Long-Range Interaction Models and Yangian Symmetry

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Abstract

The generalized Sutherland-Römer model and Yan models with internal spin degree are formulated in terms of both the Polychronakos’ approach and RTT relation associated to Yang-Baxter equation in consistent way. The Yangian symmetry is shown to generate both the models. We finally introduce the reflection algebra $K(u)$ to long range interaction models.

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I Introduction

In the last few years, a number of one-dimensional long-range interaction models have been studied [1-10]. The typical one is Calogero-Sutherland model [1, 2], then it is subsequently extended to the models with internal spin degrees of freedom [5-9]. Among them an interesting approach was proposed by Bernard-Gaudin-Haldane-Pasquier (BGHP) who made this type of models related to the RTT relation associated with Yang-Baxter equation (YBE) [10]. The BGHP approach provides a method to deal with long-range interaction models: for a given rational solution of YBE, for example, \( R(u) = u + P \), where \( P \) is the permutation and \( u \) the spectral parameter, RTT relation gives rise to the Yangian symmetry. With a particular realization of the Yangian, in general, we can generate corresponding Hamiltonian of the considered systems.

On the other hand Polychronakos had formulated the integrability in terms of the “coupled” momentum operators [3, 4]:

\[
\pi_i = p_i + i \sum_{j \neq i} V_{ij} K_{ij}
\] (1.1)

where \( p_i = -i \frac{\partial}{\partial x_i} (\hbar = 1) \), \( V_{ij} = V(x_i - x_j) \) a potential to be determined and \( K_{ij} \) the particle permutation operators. The requirements of the Hermiticity of \( \pi_i \), the absence of linear terms in \( p_i \) and that only the two-body potentials in the Hamiltonian lead to [3]

\[
V(x) = -V(-x),
\]

\[
H_0 \equiv \frac{1}{2} \sum_i \pi_i^2 = \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \left[ \frac{\partial}{\partial x_i} V_{ij} K_{ij} + V_{ij}^2 \right] - \frac{1}{6} \sum_{i \neq j \neq k \neq i} V_{ijk} K_{ijk}
\] (1.2)
where

\[
V_{ijk} = V_{ij}V_{jk} + V_{jk}V_{ki} + V_{ki}V_{ij} = W_{ij} + W_{jk} + W_{ki},
\]

\[
K_{ijk} = K_{ij}K_{jk}
\]

with \(W_{ij} = W(x_i - x_j)\) being a symmetric function. The commutation relation between \(\pi_i\) and \(\pi_j\) is found to be

\[
[\pi_i, \pi_j] = \sum_{k \neq i,j} V_{ijk} (K_{ijk} - K_{jik}).
\]

This approach can be applied to many integrable systems, especially to C-S model.

Recently, Sutherland and Römer(S-R) presented a new long-range interaction model with the Hamiltonian:

\[
H_{SR} = \frac{1}{2} \sum_i p_i^2 + \sum_{i<j} l(l - 1) \left[ \frac{P_{ij}^+}{\sinh^2 x_{ij}} - \frac{P_{ij}^-}{\cosh^2 x_{ij}} \right]
\]

where

\[
x_{ij} = x_i - x_j, \quad P_{ij}^\pm = \frac{1 \pm \sigma_i \sigma_j}{2} (\sigma_i^2 = 1)
\]

and \(a, l\) are arbitrary parameters. Sutherland and Römer had proved that eq. (1.5) is quantum integrable. In parallel to this development Yan proposed another model:

\[
H_Y = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} l\delta(x_i - x_j)P_{ij}^+
\]

that was solved in terms of Bethe Ansatz. So far both the S-R model and Yan model have not systematically been studied in terms of RTT relation.
In this paper we shall show the following points:

1) The models eq. (1.5) and eq. (1.7) are also the conclusion of Polychronakos’ approach.

2) On the basis of RTT relation the models eq. (1.5) and eq. (1.7) are related to the realization of Yangian, namely, they belong to the Yang-Baxter system. Both 1) and 2) are consistent with each other.

3) Further properties have been discussed that leads to other complicated conserved quantities.

II Sutherland-Römer Model and Yan Model

Let us first discuss the extended forms of $V_{ij}$ in eq. (1.1) that are different from those given by ref. [5, 6]. Setting

$$V_{ij} = P^+_{ij}a_{ij} + P^-_{ij}b_{ij} \quad (2.1)$$

where $P^\pm_{ij}$ are given by eq. (1.6) and $\sigma_i$ quantum operators obeying

$$\sigma_i K_{ij} = K_{ij} \sigma_j ,$$

$$\sigma_i K_{mn} = K_{mn} \sigma_i \quad (i \neq m, n) ,$$

then by substituting eq. (2.1) into eq. (1.1) and doing the parallel discussion in ref. [3], we find

$$V_{ijk} = P^+_{ijk}A_{ijk} + P^-_{ijk}A_{ijk} + P^-_{kij}A_{kij} + P^-_{jki}A_{jki} \quad (2.2)$$

where

$$P^\pm_{ijk} = P^\pm_{ij} P^\pm_{ik} ,$$
\begin{align*}
A_{ijk} &= a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij}, \\
B_{ijk} &= a_{ij}b_{jk} + b_{jk}a_{ki} + b_{ki}a_{ij}.
\end{align*}

Noting that \( P_{ijk}^+ = P_{ikj}^+ = \cdots = P_{kji}^+ \) but \( P_{ijk}^- = P_{jik}^- \) only.

The sufficient condition of the quantum integrability of eq. (1.1) is \([5, 6]\)

\[ V_{ijk} = \text{constant (or zero)} . \quad (2.3) \]

Now let us look for new solution of eq. (2.3)

\textbf{(1) When} \( A_{ijk} \neq 0, B_{ijk} \neq 0 \), a sufficient solution can be checked:

\begin{align*}
a(x) &= l \coth(a x) \quad (\text{or} \ a(x) = l \cot(ax)) , \\
b(x) &= l \tanh(a x) \quad (\text{or} \ b(x) = l \tan(ax)) \quad (2.4)
\end{align*}

where \( x \equiv x_{ij} = x_i - x_j \), \( a, l \) constants and

\[ V_{ijk} = -l^2 (P_{ijk}^+ + P_{ij}^- + P_{kij}^- + P_{jki}^-) = -l^2 . \quad (2.5) \]

Define \([6]\)

\[ H = \frac{1}{2} \sum_i \pi_i^2 - \frac{l^2}{6} \sum_{i \neq j \neq k \neq i} K_{ijk} , \quad (2.6) \]

then eq. (2.5) leads to

\[ H = \frac{1}{2} \sum_i p_i^2 + \sum_{i < j} l(l - aK_{ij}) \left[ \frac{P_{ij}^+}{\sinh^2(ax_{ij})} - \frac{P_{ij}^-}{\cosh^2(ax_{ij})} \right] . \quad (2.7) \]

Eq. (2.7) is exactly \( H_{SR} \) given by S-R \([11]\) when \( K_{ij} = \pm 1 \).

Define

\[ \bar{\pi}_i = \pi_i + il \sum_{i \neq j} K_{ij} , \quad (2.8) \]
then

\[ [\bar{\pi}_i, \bar{\pi}_j] = 2il(\bar{\pi}_i - \bar{\pi}_j)K_{ij} \]  
(2.9)

\[ [H, \pi_i] = [H, \bar{\pi}_i] = 0 . \]  
(2.10)

The conserved quantities are given by

\[ I_n = \sum_i \bar{\pi}_i^n \]  
(2.11)

which leads to

\[ [I_n, I_m] = 0 , \]  
(2.12)

\[ [H, I_n] = 0 , \]  
(2.13)

i.e. the model is quantum integrable in the sense of Polychronakos [5, 6].

(2) When \( B_{ijk} = 0 \), we consider two cases

(a) \( A_{ijk} = 0 \)

\[ a(x) = \frac{l}{x} , \quad V_{ijk} = 0 , \]

\[ H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{l(l - K_{ij})}{(x_i - x_j)^2} P_{ij}^+ \]  
(2.14)

that is well known as Calogero model when \( P_{ij}^+ \) takes the value 1.

(b) \( A_{ijk} = \beta^2 \neq 0 \)

\[ [\bar{\pi}_i, \bar{\pi}_j] = \beta \sum_{k \neq i} P_{ijk}^+ (K_{ijk} - K_{jik}) . \]  
(2.15)

Define

\[ \bar{\pi}_i = \pi_i + \beta \sum_{i \neq j} P_{ij}^+ K_{ij} , \]  
(2.16)
it is easy to prove that

$$[\bar{\pi}_i, P_{jk}^+] = 0, \quad \forall i \text{ and } j \neq k$$ \hspace{1cm} (2.17)

and

$$[\bar{\pi}_i, \bar{\pi}_j] = 2\beta P_{ij}^+(\bar{\pi}_i - \bar{\pi}_j)K_{ij}, \hspace{1cm} (2.18)$$

$$[\bar{\pi}^n_i, \bar{\pi}_j] = 2\beta P_{ij}^+(\bar{\pi}^n_i - \bar{\pi}^n_j)K_{ij}, \hspace{1cm} (2.19)$$

so that eq. (2.12) is also satisfied. Define

$$H = \frac{1}{2} \sum_i \pi_i^2 + \frac{\beta^2}{6} \sum_{i \neq j \neq k \neq i} P_{ijk}^+K_{ijk}. \hspace{1cm} (2.20)$$

With the help of eq. (2.15), one can prove

$$[H, \pi_i] = [H, \bar{\pi}_i] = [H, I_n] = 0. \hspace{1cm} (2.21)$$

For the case (b) we have two sufficient solutions of $V_{ijk}$:

(b$_1$) $a(x) = il\cot(ax)$ (or $a(x) = l\coth(ax)$)

$$V_{ijk} = -l^2P_{ijk}^+, \hspace{1cm} (2.22)$$

Eq. (2.22) is the generalization of the spin chain model considered by BGHP [10].

(b$_2$) $a(x) = l\text{sgn}(x),$

$$H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{l(l - aK_{ij})}{\sin^2 a(x_i - x_j)} P_{ij}^+. \hspace{1cm} (2.23)$$

On condition that $K_{ij} = \pm 1$, eq. (2.23) was first pointed out by Yan [12] through Bethe Ansatz, he also found the Y-operator defined by Yang [13, 14] for eq. (2.23)

$$Y_{ij}^{\alpha\beta} = \frac{1}{ik_{ij}(ik_{ij} - 2c)}[ik_{ij} - c(1 - \sigma_i\sigma_j)][-ik_{ij}P_{ij}^{\alpha\beta} + c(1 + \sigma_i\sigma_j)] \hspace{1cm} (2.24)$$
where $P$ is the permutation, $\sigma^2_i = 1$ and $Y$ satisfies
\begin{equation}
Y^{\alpha\beta}_{jk} Y^{\beta\gamma}_{ik} Y^{\alpha\beta}_{ij} = Y^{\beta\gamma}_{ij} Y^{\alpha\beta}_{ik} Y^{\beta\gamma}_{jk}
\end{equation}
and $c = l(l \pm 1)/2$ for $K_{ij} = \pm 1$. Noting that there is only $P^+_{ij}$ in the Hamiltonian eq. (2.23) for the quantum integrability.

In this section we have re-interpreted the models eq. (1.5) and eq. (1.7) from the point of view of the formulation eq. (1.1). Next we shall set up the Yangian description of models eq. (1.5) and eq. (1.7) through RTT relation.

### III RTT Relation and Long-Range Interaction Models

Let us apply the BGHP approach [10] to the S-R model and Yan model.

The solution of Yang-Baxter equation, $R$-matrix, takes the simplest form as
\begin{equation}
R(u) = u + \lambda P_{00'}
\end{equation}
and the RTT relation reads
\begin{equation}
R_{00'}(u - v)T^0(u)T^0'(v) = T^0'(v)T^0(u)R_{00'}(u - v)
\end{equation}
where $T^0(u) = T(u) \otimes 1$, $T^0' = 1 \otimes T(u)$ and $P_{00'}$ is the permutation operator exchanging the two auxiliary spaces 0 and 0'. Make the expansion [10]
\begin{equation}
T^0(u) = I + \sum_{a,b=1}^p X_{ba}^0 \sum_{n=0}^{\infty} \frac{\lambda T^{ab}_n}{u^{n+1}},
\end{equation}
\begin{equation}
P_{00'} = \sum_{a,b=1}^p X_{ba}^0 X_{ab}^{0'}.
\end{equation}
It is well known that \( \{T_{ab}^n\} \) generate the Yangian \( [15] \). Substituting eqs. (3.1), (3.3) and (3.4) into eq. (3.2) one finds

\[
\sum_{a,b} \sum_{cd} X_{ba}^0 X_{dc}^{0'} \sum_{n=0}^{\infty} \left\{ u^{-n-1} f_1^n - v^{-n-1} f_2^n + \sum_{m=0}^{n-1} u^{-n-1} v^{-m-1} f_{3,m}^{n,n} \right\} = 0 \quad (3.5)
\]

where

\[
f_1^n = \delta_{bc} T_{ad}^n - \delta_{ad} T_{cb}^n - [T_{ab}^n, T_{cd}^n],
\]

\[
f_2^n = \delta_{bc} T_{ad}^n - \delta_{ad} T_{cb}^n - [T_{ab}^0, T_{cd}^n],
\]

\[
f_{3,m}^{n,m} = \lambda(T_{ad}^n T_{cb}^m - T_{ad}^m T_{cb}^n) + [T_{ab}^n, T_{cd}^m] - [T_{ab}^0, T_{cd}^m] .
\]

For any auxiliary space \( \{X_{ab}\} \) we require \( f_1^n = f_2^n = f_{3,m}^{n,m} = 0 \). Obviously, \( f_1^n = 0 \) is equivalent to \( f_2^n = 0 \). So we need only to take

\[
f_1^n = f_{3,m}^{n,m} = 0 \quad (3.6)
\]

into account.

First from \( f_{3,0}^{n,0} = 0 \) it follows

\[
\delta_{bc} T_{n+1}^{ad} - \delta_{ad} T_{n+1}^{cb} = \lambda(T_{0}^{ad} T_{n}^{cb} - T_{ad}^{n} T_{0}^{cb}) + [T_{ab}^n, T_{cd}^n] \quad (3.7)
\]

which can be recast to

\[
T_{n+1}^{ad} = \lambda(T_{0}^{ad} T_{n}^{cc} - T_{ad}^{n} T_{0}^{cc}) + [T_{ac}^n, T_{cd}^n] \quad (a \neq d) \quad , \quad (3.8)
\]

\[
T_{n+1}^{aa} - T_{n+1}^{cc} = \lambda(T_{0}^{aa} T_{n}^{cc} - T_{aa}^{n} T_{0}^{cc}) + [T_{ac}^n, T_{ca}^n] \quad , \quad (3.9)
\]

where no summation for the repeating indices is taken. Eqs. (3.8) and (3.9) imply that \( T_{ab}^n \) can be determined by iteration for given \( T_{0}^{ab} \) and \( T_{1}^{ab} \).
Now let us set

\[ T_{0}^{ab} = \sum_{i=1}^{N} I_{i}^{ab} , \] (3.10)

\[ T_{1}^{ab} = \sum_{i=1}^{N} I_{i}^{ab} D_{i} \] (3.11)

and

\[ [I_{i}^{ab}, I_{j}^{cd}] = \delta_{ij}(\delta_{bc}I_{i}^{ad} - \delta_{ad}I_{i}^{cb}) \] (3.12)

where \( D_{i} \) are operators to be determined. Substituting eqs. (3.10)–(3.12) into \( f_{1}^{1} \) we obtain

\[ \sum_{i} \sum_{j} I_{i}^{ab}[D_{i}, I_{j}^{cd}] = 0 . \] (3.13)

Further we assume

\[ \sum_{i} I_{i}^{ab}[D_{i}, I_{j}^{cd}] = 0, \text{ for any } j \] (3.14)

with which the \( T_{2}^{ab} \) should satisfy

\[ \delta_{bc}T_{2}^{ad} - \delta_{ad}T_{2}^{cb} = \sum_{i \neq j} I_{i}^{ab} I_{j}^{cd} \left\{ \lambda \sum_{k,l} I_{i}^{kl} I_{j}^{lk}(D_{j} - D_{i}) + [D_{i}, D_{j}] \right\} + \sum_{i} (\delta_{bc}I_{i}^{ad} D_{i}^{2} - \delta_{ad}I_{i}^{cb} D_{i}^{2}) . \] (3.15)

A sufficient solution of eq. (3.15) is

\[ T_{2}^{ab} = \sum_{i} I_{i}^{ab} D_{i}^{2} \] (3.16)

with

\[ [D_{i}, D_{j}] = \lambda \sum_{a,b} I_{j}^{ab} I_{i}^{ba}(D_{i} - D_{j}) . \] (3.17)
Thus eq. (3.11) generates long-range interaction through the eq. (3.14) and (3.17).

However so far there is not simple relationship between $D_i$ and $I_{ab}^{ij}$ which should satisfy eq. (3.14). It is very difficult to determine the general relationship. Fortunately, BGHP [10] have set up the link with the help of projection. Let the permutation groups $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ be generated by $K_{ij}$, $P_{ij}$ and the product $P_{ij}K_{ij}$ respectively, where $K_{ij}$ exchange the positions of particles and $P_{ij}$ exchange the spins at position $i$ and $j$. The projection $\rho$ was defined as

$$\rho(ab) = a \quad \text{for} \quad \forall a \in \Sigma_2, b \in \Sigma_1 , \quad (3.18)$$

i.e. the wave function considered is symmetric. Let $I_{ab}^{ij}$ be the fundamental representations, then

$$P_{ij} = \sum_{a,b} I_{ab}^{ij} I_{ab}^{ba} . \quad (3.19)$$

Suppose that there exists [10]

$$D_i = \rho(\hat{D}_i), \quad D_i \in \Sigma_2, \quad \hat{D}_i \in \Sigma_1 \quad (3.20)$$

and the $\hat{D}_i$ is particle-like operators, i.e.

$$K_{ij}\hat{D}_i = \hat{D}_j K_{ij}, \quad K_{ij}\hat{D}_l = \hat{D}_l K_{ij} \quad (l \neq i, j) . \quad (3.21)$$

Define

$$T_{m}^{ab} = \sum_{i} I_{i}^{ab} \rho(\hat{D}_{i}^{m}) \quad (m \geq 0) , \quad (3.22)$$

then

(a)$$[\hat{D}_j, \hat{D}_i] = \lambda \rho^{-1}(P_{ij}(D_j - D_i)) = \lambda(\hat{D}_j - \hat{D}_i)K_{ij} . \quad (3.23)$$
(b) $T_{m}^{ab}$ satisfy eq. (3.6), i.e., RTT relation eq. (3.2).

Actually $f_1^n = 0$ is easy to be checked. By using

$$[\hat{D}_i^n, \hat{D}_j^m] = \sum_{k=0}^{n-1} \hat{D}_i^k [\hat{D}_i, \hat{D}_j^m] \hat{D}_j^{n-k-1} = \lambda \sum_{k=0}^{n-1} \hat{D}_i^k (\hat{D}_i^n - \hat{D}_j^m) \hat{D}_j^{n-k-1} K_{ij},$$

we have $f_3^{n,m} = 0$.

The projection procedure is very important for it enables us to prove that eq. (3.6) is satisfied by virtue of eq. (3.20).

With the expansion eqs. (3.3) and the projected long-range expansion eq. (3.22), the hamiltonian associated to $T(u)$ is obtained by the expansion of the deformed determinant [10]:

$$\det_q T(u) = \sum_\sigma \epsilon(\sigma) T_{1\sigma_1}(u - (p - 1)\lambda) T_{2\sigma_2}(u - (p - 2)\lambda) \cdots T_{p\sigma_p}(u). \quad (3.24)$$

A calculation gives

$$\det_q T(u) = 1 + \frac{\lambda}{u} M + \frac{\lambda}{u^2} \left[ \rho (\sum_i \hat{D}_i - \frac{\lambda}{2} \sum_{j\neq i} K_{ij}) + \frac{\lambda}{2} M(M - 1) \right]$$

$$+ \frac{\lambda}{u^3} \rho \left\{ (\sum_i \hat{D}_i - \frac{\lambda}{2} \sum_{j\neq i} K_{ij})^2 + \frac{\lambda^2}{12} \sum_{i\neq j\neq k\neq i} K_{ij} K_{jk} \right.$$  
$$+ \lambda(M - 1) \sum_i (\hat{D}_i - \frac{\lambda}{2} \sum_{j\neq i} K_{ij}) $$

$$\left. + \frac{\lambda^2}{6} M(M - 1)(M - 2) + \frac{\lambda^2}{4} M(M - 1) \right\} + \cdots. \quad (3.25)$$

One takes the Hamiltonian as

$$H = \frac{1}{2} \rho \left\{ (\sum_i \hat{D}_i - \frac{\lambda}{2} \sum_{i\neq j} K_{ij})^2 + \frac{\lambda^2}{12} \sum_{i\neq j\neq k\neq i} K_{ij} K_{jk} \right\}. \quad (3.26)$$

Therefore we define the Hamiltonian which have the Yangian symmetry given by eqs. (3.22), (3.12) and (3.17). In comparison to the known models we list the expressions for $\hat{D}_i$ satisfying eq. (3.23)
(1) \[ \hat{D}_i = p_i + \frac{1}{2} \sum_{i \neq j} [\text{sgn}(x_i - x_j) + 1]K_{ij}, \quad \lambda = 2il, \]

\[
H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} l(l - P_{ij})\delta(x_i - x_j). \tag{3.27}
\]

(2) \[ \hat{D}_i = p_i + \sum_{i \neq j} l[i \cot a(x_i - x_j) + 1]K_{ij}, \quad \lambda = 2il,
\]

\[
H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{l(l - aP_{ij})}{\sin^2 a(x_i - x_j)}. \tag{3.28}
\]

(3) \[ \hat{D}_i = p_i + il \sum_{i \neq j} [\text{coth} a(x_i - x_j)P_{ij}^+ + \tanh a(x_i - x_j)P_{ij}^- + 1]K_{ij}, \quad \lambda = 2il,
\]

\[
H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} l(l - aP_{ij}) \left( \frac{P_{ij}^+}{\sinh^2 a(x_i - x_j)} - \frac{P_{ij}^-}{\cosh^2 a(x_i - x_j)} \right). \tag{3.29}
\]

Eqs. (3.27) and (3.28) were given in ref \[5\], eq. (3.28) was studied in ref \[10\]. Eq (3.29) is the generalization of S-R model.

An alternative description of transfer matrix was given by BGHP \[10\]. Define

\[ \bar{D}_i = \hat{D}_i - \lambda \sum_{i < j} K_{ij}, \tag{3.30} \]

then

\[
[\bar{D}_i, \bar{D}_j] = 0, \tag{3.31}
\]

\[
[K_{ij}, \bar{D}_k] = 0 \quad (k \neq i, j), \tag{3.32}
\]

\[
K_{ij}\bar{D}_i - \bar{D}_jK_{ij} = \lambda. \tag{3.33}
\]

It was proved that

\[
\bar{T}_i(u) = 1 + \lambda \frac{P_{pi}}{u - \bar{D}_i}, \quad T(u) = \prod_i \bar{T}_i(u) \text{ and } \rho(T(u)) \tag{3.34}
\]

all satisfy the RTT relation.
The deformed determinant of $\bar{T}(u)$ was defined by

$$det_q \bar{T}(u) = \frac{\Delta_M(u + \lambda)}{\Delta_m(u)} , \quad \Delta_M(u) = \prod_{i=1}^{M}(u - \bar{D}_i) .$$  

(3.35)

It was proved that

$$\rho(det_q \bar{T}(u)) = det_q(T(u)) .$$  

(3.36)

To contain the model eq.(2.23), we define $\bar{D}_i$ related to the $\bar{\pi}_i$ given by eq. (2.16) as

$$\bar{D}_i = \bar{\pi}_i - \beta \sum_{j<i} P_{ij}^+ K_{ij}$$  

(3.37)

which satisfies eqs.(3.31), (3.32) and (3.34) etc. So we can put the models eqs. (2.7) and (2.23) into Yang-Baxter system.

In conclusion of this section we have shown the consistence between Yangian symmetry and the integrability of Polychronakos for long-range interaction models and given the interpretation of S-R model and Yan model from the point of view of YB system.

**IV Reflection Algebra**

The associativity of RTT relation eq. (3.2) is Yang-Baxter equation(YBE) \[13, 14\]

\[\bar{R}(u) = PR(u): \]

$$\bar{R}_{12}(u)\bar{R}_{23}(u + v)\bar{R}_{12}(v) = \bar{R}_{23}(v)\bar{R}_{12}(u + v)\bar{R}_{23}(u)$$  

(4.1)

where the subscripts indicate the spaces, namely, $1 \rightarrow 0$, $2 \rightarrow 0'$, $3 \rightarrow 0''$ in comparison to eq. (3.2).
It is well-known that for a given $\tilde{R}(u)$ satisfying eq. (4.1) there allows corresponding reflection operator $K(u)$ determined by [16]

$$\tilde{R}(u - v)K_1(u)\tilde{R}(u + v)K_1(v) = K_1(v)\tilde{R}(u + v)K_1(u)\tilde{R}(u - v)$$  \hspace{1cm} (4.2)

where $K_1(u) = K(u) \otimes 1$. Eq. (4.2) possesses the remarkable properties [16]:

(1) Suppose $K_{\pm}(u)$ are c-number solutions of eq. (4.2), so do $\tilde{K}_{\pm}(u)$

$$\tilde{K}_{\pm}(u) = T(u)K_{\pm}(u)T^{-1}(-u) .$$  \hspace{1cm} (4.3)

(2) Define

$$t(u) = \text{tr}[K_{+}(u + \lambda)T(u)K_{-}(u)T^{-1}(-u)] ,$$  \hspace{1cm} (4.4)

then

$$[t(u), t(v)] = 0 ,$$  \hspace{1cm} (4.5)

i.e. $t(u)$ forms a commuting family. In order to solve $K(u)$ in eq. (4.2) we make expansion:

$$K_0(u) = \sum_{a,b} \sum_n X_{ab}^0 K_{ab}^{(n)} u^{-n} .$$  \hspace{1cm} (4.6)

Substituting eq. (4.6) into eq. (4.2) after calculations one obtains

$$F_{ab,cd}^{n,m} = \delta_{bc}[K^{(n)}, K^{(m)}]_{ad} + \delta_{ac} \sum_e (K_{bc}^{(n+1)} K_{ed}^{(m)} + K_{be}^{(n)} K_{ed}^{(m+1)})$$

$$+ \delta_{bd} \sum_e (K_{ae}^{(m+1)} K_{ec}^{(n)} - K_{ae}^{(n)} K_{ec}^{(m+1)}) + [K_{bd}^{(n+2)}, K^{(m)}_{ad}] - [K^{(n)}_{bd}, K_{ad}^{(m+2)}]$$

$$+ \left[ K_{ac}^{(n+1)} K_{bd}^{(m)} - K_{ac}^{(m)} K_{bd}^{(n+1)} + K_{ac}^{(n)} K_{bd}^{(m+1)} - K_{ac}^{(m+1)} K_{bd}^{(n)} \right]$$  \hspace{1cm} (4.7)

$$= 0 .$$  \hspace{1cm} (4.8)
It follows

\[ [K_{ab}^{(0)}, K_{cd}^{(m)}] = 0 . \] (4.9)

Suppose \( K_{ab}^{(0)} = \delta_{ab} \), the iteration relation reads

\[
\begin{align*}
\delta_{bd} K_{ac}^{(m+2)} - \delta_{ac} K_{bd}^{(m+2)} & = \frac{1}{2} \left\{ \delta_{ac} [K^{(2)}, K^{(m)}]_{bd} - \delta_{bd} [K^{(1)}, K^{(m+1)}]_{ac} \\
& + K_{ac}^{(2)} K_{bd}^{(m)} - K_{ac}^{(m)} K_{bd}^{(2)} + K_{ac}^{(1)} K_{bd}^{(m+1)} - K_{ac}^{(m+1)} K_{bd}^{(1)} \\
& + \delta_{bc} [K^{(1)}, K^{(m)}]_{bd} + [K_{bc}^{(3)}, K_{ad}^{(m)}] \right\} \quad (m > 1) . \end{align*}
\] (4.10)

Eq. (4.10) tells that \( K^{(m)} \) can be found if \( K^{(1)}, K^{(2)} \) and \( K^{(3)} \) are given properly.

Now let us consider the simplest case where \( K(u) \) is a 2 \( \times \) 2 matrix given by eq. (4.13) (see below). Denote

\[
T(u) = \begin{bmatrix} T_{11}(u) & T_{12}(u) \\ T_{21}(u) & T_{22}(u) \end{bmatrix} , \tag{4.11}
\]

then

\[
T^{-1}(u) = [det_q T(u)]^{-1} \begin{bmatrix} T_{22}(u - \lambda) & -T_{12}(u - \lambda) \\ -T_{21}(u - \lambda) & T_{11}(u - \lambda) \end{bmatrix} . \tag{4.12}
\]

Since \( det_q T(u) \) commutes with \( T_{ab}(v) \) one does not care the common factor appearing in eq. (4.12). We consider the simplest case when \( K_{\pm} = 1 \) and denote

\[
K(u) = T(u) T^{-1}(-u) . \tag{4.13}
\]

Now let us see what happens for the long-range interaction model where \( T(u) \) is given by eq. (3.22). Noting that

\[
K_{11}(u) = T_{11}(u) T_{22}(-u - \lambda) - T_{12}(u) T_{21}(-u - \lambda) ,
\]
\[ K_{12}(u) = T_{12}(u)T_{11}(-u - \lambda) - T_{11}(u)T_{12}(-u - \lambda), \]
\[ K_{21}(u) = T_{21}(u)T_{22}(-u - \lambda) - T_{22}(u)T_{21}(-u - \lambda), \]
\[ K_{22}(u) = T_{22}(u)T_{11}(-u - \lambda) - T_{21}(u)T_{12}(-u - \lambda). \quad (4.14) \]

The \( T_{ab}(u) \) in eq. (4.14) can be expanded in the terms of eqs. (3.3) and (3.22) which give the \( T_{ab}(u) \):

\[ T_{ab}(u) = \delta_{ab} + \lambda \sum_i I_{ia}^{ba} d_i(u) \quad (4.15) \]

where \( d_i(u) = \rho \left( \frac{1}{u - D_i} \right) \). Substituting eq. (4.15) into eq. (4.14) we find

\[ K_{11}(u) = 1 + \lambda \sum_i [I_{i}^{11} d_i(u) + I_{i}^{22} d_i(-u - \lambda) - \lambda I_{i}^{22} d_i(u)d_i(-u - \lambda)] \]
\[ + \lambda^2 \sum_{i \neq j} (I_{i}^{11} I_{j}^{22} - I_{i}^{21} I_{j}^{12}) d_i(u)d_j(u - \lambda), \]
\[ K_{12}(u) = \lambda \sum_i I_{i}^{21} [d_i(u) - d_i(-u - \lambda) + \lambda d_i(u)d_i(-u - \lambda)] \]
\[ + \lambda^2 \sum_{i \neq j} (I_{i}^{21} I_{j}^{11} - I_{j}^{21} I_{i}^{11}) d_i(u)d_j(u - \lambda), \]
\[ K_{21}(u) = \lambda \sum_i I_{i}^{12} [d_i(u) - d_i(-u - \lambda) + \lambda d_i(u)d_i(-u - \lambda)] \]
\[ + \lambda^2 \sum_{i \neq j} (I_{i}^{12} I_{j}^{22} - I_{j}^{12} I_{i}^{22}) d_i(u)d_j(u - \lambda), \]
\[ K_{22}(u) = 1 + \lambda \sum_i [I_{i}^{22} d_i(u) + I_{i}^{11} d_i(-u - \lambda) - \lambda I_{i}^{11} d_i(u)d_i(-u - \lambda)] \]
\[ + \lambda^2 \sum_{i \neq j} (I_{i}^{22} I_{j}^{11} - I_{i}^{12} I_{j}^{21}) d_i(u)d_j(u - \lambda) \quad (4.16) \]

and

\[ t(u) = K_{11}(u) + K_{22}(u) \]
\[ = 2 + \frac{\lambda}{u^2} \left[ 2 \sum_i D_i + \lambda \sum_{i \neq j} P_{ij} + C_1 \right] + \frac{\lambda^2}{u^3} \left[ \sum_i D_i + \lambda \sum_{i \neq j} P_{ij} + C_2 \right] \]
\[ + \frac{\lambda}{u^4} \sum_i \left\{ 2 \rho(D_i + \frac{\lambda}{2})^3 - 2(N - 1) \lambda D_i - (N - 1) \lambda^2 D_i \right\} \]

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\begin{equation}
+ \lambda \sum_{j \neq i} \rho(\hat{D}_i \hat{D}_j) + 2 \lambda \sum_{j \neq i} P_{ij} \rho(\hat{D}_i^2) + \lambda^2 \sum_{j \neq i} P_{ij} D_i \\
+ \lambda^3 \sum_{j \neq i} P_{ij} - \lambda \sum_{j \neq i} P_{ij} \rho(\hat{D}_i \hat{D}_j) \right\} + 0(u^{-4})
\end{equation}

(4.17)

where $C_1$ and $C_2$ are constants. Obviously the second term commutes with the third one on the RHS of eq. (4.17).

Here we would like to emphasize that the $t(u)$ does not generate conserved quantities.

The physical meaning of eq. (4.17) for the long-range interaction models is not clear yet. It deserves more knowledge in this area to be explored. What we would like to say is that the simplest form of reflection matrix $K(u)$ for long-range interaction models can really be calculated. Substituting variety of the forms of $\hat{D}_i$ given in section 3, the reflection matrix $K(u)$ can explicitly be expressed by the interactions.

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