Regularity for the two-phase singular perturbation problems

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Abstract

We prove that an a priori bounded mean oscillation (BMO) gradient estimate for the two-phase singular perturbation problem implies Lipschitz regularity for the limits. This problem arises in the mathematical theory of combustion, where the reaction diffusion is modeled by the $p$-Laplacian. A key tool in our approach is the weak energy identity. Our method provides a natural and intrinsic characterization of the free boundary points and can be applied to more general classes of solutions.

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1. Introduction

A chief difficulty in working with two-phase nonlinear free boundary problems is the absence of monotonicity formulas. One question still unanswered is whether the weak or variational solutions to the two-phase free boundary problems have optimal regularity. These solutions, for instance, arise in the mathematical theory of combustion, in the models of high activation of energy.

In this paper we address this question by developing a unified approach for the nonlinear two phase free boundary problems. To elucidate our main ideas we start from the singular perturbation problem for the $p$-Laplacian.

Let $u^\varepsilon$ be a family of solutions to

$$\Delta_p u^\varepsilon = \beta_\varepsilon(u^\varepsilon), \quad |u^\varepsilon| \leq 1, \quad \text{in } B_1,$$

where $\varepsilon > 0$ is a parameter and

$$\beta_\varepsilon(t) = \frac{1}{\varepsilon} \beta\left(\frac{t}{\varepsilon}\right), \quad 0 \leq \beta \in C^\infty_0(0, 1), \quad \int_{[0,1]} \beta := M > 0. \quad (1.1)$$
The quasilinear operator $\Delta_p u^\varepsilon = \text{div}(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon), 1 < p < \infty$ is called the $p$-Laplacian. If $p > 2$, then $\Delta_p$ is degenerate elliptic, and it depicts diffusion obeying power law.

1.1. Known results

For $p = 2$, Ze’ldovich and Frank-Kamenetskii studied the one-dimensional version of $(\mathcal{P}_\varepsilon)$ in 1938, see [20, Chapter 1.4], and calculated the speed of the front which is $\sqrt{2M}$, see formula (4.30) there. In high dimensions, the one-phase problem for the Laplacian was studied by Berestycki, Caffarelli, and Nirenberg [2], Caffarelli Vazquez [7], and the two-phase problem in [4, 6] for the heat equation $\Delta u^\varepsilon - \frac{\partial u^\varepsilon}{\partial t} = \beta_\varepsilon(u^\varepsilon)$. Later Caffarelli and Kenig studied the two-phase problem for the variable coefficient case $\text{div}(a \nabla u^\varepsilon) - \frac{\partial u^\varepsilon}{\partial t} = \beta_\varepsilon(u^\varepsilon)$ [5]. The key tool the authors used in the proof of optimal regularity is the monotonicity formula of Caffarelli [3]. With its help one can establish local uniform Lipschitz estimates in the parabolic distance.

For the nonlinear operators, the extensions of this results are available only for the one-phase problem, that is, when $u^\varepsilon \geq 0$, see [8, 15, 16, 18].

1.2. Heuristic discussion

Heuristically, the limits of $u^\varepsilon$ as $\varepsilon \to 0$ are the solutions of the two-phase Bernoulli-type free boundary problem

$$\begin{cases}
\Delta_p u = 0, & \text{in } (\{u > 0\} \cup \{u < 0\}) \cap B_1, \\
|\nabla u^\varepsilon|^p - |\nabla u^\varepsilon|^p = pM, & \text{on } \partial\{u > 0\}.
\end{cases} \tag{1.2}
$$

In order to pass to the limit we need some uniform bounds for $u^\varepsilon$, say $\|u^\varepsilon\|_{L^\infty(B_1)} \leq 1$. Then from Caccioppoli’s inequality, it follows that $\sup \|u^\varepsilon\|_{W^{1,p}(B_R)} \leq C(R, n, p)$, for every $R < 1$. Assume that $p > n$, then from Sobolev’s embedding theorem, we infer that $u^\varepsilon$ is locally uniformly Hölder continuous. Consequently, if $u(x) > 0$ for some $x \in B_1$, then $u$ is $p$-harmonic in some neighborhood of $x$. Moreover, the uniform convergence also implies that $\nabla u^\varepsilon \to \nabla u$ strongly in $L^p_{\text{loc}}(B_1)$.

Using these observations it is easy to see that the limits $u$ of the singular perturbation problem satisfy the weak energy identity

$$\int |(\nabla u|^p + pB^*(x))|\text{div}X = p \int |\nabla u|^p - \nabla u \nabla X \nabla u,$$

where $X = (X^1, \ldots, X^n)$ is a $C^1$ vectorfield with support in $B_1$ and $\nabla X = \partial X^j$ is the gradient of $X$. The function $B^*$ is bounded and characterizes the concentration of the measure $\Delta_p u^\varepsilon$ on the free boundary $\partial\{u > 0\}$ as $\varepsilon \to 0$. If $\partial\{u > 0\}$ is $C^1$, then one can see that $B^* = M\chi_{\{u > 0\}}$ with $M = \int \beta$, see Lemma 7.1. In particular, every stationary point of the functional $\int_{B_1} |\nabla u|^p_\varepsilon + M\chi_{\{u > 0\}}$ satisfies the weak energy identity.

We split $\partial\{u > 0\}$ into three subsets:

- (A) $x_0 \in \partial\{u > 0\}$ is a flat point,
- (B) the Lebesgue density of $\{u < 0\}$ is small at $x_0$,
- (C) neither (A) nor (B) hold at $x_0$.

We remark that there may be solutions $u$, obtained as limits of $(\mathcal{P}_\varepsilon)$, of the form $\alpha(x - x_0)^+ + \tilde{\alpha}(x - x_0)^- + \sigma(|x - x_0|)$ near $x_0$ with $\alpha, \tilde{\alpha} \geq 0$ and at these points, the stratification argument [10] fails. This is the reason why we further split the flat points into two parts and use the Lebesgue density to identify the points where $u$ is a viscosity solution.

If $u$ fails to have linear growth at some point $x_0 \in \partial\{u > 0\}$, then the scaled functions

$$u_t(x) = \frac{u(x_0 + tx)}{\sup_{B_t(x_0)} |u|},$$
Figure 1 (colour online). The diagram schematizes the proof of Theorem 1.1 which is based on a series of dichotomies depending on the conditions in the blue boxes. The conclusion is in the yellow box, which gives the main result stated in Theorem 1.1. The constant $C$ depends only on $h_0, \delta, n, p$, and $M$.

converge to a limit function $u_0$ satisfying the identity

$$\int |\nabla v_0|^p \text{div} X = p \int |\nabla v_0|^{p-2} \nabla v_0 \nabla X \nabla v_0, \quad X \in C^1_0(\mathbb{R}^n, \mathbb{R}^n). \quad (1.3)$$

We claim that (1.3) implies that $\Delta_p u_0 = 0$ in $\mathbb{R}^n$. The converse statement is obviously true since (1.3) is the domain variation of the energy $\int |\nabla u|^p$. This is an interesting question of independent interest since (1.3), as we show in this paper, gives another characterization of the $p$-harmonic functions for $p > n$. That done, we can apply Liouville’s theorem to conclude that $u_0$ is a linear function. Combining this with the stratification argument from [10] with respect to the modulus of continuity of the slab flatness and the Lebesgue density of $\{u < 0\}$, we conclude that if either (A) or (B) hold, then $u$ has linear growth at $x_0$. For the remaining case (C), we can conclude that $u$ is a viscosity solution and $x_0$ is a flat point, so from the Harnack principle we infer that $\partial \{u > 0\}$ near $x_0$ is $C^{1,\gamma}$ smooth hypersurface, and the linear growth for this case follows from the standard boundary gradient estimates for $u$.

A schematic view of the main steps in the proof of Lipschitz regularity is illustrated in Figure 1.

1.3. Main results

**Theorem 1.1.** Let $u^{\varepsilon_j}$ be a family of solutions to $(P_\varepsilon)$ such that $u^{\varepsilon_j} \to u$ in $W^{1,p}_{\text{loc}}$. If there is a universal constant $C > 0$ such that

$$\|\nabla u\|_{BMO_{\text{loc}}(B_r(x_0))} \leq C(\|u\|_{W^{1,p}(B_{2r}(x_0))}), \quad B_{2r}(x_0) \subset B_1, \quad (1.4)$$

then $u$ is locally Lipschitz continuous.

The paper is organized as follows: Section 2 contains some well-known estimates for the solutions and subsolutions of the $p$-Laplacian.
In Section 3 we prove the weak energy identity. If the Lipschitz continuity fails, then we get that the weak energy identity simplifies. In Section 4 we show that BMO solutions satisfying this simplified identity must be \( p \)-harmonic.

Then we consider the scenarios (A), (B), and (C) in Sections 5 (Flatness vs linear growth), 6 (density of negative set), and 7 (viscosity solutions), respectively. Combining our results in Section 8 we give the proof of Theorem 1.1.

In Section 9 we study the properties of solutions with Lipschitz regularity. Section 10 is devoted to the weak solutions. Finally, in Section 11 we prove a BMO-type estimate for the tensor \( T_{in} = p |\nabla u|^{p-2} u_i u_m - |\nabla u|^p \delta_{im} \).

**Notation**

We fix some notation. The \( n \)-dimensional Euclidean space is denoted by \( \mathbb{R}^n \), \( u^+ = \max(u, 0) \) is the nonnegative part of \( u \) and similarly \( u^- = - \min(u, 0) \), so that \( u = u^+ - u^- \). The partial derivatives in \( x_i \) variable, \( i = 1, \ldots, n \) are denoted by \( \partial_i u \) or \( u_i \), so that \( \partial_i u = \frac{\partial u}{\partial x_i} \), \( (x - x_0)_i \) is the first slot function of the vector \( x - x_0 \). For every \( u \in W^{1,p}(B_1) \) we also define the tensor \( T_{in}(\nabla u) = p |\nabla u|^{p-2} u_i u_m - |\nabla u|^p \delta_{im} \). If \( X = (X^1, \ldots, X^n) \) is a \( C^1 \) vectorfield, \( X : \Omega \to \mathbb{R}^n \), then the tensor \( \nabla X = \partial_i X^j \) denotes the gradient of \( X \). Sometimes we let \( \Gamma = \Gamma_u \) to denote the free boundary \( \partial \{ u > 0 \} \) when no confusion can arise, \( \text{Vol}(E) \) denotes the \( n \)-dimensional volume of a set \( E \).

\[ \text{Preliminaries and tools: uniform estimates and compactness} \]

**Definition 2.1.** Let \( 1 < p < \infty \). A function \( u^\varepsilon \in W^{1,p}(B_1) \) is said to be a weak solution to \( \Delta_p u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \) in \( B_1 \) if for every \( \psi \in W^{1,p}_0(B_1) \) there holds

\[ -\int |\nabla u^\varepsilon|^{p-2} \langle \nabla u^\varepsilon, \nabla \psi \rangle = \int \beta_\varepsilon(u^\varepsilon) \psi. \]

If \( \Delta_p u = 0 \), then \( u \) is called \( p \)-harmonic in \( B_1 \).

We recall the well-known inequality [9]

\[ \langle |\xi|^{p-2} \xi - |\eta|^{p-2} \eta, \xi - \eta \rangle \geq \gamma|\xi - \eta|^p, p > 2. \]  

(2.1)

**2.1. Caccioppoli inequality and local compactness**

**Lemma 2.2.** Let \( u^\varepsilon \) be a family of solutions of \( (P_\varepsilon) \), and then exist a constant \( C = C(n, p) > 0 \) depending only on \( n, p \) such that for every \( R \in (0, 1) \) there holds

\[ \int_{B_R} |\nabla u^\varepsilon|^p \leq \frac{C}{(1 - R)^p} \int_{B_1} |u^\varepsilon|^p. \]

(2.2)

**Proof.** Let \( \eta \in C^\infty_0(B_1), 0 \leq \eta \leq 1, |\nabla \eta| \leq C/R \) for some \( C = C(n) > 0 \), and \( \eta = 1 \) on \( B_R \). From (1.1) it follows that \( u^\varepsilon \beta_\varepsilon(u^\varepsilon) \geq 0 \). Thus using \( u^\varepsilon \eta^p \) as a test function in the weak formulation of the equation \( \Delta_p u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \), we get

\[ 0 \leq \int |\nabla u^\varepsilon|^{p-2} \langle \nabla u^\varepsilon, \nabla u^\varepsilon \eta^p + p u^\varepsilon \eta^{p-1} \nabla \eta \rangle. \]

Rearranging the terms and using the Hölder inequality, we obtain
\[
\int |\nabla u^\varepsilon|^p \eta^p \leq -p \int |\nabla u^\varepsilon|^{p-2} \langle \nabla u^\varepsilon, \nabla \eta \rangle u^\varepsilon \eta^{p-1}
\]
\[
\leq p \int |\nabla u^\varepsilon|^{p-1} |\nabla \eta| |\nabla \eta|^{p-1}
\]
\[
\leq p \left( \int |\nabla u^\varepsilon|^{p \eta \varepsilon} \right)^{1-\frac{1}{p}} \left( \int |u^\varepsilon|^{p \eta \varepsilon} \right)^{\frac{1}{p}}.
\]

From here we see that
\[
\hat{\|\nabla u^\varepsilon\|}_p^{\eta \varepsilon} \leq p \hat{\|\nabla u^\varepsilon\|}_p, \]
and the desired estimate follows with \(C(n, p) = p^n C^p(n)\).

In the next proposition, we assume \(p > n\), then from (2.2), the assumption \(|u^\varepsilon| \leq 1\), and Sobolev’s embedding theorem, it follows that
\[
\|u^\varepsilon\|_{C^{1-\frac{2}{p}}(\overline{B_R})} \leq C(R, p, n),
\]
for fixed \(0 < R < 1\) and uniformly in \(\varepsilon\).

**Proposition 2.3.** Let \(u^\varepsilon\) be a family of solutions to \((\mathcal{P}_\varepsilon)\) and \(p > n\). Then the following statements hold true: for every sequence \(\varepsilon_k \to 0\) there is a subsequence, still labeled \(\varepsilon_k\), and a function \(u\) such that

(i) \(u^\varepsilon_k \to u\) in \(C^{1-\frac{2}{p}}(B_1) \cap W^{1,p}_0(B_1)\), and \(\Delta_p u = 0\) in \(\{u > 0\} \cup \{u < 0\}\),

(ii) \(|\nabla u^\varepsilon_k|^{p-2} \nabla u^\varepsilon_k\) weakly \(\to |\nabla u|^{p-2} \nabla u\),

(iii) \(|\nabla u^\varepsilon_k|^p \to |\nabla u|^p\) strongly in every compact of \(B_1\).

**Proof.** From Lemma 2.2 and Sobolev’s embedding theorem, it follows that
\[
\sup_k \left( \|u^\varepsilon_k\|_{C^{1-\frac{2}{p}}(\overline{B_R})} + \|u^\varepsilon_k\|_{W^{1,p}_0(B_R)} \right) \leq C(R, p, n),
\]
Thus (i) follows from a standard compactness argument.

The proof of (ii) is standard, see [9].

To prove (iii) it is enough to show that
\[
\int |\nabla u^\varepsilon|^p \psi \to \int |\nabla u|^p \psi.
\]
This and the weak convergence imply strong convergence.

It follows from (1.1) that \(u^\varepsilon \beta_\varepsilon(u^\varepsilon) \geq 0\). Using \(u^\varepsilon \psi\) as test function in the weak formulation of the equation, we get as in the proof of Lemma 2.2
\[
\int |\nabla u^\varepsilon|^p \psi \leq - \int |\nabla u^\varepsilon|^{p-2} \langle \nabla u^\varepsilon, \nabla \psi \rangle u^\varepsilon.
\]
Applying Fatou’s lemma we obtain
\[
\int |\nabla u|^p \psi \leq \liminf_{\varepsilon \to 0} \int |\nabla u^\varepsilon|^p \psi \leq - \int |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle u.
\]
Let \(s > 0\) be a small number. Then \((u - s)^+ \psi \in W^{1,p}_0(\{u > 0\})\). Therefore,
\[
\int_{\{u > s\}} |\nabla u|^p \psi = - \int |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle (u - s)^+.
\]
Similarly, \((u + s)^-\) ψ ∈ \(W^{1,p}_0(\{u < 0\})\) hence
\[
\int_{\{u < -s\}} |\nabla u|^p \psi = -\int |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle (u + s)^-.
\]
After sending \(s \to 0\) we conclude
\[
\int |\nabla u|^p \psi = \int |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle u.
\]
This and (2.9) imply
\[
- \int |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle u \leq \int |\nabla u|^p \psi \leq \liminf_{\varepsilon \to 0} \int |\nabla u^\varepsilon|^p \psi \leq - \int |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle u.
\]
Consequently,
\[
\int |\nabla u|^p \psi = \lim_{\varepsilon \to 0} \int |\nabla u^\varepsilon|^p \psi.
\]

**Remark 2.4.** The assumption \(p > n\) is technical. It allows to get the uniform continuity of \(u^\varepsilon\), see also the discussion in Section 11.

**Lemma 2.5.** There exists a constant \(C = C(n,p)\) depending only on \(n,p\) such that for every \(B_{2R}(x) \Subset B_1\), the measure \(\mu = \Delta_p u^\varepsilon\) satisfies the inequality
\[
\int_{B_R(x)} d\mu \leq CR^{2/p-1} \left( \int_{B_{2R}(x)} |\nabla u^\varepsilon|^p \right)^{1-\frac{1}{p}}.
\]  
(2.5)

**Proof.** Using the divergence theorem we get the estimate
\[
\int_{B_r(x)} \Delta_p u^\varepsilon = \int_{\partial B_r(x)} |\nabla u^\varepsilon|^{p-2} \langle \nabla u^\varepsilon, \nu \rangle \leq \int_{\partial B_r(x)} |\nabla u^\varepsilon|^{p-1}.
\]  
(2.6)
Integrating both sides of (2.6) over \(r \in [0,R]\) we obtain
\[
\int_0^R \int_{\partial B_r(x)} |\nabla u^\varepsilon|^{p-1} = \int_{B_R(x)} |\nabla u^\varepsilon|^{p-1} \leq \left( \int_{B_R(x)} |\nabla u^\varepsilon|^p \right)^{1-\frac{1}{p}} |B_R|^{1/p}.
\]  
(2.7)
On the other hand,
\[
\int_0^R \int_{B_r(x)} d\mu \geq \frac{R}{2} \int_{B_{\frac{R}{2}}(x)} d\mu.
\]
Combining this with (2.7) we get
\[
\int_{B_{\frac{R}{2}}(x)} d\mu \leq \frac{2|B_R|^{1/p}}{R} \left( \int_{B_R(x)} |\nabla u^\varepsilon|^p \right)^{1-\frac{1}{p}},
\]
and (2.5) follows.

**Proposition 2.6.** Let \(u_j\) be a sequence of solutions to \(\Delta_p u_j = \mu_j\) in \(B_1\) such that \(\sup_j \|u_j\|_{W^{1,p}(B_1)} < \infty\) and \(\mu_j\) are Radon measures in \(B_1\) such that \(\text{supp} \mu_j \subset \partial \{u_j > 0\}\). If \(u_0\) is a limit of \(u_j\), then \(\nabla u_j \to \nabla u_0\) strongly in \(L^p_{loc}(B_1)\).
Proof. Let $0 \leq \psi \in C_0^1(B_1)$ then $u_j \psi \Delta_p u_j \geq 0$ in $B_1$. Thus we have
\[
\int |\nabla u_j|^p \psi \leq -\int |\nabla u_j|^{p-2} \langle \nabla u_j, \nabla \psi \rangle u_j.
\] (2.8)

By Fatou’s lemma
\[
\int |\nabla u|^p \psi \leq \liminf_{j \to \infty} \int |\nabla u_j|^p \psi \leq -\int |\nabla u|^{p-2} \langle \nabla u_0, \nabla \psi \rangle u_0. \tag{2.9}
\]

Let $s > 0$ be a small number. Then $(u_0 - s)^+ \psi \in W_0^{1,p}(\{u_0 > 0\})$. Therefore,
\[
\int_{\{u_0 > s\}} |\nabla u_0|^p \psi = -\int |\nabla u_0|^{p-2} \langle \nabla u_0, \nabla \psi \rangle (u_0 - s)^+.
\]

Similarly, $(u_0 + s)^- \psi \in W_0^{1,p}(\{u_0 < 0\})$ hence
\[
\int_{\{u_0 < -s\}} |\nabla u_0|^p \psi = -\int |\nabla u_0|^{p-2} \langle \nabla u_0, \nabla \psi \rangle (u_0 + s)^-.
\]

After sending $s \to 0$ we conclude
\[
\int |\nabla u_0|^p \psi \geq \int_{\{u_0 \neq 0\}} |\nabla u_0|^p \psi = -\int |\nabla u_0|^{p-2} \langle \nabla u_0, \nabla \psi \rangle u_0.
\]

This and (2.9) imply
\[
-\int |\nabla u_0|^{p-2} \langle \nabla u_0, \nabla \psi \rangle u_0 \leq \int |\nabla u_0|^p \psi \leq \liminf_{j \to \infty} \int |\nabla u_j|^p \psi \leq -\int |\nabla u_0|^{p-2} \langle \nabla u_0, \nabla \psi \rangle u_0.
\]

Consequently,
\[
\int |\nabla u_0|^p \psi = \lim_{j \to \infty} \int |\nabla u_j|^p \psi.
\]

We summarize the previous results replacing $B_1$ by a general domain $D$.

**Proposition 2.7.** Let $u^\varepsilon$ be a family of solutions to $(P_\varepsilon)$ in a domain $D \subset \mathbb{R}^n$, and $p > n$. Let us assume that $\|u^\varepsilon\|_{L^\infty(D)} \leq A$ for some constant $A > 0$ independent of $\varepsilon$. For every $\varepsilon_k \to 0$ there exists a subsequence $\varepsilon_{k'} \to 0$ and $u \in C^{1-\frac{n}{p}}_{\text{loc}}(D)$, such that

(i) $u^{\varepsilon_{k'}} \to u$ uniformly on compact subsets of $D$,
(ii) $\nabla u^{\varepsilon_{k'}} \to \nabla u$ in $L^p_{\text{loc}}(D)$,
(iii) $u$ is $p$-harmonic in $D \cap \{u > 0\} \cup \{u < 0\}$.

**2.2. First and second blow-up**

Using Proposition 2.7 we can extract a sequence $u^\varepsilon_j$ for some sequence $\varepsilon_j$ such that $u^\varepsilon_j \to u$ uniformly in $B_{\frac{1}{2}}$. Let $0 < \rho_j \downarrow 0$, $x_j \in \partial\{u > 0\}$ and set $u_j(x) = \frac{u(x_j + \rho_j x)}{m_j}$, where $m_j$ are some positive numbers such that $\sup_j \rho_j / m_j < \infty$. Suppose that $u_j$ is uniformly bounded and $u_j \to U$ locally uniformly in $\mathbb{R}^n$ for some function $U$ defined on $\mathbb{R}^n$.

The function $U$ is called a blow-up limit of $u$ with respect to free boundary points $x_j$ and, in general, it depends on $\{\rho_j\}$ and $x_j$.

The two propositions to follow establish an important property of the blow-up limits, namely, that the first and second blow-ups of $u$ can be obtained from $(P_\varepsilon)$ for a suitable choice of parameter $\varepsilon$. 
If \( u^{\varepsilon_j} \) solves \((\mathcal{P}_\varepsilon)\), then the scaled functions \( \hat{u}^{\varepsilon_j}(x) = \frac{u^{\varepsilon_j}(x + \rho_j x)}{m_j} \) verify
\[
\Delta_p(\hat{u}^{\varepsilon_j}) = \frac{\rho_j^p}{m_j^{p-1}} \frac{1}{\varepsilon_j} \left( \frac{u^{\varepsilon_j}(x + \rho_j x)}{\varepsilon_j} \right) = \left[ \frac{\rho_j}{m_j} \right]^p \frac{1}{\varepsilon_j/m_j} \beta \left( \frac{\hat{u}^{\varepsilon_j}(x)}{\varepsilon_j/m_j} \right)
\]

where \( \delta_j = \frac{\varepsilon_j}{m_j} \).

If \( \delta_j = \frac{\varepsilon_j}{m_j} \to 0 \), we see that \( \hat{u}^{\varepsilon_j} \) solves \( \Delta_p \hat{u}^{\varepsilon_j} = [\frac{\rho_j}{m_j}]^p \beta_j(\hat{u}^{\varepsilon_j}) \).

**Proposition 2.8.** Let \( u^{\varepsilon_j} \) be a family of solutions to \((\mathcal{P}_\varepsilon)\) in a domain \( \mathcal{D} \subset \mathbb{R}^n \) such that \( u_{\varepsilon_j} \to u \) uniformly on \( \mathcal{D} \) and \( \varepsilon_j \to 0 \).

Let \( x_0 \in \mathcal{D} \cap \partial\{u > 0\} \) and let \( x_k \in \partial\{u > 0\} \) be such that \( x_k \to x_0 \) as \( k \to \infty \).

Let \( \rho_k \to 0 \), \( u_k(x) = \frac{1}{m_k} u(x + \rho_k x) \), \( \hat{u}^{\varepsilon_j}(x) = \frac{1}{m_k} u^{\varepsilon_j}(x + \rho_k x) \), and \( u_k \to U \) uniformly on compact subsets of \( \mathbb{R}^n \).

Then there exists \( j_k(x) \to \infty \) such that for every \( j \geq j_k \) there holds that \( \varepsilon_j/\rho_k \to 0 \) and

(i) \( \hat{u}^{\varepsilon_j}_{k} \to U \) uniformly on compact subsets of \( \mathbb{R}^n \),

(ii) \( \nabla \hat{u}^{\varepsilon_j}_{k} \to \nabla U \) in \( L^p_{loc}(\mathbb{R}^n) \),

(iii) \( \nabla u_k \to \nabla U \) in \( L^p_{loc}(\mathbb{R}^n) \).

**Proof.** Our proof closely follows that of [6, Lemma 3.2] where the case of \( \rho_k = m_k \). We estimate the difference
\[
\frac{u^{\varepsilon_j}(x + \rho_k x)}{m_k} - U(x) = \frac{u^{\varepsilon_j}(x + \rho_k x)}{m_k} - \frac{u(x + \rho_k x)}{m_k} + \frac{u(x + \rho_k x)}{m_k} - U(x) = I + II.
\]

Fix \( r > 0 \), then for every \( \delta > 0 \) and \( R < \frac{r}{\rho_k} \) there exists \( k_0 = k_0(\delta, R) \) such that for \( k > k_0 \) there holds
\[
|II| = \left| \frac{u(x + \rho_k x)}{m_k} - U(x) \right| < \delta, \quad x \in B_R.
\]

Let \( x \in B_r(x_0) \), then there is \( j(k) \) such that \( |u^{\varepsilon_j}(x) - u(x)| < \frac{m_k}{k} \) whenever \( j \geq j(k) \). This means that
\[
|I| = \left| \frac{u^{\varepsilon_j}(x + \rho_k x)}{m_k} - \frac{u(x + \rho_k x)}{m_k} \right| < \frac{1}{k}.
\]

Consequently, we have that
\[
\left| \frac{u^{\varepsilon_j}(x + \rho_k x)}{m_k} - U(x) \right| \leq |I| + |II| \leq \delta + \frac{1}{k}, \quad x \in B_R.
\]

Observe that \( j(k) \) can be chosen so large that \( \varepsilon_{j(k)}/m_k < \frac{1}{k} \). Hence recalling (2.10) and applying Proposition 2.7, we get parts (i) and (ii).

As for (iii) we use the estimate above to get \( \|\nabla \hat{u}^{\varepsilon_j} - \nabla U\|_{L^p(B_R)} < \delta \) whenever \( j > j(n) \). We have
\[
\|\nabla u_k - \nabla U\|_{L^p(B_R)} \leq \|\nabla u_k - \nabla \hat{u}^{\varepsilon_j}_{k}\|_{L^p(B_R)} + \|\nabla \hat{u}^{\varepsilon_j}_{k} - \nabla U\|_{L^p(B_R)} \leq \|\nabla u_k - \nabla \hat{u}^{\varepsilon_j}_{k}\|_{L^p(B_R)} + \delta.
\]
So it remains to estimate $\|\nabla u_k - \nabla \hat{u}_k^{\varepsilon_j}\|_{L^p(B_R)}$ for $j > j(k)$. Let us estimate

$$\|\nabla u_k - \nabla \hat{u}_k^{\varepsilon_j}\|_{L^p(B_R)}^p = \left[ \frac{\rho_k}{m_k} \right]^p \int_{B_R} |\nabla u(x_k + \rho_k x) - \nabla u^{\varepsilon_j}(x_k + \rho_k x)|^p dx$$

$$= \left[ \frac{\rho_k}{m_k} \right]^p \frac{1}{\rho_k^{p}} \int_{B_{\rho_k R}(x_k)} |\nabla u(x) - \nabla u^{\varepsilon_j}(x)|^p dx.$$  

We know that $B_{\rho_k R}(x_k) \subset B_r(x_0)$ for large $k$. Thus there is $k_0$ large such that

$$\int_{B_r(x_0)} |\nabla u(x) - \nabla u^{\varepsilon_j}(x)|^p dx \leq \delta \rho_k^n.$$  

Therefore,

$$\|\nabla u_k - \nabla U\|_{L^p(B_R)} \leq \delta + \left[ \frac{\rho_k}{m_k} \right]^p \delta \leq 2\delta,$$

and (iii) follows. \qed

Finally, recall that the result of previous proposition extends to the second blow-up.

**Proposition 2.9.** Let $u^{\varepsilon_j}$ be a solution to $(P_\varepsilon)$ in a domain $D_j \subset D_{j+1}$ and $\bigcup_j D_j = \mathbb{R}^N$ such that $u^{\varepsilon_j} \to U$ uniformly on compact sets of $\mathbb{R}^N$ and $\varepsilon_j \to 0$. Let us assume that for some choice of positive numbers $d_n$ and points $x_n \in \partial\{U > 0\}$, the sequence

$$U_{d_n}(x) = \frac{1}{d_n} U(x + d_n x)$$

converges uniformly on compact sets of $\mathbb{R}^N$ to a function $U_0$. Let

$$(u^{\varepsilon_j})_{d_n} = \frac{1}{d_n} u^{\varepsilon_j}(x + d_n x).$$

Then there exists $j(n) \to \infty$ such that for every $j_n \geq j(n)$, there holds $\varepsilon_j / d_n \to 0$ and

- $(u^{\varepsilon_{j_n}})_{d_n} \to U_0$ uniformly on compact subsets of $\mathbb{R}^N$,
- $\nabla(u^{\varepsilon_j})_{d_n} \to \nabla U_0$ in $L^2_{loc}(\mathbb{R}^N)$.

**Proof.** See Lemma 3.3 [6]. \qed

3. Weak energy identity for the solutions of $(P_\varepsilon)$ and the first domain variation

This section contains the crucial tool for the proof of our main regularity theorem, the weak energy identity which we state below. In what follows we set $B(t) = \int_0^t \beta(\tau) d\tau$.

**Lemma 3.1.** Let $u^\varepsilon$ be a family of solutions to $(P_\varepsilon)$. For every $\phi = (\phi^1, \ldots, \phi^n) \in C^1_0(B_1, \mathbb{R}^n)$ we have the identity.

$$\int (|\nabla u^\varepsilon|^p + pB(u^\varepsilon / \varepsilon)) \text{div}\phi = p \sum_{m,l=1}^n \int |\nabla u^\varepsilon|^{p-2} \partial_l u^\varepsilon \partial_m u^\varepsilon \partial_l \phi \partial_m \phi.$$  

(3.1)

**Proof.** Multiply $\Delta_p u^\varepsilon = \beta_v(u^\varepsilon)$ by $\partial_i u^\varepsilon \phi$ and integrate to get that

$$\int B(u^\varepsilon / \varepsilon) \phi_i = - \int \beta_v(u^\varepsilon) u^\varepsilon_i \phi.$$
On the other hand,
\[
\int (|\nabla u|^p - 2u_j^\varepsilon) u_j^\varepsilon \phi^i = - \int |\nabla u|^p - 2(u_j^\varepsilon (u_j^\varepsilon \phi^i + u_i \phi^i))
\]
\[= \frac{1}{p} \int |\nabla u|^p \text{div} \phi - \int |\nabla u|^p - 2u_j^\varepsilon u_j^\varepsilon \phi^i. \quad \square \]

Next we prove that (3.1) is preserved in the limit as \( \varepsilon \to 0 \).

**Lemma 3.2.** There is a bounded nonnegative function \( 0 \leq B^*(x) \leq M \) such that for every vector field \( X \in C^1_0(B_1, \mathbb{R}^n) \) we have
\[
\int (|\nabla u|^p + pB^*(x)) \text{div}X = p \int |\nabla u|^p - 2\nabla u \nabla X \nabla u. \tag{3.2}
\]

**Proof.** We have \( B(u^\varepsilon/\varepsilon) \to B^*(x) \) *-weakly in \( L^\infty \).

By strong convergence of gradients, Proposition 2.3(iii),
\[
\int (|\nabla u|^p + pB(u^\varepsilon/\varepsilon)) \text{div}X \to \int (|\nabla u|^p + pB^*(x)) \text{div}X.
\]

Let us show that the functions \( |\nabla u^\varepsilon|^p - 2\nabla u^\varepsilon \nabla X \nabla u^\varepsilon \) are equiintegrable. Given \( \sigma > 0 \), then there is \( j_0 \) and \( \delta > 0 \) such that
\[
\int_E |\nabla u^\varepsilon|^p - 2|\nabla u^\varepsilon \nabla X \nabla u^\varepsilon| < \sigma,
\]
whenever \( |E| < \delta \) and \( j > j_0 \). Indeed, we have
\[
\int_E |\nabla u^\varepsilon|^p - 2|\nabla u^\varepsilon \nabla X \nabla u^\varepsilon| \leq \|\nabla X\|_\infty \left[ \int_E |\nabla u^\varepsilon|^p - \int_E |\nabla u|^p \right] + \|\nabla X\|_\infty \int_E |\nabla u|^p.
\]

Choose \( j_0 \) so large that \( \int_E |\nabla u^\varepsilon|^p - \int_E |\nabla u|^p < \frac{\sigma}{2\|\nabla X\|_\infty} \) (which is possible thanks to Proposition 2.3 (iii)) and then by the absolute continuity of the integral of \( |\nabla u|^p \) we can choose \( \delta \) so small that \( \int_E |\nabla u|^p < \frac{\sigma}{2} \). Hence the desired result follows. \( \square \)

### 3.1. Domain variation formula for minimizers

Let \( \lambda > 0 \) be a constant. We show that the local minimizers of
\[
J_p(u) = \int \Omega |\nabla u|^p + \lambda^p \chi_{\{u > 0\}},
\]
satisfy
\[
p \int_\Omega \{ |\nabla u(y)|^{p-2} \cdot \nabla u(y) \nabla \phi(y) - |\nabla u(y)|^p + \lambda(u) \} \text{div} \phi \, dy = 0, \tag{3.3}
\]
where, for the sake of simplicity, we set \( \lambda(u) = \lambda^p \chi_{\{u > 0\}}. \) The identity (3.2) is weaker than (3.3) since we do not know the explicit form of \( B^* \).

**Lemma 3.3.** Let \( u \) be a local minimizer of \( J_p(\cdot) \), then (3.3) holds.

**Proof.** Let \( f(\xi) = |\xi|^p \). For \( \phi \in C_0^1(\Omega, \mathbb{R}^n) \) we put \( u_t(x) = u(x + t\phi(x)) \), with small \( t \in \mathbb{R} \). Then \( \phi_t(x) = x + t\phi(x) \) maps \( \Omega \) into itself. After change of variables \( y = x + t\phi(x) \) we infer
\[
\int_\Omega \left[ f(\nabla u_t(x)) + \lambda(u_t(x)) \right] \, dx = \int_\Omega \left[ f(\nabla u_t(\phi_t^{-1}(y))) + \lambda(u(y)) \right] \left[ 1 - t\text{div}(\phi(\phi_t^{-1}(y))) + o(t) \right] \, dy.
\tag{3.4}
\]
Here we used the inverse mapping theorem for \( \phi_t : x \rightarrow y \), in particular a well-known identity

\[
\left| \frac{D(x_1, \ldots, x_n)}{D(y_1, \ldots, y_n)} \right| = \left| \frac{D(y_1, \ldots, y_n)}{D(x_1, \ldots, x_n)} \right|^{-1} = \frac{1}{1 + t \text{div} \phi + o(t)}.
\]

One can easily verify that

\[
\nabla u_t(x) = \nabla u(\phi_t(x))\{I + t\nabla \phi(x)\}
\]

with \( I = \{\delta_{ij}\} \) being the identity matrix. Hence

\[
\nabla u_t(\phi_t^{-1}(y)) = \nabla u(y)\{I + t\nabla \phi(\phi_t^{-1}(y))\}.
\]

and, moreover,

\[
f(\nabla_x u_t(x)) = f(\nabla_x u(y)) + t\nabla_x f(\nabla_x u(y))\nabla_x u(y)\nabla \phi(\phi_t^{-1}(y)) + o(t).
\]

This in conjunction with (3.4) yields

\[
\int_{\Omega} \left\{ \nabla f_\xi(\nabla u(\{I + t\nabla \phi(\phi_t^{-1}(y))\}) \cdot \nabla u(y)\nabla \phi(\phi_t^{-1}(y))\right]\left[1 - t(\text{div}_x \phi)(\phi_t^{-1}(y)) + o(t)\right]
\]

\[
- \left[ f(\nabla u_t(\phi_t^{-1}(y))) + \lambda(u(y))\right][(\text{div}_x \phi)(\phi_t^{-1}(y)) + o(1)]
\]

\[
\rightarrow \int_{\Omega} \{\nabla f_\xi(\nabla u) \cdot \nabla u(y)\nabla \phi(y) - [f(\nabla u(y)) + \lambda(u)]\text{div} \phi\} dy = 0.
\]

This completes the proof of (3.3). \( \square \)

It is convenient to introduce the variational solutions of the free boundary problem

\[
\Delta_p u = 0 \text{ in } \Omega \cap \{u > 0\} \cup \{u < 0\}, \quad (3.5)
\]

\[
|\nabla u^+|^p - |\nabla u^-|^p = \frac{\lambda}{p-1} \text{ on } \Omega \cap \partial\{u > 0\}.
\]

**Definition 3.4.** Let \( f(\xi) = |\xi|^p \), then a function \( u \in W^{1,p}(\Omega) \) is said to be a variational solution of (3.5) in some domain \( \Omega \) if \( \Delta_p u = 0 \) in \( \Omega \cap \{u > 0\} \cup \{u < 0\} \) in weak sense, and for any \( \phi \in C_0^{0,1}(\Omega, \mathbb{R}^n) \)

\[
\int_{\Omega} \{\nabla f_\xi(\nabla u) \cdot \nabla u(y)\nabla \phi(y) - [f(\nabla u(y)) + \lambda(u)]\text{div} \phi\} dy = 0. \quad (3.6)
\]

**Remark 3.5.** By inspecting the proof of Theorem 8.3, one can see that the local Lipschitz estimate is valid for the variational solutions in \( B_1 \), provided that \( p > n \), the latter is a technical condition to assure uniform continuity of \( u^+ \), see the discussion in Section 2 and Remark 2.4. Also observe that every stationary point of \( J_p \) is a variational solution as the above computation shows.

4. **Viscosity solutions**

**Theorem 4.1.** Let \( u^{\varepsilon,j} \in W^{1,p}(\mathbb{R}^n) \), \( \nabla u^{\varepsilon,j} \in BMO_{\text{loc}}(\mathbb{R}^n) \) solve \( \Delta_p u^{\varepsilon,j} = \beta_j(u^{\varepsilon,j}) \) with \( \beta_j(t) = \sigma_j \frac{1}{\varepsilon_j} \beta(\frac{1}{\varepsilon_j}) \) and \( \sigma_j \downarrow 0 \). Let \( u^{\varepsilon,j} \rightarrow u \) in \( W^{1,p}(\mathbb{R}^n) \), \( \nabla u \in BMO_{\text{loc}}(\mathbb{R}^n) \) such that \( \Delta_p u = 0 \) in \( \{u > 0\} \cup \{u < 0\} \). Then \( \Delta_p u = 0 \) in \( \mathbb{R}^n \).
Proof. Let \( u(x_0) > 0 \) and let \( B_r(x_0) \subset \{ u > 0 \} \) such that \( y_0 \in \partial B_r(x_0) \) for some \( y_0 \in \partial \{ u > 0 \} \). Then \( u^- \) is bounded by the barrier \( b, \Delta \), \( b = 0 \) in \( B_{2r}(x_0) \setminus B_r(x_0) \), \( b = 0 \) on \( \partial B_r(x_0) \) and \( \max_{B_{2r}(x_0)} u^- \) on \( \partial B_{2r}(x_0) \). For any \( s \in (0, r) \) we have

\[
\int_{B_{s}(y_0)} u = \int_0^s \left[ \int_{B_{s}(y_0)} \nabla u(x) \cdot \frac{x - y_0}{|x - y_0|} \, dx \right] \, d\tau
\]

where the last equality follows from the observation that \( \int_{B_{s}(y_0)} e^{\frac{x - y_0}{|x - y_0|}} \, dx = 0 \) for any fixed vector \( e \). Consequently,

\[
\int_{B_{s}(y_0)} u^+ = \int_{B_{s}(y_0)} u^- = \int_0^s \int_{B_{s}(y_0)} \left[ \nabla u(x) - \int_{B_r(x_0)} \nabla u, \frac{(x - y_0)}{|x - y_0|} \right] \, dx \, d\tau \tag{4.1}
\]

Thus if \( u_0 \) is a blow-up \( u \) at \( y_0 \) then from [10, Lamma 4.3] (see also Lemma 7.4) either \( u_0(x) = \alpha |x_1| \), for some \( \alpha \geq 0 \) in a suitable coordinate system or \( u_0 \) is linear. Thus to show that \( u \) is a viscosity solution at \( y_0 \) it is enough to conclude that \( \alpha = 0 \).

We can use the construction in [6] of \( x_1 \) symmetric solution and assume that \( u^{\varepsilon_j} \) is \( x_1 \) symmetric. Hence, thanks to Propositions 2.8 and 2.9, from now on we assume that \( u^{\varepsilon_j} \) is a family of solutions such that \( u^{\varepsilon_j} \) that converges to \( \alpha |x_1| \) is some suitable coordinate system. We claim that \( \alpha = 0 \). To see this we observe that

\[
\partial_k(T_{1k}) = \partial_j(p|\nabla u^{\varepsilon_j}|^{p-2}\partial_k u^{\varepsilon_j} \partial_1 u^{\varepsilon_j} - \delta_{1,k} |\nabla u^{\varepsilon_j}|^p) = \partial_1 B.
\tag{4.2}
\]

Let us denote

\[
\mathcal{R}_t = \{(x_1, x') : 0 < x_1, |x'| < t\},
\]

and

\[
E_j \overset{\text{def}}{=} \int_{\mathcal{R}_t} \partial_j(p|\nabla u^{\varepsilon_j}|^{p-2}(\partial_1 u^{\varepsilon_j})^2 - |\nabla u^{\varepsilon_j}|^p) - \partial_1 B_j(u^{\varepsilon_j})
\]

\[
\overset{\text{def}}{=} -p \int_{\mathcal{R}_t} \partial_j \left[ \partial_1 u^{\varepsilon_j} \sum_{k=2}^n |\nabla u^{\varepsilon_j}|^{p-2} \partial_k u^{\varepsilon_j} \right] = -G_j. \tag{4.3}
\]

From the divergence theorem and \( \partial_1 u^{\varepsilon_j} = 0 \) on \( x_1 = 0 \), we get

\[
E_j = \int_{\{x_1 = 0\} \cap \mathcal{R}_t} (-B_j - |\nabla x' u^{\varepsilon_j}|^p)(-1) + \int_{\{x_1 = 1\} \cap \mathcal{R}_t} (p|\nabla u^{\varepsilon_j}|^{p-2}(\partial_1 u^{\varepsilon_j})^2 - B_j - |\nabla u^{\varepsilon_j}|^p)
\]

\[
\geq \int_{\{x_1 = 1\} \cap \mathcal{R}_t} (p|\nabla u^{\varepsilon_j}|^{p-2}(\partial_1 u^{\varepsilon_j})^2 - B_j - |\nabla u^{\varepsilon_j}|^p).
\]

On the other hand,

\[
-G_j = -p \int_{\{|x'| = t\}} \left[ \partial_1 u^{\varepsilon_j} \sum_{k=2}^n |\nabla u^{\varepsilon_j}|^{p-2} \partial_k u^{\varepsilon_j} \right] \nu_k \leq p \int_{\{|x'| = t\}} \left[ \partial_1 u^{\varepsilon_j} \sum_{k=2}^n |\nabla u^{\varepsilon_j}|^{p-2} |\partial_k u^{\varepsilon_j}| \right].
\]
Integrate the inequality
\[
\int_{\{x_1=1\} \cap \mathbb{R}^n} (p|\nabla u^\varepsilon|^p - B_j - |\nabla u^\varepsilon|^p)^2 \leq p \int_{\{|x'|=t\}} \left| \partial_1 u^\varepsilon \right| \sum_{k=2}^n |\nabla u^\varepsilon|^{p-2} |\partial_k u^\varepsilon| \right]
\]
over \( t \in [1 - \delta, 1] \) and using the co-area formula, we conclude that
\[
\int_{[1-\delta, 1]} dt \int_{\{x_1=1\} \cap \mathbb{R}^n} (p|\nabla u^\varepsilon|^p - B_j - |\nabla u^\varepsilon|^p)^2 \leq p \int_{\mathbb{B}_r(x_0) \setminus \mathbb{B}_{r-\delta}} \left| \partial_1 u^\varepsilon \right| \sum_{k=2}^n |\nabla u^\varepsilon|^{p-2} |\partial_k u^\varepsilon| \right].
\] (4.4)

Note that
\[
p|\nabla u^\varepsilon|^p - B_j - |\nabla u^\varepsilon|^p \rightarrow (p-1)\alpha^p,
\]
whereas
\[
\left| \partial_1 u^\varepsilon \right| \sum_{k=2}^n |\nabla u^\varepsilon|^{p-2} |\partial_k u^\varepsilon| \rightarrow 0
\]
pointwise in \( x_1 > 0 \). Hence from (4.4) it follows that
\[
(p-1)\alpha^p \leq 0,
\]
and the claim follows. Thus \( u \) is a viscosity solution of \( \Delta_p u = 0 \) in \( \mathbb{R}^n \). \( \square \)

5. **Dyadic scaling: flatness versus linear growth**

5.1. **Slab flatness**

This and next sections contain the main ingredients for the proof of the local Lipschitz estimate. The free boundary points can be characterized by the modulus of continuity \( \delta \) of the slab flatness at \( x_0 \in \partial\{u > 0\} \) and the Labasgue density \( \Theta \) of \( \{u < 0\} \). The good points for the linear growth are those where \( \delta \) and \( \Theta \) are not very small. In this section we deal with the points where \( \delta \) is not small, and we show that at such points \( u \) grows linearly.

In order to formulate the main result of this section, we introduce the notion of slab flatness for \( \partial\{u > 0\} \).

Let \( x_0 \in \partial\{u > 0\} \) and
\[
\mathcal{S}(h; x_0, \nu) := \{ x \in \mathbb{R}^n : -h < (x - x_0) \cdot \nu < h \} \quad (5.1)
\]
be the slab of height \( 2h \) in unit direction \( \nu \). Let \( h_{\min}(x_0, r, \nu) \) be the distance of two parallel planes with unit direction \( \nu \) containing the free boundary \( \partial\{u > 0\} \) in \( \mathbb{B}_r(x_0) \), that is,
\[
h_{\min}(x_0, r, \nu) := \inf\{h : \partial\{u > 0\} \cap \mathbb{B}_r(x_0) \subset \mathcal{S}(h; x_0, \nu) \cap \mathbb{B}_r(x_0)\}. \quad (5.2)
\]
Finally, let
\[
h(x_0, r) := \inf_{\nu \in \mathbb{S}^n} h_{\min}(x_0, r, \nu). \quad (5.3)
\]
Note that \( h(x_0, r) \) is **non-decreasing** in \( r \). We call \( h(x_0, r)/2 \) the slab flatness constant at scale \( r > 0 \).
5.2. Optimal growth

Proposition 5.1. Let \( u^{\epsilon,j} \) be a family of solutions to \((P_\epsilon)\), such that \( u^{\epsilon,j} \to u \) locally uniformly in \( B_1 \). Let \( x_0 \in \Gamma \cap B_{1/2} \). For any \( k \in \mathbb{N} \), set

\[
S(k, u) := \sup_{B_{2^{-k}(x_0)}} |u|.
\]

If \( h_0 > 0 \) is fixed and \( h(x_0, \frac{1}{2^n}) \geq \frac{h_0}{2^{k+1}} \) for some \( k \), then

\[
S(k + 1, u) \leq \max \left\{ \frac{L2^{-k}}{2}, \frac{S(k, u)}{2}, \ldots, \frac{S(k - m, u)}{2^{m+1}}, \ldots, \frac{S(0, u)}{2^{k+1}} \right\},
\]

for some positive constant \( L \), which is independent of \( x_0 \) and \( k \).

Proof. Suppose that the assertion of proposition is false. Then there exist integers \( k_j, j = 1, 2, \ldots \), limits \( u_j, \sup_j |u_j| \leq 1 \) (that is, \( u_j = \lim_{k \to \infty} u^{\epsilon,k(j)} \) where \( \{u^{\epsilon,k(j)}\}_{k=1}^\infty \) are solutions to \((P_\epsilon)\) for each \( j \) fixed) and points \( x_j \in \Gamma_j \cap B_1 \) such that

\[
h\left( x_j, \frac{1}{2^{k_j}} \right) \geq \frac{h_0}{2^{k_j+1}} \tag{5.5}
\]

and

\[
S(k_j + 1, u_j) > \max \left\{ \frac{j2^{-k_j}}{2}, \frac{S(k_j, u_j)}{2}, \ldots, \frac{S(k_j - m, u_j)}{2^{m+1}}, \ldots, \frac{S(0, u_j)}{2^{k_j+1}} \right\}. \tag{5.6}
\]

Therefore, from (5.6) we have that \( 1 \geq j2^{-k_j}/2 \), which implies that \( 2^{k_j} \geq j/2 \). Hence, \( k_j \) tends to \(+\infty\) when \( j \to +\infty\).

We set

\[
\sigma_j := \frac{2^{-k_j}}{S(k_j + 1, u_j)} \tag{5.7}
\]

It follows from (5.6) that

\[
\sigma_j < \frac{2}{j} \to 0 \text{ as } j \to +\infty. \tag{5.8}
\]

The basic idea of the proof is to show that the scaled functions

\[
v_j(x) := \frac{u_j(x + 2^{-k_j}x)}{S(k_j + 1, u_j)} \tag{5.9}
\]

converge to a linear function in \( \mathbb{R}^n \), which will be in contradiction with (5.5). The proof falls naturally into two parts: first establish some uniform estimates for the sequence \( \{v_j\} \) and then prove that the limit is a \( p \)-harmonic function in \( \mathbb{R}^n \).

By construction,

\[
\sup_{B_{1/2}} |v_j| = 1. \tag{5.10}
\]

Furthermore, from (5.6) we have that

\[
1 > \max \left\{ \frac{j2^{-k_j}}{2S(k_j + 1, u_j)}, \frac{1}{2} \sup_{B_1} |v_j|, \ldots, \frac{1}{2^{m+1}} \sup_{B_2} |v_j|, \ldots, \frac{1}{2^{k_j+1}} \sup_{B_{2^{k_j+1}}} |v_j| \right\}, \tag{5.9}
\]

which in turn implies that

\[
\sup_{B_{2^m}} |v_j| \leq 2^{m+1}, \text{ for any } m < 2^{k_j}. \tag{5.11}
\]
Finally, since \( u_j(x_j) = 0 \), we have that
\[
v_j(0) = 0. \tag{5.12}
\]

Next, from (5.9) we get
\[
\nabla v_j(x) = \frac{2^{-k_j}}{S(k_j + 1, u_j)} \nabla u_j(x_j + 2^{-k_j}x) = \sigma_j \nabla u_j(x_j + 2^{-k_j}x).
\]

This gives
\[
T_{lm}(\nabla u_j)(x_j + 2^{-k_j}x) = T_{lm}(\frac{1}{\sigma_j} \nabla v_j(x)) = \frac{1}{\sigma_j^p} T_{lm}(\nabla v_0(x)).
\]

Consequently, letting \( \mathcal{B}^*_j(x) := \mathcal{B}^*(x_j + 2^{-k_j}x) \) and substituting \( \nabla v_j \) into (3.2), we get the differential relation
\[
\partial_\ell(T_{lm}(\nabla v_j)) = \partial_m(\sigma_j^p \mathcal{B}^*_j(x)). \tag{5.13}
\]
Note that \( \sigma_j^p \mathcal{B}^*_j(x) \rightarrow 0 \) since \( \mathcal{B}^* \) is bounded.

Hence, from Propositions 2.3, 2.8 (with \( m_j = S(k_j + 1, u_j) \)), and 2.6, we obtain that for any \( 0 < R < 2^{k_j} \) there exists a constant \( C = C(R, p) > 0 \) independent of \( j \) such that
\[
\max\{ \| v_j \|_{C^\alpha(B_R)}, \| \nabla v_j \|_{L^p(B_R)} \} \leq C,
\]
with \( \alpha > 0 \) because \( \| \nabla v_j \|_{BMO(B_R)} \leq C \| v_j \|_{L^\infty(B_R)} \leq C 2^{m+2} \) in view of (5.11) and (1.4) (one can take \( \alpha = 1 - \frac{2}{p} \), if \( p > n \)). Therefore, by a standard compactness argument, we have that, up to a subsequence,
\[
v_j \text{ converges to some function } v \text{ as } j \rightarrow +\infty \text{ in } C^\alpha(B_R),
\]
and strongly in \( W^{1,p}(B_R) \), for any fixed \( R \).

Note that the estimate \( \| \nabla v_j \|_{BMO(B_R)} \leq C 2^{m+2} \), which holds for any fixed \( m \) satisfying \( R < 2^{m+2} \), implies that \( \| \nabla v_j \|_{L^{p'}(B_R)} \leq C(p', R') \) uniformly for any \( R' < R < 2^{m+2} \) and \( p' < \infty \). Consequently, by \( C^\alpha \) uniform continuity, the properties (5.10), (5.11), and (5.12) translate to \( v \) so that
\[
\sup_{B_{1/2}} |v| = 1, \quad \sup_{B_{2^-m}} |v| \leq 2^{m+1} \quad \text{and} \quad v(0) = 0. \tag{5.15}
\]

Thanks to Theorem 4.1, (5.13) and the strong convergence \( \nabla v_j \rightarrow \nabla v \) in \( L^p_{\text{loc}}(B_R') \) for any \( p' < \infty \), we conclude that \( \Delta_p v = 0 \) in \( \mathbb{R}^n \).

Given \( x \in \mathbb{R}^n \), take \( m \in \mathbb{N} \) such that \( 2^m \leq |x| \leq 2^{m+1} \), then utilizing the second inequality in (5.15), we get \( |v(x)| \leq \sup_{B_{2^{-m+1}}} |v| \leq C 2^{m+2} \leq 4C|x| \). Hence, from Liouville’s theorem for the \( p \)-harmonic functions in \( \mathbb{R}^n \) with linear growth, we deduce that \( v \) must be a linear function in \( \mathbb{R}^n \). After rotating the coordinate system we can take
\[
v(x) = Cx_1 \text{ for some positive constant } C. \tag{5.16}
\]
Note that \( \sup_{B_{1/2}} |v| \) in (5.15) implies that \( C \neq 0 \).

On the other hand, (5.5) implies that the following inequality holds true for the function \( v_j \):
\[
h(0, 1) \geq \frac{h_0}{2}.
\]

By the uniform convergence in (5.14), we have that for any \( \varepsilon > 0 \) there is \( j_0 \) such that
\[
|C x_1 - v_j(x)| < \varepsilon \text{ whenever } j > j_0. \text{ Since } \partial v_j > 0 \text{ is } h_0/2 \text{ thick in } B_1, \text{ it follows that there is } y_j \in \partial \{ v_j > 0 \} \cap B_1 \text{ such that } y_j = e_1 h_0/4 + t_j e', \text{ for some } t_j \in \mathbb{R}, \text{ where } e_1 \text{ is the unit direction of } x_1 \text{-axis and } e' \perp e_1. \text{ Then we have that } |C \frac{h_0}{4} - 0| = |v(y_j) - v_j(y_j)| < \varepsilon, \text{ which is a contradiction if } \varepsilon \text{ is small. This finishes the proof of } (5.4). \]
6. Density of \( \{ u < 0 \} \) versus linear growth

Define

\[
\Theta(u, x_0, r) = \frac{\text{Vol}\{ u < 0 \} \cap B_r(x_0))}{\text{Vol}(B_r)}.
\]

In this section we prove the following lemma.

**Lemma 6.1.** There is \( \delta > 0 \) such that if \( \Theta(u, x_0, r) < \delta \) for some \( B_r(x_0) \subset B_{1/2}, x_0 \in \partial\{u_0 > 0\} \), then

\[
\sup_{B_2(x_0)} |u| \leq \frac{4r}{\delta}.
\]

**Proof.** Suppose that

\[
\Theta(u, x_0, 2^{-k}) < \delta \tag{6.1}
\]

and we claim that

\[
S(k + 1) \leq \max\left\{ \frac{1}{2^{k+1}}, \frac{1}{2} S(k) \right\},
\]

where \( S(k) := \sup_{B_{2^{-k}}(x_0)} |u| \), for \( k \in \mathbb{N} \). The proof by contradiction. Suppose that (6.2) fails. Then there is a sequence of integers \( k_j \) and \( u_j \) (that is, \( u_j = \lim_{k \to \infty} u_{\varepsilon k}^{(j)} \) where \( \{u_{\varepsilon k}^{(j)}\}_k \) are solutions to \((P_\varepsilon)\) for each \( j \) fixed), with \( j = 1, 2, \ldots \), such that

\[
\Theta(u_j, x_j, 2^{-k_j}) \leq \frac{1}{j}, \quad S(k_j + 1) > \max\left\{ \frac{j}{2^{k_j + 1}}, \frac{1}{2} S(k_j) \right\}. \tag{6.3}
\]

Since \( |u_j| \leq 1 \), then (6.3) implies that \( k_j \to \infty \) as \( j \to +\infty \). Also, notice that (6.3) implies that

\[
\frac{2^{-k_j}}{S(k_j + 1)} \leq \frac{2}{j} \to 0 \quad \text{as} \quad j \to +\infty. \tag{6.4}
\]

Now, we introduce the scaled functions \( v_j(x) := \frac{u_j(x_0 + 2^{-k_j}x)}{S(k_j + 1)} \), for \( x \in B_1 \). Then, from (6.1) and (6.4), it follows that

\[
\Theta(v_j, 0, 1) \leq \frac{1}{j}, \quad v_j(0) = 0. \tag{6.5}
\]

Furthermore, (6.3) implies that

\[
\sup_{B_1} |v_j| \leq 2, \quad \text{and} \quad \sup_{B_{1/2}} |v_j| = 1. \tag{6.6}
\]

We know from Proposition 2.6 with \( m_j = S(k_j + 1) \) that \( \|v_j\|_{W^{1,p}(B_{1/2})} \) are uniformly bounded. So we can extract a converging subsequence such that \( v_j \to v_0 \) uniformly in \( B_{1/2} \) and \( \nabla v_j \to \nabla v_0 \) weakly in \( L^p(B_{1/2}) \). As in the proof of Proposition 5.1 we get that \( \Delta_p v_0 \nabla v_0 = 0 \), and consequently, \( v_0 \) is a \( p \)-harmonic function in \( \mathbb{R}^n \).

Moreover, (6.5), (6.6), and Theorem 4.1 yield

\[
\Delta_p v_0(x) = 0, \quad v_0(x) \geq 0 \text{ if } x \in B_{1/2}, \quad v_0(0) = 0, \quad \text{and} \quad \sup_{B_{1/2}} v_0 = 1,
\]

which is in contradiction with the strong minimum principle. This shows (6.2) and the proof follows. \( \square \)
7. Viscosity solutions

If \( \Theta(u, x_0, r) \) is either small or \( \partial\{u > 0\} \) is non-flat, then \( u \) has linear growth near \( x_0 \in \partial\{u > 0\} \). Thus the remaining case to be analyzed is the following:

\[
\Theta(u, x_0, r) \text{ is large and } \partial\{u > 0\} \text{ is flat near } x_0.
\]

To tackle this remaining case, we want to use the stratification argument from [10] for the viscosity solutions in order to obtain the Lipschitz continuity of \( u \). This will be done by combining the above results. To define the notion of viscosity solution we let \( \Omega^+(u) = \{u > 0\} \) and \( \Omega^-(u) = \{u < 0\} \). If the free boundary is \( C^1 \) smooth, then

\[
G(u^+, u^-) := (u_+^p) - (u_-^p) - \Lambda_0
\]

(7.1)
is called the free boundary condition, where \( u_+^p \) and \( u_-^p \) are the normal derivatives in the inward direction to \( \partial\Omega^+(u) \) and \( \partial\Omega^-(u) \), respectively. Here \( \Lambda_0 = \frac{\Lambda}{p-1} = \frac{\nu M}{p-1} \) is the Bernoulli constant with \( M = \int_0^1 \beta \).

To justify the form of the free boundary condition, we first show that for smooth free boundaries (7.1) true.

**Lemma 7.1.** Let \( u^\varepsilon \) be a family of solutions to \((\mathcal{P}_\varepsilon)\), with \( \varepsilon = \varepsilon_j \), such that \( u^\varepsilon \to u \) locally uniformly in \( B_1 \). Suppose that \( \partial\{u > 0\} \) is \( C^{1, \gamma} \), \( \gamma \in (0, 1) \) regular hypersurface and \( \Theta(u, x_0, r) > \delta \) for some \( r > 0 \). Then (7.1) holds.

**Proof.** Let \( x_0 \in \partial\{u > 0\} \), then from the boundary estimates for the \( p \)-harmonic functions, we know that \( u^\varepsilon \) are \( C^{1, \gamma} \) up to the free boundary. Let \( \nu \to 0 \) and consider \( \frac{1}{\nu} u(x_0 + \nu x) \to \alpha x^+ - \alpha x^- \) (after rotation the coordinate system), where \( \alpha, \tilde{\alpha} \) are nonnegative constants. By Proposition 2.8 there is a sequence \( \varepsilon_j \to 0 \) so that \( u^\varepsilon_j(x) \to \alpha x^+ - \alpha x^- \) uniformly on compact subsets of \( \mathbb{R}^n \) and \( \varepsilon_j/\rho \to 0 \). Moreover, \( \nabla u^\varepsilon_j \to \alpha \varepsilon_1 \chi_{\{u > 0\}} - \tilde{\alpha} \varepsilon_1 \chi_{\{u \leq 0\}} \) strongly in \( L^p \) on compact subsets of \( \mathbb{R}^n \).

Let us check that \( \mathcal{B}((u^\varepsilon_j/\rho_j)/(\varepsilon_j/\rho_j)) = \int_0^{\varepsilon_j/\varepsilon_j} \beta(t) dt \to M \chi_{\{u > 0\}} \) \( \ast \)-weakly in \( L^\infty_{\text{loc}} \). Take \( \tau > 0 \) small and fix \( R > 0 \), then

\[
\int_{B_R} \mathcal{B}((u^\varepsilon_j/\rho_j)/(\varepsilon_j/\rho_j)) = \int_{B_R \cap \{|x| < \tau\}} \mathcal{B}((u^\varepsilon_j/\rho_j)/(\varepsilon_j/\rho_j)) + \int_{B_R \cap \{|x| \geq \tau\}} \mathcal{B}((u^\varepsilon_j/\rho_j)/(\varepsilon_j/\rho_j)) = I_1 + I_2.
\]

By uniform convergence \( u^\varepsilon_j \to \alpha x_1 - \tilde{\alpha} x_1 \), we have that there exists \( j_0 \) large so that \( u^\varepsilon_j/\varepsilon_j > 1 \) in \( B_R \cap \{|x| \geq \tau\} \), thus \( I_2 = \int_{B_R \cap \{|x| \geq \tau\}} M \to M \int_{B_R \cap \{|x| \geq \tau\}} \). As for the remaining term \( I_1 \), we observe that

\[
I_1 \leq \tau R^{-n-1}.
\]

Thus first sending \( j \to \infty \) and then \( \tau \to 0 \) the desired result follows. Consequently we can apply (3.2) to \( \alpha x^+ - \tilde{\alpha} x^- \) to obtain

\[
\int_{B_1^+} (\alpha^p + pM) \text{div} X + \int_{B_1^-} (\tilde{\alpha}^p + M) \text{div} X = p \int_{B_1^+} \alpha^p X_1^1 + p \int_{B_1^-} (\tilde{\alpha}^p + M) X_1^1,
\]

where we used the notation \( B_1^{\pm} = B_1 \cap \{\pm x_1 > 0\} \). Since \( \int_{B_1^+} \text{div} X = \pm \int_{B_1 \cap \{x_1 = 0\}} X_1 \) and \( \int_{B_1^\pm} X_1^1 = \pm \int_{B_1 \cap \{x_1 = 0\}} X_1 \), we get

\[
-(\alpha^p + pM) + \tilde{\alpha}^p = -p\alpha^p + p\tilde{\alpha}^p,
\]
or equivalently \( (\alpha^p - \tilde{\alpha}^p)(p - 1) = pM \) which is (7.1). \( \square \)
**Definition 7.2.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) and let \( u \) be a continuous function in \( \Omega \). We say that \( u \) is a viscosity solution in \( \Omega \) if

(i) \( \Delta_p u = 0 \) in \( \Omega^+(u) \) and \( \Omega^-(u) \),

(ii) along the free boundary \( \Gamma, u \) satisfies the free boundary condition, in the sense that:

(a) if at \( x_0 \in \Gamma \), there exists a ball \( B \subset \Omega^+(u) \) such that \( x_0 \in \partial B \) and

\[
    u^+(x) \geq \alpha(x - x_0, \nu)^+ + o(|x - x_0|), \quad \text{for } x \in B,
\]

\[
    u^-(x) \leq \beta(x - x_0, \nu)^- + o(|x - x_0|), \quad \text{for } x \in B^c,
\]

for some \( \alpha > 0 \) and \( \beta \geq 0 \), with equality along every non-tangential domain, then the free boundary condition is satisfied

\[
    G(\alpha, \beta) = 0,
\]

(b) if at \( x_0 \in \Gamma \), there exists a ball \( B \subset \Omega^-(u) \) such that \( x_0 \in \partial B \) and

\[
    u^+(x) \leq \alpha(x - x_0, \nu)^+ + o(|x - x_0|), \quad \text{for } x \in B,
\]

\[
    u^-(x) \geq \beta(x - x_0, \nu)^- + o(|x - x_0|), \quad \text{for } x \in B^c,
\]

for some \( \alpha \geq 0 \) and \( \beta > 0 \), with equality along every non-tangential domain, then

\[
    G(\alpha, \beta) = 0.
\]

The main result of this section is the following.

**Theorem 7.3.** Let \( u^\varepsilon \) be a family of solutions to \( (P_\varepsilon) \), such that \( u^\varepsilon \to u \) locally uniformly in \( B_1 \). Then \( u \) is a viscosity solution in \( \Omega \) in the sense of Definition 7.2.

The proof of Theorem 7.3 will follow from Lemmas 7.4 and 7.7 below. For the proof of Lemma 7.4 see [10, Appendix].

**Lemma 7.4.** Let \( 0 \leq u \in W^{1,p}(\Omega) \) be a solution of \( \Delta_p u = 0 \) in \( \Omega \) and \( x_0 \in \partial \Omega \). Suppose that \( u \) continuously vanishes on \( \partial \Omega \cap B_1(x_0) \). Then

(a) if there exists a ball \( B \subset \Omega \) touching \( \partial \Omega \) at \( x_0 \), then either \( u \) grows faster than any linear function at \( x_0 \), or there exists a constant \( \alpha > 0 \) such that

\[
    u(x) \geq \alpha(x - x_0, \nu)^+ + o(|x - x_0|) \quad \text{in } B,
\]

where \( \nu \) is the unit normal to \( \partial B \) at \( x_0 \), inward to \( \Omega \). Moreover, equality holds in (7.4) in any non-tangential domain.

(b) if there exists a ball \( B \subset \Omega^c \) touching \( \partial \Omega \) at \( x_0 \), then there exists a constant \( \beta \geq 0 \) such that

\[
    u(x) \leq \beta(x - x_0, \nu)^+ + o(|x - x_0|) \quad \text{in } B^c,
\]

with equality in any non-tangential domain.

With this, we are able to prove Theorem 7.3 by utilizing the following anisotropic scaling argument.

**Theorem 7.5.** Let \( B \subset \Omega^+ \) be a touching ball to \( \Gamma \) from \( \{u > 0\} \) (respectively, \( \{u < 0\} \)), then in the asymptotic expansions (7.4) and (7.5) both \( \alpha \) and \( \beta \) are finite and uniformly bounded.
Remark 7.6. From Theorem 7.5 it follows that the limit $u$ is a viscosity solution in the sense of Definition 7.2.

We recapitulate the statement of Theorem 7.5 and amplify it by proving a more quantitative result. It can be proven in much the same way as Lemma 6.1. We give only the main ideas of the proof.

**Lemma 7.7.** Let $u^{\varepsilon_j}$ be a family of solutions to $\mathcal{P}_\varepsilon$, such that $u^{\varepsilon_j} \to u$ locally uniformly in $B_1$. Let $x_0 \in \partial\{u > 0\}$ and $r > 0$ small such that $B_r(x_0) \subset \Omega$. Assume that $\sup_{B_r(x_0)} u^- \leq C_0 r$ (respectively, $\sup_{B_r(x_0)} u^+ \leq C_0 r$) $\forall r \in (0, r_0)$, for some constant $C_0$ depending on $x_0$ and $r_0$ small.

Then there exists a constant $\sigma > 0$ such that $\sup_{B_r(x_0)} u^+ \leq (1 + \sigma C_0) r$ (respectively, $\sup_{B_r(x_0)} u^- \leq (1 + \sigma C_0) r$).

**Remark 7.8.** Lemma 7.7 implies that $u^+$ and $u^-$ have coherent growth. This implies that if $u$ is as in Theorem 7.5, then the scaled functions $\frac{u(x_0 + r x)}{r}$ converge to the half plane solution $\alpha x_1^+ - \beta x_1^-$ in an appropriate coordinate system.

**Proof.** We will show only one of the claims, the other can be proved analogously. Suppose that

$$\sup_{B_r(x_0)} u^- \leq C_0 r, \tag{7.6}$$

and we claim that

$$S(k + 1) \leq \max \left\{ \frac{1 + \sigma C_0}{2} \cdot \frac{1}{2}, \frac{1}{2} S(k) \right\}, \tag{7.7}$$

where $S(k) := \sup_{B_{2^{-k}}(x_0)} |u|$, for any $k \in \mathbb{N}$. To prove this, we argue by contradiction and we suppose that (7.7) fails. Then there is a sequence of integers $k_j$, with $j = 1, 2, \ldots$, such that

$$S(k_j + 1) > \max \left\{ \frac{j}{2(k_j + 1)}, \frac{1}{2} S(k_j) \right\}. \tag{7.8}$$

From the bound $\|u\|_\infty \leq 1$ and (7.8), it follows that $k_j \to \infty$ as $j \to +\infty$. Also, notice that (7.8) implies that

$$\sigma_j := \frac{2^{-k_j}}{S(k_j + 1)} \leq \frac{2}{j} \to 0 \quad \text{as } j \to +\infty. \tag{7.9}$$

Now, we introduce the scaled functions $v_j(x) := \frac{u(x_0 + 2^{-k_j} x)}{S(k_j + 1)}$, for $x \in B_1$. Then, from (7.6) and (7.9), it follows that

$$v_j(0) = 0 \quad \text{and} \quad v_j(x) = \frac{u^-(x_0 + 2^{-k_j} x)}{S(k_j + 1)} \leq \frac{2^{-k_j} C_0}{S(k_j + 1)} \frac{2 C_0}{j} \to 0 \quad \text{as } j \to +\infty. \tag{7.10}$$

Furthermore, it is not difficult to see that (7.8) implies that

$$\sup_{B_1} |v_j| \leq 2, \quad \text{and} \quad \sup_{B_{\frac{1}{2}}} |v_j| = 1. \tag{7.11}$$

$$\int_{B_1} \left[ |\nabla v_j|^p + \sigma_j^p p B^*(S(k_j + 1) 2^{k_j} v_j) \right] \text{div} \psi = p \int_{B_1} |\nabla v_j|^{p-2} \partial_i v_j \partial_m v_j \psi_i^m. \tag{7.12}$$

The same compactness argument as in the proof of Lemma 6.1 gives that $\|v_j\|_{W^{1,p}(B_{\frac{1}{2}})}$ are uniformly bounded. Also, it implies (with the help of Proposition 2.6) that we can extract a
converging subsequence such that \( v_j \to v_0 \) uniformly in \( B_{\frac{3}{4}} \) and \( \nabla v_j \to \nabla v_0 \) strongly in \( L^p(B_{\frac{1}{4}}) \). Moreover, (7.10), (7.11), and Theorem 4.1 give that
\[
\Delta_{\tau} v_0(x) = 0, \quad v_0(x) \geq 0 \text{ if } x \in \overline{B_{\frac{3}{4}}}, \quad v_0(0) = 0, \quad \text{and } \sup_{B_{\frac{1}{4}}} v_0 = 1,
\]
which is in contradiction with the strong minimum principle. This shows (7.7) and finishes the proof. \( \square \)

8. Lipschitz continuity of \( u \): proof of Theorem 1.1

Proposition 5.1 and Lemma 6.1 can be summarized by saying that if at \( x_0 \in \partial\{u > 0\} \) the free boundary is neither flat nor the set is \( \{u < 0\} \) is thick, then we have uniform linear growth at \( x_0 \). Thus we only have to look at those free boundary points where \( u < 0 \) is nontrivial, since in its complement we know that \( u \) is Lipschitz.

We begin by introducing another notion of flatness, suitable for the viscosity solutions, in terms of the \( \varepsilon \)-monotonicity of \( u \). More precisely, we give the following definitions.

**Definition 8.1.** We say that \( u \in C(B_1) \) is \( \varepsilon \)-monotone in \( B_{1-\varepsilon} \) if there are a unit vector \( e \) and an angle \( \theta_0 \) with \( \theta_0 > \frac{\pi}{4} \) (say) and \( \varepsilon > 0 \) (small) such that, for every \( \varepsilon' \geq \varepsilon \),
\[
\sup_{B_{\varepsilon'} \cap \sin \theta_0(\varepsilon)} u(y - \varepsilon' e) \leq u(x).
\] (8.1)

We denote by \( \Gamma(\theta_0, e) \) the cone with axis \( e \) and opening \( \theta_0 \).

**Definition 8.2.** Let \( u \) be a viscosity solution in \( B_1(x) \), with \( x \in \partial\{u > 0\} \). We say that \( u \) is \( \varepsilon \)-monotone in the cone \( \Gamma(\theta_0, e) \) if it is \( \varepsilon \)-monotone in any direction \( \tau \in \Gamma(\theta_0, e) \).

Furthermore, we say that \( u \) is \( \varepsilon \)-monotone in the cone \( \Gamma(\theta_0, e) \) in \( B_r(x) \) if the function \( U(y) = \frac{u(x+y)}{r} \), with \( y \in B_1 \), is so in the cylinder \( B'_{\varepsilon} \times (-\frac{1}{\sqrt{2}} + \varepsilon, \frac{1}{\sqrt{2}} - \varepsilon) \subset B_1 \), where \( B'_{\varepsilon} \) denotes the ball with radius \( r \) of codimension 1.

One can interpret the \( \varepsilon \)-monotonicity of \( u \) as closeness of the free boundary to a Lipschitz graph with Lipschitz constant sufficiently close to 1 if we leave the free boundary in directions \( e \) at distance \( \varepsilon \) and larger. The exact value of the Lipschitz constant is given by \( (\tan \frac{\theta_0}{2})^{-1} \). Then for suitable \( \varepsilon \) and \( \theta_0 \), which we call critical flatness constants, the ellipticity propagates to the free boundary via Harnack’s inequality giving that \( \Gamma \) is Lipschitz. Furthermore, Lipschitz free boundaries are, in fact, \( C^{1, \alpha} \) regular. Therefore we have the following theorem.

**Theorem 8.3.** Let \( x_0 \in \partial\{u > 0\} \) such that \( h(x_0, r) < rh_0 \) and \( \Theta(u, x_0, r) \geq \delta \) with \( \delta > 0 \) as in Lemma 6.1. Then there is a constant \( C = C(n, M, \delta, h_0) \) such that
\[
|u(x)| \leq C|x - x_0|, \quad x \in B_{\frac{1}{4}}(x_0).
\]

The proof is a slight modification of [10, Theorem A], since the condition \( \Theta(u, x_0, r) \geq \delta \) implies that there is a negative phase and \( u \) is a viscosity solution.

9. Behavior near free boundary

With Lipschitz continuity we can show that the results in [6] hold for the nonlinear problem \( (P_\varepsilon) \). With minor modifications the following theorem follows from the results of [6].
Theorem 9.1. Let \( u^{\varepsilon_j} \) be solutions to (\( P_\varepsilon \)) in a domain \( D \subset \mathbb{R}^n \). Let \( x_0 \in D \) and suppose that \( u^{\varepsilon_j} \) converge to \( u_0 \) uniformly on compact subsets of \( D \) as \( \varepsilon_j \to 0 \). Then the following holds.

(i) If \( u_0 = \alpha (x - x_0)^+_1 - \gamma (x - x_0)^-_1 \) with \( \alpha \geq 0, \gamma > 0 \), then
\[
\alpha^p - \gamma^p = pM.
\]

(ii) If \( u_0 = \alpha (x - x_0)^+_1 \alpha \in \mathbb{R} \), then
\[
0 \leq \alpha \leq (pM)^{\frac{1}{p}}.
\]

(iii) If \( u = \alpha (x - x_0)^+_1 + \alpha(x - x_0)^-_1 \alpha > 0, \alpha > 0 \), then
\[
\alpha = \bar{\alpha} \leq (pM)^{\frac{1}{p}}.
\]

The next two theorems exhibit the behavior of \( u \) near the free boundary.

Theorem 9.2. Let \( u^{\varepsilon_j} \) be solutions to (\( P_\varepsilon \)) in a domain \( D \subset \mathbb{R}^n \) such that \( u^{\varepsilon_j} \to u \) uniformly on compact subsets of \( D \) and \( \varepsilon_j \to 0 \). Let \( x_0 \in D \cap \partial \{ u > 0 \} \) and let \( \gamma \geq 0 \) be such that
\[
\limsup_{x \to x_0} |\nabla u^- (x)| \leq \gamma.
\]

Then,
\[
\limsup_{x \to x_0} |\nabla u^+ (x)| \leq (pM + \gamma^p)^{\frac{1}{p}}. \tag{9.1}
\]

Proof. We have divided the proof into six steps.

Step 1) Let
\[
\alpha := \limsup_{x \to x_0} |\nabla u(x)|, \quad u(x) > 0
\]

By Theorem 8.3 \( u \) is Lipschitz continuous, therefore \( \alpha \) is finite. If \( \alpha = 0 \), then we are done. Thus let us assume that \( \alpha > 0 \). There is a sequence \( x_k \in \{ u > 0 \} \) such that \( x_k \to x_0 \) and
\[
\lim_{k \to \infty} |\nabla u(x_k)| = \alpha. \quad \text{Denote } d_k = \dist(x_k, \partial \{ u > 0 \}), \text{ then we know that there is } z_k \in \partial \{ u > 0 \} \text{ such that } d_k = |x_k - z_k|.
\]

Step 2) Let
\[
u_{d_k}(x) = \frac{1}{d_k} u(z_k + d_k x).
\]
We have that \( |\nabla u_{d_k}(x)| = |\nabla u(z_k + d_k x)| \leq C \) because \( u \in C^{0,1}_{\text{loc}}(D) \) by Theorem 8.3. Consequently, \( u_{d_k}(x) \) are uniformly bounded on compact sets of \( \mathbb{R}^n \) since \( u_{d_k}(0) = 0 \). Therefore there is a subsequence (still labeled \( u_{d_k}(x) \)) such that \( u_{d_k} \to u_0 \) uniformly on the compact subsets of \( \mathbb{R}^n \) and the limit \( u_0 \) is Lipschitz continuous on the compact subsets of \( \mathbb{R}^n \).

Step 3) Consider \( \bar{x}_k = \frac{x_k - z_k}{|x_k - z_k|} \), pointing into \( \{ u_{d_k} > 0 \} \). Note that \( x_k \in \partial B_1 \) and \( B_1(\bar{x}_k) \subset \{ u_{d_k} > 0 \} \). We can extract a subsequence, still labeled \( \bar{x}_k \), such that \( \bar{x}_k \to \bar{x} \) such that \( u_0(\bar{x}) \geq 0 \) in \( B_1(\bar{x}) \) and \( \Delta_p u_0 = 0 \) in \( B_1(\bar{x}) \).

We can also extract a converging subsequence from the sequence of unit vectors
\[
n_k := \frac{\nabla u_{d_k}(\bar{x})}{|\nabla u_{d_k}(\bar{x})|}.
\]
still labeled \( \nu_k \) such that \( \nu_k \rightarrow \nu \). We claim that
\[
|\nabla u(x_k)| \rightarrow \frac{\partial u_0}{\partial \nu}(\bar{x}).
\] (9.2)
Note that \( \nabla u(x_k) = \nabla u_d(x_k) \). Hence it is enough to show check that
\[
\nabla u_d \rightarrow \nabla u_0 \quad \text{on compact subsets of } B_1(\bar{x}).
\] (9.3)
To see this we first note that \( \psi(u_d_k - u_{d_m}) \in W^{1,p}_0(B_1(\bar{x})) \) for given \( 0 \leq \psi \in C^\infty_0(B_1(\bar{x})) \) for sufficiently large \( k, m \). Therefore
\[
0 = \int (|\nabla u_d_k|^p - 2 \nabla u_d_k - |\nabla u_{d_m}|^p)\nabla u_d_k \psi + (u_{d_k} - u_{d_m}) \nabla \psi.
\]
\[
= \int (|\nabla u_d_k|^p - 2 \nabla u_d_k - |\nabla u_{d_m}|^p)\nabla (u_{d_k} - u_{d_m}) \psi
\]
\[
+ \int (|\nabla u_d_k|^p - |\nabla u_{d_m}|^p)\nabla \psi(u_{d_k} - u_{d_m})
\]
\[
\geq \gamma \int |\nabla u_d_k - \nabla u_{d_m}|^p \psi
\]
\[
- \sup_{B_2} |\nabla \psi(u_{d_k} - u_{d_m})| \int |\nabla u_{d_k}|^{p-1} + |\nabla u_{d_m}|^{p-1},
\] (9.4)
where the last inequality follows from a well-known estimate (2.1) with \( \gamma \) depending only on \( n, p \). Thus for an appropriate choice of \( \psi \equiv 0 \) we get from (9.4) that
\[
\gamma \int_B |\nabla u_d_k - \nabla u_{d_m}|^p \leq 2 \|\nabla u_{d_k}\|^{p-1}_{\infty} \text{Vol}(B_2) \sup_{2B} |u_{d_k} - u_{d_m}|
\] (9.5)
for every ball \( B \) satisfying \( 2B \in B_1(\bar{x}) \) for sufficiently large \( k, m \). Here we assume that \( 2B \) is the ball with the same center as \( B \) and of radius equal to the diameter of \( B \). On the other hand,
\[
|\nabla u_{d_k}(x) - \nabla u_{d_m}(x)| \leq \left| \nabla u_{d_k}(x) - \int_{B_r(x)} \nabla u_{d_k} \right| + \left| \int_{B_r(x)} \nabla u_{d_k} - \int_{B_r(x)} \nabla u_{d_m} \right|
\]
\[
+ \left| \int_{B_r(x)} \nabla u_{d_m} - \nabla u_{d_m}(x) \right|
\]
\[
\leq 2C \beta^2 + 2 \|\nabla u_{d_k}\|^{p-1}_{\infty} \text{Vol}(B_{2r}(x))2^n \|u_{d_k} - u_{d_m}\|_{L^\infty(B_{2r}(x))},
\]
where the last line follows from the \( \beta \)-Hölder estimate for gradient (see [9]) and (9.5). Since \( r \) is arbitrary and \( \|u_{d_k} - u_{d_m}\|_{L^\infty(B_{2r}(x))} \rightarrow 0 \), if \( k, m \) are sufficiently large, it follows that \( \nabla u_{d_k} \rightarrow \nabla u_0 \) uniformly in some uniform neighborhood of \( \bar{x} \). As result we get that
\[
\alpha \leftarrow |\nabla u(x_k)| = |\nabla u_{d_k}(\bar{x}_k)| = (\nabla u_{d_k}(\bar{x}_k), \nu_k) \rightarrow \frac{\partial u_0}{\partial \nu}(\bar{x})
\] (9.6)
and (9.2) follows.

Step 4) We claim that \( |\nabla u_0^+| \leq \alpha, |\nabla u_0^-| \leq \gamma \) in \( \mathbb{R}^n \). For every \( \tau > 0 \) there is \( \delta > 0 \) such that \( \sup_{B_{\tau}(x_0)} |\nabla u^+| < \alpha + \delta \). For fix \( R > 0, \nabla u_{d_k}(x) = \nabla u(z_k + d_k x) \) \( < \alpha + \delta \) if \( d_k \) is sufficiently small so that \( |x_0 - (z_k + d_k x)| \leq |\nabla u_{d_k}(x_0)| \rightarrow (1 + R)d_k \tau < \tau \). Thus \( \sup_{B_{\tau}} |\nabla u_{d_k}| \leq \alpha + \delta \) and hence \( \sup_{B_{\tau}} |\nabla u_0| \leq \alpha + \delta \). Since \( \delta > 0 \) is arbitrary, the claim follows. By a similar argument we can prove that \( |\nabla u^-| \leq \gamma \).

Step 5) Let \( v = \frac{\partial u_0}{\partial \nu} \). Then differentiating \( \Delta_\nu u_0 = 0 \) in \( \nu \) direction, we get that \( \text{div}(a(\nabla u_0)\nabla v) = 0 \) in \( B_1(\bar{x}) \), where \( a(\nabla u_0) \) is a matrix with \( p \)-laplacian type growth. Since by
(9.6) $\nabla u_0 \neq 0$ near $\tilde{x}$, it follows that $v$ solves a uniformly elliptic equation in $B_R(\tilde{x})$ for some $R > 0$ small. Since $v$ attains local maximum at $\tilde{x}$, then it follows that $v = \alpha$ in $B_R(\tilde{x})$. For the sake of simplicity we assume that $\nu = e_1$, thus $u = \alpha x_1 + g(x')$, $x' = (0, x_2, \ldots, x_n)$ for some function $g$. Form $|\nabla u_0| \leq \alpha$ it follows that $g$ must be constant. From the unique continuation theorem [14], it readily follows that there is a point $\tilde{x}$ such that

$$u_0(x) = \alpha(x - \tilde{x})_1^+ \quad \text{in} \quad (x - \tilde{x})_1 > 0,$$

and

$$|\nabla u_0| \leq \gamma \quad \text{in} \quad \mathbb{R}^n.$$

On the other hand, from the asymptotic expansion [10] we have that there are $\bar{\alpha}, \bar{\gamma}$ such that

$$u_0^+(x) = \bar{\alpha}(x - \tilde{x})_1^- + o(|x - \tilde{x}|), \quad \text{in} \quad (x - \tilde{x})_1 < 0,$$

$$u_0^-(x) = \bar{\gamma}(x - \tilde{x})_1^- + o(|x - \tilde{x}|), \quad \text{in} \quad (x - \tilde{x})_1 < 0.$$

Step 6) To finish the proof we blow up $u_0$ one more time. Let $u_{0\lambda}(x) = \frac{1}{\lambda} u_0(\tilde{x} + \lambda x)$. From Proposition 2.9 it follows that there is a sequence $\epsilon_j^{\bar{\alpha}}$ such that $u_j^{\bar{\alpha}}$ are solutions to $(\mathcal{P}_\epsilon)$ and $u_j^{\bar{\alpha}} \to u_{0\lambda} = \alpha x_1 + \mu x_1^-$. If $\mu = 0$, then Theorem 8 (ii) gives (9.1). If $\mu > 0$ then from Theorem 8 (iii). If $\mu > 0$, then since $\nabla u_{\lambda}$ weakly in $L^\infty_{loc}$ and $|\nabla u^-| \leq \gamma$ it follows that $|\mu| \leq \gamma$ and we can apply Theorem 8 (1).

**Theorem 9.3.** Let $u^{\epsilon_j}$ be a solution to $P\epsilon_j$ in a domain $D_j \subset \mathbb{R}^n$ such that $D_j \subset D_{j+1}$ and $\cup_j D_j = \mathbb{R}^n$. Let us assume that $u^{\epsilon_j}$ converges to a function $U$ uniformly on compact sets of $\mathbb{R}^n$ and $\epsilon_j \to 0$. Assume, in addition, that $U \in Lip(1,1)$ in $\mathbb{R}^n$ and $\partial U > 0 = \emptyset$. If $\gamma > 0$ is such that $|\nabla U^-| \leq \gamma$ in $\mathbb{R}^n$, then

$$|\nabla U^+| \leq \sqrt{2M + \gamma^2} \quad \text{in} \quad \mathbb{R}^n.$$

Proof follows from minor modifications of the previous one.

**10. Application: weak solutions**

In this section we study the set of singular points of weak solutions, a subclass of variational solutions. The main result of this section states that the weak energy identity also holds for the weak solutions; hence, if (1.4) holds, one can prove their local Lipschitz regularity, as in Theorem 1.1. We begin with the following definition of the weak solutions [1, 19].

**Definition 10.1.** A function $u$ is said to be a weak solution of our free boundary problem if the following is satisfied:

(1) $u \in W^{1,p}(\Omega)$ is continuous in $\Omega$ and $p$-harmonic in $(\{u > 0\} \cup \{u < 0\}) \cap \Omega$,

(2) for $D \Subset \Omega$, $\{u > 0\} \cap D$ is a set of finite perimeter, and

$$(\ast_1) \quad \partial_1 \{u > 0\} \text{ is open relative } \partial \{u > 0\},$$

$$(\ast_2) \quad \partial_2 \{u > 0\} \text{ is smooth},$$

$$(\ast_3) \quad \mathcal{H}^{n-1}(\partial \{u > 0\} \setminus \partial_\text{red} \{u > 0\}) = 0.$$  

$\partial_\text{red} \{u > 0\}$ is the reduced boundary of $\{u > 0\}$, see [11, 4.5.5] for definition.

(3) On $\partial_\text{red} \{u > 0\}$ we have the free boundary condition satisfied

$$(p - 1)(|\nabla u^+|^p - |\nabla u^-|^p) = \lambda^p.$$
Lemma 10.2. Let $u$ be a weak solution in the sense of Definition 10.1. Then

$$
\int (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) \text{div} \phi - p|\nabla u|^{p-2}(\nabla uD\phi) \cdot \nabla u = 0, \quad \forall \phi \in C^1_0(\Omega, \mathbb{R}^n).
$$

Proof. For a given test function $\phi \in C^0(\Omega; \mathbb{R}^n)$, the set $\text{supp} \phi \cap (\partial\{u>0\} \setminus \partial_{\text{rel}}\{u>0\})$ is compact. Consider a covering of this set by $B_r(x_i)$ such that

$$
\text{supp} \phi \cap (\partial\{u>0\} \setminus \partial_{\text{rel}}\{u>0\}) \subset \bigcup_{i=1}^\infty B_r(x_i)
$$

and $\sum_{i=1}^\infty r_i^{n-1} < \delta$. Then there is a finite subcovering $\mathscr{F} = \bigcup_{i=1}^{N(\delta)} B_r(x_i)$ such that $\text{supp} \phi \cap (\partial\{u>0\} \setminus \partial_{\text{rel}}\{u>0\}) \subset \mathscr{F}$ and $\sum_{i=1}^{N(\delta)} r_i^{n-1} < \delta$. Splitting the integral into two parts, we have to show that

$$
\int_{\text{supp} \phi} (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) \text{div} \phi - p|\nabla u|^{p-2}(\nabla uD\phi) \cdot \nabla u = \int_{\mathscr{F}} + \int_{\text{supp} \phi \setminus \mathscr{F}} = 0.
$$

By (2) Definition 10.1 $\{u>0\} \cap (\text{supp} \phi \setminus \mathscr{F})$ is of finite perimeter. By [13, Theorem 1.24 and Remark 1.27] there are sets $E_j^+ \subset \{u>0\}$ with $C^\infty$ boundaries which approximate $E = (\text{supp} \phi \setminus \mathscr{F}) \cap \{u>0\}$ from inside. After partial integration we obtain

$$
\sum_j \int_{E_j^+} |\nabla u|^p \phi_j - p \sum_{lm} \int_{E_j^+} |\nabla u|^{p-2} u_m \phi_m^l \nu_l = \int_{\partial E_j^+} |\nabla u|^p \phi \cdot \nu d\mathcal{H}^{n-1}
$$

where to get the last line we used $\Delta_p u = 0$ in $E_j^+$. Note that the integrals in above computation involving the second-order derivatives of $u$ are well defined thanks to the weighted local $W^{2,2}$ estimates for $p$-harmonic functions. Thus,

$$
\int_{E_j^+} (|\nabla u|^p + \lambda^p \chi_{\{u>0\}}) \text{div} \phi - p|\nabla u|^{p-2}(\nabla uD\phi) \cdot \nu = 0,
$$

$$
\int_{\partial E_j^+} \left\{ |\nabla u|^p \phi \cdot \nu - p \sum_m |\nabla u|^{p-2} u_m \phi_m^l \nu_l \right\} d\mathcal{H}^{n-1}.
$$

Using a similar approximation argument with $E_j^- \subset \{u<0\}$, we infer

$$
\int_{E_j^-} |\nabla u|^p \text{div} \phi - p|\nabla u|^{p-2}(\nabla uD\phi) \cdot \nu = 0,
$$

$$
\int_{\partial E_j^-} \left\{ |\nabla u|^p \phi \cdot \nu - p \sum_m |\nabla u|^{p-2} u_m \phi_m^l \nu_l \right\} d\mathcal{H}^{n-1}.
$$
Since $\partial\{u > 0\} \setminus \mathcal{F} \subset \partial_{\text{red}}\{u > 0\}$, on $\partial_{\text{red}}\{u > 0\}$ we have $u_m = -|\nabla u|\nu_m$ and the free boundary condition $(p - 1)(|\nabla u^+|^p - |\nabla u^-|^p) = \lambda^p$ is satisfied, we conclude

$$\lim_{j \to \infty} \int_{(E_j^+ \cup E_j^-) \setminus \mathcal{F}} |\nabla u|^p + \lambda^p \chi_{\{u > 0\}} \text{div}\phi - p|\nabla u|^p \nabla u D\phi \nabla u$$

$$= \int_{\partial_{\{u > 0\}} \setminus \mathcal{F}} \left(\lambda^p - (p - 1)(|\nabla u^+|^p - |\nabla u^-|^p)\right) \phi \cdot \nu d\mathcal{H}^{n-1} = 0.$$

Thus the integral over $\text{supp} \ \phi \ \setminus \mathcal{F}$ is 0. The remaining integral

$$\left|\int_{\mathcal{F}} |\nabla u|^p + \lambda^p \chi_{\{u > 0\}} \text{div}\phi - p|\nabla u|^p \nabla u D\phi \nabla u\right| \leq (p + 1)\|D\phi\|_{\infty} \int_{\mathcal{F}} |\nabla u|^p$$

tends to zero as $\delta \to 0$ since $\sum_{i=1}^{N(\delta)} r_i^n < \delta^2$ and we can utilize the absolute continuity of the integral. Sending $\delta$ to zero the result follows. \hfill \Box

11. A BMO estimate

In [10] we have proven that the gradient of a minimizer is locally BMO, provided that $p > 2$. A weaker estimate holds for the solutions of $(P_\varepsilon)$. More precisely, in this section we show that the tensor $T_{lm}(\nabla u^\varepsilon) = p|\nabla u^\varepsilon|^p u_l^\varepsilon u_m^\varepsilon - |\nabla u^\varepsilon|^p \delta_{lm}$ can be decomposed to a sum of divergence free and BMO tensors.

**Theorem 11.1.** Let $u^\varepsilon$ be a family of solutions to $(P_\varepsilon)$. Then the tensor $T_{lm}(\nabla u^\varepsilon) = p|\nabla u^\varepsilon|^p u_l^\varepsilon u_m^\varepsilon - |\nabla u^\varepsilon|^p \delta_{lm}$ admits the following decomposition:

$$T_{ij} = T_{ij}^0 + \hat{T}_{ij},$$

where $\partial_j(T_{ij}^0(\nabla u^\varepsilon)) = 0$ and $\hat{T}_{ij}(\nabla u^\varepsilon) \in BMO_{\text{loc}}(B_1)$, uniformly in $\varepsilon$, such that $\sup_{\varepsilon}\|\hat{T}\|_{BMO(\mathcal{C})} < \infty$ for every compact $\mathcal{C} \subset B_1$.

**Proof.** We recall Bogovski’s formula [12] III 3.9: for every $\omega \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp} \omega \subset B_1(0)$ and $\int_{B_1} \omega = 1$, consider the vectorfield

$$v(x) = \int_{\Omega} f(y) \frac{x - y}{|x - y|^n} \left[\int_{1}^{+\infty} \omega(y + r(x - y)) r^{n-1} dr \right] dy,$$

where $\Omega$ is a bounded domain and $f \in L^q(\Omega), q > 1$ such that $\int_{\Omega} f = 0$. Then $v \in W^{1,q}$ and

$$\|v\|_{1,q} \leq C\|f\|_q.$$

Furthermore,

$$\text{div} v = f \quad \text{in } \Omega.$$

Note that in Bogovski’s formula, $v$ has the form $v(x) = \int_{\Omega} k(x, y) f(y)$ with a singular kernel $k(x, y)$, and the derivatives of $k$ behave like Calderon–Zygmund kernels [12] III 3.15–3.17.

Let $\eta$ be a cut off function of some ball $B \Subset B_1$. Localizing $T_{ij}$ we have that $\partial_j(\eta T_{ij}) = f^i$ where

$$f^i = \eta\partial_i B^* + \eta_i T_{ij},$$

with $\int f^j = 0$. Hence from Bogovski’s formula and the estimates for the Calderon–Zygmund operators, we get that there is $\hat{T}_{ij} \in BMO(B)$ such that
where $\partial_j(T_{ij}^0) = 0$.

**Remark 11.2.** Using the techniques from [10, 17] one can show that if $u^\varepsilon$ are minimizers of $\int_{B_1} \left| \nabla u^\varepsilon \right|^p + pB(p/\varepsilon)$, then $T^0 \in BMO_{loc}$ locally uniformly for every $p > 1$. Note that $\text{Trace}T_{nm}^0 = (p-n)|\nabla u^\varepsilon|^{p}$, thus if $T^0 \in BMO_{loc}(B_1)$, then $BMO$ estimate translate to $T$ provided that $p \neq n$.

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