Visualising the arithmetic of quadratic imaginary fields

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Abstract. We study the orbit of $\mathbb{R}$ under the Bianchi group $\text{PSL}_2(\mathcal{O}_K)$, where $K$ is an imaginary quadratic field. The orbit, called a Schmidt arrangement $S_K$, is a geometric realisation, as an intricate circle packing, of the arithmetic of $K$. This paper presents several examples of this phenomenon. First, we show that the curvatures of the circles are integer multiples of $\sqrt{-\Delta}$ and describe the curvatures of tangent circles in terms of the norm form of $\mathcal{O}_K$. Second, we show that the circles themselves are in bijection with certain ideal classes in orders of $\mathcal{O}_K$, the conductor being a certain multiple of the curvature. This allows us to count circles with class numbers. Third, we show that the arrangement of circles is connected if and only if $\mathcal{O}_K$ is Euclidean. These results are meant as foundational for a study of a new class of thin groups generalising Apollonian groups, in a companion paper.

1. Introduction

Schmidt arrangements. The matrix groups $\text{PSL}_2(\mathcal{O}_K)$ are called Bianchi groups when $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field $K$. They are named for Bianchi’s initial study in the 1890’s [2], as a natural generalisation of the modular group, $\text{PSL}_2(\mathbb{Z})$, and as a natural family of discrete subgroups of $\text{PSL}_2(\mathbb{C})$, which can be realised as the group of isometries of hyperbolic 3-space.

The Bianchi groups have a beautiful visual manifestation as Möbius transformations of the extended complex plane, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We will identify the group $\text{Möb}_+(2)$ of orientation-preserving Möbius transformations acting on $\hat{\mathbb{C}}$ with $\text{PSL}_2(\mathbb{C})$, via the association

$$
\left( z \mapsto \frac{\alpha z + \gamma}{\beta z + \delta} \right) \leftrightarrow \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}.
$$

Möbius transformations act transitively on the collection of circles in $\hat{\mathbb{C}}$ (where straight lines are circles through $\infty$).

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Figure 1. Schmidt arrangement $\mathcal{S}_{\mathbb{Q}(\sqrt{-3})}$. The image includes those circles of curvature bounded by 20 within the fundamental parallelogram of the ring of integers.

Figure 2. Schmidt arrangements $\mathcal{S}_{\mathbb{Q}(\sqrt{-1})}$. The image includes those circles of curvature bounded by 20 within the fundamental parallelogram of the ring of integers.

Of particular beauty is the orbit of the real line $\mathbb{R}$ under a Bianchi group, which takes the form of an intertwined collection of circles dense in $\hat{\mathbb{C}}$, each having curvature (inverse radius) an integer multiple of $\sqrt{-\Delta}$ (where $\Delta$ is
Figure 3. Schmidt arrangements $S_K$ of quadratic imaginary fields $K$. Clockwise from top left: $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-7})$, $\mathbb{Q}(\sqrt{-15})$, $\mathbb{Q}(\sqrt{-11})$. In each case, the image includes those circles of curvature bounded by 20 within the fundamental parallelogram of $\mathcal{O}_K$.

The circles may intersect only at angles determined by the unit group of $\mathcal{O}_K$; only tangentially in the case of unit group $\{\pm 1\}$ (see Sections 4 and 5 for these and other basic properties). See Figures 2 and 3 for examples. We call the collection a Schmidt arrangement, for its appearance, in the cases of discriminant $-3, -4, -7, -8, -11$ in Asmus Schmidt’s work on complex continued fractions [19, 20, 21, 22, 23].
Definition 1.1. Let $K$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. The Schmidt arrangement of $K$, denoted $S_K$, is the orbit of the real line $\mathbb{R}$ in $\hat{\mathbb{C}}$ under the group $\text{PSL}_2(\mathcal{O}_K) \leq \text{PSL}_2(\mathbb{C})$. The circles in the orbit, i.e. the individual images of $\mathbb{R}$, are called $K$-Bianchi circles.

The purpose of this paper is to explore the ways in which the arithmetic of $\mathcal{O}_K$ shines through in the structure of $S_K$.

In fact, Schmidt’s work is the first example of the phenomenon. The Schmidt arrangement $S_K$ was envisioned by Schmidt as a generalization of the theory of Farey fractions and continued fractions to imaginary quadratic fields. The relation between Bianchi groups and such complex continued fractions is in direct analogy to the connection between $\text{PSL}_2(\mathbb{Z})$ and real continued fraction expansions, and in this author’s opinion is the most natural such generalization she has encountered\(^2\).

The Schmidt arrangement is also closely related to the action of the Bianchi groups as isometries of the hyperbolic space $\mathbb{H}^3$. The extended complex plane $\hat{\mathbb{C}}$ is identified with the boundary of $\mathbb{H}^3$, and a fundamental domain for $\text{PSL}_2(\mathcal{O}_K)$ in $\mathbb{H}^3$ has $h(K)$ cusps on the boundary, where $h(K)$ is the class number of $K$. One can imagine the Schmidt arrangement as a ‘shadow’ of the tesselation of $\mathbb{H}^3$ by fundamental domains.

It turns out that the Bianchi groups, among groups $\text{PSL}_2(\mathcal{O}_K)$ for all rings of integers $\mathcal{O}_K$, have exceptional properties. For instance, $\text{PSL}_2(\mathcal{O}_K)$ contains non-congruence subgroups of finite index when $K = \mathbb{Q}$ or $K$ is quadratic imaginary (in contrast to all other $K$) [14, 24]. Fine, who showed that for $\Delta \neq -3$ the Bianchi groups are non-trivial free products with amalgamation [8], points out that the structure of the Bianchi groups is greatly influenced by the number theory of the ring of integers, specifically by whether the ring is Euclidean [7]. Alone among the class of $\text{PSL}_2(\mathcal{O}_K)$ for number fields $K$, the Bianchi groups for non-Euclidean quadratic imaginary $\mathcal{O}_K$ fail to be generated by elementary matrices [16].

Schmidt arrangements attracted the author’s interest in connection with Apollonian circle packings: $S_K$ also appears as an ‘Apollonian superpacking’ in [12]. Some of the results of this paper generalise results about Apollonian packings to be found in [10, 12]. This relationship is generalised in a companion paper, which is devoted to the study of $K$-Apollonian packings living in $S_K$ and analogues of the Apollonian group [26]. The results of this paper will prove useful in that analysis.

The main results of the present paper form three examples of the arithmetic of $\mathcal{O}_K$ appearing in the structure of $S_K$.

\(^1\)We will use the notation $S_K$ for a subset of $\hat{\mathbb{C}}$ and for a collection of circles, without fear of confusion.

\(^2\)For example, the points of tangency of the collection of circles with curvatures bounded above by $N$ are exactly those elements of $K$ which can be expressed with a denominator bounded in norm by $2N$; this is an analogue of the Farey sequence.
1.1. **First example: curvatures and norms.** Our first example of the arithmetic in \( S_K \) is modest: we illuminate the recursive structure of the circle packing in terms of the norm form. If \( K \neq \mathbb{Q}(\sqrt{-3}) \), then the \( K \)-Bianchi circles of \( S_K \) are either tangent or disjoint (see Section 5). Fixing one circle in the packing, one can describe the curvatures of all tangent circles using the norm form of \( \mathcal{O}_K \).

**Theorem 1.2** (Introductory Version of Theorem 6.3). *If \( C \) and \( C' \) are immediately tangent\(^3\) \( K \)-Bianchi circles, then the sum of their curvatures is \( \sqrt{-\Delta N(\beta)} \) where \( \beta \) is the reduced denominator of the point of tangency.)*

That the circles tangent to one fixed circle in an Apollonian circle packing had curvatures that were values of a translated quadratic form was observed in [10, 17, 18] and used as a principle tool in several major results toward conjectures on curvatures in Apollonian packings [3, 4]. An excellent exposition is contained in [9].

See Sections 5 and 6 for details.

1.2. **Second example: circles and ideal classes.** The second example provides a deeper explanation for the first: one can associate certain ideal classes of orders of \( \mathcal{O}_K \) to the circles in \( S_K \). Denote the order of conductor \( f \) in \( \mathcal{O}_K \) by \( \mathcal{O}_f \). Write \( \text{Pic} (\mathcal{O}_f) \) for the class group of \( \mathcal{O}_f \), the quotient of its invertible fractional ideals by principal fractional ideals. There is a homomorphism of groups

\[
\theta_f : \text{Pic} (\mathcal{O}_f) \to \text{Pic} (\mathcal{O}_K)
\]

which is defined by \( \theta_f([a]) = [a\mathcal{O}_K] \) for an ideal \( a \) of \( \mathcal{O}_f \).

**Definition 1.3.** Two \( K \)-Gaussian circles are *equivalent* (denoted \( \sim \)) if one can be transformed to the other by \( \mathcal{O}_K \)-translations combined with maps of the form \( z \mapsto uz \) for a unit \( u \) of \( \mathcal{O}_K \).

Then our second exhibit is a natural bijection between the \( K \)-Bianchi circles modulo equivalence and certain ideal classes.

**Theorem 1.4.** There is a bijection of sets

\[
S_K / \sim \leftrightarrow \bigcup_{f \in \mathbb{Z}} \theta_f^{-1}(\mathcal{O}_K)
\]

in such a way that the curvature of the circle is equal to \( \sqrt{-\Delta} \) times the conductor of the ideal.

The relationship between curvatures and conductors allows one to count the circles of given curvature by comparing to the ideal class number of the corresponding order. In the case of \( K = \mathbb{Q}(i) \), this recovers Theorem 4.2 on Apollonian circle packings of [10] in the context of Theorem 6.3 of [12].

\(^3\)See Section 6 for the definition of ‘immediately.’
It is tantalizing to ask what ‘tangency’ of ideal classes might imply for the arithmetic of \( \mathcal{O}_K \); at the moment the author has no particularly compelling answer to this.

1.3. Third example: connectedness and Euclideanity. Our final example concerns the topological structure of \( \mathcal{S}_K \).

**Theorem 1.5** (Introductory version of Theorem 10.1). Suppose \( K \) is a quadratic imaginary field. Then \( \mathcal{O}_K \) is Euclidean if and only if \( \mathcal{S}_K \) is connected.

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**A note on the figures.** The images in this paper were produced with Sage Mathematics Software [27].

2. Notations

Throughout the paper, \( K \) is a quadratic imaginary field with discriminant \( \Delta < 0 \) and ring of integers \( \mathcal{O}_K \). The ring \( \mathcal{O}_K \) has an integral basis \( 1, \tau \), where

\[
\tau^2 = \begin{cases} 
\Delta/4 & \Delta \equiv 0 \pmod{4} \\
\tau + (\Delta - 1)/4 & \Delta \equiv 1 \pmod{4}
\end{cases}
\]

It is convenient to write

\[
\epsilon = \text{Tr}(\tau) = \begin{cases} 
0 & \Delta \equiv 0 \pmod{4} \\
1 & \Delta \equiv 1 \pmod{4}
\end{cases}
\]

In other words,

\[
\tau, \overline{\tau} = \frac{\epsilon}{2} \pm \frac{\sqrt{\Delta}}{2}
\]

Write \( N : K \to \mathbb{Q} \) for the norm map \( \alpha \mapsto \alpha \overline{\alpha} \). In particular,

\[
N(\tau) = -\frac{\Delta - \epsilon}{4}.
\]

We follow [11] in writing \( \text{M"ob}(2) \) for the conformal group, the group of conformal maps of \( \hat{\mathbb{C}} \), including Möbius transformations and reflections (i.e. allowing complex conjugation); and writing \( \text{M"ob}_+(2) \) for the group of Möbius transformations without reflections.
3. Oriented $K$-Bianchi circles

Although the main results in the introduction are phrased in terms $K$-Bianchi circles, it will be more convenient to use the notion of an oriented $K$-Bianchi circle. This is most easily accomplished by considering images $M(\mathbb{R})$ for $M \in \text{PGL}_2(O_K)$ (instead of $\text{PSL}_2$), while still considering these modulo right multiplication by $\text{PSL}_2(\mathbb{Z})$, as we discuss below. (Note, however, that the field $\mathbb{Q}(i)$ behaves slightly differently, as explained below.)

**Definition 3.1.** The orientation of a circle $M(\mathbb{R})$, $M \in \text{PGL}_2(O_K)$, is **positive** if $M(0)$, $M(1)$ and $M(\infty)$ appear in (some cyclic permutation of) that order when the circle is traversed counterclockwise. It is **negative** otherwise. The orientation of $\mathbb{R}$ under the identity map is positive, or, equivalently, to travel $\mathbb{R}$ counterclockwise is to travel to the right. The **interior** of an oriented circle is the area to your left as you travel along the circle according to its orientation.

Note that these conventions differ from [25].

**Definition 3.2.** An **oriented $K$-Bianchi circle** is any circle $C$ of the form $M(\mathbb{R})$ for $M \in \text{PGL}_2(O_K)$, considered together with its orientation. Write $\hat{S}_K$ for the collection of oriented $K$-Bianchi circles.

Then
\[
\text{PGL}_2(O_K) / \text{PSL}_2(\mathbb{Z})
\] (1)
parametrizes the oriented $K$-Bianchi circles; we will call this collection $\hat{S}_K$ also. This space is well-defined because $\text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})/\{\pm I\}$ can be considered a subgroup of $\text{PGL}_2(O_K) = \text{GL}_2(O_K)/\{U_K I\}$, where $U_K$ represents the units in $O_K$ (the relevant fact is that $u \in U_K$, $u^2 = 1$ implies $u = \pm 1$; therefore inequivalent matrices in $\text{PSL}_2(\mathbb{Z})$ do not become equivalent modulo $U_K$). Multiplication by $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the right swaps orientation while preserving the circle.

Suppose $K \neq \mathbb{Q}(i)$. There is a 2-to-1 map
\[
\hat{S}_K \rightarrow S_K
\]
‘forgetting orientation’: that is, each class of $\text{PGL}_2(O_K)/\text{PSL}_2(\mathbb{Z})$ has a representative of determinant 1 or $-1$ (but not both); multiplying on the right by $N$ if necessary returns you to $\text{PSL}_2(O_K)/\text{PSL}_2(\mathbb{Z})$ (but does not change the image $M(\mathbb{R})$ as a subset of $\hat{C}$). In particular, the circles $M(\mathbb{R})$ for $M \in \text{PGL}_2(O_K)$ have orientation equal to the sign of their determinant.

Now we return to the case of $K = \mathbb{Q}(i)$, where there is a wrinkle. The images $M(\mathbb{R})$ for $M \in \text{PGL}_2(\mathbb{Z}[i])$ include more circles than $S_K$, and the sign of the determinant is no longer meaningful on an equivalence class of matrices. By passing from $\text{PSL}$ to $\text{PGL}$, we obtain two orthogonal copies of $S_K$. See Figure 4. The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}[i])$ but not
in $\text{PSL}_2(\mathbb{Z})$, and when multiplied on the right, this swaps orientation while preserving the circle. With this in mind, we will define $S'_K$ to be the circles of $S_K$, considered up to orientation, parametrised by 

$$
\text{PSL}_2(\mathbb{Z}[i]) / \text{PSL}_2(\mathbb{Z}).
$$

If $K \neq \mathbb{Q}(i)$, then we will define $S'_K = \hat{S}_K$.

To summarize:

$$
S_K = \{ K\text{-Bianchi circles} \} = \text{PSL}_2(\mathcal{O}_K) / \text{PSL}_2(\mathbb{Z})
$$

$$
\hat{S}_K = \{ \text{oriented } K\text{-Bianchi circles} \} = \text{PGL}_2(\mathcal{O}_K) / \text{PSL}_2(\mathbb{Z})
$$

$$
S'_K = \begin{cases} 
\hat{S}_K & K \neq \mathbb{Q}(i) \\
S_K & K = \mathbb{Q}(i)
\end{cases}
$$

**Definition 3.3.** The *curvature* (elsewhere sometimes called a *bend*) of a circle $C \in \hat{S}_K$ is the reciprocal of its radius, with a sign determined by its orientation (‘+’ for positive orientation, ‘−’ for negative). The *co-curvature* of $C$ is defined as the curvature of a new circle obtained from $C$ by $z \mapsto 1/z$. The *curvature-centre* of $C$ when not passing through $\infty$ is its centre times its curvature. If $C$ passes through $\infty$, then $C$ is a straight line perpendicular to some unit vector $v$ pointing away from its exterior and toward its interior; take its curvature-centre to be $z \in \mathbb{C}$ such that the vector from 0 to $z$ is $v$.

### 4. Properties of a $K$-Bianchi Circle

The following proposition appears for the case of $\mathbb{Z}[i]$ in [25].

**Proposition 4.1.** Consider an oriented $K$-Bianchi circle expressed as the image of $\mathbb{R}$ under a transformation of the form

$$
M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathcal{O}_K, \quad \alpha \delta - \beta \gamma \in \mathcal{O}_K^*.
$$

The curvature of the circle is given by

$$
i(\beta \delta - \gamma \delta),
$$

the co-curvature of the circle is given by

$$
i(\alpha \gamma - \beta \gamma),
$$

and the curvature-centre is given by

$$
i(\alpha \delta - \gamma \beta).
$$

Finally, if there exists an oriented $K$-Bianchi circle of curvature $b$, co-curvature $b'$ and curvature-centre $z$, then

$$
b'b = zz - 1. \quad (2)
$$
Figure 4. $\hat{S}_K$ for $K = \mathbb{Q}(i)$, showing the region $\{s + it : 0 \leq s, t \leq 1\}$. Red indicates images under $\text{PSL}_2(\mathcal{O}_K)$, while blue indicates images under the non-trivial coset of $\text{PSL}_2(\mathcal{O}_K)$ in $\text{PGL}_2(\mathcal{O}_K)$.

Note that the curvature and co-curvature are the real parts of elements of $2i\mathcal{O}_K$, so they are integer multiples of $\sqrt{-\Delta}$; the integer alone will be referred to as the reduced curvature or reduced co-curvature, respectively.

Proof of Proposition 4.1. Define the following functions on matrices $M \in \text{PGL}_2(\mathbb{C})$:

$$H_1(M) = 2 \text{Im}(\beta\delta)/|\det(M)|^{1/2},$$
$$H_2(M) = 2 \text{Im}(\alpha\gamma)/|\det(M)|^{1/2},$$
$$H_3(M) = -\text{Im}(\beta\gamma + \alpha\delta)/|\det(M)|^{1/2},$$
$$H_4(M) = \text{Im}(i\alpha\delta - i\beta\gamma)/|\det(M)|^{1/2}.$$

(Exponent $1/2$ denotes the positive square root and Im denotes the imaginary part.) We will show that the image of the real line under an $M \in$
PGL\(_2(\mathbb{C})\) has curvature \(H_1(M)\), co-curvature \(H_2(M)\) and curvature-centre \(H_3(M) + iH_4(M)\). These are exactly the formulae of the statement.

The first key observation is that each \(H_i\) is invariant under replacing such an \(M\) by any other transformation \(M' = MP\) for some \(P \in \text{PSL}_2(\mathbb{R})\). To see this, we choose representative matrices of unit determinant, so that each \(H_i\) is the imaginary part of a Hermitian form on the column vectors of the matrix \(M\). The imaginary part \(H\) of any Hermitian form satisfies
\[
H(ax + by, cx + dy) = (ad - bc)H(x, y)
\]
whenever \(a, b, c, d \in \mathbb{R}\).

Since \(M, M' \in \text{PGL}_2(\mathbb{C})\) map \(\mathbb{R}\) to the same oriented circle if and only if \(M' = MP\) for some \(P \in \text{PSL}_2(\mathbb{R})\), it suffices to prove the statement for any \(M'\) taking \(\mathbb{R}\) to the oriented circle in question.

Every positively oriented circle not passing through \(\infty\) can be expressed as an image of \(\mathbb{R}\) under the transformation
\[
M = \left( \begin{array}{cc} \frac{i-k}{\sqrt{2}k} & \frac{k+b}{\sqrt{2}k} \\ \frac{1}{\sqrt{2}k} & \frac{1}{\sqrt{2}k} \end{array} \right), \quad b \in \mathbb{C}, k \in \mathbb{R}^>0,
\]
since this is a composition of the transformation \(\left( \begin{array}{cc} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{array} \right)\) taking \(\mathbb{R}\) to the positively oriented unit circle, with a dilation \(\left( \begin{array}{cc} \sqrt{k} & 0 \\ 0 & 1/\sqrt{k} \end{array} \right)\) and a translation \(\left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right)\). The image of the real axis under this \(M\) is the positively oriented circle with centre \(b\) and radius \(k\), and \(M\) satisfies \(H_1(M) = 1/k\), and \(H_3(M) + iH_4(M) = b/k\). The case of circles through \(\infty\) is similar and simpler.

Inversion in the unit circle is accomplished by the map \(z \mapsto \frac{1}{z}\), and takes a circle with co-curvature \(b\) to a circle with curvature \(b\). The curvature of the image of \(M\) under the Möbius transformation \(z \mapsto \frac{1}{z}\) is \(H_2(M)\).

If we change the orientation of the circle, replacing \(M\) with \(M \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)\), then the values of the \(H_i\) change sign. \(\Box\)

There is a natural way to view circles as points in Minkowski space [11, 13]. Let \(\mathbb{M}\) be Minkowski space, that is, the vector space \(\mathbb{R}^4\) endowed with inner product
\[
\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle = -x_1y_2 - x_2y_1 + x_3y_3 + x_4y_4.
\]
Define the Pedoe map
\[
\pi : \mathcal{S}_K \rightarrow \mathbb{M}
\]
by letting \(\pi(C) = (b, b', x, y)\) where \(b\) and \(b'\) are the curvature and co-curvature of \(C\), and \(x + iy = z\) is the curvature-centre. By Proposition 4.1, \(\langle \pi(C), \pi(C) \rangle = 1\), so that the image of the Pedoe map lies in the hypersurface \(Q_M = 1\) where \(Q_M\) is the quadratic form of \(\mathbb{M}\). Then the inner product gives an inner product on circles carrying geometric information.
Proposition 4.2 (Proposition 2.4 of [13]). Let \( v_i = \pi(C_i) \) for two circles \( C_1, C_2 \) which are not disjoint. Then \( \langle v_1, v_2 \rangle = \cos \theta \), where \( \theta \) is the angle between the two circles as in Figure 5. In particular,

1. \( \langle v_1, v_2 \rangle = -1 \) if and only if the circles are tangent externally

2. \( \langle v_1, v_2 \rangle = 1 \) if and only if the circles are tangent internally

3. \( \langle v_1, v_2 \rangle = 0 \) if and only if the circles are mutually orthogonal

Note that the Pedoe product of two \( K \)-Bianchi circles lies in \( \frac{1}{2} \mathbb{Z} \) (see Proposition 4.1). This indicates that the angles of intersection of \( K \)-Bianchi circles will be restricted, as we will see in the next section.

5. The intersection of circles

The delicate arrangement of circles in \( S_K \) is controlled by the number theory of \( K \). In this section, we examine some of their basic properties:

1. We show that oriented \( K \)-Bianchi circles intersect only at \( K \)-points and only in angles prescribed by the unit group of \( \mathcal{O}_K \) (usually only tangentially).

2. We describe exactly the circles passing through any point \( z \in K \).

3. We give some sufficient and necessary conditions for circles of various curvatures to be tangent or pass through a given point.

Proposition 5.1. Two oriented \( K \)-Bianchi circles intersect only at \( K \)-points.

Proof. Without loss of generality, we may consider one of the two circles to be the real line. Let the other circle \( C \) be \( M(\mathbb{R}) \) for some \( M \in \text{PGL}_2(\mathcal{O}_K) \). Then any intersection point \( t \) of \( C \) with \( \mathbb{R} \) satisfies \( M(t) \in \mathbb{R} \) for \( t \in \mathbb{R} \). In other words, \( M(t) = \overline{M}(\overline{t}) = M(t) \). Suppose

\[
M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}
\]

Then \( M(t) = \overline{M}(t) \) becomes a quadratic equation

\[
\text{Im}(\alpha\overline{\beta})t^2 + \text{Im}(\gamma\overline{\beta} + \alpha\delta)t + \text{Im}(\gamma\overline{\delta}) = 0.
\]
Let \( M' \in \text{PGL}_2(\mathcal{O}_K) \) have curvature-centre \( z = x + iy \), curvature \( b \) and co-curvature \( b' \). Then the coefficients of the quadratic are, respectively, \( b' \), \( x \), and \( b \); these are all in \( \sqrt{-\Delta \mathbb{Z}} \). The discriminant of this equation is (using (2))

\[
x^2 - 4bb' = 4 - 3(x^2 + y^2) - y^2
\]

Since \( t \in \mathbb{R} \), this discriminant is non-negative. As \( z = x + iy \in i\mathcal{O}_K \), we have \( x^2 + y^2 \in \mathbb{Z} \). Thus the possible non-negative values of the discriminant are:

\[
4, 1, \frac{1}{4}, 0,
\]

and in any case, \( t \) must be a rational root. We return to the general case by applying a Möbius transformation with coefficients in \( K \) and find that the intersection points lie in \( K \).

**Proposition 5.2.** Two oriented \( K \)-Bianchi circles may intersect at an angle of \( \theta \) only if \( e^{i\theta} \in \mathcal{O}_K \). In particular, if the unit group of \( \mathcal{O}_K \) is \( \{\pm 1\} \), then circles may only be disjoint or tangent.

**Proof.** Consider the circles passing through 0. These are all images of \( \mathbb{R} \) under Möbius transformations

\[
\begin{pmatrix}
0 & \gamma \\
1 & \delta
\end{pmatrix} \in \text{PGL}_2(\mathcal{O}_K),
\]

where \( \gamma \) is a unit in \( \mathcal{O}_K \). These have curvature-centre \( i\gamma \). Therefore the centre lies on the lines through 0 which pass through the units of \( \mathcal{O}_K \). Hence the circles through 0 may meet at angles \( \theta \) for which \( e^{i\theta} \in \mathcal{O}_K \).

Since the intersection points of any two \( K \)-Bianchi circles lie in \( K \) (by Proposition 5.1), and Möbius transformations preserve angles, we may transport the point of intersection to the origin to prove the general statement. \(\square\)

We say that \( \alpha, \beta \in \mathcal{O}_K \) are coprime if the ideals they generate are coprime, i.e. \( (\alpha) + (\beta) = (1) \).

**Proposition 5.3.** Let \( \alpha/\beta \in K \) be such that \( \alpha \) and \( \beta \) are coprime. Suppose the unit group of \( K \) has size \( |U_K| = n \). Then the oriented \( K \)-Bianchi circles passing through \( \alpha/\beta \) form \( n \) generically different \( \mathbb{Z} \)-families, namely, for each \( u \in U_K \), the images of \( \mathbb{R} \) under the transformations

\[
\begin{pmatrix}
\alpha & u\gamma + k\tau\alpha \\
\beta & u\delta + k\tau\beta
\end{pmatrix}, \quad k \in \mathbb{Z},
\]

where \( \gamma, \delta \) is a particular solution to \( \alpha\delta - \beta\gamma = 1 \). The curvatures of the circles in one family are all congruent modulo \( \sqrt{-\Delta N(\beta)} \). The centres of the circles in a given family lie on a single line through \( \alpha/\beta \). The family given by unit \( u \) contains the same circles as the family given by \( -u \), but with opposite orientations.
Proof. Any circle passing through $\alpha/\beta$ has a representative matrix $M$ with first column $(\alpha, \beta)^T$, i.e. of the form
\[
\begin{pmatrix}
\alpha & \gamma \\
\beta & \delta
\end{pmatrix}
\]
where $\alpha \delta - \gamma \beta$ is a unit in $\mathcal{O}_K$ (and $\alpha, \beta, \gamma, \delta \in \mathcal{O}_K$). There is a particular solution, call it $\gamma, \delta$, for the unit 1, and the homogeneous solutions are $\gamma + \eta \alpha, \delta + \eta \beta$ for $\eta \in \mathcal{O}_K$. Since $\eta \in \mathbb{Z}$ doesn’t change the circle, we obtain a family of circles
\[
\begin{pmatrix}
\alpha & \gamma + k\tau \alpha \\
\beta & \delta + k\tau \beta
\end{pmatrix}, \quad k \in \mathbb{Z}.
\]
The curvatures of this family are $i(\beta \delta - \delta \beta) + k(\sqrt{-\Delta})N(\beta)$ and therefore they are all distinct circles. Their curvature-centres are $i(\gamma \beta - \alpha \delta) + k(\sqrt{-\Delta})\alpha \beta$. The difference between two centres ($k = 0$ and $k = k$) has the form $\begin{pmatrix} kX \\ Y(k) \end{pmatrix}$, where $X$ doesn’t depend on $k$, and $Y(k)$ is real. Therefore, the circles in one family all lie with their centres on a single line. Since their curvatures approach $\infty$, this line must pass through $\alpha/\beta$. To justify the final sentence of the statement, note that changing the sign of the unit changes the signs of the curvature and co-curvature but does not change the collection of centres. □

Note that replacing $\gamma, \delta$ with $u\gamma, u\delta$ for a unit $u$ may change the congruence class modulo $N(\beta)$ of the curvatures. From the proof of Proposition 5.2, one learns that the unit $u = e^{i\theta}$ acts by rotation by $\theta$ on the line of centres for a family.

**Proposition 5.4.** If $K \neq \mathbb{Q}(\sqrt{-3})$ then the circles of $\mathcal{S}_K$ may only intersect tangently.

*Proof.* For $K \neq \mathbb{Q}(i)$, this is immediate from Proposition 5.2. For $K = \mathbb{Q}(i)$, oriented $K$-Bianchi circles may intersect orthogonally. However, from Proposition 5.3, two orthogonal families at a point have determinants $\{\pm 1\}$ and $\{\pm i\}$. In an equivalence class of $\text{PGL}_2(\mathcal{O}_K)$, the determinants are either $\{\pm 1\}$ or $\{\pm i\}$; the subgroup $\text{PSL}_2(\mathcal{O}_K)$ consists of those having the former shape. Therefore, by restricting to circles in $\mathcal{S}_K$, one considers only $M(\mathbb{R})$ for $M \in \text{PGL}_2(\mathcal{O}_K)$ of determinant $\pm 1$. At a single point, we obtain only two of the four families described in Proposition 5.3 and these all intersect tangently. □

For $K = \mathbb{Q}(\sqrt{-3})$, the circles of $\mathcal{S}_K$ do intersect at angles of $\pi/3$ and $2\pi/3$, for which reason this field presents extra difficulties.

6. **Immediate tangency**

The following definition will be key to describing the tangency structure of $\mathcal{S}_K$. 

Definition 6.1. A collection $\mathcal{P} \subset S_K$ of circles straddles a circle $C$ if it intersects both the interior and exterior of $C$. Two oriented $K$-Bianchi circles $C_1, C_2 \in S'_K$ are immediately tangent if they are externally tangent in such a way that the pair straddles no circles of $S'_K$.

Proposition 6.2. Let $C \in S'_K$ be an oriented $K$-Bianchi circle with $K$-rational point $x$. Then there exists

$$M_C = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \text{PGL}_2(\mathcal{O}_K)$$

such that $C = M_C(\mathbb{R})$ and $x = \alpha/\beta$. Furthermore, there exists exactly one oriented $K$-Bianchi circle $C' \in S'_K$ immediately tangent to $C$ at $x$, given by $C' = M_{C'}(\mathbb{R})$ where

$$M_{C'} = \begin{pmatrix} \alpha & -\gamma + \tau \alpha \\ \beta & -\delta + \tau \beta \end{pmatrix} = M_C \begin{pmatrix} 1 & \tau \\ 0 & -1 \end{pmatrix} \in \text{PGL}_2(\mathcal{O}_K).$$

Proof. Let $M$ be some matrix such that $M(\mathbb{R}) = C$. Then the $K$-rational points of $C$ are given by $\alpha/\beta$ where $(\alpha \beta)^T$ is a $\mathbb{Z}$-linear combination of the column vectors of $M$. Hence, some $N \in \text{PSL}_2(\mathbb{Z})$ will give the desired $M_C = MN$.

Proposition 5.3 lists the circles tangent to $C$ at $x$. There are two families of circles tangent to $C$: those whose centres lie on the line passing through $z$ and the centre of $C$ (given by $u = \pm 1$ in the language of Proposition 5.3). The circle $C'$ must be a member of one of these two families. The way each family is described, all the circles are given with an orientation such that they pass through $z$ in the same direction. With this orientation, the circles of each family are ordered by inclusion. The circles $C$ and $C'$ must be `consecutive' in the unoriented family obtained from these two families that differ only by orientation (otherwise $C$ and $C'$ will straddle another circle). That is, $k = \pm 1$, $u = -1$. The sign of $k$ is determined by the requirement that the circles have disjoint interiors (using the wrong sign will mean disjoint exteriors). \hfill \Box

Theorem 6.3. Fix an oriented $K$-Bianchi circle $C \in S'_K$ of reduced curvature $b$, given by $M(\mathbb{R})$ for

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}.$$

Let

$$\Lambda = \beta \mathbb{Z} + \delta \mathbb{Z} \subset \mathcal{O}_K.$$

Then the reduced curvatures of the collection of oriented $K$-Bianchi circles immediately tangent to $C$ are exactly the primitive values of the translated integral binary quadratic form

$$N(x) - b, \quad x \in \Lambda.$$
Proof. Immediate from Proposition 6.2 and the fact that the $K$-rational points on $C$ have denominators chosen from $\Lambda$.

7. LATTICES AND CONDUCTORS

Tacit in the results of the last sections is a study of rank two $\mathbb{Z}$-sublattices of $\mathcal{O}_K$; this is exactly the study of ideals of orders $\mathcal{O}_f$ of $K$, as we will presently explain.

We begin with some background on lattices. If one lattice is contained in another, $\Lambda \subseteq \Lambda'$, then the index of $\Lambda$ in $\Lambda'$, denoted $[\Lambda' : \Lambda]$, is defined to be the size of the quotient $\Lambda'/\Lambda$. In this paper, we consider rank two $\mathbb{Z}$-lattices in $\mathcal{O}_K$, and the index in $\mathcal{O}_K$ will be called the covolume. If a basis for $\Lambda$ is given by $a + b\omega, c + d\omega$, then its covolume is $|ad - bc|$.

In the case that $\Lambda$ is equal to an integral ideal of $\mathcal{O}_K$, the covolume is equal to the norm of the ideal. If it is an integral ideal of an order $\mathcal{O}_f$ of conductor $f$, then its covolume is $f$ times the norm. A fractional ideal class of some $\mathcal{O}_f$ is a class of lattices up to homothety (defined as scaling by an element of $\mathcal{O}_K$).

The order of any rank two $\mathbb{Z}$-lattice $\Lambda \in \mathcal{O}_K$ is defined as

$$\text{Ord}(\Lambda) = \{\alpha \in \mathcal{O}_K : \alpha\Lambda \subseteq \Lambda\}.$$ 

The lattice $\Lambda$ is always a representative lattice for some fractional ideal class of $\text{Ord}(\Lambda)$, which is an order of $K$. Homothety preserves the order of a lattice.

Lemma 7.1. Let $\Lambda$ be a rank two $\mathbb{Z}$-lattice contained in $\mathcal{O}_K$. The following are equivalent:

1. $\Lambda$ has a basis of coprime elements.
2. $\Lambda$ has conductor equal to its covolume.

Furthermore, an ideal class $[a]$ satisfies $\theta([a]) = [\mathcal{O}_K]$ if and only if it has a representative lattice $\Lambda$ satisfying the conditions above. In this case, $\Lambda$ is unique, up to homothety by a unit of $\mathcal{O}_K$. Further, if $\beta, \delta$ form a basis for $\Lambda$, then the covolume is equal to $\frac{1}{\sqrt{-\Delta}} |\beta\delta - \beta\delta|$.

We will call such a lattice primitive.

Proof. Two lattices in the same fractional ideal class of $\mathcal{O}_f$, both lying in $\mathcal{O}_K$ and having covolume $f$ are necessarily related by homothety by a unit of $\mathcal{O}_K$.

If $\Lambda$ satisfies the first of the two conditions, then its $\mathcal{O}_K$-span is $\mathcal{O}_K$, i.e. $\theta([\Lambda]) = [\mathcal{O}_K]$. Conversely, if $\theta([\Lambda]) = [\mathcal{O}_K]$, then some homothety of $\Lambda$ generates $\mathcal{O}_K$, that is, the homothetic lattice satisfies the first condition.

Now suppose $\Lambda$ has a basis $\beta = a + b\omega, \delta = c + d\omega$ of coprime elements. From the basis, the covolume of $\Lambda$ is $|ad - bc|$. It is easy to check that

$$\beta\delta - \beta\delta = \sqrt{\Delta}(bc - ad),$$

(3)
from which the last sentence of the Lemma statement follows.

We will now compute the conductor of $\Lambda$ in terms of its covolume.

Let $O_f$ be the order of $\Lambda$, having conductor $f$. The inverse of the class of $\Lambda$ is its conjugate $\bar{\Lambda}$ in the ideal classes of $O_f$. In other words, the product lattice is homothetic to the order itself:

$$\Lambda \bar{\Lambda} = \langle \lambda \mu : \lambda, \mu \in \Lambda \rangle = \nu O_f,$$

for some $\nu \in O_K$. The order $O_f$ can be characterized as those elements of $O_K$ having imaginary part an integer multiple of $f \sqrt{\Delta}/2$. Now,

$$\Lambda \bar{\Lambda} = \langle N(\beta), N(\delta), \beta \delta, \bar{\beta}\delta \rangle.$$

These generators have imaginary parts $0$ and $\pm [O_K : \Lambda] \sqrt{\Delta}/2$ by (3). Furthermore, as the rational integers

$$N(\beta), N(\delta), \beta \delta + \bar{\beta}\delta$$

have no common factor above $1$, we find $1 \in \Lambda \bar{\Lambda}$, so $\nu = \pm 1$. Consequently, $f = [O_K : \Lambda]$. $\square$

8. **Proof of Theorem 1.4**

The maps which witness this bijection are quite simply defined. Denote them

$$I : S_K / \sim \to \bigcup_{f \in \mathbb{Z}} \theta_f^{-1}(O_K).$$

and

$$S : \bigcup_{f \in \mathbb{Z}} \theta_f^{-1}(O_K) \to S_K / \sim.$$

Given $C \in S_K$, let $M \in PSL_2(O_K)$ take $\mathbb{R}$ to $C$. Write

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}.$$

Then $I(C)$ is the ideal class of the lattice $\mathbb{Z}$-generated by $\beta$ and $\delta$ in $O_K$.

For the reverse map, one chooses a representative lattice of the ideal class $a$ which has covolume equal to its conductor, and selects a $\mathbb{Z}$-basis $\beta, \delta$ of integers. Solving for $\alpha, \gamma \in O_K$ such that

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix},$$

is an element of $PGL_2(O_K)$, then $S(a)$ is the $K$-Bianchi circle $M(\mathbb{R})$.

**Proof of Theorem 1.4.** The matrices

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \text{ and } M' = \begin{pmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{pmatrix} \in PGL_2(O_K)$$
map \( \mathbb{R} \) to equivalent unoriented circles if and only if \( M' = T MS \) where \( S \in \text{PGL}_2(\mathbb{Z}) \), and \( T = \begin{pmatrix} \epsilon & \eta \\ 0 & 1 \end{pmatrix} \) (a translation by \( \eta \in \mathcal{O}_K \) followed by a multiplication by a unit \( \epsilon \in \mathcal{O}_K \)). This occurs if and only if \( \beta, \delta \) and \( \beta', \delta' \) are bases, each consisting of a pair of coprime elements, for the same lattice, up to homothety by a unit. Every lattice has a conductor and hence is a representative lattice for some ideal of some order which maps the trivial class in \( \mathcal{I}(\mathcal{O}_K) \) by Lemma 7.1. These observations suffice to show that the maps \( I \) and \( G \) are well-defined and inverses. By Proposition 4.1, the oriented curvature of the circle \( M(\mathbb{R}) \) is \( i(\beta \delta - \beta' \delta') \), while by Lemma 7.1, the conductor of the lattice is \((-\Delta)^{-\frac{1}{2}} |\beta \delta - \beta' \delta'|. \)

9. Counting circles

Let \( h_K \) denote the class number of \( K \). Let \( h_f \) denote the class number of an order \( \mathcal{O}_f \) of conductor \( f \) inside \( K \). We define \( h_1 = h_K \). Define

\[ \mathcal{U}_f = \mathcal{O}_K^*/\mathcal{O}_f^*. \]

Let \( u = \mid \mathcal{U}_f \mid. \) This group is trivial except for \( \Delta = -3, -4 \). We have

\[ u = \begin{cases} 2 & K = \mathbb{Q}(\sqrt{-1}) \\ 3 & K = \mathbb{Q}(\sqrt{-3}) \\ 1 & \text{otherwise} \end{cases} . \]

There is an exact sequence

\[ 1 \rightarrow (\mathcal{O}_K/f)^*/(\mathbb{Z}/f)^* \mathcal{U}_f \rightarrow \mathcal{P}ic(\mathcal{O}_f) \rightarrow \mathcal{P}ic(\mathcal{O}_K) \rightarrow 1 \]

Consequently, the following standard result gives the size \( h_f \) of \( \mathcal{P}ic(\mathcal{O}_f) \).

**Theorem 9.1** (See, for example, [5] or [15]). If \( f > 1 \),

\[ h_f = \frac{h_K}{u} f \prod_{p \mid f \text{ prime}} \left( 1 - \frac{1}{p} \left( \frac{\Delta}{p} \right) \right). \]

As a corollary to Theorem 1.4, we obtain

**Corollary 9.2** (Corollary to Theorem 1.4). The number of \( K \)-Gaussian circles of curvature \( \sqrt{-\Delta} f \), considered up equivalence, is \( h_f \). In particular, the number of \( K \)-Gaussian circles of curvature \( \sqrt{-\Delta} f \) with centres in the fundamental parallelogram

\[ \{a + b\omega : 0 \leq a < 1, 0 \leq b < 1\} \]

is \( 2 h_f \), unless \( f = 1 \) and \( \Delta = -4 \), in which case it is 1.

**Proof.** Follows immediately, except to observe that only in the case of \( \Delta = -4, f = 1 \), do we have circles which reflect through the origin to themselves modulo \( \mathcal{O}_K \). \( \square \)
Let $\mathcal{L}$ be the set of primitive lattices for the order $\mathcal{O}_f$. The units taking $\Lambda \in \mathcal{L}$ to itself are exactly $\mathcal{O}_f^\times$. Therefore, the kernel of $\theta_f$ has a set of coset representatives given by $\mathcal{L}$ modulo $\mathcal{U}_f$.

We’ve also seen the exact sequence

$$1 \to (\mathcal{O}_K/f)^*/(\mathbb{Z}/f)^* \mathcal{U}_f \to \mathcal{P}ic(\mathcal{O}_f) \to \theta \mathcal{P}ic(\mathcal{O}_K) \to 1$$

which gives another description of the kernel of $\theta$. See for example [15, §I.12].

In fact, we have a slightly stronger result:

**Theorem 9.3.** There is a bijection

$$\mathcal{L} \to (\mathcal{O}_K/f)^*/(\mathbb{Z}/f)^*$$

The map is given by considering $\Lambda \in \mathcal{L}$ in $\mathcal{O}_K$ and taking the indicated quotients. The inverse is given by

$$\beta \mapsto f\mathcal{O}_K + \beta\mathbb{Z}.$$ 

**Proof.** It follows from the fact that $\Lambda$ is a locally principal ideal that its image consists of just one $4$ element of $(\mathcal{O}_K/f)^*/(\mathbb{Z}/f)^*$, so that the map is well-defined. It is immediate to check that the maps are inverses (note that all primitive lattices have the form $f\mathcal{O}_K + \beta\mathbb{Z}$).

Therefore, taking the quotient on either side by $\mathcal{U}_f$, we obtain a collection in bijection to $\mathcal{S}_K/\sim$ via Theorem 1.4.

10. **Connectedness and Euclideanity**

The Schmidt arrangement is quite different in the cases of Euclidean and non-Euclidean rings of integers $\mathcal{O}_K$. Our purpose in this section is to prove the following.

**Theorem 10.1.** The Schmidt arrangement $\mathcal{S}_K$ is connected if and only if $\mathcal{O}_K$ is a Euclidean domain. If $\mathcal{S}_K$ is disconnected, it has infinitely many connected components.

Write $E_2(\mathcal{O}_K)$ for the subgroup of $\text{SL}_2(\mathcal{O}_K)$ generated by elementary matrices. That $\mathcal{S}_K$ is connected for Euclidean domains is a consequence of a relationship between $E_2$ and Euclideanity due to Cohn.

**Theorem 10.2** (Cohn, [6]). Let $K$ be a quadratic imaginary field. Then $\mathcal{O}_K$ is Euclidean if and only if $\text{SL}_2(\mathcal{O}_K)$ is generated by the elementary matrices.

For number fields besides quadratic imaginary fields, $\text{SL}_2(\mathcal{O}_K)$ is always generated by the elementary matrices [1, 28]. Nica strengthened Cohn’s result.

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4 More directly, one could choose a Hermite basis for $\Lambda$ in terms of $1, \omega$; in this case one element of the basis is in $f\mathcal{O}_K$, from which this observation follows.
Theorem 10.3 (Nica, [16]). Let $K$ be a quadratic imaginary field. The subgroup $E_2(O_K)$ of $\text{SL}_2(O_K)$ generated by the elementary matrices is either the whole group or else it is a non-normal infinite-index subgroup.

For more on this and surrounding results, the reader is also directed to the very nice exposition of [16].

Definition 10.4. Define the tangency graph of $S_K$ to be the graph whose vertices are the $K$-Bianchi circles of $S_K$ and whose edges represent tangencies. Two circles are tangency-connected if their vertices are in the same connected component of this graph. We will refer to the subsets of $S_K$ corresponding to connected components of the tangency graph as tangency-connected components of $S_K$.

Cohn’s theorem is enough to show that $S_K$ is tangency-connected if and only if $O_K$ is Euclidean, but tangency-connectedness is a stronger notion than connectedness. To show that $S_K$ is disconnected when $O_K$ is not Euclidean, we do not use Cohn’s result, and instead prove it directly. As a consequence, this provides a new proof of one direction of Cohn and Nica’s results: If $O_K$ is not Euclidean, then the subgroup of $\text{PSL}_2(O_K)$ generated by elementary matrices is of infinite index. That is, we will see below that the orbit of one $K$-Bianchi circle under elementary matrices lies in a single tangency-connected component of $S_K$, hence in one connected component of $S_K$. However, we will show that there are infinitely many connected components.

In what follows, we will first discuss tangency-connectedness, which is fairly simple, before addressing the more difficult issue of connectedness.

Proposition 10.5. The Schmidt arrangement $S_K$ is tangency-connected if and only if $O_K$ is a Euclidean domain. If it is tangency-disconnected, it has infinitely many tangency-connected components.

Proof. First, we claim that the following three matrices (multiplied on the right) take a circle to another in the same connected component of $S_K$:

\[
\begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

The first is an example of moving to a tangent circle, according to Proposition 5.3. The second leaves the circle unchanged, and the third reverses orientation. But note that

\[
\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Therefore, the orbit of $\mathbb{R}$ under elementary matrices is contained in the connected component of $\mathbb{R}$. By Theorem 10.2, then, we have shown that if $K$ is Euclidean, then $S_K$ is connected.

Now, suppose $K$ is not Euclidean (in particular, $K \neq \mathbb{Q}(\sqrt{-3})$).
The circles tangent to $C$ are given by right multiplication by an elementary matrix (after changing orientation if necessary), so all circles which are in the tangency-connected component of $\mathbb{R}$ are in the orbit of $E_2(\mathcal{O}_K)$. By Theorem 10.2, $S_K$ is therefore tangency-disconnected.

The final statement is a consequence of Theorem 10.3. □

Tangency-disconnectedness is weaker than disconnectedness. To prove the latter, we define a special circle which lies in the complement of the Schmidt arrangement.

**Definition 10.6.** The *ghost circle* for a quadratic imaginary field $K$ is the circle orthogonal to the unit circle having center

$$\frac{1}{2} + \frac{\sqrt{\Delta}}{4\sqrt{\Delta}} \quad \Delta \equiv 0 \pmod{4}$$

$$\frac{1}{2} - \frac{\Delta - 1}{4\sqrt{\Delta}} \quad \Delta \equiv 1 \pmod{4},$$

and positive orientation. (If no such circle exists, we say $K$ has no ghost circle.)

**Lemma 10.7.** If $\Delta > -11$, then $K$ has no ghost circle. Otherwise, it has one, and its curvature is

$$\begin{cases} 
\frac{4}{\sqrt{-\Delta - 12}} & \Delta \equiv 0 \pmod{4} \\
\frac{4\sqrt{-\Delta}/\sqrt{\Delta^2 + 14\Delta + 1}}{} & \Delta \equiv 1 \pmod{4}
\end{cases}$$

**Proof.** Let $G$ be the ghost circle, of curvature $B$. As it is orthogonal to the unit circle, its curvature and co-curvature are equal. We use fact (2), which in the case $\Delta \equiv 0 \pmod{4}$ has the form

$$B^2 = \frac{B^2\Delta}{4} + \frac{B^2\Delta - 16}{16\Delta} - 1 \implies B = \pm \frac{4}{\sqrt{-\Delta - 12}}.$$

and in the case $\Delta \equiv 1 \pmod{4}$ has the form

$$B^2 = \frac{B^2}{4} + \frac{B^2(-\Delta - 1)^2}{16\Delta} - 1 \implies B = \pm \frac{4\sqrt{-\Delta}}{\sqrt{\Delta^2 + 14\Delta + 1}}.$$

As $G$ is positively oriented, we choose the positive square root. These roots are real for fundamental discriminants $< -11$ (the first such is $\Delta = -15$). □

**Lemma 10.8.** Suppose $K$ has a ghost circle. Then no $K$-Bianchi circle intersects the ghost circle.

**Proof.** We will show that the Pedoe product of the ghost circle $G$ with any $K$-Bianchi circle $C$ is greater than 1 in absolute value. By Proposition 4.2, this implies that the circles do not intersect. The proof is somewhat technical, relying on the fact that the image of $\pi(C)$ is restricted in certain ways (for example, not all elements of $i\mathcal{O}_K$ can occur as centre-curvatures).
Let $B$ be the curvature of $G$. First, assume $\Delta \equiv 0 \pmod{4}$. Then

$$\pi(G) = \left( B, B \cdot \frac{\sqrt{-\Delta} B}{4} \right) = \frac{B}{\sqrt{-\Delta}} \left( \sqrt{-\Delta}, \sqrt{-\Delta}, \frac{\sqrt{-\Delta}}{2}, -\Delta \right).$$

By Proposition 4.1, we know that

$$\pi(C) \in \sqrt{-\Delta} \mathbb{Z} \times \sqrt{-\Delta} \mathbb{Z} \times \sqrt{-\Delta} \mathbb{Z} \times \mathbb{Z}.$$
Hence, taking the product,
\[ \langle G, C \rangle \in \frac{B\sqrt{-\Delta}}{4} \mathbb{Z}. \]

But
\[ \frac{B\sqrt{-\Delta}}{4} = \frac{-\Delta}{\sqrt{-\Delta} - 12} > 1. \]

Hence it suffices to show that \( \langle G, C \rangle \neq 0 \). Suppose \( \pi(C) = (b, b', x, y) \). By \( (2) \),
\[ bb' - x^2 - y^2 + 1 = 0 \]

The first two terms are divisible by \( \Delta \), hence \( y \) is odd. The equation \( \langle G, C \rangle = 0 \) is equivalent to
\[ -4b + b' \sqrt{-\Delta} + 2x \sqrt{-\Delta} - y = 0 \]

which is impossible for odd \( y \), as each of the displayed fractions is an integer.

The case \( \Delta \equiv 1 \pmod{4} \) is slightly more involved, but similar in spirit. In this case,
\[ \pi(G) = \frac{B}{\sqrt{-\Delta}} \left( \sqrt{-\Delta}, \sqrt{-\Delta}, \frac{-\Delta - 1}{2}, \frac{-\Delta - 1}{4} \right). \]

Suppose that \( C \) has centre-curvature \( z = i\frac{a+b\sqrt{\Delta}}{2}, a, b \in \mathbb{Z} \), curvature \( c\sqrt{-\Delta} \) and co-curvature \( c'\sqrt{-\Delta} \), where \( z \in i\mathcal{O}_K \) and \( c, c' \in \mathbb{Z} \) by Proposition 4.1. By \( (2) \), \( z\overline{z} = \Delta cc' + 1 \equiv 1 \pmod{\Delta} \). This implies that \( a^2 + b^2 \Delta \equiv 4 \pmod{\Delta} \), therefore \( a \equiv \pm 2 \pmod{\Delta} \), say \( a = a'\Delta + 2 \).

With this notation,
\[ \pi(C) = \left( c\sqrt{-\Delta}, c'\sqrt{-\Delta}, -b\sqrt{-\Delta}/2, a/2 \right). \]

Now we compute
\[ 4 \frac{\sqrt{-\Delta}}{B} \langle G, C \rangle = 4(c + c')\Delta + b\Delta - \Delta a'\Delta + 1 \pm (\Delta + 1). \]

This quantity lies in \( \Delta \mathbb{Z} \pm 1 \). That is,
\[ \frac{4}{B\sqrt{-\Delta}} \langle G, C \rangle = k \pm \frac{1}{\Delta}, \]

where \( k \in \mathbb{Z} \). To conclude that \( |\langle G, C \rangle| > 1 \), it suffices to show that \( k \neq 0 \) and that
\[ \frac{4}{B\sqrt{-\Delta}} < 1 - \frac{1}{\Delta}. \quad (4) \]

To show that \( k \neq 0 \), recall that \( \Delta \equiv -1 \pmod{4} \) so that \( (\Delta + 1)/2 \) is odd. Also recall that \( a \equiv b \pmod{2} \) which shows that \( a' \equiv b \pmod{2} \). Hence \( k \) is odd, hence non-zero. The inequality \( (4) \) is equivalent to
\[ 14\Delta^3 + 3\Delta^2 - 1 < 0 \]
which certainly holds in the range $\Delta < -11$, and this completes the proof. \hfill $\Box$

**Proof of Theorem 10.1.** Proposition 10.5 shows that if $\mathcal{O}_K$ is Euclidean then $\mathcal{S}_K$ is tangency-connected and therefore connected. If $\mathcal{O}_K$ is non-Euclidean, then $\Delta < -12$ and therefore there exists a ghost circle $G$ in the complement of $\mathcal{S}_K$ by Lemma 10.8. But since there is a circle of $\mathcal{S}_K$ passing through every point $z \in K$, the circles of $\mathcal{S}_K$ are dense and therefore populate both the interior and exterior of $G$. However, they do not intersect $G$ itself. But the interior and exterior are not connected, so that $\mathcal{S}_K$ must be disconnected.

Finally, any element of $\text{PSL}_2(\mathcal{O}_K)$ stabilizes $\mathcal{S}_K$ and preserves intersections, therefore the orbit of $G$ under $\text{PSL}_2(\mathcal{O}_K)$ lies in the complement of $\mathcal{S}_K$. This implies that $\mathcal{S}_K$ falls into infinitely many connected components (for example, take all translations of $G$ by $\mathcal{O}_K$). \hfill $\Box$

**References**

[1] H. Bass, J. Milnor, and J.-P. Serre. Solution of the congruence subgroup problem for $\text{SL}_n (n \geq 3)$ and $\text{Sp}_{2n} (n \geq 2)$. *Inst. Hautes Études Sci. Publ. Math.*, (33):59–137, 1967.

[2] Luigi Bianchi. Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginari. *Math. Ann.*, 40(3):332–412, 1892.

[3] Jean Bourgain and Elena Fuchs. A proof of the positive density conjecture for integer Apollonian circle packings. *J. Amer. Math. Soc.*, 24(4):945–967, 2011.

[4] Jean Bourgain and Alex Kontorovich. On the local-global conjecture for integral Apollonian gaskets. *Invent. Math.*, 196(3):589–650, 2014. With an appendix by Péter P. Varjú.

[5] Harvey Cohn. *Advanced number theory*. Dover Publications, Inc., New York, 1980. Reprint of it A second course in number theory, 1962, Dover Books on Advanced Mathematics.

[6] P. M. Cohn. On the structure of the $\text{GL}_2$ of a ring. *Inst. Hautes Études Sci. Publ. Math.*, (30):5–53, 1966.

[7] Benjamin Fine. *Algebraic theory of the Bianchi groups*, volume 129 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1989.

[8] Charles Frohman and Benjamin Fine. Some amalgam structures for Bianchi groups. *Proc. Amer. Math. Soc.*, 102(2):221–229, 1988.

[9] Elena Fuchs. Counting problems in Apollonian packings. *Bull. Amer. Math. Soc. (N.S.)*, 50(2):229–266, 2013.

[10] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: number theory. *J. Number Theory*, 100(1):1–45, 2003.

[11] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: geometry and group theory. I. The Apollonian group. *Discrete Comput. Geom.*, 34(4):547–585, 2005.

[12] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: geometry and group theory. II. Super-Apollonian group and integral packings. *Discrete Comput. Geom.*, 35(1):1–36, 2006.

[13] Jerzy Kocik. A theorem on circle configurations, 2007. arXiv:0706.0372.
[14] Alexander Lubotzky. Free quotients and the congruence kernel of $SL_2$. *J. Algebra*, 77(2):411–418, 1982.

[15] Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by G. Harder.

[16] Bogdan Nica. The unreasonable slightness of $E_2$ over imaginary quadratic rings. *Amer. Math. Monthly*, 118(5):455–462, 2011.

[17] S. Northshield. On integral Apollonian circle packings. *J. Number Theory*, 119(2):171–193, 2006.

[18] Peter Sarnak. Letter to Lagarias. [http://www.math.princeton.edu/sarnak](http://www.math.princeton.edu/sarnak).

[19] Asmus L. Schmidt. Farey triangles and Farey quadrangles in the complex plane. *Math. Scand.*, 21:241–295 (1969), 1967.

[20] Asmus L. Schmidt. Diophantine approximation of complex numbers. *Acta Math.*, 134:1–85, 1975.

[21] Asmus L. Schmidt. Diophantine approximation in the field $\mathbb{Q}((11^{1/2}))$. *J. Number Theory*, 10(2):151–176, 1978.

[22] Asmus L. Schmidt. Diophantine approximation in the Eisensteinian field. *J. Number Theory*, 16(2):169–204, 1983.

[23] Asmus L. Schmidt. Diophantine approximation in the field $\mathbb{Q}(i\sqrt{2})$. *J. Number Theory*, 131(10):1983–2012, 2011.

[24] Jean-Pierre Serre. Le problème des groupes de congruence pour SL2. *Ann. of Math. (2)*, 92:489–527, 1970.

[25] Katherine E. Stange. The sensual Apollonian circle packing, 2012. [arXiv:1208.4836](http://arxiv.org/abs/1208.4836).

[26] Katherine E. Stange. The apollonian structure of Bianchi groups, 2014.

[27] W. A. Stein et al. *Sage Mathematics Software (Version 4.8)*. The Sage Development Team, 2012. [http://www.sagemath.org](http://www.sagemath.org).

[28] L. N. Vaserstein. The group $SL_2$ over Dedekind rings of arithmetic type. *Mat. Sb. (N.S.)*, 89(131):313–322, 351, 1972.

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