t} \quad (N = 0, 1, \ldots)
\end{align}
subject to the initial conditions

\begin{align}
(3.30) \quad p_0 &= 0, \quad q_0 = \left. \frac{d}{dt} \log t^{-\mu/2} I_\mu(\sqrt{t}) \right|_{t\to 4t}, \\
(3.31) \quad \tau_{III}[0] &= 1, \quad \left. \tau_{III}[1] \right|_{t\to 4t} = I_\mu(2\sqrt{t}).
\end{align}

Our interest is in the relationship between the variables \(p_N, q_N\) in these recurrences and the reflection coefficients of Prop.3.1. This is given by the following result.

**Proposition 3.3.** The Hamiltonian variables \(q_N, p_N\) in Prop.3.2 are related to the reflection coefficients in Prop.3.1 by

\begin{align}
(3.32) \quad p_N &= 1 - r_N \bar{r}_N \left|_{t\to 4t} \right., \\
(3.33) \quad q_N &= -\sqrt{r^2_{N+1} - 1} \left|_{t\to 4t} \right. .
\end{align}

**Proof.** The equation (3.32) follows immediately upon comparing (3.27) with (2.9). Substituting (3.32) in (3.28) we see that (3.18) results if we also substitute for \(q_N\) according to (3.33). Furthermore, if we combine (3.28) and (3.29) into

\begin{align}
(3.34) \quad q_N + \frac{t}{q_{N-1}} &= \frac{N}{p_N},
\end{align}

we see that the substitutions (3.32) and (3.33) in this equation yields (3.5). □

Thus the structure of the recurrences is such that the transformation equations (3.32) and (3.33) can essentially be determined by inspection. However when we come to study the analogous recurrences for the \(P_{\nu}\) \(\tau\)-function 140 this is not possible and a more systematic procedure is called for. With this in mind, let us then present a more systematic approach to the derivation of (3.35).

For this purpose, in addition to the shift operator \(T_1\) which increments \(N\) in \(\tau_{III}[N]\), we introduce the other fundamental Schlesinger transformation of the \(P_{III}\) system \(T_2\) with the action on the parameters \(T_2 \cdot (v_1, v_2) = (v_1 + 1, v_2 - 1).\) Recalling that in \(\tau_{III}[N]\) \((v_1^0, v_2^0) = (\mu, -\mu),\) this operator then increments \(\mu\) by unity. From 110 we know that for general parameters

\begin{align}
(3.35) \quad T_1 \cdot tH_{III} = tH_{III} - q(p - 1), \quad T_2 \cdot tH_{III} = tH_{III} - qp,
\end{align}

and thus in particular

\begin{align}
(3.36) \quad q_N &= \left. \frac{d}{dt} \log \frac{T_1 \cdot \tau_N}{T_2 \cdot \tau_N} \right|_{t\to 4t}, \\
(3.37) \quad &= \frac{1}{2} (N - \mu) - \left. \frac{d}{dt} \log \kappa^2_N \tau_N \right|_{t\to 4t},
\end{align}

where to obtain the second equality use has been made of (3.25), (2.8) and (2.19). To determine the \(t\)-derivatives of the orthogonal polynomial coefficients we find the \(t\)-derivatives of the polynomials themselves. Differentiating the orthonormality condition (2.4) with \(w(z)\) given by (3.1) we have

\begin{align}
(3.38) \quad 0 &= -\frac{\bar{w}_0}{w_0} \delta_{mn} + \int_T \frac{dz}{2\pi i} \bar{w}(\phi_m + 1/2 z \phi_n) \phi_m + \int_T \frac{dz}{2\pi i} \bar{w}(\phi_m + 1/2 z \phi_n),
\end{align}
with \( \cdot \) represents differentiation with respect to \( t \). Now \( \phi_n + \frac{1}{2} \frac{z}{\phi_n} \) is of degree \( n + 1 \) in \( z \) and has no components in \( \phi_m \) for \( m \geq n + 2 \). With this established it follows from (3.38) that \( \phi_n + \frac{1}{2} \frac{z}{\phi_n} \) has no components in \( \phi_m \) for \( m \leq n - 2 \) either and so

\[
\phi_n + \frac{1}{2} \frac{z}{\phi_n} = \bar{a}_n \phi_{n+1} + \bar{b}_n \phi_{n+1} + \bar{c}_n \phi_{n-1}.
\]

Equating coefficients of the highest monomial in (3.38) and recalling (4.4) gives \( \bar{a}_n = \kappa_n/2\kappa_{n+1} \) while (3.38) in the case \( m = n - 1 \) yields \( \bar{c}_n = -\kappa_{n-1}/2\kappa_n \). Finally, setting \( m = n \) in (3.38) and recalling (3.2) shows

\[
\bar{b}_n + \bar{b}_n = \frac{f_{\mu}}{I_{\mu}}.
\]

Substituting for \( \bar{a}_n \) and \( \bar{c}_n \) in (3.38), and using the three-term recurrence (2.12) to substitute for \( \bar{a}_n \phi_{n+1} + 1/2 z \phi_n \) we deduce

\[
\phi_n = \left[ \bar{b}_n + 1/2 \frac{r_{n+1}}{r_n} \right] \phi_n - 1/2 \frac{\kappa_{n-1}}{\kappa_n} \left[ 1 + \frac{r_{n+1}}{r_n} \right] \phi_{n-1}.
\]

Recalling (2.6) it follows that

\[
\frac{\kappa_n^2}{\kappa_n} = \frac{f_{\mu}}{I_{\mu}} + 1/2 \left( r_{n+1} \bar{r}_n + \bar{r}_{n+1} \right) r_n
\]

\[
\frac{\bar{r}_n}{r_n} = 1/2 \left( r_{n+1} - r_n \right) \left( \frac{1}{r_n} - \bar{r}_n \right)
\]

\[
\frac{\phi_n}{\bar{r}_n} = 1/2 \left( \bar{r}_{n+1} - \bar{r}_{n} \right) \left( \frac{1}{\bar{r}_n} - \bar{r}_n \right).
\]

Making use of the first two of these relations in (3.39) and simplifying reclaims (3.38).

4. The P\( V \) System

The weight function

\[
w(z) = (1 + z)^n (1 + 1/z)^m z^k e^{t/z},
\]

is analytic in the cut plane \( z \in \mathbb{C} \setminus (-\infty, -1] \). For \( \Re(\mu + \nu + 1) > 0 \) the singularity is integrable at \( z = -1 \) so with this restriction there is no need to deform \( \mathbb{T} \) in (2.1). The transition from the P\( V \) to the P\( \mu \mu \) system can be achieved by making the replacements

\[
\nu \rightarrow \nu - \mu, \quad z_1 \rightarrow 2\mu z_1 / \sqrt{t}, \quad t \rightarrow t/4\nu
\]

with \( t, N, \mu \) fixed and then taking the limit \( \nu \rightarrow \infty \). In this way the Toeplitz determinant (1.3) reduces to (1.2). In [9] we have shown that with \( w \) given by (1.1)

\[
\int_{\mathbb{T}} \frac{dz}{2\pi i z} w(z) = \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu + 1 - n) \Gamma(\nu + 1 + m)} F_1(-n - \nu; \mu - n + 1; -t), \quad n \in \mathbb{Z},
\]

and thus \( I_N[w] \) as specified by (2.2) can be made explicit. In particular it follows from (1.3), (1.2) and (2.8) that

\[
r_1 = -\frac{\nu}{\mu + 1} F_1(-\nu + 1; \mu + 2; -t), \quad \bar{r}_1 = -\frac{\mu}{\nu + 1} F_1(-\nu - 1; \mu + 1; -t).
\]

As with the weight (1.1), recurrences for the reflection coefficients \( r_N, \bar{r}_N \) in the case of the weight (1.1) can be deduced from the work of Adler and van Moerbeke [4]. Thus in their first case one specialises their parameters \( \gamma_0^\mu = \gamma_0^\nu = d_2 = 0, d_1 = -1, \gamma = -\nu, \gamma_1 = \mu + \nu \) which gives their \((a, b, c) = (1, 1, 0)\), and one sets \( P_1(z) = tz, P_2(z) = 0. \)
With the identification \(x_N = r_N, y_N = \bar{r}_N\), from (4.1) eq.(0.0.14) we then read off the recurrence

\[
-\nu r_{N+1} r_N + \nu r_{N} r_{N-1} + (N + 1 + \nu) r_N \bar{r}_{N+1} - (N - 1 + \nu) r_{N-1} \bar{r}_N = 0,
\]

which is of order \(2/2\). Also (4.1) eq.(0.0.15) reads

\[
v_{N+1} [N + 1 + \nu + \nu r_{N+2} \bar{r}_N] - v_N [N + \nu + \nu r_{N+1} \bar{r}_{N-1}] + r_{N+1} \bar{r}_N
\]

\[
= v_1 (1 + \nu + \nu r_2 \bar{r}_0) + r_1 \bar{r}_0,
\]

where \(v_N := 1 - r_N \bar{r}_N\) which is also of order \(2/2\). The right hand side of (4.6) can be simplified. First we note that it can be written in terms of the Fourier components of (4.1) according to

\[
(\nu + 1)(1 - w_{-1} w_1) + t(w_{-1}^2 - w_{-2}) - w_{-1}
\]

and thus in the light of (4.8) in terms of the confluent hypergeometric function. Utilising the contiguous relation for the confluent hypergeometric function \(_1F_1(a; b; -t)\), (2.20) etc. (p.507) in the two cases \(a = -\nu, b = \mu + 1\) and \(a = -\nu + 1, b = \mu + 2\) one can show this is precisely unity, and so (4.8) simplifies to

\[
v_{N+1} [N + 1 + \nu + \nu r_{N+2} \bar{r}_N] - v_N [N + \nu + \nu r_{N+1} \bar{r}_{N-1}] + r_{N+1} \bar{r}_N = 1.
\]

We can use an orthogonal polynomial approach to derive recurrences for \(r_N, \bar{r}_N\), and from equations used in the derivation (4.5) and (4.8) can be reclaimed.

**Proposition 4.1.** The reflection coefficients \(r_N, \bar{r}_N\) for the orthogonal polynomial system on the unit circle with the weight (4.1) satisfy the coupled system of \(2/1\) and \(1/2\) order recurrence relations

\[
(N - 1 + \nu) r_{N-1} + (N + \mu + t) r_N = -t(1 - r_N \bar{r}_N) r_{N+1} + \nu \bar{r}_N, \quad (4.9)
\]

\[
(N + 1 + \nu) \bar{r}_{N+1} + (N + \mu + t) \bar{r}_N = -t(1 - r_N \bar{r}_N) \bar{r}_{N-1} + \nu r_N, \quad (4.10)
\]

subject to the initial conditions (4.5).

**Proof.** As with the derivation of the recurrences of Proposition 3.1, we will take the Freud approach, but now working with two integrals for each recurrence rather than the single integral for each recurrence required to derive the recurrences of Proposition 3.1. The first integral we consider is

\[
J_1 := \int_{\mathbb{T}} \frac{dz}{2\pi i z} (1 + z) w(z) \phi_N(z) \bar{\phi}_N(z).
\]

Integrating this by parts, employing (2.12) and the orthogonality condition shows

\[
J_1 := N \frac{\hat{I}_N}{\hat{K}_N} - (N + 1) \frac{\hat{I}_{N+1}}{\hat{K}_{N+1}}.
\]

On the other hand we note from (4.1) that

\[
\frac{w'}{w} = \frac{\mu + \nu}{1 + \nu} = \frac{\mu}{z} + t,
\]

substituting this in (4.11), an analogous calculation shows

\[
J_1 = \mu - \nu \left( \frac{\hat{I}_N}{\hat{K}_N} - \frac{\hat{I}_{N+1}}{\hat{K}_{N+1}} \right) + t \left( \frac{\hat{I}_N}{\hat{K}_N} - \frac{\hat{I}_{N+1}}{\hat{K}_{N+1}} \right) + t.
\]
Upon employing (4.24) one can equate (4.22) and (4.23) and solve for \( \bar{I}_{N+1} \) to obtain
\[
\frac{\bar{I}_{N+1}}{\kappa_{N+1}} = -\mu - t(1 - \bar{r}_N r_{N+1}) - (N + \nu)r_N \bar{r}_{N+1}.
\]
Note that one could perform another differencing at this point and eliminate \( \bar{I}_N \) in favour of \( r_N, \bar{r}_N \) but then \( \mu \) would disappear from the ensuing relations. This is a clear indication that the recurrence system would be raised unnecessarily in order, so we seek another relation for \( \bar{I}_N \), and this is found by considering the following integral
\[
J_2 := \int_\mathbb{T} \frac{dz}{2\pi i z} (1 + z) w'(z) \phi_N(z) \overline{\phi_{N+1}(z)}.
\]
The methods of evaluation used to derive (4.12) and (4.14) now yield
\[
J_2 = (N + 1) \frac{\kappa_{N+1}}{\kappa_N} - (N + 1) \frac{\kappa_N}{\kappa_{N+1}} - \frac{\bar{I}_{N+1}}{\kappa_{N+1}},
\]
and
\[
J_2 = \mu \frac{\kappa_N}{\kappa_{N+1}} + t \left( \frac{\kappa_N}{\kappa_{N+1}} + \frac{\bar{I}_N}{\kappa_{N+1}} - \frac{\bar{I}_{N+1}}{\kappa_{N+1}} \right),
\]
respectively. Equating (4.17) and (4.18), and again employing (2.17), it follows
\[
\frac{\bar{I}_{N+1}}{\kappa_{N+1}} = N + 1 - [N + 1 + \mu + t - t(r_{N+2} \bar{r}_{N+1} + r_{N+1} \bar{r}_N)] (1 - r_{N+1} \bar{r}_{N+1}).
\]
Equating (4.17) and (4.19) gives (4.10). The second recurrence can be found in a similar manner by eliminating \( \bar{I}_{N+1} \) from expressions arising from evaluation of the integrals
\[
J_3 := \int_\mathbb{T} \frac{dz}{2\pi i z} (1 + z) w'(z) \phi_{N+1}(z) \overline{\phi_N(z)},
\]
and
\[
J_4 := \int_\mathbb{T} \frac{dz}{2\pi i z} (1 + z) w'(z) \phi_N(z) \overline{\phi_{N+1}(z)}.
\]
The two expressions for \( \bar{I}_{N+1} \) are respectively
\[
\frac{\bar{I}_{N+1}}{\kappa_{N+1}} = (N + 1) r_{N+1} \bar{r}_{N+1} - (\nu + tr_{N+2} \bar{r}_N)(1 - r_{N+1} \bar{r}_{N+1})
\]
\[= -\nu - (N + \mu + t)r_{N+1} \bar{r}_N - t(1 - r_{N} \bar{r}_{N})r_{N+1} \bar{r}_{N-1}
\]
\[= -t (1 - r_{N+1} \bar{r}_{N+1}) r_{N+2} \bar{r}_N + tr_{N+2} \bar{r}_N,
\]
and this yields (4.10). □

Let us now show how we can derive the recurrences of Adler and van Moerbeke (4.5), (4.8) from the workings of the proof of Proposition 4.1. As already remarked, subtracting (4.15) the same equation with \( N \) replaced by \( N - 1 \) and recalling (2.14) gives a difference equation in \( r_N, \bar{r}_N \). This difference equation is in fact precisely (4.5). Note that with \( \Delta_N \) the forward difference operator with respect to \( N \), (4.15) itself can be written
\[
\Delta_N [ (N + \nu) \frac{\bar{I}_N}{\kappa_N} - t \frac{\bar{I}_N}{\kappa_N} ] = -\mu - t.
\]
Summing this over \( N \) shows
\[
(N + \nu) \frac{\bar{I}_N}{\kappa_N} - t \frac{\bar{I}_N}{\kappa_N} = -(\mu + t)N,
\]
which then is the basic relation underlying (4.5). The recurrence (4.8) is obtained by subtracting from (4.20) the same equation with \( N \) replaced by \( N - 1 \), and recalling (2.17).
We consider next formulae for \( \tau^V[N](t; \mu, \nu) \), \( r_N, \bar{r}_N \) in terms of generalised hypergeometric functions. From earlier work \cite{14}, \cite{11} we know

\begin{equation}
\tau^V[N](t; \mu, \nu) = \sum_{j=0}^{N-1} \left[ \frac{\Gamma(\mu + \nu + 1 + j)\Gamma(1 + j)}{(\mu + 1 + j)\Gamma(\nu + 1 + j)} \right] F_1^{(1)}(-\nu; N + \mu; t_1, \ldots, t_N) \right|_{t_1=\ldots=t_N=-t}.
\end{equation}

Noting from \cite{20}, \cite{21} and \cite{22} that for the weight \cite{23}

\begin{equation}
I_N[w] = \tau^V[N](t; \mu + 1, \nu - 1), \quad I_N^{-1}[w] = \tau^V[N](t; \mu - 1, \nu + 1),
\end{equation}

it follows from this and \cite{23} that

\begin{equation}
r_N = (-1)^N \frac{(\nu)_{N}}{(\mu + 1)_N} F_1^{(1)}(-\nu + 1; N + 1 + \mu; t_1, \ldots, t_N) \bigg|_{t_1=\ldots=t_N=-t},
\end{equation}

\begin{equation}
\bar{r}_N = (-1)^N \frac{(\mu)_N}{(\nu + 1)_N} F_1^{(1)}(-\nu - 1; N - 1 + \mu; t_1, \ldots, t_N) \bigg|_{t_1=\ldots=t_N=-t}.
\end{equation}

An immediate consequence of \cite{24} and \cite{25} is the explicit values at \( t = 0, \)

\begin{equation}
r_N = (-1)^N \frac{(\nu)_N}{(\mu + 1)_N}, \quad \bar{r}_N = (-1)^N \frac{(\mu)_N}{(\nu + 1)_N}.
\end{equation}

\begin{equation}
\frac{I_N}{\kappa_N} = - \frac{N\nu}{N + \mu}, \quad \frac{\bar{I}_N}{\bar{\kappa}_N} = - \frac{N\mu}{N + \nu}.
\end{equation}

Again we see that the transition from the generalised hypergeometric function in the \( P_V \) system \cite{26} to that of the \( P_{1IV} \) system \cite{27} is facilitated by making the replacements \cite{28} and taking the limit \( \nu \to \infty \)

\begin{equation}
r_N \to \nu^{-1} F_1^{(1)}(\sigma - \nu; N + \mu; t_1, \ldots, t_N) \bigg|_{t_1=\ldots=t_N=-t/4\nu} \quad \mu \to \infty \to q F_1^{(1)}(\sigma; N + \mu; t_1, \ldots, t_N) \bigg|_{t_1=\ldots=t_N=-t/4},
\end{equation}

for all fixed \( \sigma, N + \mu, t \). Thus the corresponding limiting forms for the reflection coefficients are

\begin{equation}
r_N \sim \nu^{-1} \left( \frac{2\nu}{\sqrt{\nu}} \right)^N r_N^{[IV]}, \quad \bar{r}_N \sim \nu^{-1} \left( \frac{\sqrt{\nu}}{2\nu} \right)^N \bar{r}_N^{[IV]},
\end{equation}

where we distinguish the two systems only when some confusion could arise. Another consequence of this limiting process is that the recurrences in \( r_N, \bar{r}_N \) \cite{29}, \cite{10} reduce to a sum of \cite{30} and \cite{31} and of \cite{32} and \cite{33} respectively. In addition we find that \cite{4} reduces to \cite{5}.

It remains to compare the recurrences \cite{6}, \cite{10} determining \( \tau^V[N] \) through the recurrence \cite{29}, and recurrences satisfied by the Hamiltonian variables \( q_N, p_N \) in the Painlevé systems approach to \( P_V \) which also determine \( \tau^V[N] \) through a recurrence \cite{9}. In relation to the latter, let \( v_1 + v_2 + v_3 + v_4 = 0 \) and introduce the Hamiltonian

\begin{equation}
tH^V = q(q - 1)p(p + t) - (v_2 - v_1 + v_3 - v_4)qp + (v_2 - v_1)p + (v_1 - v_3)tq.
\end{equation}

Starting with a special solution when \( v_3 = v_4 = 0 \), \( q_0 = 1, p = p_0 \) for some particular \( p_0 \) \cite{34}, a sequence of Hamiltonians is constructed by application of the
Schlesinger transformation $T_0^{-1}$ with the action on the parameters $T_0^{-1} \cdot (v_1, v_2, v_3, v_4) = (v_1 - 1/4, v_2 - 1/4, v_3 - 1/4, v_4 + 3/4)$ to obtain

\[(4.41)\]
\[tH_n^\tau = tH_n^\nu\bigg|_{v \to (v_1-\nu/4,v_2-\nu/4,v_3-\nu/4,v_4+3\nu/4)}.
\]

The corresponding sequence of $\tau$-functions $\tau_n^\nu$ are specified so that

\[(4.42)\]
\[H_n^\nu = \frac{d}{dt} \log \tau_n^\nu.
\]

We know from [9] that with $v_1^{(0)} - v_2^{(0)} = \mu, v_3^{(0)} - v_4^{(0)} = -\nu, v_3^{(0)} - v_4^{(0)} = 0$ the sequence $\{\tau_n^\nu\}_{n=0,1,...}$ is realised by

\[(4.43)\]
\[e^{t\tau_n^\nu} = \left[\frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)}\right]^{n} \prod_{l=0}^{n-1} \Gamma(\nu + l + 1) \tau_l^\nu(t; \mu, \nu).
\]

Furthermore $\{\tau^\nu[N]\}_{N=2,3,...}$ is determined by the following recurrence scheme [9].

**Proposition 4.2.** Let

\[(4.44)\]
\[x_N = (p_N + t)q_N + 1/2 \mu, \quad y_N = \frac{1}{q_N}.
\]

The sequences $\{\tau^\nu[N]\}_{N=0,1,...}, \{x_N\}_{N=0,1,...}, \{y_N\}_{N=0,1,...}$ satisfy the coupled recurrences

\[(4.45)\]
\[(N + \nu)\frac{\tau^\nu[N + 1] + \tau^\nu[N - 1]}{(\tau^\nu[N])^2} = \left(x_N - \frac{t}{y_N} - \nu - \frac{1}{2} \mu\right) \left(\frac{1}{y_N} - 1\right) + N
\]

\[(4.46)\]
\[x_N + x_{N-1} = \frac{t}{y_N} - \frac{N}{1 - y_N},
\]

\[(4.47)\]
\[y_N y_{N+1} = \frac{t x_N + N + 1 + \nu + \frac{1}{2} \mu}{t x_N - \frac{1}{2} \mu^2},
\]

subject to the initial conditions

\[(4.48)\]
\[x_0 = t + \frac{1}{2} \mu + t \frac{d}{dt} \log_1 F_1(-\nu; \mu + 1; -t), \quad y_0 = 1
\]

\[(4.49)\]
\[\tau^\nu[0] = 1, \quad \tau^\nu[1] = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)} F_1(-\nu; \mu + 1; -t).
\]

By adopting a strategy analogous to the method of derivation of [9,13], [9,14] given below the proof of Proposition [9,15] the relationship between the variables $x_N, y_N$ and $\tau_N, \bar{\tau}_N$ can be deduced.

**Proposition 4.3.** The Hamiltonian variables $x_N, y_N$ and the reflection coefficients $\tau_N, \bar{\tau}_N$ are related by the equations

\[(4.50)\]
\[\left[\frac{1}{y_N} - 1\right] \left[x_N - \frac{t}{y_N} - \nu - \frac{1}{2} \mu\right] + N = (N + \nu)(1 - \tau_N \bar{\tau}_N)
\]

\[\left[x_N - \frac{t}{y_N} - \nu - \frac{1}{2} \mu\right] \left(1 + \frac{N + \nu}{x_N - \frac{t}{y_N} - \frac{1}{2} \mu} \left(\frac{1}{y_N} - \nu\right)\right) - N
\]

\[= \frac{\Gamma(N + 1)}{\Gamma(\nu + 1)} F_1(-\nu; \mu + 1; -t).
\]

\[(4.52)\]
\[(1 - y_N) \left[x_N - \frac{t}{y_N} + \frac{1}{2} \mu\right] + N = -t \frac{\Gamma(N - 1)}{\bar{\tau}_N} (1 - \tau_N \bar{\tau}_N).
\]
Consequently

\[ y_N = \frac{\nu + tr_{N+1}F_N}{r_N[tr_{N+1} + (N + \nu)r_N]} \]

\[ x_N = \frac{t}{y_N} - \frac{1}{2} \mu = -tr_{N+1}F_N. \]

**Proof.** The equation (4.44) follows immediately upon substituting for \( r^V[N + 1]/r^V[N - 1]/(r^V[N]^2 \) according to the right-hand side of the general relation (2.8) in [8]. For the remaining two equations, adapting the method of derivation of (3.33), (3.32) given below the proof of Proposition 32, we require a shift operator that has the action \( \mu \mapsto \mu + 1, \nu \mapsto \nu - 1 \) and leaves \( N \) fixed. Such an operator is the Schlesinger transformation \( T^{-1}_2 \) with the action on the parameters \( T^{-1}_2 \cdot (v_1, v_2, v_3, v_4) = (v_1 + 3/4, v_2 - 1/4, v_3 - 1/4, v_4 + 1/4) \) which was studied in [10]. From the explicit form of \( T^{-1}_2 \) in terms of operators associated with the root lattice \( A_3 \) and the actions of these operators on the Hamiltonian and associated variables in [11], we can compute that

\[ T^{-1}_2 \cdot tH^V = tH^V + \left( q + \frac{v_1 - v_3}{p} \right) \left( p - \frac{v_1 - v_4}{q - 1 + \frac{2}{p}} \right) + t + v_3 - v_4 \]

\[ T^{-1}_2 \cdot tH^V = tH^V + (q - 1) \left( p - \frac{v_2 - v_1}{q} \right) - t - (v_3 - v_4) \]

Recalling (4.34), (4.35) and (4.36) it follows that

\[ x_N - \frac{t}{y_N} - \nu - \frac{1}{2} \mu \left( 1 + \frac{N + \nu}{(x_N - \frac{t}{y_N} - \nu)(1 - y_N) - N} \right) - N = t \frac{d}{dt} \log r_n \]

\[ (1 - y_N) \left[ x_N - \frac{t}{y_N} + \frac{1}{2} \mu \right] + N = t \frac{d}{dt} \log \bar{r}_n. \]

To find the \( t \)-derivatives we differentiate the orthonormality relation

\[ 0 = \frac{\bar{w}_0}{w_0} \delta_{mn} + \int \frac{dz}{2\pi i} \bar{w}[\phi_m + z\phi_m] \overline{\phi_n} + \int \frac{dz}{2\pi i} \bar{w} \phi_m \overline{\phi_n} \]

and the case \( n \leq m - 2 \) indicates that \( \dot{\phi}_m + z\phi_m \) has no components in \( \phi_n \) so

\[ \dot{\phi}_m + z\phi_m = \bar{a}_m \phi_{m+1} + \bar{d}_m \phi_m + \bar{e}_m \phi_{m-1} \]

Consideration of the coefficients of the highest power in \( z \) gives \( \bar{a}_m = \kappa_m/\kappa_{m+1} \), and the case \( n = m - 1 \) shows \( \bar{e}_m = 0 \). The \( t \)-derivative of the orthogonal polynomial is

\[ \dot{\phi}_n = \left[ \bar{a}_n + \frac{\kappa_n \phi_{n+1}(0)}{\kappa_{n+1}(0)} \right] \phi_n - \frac{\kappa_{n-1} \phi_{n+1}(0)}{\kappa_{n+1}(0)} z \phi_{n-1} \]

and this implies

\[ \frac{\dot{r}_n}{r_n} = r_{n+1} \left( \frac{1}{r_n} - \bar{r}_n \right). \]

Next we examine the case \( n \geq m + 2 \) and find

\[ \dot{\phi}_n = b_n \phi_n + c_n \phi_{n-1} \].
Taking the case \( n = m + 1 \) we infer \( c_n = -\kappa_{n-1}/\kappa_n \). The \( t \)-derivative of the orthogonal polynomial can then be written as

\[
\dot{\phi}_n = b_n \phi_n - \frac{\kappa_{n-1}}{\kappa_n} \phi_{n-1},
\]

from which we deduce

\[
\frac{\dot{r}_n}{r_n} = -\tilde{r}_{n-1} \left( \frac{1}{\tilde{r}_n} - r_n \right).
\]

Substituting (4.54), (4.57) in (4.49), (4.50) gives (4.43, 4.44) respectively.

To derive (4.45), multiply both sides of (4.43) by \((1 - y_N)/y_N\), then substitute for

\[
X \frac{1 - y_N}{y_N}, \quad X := x_N - \frac{t}{y_N} - \nu - \nu/2, \mu,
\]

using (4.42) to deduce

\[
[(N + \nu)(1 - r_N \tilde{r}_N) - N] \left( 1 - \frac{1}{y_N} \right) = N \frac{1 - y_N}{y_N} = \frac{r_{N+1}}{r_N} (1 - r_N \tilde{r}_N) \frac{1 - y_N}{y_N}.
\]

Solving this equation for \( y_N \) gives (4.44). For the derivation of (4.45) we write \( V \equiv \mu \) in the form

\[
(X + \nu)(1 - y_N) = -(N + \nu) r_N \tilde{r}_N y_N + \nu.
\]

Substituting (4.44) for \( y_N \) and simplifying we obtain (4.46).

The Hamiltonian variables \( q_N, p_N \) or \( x_N, y_N \) in the P\(_V\) theory go over to those in the P\(_{III}\) theory under the replacements (4.2) and upon taking the limit \( \nu \to \infty \)

\[
q_N \underset{\nu \to \infty}{\rightarrow} 1 - p_N^{\nu'}, \quad p_N^{\nu'} \underset{\nu \to \infty}{\rightarrow} q_N^{\nu'}
\]

(4.58)

\[
x_N \underset{\nu \to \infty}{\rightarrow} \frac{1}{2} \mu + q_N^{\nu'} (1 - p_N^{\nu'}), \quad y_N \underset{\nu \to \infty}{\rightarrow} \frac{1}{1 - p_N^{\nu'}}.
\]

Using these transitions we find that the recurrence relations (4.53, 4.54) reduce to (4.38) upon using (3.27) (which follows from (3.29) and (3.28)), and to (3.28) respectively.

In addition to the formula (4.45) for \( y_N \), we obtain a different formula by making use of (4.44). This allows a pair of 1/1 order difference equations for \( r_N, \tilde{r}_N \) to be deduced, thus reducing the 2/1 system of Proposition 4.1 down to the same order as the coupled system (4.35, 4.36) satisfied by \( x_N, y_N \).

**Theorem 4.1.** The reflection coefficients satisfy the coupled 1/1 order recurrence relations

\[
(1 - r_N \tilde{r}_N) [t r_{N+1} + (N + \nu) r_N] [t \tilde{r}_{N-1} + (N + \nu) \tilde{r}_N] = [(N + \nu) r_N \tilde{r}_N + \mu] [t - (N + \nu) r_N \tilde{r}_N]
\]

(4.59)

\[
t^2 r_N^2 \tilde{r}_{N-1} + t(\nu - \mu - t)r_N \tilde{r}_{N-1} - (N + \nu)(N - 1 + \nu) \tilde{r}_N r_{N-1}
\]

\[
-(N - 1 + \nu) \tilde{r}_{N-1} r_{N-1} - (N + \nu) \tilde{r}_N r_{N-1} - \nu = 0
\]

**Proof.** Multiplying both sides of (4.44) by \((1 - y_N)/y_N\) and substituting for \( X(1 - y_N)/y_N \) using (4.42) then solving for \( y_N \) shows

\[
y_N = \frac{N + \mu + \nu + t \tilde{r}_{N-1}(1 - r_N \tilde{r}_N)}{\mu + (N + \nu) r_N \tilde{r}_N}.
\]

Equating this with (4.46) and solving for \( r_{N+1} \) gives (4.60). The second recurrence follows by eliminating \( r_{N+1} \) between (4.60) and (4.9). \( \square \)
As with the other recurrences for \( r_N, \bar{r}_N \) in the \( P_V \) system, we find (4.60), (4.61) under the replacements (4.2) and taking the limit \( \nu \to \infty \) assume the forms of the recurrences in the \( P_{III'} \) system (3.8), (3.18) respectively.

5. CONCLUDING REMARKS

In our work [9], in addition to the \( U(N) \) averages (1.2), (1.3), an \( N \)-recurrence was also obtained for

\[
\tau^{VI}[N](t; \mu, \omega_1, \omega_2; \xi) = \left( \prod_{l=1}^{N} (1 - \xi \chi(l)) e^{\omega_1 |1+z_l|^{2\omega_1}} \left( \frac{1}{t_{2l}} \right) ^{\mu} (1 + t_{2l})^{2\mu} \right)_{U(N)},
\]

where \( \chi(l) = 1 \) for \( \theta_l \in J, \chi(l) = 0 \) otherwise. As the notation suggests, this is a known \( \tau \)-function for a \( P_{VI} \) system [11]. Both \( \tau^{III'}[N] \) and \( \tau^V[N] \) can be obtained as degenerations of (5.1), or equivalently the weights (1.7) can be obtained as limiting cases of the "master" semi-classical weight function underlying (5.1). For general parameters, a recurrence scheme based on the discrete Painlevé equation associated with the degeneration of the rational surface \( D(1)^{(4)} \rightarrow D(1)^{(5)} \) (discrete Painlevé V), was given as a consequence of the Painlevé system theory of \( P_{VI} \) (recurrence schemes for special cases of the parameters have also been given in [6], [7]). It is also true that an \( N \)-recurrence for (5.1) can be deduced from the Toeplitz lattice approach of Adler and van Moerbeke [4]. But here we have found that in the cases of the \( U(N) \) averages (1.2), (1.3) the approach of [4] leads to equivalent results as do those obtained from an orthogonal polynomial approach. One therefore suspects the same will be true in relation to (5.1), and that furthermore the corresponding recurrences are transformed versions of the discrete Painlevé V equation found in [11]. This is indeed the case, but the details do not fit well with the scheme of the present paper (in particular the Freud approach to the recurrences for \( r_n, \bar{r}_n \) is now inadequate) so will be reported elsewhere.

Another point of interest relates to the \( N \)-recurrences for Hermitian matrix (as opposed to unitary matrix) averages in which the weight function is a \( q \)-generalisation of a classical weight function. In [7] it is shown that the method of Borodin leads to \( q \)-discrete Painlevé equations. Can one obtain the \( q \)-discrete Painlevé equations from an orthogonal polynomial approach?

Acknowledgment. This research has been supported by the Australian Research Council.

REFERENCES

1. Oeuvres de Laguerre. Tome I, Chelsea Publishing Co., Bronx, N.Y., 1972, Algèbre. Calcul intégral, Rédigées par Ch. Hermite, H. Poincaré et E. Rouché, Réimpression de l’édition de 1898. MR 52 #13292
2. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, 7th ed., Dover Publications, Inc., New York, 1970.
3. M. Adler, P. J. Forrester, T. Nagao, and P. van Moerbeke, Classical skew orthogonal polynomials and random matrices, J. Statist. Phys. 99 (2000), no. 1-2, 141-170, solv-int/9907001. MR 2001k:82046
4. M. Adler and P. van Moerbeke, Recursion relations for unitary integrals, combinatorics and the Toeplitz lattice, math-ph/0201063, 2002.
5. A. Borodin, *Riemann-Hilbert problem and the discrete Bessel kernel*, Internat. Math. Res. Notices (2000), no. 9, 467–494. MR 1 756 945
6. _____, *Discrete gap probabilities and discrete Painlevé equations*, math-ph/0111008, 2001.
7. A. Borodin and D. Boyarchenko, *Distribution of the first particle in discrete orthogonal polynomial ensembles*, math-ph/0204001, 2002.
8. P. J. Forrester, *Log Gases and Random Matrices*, http://www.ms.unimelb.edu.au/~matpjf/matpjf.html.
9. P. J. Forrester and N. S. Witte, *Discrete Painlevé Equations and n-recurrences for spectral averages of n x n random hermitian matrices.
10. _____, *Application of the τ-function theory of Painlevé equations to random matrices: PV, PHI, the LUE, JUE and CUE*, Commun. Pure Appl. Math. 55 (2002), 679–727.
11. _____, *Application of the τ-function theory of Painlevé equations to random matrices: PVI, the JUE, CyUE, cJUE and scaled limits*, math-ph/0204008, 2002.
12. Géza Freud, *On the coefficients in the recursion formulae of orthogonal polynomials*, Proc. Roy. Irish Acad. Sect. A 76 (1976), no. 1, 1–6. MR 54 #7913
13. Mourad E. H. Ismail and Nicholas S. Witte, *Discriminants and functional equations for polynomials orthogonal on the unit circle*, J. Approx. Theory 110 (2001), no. 2, 200–228. MR 2002e:33011
14. J. Kaneko, *Selberg integrals and hypergeometric functions associated with Jack polynomials*, SIAM J. Math. Anal. 24 (1993), no. 4, 1086–1110. MR 94h:33010
15. I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications. MR 96h:05207
16. A. Magnus, *MAPA3072A Special topics in approximation theory 1999-2000: Semi-classical orthogonal polynomials on the unit circle*, http://www.math.ucl.ac.be/~magnus/.
17. H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. 220 (2001), no. 1, 165–229. MR 1 882 403
18. G. Szegő, *Orthogonal polynomials*, third ed., Colloquium Publications 23, American Mathematical Society, Providence, Rhode Island, 1967.
19. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 2nd ed., Cambridge University Press, Cambridge, 1965.
20. Z. M. Yan, *A class of generalized hypergeometric functions in several variables*, Canad. J. Math. 44 (1992), no. 6, 1317–1338. MR 94c:33026