Doppelgangers: the Ur-Operation and Posets of Bounded Height

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Abstract

In the early 1970’s, Richard Stanley and Kenneth Johnson introduced and laid the groundwork for studying the order polynomial of partially ordered sets (posets). Decades later, Hamaker, Patrias, Pechenik, and Williams introduced the term “doppelgangers”: equivalence classes of posets under the equivalence relation given by equality of the order polynomial [6]. We provide necessary and sufficient conditions on doppelgangers through application of both old and novel tools, including new recurrences and the Ur-operation: a new generalized poset operation. In addition, we prove that the doppelgangers of posets $P$ of bounded height $|P| - k$ may be classified up to systems of $k$ diophantine equations in $2^{O(k^2)}$ time, and similarly that the order polynomial of such posets may be computed in $O(|P|)$ time.

1 Introduction

1.1 Background

Richard Stanley introduced the order polynomial $F_P(m)$ of an unlabeled partially ordered set (poset) in 1970 as an analog to chromatic polynomials. In the brief paper, Stanley offered a number of seminal results including a combinatorial interpretation of the coefficients of $F_P(m)$ and poset reciprocity [10]. Soon after, Johnson introduced a recurrence relation on the order polynomial of unlabeled posets [7] which Stanley expanded upon through the introduction of induction on incomparable elements, a powerful tool for studying posets. Computing the order polynomial is difficult. For instance, Brightwell and Winkler proved that computing even the first coefficient of the order polynomial (counting linear extensions) is $\#P$-complete [2]. Despite this, Faigle and Schrader proved that the order polynomial of special families, series-parallel posets and posets of bounded (constant) width, may be computed in polynomial time [3].

More recently, Feray and Reiner published a paper examining posets from a new algebraic and geometric perspective by studying the ring of weak P-partitions in order to count the number of linear extensions of forests with duplications [4]. They continued this work with Boussicault and Lascoux, studying linear extension sums as valuations over polyhedral cones [1]. In their second work, Feray et al. re-introduce induction on incomparable elements, extending a simple recurrence on linear extensions to valuations.

In 2014, McNamara and Ward [9] set out to classify the equivalence classes of the multivariate generating function $K_{P,\omega}$, a function introduced by Gessel in 1983 [5], and closely related to the labeled order polynomial $\Omega_{P,\omega}(m)$. In their work, McNamara and Ward prove a number of important poset invariants for $K_{P,\omega}(m)$, and offer several conjectures and unexplained equivalences—one of which we explain in section 3.1. Later, Hamaker, Patrias, Pechenik and Williams coined the term doppelgangers for unlabeled posets with the same order polynomial, and demonstrated several examples related to the K-theory of miniscule varieties. [6]. Their paper focuses on infinite families of grid-like doppelgangers, which raises the natural question of the existence and importance of other such families.

We apply Johnson’s initial recurrence to $F_P(m)$ as well as a new recurrence on both $\Omega_{P,\omega}(m)$ and $K_{P,\omega}$ similar to that used by Feray et al. in order to further study doppelgangers. We prove existence of many more infinite families as well as providing classification and necessary and sufficient conditions for the special cases of the families considered by Faigle and Schrader.
1.2 Results

Our work begins with an exploration of the interaction between doppelgangers and the standard poset operations disjoint union and ordinal sum, the operations used to build series-parallel posets. To this end, we re-introduce Johnson’s unlabeled recurrence and consider novel recurrences on $\Omega_{P,\omega}$ and $K_{P,\omega}$, which are closer in form to that used by Feray et al. [4]. Here, for incomparable elements $x, y$, let $P|x \leq y$ be the poset with added cover relation $x \leq y$ and all further relations required by transitivity, and $P|x = y$ be $P$ with $x$ and $y$ identified.

**Lemma 1.1.** The order polynomial and multivariate generating function admit the following recurrences:

\[
F_P = F_{P|x \leq y} + F_{P|y \leq x} - F_{P|x = y} \tag{1}
\]

\[
\Omega_{P,\omega} = \Omega_{P|x \leq y,\omega} + \Omega_{P|y \leq x,\omega} \tag{2}
\]

\[
K_{P,\omega} = K_{P|x < y,\omega} + K_{P|y \leq x,\omega} \tag{3}
\]

Recurrence (3) may produce objects which are not posets, but they remain valid for the purpose of calculating $K_{P,\omega}$.

While we use these recurrences to examine ordinal sum, they provide further results on doppelgangers as well. For instance, just a single step of recurrence (1) explains the doppelganger family in Figure 1.

**Example 1.2.** For each $n \geq 1$, the posets $P_1$ and $P_2$ in Figure 1 are doppelgangers.

![Figure 1: An infinite family of doppelgangers following from the unlabeled recurrence](image)

**Proof.** After applying Equation (1) to $P_1$ and $P_2$, the three resulting posets are equal.

In fact, taking $n = 1$, Example 1.2 recovers the Nicomachus formula:

\[
\sum_{k=1}^{m} k^3 = F_{P_1} = F_{P_2} = \left( \sum_{k=1}^{m} k \right)^2
\]

In their work, McNamara and Ward offer four pairs of posets with equivalent $K_{P,\omega}$ which their methods do not explain as a springboard for further investigation [9]. We prove that our improper recurrence easily shows the first of these pairs, shown in Figure 2, have equivalent $K_{P,\omega}$. We expect Lemma 1.1 has far reaching consequences for $K_{P,\omega}$.

In order to study the interaction of doppelgangers and the ordinal sum, we combine these recurrences with Stanley’s method of induction on incomparable elements [11], re-introduced recently by Feray et al. [4]. As an introduction to this technique, we provide an elegant proof of Stanley’s poset reciprocity theorem [10] (see Theorem 3.1). Through this method, we provide a basis for the order polynomial which interacts well with ordinal sum (see Proposition 3.2 and more generally Lemma 3.3), leading to the following results. Recall that the ordinal sum of $P$ and $Q$, $P \oplus Q$, follows from stacking the Hasse diagrams of $P$ and $Q$. 

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Corollary 1.3. For labeled posets \((P, \omega), (P', \omega'), (Q, \psi), (Q', \psi')\), any two conditions imply the third:
1) \((P, \omega) \sim (P', \omega')\)
2) \((Q, \psi) \sim (Q', \psi')\)
3) \((P \oplus Q, \omega \oplus \psi) \sim (P' \oplus Q', \omega' \oplus \psi')\)

Corollary 1.4. For all labeled posets \((P, \omega), (Q, \psi)\),
\[(P \oplus Q, \omega \oplus \psi) \sim (Q \oplus P, \psi \oplus \omega)\].

Here, \(P \sim Q\) denotes that the posets \(P\) and \(Q\) are doppelgangers, and \((P \oplus Q, \omega \oplus \psi)\) is a generalization of \(\oplus\) to labeled posets defined in Section 3.2.

While Lemma 1.1 and Corollaries 1.3 and 1.4 explain a large number of small and series-parallel doppelgangers, there are examples of size \(\geq 6\) (see Figure 5) they cannot explain. To this end, we introduce a new poset operation to generalize Corollaries 1.3 and 1.4.

Definition 1.5. For a poset \(\mathcal{P} = \{x_1, \ldots, x_n\}\) and a sequence of posets \(\{P_1, \ldots, P_n\}\), let \(\mathcal{P}[x_k \mapsto P_k]_{k=1}^n\) be the poset on \(\bigcup_k P_k\) with the following operation:

\[
\text{For } p \in P_j, q \in P_k, p \leq q \text{ when } \begin{cases} 
    p \leq q, & j = k \\
    x_j \leq x_k, & j \neq k.
\end{cases}
\]

We denote this as the Ur-operation on \(\mathcal{P}\) by \(\{P_1, \ldots, P_n\}\). All \(P_k\) are assumed to be \(C_1\) if not specified.

This operation provides a simple generalization of the standard poset operations disjoint union, ordinal sum, and direct product. Further, using the operation we prove a generalization of Corollary 1.3.

Theorem 1.6. For a poset \(\mathcal{P} = \{x_1, \ldots, x_n\}\) and two sequences of posets \(\{P_1, \ldots, P_n\}\) and \(\{Q_1, \ldots, Q_n\}\) such that \(P_i \sim Q_i\), we have that \(\mathcal{P}[x_k \mapsto P_k]_{k=1}^n \sim \mathcal{P}[x_k \mapsto Q_k]_{k=1}^n\).

Theorem 1.6 shows that elements of the same poset may be exchanged for doppelgangers while preserving equivalence. This raises the natural question of when distinct elements may be exchanged with the same result.

Definition 1.7. We say \(x \in P, y \in Q\) are Ur-equivalent when \(P[x \mapsto R] \sim Q[y \mapsto S]\) for all posets \(R \sim S\).

Theorem 1.6 implies that \(x \in P\) is always Ur-equivalent to itself, proving that Ur-equivalence is an equivalence relation. In Corollary 1.10 and Conjecture 4.11, we offer a basic necessary and sufficient condition for Ur-equivalence, and conjecture a strengthening of the result.

We provide an appendix with more information on the operation including the Ur-decomposition, a generalization of the series-parallel poset decomposition which provides easy ways to build doppelgangers of any given poset \(P\).

Finally, we move to classifying infinite families of doppelgangers. In general, such calculations are beyond hope. Even computing the leading coefficient of the order polynomial is \#P-complete [2]. However, Faigle and Schrader proved that for posets with bounded width \(k\), the order polynomial may be computed in \(O(|P|^{2k+1})\) time. However, any effective algorithm to classify infinite families of doppelgangers must be constant with respect to the size of the poset. We provide such an algorithm for posets of height \(|P| - k\), a subfamily of Faigle and Schrader’s posets of bounded width.

Theorem 1.8. For constant \(k\), the doppelgangers among posets of height \(|P| - k = n - k\) are completely determined by sets of \(k\) diophantine equations computable in \(2^{O(k^2)}\) time. In addition, the order polynomial of any such poset is computable in \(O(n)\) time, and for \(k = O\left(\frac{\log(n)}{\log(\log(n))}\right)\), the time is polynomial in \(n\).

Theorem 1.8 takes advantage of several invariants on doppelgangers we will introduce in Section 3.1 as well as the rigid structure of posets of bounded height. The improvement this structure brings from \(O(n^{2k+1})\) to \(O(n)\) allows us to extend our family of bounded height past the constant restriction imposed by Faigle and Schrader on posets of bounded width.

As an example, we provide the diophantine equations for \(k = 1, 2\) in Table 1 along with general solutions where possible and descriptions of infinite families of doppelgangers following from others such as in Figure 5.
2 Doppelgangers and the Order Polynomial

For a poset $P$, let $F_P(n)$ denote the number of order-preserving maps $f$ from $P$ to $\{1, 2, \ldots, n\}$ — that is, maps which satisfy $f(x) \leq f(y)$ whenever $x \leq y$ in $P$. Thus the numbers $F_P(n)$ provide a measure of how far the poset $P$ is from a total order. If two posets $P$ and $Q$ satisfy the equivalence $F_P(n) = F_Q(n)$ for all $n$, we will call them doppelgangers, and we denote this fact by $P \sim Q$. In this paper we establish certain structural properties of a pair of of posets $(P, Q)$ which are either necessary or sufficient conditions for $P \sim Q$.

Stanley offered many seminal necessary conditions in his early work and later as exercises in Enumerative Combinatorics \cite{EnumCombinatorics}. We provide some simple but important examples from these to aid intuition.

**Proposition 2.1.** If $P$ and $Q$ are doppelgangers, then they have the same number of elements.

**Proof.** Let $a_k$ be the number of surjective order-preserving maps $f$ from $P$ to $\{1, 2, \ldots, k\}$. Then we have

$$F_P(n) = \sum_{k=1}^{\lfloor P \rfloor} a_k \binom{n}{k}.$$

In particular, $F_P$ is a polynomial of degree $|P|$ (indeed, $F_P$ is called the order polynomial of $P$ \cite{EnumCombinatorics}). \hfill $\square$

We recall several operations on posets and show that they behave well in relation to order polynomials. Let $P$ and $Q$ be posets, and let $1$ denote the poset with a single element.

- The dual of a poset $P$, denoted $P^*$, is constructed by reversing the direction of all relations in $P$.
- The disjoint union of $P$ and $Q$, denoted $P \sqcup Q$, is constructed by taking the union of the elements of $P$ and $Q$ and inheriting the relations from $P$ and $Q$ (thus the elements from $P$ remain incomparable with the elements from $Q$). For example, $1 \sqcup 1 \sqcup 1$ is the anti-chain of size 3, in which no two distinct elements are comparable.
- The ordinal sum of $P$ and $Q$, denoted $P \otimes Q$, is constructed by first taking $P + Q$, and then imposing the relation $x \leq y$ for every $x \in P$ and $y \in Q$. For example, $1 \otimes 1 \otimes 1$ is the chain of size 3, a total order.
- The ordinal product of $P$ and $Q$, denoted $P \otimes Q$, is constructed by taking the Cartesian product $P \times Q$ and imposing relations $(r, s) \leq (r', s')$ if $r < r'$ in $P$ or $r = r'$ in $P$ and $s \leq s'$ in $Q$. For example, $(1 + 1 + 1) \otimes (1 \otimes 1 \otimes 1)$ is $(1 \otimes 1 \otimes 1) + (1 \otimes 1 \otimes 1) + (1 \otimes 1 \otimes 1)$

**Proposition 2.2.** $P \sim P^*$.

**Proof.** Consider the bijection which sends an order-preserving mapping $f : P \to \{1, 2, \ldots, n\}$ to the mapping $g$, where $g(x) := n + 1 - f(x)$. The mapping $g$ is order-preserving on $P^*$. Thus $F_P(n) = F_{P^*}(n)$. \hfill $\square$

**Proposition 2.3.** $F_{P+Q}(n) = F_P(n)F_Q(n)$.

**Proof.** Since the elements from $P$ and the elements from $Q$ are incomparable in $P + Q$, every choice of order-preserving maps $f$ on $P$ and $g$ on $Q$ gives rise to an order-preserving mapping $h(x) := \begin{cases} f(x) & \text{if } x \in P \\ g(x) & \text{if } x \in Q \end{cases}$ on $P + Q$, and it is not hard to see that every order-preserving map on $P + Q$ is of this form. \hfill $\square$

These operations can be used to generate larger, more complicated pairs of doppelgangers out of smaller pairs. For example, if $Q \sim R$, then we get that

$$F_{P+Q} = F_PF_Q = F_PF_R = F_{P+R},$$
and so \( P + Q \) and \( P + R \) are doppelgangers for all posets \( P \). Analogously, in Corollary 3.6 we will also see that \( P \oplus Q \) and \( P \boxplus R \) are doppelgangers whenever \( Q \sim R \). In fact, the Ur-operation provides a direct generalization of this property, given in Theorem 4.6. While we do not provide results on the ordinal product, we propose a new operation which generalizes the operation along with direct and ordinal sum.

The term doppelganger originally referred to unlabeled posets, but extends easily to labeled posets \((P, \omega)\). A labeled poset \((P, \omega)\) is a poset \( P \) equipped with a bijective labeling \( \omega: P \to [\# P] \). In this case, a map \( f: (P, \omega) \to [m] \) is order-preserving when \( f(x) \leq f(y) \) whenever \( x \leq y \), and \( f(x) < f(y) \) whenever \( x < y \) and \( \omega(x) > \omega(y) \). The number of such maps is the order polynomial of \((P, \omega)\), denoted \( \Omega_{P,\omega}(m) \). In fact, every unlabeled poset \( P \) may be written as a labeled poset \((P, \omega)\) where \( \omega \) is a natural labeling or a linear extension of \( P \), that is when \( \omega(x) < \omega(y) \) whenever \( x < y \). In this case \( F_P = \Omega_{P,\omega} \). In fact, labeled posets admit an interesting generalization of the order polynomial studied recently by McNamara and Ward [9].

The multivariate generating function of \((P, \omega)\) is

\[
K_{P,\omega}(x) = \sum_{f \in \text{(P,}\omega)-\text{partitions}} x_1^{f^{-1}(1)} x_2^{f^{-1}(2)} \cdots
\]

and related to \( \Omega_{P,\omega} \) by

\[
\Omega_{P,\omega}(m) = K_{P,\omega}(1,\ldots,1,0,\ldots).
\]

Here, \((P, \omega)\)-partitions differ from order preserving maps only in that they map to the positive integers rather than \([m]\).

3 Order Polynomial Recurrence

3.1 The Recurrence Relations

We begin by presenting the useful recurrence relations on \( F_P \), \( \Omega_{P,\omega} \), and \( K_{P,\omega} \) from Lemma 17. These are by no means the only recurrence relations on order polynomials of posets; we leave out, for instance, the strict recurrence discussed by Stanley [11]. However, the recurrences described below are enough to provide results on both doppelgangers and the multivariate generating function. In addition, we re-introduce the method of induction on incomparable elements, first used by Stanley to prove a theorem similar to Proposition 3.2 below.

Given a poset \( P \) with incomparable elements \( x \) and \( y \), we can define the poset \( P|x \leq y \) to be the result of adding the cover relation \( x \leq y \) and all other relations implied by transitivity. We can define the poset \( P|x = y \) to be the result of identifying \( x \) and \( y \). Stanley considers these constructions in his paper [11] where he mentions the following recurrence relation

\[
F_P = F_{P|x \leq y} + F_{P|y \leq x} - F_{P|x = y}.
\]

Proof. In this relation, an order-preserving map \( f: P \to [n] \) either has \( f(x) < f(y) \) in which case it is counted by the first term, \( f(x) > f(y) \) in which case it is counted by the second term, or \( f(x) = f(y) \) in which case it is counted by all three terms.

\( \Omega_{P,\omega} \) admits a similar recurrence to \( F_P \):

\[
\Omega_{P,\omega} = \Omega_{P|x \leq y,\omega} + \Omega_{P|y \leq x,\omega}.
\]

Proof. To see this, suppose without loss of generality that \( \omega(x) < \omega(y) \). Let \( f: P \to [m] \) be order preserving. If \( f(x) \leq f(y) \), then \( f \) is only counted by \( \Omega_{P|x \leq y,\omega} \). If \( f(y) < f(x) \), then \( f \) is only counted by \( \Omega_{P|y \leq x,\omega} \).

Note that labeled posets can be viewed as an assignment of strict and weak edges. For incomparable \( x, y \in P \), let \( P|x < y \) be the poset with the added relation that \( x < y \) and all other relations implied by transitivity. This restriction might not result in a valid labeled poset. However, order preserving functions,
and thus the order polynomial and multivariate generating functions are still well-defined on these improper posets.

Similarly to above, \( K_{(P,\omega)} \) admits the following improper recurrence:

\[
K_{P,\omega} = K_{P|x<y,\omega} + K_{P|y\leq x,\omega}.
\]

In their work [9], McNamara and Ward offer four unexplained equivalences of size 5 as a spring board for further exploration. This improper recurrence explains their first example as demonstrated in Figure 2. This ends our discussion of \( K_{P,\omega} \) but application of our methodology to the function is a possible direction of further research.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Equivalence of \( K_{P,\omega} \) and \( K_{Q,\omega} \). Double edges denoted strict order relations.}
\end{figure}

### 3.2 Induction on Incomparable Elements

We use the recurrences from Section 3.1 to prove results involving order polynomials by strong induction on the number of incomparable pairs of elements. In particular, each of the terms of the recurrences have fewer pairs of incomparable elements than the original poset. In his early work [11], Stanley uses this technique to prove the well-known expression for the strict order polynomial in the \( \left( \begin{array}{c} m \\ k \end{array} \right) \) basis. We will apply this technique to provide a novel and short proof of Stanley’s poset reciprocity theorem and an expression for the order polynomial for the standard poset operation ordinal sum.

For a labeling \( \omega \) of a poset \( P \), let \( \overline{\omega} \) be the dual labeling to \( \omega \) given by \( \overline{\omega}(x) = |P| + 1 - \omega(x) \). In what follows, we will use the binomial reciprocity theorem, which states that

\[
\binom{-n}{p} = \frac{(-n)(-n-1)\ldots(-n-(p-1))}{p!} = (-1)^p \frac{n(n+1)\ldots(n+p-1)}{p!} = (-1)^p \binom{n+p-1}{p}.
\]

We provide this proof of Theorem 3.1 as a simple introduction to how the recurrence relations are used in practice. In particular, a result is first proved for posets of the form \( (C_k,\omega) \) and is then extended to all posets by strong induction on the number of pairs of incomparable elements and the recurrence relation.
Theorem 3.1 (Poset Reciprocity). For all labeled posets, \((P, \omega)\),
\[
\Omega_{P,\omega}(m) = (-1)^{|P|} \Omega_{P,\omega}(-m).
\]

Proof. We shall proceed by strong induction on the number of pairs of incomparable elements in \(P\). For the base case where \(P\) has no pairs of incomparable elements, \(P\) is a chain. Then \((P, \omega)\) can be thought to be a chain with \(i\) strict edges and \(j\) non-strict edges where \(i + j = |P| - 1\). Using a modified stars and bars technique, we get that \(\Omega_{P,\omega}(m) = \binom{m+j}{|P|}\). Since \((P, \omega)\) is a chain with \(j\) strict edges and \(i\) non-strict edges, \(\Omega_{P,\omega}(m) = \binom{m+i}{|P|}\). Then by the binomial reciprocity theorem,
\[
\Omega_{P,\omega}(m) = \binom{m+i}{|P|} = (-1)^{|P|} \binom{-m+i + |P| - 1}{|P|} = (-1)^{|P|} \binom{-m+j}{|P|} = (-1)^{|P|} \Omega_{P,\omega}(-m)
\]
which shows the base case. Now suppose that the result holds for all posets with fewer than 7 pairs of incomparable elements and suppose that \(P\) has \(n\) pairs of incomparable elements. Then let \(x, y \in P\) be incomparable. By our inductive assumption,
\[
\Omega_{P,\omega}(m) = \Omega_{P|x \leq y, \omega}(m) + \Omega_{P|y \leq x, \omega}(m)
\]
\[
= (-1)^{|P|} \Omega_{P|x \leq y, \omega}(-m) + (-1)^{|P|} \Omega_{P|y \leq x, \omega}(-m)
\]
\[
= (-1)^{|P|} \Omega_{P,\omega}(-m)
\]
which shows the inductive step and completes the proof. □

It is clear from repeated applications Johnson’s recurrence that the order polynomial of any poset should have an expression as the sum of the order polynomial of total orders, or chains, with \(F_{C_k} = \binom{m+k-1}{k}\) where \(C_k\) is a chain of cardinality \(k\). Indeed, as a consequence of poset reciprocity we can easily derive the expression for the order polynomial in the \(\binom{m+k-1}{k}\) or chain basis.

Proposition 3.2. For all posets \(P\), there exist \(c_k \in \mathbb{N}\) such that
\[
F_P(m) = (-1)^{|P|} \sum_{k=h(P)}^{P} (-1)^k c_k \binom{m+k-1}{k}.
\]

Proof. Let \(\omega\) be a natural labeling for \(P\). By the poset and binomial reciprocity theorems, it suffices to show that there exist \(c_k \in \mathbb{N}\) such that \(\Omega_{P,\omega}(m) = \sum_{k=h(P)}^{P} c_k \binom{m+k}{k}\). It is straightforward to verify that we can let \(c_k\) be the number of surjective strict order-preserving maps \(f: P \to [k]\). □

In fact, in the chain basis, ordinal sum interacts with the order polynomial just as disjoint union interacts with the order polynomial in the standard basis. That is, the coefficients of \(F_{P \oplus Q}\) in the chain basis are given by the convolution of the coefficients of \(P\) with those of \(Q\). Further, this extends to labeled posets and beyond the chain basis. In particular, we generalize \(\oplus\) to labeled posets in the following way: given labeled posets \((P, \omega)\) and \((Q, \psi)\), let \(\omega \oplus \psi\) be a labeling on \(P \oplus Q\) given by
\[
(\omega \oplus \psi)(x) = \begin{cases} 
\omega(x) & x \in P \\
|P| + \psi(x) & x \in Q
\end{cases}
\]
Then \((P \oplus Q, \omega \oplus \psi)\) is the labeled poset where every element of \(P\) is weakly less than every element of \(Q\). The following result gives a formula for the order polynomial of an ordinal sum.
Lemma 3.3. For all labeled posets \((P, \omega), (Q, \psi)\),
\[
L(\Omega_{P \oplus Q, \omega \oplus \psi}) = L(\Omega_{P, \omega})L(\Omega_{Q, \psi})
\]
for any linear transformation \(L\) on the polynomials in \(m\) such that
\[
L\left(\binom{m+c+d-1}{c+d}\right) = L\left(\binom{m+c-1}{c}\right) L\left(\binom{m+d-1}{d}\right)
\]
for all integer \(c, d \geq 0\).

Proof. We first show the result in the case where \(P\) and \(Q\) are chains. Suppose that \((P, \omega)\) has \(i\) strict edges and \(j\) non-strict edges and suppose that \((Q, \psi)\) has \(k\) strict edges and \(l\) non-strict edges. Then \((P \oplus Q, \omega \oplus \psi)\) has \(i+k+1\) strict edges and \(j+l\) non-strict edges. Then it suffices to show that
\[
L\left(\binom{m+j}{|P|}\right) L\left(\binom{m+l}{|Q|}\right) = L\left(\binom{m+j+l+1}{|P|+|Q|}\right)
\]
where \(j \leq |P|-1\), \(l \leq |Q|-1\).

We shall proceed by induction on \(|P| - 1 - j + |Q| - 1 - l\). For the base case of \(|P| - 1 - j + |Q| - 1 - l = 0\), \(j = |P| - 1\) and \(l = |Q| - 1\) in which case the result reduces to the hypothesis on \(L\). Now suppose that \(|P| - 1 - j + |Q| - 1 - l > 0\) and that the result holds for smaller values of \(|P| - 1 - j + |Q| - 1 - l\). Then without loss of generality, \(j < |P| - 1\) and
\[
L\left(\binom{m+j}{|P|}\right) = L\left(\binom{m+j+1}{|P|} - \binom{m+j}{|P|-1}\right) L\left(\binom{m+l}{|Q|}\right)
\]
which shows the inductive step and completes the proof of the case where \(P\) and \(Q\) are chains. For the general result, we shall proceed by strong induction on the number of pairs of incomparable elements in \(P \oplus Q\). For the base case where \(P \oplus Q\) has no pairs of incomparable elements, \(P\) and \(Q\) are chains which was dealt with above. Now suppose that the result holds for all posets where \(P \oplus Q\) has fewer than \(n\) pairs of incomparable elements and suppose that \(P \oplus Q\) has \(n\) pairs of incomparable elements. Then without loss of generality, \(P\) has an incomparable pair of elements, \(x\) and \(y\). Then by the linearity of \(L\) and our inductive assumption,
\[
L(\Omega_{P \oplus Q, \omega \oplus \psi}) = L(\Omega_{P \oplus Q,x \leq y, \omega \oplus \psi}) + \Omega_{P \oplus Q,y \leq x, \omega \oplus \psi})
\]
which shows the inductive step and completes the proof.

This implies our desired result in the chain basis.

Corollary 3.4. If \(F_P(n) = \sum_{i=1}^{|P|} a_i \binom{m+k-1}{k}\) and \(F_Q(n) = \sum_{j=1}^{|Q|} b_j \binom{m+k-1}{k}\), then
\[
F_{P \oplus Q}(n) = \sum_{k=1}^{|P|+|Q|} \left( \sum_{i=1}^k a_i b_{k-i} \right) \binom{m+k-1}{k}.
\]
Thus the coefficients of \( F_{P \oplus Q} \) in the chain basis are given by the Cauchy product of the coefficients of \( F_P \) and the coefficients of \( F_Q \) in the chain basis.

**Proof.** Since the polynomials \( \binom{m+k+n}{k} \) form a basis for the polynomials in \( m \), the linear transformation sending \( \binom{m+k+n}{k} \to x^k \) satisfies the requirements of Lemma 3.3. 

Further, Lemma 3.3 provides immediate results on doppelgangers with respect to ordinal sum.

**Corollary 3.5.** For labeled posets \((P, \omega), (P', \omega'), (Q, \psi), (Q', \psi')\), any two conditions imply the third:
1) \((P, \omega) \sim (P', \omega')\)
2) \((Q, \psi) \sim (Q', \psi')\)
3) \((P \oplus Q, \omega \oplus \psi) \sim (P' \oplus Q', \omega' \oplus \psi')\)

**Corollary 3.6.** For all labeled posets \((P, \omega), (Q, \psi)\),

\[(P \oplus Q, \omega \oplus \psi) \sim (Q \oplus P, \psi \oplus \omega)\].

Note that if \( \omega \) is a natural labeling on \( P \) and if \( \psi \) is a natural labeling on \( Q \) then \( \omega \oplus \psi \) is a natural labeling on \( P \oplus Q \). As a consequence, Corollaries 2.8 and 2.9 also hold for unlabeled posets. In Section 4, we will extend the unlabeled version of Corollary 3.5 to the Ur-operation. Despite its simplicity, Corollary 3.6 has merit on its own. For instance, it immediately recovers one of the doppelganger pairs discussed by Hamaker et al. [6].

**Example 3.7.** Hamaker et al. show that \( C_{n-1} \oplus A_2 \oplus C_{n-1} = A_{Q^{2n}} = \Phi_{Q^{2n}} = A_2 \oplus C_{n-1} \oplus C_{n-1} \) (see Figure 1 in [6]). This pair of doppelgangers is an immediate consequence of Corollary 3.6.

### 4 The Ur-Operation

Section 3.2 details the interactions of the order polynomial and standard poset operations. By considering a generalization of these operations, it is possible in turn to extend our results. The operation itself is simple: consider replacing some subset of points in a poset \( \mathcal{P} \) by a corresponding set of posets \( \{P_1, \ldots, P_k\} \).

**Definition 4.1.** For a poset \( \mathcal{P} = \{x_1, \ldots, x_n\} \) and a sequence of posets \( \{P_1, \ldots, P_n\} \), let \( \mathcal{P}[x_k \rightarrow P_k]_{k=1}^n \) be the poset on \( \bigcup_k P_k \) with the following operation:

\[
\text{For } p \in P_j, q \in P_k, \ p \leq q \text{ when } \begin{cases} \ p \leq q & j = k, \\
\ x_j \leq x_k & j \neq k. \end{cases}
\]

We denote this as the Ur-operation on \( \mathcal{P} \) by \( \{P_1, \ldots, P_n\} \). All \( P_k \) are assumed to be \( C_1 \) if not specified.

Note that the disjoint sum operation denoted by \( P_1 \oplus P_2 \) can be expressed as \( A_2[x_k \rightarrow P_k]_{k=1}^2 \), the ordinal sum operation denoted by \( P_1 \oplus P_2 \) can be expressed as \( C_2[x_k \rightarrow P_k]_{k=1}^2 \), and the ordinal product can be expressed as \( P[x_k \rightarrow Q]_{k=1}^n \).

The order polynomial of the Ur-operation relies heavily on the structure \( \mathcal{P} \). Therefore it is convenient throughout the rest of this section to have the following definition

**Definition 4.2.** For a poset \( P \) and \( x \in P \), define \( g^P_x(n, m) \) to be the number of order-preserving maps \( f: \mathcal{P}[x \rightarrow \mathbb{Z}] \rightarrow [m] \) such that \( 1 + \min_{x \leq y} f(y) - \max_{y \leq x} f(y) = n \), where the \( \min \) and \( \max \) are taken to be \( m \) and 1 respectively if not well defined.

**Example 4.3.** For a chain \( C_7 \) and its 4th smallest (middle) element \( e_4 \):

\[
g_{C_7}^e(n, m) = \sum_{i=1}^{m-n} F_{C_2}(i) F_{C_2}(m - i - n + 2)
\]
4.1 The Order Polynomial

With this in hand, we offer a simple formula for the order polynomial of a single substitution. The polynomial for the general operation may be given by repeated application

**Proposition 4.4.** For a poset $\mathcal{P}$ with $x \in \mathcal{P}$, a poset $Q$, and $m \geq 1$,

$$F_{\mathcal{P}[x \to Q]}(m) = \sum_{n=1}^{m} g_{x}^{\mathcal{P}}(n, m) F_{Q}(n).$$

*Proof.* The result follows from summing over all order preserving functions $f : \mathcal{P}[x \to \emptyset] \to [m]$ and counting at each step the possible order preserving functions on $Q$ that satisfy the arrangement. \(\square\)

As expected, the formulae for direct and ordinal sum follow immediately from our generalization:

**Corollary 4.5.** $F_{P+Q}(m) = F_{P}(m) F_{Q}(m)$, and $F_{P\oplus Q} = \sum_{i=1}^{m} F_{Q}(m+1-i)(F_{P}(i) - F_{P}(i-1))$ where $F_{P}(0)$ is defined to be 0.

*Proof.* By Proposition 4.4

$$F_{P+Q}(m) = F_{(P+e)[x \to Q]}(m) = \sum_{n=1}^{m} g_{x}^{P+e}(n, m) F_{Q}(n) = F_{P}(m) F_{Q}(m).$$

where we leverage $g_{x}^{P+e}(n, m) = \begin{cases} F_{P}(m) & n = m \\ 0 & n \neq m \end{cases}$

Similarly,

$$F_{P\oplus Q}(m) = F_{(P\oplus e)[x \to Q]}(m) = \sum_{n=1}^{m} (F_{P}(m+1-n) - F_{P}(m-n)) F_{Q}(n)$$

where the result follows by replacing $n$ by $i = 1 + m + 1 - n$. \(\square\)

Moreover, the following result shows that the Ur-operation generalizes the nice relation between ordinal sum and doppelgangers given by Corollaries 3.5 and 3.6.

**Theorem 4.6.** For a poset $\mathcal{P} = \{x_{1}, \ldots, x_{n}\}$ and two sequences of posets $\{P_{1}, \ldots, P_{n}\}$ and $\{Q_{1}, \ldots, Q_{n}\}$ such that $P_{k} \sim Q_{k}$, we have that $\mathcal{P}[x_{k} \to P_{k}]_{k=1}^{n} \sim \mathcal{P}[x_{k} \to Q_{k}]_{k=1}^{n}$.

*Proof.* For a poset $P$, let $S_{P}(n)$ denote the number of strict surjective order preserving maps $f : P \to [n]$. By Proposition 3.2 it suffices to show that $S_{\mathcal{P}[x_{k} \to P_{k}]}_{k=1}^{n} = S_{\mathcal{P}[x_{k} \to Q_{k}]}_{k=1}^{n}$. We call an collection of intervals $\{[a_{k}, b_{k}]\}_{k=1}^{n}$ nice if they cover $[n]$ and if $b_{j} < a_{k}$ whenever $x_{j} < x_{k}$. Let $\mathcal{N}$ denote the set of nice collections of intervals. Then

$$S_{\mathcal{P}[x_{k} \to P_{k}]}_{k=1}^{n} = \prod_{k=1}^{n} S_{P_{k}}(b_{k} - a_{k} + 1) = \prod_{k=1}^{n} S_{Q_{k}}(b_{k} - a_{k} + 1) = S_{\mathcal{P}[x_{k} \to Q_{k}]}_{k=1}^{n},$$

where the middle equality uses the fact that $S_{P_{k}} = S_{Q_{k}}$ which follows from Proposition 3.2 and the fact that a representation in the $\binom{x+m}{m-1}$ basis is unique. \(\square\)
For ease of computation, note that we need only compute \( S_{P_k} \) for nice intervals such that \( h(P_k) = h(Q_k) \leq b_k - a_k + 1 \leq |P_k| = |Q_k| \). If any \( b_k - a_k + 1 < h(P_k) = h(Q_k) \) or \( b_k - a_k + 1 > |P_k| = |Q_k| \) then the corresponding term in the product will be 0.

**Example 4.7.** The posets in Figures 3(c) and 3(f) are doppelgangers by Theorem 4.6. Due to the underlying non-series-parallel structure of \( P \), this does not follow from Corollaries 3.5 or 3.6, nor does it follow from a single application of Johnson’s recurrence.

We should mention that Theorem 4.6 can also be proved with the recurrence. In particular, since the recurrence commutes with the \( \text{Ur} \)-operation we can perform a full chain decomposition on each \( P_i \) and each \( Q_i \) independently. This will reduce the order polynomial of the \( P[x \rightarrow R]_k \) to a sum of order polynomials of posets where each point of \( \mathcal{P} \) is replaced by a chain. Since each \( P_1 \sim Q_1 \), these resulting posets will be isomorphic. Theorem 4.6 allows us to build new doppelgangers out of an arbitrary poset by iteratively replacing points with corresponding doppelgangers. We know as well, however, that one can construct doppelgangers by replacing different points of some poset \( P \) with corresponding doppelgangers, such as in \( C_k \) or \( A_k \). It is natural then to ask about a generalization of this occurrence. For posets \( P \) and \( Q \) with \( x \in P \) and \( y \in Q \), when do we have \( P[x \rightarrow R] \sim Q[y \rightarrow S] \) for all doppelgangers \( R \sim S \)?

### 4.2 Ur-Equivalence

**Definition 4.8.** We say \( x \in P, y \in Q \) are Ur-equivalent when \( P[x \rightarrow R] \sim Q[y \rightarrow S] \) for all posets \( R \sim S \).

In fact, Ur-equivalence relies on exactly the same structure the order polynomial does: on the values \( g^P_x(n,m) \) and \( g^Q_y(n,m) \).

**Proposition 4.9.** For \( x \in P \) and \( y \in Q \), \( x \) and \( y \) are Ur-equivalent if and only if \( g^P_x = g^Q_y \)

**Proof.** The backward direction is immediate from Proposition 4.4. If \( x \in P, y \in Q \) are Ur-equivalent then we have \[ \sum_{i=1}^{m+1} g^P_x(i,m) \cdot F_R(i) = \sum_{i=1}^{m+1} g^Q_y(i,m) \cdot F_R(i) \] for all posets \( R \). For any \( m \), consider applying this equation...
to any set of posets $S_1, \ldots, S_m$ such that $|P_i| = i$. Let $g^P_x(i, m) - g^Q_y(i, m) = c(i, m)$. This gives the system of equations

$$
\begin{bmatrix}
F_{S_1}(1) & \cdots & F_{S_1}(n) \\
\vdots & \ddots & \vdots \\
F_{S_m}(1) & \cdots & F_{S_m}(n)
\end{bmatrix}
\begin{bmatrix}
c(1, m) \\
\vdots \\
c(m, m)
\end{bmatrix} = 0
$$

This matrix is invertible due to the fact that the $F_{S_i}$ are linearly independent. Thus the $c(i, m)$ are 0, for $n \leq m$ $g^P_x(n, m) = g^Q_y(n, m)$, and by definition for $n > m$ both values are 0.

**Corollary 4.10.** For $x \in P$ and $y \in Q$ with $|P| = |Q| = n$, $x$ and $y$ are Ur-equivalent if and only if there exist posets $\{S_1, \ldots, S_m\}$ with $|S_i| = i$ such that $P[x \rightarrow S_i] \sim Q[y \rightarrow S_i], \forall i \in [n]$

Unfortunately, while $g^P_x$ reveals the structure behind Ur-equivalence, in general it is too difficult to calculate to be of practical use. However, one may note that $g^P_x$ is totally determined by the structure of $P[x \rightarrow \emptyset]$ and its relation to $P$. Let $A_k$ be the anti-chain of size $k$–a set with no order relations. The structure of $g^P_x$ suggests that we may be able to strengthen the Corollary 4.10.

**Conjecture 4.11.** For $x \in P$ and $y \in Q$, $x$ and $y$ are Ur-equivalent if and only if $P[x \rightarrow A_k] \sim Q[y \rightarrow A_k]$ for $k = 0, 1$.

While this conjecture may seem unlikely with the above information alone, like the order polynomial, $g^P_x$ has significant extra structure that is not understood. In fact, the conjecture holds for small posets, and further is equivalent to a number of simpler statements. For instance,

**Proposition 4.12.** If for all posets $P,Q$ and $x \in P$ and $y \in Q$, $P[x \rightarrow A_k] \sim Q[y \rightarrow A_k]$ for $k = 0, 1$ implies that $P[x \rightarrow A_2] \sim Q[y \rightarrow A_2]$, then Conjecture 4.11 holds.

**Proof.** We will prove that $P[x \rightarrow A_k] \sim Q[y \rightarrow A_k]$ for $\forall k \in \mathbb{N}$ by induction on $k$. The RHS of Conjecture 4.11 is exactly the base case of our induction. Assume $P[x \rightarrow A_i] \sim Q[y \rightarrow A_i]$ for $0 \leq i \leq k$, and let $P' = P[x \rightarrow A_k] \sim Q'[y \rightarrow A_k]$. $P'[x \rightarrow \emptyset] = P[x \rightarrow A_{k-1}] \sim Q'[y \rightarrow A_{k-1}] = Q'[x \rightarrow \emptyset]$. Then our initial assumption gives that $P'[x \rightarrow A_2] = P'[x \rightarrow A_{k+1}] \sim Q'[y \rightarrow A_2] = Q'[y \rightarrow A_{k+1}]$. This concludes our induction and the result then follows from Corollary 4.10.

While we do not know whether the assumption in Proposition 4.12 is true for all posets $P,Q$, we do know it is true for certain families of posets, such as $C_k$ or $A_k$, and it gives a simpler formulation of Conjecture 4.11.

## 5 Posets of Bounded Height

Due to the computational complexity of the order polynomial, a general classification of doppelgangers seems hopeless. However, there are certain large families for which the order polynomial is computable in polynomial time. For instance, Faigle and Schrader showed that the order polynomial of $P \in \mathcal{W}_k$, the set $\{P \in \mathcal{P}_n \mid w(P) \leq k\}$ may be computed in $O(n^{2k+1})$. While this set does not have a rigid enough structure to permit classification, a special subset does. Consider $\mathcal{H}_k \subset \mathcal{W}_k$, the set $\{P \in \mathcal{P}_n \mid h(P) = n - k\}$. We will leverage invariants on doppelgangers and the rigid structure of $\mathcal{H}_k$ to prove that one may reduce classification of doppelgangers to a number of diophantine equations in time dependent on $k$. In addition, we show that for constant $k$ the order polynomial of posets in this class has time complexity $O(n)$, and is computable in polynomial time for $k = O(\log(n)/\log(\log(n)))$.

### 5.1 Invariants

Proposition 3.2 introduces an important restriction on the roots of the order polynomial, first shown by Stanley [10]. Recall that the height of $P$, $h(P)$, is the cardinality of the largest total ordering contained in $P$. 

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Corollary 5.1. For all posets $P$, $F_P(x)$ vanishes at $x = 0, -1, \ldots, -h(P) + 1$ but not at $-h(P), -h(P) - 1, \ldots$

In particular, this means that height is a necessary condition, or invariant, for doppelgangers. Such invariants can be easily calculated allow for classification of doppelgangers of certain families of posets. Lemma 5.2 presents four such invariants that have simple recursive formulas over the operations of disjoint union and ordinal sum. One of these invariants is $e(P)$, the number of linear extensions of $P$. A linear extension of a poset $P$ is an order preserving bijection $P \rightarrow [\lvert P \rvert]$.

Lemma 5.2. If $P \sim Q$ then $\lvert P \rvert = \lvert Q \rvert$, $F_P(2) = F_Q(2)$, $h(P) = h(Q)$, $e(P) = e(Q)$. Additionally,

$$\begin{align*}
\lvert P + Q \rvert &= \lvert P \rvert + \lvert Q \rvert & (4) \\
\lvert P \oplus Q \rvert &= \lvert P \rvert + \lvert Q \rvert & (5) \\
F_{P+Q}(2) &= F_P(2)F_Q(2) & (6) \\
F_{P\oplus Q}(2) &= F_P(2) + F_Q(2) - 1 & (7) \\
h(P + Q) = \max(h(P), h(Q)) & (8) \\
h(P \oplus Q) = h(P) + h(Q) & (9) \\
e(P + Q) = \left(\frac{\lvert P \rvert + \lvert Q \rvert}{\lvert P \rvert}\right)e(P)e(Q) & (10) \\
e(P \oplus Q) = e(P)e(Q) & (11)
\end{align*}$$

for all posets $P, Q$.

Proof. By Proposition 3.2 $\lvert P \rvert = \deg F_P$, $h(P)$ is the index of the first nonzero term of $F_P$ in the $(x^k + k^{-1})$ basis, and $e(P)$ is $(\deg F_P)!$ times the leading coefficient of $F_P$ (since $c_{\lvert P \rvert} = e(P)$ in the notation of Proposition 5.2). Then all four invariants depend only on $F_P$ which shows the first part of the lemma. The recursive formulas follow from elementary combinatorial arguments.

Corollary 5.1 states that $F_P$ has roots at $0, -1, \ldots, -h(P) + 1$. This allows for the following necessary and sufficient condition for two posets to be doppelgangers.

Proposition 5.3. $P \sim Q$ if and only if $\lvert P \rvert = \lvert Q \rvert$, $h(P) = h(Q)$, $e(P) = e(Q)$, and $F_P$ and $F_Q$ agree at $\lvert P \rvert - h(P) - 1$ distinct points (not counting the trivial agreement at $1, 0, -1, \ldots, -h(P)$).

Proof. The first statement of Lemma 5.2 shows the forward direction. For the converse, note that $F_P$ and $F_Q$ have the same leading coefficient of $e(P)/\lvert P \rvert!$. Then subtracting $F_P$ and $F_Q$ results in a polynomial of degree at most $\lvert P \rvert - 1$ which vanishes at $\lvert P \rvert$ points (namely, at $x = 1, 0, -1, \ldots, -h(P) + 1$ and the $\lvert P \rvert - h(P) - 1$ other points). Then $F_P - F_Q$ is identically zero and $P \sim Q$.

Corollary 5.4. If $h(P) = \lvert P \rvert - 1$ then $P \sim Q$ if and only if $\lvert P \rvert = \lvert Q \rvert$, $h(P) = h(Q)$, and $e(P) = e(Q)$.

Corollary 5.5. If $h(P) = \lvert P \rvert - 2$ then $P \sim Q$ iff $\lvert P \rvert = \lvert Q \rvert$, $F_P(2) = F_Q(2)$, $h(P) = h(Q)$, and $e(P) = e(Q)$.

In the section that follows we will not use the $e(P)$ invariant, but it is particularly useful for enumerating $\mathcal{H}_1$ and $\mathcal{H}_2$.

5.2 Classifying $\mathcal{H}_k$

The height invariant, Corollary 5.1, and the underlying structure of $\mathcal{H}_k$ allow us to theoretically classify all its doppelgangers in time dependent on $k$. In addition, leveraging this same structure allows us to efficiently compute the order polynomial of posets in $\mathcal{H}_k$ in time $O(n)$.

Lemma 5.6. If $x_1 \leq \cdots \leq x_h$ is a chain in $P$ and $x$ is some other element of $P$, then there exist nonnegative integers $a + b + c = h$ such that $x$ is greater than $x_1, \cdots, x_a$, $x$ is incomparable to $x_{a+1}, \cdots, x_{a+b}$, and $x$ is less than $x_{a+b+1}, \cdots, x_{a+b+c}$.
Proof. Let \( m_1 \) be maximal such that \( x_{m_1} \leq x \), let \( m_2 \) be minimal such that \( x \leq x_{m_1+m_2+1} \), and let \( m_3 = n-m_1-m_2 \). Then by transitivity, \( x \) is greater than \( x_1, \ldots, x_{m_1} \) and \( x \) is less than \( x_{m_1+m_2+1}, \ldots, x_{m_1+m_2+m_3} \). Additionally, \( x \) is neither less than nor greater than \( x_{m_1+1}, \ldots, x_{m_1+m_2} \).

**Lemma 5.7.** Let \( P \) be a finite poset consisting of a chain \( x_1 \leq \ldots \leq x_{h(P)} \) and \( k = |P| - h(P) \) other elements off the chain \( y_1, \ldots, y_k \). Consider applying Lemma 5.6 to each \( y_i \), resulting in values \( a \) and \( a + b \) for each term. For convenience, we define \( a_1 \leq \ldots \leq a_{2k} \) to be the ordering of these \( 2k \) values, and further define \( a_0 = 0 \leq a_1 \), and \( a_{2k+1} = h(P) + 1 \geq a_{2k} \). Let \( d_i \), \( 0 \leq i \leq 2k \), be the difference between the \( i \) and \( i+1 \)st terms in this sequence, i.e. \( d_i = a_{i+1} - a_i \). The value of \( F_P(m) \) is a polynomial in the \( d_i \) and can be computed in \( O(m^{3k+1}) \).

Proof. To count the number of order preserving \( f : P \rightarrow [m] \), sum over at most \( m^k \) possible choices of the values of \( f \) on the \( y_i \). Note that the values of \( f \) on the \( x_i \) are completely determined by the locations of the \( m-1 \) locations where the value of \( f \) increases (note that these increases may occur before \( x_1 \) or after \( x_{h(P)} \)). Then for each choice of the value of \( f \) on the \( y_i \), we sum over the possible choices for how many times \( f \) increases between each \( x_a \) and \( x_{a+1} \). There are \( 2k+1 \) such pairs and \( m-1 \) increases so a stars and bars argument gives that there are \( (2k+1)^m \) possible choices for how many times \( f \) increases between each pair of consecutive \( x_a \). Then we are summing over at most \( m^k \binom{2k+m}{2k} \) ways to choose the values of \( f \) on the \( y_i \) and the locations of the increases of \( f \) on the chain, relative to the \( a_i \). Finally, each summand will be a product of \( \binom{d_{i+j+1}}{j} \), where \( j \) is the number of increases in between \( x_a \) and \( x_{a+1} \). Thus by symmetry there are a total of \( m^k \binom{2k+m}{2k} \) steps, and as \( \binom{2k+m}{2k} \) is a polynomial in \( m \) of degree at most \( 2k+1 \), this is \( O(m^{3k+1}) \).

**Lemma 5.8.** Given a poset \( P \), \( |P| = n \), with \( h(P) = n - k \), computing the \( a_i \) and \( d_i \) of Lemma 5.7 takes \( O(n) \) time.

Proof. Once we have identified our maximal chain, it is easy to compute \( a_i \) and \( d_i \) in linear time. Thus, we first prove that we may identify this chain of \( P \) in linear time. We require that \( P \) be given as a Hasse Diagram, a directed acyclic graph (DAG) \( G(V,E) \) of cover relations. Given our restriction \( h(P) = n - k \), we first wish to bound \( |E| \). For analysis, we will partition \( V \) into the sets \( C \), \( n-k \) nodes on our maximal chain, and \( Q \), the \( k \) “off-chain” nodes, and count the edges within and between \( C \) and \( Q \). \( C \) is our chain, and thus has exactly \( n-k \) internal edges. \( Q \) could be any poset of size \( k \), but because \( G \) is a DAG, there can be at most \( \frac{k(k-1)}{2} \) internal edges. Finally, by the structure outlined in Lemma 5.7 there can be at most \( 2k \) edges between \( C \) and \( Q \). Together, these give the bound

\[
|E| \leq n + \frac{(k+1)k}{2}.
\]

This allows us to run Depth First Search (DFS) on \( G \) in \( O(n) \) time. Treating \( G \) as an undirected graph, we may find all local minima and maxima of \( P \) (sinks of \( G \)) by running a DFS from any node along each connected component. Because \( h(P) = n - k \), there can be at most \( k+1 \) local minima and \( k+1 \) local maxima. From here finding \( C \) is a simple matter of finding the longest path between any minima and maxima, which has a well known linear solution in \( O(|V| + |E|) = O(n) \). The complement of \( C \) gives \( Q \), and the at most \( 2k \) edges between \( C \) and \( Q \) recover the \( a_i \) and \( d_i \) of Lemma 5.7 in linear time.

We are now prepared to prove Theorem 1.8 which we split into three components.

**Theorem 5.9.** For \( |P| = n \) and constant \( k \), the order polynomial of \( P \in \mathcal{H}_k \) can be computed in \( O(n) \) time.

Proof. We claim that given all \( a_i \) and \( d_i \), computing the order polynomial is poly-logarithmic. Lemma 5.1 allows us to compute the factored form of the order polynomial by polynomial interpolation and factorization of the remaining \( k \) roots. Let this polynomial be \( f_k(x) \), then

\[
F_P(x) = f_k(x) \prod_{i=0}^{n-(k+1)} (x+i)
\]
We may compute $f_k(x)$ by polynomial interpolation on $F_P(1) = 1, F_P(2), \ldots, F_P(k+1)$. With these values in hand, interpolation and factorization is polynomial in $k$ by the “LLL” algorithm [8]. Assuming we know the structure of $P$, Lemma 5.7 shows each $F_P(i)$ as a polynomial in the gaps between adjacent $a_i$ may be computed in $O(i^{3k+1})$. Given these values, evaluating the polynomial requires summing $O(i^{3k+1})$ products, each with $i−1$ terms of at most $\log(n)$ bits. For simplicity, let $n^2$ be the complexity of n-bit multiplication, then computing the products takes $O(i^{3k+1}i\log^2(n))$. The evaluated products are of at most $i\log(n)$ bits. Summing these then takes $i^{3k+1}*(i^{3k}+i\log(n))$, so computing and evaluating all $F_P(i)$ takes poly-log time. Thus Lemma 5.8 shows that computing the $a_i$ and $d_i$ is our bottleneck, and computing the order polynomial takes $O(n)$ time.

For constant $k$, we have shown that our subfamily of Faigle and Schrader’s $\mathcal{H}_k$ may be computed in $O(n)$ time, and thus does not grow asymptotically in $k$ as their $O(n^{2k+1})$ bound does. This allows us to extend our family to non-constant values of $k$.

**Corollary 5.10.** For $|P| = n$ and $k = O(\frac{\log(n)}{\log(\log(n))})$, the order polynomial of $P \in \mathcal{H}_k$ may be computed in polynomial time.

**Proof.** Proposition 5.9 showed that we can compute the order polynomial in $O(n + k^{3k+2}(k^{3k} + k^{2}\log^2(n)))$ time. Setting $k = \frac{\log(n)}{\log(\log(n))}$ gives $k^k = O(n^c)$, and thus the leading term becomes $k^{6k+2}$ which is polynomial for $k = O(\frac{\log(n)}{\log(\log(n))})$.

**Proposition 5.11.** The doppelgangers of posets in $\mathcal{H}_k$ can be completely classified up to sets of $k$ diophantine equations in $2^{O(k^2)}$ time.

**Proof.** First note that by Lemma 5.2 doppelgangers of posets in $\mathcal{H}_k$ will themselves be posets in $\mathcal{H}_k$ of the same height and size. Thus, it suffices to classify doppelgangers within $\mathcal{H}_k$. If $P \in \mathcal{H}_k$ then $P$ consists of a chain $C_{h(P)} = \{x_1 \leq \ldots \leq x_{h(P)}\}$ and $k$ other elements off the chain $Q = \{y_1, \ldots, y_k\}$. For each $y_i$, let $a_i$ denote the number of $x_j \leq y_i$ and let $b_i$ denote the number of $x_j \not\leq y_i$. We call a consistent choice of both a $Q$ and a relative ordering between the $a_i$ and the $b_i$ a family in $\mathcal{H}_k$. Enumerating all choices of $Q$ simply corresponds to enumerating all posets of size $k$, which takes $2^{O(k^2)}$ time. There are at most $(2k)!$ relative orderings between the $a_i$ and $b_i$, and given a choice of $Q$, checking any given ordering is consistent takes time polynomial in $k$. For each family in $\mathcal{H}_k$, Lemma 5.7 shows the values of $F_P(2), \ldots, F_P(k+1)$ can be written as a polynomial function of the distances between the $a_i$ and $b_i$ in $O(k^{3k+1})$ time. By Corollary 5.1 if $P, Q$ are posets with $|P| = |Q| = h(P) + k = h(Q) + k$ then $P \sim Q$ if and only if $F_P(i) = F_Q(i)$ for $i = 2, \ldots, k+1$. Thus, the doppelgangers of posets in $\mathcal{H}_k$ can be completely classified up to sets of $k$ diophantine equations in $2^{O(k^2)}$ time.

**5.3 Example: $\mathcal{H}_1$ and $\mathcal{H}_2$**

While for large $k$, classifying the $\mathcal{H}_k$ may be computationally intractable, $\mathcal{H}_1$ and $\mathcal{H}_2$ are simple enough to compute by hand. We provide a classification of these families as an example of the above method, and show how the diophantine equations lead to new infinite families of doppelgangers. Note however that we choose to use the number of linear extensions $e(P)$ rather than $F_P(3)$, as described in Corollaries 5.4 and 5.5.

We begin by enumerating the families of $\mathcal{H}_1$ and $\mathcal{H}_2$.

**Proposition 5.12.** All posets $P$ with $|P| - h(P) = 1$ are isomorphic to a poset depicted by Figure 3(a).

**Proof.** Let $C$ be a maximal chain in $P$ and let $x$ be the remaining element of $P$. Let $m_1, m_2, m_3$ be the result of applying Lemma 5.4 to $C$ and $x$. Then $P \cong Tri(m_1, m_2, m_3)$.

**Proposition 5.13.** All posets $P$ with $|P| - h(P) = 2$ are isomorphic to poset depicted by Figures 4(b–e).
Proof. Let $C$ be a maximal chain in $P$ and let $x, y$ be the two remaining elements of $P$. Let $m_1, m_2, m_3$ be the result of applying Lemma 5.6 to $C$ and $x$ and let $n_1, n_2, n_3$ be the result of applying Lemma 5.6 to $C$ and $y$. Then

\[
P \cong \begin{cases} 
  Ntri(m_1, n_1 - m_1, n_2, n_3 - m_3, m_3) & m_1 \leq n_1, m_3 \leq n_3, x \text{ and } y \text{ incomparable} \\
  Ntri(m_1, n_1 - m_1, m_2, n_3 - n_3) & m_1 \geq n_1, m_3 \geq n_3, x \text{ and } y \text{ incomparable} \\
  Xdis(m_1, m_1 - m_1, m_2 - n_1, m_3 - n_3, n_3) & m_1 \leq n_1, m_3 \geq n_3, x \text{ and } y \text{ incomparable} \\
  Xdis(n_1, m_1 - n_1, n_1 + n_2 - m_1, n_3 - m_3, m_3) & m_1 \geq n_1, m_3 \leq n_3, x \text{ and } y \text{ incomparable} \\
  Xcon(m_1, n_1 - m_1, m_2 - n_1, m_3 - n_3, n_3) & m_1 + m_2 - n_1 \geq 0, x \leq y \\
  Xcon(n_1, m_1 - n_1, n_1 + n_2 - m_1, n_3 - m_3, m_3) & n_1 + n_2 - m_1 \geq 0, y \leq x \\
  Dtri(m_1, m_2, n_1 - m_1 - m_2, n_2, n_3) & n_1 - m_1 - m_2 \geq 0, x \leq y \\
  Dtri(n_1, n_2, m_1 - n_1 - n_2, m_2, m_3) & m_1 - n_1 - n_2 \geq 0, y \leq x 
\end{cases}
\]

The values of the invariants for the Posets in Figure 4 are given in Table 1 and the computation of these values can be found in the appendix. The result of this table is that we can compute all doppelgangers among posets of height at most $|P| - 2$ by solving various pairs of Diophantine equations. All pairs lead to infinite families of doppelgangers such as that depicted in Figure 4.

Example 5.14. In the below, we drop variables which do not appear in $e(P)$ or $F_P(2)$. For instance, $Dtri(m_1, m_2, m_3, m_4)$ becomes $Dtri(m_2, m_4)$. Further, variables are assumed to be constrained in such a manner that every $m_i$ is $> 0$.

1. $Dtri(a(a + 2), a - 1) \sim Ntri(a^2 - (b + 2), a, b)$
2. $Dtri(4a, a - 1) \sim Xdis(a - 2, a, a)$
3. $Dtri(\frac{1}{2}(10a - 2n + 2), d - n) \sim Xcon(a - n, 2a, 2a - n + 1)$
4. $Ntri(3a - b - 5c, 2c, b) \sim Xdis(3c - a, a, a - 2c)$
5. $Ntri(1/2(-4 + (-3 + b)b - 2c + n + 2bn - n^2), b, c \leq \frac{b(b - 1)}{2} - 2) \sim Xcon((1/2(-2 + (-1 + b)b + 3n + 2bn - n^2), b - n, -1))$
6. $Xdis(\frac{1}{2}((1 + b)(-2 + b + 2f) - n - n^2), b, f - (n - 1)) \sim Xcon(\frac{1}{2}((1 + b)(b + 2f) + n - n^2), b - n, f)$
7. $Dtri(a, b) \sim Dtri(b, a)$
8. $Ntri(d + e - c, b, c) \sim Ntri(d, b, e)$
9. $Ntri(3b - c - 2(d + e), b, c) \sim Ntri(d, 2b - d - e, e)$
10. $Xcon(a^2 + b - b^2, a, b + 1) \sim Xcon(a, b, 1 + a(2 + a) - b(1 + b))$
11. $Xdis(a - b + d, b, c) \sim Xdis(a, b, d)$

Proposition 5.15. Equation 7 completely characterizes doppelgangers between $Dtri$ and $Dtri$ up to shifting $m_1, m_3,$ and $m_5$.

Proof. Consider $Dtri(a, b)$ and $Dtri(c, d)$. The equation for $F_P(2)$ gives $a = c + d - b$. Plugging this into our $e(P)$ equation gives $(b - c)(b - d) = 0$. Thus the only solutions are $b = c$ as shown in 7 or $b = d$ which gives two identical posets.

Proposition 5.16. Equations 8 and 2 completely characterize doppelgangers between $Ntri$ and $Ntri$ up to shifting of $m_1$ and $m_5$. 

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Figure 4: Five infinite families of posets.

| Infinite Family | $e(P)$ | $F_P(2)$ |
|-----------------|--------|----------|
| $Tri(m_1, m_2, m_3)$ | $(m_2 + 1)(m_4 + 1)$ | $|P| + m_2 + 1$ |
| $Dtri(m_1, m_2, m_3, m_4, m_5)$ | $(m_2 + m_3 + m_4 + 2)(m_3 + 1)$ | $|P| + m_2 + m_4 + 1$ |
| $Ntri(m_1, m_2, m_3, m_4, m_5)$ | $(m_2 + m_3 + 1)(m_3 + m_4 + 1) + m_3 + 1$ | $|P| + m_2 + 3m_3 + m_4 + 2$ |
| $Xdis(m_1, m_2, m_3, m_4, m_5)$ | $(m_2 + m_3 + 1)(m_3 + m_4 + 1) - \frac{1}{2}m_3(m_3 + 1)$ | $|P| + m_2 + 2m_3 + m_4 + 1$ |
| $Xcon(m_1, m_2, m_3, m_4, m_5)$ | | |

Table 1: Values of $e(P)$ and $F_P(2)$ for the five infinite families in Figure 1.

Proof. Plugging the $F_P(2)$ constraint into $e(P)$ constraint gives the equation $(-b + e)(-2b + d + e + f) = 0$. There are two possible cases:

1. $b = e$: Plugging $b = e$ into our initial equations gives the system:

$$0 = (1 + b)(a + c - d - f)$$
$$a = d + f - c$$

Thus $a$ is determined, and any $b, d, f, c \in \mathbb{N}$ s.t. $d + f - c > 0$ gives a solution.

2. $e = 2b − d − f$: This gives the system

$$0 = (1 + b)(a - 3b + c + 2(d + f))$$
$$a = 3b - c - 2(d + f)$$

Our only constraint is $3b - c - 2(d + f) > 0$.

We have enumerated all possible solutions to the equation.

6 Further directions

6.1 The Multivariate Generating Function

Perhaps the most obvious extension of this paper would be to more carefully investigate the implications of both the proper and improper recurrence relations on the multivariate generating function. Induction on incomparable elements led to a number of nice results over order polynomials, and such a tool could potentially be used to approach some of the questions offered in the end of McNamara and Ward’s paper [9].
Figure 5: Two Infinite Families of Doppelgangers which follow from Table 1: (a)∼(b) and (c)∼(d)

6.2 Single Step Chain Decomposition

We briefly touched on how our recurrences may be useful, even in only a single application, both in explaining one of McNamara and Ward’s posets as well as easily constructing an infinite family of doppelgangers for $C_n + C_n$. What other doppelgangers can be explained through a single set of chain decomposition? $C_n + C_n$ has high structural symmetry and simplicity. In a similar vein, Stanley suggested classifying doppelgangers which cannot be shown in a single step of a recurrence.

6.3 Closed Families

Our analysis of $H_k$ initially stemmed from the fact that the family is closed under Johnson’s recurrence. Further, series-parallel posets are closed under recurrence as well. This closure allows for easy recursive calculation of important invariants, and it is no coincidence that our results focus on these families. In fact, Faigle and Schrader’s $W_k$ is also such a closed family. However, the decomposition is complicated, and width is not an invariant on the order polynomial. However, the idea could carry over to the multivariate generating function where width and height do create invariants on naturally labeled posets.

6.4 Explaining Non-Series Parallel Doppelgangers with the Ur-Decomposition

Since the Ur-Decomposition generalized the series-parallel decomposition and exists for all posets, one would hope that the Ur-Decomposition could be used to prove some of the non-series parallel doppelgangers produced in [6]. However, many of the posets considered in [6] are grid-like and do not decompose well under the Ur-Decomposition. For example, the grid poset $C_n \times C_m$ decomposes as $C_1 \oplus P \oplus C_1$ where $P$ is indecomposable under the Ur-operation (this can be seen by noting that every RAP, Definition 7.2 is a singleton).

7 Appendix

7.1 Computation of Invariants for Posets of large height

Proposition 7.1. The values of $e(P)$ and $F_{P}(2)$ of the posets depicted in Figure 4 are given by Table 1.

Proof. The values of the invariants can be computed in two different ways. For simple posets such as the posets depicted in Figure 4, the values of the invariants can be computed combinatorially. For more
complicated examples where the values of the invariants cannot be computed combinatorially, a combination of recurrence relations and the formulas in Lemma 5.2 will still work. For example, the Tri, Dtri, and Ntri posets are all series parallel and so Lemma 5.2 can be used to determine the values of the invariants. The Xcon posets can be inductively reduced (see Figure 7) using the recurrence to previously computed examples and then the Xdis poset can also be reduced (see Figure 8) to previously computed examples. This demonstrates the utility of working with a family of posets that is closed under the recurrence. We now give the combinatorial computation of the values of the invariants for the posets depicted in Figure 4.

The number of order-preserving maps $f: P \to \{1, 2\}$ can be determined by first choosing the values of $f$ on the elements not on the maximal chain. Then the computation of the values of $F_P(2)$ is straightforward. To compute the values of $e(P)$, it suffices to choose the values of the elements not on the maximal chain. This is straightforward for the Tri, Dtri, and Ntri families. For the Xdis and Xcon families, let $x$ be the larger element not on the maximal chain and let $y$ be the smaller element not on the maximal chain. Firstly, consider the Xdis case. For $x \in \{m_1 + 1, \ldots, m_1 + m_2 + m_3 + 1\}$ there are $m_3 + m_4 + 1$ possible choices for $y$ and for $x = m_1 + m_2 + m_3 + 2$ there are $m_3 + 1$ choices for $y$. Secondly, consider the
Xcon case which is given by taking the previous case an setting \( x \geq y \). This eliminates the \( \binom{m_3}{2} \) ways to choose \( x, y \in \{ m_1 + m_2 + 2, \ldots, m_1 + m_2 + m_3 + 1 \} \) with \( y \geq x \) and the \( m_3 + 1 \) ways to choose \( y \) with \( x = m_1 + m_2 + m_3 + 2 \). We leave it to the reader to fill in further details.

7.2 Ur-Decomposition

Every poset has a unique decomposition in terms of the Ur-operation, a generalization of the series-parallel decomposition, but the statement and proof of this decomposition requires slightly more machinery than our paper presents. To begin, we must define a notion analogous to prime posets. Recall that a poset \( P, |P| > 1 \), is called prime if it cannot be expressed as the ordinal sum or disjoint union of two posets. The decomposition of posets into primes by these two operations is known as the series-parallel decomposition. Similarly, a poset \( P, |P| > 2 \), is a strong prime if it cannot be expressed as a result of a non-trivial Ur-Operation. Note that a poset is prime if it is a strong prime, but the converse does not hold.

\[ \begin{align*}
\text{(a)} & & \text{(b)}
\end{align*} \]

Figure 8: A prime and its corresponding strong prime

Consider when some poset \( P \) could have been created via the Ur-Operation. This can only be the case if we can find some subposet which is reducible to a point, more formally

**Definition 7.2.** A subset of a poset \( S = \{ x_k \}_{k=1}^m \subset P \) is reducible to a point (an RAP) when for every \( y \in P - S \), either \( y \leq \{ x_k \}, \{ x_k \} \leq y \), or \( \{ x_k \} \) and \( y \) are incomparable for all \( k \). An RAP \( S \) of \( P \) is maximal when it is neither \( P \) nor a subset of any other RAPs other than \( P \).

Notably, an Ur-operation is an expression of the form \( \mathcal{P}[x_k \rightarrow P_k]_{k=1}^n \) where each \( P_k \) is an RAP. RAPs should be considered under two circumstances. First, if there do not exist Non-trivially intersecting RAPs such that their union is the entire poset, then RAPs are closed under union and intersection. In this case, RAPs partition the poset. Second, we note the above is not true exactly when a poset can be expressed as an ordinal sum or disjoint union. Thus RAPs may be used to further decompose prime posets, and give a more geometric description of strongly prime posets.

**Proposition 7.3.** Any poset \( P, |P| > 2 \) is a strong prime if and only if it does not contain any non-trivial RAPs.

**Proof.** We will show that a poset is not a strong prime if and only if it contains an RAP. For the forward direction, suppose that \( P \) is not a strong prime. By definition, \( P = \mathcal{P}[x_k \rightarrow P_k] \). Then \( P_1 \) is an RAP of \( P \). For the reverse direction, suppose that \( L \) is an RAP of \( P \). Let \( \mathcal{P} \) be the poset given by reducing \( L \) to a point \( x \). Then \( P = \mathcal{P}[x \rightarrow L] \) and \( P \) is not a strong prime.

**Lemma 7.4.** If \( X \) and \( Y \) are RAPs of \( P \) with \( X \cap Y \neq \emptyset \), then \( X \cap Y \) and \( X \cup Y \) are RAPs of \( P \).
Proof. Suppose that \( z \in P - X \cup Y \). Then \( XRZ \) and \( YQZ \) for some relations \( R, Q \). Since \( X \cap Y \neq \emptyset \), \( R = Q \) and \((X \cup Y)Rz\) which shows that \( X \cup Y \) is an RAP. Since \( X \cap Y \subseteq X \), \( X \cap Y \) is RAP to \( P - X \).
Since \( X \cap Y \subseteq Y \), \( X \cap Y \) is RAP to \( P - Y \). Then \( X \cap Y \) is RAP to \( (P - X) \cup (P - Y) = P - (X \cap Y) \).

**Proposition 7.5.** For any prime poset \( P \), the maximal RAPs of \( P \) partition \( P \).

Proof. Since every point is in an RAP, every point is in a maximal RAP. Suppose that \( x \) and \( y \) are two maximal RAPs with \( x \cap y \neq \emptyset \). If \( x \cup y \neq P \) then by Lemma 27 \( x \cup y \) would be an RAP which contradicts the maximality of \( x \) and \( y \). Then \( x \cup y = P \) and either \( x \leq y \), \( y \leq x \), or \( x \) and \( y \) are incomparable. In these cases, \( P = x \oplus y \), \( P = y \oplus x \), and \( P = x + y \) respectively. Since these contradict the primality of \( P \), such an \( x \) and \( y \) don’t exist and all maximal RAPs are disjoint. Assume there exist two such partitions, then there must exist distinct RAPs \( S, T \) where \( S \cap T \neq \emptyset \), but this violates maximality by the same argument as above.

Finally we are ready to introduce the Ur-decomposition. We say a poset \( P \) is Ur-decomposable if \( |P| = 1 \) or if \( P \) can be expressed as \( P = \mathcal{P}[x_k \rightarrow P_k]_{k=1}^n \), where \( \mathcal{P} \) is a strong-prime, chain, or antichain and where each \( P_k \) is Ur-decomposable. In the case that \( \mathcal{P} \) is a chain or antichain, we additionally insist that each \( P_k \) is maximal, in that it cannot be expressed as the result of an ordinal or direct sum respectively. Such a decomposition is called an Ur-decomposition.

**Theorem 7.6.** All posets have a unique Ur-decomposition.

Proof. It suffices to show that each nontrivial poset \( P \) can be uniquely expressed as \( P = \mathcal{P}[x_k \rightarrow P_k]_{k=1}^n \) where \( \mathcal{P} \) is a strong-prime, chain, or antichain.

Assume \( P \) is prime, let \( \{P_k\} \) be the set of maximal RAPs which partition \( P \), and let \( \mathcal{P} \) be the poset defined on the RAPs. Assume \( \mathcal{P} \) is not a strong-prime, then by Proposition 2 \( \mathcal{P} \) contains a non-trivial RAP \( S \). Then the RAPs associated with \( S \) from an RAP in \( P \), which violates the maximality of RAPs in our partition. Furthermore, \( P \) is prime, and thus cannot be decomposed into a chain or anti-chain, and the RAP partition of \( P \) is unique.

Assume \( P \) is not prime, then \( P \) is expressible as the result of a direct or ordinal sum, and existence of an Ur-decomposition as a chain or anti-chain is immediate. Furthermore, these options are exclusive, and the insistance on maximal chains gives uniqueness within each class. Then it must only be shown that such a \( P \) cannot be decomposed into a strong prime \( \mathcal{P} \).

Let \( P = \mathcal{P}[x_k \rightarrow P_k]_{k=1}^n \), because \( P \) is not prime, there exists a subposet \( S \) s.t. \( S \) has the same relation to every element in \( P - S \). \( S \) must be contained at least partially in some \( P_i \), but since \( P_i \) is a RAP it must have the same relation to every element in \( P - P_i \). Furthermore, this implies \( x_i \in S \) has the same relation to \( P - x_i \). Then either \( \mathcal{P} = A_2 \) or \( C_2 \), or \( |\mathcal{P}| > 2 \) and must contain a non-trivial RAP. In either case, \( \mathcal{P} \) cannot be a strong prime.

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