Analytic approach to transport in Josephson junctions beyond the Andreev approximation: General theory and applications to the BEC-BCS crossover

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Transport in Josephson junctions is commonly described using a simplifying assumption called the Andreev approximation, which assumes that excitations are fixed at the Fermi momentum and only Andreev reflections occur at interfaces (and no normal reflections). This approximation is appropriate for BCS-type superconductors, where the chemical potential vastly exceeds the pairing gap, but it breaks down for superconductors with low carrier density, such as topological superconductors, doped semiconductors, or superfluid quantum gases. Here, we present a generic analytical framework for calculating transport in Josephson junctions that lifts up the requirement of the Andreev approximation. Using this general framework, we study in detail transport in Josephson junctions across the BCS-BEC crossover, which describes the evolution from a BCS-type superconductor with loosely-paired Cooper pairs to a BEC of tightly-paired dimers. As the interaction is tuned from the BCS to the BEC regime, we find that the overall subgap current caused by multiple Andreev reflections decreases, but nonlinearities in the current-voltage characteristic called the subharmonic gap structure become more pronounced near the intermediate unitary limit, giving rise to sharp peaks and dips in the differential conductance with even negative conductance at specific voltages.

Transport measurements are powerful and commonly employed tools to probe quantum properties of matter. In particular, transport experiments with superconductors that are connected through a tunnel barrier or point contact [see Fig. 1(a)] were among the first to characterize the superconducting gap [1, 2]. They are now routinely used to probe subgap states, such as Andreev–Saint-James [3–5], Yu-Shiba-Rusinov [6, 7], or the celebrated Majorana bound states [8, 9], all of which give rise to a subgap tunneling current. Away from the tunneling limit, even without bound states a nonlinear subgap current arises through a process called multiple Andreev reflection (MAR) [10, 11]. Figure 1(b) shows such a MAR process, which begins with a quasiparticle entering the junction and increasing in energy by the voltage bias V. If the energy is not sufficient to overcome the excitation gap and enter the opposite reservoir, it is Andreev reflected [12–14] as a hole [blue line in Fig. 1(b)], creating a Cooper pair in the superconductor. The hole traverses the junction in the opposite direction, picking up the same voltage difference. The particle/hole undergoes MARs until it is transmitted into either reservoir. The resulting current has a highly nonlinear form—called subharmonic gap structure (SGS)—as a function of the bias voltage with resonances due to n-fold Andreev-reflection processes.

A quantitative theoretical description of transport across Josephson junctions is obtained by two different methods: The first is the Landauer-Büttiker scattering formalism [15–17] that generalizes the treatment of scattering at a single normal-superconducting (NS) boundary developed by Blonder, Tinkham, and Klapwijk (BTK) [18], and the second is the Keldysh Green’s function formalism [19–22] applied to a tunneling Hamiltonian [23, 24]. The formidable complexity of these calculations is at least partially reduced by making one crucial assumption known as the Andreev approximation (AA) [13, 25], which assumes that the chemical potentials of the superconductors are much larger than their pairing

FIG. 1. (a) Schematics of an SNS junction with a tunnel barrier (brown vertical bar). Shown are also the scattering amplitudes for particles (p, orange) and holes (h, magenta) with arrows indicating the direction of propagation. (b) MAR process. A quasiparticle state enters the junction from the left reservoir and picks up energy through MARs until it is transmitted into a reservoir. (c) Scattering processes at an NS interface with μ > 0 and μ ≈ ±Δ (upper panels) and μ < 0 with |μ| ≈ Δ (lower panels). Left panels show the Bogoliubov dispersion and examples of quasiparticle states, the right panel shows the amplitudes of Andreev reflection [A(E)], blue line], normal reflection of particles [Np(E), red] and holes [Nh(E), green]. The AA results are shown as dashed lines in upper right panel.
gaps. This approximation fixes excitations at the Fermi momentum and neglects normal reflections at NS interfaces [red and green lines in Fig. 1(b)], which is appropriate for BCS-type superconductors [see Fig. 1(c) and the following discussion]. Indeed, the SGS in point contacts of (BCS-type) materials like niobium is very accurately described in this way [26].

The AA, however, no longer applies to superconductors with a gap comparable to their chemical potential, such as topological superconductors [27–32], high-\(T_c\) superconductors [33], superconducting semiconductors like strontium titanate [34–36], or iron chalcogenides [37–40], where transport experiments are routinely performed. Moreover, the AA is not expected to apply to ultracold Fermi gases, for which two-terminal transport is probed using optically created point contacts [41–46]. The description of these systems requires a more general formulation of transport theory that goes beyond the AA. This is the purpose of the current paper.

In this Letter, we use the scattering approach to formulate a generic analytical framework for transport in superconducting/superfluid junctions without making the AA. To demonstrate the effectiveness of our approach, we describe particle transport along the universal BEC-BCS crossover [47] in point contacts between two-component Fermi gases. The reservoir interaction strength in these systems is parametrized by an inverse \(s\)-wave scattering length \(a^{-1}\), which continuously interpolates between a BCS superfluid with \(\mu \gg \Delta\) (for which the AA holds), and a regime where the AA no longer applies, a unitary regime with \(\mu \simeq \Delta\) followed by a BEC superfluid of dimers with negative chemical potential \(\mu < 0\). Our results indicate that the interplay of Andreev and normal reflections leads to a significantly more pronounced SGS in the unitary regime compared to the BCS limit, with sharp peaks and dips at specific voltages leading to negative differential conductance.

We model a superfluid-normal-superfluid (SNS) junction with a single channel as a one-dimensional constriction that connects two superfluid reservoirs through a normal region at \(x = 0\), where geometric details of the junction are accounted for by a tunnel barrier [see Fig. 1(a)]. Given a bias \(V = \mu_L - \mu_R\) between the reservoir chemical potentials, the aim is to compute the expectation value of the current

\[
\hat{I}(x) = -\frac{ie}{2m} \sum_{\sigma = \uparrow, \downarrow} \left( \hat{\psi}^\dagger_\sigma (\nabla_x \hat{\psi}_\sigma) - (\nabla_x \hat{\psi}^\dagger_\sigma) \hat{\psi}_\sigma \right)
\]

(1)

across the junction, where \(\hat{\psi}^\dagger_\sigma\) is the creation operator of a particle with mass \(m\) and spin projection \(\sigma\). The Landauer-Büttiker formalism expands these particle operators in a basis of scattering states across the junction, which are in equilibrium with their reservoir of origin. The dc current is then the difference

\[
I_{dc}(V) = I_{dc}^<(V) - I_{dc}^>(V)
\]

(2)

between a current \(I_{dc}^<\) due to right-moving states originating in the left reservoir and a current \(I_{dc}^>\) due to left-moving states from the right reservoir. The challenge is to compute the scattering states needed to evaluate the current in Eq. (2). To make the problem tractable, we make a Bogoliubov approximation for the reservoir states [48] and assume sharp boundaries, i.e., the chemical potential and pairing gaps in the SNS junction are step functions with \(\mu(x) = \mu_L \Theta(-x) + \mu_R \Theta(x)\) and \(\Delta(x) = \Delta_L \Theta(-x - \ell/2) + \Delta_R \Theta(x - \ell/2)\), respectively, where \(\ell\) is the length of the normal region (here we consider short junctions, \(\ell \to 0\)). Scattering states are then obtained by solving the Bogoliubov equation with potentials \(\mu(x)\) and \(\Delta(x)\). Within each region, where \(\mu\) and \(\Delta\) are constant, solutions take a standard plane-wave form with Bogoliubov dispersion (for an \(s\)-wave gap) \(E^2 = (\varepsilon_q - \mu)^2 + |\Delta|^2\) with \(\varepsilon_q = \hbar^2 q^2/2m\). Figure 1(c) shows examples of energy dispersions in both superfluid and normal regions for two values \(\mu = 1.25\Delta\) (upper panel) and \(\mu = -\Delta\) (lower panel). Arrows indicate the propagation direction of an excitation, which is set by its group velocity \(\vec{v}(E) = \partial E(q)/\partial q\). Here, we distinguish particle-type (orange) and hole-type (magenta) excitations, for which the direction of group velocity and wave number are the same (or opposite). Note that within the AA, the whole structure simplifies considerably: All excitations are fixed at the Fermi momentum \(\hbar k_F = \sqrt{2m\mu}\), such that the momentum essentially drops out as a variable. The explicit form of a particle/hole-like (\(\kappa = p, h\)) Bogoliubov state with energy \(E\) (normalized such that it has unit probability current) in the left (\(L\)) and right (\(R\)) reservoir is

\[
\Psi_{\kappa,L}^S(E) = e^{-i\frac{\epsilon_q}{2} \xi} \sqrt{\frac{m}{\hbar^2 p_L(E)}} \frac{\left( u_{\kappa,L}(E) \right)}{\left( v_{\kappa,L}(E) \right)} e^{i\kappa L x},
\]

(3)

\[
\Psi_{\kappa,R}^S(E) = e^{-i\frac{\epsilon_q}{2} \xi} \sqrt{\frac{m}{\hbar^2 p_R(E)}} \frac{\left( u_{\kappa,R}(E) e^{-i\frac{\epsilon_q}{2} \xi} \right)}{\left( v_{\kappa,R}(E) e^{i\frac{\epsilon_q}{2} \xi} \right)} e^{i\kappa R x},
\]

(4)

with Bogoliubov factors \(u_{pL/R}^2 v_{pL/R}^2 = |1 \pm (\varepsilon_q - \mu_{L/R})/E|/2\), where \(\varepsilon_q = \hbar^2 q^2/2m\). The time dependence \(e^{i\xi q}\) in Eq. (4) appears because here the state is written with reference to the chemical potential of the left reservoir [49]. Due to MARs at finite voltage bias \(V\), the scattering state is a superposition of all Bogoliubov states with energy \(E_n = E + nV\), with amplitudes that must be matched at the NS boundaries and tunnel barrier. Computing the scattering solution is thus more intricate than solving a one-dimensional scattering problem without voltage bias.

In the following, we outline the matching of coefficients, where for notational simplicity, we consider quasi-particle states that are injected from the left reservoir and thus contribute to \(I_{dc}^<\) (the reverse current \(I_{dc}^>\) is obtained by interchanging the reservoir indices). We use \(a_{L/R,n}\), \(b_{L/R,n}\), \(c_{L/R,n}\), and \(d_{L/R,n}\) to denote the probability amplitudes of right-moving particles, left-moving
holes, right-moving holes and left-moving particles, respectively, with energy $E_n$ in the left ($L$) and right ($R$) normal region [see Fig. 1(a)]. These amplitudes are connected by three elementary scattering processes. First, at the left $S_L$-$N_L$ boundary

$$\begin{aligned}
(a_{L,n}, c_{L,n}) &= \left(N_{L,n}^P A_{L,n} N_{L,n}^h \right) \left(d_{L,n}, b_{L,n} \right) + \delta_{n0} \left(J_{L,n}^P J_{L,n}^h \right),
\end{aligned} \tag{5}$$

where $A_{L,n}, N_{L,n}^P$, and $N_{L,n}^h$ are the (energy-dependent) Andreev reflection amplitude as well as the normal reflection amplitude for particles and holes, respectively, and $J_{L,n}^P$ and $J_{L,n}^h$ are the transmission amplitudes of injected quasiparticles from the left ($S_L$) reservoir into particles and holes, respectively, in the left-normal region ($N_L$).

Second, the analogous process at the right $N_R$-$S_R$ boundary is

$$\begin{aligned}
(d_{R,n}, b_{R,n}) &= \left(N_{R,n}^P A_{R,n} N_{R,n}^h \right) \left(a_{R,n}, c_{R,n} \right).
\end{aligned} \tag{6}$$

The scattering processes in Eqs. (5) and (6) are illustrated in Figs. 1(b) and 1(c), where Andreev reflections are sketched in blue, normal reflection of particles in red and of holes in green. Analytical expressions for these amplitudes are obtained by matching the wave functions at the NS boundaries [50]. Note that within the AA, a quasiparticle impinging from a reservoir is always transmitted into the normal region as a particle, such that $J_{L,n}^P = 1$ and $J_{L,n}^h = 0$; the opposite is true for a quasihole. In addition, normal reflection is absent and there is perfect Andreev reflection $|A_{L,n}|^2 = 1$ for energies below the pairing gap [blue dashed line in Fig. 1(c)]. As is apparent from the figure, the AA is not reliable once $\mu \sim \Delta$.

Third, the scattering matrices for particles and holes at the tunnel barrier, where we apply the voltage bias, are

$$\begin{aligned}
&\begin{pmatrix}
 d_{L,n} \\
 a_{R,n+1}
\end{pmatrix} = \begin{pmatrix}
 r_{p,n} t_{p,n} & t_{p,n} t_{p,n}^{*} \\
 t_{p,n} t_{p,n}^{*} t_{p,n} & r_{p,n} t_{p,n}^{*}
\end{pmatrix} \begin{pmatrix}
 a_{L,n} \\
 d_{L,n+1}
\end{pmatrix}, \tag{7a}
&\begin{pmatrix}
 b_{L,n} \\
 c_{R,n-1}
\end{pmatrix} = \begin{pmatrix}
 r_{h,n} t_{h,n} & t_{h,n} t_{h,n}^{*} \\
 t_{h,n} t_{h,n}^{*} t_{h,n} & r_{h,n} t_{h,n}^{*}
\end{pmatrix} \begin{pmatrix}
 c_{L,n} \\
 b_{L,n-1}
\end{pmatrix}, \tag{7b}
\end{aligned}$$

with transmission ($t_{p,n}, t_{h,n}$) and reflection coefficients ($r_{p,n}, r_{h,n}$). The various coefficients are successively eliminated using Eqs. (5)-(7), which reduce to a recurrence relation for a single set of coefficients, for example ($d_{L,n}$):

$$\begin{aligned}
&&\begin{aligned}
&\begin{pmatrix}
 \alpha_n d_{L,n+2} + \beta_n d_{L,n} + \gamma_n d_{L,n-2} = S_{L,n}^P \delta_{n0} + S_{L,n}^h \delta_{n,-2},
\end{pmatrix}
&\end{aligned}
\end{aligned} \tag{8}$$

where $\alpha_n, \beta_n, \gamma_n, S_{L,n}^P$ and $S_{L,n}^h$ are functions of the scattering coefficients in Eqs. (5)-(7), for which we obtain a closed analytical form even without the AA [50]. The recurrence relation in Eq. (8) is solved using the modified Lenz method [17, 51]. Remaining coefficients are then obtained by substituting back into Eqs. (5)-(7).

Having obtained the scattering states for all energies, we evaluate the current [Eq. (1)] as

$$\begin{aligned}
I_{dc}(V) &= \frac{2}{h} \int_{-\infty}^{\infty} dE D_L(E) \\
&\times \left[ f(E) T_{p \rightarrow R}^{\text{(0)}}(E) + (1-f(E)) T_{h \rightarrow R}^{\text{(0)}}(E) \right], \tag{9}
\end{aligned}$$

where $D_L(E)$ is the quasiparticle density of states (DOS) of the left superfluid, $f(E) = 1/[1+\exp(E/k_B T)]$ (in this paper, we set the temperature $T = 0$), and

$$\begin{aligned}
&T_{p \rightarrow R}^{\text{(0)}}(E) = \sum_{n=-\infty}^{\infty} \left( |a_{L,n}|^2 - |d_{L,n}|^2 \right) \Theta(\mu_L + E_n), \tag{10a}
&T_{h \rightarrow R}^{\text{(0)}}(E) = \sum_{n=-\infty}^{\infty} \left( |b_{L,n}|^2 - |c_{L,n}|^2 \right) \Theta(\mu_L - E_n). \tag{10b}
\end{aligned}$$

are the dimensionless particle and hole current densities at energy $E$ due to quasiparticles that are transmitted into the normal region as particles [Eq. (10a)] and holes [Eq. (10b)], respectively. Due to particle-hole symmetry, the current from quasihole injections is the same as that from quasiparticles, hence the factor of 2 in Eq. (9).

Within our formalism, we obtain an analytic result for the current in the tunneling limit $|t_{p,n}| = |t_{h,n}| = t \rightarrow 0$ [50]:

$$\begin{aligned}
I_{dc}(V) &= \frac{2}{h} t^2 \int_{-\infty}^{\infty} dE \rho_L(E) \rho_R(E+V) [f(E) - f(E+V)], \tag{11}
\end{aligned}$$

where $\rho_L/R$ is the particle tunneling DOS of the left and right reservoir. Equation (11) is obtained by keeping terms up to $O(t^2)$ in Eq. (8), which corresponds to truncating the recurrence relation at $|n| \leq 2$. The tunneling current arises due to a direct transmission from the occupied band of one reservoir to the empty band of the
other reservoir and therefore is proportional the DOS of these two bands [52]. It flows only when $|V| \geq \Delta_L + \Delta_R$ where $\Delta_{L/R}$ is the spectral gap with $\Delta_{L/R} = \Delta_{L/R}$ for $\mu_{L/R} > 0$ and $\Delta_{L/R} = \left| \mu_{L/R} + \Delta_{L/R} \right|^{1/2}$ for $\mu_{L/R} \leq 0$. At larger junction transparencies, MAR processes contribute, which give rise to the SGS.

We now apply our general scattering formalism to study transport in SNS junctions across the BEC-BCS crossover, which is characterized by three parameters: the density imbalance between the two superfluids, $\nu = (n_{L} - n_{R})/(n_{L} + n_{R})$, which determines the voltage $V = \mu_{L} - \mu_{R}$ across the junction; the overall interaction strength $1/k_{F}a$ with $k_{F} = (3\pi^{2}n)^{1/3}$ (where $n = (n_{L} + n_{R})/2$); and the tunnel-barrier transparency $\mathcal{T}$, for which we assume a delta-function scatterer with energy-independent scattering coefficients in Eq. (7) with $t_{p,n} = t_{h,n} = \sqrt{\mathcal{T}} e^{i\eta}$ and $r_{p,n} = r_{h,n} = -i\sqrt{1 - \mathcal{T}} e^{i\eta}$, where $\eta = -\arctan(\sqrt{1/\mathcal{T} - 1})$ [53, 54]. We relate $a$ and $\nu$ to the reservoir chemical potentials and pairing gaps using the bulk mean-field equation of state [55], although this choice is not specific to our method and these parameters may be inferred from other many-body calculations [56].

Figure 2 shows the evolution of the SGS across the BEC-BCS crossover. Shown is the dc current (left panel) and the differential conductance (right panel) as a function of density imbalance $\nu$ (where $\nu$ is proportional to the voltage bias $V$ for $\nu \lesssim 0.8$) with different scattering lengths and perfect barrier transparency $\mathcal{T} = 1$. In the BCS limit ($1/k_{F}a \ll 0$), where the AA applies, our results reproduce standard BTK-type behavior [10, 11, 15–17, 19–21] with a weak SGS and a strong current due to MAR. As the system is tuned away from the BCS toward the BEC limit, the current decreases, indicating that the reservoirs become more insulating with an increase in normal reflections and a simultaneous decrease in MARs. The SGS, however, becomes sharper and develops dip-like features with increasingly pronounced peaks and dips in the conductance. This behavior is not captured by the AA, which breaks down even for moderate deviations from the strict BCS limit. Near unitarity ($1/k_{F}a \rightarrow 0$), the SGS is so strongly nonlinear that the conductance becomes negative. This feature can serve as a signature of the unitary limit in mesoscopic transport experiments. Finally, as the scattering length is tuned to the BEC side of the crossover ($1/k_{F}a \gg 0$), MAR processes are weakened, which strongly suppresses the current below the tunneling gap [57].

Besides tuning the BEC-BCS crossover, MARs and the SGS can also be suppressed by decreasing the barrier transparency. To compare these two effects, we extend our results in Fig. 3, where we show the dc current $I_{dc}$ (upper panels) and conductance (lower panels) as a function of density imbalance $\nu$ across the BEC-BCS crossover (left to right panels) for different barrier transparencies, interpolating between a tunnel junction limit ($\mathcal{T} \rightarrow 0$) with a current given by Eq. (11) and the perfectly transparent barrier ($\mathcal{T} \rightarrow 1$) discussed above. The current generally decreases with decreasing barrier transparency $\mathcal{T}$. In the BCS regime, we recover BTK-type behavior [15, 17, 19, 21], where the SGS becomes more pronounced at intermediate transparencies with kinks in the current and resonances in the conductance whenever the voltage exceeds the threshold for additional Andreev reflections [58]. Moreover, the SGS has smooth resonances and no negative conductance, which is visibly different
from the unitarity limit shown in Figs. 2 and 3(d,i). It is therefore inaccurate to model an SNS junction in the BEC-BCS crossover phenomenologically by assuming the AA with a finite barrier strength \[11, 59, 60\]. Again, on the BEC side [Figs. 3(e) and 3(j)], no current flows below the tunneling gap due to the absence of MAR.

We finally note that the current across a superfluid point contact at unitarity has been studied in quantum gas experiments \[43, 61\]. Interestingly, the conductance at small voltage bias in these experiments is larger than in the BCS limit, which can be attributed to fluctuations and geometric effects in the reservoir \[62–66\] that are not modeled by our theory. Despite these differences, it is noticeable that the measured current [Fig. 2(b) of Ref. [43]] has a strongly oscillatory structure with possible negative conductance, consistent with the single-channel model predictions discussed here.

In summary, we have presented a general scattering theory of transport in SNS junctions. Our approach is analytical and goes beyond the AA assumed in previous analytical works \[15–17\]. While our framework applies to any kind of SNS junctions, here we specifically use it to study transport in the BEC-BCS crossover. We show that near the unitary limit, the SGS becomes more pronounced with sharp peaks and dips in the current and even negative differential conductance, which can be used as a signature of the BEC-BCS crossover.

We thank Sriram Ganeshan and Alejandro Lobos for discussions. This work is supported by the Army Research Office Grant no. W911NF-19-1-0328 [FS] and Vetenskapsrådet (grant number 2020-04239) [JH].

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that are expressed as a harmonic series of the pairing gaps $[16, 17]$:

(a) $|V| = \Delta_L/m$ with $0 < m \leq \Delta_L/\Delta_R$, (b) $|V| = \Delta_R/m$ with $m > 0$, and (c) $|V| = (\Delta_L + \Delta_R)/(2m + 1)$ with $0 < m \leq \Delta_R/\Delta_L$, where the upper bounds for $m$ in (a) and (c) are derived by considering only Andreev reflections inside the gap.

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Supplemental Material for “Analytic approach to transport in Josephson junctions beyond the Andreev approximation: General theory and applications to the BEC-BCS crossover”

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(Dated: August 25, 2021)

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This supplemental material expands on the discussion in the main text and presents a self-contained derivation of the Landauer-Büttiker formalism for SNS junctions. We focus in particular on complications introduced when going beyond the Andreev approximation (AA). General introductions to the Landauer-Büttiker formalism applied to superconducting junctions are found, for example, in Refs. [S1, S2]. Earlier references that discuss the transport across SNS junctions using the Andreev approximation are Refs. [S3–S6].

\section{Setup and Landauer-Büttiker Formalism}

Consider two superfluid reservoirs. The reservoir on the left-hand side (right-hand side) is defined by a chemical potential \( \mu_L \) (\( \mu_R \)) and a pairing gap \( \Delta_L \) (\( \Delta_R e^{i\phi} \)). There is a confining potential \( U(x, r_\perp) \) such that both reservoirs are connected through a small contact at \( x = 0 \) that allows particle exchange and hence a particle current. We want to determine this current, i.e., we want to compute the expectation value of the operator

\[ \hat{I}_x (r, x) = \frac{-i\hbar}{2m} \sum_{\sigma = \uparrow, \downarrow} \int dr_\perp \left[ \hat{\psi}_\sigma^\dagger (\nabla \hat{\psi}_\sigma) - (\nabla \hat{\psi}_\sigma^\dagger) \hat{\psi}_\sigma \right], \]  

where \( \hat{\psi}_\sigma \) is a fermion field operator for a particle of mass \( m \) and spin projection \( \sigma \). The Landauer-Büttiker formalism assumes no particle interactions inside the junction (for a study of interaction effects in a quantum dot across the BEC-BCS crossover using other methods, see Ref. [S7]) and describes transport in terms of scattering solutions for Bogoliubov excitations. In principle, the condensate depends self-consistently on the Bogoliubov modes through a gap equation,

\[ \Delta (r) = -\frac{g}{2} \sum_m f(E_m) \Phi_m^\dagger (r) \sigma_x \Phi_m (r), \]  

where \( g \) is the interaction strength, \( f(E) = 1/[1 + \exp(E/k_B T)] \) is the Fermi function, \( m \) runs over all Bogoliubov eigenstates with excitation energy \( E_m \), \( \sigma_x \) is the \( x \)-Pauli matrix in the Nambu space, and \( \Phi_m \) solves the Bogoliubov-de Gennes equation

\[ E_m \Phi_m (r) = \begin{pmatrix} H_0 - \mu (r) & \Delta (r) \\ \Delta^* (r) & -(H_0^* - \mu (r)) \end{pmatrix} \Phi_m (r). \]  

Here, \( H_0 = -\hbar^2 \nabla^2 / 2m + U(x, r_\perp) \) is the Hamiltonian of a single particle in the potential \( U(x, r_\perp) \). If the confining potential \( U \) varies slowly along the \( x \)-direction, we may separate the wave function in a transverse part \( \lambda_\alpha (r_\perp) \) and a longitudinal part \( \Psi_\alpha (x) \) as \( \Phi_\alpha (r) = \Psi_\alpha (x) \lambda_\alpha (r_\perp) \), where \( \alpha \) is the transverse band index. In the following, we consider a single transverse band. We assume that the chemical potential is given by a step-like function

\[ \mu (x) = \begin{cases} \mu_L, & x < 0, \\ \mu_R, & x > 0. \end{cases} \]  

Furthermore, we note that in the constriction, the self-consistent gap is smaller than the bulk value [S8] and fluctuations beyond mean-field theory in this confined geometry further decreases the gap [S9]. In this paper, we make the standard approximation [S1, S2] to forgo the self-consistent solution of Eq. (S2) and instead choose a step-like function for the pairing gap

\[ \Delta (x) = \begin{cases} \Delta_L, & x \leq -\ell/2, \\ 0, & -\ell/2 < x < \ell/2, \\ \Delta_R e^{i\phi}, & x \geq \ell/2, \end{cases} \]  

where the pairing gaps in both reservoirs are constant (in this paper, the superconducting phase difference is \( \phi = 0 \)) and equal to their bulk values, while the constriction is assumed to be a normal region of length \( \ell \). In addition, we include a tunnel barrier at \( x = 0 \). While the scattering matrix formalism that we derive in later sections holds for any length \( \ell \), for simplicity we present results for \( \ell \to 0 \) throughout the main paper. The schematics of the SNS junction defined by the pairing potential in Eq. (S5) is sketched in Fig. 1 of the main text.
The Landauer-Büttiker formalism makes two central assumptions:

(a) In the presence of the junction, fermion operators may be expanded in a basis set of Lippmann-Schwinger scattering states across the potential [Eqs. (S4) and (S5)]. Restricting to a single transverse channel \( \alpha = 0 \) with transverse eigenmode \( \lambda_0(\mathbf{r}_\perp) \), fermion operators are

\[
\begin{aligned}
\begin{bmatrix}
\hat{\psi}_+^l(\tau, \mathbf{r}) \\
\hat{\psi}_-^l(\tau, \mathbf{r})
\end{bmatrix} &= \lambda_0(\mathbf{r}_\perp) \sum_{\kappa=p,h} \sum_{j=L,R} \int_{-\infty}^{\infty} \frac{dE}{\sqrt{2\pi\tilde{v}_{\kappa,j}(E)}} \frac{\sqrt{h q_{\kappa,j}(E)}}{m} \left\{ \Psi_{\kappa,j}^S(E, \tau, x) \hat{a}_{\kappa,j}^-(E) + \Psi_{\kappa,j}^{S,\tau}(E, \tau, x) \hat{a}_{\kappa,j}^-(E) \right\},
\end{aligned}
\]

where \( \Psi_{\kappa,j}^{S,\xi}(E, \tau, x) \) describe the time-dependent scattering states in the longitudinal direction for reservoir \( j = L, R \) that are asymptotically described by a right-moving (\( \xi = \rightarrow \)) or left-moving (\( \xi = \leftarrow \)) incoming Bogoliubov excitation with energy \( E \), and \( \kappa = p, h \) runs over different particle \( (p) \) and hole \( (h) \) Bogoliubov modes for a given energy. Furthermore, the fermionic Bogoliubov operators satisfy

\[
\{ \hat{a}_{\kappa,j}^S(E), \hat{a}_{\kappa',j'}^{C,\tau}(E') \} = \delta_{\kappa\kappa'} \delta_{jj'} \delta_{\xi\xi'} \delta(E - E'),
\]

and the factor

\[
\tilde{v}_{\kappa,j}(E) = \left| \frac{dq_{\kappa,j}(E)}{dE} \right|^{-1}
\]

is the residual Jacobian of the transformation from the wave number to energy integration. By convention, we separate a factor \( \sqrt{h q_{\kappa,j}(E)/m} \) in Eq. (S6) from the Lippmann-Schwinger wave functions such that the Bogoliubov excitations carry unit current. The central task is to compute the scattering states \( \Psi_{\kappa,j}^{S,\xi}(E, \tau, x) \); the explicit form of which is constructed in Sec. III and following.

(b) It is assumed that the scattering states are in equilibrium with their respective reservoirs, i.e.,

\[
\langle \hat{a}_{\kappa,j}^{S,\xi}(E) \hat{a}_{\kappa',j'}^{C,\tau}(E') \rangle = \delta_{\kappa\kappa'} \delta_{jj'} \delta_{\xi\xi'} \delta(E - E') f(E).
\]

Note that our formalism holds generally for any temperature, the paper presents results zero temperature \((T = 0)\) for simplicity.

The expectation value of the current [Eq. (S1)] then consists of a contribution from quasiparticles and quasiholes injected from the left reservoir \( j = L \) (calculated using the scattering states \( \Psi_{\kappa,j}^S \)) and a current due to quasiparticles and quasiholes injected from the right reservoir \( j = R \) (calculated using the state \( \Psi_{\kappa,j}^{S,\tau} \)). Since states injected from the left/right reservoir that impinge on the junction are right/left moving, we will use \( j = L, R \) and suppress the index \( \equiv \) whenever redundant. Because of the chemical potential mismatch between the left and right reservoirs, states at different energies may be related by multiple Andreev reflection (MAR) in the Josephson junction. Hence, we expand the Lippmann-Schwinger states as

\[
\Psi_{\kappa,j}^S(E, \tau, x) = \sum_{n=-\infty}^{\infty} e^{-iE_n \tau/h} \begin{pmatrix} \phi_{\kappa,j}^{(n)}(x) \\ \chi_{\kappa,j}^{(n)}(x) \end{pmatrix},
\]

where \( E_n = E + nV \) with \( V = \mu_L - \mu_R \) being the chemical potential difference between the two reservoirs (the bias voltage). The current [Eq. (S1)] is then written as

\[
I = I_{dc}(V) + 2 \sum_{m=1}^{\infty} \left[ A_m(V) \cos \omega_m \tau + B_m(V) \sin \omega_m \tau \right],
\]

with \( \omega_m \equiv m V/h \) and

\[
A_m(V) = \frac{2}{h} \Re \int_{-\infty}^{\infty} dE \left\{ D_L(E) \left[ f(E) T_{p \rightarrow}^{(m)}(E) + (1 - f(E)) T_{p \rightarrow}^{h(m)}(E) \right] - D_R(E) \left[ f(E) T_{p \rightarrow}^{(m)}(E) + (1 - f(E)) T_{p \rightarrow}^{h(m)}(E) \right] \right\},
\]

(S12a)
\[ B_m(V) = \frac{2}{\hbar} \text{Im} \int_{-\infty}^{\infty} dE \left\{ D_L(E) \left[ f(E)T_{p,m}(E) + (1 - f(E))T_{p,m}^*(E) \right] \right. \\
\left. - D_R(E) \left[ f(E)T_{p,m}(E) + (1 - f(E))T_{p,m}^*(E) \right] \right\}, \tag{S12b} \]

\[ I_{dc}(V) \equiv A_0(V), \tag{S12c} \]

where we define an effective quasiparticle density of states in the \( j = L, R \) reservoir by

\[ D_j(E) = \frac{\hbar^2 q_{p,j}(E)}{m \bar{v}_{p,j}(E)}. \tag{S13} \]

In addition, \( T_{p,\zeta}^{(m)}(E) \) and \( T_{p,\zeta}^{(h)}(E) \) are the dimensionless \( m \)-th Fourier components of the particle and hole current density due to quasiparticle injections at energy \( E \) from the left (\( \zeta = \rightarrow \)) or the right (\( \zeta = \leftarrow \)) superfluids:

\[ T_{p,\zeta}^{(m)}(E) = -\frac{i\hbar}{2m} \sum_{n=-\infty}^{\infty} \left( \varphi_{p,L,R}^{(n+m)*}(x)[\nabla \varphi_{p,L/R}^{(n)}(x)] - [\nabla \varphi_{p,L/R}^{(n+m)*}(x)]\varphi_{p,L/R}^{(n)}(x) \right), \tag{S14a} \]

\[ T_{p,\zeta}^{(h)}(E) = -\frac{i\hbar}{2m} \sum_{n=-\infty}^{\infty} \left( \chi_{p,L,R}^{(n)}(x)[\nabla \chi_{p,L,R}^{(n+m)*}(x)] - [\nabla \chi_{p,L,R}^{(n)}(x)]\chi_{p,L,R}^{(n+m)*}(x) \right). \tag{S14b} \]

As is shown below, the quasihole contribution to the current is equal to the quasiparticle current, hence the overall factor of 2 in Eq. (S12).

**II. BOGOLIUBOV STATES**

To set the notation, consider first Bogoliubov excitations propagating in the \( x \)-direction deep in a single reservoir, where \( \Delta(r) \) is independent of \( x \) and equal to the bulk values \( \Delta_L \) and \( \Delta_R e^{i\phi} \), respectively. The explicit result for the wave function of an excitation with momentum \( q \) along the \( x \) direction (recall that the prefactor is chosen such that for real \( q \), the state has unit probability current)

\[ \Psi(q) = \sqrt{\frac{m}{\hbar q}} \begin{pmatrix} u(q) \\ v(q) \end{pmatrix} e^{-iE_q \tau / \hbar} e^{iqx}, \tag{S15} \]

with two energy branches (for an \( s \)-wave gap)

\[ E_q = \pm \sqrt{(\varepsilon_q - \mu)^2 + |\Delta|^2}, \tag{S16} \]

where \( \varepsilon_q = \frac{\hbar^2 q^2}{2m} \), and the Bogoliubov coefficients are

\[ u^2(q) = \frac{1}{2} \left( 1 + \frac{\varepsilon_q - \mu}{E_q} \right), \tag{S17a} \]

\[ v^2(q) = \frac{1}{2} \left( 1 - \frac{\varepsilon_q - \mu}{E_q} \right). \tag{S17b} \]

For further reference, recall that Bogoliubov excitations are superpositions of a spin-up particle and a spin-down hole, such that a change in the reference potential (for example, due to the chemical potential mismatch between reservoirs) will affect the phase of the two Bogoliubov components in Eq. (S15) in the opposite way [S1].

The Bogoliubov spectrum (S16) is illustrated in Fig. S1 for (a) positive and (b) negative chemical potential. We call states

- **particle-like** if their group velocity \( \tilde{v}(q) = (\partial E_q / \partial q)/\hbar \) has the same sign as \( q \) (red and blue line in Fig. S1, orange lines in Fig. 1 of the main text),

- **hole-like** if the group velocity and momentum have opposite sign (orange and green line in Fig. S1, pink lines in Fig. 1 of the main text).
Supplementary Figure S1. Energy spectrum Eq. (S16) for (a) \( \mu > 0 \) (corresponding to the BCS-side of the crossover) and (b) \( \mu < 0 \) (corresponding to the BEC-side of the crossover).

Furthermore, we call a state

- **right-moving** if it has positive group velocity (orange and red line in Fig. S1),
- **left-moving** if it has negative group velocity (blue and green line in Fig. S1).

For reference, we note the solution for a given energy for the momentum of the right-moving particle-like excitation (red line in Fig. S1):

\[
q_p(E) = \frac{\sqrt{2m}}{h} \begin{cases} 
\left( \frac{i}{\sqrt{2}} \sqrt{E^2 - |\Delta|^2 - \mu} \right), & E < -\sqrt{\mu^2 + |\Delta|^2}, \\
\left( \frac{i}{\sqrt{2}} \sqrt{\mu + i0 - \sqrt{E^2 - |\Delta|^2} - \mu} \right), & -\sqrt{\mu^2 + |\Delta|^2} < E < -|\Delta|, \\
\left( \frac{i}{\sqrt{2}} \sqrt{\mu + i\sqrt{|\Delta|^2 - E^2}} \right), & -|\Delta| < E < |\Delta|, \\
\left( \frac{i}{\sqrt{2}} \sqrt{\mu + \sqrt{E^2 - |\Delta|^2} - \mu} \right), & |\Delta| < E < \sqrt{\mu^2 + |\Delta|^2}, \\
\left( \frac{i}{\sqrt{2}} \sqrt{-\mu + \sqrt{E^2 + |\Delta|^2}} \right), & \sqrt{\mu^2 + |\Delta|^2} < E.
\end{cases}
\]

(S18)

and for the hole-like left-moving excitation (green line in Fig. S1)

\[
q_h(E) = \frac{\sqrt{2m}}{h} \begin{cases} 
\left( \frac{1}{\sqrt{2}} \sqrt{E^2 - |\Delta|^2 - \mu} \right), & E < -\sqrt{\mu^2 + |\Delta|^2}, \\
\left( \frac{1}{\sqrt{2}} \sqrt{\mu - i0 + \sqrt{E^2 - |\Delta|^2} - \mu} \right), & -\sqrt{\mu^2 + |\Delta|^2} < E < -|\Delta|, \\
\left( \frac{1}{\sqrt{2}} \sqrt{\mu - i\sqrt{|\Delta|^2 - E^2}} \right), & -|\Delta| < E < |\Delta|, \\
\left( \frac{1}{\sqrt{2}} \sqrt{\mu - i0 - \sqrt{E^2 - |\Delta|^2} - \mu} \right), & |\Delta| < E < \sqrt{\mu^2 + |\Delta|^2}, \\
\left( \frac{1}{\sqrt{2}} \sqrt{-\mu - \sqrt{E^2 + |\Delta|^2}} \right), & \sqrt{\mu^2 + |\Delta|^2} < E.
\end{cases}
\]

(S19)

The imaginary part is chosen such that the right-moving excitations with momenta \( q_p(E) \) and \( -q_h(E) \) decays at positive spatial infinity, and left-moving excitations with momenta \( -q_p(E) \) and \( q_h(E) \) with negative imaginary parts decay at negative spatial infinity. Corresponding momenta with negative real part of the left-moving particle-like excitation are given by \( -q_p(E) \) and of the right-moving hole-like excitation by \( -q_h(E) \), respectively. Note that the Andreev approximation neglects the energy-dependence of the momenta completely and assumes modes propagating with fixed wave number \( k_F = \sqrt{2m\mu}/h \). The Bogoliubov coefficients for particle-like states are

\[
u_p(E) = \begin{cases} 
\text{sgn}(E) \left( \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\sqrt{E^2 - |\Delta|^2}}{|E|}} \right), & |E| \geq |\Delta|, \\
\left( \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\sqrt{\Delta^2 - E^2}}{E}} \right), & |E| < |\Delta|, \\
\end{cases}
\]

(S20a)

\[
u_p(E) = \begin{cases} 
\text{sgn}(E) \left( \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\sqrt{E^2 - |\Delta|^2}}{|E|}} \right), & |E| \geq |\Delta|, \\
\left( \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\sqrt{\Delta^2 - E^2}}{E}} \right), & |E| < |\Delta|, \\
\end{cases}
\]

(S20b)

Since \( u_{p/h}^2 = \frac{1}{2} [1 + (\varepsilon_{p/h}(E) - \mu)/|E|] \) and \( v_{p/h}^2 = \frac{1}{2} [1 - (\varepsilon_{p/h}(E) - \mu)/|E|] \) where \( q_p(E) = q_p^*(E) \) and \( q_h^2(E)/(2m) - \mu = -|q_h^2(E)/(2m)| - \mu \), we then have \( u_p(E) = v_h^*(-E) = u_p^*(-E) \) and \( v_p(E) = -u_h^*(-E) = -v_p^*(-E) \). For
III. SCATTERING STATES

In this section, we state the explicit form of the scattering states across the SNS junctions in terms of right- and left-moving Bogoliubov states in the superfluid \((S_L, S_R)\) and normal regions \((N_L, N_R)\). For an illustration, Fig. S2 shows an example Bogoliubov spectrum across the SNS junction, with the same color coding for the different branches as in Sec. II. We list states with reference to the energy in the left reservoir, such that states in \(N_R\) and \(S_R\) carry an additional phase factor \(e^{\mp iV/\hbar}\) as discussed in Sec. II. For completeness, we state the wave functions for junctions of finite length \(\ell\).

![Supplementary Figure S2. Example energy spectrum of the Bogoliubov excitations for the \(S_L-N_L-I-N_R-S_R\) junction. Here, we choose \(\mu_L > 0\) and \(\mu_R < 0\).](image)

The scattering state in the left superfluid region \(S_L\) reads

\[
\Psi^{S_L,\zeta}_L(E) = \sum_n e^{-iE_n\tau/\hbar} \left\{ \delta_{\Delta_0, \Delta_L} \sqrt{\frac{m}{\hbar g_{pL}(E_n)}} \left( \frac{u_{L}(E_n)}{v_{L}(E_n)} \right) e^{i\eta_{pL}(E_n)(x+\ell/2)} + a_{1,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{pL}(E_n)}} \left( \frac{u_{L}(E_n)}{v_{L}(E_n)} \right) e^{i\eta_{pL}(E_n)(x+\ell/2)} + c_{1,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{hL}(E_n)}} \left( \frac{v_{L}(E_n)}{u_{L}(E_n)} \right) e^{-i\eta_{hL}(E_n)(x+\ell/2)} + d_{1,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{hL}(E_n)}} \left( \frac{v_{L}(E_n)}{u_{L}(E_n)} \right) e^{-i\eta_{hL}(E_n)(x+\ell/2)} \right\}.
\] (S21)

In the right superfluid region \(S_R\), we have

\[
\Psi^{S_R,\zeta}_R(E) = \sum_n e^{-iE_n\tau/\hbar} \left\{ \delta_{\Delta_0, \Delta_R} \sqrt{\frac{m}{\hbar g_{pR}(E_n)}} \left( \frac{u_{R}(E_n)e^{i\phi-iV/\hbar}}{v_{R}(E_n)e^{iV/\hbar}} \right) e^{-i\eta_{pR}(E_n)(x-\ell/2)} + a_{2,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{pR}(E_n)}} \left( \frac{u_{R}(E_n)e^{i\phi-iV/\hbar}}{v_{R}(E_n)e^{iV/\hbar}} \right) e^{i\eta_{pR}(E_n)(x-\ell/2)} + c_{2,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{hR}(E_n)}} \left( \frac{v_{R}(E_n)e^{i\phi-iV/\hbar}}{u_{R}(E_n)e^{iV/\hbar}} \right) e^{-i\eta_{hR}(E_n)(x-\ell/2)} + d_{2,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{hR}(E_n)}} \left( \frac{v_{R}(E_n)e^{i\phi-iV/\hbar}}{u_{R}(E_n)e^{iV/\hbar}} \right) e^{-i\eta_{hR}(E_n)(x-\ell/2)} \right\},
\] (S22)

where the phase factors \(e^{\mp iV/\hbar}\) account for the chemical potential mismatch between the reservoirs. Likewise, the wave function in the left normal region \(N_L\) is

\[
\Psi^{N_L,\zeta}_L(E) = \sum_n e^{-iE_n\tau/\hbar} \left\{ a_{1,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{pL}(E_n)}} \left( \frac{1}{0} \right) e^{ik_{pL}(E_n)(x+\ell/2)} + a_{L,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{pL}(E_n)}} \left( \frac{1}{0} \right) e^{-ik_{pL}(E_n)(x+\ell/2)} \right\} + \sum_n e^{-iE_n\tau/\hbar} \left\{ b_{1,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{hL}(E_n)}} \left( \frac{0}{1} \right) e^{ik_{hL}(E_n)(x+\ell/2)} + b_{L,n}^{\zeta} \sqrt{\frac{m}{\hbar g_{hL}(E_n)}} \left( \frac{0}{1} \right) e^{-ik_{hL}(E_n)(x+\ell/2)} \right\},
\] (S23)
and in the right normal region $N_R$, we have

$$
\Psi^N_{R, \zeta}(E) = \sum_n e^{-i(E_n - V)\tau/\hbar} \left\{ a^\zeta_{R,n} \sqrt{\frac{m}{\hbar_\ell R(E_n)}} (0) e^{ik_{\ell R}(E_n)(x-\ell/2)} + d^\zeta_{R,n} \sqrt{\frac{m}{\hbar_\ell R(E_n)}} (1) e^{-ik_{\ell R}(E_n)(x-\ell/2)} \right\} 
+ \sum_n e^{-i(E_n + V)\tau/\hbar} \left\{ b^\zeta_{R,n} \sqrt{\frac{m}{\hbar_\ell R(E_n)}} (0) e^{ik_{\ell R}(E_n)(x+\ell/2)} + c^\zeta_{R,n} \sqrt{\frac{m}{\hbar_\ell R(E_n)}} (1) e^{-ik_{\ell R}(E_n)(x+\ell/2)} \right\},
$$

(S24)

with momenta

$$
k_{p,L/R}(E) = \frac{\sqrt{2m}}{\hbar} \sqrt{E + \mu_{L/R} + i0},
$$

(S25a)

$$
k_{h,L/R}(E) = \frac{\sqrt{2m}}{\hbar} \sqrt{-E + \mu_{L/R} - i0},
$$

(S25b)

where the chemical potential of the left ($N_L$) and right ($N_R$) normal region are taken to be $\mu_L$ and $\mu_R$ which are the same as the chemical potential of the left ($S_L$) and right ($S_R$) reservoir, respectively. The coefficients in the normal region are shown in Fig. 1 of the main text.

In terms of the scattering state in the normal region, Eq. (S23), the expression Eq. (S14) for the dimensionless current density becomes (we evaluate the current in the left normal region)

$$
T^{p(m)}_{\nu, \zeta}(E) = \frac{k_{p,L}(E) + k^*_{p,L}(E_{n+m})}{2\sqrt{k_{p,L}(E_n)k^*_{p,L}(E_{n+m})}} \left[ (a^\zeta_{L,n+m})^* a^\zeta_{L,n} - (d^\zeta_{L,n+m})^* d^\zeta_{L,n} \right] 
+ \frac{k_{p,L}(E_n) - k^*_{p,L}(E_{n+m})}{2\sqrt{k_{p,L}(E_n)k^*_{p,L}(E_{n+m})}} \left[ (d^\zeta_{L,n+m})^* a^\zeta_{L,n} - (a^\zeta_{L,n+m})^* d^\zeta_{L,n} \right],
$$

(S26a)

$$
T^{h(m)}_{\nu, \zeta}(E) = \frac{k_{h,L}(E) + k^*_{h,L}(E_{n+m})}{2\sqrt{k_{h,L}(E_n)k^*_{h,L}(E_{n+m})}} \left[ (b^\zeta_{L,n+m})^* b^\zeta_{L,n} - (c^\zeta_{L,n+m})^* c^\zeta_{L,n} \right] 
+ \frac{k_{h,L}(E_n) - k^*_{h,L}(E_{n+m})}{2\sqrt{k_{h,L}(E_n)k^*_{h,L}(E_{n+m})}} \left[ (c^\zeta_{L,n+m})^* b^\zeta_{L,n} - (b^\zeta_{L,n+m})^* c^\zeta_{L,n} \right].
$$

(S26b)

For a dc-current, which corresponds to $m = 0$, Eq. (S26) reduces to Eq. (10) of the main text. The challenge is to determine the scattering coefficients $\{a^\zeta_{L,n}\}, \{b^\zeta_{L,n}\}, \{c^\zeta_{L,n}\},$ and $\{d^\zeta_{L,n}\}$ from a solution of the scattering problem, which is done in the next Sec. IV.

Finally, using $e^{-iq_p(E)x} = [e^{iq_p(E)x}]^*$ and $(u(E), v(E)) = (v(-E), -u(-E))^*$, it follows that that the current due to quasihole injections is equal to the current due to quasiparticle injections, as discussed above. As a result, the total current is equal to twice the current due to quasiparticle injections, which justifies the factor of 2 in Eq. (S12).

IV. SCATTERING COEFFICIENTS AND WAVE FUNCTION MATCHING

The scattering states states listed in the previous section contain a large number of scattering amplitudes $\{a^\zeta_{j,n}\}, \{b^\zeta_{j,n}\}, \{c^\zeta_{j,n}\}, \{d^\zeta_{j,n}\}$ in each region $j = 1, L, R, 2$. As illustrated in Fig. 1 of the main text and in Fig. S3, these amplitudes are linked by various individual scattering processes, which are determined by a solution of the scattering problem in the potential given by Eqs. (S4) and (S5), i.e., by matching the scattering wave functions and their derivatives at the NS boundaries and at the tunnel barrier.

1. Left S-N boundary

Particles or holes in the normal region that propagate away from the NS boundary are created either by transmitting a Bogoliubov excitation across the boundary, normal-reflecting a particle or hole that impinges on the NS junction, or by Andreev-reflecting an impinging excitation to an excitation of opposite type. This implies the following relation
Supplementary Figure S3. Scattering processes across the SNS junction, where solid vertical lines mark the NS boundaries and the dashed line denotes the tunnel barrier. Shown is the insertion of particle ($J_{p,n}^L$) and hole ($J_{h,n}^L$) states into the left normal region due to the injection of a quasiparticle from the left superfluid reservoir with energy $E$, and subsequent scattering processes. Normal reflections of particles ($N_{L,n}^p$) and holes ($N_{L,n}^h$) at the NS boundary are indicated by red and green arrows, respectively, while Andreev ($A_{L/R,n}$) reflections are denoted by blue arrows [the same convention is used in Fig. 1 of the main text]. The magenta- and orange-colored horizontal arrows correspond to the scattering states for particles and holes, respectively, where in our convention the voltage drop across the reservoirs is absorbed in a time-dependent tunnel barrier that changes the energy of particles and holes. At the tunnel barrier, each transmission event changes the Floquet index $n$ of scattering states by ±1 where the scattering matrices at the tunnel barrier are denoted by $S_e$ for electrons and $S_h$ for holes.

between the amplitudes of right-moving particle and hole states $a_{L,n}$ and $c_{L,n}$ and left-moving states $b_{L,n}$ and $d_{L,n}$ [Eq. (5) of the main text]:

$$\begin{pmatrix}
    a_{L,n}^c \\
    c_{L,n}^c
\end{pmatrix} = \begin{pmatrix}
    N_{L,n}^p & A_{L,n} \\
    A_{L,n} & N_{L,n}^h
\end{pmatrix} \begin{pmatrix}
    a_{L,n}^p \\
    b_{L,n}^p
\end{pmatrix} + \delta_{n0} \delta_{\mu \mu_L} \begin{pmatrix}
    J_{L,n}^p \\
    J_{L,n}^h
\end{pmatrix}. \tag{S27}
$$

Here, $N_{L,n}^p$ and $N_{L,n}^h$ are the normal-reflection amplitudes for particles ($p$) and holes ($h$), respectively, $A_{L,n}$ is the Andreev-reflection coefficient, and $J_{L,n}^p$ and $J_{L,n}^h$ are the amplitudes for the transmission of a Bogoliubov quasiparticle into a particle ($p$) and a hole ($h$), respectively. Matching the scattering wave function at the boundary gives the following expression for these coefficients:

$$N_{L,n}^p = \frac{u_L^2(E_n)[k_{hL}(E_n) + q_{hL}(E_n)][k_{pL}(E_n) - q_{pL}(E_n)] - v_L^2(E_n)[q_{hL}(E_n) + k_{pL}(E_n)][k_{hL}(E_n) - q_{pL}(E_n)]}{u_L^2(E_n)[k_{hL}(E_n) + q_{hL}(E_n)][k_{pL}(E_n) + q_{pL}(E_n)] + v_L^2(E_n)[q_{hL}(E_n) - k_{pL}(E_n)][k_{hL}(E_n) - q_{pL}(E_n)]} \xrightarrow{AA} 0, \tag{S28a}
$$

$$N_{L,n}^h = \frac{u_L^2(E_n)[k_{hL}(E_n) - q_{hL}(E_n)][k_{pL}(E_n) + q_{pL}(E_n)] + v_L^2(E_n)[q_{hL}(E_n) - k_{pL}(E_n)][k_{hL}(E_n) + q_{pL}(E_n)]}{u_L^2(E_n)[k_{hL}(E_n) + q_{hL}(E_n)][k_{pL}(E_n) + q_{pL}(E_n)] + v_L^2(E_n)[q_{hL}(E_n) - k_{pL}(E_n)][k_{hL}(E_n) - q_{pL}(E_n)]} \xrightarrow{AA} 0, \tag{S28b}
$$

$$A_{L,n} = \frac{2\sqrt{k_{pL}(E_n)k_{hL}(E_n)}u_L(E_n)v_L(E_n)(q_{hL}(E_n) + q_{pL}(E_n))}{u_L^2(E_n)[k_{hL}(E_n) + q_{hL}(E_n)][k_{pL}(E_n) + q_{pL}(E_n)] + v_L^2(E_n)[q_{hL}(E_n) - k_{pL}(E_n)][k_{hL}(E_n) - q_{pL}(E_n)]} \xrightarrow{AA} \frac{u_L(E_n)}{v_L(E_n)}, \tag{S28c}
$$

and

$$J_{L,n}^p = \frac{2\sqrt{k_{pL}(E_n)q_{pL}(E_n)[q_{hL}(E_n) + k_{hL}(E_n)][u_L(E_n)(u_L^2(E_n) - v_L^2(E_n))]}}{u_L^2(E_n)[k_{hL}(E_n) + q_{hL}(E_n)][k_{pL}(E_n) + q_{pL}(E_n)] + v_L^2(E_n)[q_{hL}(E_n) - k_{pL}(E_n)][k_{hL}(E_n) - q_{pL}(E_n)]} \xrightarrow{AA} \frac{u_L^2(E_n) - v_L^2(E_n)}{u_L(E_n)}; \tag{S29a}
$$

$$J_{L,n}^h = \frac{2\sqrt{k_{hL}(E_n)q_{hL}(E_n) - k_{pL}(E_n)v_L(E_n)(u_L^2(E_n) - v_L^2(E_n))}}{u_L^2(E_n)[k_{hL}(E_n) + q_{hL}(E_n)][k_{pL}(E_n) + q_{pL}(E_n)] + v_L^2(E_n)[q_{hL}(E_n) - k_{pL}(E_n)][k_{hL}(E_n) - q_{pL}(E_n)]} \xrightarrow{AA} 0. \tag{S29b}
$$

In the above equations, we also include the limiting form of the scattering coefficients when using the Andreev approximation (AA). Results in the AA limit agree with Refs. [S10, S11]. Taking into account that there are no
Supplementary Figure S4. Plots of dimensionless scattering coefficients at an NS interface for different ratios of chemical potential $\mu$ and gap $\Delta$. [Upper Panel: (a)-(e)] Coefficients for Andreev reflection $[A(E)]$, normal reflections for particles $[N^p(E)]$ and holes $[N^h(E)]$. Note that the magnitude of the normal reflection coefficients $N^p$ and $N^h$ for energies below the gap ($|E| < \Delta$) increases from 0 to 1 as the chemical potential changes from large positive values ($\mu \gg \Delta$) to negative values or as we move away from the AA regime. [Lower Panel: (f)-(j)] Transmission amplitudes of quasiparticle injections from the reservoir into particle $[J^p(E)]$ and hole $[J^h(E)]$ in the normal region, multiplied by the quasiparticle density of states $D(E)$. Note that $J^h(E)$ becomes finite as we move away from the AA regime.

Figure S4 shows the scattering coefficients for different ratios of the chemical potential $\mu$ and gap $\Delta$, where the results in the Andreev approximation regime are shown in Figs. S4(a) and S4(f). Figures S4(c) and S4(e) are identical with right upper and lower panels in Fig. 1(c) of the main text. To expand on the discussion in the main text, it is apparent from the analytical results [Eqs. (S28) and (S29)] and Fig. S4 that the expressions simplify considerably in AA limit: in particular, there is no normal reflection for particles and holes, and Bogoliubov quasiparticles inserted into the normal region are always transmitted as particles, never as holes. Furthermore, we note that the Andreev approximation becomes unreliable even for a moderate increase in the gap [see Figs. S4(b) and S4(g)] and no longer gives the correct scattering form as we deviate further from the AA regime [see Figs. S4(c)-(e) and S4(h)-(j)].

2. Right N-S boundary

Matching the wave function and current at the right N_R-S_R interface, we obtain an expression for the coefficients $b_{R,n}$ and $d_{R,n}$ of the left-moving components in terms of the coefficients of the right-moving parts [Eq. (6) of the main text]:

$$\begin{pmatrix} d_{R,n} \\ \dot{b}_{R,n} \end{pmatrix} = \begin{pmatrix} N^p_{R,n} & e^{i\phi} A_{R,n} \\ e^{-i\phi} A_{R,n} & N^h_{R,n} \end{pmatrix} \begin{pmatrix} \dot{c}_{R,n} \\ \dot{\xi}_{R,n} \end{pmatrix} + \delta_{n0} \delta_{\xi} \begin{pmatrix} J^p_{R,n} \\ J^h_{R,n} \end{pmatrix},$$

(S31)

where we include the transmission amplitude of particles from the right reservoir. The expression for the coefficients $N^p_{R,n}, N^h_{R,n}, A_{R,n}, J^p_{R,n}, J^h_{R,n}$ are obtained from the corresponding coefficients at the left S-N boundary [Eq. (S28) of the main text].
and \((S29)\) by simply replacing the subscript \(L\) by \(R\).

3. Tunnel barrier

The last ingredient is the scattering matrices at the tunnel barrier connecting the coefficients in the left normal region \((N_L)\) with those of the right normal region \((N_R)\). The outgoing particle-components are related to the incoming particle components through the scattering matrix \([Eg. (7a)\) of the main text]:

\[
\begin{pmatrix}
    a_{L,n}^c \\
    d_{R,n+1}^c
\end{pmatrix} = S_p(E_n) \begin{pmatrix}
    a_{L,n}^c \\
    d_{R,n+1}^c
\end{pmatrix} = \begin{pmatrix}
    t_{p,n} & t_{p,n}^* \\
    t_{p,n} & -t_{p,n}^*
\end{pmatrix} \begin{pmatrix}
    a_{L,n}^c \\
    d_{R,n+1}^c
\end{pmatrix}
\]

\((S32)\)

The same holds for the hole-components \([Eg. (7b)\) of the main text]:

\[
\begin{pmatrix}
    b_{L,n}^c \\
    c_{R,n-1}^c
\end{pmatrix} = S_h(E_n) \begin{pmatrix}
    b_{L,n}^c \\
    c_{R,n-1}^c
\end{pmatrix} = \begin{pmatrix}
    t_{h,n} & t_{h,n}^* \\
    t_{h,n} & -t_{h,n}^*
\end{pmatrix} \begin{pmatrix}
    b_{L,n}^c \\
    c_{R,n-1}^c
\end{pmatrix}
\]

\((S33)\)

In this paper, we assume a delta-function barrier \([S2, S12]\) with energy-independent reflection and transmission coefficients, i.e., \(t_{p,n} = t_{h,n}^* = \text{te}^{i\eta}\) and \(r_{p,n} = r_{h,n}^* = -i\text{te}^{i\eta}\sqrt{1-T^2}\) where \(\eta = -\text{arctan}(Z)\). Here, \(Z\) is the dimensionless barrier strength as defined in the Blonder-Tinkham-Klapwijk (BTK) theory \([S10]\) which is related to the barrier transparency \(T = t^2\) by \(Z = \sqrt{(1/T) - 1}\).

V. RECURRANCE RELATION FOR THE SCATTERING AMPLITUDES

In this section, we simplify the constraints derived in the previous section to obtain a closed-form expression for the scattering amplitudes in the normal region \([Eg. (8)\) of the main text] needed to compute the current density \(Eq. (S26)\) across the junction. The starting point is the following set of equations that contains only the coefficients of normal-region modes propagating away from the tunnel barrier, which are obtained by expressing the incoming modes \(Eq. (S26)\) across the junction. The starting point is the following set of equations that contains only the coefficients of the scattering amplitudes in the normal region \([Eq. (8)\) of the main text]:

\[
\begin{align*}
    d_{L,n}^- &= r_{p,n}\left[N_{L,n}^p d_{L,n}^c + A_{L,n} b_{L,n}^c + \delta_{n0} J_{L,n}^p\right] + t_{p,n}\left[N_{R,n+1}^p a_{R,n+1}^c + e^{i\phi} A_{R,n+1} c_{R,n+1}^c\right], \\
    a_{R,n+1}^- &= t_{p,n}\left[N_{L,n}^p d_{L,n}^c + A_{L,n} b_{L,n}^c + \delta_{n0} J_{L,n}^p\right] - \frac{t_{p,n}}{t_{p,n}} r_{p,n}\left[N_{R,n+1}^p a_{R,n+1}^c + e^{i\phi} A_{R,n+1} c_{R,n+1}^c\right], \\
    b_{L,n}^- &= r_{h,n}\left[A_{L,n} d_{L,n}^c + N_{L,n}^h b_{L,n}^c + \delta_{n0} J_{L,n}^h\right] + t_{h,n}\left[e^{-i\phi} A_{R,n-1} a_{R,n-1}^c + N_{R,n-1}^h c_{R,n-1}^c\right], \\
    c_{R,n-1}^- &= t_{h,n}\left[A_{L,n} d_{L,n}^c + N_{L,n}^h b_{L,n}^c + \delta_{n0} J_{L,n}^h\right] - \frac{t_{h,n}}{t_{h,n}} r_{h,n}\left[e^{-i\phi} A_{R,n-1} a_{R,n-1}^c + N_{R,n-1}^h c_{R,n-1}^c\right].
\end{align*}
\]

\((S34a)-(S34d)\)

Next, we solve the last two equations \([Eqs. (S34c)\) and \((S34d)\)] for the coefficients \(\{a_{R,n}^c\}\) and \(\{c_{R,n}^c\}\) of the right-normal region and substitute the result for \(a_{R,n-1}^c\) and \(c_{R,n-1}^c\) in the remaining two constraint equations \([Eqs. (S34a)\) and \((S34b)\)], which only leaves the coefficients \(\{b_{L,n}^c\}\) and \(\{d_{L,n}^c\}\) as unknown variables. Once these are known, the remaining coefficients \(a_{L,n}^c\) and \(c_{L,n}^c\) in the left normal region follow from \(Eq. (S27)\). We solve the constraint equations \([Eqs. (S34a)\) and \((S34b)\)] for \(b_{L,n}^c\) and expressed the \(\{b_{L,n}^c\}\) in terms of the \(d_{L,n}^-\) coefficients as

\[
\begin{align*}
    b_{L,n+2}^- &= \frac{\delta_{n+1}^L}{r_{h,n+2}^L} \left|t_{p,n}^2 A_{R,n+1} d_{L,n}^c |t_{h,n+2}^L\right|^2 \\
    &+ \left[t_{h,n+2} \left(t_{p,n} \left(N_{R,n+1}^h N_{R,n+1}^p - A_{R,n+1}^2\right) + r_{p,n} t_{p,n}^* N_{R,n+1}^p b_{L,n+2}^c\right) + r_{h,n+2} t_{h,n+2}^* \left(t_{p,n} N_{R,n+1}^p + r_{p,n} t_{p,n}^*\right)\right] \frac{A_{L,n+2}}{|t_{h,n+2}|^2 d_{L,n+2}^c} \\
    &+ \left[t_{h,n+2} \left(t_{p,n} \left(N_{R,n+1}^h N_{R,n+1}^p - A_{R,n+1}^2\right) + r_{p,n} t_{p,n}^* N_{R,n+1}^p b_{L,n+2}^c\right) + r_{h,n+2} t_{h,n+2}^* \left(t_{p,n} N_{R,n+1}^p + r_{p,n} t_{p,n}^*\right)\right] \frac{J_{L,n+2}}{|t_{h,n+2}|^2 \delta_{n+2}^L},
\end{align*}
\]

\((S35)\)
with
\[
zn + r_{h,n+1}^* |t_{h,n+1}|^2 \left( t_{h,n+1}(N^h_{n+1} - r_{h,n+1}^*) \left[ t_{p,n-1}(A^2_{R,R,n} - N^h_{R,n}N^p_{R,n}) - r_{p,n-1}t_{p,n-1}^* N^h_{R,n} \right] \right.
+ t_{h,n+1}^*(1 - r_{h,n+1}N^h_{L,n+1})(t_{p,n-1}N^p_{R,n} + r_{p,n-1}t_{p,n-1}^*) \right)^{-1}.
\]

Finally, using this result in the remaining constraint equation [Eq. (S34a)] gives a closed-form matrix equation for the \( \{d_{L,n}\} \), which must be solved first to determine all other scattering amplitudes [Eq. (8) of the main text]
\[
\alpha_n^\rightarrow d_{L,n+2} + \beta_n^\rightarrow d_{L,n} + \gamma_n^\rightarrow d_{L,n+2} = S_{L,n}^p\delta_n0 + S_{L,n}^h\delta_{n+2,0},
\]
with coefficients
\[
\alpha_n^\rightarrow = -\frac{r_{p,n}}{r_{h,n+2}}|t_{p,n}|^2 A_{L,n+2}A_{R,n+1}zn_{n+1},
\]
\[
\beta_n^\rightarrow = 1 - r_{p,n}N^p_{L,n} + \frac{r_{p,n}}{r_{h,n}^*} A^2_{L,n} \left[ t_{p,n-2} \left( t_{h,n} \left( A^2_{R,R,n-1} - N^h_{R,n-1}N^p_{R,n-1} \right) - t_{h,n}^* r_{h,n}N^p_{R,n-1} \right) \right. \\
- t_{p,n-2}t_{p,n-2}^* \left( t_{h,n}N^h_{R,n-1} + t_{h,n}^*r_{h,n} \right) \left\{ \frac{zn_{n-1}}{|t_{h,n}|^2} - \frac{t_{p,n}^*}{r_{h,n+2}^*} t_{p,n}^* \left[ t_{p,n+2}N^p_{R,n+1} \left( 1 - r_{h,n+2}N^h_{L,n+2} \right) \right. \\
+ t_{h,n+2} \left( N^h_{L,n+2} - r_{h,n+2}^* \right) \left( A^2_{R,n+1} - N^h_{R,n+1}N^p_{R,n+1} \right) \right\} \left\{ \frac{zn_{n+1}}{|t_{h,n+2}|^2} \right\},
\]
\[
\gamma_n^\rightarrow = -\frac{r_{p,n}}{r_{h,n}}|t_{p,n-2}|^2 A_{L,n}A_{R,n-1}zn_{n-1},
\]
\[
S_{L,n}^p\rightarrow = r_{p,n}J^p_{L,n} - \frac{r_{p,n}}{r_{h,n}^*} A^2_{L,n} \left[ t_{p,n-2} \left( t_{h,n} \left( A^2_{R,n-1} - N^h_{R,n-1}N^p_{R,n-1} \right) - t_{h,n}^* r_{h,n}N^p_{R,n-1} \right) \right. \\
- t_{p,n-2}t_{p,n-2}^* \left( t_{h,n}N^h_{R,n-1} + t_{h,n}^*r_{h,n} \right) \left\{ \frac{zn_{n-1}}{|t_{h,n}|^2} \right\},
\]
\[
S_{L,n}^h\rightarrow = \frac{r_{p,n}}{r_{h,n+2}^*}|t_{p,n}|^2 A_{L,n+2}A_{R,n+1}zn_{n+1},
\]

For quasiparticles are injected from the right reservoir (\( \zeta = -\)), we derive a similar recurrence relation
\[
\alpha_n^\rightarrow a_{R,n-2} + \beta_n^\rightarrow a_{R,n} + \gamma_n^\rightarrow a_{R,n+2} = S_{R,n}^{p,\rightarrow} \delta_n0 + S_{R,n}^{h,\rightarrow} \delta_{n+2,0},
\]
where the coefficients \( \alpha_n^\rightarrow, \beta_n^\rightarrow, \gamma_n^\rightarrow, \) and \( S_{R,n}^{p,\rightarrow}, S_{R,n}^{h,\rightarrow} \) are obtained from the corresponding quantities in Eq. (S38) by replacing \( L \rightarrow R, n + 1 \rightarrow n - 1 \), and \( n + 2 \rightarrow n + 2 \).

As a check of our results, consider the Andreev approximation: Here, Eqs. (S36) and (S38) simplify considerably and reduce to
\[
zn = \left[ \frac{t_{p,n-1}t_{p,n-1}^*}{t_{h,n+1}t_{h,n+1}^*} \frac{A^2_{R,R,n}}{A^2_{R,R,n}} \right]^{-1},
\]
and
\[
\alpha_n^\rightarrow = -\frac{r_{p,n}}{r_{h,n+2}^*}|t_{p,n}|^2 A_{L,n+2}A_{R,n+1}zn_{n+1},
\]
\[
\beta_n^\rightarrow = 1 - r_{p,n}A^2_{L,n}zn_{n-1} \left[ \frac{t_{p,n-2}t_{h,n}t_{p,n-2}^*}{t_{h,n}^*} - \frac{t_{p,n-2}^*}{t_{h,n}^*} A^2_{R,n-1} \right] + \frac{t_{p,n}}{t_{h,n+2}^*}|t_{p,n}|^2 A^2_{R,n+1}zn_{n+1},
\]
\[
\gamma_n^\rightarrow = -\frac{r_{p,n}}{r_{h,n}}|t_{p,n-2}|^2 A_{L,n}A_{R,n-1}zn_{n-1},
\]
\[
S_{L,n}^{p,\rightarrow} = r_{p,n}J^p_{L,n},
\]
\[
S_{L,n}^{h,\rightarrow} = 0,
\]
respectively, which agrees with the literature [S6].
In this section, we discuss the solution of the infinite-dimensional matrix equation (S37), for which general solution methods exist that solve a continued-fraction representation of the constraint equations (see Refs. [S3, S4, S6]), which is solved using a “modified Lentz method” [S13]. Other scattering amplitudes then follow by direct substitution as discussed in the previous section. Define

$$ x_n^\rightarrow = \begin{cases} \frac{d_{L,n}^\rightarrow}{d_{L,n-2}^\rightarrow}, & n > 2, \\ \frac{d_{L,n}^\rightarrow}{d_{L,n+2}^\rightarrow}, & n < 0, \end{cases} (S42) $$

and rewrite Eq. (S37) as

$$ \alpha_n^\rightarrow x_{n+2}^\rightarrow + \beta_n^\rightarrow + \gamma_n^\rightarrow x_n^\rightarrow = 0, \quad n > 0, $$

$$ \alpha_n^\rightarrow x_{n}^\rightarrow + \beta_n^\rightarrow + \gamma_n^\rightarrow x_{n-2}^\rightarrow = 0, \quad n < -2, $$

with two additional equations containing the source terms for $n = 0$ and for $n = -2$, i.e.,

$$ (\beta_0^\rightarrow + \alpha_0^\rightarrow x_2^\rightarrow) d_{L,0}^\rightarrow + \gamma_0^\rightarrow d_{L,-2}^\rightarrow = S_{L,0}^p^\rightarrow, $$

$$ \alpha_{-2}^\rightarrow d_{L,0}^\rightarrow + (\beta_{-2}^\rightarrow + \gamma_{-2}^\rightarrow x_{-4}^\rightarrow) d_{L,-2}^\rightarrow = S_{L,-2}^h^\rightarrow. $$

First, we rewrite Eq. (S44) in matrix form as

$$ \begin{pmatrix} d_{L,0}^\rightarrow \\ d_{L,-2}^\rightarrow \end{pmatrix} = \frac{1}{(\beta_0^\rightarrow + \alpha_0^\rightarrow x_2^\rightarrow)(\beta_{-2}^\rightarrow + \gamma_{-2}^\rightarrow x_{-4}^\rightarrow) - \alpha_{-2}^\rightarrow \alpha_0^\rightarrow} \begin{pmatrix} \beta_{-2}^\rightarrow + \gamma_{-2}^\rightarrow x_{-4}^\rightarrow & -\gamma_0^\rightarrow \\ -\alpha_{-2}^\rightarrow & \alpha_0^\rightarrow + \alpha_0^\rightarrow x_2^\rightarrow \end{pmatrix} \begin{pmatrix} S_{L,0}^p^\rightarrow \\ S_{L,-2}^h^\rightarrow \end{pmatrix}, $$

which is solved for the coefficients $d_{L,0}^\rightarrow$ and $d_{L,-2}^\rightarrow$ once the $\{x_n^\rightarrow\}$ are known. Having solved for $d_{L,0}^\rightarrow$ and $d_{L,-2}^\rightarrow$, we obtain the remaining coefficients $\{d_{L,n}^\rightarrow\}_{n \neq 0,-2}$ from Eq. (S42),

$$ d_{L,n}^\rightarrow = \begin{cases} x_n^\rightarrow x_{n-2}^\rightarrow \ldots x_2^\rightarrow d_{L,0}^\rightarrow, & n > 0, \\ x_n^\rightarrow x_{n+2}^\rightarrow \ldots x_{-4}^\rightarrow d_{L,-2}^\rightarrow, & n < -2. \end{cases} (S46) $$

To determine the values of $\{x_n^\rightarrow\}$, we rewrite Eqs. (S43a) and (S43b) as continued fraction expansions, which express $x_n^\rightarrow$ in terms of higher-index coefficients $x_m^\rightarrow$ with $|m| = |n| + 2$:

$$ x_n^\rightarrow = \begin{cases} -\gamma_n^\rightarrow & n > 0, \\ \beta_n^\rightarrow + \alpha_n^\rightarrow x_{n+2}^\rightarrow & n < -2. \end{cases} (S47) $$

Recasting this in the general form of a continued fraction gives

$$ x_n^\rightarrow = \frac{f_{n,1}^\rightarrow}{g_{n,1}^\rightarrow + \frac{f_{n,2}^\rightarrow}{g_{n,2}^\rightarrow + \frac{f_{n,3}^\rightarrow}{g_{n,3}^\rightarrow + \ldots}}} (S48) $$

with the following coefficients for $n > 0$

$$ f_{n,m}^\rightarrow = \begin{cases} -\gamma_n^\rightarrow, & m = 1, \\ -\alpha_{n+2(m-2)}^\rightarrow \gamma_{n+2(m-1)}^\rightarrow, & m > 1. \end{cases} (S49) $$
The reflection and transmission coefficients of a delta-function tunnel barrier are [S2, S12]

\[
\begin{align*}
    r_{p,n} &= r_{h,n}^* = -ie^{in} \sqrt{1 - t^2}, \\
    t_{p,n} &= t_{h,n}^* = te^{in},
\end{align*}
\]

In principle, the continued fraction expansion (S48) could be solved by introducing an index \( n_{\text{MAX}} \) with \( x_{n_{\text{MAX}}} \rightarrow 0 \) and iterating lower-index coefficients \( x_{n} \), which however introduces a systematic error. Instead, we employ the “modified Lenz method” [S13] to solve continued fraction systems (S48), which we outline in the remainder of the section (restricting to the case \( n > 0 \) for notational simplicity). First, we define by \( x_{n_{\text{MAX}}}^\rightarrow (m) \) (with \( m > 0 \)) as the partial evaluation of the coefficient \( x_{n}^\rightarrow \) obtained by setting \( x_{n+m}^\rightarrow = 0 \) in Eq. (S50). Formally, this is written as

\[
x_{n_{\text{MAX}}}^\rightarrow (m) = \frac{F_m}{G_m},
\]

with \( F_m = g_m F_{m-1} + f_m F_{m-2} \) and \( G_m = g_m G_{m-1} + f_m G_{m-2} \), where \( F_0 = g_0 \), \( F_1 = 1 \), \( G_0 = 1 \), and \( G_1 = 0 \) as well as \( f_1^\rightarrow = -\alpha_n \), \( g_n^\rightarrow > 2 = -\gamma_n 2 (m-1) \alpha_n + 2 m \) and \( g_0^\rightarrow = 0 \). \( g_{n_{\text{MAX}}}^\rightarrow > 1 = \beta_n 2 (m-1) \). The modified Lenz method introduces the ratios \( W_m = F_m / F_{m-1} \) and \( Y_m = G_{m-1} / G_m \), such that \( x_{n_{\text{MAX}}}^\rightarrow (m) = x_{n_{\text{MAX}}}^\rightarrow (m-1) W_m Y_m \), and then iterates with initial conditions \( W_0^\rightarrow = x_{n_{\text{MAX}}}^\rightarrow (0) \) and \( Y_0^\rightarrow = 0 \) as follows:

\[
\begin{align*}
    Y_m^\rightarrow &= \frac{1}{g_m + f_m Y_{m-1}}, \\
    W_m^\rightarrow &= g_m + f_m / W_{m-1}.
\end{align*}
\]

This is done until convergence is reached, i.e., \( x_{n_{\text{MAX}}}^\rightarrow (m) \) does not change within the numerical resolution. Whenever \( W_m^\rightarrow \) or \( (Y_m^\rightarrow)^{-1} \) are zero, they should be shifted by an infinitesimal amount. Similarly, by using the substitution \( d_{h,n}^\rightarrow \rightarrow a_{h,n}^* \) and \( d_{h,n+2}^\rightarrow \rightarrow a_{h,n+2}^* \), we use the above method to solve the recurrence relation (S39) corresponding to the case where quasiparticles are injected from the right superfluid.

**VII. TUNNELING LIMIT**

For tunnel junctions with small transparency \( t^2 \ll 1 \), the current arises due to direct transmission from the occupied states in one reservoir to empty states of the other reservoir. Standard calculations using a tunneling Hamiltonian predict a current [S14]

\[
I_{dc}(V) = \frac{2}{h} t^2 \int_{-\infty}^{\infty} dE \rho_L(E) \rho_R(E + V) [f(E) - f(E + V)],
\]

which is proportional to the product of the particle tunneling density of states \( \rho_{L/R} \) in both reservoirs. Since the tunneling current is due to direct transmission instead of MAR, it can only flow if the voltage is greater than the energy difference between the occupied band of one reservoir and the empty band of the other reservoir, i.e., \( |V| \geq \Delta_L + \Delta_R \) where \( \Delta_L/R \) is the spectral gap with \( \Delta_L/R = \Delta_L/R \) for \( \mu_{L/R} > 0 \) and \( \Delta_L/R = \mu_{L/R} + \Delta_{L/R}^2 \) for \( \mu_{L/R} \leq 0 \). In this section, we derive an analytic expression from our result for the current in the tunneling limit \( (T = t^2 \ll 1) \) and show that it takes the form (S56). Note that since the Landauer-Büttiker formalism does not take into account the density of states in the final reservoir, this is a nontrivial check of our results.

The reflection and transmission coefficients of a delta-function tunnel barrier are [S2, S12]
where \( t \equiv |t_{p,n}| = |t_{h,n}| \) and \( \eta = -\arctan(Z) \) with \( Z \) being the barrier strength which is related to the barrier transparency \( T = t^2 \) by \( Z = \sqrt{(1/T) - 1} \). In the tunneling limit \( t^2 \ll 1 \), we have \( \eta = -\pi/2 \) which gives \( r_{p,n} = r_{h,n}^* = -(1-t^2/2) + O(t^4) \) and \( t_{p,n} = t_{h,n}^* = -it \). In particular, states at energy \( E_n \) are transmitted across the tunnel barrier at least \( n \) times and thus its transmission amplitudes are of the order of \( O(t^n) \). For the case of quasiparticles injected from the left reservoir, we find the following relations between the scattering amplitudes in the tunneling limit up to the order \( O(t^2) \):

\[
\begin{align*}
& a_{L,2} = N_{L,2}^p d_{L,2} + A_{L,2} b_{L,2}, \\
& b_{L,2} = -c_{L,2}, \\
& c_{L,2} = N_{L,2}^h d_{L,2} + A_{L,2} b_{L,2}, \\
& d_{L,2} = -a_{L,2}, \\
& a_{L,0} = J_{L,0}^p + N_{L,0}^p d_{L,0} + A_{L,0} b_{L,0}, \\
& b_{L,0} = -(1-t^2/2) c_{L,0} + it d_{L,0}, \\
& c_{L,0} = J_{L,0}^h + N_{L,0}^h d_{L,0} + A_{L,0} b_{L,0}, \\
& d_{L,0} = -(1-t^2/2) a_{L,0} - it d_{L,0}, \\
& a_{L,-2} = N_{L,-2}^p d_{L,-2} + A_{L,-2} b_{L,-2}, \\
& b_{L,-2} = -c_{L,-2}, \\
& c_{L,-2} = N_{L,-2}^h d_{L,-2} + A_{L,-2} b_{L,-2}, \\
& d_{L,-2} = -a_{L,-2}.
\end{align*}
\]

(S58)

where higher coefficients do not contribute to the perturbative current. Solving this set of equations, we obtain the following results for the dimensionless current density:

\[
T_{p\rightarrow l}^{(0)}(E) = \langle |a_{L,0}|^2 - |d_{L,0}|^2 \rangle \Theta(E + \mu_L),
\]

\[
= t^2 \left[ u_1^2(E) q_{pL}(E) \right] D_R(E_1) \left[ \frac{q_{pR}(E_1) u_2^2(E_1) + q_{hR}(E_1) v_2^2(E_1)}{k_{pR}(E_1)} \right] \Theta(E_1 + \mu_R) \Theta(E + \mu_L), \quad \text{and} \quad \text{(S59)}
\]

\[
T_{h\rightarrow l}^{(0)}(E) = \langle |c_{L,0}|^2 - |b_{L,0}|^2 \rangle \Theta(-E + \mu_L),
\]

\[
= t^2 \left[ v_1^2(E) q_{hL}(E) \right] D_R(E_{-1}) \left[ \frac{q_{hR}(-E_1) u_2^2(-E_1) + q_{pR}(-E_1) v_2^2(-E_1)}{k_{hR}(-E_1)} \right] \Theta(-E_1 + \mu_R) \Theta(-E + \mu_L). \quad \text{(S60)}
\]

In going to the last lines of Eqs. (S59) and (S60), we have used the quasiparticle density of states

\[
D_j(E) = \frac{1}{u_j^2(E) - v_j^2(E)} \times \begin{cases} 
\Theta(\vert E \vert - \Delta_j) \Theta(E + \sqrt{\mu_j^2 + \Delta_j^2}), & \mu_j > 0, \\
\Theta(\vert E \vert - \Delta_j) \Theta(E - \sqrt{\mu_j^2 + \Delta_j^2}), & \mu_j \leq 0,
\end{cases}
\]

\[
= \frac{\vert E \vert}{E^2 - \Delta_j^2} \times \begin{cases} 
\Theta(\vert E \vert - \Delta_j) \Theta(E + \sqrt{\mu_j^2 + \Delta_j^2}), & \mu_j > 0, \\
\Theta(\vert E \vert - \Delta_j) \Theta(E - \sqrt{\mu_j^2 + \Delta_j^2}), & \mu_j \leq 0.
\end{cases}
\]

(S61)

for \( j = R \). The dc current due to the quasiparticle injections from the left reservoir is

\[
I_{dc}(V) = \frac{2}{\hbar} t^2 \int_{-\infty}^{\infty} dE D_L(E) \left\{ \left[ \frac{u_1^2(E) q_{pL}(E)}{k_{pL}(E)} \right] \Theta(E + \mu_L) f(E) D_R(E + V) \Theta(E_1 + \mu_R) \left[ \frac{q_{pR}(E_1) u_2^2(E_1) + q_{hR}(E_1) v_2^2(E_1)}{k_{pR}(E_1)} \right] \right\} \\
+ \left[ \frac{v_1^2(E) q_{hL}(E)}{k_{hL}(E)} \right] \Theta(-E + \mu_L) (1 - f(E)) D_R(E - V) \Theta(-E_1 + \mu_R) \left[ \frac{q_{hR}(-E_1) u_2^2(-E_1) + q_{pR}(-E_1) v_2^2(-E_1)}{k_{hR}(-E_1)} \right].
\]

(S62)
Changing the integration $E \to -E$ for the hole part and using $1 - f(-E) = f(E)$, $D_{L/R}(-E) = D_{L/R}(E)$, and $k_{hL}(-E) = k_{pL}(E)$ gives the expression

$$I_{dc}^+ (V) = \frac{2}{\hbar} \frac{e^2}{h^2} \int_{-\infty}^{\infty} dE \left\{ \begin{array}{l} D_L(E) \Theta(E + \mu_L) \left[ \frac{q_{pL}(E)u^2_L(E) + q_{hL}(E)v^2_L(E)}{k_{pL}(E)} \right] \\ \times D_R(E + V) \Theta(E + \mu_R) \left[ \frac{q_{pR}(E)u^2_R(E) + q_{hR}(E)v^2_R(E)}{k_{pR}(E)} \right] f(E) \end{array} \right\}$$

$$= \frac{2}{\hbar} \frac{e^2}{h^2} \int_{-\infty}^{\infty} dE \rho_L(E) \rho_R(E + V) f(E)$$  \hspace{1cm} (S63)

with the particle tunneling density of states ($j = L, R$)

$$\rho_j(E) \equiv D_j(E) \left[ \frac{q_{pj}(E)u^2_j(E) + q_{hj}(E)v^2_j(E)}{k_{pj}(E)} \right] \Theta(E + \mu_j).$$  \hspace{1cm} (S64)

Note that in the Andreev approximation regime, $\rho_j(E) = D_j(E)$. Subtracting the current due to quasiparticle injections from the right reservoir $I_{dc}^- (V)$ gives the tunneling current (S56).

**VIII. BULK MEAN-FIELD EQUATIONS FOR THE PAIRING GAPS AND CHEMICAL POTENTIALS**

The main text presents results for the current across a Josephson junction across the BEC-BCS crossover. Here, we discuss the derivation of the reservoir pairing gaps $\Delta_L$ and $\Delta_R$ and chemical potentials $\mu_L$ and $\mu_R$.

We describe a particular configuration in terms of the density imbalance between the reservoirs

$$\nu = \frac{n_L - n_R}{n_L + n_R},$$  \hspace{1cm} (S65)

and the interaction strength

$$\frac{1}{k_F a},$$  \hspace{1cm} (S66)

where $a$ is a three-dimensional scattering length and the Fermi momentum $k_F = (3\pi^2 n)^{1/3}$ is defined in terms of the average density of both reservoirs, $n = (n_L + n_R)/2$, with a corresponding unit of energy $\varepsilon_F = \frac{k_F^2 k_F^2}{2m}$. Note that our definition implies $n_{L/R} = (1 + \nu)n$ for the reservoir densities and thus, $k_{F,L/R} = (1 + \nu)^{1/3}k_F$ and $\varepsilon_{F,L/R} = (1 + \nu)^{2/3}\varepsilon_F$.

In particular, results given in units of $\varepsilon_{F,L/R}$ are expressed in terms of the common Fermi energy $\varepsilon_F$ as

$$\frac{\mu_{L/R}}{\varepsilon_F} = (1 + \nu)^{2/3} \frac{\mu_{L/R}}{\varepsilon_{F,L/R}},$$  \hspace{1cm} (S67a)

$$\frac{\Delta_{L/R}}{\varepsilon_F} = (1 + \nu)^{2/3} \frac{\Delta_{L/R}}{\varepsilon_{F,L/R}}.$$  \hspace{1cm} (S67b)

For a given interaction strength and imbalance, the reservoir parameters should be taken from any calculation the bulk equation of state. Here, we choose for simplicity and ease of replicability a simple mean-field approximation for a bulk reservoir in three dimensions, although we emphasize that the Landauer-Büttiker framework derived in this paper is independent of this choice. The mean-field equations are written in compact form as [S15–S18]

$$\frac{\Delta}{\varepsilon_F} = \left[ \frac{2}{3I_2(y)} \right]^{2/3}, \quad \frac{1}{k_F a} = -\frac{2}{\pi} \left[ \frac{2}{3I_2(y)} \right]^{1/3} I_1(y),$$  \hspace{1cm} (S68a)

with

$$I_1(y) = \int_0^{\infty} dx \, x^2 \left( \frac{1}{E_x} - \frac{1}{x^2} \right), \quad I_2(y) = \int_0^{\infty} dx \, x^2 \left( 1 - \frac{\varepsilon_x}{E_x} \right),$$  \hspace{1cm} (S69)
Supplementary Figure S5. Left panel: (I.a) Mean-field gap and (I.b) chemical potential in units of the Fermi energy \( \varepsilon_F \) as a function of interaction strength \( 1/k_F a \) across the BEC-BCS crossover. (II.a-II.e) Pairing gaps of the left (\( \Delta_L \), blue line) and right (\( \Delta_R \), orange) reservoirs. (II.f-II.i) Chemical potential of the left reservoir (\( \mu_L \), blue), chemical potential of the right reservoir (\( \mu_R \), orange) and bias voltage across the SNS junction (\( V = \mu_L - \mu_R \), green). All quantities are plotted in unit of the Fermi energy \( \varepsilon_F \) as a function of density imbalance \( \nu \) for different interaction strengths \( 1/k_F a \) along the BEC-BCS crossover.

and we define the dimensionless variables

\[
x^2 = \frac{\hbar^2 k^2}{2m\Delta}, \quad y = \frac{\mu}{\Delta}, \quad \xi_x = \frac{\xi_k}{\Delta} = x^2 - y, \quad E_x = \frac{E_k}{\Delta} = \sqrt{\xi_x^2 + 1}.
\]

Both integrals are evaluated in terms of complete elliptic integrals of the first and second kind. Figures S5(I.a) and S5(I.b) show the mean-field result for the gap and chemical potential of a single reservoir as a function of scattering length across the BEC-BCS crossover. As apparent from the figure, the condition \( \Delta < \mu \) for the Andreev approximation in only satisfied in the BCS limit \( 1/(k_F a) \ll 0 \). In the unitary limit, gap and chemical potential are of comparable magnitude, and on the BEC-side the chemical potential even turns negative. Finally, for two reservoirs, Fig. S5 shows the mean-field gap \( \Delta_{L/R} \) [upper panel: (II.a-II.e)] and chemical potential \( \mu_{L/R} \) [lower panel: (II.f-II.i)] as a function of density imbalance \( \nu \) for five different interaction strengths \( (k_F a)^{-1} = -1.5, -1, -0.5, 0, \) and \( 0.5 \), corresponding to the parameter values in Fig. 3 of the main text. In addition, the green line indicates the chemical potential difference or bias voltage: \( V = \mu_L - \mu_R \). The Andreev approximation becomes uncontrolled even for a moderate deviations from the BCS regime, e.g., at \( (k_F a)^{-1} = -1 \) [see Figs. S5(II.b) and S5(II.g)]. Note that while present results in terms of the density imbalance instead of the bias voltage, both quantities are essentially proportional, at least for \( \nu \lesssim 0.8 \) [see green curve in Figs. S5(II.f-II.i)].
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