Lepton flavor violation, lepton \((g - 2)_{\mu,e}\) and electron EDM in the modular symmetry

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ABSTRACT: We study the lepton flavor violation (LFV), the leptonic magnetic moments \((g - 2)_{\mu,e}\) and the electric dipole moment (EDM) of the electron in the Standard-Model Effective Field Theory with the \(\Gamma_N\) modular flavor symmetry. We employ the stringy Ansatz on coupling structure that 4-point couplings of matter fields are written by a product of 3-point couplings of matter fields. We take the level 3 finite modular group, \(\Gamma_3\) for the flavor symmetry, and discuss the dipole operators at nearby fixed point \(\tau = i\), where observed lepton masses and mixing angles are well reproduced. Suppose the anomaly of the anomalous magnetic moment of the muon to be evidence of the new physics (NP), we have related it with \((g - 2)_e\), LFV decays, and the electron EDM. It is found that the NP contribution to \((g - 2)_e\) is proportional to the lepton masses squared likewise the naive scaling. We also discuss the correlations among the LFV processes \(\mu \to e\gamma\), \(\tau \to \mu\gamma\) and \(\tau \to e\gamma\), which are testable in the future. The electron EDM requires the tiny imaginary part of the relevant Wilson coefficient in the basis of real positive charged lepton masses, which is related to the \(\mu \to e\gamma\) transition in our framework.

KEYWORDS: Lepton Flavour Violation (charged), Electric Dipole Moments, Flavour Symmetries

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1 Introduction

The electric and magnetic dipole moments of the electron and the muon are low-energy probes of New Physics (NP) beyond the Standard Model (SM). The recent experimental measurement of the anomalous magnetic moment of the muon, $a_\mu$, has indicated the discrepancy with the SM prediction in refs. [1–3] (see also [4–13]).\textsuperscript{1} Therefore, this experimental result has stimulated an interest in new physics contributions to this observable. Indeed, comprehensive analyses of the electric and magnetic dipole moments of leptons are given in the SM Effective Field Theory (SMEFT), i.e., under the hypothesis of new degrees of freedom above the electroweak scale [15–18]. The phenomenological discussion of NP has appeared in the light of the anomaly of the muon $(g - 2)_\mu$ in the SMEFT [19].

\textsuperscript{1}There are arguments on the precise value of the SM prediction $a_\mu^{\text{SM}}$, for example, in ref. [14].
Several years after the pioneering analysis in ref. [20], the first complete list of the non-redundant SMEFT Lagrangian terms up to dimension-six has been presented in ref. [21]. When all the possible flavor structures are taken into account in the absence of any flavor symmetry, a large proliferation in the number of independent coefficients in the SMEFT occurs; there are 1350 CP-even and 1149 CP-odd independent coefficients for the dimension-six operators [22]. The flavor symmetry reduces the number of independent parameters of the flavor sector. Above all, the flavor symmetries \( U(3)^5 \) and \( U(2)^5 \) have been successfully applied to the SMEFT [23]. The \( U(3)^5 \) flavor symmetry is the maximal flavor symmetry allowed by the SM gauge sector, while \( U(2)^5 \) is the corresponding subgroup acting only on the first two (light) families [24–26]. The \( U(3)^5 \) allows us to apply the Minimal Flavor Violation (MFV) hypothesis [27, 28], which is the most restrictive hypothesis consistent with the SMEFT, and suppress non-standard contributions to flavor-violating observables [28].

On the other hand, we have already discussed the SMEFT with the modular flavor symmetry [29, 30]. Indeed, the well-known finite groups \( S_3, A_4, S_4 \) and \( A_5 \) are isomorphic to the finite modular groups \( \Gamma_N \) for \( N = 2, 3, 4, 5 \), respectively [31]. The lepton mass matrices have been given successfully in terms of \( \Gamma_3 \simeq A_4 \) modular forms [32]. Modular invariant flavor models have also been proposed on the \( \Gamma_2 \simeq S_3 \), \( \Gamma_4 \simeq S_4 \) and \( \Gamma_5 \simeq A_5 \) [35, 36]. Other finite groups are also derived from magnetized D-brane models [37]. By using these modular forms, the flavor mixing of quarks and leptons has been discussed successfully in these years since the non-Abelian finite groups are long familiar in quarks and leptons [38–48].

In the modular invariant theories of the finite group, the quark and lepton mass matrices are written in terms of modular forms, which are holomorphic functions of the modulus \( \tau \). The arbitrary symmetry breaking sector of the conventional models based on flavor symmetries is replaced by the moduli space, and then Yukawa couplings are given by modular forms. Phenomenological studies of the lepton flavors have been done based on \( A_4 \) [49–51], \( S_4 \) [52–54] and \( A_5 \) [35, 36]. A clear prediction of the neutrino mixing angles and the Dirac CP phase was given in the simple lepton mass matrices with the \( A_4 \) modular symmetry [50]. The Double Covering groups \( T' \) [55, 56] and \( S'_4 \) [57, 58] were also realized in the modular symmetry. Furthermore, phenomenological studies have been developed in many works [59–133].

In this paper, we focus on the lepton flavor violation (LFV), lepton \((g - 2)_{\mu,e}\) and the electric dipole moment (EDM) of the electron. Since we make the assumption that NP is heavy and can be given by the SMEFT Lagrangian, we discuss the dipole operators of leptons and their Wilson coefficients at the electroweak scale. As the flavor symmetry, we take the level 3 finite modular group, \( \Gamma_3 \) since the property of \( A_4 \) flavor symmetry has been well known [134–140]. Although modular flavor models have been constructed in the supersymmetric framework so far, the modular invariant SMEFT will be realized in the so-called moduli-mediated supersymmetry breaking scenario. Indeed, the soft supersymmetry breaking terms are generated in a modular invariant way [141]. Furthermore, higher-dimensional operators also keep the modular invariance in a certain class of the string EFT in which \( n \)-point couplings of matter fields are written by a product of 3-point couplings. We adopt this Ansatz (called stringy Ansatz) to constrain the higher-dimensional operators.
in the SMEFT. This meets the MFV hypothesis [29]. Based on the tensor decomposition of $A_4$ modular group, we discuss the bilinear operators of leptons at nearby fixed point $\tau = i$, where observed lepton masses and mixing angles are well reproduced.

The paper is organized as follows. In section 2, we present our framework, that is the stringy Ansatz. In section 3, we discuss the flavor structure of the Wilson coefficients of the leptonic dipole operator in mass basis. In section 4, we discuss the phenomenology of $(g-2)_{e,\mu}$, LFV and the electron EDM. Section 5 is devoted to the summary. In appendix A, we present the experimental constraints on the leptonic dipole operators. In appendix B, we summarize $A_4$ modular symmetry briefly. In appendix C, we present a flavor model in $A_4$ modular symmetry. In appendix D, we show the charged lepton mass matrix explicitly at the nearby $\tau = i$.

2 Stringy Ansatz

In this section, we comment on an ultraviolet origin of the SMEFT with the $\Gamma_N$ modular flavor symmetry, following refs. [29, 30]. In particular, we focus on the superstring theory for physics beyond the SM. It was known that $n$-point couplings $y^{(n)}$ of matter fields are written by products of 3-point couplings $y^{(3)}$, i.e., $y^{(n)} = (y^{(3)})^{n-2}$ in a certain class of string compactifications. For instance, 4-point couplings $y^{(4)}_{ijkl}$ of matter fields are given by

$$y^{(4)}_{ijkl} = \sum_m y^{(3)}_{ijm}y^{(3)}_{mk\ell},$$

up to an overall factor. Here, the virtual modes $m$ are light or heavy modes, depending on the compactifications. It indicates that the flavor structure of 3-point couplings and higher-dimensional operators has a common origin in string compactifications. Since the flavor symmetry in 3-point couplings controls the structure of four-dimensional EFT, it satisfies the criterion of the MFV hypothesis [29]. As pointed out in ref. [29], such a relation holds at the low-energy scale below the compactification scale.

In the string-derived EFT, these $n$-point couplings depend on moduli fields, reflecting the geometric symmetry of compact six-dimensional space. For instance, the $SL(2,\mathbb{Z})$ modular symmetry is the geometric symmetry of the torus $T^2$ as well as the orbifold $T^2/\mathbb{Z}_2$. Recalling that the transformation of matter zero-modes on toroidal backgrounds is also given by the finite subgroup of the modular symmetry (see, e.g., for heterotic string theory [144–146] and for magnetized brane models [147–152]), the flavor symmetry of matter zero-modes is determined by the modular symmetry in the low-energy effective action. Furthermore, the modular symmetry restricts the form of $n$-point couplings in a modular symmetric way. Much larger symplectic modular symmetries are possible in Calabi-Yau compactifications [157,158] whose phenomenological aspects were studied in refs. [159, 160]. As a result, the flavor structure of Yukawa couplings and higher-dimensional operators are controlled by the modular flavor symmetry in the various class of string compactifications.

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2 The modular symmetries on higher-dimensional toroidal orbifolds were also discussed in ref. [143].
3 See also [153–156].
Note that the supersymmetry breaking sector also respects the flavor symmetry as seen in the soft supersymmetry breaking terms induced by the moduli fields \[141\].

Let us ignore the dynamics of moduli fields, meaning that moduli-dependent couplings are considered spurions under the modular symmetry. Then, the modular symmetry plays an important role in the concept of the MFV. In the original MFV scenario, Yukawa couplings behave as \((3, 3, 1, 1, 1), (3, 1, 3, 1, 1),\) and \((1, 1, 1, 3, 3)\) in the \(U(3)^5 = U(3)^Q \otimes U(3)_U \otimes U(3)_D \otimes U(3)_L \otimes U(3)_E\) flavor symmetry. On the other hand, in the string EFT at the leading order, \(U(2)^5\) flavor symmetry is realized due to the rank 1 Yukawa couplings of matter fields \[142\]. It is interesting to analyze the phenomenological aspects of string-derived low-energy effective action with some modular symmetries which would be realized in toroidal as well as Calabi-Yau compactifications. Indeed, the modular symmetry and the Ansatz eq. (2.1) are powerful to predict the leptonic phenomena of flavors, as will be discussed in the next section. In this paper, for concreteness, we study the SMEFT with the level 3 finite modular group \(Γ_3\) for the flavor symmetry by imposing the stringy Ansatz eq. (2.1) on the higher-dimensional operators. Remarkably, the lepton masses and mixing angles are well fitted with the observed data when the modulus field \(τ\) is close to the fixed point \(τ = i\) in the \(SL(2, \mathbb{Z})\) moduli space. In subsequent sections, we discuss the higher-dimensional operators relevant to the lepton sector in more detail.

3 Wilson coefficients of dipole operator in mass basis

We take the assumption that NP is heavy and can be given by the SMEFT Lagrangian. Let us focus on the dipole operators of leptons and their Wilson coefficients at the weak scale as:

\[
O_{\gamma L} = \frac{v}{\sqrt{2}} E_L \sigma^{\mu\nu} E_R F_{\mu\nu}, \quad C'_{\gamma L} = \begin{pmatrix}
C'_{\gamma e} & C'_{\gamma e} & C'_{\gamma e} \\
C'_{\gamma e} & C'_{\gamma e} & C'_{\gamma e} \\
C'_{\gamma e} & C'_{\gamma e} & C'_{\gamma e}
\end{pmatrix},
\]

\[
O_{\gamma R} = \frac{v}{\sqrt{2}} E_R \sigma^{\mu\nu} E_L F_{\mu\nu}, \quad C'_{\gamma R} = \mu C'_{\gamma L},
\]

where \(E_L\) and \(E_R\) denote three flavors of left-handed and right-handed leptons, respectively, and \(v\) denotes the vacuum expectation value (VEV) of the Higgs field \(H\). Here the prime of the Wilson coefficient indicates the flavor basis corresponding to the mass-eigenstate basis of charged leptons. The relevant effective Lagrangian is written as:

\[
L_{\text{dipole}} = \frac{1}{\Lambda^2} \left( C'_{\gamma L} O_{\gamma L} + C'_{\gamma R} O_{\gamma R} \right),
\]

where \(\Lambda\) is a certain mass scale of NP in the effective theory.

In the following discussions, we take the \(A_4\) modular symmetry for leptons. Most of modular flavor models are supersymmetric models. Since we study the model below the supersymmetry breaking scale, the light modes are exactly the same as the SM with
two doublet Higgs models. Note that the modular symmetry is still a symmetry of the low-energy effective action below the supersymmetry breaking scale, as confirmed in the moduli-mediated supersymmetry breaking scenario.

3.1 Representation of charged leptons in $A_4$ modular invariant model

We take a simple $A_4$ modular-invariant flavor model of leptons, which is successful in reproducing neutrino masses and mixing angles, as shown explicitly in appendix C. In the model, the left-handed charged leptons compose a $A_4$ triplet $\mathbf{3}$ and the three right-handed ones are $A_4$ three different singlets. Then, those are expressed as follows:

$$\begin{pmatrix}
e_L \\
\mu_L \\
\tau_L
\end{pmatrix}, \quad \bar{E}_L = \begin{pmatrix}
\bar{e}_L \\
\bar{\mu}_L \\
\bar{\tau}_L
\end{pmatrix}, \quad (e_R^e, \mu_R^e, \tau_R^e) = (1, 1', 1''), \quad (e_R, \mu_R, \tau_R) = (1, 1', 1'').$$ (3.3)

It is noticed that leptons of second and third families are exchanged each other in $\bar{E}_L$. As seen in the table of appendix C, both $E_L$ and $\bar{E}_L$ have the same modular weight, $-k = -2$. On the other hand, $k = 0$ for $e_R^e$, $e_R$, etc.

The holomorphic and anti-holomorphic modular forms of weight 2 compose the $A_4$ triplet:

$$Y(\tau) = \begin{pmatrix}
Y_1(\tau) \\
Y_2(\tau) \\
Y_3(\tau)
\end{pmatrix}, \quad \bar{Y}(\tau) \equiv Y^*(\tau) = \begin{pmatrix}
Y_1^*(\tau) \\
Y_2^*(\tau) \\
Y_3^*(\tau)
\end{pmatrix},$$ (3.4)

where modular forms are given explicitly in appendix B.2.

3.2 $[\bar{E}_R \Gamma E_L]$ and $[\bar{E}_L \Gamma E_R]$ bilinears in the flavor space

In order to investigate the flavor structure of the Wilson coefficient of the dipole operator, let us begin with discussing the holomorphic operator of charged leptons, $[\bar{E}_R \Gamma E_L]$ and anti-holomorphic operator $[\bar{E}_L \Gamma E_R]$ in the flavor space, where $\Gamma$ denotes the relevant Lorentz structure. The magnitudes of $LR$ couplings are proportional to modular forms. Taking account of $E_R = (e^e, \mu^e, \tau^e)$, we can decompose the operator in terms of the $A_4$ triplet holomorphic and anti-holomorphic modular forms of weight 2 in the basis of eq. (B.9) for $S$ and $T$. The result has been given in ref. [30] as:

$$[\bar{E}_R \Gamma E_L] \Rightarrow [\bar{E}_R \Gamma Y(\tau) E_L]_1$$

$$= [\alpha_e e_R \Gamma (Y_1(\tau) e_L + Y_2(\tau) \tau_L + Y_3(\tau) \mu_L) + \beta_e \bar{\mu}_R \Gamma (Y_3(\tau) \tau_L + Y_1(\tau) \mu_L + Y_2(\tau) e_L)_{1\nu}$$

$$+ \gamma_e \tau_R \Gamma (Y_2(\tau) \mu_L + Y_1(\tau) \tau_L + Y_3(\tau) e_L)_{1\nu}],$$

$$[\bar{E}_L \Gamma E_R] \Rightarrow [\bar{E}_L \Gamma Y^*(\tau) E_R]_1$$

$$= [\alpha_e^* e_R \Gamma (Y_1^*(\tau) \bar{e}_L + Y_2^*(\tau) \bar{\tau}_L + Y_3^*(\tau) \bar{\mu}_L) + \beta_e^* \bar{\mu}_R \Gamma (Y_3^*(\tau) \bar{\tau}_L + Y_1^*(\tau) \bar{\mu}_L + Y_2^*(\tau) \bar{e}_L)_{1\nu}$$

$$+ \gamma_e^* \tau_R \Gamma (Y_2^*(\tau) \bar{\mu}_L + Y_1^*(\tau) \bar{\tau}_L + Y_3^*(\tau) \bar{e}_L)_{1\nu}].$$ (3.5)

where the subscript $1, 1', 1''$ denote three $A_4$ singlets, respectively. The parameters $\alpha_e, \beta_e$ and $\gamma_e$ are constants.
These expressions are written in the matrix representation as

\[
\begin{align*}
[ \bar{E}_R \Gamma Y(\tau) E_L ]_1 &= (\bar{e}_R, \bar{\mu}_R, \bar{\tau}_L) \Gamma 
\begin{pmatrix}
\alpha_e & 0 & 0 \\
0 & \beta_e & 0 \\
0 & 0 & \gamma_e
\end{pmatrix} 
\begin{pmatrix}
Y_1(\tau) & Y_2(\tau) & Y_3(\tau) \\
Y_2(\tau) & Y_1(\tau) & Y_3(\tau) \\
Y_3(\tau) & Y_2(\tau) & Y_1(\tau)
\end{pmatrix} 
\begin{pmatrix}
\epsilon_L \\
\mu_L \\
\tau_L
\end{pmatrix}, \\
[ \bar{E}_L Y^*(\tau) \Gamma E_R ]_1 &= (\bar{e}_L, \bar{\mu}_L, \bar{\tau}_L) \Gamma 
\begin{pmatrix}
Y_1^*(\tau) & Y_2^*(\tau) & Y_3^*(\tau) \\
Y_2^*(\tau) & Y_1^*(\tau) & Y_3^*(\tau) \\
Y_3^*(\tau) & Y_2^*(\tau) & Y_1^*(\tau)
\end{pmatrix} 
\begin{pmatrix}
\alpha_e^* & 0 & 0 \\
0 & \beta_e^* & 0 \\
0 & 0 & \gamma_e^*
\end{pmatrix} 
\begin{pmatrix}
\epsilon_R \\
\mu_R \\
\tau_R
\end{pmatrix}.
\end{align*}
\]

It is useful to compare them with the charged lepton mass matrix \( M_E \) in appendix C. The mass matrix is given in terms of weight 2 modular forms as:

\[
M_E = v_d \begin{pmatrix}
\alpha_{e(m)} & 0 & 0 \\
0 & \beta_{e(m)} & 0 \\
0 & 0 & \gamma_{e(m)}
\end{pmatrix} 
\begin{pmatrix}
Y_1(\tau) & Y_2(\tau) & Y_3(\tau) \\
Y_2(\tau) & Y_1(\tau) & Y_3(\tau) \\
Y_3(\tau) & Y_2(\tau) & Y_1(\tau)
\end{pmatrix}_{RL},
\]

where the VEV of the Higgs field \( H_d \) is denoted by \( v_d \). Parameters \( \alpha_{e(m)}, \beta_{e(m)}, \gamma_{e(m)} \) can be taken to be real constants. Since the bilinear operators appear in four-field operators corresponding to eq. (3.1) by replacing \( v \) with \( H \), it is reasonable to assume

\[
\alpha_e = \kappa \alpha_{e(m)}, \quad \beta_e = \kappa \beta_{e(m)}, \quad \gamma_e = \kappa \gamma_{e(m)},
\]

from the viewpoint of the Ansatz eq. (2.1), where the mode \( m \) may correspond to \( H_d \). Here, \( \kappa \) is a common constant. Hereafter, we set \( \kappa = 1 \) by taking the relevant normalization. In this case, the matrix structure of bilinear operators \( [ \bar{E}_R \Gamma E_L ] \) is exactly the same as the mass matrix. Obviously, the bilinear operator matrix is diagonal in the basis for mass eigenstates. The flavor changing (FC) processes such as \( \mu \rightarrow e, \tau \rightarrow \mu \) and \( \tau \rightarrow e \) never happen. Hence, we obtain the very clear results in the modular symmetric SMEFT with the Ansatz eq. (2.1).

However, additional unknown modes in eq. (2.1) may cause the flavor violation. If the relation of eq. (3.9) is violated, the situation would change drastically. Consider the case that such violations are small in the following discussions:

\[
\alpha_e - \alpha_{e(m)} \ll \alpha_e, \quad \beta_e - \beta_{e(m)} \ll \beta_e, \quad \gamma_e - \gamma_{e(m)} \ll \gamma_e.
\]

We summarize the \( A_4 \) flavor coefficients of eqs. (3.6) and (3.7) in table 1 for relevant bilinear operators of charged leptons [30], where the overall strength of the NP effect is not included.\(^4\)

It is noticed that the above operators are given in the flavor basis. In order to obtain the mass eigenstate of leptons, we must fix the modulus \( \tau \). The value of \( \tau \) depends on models. In the model in appendix C, \( \tau \) is close to the fixed point of the modulus, \( \tau = i \).

\(^4\)The overall strength of the NP effect is also omitted in coefficients of other tables.
\[ \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{RL operators} & \tilde{\mu}_R \Gamma_{\tau_L} & \tilde{e}_R \Gamma_{\tau_L} & \tilde{\mu}_L \Gamma_{\tau_R} & \tilde{e}_L \Gamma_{\tau_R} \\
\hline
\text{LR operators} & \bar{\mu}_R \Gamma_{\tau_L} & \bar{e}_R \Gamma_{\tau_L} & \bar{\mu}_L \Gamma_{\tau_R} & \bar{e}_L \Gamma_{\tau_R} \\
\hline
\text{Flavor Coefficients} & \beta_e Y_3(\tau_e) & \alpha_e Y_2(\tau_e) & \alpha_e Y_3(\tau_e) & \alpha_e Y_2(\tau_e) & \beta_e Y_1(\tau_e) & \gamma_e Y_1(\tau_e) \\
\hline
\end{array} \]

Table 1. Flavor coefficients of the bilinear operators of charged leptons in the flavor basis.

### 3.2.1 Mass eigenstate at nearby \( \tau = i \)

At the fixed point \( \tau = i \), the flavor structure is too simple to reproduce the observed lepton mixing angles. The modulus \( \tau \) is deviated from the fixed points \( \tau = i \) in order to get the observed lepton masses and mixing angles. Indeed, the successful charged lepton mass matrix has been obtained at nearby \( \tau = i \) [61]. By using a small dimensionless parameter \( \epsilon \), we put the modulus value as \( \tau = i + \epsilon \). Then, approximate behaviors of the ratios of modular forms are [98]:

\[
\frac{Y_2(\tau)}{Y_1(\tau)} \approx (1 + \epsilon_1)(1 - \sqrt{3}), \quad \frac{Y_3(\tau)}{Y_1(\tau)} \approx (1 + \epsilon_2)(-2 + \sqrt{3}), \quad \epsilon_1 = \frac{1}{2} \epsilon_2 \approx 2.05 i \epsilon. \quad (3.11)
\]

These approximate forms are agreement with exact numerical values within 0.1% for \( |\epsilon| \leq 0.05 \).

The charged lepton mass matrix is diagonalized by the following transformation which is also shown in appendix D:

\[
\begin{align*}
E_L &\rightarrow E_L^m \equiv U_{Lme}^\dagger U_S E_L, \\
E_R &\rightarrow E_R^m \equiv U_{Rme}^\dagger U_{12} E_R,
\end{align*}
\]

where

\[
U_S = \frac{1}{2\sqrt{3}} \begin{pmatrix}
2 & 2 & 2 \\
\sqrt{3} + 1 & -2 & \sqrt{3} - 1 \\
\sqrt{3} - 1 & -2 & \sqrt{3} + 1
\end{pmatrix}, \quad U_{12} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}. \quad (3.13)
\]

The mixing matrices are parametrized as:

\[
U_{Lme} \simeq P_e^* \begin{pmatrix}
1 & s_{L12}^* & s_{L13}^* \\
-s_{L12}^* & 1 & s_{L23}^* \\
s_{L12}^* s_{L23}^* - s_{L13}^* & -s_{L23}^* & 1
\end{pmatrix}, \quad U_{Rme} \simeq \begin{pmatrix}
1 & s_{R12}^* & s_{R13}^* \\
-s_{R12}^* & 1 & s_{R23}^* \\
s_{R12}^* s_{R23}^* - s_{R13}^* & -s_{R23}^* & 1
\end{pmatrix}, \quad (3.14)
\]

where the phase matrix \( P_e \) is

\[
P_e = \begin{pmatrix}
e^{i\eta_e} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \eta_e = \arg \epsilon_1. \quad (3.15)
\]
The mixing angles are given as seen in appendix D:

\[
\begin{align*}
    s^e_{L12} & \simeq -|\epsilon_1^e|, \\
    s^e_{L23} & \simeq -\frac{\sqrt{3}}{2} \frac{\alpha_c}{\gamma_c}, \\
    s^e_{L13} & \simeq \frac{1}{3} |\epsilon_1^e|, \\
    s^e_{R12} & \simeq -\frac{\bar{\beta}_c}{\alpha_c}, \\
    s^e_{R23} & \simeq -\frac{1}{2} \frac{\alpha_c}{\gamma_c}, \\
    s^e_{R13} & \simeq -\frac{1}{2} \frac{\bar{\beta}_c}{\gamma_c},
\end{align*}
\]

(3.16)

where \( \alpha_c(e) = (6 - 3\sqrt{3})Y_1(i)\alpha_c, \) \( \bar{\beta}_c(e) = (6 - 3\sqrt{3})Y_1(i)\beta_c(e) \) and \( \gamma_c(e) = (6 - 3\sqrt{3})Y_1(i)\gamma_c(e) \). Indeed, the numerical fit was succeeded in the case of \( \bar{\gamma}_c(e) \Rightarrow \alpha_c(e) \Rightarrow \beta_c(e) \) [61]. In the mass eigenstate, the \( A_4 \) flavor coefficients of charged lepton bilinear operators are given in terms of mixing angles \( s^e_{L2}, s^e_{L3} \) and \( |\epsilon_1^e| \) at \( \tau = i + \epsilon \) in table 2.

\[\begin{tabular}{|c|c|c|}
\hline
\( m_\ell^2 \) & \( m_\ell R \) & \( m_\ell L \) \\
\hline
\( (\sqrt{3}s_{23} \alpha_2 + s_{12} \beta_2) \gamma_2 \) & \( \sqrt{3}s_{23} \beta_2 + 2(s_{12} \beta_2 - s_{12} \gamma_2) \) & \( \frac{1}{2}(\beta_2 + s_{12} \gamma_2) \) \\
\hline
\end{tabular}\]

Table 2. \( A_4 \) flavor coefficients of the FC lepton bilinear operators at \( \tau = i + \epsilon \), where \( O(|\epsilon|^2) \) is neglected because the modular forms are expanded in \( O(|\epsilon|) \), and \( \bar{\alpha}_c = (6 - 3\sqrt{3})Y_1(i)\alpha_c, \) \( \bar{\beta}_c = (6 - 3\sqrt{3})Y_1(i)\beta_c, \) \( \bar{\gamma}_c = (6 - 3\sqrt{3})Y_1(i)\gamma_c. \) A common overall factor \( (1 - \sqrt{3}) \) is omitted in the coefficients.

It is easily noticed that coefficients of \( \bar{\mu}_L \Gamma_T R, \bar{e}_R \Gamma_T L \) and \( \bar{e}_R \Gamma_M L \) in table 2 are much suppressed in spite of \( \alpha_c \neq \alpha_c(e), \beta_c \neq \beta_c(e), \gamma_c \neq \gamma_c(e) \), by inputting mixing angles of eq. (3.16) into them. Indeed, we find that those coefficients are \( O(|\epsilon_1^e|^2/\gamma_c(e)) \) for \( \bar{\mu}_L \Gamma_T R \) and \( \bar{e}_L \Gamma_T R \) while \( O(\beta_c^2/\alpha_c(e)) \) for \( \bar{e}_L \Gamma_M L \) after calculations of the next-to-leading terms. Numerical values of these parameter are given at the best fit point as follows [61]:

\[
\tau = -0.080 + 1.007i, \ |\epsilon_1^e| = 0.165, \ \frac{\bar{\alpha}_c(e)}{\gamma_c(e)} \simeq 6.82 \times 10^{-2}, \ \frac{\bar{\beta}_c(e)}{\gamma_c(e)} \simeq 1.50 \times 10^{-2}.
\]

(3.17)

However, the coefficients of the bilinear \( RL \) operators \( \bar{\mu}_R \Gamma_T L, \bar{e}_R \Gamma_T L \) and \( \bar{e}_R \Gamma_M L \) are not so suppressed. Those operators may lead to the sizable LFV decays. Including a common overall factor \( (1 - \sqrt{3}) \), which is omitted in table 2, the coefficients are given as:

\[\begin{align*}
    C_{e\gamma}^{\mu} & = \frac{3}{2} (1 - \sqrt{3}) \bar{\alpha}_c \left( 1 - \frac{\bar{\gamma}_c}{\gamma_c(e)} \frac{\bar{\alpha}_c(e)}{\bar{\alpha}_c(e)} \right), \\
    C_{e\gamma}^{\tau} & = \frac{3}{2} (1 - \sqrt{3}) \bar{\beta}_c \left( 1 + \frac{\bar{\alpha}_c}{\bar{\alpha}_c(e)} \frac{\bar{\beta}_c(e)}{\bar{\beta}_c(e)} - 2 \frac{\bar{\alpha}_c(e)}{\bar{\alpha}_c(e)} \frac{\bar{\gamma}_c(e)}{\bar{\gamma}_c(e)} \right), \\
    C_{e\gamma}^{\mu} & = \frac{3}{2} (1 - \sqrt{3}) \bar{\beta}_c \left( 1 - \frac{\bar{\alpha}_c(e)}{\bar{\alpha}_c(e)} \frac{\bar{\beta}_c(e)}{\bar{\beta}_c(e)} \right),
\end{align*}\]

(3.18)

where \( \bar{\alpha}_c/\bar{\alpha}_c(e), \bar{\beta}_c/\bar{\beta}_c(e) \) and \( \bar{\gamma}_c/\bar{\gamma}_c(e) \) are close to 1 due to the condition of eq. (3.10).

\[5\text{These results are different from ones in the previous our works}[30]. The previous result was obtained in a flavor basis of leptons where the right-handed leptons are not rotated. However, the previous results are also justified approximately due to the different condition from eq. (3.10), such as } \alpha_c - \alpha_c(e) \sim \bar{\alpha}_c, \text{ etc.}
On the other hand, the diagonal coefficients of the bilinear $RL$ operators $\bar{e}_R\Gamma e_L$, $\bar{\mu}_R\Gamma\mu_L$ and $\bar{\tau}_R\Gamma\tau_L$ are given as:

$$C'_{e\gamma}^{ee} = 3 \left(1 - \sqrt{3}\right) \tilde{\beta}_e |\epsilon_1^*|$$
$$C'_{e\gamma}^{\mu\mu} = \frac{3}{2} \left(1 - \sqrt{3}\right) \tilde{\alpha}_e$$
$$C'_{e\gamma}^{\tau\tau} = \sqrt{3} \left(1 - \sqrt{3}\right) \tilde{\gamma}_e$$

(3.19)

where the phase of $\epsilon_1$ is rotated away. These give the anomalous magnetic moment of leptons.

4 Phenomenology of $(g - 2)_{\mu, e}$, LFV and EDM

The anomalous magnetic moment of the muon, $a_\mu = (g - 2)_\mu / 2$, is a powerful probe beyond the SM. The recent experimental measurement of $a_\mu$ by the E989 experiment at FNAL [1], combined with the previous BNL result [2], has indicated the discrepancy with the SM prediction reported in ref. [3]. If this result is evidence of NP, we can relate it with other phenomena, $(g - 2)_e$, LFV processes and the electron EDM in the framework of the stringy Ansatz eq. (2.1) with the modular symmetry. We study the correlations among them in this section.

4.1 $(g - 2)_\mu$ and $(g - 2)_e$

The NP of $(g - 2)_\mu$ and $(g - 2)_e$ appears in the diagonal components of the Wilson coefficient of the dipole operator at the mass basis. We have the ratios of the diagonal coefficients from eq. (3.19) as:

$$\frac{C'_{e\gamma}^{ee}}{C'_{e\gamma}^{\mu\mu}} = 2 \left(1 - \sqrt{3}\right) \tilde{\beta}_e / \tilde{\alpha}_e |\epsilon_1^*| \simeq 4.9 \times 10^{-3},$$
$$\frac{C'_{e\gamma}^{\mu\mu}}{C'_{e\gamma}^{\tau\tau}} = \frac{\sqrt{3}}{2} \tilde{\alpha}_e / \tilde{\gamma}_e \simeq 5.9 \times 10^{-2},$$

(4.1)

where numerical values of eq. (3.17) are put for $\tilde{\beta}_e / \tilde{\alpha}_e$, $\tilde{\alpha}_e / \tilde{\gamma}_e$ and $|\epsilon_1^*|$. These predicted ratios are almost agree with the charged lepton mass ratios $m_e / m_\mu = 4.84 \times 10^{-3}$ and $m_\mu / m_\tau = 5.95 \times 10^{-2}$.

If this dipole operator is responsible for the observed anomaly of $(g - 2)_\mu$, the magnitude of its Wilson coefficient can be estimated as shown in appendix A. By inputting the experimental value of eq. (A.2), the real part of the Wilson coefficient of the muon $C'_{e\gamma}^{\mu\mu}$ has been obtained as seen in eq. (A.4) [19]. Now, we can estimate the magnitude of the electron $(g - 2)_e$ anomaly by using the relation in eq. (4.1) as:

$$\Delta a_e = \frac{4m_e}{e} \frac{v}{\Lambda^2} \left| C'_{e\gamma}^{ee} \right| \simeq 5.8 \times 10^{-14},$$

(4.2)

where $\Lambda$ denotes a certain mass scale of NP. It is easily seen that $\Delta a_e$ and $\Delta a_\mu$ are proportional to the lepton masses squared. This result is agreement with the naive scaling $\Delta a_\ell \propto m_\ell^2$ [161].

In the electron anomalous magnetic moment, the experiments [162] give

$$a_{e}^{\text{Exp}} = 1 159 652 180.73(28) \times 10^{-12},$$

(4.3)
while the SM prediction crucially depends on the input value for the fine-structure constant \( \alpha \). Two latest determination [163, 164] based on Cesium and Rubidium atomic recoils differ by more than 5\( \sigma \). Those observations lead to the difference from the SM prediction

\[
\frac{\Delta \alpha_{\text{e}}^{\text{CS}}}{\Delta \alpha_{\text{e}}^{\text{SM,Rb}}} = \left( -8.8 \pm 3.6 \right) \times 10^{-13} ,
\]

\[
\Delta \alpha_{\text{e}}^{\text{RS}} = \left( 4.8 \pm 3.0 \right) \times 10^{-13} .
\]

(4.4)

Our predicted value is small of one order compared with the present observed one at present. We wait for the precise observation of the fine structure constant to test our framework.

4.2 \((g-2)_\mu\) and \(\mu \to e\gamma\)

The NP in the LFV process is severely constrained by the experimental bound \(\mathcal{B}(\mu^+ \to e^+\gamma) < 4.2 \times 10^{-13}\) in the MEG experiment [165]. We can discuss the correlation between the anomaly of the muon \((g-2)_\mu\) and the LFV process \(\mu \to e\gamma\) by using the Wilson coefficients in eqs. (3.18) and (3.19). The ratio is given as:

\[
\frac{\left| \frac{N_{\mu e}}{N_{e\mu}} \right|}{\left| \frac{N_{e\gamma}}{N_{\mu \gamma}} \right|} = \beta_{e} \left| 1 - \frac{\alpha_{e}}{\alpha_{e(m)}} \frac{\beta_{e(m)}}{\beta_{e}} \right| .
\]

(4.5)

Let us introduce small parameters \(\delta_\alpha\), \(\delta_\beta\) and \(\delta_\gamma\) as follows:

\[
\frac{\beta_{e}}{\beta_{e(m)}} = 1 + \frac{c_{\beta}}{\beta_{e(m)}} \equiv 1 + \delta_\beta ,
\]

\[
\frac{\alpha_{e}}{\alpha_{e(m)}} = 1 + \frac{c_{\alpha}}{\alpha_{e(m)}} \equiv 1 + \delta_\alpha ,
\]

\[
\frac{\gamma_{e}}{\gamma_{e(m)}} = 1 + \frac{c_{\gamma}}{\gamma_{e(m)}} \equiv 1 + \delta_\gamma .
\]

(4.6)

where \(c_{\alpha}\), \(c_{\beta}\) and \(c_{\gamma}\) are tiny contributions from the unknown mode of \(m\) in eq. (2.1). Putting the experimental bound of this ratio in eq. (A.7) with eq. (3.17), we obtain

\[
\left| 1 - \frac{\alpha_{e}}{\alpha_{e(m)}} \frac{\beta_{e(m)}}{\beta_{e}} \right| \simeq |\delta_\beta - \delta_\alpha| < 1.4 \times 10^{-3} ,
\]

(4.7)

which suggests

\[
|\delta_\alpha| < \mathcal{O}(10^{-3}) , \quad |\delta_\beta| < \mathcal{O}(10^{-3}) ,
\]

(4.8)

without tuning between \(\delta_\alpha\) and \(\delta_\beta\). Thus, additional contributions to \(\alpha_{e(m)}\) and \(\beta_{e(m)}\) are at most \(\mathcal{O}(10^{-3})\).

It is emphasized that the NP signal of the \(\mu \to e\gamma\) process comes from the operator \(\bar{e}R\sigma_{\mu\mu\mu L}\) mainly in our scheme. The angular distribution with respect to the muon polarization can distinguish between \(\mu^+ \to e^+_L \gamma\) and \(\mu^+ \to e^+_R \gamma\) [166].

Let us consider the correlation among the LFV processes \(\mu \to e\gamma\), \(\tau \to \mu\gamma\) and \(\tau \to e\gamma\). Since it depends on \(\delta_\alpha\), \(\delta_\beta\) and \(\delta_\gamma\), we consider two cases for these parameters. The first
one is the case that the additional unknown mode of $m$ is the Higgs-like mode, that is, $\delta_\alpha \sim \delta_\beta \sim \delta_\gamma$. Then, we obtain ratios of the Wilson coefficients by using eq. (3.18) as:

$$\frac{C_{e\gamma}^{\prime}}{C_{e\gamma}^{\prime \mu e}} = \frac{1}{\sqrt{3}} \times O(1), \quad \frac{C_{\tau e}^{\prime}}{C_{\tau e}^{\prime \mu e}} = \tilde{\beta}_e \times O(1) \sim 10^{-2}, \quad (4.9)$$

where the numerical value in eq. (3.17) is put. The decay rates are calculated in terms of Wilson coefficients as seen in eq. (A.5). In this case, we have $B(\tau \rightarrow \mu \gamma) : B(\mu \rightarrow e \gamma) \sim 10^4 : 1 : 10$, where we take account of the kinematical factor. Since the present upper bounds of $B(\tau \rightarrow e \gamma)$ and $B(\tau \rightarrow \mu \gamma)$ are $3.3 \times 10^{-8}$ and $4.4 \times 10^{-8}$ [167], respectively, we expect the experimental test of this prediction for $\tau \rightarrow \mu \gamma$ in the future.

Another case is that unknown mode of $m$ is the flavor blind one, that is $c_\alpha = c_\beta = c_\gamma = c$ in eq. (4.6). Therefore, we have $|\delta_\beta| \gg |\delta_\alpha| \gg |\delta_\gamma|$ due to the hierarchy of $\tilde{\beta}_e(m) \ll \tilde{\alpha}_e(m) \ll \tilde{\gamma}(m)$. We obtain the Wilson coefficients by using eq. (3.18):

$$C_{e\gamma}^{\prime \mu e} = \frac{\sqrt{3}}{2} (1 - \sqrt{3}) \tilde{\alpha}_e \left( 1 - \frac{1 + \delta_\gamma}{1 + \delta_\alpha} \right) \simeq \frac{\sqrt{3}}{2} (1 - \sqrt{3}) \tilde{\alpha}_e \delta_\alpha,$$

$$C_{\tau e}^{\prime \mu e} = \frac{\sqrt{3}}{2} (1 - \sqrt{3}) \tilde{\beta}_e \left( 1 + \frac{1 + \delta_\alpha}{1 + \delta_\beta} \right) \simeq \frac{\sqrt{3}}{2} (1 - \sqrt{3}) \tilde{\beta}_e (\delta_\alpha + \delta_\beta),$$

$$C_{\tau e}^{\prime \mu e} = 3 \frac{2}{3} (1 - \sqrt{3}) \tilde{\beta}_e \left( 1 - \frac{1 + \delta_\alpha}{1 + \delta_\beta} \right) \simeq 3 \frac{2}{3} (1 - \sqrt{3}) \tilde{\beta}_e (\delta_\beta - \delta_\alpha). \quad (4.10)$$

Therefore, ratios of the Wilson coefficients are expected as:

$$\frac{C_{e\gamma}^{\prime \mu e}}{C_{e\gamma}^{\prime \mu e}} \simeq \frac{1}{\sqrt{3}}, \quad \frac{C_{\tau e}^{\prime \mu e}}{C_{\tau e}^{\prime \mu e}} \simeq \frac{\tilde{\beta}_e \delta_\beta}{\tilde{\alpha}_e \delta_\alpha} \simeq \frac{\tilde{\beta}_e c}{\tilde{\alpha}_e c} \simeq 1. \quad (4.11)$$

It results in $B(\tau \rightarrow \mu \gamma) : B(\tau \rightarrow e \gamma) : B(\mu \rightarrow e \gamma) \sim 1 : 1 : 10$ for the case of $c_\alpha = c_\beta = c_\gamma = c$.

### 4.3 RG evolution contribution of the leptonic dipole operator

In the previous subsection, our discussion does not include the renormalization group (RG) contribution. Let us briefly summarize the RG evolution contribution of the leptonic dipole operators in ref. [19] to discuss the RG effect on the numerical results in eqs. (4.9) and (4.11) at the low-energy. Adopting the SMEFT Warsaw basis [21] for the dimension 6 effective operators, the relevant terms are decomposed as

$$\Delta L_{\text{unbroken}} = \Delta L_H + \Delta L_{4f} + h.c., \quad (4.12)$$

where

$$\Delta L_H = -[Y_{\ell \mu}]_{pr} (\tilde{\ell}_p \epsilon_{r}) H + \frac{1}{A^2} \left[ C_{pr}^{\mu} (\tilde{\ell}_p \epsilon_r) (H^I H^I) + C_{pr}^{\mu} (\tilde{\ell}_p \sigma_{\mu \nu} \epsilon_r) \tau^I H W_{\mu \nu} + C_{pr}^{\mu} (\tilde{\ell}_p \sigma_{\mu \nu} \epsilon_r) H B_{\mu \nu} \right],$$

$$\Delta L_{4f} = \frac{1}{A^2} \left[ C_{pr}^{(3)} (\tilde{\ell}_p \sigma_{\mu \nu} \epsilon_r) \epsilon_{jk} (\tilde{q}_k \sigma_{\mu \nu} u_t) + C_{pr}^{(1)} (\tilde{\ell}_p \epsilon_r) \epsilon_{jk} (\tilde{q}_k u_t) + C_{pr}^{(2)} (\tilde{\ell}_p \epsilon_r) (\tilde{d}_k q_{t j}) \right]. \quad (4.13)$$

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- 11 -
After the spontaneous breaking of the Higgs field $H$, $\Delta \mathcal{L}_H$ is rewritten as:

$$\Delta \mathcal{L}_{H \text{ broken}} = - [\mathcal{Y}_e |_{\mu R} \nu \sqrt{2} (\bar{\ell}_L p e R_R) + \frac{1}{\Lambda^2} \left[ - [\mathcal{Y}_{he}]_{\mu R} \frac{1}{\sqrt{2}} (\bar{\ell}_L p e R_R) + \mathcal{C}_{\gamma} (\bar{\ell}_L p e R_R) F_{\mu \nu} \right] + \mathcal{C}_Z (\bar{\ell}_L p e R_R) + \mathcal{O}(h^2, h F_{\mu \nu}, h Z_{\mu \nu})],$$  \hspace{1cm} (4.14)

where $Z_{\mu \nu}$ is the field strength tensor for the $Z$ boson and $h$ is the physical Higgs boson. The relations between terms in the broken and unbroken phases are

$$\begin{pmatrix} C_{\gamma} \\ C_Z \end{pmatrix} |_{\mu R} = \begin{pmatrix} c_{\theta} & - s_{\theta} & 0 \\ - s_{\theta} & c_{\theta} & 0 \end{pmatrix} \begin{pmatrix} C_{\gamma} \\ C_Z \end{pmatrix} |_{\mu R},$$  \hspace{1cm} (4.15)

with

$$c_{\theta} = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} = \frac{e}{g_1}, \quad s_{\theta} = \frac{g_1}{\sqrt{g_1^2 + g_2^2}} = \frac{e}{g_2},$$  \hspace{1cm} (4.16)

where $g_1$ and $g_2$ are U(1)\nu and SU(2) gauge couplings.

The solution to RG equations of the electromagnetic dipole operators and Yukawa couplings are given at the one-loop level as follows:

$$\mathcal{C}_{\gamma} (\mu_L) = [1 - 3 \hat{L} (y_{\ell}^2 + y_{H}^2)] \mathcal{C}_{\gamma} (\mu_H) - [16 \hat{L} y_{\ell} e C_{\text{equ}}^{(2)} Y_{\text{equ}}^{(3)} (\mu_H)],$$  \hspace{1cm} (4.17)

$$\mathcal{Y}_{\text{equ}}^{(3)} (\mu_H) = \mathcal{Y}_{\text{equ}}^{(3)} (\mu_H) - \frac{\nu^2}{2 \Lambda^2} \mathcal{C}_{\text{equ}} (\mu_H)$$  \hspace{1cm} (4.18)

where

$$\hat{L} = \frac{1}{16 \pi^2} \log \left( \frac{\mu_H}{\mu_L} \right),$$

and $\mu_H (L)$ denotes the higher (lower) mass scale. The coefficient in front of $\mathcal{C}_{\text{equ}}$ is numerically small. The coefficients $\mathcal{C}_{\text{equ}}$ controls the $\mu$-$e$ flavor violating coupling of the physical Higgs boson, which is tightly constrained by other observables [168, 169] and can be safely ignored in the present analysis.

Finally, approximate evolutions are obtained as follows:

$$\mathcal{C}_{\gamma} (\mu_L) = [1 - 3 \hat{L} (y_{\ell}^2 + y_{H}^2)] \mathcal{C}_{\gamma} (\mu_H) - 16 \hat{L} y_{\ell} e C_{\text{equ}}^{(2)} Y_{\text{equ}}^{(3)} (\mu_H),$$  \hspace{1cm} (4.19)

It is emphasized that the leptonic flavor structures of $C_{\text{equ}}^{(2)} (\mu_H)$ and $C_{\text{equ}}^{(3)} (\mu_H)$ are just the same ones as $C_{\gamma} (\mu_H)$ because these Wilson coefficients of the 4-fermion operator are written by a product of 3-point coupling of leptons and that of quarks in our Ansatz eq. (2.1). Therefore, the RG contributions do not change the flavor structure of $C_{\gamma} (\mu_H)$ apart from the overall factor at low-energy.
On the other hand, $[\mathcal{V}_e]_{rs}(\mu_L)$ has non-trivial RG contribution to the flavor structure due to $C_{\text{equ}}^{(1)}_{rs33}(\mu_H)$, which has the same flavor structure of $C_{\gamma L}^{[3]}(\mu_H)$. If magnitudes of $C_{\text{equ}}^{(1)}_{rs33}(\mu_H)$ and $C_{\text{equ}}^{(3)}_{rs33}(\mu_H)$ are comparable, we have the relation by using the numerical value of (A.4),

$$\left\vert \frac{6\lambda^2\sqrt{2}y_t^2 C_{\text{equ}}^{(1)}_{rs33}(\mu_H)}{m_\mu/v} \right\vert \simeq 10^{-3} \times \left\vert \frac{16\hat{\lambda}_tC_{\gamma L}^{(3)}_{rs33}(\mu_H)}{C_{\gamma L}^{[3]}(\mu_L)} \right\vert,$$

(4.20)

where both sides denote relative contributions of the RG versus diagonal (2,2) components of $C_{\gamma L}^{[3]}(\mu_L)$ and $[\mathcal{V}_e]_{rs}(\mu_L)$, respectively. Thus, the impact of the term $6\lambda^2\sqrt{2}y_t^2 C_{\text{equ}}^{(1)}_{rs33}(\mu_H)$ on the flavor structure is minor in eq. (4.19) as far as the RG terms are next-to-leading ones. Therefore, our numerical result in section 4 is still available even if the RG effect is included.

4.4 EDM of the electron

The current experimental limit for the electric dipole moment of the electron is given by ACME collaboration [170]:

$$|d_e/e| \lesssim 1.1 \times 10^{-29} \text{ cm} = 5.6 \times 10^{-13} \text{ TeV}^{-1},$$

(4.21)

at 90% confidence level. Precise measurements of the electron EDM are rapidly being updated. The future sensitivity at ACME III is [171, 172]:

$$|d_e/e| \lesssim 0.3 \times 10^{-30} \text{ cm} = 1.5 \times 10^{-14} \text{ TeV}^{-1}.$$  (4.22)

The EDM of the electron $d_e$ is defined in the operator:

$$\mathcal{O}_{\text{edm}} = -\frac{i}{2} d_e(\mu) \sigma^{\mu\nu} \gamma_5 e F_{\mu\nu},$$

(4.23)

where $d_e = d_e(\mu = m_e)$. Therefore, the EDM of the electron is extracted from the effective Lagrangian

$$\mathcal{L}_{\text{EDM}} = \frac{1}{\Lambda^2} C_{ee}^{\gamma} \mathcal{O}_{ee}^{\gamma} = \frac{1}{\Lambda^2} C_{ee}^{\gamma} \frac{v}{\sqrt{2}} \bar{e}_L \sigma^{\mu\nu} e_R F_{\mu\nu},$$

(4.24)

which leads to

$$d_e = -\sqrt{2} \frac{v}{\Lambda^2} \text{Im} [C_{ee}^{\gamma}],$$

(4.25)

at tree level, where the small effect of running below the electroweak scale is neglected. The experimental upper bound in eq. (4.21) leads to:

$$\frac{1}{\Lambda^2} \text{Im} [C_{ee}^{\gamma}] < 1.6 \times 10^{-12} \text{ TeV}^{-2},$$

(4.26)

and it may be compared with its real part of

$$\frac{1}{\Lambda^2} \text{Re} [C_{ee}^{\gamma}] = 4.9 \times 10^{-8} \text{ TeV}^{-2},$$

(4.27)
which is derived by eqs. (4.1) and (A.4). What is the origin of the tiny imaginary part of $C'_{ee}$? The coefficient $C'_{ee}$ in eq. (3.19) is rewritten in a term of small parameter $\delta_\beta$ likewise eq. (4.6)

$$C'_{ee} = 3 (1 - \sqrt{3})\tilde{\beta}_e |\epsilon_1^e| = 3 (1 - \sqrt{3})\tilde{\beta}_{e(m)} (1 + \delta_\beta) |\epsilon_1^e|,$$  

(4.28)

where $\beta_{e(m)}$ is taken to be real positive by the redefinition of the right-handed charged lepton field in order to reproduce real positive charged lepton mass. However, $\delta_\beta$, which is originated from the unknown mode of $m$, is complex in general. The small parameter $\delta_\beta$ could be related to both the $\mu \to e\gamma$ transition and the electron EDM. Eqs. (4.10) and (4.28) lead to

$$\text{Im}[C'_{ee}] \simeq 3 (1 - \sqrt{3})\tilde{\beta}_{e(m)} (\text{Im} \delta_\beta) |\epsilon_1^e|, \quad C'_{\mu e} \simeq \frac{3}{2} (1 - \sqrt{3})\tilde{\beta}_{e(m)} (\delta_\beta - \delta_\alpha).$$  

(4.29)

Putting the constraints of experiments in eqs. (4.21) and (4.27), we obtain

$$\frac{\text{Im}[C'_{ee}]}{\text{Re}[C'_{ee}]} \simeq (\text{Im} \delta_\beta) < \frac{1.6 \times 10^{-12}}{4.9 \times 10^{-8}} = 3.3 \times 10^{-5}. \quad (4.30)$$

Suppose $|\text{Im} \delta_\beta| \simeq |\delta_\beta|$ and $|\delta_\alpha| \simeq |\delta_\beta|$ (or $|\delta_\alpha| \ll |\delta_\beta|$), then, this bound is stronger than $1.4 \times 10^{-3}$ from the $\mu \to e\gamma$ in eq. (4.7). Indeed, the upper bound of the electron EDM forces the branching ratio of $\mu \to e\gamma$ to be $B(\mu^+ \to e^+\gamma) < 2.3 \times 10^{-16}$.

The muon and the tauon EDM may be interesting in high-energy model building [173]. We can also estimate them by using eq. (3.19). It is easily found that the predicted value increases at most proportional to its mass. The muon EDM is predicted to be far smaller than the present upper bound [174].

Another origin of the electron EDM is possible at one-loop in the dimension-6 SMEFT. The contributions have been studied comprehensively by discussing the five types of operators $\psi^2 H F$, $H^2 F^2$, $F^3$, $\psi^4$ and $\psi^2 \bar{\psi}^2$ in ref. [18]. In our scheme of the modular symmetry, the finite contribution comes from the operator [18]

$$O_{\ell e ee} = (E^a_L \gamma_\mu E^b_L)(\bar{E}^c_R \gamma_\mu E^d_R) = 2(\bar{E}^a_L E^b_R)(E^c_R E^d_L) + E_{LR}^{(2)},$$  

(4.31)

where $E_{LR}^{(2)}$ is an evanescent operator that vanishes in 4 dimensions. Then, we have

$$\frac{d_e}{e} = -Q_e \frac{1}{\Lambda^2} \sum_{i=\mu,\tau} \frac{m_i}{8\pi^2} \text{Im}[C'_{\ell e}],$$  

(4.32)

where $m_i$ is the muon or tauon mass and Im $[C'_{\ell e}]$ is the Wilson coefficients corresponding to the operator $O_{\ell e}$. Some part of Im $[C'_{\ell e}]$ is calculable in our scheme. After omitting $E_{LR}^{(2)}$, the Wilson coefficient of the operator $2(\bar{E}^a_L E^b_R)(\bar{E}^c_R E^d_L)$ is calculable under our Ansatz eq. (2.1) because 4-point couplings of matter fields are written by a product of 3-point
couplings of matter fields. The following $A_4$ decomposition gives the relevant Wilson coefficients:

$$2 \left( |\bar{E}_1 Y^*(\tau)E_R|_1 \otimes |\bar{E}_R Y(\tau)E_L|_1 + |\bar{E}_L Y^*(\tau)E_R|_{1'} \otimes |\bar{E}_R Y(\tau)E_L|_{1'} \right)$$

$$+ |\bar{E}_L Y^*(\tau)E_R|_{1''} \otimes |\bar{E}_R Y(\tau)E_L|_{1''} \right) + \left\{ |\bar{E}_L Y^*(\tau)E_R|_{3a,3a} \otimes |\bar{E}_R Y(\tau)E_L|_{3a,3a} \right\} \right) ,$$

where the subscripts 1, 1', 1'', 3a and 3a denote the $A_4$ representations, respectively.

The first term is given by eqs. (3.6) and (3.7). In the mass basis of the charged leptons, eq. (3.19) gives

$$C'_{\ell e} = C'_{ee} \times C'_{\mu\mu} = \frac{2}{9} (1-\sqrt{3})^2 \alpha_e \beta_e |\epsilon_1|^2 , \quad C'_{\ell e} = C'_{ee} \times C'_{\tau\tau} = 3\sqrt{3} (1-\sqrt{3})^2 \gamma_e \beta_e |\epsilon_1|^2 . \quad (4.34)$$

In order to calculate the magnitude of these coefficients, which is the NP contribution, we estimate $\alpha_e$ up to the overall strength by combining eqs. (3.19) and (A.4) as:

$$\frac{1}{L^2} \text{Re}[C'_{\rho\rho}] = \frac{1}{L^2} \frac{3}{2} (\sqrt{3}-1) \text{Re}\alpha_e = 1.0 \times 10^{-5} \text{TeV}^{-2} \Rightarrow \text{Re}\alpha_e = \frac{2}{3(\sqrt{3}-1)} \times 10^{-5} \left( \frac{\Lambda}{1 \text{TeV}} \right)^2 . \quad (4.35)$$

The magnitudes of $\beta_e$, $\gamma_e$ and $|\epsilon_1|^2$ are obtained by the relations in eq. (3.17) since $\alpha_e$, $\beta_e$, $\gamma_e$ are almost real. Then, we obtain the tauon contribution as:

$$C'_{\ell e} = 8.6 \times 10^{-12} \left( \frac{\Lambda}{1 \text{TeV}} \right)^4 e^{i\phi} . \quad (4.36)$$

where $\phi$ is the phase originated from small imaginary parts of $\alpha_e$, $\beta_e$, $\gamma_e$. On the other hand, the muon contribution $C'_{\ell e}$ is much smaller than that. Then, we have

$$\frac{d_e}{e} \simeq -Q_e \frac{1}{L^2} \frac{m_{\tau}}{8\pi^2} \text{Im}[C'_{\ell e}] \simeq 2 \times 10^{-16} \left( \frac{\Lambda}{1 \text{TeV}} \right)^2 \sin \phi \ [\text{TeV}^{-1}] , \quad (4.37)$$

where $\sin \phi$ is at most $10^{-5}$ as discussed in eq. (4.30). Here, the cutoff scale $\Lambda$ is upper bounded by $O(100 \text{TeV})$; otherwise the magnitude of $\gamma_e$ exceeds $O(1)$ due to the relation (A.4).

However, the second and third terms in eq. (4.33) could include the phase of $O(1)$ because they do not relate with the mass matrix of charged leptons. In the basis of real positive diagonal masses, there are complex in general. Those terms are obtained by the cyclic permutation $1 \rightarrow 1'$, $1' \rightarrow 1''$ and $1'' \rightarrow 1$ in eq. (3.6) and taking its dagger, in which $(\alpha_e, \beta_e, \gamma_e)$ are replaced with complex new parameters $(\tilde{\alpha}'_e, \tilde{\beta}'_e, \tilde{\gamma}'_e)$ and $(\tilde{\alpha}''_e, \tilde{\beta}''_e, \tilde{\gamma}''_e)$, respectively. Although new parameters are unknown ones, we can assume reasonably them to be comparable to $(\alpha_e, \beta_e, \gamma_e)$. Therefore, the electron EDM is estimated roughly by eq. (4.37) by putting $\sin \phi \sim 1$. Then, $d_e/e$ is expected to be $2 \times 10^{-14} (5 \times 10^{-13}) \text{TeV}^{-1}$ for $\Lambda = 10 (50) \text{TeV}$. These are consistent with the present upper-bound $d_e/e < 5.6 \times 10^{-13} \text{TeV}^{-1}$. It is noted that the last term of $3 \otimes 3$ in eq. (4.33) is unable to be estimated because this term has many unknown parameters.
5 Summary

We have studied the LFV decays, the anomalous magnetic moments \((g-2)_{\mu,e}\) and the EDM of the electron in the SMEFT with the \(\Gamma_N\) modular flavor symmetry. We employ the relation eq. (2.1) as Ansatz in the SMEFT. Through this Ansatz, higher-dimensional operators are related with 3-point couplings. We take the level 3 finite modular group, \(\Gamma_3\) for the flavor symmetry, and discuss the dipole operators at nearby fixed point \(\tau = i\), where observed lepton masses and mixing angles are well reproduced.

Suppose the anomaly of the anomalous magnetic moment of the muon, \(\Delta a_\mu\) to be evidence of NP, we have related it with the electron \(\Delta a_e\), the LFV decays and the electron EDM. It is found that the NP contribution to \(a_e\) and \(a_\mu\) is proportional to the lepton masses squared likewise the naive scaling \(\Delta a_\ell \propto m_\ell^2\). The predicted value of the anomaly of \((g-2)_e\) is small of one order compared with the observed one at present. The precise observation of the fine structure constant will provide the test of our framework.

We have also predicted the correlations among the LFV processes: \(\mu \rightarrow e\gamma\), \(\tau \rightarrow \mu\gamma\) and \(\tau \rightarrow e\gamma\). They depend on the additional unknown mode of \(m\) in eq. (2.1). If the unknown mode is Higgs-like, we have predicted \(B(\tau \rightarrow \mu\gamma) : B(\tau \rightarrow e\gamma) : B(\mu \rightarrow e\gamma) \sim 10^4 : 1 : 10\). On the other hand, if the unknown mode is the flavor blind, \(B(\tau \rightarrow \mu\gamma) : B(\tau \rightarrow e\gamma) : B(\mu \rightarrow e\gamma) \sim 1 : 1 : 10\) is predicted. The experimental test will be available in the future. It is also remarked that our numerical result of the LFV processes is acceptable even if the RG effect is included.

In order to realize the electron EDM, we need the tiny imaginary part of the Wilson coefficient \(C'_{ee}\). In the basis of real positive charged lepton masses, the imaginary part originates from the unknown mode of \(m\), which is complex in general. The parameters of the unknown mode could be related to both the \(\mu \rightarrow e\gamma\) transition and the electron EDM. It is found that the upper bound of the electron EDM forces the \(\mu \rightarrow e\gamma\) decay to be \(B(\mu^+ \rightarrow e^+\gamma) < 2.3 \times 10^{-16}\).

Another origin of the electron EDM is possible at one-loop level in the dimension-6 SMEFT. The finite contribution comes from the Wilson coefficient of the operator \(2(\bar{E}_L^e E_R^e)(\bar{E}_R^i E_L^i)\). Then, we have obtained \(d_e/e \simeq 2 \times 10^{-16} \Lambda^2 \sin \phi\), where \(\Lambda\) denotes a certain mass scale, and \(\phi\) is an unknown phase of \(O(1)\) in the Wilson coefficient. Then, \(d_e/e\) is expected to be \(2 \times 10^{-14} (5 \times 10^{-13}) \text{ TeV}^{-1}\) for \(\Lambda = 10 (50) \text{ TeV}\). These are consistent with the present upper bound \(d_e/e < 5.6 \times 10^{-13} \text{ TeV}^{-1}\).

Thus, our Ansatz in the SMEFT with the modular symmetry of flavors is powerful to study the leptonic phenomena of flavors comprehensively.

Acknowledgments

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A Experimental constraints on the dipole operators

We summarize briefly the experimental constraints on the dipole operators given by ref. [19]. Below the scale of electroweak symmetry breaking, the leptonic dipole operators are given as:

\[ O_{\gamma r s} = \frac{v}{\sqrt{2}} \sigma_{\mu\nu} r L_r e^e F_{\mu\nu}, \]  

(A.1)

where \( \{r, s\} \) are flavor indices \( e, \mu, \tau \) and \( F_{\mu\nu} \) is the electromagnetic field strength tensor. The corresponding Wilson coefficient is denoted by \( C'_{\gamma r s} \) in the mass basis of leptons.

The combined result from the E989 experiment at FNAL [1] and the E821 experiment at BNL [2] on the \( a_\mu = (g - 2)/2 \), together with the SM prediction in [3], implies

\[ \Delta a_\mu = a_\mu^{\text{Exp}} - a_\mu^{\text{SM}} = (251 \pm 59) \times 10^{-11}. \]  

(A.2)

The tree-level expression for \( \Delta a_\mu \) in terms of the Wilson coefficient of the dipole operator is

\[ \Delta a_\mu = \frac{4m_\mu v}{e} \frac{1}{\Lambda^2} \text{Re} [C'_{\gamma e \mu}], \]  

(A.3)

where \( v \approx 246 \text{ GeV} \) and \( \Lambda \) is a certain mass scale of NP. Here the Wilson coefficient is understood to be evaluated at the weak scale (we neglect the small effect of running below the weak scale), and the prime of the Wilson coefficient indicates the flavor basis corresponding to the mass-eigenstate basis of charged leptons.\(^6\) Inputting the experimental results leads to

\[ \frac{1}{\Lambda^2} \text{Re} [C'_{\gamma e \mu}] \approx 1.0 \times 10^{-5} \text{ TeV}^{-2}. \]  

(A.4)

The tree-level expression of a radiative LFV rate in terms of the Wilson coefficients is

\[ B(\ell_r \to \ell_s \gamma) = \frac{m_\ell_r^3 v^2}{8\pi \Gamma_{\ell_r}} \frac{1}{\Lambda^4} \left( |C'_{\gamma r s}|^2 + |C'_{\gamma r s}|^2 \right). \]  

\[ B(\mu^+ \to e^+ \gamma) < 4.2 \times 10^{-13} \text{ (90\% C.L.)} \]  

obtained by the MEG experiment [165] can be translated into the upper bound

\[ \frac{1}{\Lambda^2} |C'_{e \mu (e \mu)}| < 2.1 \times 10^{-10} \text{ TeV}^{-2}. \]  

(A.6)

Using this expression, the experimental bound \( B(\mu^+ \to e^+ \gamma) < 4.2 \times 10^{-13} \text{ (90\% C.L.)} \) obtained by the MEG experiment [165] can be translated into the upper bound

\[ \frac{1}{\Lambda^2} |C'_{e \mu (e \mu)}| < 2.1 \times 10^{-10} \text{ TeV}^{-2}. \]

Taking into account eqs. (A.4) and (A.6), we have the ratio:

\[ \left| \frac{C'_{e \gamma e \mu}}{C'_{\gamma e \mu}} \right| < 2.1 \times 10^{-5}. \]  

(A.7)

\(^6\)The one-loop relation can be found in [175].
B  $A_4$ modular symmetry

B.1  Modular flavor symmetry

We briefly review the models with $A_4$ modular symmetry. The modular group $\bar{\Gamma}$ is the group of linear fractional transformations $\gamma$ acting on the modulus $\tau$, belonging to the upper-half complex plane as:

$$\tau \longrightarrow \gamma \tau = \frac{a\tau + b}{c\tau + d}, \quad \text{where } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1, \quad \text{Im}[\tau] > 0 , \quad (B.1)$$

which is isomorphic to $\text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\{I, -I\}$ transformation. This modular transformation is generated by $S$ and $T$,

$$S: \tau \longrightarrow -\frac{1}{\tau}, \quad T: \tau \longrightarrow \tau + 1 , \quad (B.2)$$

which satisfy the following algebraic relations,

$$S^2 = I, \quad (ST)^3 = I \quad . \quad (B.3)$$

We introduce the series of groups $\Gamma(N)$, called principal congruence subgroups, where $N$ is the level $1, 2, 3, \ldots$. These groups are defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{mod} N) \right\} . \quad (B.4)$$

For $N = 2$, we define $\bar{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\}$. Since the element $-I$ does not belong to $\Gamma(N)$ for $N > 2$, we have $\bar{\Gamma}(N) = \Gamma(N)$. The quotient groups defined as $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$ are finite modular groups. In these finite groups $\Gamma_N$, $T^N = I$ is imposed. The groups $\Gamma_N$ with $N = 2, 3, 4, 5$ are isomorphic to $S_3, A_4, S_4$ and $A_5$, respectively [31].

Modular forms $f_i(\tau)$ of weight $k$ are the holomorphic functions of $\tau$ and transform as

$$f_i(\tau) \longrightarrow (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau), \quad \gamma \in \bar{\Gamma} , \quad (B.5)$$

under the modular symmetry, where $\rho(\gamma)_{ij}$ is a unitary matrix under $\Gamma_N$.

Under the modular transformation of eq. (B.1), chiral superfields $\psi_i$ ($i$ denotes flavors) with weight $-k$ transform as [176]

$$\psi_i \longrightarrow (c\tau + d)^{-k} \rho(\gamma)_{ij} \psi_j . \quad (B.6)$$

We study global SUSY models. The superpotential which is built from matter fields and modular forms is assumed to be modular invariant, i.e., to have a vanishing modular weight. For given modular forms, this can be achieved by assigning appropriate weights to the matter superfields.

The kinetic terms are derived from a Kähler potential. The Kähler potential of chiral matter fields $\psi_i$ with the modular weight $-k$ is given simply by

$$\frac{1}{i(\bar{\tau} - \tau)^k} \sum_i |\psi_i|^2 , \quad (B.7)$$
where the superfield and its scalar component are denoted by the same letter, and \( \bar{\tau} = \tau^* \) after taking VEV of \( \tau \). The canonical form of the kinetic terms is obtained by changing the normalization of parameters [50]. The general Kähler potential consistent with the modular symmetry possibly contains additional terms [177]. However, we consider only the simplest form of the Kähler potential, as naturally realized in the effective action of higher-dimensional theory such as the string theory.

For \( \Gamma_3 \simeq A_4 \), the dimension of the linear space \( \mathcal{M}_k(\Gamma(3)) \) of modular forms of weight \( k \) is \( k + 1 \) [178–180], i.e., there are three linearly independent modular forms of the lowest non-trivial weight 2, which form a triplet of the \( A_4 \) group. These modular forms have been explicitly given [32] in the symmetric base of the \( A_4 \) generators \( S \) and \( T \) for the triplet representation as shown in the next subsection.

**B.2 Modular forms**

The holomorphic and anti-holomorphic modular forms of weight 2 compose the \( A_4 \) triplet as:

\[
Y(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \quad \overline{Y}(\tau) \equiv Y^*(\tau) = \begin{pmatrix} Y_1^*(\tau) \\ Y_2^*(\tau) \\ Y_3^*(\tau) \end{pmatrix}.
\]  

(B.8)

In the representation of the generators \( S \) and \( T \) for \( A_4 \) triplet:

\[
S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
\]

(B.9)

where \( \omega = e^{\frac{i\pi}{3}} \), modular forms are given explicitly [32]:

\[
Y_1(\tau) = \frac{i}{2\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'(\tau + 1)/3}{\eta((\tau + 1)/3)} + \frac{\eta'(\tau + 2)/3}{\eta((\tau + 2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right),
\]

\[
Y_2(\tau) = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^{2} \frac{\eta'(\tau + 1)/3}{\eta((\tau + 1)/3)} + \frac{\eta'(\tau + 2)/3}{\eta((\tau + 2)/3)} \right),
\]

\[
Y_3(\tau) = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'(\tau + 1)/3}{\eta((\tau + 1)/3)} + \omega^{2} \frac{\eta'(\tau + 2)/3}{\eta((\tau + 2)/3)} \right),
\]

(B.10)

where \( \eta(\tau) \) is the Dedekind eta function,

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = \exp(2\pi i \tau).
\]

(B.11)

Those are also expressed in the expansions of \( q = \exp(2\pi i \tau) \):

\[
\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \ldots \\ -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \end{pmatrix}.
\]

(B.12)
Table 3. Representations and weights $k$ for MSSM fields and modular forms of weight 2 and 4.

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & L_L & (e_R^c, \mu_R^c, \tau_R^c) & H_d, H_u & Y_R^{(2)}, Y_R^{(4)} \\
\hline
SU(2) & 2 & 1 & 2 & 1 \\
A_4 & 3 & (1, 1'', 1') & 1 & 3 \{3, 1, 1'\} \\
k & 2 & (0, 0, 0) & 0 & 2 \quad 4 \\
\hline
\end{array}
\]

\section{A model of mass matrices in $A_4$ modular symmetry}

We present a model for leptons [61], where the neutrino mass matrix is given in terms of weight 4 modular forms by using Weinberg operator. The prediction of this model is consistent with the NuFIT 5.0 data [181]. The assignments of representations and modular weights to the lepton fields are presented in table 3.

The charged lepton mass matrix and neutrino ones are given as:

\[
M_E = v_d \begin{pmatrix}
\alpha_{e(m)} & 0 & 0 \\
0 & \beta_{e(m)} & 0 \\
0 & 0 & \gamma_{e(m)}
\end{pmatrix}
\begin{pmatrix}
Y_1 & Y_3 & Y_2 \\
Y_2 & Y_1 & Y_3 \\
Y_3 & Y_2 & Y_1
\end{pmatrix}_{RL},
\]

\[
M_\nu = \frac{v_u^2}{\Lambda} \left[ 2Y_1^{(4)} - Y_3^{(4)} - Y_2^{(4)} \\
- Y_3^{(4)} & 2Y_2^{(4)} & - Y_1^{(4)} \\
- Y_2^{(4)} & - Y_1^{(4)} & 2Y_3^{(4)}
\right] + g_1^{\nu} Y_1^{(4)} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} + g_2^{\nu} Y_1^{(4)} \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix},
\]

respectively, where $\alpha_{e(m)}$, $\beta_{e(m)}$, and $\gamma_{e(m)}$ are real and $g_1^{\nu}$, $g_2^{\nu}$ are supposed to be also real. The parameters $v_d$ and $v_u$ denote the VEV of Higgs fields $H_d$ and $H_u$, respectively.

The modular forms of higher weights are obtained by tensor products of $Y_3^{(2)}(\tau)$. The modular forms of weight $k = 4$ have the dimension $d_4 = 5$. They decompose to $3, 1$, and $1'$ and are written explicitly by

\[
Y_3^{(4)}(\tau) = \begin{pmatrix}
Y_1^{2} - Y_2 Y_3 \\
Y_3^{2} - Y_1 Y_2 \\
Y_2^{2} - Y_1 Y_3
\end{pmatrix},
\]

\[
Y_1^{(4)} = Y_1^{2} + 2Y_2 Y_3, \quad Y_1^{(4)} = Y_3^{2} + 2Y_1 Y_2.
\]

\section{Charged lepton mass matrix at nearby $\tau = i$}

Residual symmetries arise whenever the VEV of the modulus $\tau$ breaks the modular group $\Gamma$ only partially. There are only 2 inequivalent finite points in the fundamental domain of $\Gamma$, namely, $\tau = i$ and $\tau = \omega = -1/2 + i\sqrt{3}/2$. The first point is invariant under the $S$ transformation $\tau = -1/\tau$. In the case of $A_4$ symmetry, the subgroup $Z_S^2 = \{I, S\}$ is preserved at $\tau = i$.
If a residual symmetry of $S$ in $A_4$ is preserved in mass matrices of leptons, we have commutation relations between the mass matrices and the generator $S$ as:

$$[M_E^1, M_E, S] = 0.$$  \hspace{1cm}  \text{(D.1)}

Therefore, $M_E^1 M_E$ could be diagonal in the diagonal basis of the generator $S$.

Let us consider the fixed point $\tau = i$, where holomorphic and anti-holomorphic modular forms of weight 2 are given as:

$$Y(\tau = i) = Y_1(i) \begin{pmatrix} 1 \\ 1 - \sqrt{3} \\ -2 + \sqrt{3} \end{pmatrix}, \quad Y^*(\tau = i) = Y_1(i) \begin{pmatrix} 1 \\ -2 + \sqrt{3} \\ 1 - \sqrt{3} \end{pmatrix},$$

$$Y(\tau_e = i) = Y_1(i) \begin{pmatrix} 1 \\ 1 - \sqrt{3} \\ -2 + \sqrt{3} \end{pmatrix}, \quad Y(\tau_\tau = i) = Y_1(i) \begin{pmatrix} 1 \\ -2 + \sqrt{3} \\ 1 - \sqrt{3} \end{pmatrix}, \quad (D.2)$$

in the basis of eq. (B.9).

The charged lepton mass matrix of eq. (C.1) is given in terms of weight 2 modular forms. Perform the unitary transformation of $E_L \rightarrow U_S E_L$, where

$$U_S = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 & 2 & 2 \\ \sqrt{3} + 1 & -2 & \sqrt{3} - 1 \\ \sqrt{3} - 1 & -2 & \sqrt{3} + 1 \end{pmatrix}. \quad (D.3)$$

It is noticed that $U_S MU_S^\dagger$ is diagonal. Then, the charged lepton mass matrix at $\tau = i$ in eq. (C.1) is simply given as:

$$M_E = \frac{1}{2} v_d \begin{pmatrix} 0 & 3(\sqrt{3} - 1)\tilde{\alpha}_{e(m)} & -(3 - \sqrt{3})\tilde{\alpha}_{e(m)} \\ 0 & -3(\sqrt{3} - 1)\tilde{\beta}_{e(m)} & -(3 - \sqrt{3})\tilde{\beta}_{e(m)} \\ 0 & 2(3 - \sqrt{3})\tilde{\gamma}_{e(m)} & 0 \end{pmatrix}_{RL}, \quad (D.4)$$

$$M_E^1 M_E = \frac{1}{2} v_d \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9(2 - \sqrt{3})(\tilde{\alpha}_{e(m)}^2 + \tilde{\beta}_{e(m)}^2) + 3(2 - \sqrt{3})(\tilde{\alpha}_{e(m)}^2 - \tilde{\beta}_{e(m)}^2) & 0 \\ 0 & 3(3 - 2\sqrt{3})(\tilde{\alpha}_{e(m)}^2 + \tilde{\beta}_{e(m)}^2) + 3(2 - \sqrt{3})(\tilde{\alpha}_{e(m)}^2 + \tilde{\beta}_{e(m)}^2 + 4\tilde{\gamma}_{e(m)}^2) & 2\sqrt{3} \end{pmatrix}_{LL},$$

where $\tilde{\alpha}_{e(m)} = (6 - 3\sqrt{3})Y_1(i)\alpha_{e(m)}$, $\tilde{\beta}_{e(m)} = (6 - 3\sqrt{3})Y_1(i)\beta_{e(m)}$, and $\tilde{\gamma}_{e(m)} = (6 - 3\sqrt{3})Y_1(i)\gamma_{e(m)}$, and the magnitudes of $\tilde{\gamma}_{e(m)}$ is supposed to be much larger than $\tilde{\alpha}_{e(m)}$ and $\tilde{\beta}_{e(m)}$. Since two eigenvalues of $S$ are degenerate such as $(1, -1, -1)$, there is still a freedom of the 2–3 family rotation. Therefore, $M_E^1 M_E$ could be diagonal after the small 2–3 family rotation of $O(\tilde{\alpha}_{e(m)}^2 \tilde{\gamma}_{e(m)}^2, \tilde{\beta}_{e(m)}^2 / \tilde{\gamma}_{e(m)}^2)$.

Since the lepton mass matrices cannot reproduce the observed PMNS matrices at fixed points as discussed in ref. [98]. Therefore, the deviations from the fixed points are required to realize observed masses and mixing angles. Let us consider the small deviation from $\tau = \tau = \tau$.
Then, the charged lepton mass matrix in eq. (D.4) turns to:

\[
M_{\nu}^{\nu} \approx \nu_d^2 \left( \begin{array}{ccc}
2(2-\sqrt{3})(\hat{\gamma}_{e(m)}^2 + \hat{\alpha}_{e(m)}^2 + \hat{\beta}_{e(m)}^2) |\epsilon_1|^2 & \frac{1}{2} (3-2\sqrt{3}) (\hat{\alpha}_{e(m)}^2 - \hat{\beta}_{e(m)}^2) |\epsilon_1|^2 & \frac{1}{2} (3-2\sqrt{3}) (\hat{\alpha}_{e(m)}^2 - \hat{\beta}_{e(m)}^2) |\epsilon_1|^2 \\
(3-2\sqrt{3}) (\hat{\alpha}_{e(m)}^2 - \hat{\beta}_{e(m)}^2) |\epsilon_1|^2 & \frac{1}{2} (3-2\sqrt{3}) (\hat{\gamma}_{e(m)}^2 + \hat{\beta}_{e(m)}^2) |\epsilon_1|^2 & \frac{1}{2} (3-2\sqrt{3}) (\hat{\gamma}_{e(m)}^2 + \hat{\beta}_{e(m)}^2) |\epsilon_1|^2 \\
(3-2\sqrt{3}) (\hat{\gamma}_{e(m)}^2 + \hat{\beta}_{e(m)}^2) |\epsilon_1|^2 & \frac{1}{2} (3-2\sqrt{3}) (\hat{\gamma}_{e(m)}^2 + \hat{\beta}_{e(m)}^2) |\epsilon_1|^2 & \frac{1}{2} (3-2\sqrt{3}) (\hat{\gamma}_{e(m)}^2 + \hat{\beta}_{e(m)}^2) |\epsilon_1|^2
\end{array} \right)
\] (D.5)

where \( \gamma_{e(m)}^2 \gg \alpha_{e(m)}^2 \) is taken following from the numerical result in ref. [61], and \( \epsilon_2 = 2\epsilon_1 \) in eq. (3.11) is put. The phase matrix \( P_e \) is given as

\[
P_e = \left( \begin{array}{ccc}
e^{i\eta_e} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \quad \eta_e = \arg [\epsilon_1].
\] (D.6)

The matrix of eq. (D.5) is a rank one at the limit of \( \alpha_{e(m)}^2 = \beta_{e(m)}^2 = 0 \). Putting small relevant values of \( \alpha_{e(m)}^2/\gamma_{e(m)}^2, \beta_{e(m)}^2/\gamma_{e(m)}^2 \) and \( |\epsilon_1| \), observed charged lepton masses could be obtained.

The mixing matrix \( U_{Lme} \) to diagonalize \( M_{\nu}^{\nu} \) such as \( U_{Lme}^\dagger M_{\nu}^{\nu} U_{Lme} = \text{diag}(m_e^2, m_{\mu}^2, m_{\tau}^2) \) is given as:

\[
U_{Lme} \simeq P_e \left( \begin{array}{ccc}
1 & s_{L12}^e & s_{L13}^e \\
-s_{L12}^e & 1 & s_{L23}^e \\
s_{L12}^e s_{L23}^e - s_{L13}^e & -s_{L12}^e & 1
\end{array} \right),
\] (D.7)

where

\[
s_{L12}^e \simeq -|\epsilon_1|, \quad s_{L13}^e \simeq -\frac{\sqrt{3}}{2} \frac{\hat{\alpha}_{e(m)}^2}{\gamma_{e(m)}^2} |\epsilon_1|, \quad s_{L23}^e \simeq -\frac{\sqrt{3}}{2} |\epsilon_1|.
\] (D.8)

We can also obtain the mixing angles of the right-handed sector. Performing the unitary transformation \( E_R = U_{12}^T E_R \) and \( \bar{E}_R \rightarrow \bar{E}_R U_{12} \) for the case of \( \hat{\gamma}_{e(m)} \gg \hat{\alpha}_{e(m)} \gg \hat{\beta}_{e(m)} \), where

\[
U_{12} = \left( \begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array} \right).
\] (D.9)

Then, the charged lepton mass matrix in eq. (D.4) turns to:

\[
M_E = \frac{1}{2} \nu_d \left( \begin{array}{ccc}
0 -3(\sqrt{3} - 1)\hat{\beta}_{e(m)} - (3 - \sqrt{3})\hat{\beta}_{e(m)} \\
0 & 3(\sqrt{3} - 1)\hat{\alpha}_{e(m)} - (3 - \sqrt{3})\hat{\alpha}_{e(m)} \\
0 & 0 & 2(3 - \sqrt{3})\hat{\gamma}_{d(m)}
\end{array} \right)_{RL}
\] (D.10)
at $\tau = i$. The matrix $M_E M_E^\dagger$ is given at $\tau = i + \epsilon$ as:

$$M_E M_E^\dagger \simeq 3(2 - \sqrt{3}) \left( 1 + \frac{2}{3} (3 - \sqrt{3}) \epsilon_1 \right) v_d^2 \left( \begin{array}{ccc} 2 \beta_{e(m)} & -\alpha_{e(m)} & -\beta_{e(m)} \tilde{\gamma}_{e(m)} \\ -\alpha_{e(m)} \beta_{e(m)} & 2 \beta_{e(m)} & -\alpha_{e(m)} \tilde{\gamma}_{e(m)} \\ -\beta_{e(m)} \tilde{\gamma}_{e(m)} & -\alpha_{e(m)} \tilde{\gamma}_{e(m)} & 2 \tilde{\gamma}_{e(m)} \end{array} \right),$$

where $\tilde{\gamma}_{e(m)} \gg \tilde{\alpha}_{e(m)} \gg \beta_{e(m)}$.

The right-handed mixing angles are given as:

$$U_{Rme}^\dagger M_E M_E^\dagger U_{Rme} = \text{diag}(m_e^2, m_\mu^2, m_m^2), \quad U_{Rme} \simeq \begin{pmatrix} 1 & s_{R12}^e & s_{R13}^e \\ -s_{R12}^e & 1 & s_{R23}^e \\ s_{R12}^e s_{R23}^e - s_{R13}^e s_{R23}^e & -s_{R13}^e & 1 \end{pmatrix},$$

where

$$s_{R12}^e \simeq -\frac{\beta_{e(m)}}{\alpha_{e(m)}}, \quad s_{R23}^e \simeq -\frac{1}{2} \frac{\tilde{\alpha}_{e(m)}}{\tilde{\gamma}_{e(m)}}, \quad s_{R13}^e \simeq -\frac{1}{2} \frac{\beta_{e(m)}}{\tilde{\gamma}_{e(m)}}.$$

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