CONTROLLABILITY AND OBSERVABILITY OF SOME COUPLED STOCHASTIC PARABOLIC SYSTEMS

LINGYANG LIU
School of Mathematics and Statistics
Northeast Normal University
Changchun 130024, China

XU LIU
Key Laboratory of Applied Statistics of MOE
School of Mathematics and Statistics
Northeast Normal University
Changchun 130024, China

Abstract. This paper is devoted to a study of controllability and observability problems for some stochastic coupled linear parabolic systems only by one control and through an observer, respectively. In order to get a null controllability result, the Lebeau-Robbiano technique is adopted. The key point is to prove an observability inequality for certain stochastic coupled backward parabolic system by an iteration, when terminal values belong to a finite dimensional space. Different from deterministic systems, Kalman-type rank conditions for the controllability of stochastic coupled parabolic systems do not hold any more. Meanwhile, based on the Carleman estimates method, an observability inequality and unique continuation property for general stochastic linear coupled parabolic systems through an observer are derived.

1. Introduction. Let $n$ and $m$ be two positive integers. $T > 0$ and $G$ is a nonempty bounded domain in $\mathbb{R}^m$ with a smooth boundary $\Gamma$. Assume that $G_0$ and $G_1$ are two nonempty open subsets of $G$ satisfying $G_1 \subset G_0$. Denote by $\chi_{G_0}$ the characteristic function of $G_0$. Put $Q = G \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space, on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined, so that $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by $W(\cdot)$, augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$. Let $H$ be a Banach space. Denote by $L^2_F(0, T; H)$ the Banach space consisting of all $H$-valued $\mathcal{F}$-adapted processes $X(\cdot)$ satisfying $\mathbb{E}(\|X(\cdot)\|_{L^2(0, T; H)}^2) < \infty$; by $L^\infty_F(0, T; H)$ the Banach space consisting of all $H$-valued $\mathcal{F}$-adapted bounded processes; and by $L^2_F(\Omega; C([0, T]; H))$ the Banach space consisting of all $H$-valued $\mathcal{F}$-adapted continuous processes $X(\cdot)$ satisfying $\mathbb{E}(|X(\cdot)|^2_{C([0, T]; H)}) < \infty$. Also, for a real-valued matrix $P$, denote by $P^\top$ and $|P|$ its conjugate matrix and determinant, respectively. For a
linear operator $A$, whose domain is dense in a Hilbert space, denote by $A^*$ the conjugate operator of it. In the sequel, $C$ is used to denote a generic positive constant and for simplicity, all zero vectors are denoted by 0.

Consider the following stochastic coupled linear parabolic system:

$$
\begin{align*}
\left\{ \begin{array}{ll}
\frac{dY}{dt} = AY dt + B(t)Y dt + h_0 \chi_{G_0} u dt + D(t) Y dW(t) & \text{in } Q, \\
Y(0) = Y_0 & \text{in } G,
\end{array} \right. \\
\end{align*}
$$

where $u$ is the control variable, $Y = (y_1, y_2, \cdots, y_n)^\top$ is the state variable, $Y_0 = (y_0^1, \cdots, y_0^n)^\top$ is any given initial value and $h_0 = (1, 0, \cdots, 0)^\top$. Then all solution components of the system (1) will be influenced only by one control $u$. Also, $A$ is the following linear operator on $L^2(G)$:

$$
Ay = \sum_{i,j=1}^m (a_{ij}y_{x_i})x_j, \quad \forall \ y \in \mathcal{D}(A) = H^1_0(G) \cap H^2(G),
$$

Moreover, the coupling coefficients matrices $B(\cdot)$ and $D(\cdot)$ are as follows:

$$
\begin{align*}
B(t) &= \begin{pmatrix}
b_{11}(t) & b_{12}(t) & \cdots & b_{1n}(t) \\
b_{21}(t) & b_{22}(t) & \cdots & b_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1}(t) & b_{n2}(t) & \cdots & b_{nn}(t)
\end{pmatrix}, \\
D(t) &= \begin{pmatrix}
d_{11}(t) & d_{12}(t) & \cdots & d_{1n}(t) \\
d_{21}(t) & d_{22}(t) & \cdots & d_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
d_{n1}(t) & d_{n2}(t) & \cdots & d_{nn}(t)
\end{pmatrix},
\end{align*}
$$

where $b_{ij}(\cdot), d_{ij}(\cdot) \in L^\infty(0, T)$, $i, j = 1, \cdots, n$.

In order to get the null controllability of the system (1), assume that there exist a subinterval $[t_1, t_2]$ of $[0, T]$ and positive constant $d_0$, so that the following conditions hold for $t \in (t_1, t_2)$:

(H$_1$) $b_{i, i-1}(t) \geq d_0$ or $b_{i, i-1}(t) \leq -d_0$ ($i = 2, \cdots, n$);

(H$_2$) $b_{ij}(t) = 0$ ($i, j = 1, \cdots, n$; $i > j + 1$);

(H$_3$) and $d_{ij}(t) = 0$ ($i > j$).

For example, when $n = 3$, (1) is a system governed by three stochastic parabolic equations. The above assumptions (H$_1$)-(H$_3$) become that for $t \in (t_1, t_2)$,

$$
\begin{align*}
B(t) &= \begin{pmatrix}
b_{11}(t) & b_{12}(t) & b_{13}(t) \\
b_{21}(t) & b_{22}(t) & b_{23}(t) \\
0 & b_{32}(t) & b_{33}(t)
\end{pmatrix}, \\
D(t) &= \begin{pmatrix}
d_{11}(t) & d_{12}(t) & d_{13}(t) \\
0 & d_{22}(t) & d_{23}(t) \\
0 & 0 & d_{33}(t)
\end{pmatrix},
\end{align*}
$$

where $b_{23}(t), b_{32}(t) \geq d_0$ or $b_{21}(t), b_{31}(t) \leq -d_0$, for a positive constant $d_0$.

The first result of this paper is stated as follows.

**Theorem 1.1.** Suppose that the conditions (H$_1$)-(H$_3$) hold. Then for any $T > 0$, the system (1) is null controllable at the time $T$, i.e., for any $Y_0 \in (L^2(G))^n$, there is a control $u \in L^2(0; T; L^2(G))$ satisfying $\text{supp} u \subseteq G_0 \times [0, T]$, so that the
corresponding solution $Y \in (L^2_\mathbb{F}(\Omega; C([0,T]; L^2(G))))^n$ of (1) satisfies that $Y(T) = 0$ in $G$, $\mathbb{P}$-a.s. Moreover,

$$|u|_{L^2(0,T; L^2(G))} \leq C|Y_0|_{(L^2(G))^n}.$$  

Remark 1. If one of the conditions (H1)-(H3) does not hold, the null controllability result in Theorem 1.1 may be untrue. In fact, some counterexamples on (H1) and (H3) were given in [11] for the case of $n = 2$. In section 3, an example on the assumption (H2) will be presented for $n = 3$.

Based on the result of Theorem 1.1, consider the following stochastic coupled linear parabolic system:

$$\begin{cases}
    dY = AY dt + BY dt + h \chi_{G_0} u dt + D(t)Y dW(t) & \text{in } Q, \\
    Y = 0 & \text{on } \Sigma, \\
    Y(0) = Y_0 & \text{in } G,
\end{cases}
$$

where $h \in \mathbb{R}^n$ is a given vector and $B$ is constant.

Set $N = (h Bh \cdots B^{n-1}h)$. Then similar to [1], one can get the following controllability result for (3).

Corollary 1. If $\text{rank} N = n$ and $N^{-1}D(t)N$ is an upper triangular matrix on a subinterval $(t_1, t_2)$ of $(0, T)$, $\mathbb{P}$-a.s., the system (3) is null controllable.

Remark 2. Notice that in [1], a Kalman-type rank condition is given for assessing the null controllability of deterministic coupled linear parabolic systems. Especially, when $D(t) \equiv 0$ and (3) degenerates to a deterministic parabolic system, its null controllability is equivalent to $\text{rank} N = n$. However, by Example 3 in [11] and section 3 of this paper, the null controllability of stochastic coupled parabolic systems is not robust with respect to small perturbations of some coefficients. Therefore, one cannot get an equivalent Kalman rank condition for the controllability of stochastic coupled parabolic systems, different from deterministic systems.

Remark 3. The stochastic coupled linear parabolic system (1) in Theorem 1.1 is very special. In each equation of it, the elliptic operators of principal parts are the same. Also, the coefficients in principal operators depend only on the spatial variable, while those in the lower order terms are independent of it. Moreover, the system has no first order operators with respect to the spatial variable. These restrictions are technical. However, how to remove them is still an open problem and remains to be done.

Remark 4. When the first equation in the system (1) becomes

$$dY = AY dt + B(t)Y dt + h_0 \chi_{G_0} u dt + [D(t)Y + D_2(t)u] dW(t),$$

where $D_2(\cdot) \in L^\infty(0,T; \mathbb{R}^n)$, the controllability of the associated coupled system seems difficult and remains to be done. This is because we do not know how to prove the following estimate for any solution $(Z, \tilde{Z})$ of the system (7) on $Q$:

$$|Z(0)|^2_{(L^2(G))^n} \leq CE \int_Q |h_0 \chi_{G_0} Z + D_2^T(t)\tilde{Z}|^2 dx dt, \forall Z(T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; (L^2(G))^n).$$
On the other hand, consider the following general linear coupled stochastic parabolic system:

\[
\begin{cases}
  dY = \bar{A}(t)Y dt + \bar{B}(x,t)Y dt + \bar{D}(x,t)Y dW(t) & \text{in } Q, \\
  Y = 0 & \text{on } \Sigma, \\
  Y(0) = Y_0 & \text{in } G,
\end{cases}
\]

where the operator \( \bar{A}(t) \) is as follows: \( \forall \bar{Y} = (\bar{y}_1, \cdots, \bar{y}_n)^\top \in (H^1_0(G) \cap H^2(G))^n, \)

\[
\bar{A}(t)\bar{Y} = \begin{pmatrix}
  - \sum_{i,j=1}^{m} (\hat{a}^1_{ij}\bar{y}_{1,x_i})_{x_j} & 0 & \cdots & 0 \\
  0 & - \sum_{i,j=1}^{m} (\hat{a}^2_{ij}\bar{y}_{2,x_i})_{x_j} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & - \sum_{i,j=1}^{m} (\hat{a}^n_{ij}\bar{y}_{n,x_i})_{x_j}
\end{pmatrix}.
\]

Here \( \hat{a}^k_{ij} \in L^\infty_\mathbb{P}(\Omega; C^1([0,T]; W^1,\infty(G))) \) \( (i,j = 1, \cdots, m; k = 1, \cdots, n) \) satisfy the conditions:

1' \( \hat{a}^k_{ij}(x,t) = \hat{a}^k_{ji}(x,t), \quad \forall (x,t) \in \bar{Q}; \)

2' for any \( \varepsilon > 0, \) there exists a positive constant \( \delta, \) so that

\[
|\hat{a}^k_{ij}(\omega,x_1,t) - \hat{a}^k_{ij}(\omega,x_2,t)| \leq \varepsilon, \quad \forall t \in [0,T], \ x_1, x_2 \in \bar{G}: |x_1 - x_2| \leq \delta, \mathbb{P}\text{-a.s.};
\]

3' and there exists a positive constant \( \rho_1, \) so that for any \( k = 1, \cdots, n, \)

\[
\sum_{i,j=1}^{m} \hat{a}^k_{ij}(\omega,x,t)\zeta_i\zeta_j \geq \rho_1|\zeta|^2, \quad \forall (\omega,x,t,\zeta) = (\omega,x,t,\zeta_1,\cdots,\zeta_m) \in \Omega \times \bar{Q} \times \mathbb{R}^m.
\]

Also, the coefficients matrices \( \bar{B}(\cdot) \) and \( \bar{D}(\cdot) \) are as follows:

\[
\bar{B}(\cdot) = \begin{pmatrix}
  \bar{b}_{11}(\cdot) & \bar{b}_{12}(\cdot) & \cdots & \bar{b}_{1n}(\cdot) \\
  \bar{b}_{21}(\cdot) & \bar{b}_{22}(\cdot) & \cdots & \bar{b}_{2n}(\cdot) \\
  \vdots & \vdots & \ddots & \vdots \\
  \bar{b}_{n1}(\cdot) & \bar{b}_{n2}(\cdot) & \cdots & \bar{b}_{nn}(\cdot)
\end{pmatrix}, \quad \bar{D}(\cdot) = \begin{pmatrix}
  \bar{d}_{11}(\cdot) & \bar{d}_{12}(\cdot) & \cdots & \bar{d}_{1n}(\cdot) \\
  \bar{d}_{21}(\cdot) & \bar{d}_{22}(\cdot) & \cdots & \bar{d}_{2n}(\cdot) \\
  \vdots & \vdots & \ddots & \vdots \\
  \bar{d}_{n1}(\cdot) & \bar{d}_{n2}(\cdot) & \cdots & \bar{d}_{nn}(\cdot)
\end{pmatrix},
\]

where \( \bar{b}_{ij}(\cdot), \bar{d}_{ij}(\cdot) \in L^\infty_{\mathbb{P}}(0,T; L^\infty(G)) \) \( (i,j = 1, \cdots, n). \) Then by [8] and [15], for any \( Y_0 \in (L^2(G))^n, \) (4) admits a unique solution in the class of

\[
Y \in \left( L^2_{\mathbb{P}}(\Omega; C([0,T]; L^2(G))) \right)^n \cap L^2_{\mathbb{P}}(0,T; H^1_0(G)).
\]

In order to study the observability of general linear coupled stochastic parabolic system (4), assume that there exist a nonempty open subset \( G_\ast \) of \( G_0 \) and positive constant \( b_0, \) so that the following conditions hold for \( (x,t) \in G_\ast \times (0,T): \)

1' \( \bar{b}_i(x,t) \geq b_0 \) or \( \bar{b}_i(x,t) \leq -b_0 \) \( (i = 2, \cdots, n); \)

2' \( \bar{b}_i(x,t) = 0 \) \( (i,j = 1, \cdots, n, i > j + 1) \) and \( \bar{d}_i(x,t) = 0 \) \( (i > j); \)

The other main result of this paper is stated as follows.

**Theorem 1.2.** If the conditions (P1)-(P2) hold, then there exists a positive constant \( C, \) so that any solution \( Y = (y_1, \cdots, y_n)^\top \in \left( L^2_{\mathbb{P}}(\Omega; C([0,T]; L^2(G))) \right)^n \) of (4)
satisfies that for any \( Y_0 \in (L^2(G))^n \),
\[
E \int_G |Y(x,T)|^2 dx = \sum_{i=1}^n E \int_G |y_i(x,T)|^2 dx \leq C E \int_0^T \int_{G_0} y_n^2(x,t) dx dt. \tag{5}
\]

In addition, the following unique continuation property holds:

\[
\text{if } y_n = 0 \text{ in } G_0 \times (0,T), \quad \text{then } Y \equiv 0 \text{ in } Q. \tag{6}
\]

The above observability results mean that under the assumptions \((P_1)-(P_2)\), the information on a component \( y_n \) of \( Y \) in a local domain \( G_0 \) can determine the whole information of solution vector \( Y \) uniquely.

**Remark 5.** Similar to Example 1 in [11], it is easy to show that if the assumption \((P_1)\) does not hold, Theorem 1.2 may be untrue. However, the assumption \((P_2)\) seems only technical.

**Remark 6.** The result of Theorem 1.2 can be generalized to the following more general linear coupled stochastic parabolic systems with first order operators with respect to the spatial variable:

\[
dY = \tilde{A}(t)Y dt + [\tilde{B}(x,t)Y + M(x,t) \cdot \nabla Y] dt + \tilde{D}(x,t)Y dW(t),
\]

where \( M = (M_{ij})_{1 \leq i,j \leq n} \) and \( M_{ij} \in L_2^\infty(0,T; (L_\infty(G))^n) \). Similar to arguments in [7], \( M_{ij} \) are required to satisfy the same condition \((P_2)\) as \( \tilde{d}_{ij} \).

Up to now, there have been numerous works addressing controllability and observability problems of deterministic parabolic equations/systems (see, e.g., [2], [4], [5], [6], [17], [18], [19], and the references therein). However, very little is known about the controllability and observability of stochastic parabolic equations/systems. Let us recall some known results in this respect. In [3], [14] and [16], the controllability and observability for some forward and backward stochastic parabolic (single) equations were studied, respectively. In [11], the controllability of some coupled systems governed by two stochastic parabolic equations (the special case of \( n = 2 \) in Theorem 1.1) was considered. Hence, only the assumptions \((H_1)\) and \((H_3)\) in the controllability results of [11] were involved. In this paper, we generalize the null controllability result in [11] to more general case for any integer \( n \geq 2 \). The key point is to choose suitable weight functions in the proof of an observability inequality for some backward stochastic coupled parabolic systems (Proposition 1) by an iteration method. Also, further explanations on the necessity of \((H_2)\) are given in section 3 for \( n = 3 \). On the other hand, in [10], a null controllability result for some coupled systems by two backward stochastic parabolic equations by one control was obtained. By a duality technique, this is indeed a special case of Theorem 1.2 for \( n = 2 \), since the controllability result in [10] is equivalent to the observability for a coupled system of two forward stochastic parabolic equations through only one observer. Compared to the known results in [10], the coupling appears not only in drift terms, but also in diffusion terms in our paper. Also, the requirement for regularity on coefficients of principal parts may be relaxed to \( W^{1,\infty}(G) \), while the coefficients in diffusion terms are required to be in \( L^\infty(G) \), rather than \( W^{1,\infty}(G) \) in [10].

The rest of this paper is organized as follows. In section 2, an observability inequality for some coupled backward stochastic parabolic systems is derived, when terminal values are in a finite dimensional space. Section 3 is devoted to proving the controllability result in Theorem 1.1, based on the Lebeau-Robbiano technique.
Also, some examples on the assumptions are given. In section 4, the observability result in Theorem 1.2 is derived by the Carleman estimates method.

2. An observability inequality in finite dimensional spaces. This section is devoted to establishing an observability inequality for some coupled backward stochastic parabolic systems, based on the Lebeau-Robbiano inequality. To this aim, for the operator $A$ defined in (2), denote by $\{\lambda_i\}_{i=1}^{\infty}$ and $\{e_i\}_{i=1}^{\infty}$ eigenvalues and the corresponding eigenfunctions of $-A$, respectively. Also, $|a_i|_{L^2(G)} = 1$, for $i = 1, 2, \cdots$. For any positive integer $k$, set $X_k = \text{span}\{e_1, e_2, \cdots, e_k\}$ and denote by $P_k$ the orthogonal projection from $L^2(G)$ to $X_k$. By [9] and [13], one has the following Lebeau-Robbiano inequality.

**Lemma 2.1.** For any nonempty open subset $G_0$ of $G$, there exists a positive constant $C$, such that for any positive integer $k$ and complex numbers $a_i$ ($i = 1, 2, \cdots, k$), it holds that

$$\sum_{i=1}^{k} |a_i|^2 \leq Ce^{C\sqrt{k}} \int_{G_0} \left| \sum_{i=1}^{k} a_i e_i(x) \right|^2 dx.$$  

Consider the following coupled backward stochastic parabolic system:

$$\begin{cases}
dZ = -AZdt - B^\top(t)Zdt - D^\top(t)\tilde{Z}dt + \tilde{Z}dW(t) & \text{in } Q_1 = G \times (t_1, t_2), \\
Z = 0 & \text{on } \Sigma_1 = \partial G \times (t_1, t_2), \\
Z(t_2) = Z_{t_2} & \text{in } G,
\end{cases}$$

where $A, B(\cdot)$ and $D(\cdot)$ are those given in (1), $(t_1, t_2)$ is the interval in the assumptions $(H_1)$-$(H_3)$, and $Z_{t_2} \in L^2(\Omega, \mathcal{F}_{t_2}, \mathbb{P}; (X_k)^n)$. Also, define

$$\tau = 1 + 2n \sum_{i,j=1}^{n} (|b_{ij}|_{L_p^\infty(0,T)} + |d_{ij}|_{L_p^\infty(0,T)})^2.$$  

Then one has the following observability result for (7).

**Proposition 1.** Assume that $(H_1)$-$(H_3)$ hold. Then there exists a positive constant $C$, independent of $t_1$ and $t_2$, such that for any positive integer $k$ and $Z_{t_2} \in L^2(\Omega, \mathcal{F}_{t_2}, \mathbb{P}; (X_k)^n)$, the corresponding solution $(Z, \tilde{Z}) = (z_1, \cdots, z_n, \tilde{z}_1, \cdots, \tilde{z}_n)^\top$ of (7) satisfies

$$\mathbb{E}[Z(t_1)^2_{(L^2(G))}^n] \leq Ce^{C\sqrt{k}} \int_{0}^{t_2} \int_{G_0} z_i^2(x, t) dx dt.$$  

**Proof.** The whole proof is divided into five parts.

**Step 1.** First, for any $Z_{t_2} = (z_{1,t_2}, \cdots, z_{n,t_2})^\top \in L^2(\Omega, \mathcal{F}_{t_2}, \mathbb{P}; (X_k)^n)$, set $z_{i,t_2} = \sum_{j=1}^{k} z_{i,j,t_2} e_j(x)$ ($i = 1, \cdots, n$), where $z_{i,j,t_2}$ are $\mathcal{F}_{t_2}$-measurable random variables. Then the corresponding solution $(Z, \tilde{Z}) = (z_1, \cdots, z_n, \tilde{z}_1, \cdots, \tilde{z}_n)^\top$ of the system (7) can be represented as follows:

$$z_i(x, t) = \sum_{j=1}^{k} z_{i,j}(t) e_j(x) \quad \text{and} \quad \tilde{z}_i(x, t) = \sum_{j=1}^{k} \tilde{z}_{i,j}(t) e_j(x), \quad \text{for } i = 1, \cdots, n,$$

where $z_{i,j}(\cdot) \in L^2(\Omega; C([t_1, t_2]))$ and $\tilde{z}_{i,j}(\cdot) \in L^2([t_1, t_2])$. For any $j = 1, \cdots, k$, write

$$Z_j(t) = (z_{j,1}(t), z_{j,2}(t), \cdots, z_{j,k}(t))^\top \quad \text{and} \quad \tilde{Z}_j(t) = (\tilde{z}_{j,1}(t), \tilde{z}_{j,2}(t), \cdots, \tilde{z}_{j,k}(t))^\top.$$
Then
\[
\begin{cases}
  dZ_j = [\lambda_j Z_j - B^T(t) Z_j - D^T(t) \dot{Z}_j] dt + \dot{Z}_j dW(t) \text{ in } (t_1, t_2), \\
  Z_j(t_2) = (z_{j,1}^{i_1}, z_{j,2}^{i_2}, \ldots, z_{j,i_2}^{i_2})^T.
\end{cases}
\]

For the first component \(z_1\) of \((Z, \dot{Z})\), by Lemma 2.1, it holds that
\[
\int_G z_1^2(x_1,t) dx \leq C e^{C \sqrt{\lambda_k}} \int_G z_1^2(x_1,t) dx, \text{ a.e. } (t, \omega) \in (t_1, t_2) \times \Omega. \tag{9}
\]

**Step 2.** We give a weighted energy estimate on the solution component \(z_n\). First, for the component \(z_i\) \((i = 1, \ldots, n)\) of \(Z\), by Itô's formula, \(d(e^{\tau z_i^2}) = \tau e^{\tau z_i^2} dt + 2e^{\tau z_i} dZ_i + e^{\tau}(dZ_i)^2\). For any \(t \in (t_1, t_2)\), by (7), it follows that
\[
e^{\tau t} E \int_G z_i^2(x, t) dx - e^{\tau t_1} E \int_G z_i^2(x, t_1) dx
\]
\[
= E \int_{t_1}^t e^{\tau s} \left\{ \sum_{i,j=1}^n a_{i,j} z_i z_j - b_i z_i^2 - d_i \dot{z}_i \right\} dx ds
\]
\[
= 2E \int_{t_1}^t e^{\tau s} \left\{ \sum_{i,j=1}^n \lambda_i z_i^2 \right\} dx ds + E \int_{t_1}^t \int_G e^{\tau s} \left[ 2z_i b_i z_i + \tau z_i^2 \right] dx ds
\]
\[
\geq 2\lambda_i E \int_{t_1}^t e^{\tau s} \sum_{i=1}^n \lambda_i z_i^2 dx ds + \tau E \int_{t_1}^t \int_G e^{\tau s} \sum_{i=1}^n z_i^2 dx ds
\]
\[
\geq E \int_{t_1}^t \int_G e^{\tau s} \left( \sum_{i,j=1}^n |b_{ij}| z_i^2 + \sum_{i=1}^n |b_{ij}| z_i^2 + n \sum_{j=1}^n |d_{ij}|^2 z_i^2 + \frac{1}{n} \sum_{j=1}^n \sum_{j=1}^n \sum_{i=1}^n |d_{ij}|^2 z_i^2 \right) dx ds.
\]

Summing the above inequalities with respect to \(i\) from 1 to \(n\), by the definition of \(\tau\), one gets that
\[
\sum_{i=1}^n e^{\tau t} E \int_G z_i^2(x, t) dx - \sum_{i=1}^n e^{\tau t_1} E \int_G z_i^2(x, t_1) dx
\]
\[
\geq 2\lambda_i E \int_{t_1}^t e^{\tau s} \sum_{i=1}^n z_i^2 dx ds + \tau E \int_{t_1}^t \int_G e^{\tau s} \sum_{i=1}^n z_i^2 dx ds
\]
\[
- E \int_{t_1}^t \int_G e^{\tau s} \left( \sum_{i,j=1}^n |b_{ij}| z_i^2 + \sum_{i=1}^n |b_{ij}| z_i^2 + n \sum_{j=1}^n |d_{ij}|^2 z_i^2 + \frac{1}{n} \sum_{j=1}^n \sum_{j=1}^n \sum_{i=1}^n |d_{ij}|^2 z_i^2 \right) dx ds \geq 0.
\]

This implies that
\[
\sum_{i=1}^n E \int_G z_i^2(x_1, t) dx \leq e^{\tau(t_2 - t_1)} \sum_{i=1}^n E \int_G z_i^2(x_1, t) dx, \quad \forall t \in [t_1, t_2]. \tag{10}
\]

Next, without loss of generality, assume that \(T < 1\), and \(b_{i,i-1}(t) \geq d_0\) \((i = 2, \ldots, n)\) in the assumption (H1). Also, introduce a function \(\xi(t) = (t-t_1)^{n(t_2-t)}\) on \((t_1, t_2)\). Then, it is easy to find that \(0 \leq \xi(t) \leq 1\) in \((t_1, t_2)\),
\[
\xi(t_1) = \xi(t_2) = \cdots = \xi^{(n-1)}(t_1) = \xi^{(n-1)}(t_2) = 0 \quad \text{and} \quad \left| \frac{\xi_t}{\xi} \right| \leq 2n.
\]
By Itô’s formula, and the \((n-1)\)-th and \(n\)-th equations in (7) \((n \geq 2)\), it holds
\[
\begin{align*}
d(\xi^{-1}z_{n-1}z_n) &= (n-1)\xi^{-2}\xi t z_{n-1}z_n dt + \xi^{-1}z_{n-1}dz_n + \xi^{-1}s_{n-1}d\xi z_{n-1}d\xi_n \\
&= (n-1)\xi^{-2}\xi t z_{n-1}z_n dt \\
&+ \xi^{-1}z_{n-1} \left[ -\sum_{\nu', j'=1} (a_{\nu', j'}z_{n-1, x_{\nu'}})_{x_{\nu'}} - b_{1n}z_1 - b_{2n}z_2 - \cdots - b_{n-1}z_{n-1} \\
&- b_{nn}z_n - d_{1n}z_1 - d_{2n}z_2 - \cdots - d_{nn}z_n \right] dt + \xi^{-1}z_{n-1}z_n dW(t) \\
&+ \xi^{-1}z_n \left[ -\sum_{\nu', j'=1} (a_{\nu', j'}z_{n-1, x_{\nu'}})_{x_{\nu'}} - b_{1n}z_1 - \cdots - b_{n-1}z_{n-1} \\
&- b_{nn}z_n - d_{1n}z_1 - \cdots - d_{n-1}z_{n-1} - d_{nn}z_n \right] dt \\
&+ \xi^{-1}z_{n-1}z_n dW(t) + \xi^{-1}d_{n-1}z_n dt.
\end{align*}
\]
By the assumptions (H₂) and (H₃), the above equality implies that
\[
\begin{align*}
\mathbb{E} \int_{Q_t} b_{n-1}^2 \xi^{-1}z_n^2 d\xi \int_{Q_t} E \left[ \xi^{-1}z_{n-1}z_n dz_n \right] dt \\
&= (n-1)\mathbb{E} \int_{Q_t} \xi^{-2}\xi t z_{n-1}z_n dz_n dt \\
&- \mathbb{E} \int_{Q_t} \left[ \xi^{-1}z_{n-1} \sum_{\nu', j'=1} (a_{\nu', j'}z_{n-1, x_{\nu'}})_{x_{\nu'}} \right] dz_n dt \\
&+ \mathbb{E} \int_{Q_t} \xi^{-1}z_{n-1} \left( -b_{1n}z_1 - \cdots - b_{nn}z_n - d_{1n}z_1 - \cdots - d_{nn}z_n \right) dz_n dt \\
&+ \mathbb{E} \int_{Q_t} \xi^{-1}z_n \left( -b_{1n}z_1 - \cdots - b_{n-1}z_{n-1} - d_{1n}z_1 - \cdots - d_{nn}z_n \right) dz_n dt \\
&- \mathbb{E} \int_{Q_t} \left[ \xi^{-1}z_{n-1} \sum_{\nu', j'=1} (a_{\nu', j'}z_{n-1, x_{\nu'}})_{x_{\nu'}} \right] dz_n dt.
\end{align*}
\]
Also, by Hölder’s inequality, we find that for any \(\varepsilon > 0\),
\[
\begin{align*}
\mathbb{E} \int_{Q_t} \xi^{-2}\xi t z_{n-1}z_n dx dt &= \mathbb{E} \int_{Q_t} \xi^{-1} \frac{1}{\xi} z_{n-1}z_n dx dt \\
&\leq \varepsilon \mathbb{E} \int_{Q_t} \xi^{-1}z_n^2 dx dt + C \mathbb{E} \int_{Q_t} \xi^{-2}z_n^2 dx dt;
\end{align*}
\]
\[
\begin{align*}
\mathbb{E} \int_{Q_t} \xi^{-1}z_{n-1}z_n dx dt &\leq \varepsilon \mathbb{E} \int_{Q_t} \xi^{2n-1}z_n^2 dx dt + C \mathbb{E} \int_{Q_t} \xi^{2n-2}z_n^2 dx dt;
\end{align*}
\]
\[
\begin{align*}
\mathbb{E} \int_{Q_t} \xi^{-1}z_{n-1}z_n dx dt &\leq C \mathbb{E} \int_{Q_t} \xi^{-2}z_n^2 dx dt + \varepsilon \mathbb{E} \int_{Q_t} \xi^{2n-1}z_n^2 dx dt;
\end{align*}
\]
and
\[
-\mathbb{E} \int_{Q_t} \left[ \xi^{-1}z_{n-1} \sum_{\nu', j'=1} (a_{\nu', j'}z_{n-1, x_{\nu'}})_{x_{\nu'}} \right] dz_n dt + \mathbb{E} \int_{Q_t} \xi^{-1}z_n \left[ \sum_{\nu', j'=1} (a_{\nu', j'}z_{n-1, x_{\nu'}})_{x_{\nu'}} \right] dz_n dt.
\]
By Hölder’s inequality again, for any $H$, hence, by the assumption (H), together with the above inequalities, indicates

$$\mathbb{E} \int_{Q_1} b_n z_n^2 \, dx \, dt$$

$$\leq \varepsilon \mathbb{E} \int_{Q_1} \xi^{n-1} z_n^2 \, dx \, dt + \varepsilon \mathbb{E} \int_{Q_1} \xi^{2n-3} \zeta_n^2 \, dx \, dt$$

$$+ C \sum_{j=1}^{n-1} \mathbb{E} \int_{Q_1} (\xi^{j-1} z_j^2 + \xi^{2j-2} z_j^2) \, dx \, dt$$

$$+ \varepsilon \mathbb{E} \int_{t_1}^{t_2} \xi^{2n-3} \sum_{j=1}^{k} \lambda_j (z_j^*)^2 (t) \, dt + C \mathbb{E} \int_{t_1}^{t_2} \xi^{2n-3} \sum_{j=1}^{k} \lambda_j (z_j^{n-1})^2 (t) \, dt.$$  \tag{12}

Hence, by the assumption $(H_1)$, it follows that

$$\mathbb{E} \int_{Q_1} \xi^{n-1} z_n^2 \, dx \, dt$$

$$\leq \varepsilon \mathbb{E} \int_{Q_1} \xi^{2n-3} \zeta_n^2 \, dx \, dt + C \sum_{j=1}^{n-1} \mathbb{E} \int_{Q_1} (\xi^{j-1} z_j^2 + \xi^{2j-2} z_j^2) \, dx \, dt$$

$$+ \varepsilon \mathbb{E} \int_{t_1}^{t_2} \xi^{2n-3} \sum_{j=1}^{k} \lambda_j (z_j^*)^2 (t) \, dt + C \mathbb{E} \int_{t_1}^{t_2} \xi^{2n-3} \sum_{j=1}^{k} \lambda_j (z_j^{n-1})^2 (t) \, dt.$$ \tag{13}

Further, we give an estimate on $\mathbb{E} \int_{Q_1} \xi^{2n-3} \zeta_n^2 \, dx \, dt$. By Itô’s formula,

$$d(\xi^{2n-3} \zeta_n^2) = 2n-1 \xi^{2n-3} \xi_t z_n^2 \, dt + 2 \xi^{2n-3} \xi_t z_n \xi_t dW(t) + \xi^{2n-3} (dz_n)^2$$

$$= 2n-1 \xi^{2n-3} \xi_t z_n^2 \, dt + 2 \xi^{2n-3} \xi_t z_n \left[ - \sum_{i,j=1}^{m} (a_{ij} z_{n,x_i} x_j) + b_n z_1 + \cdots - b_n z_n \right] \, dt + 2 \xi^{2n-3} \xi_t z_n dW(t) + \xi^{2n-3} (dz_n)^2.$$  \tag{14}

By Hölder’s inequality again, for any $\varepsilon > 0$, it holds that

$$\mathbb{E} \int_{Q_1} \xi^{2n-3} \zeta_n^2 \, dx \, dt$$

$$= - \frac{2n-1}{2} \mathbb{E} \int_{Q_1} \xi^{n-\frac{3}{2}} \frac{\xi_t}{\xi} \zeta_n^2 \, dx \, dt - 2 \mathbb{E} \int_{t_1}^{t_2} \xi^{2n-3} \sum_{j=1}^{k} \lambda_j (z_j^*)^2 (t) \, dt$$

$$+ 2 \mathbb{E} \int_{Q_1} \xi^{2n-3} \xi_t (b_1 z_1 + \cdots + b_n z_n) \, dx \, dt.$$  \tag{15}
Hence,
\[
E \int_{Q_1} \xi^{2n-1} z_n^2 dx dt \\
\leq C \sum_{j=1}^{n} E \int_{Q_1} \xi^{j-1} z_j^2 dx dt + C \sum_{j=1}^{n-1} E \int_{Q_1} \xi^{j-1} z_j^2 dx dt + \varepsilon E \int_{Q_1} \xi^{2n-1} z_n^2 dx dt \\
-2E \int_{t_1}^{t_2} \xi^{2n-1} \sum_{j=1}^{k} \lambda_j [z_j^2(t)]^2 dt.
\]
This implies that
\[
E \int_{Q_1} \xi^{2n-1} z_n^2 dx dt + 2E \int_{t_1}^{t_2} \xi^{2n-1} \sum_{j=1}^{k} \lambda_j [z_j^2(t)]^2 dt \\
\leq C \sum_{j=1}^{n} E \int_{Q_1} \xi^{j-1} z_j^2 dx dt + C \sum_{j=1}^{n-1} E \int_{Q_1} \xi^{j-1} z_j^2 dx dt. \tag{14}
\]
Combining (13) with (14), we obtain that
\[
E \int_{Q_1} \xi^{n-1} z_n^2 dx dt \\
\leq C E \int_{Q_1} \sum_{j=1}^{n-1} (\xi^{j-1} z_j^2 + \xi^{2n-1} z_j^2) dx dt + C E \int_{t_1}^{t_2} \xi^{2n-1} \sum_{j=1}^{k} \lambda_j (z_j^{n-1})^2 dt. \tag{15}
\]
Step 3. We give an estimate on \( E \int_{Q_1} \xi^{n-2} z_{n-1}^2 dx dt \ (n \geq 3) \). By Itô’s formula, and the \((n - 2)\)-th and \((n - 1)\)-th equations of (7), it follows that
\[
d(\xi^{n-2} z_{n-2} z_{n-1}) \\
= (n - 2)\xi^{n-3} \xi_t z_{n-2} \tilde{z}_{n-1} dt + \xi^{n-2} z_{n-2} d z_{n-1} + \xi^{n-2} z_{n-2} d z_{n-2} \\
+ \xi^{n-2} d z_{n-1} d z_{n-2} \\
= (n - 2)\xi^{n-3} \xi_t z_{n-2} \tilde{z}_{n-1} dt + \xi^{n-2} z_{n-2} \left[ - \sum_{i,j=1}^{m} (a_{ij} z_{n-1,x_i}) x_j \\
- b_{1,n-1} \tilde{z}_1 - \cdots - b_{n,n-1} z_n - d_{1,n-1} \tilde{z}_1 - \cdots - d_{n,n-1} \tilde{z}_n \right] dt \\
+ \xi^{n-2} z_{n-2} \tilde{z}_{n-1} dW(t) \\
+ \xi^{n-2} z_{n-1} \left[ - \sum_{i,j=1}^{m} (a_{ij} z_{n-2,x_i}) x_j \\
- b_{n-1,n-2} z_{n-1} - b_{n,n-2} z_n - d_{1,n-2} \tilde{z}_1 - \cdots - d_{n,n-2} \tilde{z}_n \right] dt \\
+ \xi^{n-2} z_{n-1} \tilde{z}_{n-2} dW(t) + \xi^{n-2} d z_{n-2} d z_{n-1}.
\]
By the assumptions (H2)-(H3), similar to (12), it is easy to check that for any \( \varepsilon > 0 \),
follows that

This, together with the assumption (H1) and (15), shows that

Next, we give an estimate on $E \int_{Q_1} \xi^{n-2} \frac{n-2}{z_{n-1}^2} dx dt$. By Itô’s formula again, it follows that

Then it is easy to show that for any $\varepsilon > 0$,
Step 4. Combining (16) with (17), we obtain that

\[
\begin{align*}
\mathbb{E} & \leq \int_{Q_1} (\xi^{n-1} z_n^2 + \xi^{2n-3} z_{n-1}^2) dxdt \\
& \quad + b_n n^{-1} z_n + d_1 n^{-1} z_1 + \cdots + d_{n-1} n^{-1} z_{n-1}) dxdt \\
& \leq \varepsilon \int_{Q_1} (\xi^{n-1} z_n^2 + \xi^{2n-3} z_{n-1}^2) dxdt \\
& \quad + C \sum_{j=1}^{n-1} \mathbb{E} \int_{Q_1} \xi^{j-1} z_j^2 dxdt + C \sum_{j=1}^{n-2} \mathbb{E} \int_{Q_1} \xi^{j-1} \tilde{z}_j^2 dxdt.
\end{align*}
\]

This implies that

\[
\begin{align*}
\mathbb{E} \int_{Q_1} \xi^{n-2} \tilde{z}_n^2 dxdt & + 2\mathbb{E} \int_{t_1}^{t_2} \xi^{2n-3} \sum_{j=1}^{k} \lambda_j [\xi^{n-1}(t)]^2 dt \\
& \leq \varepsilon \int_{Q_1} \xi^{n-1} z_n^2 dxdt + C \sum_{j=1}^{n-1} \mathbb{E} \int_{Q_1} \xi^{j-1} z_j^2 dxdt \\
& \quad + C \sum_{j=1}^{n-2} \mathbb{E} \int_{Q_1} \xi^{j-1} \tilde{z}_j^2 dxdt.
\end{align*}
\]

Combining (16) with (17), we obtain that

\[
\begin{align*}
\mathbb{E} \int_{Q_1} \xi^{n-2} \tilde{z}_n^2 dxdt & - \int_{t_1}^{t_2} \xi^{2n-3} \sum_{j=1}^{k} \lambda_j [\xi^{n-1}(t)]^2 dt \\
& \leq \varepsilon \int_{Q_1} \xi^{n-1} z_n^2 dxdt + C \sum_{j=1}^{n-1} \mathbb{E} \int_{Q_1} \xi^{j-1} z_j^2 dxdt \\
& \quad + C \sum_{j=1}^{n-2} \mathbb{E} \int_{Q_1} \xi^{j-1} \tilde{z}_j^2 dxdt.
\end{align*}
\]

(15), together with (17) and (18), indicates

\[
\begin{align*}
\mathbb{E} \int_{Q_1} \xi^{n-1} z_n^2 dxdt & \leq C \int_{t_1}^{t_2} \xi^{2n-3} \sum_{j=1}^{k} \lambda_j [\xi^{n-2}(t)]^2 dt + C \sum_{j=1}^{n-2} \mathbb{E} \int_{Q_1} \left(\xi^{j-1} z_j^2 + \xi^{2j-1} \tilde{z}_j^2\right) dxdt.
\end{align*}
\]

Therefore,

\[
\begin{align*}
\mathbb{E} \int_{Q_1} \xi^{n-2} \tilde{z}_n^2 dxdt & + \mathbb{E} \int_{Q_1} \xi^{n-2} \tilde{z}_{n-1}^2 dxdt \\
& \leq C \int_{t_1}^{t_2} \xi^{2n-3} \sum_{j=1}^{k} \lambda_j [\xi^{n-2}(t)]^2 dt + C \sum_{j=1}^{n-2} \mathbb{E} \int_{Q_1} \left(\xi^{j-1} z_j^2 + \xi^{2j-1} \tilde{z}_j^2\right) dxdt.
\end{align*}
\]

Step 4. We will use an inductive method to prove that for any \( k' = 2, \cdots, n - 1, \)

\[
\begin{align*}
\mathbb{E} \int_{Q_1} \sum_{i=1}^{k'} \xi^{n-i} z_{n-i+1} dxdt & \leq C \int_{Q_1} \sum_{j=1}^{n-k'} \left(\xi^{j-1} z_j^2 + \xi^{2j-1} \tilde{z}_j^2\right) dxdt
\end{align*}
\]
Hence,

$$\sum_{j=1}^{k} \lambda_j [z_j^{n-k'}(t)]^2 dt.$$ 

By (19), (20) holds for $k' = 2$. Now, assume that (20) holds for $k' = 2, \cdots, \ell$ ($\ell < n - 1$). Then,

$$\mathbb{E} \int_{t_1}^{t_2} \xi^{\frac{2(n-k')-1}{2}} \sum_{j=1}^{k} \lambda_j [z_j^{n-k'}(t)]^2 dt.$$ 

By (19), (20) holds for $k' = 2, \cdots, \ell$ ($\ell < n - 1$). Then,

$$\mathbb{E} \int_{Q_1}^{t_1} \sum_{i=1}^{\ell} \xi^{c-i} \int_{n-i+1}^{d} dx dt$$

$$\leq \mathbb{E} \int_{Q_1}^{t_1} \sum_{i=1}^{\ell} \xi^{c-i} \sum_{n-i+1}^{d} dx dt + \mathbb{E} \int_{Q_1}^{t_1} \xi^{c-i} \sum_{n-i+1}^{d} dx dt$$

By (19), (20) holds for $k' = 2$, Now, assume that (20) holds for $k' = 2, \cdots, \ell$ ($\ell < n - 1$). Then,

$$\mathbb{E} \int_{Q_1}^{t_1} \sum_{i=1}^{\ell} \xi^{c-i} \sum_{n-i+1}^{d} dx dt$$

By (19), (20) holds for $k' = 2$, Now, assume that (20) holds for $k' = 2, \cdots, \ell$ ($\ell < n - 1$). Then,
\[ E \int_{Q_1} 2(n - \ell - 1) - 1 \xi^{2(n-\ell-1)} \xi_t z_{n-\ell}^2 dxdt + E \int_{Q_1} 2\xi^{2(n-\ell-1)} z_{n-\ell} \left( b_t n - \ell z_1 + \cdots + b_{n-\ell} n - \ell z_{n-\ell} + b_{n-\ell+1} n - \ell z_{n-\ell+1} + d_{n-\ell} n - \ell z_{n-\ell} \right) dxdt. \]

Hence, it is easy to show that for any \( \varepsilon > 0 \),

\[
E \int_{Q_1} \xi^{2(n-\ell-1)} \bar{z}_{n-\ell}^2 dxdt + 2E \int_{t_1}^{t_2} \xi^{2(n-\ell-1)} \sum_{j=1}^k \lambda_j [z_{n-\ell}^{(\ell)}(t)]^2 dt \
\leq C E \int_{Q_1} \sum_{j=1}^{n-\ell-1} (\xi^{j-1} z_j^2 + \xi^{2j-1} \tilde{z}_j^2) dxdt \quad (22)
\]

Using Itô’s formula again for the \( (n - \ell) \)-th and \( (n - \ell - 1) \)-th equations of (7), we obtain that

\[
d(\xi^{n-\ell-1} z_{n-\ell-1} z_{n-\ell}) = (n - \ell - 1) \xi^{n-\ell-2} \xi_t z_{n-\ell-1} z_{n-\ell} dt + \xi^{n-\ell-1} z_{n-\ell-1} d z_{n-\ell} \\
+ \xi^{n-\ell-1} z_{n-\ell} d z_{n-\ell-1} + \xi^{n-\ell-1} z_{n-\ell-1} d z_{n-\ell}.
\]

By the assumptions \((H_2)-(H_3)\), it follows that

\[
0 = E \int_{Q_1} (n - \ell - 1) \xi^{n-\ell-2} \xi_t z_{n-\ell-1} z_{n-\ell} dxdt \\
+ E \int_{Q_1} \xi^{n-\ell-1} z_{n-\ell-1} \left[ - \sum_{i,j=1}^m (a_{ij} z_{n-\ell-1} x_i) x_j - b_t n - \ell z_1 - \cdots - b_{n-\ell} n - \ell z_{n-\ell} \\
- b_{n-\ell+1} n - \ell z_{n-\ell+1} - d_{n-\ell} n - \ell \tilde{z}_{n-\ell} \right] dxdt \\
+ E \int_{Q_1} \xi^{n-\ell-1} z_{n-\ell} \left[ - \sum_{i,j=1}^m (a_{ij} z_{n-\ell} x_i) x_j - b_t n - \ell z_1 - \cdots \\
- b_{n-\ell} n - \ell z_{n-\ell} - d_{n-\ell-1} n - \ell - 1 \tilde{z}_{n-\ell} \right] dxdt \\
+ E \int_{Q_1} \xi^{n-\ell-1} \bar{z}_{n-\ell-1} \bar{z}_{n-\ell} dxdt.
\]

The above equality, together with \((H_1)\), implies that for any \( \varepsilon > 0 \),

\[
E \int_{Q_1} \xi^{n-\ell-1} z_{n-\ell}^2 dxdt \\
\leq \varepsilon E \int_{Q_1} \xi^{2(n-\ell-1)} \bar{z}_{n-\ell}^2 + \xi^{n-\ell} z_{n-\ell+1}^2 dxdt \quad (23)
\]

\[ + C E \int_{Q_1} \sum_{j=1}^{n-\ell-1} (\xi^{j-1} z_j^2 + \xi^{2j-1} \tilde{z}_j^2) dxdt. \]
Taking Step 5. Combining (22) with (23), we get the desired estimate (21). Hence, (20) is arrived.

\[ \xi \sum_{j=1}^{k} \lambda_{j}(z_{j}^{n-\ell})^{2}(t)dt \]

Combining (22) with (23), we get the desired estimate (21). Hence, (20) is arrived.

**Step 5.** Taking \( k' = n - 1 \) in (20), one has that

\[
E \int_{Q_{1}} \left( \xi^{n-1} z_{n}^{2} + \xi^{n-2} z_{n-1}^{2} + \cdots + \xi z_{2}^{2} \right) dx dt
\leq C E \int_{Q_{1}} \left( z_{1} + \xi^{\frac{1}{2}} z_{1}^{2} \right) dx dt + C E \int_{t_{1}}^{t_{2}} \xi^{\frac{1}{2}} \sum_{j=1}^{k} \lambda_{j}(z_{j}^{1})^{2}(t) dt. \tag{24} \]

In the following, we estimate the last two terms in (24). By Itô’s formula,

\[ d(\xi^{\frac{1}{2}} z_{1}^{2}) = \frac{1}{2} \xi^{-\frac{1}{2}} \xi_{t} z_{1}^{2} dt + 2 \xi^{\frac{1}{2}} z_{1} dz_{1} + \xi^{\frac{1}{2}} (dz_{1})^{2}. \]

Then by the first equation of (7), it is easy to show that for any \( \varepsilon > 0 \),

\[
E \int_{Q_{1}} \xi^{\frac{1}{2}} z_{1}^{2} dx dt + 2 E \int_{t_{1}}^{t_{2}} \xi^{\frac{1}{2}} \sum_{j=1}^{k} \lambda_{j}(z_{j}^{1})^{2}(t) dt
= -E \int_{Q_{1}} \frac{1}{2} \xi^{-\frac{1}{2}} \xi_{t} z_{1}^{2} dx dt + 2 E \int_{Q_{1}} \xi^{\frac{1}{2}} z_{1} (b_{11} z_{1} + b_{21} z_{2} + d_{11} \tilde{z}_{1}) dx dt
\leq C E \int_{Q_{1}} z_{1}^{2} dx dt + \varepsilon E \int_{Q_{1}} \xi z_{1}^{2} dx dt + \varepsilon E \int_{Q_{1}} \xi^{\frac{1}{2}} z_{1}^{2} dx dt.
\]

This, together with (24), indicates

\[
E \int_{Q_{1}} \left( \xi^{n-1} z_{n}^{2} + \xi^{n-2} z_{n-1}^{2} + \cdots + \xi z_{2}^{2} \right) dx dt \leq C E \int_{Q_{1}} z_{1}^{2} dx dt. \tag{25}
\]

Put

\[ t_{3} = t_{1} + \frac{t_{2} - t_{1}}{4} \quad \text{and} \quad t_{4} = t_{2} - \frac{t_{2} - t_{1}}{4}. \]

Since \( \xi(t) \geq \left( \frac{t_{2} - t_{1}}{4} \right)^{2n} \) in \([t_{3}, t_{4}]\), combining (10) and (25) with (9), we get that

\[
\sum_{i=1}^{n} \int_{G} z_{i}^{2}(x, t_{1}) dx \leq \frac{e^{\tau(t_{2} - t_{1})}}{t_{4} - t_{3}} \sum_{i=1}^{n} \int_{t_{3}}^{t_{4}} \int_{G} z_{i}^{2}(x, t) dx dt
\leq \frac{2 e^{\tau(t_{2} - t_{1})}}{t_{2} - t_{1}} \left( \frac{4}{2} \right)^{2n(n-1)} E \int_{t_{3}}^{t_{4}} \int_{G} (z_{1}^{2} + \xi z_{2}^{2} + \cdots + \xi^{n-1} z_{n}^{2}) dx dt
\leq \frac{C e^{\tau(t_{2} - t_{1})}}{(t_{2} - t_{1})^{2n(n-1)+1}} E \int_{Q_{1}} z_{1}^{2} dx dt \leq \frac{C e^{\tau(t_{2} - t_{1})} + C \sqrt{h}}{(t_{2} - t_{1})^{2n(n-1)+1}} E \int_{t_{1}}^{t_{2}} \int_{G_{0}} z_{1}^{2} dx dt.
\]

Hence, we get the desired estimate (8) in Proposition 1. \( \square \)
3. Controllability of coupled stochastic parabolic systems. In this section, similar to [14], by the Lebeau-Robbiano technique, one can get the desired null controllability result for (1) in Theorem 1.1.

First, by the duality, the observability estimate (8) in Proposition 1 implies the following controllability result.

**Proposition 2.** Assume that \( H_1 \)-(H3) hold. Then for any positive integer \( k \) and \( Y(t_1) \in L^2(\Omega, \mathcal{F}_{t_1}, \mathbb{P}; (L^2(G))^n) \), one can find a control \( u_k \in L^2(t_1, t_2; L^2(G)) \), such that \( \text{supp} u_k \subseteq G_0 \times [t_1, t_2] \), and the corresponding solution \( Y(\cdot) \) of (1) in \( Q_1 \) satisfies that

\[
P_k(Y(t_2)) = 0 \quad \text{in} \quad G, \ \mathbb{P}\text{-a.s.}
\]

Moreover, there exists a positive constant \( C \), independent of \( t_1 \) and \( t_2 \), such that

\[
|u_k|^2_{L^2(t_1, t_2; L^2(G))} \leq C e^{C\sqrt{n_1 + \tau(t_2 - t_1)}} \left( t_2 - t_1 \right)^{2n(n-1)+1} E|Y(t_1)|^2_{(L^2(G))^n},
\]

and

\[
E|Y(t_2)|^2_{(L^2(G))^n} \leq C \left[ e^{C\sqrt{n_1 + \tau(t_2 - t_1)}} \left( t_2 - t_1 \right)^{2n(n-1)+1} + 1 \right] e^{\tau(t_2 - t_1)} E|Y(t_1)|^2_{(L^2(G))^n}.
\]

On the other hand, it is easy to show the following decay property for solutions of (1).

**Proposition 3.** Assume that \( u \equiv 0 \) in \( Q_1 \). Then for any positive integer \( k \) and \( Y(t_1) \in L^2(\Omega, \mathcal{F}_{t_1}, \mathbb{P}; (L^2(G))^n) \), if \( P_k(Y(t_1)) = 0 \), the corresponding solution \( Y(\cdot) \) of (1) in \( Q_1 \) satisfies that \( P_0(Y(t)) = 0 \) and

\[
E|Y(t)|^2_{(L^2(G))^n} \leq e^{-(2n+1-\tau)(t-t_1)} E|Y(t_1)|^2_{(L^2(G))^n}, \quad \text{for any} \ t \in [t_1, t_2].
\]

By the classical Lebeau-Robbiano technique, by Propositions 2 and 3, one can get the null controllability result of (1) in Theorem 1.1. Now, we give a sketch of its proof.

**Sketch of a proof of Theorem 1.1.** The whole proof is divided into four parts.

**Step 1.** First, introduce some notations. For any \( \ell \in \mathbb{N} \), write

\[
T_{2\ell} = t_1 + \frac{(2^\ell - 1)(t_2 - t_1)}{2^\ell}, \quad T_{2\ell+1} = t_1 + \frac{(2^{\ell+2} - 3)(t_2 - t_1)}{2^{\ell+2}}.
\]

\[
I_\ell = [T_{2\ell}, T_{2\ell+1}] \quad \text{and} \quad J_\ell = [T_{2\ell+1}, T_{2\ell+2}].
\]

Then it is easy to check that

\[
T_{2\ell+1} - T_{2\ell} = T_{2\ell+2} - T_{2\ell+1} = \frac{t_2 - t_1}{2^{\ell+2}} \quad \text{and} \quad [t_1, t_2] = \bigcup_{\ell \in \mathbb{N}} (I_\ell \cup J_\ell).
\]

Let \( \{\sigma_\ell\}_{\ell \in \mathbb{N}} \) be a sequence of sufficiently large positive constants, which will be specified later.

For any \( Y_0 \in (L^2(G))^n \), take a control \( u \equiv 0 \) in \( G \times [0, t_1] \). For the corresponding solution \( Y(\cdot) \) of (1) in \( G \times [0, t_1] \), set \( Y_1 = Y(t_1) \).

**Step 2.** First, consider the system (1) on the interval \( I_0 = [T_0, T_1] \):

\[
\begin{cases}
  dY = AY dt + B(t)Y dt + h_0 \chi_{G_0} u dt + D(t)Y dW(t) & \text{in} \ G \times I_0, \\
  Y = 0 & \text{on} \ \Sigma, \\
  Y(T_0) = Y_1 & \text{in} \ G.
\end{cases}
\]

(26)
By Proposition 2, for $\sigma_1 \in \mathbb{N}$, there exists a control $u_1 \in L^2_{T_0}(\mathcal{I}_0; L^2(G_0))$, such that the corresponding solution $Y_{11}$ of (26) satisfies $P_{\sigma_1}(Y_{11}(T_1)) = 0$ in $G$, $\mathbb{P}$-a.s. Moreover,

$$|u_1|^2_{L^2_{T_0}(\mathcal{I}_0; L^2(G_0))} \leq C e^{C\sqrt{\lambda_{\sigma_1} + \tau(T_1-\lambda_{T_0})}} \mathbb{E}[Y_{11}]^2_{L^2(G)} |	imes|$$

and

$$\mathbb{E}[Y_{11}(T_1)]^2_{L^2(G)} \leq C \mathbb{E} \left[ e^{C\sqrt{\lambda_{\sigma_1} + \tau(T_1-\lambda_{T_0})}} (T_1-\lambda_{T_0})^{2n(n+1)+1} \right] \mathbb{E}[Y_{11}]^2_{L^2(G)} |	imes|$$

On the other hand, consider the system (1) with $u \equiv 0$ on the interval $J_0 = [T_1, T_2]$:

$$dY = AYdt + B(t)Ydt + D(t)YdW(t) \quad \text{ in } G \times J_0,$$

$$Y = 0 \quad \text{ on } \Sigma,$$

$$Y(T_1) = Y_{11}(T_1) \quad \text{ in } G.$$ 

By Proposition 3, the solution $Y_{12}$ of (27) satisfies

$$\mathbb{E}[Y_{12}(T_2)]^2_{L^2(G)} \leq e^{-2\lambda_{\sigma_1} + \tau(T_2-\lambda_{T_1})} \mathbb{E}[Y_{12}(T_1)]^2_{L^2(G)} |	imes|$$

and

$$P_{\sigma_1}(Y_{12}(t)) = 0 \quad \text{ in } G, \quad \mathbb{P} \text{-a.s., for any } t \in J_0.$$

**Step 3.** Similarly, for any positive integer $\ell$, consider the system (1) on $I_{\ell-1} = [T_{2\ell-2}, T_{2\ell-1}]$. By Proposition 2, there exists a control $u_\ell \in L^2_{T_0}(\mathcal{I}_{\ell-1}; L^2(G_0))$, such that the solution $Y$ of (1) in $G \times I_{\ell-1}$ satisfies $P_{\sigma_\ell}(Y(T_{2\ell-1})) = 0$ in $G$, $\mathbb{P}$-a.s. Moreover,

$$|u_\ell|^2_{L^2_{T_0}(\mathcal{I}_{\ell-1}; L^2(G_0))} \leq C e^{C\sqrt{\lambda_{\sigma_\ell} + \tau(T_{2\ell-1}-T_{2\ell-2})}} \mathbb{E}[Y(T_{2\ell-2})]^2_{L^2(G)} |	imes|$$

and

$$\mathbb{E}[Y(T_{2\ell-1})]^2_{L^2(G)} \leq C \mathbb{E} \left[ e^{C\sqrt{\lambda_{\sigma_\ell} + \tau(T_{2\ell-1}-T_{2\ell-2})}} (T_{2\ell-1}-T_{2\ell-2})^{2n(n+1)+1} \right] \mathbb{E}[Y(T_{2\ell-2})]^2_{L^2(G)} |	imes|$$

On the other hand, consider the system (1) with $u \equiv 0$ on $J_{\ell-1} = [T_{2\ell-1}, T_{2\ell}]$. By Proposition 3, the solution $Y$ of (1) in $G \times J_{\ell-1}$ satisfies

$$\mathbb{E}[Y(T_{2\ell})]^2_{L^2(G)} \leq e^{-2\lambda_{\sigma_\ell} + \tau(T_{2\ell}-T_{2\ell-1})} \mathbb{E}[Y(T_{2\ell-1})]^2_{L^2(G)} |	imes|$$

and

$$P_{\sigma_\ell}(Y(t)) = 0 \quad \text{ in } G, \quad \mathbb{P} \text{-a.s., for any } t \in J_{\ell-1}.$$

**Step 4.** By (29) and (30), it is easy to show that

$$\mathbb{E}[Y(T_{2\ell})]^2_{L^2(G)} \leq C e^{-2\lambda_{\sigma_\ell} + \tau(T_{2\ell}-T_{2\ell-1})} \mathbb{E}[Y(T_{2\ell-1})]^2_{L^2(G)} |	imes|$$

$$\mathbb{E}[Y(T_{2\ell})]^2_{L^2(G)} \leq C e^{-2\lambda_{\sigma_\ell} + \tau(T_{2\ell}-T_{2\ell-1})} \mathbb{E}[Y(T_{2\ell-1})]^2_{L^2(G)} |	imes|$$
By Weyl’s formula, for sufficiently large $\sigma_\ell$, it holds that
\[
\mathbb{E}|Y(T_{2\ell})|^2_{L^2(G)}^n \leq C2^{[2n(n-1)+1]}e^{-C2^{-\ell}}a_\ell^2 \cdot e^{C\sigma_\ell} \cdot e^{C\tau^2 \ell} \mathbb{E}|Y(T_{2\ell-2})|^2_{L^2(G)}^n \leq \cdots \leq e^{-C4\ell} |Y_0|^2_{L^2(G)}^n.
\]
By a similar argument, by (28), it follows that
\[
|u_\ell|^2_{L^2((t_{\ell-1}; L^2(G_0)))} \leq C e^{-C4\ell} |Y_0|^2_{L^2(G)}.
\]
Choose a control function $u$ as follows:
\[
u(x, t) = \begin{cases} 
0 & \text{in } G \times (0, t_1), \\
u_\ell(x, t) & \text{in } G \times I_{\ell-1} (\ell \geq 1), \\
0 & \text{in } G \times J_{\ell-1} (\ell \geq 1), \\
0 & \text{in } G \times (t_2, T). 
\end{cases}
\]
By (28), \[|u|^2_{L^2(0, T; L^2(G_0))} = \sum_{\ell=1}^{\infty} |u_\ell|^2_{L^2((t_{\ell-1}; L^2(G_0)))} \leq C |Y_0|^2_{L^2(G)}.
\]
Also, $Y(T) = 0$ in $G$, $\mathbb{P}$-a.s. The proof of Theorem 1.1 is completed.

Next, we give a proof of Corollary 1.

**Proof of Corollary 1.** Assume that $Y$ is a solution of (3) associated to a control $u$, and set $\hat{Y} = N^{-1}Y$. Then it is easy to check that $\hat{Y}$ satisfies the following system:
\[
\begin{align*}
\dot{\hat{Y}} &= A\hat{Y} dt + B\hat{Y} dt + h_0 \chi_{G_0} u dt + \bar{D}(t)\hat{Y} dW(t) & \text{in } Q, \\
\hat{Y} &= 0 & \text{on } \Sigma, \\
\hat{Y}(0) &= \hat{Y}_0 & \text{in } G,
\end{align*}
\]
where $\bar{D}(t) = N^{-1}D(t)N$, $\bar{B} = \begin{pmatrix}
0 & 0 & \cdots & -a_n \\
1 & 0 & \cdots & -a_{n-1} \\
0 & 1 & \cdots & -a_{n-2} \\
\vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 1 - a_1
\end{pmatrix}$ ($a_i \in \mathbb{R}, i = 1, \cdots, n$), and $\hat{Y}_0 = N^{-1}Y_0$. Since the assumptions $(H_1)$-$(H_3)$ hold for the system (31), by Theorem 1.1, one can get the null controllability for (31) and (3).

Finally, we give some counterexamples to show that the null controllability of the coupled stochastic parabolic system (1) may be untrue, if the assumption $(H_1)$, $(H_2)$ or $(H_3)$ for $n = 3$ does not hold.

**Example 1.** Choose $n = 3$, $A = \Delta$, $B$ is an identity matrix, and $D = 0$ in (1). Then the assumption $(H_1)$ fails, but the others hold. It is easy to find that the corresponding deterministic coupled parabolic system is not null controllable.

**Example 2.** In (1), choose $n = 3$, $A = \Delta$, $D$ is an identity matrix, and
\[
B = \begin{pmatrix}
0 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0
\end{pmatrix}.
\]
Then the assumption \((H_2)\) fails, but the others hold. Also, (1) becomes
\[
\begin{aligned}
&dy_1 - \Delta y_1 dt = \chi_{G_0} u dt + y_1 dW(t) \quad \text{in } Q, \\
&dy_2 - \Delta y_2 dt = (y_1 + y_2) dt + y_2 dW(t) \quad \text{in } Q, \\
&dy_3 - \Delta y_3 dt = (y_1 + y_2) dt + y_3 dW(t) \quad \text{in } Q, \\
y_1 = y_2 = y_3 = 0 \quad \text{on } \Sigma, \\
y_1(0) = y_1^0, \ y_2(0) = y_2^0, \ y_3(0) = y_3^0 \quad \text{in } G,
\end{aligned}
\]
where \((y_1^0, y_2^0, y_3^0) \in (L^2(G))^3\) is any given initial value. Consider the following coupled backward stochastic parabolic system:
\[
\begin{aligned}
dz_1 + \Delta z_1 dt &= -(z_2 + z_3 + \tilde{z}_1) dt + \tilde{z}_1 dW(t) \quad \text{in } Q, \\
dz_2 + \Delta z_2 dt &= -(z_2 + z_3 + \tilde{z}_2) dt + \tilde{z}_2 dW(t) \quad \text{in } Q, \\
dz_3 + \Delta z_3 dt &= -z_3 dt + \tilde{z}_3 dW(t) \quad \text{in } Q, \\
z_1 = z_2 = z_3 = 0 \quad \text{on } \Sigma, \\
z_1(T) = z_1^T, \ z_2(T) = z_2^T, \ z_3(T) = z_3^T \quad \text{in } G.
\end{aligned}
\]
Then by the duality, the null controllability of (32) is equivalent to the following observability estimate for (33):
\[
|z_1(0)|^2_{L^2(G)} + |z_2(0)|^2_{L^2(G)} + |z_3(0)|^2_{L^2(G)} \\
\leq CE \int_0^T \int_{G_0} z_i^2(x,t) dx dt, \quad \forall z_i^T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(G)) \ (i = 1, 2, 3).
\]
(34)

Consider the following stochastic differential equation:
\[
\begin{aligned}
d\zeta - \lambda_1 \zeta dt &= -\zeta dt + \zeta dW(t) \quad \text{in } (0,T), \\
\zeta(0) &= 1.
\end{aligned}
\]
Put \(\hat{z}_2 = \zeta(t)e_1\) and \(\hat{z}_3 = -\zeta(t)e_1\). Then it is easy to show that \(\hat{z}_2\) and \(\hat{z}_3\) satisfy that
\[
\begin{aligned}
d\hat{z} + \Delta \hat{z} dt &= -\hat{z} dt + \hat{z} dW(t) \quad \text{in } Q, \\
\hat{z} &= 0 \quad \text{on } \Sigma,
\end{aligned}
\]
and \(\hat{z}_2(0) = -\hat{z}_3(0) = e_1\). Choose \(z_1^T = 0, \ z_2^T = \zeta(T)e_1\) and \(z_3^T = -\zeta(T)e_1\).
Then in (33), \(z_2 = \hat{z}_2 = -\hat{z}_3 = -\hat{z}_3\) and \(z_1 = \hat{z}_1 = 0\) in \(Q\). Also, \(|z_2(0)|_{L^2(G)} = |z_3(0)|_{L^2(G)} = 1\). Hence, (34) fails.

**Example 3.** In (1), for any given \(\varepsilon_i > 0 \ (i = 1, 2, 3)\), choose \(n = 3, \ A = \Delta, \)
\[
B = \begin{pmatrix}
0 & 0 & 0 \\
1 & -1 & 0 \\
0 & 1 & -1
\end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix}
0 & 0 & 0 \\
\varepsilon_1 & 0 & 0 \\
\varepsilon_2 & \varepsilon_3 & 0
\end{pmatrix}.
\]
Furthermore, consider the following backward stochastic parabolic equation:

\[
\begin{aligned}
dy_1 - \Delta y_1 dt &= \chi_{G_0} ud t & \text{in } Q, \\
dy_2 - \Delta y_2 dt &= (y_1 - y_2) dt + \varepsilon_1 y_1 dW(t) & \text{in } Q, \\
y_1(0) &= y_1^0, \ y_2(0) = y_2^0 & \text{in } G, \\
y_1(0) &= y_1^0, \ y_2(0) = y_2^0, \ y_3(0) = y_3^0 & \text{on } \Sigma, \\
y_1 &= y_2 = y_3 = 0 & \text{in } \Omega, \\
y_1 &= y_2 = y_3 = 0 & \text{on } \Sigma, \\
\end{aligned}
\]

Then the assumption \((H_3)\) fails, but the others hold. Also, \((1)\) becomes

\[
\begin{aligned}
dz_1 + \Delta z_1 dt &= -(z_2 + \varepsilon_1 \tilde{z}_2 + \varepsilon_2 \tilde{z}_3) dt + \tilde{z}_1 dW(t) & \text{in } Q, \\
dz_2 + \Delta z_2 dt &= (z_2 - z_3 - \varepsilon_3 \tilde{z}_3) dt + \tilde{z}_2 dW(t) & \text{in } Q, \\
dz_3 + \Delta z_3 dt &= z_3 dt + \tilde{z}_3 dW(t) & \text{in } Q, \\
z_1 &= z_2 = z_3 = 0 & \text{on } \Sigma, \\
z_1(T) &= z_1^T, \ z_2(T) = z_2^T, \ z_3(T) = z_3^T & \text{in } G. \\
\end{aligned}
\]

where \((y_1^0, y_2^0, y_3^0) \in (L^2(G))^3\) is any given initial value. Furthermore, consider the following coupled backward stochastic parabolic system:

\[
\begin{aligned}
dz_1 + \Delta z_1 dt &= -(z_2 + \varepsilon_1 \tilde{z}_2 + \varepsilon_2 \tilde{z}_3) dt + \tilde{z}_1 dW(t) & \text{in } Q, \\
dz_2 + \Delta z_2 dt &= (z_2 - z_3 - \varepsilon_3 \tilde{z}_3) dt + \tilde{z}_2 dW(t) & \text{in } Q, \\
dz_3 + \Delta z_3 dt &= z_3 dt + \tilde{z}_3 dW(t) & \text{in } Q, \\
z_1 = z_2 = z_3 = 0 & \text{on } \Sigma, \\
z_1(T) = z_1^T, \ z_2(T) = z_2^T, \ z_3(T) = z_3^T & \text{in } G. \\
\end{aligned}
\]

Then by the duality, the null controllability of \((35)\) is equivalent to the following observability estimate for \((36)\):

\[
|z_1(0)|^2_{L^2(G)} + |z_2(0)|^2_{L^2(G)} + |z_3(0)|^2_{L^2(G)} \\
\leq CE \int_0^T \int_{G_0} z_i^2(x, t) dx dt, \quad \forall z_i^T \in L^2(\Omega, F_T, P; L^2(G)) (i = 1, 2, 3). \tag{37}
\]

The following arguments will show that the estimate \((37)\) indeed does not hold.

**Case 1.** Assume that \(\varepsilon_1 \neq \varepsilon_3\).

Consider the following stochastic differential equation:

\[
\begin{aligned}
d\zeta_1 - \lambda_1 \zeta_1 dt &= \zeta_1 dt - \frac{1}{\varepsilon_3} \zeta_1 dW(t) & \text{in } (0, T), \\
\zeta_1(0) &= 1. \\
\end{aligned}
\]

Put \(\hat{z}_3 = \zeta_1(t)e_1\). Then it is easy to show that \(\hat{z}_3\) satisfies that

\[
\begin{aligned}
d\hat{z}_3 + \Delta \hat{z}_3 dt &= \hat{z}_3 dt - \frac{1}{\varepsilon_3} \hat{z}_3 dW(t) & \text{in } Q, \\
\hat{z}_3 &= 0 & \text{on } \Sigma, \\
\hat{z}_3(0) &= e_1 & \text{in } G. \\
\end{aligned}
\]

Furthermore, consider the following backward stochastic parabolic equation:

\[
\begin{aligned}
dz_3 + \Delta z_3 dt &= z_3 dt + \tilde{z}_3 dW(t) & \text{in } Q, \\
z_3 &= 0 & \text{on } \Sigma, \\
z_3(T) &= \zeta_1(T)e_1 & \text{in } G. \\
\end{aligned}
\]

Then it is easy to show that \(z_3 = \hat{z}_3\) and \(\tilde{z}_3 = -\frac{1}{\varepsilon_3} \hat{z}_3\). Hence, \(z_3 + \varepsilon_3 \tilde{z}_3 = 0\) in \(Q\) and \(|z_3(0)|_{L^2(G)} = 1\).
On the other hand, take $z_2^T = \frac{\epsilon_2}{\epsilon_3 - \epsilon_1} \zeta_1(T)e_1$, $z_2 = \frac{\epsilon_2}{\epsilon_3 - \epsilon_1} z_3$ and $\hat{z}_2 = \frac{\epsilon_2}{\epsilon_3 - \epsilon_1} \hat{z}_3$.

Then it is easy to show that $(z_2, \hat{z}_2)$ satisfies

$$
\begin{cases}
    dz_2 + \Delta z_2 dt = z_2 dt + \hat{z}_2 dW(t) & \text{in } Q, \\
    z_2 = 0 & \text{on } \Sigma, \\
    z_2(T) = \frac{\epsilon_2}{\epsilon_3 - \epsilon_1} \zeta_1(T)e_1 & \text{in } G,
\end{cases}
$$

and $z_2 + \epsilon_1 \hat{z}_2 + \epsilon_2 \hat{z}_3 = 0$ in $Q$. Hence, for $(z_1^T, z_2^T, z_3^T) = (0, \frac{\epsilon_2}{\epsilon_3 - \epsilon_1} \zeta_1(T)e_1, \zeta_1(T)e_1)$, (36) becomes:

$$
\begin{cases}
    dz_1 + \Delta z_1 dt = \hat{z}_1 dW(t) & \text{in } Q, \\
    dz_2 + \Delta z_2 dt = z_2 dt + \hat{z}_2 dW(t) & \text{in } Q, \\
    dz_3 + \Delta z_3 dt = z_3 dt + \hat{z}_3 dW(t) & \text{in } Q, \\
    z_1 = z_2 = z_3 = 0 & \text{on } \Sigma, \\
    z_1(T) = 0, z_2(T) = \frac{\epsilon_2}{\epsilon_3 - \epsilon_1} \zeta_1(T)e_1, z_3(T) = \zeta_1(T)e_1 & \text{in } G.
\end{cases}
$$

Since $|z_3(0)|_{L^2(G)} = 1$, $|z_2(0)|_{L^2(G)} = \frac{\epsilon_2}{\epsilon_3 - \epsilon_1}$ and $z_1 \equiv 0$ in $Q$, (37) does not hold.

**Case 2.** Assume that $\epsilon_1 = \epsilon_3$.

Choose $z_1(T) = z_3(T) = 0$ and $z_2(T) = \zeta_1(T)e_1$. Then (36) becomes:

$$
\begin{cases}
    dz_1 + \Delta z_1 dt = \hat{z}_1 dW(t) & \text{in } Q, \\
    dz_2 + \Delta z_2 dt = z_2 dt + \hat{z}_2 dW(t) & \text{in } Q, \\
    dz_3 + \Delta z_3 dt = z_3 dt + \hat{z}_3 dW(t) & \text{in } Q, \\
    z_1 = z_2 = z_3 = 0 & \text{on } \Sigma, \\
    z_1(T) = 0, z_2(T) = \zeta_1(T)e_1, z_3(T) = 0 & \text{in } G.
\end{cases}
$$

Since $z_1 \equiv 0$ in $Q$ and $|z_2(0)|_{L^2(G)} = 1$, (37) does not hold.

4. **Observability of coupled stochastic parabolic systems.** This section is devoted to observability problems for the coupled stochastic parabolic system (4).

To this aim, we recall a known Carleman estimate for single stochastic parabolic equations. First, introduce some auxiliary functions. By [6], there exists a function $\psi(\cdot) \in C^4(\bar{G})$, such that

$$
\psi(x) \geq 0 \text{ in } G, \quad \psi(x) = 0 \text{ on } \Gamma, \quad \text{and } \quad |\nabla \psi(x)| \geq 0 \text{ in } G \setminus G_1.
$$

Let $\beta \geq 1$ and $\lambda \geq 1$ be two parameters. For any positive integer $k$, define

$$
\gamma(t) = \frac{1}{t^k(T-t)^k}, \quad \alpha(x,t) = \frac{e^{\beta \psi(x)} - e^{2\beta |\psi|_{C(\partial)}}}{t^k(T-t)^k} \text{ and } \theta(x,t) = e^{\lambda \alpha(x,t)}.
$$
Consider the following stochastic parabolic equation:

\[
\begin{aligned}
&\begin{cases}
  dy - \sum_{i,j=1}^{m} (\tilde{a}_{ij} y_{x_i})_{x_j} dt = f dt + gdW(t) & \text{in } Q,
  \\
y = 0 & \text{on } \Sigma,
  \\
y(0) = y^0 & \text{in } G,
\end{cases}
\end{aligned}
\]  

(38)

where \(\tilde{a}_{ij}\) is the function given in (4), \(y^0 \in L^2(G)\) and \(f, g \in L^2_0(0,T; L^2(G))\). Then by [12], the following Carleman estimate holds for solutions of (38).

**Lemma 4.1.** There exist positive constants \(\beta_1, \lambda_1\) and \(C\), such that for any \(\beta \geq \beta_1\) and \(\lambda \geq \lambda_1\), solutions of (38) satisfy that for any \(y^0 \in L^2(G)\),

\[
\begin{aligned}
&\mathbb{E} \int_Q \theta^2(\lambda^3 \gamma^2 y^2 + \lambda \gamma |\nabla y|^2)dxdt \\
&\leq C \left( \mathbb{E} \int_0^T \int_{\Omega_1} \theta^2 \lambda^3 \gamma^2 y^2 dxdt + \mathbb{E} \int_Q \theta^2 f^2 dxdt + \mathbb{E} \int_Q \theta^2 \lambda^2 \gamma^2 y^2 dxdt \right).
\end{aligned}
\]

Let \(\Omega_0\) and \(\Omega_1\) be any two nonempty open subsets of \(G_*\), satisfying that \(G_1 \subseteq \Omega_1\), \(\Omega_1 \subseteq \Omega_0\) and \(\overline{\Omega}_0 \subseteq G_*\). In order to establish an observability estimate for (4), borrowing ideas from [7], we derive the following Carleman estimate for the system (4).

**Proposition 4.** Assume that \((\text{P}_1)-(\text{P}_2)\) hold. Then for any \(i = 1, \cdots, n-1,\) \(L \geq 3\) and \(\varepsilon > 0\), one can find positive integers \(k_{L,j}\) \((j = i+1, \cdots, n)\) and a positive constant \(C\), such that solutions \(Y = (y_1, \cdots, y_n)^T\) of (4) satisfy

\[
\begin{aligned}
&\mathbb{E} \int_0^T \int_{\Omega_1} \theta^2 \lambda^L \gamma^{L-i} y^2 dxdt \\
&\leq \varepsilon \mathbb{E} \int_Q \theta^2(\lambda^3 \gamma^2 y^2 + \lambda \gamma |\nabla y|^2)dxdt + \varepsilon \mathbb{E} \int_Q \theta^2 \lambda^3 \gamma^2 y^2 dxdt \\
&\quad + C \sum_{j=i+1}^{n} \mathbb{E} \int_0^T \int_{\Omega_0} \theta^2 \lambda^{L-j} \gamma^{k_{L,j}} y^2 dxdt, \quad \forall \ Y_0 \in (L^2(G))^n.
\end{aligned}
\]  

(39)

Also, for \(i = 0\), set \(y_0 = 0\).

**Proof.** First, choose a cut-off function \(\zeta \in C_0^\infty(\Omega_0)\), such that

\[
0 \leq \zeta \leq 1 \text{ in } \Omega_0, \quad \zeta = 1 \text{ in } \Omega_1 \quad \text{and} \quad |\nabla \zeta| \leq C \zeta^{\frac{1}{2}} \text{ in } \Omega_0.
\]

For the \(i\)-th and \((i+1)\)-th equations of (4) \((i = 1,2,\cdots, n-1)\), by Itô’s formula, we have that

\[
d(\zeta^2 \theta^2 \lambda^L \gamma^L y_{i+1}) = (\zeta^2 \theta^2 \lambda^L \gamma^L)_{i} y_{i+1} dt + \zeta^2 \theta^2 \lambda^L \gamma^L y_{i+1} dy_{i+1} \\
+ \zeta^2 \theta^2 \lambda^L \gamma^L y_{i+1} dy_{i} + \zeta^2 \theta^2 \lambda^L \gamma^L dy_{i} dy_{i+1}.
\]

By the assumption \((\text{P}_2)\), it is easy to check that

\[
\mathbb{E} \int_Q \tilde{b}_{i+1} i \zeta^2 \theta^2 \lambda^L \gamma^L y_{i+1}^2 dxdt \\
= -\mathbb{E} \int_Q (2\zeta^2 \theta^2 |\lambda^L \gamma^L y_{i+1} + L \zeta^2 \theta^2 \lambda^L \gamma^{L-1} \gamma y_{i+1}| dxdt
\]
Similarly, By Hölder’s inequality, for any \(\varepsilon > 0\),

\[
-\mathbb{E} \int_Q \left\{ \zeta^2 \theta^2 \lambda \gamma^2 y_i \left[ \sum_{i', j' = 1}^m (\tilde{a}_{i', j'}^{1+1} y_{i+1, x_{j'}})_{x_{j'}} + \tilde{b}_{i+1} + \cdots + \tilde{b}_{i+1} n y_n \right] \right. \\
+ \left. \zeta^2 \theta^2 \lambda \gamma^2 y_{i+1} \left[ \sum_{i', j' = 1}^m (\tilde{a}_{i', j'}^{1} y_{i, x_{j'}})_{x_{j'}} + \tilde{b}_{i} i-1 y_{i, i-1} + \tilde{b}_{i} y_{i} + \cdots \\
+ \tilde{b}_{in} y_n \right] \right\} dx dt \\
-\mathbb{E} \int_Q \zeta^2 \theta^2 \lambda \gamma^2 \left[ (\tilde{a}_{i} i x + \cdots + \tilde{d}_{in} y_n) (\tilde{a}_{i+1} i_1 y_{i+1} + \cdots + \tilde{d}_{i+1} n y_n) \right] dx dt.
\]

and

\[
-\mathbb{E} \int_Q \left( \zeta^2 \theta^2 \lambda \gamma^2 y_i \sum_{i', j' = 1}^m (\tilde{a}_{i', j'}^{1+1} y_{i+1, x_{j'}})_{x_{j'}} \right) dx dt \\
= \mathbb{E} \int_Q \left( \sum_{i', j' = 1}^m (\zeta^2 \theta^2 \lambda \gamma^2 y_i)_{x_{j'}} \tilde{a}_{i', j'}^{1+1} y_{i+1, x_{j'}} \right) dx dt \\
= \mathbb{E} \int_Q \left( \sum_{i', j' = 1}^m (2\zeta x_{j'} \theta^2 \lambda \gamma^2 \tilde{a}_{i', j'}^{1+1} y_{i+1, x_{j'}} + 2\zeta^2 \theta^2 \gamma^2 \lambda \gamma^{i+1} y_{i+1, x_{j'}} + \zeta^2 \theta^2 \gamma^2 \lambda \gamma^{i+1} y_{i+1, x_{j'}} \right) dx dt \\
\leq \varepsilon \mathbb{E} \int_Q \zeta \theta^2 \lambda \gamma^2 y_i^2 dx dt + \mathbb{E} \int_Q \zeta \theta^2 \lambda \gamma^2 \left| \nabla y_i \right|^2 dx dt.
\]

Similarly,

\[
-\mathbb{E} \int_Q \left( \zeta^2 \theta^2 \lambda \gamma^2 y_{i+1} \sum_{i', j' = 1}^m (\tilde{a}_{i', j'}^{1} y_{i, x_{j'}})_{x_{j'}} \right) dx dt \\
\leq \mathbb{E} \int_Q \zeta \theta^2 \lambda \gamma^2 \left| \nabla y_{i+1} \right|^2 dx dt + \mathbb{E} \int_Q \zeta \theta^2 \lambda \gamma^2 \left| \nabla y_{i+1} \right|^2 dx dt \\
+ \varepsilon \mathbb{E} \int_Q \zeta \theta^2 \lambda \gamma \left| \nabla y_i \right|^2 dx dt.
\]
Therefore, we have that
\[
E \int_Q b_{i+1} i \zeta^2 \theta^2 \lambda^{2L-1} y_i^2 dx dt \\
\leq \varepsilon E \int_Q (\zeta^2 \theta^2 \lambda^{2L-1} y_i^2 + \zeta^2 \theta^2 \lambda \gamma |\nabla y_i|^2) dx dt + \varepsilon E \int_Q \zeta^2 \theta^2 \lambda^{2L-1} y_{i-1}^2 dx dt \\
+ C \varepsilon \int_Q \zeta^2 \theta^2 \lambda^{2L-1} - \gamma^{2L-1} |\nabla y_{i+1}|^2 dx dt + C E \sum_{j=i+1}^n \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_j^2 dx dt.
\]
(40)

Next, we give an estimate on \( E \int_Q \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} |\nabla y_{i+1}|^2 dx dt \). By Itô’s formula again, we find
\[
d(\zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_{i+1}^2) = (\zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1}_t y_{i+1}^2) dt \\
+ 2 \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_{i+1} dy_{i+1} + \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} (dy_{i+1})^2.
\]
By the assumption \((P_2)\), it follows that
\[
-2E \int_Q \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_{i+1} \sum_{i', j'=1}^m (\tilde{a}_{i,j} y_{i+1, x_{i'}})_{x_{i'}} dx dt \\
= E \int_Q \left[ 2 \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_{i+1}^2 + (2L - 1) \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-2} \gamma^{2L-2} y_{i+1}^2 \\
+ 2 \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_{i+1} (\tilde{b}_{i+1} y_i + \tilde{b}_{i+1} + i_{i+1} y_{i+1} + \cdots + \tilde{b}_{i+1} + y_{i+1}) \\
+ \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} (\tilde{d}_{i+1} y_{i+1} + \cdots + \tilde{d}_{i+1} y_{2L-1} y_{i+1})^2 \right] dx dt.
\]
This implies that for any \( \varepsilon > 0 \),
\[
E \int_Q \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} |\nabla y_{i+1}|^2 dx dt \\
\leq C E \int_Q \left( \zeta^2 \theta^2 \lambda^{2L-1} y_{i+1}^2 + \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} |\nabla y_{i+1}|^2 \right) dx dt \\
\leq \varepsilon E \int_Q \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} |\nabla y_{i+1}|^2 dx dt + C E \int_Q \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_{i+1}^2 dx dt \\
+ \varepsilon E \int_Q \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} |\nabla y_{i+1}|^2 dx dt + C E \int_Q \sum_{j=i+1}^n \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_j^2 dx dt.
\]
Therefore,
\[
E \int_Q \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} |\nabla y_{i+1}|^2 dx dt \\
\leq \varepsilon E \int_Q \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_{i+1}^2 dx dt + C E \int_Q \sum_{j=i+1}^n \zeta^2 \theta^2 \lambda^{2L-1} \gamma^{2L-1} y_j^2 dx dt.
\]
(41)

Combining (40) and (41) with \((P_1)\), we get the desired Carleman estimate (39).
Finally, we give a proof of Theorem 1.2.

**Proof of Theorem 1.2.** Suppose that \( O^i (i = 1, \cdots, n) \) are nonempty open subsets of \( G^* \), satisfying \( G_1 \subseteq O^1, \overline{O^i} \subseteq O^{i+1} \) and \( \overline{O^n} \subseteq G^* \). First, applying Lemma 4.1 to the \( i \)-th equation in (4), and summing them with respect to \( i \) from 1 to \( n \), we can get that for a sufficiently large \( \lambda \),

\[
\sum_{i=1}^{n} \mathbb{E} \int_Q \theta^2 (\lambda^3 \gamma^3 y_i^2 + \lambda \gamma |\nabla y_i|^2) dx dt \leq C \sum_{i=1}^{n} \mathbb{E} \int_0^T \int_{O^i} \theta^2 \lambda^3 \gamma^3 y_i^2 dx dt. \tag{42}
\]

In Proposition 4, take \( O_0 = O^{i+1} \) and \( O_1 = O^i (i = 1, \cdots, n) \). Then by an iteration for the terms in the right side of (42), there always exists a positive integer \( L^* \), such that

\[
\sum_{i=1}^{n} \mathbb{E} \int_Q \theta^2 (\lambda^3 \gamma^3 y_i^2 + \lambda \gamma |\nabla y_i|^2) dx dt \leq C \mathbb{E} \int_0^T \int_{G_0} \theta^2 \lambda^{L^*} \gamma^{L^*} y_i^2 dx dt. \tag{43}
\]

Similar to arguments in [16], combining (43) with an energy estimate for (4), we can get the desired observability estimate (5). Also, (43) implies (6). \( \square \)

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E-mail address: liuly938@nenu.edu.cn
E-mail address: liux216@nenu.edu.cn