VISUALIZE GEOMETRIC SERIES

HÙNG VIỆT CHU

ABSTRACT. We review Mabry’s, Edgar’s, and the Viewpoints 2000 Group’s proofs without words for the geometric series formula. Mabry and Edgar proved without words that
\[
\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{3}{4} \quad \text{and} \quad \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \cdots = \frac{4}{5},
\]
respectively. We show that their proofs satisfy certain requirements that make them unique. We then illustrate a common idea between their and the Viewpoints 2000 Group’s proofs.

1. INTRODUCTION

Probably all calculus students have encountered the following formula to compute the geometric series \( a + ar + ar^2 + ar^3 + \cdots \) when \(|r| < 1\):
\[
a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}. \tag{1.1}
\]
It is one of a few precious tools to find the exact sum of a convergent series. Other well-known tools include telescoping series and the Weierstrass factorization theorem, used by Euler in computing \(\sum_{n=1}^{\infty} \frac{1}{n^2}\). It is worth emphasizing that (1.1) computes all geometric series as long as the common ratio has a magnitude less than 1, but its usefulness does not stop there. For example, differentiating both sides of (1.1) with respect to \(r\) gives another formula:
\[
\sum_{n=1}^{\infty} anr^{n-1} = a + 2ar + 3ar^2 + \cdots = \frac{a}{(1-r)^2}, \text{ for } |r| < 1.
\]
While (1.1) is simply the Maclaurin series for \(a(1 - r)^{-1}\), a more elegant proof is to find a closed-form formula for the partial sums \(s_n\) of the geometric series and then let \(n\) go to infinity. In particular, observe that for \(n \geq 1\),
\[
(1-r)(a + ar + ar^2 + \cdots + ar^{n-1}) = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^n) = a - ar^n.
\]
Hence,
\[
s_n := a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}.
\]
Letting \(n \to \infty\) and noting that \(\lim_{n \to \infty} r^n = 0\) because \(|r| < 1\), we obtain the desired formula. Though the proof is elegant and explanatory, there is still the need for more...
intuition behind the formula. As a result, many clever proofs without words have been devised.

In 1999, Mabry \cite{2} provided Figure 1 as a visual proof of

\[ \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}. \]  

(1.2)

\begin{figure}[h]
  \centering
  \includegraphics[width=0.4\textwidth]{fig1}
  \caption{Mabry’s proof.}
\end{figure}

Though Mabry’s proof is only for the case \( a = r = 1/4 \), it provides a nice intuition. If the area of the outermost triangle is 1, then since the shaded triangles cover a third of each layer, their total area is \( 1/3 \). Another way to compute the total shaded area is by summing up individual triangles: \( 1/4 + (1/4)^2 + (1/4)^3 + \cdots \). The two ways must give the same answer, so we have (1.2).

With a similar idea, Edgar \cite{1} used Figure 2 to prove

\[ \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \cdots = \frac{4}{5}, \]

which is (1.1) with \( a = r = 4/9 \).

\begin{figure}[h]
  \centering
  \includegraphics[width=0.4\textwidth]{fig2}
  \caption{Edgar’s proof.}
\end{figure}

Edgar asked, “Is it possible to determine which other series allow analogous proof without words?” The answer is, “It depends on what we mean by “analogous” proofs!”. If the question asked for a series with \( r = a \) as in Mabry’s and Edgar’s proofs, then we shall see that Figures 1 and 2 are unique of their kinds; if \( r \) is allowed to be different from \( a \), then there are many others.
We conclude this section with the beautiful proof without words by the Viewpoints 2000 Group [3]. Figure 3 proves (1.1) for all $r \in (0, 1)$ rather than for special cases as Figures 1 and 2; however, we shall show that all of them share a common proof idea.

![Figure 3. Viewpoints 2000 Group’s proof: $a + ar + ar^2 + \cdots = a/(1-r)$, for $0 < r < 1$.](image)

2. GENERALIZING MABRY’S AND EDGAR’S PROOF

We shall analyze Mabry’s and Edgar’s proofs to generalize them to other values of $a$ and $r$. Interestingly, their proofs are the only of their kinds if we ask for the following properties:

(P1) Parallel line segments with the triangle’s base partition the outermost triangle into adjacent layers. Every layer consists of the same number of equal equilateral triangles, the same number of which are shaded in every layer. Furthermore, the height of these triangles is equal to the height of the layer that contains them.

(P2) The lengths of the parallel line segments including the triangle’s base form a geometric progression. In Mabry’s and Edgar’s proof, consecutive lengths maintain a constant ratio of $1/2$ and $2/3$, respectively.

(P3) The first term of the geometric series to be computed is equal to the common ratio, i.e., $r = a$.

Figure 4 is the skeleton of Figures 1 and 2. The area of $\triangle ABC = 1$, thus the length $|BC| = 2 \cdot 3^{-1/4}$. Let $n$ and $k$ be the number of triangles and of shaded triangles in each layer, respectively. Then $1 \leq k < n$. Let $s \in (0, 1)$ be the ratio of the lengths of consecutive parallel line segments. For example, $s$ is $1/2$ and $2/3$ in Figures 1 and 2, respectively.
We compute the shaded area, denoted by $T$, in two ways. In each layer, there are $n$ triangles, $k$ of which are shaded, so $T = k/n$.

Another way to compute $T$ is to sum up the areas of the shaded triangles. The area of the 1st layer is $1 - s^2$, meaning each triangle in the 1st layer has area $u := (1 - s^2)/n$. Due to similar triangles, each triangle in the $n$th layer has area $us^{2(n-1)}$ thanks to Property (P2). Therefore, the total shaded area is (recall that in each layer, $k$ triangles are shaded)

$$T = ku + kus^2 +kus^4 + \cdots = ku \sum_{j=0}^{\infty} s^{2j}.$$

As the two ways of computing $T$ must produce the same result, we have the identity

$$ku + kus^2 +kus^4 + \cdots = \frac{k}{n}. \quad (2.1)$$

We now employ (2.1) to obtain the two identities proven by Mabry and Edgar. In Mabry’s proof, we have $(n, k, s) = (3, 1, 1/2)$. Then $u = 1/4$, and (2.1) gives

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}.$$

In Edgar’s proof, we have $(n, k, s) = (5, 4, 2/3)$. Then $u = 1/9$, and (2.1) gives

$$\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \cdots = \frac{4}{5}.$$

We are now ready to show that Mabry’s and Edgar’s proofs are the only ones that satisfy all the Properties (P1), (P2), and (P3). Indeed, their proofs require two conditions on $s$, which are simultaneously satisfied only when $s$ is 1/2 or 2/3.

First, to satisfy Property (P3), we need

$$kus^2 = k^2u^2,$$

which is equivalent to

$$\frac{s^2}{1 - s^2} = \frac{k}{n}. \quad (2.2)$$

Since $k/n < 1$, we know that $0 < s < 1/\sqrt{2}$. This is the first restriction on $s$. 
Furthermore, each triangle in the 1st layer has area \((1 - s)^2\), while the 1st layer is 1 – \(s^2\). Therefore, Property (P1) asks that \(n = (1 - s^2)/(1 - s)^2 \in \mathbb{N}\), so \(2/(1 - s) \in \mathbb{N}\). Write \(s = 1 - 2/m\) for some integer \(m \geq 3\). Then the number of triangles in each layer is \(n = m - 1\). Plugging this back to (2.2) gives

\[
\frac{(1 - 2/m)^2}{1 - (1 - 2/m)^2} = \frac{k}{m - 1}.
\]

We obtain \(k = m^2/4 + 1 - m\). Hence, \(m^2/4\) is an integer, implying that \(m = 2m'\) for some \(m' \in \mathbb{N}\). As a result, \(s = 1 - 1/m'\). This is the second restriction on \(s\).

The two restrictions on \(s\) imply that \(s \in \{1/2, 2/3\}\). We conclude that Mabry’s and Edgar’s proofs are the only ones that satisfy all of the Properties (P1), (P2), and (P3).

If we drop Property (P3), then we have many other similar proofs to Figures 1 and 2.

From the above analysis, we need only to choose \(s = 1 - 2/m\) for some \(m \geq 3\) and partition each layer into \(m - 1\) equal triangles. However, this is only true when \(m\) is even. Figure 5 considers the case \(m = 3\), a representative for odd \(m\).

![Figure 5](image)

**Figure 5.** The case \(m = 3\). No layer can be partitioned into two equal triangles.

To fix this problem, Figure 6 partitions each layer into right (instead of equilateral) triangles.

![Figure 6](image)

**Figure 6.** Proof that \(\frac{4}{9} + \frac{4}{9} \left(\frac{1}{9}\right) + \frac{4}{9} \left(\frac{1}{9}\right)^2 + \frac{4}{9} \left(\frac{1}{9}\right)^3 + \cdots = \frac{1}{2}\).

Let us consider \(m = 8\).
FIGURE 7. The case $m = 8$. Each layer is partitioned into seven equal equilateral triangles with the same height as the layer. The picture proves that $\frac{1}{4} + \frac{1}{4} \left( \frac{9}{16} \right) + \frac{1}{4} \left( \frac{9}{16} \right)^2 + \cdots = \frac{4}{7}$.

3. A COMMON IDEA AMONG THE THREE PROOFS WITHOUT WORDS

At first glance, Figures 1, 2, and 3 are little related. However, they are based on the same idea. To illustrate this, we reposition the Viewpoints 2000 Group’s graph as in Figure 8.

FIGURE 8. Viewpoints 2000 Group’s graph repositioned.

We can now apply Mabry’s and Edgar’s idea to prove the formula for geometric series. The shaded area is clearly

$$\frac{1}{2} + \frac{1}{2}r + \frac{1}{2}r^2 + \frac{1}{2}r^3 + \cdots.$$
On the other hand, the shaded area covers $(1 + \sqrt{r})^{-1}$ of the area of each layer. Hence, the shaded area covers $(1 + \sqrt{r})^{-1}$ of the area of $\triangle ABC$. Therefore, we have
\[
\frac{1}{2} + \frac{1}{2}r + \frac{1}{2}r^2 + \frac{1}{2}r^3 + \cdots = \frac{1}{1 + \sqrt{r}} \cdot \frac{1}{2(1 - \sqrt{r})},
\]
which is simplified to
\[
1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}.
\]

REFERENCES

[1] T. Edgar, Proof without words: sum of powers of $\frac{1}{9}$, *Math. Mag.* 89 (2016), 191.
[2] R. Mabry, Proof without words: $\frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \cdots = \frac{1}{4}$, *Math. Mag.* 72 (1999), 63.
[3] The Viewpoints 2000 Group, Proof without words: geometric series, *Math. Mag.* 74 (2001), 320.

Email address: hungchul@tamu.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA