Towards the strong Viterbo conjecture

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Abstract

This paper is a step towards the strong Viterbo conjecture on the coincidence of all symplectic capacities on convex domains. Our main result is a proof of this conjecture in dimension 4 for the classes of convex and concave toric domains. The second result is that, in any dimension, $c_{1}^{	ext{Ekeland-Hofer}}(W) = c_{1}^{	ext{ECH}}(W) = c_{\text{Viterbo}}(W)$ for all convex domains $W \subset \mathbb{R}^{2n}$. Moreover, if $W$ is a convex or concave toric domain $W = \mathbb{R}^n \subset \mathbb{R}^{2n}$, then $c_{1}^{	ext{ECH}}(\mathbb{R}^n) = c_{\text{Gromov}}(\mathbb{R}^n)$.

1 Introduction

Given two symplectic manifolds, $(X,\omega_X)$ and $(U,\omega_U)$, a symplectic embedding is a smooth embedding $\varphi : X \hookrightarrow U$ such that $\varphi^*\omega_U = \omega_X$. An important problem in symplectic topology is to classify the symplectic embeddings between two symplectic manifolds $X$ and $U$ of the same dimension. This problem originated in the non-squeezing theorem of Gromov stating that the ball $B^{2n}(r) = \{ z \in \mathbb{R}^{2n} \mid \pi|z|^2 \leq r \}$ symplectically embeds into the cylinder $Z^{2n}(R) := B^2(R) \times \mathbb{R}^{2n-2}$ if and only if $r \leq R$. Even simple questions about embeddings are wildly open.

If there exists a symplectic embedding from $(X,\omega_X)$ into $(U,\omega_U)$, we write $(X,\omega_X) \hookrightarrow (U,\omega_U)$. If $X$ and $U$ are subsets of $\mathbb{R}^{2n}$ we endow them with the standard symplectic form and we denote the existence of a symplectic embedding by $X \hookrightarrow U$. One often proves the non-existence of symplectic embeddings using various kinds of symplectic capacities. Definitions of the latter term vary; here we define a symplectic capacity to be a function $c$ which assigns to each symplectic manifold $(X,\omega)$, possibly in some restricted class, a number $c(X,\omega) \in [0, \infty]$, satisfying the following axioms:

(Monotonicity) If $(X,\omega)$ and $(X',\omega')$ have the same dimension, and if there exists a symplectic embedding $(X,\omega) \hookrightarrow (X',\omega')$, then $c(X,\omega) \leq c(X',\omega')$.

(Conformality) If $r$ is a positive real number then $c(X,r\omega) = rc(X,\omega)$.

We say that a capacity $c$ is normalized if

- $c(B^{2n}(1)) = c(Z^{2n}(1)) = 1$.

Examples of normalized capacities include the Gromov width [9]

$$c_{\text{Gr}}(X,\omega) = \sup \{ r \mid \exists \text{ a symplectic embedding } (B^{2n}(r),\omega_{st}) \hookrightarrow (X,\omega) \},$$

the first Ekeland-Hofer capacity, $c_{1}^{\text{EH}}$ [7, 8], Viterbo capacity, $c_{\text{V}}$ [14], the first capacity from $S^1$-equivariant symplectic homology, $c_{1}^{\text{ECH}}$ [22], the first capacity from $S^1$-equivariant symplectic homology, $c_{1}^{\text{ECH}}$ [22], the first capacity from $S^1$-equivariant symplectic homology, $c_{1}^{\text{ECH}}$ [22], the first capacity from $S^1$-equivariant symplectic homology, $c_{1}^{\text{ECH}}$ [22].

An important conjecture in symplectic topology is the following statement.

Conjecture 1.1. All normalized symplectic capacities coincide on convex sets in $\mathbb{R}^{2n}$.

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Note that the statement above is a stronger version of the original conjecture posed in [23], usually known as the Viterbo conjecture.

**Conjecture 1.2** (Viterbo). If \( X \subset \mathbb{R}^{2n} \) is a convex set, then

\[
c(X) \leq (n! \text{Vol}(X))^\frac{1}{n}
\]

Conjecture 1.2 is true for the Gromov width \( c_{Gr} \), by monotonicity. Thus Conjecture 1.1 implies Conjecture 1.2. The Viterbo conjecture recently gained more attention as it was shown in [2] that the Mahler conjecture [15] is a consequence thereof. Recall that the Mahler conjecture in convex geometry states that for any \( n \)-dimensional normed space \( V \), we have

**Conjecture 1.3** (Mahler).

\[
\text{Vol}(B_V) \text{Vol}(B_{V^*}) \geq \frac{4^n}{n!}
\]

where \( B_V \) denote the unit ball of \( V \) and \( B_{V^*} \) the unit ball of the dual space \( V^* \).

The main result of this paper is a proof of Conjecture 1.1 for a large class of examples.

**Theorem 1.4.** For a 4-dimensional convex or concave toric domain \( X_\Omega \) all symplectic capacities coincide.

We remark that many but not all concave toric domains are convex sets. We also recall that there are 4-dimensional convex sets which are nontrivially symplectomorphic to convex or concave toric domains. This includes many lagrangian products and even the \( \ell^p \)-sum of two disks, see [17, 18, 16]. So this result applies to an even larger class of convex sets.

In higher dimensions, we lack a result such as [6, Theorem 1.2] on necessary and sufficient conditions on the existence of a symplectic embedding to extend our main result. Nonetheless we can prove Conjecture 1.1 for some well-known normalized symplectic capacities.

**Theorem 1.5.** For any convex domain \( X \) in \( \mathbb{R}^{2n} \), we have

\[
c_1^{EH}(X) = c_1^{CH}(X) = c_{SH}(X).
\]

Moreover, if \( X = X_\Omega \) is a convex or concave toric domain, then

\[
c_1^{EH}(X_\Omega) = c_1^{CH}(X_\Omega) = c_{SH}(X_\Omega) = c_{Gr}(X_\Omega).
\]

Note that the equality \( c_1^{EH}(X) = c_{SH}(X) \) was already known [13, 7, 11, 1] as well as the coincidence with the Hofer-Zehnder capacity [13] on all convex domains. This equality depend on the known fact that for a convex domain \( X \), \( c_1^{EH}(X) \) is the smallest period of a periodic Reeb orbit on the boundary. We could not find a complete proof of this fact and thus give one here.

**Organization of the paper**

Section 2 is devoted to the definition of the various capacities and proving (1.1). We prove the second part of the main result in Section 3, and our main result is proven in Section 4.

**2 Capacities**

In this section we prove that the first Ekeland-Hofer capacity coincides with the first equivariant capacity for convex sets. Throughout this section \( W \subset \mathbb{R}^{2n} \) will denote a compact star-shaped domain with smooth boundary.

We start by recalling the main definitions. Let \( E = H^{1/2}(S^1, \mathbb{R}^{2n}) \), i.e., if \( x \in L^2(S^1, \mathbb{R}^{2n}) \) can be written as a Fourier series as \( x = \sum_{k \in \mathbb{Z}} e^{2\pi ik \theta} x_k \) where \( x_k \in \mathbb{R}^{2n} \), then

\[
x \in E \iff \sum_{k \in \mathbb{Z}} |k| |x_k|^2 < \infty.
\]
Recall that there is an orthogonal splitting $E = E^+ \oplus E^0 \oplus E^-$ and orthogonal projections $P^\circ : E \to E^\circ$ where $\circ = +, 0, -$. The symplectic action of $x \in E$ is defined to be

$$A(x) = \frac{1}{2} \left( \|P^+ x\|_{H^1/2}^2 - \|P^- x\|_{H^1/2}^2 \right).$$

It follows from a simple calculation that if $x$ is smooth, then $A(x) = \int_x \lambda_0$, where $\lambda_0$ denotes the standard Liouville form on $\mathbb{R}^{2n}$.

Let $\mathcal{H}$ denote the set of $H \in C^\infty(\mathbb{R}^{2n})$ such that

- $H|_U \equiv 0$ for some $U \subset \mathbb{R}^{2n}$ open,
- $H(z) = c|z|^2$ for $z \gg 0$ where $c \not\in \{\pi, 2\pi, 3\pi, \ldots\}$.

For $H \in \mathcal{H}$, the action functional $A_H : H^{1/2}(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ is defined as

$$A_H(x) = A(x) - \int_0^1 H(x(t)) dt. \quad (2.1)$$

Note that the natural action of $S^1$ on itself induces an $S^1$-action on $E$. Let $\Gamma$ be the set of homeomorphisms $h : E \to E$ such that $h$ can be written as

$$h(x) = e^{\gamma_+(x)} P^+ x + P^0 x + e^{\gamma_-(x)} P^- x + K(x),$$

where $\gamma_+, \gamma_- : E \to \mathbb{R}$ are continuous, $S^1$-invariant and map bounded sets to bounded sets, and $K : E \to E$ is continuous, $S^1$-equivariant and maps bounded sets to precompact sets. Let $S^+$ denote the unit sphere in $E^+$ with respect to the $H^{1/2}$ norm. The first Ekeland-Hofer capacity is defined in [8] as

$$c_{1EH}^1(W) = \inf \{ c_{H,1} \mid H \in \mathcal{H}, W \subset \text{supp } H \},$$

where

$$c_{H,1} = \inf \{ \sup_{\gamma \in E} A_H(\xi) \mid \xi \in E \text{ is } S^1\text{-invariant}, \text{ and } \forall h \in \Gamma : h(\xi) \cap S^+ \neq \emptyset \}.$$
Proof of Theorem 2.1. We assume that $\partial W$ is smooth. By monotonicity of the capacities the result follows for all convex domains.

We will use the following commutative diagram [3, Theorem 1.2]

\[
\begin{array}{cccccc}
\SH_k(W) & \rightarrow & \SH_k^L(W) & \rightarrow & \SH_k^S(W) & \rightarrow & \SH_k^{S,L}(W) \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \delta & & \downarrow \gamma \\
\SH_{k-2}(W) & \rightarrow & \SH_{k-2}^L(W) & \rightarrow & \SH_{k-2}^S(W) & \rightarrow & \SH_{k-2}^{S,L}(W) \\
\end{array}
\]

(2.2)

By [10, Lemma 4.7], the first equivariant capacity is given by

\[
c_1^{CH}(W) = \inf \{ L \mid CH_{n+1}^L(W) \neq 0 \}. \quad (2.3)
\]

Since $W$ is convex, $\partial W$ is dynamically convex, which implies that the three elements of the lower triangle in Equation (2.2) vanish in degrees $n-1$ and $n-2$.

\[
\begin{align*}
SH_{n-2}^{S,L}(W) &= 0 = SH_{n-2}^{S}(W) \\
CH_{n-2}^L(W) &= 0 = CH_{n-1}^L(W)
\end{align*}
\]

Thus, in degree $n$, the maps $a$, $b$ and $c$ are isomorphisms. Therefore

\[
\inf \{ L \mid CH_{n+1}^L(W) \neq 0 \} = \inf \{ L \mid SH_{n+1}^{S,L}(W) \neq 0 \}. \quad (2.4)
\]

From Viterbo’s isomorphism ([22, Proposition 1.4]) we know that $SH_{n+1}^S(W) = 0$ and $SH_{n+1}^L(W) = Q$. So from the upper triangle in Equation (2.2)we obtain the following exact sequence:

\[
\begin{array}{cccccc}
0 & \rightarrow & SH_{n+1}^L(W) & \rightarrow & SH_{n+1}^{S,L}(W) & \rightarrow & SH_{n}^L(W) \\
\end{array}
\]

Now recall from [1, Main Corolary] that $c_{SH}(W) = l_{\min}(\partial W)$. If the map $SH_{n}^{S,L}(W) \rightarrow SH_{n}^{S,L}(W)$ is zero, then the map $\delta$ is surjective, in particular $SH_{n+1}^{S,L}(W) \neq 0$. So

\[
l_{\min}(\partial W) = c_{SH}(W) = \inf \{ L > \varepsilon \mid SH_{n}^{S,L}(W) \rightarrow SH_{n}^{S,L}(W) \text{ is zero} \} \geq \inf \{ L \mid SH_{n+1}^{S,L}(W) \neq 0 \}. \quad (2.5)
\]

It now follows from Theorem 2.2, (2.3), (2.4) and (2.5) that

\[
c_1^{CH}(W) = l_{\min}(\partial W) = c_{SH}(W) \geq c_1^{CH}(W) \geq l_{\min}(\partial W).
\]

Therefore

\[
c_1^{EH}(W) = c_1^{CH}(W) = c_{SH}(W).
\]
We now give a proof of Theorem 2.2.

**Proof.** Recall that there exists a differentiable function \( r : \mathbb{R}^{2n} \to \mathbb{R} \) which is \( C^\infty \) in \( \mathbb{R}^{2n} \setminus \{0\} \) satisfying \( r(cz) = c^2 r(z) \) for \( c \geq 0 \) such that
\[
W = \{ z \in \mathbb{R}^{2n} \mid r(z) \leq 1 \},
\]
\[
\partial W = \{ z \in \mathbb{R}^{2n} \mid r(z) = 1 \}.
\]
Let \( \alpha = l_{\min}(\partial W) \) and fix \( \varepsilon > 0 \). Let \( f \in C^\infty_{\geq 0}(\mathbb{R}) \) be a convex function such that \( f(r) = 0 \) for \( r \leq 1 \) and \( f(r) = (\alpha + \varepsilon)(r - 1) \) for \( r \geq 2 \). In particular,
\[
f(r) \geq (\alpha + \varepsilon)(r - 1), \quad \text{for all } r.
\]
We now choose a convex function \( H \in C^\infty(\mathbb{R}^{2n}) \) such that
\[
\begin{align*}
H(z) &= f(r(z)), \quad \text{if } r(z) \leq 2, \\
H(z) &\geq f(r(z)), \quad \text{for all } z \in \mathbb{R}^{2n}, \\
H(z) &= c|z|^2, \quad \text{if } z >> 0 \text{ for some } c \in \mathbb{R}_{>0} \setminus \pi \mathbb{Z}.
\end{align*}
\]
Let \( x_0 \in E \) be a minimizing characteristic of \( \partial W \), i.e., \( A(x_0) = \alpha \) and \( r(x_0) \equiv 1 \). So \( \dot{x}_0 = \alpha J \nabla r(x_0) \). From a simple calculation we deduce that \( x_0 \) is a critical point of the functional \( \Psi : E \to \mathbb{R} \) defined as
\[
\Psi(x) = A(x) - \alpha \int_0^1 r(x(t))\, dt.
\]
Observe that \( \Psi(cx) = c^2 \Psi(x) \) for \( c \geq 0 \). So \( sx_0 \) is a critical point of \( \Psi \) for all \( s \geq 0 \). Let \( \xi = [0, \infty) \cdot (P^+ x_0 \oplus E^0 \oplus E^-) \).

We now claim that \( \Psi(x) \leq 0 \) for all \( x \in \xi \). To prove that we let \( \xi_s = sP^+ x_0 \oplus E^0 \oplus E^- \).

We note that \( \Psi_{\xi_s} \) is a concave function. Since \( sx_0 \) is a critical point of \( \Psi_{\xi_s} \) it follows that \( \max \Psi(\xi_s) = \Psi(sx_0) = s^2 \Psi(x_0) = 0 \).

From (2.1), (2.6), (2.7) and (2.8) we obtain
\[
A_H(x) \leq \Psi(x) + \alpha + \varepsilon - \varepsilon \int_0^1 r(x(t))\, dt \leq \alpha + \varepsilon.
\]
It is clear that \( \xi \) is \( S^1 \)-invariant. Moreover it is proven in [7] that \( h(\xi) \cap S^+ \neq \emptyset \) for all \( h \in \Gamma \). So \( c_{H,1} \leq \alpha + \varepsilon \). Hence \( c^E_{FH}(W) \leq \alpha + \varepsilon \) for all \( \varepsilon > 0 \). Therefore
\[
c^E_{FH}(W) \leq \alpha.
\]

For the converse inequality, recall that it was proven in [8, Proposition 2] that \( c^E_{FH}(W) \) is the sympletic action of a closed characteristic on \( \partial W \). So
\[
c^E_{FH}(W) \geq \alpha.
\]

\[ \square \]

## 3 Capacities on toric domains

In this section, we prove that the first equivariant capacity coincides with the Gromov width on all convex and concave toric domains. A toric domain is a set of the form \( X_\Omega = \mu^{-1}(\Omega) \) where \( \Omega \) is a domain\(^1\) in \([0, \infty)^n \) and \( \mu : \mathbb{C}^n \to [0, \infty)^n \) is defined by \( \mu(z_1, \ldots, z_n) = (|z_1|^2, \ldots, |z_n|^2) \).

The toric domain \( X_\Omega \) is concave if \([0, \infty)^n \setminus \Omega \) is convex and \( X_\Omega \) is convex if \( \Omega \) is a convex domain in \( \mathbb{R}^n \) where
\[
\Omega = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid (|x_1|, \ldots, |x_n|) \in \Omega \}.
\]

\(^1\) A "domain" always denotes the closure of a bounded open subset of \( \mathbb{R}^n \) or \( \mathbb{C}^n \); in particular the domains are compact.
If \( n = 2 \), then \( X_\Omega \) is a convex toric domain if and only if
\[
\Omega = \{(x_1, x_2) \mid 0 \leq x_1 \leq A, \ 0 \leq x_2 \leq g(x_1)\}
\]
(3.1)
where
\[ g : [0, A] \to \mathbb{R}_{\geq 0} \]
is a nonincreasing concave function.

If \( n = 2 \), then \( X_\Omega \) is a concave toric domain if and only if \( \Omega \) is given by (3.1) where \( g : [0, A] \to \mathbb{R}_{\geq 0} \) is a nonincreasing convex function with \( g(A) = 0 \).

**Theorem 3.1.** For all convex and concave toric domain \( X_\Omega \), we have
\[ c_{Gr}(X_\Omega) = c_1^{CH}(X_\Omega). \]

**Proof.** The inequality \( c_{Gr}(X_\Omega) \leq c_1^{CH}(X_\Omega) \) holds in general (for any symplectic manifold) [5, Equation (8)]. Therefore we have to prove the other inequality. The concave case was done in [10, Corollary 1.16]. For the convex case, recall from [10, Theorem 1.6] that
\[ c_1(X_\Omega) = \min \left\{ \|v\|_\Omega^* \mid v = (v_1, \ldots, v_n) \in \mathbb{N}^n, \sum_{i=1}^n v_i = 1 \right\} \]
where
\[ \|v\|_\Omega^* = \max\{\langle v, w \rangle \mid w \in \Omega\}. \]
The “\( v \)” in consideration are the standard basis elements \( e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \) with only the \( i \)-th component non-zero, and equal to 1.

Let \( \Sigma \) denote the closure of \( \partial \Omega \cap [0, \infty[^n \). By convexity of \( \Omega \), we have
\[ \|e_i\|_\Omega^* = \|\Sigma \cap R(e_i)\|. \]
Thus \( \min\{\|e_i\|_\Omega^* \} \) is less than the size of the largest simplex\(^2 \) included in \( \Omega \) which is at least the Gromov width. \( \square \)

## 4 The Viterbo conjecture for 4-dimensional convex and concave toric domains

In dimension 4 there is another set of capacities which were constructed by Hutchings in [14] using embedded contact homology. Similarly to the Ekeland-Hofer and the equivariant capacities, they are nondecreasing and the first one is normalized. These capacities are sharp for many symplectic embedding problems, in particular, when embedding concave toric domains into convex toric domains, as proven by Cristofaro-Gardiner in [6]. We can now prove Theorem 1.4.

**Proof of Theorem 1.4.** Let \( c_Z(X) \) denote the cylindrical capacity which is defined by
\[ c_Z(X) = \inf\{r \mid X \hookrightarrow Z^{2n}(r)\}. \]
If \( c \) is any other normalized capacity, it follows from the axioms that
\[ X \hookrightarrow Z^{2n}(r) \Rightarrow c(X) \leq c(Z^{2n}(r)) = r. \]
So
\[ c_{Gr}(X) \leq c(X) \leq c_Z(X). \]

\(^2\) The simplex is the convex hull of \( \{0, se_1, \ldots, se_n\} \) and its size is \( s \)
Hence in order to show that all capacities coincide it is enough to prove that
\[ c_{G^r}(X) = c_Z(X). \]  
(4.1)

We now show (4.1) for all 4-dimensional convex or concave toric domain \( X_\Omega \).

First suppose that \( X_\Omega \) is a 4-dimensional convex toric domain and let \( g : [0, A] \to \mathbb{R} \) be the function appearing in (3.1). Let \( r = \min\{A, g(0)\} \). Since \( \Omega \) is convex, the line segment connecting \((A,0)\) with \((0,g(0))\) is contained in \( \Omega \). So the triangle with vertices \((0,0), (r,0)\) and \((0,r)\) is also contained in \( \Omega \). Hence \( B^4(r) \subset X_\Omega \). Moreover \( \Omega \) is contained either in \([0,r] \times [0,\infty)\) or in \([0,\infty) \times [0,r] \), whose corresponding toric domains are both symplectomorphic to \( Z^4(r) \). Hence
\[ X_\Omega \hookrightarrow Z^4(r). \]

Therefore \( c_{G^r}(X_\Omega) = c_Z(X_\Omega) = r \).

Now suppose that \( X_\Omega \) is a 4-dimensional concave toric domain. The weight sequence of \( X_\Omega \) is a sequence of positive numbers \((w_1, w_2, \ldots)\) defined as follows. Let \( w_1 \) be the largest \( r \) such that the triangle \( T(r) \) with vertices \((0,0), (r,0)\) and \((0,r)\) is contained in \( \Omega \). We can write the set \( \Omega \setminus T(r) \) as \( \Omega_1 \sqcup \Omega_2 \), where \( \Omega_1 \) does not intersect the \( y \)-axis and \( \Omega_2 \) does not intersect the \( x \)-axis. It is possible that \( \Omega_1 \) or \( \Omega_2 \) is empty. After translating the closures of \( \Omega_1 \) or \( \Omega_2 \) by \((-r,0)\) and \((0,-r)\) and multiplying them by the matrices \[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\]
respectively, we obtain two new concave toric domains \( \Omega_1 \) and \( \Omega_2 \). We then continue this process inductively which yields a (possibly infinite) set of numbers. After reordering this sequence, we obtain the so-called weights
\[ w_1 \geq w_2 \geq \ldots. \]

We can think of \( \{w_1, w_2, \ldots\} \) as a multiset, since there could be repetitions. For a more thorough explanation of this construction, see [4, 17]. Using a result from Traynor [20], we construct an embedding
\[ \bigsqcup_{i=1}^{\infty} \text{int}(B^4(w_i)) \hookrightarrow X_\Omega. \]  
(4.2)

This is explained in detail in [4]. We observe that the height of \( \Omega_1 \) and the width of \( \Omega_2 \) are less than \( w_1 \). This means that \( \Omega_1 \subset [0,\infty) \times [0, w_1] \) and that \( \Omega_2 \subset [0, w_1] \times [0,\infty) \). We can write
\[ \{w_2, w_3, w_4, \ldots\} = \{w_1, w_1, \ldots\} \cup \{w_2, w_2, \ldots\} \]
as multisets, where \((w_{i_1}, w_{i_2}, \ldots)\) and \((w_{j_1}, w_{j_2}, \ldots)\) are the weight sequences of \( \Omega_1 \) and \( \Omega_2 \), respectively. Let \( \Omega_1' \) be the image of \( \Omega_1 \) under the symplectomorphism \((z_1, z_2) \mapsto (z_2, z_1) \). Then a translate of \( \Omega_1' \) is contained in the complement of \( T(r) \cup \Omega_2 \subset [0, w_1] \times [0,\infty) \). So
\[ T(w_1) \cup \Omega_2 \cup \Omega_1 \hookrightarrow [0, w_1] \times [0,\infty], \]
for some \( R >> 0 \). It follows from (4.2) applied to \( \Omega_1 \) and \( \Omega_2 \) that
\[ \bigsqcup_{i=1}^{\infty} \text{int}(B^4(w_i)) \hookrightarrow \text{int}(B^4(w_1)) \cup \Omega_2 \cup \Omega_1 \hookrightarrow X_{[0,w_1] \times [0,R]} =: P(w_1, R). \]  
(4.3)

It was proven in [6] that if \( X_\Omega \) is a concave toric domain with weight sequence \((w_1, w_2, \ldots)\) and if \( X_{\Omega'} \) is a convex toric domain. Then
\[ \bigsqcup_{i=1}^{\infty} \text{int}(B^4(w_i)) \hookrightarrow X_{\Omega'} \Rightarrow X_\Omega \hookrightarrow X_{\Omega'}. \]  
(4.4)

It follows from (4.3) and (4.4) that \( X_\Omega \hookrightarrow P(w_1, R) \) and hence
\[ B^4(w_1) \subset X_\Omega \hookrightarrow P(w_1, R) \subset Z^4(w_1). \]

Therefore
\[ c_{G^r}(X_\Omega) = c_Z(X_\Omega) = w_1. \]
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