On the construction of heat wave in symmetric case

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Abstract. A nonlinear second-order parabolic equation with two variables is considered. Under additional conditions, this equation can be interpreted as the porous medium equation in case of dependence of the unknown function on two variables: time and distance from the origin. The equation has a wide variety of applications in continuum mechanics, for example, it is applicable for mathematical modeling of filtration of ideal polytropic gas in porous media or heat conduction. The authors deal with a special solutions which are usually called heat waves. A special feature of such solution is that it consists of two continuously joined solutions. The first of them is trivial and the second one is nonnegative. The heat wave solution can have discontinuous derivatives on the line of joint which is called the front of heat wave, i.e. smoothness of the solution, generally speaking, is broken. The most natural problem which has such solutions is the so-called “the Sakharov problem of the initiation of a heat wave”. New solutions of the problem in the form of multiple power series for physical variables are constructed. The coefficients of the series are obtained from tridiagonal systems of linear algebraic equations. Herewith, the elements of matrices of this systems depend on the matrix order and the condition of the diagonal dominance is not fulfilled. The recurrent formulas for the coefficients are suggested.

1. Introduction
In this paper we consider the nonlinear porous medium equation [1] having the form

\[ T_t = T \Delta T + \frac{1}{\sigma}(\nabla T)^2. \]  

(1)

This differential equation represents two fundamental laws of physics: the Fourier law for heat conductivity [2] and the Darcy law for filtering [3] provided the dependencies of the heat conductivity coefficient on temperature and of the filtration coefficient on the gas density which are exponential.

In the case of one independent variable eq. (1) can be represented as

\[ u_t = \sigma u_{pp} + \frac{1}{\sigma} u_p^2 + \frac{\nu}{\rho} u_p. \]  

(2)

Here \( u(t, \rho) \) is the unknown function; \( t \) is the time, \( \rho = \sqrt{x_1^2 + \ldots + x_{\nu+1}^2} \) is the spatial coordinate; \( \sigma > 0 \) is a constant characterizing the properties of the medium; the parameter \( \nu \) is equal to 0,1 or 2 in the cases of flat, axial or central symmetry, respectively.

Eq. (1) is a second-order nonlinear equation of parabolic type [4]. Note that (1) can be considered as an independent mathematical object (with no connection to applications), in which case the constants \( \sigma, \nu \) may admit any real values (except \( \sigma = 0 \)).
The heat waves (wave filtering) propagating on cold (zero temperature) background with a finite velocity are one of the interesting classes of solutions of the heat equation. Geometrically the solution of heat-wave type solution consists of two surfaces (perturbed solution $u(t, \bar{x}) > 0$ and zero temperature background $u \equiv 0$), which are continuously joined along some sufficiently smooth line $x = b(t)$, called the front. In the linear case such solutions are apparently known since the time of J. Fourier.

A simple example of the heat wave for a linear heat equation $u_t = u_{xx}$ is given by

$$u(t, x) = \begin{cases} \exp\left(-x/\sqrt{2}\right) \sin \left(t - x/\sqrt{2}\right), & 0 \leq x < t\sqrt{2}, \\ 0, & x \geq t\sqrt{2}, \end{cases}$$

where $0 \leq t \leq \pi/2$. It is easy to see that the front here is the line $x = t\sqrt{2}$.

Note that in this case the question of uniqueness is not addressed: for the unique solvability we need to assume an additional condition (for example, to determine the derivative with respect to the spatial coordinate at $x = 0$, i.e. to consider the Cauchy problem).

In relation to Eq. (1) the terms “heat wave” and the “wave filter”, have been previously used as well. For example, in the papers [5, 6] the composite solution:

$$u_*(t, \bar{x}) = \begin{cases} u(t, \bar{x}) > 0, & a(t, x_2, \ldots, x_n) > x_1, \\ 0, & x_1 \geq a(t, x_2, \ldots, x_n), \end{cases}$$

where $a(t, x_2, \ldots, x_n) > 0$ was understood in this sense. For the problems discussed in the paper, such definition is not quite convenient, because it assumes the specification of the front in the form of a sufficiently smooth function. So, we define the heat wave in the following (more general) way.

**Definition 1.** Let $u(t, \bar{x})$ be a continuous, non-negative function defined for $t \in [t_*, t^*), \bar{x} \in X \subseteq \mathbb{R}^n$, with a compact simply connected support $\text{supp} u = \overline{D}$, where $D = \{(t, \bar{x}) \mid u(t, \bar{x}) > 0\}$.

We call the function $u(t, \bar{x})$ a heat wave, if it is

(i) twice continuously differentiable in $D$ with respect to the spatial variables $\bar{x}$ and continuously differentiable with respect to the time $t$;

(ii) satisfies Eq. (1) in $D$;

(iii) the domain $D$ has the property: if $t_* \leq t_1 < t_2 < t^*$, then $D(t_1) \subset D(t_2)$, where $D(t_i)$ is the projection of the section of $D$ by the hyperplane $t = t_i, i = 1, 2$ to $\mathbb{R}^n$.

In the case when the function $u(t, \bar{x})$ is analytical in $D$, we consider the analytical heat wave. The boundary $\Gamma = \overline{D} \setminus D$ of $D$ will be called the front of a heat wave or a zero temperature front.

Since the function $u \equiv 0$ obviously satisfies (2), the heat wave is a classical (smooth) solution of Eq. (1) everywhere in the domain of definition, except perhaps for a set $\Gamma$, where the derivatives (but not unknown function) are allowed to be discontinuous.

The simplest example of the solution of heat-wave type for Eq. (1) in the case of a plane symmetry and when $\nu = 0$, $\rho = x$ is the piecewise linear function

$$u(t, x) = \begin{cases} \alpha_1 t - \sqrt{\alpha_1} x, & x < b(t) = \alpha_1 t/\sqrt{\alpha_1}, \\ 0, & x \geq \alpha_1 t/\sqrt{\alpha_1}, \alpha_1 = \text{const} > 0. \end{cases}$$

It was Ya.B. Zeldovich, who first obtained solutions of (1), in the form of a heat wave for nonlinear heat conductivity problem [7]. In [3] similar results were obtained for filtration problems by G.I. Barenblatt. In the paper [8] by O.A. Oleinik and co-authors the boundary problems, in which the speed of propagation of the front filter is supposed to be finite, are studied in abstract functional spaces. Heat waves in the class of piecewise-analytic functions
were first considered by A.F. Sidorov [9, 10]. The method of power series [11] for the porous medium equation with special boundary conditions is applied in [12].

The present paper discusses the solution of heat-wave type for Eq. (1). These solutions are constructed in the class of analytic functions (analytical heat wave) as power series with coefficients, which are recurrently determined by the tridiagonal systems of linear algebraic equations [13]. At the same time, unlike previously published papers of the A.F. Sidorov’s scientific school [6, 10, 14, 15], the solution is constructed in an explicit form. Until now similar results for Eq. (2) were obtained only in the case of $\nu = 0$ [5] and $\nu = 1$ [15].

The constructed power series are compared with the exact solutions of the equation, which are obtained by the methods of the group analysis [16, 17] of differential equations. The comparison shows that the first coefficients of the Taylor series are the same, which confirms the correctness of the results.

2. Boundary condition and existence theorem

Let for Eq. (2) the following boundary condition

$$u(t, \rho)|_{\rho = \nu R} = f(t), \ f(0) = 0, \ f'(0) > 0. \quad (3)$$

This is be given the most natural problem, in which the solutions described in the previous section arise. In the Russian scientific literature it is called “The Sakharov problem of the initiation of a heat wave”.

Note that (3) contains only one boundary condition for second-order equations.

Nevertheless due to the presence of degeneration that is associated with the vanishing factor to the highest derivative, Eq. (2) has the specific properties characteristic to first-order equations. So, for the problem (2), (3) we have the following theorem of existence and uniqueness of solutions.

**Theorem 1.** Suppose that $f = f(t)$ is analytical in some neighborhood of $t = 0$. Then the problem (2), (3) has a unique analytical solution in a full neighborhood of $t = 0$, $\rho = \nu R$, if the sign $u_{\rho}|_{t=0}, \rho = \nu R$ is choosen.

The proof of it is very cumbersome. So, here we present just the general scheme.

(i) We make some nondegenerate analytic changes of variables (A–D) that allow to bring the problem (2), (3) to a special (characteristic) form.

(A) The first substitution allows to transforms not yet known front of the heat wave in the coordinate axis. This adds to the problem an extra unknown function and an additional boundary condition ($u$ is equal to zero on the front).

(B) The second substitution is a special analog of the hodograph transformation and changes the roles of the unknown function $u$ and one of the independent variables. It allows to get rid of the unknown function $b$ and one of the boundary conditions.

(C) Next substitution brings the curve $u = f(t)$ to the coordinate axis.

(D) The last substitution is a partial decomposition of the unknown function in a Taylor series, and the boundary condition is introduced into the equation.

(ii) The solution of transformed problem is constructed in the form of the characteristic power series of one variable, whose coefficients are uniquely determined (if we initiate the sign of the first one).

(iii) For the solution of the transformed problem we construct a majorizing series.

(iv) We prove the convergence of the majorizing series by constructing one more majorant.
3. Heat wave construction

We construct a solution of the problem (2), (3) in the form of a multiple power series

\[
u = \sum_{l,m=0}^{\infty} u_{l,m} \frac{t^l}{l!} \frac{(\rho - \nu R)^m}{m!}, \quad u_{l,m} = \frac{\partial^{l+m} u(t, \rho)}{\partial t^l \partial \rho^m} \bigg|_{t=0, \, \rho=\nu R}.
\]

Using the condition (3) it is possible to find the coefficients \(u_{l,0}, \, l = 0, 1, 2, \ldots\). In fact, by the assumption of the theorem 1, for the function \(f(t)\) in some neighborhood of \(t = 0\) we have the expansion

\[f(t) = \sum_{l=0}^{\infty} f_l t^l,
\]

where \(f_l = f^{(l)}(0)\). From (3) and (4) there follows the equality

\[u(t, \rho)|_{\rho=\nu R} = \sum_{l=0}^{\infty} u_{l,0} \frac{t^l}{l!} = \sum_{l=0}^{\infty} f_l t^l.
\]

We can see, that \(u_{l,0} = f_l\), in particular, \(u_{0,0} = u(0, \nu R) = f_0 = 0, \quad u_{1,0} = f_1 = f'(0) > 0\). The remaining coefficients of the series (4) are found by induction on the total differentiation order \(l+m\).

Let us assume in (2) that \(t = 0, \, \rho = \nu R\). Taking into account that the values \(u_{0,0} = 0, \quad u_{1,0} > 0\) have already been found, we obtain the quadratic equation for \(u_{0,1}\), which shows that \(u_{0,1}\) is defined in two ways by the formula \(u_{0,1} = \pm \sqrt{\sigma f_1}\). Thus, the base of induction is found.

Suppose now that all the coefficients up to order \(n\) are found, i.e. \(m+l = 1, \ldots, n\). Then, the coefficients of the order \(n+1\) are found by solving a system of linear algebraic equations (SLAE) with a tridiagonal matrix, where the main diagonal elements are \(a_i = -(i+2/\sigma) u_{0,1}, \quad i = 0, \ldots, n\), the superdiagonal elements are \(b_j = -ju_{1,0} < 0, \quad j = n, \ldots, 1\) and the subdiagonal elements are equal to one [15]. Obviously, in general, such matrix is not diagonally dominant.

The resolution of the SLAE will be described below. But first we need to introduce some additional sequences

\[
\lambda_0 = 1, \quad \lambda_1 = a_n, \quad \lambda_{k+1} = a_{n-k} \lambda_k - b_k \lambda_{k-1};
\]

\[
\eta_0 = 1, \quad \eta_1 = a_0, \quad \eta_{k+1} = a_k \eta_k - b_{n+1-k} \eta_{k-1}; \quad k = 1, \ldots, n.
\]

By induction on \(k\) we can show that all \(\lambda_k \neq 0, \, \eta_k \neq 0\) (regardless of the sign of \(u_{0,1}\)).

Let \(|A|\) be the determinant of the matrix SLAE. Then \(|A| = \eta_{n+1} = \lambda_{n+1}\. In view of the introduced notations, \(n + 1\) order coefficients of the series (4) are determined by the following formulas

\[
u_{n,1} = \frac{1}{|A|} \lambda_n (L_{n,0} - f_{n+1}) + \frac{1}{|A|} \sum_{j=2}^{n+1} \eta_0 \lambda_{n+1-j} \prod_{l=j-1}^{1} (-b_{n+1-l}) L_{n+1-j, l-1},
\]

\[
u_{n-1,2} = -\frac{1}{|A|} \eta_0 \lambda_{n-1} (L_{n,0} - f_{n+1}) + \frac{1}{|A|} \eta_1 \lambda_{n-1} L_{n-1, 1} + \frac{1}{|A|} \sum_{j=3}^{n+1} \eta_1 \lambda_{n+1-j} \prod_{l=j-1}^{2} (-b_{n+1-l}) L_{n+1-j, l-1}.
\]

\[
u_{n+1-k, k} = \frac{1}{|A|} \eta_0 \lambda_{n+1-k} (-1)^{k-1} (L_{n,0} - f_{n+1}) + \]
for the solution \( b \)

the new coordinate

the sign of the existence and uniqueness of an analytic solution of the problem (2), (3) with the choice of

ensures the uniqueness. We show now that this implies the validity of the above statement.

\[ u_{1,n} = \frac{1}{|A|} \eta_0 \lambda_1(-1)^{n-1}(L_{n,0} - f_{n+1}) + \]

\[ + \frac{1}{|A|} \sum_{j=2}^{n} \eta_{j-1}\lambda_1(-1)^{n-j} L_{n+1-j,j-1} - \frac{1}{|A|} \eta_{n-1}\lambda_0 b_1 L_{0,n}, \]

\[ u_{0,n+1} = \frac{1}{|A|} \eta_0 \lambda_0(-1)^{n}(L_{n,0} - f_{n+1}) + \frac{1}{|A|} \sum_{j=2}^{n+1} \eta_{j-1}\lambda_0(-1)^{n+1-j} L_{n+1-j,j-1}. \]

Here \( L_{n-k,k} \) are known by the induction hypothesis (since it depends on the coefficients of the series of the order less than \( n \)):

\[ L_{n-k,k} = \sum_{i=0}^{n-k} \sum_{j=0}^{k} \frac{C_{n-k}^i C_{k}^j u_{i,j}}{\sigma} u_{n-k-i,k-j+2} + \]

\[ + \frac{1}{\sigma} \sum_{i=0}^{n-k} \sum_{j=0}^{k} \frac{C_{n-k}^i C_{k}^j u_{i,j+1} u_{n-k-i,k-j+1}}{R^j} + \]

\[ + \nu \sum_{i=0}^{n-k} \sum_{j=0}^{k} \frac{C_{n-k}^i C_{k}^j \left( \sum_{l=0}^{j} \frac{(-1)^{j-1}(j-l)!}{R^{j-l+1}} \right)}{u_{n-k-i,k-j+1}}. \]

Thus, the series (4) is constructed. Its convergence follows from Theorem 1, namely, from the existence and uniqueness of an analytic solution of the problem (2), (3) with the choice of the sign of \( u_{0,1} \).

The above construction of the series allows to prove the following statement. But first we make a shift of the spatial coordinate so that the boundary condition is set at the zero value of the new coordinate \( r \). Let \( r = \rho - \nu \).

**Theorem 2.** Under the conditions of the theorem 1 the problem

\[ (r + \nu R)^2 u_t = u \left[ \nu(r + \nu R) u_r + (r + \nu R)^2 u_{rr} \right] + \frac{1}{\sigma} (r + \nu R)^2 u_r^2, \]

\[ u(t,r)|_{r=0} = f(t), \quad u(t,r)|_{t=0} = 0, \]

has piecewise analytical solution which, if \( \nu = 0, 1, 2, \ldots \), is an analytical heat wave in the neighborhood of \( t = 0, r = 0 \). The choice of the direction of wave motion (to the singular point \( r = \nu R \) or from it) ensures the uniqueness.

**Proof.** From the theorem 1 and the results of this section there follows the existence of two piecewise analytical solutions of the boundary problem (2), (3). The choice of the sign of \( u_{0,1} \) ensures the uniqueness. We show now that this implies the validity of the above statement.

As shown in the beginning of this section \( u_{1,0} = f_1 > 0, u_{0,1} = \pm \sqrt{\sigma} f_1 \). Hence, we have that for the solution \( u = u_{+}(t,r) \) corresponding to a positive value of \( u_{0,1} \), there is a line \( r = b_+(t) \), \( b_+(0) = 0, b_+'(0) < 0 \), in the plane of the variables \( t, r \) on which the condition \( u_{+}|_{r=b_+(t)} = 0 \) is
satisfied in a neighborhood of \( t = 0, r = 0 \), if \( t > 0 \). For the solution \( u = u_-(t, r) \) corresponding to a negative value of \( u_{0,1} \), there is a line \( r = b_-(t) \), \( b_-(0) = 0 \), \( b'_-(0) > 0 \), in the plane of the variables \( t, r \) on which the analogous condition \( u_--|_{r=b_-(t)} = 0 \) is satisfied in a neighborhood of \( t = 0, r = 0 \), if \( t > 0 \).

Since \( b'_+(0) < 0 \), \( u_{1,0} > 0 \), the solution \( u = u_+(t, r) \) allows us to find the heat wave

\[
  u(t, x) = \begin{cases} 
    u_+ > 0, & b_+(t) < r \leq 0, \\
    0, & r \leq b_+(t),
  \end{cases}
\]

which is moving into the inner area towards the singular point \( r = \nu R \).

On the other hand, \( b'_- (0) > 0 \), \( u_{1,0} > 0 \) and the solution \( u = u_-(t, r) \) allows to determine the heat wave

\[
  u(t, x) = \begin{cases} 
    u_- > 0, & b_-(t) > r \geq 0, \\
    0, & r \geq b_-(t),
  \end{cases}
\]

which is moving to the external area.

It is easy to see that as \( b_+(0) = 0 \), then in both cases the condition \( u|_{t=0} = 0 \) is satisfied and also on the zero front we have discontinuities of the derivatives. The value of \( t^* \) (see the definition) is determined by the radius of convergence for the series (4). So, the constructed composite solutions, indeed, fall under the definition of an analytical heat wave. The choice of the sign of \( u_{0,1} \) (which, recall, provides the uniqueness) is equivalent to the choice of the front of the heat wave motion direction. This proves the theorem.

4. Comparison

As already noted, one of the main results of the present study is that for the first time the authors constructed solutions of the heat wave initiation problem by given boundary conditions in a constructive way. In this section, the constructed series are compared with the exact solutions. We further assume that \( \nu \neq 0 \), as the case \( \nu = 0 \) was considered previously in [5].

Let us obtain the first coefficients of the series. Already obtained in the previous section

\[
  u_{0,0} = 0, \quad u_{l,0} = f_l, \quad l = 1, 2, 3, ..., \quad u_{0,1} = \pm \sqrt{\sigma f_1},
\]

Using (5), we can show that

\[
  u_{1,1} = \frac{\pm(1 + 2/\sigma) \sqrt{\sigma f_1} f_2 - 2 f_1^2}{(3 + 4/\sigma) R f_1}, \quad u_{0,2} = \frac{R f_2 - 3 f_1 \sqrt{\sigma f_1}}{(3 + 4/\sigma) R f_1}.
\]

Note that if the boundary conditions are set in the form (3), then the parameter \( \nu \) specifies the point where the condition is set. At the same time \( \nu \) vanishes in the formulas for the coefficients.

The coefficients of the series (4) are compared with the coefficients of Taylor series for the exact solutions of Eq. (2) in a neighborhood of zero.

\[
  u = \pm \frac{\rho^2}{C \mp \alpha t} \mp \frac{C^{3-1}(\nu R)^2}{(C \mp \alpha t)^3},
\]

where \( \alpha = 2 + 2\nu + 4/\sigma, \quad \beta = (2 + 2\nu)/\alpha; \quad C > 0 \) is an arbitrary constant. The upper sign corresponds to the heat wave that moves from \( \rho = R\nu \) in the direction of increasing of the coordinate \( \rho \). The lower one corresponds to the movement in the direction of decreasing of \( \rho \) (to the singular point \( \rho = 0 \)). This solution was obtained by using the group theory method [17].
We obtain the coefficients of Taylor expansion for the exact solution by a direct
differentiation. For definiteness assume $C = 1, \nu = 1, R = 1$ and consider the movement of
the heat wave in the the outer area.

$$
\begin{align*}
    u_{1,0} &= f_1 = \alpha(1 - \beta) = \frac{4}{\sigma}, \\
    u_{0,1} &= -2, \\
    u_{2,0} &= f_2 = -\frac{16(3\sigma + 2)}{\sigma^2}, \\
    u_{1,1} &= \frac{8(\sigma + 1)}{\sigma}, \\
    u_{0,2} &= -2.
\end{align*}
$$

(9)

It is not difficult to make sure by substituting the absolute values that formula (7) takes
the form of (9). It should be observed that the high-order coefficients of the constructed series
coincide with the respective Taylor expansion coefficients of (8) as well.

Note, that proposed power series is applicable for verification of numerical results [6, 18].

5. Conclusion

Summarizing the research, note that the boundary value problem with singularity for a nonlinear
parabolic equation is considered. This equation becomes the one-dimensional porous medium
equation in the case the parameter $\nu$ is a non-negative integer. At the same time it can be
interpreted as a problem of the initiation of the heat wave by the boundary condition, which is
specified in the sphere in $\mathbb{R}^{\nu+1}$.

For the considered problem the existence and uniqueness theorem in the class of analytical
functions is proved. This theorem is an analogue of the classic Cauchy-Kovalevskaya theorem.
Authors construct solution in the form of multiple series with respect to powers of the physical
variables. Its coefficients are determined by solving the tridiagonal systems of linear algebraic
equations without diagonal dominance. A comparison of the constructed series coefficients and
the coefficients of Taylor series for exact solutions showed their identity.

Acknowledgments

The authors are grateful to P. Kuznetsov for collaboration in consideration of some particular
problems of this paper.

The reported study was particulary funded by RFBR according to the research projects
No. 16-01-00608 and No. 16-31-00291.

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