THE BV FORMALISM FOR $L_\infty$-ALGEBRAS

DENIS BASHKIROV AND ALEXANDER A. VORONOV

Abstract. The notions of a BV$_\infty$-morphism and a category of BV$_\infty$-algebras are investigated. The category of $L_\infty$-algebras with $L_\infty$-morphisms is characterized as a certain subcategory of the category of BV$_\infty$-algebras. This provides a Fourier-dual, BV alternative to the standard characterization of the category of $L_\infty$-algebras as a subcategory of the category of dg cocommutative coalgebras or formal pointed dg manifolds. The functor assigning to a BV$_\infty$-algebra the $L_\infty$-algebra given by higher derived brackets is also shown to have a left adjoint.

Introduction

One way to approach the notion of a strongly homotopy Lie algebra is via the language of formal geometry, see [KS06]. Namely, it is known that the data of an $L_\infty$-algebra $\mathfrak{g}$ is equivalent to that of a formal pointed differential graded (dg) manifold $\mathfrak{g}[1]$. The corresponding $L_\infty$ structure is then encoded in the cofree dg cocommutative coalgebra $S(\mathfrak{g}[1])$ of distributions on $\mathfrak{g}[1]$ supported at the basepoint. The idea of Batalin-Vilkovisky (BV) formalism in physics suggests that it might be useful to study what the $L_\infty$ structure looks like from a Fourier-dual perspective [Los07], namely, the point of view of the standard dg commutative algebra structure on $S(\mathfrak{g}[1])$. In fact, we show that an $L_\infty$ structure on $\mathfrak{g}$ translates into a special kind of commutative homotopy Batalin-Vilkovisky (BV$_\infty$) structure on $S(\mathfrak{g}[-1])$ and, moreover, does it in a functorial way. Geometrically, we can say that we describe a formal pointed dg manifold $\mathfrak{g}[1]$ as a pointed BV$_\infty$-manifold $(\mathfrak{g}[-1])^*$ of a special kind. We also show that the functor that assigns to an $L_\infty$-algebra $\mathfrak{g}$ the BV$_\infty$-algebra $S(\mathfrak{g}[-1])$ is left adjoint to a “functor” that assigns to a BV$_\infty$-algebra the $L_\infty$-algebra given by higher derived brackets. This fact may be interpreted geometrically as a statement that the functor $\mathfrak{g}[1] \mapsto (\mathfrak{g}[-1])^*$ from formal pointed dg manifolds to pointed BV$_\infty$-manifolds has a right adjoint.

The correspondence between $L_\infty$ and BV$_\infty$ structures that we establish in the paper is to a large extent motivated by the technique of higher derived brackets. The origins of higher derived brackets can be traced back to the iterated commutators of A. Grothendieck, see Exposé VII A by P. Gabriel in [SGA3], and J.-L. Koszul [Kos85], used in the algebraic study of differential operators, though the subject really flourished later in physics under the name of higher “antibrackets” in the works of J. Alfaro, I. A. Batalin, K. Bering, P. H. Damgaard and R. Marnelius [AD96, BBD97, BM98, BM99a, BM99b, BDA96, Ber07] on the BV formalism. A mathematically friendly approach was developed by F. Akman’s [Akm97, Akm00] and generalized further by T. Voronov [Vor05a, Vor05b], who described $L_\infty$ brackets.
derived by iterating a binary Lie bracket not necessarily given by the commutator. There have been various versions of higher derived brackets introduced in other contexts, such as the ternary bracket of [CM07, Get10] for Courant algebroids and its higher-bracket generalization of [Bor14] for dg associative algebras and the twisted $L_\infty$ brackets and $A_\infty$ products of [Mar13]. The notion of a homotopy BV algebra was studied by K. Bering and T. Lada [BL09], K. Cieliebak and J. Latschev [CL07], I. Gálvez-Carrillo, A. Tonks and B. Vallette [GCTV12], O. Kravchenko [Kra00], and D. Tamarkin and B. Tsygan [TT00].

The BV formalism as a replacement of the dg-coalgebra language seems to be even more natural for studying Lie-Rinehart pairs ($g, A$), see [Hue98, Vit13].

We review the notion of a BV$_\infty$-algebra in Section 1 and describe the BV$_\infty$-structure on $S(g[−1])$ in Section 2. A characterization of BV$_\infty$-algebras arising this way is presented in Section 3. In Section 4 we prove the first main result of the paper: a description of the category of $L_\infty$-algebras as a certain subcategory of the category of BV$_\infty$-algebras. Section 5 is dedicated to the second main result, the adjunction theorem.

Conventions and Notation. We will work over a ground field $k$ of characteristic zero. A differential graded (dg) vector space $V$ will mean a complex of $k$-vector spaces with a differential of degree one. The degree of a homogeneous element $v \in V$ will be denoted by $|v|$. In the context of graded algebra, we will be using the Koszul rule of signs when talking about the graded version of notions involving symmetry, including commutators, brackets, symmetric algebras, derivations, etc., often omitting the modifier graded. For any integer $n$, we define a translation (or n-fold desuspension) $V[n]$ of $V$: $V[n]^p := V^{n+p}$ for each $p \in \mathbb{Z}$. For a dg vector space $V$, we will also consider the dg $k[[\hbar]]$-module $V[[\hbar]]$ of formal power series in a variable $\hbar$ of degree 2 with values in $V$. We will also sometimes refer to differential operators of order $\leq n$ as differential operators of order $n$.

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1. Homotopy BV algebras

We will utilize a strictly commutative version of the notion of a homotopy BV algebra, also known as a generalized BV algebra, due to Kravchenko [Kra00], which is less general than the full-fledged homotopy versions of [TT00] and [GCTV12]. Nevertheless, we will take the liberty to use the term BV$_\infty$-algebra, following a trend set by several authors [CL07, TTT01, BL13, Vit13]. The following definition gives a graded version of Grothendieck’s notion of a differential operator in commutative algebra.

Definition 1.1. Let $n \geq 0$ be an integer. A $k$-linear operator $D : V \to V$ on a graded commutative algebra $V$ is said to be a differential operator of order $\leq n$ if for any $n + 1$ elements $a_0, \ldots, a_n \in V$, we have

$$[[\ldots [D, L_{a_0}], \ldots], L_{a_n}] = 0,$$
where the \( L_a \) is the left-multiplication operator
\[
L_a(b) := ab
\]
on \( V \) and the bracket \([-,-]\) is the graded commutator of two \( k \)-linear operators.

**Definition 1.2.** A BV\(_\infty\)-algebra is a graded commutative algebra \( V \) over \( k \) with a \( k \)-linear map \( \Delta : V \to V[[\hbar]] \) of degree one satisfying the following properties:
\[
\Delta = \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \Delta_n,
\]
where \( \Delta_n \) is a differential operator of order (at most) \( n \) on \( V \),
\[
\Delta^2 = 0, \quad \text{and} \quad \Delta(1) = 0,
\]
The continuous (in the \( \hbar \)-adic topology), \( k[[\hbar]] \)-linear extension of \( \Delta \) to \( V[[\hbar]] \) will also be denoted \( \Delta : V[[\hbar]] \to V[[\hbar]] \) and called a BV\(_\infty\) operator.

Recall that we assumed that \(|\hbar| = 2\), thus \(|\Delta_1| = 1\), \(|\Delta_2| = -1\), and generally, \(|\Delta_n| = 3 - 2n\) for \( n \geq 1 \). Note that \( \Delta_1 \) will automatically be a differential in the usual sense, i.e., define the structure of a dg commutative algebra on \( V \). If \( \Delta_n = 0 \) for \( n \geq 3 \), we recover the notion of a dg BV algebra, see [KS95, AKm97, BK98, Hmu98, Man99, Kra00, TT00]. If moreover \( \Delta_1 = 0 \), we get the notion of a BV algebra, also known as a Beilinson-Drinfeld algebra, see [BD04, Gwi12, CG14].

BV algebras arose as part of the BV formalism in physics. A basic geometric example of (a \( \mathbb{Z}/2\mathbb{Z} \)-graded version of) a BV algebra is the algebra of functions on a smooth supermanifold with an odd symplectic form and a volume density, see [Sch93, Get94]. An example of such a supermanifold is the odd cotangent bundle \( \Pi T^* M \) of a classical manifold \( M \) with a volume form, where \( \Pi T^* M \) denotes the translation \( T^*[−1] M \) modulo 2 of the cotangent bundle. A Lie-theoretic version of this example is the graded symmetric algebra \( S(g[−1]) \), also known as the exterior algebra \( \Lambda(g) \), of a Lie algebra \( g \), with the Chevalley-Eilenberg differential as \( \Delta_2 \). We will describe an \( L_\infty \) generalization of this example is Section 2. More generally, one can view the BV\(_\infty\) structure considered in this paper as a homotopy version of the algebraic structure arising in BV geometry.

**Example 1.3.** ([KV08] [BL13] [Vit13]) Let \( M \) be a smooth graded manifold and \( C^\infty(M,S(T[−1]M)[1]) \) be the graded Lie algebra of (global, smooth) multivector fields on \( M \) with respect to the Schouten bracket. When \( M \) is a usual, ungraded manifold, \( S(T[−1]M)[1] \) is the exterior-algebra bundle \( \Lambda TM \), in which a \( k \)-vector field, or a section of \( \Lambda^k TM \), has degree \( k − 1 \). A generalized Poisson structure on a graded manifold \( M \) is a multivector field \( P \) of degree one such that \([P,P] = 0\). A generalized Poisson structure on \( M \) may be expanded as \( P = P_0 + P_1 + \cdots \) with \( P_n \in C^\infty(M,S^n(T[−1]M)[1]) \). For \( n \geq 1 \), the generalized Lie derivative \( \Delta_n = [d, i_{P_n}] \), where \( i_{(-)} \) is the interior product, defines an \( n \)-th order differential operator of degree \( 3 − 2n \) on the de Rham algebra \( \Omega(M),d \), where \( \Omega(M) := C^\infty(M,S(T[−1]M)) \). If we assume that \( P_0 = 0 \) to avoid differential operators of order zero, then \( \Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_n \hbar^{n−1} + \cdots : \Omega(M) \to \Omega(M)[[\hbar]] \) defines a BV\(_\infty\) structure on \( \Omega(M) \), known as the de Rham-Koszul BV\(_\infty\) structure.
2. From homotopy Lie algebras to homotopy BV algebras

The construction of this section belongs essentially to C. Braun and A. Lazarev, see [BL13, Example 3.12]. Consider an $L_\infty$-algebra $\mathfrak{g}$, i.e., a graded vector space $\mathfrak{g}$ and a coderivation on the cofree graded cocommutative coalgebra $S(\mathfrak{g}[1])$ on $\mathfrak{g}[1]$ with respect to the “shuffle” comultiplication:

$$\delta(x_1 \ldots x_m) := \sum_{n=0}^{m} \sum_{\sigma \in Sh_{n,m-n}} (-1)^{|\sigma|} (x_{\sigma(1)} \ldots x_{\sigma(n)}) \otimes (x_{\sigma(n+1)} \ldots x_{\sigma(m)}),$$

where $x_1, \ldots, x_m \in \mathfrak{g}[1]$, $Sh_{n,m-n}$ is the set of $(n, m-n)$ shuffles: permutations $\sigma$ of $\{1, 2, \ldots, m\}$ such that $\sigma(1) < \cdots < \sigma(2) < \cdots < \sigma(n)$ and $\sigma(n+1) < \cdots < \sigma(n+2) < \cdots < \sigma(m)$, and $(-1)^{|\sigma|}$ is the Koszul sign of the permutation $\sigma$.

Here a codifferential is a coderivation $D : S(\mathfrak{g}[1]) \to S(\mathfrak{g}[1])$ of degree one such that $D^2 = 0$ and $D(1) = 0$. Given that a coderivation is determined by its projection to the cogenerators, we can write

$$D = D_1 + D_2 + D_3 + \ldots,$$

where $D_n$ is the extension as a coderivation of the $n$th symmetric component $l_n : S^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ of the projection $S(\mathfrak{g}[1]) \xrightarrow{D} S(\mathfrak{g}[1]) \to \mathfrak{g}[1]$. An explicit relation between $D_n$ and $l_n$ will be useful: for $x_1, \ldots, x_m \in \mathfrak{g}[1]$

\begin{equation}
D_n(x_1 \ldots x_m) = \sum_{\sigma \in Sh_{n,m-n}} (-1)^{|\sigma|} l_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) x_{\sigma(n+1)} \ldots x_{\sigma(m)},
\end{equation}

if $m \geq n$, and $D_n(x_1 \ldots x_m) = 0$ otherwise.

**Theorem 2.1** (C. Braun and A. Lazarev). Given an $L_\infty$-algebra $\mathfrak{g}$, the free graded commutative algebra $S(\mathfrak{g}[-1])$ becomes a BV$_\infty$-algebra under the BV$_\infty$ operator

\begin{equation}
\Delta := \frac{1}{h} \sum_{n=1}^{\infty} h^n D_n.
\end{equation}

**Proof.** Since $D_n : S^n(\mathfrak{g}[1]) \to S^{n-n+1}(\mathfrak{g}[1])$ is a degree one map, it turns into a map $D_n : S^n(\mathfrak{g}[-1]) \to S^{n-n+1}(\mathfrak{g}[-1])$ of degree $3 - 2n$ under the new grading.$^3$

For each $n$

$$\sum_{i+j=n} D_i D_j = 0,$$

because this sum is exactly the component of $D^2$ which maps $S^m(\mathfrak{g}[1])$ to $S^{m-n+2}(\mathfrak{g}[1])$. The map $D_n$ will also be a differential operator of order $n$, because of the following lemma, which may be observed directly from Equation (1).

**Lemma 2.2.** The coderivation of the coalgebra $S(\mathfrak{g}[1])$ extending a linear map $S^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ becomes a differential operator of order $\leq n$ on the algebra $S(\mathfrak{g}[-1])$.

\[\square\]

The construction of this section seems to be math-physics folklore in the case when $(\mathfrak{g}, d, [-,-])$ is a dg Lie algebra: the differential $\Delta = D_1 + hD_2$ defines a dg

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$^3$Strictly speaking, the use of $D_n$ to denote the two maps is abuse of notation, because they differ by powers of the double suspension operator $\mathfrak{g}[1] \to \mathfrak{g}[-1]$, but we prefer to keep it this way, because double suspension does not affect signs.
BV algebra structure on $S(\mathfrak{g}[-1])$. The operator $\Delta$ is essentially the homological Chevalley-Eilenberg differential:

$$\Delta(x_1 \ldots x_m) = \sum_{i=1}^{m} (-1)^{|x_1| \cdots |x_{i-1}|} x_1 \ldots dx_i \ldots x_m$$

$$+ \hbar \sum_{1 \leq i < j \leq m} (-1)^{|x_{\sigma(i,j)}|} |x_i|x_j x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_m,$$

where $\sigma(i,j)$ is the corresponding shuffle, the $x_i$'s in $\mathfrak{g}$ are treated as elements of $\mathfrak{g}[-1]$, and, following standard conventions, $d = l_1$ and $[x_i, x_j] = (-1)^{|x_i|} l_2(x_i, x_j)$.

**Remark.** An $A_\infty$-analogue of the above construction has been proposed by J. Terilla, T. Tradler, and S. Wilson in [TTW11]: for an $A_\infty$-algebra $V$, the tensor algebra $T(V[-1])$ (equipped with the shuffle product) is provided with a BV$_\infty$-structure. There is also an interesting generalization to the $\infty$-version of a Lie-Rinehart pair considered by L. Vitagliano [Vit13].

**Remark.** We will also need a certain $\hbar$-enhancement of the construction of a BV$_\infty$-algebra from an $L_\infty$-algebra. Suppose the graded $k[[\hbar]]$-module $\mathfrak{g}[[\hbar]]$ for a graded vector space $\mathfrak{g}$ is provided with the structure of a topological $L_\infty$-algebra over $k[[\hbar]]$ with respect to $\hbar$-adic topology. Then the same formula (2) defines a BV$_\infty$-structure on $S(\mathfrak{g}[-1])$ over $k$. There is a subtlety, though: each operator $D_n$ is a formal power series in $\hbar$ now, and in the $\hbar$-expansion

$$\Delta = \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \Delta_n,$$

there are contributions to $\Delta_n$ from $D_1, D_2, \ldots, D_n$. This still guarantees that $\Delta_n$ is a differential operator of order at most $n$ on $S(\mathfrak{g}[-1])$ satisfying the conditions of Definition 1.2.

To summarize, given an $L_\infty$-algebra $\mathfrak{g}$, we obtain a canonical BV$_\infty$-algebra structure on $S(\mathfrak{g}[-1])$. There is also a construction going in the opposite direction.

### 3. FROM HOMOTOPY BV ALGEBRAS TO HOMOTOPY LIE ALGEBRAS

Suppose we have a BV$_\infty$-algebra $V$. Then for each $n \geq 1$, the following higher derived brackets

$$l^h_n(a_1, \ldots, a_n) := \lim_{\hbar \to 0} \frac{1}{\hbar^{n-1}} l_n(a_1, \ldots, a_n)$$

on $V[[\hbar]]$, their $\hbar$-modification

$$L_n := \frac{1}{\hbar^{n-1}} l_n,$$

and their “semiclassical limit”

$$l_n(a_1, \ldots, a_n) := \lim_{\hbar \to 0} L_n(a_1, \ldots, a_n)$$

on $V[[\hbar]]$. This still guarantees that $L_n$ is a differential operator of order at most $n$ on $S(\mathfrak{g}[-1])$ satisfying the conditions of Definition 1.2.
on $V$ turn out to be $L_\infty$ brackets, according to the results in this section below. Just for comparison, note that $l^h_1 = L_1 = \Delta$, whereas $l_1 = \Delta_1$. Observe also that we have a linear (or strict) $L_\infty$-morphism

$$(V[[\hbar]][-1], l^h_1) \to (V[[\hbar]][1], L_n),$$

$v \mapsto \hbar v,$

which becomes an $L_\infty$-isomorphism after localization in $\hbar$. Thus, we can think of the $L_\infty$ structure given by the brackets $L_n$ as an $\hbar$-translation of the $L_\infty$ structure given by $l^h_n$.

One can express $\Delta$ through $l^h_n$ via the following useful formula

$$(\Delta(a_1 \ldots a_n) = \sum_{j=1}^n \sum_{\sigma \in S_{j,n-j}} (-1)^{|a_\sigma|} l^h_j(a_{\sigma(1)} \ldots ,a_{\sigma(j)})a_{\sigma(j+1)} \ldots a_{\sigma(n)}$$

for $a_1, \ldots ,a_n \in V$, which is easy to prove by induction on $n$ using Equation (7) below, starting with $n = 1$ for $l^h_1 = \Delta$.

**Theorem 3.1** (Bering-Damgaard-Alfaro). For a BV$_\infty$-algebra $V$, the higher brackets $l^h_n$, $n \geq 1$, defined by (8) endow the suspension $V[[\hbar]][-1]$ with the structure of an $L_\infty$-algebra over $k[[\hbar]]$. Moreover, the bracket $l^h_{n+1}$ measures the deviation of $l^h_n$ from being a coderivation with respect to multiplication.

Remark. This result was first observed by the physicists [BDA96] and proven in a more general context by T. Voronov [Vor05a, Vor05b]. The $L_\infty$ structure was also rediscovered by O. Kravchenko in [Kra00].

**Proof.** Using the Jacobi identity for the commutator of linear operators along with the fact that $L_a$ and $L_b$ (graded) commute, it is easy to check that the higher brackets $l^h_n$ are symmetric on $V[[\hbar]]$:

$$l^h_n(a_{\pi(1)} \ldots ,a_{\pi(n)}) = (-1)^{|a_\pi|} l^h_n(a_1 \ldots ,a_n)$$

for all $a_1, \ldots ,a_n \in V[[\hbar]]$, where $(-1)^{|a_\pi|}$ is the Koszul sign, see Section 2. Since $|\Delta| = 1$, the degree of $l^h_n$ as a bracket on $V[[\hbar]]$ will be the same. We can extend the $k[[\hbar]]$-linear operators $l^h_n : S^n(V)[[\hbar]] \to V[[\hbar]]$ to coderivations $D_n : S(V)[[\hbar]] \to S(V)[[\hbar]]$ and consider the total coderivation

$$D = D_1 + D_2 + \ldots .$$

on $S(V)[[\hbar]]$. The differential property $D^2 = 0$ for this coderivation is equivalent to the series of higher Jacobi identities:

$$\sum_{n=1}^m \sum_{\sigma \in S_{h,n-m}} (-1)^{|a_\sigma|} l^h_{m-n+1}(l^h_n(a_{\sigma(1)} \ldots ,a_{\sigma(n)}), a_{\sigma(n+1)} \ldots a_{\sigma(m)}) = 0$$

for all $a_1, \ldots ,a_m \in V[[\hbar]]$, $m \geq 1$. The physicists [BDA96] and T. Voronov [Vor05a, Vor05b] in a more general situation checked these identities using the following key observation for an arbitrary odd operator $\Delta$ on $V[[\hbar]]$, not necessarily squaring to zero:

$$\sum_{n=1}^m \sum_{\sigma \in S_{h,n-m}} (-1)^{|a_\sigma|} l^h_{m-n+1}(l^h_n(a_{\sigma(1)} \ldots ,a_{\sigma(n)}), a_{\sigma(n+1)} \ldots a_{\sigma(m)})$$

$$= [[\ldots [\Delta^2, L_{a_1}], \ldots , L_{a_m}], 1].$$

Given that $\Delta^2 = 0$, the higher Jacobi identities follow.
The deviated multiderivation property, more precisely,

\[ l_n^h(a_1, \ldots, a_i, a_{i+1}, \ldots, a_{n+1}) = l_n(a_1, \ldots, a_i \cdot a_{i+1}, \ldots, a_{n+1}) \]
\[ - (−1)^{(1+|a_1|+\cdots+|a_{i−1}|)|a_i|} l_n^h(a_1, \ldots, a_{i−1}, a_{i+1}, \ldots, a_{n+1}) \]
\[ - (−1)^{(1+|a_1|+\cdots+|a_{i−1}|)|a_{i+1}|} a_i l_n^h(a_{i+1}, \ldots, a_{i+2}, \ldots, a_{n+1}) \]

of the higher brackets may be derived from the identity

\[ [Q, L_{ab}] = [[Q, L_a], L_b] + (−1)^{|Q|+|a|} L_a [Q, L_b] + (−1)^{(1|Q|+|a|)|b|} L_b [Q, L_a] \]

for an arbitrary (homogeneous) linear operator \( Q \) on \( V[[\hbar]] \). Applying this to \( Q = [[\ldots[\Delta, L_{a_1}], \ldots], L_{a_n}] \), we see that \( l_n^h \) measures the deviation of \( l_n^h \) from being a derivation in each variable. Since the higher brackets are symmetric, we obtain the same property in each variable.

**Corollary 3.2** (T. Voronov [Vor05b]). Given a BV\(_\infty\)-algebra \( V \), the brackets \( L_n \), \( n \geq 1 \), defined by (1) endow the graded \( k[[\hbar]] \)-module \( V[[\hbar]]1 \) with the structure of an \( L_{\infty} \)-algebra over \( k[[\hbar]] \). Likewise, the brackets \( l_n \), \( n \geq 1 \), defined by (5) endow the graded vector space \( V[1] \) with the structure of an \( L_{\infty} \)-algebra over \( k \). Moreover, the brackets \( l_n \) are multiderivations of the graded commutative algebra structure.

**Proof.** The statements of the corollary are obtained as the “semiclassical limit” of the statements of Theorem 3.1 and so is the proof. Note the change of suspension from the theorem to the corollary. This corresponds intuitively to the statement that the semiclassical limit of the space \( V[[\hbar]]1 \) is \( \hbar V[−1] = V[1] \). Concretely, the desuspension guarantees that the degree of the \( n \)th higher bracket \( l_n \) on \( V[2] \) is still one: indeed \( |\Delta_n| = 3 − 2n \), when \( \Delta_n \) is considered as an operator on \( V \); therefore, the degree of \( l_n \) as a multilinear operation on \( V[2] \) will be \( 3 − 2n + 2(n − 1) = 1 \).

The multiderivation property is obtained by dividing (7) by \( \hbar^{n−1} \) and noticing that the left-hand side will not survive the limit as \( \hbar \to 0 \), because it has \( \hbar \) as a factor.

**Remark.** The algebraic structure which combines the graded commutative multiplicity with the \( L_{\infty} \) structure given by the brackets \( l_n \) is a particular case of the \( G_{\infty} \)-algebra structure, see [GJ94, Tam08, Tam99, Vor00].

**Remark.** The construction given by the higher brackets \( l_n \) obviously induces an operad morphism \( sL_{\infty} \to BV_{\infty} \), where \( s \) denotes the operadic suspension, see, e.g., [MSS02]. Here \( BV_{\infty} \) stands for the operad describing commutative BV\(_\infty\)-algebras, as opposed to the full BV\(_\infty\) operad of [TT00, GCTV12, DCV13]. This operad morphism immediately gives a functor from the category of BV\(_\infty\)-algebras to that of \( L_{\infty} \)-algebras, provided we restrict ourselves to morphisms of algebras over operads, i.e., consider only linear (strict) morphisms. However, we will focus on nonlinear morphisms in the subsequent sections of the paper.

**Example 3.3.** The \( L_{\infty} \) structure given by the brackets \( l_n \) coming from the BV\(_\infty\) structure of Example 1.3 is known as the de Rham-Koszul \( L_{\infty} \) structure and generalizes the Koszul brackets on the de Rham complex of a manifold, [KV08, BL13, VIL].

**Example 3.4.** Let \( g \) be an \( L_{\infty} \)-algebra. Then by the construction of Section 2 we get the structure of a BV\(_\infty\)-algebra on \( S(g[−1]) \). If we apply the “semiclassical”
derived brackets \( l_n \) of this section to the BV\(_\infty\)-algebra \( S(\mathfrak{g}[-1]) \), we will get the structure of an \( L_\infty \)-algebra on \( S(\mathfrak{g}[-1])[1] \). Later we show in Theorem 3.6(3) that the \( L_\infty \)-algebra \( \mathfrak{g} \) becomes an \( L_\infty \)-subalgebra of \( S(\mathfrak{g}[-1])[1] \). The higher brackets on \( S(\mathfrak{g}[-1])[1] \) may be viewed as extensions of the higher brackets on \( \mathfrak{g} \) as multiderivations. These higher brackets generalize the Schouten bracket on the exterior algebra of a Lie algebra to the \( L_\infty \) case.

Our goal is to characterize those BV\(_\infty\)-algebras which come from \( L_\infty \)-algebras as in Section 2. Note that such a BV\(_\infty\)-algebra is free as a graded commutative algebra by construction: \( V = S(U) \), and that for each \( n \geq 1 \), the \( n \)th component \( \Delta_n \) of the BV\(_\infty\) operator maps \( S^m(U) \) to \( S^{m-n+1}(U) \) for \( m \geq n \) and to 0 for \( 0 \leq m < n \), because of Equation (1). Since an \( n \)th-order differential operator on a free algebra \( S(U) \) is determined by its restriction to \( S^{\leq n}(U) \), this condition on \( \Delta_n \) is equivalent to the condition that \( \Delta_n \) maps \( S^n(U) \) to \( U \) and \( S^{<n}(U) \) to 0. Interpreting differential operators on \( S(U) \) as linear combinations of partial derivatives with polynomial coefficients, differential operators of the above type may also be characterized as differential operators of order \( n \) with linear coefficients.

**Definition 3.5.** A pure BV\(_\infty\)-algebra is the free graded commutative algebra \( S(U) \) on a graded vector space \( U \) with a BV\(_\infty\) operator \( \Delta : S(U) \to S(U)[[h]] \) such that, for each \( n \geq 1 \), \( \Delta_n \) maps \( S^n(U) \) to \( U \) and \( S^{<n}(U) \) to 0.

The following theorem (Parts (1) and (2)) shows that freeness and purity are not only necessary but also sufficient conditions for a BV\(_\infty\)-algebra to arise from an \( L_\infty \)-algebra.

**Theorem 3.6.**

1. Given a pure BV\(_\infty\) algebra \( (V = S(U), \Delta) \), the restriction of the brackets \( l_n \) to \( U[1] \subset S(U)[1] \) provides \( U[1] \) with the structure of an \( L_\infty \)-subalgebra.
2. The original pure BV\(_\infty\) structure on \( S(U) \) coincides with the BV\(_\infty\) structure of Section 2 coming from the derived \( L_\infty \) structure on \( U[1] \).
3. If we start with an \( L_\infty \) structure on a graded vector space \( U[1] \) and construct the BV\(_\infty\)-algebra \( S(U) \) as in Section 2 then the derived brackets \( l_n \) on \( U[1] \subset S(U)[1] \) return the original \( L_\infty \) structure on \( U[1] \).

**Proof.** The first statement we need to check is that \( l_n(a_1, \ldots, a_n) \) is in \( U \) whenever \( a_1, \ldots, a_n \in U \) and \( n \geq 1 \), as a priori all we know is that \( l_n(a_1, \ldots, a_n) \in S(U) \). The condition that \( \Delta_n \) maps \( S^m(U) \) to 0 for \( 0 \leq m < n \) implies by (1) that \( l_n(a_1, \ldots, a_n) = \Delta_n(a_1 \ldots a_n) \), which must be in \( U \), because of the condition \( \Delta_n : S^n(U) \to S^1(U) = U \).

For the second statement, we need to check that the \( n \)th-order differential operator \( \Delta_n \), the \( n \)th component of the given BV\(_\infty\) structure, is equal to the coderivation \( D_n \) defined by (1). Recall that on the free algebra \( S(U) \), an \( n \)th-order differential operator is determined by its restriction to \( S^{\leq n}(U) \). Given the assumption that \( \Delta_n \) vanishes on \( S^{<n}(U) \), it follows that \( \Delta_n \) on \( S(U) \) is determined by its restriction to \( S^n(U) \). By the previous paragraph, its restriction to \( S^n(U) \) is equal to \( l_n \). On the other hand, this is also the restriction of the coderivation \( D_n \) to \( S^n(U) \), as per formula (1). Lemma 2.2 shows that the coderivation \( D_n \) is also an \( n \)th-order differential operator. Thus, it is also determined by its restriction to \( S^n(U) \).

Finally, let \( \bar{l}_n \) be the \( L_\infty \) brackets on an \( L_\infty \)-algebra \( U[1] \) and \( \bar{l}_n \) be the higher derived brackets produced on the pure BV\(_\infty\)-algebra \( S(U) \) by formula (1) for \( n = 1 \),
2, . . . We claim that \( \tilde{l}_n(x_1, \ldots, x_n) = l_n(x_1, \ldots, x_n) \) for all \( n \) and \( x_1, \ldots, x_n \in U \). Indeed, 

\[
\tilde{l}_n(x_1, \ldots, x_n) = [[\ldots [D_n, L_{x_1}], \ldots], L_{x_n}],
\]

where \( D_n \) is the extension of \( l_n \) to \( S(U) \) as a coderivation, see Equation (1). The same equation implies that \( D_n : S(U) \rightarrow S(U) \) is zero on \( S^\infty(U) \). Hence all but one term \( (D_n \circ L_{x_1} \circ \cdots \circ L_{x_n})(1) \) of this iterated commutator vanish. It remains to observe that by (1) this is nothing but \( l_n(x_1, \ldots, x_n) \).

\[\square\]

**Remark.** A general, not necessarily pure \( BV_\infty \)-structure on \( S(U) \) leads to an interesting algebraic structure on \( U[1] \), called an involutive \( L_\infty \)-bialgebra, see [CFL13]. From the properadic, rather than BV prospective, this structure is described in [Val07] and [DCTT10]. The BV formalism for ordinary \( L_\infty \)-bialgebras seems to be subtler: apparently, one needs to weaken the definition of a \( BV_\infty \) structure on \( S(U) \) by requiring that for each \( n \geq 1 \), the coefficient by \( \hbar^{n-1} \) in the expansion of \( \Delta^2 = \frac{1}{2}[\Delta, \Delta] \) be of order \( \leq n-1 \), rather than the expected order \( \leq n \), instead of asking for vanishing of each coefficient of \( \Delta^2 \) in its expansion in \( \hbar \).

### 4. Functoriality

The correspondence between \( BV_\infty \)-algebras and \( L_\infty \)-algebras that we studied above has remarkable functorial properties with a suitable notion of a morphism between \( BV_\infty \)-algebras.

First of all, recall the definition of a morphism between \( L_\infty \)-algebras.

**Definition 4.1.** An \( L_\infty \)-morphism \( \mathfrak{g} \rightarrow \mathfrak{g}' \) between \( L_\infty \)-algebras is a morphism \( S(\mathfrak{g}[1]) \rightarrow S(\mathfrak{g}'[1]) \) of codifferential graded coalgebras, i.e., a morphism of graded coalgebras commuting with the structure codifferentials, such that \( 1 \in S^0(\mathfrak{g}[1]) \) maps to \( 1 \in S^0(\mathfrak{g}'[1]) \).

**Remark.** Since we deal with counital coalgebras, we assume that \( L_\infty \)-morphisms respect the counits. The extra condition \( 1 \mapsto 1 \) means that we are talking about “pointed” morphisms, if we invoke the interpretation of \( L_\infty \)-morphisms as morphisms between formal pointed dg manifolds, see [KS06].

Now we will consider the corresponding notion of a morphism between \( BV_\infty \)-algebras. We will only need this notion for \( BV_\infty \)-algebras of Theorem 3.6, that is to say, \( BV_\infty \)-algebras which are pure. Somewhat more generally, we will give a definition in the case when the source \( BV_\infty \)-algebra is just free. A more general notion of a \( BV_\infty \)-morphism for more general \( BV_\infty \)-algebras can be found in [LT00]. We use the definition of a \( BV_\infty \)-morphism by Cieliebak-Latchev [CL07].

Before giving the definition, we need to recall a few more notions. Fix a morphism \( f : A \rightarrow A' \) between graded commutative algebras. We say that a \( k \)-linear map \( D : A \rightarrow A' \) is a differential operator of order \( \leq n \) over \( f : A \rightarrow A' \) or simply a relative differential operator of order \( \leq n \) if for any \( n+1 \) elements \( a_0, \ldots, a_n \in A \), we have

\[
[[\ldots [D, L_{a_0}], \ldots], L_{a_n}] = 0,
\]

where \([D, L_a]\) is understood as the map \( A \rightarrow A' \) defined by

\[
[D, L_a](b) := D(ab) - (-1)^{|a||D|}f(a)D(b).
\]

For \( f = id \) we recover the standard definition Def. 1.1 of a differential operator on a graded commutative algebra.
Let $V = S(U)$ be a free graded commutative algebra and $V'$ an arbitrary graded commutative algebra. Given a $k$-linear map $\varphi : S(U) \to V'[[h]]$ of degree zero such that $\varphi(1) = 0$, define a degree-zero, continuous $k[[h]]$-linear map $\exp(\varphi) : S(U)[[h]] \to V'[[h]]$, called the exponential, by
\[
\exp(\varphi)(x_1 \ldots x_m) := \sum_{\substack{k=1 \\text{or} \\kappa}} \frac{1}{k!} \sum_{i_1, \ldots, i_k = 1}^{m} \frac{1}{i_1! \ldots i_k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \varphi(x_{\sigma(1)} \ldots x_{\sigma(i_1)}) \ldots \varphi(x_{\sigma(m-i_k+1)} \ldots x_{\sigma(m)}),
\]
where $S_m$ denotes the symmetric group, $x_1, \ldots, x_m$ are in $U$, and $(-1)^{|x_\sigma|}$ is the Koszul sign of the permutation of $x_1 \ldots x_m$ to $x_{\sigma(1)} \ldots x_{\sigma(m)}$ in $S(U)$. By convention, we set $\exp(\varphi)(1) := 1$. The reason for the exponential notation, introduced by Cieliebak and Latschev [CL07], is, perhaps, the following statement, which they might have been aware of, cf. [CFL13]. The proof is a straightforward computation.

**Lemma 4.2.** If $S \in \lambda U[[h]]^0[[\lambda]]$ or $\lambda U((h))^0[[\lambda]]$, where $\lambda$ is another, degree-zero formal variable, then
\[
\exp(\varphi)(e^S) = e^{\varphi(e^S)}.
\]
Here we have extended $\varphi$ and $\exp(\varphi)$ to $\lambda S(U)((h))[[\lambda]]$ by $h^{-1}$- and $\lambda$-linearity and continuity.

**Remark.** The extra formal variable $\lambda$ in the lemma guarantees “convergence” of the exponential $e^S$. We could have achieved the same goal, if we considered completions of our algebras or assumed that $\lambda$ was a nilpotent variable, varying over the maximal ideals of finite-dimensional local Artin algebras. Informally speaking, given the way the space $S(U)[\lambda, h, h^{-1}]$ of $S(U)$-valued polynomials in $\lambda$ and Laurent polynomials in $h$ is completed: $S(U)((h))[[\lambda]]$, we could think of $\lambda$ as being “much smaller” than $h$.

The exponential, not surprisingly, has an inverse, called the logarithm, which we will use a little later. Given a $k$-linear map $\Phi : S(U) \to V'[[h]]$ of degree zero such that $\Phi(1) = 1$, define a degree-zero, continuous $k[[h]]$-linear map $\log(\Phi) : S(U)[[h]] \to V'[[h]]$ by
\[
\log(\Phi)(x_1 \ldots x_m) := \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{\substack{i_1, \ldots, i_k = 1 \\text{or} \\kappa}}^{m} \frac{1}{i_1! \ldots i_k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \Phi(x_{\sigma(1)} \ldots x_{\sigma(i_1)}) \ldots \Phi(x_{\sigma(m-i_k+1)} \ldots x_{\sigma(m)})
\]
under the same notation as for the exponential. By convention, we set $\log(\Phi)(1) := 0$.

**Definition 4.3** (Cieliebak-Latchev [CL07]). A **$BV_{\infty}$-morphism** from a $BV_{\infty}$-algebra $(V = S(U), \Delta)$ to a $BV_{\infty}$-algebra $(V', \Delta')$ is a $k$-linear map $\varphi : V \to V'[[h]]$ of degree zero satisfying the following properties:

1. $\varphi(1) = 0$,
2. $\exp(\varphi) \Delta = \Delta' \exp(\varphi)$, and
(3) $\varphi$ admits an expansion

$$
\varphi = \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \varphi_n,
$$

where $\varphi_n : V \to V'$ is a differential operator of order $\leq n$ over the morphism $S(U) \to V'$ induced by the zero linear map $U \overset{0}{\to} V'$, i.e., $\varphi_n$ maps $S^{>n}(U)$ to 0.

We will use the same notation for the continuous, $k[[\hbar]]$-linear extension $\varphi : V[[\hbar]] \to V'[[[\hbar]]]$ of the $k$-linear map $\varphi : V \to V'[[[\hbar]]]$, as well as for the corresponding BV$_\infty$-morphism $\varphi : (V, \Delta) \to (V', \Delta')$.

Remark. A BV$_\infty$-morphism can be regarded as a quantization of a morphism of dg commutative algebras. Indeed, $\varphi_1$ must be nonzero only on $U = S^1(U) \subset S(U)$ and by construction $\exp(\varphi_1)$ will be a graded algebra morphism. The equation $\exp(\varphi)\Delta = \Delta' \exp(\varphi)$ at $\hbar = 0$ reduces to $\exp(\varphi_1)\Delta_1 = \Delta'_1 \exp(\varphi_1)$, which implies that $\exp(\varphi_1)$ is a morphism of dg algebras with respect to the “classical limits” $\Delta_1$ and $\Delta'_1$ of the BV$_\infty$ operators.

Example 4.4. A nice example of a BV$_\infty$-morphism $S(V) \to V$ may be obtained from the projection $p_1 : S(V) \to V$ of the symmetric algebra $S(V)$ to its linear component $V = S^1(V)$ for any BV$_\infty$-algebra $V$. Before talking about morphisms, we need to provide $S(V)$ with the structure of a BV$_\infty$-algebra. To do that, we take the $L_\infty$ structure on $V[[\hbar]][1]$ over $k[[\hbar]]$ given by the brackets $L_n$, see (4), and then the BV$_\infty$ structure on $S(V)$ from the remark at the end of Section 2. To regard $p_1$ as a BV$_\infty$-morphism, we compose it with the inclusion $V \subset V[[\hbar]]$ and get a $k$-linear map $\varphi = \varphi_1 : S(V) \to V[[\hbar]]$. By construction, $\varphi(1) = 0$. One can easily check that $\exp(\varphi) = m$, the multiplication operator $S(V) \to V$. To see that $\exp(\varphi)$ commutes with the BV$_\infty$ operators, we observe that, for $a_1, \ldots, a_n \in V$, the value of the BV$_\infty$ operator coming from the brackets $L_j$ on the product $a_1 \otimes \cdots \otimes a_n \in S(V)$ is equal to

$$
\sum_{j=1}^{n} \hbar^{j-1} \sum_{\sigma \in S_{h-j,n-j}} (-1)^{|\sigma|} L_j(a_{\sigma(1)}, \ldots, a_{\sigma(j)}) \otimes a_{\sigma(j+1)} \otimes \cdots \otimes a_{\sigma(n)},
$$

because of Equations (11) and (12). When we apply $m$ to that, the tensor product (multiplication in $S(V)$) will change to multiplication in $V$. The result will just be equal to $(\Delta m)(a_1 \otimes \cdots \otimes a_n)$ in view of Equation (13).

Another feature of a BV$_\infty$-morphism $\varphi : S(U)[[\hbar]] \to V'[[[\hbar]]]$ is that it propagates solutions $S \in \mathcal{M}U((\hbar))^2[[\lambda]]$ of the Quantum Master Equation (QME)

(8)

$$
\Delta e^{S/\hbar} = 0
$$

to solutions of the QME in $\lambda V'(((\hbar))^2[[\lambda]]$.

Proposition 4.5. If $\varphi : S(U) \to V'$ is a BV$_\infty$-morphism and $S \in \lambda U((\hbar))^2[[\lambda]]$ is a solution of the QME (8), then

$$
S' := \hbar \varphi(e^{S/\hbar}) \in \lambda V'(((\hbar))^2[[\lambda]]
$$

is a solution of the QME

$$
\Delta' e^{S'/\hbar} = 0.
$$
Proposition 4.7. The composition $\phi \circ \psi : S(U) \to S(U')$ between BV-infinity-algebras is a pure BV-infinity-morphism if and only if

$$\Delta' e^{\phi(e^{S/h})} = \Delta' \exp(\phi)(e^{S/h}) = \exp(\phi) \Delta e^{S/h} = 0.$$

Now we are ready to study functorial properties of the correspondence between $L_\infty$-algebras and BV-infinity-algebras from Theorem 3.13. Since the BV-infinity-algebra corresponding to an $L_\infty$-algebra is pure, we would like to concentrate on BV-infinity-morphisms between such BV-infinity-algebras. Among these BV-infinity-morphisms, those of the following type turn out to form an interesting category.

**Definition 4.6.** We will call a BV-infinity-morphism $\phi : S(U) \to S(U')$ between BV-infinity-algebras which are free as graded commutative algebras pure, if $\phi_n$ maps $S^n(U)$ to $U' \subseteq S(U')[[\hbar]]$ and all other symmetric powers $S^k(U)$ to 0. In other words, one can say that $\phi_n$ is a differential operator of order $n$ with linear coefficients over the morphism $S(U) \to S(U')$ induced by the zero map $U \xrightarrow{0} S(U')$.

**Proposition 4.7.** The composition $\psi \circ \phi$ of any pure BV-infinity-morphisms is a pure BV-infinity-morphism.

**Proof.** First of all, we need to see that the properties (1)-(3) of a BV-infinity-morphism are satisfied. Property (1) is satisfied because of our conventions on the values of exponentials and logarithms of maps at 1. Property (2) is obvious by construction. Property (3) may be established from the formula

$$\psi \circ \phi(x_1 \ldots x_m) = \frac{1}{m!} \sum_{i_1, \ldots, i_m=1} \frac{1}{1! \ldots k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \psi(\phi(x_{\sigma(1)} \ldots x_{\sigma(i_1)}) \ldots \phi(x_{\sigma(m-i_k+1)} \ldots x_{\sigma(m)})),$$

which is easily verified by exponentiating it and comparing it to $\exp(\psi) \circ \exp(\phi)$. Indeed, the coefficient $(\psi \circ \phi)_n(x_1 \ldots x_m)$ by $\hbar^{n-1}$ on the right-hand side will be coming from terms

$$\psi_j \phi_j(x_{\sigma(1)} \ldots x_{\sigma(i_1)}) \ldots \phi_{j_k}(x_{\sigma(m-i_k+1)} \ldots x_{\sigma(m)})$$

with $j - 1 + \sum_{p=1}^k (j_p - 1) = j - 1 + \sum_{p=1}^k j_p - k = n - 1$. Observe that because of Property (3) for $\psi$ and $\phi$, for such a term not to vanish, it is necessary that $j \geq k$ and $j_p \geq i_p$ for each $p$. Thus, we will have $n = j + \sum_{p=1}^k j_p - k \geq k + \sum_{p=1}^k t_p - k = m$, which is Property (3) for $\psi \circ \phi$. The fact that the composite BV-infinity-morphism is pure is obvious from Eq. (9) and purity of $\psi$. \qed
Theorem 4.8. The correspondence \( g \mapsto S(g[-1]) \) of Section 2 from \( L_\infty \)-algebras to \( BV_\infty \)-algebras is functorial. This functor establishes an equivalence between the category of \( L_\infty \)-algebras and the full subcategory of pure \( BV_\infty \)-algebras of the category of \( BV_\infty \)-algebras free as graded commutative algebras with pure morphisms. The functor \( V = S(U) \mapsto U[1] \) of Theorem 3.6.1 provides a weak inverse to this equivalence.

Restricting this theorem to the world of dg Lie algebras and dg BV algebras, we obtain the following corollaries, which, surprisingly, seem to be new.

Corollary 4.9. The functor \( g \mapsto S(g[-1]) \) from dg Lie algebras to dg BV algebras establishes an equivalence between the category of dg Lie algebras with \( L_\infty \)-morphisms and the category of dg BV algebras \((V, \Delta_1, \Delta_2)\), free as graded commutative algebras \( V = S(U) \) and whose BV structure is pure: \( \Delta_2 \) maps \( U \) to 0 and \( S^2(U) \) to \( U \), with \( BV_\infty \)-morphisms \( S(U) \rightarrow S(U') \) satisfying the purity condition.

Corollary 4.10. The functor \( g \mapsto S(g[-1]) \) from dg Lie algebras to dg BV algebras establishes an equivalence between the category of dg Lie algebras (with dg Lie morphisms) and the category of dg BV algebras \((V, \Delta_1, \Delta_2)\), free as graded commutative algebras \( V = S(U) \) and whose BV structure is pure: \( \Delta_2 \) maps \( U \) to 0 and \( S^2(U) \) to \( U \), with morphisms defined as morphisms \( \Phi : S(U) \rightarrow S(U') \) of graded algebras respecting the differentials \( \Delta_1 \) and \( \Delta_2 \) and satisfying the purity condition: \( \Phi \) maps \( U \) to \( U' \).

Now let us prove the theorem.

Proof. We need to see that an \( L_\infty \)-morphism \( g \rightarrow g' \) induces a \( BV_\infty \)-morphism \( S(g[-1]) \rightarrow S(g'[1]) \). By definition an \( L_\infty \)-morphism is graded coalgebra morphism \( \Phi : S(g[1]) \rightarrow S(g'[1]) \) respecting the codifferentials and such that \( \Phi(1) = 1 \). As a coalgebra morphism, \( \Phi \) is determined by its projection \( \varphi : S(g[1]) \rightarrow g'[1] \) to the cogenerators \( g'[1] \) via the following formula:

\[
\Phi(x_1 \ldots x_m) = \sum_{k=1}^m \frac{1}{k!} \sum_{\substack{i_1, \ldots, i_k=1 \\
i_1 + \ldots + i_k = m}} (-1)^{|x_{i_1}|} \varphi(x_{\sigma(1)} \ldots x_{\sigma(i_1)}) \ldots \varphi(x_{\sigma(m-i_k+1)} \ldots x_{\sigma(m)}),
\]

where \( Sh_{i_1, \ldots, i_k} \) denotes the set of \( (i_1, \ldots, i_k) \) shuffles, \( x_1, \ldots, x_m \) are in \( g[1] \), and \((-1)^{|x_{i_1}|} \) is the Koszul sign of the permutation of \( x_1 \cdots x_m \) to \( x_{\sigma(1)} \cdots x_{\sigma(m)} \) in \( S(g[1]) \). (For \( m = 0 \), we just have \( \Phi(1) = 1 \) and \( \varphi(1) = 0 \).) The above formula follows from iteration of the coalgebra morphism property:

\[
\delta^{k-1} \Phi = \Phi \circ k \delta^{k-1}
\]

along with its projection to \( (g'[1])^{\otimes k} \) for each \( k = 1, \ldots, m \). To turn \( \varphi \) into a \( BV_\infty \)-morphism, we need to rewrite it as a power series in \( \hbar \):

\[
\varphi_h := \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \varphi_n,
\]

where \( \varphi_n : S(g[-1]) \rightarrow S(g'[1]) \) maps all symmetric powers to 0, except for \( S^n(g[-1]) \), on which \( \varphi_n \) is the restriction of \( \varphi : S(g[1]) \rightarrow g'[1] \) to \( S^n(g[1]) \) along with an appropriate shift in degree to make it into a linear map \( S^n(g[-1]) \rightarrow g'[-1] \).

Note that the degree of \( \varphi \) was supposed to be zero, as it was a projection of the
morphism \( \Phi \) of graded coalgebras. In terms of grading on \( S^n(g[-1]) \) and \( g'[−1] \), the degree of shifted \( \varphi_n \) is \( 2 − 2n \). Multiplication by \( h^{n−1} \) shifts that degree back to 0, thus we see that the degree of \( \varphi_h \) is zero as well.

Note that by construction, the purity condition on \( \varphi_h \) is satisfied, and thereby we have

\[
\exp(\varphi_h)(x_1 \ldots x_m)
= \sum_{k=1}^{m} \frac{h^{m-k}}{k!} \sum_{i_1, \ldots, i_k=1}^{m} \frac{1}{i_1! \ldots i_k!} \sum_{\sigma \in S_m} \sum_{\sigma(i_1) \ldots \sigma(i_k)} (-1)^{\sigma(i_1)} \varphi(x_{\sigma(1)} \ldots x_{\sigma(i_k)}) \ldots
\]

whence, comparing this to the right-hand side of (10), we get

\[
\exp(\varphi_h) = \sum_{m=0}^{\infty} h^m \Phi_m,
\]

where \( \Phi_m \) is the component of \( \Phi \) of degree \(-m\) in the grading given by the symmetric power, so as

\[
\Phi = \sum_{m=0}^{\infty} \Phi_m.
\]

We know that \( \Phi \) is compatible with the structure codifferentials \( D \) and \( D' \) of \( g \) and \( g' \): \( \Phi D = D' \Phi \). The \( BV_\infty \) operator on \( S(g[-1]) \) was defined as \( \Delta = \sum_{m=1}^{\infty} h^{m-1} D_m \), where \( D_m \) maps each \( S^n(g[1]) \) to \( S^{n-m+1}(g[1]) \); likewise for \( S(g'[−1]) \), see (2). Thus, the equation \( \exp(\varphi_h) \Delta = \Delta' \exp(\varphi_h) \) is satisfied, being just a weighted sum of the components of the equation \( \Phi D = D' \Phi \), where the component shifting the symmetric power down by \( n \geq 0 \) is being multiplied by \( h^n \). This completes verification of the fact that \( \varphi_h \) is a pure \( BV_\infty \)-morphism.

Conversely, we need to see that every pure \( BV_\infty \)-morphism comes from an \( L_\infty \)-morphism. By Theorem 3.6 we can assume that the source and the target of this \( BV_\infty \)-morphism are the \( BV_\infty \)-algebras \( S(g[-1]) \) and \( S(g'[−1]) \) coming from some \( L_\infty \)-algebras \( g \) and \( g' \). Every \( BV_\infty \)-morphism is given by a formal \( h \)-series like (11) satisfying the three conditions of Definition 4.3. Since the morphism is pure, we can “drop” the \( h \) from \( \varphi_h \) and note that the formal series

\[
\varphi := \sum_{n=1}^{\infty} \varphi_n
\]

will produce a well-defined linear map \( S(g[1]) \to g'[1] \). Dropping the \( h \) results in this map also having degree zero. Now we can generate a unique morphism \( \Phi : S(g[1]) \to S(g'[1]) \) of coalgebras by the linear map \( \varphi \). This morphism \( \Phi \) will be given by formula (12). Since \( \varphi_h(1) = 0 \), we get \( \varphi(1) = 0 \) and \( \Phi(1) = 1 \) by the same formula. We just need to check that this morphism \( \Phi \) respects the codifferentials \( D \) and \( D' \) on these two coalgebras, respectively. As in the first part of the proof, we see that the equation \( \exp(\varphi_h) \Delta = \Delta' \exp(\varphi_h) \) implies \( \Phi D = D' \Phi \). Thus, \( \Phi \) is an \( L_\infty \)-morphism.

We also need to check the functoriality properties of the correspondence \( g \mapsto S(g[−1]) \). The fact that \( id_g \) maps to the identity morphism is obvious. Now, if we have two \( L_\infty \)-morphisms \( g \to g' \to g'' \) given by dg coalgebra morphisms \( S(g[1]) \xrightarrow{\Phi} S(g'[1]) \xrightarrow{\Psi} S(g''[1]) \) with \( \Phi = \sum_{m=0}^{\infty} \Phi_m \) and \( \Psi = \sum_{m=0}^{\infty} \Psi_m \), we note
that the exponentials \( \exp(\varphi_h) = \sum_{m=0}^{\infty} h^m \Phi_m \) and \( \exp(\psi_h) = \sum_{m=0}^{\infty} h^m \Psi_m \) of the respective BV\( \infty \)-morphisms will compose in the same way as \( \Phi \) and \( \Psi \), the only difference being that the component decreasing the symmetric power by \( m \) gets multiplied by \( h^m \).

\[ \square \]

5. Adjunction

In this section, we prove a certain “adjunction” property. The quotation marks are due to the fact that in our setting, arbitrary BV\( \infty \)-algebras do not even make up a category. However, the theorem below makes sense for arbitrary BV\( \infty \)-algebras and BV\( \infty \)-morphisms.

Recall that given an \( L_\infty \)-algebra \( g \), we have constructed a BV\( \infty \)-algebra \( S(g[-1]) \) in Section \( \ref{BVAlg} \). Conversely, given a BV\( \infty \)-algebra \( V \), we have used the higher derived brackets \( L_n \) to induce an \( L_\infty \)-structure on \( V[[h]][1] \) over \( k[[h]] \) as in Corollary \( \ref{CorBVAlg} \).

Note that both constructions are functorial. The fact that \( g \mapsto S(g[-1]) \) defines a functor is the first statement of Theorem \( \ref{thm1} \). We need to see that the construction assigning to a BV\( \infty \)-algebra \( V \) the \( L_\infty \)-algebra \((V[[h]][1], L_n)\) is also functorial. Given a BV\( \infty \)-morphism \( \varphi : V = S(U) \rightarrow V' \), we need to construct an \( L_\infty \)-morphism \( V[[h]][1] \rightarrow V'[[h]][1] \). This construction will be accomplished in two steps.

**Step 1.** Compose the BV\( \infty \)-morphism \( \varphi : V \rightarrow V' \) with the BV\( \infty \)-morphism \( p_1 : S(V) \rightarrow V \) of Example \( \ref{ex1} \) to get a BV\( \infty \)-morphism \( \varphi \circ p_1 : S(V) \rightarrow V' \).

**Step 2.** Given an \( L_\infty \)-algebra \( g[[h]] \) over \( k[[h]] \) and a BV\( \infty \)-morphism \( \psi : S(g[-1]) \rightarrow V' \), where \( S(g[-1]) \) is provided with the BV\( \infty \) structure of the remark at the end of Section \( \ref{BVAlg} \) we will construct a canonical \( L_\infty \)-morphism \( g[[h]] \rightarrow V'[[h]][1] \). Then we will just apply this construction to the BV\( \infty \)-morphism \( S(V) \rightarrow V' \) of Step 1.

In order to construct an \( L_\infty \)-morphism \( g[[h]] \rightarrow V'[[h]][1] \), take the graded \( k[[h]] \)-coalgebra morphism, continuous in the \( h \)-adic topology,

\[
F : S(g[1])[[h]] \rightarrow S(V'[2])[[h]]
\]

induced by the \( k[[h]] \)-linear map

\[
f : S(g[1])[[h]] \rightarrow V'[[h]][2]
\]

whose restriction \( f|_{S^k(g[1])[[h]]} : S^k(g[1])[[h]] \rightarrow V'[[h]][2] \) is the restriction of \( h^{1-k} \psi \) to \( S^k(g[1])[[h]] \) for \( k \geq 0 \):

\[
f|_{S^k(g[1])[[h]]} = h^{1-k} \psi|_{S^k(g[1])[[h]]}.
\]

This map takes values in \( V'[[h]][2] \), despite the division by a power of \( h \), because the restriction of \( \psi \) to \( S^k(g[-1]) \) is in fact equal to \( \sum_{n=k}^{\infty} h^{n-1} \psi_n = h^{k-1} \sum_{n=0}^{\infty} h^n \psi_{n+k} \). Note that since \( \psi \) is of degree zero, \( f \) will also have degree zero.

We need to check that \( F \) defines an \( L_\infty \)-morphism. It is easy to see that \( F(1) = 1 \), because \( \psi(1) = 0 \). What is far less trivial is the fact that \( F \) respects the codifferentials, the structure codifferential \( D \) on \( S(g[1])[[h]] \) and the codifferential \( D' \) on \( S(V'[2])[[h]] \) induced as a continuous coderivation, see \( \ref{BTCodiff} \), by the sum of the brackets \( \ref{BTCodiff} \):

\[
L_n : S^n(V'[2])[[h]] \rightarrow V'[[h]][2].
\]

What we know is \( \Delta' \exp(\psi) = \exp(\psi) \Delta \), where \( \Delta' \) is the BV\( \infty \) operator on \( V' \) and \( \Delta \) is the structure codifferential \( D \) on \( S(g[1])[[h]] \) enhanced by \( h \), as in the remark
at the end of Section 2. To see that this implies the equation $D'F = FD$, we need to develop some BV calculus and compare it to coalgebra calculus.

Let us start with coalgebra calculus. Each side of the equation is a continuous coderivation over the coalgebra morphism $F$ and as such determined by the projection $p_1 : S(V'[2])[\hbar] \to V'[\hbar][2]$ to the cogenerators $V'[\hbar][2]$ of the range. Thus, all we need to show is that $p_1 D'F = p_1 FD$, after projecting to the cogenerators. Now, for a monomial $x_1 \ldots x_m \in S^m(\mathfrak{g}[1])$, we have

$$p_1 D'F(x_1 \ldots x_m) = \sum_{k=1}^{m} \frac{1}{k!} \sum_{i_1, \ldots, i_k=1}^{\infty} \sum_{i_1 + \cdots + i_k = m} (-1)^{|x_{i_1}|} \ell_k f(x_{(i_1)} \ldots x_{(i_k)}), \ldots, f(x_{(m-i_k+1)} \ldots x_{(m)}),$$

using the shuffle notation, see Equation (10), as well as

$$p_1 FD(x_1 \ldots x_m) = f(D(x_1 \ldots x_m)).$$

We need to show that the right-hand sides of these equations are equal, based on the equation $\Delta' \exp(\psi) = \exp(\psi) \Delta$. We will do that after we develop some BV calculus.

Turning to BV calculus, we have

$$\Delta(x_1 \ldots x_m) = \sum_{k=1}^{m} h^{k-1} \sum_{\tau \in Sh_{k,m-k}} (-1)^{|x_{\tau(1)}|} \ell_k (x_{\tau(1)}, \ldots, x_{\tau(k)} x_{\tau(k+1)} \ldots x_{\tau(m)}),$$

where $\ell_k$'s are the $L_\infty$ brackets on $\mathfrak{g}$, because of Equation (11). Now apply $\exp(\psi)$ to both sides, reassemble products of $\psi$'s not containing $\ell_k$'s into $\exp(\psi)$, and use (11) again to pass from $\ell_k$'s back to $\Delta$ and get

$$\exp(\psi) \Delta(x_1 \ldots x_m) = \sum_{n=1}^{m} \sum_{\sigma \in Sh_{n,m-n}} (-1)^{|x_{\sigma}|} \psi(\Delta(x_{\sigma(1)} \ldots x_{\sigma(n)})) \exp(\psi)(x_{\sigma(n+1)} \ldots x_{\sigma(m)}).$$

Move on to computation of $\Delta' \exp(\psi)$:

$$\Delta' \exp(\psi)(x_1 \ldots x_m) = \sum_{n=1}^{m} \sum_{\sigma \in Sh_{n,m-n}} (-1)^{|x_{\sigma}|} \psi(\Delta(x_{\sigma(1)} \ldots x_{\sigma(n)})) \exp(\psi)(x_{\sigma(n+1)} \ldots x_{\sigma(m)}),$$

which follows from the definition of $\exp(\psi)$ and the identity (6).

Now let us compare (15) with (16), which are equal by assumption. One can show by induction on $m$ that the top, $n = m$ terms of the two formulas must also
be equal:
\[ \psi(\Delta(x_1 \ldots x_m)) = \sum_{k=1}^{m} \frac{1}{k!} \sum_{i_1, \ldots, i_k = 1}^{m} \sum_{\tau \in \text{Sh}_{i_1, \ldots, i_k}} (-1)^{|\tau|} l_k^h(\psi(x_{\tau(1)} \ldots x_{\tau(i_k)})), \ldots, \psi(x_{\tau(m-i_k+1)} \ldots x_{\tau(m)})]. \]

It remains to pass from \( \psi, \Delta, \) and \( l_k^h \) to \( f, D, \) and \( L_k \), respectively, in this equation, with appropriate powers of \( h \), resulting in the equation
\[ h^{m-1} f(D(x_1 \ldots x_m)) = \sum_{k=1}^{m} \frac{1}{k!} \sum_{i_1, \ldots, i_k = 1}^{m} \sum_{\tau \in \text{Sh}_{i_1, \ldots, i_k}} (-1)^{|\tau|} L_k(f(x_{\tau(1)} \ldots x_{\tau(i_k)})), \ldots, f(x_{\tau(m-i_k+1)} \ldots x_{\tau(m)})]. \]

In view of (12) and (13), we see that \( D'F = FD \). This completes Step 2.

**Theorem 5.1.** Suppose \( \mathfrak{g} \) is an \( L_{\infty} \)-algebra and \( V \) is a \( BV_{\infty} \)-algebra. There exists a canonical bijection
\[ \text{Hom}_{BV_{\infty}}(S(\mathfrak{g}[-1]), V) \cong \text{Hom}_{L_{\infty}}(\mathfrak{g}, V[[h]][1]), \]
where the \( L_{\infty} \)-structure on \( V[[h]][1] \) is given by the modified brackets \( L_n \). This bijection is natural in the \( L_{\infty} \)-algebra \( \mathfrak{g} \) and in the \( BV_{\infty} \)-algebra \( V \).

**Proof.** A correspondence from the \( BV_{\infty} \)-morphisms on the left-hand side to the \( L_{\infty} \)-morphisms on the right-hand side was constructed in Step 2 before the theorem in a more general case of an \( L_{\infty} \)-algebra over \( k[[h]] \).

Conversely, given an \( L_{\infty} \)-morphism \( F : S(\mathfrak{g}[1]) \to S(V[2])[[h]] \), we use the same conversion formula
\[ \varphi|_{S(k(\mathfrak{g}[1])[[h]])} = h^{k-1} f|_{S(k(\mathfrak{g}[1])[[h]])}, \]
\( f \) being the projection of \( F \) to the cogenerators \( V[2][[h]] \), for \( k \geq 0 \), as in Step 2 before the theorem, to get a \( BV_{\infty} \)-morphism \( \varphi : S(\mathfrak{g}[-1]) \to V \). Tracing the argument there backward, we see that \( \varphi \) is indeed a \( BV_{\infty} \)-morphism. This establishes a bijection in the adjunction formula.

The naturality of the construction follows from the fact that, in view of (17), \( F \) and \( \exp(\varphi) \) are given by almost identical formulas, with the only difference coming from insertion of powers of \( h \), which plays the role of grading shift. \( \square \)

**Corollary 5.2.** The functor \( \mathfrak{g} \mapsto S(\mathfrak{g}[-1]) \) of Sections 2 and 3 from the category of \( L_{\infty} \)-algebras to the category of \( BV_{\infty} \)-algebras free as graded commutative algebras with pure morphisms has a right adjoint, which is given by the functor of modified higher derived brackets \( L_n \).

**Remark.** This corollary generalizes the construction of a pair of adjoint functors by Beilinson and Drinfeld [BD04, 4.1.8] from the case of dg Lie and BV algebras to the case of \( L_{\infty} \)- and \( BV_{\infty} \)-algebras.
Remark. The results of this section extend easily to the case when an $L_\infty$-algebra $g$ is replaced with a topological $L_\infty$-algebra $\mathfrak{g}[[\hbar]]$ over $k[[\hbar]]$ and we use the BV$_\infty$ structure on $S(\mathfrak{g}[-1])$ described in the remark at the end of Section 2. In particular, there is a natural bijection
\[ \text{Hom}_{\text{BV}_\infty}(S(\mathfrak{g}[-1]), V) \cong \text{Hom}_{L_\infty}(\mathfrak{g}[[\hbar]], V[[\hbar]][1]), \]
where on the right-hand side, we consider continuous $L_\infty$-morphisms over $k[[\hbar]]$.

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*E-mail address*: bashk003@umn.edu

**School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA**

*E-mail address*: voronov@umn.edu

**School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA, and Kavli IPMU (WPI), University of Tokyo, Kashiwa, Japan**