Scalar hidden-charm tetraquark states with QCD sum rules

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Abstract

In this article, we study the masses and pole residues of the pseudoscalar-diquark-pseudoscalar-antidiquark type and vector-diquark-vector-antidiquark type scalar hidden-charm \(\bar{c} \bar{c} \bar{d} (\bar{c} \bar{u})\) tetraquark states with QCD sum rules by taking into account the contributions of the vacuum condensates up to dimension-10 in the operator product expansion. The predicted masses can be confronted with the experimental data in the future. Possible decays of those tetraquark states are also discussed.

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1 Introduction

In recent years, a number of charged charmonium-like (bottomonium-like) exotic states have been observed, such as the \(Z_c(3900), Z_c(4020), Z_c(4050), Z_c(4055), Z_c(4200), Z_c(4240), Z_c(4250), Z(4430), Z_b(10610)\) and \(Z_b(10650)\). If those charged charmonium-like (bottomonium-like) states are resonances indeed, their quark constituents must be \(\bar{c} \bar{c} \bar{d} (\bar{c} \bar{u})\), irrespective of the diquark-antidiquark type or meson-meson type substructures.

Those exotic states cannot be accommodated within the naive quark model, and represent a new facet of QCD and provide a new opportunity for a deeper understanding of the non-perturbative QCD. The QCD sum rules method is a powerful tool in studying the hidden-charm (bottom) tetraquark or molecular states and hidden-charm pentaquark states. In the QCD sum rules, the operator product expansion is used to expand the time-ordered currents into a series of quark and gluon condensates which parameterize the non-perturbative properties of the QCD vacuum. Based on the quark-hadron duality, we can obtain copious information about the hadronic parameters at the phenomenological side.

The diquarks \(\epsilon^{ijk} q^T_j C q^T_k\) have five structures in Dirac spinor space, where \(CT = C\gamma_5, C, C\gamma_\mu\gamma_5, C\gamma_\mu\) and \(C\sigma_{\mu\nu}\) for the scalar, pseudoscalar, vector, axialvector and tensor diquarks, respectively. In this expression, \(q_j\) denotes the quark field; \(i, j\) and \(k\) are color indexes; \(C\) is the charge conjugation matrix; and the superscript \(T\) denotes the transpose of the Dirac indexes. The attractive interactions of one-gluon exchange favor formation of the diquarks in color antitriplet, flavor antitriplet and spin singlet, while the favored configurations are the scalar \((C\gamma_5)\) and axialvector \((C\gamma_\mu)\) diquark states. We can construct the diquark-antidiquark type hidden charm tetraquark states

\[
\begin{align*}
C\gamma_5 & \otimes \gamma_5 C, \\
C\gamma_\mu & \otimes \gamma^\mu C,
\end{align*}
\]

(1)

to study the lowest scalar tetraquark states, or construct the

\[
\begin{align*}
C & \otimes C, \\
C\gamma_\mu\gamma_5 & \otimes \gamma_5\gamma^\mu C,
\end{align*}
\]

(2)

to study the scalar tetraquark states having larger masses.

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In this article, we construct $C \otimes C$ and $C\gamma_\mu\gamma_5 \otimes \gamma_5 \gamma^\mu C$ type currents to explore the charged scalar hidden-charm tetraquark states by calculating the contributions of the vacuum condensates up to dimension-10 in a consistent way.

This article is organized as follows: in section 2, we derive the QCD sum rules to extract the masses and pole residues of the charged scalar $cu\bar{c}d$ ($cu\bar{c}s$) tetraquark states; in section 3, we present the numerical results and discussions; section 4 is reserved for conclusion.

## 2 QCD sum rules for the scalar hidden-charm tetraquark states

In QCD sum rules, we consider the two-point correlation functions $\Pi_{1,2,3,4}(p)$,

$$\Pi_{1,2,3,4}(p) = i \int d^4xe^{ipx}\langle0|T\left\{J_{1,2,3,4}(x)J_{1,2,3,4}^\dagger(0)\right\}|0\rangle,$$

where the $J_{1,2,3,4}(x)$ are the interpolating currents with the same quantum numbers as the tetraquark states we want to study. Those currents are constructed in the diquark model and can be written as,

$$J_1(x) = \epsilon^{ijk}\epsilon^{imn}u_i^{T}(x)CC^k\bar{d}^m(x)C\bar{e}n^T(x),$$

$$J_2(x) = \epsilon^{ijk}\epsilon^{imn}u_i^{T}(x)C\gamma_\mu\gamma_5c^k\bar{d}^m(x)\gamma_5\gamma^\nu\gamma^\mu C\bar{e}n^T(x),$$

$$J_3(x) = \epsilon^{ijk}\epsilon^{imn}u_i^{T}(x)C\gamma_\nu\gamma_5c^k\bar{s}^m(x)C\bar{e}n^T(x),$$

$$J_4(x) = \epsilon^{ijk}\epsilon^{imn}u_i^{T}(x)C\gamma_\mu\gamma_5c^k\bar{s}^m(x)\gamma_5\gamma^\mu C\bar{e}n^T(x).$$

On the phenomenological side, we insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators $J_{1,2,3,4}(x)$ into the correlation functions $\Pi_{1,2,3,4}(p)$ to obtain the hadronic representations. After isolating the ground state contributions of the scalar tetraquark states, we get the following results,

$$\Pi_{1,2,3,4}(p) = \frac{\lambda^2_{Z_{1,2,3,4}}}{M_{Z_{1,2,3,4}}^2 - p^2} + \cdots,$$

where the pole residues $\lambda_{Z_{1,2,3,4}}$ are defined by $\langle0|J_{1,2,3,4}(0)|Z_{1,2,3,4}(p)\rangle = \lambda_{Z_{1,2,3,4}}$.

At the quark level, the two-point correlation functions $\Pi_{1,2,3,4}(p)$ can be evaluated via the operator product expansion method. We contract the $u$, $d$, $c$ and $s$ quark fields with the wick theorem and obtain the following results:

$$\Pi_1(p) = i\epsilon^{ijk}\epsilon^{imn}\epsilon^{i'j'k'}\epsilon^{i'm'n'} \int d^4xe^{ipx}$$

$$\text{Tr}\left[C^{kk'}(x)CU^{jj'}T(x)C\right]\text{Tr}\left[C^{nn'}(-x)C D^{m'm'}T(-x)C\right],$$

$$\Pi_2(p) = i\epsilon^{ijk}\epsilon^{imn}\epsilon^{i'j'k'}\epsilon^{i'm'n'} \int d^4xe^{ipx}$$

$$\text{Tr}\left[\gamma_\mu\gamma_5C^{kk'}(x)\gamma_5\gamma_\nu CU^{jj'}T(x)C\right]\text{Tr}\left[\gamma_5\gamma^\nu C^{nn'}(-x)\gamma_\mu\gamma_5CD^{m'm'}T(-x)C\right],$$

$$\Pi_3(p) = i\epsilon^{ijk}\epsilon^{imn}\epsilon^{i'j'k'}\epsilon^{i'm'n'} \int d^4xe^{ipx}$$

$$\text{Tr}\left[C^{kk'}(x)CU^{jj'}T(x)C\right]\text{Tr}\left[C^{nn'}(-x)CS^{m'm'}T(-x)C\right],$$

$$\Pi_4(p) = i\epsilon^{ijk}\epsilon^{imn}\epsilon^{i'j'k'}\epsilon^{i'm'n'} \int d^4xe^{ipx}$$

$$\text{Tr}\left[\gamma_\mu\gamma_5C^{kk'}(x)\gamma_5\gamma_\nu CU^{jj'}T(x)C\right]\text{Tr}\left[\gamma_5\gamma^\nu C^{nn'}(-x)\gamma_\mu\gamma_5CS^{m'm'}T(-x)C\right].$$
where the $U_{ij}(x)$, $D_{ij}(x)$, $S_{ij}(x)$ and $C_{ij}(x)$ are the full $u$, $d$, $s$ and $c$ quark propagators, respectively. For simplicity, the $U_{ij}(x)$ and $D_{ij}(x)$ can be written as $P_{ij}(x)$.

\[
\begin{align*}
P_{ij}(x) &= \frac{i\delta_{ij}x - \delta_{ij}\langle \bar{q}q \rangle}{2\pi^2 x^4} - \frac{\delta_{ij}x^2\langle \bar{q}g_i\sigma Gq \rangle}{192} - \frac{\delta_{ij}x^2\langle \bar{q}g_i\sigma Gq \rangle^2}{7776} - \frac{ig_sG^n_{\alpha\beta}t^n_{ij}(\bar{q}\sigma\alpha\beta + \sigma\alpha\beta\bar{q})}{32\pi^2 x^2} \\
S_{ij}(x) &= \frac{i\delta_{ij}x - \delta_{ij}m_s}{2\pi^2 x^4} - \frac{\delta_{ij}\langle \bar{s}s \rangle}{12} + \frac{\delta_{ij}\bar{x}\langle \bar{s}g_sG \rangle}{48} - \frac{\delta_{ij}\bar{x}^2\langle \bar{s}g_sG \rangle}{192} \\
C_{ij}(x) &= \frac{i}{(2\pi)^4} \int d^4k e^{-ikx} \left\{ \frac{k + m_c}{k^2 - m_c^2} \delta_{ij} - g_s\bar{t}n_{ij}G^n_{\alpha\beta}(\bar{k} + m_c) + \frac{g_s\bar{t}n_{ij}D^n_{\alpha}}{3(k^2 - m_c^2)^2} \\
&\quad+ g_s\bar{t}n_{ij}D^n_{\alpha}(f^{\lambda\alpha\beta} + f^{\lambda\beta\alpha}) - g_s^2(t^n\bar{t}^n)_{ij}G_{\alpha\beta\gamma\delta}(f^{\beta\gamma\lambda\delta} + f^{\beta\gamma\lambda\delta}) \right\} + \cdots,
\end{align*}
\]

\[
f^{\lambda\alpha\beta} = \frac{(\bar{k} + m_c)\gamma^\lambda(\bar{k} + m_c)\gamma^\alpha(\bar{k} + m_c)\gamma^\beta(\bar{k} + m_c)},
\]

\[
f^{\alpha\beta\gamma\delta} = \frac{(\bar{k} + m_c)\gamma^\alpha(\bar{k} + m_c)\gamma^\beta(\bar{k} + m_c)\gamma^\gamma(\bar{k} + m_c)\gamma^\delta(\bar{k} + m_c)},
\]

and $t^n = \frac{\lambda^n}{2}$, the $\lambda^n$ is the Gell-Mann matrix, $D_\alpha = \partial_\alpha - ig_sG^n_{\alpha\beta}t^n$ \[1\]. In Eqs. [13]–[14], we retain the terms $\langle \bar{q}j\sigma\mu\nu q \rangle$, $\langle \bar{s}j\sigma\mu\nu s \rangle$, $\langle \bar{q}j\gamma_\mu q \rangle$ and $\langle \bar{s}j\gamma_\mu s \rangle$ originate from the Fierz re-arrangement of the $\langle q_iq_j \rangle$, $\langle s_is_j \rangle$ to absorb the gluons emitted from the heavy quark lines so as to extract the mixed condensates and four-quark condensates $\langle \bar{q}g_sG \rangle$, $\langle \bar{s}g_sG \rangle$, $g_s^2\langle \bar{q}q \rangle^2$ and $g_s^2\langle \bar{s}s \rangle^2$, respectively. We compute the integrals both in the coordinate and momentum spaces by taking into account the contributions of the vacuum condensates up to dimension-10. Then, we obtain the QCD spectral densities from the imaginary parts of the correlations.

After getting the QCD spectral densities, we take the quark-hadron duality below the continuum thresholds $s_0$ and perform the Borel transformation with respect to the variable $P^2 = -p^2$ to obtain the QCD sum rules,

\[
\lambda_{Z,1,2,3,4}^2 \exp \left(-\frac{M_{Z,1,2,3,4}^2}{T^2} \right) = \int_{4m_c^2}^{1,2,3,4} ds \rho_{1,2,3,4}^{}(s) \exp \left(-\frac{s}{T^2} \right),
\]

where

\[
\rho_{1,2,3,4}^{}(s) = \rho_0_{1,2,3,4}^{}(s) + \rho_3_{1,2,3,4}^{}(s) + \rho_4_{1,2,3,4}^{}(s) + \rho_5_{1,2,3,4}^{}(s) + \rho_6_{1,2,3,4}^{}(s) + \rho_7_{1,2,3,4}^{}(s) + \rho_8_{1,2,3,4}^{}(s) + \rho_9_{1,2,3,4}^{}(s),
\]

the explicit expressions of the spectral densities $\rho_{1,2,3,4}^{}(s)$ are given in the appendix. The subscripts 0, 3, 4, 5, 6, 7, 8, 10 denote the dimensions of the vacuum condensates. We take into account the vacuum condensates which are vacuum expectations of the operators of the orders $O(\alpha_s^k)$ with $k \leq 1$ consistently.

We derive Eq. [17] with respect to $\frac{1}{T^2}$, then eliminate the pole residues $\lambda_{Z,1,2,3,4}^{}$ and obtain the expressions for the masses of the scalar tetraquark states,

\[
M_{Z,1,2,3,4}^2 = \frac{\int_{4m_c^2}^{1,2,3,4} ds \rho_{1,2,3,4}^{}(s) \exp \left(-\frac{s}{T^2} \right)}{\int_{4m_c^2}^{1,2,3,4} ds \rho_{1,2,3,4}^{}(s) \exp \left(-\frac{s}{T^2} \right)}.
\]

3
Once the masses are obtained, we can take them as input parameters and obtain the pole residues from the QCD sum rules in Eq. (17).

3 Numerical results and discussions

In this section, we perform the numerical analysis, and choose the reasonable QCD parameters for the quark masses and vacuum condensates. The vacuum condensates are taken to be the standard values $\langle \bar{q}q \rangle = -(0.24 \pm 0.01 \text{ GeV})^3$, $\langle \bar{s}s \rangle = (0.8 \pm 0.1)\langle \bar{q}q \rangle$, $\langle \bar{q}g_sGq \rangle = m_q^2\langle \bar{q}q \rangle$, $\langle \bar{q}g_sGs \rangle = m_0^2\langle \bar{q}q \rangle$, $m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2$, $\langle \alpha_s G^2 \rangle = (0.33 \text{ GeV})^4$ at the energy scale $\mu = 1 \text{ GeV}$ [8, 9, 11]. The quark condensates and mixed quark condensates evolve with the renormalization group equation, $\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle(Q)[\alpha_s(Q)/\alpha_s(\mu)]^{\frac{1}{2}}$, $\langle \bar{s}s \rangle(\mu) = \langle \bar{s}s \rangle(Q)[\alpha_s(Q)/\alpha_s(\mu)]^{\frac{1}{2}}$, $\langle \bar{q}g_sGq \rangle(\mu) = \langle \bar{q}g_sGq \rangle(Q)[\alpha_s(Q)/\alpha_s(\mu)]^{\frac{1}{2}}$ and $\langle \bar{q}g_sGs \rangle(\mu) = \langle \bar{q}g_sGs \rangle(Q)[\alpha_s(Q)/\alpha_s(\mu)]^{\frac{1}{2}}$. In addition, we take the values $m_u(\mu = 1 \text{ GeV}) = m_d(\mu = 1 \text{ GeV}) = m_q(\mu = 1 \text{ GeV}) = m_q(\mu = 1 \text{ GeV}) = 0.066 \text{ GeV}$ from the Gell-Mann-Oakes-Renner relation, and choose the $\overline{\text{MS}}$ mass $m_c(m_c) = (1.275 \pm 0.025) \text{ GeV}$ and $m_s(\mu = 2 \text{ GeV}) = (0.955 \pm 0.005) \text{ GeV}$ from the Particle Data Group [11], and take into account the energy-scale dependence of the $\overline{\text{MS}}$ masses from the renormalization group equation,

$$
m_q(\mu) = m_q(1 \text{ GeV}) \left[ \frac{\alpha_s(\mu)}{\alpha_s(1 \text{ GeV})} \right]^{\frac{1}{2}} ,
$$

$$
m_s(\mu) = m_s(2 \text{ GeV}) \left[ \frac{\alpha_s(\mu)}{\alpha_s(2 \text{ GeV})} \right]^{\frac{1}{2}} ,
$$

$$
m_c(\mu) = m_c(m_c) \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{3}{2}} ,
$$

$$
\alpha_s(\mu) = \frac{1}{b_0 t} \left[ 1 - b_1 \log t \frac{b_0}{b_2} \left( \log^2 t - \log t - 1 \right) + b_0 b_2 \right] ,
$$

where $t = \log \frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33 - 2n_f}{12\pi}$, $b_1 = \frac{153 - 19n_f}{24\pi^2}$, $b_2 = \frac{2857 - 603n_f + 22n_f^2}{128\pi^4}$, $\Lambda = 213 \text{ MeV}$, $296 \text{ MeV}$ and $339 \text{ MeV}$ for the flavors $n_f = 5, 4$ and 3, respectively [11].

The energy scale $\mu$ is considered as a variable. In Refs. [3, 4], the energy scale dependence of the QCD sum rules for the hidden-charm tetraquark states and molecular states is studied in details for the first time, and an energy scale formula,

$$
\mu = \sqrt{M_{X/Y/Z}^2 - 2M_c^2} ,
$$

with the effective c-quark mass $M_c$ is suggested. The formula works well for the $X(3872)$, $Z_c(3885/3900)$, $X^+(3860)$, $Y(3915)$, $Z_c(4020/4025)$, $Z(4430)$, $X(4500)$, $Y(4630/4660)$, $X(4700)$ in the scenario of tetraquark states. In this article, we take the updated value of the effective c-quark mass $M_c = 1.82 \text{ GeV}$ to determine the energy scales of the QCD spectral densities [12].

The mass gaps between the ground states and the first radial excited states are usually taken as $(0.4 - 0.6) \text{ GeV}$. For examples, the $Z(4430)$ is tentatively assigned to be the first radial excitation of the $Z_c(3900)$ according to the analogous decays, $Z_c(3900) \rightarrow J/\psi \pi^\pm$, $Z(4430) \rightarrow \psi' \pi^\pm$ and the mass differences $M_{Z(4430)} - M_{Z_c(3900)} = 576 \text{ MeV}$, $M_{\psi'} - M_{J/\psi} = 589 \text{ MeV}$ [13, 14, 15], the $X(3915)$ and $X(4500)$ are assigned to be the ground state and the first radial excited state of the axialvector-diquark-axialvector-antidiquark type scalar $csc\bar{s}$ tetraquark states, respectively, and their mass difference is $M_X(4500) - M_X(3915) = 588 \text{ MeV}$ [16]. The relation

$$
\sqrt{s_0} = M_{X/Y/Z} + (0.4 - 0.6) \text{ GeV} ,
$$

serves as another constraint on the masses of the hidden-charm tetraquark states. In calculations, we observe that the values of the masses $M_Z$ decrease with increase of the energy scales $\mu$ from
Figure 1: The pole contributions with variations of the Borel parameters $T^2$ and threshold parameters $s_0$. 
QCD sum rules in Eq. (19). While Eq. (21) indicates that the value of the masses $M_Z$ increase when the energy scales $\mu$ increase. There exist optimal energy scales, which lead to reasonable masses $M_Z$.

We should obey two criteria to choose the Borel parameters $T^2$ and threshold parameters $s_0$ in numerical calculations. The first criterion is the pole dominance on the phenomenological side. The pole contribution (PC) is defined by,

$$PC = \frac{\int_{4m_c^2}^{s_0} ds \rho_1^{1,2,3,4}(s) \exp\left(-\frac{s}{M^2}\right)}{\int_{4m_c^2}^{s_0} ds \rho^{1,2,3,4}(s) \exp\left(-\frac{s}{M^2}\right)},$$

(23)

The second criterion is the convergence of the operator product expansion. To judge the convergence, we calculate the contributions $D_i$ in the operator product expansion with the formula,

$$D_i = \frac{\int_{4m_c^2}^{s_0} ds \rho_1^{1,2,3,4}(s) \exp\left(-\frac{s}{M^2}\right)}{\int_{4m_c^2}^{s_0} ds \rho^{1,2,3,4}(s) \exp\left(-\frac{s}{M^2}\right)},$$

(24)

where the index $i$ denotes the dimension of the vacuum condensates.

Figure 2: The contributions of different terms in the operator product expansion with variations of the Borel parameters $T^2$, where the 0, 3, 4, 5, 6, 7, 8, 10 denote the dimensions of the vacuum condensates.

In Fig. 2, the contributions of the pole terms are plotted with variations of the Borel parameters $T^2$ for different values of the threshold parameters $s_0$, where the values of energy scales are taken as $\mu = 4.00\,\text{GeV}, 2.90\,\text{GeV}, 4.05\,\text{GeV}$ and $2.95\,\text{GeV}$ for the tetraquark states $Z_1$, $Z_2$, $Z_3$, $Z_4$, respectively. From the figure, we can see that the continuum thresholds $\sqrt{s_0} \leq 5.70\,\text{GeV}, \leq$
play an important role, while the other condensate terms play a minor important role. At the tetraquark states $Z_i$, with increase of the $T^2$ parameter, the perturbative term decreases monotonously and quickly to result in reasonable Borel windows.

In Figs. 2 and 3 the contributions of different condensate terms in the operator product expansion are plotted with the Borel parameters $T^2$ for the continuum thresholds $\sqrt{s_0} = 5.90$ GeV, $5.20$ GeV, $5.95$ GeV and $5.25$ GeV at the energy scales $\mu = 4.00$ GeV, $2.90$ GeV, $4.05$ GeV and $2.95$ GeV for the tetraquark states $Z_1$, $Z_2$, $Z_3$, $Z_4$, respectively. From the figure, we can see the contributions $D_i$ explicitly and choose reasonable Borel parameters. We take the $Z_1$ state as an example to illustrate the procedure. In that case, we observe that the perturbative term and the $\langle \bar{q} q \rangle$ term play an important role, while the other condensate terms play a minor important role. At the value $T^2 \leq 4.9$ GeV$^2$, the perturbative term and $\langle \bar{q} q \rangle$ term decrease monotonously and quickly with increase of the $T^2$, which cannot lead to stable masses and pole residues. At the value $T^2 = (5.0 - 5.4)$ GeV$^2$, the convergent behavior in the operator product expansion is very good and the perturbative term makes the main contribution. We present the optimal energy scales $\mu$, ideal Borel parameters $T^2$, continuum threshold parameters $s_0$ and pole contributions in Table 1. From the Table, we can see that the two criteria of the QCD sum rules can be satisfied.

We take into account all uncertainties of the input parameters, and obtain the masses and pole residues of the hidden-charm tetraquark states, which are shown explicitly in Figs. 3 and 4 and Table 1. From Figs. 3 and 4, we can see that the Borel plateau exist. On the other hand, from Table 1, we can see that the energy scale formula $\mu = \sqrt{M^2_{X/Z} - (2M_c)^2}$ is well satisfied. We expect to make reasonable predictions, the present predictions can be confronted with the experimental data in the future.

In the following, we discuss the possible hadronic decay patterns of the $cu\bar{c}d$ and $cu\bar{c}s$ scalar tetraquark states. Being composed of a diquark and antidiquark pair, a hidden-charm tetraquark state can decay very easily into a pair of open-charm $D$ mesons or one charmonium state plus...
a light meson through quark rearrangement. The strong decays are Okubo-Zweig-Iizuka superallowed. Under the restrictions of the symmetries and the masses of the studied scalar tetraquark states obtained in Table 1, the possible two-body strong decay channels are

\[
\begin{align*}
Z_1 & \rightarrow \chi_{c0}a_0^+(980), \eta_c\pi^+, J/\psi\rho^+(770), \psi(3770)\rho^+(770), \chi_{cl}a_1^+(1260), \\
& \quad \bar{D}_0^+(2430)D_s^+(2430), \bar{D}^0D^+, \bar{D}^*o(2007)D^{*+}(2010), \bar{D}_0^o(2420)D_s^+(2420), \\
& \quad \bar{D}_0^o(2430)D_s^+(2430), \\
Z_2 & \rightarrow \chi_{c0}a_0^+(980), \eta_c\pi^+, J/\psi\rho^+(770), \bar{D}^0D^+, \bar{D}^*o(2007)D^{*+}(2010), \\
Z_3 & \rightarrow \chi_{c0}K^+\pi^+(892), \psi(3770)K^{*+}(892), \\
& \quad \bar{D}_0^*(2400)D_s^{*+}(2317), \bar{D}^0D_s^+, \bar{D}^*o(2007)D_s^{*+}, \bar{D}_0^o(2420)D_s^{*+}(2536), \\
& \quad \bar{D}_0^o(2430)D_s^{*+}(2460), \\
Z_4 & \rightarrow \chi_{c0}K^{*+}(800), \eta_cK^+, J/\psi K^{*+}(892), \bar{D}^0D_s^+, \bar{D}^*o(2007)D_s^{*+},
\end{align*}
\]

Table 1: The energy scales, Borel parameters, continuum threshold parameters, pole contributions, masses and pole residues for the scalar tetraquark states.

```latex
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$J^{PC}$ & $\mu$(GeV) & $T^2$(GeV$^2$) & $\sqrt{s_0}$(GeV) & pole & $M_Z$(GeV) & $\lambda_Z$(GeV$^2$) \\
\hline
$0^+\pi^+(Z_1)$ & 4.00 & 5.0 – 5.4 & 5.90 ± 0.10 & (42 – 57) % & 5.43$^{+0.15}_{-0.09}$ & (1.88$^{+0.22}_{-0.19}$) $\times 10^{-1}$ \\
$0^+\pi^+(Z_2)$ & 2.90 & 3.6 – 4.0 & 5.20 ± 0.10 & (42 – 61) % & 4.64$^{+0.09}_{-0.09}$ & (1.31$^{+0.22}_{-0.21}$) $\times 10^{-1}$ \\
$0^+\pi^+(Z_3)$ & 4.05 & 5.1 – 5.5 & 5.95 ± 0.10 & (43 – 57) % & 5.45$^{+0.12}_{-0.09}$ & (1.92$^{+0.24}_{-0.19}$) $\times 10^{-1}$ \\
$0^+\pi^+(Z_4)$ & 2.95 & 3.7 – 4.1 & 5.25 ± 0.10 & (43 – 61) % & 4.67$^{+0.08}_{-0.09}$ & (1.39$^{+0.23}_{-0.22}$) $\times 10^{-1}$ \\
\hline
\end{tabular}
\caption{The pole residues with variations of the Borel parameters $T^2$.}
\end{table}
```
which are kinematically allowed. Theoretically, the mass of the ground state is lighter than the counterpart of its excited state, and the corresponding phase space is larger, thus the decay width of the resonance to the ground state is wider. This means that the ground state decay modes of the resonance can take place more easily. Besides, the excited state is unstable, which bring some difficulty to the observation of the decay process for the resonance state. Consequently, the dominant decay modes of the scalar tetraquark states are $Z_1(Z_2) \rightarrow \eta_c \pi^+$, $J/\psi \rho^+$ (770), $D^0 \bar{D}^+$, $Z_3(Z_4) \rightarrow \eta_c K^+$, $J/\psi K^{*+}$ (892), $D^0 \bar{D}^{*+}$. We can search for those scalar hidden-charm tetraquark states in those decay channels.

4 Conclusion

In this article, we study the pseudoscalar-diquark-pseudoscalar-antidiquark type and vector-diquark-vector-antidiquark type scalar hidden-charm $cuc\bar{d}$ ($cu\bar{c}s$) tetraquark states with the QCD sum rules by calculating the contributions of the vacuum condensates up to dimension-10 in the operator product expansion. In numerical calculations, we use the energy scale formula $\mu = \sqrt{M_{\chi}/Y_{\rho} - (2M_{\rho})^2}$ to determine the ideal energy scales of the QCD spectral densities and search for the optimal Borel parameters $T^2$ and continuum thresholds $s_0$ to satisfy the two criteria of the QCD sum rules (i.e. pole dominance on the phenomenological side and convergence of the operator product expansion). We obtain the masses and pole residues of the scalar hidden-charm $cuc\bar{d}$ ($cu\bar{c}s$) tetraquark states. The predicted masses are around 5.43–5.45 GeV for the $C \otimes C$ type tetraquark states and 4.64–4.67 GeV for the $C\gamma_5 \otimes \gamma_5 \gamma^\mu C$ type ones, which can be confronted with the experimental data in the future. Moreover, we discuss the possible hadronic decay patterns of the two types of tetraquark states, and list their dominant decays. As the predicted masses of the $C\gamma_5 \otimes \gamma_5 \gamma^\mu C$ type tetraquark states are lighter than the counterparts of the $C \otimes C$ type ones and the two types of tetraquark states have the same dominant decays, the widths of the $C\gamma_5 \otimes \gamma_5 \gamma^\mu C$ type tetraquark states are narrower. Therefore, the $C\gamma_5 \otimes \gamma_5 \gamma^\mu C$ type tetraquark states can be observed more easily, which can be testified in the future at the BESIII, LHCb and Belle-II.

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Appendix

The explicit expressions of the QCD spectral densities $\rho^{1,2,3,4}(s)$,

$$
\rho^1(s) = \rho^3(s) = \rho^3(s)_{m_s \rightarrow 0, (\bar{s}s) \rightarrow (\bar{q}q), (\bar{s}g, \sigma G_s) \rightarrow (\bar{q}g, \sigma G_q)},
\rho^2(s) = \rho^4(s)_{m_s \rightarrow 0, (\bar{s}s) \rightarrow (\bar{q}q), (\bar{s}g, \sigma G_s) \rightarrow (\bar{q}g, \sigma G_q)}.
$$

(26)

$$
\rho^3_0(s) = \frac{1}{512\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \ yz(1-y-z)^3(s-\bar{m}_c^2)^2(7s^2-6s\bar{m}_c^2+\bar{m}_c^4)
-\frac{m_s\bar{m}_c}{512\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \ (y+z)(1-y-z)^3(s-\bar{m}_c^2)^2(5s-2\bar{m}_c^2),
$$

(27)
\[ \rho_0^4(s) = \frac{1}{256\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, yz(1-y-z)^3(s-\hat{m}^2_c)^2(7s^2-6s\hat{m}^2_c+\hat{m}^4_c) \]
\[ + \frac{1}{256\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, yz(1-y-z)^2(s-\hat{m}^2_c)^3(3s-\hat{m}^2_c) \]
\[ - \frac{m_s m_c}{256\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, (y+z)(1-y-z)^2(s-\hat{m}^2_c)^2(5s-2\hat{m}^2_c) \]  \( (28) \)

\[ \rho_3^4(s) = \frac{m_s (\langle \bar{q}q \rangle + \langle \bar{s}s \rangle)}{32\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, (y+z)(1-y-z)(s-\hat{m}^2_c)(2s-\hat{m}^2_c) \]
\[ + \frac{m_s \langle \bar{s}s \rangle}{32\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, yz(1-y-z)(10s^2-12s\hat{m}^2_c+3\hat{m}^4_c) \]
\[ - \frac{m_s m_s^2 \langle \bar{q}q \rangle}{16\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, (s-\hat{m}^2_c) \], \( (29) \)

\[ \rho_4^4(s) = \frac{m_s (\langle \bar{q}q \rangle + \langle \bar{s}s \rangle)}{16\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, (y+z)(1-y-z)(s-\hat{m}^2_c)(2s-\hat{m}^2_c) \]
\[ - \frac{m_s m_s^2 \langle \bar{q}q \rangle}{4\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, (s-\hat{m}^2_c) \]
\[ + \frac{m_s \langle \bar{s}s \rangle}{16\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, yz(1-y-z)(10s^2-12s\hat{m}^2_c+3\hat{m}^4_c) \]
\[ + \frac{m_s \langle \bar{s}s \rangle}{16\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, yz(s-\hat{m}^2_c)(2s-\hat{m}^2_c) \]  \( (30) \)

\[ \rho_3^4(s) = - \frac{m_s^2}{384\pi^4} \frac{\alpha_{GG}}{\pi} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, \left( \frac{y}{z^2} + \frac{z}{y^2} \right) (1-y-z)^3 \left\{ (2s-\hat{m}^2_c) + \frac{\hat{m}^2_c}{6} \delta(s-\hat{m}^2_c) \right\} \]
\[ + \frac{1}{512\pi^4} \frac{\alpha_{GG}}{\pi} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, (y+z)(1-y-z)^2(10s^2-12s\hat{m}^2_c+3\hat{m}^4_c) \]
\[ - \frac{m_s m_s}{512\pi^4} \frac{\alpha_{GG}}{\pi} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, \left( \frac{y}{z^2} + \frac{z}{y^2} \right) (1-y-z)^2(3s-2\hat{m}^2_c) \]
\[ - \frac{m_s m_s}{768\pi^4} \frac{\alpha_{GG}}{\pi} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, \left( \frac{y}{z^3} + \frac{z}{y^3} + \frac{1}{z^2} + \frac{1}{y^2} \right) (1-y-z)^2 \left\{ 1 + \frac{\hat{m}^2_c}{2} \delta(s-\hat{m}^2_c) \right\} \]
\[ - \frac{m_s m_s}{256\pi^4} \frac{\alpha_{GG}}{\pi} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \, (1-y-z)(3s-2\hat{m}^2_c) \]  \( (31) \)
\[ \rho_4^2(s) = -m_c^2 \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left\{ (1-y-z)^3 \left\{ (2s - \hat{m}_c^2) + \frac{\hat{m}_c^4}{6} \delta(s - \hat{m}_c^2) \right\} \right. \\
- \frac{m_c^2}{384\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left( y + z \right) (1-y-z)^2 \left( 10s^2 - 12s\hat{m}_c^2 + 3\hat{m}_c^4 \right) \\
+ \frac{1}{768\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left( y + z \right) (1-y-z)(s - \hat{m}_c^2)(2s - \hat{m}_c^2) \\
+ \frac{1}{3456\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left( y + z \right)(1-y-z)(s - \hat{m}_c^2)(2s - \hat{m}_c^2) \\
+ \frac{1}{576\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left( y + z \right)(1-y-z)(s - \hat{m}_c^2)(2s - \hat{m}_c^2) \\
+ \frac{1}{288\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz (s - \hat{m}_c^2)(2s - \hat{m}_c^2) \\
- \frac{m_c m_c}{256\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left( y + z \right)(1-y-z)^2 (3s - 2\hat{m}_c^2) \\
+ \frac{m_c}{384\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left( y + z \right)(1-y-z)^2 (3s - 2\hat{m}_c^2) \\
+ \frac{m_c}{128\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz (1-y-z)(3s - 2\hat{m}_c^2) \\
- \frac{m_c}{384\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left( y + z \right)(1-y-z)(3s - 2\hat{m}_c^2) \\
+ \frac{m_c}{768\pi^4} \left( \alpha_{GG} \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left( y + z \right)(1-y-z)(3s - 2\hat{m}_c^2) \\ 
\right\} \\
+ \frac{m_c}{128\pi^4} \left( \alpha_{qGq} + \langle s\bar{q}Gs \rangle \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz (y + z)(3s - 2\hat{m}_c^2) \\
- \frac{m_c}{128\pi^4} \left( \alpha_{qGq} + \langle s\bar{q}Gs \rangle \right) \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz (y + z)(3s - 2\hat{m}_c^2) \\
- \frac{m_c}{32\pi^4} \langle s\bar{q}Gs \rangle \prod_{y_i} \frac{dy}{y_i} \int_{z_i}^{1-y} dz \left\{ (2s - \hat{m}_c^2) + \frac{\hat{m}_c^4}{6} \delta(s - \hat{m}_c^2) \right\} \\
+ \frac{m_c^2}{64\pi^4} \langle \bar{q}qGq \rangle \prod_{y_i} \frac{dy}{y_i} - \frac{m_c^2}{128\pi^4} \langle \bar{q}qGq \rangle \prod_{y_i} \frac{dy}{y_i} \\
\left\{ \left( y + z \right) \right\} \left\{ (1-y-z)^2 (3s - 2\hat{m}_c^2) \right\} , \quad (32) \\
\]
\[ \rho_5^A(s) = -\frac{m_c (\langle \bar{q}g_s G q \rangle + \langle \bar{s}g_s G s \rangle)}{96 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ (y+z)(3s-2\tilde{m}_c^2) \]
\[ + \frac{m_c (\langle \bar{q}g_s G q \rangle + \langle \bar{s}g_s G s \rangle)}{96 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ (1-y-z)(3s-2\tilde{m}_c^2) \]
\[ - \frac{m_s (\bar{s}g_s G s)}{16 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ y z \left\{ (2s-\tilde{m}_c^2) + \frac{\tilde{m}_c^4}{6} \delta(s-\tilde{m}_c^2) \right\} \]
\[ + \frac{m_s m_c^2 (\langle \bar{q}g_s G q \rangle)}{96 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ \left( \frac{1}{y} + \frac{1}{z} \right) , \] (34)

\[ \rho_6^3(s) = \frac{m_s^2 \langle \bar{q}q \rangle \langle \bar{s}s \rangle}{12 \pi^2} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ \left\{ (2s-\tilde{m}_c^2) + \frac{\tilde{m}_c^4}{6} \delta(s-\tilde{m}_c^2) \right\} \]
\[ - \frac{7g_s^2 (\langle \bar{q}q \rangle^2 + \langle \bar{s}s \rangle^2)}{2592 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ \left( \frac{y}{z} + \frac{z}{y} \right) (1-y-z)(3s-2\tilde{m}_c^2) \]
\[ - \frac{5m_s^2 g_s^2 (\langle \bar{q}q \rangle^2 + \langle \bar{s}s \rangle^2)}{1944 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ \left( \frac{y}{z^2} + \frac{z}{y^2} \right) (1-y-z) \left\{ 1 + \frac{\tilde{m}_c^2}{2} \delta(s-\tilde{m}_c^2) \right\} \]
\[ - \frac{g_s^2 (\langle \bar{q}q \rangle^2 + \langle \bar{s}s \rangle^2)}{648 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ \left( y+z \right)(1-y-z) \left\{ (2s-\tilde{m}_c^2) + \frac{\tilde{m}_c^4}{6} \delta(s-\tilde{m}_c^2) \right\} \]
\[ + \frac{m_s m_c^2 \langle \bar{q}q \rangle \langle \bar{s}s \rangle}{48 \pi^2} \int_{y_1}^{y_f} dy \left\{ 2 + \tilde{m}_c^2 \delta(s-\tilde{m}_c^2) \right\} - \frac{m_s m_c g_s^2 (\langle \bar{q}q \rangle^2)}{2592 \pi^4} \int_{y_1}^{y_f} dy \ \left\{ 2 + \tilde{m}_c^2 \delta(s-\tilde{m}_c^2) \right\} \]
\[ + \frac{m_s m_c^3 g_s^3 (\langle \bar{q}q \rangle^2)}{48 \pi^2} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ \left( \frac{1}{y^2} + \frac{1}{z^2} \right) \delta(s-\tilde{m}_c^2) \]
\[ + \frac{m_s m_c^3 g_s^3 (\langle \bar{q}q \rangle^2)}{432 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ \left( \frac{1}{y} + \frac{1}{z} \right) \]
\[ + \frac{m_s m_c^3 g_s^3 (\langle \bar{q}q \rangle^2)}{2592 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \ \left( \frac{z}{y} + \frac{y}{z} \right) \left\{ 2 + \tilde{m}_c^2 \delta(s-\tilde{m}_c^2) \right\} , \] (35)
\[ \rho_0(s) = \frac{m_c^2\langle q\bar{q}\rangle\langle s\bar{s}\rangle}{3\pi^2} \int_{y_i}^{y_f} dy + \frac{g_2^2\langle(q\bar{q})^2 + \langle s\bar{s}\rangle^2\rangle}{108\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left\{ (2s - \hat{m}_c^2) + \frac{\hat{m}_c^4}{6}\delta(s - \hat{m}_c^2) \right\} \\
+ \frac{g_2^2\langle(q\bar{q})^2 + \langle s\bar{s}\rangle^2\rangle}{648\pi^4} \int_{y_i}^{y_f} dy \left( y(1 - y)(3s - 2\hat{m}_c^2) \right) \\
- \frac{5m_c^2g_2^2\langle(q\bar{q})^2 + \langle s\bar{s}\rangle^2\rangle}{972\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y}{z^2} + \frac{z}{y^2} \right) (1 - y - z) \left\{ 1 + \frac{\hat{m}_c^2}{2}\delta(s - \hat{m}_c^2) \right\} \\
- \frac{g_2^2\langle(q\bar{q})^2 + \langle s\bar{s}\rangle^2\rangle}{162\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y}{z} + \frac{z}{y} \right) (1 - y - z)(3s - 2\hat{m}_c^2) \\
- \frac{g_2^2\langle(q\bar{q})^2 + \langle s\bar{s}\rangle^2\rangle}{81\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( y + z \right)(1 - y - z) \left\{ (2s - \hat{m}_c^2) + \frac{\hat{m}_c^4}{6}\delta(s - \hat{m}_c^2) \right\} \\
+ \frac{m_s\alpha_c\langle(q\bar{q})\langle s\bar{s}\rangle\rangle}{24\pi^2} \int_{y_i}^{y_f} dy \left\{ 2 + \hat{m}_c^2\delta(s - \hat{m}_c^2) \right\} - \frac{m_s\alpha_c\langle q\bar{q}\rangle^2}{1296\pi^4} \int_{y_i}^{y_f} dy \left\{ 2 + \hat{m}_c^2\delta(s - \hat{m}_c^2) \right\} \\
+ \frac{m_s^2\alpha_c\langle q\bar{q}\rangle\langle s\bar{s}\rangle\rangle}{432\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^2} + \frac{1}{z^2} \right) \delta(s - \hat{m}_c^2) \\
+ \frac{m_s^2\alpha_c\langle q\bar{q}\rangle\langle s\bar{s}\rangle\rangle}{216\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y} + \frac{1}{z} \right) \\
+ \frac{m_s^2\alpha_c\langle q\bar{q}\rangle\langle s\bar{s}\rangle\rangle}{1296\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( z + \frac{y}{z} \right) \left\{ 2 + \hat{m}_c^2\delta(s - \hat{m}_c^2) \right\} , \tag{36} \]

\[ \rho_3(s) = -\frac{m_c^3\langle q\bar{q} \rangle^2}{576\pi^2} \langle \alpha_s GG \rangle_\pi \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^2} + \frac{1}{z^2} + \frac{y}{z^3} + \frac{z}{y^3} \right) (1 - y - z) \\
\left( 1 + \frac{\hat{m}_c^2}{T^2} \right) \delta(s - \hat{m}_c^2) \\
+ \frac{m_c\langle(q\bar{q}) + \langle s\bar{s}\rangle\rangle_\alpha_s GG}{192\pi^2} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y}{z^2} + \frac{z}{y^2} \right) (1 - y - z) \left\{ 2 + \hat{m}_c^2\delta(s - \hat{m}_c^2) \right\} \\
+ \frac{m_c\langle(q\bar{q}) + \langle s\bar{s}\rangle\rangle_\alpha_s GG}{192\pi^2} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left\{ 2 + \hat{m}_c^2\delta(s - \hat{m}_c^2) \right\} \\
+ \frac{m_c\langle(q\bar{q}) + \langle s\bar{s}\rangle\rangle_\alpha_s GG}{1152\pi^2} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y + z} \right) \left( 1 + \frac{\hat{m}_c^2}{T^2} + \frac{\hat{m}_c^4}{2T^4} \right) \delta(s - \hat{m}_c^2) \\
+ \frac{m_s\alpha_c\langle q\bar{q}\rangle^2}{288\pi^2} \langle \alpha_s GG \rangle_\pi \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^2} + \frac{1}{z^2} \right) \delta(s - \hat{m}_c^2) \\
+ \frac{m_s\alpha_c\langle q\bar{q}\rangle^2}{96\pi^2} \langle \alpha_s GG \rangle_\pi \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^3} + \frac{1}{z^3} \right) \delta(s - \hat{m}_c^2) \\
+ \frac{m_s\langle s\bar{s}\rangle_\alpha_s GG}{288T^2\pi^2} \langle \alpha_s GG \rangle_\pi \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y + z) \left\{ 6 + 4\hat{m}_c^2\delta(s - \hat{m}_c^2) + \frac{\hat{m}_c^4\delta(s - \hat{m}_c^2)}{T^2} \right\} \\
+ \frac{m_s\alpha_c\langle q\bar{q}\rangle^2}{768\pi^2} \langle \alpha_s GG \rangle_\pi \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( 1 + \frac{\hat{m}_c^2}{T^2} \right) \delta(s - \hat{m}_c^2) , \tag{37} \]
\[
\rho_7^2(s) = - \frac{m_c^2 \langle \bar{q}q \rangle + \langle ss \rangle}{288\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^2} + \frac{1}{z^2} + \frac{y}{z^2} + \frac{z}{y^2} \right) (1-y-z) \\
\left( 1 + \hat{m}_c^2 \right) \delta(s - \hat{m}_c^2) \\
+ \frac{m_c \langle \bar{q}q \rangle + \langle ss \rangle}{96\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} \left( \frac{y}{z} + \frac{z}{y} \right) \left( 1 - y - z \right) \left( 2 + \hat{m}_c^2 \delta(s - \hat{m}_c^2) \right) \\
- \frac{m_c \langle \bar{q}q \rangle + \langle ss \rangle}{288\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^3} + \frac{1}{z^3} \right) \left( 1 - y - z \right) \left( 2 + \hat{m}_c^2 \delta(s - \hat{m}_c^2) \right) \\
+ \frac{m_c \langle \bar{q}q \rangle + \langle ss \rangle}{288\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^3} + \frac{1}{z^3} \right) \left( 1 - y - z \right) \left( 2 + \hat{m}_c^2 \delta(s - \hat{m}_c^2) \right) \\
+ \frac{m_c \langle \bar{q}q \rangle + \langle ss \rangle}{576\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^3} + \frac{1}{z^3} \right) \left( 1 - y - z \right) \left( 2 + \hat{m}_c^2 \delta(s - \hat{m}_c^2) \right) \\
- \frac{m_c m_c \langle \bar{q}q \rangle}{24\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^2} + \frac{1}{z^2} \right) \delta(s - \hat{m}_c^2) \\
+ \frac{m_c m_c \langle \bar{q}q \rangle}{72\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^3} + \frac{1}{z^3} \right) \delta(s - \hat{m}_c^2) \\
- \frac{m_c m_c \langle \bar{q}q \rangle}{144\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y}{z^2} + \frac{z}{y^2} \right) \left( 1 - y - z \right) \left( 1 + \hat{m}_c^2 + \frac{\hat{m}_c^4}{2T^4} \right) \delta(s - \hat{m}_c^2) \\
- \frac{m_c m_c \langle \bar{q}q \rangle}{144\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y}{z^2} + \frac{z}{y^2} \right) \left( 1 + \hat{m}_c^2 \right) \delta(s - \hat{m}_c^2) \\
- \frac{11 m_c m_c \langle \bar{q}q \rangle}{3456\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( 2 + \hat{m}_c^2 \delta(s - \hat{m}_c^2) \right) \\
- \frac{m_c m_c \langle \bar{q}q \rangle}{72\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^2} \right) \delta(s - \hat{m}_c^2) \\
- \frac{m_c m_c \langle \bar{q}q \rangle}{144\pi^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( 1 + \hat{m}_c^2 \right) \delta(s - \hat{m}_c^2),
\]
\[
\rho_8^4(s) = -\frac{m^2}{16\pi^2} \left( \langle \bar{q}q \rangle \langle \bar{g}_s \sigma G_s \rangle + \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma G_q \rangle \right) \int_{y_i}^{y_f} dy \left( 1 + \frac{\bar{m}_c^2}{T^2} \right) \delta(s - \bar{m}_c^2) \\
+ \frac{m^2}{96\pi^2} \left( \langle \bar{q}q \rangle \langle \bar{g}_s \sigma G_s \rangle + \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma G_q \rangle \right) \int_{y_i}^{y_f} dy \frac{1}{y(1-y)} \delta(s - \bar{m}_c^2) \\
- \frac{m_c m_c (2\bar{q}q \langle \bar{g}_s \sigma G_s \rangle + 3\langle \bar{s}s \rangle \langle \bar{q}g_s \sigma G_q \rangle)}{288\pi^2} \int_{y_i}^{y_f} dy \left( 1 + \frac{\bar{m}_c^2}{T^2} + \frac{\bar{m}_c^4}{2T^4} \right) \delta(s - \bar{m}_c^2) \\
+ \frac{m_c m_c \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma G_q \rangle}{192\pi^2} \int_{y_i}^{y_f} dy \left( \frac{1-y}{y} + \frac{y}{(1-y)} \right) \left( 1 + \frac{\bar{m}_c^2}{T^2} + \frac{\bar{m}_c^4}{2T^4} \right) \delta(s - \bar{m}_c^2) , \quad (39)
\]

\[
\rho_8^4(s) = -\frac{m^2}{12\pi^2} \left( \langle \bar{q}q \rangle \langle \bar{g}_s \sigma G_s \rangle + \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma G_q \rangle \right) \int_{y_i}^{y_f} dy \left( 1 + \frac{\bar{m}_c^2}{T^2} \right) \delta(s - \bar{m}_c^2) \\
+ \frac{m^2}{72\pi^2} \left( \langle \bar{q}q \rangle \langle \bar{g}_s \sigma G_s \rangle + \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma G_q \rangle \right) \int_{y_i}^{y_f} dy \frac{1}{y(1-y)} \delta(s - \bar{m}_c^2) \\
- \frac{m_c m_c (2\bar{q}q \langle \bar{g}_s \sigma G_s \rangle + 3\langle \bar{s}s \rangle \langle \bar{q}g_s \sigma G_q \rangle)}{144\pi^2} \int_{y_i}^{y_f} dy \left( 1 + \frac{\bar{m}_c^2}{T^2} + \frac{\bar{m}_c^4}{2T^4} \right) \delta(s - \bar{m}_c^2) \\
+ \frac{m_c m_c \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma G_q \rangle}{144\pi^2} \int_{y_i}^{y_f} dy \left( 1 + \frac{\bar{m}_c^2}{T^2} \right) \delta(s - \bar{m}_c^2) , \quad (40)
\]

\[
\rho_{10}^3(s) = -\frac{m_c^2}{192\pi^2 T^6} \left( \langle \bar{q}q \sigma G_q \rangle \langle \bar{g}_s \sigma G_s \rangle \right) \int_{y_i}^{y_f} dy \bar{m}_c^3 \delta(s - \bar{m}_c^2) \\
- \frac{m_c^2}{216T^4} \left( \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle \alpha_s GG \rangle \right) \int_{y_i}^{y_f} dy \frac{1}{y^3} + \frac{1}{(1-y)^3} \delta(s - \bar{m}_c^2) \\
+ \frac{m_c^2}{72T^2} \left( \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma G_q \rangle \langle \alpha_s GG \rangle \right) \int_{y_i}^{y_f} dy \frac{1}{y^2} + \frac{1}{(1-y)^2} \delta(s - \bar{m}_c^2) \\
- \frac{m_c^2}{192\pi^2 T^4} \left( \langle \bar{q}q \sigma G_q \rangle \langle \bar{g}_s \sigma G_s \rangle \right) \int_{y_i}^{y_f} dy \frac{1}{y(1-y)} \bar{m}_c^2 \delta(s - \bar{m}_c^2) \\
+ \frac{m_c^2}{128\pi^2 T^2} \left( \langle \bar{q}q \rangle \langle \bar{g}_s \sigma G_s \rangle \right) \int_{y_i}^{y_f} dy \frac{1}{y(1-y)} \delta(s - \bar{m}_c^2) \\
+ \frac{m_c^2}{216T^6} \left( \langle \bar{s}s \rangle \langle \alpha_s GG \rangle \right) \int_{y_i}^{y_f} dy \bar{m}_c^4 \delta(s - \bar{m}_c^2) \\
+ \frac{m_c m_c \langle \bar{q}q \sigma G_q \rangle \langle \bar{g}_s \sigma G_s \rangle}{1152\pi^2 T^8} \int_{y_i}^{y_f} dy \bar{m}_c^5 \delta(s - \bar{m}_c^2) \\
+ \frac{m_c m_c^2 \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle \alpha_s GG \rangle}{864T^4} \int_{y_i}^{y_f} dy \frac{1}{y^3} + \frac{1}{(1-y)^3} \left( 1 - \frac{\bar{m}_c^2}{T^2} \right) \delta(s - \bar{m}_c^2) \\
+ \frac{m_c m_c \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle \alpha_s GG \rangle}{288T^4} \int_{y_i}^{y_f} dy \left( \frac{1-y}{y^2} + \frac{y}{(1-y)^2} \right) \bar{m}_c^2 \delta(s - \bar{m}_c^2) \\
- \frac{m_c m_c \langle \bar{q}q \sigma G_q \rangle \langle \bar{g}_s \sigma G_s \rangle}{1152\pi^2 T^8} \int_{y_i}^{y_f} dy \left( \frac{1-y}{y} + \frac{y}{(1-y)} \right) \bar{m}_c^4 \delta(s - \bar{m}_c^2) \\
+ \frac{m_c m_c \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle \alpha_s GG \rangle}{1728T^8} \int_{y_i}^{y_f} dy \bar{m}_c^5 \delta(s - \bar{m}_c^2) , \quad (41)
\]
\[
\rho_{10}^4(s) = \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{48\pi^2 T^6} \int_{y_i}^{y_f} dy \, \tilde{m}_c^4 \delta(s - \tilde{m}_c^2) \\
- \frac{m_c^4 \langle \bar{q}q \rangle \langle \bar{s}s \rangle \alpha_s GG}{54T^4} \int_{y_i}^{y_f} dy \, \left\{ \frac{1}{y^3} + \frac{1}{(1-y)^3} \right\} \delta(s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{18T^2} \int_{y_i}^{y_f} dy \, \left\{ \frac{1}{y^2} + \frac{1}{(1-y)^2} \right\} \delta(s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{q}q \rangle \langle \bar{s}s \rangle \alpha_s GG}{54T^4} \int_{y_i}^{y_f} dy \, \frac{1}{y(1-y)} \delta(s - \tilde{m}_c^2) \\
- \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{144\pi^2 T^4} \int_{y_i}^{y_f} dy \, \frac{1}{y(1-y)} \tilde{m}_c^2 \delta(s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{32\pi^2 T^2} \int_{y_i}^{y_f} dy \, \frac{1}{y(1-y)} \delta(s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{54T^4} \int_{y_i}^{y_f} dy \, \tilde{m}_c^4 \delta(s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{576\pi^2 T^8} \int_{y_i}^{y_f} dy \, \tilde{m}_c^6 \delta(s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{144T^4} \int_{y_i}^{y_f} dy \, \left\{ \frac{1}{y^3} + \frac{1}{(1-y)^3} \right\} \tilde{m}_c^2 \delta(s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{432T^4} \int_{y_i}^{y_f} dy \, \left\{ \frac{1}{y^2} + \frac{1}{(1-y)^2} \right\} \left(1 - \tilde{m}_c^2 \right) \delta(s - \tilde{m}_c^2) \\
- \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{216T^4} \int_{y_i}^{y_f} dy \, \frac{1}{y(1-y)} \tilde{m}_c^2 \delta(s - \tilde{m}_c^2) \\
- \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{864\pi^2 T^6} \int_{y_i}^{y_f} dy \, \tilde{m}_c^4 \delta(s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{q}gGq \rangle \langle \bar{s}gGs \rangle}{864T^8} \int_{y_i}^{y_f} dy \, \tilde{m}_c^6 \delta(s - \tilde{m}_c^2) \\
\tag{42}
\]

where \( y_f = \frac{1 + \sqrt{1 - 4m_c^2/s}}{2}, y_i = \frac{1 - \sqrt{1 - 4m_c^2/s}}{2}, z_i = \frac{m_c^2}{y^2 - m_c^2}, \tilde{m}_c^2 = \frac{(y+z)m_c^2}{yz}, \tilde{m}_c^2 = \frac{m_c^2}{y(1-y)}, \int_{y_i}^{y_f} dy \rightarrow \int_0^{1-y} dz \rightarrow \int_0^{1-y} dz \), when the \( \delta \) functions \( \delta(s - \tilde{m}_c^2) \) and \( \delta(s - \tilde{m}_c^2) \) appear.

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