CENTRAL LIMIT THEOREM AND MODERATE DEVIATIONS FOR
A CLASS OF SEMILINEAR SPDES

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Abstract. In this paper we prove a central limit theorem and a moderate deviation principle for a class of semilinear stochastic partial differential equations, which contain Burgers’ equation and the stochastic reaction-diffusion equation. The weak convergence method plays an important role.

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1. INTRODUCTION

For any $\varepsilon > 0$, consider the following semilinear stochastic partial differential equation

$$
\frac{\partial U^\varepsilon}{\partial t}(t, x) = \frac{\partial^2 U^\varepsilon}{\partial x^2}(t, x) + \sqrt{\varepsilon} \sigma(t, x, U^\varepsilon(t, x)) \frac{\partial^2 W}{\partial t \partial x}(t, x)
+ \frac{\partial}{\partial x} g(t, x, U^\varepsilon(t, x)) + f(t, x, U^\varepsilon(t, x)),
$$

(1.1)

for all $(t, x) \in [0, T] \times [0, 1]$, with Dirichlet boundary conditions $(U^\varepsilon(t, 0) = U^\varepsilon(t, 1) = 0)$ and initial condition $U^\varepsilon(0, x) = \eta(x) \in L^p([0, 1]), p \geq 2$; $W$ denotes the Brownian sheet defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$; the coefficients $f = f(t, x, r), g = g(t, x, r), \sigma = \sigma(t, x, r)$ are Borel functions of $(t, x, r) \in \mathbb{R}_+ \times [0, 1] \times \mathbb{R}$. See Section 2 for details. This family of semilinear equations contains both the stochastic Burgers’ equation and the stochastic reaction-diffusion equation, See Gyöngy [11] for details.

Intuitively, as the parameter $\varepsilon$ tends to zero, the solutions $U^\varepsilon$ of (1.1) will tend to the solution of

$$
\frac{\partial U^0}{\partial t}(t, x) = \frac{\partial^2 U^0}{\partial x^2}(t, x) + \frac{\partial}{\partial x} g(t, x, U^0(t, x)) + f(t, x, U^0(t, x)),
$$

(1.2)

for all $(t, x) \in [0, T] \times [0, 1]$, with the Dirichlet’s boundary conditions.

It is always interesting to investigate deviations of $U^\varepsilon$ from the deterministic solution $U^0$, as $\varepsilon$ decreases to 0, that is, the asymptotic behavior of the trajectory,

$$
X^\varepsilon(t, x) := \frac{1}{\sqrt{\varepsilon \lambda(\varepsilon)}} \left( U^\varepsilon - U^0 \right)(t, x), \quad (t, x) \in [0, T] \times [0, 1],
$$

where $\lambda(\varepsilon)$ is some deviation scale which strongly influences the asymptotic behavior of $X^\varepsilon$. 

1
The case $\lambda(\varepsilon) = 1/\sqrt{\varepsilon}$ provides some large deviations estimates. Cardon-Weber studied the large deviations for the small noise limit of stochastic semilinear PDEs by the exponential approximations. Very recently, Foondun and Setayeshgar extended Cardon-Weber’s result to a less restrictive case by using the weak convergence approach.

If $\lambda(\varepsilon)$ is identically equal to 1, we are in the domain of the central limit theorem (CLT). We will show that $(U^\varepsilon - U^0)/\sqrt{\varepsilon}$ converges as $\varepsilon \downarrow 0$ to a random field.

When the deviation scale satisfies
\[
\lambda(\varepsilon) \to +\infty, \quad \sqrt{\varepsilon} \lambda(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0,
\] it is the moderate deviations, see [5]. Throughout this paper, we assume (1.3) is in place.

The moderate deviation principle (MDP) enables us to refine the estimates obtained through the central limit theorem. It provides the asymptotic behavior for \(\mathbb{P}(\|U^\varepsilon - U^0\| \geq \delta \sqrt{\varepsilon} h(\varepsilon))\) while the central limit theorem gives asymptotic bounds for \(\mathbb{P}(\|U^\varepsilon - U^0\| \geq \delta \sqrt{\varepsilon})\).

Like large deviations, the moderate deviations arise in the theory of statistical inference quite naturally. The moderate deviation principle can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals, refer to the recent works [8], [9] and the references therein. Usually, the quadratic form of the MDP’s rate function allows for the explicit minimization and in particular, it allows to obtain an asymptotic evaluation for the exit time, see [14]. Quite recently, the study of the MDP estimates for stochastic (partial) differential equation has been carried out as well, see [2], [6], [10], [13], [16], [17] etc. Especially, Belfadli R. et al. proved a moderate deviation principle for the law of a stochastic Burgers equation and established some useful estimates toward a central limit theorem.

In this paper, we shall study the problems of the central limit theorem and the moderate deviation principle for the stochastic semilinear SPDE which contains Burgers’ equation and the stochastic reaction-diffusion equation. We generalize the moderate deviation result in [1] and prove the central limit theorem.

The rest of this paper is organized as follows. In Section 2, we give the framework of the stochastic semilinear SPDEs, and state the main results of this paper. In Section 3, we prove some convergence results. Section 4 is devoted to the proof of central limit theorem. In Section 5, we prove the moderate deviation principle by using the weak convergence method.

Throughout the paper, $C(p)$ is a positive constant depending on the parameter $p$, and $C$ is a constant depending on no specific parameter (except $T$ and the Lipschitz constants), whose value may be different from line to line by convention.

2. Framework and main results

2.1. Framework. Let us give the framework taken from [7] and [11].

For any $T > 0$, assume that the coefficients $f = f(t, x, r), g = g(t, x, r), \sigma = \sigma(t, x, r)$ in (1.1) are Borel functions of $(t, x, r) \in [0, T] \times [0, 1] \times \mathbb{R}$ and there exist positive constants $K, L$ satisfying the following conditions:

\[
\int_0^T \int_0^1 \int_{\mathbb{R}} (|f(t, x, r)| + |g(t, x, r)| + |\sigma(t, x, r)|)^2 \, dr \, dx \, dt < \infty.
\]
(H1) for all \((t, x, r) \in [0, T] \times [0, 1] \times \mathbb{R}\), it holds that
\[|f(t, x, r)| \leq K(1 + |r|).\]

(H2) the function \(g\) is of the form \(g(t, x, r) = g_1(t, x, r) + g_2(t, r)\), where \(g_1\) and \(g_2\) are Borel functions satisfying that
\[|g_1(t, x, r)| \leq K(1 + |r|) \quad \text{and} \quad |g_2(t, r)| \leq K(1 + |r|^2).\]

(H3) \(\sigma\) is bounded and for any \((t, x, p, q) \in [0, T] \times [0, 1] \times \mathbb{R}^2\),
\[|\sigma(t, x, p) - \sigma(t, x, q)| \leq L|p - q|.

Furthermore, \(f\) and \(g\) are locally Lipschitz with linearly growing Lipschitz constant, i.e.,
\[|f(t, x, p) - f(t, x, q)| \leq L(1 + |p| + |q|)|p - q|,
\[|g(t, x, p) - g(t, x, q)| \leq L(1 + |p| + |q|)|p - q|.

**Definition 2.1 (Mild Solution).** A random field \(U^\varepsilon = \{U^\varepsilon(t, x) : t \in [0, T], x \in [0, 1]\}\) is called a mild solution of (1.1) with initial condition \(\eta\) if \(U^\varepsilon(t, x)\) is \(\mathcal{F}_t\)-measurable, \((t, x) \mapsto U^\varepsilon(t, x)\) is continuous a.s., and
\[
U^\varepsilon(t, x) = \int_0^1 G_t(x, y)\eta(y)dy + \sqrt{\varepsilon}\int_0^t \int_0^1 G_{t-s}(x, y)\sigma(s, U^\varepsilon(s))(y)W(dy, ds)
- \int_0^t \int_0^1 \partial_y G_{t-s}(x, y)g(s, U^\varepsilon(s))(y)dyds
+ \int_0^t \int_0^1 G_{t-s}(x, y)f(s, U^\varepsilon(s))(y)dyds. \tag{2.1}
\]

Here \(G_t(\cdot, \cdot)\) is the Green kernel associated with the heat operator \(\partial/\partial t - \partial^2/\partial x^2\) with the Dirichlet’s boundary conditions.

Gyöngy [11] proved the following result for the existence and uniqueness of the solution to Eq.(1.1).

**Theorem 2.1.** ([11] Theorem 2.1). Under conditions (H1)-(H3), for any \(\eta \in L^p([0, 1])\), \(p \geq 2\), there exists a measurable functional
\[\xi^\varepsilon : L^p([0, 1]) \times C([0, T] \times [0, 1]; \mathbb{R}) \to C([0, T]; L^p([0, 1])),\]
such that \(U^\varepsilon = \xi^\varepsilon(\eta, \sqrt{\varepsilon}W)\) is the unique mild solution of (1.1).

Furthermore, from the proof of [11] Theorem 2.1, we know that \(\sup_{t \in [0, T]} \|U^\varepsilon(t, \cdot)\|_2\) is bounded in probability, i.e.,
\[
\lim_{M \to \infty} \sup_{\varepsilon \in (0, 1]} \mathbb{P} \left( \sup_{t \in [0, T]} \|U^\varepsilon(t, \cdot)\|_2 > M \right) = 0. \tag{2.2}
\]
Particularly, taking ε = 0 in (1.1), we know that the determinate equation (1.2) admits a unique solution $U^0_0 \in C([0, T]; L^2([0, 1]))$, given by

$$U^0_0(t, x) = \int_0^1 G_t(x, y)\eta(y)dy - \int_0^t \int_0^1 \partial_y G_{t-s}(x, y)g(U^0_0)(s, y)dyds$$
$$+ \int_0^t \int_0^1 G_{t-s}(x, y)f(U^0_0)(s, y)dyds,$$

and

$$\sup_{t \in [0, T]} \|U^0_0(t, \cdot)\|_2 < \infty. \quad (2.4)$$

2.2. Main results. To study the central limit theorem and moderate deviation principle, we furthermore suppose that (H4) the coefficients $f$ and $g$ are differentiable with respect to the last variable, and the derivatives $f'$ and $g'$ are also uniformly Lipschitz with respect to the last variable, more precisely, there exists a positive constant $K'$ such that

$$|f'(t, x, y) - f'(t, x, z)| \leq K'|y - z|, \quad |g'(t, x, y) - g'(t, x, z)| \leq K'|y - z| \quad (2.5)$$

for all $(t, x) \in [0, T] \times [0, 1], y, z \in \mathbb{R}$.

Combined with the growth condition (H3), we conclude that

$$|f'(t, x, r)| \leq L(1 + 2|r|), \quad |g'(t, x, r)| \leq L(1 + 2|r|). \quad (2.6)$$

Our first main result is the following functional central limit theorem.

**Theorem 2.2.** Under conditions (H1)-(H4), for any $T > 0$, the process $(U^\varepsilon - U^0_0)/\varepsilon$ converges to a random field $V$ in probability on $C([0, T]; L^2([0, 1]))$, determined by

$$V(t, x) = \int_0^t \int_0^1 G_{t-s}(x, y)f'(s, y, U^0_0(s, y))V(s, y)dyds$$
$$- \int_0^t \int_0^1 \partial_y G_{t-s}(x, y)g'(s, y, U^0_0(s, y))V(s, y)dyds$$
$$+ \int_0^t \int_0^1 G_{t-s}(x, y)\sigma(s, y, U^0_0(s, y))W(dy, ds). \quad (2.7)$$

In view of the assumption (1.3) and (2.5), by the large deviation principle for stochastic partial differential equation (see [4]), one can obtain that $V/\lambda(\varepsilon)$ obeys an LDP on $C([0, T]; L^2([0, 1]))$ with the speed $\lambda^2(\varepsilon)$ and with the good rate function:

$$I(f) = \inf \left\{ \frac{1}{2} \int_0^T \int_0^1 |\dot{h}(t, x)|^2 dt dx : \ h \in \mathcal{H}, X^f = h \right\}, \quad (2.8)$$
where the function $X^h$ is the solution of the following deterministic partial differential equation

$$
X^h(t, x) = \int_0^t \int_0^1 G_{t-s}(x, y)f'(s, y, U^0(s, y))X^h(s, y)dyds - \int_0^t \int_0^1 \partial_y G_{t-s}(x, y)g'(s, y, U^0(s, y))X^h(s, y)dyds + \int_0^t \int_0^1 G_{t-s}(x, y)\sigma(s, y, U^0(s, y))\dot{h}(s, y)dyds. \tag{2.9}
$$

Our second main result reads as follows:

**Theorem 2.3.** Under conditions (H1)-(H4), $(U^\varepsilon - U^0)/\sqrt{\varepsilon \lambda(\varepsilon)}$ obeys an LDP on $C([0, T]; L^2([0, 1]))$ with the speed $\lambda^2(\varepsilon)$ and with the rate function $I$ given by $(2.8)$.

### 3. Some preliminary estimates

#### 3.1. Some preliminary estimates

The following estimates of Green function $G$ hold, (see Cardon-Weber [4], Walsh [15], Gyöngy [11]). There exist positive constants $K, a, b, d$ such that for all $x, y \in [0, 1]$ and $0 \leq s < t \leq T$,

1. $|G_{t-s}(x, y)| \leq K \frac{1}{\sqrt{t-s}} \exp \left\{ -a \frac{(x - y)^2}{t-s} \right\}$, \tag{3.1}

2. $\left| \frac{\partial}{\partial x} G_{t-s}(x, y) \right| \leq K \frac{1}{(t-s)^{3/2}} \exp \left\{ -b \frac{(x - y)^2}{t-s} \right\}$, \tag{3.2}

3. $\left| \frac{\partial}{\partial t} G_{t-s}(x, y) \right| \leq K \frac{1}{|t-s|^2} \exp \left\{ -d \frac{(x - y)^2}{t-s} \right\}$, \tag{3.3}

4. $\sup_{s \in [0, T]} \int_0^s \int_0^1 |G_u(x, z) - G_u(y, z)|^p dz du \leq K|x - y|^{3-p} \frac{3}{2} < p < 3$, \tag{3.4}

5. $\sup_{x \in [0, 1]} \int_0^s \int_0^1 |G_{t-u}(x, z) - G_{s-u}(x, z)|^p dz du \leq K|t-s|^{(3-p)/2}, 1 < p < 3$, \tag{3.5}

6. $\sup_{x \in [0, 1]} \int_s^t \int_0^1 |G_u(x, z)|^p dz du \leq K|t-s|^{(3-p)/2}, 1 < p < 3$. \tag{3.6}

For any $\tilde{\alpha} = \frac{1-\gamma}{2\gamma}$ with $\gamma \in (1, \infty)$, $\alpha < \tilde{\alpha}$, there exists a constant $\tilde{K}(\alpha)$ such that for all $0 < s < t < T$, $x, y \in [0, 1]$,

7. $\int_0^s \int_0^1 |G_{t-u}(x, z) - G_{s-u}(y, z)|^2 dz du \leq \tilde{K}(\alpha) \rho((t, x), (s, y))^{2\alpha}$, \tag{3.7}
where $\rho$ is the Euclidean distance in $[0, T] \times [0, 1]$.

For any $v \in L^\infty([0, T], L^1([0, 1]))$, $t \in [0, T]$, $x \in [0, 1]$, let $J$ be a linear operator defined by

$$J(v)(t, x) := \int_0^t \int_0^1 H(r, t; x, y)v(r, y)dy dr,$$

with $H(t, s, x, y) = G_{t-s}(x, y), G_{t-s}^2(x, y)$ or $\partial_y G_{t-s}(x, y)$. Recall the following regularity of $J(v)(t, x)$ from Gyöngy [11].

**Lemma 3.1.** ([11] Lemma 3.1). For any $\rho \in [1, \infty], q \in [1, \rho], K := 1 + 1/\rho - 1/q$, it holds that $J$ is a bounded linear operator from $L^\gamma([0, T]; L^q[0, 1])$ into $C([0, T]; L^\rho[0, 1])$ for any $\gamma > 2K^{-1}$. Moreover, $J$ satisfies the following inequalities:

1. For every $t \in [0, T]$ and $\gamma > 2K^{-1}$,

$$\|J(v)(t, \cdot)\|_\rho \leq C_1 \int_0^t (t - r)^{(1/2)K^{-1}}\|v(r, \cdot)\|_q dr$$

$$\leq C_2 t^{K^{-1} - \frac{1}{\gamma}} \left( \int_0^t \|v(r, \cdot)\|_q^\gamma dr \right)^{\frac{1}{\gamma}}.$$

2. For every $0 < \alpha < K/2$ and $\gamma > (K/2 - \alpha)^{-1}$, there exists a constant $C > 0$ such that for all $0 \leq s \leq t \leq T$,

$$\|J(v)(t, \cdot) - J(v)(s, \cdot)\|_\rho \leq C(t - s)^\alpha \left( \int_0^t \|v(r, \cdot)\|_q^\gamma dr \right)^\frac{1}{\gamma}.$$

### 3.2. The convergence of $U^\varepsilon$

This section is concerned with the convergence of $U^\varepsilon$ to $U^0$ as $\varepsilon \to 0$.

For any $M > 0$, define the stopping time

$$\tau^{M, \varepsilon} := \inf \{ t \geq 0; \|U^\varepsilon(t, \cdot)\|_2 \geq M \}.$$

By (2.2), we know that

$$\lim_{M \to \infty} \sup_{\varepsilon \in [0, 1]} \mathbb{P} (\tau^{M, \varepsilon} \leq T) = 0. \quad (3.8)$$

**Proposition 3.2.** Under conditions (H1)-(H3), there exists some constant $C(M, L, \sigma, T)$ depending on $M, L, \sigma, T$ such that

$$\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau^{M, \varepsilon}]} \|U^\varepsilon(t, \cdot) - U^0(t, \cdot)\|_2^2 \right] \leq \varepsilon C(M, L, \sigma, T) \to 0, \quad \text{as } \varepsilon \to 0. \quad (3.9)$$

**Proof.** Since

$$U^\varepsilon(t, x) - U^0(t, x) = -\int_0^t \int_0^1 \partial_y G_{t-s}(x, y) \left[ g(U^\varepsilon) - g(U^0) \right](s, y) dy ds$$

$$+ \int_0^t \int_0^1 G_{t-s}(x, y) \left[ f(U^\varepsilon) - f(U^0) \right](s, y) dy ds$$

$$+ \sqrt{\varepsilon} \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(U^\varepsilon)(s, y) W(dy, ds)$$

$$=: I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon. \quad (3.10)$$
By (H3) and Cauchy-Schwarz’s inequality, we have that for any \((t, x) \in [0, T] \times [0, 1],\)

\[
\int_0^1 \left| g(U^\varepsilon)(s, y) - g(U^0)(s, y) \right| dy \\
\leq L \int_0^1 \left( 1 + |U^\varepsilon(s, y)| + |U^0(s, y)| \right) \cdot |U^\varepsilon(s, y) - U^0(s, y)| dy \\
\leq L \left( \int_0^1 \left( 1 + |U^\varepsilon(s, y)| + |U^0(s, y)| \right)^2 dy \cdot \int_0^1 |U^\varepsilon(s, y) - U^0(s, y)|^2 dy \right)^{\frac{1}{2}} \\
\leq C(L) \left( \int_0^1 \left( 1 + |U^\varepsilon(s, y)|^2 + |U^0(s, y)|^2 \right) dy \cdot \int_0^1 |U^\varepsilon(s, y) - U^0(s, y)|^2 dy \right)^{\frac{1}{2}}. \tag{3.11}
\]

Hence, applying Lemma 3.1 with \(\rho = 2, q = 1\) and by Cauchy-Schwarz’s inequality, we have

\[
\|I^\varepsilon_1(s, \cdot)\|_2^2 \\
\leq C \left( \int_0^s (s - r)^{-\frac{2}{3}} \left\| g(U^\varepsilon) - g(U^0)(r, \cdot) \right\|_1 dr \right)^2 \\
\leq C(L) \left( \int_0^s (s - r)^{-\frac{2}{3}} \left( 1 + \|U^\varepsilon(r, \cdot)\|_2^2 + \|U^0(r, \cdot)\|_2^2 \right)^{\frac{1}{2}} \cdot \|U^\varepsilon(r, \cdot) - U^0(r, \cdot)\|_2 dr \right)^2 \\
\leq C(L) \int_0^s (s - r)^{-\frac{2}{3}} \left( 1 + \|U^\varepsilon(r, \cdot)\|_2^2 + \|U^0(r, \cdot)\|_2^2 \right) dr \cdot \int_0^s (s - r)^{-\frac{2}{3}} \|U^\varepsilon(r, \cdot) - U^0(r, \cdot)\|_2^2 dr.
\]

First taking the supremum of time over \([0, t \wedge \tau^{M, \varepsilon}]\), and then taking expectation, by the definition of \(\tau^{M, \varepsilon}\), we obtain that

\[
\mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^{M, \varepsilon}]} \|I^\varepsilon_1(s, \cdot)\|_2^2 \right] \\
\leq C(L) \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^{M, \varepsilon}]} \int_0^s (s - r)^{-\frac{2}{3}} \left( 1 + \|U^\varepsilon(r, \cdot)\|_2^2 + \|U^0(r, \cdot)\|_2^2 \right) dr \right. \\
\times \left. \int_0^s (s - r)^{-\frac{2}{3}} \|U^\varepsilon(r, \cdot) - U^0(r, \cdot)\|_2^2 dr \right] \\
\leq C(M, L) \mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^{M, \varepsilon}]} \int_0^s (s - r)^{-\frac{2}{3}} \|U^\varepsilon(r, \cdot) - U^0(r, \cdot)\|_2^2 dr \right] \\
\leq C(M, L) \int_0^t (t - r)^{-\frac{2}{3}} \mathbb{E} \left[ \sup_{s \in [0, r \wedge \tau^{M, \varepsilon}]} \|U^\varepsilon(s, \cdot) - U^0(s, \cdot)\|_2^2 \right] dr, \tag{3.12}
\]

where we have used the Fubini’s theorem in the last inequality.
For the term $I_2^\epsilon$, similarly to the proof of (3.12), we have
\[
\mathbb{E} \left[ \sup_{s \in [0,t \wedge \tau^{M,\epsilon}]} \| I_2^\epsilon (s, \cdot) \|^2 \right] \\
\leq C(M, L) \int_0^t (t - r)^{-\frac{3}{4}} \mathbb{E} \left[ \sup_{s \in [0,r \wedge \tau^{M,\epsilon}]} \| U^\epsilon(s, \cdot) - U^0(s, \cdot) \|^2 \right] dr. \tag{3.13}
\]

For the term $I_3^\epsilon$, first taking the supremum of time over $[0, t]$, and then taking expectation, by Burkholder’s inequality for stochastic integrals against Brownian sheets (see [12]) and the boundedness of $\sigma$, we have
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} \| I_3^\epsilon(s, \cdot) \|^2 \right] \\
\leq \varepsilon \mathbb{E} \left[ \int_0^t \int_0^1 G^2_{t-s}(x,y) \sigma^2(U^\epsilon(s,y)) dy ds \right] \\
\leq \varepsilon C(M, \sigma). \tag{3.14}
\]

Putting (3.10), (3.12), (3.13), (3.14) together, we have
\[
\mathbb{E} \left[ \sup_{s \in [0,t \wedge \tau^{M,\epsilon}]} \| U^\epsilon(s, \cdot) - U^0(s, \cdot) \|^2 \right] \\
\leq \varepsilon C(M, \sigma) + C(M, L) \int_0^t (t - r)^{-\frac{3}{4}} \mathbb{E} \left[ \sup_{s \in [0,r \wedge \tau^{M,\epsilon}]} \| U^\epsilon(s, \cdot) - U^0(s, \cdot) \|^2 \right] dr.
\]

By Gronwall’s inequality (see [18]), we know that there exists a constant $C(M, L, \sigma, T)$ such that
\[
\mathbb{E} \left[ \sup_{s \in [0,t \wedge \tau^{M,\epsilon}]} \| U^\epsilon(s, \cdot) - U^0(s, \cdot) \|^2 \right] \leq \varepsilon C(M, L, \sigma, T).
\]

The proof is complete. \hfill \square

4. PROOF OF THE THEOREM 2.2

Proof of Theorem 2.2. Let
\[
V^\epsilon := (U^\epsilon - U^0)/\sqrt{\varepsilon}.
\]
We will prove that for any $\delta > 0$,
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{t \in [0,T]} \| V^\epsilon(t, \cdot) - V(t, \cdot) \|^2 \geq \delta \right) = 0. \tag{4.1}
\]

Recall the stopping time defined by
\[
\tau^{M,\epsilon} = \inf \{ t \geq 0; \| U^\epsilon(t, \cdot) \|^2 \geq M \}.
\]

Since
\[
\mathbb{P} \left( \sup_{t \in [0,T]} \| V^\epsilon(t, \cdot) - V(t, \cdot) \|^2 \geq \delta \right) \\
\leq \mathbb{P} \left( \sup_{t \in [0,T \wedge \tau^{M,\epsilon}]} \| V^\epsilon(t, \cdot) - V(t, \cdot) \|^2 \geq \delta \right) + \mathbb{P}(\tau^{M,\epsilon} \leq T),
\]

...
to prove (4.1), by (3.8), it is sufficient to prove that for any $M > 0$ large enough

$$
\lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{t \in [0, T \wedge \tau^{M, \varepsilon}]} \| V^\varepsilon(t, \cdot) - V(t, \cdot) \|_2 \geq \delta \right) = 0. \tag{4.2}
$$

By the definition of $V^\varepsilon$ and $V$, we have

$$
V^\varepsilon(t, x) - V(t, x) = \int_0^t \int_0^1 G_{t-s}(x, y) \left[ \frac{1}{\sqrt{\varepsilon}} \left( f(U^0 + \sqrt{\varepsilon} V^\varepsilon) - f(U^0) \right) - f'(U^0) V^\varepsilon \right] (s, y)dyds
- \int_0^t \int_0^1 \partial_y G_{t-s}(x, y) \left[ \frac{1}{\sqrt{\varepsilon}} \left( g(U^0) - g(U^0) \right) - g'(U^0) V^\varepsilon \right] (s, y)dyds
+ \int_0^t \int_0^1 G_{t-s}(x, y) \left[ \sigma(U^0) - \sigma(U^0) \right] (s, y)W(dy, ds)
=: A^\varepsilon_1(t, x) + A^\varepsilon_2(t, x) + A^\varepsilon_3(t, x). \tag{4.3}
$$

The first term $A^\varepsilon_1$ is further divided into two terms:

$$
A^\varepsilon_1(t, x) = \int_0^t \int_0^1 G_{t-s}(x, y) \left[ \frac{1}{\sqrt{\varepsilon}} \left( f(U^0 + \sqrt{\varepsilon} V^\varepsilon) - f(U^0) \right) - f'(U^0) V^\varepsilon \right] (s, y)dyds
+ \int_0^t \int_0^1 G_{t-s}(x, y) \left[ f'(U^0)(V^\varepsilon - V) \right] (s, y)dyds
=: A^\varepsilon_{11}(t, x) + A^\varepsilon_{12}(t, x). \tag{4.4}
$$

By Taylor’s formula, there exists a random field $\eta^\varepsilon$ taking values in $(0, 1)$ such that

$$
f(U^0 + \sqrt{\varepsilon} V^\varepsilon) - f(U^0) = \sqrt{\varepsilon} f' \left( U^0 + \sqrt{\varepsilon} \eta^\varepsilon V^\varepsilon \right) V^\varepsilon.
$$

Since $f'$ is Lipschitz continuous, we have

$$
\left| \frac{1}{\sqrt{\varepsilon}} \left[ f(U^0 + \sqrt{\varepsilon} V^\varepsilon) - f(U^0) \right] - f'(U^0) V^\varepsilon \right| \leq \sqrt{\varepsilon} K' |V^\varepsilon|^2.
$$

Applying Lemma 3.1 for $A^\varepsilon_{11}$ with $\rho = 2, q = 1$, we have

$$
\| A^\varepsilon_{11}(s, \cdot) \|_2 \leq \sqrt{\varepsilon} K' \int_0^s (s - r)^{-\frac{3}{2}} \| V^\varepsilon(r, \cdot) \|_2^2 dr.
$$

First taking the supremum of $s$ over $[0, t \wedge \tau^{M, \varepsilon}]$, and then taking expectation, by Proposition 3.2 we have

$$
\mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^{M, \varepsilon}]} \| A^\varepsilon_{11}(s, \cdot) \|_2 \right] \leq \sqrt{\varepsilon} K' \int_0^t (t - r)^{-\frac{3}{2}} \mathbb{E} \left[ \sup_{s \in [0, r \wedge \tau^{M, \varepsilon}]} \| V^\varepsilon(s, \cdot) \|_2^2 \right] dr
\leq \sqrt{\varepsilon} C(K', K', L). \tag{4.5}
$$
Applying Lemma 3.1 for $A_{12}^\varepsilon$ with $\rho = 2$, $q = 1$, by Cauchy-Schwarz’s inequality, we have

$$\|A_{12}^\varepsilon(s, \cdot)\|_2 \leq C \int_0^t (s - r)^{-\frac{3}{2}} \|f'(U^0)(V^\varepsilon - V)(r, \cdot)\|_4 dr$$

$$\leq C \int_0^t (s - r)^{-\frac{3}{2}} \|f'(U^0)(r, \cdot)\|_2 \cdot \|(V^\varepsilon - V)(r, \cdot)\|_2 dr.$$  

First taking the supremum of $s$ over $[0, t \wedge \tau^{M, \varepsilon}]$, and then taking expectation, we have

$$\mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^{M, \varepsilon}]} \|A_{12}^\varepsilon(s, \cdot)\|_2 \right]$$

$$\leq C \int_0^t (t - r)^{-\frac{3}{2}} \|f'(U^0)(r, \cdot)\|_2 \cdot \mathbb{E} \left[ \sup_{s \in [0, r \wedge \tau^{M, \varepsilon}]} \|(V^\varepsilon - U)(s, \cdot)\|_2 \right] dr. \quad (4.6)$$

Putting (4.5) and (4.6), we have

$$\mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^{M, \varepsilon}]} \|A_{12}^\varepsilon(s, \cdot)\|_2 \right]$$

$$\leq \sqrt{\varepsilon} C_1 + C_2 \int_0^t (t - r)^{-\frac{3}{2}} \|f'(U^0)(r, \cdot)\|_2 \cdot \mathbb{E} \left[ \sup_{s \in [0, r \wedge \tau^{M, \varepsilon}]} \|(V^\varepsilon - V)(s, \cdot)\|_2 \right] dr. \quad (4.7)$$

For the term $A_2^\varepsilon$, similarly to the proof of (4.7), we have

$$\mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^{M, \varepsilon}]} \|A_2^\varepsilon(s, \cdot)\|_2 \right]$$

$$\leq \sqrt{\varepsilon} C_1 + C_2 \int_0^t (t - r)^{-\frac{3}{2}} \|g'(U^0)(r, \cdot)\|_2 \cdot \mathbb{E} \left[ \sup_{s \in [0, r \wedge \tau^{M, \varepsilon}]} \|(V^\varepsilon - V)(s, \cdot)\|_2 \right] dr. \quad (4.8)$$

For the term $A_3^\varepsilon$, first taking the supremum of time over $[0, t \wedge \tau^{M, \varepsilon}]$, and then taking expectation, by Burkholder’s inequality for stochastic integrals against Brownian sheets (see [14]), the Lipschitz continuity of $\sigma$ and Proposition 3.2, we have

$$\mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^{M, \varepsilon}]} \|A_3^\varepsilon(s, \cdot)\|_2 \right]$$

$$\leq C \left( \int_0^t (t - r)^{-\frac{3}{2}} \mathbb{E} \left[ \sup_{s \in [0, r \wedge \tau^{M, \varepsilon}]} \|(U^\varepsilon - U^0)(s, \cdot)\|_2 \right] dr \right)^{\frac{1}{2}}$$

$$\leq \sqrt{\varepsilon} C(M, L, \sigma, T). \quad (4.9)$$

Putting (4.3), (4.7)-(4.9) together, we have

$$\mathbb{E} \left[ \sup_{s \in [0, t \wedge \tau^{M, \varepsilon}]} \|(V^\varepsilon - V)(s, \cdot)\|_2 \right]$$

$$\leq C_2 \int_0^t (t - r)^{-\frac{3}{2}} \left( \|f'(U^0)(r, \cdot)\|_2 + \|g'(U^0)(r, \cdot)\|_2 \right) \mathbb{E} \left[ \sup_{s \in [0, r \wedge \tau^{M, \varepsilon}]} \|(V^\varepsilon - V)(s, \cdot)\|_2 \right] dr$$

$$+ \sqrt{\varepsilon} C_1(M, L, \sigma, T).$$
By Gronwall’s inequalities (Theorem 1), (2.4) and (2.6), we have
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{t \in [0, T \wedge T, \varepsilon]} \| (V^\varepsilon - V)(t, \cdot) \|_2 \right] = 0,
\]
which implies (4.2).

The proof is complete. \qed

5. Proof of the Theorem 2.3

5.1. Weak convergence approach in LDP. First, recall the definition of large deviation principle (c.f. [5]). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with an increasing family \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) of the sub-\(\sigma\)-fields of \(\mathcal{F}\) satisfying the usual conditions. Let \(\mathcal{E}\) be a Polish space with the Borel \(\sigma\)-field \(\mathcal{B}(\mathcal{E})\).

**Definition 5.1.** A function \(I : \mathcal{E} \to [0, \infty]\) is called a rate function on \(\mathcal{E}\), if for each \(M < \infty\), the level set \(\{x \in \mathcal{E} : I(x) \leq M\}\) is a compact subset of \(\mathcal{E}\). A family of positive numbers \(\{h(\varepsilon)\}_{\varepsilon > 0}\) is called a speed function if \(h(\varepsilon) \to +\infty\) as \(\varepsilon \to 0\).

**Definition 5.2.** \(\{X^\varepsilon\}\) is said to satisfy the large deviation principle on \(\mathcal{E}\) with rate function \(I\) and with speed function \(\{h(\varepsilon)\}_{\varepsilon > 0}\), if the following two conditions hold:

(a) (Upper bound) For each closed subset \(F\) of \(\mathcal{E}\),
\[
\limsup_{\varepsilon \to 0} \frac{1}{h(\varepsilon)} \log \mathbb{P}(X^\varepsilon \in F) \leq -\inf_{x \in F} I(x);
\]

(b) (Lower bound) For each open subset \(G\) of \(\mathcal{E}\),
\[
\liminf_{\varepsilon \to 0} \frac{1}{h(\varepsilon)} \log \mathbb{P}(X^\varepsilon \in G) \geq -\inf_{x \in G} I(x).
\]

The Cameron-Martin space associated with the Brownian sheet \(\{W(t, x); t \in [0, T], x \in [0, 1]\}\) is given by
\[
\mathcal{H} := \left\{ h(t, x) = \int_0^T \int_0^x \dot{h}(s, z) dz ds; \dot{h} \in L^2([0, T] \times [0, 1]) \right\}.
\]

The space \(\mathcal{H}\) is a Hilbert space with inner product
\[
\langle h_1, h_2 \rangle_{\mathcal{H}} := \int_0^T \int_0^1 \dot{h}_1(s, z) \dot{h}_2(s, z) dz ds.
\]
The Hilbert space \(\mathcal{H}\) is endowed with the norm \(\|h\|_{\mathcal{H}} := (\langle h, h \rangle_{\mathcal{H}})^{1/2}\).

Let \(\mathcal{A}\) denote the class of real-valued \(\{\mathcal{F}_t\}\)-predictable processes \(\phi\) belonging to \(\mathcal{H}\) a.s., and let
\[
S_N := \{h \in \mathcal{H}; \|h\|_{\mathcal{H}} \leq N\}.
\]
The set \(S_N\) endowed with the weak topology is a Polish space. Define
\[
\mathcal{A}_N := \{\phi \in \mathcal{A}; \phi(\omega) \in S_N, \mathbb{P}\text{-a.s.}\}.
\]
Recall the following result from [3].
Theorem 5.1. (Theorem 6). For any \( \varepsilon > 0 \), let \( \Gamma^\varepsilon \) be a measurable mapping from \( C([0, T]; \mathbb{R}) \) into \( \mathcal{E} \). Suppose that \( \{\Gamma^\varepsilon\}_{\varepsilon > 0} \) satisfies the following assumptions: there exists a measurable map \( \Gamma^0 : C([0, T]; \mathbb{R}) \rightarrow \mathcal{E} \) such that

(a) for any \( N < +\infty \) and family \( \{h^\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_N \) satisfying that \( h^\varepsilon \) converge in distribution as \( S_N \)-valued random elements to \( h \) as \( \varepsilon \to 0 \), \( \Gamma^0 \left( \int_0^T \int_0^1 \hat{h}(s, y)dyds \right) \) converges in distribution to \( \Gamma^0 \left( \int_0^T \int_0^1 \hat{h}(s, y)dyds \right) \) as \( \varepsilon \to 0 \);

(b) for every \( N < +\infty \), the set \( \left\{ \Gamma^0 \left( \int_0^T \int_0^1 \hat{h}(s, y)dyds \right); h \in S_N \right\} \) is a compact subset of \( \mathcal{E} \).

Then the family \( \{\Gamma^\varepsilon(W)\}_{\varepsilon > 0} \) satisfies a large deviation principle in \( \mathcal{E} \) with the rate function \( I \) given by

\[
I(g) := \inf_{\left\{ h \in \mathcal{H}; g = \Gamma^0 \left( \int_0^T \int_0^1 \hat{h}(s, y)dyds \right) \right\}} \frac{1}{2} \|h\|_H^2, \quad g \in \mathcal{E},
\]

with the convention \( \inf \emptyset = +\infty \).

5.2. The skeleton equation. The purpose of this part is to study the skeleton equation, which will be used in the weak convergence approach.

Recall the skeleton equation defined in (2.9). Using the same strategy in the proof of the existence and uniqueness for the solution to Eq. (1.1), we know that

Proposition 5.2. Under conditions (H1)-(H4), there exists a unique solution to Eq. (2.9) satisfying that

\[
\sup_{h \in \mathcal{H}} \sup_{t \in [0, T]} \left\| X^h(t, \cdot) \right\|_2 < +\infty.
\]

For any \( h \in \mathcal{H} \), set

\[
\Gamma^0 \left( \int_0^T \int_0^1 \hat{h}(s, y)dyds \right) := X^h,
\]

where \( X^h \) is the solution of (2.9).

Theorem 5.3. Under conditions (H1)-(H4), the mapping \( h : S_N \rightarrow X^h \in C([0, T]; L^2([0, 1])) \) is continuous with respect to the weak topology.

Proof. Let \( \{h, (h_n)_{n \geq 1}\} \subset S_N \) such that for any \( g \in \mathcal{H} \),

\[
\lim_{n \to \infty} \langle h_n - h, g \rangle_\mathcal{H} = 0.
\]

We need to prove that

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \left\| X^{h_n}(t, \cdot) - X^h(t, \cdot) \right\|_2 = 0.
\]
Thus, the function \( (s, y) \mapsto f'(s, y, U^0(s, y)) (X^{h_n}(s, y) - X^h(s, y)) \) belongs to \( L^\infty([0, T]; L^1([0, 1])) \). Applying Lemma 3.1 with \( \rho = 2, q = 1 \) and by the Hölder inequality, we have

\[
\|I_1^n(t, \cdot)\|_2 \leq C \int_0^t (t - s)^{-\frac{1}{4}} \cdot \|X^{h_n}(s, \cdot) - X^h(s, \cdot)\|_1 ds \\
\leq C \int_0^t (t - s)^{-\frac{1}{4}} \cdot \|X^{h_n}(s, \cdot) - X^h(s, \cdot)\|_2 ds.
\]

Similarly, we obtain that

\[
\|I_2^n(t, \cdot)\|_2 \leq C \int_0^t (t - s)^{-\frac{1}{4}} \cdot \|X^{h_n}(s, \cdot) - X^h(s, \cdot)\|_2 ds.
\]

Since \( \sigma \) is bounded, for any fixed \((t, x) \in [0, T] \times [0, 1]\), the function \( G_{t-x}(s, y) \sigma(s, y, U^0(s, y)) : (s, y) \in [0, t] \times [0, 1] \to \mathbb{R} \) belongs to \( L^2([0, T] \times [0, 1]; \mathbb{R}) \). As \( \hat{h}_n \to \hat{h} \) weakly in \( L^2([0, T] \times [0, 1]; \mathbb{R}) \), it holds that

\[
I_3^n(t, x) = \int_0^t \int_0^1 G_{t-x}(s, y) \sigma(s, y, U^0(s, y)) \left( \hat{h}_n(s, y) - \hat{h}(s, y) \right) dyds \to 0. \]
For any $0 \leq s \leq t \leq T$, applying formulas (3.1), (3.5) and (3.6), by the boundedness of $\sigma$ and Hölder’s inequality, we obtain that

$$|I_3^n(t, x) - I_3^n(s, x)|$$

$$\leq \left| \int_0^s \int_0^1 (G_{t-u}(x, y) - G_{s-u}(x, y)) \sigma(u, y, U^0(u, y)) (\dot{h}_n(u, y) - \dot{h}(u, y)) dy du \right|$$

$$+ \left| \int_s^t \int_0^1 G_{t-u}(x, y) \sigma(u, y, U^0(u, y)) (\dot{h}_n(u, y) - \dot{h}(u, y)) dy du \right|$$

$$\leq C(N, \sigma) \left( \left( \int_0^s \int_0^1 |G_{t-u}(x, y) - G_{s-u}(x, y)|^2 dy du \right)^{\frac{1}{2}} + \left( \int_s^t \int_0^1 G_{t-u}(x, y)^2 dy du \right)^{\frac{1}{2}} \right)$$

$$\leq C(N, \sigma) (t - s)^{\frac{1}{2}}.$$ 

Then

$$\|I_3^n(t, \cdot) - I_3^n(s, \cdot)\|_2 \leq C(N, \sigma) (t - s)^{\frac{1}{2}}.$$ 

Particularly, taking $s = 0$, we obtain that

$$\|I_3^n(t, \cdot)\|_2 \leq C(N, \sigma) t^{\frac{1}{2}}.$$ 

(5.10)

Hence, the functions $\{I_3^n\}_{n \geq 1}$ are uniformly bounded and equi-continuous in $C([0, T]; L^2([0, 1]))$. According to Arzela-Ascoli theorem, the functions $t \mapsto \{I_3^n(t, \cdot)\}_{n \geq 1}$ is pre-compact in $C([0, T]; L^2([0, 1]))$. Thus, by (5.9), we obtain that

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \|I_3^n(t, \cdot)\|_2 = 0.$$ 

(5.11)

Set $\zeta^n(t) = \sup_{0 \leq s \leq t} \|X^{h_n}(s, \cdot) - X^h(s, \cdot)\|_2$. By (5.6)-(5.8), we have

$$\zeta^n(t) \leq C \int_0^t (t - s)^{-\frac{3}{2}} \zeta^n(s) ds + \sup_{s \in [0, t]} \|I_3^n(s, \cdot)\|_2.$$ 

Hence, by a generalized Gronwall lemma (eg. [18, Theorem 1]), we have

$$\zeta^n(t) \leq C \sup_{s \in [0, t]} \|I_3^n(s, \cdot)\|_2,$$ 

which, together with (5.11), implies the desired estimate (5.3).

The proof is complete. \(\square\)

5.3. The proof of Theorem 2.3. Recall that $X^{\varepsilon} = (U^{\varepsilon} - U^0)/(\sqrt{\varepsilon}\lambda(\varepsilon))$. By (2.1) and (2.3), we know that

$$X^{\varepsilon}(t, x)$$

$$= \frac{1}{\lambda(\varepsilon)} \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(s, y, U^0(s, y) + \sqrt{\varepsilon}\lambda(\varepsilon) X^{\varepsilon}(s, y)) W(dy, ds)$$

$$+ \frac{1}{\sqrt{\varepsilon}\lambda(\varepsilon)} \int_0^t \int_0^1 G_{t-s}(x, y) \left( f(s, y, U^0(s, y) + \sqrt{\varepsilon}\lambda(\varepsilon) X^{\varepsilon}(s, y)) - f(s, y, U^0(s, y)) \right) dy ds$$

$$- \frac{1}{\sqrt{\varepsilon}\lambda(\varepsilon)} \int_0^t \int_0^1 \partial_y G_{t-s}(x, y) \left( g(s, y, U^0(s, y) + \sqrt{\varepsilon}\lambda(\varepsilon) X^{\varepsilon}(s, y)) - g(s, y, U^0(s, y)) \right) dy ds.$$ 

(5.12)
This equation admits a unique strong solution

\[ X^\varepsilon := \Gamma^\varepsilon(W), \quad (5.13) \]

where \( \Gamma^\varepsilon \) stands for the solution functional from \( C([0, T] \times [0, 1]; \mathbb{R}) \) into \( C([0, T] \times [0, 1]; \mathbb{R}) \).

The following lemma is a direct consequence of Girsanov’s theorem, refer to [7, Theorem 3.2].

**Lemma 5.4.** For every fixed \( N \in \mathbb{N} \), let \( v \in A_N \) and \( \Gamma^\varepsilon \) be given by (5.13). Then \( X^{\varepsilon,v} := \Gamma^\varepsilon(W + \lambda(\varepsilon) v) \in C([0, T]; L^2([0, 1])) \) solves the following equation:

\[
X^{\varepsilon,v}(t, x) = \frac{1}{\lambda(\varepsilon)} \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(s, y, U^0(s, y) + \sqrt{\varepsilon} \lambda(\varepsilon) X^{\varepsilon,v}(s, y)) W(dy, ds) \\
+ \frac{1}{\sqrt{\varepsilon} \lambda(\varepsilon)} \int_0^t \int_0^1 G_{t-s}(x, y) \left(f(s, y, U^0(s, y) + \sqrt{\varepsilon} \lambda(\varepsilon) X^{\varepsilon,v}(s, y)) - f(s, y, U^0(s, y))\right) dy ds \\
- \frac{1}{\sqrt{\varepsilon} \lambda(\varepsilon)} \int_0^t \int_0^1 \partial_y G_{t-s}(x, y) \left(g(s, y, U^0(s, y) + \sqrt{\varepsilon} \lambda(\varepsilon) X^{\varepsilon,v}(s, y)) - g(s, y, U^0(s, y))\right) dy ds \\
+ \int_0^t \int_0^1 G_{t-s}(x, y) \left(\sigma(s, y, U^0(s, y) + \sqrt{\varepsilon} \lambda(\varepsilon) X^{\varepsilon,v}(s, y)) \dot{v}(s, y)\right) dy ds. \quad (5.14) 
\]

Furthermore,

\[
\lim_{M \to \infty} \sup_{\varepsilon \leq 1} \sup_{v \in A_N} \mathbb{P}\left( \sup_{t \in [0, T]} \|X^{\varepsilon,v}(t, \cdot)\|_2^2 \geq M \right) = 0. \quad (5.15) 
\]

We are now ready to state our main result. Recall the mapping \( \Gamma_0 \) given by (5.4). For any \( g \in C([0, T]; L^2([0, 1])) \), let

\[
I(g) := \inf_{\{h \in \mathcal{H} : g = \Gamma_0(\int_0^T h(s, y) dy ds)\}} \frac{1}{2} \|h\|^2_{\mathcal{H}}. \quad (5.16) 
\]

**Proof.** According to Theorem 5.1, we need to prove that the following two conditions are fulfilled:

(a) the set \( \{X^h : h \in S_N\} \) is a compact set of \( C([0, T]; L^2([0, 1])) \), where \( X^h \) is the solution of Eq. (2.9).

(b) for any family \( \{v^\varepsilon : \varepsilon > 0\} \subset A_N \) which converges in distribution as \( \varepsilon \to 0 \) to \( v \in A_N \), as \( S_N \)-valued random variables, we have

\[
\lim_{\varepsilon \to 0} X^{\varepsilon,v^\varepsilon} = X^v \quad \text{in distribution},
\]

as \( C([0, T]; L^2([0, 1])) \)-valued random variables, where \( X^v \) denotes the solution of Eq. (2.9) corresponding to the \( S_N \)-valued random variable \( v \) (instead of a deterministic function \( h \)).

Condition (a) follows from the continuity of the mapping \( h : S_N \to X^h \in C([0, T]; L^2([0, 1])) \), which has been established in Theorem 5.3. The verification of condition (b) will be given by Proposition 5.5 below.

The proof is complete. \( \Box \)
Proposition 5.5. Assume (H1)-(H4). For every fixed $N \in \mathbb{N}$, let $v^\varepsilon, v \in \mathcal{A}_N$ be such that $v^\varepsilon$ converges in distribution to $v$ as $\varepsilon \to 0$. Then

$$\Gamma^\varepsilon (W + \lambda(\varepsilon)v^\varepsilon)$$

converges in distribution to $\Gamma^0(v)$, in $C([0,T]; L^2([0,1]))$ as $\varepsilon \to 0$.

Proof. By Skorokhod representation theorem, there exist a probability basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \bar{\mathbb{P}})$, and, on this basis, a sequence of independent Brownian sheets $\bar{W} = (\bar{W}_k)_{k \geq 1}$ and also a family of $\mathcal{F}_t$-predictable processes $\{\bar{v}^\varepsilon; \varepsilon > 0\}$, $\bar{v}$ belonging to $L^2(\Omega \times [0,T]; \mathcal{H})$ taking values on $\mathcal{S}_N$, $\bar{\mathbb{P}}$-a.s., such that the joint law of $(v^\varepsilon, v, W)$ under $\mathbb{P}$ coincides with that of $(\bar{v}^\varepsilon, \bar{v}, \bar{W})$ under $\bar{\mathbb{P}}$ and

$$\lim_{\varepsilon \to 0} \langle \bar{v}^\varepsilon - \bar{v}, g \rangle_{\mathcal{H}} = 0, \quad \forall g \in \mathcal{H}, \bar{\mathbb{P}}\text{-a.s.}$$

Let $\bar{X}^{\varepsilon,\bar{v}^\varepsilon}$ be the solution to a similar equation as (5.14) replacing $v$ by $\bar{v}^\varepsilon$ and $W$ by $\bar{W}$. Thus, to prove this proposition, it is sufficient to prove that

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \|\bar{X}^{\varepsilon,\bar{v}^\varepsilon} - \bar{X}^\varepsilon\|_2 = 0, \quad \text{in probability.}$$

(5.17)

From now on, we drop the bars in the notation for the sake of simplicity, and we denote

$$Y^{\varepsilon,v^\varepsilon,v} := X^{\varepsilon,v^\varepsilon} - X^v.$$

Notice that

$$X^{\varepsilon,v^\varepsilon}(t,x) - X^v(t,x)$$

$$= \frac{1}{\lambda(\varepsilon)} \int_0^t \int_0^1 G_{t-s}(x,y) \sigma(s,y,U^0(s,y)) + \sqrt{\varepsilon \lambda(\varepsilon)} X^{\varepsilon,v^\varepsilon}(s,y)) W(dy,ds)$$

$$+ \int_0^t \int_0^1 G_{t-s}(x,y) \left[ \frac{1}{\sqrt{\varepsilon \lambda(\varepsilon)}} (f(s,y,U^0(s,y)) + \sqrt{\varepsilon \lambda(\varepsilon)} X^{\varepsilon,v^\varepsilon}(s,y)) - f(s,y,U^0(s,y))) \right] dyds$$

$$- f'(s,y,U^0(s,y)) X^v(t,x) dyds$$

$$- \int_0^t \int_0^1 \partial_y G_{t-s}(x,y) \left[ \frac{1}{\sqrt{\varepsilon \lambda(\varepsilon)}} (g(s,y,U^0(s,y)) + \sqrt{\varepsilon \lambda(\varepsilon)} X^{\varepsilon,v^\varepsilon}(s,y)) - g(s,y,U^0(s,y))) \right] dyds$$

$$- g'(s,y,U^0(s,y)) X^v(t,x) dyds$$

$$+ \int_0^t \int_0^1 G_{t-s}(x,y) \left[ \sigma(s,y,U^0(s,y)) + \sqrt{\varepsilon \lambda(\varepsilon)} X^{\varepsilon,v^\varepsilon}(s,y)) \right] dyds$$

$$- \sigma(s,y,U^0(s,y)) \hat{v}(s,y) dyds$$

$$= : A_1(t,x) + A_2(t,x) + A_3(t,x) + A_4(t,x).$$

(5.18)
Since $\sigma$ is bounded, by Burkholder's inequality for stochastic integrals against Brownian sheets (see [12]), we have
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|A_2^\varepsilon(t, \cdot)\|_2^2 \right] \leq \frac{1}{\lambda^2(\varepsilon)} C(\sigma, T) \int_0^T \int_0^1 G_{t-s}^2(x, y) dy ds \to 0, \text{ as } \varepsilon \to 0. \quad (5.19)
\]

**Terms $A_2^\varepsilon$ and $A_3^\varepsilon$.** Notice that
\[
A_2^\varepsilon(t, x) = \int_0^t \int_0^1 G_{t-s}(x, y) \left[ \frac{1}{\sqrt{\varepsilon \lambda(\varepsilon)}} \left( f(s, y, U^0(s, y) + \sqrt{\varepsilon \lambda(\varepsilon)} X^{\varepsilon, \nu}(s, y)) - f(s, y, U^0(s, y)) \right) 
- f'(s, y, U^0(s, y)) X^{\varepsilon, \nu}(s, y) \right] dy ds 
+ \int_0^t \int_0^1 G_{t-s}(x, y) \left[ f'(s, y, U^0(s, y))(X^{\varepsilon, \nu}(s, y) - X^v(s, y)) \right] dy ds 
=: A_{21}^\varepsilon(t, x) + A_{22}^\varepsilon(t, x). \quad (5.20)
\]

By Taylor’s formula, there exists a random field $\eta^\varepsilon(s, y)$ taking values in $(0, 1)$ such that
\[
\begin{align*}
&f\left(s, y, U^0(s, y) + \sqrt{\varepsilon \lambda(\varepsilon)} X^{\varepsilon, \nu}(s, y)\right) - f\left(s, y, U^0(s, y)\right) \\
= &\sqrt{\varepsilon \lambda(\varepsilon)} f'(s, y, U^0(s, y)) + \sqrt{\varepsilon \lambda(\varepsilon)} \eta^\varepsilon(s, y) X^{\varepsilon, \nu}(s, y) X^{\varepsilon, \nu}(s, y).
\end{align*}
\]

Since $f'$ is also Lipschitz continuous, we have
\[
\left| f'(s, y, U^0(s, y) + \sqrt{\varepsilon \lambda(\varepsilon)} \eta^\varepsilon(s, y) X^{\varepsilon, \nu}(s, y)) X^{\varepsilon, \nu}(s, y) - f'(s, y, U^0(s, y)) X^{\varepsilon, \nu}(s, y) \right| 
\leq K \sqrt{\varepsilon \lambda(\varepsilon)} \left| X^{\varepsilon, \nu}(s, y) \right|^2.
\]

Define the stopping time
\[
\tau_{M, \varepsilon} := \inf \left\{ t \geq 0; \|X^{\varepsilon, \nu}(t, \cdot)\|_2 \vee \|X^v(t, \cdot)\|_2 \geq M \right\},
\]
where $M$ is some constant large enough.

Applying Lemma 3.3 for $A_{21}^\varepsilon$ with $\rho = 2, q = 1$, we have that for all $t \in [0, T],$
\[
\|A_{21}^\varepsilon(t, \cdot)\|_2 \leq \sqrt{\varepsilon \lambda(\varepsilon)} C(K) \int_0^t (t - s)^{-\frac{3}{2}} \left\| X^{\varepsilon, \nu}(s, \cdot) \right\|_2^2 ds.
\]

First taking the supremum of time over $[0, t \wedge \tau_{M, \varepsilon}], \text{ and then taking expectation, we obtain that}$
\[
\begin{align*}
&\mathbb{E}\left[ \sup_{s \in [0, t \wedge \tau_{M, \varepsilon}]} \|A_{21}^\varepsilon(s, \cdot)\|_2 \right] \\
&\leq \sqrt{\varepsilon \lambda(\varepsilon)} C(K) \int_0^t (t - s)^{-\frac{3}{2}} \mathbb{E}\left[ \sup_{0 \leq r \leq t \wedge \tau_{M, \varepsilon}} \|X^{\varepsilon, \nu}(r, \cdot)\|_2^2 \right] ds \\
&\leq \sqrt{\varepsilon \lambda(\varepsilon)} C(K, T, M). \quad (5.21)
\end{align*}
\]
For the term $A_{22}^\varepsilon$, similarly to the proof of (5.7), we have

$$
E \left[ \sup_{s \in [0, t \wedge T, M, \varepsilon]} \| A_{22}^\varepsilon (s, \cdot) \|_2 \right] 
\leq C \int_0^t (t - s)^{-\frac{3}{4}} \cdot E \left[ \sup_{0 \leq r \leq s \wedge T, M, \varepsilon} \| X^{\varepsilon, v^\varepsilon} (r, \cdot) - X^v (r, \cdot) \|_2 \right] ds. \tag{5.22}
$$

Putting (5.21) and (5.22) together, we have

$$
E \left[ \sup_{s \in [0, t \wedge T, M, \varepsilon]} \| A_{22}^\varepsilon (s, \cdot) \|_2 \right] 
\leq \sqrt{\varepsilon} \lambda (\varepsilon) C (K, T, M) + C \int_0^t (t - s)^{-\frac{3}{4}} \cdot E \left[ \sup_{0 \leq r \leq s \wedge T, M, \varepsilon} \| X^{\varepsilon, v^\varepsilon} (r, \cdot) - X^v (r, \cdot) \|_2 \right] ds. \tag{5.23}
$$

Similarly to the proof of (5.23), we obtain the following estimate for $A_{32}^\varepsilon$:

$$
E \left[ \sup_{s \in [0, t \wedge T, M, \varepsilon]} \| A_{32}^\varepsilon (s, \cdot) \|_2 \right] 
\leq \sqrt{\varepsilon} \lambda (\varepsilon) C (K, T, M) + C \int_0^t (t - s)^{-\frac{3}{4}} \cdot E \left[ \sup_{0 \leq r \leq s \wedge T, M, \varepsilon} \| X^{\varepsilon, v^\varepsilon} (r, \cdot) - X^v (r, \cdot) \|_2 \right] ds. \tag{5.24}
$$

**Term $A_{41}^\varepsilon$.** Notice that

$$
A_{41}^\varepsilon (t, x) 
= \int_0^t \int_0^1 G_{t-s}(x, y) \left[ \sigma (s, y, U^0 (s, y) + \sqrt{\varepsilon} \lambda (\varepsilon) X^{\varepsilon, v^\varepsilon} (s, y)) (v^\varepsilon (s, y) - v (s, y)) \right] dy ds 
+ \int_0^t \int_0^1 G_{t-s}(x, y) \left\{ \left[ \sigma (s, y, U^0 (s, y) + \sqrt{\varepsilon} \lambda (\varepsilon) X^{\varepsilon, v^\varepsilon} (s, y)) - \sigma (s, y, U^0 (s, y)) \right] \dot{v} (s, y) \right\} dy ds 
=: A_{41}^\varepsilon (t, x) + A_{42}^\varepsilon (t, x). \tag{5.25}
$$

Using the argument as that in the proof of (5.11), we obtain that

$$
\lim_{\varepsilon \to 0} E \left[ \sup_{t \in [0, T]} \| A_{41}^\varepsilon (t, \cdot) \|_2 \right] = 0. \tag{5.26}
$$

By the Lipschitz continuity of $\sigma$, we know that

$$
|A_{42}^\varepsilon (t, x)| \leq \sqrt{\varepsilon} \lambda (\varepsilon) K \int_0^t \int_0^1 G_{t-s}(x, y) |X^{\varepsilon, v^\varepsilon} (s, y)| \cdot |\dot{v} (s, y)| dy ds.
$$
Applying Lemma 3.1 for $A^{\varepsilon}_{42}$ with $\rho = 2, q = 1$ and by Hölder’s inequality, we obtain that for all $t \in [0, T]$, 
\[
\|A^{\varepsilon}_{42}(t, \cdot)\|_2 \leq \sqrt{\varepsilon} \lambda(\varepsilon) C(K) \int_0^t (t - r)^{-\frac{3}{4}} \|X^{\varepsilon, v} (r, \cdot) \dot{v}(r, \cdot)\|_1 dr \leq \sqrt{\varepsilon} \lambda(\varepsilon) C(K) \int_0^t (t - r)^{-\frac{3}{4}} \|X^{\varepsilon, v} (r, \cdot)\|_2 \cdot \|\dot{v}(r, \cdot)\|_2 dr.
\]
First taking the supremum of $t$ over $[0, T \wedge \tau_{M, \varepsilon}]$, and then taking expectation, we obtain that 
\[
E \left[ \sup_{t \in [0, T]} \|A^{\varepsilon}_{42}(t \wedge \tau_{M, \varepsilon}, \cdot)\|_2 \right] \leq \sqrt{\varepsilon} \lambda(\varepsilon) C(K) \int_0^t (t - r)^{-\frac{3}{4}} E \left[ \sup_{0 \leq r \leq s \leq \tau_{M, \varepsilon}} \|X^{\varepsilon, v} (r, \cdot)\|_2 \cdot \|\dot{v}(r, \cdot)\|_2 \right] dr \leq \sqrt{\varepsilon} \lambda(\varepsilon) C(K, T, N, M). \quad (5.27)
\]
According to (5.25)–(5.27), we have that 
\[
\lim_{\varepsilon \to 0} E \left[ \sup_{t \in [0, T]} \|A^{\varepsilon}_{42}(t \wedge \tau_{M, \varepsilon}, \cdot)\|_2 \right] = 0. \quad (5.28)
\]
Putting (5.18), (5.19), (5.23), (5.24) and (5.28) together and by Gronwall’s inequality ([18, Theorem 1]), we have
\[
\lim_{\varepsilon \to 0} E \left[ \sup_{t \in [0, T]} \|X^{\varepsilon, v} (s \wedge \tau_{M, \varepsilon}, \cdot) - X^{v}(t \wedge \tau_{M, \varepsilon}, \cdot)\|_2 \right] = 0. \quad (5.29)
\]
By Chebychev’s inequality, we have
\[
\lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \|X^{\varepsilon, v} (s \wedge \tau_{M, \varepsilon}, \cdot) - X^{v}(t \wedge \tau_{M, \varepsilon}, \cdot)\|_2 = 0, \text{ in probability.}
\]
Letting $M \to \infty$, by (5.15) we get the desired result (5.17).

The proof is complete.

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