A comment on the dual field in the AdS-CFT correspondence

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Abstract

In the perturbative AdS-CFT correspondence, the dual field whose
source are the prescribed boundary values of a bulk field in the func-
tional integral, and the boundary limit of the quantized bulk field are
the same thing. This statement is due to the fact that Witten graphs
are boundary limits of the corresponding Feynman graphs for the bulk
fields, and hence the dual conformal correlation functions are limits of
bulk correlation functions. This manifestation of duality is analyzed
in terms of the underlying functional integrals of different structure.

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1 Introduction

The AdS-CFT correspondence [8, 19] as a concretization of Maldacena’s
conjecture [13] owes much of its fascination to the fact that it produces
conformal correlation functions for the “dual” conformal field \(\mathcal{O}\), using as
the generating functional a functional integral of highly non field theoretical
appearance, of the form

\[
\langle e^{\mathcal{O}(f)} \rangle = \frac{Z(f)}{Z(0)} \quad \text{with} \quad Z(f) = \int D\phi \, e^{-I(\phi)} \delta(\phi_0 - f),
\]

(1.1)

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where $\phi_0$ is the boundary limit of the functional variable $\phi$, and $\mathcal{O}$ is the (Euclidean) dual field. The use of prescribed values of the functional integration variable $\phi$ as the source for a quantum field, is deeply inspired from string theory [8, 19] and has no precedent in field theory. We shall refer to (1.1) as the “dual prescription”.

It was noticed soon (e.g., [1]) that the bulk-to-boundary propagators derived from the dual prescription are limits of the bulk-to-bulk propagators as one of the bulk coordinates approaches the boundary. This implies that the dual Green functions are boundary limits of bulk Green functions, and hence the dual conformal fields themselves are boundary limits, or “restrictions”, of the bulk fields in the sense of eqs. (1.7), (4.1) below.

The recognition of the dual field as a boundary limit of the bulk field perfectly complies with two results on the AdS-CFT correspondence derived in axiomatic frameworks. In the Wightman axiomatic framework is has been shown [3] that the boundary limits of AdS correlation functions inherit the properties of locality, covariance, energy positivity and Hilbert space positivity (unitarity), and fulfill the physical requirements of a local conformal QFT in Minkowski space. A similar conclusion can be drawn from the algebraic treatment in [17] where the local observables of a QFT on AdS and a corresponding conformal QFT are identified, such that the sharply localized conformal observables coincide with the AdS observables close to the boundary. (The algebraic treatment also allows to characterize and define the observables in the interior of AdS in terms of conformal observables.)

Indeed, many tests of the field theoretic properties of correlation functions computed with the dual prescription have produced perfectly sensible results (including operator product expansions, Ward identities, and positivity [7, 10, 14]).

The generating functional for the conformal boundary correlations should therefore be as well representable as a functional integral where the field is coupled in the usual field theoretic way to a source, with the specification that the source is confined to the boundary,

$$\left< e^{\phi_0(f)} \right> = \frac{\tilde{Z}(f)}{\tilde{Z}(0)} \quad \text{with} \quad \tilde{Z}(f) = \int D\phi \, e^{-I(\phi)} e^{\int \phi_0 f}. \quad (1.2)$$

On the right-hand side of this formula, $\phi_0$ stands for the boundary limit

\footnote{The boundary limit of the bulk field is a special case of the restriction of a quantum field to a time-like hypersurface, which is well defined [5] and yields (non-Lagrangian) quantum fields in one dimension less. In contrast, restrictions of quantum fields to space-like surfaces (“time zero fields”) [18] or light fronts [18] are generally too singular to exist.}
Dual fields in AdS-CFT

of the integration variable $\phi$, while on the left-hand side it denotes the
Euclidean bulk quantum field, restricted to the boundary.

It is the purpose of this letter to understand the way how the two com-
peting functional integrals (1.1) and (1.2) of drastically different appearance
can provide the same results. $\tilde{Z}(i\phi)$ is actually the functional Fourier trans-
form of $Z(\phi)$. Coincidence of the Schwinger functions generated by the two
functional integrals seems to imply that the integrals coincide and hence
must be essentially their own functional Fourier transforms, for any form of
the action $I(\phi)$. This looks like a straight absurdity.

The apparent conflict is resolved by the fact that we shall have to specify
the function spaces to which the respective functional meas ures $D\phi$ apply,
which is equivalent to the choice of the propagators $G_+, G_-$, formally giving
rise to two different functional integrals $Z^\pm(\phi)$, $\tilde{Z}^\pm(\phi)$ of either type (1.1),
(1.2), the superscript distinguishing the two measures. Then we observe
that in the dual case, the implementation of the $\delta$-function in (1.1) leads
to an effective modification of the propagator. The total dual bulk-to-bulk
propagator $\Gamma_-$ turns out to coincide with the field theoretic propagator $G_+$,

$$\Gamma_-(z, x; z', x') = G_+(z, x; z', x'). \tag{1.3}$$

Likewise, we shall analyze the consequences of the dual prescription for the
bulk-to-boundary propagator $K_-$ and for the tree level 2-point function (the
only connected graph without vertices), with the result that

$$K_-(z, x; x') = c \cdot \lim_{z' \to 0} z'\Delta_+ G_+(z, x; z', x') \tag{1.4}$$

where $\Delta_+$ is the scaling dimension of the boundary field and $c$ a numerical
coefficient, and the tree level 2-point function equals

$$c^2 \cdot \lim_{z' \to 0} z'\Delta_+ \lim_{z \to 0} z\Delta_+ G_+(z, x; z', x'). \tag{1.5}$$

The right-hand sides of eqs. (1.4), (1.5) are the appropriate limits of the field
theoretical propagator when the source is confined to the boundary. Hence
eqs. (1.3–5) imply

$$Z^-(\phi) = \tilde{Z}^+(c \cdot \phi), \tag{1.6}$$

valid graph by graph in the formal Euclidean perturbation series. This in
turn implies that the conformal field $O^-$ defined by the dual prescription
(1.1) coincides with $\phi_0^+$ defined by the restriction prescription (1.2),

$$O^-(x) = c \cdot \phi_0^+(x) \equiv c \cdot \lim_{z \to 0} z^{-\Delta_+} \phi^+(z, x). \tag{1.7}$$
We believe that these facts, which we systematically establish for the most general scalar and vector fields, are a non-trivial manifestation of duality in the AdS-CFT correspondence. They pertain to the approximation of the holographic AdS-CFT correspondence in which string effects are suppressed and gravity is treated as a tensor field in a fixed background. Indeed, the action may be any local functional involving a finite number of tensor fields. Supersymmetry or “large $N$” is not assumed.

We emphasize that these results concern the formal perturbative expansions of the Euclidean boundary field theories in question, subject to the well-known difficulties encountered in the Euclidean functional integral approach. Clearly, individual graphs require renormalization, and the entire series diverges. Moreover, the correlation functions may fail to satisfy the Osterwalder-Schrader (OS) positivity condition \[16\], which is crucial in order to qualify as Schwinger functions of an associated real-time QFT. Only OS positivity guarantees Einstein causality, Hilbert space positivity and positivity of the energy. The positivity property of the functional integral inherited from the Gaussian measure is not sufficient in this respect.

The graph-by-graph identification (1.6) is not affected by renormalization (if the same renormalization conditions are imposed) and analytic continuation. One may therefore expect that a proper renormalized real-time interpretation of (1.1) also coincides with the real-time perturbation theory for a bulk field with subsequent restriction.

2 A discrete model

We want to emphasize the basically algebraic nature of the relations among the relevant propagators, pertaining to the passage between source terms of the respective forms $e^\phi_0 f$ and $\delta(\phi_0 - f)$. For this purpose we first consider finite-dimensional Gaussian integrals, replacing anti-de Sitter space by a lattice. In the finite-dimensional case, formal manipulations with Gaussian integrals are exact. In particular, there is no room for further specifications of propagators, and the generating functionals $Z(if)$ and $\tilde{Z}(f)$ are definitely distinct. We shall see that the difference resides entirely in the propagators. Their algebraic characterizations established in this section will be exploited in the next sections for the continuum case.

We replace the real functions $\phi_\alpha(x)$ by an $N$-tuple of integration variables $\phi \equiv (\phi_i)_{i=1...N} \in \mathbb{R}^N$ where the index $i$ labels both the lattice points and the Lorentz (multi)indices $\alpha$ in the case of a tensor field. We arrange the numbering such that $i = 1...n$ label the boundary variables (boundary
values of the field), while the remaining ones label the bulk variables.

We denote by \( e : \mathbb{R}^n \to \mathbb{R}^N \) the corresponding embedding of the spaces of integration variables, and by \( e^t : \mathbb{R}^N \to \mathbb{R}^n \) its adjoint. The boundary variables are thus \( \phi_0 \equiv e^t \phi \in \mathbb{R}^n \).

The quadratic part \( \frac{1}{2}(\phi, A\phi) \) of the action is given by a symmetric matrix \( A \in \text{Mat}_N(\mathbb{R}) \). The total action is of the form

\[
I(\phi) = \frac{1}{2}(\phi, A\phi) + V(\phi)
\]

with a local polynomial potential \( V(\phi) = \sum_i v(\phi_i) \). We proceed in the usual perturbative way by expanding \( \exp -V(\phi) \) as a power series, and performing the Gaussian integrals.

The integral \( \tilde{Z}(f) \), \( f \in \mathbb{R}^n \), involving the source term \( \exp(\phi_0, f) \equiv \exp(e^t\phi, f) \) is computed as usual by completing the square and shifting the integration variable \( \phi \to \phi + A^{-1}e f \). This yields

\[
\tilde{Z}(f) = e^{-\frac{1}{2}(f, \alpha f)} \cdot \int D\phi \, e^{-\frac{1}{2}(\phi, A\phi)} \exp -V(\phi + A^{-1}e f).
\]

The Gaussian prefactor comes from \( (e f, A^{-1}e f) = (f, \alpha f) \), where \( \alpha \) is the \( n \times n \) matrix

\[
\alpha := e^t A^{-1} e \in \text{Mat}_n(\mathbb{R}).
\]

For the integral \( Z(f) \), \( f \in \mathbb{R}^n \), with the source term \( \delta(\phi_0 - f) \) we use the projections \( E = ee^t \) (“boundary”) and its complement \( E^\perp = 1_N - E \) (“bulk”) to separate the boundary variables from the bulk variables:

\[
\phi = E\phi + E^\perp \phi \equiv e\phi_0 + E^\perp \phi,
\]
and perform the obvious integration over the boundary variables \( \phi_0 \), thus \( \phi = e f + E^\perp \phi \). In order to decouple the external variables \( f \) from the integration variables \( E^\perp \phi \), we shift the latter by \( E^\perp A^{-1}(e\alpha^{-1} f) \) such that \( \phi = E^\perp \phi' + e f + E^\perp A^{-1}(e\alpha^{-1} f) \). Writing \( e f = EA^{-1}(e\alpha^{-1} f) \), we get \( \phi = E^\perp \phi' + A^{-1}(e\alpha^{-1} f) \). The quadratic term decouples as desired:

\[
(\phi, A\phi) = (f, \alpha^{-1} f) + (E^\perp \phi', A E^\perp \phi'),
\]
and the functional integral becomes (suppressing the prime)

\[
Z(f) = e^{-\frac{1}{2}(f, \alpha^{-1} f)} \times \int D(E^\perp \phi) \, e^{-\frac{1}{2}(E^\perp \phi, A E^\perp \phi)} \exp -V(E^\perp \phi + A^{-1}(e\alpha^{-1} f)).
\]
From these formulae (2.2), (2.6), we read off the diagrammatic rules. The vertices, given by the polynomial structure of the potential \( v \), are common to both functional integrals. They involve a summation over the lattice index \( i \). Due to the respective shifts of the variable \( \phi_i \), there are inner lines (corresponding to the integration variables) and outer lines (corresponding to the external variables \( f \)) attached to each vertex.

The “bulk-to-bulk propagator” for the inner lines connecting two vertices is the inverse of the Gaussian covariance matrix of the respective integral. The “bulk-to-boundary” propagator for the outer lines is the \((N \times n)\) matrix-valued coefficient of \( f \) in the shifted argument of \( V \).

For \( \tilde{Z}(f) \) with the exponential insertion, eq. (2.2), we read off the bulk-to-bulk propagator

\[
G = A^{-1}
\]

and the bulk-to-boundary propagator which is the right boundary restriction of \( G \), while \( \alpha \) giving the leading Gaussian is its two-sided restriction,

\[
H = Ge, \quad \alpha = e^tGe = e^tH.
\]

For \( Z(f) \) with the \( \delta \) function insertion, eq. (2.6), the bulk-to-bulk propagator is obtained as follows. Since only the bulk variables \( E^\perp \phi \) propagate, \( \Gamma \) should have vanishing \( \mathbb{R}^n \) (boundary) components. On the orthogonal (bulk) subspace \( E^\perp \mathbb{R}^N \), \( \Gamma \) should be the inverse of \( A \). Hence

\[
ET = 0 = \Gamma E \quad \text{and} \quad E^\perp A\Gamma = E^\perp = \Gamma A E^\perp.
\]

This pair of algebraic conditions determines the matrix \( \Gamma \) uniquely as

\[
\Gamma = G - Ge \alpha^{-1} e^tG.
\]

The bulk-to-boundary propagator is

\[
K = Ge \alpha^{-1} = H \alpha^{-1},
\]

and can be uniquely characterized by the pair of algebraic conditions

\[
EK = e \iff e^tK = 1_n \quad \text{and} \quad E^\perp A K = 0.
\]

We recognize in (2.9) a discrete version of Dirichlet boundary conditions for the inverse of \( A \) on the bulk. This property will be crucial when we pass to the continuum in the next section.\footnote{It is also instructive to pass to the other extreme in which the lattice consists of only two points 1 (the boundary) and 2 (the bulk), i.e., \( n = N - n = 1 \). In this case, with \( A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) and \( A^{-1} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \), one has \( \Gamma = \begin{pmatrix} 0 & 0 \\ 0 & 1/c \end{pmatrix} \) where \( 1/c = \gamma - \beta^2/\alpha \).}
We conclude that the two functional integrals $Z(f)$ and $\tilde{Z}(f)$ are obtained as sums over the same sets of graphs but with different prescriptions for the propagators to be inserted for the internal and external lines, and with different leading Gaussian prefactors.

3 Scalar propagators on AdS

Passing to Euclidean field theory on $d + 1$-dimensional anti-deSitter space, we substitute the real scalar field $\phi(z, x)$ (in the usual coordinates $z \in \mathbb{R}_+, \ x \in \mathbb{R}^d$) for the vector $\phi$, and the Klein-Gordon operator for the matrix $A$:

$$A = -\Box_g + M^2 = -z^{1+d}\partial_z z^{1-d}\partial_z - z^2 \sum_{i=1}^d \partial_i^2 + M^2. \quad (3.1)$$

The inner product $(\phi, A\phi)$ is the bulk integration with measure $dz\,d^dx/\sqrt{g}$,

$$\sqrt{g} = z^{-1-d}. \quad \text{The potential has the form } V(\phi) = \int dz\,d^dx/\sqrt{g} \, v(\phi(z, x)) \text{ with some polynomial density } v(\phi).$$

The inverse $G = A^{-1}$ is the Green function solving

$$(-\Box_g + M^2) G(z, x; z', x') = z^{1+d}\delta(z-z')\delta^d(x-x'). \quad (3.2)$$

There are two linearly independent AdS-invariant solutions,

$$G_\pm(z, x; z', x') = \gamma_\pm \cdot (2u)^{-\Delta_\pm} F_1(\Delta_\pm, \Delta_\pm + \frac{1-d}{2}, 2\Delta_\pm + 1 - d; -2u^{-1}) \quad (3.3)$$

where $u = \frac{(z-z')^2 + (x-x')^2}{2z^2}$, $\Delta_\pm$ are the two solutions of $\Delta(\Delta - d) = M^2$, $\Delta_\pm = \frac{d^2}{2} \pm \frac{1}{2} \sqrt{d^2 + 4M^2}$, (3.4)

and the normalization coefficients are $\gamma_\pm = \frac{\pi^{-\frac{d}{2}}\Gamma(\Delta_\pm)}{2\Gamma(\Delta_\pm + 1 - \frac{d}{2})}$. The two solutions are distinguished by the boundary behaviour

$$G_\pm \sim z^{\Delta_\pm} \quad \text{as} \quad z \to 0 \quad (3.5)$$

(and likewise for $z'$). The choice of either of them therefore specifies the functional integration measure to extend formally over spaces of functions $\phi^\pm(z, x)$ with the corresponding boundary behaviour $\sim z^{\Delta_\pm}$. We denote the corresponding integrals (1.1) and (1.2) by $Z^\pm(f)$ and $\tilde{Z}^\pm(f)$.

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3. The Klein-Gordon operator (3.1) is homogeneous in $z$ near the boundary and hence preserves spaces of functions $\phi$ which behave like $\sim z^\Delta$ near $z = 0$. The integral $(\phi, A\phi) = \int z^{-1-d}dz\,dx \, (\phi(-\Box_g + M^2)\phi$ converges at $z = 0$ and is symmetric as a quadratic form only in the case of $\Delta_+ > \frac{d}{2}$. Nevertheless, proceeding formally also in the case of $Z^-$ where $\Delta_- < \frac{d}{2}$, will turn out to be justified (due to the suppression of the boundary functional integration variables by the $\delta$ function), and in fact match the perturbative rules adopted in the literature [7, 8, 12, 19].
We retain from the discrete model the diagrammatical rules. The field theoretical integrals $\tilde{Z}^\pm(f)$ have Gaussian prefactors $\exp \frac{1}{2}(f, \alpha \pm f)$ and involve propagators $G_\pm$ (bulk-to-bulk) and $H_\pm$ (bulk-to-boundary), while the AdS-CFT dual integrals $Z^\pm(f)$ have prefactors $\exp -\frac{1}{2}(f, \alpha \mp f)$ and involve propagators $\Gamma_\pm$ and $K_\pm$. The various propagators are obtained from $G_\pm$ via the algebraic relations (2.3) and (2.7)–(2.12), to be understood as relations among integration kernels $G_\pm, \Gamma_\pm, H_\pm, K_\pm, \alpha_\pm$ (some of which will require regularization) rather than matrices, and $e_\pm$ stand for rescaled limits of the form

$$ (F_\pm e_\pm)(x) := \lim_{z \to 0} z^{-\Delta_\pm} F_\pm(z, x). \quad (3.6) $$

The graphs to be summed in both integrals are the same, with the same vertices, but different propagators.

For the field theoretical integrals $\tilde{Z}^\pm(f)$, the bulk-to-bulk propagators are $G_\pm(z, x; z', x')$ as in eq. (3.3). The bulk-to-boundary propagators are, according to eq. (2.8), the limits

$$ H_\pm(z, x; x') = \lim_{z' \to 0} z'^{-\Delta_\pm} G_\pm(z, x; z', x') = \gamma_\pm \cdot \left( \frac{z}{z^2 + (x - x')^2} \right)^{\Delta_\pm} \quad (3.7) $$

and likewise, according to eq. (2.3), the tree level 2-point functions are

$$ \alpha_\pm(x, x') = \lim_{z \to 0} z^{-\Delta_\pm} H_\pm(z, x; x') = \gamma_\pm \cdot (x - x')^{-2\Delta_\pm}. \quad (3.8) $$

Thus, the boundary fields $\phi_0^\pm$ have the scaling dimensions $\Delta_\pm$ (at tree level).

To compute the propagators $\Gamma \equiv G - H \alpha^{-1} H^t$ and $K = H \alpha^{-1}$ for the dual integrals $Z^\pm(f)$ according to (2.10) and (2.11), would involve the determination of, and multiplication with inverse integral kernels $\alpha_\pm^{-1}$. It turns out advantageous to exploit instead the algebraic characterizations (2.9) and (2.12) of the dual propagators, worked out in Sect. 2.

Translated into the continuum context, (2.9) states that $\Gamma_\pm$ solve Green’s differential equation in the bulk, and vanish on the boundary. In other words, they are the Green functions with Dirichlet conditions with respect to the restrictions given by the limits $e_\pm$. Now, by (3.5) and $\Delta_+ > \Delta_-$, the Green function $G_+$ vanishes faster than $G_-$ and hence satisfies the Dirichlet condition with respect to the limit $e_-$. We conclude that

$$ \Gamma_- = G_. \quad (3.9) $$

Likewise, (2.12) translates into the conditions that $K_\pm$ solve the Klein-Gordon equation in the bulk, and approach $\delta^d(x - x')$ in the limits $e_\pm$. 
By definition, the first condition for $K_\pm$ is fulfilled by $H_\pm$. By virtue of a simple scaling argument \[13\] based on the relation 
$$\Delta_+ + \Delta_- = d, \quad (3.10)$$

$H_+$ also fulfills the second condition for $K_-$ up to a normalization,

$$(e^t H_+)(z, x; x') \equiv \lim_{z \to 0} z^{-\Delta} H_+(z, x; x') = c^{-1} \cdot \delta(x - x'). \quad (3.11)$$

The constant is computed as $c = 2\Delta_+ - d =: c(\Delta_+)$. Hence

$$K_- = c \cdot H_+. \quad (3.12)$$

By (2.11), this implies the integral identity

$$c \cdot H_+ \alpha_- = H_- \quad (3.13)$$

involving a perfectly regular $\mathbb{R}^d$ integration. In contrast, replacing $\Delta_\pm$ by $\Delta_\mp$ everywhere, $H_+ \alpha_+$ is UV-divergent. We UV-regularize $\alpha_+$ by analytic continuation of (3.13), such that also

$$-c \cdot H_- \alpha_+ = H_+ \quad (3.14)$$

holds. Applying the limit $e_-$ to both sides of (3.14), using $e_+ H_+ = \alpha_-$ and $c \cdot e_- H_+ = 1$, we get $-c^2 \alpha_- \alpha_+ = 1$ or

$$\alpha_-^{-1} = -c^2 \alpha_+. \quad (3.15)$$

As $-\alpha_\pm^{-1}(x, x')$ are the tree level 2-point functions of the dual fields $O_\pm$, the latter have the scaling dimensions $\Delta_\mp$. The regularization of $\alpha_+$ implicit in (3.14) is in agreement with the one adopted in \[12\], and the absolute normalization $c_2^2 \gamma_\mp$ of the 2-point functions of $O_\pm$ inferred from (3.15) is in agreement with the correction advocated in \[7, 12\].

Now, by virtue of the identifications (3.9), (3.12) and (3.15), the propagators involved in $Z^-(f)$ and $\tilde{Z}^+(f)$ are the same, up to the numerical factors. This proves the assertion (1.6), and hence (1.7), with $c = \sqrt{d^2 + 4M^2}$.

Scrutinizing the above argument, we observe that most of it follows from the algebraic characterizations of the propagators obtained in Sect. 2. The

\[\text{Regarding} \ H_\pm, \ \alpha_\pm \ \text{as functions of} \ \Delta_\pm, \ \text{and} \ \Delta_- \ \text{as a function of} \ \Delta_+, \ (3.13) \text{is an equality} \ c(\Delta) \cdot H(\Delta) \alpha(d-\Delta) = H(d-\Delta) \text{of two meromorphic functions, valid at} \ Re \ \Delta > \frac{d}{2} \ (\text{hence at} \ \Delta = \Delta_+). \ \text{The right-hand side being analytic also at} \ Re \ \Delta < \frac{d}{2} + 1, \ \text{it defines the analytic continuation of the left-hand side to} \ \Delta = \Delta_- \text{. At this point,} \ H(\Delta_-) = H_-, \ \ H(d-\Delta_-) = H_+, \ \ \alpha(d-\Delta_-) = \alpha_+, \ \text{and} \ c(\Delta_-) = -c.\]
only independent information was the validity of the limit (3.11) which in turn followed by a scaling argument from the relation (3.10), along with the analytic property of the coefficient as a function of $\Delta_+$

\[ c(d - \Delta_+) = -c(\Delta_+) \] (3.16)

ensuring the correct relative normalizations of the coefficients in (1.4), (1.5).

4 Vector fields

We want to generalize the previous argument to vector fields $\phi_\mu(z, x)$, $\mu = z, 0, \ldots d - 1$. For vector fields, the restriction maps $e_{\pm}$ involve rescaled limits and the projection onto the transverse components $\phi_i$, $i = 0, \ldots d - 1$. We shall establish the identity

\[ O^-_i(x) = c \cdot (\phi^+_0)_i(x) \equiv c \cdot \lim_{z \to 0} z^{1-\Delta_+} \phi^+_i(z, x). \] (4.1)

The dimensions $\Delta_\pm$ will be determined from the quadratic part of the action, and satisfy again (3.10). Adapting the remark at the end of the previous section, we observe that we only have to compute the coefficient $c$ in

\[ \lim_{z \to 0} z^{1-\Delta_-} \left( \lim_{z' \to 0} z'^{1-\Delta_+} G_{+,ij}(z, x; z', x') \right) = c^{-1} \cdot \delta_{ij} \delta(x - x'), \] (4.2)

as a function of $\Delta_+$ and verify that it again satisfies eq. (3.16).

We shall only sketch the computation. The most general quadratic action

\[ \int dz \, \sqrt{g} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \lambda (D^\mu \phi_\mu)^2 + \frac{1}{2} M^2 \phi_\mu \phi_\mu \right) \] (4.3)

where $F_{\mu\nu} = D_\mu \phi_\nu - D_\nu \phi_\mu$, gives rise to Green’s differential equation

\[ A_{\mu \nu} G_{\nu \alpha}(z, x; z', x') = g_{\mu \alpha} z^{1+d} \delta(z - z') \delta^d(x - x') \] (4.4)

with the differential operator

\[ A_{\mu \nu} = (-D_\alpha D^\kappa + M^2 - d) g_{\mu \nu} + (1 - \lambda) D_\mu D_\nu \] (4.5)

(the shift in the mass being due to the curvature). We make an ansatz with the most general AdS-covariant bivector (with $u = \frac{(z-z')^2 + (x-x')^2}{2z^2}$ as before)

\[ G_{\mu \alpha}(z, x; z', x') = -g_1(u) \cdot (\partial_\mu \partial'_\alpha u) - g_2(u) \cdot (\partial_\alpha u)(\partial'_\mu u). \] (4.6)
The singularity in (4.4) requires the short-distance behaviour of the functions $g_1 \approx \gamma_1' u^{-\frac{d+1}{2}}$ and $g_2 \approx \gamma_2' u^{-\frac{d+1}{2}}$ at $u \approx 0$, and

$$(d-1)\gamma_1' + 2\gamma_2' = (2\pi)^{-\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right). \quad (4.7)$$

At $u \neq 0$, (4.4) yields two differential equations of second order, which can be decoupled with the help of $f = g_2 - g_1'$:

$$u(u+2)f'' + (d+3)(u+1)f' + (2d-M^2)f = 0,$$  

$$u(u+2)g_1'' + (d+3)(u+1)g_1' + (d+1-M^2/\lambda)g_1 = \frac{1-\lambda}{\lambda} u(u+2)f' + (\frac{1-\lambda}{\lambda} d-2)(u+1)f. \quad (4.8)$$

The homogeneous equation (4.7) for $f(u)$ is solved by

$$f_\pm(u) = \gamma_\pm \cdot u^{-\Delta_\pm-1} F_1(\Delta_\pm + 1, \Delta_\pm + \frac{1-d}{2}, 2\Delta_\pm + 1 - d; -2u^{-1}) \quad (4.10)$$

with

$$\Delta_\pm = \frac{d}{2} \pm \frac{1}{2} \sqrt{(d-2)^2 + 4M^2}. \quad (4.11)$$

Both solutions have the same short-distance behavior $f \approx \gamma' u^{-\frac{d+1}{2}}$. Because of $f = g_2 - g_1'$, the coefficient is $\gamma' = \gamma_2' + \frac{d+1}{2} \gamma_1'$, which is determined by (4.7). This fixes the absolute normalizations in (4.10),

$$\gamma_\pm = 2^{-\Delta_\pm-1} \pi^{-\frac{d}{2}} \frac{\Gamma(\Delta_\pm+1)}{\Gamma(\Delta_\pm+1-\frac{d}{2})}. \quad (4.12)$$

We choose either $f_+$ or $f_-$ and suppress the subscript for the moment. The function $g_1 \equiv g_{1\pm}$ is determined by the inhomogeneous equation (4.9) up to a solution of the corresponding homogeneous equation. One of these is too singular at short distance and can be excluded. The less singular solution

$$2F_1(q_+, q_-, \frac{d+2}{2}; -u/2), \quad q_\pm = \frac{d+2}{2} \pm \frac{1}{2} \sqrt{d^2 + 4M^2/\lambda} \quad (4.13)$$

behaves like $u^{-q_-}$ at large $u$ (small $z$, $z'$). If this solution would dominate the behavior of $g_1$ at large $u$ (in case $q_- < \Delta$), then $g_1'$ would dominate $f$, and $g_2 \approx g_1'$. This would produce a boundary 2-point function of scaling dimension $q_-$, which violates conformal invariance, however. Thus we must seek the special solution $g_1$ whose behavior at large $u$ is determined not by (4.13), but by the inhomogeneity of (4.9). For this solution, $g_1 \approx \gamma_1 u^{-\Delta}$ and $g_2 = g_1' - f \approx \gamma_2 u^{-\Delta-1}$, with

$$\gamma_1 = \frac{\gamma}{\Delta-1} = \frac{2^{-\Delta-1} \pi^{-\frac{d}{2}} \Gamma(\Delta+1)}{\Gamma(\Delta+1-\frac{d}{2})}, \quad \gamma_2 = -\Delta \gamma_1 - \gamma = -\gamma_1. \quad (4.14)$$
This controls the boundary behavior of the Green function. With
\[ H_{\mu j}(z, x; x') = \lim_{z' \to 0} z'^{1-\Delta} G_{\mu j}(z, x; x') = \gamma_1 \left( \frac{2z}{z^2 + (x-x')^2} \right)^{\Delta} \partial_\mu \frac{(x-x')_j}{z} + \gamma_2 \left( \frac{2z}{z^2 + (x-x')^2} \right)^{\Delta+1} \partial_\mu (x-x')_j \frac{z^2 + (x-x')^2}{2z}, \]
we obtain the conformally invariant tree level 2-point function
\[ \alpha_{ij}(x; x') = \lim_{z \to 0} z^{1-\Delta} H_{ij}(z, x; x') = 2\Delta \gamma_1 \frac{\delta_{ij} - 2(x-x')_i(x-x')_j}{(x-x')^{2\Delta}}. \] (4.16)
In particular, \( \Delta \equiv \Delta_\pm \) given in (4.11) are the scaling dimensions of the boundary fields, satisfying (3.10) as announced.

We can now also compute the limit \( e^c_+ H_+ \) and find
\[ \lim_{z \to 0} z^{1-\Delta} H_{+ij}(z, x; x') = (2\Delta_+ - d)^{-1} \cdot \delta_{ij} \delta^d (x - x'). \] (4.17)
Thus, \( c = 2\Delta_+ - d \) is the same analytic function of \( \Delta_+ \) as for scalar fields, satisfying (3.16) as desired.

As explained above, this completes the proof for the validity of (1.6) also for vector fields. More explicitly, eq. (4.1) holds with \( c = \sqrt{(d-2)^2 + 4M^2} \). The case of massless gauge fields corresponds to \( M^2 = 0 \), \( \Delta_+ = d - 1 \) (independent of the gauge parameter \( \lambda \)), so that (4.16) is the 2-point function of a conserved current. For tensor fields of higher spin, the most general bicovariant ansatz for the Green functions involves more unknown functions, which complicates the analysis. But since only (3.10) and (3.16) need to be verified, the comparison of (3.4) with (4.10) and the coincidence of the function \( c(\Delta_+) \) both for scalar and vector fields, lend support to the expectation that the result generalizes to tensor fields of arbitrary rank.

5 Conclusion

The perturbative expansion of the dual field \( O^- \) in terms of “Witten graphs” matches the canonical (field theoretical) expansion of the interacting field \( \phi^+ \) in the bulk of AdS, with subsequent restriction to the boundary. We have presented a structural analysis of this fact (which was previously observed, e.g., by [1]) in terms of the formal identification (1.6), graph by graph, of the generating functionals for the respective Euclidean correlation functions. The relations between the relevant propagators, pertaining to the passage between source terms of the respective forms \( e^{\phi_0 f} \) and \( \delta(\phi_0 - f) \), are basically
algebraic, and independent of any specific geometry. The solution (1.3–5) to these relations, in contrast, is largely due to $SO(1, d + 1)$ symmetry, while the only piece of the argument which seems not automatic, is the analytic property (3.16) of the coefficient function $c(\Delta_+)$ appearing in (3.11).

In the free case, $V(\phi) = 0$, both integrals (1.2) and (1.1) give rise to purely Gaussian boundary fields with scaling dimensions $\Delta_+ = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4M^2}$. Their Euclidean correlations satisfy OS positivity provided the dimension satisfies the unitarity bound $\Delta \geq \frac{d}{2} - 1$. Thus, $\phi^+_0$ is always related to a real-time quantum field, and so is $\phi^-_0$ provided $\frac{1}{2}\sqrt{d^2 + 4M^2} \leq 1$.

These are Gaussian fields with non-canonical dimension $\Delta$. Such fields belong to the class of “generalized free fields” [11, Ch. 2.6] which were first introduced in [15] as asymptotic fields appropriate when a particle interpretation breaks down (e.g., in conformal theories). The $n$-point functions of a generalized free field factorize into 2-point functions, and its commutator is a numerical distribution, but there is no Lagrangian description with an equation of motion because the Källen-Lehmann measure $\rho (m^2)$

$$\langle \Omega, \varphi(x) \varphi(y) \Omega \rangle = \int_0^\infty dm^2 \rho (m^2) \Delta^+_{\text{ret}} \Delta^-_{\text{ret}}$$

may cover a continuum of masses. (Specifically, for $\phi^+_0$, $\rho (m^2) \sim m^{2\Delta_+ - d}$.) Thus a generalized free field can have the same 2-point function as an interacting field which necessarily covers a continuum of masses extending to $\infty$ [11, Ch. 6.1]. It will be shown elsewhere [6] that in spite of the absence of an equation of motion, a stress-energy tensor can be defined for generalized free fields, which is more singular than a Wightman field but still is a local density for the generators of space-time symmetries.

The (real time) AdS-CFT correspondence thus amounts to a perturbation around a conformal generalized free field whose non-canonical dimension is not itself a perturbative effect, unlike an anomalous dimension.

We notice that a standard perturbation theory around generalized free fields has not been formulated so far, and is expected to suffer from aggravated renormalization problems: e.g., in the case of $\phi^+_0$ already the integration for the retarded propagator

$$[\phi^+_0(x), \phi^+_0(x')] \theta (x^0 - x'^0) = \int_0^\infty dm^2 m^{2\nu} \Delta^\text{ret}_{\text{m}} (x - x')$$

is UV divergent. Thus, the free propagator itself requires renormalization, i.e., its distributional extension to the diagonal $x = x'$ is non-unique [6].
Moreover, the propagator entering the power counting argument with a larger scaling dimension affects renormalizability always for the worse.

But perturbation theory around free Klein-Gordon fields on curved spacetime is well-defined [4], subject to the same UV limitations as in flat spacetime. Applied to AdS, the interacting fields may be restricted to the boundary (in the sense of limits of correlation functions [2]). Thus, canonical bulk perturbation theory with subsequent restriction provides a new perturbative scheme around non-canonical free fields.

Let us return to the Euclidean functional integrals. In the free case, an identification between \( Z^{-}(f) \) and \( \tilde{Z}^{+}(f) \) as in (1.6) also holds symmetrically between \( Z^{+}(f) \) and \( \tilde{Z}^{-}(f) \). The latter both yield the Gaussian Euclidean field with dimension \( \Delta_{-} \), and, unless the AdS mass parameter \( M \) exceeds the unitarity bound, the real-time generalized free field with the same dimension. But in the presence of an interaction, the generating functionals \( Z^{+} \) seems to be ill defined because a Green function \( \Gamma_{+} \) decaying faster than \( z^{\Delta_{+}} \) does not exist. Eq. (1.6), however, suggests to define \( Z^{+}(f) \) as the functional Fourier transform of \( Z^{-}(if/c) \). This qualifies and extends the observation that in the free case the corresponding connected functionals \( \log Z^{-} \) and \( \log Z^{+} \) are each other’s Legendre transforms [2]. One may doubt, however, that the Fourier transform of the generating functional respects OS positivity.

On the other hand, we see no a priori obstruction against a field theoretical perturbation around the canonical free bulk field \( \phi^{-} \) (provided \( \Delta_{-} > \frac{d-2}{2} \)), which then admits a sensible restriction \( \phi_{0}^{-} \). Its generating functional \( \tilde{Z}^{-}(f) \), however, would have no interpretation as an AdS-CFT functional integral with \( \delta \) function insertion.

We finally notice that we had to chose an (implicit) UV-regularization in (3.14). Our choice seems to be the most natural one, and it gives automatically the correct normalization required by Ward identities when the scalar field is coupled to a massless vector field [7, 12]. The identification (1.6) suggests that the fulfillment of Ward identities is another feature which is inherited upon restriction from the bulk QFT.

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