Weyl-Wigner-Moyal Formalism for Fermi Classical Systems

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Abstract

The Weyl-Wigner-Moyal formalism of fermionic classical systems with a finite number of degrees of freedom is considered. This correspondence is studied by computing the relevant Stratonovich-Weyl quantizer. The Moyal $\star$-product, Wigner functions and normal ordering are obtained for generic fermionic systems. Finally, this formalism is used to perform the deformation quantization of the Fermi oscillator and the supersymmetric quantum mechanics.

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I. INTRODUCTION

The formalism introduced by H. Weyl [1], E.P. Wigner [2], A. Groenewold [3] and J.E. Moyal [4] (or WWM formalism), establishes an isomorphism between the Heisenberg operator algebra and the corresponding algebra of symbols of these operators through the so called WWM correspondence. The operator product is mapped to an associative and noncommutative product called the Moyal $\star$-product. Eventually, the theory of deformation quantization (from which the WWM correspondence is an example) was introduced in 1978 by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer (BFFLS) [5] (for some recent reviews, see [6, 7, 8]). In BFFLS paper, deformation quantization was introduced as an alternative procedure to the canonical quantization and the path integral quantization in quantum mechanics [5]. Similarly to the path integral formalism, deformation quantization uses the algebraic structure of classical systems instead of operator theory.

This formalism has been employed also to quantize several physical systems from particles to strings and recently used in the quantization of dissipative systems [9]. The philosophy of deformation quantization is based on the fact that quantization of classical systems can be regarded equivalently as deformations of the algebraic structures associated with these classical systems [10]. Recently, WWM formalism has received a great deal of interest motivated by the fact that it is contained inside string theory. The presence of a non-zero constant $B$-field on the worldvolume of a D-brane deforms the product of functions (or classical fields) on the D-brane, and the ordinary effective gauge theory on the brane turns out into a noncommutative field theory with the usual Moyal $\star$-product [11].

In deformation quantization, the quantization is understood as a deformation of the algebra of classical observables and not as a radical change of the mathematical nature of them. The ordinary product of functions is then deformed into the Moyal $\star$-product and the Poisson brackets turn into the Moyal brackets. More generally by explicit construction $\star$-product has been proved to exist for any finite dimensional symplectic manifold [12, 13]. One of the remarkable works is that of Fedosov [13], in where he was able to find a general star product for any symplectic manifold with symplectic geometry. Recently the geometric origin of the Fedosov’s star product, was elucidated in [14]). Then in 1997 Kontsevich [15] proved the existence of the star product for any Poisson manifold, and the explicit construction was done for $\mathbb{R}^n$.

Deformation quantization formalism has a firm mathematical basis, however its application to the quantization of arbitrary physical systems presents still great challenges (see [8]). Most of the cases studied by the deformation quantization for systems with a finite number of degrees of freedom deal with bosonic variables. However the analysis of some classical physical systems requires the description of fermionic degrees of freedom which involve Grassmann variables. These systems has been discussed in the literature for some years [16, 17, 18, 19, 20, 21]. The canonical quantization of these systems by
using fermionic quantum mechanics was studied in \cite{22}. Very recently some authors started to applied
the deformation quantization for these classical fermionic systems \cite{23, 24, 25, 26, 27, 28}. In particular in
Refs. \cite{27, 28} the authors show concretely how the deformation quantization program can be carried over
to specific physical systems. The same techniques have been applied recently for the noncommutative
superspace \cite{29}.

In the present paper we study another approach to deformation quantization for fermionic systems
by employing the Weyl-Wigner-Moyal correspondence \cite{30, 31, 32}. We find that the super-Hilbert space
structure gives some similar formulas to the ones obtained within the WWM formalism for the bosonic
case. Moreover we will be able to obtain some of the proposed basic formulas of the deformation quan-
tization formalism described in Ref. \cite{27} and generalize them. In this respect, the present paper is
complementary and provides some support to Ref. \cite{27}.

Our paper is organized as follows. In sec. II we give general prescriptions of quantum mechanical
systems of fermions described by $2n$ (odd) Grassmann variables. Sec. III is devoted to construct the
Stratonovich-Weyl (SW) quantizer which is the main object to determine the Weyl correspondence be-
tween operators and functions on the fermionic phase space. Then the SW quantizer for Fermi systems
is defined and its main properties are found. Using a modified notion of the trace of an operator we show
that these properties have a similar form as the respective properties in the bosonic case. The Moyal
$\star$-product is found in sec. IV and in sec. V the Wigner function for Fermi systems is also obtained. Sec.
VI is devoted to study the normal ordering for generic fermionic systems. In sec. VII we discuss in detail
two examples: the Fermi oscillator and the supersymmetric harmonic oscillator. Finally, in sec. VIII,
some concluding remarks close the paper.

II. PRELIMINARIES OF QUANTUM MECHANICS FOR FERMIONIC SYSTEMS

In this section we review some important facts concerning the quantization of fermionic classical
systems with a finite number of degrees of freedom. Our aim is not to provide an extensive review but
briefly recall some of their relevant properties. We refer the reader to Weinberg’s book \cite{33} for details.

Let $\psi = (\psi_1, \cdots, \psi_n)$ and $\pi = (\pi_1, \cdots, \pi_n)$ be complex Grassmann coordinates on the phase space
$\Gamma_{2n}^F$ of the relevant purely fermionic classical system. The momenta $\pi_j$ are given by
\begin{equation}
\pi_j = i\psi^*_j, \quad j = 1, \cdots, n.
\end{equation}
(In what follows $j, k, l$ run from 1 to $n$.)
Quantization establishes the rules

\[ [\hat{\psi}_j, \hat{\pi}_k]_+ = i\hbar\delta_{jk}, \]
\[ [\hat{\psi}_j, \hat{\psi}_k]_+ = 0 = [\hat{\pi}_j, \hat{\pi}_k]_+, \]
\[ [\hat{\psi}_j, \hat{\psi}^*_k]_+ = \hbar\delta_{jk}, \]

where \([\cdot, \cdot]_+\) stands for the anticommutator and \([\cdot, \cdot]_-\) defines the commutator, i.e. \([\hat{A}, \hat{B}]_\pm := \hat{A}\cdot\hat{B} \pm \hat{B}\cdot\hat{A}\).

Define also

\[ \hat{b}_j := \frac{\hat{\psi}_j}{\sqrt{\hbar}}, \quad \hat{b}^*_j := \frac{\hat{\psi}^*_j}{\sqrt{\hbar}}. \]  

There exists a vacuum state given by the ket vector \(|0\rangle\) or a bra vector \langle 0| defined by

\[ \hat{b}_j|0\rangle = 0, \quad \langle 0|\hat{b}^*_j = 0, \quad \forall j, \]

satisfying the normalization condition \(\langle 0|0\rangle = 1\). The basis of all states can be constructed from excitations of the vacuum state \(|0\rangle\) and they are given by

\[ |j, k, l, \cdots\rangle := \hat{b}^*_j\hat{b}^*_k\hat{b}^*_l\cdots|0\rangle. \]

Then if \(l \notin \{j, k, \cdots\}\) one gets

\[ \hat{b}_l|j, k, \cdots\rangle = 0, \quad \hat{b}^*_l|j, k, \cdots\rangle = |l, j, k, \cdots\rangle. \]

Moreover

\[ \hat{b}_l|j, k, \cdots\rangle = |j, k, \cdots\rangle, \quad \hat{b}^*_l|j, k, \cdots\rangle = 0. \]

The dual basis reads

\[ \langle j, k, l, \cdots | := \langle 0|\cdots\hat{b}_l\hat{b}_k\hat{b}_j. \]

One quickly finds that

\[ \langle j_1, k_1, l_1, \cdots | j_2, k_2, l_2, \cdots \rangle = \begin{cases} 0 & \text{if } \{j_1, k_1, l_1, \cdots\} \neq \{j_2, k_2, l_2, \cdots\} \\ 1 & \text{if } j_1 = j_2, k_1 = k_2, l_1 = l_2, \cdots \end{cases} \]

Now we look for the state \(|\psi_1, \cdots, \psi_n\rangle \equiv |\psi\rangle\) satisfying the following condition

\[ \hat{\psi}_j|\psi\rangle = \psi_j|\psi\rangle, \quad \forall j. \]

It is an easy matter to show that \(|\psi\rangle\) has the following form

\[ |\psi\rangle = \exp\left\{-\frac{i}{\hbar} \sum_{j=1}^n \hat{\pi}_j\psi_j\right\}|0\rangle. \]
Indeed (10)

\[ \hat{\psi}_k |\psi\rangle = \hat{\psi}_k \exp \left\{ - \frac{i}{\hbar} \pi_k \psi_k \right\} \cdot \exp \left\{ - \frac{i}{\hbar} \sum_{j \neq k} \pi_j \psi_j \right\} |0\rangle = \psi_k \exp \left\{ - \frac{i}{\hbar} \sum_{j \neq k} \pi_j \psi_j \right\} |0\rangle = \psi_k |\psi\rangle. \] (12)

In order to obtain the above result we have used the facts that \( \hat{\psi}_k |0\rangle = 0 \) and

\[ \hat{\psi}_k \exp \left\{ - \frac{i}{\hbar} n \sum_{j \neq k} \pi_j \psi_j \right\} \hat{\psi}_k, \psi_k \exp \left\{ - \frac{i}{\hbar} \sum_{j \neq k} \pi_j \psi_j \right\} = \psi_k \exp \left\{ - \frac{i}{\hbar} \sum_{j \neq k} \pi_j \psi_j \right\}, \] (13)

Thus Eq. (11) holds true.

Then it is not difficult to get the crucial formula

\[ \exp \left\{ - \frac{i}{\hbar} \sum_{j=1}^n \pi_j \xi_j \right\} |\psi\rangle = |\psi + \xi\rangle, \] (14)

which has the same form as the corresponding formula for bosonic degrees of freedom \[30, 31, 32\], but in the present case the order of \( \pi_j \xi_j \) should be respected.

Define

\[ \langle \psi \rangle := \langle 0 | \hat{\psi}_1 \cdot \cdot \cdot \hat{\psi}_n \exp \left\{ - \frac{i}{\hbar} \sum_{j=1}^n \psi_j \pi_j \right\} = \langle 0 | (\prod_{j=1}^n \hat{\psi}_j) \exp \left\{ i \frac{\pi}{n} \sum_{j=1}^n \pi_j \psi_j \right\}. \] (15)

The ordering of \( \hat{\psi}_j \) can be arbitrary chosen but then must be fixed. We choose the ordering \( \hat{\psi}_1 \cdot \cdot \cdot \hat{\psi}_n \). One finds (compare with (12)):

\[ \langle \psi \rangle \hat{\psi}_j = \langle \psi \rangle |\psi\rangle, \quad \forall j. \] (16)

Comparing Eqs. (11) and (15) one can see that in contrary to the bosonic case we have, \( \langle \psi \rangle \neq (|\psi\rangle)^* \).

We note also the following important point: The ket \( |\psi\rangle \), by its very definition (11), commutes with all Grassmann numbers \( \eta \), i.e., \( \eta |\psi\rangle = |\psi\rangle \eta \). But for the bra \( \langle \psi \rangle \) defined by (15) one has

\[ \eta \langle \psi \rangle = (-1)^{\varepsilon_\eta \cdot n} \langle \psi | \eta, \] (17)

where \( \varepsilon_\eta = 1 \) for odd Grassmann numbers or \( \varepsilon_\eta = 0 \) for even Grassmann numbers. The inner product \( \langle \psi' | \psi \rangle \), after some minor computations, is found to read

\[ \langle \psi' | \psi \rangle = \langle 0 | \hat{\psi}_1 \cdot \cdot \cdot \hat{\psi}_n \exp \left\{ \frac{i}{\pi} \sum_{j=1}^n \pi_j (\psi'_j - \psi_j) \right\} |0\rangle = \prod_{j=1}^n (\psi_j - \psi'_j) =: \delta(\psi - \psi'). \] (18)
The analogous procedure can be performed for $|\pi\rangle$. We look for $|\pi\rangle$ and $\langle \pi |$ such that

$$
\hat{\pi}_j |\pi\rangle = \pi_j |\pi\rangle,
\langle \pi | \hat{\pi}_j = \langle \pi | \pi_j, \quad \forall j.
$$

(19)

It is an easy matter to show that

$$
|\pi\rangle = \exp \left\{ - \frac{i}{\hbar} \sum_{j=1}^{n} \hat{\psi}_j \pi_j \right\} \prod_{j=1}^{n} \hat{\pi}_j |0\rangle
$$

and

$$
\langle \pi | = \langle 0 | \exp \left\{ - \frac{i}{\hbar} \sum_{j=1}^{n} \pi_j \hat{\psi}_j \right\} = \langle 0 | \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{n} \pi_j \hat{\psi}_j \right\}.
$$

(20)

(21)

Of course $\langle \pi | \neq (|\pi\rangle)^\ast$. Then the inner product $\langle \pi' | \pi \rangle$ reads

$$
\langle \pi' | \pi \rangle = \prod_{j=1}^{n} (\pi'_j - \pi_j) = \delta(\pi' - \pi).
$$

(22)

(Compare the formulas (18) and (22)).

From Eqs. (11) and (21) we get

$$
\langle \pi | \psi \rangle = \exp \left\{ - \frac{i}{\hbar} \sum_{j=1}^{n} \pi_j \psi_j \right\}.
$$

(23)

In a similar way using Eqs. (15) and (20) one obtains

$$
\langle \psi | \pi \rangle = (i^n \hbar)^n \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{n} \pi_j \psi_j \right\}.
$$

(24)

In this paper we use the following convention for integrals

$$
\int \psi_j d\psi_j = - \int d\psi_j \psi_j = 1, \quad \int \pi_j d\pi_j = - \int d\pi_j \pi_j = 1, \quad \forall j.
$$

(25)

This yields

$$
\int |\psi\rangle D\psi \langle \psi | = 1,
$$

(26)

where $D\psi = d\psi_n \cdots d\psi_1$.

[Note the ordering. In Ref. [33] $D\psi$ would be denoted by $\prod_{j=1}^{n} d\psi_j$. For instance in [33], Eq. (26) takes the form $\int |\psi\rangle((-1)^n D\psi)\langle \psi| = 1$ but the convention for integrals used in Weinberg’s book is $\int d\psi_j \psi'_j = - \int \psi_j d\psi'_j = 1$.]

Analogously one proves that

$$
\int |\pi\rangle(-1)^n D\pi \langle \pi | = 1.
$$

(27)

After this brief survey we are ready to implement the WWM formalism. Let’s start with the Stratonovich-Weyl quantizer.
III. THE STRATONOVICH-WEYL QUANTIZER

Let \( f = f(\pi, \psi) \) be a classical observable on the Grassmann phase space \( \Gamma_{F}^{2n} \). The Fourier transform of \( f \) is defined by

\[
\tilde{f}(\lambda, \mu) := \int f(\pi, \psi) \exp\left\{ -i \sum_{j=1}^{n} (\pi_j \lambda_j + \psi_j \mu_j) \right\} \prod d\pi d\psi,
\]

where \( \prod d\pi d\psi := d\pi_1 d\psi_1 \cdots d\pi_n d\psi_n \).

Then it is an easy matter to show that

\[
f(\pi, \psi) := \int \tilde{f}(\lambda, \mu) \exp\left\{ i \sum_{j=1}^{n} (\pi_j \lambda_j + \psi_j \mu_j) \right\} \prod d\lambda d\mu.
\]

The Fourier transform of \( f(\pi, \psi) = 1 \) reads

\[
\tilde{1}(\lambda, \mu) = \int \exp\left\{ -i \sum_{j=1}^{n} (\pi_j \lambda_j + \psi_j \mu_j) \right\} \prod d\pi d\psi
\]

\[
= \int \mu_1 \psi_1 \pi_1 \cdots \mu_n \psi_n \pi_n \prod d\pi d\psi
\]

\[
= \mu_1 \psi_1 \cdots \mu_n \psi_n = \delta(\mu, \lambda),
\]

where \( \delta(\mu, \lambda) := (-1)^{\frac{n(n-1)}{2}} \delta(\mu)\delta(\lambda) \). Thus we have

\[
f(\lambda, \mu) = \int f(\lambda', \mu') \delta(\mu - \mu', \lambda - \lambda') \prod d\lambda' d\mu'.
\]

Now we are at the position to consider the Stratonovich-Weyl quantizer.

Let \( f = f(\pi, \psi) \) be a smooth function on \( \Gamma_{F}^{2n} \). Then the Weyl quantization rule is given by

\[
\hat{f} := \int \tilde{f}(\lambda, \mu) \exp\left\{ i \sum_{j=1}^{n} (\hat{\pi}_j \lambda_j + \hat{\psi}_j \mu_j) \right\} \prod d\lambda d\mu,
\]

where \( \tilde{f}(\lambda, \mu) \) is defined by Eq. (28). In another form

\[
\hat{f} = \int f(\pi, \psi) \hat{\Omega}(\pi, \psi) \prod d\pi d\psi,
\]

where

\[
\hat{\Omega}(\pi, \psi) = \int \exp\left\{ i \sum_{j=1}^{n} \left[ (\hat{\pi}_j - \pi_j) \lambda_j + (\hat{\psi}_j - \psi_j) \mu_j \right] \right\} \prod d\lambda d\mu,
\]

is called the Stratonovich-Weyl (SW) quantizer (compare with Ref. [19]).

The SW quantizer can be rewritten in the following form

\[
\hat{\Omega}(\pi, \psi) = \int \exp\left\{ -i \sum_{j=1}^{n} (\pi_j \lambda_j + \psi_j \mu_j) \right\} \exp\left\{ i \sum_{j=1}^{n} \hat{\pi}_j \lambda_j \right\}
\]

\[
= \exp\left\{ -i \sum_{j=1}^{n} \pi_j \lambda_j \right\} \exp\left\{ i \sum_{j=1}^{n} \hat{\pi}_j \lambda_j \right\} \prod d\lambda d\mu,
\]
\[ \times \exp \left\{ i \sum_{j=1}^{n} \hat{\psi}_j \mu_j \right\} \exp \left\{ -\frac{i\hbar}{2} \sum_{j=1}^{n} \lambda_j \mu_j \right\} \prod d\lambda d\mu \]

\[ = \int \exp \left\{ -i \sum_{j=1}^{n} (\pi_j \lambda_j + \psi_j \mu_j) \right\} \exp \left\{ i \sum_{j=1}^{n} \hat{\psi}_j \mu_j \right\} \exp \left\{ i \sum_{j=1}^{n} \hat{\pi}_j \lambda_j \right\} \exp \left\{ \frac{i\hbar}{2} \sum_{j=1}^{n} \lambda_j \mu_j \right\} \prod d\lambda d\mu. \tag{34} \]

Using Eqs. (14) and (20) we reexpress the above equation in the form

\[ \hat{\Omega}(\pi, \psi) = \int \exp \left\{ -i \sum_{j=1}^{n} \pi_j \lambda_j \right\} i^n \left[ \psi_1 - (\psi_1 + \frac{\hbar \lambda_1}{2}) \right] \cdots \left[ \psi_n - (\psi_n + \frac{\hbar \lambda_n}{2}) \right] \]

\[ \times (-1)^{\frac{n(n-1)}{2}} \cdot (-1)^{\frac{n(n+1)}{2}} \mu_1 \cdots \mu_n d\mu_1 \cdots d\mu_n d\lambda_n \cdots d\lambda_1 |\psi' - h\lambda\rangle D\psi' \langle \psi'| \]

\[ = \int D\lambda \exp \left\{ -i \sum_{j=1}^{n} \pi_j \lambda_j \right\} \left[ (\psi_1 + \frac{\hbar \lambda_1}{2}) - \psi' \right] \cdots \left[ (\psi_n + \frac{\hbar \lambda_n}{2}) - \psi' \right] d\psi' \cdots d\psi' (i)^n |\psi' - h\lambda\rangle \langle \psi'| \]

\[ = i^n \int D\lambda \exp \left\{ -i \sum_{j=1}^{n} \pi_j \lambda_j \right\} |\psi - \frac{\hbar \lambda}{2}\rangle \langle \psi + \frac{\hbar \lambda}{2}|, \tag{35} \]

where \( D\lambda := d\lambda_n \cdots d\lambda_1 \). Thus we arrive at a result very similar to that of bosonic case.

Analogously we have from Eqs. (20), (27) and (34) that

\[ \hat{\Omega}(\pi, \psi) = (-1)^n \int \hat{\Omega}(\pi, \psi)|\pi'\rangle D\pi' \langle \pi'| \]

\[ = (-i)^n \int D\mu \exp \left\{ -i \sum_{j=1}^{n} \psi_j \mu_j \right\} |\pi - \frac{\hbar \mu}{2}\rangle \langle \pi + \frac{\hbar \mu}{2}|, \tag{36} \]

where \( D\mu := d\mu_n \cdots d\mu_1 \). The formulas (35) and (36) are similar to the corresponding formulas for the bosonic case [30, 31, 32].

Define the following mapping ‘tr’ which will be called the “trace” as

\[ \text{tr}\{ \hat{A} \} := c \int D\psi \langle \psi | \hat{A} |\psi\rangle, \tag{37} \]

where \( c \) is to be determined from the condition

\[ \text{tr}\{ \hat{\Omega}(\pi, \psi) \} = 1. \tag{38} \]

By using Eqs. (35), (37) and (38) one finds that \( c = (i\hbar)^{-n} \). Finally, the “trace” of an operator \( \hat{A} \) is

\[ \text{tr}\{ \hat{A} \} = (i\hbar)^{-n} \int D\psi \langle \psi | \hat{A} |\psi\rangle. \tag{39} \]

In terms of the \( \langle \pi\rangle \)-representation one has

\[ \text{tr}\{ \hat{A} \} = (i\hbar)^{-n} \int D\pi \langle \pi | \hat{A} |\pi\rangle. \tag{40} \]

With the above definition, Eq. (38) is also satisfied. In addition, with the aid of Eq. (35) and after laborious computations we find the following important property of the Stratonovich-Weyl quantizer

\[ \text{tr}\left\{ \hat{\Omega}(\pi', \psi')\hat{\Omega}(\pi'', \psi'') \right\} = (\psi'_1 - \psi''_1)(\pi'_1 - \pi''_1) \cdots (\psi'_n - \psi''_n)(\pi'_n - \pi''_n) \]

\[ \times \exp \left\{ i \sum_{j=1}^{n} \hat{\psi}_j \mu_j \right\} \exp \left\{ -\frac{i\hbar}{2} \sum_{j=1}^{n} \lambda_j \mu_j \right\} \prod d\lambda d\mu. \]
\[ \delta(\psi' - \psi'', \pi' - \pi''). \] (41)

This is a crucial formula which leads to the realization of the Weyl correspondence in the case of classical systems of fermions. We consider the Weyl correspondence \( \hat{f} = W^{-1}(f) \) as given by

\[ \hat{f} = \int f(\pi', \psi') \hat{\Omega}(\pi', \psi') \prod \, d\pi' d\psi'. \] (42)

Then, multiplying this equation by \( \hat{\Omega}(\pi, \psi) \), taking its “trace” and using Eq. (41) we obtain

\[ \text{tr}\{ \hat{f} \hat{\Omega}(\pi, \psi) \} = \int \text{tr}\{ \hat{\Omega}(\pi', \psi') \hat{\Omega}(\pi, \psi) \} \prod \, d\pi' d\psi' f(\pi', \psi') \]

\[ = \int \delta(\psi' - \psi, \pi' - \pi) \prod \, d\pi' d\psi' f(\pi', \psi') = f(\pi, \psi). \]

Finally

\[ f(\pi, \psi) = \text{tr}\{ \hat{\Omega}(\pi, \psi) \} \] (43)

and this is exactly the same expression as in the bosonic case.

[ Remark: Similarly to the bosonic case one introduces the oscillator variables \( Q_j \) and \( P_j \)

\[ Q_j := \frac{1}{\sqrt{2}}(\psi_j - i\pi_j) = \frac{1}{\sqrt{2}}(\psi_j + \psi_j^*), \]

\[ P_j := \frac{1}{\sqrt{2}}(\pi_j - i\psi_j) = \frac{i}{\sqrt{2}}(\psi_j^* - \psi_j), \] (44)

which form \( 2n \) real coordinates of \( \Gamma^{2n}_{F_2} \). From Eqs. (25) and (44) we find the integrals

\[ \int Q_j dQ_j = - \int dQ_j Q_j = 1, \quad \int P_j dP_j = - \int dP_j P_j = 1, \quad \forall j, \]

\[ \prod \, d\pi d\psi = \prod \, dPdQ. \] (45)

One can rewrite all the formulas in terms of \( Q, P \). In particular the Fourier integral is given by

\[ \tilde{f}(\alpha, \beta) = \int f(P, Q) \exp\left\{ -i \sum_{j=1}^{n} (P_j \alpha_j + Q_j \beta_j) \right\} \prod \, dPdQ, \] (46)

where

\[ f(P, Q) = \int \tilde{f}(\alpha, \beta) \exp\left\{ i \sum_{j=1}^{n} (P_j \alpha_j + Q_j \beta_j) \right\} \prod \, d\alpha d\beta. \] (47)

While the Weyl correspondence in these variables reads

\[ \hat{f} = \int f(\pi(P, Q), \psi(P, Q)) \hat{\Omega}(\pi(P, Q), \psi(P, Q)) \prod \, dPdQ \] (48)

and

\[ f(P, Q) = \text{tr}\{ \hat{\Omega}(P, Q) \}. \] (49)

Some of the above formulas will be used in Sec. VI. ]
Consider now \( \text{tr}\{\hat{\Omega}(\pi', \psi')\hat{\Omega}(\pi'', \psi'')\hat{\Omega}(\pi, \psi)\} \). As will be seen in the next section this is the main point to find the Moyal product. First
\[
\hat{\Omega}(\pi', \psi')\hat{\Omega}(\pi'', \psi'')\hat{\Omega}(\pi, \psi) = \frac{i\hbar}{2} \int \text{D}\psi'' \exp \left\{-\frac{2i}{\hbar} \left(\pi'\lambda' + \pi''\lambda'' + \pi\lambda\right)\right\} |\psi' - \lambda'\rangle |\psi'' + \lambda''\rangle |\psi + \lambda\rangle |\mathcal{D}\lambda|\mathcal{D}\lambda'\mathcal{D}\lambda''\mathcal{D}\lambda.
\]

Then
\[
\text{tr}\left\{\hat{\Omega}(\pi', \psi')\hat{\Omega}(\pi'', \psi'')\hat{\Omega}(\pi, \psi)\right\} = (i\hbar)^{-n} \left(\frac{i\hbar}{2}\right)^{3n} \int \text{D}\psi'' \exp \left\{-\frac{2i}{\hbar} \left(\pi'\lambda' + \pi''\lambda'' + \pi\lambda\right)\right\} |\psi''|\psi'\rangle \langle \psi' + \lambda'|\psi'' - \lambda''\rangle |\psi'' - \lambda''\rangle |\psi + \lambda\rangle |\mathcal{D}\lambda|\mathcal{D}\lambda'\mathcal{D}\lambda''|\mathcal{D}\lambda'
\]
\[
= (i\hbar)^{-n} \left(\frac{i\hbar}{2}\right)^{3n} \int \text{D}\psi'' \exp \left\{-\frac{2i}{\hbar} \left(\pi'\lambda' + \pi''\lambda'' + \pi\lambda\right)\right\} |\psi''|\psi'\rangle \langle \psi' + \lambda'|\psi'' - \lambda''\rangle |\psi'' - \lambda''\rangle |\psi + \lambda\rangle |\mathcal{D}\lambda|\mathcal{D}\lambda'\mathcal{D}\lambda''|\mathcal{D}\lambda'
\]
\[
= (i\hbar)^{2n} \exp \left\{-\frac{2i}{\hbar} \left(\pi'\psi'' - \psi'\right) + \pi''\left(\psi' - \psi''\right) + \pi\left(\psi'' - \psi'\right)\right\}.
\]

(50)

Employing (26) and (39) one gets
\[
\text{tr}\left\{\hat{A}\hat{B}\right\} = (i\hbar)^{-n} \int \text{D}\psi |\hat{A}\hat{B}|\psi\rangle \langle \psi\right\}
\]
\[
= (i\hbar)^{-n} \int \text{D}\psi |\hat{A}|\psi'\rangle \text{D}\psi' |\hat{B}|\psi\rangle
\]
\[
= (i\hbar)^{-n} \left(-1\right)^{\varepsilon_A \varepsilon_B + 2n\varepsilon_B} \left(-1\right)^{n\varepsilon_B} \int \text{D}\psi' |\hat{B}|\psi\rangle \text{D}\psi\langle \psi' |\hat{A}|\psi'\right\}
\]
\[
= \left(-1\right)^{\varepsilon_A \varepsilon_B} \text{tr}\left\{\hat{B}\hat{A}\right\}.
\]

Thus we have the following relation
\[
\text{tr}\left\{\hat{A}\hat{B}\right\} = \left(-1\right)^{\varepsilon_A \varepsilon_B} \text{tr}\left\{\hat{B}\hat{A}\right\},
\]

(51)

where \(\varepsilon_A, \varepsilon_B\) are the Grassmann parity of \(\hat{A}\) and \(\hat{B}\), respectively. \(\varepsilon_A = 0, 1, \varepsilon_B = 0, 1\). By the definition (33), SW quantizer is an even operator, i.e., \(\varepsilon_{\hat{\Omega}} = 0\), thus one gets
\[
f(\pi, \psi) = \text{tr}\left\{\hat{f}\hat{\Omega}(\pi, \psi)\right\} = \text{tr}\left\{\hat{\Omega}(\pi, \psi)\hat{f}\right\}.
\]

(52)
In this section we find the Moyal $\star$-product. Let $f = f(\pi, \psi)$ and $g = g(\pi, \psi)$ be any pair of functions defined on our fermionic phase space $\Gamma_{F}^{2n}$ and let $f = W^{-1}(\hat{f})$ and $g = W^{-1}(\hat{g})$ be the corresponding operators via the Weyl correspondence. Then we are looking for the product $(f \star g)(\pi, \psi)$ which corresponds to the product of operators $\hat{f} \cdot \hat{g}$ through the Weyl correspondence

$$(f \star g)(\pi, \psi) = \text{tr}\left\{\hat{f}\hat{g}\Omega(\pi, \psi)\right\}.$$ \hspace{1cm} (53)

Substituting (32) into (53) and using (50), we get

$$(f \star g)(\pi, \psi) = \left(i\hbar\right)^{2n} \int f(\pi', \psi') g(\pi'', \psi'') \exp \left\{-\frac{2i}{\hbar} \left[i' \left(\psi'' - \psi''\right) + \pi'' \left(\psi - \psi'\right) + \pi' \left(\psi' - \psi''\right)\right]\right\} D\pi' D\psi' D\pi'' D\psi''. \hspace{1cm} (54)$$

By changing the variables: $\Psi' = \psi' - \psi$, $\Pi' = \pi' - \pi$, $\Psi'' = \psi'' - \psi$, $\Pi'' = \pi'' - \pi$, then the Moyal product take the form

$$(f \star g)(\pi, \psi) = \left(i\hbar\right)^{2n} \int f(\Pi' + \pi, \Psi' + \psi) g(\Pi'' + \pi, \Psi'' + \psi) \exp \left\{-\frac{2i}{\hbar} \left[\Pi' \Psi'' - \Pi'' \Psi'\right]\right\} D\Pi' D\Psi' D\Pi'' D\Psi''.$$ \hspace{1cm} (54)

Expanding $f$ and $g$ into the Taylor series and performing some manipulations one arrives to the main formula

$$(f \star g)(\pi, \psi) = f(\pi, \psi) \exp \left\{\frac{i\hbar}{2} \hat{\mathcal{P}}_{F}\right\} g(\pi, \psi),$$ \hspace{1cm} (55)

where

$$\hat{\mathcal{P}}_{F} = \frac{\hat{\partial}}{\hat{\partial}\pi} \frac{\hat{\partial}}{\hat{\partial}\psi} + \frac{\hat{\partial}}{\hat{\partial}\psi} \frac{\hat{\partial}}{\hat{\partial}\pi}. \hspace{1cm} (56)$$

with $\hat{\partial}$ and $\hat{\partial}$ stand for the right derivative and left derivative, respectively. This is the Moyal $\star$-product for the fermionic part of the super-Poisson bracket discussed in Refs. [17, 18, 19, 20, 21, 23, 24, 27].

V. THE WIGNER FUNCTION

From Eq (52) one can conclude that the Wigner function in the fermionic case should be defined simply by

$$\rho_{W}(\pi, \psi) = \text{tr}\left\{\hat{\rho}\Omega(\pi, \psi)\right\},$$ \hspace{1cm} (57)

where $\hat{\rho}$ stands for the density operator. Observe that the formula (57) is almost the same as in the bosonic case [30]. In the fermionic case we don’t have a factor $\frac{1}{(2\pi\hbar)^{n}}$. However, this factor in a modified form appears in the definition of trace ”tr” by Eqs. (39) or (40). Substituting Eqs. (53) and (39) into...
after some computations we get

\[ \rho_W(\pi, \psi) = (\hbar)^{-n} \int \mathcal{D}\psi' \langle \psi' | \hat{\rho} \hat{\Omega}(\pi, \psi) | \psi' \rangle \]

\[ = \hbar^{-n} \int \mathcal{D}\psi \mathcal{D}\lambda \exp \left\{ -i \sum_{j=1}^{n} \pi_j \lambda_j \right\} (-1)^{n+n-\varphi} \langle \psi' | \hat{\rho} | \psi - \frac{\hbar \lambda}{2} \rangle \delta(\psi' - (\psi + \frac{\hbar \lambda}{2})) \]

\[ = \hbar^{-n} \int \mathcal{D}\lambda \exp \left\{ -i \sum_{j=1}^{n} \pi_j \lambda_j \right\} \langle \psi + \frac{\hbar \lambda}{2} | \hat{\rho} | \psi - \frac{\hbar \lambda}{2} \rangle. \]

Thus, finally

\[ \rho_W(\pi, \psi) = \hbar^{-n} \int \mathcal{D}\lambda \exp \left\{ -i \sum_{j=1}^{n} \pi_j \lambda_j \right\} \langle \psi + \frac{\hbar \lambda}{2} | \hat{\rho} | \psi - \frac{\hbar \lambda}{2} \rangle. \] (58)

VI. NORMAL ORDERING

Assume first \( n = 1 \) and consider \( f := b^* b \) and \( g := bb^* \), where \( b, b^* \) are defined by \( b = \frac{\psi}{\sqrt{\hbar}} \) and \( b^* = \frac{\psi^*}{\sqrt{\hbar}} \).

Then the quantization of \( f \) and \( g \) in the Berezin-Wick (B-W) or normal ordering gives

\[ f = b^* b \xrightarrow{B-W} \hat{b}^* \hat{b}, \quad g = bb^* \xrightarrow{B-W} - \hat{b}^* \hat{b}. \] (59)

It is an easy matter to find that the same results are obtained if one uses the Weyl rule (32) or (48) changing \( f \) and \( g \) by introducing the operator of normal ordering \( \hat{\mathcal{N}} \). Thus

\[ \hat{b}^* \hat{b} = \int (\hat{\mathcal{N}} b^* b) \hat{\Omega}(\pi, \psi) d\pi d\psi, \] (60)

\[ -\hat{b}^* \hat{b} = \int (\hat{\mathcal{N}} bb^*) \hat{\Omega}(\pi, \psi) d\pi d\psi, \] (61)

where

\[ \hat{\mathcal{N}} := \exp \left\{ \frac{ih}{2} \frac{\partial^2}{\partial \psi \partial \pi} \right\}. \] (62)

By induction one arrives at the general result for any \( n \). Let \( f(\pi, \psi) \) be any function on \( \Gamma^{2n} \). Then the Berezin-Wick quantization (normal quantization) of \( f \) is done by the Weyl quantization of a modified \( f \) according to

\[ f_N := \hat{\mathcal{N}} f, \] (63)

where

\[ \hat{\mathcal{N}} = \exp \left\{ \frac{ih}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial \psi_j \partial \pi_j} \right\} = \exp \left\{ \frac{\hbar}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial \psi_j \partial \psi_j^*} \right\} \]

\[ = \exp \left\{ \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial b_j \partial b_j^*} \right\} = \exp \left\{ \frac{ih}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial Q_j \partial P_j} \right\}. \] (64)
Then the Weyl correspondence reads

$$\hat{f}_N = \int (\hat{N} f) \hat{\Omega} \prod d\pi d\psi.$$  \hfill (65)

In the next section we will analyze two simple examples to see how the WWM formalism works for systems involving fermions.

VII. EXAMPLES

A. Fermi Oscillator

As an example of our WWM formalism consider the fermionic oscillator for one degree of freedom ($n = 1$). The Lagrangian of such an oscillator reads

$$L = i\psi^* \dot{\psi} - \omega \psi^* \psi.$$ \hfill (66)

Then the momentum conjugate to $\psi$ is $\pi = \partial L / \partial \dot{\psi} = -i\psi^*$, while the Hamiltonian $H$ is given by

$$H = \dot{\psi} \pi - L = i\omega \pi \psi.$$ \hfill (67)

Here $\partial$ stands for the left-derivative $[16, 27]$.

Now is convenient to use $\pi := -\pi$ and $\pi = i\psi^*$ instead of $\pi$. Thus in terms of $\pi$ the Hamiltonian \hfill (67) reads

$$H = -i\omega \pi \psi = \omega \psi^* \psi.$$ \hfill (68)

The coherent state $|\psi\rangle$ \hfill (11) is given by

$$|\psi\rangle = \exp \left\{-\frac{i}{\hbar} \pi \psi\right\} |0\rangle = \exp \left\{\frac{\psi^* \psi}{\hbar}\right\} |0\rangle,$$

where $|0\rangle$ is the ground state and $\pi$ is the momentum operator. The quantum operators satisfy the well known anti-commutation relations $[\hat{\psi}, \pi]_+ = i\hbar$, or $[\hat{\psi}, \hat{\psi}^*]_+ = \hbar$.

In the matrix representation

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\psi} = \sqrt{\hbar} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\psi}^* = \sqrt{\hbar} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ \hfill (70)

Then the vector $|1\rangle$ is defined by $|1\rangle = \frac{1}{\sqrt{\hbar}} \hat{\psi}^* |0\rangle$ or in the matrix representation $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In fact it is a supervector with Grassmann parity $\varepsilon_{ij} = 1$ or an “odd supervector”. Obviously the state $|0\rangle$ has
parity \( \varepsilon_{(0)} = 0 \). In other words it is an “even supervector”. The state \(|\psi\rangle\) has also the parity \( \varepsilon_{(\psi)} = 0 \) (it is an even supervector). By (69) and (70) we have

\[
|\psi\rangle = \exp \left\{ \frac{\hat{\psi}^* \psi}{\hbar} \right\} |0\rangle = \left( 1 - \frac{\psi \hat{\psi}^*}{\hbar} \right) |0\rangle = \left( \frac{1}{\sqrt{\hbar}} \right) |0\rangle - \frac{\psi}{\sqrt{\hbar}} |1\rangle.
\]

(71)

The dual vector is given by \( \langle \psi | = \sqrt{\hbar} (0, 1) - \psi (1, 0) = \sqrt{\hbar} |1\rangle - \psi |0\rangle \). Remember that \( \langle \psi | \psi' \rangle = \psi' - \psi = \delta (\psi' - \psi) \). Then the complementary states are

\[
\langle \pi | = \langle \psi^* | := (|\psi\rangle)^* = (1, \ - \frac{\psi^*}{\sqrt{\hbar}}) = \langle 0 | - \frac{\psi^*}{\sqrt{\hbar}} |1\rangle.
\]

(72)

Of course \( \langle 0 | = (|0\rangle)^* = (1, 0) \), and \( \langle 1 | = (|1\rangle)^* = (0, 1) \). We have also from Eq. (20) (for \( n = 1 \)) that

\[
|\pi\rangle = \exp \left\{ - \frac{i}{\hbar} \hat{\psi} \pi \right\} \tilde{\pi} |0\rangle = i \sqrt{\hbar} |1\rangle - i \psi^* |0\rangle = i \sqrt{\hbar} \begin{pmatrix} 0 \\ \psi \end{pmatrix} - i \psi^* \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

(73)

It is also an easy matter to see that the state \(|\pi\rangle\) has odd parity and therefore \( \varepsilon_{|\pi\rangle} = 1 \). Simple calculations show that \( \int |\psi\rangle d\psi \langle \psi | = 1 \) and \( \int |\pi\rangle (-1) d\pi \langle \pi | = 1 \), as expected.

Stratovovich-Weyl Quantizer

According to our previous considerations the Stratovovich-Weyl quantizer is given by Eq. (35)

\[
\hat{\Omega}(\pi, \psi) = i \int d\lambda \exp \left\{ - i \pi \lambda \right\} |\psi - \frac{\hbar \lambda}{2} \rangle \langle \psi + \frac{\hbar \lambda}{2} |\psi\rangle
\]

\[
= i \int d\lambda (1 + \psi^* \lambda) \begin{pmatrix} 1 & \left( \psi - \frac{\hbar \lambda}{2 \sqrt{\hbar}} \right) \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{\hbar} & (0, 1) - (\psi + \frac{\hbar \lambda}{2}) (1, 0) \end{pmatrix}
\]

Integrating over \( d\lambda \) we get

\[
\hat{\Omega}(\pi, \psi) = i \begin{pmatrix} \psi \psi^* & \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \psi^* \sqrt{\hbar} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \psi \sqrt{\hbar} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}
\]

\[
= i \begin{pmatrix} \psi \psi^* - \psi \hat{\psi}^* + \psi^* \hat{\psi} - \hat{\psi}^* \hat{\psi} + \frac{\hbar}{2} \end{pmatrix}.
\]

(74)

\[
\text{Remember that } \varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then by Eqs. (71) and (74) one quickly finds that

\[
\text{tr} \left\{ \hat{\Omega}(\pi, \psi) \right\} = \frac{1}{i \hbar} \int d\psi' \langle \psi | \hat{\Omega}(\pi, \psi) |\psi'\rangle = 1
\]

(75)
as should be. From (74) we obtain that \( \hat{\Omega} \) satisfies the relation

\[
\hat{\Omega}'(\pi, \psi) = - \hat{\Omega}(\pi, \psi).
\]

(76)
One can also check that
\[
\text{tr}\left\{ \widehat{\Omega}(\pi', \psi') \widehat{\Omega}(\pi'', \psi'') \right\} = \delta(\psi' - \psi'', \pi' - \pi'').
\] (77)

**The Moyal ⋆-Product**

The Moyal ⋆-product is given by
\[
\star = \exp \left\{ \frac{i \hbar}{2} \left( \frac{\partial}{\partial \psi} \frac{\partial}{\partial \pi} + \frac{\partial}{\partial \pi} \frac{\partial}{\partial \psi} \right) \right\} = \exp \left\{ \frac{\hbar}{2} \left( \frac{\partial}{\partial \psi} \frac{\partial}{\partial \psi^*} + \frac{\partial}{\partial \pi} \frac{\partial}{\partial \pi} \right) \right\} = \exp \left\{ \frac{\hbar}{2} \left( \frac{\partial}{\partial Q} \frac{\partial}{\partial Q^*} + \frac{\partial}{\partial P} \frac{\partial}{\partial P^*} \right) \right\},
\] (78)

where \(Q\) and \(P\) are the oscillator variables (43).

**The Wigner Function**

Consider the eigenvalue equation for the Hamiltonian \(H\) given by (68)
\[
H \star \rho_W = E \rho_W,
\] (79)

where \(\rho_W\) stands for the Wigner function. We need also that \(\rho_W\) be a real Grassmann superfunction such that
\[
\rho_W^* = \rho_W.
\] (80)

Therefore it expands as
\[
\rho_W = A_0 + A_1 \psi^\ast \psi,
\] (81)

where \(A_0\) and \(A_1\) are real numbers. Substituting (68), (78) and (81) into (79) we get two linear equations for \(A_0\) and \(A_1\)
\[
EA_0 - \frac{\hbar^2}{4} \omega A_1 = 0,
\]
\[
-\omega A_0 + EA_1 = 0.
\] (82)

As \(|A_0| + |A_1| \neq 0\) one finds
\[
\det \begin{pmatrix} E & -\frac{\hbar^2}{4} \omega \\ -\omega & E \end{pmatrix} = 0,
\] (83)

with the solutions
\[
E = \mp \frac{\hbar \omega}{2}.
\] (84)

Then (82) and (83) lead to the following energy eigenvalues
\[
E^{(-)} = -\frac{\hbar \omega}{2}, \quad E^{(+)} = \frac{\hbar \omega}{2},
\] (85)
with corresponding eigenfunctions

\[ \rho_w^{(-)} = A_0 (1 - \frac{2}{\hbar} \psi^* \psi), \quad \rho_w^{(+)} = A_0 (1 + \frac{2}{\hbar} \psi^* \psi). \]  \hspace{1cm} (86)

As is known the density operator is given by

\[ \hat{\rho} = \int \rho_w(\pi, \psi) \hat{\Omega}(\pi, \psi) d\pi d\psi. \]  \hspace{1cm} (87)

Then (with the usual trace Tr) we have

\[ \text{Tr}\{\hat{\rho}\} = \int \rho_w(\pi, \psi) \left[ \text{Tr}\left\{\hat{\Omega}(\pi, \psi)\right\}\right] d\pi d\psi \]
\[ = \int \rho_w(\pi, \psi) 2\delta(\psi)\delta(\pi) d\pi d\psi \]  \hspace{1cm} (88)
\[ = 2\rho_w(0, 0). \]

But since \( \text{Tr}\{\hat{\rho}\} = 1 \), this implies that \( A_0 = \rho_w(0, 0) = \frac{1}{2} \). Substituting this value of \( A_0 \) into (86) we get

\[ \rho_w^{(-)} = \frac{1}{2} (1 - \frac{2}{\hbar} \psi^* \psi) = \frac{1}{2} \exp \left\{ - \frac{2}{\hbar} \psi^* \psi \right\}, \quad \rho_w^{(+)} = \frac{1}{2} (1 + \frac{2}{\hbar} \psi^* \psi) = \frac{1}{2} \exp \left\{ \frac{2}{\hbar} \psi^* \psi \right\}. \]  \hspace{1cm} (89)

Finally, from Eqs. (74) and (89) after some algebra we find

\[ \hat{\rho}_w^{(-)} = \int \rho_w^{(-)}(\pi, \psi) \hat{\Omega}(\pi, \psi) d\pi d\psi \]
\[ = \frac{1}{2} \int \left( \psi \psi^* - \frac{2}{\hbar} \psi^* \psi + \frac{2}{\hbar} \psi \psi^* \psi \right) \psi^* d\psi \]
\[ = 1 - \frac{1}{\hbar} \hat{\psi}^* \hat{\psi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle\langle 0|. \]  \hspace{1cm} (90)

Analogously for \( \rho_w^{(+)}(\pi, \psi) \) we get

\[ \hat{\rho}_w^{(+)} = \frac{1}{\hbar} \hat{\psi}^* \hat{\psi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle\langle 1|. \]  \hspace{1cm} (91)

**Normal Ordering**

The Weyl correspondence of the hamiltonian \( H = \omega \psi^* \psi = i\omega QP \) leads to the operator

\[ \hat{H} = \int H(\pi, \psi) \hat{\Omega}(\pi, \psi) d\pi d\psi = i\omega \hat{Q} \hat{P} \]
\[ = \frac{1}{2} \omega \left( \hat{\psi}^* \hat{\psi} - \hat{\psi} \hat{\psi}^* \right) = \omega \left( \hat{\psi}^* \hat{\psi} - \frac{\hbar}{2} \right) \]
\[ = \frac{\hbar \omega}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]  \hspace{1cm} (92)
It is evident that
\[ \hat{H}|0\rangle = -\frac{1}{2}\hbar\omega|0\rangle, \quad \hat{H}|1\rangle = \frac{1}{2}\hbar\omega|1\rangle. \] (93)

Then we can compute the normal ordered hamiltonian to be
\[ \hat{H}_N \equiv \hat{H} := \int (\hat{N}H)(\pi, \psi)\hat{\Omega}(\pi, \psi)d\pi d\psi \]
\[ = \int \left[ \exp\left\{ \frac{\hbar}{2}\frac{\partial}{\partial\psi^*\psi} \right\} \omega\psi^*\psi \right] \hat{\Omega}(\pi, \psi)d\pi d\psi \]
\[ = \int \left( \omega\psi^*\psi + \frac{\hbar\omega}{2} \right) \hat{\Omega}(\pi, \psi)d\pi d\psi \]
\[ = i\omega\hat{Q}\hat{P} + \frac{\hbar\omega}{2} = \omega\hat{\psi}^*\hat{\psi} - \frac{\hbar}{2} + \frac{\hbar\omega}{2} = \omega\hat{\psi}^*\hat{\psi} \]
\[ = \hbar\omega \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \] (94)

Consequently, the eigenvalue equation
\[ \hat{H}_N \rho' = E'\rho' \] (95)
is equivalent to
\[ (\hat{N}H) \star \rho' = E'\rho'. \] (96)

Write \( \rho' = A'_0 + A'_1\psi^*\psi \). Then the equation (96) leads to the system of equations (compare with (79)-(86)):
\[ \left( E' - \frac{\hbar\omega}{2} \right) A'_0 - \frac{\hbar^2\omega}{4} A'_1 = 0, \]
\[ -\omega A'_0 + \left( E' - \frac{\hbar\omega}{2} \right) A'_1 = 0. \] (97)

Or equivalently
\[ \det \begin{pmatrix} E' - \frac{\hbar\omega}{2} & -\frac{\hbar^2\omega}{4} \\ -\omega & E' - \frac{\hbar\omega}{2} \end{pmatrix} = 0. \] (98)

Thus the eigenvalues of \( H_N \) read
\[ E'^{(0)} = 0, \quad E'^{(+)}) = \hbar\omega. \] (99)

The corresponding Wigner functions are
\[ \rho'^{(0)} = \frac{1}{2}\left( 1 - \frac{2}{\hbar}\psi^*\psi \right), \quad \rho'^{(+)}) = \frac{1}{2}\left( 1 + \frac{2}{\hbar}\psi^*\psi \right). \] (100)

Finally, it is interesting to deal with the normal star product which is defined by
\[ f \star_{(N)} g = \hat{N}^{-1}(\hat{N}f \star \hat{N}g). \] (101)
Consider the following eigenvalue equation

\[ H \star_{\langle N \rangle} \rho'' = E'' \rho''. \]  

(102)

According to (101) one has

\[ \hat{N}^{-1} (\hat{N} H \star \hat{N} \rho'') = E'' \rho''. \]  

(103)

Hence \( \hat{N} H \star \hat{N} \rho'' = E'' \hat{N} \rho'' \) implies that

\[ H_N \star \hat{N} \rho'' = E'' \hat{N} \rho''. \]  

(104)

Comparing with (96) one quickly finds that \( E'' = E' \) and \( \rho'' \sim \hat{N}^{-1} \rho' \). Consequently we find that the eigenvalues \( E''^{(0)} = 0 \), \( E''^{(+)} = \hbar \omega \), correspond to the Wigner functions

\[ \rho''^{(0)} \sim \exp \left\{ -\frac{\hbar^2}{2} \frac{\partial^2}{\partial \psi \partial \psi^*} \right\} \rho^{(0)} = 1 - \frac{1}{\hbar} \psi^* \psi, \quad \rho''^{(+)} \sim 1 + \frac{1}{\hbar} \psi^* \psi, \]  

(105)

respectively.

**B. Supersymmetric Weyl-Wigner-Moyal Formalism: The Susy Harmonic Oscillator**

In this subsection we construct a formalism for supersymmetric quantum mechanics. To be more precise we construct all the ingredients of the WWM formalism for a system defined on the supersymmetric phase space of \( 2n \times 2N \) degrees of freedom (or super-phase-space) \( \Gamma_S = \{ (\vec{p}, \vec{x}, \vec{\pi}, \vec{\theta}) \} = \Gamma_B^{2n} \times \Gamma_F^{2N} \); here \( \Gamma_B^{2n} = \{ (\vec{p}, \vec{x}) \} \) such that \( [\vec{x}_j, \vec{p}_k]_- = i\hbar \delta_{jk} \) and \( \Gamma_F^{2N} = \{ (\vec{\pi}, \vec{\theta}) \} \) with \( [\vec{\theta}_j, \vec{\pi}_k]_+ = i\hbar \delta_{jk} \). We assume that \( (\vec{p}, \vec{x}) \) are real bosonic variables and \( (\vec{\pi}, \vec{\theta}) \) are complex fermionic (Grassmann) variables. We will find the Stratonovich-Weyl quantizer, the Moyal \( \star \)-product and the Wigner function for the supersymmetric systems. In particular one finds that the supersymmetric Moyal \( \star \)-product is the tensor product of the corresponding Moyal \( \star \)-products for the bosonic and fermionic systems. Then the supersymmetric WWM machinery will be applied to the bosonic and fermionic harmonic oscillators with \( n = 1 \) and \( N = 1 \).

Let \( |x, \theta⟩ = |x⟩ \otimes |\theta⟩ \in \mathcal{H} \) be a coherent state of a quantum system in the respective super Hilbert space \( \mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_F \). Define the normalized vacuum state \( |0_B, 0_F⟩ = |0_B⟩ \otimes |0_F⟩ \) as

\[ \hat{\theta}_j |0_B, 0_F⟩ = 0, \quad \hat{x}_j |0_B, 0_F⟩ = 0. \]  

(106)

Analogously to (11) we have
\[
|x, \theta\rangle = \exp\left\{ -\frac{i}{\hbar} \left[ \sum_{\alpha=1}^{n} \hat{p}_\alpha x_\alpha + \sum_{j=1}^{N} \hat{\pi}_j \theta_j \right] \right\} |0_B, 0_F\rangle \\
= \exp\left\{ -\frac{i}{\hbar} \sum_{\alpha=1}^{n} \hat{p}_\alpha x_\alpha \right\} |0_B, \theta\rangle \\
= \exp\left\{ -\frac{i}{\hbar} \sum_{j=1}^{N} \hat{\pi}_j \theta_j \right\} |x, 0_F\rangle.
\]

(107)

This state satisfies the following eigenvalue equations
\[
\hat{x}_k |x, \theta\rangle = x_k |x, \theta\rangle, \\
\hat{\theta}_k |x, \theta\rangle = \theta_k |x, \theta\rangle.
\]

(108)

Hence, for any superfunction \(\hat{f} = f(\hat{x}, \hat{\theta})\) we have
\[
\hat{f}|x, \theta\rangle = f(\hat{x}, \hat{\theta})|x, \theta\rangle = f(x, \theta)|x, \theta\rangle.
\]

(109)

Another useful result is the following (see (114))
\[
\exp\left\{ -\frac{i}{\hbar} \left[ \sum_{\alpha=1}^{n} \hat{p}_\alpha y_\alpha + \sum_{j=1}^{N} \hat{\pi}_j \eta_j \right] \right\} |x, \theta\rangle = |x + y, \theta + \eta\rangle.
\]

(110)

One can construct the corresponding bras \(\langle x, \theta|\) such that
\[
\langle x_\alpha, \theta_j | x_\beta, \theta_k \rangle = \langle x_\alpha | x_\beta \rangle \langle \theta_j | \theta_k \rangle \\
= \delta(x_\alpha - x_\beta) \delta(\theta_k - \theta_j).
\]

(111)

The completeness relation reads
\[
\int |x, \theta\rangle DxD\theta \langle x, \theta| = 1.
\]

(112)

Note that the parity of the superstates is given by
\[
\varepsilon_{|x\rangle} = \varepsilon_{|\theta\rangle} = 0 \\
\Rightarrow \varepsilon_{|x, \theta\rangle} = 0, \\
\varepsilon_{\langle x|} = 0, \varepsilon_{\langle \theta|} = \begin{cases} 0 & \text{for } N \text{ even} \\ 1 & \text{for } N \text{ odd} \end{cases} \\
\Rightarrow \varepsilon_{\langle x, \theta|} = \begin{cases} 0 & \text{for } N \text{ even} \\ 1 & \text{for } N \text{ odd} \end{cases}.
\]

(113)

**Stratonovich-Weyl Quantizer**

In what follows we restrict ourselves to the simplest case of \(n = 1\) and \(N = 1\). The generalization to any \(n\) and any \(N\) is obvious. Let \(f = f(p, x, \pi, \theta)\) be a function on the super-phase-space \(\Gamma_S\); its Fourier transform is defined by
\[
\tilde{f}(\omega, \kappa, \lambda, \mu) := \int f(p, x, \pi, \theta) \exp\{-i(p\omega + x\kappa + \pi\lambda + \theta\mu)\} dp dx d\pi d\theta.
\]

(114)
Inserting this into the Weyl quantization rule, the corresponding operator \( \hat{f} \) is given by
\[
\hat{f} := (2\pi)^{-2} \int \hat{f}(\omega, \kappa, \lambda, \mu) \exp\{i(\hat{p}\omega + \hat{x}\kappa + \hat{\pi}\lambda + \hat{\theta}\mu)\} d\omega d\kappa d\lambda d\mu.
\]
(115)

After straightforward computations we obtain
\[
\hat{f} := (2\pi\hbar)^{-1} \int f(p, x, \pi, \theta) \hat{\Omega}(p, x, \pi, \theta) dpdxd\pi d\theta,
\]
(116)
where \( \hat{\Omega}(p, x, \pi, \theta) \) is the supersymmetric Stratonovich-Weyl quantizer
\[
\hat{\Omega}(p, x, \pi, \theta) = (2\pi)^{-1} \hbar \int \exp \left\{ i \left[ (\hat{p}-p)\omega + (\hat{x}-x)\kappa + (\hat{\pi}-\pi)\lambda + (\hat{\theta}-\theta)\mu \right] \right\} d\omega d\kappa d\lambda d\mu.
\]
(117)

From its structure it is obvious that this is an even operator and it can be rewritten in the form
\[
\hat{\Omega}(p, x, \pi, \theta) = (2\pi)^{-1} \hbar \int \exp \left\{ i \left[ (\hat{p}-p)\omega + (\hat{x}-x)\kappa \right] \right\} dpd\pi d\theta
\]
\[
= \hat{\Omega}_B(p, x) \otimes \hat{\Omega}_F(\pi, \theta).
\]
(118)

Another useful form of the supersymmetric Stratonovich-Weyl quantizer is
\[
\hat{\Omega}(p, x, \pi, \theta) = \hbar \int d\omega d\lambda \exp\{-\frac{2i}{\hbar} (p\omega + \pi\lambda)\} |x - \omega, \theta - \lambda\rangle \langle x + \omega, \theta + \lambda|
\]
\[
= \hbar \int d\omega d\lambda \exp\{-\frac{2i}{\hbar} (p\omega + \pi\lambda)\} (|x - \omega\rangle \otimes |\theta - \lambda\rangle)(\langle x + \omega| \otimes \langle \theta + \lambda|)
\]
\[
= \hbar \int d\omega d\lambda \exp\{-\frac{2i}{\hbar} (p\omega + \pi\lambda)\} |x - \omega\rangle \langle x + \omega| \otimes |\theta - \lambda\rangle \langle \theta + \lambda|
\]
\[
= \hat{\Omega}_B(p, x) \otimes \hat{\Omega}_F(\pi, \theta).
\]
(119)

The inverse Weyl correspondence is given by
\[
f(p, x, \pi, \theta) = tr\{\hat{\Omega}(p, x, \pi, \theta) \hat{f}\},
\]
(120)
where the “trace” \( tr \) is defined by
\[
tr\{\cdot\} := (i\hbar)^{-1} \int dx d\theta \langle x, \theta | \cdot | x, \theta \rangle.
\]
(121)

The Wigner function corresponding to the density \( \hat{\rho} = |\Psi\rangle \langle \Psi| \) associated with the superstate \( |\Psi\rangle \)
\[
\rho(p, x, \pi, \theta) = (2\pi\hbar)^{-1} tr\{\hat{\Omega}(p, x, \pi, \theta) \hat{\rho}\}
\]
\[
= (2\pi\hbar)^{-1} (i\hbar)^{-1} \int dx' d\theta' \langle x', \theta' | \hat{\Omega}(p, x, \pi, \theta) \hat{\rho} | x', \theta' \rangle
\]
\[
= (2\pi\hbar)^{-1} \int dx' d\theta' d\omega d\lambda (x', \theta') \exp\{-\frac{2i}{\hbar} (p\omega + \pi\lambda)\} |x - \omega, \theta - \lambda\rangle \langle x + \omega, \theta + \lambda| \Psi \rangle\langle \Psi| x', \theta' \rangle
\]
\[
= (2\pi\hbar)^{-1} \int d\omega d\lambda \exp\{-\frac{2i}{\hbar} (p\omega + \pi\lambda)\} \Psi(x + \omega, \theta + \lambda) \Psi^\dagger(x - \omega, \theta - \lambda)
\]
(122)
where \( \Psi(x, \theta) = \langle x, \theta | \Psi \rangle \) and \( \Psi^\dagger(x, \theta) = \langle \Psi^* | x, \theta \rangle \).
In the case of a supersymmetric pure state $|\Psi\rangle = |\phi, \psi\rangle$ the Wigner function factorizes as follows

$$
\rho(p, x, \pi, \theta) = (2\pi\hbar)^{-1} \int d\omega d\lambda \exp\left\{-\frac{i\hbar}{\hbar} (p\omega + \pi \lambda)\right\} \Psi(x + \omega, \theta + \lambda) \Psi^\dagger(x - \omega, \theta - \lambda) \\
= (2\pi\hbar)^{-1} \int d\omega d\lambda \exp\left\{-\frac{i\hbar}{\hbar} (p\omega + \pi \lambda)\right\} \phi(x + \omega) \psi(\theta + \lambda) \phi^\dagger(x - \omega) \psi^\dagger(\theta - \lambda) \\
= \rho_B(p, x) \cdot \rho_F(\pi, \theta).
$$

(123)

**Supersymmetric Moyal Product**

The Moyal $\star$-product for any two functions in the super-phase space is defined by

$$
\star_S := \exp\left\{ \frac{i\hbar}{2} \hat{P}_S \right\}
$$

where $\hat{P}_S$ is given by the relation

$$
\hat{P}_S = \hat{P}_B + \hat{P}_F \\
= \left( \frac{\partial}{\partial x} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial x} \right) + \left( \frac{\partial}{\partial \pi} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \theta} \frac{\partial}{\partial \pi} \right).
$$

(124)

This is exactly the super-Poisson bracket operator used in Refs. [16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28].

**Normal Ordering**

Normal ordering corresponding to a supersymmetric system is the composition of the normal ordering for bosons, and the normal ordering for fermions i.e.,

$$
\hat{N}_S = \hat{N}_B \hat{N}_F = \exp\left\{-\frac{1}{2} \left( \frac{\partial^2}{\partial a \partial a^*} - \frac{\partial^2}{\partial b \partial b^*} \right) \right\},
$$

(125)

where $a, a^*, b, b^*$ are the oscillator variables, which will be defined later.

**The System**

To begin with we define the lagrangian for a supersymmetric harmonic oscillator of one variable and of a frequency $\omega_B$ for the bosonic oscillator and $\omega_F$ for the Fermi one

$$
L = \frac{1}{2} \dot{\vartheta}^2 - \frac{1}{2} \omega_B^2 x^2 + i\theta^* \dot{\theta} - \omega_F \theta^* \theta.
$$

(126)

Then the hamiltonian reads

$$
H = \frac{1}{2} \dot{\vartheta}^2 + \frac{1}{2} \omega_B^2 x^2 + \omega_F \theta^* \theta.
$$

(127)

and the respective hamiltonian operator is

$$
\hat{H} = \frac{1}{2} \hat{\vartheta}^2 + \frac{1}{2} \omega_B \hat{x}^2 + \frac{\omega_F}{2} (\hat{\theta}^* \hat{\theta} - \hat{\theta} \hat{\theta}^*).
$$

(128)
Here $\hat{x}, \hat{p}, \hat{\theta}, \hat{\theta}^*$ obey the standard rules

$$[\hat{x}, \hat{p}]_- = i\hbar, \quad [\hat{x}, \hat{x}]_- = [\hat{p}, \hat{p}]_- = 0,$$

$$[\hat{\theta}, \hat{\theta}^*]_+ = \hbar, \quad [\hat{\theta}, \hat{\theta}]_+ = [\hat{\theta}^*, \hat{\theta}^*]_+ = 0.$$  

According to the standard procedure let us define:

$$\hat{a} = \sqrt{\frac{\omega_B}{2\hbar}} (\hat{x} + \frac{i\hat{p}}{\omega_B}), \quad \hat{a}^* = \sqrt{\frac{\omega_B}{2\hbar}} (\hat{x} - \frac{i\hat{p}}{\omega_B});$$

$$\hat{b} = \frac{\hat{\theta}}{\sqrt{\hbar}}, \quad \hat{b}^* = -\frac{i}{\sqrt{\hbar}} \hat{\pi} = \frac{\hat{\theta}^*}{\sqrt{\hbar}}$$

or

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega_B}} (\hat{a}^* + \hat{a}), \quad \hat{p} = i\sqrt{\frac{\omega_B}{2}} (\hat{a}^* - \hat{a})$$

$$\hat{\theta} = \sqrt{\hbar} \hat{b}, \quad \hat{\theta}^* = -i\hat{\pi} = \sqrt{\hbar} \hat{b}^*;$$

Then

$$[\hat{a}, \hat{a}^*]_- = 1, \quad [\hat{a}, \hat{a}]_- = [\hat{a}^*, \hat{a}^*]_- = 0,$$

$$[\hat{b}, \hat{b}^*]_+ = 1, \quad [\hat{b}, \hat{b}]_+ = [\hat{b}^*, \hat{b}^*]_+ = 0.$$  

With the use of these operators one gets

$$\hat{H} = \frac{\hbar}{2} [\hat{a}^*, \hat{a}]_+ + \frac{\hbar}{2} [\hat{b}^*, \hat{b}]_- , \quad (129)$$

and finally

$$\hat{H} = \hbar \omega (\hat{a}^* \hat{a} + 1/2 + \hat{b}^* \hat{b} - 1/2),$$

$$= \hbar \omega (\hat{n}_B + \hat{n}_F), \quad (130)$$

where due to the condition of unbroken supersymmetry (i.e. the total energy of the vacuum vanishes) we put $\omega_B = \omega_F = \omega$ and the normal ordering is no longer necessary in this case; $\hat{n}_B = \hat{a}^* \hat{a}$ and $\hat{n}_F = \hat{b}^* \hat{b}$ are the number operator for bosons and fermions, respectively.

On the other hand, in the spirit of supersymmetric quantum mechanics, we can construct the supercharges as follows

$$\hat{Q} = \sqrt{2\omega} \hat{a}^* \hat{b}, \quad \hat{Q}^* = \sqrt{2\omega} \hat{a} \hat{b}^*; \quad (131)$$

In terms of these supercharges

$$\hat{H} = \frac{\hbar}{2} (\hat{Q} \hat{Q}^* + \hat{Q}^* \hat{Q}). \quad (132)$$
The commutation relations read
\[
[\hat{Q}, \hat{Q}]_+ = [\hat{Q}^*, \hat{Q}^*]_+ = 0,
\]
\[
[\hat{Q}, \hat{H}]_- = [\hat{Q}, \hat{H}]_+ = 0.
\] (133)

The super-Weyl correspondence gives
\[
a = tr\{\hat{\Omega}_S \hat{a}\} = \sqrt{\frac{\omega}{2\hbar}} (x + \frac{ip}{\omega}), \quad a^* = tr\{\hat{\Omega}_S \hat{a}^*\} = \sqrt{\frac{\omega}{2\hbar}} (x - \frac{ip}{\omega}),
\]
\[
b = tr\{\hat{\Omega}_S \hat{b}\} = \frac{\theta}{\sqrt{\hbar}}, \quad b^* = tr\{\hat{\Omega}_S \hat{b}^*\} = -\frac{i}{\sqrt{\hbar}} \pi = \frac{\theta^*}{\sqrt{\hbar}}. \tag{134}
\]

and
\[
H = \hbar \omega (a^* \star_S a + b^* \star_S b) = \hbar \omega (a^* a + b^* b). \tag{135}
\]

where \(\star_S = \exp\{\frac{\hbar}{2} \hat{P}_S\}\) and
\[
\hat{P}_s = -\frac{i}{\hbar} \left( -\frac{\hat{\partial}}{a} \frac{\hat{\partial}}{a^*} + \frac{\hat{\partial}}{a} \frac{\hat{\partial}}{a^*} + \frac{\hat{\partial}}{b} \frac{\hat{\partial}}{b^*} + \frac{\hat{\partial}}{b} \frac{\hat{\partial}}{b^*} \right) \tag{136}
\]

The hamiltonian splits into the bosonic and fermionic parts
\[
H = H_B + H_F. \tag{137}
\]

For the supercharge operators the super-Weyl correspondence gives
\[
Q = tr\{\sqrt{2\omega} \hat{a}^* \hat{\tilde{b}}\} = \sqrt{2\omega} a^* \star_S b = \sqrt{2\omega} a^* b,
\]
\[
Q^* = tr\{\sqrt{2\omega} \hat{a} \hat{\tilde{b}}^*\} = \sqrt{2\omega} a \star_S b^* = \sqrt{2\omega} ab^*, \tag{138}
\]

and then
\[
Q \star_S Q = 0, \quad Q^* \star_S Q^* = 0,
\]
\[
h(Q \star_S Q^* + Q^* \star_S Q) = 2h\omega(a^* a + b^* b) = 2H. \tag{139}
\]

Finally the Wigner function of the ground state satisfies the equation
\[
H \star_S \rho_{W_0} = E_0 \rho_{W_0} = 0. \tag{140}
\]

One can easily find that the solution of this equation is given by
\[
\rho_{W_0} = \rho_{WB_0} \rho_{WF_0} \propto \exp\{-2(a^* a + b^* b)\}. \tag{141}
\]
VIII. FINAL REMARKS

In this paper we have studied the Weyl-Wigner-Moyal formalism for fermionic systems with a finite number of degrees of freedom.

The relevant objects involved in the WWM-formalism as the Stratonovich-Weyl quantizer, the Moyal $\star$-product, the Wigner functions and the normal ordering have been found. Two examples have been discussed in detail: the Fermi oscillator and the supersymmetric quantum mechanics. The relation to the results of Ref. [27] is also explicitly given. In this respect, our results are complementary to those of [27].

The extension of our present considerations to the fermionic systems with an infinite number of degrees of freedom will be reported in a future communication [34]. In particular we are going to deal with deformation quantization of fermionic fields coupled to electromagnetic fields [35], [36].

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[1] H. Weyl, Group Theory and Quantum Mechanics, (Dover, New York, 1931).
[2] E.P. Wigner, Phys. Rev. 40, 749 (1932).
[3] A. Groenewold, Physica 12 (1946) 405-460.
[4] J.E. Moyal, Proc. Camb. Phil. Soc. 45, 99 (1949).
[5] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. 111, 61 (1978); Ann. Phys. 111, 111 (1978).
[6] C.K. Zachos, “Deformation Quantization: Quantum Mechanics Lives and Works in Phase Space”, Int. J. Mod. Phys. A 17 (2002) 297, hep-th/0110114.
[7] A.C. Hirshfeld and P. Henselder, “Deformation Quantization in the Teaching for Quantum Mechanics”, Am. J. Phys. 70 (2002) 537.
[8] G. Dito and D. Sternheimer, “Deformation Quantization: Genesis, Developments and Metamorphoses, Deformation Quantization (Strasbourg 2001) Lect. Math. Theor. Phys. 1 Ed. de Gruyter, Berlin, IRMA (2002) pp. 9-54.
[9] G. Dito and F. J. Turrubiates, “The damped harmonic oscillator in deformation quantization,” Phys. Lett. A 352, 309 (2006) [arXiv:quant-ph/0510150].

[10] D. Sternheimer, “Quantization is Deformation”, talk given at the conference, *Topics in Deformation Quantization and Noncommutative Structures*, September 2005, Cinvestav, Mexico City, [http://www.fis.cinvestav.mx/~compean/kontsevich.html](http://www.fis.cinvestav.mx/~compean/kontsevich.html).

[11] N. Seiberg and E. Witten, “String theory and Noncommutative Geometry,” JHEP 9909, 032 (1999) [arXiv:hep-th/9908142].

[12] M. De Wilde and P.B.A. Lecomte, Lett. Math. Phys. 7 (1983) 487; H. Omori, Y. Maeda and A. Yoshioka, Adv. Math. 85 (1991) 224.

[13] B. Fedosov, J. Diff. Geom. 40, 213 (1994); *Deformation Quantization and Index Theory* (Akademie Verlag, Berlin, 1996).

[14] M. Gadella, M. A. del Olmo and J. Tosiek, “Geometrical origin of the *-product in the Fedosov formalism,” J. Geom. Phys. 55, 316 (2005) [arXiv:hep-th/0405157].

[15] M. Kontsevich, “Deformation Quantization of Poisson Manifolds I”, [q-alg/9709040] Lett. Math. Phys. 48, 35 (1999).

[16] F.A. Berezin, *The Method of Second Quantization*, (Academic Press, New York, 1966).

[17] R. Casalbuoni, “The Classical Mechanics for Bose- Fermi Systems”, Nuovo Cimmento 33 (1976) 389.

[18] F.A. Berezin and M.S. Marinov, “Particle Spin Dynamics as the Grassmann Variant of Classical Mechanics”, Ann. Phys. 104 (1977) 336.

[19] F.A. Berezin, *Introduction to Superanalysis*, D. Reidel Publishing Company, Dordrecht (1987).

[20] B. deWitt, *Supermanifolds*, Second Edition Cambridge University Press, Cambridge (1992).

[21] A. Lahiri, P. Kumar Roy and B. Bagchi, “Supersymmetry in Quantum Mechanics”, Int. J. Mod. Phys. A 5 (1990) 1383.

[22] R. Marnelius, “Half-Integer Ghost States and Simple BRST Quantization”, Nucl. Phys. B 294 (1987) 671; “Fermionic Quantum Mechanics and Superfields”, Int. J. Mod. Phys. A 5 (1990) 329.

[23] M. Bordemann, “The Deformation Quantization of Certain Super-Poisson Brackets and BRST-Cohomology”, in *Quantization, Deformations, and Symmetries*, Vol. II, Eds. G. Dito and D. Sternheimer, Kluwer Academic Publishers, Dordrecht (2000).

[24] M. Bordemann, H.-C. Herbig and S. Walmann, Commun. Math. Phys. 210 (2000) 107.

[25] M. Duetsch and K. Fredenhagen, “Deformation Stability of BRST-quantization,” AIP Conf. Proc. 453, 324 (1998) [arXiv:hep-th/9807215].

[26] C. Zachos, J. Math. Phys. 41 (2000) 5129.

[27] A.C. Hirshfeld and P. Henselder, “Deformation Quantization for Systems with Fermions”, Ann. Phys. (N.Y.) 302 (2002) 59.
[28] A.C. Hirshfeld and P. Henselder, “Clifforddization, Spin and Fermionic Star Products”, Annals Phys. 314 (2004) 75, quant-ph/0404168.

[29] N. Seiberg, “Noncommutative Superspace, $\mathcal{N} = \frac{1}{2}$ Supersymmetry, Field Theory and String Theory”, JHEP 0306, 010 (2003), arXiv:hep-th/0305248.

[30] R.L. Stratonovich, Sov. Phys. JETP 31, 1012 (1956); A. Grossmann, Commun. Math. Phys. 48, 191 (1976); J.M. Gracia Bondía and J.C. Varilly, J. Phys. A: Math. Gen. 21, L879 (1988), Ann. Phys. 190, 107 (1989); J.F. Cariñena, J.M. Gracia Bondía and J.C. Varilly, J. Phys. A: Math. Gen. 23, 901 (1990); M. Gadella, M.A. Martín, L.M. Nieto and M.A. del Olmo, J. Math. Phys. 32, 1182 (1991); J.F. Plebański, M. Przanowski and J. Tosiek, Acta Phys. Pol. B27 1961 (1996).

[31] W.I. Tatarskii, Usp. Fiz. Nauk 139, 587 (1983).

[32] M. Hillery, R.F. O’Connell, M.O. Scully and E.P. Wigner, Phys. Rep. 106, 121 (1984).

[33] S. Weinberg, The Quantum Theory of Fields Vol. I, (Cambridge University Press, Cambridge, 1995).

[34] I. Galaviz, H. García-Compeán, M. Przanowski and F.J. Turrubiates, “Deformation Quantization of Fermi Fields”, to appear (2006).

[35] H. García-Compeán, J. F. Plebański, M. Przanowski and F. J. Turrubiates, “Deformation quantization of classical fields,” Int. J. Mod. Phys. A 16, 2533 (2001) arXiv:hep-th/9909206.

[36] I. Galaviz, H. García-Compeán, M. Przanowski and F.J. Turrubiates, “Deformation Quantization of Spinor Electrodynamics”, to appear (2006).