Superenergy tensors and their applications

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Abstract

In Lorentzian manifolds of any dimension the concept of causal tensors is introduced. Causal tensors have positivity properties analogous to the so-called “dominant energy condition”. Further, it is shown how to build, from any given tensor \( A \), a new tensor quadratic in \( A \) and “positive”, in the sense that it is causal. These tensors are called superenergy tensors due to historical reasons because they generalize the classical energy-momentum and Bel-Robinson constructions. Superenergy tensors are basically unique and with multiple and diverse physical and mathematical applications, such as: a) definition of new divergence-free currents, b) conservation laws in propagation of discontinuities of fields, c) the causal propagation of fields, d) null-cone preserving maps, e) generalized Rainich-like conditions, f) causal relations and transformations, and g) generalized symmetries. Among many others.

1 Causal tensors

In this contribution \( V \) will denote a differentiable \( N \)-dimensional manifold \( V \) endowed with a metric \( g \) of Lorentzian signature \( N-2 \). The solid Lorentzian cone at \( x \) will be denoted by \( \Theta^\pm_x = \Theta^+_x \cup \Theta^-_x \), where \( \Theta^\pm_x \subset T^*_x(V) \) are the future (+) and past (−) half-cones. The null cone \( \partial \Theta_x \) is the boundary of \( \Theta_x \) and its elements are the null vectors at \( x \). An arbitrary point \( x \in V \) is usually taken, but all definitions and results translate immediately to tensor fields if a time orientation has been chosen. The \( x \)-subscript is then dropped.

Definition 1.1 \( ^\text{1} \) A rank-\( r \) tensor \( T \) has the dominant property at \( x \in V \) if

\[
T(\vec{u}_1, \ldots, \vec{u}_r) \geq 0 \quad \forall \vec{u}_1, \ldots, \vec{u}_r \in \Theta^+_x.
\]

The set of rank-\( r \) tensor (fields) with the dominant property will be denoted by \( \mathcal{D}P^+_r \). We also put \( \mathcal{D}P^-_r \equiv \{ T : -T \in \mathcal{D}P^+_r \} \), \( \mathcal{D}P_r \equiv \mathcal{D}P^+_r \cup \mathcal{D}P^-_r \), \( \mathcal{D}P^\pm \equiv \bigcup_r \mathcal{D}P_r^\pm \), \( \mathcal{D}P \equiv \mathcal{D}P^+ \cup \mathcal{D}P^- \).

\(^1\)It is worthwhile to check also my joint contribution with García-Parrado, as well as that of Bergqvist’s, in this volume, with related results. Notice that signature convention here is opposite to those contributions and to \( ^\text{1} \).
By a natural extension $\mathbb{R}^+ = \mathcal{D}P_0^+ \subset \mathcal{D}P^+$. Rank-1 tensors in $\mathcal{D}P^+$ are simply the past-pointing causal 1-forms (while those in $\mathcal{D}P_1^-$ are the future-directed ones). For rank-2 tensors, the dominant property was introduced by Plebański [2] in General Relativity and is usually called the dominant energy condition [3] because it is a requirement for physically acceptable energy-momentum tensors. The elements of $\mathcal{D}P$ will be called “causal tensors”. As in the case of past- and future-pointing vectors, any statement concerning $\mathcal{D}P^+$ has its counterpart concerning $\mathcal{D}P^-$, and they will be generally omitted. Trivially one has

**Property 1.1** If $T^{(i)} \in \mathcal{D}P^+_r$ and $\alpha_i \in \mathbb{R}^+ (i = 1, \ldots, n)$ then $\sum_{i=1}^n \alpha_i T^{(i)} \in \mathcal{D}P^+_r$. Moreover, if $T^{(1)}, T^{(2)} \in \mathcal{D}P^+$ then $T^{(1)} \otimes T^{(2)} \in \mathcal{D}P^+$.

This tells us that $\mathcal{D}P^+$ is a graded algebra of cones. For later use, let us introduce the following notation

$$T^{(i)} \times_j T^{(2)} \equiv C^1_i C^2_{r+j} \left( g^{-1} \otimes T^{(1)} \otimes T^{(2)} \right)$$

that is to say, the contraction (via the metric) of the $i^{th}$ entry of the first tensor (which has rank $r$) with the $j^{th}$ of the second. There are of course many different products $\times_j$ depending on where the contraction is made.

Several characterizations of $\mathcal{D}P^+$ can be found. For instance [1, 4]

**Proposition 1.1** The following conditions are equivalent:

1. $T \in \mathcal{D}P^+_r$.
2. $T(\vec{k}_1, \ldots, \vec{k}_r) \geq 0 \ \forall \vec{k}_1, \ldots, \vec{k}_r \in \partial \Theta^+_x$.
3. $T(\vec{u}_1, \ldots, \vec{u}_r) > 0 \ \forall \vec{u}_1, \ldots, \vec{u}_r \in \text{int} \Theta^+_x, (T \neq 0)$.
4. $T(\vec{e}_0, \ldots, \vec{e}_0) \geq |T(\vec{e}_{\alpha_1}, \ldots, \vec{e}_{\alpha_r})| \ \forall \alpha_1, \ldots, \alpha_r \in \{0, 1, \ldots, N-1\}$, in all orthonormal bases $\{\vec{e}_0, \ldots, \vec{e}_{N-1}\}$ with a future-pointing timelike $\vec{e}_0$.
5. $T_i \times_j \tilde{T} \in \mathcal{D}P^+_{r+s-2}$, $\forall \tilde{T} \in \mathcal{D}P^-_s$, $\forall i = 1, \ldots, r$, $\forall j = 1, \ldots, s$.

**Proposition 1.2** Similarly, some characterizations of $\mathcal{D}P$ are [1]

1. $0 \neq T_i \times_i T \in \mathcal{D}P^-$ for some $i \implies T_i \times_j T \in \mathcal{D}P^-$ for all $i, j \implies T \in \mathcal{D}P$.
2. $T_i \times_i T = 0$ for all $i \iff T = k_1 \otimes \ldots \otimes k_r$ where $k_i$ are null $\implies T \in \mathcal{D}P$. 
2 Superenergy tensors

In this section the questions of how general is the class $\mathcal{DP}$ and how one can build causal tensors are faced. The main result is that:

Given an arbitrary tensor $A$, there is a canonical procedure (unique up to permutations) to construct a causal tensor quadratic in $A$.

This procedure was introduced in [5] and extensively considered in [4], and the causal tensors thus built are called “super-energy tensors”. The whole thing is based in the following

Remark 2.1 Given any rank-$m$ tensor $A$, there is a minimum value $r \in \mathbb{N}$, $r \leq m$ and a unique set of $r$ numbers $n_1, \ldots, n_r \in \mathbb{N}$, with $\sum_{i=1}^{r} n_i = m$, such that $A$ is a linear map on $\Lambda^{n_1} \times \ldots \times \Lambda^{n_r}$.

Here $\Lambda^p$ stands for the vector space of “contravariant $p$-forms” at any $x \in V$. In other words, $\exists$ a minimum $r$ such that $\tilde{A} \in \Lambda_{n_1} \otimes \ldots \otimes \Lambda_{n_r}$, where $\tilde{A}$ is the appropriate permuted version of $A$ which selects the natural order for the $n_1, \ldots, n_r$ entries. Tensors seen in this way are called $r$-fold $(n_1, \ldots, n_r)$-forms.

Some simple examples are: any $p$-form $\Omega$ is trivially a single (that is, 1-fold) $p$-form, while $\nabla \Omega$ is a double $(1,p)$-form; the Riemann tensor $R$ is a double $(2,2)$-form which is symmetric (the pairs can be interchanged), while $\nabla R$ is a triple $(1,2,2)$-form; the Ricci tensor $\text{Ric}$ is a double symmetric $(1,1)$-form and, in general, any completely symmetric $r$-tensor is an $r$-fold $(1,1,\ldots,1)$-form. A 3-tensor $A$ with the property $A(\vec{x}, \vec{y}, \vec{z}) = -A(\vec{z}, \vec{y}, \vec{x})$ is a double $(2,1)$-form and the corresponding $\tilde{A}$ is clearly given by $\tilde{A}(\vec{x}, \vec{y}, \vec{z}) \equiv A(\vec{x}, \vec{z}, \vec{y})$, $\forall \vec{x}, \vec{y}, \vec{z}$.

For $r$-fold forms, the interior contraction can be generalized in the obvious way $i_{\vec{x}_1,\ldots,\vec{x}_r} : \Lambda_{n_1} \otimes \ldots \otimes \Lambda_{n_r} \rightarrow \Lambda_{n_1-1} \otimes \ldots \otimes \Lambda_{n_r-1}$ by means of

$$i_{\vec{x}_1,\ldots,\vec{x}_r} A = C_1^1 C_{n_1+1}^2 \ldots C_{n_1+\ldots+n_r-1+1}^r (\vec{x}_1 \otimes \vec{x}_2 \otimes \ldots \otimes \vec{x}_r \otimes \tilde{A})$$

which is simply the interior contraction of each vector with each antisymmetric block. Similarly, by using the canonical volume element of $(V, g)$ one can define the multiple Hodge duals as follows:

$$*_{\mathcal{P}} : \Lambda_{n_1} \otimes \ldots \otimes \Lambda_{n_r} \rightarrow \Lambda_{n_1+\epsilon_1(N-2n_1)} \otimes \ldots \otimes \Lambda_{n_r+\epsilon_r(N-2n_r)}$$

where $\epsilon_i \in \{0,1\}$ $\forall i = 1, \ldots, r$ and the convention is taken that $\epsilon_i = 1$ if the $i^{th}$ antisymmetric block is dualized and $\epsilon_i = 0$ otherwise, and where $\mathcal{P} = 1, \ldots, 2^r$ is defined by $\mathcal{P} = 1 + \sum_{i=1}^{r} 2^i \epsilon_i$. Thus, there are $2^r$ different Hodge duals for any $r$-fold form $A$ and they can be adequately written as $A_{\mathcal{P}} \equiv *_{\mathcal{P}} A$. One also needs a product $\odot$ of $A$ by itself resulting in a $2r$-covariant tensor, given
by

\[(A \odot A) (\vec{x}_1, \vec{y}_1, \ldots, \vec{x}_r, \vec{y}_r) \equiv \left( \prod_{i=1}^{r} \frac{1}{(n_i - 1)!} \right) g \left( i_{\vec{x}_1, \ldots, \vec{x}_r} A, i_{\vec{y}_1, \ldots, \vec{y}_r} A \right) \]

where for any tensor \(B\) we write \(g (B, B) \in \mathbb{R}\) for the complete contraction in all indices in order.

**Definition 2.1** [4, 5] The basic superenergy tensor of \(A\) is defined to be

\[T \{ A \} = \frac{1}{2} \sum_{P=1}^{2^r} A_P \odot A_P.\]

Here the word basic is used because linear combinations of \(T \{ A \}\) with its permutted versions maintain most of its properties; however, the completely symmetric part is unique (up to a factor of proportionality) [4]. It is remarkable that one can provide an explicit expression for \(T \{ A \}\) which is independent of the dimension \(N\), see [4]. In the case of a general \(p\)-form \(\Omega\), its rank-2 superenergy tensor becomes

\[T \{ \Omega \} (\vec{x}, \vec{y}) = \frac{1}{(p-1)!} \left[ g \left( i_{\vec{x}} \Omega, i_{\vec{y}} \Omega \right) - \frac{1}{2p} g (\Omega, \Omega) g (\vec{x}, \vec{y}) \right]. \tag{1} \]

In Definition 2.1 we implicitly assumed that the \(r\)-fold form \(A\) has no antisymmetric blocks of maximum degree \(N\). Nevertheless, the above expression (1) is perfectly well defined for \(N\)-forms: if \(\Omega = f \eta\) where \(\eta\) is the canonical volume form and \(f\) a scalar, then (1) gives \(T \{ \Omega \} = -\frac{1}{2} f^2 g\). Using this the Definition 2.1 is naturally extended to include \(N\)-blocks, see [4] for details.

In \(N = 4\), the superenergy tensor of a 2-form \(F\) is its Maxwell energy-momentum tensor, and the superenergy tensor of an exact 1-form \(d\phi\) has the form of the energy-momentum tensor for a massless scalar field \(\phi\). If we compute the superenergy tensor of \(R\) we get the so-called Bel tensor [6]. The superenergy tensor of the Weyl curvature tensor is the well-known Bel-Robinson tensor [7, 8, 9]. The main properties of \(T \{ A \}\) are [4]:

**Property 2.1** If \(A\) is an \(r\)-fold form, then \(T \{ A \}\) is a \(2r\)-covariant tensor.

**Property 2.2** \(T \{ A \}\) is symmetric on each pair of entries, that is, for all \(i = 1, \ldots, r\) one has

\[T \{ A \} (\vec{x}_1, \ldots, \vec{x}_{2i-1}, \vec{x}_{2i}, \ldots, \vec{x}_{2r}) = T \{ A \} (\vec{x}_1, \ldots, \vec{x}_{2i}, \vec{x}_{2i-1}, \ldots, \vec{x}_{2r}).\]
Property 2.3 $T\{A\} = T\{A_P\}$ $\forall P = 1, \ldots, 2^r$.

Property 2.4 $T\{A\} = T\{-A\}$; $T\{A\} = 0 \iff A = 0$.

Property 2.5 $T\{A \otimes B\} = T\{A\} \otimes T\{B\}$.

Property 2.6 $T\{A\} \in \mathcal{D}P^+$.

Observe that property 2.6 is what we were seeking, so that $T\{A\}$ is the “positive square” of $A$ in the causal sense.

Property 2.7 $T\{A\}(\vec{e}_0, \ldots, \vec{e}_0) = \frac{1}{2} \sum_{\alpha_1, \ldots, \alpha_m = 0}^{N-1} (A(\vec{e}_{\alpha_1}, \ldots, \vec{e}_{\alpha_m}))^2$ in orthonormal bases $\{\vec{e}_0, \ldots, \vec{e}_{N-1}\}$.

The set of superenergy tensors somehow build up the class $\mathcal{D}P$; in fact, in many occasions the rank-2 superenergy tensors (that is, those for single $p$-forms) are the basic building blocks of the whole $\mathcal{D}P$ \footnote{This can be seen as follows.}

Definition 2.2 An $r$-fold form $A$ is said to be decomposable if there are $r$ forms $\Omega_i$ ($i = 1, \ldots, r$) such that $\tilde{A} = \Omega_1 \otimes \ldots \otimes \Omega_r$.

From this and Property 2.5 one derives

Corollary 2.1 If $A$ is decomposable, then $T\{A\} = T\{\Omega_1\} \otimes \ldots \otimes T\{\Omega_r\}$.

Notice that each of the $T\{\Omega_i\}$ on the righthand side is a rank-2 tensor. We now have

Theorem 2.1 Any symmetric $T \in \mathcal{D}P^+_2$ can be decomposed as

$$T = \sum_{p=1}^{N} T\{\Omega_p\}$$

where $\Omega_p$ are simple $p$-forms such that for $p > 1$ they have the structure $\Omega_p = k_1 \wedge \ldots \wedge k_p$ where $k_1, \ldots, k_p$ are appropriate null 1-forms and $\Omega_1 \in \mathcal{D}P_1$.

See \footnote{From this point on I shall use the standard notation $T^2$ instead of $T_1 \times_1 T = T_2 \times_2 T = T_2 \times_1 T = T_1 \times_2 T$ for the case of rank-2 symmetric tensors $T$. $T^2$ is symmetric.} for the detailed structure of the above decomposition and for the relation between $T$ and the null 1-forms. From this one obtains\footnote{This can be seen as follows.}
Theorem 2.2 A symmetric rank-2 tensor $T$ satisfies $T^2 = g$ if and only if $T = \pm T\{\Omega_p\}$, i.e., if $T$ is up to sign the superenergy tensor of a simple $p$-form $\Omega_p$. Moreover, the rank $p$ of the $p$-form is given by $\pm \text{tr} T = 2p - N$.

This important theorem allows to classify all Lorentz transformations and, in more generality, all maps which preserve the null cone $\partial \Theta_x$ at $x \in V$, see [1].

3 Applications

In this section several applications of superenergy and causal tensors are presented. They include both mathematical and physical ones.

3.1 Rainich’s conditions

The classical Rainich conditions [10, 11] are necessary and sufficient conditions for a spacetime to originate via Einstein’s equations in a Maxwell electromagnetic field. They are of two kinds: algebraic and differential. Here I am only concerned with the algebraic part which nowadays are presented as follows (see, e.g., [12]):

(Classical Rainich’s conditions) The Einstein tensor $G = \text{Ric} - \frac{1}{2} S g$ of a 4-dimensional spacetime is proportional to the energy-momentum tensor of a Maxwell field (a 2-form) if and only if $G^2 \propto g$, $\text{tr} G = 0$ and $G \in \mathcal{D}P^+_2$.

In fact, from theorem 2.2 one can immediately improve a little this classical result

Corollary 3.1 In 4 dimensions, $G$ is algebraically up to sign proportional to the energy-momentum tensor of a Maxwell field if and only if $G^2 \propto g$, $\text{tr} G = 0$. Furthermore, $G$ is proportional to the energy-momentum tensor of (possibly another) Maxwell 2-form which is simple.

The last part of this corollary is related to the so-called duality rotations of the electromagnetic field [12]. Observe that this is clearly a way of determining physics from geometry because, given a particular spacetime, one only has to compute its Einstein tensor and check the above simple conditions. If they hold, then the energy-momentum tensor is that of a 2-form (and for a complete result the Rainich differential conditions will then be needed).

The classical Rainich conditions are based on a dimensionally-dependent identity, see [13], valid only for $N = 4$. However, theorem 2.2 has universal validity and can be applied to obtain the generalization of Rainich’s conditions in many cases. For instance, we were able to derive the following results [4].

Corollary 3.2 In $N$ dimensions, a rank-2 symmetric tensor $T$ is algebraically the energy-momentum tensor of a minimally coupled massless scalar field $\phi$ if
and only if $T^2 \propto g$ and $\text{tr}T = \beta \sqrt{\text{tr}T^2/N}$ where $\beta = \pm (N-2)$. Moreover, $d\phi$ is spacelike if $\beta = 2 - N$ and $\text{tr}T \neq 0$, timelike if $\beta = N - 2$ and $\text{tr}T \neq 0$, and null if $\text{tr}T = 0 = \text{tr}T^2$.

**Corollary 3.3** In $N$ dimensions, a rank-2 symmetric tensor $T$ is the energy-momentum tensor of a perfect fluid satisfying the dominant energy condition if and only if there exist two positive functions $\lambda, \mu$ such that

$$T^2 = 2\lambda T + (\mu^2 - \lambda^2) g, \quad \text{tr}T = (N-2)\mu - \lambda N.$$ 

This is an improvement and a generalization to arbitrary $N$ of the conditions in [14] for $N = 4$. In particular, the case of dust can be deduced from the previous one by setting the pressure of the perfect fluid equal to zero.

**Corollary 3.4** In $N$ dimensions, a symmetric tensor $T$ is algebraically the energy-momentum tensor of a dust satisfying the dominant energy condition if and only if

$$T^2 = (\text{tr}T)T, \quad \text{tr}T < 0.$$ 

### 3.2 Causal propagation of fields

Following a classical reasoning appearing in [3], the causal propagation of arbitrary fields can be studied by simply using its superenergy tensor, see [15]. Let $\zeta$ be any closed achronal set in $V$ and $D(\zeta)$ its total Cauchy development (an overbar over a set denotes its closure, see [3, 16] for definitions and notation).

**Theorem 3.1** If the tensor $T\{A\}$ satisfies the following divergence inequality

$$\text{div}T\{A\}(\vec{x},\ldots,\vec{x}) \leq fT\{A\}(\vec{x},\ldots,\vec{x})$$

where $f$ is a continuous function and $\vec{x} = g^{-1}(\vec{x},-d\tau)$ is any timelike vector foliating $D(\zeta)$ with hypersurfaces $\tau = \text{const.}$, then

$$A|_{\zeta} = 0 \quad \implies \quad A|_{\overline{D(\zeta)}} = 0.$$ 

This theorem proves the causal propagation of the field $A$ because if $A \neq 0$ at a point $x \notin \overline{D(\zeta)}$ arbitrarily close to $\overline{D(\zeta)}$, then $A$ will propagate in time from $x$ according to its field equations, but it will never be able to enter into $\overline{D(\zeta)}$, showing that $A$ cannot travel faster than light.

The divergence condition in the theorem, being an inequality, is very mild and it is very easy to check whether or not is valid for a given field satisfying
field equations. In general, of course, it will work for linear field equations, and for many other cases too. It has been used to prove the causal propagation of gravity in vacuum \([17]\) or in general \(N\)-dimensional Lorentzian manifolds conformally related to Einstein spaces \([15]\), and also for the massless spin-\(n/2\) fields in General Relativity \([15]\). It must be stressed that in many occasions the standard energy-momentum tensor of the field does not allow to prove the same result, so that the universality of the superenergy construction reveals itself as essential in this application.

3.3 Propagation of discontinuities: conserved quantities

Several ways to derive conserved quantities and exchange of superenergy properties have been pursued. One of them is the construction of divergence-free vector fields, called currents. This has been successfully achieved in the case of a minimally coupled scalar field if the Einstein-Klein-Gordon field equations hold, see \([4, 18]\). In this subsection the propagation of discontinuities of the electromagnetic and gravitational fields will be analyzed from the superenergy point of view. This will be enough to prove the interchange of superenergy quantities between these two physical fields and some conservation laws arising naturally when the field has a ‘wave-front’, see \([4, 18]\).

To that end, we need to recall some well-known basic properties of the wave-fronts, which are null hypersurfaces. Let \(\sigma\) be such a null hypersurface and \(n\) a 1-form normal to \(\sigma\). Obviously, \(n\) is null \(g^{-1}(n, n) = 0\) and therefore \(\tilde{n} \equiv g^{-1}(\cdot, n)\) is in fact a vector tangent to \(\sigma\), see e.g. \([19]\), and \(n\) cannot be normalized so that it is defined up to a transformation of the form

\[
 n \longrightarrow \rho n, \quad \rho > 0. \tag{2}
\]

The null curves tangent to \(\tilde{n}\) are null geodesics \(\nabla_{\tilde{n}} \tilde{n} = \Psi \tilde{n}\), called ‘bicharacteristics’, contained in \(\sigma\). Let \(\bar{g}\) denote the first fundamental form of \(\sigma\), which is a degenerate metric because \(\bar{g}(\tilde{n}, \cdot) = 0\) \([3, 19, 16]\). The second fundamental form of \(\sigma\) relative to \(n\) can be defined as

\[
 K \equiv \frac{1}{2} \mathcal{L}_{\tilde{n}} \bar{g}
\]

where \(\mathcal{L}_{\tilde{n}}\) denotes the Lie derivative with respect to \(\tilde{n}\) within \(\sigma\). \(K\) is intrinsic to the null hypersurface \(\sigma\) and shares the degeneracy with \(\bar{g}\): \(K(\tilde{n}, \cdot) = 0\) \([19]\). Because of this, and even though \(\bar{g}\) has no inverse, one can define the “trace” of \(K\) by contracting with the inverse of the metric induced on the quotient spaces \(T(\sigma)/<\tilde{n}>\). This trace will be denoted by \(\vartheta\) and has the following interpretation: if \(s \subset \sigma\) denotes any spacelike cut of \(\sigma\), that is, a spacelike \((N - 2)\)-surface orthogonal to \(n\) and within \(\sigma\), then \(\vartheta\) measures the volume
expansion of $s$ along the null generators of $\sigma$. In fact, $\vartheta$ can be easily related to the derivative along $\vec{n}$ of the $(N - 2)$-volume element of $s$ \cite{16}.

Let us consider the case when there is an electromagnetic field (a 2-form $F$) propagating in a background spacetime so that there is a null hypersurface of discontinuity $\sigma$ \cite{20, 21}, called a ‘characteristic’. Let $[E]_\sigma$ denote the discontinuity of any object $E$ across $\sigma$. Using the classical Hadamard results \cite{22, 20, 21}, one can prove the existence of a 1-form $c$ on $\sigma$ such that

$$[F]_\sigma = n \wedge c, \quad g^{-1}(n, c) = 0.$$ 

Observe that $c$ transforms under the freedom \cite{2} as $c \rightarrow c/\rho$. From Maxwell’s equations for $F$ considered in a distributional sense one derives a propagation law \cite{21, 18}, or ‘transport equation’ \cite{20},

$$\vec{n} \left( |c|^2 \right) + |c|^2 (\vartheta + 2\Psi) = 0, \quad |c|^2 \equiv g^{-1}(c, c) \geq 0.$$ 

This propagation equation implies that if $c|_x = 0$ at any point $x \in \sigma$, then $c = 0$ along the null geodesic originated at $x$ and tangent to $\vec{n}$. Moreover, for arbitrary conformal Killing vectors $\vec{\zeta}_1, \vec{\zeta}_2$, the above propagation law allows to prove that \cite{20, 18}

$$\int_s |c|^2 n(\vec{\zeta}_1)n(\vec{\zeta}_2) \omega|_s$$

are conserved quantities along $\sigma$, where $\omega|_s$ is the canonical volume element $(N - 2)$-form of $s$, in the sense that the integral is independent of the cut $s$ chosen. Notice also that \cite{3} are invariant under the transformation \cite{2}. These conserved quantities can be easily related to the energy-momentum properties of the electromagnetic field because $T \{ [F]_\sigma \} = |c|^2 n \otimes n$, so that the Maxwell tensor of the discontinuity $[F]_\sigma$ contracted with the conformal Killing vectors $\vec{\zeta}_1, \vec{\zeta}_2$ gives the function integrated in \cite{3}.

However, the integral \cite{3} vanishes when $[F]_\sigma = 0 \iff c = 0$. Using again Hadamard’s theory, now there exist a 2-covariant symmetric tensor $B$ and a 1-form $f$ defined only on $\sigma$ such that \cite{8, 23, 21, 13}

$$[R]_\sigma = n \wedge B \wedge n, \quad B(\vec{n}, \ ) + \text{tr} B n = 0,$$

$$[\nabla F]_\sigma = n \otimes (n \wedge f), \quad g^{-1}(n, f) = 0.$$ 

These objects transform under \cite{4} according to $f, B \rightarrow f/\rho^2, B/\rho^2$. Assuming that the Einstein-Maxwell equations hold Lichnerowicz deduced the propagation laws for $f, B$ in \cite{21}, and in particular he found the following transport equation \cite{21, 4}

$$\vec{n} \left( |B|^2 + |f|^2 \right) + (|B|^2 + |f|^2) (\vartheta + 4\Psi) = 0,$$
where \(|f|^2 \equiv g^{-1}(f,f) \geq 0, |B|^2 \equiv \text{tr} B^2 \geq 0\). Once again, with the help of any conformal Killing vectors \(\zeta_1, \ldots, \zeta_4\), the following quantities

\[
\int_s (|B|^2 + |f|^2) \, n(\zeta_1)n(\zeta_2)n(\zeta_3)n(\zeta_4) \, \omega|_s
\]

are conserved along \(\sigma\) in the sense that the integral is independent of the spacelike cut \(s\). Two important points can be derived from this relation: first, both the electromagnetic and gravitational contributions are necessary, so that neither the integrals involving only \(|B|^2\) or only \(|f|^2\) are equal for different cuts \(s\) in general. Second, the integrand in (4) is related to superenergy tensors because

\[
\frac{1}{2} (T\{[R]_\sigma\} + T\{[\nabla F]_\sigma\}) (\zeta_1, \zeta_2, \zeta_3, \zeta_4)
\]

which demonstrates the interplay between superenergy quantities of different fields, in this case the electromagnetic and gravitational ones. Observe that the above tensors are completely symmetric in this case, and that they together with the conserved quantity (4) are invariant under the transformation (2).

### 3.4 Causal relations

The fact that the tensors in \(\mathcal{DP}_2\) can be seen as linear mappings preserving the Lorentz cone leads in a natural way to consider the possibility of relating different Lorentzian manifolds at their corresponding causal levels, even before the metric properties are taking into consideration. To that end, using the standard notation \(\varphi^*\) and \(\varphi_*\) for the push-forward and pull-back mappings, respectively, we give the next

**Definition 3.1** Let \(\varphi : V \to W\) be a global diffeomorphism between two Lorentzian manifolds. \(W\) is said to be properly causally related with \(V\) by \(\varphi\), denoted \(V \prec_\varphi W\), if \(\forall \, \bar{u} \in \Theta^+(V), \varphi^* \bar{u} \in \Theta^+(W)\). \(W\) is simply said to be properly causally related with \(V\), denoted by \(V \preceq W\), if \(\exists \varphi\) such that \(V \prec_\varphi W\).

In simpler terms, what one demands here is that the solid Lorentz cones at all \(x \in V\) are mapped by \(\varphi\) to sets contained in the solid Lorentz cones at \(\varphi(x) \in W\) keeping the time orientation: \(\varphi^* \Theta^+_x \subseteq \Theta^+_{\varphi(x)}\), \(\forall \, x \in V\).

Observe that two Lorentzian manifolds can be properly causally related by some diffeomorphisms but not by others. As a simple example, consider \(\mathbb{L}\).
with typical Cartesian coordinates $x^0, \ldots, x^{N-1}$ (the 0-index indicating the time coordinate) and let $\varphi_q$ be the diffeomorphisms

$$
\varphi_q : \mathbb{L} \rightarrow \mathbb{L} \\
(x^0, \ldots, x^{N-1}) \rightarrow (q x^0, \ldots, x^{N-1})
$$

for any constant $q \neq 0$. It is easily checked that $\varphi_q$ is a proper causal relation for all $q \geq 1$, but is not for all $q < 1$. Thus $\mathbb{L} \prec \mathbb{L}$ but, say, $\mathbb{L} \not\prec_{1/2} \mathbb{L}$. Notice also that for $q \leq -1$ the diffeomorphisms $\varphi_q$ change the time orientation of the causal vectors, but still $\varphi' \Theta_x \subseteq \Theta_{\varphi(x)}$ (with $\varphi' \Theta_x^+ \subseteq \Theta_{\varphi(x)}^-$).

Proper causal relations can be easily characterized by some equivalent simple conditions.

**Theorem 3.2** The following statements are equivalent:

1. $V \prec \mathbb{W}$.
2. $\varphi^* (\mathcal{D} \mathcal{P}_r^+(W)) \subseteq \mathcal{D} \mathcal{P}_r^+(V)$ for all $r \in \mathbb{N}$.
3. $\varphi^* (\mathcal{D} \mathcal{P}_1^+(W)) \subseteq \mathcal{D} \mathcal{P}_1^+(V)$.
4. $\varphi^* h \in \mathcal{D} \mathcal{P}_2^-(V)$ where $h$ is the metric of $W$ up to time orientation.

**Proof:**

1 $\Rightarrow$ 2: let $T \in \mathcal{D} \mathcal{P}_r^+(W)$, then $(\varphi^* T)(\vec{u}_1, \ldots, \vec{u}_r) = T(\varphi' \vec{u}_1, \ldots, \varphi' \vec{u}_r) \geq 0$

for all $\vec{u}_1, \ldots, \vec{u}_r \in \Theta^+(V)$ given that $\varphi' \vec{u}_1, \ldots, \varphi' \vec{u}_r \in \Theta^+(W)$ by assumption. Thus $\varphi^* T \in \mathcal{D} \mathcal{P}_r^+(V)$.

2 $\Rightarrow$ 3: Trivial.

3 $\Rightarrow$ 4: Choose any basis $w^1, \ldots, w^N$ of $T^*(W)$ such that $w^1, \ldots, w^N \in \mathcal{D} \mathcal{P}_1^+(W)$. Then $h = \sum_{\mu, \nu=1}^N h_{\mu \nu} w^\mu \otimes w^\nu$ with $h_{\mu \nu} < 0$ for all $\mu, \nu$. Thus $\varphi^* h = \sum_{\mu, \nu=1}^N (\varphi^* h_{\mu \nu}) \varphi^* w^\mu \otimes \varphi^* w^\nu$ with $\varphi^* w^\mu \in \mathcal{D} \mathcal{P}_1^+(V)$ by hypothesis and $\varphi^* h_{\mu \nu} = h_{\mu \nu}(\varphi) < 0$. Then, $\varphi^* h \in \mathcal{D} \mathcal{P}_2^-(V)$.

4 $\Rightarrow$ 1: For every $\vec{u} \in \Theta^+(V)$ we have that $(\varphi^* h)(\vec{u}, \vec{u}) = h(\varphi' \vec{u}, \varphi' \vec{u}) \leq 0$ and hence $\varphi' \vec{u} \in \Theta(W)$. Besides, for any other $\vec{v} \in \Theta^+(V)$, $h(\varphi' \vec{u}, \varphi' \vec{v}) \leq 0$ so that every two vectors with the same time orientation are mapped to vectors with the same time orientation. However, it could happen that $\Theta^+(V)$ is actually mapped to $\Theta^-(W)$, and $\Theta^-(V)$ to $\Theta^+(W)$.

By changing the time orientation of $W$, if necessary, the result follows.

Condition 4 in this theorem can be replaced by

4'. $\varphi^* h \in \mathcal{D} \mathcal{P}_2^-(V)$ and $\varphi' \vec{u} \in \Theta^+(W)$ for at least one $\vec{u} \in \Theta^+(V)$.

Leaving this time-orientation problem aside (in the end, condition 4 just means that $W$ with one of its time orientations is properly causally related with $V$),
let us stress that the condition 4 (or 4') is very easy to check and thereby extremely valuable in practical problems: first, one only has to work with one tensor field \( h \), and also as we saw in proposition 1.1 there are several simple ways to check whether \( \varphi^* h \in \mathcal{DP}_2^-(V) \) or not.

The combination of theorem 2.1 and condition 4 in theorem 3.2 provides a classification of the proper causal relations according to the number of independent null vectors which are mapped by \( \varphi \) to null vectors at each point. The key result here is that

**Proposition 3.1** Let \( V \prec \varphi W \) and \( \vec{u} \in \Theta_x^+ \), \( x \in V \). Then \( \varphi' \vec{u} \) is null at \( \varphi(x) \in W \) if and only if \( \vec{u} \) is a null eigenvector of \( \varphi^* h |_{x} \).

**Proof:** See [24]. Recall that \( \vec{u} \) is called an “eigenvector” of a 2-covariant tensor \( T \) if \( T(\vec{u},\vec{v}) = \lambda g(\vec{u},\vec{v}) \) and \( \lambda \) is then the corresponding eigenvalue.

The vectors which remain null under the causal relation \( \prec \) are called its canonical null directions, and there are at most \( N \) of them linearly independent. Hence, using theorem 2.1 one can see that essentially are \( N \) different types of proper causal relations, and that the conformal relations are included as the particular case in which all null directions are canonical [24].

Clearly \( V \prec V \) for all \( V \) (by just taking the identity mapping). Moreover

**Proposition 3.2** \( V \prec W \) and \( W \prec U \Longrightarrow V \prec U \).

**Proof:** Consider any \( \vec{u} \in \Theta^+(V) \). Since there are \( \varphi, \psi \) such that \( V \prec_\varphi W \) and \( W \prec_\psi U \), then \( \varphi' \vec{u} \in \Theta^+(W) \) and \( \psi' [\varphi' \vec{u}] \in \Theta^+(U) \) so that \( (\psi \circ \varphi)' \vec{u} \in \Theta^+(U) \) from where \( V \prec U \).

It follows that the relation \( \prec \) is a preorder for the class of all Lorentzian manifolds. This is not a partial order as \( V \prec W \) and \( W \prec V \) does not imply that \( V = W \) and, actually, it does not even imply that \( V \) is conformally related to \( W \). The point here is that \( V \prec_\varphi W \) and \( W \prec_\psi V \) can perfectly happen with \( \psi \neq \varphi^{-1} \). In the case that \( V \prec_\varphi W \) and \( W \prec_\varphi^{-1} V \) then necessarily \( \varphi \) is a conformal relation and \( \varphi^* h = e^{2f} g \), but we are dealing with more general and basic causal equivalences.

**Definition 3.2** Two Lorentzian manifolds \( V \) and \( W \) are called causally isomorphic, denoted by \( V \sim W \), if \( V \prec W \) and \( W \prec V \).

If \( V \sim W \) then the causal structures in both manifolds are somehow the same. Clearly, \( \sim \) is an equivalence relation, and now one can obtain a partial order \( \preceq \) for the corresponding classes of equivalence \( \text{coset}(V) \equiv \{ U : V \sim U \} \), by setting

\[
\text{coset}(V) \preceq \text{coset}(W) \iff V \prec W.
\]
This partial order provides chains of (classes of equivalence of) Lorentzian manifolds which keep “improving” the causal properties of the spacetimes. To see this, we need the following (see [3, 16] for definitions)

**Proposition 3.3** Let $V \prec W$. Then, if $V$ violates any of the following

1. the chronology condition,
2. the causality condition,
3. the future-distinguishing condition (or the past one),
4. the strong causality condition,
5. the stable causality condition,
6. the global hyperbolicity condition,

so does $W$.

**Proof:** For 1 to 4, let $\gamma$ be a causal curve responsible for the given violation of causality (that is, a closed timelike curve for 1, or a curve cutting any neighbourhood of a point in a disconnected set for 4, and so on). Then, $\varphi(\gamma)$ has the corresponding property in $W$. To prove 5, if there were a function $f$ in $W$ such that $-df \in \mathcal{DP}_1^+(W)$, from theorem 3.2 point 3 it would follow that $d(\varphi^*f) = \varphi^*df \in \mathcal{DP}_1^-(V)$ so that $\varphi^*f$ would also be a time function. Finally, 6 follows from corollary 3.1 in [24].

With this result at hand, we can build the afore-mentioned chains of spacetimes, such as

$$\text{coset}(V) \preceq \ldots \preceq \text{coset}(W) \preceq \ldots \preceq \text{coset}(U) \preceq \ldots \preceq \text{coset}(Z)$$

where the spacetimes satisfying stronger causality properties are to the left, while those violating causality properties appear more and more to the right. This is natural because the light cones “open up” under a causal mapping. The actual properties depend on the particular chain and its length, but an optimal one would start with a $V$ which is globally hyperbolic, and then it could pass through a $W$ which is just causally stable, then $U$ could be causal, say, and the last step $Z$ could be a totally vicious spacetime [16].

Perhaps the above results can be used to give a first fundamental characterization of asymptotically equivalent spacetimes, at a level prior to the existence of the metric, which might then be included in a subsequent step. This could be accomplished by means of the following tentative definitions, which may need some refinement.
Definition 3.3 An open set \( \zeta \subset V \) is called a neighbourhood of

1. the causal boundary of \( V \) if \( \zeta \cap \gamma \neq \emptyset \) for all endless causal curves \( \gamma \);
2. a singularity set \( S \) if \( \zeta \cap \gamma \neq \emptyset \) for all endless causal curves \( \gamma \) which are incomplete towards \( S \);
3. the causal infinity if \( \zeta \cap \gamma \neq \emptyset \) for all complete causal curves \( \gamma \).

(See [16] for definition of boundaries, singularity sets, etcetera).

Definition 3.4 \( W \) is said to be causally asymptotically \( V \) if any two neighbourhoods of the causal infinity \( \zeta \subset V \) and \( \tilde{\zeta} \subset W \) contain corresponding neighbourhoods \( \zeta' \subset \zeta \) and \( \tilde{\zeta}' \subset \tilde{\zeta} \) of the causal infinity such that \( \zeta' \sim \tilde{\zeta}' \).

Similar definitions can be given for \( W \) having causally the singularity structure of \( V \), or the causal boundary of \( V \), replacing in the given definition the neighbourhoods of the causal infinity by those of the singularity and of the causal boundary, respectively. The usefulness of these investigations is still unclear.

3.5 Causal transformations and generalized symmetries

Here the natural question of whether the causal relations can be used to define a generalization of the group of conformal motions is analyzed. To start with, we need a basic concept.

Definition 3.5 A transformation \( \varphi : V \rightarrow V \) is called causal if \( V \prec \varphi V \).

The set of causal transformations of \( V \) is written as \( \mathcal{C}(V) \).

\( \mathcal{C}(V) \) is a subset of the group of transformations of \( V \). In fact, from the proof of proposition 3.2 follows that \( \mathcal{C}(V) \) is closed under the composition of diffeomorphisms. As the identity map is also clearly in \( \mathcal{C}(V) \) its algebraic structure is that of a submonoid, see e.g. [23], of the group of diffeomorphisms of \( V \). However, \( \mathcal{C}(V) \) generically fails to be a subgroup, because (see [24] for the proof):

Proposition 3.4 Every subgroup of causal transformations is a group of conformal motions.

From standard results, see [23], one identifies \( \mathcal{C}(V) \cap \mathcal{C}(V)^{-1} \) as the group of conformal motions of \( V \) and there is no other subgroup of \( \mathcal{C}(V) \) containing \( \mathcal{C}(V) \cap \mathcal{C}(V)^{-1} \). The transformations in \( \mathcal{C}(V) \setminus (\mathcal{C}(V) \cap \mathcal{C}(V)^{-1}) \) are called proper causal transformations.
Take now a one-parameter group of causal transformations \( \{ \varphi_t \}_{t \in \mathbb{R}} \). From proposition 3.4 it follows that \( \{ \varphi_t \} \) must be in fact a group of conformal motions, and its infinitesimal generator is a conformal Killing vector, so that nothing new is found here. Nevertheless, one can generalize naturally the conformal Killings by building one-parameter groups of transformations \( \{ \varphi_t \} \) such that only part of them are causal transformations. Given that the problem arises because both \( \varphi_t \) and \( \varphi_{-t} = \varphi_t^{-1} \) belong to the family and thus they would both be conformal if they are both causal, one readily realizes that the natural generalization is to assume that \( \{ \varphi_t \}_{t \in \mathbb{R}} \) is such that either \( \{ \varphi_t \}_{t \in \mathbb{R}^+} \) or \( \{ \varphi_t \}_{t \in \mathbb{R}^-} \) is a subset of \( \mathcal{C}(V) \), but only one of the two. Any group \( \{ \varphi_t \}_{t \in \mathbb{R}} \) with this property is called a maximal one-parameter submonoid of proper causal transformations. Of course, the one-parameter submonoid can just be a local one so that the transformations are defined only for some interval \( I = (-\epsilon, \epsilon) \in \mathbb{R} \) and only those with \( t \in (0, \epsilon) \) (or \( t \in (-\epsilon, 0) \)) are proper causal transformations.

Let then \( \{ \varphi_t \}_{t \in I} \) be a local one-parameter submonoid of proper causal transformations, and assume that \( t \geq 0 \) provides the subset of proper causal transformations (otherwise, just change the sign of \( t \)). The infinitesimal generator of \( \{ \varphi_t \}_{t \in I} \) is defined as the vector field
\[
\xi = \left. \frac{d \varphi_t}{dt} \right|_{t=0}
\]
so that for every covariant tensor field \( T \) one has
\[
\left. \frac{d(\varphi_t^* T)}{dt} \right|_{t=0} = \mathcal{L}_\xi T
\]
where \( \mathcal{L}_\xi \) denotes the Lie derivative with respect to \( \xi \). As \( \{ \varphi_t \}_{t \geq 0} \) are proper causal transformations, and using point 2 in theorem 3.2, one gets \( \varphi_t^* T \in \mathcal{D} \mathcal{P}_r^+ \) for \( t \geq 0 \) and for all tensor fields \( T \in \mathcal{D} \mathcal{P}_r^+ \). In particular,
\[
\varphi_t^* T (\vec{u}_1, \ldots, \vec{u}_r) \geq 0, \quad \forall \vec{u}_1, \ldots, \vec{u}_r \in \Theta^+, \quad \forall T \in \mathcal{D} \mathcal{P}_r^+, \quad t \geq 0,
\]
(5)
from where we can derive the next result.

**Lemma 3.1** Let \( T \in \mathcal{D} \mathcal{P}_r^+ \) and \( \vec{k} \in \Theta^+ \) be such that \( T(\vec{k}, \ldots, \vec{k}) = 0 \). If \( \varphi_t \in \mathcal{C}(V) \) for \( t \in [0, \epsilon) \), then
\[
(\mathcal{L}_\xi T)(\vec{k}, \ldots, \vec{k}) \geq 0.
\]
Superenergy tensors

Proof: Under the conditions of the lemma, and due to points 2 and 3 of proposition [1.1], it is necessary that $\vec{k}$ is null, that is, $\vec{k} \in \partial \Theta^+$. From formula (3) one obtains $\varphi_t^* T(\vec{k}, \ldots, \vec{k}) \geq 0$ for all $t \in [0, \epsilon)$. But $\varphi_0$ is the identity transformation, so $\varphi_0^* T(\vec{k}, \ldots, \vec{k}) = T(\vec{k}, \ldots, \vec{k}) = 0$, from where necessarily follows that $\varphi_t^* T(\vec{k}, \ldots, \vec{k})$ is a non-decreasing function of $t$ at $t = 0$, that is to say, $(d/dt)(\varphi_t^* T(\vec{k}, \ldots, \vec{k}))|_{t=0} \geq 0$.

Corollary 3.5 Let $\vec{\xi}$ be the infinitesimal generator of a local one-parameter submonoid of proper causal transformations $\{\varphi_t\}_{t \in I}$ and choose the sign of $t$ such that $\{\varphi_t\}_{t \geq 0} \subset C(V)$. Then

$$(\mathcal{L}_{\vec{\xi}} g)(\vec{k}, \vec{k}) \leq 0, \quad \forall \vec{k} \in \partial \Theta$$

Proof: Obviously $g(\vec{k}, \vec{k}) = 0$ for all null $\vec{k}$, and also $g \in \mathcal{DP}_2^-$, so lemma 3.1 can be applied to $-g$ and the result follows.

This result is a generalization of the condition for conformal Killing vectors $(\mathcal{L}_{\vec{\xi}} g \propto g)$ and can be analyzed in a similar manner. As a matter of fact, the application of the decomposition theorem 2.1 to $\varphi_t^* g \in \mathcal{DP}_2^-$ leads to a much stronger result which allows for a complete characterization of the vector fields $\vec{\xi}$ and their properties.

Theorem 3.3 Let $\vec{\xi}$ be the infinitesimal generator of a local one-parameter submonoid of proper causal transformations $\{\varphi_t\}_{t \in I}$ and choose the sign of $t$ such that $\{\varphi_t\}_{t \geq 0} \subset C(V)$. Then there is a function $\psi$ such that

$$\left[\mathcal{L}_{\vec{\xi}} g - 2\psi g\right] \in \mathcal{DP}_2^-.$$

Proof: From theorem 2.1 and given that $\varphi_t^* g \in \mathcal{DP}_2^-$ for $t \in [0, \epsilon)$ one has

$$\varphi_t^* g = - \sum_{p=1}^N T_t\{\Omega_p\} = - \sum_{p=1}^{N-1} T_t\{\Omega_p\} + \Psi_t^2 g$$

where $T_t\{\Omega_p\}$ are superenergy tensors of simple $p$-forms for all values of $t \in [0, \epsilon)$ and $\Psi_t$ are functions on $V$ with $\Psi_0 = 1$. Then we have $\varphi_t^* g(\vec{u}, \vec{v}) \leq \Psi_t^2 g(\vec{u}, \vec{v}) \leq 0$ for all $\vec{u}, \vec{v} \in \Theta^+$, or equivalently,

$$\Psi_t^{-2} \varphi_t^* g(\vec{u}, \vec{v}) \leq g(\vec{u}, \vec{v}) = \Psi_0^{-2} \varphi_0^* g(\vec{u}, \vec{v}) \leq 0$$
from where a reasoning similar to that in lemma 3.1, by taking the derivative with respect to \( t \) at \( t = 0 \), gives

\[
\left[ \mathcal{L}_\xi g - 2\Phi g \right](\bar{u}, \bar{v}) \leq 0, \quad \forall \bar{u}, \bar{v} \in \Theta^+
\]

where \( \Phi \equiv d\Psi_t/dt|_{t=0} \).

This set of vector fields generalize the traditional (conformal) symmetries and the previous theorem together with theorem 2.1 provides first a definition of generalized symmetries, and second its full classification because \( \mathcal{L}_\xi g - 2\psi g \) itself can be written as a sum of superenergy tensors of simple \( p \)-forms. The number of independent null eigenvectors of \( \mathcal{L}_\xi g - 2\psi g \) (ranging from 0 to \( N \)) gives the desired classification, where \( N \) corresponds to the conformal Killing vectors. This is under current investigation. It must be remarked that the above theorem does not provide a sufficient condition for a vector field to generate locally a one-parameter submonoid of causal transformations.

Several examples of generalized Killing vectors in this sense can be presented. One of them is a particular case of a previous partial generalization of isometries considered in [26] and called Kerr-Schild vector fields. They are vector fields which satisfy \( \mathcal{L}_\xi g \propto \ell \otimes \ell \) and \( \mathcal{L}_\xi \ell \propto \ell \) where \( \ell \) is a null 1-form. Obviously, as \( \mathcal{L}_\xi g \in \mathcal{D}P_2 \) this can give rise to a one-parameter submonoid of causal transformations. See Example 4 in [24] for an explicit case of this.

Another interesting example arises by considering the typical Robertson-Walker spacetimes \( \mathbb{R}W \), the manifold being \( I \times M_{N-1}(\kappa) \) where \( I \subset \mathbb{R} \) is an open interval of the real line with coordinate \( x^0 \) and \( M_{N-1}(\kappa) \) is the \((N-1)\)-dimensional Riemannian space of constant curvature \( \kappa \), its canonical positive-definite metric being denoted here by \( g_\kappa \). The Lorentzian metric in \( \mathbb{R}W \) is the warped product

\[
g = -dx^0 \otimes dx^0 + a^2(x^0) \ g_\kappa
\]

where \( a(x^0) > 0 \) is a \( C^2 \) function on \( I \). Take the diffeomorphisms \( \varphi_t : \mathbb{R}W \to \mathbb{R}W \) which leave \( M_{N-1}(\kappa) \) invariant (they are the identity on \( M_{N-1}(\kappa) \)) and act on \( I \) as \( x^0 \to x^0 + t \). It is immediate that

\[
\varphi_t^* g = -dx^0 \otimes dx^0 + a^2(x^0 + t) \ g_\kappa
\]

so that \( \varphi_t^* g \in \mathcal{D}P_2^- (\mathbb{R}W) \) if and only if \( a(x^0 + t) \leq a(x^0) \), and therefore \( \varphi_t^* g \in \mathcal{D}P_2^+ (\mathbb{R}W) \) for \( t \in [0, \epsilon) \) if and only if \( a \) is a non-increasing function. Physically this means that \( \{\varphi_t\}_{t \in I} \) is a one-parameter submonoid of proper causal transformations in \( \mathbb{R}W \) if and only if the Robertson-Walker spacetime is non-expanding. Naturally, the non-contracting case, perhaps of more physical importance, can be studied analogously by simply changing the sign of \( t \).
The infinitesimal generator of this one-parameter group is

\[ \xi(t) = \frac{d\varphi_t}{dt} \bigg|_{t=0} = \frac{\partial}{\partial x^0} \]

and the deformation of the metric tensor reads

\[ \mathcal{L}_\xi g = 2\dot{a} \dot{a} g_{\alpha} = \frac{2\dot{a}}{a} (g + \xi \otimes \xi) \]

where \( \dot{a} \) is the derivative of \( a \) and \( \xi = g(\cdot, \xi) = dx^0 \). Observe that,

\[ \mathcal{L}_\xi g = \frac{2\dot{a}}{a} T\{\xi\} + \frac{\dot{a}}{a} g \]

where \( T\{\xi\} \) is the superenergy tensor of \( \xi \). Obviously, the sign of \( \dot{a} \) is determinant here for \( \mathcal{L}_\xi g - (\dot{a}/a) g \) to be in \( DP_2 \), in accordance with the previous reasoning and the theorem \([\text{3.3}]\). In fact, in this explicit case, as \( g_{\alpha} \) is a positive-definite metric, one can prove

\[ (\mathcal{L}_\xi g)(\bar{x}, \bar{x}) = 2\dot{a} \dot{a} g_{\alpha}(\bar{x}, \bar{x}), \quad \forall \bar{x} \in T(\mathbb{R}^N) \]

which has the sign of \( \dot{a} \) for all vector fields \( \bar{x} \). This same property is shared by the Example 4 of \([\text{24}]\).

All in all, the deformation \( \mathcal{L}_\xi g \) produced by one-parameter local submonoids of causal transformations has been shown to be controllable and the generalized symmetries thereby defined can be attacked using traditional techniques.

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