Noise Induced Phenomena in Population Dynamics

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Abstract. Noise through its interaction with the nonlinearity of the living systems can give rise to counter-intuitive phenomena. In this paper we shortly review the noise induced effects in different ecosystems. The transient dynamics of these ecosystems are analyzed through generalized Lotka-Volterra equations in the presence of multiplicative noise, which models the interaction between the species and the environment. We find noise induced phenomena such as quasi-deterministic oscillations, stochastic resonance, noise delayed extinction, and noise-induced pattern formation. The spatial pattern of two competing species, obtained with the coupled map lattice, is compared with real data. Finally, the experimental dynamical behavior of two competing bacterial populations in a meat product and the probability distribution at long times of one of them are well reproduced by a stochastic microbial predictive model.

Key words: noise induced phenomena, population dynamics, Langevin equation, multiplicative noise, stochastic resonance, predictive microbiology
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1. Introduction

During the last years the role of noise in population dynamics has been the subject of several theoretical studies [1]-[24]. Natural systems are a typical example of open systems due to the continuous presence of deterministic and stochastic forces coming from the environment, which affect the dynamics of these systems.

The study of the effects of noise is a well established subject in several different disciplines ranging from physics, to chemistry and biology. However, the essential role of the noise in theoretical ecology has been recently recognized. Some key questions in population ecology are related to the comprehension of the role that noise, climatic forcing and nonlinear interactions among individuals of the same or different species play on the dynamics of the ecosystems [21]-[23]. More recently noise induced effects on population dynamics have been investigated [2], [17]-[20], [24].

In particular several works have studied the effects of random fluctuations on the stability of ecological systems [21,22], showing the presence of counterintuitive phenomena, such as noise enhanced stability [25]-[29], stochastic resonance [30], [31]-[34], and noise delayed extinction [17]-[20], [35],[36]. The origin of these effects is the interplay between nonlinear interactions typical of natural systems and random fluctuations coming from environment due to their intrinsic characteristic of open systems. The contemporary presence of noise and nonlinear interaction in population dynamics causes indeed an increase of complexity respect to other noise-driven systems, such as financial markets [37]-[40], or many physical and chemical processes described by deterministic dynamics [41].

Therefore the intrinsic nonlinearity can cause ecological systems to critically depend on initial conditions, and both deterministic and random perturbations coming from the environment. As a consequence, the understanding of the role played by the noise in the dynamics of nonlinear systems is a crucial point for a deeper comprehension and successive modeling of the open systems governed by nonlinear dynamics, which are known as complex systems.

During the last years the deterministic mathematical models were able to describe ecosystems showing the presence of several phenomena, such as chaotic dynamics and spatial patterns. However, due to their deterministic nature these models can not nor reproduce neither explain the effects of random fluctuations, which come from the intrinsic stochastic nature of open systems. Recently researchers devoted more interest and attention to explain the role of noise in different fields of biology [1], [7], [10]-[13], [34], [41]-[50].

Moreover we recall that the study of spatial distributions of species densities is a major element to get a correct description in population dynamics. An accurate analysis of spatio-temporal patterns represents in fact a crucial point to devise predictive models. As a consequence the comprehension of the role played by the simultaneous presence of random fluctuations, deterministic forces and nonlinear interaction, typical of natural systems, is fundamental to effectively describe spatio-temporal dynamics of biological populations [17]-[20], [24], [51]-[54].

In this work we review some recent findings in biological context based on a generalization of Lotka-Volterra equations in which external random fluctuations are modeled by inserting terms of multiplicative noise [10]-[12]. As a result we find that: (a) in a single compartment ecosystem with two species the presence of a driving force causes stochastic resonance, which results in quasi-
periodic oscillations of population densities; b) in the same ecosystem it is possible to observe a nonmonotonic behavior of the average extinction time of one species as a function of the noise intensity, i.e. noise delayed extinction; (c) a two-dimensional spatio-temporal model is able to reproduce fish concentrations in a real marine ecosystem located in the Mediterranean Sea; (d) in a food product the presence of randomly fluctuating environmental variables such as temperature, pH, and available water is taken into account in modeling bacterial growth, allowing to obtain a better agreement between experimental data and theoretical results respect to the corresponding deterministic approach.

2. Stochastic dynamics of two competing species: single compartment model

In this section we study the effect of random fluctuations in the dynamics of two competing species. The model consists of generalized Lotka-Volterra equations with terms of multiplicative noise, which mimics the random fluctuations of environmental variables. The two populations, $x$ and $y$, interact through the term $-\beta xy$, where $\beta$ is the coupling constant, which regulates the interaction strength and is a stochastic process. More in detail the interaction coefficient $\beta(t)$ can be represented as a virtual particle moving along a bistable potential and subject to both a periodic driving term, which models seasonal changes in temperature, and a term of additive noise which describes the effects due to the noisy behaviour of the environment.

2.1. The model

The stochastic dynamics of our ecosystem is given by the two following generalized Lotka-Volterra equations \[ \frac{dx}{dt} = \mu_1 x (\alpha_1 - x - \beta_1(t)y) + x \xi_x(t) \] \[ \frac{dy}{dt} = \mu_2 y (\alpha_2 - y - \beta_2(t)x) + y \xi_y(t), \] which are stochastic differential equations in Ito sense, with $\xi_x(t)$ and $\xi_y(t)$ statistically independent Gaussian white noises with zero mean and correlation function $\langle \xi_i(t)\xi_j(t') \rangle = \sigma \delta(t - t') \delta_{ij}$ ($i, j = x, y$), and $\sigma$ the multiplicative noise intensity. To get the time evolution for the two species densities the parameters are set as follows: $\alpha_1 = \alpha_2 = \alpha$, $\beta_1(t) = \beta_2(t) = \beta(t)$. We recall that $\beta < 1$ determines the coexistence regime (both species survives), while $\beta > 1$ corresponds to the exclusion regime (one of the two species disappears after some time). Coexistence and exclusion of one of the two species represent indeed stable states of the Lotka-Volterra’s deterministic model [57]. As previously said, natural systems are affected by a continuous exchange with a noisy nonstationary environment. This implies that also the interaction parameter is subject to both random fluctuations and deterministic external signal such as the periodical changes in temperature. The competition rate $\beta(t)$, continuously varying between exclusion and coexistence regime...
due the interplay between two main factors such as limiting resources and noisy environment, is responsible for a random competition between the two populations. As a consequence, the simultaneous presence of noise and periodic driving causes the system to pass from a dynamical regime ($\beta < 1$, coexistence) to the other one ($\beta > 1$, exclusion) and vice versa. This random process can be described by an Ito stochastic differential equation, which reproduces, as previously noted, the dynamics of a virtual particle moving along a bistable potential, in the presence of a periodical driving force and an additive noise term

$$\frac{d\beta(t)}{dt} = -\frac{dU(\beta)}{d\beta} + \gamma \cos(\omega_0 t) + \xi(\beta)(t),$$

(2.3)

where $U(\beta)$ is the bistable potential shown in Fig. 1 and given by

$$U(\beta) = h(\beta - (1 + \rho))^4/\eta^4 - 2h(\beta - (1 + \rho))^2/\eta^2,$$

(2.4)

Here $h$ is the height of the potential barrier. The periodic term account for seasonal variations of environmental temperature, with $\gamma = 10^{-1}$ and $\omega_0/(2\pi) = 10^{-3}$. In Eq. (2.3) $\xi(\beta)(t)$ is a Gaussian white noise with the usual statistical properties $\langle \xi(\beta)(t) \rangle = 0$ and $\langle \xi(\beta)(t)\xi(\beta)(t') \rangle = \sigma_\beta \delta(t - t')$, with $\sigma_\beta$ the additive noise intensity. According to the form of the potential, one can expect that in the deterministic case ($\sigma_\beta = 0$) the coexistence regime takes place when the virtual particle (initial value of $\beta$) is placed in the left well.
2.2. Stochastic resonance

As a first step we study how the noise affects the dynamics of the two populations, whose dynamical regime is strongly dependent on the interaction parameter. Therefore we analyze the time evolution of $\beta(t)$ fixing different values of the additive noise intensity. Setting $\sigma_\beta = 0$

Figure 2: Time evolution of the interaction parameter for different values of the additive noise intensity. (a) $\sigma_\beta = 0$; (b) $\sigma_\beta = 1.78 \cdot 10^{-4}$; (c) $\sigma_\beta = 1.78 \cdot 10^{-3}$; (d) $\sigma_\beta = 1.78 \cdot 10^{-2}$. The values of the parameters are $\gamma = 10^{-1}$, $\omega_0/(2\pi) = 10^{-3}$.

(deterministic regime), $\beta(t)$ undergoes a periodical behavior with the system remaining in the coexistence regime (see Fig. 2a). Increasing the noise ($\sigma_\beta \ll h$) the periodical behavior appears slightly perturbed due to the presence of random fluctuations (Fig. 2c). As the noise intensity increases ($\sigma_\beta \simeq h$), the virtual particle, i.e. the value of the interaction parameter, jumps between $\beta = 0.94$ and $\beta = 1.04$, which correspond to the coexistence ($\beta < 1$) and exclusion regime ($\beta > 1$), respectively. In Fig. 2b the typical picture of stochastic resonance is shown. However, for higher values of the noise intensity $\sigma_\beta$, a loss of coherence is observed and the system dynamics is mainly driven by the source of random fluctuations (see Fig. 2d). This behavior can be interpreted as a cooperation between the periodical driving of the temperature, due to some geological cause, and the environmental noise [58]-[60] for intermediate values of the noise intensity. In this case the noise results to be tuned with the deterministic oscillating external perturbation, causing the well known stochastic resonance phenomenon. The noise intensity, $\sigma_\beta = 1.78 \cdot 10^{-3}$, which causes the synchronization shown in Fig. 2c, can be easily obtained by the formula [61, 62] setting

$$\tau_k = T_0/2,$$

(2.5)
Figure 3: Time evolution of both populations at different levels of the multiplicative noise: (a) $\sigma = 0$; (b) $\sigma = 10^{-11}$; (c) $\sigma = 10^{-10}$; (d) $\sigma = 10^{-9}$; (e) $\sigma = 10^{-4}$; (f) $\sigma = 10^{-1}$. The values of the parameters are $\mu = 1$, $\alpha = 1$, $\gamma = 10^{-1}$, $\omega_0/2\pi = 10^{-3}$. The intensity of the additive noise is fixed at the value $\sigma_\beta = 1.78 \cdot 10^{-3}$. The initial values of the two species are $x(0) = y(0) = 1$.

where $\tau_k$ is the Kramers time \[63\]

\[
\tau_k = \frac{2\pi}{\sqrt{|U''(0.99)|U''(0.94)}} \exp\left[2h/\sigma_\beta\right], \tag{2.6}
\]

and $T_0$ is the period of the driving force. In Eq. (2.6) $U''(0.99)$ and $U''(0.94)$ are the second derivative calculated in the unstable and stable states of the potential, respectively. Setting $\sigma_\beta = 1.78 \cdot 10^{-3}$, which gives an alternated regime with quasi-periodical jumps between coexistence.
and exclusion, it is possible to get the ecosystem dynamics when the stochastic resonance (SR) condition is present, varying the magnitude $\sigma$ of the multiplicative noise sources which act directly on the two species. Using as initial conditions $x(0) = y(0) = 1$, we note that, after a short transient, both populations take on the same stationary value $x_{st} = y_{st} = \alpha/(1 + \beta) \approx 1/2$, around which the population densities perform quasi-periodic oscillations whose amplitudes depend on the magnitude of the multiplicative noise.

We observe that in the absence of multiplicative noise, i.e. $\sigma = 0$ (see Fig. 3a), and for low noise intensity, i.e. $\sigma = 10^{-12}$ (see Fig. 3b), a dynamical coexistence regime characterized by correlated oscillations of the two species densities is established. We observe that this behaviour is connected with the symmetry of the Lotka-Volterra equations due to the choice of the parameter values and initial conditions (the same for the two populations). As a consequence the species undergoes correlated oscillations around the stationary value $\alpha/(1 + \beta)$ (see panels a, b of Fig. 3), even if the ecosystem, driven by the quasi-periodic behaviour of the interaction parameter (SR effect), is in the exclusion regime during the 50% of the time. The symmetry condition is broken as the multiplicative noise intensity increases, causing the appearance of anti-correlated oscillations (see panels c, d, e of Fig. 3). The amplitude of these anti-correlated oscillations increases for higher intensity of the multiplicative noise ($\sigma = 10^{-2}$), causing a degradation of the quasi-periodical behaviour in the time series of the populations (see Fig. 3f): now the ecosystem is mainly driven by the noise which tends to suppress the effect of the periodical signal. The presence of the multiplicative noise indeed breaks the symmetric dynamical behaviour of the ecosystem. For $\beta > 1$, i. e. in the exclusion regime, this symmetry breaking determines different behaviour in the time series of the two species: one tends to survive, while the other one tends to extinguish. Fig. 3 indicates that the effect of quasi-periodic signal can be amplified by higher intensities of the multiplicative noise. The periodicity of the noise-induced oscillations in the time behavior of the population densities shown in Fig. 3e is the same of the driving periodic term of Eq. (2.3). This is the signature

![Figure 4: Log-Log plot of SNR as a function of the multiplicative noise intensity. The SNR corresponds to the squared difference of population densities $(x - y)^2$. The values of the parameters are the same of Fig. 3](image)
of stochastic resonance phenomenon. This second SR phenomenon can be quantitatively investigated calculating the signal-to-noise ratio (SNR) of the squared difference of population densities $(x - y)^2$. This quantity is shown in Fig. 4 as a function of the multiplicative noise intensity $\sigma$, for $\sigma_\beta = 1.78 \cdot 10^{-3}$. We note that a maximum is found for $\sigma = 10^{-4}$. This indicates that the additive noise determines the conditions for the different dynamical regimes of the two species, while the multiplicative noise is responsible for a coherent response of the system, breaking the initial condition of symmetry of the ecosystem.

2.3. Noise delayed extinction

In this paragraph we study how one of the two populations can vanish due to the interplay between the periodical signal and the additive noise $\xi(t)$, which drive the ecosystem through the two different regimes, i.e. coexistence and exclusion.

For this purpose we introduce the average extinction time of one species and calculate it for different values of the noise intensity $\sigma_\beta$, setting the multiplicative noise intensity at a small value, so that the dynamics is weakly perturbed by the noise and the ecosystem remains far from the SR regime. Fixing $\sigma = 10^{-9}$ and $\beta(0) = 0.94$ as initial condition, we integrate Eqs. (1) and (2) by performing 200 numerical realizations, and obtaining the behaviour of the mean extinction time (MET) of one species as a function of the additive noise intensity $\sigma_\beta$ (see Fig. 5a).

In these conditions the ecosystem is in the coexistence regime, that is the deterministic extinction time of both species is infinite. By introducing noise, exclusion takes place and a finite mean extinction time (MET) appears.

We note that for small and large values of the additive noise intensity (see regions 1 and 3 in Fig. 5a), the system is in a dynamical regime which favours the coexistence of the two species, so that no extinction occurs (see time series shown in panels b and d of Fig. 5). Conversely, for intermediate values of $\sigma_\beta$, exclusion of one species is found with different extinction times. In particular, a minimum MET is observed for $\sigma_\beta = 2.75 \cdot 10^{-3}$, which is of the same order of magnitude of the potential barrier height $h$. The Kramers time corresponding to this noise intensity is $\tau_k = 41.6$, a value approximately equal to $\tau_{min}$. This behaviour reproduces the effect of random modifications of the environmental conditions responsible for the delayed extinction of biological species in real ecosystems, as it is observed in experimental data of populations in a very long time interval [64, 65]. Starting from a situation in which one of the two population undergoes rapidly extinction (region 2 in Fig. 5a), varying the noise intensity (decrease or increase of $\sigma_\beta$) the ecosystem moves towards regions where the MET becomes larger (see the region between 1 and 2 and that between 2 and 3 in Fig. 5a). From a physical point of view this effect can be explained observing that for small values of $\sigma_\beta$, the initial condition ($\beta(0) = 0.94$) is maintained for low noise intensity (the Kramers time is very long and the virtual particle takes very long times to cross the barrier and reach the right well corresponding to the exclusion regime). On the other side, for higher noise intensities there is a strong decrease of the Kramers time and the values of $\beta(t)$ switch very rapidly between the two wells, determining an alternate regime coexistence/exclusion which avoids the extinction. Conversely, for intermediate values of the additive noise intensity (see region 2 in Fig. 5a), after an initial permanence in the left well $\beta(t)$ reaches the right well, where it remains
Figure 5: (a) Mean extinction time of one species as a function of the noise intensity $\sigma_{\beta}$. Time evolution of both species for different levels of additive noise: (b) $\sigma_{\beta} = 10^{-4}$, (c) $\sigma_{\beta} = 2 \cdot 10^{-3}$, (d) $\sigma_{\beta} = 10^{-1}$. The values of the parameters are $\mu = 1$, $\alpha = 45$, $\gamma = 10^{-1}$, $\omega_0/2\pi = 10^{-3}$. The intensity of the multiplicative noise is fixed at the value $\sigma = 10^{-9}$. The initial values of the two species are $x(0) = y(0) = 1$.

for a time long enough to cause the extinction of one of the two species (see time series shown in panel c of Fig. 5).

3. Stochastic dynamics of two competing species: spatially extended model

In this section we consider two populations, distributed in a two-dimensional spatial domain, subject to a source of multiplicative noise, in the presence of an external periodic signal. The multiplicative noise source mimics the effects of random fluctuations of the environmental variables.
3.1. The model

The spatio-temporal dynamics of the two populations is given by a discrete time evolution model, based on the coupled map lattice (CML) approach [66], and is the discrete version of the Lotka-Volterra equations, where diffusive terms were added [67].

\[
x_{i,j}^{n+1} = \mu x_{i,j}^n (1 - x_{i,j}^n - \beta^n y_{i,j}^n) + \sqrt{\sigma_x} x_{i,j}^n X_{i,j}^n + D \sum_\gamma (x_{i,j}^n - x_{\gamma i,j}^n)
\]

\[
y_{i,j}^{n+1} = \mu y_{i,j}^n (1 - y_{i,j}^n - \beta^n x_{i,j}^n) + \sqrt{\sigma_y} y_{i,j}^n Y_{i,j}^n + D \sum_\gamma (y_{i,j}^n - y_{\gamma i,j}^n),
\]

with \(x_{i,j}^n\) and \(y_{i,j}^n\) densities of the two populations in the site \((i,j)\) at the time step \(n\). Here \(\beta^n\) is the interaction parameter at the same time step, \(\mu\) is the growth rate, \(D\) is the diffusion constant and \(\sum_\gamma\) represents the sum over the four nearest neighbors. \(X_{i,j}^n\) and \(Y_{i,j}^n\) are independent Gaussian random variables with zero mean and variance unit. Moreover, \(\sigma_x\) and \(\sigma_y\) are the intensities of the two multiplicative noise sources. The interaction parameter \(\beta^n\) is the same stochastic process \(\beta(t)\) as in previous section (see Eq. (2.3)), where \(U(\beta)\) is the bistable potential shown in Fig. 1. Therefore, also in this spatially extended model the switching of \(\beta(t)\) between the exclusion and coexistence regime occurs randomly. This mimics the effect of the noisy environment on limiting factors such as food resources, with the periodical force accounting for the periodical (seasonal) oscillations of temperature. According to the analysis performed for the single compartment system, the two stable states (left and right wells of the potential \(U(\beta)\)) correspond to coexistence and exclusion of one of the two species of the Lotka-Volterra deterministic model [17]-[20], [24, 36].

3.2. Results

By numerical integration of Eqs. (3.1)-(2.3) we obtain the spatio-temporal evolution of the population densities for the two theoretical species. The results are given in Fig. 6. Here the spatial distribution of the species \(x\) and \(y\) are shown at a certain time in the left and right panel, respectively. Light and dark grey zones represent low and high densities, respectively. These theoretical spatial distribution are compared with real data, for anchovies and sardines, collected along the acoustic transects tracked by the Interdisciplinary Group of Oceanography of IAMC-CNR of Mazara del Vallo [68], during the oceanographic campaign "ANCHEVA '02" in the Strait of Sicily. In particular, in Fig. 6 (left panel) the anchovy abundance, estimated experimentally along the acoustic transects, from point 42 to point 50 is shown. Analogously in Fig. 6 (right panel) the sardine abundance, estimated experimentally along the same acoustic transects, is shown. White and black tracts indicate small and large values, respectively, of anchovy and sardine abundances.
Figure 6: Comparison between spatial distribution of species $x$ and anchovy abundance (left panel) and species $y$ and sardine abundance (right panel). The values for $x$ and $y$ were obtained from the model at time step $n = 600$. The anchovy and sardine abundances were estimated experimentally along the acoustic transect, from point 42 to point 50, tracked during the oceanographic campaign "ANCHEVA '02" by the Interdisciplinary Group of Oceanography of IAMC-CNR of Mazara del Vallo. The spatial distribution of $x$ and $y$ are drawn by light and dark grey zones which represent, respectively, low and high density of the two species. The values of the parameters are $\mu = 2$, $\nu = \omega_0/(2\pi) = 0.34$, $\gamma = 10^{-5}$, $\sigma_x = \sigma_y = 10^{-8}$, $\sigma_\beta = 10^{-12}$, $D = 0.05$, $\beta(0) = 0.95$. The initial values for the spatial distributions of $x$ and $y$ are $x_{i,j}^{\text{init}} = y_{i,j}^{\text{init}} = 0.5$ for all sites $(i,j)$. Concerning the experimental distributions, white and black tracts indicate, respectively, small and large values of anchovy (left panel) and sardine (right panel) abundances estimated experimentally during the oceanographic campaign [68].

4. Predictive microbiology

Predictive microbiology exploits mathematical models to describe bacterial dynamics in different products of food industry. The models take into account the role played by environmental variables, whose variations can affect, sometimes dramatically, the quality and safety of the food products. Predictive models belong to three different types: primary, secondary and tertiary [69]. The first class of models allows to obtain the time evolution of microbial populations. The models belonging to the second type give information on the relationship between parameters which appear in primary models, and physical and chemical variables such as $T$ (temperature), $pH$ (hydrogen ion concentration), and $aw$ (activity water). The third class of models puts together the primary and secondary ones, letting the evolution of physical and chemical variables be considered, when analysis and prediction of the concentration of spoiling or pathogen bacteria of the food are performed [70].

A well-known method for the theoretical analysis of microbial growth exploits generalized Lotka-Volterra (LV) equations [55, 56], which allow to describe the dynamics of two competing
bacterial populations in different food products. A prototype model structure for mixed microbial populations in food products was proposed by Dens et al. [71]. A similar approach indicated that experimental data for Escherichia coli O157:H7 in ground beef could be well reproduced by an interspecific competition model for two bacterial populations. In the same work the effects of random fluctuations were considered using growth rates whose values are obtained from uniform random distributions [72]. An extensive review on predictive microbiology showed that in general a stochastic approach provides predictions which exclude the worst-case scenario [73]. In particular, stochastic terms were introduced to reproduce and predict bacterial dynamics, exploiting an approach based on primary and secondary growth models [74]. Moreover other authors presented a stochastic model which interprets the bacterial growth as the average evolution of many cells: measured values of the growth rate for many different cells allow to describe the theoretical growth rate used in the model as a stochastic variable with a corresponding probability distribution [75]-[77]. Finally, a stochastic ecological model, based on the Verhulst logistic differential equation, was devised [78].

The previous models however do not include explicitly stochastic terms in the equations of motion of the systems analyzed. In other words, the models used in predictive microbiology are not usually based on stochastic differential equations.

In the following we analyze how predictions for bacterial dynamics are affected by the three following features: (i) use of differential equations (dynamical approach); (ii) presence of interactions among bacterial populations; (iii) introduction of stochastic terms, i.e. noise sources, which mimic the random fluctuations of environmental variables.

4.1. Bacterial growth in meat products: single compartment dynamics of two interacting populations

In this section we introduce a model for the dynamics of two competing bacterial populations, Listeria monocytogenes and lactic acid bacteria (LAB), present in a meat product, i.e. a traditional Sicilian salami (Salame S. Angelo PGI (Protected Geographical Indication)) very important from the point of view of the Italian food industry. Specifically, L. monocytogenes is a microbial agent of foodborne disease, while LAB constitute the normal bacterial flora of the substrate. The theoretical approach is based on generalized Lotka-Volterra (LV) equations [71, 72], in which the bacterial growth rates depend on environmental variables, such as temperature, pH, and activity water, whose randomly fluctuating behaviour can be modeled by inserting terms of additive white Gaussian noise:

\[ \frac{dN_{Lmo}}{dt} = \mu_{Lmo}^\text{max} N_{Lmo} Q_{Lmo} \left( 1 + \frac{Q_{Lmo}}{N_{Lmo}^\text{max}} \left( 1 - \frac{N_{Lmo} + \beta_{Lmo/LAB} N_{LAB}}{N_{Lmo}^\text{max}} \right) \right) \] \hspace{1cm} (4.1)

\[ \frac{dQ_{Lmo}}{dt} = \mu_{Lmo}^\text{max} Q_{Lmo} \] \hspace{1cm} (4.2)

\[ \frac{dN_{LAB}}{dt} = \mu_{LAB}^\text{max} N_{LAB} Q_{LAB} \left( 1 + \frac{Q_{LAB}}{N_{LAB}^\text{max}} \left( 1 - \frac{N_{LAB} + \beta_{LAB/Lmo} N_{Lmo}}{N_{LAB}^\text{max}} \right) \right) \] \hspace{1cm} (4.3)

\[ \frac{dQ_{LAB}}{dt} = \mu_{LAB}^\text{max} Q_{LAB}. \] \hspace{1cm} (4.4)
Here, \( N_{Lmo} \) and \( N_{LAB} \) are the population concentrations of \( L.\ monocytogenes \) and LAB, respectively; \( \mu_{Lmo} \) and \( \mu_{LAB} \) represent the maximum specific growth rates of the two populations, and \( N^{max}_{Lmo} \) and \( N^{max}_{LAB} \) are the theoretical maximum population concentrations. The coefficients \( \beta_{Lmo/LAB} \) and \( \beta_{LAB/Lmo} \) are the interspecific competition parameters of LAB on \( L.\ monocytogenes \) and vice-versa. \( Q_{Lmo} \) and \( Q_{LAB} \) represent the physiological state of the two populations.

To solve Eqs. (4.1)–(4.4) it is necessary to set how \( \mu_{max}^{Lmo} \) and \( \mu_{max}^{LAB} \) vary. This can be done by introducing for the maximum growth rates the following secondary model

\[
\begin{align*}
\mu_{max}^{Lmo} &= 0.14776 (T_0 - 0.88) \cdot (1 - \exp(0.536 (T - 41.4))) \cdot \sqrt{aw - 0.923} \\
&\quad \cdot \sqrt{1 - 10^{4.97-pH}} \cdot \sqrt{1 - \frac{LAC}{3.79 (1 + 10^{pH-3.86})} 10^{4.97-pH}} \cdot \frac{350 - NIT}{350} \\
\mu_{max}^{LAB} &= 0.00234 (aw - 0.928) \cdot (pH - 4.24) \cdot (pH - 9.53) \cdot (T - 3.63).
\end{align*}
\]

obtained by a phenomenological approach (see Ref. [79] and references therein). Here, NIT is nitrite concentration in \( ppm \) and LAC is lactic acid concentration in \( gl^{-1} \). The values 0.88, 41.4, 0.923, 4.97, and 350 represent \( T_{min} \ (\degree C) \), \( T_{max} \ (\degree C) \), \( aw_{min} \), \( pH_{min} \), and \( NIT_{max} \), respectively. Temperature, pH, and activity water are described as stochastic processes. In particular, their dynamics is given by two different contributions: (i) a linearly decreasing deterministic behaviour within a time interval of 168 h, according to the procedure followed in the production process (a fermentation period of 7 days); (ii) terms of additive white Gaussian noise, which account for the presence of random fluctuations due to environmental perturbations. By this way the following system of three stochastic differential equations is obtained [79]

\[
\begin{align*}
\frac{dT(t)}{dt} &= k_T t + \xi_T(t) \\
\frac{dpH(t)}{dt} &= k_{pH} t + \xi_{pH}(t) \\
\frac{daw(t)}{dt} &= k_{aw} t + \xi_{aw}(t),
\end{align*}
\]

where \( \xi_i(t) \), with \( i = T, pH, aw \), are statistically independent Gaussian white noises with the following properties

\[
\begin{align*}
< \xi_i(t) > &= 0 \\
< \xi_i(t) \xi_i(t') > &= \sigma_i \delta(t - t'),
\end{align*}
\]

and \( \sigma_i \) are the noise intensities. Eqs. (4.1)–(4.9) have been solved numerically within the Ito scheme, performing 1000 realizations and obtaining the mean growth curves in absence of noise \((\sigma_T = 0, \sigma_{pH} = 0, \sigma_{aw} = 0)\). The initial concentrations of the two populations, however, have been set randomly. Specifically, in each realization the initial values of \( N_{Lmo} \) and \( N_{LAB} \) have been extracted from two Gaussian distributions, whose mean values and standard deviations were equal to those of the distributions experimentally observed [80]. The results, obtained for suitable values of the interaction parameters \((\beta_{Lmo/LAB} = 0.656, \beta_{LAB/Lmo} = 0)\), are shown in Fig. 7.
Here we note that the theoretical curves of \textit{L. monocytogenes} (dashed black line) and LAB (full black line) fit very well the corresponding experimental data (black squares for \textit{L. monocytogenes}, black circles for LAB). This indicates that the interaction, present in the model, between the two bacterial populations reproduces a feature of the real biological system [67, 81, 82]. In particular, we note that the condition $\beta_{LAB/Lmo} = 0$ implies the absence of any direct effects of \textit{L. monocytogenes} on the dynamics of LAB. On the other side, the limiting effect of LAB on the growth of \textit{L. monocytogenes}, obtained for a suitable positive value of the other interaction parameter ($\beta_{Lmo/LAB} = 0.665$), determines conditions for the coexistence of the two populations, according to empirical data [80, 83, 84]. To analyze the role of the random fluctuations on the dynamics of the system, we solve Eqs. (4.1)–(4.9) both in deterministic regime and for three different values of the noise intensities $\sigma_T$, $\sigma_{pH}$ and $\sigma_{aw}$. We obtain the theoretical probability distributions of \textit{L. monocytogenes} concentration at the end of the fermentation period (168 hours). Predicted results, together with observed data, are shown in Fig. 8. Here, the histograms indicate that the best agreement between the theoretical distribution (white bars) and experimental one (black bars) is observed when the bacterial dynamics is obtained for values of the noise intensities different from zero (stochastic dynamics), and in particular for $\sigma_T = 2 \cdot 10^{-1}$, $\sigma_{pH} = 10^{-4}$, $\sigma_{aw} = 1.5 \cdot 10^{-4}$ (panel c). This result accords with the complex nature of the system analyzed, in which random fluctuations of environmental variables such as temperature, pH and activity water, are present.
Figure 8: Theoretical distributions (white bars) of the *L. monocytogenes* concentration at 168 hours for (a) $\sigma_T = 0$, $\sigma_{pH} = 0$, $\sigma_{aw} = 0$, (b) $\sigma_T = 10^{-2}$, $\sigma_{pH} = 5 \cdot 10^{-4}$, $\sigma_{aw} = 10^{-5}$, (c) $\sigma_T = 2 \cdot 10^{-1}$, $\sigma_{pH} = 10^{-4}$, $\sigma_{aw} = 1.5 \cdot 10^{-4}$, (d) $\sigma_T = 5 \cdot 10^{-1}$, $\sigma_{pH} = 5 \cdot 10^{-4}$, $\sigma_{aw} = 10^{-5}$. Black bars represent the corresponding experimental distribution.

5. Conclusions

In this paper we reviewed some recent results on the effects of noise in the dynamics of two competing populations, whose interaction depends on the parameter $\beta$.

First we studied the dynamics of an ecosystem affected by two sources of random fluctuations: a multiplicative noise and an additive noise. The latter governs the dynamics of the interaction parameter, which is therefore a stochastic process. Due to the presence of an oscillating bistable potential, the interaction parameter can change its values switching between the two wells of the bistable potential, which correspond to different conditions for the ecosystem, i.e. coexistence of both species (left side well) and exclusion of one of them (right side well). The additive noise induces a coherent time behavior and oscillating time series of the two species densities. An enhancement of the response of the system through stochastic resonance phenomenon is also observed as a function of the multiplicative noise intensity. More in detail the switching between the coexistence and the exclusion regimes is governed by the additive noise, while the multiplicative noise is responsible for coherent oscillations of the two species. The stochastic resonance observed in the dynamics of $\beta$ induces a similar effect in the two populations. The model reproduces the dynamics of ecosystems subject to both deterministic oscillating changes and random modifications of environmental variables, such as variations of temperature: the interplay between the deterministic and
random signals and the nonlinearity can determine a coherent response of the ecosystem. Finally, additive noise causes also a delayed extinction of one of the two populations: a nonmonotonic behaviour of the mean extinction time, with a minimum, is found as a function of the additive noise intensity. These noise induced phenomena can contribute to understand population dynamics in real ecosystems, which are complex systems due to their intrinsic nonlinearity and continuous exchange with the environment through deterministic and random perturbations. [17]-[22], [64, 65].

Afterwards we presented an application of population dynamics to areal ecosystem. Specifically we analyzed the spatio-temporal dynamics of two competing species. For this purpose we used a discrete time evolution model, based on the coupled map lattice (CML), obtaining at different times the distributions of the two population densities in a two-dimensional spatial domain. Results obtained from the CML model were compared with those recorded for the spatial distributions of two pelagic fish populations, i.e. anchovies and sardines. In the marine areas analyzed we observed strong correlations between theoretical and experimental distributions for both populations. Slight discrepancies appearing in few zones can be considered as an effect of different spatial coarse grained scale and different resolution between spatial distribution data.

Finally we discussed a predictive microbiological model which allows to describe microbial evolution in food products as a function of environmental conditions. Our findings indicate that interspecific bacterial interaction and environmental random fluctuations are essential for a more precise and reliable prediction of the bacterial dynamics. In particular we note that the time evolution of the microbial concentration (*L. monocytogenes*) in a meat product shows a slow increase during the first part (lag-time) of the dynamics; afterwards a faster increase (log-phase) of the bacterial concentration takes place until the curve reaches a saturation value (stationary phase), which corresponds to the maximum bacterial concentration measured. However, in our experiment the microbial growth is affected by adverse environmental conditions and strong LAB interaction. As a consequence, the log phase shows an increase slower than in usual bacterial dynamics. These results could play a key role in view of incorporating stochastic microbial predictive models into a risk assessment process, contributing to improve the precision of the expected concentrations of a foodborne disease agent.

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**References**

[1] E. M. Hart, L. Avilés. *Reconstructing Local Population Dynamics in Noisy Metapopulations - The Role of Random Catastrophes and Allee Effects*. PLoS One, 9 (2014), e110049.
[2] A. La Cognata, D. Valenti, A. A. Dubkov, B. Spagnolo. Dynamics of two competing species in the presence of Lévy noise sources. Phys. Rev. E, 82 (2010), 011121.

[3] L. Ruokolainen, A. Lindén, V. Kaitala, M. S. Fowler. Ecological and evolutionary dynamics under coloured environmental variation. Trends Ecol. Evol., 24 (2009), 555–563.

[4] B. Spagnolo, S. Spezia, L. Curcio, N. Pizzolato, A. Fiasconaro, D. Valenti, P. Lo Bue, E. Peri, S. Colazza. Noise Effects in two different Biological Systems. Eur. Phys. J. B, 69 (2009), 133–146.

[5] L. Ruokolainen, E. Ranta, V. Kaitala, M. S. Fowler. Community stability under different correlation structures of species environmental responses. J. Theor. Biol., 261 (2009), 379–387.

[6] A. A. Dubkov, B. Spagnolo. Verhulst model with Lévy white noise excitation. Eur. Phys. J. B, 65 (2008), 361–367.

[7] M. Hoffmann, H. H. Chang, S. Huang, D. E. Ingber, M. Loeffler, J. Galle. Noise-Driven Stem Cell and Progenitor Population Dynamics. PLoS One, 3 (2008), e2922.

[8] D. A. Vasseur, J. W. Fox. Environmental fluctuations can stabilize food web dynamics by increasing synchrony. Ecol. Lett., 10 (2007), 1066–1074.

[9] M. Scheffer, S. Carpenter, J. A. Foley, C. Folke, B. Walker. Catastrophic shifts in ecosystems. Nature, 413 (2001), 591–596.

[10] S. Ciuchi, F. de Pasquale, B. Spagnolo. Self Regulation Mechanism of an Ecosystem in a Non-Gaussian Fluctuation Regime. Phys. Rev. E, 53 (1996), 706–716.

[11] S. Ciuchi, F. de Pasquale, B. Spagnolo. Nonlinear Relaxation in the presence of an Absorbing Barrier. Phys. Rev. E, 47 (1993), 3915–3926.

[12] P. Barrera, S. Ciuchi, B. Spagnolo. Generating Function for a Multiplicative Noise with Group Analysis. J. Phys. A: Math. Gen., 26 (1993), L559–L565.

[13] J. M. G. Vilar, R. V. Solé. Effects of Noise in Symmetric Two-Species Competition. Phys. Rev. Lett., 80 (1998), 4099–4102.

[14] K. Staliunas. Spatial and Temporal Noise Spectra of Spatially Extended Systems with Order-Disorder Phase Transitions. Int. J. Bifurcat. Chaos, 11 (2001), 2845–2852.

[15] A. F. Rozenfeld, E. Albano. Study of a lattice-gas model for a prey-predator system. Physica A, 266 (1999), 322–329.

[16] A. F. Rozenfeld, C. J. Tessone, E. Albano, H.S. Wio. On the influence of noise on the critical and oscillatory behavior of a predator-prey model: coherent stochastic resonance at the proper frequency of the system. Phys. Lett. A, 280 (2001), 45–52.
[17] B. Spagnolo, M. Cirone, A. La Barbera, F. de Pasquale. *Noise Induced Effects in Population Dynamics*. J Phys-Condens. Mat., 14 (2002), 2247–2255.

[18] A. La Barbera, B. Spagnolo. *Spatio-Temporal Patterns in Population Dynamics*. Physica A, 314 (2002), 693–697.

[19] B. Spagnolo, A. La Barbera. *Role of the noise on the transient Dynamics of an ecosystem of interacting species*. Physica A, 315 (2002), 201–211.

[20] M. A. Cirone, F. de Pasquale, B. Spagnolo. *Nonlinear Relaxation in Population Dynamics*. Fractals, 11 (2003), 217-226.

[21] C. Zimmer. *Life after chaos*. Science, 284 (1999), 83-86. See also the special section on "Complex Systems", Science 284 (1999), 79–107.

[22] O. N. Bjornstad, B. T. Grenfell. *Noisy clockwork: time series analysis of population fluctuations in animals*. Science, 293 (2001), 638–643.

[23] M. Droz, A. Pekalski. *Dynamics of populations in a changing environment*. Phys. Rev. E, 65 (2002), 051911.

[24] B. Spagnolo, D. Valenti, A. Fiasconaro. *Noise in Ecosystems: A Short Review*. Math. Biosci. Eng., 1 (2004), 185–211.

[25] N. Agudov, B. Spagnolo. *Noise enhanced stability of periodically driven metastable states*. Phys. Rev. E, 64 (2001), 035102(R).

[26] R. N. Mantegna, B. Spagnolo. *Noise Enhanced Stability in an Unstable System*. Phys. Rev. Lett., 76 (1996), 563–566.

[27] A. A. Dubkov, N. V. Agudov, B. Spagnolo. *Noise enhanced stability in fluctuating metastable states*. Phys. Rev. E, 69 (2004), 061103.

[28] B. Spagnolo, A. A. Dubkov, N. V. Agudov. *Noise Enhanced Stability*. Acta Phys. Pol. B, 35 (2004), 1419–1436.

[29] B. Spagnolo, A. A. Dubkov, A. L. Pankratov, E. V. Pankratova, A. Fiasconaro, A. Ochab-Marcinek. *Lifetime of metastable states and suppression of noise in Interdisciplinary Physical Models*. Acta Phys. Pol. B, 38 (2007), 1925–1950.

[30] L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni. *Stochastic resonance*. Rev. Mod. Phys., 70 (1998), 223–287.

[31] R.N. Mantegna, B. Spagnolo. *Stochastic Resonance in a Tunnel Diode*. Phys. Rev. E, 49 (1994), R1792–R1795.

[32] R.N. Mantegna, B. Spagnolo, M. Trapanese. *Linear and Nonlinear Experimental Regimes of Stochastic Resonance*. Phys. Rev. E, 63 (2001), 011101.
[33] E. Lanzara, R.N. Mantegna, B. Spagnolo, R. Zangara. Experimental Study of a Nonlinear System in the Presence of Noise: The Stochastic Resonance. Am. J. Phys, 65 (1997), 341–349.

[34] A. Fiasconaro, A. Ochab-Marcinek, B. Spagnolo, E. Gudowska-Nowak. Monitoring noise-resonant effects in cancer growth influenced by spontaneous fluctuations and periodic treatment. Eur. Phys. J. B, 65 (2008), 435–442.

[35] B. Spagnolo, A. Fiasconaro, D. Valenti. Noise Induced Phenomena in Lotka-Volterra Systems. Fluct. Noise Lett., 3 (2003), L177–L185.

[36] D. Valenti, A. Fiasconaro, B. Spagnolo. Stochastic resonance and noise delayed extinction in a model of two competing species. Physica A, 331 (2004), 477–486.

[37] G. Bonanno, D. Valenti, B. Spagnolo. Role of Noise in a Market Model with Stochastic Volatility. Eur. Phys. J. B, 53 (2006), 405–409.

[38] G. Bonanno, D. Valenti, B. Spagnolo. Mean Escape Time in a System with Stochastic Volatility. Phys. Rev. E, 75 (2007), 016106.

[39] D. Valenti, B. Spagnolo, G. Bonanno. Hitting Time Distributions in Financial Markets. Physica A, 382 (2007), 311–320.

[40] B. Spagnolo, D. Valenti. Volatility Effects on the Escape Time in Financial Market Models. Int. J. Bifurcat. Chaos, 18 (2008), 2775–2786.

[41] See the special section on “Ecology Through Time”, Science, 293 (2001), 623–657.

[42] N. Goldenfeld, L. P. Kadanoff. Simple lessons from complexity. Science, 284 (1999), 87–89.

[43] O. N. Bjørnstad, J. M. Fromentin, N. C. Stenseth, J. Gjøsæter. Cycles and trends in cod populations. Proc. Natl. Acad. Sci. U.S.A., 96 (1999), 5066–5071.

[44] K. Higgins, A. Hastings, J. N. Sarvela, L. W. Botsford. Stochastic Dynamics and Deterministic Skeletons: Population Behavior of Dungeness Crab. Science, 276 (1997), 1431–1435.

[45] J. A. Freund, T. Pöschel (Eds.). Stochastic Processes in Physics, Chemistry, and Biology. Lecture Notes in Physics 557, Springer, Berlin, 2000.

[46] J. H. Brown, T. G. Whitham, S. K. M. Ernest, C. A. Gehring. Complex species interactions and the dynamics of ecological systems: long-term experiments. Science, 293 (2001), 643–650.

[47] P. Turchin, L. Oksansen, P. Ekerholm, T. Oksanen, H. Henttonen. Are lemmings prey or predators? Nature, 405 (2000), 562–565.

[48] O. Chichigina, D. Valenti, B. Spagnolo. A Simple Noise Model with Memory for Biological Systems. Fluct. Noise Lett., 5 (2005), L243–L250.
[49] A. A. Dubkov, B. Spagnolo. *Acceleration of Diffusion in Randomly Switching Potential with Supersymmetry*. Phys. Rev. E, 72 (2005), 041104.

[50] D. Valenti, G. Augello, B. Spagnolo. *Dynamics of a FitzHugh-Nagumo system subjected to autocorrelated noise*. Eur. Phys. J. B, 65 (2008), 443–451.

[51] H. Zhonghuai, Y. Lingfa, X. Zuo, X. Houwen. *Noise Induced Pattern Transition and Spatio-temporal Stochastic Resonance*. Phys. Rev. Lett., 81 (1998), 2854–2857.

[52] D. F. Russel, L. A. Wilkens, F. Moss. *Use of behavioural stochastic resonance by paddle fish for feeding*. Nature, 402 (2000), 291–294.

[53] A. A. King, W. M. Schaffer. *The geometry of a population cycle: A mechanistic model of snowshoe hare demography*. Ecology, 82 (2001), 814–830.

[54] B. Blasius, A. Huppert, L. Stone. *Complex dynamics and phase synchronization in spatially extended ecological systems*. Nature, 399 (1999), 354–359.

[55] A.J. Lotka. *Analytical Note on Certain Rhythmic Relations in Organic Systems*. Proc. Nat. Acad. Sci. U.S. A., 6 (1920), 410–415.

[56] V. Volterra. *Variazioni e fluttuazioni del numero d’individui in specie animali conviventi*. Mem. Accad. Nazionale Lincei Ser. 6, 2 (1926), 31–113.

[57] A. D. Bazykin. Nonlinear Dynamics of Interacting Populations. World Sc. series on Nonlinear Science, Series A vol.11, Singapore, 1998.

[58] R. Benzi, A. Sutera, A. Vulpiani. *The mechanism of stochastic resonance*. J. Phys. A: Math Gen., 14 (1981), L453–L457.

[59] R. Benzi, G. Parisi, A. Sutera, A. Vulpiani. *Stochastic resonance in climatic change*. Tellus, 34 (1982), 10–16.

[60] R. Alley, S. Anandakrishnan, P. Jung. *Stochastic resonance in the North Atlantic*. Paleoceanography, 16 (2001), 190–198.

[61] P. Jung, P. Hänggi. *Amplification of small signals via stochastic resonance*. Phys. Rev. A, 44 (1991), 8032–8042.

[62] P. Jung, P. Hänggi. *Stochastic nonlinear dynamics modulated by external periodic forces*. Europhys. Lett., 8 (1989), 505–510.

[63] P. Hänggi, P. Talkner, M. Borkovec. *Reaction rate theory: fifty years after Kramers*. Rev. Mod. Phys., 62 (1990), 251–342.

[64] J. Garcia Lafuente, A. Garcia, S. Mazzola, L. Quintanilla, J. Delgado, A. Cuttitta, B. Patti. *Hydrographic phenomena influencing early life stages of the Sicilian Channel anchovy*. Fish. Oceanogr., 11 (2002), 31–44.
[65] A. Caruso, M. Sprovieri, A. Bonanno, R. Sprovieri. *Astronomical calibration of the Serravallian-Tortonian Case Pelacani section (Sicily, Italy).* In: Iaccarino S. (Ed.), *Integrated stratigraphy and paleoceanography of the Mediterranean Middle Miocene.* Riv. Ital. Paleont. e Strat., 108 (2002), 297–306.

[66] Special issue CML models, edited by K. Kaneko. *Chaos,* 2 (1992), 279–460.

[67] D. Valenti, A. Fiasconaro, B. Spagnolo. *Pattern formation and spatial correlation induced by the noise in two competing species.* Acta Phys. Pol. B, 35 (2004), 1481–1489.

[68] A. Bonanno. *Rapporto finale della campagna oceanografica “ANCHEVA ’02”,* Stretto di Sicilia ed Isola di Lampedusa, IAMC-CNR, Mazara del Vallo, 04-15 luglio 2002.

[69] R. C. Whiting, R. L. Buchanan. *A classification of models for predictive microbiology.* Food Microbiol., 10 (1993), 175–177.

[70] P. Dalgaard, P. Buch, S. Silberg. *Seafood Spoilage Predictor—development and distribution of a product specific application software.* Int. J. Food Microbiol., 73 (2002), 343–349.

[71] E. J. Dens, K. M. Vereecken, J. F. Van Impe. *A prototype model structure for mixed microbial populations in homogeneous food products.* J. Theor. Biol., 201 (1999), 159–170.

[72] M. Powell, W. Schlosser, E. Ebel. *Considering the complexity of microbial community dynamics in food safety risk assessment.* Int. J. Food Microbiol., 90 (2002), 171–179.

[73] T. A. McMeekin, J. Olley, D. A. Ratkowsky, T. Ross. *Predictive microbiology: towards the interface and beyond.* Int. J. Food Microbiol., 73 (2002), 395–407.

[74] M. J. Nauta. *Separation of uncertainty and variability in quantitative microbial risk assessment models.* Int. J. Food Microbiol., 57 (2000), 9–18.

[75] J. Baranyi. *Comparison of Stochastic and Deterministic Concepts of Bacterial Lag.* J. Theor. Biol., 192 (1998), 403–408.

[76] J. Baranyi, C. Pin. *A Parallel Study on Bacterial Growth and Inactivation.* J. Theor. Biol., 210 (2001), 327–336.

[77] I. A. M. Swinnen, K. Bernaerts, E. J. J. Dens, A. H. Geeraerd, J. F. Van Impe. *Predictive modelling of the microbial lag phase: a review.* Int. J. Food Microbiol., 94 (2004), 137–159.

[78] J. M. Ponciano, F. P. J. Vandecasteele, T. F. Hess, L. J. Forney, R. L. Crawford, P. Joyce. *Use of stochastic models to assess the effect of environmental factors on microbial growth.* Appl. Environ. Microbiol., 71 (2005), 2355–2364.

[79] A. Giuffrida, D. Valenti, G. Ziino, B. Spagnolo, A. Panebianco. *A stochastic interspecific competition model to predict the behaviour of Listeria monocytogenes in the fermentation process of a traditional Sicilian salami.* Eur. Food Res. Technol., 228 (2009), 767–775.
[80] A. Giuffrida, G. Ziino, F. Rago, F. Giarratana, A. Taviano, A. Panebianco, Behaviour of *Lysteria monocytogenes* during the Seasoning of a Typical Sicilian Dried Salami”. In Proceedings of INTRADFOOD 2005 (Innovation in Traditional Food), Valencia, Spain, 25-28 October 2005, pp. 537–540.

[81] D. Valenti, L. Tranchina, M. Brai, A. Caruso, C. Cosentino, B. Spagnolo, *Environmental metal pollution considered as noise: Effects on the spatial distribution of benthic foraminifera in two coastal marine areas of Sicily (Southern Italy)*. Ecol. Model., 213 (2008), 449–462.

[82] A. Fiasconaro, D. Valenti, B. Spagnolo. *Nonmonotonic Behaviour of Spatiotemporal Pattern Formation in a Noisy Lotka-Volterra System*. Acta Phys. Pol. B, 35 (2004), 1491–1500.

[83] M. Campanini, I. Pedrazzoni, S. Barbuti, P. Baldini. *Behaviour of Listeria monocytogenes during the maturation of naturally and artificially contaminated salami: effect of lactic-acid bacteria starter cultures*. Int. J. Food Microbiol., 20 (1993), 169–175.

[84] M. Trüssel, T. Jemmi. *The behaviour of Listeria monocytogenes during the ripening and storage of artificially contaminated salami and Mettwurst*. Fleischwirtschaft, 69 (1989), 1586–1592.