Queue Stability and Probability 1 Convergence via Lyapunov Optimization

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Abstract

Lyapunov drift and Lyapunov optimization are powerful techniques for optimizing time averages in stochastic queueing networks subject to stability. However, there are various definitions of queue stability in the literature, and the most convenient Lyapunov drift conditions often provide stability and performance bounds only in terms of a time average expectation, rather than a pure time average. We extend the theory to show that for quadratic Lyapunov functions, the basic drift condition, together with a mild bounded fourth moment condition, implies all major forms of stability. Further, we show that the basic drift-plus-penalty condition implies that the same bounds for queue backlog and penalty expenditure that are known to hold for time average expectations also hold for pure time averages with probability 1. Our analysis combines Lyapunov drift theory with the Kolmogorov law of large numbers for martingale differences with finite variance.

Index Terms

Network utility maximization, Wireless networks, dynamic scheduling, stochastic network optimization

I. INTRODUCTION

Lyapunov optimization is a powerful technique for optimizing time averages in stochastic queueing networks (see [1]-[13]). Work in [1] presents a drift-plus-penalty theorem that provides a methodology for designing control algorithms to maximize time average network utility subject to queue stability. The theorem also provides explicit performance tradeoffs between utility maximization and average queue backlog. Example applications include maximizing network throughput subject to average power constraints, minimizing average power expenditure subject to network stability, and maximizing network throughput-utility subject to network stability [1]-[5]. The drift-plus-penalty theorem provides performance bounds in terms of time average expectations. Time average expectations are the same as pure time averages (with probability 1) in certain cases, such as when the system evolves on an irreducible and positive recurrent Markov chain with a finite or countably infinite state space (and when some additional mild assumptions are satisfied). However, many systems have an uncountably infinite state space and/or do not have the required Markov structure. It is not clear if pure time averages satisfy the same guarantees in general. This paper proves a sample path version of the drift-plus-penalty theorem, showing that if fourth moment boundedness conditions are satisfied, then the same guarantees hold for pure time averages with probability 1.

To understand this result and the systems it can be applied to, we consider a stochastic queueing network that evolves in discrete time with unit timeslots \( t \in \{0, 1, 2, \ldots \} \). Suppose there are \( K \) queues, and let \( Q(t) = (Q_1(t), \ldots, Q_K(t)) \) represent the vector of current queue backlogs. Random events, such as random channel conditions and packet arrivals, can take place every slot. A network controller reacts to the random events by choosing a control action every slot. The control action affects queue arrival and service variables on slot \( t \), and also incurs a vector of penalties \( y(t) = (y_0(t), y_1(t), \ldots, y_M(t)) \). The goal
is to stabilize all network queues while minimizing the time average of $y_0(t)$ subject to the time averages of $y_m(t)$ being less than or equal to 0:

Minimize: \[ \overline{y}_0 \] \hspace{1cm} (1)
Subject to: \begin{align*}
(1) & \quad \overline{y}_m \leq 0 \quad \forall m \in \{1, \ldots, M\} \\
(2) & \quad \text{All queues are stable} \hspace{1cm} (3)
\end{align*}

Assuming that the problem is feasible and that a certain drift-plus-penalty condition is met, the existing drift-plus-penalty theory in [1] can solve this problem by specifying a class of algorithms, parameterized by a constant $V \geq 0$ chosen as desired, to yield:

\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \mathbb{E}\{y_0(\tau)\} \leq y_0^* + O(1/V) \] \hspace{1cm} (4)
\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \mathbb{E}\{y_m(\tau)\} \leq 0 \quad \forall m \in \{1, \ldots, M\} \] \hspace{1cm} (5)
\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} \mathbb{E}\{|Q_k(\tau)|\} \leq O(V) \quad \forall k \in \{1, \ldots, K\} \] \hspace{1cm} (6)

where $y_0^*$ is the infimum time average of $y_0(t)$ over all algorithms that can satisfy the desired constraints. The guarantee (6) implies that the lim sup time average expected queue backlog is finite for all queues, and is a condition often called strong stability. The above bounds say that the time average constraints $\overline{y}_m \leq 0$ are satisfied for all $m \in \{1, \ldots, M\}$ in a time average expected sense, that all queues $Q_k(t)$ are strongly stable with average backlog $O(V)$, and time average expected penalty is within $O(1/V)$ of optimality. The $O(1/V)$ penalty gap can be made arbitrarily small by choosing a suitably large $V$ parameter, at the expense of increasing the average backlog bound linearly with $V$.

We would like to know when we can also claim that:

\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} y_0(\tau) \leq y_0^* + O(1/V) \quad (w.p.1) \] \hspace{1cm} (7)
\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} y_m(\tau) \leq 0 \quad (w.p.1) \quad \forall m \in \{1, \ldots, M\} \] \hspace{1cm} (8)
\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} |Q_k(\tau)| \leq O(V) \quad (w.p.1) \quad \forall k \in \{1, \ldots, K\} \] \hspace{1cm} (9)

where “w.p.1” stands for “with probability 1.” This paper shows that (7)-(9) hold if a similar drift-plus-penalty condition holds, and additionally if the $y_m(t)$ penalties and the one-slot changes in queue backlogs have conditionally bounded fourth moments given the past.

We note that related problems of minimizing convex functions of time averages, rather than minimizing time averages themselves, can be transformed into problems of the type (1)-(3) using a technique of auxiliary variables [3][1][8][14]. Hence, these extended problems can also be treated using the framework of this paper. However, for brevity we limit attention to problems of the type (1)-(3).

A. On relationships between time average expectations and time averages

It is known by Fatou’s Lemma that if a random process is deterministically lower-bounded (such as being non-negative) and has time averages that converge to a constant with probability 1, then this constant must be less than or equal to the $\liminf$ time average expectation [15]. Thus, the inequalities (4)-(6) imply (7)-(9) when the $y_m(t)$ and $|Q_k(t)|$ processes are deterministically lower bounded and have convergent
time averages with probability 1. Systems that evolve on positive recurrent irreducible Markov chains with finite or countably infinite state space can often be shown to have convergent time average penalties. Further, if the Markov chain is irreducible and has a finite or countably infinite state space with the property that the event \( \{ \sum_{k=1}^{K} |Q_k| > \theta \} \) corresponds to only a finite number of states for each real number \( \theta \), then the condition (6) implies positive recurrence. However, in addition to the actual network queues, the drift-plus-penalty method introduces virtual queues to enforce the desired time average constraints. These queues typically give the overall system an uncountably infinite state space. Time average convergence can be shown using generalized Harris recurrence theory for Markov chains with uncountably infinite state space, provided that certain generalized irreducibility assumptions and petite set assumptions are satisfied [16]. However, it is often difficult to check if these assumptions hold for the particular systems of interest.

Strong stability of a queue \( Q(t) \), together with either deterministically bounded arrival or server rate processes, implies rate stability [17]. Rate stability of \( Q(t) \) means that \( \lim_{t \to \infty} Q(t)/t = 0 \) with probability 1. This result can be used to prove that (8) holds if the \( y_m(t) \) processes are suitably deterministically bounded on each slot \( t \). However, this does not ensure the constraints (7) or (9) hold.

Certain types of systems, such as networks with flow control, often have a structure that yields deterministically bounded queues [4][18], which can be used to ensure constraints (8)-(9) hold for those systems. However, this requires special structure, and it also does not ensure (7) holds unless suitable Markov chain assumptions are met.

B. Alternative algorithms

A dual-based algorithm related to the drift-plus-penalty method is considered for a wireless downlink with “infinite backlog” in [7], and convergence to near-optimal utility is shown using a countable state space Markov chain assumption. Stochastic approximation algorithms are used in [19], and diminishing stepsize convex programming is used in [20] to treat problems that are more deterministic in structure. The works [7][19][20] do not show the \([O(1/V), O(V)]\) performance-backlog tradeoff.

Primal-dual algorithms are considered for scheduling in wireless systems with “infinite backlog” in [21][22] and shown to converge to a utility optimal operating point, although this work does not consider queueing or time average constraints. A related primal-dual algorithm is treated in [6] for systems with queues. A fluid version of the system is shown to have a utility-optimal trajectory, and it is conjectured that the actual system has a near-optimal utility. Recent work in [13] considers fluid analysis of primal-dual updates and proves near-optimal utility of the actual system with probability 1. It also treats a more general class of objective functions that have time varying parameters. However, it considers only rate stability for queues and does not specify the \([O(1/V), O(V)]\) tradeoff. Work in [23] considers stochastic queues with a non-convex objective function, and shows that if the throughput vector converges, it converges to a near-local optimum or a critical point with a \([O(1/V), O(V)]\) utility-delay tradeoff (where a near-local optimum is a near-global optimum in the special case of convex problems).

C. Paper Outline

In the next section we review the basic drift-plus-penalty theorem and discuss the performance bounds it provides, which are in terms of time average expectations. We then state the main theorem of this paper, which shows the same bounds hold as pure time averages with probability 1. A key special case of this theorem is that if a certain quadratic Lyapunov drift condition is satisfied, then the network queues satisfy all of the six major forms of queue stability. Section III provides background on the Kolmogorov law of large numbers needed in our analysis, and derives a simple but useful generalized drift-plus-penalty theorem. Section IV shows that the conditions required for the generalized drift-plus-penalty theorem to hold are satisfied under quadratic Lyapunov functions if certain boundedness properties hold. Section V uses this result in queueing networks to derive bounds of the form (4)-(6) for those systems.
II. The Drift-Plus-Penalty Theorem

Let \( Q(t) \triangleq (Q_1(t), Q_2(t), \ldots, Q_K(t)) \) be a stochastic vector with real-valued components, and let \( p(t) \) be a real-valued stochastic process on the same probability space as \( Q(t) \). These processes evolve in discrete time with unit time slots \( t \in \{0, 1, 2, \ldots \} \). The vector \( Q(t) \) can represent queue backlogs in a network of \( K \) queues. The process \( p(t) \) can represent a penalty process, where \( p(t) \) is a real-valued penalty (such as power expenditure) incurred by some control action taken by the system on slot \( t \). While typical queue backlogs and penalties are non-negative, for generality we allow them to possibly take negative values.

For each slot \( t \), define \( \mathcal{H}(t) \) as the history of past \( Q(\tau) \) and \( p(\tau) \) values, where \( Q(\tau) \) values are taken up to and including slot \( t \), and \( p(\tau) \) values are taken up to but not including slot \( t \). Specifically, define \( \mathcal{H}(0) \triangleq \{Q(0)\} \), and for each \( t > 0 \) define:

\[
\mathcal{H}(t) \triangleq \{Q(0), Q(1), \ldots, Q(t), p(0), p(1), \ldots, p(t-1)\}
\]

As a scalar measure of the size of the \( Q(t) \) vector, define the following quadratic Lyapunov function:

\[
L(Q(t)) \triangleq \frac{1}{2} \sum_{k=1}^{K} w_k Q_k(t)^2
\]

where the constants \( w_k \) are positive weights. Define \( \Delta(\mathcal{H}(t)) \) as the conditional Lyapunov drift:

\[
\Delta(\mathcal{H}(t)) \triangleq \mathbb{E}\{L(Q(t+1)) - L(Q(t))|\mathcal{H}(t)\}
\]

Note that \( \mathcal{H}(t) \) includes \( Q(t) \), and so the above conditional expectation is with respect to the conditional distribution of \( Q(t+1) \) given \( Q(0), \ldots, Q(t), p(0), \ldots, p(t-1) \).

The drift-plus-penalty algorithm for minimizing the time average expected penalty \( p(t) \) subject to queue stability operates as follows: Every slot \( t \) the network controller observes the current \( \mathcal{H}(t) \) and chooses a control policy that minimizes a bound on the following expression:\(^1\)

\[
\Delta(\mathcal{H}(t)) + V \mathbb{E}\{p(t)|\mathcal{H}(t)\}
\]

where \( V \) is a non-negative control parameter that is chosen to affect a desired tradeoff between the average penalty and the average queue backlog. A version of this algorithm was developed in [2] for maximizing throughput-utility subject to stability, and simple modifications were presented for other contexts in [1][4]. This is a useful algorithm for queueing networks because it can typically be implemented based only on \( Q(t) \), without keeping a memory of the full history and without requiring knowledge of traffic rates or channel probabilities (see applications in Section V). Such a control policy often gives rise to stochastic processes \( Q(t) \) and \( p(t) \) that satisfy the following drift-plus-penalty condition for all slots \( t \in \{0, 1, 2, \ldots\} \) and all possible \( \mathcal{H}(t) \):

\[
\Delta(\mathcal{H}(t)) + V \mathbb{E}\{p(t)|\mathcal{H}(t)\} \leq B + V p^* - \epsilon \sum_{k=1}^{K} |Q_k(t)|
\]

where \( B, p^*, \epsilon \) are finite constants, with \( \epsilon > 0 \). The value \( p^* \) represents a target value for the time average expectation of the penalty process \( p(t) \). The following theorem from [1][2][4] shows that this condition immediately implies the time average expectation of \( p(t) \) is either above the target \( p^* \), or is within a distance of at most \( O(1/V) \) from \( p^* \), while ensuring time average expected queue backlog is \( O(V) \).

\(^1\)Strictly speaking, the prior work in [1] defines the Lyapunov drift by conditioning only on \( Q(t) \), rather than on the full history \( \mathcal{H}(t) \). We condition on \( \mathcal{H}(t) \) in this paper because such conditioning is needed for application of the Kolmogorov law of large numbers.
Theorem 1: (Lyapunov Optimization with Expectations [1][2][4]) Assume that $\mathbb{E}\{L(Q(0))\} < \infty$, and that the condition (14) holds for some finite constants $B$, $p^*$, $V > 0$, and $\epsilon > 0$. If there is a finite (and possibly negative) constant $p_{\text{min}}$ such that $\mathbb{E}\{p(t)\} \geq p_{\text{min}}$ for all slots $t \in \{0, 1, 2, \ldots\}$, then:

$$\limsup_{M \to \infty} \frac{1}{M} \sum_{t=0}^{M-1} \mathbb{E}\{p(t)\} \leq p^* + \frac{B}{V}$$

(15)

$$\limsup_{M \to \infty} \frac{1}{M} \sum_{t=0}^{M-1} \sum_{k=1}^{K} \mathbb{E}\{|Q_k(t)\}| \leq \frac{B + V(p^* - p_{\text{min}})}{\epsilon}$$

(16)

Further, if (14) holds in the case $V = 0$, then inequality (16) still holds. Likewise, if (14) holds in the case $\epsilon = 0$, then inequality (15) still holds.

The proof of Theorem 1 requires only three lines and is repeated below to provide intuition: Taking expectations of (14) and using the law of iterated expectations yields the following for all $t \in \{0, 1, 2, \ldots\}$:

$$\mathbb{E}\{L(Q(t+1))\} - \mathbb{E}\{L(Q(t))\} + V \mathbb{E}\{p(t)\} \leq B + Vp^* - \epsilon \sum_{k=1}^{K} \mathbb{E}\{|Q_k(t)\}|$$

Summing the above over $t \in \{0, \ldots, M-1\}$ for some integer $M > 0$ and dividing by $M$ yields:

$$\frac{\mathbb{E}\{L(Q(M))\} - \mathbb{E}\{L(Q(0))\}}{M} + V \frac{1}{M} \sum_{t=0}^{M-1} \mathbb{E}\{p(t)\} \leq B + Vp^* - \epsilon \frac{1}{M} \sum_{t=0}^{M-1} \sum_{k=1}^{K} \mathbb{E}\{|Q_k(t)\}|$$

Rearranging terms in the above inequality and using the fact that $\mathbb{E}\{L(Q(M))\} \geq 0$ and $\mathbb{E}\{p(t)\} \geq p_{\text{min}}$ for all $t$ immediately leads to the following two inequalities:

$$\frac{1}{M} \sum_{t=0}^{M-1} \mathbb{E}\{p(t)\} \leq p^* + \frac{B}{V} + \frac{\mathbb{E}\{L(Q(0))\}}{VM}$$

$$\frac{1}{M} \sum_{t=0}^{M-1} \sum_{k=1}^{K} \mathbb{E}\{|Q_k(t)\}| \leq \frac{B + V(p^* - p_{\text{min}})}{\epsilon} + \frac{\mathbb{E}\{L(Q(0))\}}{\epsilon M}$$

Taking a limit of the above inequalities as $M \to \infty$ yields (15)-(16).

A. Main Result of This Paper

Theorem 1 illustrates an important tradeoff between time average expected penalty and the resulting time average expected queue backlog. However, one may wonder if the same bounds hold with probability 1 for pure time averages (without the expectations). To address this question, we impose the following additional boundedness assumptions:

- The second moments $\mathbb{E}\{p(t)^2\}$ are finite for all $t \in \{0, 1, 2, \ldots\}$ and satisfy:

$$\sum_{\tau=1}^{\infty} \frac{\mathbb{E}\{p(\tau)^2\}}{\tau^2} < \infty$$

(17)

- There is a finite (possibly negative) constant $p_{\text{min}}$ such that for all slots $t$ and all possible $H(t)$:

$$\mathbb{E}\{p(t)|H(t)\} \geq p_{\text{min}}$$

(18)

- There is a finite constant $D > 0$ such that for all slots $t$, all possible $Q(t)$, and all $k \in \{1, \ldots, K\}$ the conditional fourth moments of queue changes are bounded as follows:

$$\mathbb{E}\{(Q_k(t+1) - Q_k(t))^4|Q(t)\} \leq D$$

(19)
Note that condition (17) holds whenever $E \{ p(t)^2 \} \leq C$ for all $t$ for some finite constant $C > 0$. The following theorem is the main result of this paper:

**Theorem 2**: (Lyapunov Optimization with Pure Time Averages) Assume the boundedness assumptions (17)-(19) hold. Let $L(Q(t))$ be a quadratic Lyapunov function of the form (11), and assume the initial queue backlog $Q(0)$ is finite with probability 1. If the drift-plus-penalty condition (14) is satisfied for all slots $t$ and all possible $H(t)$ (with finite constants $B, p^*, V > 0, \epsilon > 0$), then:

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} p(\tau) \leq p^* + \frac{B}{V} \text{ (w.p.1)}$$

(20)

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} |Q_k(\tau)| \leq \frac{B + V(p^* - p_{\min})}{\epsilon} \text{ (w.p.1)}$$

(21)

where (w.p.1) stand for “with probability 1.” Further, for all $k \in \{1, \ldots, K\}$ we have:

$$\lim_{t \to \infty} \frac{Q_k(t)}{t} = 0 \text{ (w.p.1)}$$

(22)

Finally, if (14) holds in the case $V = 0$, then inequality (21) and equality (22) still hold.

A more detailed upper bound on time average queue backlog is provided in (52) of the proof.

**B. Queue Stability**

A special case of Theorem 2 is when the fourth moment condition (19) is satisfied and when the following drift condition holds for all $t$ and all $H(t)$:

$$\Delta(H(t)) \leq B - \epsilon \sum_{k=1}^{K} |Q_k(t)|$$

(23)

where $B > 0$ and $\epsilon > 0$. This is a special case of (14) with $V = 0$ and $p(t) = p^* = 0$. In this case we have that all queues $Q_k(t)$ in the system satisfy:

$$\limsup_{M \to \infty} \frac{1}{M} \sum_{t=0}^{M} E \{ |Q_k(t)| \} \leq B/\epsilon$$

(24)

$$\limsup_{M \to \infty} \frac{1}{M} \sum_{t=0}^{M} |Q_k(t)| \leq B/\epsilon \text{ (w.p.1)}$$

(25)

$$\lim_{q \to \infty} \left[ \limsup_{M \to \infty} \frac{1}{M} \sum_{t=0}^{M-1} Pr[|Q_k(t)| > q] \right] = 0$$

(26)

$$\lim_{q \to \infty} \left[ \limsup_{M \to \infty} \frac{1}{M} \sum_{t=0}^{M-1} 1\{|Q_k(t)| > q\} \right] = 0 \text{ (w.p.1)}$$

(27)

$$\lim_{t \to \infty} \frac{E \{ |Q_k(t)| \}}{t} = 0$$

(28)

$$\lim_{t \to \infty} Q_k(t)/t = 0 \text{ (w.p.1)}$$

(29)

where $1\{|Q_k(t)| > q\}$ is an indicator function that is 1 if $|Q_k(t)| > q$, and 0 else. The above are 6 major forms of queue stability. The inequality (24) is often called *strong stability*, and holds by Theorem 1. Its sample path version is inequality (25), and this holds by Theorem 2. The inequality (24) can easily be used to prove (26) via the fact that $|Q_k(t)| \geq q1\{|Q_k(t)| > q\}$, and the same fact can easily prove that (25) implies (27). The stability definition (28) is called *mean rate stability*, and does not follow from any of the above results, but follows from Theorem 3 given below. The stability definition (29) is a sample path.
version called rate stability, and is implied by Theorem 2. Relationships between these various stability
definitions are discussed in [17]. In summary, if changes in queue backlogs have uniformly bounded
conditional fourth moments (so that (19) holds), and if the Lyapunov drift condition (23) holds for a
quadratic Lyapunov function, then all queues in the network satisfy all of the major forms of stability.

The following useful theorem shows that in the special case $\varepsilon = 0$, the condition (23) still implies rate
stability and mean rate stability, regardless of whether or not conditional fourth moments are bounded.

**Theorem 3:** (Rate Stability and Mean Rate Stability) Let $L(Q(t))$ be a quadratic Lyapunov function of
the form (11). Suppose there is a finite constant $B > 0$ such that for all $\tau \in \{0, 1, 2, \ldots\}$ and all possible $H(\tau)$, we have:

$$\Delta(H(\tau)) \leq B$$

Then:
(a) If $E\{L(Q(0))\} < \infty$, then $Q_k(t)$ is mean rate stable for all $k \in \{1, \ldots, K\}$. That is:

$$\lim_{t \to \infty} E\{|Q_k(t)|/t = 0$$

(b) If $Q(0)$ is finite with probability 1, and if there is a finite constant $D > 0$ such that for all $t \in \{0, 1, 2, \ldots\}$ and all $k \in \{1, \ldots, K\}$ we have:

$$E\{(Q_k(t+1) - Q_k(t))^2\} \leq D$$

then $Q_k(t)$ is rate stable for all $k \in \{1, \ldots, K\}$. That is:

$$\lim_{t \to \infty} Q_k(t)/t = 0 \ (w.p.1)$$

**Proof:** See Appendix E. \qed

Theorem 3 only requires the (unconditional) second moment of queue changes to be bounded, whereas
Theorem 2 requires (conditional) fourth moments to be bounded.

III. CONVERGENCE OF TIME AVERAGES

This section reviews basic convergence definitions and results needed in the proof of Theorem 2. It
then develops a generalized drift-plus-penalty result for processes with a certain variance property.

A. Discussion of Convergence With Probability 1

Let $Y(t)$ be a real-valued stochastic process defined on $t \in \{0, 1, 2, \ldots\}$. To say that $Y(t)$ converges
to a constant $\alpha \in \mathbb{R}$ “with probability 1” (or “almost surely”), we use the notation:

$$\lim_{t \to \infty} Y(t) = \alpha \ (w.p.1) \quad (30)$$

It is well known that (30) holds if and only if for all $\varepsilon > 0$ we have:

$$\lim_{n \to \infty} Pr[\cup_{t \geq n}\{|Y(t) - \alpha| \geq \varepsilon\}] = 0 \quad (31)$$

Probabilities of the type (31) can be bounded via the union bound:

$$0 \leq Pr[\cup_{t \geq n}\{|Y(t) - \alpha| \geq \varepsilon\}] \leq \sum_{t=n}^{\infty} Pr[|Y(t) - \alpha| \geq \varepsilon] \quad (32)$$

It follows that (31) holds if the infinite sum on the right-hand-side of (32) is the tail of a convergent
series. Bounds on each term of the series can be obtained via the well known Chebyshev inequality:

$$Pr[|Y(t) - \alpha| \geq \varepsilon] \leq \frac{E\{(Y(t) - \alpha)^2\}}{\varepsilon^2} \quad (33)$$

2The same results for Theorem 3 hold if the requirement “$\Delta(H(t)) \leq B$” (which conditions on the full history $H(t)$), is replaced with
“$E\{L(Q(t+1)) - L(Q(t))|Q(t)\} \leq B$” (which conditions only on $Q(t)\)
The above discussion explains the following well known lemma:

**Lemma 1:** If \( Y(t) \) satisfies the following:

\[
\sum_{t=1}^{\infty} \mathbb{E}\{(Y(t) - \alpha)^2\} < \infty
\]

then (30) holds, that is, the variables \( Y(t) \) converge to \( \alpha \) with probability 1.

**Corollary 1:** (Rate Stability in Queues with Finite Variance) If \( Q(t) \) is a real-valued stochastic process defined over slots \( t \in \{0, 1, 2, \ldots\} \) that satisfies:

\[
\sum_{t=1}^{\infty} \frac{\mathbb{E}\{Q(t)^2\}}{t^2} < \infty
\]

then:

\[
\lim_{t \to \infty} \frac{Q(t)}{t} = 0 \ (w.p.1)
\]

In particular, this holds whenever there is a finite constant \( C > 0 \) such that \( \mathbb{E}\{Q(t)^2\} \leq C \) for all \( t \).

**Proof:** This corollary follows as an immediate consequence of Lemma 1 by defining \( Y(t) \triangleq Q(t)^2/t^2 \) and \( \alpha = 0 \). The special case when \( \mathbb{E}\{Q(t)^2\} \leq C \) follows because \( \sum_{t=1}^{\infty} \frac{C}{t^2} < \infty \).

\( \square \)

**B. Time Averages and the Kolmogorov Strong Law for Martingale Differences**

Let \( X(t) \) be a real-valued stochastic process defined over timeslots \( t \in \{0, 1, 2, \ldots\} \). Define the history \( \mathcal{H}_X(t) \) to be the set of values of the process before slot \( t \), so that \( \mathcal{H}_X(0) \) is the empty set, and for all slots \( t > 0 \) we have:

\[
\mathcal{H}_X(t) \triangleq \{X(0), X(1), \ldots, X(t-1)\}
\]  

(33)

We first assume the process \( X(t) \) has the property \( \mathbb{E}\{X(t)|\mathcal{H}_X(t)\} = 0 \) for all \( t \) and all possible \( \mathcal{H}_X(t) \). Such processes are called *martingale differences*. The following theorem is a well known variation on the Kolmogorov strong law of large numbers.

**Theorem 4:** (Kolmogorov strong law for martingale differences [15][24][25]) Suppose that \( X(t) \) is a stochastic process over \( t \in \{0, 1, 2, \ldots\} \) such that:

- \( \mathbb{E}\{X(t)|\mathcal{H}_X(t)\} = 0 \) for all \( t \) and all \( \mathcal{H}_X(t) \), where \( \mathcal{H}_X(t) \) is defined in (33).
- The second moments \( \mathbb{E}\{X(t)^2\} \) are finite for all \( t \) and satisfy:

\[
\sum_{t=1}^{\infty} \frac{\mathbb{E}\{X(t)^2\}}{t^2} < \infty
\]

(34)

Then:

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} X(\tau) = 0 \ (w.p.1)
\]

The following corollary follows easily from the Kolmogorov strong law given above.

**Corollary 2:** Let \( X(t) \) be a stochastic process defined over slots \( t \in \{0, 1, 2, \ldots\} \), and suppose that:

- There is a finite constant \( B \) such that \( \mathbb{E}\{X(t)|\mathcal{H}_X(t)\} \leq B \) for all \( t \) and all \( \mathcal{H}_X(t) \), where the history \( \mathcal{H}_X(t) \) is defined in (33).
- The second moments \( \mathbb{E}\{X(t)^2\} \) are finite for all \( t \) and satisfy:

\[
\sum_{t=1}^{\infty} \frac{\mathbb{E}\{X(t)^2\}}{t^2} < \infty
\]

Then:

\[
\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} X(\tau) \leq B \ (w.p.1)
\]

**Proof:** The idea is to define the process \( \tilde{X}(t) \triangleq X(t) - \mathbb{E}\{X(t)|\mathcal{H}_X(t)\} \), and then apply the result of Theorem 4 to the process \( \tilde{X}(t) \). This is shown in Appendix A for completeness.

\( \square \)
C. A Generalized Drift-Plus-Penalty Theorem

Now let $\Psi(t)$ be a non-negative stochastic process defined over slots $t \in \{0, 1, 2, \ldots\}$, and let $\beta(t)$ be another stochastic process defined on the same probability space and whose time average we want to show is non-negative. The $\Psi(t)$ process can represent the values of a general Lyapunov function over time $t \in \{0, 1, 2, \ldots\}$. Define $\delta(t) \triangleq \Psi(t+1) - \Psi(t)$ as the difference process. Define the history $\mathcal{H}(t)$ for this system by:

$$\mathcal{H}(t) \triangleq \{\Psi(0), \ldots, \Psi(t), \beta(0), \ldots, \beta(t-1)\}$$

(35)

**Theorem 5:** (Generalized Drift-Plus-Penalty) Suppose $\Psi(0)$ is finite with probability 1, that $\mathbb{E}\{\delta(t)^2\}$ and $\mathbb{E}\{\beta(t)^2\}$ are finite for all $t$, and that:

$$\sum_{t=1}^{\infty} \frac{\mathbb{E}\{\delta(t)^2\} + \mathbb{E}\{\beta(t)^2\}}{t^2} < \infty$$

Further suppose that the following drift-plus-penalty condition holds for all $t$ and all possible $\mathcal{H}(t)$:

$$\mathbb{E}\{\delta(t) + \beta(t)|\mathcal{H}(t)\} \leq 0$$

(36)

Then:

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \beta(\tau) \leq 0 \ (w.p.1)$$

**Proof:** Define $X(t) \triangleq \delta(t) + \beta(t)$. The idea is to apply Corollary 2 to the process $X(t)$. To this end, we simply need to show that $X(t)$ satisfies the assumptions needed in Corollary 2. Note that the history $\mathcal{H}(t)$ contains more information that the history $\mathcal{H}_X(t)$, defined:

$$\mathcal{H}_X(t) \triangleq \{X(0), X(1), \ldots, X(t-1)\}$$

Indeed, $\mathcal{H}_X(t)$ can be ascertained with knowledge of the more detailed history $\mathcal{H}(t)$. Thus, we can write $\mathcal{H}(t) = \mathcal{H}(t) \cup \mathcal{H}_X(t)$, as adding the information $\mathcal{H}_X(t)$ does not create any new information. Thus, using iterated expectations yields:

$$\mathbb{E}\{\mathbb{E}\{X(t)|\mathcal{H}(t)\} | \mathcal{H}_X(t)\} = \mathbb{E}\{\mathbb{E}\{X(t)|\mathcal{H}(t) \cup \mathcal{H}_X(t)\} | \mathcal{H}_X(t)\} = \mathbb{E}\{X(t)|\mathcal{H}_X(t)\}$$

On the other hand, by (36) we have:

$$\mathbb{E}\{\mathbb{E}\{X(t)|\mathcal{H}(t)\} | \mathcal{H}_X(t)\} = \mathbb{E}\{\mathbb{E}\{\delta(t) + \beta(t)|\mathcal{H}(t)\} | \mathcal{H}_X(t)\} \leq \mathbb{E}\{0|\mathcal{H}_X(t)\} = 0$$

Therefore, for all $t$ and all possible $\mathcal{H}_X(t)$ we have:

$$\mathbb{E}\{X(t)|\mathcal{H}_X(t)\} \leq 0$$

It remains only to show that:

$$\sum_{t=1}^{\infty} \frac{\mathbb{E}\{X(t)^2\}}{t^2} < \infty$$

Because $(\delta(t) + \beta(t))^2 \leq 2\delta(t)^2 + 2\beta(t)^2$, we have:

$$\sum_{t=1}^{\infty} \frac{\mathbb{E}\{X(t)^2\}}{t^2} = \sum_{t=1}^{\infty} \frac{\mathbb{E}\{(\delta(t) + \beta(t))^2\}}{t^2} \leq 2 \sum_{t=1}^{\infty} \frac{\mathbb{E}\{\delta(t)^2 + \beta(t)^2\}}{t^2} < \infty$$
Thus, by Corollary 2 we have:
\[
\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} X(\tau) \leq 0 \ (w.p.1) \tag{37}
\]

However, recalling that \( X(t) \triangleq \Psi(t+1) - \Psi(t) + \beta(t) \), we have:
\[
\sum_{\tau=0}^{t-1} X(\tau) = \Psi(t) - \Psi(0) + \sum_{\tau=0}^{t-1} \beta(\tau)
\geq -\Psi(0) + \sum_{\tau=0}^{t-1} \beta(\tau)
\]
where the final inequality holds because \( \Psi(t) \geq 0 \). Dividing the above inequality by \( t \) yields:
\[
\frac{1}{t} \sum_{\tau=0}^{t-1} X(\tau) \geq \frac{-\Psi(0)}{t} + \frac{1}{t} \sum_{\tau=0}^{t-1} \beta(\tau)
\]

Taking a \( \limsup \) of the above as \( t \to \infty \) and using (37) proves the result. \( \square \)

IV. THE LYAPUNOV OPTIMIZATION THEOREM — PROVING THEOREM 2

Consider the stochastic processes \( Q(t) = (Q_1(t), \ldots, Q_K(t)) \) and \( p(t) \) as described in Section II. Consider the quadratic Lyapunov function \( L(Q(t)) \) defined in (11), repeated again here for convenience:
\[
L(Q(t)) = \frac{1}{2} \sum_{k=1}^{K} w_k Q_k(t)^2
\]
where \( w_k > 0 \) for all \( k \). Define \( ||Q(t)|| \) by:
\[
||Q(t)|| \triangleq \sqrt{L(Q(t))} = \sqrt{\frac{1}{2} \sum_{k=1}^{K} w_k Q_k(t)^2}
\]
It is not difficult to show that:
\[
\sum_{k=1}^{K} \frac{\sqrt{w_k}}{\sqrt{2}} |Q_k(t)| \geq ||Q(t)|| \tag{38}
\]
Further, for any vectors \( a, b \) we have:
\[
||a + b|| \leq ||a|| + ||b|| \tag{39}
\]

Define the drift \( \Delta(H(t)) \) according to (12), where the history \( H(t) \) is defined in (10). Define the Lyapunov difference process \( \delta(t) \triangleq L(Q(t+1)) - L(Q(t)) \), and note by definition that:
\[
\mathbb{E} \{ \delta(t) | H(t) \} = \Delta(H(t)) \tag{40}
\]
Define \( d_k(t) \) as the queue \( k \) difference process:
\[
d_k(t) \triangleq Q_k(t + 1) - Q_k(t)
\]
We will bound the time averages of \( p(t) \) and \( Q_k(t) \) when the following drift-plus-penalty condition holds for all \( t \) and all \( H(t) \):
\[
\Delta(H(t)) + V \mathbb{E} \{ p(t) | H(t) \} \leq B + V p^* - \epsilon \sum_{k=1}^{K} |Q_k(t)| \tag{41}
\]
for some finite constants $B$, $p^*$, $V$, $\epsilon$. To this end, we define $\Psi(t) \triangleq L(Q(t))$ and $\beta(t)$ as follows:

$$\beta(t) \triangleq Vp(t) - B - Vp^* + \epsilon \sum_{k=1}^{K} |Q_k(t)|$$  \hspace{1cm} (42)

The idea is to show that the assumptions needed in the generalized drift-plus-penalty theorem (Theorem 5) hold for these definitions of $\Psi(t)$ and $\beta(t)$.

**Theorem 6:** Suppose that the boundedness assumptions (17) and (19) hold. Suppose that $\mathbb{E} \{Q_k(t)^2\}$ is finite for all $k$ and all $t$, and that for all $k \in \{1, \ldots, K\}$ we have:

$$\sum_{t=1}^{\infty} \frac{\mathbb{E} \{Q_k(t)^2\}}{t^2} < \infty$$  \hspace{1cm} (43)

Define the quadratic Lyapunov function $L(Q(t))$ as in (11), and suppose there are constants $B$, $p^*$, $V \geq 0$, $\epsilon \geq 0$ for which the drift-plus-penalty condition (41) holds for all $t$ and all possible $\mathcal{H}(t)$. Then:

a) If $V > 0$ we have:

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} p(\tau) \leq p^* + B/V \quad (w.p.1)$$  \hspace{1cm} (44)

b) If $\epsilon > 0$, we have:

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} |Q_k(\tau)| \leq \frac{B}{\epsilon} + \frac{V}{\epsilon} \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} |p^* - p(\tau)| \quad (w.p.1)$$  \hspace{1cm} (45)

**Proof:** Define $\Psi(t) \triangleq L(Q(t))$, $\delta(t) \triangleq L(Q(t+1)) - L(Q(t))$ and define $\beta(t)$ as in (42). For all $t$ and all $\mathcal{H}(t)$ we have:

$$\mathbb{E} \{\delta(t) + \beta(t) | \mathcal{H}(t)\} = \Delta(\mathcal{H}(t)) + \mathbb{E} \{\beta(t) | \mathcal{H}(t)\}$$

$$= \Delta(\mathcal{H}(t)) + V \mathbb{E} \{p(t) | \mathcal{H}(t)\} - B - Vp^* + \epsilon \sum_{k=1}^{K} |Q_k(t)|$$

$$\leq 0$$  \hspace{1cm} (47)

where (46) follows from (40), (47) follows by definition of $\beta(t)$ and the fact that $\mathbb{E} \{|Q_k(t)| | \mathcal{H}(t)\} = |Q_k(t)|$, and (48) follows from (44).

**Claim 1:**

$$\sum_{t=1}^{\infty} \frac{\mathbb{E} \{\delta(t)^2 + \beta(t)^2\}}{t^2} < \infty$$

This claim is proven in Appendix B. Assuming the result of the claim, we know that all conditions for the $\Psi(t)$ and $\beta(t)$ processes needed to apply Theorem 5 hold. We thus conclude:

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \beta(\tau) \leq 0 \quad (w.p.1)$$

That is:

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \left[ Vp(\tau) - B - Vp^* + \epsilon \sum_{k=1}^{K} |Q_k(\tau)| \right] \leq 0 \quad (w.p.1)$$  \hspace{1cm} (49)

First assume that $V > 0$. Neglecting the non-negative term $\epsilon \sum_{k=1}^{K} |Q_k(\tau)|$ from (49) and dividing by $V$ yields:

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} |p(\tau) - B/V - p^*| \leq 0 \quad (w.p.1)$$
This proves (44).

Now note that for any functions \( f(t), g(t) \), we have
\[
\limsup_{t \to \infty} |f(t) - g(t)| \leq 0 \implies \limsup_{t \to \infty} f(t) \leq \limsup_{t \to \infty} g(t)
\]
Defining \( f(t) = \frac{\epsilon}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} |Q_k(\tau)| \) and \( g(t) = \frac{1}{t} \sum_{\tau=0}^{t-1} [B + V(p^* - p(\tau))] \), it follows from (49) that:
\[
\limsup_{t \to \infty} \frac{\epsilon}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} |Q_k(\tau)| \leq \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} [B + V(p^* - p(\tau))] \quad (w.p.1)
\]
If \( \epsilon > 0 \), we can divide the above by \( \epsilon \) to prove (45). \( \square \)

**Theorem 7:** Suppose we have a quadratic Lyapunov function \( L(Q(t)) \) as defined in (11), and that assumption (19) holds, so that \( \mathbb{E} \{ d_k(t)^4 |Q(t)| \} \leq D \) for all \( t \) and for some finite constant \( D \), where \( d_k(t) = Q_k(t+1) - Q_k(t) \). Suppose that \( \mathbb{E} \{ ||Q(0)||^4 \} < \infty \). Suppose that there is an \( \epsilon > 0 \) and a constant \( B > 0 \) such that:
\[
\Delta(\mathcal{H}(t)) \leq \tilde{B} - \epsilon \sum_{k=1}^{K} |Q_k(t)| \quad (50)
\]
Then:

a) There are constants \( c > 0 \) and \( a > 0 \) such that whenever \( ||Q(t)|| \geq a \), we have:
\[
\mathbb{E} \left\{ ||Q(t+1)|| |Q(t)| \right\} \leq ||Q(t)|| - c
\]
b) There is a finite constant \( b > 0 \) such that for all \( M \in \{1, 2, \ldots\} \) we have:
\[
\frac{1}{M} \sum_{t=0}^{M-1} \mathbb{E} \{ ||Q(t)||^3 \} \leq b
\]
c) For all \( k \in \{1, \ldots, K\} \) we have:
\[
\sum_{t=1}^{\infty} \frac{\mathbb{E} \{ |Q_k(t)|^2 \} t^2}{t^2} < \infty
\]
d) For all \( k \in \{1, \ldots, K\} \) we have:
\[
\lim_{t \to \infty} \frac{Q_k(t)}{t} = 0 \quad (w.p.1)
\]
e) We have:
\[
\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} |Q_k(\tau)| \leq \frac{\tilde{B}}{\epsilon} \quad (w.p.1)
\]
**Proof:** The proof of parts (a) and (b) closely follow a similar result derived for exponential Lyapunov functions with deterministically bounded queue changes in [26], and are provided in Appendix C. To prove parts (c), (d), (e), we have from part (b) that for all \( M \in \{1, 2, 3, \ldots\} \):
\[
\sum_{t=0}^{M-1} \mathbb{E} \{ ||Q(t)||^3 \} \leq bM \quad (51)
\]
However, we have \( ||Q(t)||^3 \geq ||Q(t)||^2 - 1 \). Using this with (51) gives:
\[
\sum_{t=0}^{M-1} (\mathbb{E} \{ ||Q(t)||^2 \} - 1) \leq bM
\]
\footnote{This follows by: \( \limsup_{t \to \infty} f(t) = \limsup_{t \to \infty} [g(t) + (f(t) - g(t))] \leq \limsup_{t \to \infty} g(t) + \limsup_{t \to \infty} (f(t) - g(t)). \)}
and so:

\[ \sum_{t=0}^{M-1} \mathbb{E} \{ \|Q(t)\|^2 \} \leq (b + 1)M \]

Using \( \frac{w_k}{2} Q_k(t)^2 \leq \|Q(t)\|^2 \) in the above inequality proves that there is a finite constant \( C > 0 \) such that for all \( k \in \{1, \ldots, K\} \) we have:

\[ \sum_{t=0}^{M-1} \mathbb{E} \{ Q_k(t)^2 \} \leq CM \forall M \in \{1, 2, 3, \ldots\} \]

Lemma 4 in Appendix D shows that the above inequality implies the result of part (c).

Part (d) follows immediately from the result of part (c) together with Corollary 1. To prove part (e), we note that the result of part (c) implies that the conditions for Theorem 6 are met for the case \( \epsilon > 0, p(t) = p^* = V = 0, B = \tilde{B} \), which yields the result.

**A. Completing the proof of Theorem 2**

Suppose now the assumptions of Theorem 2 hold, so that the drift-plus-penalty condition (14) is satisfied for all \( t \) and all \( \mathcal{H}(t) \), and the boundedness assumptions (17)-(19) hold. We temporarily also assume that the initial state \( Q(0) \) is deterministically given as some constant vector (so that \( \mathbb{E} \{ \|Q(0)\|^4 \} = \|Q(0)\|^4 < \infty \)). The condition (14) together with the fact that \( \mathbb{E} \{ p(t) | \mathcal{H}(t) \} \geq p_{\min} \) implies:

\[ \Delta(\mathcal{H}(t)) \leq B + V (p^* - p_{\min}) - \epsilon \sum_{k=1}^{K} |Q_k(t)| \]

Defining \( \tilde{B} = B + V (p^* - p_{\min}) \), by Theorem 7 we know all queues are rate stable, that is, \( \lim_{t \to \infty} Q_k(t)/t = 0 \) with probability 1. We also know by Theorem 7 that:

\[ \sum_{t=1}^{\infty} \frac{\mathbb{E} \{ Q_k(t)^2 \}}{t^2} < \infty \]

Then all assumptions are satisfied to apply Theorem 6 and so we have that:

\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} p(\tau) \leq p^* + \frac{B}{V} \quad (w.p.1) \]

\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} |Q_k(\tau)| \leq \frac{B}{\epsilon} + \frac{V}{\epsilon} \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} [p^* - p(\tau)] \quad (w.p.1) \] (52)

Because \( \mathbb{E} \{ -p(t) | \mathcal{H}(t) \} \leq -p_{\min} \) for all \( t \) and all \( \mathcal{H}(t) \), and \( \sum_{t=1}^{\infty} \mathbb{E} \{ p(t)^2 \}/t^2 < \infty \), we know by Corollary 2 that:

\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} [p^* - p(\tau)] \leq p^* - p_{\min} \]

This together with (52) proves (21). Thus, all desired performance bounds hold with probability 1 under the assumption that the initial queue vector is some finite value \( Q(0) \). Because these bounds do not depend on \( Q(0) \), it follows that these same bounds hold (with probability 1) if \( Q(0) \) is chosen randomly, provided that \( Q(0) \) is finite with probability 1.
B. Variations on Theorem 2

Suppose there are processes $B(t), p(t), p^*(t), Q(t)$ and constants $V \geq 0, \epsilon > 0$ such that for all $t$ and all possible $H(t)$, we have:

$$
\Delta(H(t)) + V \mathbb{E} \{p(t)|H(t)\} \leq \mathbb{E} \{B(t)|H(t)\} + V \mathbb{E} \{p^*(t)|H(t)\} - \epsilon \sum_{k=1}^{K} |Q_k(t)| \tag{53}
$$

This is a variation on the drift-plus-penalty condition (14) that uses a time-varying $p^*(t)$ and $B(t)$. Suppose that $Q(0)$ is finite with probability 1, and that:

- Second moments of $p(t), B(t), \text{and} p^*(t)$ are finite for all $t$, and:

$$
\sum_{t=1}^{\infty} \mathbb{E} \left\{ \frac{(V(p(t) - p^*(t)) - B(t))^2}{t^2} \right\} < \infty
$$

- There is a constant $\beta_{min}$ such that for all $t$ and all $H(t)$:

$$
\mathbb{E} \{V(p(t) - p^*(t)) - B(t)|H(t)\} \geq \beta_{min}
$$

- There is a constant $D > 0$ such that for all $k \in \{1, \ldots, K\}, \text{all} t$, and all possible $Q(t)$:

$$
\mathbb{E} \{(Q_k(t + 1) - Q_k(t))^4|Q(t)\} \leq D
$$

Then we can define $\tilde{B} \triangleq 0, \tilde{V} = 1, (\beta(t) \triangleq V(p(t) - p^*(t)) - B(t), \beta^* = 0$ to find:

$$
\Delta(H(t)) + \mathbb{E} \{\beta(t)|H(t)\} \leq 0 - \epsilon \sum_{k=1}^{K} |Q_k(t)|
$$

Then the conditions of Theorem 2 hold for $\beta(t)$ and $\beta^*$, and so we conclude (using (52)):

$$
\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \beta(\tau) \leq 0 \text{ (w.p.1)}
$$

$$
\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} |Q_k(\tau)| \leq \frac{1}{\epsilon} \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} [\beta(\tau)] \text{ (w.p.1)}
$$

Thus:

$$
\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} [p(\tau) - p^*(\tau)] \leq \frac{1}{V} \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} B(\tau) \text{ (w.p.1)}
$$

$$
\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \sum_{k=1}^{K} |Q_k(\tau)| \leq \frac{1}{\epsilon} \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} [B(\tau) + V(p^*(\tau) - p(\tau))] \text{ (w.p.1)}
$$

V. APPLICATIONS

Here we illustrate an important application of Theorem 2 to optimization of time averages in stochastic queueing networks. This is the same scenario treated in [1]. However, while the work in [1] obtains bounds on the time average expectations via Theorem 1 here we obtain bounds on the pure time averages via Theorem 2.

Consider a $K$ queue network with queue vector $Q(t) = (Q_1(t), \ldots, Q_K(t))$ that evolves in slotted time $t \in \{0, 1, 2, \ldots\}$ with update equation:

$$
Q_k(t + 1) = \max [Q_k(t) - b_k(t) + a_k(t), 0] \quad \forall k \in \{1, \ldots, K\} \tag{54}
$$
where \( a_k(t) \) and \( b_k(t) \) are arrival and service variables, respectively, for queue \( k \). These are determined on slot \( t \) by general functions \( \hat{a}_k(\alpha(t), \omega(t)) \), \( \hat{b}_k(\alpha(t), \omega(t)) \) of a network state \( \omega(t) \) and a control action \( \alpha(t) \):
\[
a_k(t) = \hat{a}_k(\alpha(t), \omega(t)), \quad b_k(t) = \hat{b}_k(\alpha(t), \omega(t))
\]
where the control action \( \alpha(t) \) is make every slot \( t \) with knowledge of the current \( \omega(t) \) and is chosen within some abstract set \( \mathcal{A}_{\omega(t)} \). The \( \omega(t) \) value can represent random arrival and channel state information on slot \( t \), and \( \alpha(t) \) can represent a resource allocation decision. For simplicity, assume the \( \omega(t) \) process is i.i.d. over slots.

The goal is to choose control actions \( \alpha(t) \in \mathcal{A}_{\omega(t)} \) over time to solve the following stochastic network optimization problem:

Minimize: \[
\limsup_{t \to \infty} \bar{y}_0(t)
\]
Subject to: 1) \[
\limsup_{t \to \infty} \bar{Q}_k(t) < \infty \quad \forall k \in \{1, \ldots, K\}
\]
2) \[
\limsup_{t \to \infty} \bar{y}_m(t) \leq 0 \quad \forall m \in \{1, \ldots, M\}
\]
3) \[
\alpha(t) \in \mathcal{A}_{\omega(t)} \quad \forall t \in \{0, 1, 2, \ldots\}
\]

Typical penalties can represent power expenditures. For example, suppose \( y_m(t) \geq p_m(t) - p_m^{av} \), where \( p_m(t) \) is the power incurred in component \( m \) of the network on slot \( t \), and \( p_m^{av} \) is a required time average power expenditure. Then ensuring \( \limsup_{t \to \infty} \bar{y}_m(t) \leq 0 \) ensures that \( \limsup_{t \to \infty} \bar{y}_m(t) \leq p_m^{av} \), so that the desired time average power constraint is met [4].

To ensure the time average penalty constraints are met, for each \( m \in \{1, \ldots, M\} \) we define a virtual queue \( Z_m(t) \) as follows:
\[
Z_m(t + 1) = \max[Z_m(t) + y_m(t), 0]
\]

It is easy to see that for any \( t > 0 \) we have:
\[
Z_m(t) - Z_m(0) \geq \sum_{\tau=0}^{t-1} y_m(\tau)
\]
and therefore, dividing by \( t \) and rearranging terms yields:
\[
\frac{1}{t} \sum_{\tau=0}^{t-1} y_m(\tau) \leq \frac{Z_m(t)}{t} - \frac{Z_m(0)}{t}
\]

It follows that if \( Z_m(t) \) is rate stable for all \( m \), so that \( Z_m(t)/t \to 0 \) with probability 1, then the constraint (57) is satisfied with probability 1.
Now define $\Theta(t) \triangleq [Q(t), Z(t)]$ as the combined queue vector, and define the Lyapunov function:

$$L(\Theta(t)) \triangleq \frac{1}{2} \left[ \sum_{k=1}^{K} Q_k(t)^2 + \sum_{m=1}^{M} Z_m(t)^2 \right]$$

The system history $\mathcal{H}(t)$ is defined:

$$\mathcal{H}(t) \triangleq \{ \Theta(0), \Theta(1), \ldots, \Theta(t), y_0(0), y_0(1), \ldots, y_0(t-1) \}$$

The drift-plus-penalty algorithm thus seeks to minimize a bound on:

$$\Delta(\mathcal{H}(t)) + V \mathbb{E} \{ \hat{y}_0(\alpha(t), \omega(t)) | \mathcal{H}(t) \}$$

**A. Computing the Drift-Plus-Penalty Inequality**

Assume the functions $\hat{a}_k(\cdot), \hat{b}_k(\cdot), \hat{y}_0(\cdot)$ satisfy the following for all possible $\omega(t)$ and all possible $\alpha(t) \in \mathcal{A}_\omega(t)$:

$$0 \leq \hat{a}_k(\alpha(t), \omega(t)), 0 \leq \hat{b}_k(\alpha(t), \omega(t)), \hat{y}_0(\alpha(t), \omega(t)) \geq y_0^{\text{min}}$$

where $y_0^{\text{min}}$ is a deterministic lower bound on $y_0(t)$ for all $t$. Also assume that there is a finite constant $D > 0$ such that for all (possibly randomized) choices of $\alpha(t)$ in reaction to the i.i.d. $\omega(t)$ we have:

$$\mathbb{E} \{ \hat{a}_k(\alpha(t), \omega(t))^4 \} \leq D \ \forall k \in \{1, \ldots, K\} \quad (60)$$

$$\mathbb{E} \{ \hat{b}_k(\alpha(t), \omega(t))^4 \} \leq D \ \forall k \in \{1, \ldots, K\} \quad (61)$$

$$\mathbb{E} \{ \hat{y}_m(\alpha(t), \omega(t))^4 \} \leq D \ \forall m \in \{1, \ldots, M\} \quad (62)$$

$$\mathbb{E} \{ \hat{y}_0(\alpha(t), \omega(t))^2 \} \leq D \quad (63)$$

where the expectations are taken with respect to the distribution of the i.i.d. $\omega(t)$ process, and the possibly randomized decisions $\alpha(t) \in \mathcal{A}_\omega(t)$.

By squaring (54) and (59) it is not difficult to show that the drift-plus-penalty expression satisfies the following bound (see [1]):

$$\Delta(\mathcal{H}(t)) + V \mathbb{E} \{ \hat{y}_0(\alpha(t), \omega(t)) | \mathcal{H}(t) \} \leq B + V \mathbb{E} \{ \hat{y}_0(\alpha(t), \omega(t)) | \mathcal{H}(t) \}$$

$$+ \sum_{k=1}^{K} Q_k(t) \mathbb{E} \{ \hat{a}_k(\alpha(t), \omega(t)) - \hat{b}_k(\alpha(t), \omega(t)) | \mathcal{H}(t) \}$$

$$+ \sum_{m=1}^{M} Z_m(t) \mathbb{E} \{ \hat{y}_m(\alpha(t), \omega(t)) | \mathcal{H}(t) \}$$

(64)

for some finite constant $B > 0$, representing a sum on the second moment bounds of the $a_k(t), b_k(t),$ and $y_m(t)$ processes.

**B. The Dynamic Drift-Plus-Penalty Algorithm**

It is easy to show that the right-hand-side of the inequality (64) is minimized by the policy that, every slot $t$, observes only the current queue values $Q(t), Z(t)$ and the current $\omega(t)$ and chooses $\alpha(t) \in \mathcal{A}_\omega(t)$ to minimize the following expression:

$$V \hat{y}_0(\alpha(t), \omega(t)) + \sum_{k=1}^{K} Q_k(t)[\hat{a}_k(\alpha(t), \omega(t)) - \hat{b}_k(\alpha(t), \omega(t))] + \sum_{m=1}^{M} Z_m(t)\hat{y}_m(\alpha(t), \omega(t))$$

Then update the actual queues $Q_k(t)$ according to (54) and the virtual queues $Z_m(t)$ according to (59). This policy does not require knowledge of the probability distribution for $\omega(t)$. One difficulty is that
it may not be possible to achieve the infimum of the above expression over the set $A_{\omega(t)}$, because we are using general (possibly non-continuous) functions $\hat{a}_k(\alpha(t), \omega(t))$, $\hat{b}_k(\alpha(t), \omega(t))$, $\hat{y}_m(\alpha(t), \omega(t))$ and a general (possibly non-compact) set $A_{\omega(t)}$. Thus, we simply assume there is a finite constant $C \geq 0$ such that our algorithm chooses $\alpha(t) \in A_{\omega(t)}$ to come within an additive constant $C$ of the infimum on every slot $t$, so that:

$$V \hat{y}_0(\alpha(t), \omega(t)) + \sum_{k=1}^K Q_k(t)[\hat{a}_k(\alpha(t), \omega(t)) - \hat{b}_k(\alpha(t), \omega(t))] + \sum_{m=1}^M Z_m(t)\hat{y}_m(\alpha(t), \omega(t))$$

$$\leq C + \inf_{\alpha \in A_{\omega(t)}} \left[ V \hat{y}_0(\alpha, \omega(t)) + \sum_{k=1}^K Q_k(t)[\hat{a}_k(\alpha, \omega(t)) - \hat{b}_k(\alpha, \omega(t))] + \sum_{m=1}^M Z_m(t)\hat{y}_m(\alpha, \omega(t)) \right]$$

Such a choice of $\alpha(t)$ is called a $C$-additive approximation. The case $C = 0$ corresponds to achieving the exact infimum every slot.

### C. $\omega$-only policies

Define a $\omega$-only policy to be one that chooses $\alpha(t) \in A_{\omega(t)}$ every slot $t$ according to a stationary and randomized decision based only on the observed $\omega(t)$ (in particular, being independent of $H(t)$). Assume there exists an $\epsilon > 0$ and a particular $\omega$-only policy $\alpha^*(t)$ that yields the following:

$$\mathbb{E} \left\{ \hat{a}_k(\alpha^*(t), \omega(t)) - \hat{b}_k(\alpha^*(t), \omega(t)) \right\} \leq -\epsilon \quad \forall k \in \{1, \ldots, K\}$$

$$\mathbb{E} \left\{ \hat{y}_m(\alpha^*(t), \omega(t)) \right\} \leq -\epsilon \quad \forall m \in \{1, \ldots, M\}$$

Under this assumption, it can be shown that the algorithm that uses the $\omega$-only decisions $\alpha^*(t)$ every slot $t$ satisfies the constraints (56)-(58) and hence the problem (55)-(57) is feasible (meaning that its constraints are possible to satisfy). Further, this assumption (similar to a Slater assumption in convex optimization theory [27]) is only slightly stronger than what is required for feasibility. Indeed, it can be shown that if the problem (55)-(57) is feasible, then for all $\delta > 0$ there must be an $\omega$-only algorithm that satisfies [28]:

$$\mathbb{E} \left\{ \hat{a}_k(\alpha^*(t), \omega(t)) - \hat{b}_k(\alpha^*(t), \omega(t)) \right\} \leq \delta \quad \forall k \in \{1, \ldots, K\}$$

$$\mathbb{E} \left\{ \hat{y}_m(\alpha^*(t), \omega(t)) \right\} \leq \delta \quad \forall m \in \{1, \ldots, M\}$$

Define $\epsilon_{\text{max}}$ as the supremum of all $\epsilon$ values for which an $\omega$-only policy exists and satisfies (65)-(66). For $0 \leq \epsilon \leq \epsilon_{\text{max}}$, define $y_0^{\text{opt}}(\epsilon)$ as the infimum value of $y$ such that for all $\delta > 0$, there exists an $\omega$-only policy $\alpha^*(t)$ that satisfies the following constraints:

$$\mathbb{E} \left\{ \hat{y}_0(\alpha^*(t), \omega(t)) \right\} \leq y + \epsilon$$

$$\mathbb{E} \left\{ \hat{a}_k(\alpha^*(t), \omega(t)) - \hat{b}_k(\alpha^*(t), \omega(t)) \right\} \leq -\epsilon + \delta \quad \forall k \in \{1, \ldots, K\}$$

$$\mathbb{E} \left\{ \hat{y}_m(\alpha^*(t), \omega(t)) \right\} \leq -\epsilon + \delta \quad \forall m \in \{1, \ldots, M\}$$

It is not difficult to show that:

- These constraints are feasible whenever $0 \leq \epsilon \leq \epsilon_{\text{max}}$.
- The function $y_0(\epsilon)$ is finite, continuous, and non-decreasing on the interval $0 \leq \epsilon \leq \epsilon_{\text{max}}$.
- The set of all such $y$ values that satisfy the above constraints is closed.

Thus, whenever $0 \leq \epsilon \leq \epsilon_{\text{max}}$, for any $\delta > 0$ there exists an $\omega$-only algorithm $\alpha^*(t)$ such that:

$$\mathbb{E} \left\{ \hat{y}_0(\alpha^*(t), \omega(t)) \right\} \leq y_0^{\text{opt}}(\epsilon) + \delta$$

$$\mathbb{E} \left\{ \hat{a}_k(\alpha^*(t), \omega(t)) - \hat{b}_k(\alpha^*(t), \omega(t)) \right\} \leq -\epsilon + \delta \quad \forall k \in \{1, \ldots, K\}$$

$$\mathbb{E} \left\{ \hat{y}_m(\alpha^*(t), \omega(t)) \right\} \leq -\epsilon + \delta \quad \forall m \in \{1, \ldots, M\}$$

It can be shown that $y_0^{\text{opt}}(0)$ is the infimum time average penalty for $y_0(t)$ over all algorithms that meet the constraints (56)-(58) (not just $\omega$-only algorithms) [4][28]. Thus, we define $y_0^{\text{opt}} = y_0^{\text{opt}}(0)$. 
D. Performance Bounds

Because our policy $\alpha(t)$ comes within $C \geq 0$ of minimizing the right-hand-side of (64) every slot $t$ (given the observed $H(t)$), we have for all $t$ and all possible $H(t)$:

$$
\Delta(H(t)) + V E \{ \hat{y}_0(\alpha(t), \omega(t)) | H(t) \} \leq B + C + V E \{ \hat{y}_0(\alpha^*(t), \omega(t)) | H(t) \} + \sum_{k=1}^{K} Q_k(t) E \{ \hat{a}_k(\alpha^*(t), \omega(t)) - \hat{b}_k(\alpha^*(t), \omega(t)) | H(t) \} + \sum_{m=1}^{M} Z_m(t) E \{ \hat{y}_m(\alpha^*(t), \omega(t)) | H(t) \}
$$

where $\alpha^*(t)$ is any other decision that can be implemented on slot $t$. Now fix $\epsilon$ in the interval $0 < \epsilon \leq \epsilon_{\text{max}}$. Fix any $\delta > 0$. Using the policy $\alpha^*(t)$ designed to achieve (67)-(69) and noting that this policy makes decisions independent of $H(t)$ yields:

$$
\Delta(H(t)) + V E \{ \hat{y}_0(\alpha(t), \omega(t)) | H(t) \} \leq B + C + V y_0^{\text{opt}}(\epsilon) + \delta - (\epsilon - \delta) \sum_{k=1}^{K} Q_k(t) - (\epsilon - \delta) \sum_{m=1}^{M} Z_m(t)
$$

The above holds for all $\delta > 0$. Taking a limit as $\delta \to 0$ yields:

$$
\Delta(H(t)) + V E \{ \hat{y}_0(\alpha(t), \omega(t)) | H(t) \} \leq B + C + V y_0^{\text{opt}}(\epsilon) - \epsilon \sum_{k=1}^{K} Q_k(t) - \epsilon \sum_{m=1}^{M} Z_m(t) \quad (70)
$$

where for simplicity we have substituted $y_0(t) = \hat{y}_0(\alpha(t), \omega(t))$ on the left-hand-side. Inequality (70) is in the exact form of the drift-plus-penalty condition (14). Recall that the penalty $y_0(t)$ is deterministically lower bounded by some finite (possibly negative) value $y_0^{\text{min}}$. Further, the moment bounds (60)-(63) can easily be shown to imply that the boundedness assumptions (17)-(19) hold. Thus, we can apply Theorem 2 to conclude that all queues are rate stable (in particular $Z_m(t)/t \to 0$ with probability 1 for all $k$, so that the constraints (57) are satisfied:

$$
\lim_{t \to \infty} \sup_{m \in \{1, \ldots, M\}} \frac{Z_m(t)}{t} \leq 0 \quad (w.p.1)
$$

Further:

$$
\lim_{t \to \infty} \sup_{t} \frac{1}{t} \sum_{\tau=0}^{t-1} y_0(\tau) \leq y_0^{\text{opt}}(\epsilon) + (B + C)/V \quad (w.p.1)
$$

$$
\lim_{t \to \infty} \sup_{t} \frac{1}{t} \sum_{\tau=0}^{t-1} \left[ \sum_{k=1}^{K} Q_k(\tau) + \sum_{m=1}^{M} Z_m(\tau) \right] \leq \frac{B + C + V y_0^{\text{opt}}(\epsilon) - y_0^{\text{min}}}{\epsilon} \quad (w.p.1)
$$

However, the above two bounds hold for all $\epsilon$ such that $0 < \epsilon \leq \epsilon_{\text{max}}$, and hence the two performance bounds can be optimized separately over this interval. Taking a limit as $\epsilon \to 0$ in the first bound and noting by continuity that $\lim_{\epsilon \to 0} y_0^{\text{opt}}(\epsilon) = y_0^{\text{opt}}(0) \geq y_0^{\text{opt}}$ yields:

$$
\lim_{t \to \infty} \sup_{t} \frac{1}{t} \sum_{\tau=0}^{t-1} y_0(\tau) \leq y_0^{\text{opt}}(0) + (B + C)/V \quad (w.p.1) \quad (71)
$$

Using $\epsilon = \epsilon_{\text{max}}$ in the second bound yields:

$$
\lim_{t \to \infty} \sup_{t} \frac{1}{t} \sum_{\tau=0}^{t-1} \left[ \sum_{k=1}^{K} Q_k(\tau) + \sum_{m=1}^{M} Z_m(\tau) \right] \leq \frac{B + C + V y_0^{\text{opt}}(\epsilon_{\text{max}}) - y_0^{\text{min}}}{\epsilon_{\text{max}}} \quad (w.p.1) \quad (72)
$$
Thus, this simple dynamic algorithm satisfies the desired time average penalty constraints, stabilizes all queues $Q_k(t)$, and yields a time average penalty for $y_0(t)$ that is within $B/V$ of the optimal value $y_0^{\text{opt}}$. The performance gap $B/V$ can be made arbitrarily small by choosing the $V$ parameter large (as shown by (71)). The tradeoff is a time average queue backlog that is $O(V)$ (as shown by (72)).

By (52), the bound (72) can be improved, at the expense of sometimes making it less easy to compute, by replacing “$-y_{\text{min}}$” on the right-hand-side with “$-\lim \inf_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} y_0(\tau)$.” Further, we note that the concept of place-holder backlog from [5] is compatible with this analysis and can often be used together with the above to provide improved backlog bounds.

VI. CONCLUSIONS

This work derives an extended drift-plus-penalty theorem for discrete time queueing systems. The theorem ensures all queues satisfy all major forms of stability, and that time averages meet desired constraints with probability 1. This extends prior results that were known to hold only for time average expectations. The boundedness conditions required for the theorem are mild and easily checked. In particular, the theorem applies to systems with an uncountably infinite number of possible events, to Markov systems with an uncountably infinite state space (possibly neither irreducible nor aperiodic), and to non-Markov systems. Our analysis combined the Kolmogorov law of large numbers for martingale differences with the drift-plus-penalty method from Lyapunov optimization. The results are applicable to a broad class of stochastic queueing networks, and are also useful in other contexts.

APPENDIX A — PROOF OF COROLLARY 2

Suppose the assumptions of Corollary 2 hold, so that $E \{X(t) | H_X(t)\} \leq B$ for all $t$ and all $H_X(t)$, and:

$$\sum_{t=1}^{\infty} \frac{E\{X(t)^2\}}{t^2} < \infty$$

Define $\tilde{X}(t) \triangleq X(t) - E\{X(t) | H_X(t)\}$. Clearly $E\{\tilde{X}(t) | H_X(t)\} = 0$ for all $t$ and all $H_X(t)$. Now define $H_{\tilde{X}}(t)$ as the history of the $\tilde{X}(t)$ process:

$$H_{\tilde{X}}(t) = \{\tilde{X}(0), \ldots, \tilde{X}(t-1)\}$$

It is easy to see that conditioning on $H_{\tilde{X}}(t)$ is the same as conditioning on $H_X(t)$, because these provide the same information. Thus $E\{\tilde{X}(t) | H_{\tilde{X}}(t)\} = 0$ for all $t$ and all possible $H_{\tilde{X}}(t)$. To apply the result of Theorem 4, we show that the second moment of $\tilde{X}(t)$ satisfies the condition (54). We have for all $t$:

$$E\{\tilde{X}(t)^2\} = E\{ (X(t) - E\{X(t) | H_X(t)\})^2\}$$

$$\quad = E\{ X(t)^2\} + E\{ X(t) | H_X(t)\}^2\} - 2E\{ X(t) E\{X(t) | H_X(t)\}\}$$

$$\quad \leq E\{ X(t)^2\} + E\{ X(t)^2 | H_X(t)\}^2\} - 2E\{ X(t) E\{X(t) | H_X(t)\}\}$$

$$\quad = 2E\{ X(t)^2\} - 2E\{ X(t) E\{X(t) | H_X(t)\}\}$$

$$\quad \leq 2E\{ X(t)^2\} + 2\sqrt{E\{ X(t)^2\} E\{ X(t)^2 | H_X(t)\}^2\}$$

$$\quad \leq 2E\{ X(t)^2\} + 2\sqrt{E\{ X(t)^2\} E\{ X(t)^2 | H_X(t)\}}$$

$$\quad \leq 2E\{ X(t)^2\} + 2\sqrt{E\{ X(t)^2\} E\{ X(t)^2\}} = 4E\{ X(t)^2\}$$

where (73) follows by Jensen’s inequality, and (74) follows by the Cauchy-Schwartz inner product inequality. It follows that:

$$\sum_{t=1}^{\infty} \frac{E\{\tilde{X}(t)^2\}}{t^2} \leq \sum_{t=1}^{\infty} \frac{4E\{X(t)^2\}}{t^2} < \infty$$
Thus, the result of Theorem 4 holds for the process \( \tilde{X}(t) \), and so:

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \tilde{X}(\tau) = 0 \quad (w.p.1)
\]

That is:

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} [X(\tau) - \mathbb{E} \{X(\tau) | \mathcal{H}(\tau)\}] = 0 \quad (w.p.1)
\]  

(75)

Using the fact that \( \mathbb{E} \{X(\tau) | \mathcal{H}(\tau)\} \leq B \) yields:

\[
\frac{1}{t} \sum_{\tau=0}^{t-1} [X(\tau) - \mathbb{E} \{X(\tau) | \mathcal{H}(\tau)\}] \geq \frac{1}{t} \sum_{\tau=0}^{t-1} [X(\tau) - B]
\]

Taking a \( \lim \sup \) of the above as \( t \to \infty \) and using (75) yields:

\[
\lim_{t \to \infty} \lim \sup \frac{1}{t} \sum_{\tau=0}^{t-1} [X(\tau) - B] \leq 0 \quad (w.p.1)
\]

This proves the result.

**APPENDIX B — PROOF OF CLAIM 1 IN THEOREM 6**

Here we prove the Claim 1 needed in Theorem 6. Recall that \( \delta(t) \triangleq L(Q(t+1)) - L(Q(t)) \), where \( L(Q(t)) \) is defined in (11) with any weights \( w_k > 0 \). We prove Claim 1 with two lemmas.

**Lemma 2:** Suppose there is a finite constant \( D > 0 \) such that for all \( t \) and all possible \( Q(t) \) we have:

\[
\mathbb{E} \{d_k(t)^4 | Q(t)\} \leq D \quad \forall k \in \{1, \ldots, K\}, \forall t \in \{0, 1, 2, \ldots\}
\]

Further suppose that:

\[
\sum_{t=1}^{\infty} \frac{\mathbb{E} \{Q_k(t)^2\}}{t^2} < \infty
\]  

(76)

Then:

\[
\sum_{t=1}^{\infty} \frac{\mathbb{E} \{\delta(t)^2\}}{t^2} < \infty
\]

**Proof:** We have:

\[
\delta(t) \triangleq \sum_{k=1}^{K} \frac{w_k}{2} [Q_k(t+1)^2 - Q_k(t)^2] = \sum_{k=1}^{K} \frac{w_k}{2} [(Q_k(t) + d_k(t))^2 - Q_k(t)^2] = \sum_{k=1}^{K} \frac{w_k}{2} [2Q_k(t)d_k(t) + d_k(t)^2]
\]

Thus:

\[
\mathbb{E} \{\delta(t)^2\} = \sum_{k=1}^{K} \sum_{i=1}^{K} \frac{w_kw_i}{4} \mathbb{E} \{(2Q_k(t)d_k(t) + d_k(t)^2)(2Q_i(t)d_i(t) + d_i(t)^2)\}
\]
Further:
\[
\mathbb{E}\left\{ (2Q_k(t)d_k(t) + d_k(t)^2)(2Q_i(t)d_i(t) + d_i(t)^2) \right\} = 4\mathbb{E}\left\{ Q_k(t)Q_i(t)d_k(t)d_i(t) \right\} + \mathbb{E}\left\{ d_k(t)^2d_i(t)^2 \right\} + 2\mathbb{E}\left\{ Q_k(t)d_k(t)d_i(t)^2 \right\} + 2\mathbb{E}\left\{ Q_i(t)d_i(t)d_k(t)^2 \right\}
\leq 4\sqrt{\mathbb{E}\left\{ Q_k(t)^2d_k(t)^2 \right\} \mathbb{E}\left\{ Q_i(t)^2d_i(t)^2 \right\}} + \sqrt{\mathbb{E}\left\{ d_k(t)^4 \right\} \mathbb{E}\left\{ d_i(t)^4 \right\}} + 2\sqrt{\mathbb{E}\left\{ Q_k(t)^2d_k(t)^2 \right\} \mathbb{E}\left\{ d_i(t)^4 \right\}} + 2\sqrt{\mathbb{E}\left\{ Q_i(t)^2d_i(t)^2 \right\} \mathbb{E}\left\{ d_k(t)^4 \right\}}
\]

Because \(\mathbb{E}\left\{ d_k(t)^4|Q(t)\right\} \leq D\) for all possible \(Q(t)\), we have from iterated expectations that for all \(k \in \{1, \ldots, K\}\)
\[
\mathbb{E}\left\{ d_k(t)^4 \right\} \leq D
\]

Further, for all \(k \in \{1, \ldots, K\}\) we have:
\[
\mathbb{E}\left\{ Q_k(t)^2d_k(t)^2 \right\} = \mathbb{E}\left\{ \mathbb{E}\left\{ Q_k(t)^2d_k(t)^2|Q(t)\right\} \right\} = \mathbb{E}\left\{ Q_k(t)^2 \mathbb{E}\left\{ d_k(t)^2|Q(t)\right\} \right\} \leq \mathbb{E}\left\{ Q_k(t)^2 \sqrt{\mathbb{E}\left\{ d_k(t)^4|Q(t)\right\}} \right\} \leq \mathbb{E}\left\{ Q_k(t)^2 \right\} \leq D\mathbb{E}\left\{ Q_{max}(t)^2 \right\}
\]

where we define \(Q_{max}(t)^2 \triangleq \max_{k \in \{1, \ldots, K\}} Q_k(t)^2\). Thus:
\[
\mathbb{E}\left\{ (2Q_k(t)d_k(t) + d_k(t)^2)(2Q_i(t)d_i(t) + d_i(t)^2) \right\} \leq 4D\mathbb{E}\left\{ Q_{max}(t)^2 \right\} + D + 4D\sqrt{\mathbb{E}\left\{ Q_{max}(t)^2 \right\}} \leq D_1\mathbb{E}\left\{ Q_{max}(t)^2 \right\} + D_2
\]

for some positive constants \(D_1, D_2\). Thus:
\[
\mathbb{E}\left\{ \delta(t)^2 \right\} \leq (D_1\mathbb{E}\left\{ Q_{max}(t)^2 \right\} + D_2) \sum_{k=1}^{K} \sum_{i=1}^{K} \frac{w_kw_i}{4} \leq D_3 + D_4 \sum_{k=1}^{K} \mathbb{E}\left\{ Q_k(t)^2 \right\}
\]

for some positive constants \(D_3, D_4\). Thus:
\[
\sum_{t=1}^{\infty} \frac{\mathbb{E}\left\{ \delta(t)^2 \right\}}{t^2} \leq \sum_{t=1}^{\infty} \frac{D_3}{t^2} + D_4 \sum_{k=1}^{K} \sum_{t=1}^{\infty} \frac{\mathbb{E}\left\{ Q_k(t)^2 \right\}}{t^2} < \infty \quad (77)
\]

Now fix any constants \(V, B, p^*, \epsilon\), and recall that \(\beta(t)\) is defined:
\[
\beta(t) \triangleq Vp(t) - B - Vp^* + \epsilon \sum_{k=1}^{K} |Q_k(t)|
\]

**Lemma 3:** Suppose that for all \(k \in \{1, \ldots, K\}\) we have:
\[
\sum_{t=1}^{\infty} \frac{\mathbb{E}\left\{ Q_k(t)^2 \right\}}{t} < \infty \quad (78)
\]
\[
\sum_{t=1}^{\infty} \frac{\mathbb{E}\left\{ p(t)^2 \right\}}{t^2} < \infty \quad (79)
\]
Then:
\[
\sum_{t=1}^{\infty} \frac{\mathbb{E} \{ \beta(t)^2 \}}{t^2} < \infty
\]

Note that Lemmas 2 and 3 together prove Claim 1. It remains only to prove Lemma 3.

**Proof:** (Lemma 3) We have:

\[
\mathbb{E} \{ \beta(t)^2 \} = \mathbb{E} \{ (Vp(t) - B - Vp^*)^2 \} + \epsilon^2 \sum_{k=1}^{K} \sum_{i=1}^{K} \mathbb{E} \{ |Q_k(t)||Q_i(t)| \}
\]

\[
+ \epsilon \sum_{k=1}^{K} \mathbb{E} \{ (Vp(t) - B - Vp^*)|Q_k(t)| \}
\]

\[
\leq \mathbb{E} \{ (Vp(t) - B - Vp^*)^2 \} + \epsilon^2 \sum_{k=1}^{K} \sum_{i=1}^{K} \sqrt{\mathbb{E} \{ Q_k(t)^2 \} \mathbb{E} \{ Q_i(t)^2 \}}
\]

\[
+ \epsilon \sum_{k=1}^{K} \sqrt{\mathbb{E} \{ (Vp(t) - B - Vp^*)^2 \} \mathbb{E} \{ Q_k(t)^2 \}}
\]

However, because \(|ab| \leq \frac{1}{2}[a^2 + b^2] \) for all real numbers \(a, b\), we have:

\[
\sqrt{\mathbb{E} \{ (Vp(t) - B - Vp^*)^2 \} \mathbb{E} \{ Q_k(t)^2 \}} \leq \frac{1}{2} \mathbb{E} \{ (Vp(t) - B - Vp^*)^2 \} + \frac{1}{2} \mathbb{E} \{ Q_k(t)^2 \}
\]

Thus:

\[
\mathbb{E} \{ \beta(t)^2 \} \leq \mathbb{E} \{ (Vp(t) - B - Vp^*)^2 \} + \epsilon^2 \sum_{k=1}^{K} \sum_{i=1}^{K} \sqrt{\mathbb{E} \{ Q_k(t)^2 \} \mathbb{E} \{ Q_i(t)^2 \}}
\]

\[
+ \frac{\epsilon}{2} \sum_{k=1}^{K} \mathbb{E} \{ (Vp(t) - B - Vp^*)^2 \} + \frac{\epsilon}{2} \sum_{k=1}^{K} \mathbb{E} \{ Q_k(t)^2 \}
\]

\[
\leq (1 + \frac{\epsilon K}{2}) \mathbb{E} \{ (Vp(t) - B - Vp^*)^2 \} + (\epsilon^2 K^2 + \epsilon K/2) \mathbb{E} \{ Q_{max}(t)^2 \}
\]

where we define \(Q_{max}(t)^2 = \max_{k \in \{1, \ldots, K\}} Q_k(t)^2\). It follows that there are finite constants \(D_1, D_2, D_3\) such that:

\[
\mathbb{E} \{ \beta(t)^2 \} \leq D_1 + D_2 \mathbb{E} \{ p(t)^2 \} + D_3 \mathbb{E} \{ Q_{max}(t)^2 \}
\]

Because \(Q_{max}(t)^2 \leq \sum_{k=1}^{K} Q_k(t)^2\), we have:

\[
\mathbb{E} \{ \beta(t)^2 \} \leq D_1 + D_2 \mathbb{E} \{ p(t)^2 \} + D_3 \sum_{k=1}^{K} \mathbb{E} \{ Q_k(t)^2 \}
\]

Thus, from (78)-(79) we have:

\[
\sum_{t=1}^{\infty} \frac{\mathbb{E} \{ \beta(t)^2 \}}{t^2} \leq \sum_{t=1}^{\infty} \frac{D_1}{t^2} + D_2 \sum_{t=1}^{\infty} \frac{\mathbb{E} \{ p(t)^2 \}}{t^2} + D_3 \sum_{k=1}^{K} \sum_{t=1}^{\infty} \frac{\mathbb{E} \{ Q_k(t)^2 \}}{t^2} < \infty
\]

which proves the result. \(\square\)
APPENDIX C — PROOF OF THEOREM 7 PARTS (A) AND (B)

Proof: (Theorem 7 part (a)) The proof closely follows a similar result derived for exponential Lyapunov functions with deterministically bounded queue changes in [26]. From (50) we have:

$$
\mathbb{E} \{ L(Q(t+1))|Q(t) \} \leq L(Q(t)) + \tilde{B} - \epsilon \sum_{k=1}^{K} |Q_k(t)|
$$

Therefore:

$$
\mathbb{E} \{ \|Q(t+1)\|^2 |Q(t)\} \leq \|Q(t)\|^2 + \tilde{B} - \epsilon \sum_{k=1}^{K} |Q_k(t)|
$$

$$
\leq \|Q(t)\|^2 + \tilde{B} - \frac{\epsilon}{\sqrt{w_{\text{max}}}} \sum_{k=1}^{K} \sqrt{w_k} |Q_k(t)|
$$

$$
\leq \|Q(t)\|^2 + \tilde{B} - \frac{\epsilon}{\sqrt{w_{\text{max}}}} \|Q(t)\|
$$

$$
= \|Q(t)\|^2 + \tilde{B} - 4c \|Q(t)\|
$$

where $w_{\text{max}} \triangleq \max_{k \in \{1, \ldots, K\}} w_k$ and $c \triangleq \sqrt{2}/(4\sqrt{w_{\text{max}}})$. The third inequality above follows by (38). Now suppose that $\|Q(t)\| \geq \tilde{B}/(2c)$. It follows that:

$$
\mathbb{E} \{ \|Q(t+1)\| |Q(t)\} \leq \|Q(t)\| + 2B - 2c \|Q(t)\| - 2c \|Q(t)\|
$$

$$
\leq \|Q(t)\| - 2c \|Q(t)\|
$$

$$
\leq \|Q(t)\| - 2c \|Q(t)\| + c^2
$$

$$
= (\|Q(t)\| - c)^2
$$

However, we have by Jensen’s inequality:

$$
\mathbb{E} \{ \|Q(t+1)\| |Q(t)\}^2 \leq \mathbb{E} \{ \|Q(t+1)\|^2 |Q(t)\}
$$

Therefore:

$$
\mathbb{E} \{ \|Q(t+1)\| |Q(t)\}^2 \leq (\|Q(t)\| - c)^2
$$

Assume now that $\|Q(t)\| \geq \max[\tilde{B}/(2c), c]$, so that we have both that $\|Q(t)\| - c \geq 0$ and $\|Q(t)\| \geq \tilde{B}/(2c)$. Taking square roots of the above inequality then proves that whenever $\|Q(t)\| \geq \max[\tilde{B}/(2c), c]$ we have:

$$
\mathbb{E} \{ \|Q(t+1)\| |Q(t)\} \leq \|Q(t)\| - c
$$

Defining $a \triangleq \max[\tilde{B}/(2c), c]$ proves part (a).

Proof: (Theorem 7 part (b)) We have $Q(t+1) = Q(t) + d(t)$, where $d(t) \triangleq (d_1(t), \ldots, d_K(t))$. Define $\gamma(t) \triangleq \|Q(t+1)\| - \|Q(t)\|$. Then $|\gamma(t)| \leq \|d(t)\|$ (by (39)), and we have:

$$
\|Q(t+1)\|^4 = \|Q(t)\|^4 + 4\|Q(t)\|^3 \gamma(t) + 6\|Q(t)\|^2 \gamma(t)^2 + 4\|Q(t)\| \gamma(t)^2 + \gamma(t)^4
$$

(80)

However, note by part (a) that $\mathbb{E} \{ \gamma(t)|Q(t)\} \leq -c$ whenever $\|Q(t)\| \geq a$ (for some constants $c > 0$, $a > 0$). Thus:

$$
4\|Q(t)\|^3 \mathbb{E} \{ \gamma(t)|Q(t)\} \leq \begin{cases} 
-4c\|Q(t)\|^3 & \text{if } \|Q(t)\| \geq a \\
4a^3 \mathbb{E} \{\|d(t)\||Q(t)\} & \text{otherwise}
\end{cases}
$$
Hence:

$$4||Q(t)||^3 \mathbb{E} \{ \gamma(t) | Q(t) \} \leq -4c||Q(t)||^3 + 4a^3 \mathbb{E} \{ ||d(t)||Q(t) \} + 4ca^3$$

Taking conditional expectations of (80) and substituting the above yields:

$$\mathbb{E} \{ ||Q(t+1)||^4 | Q(t) \} \leq ||Q(t)||^4 - 4c||Q(t)||^3 + 4a^3 \mathbb{E} \{ ||d(t)||Q(t) \} + 4ca^3$$

$$+ 6||Q(t)||^2 \mathbb{E} \{ ||d(t)||^2 | Q(t) \} +$$

$$4||Q(t)|| \mathbb{E} \{ ||d(t)||^3 | Q(t) \} + \mathbb{E} \{ ||d(t)||^4 | Q(t) \}$$

(81)

Because $||d(t)|| \leq g \sum_{k=1}^{K} |d_k(t)|$ (where $g = \sqrt{w_{max}/2}$, with $w_{max}=\max_{k \in \{1,\ldots,K\}} w_k$), we have:

$$\mathbb{E} \{ ||d(t)||^4 | Q(t) \} \leq g^4 \sum_{k=1}^{K} \sum_{j=1}^{K} \sum_{l=1}^{K} \mathbb{E} \{ |d_k(t)||d_i(t)||d_j(t)||d_l(t)||Q(t) \}$$

However, by repeated application of Cauchy-Schwartz and the fact that $\mathbb{E} \{ d_k(t)||Q(t) \} \leq D$, we have:

$$\mathbb{E} \{ |d_k(t)||d_i(t)||d_j(t)||d_l(t)||Q(t) \} \leq D$$

Thus:

$$\mathbb{E} \{ ||d(t)||^4 | Q(t) \} \leq g^4 K^4 D$$

(82)

Further, by Jensen’s inequality:

$$\mathbb{E} \{ ||d(t)||^3 | Q(t) \} \leq \mathbb{E} \{ ||d(t)||^4 | Q(t) \}^{3/4} \leq D^{3/4}$$

(83)

$$\mathbb{E} \{ ||d(t)||^2 | Q(t) \} \leq \mathbb{E} \{ ||d(t)||^4 | Q(t) \}^{1/2} \leq D^{1/2}$$

(84)

$$\mathbb{E} \{ ||d(t)|| | Q(t) \} \leq \mathbb{E} \{ ||d(t)||^4 | Q(t) \}^{1/4} \leq D^{1/4}$$

(85)

Substituting (82)–(85) into (81) yields:

$$\mathbb{E} \{ ||Q(t+1)||^4 | Q(t) \} - ||Q(t)||^4 \leq -4c||Q(t)||^3 + 4a^3 D^{1/4} + 4ca^3$$

$$+ 6||Q(t)||^2 D^{1/2} + 4||Q(t)|| D^{3/4} + D$$

(86)

Because the term $-4c||Q(t)||^3$ is the dominant term on the right-hand-side above (for $||Q(t)||$ large), there must be a constant $b_1 > 0$ such that:

$$-2c||Q(t)||^3 + 4a^3 D^{1/4} + 4ca^3 + 6||Q(t)||^2 D^{1/2} + 4||Q(t)|| D^{3/4} + D \leq 0$$

whenever $||Q(t)|| \geq b_1$. Thus, the right-hand-side of (86) is less than or equal to $-2c||Q(t)||^3$ whenever $||Q(t)|| \geq b_1$, and is less than or equal to $4a^3 D^{1/4} + 4ca^3 + 6b_1^2 D^{1/2} + 4b_1 D^{3/4} + D$ otherwise. It follows that there are constants $b_2 > 0$, $c > 0$ such that for all $t$ and all $Q(t)$ we have:

$$\mathbb{E} \{ ||Q(t+1)||^4 | Q(t) \} - ||Q(t)||^4 \leq b_2 - 2c||Q(t)||^3$$

Taking expectations of the above yields:

$$\mathbb{E} \{ ||Q(t+1)||^4 \} - \mathbb{E} \{ ||Q(t)||^4 \} \leq b_2 - 2c \mathbb{E} \{ ||Q(t)||^3 \}$$

Summing the above over $t \in \{0, \ldots, M-1\}$ and dividing by $M$ yields:

$$\frac{\mathbb{E} \{ ||Q(M)||^4 \} - \mathbb{E} \{ ||Q(0)||^4 \}}{M} \leq b_2 - 2c \frac{1}{M} \sum_{t=0}^{M-1} \mathbb{E} \{ ||Q(t)||^3 \}$$

Rearranging terms and using the fact that $||Q(M)||^4 \geq 0$ yields:

$$\frac{1}{M} \sum_{t=0}^{M-1} \mathbb{E} \{ ||Q(t)||^3 \} \leq \frac{b_2}{2c} + \frac{\mathbb{E} \{ ||Q(0)||^4 \}}{2cM} \leq \frac{b_2}{2c} + \frac{\mathbb{E} \{ ||Q(0)||^4 \}}{2c}$$

This completes the proof of part (b).
Lemma 4: Suppose \( \{x_i\}_{i=1}^{\infty} \) is an infinite sequence of non-negative real numbers such that there are constants \( C > 0 \) and \( 0 \leq \theta < 1 \) such that:

\[
\sum_{i=1}^{M} x_i \leq CM^{1+\theta} \ \forall M \in \{1, 2, 3, \ldots \}
\]

Then:

\[
\sum_{i=1}^{\infty} \frac{x_i}{M^2} < \infty
\]

Proof: For \( M \in \{1, 2, 3, \ldots \} \), define \( \phi(M) \) as:

\[
\phi(M) = \frac{1}{M^2} \sum_{i=1}^{M} x_i
\]

Then clearly:

\[
\phi(M) \leq \frac{C}{M^{1-\theta}} \ \forall M \in \{1, 2, 3, \ldots \} \quad (87)
\]

On the other hand, from the definition of \( \phi(M) \) we have for all \( M \in \{1, 2, 3, \ldots \} \):

\[
\phi(M + 1) = \frac{M^2}{(M + 1)^2} \phi(M) + \frac{x_{M+1}}{(M + 1)^2}
\]

So:

\[
\phi(M + 1) = \phi(M) \left[ 1 - \frac{2}{M + 1} + \frac{1}{(M+1)^2} \right] + \frac{x_{M+1}}{(M + 1)^2}
\]

Thus:

\[
\frac{x_{M+1}}{(M + 1)^2} = \phi(M + 1) - \phi(M) + \frac{2\phi(M)}{M + 1} - \frac{\phi(M)}{(M+1)^2}
\]

\[
\leq \phi(M + 1) - \phi(M) + \frac{2\phi(M)}{M + 1}
\]

where the final inequality holds because \( \phi(M) \geq 0 \). Summing the above over \( M \in \{1, \ldots, G\} \) for some positive integer \( G \) yields:

\[
\sum_{M=1}^{G} \frac{x_{M+1}}{(M + 1)^2} \leq \phi(G + 1) - \phi(1) + 2 \sum_{M=1}^{G} \frac{\phi(M)}{M + 1}
\]

Because \( \phi(1) = x_1 \), rearranging the above yields:

\[
\sum_{M=1}^{G+1} \frac{x_M}{M^2} \leq \phi(G + 1) + 2 \sum_{M=1}^{G} \frac{\phi(M)}{M + 1}
\]

\[
\leq \frac{C}{(G + 1)^{1-\theta}} + 2 \sum_{M=1}^{G} \frac{C}{M^{1-\theta}(M + 1)} \quad (88)
\]

where (88) follows from (87). Because \( \theta < 1 \), the first term on the right-hand-side of (88) goes to 0 as \( G \to \infty \), and the second term is a summable series and hence is less than a bounded constant as \( G \to \infty \). Thus:

\[
\lim_{G \to \infty} \sum_{M=1}^{G+1} \frac{x_M}{M^2} < \infty
\]

\[\square\]
APPENDIX E — PROOF OF THEOREM 3 ON RATE STABILITY

We prove Theorem 3 with the help of two preliminary lemmas. Let \( Q(t) \) be a non-negative stochastic process defined over \( t \in \{0, 1, 2, \ldots\} \). Fix \( \delta > 0 \), and for each non-negative integer \( n \) define \( t_n(\delta) \) by:

\[
t_n(\delta) \triangleq \lceil n^{1+\delta} \rceil
\]

where \( \lceil x \rceil \) represents the smallest integer greater than or equal to \( x \). The sequence \( \{t_n(\delta)\}_{n=0}^\infty \) is a (sparse) subsequence of the non-negative integers that increases super-linearly with \( n \). Lemma 5 below shows that if \( \mathbb{E}\{Q(t)^2\} \) grows at most linearly with \( t \), then \( Q(t) \) is rate stable when sampled over the subsequence \( \{t_n(\delta)\}_{n=0}^\infty \). We note that rate stability over this sparse sampling is not as strong as ordinary rate stability. This is because \( Q(t)/t \) may not converge to zero, even though it converges to 0 over the sparse sampling. However, Lemma 6 below shows that rate stability over the sparse sampling, together with an additional second moment bound on changes in \( Q(t) \), is sufficient to ensure ordinary rate stability.

**Lemma 5:** Suppose there is a finite constant \( C > 0 \) and a positive integer \( t^* \) such that:

\[
\mathbb{E}\{Q(t)^2\} \leq Ct \quad \forall t \geq t^*
\]

Then for any \( \delta > 0 \), \( Q(t) \) is rate stable when sampled over the subsequence of times \( \{t_n(\delta)\}_{n=0}^\infty \). That is:

\[
\lim_{n \to \infty} \frac{Q(t_n(\delta))}{t_n(\delta)} = 0 \quad (w.p.1)
\]

**Proof:** Fix \( \epsilon > 0 \). It suffices to show that:

\[
\lim_{M \to \infty} \mathbb{P}[\bigcup_{n \geq M} \{Q(t_n(\delta))/t_n(\delta) > \epsilon\}] = 0 \tag{90}
\]

To this end, note by the Markov inequality that for any slot \( t \geq t^* \):

\[
\mathbb{P}[Q(t)/t > \epsilon] = \mathbb{P}[Q(t)^2 > \epsilon^2 t^2] \leq \frac{\mathbb{E}\{Q(t)^2\}}{\epsilon^2 t^2} \leq \frac{C}{\epsilon^2 t^2}
\]

Substituting \( t = t_n(\delta) \) into the above inequality (assuming that \( t_n(\delta) \geq t^* \)) yields:

\[
\mathbb{P}[Q(t_n(\delta))/t_n(\delta) > \epsilon] \leq \frac{C}{\epsilon^2 t_n(\delta)} \leq \frac{C}{\epsilon^2 n^{1+\delta}}
\]

Therefore, by the union bound, we have for any positive integer \( M \) such that \( t_M(\delta) \geq t^* \):

\[
0 \leq \mathbb{P}[\bigcup_{n \geq M} \{Q(t_n(\delta))/t_n(\delta) > \epsilon\}] \leq \sum_{n=M}^\infty \mathbb{P}[Q(t_n(\delta))/t_n(\delta) > \epsilon] \\
\leq \sum_{n=M}^\infty \frac{C}{\epsilon^2 n^{1+\delta}} < \infty
\]

Thus, the probability on the left-hand-side of the above chain of inequalities is bounded by the tail of a convergent series, and so (90) holds.

**Lemma 6:** Suppose there is a finite constant \( C > 0 \) and a positive integer \( t^* \) such that:

\[
\mathbb{E}\{Q(t)^2\} \leq Ct \quad \forall t \geq t^*
\]

Further suppose there is a finite constant \( D > 0 \) such that for all \( t \in \{0, 1, 2, \ldots\} \) we have:

\[
\mathbb{E}\{(Q(t+1) - Q(t))^2\} \leq D
\]

Then \( Q(t) \) is rate stable.
Thus:

\[ \text{such that} \]

for all \( n \) such that \( t_n \) is defined in (89). For simplicity of notation, below we write “\( t_n \)” in replacement for “\( t_n(\delta) \).” Thus, \( t_n \geq \lceil n^{(1+\delta)} \rceil \), and:

\[
\lim_{n \to \infty} \frac{Q(t_n)}{t_n} = 0 \quad (w.p.1)
\]

Now note by the Markov inequality that for all \( t \geq 0 \):

\[
Pr[|Q(t+1) - Q(t)| \geq t^{3/4}] = Pr[(Q(t+1) - Q(t))^2 \geq t^{3/2}] \leq \frac{D}{t^{3/2}}
\]

Thus, for any integer \( M > 0 \):

\[
Pr[\bigcup_{t \geq M} \{|Q(t+1) - Q(t)| \geq t^{3/4}\}] \leq \sum_{t=M}^{\infty} \frac{D}{t^{3/2}} < \infty
\]

Thus:

\[
\lim_{M \to \infty} Pr[\bigcup_{t \geq M} \{|Q(t+1) - Q(t)| \geq t^{3/4}\}] = 0
\]

It follows that, with probability 1, there is some positive random integer \( K \) such that \( |Q(t+1) - Q(t)| < t^{3/4} \) for all \( t \geq K \).

Now for any integer \( t > 0 \), define \( n(t) \) as the integer such that \( t_n(t) \leq t < t_n(t)+1 \). Then for any \( t > 0 \) such that \( t_n(t) \geq K \), we have:

\[
Q(t) \leq Q(t_n(t)) + [t_n(t)+1 - t_n(t)]^{3/4}_{n(t)+1} \\
\leq Q(t_n(t)) + [t_n(t)+1 - t_n(t)][(n(t) + 1)^{(3/4)(1+\delta)} + 1]
\]

Thus:

\[
\frac{Q(t)}{t} \leq \frac{Q(t_n(t)) + [t_n(t)+1 - t_n(t)][(n(t) + 1)^{(3/4)(1+\delta)} + 1]}{t_n(t)}
\]  

(91)

On the other hand, for any \( n > 0 \) we have by a Taylor expansion

\[
t_{n+1} \leq 1 + (n + 1)^{1+\delta} \\
\leq 1 + n^{1+\delta} + (1 + \delta)n^\delta + \frac{(1 + \delta)^2}{2}n^{\delta-1} \\
\leq a + n^{1+\delta} + (1 + \delta)n^\delta
\]

where \( a \geq 1 + (1 + \delta)^{\delta/2} \). Thus, for any \( n(t) > 0 \) we have:

\[
t_n(t)+1 - t_n(t) \leq t_n(t)+1 - n(t)^{1+\delta} \leq a + (1 + \delta)n(t)^\delta
\]

Using this in (91) yields:

\[
0 \leq \frac{Q(t)}{t} \leq \frac{Q(t_n(t)) + [a + (1 + \delta)n(t)^\delta][(n(t) + 1)^{(3/4)(1+\delta)} + 1]}{t_n(t)}
\]  

(92)

\[
\leq \frac{Q(t_n(t)) + a[(n(t) + 1)^{(3/4)(1+\delta)} + 1] + (1 + \delta)n(t)^\delta[(n(t) + 1)^{(3/4)(1+\delta)} + 1]}{t_n(t)^{1+\delta}}
\]  

(93)

Taking limits and using the fact that \( Q(t_n(t))/t_n(t) \to 0 \) with probability 1, and the fact that \( \delta + (3/4)(1 + \delta) < 1 \), yields:

\[
0 \leq \lim_{t \to \infty} \frac{Q(t)}{t} \leq 0 \quad (w.p.1)
\]
We now prove Theorem 3. Let \( Q(t) = (Q_1(t), \ldots, Q_K(t)) \) be a stochastic vector defined over \( t \in \{0, 1, 2, \ldots\} \). Assume \( Q(t) \) has real-valued entries. Define the quadratic Lyapunov function \( L(Q(t)) \) as in (11) and define the drift \( \Delta(H(t)) \) as in (12). Suppose there is a finite constant \( B > 0 \) such that for all \( \tau \in \{0, 1, 2, \ldots\} \) and all possible \( H(t) \), we have:

\[
\Delta(H(t)) \leq B
\]  

(94)

**Proof:** (Theorem 3 part (a)) Assume that \( \mathbb{E} \{L(Q(0))\} < \infty \). Fix a slot \( \tau \geq 0 \). Taking expectations of (94) yields:

\[
\mathbb{E} \{L(Q(\tau + 1))\} - \mathbb{E} \{L(Q(\tau))\} \leq B
\]

Summing the above over \( \tau \in \{0, 1, \ldots, t - 1\} \) for some integer \( t > 0 \) yields:

\[
\mathbb{E} \{L(Q(t))\} - \mathbb{E} \{L(Q(0))\} \leq Bt
\]

(95)

Substituting the definition of \( L(Q(t)) \) in (11) into the above inequality yields:

\[
\frac{1}{2} \sum_{k=1}^{K} w_k \mathbb{E} \{|Q_k(t)|^2\} \leq Bt + \mathbb{E} \{L(Q(0))\}
\]

(96)

It follows from (95) that for each \( k \in \{1, \ldots, K\} \):

\[
\mathbb{E} \{|Q_k(t)|\}^2 \leq \mathbb{E} \{Q_k(t)^2\} \leq \frac{2Bt + 2\mathbb{E} \{L(Q(0))\}}{w_k}
\]

and so:

\[
\mathbb{E} \{|Q_k(t)|\} \leq \sqrt{\frac{2Bt}{w_k} + \frac{2\mathbb{E} \{L(Q(0))\}}{w_k}}
\]

Dividing the above by \( t \) and taking limits as \( t \to \infty \) shows that \( Q_k(t) \) is mean rate stable, proving part (a).

**Proof:** (Theorem 3 part (b)) First assume that \( Q(0) \) is a given finite constant (with probability 1), so that \( \mathbb{E} \{L(Q(0))\} = L(Q(0)) \). We have from (96) that for all \( t \geq 1 \) and all \( k \in \{1, \ldots, K\} \):

\[
\mathbb{E} \{Q_k(t)^2\} \leq \frac{2B + 2L(Q(0))}{w_k} t
\]

Furthermore, it can be shown that \( \mathbb{E} \{(Q_k(t+1) - Q_k(t))^2\} \leq D \) implies \( \mathbb{E} \{(|Q_k(t+1)| - |Q_k(t)|)^2\} \leq D \). Thus, the conditions required to apply Lemma 6 hold (using \( Q(t) = |Q_k(t)| \), \( t^* = 1 \) and \( C = [2B + 2L(Q(0))] / w_k \)). Then Lemma 6 ensures \( |Q_k(t)| \) is rate stable for all \( k \in \{1, \ldots, K\} \), and hence \( Q_k(t) \) is rate stable for all \( k \in \{1, \ldots, K\} \). The above holds whenever the initial condition \( Q(0) \) is any given finite constant, and hence it holds whenever \( Q(0) \) is finite with probability 1. \( \square \)

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