Griffiths inequalities for the $O(N)$-spin model

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Abstract

We prove Griffiths inequalities for the $O(N)$-spin model with inhomogeneous coupling constants and external magnetic field for any $N \geq 2$. This is achieved by using a representation of $O(N)$-spins in terms of random paths that reduces to the random current representation of the Ising model for $N = 1$ and an identity that is analogous to the switching lemma for random currents.

1 Introduction

Correlation inequalities for models in statistical mechanics were first presented by Griffiths [13] for the Ising model. These Griffiths inequalities were extended by Ginibre [12] to a more general framework that also includes quantum spin systems. This framework includes the classical XY (or $O(2)$-spin) model [12, 14] and the spin-$\frac{1}{2}$ XY model [6]. We direct the reader to [7] for an overview of results in these cases. A Griffiths inequality was also proven for the classical Heisenberg (or $O(3)$-spin) ferromagnet in the case of homogeneous rotations and for four-component spin systems such as the $|\phi|^4$ lattice euclidean field [11, 14, 18]. Griffiths inequalities are useful tools for proofs of the existence of the infinite volume limit of correlation functions and monotonicity of spontaneous magnetisation. They also allow comparisons of aspects, such as the critical temperature, of models with different spin and/or spatial dimension [7]. The extension of these inequalities to the $O(N)$-spin model and to higher spin quantum XY models or quantum Heisenberg models has been an important problem.

The representation of spin models in terms of random geometric objects has enjoyed increasing interest in recent years and has led to new results and new proofs for several important systems. The idea finds its origins in the work of Symanzik [20] and Brydges, Fröhlich and Spencer [8] for the random walk expansion of spin systems. Later, Aizenman [1] developed techniques for the random current representation of the Ising model, including the famous and powerful switching lemma for this model. There are analogous representations for quantum models, such as the representation of the Heisenberg ferromagnet in terms of sequences of random transpositions which was used by Tóth to bound the pressure of this model, or the representation of the Heisenberg anti-ferromagnet as a type of random cluster model by Aizenman and Nachtergaele [4]. It was later shown by Ueltschi [22] that these two representations can be combined into a random loop model that reduces to the previous models for a particular choice of the model’s parameters. This representation has enjoyed significant interest in the literature.

In this article we will make use of a representation of the $O(N)$-spin model in terms of random paths (walks and loops) that was introduced in [15] and is closely related to the model introduced in [5]. The difference with previous representations is that realisations can be defined purely in terms of objects local to individual edges and vertices. This distinction has allowed the use important tools such as reflection positivity [15, 19, 21], a proof of exponential decay of transverse correlations in the presence of an external magnetic field [16] that goes beyond results proven using the well-known Lee-Yang method, and a new proof of the BKT phase transition for the $O(2)$-spin model in two dimensions [17].

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For the case \( N = 1 \) the representation reduces to the random current representation of the Ising model. This representation has been of enormous use to the study of the Ising model, for example the phase transition of the Ising model coincides with a percolation transition in a system of currents [1]. One especially powerful tool is the switching lemma which, roughly speaking, allows the movement of sources (vertices with odd incoming/outgoing current) between different configurations of the random current model. This approach has allowed the (re)proof of many important results such as a proof of sharpness of the phase transition [2] and continuity of the spontaneous magnetisation [3]. We refer the reader to [9, 10] for an overview.

We prove a Griffiths inequality for the \( O(N) \)-spin model for any \( N \geq 2 \) on an arbitrary simple graph with inhomogeneous ferromagnetic coupling constants that may vary in different spin-directions. To the author's knowledge, this is the first proof of a Griffiths inequality for the \( O(N) \)-spin model with \( N > 4 \) and includes the cases of inhomogeneous coupling constants at each spin and in each spin direction (inhomogeneous rotations of spins), an inhomogeneous external field, and general boundary conditions. This is achieved by using the connection of this system with the random path model. By presenting the random path model in a slightly different way to the presentation in [15] we obtain a Griffiths inequality by using the proof of the switching lemma for the random current model presented in [1] as a template. The result of this is Lemma 3.1 which is an analogue of the switching lemma for the random current model. This approach has been of enormous use to the study of the Ising model, for example the phase transition of the Ising model coincides with a percolation transition in a system of currents [1].

1.1 Model and main result

Here we define the \( O(N) \)-spin model on an arbitrary graph with inhomogeneous coupling constants. In Section 4 we introduce the model with an external field. Consider a finite simple graph \( G \).

\[ H^\text{free}_{G,N,J}(\varphi) = - \sum_{\{x,y\} \in E} \sum_{i=1}^{N} J^i_{\{x,y\}} \varphi^i_x \varphi^i_y. \tag{1.1} \]

We define the expectation operator \( \langle \cdot \rangle^\text{free}_{G,N,J} \) acting on \( f : \Omega_{G,N} \to \mathbb{R} \) by

\[ \langle f \rangle^\text{free}_{G,N,J} = \frac{1}{Z^\text{free}_{G,N,J}} \int_{\Omega_{G,N}} d\varphi \ f(\varphi) e^{-H^\text{free}_{G,N,J}(\varphi)}, \tag{1.2} \]

where \( d\varphi = \prod_{x \in V} d\varphi_x \) is a product measure with \( d\varphi_x \) the uniform measure on \( \mathbb{S}^{N-1} \) and \( Z^\text{free}_{G,N,J} \) is a normalising constant that ensures \( \langle 1 \rangle_{G,N,\beta} = 1 \). Note that this definition includes the case of free and periodic boundary conditions (for example a torus \( \mathbb{Z}^d/L\mathbb{Z}^d \) with nearest neighbour edges). We may also consider other boundary conditions such as the “+” boundary condition which we define here in analogy with the Ising model.

Suppose that \( G \) is a subgraph of some larger (possibly infinite) graph \( \mathcal{G} \). Let \( \partial V \subset V \) be the set of vertices that share an edge with some vertex of \( \mathcal{G} \setminus G \) and let \( \partial E \) be this set of edges. We also denote by \( \partial^x V \subset V^c \) the set of vertices in \( V^c \) that share an edge with a vertex in \( V \) (the exterior vertex boundary). We define the hamiltonian with + boundary conditions as

\[ H^+_{G,N,J}(\varphi) = - \sum_{\{x,y\} \in E} \sum_{i=1}^{N} J^i_{\{x,y\}} \varphi^i_x \varphi^i_y - \sum_{\{x,y\} \in \partial E} J^1_{\{x,y\}} \varphi^1_x. \tag{1.3} \]

We interpret this as having all spins outside of \( V \) fixed as \( (1,0,\ldots,0) \), i.e. fully aligned along the first spin-direction, so that \( \sum_{i=1}^{N} J^i_{\{x,y\}} \varphi^i_x \varphi^i_y = J^1_{\{x,y\}} \varphi^1_x \) for edges \( \{x,y\} \in \partial E \) with \( x \in V, \ y \in V^c \). We
have analogous expectations to the case of free or periodic boundary. For $f : \Omega_{G,N} \rightarrow \mathbb{R}$

$$\langle f \rangle_{G,N,J}^+ = \frac{1}{Z_{G,N,J}^+} \int_{\Omega_{G,N}} d\varphi \ f(\varphi) e^{-H_{G,N,J}^+ (\varphi)}.$$  \hspace{1cm} (1.4)$$

We can think of the free boundary case as the case of + boundary with $J_e^i = 0$ for every $e \in \partial E$, we encourage the reader to adopt this point of view if it is helpful.

Our main result concerns correlations between the components of spins. For any $x \in V$, define the random variable $S_x : \Omega_{G,N} \mapsto \mathbb{S}^{N-1}$ representing the spin at $x$ as, $S_x(\varphi) := \varphi_x$, moreover we represent its components as $S_x = (S_1^x, \ldots, S_N^x)$. So, for example, $S_i^x(\varphi) = \varphi_i^x$ for $i \in [N] = \{1, \ldots, N\}$.

**Theorem 1.1.** Let $G = (V,E)$ be a finite simple graph and $N \in \mathbb{N}_{>1}$. Assume that $J_e^i \geq 0$ for every $e \in E$, $i \in [N]$. For $A,B \subset V$ and $\eta \in \{\text{free,} +\}$ we have

$$\langle \prod_{x \in A} S_1^x \prod_{y \in B} S_1^y \rangle_{G,N,J}^\eta \geq \langle \prod_{x \in A} S_1^x \rangle_{G,N,J}^\eta \langle \prod_{y \in B} S_1^y \rangle_{G,N,J}^\eta.$$  

This holds, for example, on $\mathbb{Z}^d$ with finite range (not necessarily translation invariant) coupling constants. The following monotonicity result is a straightforward corollary of Theorem 1.1.

**Corollary 1.2.** Let $G = (V,E)$ be a finite simple graph and $N \in \mathbb{N}_{>1}$. Assume that $J_e^i \geq 0$ for every $e \in E$, $i \in [N]$ and $\eta \in \{\text{free,} +\}$. For $A \subset V$ and $e \in E$ we have

$$\frac{\partial}{\partial J_e^i} \langle \prod_{x \in A} S_1^x \rangle_{G,N,J}^\eta \geq 0.$$

**Remark 1.3.** Theorem 1.1 holds for more general boundary conditions, the proof is very similar to the case of an inhomogeneous external magnetic field considered in Section 4. The random path model used for the proof can easily be extended to this case [16]. Because we already need to make modifications to the proof for the cases $\eta = \text{free}$ or $\eta = +$ and the cases $A \cap B = \emptyset$ or $A \cap B \neq \emptyset$, we will describe the $O(N)$-spin model with external field, the corresponding changes to the random path model, and the differences in the proof in Section 4. In this way we avoid the need for the extra modifications to be present throughout the article and can focus on the case of no external field, which already contains all the important ideas of the proof.

### 1.2 Proof method

To prove Theorem 1.1 we will use the connection between the $O(N)$-spin model and the random path model [15]. We will present the random path model in a slightly different way that is convenient for proofs. Both presentations are equivalent and hence reduce to the random current representation of the Ising model for $N = 1$. With this in mind, we use the proof of the switching lemma for random currents and a template. The presence of pairings makes things more delicate. This leads to Lemma 3.1 from which Theorem 1.1 follows easily.
Notation
\[
\begin{align*}
\mathbb{N} & \quad \{0,1,\ldots\} \\
\mathbb{N}_{>0} & \quad \{1,2,\ldots\} \\
2\mathbb{N} & \quad \text{the set of even integers in } \mathbb{N} \\
2\mathbb{N} + 1 & \quad \text{the set of odd integers in } \mathbb{N} \\
G = (V,E) & \quad \text{an undirected, simple, finite graph} \\
A \Delta B & \quad \text{the symmetric difference of sets } A \text{ and } B \\
N \in \mathbb{N}_{>0} & \quad \text{the number of colours or spin components} \\
[N] & \quad \{1,\ldots,N\} \\
\mathcal{M}_G^{N} & \quad \text{the set of } (m^1,\ldots,m^N) \in \mathbb{N}^{E \times [N]} \\
m & \quad \text{an element } (m^1,\ldots,m^N) \text{ of } \mathcal{M}_G^{N} \\
\mathcal{P}_G(m) & \quad \text{the set of pairings of } m = (m^1,\ldots,m^N) \\
n^z_x(m) & \quad \sum_{i=1}^N n^z_x(m) \\
m^z_x & \quad \sum_{e \in z} m^z_e \\
Z_{G,N,J,U}(A) & \quad \text{the measure of configurations such that } m^z_e \text{ is odd iff } j = 1 \text{ and } z \in A \\
\mathcal{G}_{G,N,J,U}(A) & \quad \text{the correlation function } Z_{G,N,J,U}(A) / Z_{G,N,J,U}(\emptyset) \text{ for } A \subset V
\end{align*}
\]

2 The Random Path Model

In this section we introduce the Random Path Model (RPM) in two equivalent ways. The first way was introduced in [15] and was shown to include many interesting models for specific choices of its parameters. The second way differs only slightly and will be more convenient for proofs. In both cases we will introduce the model for the case of free or periodic boundary conditions and then comment on the very minor difference needed to make the connection with + boundary conditions.

The RPM differs from previous representations of \(O(N)\)-spins in terms of loops and walks, such as the one introduced by Symanzik [20] and studied by Brydges, Fröhlich and Spencer [8], due to the presence of colourings of our basic objects into \(N\) colours and pairings.

The first description of the Random Path Model introduced in [15] has colourings and pairings “built-in”. The purpose of introducing this representation is to allow us to take advantage of the proven connections between this model and the \(O(N)\)-spin model that we require. The reader is encouraged to consult [15] or [16] for an alternative introduction of the model.

The case \(N = 1\) reduces to the random current representation of the Ising model where pairings are not present (as we will see below, the pairings are perfectly cancelled by the vertex weight function in this case).

2.1 The Random Path Model with Colourings and Pairings

Let \(G = (V,E)\) be a finite simple graph and \(N \in \mathbb{N}_{>0}\) be the number of colours. RPM configurations will correspond to undirected walks and loops, collectively referred to as paths, taking colours in \([N]\).

First, define \(\mathcal{M}_G := \mathbb{N}^E\). An element \(m = (m_e)_{e \in E} \in \mathcal{M}_G\) will be thought of as a collection of links on \(E\) with \(m_e\) referred to as the number of links on \(e \in E\). We say a link is incident to \(x \in V\) if it is on an edge incident to \(x\).

For \(m \in \mathcal{M}_G\), a colouring \(c = (c_e)_{e \in E}\) is a collection of functions, one for each \(e \in E\), with \(c_e : \{1,\ldots,m_e\} \rightarrow [N]\) interpreted as an assignment of an integer (colour) in \([N]\) to each link. More precisely, if we use \((e,p)\) to represent the \(p^{th}\) link on \(e\) then \(c((e,p)) \in [N]\) is the colour of the \(p^{th}\) link on \(e \in E\). A link with colour \(i \in [N]\) is called an \(i\)-link and we denote by \(m^z_e = m^z_e(m,c)\) the number of \(i\)-links on \(e\) for a pair \((m,c)\) of a collection of links and a colouring of those links. We denote by \(C_G(m)\) the set of all possible colourings \(c = (c_e)_{e \in E}\) for \(m\).
Given a collection of links $m \in \mathcal{M}_G$, and a colouring $c \in \mathcal{C}_G(m)$, we say $\pi = (\pi_x)_{x \in V}$ is a pairing of $(m,c)$ if, for each $x \in V$, $\pi_x(m,c)$ is a partition of the links incident to $x$ into sets with at most two links each such that if two links are in the same set of the partition, then they have the same colour. If a set contains two links then we say those links are paired at $x$ and think of them being joined (or wired) together at $x$ to form part of a continuous path passing through $x$. If a set contains only one link then we say this link is unpaired at $x$. A link can be paired to at most two other links, one at each end point of its edge. We denote by $\mathcal{P}_G(m,c)$ the set of all pairings for $m \in \mathcal{M}_G$ and $c \in \mathcal{C}_G(m)$.

We remark here that, in general, there are many possible colourings of $m \in \mathcal{M}_G$ and given a colouring $c \in \mathcal{C}_G(m)$ there are many possible pairings of $(m,c)$, in particular the “empty pairing” that leaves all links unpaired at every vertex.

A configuration for the random path model is a triple $w = (m,c,\pi)$ such that $m \in \mathcal{M}_G$, $c \in \mathcal{C}_G(m)$, and $\pi \in \mathcal{P}_G(m,c)$. Let $\mathcal{W}_G$ be the set of all random path configurations on $G$. As we can see from the example in Figure 2.1, any $w \in \mathcal{W}_G$ can be viewed as a collection of closed or open paths. A path is a maximal set of links such that each link in the path is paired to at least one other link in the path. It follows that all links in a path are the same colour. For each path, either every link in the path is paired at both end points, in which case we call the path closed, or there are two link end points that are not paired in which case we call the path open. We will often refer to closed paths as loops and open paths as walks, we also refer to the vertices at which an open path has unpaired links as its end points. We note that an open path may consist of a single link, unpaired at both end points, or start and end at the same vertex $x \in V$, in which case $x$ is counted as both end points, this case is distinct from the case when the unpaired links at $x$ are paired to form a loop passing through $x$. We refer to paths/loops/walks whose links have colour $i \in [N]$ as $i$-paths/loops/walks, respectively. By a slight abuse of notation, we will also view $m : \mathcal{W}_G \mapsto \mathcal{M}_G$ as a function such that, for $w' = (m',c',\pi')$, $m(w') = m'$.

Let $u_{x}^{i}(w)$ be the number of $i$-links incident to $x$ which are unpaired at $x$ (this is also the number of $i$-walk end-points at $x$). Let $v_{x}^{i}(w)$ be the number of pairs of $i$-links incident to $x$. Note that $2v_{x}^{i}(w) + u_{x}^{i}(w)$ is the number of $i$-links incident to $x$ in $w$.

Moreover, let

$$n_{x}^{i}(w) := v_{x}^{i}(w) + u_{x}^{i}(w)$$

(2.1)

be the local time of $i$-paths at $x$. Unpaired end-points of links incident to $x$ and pairs of paired links incident to $x$ both contribute +1 to the local time. In words, $n_{x}^{i}(w)$ is the number of times an $i$-path visits $x$, where a visit means that a path (open or closed) passes through $x$ or an open path ends at
We also define
\[
n_x(w) := \sum_{i=1}^N n_x^i(w)
\]
(2.2)
to be the local time of paths of all colours at \(x\).

We are interested in a particular subset of \(W_G\) which is relevant for understanding the correlations of the \(O(N)\)-spin model. Consider the set of configuration \(W_G' \subset W_G\) such that the only unpaired links have colour 1. Hence, any configuration in \(W_G'\) consists of open paths (walks) of colour 1, and closed paths (loops) of any colour.

We may also consider a model with boundary conditions. For \(G \supset G\) we define random path configurations on \(G \cup \partial E\) as above except that we do not specify pairings on the end points of \(\partial E\) that lie in \(V^c\). Intuitively, we think of random path configurations where paths are allowed to leave \(G\) at the boundary, but we do not track where they go after this. We denote by \(W_{G,\partial E}\) the set of all such configurations where only 1-links may be unpaired on \(V\) and no links are paired on \(V^c\). In the definitions, to make the connection with + boundary conditions for the \(O(N)\)-spin model, we will make the restriction that only 1-paths may leave \(G\).

Now we are ready to define our measure on configurations.

**Definition 2.1.** Let \(W_G'\) be the set of configurations \(w \in W_G\) such that \(u_x^1(w) = \ldots = u_x^N(w) = 0\) for every \(x \in V\). Given \(N \in \mathbb{N}_{>0}\) and non-negative coupling constants \(\mathbf{J} = (J_e^1, \ldots, J_e^N)_{e \in E}\) define the measure \(\tilde{\mu}_{G,N,\mathbf{J},U}^{\text{free}}\) on \(W_G'\) by
\[
\tilde{\mu}_{G,N,\mathbf{J},U}^{\text{free}}(w) := \prod_{e \in E} \prod_{i=1}^N \frac{\prod_{x \in V} U_x(w)^{m_e^i}}{m_e^{i!}} \prod_{x \in V^c} U_x(w), \quad w \in W_G',
\]
where \(U_x(w)\) is a vertex weight function depending only on the links adjacent to \(x\) and their pairing.

In the case where \(G \subset G\), we also define a measure on configurations on \(G \cup \partial E\) that is related to the case of + boundary conditions
\[
\tilde{\mu}_{G,N,\mathbf{J},U}^{+}(w) := \prod_{e \in E \cup \partial E} \prod_{i=1}^N \frac{\prod_{x \in V} U_x(w)^{m_e^i}}{m_e^{i!}} \prod_{x \in V} U_x(w) \prod_{e \in \partial E} \prod_{i=1}^N 1_{\{m_e^i = 0\}}, \quad w \in W_{G,\partial E}'\).
\]
The difference with \(\tilde{\mu}_{G,N,\mathbf{J},U}^{\text{free}}\) is that we allow 1-links on edges of \(\partial E\) (and hence for 1-paths to leave \(G\)). However we do not allow any links of colour 2, \ldots, \(N\) on \(\partial E\).

The central quantities of interest are correlation functions. We define this quantity for the first description of the random path model. Below we will define the analogous quantity for the second description of the random path model.

**Definition 2.2.** For \(A \subset V\), define \(\tilde{S}(A)\) to be the set of configurations \(w \in W_G'\) for the free or periodic boundary case or in \(W_{G,\partial E}'\) for the + boundary case such that \(u_x^1(w) = 1\) for every \(x \in A\) and \(u_x^1(w) = 0\) for every \(x \in V \setminus A\).

Further, for \(B \subset V\) define \(\tilde{S}(A; B)\) to be the set of configurations \(w \in W_G'\) for the free or periodic boundary case or in \(W_{G,\partial E}'\) for the + boundary case such that \(u_x^1(w) = 1\) for every \(x \in A \cup B\), \(u_x^1(w) = 2\) for every \(x \in A \cap B\), and \(u_x^1(w) = 0\) for every \(x \in V \setminus (A \cup B)\).

For \(\eta \in \{\text{free}, +\}\), we define \(\tilde{Z}_{G,N,\mathbf{J},U}^{\eta}(A) = \tilde{\mu}_{G,N,\mathbf{J},U}^{\eta}(\tilde{S}(A))\), \(\tilde{Z}_{G,N,\mathbf{J},U}^{\eta}(A; B) = \tilde{\mu}_{G,N,\mathbf{J},U}^{\eta}(\tilde{S}(A; B))\) and \(\tilde{Z}_{G,N,\mathbf{J},U}^{\text{loop,}\eta} = \tilde{\mu}_{G,N,\mathbf{J},U}^{\eta}(\tilde{S}(\emptyset))\). Finally, we define the correlation functions by
\[
\mathbb{G}_{G,N,\mathbf{J},U}^{\eta}(A) := \frac{\tilde{Z}_{G,N,\mathbf{J},U}^{\eta}(A)}{\tilde{Z}_{G,N,\mathbf{J},U}^{\text{loop,}\eta}}, \quad \mathbb{G}_{G,N,\mathbf{J},U}^{\eta}(A; B) := \frac{\tilde{Z}_{G,N,\mathbf{J},U}^{\eta}(A; B)}{\tilde{Z}_{G,N,\mathbf{J},U}^{\text{loop,}\eta}}.
\]
We call the cases where \(|A| = 2\) two-point functions. Note that if \(\eta = \text{free}\) and \(|A|\) is odd then \(\tilde{S}(A) = 0\). If \(A \cap B = \emptyset\) then \(G^\eta_{G,N,J,U}(A; B) = G^\eta_{G,N,J,U}(A \cup B)\).

For \(A \subset V\), \(\tilde{Z}_{G,N,J,U}^\eta(A)\) corresponds to the total measure of configurations with coloured loops and 1-paths with end points at vertices of \(A\), such that each \(x \in A\) has precisely one 1-walk end point. In particular, the case \(A = \{x, y\}\) with \(x \neq y\) and \(\eta = \text{free}\) is the measure of configurations with coloured loops and a single 1-walk with end points \(x\) and \(y\). This quantity is relevant for the correlation \(\langle \prod_{x \in A} S^1_x \prod_{y \in B} S^1_y \rangle_{G,N,J}\). For \(A, B \subset V\), \(\tilde{Z}_{G,N,J,U}^\eta(A; B)\) corresponds to the total measure of configurations with coloured loops and 1-paths with a single 1-walk end point at vertices of \(A\), and two 1-walk end points at vertices of \(A \cap B\). In the case \(A = B = \{x\}\) this reduces to \(\mu_{G,N,J,U}(R_x)\) as defined in Definition 2.2 of [15]. This quantity is relevant for the correlation \(\langle \prod_{x \in A} S^1_x \prod_{y \in B} S^1_y \rangle_{G,N,J}\) when \(A \cap B \neq \emptyset\).

The next proposition connects the correlation functions defined above to the spin correlations of the \(O(N)\)-spin model. For \(\langle \prod_{x \in A} S^1_x \rangle_{G,N,J}\), the case of homogeneous coupling constants \(J^e_i = \beta \geq 0\) for all \(e \in E\), \(i \in [N]\) and free or periodic boundary conditions, the proof can be found in [15] for the case \(|A| = 2\) and in [16] for the general case, even with an external field. The extension to inhomogeneous coupling and + boundary condition is straightforward. For \(\langle \prod_{x \in A} S^1_x \prod_{y \in B} S^1_y \rangle_{G,N,J}\), this is a straightforward extension of the previous case by using the case \(\langle (S^1_x)^2 \rangle_{G,N,J}\), dealt with in [15], as a guide.

**Proposition 2.3** (Lees-Taggi 2020). Let \(G = (V, E)\) be a finite, simple graph. Let \(N \in \mathbb{N}_{>0}\) and \(J = (J^e_1, \ldots, J^e_N)\) be non-negative coupling constants. Let the vertex weight function introduced in Definition 2.2 be given by \(U^N_x(w) := U^N(\eta_x(w))\) with

\[
U^N(\eta)(r) := \frac{\Gamma\left(\frac{N}{2}\right)}{2^r \Gamma\left(r + \frac{N}{2}\right)}, \quad r \in \mathbb{N}.
\]

(2.5)

For \(A, B \subset V\) and \(\eta \in \{\text{free}, +\}\) we have that

\[
G^\eta_{G,N,J,U^N}(A) = \left\langle \prod_{x \in A} S^1_x \right\rangle_{G,N,J}^\eta, \quad G^\eta_{G,N,J,U^N}(A; B) \leq \left\langle \prod_{x \in A} S^1_x \prod_{y \in B} S^1_y \right\rangle_{G,N,J}^\eta.
\]

(2.6)

The inequality for \(G^\eta_{G,N,J,U^N}(A; B)\) is due to the possible presence of “degenerate paths” with 0 edges and hence two end-points on the same vertex of \(A \cap B\) when expanding the right side as in [15, 16].

**Remark 2.4.** In order to obtain a Griffiths inequality when \(A \cap B \neq \emptyset\) we do not need to worry about the degenerate paths mentioned above as we prove the inequality \(G^\eta_{G,N,J,U^N}(A)G^\eta_{G,N,J,U^N}(B) \leq G^\eta_{G,N,J,U^N}(A; B)\), which is sufficient.

In the case \(N = 1\) we find that

\[
U^N(\eta)(r) = 1/(2r - 1)!!.
\]

where \((2r - 1)!!\) is the number of ways to pair \(2r\) objects, this cancels out the contribution from pairings and leads to the random current representation of the Ising model.

### 2.2 The Random Path Model with pre-coloured links

We now introduce a different description of the RPM, there are many similarities with the previous description. We will focus on the case relevant for \(\langle \prod_{x \in A} S^1_x \rangle_{G,N,J}\) and then in Remark 2.8 comment on the changes needed to make the connection with \(\langle \prod_{x \in A} S^1_x \prod_{y \in B} S^1_y \rangle_{G,N,J}\). The changes are conceptually simple but result in cumbersome extra notation which, in the author’s opinion, detracts too much from clarity.
Let $G = (V, E)$ be a finite simple graph and $N \in \mathbb{N}_{>0}$ be the number of colours. We introduce “pre-coloured” link configurations by defining $\mathcal{M}_G^N := \mathbb{N}^{E \times [N]}$. An element $m = (m^1, \ldots, m^N) \in \mathcal{M}_G^N$ is a tuple of $N$ collections of links on $E$ with $m^i = (m^i_e)_{e \in E}$ the collection of i-links and $m^i_e$ the number of i-links on $e \in E$. As above, we say a link is incident to $z \in V$ if it is on an edge incident to $z$. We also define

$$m_e = \sum_{i=1}^N m^i_e. \quad (2.7)$$

Given a collection $m = (m^1, \ldots, m^N) \in \mathcal{M}_G^N$ we again define a pairing $\pi = (\pi_z)_{z \in V}$ to be a collection of functions such that $\pi_z(m)$ is a partition of the links incident to $z$ into sets of at most two links each such that if two links are in the same set, then they belong to the same $m^i$. As above we say that two links are paired at $z \in V$ if they are in the same set of $\pi_z(m)$ and links are unpaired at $z$ if they are in a set by themselves in $\pi_z(m)$. Denote by $\mathcal{P}_G(m)$ the set of all pairings of $m \in \mathcal{M}_G^N$. As above, a pair $(m, \pi)$ where $m \in \mathcal{M}_G^N$ and $\pi \in \mathcal{P}_G(m)$ has a natural interpretation as a collection of open and closed paths taking colours in $[N]$.

Configurations in this description are simply the tuples $m = (m^1, \ldots, m^N) \in \mathcal{M}_G^N$, they do not have a specific pairing “built in” and so to make the connection between correlations for the two descriptions a sum over pairings will appear in the measure of the model. For this reason we need to immediately restrict the types of pairings we will sum over in order to recover correlations of the $O(N)$-spin model.

Define $P_G(m) \subset \mathcal{P}_G(m)$ to be the set of pairings such that at most one link of each colour is unpaired at each $z \in V$. We see from this that for a pairing in $P_G(m)$ there is an unpaired i-link at $z \in V$ if and only if $\sum_{e \ni z} m^i_e \in 2\mathbb{N} + 1$. It hence makes sense to define, for $z \in V$, $i \in [N],$

$$m^i_z := \sum_{e \ni z} m^i_e, \quad m_z = \sum_{i=1}^N m^i_z. \quad (2.8)$$

Pairings in $P_G(m)$ are maximal in the sense that they have as many paired links as possible at each vertex. Because we will only consider pairings in $P_G(m)$ we can make an alternate definition of local time which is consistent with the definition above in this case. For $z \in V$ and $i \in [N]$ we define

$$n^i_z(m) := \left\lfloor \frac{1}{2} m^i_z \right\rfloor, \quad n_z(m) = \sum_{i=1}^N n^i_z(m). \quad (2.9)$$

When it is not ambiguous, we will write $n^i_z = n^i_z(m)$ and $n_z = n_z(m)$. As above, we are interested in a particular subset of configurations (in this case - a subset of $\mathcal{M}_G^N$). We consider the set $\mathcal{M}_G^N(1) \subset \mathcal{M}_G^N$ of configurations $m$ such that $m^i_e \in 2\mathbb{N}$ for every $i \in \{2, \ldots, N\}$ and every $z \in V$. Configurations $m \in \mathcal{M}_G^N(1)$ can only have an odd number of incident i-links at a vertex if $i = 1$ and hence when pairing these configurations with an element of $P_G(m)$ the only walks present (if any) will be 1-walks.

As above, we may also consider the model with boundary conditions. For $\mathcal{G} \supseteq G$ we define random path configurations on $G \cup \partial E$ as above, allowing links on edges of $\partial E$. In this case we do not specify pairings on the end points of $\partial E$ that lie in $V^c$ so that $\mathcal{P}_{G,\partial E}(m)$ is only the set of pairings of links incident to each $z \in V$. We denote by $\mathcal{M}_G^{N,\partial E}(1)$ the set of all such configurations where $m^i_e \in 2\mathbb{N}$ for every $i \in \{2, \ldots, N\}$ and every $z \in V$. In the definitions, to make the connection with + boundary conditions for the $O(N)$-spin model, we make the restriction that only 1-links may be present on $\partial E$.

We are now ready to define our measure on these configurations.

**Definition 2.5.** Consider the set $\mathcal{M}_G^N(1) \subset \mathcal{M}_G^N$ of link configurations such that $m^i_e \in 2\mathbb{N}$ for every $i \in \{2, \ldots, N\}$ and every $z \in V$. Given $N \in \mathbb{N}_{>0}$ and non-negative coupling constant $J = (J^1_e, \ldots, J^N_e)_{e \in E}$
define the measure $\mu_{G,N,J,U}^{\text{free}}$ on $\mathcal{M}_{G}^{N}(1)$ by

$$
\mu_{G,N,J,U}^{\text{free}}(m) = \prod_{e \in E} \prod_{i=1}^{N} \frac{(J_{e})^{m_{i}}}{m_{i}!} \prod_{z \in V} U_{z}(m)|P_{G}(m)|, \quad m \in \mathcal{M}_{G}^{N}(1),
$$

where $U_{z}(m)$ is a vertex weight function depending only on the links incident to $z$. In the case where $G \subset \mathcal{G}$, we also define a measure on configurations on $G \cup \partial E$

$$
\mu_{G,N,J,U}^{\text{free}}(m) := \prod_{e \in E \cup \partial E} \prod_{i=1}^{N} \frac{(J_{e})^{m_{i}}}{m_{i}!} \prod_{z \in V} U_{z}(m)|P_{G \cup \partial E}(m)| \prod_{e \in \partial E} \prod_{i=2}^{N} \mathbb{I}(m_{i}=0), \quad m \in \mathcal{M}_{G \cup \partial E}^{N}(1).
$$

The difference with $\mu_{G,N,J,U}^{\text{free}}$ is that we allow 1-links on edges of $\partial E$. However we do not allow any links of colour 2,\ldots,N on $\partial E$.

In order to make the connection with the previous presentation of the random path model, and therefore with $O(N)$-spins, we introduce correlations for this measure in an analogous way to Definition 2.2. We will use the same notation to denote correlations as the correlations are indeed equal to those defined in 2.2, this is proven in Proposition 2.7.

**Definition 2.6.** For $A \subset V$, define $\mathcal{S}(A)$ to be the set of configurations $m = (m_{1},\ldots,m_{N})$ in $\mathcal{M}_{G}^{N}(1)$ for the free or periodic boundary case or in $\mathcal{M}_{G,\partial E}^{N}(1)$ for the + boundary case such that, for $z \in V$, $m_{i} \in 2\mathbb{N}+1$ if and only if $z \in A$. For $\eta \in \{\text{free, +}\}$, we define $Z_{G,N,J,U}^{\eta}(A) = \mu_{G,N,J,U}^{\eta}(\mathcal{S}(A))$ and $Z_{G,N,J,U}^{\text{loop,}\eta} = \mu_{G,N,J,U}^{\eta}(\mathcal{S}(0))$. Finally, the correlation functions are given by

$$
G_{G,N,J,U}^{\eta}(A) := \frac{Z_{G,N,J,U}^{\eta}(A)}{Z_{G,N,J,U}^{\text{loop,}\eta}}.
$$

The next proposition justifies this re-use of notation for correlations.

**Proposition 2.7.** Let $N \in \mathbb{N}_{>0}$, $J$ such that $J_{e}^{i} \geq 0$ for all $e \in E$ and $i \in [N]$, and vertex weight functions $U_{z}^{(N)}$ be fixed as in Proposition 2.3. For any $A \subset V$ and $\eta \in \{\text{free, +}\}$ we have that

$$
\tilde{Z}_{G,N,J,U}^{\eta}(A) = Z_{G,N,J,U}^{\eta}(A).
$$

**Proof.** The proof is a simple calculation. We present the proof for free or periodic boundary conditions, given this the adaptation to + boundary conditions is trivial, we simply insert indicators that no i-links are present on $\partial E$ for $i = 2,\ldots,N$. We have

$$
\tilde{Z}_{G,N,J,U}^{\text{free}}(A)
= \sum_{(m,c,\pi) \in \tilde{\mathcal{S}}(A)} \prod_{e \in E} \prod_{i=1}^{N} \frac{(J_{e})^{m_{i}}}{m_{i}!} \prod_{z \in V} U_{z}^{(N)}(m,c,\pi)
= \sum_{m \in \mathcal{M}_{G}} \sum_{c \in \mathcal{C}_{G}(m)} \sum_{\pi \in \mathcal{P}_{G}(m,c)} \mathbb{I}(m,c,\pi) \in \tilde{\mathcal{S}}(A) \prod_{e \in E} \prod_{i=1}^{N} \frac{(J_{e})^{m_{i}}}{m_{i}!} \prod_{z \in V} U_{z}^{(N)}(m,c,\pi)
= \sum_{(m_{1},\ldots,m_{N}) \in \tilde{\mathcal{S}}(A)} \sum_{m \in \mathcal{M}_{G}} \sum_{c \in \mathcal{C}_{G}(m)} \left( \prod_{e \in E} \prod_{i=1}^{N} \mathbb{I}(m_{i},c) = m_{i} \right) \sum_{\pi \in \mathcal{P}_{G}(m,c)} \prod_{e \in E} \prod_{i=2}^{N} m_{i}! \prod_{z \in V} U_{z}^{(N)}(m)
= \sum_{(m_{1},\ldots,m_{N}) \in \tilde{\mathcal{S}}(A)} \prod_{e \in E} \prod_{i=1}^{N} \frac{(J_{e})^{m_{i}}}{m_{i}!} \prod_{z \in V} U_{z}^{(N)}(m)|P_{G}(m)|
$$

(2.11)
where the third equality used that the definitions of \( n_z \) coincide on \( \tilde{S}(A) \) and \( S(A) \) and the last equality used that there are \( (m^1, \ldots, m^N) \) ways to colour \( m_e \) objects so that there are \( m^i_e \) objects of colour \( i \), for \( i = 1, \ldots, N \). The last expression is \( Z_{G,N,J,U(\eta)}^{(1)}(A) \), as desired.

\[ \square \]

**Remark 2.8.** For the case \( \langle \prod_{x \in A} S_x \prod_{y \in B} S_y \rangle_{G,N,J} \) with \( A \cap B \neq \emptyset \) we consider \( m \) such that \( m^i_1 \in 2\mathbb{N} + 1 \) for \( z \in A \Delta B \) and \( m^i_1 \in 2\mathbb{N} \) otherwise. We also consider pairings such that precisely one \( 1 \)-link is unpaired at each \( z \in A \Delta B \) and two \( 1 \)-links are unpaired at each \( z \in A \cap B \). \( n_z(m) \) is increased by one for \( z \in A \cap B \) as the two links incident to \( z \) that correspond to \( 1 \)-walk end points in some fixed pairing now contribute one each to the local time. No other links are unpaired. Defining \( Z_{G,N,J,U(\eta)}^{(1)}(A;B) \) with these changes we can show \( Z_{G,N,J,U(\eta)}^{(1)}(A;B) = Z_{G,N,J,U(\eta)}^{(1)}(A;B) \) with an identical proof to above.

### 3 Proof of Theorem 1.1

During the proof we will consider, \( G, N, J \) and \( U(\eta) \) fixed and denote \( \mu_{G,N,J,U(\eta)}^{\eta} \) by \( \mu_N \). The proof of Theorem 1.1 follows easily from Lemma 3.1 as explained immediately below the lemma. Before proceeding to the statement and proof of the lemma, we introduce some simple concepts that are necessary. To avoid repeated duplication of expressions and definitions, throughout this section we will stick to the notation for free or periodic boundary conditions unless the required change for + boundary is not immediate. We will also consider the case \( A \cap B = \emptyset \) and comment on required changes where necessary.

For \( m, \bar{m} \in M_G^N(1) \) we define the sum of configurations \( m + \bar{m} = (m^1 + \bar{m}^1, \ldots, m^N + \bar{m}^N) \) where \( m^i + \bar{m}^i = (m^i_1 + \bar{m}^i_1)_{e \in E} \). In other words, \( m + \bar{m} \) is the configuration with \( m^i_1 + \bar{m}^i_1 \) \( i \)-links on \( e \).

We say that \( \bar{m} \leq m \) if \( \bar{m}^i_1 \leq m^i_1 \) for every \( e \in E \) and \( i \in [N] \). When \( \bar{m} \leq m \) we can define their difference by \( m - \bar{m} = (m^1 - \bar{m}^1, \ldots, m^N - \bar{m}^N) \) where \( m^i - \bar{m}^i = (m^i_1 - \bar{m}^i_1)_{e \in E} \). In the sequel we will make use of the fact that, for \( A, B \subset V \) with \( A \cap B = \emptyset \), if we take \( m \in S(A) \) and \( \bar{m} \in S(B) \) then \( n_z(m) + n_z(\bar{m}) = n_z(m + \bar{m}) \).

We may also interpret \( m \in M_G^N(1) \) as a collection of multigraphs \((m^1, \ldots, m^N)\) each with vertex set \( V \) and the \( i^{th} \) multigraph having \( m^i_{(x,y)} \) edges between \( x, y \in V \) labelled \( 1, \ldots, m^i_{(x,y)} \). This interpretation was used in [17]. Consider a collection of multigraphs \( \bar{m} = (\bar{m}^1, \ldots, \bar{m}^N) \) such that \( \bar{m}^i \) is a subgraph of \( m^i \) for each \( i \in [N] \) (possibly \( \bar{m} = m \)). We can define pairings of the edges of these multigraphs in a completely analogous way to pairings for elements of \( M_G^N \) and by a small abuse of notation denote by \( P_G(\bar{m}) \) the set of pairings of the multigraph \( \bar{m} \) that leave at most one edge of each multigraph in the collection unpaired at each \( z \in V \). For \( A \subset V \) we commit another slight abuse of notation and say that a collection of multigraphs \( m \in S(\eta) \) if \( m^i_1 \in 2\mathbb{N} + 1 \) for \( i = 1 \) and \( z \in A \) and \( m^i_1 \in 2\mathbb{N} \) otherwise.

For \( A \subset V \), define \( F_A \subset M_G^N(1) \) to be the set of \( m \in M_G^N(1) \) such that each connected component of \( m \) (when considered as a multigraph) contains an even number (possibly 0) of vertices from \( A \). Further, for the case with boundary, define \( F_A^+ \subset M_G^N(1) \) to be the set of \( m \in M_G^N(1) \) such that each connected component of \( m \) (when considered as a multigraph) either contains an even number (possibly 0) of vertices from \( A \) or is connected to \( \partial E \). For \( A, B \subset V \) with \( A \cap B = \emptyset \) and \( m \in S(A \cup B) \) we define \( P_{A,B}(m) \subset P_G(m) \) to be those pairings \( \pi \) of \( m \) for which the collection of 1-paths defined by the pair \((m, \pi)\) is such no walk has an endpoint in \( A \) and an endpoint in \( B \).

We have the following lemma.

**Lemma 3.1.** Let \( A, B \subset V \) with \( A \cap B = \emptyset \), let vertex weight functions \( U_{x}^{(N)} \) be fixed as in Proposition 2.3 and \( \eta = \text{free} \). For any \( N \in \mathbb{N} \), non-negative couple constants \( J = (J^e_1, \ldots, J^e_N)_{e \in E} \) and \( F : \)
First we note that, for \( N \) even, we have

\[
\mathcal{M}_G^N \rightarrow \mathbb{R}_{\geq 0} \text{ we have }
\sum_{m \in \mathcal{S}(A)} F(m + \bar{m}) \mu_N(m) \mu_N(\bar{m}) = 
\sum_{m \in \mathcal{S}(A \cup B)} F(m + \bar{m}) \mu_N(m) \mu_N(\bar{m}) \mathcal{F}_B(m + \bar{m}) |P_{A,B}(m)| \frac{|P_G(m)|}{|P_G(m)|}.
\]

This holds for \( \eta = + \) if we replace \( \mathcal{F}_B \) on the right side by \( \mathcal{F}_B^+ \).

For \( A \cap B \neq \emptyset \) we replace \( \mathcal{S}(A \cup B) \) with \( \mathcal{S}(A \Delta B) \) and require that pairings for \( m \) on the right side have two unpaired 1-links on each \( z \in A \cap B \). Vertices in \( A \cap B \) count twice in the definition of \( \mathcal{F}_B^{(+)}. \)

Theorem 1.1 follows from Lemma 3.1 and Proposition 2.3 when we set \( F \equiv 1 \), bound the indicator and ratio on the right side of the inequality by 1 and divide both sides by \( \mu_N(\mathcal{S}(\emptyset))^2 \).

**Remark 3.2.** The proof of Lemma 3.1 is slightly different depending on whether \( N \) is even or odd. In both cases we use the proof of the switching lemma for \( N = 1 \) as a template. By adding configurations in \( \mathcal{S}(A) \) and \( \mathcal{S}(B) \) we obtain an element \( m \in \mathcal{S}(A \cup B) \), when summing over these configuration we have a sum over \( \bar{m} \in \mathcal{S}(B) \) such that \( \bar{m} \leq m \) which we can give a geometric interpretation. The proof for the case \( A \cap B \neq \emptyset \) is essentially identical in both cases once we note the differences mentioned in the statement of the lemma and the requirement in the definition of \( \mathcal{S}(A \Delta B) \) that two 1-links are unpaired at each \( z \in A \cap B \).

The identity in the lemma seems to be the analogous equality to the switching lemma for the case \( N > 1 \). The presence of the specific set of pairings \( P_{A,B}(m) \) is not surprising as pairings are not cancelled by \( U_{<}(N) \) for \( N > 1 \). This equality should have further useful applications.

For the proof, we will make use of the following ordering on the pairs of a pairing at each \( z \in V \).

Given a pairing of \( m \in \mathcal{M}_G^N \) we may order the pairs at each \( z \in V \) as follows. For pairs \( p_1, p_2 \) consisting of one or two edges each, \( p_1 < p_2 \) if \( p_1 \) contains edges of \( m^i \) and \( p_2 \) contains edges of \( m^j \) for \( i < j \). If \( p_1 \) and \( p_2 \) both contain edges of \( m^i \) then \( p_1 < p_2 \) if the the lowest label of edges in \( p_1 \cup p_2 \) belongs to an edge in \( p_1 \), with arbitrary tie breaking decided by some ordering on \( E \) (or \( E \cup \partial E \)) if the lowest label of each pairing is the same. Given a pairing of \( m \) considered as a multigraph we may order the pairs at each \( z \in V \) in the same way.

We are now ready to prove Lemma 3.1, we begin with the proof for \( N \) even.

### 3.1 Proof of Lemma 3.1 for \( N \) even

First we note that, for \( N \) even, we have that

\[
\mathcal{U}^{(N)}(r) = \frac{\Gamma\left(\frac{N}{2}\right)}{2^r (r + \frac{N-2}{2})!}.
\]

and hence for \( k, r \in \mathbb{N} \) with \( r \leq k \)

\[
\mathcal{U}^{(N)}(k - r) \mathcal{U}^{(N)}(r) = \frac{\Gamma\left(\frac{N}{2}\right)^2}{2^k (k + N - 2)!} \left(\frac{N - 2}{2}\right)! \left(\frac{k + N - 2}{2}\right).
\]

(3.1)
Let $A, B \subset V$ such that $A \cap B = \emptyset$. A calculation similar to that in [? , Lemma 4.3] gives

\[
\sum_{m \in \mathcal{S}(A)} F(m + \bar{m}) \mu_N(m) \mu_N(\bar{m})
\]

\[
= \sum_{m \in \mathcal{S}(A)} F(m + \bar{m}) \prod_{e \in E} \prod_{i=1}^{N} \frac{(J^e)^{m^e_i + \bar{m}^e_i}}{m^e_i! \bar{m}^e_i!} \prod_{z \in V} \frac{U_z(N)(m)U_z(N)(\bar{m})|P_G(m)||P_G(\bar{m})|}{|P_G(m)|\prod_{z \in V} (n_z(m) + N - 2)!},
\]

where the last equality used (3.1) and we inserted an extra factor of $(n_z(m) + N - 2)!$ for each $z \in V$. We note that the set \{ $\bar{m} \in \mathcal{S}(B) : \bar{m} \leq m$ \} is non-empty (and therefore the corresponding sum non-zero) if and only if $m \in F_B$ when $\eta = \text{free}$, or $m \in F^\eta_B$ when $\eta = \text{free}$. Indeed, for the case $\eta = \text{free}$ there can only be an odd number of i-links incident to every vertex of $B$ for $m \leq \bar{m}$ if vertices of $B$ can be paired off in some way within each connected component of $m$. This corresponds to having a walk between these pairs. For $\eta = \text{free}$ the vertices of $B$ do not have to be paired off each other as long as they can be connected to the boundary.

Consider a subgraph $m \in \mathcal{S}(B)$ of the multigraph defined by $m \in \mathcal{S}(A \cup B)$ and pairings $\vec{\pi}$ of $m$ and $\pi$ of $m \setminus \bar{m}$. Now replace each vertex $z \in V$ with the $n_z(m) + N - 2$ vertices \{ $z_1, \ldots, z_{n_z(m)+N-2}$ \}. For each $z \in V$, select a subset, $V_z(\bar{m})$, of these vertices of size $n_z(m) + N - 2$ and take an ordering, $p_z$, of these $n_z(m) + N - 2$ vertices. The sum over $m \in \mathcal{S}(B)$ above is then the number of choices of such a subgraph $\bar{m}$, a pairing of $\bar{m}$ and $m \setminus \bar{m}$, a set, $V_z(\bar{m})$, of size $n_z(m) + \frac{N-2}{2}$ at each $z \in V$ and an ordering, $p_z$, of the $n_z(m) + N - 2$ vertices for each $z \in V$. Encode these choices in a tuple $(\bar{m}, \vec{\pi}, \vec{\pi}, (V_z(\bar{m}))_{z \in V}, (p_z)_{z \in V})$.

We will do a bijection from the set of tuples $(\bar{m}, \vec{\pi}, \vec{\pi}, (V_z(\bar{m}))_{z \in V}, (p_z)_{z \in V})$ with $\bar{m} \in \mathcal{S}(B)$ into the set of tuples $(\bar{m}, \vec{\pi}, \vec{\pi}, (V_z(\bar{m}))_{z \in V}, (p_z)_{z \in V})$ with $\bar{m} \in \mathcal{S}(\emptyset)$ by using the ordering on pairs defined above.

**Remark 3.3.** Informally speaking, we want to identify the walks defined by $\bar{m}$ and $\vec{\pi}$ and some corresponding subset of $V_z(\bar{m})$. The mapping will then move these walks and the corresponding number of vertices from $\bar{m}$, $\vec{\pi}$ and $(V_z(\bar{m}))_{z \in V}$ to $m \setminus \bar{m}$, $\pi$, and $(V_z(\bar{m}))_{z \in V}$. To do this, we need a deterministic choice to decide which vertices in $V_z(\bar{m})$ “belong” to the walks and a way to retain this information to uniquely obtain an image tuple. This deterministic choice is provided by the ordering on pairs defined above Lemma 3.1 and the orderings $p_z$ of vertices.
on pairs of \( \pi, \pi' \), and the ordering \( p_z \) of vertices \( \{z_1, \ldots, z_{n_z(m)+N-2}\} \) at each \( z \in V \).

Consider the connected components of this graph corresponding to the walks with end points on vertices associated to \( B \). Let \( \bar{m}^B \subset \bar{m} \) be the corresponding set of edges, let \( \bar{\pi}^B \) respectively \( \bar{\pi}^{\text{loops}} \) be the pairing of \( \bar{m}^B \) respectively \( \bar{m} \setminus \bar{m}^B \) coming from the restriction of \( \bar{\pi} \) to \( \bar{m}^B \) respectively \( \bar{m} \setminus \bar{m}^B \), and let \( V_z(\bar{m}^B) \subset V_z(\bar{m}) \) be the set of vertices attached to pairs of \( \bar{\pi}^B \). Note that identifying \( \bar{m}^B \) requires knowledge of \( \bar{\pi} \) and that the pair \((\bar{m} \setminus \bar{m}^B, \bar{\pi}^{\text{loops}})\) can be interpreted as a collection of coloured loops. We define our injection by mapping

\[
(\bar{m}, \bar{\pi}, \pi, (V_z(\bar{m}))_{z \in V}, (p_z)_{z \in V}) \mapsto (\bar{m} \setminus \bar{m}^B, \bar{\pi}^B, \pi \sqcup \bar{\pi}^B, (V_z(\bar{m}) \setminus V_z(\bar{m}^B))_{z \in V}, (\tilde{p}_z)_{z \in V})
\] (3.3)

where \( \pi \sqcup \bar{\pi}^B \) is the pairing of \((\bar{m} \setminus \bar{m}) \cup \bar{m}^B\) that pairs the edges of \( \bar{m} \setminus \bar{m}^B \) according to \( \pi \) and the edges of \( \bar{m}^B \) according to \( \bar{\pi}^B \) and \( \tilde{p}_z \) is the ordering on \( \{z_1, \ldots, z_{n_z(m)+N-2}\} \) that results in the same vertex-pair attachments and same relative ordering of isolated vertices when constructing a simple graph from the image tuple (analogously to above) as \( p_z \) gives from the original tuple. The image of this map consists of a tuple of an element \( \bar{m} = \bar{m} \setminus \bar{m}^B \in S(\emptyset) \), a pairing of \( \bar{m} \), a pairing of \( \bar{m} \setminus \bar{m} \) and a collection of sets \((V_z(\bar{m}))_{z \in V}\) such that \(|V_z(\bar{m})| = n_z(\bar{m}) + N_2\) and an ordering \( \tilde{p}_z \) of the vertices corresponding to each \( z \in V \). This can again be interpreted as a graph whose connected components are isolated vertices and paths defined by the triple \( \bar{m}, \bar{\pi}^B \) and \((V_z(\bar{m}))_{z \in V}\) or the triple \( m \setminus m, \pi \sqcup \bar{\pi}^B, (V_z(m))_{z \in V}\) where pairs are attached to vertices according to the orderings on pairs and \((\tilde{p}_z)_{z \in V}\). The new collection of orderings \((\tilde{p}_z)_{z \in V}\) is the unique collection such that, when attaching vertices and pairs as described above (without reference to \( V_z(\bar{m}) \setminus V_z(\bar{m}^B) \), i.e. taking it to be the empty set) the connected components of this graph have all the same vertex-pair attachments as the graph constructed by the pre-image tuple and the isolated vertices are in the same relative order. The selected sets \( V_z(\bar{m}) \setminus V_z(\bar{m}^B) \) at each vertex then correspond to the vertices of the triple \((\bar{m}, \bar{\pi}, (V_z(\bar{m}))_{z \in V})\) with the walks ending on \( B \) (and their vertices) removed, as expected. Described somewhat informally, the mapping takes the graph obtained from \((\bar{m}, \bar{\pi}, \pi, (V_z(\bar{m}))_{z \in V}, (p_z)_{z \in V})\) and moves the walks with end points in \( B \) from the first triple to the second without disturbing any vertex-pair attachments. The new orderings \((\tilde{p}_z)_{z \in V}\) allows us to interpret this graph as a new tuple whose first element is in \( S(\emptyset) \).

We note that every tuple in the image of this map involved pairings such that no walk has an end point on \( A \) and an end point on \( B \). In other words these pairings are elements of \( P_G(\bar{m} \setminus \bar{m}^B) \) and \( P_{A,B}( (\bar{m} \setminus \bar{m}) \cup \bar{m}^B) \), respectively. The map is an injection. Indeed, the tuple \((\bar{m}, \bar{\pi}, \pi, (V_z(\bar{m}))_{z \in V}, (p_z)_{z \in V})\) together with the ordering on pairings defines uniquely the graph described below Remark 3.3 whose connected components are isolated vertices and paths labelled by either the triple \( \bar{m}, \bar{\pi}, (V_z(\bar{m}))_{z \in V} \) or the triple \( m \setminus m, \pi, (V_z(m))_{z \in V} \) with vertices of each component decided by \((p_z)_{z \in V}\) and the ordering on pairs. The path structure is unchanged by the mapping, so tuples that differ in their first three entries differ in their image. If two tuples differ only in their last two entries then there are two possibilities. The first possibility is that the difference results in one or more vertex-pair attachments being different between the two tuples, this results in different image tuples (the mapping maintains all vertex-pairs attachments). The second possibility is that the difference does not cause any vertex-pair attachments to be different between tuples. In this case the difference is in the choice or order of the isolated vertices for one or more \( z \in V \), if the choice differs then there is also a difference in the image tuples (these vertices are not changed by the map), if the orders differ then they will also differ in the image (the map does not change the order of these vertices relative to each other).

The map is also a surjection onto tuples whose first entry \( \bar{m} \) is a subset of edges in \( S(\emptyset) \) and whose third entry \( \pi \) is a pairing in \( P_{A,B}(m \setminus \bar{m}) \). Indeed, suppose we have desired image tuple \((\bar{m}, \bar{\pi}, \pi, (V_z(\bar{m}))_{z \in V}, (p_z)_{z \in V})\) with \( \bar{m} \in S(\emptyset) \) and \( \pi \in P_{A,B}(m \setminus \bar{m}) \). Let \( m^B, \bar{\pi}^B \), and \( V_z(m^B) \subset V_z(m^B) \), \( z \in V \), be the edges, pairs and vertices of the walks with end points at vertices associated to \( B \). Let \( \pi^B \) and \( \pi' \) be the pairings coming from the restriction of \( \pi \) to \( m^B \) and \((m \setminus \bar{m}) \setminus m^B \).
respectively. Let $\tilde{\pi} \cup \pi^B$ be the pairing of $\tilde{m} \cup m^B$ that pairs edges of $\tilde{m}$ according to $\tilde{\pi}$ and edges of $m^B$ according to $\pi^B$. Finally, let $(p_i^j)_{i \in V}$ be the collection of orderings of vertices such that the tuple $(\tilde{m} \cup m^B, \tilde{\pi} \cup \pi^B, \pi', (V_i(\tilde{m}) \cup V_i(m^B)))_{i \in V}$, $(p_i^j)_{i \in V}$ has the same vertex-pair attachments and relative order of isolated vertices as the desired image tuple when constructing a simple graph from it as described above. The image of this tuple is the desired tuple.

Putting this together we have that

$$
\sum_{m \in S(A)} F(m + \tilde{m}) \mu_N(m) \mu_N(\tilde{m})
= \sum_{m \in S(A \cup B)} F(m) \prod_{e \in E} \prod_{i=1}^N (J_i)^{m_i^e} \prod_{z \in V} \frac{\Gamma\left(\frac{N}{2}\right)^2}{2^{n_z(m)}((n_z(m) + N - 2)!)^2} \mathbb{I}_{\mathcal{F}_m}(m)
\times \sum_{m \in S(\emptyset) \cap \mathbb{N} \leq m} \prod_{m \in S(\emptyset)} \prod_{e \in E} \prod_{i=1}^N \left(m_i^e\right) \prod_{z \in V} \left(n_z(m) + N - 2\right)! \left|P_{A,B}(m - \tilde{m})\right| \left|P_G(\tilde{m})\right| \prod_{z \in V} \left(n_z(m) + N - 2\right)!
$$

Now by tracing the first calculation backwards we obtain Lemma 3.1 for $N$ even.

### 3.2 Proof of Lemma 3.1 for $N \geq 3$ odd

The proof of $N$ odd differs slightly from that of $N$ even due to the presence of a half-integer in the gamma functions in the definition of $U(3)^N$. Other than this difference, and hence the need for a slightly different geometric interpretation of the sum over $\tilde{m} \in S(B)$ to the one for $N$ even, the proofs are very similar. We include the details here for completeness.

For $N \geq 3$ odd, a simple calculation shows that

$$
\mathcal{U}^{(N)}(r) = 2^{N-1} \frac{\Gamma\left(\frac{N}{2}\right)}{(2r + N - 2)!! \sqrt{\pi}}
$$

where $(2k - 1)!! = (2k - 1)(2k - 3) \cdots 3 = (2k)!/2^k k!$ is the double factorial. Hence, for $k, r \in \mathbb{N}$ with $r \leq k$ another simple calculation gives

$$
\mathcal{U}^{(N)}(k - r) \mathcal{U}^{(N)}(r) = \frac{2^{k+2N-2} \Gamma\left(\frac{N}{2}\right)^2}{\pi(2k + 2N - 2)!} \left(\frac{2k + 2N - 2}{2r + N - 1}\right) \left(\frac{r + N - 1}{2}\right)! \left(k - r + \frac{N - 1}{2}\right)!
$$

Using this identity a similar calculation to (3.2) gives

$$
\sum_{m \in S(A \cup B)} F(m + \tilde{m}) \mu_N(m) \mu_N(\tilde{m}) = \sum_{m \in S(A \cup B)} F(m) \prod_{e \in E} \prod_{i=1}^N (J_i)^{m_i^e} \prod_{z \in V} \frac{2^{n_z(m) + 2N - 2} \Gamma\left(\frac{N}{2}\right)^2}{\pi((2n_z(m) + 2N - 2)!)^2} \mathbb{I}_{\mathcal{F}_m}(m)
\times \sum_{m \in S(B) \cap \mathbb{N} \leq m} \prod_{m \in S(\emptyset)} \prod_{e \in E} \prod_{i=1}^N \left(m_i^e\right) \prod_{z \in V} \left(2n_z(m) + 2N - 2\right)! \left(2n_z(\tilde{m}) + N - 1\right)
\left(n_z(m) + N - 1\right)! \left(n_z(m - \tilde{m}) + N - 1\right)! \left(2n_z(\tilde{m}) + 2N - 2\right)! \left|P_G(m - \tilde{m})\right| \left|P_G(\tilde{m})\right|.
$$

The set $\{m \in S(B) : m \leq m\}$ is non-empty (and therefore the corresponding sum non-zero) if and only if $m \in \mathcal{F}_B$ when $\eta = \text{free}$, or $m \in \mathcal{F}_B^+$ when $\eta =$. Indeed, for the case $\eta = \text{free}$ there can only be an odd number of $i$-links incident to every vertex of $B$ for $m \leq m$ if vertices of $B$ can be paired off in some way within each connected component of $m$. This corresponds to having a walk between
these pairs. For \( \eta = + \) the vertices of \( B \) do not have to be paired off with each other as long as they can be connected to the boundary.

Consider a subgraph \( \bar{m} \in S(B) \) of the multigraph defined by \( m \in S(A \cup B) \) and pairings \( \bar{\pi} \) of \( m \) and \( \pi \) of \( m \). Now replace each vertex \( z \in V \) with the 2\( n_z(m) \) + 2N - 2 vertices \( \{z_1, \ldots, z_{2n_z(m) + 2N - 2}\} \). At each \( z \in V \), select a subset, \( V_z(\bar{m}) \), of these vertices of size 2\( n_z(\bar{m}) \) + N - 1. Now select an ordering, \( q_z \), of these 2\( n_z(m) \) + 2N - 2 vertices and then select a labelling, \( \bar{p}_z \), of the first (according to \( q_z \)) \( n_z(m) + \frac{N - 1}{2} \) in this set with labels in \( \{1, \ldots, n_z(\bar{m}) + \frac{N - 1}{2}\} \). Further select a labelling, \( p_z \), of the first (according to \( q_z \)) \( n_z(m) - n_z(\bar{m}) + \frac{N - 2}{2} \) vertices in the complementary set \( V_z(\bar{m})^c \) with labels in \( \{1, \ldots, n_z(\bar{m}) - n_z(\bar{m}) + \frac{N - 1}{2}\} \). The sum over \( \bar{m} \in S(B) \) above is then the number of choices of such an ordering, a subgraph \( \bar{m} \), a pairing of \( \bar{m} \) and \( m \setminus \bar{m} \), a set of size 2\( n_z(\bar{m}) \) + N - 1 at each \( z \in B \) which we will denote by \( V_z(\bar{m}) \), a labelling \( \bar{p}_z \) of the first half of \( V_z(\bar{m}) \), and a labelling \( p_z \) of the first half of \( V_z(\bar{m})^c \). Encode these choices in a tuple \( (\bar{m}, \bar{\pi}, \pi, (V_z(\bar{m})))_{z \in V}, (q_z)_{z \in V}, (\bar{p}_z)_{z \in V}, (p_z)_{z \in V} \).

Note that we have interpreted the term \( (n_z(\bar{m}) + \frac{N - 1}{2})! \) as the number of ways to label the first half of \( V_z(\bar{m}) \), but we could also replace this with a labelling of any other deterministically chosen subset of \( V_z(\bar{m}) \) that has size \( n_z(\bar{m}) + \frac{N - 1}{2} \) and have a valid interpretation of the term \( (n_z(\bar{m}) + \frac{N - 1}{2})! \). This will be helpful below.

We will define a bijection from the set of tuples \( (m, \bar{\pi}, \pi, (V_z(\bar{m})))_{z \in V}, (q_z)_{z \in V}, (\bar{p}_z)_{z \in V}, (p_z)_{z \in V} \) with \( \bar{m} \in S(B) \) into the set of tuples \( (m, \bar{\pi}, \pi, (V_z(\bar{m})))_{z \in V}, (q_z)_{z \in V}, (\bar{p}_z)_{z \in V}, (p_z)_{z \in V} \) with \( m \in S(\emptyset) \) by extending the labellings \( \bar{p}_z \) and \( p_z \) to all of \( V_z(\bar{m}) \) and \( V_z(\bar{m})^c \), respectively, and then using the ordering on pairs defined above. First we describe how we extend the labellings.

Given a labelling of the first half (according to \( q_z \)) of size 2\( n_z(\bar{m}) \) + N - 1 we extend the labelling to all vertices by repetition as follows. If the vertex in \( V_z(\bar{m}) \) with label \( k, k \leq n_z(\bar{m}) + \frac{N - 1}{2} \), is in position \( \ell \) of the ordering \( q_z \) then the \((\ell + n_z(\bar{m}) + \frac{N - 1}{2})^{th} \) vertex of \( V_z(\bar{m}) \) is given label \( k + n_z(\bar{m}) + \frac{N - 1}{2} \). We extend the labelling of the first \( n_z(\bar{m}) - n_z(\bar{m}) + \frac{N - 2}{2} \) vertices of \( V_z(\bar{m})^c \) to all of \( V_z(\bar{m})^c \) in the same way. Now, for a tuple \( (m, \bar{\pi}, \pi, (V_z(\bar{m})))_{z \in V}, (q_z)_{z \in V}, (\bar{p}_z)_{z \in V}, (p_z)_{z \in V} \) with \( m \in S(B) \) we consider the paths defined by \( \bar{m} \) and \( \bar{\pi} \) and attach the pair of \( \bar{\pi}_z \) with label \( j \) to the \( j^{th} \) and \((j + n_z(\bar{m}) + \frac{N - 1}{2})^{th} \) vertices of \( V_z(\bar{m}) \) according to \( q_z \). Similarly, we attach the pair of \( \pi_z \) with label \( j \) to the \( j^{th} \) and \((j + n_z(\bar{m}) - n_z(\bar{m}) + \frac{N - 1}{2})^{th} \) vertices of \( V_z(\bar{m})^c \) according to \( q_z \). In the sequel, when referring to \( \bar{p}_z \) or \( p_z \), we will mean the extended labelling.

This results in a “graph” (edges having two vertices at each end point mean this is not really a graph, but for convenience we will refer to it as such) with \( m_e \) edges for each \( e \in E \) and 2\( n_z(m) \) + 2N - 2 vertices for each \( z \in V \) such that each connected component of the graph is either an isolated vertex (2N - 2 isolated vertices for every \( z \in V \)) or a “path” defined by \( \bar{m}, \bar{\pi}, (V_z(\bar{m}))_{z \in V} \) and \( (\bar{p}_z)_{z \in V} \) or by \( m \setminus \bar{m}, \pi, (V_z(\bar{m}))_{z \in V} \) and \( (p_z)_{z \in V} \) (and both tuples are with respect to a fixed “reference” order, \( q_z \) for each \( z \in V \)) such that each paired pair of edges has two vertices “attached” to it.

Informally speaking, the mapping we define below works almost identically to the mapping described in Remark 3.3. The difference with the case of \( N \) even is that for \( N \) even we had an ordering of all vertices to track the vertex-pair connections and maintain them after the mapping. For \( N \) odd we also maintain vertex-pair connections but also have the labellings \( q_z \) and \( q_z \). There are also extra vertices that will be moved by the mapping by referring to the extended labelling of vertices.

Consider the connected components of this graph corresponding to the walks with end points on vertices associated to \( B \). Let \( m^B \subset m \) be the corresponding set of edges and let \( \bar{\pi}^B \) respectively \( \pi^B \) be the pairing of \( m^B \) respectively \( m \setminus m^B \) coming from the restriction of the partition defined by \( \bar{\pi} \) to \( m^B \) respectively \( m \setminus m^B \). Let \( V_z(m^B) \subset V_z(m) \) be the set of vertices that are attached to pairs of \( \bar{\pi}^B \). Finally we consider the labellings \( \bar{p}_z \) and \( p_z \). Let \( \bar{p}_z^{loop} \) be the labelling on \( V_z(m) \setminus V_z(m^B) \) induced by \( \bar{p}_z \) (i.e. \( \bar{p}_z^{loop} \) maintains the same relative order of labels on \( V_z(m) \setminus V_z(m^B) \)). Let \( p_z \cup p_z^{loop} \) be the labelling on \( V_z(m)^c \cup V_z(m^B) \) induced by the labelling of \( V_z(m)^c \) coming from \( p_z \) and the labelling of \( V_z(m^B) \) coming from \( p_z^{loop} \). In other words, it is the labelling obtained by taking the union
of $V_z(m)^c$ labelled according to $p_z$ and $V_z(m^b)$ labelled according to $\bar{p}_z$, maintaining the same pair-
vertex connections of these vertices within $\{1, \ldots, 2n_z(m) + 2N + 2\}$ with respect to the reference order $q_z$ if the labelling were of $V_z(m)^c \cup V_z(m^b)$ and connections were made according to the same procedure as above. We note that $|V_z(m^b)| = 2n_z(m^b)$. Also note that identifying $\bar{m}^b$ requires knowledge of $\bar{\pi}$ and that the pair $(\bar{m} \setminus \bar{m}^b, \bar{\pi}^{\text{loop}})$ can be interpreted as a collection of coloured loops. We define our injection by the mapping

$$
(\bar{m}, \bar{\pi}, (V_z(\bar{m}))_{z \in V}, (q_z)_{z \in V}, (\bar{p}_z)_{z \in V}, (p_z)_{z \in V})
\mapsto (m \setminus m^b, \bar{\pi}^{\text{loop}}, \pi \sqcup \bar{\pi}^B, (V_z(m) \setminus V_z(m^b))_{z \in V}, (\bar{q}_z)_{z \in V}, (\bar{p}_z^\text{loop})_{z \in V}, (p_z \sqcup \bar{p}_z^B)_{z \in V}),
$$

where $\bar{q}_z$ is the ordering of $\{1, \ldots, 2n_z(m) + 2N + 2\}$ such that when constructing a “graph” from the image tuple as described above (ignoring the selected sets $V_z(\bar{m}) \setminus V_z(\bar{m}^b)$ for now, as for $N$ even, all of the vertex-pair connections and relative order of isolated vertices and attached vertices are the same as for the original tuple and the two attached vertices for each pair differ by $n_z(m) + \frac{N-1}{2}$ respectively $n_z(m) - n_z(m) + \frac{N-1}{2}$ places in the order $\bar{q}_z$ restricted to $V_z(m) \setminus V_z(m^b)$ respectively $V_z(m^b) \cup V_z(m)^c$. Described informally, this order is the one obtained from $q_z$ by moving the first attached vertex of each pair into the first half of their respective subset and the second attach vertex into the second half, while maintaining the relative order of vertices in each half with respect to each other.

The image of this map consists of a tuple of a collection of orderings $(\bar{q}_z)_{z \in V}$, an element $\bar{m} = m \setminus m^b \in S(\emptyset)$, a pairing of $\bar{m}$, a pairing of $m \setminus m$, a collection of sets $(V_z(\bar{m}))_{z \in V}$ such that $|V_z(\bar{m})| = 2n_z(m) + N - 1$, a labelling of the first (according to $\bar{q}_z$) $n_z(m) + \frac{N-1}{2}$ vertices of $V_z(\bar{m})$ for each $z \in V$ and a labelling of the first (according to $\bar{q}_z$)$n_z(m) - n_z(m) + \frac{N-1}{2}$ vertices of $V_z(\bar{m})^c$ for each $z \in V$. This can again by interpreted as a graph whose connected components are isolated vertices and paths defined by the triple $\bar{m}$, $\bar{\pi}^{\text{loop}}$ and $(V_z(\bar{m}))_{z \in V}$ or the triple $m \setminus m$, $\bar{\pi} \sqcup \bar{\pi}^B$, $(V_z(m)^c)_{z \in V}$ where pairs are attached to vertices according to a fixed ordering and fixed labellings. Described somewhat informally, the mapping takes the graph obtained from the tuple and moves the walks with end points in $B$ from the first triple of paths to the second.

As for the case of $N$ even, this map is an injection. Indeed, the graph structure is unchanged and the ordering and labelling on pairs determines the graph. If distinct tuples have the same graph structure they must have different orderings and differ in one or both labellings. Because the map also maintains the relative order of labels there are three cases to consider. The first case is that the labels differ at a vertex of the paths ending on $B$ and a vertex of a path that does not end on $B$, both associated to the same $z \in V$. In this case the vertices are in different positions relative to each other in the ordering and this difference is maintained by the new ordering. The result is distinct tuples in the image. The second and third case is that the labels differ on two vertices of paths ending on $B$ or two vertices of paths that do not end on $B$, respectively. In this case the new labellings will differ at these vertices and hence the image tuples are distinct.

The map is also a surjection onto tuples whose first entry $\bar{m}$ is a subset of edges in $S(\emptyset)$ and whose third entry $\pi$ is a pairing in $P_{A,B}(m \setminus \bar{m})$, as can be shown analogously to the case of $N$ even. We define the analogous mapping that moves the walks ending on $B$ to the triple $(\bar{m}, \bar{\pi}, (V_z(\bar{m}))_{z \in V})$ (where now $\bar{m} \in S(\emptyset)$ ) from the complimentary triple, this is the inverse map.
Putting this together we have that

\[
\sum_{m \in S(A)} F(m + \bar{m}) \mu_N(m) \mu_N(\bar{m}) = \sum_{m \in \tilde{S}(A \cup B)} F(m) \prod_{e \in E} \prod_{i=1}^{N} \left( \frac{(J_i)^{m_e}}{m_e!} \prod_{x \in V} (2n_z(m) + 2N - 2)! \right)^{\nu_n(m) + 2N - 2} ||f_{\bar{B}}(m) |
\times \sum_{m \in \tilde{S}(B)} N \prod_{e \in E} \prod_{i=1}^{N} \left( \frac{m_i^{e}}{m_i^{e}!} \prod_{x \in V} (2n_z(\bar{m}) + 2N - 2)! \right)^{\nu_n(\bar{m}) + 2N - 2} ||f_{\bar{B}}(\bar{m}) |
\]

\[
(n_z(m) + \frac{N-1}{2})!(n_z(m - \bar{m}) + \frac{N-1}{2})!(2n_z(m) + 2N - 2)! ||P_{A,B}(m - \bar{m}) || ||P_G(\bar{m}) ||
\]

(3.8)

4 Extension to the case of external magnetic field

In this section we introduce the $O(N)$-spin model with external field and the corresponding RPM. The setting is as in Section 1.1 except in addition we introduce an inhomogeneous external field $h = (h_x^1, \ldots, h_x^N)_{x \in V}$ with $h_x^i \geq 0$ for each $x \in V, i \in [N]$. For $\eta \in \{\text{free}, +\}$ and $\varphi \in \Omega_{G,N}$ we define

\[
H^\eta_{G,N,J,h}(\varphi) = H^\eta_{G,N,J}(\varphi) - \sum_{x \in V} \sum_{i=1}^{N} h_x^i \varphi_x^i.
\]

(4.1)

This corresponds to having an external magnetic field on $V$ whose magnitude and direction on $x \in V$ are given by $|h_x| = (h_x^1, \ldots, h_x^N)$ and $\langle h_x^1, \ldots, h_x^N | (\varphi_x^1, \ldots, \varphi_x^N) \rangle$, respectively. Note that the magnetic field term looks similar to the boundary term for $\eta = +$. Indeed, we can think of this term as a boundary term coming from edges connected to ghost vertices. We use this interpretation when introducing the modification of the random path model. This means that the extension to this case can naturally be arrived at from the case with boundary. Nevertheless we make the effort to present the necessary changes. The changes required to prove Theorems 1.1 and 4.1 for a general boundary condition (other than $+$) are virtually identical to the changes required for an external magnetic field.

Expectations are defined analogously to Section 1.1. For $f : \Omega_{G,N} \rightarrow \mathbb{R}$

\[
\langle f \rangle^\eta_{G,N,J,h} = \frac{1}{Z^\eta_{G,N,J,h}} \int_{\Omega_{G,N}} d\varphi f(\varphi) e^{-H^\eta_{G,N,J,h}(\varphi)}
\]

(4.2)

where $Z^\eta_{G,N,J,h}$ is a normalising constant. We also have the analogous theorem and corollary to the case of no external field. For $x \in V$, define $S_x : \Omega_{G,N} \rightarrow S^{N-1}$ by $S_x(\varphi) = \varphi_x$ and its components $S_x = (S^1_x, \ldots, S^N_x)$.

Theorem 4.1. Let $G = (V, E)$ be a finite simple graph and $N \in \mathbb{N}_{\geq 1}$. Assume that $J_i^e \geq 0$ for every $e \in E, i \in [N]$ and $h_x^i \geq 0$ for every $x \in V, i \in [N]$. For $A \subset V$ and $\eta \in \{\text{free}, +\}$ we have

\[
\left( \prod_{x \in A} S^1_x \prod_{y \in B} S^1_y \right)^\eta_{G,N,J,h} \geq \left( \prod_{x \in A} S^1_x \right)^\eta_{G,N,J,h} \left( \prod_{y \in B} S^1_y \right)^\eta_{G,N,J,h}.
\]

Corollary 4.2. Let $G = (V, E)$ be a finite simple graph and $N \in \mathbb{N}_{\geq 1}$. Assume that $J_i^e \geq 0$ for every $e \in E, i \in [N]$ and $h_x^i \geq 0$ for every $x \in V, i \in [N]$. Let $\eta \in \{\text{free}, +\}$. For $A \subset V$ and $e \in E$ we have

\[
\frac{\partial}{\partial J_i^e} \left( \prod_{x \in A} S^1_x \right)^\eta_{G,N,J,h} \geq 0.
\]

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For the proof we will stick to the case \( A \cap B = \emptyset \), the modifications for \( A \cap B \neq \emptyset \) are completely analogous to the modifications for the case of no external field.

In order to prove this result, we use a slight modification of the random path model. As in [16] we do this by introducing ghost vertices. We introduce \( N \) ghost vertices, one for each colour. The case where \( h^i_x = 0 \) for \( i \in \{1, \ldots, N-1\} \) for every \( x \in V \) corresponds to having one ghost vertex as considered in [16], the extension to multiple ghost vertices is straightforward. Let \( g_1, \ldots, g_N \) be vertices not in \( G \) (or in the case with boundary, not in \( G \)). Further define edge set \( \tilde{E}_g = \{ \{x, g_1\}, \ldots, \{x, g_N\} : x \in V \} \).

We now define a modified graph \( G_g = (V_g, E_g) \) where \( V_g = V \cup \{g_1, \ldots, g_N\} \) and \( E_g = E \cup \tilde{E}_g \). \( G_g \) consists of the original graph \( G \) together with \( N \) ghost vertices and an edge from each vertex of \( V \) to each ghost vertex.

Now consider the random path model on \( G_g \) (\( G_g \cup \partial E \) for the case with boundary). \( M^N_{G_g}(1) \subset M^N_{G,g} = \mathbb{N}^{E_g \times [N]} \) \( (M^N_{G,g,\partial E}(1)) \) consists of tuples \( m = (m^1, \ldots, m^N) \) of \( N \) collections of links on \( E_g \) (\( E_g \cup \partial E \)) such that \( m^i_x \in 2\mathbb{N} \) for every \( i \in \{2, \ldots, N\} \) and \( x \in V \) and \( m^i_{\{x,g_j\}} = 0 \) whenever \( i \neq j \) for every \( x \in V \) (i.e. no \( i \)-links incident to the \( j \)th ghost vertex for \( i \neq j \)). Note that we place no restriction on the parity of \( m^i_x \). For \( m \in M^N_{G_g}(1) \) \( (m \in M^N_{G,g,\partial E}(1)) \) let \( P_{G_g} \) \( (P_{G_g,\partial E}) \) be the set of pairings of links incident to each \( z \in V \). Links incident to \( g_j \) for \( j \in [N] \) (and \( z \in \partial \text{ext} V \)) are not paired. Let \( P_{G_g}(m) \) \( (P_{G_g,\partial E}(m)) \) be the set of pairings such that at most one link of each colour is unpaired at each \( z \in V \). Recall that when considering \( (\prod_{i \in A} S^1_i \prod_{y \in B} s^y_{g_i})_{g_i \in G,N,J} \) this set of pairings is relevant when \( A \cap B = \emptyset \), when \( A \cap B \neq \emptyset \) we consider pairings where one 1-link is unpaired at each \( x \in A \Delta B \) and two 1-links are unpaired at each \( x \in A \cap B \).

We see that a tuple \((m, \pi)\) with \( m \in M^N_{G_g}(1) \) \( (m \in M^N_{G,g,\partial E}(1)) \) and \( \pi \in P_{G_g}(m) \) \( (P_{G_g,\partial E}(m)) \) can be interpreted as a collection of loops with colours in \([N]\), 1-paths with end points on \( V \), and \( i \)-paths with both end points on \( g_i \) for \( i \in [N] \) (in addition these tuples may also have paths leaving \( G_g \) for the case \( G_g \cup \partial E \)).

We now define our measure on these new configurations. For \( \eta = \text{free} \) the measure of \( m \in M^N_{G_g}(1) \) is
\begin{equation}
\mu^\eta_{G_N,N,J,h,U}(m) = \prod_{e \in E} \prod_{i=1}^N \frac{(j^i_e)^{m^i_e}}{m^i_e!} \prod_{z \in V} \prod_{i=1}^N \frac{(h^i_z)^{m_{\{x,g_i\}}}}{m_{\{x,g_i\}}!} \prod_{z \in V} U_z(m) |P_{G_g}(m)| \prod_{e \in \partial E} \prod_{i=1}^N \mathbb{I}_{\{m^i_e = 0\}}. \tag{4.3}
\end{equation}

For \( \eta = + \) the measure of \( m \in M^N_{G,g,\partial E}(1) \) is
\begin{equation}
\mu^\eta_{G,N,J,h,U}(m) = \prod_{e \in E, \partial E} \prod_{i=1}^N \frac{(j^i_e)^{m^i_e}}{m^i_e!} \prod_{z \in V} \prod_{i=1}^N \frac{(h^i_z)^{m_{\{x,g_i\}}}}{m_{\{x,g_i\}}!} \prod_{z \in V} U_z(m) |P_{G_g}(m)| \prod_{e \in \partial E} \prod_{i=1}^N \mathbb{I}_{\{m^i_e = 0\}}. \tag{4.4}
\end{equation}

For \( A \subset V \) we define \( S(A) \) to be the set of \( m \in M^N_{G_g}(1) \) \( (m \in M^N_{G,g,\partial E}(1)) \) such that, for \( z \in V \), \( m^i_z \in 2\mathbb{N} + 1 \) if and only if \( z \in A \). For \( \eta \in \{\text{free}, +\} \) define \( Z^\eta_{G,N,J,h,U}(A) = \mu^\eta_{G,N,J,h,U}(S(A)) \) and
\begin{equation}
Z^\eta_{G,N,J,h,U} = \mu^\eta_{G,N,J,h,U}(S(\emptyset)). \tag{4.5}
\end{equation}

Of course, in the case with ghost vertices configurations in \( S(\emptyset) \) contain not only coloured loops, but also coloured walks ending at the matching ghost vertex (and paths leaving the boundary) but we still use the superscript \( \text{loop} \) to be consistent with the notation above. Correlations are given by
\begin{equation}
G^\eta_{G,N,J,h,U}(A) := \frac{Z^\eta_{G,N,J,h,U}(A)}{Z^\eta_{G,N,J,h,U}}. \tag{4.5}
\end{equation}

These correlations are equal to the correlations presented in [16] for the first description of the RPM in Section 2.1. The proof that the description above leads to the same correlation functions is identical to the proof of Proposition 2.7. The proof of the following proposition can be found in [16] for the case of homogeneous couplings and external field. As explained in [16, Section 5.2], the extension to the inhomogeneous case is straightforward.

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Proposition 4.3. Let \( N \in \mathbb{N}_{>0} \), \( J \) such that \( J^e_i \geq 0 \) for all \( e \in E \) and \( i \in [N] \), \( h \) such that \( h^x_i \geq 0 \) for all \( x \in V \) and \( i \in [N] \), and vertex weight functions \( U^{(N)}_x \) as in Proposition 2.3. For any \( A \subset V \) and \( \eta \in \{ \text{free, +} \} \) we have that

\[
\mathcal{G}^\eta_{G,N,J,h,U^{(N)}}(A) = \left\langle \prod_{x \in A} S^1_x \right\rangle_{G,N,J,h}^{\eta}.
\]

Now that we have this proposition the proof of Theorem 4.1 can proceed as in the case of no external field. Indeed, we can think of \( \tilde{E}_g \) as being boundary edges of \( G \subset G = G_g \).

We can define the sum and difference of configurations, and the partial ordering as in Section 3, we can also interpret configurations as collections of multigraphs and define \( F_A \subset \mathcal{M}^N_{G_g}(1) (F_A^+ \subset \mathcal{M}^N_{G_g \cup \partial E}(1)) \) as in Section 3 with the difference that connected components must contain an even number of vertices from \( A \) or be connected to \( g_1 \) (or \( \partial E \)). Indeed, the ghost vertices act as boundary vertices, as mentioned above, so 1-walks may leave \( G \) by going to \( g_1 \). We also define \( P_{A,B}(m) \) to be pairings such that no 1-walk has an end point on \( A \) and an end point on \( B \).

Theorem 4.1 is a consequence of the following lemma, in the same way that Theorem 1.1 follows from Lemma 3.1.

Lemma 4.4. Let \( A,B \subset V \) with \( A \cap B = \emptyset \) and let vertex weight functions \( U^{(N)}_x \) be fixed as in Proposition 2.3 and \( \eta = \text{free} \). For any \( N \in \mathbb{N}_{>1} \), non-negative couple constants \( J = (J^1_e, \ldots, J^N_e)_{e \in E} \), non-negative external field \( h = (h^1_x, \ldots, h^N_x)_{x \in V} \), and \( F : \mathcal{M}^N_G \to \mathbb{R}_{>0} \) we have

\[
\sum_{m \in S(A)} F(m + \bar{m}) \mu_N(m) \mu_N(\bar{m}) = \sum_{m \in S(A \cup B)} F(m + \bar{m}) \mu_N(m) \mu_N(\bar{m}) \|_{F_B}(m + \bar{m}) \frac{P_{A,B}(m)}{P_G(m)}.
\]

This holds for \( \eta = + \) if we replace \( F_B \) on the right side by \( F_B^+ \).

The proof of this lemma is identical to the proof of Lemma 3.1 and gives us Theorem 4.1.

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