Shell-model descriptions of mass 16-19 nuclei with chiral two- and three-nucleon interactions

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Shell-model calculations for several mass 16-19 nuclei are performed using the N\textsuperscript{3}LO two-nucleon potential $V_{2N}$ with and without the addition of an in-medium three-nucleon potential $V_{3N}^{med}$, which is a density-dependent effective two-nucleon potential recently derived from the leading-order chiral three-nucleon force $V_{3N}$ by Holt, Kaiser, and Weise. We first calculate the $V_{low-k}$ low-momentum interactions from $V_{2N}$ and $V_{3N}^{med}$. The shell-model effective interactions for both the sd one-shell and $sdpf$ two-shell model spaces are then obtained from these low-momentum interactions using respectively the Lee-Suzuki and the recently developed Okamoto and Suzuki iteration methods. The effects of $V_{3N}^{med}$ to the low-lying states of $^{18}O$, $^{19}F$, $^{19}O$ and $^{19}F$ are generally small and attractive, mainly lowering the ground-state energies of these nuclei and making them in better agreements with experiments than those calculated with $V_{2N}$ alone. The excitation spectra of these nuclei are not significantly affected by $V_{3N}^{med}$. The low-lying spectra of these nuclei calculated with the sd and $sdpf$ model spaces are closely similar to each other. Our shell-model calculations for $^{16}O$ indicate that the $V_{3N}^{med}$ interaction is important and desirable for the binding energy of this nucleus.

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I. INTRODUCTION

The nuclear shell model has been very successful in microscopic descriptions of nuclear structure, and in this approach the shell-model effective interaction $V_{eff}$ has played an important role and its determination has been extensively studied \textsuperscript{[1–4]}. As discussed in these references, $V_{eff}$ may be determined using either an empirical approach where it is required to reproduce selected experimental data or a microscopic one where $V_{eff}$ is derived from realistic nuclear-nucleon interactions using many-body methods. The interactions used in such microscopic calculations have been mostly the two-nucleon (NN) interaction $V_{2N}$ \textsuperscript{[3–4]}. Should $V_{eff}$ have also contributions from the three-nucleon (NNN) force $V_{3N}$ in addition to the two-nucleon one? In fact the need of $V_{3N}$ in nuclear many-body problems has long been recognized. The use of $V_{2N}$ alone has been inadequate in reproducing the empirical nuclear matter saturation properties (see e.g. \textsuperscript{[5,6]} and references quoted therein). The inclusion of $V_{3N}$ has been of essential importance in describing the binding energies and low-lying spectra of light nuclei \textsuperscript{[7–9]} and in explaining the long half-life of the $^{14}C \rightarrow ^{14}N$ $\beta$-decay \textsuperscript{[10,11]}. Otsuka \textit{et al.} \textsuperscript{[12]} have shown that the inclusion of $V_{3N}$ plays a crucial role in describing the oxygen isotopes near the drip line.

In the present work, we shall calculate the shell-model effective interactions for the sd and $sdpf$ shells using the chiral $N^3LO$ two-nucleon potential $V_{2N}$ \textsuperscript{[13]} with and without the inclusion of the in-medium NNN force $V_{3N}^{med}$, which is a density-dependent two-nucleon potential recently derived from the leading-order chiral NNN force $V_{3N}$ by Holt, Kaiser and Weise \textsuperscript{[10,11]}. We shall apply these effective interactions to shell-model calculations for nuclei $^{18}O$, $^{19}F$, $^{19}O$ and $^{19}F$, to study the effects of $V_{3N}^{med}$ in these nuclei. We shall first calculate the low-momentum $V_{low-k}$ interactions $V_{2N}$ and $V_{3N}^{med}$. Our shell model effective interactions $V_{eff}$ will then be calculated from these $V_{low-k}$ interactions using a folded-diagram formalism \textsuperscript{[3–4,17,18]}. In this formalism, $V_{eff}$ is given as a folded-diagram expansion. For the degenerate sd case, this expansion can be summed up using the commonly employed Lee-Suzuki (LS) iteration method \textsuperscript{[19,20]}. For the non-degenerate $sdpf$ case we shall sum up the expansion using the extended Krenciglowa-Kuo iteration method recently developed by Okamoto, Suzuki, Kumagai and Fujii (EKKO) \textsuperscript{[21,22]}. The EKKO method is efficient for deriving the effective interactions of non-degenerate model spaces such as the $sdpf$ two-shell space, and it can also be conveniently applied to degenerate model spaces such as the sd one-shell one. \textsuperscript{[22]}

As mentioned earlier, the in-medium NNN potential $V_{3N}^{med}$ is dependent on the nuclear medium density $\rho$. We shall study this $\rho$ dependence for both low and moderately high densities. Clearly, at zero density $V_{3N}^{med}$ vanishes. Nuclear matter calculations using $V_{2N}$ alone have not been able to satisfactorily reproduce the empirical nuclear matter saturation properties \textsuperscript{[5,6]}. It may be useful to study if the $\rho$ dependence of $V_{3N}^{med}$ near the nuclear matter saturation density $\rho_0$ (0.16 fm\textsuperscript{-3}) may play an important role for nuclear matter saturation. We shall do so by carrying out nuclear matter calculations with and without the inclusion of $V_{3N}^{med}$. Similarly this dependence may also be important for closed-shell nuclei, such as $^{16}O$, whose nucleons are embedded in a nuclear
medium of densities near \( \rho_0 \). We shall calculate closed-shell nucleus \( ^{16}\text{O} \) with and without the inclusion of \( V_{3N}^{\text{med}} \), as a further study of its effect near \( \rho_0 \). The valence nucleons, such as those of \( ^{18}\text{O} \), are in a medium of densities much less than \( \rho_0 \). Thus our shell-model calculations for valence nuclei mentioned earlier are a study of \( V_{3N}^{\text{med}} \) at low densities.

The organization of the present paper is as follows. We shall describe some details about the derivation of \( V_{3N}^{\text{med}} \) from \( V_{3N} \) in section II. It may be noted that \( V_{3N}^{\text{med}} \) is an effective density-dependent two-nucleon interaction, which is more convenient than its underlying three-nucleon potential for nuclear many-body calculations. The methods we shall employ for the derivation of the shell-model effective interactions \( V_{\text{eff}} \) from \( V_{2N} \) and \( V_{3N}^{\text{med}} \) will be outlined there. In section III we shall present first our results of nuclear matter calculations using \( V_{2N} \) with and without the inclusion of \( V_{3N}^{\text{med}} \). The ring-diagram formalism \( \cite{13, 14} \) employed for our nuclear matter calculations will also be outlined. Next we shall report the results of a similar ring-diagram calculation for \( ^{16}\text{O} \). Results of our shell-model calculations of \( ^{18}\text{O} \), \( ^{18}\text{F} \), \( ^{19}\text{O} \) and \( ^{19}\text{F} \) using \( V_{2N} \) with and without the inclusion of \( V_{3N}^{\text{med}} \) will be presented and discussed in this section. A summary and conclusion is presented in section IV.

II. FORMALISM

We first describe how we include the effects of the leading-order chiral three-nucleon interaction, \( V_{3N} \), in our calculations. We consider the nuclear interaction \( V \) as given by

\[
V = \left( V_{2N} + V_{3N}^{\text{med}} \right),
\]

where \( V_{2N} \) is the \( \text{N}^3\text{LO} \) Idaho two-nucleon potential \( \cite{13} \) and \( V_{3N}^{\text{med}} \) is a density-dependent two-body interaction obtained from the chiral three-nucleon force by closing one pair of external lines and summing over the filled Fermi sea of nucleons. The leading contribution to \( V_{3N} \) occurs at \( \text{N}^2\text{LO} \) in the chiral power counting and is composed of a long-range two-pion exchange component \( V_{3N}^{2\pi} \), a medium-range one-pion exchange term \( V_{3N}^{2\pi} \), and a pure contact interaction \( V_{3N}^{\text{ct}} \):

\[
V_{3N}^{(2\pi)} = \sum_{i \neq j \neq k} \frac{g_A^2}{8\pi^4 (q_i^2 + m_n^2)(q_j^2 + m_n^2)} F_{ij}^\alpha F_{jk}^\beta \sigma_i^\alpha \sigma_j^\beta \sigma_k^\beta, \tag{1}
\]

\[
V_{3N}^{(1\pi)} = -\sum_{i \neq j \neq k} \frac{g_A^{\alpha CD}}{8\pi^4 \Lambda_X (q_i^2 + m_n^2)(q_j^2 + m_n^2)} \sigma_i^\alpha \cdot \sigma_j^\alpha \sigma_k^\alpha \cdot \sigma_j^\alpha, \tag{2}
\]

\[
V_{3N}^{\text{ct}} = \sum_{i \neq j \neq k} \frac{c_E}{2\pi^4 \Lambda_X} \tau_i \cdot \tau_j, \tag{3}
\]

where \( g_A = 1.29, f_\pi = 92.4\text{MeV}, \Lambda_X = 700\text{MeV}, m_n = 138.04\text{MeV}/c^2 \) is the average pion mass, \( \hat{q}_i = \vec{p}_i - \vec{p}_k \) is the difference between the final and initial momentum of nucleon \( i \) and

\[
F_{ij}^{\alpha \beta} = \delta^{\alpha \beta} \left( -4c_1 m_n^2 + 2c_3 q_i \cdot q_j \right) + c_4 \epsilon^{\alpha \beta \gamma \delta} \sigma_k \cdot (q_i \times q_j). \tag{4}
\]

The low-energy constants \( c_1 = -0.76\text{GeV}^{-1}, c_3 = -4.78\text{GeV}^{-1} \), and \( c_4 = 3.96\text{GeV}^{-1} \) appear already in the \( \text{N}^2\text{LO} \) two-nucleon potential and are therefore constrained by low-energy NN phase shifts \( \cite{22} \). The low-energy constants \( c_D \) and \( c_E \) are typically fit to reproduce the properties of light nuclei \( \cite{3, 4} \).

A general three-body force may be written in second quantization as

\[
\hat{V}_{3N}^{\text{med}} = \frac{1}{36} \sum_{123456} V([123],[456]) b_1^\dagger b_3^\dagger b_4 b_5 b_6 b_4, \tag{5}
\]

where the antisymmetrized matrix element is

\[
V([123],[456]) \equiv \langle 123|V_{3N}|456 + 465 - 654 - 564 - 465 \rangle. \tag{6}
\]

Here \( \langle 123|V_{3N}|456 \rangle \) is a simple product matrix element, and \( b^\dagger \) and \( b \) are creation and destruction operators defined with respect to the particle-hole vacuum \( |C \rangle \) with \( b_k|C \rangle = 0 \) for all \( k \). From eqs. \( \cite{3, 4} \) we can write \( V_{3N} = V_{3N}^{(1)} + V_{3N}^{(2)} + V_{3N}^{(3)} \), where \( V_{3N}^{(1)} \) is the component of \( V_{3N} \) that is symmetric with respect to the interchange \( j \leftrightarrow k \). Now we contract one pair of the \( b^\dagger \) and \( b \) operators of the above \( V_{3N} \) (both operators must be holes), and this leads to an effective two-body force

\[
\hat{V}_{3N}^{\text{med}} = \frac{1}{4} \sum_{12} D([12],[45]) b_1^\dagger b_2^\dagger b_5 b_6 b_4, \tag{7}
\]

with

\[
D([12],[45]) = \sum_{i \leq k} \left[ \langle 12|V_{3N}^{(2)}|i5 \rangle + \langle 12|V_{3N}^{(2)}|45 \rangle + \langle 12|V_{3N}^{(2)}|45 \rangle - \langle 12|V_{3N}^{(2)}|45 \rangle - \langle 12|V_{3N}^{(2)}|45 \rangle + \langle 12|V_{3N}^{(2)}|45 \rangle - \langle 12|V_{3N}^{(2)}|45 \rangle - \langle 12|V_{3N}^{(2)}|45 \rangle - \langle 12|V_{3N}^{(2)}|45 \rangle \right], \tag{8}
\]

where \( 4 \leftrightarrow 5 \) denotes the nine exchange terms. The above result is unchanged when \( V_{3N}^{(2)} \) is replaced by either \( V_{3N}^{(1)} \) or \( V_{3N}^{(3)} \). In our calculations we consider a background medium of symmetric nuclear matter at constant density characterized by a Fermi momentum \( k_F \). In this way analytic expressions can be obtained for \( V_{3N}^{\text{med}} \), as shown in refs. \( \cite{14, 11} \). The above is a density dependent effective ‘two-nucleon’ interaction which, unlike its underlying three-nucleon force, can be readily used in many-body problems. As detailed in \( \cite{11} \), the partial-wave potentials of the above \( V_{3N}^{\text{med}} \) have been derived from the lowest-order three-nucleon force. We shall use
them in our calculations, namely we shall consider the nucleon interaction as given by \((V_{2N} + V_{3N}^{med})\).

We use the folded-diagram theory \[19, 20\] to calculate the effective interaction. Briefly speaking, in this theory \(V_{eff}\) is given by a folded-diagram series

\[
V_{eff} = \hat{Q} - \hat{Q}' \int \hat{Q} + \hat{Q}' \int \hat{Q} - \hat{Q}' \int \hat{Q} \int \hat{Q} \int \hat{Q} \cdots,
\]

(9)

where \(\hat{Q}\) represents a so-called \(\hat{Q}\)-box consisted of irreducible diagrams, as illustrated by the 1st- and 2nd-order diagrams of Fig. 1. (The \(\hat{Q}'\)-box is the same as the \(\hat{Q}\)-box except that \(\hat{Q}'\) does not have diagrams 1st-order in the interaction.) Suppose we include only \(V_{2N}\). Then each vertex in the diagrams shown represents a \(V_{low-k}\) low-momentum interaction \[14, 10\] derived from \(V_{2N}\). When we include also \(V_{3N}^{med}\) in the calculation of \(V_{eff}\), the \(\hat{Q}\)-box diagrams for effective interactions will have a specific type of \(V_{3N}^{med}\) vertices as illustrated in Fig. 2. Recall that \(V_{low-k}\) interaction derived from \(V_{3N}^{med}\) is singular, overcoming the interaction of two valence nucleons via \(V_{2N}\). This diagram becomes diagram (a) of Fig. 2 if its \(V_{2N}\) is replaced by \(V_{3N}^{med}\). Here the two valence nucleons have to have the participation of a sea-nucleon (below Fermi sea) in order to activate \(V_{3N}^{med}\), as indicated by the hole-line loop in the diagram. (Note each \(V_{3N}^{med}\) vertex of Fig. 2 represents a \(V_{low-k}\) interaction derived from \(V_{3N}^{med}\).) Similarly diagram d1 of Fig. 1 represents the interaction of a valence nucleon with a sea-nucleon via the two-nucleon interaction \(V_{2N}\). If this interaction is replaced by \(V_{3N}^{med}\), this diagram becomes diagram (b) of Fig. 2. \(V_{3N}^{med}\) must involve three nucleons, and hence here the valence nucleon interacts with ‘two’ sea-nucleons as indicated by the two hole-line loops attached to \(V_{3N}^{med}\). Diagram (c) is the \(V_{3N}\) core polarization diagram corresponding to diagram d7 of Fig. 1. Again here one hole-line loop is needed for each vertex to activate \(V_{3N}^{med}\). As seen from Eqs. (31) and (32), \(V_{3N}^{med}\) requires the involvement of at least one sea-nucleon, and it is this requirement which is reflected by the hole-line loops attached to the \(V_{3N}^{med}\) vertices in Fig. 2. To summarize, the \(\hat{Q}\)-box diagrams of Fig. 1 are used for calculating \(V_{eff}\) when we use only \(V_{2N}\), while additional diagrams as illustrated by Fig. 2 are also included when the interaction is consisted of both \(V_{3N}^{med}\) and \(V_{3N}\).

After obtaining the \(\hat{Q}\)-box, we calculate the effective interaction of Eq. (9) using iteration methods. The LS method \[19, 20\] is convenient for degenerate model spaces, and it is used to calculate the sd one-shell effective interactions. For the non-degenerate \(sdpf\) two-shell effective interactions, one can use either the Krenciglowa-Kuo iteration method (KK) \[23, 24\] or the recent extended KK method (EKKO) of Okamoto et al. \[21\]. Both the EKKO and KK methods are convenient for calculating the effective interaction for non-degenerate model spaces, while the EKKO method being more efficient (faster converging rate). \[25\]. The main difference between the EKKO and KK methods is the following.

\[
\hat{Q}'(\omega) = [PVP + PVPQ_1(\omega)\frac{1}{\omega - QHQ}QVP]_L
\]

(10)

where \(V\) represents the NN interaction, \(P\) denotes the model-space projection operator, and \(Q \equiv (1 - P)\). The energy variable \(\omega\) is determined self-consistently \[13, 20\]. Note that \(Q\) contains only valence-linked diagrams as indicated by the subscript \(L\). In the EKKO method, the above \(\hat{Q}\)-box is replaced by the \(\hat{Z}\)-box defined by \[21, 22\]

\[
\hat{Z}(\omega) = \frac{1}{1 - \hat{Q}_1(\omega)}[\hat{Q}(\omega) - \hat{Q}_1(\omega)P(\omega - H_0)P],
\]

(11)

where \(\hat{Q}_1\) is the first derivative \(d\hat{Q}/d\omega\). When the P- and Q-space are not sufficiently separated, the above \(\hat{Q}\)-box may have singularities. An important advantage of the EKKO method is that the \(\hat{Z}\)-box is well behaved even when the corresponding \(\hat{Q}\)-box is singular, overcoming the above singularity difficulty and making the EKKO method more efficient \[21, 22\]. We shall calculate the \(sdpf\) two-shell effective interactions using both the EKKO and KK iteration methods, to cross check their convergences.
III. RESULTS AND DISCUSSION

As described in section II, we shall calculate the shell-model effective interactions with the inclusion of the medium-dependent three-nucleon force $V_{3N}^{med}$ which is obtained from a chiral NNN force by integrating one participating sea-nucleon over the Fermi sea. The $V_{3N}^{med}$ interaction is a density dependent effective interaction, and we shall first carry out nuclear matter calculations with the inclusion of $V_{3N}^{med}$, to study the properties of this interaction at various densities near the nuclear matter saturation density $\rho_0$. Before doing so, we need to choose or decide the low-energy constant of $V_{3N}^{med}$ (see Eqs. (1-3)) to be used in our calculations. The low-energy constants $c_1$, $c_3$ and $c_4$ of $V_{3N}^{(2n)}$ are well determined; they are constrained by low-energy NN scattering data [27]. Their values so determined (section II) will be used in the present work. But the low-energy constants $c_D$ and $c_E$ of $V_{3N}^{(1\pi)}$ and $V_{3N}^{(ct)}$ respectively, are less well known; their values determined from properties of light nuclei exhibit considerable variations [7-10]. The $V_{3N}^{med}$ interaction depends explicitly on the Fermi momentum $k_F$ which is well defined for nuclear matter. But $k_F$ is not well defined for finite nuclei, causing uncertainty in determining the low-energy constants of $V_{3N}^{med}$ from properties of finite nuclei. But for nuclear matter, $k_F$ is well defined; we have thus chosen to study these constants by way of nuclear matter calculations with $V_{3N}^{med}$ included.

![FIG. 3: Nuclear matter ring diagrams with vertices from $V_{2N}$ (wavy line) and $V_{3N}^{med}$ (narrow band with hole-line-loop).](image)

We calculate the equation of state (EOS) for symmetric nuclear matter using a ring-digram formalism [3, 4]. A brief description of this formalism is presented below, to outline how we include $V_{3N}^{med}$ in our calculations. Using familiar renormalization procedures [14, 16], we first calculate the low-momentum interactions $V_{2N}^{low-k}(A)$ and $V_{3N}^{low-k}(A)$ respectively from $V_{2N}$ [13] and $V_{3N}^{med}$. A common decimation scale of $\Lambda = 2.1\text{fm}^{-1}$ is employed. This value is chosen because at this scale the low-momentum interactions derived from different NN potentials [13, 29-31] are remarkably close to each other [16], leading to a nearly unique low-momentum interaction. Using the above ring-diagram framework, the ground-state energy shift $\Delta E$ is given by the all-order sum of the $pphh$ ring diagrams as illustrated in Fig. 3. ($\Delta E_0$ is defined as $(E_0 - E_0^{free})$ where $E_0$ is the true ground state energy and $E_0^{free}$ that for the non-interacting system.) As shown in Fig.3, diagram (1) is a 1st-order ring diagram with its vertex $V_{2N}^{low-k}$ calculated from the N$^3$LO chiral NN potential [13]. Similarly diagram (2) is a 1st-order ring diagram with $V_{3N}^{low-k}$ obtained from the lowest order chiral $V_{3N}$ as described in section II. Diagram (3) has vertices from both $V_{2N}^{low-k}$ and $V_{3N}^{low-k}$. Similar to Fig. 2, to have the pair of nucleons in the ring diagrams interact with $V_{3N}$ there must be the participation of a third nucleon. Thus the $V_{3N}$ vertices in Fig. 3 all have a one-hole-line loop attached to them.

With these ring diagram summed to all orders, the ground-state energy shift for nuclear matter is given as

$$\Delta E_0 = \int_0^1 d\lambda \sum_m \sum_{ijkl<\Lambda} Y_m(ij,\lambda)Y_m^*(kl,\lambda) \langle ij|V_{2N}^{low-k}(\Lambda)+V_{3N}^{low-k}(\Lambda)|kl\rangle.\quad(12)$$

The transition amplitudes above are $Y_m^*(kl,\lambda) = \langle \Psi_m(\lambda, A-2)|\beta(k)|\Psi_0(\lambda, A)\rangle$, where $\Psi_0(\lambda, A)$ denotes the true ground state of nuclear matter which has $A$ particles, $\Psi_m(\lambda, A-2)$ the mth true eigenstate of the ($\Lambda$-2) system, and $\beta(k) = b_{hk}$ if $(k,l) > k_F$ and = $b_{ik}^*b_{lj}$ if $(k,l) \leq k_F$. Note that $\lambda$ is a strength parameter, integrated from 0 to 1. The amplitudes $Y$ are calculated from a RPA equation [3, 4, 26] based on the $(V_{2N}+V_{3N}^{med})$ interactions. Note that our calculation is the same as the usual Hartree-Fock (HF) one if we include only the 1st-order ring diagrams (1) and (2) of Fig. 4, and in this case the above $Y$ amplitude becomes $Y(ij) = n_in_j$ where $n_i=1$ for $i < k_F$ and $=0$ otherwise. Our ring-diagram calculations include particle-hole fluctuations of the Fermi sea, while not so for the HF calculations.

![FIG. 4: Ring-diagram equation of state for symmetric nuclear matter calculated with (w/ 3NF) and without (no 3NF) $V_{3N}^{med}$.](image)

We have carried out ring-diagram calculations for symmetric nuclear matter using a wide range of values for $c_D$ and $c_E$. Our results with $c_D = -2.7$ and $c_E = 0.7$ are displayed in Fig. 4; they give $E_0/A \simeq -16\text{MeV}$ and $\rho_0 \simeq 0.16\text{fm}^{-3}$, both in satisfactory agreement with the empirical values. Two curves are shown in the figure,
one with $V_{2N}$ alone and the other with the addition of $V_{3N}^{med}$. Comparing with the $V_{2N}$ curve, it is of interest that the effect of $V_{3N}^{med}$ is slightly attractive for low densities ($\sim 2\rho_0/3$) while becomes strongly repulsive at high densities. Recall that we have used a common decimation scale of $\Lambda = 2.1\text{fm}^{-1}$ [16] for both $V_{2N}^{low-k}$ and $V_{3N}^{low-k}$. It is of interest that in the ring-diagram calculations with Brown-Rho scaling [8, 9] a large $\Lambda$ of $\sim 3.0\text{fm}^{-1}$ is needed for obtaining satisfactory nuclear matter saturation properties, while in the present calculation with $V_{3N}^{med}$ satisfactory results can be obtained with the use of a smaller $\Lambda$ (2.1 fm$^{-1}$).

We have calculated the HF s.p. spectrum $\epsilon_k$ in nuclear matter with the inclusion of both $V_{2N}$ and three-nucleon force $V_{3N}^{med}$. The inclusion of the latter has been found to have significant effect to the spectrum. $\epsilon_k$ can be well fitted by the quadratic expression $(k^2\hbar^2/2m^*) + \Delta$ where $m^*$ is the effective mass and $\Delta$ is a well-depth parameter representing the s.p. energy at zero momentum. In Fig. 5 we present our results for $\Delta$ and $m/m^*$ for various densities. Two curves are shown: the lower one without and the top one with the inclusion of $V_{3N}^{med}$. The densities for the 7 data points of each curve are, from left to right, (0.25, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0 $\rho_0$) respectively. As shown by the lower curve of the figure, $\Delta$ and $m/m^*$ both vary monotonically with the density when only $V_{2N}$ is employed. With increasing density, $\Delta$ becomes increasingly more negative and $m/m^*$ increasingly larger, exhibiting no saturation (up to 3.0 $\rho_0$). The trend is quite different for the upper curve where the three-nucleon force is included. Here $m/m^*$ still monotonically increases with density, but $\Delta$ arises after saturation, approaching zero at some higher density. It is of interest that $V_{3N}^{med}$ has a large effect in raising the chemical potential of nucleons in high density nuclear matter; this raise of the chemical potential may play an important role in enabling such nucleons decaying into other baryons such as hyperons.

As indicated by Fig. 4, our ring-diagram calculations with the inclusion of the NNN force $V_{3N}^{med}$ have described well the nuclear matter saturation properties. It would be useful and of interest to check if this approach is also satisfactory for describing closed-shell nuclei whose nucleons are mainly embedded in a similar nuclear medium of densities near $\rho_0$, the nuclear matter saturation density. To study this possibility, we have extended the above ring-diagram nuclear matter calculation to the closed shell nucleus $^{16}\text{O}$, using the same low-energy parameters $C_D$ and $C_E$ (-2.70 and 0.70) determined earlier. Note that these parameters will also be used later in our $sd$ and $sdpf$ shell-model calculations. Our ring-diagram calculation for $^{16}\text{O}$ is quite similar to the nuclear matter one. The ground-state energy shift for $^{16}\text{O}$ is also given by the all-order sum of the ring diagrams illustrated in Fig. 3, except for the following differences. For nuclear matter, the s.p. states are plane-wave states and each particle line in the figure denotes a plane-wave particle with momentum $k > k_F$ and each hole line with $k < k_F$. For $^{16}\text{O}$ we use harmonic oscillator s.p. wave functions ($\hbar\omega$=14 MeV), and each particle line in the figure denotes a nucleon in the particle shells ($1s0d1p0f$), and each hole line nucleon in the $0s0p$ shells. For nuclear matter, we use momentum model space where all particles have momentum $k < \Lambda$. For $^{16}\text{O}$, we use an oscillator model space composed of the s.p. orbits of the $0s0p1s0d1p0f$ shells. Indeed the ring-diagram calculations for nuclear matter and $^{16}\text{O}$ are quite similar. The ground-state energy shift for $^{16}\text{O}$ is also given by Eq.(12), except that the orbits $(ijkl)$ now all refer to shell-model orbits and the summation restriction of $(ijkl < \Lambda)$ is replaced by that of $ijkl$ belonging to the above oscillator model space.

With the same $V_{2N}$ and $V_{3N}^{med}$ interactions used for our nuclear matter calculations, we have carried out ring-diagram calculations for the ground-state energy of $^{16}\text{O}$. Since $V_{3N}^{med}$ is dependent on the nuclear density $\rho$, we need to have the value of $\rho$ to carry on the calculation. For nuclear matter, this $\rho$ is well defined, but not so for finite nuclei. A preliminary estimate for the density for $^{16}\text{O}$ can be made by considering it as a uniform sphere [33]; in this way the average density for $^{16}\text{O}$ is estimated as $\sim 0.85\rho_0$. A more realistic estimate for this density should be lower, as physically the nuclear surface is diffused (not sharp). In Table I we present results for the ground-state energy per nucleon of $^{16}\text{O}$ for several densities, $\rho/\rho_0 = 0, 0.5, 0.70$ and 0.75. As seen, the binding energy calculated with $\rho = 0$ (i.e. no $V_{3N}^{med}$) is clearly not adequate. The inclusion of $V_{3N}^{med}$ lowers the ground-state energy per nucleon significantly; for example the inclusion of $V_{3N}^{med}$ at $\rho/\rho_0=0.75$ has increased the ring-diagram ground-state energy from -6.97 (only $V_{2N}$) to -8.01 MeV (with $V_{3N}^{med}$) per nucleon, in good agreement with experiment. Since we have not included the Coulomb interaction in our calculations, an empirical Coulomb energy of 1.14 MeV per nucleon [33] has been included in the above calculation for the ground-state energy of $^{16}\text{O}$. It
may be noted that our results for $\rho/\rho_0 = 0.70$ and 0.75 are rather close to each other. We have calculated the potential energy (PE) from the $(V_{2N} + V_{3N}^{med})$ interaction. Results given by the 1st- and 2nd- and all-order ring diagrams, denoted by PE-1st, PE-2nd and PE-ring respectively, are listed in the Table. It may be noted that PE-2nd is much smaller than PE-1st. And the difference between (PE-1st+PE-2nd) and PE-ring is rather small, about 5%. Bogner et al. \[34\] have found that $V_{low-k}$ is suitable for perturbative nuclear matter calculations. The good convergence shown in Table I suggests that this interaction is also suitable for perturbative calculations of finite nuclei.

Having seen that the medium-dependent three-nucleon force $V_{3N}^{med}$ has given desirable results for nuclear matter and closed-shell nucleus $^{16}O$, we shall now study its effects on the nuclear effective interactions for the degenerate sd one-shell and the non-degenerate sd$fp$ two-shell model spaces. We use the same methods as described in \[25\] for the calculation and application of these effective interactions, except for the following difference: In \[25\] only the two-nucleon interaction $V_{2N}$ was employed. In the present work we include both $V_{2N}$ and $V_{3N}^{med}$. We first calculate the low-momentum interactions from these interactions, and using them to obtain the $Q$-box diagrams as shown in Figs. 1 and 2. The same $V_{2N}$ and $V_{3N}^{med}$ interactions we used earlier in our calculations for nuclear matter and $^{16}O$ are employed. The degenerate sd and non-degenerate sd$fp$ effective interactions are obtained from the $Q$-box using respectively the LS iteration method \[19, 20\] and the EKKO iteration method \[21, 25\]. The nondegenerate sd$fp$ effective interactions can also be calculated using the KK iteration method \[23, 25\].

We first apply the above effective interactions with $V_{3N}^{med}$ to nuclei $^{18}O$ and $^{18}F$. Since $V_{3N}^{med}$ is density dependent, we need to know the ‘local density’ $\rho_v$ felt by the valence nucleons in these nuclei. It should be small, but its precise value is rather uncertain. In the present work, we shall estimate $\rho_v$ by comparing the density profile of $^{16}O$ and that for the valence nucleons. In Fig. 6 we plot the density profile $\rho_{core}(r)$ of $^{16}O$ obtained with its wave function assumed to be a closed shell-model $s^4p^6$ core. We also plot the distribution $\Phi(r) = \xi \phi(r)^4$ where $\phi(r)$ is the radial harmonic oscillator wave function for $d_{5/2}$, with the scaling parameter $\xi = 0.1$. As seen from the plots, the pair of $d_{5/2}$ nucleons reside primarily in the low-density region of $\rho_{core}$. Depending on the averaging procedure employed, we have estimated $\rho_v$ from $\rho_{core}(r)$ and $\phi(r)$, obtaining values ranging from $\sim 0.015$ to $\sim 0.030 \text{fm}^{-3}$. We have considered another scheme to estimate $\rho_v$. The rms radius $<r_{val}>$ for the shell-model $d_{5/2}$ orbit is indicated by an arrow in the figure. A simple scheme to estimate $\rho_v$ is to let it equal to $\rho_{core}(<r_{val}>)$, the value obtained in this way being $\sim 0.025 \text{fm}^{-3}$. This value may be reduced if realistic s.p. wave functions are employed. The $d_{3/2}$ orbit is nearly unbound, and its rms radius should be considerably larger than that for $d_{5/2}$. Assuming $\hbar \omega =$10 MeV for the $d_{3/2}$ orbit, its $<r_{val}>$
would be $\sim 3.8 fm$ giving $\rho_v \simeq 0.01 fm^{-3}$ for this orbit. We believe that a suitable range for $\rho_v$ is from $\sim 0.1$ to $\sim 0.2\rho_0$ ($\rho_0 \approx 0.16 fm^{-3}$). In the present work, we adopt $\rho_v = 0.15\rho_0$, to illustrate the effects of $V_{3N}^{med}$.

In Fig. 7 the matrix elements of the sd-shell effective interactions calculated from $V_{2N}$ with and without $V_{3N}^{med}$, denoted respectively as 'NN+NNN' and 'NN', are compared. As seen, the two sets of matrix elements are rather similar to each other, with the magnitudes of those including $V_{3N}^{med}$ being slightly larger. The low-lying spectrum of $^{18}O$ and $^{18}F$ calculated with the above interactions are presented in Figs. 8 and 9. The main effect of the 'NNN' force is a small downward shift for the ground states of $^{18}O$ and $^{18}F$, while leaving the other states largely unchanged. This trend is consistent with what we have observed for nuclear matter calculations (see Fig. 4), namely the effect of $V_{3N}^{med}$ is slightly attractive at low densities. With the inclusion of the 'NNN' force, the ground-state energy of $^{18}F$ is in good agreement with experiment while that for $^{18}O$ is slightly overbound. The agreements of the other states with experiments for $^{18}O$ are better than $^{18}F$.

Using the method outlined in section II and [22], we have also calculated the effective interaction for a sdpf two-shell space using $V_{2N}$ with and without the inclusion of $V_{3N}^{med}$. In Figs. 8 and 9, the spectra of $^{18}O$ and $^{18}F$ obtained with the former effective interaction are also presented (‘two-shell’, 'NN+NNN'). As seen, the ‘two-shell’ and ‘one-shell’ results are rather similar to each other. To further study the effects of the three-nucleon force $V_{3N}^{med}$, we have extended the calculations of Figs. 8 and 9 to nuclei $^{19}O$ and $^{19}F$, with results presented in Figs. 10 and 11. Here, similar to $^{18}O$ and $^{18}F$, the ground-state energies of $^{19}O$ and $^{19}F$ obtained with $V_{2N}$ only are both too high compared with experiments (see the 'NN' columns). As shown by the 'NN+NNN' columns, the inclusion of $V_{3N}^{med}$ slightly lowers these energies. The 'one-shell' and 'two-shell' results are rather close to each other for $^{19}O$. But they are slightly different for $^{19}F$, with the 'NN+NNN' ground-state energy noticeably lower than the 'NN' one.

As shown in Figs. 8-11, although there are qualitative agreements between the calculated and experimental spectra of $^{18}O$, $^{18}F$, $^{19}O$ and $^{19}F$, there are significant disagreements between them. For example, the orderings of the calculated low-lying states of $^{19}O$ are in fair agreement with the experimental ones, but their relative spacings are not. As shown by the figures, the effects from $V_{3N}^{med}$ are generally small and attractive, mainly slightly lowering their ground-state energies while having nearly no influence on the relative spacings of the calculated spectrum. It should be useful to further study how to reduce the above disagreements. We have employed a low-order approximation for the $Q$-box, including only 1st- and 2nd-order diagrams. The inclusion of higher-order diagrams may alter our results. As carried out by Holt et al. [33], certain classes of planar diagrams of the $Q$-box can be summed up to all orders using the Kirson-Babu-Brown induced interaction method. We plan to extend our present calculations by including these planar diagrams, and study their effects to the shell-model description of these nuclei.

IV. SUMMARY AND CONCLUSION

We have carried out shell-model calculations for nuclei $^{18}O$, $^{18}F$, $^{19}O$, $^{19}F$ and $^{18}O$ using effective interactions derived from the chiral two- and three-nucleon potentials. We start from the $N^3LO$ two-nucleon potential $V_{2N}$ together with an in-medium three-nucleon potential $V_{3N}^{med}$ which is a density-dependent effective interaction derived from the lowest-order chiral three-nucleon force $V_{3N}$ by Holt, Kaiser and Weise [10, 11]. The $V_{low-k}$ low-momentum interactions are then derived from these potentials, and are used in a folded-diagram expansion.
for calculating the $V_{eff}$ interactions for both the degenerate $sd$ and non-degenerate $sdpf$ model spaces. The well-knownLee-Suzuki iteration method [19, 20] is used for the derivation of the $sd$ effective interactions, while the recently developed iteration method of Okamoto et al. [21] is employed for the $sdpf$ ones.

We have studied the effects of $V^{med}_{3N}$, which is an effective two-body interaction depending on the medium density $\rho$, at various densities near and below the nuclear matter saturation density $\rho_0$. We first apply the $V^{med}_{3N}$ interaction to nuclear matter calculations for which the medium density is uniform and well defined. A ring-diagram method for nuclear matter has been employed. The low-energy constants $C_D$ and $C_E$, of respectively the one-pion and contact terms of the leading-order chiral three-nucleon potential $V_{3N}$, are not well known. We have chosen these constants by requiring that their use in our nuclear matter calculations with $V_{2N}$ and $V^{med}_{3N}$, which is derived from $V_{3N}$, satisfactorily reproduces the saturation properties of nuclear matter. As indicated in Fig. 4, the inclusion of $V^{med}_{3N}$ has largely improved our nuclear matter results comparing with those given by $V_{2N}$ alone, and the resulting nuclear matter saturation properties are in good agreement with the empirical values.

We then calculate the $sd$ and $sdpf$ effective interactions with the inclusion of the above $V^{med}_{3N}$, and apply them to nuclei $^{18}O$, $^{19}F$, $^{19}O$ and $^{19}F$. An ambiguity here is the determination of the local density $\rho_v$ felt by the valence nucleons of these nuclei. This density should be small compared with the saturation density $\rho_0$ of nuclear matter, but it is difficult to determine its value precisely. We have estimated this value based on the density profile of $^{16}O$ given by a simple $s^4p^{12}$ shell-model wave function. This is an approximation which should be further investigated and improved upon. In our calculations for these nuclei, we have adopted the estimation of $\rho_v/\rho_0=0.15$. As indicated by Figs. 8-11, the main effect of $V^{med}_{3N}$ is to lower the ground-state energies of these nuclei slightly, making them in better agreements with experiments than the energies given by $V_{2N}$ alone. The effects of $V^{med}_{3N}$ to these nuclei are general small, mainly because their valence nucleons are in a low-density medium.

The above $V^{med}_{3N}$ has also been applied to a shell-model calculation of closed-shell nucleus $^{16}O$. The nucleons in closed-shell nuclei are embedded in nuclear medium of much larger density than that for the valence nucleons mentioned above. For them the effect of $V^{med}_{3N}$ is expected to be more important. Indeed this is confirmed by our calculations. As illustrated in Table I, the inclusion of this interaction has been essential for satisfactorily reproducing the empirical binding energy of $^{16}O$. It should be useful and of interest to further study this in-medium three-nucleon potential by applying it to closed-shell nuclei and those with a few valence holes.

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**Figure 10**: Same as Fig. 8 except for $^{19}O$.

**Figure 11**: Same as Fig. 8 except for $^{19}F$.

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