Independent and Decentralized Learning in Markov Potential Games

Chinmay Maheshwari†, Manxi Wu†, Druv Pai, and Shankar Sastry ‡

Abstract

We propose a multi-agent reinforcement learning dynamics, and analyze its convergence in infinite-horizon discounted Markov potential games. We focus on the independent and decentralized setting, where players do not have knowledge of the game model and cannot coordinate. In each stage, players update their estimate of Q-function that evaluates their total contingent payoff based on the realized one-stage reward in an asynchronous manner. Then, players independently update their policies by incorporating an optimal one-stage deviation strategy based on the estimated Q-function. A key feature of the learning dynamics is that the Q-function estimates are updated at a faster timescale than the policies. We prove that the policies induced by our learning dynamics converge to the set of stationary Nash equilibria in Markov potential games with probability 1. Our results highlight the efficacy of simple learning dynamics in reaching to the set of stationary Nash equilibrium even in environments with minimal information available.

1 Introduction

Multi-Agent Reinforcement Learning (MARL) focuses on analyzing strategic interactions among multiple players in a dynamic environment, where the utilities and state transitions are jointly determined by players’ actions. Such interactions are common in many important
applications including autonomous driving Shalev-Shwartz et al. (2016), adaptive traffic control Prabuchandran et al. (2014); Bazzan (2009), e-commerce Kutschinski et al. (2003), and AI training in real-time strategy games Vinyals et al. (2019); Brown and Sandholm (2018). In these applications, players often need to act in an independent and decentralized manner, adapting to the information received through interactions in uncertain and dynamic environments. Coordination and communication may be absent, and players may not have the knowledge of the existence of other players or their payoffs. This leads to the following important question:

How can players learn a stationary Nash equilibrium through simple updates without coordination or the knowledge of the game?

In this article, we affirmatively answer the above question in the context of Markov Potential Games (MPG), which have been extensively studied in recent works Leonardos et al. (2021); Zhang et al. (2021); Song et al. (2021); Mao et al. (2021); Ding et al. (2022); Fox et al. (2022); Zhang et al. (2022); Marden (2012); Macua et al. (2018). In MPGs, the change of utility of any player from unilaterally deviating their policy can be evaluated by the change of the potential function, and a stationary Nash equilibrium policy can be solved as the global optimum of the potential function. Previous studies have mainly focused on developing gradient-based algorithms to compute stationary Nash equilibrium in both the discounted infinite horizon settings Zhang et al. (2021); Leonardos et al. (2021); Ding et al. (2022); Fox et al. (2022); and the finite horizon episodic settings Mao et al. (2021); Song et al. (2021). However, gradient computation requires coordination among agents since the computation of value function and its derivatives require agents to have the access to simulator/oracle or to coordinate and communicate their strategies and payoffs Leonardos et al. (2021); Fox et al. (2022); Daskalakis et al. (2020); Ding et al. (2022). Such communication and coordination may be limited in many real-world multi-agent systems due to communication constraints and privacy concerns Kutschinski et al. (2003); Nuti et al. (2011).

We propose an independent and decentralized multi-agent reinforcement learning dynamics, and prove its convergence to the set of stationary Nash equilibria in infinite-horizon discounted MPGs. In our setting, players do not know the existence of other players participating in the game, and do not have knowledge of state transition probability, their own utility functions or the opponents’ utility functions. Each player only observes the realized state, and their own realized reward in each stage.

The main challenge of developing convergent algorithms in independent and decentralized learning setting is the non-stationarity induced in the environment as players update their
policies while learning about the environment. Indeed, naively designed algorithms to obtain independent and decentralized multi-agent learning often fail to converge to Nash equilibrium \cite{Tan1993,Matignon2012,Mazumdar2020}. One approach to reinstate the stationarity in the learning process is to enable players to update their policies at a slower rate than value function estimate, akin to actor-critic algorithms \cite{Konda1999}. Our proposed learning dynamics builds on the actor-critic algorithms for single agent reinforcement learning, and extend them to prove the convergence to Nash equilibrium in MPGs. In particular, our proposed learning dynamics has the following key features:

(i) the dynamics have \textit{two timescales}: each player updates a q-estimate of their contingent payoff (represented as the Q-function defined in Sec. 2) at a faster timescale, and update their policies at a slower timescale.

(ii) players are \textit{self-interested} in that their updated policy incorporates an \textit{optimal one-stage deviation} that maximizes the expected contingent payoff derived from the current q-estimate.

(iii) learning is \textit{asynchronous} and \textit{heterogeneous} among players. In every stage, only the q-estimate of the realized state-action pair, and the policy corresponding to the realized state are updated. The remaining elements in q-estimate and policy remain unchanged. Furthermore, the stepsizes of updating the element corresponds to each state and action are heterogeneous across players, and are asynchronously adjusted according to the number of times a state and that player’s action are realized. counters.

(iv) In every stage, the players take actions by mixing their strategy with a uniformly random strategy over their action space to promote exploration in order to learn the value functions at all states.

We prove that our learning dynamics leads to the set of Nash equilibria with probability 1. This convergence result indicates that players learn a stationary Nash equilibrium in MPGs by adopting an actor-critic learning dynamics in a decentralized and independent manner.

Our approach involves non-trivial extensions of the single-agent reinforcement learning methods to MPG setting, and developing game-theoretic tools to utilize the equilibrium condition of the underlying Markov game. Specifically, we study the convergence properties of discrete time dynamics using an associated continuous time dynamical system by exploiting the timescale separation between the updates of q-estimate and policy \cite{Borkar2009,Tsitsiklis1994,Perkins2013}. In this dynamical system, the fast dynamics
– update of the q-estimates – can be analyzed while viewing the policy updates (the slow dynamics) as static, and thus the q-estimate of each player converges to their Q-function. Additionally, we show that the potential function serves as the Lyapunov function of the continuous time dynamical system associated with the policy updates at the slow timescale. We show that the trajectories of the continuous time dynamical system associated with policy updates converge to an invariant set, which is contained in the set of Nash equilibrium of the game.

1.1 Related Works

Apart from learning in MPGs, another line of work in multi-agent reinforcement learning focuses on the fully competitive setting of Markov zero-sum games [Daskalakis et al. (2020); Sayin et al. (2022a, 2021); Alacaoglu et al. (2022); Guo et al. (2021); Perolat et al. (2015)]. Most articles in this line of works require players to either observe the opponents’ rewards or actions [Alacaoglu et al. (2022); Sayin et al. (2022a,b)], or to coordinate in policy updates [Daskalakis et al. (2020); Guo et al. (2021)]. Reference Sayin et al. (2021) proposed an independent and decentralized learning dynamics, and showed its convergence in Markov zero-sum games. The algorithm in Sayin et al. (2021) also has timescale separation between policy update and value update, but is in reversed ordering compared to ours. That is, their dynamics update value function at a slower timescale, and update policies at a faster timescale, while our policy update is slower compared to the q-estimate update. Another difference is that our learning dynamics adopts a different stepsize adjustment procedure that allows players to update their step-sizes based on their own counters of states and actions heterogeneously. We emphasize that the convergence analysis in our paper is different from that in Sayin et al. (2021) due to the differences in the two learning algorithms and the inherent difference between Markov zero-sum games and Markov potential games.\[\text{1}\]

Two-timescale based algorithms have also been studied in other non-zero sum games [Borkar (2002); Perolat et al. (2018); Prasad et al. (2015); Arslan and Yüksel (2016)]. Specifically, Borkar (2002) proposed an actor-critic algorithm, and showed that certain weighted empirical distribution of realized actions converges to a generalized Nash equilibrium, which may not be a Nash equilibrium. In Arslan and Yüksel (2016), the authors presented an algorithm in the setting of acyclic Markov games, which subsume Markov team games. However, the proposed algorithm require coordination amongst agents. The paper Prasad et al. (2015); Perolat et al. (2018) proposed actor-critic algorithms with a fast value function up-

\[\text{1}\] Our convergence proof builds on the existence of potential function and the convergence of fast q-learning. On the other hand, the proof of convergence in zero-sum Markov games depends on the Shapley iteration convergence.
date – based on temporal difference learning – and a slow policy update. In Prasad et al. (2015), the gradient-based policy update requires the knowledge of opponents' rewards. The paper Perolat et al. (2018) adopted a best-response based policy update that is similar to our learning dynamics, and proved its convergence in multistage games, which are a class of generalized normal form games with tree structures. Our algorithm differs from that in Perolat et al. (2018) from two aspects: (a) we consider updates with asynchronous step sizes that are adjusted based on counters of each state and each state-action pair while Perolat et al. (2018) considers homogeneous step sizes; (b) the q-estimate update in our dynamics introduces a reward perturbation in each stage. It turns out that these two differences are crucial for us to achieve equilibrium convergence in a much more general setting – MPGs with infinite stages and no restrictions on state transition other than irreducibility and aperiodicity. Furthermore, since the proof techniques developed in Perolat et al. (2018) exploits the special tree structure of multistage games, they cannot be applied in our setting.

Finally, our results also advance the rich literature of learning in stateless potential game that includes continuous and discrete time best response dynamics Monderer and Shapley (1996b); Swenson et al. (2018), fictitious play Monderer and Shapley (1996a); Hofbauer and Sandholm (2002); Marden et al. (2009), replicator dynamics Panageas and Piliouras (2016); Hofbauer and Sigmund (2003), no-regret learning Heliou et al. (2017); Krichene et al. (2014), and payoff-based learning Cominetti et al. (2010). In particular, our learning dynamics share similar spirit with the payoff-based learning dynamics in stateless potential games Cominetti et al. (2010). In payoff-based learning, players update their payoff estimates based only on their own payoffs and adjust their mixed strategy using a best response. In MPGs, the payoff estimates of different state-action pairs are updated asynchronously, and the best response becomes an optimal one-stage deviation policy. Therefore, our result is not a direct extension of stateless potential games as it involves using reinforcement learning tools to prove policy convergence.

Outline

Section 2 presents Markov potential games. We present our independent and decentralized learning dynamics, and the convergence results in Section 3 and conclude our work in Section 4. We include the proofs of technical lemmas in the appendix.

2 Model

Consider a Markov game $G$, where a finite set of players interact with one another in a dynamic environment with discrete and infinite time horizon. We formally characterize the
Markov game $G$ by tuple $G = (I, S, (A_i)_{i \in I}, (u_i)_{i \in I}, P, \delta)$, where

- $I$ is a finite set of players;
- $S$ is a finite set of states;
- $A_i$ is a finite set of actions with generic member $a_i$ for each player $i \in I$, and $a = (a_i)_{i \in I} \in A = \times_{i \in I} A_i$ is the action profile of all players;
- $u_i(s, a) : S \times A \to \mathbb{R}$ is the one-stage payoff of player $i$ with state $s \in S$, and action profile $a \in A$;
- $P = (P(s'|s, a))_{s,s',a \in A}$ is the state transition matrix and $P(s'|s, a)$ is the probability that state changes from $s$ to $s'$ with action profile $a$;
- $\delta \in (0, 1)$ is the discount factor.

We denote a stationary (Markov) policy $\pi_i = (\pi_i(s, a_i))_{s \in S, a_i \in A_i} \in \Pi_i = \Delta(A_i)^{|S|}$, where $\pi_i(s, a_i)$ is the probability that player $i$ chooses action $a_i$ given state $s$. For each $i \in I$ and each $s \in S$, we denote $\pi_i(s) = (\pi_i(s, a_i))_{a_i \in A_i}$. The joint policy profile is denoted as $\pi = (\pi_i)_{i \in I} \in \Pi = \times_{i \in I} \Pi_i$. We also use the notation $\pi_{-i} = (\pi_j)_{j \in I \setminus \{i\}} \in \Pi_{-i} = \times_{j \in I \setminus \{i\}} \Pi_j$ to refer to the joint policy of all players except for player $i$.

The game proceeds in discrete-time stages indexed by $k = \{0, 1, \ldots\}$. At $k = 0$, the initial state $s^0$ is sampled from a distribution $\mu$. At every time step $k$, given the state $s^k$, each player’s action $a^k_i \in A_i$ is sampled from the policy $\pi_i(s^k)$, and the joint action profile is $a^k = (a^k_i)_{i \in I}$. The state of the next stage $s^{k+1}$ is realized according to the transition matrix $P(\cdot|s^k, a^k)$ based on the current state $s^k$ and action profile $a^k$. Given an initial state distribution $\mu$, and a stationary policy profile $\pi$, the expected total discounted payoff of each player $i \in I$ is given by:

$$V_i(\mu, \pi) = \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k u_i(s^k, a^k) \right],$$

where $s^0 \sim \mu$, $a^k \sim \pi(s^k)$, and $s^k \sim P(\cdot|s^{k-1}, a^{k-1})$. For the rest of the article, with slight abuse of notation, we use $V_i(s, \pi)$ to denote the expected total utility of player $i$ when the initial state is a fixed state $s \in S$. Thus, we have $V_i(\mu, \pi) = \sum_{s \in S} \mu(s)V_i(s, \pi)$.

**Definition 2.1** (Markov potential games [Leonardos et al. (2021)]). A Markov game $G$ is a Markov potential game (MPG) if there exists a state-dependent potential function $\Phi :
\(S \times \Pi \rightarrow \mathbb{R}\) such that for every \(s \in S, i \in I, \pi_i, \pi'_i \in \Pi_i, \pi_{-i} \in \Pi_{-i},\)

\[\Phi(s, \pi'_i, \pi_{-i}) - \Phi(s, \pi_i, \pi_{-i}) = V_i(s, \pi'_i, \pi_{-i}) - V_i(s, \pi_i, \pi_{-i}).\]

Moreover, given an initial state distribution \(\mu \in \Delta(S),\) the potential function \(\Phi(\mu, \pi) := \sum_{s \in S} \mu(s) \Phi(s, \pi)\) satisfies

\[\Phi(\mu, \pi'_i, \pi_{-i}) - \Phi(\mu, \pi_i, \pi_{-i}) = V_i(\mu, \pi'_i, \pi_{-i}) - V_i(\mu, \pi_i, \pi_{-i}),\]

for every \(i \in I, \pi_i, \pi'_i \in \Pi_i, \pi_{-i} \in \Pi_{-i}.\)

That is, in a MPG, the change of a single deviator’s value function can be characterized by the change of the value of the potential function. We next present the definition of stationary Nash equilibrium, and \(\epsilon\)-stationary Nash equilibrium.

**Definition 2.2** (Stationary Nash equilibrium policy). A policy profile \(\pi^*\) is a stationary Nash equilibrium of \(G\) if for any \(i \in I,\) any \(\pi_i \in \Pi_i,\) and any \(\mu \in \Delta(S),\)

\[V_i(\mu, \pi^*_i, \pi^*_{-i}) \geq V_i(\mu, \pi_i, \pi^*_{-i}).\]

**Definition 2.3** (\(\epsilon\)-Stationary Nash equilibrium policy). For any \(\epsilon \geq 0,\) a policy profile \(\pi^*\) is an \(\epsilon\)-stationary Nash equilibrium of \(G\) if for any \(i \in I,\) any \(\pi_i \in \Pi_i,\) and any \(\mu \in \Delta(S),\)

\[V_i(\mu, \pi^*_i, \pi^*_{-i}) \geq V_i(\mu, \pi_i, \pi^*_{-i}) - \epsilon.\]

Any \(\epsilon\)-Nash equilibrium with \(\epsilon = 0\) is a Nash equilibrium.

Both stationary Nash equilibrium and \(\epsilon\)-stationary Nash equilibrium exist in Markov games with finite states and actions [Fudenberg and Tirole (1991)]. In a MPG, if there exists a policy \(\pi^*\) such that \(\pi^* = \arg \max_{\pi \in \Pi} \Phi(s, \pi)\) for every \(s \in S\) then \(\pi^*\) is a stationary Nash equilibrium policy of the MPG. However, computing the Nash equilibrium as the maximizer of \(\Phi(s, \cdot)\) is challenging due to the fact that \(\Phi(s, \cdot)\) can be non-linear and non-concave. Also, optimization tools that are used for computing the global maximizer requires the knowledge of the model, and centralized coordination among agents.

## 3 Independent and Decentralized Learning Dynamics

## 4 Independent and Decentralized Learning Dynamics

In this section we present the independent and decentralized learning dynamics, and the convergence analysis. We emphasize that the learning only assumes that each player \(i \in I\)
knows the state set \( S \), and their own action set \( A_i \). Players do not know the state transition probability matrix \( P \), their own or others’ utility functions \((u_i)_{i \in I}\). Players do not even know the existence of other players. In each stage \( k = 0, 1, 2, \ldots \), players observe the realized state \( s^k \) that they use to compute the action \( a^k \) and in turn obtain the reward \( r^k_i = u_i(s^k, a^k) \). Players do not observe the actions or the rewards of their opponents.

Before delineating the learning dynamics, we define some important definitions which would lay foundation for presenting the dynamics. Given any policy \( \pi \in \Pi \), and any initial state \( s \in S \), we define the following \textit{Q-function} for each player \( i \in I \) and action \( a_i \in A_i \):

\[
Q_i(s, a_i; \pi) = \sum_{a_{-i} \in A_{-i}} \pi_{-i}(s, a_{-i}) \left( u_i(s, a_i, a_{-i}) + \delta \sum_{s' \in S} P(s'|s, a_i, a_{-i}) V_i(s', \pi) \right).
\]  

(2)

In (2), player \( i \)'s expected utility in the first stage with state \( s \) is derived from playing action \( a_i \) and her opponents choosing policy \( \pi_{-i} \). The expected total utility starting from stage 2 is derived from all players following policy \( \pi \). Therefore, the Q-function \( Q_i(s, a_i; \pi) \) represents player \( i \)'s expected discounted utility when the game starts in state \( s \), and player \( i \) deviates for one-stage (namely, the first stage) from her policy to play \( a_i \). Furthermore, we define \textit{optimal one-stage deviation} from policy \( \pi \in \Pi \) in state \( s \in S \) as

\[
br_i(s; \pi) = \arg \max_{\tilde{\pi}_i \in \Delta(A_i)} \left( \sum_{a_i \in A_i} \tilde{\pi}_i(a_i) Q_i(s, a_i; \pi) \right).\]

(3)

### 4.1 Learning Dynamics

The proposed learning happens in iterates, denoted by \( t \). In every iterate \( t \), each player \( i \in I \) updates four components \( n^t_i, \tilde{n}^t_i, q^t_i, \pi^t_i \). In particular, \( n^t_i = (n^t_i(s))_{s \in S} \) is the vector of state counters, where \( n^t_i(s) \) is the number of times state \( s \) is realized before iterate \( t \). For each player \( i \in I \), \( \tilde{n}^t_i = (\tilde{n}^t_i(s, a_i))_{s \in S, a_i \in A_i} \) is the counter of state-action tuple, where \( \tilde{n}^t_i(s, a_i) \) is the number of times that player \( i \) has played action \( a_i \) in state \( s \) before iterate \( t \).

Additionally, \( q^t_i = (q^t_i(s, a_i))_{s \in S, a_i \in A_i} \) is player \( i \)'s estimate of her Q-function, and \( \pi^t_i = (\pi^t_i(s, a_i))_{s \in S, a_i \in A_i} \) is player \( i \)'s policy in iterate \( t \). Given the state-(local)action tuple of any player \( i \), \( (s^{t-1}, a_i^{t-1}) \), the estimate \( q^t_i(s^{t-1}, a_i^{t-1}) \) is updated in (4) as a linear combination of the estimate \( q_i^{t-1}(s^{t-1}, a_i^{t-1}) \) in the previous stage, and a new estimate that is comprised of the realized one-stage payoff \( r_i^{t-1} \) and the estimated total discounted payoff from the next iterate.

The policy \( \pi^t_i(s^{t-1}) \) updated in (5) is a linear combination of the policy \( \pi_i^{t-1}(s^{t-1}) \) in
the previous stage, and player \( i \)'s optimal one-stage deviation. Particularly, the optimal one-stage deviation is computed using the updated q-estimate \( q^t_i \) instead of the actual \( Q_i \), which is unknown to the player. At the end of every iterate \( t \), each player chooses an action by combining the policy \( \pi_i^{(t)} \) with uniform exploration as described in (6). Note that the exploration parameter \( \theta_i \in (0, 1) \) can be different for different players.

Note that updates of \( q^t_i \) (resp. \( \pi_i^{(t)} \)) in each iterate only change the element that corresponds to the realized state and action \((s^{t-1}, a^{t-1})\) (resp. state \( s^t \)), and the remaining elements stay unchanged. Furthermore, the update speed of \( q^t_i(s^{t-1}, a^{t-1}) \) (resp. \( \pi_i^{(t)}(s^{t-1}) \)) is governed by the step size sequence \( \alpha_i(n) \) (resp. \( \beta_i(n) \)) corresponds to the state-action counter \( n = n_i^t(s^{t-1}, a^{t-1}) \) (resp. state counter \( n = n^t(s^{t-1}) \)). Therefore, the update is asynchronous in that the stepsizes are different across the elements associated with different states and actions, and stepsizes are different for different players.

### 4.2 Convergence Analysis

This section presents the main result of this paper which establishes convergence of learning dynamics (\( P \)) to Nash equilibrium. We first introduce the following two assumptions:

**Assumption 4.1.** The initial state distribution \( \mu(s) > 0 \) for all \( s \in S \). Additionally, there exists a joint action \( a \in A \) such that the induced Markov chain with probability transition \( \left(P(s'|s, a) \right)_{s, s' \in S} \) is irreducible and aperiodic.

Assumption 4.1 is a standard assumption in RL [Konda and Tsitsiklis (1999)] and MARL [Savin et al. (2021)] to ensure that every state is visited infinitely often so that players can effectively learn the value functions in every state. Furthermore, we make the following assumption on the stepsizes \( \{\{\alpha_i(n)\}_{i \in I}\}_{n=0}^{\infty}, \{\{\beta_i(n)\}_{i \in I}\}_{n=0}^{\infty} \):

**Assumption 4.2.** The step sizes \( \{\alpha_i(n) \in (0, 1)\}_{n=0, i \in I} \) and \( \{\beta_i(n) \in (0, 1)\}_{n=0, i \in I} \) satisfy

(i) For all \( i \in I \), \( \sum_{n=0}^{\infty} \alpha_i(n) = \infty, \sum_{n=0}^{\infty} \beta_i(n) = \infty \), \( \lim_{n \to \infty} \alpha_i(n) = \lim_{n \to \infty} \beta_i(n) = 0 \) and \( \{\alpha_i(n)\}, \{\beta_i(n)\} \) are non-increasing in \( n \);

(ii) For every \( i \in I \), there exist some \( q, q' \geq 2 \), \( \sum_{n=0}^{\infty} \alpha_i(n)^{1+q/2} < \infty \) and \( \sum_{n=0}^{\infty} \beta_i(n)^{1+q'/2} < \infty \);

(iii) For every \( i \in I \), \( \sup_n \alpha_i([xn]/\alpha_i(n)) < \infty \), \( \sup_n \beta_i([xn]/\beta_i(n)) < \infty \) for all \( x \in (0, 1) \);

(iv) For every \( i, j \in I \), \( \lim_{n \to \infty} \beta_i(n)/\alpha_j(n) = 0 \);

(v) For every \( i, j \in I \), there exists \( 0 < \xi^{\alpha}_{ij} < \zeta^{\alpha}_{ij} < \infty \), and \( 0 < \xi^{\beta}_{ij} < \zeta^{\beta}_{ij} < \infty \) such that \( \frac{\alpha_i(n)}{\alpha_j(n)} \in [\xi^{\alpha}_{ij}, \zeta^{\alpha}_{ij}] \) and \( \frac{\beta_i(n)}{\beta_j(n)} \in [\xi^{\beta}_{ij}, \zeta^{\beta}_{ij}] \) for all \( n \).
In every iterate

\begin{algorithm}
\begin{algorithmic}
\textbf{Initialization:} \( n^0(s) = 0, \forall s \in S; \tilde{n}^0_i(s, a_i) = 0, q^0_i(s, a_i) = 0, \pi^0_i(s, a_i) = 1/|A_i|, \forall (i, a_i, s), \) and \( \theta_i \in (0, 1). \)

In stage 0, each player observes \( s^0 \), choose their action \( a_i^0 \sim \pi^0_i(s^0) \), and observe \( r_i^0 = u_i(s^0, a^0) \).

\textbf{In every iterate} \( t = 1, 2, \ldots \), each player observes \( s^t \), and independently updates \( \{n^t, \tilde{n}^t_i, q^t_i, \pi^t_i\} \).

\textbf{Update} \( n^t, \tilde{n}^t_i \):

\[
\begin{align*}
n^t(s^{t-1}) &= n^{t-1}(s^{t-1}) + 1, \\
\tilde{n}^t_i(s^{t-1}, a_i) &= \tilde{n}^{t-1}_i(s^{t-1}, a_i) + 1,
\end{align*}
\]

Furthermore, for \( s \neq s^{t-1}, a_i \neq a_i^{t-1} \)

\[
\tilde{n}^t_i(s, a_i) = \tilde{n}^{t-1}_i(s, a_i), \quad n^t(s) = n^{t-1}(s).
\]

\textbf{Update} \( q^t_i \):

\[
q^t_i(s^{t-1}, a_i) = q^{t-1}_i(s^{t-1}, a_i) + \alpha_i(\tilde{n}^t_i(s^{t-1}, a_i)) \\
\cdot \left( r_i^{t-1} + \delta \sum_{a_i \in A_i} \pi^{t-1}_i(s^{t}, a_i) - q^{t-1}_i(s^{t-1}, a_i) \right),
\]

\[
q^t_i(s, a_i) = q^{t-1}_i(s, a_i), \quad \forall (s, a_i) \neq (s^{t-1}, a_i^{t-1}),
\]

where \( r_i^{t-1} = u_i(s^{t-1}, a^t). \)

\textbf{Update} \( \pi^t_i \): For every \( a_i \in A_i \),

\[
\pi^t_i(s^{t-1}, a_i) = \pi^{t-1}_i(s^{t-1}, a_i) + \beta_i(n^t(s^{t-1})).
\]

\[
\begin{align*}
\left( \bar{b}r_i^{t-1}(s^{t-1}, a_i) - \pi^{t-1}_i(s^{t-1}, a_i) \right), \\
\pi^t_i(s, a_i) = \pi^{t-1}_i(s, a_i), \quad \forall s \neq s^{t-1},
\end{align*}
\]

where \( \bar{b}r_i(s) \in \arg \max_{\pi_i \in \Delta(A_i)} \sum_{a_i \in A_i} \pi_i(a_i) q^t_i(s, a_i) \).

At the end of iterate \( t \), each player chooses their action

\[ a_i^t = \arg \max (1 - \theta_i) \pi_i^t(s^t) + (\theta_i/|A_i|) 1, \]

and observes their own realized reward \( r_i^t = u_i(s^t, a^t) \).
\end{algorithmic}
\end{algorithm}
Assumption 4.2(i)-(ii) are standard in stochastic approximation theory Borkar (2009); Benaïm (1999); Tsitsiklis (1994), and Assumption 4.2(iii) is a technical condition required in asynchronous updates Perkins and Leslie (2013). Assumption 4.2(iv) implies that our learning dynamics have two timescales: the update of \( \{q^t_i\}_t \) is asymptotically faster than the update of \( \{\pi^t_i\}_t \). Assumption 4.2(v) implies that step sizes can be heterogeneous between players as long as the ratio between their stepsizes is bounded between zero and a finite number for all steps. One example of stepsizes that satisfies Assumption 4.2 is \( \alpha_i(n) = z_i n^{-c_1} \) and \( \beta_i(n) = y_i n^{-c_2} \) where \( 0 < c_1 \leq c_2 \leq 1 \) and \( y_i, z_i \) are player specific positive scalars.

We state the main result of this paper formally below:

**Theorem 4.3.** Under Assumptions 4.1 and 4.2, for every \( s \in S, a_i \in A_i \), and \( i \in I \), the q-estimate updates \( \{q^t_i\}_t \) satisfies \( \lim_{t \to \infty} |q^t_i(s,a_i) - Q_i(s,a_i;\pi^t)\| = 0 \). Furthermore, the policy updates \( \{\pi^t_i\}_t \) induced by Algorithm 1 converges to the set of Nash equilibria of the Markov potential game \( G \) with probability 1.

To prove Theorem 4.3, we use the two-timescale asynchronous stochastic approximation theory Perkins and Leslie (2013) which states that convergence of coupled q-estimate-policy updates can be argued by convergence of continuous-time decoupled dynamical systems. The convergence of (fast) q-estimate updates, \( \{q^t_i\}_t \), is based on Lemma 4.4. Meanwhile, convergence of (slow) policy updates is based on Lemma 4.5-4.7.

**Lemma 4.4.** For any \( s \in S, a_i \in A_i \), and \( i \in I \), \( \lim_{t \to \infty} |q^t_i(s,a_i) - Q_i(s,a_i;\pi^t)\| = 0 \) with probability 1.

The proof of Lemma 4.4 is based on two crucial steps. First, we show that under Assumption 4.1 and 4.2, our learning dynamics satisfies the set of conditions introduced in Perkins and Leslie (2013) (restated in Appendix A) so that convergence of q-estimates can be analyzed by the associated continuous time differential equation where the the policy drifts are treated as asymptotically negligible errors. Second, we argue the global convergence of the continuous time dynamical system using the contraction property of Bellman operator. The detailed proof of Lemma 4.4 is deferred to Appendix.

Next, we analyze the policy updates with respect to the convergent values of the q-estimates provided by the fast dynamics as in Lemma 4.4. Particularly, the policy \( \pi^t_i(s^{t-1}) \) in 4.3 becomes a linear combination of \( \pi^{t-1}_i(s^{t-1}) \), and the smoothed optimal one-stage deviation \( br_i(s^{t-1},\pi^{t-1}) \) based on the actual Q-function as in (3). Under Assumption 4.1, the convergence properties of \( \{\pi^t_i\}_t \) can be analyzed using the following continuous time
differential inclusion, where \( \tau \in [0, \infty) \) is a continuous time index, and \( \varpi^\tau \in \Pi \) is a continuous time policy function:

\[
\frac{d}{d\tau} \varpi^\tau_i(s, a_i) = \gamma_i(s) \left( b r_i(s, a_i; \varpi^\tau) - \varpi^\tau_i(s, a_i) \right),
\]

for every \( i \in I, (s, a_i) \in S \times A_i \), and \( \gamma_i(s) \) captures the asynchronous updates of policies in different states (cf. (5)) and is lower bounded by a positive number \( \eta > 0 \) for all \( s \in S \). Note that the correspondence \( b r_i(\cdot; \cdot) \) satisfies the conditions necessary for existence of an absolutely continuous solution of (8), \( \varpi^\tau \) for every \( i \in I \) (Clarke et al. (2008)). For every \( i \in I, (s, a_i) \in S \times A_i \), let \( \tilde{b} r^\tau_i(s) \in b r_i(s; \varpi^\tau) \) be such that

\[
\frac{d}{d\tau} \varpi^\tau_i(s, a_i) = \gamma_i(s) \left( \tilde{b} r^\tau_i(s, a_i) - \varpi^\tau_i(s, a_i) \right).
\]

To establish the convergence of (8), we define a Lyapunov function \( \phi : [0, \infty) \to \mathbb{R} \) as follows:

\[
\phi(\tau) = \max_{\varpi \in \Pi} \sum_{s \in S} \mu(s) \Phi(s, \varpi) - \sum_{s \in S} \mu(s) \Phi(s, \varpi^\tau),
\]

which is the difference of the potential function at its maximizer with that of its value at \( \varpi^\tau \). We show that \( \phi(\tau) \) is weakly decreasing in \( \tau \). Additionally, \( \phi(\tau) \) decreases strictly as long as \( \varpi^\tau \) is not a Nash equilibrium policy.

Lemma 4.5. \( \frac{d}{d\tau}(\phi(\tau)) \leq 0 \). If \( \varpi^\tau \) is not a Nash equilibrium of \( G \) then \( \frac{d}{d\tau}(\phi(\tau)) < 0 \). Furthermore, if \( \varpi^\tau \) is a Nash equilibrium of \( G \) then \( \frac{d}{d\tau}(\phi(\tau)) = 0 \). Consequently, \( \lim_{\tau \to \infty} \varpi^\tau \) lies in the set of Nash equilibria of \( G \).

Since the potential function is non-concave in each player’s policy, we develop a new approach to demonstrate that \( \phi(\tau) \) decreases along the trajectory of \( \varpi^\tau \). To establish the decrement property of \( \phi(\tau) \), we prove the following technical lemma which comprises of extensions of RL tools in the context of games and equivalence between set of Nash equilibrium of \( G \) and stationary points of (8). This lemma is of independent interest beyond this work.

Lemma 4.6. (a) Policy gradient theorem: For any \( \mu \in \Delta(S), s \in S, \pi \in \Pi, i \in I, a_i \in A_i \),

\[
\frac{\partial V_i(\mu, \pi)}{\partial \pi_i(s, a_i)} = \frac{1}{1 - \delta} d_{\pi}^\mu(s) Q_i(s, a_i; \pi),
\]

where \( d_{\mu}^\pi(s) \triangleq (1 - \delta) \sum_{s^0 \in S} \mu(s^0) \sum_{k=0}^{\infty} \delta^k \Pr(s^k = s | s^0) \).

12
(b) Multi-agent performance difference lemma: For any policy \( \pi = (\pi_i, \pi_{-i}) \), \( \pi' = (\pi'_i, \pi_{-i}) \) \( (\pi'_i, \pi_{-i}) \) \( \in \) \( \Pi \) and any \( \mu \in \Delta(S) \),

\[
V_i(\mu, \pi) - V_i(\mu, \pi') = \frac{1}{1-\delta} \sum_{s'} d^\mu_{\pi}(s') \Gamma_i(s', \pi_i; \pi') ,
\]

where \( \Gamma_i(s, a_i; \pi) \) is the advantage function given by

\[
\Gamma_i(s, a_i; \pi) \triangleq Q_i(s, a_i; \pi) - V_i(s, \pi) ,
\]

for every \( i \in I \), \( \forall s \in S \), \( \forall a_i \in A_i \), \( \pi \in \Pi \).

(c) Characterization of set of Nash equilibrium: A policy \( \text{eq} \in \Pi \) is a Nash equilibrium of \( G \) if and only if \( \text{eq}(s) = \text{br}_i(s; \text{eq}) \) for all \( i \in I \) and all \( s \in S \).

The proof of Lemma 4.6 is presented in Appendix B. We are now ready to proof Lemma 4.5 based on Lemma 4.6.

**Proof of Lemma 4.5** We compute the derivative of \( \phi(\tau) \) with respect to \( \tau \), where \( \phi(\tau) \) is given by (9).

\[
\frac{d}{d\tau} \phi(\tau) = -\sum_i \sum_s \sum_{a_i} \frac{\partial \Phi(\mu, \overline{\omega}^\tau) d\overline{\omega}^\tau_i(s, a_i)}{\partial \omega_i(s, a_i)} \frac{d\omega_i(s, a_i)}{d\tau} \\
= (a) - \sum_{i, s, a_i} \frac{\partial V_i(\mu, \overline{\omega}^\tau) d\overline{\omega}^\tau_i(s, a_i)}{\partial \omega_i(s, a_i)} \frac{d\omega_i(s, a_i)}{dt} \\
= (b) - \sum_{i, s, a_i} \frac{d\overline{\omega}^\tau(s)}{1-\delta} \cdot \gamma_i(s) \left( \text{br}_i^\tau(s, a_i) - \omega_i^\tau(s, a_i) \right) ,
\]

where (a) is due to the fact that \( \Phi \) is a potential function of the game, and (b) is due to the policy gradient theorem for the game (Lemma 4.6(a)) and (8).

Recall that \( \text{br}_i(s; \pi) = \arg \max_{\hat{\pi}_i; \pi} f(\hat{\pi}_i; s, \pi) \), where \( f(\hat{\pi}_i; s, \pi) = \left( \sum_{a_i \in A_i} \hat{\pi}_i(a_i) Q_i(s, a_i, \pi) \right) \) is linear in \( \hat{\pi}_i \). From (12), Lemma 4.6(c) and the fact that \( \gamma_i(s) > \eta \) for all \( i \in I \), \( s \in S \), we have

\[
\frac{d}{d\tau} \phi(\tau) \leq -\eta \frac{1}{1-\delta} \sum_{i, s, a_i} \frac{d\overline{\omega}^\tau(s)}{1-\delta} \cdot \left( \text{br}_i^\tau(s, a_i) - \omega_i^\tau(s, a_i) \right) \\
= -\eta \frac{1}{1-\delta} \sum_{i, s, a_i} \frac{d\overline{\omega}^\tau(s)}{1-\delta} \cdot \left( \text{br}_i^\tau(s, a_i) - \omega_i^\tau(s, a_i) \right) \\
\]

13
where the last inequality is due to definition of $f$.

Additionally, let $\pi^*_i \in \arg \max_{\pi_i} V_i(\mu, \pi_i; \omega^\tau_i)$ be a best response of player $i$ if the joint strategy of other players is $\omega^\tau_i$. Note that $\pi^*_i$ maximizes the total payoff instead of just maximizing the payoff of one-stage deviation. Therefore, $\pi^*_i$ is different from the optimal one-stage deviation policy. We drop the dependence of $\pi^*_i$ on $\omega^\tau_i$ for notational simplicity.

We define $D = \frac{1}{1-\delta} \max_i \left\| \frac{\pi_i^{\omega^\tau} - \pi_i^{\omega^\tau_i}}{\mu} \right\|_{\infty}$. We note that $D$ is finite under the assumption that $\mu$ has full support (Assumption 4.1). Additionally, we have $d^\omega_{\mu}(s) \geq (1-\delta)\mu(s)$. Consequently,

$$\sum_{i,s} d^\mu_i(s) \max_{\pi_i \in \Delta(A_i)} \left( f(\hat{\pi}_i; s, \omega^\tau) - f(\omega^\tau_i(s); s, \omega^\tau) \right) \leq \frac{1}{1-\delta} \sum_{i,s} d^\omega_{\mu}(s) \left\| \frac{\pi_i^{\omega^\tau} - \pi_i^{\omega^\tau_i}}{\mu} \right\|_{\infty} \cdot \max_{\pi_i \in \Delta(A_i)} \left( f(\hat{\pi}_i; s, \omega^\tau) - f(\omega^\tau_i(s); s, \omega^\tau) \right)$$

$$= D \sum_{i,s} d^\omega_{\mu}(s) \max_{\pi_i \in \Delta(A_i)} \left( f(\hat{\pi}_i; s, \omega^\tau) - f(\omega^\tau_i(s); s, \omega^\tau) \right)$$

$$= D \sum_{i,s} d^\omega_{\mu}(s) \left( f(\tilde{\omega}^\tau_i(s); s, \omega^\tau) - f(\omega^\tau_i(s); s, \omega^\tau) \right), \quad (14)$$

Then, from (13) and (14), we have

$$\frac{d}{dt} \phi(\tau) \leq -\frac{\eta}{1-\delta} \sum_{i,s} d^\mu_i(s) \left( f(\tilde{\omega}^\tau_i(s); s, \omega^\tau) - f(\omega^\tau_i(s); s, \omega^\tau) \right)$$

$$\leq -\frac{\eta}{D(1-\delta)} \sum_{i,s} d^\mu_i(s) \left( f(\tilde{\omega}^\tau_i(s); s, \omega^\tau) - f(\omega^\tau_i(s); s, \omega^\tau) \right)$$
\[
\max_{\pi_i \in \Delta(A_i)} (f(\pi_i; s, \omega^r) - f(\omega_i^r(s); s, \omega^r))
\leq -\eta \frac{D}{(1-\delta)} \sum_{i,s} d^\pi_{i,\omega_i^r} (s) \left( f(\pi_i^\dagger(s); s, \omega^r) - f(\omega_i^r(s); s, \omega^r) \right)
= -\frac{\eta}{D} \sum_{i,s} d^\pi_{i,\omega_i^r} (s) \left( Q_i(s, \pi_i^\dagger; \omega^r) - Q_i(s, \omega_i^r; \omega^r) \right),
\]  

(15)

where the last equation follows from the definition of function \( f \). We now analyze the right-hand-side of (15),

\[
Q_i(s, \pi_i^\dagger; \omega^r) - Q_i(s, \omega_i^r; \omega^r)
= \sum_{a_i} Q_i(s, a_i; \omega^r) (\pi_i^\dagger(s, a_i) - \omega_i^r(s, a_i))
= \sum_{a_i} (Q_i(s, a_i; \omega^r) - V_i(s, \omega^r)) (\pi_i^\dagger(s, a_i) - \omega_i^r(s, a_i))
= \sum_{a_i} \Gamma_i(s, a_i; \omega^r) \pi_i^\dagger(s, a_i)
= \Gamma_i(s, \pi_i^\dagger; \omega^r),
\]  

(16)

where the second to last equality is due to (11). Combining (15) and (16), we have

\[
\frac{d\phi(\tau)}{d\tau} \leq -\eta \frac{D}{(1-\delta)} \sum_{i,s} d^\pi_{i,\omega_i^r} (s) \Gamma_i(s, \pi_i^\dagger; \omega^r)
= -\frac{\eta}{D} \sum_i \left( V_i(\mu, \pi_i^\dagger, \omega_i^r) - V_i(\mu, \omega^r) \right),
\]

where the equality is due to the multi agent performance difference lemma (Lemma 4.6(b)).

Note that since \( \pi_i^\dagger \) is a best response corresponding to \( \omega^r \) this means \( V_i(\mu, \pi_i^\dagger, \omega_i^r) - V_i(\mu, \omega^r) \geq 0 \) for all \( i \). Thus \( d\phi(\tau)/d\tau \leq 0 \). Furthermore, note that the function \( \phi \) is locally Lipschitz which shows that \( \phi \) is weakly increasing. Moreover, if \( \omega^r \) is not a Nash equilibrium then there exists \( i \) such that

\[
V_i(\mu, \pi_i^\dagger, \omega_i^r) - V_i(\mu, \omega^r) > 0
\]

which implies \( d\phi(\tau)/d\tau < 0 \). Equivalently, if \( d\phi(\tau)/d\tau = 0 \) then \( \omega^r \) is a Nash equilibrium. Moreover, if \( \omega^r \) is a Nash equilibrium then from Lemma 4.6(c) we observe that \( d\phi(\tau)/d\tau = 0 \) as \( d\omega_i^r(s)/d\tau = 0 \) for every \( i \in I, s \in S \). Therefore, the limit set of every solution of (8) is a connected subset of Nash equilibrium [Benaim et al. (2005)].

Based on asynchronous stochastic approximation theory (Perkins and Leslie, 2013, Theorem 4.7), \( \{\pi^\dagger\} \) follows the trajectories of (8) in the limit. From Lemma 4.5 we know that
the trajectories of (8) converge to the set of Nash equilibria. Consequently, the following Lemma holds.

**Lemma 4.7.** The updates $\pi^t$ in Algorithm 1 almost surely converges to the set of Nash equilibria of the game $\mathcal{G}$.

We are now ready to prove Theorem 4.3 which builds on Lemma 4.4 - 4.7.

**Proof of Theorem 4.3.** Note that the convergence of q-estimate follows directly from Lemma 4.4. Additionally, Lemma 4.7 guarantees that the policy updates $(\pi^t)_{t=0}^\infty$ from Algorithm 1 converges to the set of Nash equilibria of the game $\mathcal{G}$. □

5 Conclusions

We proposed an independent and decentralized learning dynamics which converges to the set of Nash equilibria of the game $\mathcal{G}$ for Markov potential games in infinite-horizon discounted reward setting. Some interesting directions of future research include (i) analyzing the finite-time convergence rate of the proposed dynamics in Markov potential games; (ii) analyzing the convergence property of our learning dynamics beyond the scope of Markov potential games; (iii) analyzing the outcome of our learning dynamics when agents do not have complete information of the state realization in the system.

References

Alacaoglu, A., Viano, L., He, N., and Cevher, V. (2022). A natural actor-critic framework for zero-sum Markov games. In *International Conference on Machine Learning*, pages 307–366. PMLR.

Arslan, G. and Yüksel, S. (2016). Decentralized Q-learning for stochastic teams and games. *IEEE Transactions on Automatic Control*, 62(4):1545–1558.

Bazzan, A. L. (2009). Opportunities for multiagent systems and multiagent reinforcement learning in traffic control. *Autonomous Agents and Multi-Agent Systems*, 18(3):342–375.

Benaïm, M. (1999). Dynamics of stochastic approximation algorithms. In *Seminaire de Probabilites XXXIII*, pages 1–68. Springer.

Benaïm, M., Hofbauer, J., and Sorin, S. (2005). Stochastic approximations and differential inclusions. *SIAM Journal on Control and Optimization*, 44(1):328–348.
Borkar, V. S. (2002). Reinforcement learning in Markovian evolutionary games. *Advances in Complex Systems*, 5(01):55–72.

Borkar, V. S. (2009). *Stochastic Approximation: A Dynamical Systems Viewpoint*, volume 48. Springer.

Brown, N. and Sandholm, T. (2018). Superhuman AI for heads-up no-limit poker: Libratus beats top professionals. *Science*, 359(6374):418–424.

Clarke, F. H., Ledyaev, Y. S., Stern, R. J., and Wolenski, P. R. (2008). *Nonsmooth analysis and control theory*, volume 178. Springer Science & Business Media.

Cominetti, R., Melo, E., and Sorin, S. (2010). A payoff-based learning procedure and its application to traffic games. *Games and Economic Behavior*, 70(1):71–83.

Daskalakis, C., Foster, D. J., and Golowich, N. (2020). Independent policy gradient methods for competitive reinforcement learning. *Advances in neural information processing systems*, 33:5527–5540.

Ding, D., Wei, C.-Y., Zhang, K., and Jovanovic, M. (2022). Independent policy gradient for large-scale Markov potential games: Sharper rates, function approximation, and game-agnostic convergence. In *International Conference on Machine Learning*, pages 5166–5220. PMLR.

Fox, R., Mcalleer, S. M., Overman, W., and Panageas, I. (2022). Independent natural policy gradient always converges in Markov potential games. In *AISTATS*, pages 4414–4425. PMLR.

Fudenberg, D. and Tirole, J. (1991). *Game Theory*. MIT press.

Guo, H., Fu, Z., Yang, Z., and Wang, Z. (2021). Decentralized single-timescale actor-critic on zero-sum two-player stochastic games. In *International Conference on Machine Learning*, pages 3899–3909. PMLR.

Heliou, A., Cohen, J., and Mertikopoulos, P. (2017). Learning with bandit feedback in potential games. *Advances in Neural Information Processing Systems*, 30.

Hofbauer, J. and Sandholm, W. H. (2002). On the global convergence of stochastic fictitious play. *Econometrica*, 70(6):2265–2294.

Hofbauer, J. and Sigmund, K. (2003). Evolutionary game dynamics. *Bulletin of the American Mathematical Society*, 40(4):479–519.
Konda, V. and Tsitsiklis, J. (1999). Actor-critic algorithms. *Advances in Neural Information Processing Systems, 12.*

Krichene, W., Drighès, B., and Bayen, A. (2014). On the convergence of no-regret learning in selfish routing. In *International Conference on Machine Learning*, pages 163–171. PMLR.

Kutschinski, E., Uthmann, T., and Polani, D. (2003). Learning competitive pricing strategies by multi-agent reinforcement learning. *Journal of Economic Dynamics and Control, 27*(11-12):2207–2218.

Leonardos, S., Overman, W., Panageas, I., and Piliouras, G. (2021). Global convergence of multi-agent policy gradient in Markov potential games. *arXiv preprint arXiv:2106.01969.*

Macua, S. V., Zazo, J., and Zazo, S. (2018). Learning parametric closed-loop policies for Markov potential games. *arXiv preprint arXiv:1802.00899.*

Mao, W., Başar, T., Yang, L. F., and Zhang, K. (2021). Decentralized cooperative multi-agent reinforcement learning with exploration. *arXiv preprint arXiv:2110.05707.*

Marden, J. R. (2012). State based potential games. *Automatica, 48*(12):3075–3088.

Marden, J. R., Arslan, G., and Shamma, J. S. (2009). Joint strategy fictitious play with inertia for potential games. *IEEE Transactions on Automatic Control, 54*(2):208–220.

Matignon, L., Laurent, G. J., and Le Fort-Piat, N. (2012). Independent reinforcement learners in cooperative Markov games: A survey regarding coordination problems. *The Knowledge Engineering Review, 27*(1):1–31.

Mazumdar, E., Ratliff, L. J., Jordan, M. I., and Sastry, S. S. (2020). Policy-gradient algorithms have no guarantees of convergence in linear quadratic games. In *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS ’20*, page 860–868.

Monderer, D. and Shapley, L. S. (1996a). Fictitious play property for games with identical interests. *Journal of Economic Theory, 68*(1):258–265.

Monderer, D. and Shapley, L. S. (1996b). Potential games. *Games and Economic Behavior, 14*(1):124–143.

Nuti, G., Mirghaemi, M., Treleaven, P., and Yingsaeree, C. (2011). Algorithmic trading. *Computer, 44*(11):61–69.
Panageas, I. and Piliouras, G. (2016). Average case performance of replicator dynamics in potential games via computing regions of attraction. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, pages 703–720.

Perkins, S. and Leslie, D. S. (2013). Asynchronous stochastic approximation with differential inclusions. *Stochastic Systems*, 2(2):409–446.

Perolat, J., Piot, B., and Pietquin, O. (2018). Actor-critic fictitious play in simultaneous move multistage games. In *International Conference on Artificial Intelligence and Statistics*, pages 919–928. PMLR.

Perolat, J., Scherrer, B., Piot, B., and Pietquin, O. (2015). Approximate dynamic programming for two-player zero-sum Markov games. In *International Conference on Machine Learning*, pages 1321–1329. PMLR.

Prabuchandran, K., AN, H. K., and Bhatnagar, S. (2014). Multi-agent reinforcement learning for traffic signal control. In *17th International IEEE Conference on Intelligent Transportation Systems (ITSC)*, pages 2529–2534. IEEE.

Prasad, H., LA, P., and Bhatnagar, S. (2015). Two-timescale algorithms for learning nash equilibria in general-sum stochastic games. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, pages 1371–1379.

Sayin, M., Zhang, K., Leslie, D., Basar, T., and Ozdaglar, A. (2021). Decentralized q-learning in zero-sum Markov games. *Advances in Neural Information Processing Systems*, 34.

Sayin, M. O., Parise, F., and Ozdaglar, A. (2022a). Fictitious play in zero-sum stochastic games. *SIAM Journal on Control and Optimization*, 60(4):2095–2114.

Sayin, M. O., Zhang, K., and Ozdaglar, A. (2022b). Fictitious play in markov games with single controller. In *Proceedings of the 23rd ACM Conference on Economics and Computation*, pages 919–936.

Shalev-Shwartz, S., Shammah, S., and Shashua, A. (2016). Safe, multi-agent, reinforcement learning for autonomous driving. *arXiv preprint arXiv:1610.03295*.

Song, Z., Mei, S., and Bai, Y. (2021). When can we learn general-sum Markov games with a large number of players sample-efficiently? *arXiv preprint arXiv:2110.04184*.
Swenson, B., Murray, R., and Kar, S. (2018). On best-response dynamics in potential games. *SIAM Journal on Control and Optimization, 56*(4):2734–2767.

Tan, M. (1993). Multi-agent reinforcement learning: Independent vs. cooperative agents. In *Proceedings of the tenth international conference on machine learning*, pages 330–337.

Tsitsiklis, J. N. (1994). Asynchronous stochastic approximation and Q-learning. *Machine Learning, 16*(3):185–202.

Vinyals, O., Babuschkin, I., Czarnecki, W. M., Mathieu, M., Dudzik, A., Chung, J., Choi, D. H., Powell, R., Ewalds, T., Georgiev, P., et al. (2019). Grandmaster level in starcraft II using multi-agent reinforcement learning. *Nature, 575*(7782):350–354.

Zhang, R., Mei, J., Dai, B., Schuurmans, D., and Li, N. (2022). On the global convergence rates of decentralized softmax gradient play in Markov potential games. In *Advances in Neural Information Processing Systems*.

Zhang, R., Ren, Z., and Li, N. (2021). Gradient play in stochastic games: stationary points, convergence, and sample complexity. *arXiv preprint arXiv:2106.00198.*
Appendix

The appendix is organized as follows. In Sec A we review the theory of two-timescale asynchronous stochastic approximation from Perkins and Leslie (2013). In Sec B we present the proofs of technical lemmas presented in Sec 3.

A Review of Two-timescale asynchronous stochastic approximation

In this section we review the results from Perkins and Leslie (2013) on two-timescale asynchronous stochastic approximation. Note that we do not state their results in full generality but only to the extent necessary for this paper.

Let \( \{x_t\}_{t=1}^{\infty}, \{y_t\}_{t=1}^{\infty} \) be the stochastic approximation updates. Let \( x_t \in \mathbb{R}^X, y_t \in \mathbb{R}^Y \) for all \( t \in \{1, 2, \ldots\} \). Let \( \tilde{X} \subset [X] \) (resp. \( \tilde{Y} \subset [Y] \)) be the elements of \( x \) update (resp. \( y \) update) that have positive probability of being updated in the asynchronous update process. At iterate \( t \), let \( \tilde{X}^t \subset \tilde{X} \) and \( \tilde{Y}^t \subset \tilde{Y} \) be the elements that are updated. Let

\[
\tilde{n}_t(i) = \sum_{p=1}^{t} 1(i \in \tilde{X}^p), \quad n_t(j) = \sum_{p=1}^{t} 1(j \in \tilde{Y}^p),
\]

for every \( i \in [X] \) and \( j \in [Y] \). Consider the following asynchronous stochastic approximation updates indexed by \( t \in \{1, 2, \ldots\} \)

\[
x_t(i) \in x_{t-1}(i) + \alpha_i(\tilde{n}_t(i))1(i \in \tilde{X}^t)|F(i; x_{t-1}, y_{t-1}) + \tilde{M}_t(i) + \delta_t(i), \quad \forall i \in [X]
\]

\[
y_t(j) \in y_{t-1}(j) + \beta_j(n_t(j))1(j \in \tilde{Y}^t)|G(j; x_{t-1}, y_{t-1}) + M_t(j) + \epsilon_t(j), \quad \forall j \in [Y],
\]

where

1. for any \( x \in \mathbb{R}^X, y \in \mathbb{R}^Y, F(x, y) = (F(i; x, y))_{i \in [X]} \subset \mathbb{R}^X \) and \( G(x, y) = (G(j; x, y))_{j \in [Y]} \subset \mathbb{R}^Y \) are set-valued maps;

2. \( \{\tilde{M}_t = (\tilde{M}_t(i))_{i \in [X]}\}, \{M_t = (M_t(j))_{j \in [Y]}\} \) be martingale difference processes defined on \( \mathbb{R}^X, \mathbb{R}^Y \) respectively;
(iii) \( \{d^t = (d^t(i))_{i \in \mathbb{R}^x}, e^k = (e^k(j))_{j \in \mathbb{R}^y}\} \) are asymptotically negligible error terms;

(iv) For every \( i \in [X], j \in [Y], \{\alpha_i(n)\}_{n=0}^{\infty}, \{\beta_j(n)\}_{n=0}^{\infty} \) are the step sizes;

(v) \( x^0 \in \mathbb{R}^X, y^0 \in \mathbb{R}^Y \) are initialized at some values.

For every \( t \in \{1, 2, \ldots\}, \) define

\[
\bar{\alpha}^t = \max_{i \in \mathbb{X}^t} \alpha_i(n^t(i)), \quad \bar{\mu}^t(i) = \frac{\alpha_i(n^t(i))}{\bar{\alpha}^t} 1(i \in \mathbb{X}^t)
\]

\[
\bar{\beta}^t = \max_{j \in \mathbb{Y}^t} \beta_j(n^t(j)), \quad \bar{\sigma}^t(j) = \frac{\beta_j(n^t(j))}{\bar{\beta}^t} 1(j \in \mathbb{Y}^t)
\]

\[
\bar{D}^t = \text{diag}([\bar{\mu}^t(1), \bar{\mu}^t(2), \ldots, \bar{\mu}^t(X)]),
\]

\[
\bar{D}^t = \text{diag}([\bar{\sigma}^t(1), \bar{\sigma}^t(2), \ldots, \bar{\sigma}^t(Y)]).
\]

Using these notations we can concisely write (18) as

\[
x^t \in x^{t-1} + \bar{\alpha}^t \bar{D}^t \left( F(x^{t-1}, y^{t-1}) + \bar{M}^t + d^t \right)
\]

\[
y^t \in y^{t-1} + \bar{\beta}^t \bar{D}^t \left( G(x^{t-1}, y^{t-1}) + M^t + e^t \right).
\]

We now state some assumption that are crucial to study the asymptotic property of the stochastic approximation (19). First, we introduce some important notations. Define \( \bar{H} \subset \mathbb{X} \times \mathbb{Y} \) such that if \( i \in \mathbb{X}, j \in \mathbb{Y} \) then \((i, j) \in \bar{H}\) if and only if \( i, j \) have positive probability of occurring simultaneously. At iterate \( t, \bar{H}^t \in \bar{H} \) be the updated component in \( [X] \times [Y] \). Furthermore \( z^t = (x^t, y^t) \) be the joint update. Let \( \mathcal{F}^t = \sigma(\{\bar{H}^m\}_m, \{z^m\}_m, \{n^m(i)\}, \{n^m(j)\}) \) \( \forall m \leq t, i \in [X], j \in [Y] \) be sigma-algebra containing all information upto iterate \( t \). For any positive integer \( K \), define \( \Omega^t_K = \{\text{diag}(\omega(1), ..., \omega(K)) : \omega(i) \in [\eta, 1] \ \forall i = 1, 2, \ldots, K\} \).

Next, we present the assumptions required in Perkins and Leslie (2013) to study the asymptotic behavior of two-timescale asynchronous stochastic approximation update (18).

**Assumption A.1.** Let the following assumptions hold

(A1) For compact sets \( \tilde{S} \subset \mathbb{R}^X, S \subset \mathbb{R}^Y \), \( x^t \in \tilde{S}, y^t \in S \) for all \( t \in \{0, 1, \ldots\} \).

(A2) \( \{d^t\}, \{e^t\} \) are bounded sequence such that \( \lim_{t \to \infty} d^t = \lim_{t \to \infty} e^t = 0 \).

(A3) Following must be true for the stepsizes:

(i) For every \( i \in [X], j \in [Y], \sum_n \alpha_i(n) = \infty, \sum_n \beta_j(n) = \infty, \lim_{n \to \infty} \alpha_i(n) = \lim_{n \to \infty} \beta_j(n) = 0 \) and \( \{\alpha_i(n)\}, \{\beta_j(n)\} \) are non-increasing sequences.
(ii) For any $\theta \in (0, 1)$, $i \in [X]$ and $j \in [Y]$ it holds that $\sup_n \alpha_i([\theta n])/\alpha_i(n) < \infty$, $\sup_n \beta_i([\theta n])/\beta_i(n) < \infty$.

(iii) For every $i \in [X], j \in [Y]$ it holds that $\lim_{n \to \infty} \beta_i(n)/\alpha_i(n) = 0$.

(iv) For every $i, i' \in [X], j, j' \in [Y]$, there exists $0 < \zeta_{ii'} \in \zeta_{ii'} < \infty$, and $0 < \zeta_{jj'} < \zeta_{jj'} < \infty$ such that $\alpha^{(n)}_{ii'} \epsilon [\zeta_{ii'}, \zeta_{ii'}]$ and $\beta^{(n)}_{jj'} \epsilon [\zeta_{jj'}, \zeta_{jj'}]$ for all $n$.

(A4) The maps $F(\cdot, \cdot), G(\cdot, \cdot)$ are such that

(i) $F : \tilde{S} \times S \Rightarrow S$ is upper semi-continuous, for every $z \in \tilde{S} \times S$, $F(z)$ is non-empty, compact, convex subset of $S$, and $\sup_{t \epsilon F(z)} ||t|| \leq c(1 + ||z||)$ where $c$ is a constant independent of $z$.

(ii) $G : \tilde{S} \times S \Rightarrow \tilde{S}$ is upper semi-continuous. $G(x, \cdot)$ is non-empty, convex and compact and satisfy $\sup_{t \epsilon G(x, y)} ||t|| \leq c(1 + ||y||)$ where $c$ is a constant independent of $x, y$.

(A5) (i) for all $z \epsilon \tilde{S} \times S$ and $h^{t-1}, h^t \epsilon \tilde{H}$,

$$\Pr \left( \tilde{H}^t = h^t | \mathcal{F}^{t-1} \right) = \Pr(\tilde{H}^t = h^t | \tilde{H}^{t-1} = h^{t-1}, z^{t-1} = z)$$

(ii) For any $z \epsilon \tilde{S} \times S$ the transition probability

$$\mathcal{P}(z; h^t, h^{t-1})$$

form aperiodic, irreducible, positive recurrent Markov chain over $\tilde{H}$ and for every $i \epsilon X$ and $j \epsilon Y$ there exists $h, h' \epsilon \tilde{H}$ such that $i \epsilon h$ and $j \epsilon h'$.

(iii) the map $z \mapsto \mathcal{P}(z; h^t, h^{t-1})$ is Lipschitz.

(A6) For some $q \geq 2$, $\sum_{n} \alpha_i(n)^{1+q/2} < \infty$ and $\sup_t \mathbb{E} \left[ ||\tilde{M}^t||^q \right] < \infty$ for every $i \epsilon [X]$. For some $q' \geq 2$, $\sum_{n} \beta_i(n)^{1+q'/2} < \infty$ and $\sup_t \mathbb{E} \left[ ||M^t||^q \right] < \infty$ for every $j \epsilon [Y]$.

(A7) For all $y \epsilon S$ and every $\eta > 0$ the differential equation

$$\frac{d}{d\tau} x^\tau = \Omega_x^y \cdot F(x^\tau, y),$$

has unique global attractor $\Lambda(y)$ where $\Lambda : \mathbb{R}^Y \rightarrow \mathbb{R}^X$ is bounded, continuous and single-valued for all $y \epsilon S$.  

23
Theorem A.2 (Fast-timescale convergence). (Perkins and Leslie, 2013, Corollary 4.4) Under assumption (A1)-(A7) in Assumption A.1 with probability 1,

\[(x^t, y^t) \to \{(\Lambda(y), y) : y \in S\} \text{ as } t \to \infty.\]

Next, we present the corresponding convergence results for the slow dynamics, \{y^\tau\}. Define \(G^\Lambda : \mathbb{R}^J \to \mathbb{R}^J\) as \(G^\Lambda(y) = G(\Lambda(y), y)\). Furthermore, let \(\bar{G}^{\Lambda, \eta} = \Omega^\eta Y G^\Lambda(y)\). Consider the following differential equation

\[\dot{y}^\tau = \bar{G}^{\Lambda, \eta}(y^\tau)\]  \hspace{1cm} (21)

Theorem A.3 (Slow-timescale convergence). (Perkins and Leslie, 2013, Corollary 4.8) If for all \(\eta > 0\), there is a global attractor \(A \subset S\) for the differential equation (21) and assumptions (A1)-(A7) in Assumption A.1 are satisfied then \(\{y^\tau\}_{\tau=0}^\infty\) will almost surely converge to \(A\).

Remark A.4. Note that Perkins and Leslie (2013) assume that for every \(i, i' \in [X], j, j' \in [Y]\) it holds that \(\alpha_i(\cdot) = \alpha_{i'}(\cdot)\) and \(\beta_j(\cdot) = \beta_{j'}(\cdot)\). However, their easily generalize under the setting of heterogeneous step sizes considered here due to Assumption (A3)-(iv). Indeed, Theorem A.2 (resp. Theorem A.3) follow similar to Perkins and Leslie (2013) if we fix a \(\tilde{i} \in [X]\) (resp. \(\tilde{j} \in [Y]\)) and bound the relative evolution of step sizes at fast (resp. slow) timescale \(i \neq \tilde{i}\) (resp. \(j \neq \tilde{j}\) with respect to \(\tilde{i}, \tilde{j}\) using Assumption (A3)-(iv).
B Proof of Lemmas

For clear presentation, we define the following notations which would be used throughout the section:

\[
\begin{align*}
u_i(s, a_i, \pi_{-i}) & \triangleq \sum_{a_{-i}} \pi_{-i}(s, a_{-i})u_i(s, a_i, a_{-i}), \\
P(s'|s, a_i, \pi_{-i}) & \triangleq \sum_{a_{-i}} \pi_{-i}(s, a_{-i})P(s'|s, a_i, a_{-i}), \\
P(s'|s, \pi) & \triangleq \sum_{a_i} \sum_{a_{-i}} \pi_i(s, a_i)\pi_{-i}(s, a_{-i})P(s'|s, a_i, a_{-i}), \\
Q_i(s, \pi_{i}'; \pi) & \triangleq \sum_{a_i} \pi_i'(s, a_i)Q_i(s, a_i; \pi)
\end{align*}
\]

B.1 Proof of Lemma 4.4

The proof follows by verifying that Assumption A.1 (A1)-(A7) are satisfied and then evoking Theorem A.2. Towards that goal, first we verify Assumption A.1 (A1)-(A7).

Before verifying the conditions for two-timescale asynchronous stochastic approximation stated in Section A, we introduce some notations that helps in clear presentation. For any \(q, \pi \in \Pi, i \in I, a_i \in A_i, s \in S\), we define

\[
\begin{align*}
T_{i}^\pi q_i(s, a_i) & \triangleq u_i(s, a_i, \pi_{-i}) \\
& + \delta \sum_{s'} P(s'|s, a_i, \pi_{-i}) \sum_{a_i'} \pi_i(s', a_i')q(s', a_i'),
\end{align*}
\]

which is analogous to Bellman operator in the setup of this paper. Furthermore, let’s define

\[
\begin{align*}
\hat{T}_{i}^\pi q_i(s, a_i) & \triangleq u_i(s, a_i, a_{-i}) + \delta \sum_{a_i'} \pi_i(s', a_i')q_i(s', a_i'),
\end{align*}
\]

where \(a_{-i} \sim \pi_{-i}(s)\) and \(s' \sim \sum_{a_{-i}} \pi_{-i}(s, a_{-i})P(\cdot|s, a_i, a_{-i})\). Moreover, for any \(s \in S, i \in I, a_i \in A_i\), we define

\[
\bar{b}_i(s; q) = \arg\max_{\pi \in \Delta(A_i)} \pi^\top q_i(s),
\]

where for every \(i \in I, s \in S, q_i(s) \in \mathbb{R}^{\lvert A_i \rvert}\). Under the above introduced notations we re-write
holds for $t$ by construction. Suppose it holds for a unique element in the set $\bar{\Pi}$.

Assume $\tilde{T}_i$, joint q-estimate and policy update which are updated at any instant of time there exists a

Note that $E\tilde{T}_{i} = 1$ is a martingale difference sequence. Note that the updates (25a)-(25b) can be cast in the
framework of (18).

We now verify Assumption A.1 (A1)-(A7) one by one

(i) First we show that (A1) in Assumption A.1 is satisfied with $(q^t, \pi^t)$ update (25a)-(25b). Let $\bar{R} = \max\{u/(1-\delta), \max_i ||q_i^0||_{\infty}\}$. Then we claim that $||q_i^t||_{\infty} \leq \bar{R}$ for all $t = \{0, 1, 2, \ldots\}$. We show this by induction. It holds for $t = 0$ by construction. Suppose it holds for $t = m - 1$ for some $m$ then we show that it also holds for $t = m$. Indeed, we note from (25a) that $q_i^k$ is a convex combination$^2$ of $q_i^{t-1}$ and $\tilde{T}_i q_i^{t-1}(s, a_i) + \tilde{M}(s, a_i)$. Using (23) and (26a) we see that

$$
||\tilde{T}_i q_i^{t-1}(s, a_i) + \tilde{M}(s, a_i)|| = ||\tilde{T}_i q_i^{t-1} ||_{\infty}
\leq \bar{u} + \delta \bar{R} \leq (1-\delta)\bar{R} + \delta \bar{R} = \bar{R}.
$$

This shows that $||q_i^t||_{\infty} \leq \bar{R}$. Moreover note that $\pi^t \in \Pi$ which is product simplex and is always compact.

(ii) Since we do not have any asymptotically negligible error terms in the asynchronous updates, Assumption A.1 (A2) is immediately satisfied

$^2$This is because we assume that $\alpha(n) \in (0, 1)$ in Assumption 4.2
(iii) Next we note Assumption A.1-(A3) is satisfied due to Assumption 4.2.
(iv) Now we show Assumption A.1-(A4) is satisfied. First, we concisely write the mean fields of (25a)-(25b) as follows

\[
F((s, a_i); q, \pi) \triangleq T_i^\pi q_i(s, a_i) - q_i(s, a_i),
\]

\[
G((s, a_i); q, \pi) = \bar{r}_i(s, a_i; q) - \pi_i(s, a_i),
\]

for every \(s \in S, i \in I, a_i \in A_i\). Define \(F(q, \pi) = \left( F((s, a_i); q, \pi) \right)_{s \in S, i \in I, a_i \in A_i}\), \(G(q, \pi) = \left( G((s, a_i); q, \pi) \right)_{s \in S, i \in I, a_i \in A_i}\). We note that both \(F, G\) are continuous as demanded in Assumption A.1-(A4). Furthermore, observe that

\[
\|F(q, \pi)\|_\infty \leq \|T_i^\pi q_i\|_\infty + \|q\|_\infty
\]

\[
\leq \bar{u} + \delta \|q\|_\infty + \|q\|_\infty \leq \bar{c}(1 + \|q\|_\infty),
\]

where \(\bar{c} = \max\{\bar{u}, 1 + \delta\}\). Also note that

\[
\sup_{w \in G(q, \pi)} \|w\|_\infty \leq 1 + \|\pi\|_\infty.
\]

Thus we conclude that Assumption A.1-(A4) is satisfied.

(v) We now verify Assumption A.1-(A5). Consider \(h, h' \in \bar{H}\) such that \(h = ((s, a_1), (s, a_2), \ldots (s, a_I))\) and \(h' = ((s', a'_1), (s, a'_2), \ldots (s', a'_I))\). Moreover, let \(z = (q, \pi)\) then,

\[
P(z; h, h') = P(s'|s, a) \prod_{i \in I} \pi_i(s', a'_i),
\]

(27)

where \(a = (a_i)_{i \in I}\) and the function \(P(z; h, h')\) is defined in (20). Since \(\theta_i > 0\) for every \(i \in I\) the policy updates will always assign non-zero probability over actions. That is, for every \(k \in \mathbb{N}\) we have \(\pi_i^k(s, a_i) > \theta_i/|A_i|\) for all \(s \in S, i \in I, a_i \in A_i\). Moreover, we impose Assumption A.1 on transition matrix which ensures that every state is visited with some non-zero probability. Thus Assumption A.1-(A5)-(i) and A.1-(A5)-(ii) are satisfied. Finally Assumption A.1-(A5)-(iii) is satisfied by noting that (27) is Lipschitz in \(\pi\) and therefore in \(z\).

(vi) Assumption A.1-(A6) is satisfied by noting that (a) \(\bar{M}\) is a bounded martingale difference sequence and (b) the step size condition in Assumption 4.2-(ii) holds.

(vii) For any \(\eta > 0, \pi \in \Pi\) consider the differential equation

\[
\frac{d}{d\tau} q_i^\tau = \Omega^{\eta}_A (\mathcal{T}_i^\pi q_i^\tau - q_i^\tau), \quad \forall i \in I
\]

(28)
where $\Omega_{A_i}^\eta = \{\text{diag}(\omega(1), ..., \omega(A_i)) : \omega(A_i) \in [\eta, 1] \ \forall i = 1, 2, \ldots, A_i\}$. In order to verify Assumption A.1-(A7), we show that (28) has unique global attractor for every $\pi \in \Pi$.

We first note that $T_\pi^i$ is a contraction for every $\pi \in \Pi$. Indeed

$$
T_\pi^i q_i(s, a_i) - T_\pi^i \bar{q}_i(s, a_i) = \delta \sum_{s'} P(s'|s, a_i, \pi_{-i}) \sum_{a'_i \in A_i} \pi_i(s', a'_i) (q_i(s', a'_i) - \bar{q}_i(s', a'_i)).
$$

Thus, for every $s \in S, i \in I, a_i \in A_i$, we have

$$
|T_\pi^i q_i(s, a_i) - T_\pi^i \bar{q}_i(s, a_i)| \leq \delta \|q_i - \bar{q}_i\|_\infty.
$$

Consequently, this $T_\pi^i$ is a contraction. That is,

$$
\|T_\pi^i q_i - T_\pi^i \bar{q}_i\|_\infty \leq \delta \|q_i - \bar{q}_i\|_\infty.
$$

Since $T_\pi^i$ is a contraction, this means that (28) has unique global attractor which is the fixed point of $T_\pi^i$. In fact it follows from definition that $Q_i(\cdot, \cdot; \cdot)$ is the unique global attractor. That is,

$$
T_\pi^i Q_i(s, a_i; \pi) = Q_i(s, a_i; \pi), \quad \forall s \in S, i \in I, a_i \in A_i.
$$

Then $\|Q_i(\cdot, \cdot; \pi)\|_\infty \leq \frac{\bar{u}}{1-\delta}$. Additionally, $Q_i(\cdot, \cdot; \pi)$ is also continuous in $\pi$. Finally, the convergence of (28) follows by (Borkar, 2009, Chapter 7). Thus Assumption A.1(A7) is satisfied. Finally, the claim in Lemma 4.4 follows by Theorem A.2.

### B.2 Proof of Lemma 4.6

We prove (a)-(d) in sequence

(a) We claim that for any integer $K \geq 0, \mu \in \Delta(S), s \in S, i \in I, a_i \in A_i$,

$$
\frac{\partial V_i(\mu, \pi)}{\partial \pi_i(s, a_i)} = \mathbb{E} \left[ \sum_{k=0}^{K} \delta^k \mathbb{1}(s^k = s) Q_i(s, a_i; \pi) + \delta^{K+1} \left( \frac{\partial V_i(s^{K+1}, \pi)}{\partial \pi_i(s, a_i)} \right) \right],
$$

where $s_0 \sim \mu, a^{k-1} \sim \pi(s^{k-1}), s^k \sim P(\cdot|s^{k-1}, a^{k-1})$. We prove this claim by induction.
Indeed, this holds for $K = 0$ by noting that

$$
\frac{\partial V_i(\mu, \pi)}{\partial \pi_i(s, a_i)} = \frac{\partial}{\partial \pi_i(s, a_i)} \left( \sum_{\tilde{s} \in S} \mu(\tilde{s}) \left( \sum_{a_i \in A_i} \pi_i(\tilde{s}, a_i) Q_i(s, a_i; \pi) \right) \right)
$$

$$
= \frac{\partial}{\partial \pi_i(s, a_i)} \left( \sum_{\tilde{s}} \mu(\tilde{s}) \sum_{a_i} \pi_i(\tilde{s}, a_i) \left( u_i(\tilde{s}, a_i, \pi_{i-1}(\tilde{s})) + \delta \sum_{s'} P(s'|\tilde{s}, a_i, \pi_{i-1}(\tilde{s})) V_i(s', \pi) \right) \right)
$$

$$
= \mu(s) \left( u_i(s, a_i, \pi_{i-1}(s)) + \delta \sum_{s'} P(s'|s, a_i, \pi_{i-1}(s)) V_i(s', \pi) \right)
$$

We now suppose that the claim holds for some integer $K$ and then show that it holds for $K + 1$, that is we have

$$
\frac{\partial V_i(\mu, \pi)}{\partial \pi_i(s, a_i)} = \mathbb{E} \left[ \sum_{k=0}^{K} \delta^k \mathbb{1}(s^k = s) \right] Q_i(s, a_i; \pi)
$$

$$
+ \delta^{K+1} \mathbb{E} \left[ \frac{\partial V_i(s^{K+1}, \pi)}{\partial \pi_i(s, a_i)} \right]
$$

$$
= \mathbb{E} \left[ \sum_{k=0}^{K} \delta^k \mathbb{1}(s^k = s) \right] Q_i(s, a_i; \pi)
$$

$$
+ \delta^{K+1} \mathbb{E} \left[ \frac{\partial}{\partial \pi_i(s, a_i)} \left( \sum_{a_i} \pi_i(s^{K+1}, a_i) Q_i(s^{K+1}, a_i; \pi) \right) \right]
$$

$$
= \mathbb{E} \left[ \sum_{k=0}^{K} \delta^k \mathbb{1}(s^k = s) \right] Q_i(s, a_i; \pi)
$$

$$
+ \delta^{K+1} \mathbb{E} \left[ \mathbb{1}(s^{K+1} = s) \right] Q_i(s, a_i; \pi)
$$

$$
+ \delta^{K+1} \mathbb{E} \left[ \frac{\partial V_i(s^{K+2}, \pi)}{\partial \pi_i(s, a_i)} \right].
$$
This completes the proof of (29). Now if we let $K \to \infty$ in (29) then we obtain

$$\frac{\partial V_i(\mu, \pi)}{\partial \pi_i(s, a_i)} = \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k \mathbb{1}(s^k = s) \right] Q_i(s, a_i; \pi)$$

$$= \sum_{s^0 \in \mathcal{S}} \mu(s^0) \sum_{k=0}^{\infty} \Pr(s^k = s|s^0)Q_i(s, a_i; \pi)$$

$$= \frac{1}{1 - \delta} d^\mu_i(s)Q_i(s, a_i; \pi)$$

(b) For any initial state distribution $\mu$ and joint policy $\pi = (\pi_i, \pi_{-i}), \pi' = (\pi'_i, \pi'_{-i}) \in \Pi$, it holds that

$$V_i(\mu, \pi) - V_i(\mu, \pi')$$

$$= \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k u_i(s^k, a^k) \right] - V_i(\mu, \pi')$$

$$= \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k \left( u_i(s^k, a^k) - V_i(s^k, \pi') + V_i(s^k, \pi') \right) \right] - V_i(\mu, \pi')$$

$$= \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k \left( u_i(s^k, a^k) - V_i(s^k, \pi') \right) \right]$$

$$+ \mathbb{E} \left[ V_i(s^0, \pi') \right] + \mathbb{E} \left[ \sum_{k=1}^{\infty} \delta^k V_i(s^k, \pi') \right] - V_i(\mu, \pi'),$$

where $s^0 \sim \mu, a^{k-1} \sim \pi(s^{k-1}), s^k \sim P(\cdot|s^{k-1}, a^{k-1})$. We note that

$$\mathbb{E} \left[ V_i(s^0, \pi') \right] = V_i(\mu, \pi'),$$

$$\mathbb{E} \left[ \sum_{k=1}^{\infty} \delta^k V_i(s^k, \pi') \right] = \delta \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k V_i(s^{k+1}, \pi') \right].$$

Therefore,

$$V_i(\mu, \pi) - V_i(\mu, \pi')$$

$$= \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k \left( u_i(s^k, a^k) - V_i(s^k, \pi') \right) \right]$$

$$+ \delta \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k V_i(s^{k+1}, \pi') \right]$$

$$= \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k \left( u_i(s^k, a^k) - V_i(s^k, \pi') + \delta V_i(s^{k+1}, \pi') \right) \right]$$

$$= \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k \left( u_i(s^k, a^k) + \delta V_i(s^{k+1}, \pi') - V_i(s^k, \pi') \right) \right]$$

30
Thus, we conclude that
\[ V_i(\mu, \pi) - V_i(\mu, \pi') = E \left[ \sum_{k=0}^{\infty} \delta^k \left( Q_i(s^k, a_i^k; \pi') - V_i(s^k, \pi') \right) \right] \]
\[ = E \left[ \sum_{k=0}^{\infty} \delta^k \left( \Gamma_i(s^k, a_i^k; \pi') \right) \right] \]
\[ = \frac{1}{1 - \delta} \sum_{s'} d_{\pi_0}^\pi(s') \left( \Gamma_i(s, \pi_i; \pi') \right). \]

(c) We prove the claim in two parts – first, we show that any stationary points of (8) is a Nash equilibrium of \( G \). Next, we show that any Nash equilibrium of \( G \) is a stationary point of (8).

First, we provide an important characterization of the one-step optimal deviation which is crucial for the following proof

\[ \text{br}_i(s; \pi^*) \]
\[ = \arg \max \sum_{a_i \in A_i} \hat{\pi}_i(a_i) Q_i(s, a_i; \pi^*) \]
\[ = \arg \max \left( u_i(s, \hat{\pi}_i, \pi^*_{-i}) + \delta \sum_{s'} P(s'|s, \hat{\pi}_i, \pi^*_{-i}) V_i(s', \pi^*) \right). \] (30)

First, we prove that \( \pi^* \) is a Nash equilibrium of \( G \), we need to show that for every \( i \in I, s \in S, \pi'_i \in \Pi_i \),

\[ V_i(s, \pi^*_i, \pi^*_{-i}) \geq V_i(s, \pi'_i, \pi^*_{-i}). \] (31)

Before proving (31), we first show that for any integer \( K \geq 1 \), any \( s \in S \), any \( i \in I \), and any \( \pi'_i \in \Pi_i \),

\[ V_i(s, \pi^*_i, \pi^*_{-i}) \geq E \left[ \sum_{k=0}^{K-1} \delta^k u_i(s^k, \pi'_i, \pi^*_i) + \delta^K V_i(s^K, \pi^*) \right], \] (32)

where \( s^0 = s, a_i^k \sim \pi'_i(s^k), a_{-i}^k \sim \pi^*_i(s^k), s^k \sim P(\cdot|s^{k-1}, a^{k-1}) \). Consider \( K = 1 \), for any
\( s \in S \), any \( i \in I \), any \( \pi'_i \in \Pi_i \) we have,

\[
V_i(s, \pi^*_i, \pi^*_{-i}) = u_i(s, \pi^*_i, \pi^*_{-i}) + \delta \sum_{s'} P(s'|s, \pi^*_i, \pi^*_{-i}) V(s', \pi^*) \geq u_i(s, \pi'_i, \pi^*_{-i}) + \delta \sum_{s'} P(s'|s, \pi'_i, \pi^*_{-i}) V(s', \pi^*) = \mathbb{E} \left[ u_i(s^0, \pi'_i, \pi^*_{-i}) + \delta V_i(s^1, \pi^*) \right],
\]

where, again, \( s^0 = s, a^0_i \sim \pi'_i(s^0), a^0_{-i} \sim \pi^*_{-i}(s^0), s^1 \sim P(\cdot|s^0, a^0) \) and the inequality follows from (33) as \( \pi^*_i(s) \in \text{br}_i(s; \pi^*) \) for every \( i \in I, s \in S \).

Next, suppose that (32) holds for some integer \( K \), we consider \( K + 1 \):

\[
V_i(s, \pi^*_i, \pi^*_{-i}) \geq \mathbb{E} \left[ \sum_{k=0}^{K-1} \delta^k u_i(s^k, \pi'_i, \pi^*_{-i}) + \delta^K V_i(s^K, \pi^*) \right] \geq \mathbb{E} \left[ \sum_{k=0}^{K-1} \delta^k u_i(s^k, \pi'_i, \pi^*_{-i}) + \delta^K \left( u_i(s^K, \pi'_i, \pi^*_{-i}) + \delta \sum_{s'} P(s'|s^K, \pi'_i, \pi^*_{-i}) V_i(s', \pi^*) \right) \right] \geq \mathbb{E} \left[ \sum_{k=0}^{K} \delta^k u_i(s^k, \pi'_i, \pi^*_{-i}) + \delta^{K+1} V_i(s^{K+1}, \pi^*) \right],
\]

where (a) is by induction hypothesis, (b) is due to (33), (c) is due to (30) and (d) is by rearrangement of terms. Thus, by mathematical induction, we have established that (32) holds for all \( K \). Let \( K \to \infty \) in (32), we have

\[
V_i(s, \pi^*_i, \pi^*_{-i}) \geq \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k u_i(s^k, \pi'_i, \pi^*_{-i}) \right] = V_i(s, \pi'_i, \pi^*_{-i}),
\]

for every \( s \in S, i \in I, \pi'_i \in \Pi_i \). Thus, we have proved (31), i.e. \( \pi^* \) is a Nash equilibrium of game \( G \).

Next we show that any Nash equilibrium \( \pi^* \) of \( G \) is a stationary point of (8). That is, \( \pi^*_i(s) \in \text{br}_i(s; \pi^*) \) for every \( i \in I, s \in S \). We prove this by contradiction. Suppose there exists a player \( i \in I \), and set of states \( \bar{S} \subseteq S \) such that for every \( \bar{s} \in \bar{S} \) it holds that
\[ \pi_i^*(\bar{s}) \not\in \text{br}_i(s; \pi^*). \] Let \( \pi' \) be a policy such that for all \( s \in S, i \in I, \pi'_i(s) \in \text{br}_i(s; \pi^*) \). Without loss of generality we assume \(|\bar{S}| = 1\).

We claim that for any integer \( K \geq 1 \), any \( s \in S, i \in I \) it holds that

\[
V_i(s, \pi_i^*, \pi_{-i}^*) \leq \mathbb{E} \left[ \sum_{k=0}^{K-1} \delta^k u_i(s^k, \pi'_i, \pi^*_{-i}) + \delta^K V_i(s^K, \pi^*) \right],
\]

where \( s^0 = s, a_i^k \sim \pi'_i(s^k), a_{-i}^k \sim \pi^*_{-i}(s^k), s^k \sim P(\cdot|s^{k-1}, a^{k-1}) \) and the inequality is strict for \( s = \bar{s} \).

Consider \( K = 1 \), for any \( s \in S \), any \( i \in I \), we have

\[
V_i(s, \pi_i^*, \pi_{-i}^*) = u_i(s, \pi_i^*, \pi_{-i}^*) + \delta \sum_{s'} P(s'|s, \pi_i^*, \pi_{-i}^*) V_i(s', \pi^*) \\
\leq u_i(s, \pi_i^*, \pi_{-i}^*) + \delta \sum_{s'} P(s'|s, \pi'_i, \pi_{-i}^*) V_i(s', \pi^*) \\
= \mathbb{E} \left[ u_i(s^0, \pi'_i, \pi_{-i}^*) + \delta V_i(s^1, \pi^*) \right],
\]

where, again, \( s^0 = s, a_i^0 \sim \pi'_i(s^0), a_{-i}^0 \sim \pi^*_{-i}(s^0), s^1 \sim P(\cdot|s^0, a^0) \) and the inequality is strict for \( s^0 = \bar{s} \).

Next, suppose \( (34) \) holds for some integer \( K \), we consider \( K + 1 \):

\[
V_i(s, \pi_i^*, \pi_{-i}^*) \leq \mathbb{E} \left[ \sum_{k=0}^{K-1} \delta^k u_i(s^k, \pi'_i, \pi_{-i}^*) + \delta^K V_i(s^K, \pi^*) \right] \\
\leq \mathbb{E} \left[ \sum_{k=0}^{K-1} \delta^k u_i(s^k, \pi'_i, \pi_{-i}^*) + \delta^K \left( u_i(s^K, \pi_i^*, \pi_{-i}^*) \\
+ \delta \sum_{s'} P(s'|s^k, \pi_i^*, \pi_{-i}^*) V_i(s', \pi^*) \right) \right] \\
\leq \mathbb{E} \left[ \sum_{k=0}^{K-1} \delta^k u_i(s^k, \pi'_i, \pi_{-i}^*) + \delta^K \left( u_i(s^K, \pi'_i, \pi_{-i}^*) \\
+ \delta \sum_{s'} P(s'|s^k, \pi'_i, \pi_{-i}^*) V_i(s', \pi^*) \right) \right] \\
= \mathbb{E} \left[ \sum_{k=0}^{K} \delta^k u_i(s^k, \pi'_i, \pi_{-i}^*) + \delta^{K+1} V_i(s^{K+1}, \pi^*) \right],
\]

where \( (a) \) is by induction hypothesis, \( (b) \) is due to \( (30) \) and \( (c) \) is by rearrangement of terms. Thus, by mathematical induction, we have established that \( (34) \) holds for all \( K \). Let \( K \to \infty \).
in (34), we have

\[ V_i(s, \pi_i^*, \pi_{-i}^*) \leq \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k u_i(s^k, \pi_i', \pi_{-i}') \right] = V_i(s, \pi_i', \pi_{-i}) , \]

for every \( s \in S, i \in I, \pi_i' \in \Pi \). Furthermore,

\[ V_i(\bar{s}, \pi_i^*, \pi_{-i}^*) < \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k u_i(s^k, \pi_i', \pi_{-i}') \right] = V_i(\bar{s}, \pi_i', \pi_{-i}) , \]

This contradicts the fact that \( \pi_i^* \) is a Nash equilibrium of game \( G \).

**B.3 Proof of Lemma 4.7**

Recall from the proof of Lemma 4.4, the assumptions (A1)-(A7) (refer Appendix A) in two-timescale asynchronous stochastic approximation are satisfied. Thus, we can apply Theorem A.3 to show that the convergence of the discrete time dynamics \((\pi^t)_{t=0}^{\infty}\) induced by Algorithm \(1\) is the same as that of the continuous time dynamical system in (8). From Lemma 4.5 we know that any solution of the continuous time dynamical system (8) must converge to the set of Nash equilibria of the game \( G \). Thus, \( \pi^t \) almost surely converges to this set.