Exact quantum query complexity for total Boolean functions

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Abstract. We will show that if there exists a quantum query algorithm that exactly computes some total Boolean function \( f \) by making \( T \) queries, then there is a classical deterministic algorithm \( A \) that exactly computes \( f \) making \( O(T^3) \) queries. The best know bound previously was \( O(T^4) \) due to Beals et al. \[6\].

1 Introduction, motivation and results

The laws of the quantum world offers to construct new models of computation that possibly are more adequate to nature. The one of the most popular models of quantum computing is quantum query algorithms. In this paper we will view only quantum query algorithms computing total Boolean functions. There are some very exciting quantum query algorithms that are better than their classical analogs. The one example is Grover’s search algorithm \[11\] that computes OR function with probability \( \frac{2}{3} \) making \( O(\sqrt{n}) \) queries, where \( n \) is number of Boolean variables. The other example is exact (giving right answer with probability 1) quantum algorithm for PARITY making \( n/2 \) queries \[10\]. It is the best from known exact quantum query algorithms for total Boolean functions.

Those amazing examples show that proving nontrivial lower bounds for quantum algorithms are essentially necessary. A lot of work has been done on it, however many problems are still open.

We will focus on exact quantum query algorithms. There are two general methods how to show quantum lower bounds. The first is adversary method (the survey and the most general version can be found in paper of Laplante and Magniez \[14\]). The second is quantum query lower bound by polynomials introduced by Beals et al. \[6\]. Their power is incomparable, see for example \[3\]. Beals et al. \[6\] showed that the number of queries needed to compute a Boolean function \( f \) by a quantum algorithm exactly \( Q_E(f) \) is at least \( \deg(f)/2 \), where \( \deg(f) \) is the degree of multilinear polynomial representing \( f \). Nisan and Smolensky \[17\] showed that the number of queries needed to compute \( f \) by a deterministic algorithm \( D(f) \) is at most \( 2\deg(f)^4 \). It implies \( D(f) \leq 32Q_E(f)^4 \).

In this paper we will show that \( D(f) \leq 2\deg(f)^3 \) thus deriving \( D(f) \leq 16Q_E(f)^3 \).

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The best known result from the opposite direction is $D(f) = \deg(f)^{\log_3 6}$ by Kushilevitz [12]. The other is $D(f) = \deg(f)^{\log_2 3}$ by Nisan and Szegedy [18] and Ambainis [3].

2 Preliminaries

2.1 Quantum query algorithms

A good survey on decision tree complexity is by Buhrman and de Wolf [8]. We will give only brief summary on definition.

We consider computing a Boolean function $f(x_1, ..., x_N) : \{0, 1\}^N \to \{0, 1\}$ in the quantum query model. In this model, the input bits can be accessed by queries to an oracle $X$ and the complexity of $f$ is the number of queries needed to compute $f$. A quantum computation with $T$ queries is just a sequence of unitary transformations $U_1 \to O \to U_2 \to O \to ... \to U_{T-1} \to O \to U_T \to O$.

$U_j$ can be arbitrary unitary transformation that do not depend on the input bits $x_1, ..., x_N$. $O$ are query transformations. To define $O$, we represent basis states as $|i, b, z\rangle$ where $i$ consists of $\lceil \log N \rceil$ bits, $b$ is one bit and $z$ consists of all other bits. Then, $O$ maps $|i, b, z\rangle$ to $(-1)^{bx_i}|i, b, z\rangle$ (i.e., we change phase depending on $x_i$). The computation starts with a state $|0\rangle$. Then, we apply $U_1, O, ..., O, U_T$ and measure the final state. The result of the computation is the rightmost bit of the state obtained by the measurement. The quantum computation computes $f$ exactly if, for every $x = (x_1, ..., x_N)$, the rightmost bit of $U_T O_x ... O_2 U_1 |0\rangle$ equals $f(x_1, ..., x_N)$ with certainty. $Q_E(f)$ denotes the minimum number $T$ of queries in a quantum algorithm that computes $f$ exactly.

2.2 Quantum query lower bounds

To see quantum and randomized query lower bounds by adversary method one can start with [14]. We will use polynomials method, derived by Nisan and Szegedy [18] and Beals et al. [6]. Quite often it is used to derive quantum lower bounds, for example in [1], [2], [5], [9], [7], [13], [15], [20], [22], [21], [23].

For any Boolean function $f$, there is a unique multilinear polynomial $g$ such that $f(x_1, ..., x_N) = g(x_1, ..., x_N)$ for all $x_1, ..., x_N \in \{0, 1\}$. We say that $g$ represents $f$. Let $\deg(f)$ denote the degree of $g$. It is known that

**Theorem 1** [2] For any total Boolean $f$, $Q_E(f) \geq \deg(f)/2$.

The block sensitivity of $f$ on $x$ is the maximum number of disjoint $B_j \subseteq \{1, ..., n\}$ such that $f(x^{B_j}) \neq f(x)$, $x^{B_j}$ being $x$ with all $x_i$ for $i \in B_j$ changed to $1 - x_i$. We denote it $bs_x(f)$. Let $bs(f) = \max bs_x(f)$. It is known that

**Theorem 2** [13] For any total Boolean function $f$, $bs(f) \leq 2\deg(f)^2$. 
3 Deterministic vs. quantum exact algorithms

Now we will show that \( D(f) \) is upper bounded by \( 2\deg(f)^3 \) for every Boolean function \( f \). Our method will be quite similar to Nisan and Smolensky [17]. Sometimes we will think about Boolean function \( f \) as polynomial representing it. Here \textit{maxonomial} of polynomial \( f \) is a monomial with maximal degree.

**Lemma 3** For every word \( w \in \{0,1\}^N \) and every maxonomial \( M \) of \( f \), there is a set \( B \) of variables in \( M \) such that \( f(w^B) \neq f(w) \).

\( \square \)

**Theorem 4** For every total Boolean function \( f \) holds \( D(f) \leq 2\deg(f)^3 \).

**Proof.** The deterministic algorithm \( A \) is written in pseudo code, as function from polynomial \( f \) and word \( X \in \{0,1\}^N \) that returns value of \( f(X) \). The algorithm \( A \):

\[
\{0,1\} \text{ function ValueOf}(\text{By value } f \text{ as polynomial, by queries } X \in \{0,1\}^N)\{
1. p := f;
2. While p is not constant{
3. Pick maxonomial \( M \) in polynomial \( p \);
4. Query \( X \)-values of \( M \)'s variables;
5. Replace all queried variables in \( p \) with appropriate constants;
};
6. Return p;
};
\]

The nondeterministic "pick maxonomial" can easily be made deterministic by choosing the the first maxonomial in some fixed order.

It is easy to see that the algorithm \( A \) always returns the right result, since polynomial \( p \) always describes polynomial \( f \) on word \( X \).

We will show that the cycle executes at most \( bs_X(f) \leq bs(f) \) times. Let \( a \) denote the number of cycle executions.

We will show that \( bs_X(f) \geq a \) by induction. Bases: before cycle is executed, \( X \) has no blocks since there are no variables queried yet. Inductive assumption: after \( a - 1 \) executions of cycle \( X \) has at least \( a - 1 \) disjoint blocks that take their variables only from yet queried variables and to which \( f \) is sensitive on \( X \).

We will show that in the next cycle execution there exists a block \( B \) that takes its variables only from variables queried in this cycle (therefore is disjoint with previous ones) and to which \( f \) is sensitive on \( X \).

Let \( M \) denote maxonomial chosen in this cycle. Let \( w \) denote the word whom holds \( f(X) = p(w) \). Such exists, since \( p \) is just polynomial \( f \) where some variables are replaced with constants according to \( X \). It is easy to see that for any set of
variables $B$ holds $p(w^B) = f(X^B)$. Now Lemma 3 says that there is a set $B$ of variables in $M$ such that $p(w^B) \neq p(w)$. Since $f(X) = p(w)$ and $f(X^B) = p(w^B)$ it follows that $f(X) \neq f(X^B)$.

$bs_X(f) \geq a$ implies that $a \leq bs(f)$, thus the cycle is executed at most $bs_X(f) \leq bs(f)$ times.

It is easy to see that for every maxonomial $M$ holds $|M| = \deg(p)$ and at every moment $\deg(p) \leq \deg(f)$, thus in every cycle $A$ makes at most $\deg(f)$ queries, hence $D(f) \leq \deg(f) \cdot bs(f)$. Theorem 2 gives $D(f) \leq 2\deg(f)^3$.

\begin{theorem}
For every total Boolean function $f$ holds $D(f) \leq 16Q_E(f)^3$.
\end{theorem}

\begin{proof}
By Theorem 4 and Theorem 1.
\end{proof}

As noticed by Ronald de Wolf, our proof works also for "nondeterministic polynomials", giving $D(f) \leq bs(f) \cdot ndeg(f)$, where nondeterministic polynomial is polynomial that takes nonzero value whenever function is 1. This relation also improves some results of paper by de Wolf [23]. For example, it follows that $D(f) = O(Q_2(f)^2 NQ(f))$, where $Q_2(f)$ is bounded-error and $NQ(f)$ nondeterministic quantum query complexity for function $f$. See [23] for more precise definitions.

The same proof also works to prove average-case upper bound: average$D(f) \leq \text{average} bs(f) \cdot ndeg(f)$, since the run of algorithm $A$ on input word $X$ is bounded by $bs_X(f)$ cycles.

4 Open problems

1. It is well known that block sensitivity is not tight measure of exact quantum query complexity of all Boolean functions, see Ambainis [4], [3]. Can one somehow use it to derive better quantum lower bound for any total Boolean function?

2. Can similar arguments be used to show upper bound over degree of approximating function, for example $D(f) = O(\deg(f)^5)$? This question is related with quantum bounded-error query complexity.

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