A Note on the Order of Iterated Line Digraphs

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Abstract: Given a digraph G, we propose a new method to find the recurrence equation for the number of vertices \( n_k \) of the \( k \)-iterated line digraph \( L_k(G) \), for \( k \geq 0 \), where \( L^0(G) = G \). We obtain this result by using the minimal polynomial of a quotient digraph \( \pi(G) \) of \( G \). © 2016 Wiley Periodicals, Inc. J. Graph Theory 85: 395–399, 2017

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1. PRELIMINARIES

In this section, we recall some basic notation and results concerning digraphs and their spectra. A digraph \( G = (V, E) \) consists of a (finite) set \( V = V(G) \) of vertices and a set \( E = E(G) \) of arcs (directed edges) between vertices of \( G \). As the initial and final vertices of an arc are not necessarily different, the digraphs may have loops (arcs from a vertex to itself), and multiple arcs, that is, there can be more than one arc from each vertex to any other. If \( a = (u, v) \) is an arc from \( u \) to \( v \), then vertex \( u \) (and arc \( a \)) is adjacent to vertex \( v \), and vertex \( v \) (and arc \( a \)) is adjacent from \( u \). Let \( G^+(v) \) and \( G^-(v) \) denote the set of arcs adjacent from and to vertex \( v \), respectively. A digraph \( G \) is \( d \)-regular if \( |G^+(v)| = |G^-(v)| = d \) for all \( v \in V \).

In the line digraph \( L(G) \) of a digraph \( G \), each vertex of \( L(G) \) represents an arc of \( G \), that is, \( V(L(G)) = \{uv | (u, v) \in E(G)\} \); and vertices \( uv \) and \( wz \) of \( L(G) \) are adjacent if and only if \( v = w \), namely, when arc \((u, v)\) is adjacent to arc \((w, z)\) in \( G \). For \( k \geq 0 \), we consider the sequence of line digraph iterations \( L^0(G) = G, L(G), L^2(G), \ldots, L^K(G) = L(L^{K-1}(G)) \). \( L^0(G) = G \) is the initial digraph, and \( L(K+1)(G) = L^K(G) \) is the \( K \)-th iterate of \( G \).

Recall also that a digraph \( G \) is strongly connected if there is a (directed) walk between every pair of its vertices. If \( G \) is strongly connected, different from a directed cycle, and it has diameter \( D \), then its line digraph \( L^D(G) \) has diameter \( D + k \). See Fiol et al. [4] for more details. The interest of the line digraph technique is that it allows us to obtain digraphs with small diameter and large connectivity. For a comparison between the line digraph technique and other techniques to obtain digraphs with minimum diameter see Miller et al. [7]. Since these techniques are related to the degree/diameter problem, we refer also to the comprehensive survey on this problem by Miller and Širáň [6].

For the concepts and/or results not presented here, we refer the reader to some of the basic textbooks and papers on the subject; about digraphs see, for instance, Chartrand and Lesniak [2] or Diestel [3], and Godsil [5] about the quotient graphs.

This note is organized as follows. In Section 2, we recall the definition of regular partitions and we give some lemmas about them, and in Section 3 we prove our main result and mention some possible examples.

2. REGULAR PARTITIONS

Let \( G \) be a digraph with adjacency matrix \( A \). A partition \( \pi = (V_1, \ldots, V_m) \) of its vertex set \( V \) is called regular (or equitable) whenever, for any \( i, j = 1, \ldots, m \), the intersection numbers, \( b_{ij}(u) \) which is the cardinality of the set \( \omega^+(u, V_j) \) of arcs from \( u \in V_i \) to (the
vertices of) \(V_j\), does not depend on the vertex \(u\) but only on the subsets (usually called classes or cells) \(V_i\) and \(V_j\). In this case, such numbers are simply written as \(b_{ij}\), and the \(m \times m\) matrix \(B = (b_{ij})\) is referred to as the quotient matrix of \(A\) with respect to \(\pi\). This is also represented by the quotient (weighted) digraph \(\pi(G)\) (associated to the partition \(\pi\)), with vertices representing the cells, and an arc with weight \(b_{ij}\) from vertex \(V_i\) to vertex \(V_j\) if and only if \(b_{ij} \neq 0\). Of course, if \(b_{ii} > 0\) for some \(i = 1, \ldots, m\), the quotient digraph \(\pi(G)\) has loops.

The characteristic matrix of (any) partition \(\pi\) is the \(n \times m\) matrix \(S = (s_{ui})\) whose \(i\)-th column is the characteristic vector of \(V_i\), that is, \(s_{ui} = 1\) if \(u \in V_i\), and \(s_{ui} = 0\) otherwise. In terms of such a matrix, we have the following characterization of regular partitions.

**Lemma 2.1.** Let \(G = (V, E)\) be a digraph with adjacency matrix \(A\), and vertex partition \(\pi\) with characteristic matrix \(S\). Then \(\pi\) is regular if and only if there exists an \(m \times m\) matrix \(C\) such that \(SC = AS\). Moreover, \(C = B\), the quotient matrix of \(A\) with respect to \(\pi\).

**Proof.** Let \(C = (c_{ij})\) be an \(m \times m\) matrix. For any fixed \(u \in V_i\) and \(j = 1, \ldots, m\), we have

\[
(SC)_{uj} = \sum_{k=1}^{m} s_{uk} c_{kj} = c_{ij}, \quad (AS)_{uj} = \sum_{v \in V} a_{uv} s_{vj} = |\omega^+(u, V_j)| = b_{ij}(u),
\]

and the result follows.

Most of the results about regular partitions in graphs can be generalized for regular partitions in digraphs. For instance, using the above lemma it can be proved that all the eigenvalues of the quotient matrix \(B\) are also eigenvalues of \(A\). Moreover, we have the following result.

**Lemma 2.2.** Let \(G\) be a digraph with adjacency matrix \(A\). Let \(\pi = (V_1, \ldots, V_m)\) be a regular partition of \(G\), with quotient matrix \(B\). Then, the number of \(k\)-walks from each vertex \(u \in V_i\) to all vertices of \(V_j\) is the \(ij\)-entry of \(B^k\).

**Proof.** We use induction. The result is clearly true for \(k = 0\), since \(B^0 = I\), and for \(k = 1\) because of the definition of \(B\). Suppose that the result holds for some \(k > 1\). Then the set of walks of length \(k + 1\) from \(u \in V_i\) to the vertices of \(V_j\) is in bijective correspondence with the set of \(k\)-walks from \(u\) to vertices \(v \in V_h\) adjacent to some vertex of \(V_j\). Then, the number of such walks is \(\sum_{h=1}^{m} (B^k)_{ih} b_{hj} = (B^{k+1})_{ij}\), as claimed.

As a consequence of this lemma, the number of vertices of \(L^k(G)\) is

\[
n_k = \sum_{i=1}^{m} |V_i| \sum_{j=1}^{m} (B^k)_{ij} = s B^k j^T,
\]

where \(s = (|V_1|, \ldots, |V_m|)\) and \(j = (1, \ldots, 1)\).

### 3. MAIN RESULT

In the following result, we obtain a recurrence equation on the number of vertices \(n_k\) of the \(k\)-iterated line digraph of a digraph \(G\).
Theorem 3.1. Let $G = (V, E)$ be a digraph on $n$ vertices, and consider a regular partition $\pi = (V_1, \ldots, V_m)$ with quotient matrix $B$. Let $m(x) = x^r - \alpha_{r-1}x^{r-1} - \cdots - \alpha_0$ be the minimal polynomial of $B$. Then, the number of vertices $n_k$ of the $k$-iterated line digraph $L^k(G)$ satisfies the recurrence

$$n_k = \alpha_{r-1}n_{k-1} + \cdots + \alpha_0n_{k-r}, \quad k = r, r+1, \ldots$$

initialized with the values $n_k$, for $k = 0, 1, \ldots, r-1$, given by (2).

Proof. Since the polynomial $x^{k-r}m(x)$ annihilates $B$ for any $k \geq 0$, we have

$$B^k = \alpha_{r-1}B^{k-1} + \cdots + \alpha_0B^{k-r}.$$ 

Then, by (2), we get the recurrence

$$n_k = sB^kj^\top = \alpha_{r-1}sB^{k-1}j^\top + \cdots + \alpha_0sB^{k-r}j^\top$$

$$= \alpha_{r-1}n_{k-1} + \cdots + \alpha_0n_{k-r},$$

with the first values $n_k$, for $k = 0, \ldots, r-1$, given as claimed.

By applying this theorem, it is easy to give examples for the three possible behaviours of the sequence $n_0, n_1, n_2, \ldots$ Namely, when it is increasing, tending to a positive constant, or tending to zero. For instance, we can apply the result, respectively, to the so-called cyclic Kautz digraph, introduced by Böhmová, Dalfó and Huemer et al. in [1], the unicyclic digraphs, and the acyclic digraphs. In the first case, our method gives the enumeration of the ternary length-2 square-free words of any length, see the sequence A022089 in the On-Line Encyclopedia of Integer Sequences [9]. For more details about the behavior of digraphs under the line operator, see Prisner [8].

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