GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE DEFOCUSING ENERGY-CRITICAL NONLINEAR SCHRODINGER EQUATION IN $\mathbb{R}^{1+4}$

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Abstract. We obtain global well-posedness, scattering, uniform regularity, and global $L^{6}_{t,x}$ spacetime bounds for energy-space solutions to the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R} \times \mathbb{R}^{4}$. Our arguments closely follow those of Colliander, Hoel, et al., though our derivation of the frequency-localized interaction Morawetz estimate is somewhat simpler. As a consequence, our method yields a better bound on the $L^{6}_{t,x}$-norm.

1. Introduction. We study the following initial value problem for the cubic defocusing nonlinear Schrödinger equation in $\mathbb{R} \times \mathbb{R}^{4}$

\[
\begin{cases}
    iu_t + \Delta u = |u|^{2}u \\
u(0,x) = u_0(x)
\end{cases}
\]

(1.1)

where $u(t,x)$ is a complex-valued function in spacetime $\mathbb{R} \times \mathbb{R}^{4}$.

This equation has the Hamiltonian

\[
E(u(t)) = \int \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{4} |u(t,x)|^4 \, dx.
\]

(1.2)

Since (1.2) is preserved by the flow corresponding to (1.1) we shall refer to it as the energy and often write $E(u)$ for $E(u(t))$.

A second conserved quantity we will occasionally rely on is the mass $\|u(t)\|^{2}_{L^{2}(\mathbb{R}^{4})}$. However, since the equation is $L^{2}_{x}$-supercritical with respect to the scaling (see below), we do not have bounds on the mass that are uniform across frequencies (indeed, the low frequencies may simultaneously have small energy and large mass).

We are primarily interested in the cubic defocusing equation (1.1) since it is critical with respect to the energy norm. That is, the scaling $u \mapsto u^\lambda$ where

\[
u^\lambda(t,x) := \frac{1}{\lambda} u \left( \frac{t}{\lambda^2}, \frac{x}{\lambda} \right),
\]

(1.3)
maps a solution to (1.1) to another solution to (1.1), and \( u \) and \( u^\lambda \) have the same energy.

It is known that if the initial data \( u_0 \) has finite energy, then (1.1) is locally well-posed (see, for instance [5]). That is, there exists a unique local-in-time solution that lies in \( C^0_0 H^1_x \cap L^6_t \), and the map from the initial data to the solution is locally Lipschitz in these norms. If in addition the energy is small, it is known that the solution exists globally in time and scattering occurs; that is, there exist solutions \( u_{\pm} \) of the free Schrödinger equation \((i\partial_t + \Delta) u_{\pm} = 0\) such that \( \|u(t) - u_{\pm}(t)\|_{H^1_t} \to 0 \) as \( t \to \pm \infty \). However, for initial data with large energy, the local well-posedness arguments do not extend to give global well-posedness.

Global well-posedness in \( H^1_x(\mathbb{R}^3) \) for the energy-critical NLS in the case of large finite-energy, radially-symmetric initial data was first obtained by Bourgain ([2], [3]) and subsequently by Grillakis [14]. Tao [23] settled the problem for arbitrary dimensions (with an improvement in the final bound due to a simplification of the argument), but again only for radially symmetric data. A major breakthrough in the field was made by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao in [11] where they obtained global well-posedness for the energy-critical NLS in dimension \( n = 3 \) with arbitrary data. In dimensions \( n \geq 4 \), the problem was still open.

The main result of this paper is the global well-posedness statement for (1.1) in the energy space,

**Theorem 1.1.** For any \( u_0 \) with finite energy \( E(u_0) < \infty \), there exists a unique global solution \( u \in C^0_0 H^1_x \cap L^6_t \) to (1.1) such that

\[
\int_{-\infty}^{\infty} \int_{\mathbb{R}^4} |u(t,x)|^6 \, dx \, dt \leq C(E(u_0))
\]

for some constant \( C(E(u_0)) \) depending only on the energy.

**1.1. Outline of the proof of Theorem 1.1.** We first notice that it suffices to prove Theorem 1.1 for Schwartz functions. Indeed, if one obtains a uniform \( L^6_t (I \times \mathbb{R}^4) \) bound for all Schwartz solutions and all compact intervals \( I \), one can approximate arbitrary finite-energy initial data by Schwartz initial data and use perturbation theory to prove that the corresponding sequence of solutions to (1.1) converges in \( S^1(I \times \mathbb{R}^4) \) to a finite-energy solution of (1.1). See Sections 1.2 and 2.1 for the notation and definitions appearing in the outline of the proof.

For an energy \( E \geq 0 \) we define the quantity \( M(E) \) by

\[
M(E) := \sup \|u\|_{L^6_t (I \times \mathbb{R}^4)},
\]

where \( I \subset \mathbb{R} \) ranges over all compact time intervals, and \( u \) ranges over all Schwartz solutions to (1.1) on \( I \times \mathbb{R}^4 \) with \( E(u) \leq E \). For \( E < 0 \) we define
\( M(E) = 0 \) since, of course, there are no negative energy solutions. Our task is to show that \( M(E) < \infty \) for all \( E \).

Let us note that by the small-energy well-posedness result discussed above, we know that \( M(E) \) is finite for \( E \) sufficiently small. We will assume for a contradiction that \( M(E) \) is not always finite. From perturbation theory (see Lemma 3.3) it follows that the set \( \{ E: M(E) < \infty \} \) is open. Since it is also connected and contains zero, there exists a critical energy \( 0 < E_{\text{crit}} < \infty \) such that \( M(E_{\text{crit}}) = \infty \) but \( M(E) < \infty \) for all \( E < E_{\text{crit}} \). From the definition of \( E_{\text{crit}} \) and the \( L^6_{t,x} \) well-posedness theory (see Section 3 for details) we get:

\[
\text{LEMMA 1.2 (Induction on energy hypothesis). Let } t_0 \in \mathbb{R} \text{ and let } v(t_0) \text{ be a } \text{Schwartz function with } E(v(t_0)) \leq E_{\text{crit}} - \eta \text{ for some } \eta > 0. \text{ Then there exists a } \text{global Schwartz solution } v \text{ of (1.1) on } \mathbb{R} \times \mathbb{R}^4 \text{ with initial data } v(t_0) \text{ at time } t_0, \text{ such that}
\]
\[
\| v \|_{L^6_{t,x}(\mathbb{R} \times \mathbb{R}^4)} \leq M(E_{\text{crit}} - \eta).
\]

Moreover we have

\[
\| v \|_{S^1(\mathbb{R} \times \mathbb{R}^4)} \leq C(E_{\text{crit}}, M(E_{\text{crit}} - \eta)).
\]

We will need a few small parameters for the contradiction argument (albeit two less than were necessary in [11]). Specifically we will need

\[
1 \gg \eta_0 \gg \eta_1 \gg \eta_2 \gg \eta_3 \gg \eta_4 > 0
\]

where each \( \eta_j \) is allowed to depend on the critical energy and any of the larger \( \eta \)'s. We will choose \( \eta_j \) small enough that, in particular, it will be smaller than any constant depending on the previous \( \eta \)'s used in the argument.

As \( M(E_{\text{crit}}) \) is infinite, given any \( \eta_4 > 0 \) there exist a compact interval \( I_\ast \subset \mathbb{R} \) and a Schwartz solution \( u \) to (1.1) on \( I_\ast \times \mathbb{R}^4 \) with \( E(u) \leq E_{\text{crit}} \) but

\[
(1.5) \quad \| u \|_{L^6_{t,x}(I_\ast \times \mathbb{R}^4)} > 1/\eta_4.
\]

Note that we may assume \( E(u) \geq \frac{1}{2}E_{\text{crit}} \), since otherwise we would get

\[
\| u \|_{L^6_{t,x}(I_\ast \times \mathbb{R}^4)} \leq M\left( \frac{1}{2}E_{\text{crit}} \right) < \infty
\]

and we would be done.
This suggests we make the following definition.

**Definition 1.3.** A minimal energy blowup solution of (1.1) is a Schwartz solution \( u \) on a time interval \( I_\ast \subset \mathbb{R} \) with energy

\[
\frac{1}{2} E_{\text{crit}} \leq E(u(t)) \leq E_{\text{crit}}
\]

and huge \( L^6_{t,x} \)-norm in the sense of (1.5).

Note that (1.6) implies the kinetic energy bound

\[
\|u\|_{L^\infty_t L^3_x(I_\ast \times \mathbb{R}^4)} \sim 1,
\]

while from Sobolev embedding we obtain a bound on the potential energy,

\[
\|u\|_{L^\infty_t L^6_x(I_\ast \times \mathbb{R}^4)} \lesssim 1.
\]

In Sections 2 and 3 we recall the Strichartz estimates and the perturbation theory we will use throughout the proof of Theorem 1.1. Many of the ideas of these sections have been previously developed, and in a few cases we content ourselves with citing the relevant source (e.g., Lemmas 2.2 and 2.3).

We expect that a minimal energy blowup solution should be localized in both physical and frequency space. For if not, it could be decomposed into two essentially separate solutions, each with strictly smaller energy than the original. By Lemma 1.2 we can then extend these smaller energy solutions to all of \( I_\ast \). As each of the separate evolutions exactly solves (1.1), we expect their sum to solve (1.1) approximately. We could then use the perturbation theory results and the bounds from Lemma 1.2 to bound \( \|u\|_{L^6_t L^6_x} \) in terms of \( \eta_0, \eta_1, \eta_2, \) and \( \eta_3 \), thus contradicting the fact that \( \eta_4 \) can be chosen arbitrarily small.

This intuition will underpin the frequency localization argument we give in Section 4. The spatial concentration result follows in a similar manner, but is a bit more technical. For instance, we restrict our analysis to a subinterval \( I_0 \subset I_\ast \) and will need to use both frequency localization and the fact that the potential energy of a minimal energy blowup solution is bounded away from zero.

In Section 5 we obtain the frequency-localized Morawetz inequality (5.1), which will be used to derive a contradiction to the frequency localization and spatial concentration results just described.

A typical example of a Morawetz inequality for (1.1) (see [17]) is the bound

\[
\int_I \int_{\mathbb{R}^4} \frac{|u(t,x)|^4}{|x|} \, dx \, dt \lesssim \sup_{t \in I} \|u(t)\|_{H^{1/2} (\mathbb{R}^4)}^2,
\]

for all time intervals \( I \) and all Schwartz solutions \( u : I \times \mathbb{R}^4 \to \mathbb{C} \).
This estimate is not particularly useful for the energy-critical problem since the $H^{1/2}$ norm is supercritical with respect to the scaling (1.3). To get around this problem, Bourgain and Grillakis introduced a spatial cutoff obtaining the variant

$$
\int_I \int_{|x| \leq A|I|^{1/2}} \frac{|u(t,x)|^4}{|x|} \, dx \, dt \lesssim A|I|^{1/2} E(u),
$$

for all $A \geq 1$, where $|I|$ denotes the length of the time interval $I$. While this estimate is better suited for the critical NLS (it involves the energy on the right-hand side), it only prevents concentration of $u$ at the spatial origin $x = 0$. This is especially useful in the spherically-symmetric case $u(t,x) = u(t,|x|)$, since the spherical symmetry combined with the bounded energy assumption can be used to show that $u$ cannot concentrate at any location but the spatial origin. However, it does not provide much information about the solution away from the origin. Following [11], we circumvent this problem by using a frequency-localized interaction Morawetz inequality.

While the previously mentioned Morawetz inequalities were a priori estimates, the frequency-localized interaction Morawetz inequality we will develop is not; it only applies to minimal energy blowup solutions. Our model in obtaining the frequency-localized interaction Morawetz estimate is [11]. However, our argument is not as technical as theirs, lacking the need for spatial localization. It is this simplification that will yield an improvement in the final bound on the $L^6_{t,x}$-norm.

A corollary of (5.1) is good $L^3_{t,x}$ control over the high-frequency part of a minimal energy blowup solution. One then has to use Sobolev embedding to bootstrap this $L^3_{t,x}$ control to $L^6_{t,x}$ control. However, one needs to make sure that the solution is not shifting its energy from low to high frequencies causing the $L^6_{t,x}$-norm to blow up while the $L^3_{t,x}$-norm stays bounded. This is done in Section 6, where we prove a frequency-localized mass estimate that prevents energy evacuation to high modes. We put all these pieces together in Section 7 where the contradiction argument is concluded.

In Section 8 we comment on the tower bound we get on the $L^6_{t,x}$-norm in Theorem 1.1, and we show how this bound yields scattering, asymptotic completeness, and uniform regularity.

As will certainly become clear to the reader, our paper relies heavily on the arguments developed in [11]. One should note a few differences though, mainly related to the Strichartz norms and the frequency-localized interaction Morawetz inequality. While it is true that in higher dimensions one has more Strichartz estimates, we lack control over $L^2_t L^\infty_x$ (for which one gets a logarithmic divergence). This turns out not to be a problem most of the time, since the triangles in which the low- and high-frequency parts of the minimal energy blowup solution live are large enough in four dimensions. (Indeed, by Bernstein, the low-frequency portion of the solution has finite spacetime norm for every $L^p_t L^q_x$ in the closed
triangle with vertices $L^2_t L^\infty_x$, $L^\infty_t L^4_x$, and $L^\infty_t L^4_x$, except for the vertex $L^2_t L^\infty_x$. By interpolation, the high-frequency portion has finite spacetime norm for every $L^p_t L^q_x$ in the closed triangle with vertices $L^3_t L^\infty_x$, $L^\infty_t L^2_x$, and $L^\infty_t L^4_x$. (However, when it comes to controlling the error terms generated by the frequency localization in the interaction Morawetz inequality, we have to do something different. While in [11] the authors strive to gain better control on low frequencies, we will use multilinear operator theory (specifically a theorem of Coifman and Meyer [6], [7]) to gain control over the high frequencies. Being able to control the high frequencies in $L^3_t L^\infty_x$ will save us from having to localize the interaction Morawetz inequality in space as well. As a consequence, our argument is somewhat simpler and it yields a smaller tower bound on the $L^6_t L^\infty_x$-norm.

Acknowledgments. We thank Terence Tao for valuable comments on this paper and explanatory details related to [11].

1.2. Notation. We will often use the notation $X \lesssim Y$ whenever there exists some constant $C$, possibly depending on the critical energy but not on any other parameters, so that $X \leq CY$. Similarly we will use $X \sim Y$ if $X \lesssim Y \lesssim X$. We use $X \ll Y$ if $X \leq cY$ for some small constant $c$, again possibly depending on the critical energy. We will use the notation $X^+ := X + \epsilon$, for some $0 < \epsilon \ll 1$; similarly $X^- := X - \epsilon$. We also use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$.

We define the Fourier transform on $\mathbb{R}^4$ to be

$$\hat{f}(\xi) := \int_{\mathbb{R}^4} e^{-2\pi i x \cdot \xi} f(x) \, dx.$$ 

We will make frequent use of the fractional differentiation operators $|\nabla|^s$ defined by

$$|\nabla|^s f(\xi) := |\xi|^s \hat{f}(\xi).$$

These define the homogeneous Sobolev norms

$$\|f\|_{H^s_t} := \||\nabla|^s f\|_{L^2_t(\mathbb{R}^4)}.$$

Let $e^{it\Delta}$ be the free Schrödinger propagator. In physical space this is given by the formula

$$e^{it\Delta}f(x) = \frac{1}{(4\pi it)^2} \int_{\mathbb{R}^4} e^{i|x-y|^2/4t} f(y) \, dy,$$

while in frequency space one can write this as

$$\hat{e^{it\Delta}f}(\xi) = e^{-4\pi^2 t|\xi|^2} \hat{f}(\xi).$$

(1.9)
In particular, the propagator preserves the above Sobolev norms and obeys the dispersive inequality

\[ \| e^{it\Delta}f \|_{L^\infty_t L^s_x(\mathbb{R}^4)} \lesssim |t|^{-2} \| f \|_{L^1_t L^s_x(\mathbb{R}^4)} \]

for all times \( t \). We also recall Duhamel’s formula

\[ u(t) = e^{i(t-t_0)\Delta}u(t_0) - i \int_{t_0}^{t} e^{i(t-s)\Delta}(iu_s + \Delta u(s)) \, ds. \]

We will use the notation \( O(X) \) to denote a quantity that resembles \( X \); that is a finite linear combination of terms that look like \( X \), but possibly with some factors replaced by their complex conjugates. For example we will write

\[ |u + v|^2(u + v) = \sum_{j=0}^{3} O(u^j v^{3-j}). \]

We will occasionally use subscripts to denote spatial derivatives and will use the summation convention over repeated indices.

We will also need some Littlewood-Paley theory. Specifically, let \( \varphi(\xi) \) be a smooth bump adapted to the ball \( |\xi| \leq 2 \) equalling one on the ball \( |\xi| \leq 1 \). For each dyadic number \( N \in 2^\mathbb{Z} \) we define the Littlewood-Paley operators

\[
\begin{align*}
\hat{P}_{\leq N}f(\xi) &:= \varphi(\xi/N)\hat{f}(\xi) \\
\hat{P}_{> N}f(\xi) &:= (1 - \varphi(\xi/N))\hat{f}(\xi) \\
\hat{P}_N f(\xi) &:= (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi).
\end{align*}
\]

Similarly we can define \( P_{< N} \), \( P_{\geq N} \), and \( P_{M < \leq N} := P_{\leq N} - P_{\leq M} \), whenever \( M \) and \( N \) are dyadic numbers. We will frequently write \( f_{\leq N} \) for \( P_{\leq N}f \) and similarly for the other operators.

The Littlewood-Paley operators commute with derivative operators, the free propagator, and the conjugation operator. They are self-adjoint and bounded on every \( L^p \) and \( H^s \) space for \( 1 \leq p \leq \infty \) and \( s \geq 0 \). They also obey the following Sobolev and Bernstein estimates

\[
\begin{align*}
\|P_{\geq N}f\|_{L^p} &\lesssim N^{-s}\|\nabla^s P_{\geq N}f\|_{L^p}, \\
\|\nabla^s P_{\leq N}f\|_{L^p} &\lesssim N^s\|P_{\leq N}f\|_{L^p}, \\
\|\nabla^{\pm s} P_{N}f\|_{L^p} &\sim N^{\pm s}\|P_{N}f\|_{L^p}, \\
\|P_{\leq N}f\|_{L^q} &\lesssim N^{\frac{s}{q}}\|P_{\leq N}f\|_{L^p}, \\
\|P_{N}f\|_{L^q} &\lesssim N^{\frac{s}{q}}\|P_{N}f\|_{L^p},
\end{align*}
\]

whenever \( s \geq 0 \) and \( 1 \leq p \leq q \leq \infty \).
For instance, we can use the above Bernstein estimates and (1.7) to bound the mass of high frequencies

\[ \|P_{> M} u\|_{L^2(\mathbb{R}^4)} \lesssim \frac{1}{M} \text{ for all } M \in 2\mathbb{Z}. \]  

(1.13)

2. Strichartz numerology. In this section we recall Strichartz and bilinear Strichartz estimates in \( \mathbb{R} \times \mathbb{R}^4 \) and develop trilinear Strichartz estimates.

We use \( L^q_t L^r_x \) to denote the spacetime norm

\[ \|u\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^4)} := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^4} |u(t, x)|^r \, dx \right)^{q/r} dt \right)^{1/q}, \]

with the usual modifications when \( q \) or \( r \) is infinity, or when the domain \( \mathbb{R} \times \mathbb{R}^4 \) is replaced by some smaller spacetime region. When \( q = r \) we abbreviate \( L^q_t L^r_x \) by \( L^q x \).

2.1. Linear and bilinear Strichartz estimates. We say that a pair of exponents \( (q, r) \) is admissible if \( \frac{2}{q} + \frac{4}{r} = 2 \) and \( 2 \leq q, r \leq \infty \). If \( I \times \mathbb{R}^4 \) is a spacetime slab, we define the \( S^0(I \times \mathbb{R}^4) \) Strichartz norm by

\[ \|u\|_{S^0(I \times \mathbb{R}^4)} := \sup_{N} \left( \sum_N \|P_N u\|^2_{L^q_t L^r_x(I \times \mathbb{R}^4)} \right)^{1/2} \]  

(2.1)

where the sup is taken over all admissible pairs \( (q, r) \). For \( k = 1, 2 \) we define the \( \dot{S}^k(I \times \mathbb{R}^4) \) Strichartz norm by

\[ \|u\|_{\dot{S}^k(I \times \mathbb{R}^4)} := \|\nabla^k u\|_{S^0(I \times \mathbb{R}^4)}. \]

We observe the inequality

\[ \left( \sum_N |f_N|^2 \right)^{1/2} \|P_N u\|^2_{L^q_t L^r_x(I \times \mathbb{R}^4)} \leq \left( \sum_N \|f_N\|^2_{L^q_t L^r_x(I \times \mathbb{R}^4)} \right)^{1/2} \]  

(2.2)

for all \( 2 \leq q, r \leq \infty \) and arbitrary functions \( f_N \), which one proves by interpolating between the trivial cases \( (2, 2), (2, \infty), (\infty, 2) \), and \( (\infty, \infty) \). In particular, (2.2) holds for all admissible exponents \( (q, r) \). Combining this with the Littlewood-Paley inequality we find

\[ \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^4)} \lesssim \left( \sum_N \|P_N u\|^2 \right)^{1/2} \|f_N\|^2_{L^q_t L^r_x(I \times \mathbb{R}^4)} \]  

(2.3)
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\[ \left( \sum_N \| P_N u \|^2_{L^q_t L^r_x(I \times \mathbb{R}^4)} \right)^{1/2} \lesssim \| u \|^\delta_{S(I \times \mathbb{R}^4)}, \]

which in particular implies

\[ \| \nabla u \|^2_{L^q_t L^r_x(I \times \mathbb{R}^4)} \lesssim \| u \|^\delta_{S(I \times \mathbb{R}^4)}. \] (2.3)

In fact, by (2.3) and Sobolev embedding the \( \delta_1 \) norm controls the following spacetime norms:

**Lemma 2.1.** For any Schwartz function \( u \) on \( I \times \mathbb{R}^4 \), we have

\[ \| \nabla u \|_{L^\infty_t L^2_x} + \| \nabla u \|_{L^{12}_t L^{5/3}_x} + \| \nabla u \|_{L^4_t L^4_x} + \| \nabla u \|_{L^3_t L^2_x} + \| \nabla u \|_{L^2_t L^4_x} \lesssim \| u \|_{\delta_1(I \times \mathbb{R}^4)}. \] (2.4)

One has the following standard Strichartz estimates (see for instance [16]):

**Lemma 2.2.** Let \( I \) be a compact time interval, and let \( u : I \times \mathbb{R}^4 \to \mathbb{C} \) be a Schwartz solution to the forced Schrödinger equation

\[ i u_t + \Delta u = \sum_{m=1}^M F_m \]

for some Schwartz functions \( F_1, \ldots, F_M \). Then we have

\[ \| u \|^k_{S^k(I \times \mathbb{R}^4)} \lesssim \| u(t_0) \|_{H^k(\mathbb{R}^4)} + \sum_{m=1}^M \| \nabla^k F_m \|^q_{L^q_t L^r_x(I \times \mathbb{R}^4)} \] (2.5)

for any integer \( k \geq 0 \), any time \( t_0 \in I \), and any admissible exponents \( (q_1, r_1), \ldots, (q_m, r_m) \), where as usual \( p' \) denotes the dual exponent to \( p \), that is \( 1/p + 1/p' = 1 \).

We also recall without proof the bilinear Strichartz estimates which were obtained in [11] (see their Lemma 3.4):

**Lemma 2.3.** Let \( n \geq 2 \). For any spacetime slab \( I \times \mathbb{R}^n \), any \( t_0 \in I \), and any \( \delta > 0 \), we have

\[ \| u \|^2_{L^2_t L^{n/2}_x(I \times \mathbb{R}^n)} \leq C(\delta)(\| u(t_0) \|^n_{H^{n/2}_x} + \| (i \partial_t + \Delta) u \|^n_{L^{n/2}_t H^{n/2}_x(I \times \mathbb{R}^n)}) \]

\[ \times (\| v(t_0) \|^n_{H^{n/2}_x} + \| (i \partial_t + \Delta) v \|^n_{L^{n/2}_t H^{n/2}_x(I \times \mathbb{R}^n)}). \] (2.6)
2.2. Trilinear Strichartz estimates. We will also need the following estimate:

**Lemma 2.4.** For $k = 0, 1, 2$, any slab $I \times \mathbb{R}^4$, and any smooth functions $v_1, v_2, v_3$ on this slab, we have

\[(2.8) \quad \|\nabla^k \mathcal{O}(v_1 v_2 v_3)\|_{L^1_t L^2_x} \lesssim \sum_{\{a,b,c\} = \{1,2,3\}} \|\nabla^k v_a\|_{S_k} \text{min} (\|v_b\|_{S^1}, \|v_b\|_{L^6_{t,x}}, \|v_c\|_{S^1}),\]

where all spacetime norms are on $I \times \mathbb{R}^4$. Similarly we have

\[(2.9) \quad \|\nabla \mathcal{O}(v_1 v_2 v_3)\|_{L^2_t L^{4/3}_x} \lesssim \sum_{\{a,b,c\} = \{1,2,3\}} \|\nabla v_a\|_{L^6_t L^{12/5}_x} \|v_b\|_{L^6_{t,x}} \|v_c\|_{L^6_{t,x}},\]

\[\lesssim \prod_{j=1}^3 \|v_j\|_{S^1}.\]

**Proof.** First consider the $k = 0$ case of (2.8). The claim follows from (2.4) estimating

\[\|\mathcal{O}(v_1 v_2 v_3)\|_{L^1_t L^2_x} \lesssim \|v_1\|_{L^6_t L^1_x} \|v_2\|_{L^6_t L^1_x} \|v_3\|_{L^6_{t,x}}.\]

Applying the Leibnitz rule we see that the case $k = 1$ reduces to obtaining estimates of the form

\[\|\mathcal{O}(\nabla v_1) v_2 v_3)\|_{L^1_t L^2_x} \lesssim \|\nabla v_1\|_{L^6_t L^1_x} \|v_2\|_{L^6_t L^1_x} \|v_3\|_{L^6_{t,x}}.\]

The lemma again follows from (2.4). The $k = 2$ case of (2.8) proceeds similarly using estimates such as

\[\|\mathcal{O}(\nabla^2 v_1) v_2 v_3)\|_{L^1_t L^2_x} \lesssim \|\nabla^2 v_1\|_{L^6_t L^1_x} \|v_2\|_{L^6_t L^1_x} \|v_3\|_{L^6_{t,x}}\]

and

\[\|\mathcal{O}(\nabla v_1 (\nabla v_2) v_3)\|_{L^1_t L^2_x} \lesssim \|\nabla v_1\|_{L^6_t L^1_x} \|\nabla v_2\|_{L^6_t L^1_x} \|v_3\|_{L^6_{t,x}}.\]

Finally, the estimate (2.9) follows from (2.4) and Hölder’s inequality,

\[\|\mathcal{O}(\nabla v_1) v_2 v_3)\|_{L^2_t L^{4/3}_x} \lesssim \|\nabla v_1\|_{L^6_t L^{12/5}_x} \|v_2\|_{L^6_{t,x}} \|v_3\|_{L^6_{t,x}}.\]

The following is a variant of the above lemma adapted to the case where some factors are high frequency and others are low frequency.
LEMMA 2.5. Suppose $v_{hi}$ and $v_{lo}$ are functions on $I \times \mathbb{R}^4$ such that

$$
\|v_{hi}\|_{S^0} + \| (i \partial_t + \Delta) v_{hi} \|_{L^1_t L^4_x} \lesssim \varepsilon K
$$

$$
\|v_{hi}\|_{S^1} + \| \nabla (i \partial_t + \Delta) v_{hi} \|_{L^1_t L^2_x} \lesssim K
$$

$$
\|v_{lo}\|_{S^0} + \| \nabla (i \partial_t + \Delta) v_{lo} \|_{L^1_t L^2_x} \lesssim K
$$

$$
\|v_{lo}\|_{S^2} + \| \nabla^2 (i \partial_t + \Delta) v_{lo} \|_{L^1_t L^2_x} \lesssim \varepsilon K
$$

for some constants $K > 0$ and $0 < \varepsilon \ll 1$ (where all spacetime norms are on $I \times \mathbb{R}^4$). Then for $j = 1, 2$ and any $0 < \delta \ll 1$, we have

$$
\| \nabla \mathcal{O}(v_{hi}^{1-j} v_{lo}^{3-j}) \|_{L^{4/3}_t L^{4}_x} \lesssim \varepsilon^{1-2\delta} K^3.
$$

Remark 2.6. The importance of this lemma lies in the gain of $\varepsilon^{1-2\delta}$. The $S^0$ bound effectively restricts $v_{hi}$ to high frequencies; similarly the $S^2$ bound restricts $v_{lo}$ to low frequencies. Comparing the conclusion with (1.12) we see that the components of the nonlinearity in (1.1) arising from the interaction between high and low frequencies are rather weak. While this will be especially important for the frequency localization result (Proposition 4.4), the idea of controlling frequency interactions will appear repeatedly.

Proof. Throughout, all spacetime norms will be in $I \times \mathbb{R}^4$. We begin by normalizing $K = 1$. By Leibnitz we have

$$
\| \nabla \mathcal{O}(v_{hi}^{1-j} v_{lo}^{3-j}) \|_{L^{4/3}_t L^{4}_x} \lesssim \| \mathcal{O}(v_{hi}^{1-j} v_{lo}^{3-j}) \|_{L^{4/3}_t L^{4}_x} + \| \mathcal{O}(v_{hi}^{1-j} v_{lo}^{3-j}) \|_{L^{4/3}_t L^{4}_x}.
$$

Consider the first term. By Hölder we can bound this by

$$
\| \nabla v_{hi} \|_{L^{\infty}_t L^4_x} \| v_{hi} \|_{L^{2}_t L^4_x} \| v_{hi} \|_{L^{2}_t L^4_x} \| v_{lo} \|_{L^{2-j}_t L^4_x} \lesssim \| v_{hi} \|_{S^1} \| v_{hi} \|_{S^1} \| v_{lo} \|_{S^0} \| v_{lo} \|_{S^0} \lesssim \varepsilon^2.
$$

Now consider the second term for the case $j = 2$. We can bound

$$
\| \mathcal{O}(v_{hi} v_{lo} \nabla v_{hi}) \|_{L^{4/3}_t L^{4}_x} \lesssim \| \nabla v_{hi} \|_{L^{2}_t L^4_x} \| v_{lo} \|_{L^{2}_t L^4_x} \| v_{hi} \|_{L^{\infty}_t L^4_x} \lesssim \| v_{hi} \|_{S^1} \| v_{lo} \|_{L^{2}_t L^4_x} \| v_{hi} \|_{S^0}.
$$

Decomposing $v_{lo} = \sum_N P_N v_{lo}$ and using Bernstein we get

$$
\| v_{lo} \|_{L^{2}_t L^4_x} \lesssim \left( \sum_N \| P_N v_{lo} \|_{L^{2}_t L^4_x}^2 \right)^{1/2} \lesssim \left( \sum_N \| \nabla P_N v_{lo} \|_{L^{2}_t L^4_x}^2 \right)^{1/2} \lesssim \| v_{lo} \|_{S^2}.
$$
Hence, we obtain the bound
\[
\|O(v_h^2 \nabla v_h)\|_{L^4_t L^3_x} \lesssim \|v_h\|_{S^1} \|v_{lo}\|_{S^2} \|v_{hi}\|_{S^0} \lesssim \varepsilon^2
\]
which is acceptable.

Finally consider the second term for the case \( j = 1 \). We split the \( v_{lo} \) terms dyadically and use Hölder to bound
\[
\|O(v_{lo} \nabla v_{hi})\|_{L^4_t L^3_x} \lesssim \sum_{N_1 \geq N_2} \|O((P_{N_1} v_{lo})(P_{N_2} v_{lo}) \nabla v_{hi})\|_{L^4_t L^3_x}
\]
\[
\lesssim \sum_{N_1 \geq N_2} \|O((P_{N_2} v_{lo}) \nabla v_{hi})\|_{L^4_t L^3_x} \|P_{N_1} v_{lo}\|_{L^\infty_t L^3_x}.
\]
Now \( \|P_{N_1} v_{lo}\|_{L^\infty_t L^3_x} \lesssim \|v_{lo}\|_{S^1} \lesssim 1 \), while by Bernstein,
\[
\|P_{N_1} v_{lo}\|_{L^\infty_t L^3_x} \lesssim N_1^{-1} \|P_{N_1} \nabla v_{lo}\|_{L^\infty_t L^3_x} \lesssim N_1^{-1} \|v_{lo}\|_{S^2} \lesssim \varepsilon N_1^{-1}.
\]
So \( \|P_{N_1} v_{lo}\|_{L^\infty_t L^3_x} \lesssim \min(1, \varepsilon N_1^{-1}) \).

By Lemma 2.3 we have
\[
\|O((P_{N_2} v_{lo}) \nabla v_{hi})\|_{L^4_t L^3_x} \lesssim (\|\nabla v_{hi}(t_0)\|_{B^{1/2+\delta}_x} + \|(i\partial_t + \Delta) \nabla v_{hi}\|_{L^3_t B_{x}^{1/2+\delta}})
\times (\|P_{N_2} v_{lo}(t_0)\|_{B^{3/2-\delta}_x} + \|(i\partial_t + \Delta) P_{N_2} v_{lo}\|_{L^3_t B_{x}^{3/2-\delta}}).
\]
By interpolation we can bound
\[
\|\nabla v_{hi}(t_0)\|_{B^{1/2+\delta}_x} + \|(i\partial_t + \Delta) \nabla v_{hi}\|_{L^3_t B_{x}^{1/2+\delta}} \lesssim \varepsilon^{1/2-\delta},
\]
while by Bernstein and the \( S^1 \) and \( S^2 \) bounds we have
\[
\|P_{N_2} v_{lo}(t_0)\|_{B^{3/2-\delta}_x} + \|(i\partial_t + \Delta) P_{N_2} v_{lo}\|_{L^3_t B_{x}^{3/2-\delta}} \lesssim \min(N_2^{1/2-\delta}, \varepsilon N_2^{-1/2-\delta}).
\]
Putting these together we see that
\[
\|O(v_{lo}^2 \nabla v_{hi})\|_{L^4_t L^3_x} \lesssim \sum_{N_1 \geq N_2} \varepsilon^{1/2-\delta} \min(1, \varepsilon N_1^{-1}) \min(N_2^{1/2-\delta}, \varepsilon N_2^{-1/2-\delta}).
\]
We break the right-hand side term into three sums
\[
\sum_{N_1 \geq N_2} = \sum_{N_1 \geq N_2 \geq \varepsilon} + \sum_{N_1 \geq \varepsilon \geq N_2} + \sum_{\varepsilon \geq N_1 \geq N_2} = I + II + III
\]
which we bound as follows

\[ I = \sum_{N_1 \geq N_2 \geq \varepsilon} e^{1/2 - \delta} \varepsilon N_1^{-1} e^{N_2^{-1/2 - \delta}} \lesssim e^{1 - 2\delta} \]

\[ II = \sum_{N_1 \geq \varepsilon \geq N_2} e^{1/2 - \delta} \varepsilon N_1^{-1} N_2^{1/2 - \delta} \lesssim e^{1 - 2\delta} \]

\[ III = \sum_{\varepsilon \geq N_1 \geq N_2} e^{1/2 - \delta} N_2^{1/2 - \delta} \lesssim e^{1 - 2\delta}. \]

The lemma follows.

3. Perturbation theory. As mentioned in the introduction, the Cauchy problem for (1.1) is locally well-posed in \( H^1_t(\mathbb{R}^4) \). Indeed, this well-posedness extends to any interval where one has uniform control of the \( L^6_t \) norm (see Lemma 3.6 below). This section describes variants of the local well-posedness theory. In particular we will be interested in when we can perturb a solution (or near-solution, see (3.1) below) in the energy norm when we can control the solution in \( L^6_t \) and the error in a dual Strichartz space.

As a first step, we consider the case where the near-solution, the error, and the free evolution of the perturbation are small in spacetime norms, but allowed to be large in the energy norm.

**Lemma 3.1 (Short-time perturbations).** Let \( I \) be a compact time interval, and let \( \tilde{u} \) be a near-solution to (1.1) on \( I \times \mathbb{R}^4 \) in the sense that

\[ (i\partial_t + \Delta)\tilde{u} = |\tilde{u}|^2 \tilde{u} + e \tag{3.1} \]

for some function \( e \). Suppose that we also have the energy bound

\[ \|\tilde{u}\|_{L^\infty_t H^1_x(I \times \mathbb{R}^4)} \leq E \tag{3.2} \]

for some \( E > 0 \). Let \( t_0 \in I \) and let \( u(t_0) \) close to \( \tilde{u}(t_0) \) in the sense that

\[ \|u(t_0) - \tilde{u}(t_0)\|_{H^1} \leq E' \tag{3.3} \]

for some \( E' > 0 \). Assume also the smallness conditions

\[ \|\nabla \tilde{u}\|_{L^5_t L^{12/5}_x(I \times \mathbb{R}^4)} \leq \delta \tag{3.4} \]

\[ \|e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L^5_t L^{12/5}_x(I \times \mathbb{R}^4)} \leq \varepsilon \tag{3.5} \]

\[ \|\nabla e\|_{L^5_t L^{4/3}_x(I \times \mathbb{R}^4)} \leq \varepsilon \tag{3.6} \]

for some \( 0 < \varepsilon, \delta < \varepsilon_0 \), where \( \varepsilon_0 = \varepsilon_0(E, E') \) is a small positive constant.
Then there exists a solution \( u \) to (1.1) on \( I \times \mathbb{R}^4 \) with the specified initial data \( u(t_0) \) at \( t_0 \) satisfying

\[
\| u - \tilde{u} \|_{L^6_t L^4_x(I \times \mathbb{R}^4)} \lesssim \varepsilon \tag{3.7}
\]
\[
\| u - \tilde{u} \|_{S^1(I \times \mathbb{R}^4)} \lesssim \varepsilon + E' \tag{3.8}
\]
\[
\| u \|_{S^1(I \times \mathbb{R}^4)} \lesssim E' + E \tag{3.9}
\]
\[
\| \nabla [ (i\partial_t + \Delta)(u - \tilde{u}) + e] \|_{L^2_t L^{4/3}_x(I \times \mathbb{R}^4)} \lesssim \varepsilon. \tag{3.10}
\]

**Remark 3.2.** Notice that

\[
(3.5) \lesssim \| e^{(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0)) \|_{S^1(I \times \mathbb{R}^4)} \lesssim \| u(t_0) - \tilde{u}(t_0) \|_{H^1} \leq E',
\]
so (3.5) is redundant if \( E' = O(\varepsilon) \).

**Proof.** By the well-posedness theory it will suffice to prove (3.8)--(3.10) as \textit{a priori} estimates, that is, we will assume that the solution \( u \) already exists and is smooth on \( I \). Without loss of generality we may assume that \( t \geq t_0 \), since the proof on the \( t \leq t_0 \) portion of \( I \) is similar.

Let \( v := u - \tilde{u} \), and for \( t \in I \) define

\[
S(t) := \| \nabla [ (i\partial_t + \Delta)v + e] \|_{L^2_t L^{4/3}_x([t_0, t] \times \mathbb{R}^4)}.
\]

By Lemma 2.2, (3.5), and (3.6) we get

\[
\| \nabla v \|_{L^2_t L^{12/5}_x([t_0, t] \times \mathbb{R}^4)} \lesssim \| \nabla e^{(t-t_0)\Delta}v(t_0) \|_{L^6_t L^{12/5}_x([t_0, t] \times \mathbb{R}^4)} + \| \nabla [ (i\partial_t + \Delta)v + e] \|_{L^2_t L^{4/3}_x([t_0, t] \times \mathbb{R}^4)} + \| \nabla e \|_{L^2_t L^{4/3}_x([t_0, t] \times \mathbb{R}^4)} \lesssim S(t) + \varepsilon. \tag{3.11}
\]

By Sobolev embedding, (3.11) yields

\[
\| v \|_{L^6_t L^{12/5}_x([t_0, t] \times \mathbb{R}^4)} \lesssim \| \nabla v \|_{L^2_t L^{12/5}_x([t_0, t] \times \mathbb{R}^4)} \lesssim S(t) + \varepsilon. \tag{3.12}
\]

On the other hand, since

\[
(i\partial_t + \Delta)v = |\tilde{u} + v|^2(\tilde{u} + v) - |\tilde{u}|^2\tilde{u} - e = \sum_{j=1}^{3} \mathcal{O}(v^j \tilde{u}^{3-j}) - e
\]
we get
\[ S(t) \lesssim \| \nabla \sum_{j=1}^{3} \mathcal{O}(\nu^j \tilde{u}^{3-j}) \|_{L^4_t L^{4/3}_x ([t_0,t] \times \mathbb{R}^4)}. \]

Using the trilinear Strichartz estimate (2.9) together with (3.4), (3.11), and (3.12), one estimates
\[ S(t) \lesssim \varepsilon + (E + \varepsilon + \delta) S(t)^2 + S(t)^3. \]

A standard continuity argument shows that if we take \( \varepsilon_0 = \varepsilon_0(E, E') \) sufficiently small we obtain
\[ S(t) \lesssim \varepsilon \tag{3.13} \]
for all \( t \in I \), which implies (3.10). Using (3.12) and (3.13), one easily derives (3.7). To obtain (3.8), we use Strichartz’s inequality:
\[
\| u - \tilde{u} \|_{S(I \times \mathbb{R}^4)} \lesssim \| u(t_0) - \tilde{u}(t_0) \|_{\dot{H}^{1}_v} + \| \nabla [(i\partial_t + \Delta) u + e] \|_{L^4_t L^{4/3}_x (I \times \mathbb{R}^4)} \\
+ \| \nabla e \|_{L^4_t L^{4/3}_x (I \times \mathbb{R}^4)} \\
\lesssim E' + \varepsilon.
\]

By the triangle inequality, (3.4) and (3.7) imply \( \| u \|_{L^6_{t,x}(I \times \mathbb{R}^4)} \lesssim \varepsilon + \delta \). An application of Strichartz’s inequality yields
\[
\| u \|_{S(I \times \mathbb{R}^4)} \lesssim \| u(t_0) \|_{\dot{H}^{1}_v} + \| \nabla (|u|^2 u) \|_{L^4_t L^{4/3}_x (I \times \mathbb{R}^4)} \\
\lesssim \| \tilde{u}(t_0) \|_{\dot{H}^{1}_v} + \| u(t_0) - \tilde{u}(t_0) \|_{\dot{H}^{1}_v} + \| u \|_{L^6_{t,x}(I \times \mathbb{R}^4)}^2 \| u \|_{S(I \times \mathbb{R}^4)} \\
\lesssim E + E' + (\varepsilon + \delta)^2 \| u \|_{S(I \times \mathbb{R}^4)},
\]
which proves (3.9), provided \( \varepsilon_0 \) is chosen sufficiently small depending on \( E \) and \( E' \). \( \square \)

We will also need the following version of the above lemma that deals with near-solutions with large but finite \( L^6_{t,x} \)-norms.

**Lemma 3.3 (Long-time perturbations).** Let \( I \) be a compact time interval, and let \( \tilde{u} \) be function on \( I \times \mathbb{R}^4 \) that obeys the bounds
\[
\| \tilde{u} \|_{L^6_{t,x}(I \times \mathbb{R}^4)} \leq M \tag{3.14}
\]
\[
\| \tilde{u} \|_{L^\infty_{t} \dot{H}^{1}_v(I \times \mathbb{R}^4)} \leq E \tag{3.15}
\]
for some $M, E > 0$. Suppose also that $\tilde{u}$ is a near-solution to (1.1) in the sense of (3.1) for some function $e$. Let $t_0 \in I$ and let $u(t_0)$ close to $\tilde{u}(t_0)$ in the sense that

(3.16) \[ \|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1_x} \leq E' \]

for some $E' > 0$. Assume also the smallness conditions

(3.17) \[ \|\nabla e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L^3_t L^{3/2}_x(I \times \mathbb{R}^4)} \leq \varepsilon \]

(3.18) \[ \|\nabla e\|_{L^2_t L^{4/3}_x(I \times \mathbb{R}^4)} \leq \varepsilon \]

for some $0 < \varepsilon < \varepsilon_1$, where $\varepsilon_1 = \varepsilon_1(E, E', M)$ is a small positive constant.

Then there exists a solution $u$ to (1.1) on $I \times \mathbb{R}^4$ with the specified data $u(t_0)$ at $t_0$ satisfying

(3.19) \[ \|u - \tilde{u}\|_{L^6_t L^6_x(I \times \mathbb{R}^4)} \leq C(E, E', M)\varepsilon \]

(3.20) \[ \|u - \tilde{u}\|_{S^1(I \times \mathbb{R}^4)} \leq C(E, E', M) \]

(3.21) \[ \|u\|_{S^1(I \times \mathbb{R}^4)} \leq C(E, E', M). \]

Remark 3.4. The same computation as in Remark 3.1 shows that assumption (3.17) is redundant if one assumes $E' = O(\varepsilon)$. Also notice that if we take $e = 0$ the lemma implies local well-posedness in the energy space whenever the $L^6_t L^6_x$-norm is finite.

Proof. Without loss of generality we may assume $t_0 = \inf I$. Let $\varepsilon_0 = \varepsilon_0(E, 2E')$ be as in the previous lemma (we must replace $E'$ by $2E'$ because the kinetic energy of $u - \tilde{u}$ will grow in time).

We first note that (3.14) implies $u \in S^1(I \times \mathbb{R}^4)$. Indeed, subdividing $I$ into $N_0 \sim (1 + M/\varepsilon_0)^6$ subintervals $J_k$ such that on each $J_k$ we have

\[ \|\tilde{u}\|_{L^6_t L^6_x(J_k \times \mathbb{R}^4)} \leq \varepsilon_0, \]

and using Lemma 2.2 and Lemma 2.4, we estimate

\[ \|\tilde{u}\|_{S^1(J_k \times \mathbb{R}^4)} \lesssim \|\tilde{u}(t_0)\|_{H^1} + \|\nabla(|\tilde{u}|^2\tilde{u})\|_{L^3_t L^{3/2}_x(J_k \times \mathbb{R}^4)} + \|\nabla e\|_{L^3_t L^{3/2}_x(J_k \times \mathbb{R}^4)} \]

\[ \lesssim E + \|\tilde{u}\|_{L^3_t L^{3/2}_x(J_k \times \mathbb{R}^4)}^2 + \|\tilde{u}\|_{S^1(J_k \times \mathbb{R}^4)} + \varepsilon \]

\[ \lesssim E + \varepsilon_0^2 \|\tilde{u}\|_{S^1(J_k \times \mathbb{R}^4)} + \varepsilon. \]

Thus, for $\varepsilon_0$ sufficiently small, this yields

\[ \|\tilde{u}\|_{S^1(J_k \times \mathbb{R}^4)} \lesssim E + \varepsilon. \]
Summing these bounds over all the intervals $J_k$ we obtain

$$
\| \tilde{u} \|_{S^1(I \times \mathbb{R}^4)} \leq C(M, E, \varepsilon_0),
$$

which also implies by Lemma 2.1 that

$$
\| \nabla \tilde{u} \|_{L^6_t L^{12/5}_x(I \times \mathbb{R}^4)} \leq C(M, E, \varepsilon_0).
$$

We can now subdivide $I$ into $N_1 = N_1(M, E, \varepsilon_0)$ subintervals $I_j = [t_j, t_{j+1}]$ such that on each $I_j$ we have

$$
\| \tilde{u} \|_{L^6_t L^{12/5}_x(I_j \times \mathbb{R}^4)} \leq \varepsilon_0.
$$

Choosing $\varepsilon_1$ sufficiently small depending on $\varepsilon_0, N_1, E,$ and $E'$, Lemma 3.1 implies that for each $j$ and all $0 < \varepsilon < \varepsilon_1$,

$$
\| u - \tilde{u} \|_{L^6_t L^{12/5}_x(I_j \times \mathbb{R}^4)} \leq C(j) \varepsilon,
$$

$$
\| u - \tilde{u} \|_{S^1(I_j \times \mathbb{R}^4)} \leq C(j)(\varepsilon + E')
$$

$$
\| u \|_{S^1(I_j \times \mathbb{R}^4)} \leq C(j)(E' + E)
$$

$$
\| \nabla [(i \partial_t + \Delta)(u - \tilde{u}) + e] \|_{L^6_t L^{12/5}_x(I_j \times \mathbb{R}^4)} \leq C(j) \varepsilon,
$$

provided we can show that (3.16) and (3.17) hold when $t_0$ is replaced by $t_j$. We check this using an inductive argument. By Strichartz we have the bounds

$$
\| u(t_{j+1}) - \tilde{u}(t_{j+1}) \|_{H^s_1} \lesssim \| u(t_0) - \tilde{u}(t_0) \|_{H^s_1} + \| \nabla e \|_{L^2_t L^{12/5}_x(I \times \mathbb{R}^4)}
$$

$$
+ \| \nabla [(i \partial_t + \Delta)(u - \tilde{u}) + e] \|_{L^2_t L^{4/3}_x([t_0, t_{j+1}] \times \mathbb{R}^4)}
$$

$$
\lesssim E' + \sum_{k=0}^j C(k) \varepsilon,
$$

and similarly

$$
\| \nabla e^{(t-t_{j+1})\Delta}(u(t_{j+1}) - \tilde{u}(t_{j+1})) \|_{L^6_t L^{12/5}_x(I \times \mathbb{R}^4)}
$$

$$
\lesssim \| \nabla e^{(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0)) \|_{L^2_t L^{12/5}_x(I \times \mathbb{R}^4)} + \| \nabla e \|_{L^2_t L^{4/3}_x(I \times \mathbb{R}^4)}
$$

$$
+ \| \nabla [(i \partial_t + \Delta)(u - \tilde{u}) + e] \|_{L^2_t L^{4/3}_x([t_0, t_{j+1}] \times \mathbb{R}^4)}
$$

$$
\leq \sum_{k=0}^j C(k) \varepsilon,
$$
where $C(k)$ depends on $k, E, E'$, and $\varepsilon_0$. Choosing $\varepsilon_1$ sufficiently small depending on $\varepsilon_0, N_1, E, E'$, we can continue the inductive argument.

Remark 3.5. The dependence of $\varepsilon_1$ on the parameters $M, E,$ and $E'$ is extremely bad. Specifically, $\varepsilon_1(M, E, E') \approx \exp(-(MC(E)C(E'))^C)$. Since we’ll use this lemma frequently, the bounds in Theorem 1.1 will grow quite rapidly in $E$. For remarks on the final bound see Section 8. It is of some interest to determine better bounds for the theorem, but for us it will suffice that they remain finite.

We’ll also need the following related result concerning the persistence of $L^2, H^1_x,$ and $H^2_x$ regularity.

**Lemma 3.6 (Persistence of regularity).** Let $k = 0, 1, 2$, $I$ a compact time interval, and $u$ a finite-energy solution to (1.1) obeying

$$\|u\|_{L^6_t(x|\mathbb{R}^4)} \leq M.$$

Then if $t_0 \in I$ and $u(t_0) \in H^k_x$, we have

$$(3.22) \quad \|u\|_{L^6_t(x|\mathbb{R}^4)} \leq C(M, E(u)) \|u(t_0)\|_{H^k_x}.$$

So once we have $L^6_t$ control of a finite-energy solution, we control all Strichartz norms as well. If the initial data is $H^2_x$ then we also control the $S^2$-norm. By iterating, we can continue a local-in-time solution to any interval on which we have uniform control of the $L^6_t$-norm.

**Proof.** Again it suffices to prove (3.22) as an *a priori* bound. Let $\tilde{u} = u$, $e = 0$, and $E' = 0$ and apply Lemma 3.3 to get

$$\|u\|_{S^1(I \times \mathbb{R}^4)} \leq C(M, E).$$

By (2.8) we also have

$$(3.23) \quad \|\nabla^k O(u^3)\|_{L^1_t L^2_x(I \times \mathbb{R}^4)} \lesssim \|u\|_{L^6_t L^2_x(I \times \mathbb{R}^4)} \|u\|_{S^1(I \times \mathbb{R}^4)}.$$

Now, divide $I$ into $N \approx (1 + \frac{M}{\delta})^6$ subintervals $I_j = [T_j, T_{j+1}]$ on which

$$\|u\|_{L^6_t L^2_x(I \times \mathbb{R}^4)} \leq \delta$$

where $\delta$ will be chosen later. On each $I_j$ Strichartz and (3.23) yield

$$\|u\|_{S^k(I \times \mathbb{R}^4)} \lesssim \|u(T_j)\|_{H^k_x(\mathbb{R}^4)} + \|\nabla^k (|u|^2 u)\|_{L^1_t L^2_x(I \times \mathbb{R}^4)}$$

$$\lesssim \|u(T_j)\|_{H^k_x(\mathbb{R}^4)} + \|u\|_{L^6_t L^2_x(I \times \mathbb{R}^4)} \|u\|_{S^1(I \times \mathbb{R}^4)} \|u\|_{S^1(I \times \mathbb{R}^4)}.$$
So choosing $\delta \leq \frac{1}{2} C(M, E)^{-1}$ we get

$$\|u\|_{\dot{H}^k(I_j \times \mathbb{R}^4)} \lesssim \|u(T_j)\|_{\dot{H}^k(\mathbb{R}^4)}.$$  

(3.24)

The lemma follows from adding up the bounds (3.24) on each subinterval. \qed

4. Frequency localization and space concentration. Recall from the introduction that we expect a minimal energy blowup solution to be localized in both frequency and space. In this section we will prove that this is indeed the case (we will not actually prove that the solution is localized in space, just that it concentrates; see the discussion after the proof of Corollary 4.4). The first step is the following proposition:

**Proposition 4.1** (Frequency delocalization $\Rightarrow$ spacetime bound). Let $u$ be a solution to (1.1) on $I_* \times \mathbb{R}^4$ with $E(u) \leq E_{\text{crit}}$. Let $\eta > 0$ and suppose there exist a dyadic frequency $N_{lo} > 0$ and a time $t_0 \in I_*$ such that we have the energy separation conditions

$$\|P_{\leq N_{lo}} u(t_0)\|_{\dot{H}^1_x} \geq \eta$$  

(4.1)

and

$$\|P_{\geq K(\eta) N_{lo}} u(t_0)\|_{\dot{H}^1_x} \geq \eta.$$  

(4.2)

If $K(\eta)$ is sufficiently large depending on $\eta$ we have

$$\|u\|_{L^6_t L^{\infty}_x(I_* \times \mathbb{R}^4)} \leq C(\eta).$$  

(4.3)

**Proof.** Let $0 < \varepsilon = \varepsilon(\eta) \ll 1$ be a small number to be chosen later. If $K(\eta)$ is sufficiently large depending on $\varepsilon$ (i.e., $K(\eta)$ needs to be of the order $\varepsilon^{-2}$), then one can find $\varepsilon^{-2}$ disjoint intervals $[\varepsilon^2 N_j, \varepsilon^{-2} N_j]$ contained in $[N_{lo}, K(\eta) N_{lo}]$. By (1.7) and the pigeonhole principle, we may find an $N_j$ such that the interval $[\varepsilon^2 N_j, \varepsilon^{-2} N_j]$ has very little energy

$$\|P_{\varepsilon^2 N_j \leq \cdot \leq \varepsilon^{-2} N_j} u(t_0)\|_{\dot{H}^1_x} \lesssim \varepsilon.$$  

Since both the statement and conclusion of the proposition are invariant under the scaling (1.3), we normalize $N_j = 1$.

Define $u_{lo}(t_0) := P_{\varepsilon \leq \cdot} u(t_0)$ and $u_{hi}(t_0) = P_{\varepsilon^{-1} \leq \cdot} u(t_0)$. We claim that $u_{hi}$ and $u_{lo}$ have smaller energy than $u$.

**Lemma 4.2.** If $\varepsilon$ is sufficiently small depending on $\eta$, then we have

$$E(u_{lo}(t_0)), E(u_{hi}(t_0)) \leq E_{\text{crit}} - c\eta C.$$
Proof. We’ll prove this for $u_{lo}$; the proof for $u_{hi}$ is similar. Define $u_{hi'}(t_0) := P_{>\varepsilon}u(t_0)$ so that $u(t_0) = u_{lo}(t_0) + u_{hi'}(t_0)$, and consider the quantity

$$(4.4) \quad |E(u(t_0)) - E(u_{lo}(t_0)) - E(u_{hi'}(t_0))|.$$ 

By the definition of energy we can bound this by

$$(4.5) \quad |\langle \nabla u_{lo}(t_0), \nabla u_{hi'}(t_0) \rangle| + \left| \int (|u_{lo}(t)|^4 - |u_{hi'}(t)|^4) \, dx \right|.$$ 

We deal with the potential energy term first. By the pointwise estimate

$$| |u(t_0)|^4 - |u_{lo}(t_0)|^4 - |u_{hi'}(t_0)|^4 | \lesssim |u_{lo}(t_0)||u_{hi'}(t_0)|(|u_{lo}(t_0)| + |u_{hi'}(t_0)|)^2$$

and H"older, we can bound the potential energy term in (4.5) by

$$(4.6) \quad \|u_{lo}(t_0)\|_{L^\infty} \|u_{hi'}(t_0)\|_{L^4}^2 (\|u_{lo}(t_0)\|_{L^4}^2 + \|u_{hi'}(t_0)\|_{L^4}^2)^2.$$ 

An application of Bernstein yields

$$\|u_{lo}(t_0)\|_{L^\infty} \lesssim \varepsilon \|u_{lo}(t_0)\|_{L^4} \lesssim \varepsilon \|u_{lo}\|_{H^1} \lesssim \varepsilon.$$ 

Similarly

$$\|u_{hi'}(t_0)\|_{L^4} \lesssim \sum_{N>\varepsilon} \|P_N u(t_0)\|_{L^4} \lesssim \sum_{N>\varepsilon} N^{-1} \|P_N u(t_0)\|_{H^1}$$

$$\lesssim \sum_{N>1/\varepsilon^2} N^{-1} + \sum_{\varepsilon < N \leq 1/\varepsilon^2} N^{-1} \varepsilon \lesssim \varepsilon^2 + \varepsilon (1/\varepsilon - \varepsilon^2)$$

$$\lesssim 1.$$ 

Thus (4.6) $\lesssim \varepsilon$.

Now we deal with the kinetic part of (4.5). We estimate

$$|\langle \nabla u_{lo}(t_0), \nabla u_{hi'}(t_0) \rangle| \lesssim |\langle \nabla P_{>\varepsilon} P_{\leq \varepsilon} u(t_0), \nabla u(t_0) \rangle|$$

$$\lesssim \|\nabla P_{>\varepsilon} P_{\leq \varepsilon} u(t_0)\|_{L^2} \|\nabla u(t_0)\|_{L^2}.$$ 

As

$$\|\nabla u(t_0)\|_{L^2} \lesssim 1$$

and

$$\|\nabla P_{>\varepsilon} P_{\leq \varepsilon} u(t_0)\|_{L^2} = \| (\nabla P_{>\varepsilon} P_{\leq \varepsilon} u(t_0)) \|_{L^2}$$

we can bound the kinetic energy term in (4.5) by

$$\|\nabla u_{lo}(t_0)\|_{L^2} \|\nabla u_{hi'}(t_0)\|_{L^2} \lesssim \varepsilon.$$ 

Thus (4.5) $\lesssim \varepsilon$.
we get as a final bound on kinetic energy

\[
\left| \langle \nabla u_{lo}(t_0), \nabla u_{hi}(t_0) \rangle \right| \lesssim \varepsilon.
\]

Therefore (4.4) \( \lesssim \varepsilon \). As

\[
E(u) \leq E_{\text{crit}},
\]

and by hypothesis

\[
E(u_{hi}(t_0)) \gtrsim \| u_{hi}(t_0) \|_{\dot{H}^1_x}^2 \gtrsim \eta^2,
\]

the triangle inequality implies \( E(u_{lo}(t_0)) \leq E_{\text{crit}} - c\eta^C \).

Now since \( E(u_{lo}(t_0)), E(u_{hi}(t_0)) \leq E_{\text{crit}} - c\eta^C < E_{\text{crit}} \) we can apply Lemma 1.2 to deduce that there exist solutions \( u_{lo} \) and \( u_{hi} \) on the slab \( I_s \times \mathbb{R}^4 \) with initial data \( u_{lo}(t_0) \) and \( u_{hi}(t_0) \) such that

\[
\| u_{lo} \|_{\dot{H}^s(I_s \times \mathbb{R}^4)} \lesssim C(\eta)
\]

\[
\| u_{hi} \|_{\dot{H}^s(I_s \times \mathbb{R}^4)} \lesssim C(\eta).
\]

Define \( \tilde{u} := u_{lo} + u_{hi} \). We claim that \( \tilde{u} \) is a near-solution to (1.1).

Lemma 4.3. We have

\[
i\tilde{u}_t + \Delta \tilde{u} = |\tilde{u}|^2 \tilde{u} - e
\]

where the error \( e \) obeys the bound

(4.7) \( \| \nabla e \|_{L^2_t L^{4/3}_x(I_s \times \mathbb{R}^4)} \lesssim C(\eta)\varepsilon^{1/2} \).

Proof. By the above estimates and (1.13) we have that \( \| u_{hi}(t_0) \|_{L^2_x} \lesssim \varepsilon \), and so by (3.22) we get

(4.8) \( \| u_{hi} \|_{\dot{H}^s(I_s \times \mathbb{R}^4)} \lesssim C(\eta)\varepsilon \).

Similarly, from (1.7) and Bernstein we have

\[
\| u_{lo}(t_0) \|_{\dot{H}^2_x} \lesssim \varepsilon \| u_{lo}(t_0) \|_{\dot{H}^1_x} \lesssim \varepsilon \| u_{lo} \|_{L^\infty_t \dot{H}^1_x(I_s \times \mathbb{R}^4)} \lesssim C(\eta)\varepsilon,
\]
and so by (3.22)
\[(4.9) \quad \|u_0\|_{\mathcal{G}(I_\ast \times \mathbb{R}^4)} \lesssim C(\eta)\varepsilon.\]

From Lemma 2.4 we have the additional bounds
\[
\|\nabla(|u_{hi}|^2u_{hi})\|_{L^1_t L^2_x(I_\ast \times \mathbb{R}^4)} \lesssim C(\eta)\varepsilon
\]
\[
\|\nabla(|u_{lo}|^2u_{lo})\|_{L^1_t L^2_x(I_\ast \times \mathbb{R}^4)} \lesssim C(\eta)\varepsilon
\]
\[
\|\nabla^2(|u_{lo}|^2u_{lo})\|_{L^1_t L^2_x(I_\ast \times \mathbb{R}^4)} \lesssim C(\eta)\varepsilon.
\]

Applying Lemma 2.5 we see
\[
\|\nabla^3(u_{hi}^3-u_{lo}^3)\|_{L^1_t L^{2/3}_x(I_\ast \times \mathbb{R}^4)} \lesssim C(\eta)\varepsilon^{1/2}
\]
for \(j = 1, 2\). Since \(e = \sum_{j=1}^2 \mathcal{O}(u_{hi}^3u_{lo}^{3-j})\) the claim follows. \(\square\)

We derive estimates on \(u\) from those on \(\tilde{u}\) via perturbation theory. More precisely, we know that
\[
\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1_x} \lesssim \varepsilon
\]
and
\[
\|
\tilde{u}\|_{L^2_t L^4_x(I_\ast \times \mathbb{R}^4)} \lesssim \|u_0\|_{\dot{S}^1(I_\ast \times \mathbb{R}^4)} + \|u_{hi}\|_{\dot{S}^1(I_\ast \times \mathbb{R}^4)} \lesssim C(\eta).
\]
So if \(\varepsilon\) is sufficiently small depending on \(\eta\), we can apply Lemma 3.3 to deduce the bound (4.3). This concludes the proof of Proposition 4.1. \(\square\)

Comparing (4.3) with (1.5) gives the desired contradiction if \(u\) satisfies the hypotheses of Proposition 4.1. We therefore expect \(u\) to be localized in frequency for each \(t\). Indeed we have:

**Corollary 4.4 (Frequency localization of energy at each time).** Let \(u\) be a minimal energy blowup solution of (1.1). Then for each time \(t_0 \in I_\ast\) there exists a dyadic frequency \(N(t) \in 2\mathbb{Z}\) such that for every \(\eta_3 \leq \eta \leq \eta_0\) we have small energy at frequencies \(\ll N(t)\)
\[
(4.10) \quad \|P_{\leq C(\eta)N(t)}u(t)\|_{\dot{H}^1_x} \leq \eta
\]
small energy at frequencies \(\gg N(t)\)
\[
(4.11) \quad \|P_{\geq C(\eta)N(t)}u(t)\|_{\dot{H}^1_x} \leq \eta
\]
and large energy at frequencies $\sim N(t)$

\begin{equation}
\|P_{c(\eta)N(t)N(t)}u(t)\|_{\dot{H}^1_x} \sim 1
\end{equation}

where the values of $0 < c(\eta) \ll 1 \ll C(\eta) < \infty$ depend on $\eta$.

Proof. For $t \in I_*$ define

\[ N(t) := \sup\{N \in 2\mathbb{Z} : \|P_{\leq N}u(t)\|_{\dot{H}^1_x} \leq \eta_0 \}. \]

As $u$ is Schwartz, $N(t) > 0$; as $\|u\|_{L^\infty_t \dot{H}^1_x} \sim 1$, $N(t) < \infty$. From the definition of $N(t)$ we have that $\|P_{\leq 2N(t)}u(t)\|_{\dot{H}^1_x} > \eta_0$.

Let $\eta_3 \leq \eta \leq \eta_0$. If $C(\eta) \gg 1$ then we must have $\|P_{\geq C(\eta)N(t)}u(t)\|_{\dot{H}^1_x} \leq \eta$, since otherwise Proposition 4.1 would imply $\|u\|_{L^6_t(L^\infty_x \mathbb{R}^4)} \lesssim C(\eta)$, which would contradict $u$ being a minimal energy blowup solution.

Similarly, $\|u\|_{L^\infty_t \dot{H}^1_x} \sim 1$ implies that

\[ \|P_{c(\eta)N(t)N(t)}u(t_0)\|_{\dot{H}^1_x} \sim 1, \]

and therefore that

\[ \|P_{c(\eta)N(t)N(t)}u(t_0)\|_{\dot{H}^1_x} \sim 1 \]

for all $\eta_3 \leq \eta \leq \eta_0$. Thus, if $c(\eta) \ll 1$ then $\|P_{\leq c(\eta)N(t)}u(t)\|_{\dot{H}^1_x} \leq \eta$ for all $\eta_3 \leq \eta \leq \eta_0$, since otherwise Proposition 4.1 would again imply $\|u\|_{L^6_t(L^\infty_x \mathbb{R}^4)} \lesssim C(\eta)$.

Having shown that a minimal energy blowup solution must be localized in frequency, we turn our attention to space. While it is true that such a solution is localized in space (the methods employed in [11] carry through to the four dimensional energy-critical NLS case), we will only need the following weaker result concerning concentration of the solution (roughly, concentration will mean large at some point, while we reserve localization to mean simultaneously concentrated and small at points far from the concentration point). To obtain the concentration result, we divide the interval $I_*$ into three consecutive subintervals $I_* = I_- \cup I_0 \cup I_+$, each containing a third of the $L^6_t$ density of $u$:

\[ \int_I \int_{\mathbb{R}^4} |u(t,x)|^6 \, dx \, dt = \frac{1}{3} \int_{I_*} \int_{\mathbb{R}^4} |u(t,x)|^6 \, dx \, dt \text{ for } I = I_-, I_0, I_. \]
It is on the middle interval $I_0$ that we will show physical space concentration. The first step is:

**Proposition 4.5 (Potential energy bounded from below).** For any minimal energy blowup solution of (1.1) and all $t \in I_0$ we have

$$(4.13) \quad \|u(t)\|_{L^4_x} \geq \eta_1.$$  

**Proof.** We will use an idea of Bourgain [2]. If the linear evolution of the solution does not concentrate at some point in spacetime, then we can use the small data theory and iterate. So say the linear evolution concentrates at some point $(t_1, x_1)$. If the linear evolution is small in $L^4_t \times \mathbb{R}^4$ at time $t_0$, we show $t_0$ must be far from $t_1$. We then remove the energy concentrating at $(t_1, x_1)$ and use induction on energy.

More formally, we’ll argue by contradiction. Suppose there exists some time $t_0 \in I_0$ such that

$$(4.14) \quad \|u(t_0)\|_{L^4_x} \lesssim \eta_1.$$  

Using (1.3) we scale $N(t_0) = 1$. If the linear evolution $e^{i(t-t_0)\Delta}u(t_0)$ had small $L^6_t \times \mathbb{R}^4$-norm, then by perturbation theory the nonlinear solution would have small $L^6_t \times \mathbb{R}^4$-norm as well. Hence, we may assume $\|e^{i(t-t_0)\Delta}u(t_0)\|_{L^6_t (\mathbb{R} \times \mathbb{R}^4)} \gtrsim 1$.

On the other hand Corollary 4.4 implies that

$$\|P_{lo}u(t_0)\|_{\dot{H}^1_x} + \|P_{hi}u(t_0)\|_{\dot{H}^1_x} \lesssim \eta_0,$$

where $P_{lo} = P_{\prec (\eta_0)}$ and $P_{hi} = P_{\succ C(\eta_0)}$. Then by Strichartz

$$\|e^{i(t-t_0)\Delta}P_{lo}u(t_0)\|_{L^6_t (\mathbb{R} \times \mathbb{R}^4)} + \|e^{i(t-t_0)\Delta}P_{hi}u(t_0)\|_{L^6_t (\mathbb{R} \times \mathbb{R}^4)} \lesssim \eta_0.$$

Thus

$$\|e^{i(t-t_0)\Delta}P_{med}u(t_0)\|_{L^6_t (\mathbb{R} \times \mathbb{R}^4)} \sim 1,$$

where $P_{med} = 1 - P_{lo} - P_{hi}$. However, $P_{med}u(t_0)$ has bounded energy (by (1.7)) and Fourier support in $c(\eta_0) \lesssim \|\xi\| \lesssim C(\eta_0)$. Another application of Strichartz yields

$$\|e^{i(t-t_0)\Delta}P_{med}u(t_0)\|_{L^2_t L^3_x} \lesssim \|P_{med}u(t_0)\|_{L^2_t L^3_x} \lesssim C(\eta_0).$$

Combining these estimates with Hölder we have

$$\|e^{i(t-t_0)\Delta}P_{med}u(t_0)\|_{L^\infty_t L^3_x} \gtrsim c(\eta_0).$$
In particular there exist a time $t_1 \in \mathbb{R}$ and a point $x_1 \in \mathbb{R}^4$ so that

$$|e^{i(t-t_0)\Lambda}(P_{\text{med}}u(t_0))(x_1)| \gtrsim c(\eta_0).$$

We may perturb $t_1$ so that $t_1 \neq t_0$, and by time reversal symmetry we may take $t_1 < t_0$. Let $\delta_{x_1}$ be the Dirac mass at $x_1$. Define $f(t_1) := P_{\text{med}}\delta_{x_1}$ and for $t > t_1$ define $f(t) := e^{i(t-t_1)\Lambda}f(t_1)$. One should think of $f(t_1)$ as basically $u$ at $(t_1, x_1)$. The point is then to compare $u(t_0)$ to the linear evolution of $f(t_1)$ at time $t_0$. We will show that $f(t)$ decays rapidly in any $L^p_x$-norm for $1 \leq p \leq \infty$.

**Lemma 4.6.** For any $t \in \mathbb{R}$ and any $1 \leq p \leq \infty$ we have

$$\|f(t)\|_{L^p_x} \lesssim C(\eta_0)(t-t_1)^{-\frac{1}{2} - 2}.$$

**Proof.** We translate so that $t_1 = x_1 = 0$, then use Bernstein and the unitarity of $e^{it\Delta}$ to get

$$\|f(t)\|_{L^\infty_x} \lesssim C(\eta_0)\|f(t)\|_{L^2_x} = C(\eta_0)\|P_{\text{med}}\delta_{x_1}\|_{L^2_x} \lesssim C(\eta_0).$$

By (1.10) we also have

$$\|f(t)\|_{L^\infty_x} \lesssim |t|^{-2}\|P_{\text{med}}\delta_{x_1}\|_{L^1_x} \lesssim C(\eta_0)|t|^{-2}.$$

Combining these two we obtain

$$\|f(t)\|_{L^\infty_x} \lesssim C(\eta_0)|t|^{-2}.$$

This proves the lemma in the case $p = \infty$.

For other $p$'s we use (1.9) to write

$$f(t, x) = \int e^{2\pi i(x\xi - 2\pi|\xi|^2)}\phi_{\text{med}}(\xi) d\xi$$

where $\phi_{\text{med}}$ is the Fourier multiplier corresponding to $P_{\text{med}}$. For $|x| \gg 1 + |t|$, repeated integration by parts shows $|f(t, x)| \lesssim |x|^{-100}$. On $|x| \lesssim 1 + |t|$, one integrates using the above $L^\infty_x$-bound. \qed

From (4.14) and Hölder we have

$$|\langle u(t_0), f(t_0) \rangle| \lesssim \|f(t_0)\|_{L^{4/3}_x}\|u(t_0)\|_{L^4_x} \lesssim \eta_1 C(\eta_0)(t_1 - t_0).$$

On the other hand

$$|\langle u(t_0), f(t_0) \rangle| = |\langle e^{i(t_1-t_0)\Lambda}P_{\text{med}}u(t_0), \delta_{x_1} \rangle| \gtrsim c(\eta_0).$$
So since \( (t_1 - t_0) \gtrsim c(\eta_0)/\eta_1 \), we have that \( t_1 \) is far from \( t_0 \). In particular, the time of concentration must be far from where the \( L_1^4 \)-norm is small. Also, from Lemma 4.6 we have that \( \nabla f \) has a small \( L_4^6 L_1^{12/5} \)-norm to the future of \( t_0 \) (recall \( t_1 < t_0 \)):

\[
\|f\|_{L_4^6 L_1^{12/5}([t_0, \infty) \times \mathbb{R}^4)} \lesssim C(\eta_0) |t_1 - t_0|^{-1/6} \lesssim C(\eta_0) \eta_1^{1/6}.
\]

Now we use the induction hypothesis. Split \( u(t_0) = v(t_0) + w(t_0) \) where \( w(t_0) = \delta e^{i\theta} \Delta^{-1} f(t_0) \) for some small \( \delta = \delta(\eta_0) > 0 \) and phase \( \theta \) to be chosen later. The point of \( \Delta^{-1} \) in the definition of \( w(t_0) \) is that our inner products will be in \( \dot{H}_x^1 \) instead of the more common \( L_2^2 \). One should think of \( w(t_0) \) as the contribution coming from the point \((t_1, x_1)\) where the solution concentrates. We will show that for an appropriate choice of \( \delta \) and \( \theta \), \( v(t_0) \) has slightly smaller energy than \( u \). By the definition of \( f \) and an integration by parts we have

\[
\frac{1}{2} \int_{\mathbb{R}^4} |\nabla v(t_0)|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u(t_0) - \nabla w(t_0)|^2 \, dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u(t_0)|^2 \, dx - \delta \text{Re} \int e^{-i\theta} \nabla \Delta^{-1} f(t_0) \cdot \nabla u(t_0) \, dx
\]

\[
+ O(\delta^2 \|\Delta^{-1} f(t_0)\|_{L_2^2}^2)
\]

\[
\leq E_{\text{crit}} + \delta \text{Re} \, e^{-i\theta} \langle u(t_0), f(t_0) \rangle + O(\delta^2 C(\eta_0)).
\]

Choosing \( \delta \) and \( \theta \) appropriately we get

\[
\frac{1}{2} \int_{\mathbb{R}^4} |\nabla v(t_0)|^2 \, dx \leq E_{\text{crit}} - c(\eta_0).
\]

Again by Lemma 4.6 we have

\[
\|w(t_0)\|_{L_4^4} \lesssim C(\eta_0) \|f(t_0)\|_{L_4^4} \lesssim C(\eta_0) (t_1 - t_0)^{-1} \lesssim C(\eta_0) \eta_1.
\]

So by (4.14) and the triangle inequality we obtain

\[
\int_{\mathbb{R}^4} |v(t_0)|^4 \, dx \lesssim C(\eta_0) \eta_1^4.
\]

Combining the above two energy estimates we see that

\[
E(v(t_0)) \leq E_{\text{crit}} - c(\eta_0),
\]
and Lemma 1.2 implies there exists a global solution $v$ of (1.1) on $[t_0, \infty) \times \mathbb{R}^4$ with data $v(t_0)$ at time $t_0$ satisfying
\[
\|v\|_{L^6_t L^3_x([t_0, \infty) \times \mathbb{R}^4)} \leq M(E_{\text{crit}} - c(\eta_0)) = C(\eta_0).
\]

However, (4.15) and frequency localization give
\[
\|\nabla e^{i(t-t_0)\Delta}w(t_0)\|_{L^{12/5}_t L^{12}_x([t_0, \infty) \times \mathbb{R}^4)} \lesssim \|\nabla \Delta^{-1}f\|_{L^{12/5}_t L^{12}_x([t_0, \infty) \times \mathbb{R}^4)} \lesssim C(\eta_0) \eta_1^{1/6}.
\]

So if $\eta_1$ is sufficiently small depending on $\eta_0$, we can apply Lemma 3.3 with \(\tilde{u} = v\) and $e = 0$ to conclude that $u$ extends to all of $[t_0, \infty)$ and obeys
\[
\|u\|_{L^6_t L^3_x([t_0, \infty) \times \mathbb{R}^4)} \lesssim C(\eta_0, \eta_1).
\]

As $[t_0, \infty)$ contains $I_+$, the above estimate contradicts (1.5) if $\eta_4$ is chosen sufficiently small. This concludes the proof of Proposition 4.5.

Using (4.13) we can deduce the desired concentration result:

**Proposition 4.7** (Spatial concentration of energy at each time). For any minimal energy blowup solution of (1.1) and for each $t \in I_0$, there exists $x(t) \in \mathbb{R}^4$ such that
\[
\int_{|x-x(t)| \leq C(\eta_1)/N(t)} |\nabla u(t, x)|^2 \, dx \gtrsim c(\eta_1)
\]

and
\[
\int_{|x-x(t)| \leq C(\eta_1)/N(t)} |u(t, x)|^p \, dx \gtrsim c(\eta_1) N(t)^{p-4}
\]

for all $1 < p < \infty$, where the implicit constants depend on $p$. In particular
\[
\int_{|x-x(t)| \leq C(\eta_1)/N(t)} |u(t, x)|^4 \, dx \gtrsim c(\eta_1).
\]

Note that $u(t, x)$ is roughly of size $N(t)$ on average when $|x-x(t)| \leq N(t)^{-1}$, which is consistent with both the concentration of energy and the uncertainty principle.

**Proof.** Fix $t$ and normalize $N(t) = 1$. By Corollary 4.4 we have
\[
\|P_{\langle c(\eta_1)\rangle} u(t)\|_{\dot{H}^1_x} + \|P_{\langle c(\eta_1)\rangle} u(t)\|_{\dot{H}^1_x} \lesssim \eta_1^{100}.
\]
Sobolev embedding implies
\[ \|P_{<\epsilon_1} u(t)\|_{L^4} + \|P_{>C(\eta_1)} u(t)\|_{L^4} \lesssim \eta_1^{100} \]
and so by (4.13)
\[ \|P_{\text{med}} u(t)\|_{L^4} \gtrsim \eta_1, \]
where \( P_{\text{med}} = P_{\epsilon_1} \leq \cdot \leq C(\eta_1). \) On the other hand, by (1.7) we have
\[ \|P_{\text{med}} u(t)\|_{L^2} \lesssim C(\eta_1). \]

By Hölder
\[ \eta_1 \lesssim \|P_{\text{med}} u(t)\|_{L^4} \lesssim \|P_{\text{med}} u(t)\|_{L^\infty}^{1/2} \cdot \|P_{\text{med}} u(t)\|_{L^2}^{1/2} \lesssim C(\eta_1)\|P_{\text{med}} u(t)\|_{L^\infty}^{1/2}, \]
which implies that
\[ \|P_{\text{med}} u(t)\|_{L^\infty} \gtrsim c(\eta_1). \]

In particular, there exists a point \( x(t) \in \mathbb{R}^4 \) so that
\[ (4.19) \quad c(\eta_1) \lesssim |P_{\text{med}} u(t, x(t))|. \]

As our function is now localized both in frequency and in space, all the Sobolev norms are practically equivalent. So let’s consider the operator \( P_{\text{med}} \nabla \Delta^{-1} \) and let \( K_{\text{med}} \) denote its kernel. Then
\[
c(\eta_1) \lesssim |P_{\text{med}} u(t, x(t))| \lesssim |K_{\text{med}} \ast \nabla u(t, x(t))| \lesssim \int |K_{\text{med}}(x(t) - x)||\nabla u(t, x)| \, dx \\
\sim \int_{|x-x(t)|<C(\eta_1)} |K_{\text{med}}(x(t) - x)||\nabla u(t, x)| \, dx \\
+ \int_{|x-x(t)|\geq C(\eta_1)} |K_{\text{med}}(x(t) - x)||\nabla u(t, x)| \, dx \\
\lesssim C(\eta_1) \left( \int_{|x-x(t)|<C(\eta_1)} |\nabla u(t, x)|^2 \, dx \right)^{1/2} + \int_{|x-x(t)|\geq C(\eta_1)} \frac{|\nabla u(t, x)|}{|x-x(t)|^{100}} \, dx, \]
where in order to obtain the last inequality we used Cauchy-Schwarz and that \( K_{\text{med}} \) is a Schwartz function. Therefore, by (1.7) and possibly making \( C(\eta_1) \) larger, we have
\[
c(\eta_1) \lesssim \left( \int_{|x-x(t)|<C(\eta_1)} |\nabla u(t, x)|^2 \, dx \right)^{1/2} + C(\eta_1)^{-\alpha}
\]
for some \( \alpha > 0 \), proving (4.16).
Now let \( \tilde{K}_{med} \) be the kernel associated to \( P_{med} \), and let \( 1 < p < \infty \). As above we get

\[
c(\eta_1) \lesssim \int |\tilde{K}_{med}(x(t) - x)| |u(t,x)| \, dx \\
\sim \int_{|x-x(0)| < C(\eta_1)} |\tilde{K}_{med}(x(t) - x)| |u(t,x)| \, dx \\
+ \int_{|x-x(0)| \geq C(\eta_1)} |\tilde{K}_{med}(x(t) - x)| |u(t,x)| \, dx \\
\lesssim C(\eta_1) \left( \int_{|x-x(0)| < C(\eta_1)} |u(t,x)|^p \, dx \right)^{1/p} \\
+ \|u(t)\|_{L^2} \left( \int_{|x-x(0)| \geq C(\eta_1)} \frac{1}{|x-x(t)|^{100/3}} \, dx \right)^{3/4} \\
\lesssim C(\eta_1) \left( \int_{|x-x(0)| < C(\eta_1)} |u(t,x)|^p \, dx \right)^{1/p} + C(\eta_1)^{-\alpha}
\]

for some \( \alpha > 0 \) which, after undoing the scaling, proves (4.17) if \( C(\eta_1) \) is sufficiently large.

\[\square\]

5. Frequency-localized interaction Morawetz inequality. The goal of this section is to prove:

PROPOSITION 5.1 (Frequency-localized interaction Morawetz estimate).
Roughly speaking, this proposition states that after throwing away some low frequency components of the minimal energy blowup solution, the remainder obeys good \( L^2_t H^{-1/2} \) estimates. Assuming \( u \) is a minimal energy blowup solution of (1.1) and \( N_* < c(\eta_1) N_{min} \), we have

\[
\int_{I_0} \int_{R^4} \int_{R^4} \frac{|P_{\geq N_*} u(t,x)|^2 |P_{\geq N_*} u(t,y)|^2}{|x-y|^{3}} \, dx \, dy \, dt \lesssim \eta_1 N_*^{-3}. \tag{5.1}
\]

In order to prove the proposition we introduce an interaction potential generalization of the classical Morawetz inequality.

5.1. An interaction virial identity and a general interaction Morawetz estimate for general equations. We start by recalling the standard Morawetz action centered at a point for general equations. Let \( a \) be a function on the slab \( I \times R^n \) and \( \phi \) satisfy \( i \phi_t + \Delta \phi = N \) on \( I \times R^n \). We define the virial potential to be

\[
V_a(t) = \int_{R^n} a(x)|\phi(t,x)|^2 \, dx
\]
and the Morawetz action centered at zero to be

\[ M_0^0(t) = 2 \int_{\mathbb{R}^n} a_j(x) \Im \left( \overline{\phi(x)} \phi_j(x) \right) dx. \]

A computation shows that

\[ \partial_t V_a = M_0^0 + 2 \int_{\mathbb{R}^n} a \{ N, \phi \}_m dx, \]

where the mass bracket is defined to be \( \{ f, g \}_m = \Im (f \overline{g}) \). Note that in the particular case when \( N = F'(|\phi|^2)\phi \) one has \( M_0^0 = \partial_t V_a \).

Another calculation establishes:

**Lemma 5.2.**

\[ \partial_t M_0^0 = \int_{\mathbb{R}^n} (-\Delta \Delta a) |\phi|^2 + 4 \int_{\mathbb{R}^n} a_{jk} \Re (\overline{\phi_j} \phi_k) + 2 \int_{\mathbb{R}^n} a_j \{ N, \phi \}_p^j, \]

where we define the momentum bracket to be \( \{ f, g \}_p = \Re (f \nabla \overline{g} - g \nabla f) \) and repeated indices are implicitly summed.

Note again that when \( N = F'(|\phi|^2)\phi \) we have \( \{ N, \phi \}_p = -\nabla G(|\phi|^2) \) for \( G(x) = xF'(x) - F(x) \). In particular, in the cubic case \( \{ N, \phi \}_p = -\frac{1}{2} \nabla (|\phi|^4) \).

Now let \( a(x) = |x| \). Easy computations show that for dimension \( n \geq 4 \) we have the following identities:

\[ a_j(x) = \frac{x_j}{|x|} \]
\[ a_{jk}(x) = \frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} \]
\[ \Delta a(x) = \frac{n - 1}{|x|} \]
\[ -\Delta \Delta a(x) = \frac{(n - 1)(n - 3)}{|x|^3}. \]

For this choice of the function \( a \), one should interpret the \( M_0^0 \) as a spatial average of the radial component of the \( L^2 \)-mass current. Taking its time derivative we get

\[ \partial_t M_0^0 = (n - 1)(n - 3) \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^3} dx + 4 \int_{\mathbb{R}^n} \left( \frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} \right) \Re (\overline{\phi_j} \phi_k)(x) dx \]
\[ + 2 \int_{\mathbb{R}^n} \frac{x_j}{|x|} \{ N, \phi \}_p^j(x) dx \]
= (n - 1)(n - 3) \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^3} \, dx + 4 \int_{\mathbb{R}^n} \frac{1}{|x|} |\nabla_0 \phi(x)|^2 \, dx
+ 2 \int_{\mathbb{R}^n} \frac{x}{|x|} \{\mathcal{N}, \phi\}_{\rho}(x) \, dx,

where we use \( \nabla_0 \) to denote the complement of the radial portion of the gradient, that is \( \nabla_0 = \nabla - \frac{x}{|x|} (\frac{x}{|x|} \nabla) \).

We may center the above argument at any other point \( y \in \mathbb{R}^4 \). Choosing \( a(x) = |x - y| \), we define the Morawetz action centered at \( y \) to be

\[ M_y^a(t) = 2 \int_{\mathbb{R}^n} \frac{x - y}{|x - y|} \Im(\overline{\phi(x)} \nabla \phi(x)) \, dx. \]

The same computations now yield

\[ \partial_t M_y^a = (n - 1)(n - 3) \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x - y|^3} \, dx + 4 \int_{\mathbb{R}^n} \frac{1}{|x - y|} |\nabla_y \phi(x)|^2 \, dx
+ 2 \int_{\mathbb{R}^n} \frac{x - y}{|x - y|} \{\mathcal{N}, \phi\}_{\rho}(x) \, dx. \]

We are now ready to define the interaction Morawetz potential, which is a way of quantifying how mass is interacting with (moving away from) itself:

\[ M_{\text{interact}}(t) = \int_{\mathbb{R}^n} |\phi(t, y)|^2 M_y^a(t) \, dy \]

\[ = 2 \Im \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\phi(t, y)|^2 \frac{x - y}{|x - y|} \nabla \phi(t, x) \overline{\phi(t, x)} \, dx \, dy. \]

One gets immediately the easy estimate

\[ |M_{\text{interact}}(t)| \leq 2 \|\phi(t)\|_{L_x^3}^2 \|\phi(t)\|_{H_x^1}. \]

Calculating the time derivative of the interaction Morawetz potential we get the following virial-type identity,

(5.2) \[ \partial_t M_{\text{interact}} = (n - 1)(n - 3) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(y)|^2 |\phi(y)|^2}{|x - y|^3} \, dx \, dy \]

(5.3) \[ + 4 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(y)|^2 |\nabla_y \phi(x)|^2}{|x - y|} \, dx \, dy \]

(5.4) \[ + 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(y)|^2}{|x - y|} (x - y) \{\mathcal{N}, \phi\}_{\rho}(x) \, dx \, dy \]
\begin{equation}
(5.5) \quad + 2 \int_{\mathbb{R}^n} \partial_{y_k} \text{Im} (\bar{\phi} \partial_k (y) M_u^y) \, dy \\
(5.6) \quad + 4 \text{Im} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{N, \phi\}_m(y) \frac{x - y}{|x - y|} \nabla \phi(x) \bar{\phi}(x) \, dx \
\end{equation}

As far as the terms in the above virial-type identity are concerned, we will establish

**Lemma 5.3.** $(5.5) \geq -(5.3)$.

Thus, integrating over the compact interval $I_0$ we get:

**Proposition 5.4 (Interaction Morawetz inequality).**

\begin{equation}
(n - 1)(n - 3) \int_{I_0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(t,y)|^2 |\phi(t,x)|^2}{|x - y|^3} \, dx \
\end{equation}

\begin{equation}
\quad + 2 \int_{I_0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(t,y)|^2}{|x - y|} (x - y) \{N, \phi\}_p(t,x) \, dx \
\end{equation}

\begin{equation}
\quad \leq 2 \|\phi\|_{L_x^\infty L_t^2(I_0 \times \mathbb{R}^n)}^3 \|\phi\|_{L_x^\infty H_t^1(I_0 \times \mathbb{R}^n)}^3 \
\end{equation}

\begin{equation}
\quad + 4 \int_{I_0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\{N, \phi\}_m(t,y)| \|
abla \phi(t,x)||\phi(t,x)| \, dx \
\end{equation}

Note that in the particular case $N = |u|^2 u$, after performing an integration by parts in the momentum bracket term, the inequality becomes

\begin{equation}
(n - 1)(n - 3) \int_{I_0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(t,y)|^2 |u(t,x)|^2}{|x - y|^3} \, dx \
\end{equation}

\begin{equation}
\quad + (n - 1) \int_{I_0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(t,y)|^2 |u(t,x)|^4}{|x - y|} \, dx \
\end{equation}

\begin{equation}
\quad \leq 2 \|u\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^n)}^3 \|u\|_{L_t^\infty H_t^1(I_0 \times \mathbb{R}^n)}^3.
\end{equation}

We turn now to the proof of Lemma 5.3. We write

\begin{equation}
(5.5) = 4 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_{y_k} \text{Im} (\bar{\phi}(y) \partial_k \phi(y)) \frac{x_j - y_j}{|x - y|} \text{Im} (\bar{\phi}(x) \partial_j \phi(x)) \, dx \
\end{equation}

where we sum over repeated indices. We integrate by parts moving $\partial_{y_k}$ to the unit vector $\frac{x - y}{|x - y|}$. Using the identity

\begin{equation}
\partial_{y_k} \left( \frac{x_j - y_j}{|x - y|} \right) = - \frac{\delta_{kj}}{|x - y|} + \frac{(x_k - y_k)(x_j - y_j)}{|x - y|^3}
\end{equation}
and the notation $p(x) = 2 \text{Im} (\overline{\phi(x)} \nabla \phi(x))$ for the momentum density, we rewrite (5.5) as

$$- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left[ p(y)p(x) - \left( p(y) \frac{x-y}{|x-y|} \right) \left( p(x) \frac{x-y}{|x-y|} \right) \right] \frac{dx}{|x-y|}. $$

Note that the quantity between the square brackets represents the inner product between the projection of the momentum density $p(y)$ onto the orthogonal complement of $(x-y)$ and the projection of $p(x)$ onto the same space. But

$$|\pi_{(x-y)\perp} p(y)| = \left| p(y) - \frac{x-y}{|x-y|} \left( \frac{x-y}{|x-y|} p(y) \right) \right| = 2 |\text{Im} (\overline{\phi(y)} \nabla_x \phi(y))| \leq 2 |\phi(y)||\nabla_x \phi(y)|.$$

As the same estimate holds when we switch $y$ and $x$, we get

$$|\pi_{(x-y)\perp} p(y)| \leq 2 |\phi(y)||\nabla_x \phi(y)|.$$

5.2. Morawetz inequality: the setup. We are now ready to start the proof of Proposition 5.1. As the statement is invariant under scaling, we normalize $N^* = 1$ and define $u_{hi} = P_{>1} u$ and $u_{lo} = P_{\leq 1} u$. As we assume $1 = N_s < c(\eta_1) N_{\min}$, we get $1 < c(\eta_1) N(t), \forall t \in I_0$. Provided we choose $c(\eta_1)$ sufficiently small, the frequency localization result and Sobolev yield

$$\|u_{<\eta_1^{-1}}\|_{L_t^\infty H_x^1(I_0 \times \mathbb{R}^4)} + \|u_{\leq \eta_1^{-1}}\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^4)} \lesssim \eta_1. \quad (5.7)$$

Hence, $u_{lo}$ has small energy

$$\|u_{lo}\|_{L_t^\infty H_x^1(I_0 \times \mathbb{R}^4)} + \|u_{lo}\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^4)} \lesssim \eta_1. \quad (5.8)$$

Using (1.13) and (5.7), one also sees that $u_{hi}$ has small mass

$$\|u_{hi}\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^4)} \lesssim \eta_1. \quad (5.9)$$

Our goal is to prove

$$\int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,x)|^2 |u_{hi}(t,y)|^2}{|x-y|^3} \frac{dx}{|x-y|} dt \lesssim \eta_1. \quad (5.10)$$
Since in four dimensions convolution with $1/|x|^3$ is basically the same as the fractional integration operator $|\nabla|^{-1}$, the above estimate translates into

\[(5.11) \quad \|u_{hi}\|_{L_t^2 L_x^{1/2}(I_0 \times \mathbb{R}^4)}^2 \lesssim \eta_1^{1/2}.\]

By a standard continuity argument, it will suffice to prove (5.11) under the bootstrap hypothesis

\[(5.12) \quad \|u_{hi}\|_{L_t^2 L_x^{1/2}(I_0 \times \mathbb{R}^4)}^2 \leq C_0^{1/2} \eta_1^{1/2},\]

for a large constant $C_0$ depending on energy but not on any of the $\eta$'s. More rigorously, one needs to prove that (5.12) implies (5.10) whenever $I_0$ is replaced by a subinterval of $I_0$ in order to run the bootstrap argument correctly. However, it will become clear to the reader that the argument below works not only for $I_0$ but also for any subinterval of $I_0$.

Note that a consequence of (5.12) is

\[(5.13) \quad \|P \leq N |u_{hi}|^2\|_{L_t^2 L_x^4(I_0 \times \mathbb{R}^4)} \leq N^{1/2} C_0^{1/2} \eta_1^{1/2}.\]

**Proposition 5.5.** With the notation and assumptions above we have

\[
\int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2 |u_{hi}(t,x)|^2}{|x-y|^3} \, dx \, dy \, dt + \int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2 |u_{hi}(t,x)|^4}{|x-y|} \, dx \, dy \, dt \\
\lesssim \eta_1^3
\]

\[(5.14) \quad + \eta_1 \sum_{j=0}^2 \int_{I_0} \int_{\mathbb{R}^4} |P_{hi} O(u_{hi}^{3-j})(t,y)||u_{hi}(t,y)| \, dy \, dt
\]

\[(5.15) \quad + \eta_1 \int_{I_0} \int_{\mathbb{R}^4} |P_{lo} O(u_{lo}^3)(t,y)||u_{hi}(t,y)| \, dy \, dt
\]

\[(5.16) \quad + \eta_1^2 \sum_{j=1}^3 \int_{I_0} \int_{\mathbb{R}^4} |u_{hi}(t,x)|^2 |u_{lo}(t,x)|^3-j|\nabla u_{lo}(t,x)| \, dx \, dt
\]

\[(5.17) \quad + \eta_1^2 \int_{I_0} \int_{\mathbb{R}^4} |u_{hi} \nabla P_{lo}(|u|^2 u)(t,x) |\, dx \, dt
\]

\[(5.18) \quad + \sum_{j=1}^3 \int_{I_0} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2 O(u_{hi}^{4-j})(t,x)}{|x-y|} \, dx \, dy \, dt
\]

\[(5.19) \quad + \sum_{j=1}^3 \int_{I_0} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2 O(u_{hi}^{4-j})(t,x)}{|x-y|} \, dx \, dy \, dt.
\]
Proof. Applying Proposition 5.4 with $\phi = u_{hi}$ and $\mathcal{N} = P_{hi}(|u|^2u)$ we find

\begin{align*}
3 \int_{I_0} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)|^2}{|x-y|^3} \, dx \, dy \, dt \\
+ 2 \int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2}{|x-y|}(x-y) \{P_{hi}(|u|^2u), u_{hi}\}_p(t,x) \, dx \, dy \, dt \\
\leq 2 \|u_{hi}\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^4)}^3 \|u_{hi}\|_{L_t^\infty H_x^1(I_0 \times \mathbb{R}^4)} \\
+ 4 \int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \{P_{hi}(|u|^2u), u_{hi}\}_m(t,y) ||\nabla u_{hi}(t,x)|| u_{hi}(t,x) \, dx \, dy \, dt.
\end{align*}

Observe that (5.9) plus conservation of energy dictates

$$\|u_{hi}\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^4)}^3 \|u_{hi}\|_{L_t^\infty H_x^1(I_0 \times \mathbb{R}^4)} \lesssim \eta_1^3,$$

which is the error term (5.14).

We deal first with the mass bracket term. Exploiting cancellation we write

$$\{P_{hi}(|u|^2u), u_{hi}\}_m = \{P_{hi}(|u|^2u) - |u_{hi}|^2u_{hi}, u_{hi}\}_m.$$

Writing

$$P_{hi}(|u|^2u) - |u_{hi}|^2u_{hi} = P_{hi}(|u|^2u - |u_{hi}|^2u_{hi}) - P_{lo}(|u_{hi}|^2u_{hi})$$

$$= \sum_{j=0}^{2} P_{hi}O(u_{hi}^j u_{lo}^{3-j}) - P_{lo}O(u_{hi}^3),$$

we estimate the mass bracket term by the error terms (5.15) and (5.16) as follows:

\begin{align*}
\sum_{j=0}^{2} \int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} |u_{hi}(t,x)||\nabla u_{hi}(t,x)||P_{hi}O(u_{hi}^j u_{lo}^{3-j})(t,y)||u_{hi}(t,y)| \, dx \, dy \, dt \\
+ \int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} |u_{hi}(t,x)||\nabla u_{hi}(t,x)||P_{lo}O(u_{hi}^3)(t,y)||u_{hi}(t,y)| \, dx \, dy \, dt \\
\lesssim \eta_1 \sum_{j=0}^{2} \int_{I_0} \int_{\mathbb{R}^4} |P_{hi}O(u_{hi}^j u_{lo}^{3-j})(t,y)||u_{hi}(t,y)| \, dy \, dt \\
+ \eta_1 \int_{I_0} \int_{\mathbb{R}^4} |P_{lo}O(u_{hi}^3)(t,y)||u_{hi}(t,y)| \, dy \, dt,
\end{align*}
where in order to obtain the last inequality we used

\[ \int |uhi(t,x)||\nabla uhi(t,x)| \, dx \lesssim \|uhi\|_{L^\infty_t L^2_x} \|\nabla uhi\|_{L^\infty_t L^2_x} \lesssim \eta_1. \]

We turn now towards the momentum bracket term and write

\[
\{P_{hi}(|u|^2 u), uhi\}_p = \{|u|^2 u, uhi\}_p - \{P_{lo}(|u|^2 u), uhi\}_p \\
= \{|u|^2 u, u\}_p - \{|u|^2 u, u_{lo}\}_p - \{P_{lo}(|u|^2 u), u_{lo}\}_p \\
= \{|u|^2 u, u\}_p - \{|u|^2 u_{lo}, u_{lo}\}_p - \{|u|^2 u - |u|^2 u_{lo}, u_{lo}\}_p \\
- \{P_{lo}(|u|^2 u), uhi\}_p \\
= -\frac{1}{2} \nabla(|u|^4 - |u_{lo}|^4) - \{|u|^2 u - |u|^2 u_{lo}, u_{lo}\}_p \\
- \{P_{lo}(|u|^2 u), uhi\}_p.
\]

After an integration by parts, the first term in the momentum bracket contributes a multiple of

\[
\int_{I_0} \int_{R^4} \int_{R^4} \frac{|uhi(t,y)|^2 |uhi(t,x)|^4}{|x-y|} \, dx \, dy \, dt \\
+ \sum_{j=1}^{3} \int_{I_0} \int_{R^4} \int_{R^4} \frac{|uhi(t,y)|^2 \mathcal{O}(uhi^4_{lo} u_{lo}^{3-j})(t,x)}{|x-y|} \, dx \, dy \, dt,
\]

in which we recognize (5.19) and the left-hand side term in Proposition 5.5.

In order to estimate the contribution of the second term in the momentum bracket, we write \(\{f, g\}_p = \nabla \mathcal{O}(fg) + \mathcal{O}(f \nabla g)\) and

\[
|u|^2 u - |u|^2 u_{lo} = \sum_{j=1}^{3} \mathcal{O}(uhi^4_{lo} u_{lo}^{3-j})
\]

and we decompose

\[
\{|u|^2 u - |u|^2 u_{lo}, u_{lo}\}_p = \sum_{j=1}^{3} \nabla \mathcal{O}(uhi^4_{lo} u_{lo}^{3-j}) + \sum_{j=1}^{3} \mathcal{O}(uhi^4_{lo} u_{lo}^{3-j} \nabla u_{lo}) = I + II.
\]

In order to estimate the contribution of \(I\) to the momentum bracket term, we integrate by parts to rediscover (5.19).
Taking absolute values inside the integrals, $II$ contributes to the momentum bracket term

$$
\sum_{j=1}^{3} \int_{I_0} \int_{\mathbb{R}^4} \left| u_{hi}(t,y) \right|^2 \left| u_{hi}(t,x) \right|^j \left| u_{lo}(t,x) \right|^{3-j} \left| \nabla u_{lo}(t,x) \right| \, dx \, dy \, dt
$$

$$
\lesssim \left\| u_{hi}\right\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^4)}^2 \sum_{j=1}^{3} \int_{I_0} \int_{\mathbb{R}^4} \left| u_{hi}(t,x) \right|^j \left| u_{lo}(t,x) \right|^{3-j} \left| \nabla u_{lo}(t,x) \right| \, dx \, dt
$$

$$
\lesssim \eta_1^2 \sum_{j=1}^{3} \int_{I_0} \int_{\mathbb{R}^4} \left| u_{hi}(t,x) \right|^j \left| u_{lo}(t,x) \right|^{3-j} \left| \nabla u_{lo}(t,x) \right| \, dx \, dt,
$$

which is (5.17).

The only term left to consider is the last term in the momentum bracket. When the derivative (from the definition of the momentum bracket) falls on $P_{lo}(\left| u \right|^2 u)$, we estimate its final contribution by taking absolute values inside the integrals to get

$$
\int_{I_0} \int_{\mathbb{R}^4} \left| u_{hi}(t,y) \right|^2 |u_{hi}(t,x)| \nabla P_{lo}(\left| u \right|^2 u)(t,x) \, dx \, dy \, dt
$$

$$
\lesssim \left\| u_{hi}\right\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^4)}^2 \int_{I_0} \int_{\mathbb{R}^4} \left| \nabla P_{lo}(\left| u \right|^2 u)(t,x) \right| \, dx \, dt
$$

$$
\lesssim \eta_1^2 \int_{I_0} \int_{\mathbb{R}^4} \left| u_{hi}(t,x) \right| \nabla P_{lo}(\left| u \right|^2 u)(t,x) \, dx \, dt,
$$

in which we recognize the error term (5.18).

When the derivative falls on $u_{hi}$, we first integrate by parts to get as a final contribution

$$
\int_{I_0} \int_{\mathbb{R}^4} \left| u_{hi}(t,y) \right|^2 |u_{hi}(t,x)| \nabla P_{lo}(\left| u \right|^2 u)(t,x) \, dx \, dy \, dt
$$

(5.20)

$$
+ \left| \int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \left| u_{hi}(t,y) \right|^2 \left| u_{hi}(t,x) \right| P_{lo}(\left| u \right|^2 u)(t,x) \frac{1}{|x-y|} \, dx \, dy \right|
$$

(5.21)

$$
\lesssim \eta_1^2 \int_{I_0} \int_{\mathbb{R}^4} \left| u_{hi} \nabla P_{lo}(\left| u \right|^2 u)(t,x) \right| \, dx \, dt
$$

(5.22)

$$
+ \left| \int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \left| u_{hi}(t,y) \right|^2 \left| u_{hi}(t,x) \right| P_{lo}(\left| u \right|^2 u)(t,x) \frac{1}{|x-y|} \, dx \, dy \right|.
$$

(5.23)

Decomposing $u = u_{lo} + u_{hi}$, we bound (5.23) by (5.19) and the second term on the left-hand side of Proposition 5.5.
5.3. Strichartz control on low and high frequencies. The purpose of this section is to obtain estimates on the low and high frequency parts of $u$, which we will use to bound the error terms obtained in the previous section.

**Proposition 5.6** (Strichartz control on low frequencies). We have

\[ \|u_{lo}\|_{S^1(I_0 \times \mathbb{R}^4)} \lesssim \eta_1^2 C_0^2. \]

**Proof.** Throughout the proof all spacetime norms will be on $I_0 \times \mathbb{R}^4$. Lemma 2.2 dictates

\[ \|u_{lo}\|_{S^1} \lesssim \|u_{lo}(t_0)\|_{H^1_t} + \|\nabla P_{lo}(|u|^2 u)\|_{L^2_t L^{4/3}_x}. \]

Note that according to our assumptions, $\|u_{lo}(t_0)\|_{H^1_t} \lesssim \eta_1$.

Write $\nabla P_{lo}(|u|^2 u) = \sum_{j=0}^3 \nabla P_{lo} O(u^{j-1} u_{lo})$. The contribution coming from the $j = 0$ term can be estimated by

\[ \|\nabla P_{lo} O(u^{j-1} u_{lo})\|_{L^2_t L^{4/3}_x} \lesssim \|u_{lo}\|_{L^2_t L^4_x} \|u_{lo}\|_{L^\infty_t L^2_x} \lesssim \eta_1^2 \|u_{lo}\|_{S^1}. \]

We estimate the contribution coming from the term corresponding to $j = 1$ using Bernstein and (5.9) as

\[ \|\nabla P_{lo} O(u^2 u_{lo})\|_{L^2_t L^{4/3}_x} \lesssim \|u_{lo}\|_{L^2_t L^4_x} \|u_{lo}\|_{L^\infty_t L^2_x} \lesssim \eta_1 \|u_{lo}\|_{L^2_t L^4_x} \lesssim \eta_1 \|u_{lo}\|_{S^1}. \]

The term corresponding to $j = 2$ is estimated via Bernstein and (5.13) as

\[ \|\nabla P_{lo} O(u^3 u_{lo})\|_{L^2_t L^{4/3}_x} \lesssim \|P_{lo} O(u^3 u_{lo})\|_{L^2_t L^{4/3}_x} \lesssim \|u_{lo}\|_{L^2_t L^4_x} \|P_{\leq 10}^1 u_{lo}\|_{L^2_x} \lesssim \eta_1^2 C_0^2. \]

We now turn toward the $j = 3$ term; consider a dyadic piece $\nabla P_{lo} O(u_{N1} u_{N2} u_{N3})$ for $N_1 \geq N_2 \geq N_3 \geq 1$, and notice that we can replace $u_{N2} u_{N3}$ by $P_{\leq 10 N_1}^1 (u_{N2} u_{N3})$. Using Bernstein and (5.13) we estimate

\[ \|\nabla P_{lo} O(u^3 u_{lo})\|_{L^2_t L^{4/3}_x} \lesssim \|P_{lo} O(u_{lo})\|_{L^2_t L^4_x} \lesssim \sum_{N_1 \geq N_2 \geq N_3 \geq 1} \|u_{N1}\|_{L^\infty_t L^2_x} \|P_{\leq 10 N_1}^1 (u_{N2} u_{N3})\|_{L^2_x} \lesssim \sum_{N_1 \geq N_2 \geq N_3 \geq 1} \eta_1^2 \eta_1^2 C_0^2 \lesssim \eta_1^2 C_0^2. \]
Putting everything together we obtain
\[ \|u_{lo}\|_{S_1} \lesssim \eta_1 + \eta_1^2 \|u_{lo}\|_{S_1} + \eta_1 \|u_{lo}\|_{S_1} + \eta_1^3 C_0^{1/2} + \eta_1^4 C_0^{1/2}. \]

Choosing \( \eta_1 \) sufficiently small, we get \( \|u_{lo}\|_{S_1} \lesssim \eta_1^{1/2} C_0^{1/2}. \)

**Proposition 5.7 (Strichartz control on high frequencies).** We have

\[ \|\nabla^{-1/2} u_{hi}\|_{L^2_t L^4_x(I_0 \times \mathbb{R}^4)} \lesssim C_0^{1/2} \eta_1. \] (5.25)

**Proof.** Throughout the proof all spacetime norms will be on \( I_0 \times \mathbb{R}^4 \). Strichartz dictates

\[ \|\nabla^{-1/2} u_{hi}\|_{L^2_t L^4_x} \lesssim \|\nabla^{-1/2} u_{hi}\|_{L^2_t L^8_x} + \|\nabla^{-1/2} P_{hi}(|u|^2 u)\|_{L^2_t L^4_x}. \]

We estimate the first term crudely by

\[ \|\nabla^{-1/2} u_{hi}\|_{L^2_t L^4_x} \lesssim \|u_{hi}\|_{L^\infty_t L^2_x} \lesssim \eta_1. \]

To deal with the nonlinearity we decompose \( u \) into \( u_{lo} \) and \( u_{hi} \), and write \( \nabla^{-1/2} P_{hi}(|u|^2 u) = \sum_{j=0}^3 \nabla^{-1/2} P_{hi}(u_{hi}^j u_{lo}^{3-j}). \) The control we have gained on low frequencies in Proposition 5.6 and Bernstein allow us estimate the terms corresponding to the cases \( j = 0 \) and \( j = 1 \):

\[ \|\nabla^{-1/2} P_{hi}(u_{lo}^3)\|_{L^2_t L^8_x} \lesssim \|\nabla P_{hi}(u_{lo}^3)\|_{L^2_t L^8_x} \lesssim \|\nabla u_{lo}\|_{L^2_t L^2_x} \lesssim \eta_1^2 \|u_{lo}\|_{S_1} \lesssim C_0^{1/2} \eta_1 \]
\[ \|\nabla^{-1/2} P_{hi}(u_{hi} u_{lo}^2)\|_{L^2_t L^8_x} \lesssim \|P_{hi}(u_{hi} u_{lo}^2)\|_{L^2_t L^8_x} \lesssim \|u_{hi}\|_{L^\infty_t L^2_x} \lesssim \eta_1 \|u_{lo}\|_{S_1} \lesssim C_0 \eta_1^2. \]

In order to deal with the terms corresponding to the cases \( j = 2 \) and \( j = 3 \), we will prove the more general estimate

\[ \|\nabla^{-1/2}(\nabla^{1/2} f)(\nabla^{-1/2} g)\|_{L^4_t} \lesssim \|f\|_{L^2_t} \|g\|_{L^4_t}. \] (5.26)

In order to prove (5.26), we decompose the left-hand side into \( \pi_{hh}, \pi_{lh}, \) and \( \pi_{hl} \) which represent the projections onto high-high, low-high, and high-low frequency interactions. More precisely, for any pair of functions \((\phi, \psi)\), we
write \( \pi_{h,h}(\phi, \psi) = \sum_{N> M} P_N \phi P_M \psi \), \( \pi_{l,h}(\phi, \psi) = \sum_{N \leq M} P_N \phi P_M \psi \), and \( \pi_{h,l}(\phi, \psi) = \sum_{N \geq M} P_N \phi P_M \psi \).

The high-high and low-high frequency interactions are going to be treated in the same manner. Let’s consider for example the first one. A simple application of Sobolev embedding yields

\[
\left\| \nabla^{-\frac{1}{2}} \pi_{h,h} \{ (\nabla^{\frac{1}{2}} f)(\nabla^{-\frac{1}{2}} g) \} \right\|_{L^4_x} \lesssim \left\| \pi_{h,h} \{ (\nabla^{\frac{1}{2}} f)(\nabla^{-\frac{1}{2}} g) \} \right\|_{L^8_x}.
\]

Now we only have to notice that the multiplier associated to the operator \( T(f, g) = \pi_{h,h} \{ (\nabla^{\frac{1}{2}} f)(\nabla^{-\frac{1}{2}} g) \} \), i.e.,

\[
\sum_{N \sim M} |\xi_1|^{\frac{1}{2}} \hat{P}_N f(\xi_1)|\xi_2|^{-\frac{1}{2}} \hat{P}_M g(\xi_2),
\]

is a symbol of order one with \( \xi = (\xi_1, \xi_2) \), since then a theorem of Coifman and Meyer ([6], [7]) will conclude our claim.

To deal with the \( \pi_{h,l} \) term, we first notice that the multiplier associated to the operator \( T(f, h) = \nabla^{-\frac{1}{2}} \pi_{h,l} \{ (\nabla^{\frac{1}{2}} f)h \} \), i.e.

\[
\sum_{N \geq M} |\xi_1 + \xi_2|^{-\frac{1}{2}} |\xi_1|^{\frac{1}{2}} \hat{P}_N f(\xi_1) \hat{P}_M h(\xi_2),
\]

is an order one symbol. The result cited above yields

\[
\left\| \nabla^{-\frac{1}{2}} \pi_{h,l} \{ (\nabla^{\frac{1}{2}} f)(\nabla^{-\frac{1}{2}} g) \} \right\|_{L^4_x} \lesssim \left\| f \right\|_{L^2_x} \left\| \nabla^{-\frac{1}{2}} g \right\|_{L^4_x}.
\]

Finally, Sobolev embedding dictates the estimate \( \left\| \nabla^{-\frac{1}{2}} g \right\|_{L^4_x} \lesssim \left\| g \right\|_{L^{8/3}_x} \).

To estimate the contribution of the term corresponding to the case \( j = 2 \), we take \( f = \nabla^{-\frac{1}{2}} |u_{hi}|^2 \) and \( g = \nabla^{\frac{1}{2}} u_{lo} \) in (5.26). Using Sobolev embedding and (5.12) we get

\[
\left\| \nabla^{-\frac{1}{2}} P_{hi} O(u_{hi}^2 u_{lo}) \right\|_{L^{4/3}_x L^{4/3}_t} \lesssim \left\| \nabla^{-\frac{1}{2}} |u_{hi}|^2 \right\|_{L^2_x} \left\| \nabla^{\frac{1}{2}} u_{lo} \right\|_{L^{8/3}_x L^{8/3}_t} \lesssim \eta_1^{\frac{1}{2}} C_0^{\frac{1}{2}} \left\| \nabla u_{lo} \right\|_{L^{\infty}_t L^{2}_x} \lesssim C_0^{\frac{1}{2}} \eta_1^{\frac{1}{2}} \eta_1.
\]

Similarly, to treat the case \( j = 3 \) we take \( f = \nabla^{-\frac{1}{2}} |u_{hi}|^2 \) and \( g = \nabla^{\frac{1}{2}} u_{hi} \) in (5.26) and use Sobolev embedding and (5.12) to estimate

\[
\left\| \nabla^{-\frac{1}{2}} P_{hi} O(u_{hi}^3) \right\|_{L^{4/3}_x L^{4/3}_t} \lesssim \left\| \nabla^{-\frac{1}{2}} |u_{hi}|^2 \right\|_{L^2_x} \left\| \nabla^{\frac{1}{2}} u_{hi} \right\|_{L^{8/3}_x L^{8/3}_t} \lesssim \eta_1^{\frac{1}{2}} C_0^{\frac{1}{2}} \left\| \nabla u_{hi} \right\|_{L^{\infty}_t L^{2}_x} \lesssim \eta_1^{\frac{1}{2}} C_0^{\frac{1}{2}}.
\]
Combining the above estimates we get
\[
\|\nabla^{-\frac{1}{2}} u_{hi}\|_{L_{t}^{4}L_{x}^{4}} \lesssim \eta_{1} + C_{0}^{\frac{1}{2}} \eta_{1}^{\frac{3}{2}} + C_{0} \eta_{1}^{2} + C_{0}^{\frac{1}{2}} \eta_{1}^{\frac{3}{2}} + \eta_{1}^{2} C_{0}^{\frac{1}{2}} \lesssim \eta_{1}^{\frac{1}{2}} C_{0}^{\frac{1}{2}}.
\]
which proves the proposition. \( \square \)

**Corollary 5.8.**

\[(5.28) \quad \|u_{hi}\|_{L_{x}^{2}(I_{0} \times \mathbb{R}^{4})} \lesssim C_{0}^{\frac{1}{2}} \eta_{1}^{\frac{1}{2}}.\]

**Proof.** The claim follows interpolating between
\[
\nabla u_{hi} \in L_{t}^{\infty}L_{x}^{2}(I_{0} \times \mathbb{R}^{4})
\]
and
\[
\nabla^{-\frac{1}{2}} u_{hi} \in L_{t}^{2}L_{x}^{4}(I_{0} \times \mathbb{R}^{4}).
\]

### 5.4. Morawetz Inequality: Error Terms.

In this section we use the control on \(u_{lo}\) and \(u_{hi}\) that Strichartz won us to bound the terms appearing on the right-hand side of Proposition 5.5. For the rest of this section all spacetime norms are going to be on \(I_{0} \times \mathbb{R}^{4} \).

Let’s consider \((5.15)\). The term corresponding to \(j = 2\) can be bounded via Bernstein and \((5.28)\) by
\[
\eta_{1} \|u_{hi}\|_{L_{t}^{3}L_{x}^{3}}^{2} \|u_{lo}\|_{L_{t}^{\infty}L_{x}^{\infty}} \lesssim \eta_{1}^{3} C_{0}.
\]
We estimate the term corresponding to \(j = 1\) using Bernstein, \((5.24)\), and \((5.28)\):
\[
\eta_{1} \|u_{hi}\|_{L_{t}^{2}L_{x}^{2}}^{2} \|u_{lo}\|_{L_{t}^{\infty}L_{x}^{4}} \|u_{lo}\|_{L_{t}^{12}L_{x}^{12}} \lesssim \eta_{1}^{2} \eta_{1}^{\frac{7}{2}} C_{0}^{\frac{7}{2}}.
\]
The term coming from \(j = 0\) we again estimate using Bernstein and \((5.24)\):
\[
\eta_{1} \|u_{hi} P_{hi} \mathcal{O}(u_{lo}^{3})\|_{L_{t}^{3}L_{x}^{3}} \lesssim \eta_{1} \|u_{hi}\|_{L_{t}^{\infty}L_{x}^{4}} \|P_{hi} \mathcal{O}(u_{lo}^{3})\|_{L_{t}^{1}L_{x}^{2}} \lesssim \eta_{1}^{2} \|\nabla P_{hi} \mathcal{O}(u_{lo}^{3})\|_{L_{t}^{1}L_{x}^{2}} \lesssim \eta_{1}^{2} \|\nabla u_{lo}\|_{L_{t}^{2}L_{x}^{2}}^{2} \|u_{lo}\|_{L_{t}^{4}L_{x}^{4}}^{2} \lesssim \eta_{1}^{2} \|u_{lo}\|_{L_{t}^{3}L_{x}^{3}}^{3} \lesssim \eta_{1}^{2} \eta_{1}^{\frac{3}{2}} C_{0}^{\frac{3}{2}}.
\]
The final contribution of the error term \((5.15)\) is therefore at most
\[
\eta_{1}^{2} (\eta_{1} C_{0} + \eta_{1}^{\frac{7}{2}} C_{0}^{\frac{7}{2}} + \eta_{1}^{\frac{3}{2}} C_{0}^{\frac{3}{2}}) \lesssim \eta_{1}.
\]
We now consider the error term (5.16), which we can bound as
\[
\eta_1 \int_{I_0} \int_{\mathbb{R}^4} |P_{lo} \mathcal{O}(u_{hi}^3)(t,y)| |u_{hi}(t,y)| \, dy \, dt \lesssim \eta_1 \|u_{hi}\|_{L^\infty \mathcal{L}_t^2} \|P_{lo} \mathcal{O}(u_{hi}^3)\|_{L_t^1 L_x^2} \\
\lesssim \eta_1^2 \|P_{lo} \mathcal{O}(u_{hi})\|_{L_t^1 L_x^4} \lesssim \eta_1^2 \|u_{hi}\|_{L_t^1 L_x^2}^3 \\
\lesssim \eta_1^3 C_0.
\]

Hence, (5.16) is bounded by $\eta_1^3 C_0 \lesssim \eta_1$.

Consider (5.17). The contribution coming from $j = 1$ can be estimated via (5.24) as
\[
\eta_1^2 \|\nabla u_{lo}\|_{L_t^1 L_x^2} \|u_{hi}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{hi}\|_{L_t^1 L_x^2} \lesssim \eta_1^3 \|u_{lo}\|_{S_t^1} \lesssim \eta_1 \eta_1^2 C_0^2.
\]
Applying Bernstein, (5.24), and (5.28), we estimate the contribution from $j = 2$ by
\[
\eta_1^2 \|\nabla u_{lo}\|_{L_t^1 L_x^2} \|u_{hi}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \lesssim \eta_1 \eta_1^2 C_0^2.
\]
Using Bernstein and (5.28) we estimate the term corresponding to $j = 3$ by
\[
\eta_1^2 \|u_{hi}\|_{L_t^1 L_x^2} \|\nabla u_{lo}\|_{L_t^1 L_x^2} \lesssim \eta_1^3 C_0 \|u_{lo}\|_{L_t^1 L_x^2} \lesssim \eta_1 \eta_1^2 C_0.
\]
Thus the error term (5.17) contributes at most $\eta_1^3 (\eta_1 C_0 + \eta_1^2 C_0^2 + \eta_1^2 C_0^3) \lesssim \eta_1$.

We now turn towards the error term (5.18). Decomposing $u = u_{hi} + u_{lo}$, we write $\nabla P_{lo}(\|u\|^2 u) = \sum_{j=0}^3 \nabla P_{lo} \mathcal{O}(u_{hi}^j u_{lo}^{3-j})$. Using Bernstein, Proposition 5.6, and Corollary 5.8, we estimate
\[
\eta_1^2 \|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^3)\|_{L_t^1 L_x^2} \lesssim \eta_1^2 \|u_{hi}\|_{L_t^1 L_x^4} \|\nabla u_{lo}\|_{L_t^1 L_x^4} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \lesssim \eta_1^3 \|u_{lo}\|_{S_t^1} \\
\lesssim \eta_1^3 \eta_1^2 C_0^2 \lesssim \eta_1
\]
\[
\eta_1^2 \|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^2 u_{lo})\|_{L_t^1 L_x^2} \lesssim \eta_1^2 \|u_{hi}\|_{L_t^1 L_x^4} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \lesssim \eta_1^2 \|u_{hi}\|_{L_t^1 L_x^2} \lesssim \eta_1^2 C_0 \eta_1 \lesssim \eta_1
\]
\[
\eta_1^2 \|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^3 u_{lo})\|_{L_t^1 L_x^2} \lesssim \eta_1^2 \|u_{hi}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \|u_{lo}\|_{L_t^1 L_x^2} \lesssim \eta_1^2 \|u_{hi}\|_{L_t^1 L_x^2} \lesssim \eta_1^2 C_0 \eta_1 \lesssim \eta_1
\]
\[ \eta_1^2 \|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^3)\|_{L_t^1 L_x^3} \lesssim \eta_1^2 \|u_{hi}\|_{L_t^{\infty} L_x^4} \|\nabla P_{lo} \mathcal{O}(u_{hi})\|_{L_t^1 L_x^2} \lesssim \eta_1^3 \|u_{hi}\|_{L_t^1 L_x^4} \]
\[ \lesssim \eta_1^3 \|u_{hi}\|^3_{L_t^1 L_x^4} \lesssim \eta_1^4 \|u_{hi}\|_{L_t^1 L_x^4} \lesssim \eta_1. \]

Hence, the error term (5.18) is bounded by \( \eta_1 \).

We consider now (5.19). For the case \( j = 3 \), we split the region of integration into \( |x - y| \leq 1 \) and \( |x - y| > 1 \). On the first region we use Cauchy-Schwartz and Young’s inequality, remembering that now we are convolving with an \( L_x^2 \) function, obtaining the bound
\[ \|u_{hi} \|^2_{L_t^{\infty} L_x^4} \|u_{hi}^3 u_{lo}\|_{L_t^1 L_x^4} \lesssim \|u_{hi}\|^2_{L_t^{\infty} L_x^4} \|u_{hi}\|^3_{L_t^1 L_x^4} \|u_{lo}\|_{L_t^{\infty} L_x^4} \lesssim \eta_1^4 C_0. \]

On the second region of integration, we bound \( \frac{1}{|x - y|} < 1 \) and estimate
\[ \|u_{hi}\|^2_{L_t^{\infty} L_x^4} \|u_{hi}^3 u_{lo}\|_{L_t^1 L_x^4} \lesssim \eta_1^2 \|u_{hi}\|^3_{L_t^1 L_x^4} \|u_{lo}\|_{L_t^{\infty} L_x^4} \lesssim \eta_1^4 C_0. \]

The case \( j = 2 \) is treated analogously using
\[ \|u_{hi}^2 u_{lo}\|_{L_t^1 L_x^4} \lesssim \|u_{hi}\|^2_{L_t^{\infty} L_x^4} \|u_{hi}\|^2_{L_t^1 L_x^4} \|u_{lo}\|^2_{L_t^{\infty} L_x^4} \lesssim \eta_1^4 C_0 \|u_{lo}\|_{S_1}^2 \lesssim \eta_1^4 C_0 \]

to get the bound \( \eta_1^4 C_0^4 \) on the first region and \( \eta_1^2 \eta_1^7 C_0^4 \) on the second region.

We turn now towards the term corresponding to \( j = 1 \). We write \( u_{hi} = \nabla (\nabla^{-1} u_{hi}) \) and integrate by parts to bound this term by

\[ \int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{hi}(t, y)|^2 (\nabla^{-1} u_{hi})(t, x) u_{lo}^2(t, x) \nabla u_{lo}(t, x)}{|x - y|} \, dx \, dy \, dt \]
\[ + \int_{I_0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{hi}(t, y)|^2 u_{lo}^3(t, x) (\nabla^{-1} u_{hi})(t, x) \frac{x - y}{|x - y|^3}} \, dx \, dy \, dt. \]  

Let’s consider (5.29). We again split the domain of integration into \( |x - y| \leq 1 \) and \( |x - y| > 1 \). On the first region we proceed as in the previous two cases, i.e. use Cauchy-Schwartz, Young’s inequality, and that the function we convolve with is in \( L_t^1 \) to bound
\[ \|u_{hi}\|^2_{L_t^{\infty} L_x^4} \|V^{-1} u_{hi}\|_{L_t^{1/2} L_x^4} \nabla u_{lo}\| L_t^1 L_x^4 \]
\[ \lesssim \|u_{hi}\|^2_{L_t^{\infty} L_x^4} \|\nabla^{-1} u_{hi}\|_{L_t^{1/2} L_x^4} \|\nabla u_{lo}\|_{L_t^{1/2} L_x^4} \|u_{lo}\|_{L_t^{\infty} L_x^4} \]
\[ \lesssim \eta_1^2 \|u_{lo}\|_{S_1} \|\nabla^{-1} u_{hi}\|_{L_t^{1/2} L_x^4} \]
\[ \lesssim \eta_1^3 C_0. \]
On the second region we take the absolute values inside the integral and estimate

\[
\int_{I_0} \int_{\mathbb{R}^4} \frac{|\eta_i(t,y)|^2 (|\nabla^{-1} \eta_i|(t,x) |u_{lo}(t,x)| |\nabla u_{lo}(t,x)|)}{\langle x - y \rangle} \, dx \, dy \, dt \\
\lesssim \|\eta_i\|^2_{L_t^\infty L_x^2} \sup_{y \in \mathbb{R}^4} \int_{I_0} \int_{\mathbb{R}^4} \frac{|\nabla^{-1} \eta_i(t,x) |u_{lo}^2(t,x)| |\nabla u_{lo}(t,x)|}{\langle x - y \rangle} \, dx \, dt \\
\lesssim \eta_i^2 \|\langle x \rangle^{-1} \|\nabla^{-1} \eta_i u_{lo}^2 \nabla u_{lo}\|_{L_t^1 L_x^{1/3}} \\
\lesssim \eta_i^2 \|\nabla^{-1} \eta_i \|_{L_t^2 L_x^2} \|\nabla u_{lo}\|_{L_t^2 L_x^2} \|u_{lo}\|_{L_t^\infty L_x^6} \\
\lesssim \eta_i^4 \|u_{lo}\|_{L_t^3} \|\nabla^{-1} \eta_i \|_{L_t^2 L_x^2} \\
\lesssim \eta_i^5 C_0.
\]

Taking absolute values inside the integrals, we write (5.30) as

\[
|\langle \nabla^{-2} |u_{hi}|^2, (\nabla^{-1} u_{hi})u_{lo}^3 \rangle| = |\langle \nabla^{-\frac{3}{2}} |u_{hi}|^2, \nabla^{-\frac{1}{2}} (\nabla^{-1} u_{hi})u_{lo}^3 \rangle| \\
\lesssim \|\nabla^{-\frac{1}{2}} |u_{hi}|^2\|_{L_t^1 L_x^4} \|\nabla^{-\frac{1}{2}} (\nabla^{-1} u_{hi})u_{lo}^3\|_{L_t^1 L_x^4} \\
\lesssim \eta_i^\frac{1}{4} C_0 \int_{\mathbb{R}^4} \frac{|\nabla^{-1} \eta_i(t,x) |u_{lo}(t,x)|^3}{|x - y|^\frac{3}{2}} \, dx \|_{L_t^2 L_x^6}.
\]

To estimate this integral we again split the domain of integration into $|x - y| \leq 1$ and $|x - y| > 1$. On the first region we note that the function we convolve with is in $L_t^1$ and we estimate

\[
\| (\nabla^{-1} \eta_i)u_{lo}^3 \|_{L_t^2 L_x^6} \lesssim \| (\nabla^{-1} \eta_i) \|_{L_t^2 L_x^2} \|u_{lo}\|_{L_t^\infty L_x^8} \|u_{lo}\|_{L_t^\infty L_x^8} \lesssim \eta_i^\frac{3}{2} \eta_i \frac{1}{4} C_0 \lesssim \eta_i^\frac{7}{2} C_0^\frac{1}{4}.
\]

On the second region of integration we estimate

\[
\| (\nabla^{-1} \eta_i)u_{lo}^3 \ast \langle x \rangle^{-\frac{3}{2}} \|_{L_t^2 L_x^6} \lesssim \| (\nabla^{-1} \eta_i)u_{lo}^3 \|_{L_t^2 L_x^2} \|\langle x \rangle^{-\frac{3}{2}} \|_{L_t^\infty L_x^2} \\
\lesssim \| (\nabla^{-1} \eta_i)u_{lo}^3 \|_{L_t^2 L_x^2} \|u_{lo}\|_{L_t^\infty L_x^2} \lesssim \eta_i^\frac{3}{2} \eta_i \frac{1}{4} C_0 \lesssim \eta_i^\frac{7}{2} C_0^\frac{1}{4}.
\]

We see therefore that the error term (5.19) is also bounded by $\eta_i$.

Upon rescaling, this concludes the proof of Proposition 5.1.
6. Preventing energy evacuation. The purpose of this section is to prove:

**Proposition 6.1** (Energy cannot evacuate to high frequencies). Suppose that $u$ is a minimal energy blowup solution of (1.1). Then for all $t \in I_0$,

$$N(t) \lesssim C(\eta_3)N_{\text{min}}.$$  

**Remark 6.2.** $N_{\text{min}} > 0$. Indeed, from the frequency localization result we know that for $\forall t \in I_0$

$$\|P_{c(\eta_0)^N(t)} \lesssim C(\eta_0)\|u(t)\|_{H^1_x} \sim 1.$$  

On the other hand, from Bernstein

$$\|P_{c(\eta_0)^N(t)} \lesssim C(\eta_0)\|u(t)\|_{L^\infty_t L^2_x} \leq C(\eta_0).$$

Hence, $N(t) \geq c(\eta_0)\|u\|_{L^\infty_t L^2_x}^{-1}$ for all $t \in I_0$, which implies $N_{\text{min}} = \inf_{t \in I_0} N(t) > 0$.

**6.1. The setup.** We normalize $N_{\text{min}} = 1$. As $N(t) \in 2\mathbb{Z}$, there exists $t_{\text{min}} \in I_0$ such that $N(t_{\text{min}}) = N_{\text{min}} = 1$.

At the time $t = t_{\text{min}}$ we have a considerable amount of mass at medium frequencies:

$$\|P_{c(\eta_0)^N(t)} \lesssim C(\eta_0)\|u(t)\|_{H^1_x} \sim c(\eta_0).$$

However, by Bernstein there is not much mass at frequencies higher than $C(\eta_0)$:

$$\|P_{c(\eta_0)^N(t)} \lesssim c(\eta_0).$$

Let’s assume for a contradiction that there exists $t_{\text{evac}} \in I_0$ such that $N(t_{\text{evac}}) \gg C(\eta_3)$. By time reversal symmetry we may assume $t_{\text{evac}} > t_{\text{min}}$. As

$$\|P_{c(\eta_0)^N(t)} \lesssim \eta,$$

for every $\eta_3 \leq \eta \leq \eta_0$ and all $t \in I_0$, we see that choosing $C(\eta_3)$ sufficiently large we get very small energy at low and medium frequencies at the time $t = t_{\text{evac}}$:

$$\|P_{c(\eta_0)^N(t)} \lesssim \eta_3.$$  

We define $u_{lo} = P_{c(\eta_0)^N} u$ and $u_{hi} = P_{c(\eta_0)^N} u$. (6.2) implies that

$$\|u_{hi}(t_{\text{min}})\|_{L^2_x} \geq \eta_1.$$
Suppose we could show that a big portion of the mass sticks around until \( t = t_{\text{evac}} \), i.e.

\[
\| u_{hi}(t_{\text{evac}}) \|_{L^2_x} \geq \frac{1}{2} \eta_1 .
\]

Then, since by Bernstein

\[
\| P_{\geq C(\eta_1)} u_{hi}(t_{\text{evac}}) \|_{L^2_x} \leq c(\eta_1),
\]

the triangle inequality would imply

\[
\| P_{\leq C(\eta_1)} u_{hi}(t_{\text{evac}}) \|_{L^2_x} \geq \frac{1}{4} \eta_1 .
\]

Another application of Bernstein yields

\[
\| P_{\leq C(\eta_1)} u(t_{\text{evac}}) \|_{H^1_x} \gtrsim c(\eta_1, \eta_2),
\]

which would contradict (6.3) if \( \eta_3 \) were chosen sufficiently small.

It therefore remains to show (6.5). In order to prove it we assume that there exists a time \( t_* \) such that \( t_{\text{min}} \leq t_* \leq t_{\text{evac}} \) and

\[
\inf_{t_{\text{min}} \leq t \leq t_*} \| u_{hi}(t) \|_{L^2_x} \geq \frac{1}{2} \eta_1 .
\]

We will show that this can be bootstrapped to

\[
\inf_{t_{\text{min}} \leq t \leq t_*} \| u_{hi}(t) \|_{L^2_x} \geq \frac{3}{4} \eta_1 .
\]

Hence \( \{ t_* \in [t_{\text{min}}, t_{\text{evac}}] : (6.6) \text{ holds} \} \) is both open and closed in \( [t_{\text{min}}, t_{\text{evac}}] \), and (6.5) holds.

In order to show that (6.6) implies (6.7) we treat the \( L^2_x \)-norm of \( u_{hi} \) as an almost conserved quantity. Define

\[
L(t) = \int_{\mathbb{R}^4} |u_{hi}(t, x)|^2 \, dx .
\]

By (6.4) we have \( L(t_{\text{min}}) \geq \eta_1^2 \). Hence, by the Fundamental Theorem of Calculus it suffices to show that

\[
\int_{t_{\text{min}}}^{t_*} |\partial_t L(t)| \, dt \leq \frac{1}{100} \eta_1^2 .
\]
We have
\[ \partial_t L(t) = 2 \int_{\mathbb{R}^4} \{ P_{hi}(|u|^2 u), uhi \}_m \, dx \]
\[ = 2 \int_{\mathbb{R}^4} \{ P_{hi}(|u|^2 u) - |uhi|^2 uhi, uhi \}_m \, dx. \]

Thus, we need to show
\[ (6.8) \quad \int_{t_{\min}}^{t_{\ast}} \left| \int_{\mathbb{R}^4} \{ P_{hi}(|u|^2 u) - |uhi|^2 uhi, uhi \}_m \, dx \right| \, dt \leq \frac{1}{100} \eta_1^2. \]

In order to prove (6.8), we need to control the various interactions between low, medium, and high frequencies. In the next section we will develop the tools that will make this goal possible.

### 6.2. Spacetime estimates on low, medium, and high frequencies.

Remember that the frequency-localized Morawetz inequality implies that for \( N < c(\eta_1) \),
\[ (6.9) \quad \int_{t_{\min}}^{t_{\text{evac}}} \int_{\mathbb{R}^4} |P_{\geq N} u(t, x)|^3 \, dx \, dt \lesssim \eta_1 N^{-3}. \]

This estimate is useful for medium and high frequencies; however it is extremely bad for low frequencies since \( N^{-3} \) diverges as \( N \to 0 \). We therefore need to develop better estimates in this case. As \( u \leq \eta_2 \) has extremely small energy at \( t = t_{\text{evac}} \) (see (6.3)), we expect to have small energy at all times in \([t_{\min}, t_{\text{evac}}]\). Of course, there is energy leaking from the high frequencies to the low frequencies, but (6.9) limits this leakage. Indeed, we have

**Lemma 6.3.** Under the assumptions above, we have
\[ (6.10) \quad \| P_{\leq N} u \|_{S^1([t_{\min}, t_{\text{evac}}] \times \mathbb{R}^4)} \lesssim \eta_3 + \eta_2^{-\frac{3}{2}} N^\frac{1}{2} \]
for all \( N \leq \eta_2 \).

**Remark 6.4.** One should think of the \( \eta_3 \) factor on the right-hand side of (6.10) as the energy coming from the low modes of \( u(t_{\text{evac}}) \), and the \( \eta_2^{-\frac{3}{2}} N^\frac{1}{2} \) term as the energy coming from the high frequencies of \( u(t) \) for \( t_{\min} \leq t \leq t_{\text{evac}} \).

**Proof.** Consider the set
\[ \Omega = \{ t \in [t_{\min}, t_{\text{evac}}] \colon \| P_{\leq N} u \|_{S^1([t, t_{\text{evac}}] \times \mathbb{R}^4)} \leq C_0 \eta_3 + \eta_0 \eta_2^{-\frac{3}{2}} N^\frac{1}{2}, \forall N \leq \eta_2 \}, \]
where $C_0$ is a large constant to be chosen later and not depending on any of the $\eta$'s.

Our goal is to show that $t_{min} \in \Omega$. First we will show that $t \in \Omega$ for $t$ close to $t_{evac}$. Indeed, from Strichartz and Sobolev we get

$$
\|P_{\leq N} u\|_{S_t^1([t_{evac}, t_{evac} + 2^M t_{evac}] \times \mathbb{R}^d)} \lesssim \|\nabla P_{\leq N} u\|_{L_t^\infty L_x^2([t_{evac}, t_{evac} + 2^M t_{evac}] \times \mathbb{R}^d)} + \|\nabla u\|_{L_t^\infty L_x^4([t_{evac}, t_{evac} + 2^M t_{evac}] \times \mathbb{R}^d)}
$$

$$\lesssim \|\nabla P_{\leq N} u(t_{evac})\|_{L_t^\infty L_x^2} + C|t_{evac} - t|\|\nabla \partial_t u\|_{L_t^\infty L_x^2(t \times \mathbb{R}^d)} + |t_{evac} - t|^\frac{3}{4}\|\nabla u\|_{L_t^\infty L_x^2(t \times \mathbb{R}^d)}.
$$

As $u$ is Schwartz the last two norms are finite, so (6.3) implies

$$
\|P_{\leq N} u\|_{S_t^1([t_{evac}, t_{evac} + 2^M t_{evac}] \times \mathbb{R}^d)} \lesssim \eta_3 + C(I_0, u)|t_{evac} - t| + C(I_0, u)|t_{evac} - t|^\frac{1}{4}.
$$

Thus $t \in \Omega$ provided $|t_{evac} - t|$ is sufficiently small and we choose $C_0$ sufficiently large.

Now suppose that $t \in \Omega$. We will show that

$$
(6.11) \quad \|P_{\leq N} u\|_{S_t^1([t_{evac}, t_{evac} + 2^M t_{evac}] \times \mathbb{R}^d)} \leq \frac{1}{2} C_0 \eta_3 + \frac{1}{2} \eta_0 \eta_2^{-\frac{3}{2}} N^{\frac{1}{2}}
$$

holds for $\forall N \leq \eta_2$. Thus, $\Omega$ is both open and closed in $[t_{min}, t_{evac}]$ and we have $t_{min} \in \Omega$ as desired.

Fixing $N \leq \eta_2$, Lemma 2.2 implies

$$
\|P_{\leq N} u\|_{S_t^1([t_{evac}, t_{evac} + 2^M t_{evac}] \times \mathbb{R}^d)} \leq \|P_{\leq N} u(t_{evac})\|_{H_t^1} + \sum_{m=1}^{M} \|\nabla F_m\|_{L_t^{q_m} L_x^{r_m}([t_{evac}, t_{evac} + 2^M t_{evac}] \times \mathbb{R}^d)}
$$

for some decomposition $P_{\leq N}(u^2 u) = \sum_{m=1}^{M} F_m$ and some dual Schrödinger admissible pair $(q_m', r_m)$.

From (6.3) we have

$$
\|P_{\leq N} u(t_{evac})\|_{H_t^1} \lesssim \eta_3,
$$

which is acceptable if we choose $C_0$ sufficiently large.

We decompose $P_{\leq N}(u^2 u) = \sum_{j=0}^{3} P_{\leq N} O(u_{hi}^3 u_{lo'}^3) = \sum_{j=0}^{3} F_j$ with $u_{lo'} = u_{\leq \eta_2}$ and $u_{hi} = u_{> \eta_2}$.

Consider the $j = 3$ term. Using Bernstein and (5.27) we estimate

$$
\|\nabla P_{\leq N} O(u_{hi}^3)\|_{L_t^{4/3} L_x^4([t_{evac}, t_{evac} + 2^M t_{evac}] \times \mathbb{R}^d)} \lesssim N^{\frac{3}{2}} \|\nabla^{-\frac{3}{4}} P_{\leq N} O(u_{hi}^3)\|_{L_t^{4/3} L_x^4([t_{evac}, t_{evac} + 2^M t_{evac}] \times \mathbb{R}^d)}
$$

$$\lesssim N^{\frac{3}{2}} (\eta_1 \eta_2^{-\frac{3}{4}})^{\frac{1}{2}} = \eta_1 \eta_2^{-\frac{1}{2}} N^{\frac{3}{2}}.
$$
Now consider the term corresponding to \( j = 2 \). By Bernstein and Hölder,

\[
\| \nabla P_{\leq N} \mathcal{O}(u_{lo}^2 u_{lo}') \|_{L_t^2 L_x^{4/3}([t_1,t_2]; \mathbb{R}^4)} \lesssim N^2 \| u_{lo}^2 u_{lo}' \|_{L_t^2 L_x^3([t_1,t_2]; \mathbb{R}^4)}
\]

\[
\lesssim N^2 \| u_{lo} \|_{L_t^2 L_x^3([t_1,t_2]; \mathbb{R}^4)} \| u_{lo}' \|_{L_t^\infty L_x^2([t_1,t_2]; \mathbb{R}^4)} \| u_{lo}' \|_{L_t^\infty L_x^2([t_1,t_2]; \mathbb{R}^4)}
\]

\[
\lesssim N^2 (\eta_1 \eta_2^{-3})^{1/2} \eta_2^{-1} \| u_{lo}' \|_{S^1([t_1,t_2]; \mathbb{R}^4)}.
\]

But since \( t \in \Omega \), \( \| u_{lo}' \|_{S^1([t_1,t_2]; \mathbb{R}^4)} \leq \eta_0 \), and

\[
\| \nabla P_{\leq N} \mathcal{O}(u_{lo}^2 u_{lo}') \|_{L_t^2 L_x^{4/3}([t_1,t_2]; \mathbb{R}^4)} \lesssim \eta_0 \eta_2^{-2} N^2 \lesssim \eta_0 \eta_2^{-2} N^2.
\]

We turn now towards the \( j = 1 \) contribution. Consider first the case when \( N < \eta_1 \). The expression \( \| \nabla \mathcal{O}(u_{hi}^2 u_{lo}') \|_{L_t^{q_1} L_x^{r_1}([t_1,t_2]; \mathbb{R}^4)} \) vanishes unless one of the \( u_{lo}' \) factors has frequency \( > \eta_1 \). Writing \( u_{lo}' = P_{\leq \eta_1} u_{lo}' + P_{> \eta_1} u_{lo}' \) and recalling that the projections are bounded on the space considered, we see that we only need to control

\[
\| \nabla P_{\leq N} \mathcal{O}(u_{hi}^2 u_{lo}'(P_{> \eta_1} u_{lo}')) \|_{L_t^2 L_x^{4/3}([t_1,t_2]; \mathbb{R}^4)}.
\]

But \( P_{> \eta_1} u_{lo} \) obeys the same \( L_t^3 \) estimates as \( u_{hi} \), so controlling the above expression amounts to the manipulations of case \( j = 2 \).

Now consider the case when \( N \geq \eta_1 \). Taking \( (q_1', r_1') = (1, 2) \) and using Bernstein we have

\[
\| \nabla P_{\leq N} \mathcal{O}(u_{hi}^2 u_{lo}') \|_{L_t^{q_1} L_x^{r_1}([t_1,t_2]; \mathbb{R}^4)} \lesssim \eta_2 \| u_{hi}^2 u_{lo}' \|_{L_t^{q_1} L_x^{r_1}([t_1,t_2]; \mathbb{R}^4)}
\]

\[
\lesssim \eta_2 \| u_{hi} \|_{L_t^3([t_1,t_2]; \mathbb{R}^4)} \| u_{lo}' \|_{L_t^2 L_x^2([t_1,t_2]; \mathbb{R}^4)} \| u_{lo}' \|_{L_t^\infty L_x^2([t_1,t_2]; \mathbb{R}^4)}
\]

\[
\lesssim \eta_2 (\eta_1 \eta_2^{-3})^{1/2} \| u_{lo}' \|_{S^1([t_1,t_2]; \mathbb{R}^4)}
\]

\[
\lesssim \eta_0 \eta_2^{-2} N^2.
\]

We are left with the \( j = 0 \) term. Write \( u_{lo}' = u_{< \eta_3} + u_{\eta_3 \leq \eta_2} \). Any term containing at least one \( u_{< \eta_3} \) can be controlled using the trilinear Strichartz estimates

\[
\| \nabla P_{\leq N} \mathcal{O}(u_{lo}'^2 u_{< \eta_3}) \|_{L_t^{q_1} L_x^{r_1}([t_1,t_2]; \mathbb{R}^4)} \lesssim \| u_{lo}' \|_{S^1([t_1,t_2]; \mathbb{R}^4)} \| u_{< \eta_3} \|_{S^1([t_1,t_2]; \mathbb{R}^4)}
\]

\[
\lesssim (C_0 \eta_3 + \eta_0)^2 (C_0 \eta_3 + \eta_0 \eta_2^{1/2} \eta_3^{1/2})
\]

\[
\lesssim C_0 \eta_0^{1/2} \eta_3,
\]

which is acceptable if we choose \( \eta_3 \) sufficiently small.
We can thus discard all the terms involving $u_{< \eta_3}$ and focus on the term

$$\| \nabla P_{\leq N} \mathcal{O}(u_{\eta_3}^3 \leq \eta_2) \|_{L^1_t L^2_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)}.$$ 

Using Bernstein we estimate this as

$$\| \nabla P_{\leq N} \mathcal{O}(u_{\eta_3}^3 \leq \eta_2) \|_{L^1_t L^2_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)} \lesssim N \| \nabla P_{\leq N} \mathcal{O}(u_{\eta_3}^3 \leq \eta_2) \|_{L^1_t L^2_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)} \lesssim N^{\frac{3}{2}} \| u_{\eta_2} \|_{L^3_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)}^{\frac{1}{2}} \| u_{\eta_2} \|_{L^3_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)}^{\frac{1}{2}} \| u_{\eta_2} \|_{L^3_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)}^{\frac{1}{2}}.$$

But by Bernstein and the hypothesis $t \in \Omega$, we have

$$\| u_{\eta_3} \|_{L^3_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)} \lesssim \sum_{\eta_3 \leq M \leq \eta_2} \| P Mu \|_{L^3_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)} \lesssim \sum_{\eta_3 \leq M \leq \eta_2} M^{-1} \| \nabla P Mu \|_{L^2_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)} \lesssim \sum_{\eta_3 \leq M \leq \eta_2} M^{-\frac{1}{2}} \| \nabla P Mu \|_{L^2_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)} \lesssim \sum_{\eta_3 \leq M \leq \eta_2} M^{-\frac{1}{2}} \| P Mu \|_{L^2 ([t, t_{\text{evac}}] \times \mathbb{R}^4)} \lesssim \sum_{\eta_3 \leq M \leq \eta_2} M^{-\frac{1}{2}} (C_0 \eta_3 + \eta_0 \eta_2^{-\frac{3}{2}} M^{\frac{1}{2}}) \lesssim \eta_0 \eta_2^{-\frac{1}{2}}.$$

Hence,

$$\| \nabla P_{\leq N} \mathcal{O}(u_{\eta_3}^3 \leq \eta_2) \|_{L^1_t L^2_x ([t, t_{\text{evac}}] \times \mathbb{R}^4)} \lesssim \eta_0 \eta_2^{-\frac{3}{2}} N^{\frac{1}{2}}.$$ 

This proves (6.11) and closes the bootstrap. 

\[ \Box \]

### 6.3. Controlling the localized mass increment.

We now have good enough control over low, medium, and high frequencies to prove (6.8). Writing

$$P_{hi}(|u|^2 u) - |u_{hi}|^2 u_{hi} = P_{hi}(|u|^2 u - |u_{hi}|^2 u_{hi}) - |u_{lo}|^2 u_{lo} + P_{hi}(|u_{lo}|^2 u_{lo}) - P_{lo}(|u_{hi}|^2 u_{hi}),$$
we only have to consider the terms
\[ \int_{t_{\text{min}}}^{t_{\ast}} \left| \int_{\mathbb{R}^4} \overline{u_i} \varphi(u) (|u|^2 u - |u_i|^2 u_i - |u_{\text{lo}}|^2 u_{\text{lo}}) \, dx \right| \, dt \]
\[ \int_{t_{\text{min}}}^{t_{\ast}} \left| \int_{\mathbb{R}^4} \overline{u_i} \varphi(u_{\text{lo}})^2 \, dx \right| \, dt \]
\[ \int_{t_{\text{min}}}^{t_{\ast}} \left| \int_{\mathbb{R}^4} \overline{u_i} \varphi(u_{\text{i}})^2 \, dx \right| \, dt. \]

Take (6.12). We use the inequality
\[ ||u|^2 u - |u_i|^2 u_i - |u_{\text{lo}}|^2 u_{\text{lo}}| \lesssim |u_i|^2 |u_{\text{lo}}| + |u_i||u_{\text{lo}}|^2 \]
to estimate
\[ (6.12) \lesssim \int_{t_{\text{min}}}^{t_{\ast}} \int_{\mathbb{R}^4} |P_{\text{hi}} u_i| (|u_i||u_{\text{lo}}|^2 + |u_{\text{lo}}|^2|u_{\text{lo}}|) \, dx \, dt \]
\[ = \int_{t_{\text{min}}}^{t_{\ast}} \int_{\mathbb{R}^4} |P_{\text{hi}} u_i| |u_{\text{lo}}|^2 \, dx \, dt \]
\[ + \int_{t_{\text{min}}}^{t_{\ast}} \int_{\mathbb{R}^4} |P_{\text{hi}} u_i| |u_{\text{i}}|^2 |u_{\text{lo}}| \, dx \, dt. \]

We estimate
\[ (6.16) \lesssim \left\| P_{\text{hi}} u_i \right\|_{L^3_{t,x}(t_{\text{min}}, t_{\ast} \times \mathbb{R}^4)} \left\| u_{\text{hi}} \right\|_{L^3_{t,x}(t_{\text{min}}, t_{\ast} \times \mathbb{R}^4)} \left\| u_{\text{lo}} \right\|_{L^2_{t,x}(t_{\text{min}}, t_{\ast} \times \mathbb{R}^4)}^2 \]
\[ \lesssim \{ \eta_1 (\eta_2^{100})^{-3} \}^{\frac{2}{3}} \{ \eta_3 + \eta_2^{-\frac{1}{2}} (\eta_2^{100})^{\frac{1}{2}} \}^2 \]
\[ \lesssim \eta_1^2 \eta_2^{97} \ll \eta_1^2. \]

We now turn towards the contribution of (6.17). We decompose \( u_{\text{hi}} = u_{\eta_2^{100} \leq \eta_2} + u_{> \eta_2} \) and estimate
\[ \left\| u_{> \eta_2} \right\|_{L^3_{t,x}(t_{\text{min}}, t_{\ast} \times \mathbb{R}^4)} \lesssim (\eta_1 \eta_2^{-3})^{\frac{1}{2}} = \eta_1^{\frac{1}{2}} \eta_2^{-1} \]
and
\[ \left\| u_{\eta_2^{100} \leq \eta_2} \right\|_{L^3_{t,x}(t_{\text{min}}, t_{\ast} \times \mathbb{R}^4)} \lesssim \sum_{\eta_2^{100} \leq N \leq \eta_2} \left\| u_N \right\|_{L^3_{t,x}(t_{\text{min}}, t_{\ast} \times \mathbb{R}^4)} \]
\[ \lesssim \sum_{\eta_2^{100} \leq N \leq \eta_2} N^{-1} \left\| \nabla u_N \right\|_{L^1_{t,x}(t_{\text{min}}, t_{\ast} \times \mathbb{R}^4)} \]
\[
\lesssim \sum_{\eta_1^{100} \leq N \leq \eta_2} N^{-1} \| u_N \|_{L^6(\mathbb{R}^4)}
\]
\[
\lesssim \sum_{\eta_1^{100} \leq N \leq \eta_2} N^{-1} (\eta_3 + \eta_2^{-\frac{3}{2}} N^2)
\]
\[
\lesssim \eta_2^{-\frac{3}{2}} \eta_2^4 = \eta_2^{-1}.
\]

By Bernstein and (1.8),
\[
\| u_{lo} \|_{L^\infty_t L^3_x([t_{min}, t^*] \times \mathbb{R}^4)} \lesssim \eta_2^{100} \| u_{lo} \|_{L^\infty_t L^3_x([t_{min}, t^*] \times \mathbb{R}^4)} \lesssim \eta_2^{100}.
\]

As \( P_{hi} u_{hi} \) obeys the same estimates as \( u_{hi} \), we bound the contribution of (6.17) by
\[
\int_{t_{min}}^{t^*} \int_{\mathbb{R}^4} |u_{hi}|^3 |u_{lo}| \, dx \, dt
\]
\[
\lesssim \sum_{j=0}^{3} \| u_{\eta_2^{100} \leq N \leq \eta_2} \|_{L^6_t L^3_x([t_{min}, t^*] \times \mathbb{R}^4)} \| u_{> \eta_2} \|_{L^6_t L^3_x([t_{min}, t^*] \times \mathbb{R}^4)} \| u_{lo} \|_{L^\infty_t L^6_x([t_{min}, t^*] \times \mathbb{R}^4)}
\]
\[
\lesssim \eta_2^{-3} \eta_2^{100} \ll \eta_1^2.
\]

Hence, (6.12) \( \ll \eta_1^2 \).

We consider next (6.13). Because of the presence of \( P_{hi} \), one of the terms \( u_{lo} \) must have frequency larger than \( c\eta_3 \) or the expression vanishes. Moving \( P_{hi} \) over to \( \Pi_{hi} \), we bound (6.13) by a sum of terms that look like
\[
\int_{t_{min}}^{t^*} \int_{\mathbb{R}^4} |P_{hi} u_{hi}| \| P_{> c\eta_3} u_{lo} \| u_{lo} \| dx \, dt.
\]
As \( P_{> c\eta_3} u_{lo} = P_{lo} u_{> c\eta_3} \) behaves like \( u_{hi} \) as far as norms are concerned, this expression can be estimated by the procedure used to estimate (6.16).

We now turn to (6.14). Moving the projection \( P_{lo} \) onto \( \Pi_{hi} \) and writing \( P_{lo} u_{hi} = P_{hi} u_{lo} \), we bound the contribution of (6.14) by
\[
\int_{t_{min}}^{t^*} \int_{\mathbb{R}^4} |P_{hi} u_{lo} | u_{hi} |^3 dx \, dt.
\]
As \( P_{hi} u_{lo} \) behaves like \( u_{lo} \) as far as norms are concerned, (6.14) is bounded by the same method as (6.17).

Therefore (6.8) holds, and this concludes the proof of Proposition 6.1.

7. The contradiction argument. We now have all the information we need about a minimal energy blowup solution to conclude the contradiction argument.
We know it is localized in frequency and concentrates in space. The Morawetz inequality provides good control on the high-frequency part of \( u \) in \( L^{3}_{t,x} \). By the arguments in the previous section we have excluded the last enemy by showing that the solution can’t shift its energy from low modes to high modes causing the \( L^{6}_{t,x} \)-norm to blow up while the \( L^{3}_{t,x} \)-norm remains bounded. Hence \( N(t) \) must remain within a bounded set \([N_{\text{min}}, N_{\text{max}}]\), where \( N_{\text{max}} \leq C(\eta_{3})N_{\text{min}} \) and \( N_{\text{min}} \geq c(\eta_{0})\|u\|^{-1}_{L^{3}_{t,x}} \). Combining all these (and again relying on the Morawetz inequality) we can derive the desired contradiction. We begin with:

**Lemma 7.1.** For any minimal energy blowup solution of (1.1) we have

\[
\int_{I_{0}} N(t)^{-1} \, dt \lesssim C(\eta_{1}, \eta_{2})N_{\text{min}}^{-3}. \tag{7.1}
\]

In particular, because \( N(t) \leq C(\eta_{3})N_{\text{min}} \) for all \( t \in I_{0} \), we have

\[
|I_{0}| \lesssim C(\eta_{1}, \eta_{2}, \eta_{3}, N_{\text{min}}). \tag{7.2}
\]

**Proof.** By (5.28) we have

\[
\int_{I_{0}} \int_{\mathbb{R}^{4}} |P_{\geq N_{*}}u|^{3} \, dx \, dt \lesssim \eta_{1}N_{*}^{-3}
\]

for all \( N_{*} < c(\eta_{1})N_{\text{min}} \). Let \( N_{*} = c(\eta_{2})N_{\text{min}} \) and rewrite the above estimate as

\[
\int_{I_{0}} \int_{\mathbb{R}^{4}} |P_{\geq N_{*}}u|^{3} \, dx \, dt \lesssim C(\eta_{1}, \eta_{2})N_{\text{min}}^{-3}. \tag{7.3}
\]

On the other hand, by Bernstein and (1.8) we have

\[
\int_{|x-x(t)| \leq C(\eta_{1})/N(t)} |P_{<N_{*}}u(t)|^{3} \, dx \lesssim C(\eta_{1})N(t)^{-4} \|P_{<N_{*}}u(t)\|_{L^{\infty}}^{3}
\]

\[
\lesssim C(\eta_{1})N(t)^{-4}N(t)^{3}c(\eta_{2})\|P_{<N_{*}}u(t)\|_{L^{3}}^{3}
\]

\[
\lesssim c(\eta_{2})N(t)^{-1}. \tag{7.4}
\]

By (4.17) we also have

\[
\int_{|x-x(t)| \leq C(\eta_{1})/N(t)} |u(t)|^{3} \, dx \gtrsim c(\eta_{1})N(t)^{-1}.
\]

Combining this with (7.5) and using the triangle inequality we find

\[
c(\eta_{1})N(t)^{-1} \lesssim \int_{|x-x(t)| \leq C(\eta_{1})/N(t)} |P_{\geq N_{*}}u(t, x)|^{3} \, dx.
\]

Integrating this over \( I_{0} \) and comparing with (7.3) proves (7.1). \( \square \)
We can now (finally!) conclude the contradiction argument. It remains to prove \( \|u\|_{L^6_t(I_0 \times \mathbb{R}^4)} \lesssim C(\eta_0, \eta_1, \eta_2, \eta_3) \), which we expect since the bound (7.2) shows that the interval \( I \) is not long enough to allow the \( L^6_{t,x} \)-norm of \( u \) to grow too large.

**Proposition 7.2.** We have

\[
\|u\|_{L^6_{t,x}(I_0 \times \mathbb{R}^4)} \lesssim C(\eta_0, \eta_1, \eta_2, \eta_3).
\]

**Proof.** We normalize \( N_{\min} = 1 \). Let \( \delta = \delta(\eta_0, N_{\max}) > 0 \) be a small number to be chosen momentarily. Partition \( I_0 \) into \( O\left(\frac{|I_0|}{\delta}\right) \) subintervals \( I_1, \ldots, I_J \) with \( |I_j| \leq \delta \). Let \( t_j \in I_j \). Since \( N(t_j) \leq N_{\max} \) we have from Corollary 4.4

(7.5) \[
\|P_{\geq C(\eta_0)N_{\max}} u(t_j)\|_{H_x^1} \leq \eta_0.
\]

Let \( \tilde{u}(t) = e^{i(t-t_j)\Delta} P_{< C(\eta_0)N_{\max}} u(t_j) \) be the free evolution of the low and medium frequencies of \( u(t_j) \). The above bound becomes

\[
\|u(t_j) - \tilde{u}(t_j)\|_{H_x^1} \leq \eta_0.
\]

But from Bernstein, Sobolev embedding, and (1.7) we get

\[
\|\tilde{u}\|_{L^6_{t,x}(I_0 \times \mathbb{R}^4)} \lesssim C(\eta_0, N_{\max})^\delta^{1/6}.
\]

Similarly we have

\[
\|\nabla(|\tilde{u}|^2 \tilde{u}(t))\|_{L^{4/3}_{t,x}} \lesssim \|\nabla \tilde{u}(t)\|_{L_x^4}^2 \lesssim C(\eta_0, N_{\max})^\delta^{1/2}.
\]
From these two estimates, (1.7), and Lemma 3.1 with $e = -|\tilde{u}|^2\tilde{u}$ we see that

$$\|u\|_{L^6_t(I_j \times \mathbb{R}^4)} \lesssim 1$$

if $\delta$ is chosen small enough. Summing these bounds in $j$ and using (7.2) we get

$$\|u\|_{L^6_t(I_0 \times \mathbb{R}^4)} \lesssim C(\eta_0, N_{\text{max}})|I_0| \lesssim C(\eta_0, \eta_1, \eta_2, \eta_3).$$

\[\Box\]

### 8. Remarks.

We first note that as a consequence of the bound (1.4), one gets scattering, asymptotic completeness, and uniform regularity. Indeed, we have:

**Proposition 8.1.** Let $u_0$ have finite energy. Then there exist finite energy solutions $u_{\pm}(t, x)$ to the free Schrödinger equation $(i\partial_t + \Delta)u_{\pm} = 0$ such that

$$\|u_{\pm}(t) - u(t)\|_{H^{1}_x} \to 0$$

as $t \to \pm \infty$. Furthermore, the maps $u_0 \mapsto u_{\pm}(0)$ are homeomorphisms from $\dot{H}^1_x(\mathbb{R}^4)$ to itself. Finally, if $u_0 \in H^s_x$ for some $s \geq 1$, then $u(t) \in H^s_x$ for all times $t$, and one has the uniform bounds

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s_x} \leq C(E(u_0), s)\|u_0\|_{H^s_x}.$$

**Proof.** We will only prove the statement for $u_+$, since the proof for $u_-$ follows similarly. Let us first construct the scattering state $u_+(0)$. For $t > 0$ define $v(t) = e^{-it\Delta}u(t)$. We will show that $v(t)$ converges in $\dot{H}^1_x$ as $t \to \infty$, and define $u_+(0)$ to be that limit. Indeed, from Duhamel’s formula (1.11) we have

$$v(t) = u_0 - i \int_0^t e^{-is\Delta}(|u|^2u)(s) \, ds. \tag{8.1}$$

Therefore, for $0 < \tau < t$,

$$v(t) - v(\tau) = -i \int_\tau^t e^{-is\Delta}(|u|^2u)(s) \, ds.$$

Lemma 2.2 and Lemma 2.4 yield

$$\|v(t) - v(\tau)\|_{\dot{H}^1_x} = \|e^{it\Delta}(v(t) - v(\tau))\|_{\dot{H}^1_x} \lesssim \|\nabla(|u|^2u)\|_{L^2_t(\mathbb{R}^4)} \lesssim \|u\|^3_{L^6_t(\mathbb{R}^4)} \lesssim \|u\|^3_{L^6_x(\mathbb{R}^4)}.$$
However, Lemma 3.6 implies that \( \|u\|_{S^1} \) is finite, while the bound (1.4) implies that for any \( \varepsilon > 0 \) there exists \( t_\varepsilon \in \mathbb{R}_+ \) such that \( \|u\|_{L^6_t([t,\infty) \times \mathbb{R}^d)} \leq \varepsilon \) whenever \( t > t_\varepsilon \). Hence,

\[
\|v(t) - v(\tau)\|_{\dot{H}^1} \to 0 \quad \text{as} \ t, \tau \to \infty.
\]

In particular, this implies that \( u_*(0) \) is well defined. Also, inspecting (8.1) one easily sees that

\[
(8.2) \quad u_*(0) = u_0 - i \int_0^\infty e^{-is\Delta} \langle |u|^2 u \rangle ds
\]

and thus

\[
(8.3) \quad u_*(t) = e^{i\Delta} u_0 - i \int_0^\infty e^{i(t-s)\Delta} \langle |u|^2 u \rangle ds.
\]

By the same arguments as above, (8.3) and Duhamel’s formula (1.11) imply that \( \|u_*(t) - u(t)\|_{\dot{H}^1} \to 0 \) as \( t \to \infty \).

Similar estimates prove that the map \( u_0 \mapsto u_*(0) \) is continuous from \( \dot{H}^1(\mathbb{R}^d) \) to itself. To show that the map is also injective, let \( u_{01}, u_{02} \in \dot{H}^1 \), and let \( u_1, u_2 \) be the corresponding solutions to (1.1) with initial data \( u_{01}, u_{02} \). Assume that \( \|e^{-it\Delta} u_j(t) - u_*(0)\|_{\dot{H}^1} \to 0 \) as \( t \to \infty \) for \( j = 1, 2 \). Then, (8.3) yields

\[
(8.4) \quad u_j(t) = u_*(t) - i \int_t^\infty e^{i(t-s)\Delta} \langle |u|^2 u \rangle ds
\]

for \( t > 0 \) and \( j = 1, 2 \). Using the Strichartz estimates of Lemma 2.2 and Lemma 2.4, Picard’s fixed point argument forces \( u_1(t) = u_2(t) \) for \( t \) sufficiently large. By the uniqueness of solutions to the Cauchy problem (1.1) we obtain \( u_{01} = u_{02} \), which proves injectivity for the map \( u_0 \mapsto u_*(0) \) on \( \dot{H}^1(\mathbb{R}^d) \).

We now turn towards the construction of the wave operators, i.e. for every \( u_*(0) \in \dot{H}^1(\mathbb{R}^d) \) there exists \( u_0 \in \dot{H}^1(\mathbb{R}^d) \) such that \( \|u_*(t) - u(t)\|_{\dot{H}^1} \to 0 \) as \( t \to \infty \), and moreover the map \( u_*(0) \mapsto u_0 \) from \( \dot{H}^1(\mathbb{R}^d) \) to itself is continuous. For \( t > 0 \) take \( u_*(t) \) to be initial data for the equation (1.1) and solve the Cauchy problem backwards in time. Denote the solution at time \( t = 0 \) by \( u_{0t} \). Choosing \( t_2 > t_1 \) sufficiently large, we see that the free and the nonlinear evolutions from \( t_2 \) to \( t_1 \) are within \( \varepsilon \) of each other in the \( S^1([t_1,t_2] \times \mathbb{R}^d) \)-norm. An application of Lemma 3.3 with \( e = 0 \) yields \( \|u_{0t_1} - u_{0t_2}\|_{\dot{H}^1} \lesssim \varepsilon \), which implies that \( u_{0t} \) converges in \( \dot{H}^1(\mathbb{R}^d) \) as \( t \to \infty \). Denote the limit by \( u_0 \), and let \( u \) be the global solution to the Cauchy problem (1.1) with initial data \( u_0 \). Then at time \( t > 0 \) we have

\[
u(t) = e^{i\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} \langle |u|^2 u \rangle ds
\]
\[
\begin{align*}
&= e^{it\Delta}u_0 + e^{it\Delta}(u_0 - u_{0t}) - i \int_0^t e^{i(t-s)\Delta}(|u|^2u)(s) \, ds \\
&= u_+(t) + e^{it\Delta}(u_0 - u_{0t}),
\end{align*}
\]
which implies that \( \|u_+(t) - u(t)\|_{H^1_x} \to 0 \) as \( t \to \infty\).

The continuity of the map \( u_+(0) \mapsto u_0 \) on \( H^1_x(\mathbb{R}^4) \) follows immediately from Lemma 3.3 with \( e = 0 \).

The regularity statement follows from obvious modifications of Lemma 3.6.

We now develop the tower-type bounds for \( M(E) \). To do so we must determine the dependence of each of the \( \eta_j \)'s on the previous ones. Throughout, we will let \( c \) and \( C \) represent small and large constants, possibly depending on the energy. To begin, we need \( \eta_0 \) to be small relative to the energy. Specifically, we choose \( \eta_0 \leq cE^{-C} \), so that when we remove the low- and high-frequency portions of the solution (whose norms are bounded by \( c\eta_0^C \)) the majority of the energy remains.

Recall that the dependence of Lemma 3.3 is exponential in its parameters. That is, if \( E, E', \) and \( M \) represent the various bounds in the statement of the lemma, we need to choose \( \varepsilon_1 \lesssim \exp(-CM(E(E')C\varepsilon)) \). When we apply this lemma in the proof of Proposition 4.1, the parameters are

\[
\begin{align*}
E' &= \varepsilon \\
M &= \eta^{-C}M(E_{\text{crit}} - \eta^C)^C \\
E &= \eta^{-C}M(E_{\text{crit}} - \eta^C)^C \\
\varepsilon_1 &= \eta^{-C}M(E_{\text{crit}} - \eta^C)^C \varepsilon^{1/2},
\end{align*}
\]

where the \( \eta^{-C}M(E_{\text{crit}} - \eta^C)^C \) factors arise from an application of Lemma 1.2 earlier in the proof. Thus, we need to choose

\[
\eta^{-C}M(E_{\text{crit}} - \eta^C)^C \varepsilon^{1/2} \lesssim \exp(-\eta^{-C}M(E_{\text{crit}} - \eta^C)^C(\varepsilon)^C),
\]

or equivalently

\[
\varepsilon^{-C} \lesssim \exp(-C\eta^{-C}M(E_{\text{crit}} - \eta^C)^C).
\]

Also, in order to make the pigeonhole argument work (still in the proof of Proposition 4.1) we need the frequency separation to be

\[
K(\eta) \geq \varepsilon^{-2} \geq C \exp(C\eta^{-C}M(E_{\text{crit}} - \eta^C)^C). \]
As a result, the dependence of the constants $C(\eta)$ and $c(\eta)$ in Corollary 4.4 is quite bad:

$$C(\eta) \geq C \exp(C\eta^{-C}M(E_{\text{crit}} - \eta^C)^C)$$

$$c(\eta) \leq 1/C(\eta).$$

This motivates introducing the notation $t(\eta)$ for any term of the form

$$c \exp(-C\eta^{-C}M(E_{\text{crit}} - \eta^C)^C).$$

Now, the constant $c(\eta_0)$ appearing in the proof of Proposition 4.5 needs to be of the form $c(\eta_0) = t(\eta_0)$, due to the use of Corollary 4.4. So in order to apply Lemma 3.3 (later in the same proof) we need to choose $\eta_1 \leq t(c(\eta_0)) = t(t(\eta_0))$.

The constant $c(\eta_1)$ appearing in Proposition 5.1 is roughly of size $t(\eta_1)$. In order to apply this proposition in Section 6, we therefore must choose $\eta_2 \leq t(\eta_1)$.

We need to choose $\eta_3$ smaller than any polynomial in $\eta_2$ appearing in the proof of Proposition 6.1. For instance, $\eta_3 \leq \exp(-\eta_2^{-C})$ suffices; given the final bound we obtain, this makes $\eta_2$ and $\eta_3$ virtually equivalent.

We also see from the proof of Proposition 6.1 that $N_{\text{max}} \leq 1/t(\eta_3)$. Proposition 7.2 then implies the desired contradiction, provided we choose $\eta_4 \leq t(\eta_3)$.

Putting this all together we find a final bound of the form

$$M(E) \leq 1/t(t(t(E^{-C}))).$$

To properly simplify this expression requires some notation (the Ackerman hierarchy). Recall that multiplication is iterated addition

$$a \times b = a + \ldots + a$$

with $b$ factors on the right, and exponentiation is iterated multiplication

$$a \uparrow b = a \times \ldots \times a.$$ We define tower exponentiation as iterated exponentiation:

$$a \uparrow\uparrow b := a \uparrow (a \uparrow \ldots (a \uparrow a) \ldots)$$

with $b$ arrows on the right. Similarly, double tower exponentiation is iterated tower exponentiation

$$a \uparrow\uparrow\uparrow b := a \uparrow\uparrow (a \uparrow\uparrow \ldots (a \uparrow\uparrow a) \ldots),$$

and triple tower exponentiation is iterated double tower exponentiation, etc.
Now, we can essentially expand (8.5) as

\[ M(E) \leq \exp(M(E - t(t(E^{-C})))) \]  

(8.6)

Iterating (8.6) approximately \(1/t(t(E^{-C}))\) times, we obtain

\[ M(E) \leq C \uparrow \uparrow 1/t(t(E^{-C})). \]

We again expand this as

\[ M(E) \leq C \uparrow \uparrow \exp(M(E - t(t(E^{-C})))), \]

and iterating it we obtain

\[ M(E) \leq C \uparrow \uparrow \uparrow 1/t(E^{-C}). \]

Repeating this process one more time we get the final bound

\[ M(E) \leq C \uparrow \uparrow \uparrow \uparrow CE^C. \]

This tower bound is a far cry from the exponential bound obtained in [23], though better than that obtained in [11], which is of the form \(M(E) \leq C \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow CE^C\). It is plausible that one could obtain a polynomial bound in the energy-critical case, but to do so would require abandoning the inductive approach used in this paper.

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REFERENCES

[1] J. Bourgain, Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity, Internat. Math. Res. Notices 5 (1998), 253–283.
[2] ———, Global well-posedness of defocusing 3D critical NLS in the radial case, J. Amer. Math. Soc. 12 (1999), 145–171.
[3] ———, New Global Well-posedness Results for Non-linear Schrödinger Equations, AMS Publications, Providence, RI, 1999.
[4] ———, A remark on normal forms and the ‘I-Method’ for periodic NLS, preprint, 2003.
[5] T. Cazenave and F. B. Weissler, Critical nonlinear Schrödinger Equation, Nonlinear Anal. 14 (1990), 807–836.
[6] R. R. Coifman and Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Trans. Amer. Math. Soc. 212 (1975), 315–331.
[7] ———, Ondelettes et operateurs III, Operateurs multilinéaires, Actualites Mathematiques, Hermann, Paris, 1991.
[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Global well-posedness for the Schrödinger equations with derivative, *Siam J. Math.* 33 (2001), 649–669.

[9] ————, Existence globale et diffusion pour l’équation de Schrödinger nonlinéaire répulsive cubique sur $\mathbb{R}^3$ en dessous l’espace d’énergie, *Journées “Equations aux Dérivées Partielles”* (Forges-les-Eaux, 2002), Exp. No. X, 14, 2002.

[10] ————, Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$, *Comm. Pure Appl. Math.* 57 (2004), 987–1014.

[11] ————, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$, *Ann. of Math.* (to appear).

[12] R. T. Glassey, On the blowing up of solution to the Cauchy problem for nonlinear Schrödinger operators, *J. Math. Phys.* 8 (1977), 1794–1797.

[13] J. Ginibre and G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, *J. Math. Pure. Appl.* 64 (1985), 363–401.

[14] M. Grillakis, On nonlinear Schrödinger equations, *Comm. Partial Differential Equations* 25 nos. 9–10 (2000), 1827–1844.

[15] A. Hassell, T. Tao, and J. Wunsch, A Strichartz inequality for the Schrödinger equation on non-trapping asymptotically conic manifolds, *Comm. Partial Differential Equations* 30 (2005), 157–205.

[16] M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. Math. J.* 120 (1998), 955–980.

[17] J. Lin and W. Strauss, Decay and scattering of solutions of a nonlinear Schrödinger equation, *J. Funct. Anal.* 30 (1978), 245–263.

[18] C. Morawetz, Time decay for the nonlinear Klein-Gordon equation, *Proc. Roy. Soc. London Ser. A* 306 (1968), 291–296.

[19] K. Nakanishi, Energy scattering for non-linear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2, *J. Funct. Anal.* 169 (1999), 201–225.

[20] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[21] ————, *Harmonic Analysis*, Princeton University Press, 1993.

[22] T. Tao, On the asymptotic behavior of large radial data for a focusing non-linear Schrödinger equation, *Dynamics of PDE* 1 (2004), 1–48.

[23] ————, Global well-posedness and scattering for the higher-dimensional energy-critical non-linear Schrödinger equation for radial data, *New York J. Math.* 11 (2005), 57–80.