THE STRUCTURE FUNCTIONS OF THE
PHOTON AT LOW VIRTUALITIES

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January 2002

Abstract

The structure functions $F_1$ and $F_2$ of the real photon at low virtualities are calculated in the framework of chiral perturbation theory (ChPT) in the zero and first order of ChPT. It is assumed that the virtuality of hard projectile photon $Q^2$ is much less than the characteristic ChPT scale. In this approximation the structure functions are determined by the production of two pions in $\gamma\gamma$ collisions. The numerical results for $F_2$ and $F_1$ are presented.

1 Introduction

The chiral perturbation theory (ChPT) is the convenient tool for obtaining quantitative predictions in the low-energy region [1, 2]. The properties of the photon at low virtualities also can be described by this theory. The cross section for $\gamma\gamma \to \pi^+\pi^-$ has been calculated some years ago [3]. It is possible to extend this method to the case when both target and projectile photons are virtual. For example, the structure functions of longitudinal virtual photon were calculated in [Me]. In this paper we perform the calculations of the real photon structure functions at low $Q^2$ basing on ChPT. In order to be in the framework of ChPT it is necessary to assume that virtuality of probe photon $Q^2$ is much less than the applicability limit of ChPT $\Lambda^2 \approx m^2_{\rho} = 0.6$GeV$^2$. For the value of center of mass energy $s = (p + q)^2$ the more weak limitation is imposed: $s \sim \Lambda^2$ because $s$ explicitly enters only in the small corrections to the results. It means that the photon structure functions are not in the scaling region of $Q^2$ and our results can not be directly compared with another theoretical predictions for structure functions of real photon at large $Q^2$ [4]. However, it seems useful to have the data from the both sides of region of intermediate $Q^2$.

2 Calculation of the Structure Functions

In order to get the structure functions $F_2$ and $F_1$ of the real target photon the imaginary part of $\gamma\gamma$ forward scattering amplitude $\text{Im}T_{\mu\nu\lambda\sigma}(p,q)$ must be multiplied by the product of the photon polarization vectors in the initial and final states

$$\frac{1}{2\pi} \Sigma e_\lambda e^*_\sigma \text{Im}T_{\mu\nu\lambda\sigma}(p,q) = (-q_{\mu\nu} + q_{\mu}q_{\nu}/q^2)F_1 + (p_{\mu} - q_{\mu}q_{\nu}/q^2)(p_{\nu} - q_{\nu}q_{\nu}/q^2)F_2/q, \quad (1)$$
where $\nu = pq$. By the standard definition of the structure functions the factor $e^2$ arising from electromagnetic vertices with projectile photon should be omitted in the amplitude. For the case of real unpolarized photon the photon density matrix $\Sigma_{\lambda\lambda'}$ can be written as $-\frac{g_{\lambda\sigma}}{2}$.

In the zero order of ChPT the chiral Lagrangian reduces to the Lagrangian of scalar electrodynamics

$$L_{\text{scal.elec.}} = (\partial_\mu \pi + i A_\mu \pi)^+(\partial_\nu \pi + i A_\nu \pi) - M_\pi^2 \pi^+ \pi$$

and $\gamma\gamma$ forward scattering amplitude is determined by the two box diagrams in Fig.1 and the diagrams of Fig. 2, where two photon interact with two pions at one point - the diagrams arising from $A_\mu^2 \pi^+ \pi$ term in Lagrangian of scalar electrodynamics.

The imaginary part of the amplitude is connected with discontinuity of the amplitude in $s$-channel by:

$$\text{Im} T_{\mu\nu\lambda\sigma}(p^2, q^2, s) = \frac{1}{2i} [T_{\mu\nu\lambda\sigma}(p^2, q^2, s + i0) - T_{\mu\nu\lambda\sigma}(p^2, q^2, s - i0)].$$  

It is convenient to calculate $F_2$ and $F_1$ structure functions as the coefficient before $p_\mu p_\nu$ and $-g_{\mu\nu}$, respectively. Doing in this way we find the contributions of diagrams in Fig.1,2 to the structure functions (at $p^2 = 0$): for the direct diagram

$$F_2^{1a} = \frac{\alpha}{4\pi} \left\{ -2x^2 L + (1 + 2x - 10x^2 + 4x^3)\phi + \frac{2M_\pi^2}{\nu} x(5 - 6x) L + (-1 + 8x - 8x^2)\phi \right\},$$

$$F_1^{1a} = \frac{\alpha}{4\pi} \left\{ (1 - x)\phi + \frac{M_\pi^2}{\nu} [(3 - 2x)L + 4(1 - x)\phi] \right\},$$

for the crossing diagram

$$F_2^{1b} = \frac{\alpha}{4\pi} \left\{ (1 - 2x)[2x(1 - x)L + (6x - 6x^2 - 1)\phi] + \frac{2M_\pi^2}{\nu} [-xL + (1 - 6x + 6x^2)\phi - 2x \frac{M_\pi^2}{\nu} L] \right\}.$$  

Figure 1: The forward $\gamma\gamma$ scattering amplitude box diagrams in scalar electrodynamics – the zero order of ChPT; the solid lines correspond to pions, the wavy lines to photons, a) direct diagram; b) crossing diagram.
Figure 2: The diagrams of scalar electrodynamics for the forward $\gamma \gamma$ scattering, arising from the term $A_2^{\mu\pi^+\pi}$ in the Lagrangian.

\[
F^{1b}_1 = \frac{\alpha}{4\pi} \left\{ (1 - 3x + 2x^2)\phi + \frac{M_\pi^2}{\nu}[(1 - 2x)L - 2(1 - x)\phi - \frac{2M_\pi^2}{\nu}L], \right\} \tag{7}
\]

and for the contact diagrams

\[
F^{2a,b}_2 = \frac{\alpha}{2\pi} \left\{ x(2x - 1)L + (-4x + 6x^2)\phi - \frac{2M_\pi^2}{\nu}xL \right\}; \quad F^{2c}_2 = 0 \tag{8}
\]

\[
F^{2a,b}_1 = \frac{\alpha}{2\pi} \left\{ -(1 + x)\phi - \frac{M_\pi^2}{\nu}L \right\}; \quad F^{2c}_1 = \frac{\alpha\phi}{4\pi} \tag{9}
\]

Here $x = Q^2/2\nu$ is the Bjorken variable, $\phi$ accounts two pion phase space,

\[
\phi = \left( 1 - \frac{4M_\pi^2x}{Q^2(1 - x)} \right)^{1/2}, \quad L = \ln \left[ \frac{1 - \phi}{1 + \phi} \right]. \tag{10}\]

The next step which can be done is the calculation of ChPT corrections induced by pion self-interaction. The general method of calculation of these corrections is exposed in [1, 2]. From the general form of ChPT effective Lagrangian it can be easily shown that in the first order in ChPT only the two pions intermediate states can contribute to the imaginary part of the forward $\gamma \gamma$-scattering amplitude.

To calculate corrections to the structure functions $F_2$ and $F_1$ in the first order of ChPT one should consider the general expression for effective Lagrangian which contains all terms permitted by chiral invariance up to the order (momentum)$^4$.

In the leading order of ChPT the effective Lagrangian has the form

\[
L = \frac{F_\pi^2}{4} \Tr\{\nabla_\mu U \nabla_\mu U^\dagger\} + \Sigma \Re \Tr\{MU^\dagger\} + \cdots \tag{11}
\]

where $U$ is the unitary matrix, $U = U^0 + iU^i\tau^i$ and $U^i = \pi^i/F_\pi$ in Weinberg parameterization. $F_\pi = 93\text{MeV}$ is the pion decay constant and parameter $\Sigma$ has the meaning of quark condensate. The ChPT expansion corresponds to the expansion in inverse powers of $F_\pi^2$. So, in the first order of ChPT

\[
U^0 = 1 - \frac{1}{2}U^iU^i \tag{12}\]
The covariant derivative $\nabla_\mu U$ is defined by

$$\nabla_\mu U = \partial_\mu U + i[U, V_\mu]$$

where $V_\mu = A_\mu \tau^3/2$ is the electromagnetic field. Expressing the real pion fields $\pi^1$ and $\pi^2$ through the fields of charged pions $\pi^+$ and $\pi$

$$\pi^1 = \frac{1}{\sqrt{2}}(\pi + \pi^+)$$
$$\pi^2 = \frac{1}{\sqrt{2}i}(\pi - \pi^+)$$

and expanding (13) up to terms $\sim 1/F_\pi^2$, we have (only charged pion fields are retained)

$$L^{(2)} = L^{\text{scal.elec.}} + \frac{1}{2F_\pi^2} \{ \partial_\mu (\pi^+ \pi) \}^2 - \frac{M_\pi^2}{2F_\pi^2} [\pi^+ \pi^-]^2,$$

where $L^{\text{scal.elec.}}$ defined in (2). Second and third terms in $L^{(2)}$ corresponds to the four pion interaction and leads to appearance of the loop corrections to the structure functions which are proportional to $1/F_\pi^2$.

In the order (momentum)$^4$, i.e. $\sim 1/F_\pi^2$, there are two terms in the general effective Lagrangian of ChPT [1, 2], which are essential for us and they can be written in terms of charged pion fields

$$L^{(4)} = -\frac{2l_5}{F_\pi^2} (eF_{\mu\nu})^2 \pi^+ \pi^- - \frac{2il_6}{F_\pi^2} eF_{\mu\nu} [\partial_\mu \pi^- \partial_\nu \pi^+ + i e A_\mu \partial_\nu (\pi^+ \pi^-)],$$

where $F_{\mu\nu}$ is the electromagnetic field strength and $l_5, l_6$ are some phenomenological (infinite) constants.

The terms in the Lagrangian $L^{(4)}$ serve as the counter terms for the renormalization of loops: the infinities arising in loop calculations are absorbed in $l_5$ and $l_6$, and as a result the finite values $\bar{l}_5$ and $\bar{l}_6$ arise.

For the calculation of the first order ChPT corrections to the structure functions we should calculate effective $\gamma\pi\pi$ and $\gamma\gamma\pi\pi$ vertices, substitute ones in all zero order diagrams of ChPT and collect terms proportional to $1/F_\pi^2$.

At first let us consider effective $\gamma\pi\pi$ vertex. In the chosen parameterization of $U$ there are three diagrams which contribute to this vertex in the first order of ChPT (Fig.3).

First diagram corresponds to vertex of $\gamma\pi\pi$ interaction in the scalar electrodynamics, the second comes from $L_6$ and the third is the loop (unitary) diagram. To renormalize the loop diagram it is convenient to use the dimensional regularization. The contribution of this diagram to the $\gamma\gamma\pi$ vertex can be divided into finite part and divergent part which contains the infinite factor $\lambda$, where $\lambda$ is:

$$\lambda = \frac{2}{4-d} + \ln 4\pi + 1 - \gamma_E - \ln \frac{M_\pi^2}{\mu^2},$$

and $\mu$ is the scale of mass introduced by dimensional regularization.

In the sum of the diagrams the divergent part will be absorbed into $\bar{l}_6$ and the following result for effective $\gamma\pi\pi$ vertex in the first order of ChPT was obtained [1, 2]:

$$-\frac{1}{i} \Gamma_\mu(k,k';q) = (k + \mu) - \frac{\big(\bar{l}_6 - 1/3 + \sigma^2 \{ \sigma \ln(\frac{\sigma}{\sigma+1}) + 2 \} \big)}{48\pi^2 F_\pi^2} (q_\mu k_q - k_\mu q^2),$$

(17)
Figure 3: The diagrams for effective $\gamma\pi\pi$ vertex, a) the diagram corresponding to scalar electrodynamics; b) the diagram arising from $L_6$ term in chiral Lagrangian; c) the loop diagram corresponding to four pion interaction.

where

$$\sigma = (1 + 4M_\pi^2/Q^2)^{1/2},$$

$k$ and $k'$ are the pion initial and final momenta, $q$ is the photon momentum, $k = k' + q$. The numerical value of $\bar{l}_6$ was found in [1] from the data on electromagnetic charge radius of the pion:

$$\bar{l}_6 = 16.5 \pm 1.1$$

After substituting this effective vertex in zero order diagrams in all cases the following combination appears

$$R_6(Q^2) = \bar{l}_6 - 1/3 + \sigma^2 \left\{ \sigma \ln \left[ \frac{\sigma - 1}{\sigma + 1} \right] + 2 \right\}.$$  \hspace{1cm} (20)

The term proportional to $\sigma^2$ in (20) arises from the loop correction – the diagram Fig. 3c. Numerically, it is much smaller numerically (about 10 times) than $\bar{l}_6$. In the case of $\gamma\gamma\pi\pi$ effective vertex there are six diagrams (Fig. 4): first diagram comes from scalar electrodynamics, second and third ones correspond to $L_6$ and $L_5$, respectively, and the other diagrams are loop (unitary) corrections.

The calculation of these diagrams leads to the following expression of $\gamma\gamma\pi\pi$ effective vertex

$$\frac{1}{i} \Gamma_{\mu\lambda}(p, q) = 2g_{\mu\lambda} - \frac{(\bar{l}_6 - \bar{l}_5)}{24\pi^2 F_\pi^2} (p_\mu q_\lambda - \nu g_{\mu\lambda}) - \frac{s}{F_\pi^2} B_{\mu\lambda} -$$

$$- \frac{R_6}{48\pi^2 F_\pi^2} (q_\mu q_\lambda - g_{\mu\lambda} q^2 + p_\mu p_\lambda - g_{\mu\lambda} p^2),$$

where $\bar{l}_6 - \bar{l}_5 \approx 2.7$ [1] and

$$B_{\mu\lambda} = \frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{(2k + q)_\mu (2k - p)_\lambda}{[k^2 - M_\pi^2][(k - p)^2 - M_\pi^2][(k + q)^2 - M_\pi^2]} -

- \frac{g_{\mu\lambda}}{[(k - p)^2 - M_\pi^2][(k + q)^2 - M_\pi^2]} \right\}.$$  \hspace{1cm} (22)

In principle we can write out the tensor structure of $B_{\mu\lambda}$ at once:

$$B_{\mu\lambda} = \frac{\delta}{24\pi^2 s} (p_\mu q_\lambda - \nu g_{\mu\lambda}) + A(q_{\mu} - p_{\mu} q^2/\nu)(p_{\lambda} - q_{\lambda} p^2/\nu).$$

(23)
Figure 4: The diagrams for effective $\gamma\gamma\pi\pi$ vertex, a) the diagram coming from scalar electrodynamics; b), c) the diagrams corresponding to $L_6$ and $L_5$ terms in chiral Lagrangian, respectively; d), e), f) the loop diagrams.
This equation follows from the fact that $B_{\mu\lambda}$ is transverse in $q_\mu$ and $p_\lambda$ simultaneously and can be checked by the direct calculation.

After putting $p^2 = 0$ second term in (23) becomes proportional to the momentum of the target photon $p_\lambda$. Averaging over photon density matrix actually means contracting of two indices ($\sigma$ and $\lambda$) in the amplitude $\text{Im}T_{\mu\nu\lambda\sigma}(p,q)$ corresponding to the target real photon and contribution of the second term in (23) to the structure functions vanishes due to the transversality of $\text{Im}T_{\mu\nu\lambda\sigma}$. It means that we should find only $\delta$ which can be calculated by the standard technique

$$\delta(p^2 = 0) = 3(1 - x) \left[ 2 \int_0^1 dy \int_0^y dz \ln \frac{M^2 + Q^2 z(1 - z) - 2\nu z(1 - y)}{M^2} - \int_0^1 dy \ln \frac{M^2 - sy(1 - y)}{M^2} \right]$$

Substituting these effective vertices into all zero order diagrams and collecting the terms proportional to $1/F_2^2\pi$ we get the final results for ChPT corrections to structure functions $F_2$ and $F_1$ of the real photon

$$F_{2\text{ChPT}} = -\frac{R_6 Q^2}{48\pi^2 F_\pi^2} \left[ F_2^{(1a)} + F_2^{(1b)} + F_2^{(2a,b)}/2 \right] + \frac{\alpha Q^2(\tilde{t}_6 - \tilde{t}_5 + \delta)}{96\pi^3 F_\pi^2} \left( \phi x - \frac{M^2}{\nu L} \right)$$

and

$$F_{1\text{ChPT}} = -\frac{R_6 Q^2}{48\pi^2 F_\pi^2} \left[ F_1^{(1a)} + F_1^{(1b)} + F_1^{(2a,b)}/2 \right] + \frac{\alpha Q^2[(\tilde{t}_6 - \tilde{t}_5 + \delta) - R_6x]}{192\pi^3 F_\pi^2 x} \left( \phi x - \frac{M^2}{\nu L} \right)$$

where $F_2^{(1a,b,2a,b)}$ and $F_1^{(1a,b,2a,b)}$ correspond to formulas (4–9).

The absolute value of $\delta$ does not exceed 3 and it is much less than $\tilde{t}_6$ and $\tilde{t}_5$ but the corrections due to $\tilde{t}_5$ and $\tilde{t}_6$ have a different sign and almost equal values and therefore the account of $\delta$ is necessary.

3 Conclusion

Collecting all terms we can write final results for structure functions $F_2$:

$$F_2 = \left( 1 - \frac{R_6 Q^2}{48\pi^2 F_\pi^2} \right) \left[ F_2^{(1a)} + F_2^{(1b)} + F_2^{(2a,b)}/2 \right] + \frac{\alpha Q^2(\tilde{t}_6 - \tilde{t}_5 + \delta)}{96\pi^3 F_\pi^2} \left( \phi x - \frac{M^2}{\nu L} \right)$$

and

$$F_1 = \left( 1 - \frac{R_6 Q^2}{48\pi^2 F_\pi^2} \right) \left[ F_1^{(1a)} + F_1^{(1b)} + F_1^{(2a,b)}/2 \right] + F_1^{(2c)} +$$

$$+ \frac{\alpha Q^2[(\tilde{t}_6 - \tilde{t}_5 + \delta) - R_6x]}{192\pi^3 F_\pi^2 x} \left( \phi x - \frac{M^2}{\nu L} \right),$$

It is worth to note that we did not make any assumption about ratio $M^2/Q^2$, so these results can be used from $Q^2 = 0$. 

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Figure 5: The structure function $F_2$ as a function of $Q^2$ at fixed $x = 0.2$ and $x = 0.4$. The dashed line represents the contribution of the scalar electrodynamics (Born) term at $x = 0.4$.

At $Q^2 \approx 0.1 - 0.15$ GeV$^2$ the large first order ChPT correction comes from the factors in the square brackets in front of the first terms in the r.h.s of (27) and (28). These factors have the meaning of the squares of pion form factors in the vertices of the diagrams Figs. 1,2. Therefore, the accuracy of (27) and (28) may be improved, if these factors would be represented in the standard form of pion form factors $(1 + R_6 Q^2 / 96 \pi^2 F_\pi^2)^{-2}$. In the numerical calculations we use such a procedure.

The numerical results of the calculation at few values of parameters are represented in Fig. 5-9. In the Fig. 5,6 one can see the $Q^2$-dependence for $F_2$ and $F_1$ functions, respectively.

We see that the zero order (Born term) and the chiral corrections to $F_2$ grows with the same rate and as a result $F_2$ becomes an almost constant. For the case of $F_1$ the situations is different: the chiral corrections grows faster then zero order term and $F_1$ decreases from some value of $Q^2$. It is important to say that this behavior appear when chiral corrections are rather small and ChPT is valid. In the scaling region of high $Q^2$ numerical value of $F_1$ is approximately a few times as large as $F_1$ at low $Q^2$ and from decreasing of $F_1$ it follows that $F_1$ is not monotonous function of $Q^2$ but has a minimum at some intermediate $Q^2$.

The $x$-dependence of $F_2$ and $F_1$ structure function are represented in Fig. 7,8, respectively. At large $x$ the structure functions vanishes due to phase volume factor.

Let us now discuss the accuracy of the obtained results. Since only two terms in ChPT were calculated, only the general arguments, referring to the convergence of ChPT can be used. According to these arguments the expansion parameters are the ratios of all invariants entering in the problem - $Q^2$, and $s$ to the characteristic ChPT scale $\Lambda^2 \simeq 0.6$ GeV$^2$. In our case at $Q^2 \simeq 0.1 - 0.15$ GeV$^2$ and this ratio is of order 1/3. The total energy $s$ enters only in the correction $\delta$ due to four pion interaction and at $x \gtrsim 0.15$ does not contribute to the structure functions more than 20%. So, one may expect that the accuracy of obtained results is about $30 - 50\%$. The accuracy is better at intermediate $x \sim 0.2 - 0.3$ and worse at
Figure 6: The structure function $F_1$ as a function of $Q^2$ at fixed $x = 0.2$ and $x = 0.3$. The dashed line represents the contribution of the scalar electrodynamics (Born) term at $x = 0.3$.

Figure 7: The structure function $F_2$ as a function of $x$ at fixed $Q^2 = 0.05\text{GeV}^2$ and $Q^2 = 0.15\text{GeV}^2$. 
$x \approx 0.15$ because the ratio $s/\Lambda^2$ becomes large. It is not, however, a solid statement. From [3] we can see that ChPT predictions for the cross section for $\gamma\gamma \rightarrow \pi^+\pi^-$ process are in a good agreement with experiment up to $s \sim 1\text{GeV}^2$. The comparison of the obtained results with experiment would be important and useful for understanding the applicability domain of chiral theory, the role of higher order ChPT corrections, etc.

**Acknowledgements**

This work was done under the partial support of CRDF grant RP2-2247, INTAS grant 2000-587 and RFBR grant 00-02-17808.

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