GLOBAL MILD SOLUTIONS TO THE VLASOV-MAXWELL-FOKKER-PLANCK SYSTEM

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ABSTRACT. Motivated by Duan et al. [Global Mild Solutions of the Landau and Non-Cutoff Boltzmann Equations, Comm. Pure Appl. Math., 74(5), 932-1020], the global existence of mild solutions to the Vlasov-Maxwell-Fokker-Planck system near a global Maxwellians with small-amplitude initial data in the function space $L^1_t L^\infty_x L^2_v$ is constructed. Moreover, the exponential time decay of the solutions is obtained.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we are concerned with the Cauchy problem to the Vlasov-Maxwell-Fokker-Planck system as follows:

\begin{align}
\begin{cases}
\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = L_{FP} f, \\
\partial_t E - \nabla_x \times B = - \int_{\mathbb{R}^3} v F dv, \\
\partial_t B + \nabla_x \times E = 0, \\
\nabla_x \cdot E = \int_{\mathbb{R}^3} F dv - 1, \quad \nabla_x \cdot B = 0.
\end{cases}
\end{align}

with initial data

\begin{align}
F(0, x, v) = F_0(x, v), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x).
\end{align}

and the compatibility conditions

\begin{align}
\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} F_0(x, v) dv - 1, \quad \nabla_x \cdot B_0 = 0.
\end{align}

where $F(t, x, v) \geq 0$ is the spatially periodic distribution function for the particles at time $t \geq 0$, spatial coordinates $x = (x_1, x_2, x_3) \in \mathbb{T}^3 := [0, 2\pi]^3$ with velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The self-consistent electromagnetic field $[E(t, x), B(t, x)]$ is coupled with the distribution function $F(t, x, v)$ through the Maxwell equations. The Fokker-Planck operator $L_{FP}$ is defined by

\[ L_{FP} = \nabla_v \cdot (v F + \nabla_v F). \]

We consider the following global Maxwellian equilibrium state

\[ \mu(v) = (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{|v|^2}{2}\right) \]

be the global Maxwellian equilibrium, we define the perturbation $f = f(t, x, v)$ by

\[ F(t, x, v) = \mu + \mu^\frac{1}{2} f(t, x, v). \]

Then the Cauchy problem (1.1) and (1.2) is reformulated as

\begin{align}
\begin{cases}
\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f - \frac{1}{2} E \cdot v f - E \cdot v \mu^\frac{1}{2} = L_{FP} f, \\
\partial_t E - \nabla_x \times B = - \int_{\mathbb{R}^3} v \mu^\frac{1}{2} f dv, \\
\partial_t B + \nabla_x \times E = 0, \\
\nabla_x \cdot E = \int_{\mathbb{R}^3} \mu^\frac{1}{2} f dv, \quad \nabla_x \cdot B = 0.
\end{cases}
\end{align}

with initial data

\begin{align}
f(0, x, v) = f_0(x, v), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x).
\end{align}

Here the Fokker-Planck operator $L_{FP}$ is given by

\[ L_{FP} f = \mu^{-\frac{1}{2}} \nabla_v \cdot (\mu \nabla_v (\mu^{-\frac{1}{2}} f)) = \Delta_v f + \frac{1}{4} (6 - |v|^2) f. \]

Let us define the velocity orthogonal projection

\[ P : L^2(\mathbb{R}^3_v) \rightarrow \text{Span} \left\{ \mu^\frac{1}{2}, \ v_i \mu^\frac{1}{2} (1 \leq i \leq 3) \right\}, \]
In this paper, we will use the dissipation rate functional $D$ where the Fourier transformation of $f$ is known [2, 8], the Fokker-Planck operator $\mu$ with respect to the given global Maxwellian $\mu$. Therefore, we have the following macro-micro decomposition of solutions $f(t, x, v)$ of the Vlasov-Maxwell-Fokker-Planck system (1.4) with respect to the given global Maxwellian $\mu$ which was introduced in [16]

$$f(t, x, v) = Pf(t, x, v) + \{I - P\}f(t, x, v),$$

where $I$ denotes the identity operator, $P$ and $\{I - P\}$ are called the macroscopic and the microscopic component of $f(t, x, v)$, respectively.

Integrating (1.1) with respect to $v$, we can obtain the local balance laws

$$\partial_t \int_{\mathbb{R}^3} F dv + \nabla_x \cdot \int_{\mathbb{R}^3} vF dv = 0,$$

By using the (1.3) and (1.8), we can deduce the macroscopic system [25]

$$\partial_t a + \nabla_x \cdot b = 0,$$

Furthermore, integrating the above identity with respect to $x$, we can get the conservation law of mass

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \mu^2 f(t, x, v) dv dx = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \mu^2 f(0, x, v) dv dx.$$

Let $\nu(v) = 1 + |v|^2$ and denote the norm $| \cdot |_\nu$ by

$$|f|_\nu^2 = \int_{\mathbb{R}^3} (\nu(v)|f|^2 + |\nabla_v f|^2) dv.$$

As is known [2, 8], the Fokker-Planck operator $L_{FP}$ is coercive in the sense that there is a positive constant $\lambda_0$ such that

$$(L_{FP}f, f)_{L^2_\nu} \geq \lambda_0 \|\{I - P\}f\|_{L^2_\nu}^2 + |b|^2.$$

Motivated by [10], we use the low regularity function space $L^1_k L^\infty_{v,t} L^2_\nu$ equipped with norm

$$\|f\|_{L^1_k L^\infty_{v,t} L^2_\nu} := \int_{\mathbb{Z}^3_+} \sup_{0 \leq t \leq T} \|\hat{f}(t, k, \cdot)\|_{L^2_\nu} d\Sigma(k) < \infty,$$

where the Fourier transformation of $f(t, x, v)$ with respect to $x \in \mathbb{T}^3$ is defined by

$$\hat{f}(t, k, v) = F_x f(t, k, v) = \int_{\mathbb{T}^3} e^{-ik \cdot x} f(t, x, v) dx, \quad k \in \mathbb{Z}^3.$$

In this paper, we will use $d\Sigma(k)$ to denote the discrete measure in $\mathbb{Z}^3$ for convenience, namely,

$$\int_{\mathbb{Z}^3} g(k) d\Sigma(k) = \sum_{k \in \mathbb{Z}^3} g(k).$$

To state the result of the paper, we define the energy functional $E(f(t))$ and the corresponding energy dissipation rate functional $\mathcal{D}(f(t))$ by

$$E(f(t)) \equiv \int_{\mathbb{Z}^3} \sup_{0 \leq \tau \leq t} \left( \|\hat{f}(\tau, k, \cdot)\|_{L^2_\nu} + |\hat{E}(\tau, k)| + |\hat{B}(\tau, k)| \right) d\Sigma(k),$$

and

$$\mathcal{D}(f(t)) \equiv \int_{\mathbb{Z}^3} \left( \int_0^t |\{I - P\} \hat{f}(\tau, k, \cdot)|_{L^2_\nu} d\tau \right)^{1/2} d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_0^t |\hat{a}, \hat{b}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k)$$

$$+ \int_{\mathbb{Z}^3} \left( \int_0^t |\hat{E}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k).$$
respectively. In order to get the exponential time decay, we need the high regularity of spatial variable function space \( L^m_{k,x} L^r_{t,v} \) with norm
\[
\|f\|_{L^m_{k,x} L^r_{t,v}} := \int_{\mathbb{R}^3} \sup_{0 \leq t \leq T} \| \langle k \rangle^m \hat{f}(t,k) \|_{L^r_{t,v}} d\Sigma(k),
\] (1.17)
for any integer \( m \geq 0 \). And we also define the corresponding energy functional \( \mathcal{E}_m(f(t)) \) and the energy dissipation rate functional \( \mathcal{D}_m(f(t)) \) by
\[
\mathcal{E}_m(f(t)) = \int_{\mathbb{R}^3} \sup_{0 \leq t \leq T} \left( \| \langle k \rangle^m \hat{f}(t,k,\cdot) \|_{L^r_{t,v}} + \| \langle k \rangle^m \hat{E}(t,k) \| + \| \langle k \rangle^m \hat{B}(t,k) \| \right) d\Sigma(k)
\] (1.18)
and
\[
\mathcal{D}_m(f(t)) = \int_{\mathbb{R}^3} \left( \int_0^t \| \langle k \rangle^m \{I - P\} \hat{f}(\tau,k,\cdot) \|_{L^r_{t,v}}^2 d\tau \right)^{1/2} d\Sigma(k) + \int_{\mathbb{R}^3} \left( \int_0^t \| \langle k \rangle^m \hat{E}(\tau,k) \|_{L^r_{t,v}}^2 d\tau \right)^{1/2} d\Sigma(k)
\] (1.19)
\[
+ \int_{\mathbb{R}^3} \left( \int_0^t \| \langle k \rangle^m \hat{B}(\tau,k) \|_{L^r_{t,v}}^2 d\tau \right)^{1/2} d\Sigma(k) + \int_{\mathbb{R}^3} \left( \int_0^t \| \langle k \rangle^m \hat{B}(\tau,k) \|_{L^r_{t,v}}^2 d\tau \right)^{1/2} d\Sigma(k)
\]
respectively.

**Remark 1.1.** Compared with the dissipation functional \( \mathcal{D}(f(t)) \) defined in (1.16), \( \mathcal{D}_m(f(t))(m \geq 1) \) contains the dissipation of \( B \) which can be obtained from the Maxwell equations (3.3).

**Notations.**
- \( A \lesssim B \) means that there is a constant \( C > 0 \) such that \( A \leq CB \). \( A \sim B \) means \( A \lesssim B \) and \( B \lesssim A \).
- Denoting \( (\cdot, \cdot)_{L^2_v} \) the complex inner product over \( L^2_v \), i.e., \( (f, g)_{L^2_v} = \int_{\mathbb{R}^3} f(v) \overline{g(v)} dv \).
- \( \Re \) denotes the real part of a complex number.
- \( \langle k \rangle := \sqrt{1 + |k|^2} \) for any integer \( m \geq 0 \).

**Theorem 1.1** (Global-in-time Existence). Assume that \( f_0(x,v) = f(0,x,v) \) satisfies the conservation law of mass
\[
\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \mu \frac{3}{2} f(t,x,v) dvdx = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \mu \frac{1}{2} f(0,x,v) dvdx = 0.
\] (1.20)
Then there is \( \epsilon_0 > 0 \) such that if the initial data \( F_0(x,v) = \mu + \mu \frac{3}{2} f_0(x,v) \geq 0 \) and
\[
\mathcal{E}(f_0) = \| f_0 \|_{L^1_x L^2_v} + \| \hat{E}_0(k) \|_{L^1_k} + \| \hat{B}_0(k) \|_{L^1_k} \leq \epsilon_0,
\]
then the Cauchy problem (1.4) and (1.5) admits a unique global mild solution \( f(t,x,v) \), \( t > 0 \), \( x \in \mathbb{T}^3 \), \( v \in \mathbb{R}^3 \) satisfying \( F(t,x,v) = \mu + \mu \frac{3}{2} f(t,x,v) \geq 0 \) and the uniform estimate
\[
\mathcal{E}(f(t)) + \mathcal{D}(f(t)) \lesssim \mathcal{E}(f_0),
\] (1.21)
for any \( t > 0 \).

**Theorem 1.2** (High order spatial regularity and Time decay). Assume that \( f_0(x,v) = f(0,x,v) \) satisfies the conservation law of mass
\[
\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \mu \frac{3}{2} f(t,x,v) dvdx = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \mu \frac{1}{2} f(0,x,v) dvdx = 0.
\]
and \( B(t,x) \) is spatially periodic function, i.e. \( \int_{\mathbb{T}^3} B(t,x) dx = 0 \). Then there is \( \epsilon_0 > 0 \) such that if the initial data \( F_0(x,v) = \mu + \mu \frac{3}{2} f_0(x,v) \geq 0 \) and
\[
\mathcal{E}_1(f_0) = \| f_0 \|_{L^1_x L^2_v} + \| \langle k \rangle \hat{E}_0(k) \|_{L^1_k} + \| \langle k \rangle \hat{B}_0(k) \|_{L^1_k} \leq \epsilon_0,
\]
then the Cauchy problem (1.4) and (1.5) admits a unique global mild solution \( f(t,x,v) \), \( t > 0 \), \( x \in \mathbb{T}^3 \), \( v \in \mathbb{R}^3 \) satisfying \( F(t,x,v) = \mu + \mu \frac{3}{2} f(t,x,v) \geq 0 \), and the uniform estimate
\[
\mathcal{E}_1(f(t)) + \mathcal{D}_1(f(t)) \lesssim \mathcal{E}_1(f_0),
\] (1.22)
for any $t > 0$. Moreover, there is $\lambda > 0$ such that the solution also admits the time decay estimate

$$\|f(t)\|_{L^1} + \|(k)\dot{E}(t, k)\|_{L^1} + \|(k)\dot{B}(t, k)\|_{L^1} \lesssim e^{-\lambda t} E_1(f_0),$$

(1.23)

for any $t \geq 0$.

**Remark 1.2.**

- The energy estimates in function space $L^1_t L^\infty_x L^2_v$ are not required the $v$-derivatives and $x$-derivatives compared with the integer Sobolev space used in [25, 26], so regularity assumption on the initial data is more weaker.
- Due to the lack of the zero-order dissipation of $B$, we need the high regularity of spatial variable which contained the high order dissipation of $B$ to get the exponential time decay of the solutions.

Now we recall the related works of this manuscript. There have been a lot of studies on the Vlasov-Maxwell-Fokker-Planck equation (1.1). Wollman [24] obtained the local existence and uniqueness of the smooth solutions to the Vlasov-Maxwell system. Diperna and Lions [5] proved the stability of solutions in weak topologies and deduce from this stability result the global existence of a weak solution with large initial data for the Vlasov-Maxwell systems. For the special one and one half dimensional case, the global-in-time existence and uniqueness of classical solutions for the relativistic version of the Vlasov-Maxwell-Fokker-Planck system was proved by [20, 22]. For the various limiting problems on the Vlasov-Maxwell-Fokker-Planck system, we refer to [1, 19] and the references therein.

Recently, the nonlinear energy method developed in [16] for the Boltzmann equation provides an effective approach to establish the classical global existence solutions in the perturbative framework, which are generally called macro-micro decomposition method. In the spirit of this method, global in time solutions to many kinetic models are constructed, including the time decay rates [17, 12]. For the Vlasov-Maxwell-Boltzmann system, the global existence of solutions to the periodic initial boundary value problem near the global Maxwellian equilibrium states was firstly investigated by Guo [18] for the hard sphere model. The global in time classical solutions and the large-time behavior for the Cauchy problem with cutoff collision kernels were studied in [7, 11, 23] and the non-cutoff case in [9].

As for the Vlasov-Maxwell-Fokker-Planck equation, Yang and Yu [26] obtained the global-in-time classical solutions near Maxwellian based on the energy method by combining the compensating function. In addition, a convergence rate in time of the solution to its equilibrium is obtained. More results of the global existence of solutions to the Vlasov-Maxwell-Fokker-Planck system near Maxwellians in the whole space by utilizing the refined energy method are obtained in [3, 25].

However, these results are obtained in Sobolev space involved the $v$-derivatives or $x$-derivatives which required high regularity on the initial data. In order to obtain the global in time solutions in low regularity function space, Duan-Liu-Sakamoto-Strain introduced the space $L^1_t L^\infty_x L^2_v$ to deal with the Landau and non-cutoff Boltzmann equation, where $L^1_t$ corresponds to the Weiner algebra over a torus satisfying

$$\|fg\|_{L^1_t} \leq \|f\|_{L^1_t} \|g\|_{L^1_t}.$$

Motivated by this method, we are desired to obtain the global existence of solutions to the Vlasov-Maxwell-Fokker-Planck equation in low regularity function space.

The rest of this paper is organized as follows. In Section 2, we list some basic lemmas which will be used later. Section 3 and Section 4 are devoted to deducing the desired microscopic and macroscopic estimates, respectively. The proofs of theorem 1.1 will be given in Section 5.

2. Preliminary

In this section, we collect several fundamental results which will be used later. The first lemma concerns on the Fokker-Planck operator $L_{FP}$. Since $L_{FP}$ is a linear operator, the proofs is similar with the result in [2, 8].

**Lemma 2.1.** ([2, 8]) It holds that

$$-\mathcal{A}(L_{FP}\hat{f}, \hat{f})_{L^2_v} \geq \lambda_0 \|\{I - P\} \hat{f}(t, k, \cdot)\|_{L^2_v}^2 + \|\dot{b}(t, k)\|^2.$$

(2.1)

The following two lemmas are concerned with the estimates of the nonlinear terms. The first is the estimate on $E \cdot \nabla_v f$.

**Lemma 2.2.** There is a constant $\eta > 0$ sufficiently small such that
(i) 
\[ \int_{\mathbb{R}^3_1} \left( \int_0^t \left| \left( \tilde{E} \ast \nabla_v \tilde{f}, \tilde{f} \right) \right|_{L^2_v}^2 \, d\tau \right)^{1/2} \, d\Sigma(k) \leq \eta \int_{\mathbb{R}^3_1} \left( \int_0^t \left\| \tilde{f}(\tau, k, \cdot) \right\|_{L^2_v} \, d\tau \right)^{1/2} \, d\Sigma(k) \\
+ \mathcal{E}(f(t)) \int_{\mathbb{R}^3_1} \left( \int_0^t \left\| \nabla_v \tilde{f}(\tau, l, \cdot) \right\|_{L^2_v}^2 \, d\tau \right)^{1/2} \, d\Sigma(l), \]
\[ \leq \eta \mathcal{D}(f(t)) + \mathcal{E}(f(t)) \mathcal{D}(f(t)). \]

(ii) 
\[ \int_{\mathbb{R}^3_1} \left( \int_0^t \left( \tilde{E} \ast \nabla_v \tilde{f}, \mu \tilde{f} \right) \right|_{L^2_v}^2 \, d\tau \right)^{1/2} \, d\Sigma(k) \leq \mathcal{E}(f(t)) \int_{\mathbb{R}^3_1} \left( \int_0^t \left\| \nabla_v \tilde{f}(\tau, l, \cdot) \right\|_{L^2_v}^2 \, d\tau \right)^{1/2} \, d\Sigma(l), \]
\[ \leq \mathcal{E}(f(t)) \mathcal{D}(f(t)). \]

Proof. (i): By Fubini’s theorem, one can get
\[
\left| \left( \tilde{E} \ast \nabla_v \tilde{f}, \tilde{f} \right) \right|_{L^2_v} = \left| \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \tilde{E}(\tau, k - l) \nabla_v \tilde{f}(\tau, l, v) \, d\Sigma(l) \right) \tilde{f}(\tau, k, v) \, dv \right|
\leq \int_{\mathbb{R}^3} \tilde{E}(\tau, k - l) \left( \int_{\mathbb{R}^3} \nabla_v \tilde{f}(\tau, l, v) \tilde{f}(\tau, k, v) \, dv \right) \, d\Sigma(l)
\leq \int_{\mathbb{R}^3} \left| \tilde{E}(\tau, k - l) \right| \left\| \nabla_v \tilde{f}(\tau, l, \cdot) \right\|_{L^2_v} \left\| \tilde{f}(\tau, k, \cdot) \right\|_{L^2_v} \, d\Sigma(l).
\]
Then applying Cauchy-Schwarz’s inequality with respect to \( \int_0^t(\cdot) \, d\tau \) and using Young’s inequality, we have
\[
\int_{\mathbb{R}^3} \left( \int_0^t \left| \left( \tilde{E} \ast \nabla_v \tilde{f}, \tilde{f} \right) \right|_{L^2_v} \, d\tau \right)^{1/2} \, d\Sigma(k)
\leq \int_{\mathbb{R}^3} \left( \int_0^t \left| \tilde{E}(\tau, k - l) \right| \left\| \nabla_v \tilde{f}(\tau, l, \cdot) \right\|_{L^2_v} \left\| \tilde{f}(\tau, k, \cdot) \right\|_{L^2_v} \, d\Sigma(l) \right)^{1/2} \, d\Sigma(k)
\leq \int_{\mathbb{R}^3} \left( \int_0^t \left| \tilde{E}(\tau, k - l) \right| \left\| \nabla_v \tilde{f}(\tau, l, \cdot) \right\|_{L^2_v}^2 \, d\Sigma(l) \right)^{1/4} \, d\Sigma(k)
\times \left( \int_0^t \left\| \tilde{f}(\tau, k, \cdot) \right\|_{L^2_v}^2 \, d\tau \right)^{1/4} \, d\Sigma(k)
\leq \eta \int_{\mathbb{R}^3} \left( \int_0^t \left\| \tilde{f}(\tau, k, \cdot) \right\|_{L^2_v}^2 \, d\tau \right)^{1/2} \, d\Sigma(k)
+ \frac{1}{4\eta} \int_{\mathbb{R}^3} \left( \int_0^t \left| \tilde{E}(\tau, k - l) \right| \left\| \nabla_v \tilde{f}(\tau, l, \cdot) \right\|_{L^2_v}^2 \, d\Sigma(l) \right)^{1/2} \, d\Sigma(k),
\]
where \( \eta > 0 \) is a sufficiently small universal constant. For the second term in the above inequality, we can get
\[
\left( \int_0^t \left| \tilde{E}(\tau, k - l) \left\| \nabla_v \tilde{f}(\tau, l, \cdot) \right\|_{L^2_v} \right. \, d\tau \right)^{1/2}
\leq \int_{\mathbb{R}^3} \left( \int_0^t \left| \tilde{E}(\tau, k - l) \right|^2 \left\| \nabla_v \tilde{f}(\tau, l, \cdot) \right\|_{L^2_v}^2 \, d\tau \right)^{1/2} \, d\Sigma(l),
\]
by the Minkowski’s inequality
\[
\left\| \cdot \right\|_{L^1_v} \leq \left\| \cdot \right\|_{L^2_v}. \]
Therefore, it follows by Fubini’s theorem and translation invariance with (2.5)

\[ \int_{Z^3_k} \left( \int_0^t \left( \int_0^t |E(\tau, k - l)| \| \nabla_v \hat{f}(\tau, l, \cdot) \|_{L^2_v} \right)^2 d\tau \right)^{1/2} d\Sigma(k) \]

\[ \leq \int_{Z^3_k} \int_{Z^3_k} \left( \int_0^t |E(\tau, k - l)| \| \nabla_v \hat{f}(\tau, l, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(l) d\Sigma(k) \]

\[ \leq \int_{Z^3_k} \sup_{0 \leq \tau \leq t} |E(\tau, k - l)| \left( \int_0^t \| \nabla_v \hat{f}(\tau, l, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(l) d\Sigma(k) \]

\[ = \int_{Z^3_k} \left( \int_0^t \sup_{0 \leq \tau \leq t} |E(\tau, k - l)| d\Sigma(k) \right) \left( \int_0^t \| \nabla_v \hat{f}(\tau, l, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(l) \]

\[ \leq \left( \int_{Z^3_k} \sup_{0 \leq \tau \leq t} |E(\tau, k)| d\Sigma(k) \right) \int_{Z^3_k} \left( \int_0^t \| \nabla_v \hat{f}(\tau, l, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(l). \]  

Due to (1.12), we have

\[ \int_{Z^3_k} \left( \int_0^t \| \nabla_v (I - P) \hat{f}(\tau, k, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(k) \]

\[ \lesssim \int_{Z^3_k} \left( \int_0^t \| (I - P) \hat{f}(\tau, k, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(k), \]

and the fact that

\[ \int_{Z^3_k} \left( \int_0^t \| \nabla_v P \hat{f}(\tau, k, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(k) \lesssim \int_{Z^3_k} \left( \int_0^t |(\hat{a}, \hat{b})(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k), \]

then by the definition of \( D(f(t)) \), i.e. (1.16), we can obtain

\[ \int_{Z^3_k} \left( \int_0^t \| \nabla_v \hat{f}(\tau, k, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(k) \lesssim \int_{Z^3_k} \left( \int_0^t |(\hat{a}, \hat{b})(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]

\[ + \int_{Z^3_k} \left( \int_0^t \| \nabla_v P \hat{f}(\tau, k, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(k) \lesssim D(f(t)) \]  

(2.7)

Similarly, we have

\[ \int_{Z^3_k} \left( \int_0^t \| \hat{f}(\tau, k, \cdot) \|_{L^2_v} \right)^{1/2} d\Sigma(k) \lesssim D(f(t)) \]

(2.8)

Combining (2.4),(2.6),(2.7),(2.8) with the definition of \( \mathcal{E}(f(t)) \), i.e. (1.15), the desired result (2.2) is obtained.

(ii): Since

\[ \left| (\hat{E} \ast \nabla_v \hat{f}, \mu^k) \right|_{L^2_v} \lesssim \int_{Z^3_k} |\hat{E}(\tau, k - l)| \| \nabla_v \hat{f}(\tau, l, \cdot) \|_{L^2_v} d\Sigma(l). \]

Similarly as the estimate of (i), combining (2.5),(2.6),(2.7) and (2.8), the desired result (2.3) is obtained.

\[ \square \]

The other one is the estimate on \( v \times B \cdot \nabla_v f \).

**Lemma 2.3.** It holds that

\[ \int_{Z^3_k} \left( \int_0^t \left( v \times \hat{B} \ast \nabla_v \hat{f}, \hat{f} \right) \right)^{1/2} d\tau \]

(2.9)
and
\[
\int_{\mathbb{R}^3_k} \left( \int_0^t \left( v \times \dot{B} \ast \nabla_v \hat{f}, \hat{f} \right) \right)^2_{L^2_v} d\tau \right)^{1/2} \ d\Sigma(k) \lesssim \mathcal{E}(f(t)) \mathcal{D}(f(t)),
\]  
(2.10)

where \( \eta > 0 \) is a sufficiently small universal constant.

Proof. (i): Similarly as in Lemma 2.2, one can get
\[
\left| \left( v \times \dot{B} \ast \nabla_v \hat{f}, \hat{f} \right) \right|_{L^2_v} \lesssim \int_{\mathbb{R}^3_k} |\dot{B}(\tau, k - l)\|\nabla_v \hat{f}(\tau, l, \cdot)\|_{L^2_v} \|v\hat{f}(\tau, k, \cdot)\|_{L^2} d\Sigma(l),
\]
therefore, we have
\[
\int_{\mathbb{R}^3_k} \left( \int_0^t \left( v \times \dot{B} \ast \nabla_v \hat{f}, \hat{f} \right) \right)^2_{L^2_v} d\tau \right)^{1/2} \ d\Sigma(k)
\]
\[
\lesssim \int_{\mathbb{R}^3_k} \left( \int_0^t \int_{\mathbb{R}^3_k} |\dot{B}(\tau, k - l)\|\nabla_v \hat{f}(\tau, l, \cdot)\|_{L^2_v} \|v\hat{f}(\tau, k, \cdot)\|_{L^2} d\Sigma(l) d\tau \right)^{1/2} \ d\Sigma(k)
\]
\[
\lesssim \int_{\mathbb{R}^3_k} \left( \int_0^t \left( \int_{\mathbb{R}^3_k} |\dot{E}(\tau, k - l)\|\nabla_v \hat{f}(\tau, l, \cdot)\|_{L^2_v} d\Sigma(l) \right)^2 d\tau \right)^{1/4}
\]
\[
\times \left( \int_0^t \|v\hat{f}(\tau, k, \cdot)\|_{L^2_v}^2 d\tau \right)^{1/4} \ d\Sigma(k)
\]
\[
\lesssim \eta \int_{\mathbb{R}^3_k} \left( \int_0^t \|v\hat{f}(\tau, k, \cdot)\|_{L^2_v}^2 d\tau \right)^{1/2} \ d\Sigma(k)
\]
\[
+ \frac{1}{4\eta} \int_{\mathbb{R}^3_k} \left( \int_0^t \left( \int_{\mathbb{R}^3_k} |\dot{E}(\tau, k - l)\|\nabla_v \hat{f}(\tau, l, \cdot)\|_{L^2_v} d\Sigma(l) \right)^2 d\tau \right)^{1/2} \ d\Sigma(k)
\]
\[
\lesssim \eta \mathcal{D}(f(t)) + \mathcal{E}(f(t)) \mathcal{D}(f(t)),
\]
the last inequality holds because of \( |v||f|^2 \leq \nu(v)|f|^2 \). Other details is referred as Lemma 2.2. \( \square \)

The other one is the estimate on \( v \cdot E f \). Since the proof is similar to Lemma 2.2, we omit it.

Lemma 2.4. It holds that
\[
\int_{\mathbb{R}^3_k} \left( \int_0^t \left( v \cdot E \ast \hat{f}, \hat{f} \right) \right)^2_{L^2_v} d\tau \right)^{1/2} \ d\Sigma(k) \lesssim \eta \mathcal{D}(f(t)) + \mathcal{E}(f(t)) \mathcal{D}(f(t)),
\]  
(2.12)

and
\[
\int_{\mathbb{R}^3_k} \left( \int_0^t \left( v \cdot \dot{B} \ast \hat{f}, \mu \hat{f} \right) \right)^2_{L^2_v} d\tau \right)^{1/2} \ d\Sigma(k) \lesssim \mathcal{E}(f(t)) \mathcal{D}(f(t)).
\]  
(2.13)

where \( \eta > 0 \) is a sufficiently small universal constant.

3. Microscopic estimates

Firstly, we need the estimates of the microscopic dissipation for the solution \( f \) in (1.4).

Lemma 3.1. Assume that \( f(\tau, x, v) \) is a smooth solution to the Cauchy problem (1.4) on \([0,t] \), it holds that
\[
\int_{\mathbb{R}^3_k} \sup_{0 \leq \tau \leq t} \left( ||\dot{f}(\tau, k, \cdot)||_{L^2_v} + |\dot{E}(\tau, k)| + |\dot{B}(\tau, k)| \right) d\Sigma(k)
\]
\[
+ \int_{\mathbb{R}^3_k} \left( \int_0^t \left( |1 - P\right)|\hat{f}(\tau, k, \cdot)|^2 d\tau \right)^{1/2} \ d\Sigma(k) + \int_{\mathbb{R}^3_k} \left( \int_0^t |\hat{b}(\tau, k)|^2 d\tau \right)^{1/2} \ d\Sigma(k)
\]
\[
\lesssim ||\hat{f}_0||_{L^1_k L^2_v} + ||\dot{E}_0(k)||_{L^1_k} + ||\dot{B}_0(k)||_{L^1_k} + \eta \mathcal{D}(f(t)) + \mathcal{E}(f(t)) \mathcal{D}(f(t)),
\]
here \( \eta > 0 \) is a sufficiently small universal constant.
Proof. The Fourier transform of (1.4)\(_1\) with \(x \in \mathbb{T}^3\) gives
\[
\begin{align*}
\partial_\tau \hat{f}(\tau, k, v) + iv \cdot k \hat{f}(\tau, k, v) + [\hat{E} \ast \nabla_v \hat{f}](\tau, k, v) + [v \times \hat{B} \ast \nabla_v \hat{f}](\tau, k, v) - \frac{1}{2} [v \cdot \hat{E} \ast \hat{f}](\tau, k, v) - \hat{E}(\tau, k) \cdot v \mu^2 &= L_{FP} \hat{f}(\tau, k, v).
\end{align*}
\] (3.2)

where the convolutions are taken with respect to \(k \in \mathbb{Z}^3\):
\[
\begin{align*}
[\hat{E} \ast \nabla_v \hat{f}](\tau, k, v) &= \int_{\mathbb{Z}^3} \hat{E}(\tau, k - l) \nabla_v \hat{f}(\tau, l, v) d\Sigma(l), \\
[v \times \hat{B} \ast \nabla_v \hat{f}](\tau, k, v) &= \int_{\mathbb{Z}^3} v \times \hat{B}(\tau, k - l) \nabla_v \hat{f}(\tau, l, v) d\Sigma(l), \\
[v \cdot \hat{E} \ast \hat{f}](\tau, k, v) &= \int_{\mathbb{Z}^3} v \cdot \hat{E}(\tau, k - l) \hat{f}(\tau, l, v) d\Sigma(l).
\end{align*}
\]

Next, taking the Fourier transform in \(x \in \mathbb{T}^3\) for the Maxwell equations to obtain
\[
\begin{align*}
\begin{cases}
\partial_\tau \hat{E} - ik \times \hat{B} = -\hat{b}, \\
\partial_\tau \hat{B} + ik \times \hat{E} = 0, \\
\partial_\tau \hat{E} = \hat{a}, \quad \partial_\tau \hat{B} = 0.
\end{cases}
\end{align*}
\] (3.3)

Due to (1.8) and (3.3), it holds that
\[
(-\hat{E} \cdot v \mu^2, \hat{f})_{L^2_\tau} = -\hat{E} \cdot \hat{b} = \hat{E}(\partial_\tau \hat{E} - ik \times \hat{B}) = \hat{E} \partial_\tau \hat{E} + \hat{E} \cdot ik \times \hat{B} = \hat{E} \partial_\tau \hat{E} + \hat{B} \partial_\tau \hat{B} = \frac{1}{2} \partial_\tau (|\hat{E}|^2 + |\hat{B}|^2).
\]

Then taking the product of (3.2) with the complex conjugate of \(\hat{f}(\tau, k, v)\) and integrating the above identity with respect to \(v\), further taking the real part of the resulting equation and integrating the above identity with respect to \(\tau\), we can get
\[
\frac{1}{2} \left( \|\hat{f}(\tau, k, \cdot)\|_{L^2_\tau}^2 + |\hat{E}(\tau, k)|^2 + |\hat{B}(\tau, k)|^2 \right) - \int_0^\tau \Re (L_{FP} \hat{f}, \hat{f})_{L^2_\tau} d\tau = \frac{1}{2} \left( \|\hat{f}_0(\cdot, \cdot)\|_{L^2_\tau}^2 + |\hat{E}_0(\cdot)|^2 + |\hat{B}_0(\cdot)|^2 \right) - \int_0^\tau \Re (\hat{E} \ast \nabla_v \hat{f}, \hat{f})_{L^2_\tau} d\tau
\]
\[
- \int_0^\tau \Re (v \times \hat{B} \ast \nabla_v \hat{f}, \hat{f})_{L^2_\tau} d\tau + \int_0^\tau \Re (\frac{1}{2} v \cdot \hat{E} \ast \hat{f}, \hat{f})_{L^2_\tau} d\tau,
\] (3.4)

where \(\hat{f}_0(\cdot, \cdot) = \hat{f}(0, \cdot, \cdot)\), \(\hat{E}_0(\cdot) = \hat{E}(0, \cdot)\), \(\hat{B}_0(\cdot) = \hat{B}(0, \cdot)\). By the coercivity estimate of \(L_{FP}\) (2.1), taking the square root on both sides and using the elementary inequalities
\[
\frac{1}{\sqrt{2}} (A + B) \leq \sqrt{A^2 + B^2} \leq A + B, \quad A, B \geq 0,
\] (3.5)

We can get
\[
\frac{1}{\sqrt{2}} \left( \|\hat{f}(\tau, k, \cdot)\|_{L^2_\tau} + |\hat{E}(\tau,k)| + |\hat{B}(\tau,k)| \right) + \sqrt{\lambda_0} \left( \int_0^\tau \|\{I - \hat{P}\} \hat{f}(\tau, k, \cdot)\|_{L^2_\tau}^2 d\tau \right)^{1/2}
\]
\[
+ \sqrt{\lambda_0} \left( \int_0^\tau |\hat{b}(\tau,k)|^2 d\tau \right)^{1/2}
\]
\[
\leq \|\hat{f}_0(\cdot, \cdot)\|_{L^2_\tau} + |\hat{E}_0(\cdot)| + |\hat{B}_0(\cdot)| + \sqrt{2} \left( \int_0^\tau \Re (\hat{E} \ast \nabla_v \hat{f}, \hat{f})_{L^2_\tau} d\tau \right)^{1/2}
\]
\[
+ \sqrt{2} \left( \int_0^\tau \Re (v \times \hat{B} \ast \nabla_v \hat{f}, \hat{f})_{L^2_\tau} d\tau \right)^{1/2} + \sqrt{2} \left( \int_0^\tau \Re (\frac{1}{2} v \cdot \hat{E} \ast \hat{f}, \hat{f})_{L^2_\tau} d\tau \right)^{1/2},
\] (3.6)
Moreover, taking $\sup_{0 \leq \tau \leq t}$ on both sides of (3.6) and then integrating the resulting inequality with respect to $d\Sigma(k)$ over $\mathbb{Z}^3$, we have
\[
\int_{\mathbb{Z}^3} \sup_{0 \leq \tau \leq t} \left( \| \hat{f}(\tau, k, \cdot) \|_{L^2_k} + |\hat{E}(\tau, k)| + |\hat{B}(\tau, k)| \right) d\Sigma(k)
+ \int_{\mathbb{Z}^3} \left( \int_0^t |\hat{f}(\tau, k, \cdot)|^2 d\tau \right)^{1/2} d\Sigma(k)
+ \int_{\mathbb{Z}^3} \left( \int_0^t |\hat{b}(\tau, k)| d\tau \right)^{1/2} d\Sigma(k)
\leq \|\hat{f}_0\|_{L^1_k L^2_k} + \|\hat{E}_0(k)\| + \|\hat{B}_0(k)\|_{L^1_k} + \int_{\mathbb{Z}^3} \left( \int_0^t \left| \left( \hat{E} \ast \nabla_v \hat{f}, \hat{f} \right) \right|_{L^2_k} d\tau \right)^{1/2} d\Sigma(k)
+ \int_{\mathbb{Z}^3} \left( \int_0^t \left| \left( \frac{1}{2} v \cdot \hat{E} \ast \hat{f}, \hat{f} \right) \right|_{L^2_k} d\tau \right)^{1/2} d\Sigma(k)
\leq \|\hat{f}_0\|_{L^1_k L^2_k} + \|\hat{E}_0(k)\|_{L^1_k} + \|\hat{B}_0(k)\|_{L^1_k} + \eta D(f(t)) + E(f(t)) D(f(t)).
\]

where we have used Lemma 2.2 and Lemma 2.3. thus the proof is completed. \[\square\]

4. MACROSCOPIC ESTIMATES

In this section, we will derive the uniform a priori estimates for the macroscopic part of $\hat{a}(\tau, k)$ followed by the same strategy as in [10] by the dual argument.

**Theorem 4.1.** Under the assumptions of theorem 1.1, it holds that
\[
\int_{\mathbb{Z}^3} \left( \int_0^t \| \hat{a}(\tau, k) \|^2 d\tau \right)^{1/2} d\Sigma(k)
+ \int_{\mathbb{Z}^3} \left( \int_0^t \| \hat{E}(\tau, k) \|^2 d\tau \right)^{1/2} d\Sigma(k)
\leq E(f(t)) + \|\hat{f}_0\|_{L^1_k L^2_k} + \int_{\mathbb{Z}^3} \left( \int_0^t \left| \hat{b}(\tau, k) \right|^2 d\tau \right)^{1/2} d\Sigma(k)
+ \int_{\mathbb{Z}^3} \left( \int_0^t |\hat{f}(\tau, k, \cdot)|^2 d\tau \right)^{1/2} d\Sigma(k) + E(f(t)) D(f(t))
\]

**Proof.** In order to obtain the estimate of $a$, we take a test function as
\[\hat{\Phi}(\tau, k, v) \in C^1((0, \infty) \times \mathbb{Z}^3 \times \mathbb{R}^3),\]

Applying the Fourier transform to (1.4), taking the inner product of it and $\hat{\Phi}$ in $L^2_k$, then integrating the resultant over $[0, t]$, we have
\[
\left( \hat{f}, \hat{\Phi} \right)_{\tau=t} - \left( \hat{f}, \hat{\Phi} \right)_{\tau=0} - \int_0^t \left( \hat{f}, \partial_t \hat{\Phi} \right) d\tau - \int_0^t \left( \hat{f}, v \cdot \mu \Phi \right) d\tau - \int_0^t \left( \hat{E}, v \cdot \mu \hat{\Phi} \right) d\tau
= \int_0^t (L_{FP} \hat{f}, \hat{\Phi}) d\tau + \int_0^t (\hat{H}, \hat{\Phi}) d\tau.
\]

where
\[H = -(E + v \times B) \cdot \nabla_v f + \frac{1}{2} E \cdot v f \]

is the nonlinear terms. Here we have used the notations $(\cdot, \cdot)_{L^2_k}, (\hat{f}, \hat{\Phi})_{\tau=t} = (\hat{f}, \hat{\Phi})(t)$ and $(\hat{f}, \hat{\Phi})_{\tau=0} = (\hat{f}, \hat{\Phi})(0)$. Plugging in the macro-micro decomposition yields
\[
\int_0^t \left( \hat{P}, f \cdot ik \hat{\Phi} \right) d\tau = \left( \hat{f}, \hat{\Phi} \right)(t) - \left( \hat{f}, \hat{\Phi} \right)(0) - \int_0^t \left( \hat{f}, \partial_t \hat{\Phi} \right) d\tau - \int_0^t \left( \hat{E}, v \cdot \mu \hat{\Phi} \right) d\tau
- \int_0^t \left( \left( \hat{E}, v \cdot \mu \hat{\Phi} \right) - \int_0^t (L_{FP} \hat{f}, \hat{\Phi}) d\tau - \int_0^t (\hat{H}, \hat{\Phi}) d\tau.\]

\]
Thanks to (1.20), we can get the conservation law for mass, i.e.

\[
\int_{T^3} a(\tau, x) dx = \int_{T^3} a(0, x) dx = 0,
\]
so that \( \dot{a}(\tau, 0) = 0 \). Now we choose the test function as

\[
\hat{\Phi}(\tau, k, v) = (|v|^2 - 10)v \cdot ik \hat{\phi}_a(\tau, k) \mu^\frac{1}{2},
\]  

(4.3)

where \( \hat{\phi}_a(\tau, k) \) is a solution to

\[
|k|^2 \hat{\phi}_a(\tau, k) = \dot{a}(\tau, k).
\]

Since \( \dot{a}(\tau, 0) = 0 \), we can formally write \( \hat{\phi}_a(\tau, k) = \dot{a}(\tau, k)/|k|^2 \) for any \( k \in \mathbb{Z}^3 \) with the understanding that we define \( \hat{\phi}_a(\tau, 0) = 0 \). By this choice and with the estimate of \( a \) in [10], we can obtain

\[
\int_0^t (P \dot{f}, v \cdot ik \dot{\Phi}) d\tau = 5 \int_0^t |\dot{a}(\tau, k)|^2 d\tau,
\]

and

\[
|J_1| \leq \| \ddot{f}(t, k, \cdot) \|^2_{L^2_{t-k}} + \| \hat{f}_0(k) \|^2_{L^2_{k}} + \int_0^t \| \hat{b}(\tau, k) \|^2_{L^2_{t-k}} d\tau + \int_0^t |(I - P) \dot{f}(\tau, k)|^2_{L^2_{t-k}} d\tau,
\]

\[
|J_3| \leq \eta \int_0^t |\dot{a}(\tau, k)|^2 d\tau + C\eta \int_0^t |(I - P) \dot{f}(\tau, k)|^2_{L^2_{t-k}} d\tau.
\]

where \( \eta > 0 \) is a sufficiently small universal constant. Now we concentrate on the other terms.

For the estimate of \( J_2 \), we have

\[
J_2 = - \int_0^t (\ddot{E}, v \cdot \mu^\frac{1}{2} \hat{\Phi}) d\tau = - \int_0^t (\ddot{E}, v \cdot (|v|^2 - 10)v \cdot ik \hat{\phi}_a(\tau, k) \mu) d\tau = 5 \int_0^t (\ddot{E}, ik \hat{\phi}_a) d\tau = 5 \int_0^t (\ddot{E}, \hat{\phi}) d\tau = 5 \int_0^t (\ddot{E}, \hat{\phi}) d\tau = -5 \int_0^t (\ddot{E}, \hat{\phi}) d\tau.
\]

where we have used \( |k|^2 \hat{\phi}_a(\tau, k) = \dot{a}(\tau, k) \) and \( ik \cdot \ddot{E} = \dot{a} \) in (3.3).

For the estimate of \( J_4 \), we can get

\[
|J_4| = \left| \int_0^t (L_{FP} \dot{f}, \hat{\Phi}) d\tau \right| = \left| \int_0^t (\Delta_v \dot{f} + \frac{1}{4} (6 - |v|^2) \dot{f}, \hat{\Phi}) d\tau \right|
\]

\[
\leq \left| \int_0^t (\Delta_v \dot{f}, \hat{\Phi}) d\tau \right| + \left| \int_0^t (\frac{1}{4} (6 - |v|^2) \dot{f}, \hat{\Phi}) d\tau \right|
\]

by using (1.6). By virtue of the macro-micro decomposition (1.9) and (1.7) gives

\[
J_{4,1} \leq \left| \int_0^t (\Delta_v \mu^\frac{1}{2}, \hat{\Phi}) d\tau \right| + \left| \int_0^t (\Delta_v (I - P) \dot{f}, \hat{\Phi}) d\tau \right|
\]

\[
\leq \left| \int_0^t (\dot{a} \Delta_v \mu^\frac{1}{2}, \hat{\Phi}) d\tau \right| + \left| \int_0^t (\ddot{b} \Delta_v (v \mu^\frac{1}{2}), \hat{\Phi}) d\tau \right| + \left| \int_0^t (\Delta_v (I - P) \dot{f}, \hat{\Phi}) d\tau \right|
\]

Since \( \Delta_v \mu^\frac{1}{2} = (\frac{1}{4} v^2 - \frac{3}{2}) \mu^\frac{1}{2} \) and (4.3), we get

\[
J_{4,1} \leq \left| \int_0^t (\dot{a} \Delta_v \mu^\frac{1}{2}, (|v|^2 - 10) v \cdot ik \hat{\phi}_a(\tau, k) \mu^\frac{1}{2}) d\tau \right|
\]

\[
= \left| \int_0^t \left( \dot{a} \left( \frac{1}{4} v^2 - \frac{3}{2} \right) \mu^\frac{1}{2}, (|v|^2 - 10) v \cdot ik \hat{\phi}_a(\tau, k) \mu^\frac{1}{2} \right) d\tau \right| = 0
\]

as the integrand function is odd for \( v \). Due to \( k \in \mathbb{Z}^3 \) and \( |k|^2 \hat{\phi}_a(\tau, k) = \dot{a}(\tau, k) \), it holds that

\[
J_{4,1} \lesssim \int_0^t |\ddot{b}(\tau, k)||k||\hat{\phi}_a(\tau, k)| d\tau \lesssim \eta \int_0^t |\ddot{a}(\tau, k)|^2 d\tau + C\eta \int_0^t |\ddot{b}(\tau, k)|^2 d\tau
\]
by Young’s inequality. Similarly, we can obtain
\[ J_{3,1} = \left| \int_0^t \langle (I - P) \hat{f}, \Delta \phi \rangle d\tau \right| \]
\[ \lesssim \eta \int_0^t |\hat{a}(\tau, k)|^2 d\tau + C \eta \int_0^t |(I - P) \hat{f}(\tau, k)|^2 d\tau \]
Regarding the estimate of \( J_{4,2} \), we can also show that
\[ J_{4,2} \lesssim \eta \int_0^t |\hat{a}(\tau, k)|^2 d\tau + C \eta \int_0^t |\hat{b}(\tau, k)|^2 d\tau + C \eta \int_0^t |(I - P) \hat{f}(\tau, k)|^2 d\tau \]
For the \( J_5 \), by virtue of Young’s inequality, we have
\[ |J_5| = \int_0^t \langle \hat{H}, (|v|^2 - 10) v \cdot \bar{\phi}_d(\tau, k) \mu \rangle d\tau \]
\[ \lesssim \eta \int_0^t |\hat{a}(\tau, k)|^2 d\tau + C \eta \int_0^t |(\hat{H}(\tau, k), \mu \rangle|^2 d\tau \]
Collecting the above estimates, we obtain
\[ \int_0^t |\hat{a}(\tau, k)|^2 d\tau + \int_0^t |\hat{E}(\tau, k)|^2 d\tau \lesssim \int_0^t |\hat{f}(k, t)|^2 d\tau + \int_0^t |\hat{f}_0(k)|^2 d\tau + \int_0^t |\hat{b}(\tau, k)|^2 d\tau \]
\[ + \int_0^t |(I - P) \hat{f}(\tau, k)|^2 d\tau + \int_0^t |(\hat{H}(\tau, k), \mu \rangle|^2 d\tau \]
Furthermore, by (4.2), Lemma 2.2, Lemma 2.3 and (3.5), it holds that
\[ \int_{\mathbb{R}^3} \left( \int_0^t |\hat{a}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) + \int_{\mathbb{R}^3} \left( \int_0^t |\hat{E}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]
\[ \lesssim \mathcal{E}(f(t)) + \int_{\mathbb{R}^3} |\hat{f}_0(k)| \| \hat{f}_0 \|_{L^2} d\Sigma(k) + \int_{\mathbb{R}^3} \left( \int_0^t |\hat{b}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]
\[ + \int_{\mathbb{R}^3} \left( \int_0^t |(I - P) \hat{f}(\tau, k, \cdot)|^2 d\tau \right)^{1/2} d\Sigma(k) + \mathcal{E}(f(t)) \mathcal{D}(f(t)) \]
where we have used
\[ \int_{\mathbb{R}^3} ||\hat{f}(t, k, \cdot)||_{L^2} d\Sigma(k) \leq \int_{\mathbb{R}^3} \sup_{0 \leq \tau \leq t} ||\hat{f}(\tau, k, \cdot)||_{L^2} d\Sigma(k) \leq \mathcal{E}(f(t)) \]
thus this completes the proof the theorem. \( \square \)

5. The Proofs of Our Main Results

This section is devoted to proving our main results. The first result is the global-in-time existence to the Cauchy problem (1.4) and (1.5), i.e. Theorem 1.1.

5.1. Proof of the global-in-time existence. For this purpose, suppose that the Cauchy problem (1.4) and (1.5) admits a unique local solution \( f(t, x, v) \) defined on the time interval \( 0 \leq t \leq T \) for some \( 0 < T < \infty \) and the solution \( f(t, x, v) \) satisfies the a priori assumption
\[ \sup_{0 \leq t \leq T} \mathcal{E}(f(t)) \leq M \] (5.1)
where \( M > 0 \) is a sufficiently small positive constant. Then use the continuation argument to extend such a solution step by step to a global one, one only need to deduce certain uniform-in-time energy type estimates on \( f(t, x, v) \) such that the a priori assumption (5.1) can be closed.

Lemma 5.1. There is a positive constant \( M > 0 \) such that if
\[ \sup_{0 \leq t \leq T} \mathcal{E}(f(t)) \leq M \]
for any \( 0 < T < \infty \), then we can deduce that
\[ \mathcal{E}(f(t)) + \mathcal{D}(f(t)) \leq \mathcal{E}(f_0). \]
Proof. Choosing $\lambda > 0$ small enough and adding $\lambda \times (4.1)$ to (3.1), we can get

$$\int_{Z_k^2} \sup_{0 \leq \tau \leq t} \left( \|\hat{f}(\tau, k, \cdot)\|_{L^2} + |\hat{E}(\tau, k)| + |\hat{B}(\tau, k)| \right) d\Sigma(k)$$

$$+ \int_{Z_k^2} \left( \int_0^t \|\{I - P\} \hat{f}(\tau, k, \cdot)\|^2 d\tau \right)^{1/2} d\Sigma(k) + \int_{Z_k^2} \left( \int_0^t |\hat{b}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k)$$

$$+ \lambda \int_{Z_k^2} \left( \int_0^t |\hat{a}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) + \lambda \int_{Z_k^2} \left( \int_0^t |\hat{E}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k)$$

$$\lesssim \|\hat{f}_0\|_{L^1_k L^2} + \|\hat{E}_0(\cdot)\|_{L^1_k} + \|\hat{B}_0(\cdot)\|_{L^1_k} + \eta \mathcal{D}(f(t)) + \mathcal{E}(f(t))\mathcal{D}(f(t)).$$

Then taking $\eta > 0$ to be small enough and by using the definition $\mathcal{E}(f(t))$ and $\mathcal{D}(f(t))$, i.e. (1.15) and (1.16) yields

$$\mathcal{E}(f(t)) + \mathcal{D}(f(t)) \lesssim \mathcal{E}(f_0) + \mathcal{E}(f(t))\mathcal{D}(f(t)).$$

Then take $M > 0$ to be properly small, we have

$$\mathcal{E}(f(t)) + \mathcal{D}(f(t)) \leq \mathcal{E}(f_0).$$

This complete the proof of the lemma. \qed

The local-in-time existence and uniqueness of the solutions to the Cauchy problem (1.4) can be proved in terms of the energy functional given $\mathcal{E}(f(t))$ in (1.15). The details referred as [10] are omitted for simplicity.

Because of the smallness assumption on $\mathcal{E}(f_0)$, with Lemma 5.1 we have

$$\mathcal{E}(f(t)) + \mathcal{D}(f(t)) \lesssim \mathcal{E}(f_0).$$

Combing this with the local-in-time existence, the global mild solution and uniqueness follows immediately from the standard continuity argument. This completes the proof of the global existence and the uniform estimate (1.21).

5.2. Proof of the large-time behavior. In order to get the exponential time decay, we need the high regularity of spatial variable which contained the high order dissipation of $B$. For this purpose, taking the $L^2_v$ inner product of the Fourier transform of (1.4) with $|\vec{k}|^2 \hat{f}$, we have

$$(\partial_t \hat{f}, (|\vec{k}|^2 \hat{f})) + (iv \cdot \hat{k} \hat{f}, (|\vec{k}|^2 \hat{f})) + (\hat{E} * \nabla_v \hat{f}, (|\vec{k}|^2 \hat{f})) + (v \times \hat{B} * \nabla_v \hat{f}, (|\vec{k}|^2 \hat{f}))$$

$$- \frac{1}{2} (v \cdot \hat{E} \hat{f}, (|\vec{k}|^2 \hat{f})) - (\hat{E} \cdot v^2 \hat{f}, (|\vec{k}|^2 \hat{f})) = (L_{FP} \hat{f}, (|\vec{k}|^2 \hat{f})).$$

Then by using the same argument that was used to derive (3.1) and (4.1), we have

$$\int_{Z_k^2} \sup_{0 \leq \tau \leq t} \left( \|\langle k \rangle \hat{f}(\tau, k, \cdot)\|_{L^2} + \|\langle k \rangle \hat{E}(\tau, k)\| + \|\langle k \rangle \hat{B}(\tau, k)\| \right) d\Sigma(k)$$

$$+ \int_{Z_k^2} \left( \int_0^t \|\{I - P\} \hat{f}(\tau, k, \cdot)\|^2 d\tau \right)^{1/2} d\Sigma(k) + \int_{Z_k^2} \left( \int_0^t |\langle k \rangle \hat{b}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k)$$

$$\lesssim \|\langle k \rangle \hat{f}_0\|_{L^1_k L^2} + \|\langle k \rangle \hat{E}_0(\cdot)\|_{L^1_k} + \|\langle k \rangle \hat{B}_0(\cdot)\|_{L^1_k} + \eta \mathcal{D}_1(f(t)) + \mathcal{E}_1(f(t))\mathcal{D}_1(f(t)),$$

and

$$\int_{Z_k^2} \left( \int_0^t \|\langle k \rangle \hat{a}(\tau, k)\|^2 d\tau \right)^{1/2} d\Sigma(k) + \int_{Z_k^2} \left( \int_0^t \|\langle k \rangle \hat{E}(\tau, k)\|^2 d\tau \right)^{1/2} d\Sigma(k)$$

$$\lesssim \mathcal{E}_1(f(t)) + \|\langle k \rangle \hat{f}_0\|_{L^1_k L^2} + \int_{Z_k^2} \left( \int_0^t |\langle k \rangle \hat{b}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k)$$

$$+ \int_{Z_k^2} \left( \int_0^t \|\{I - P\} \hat{f}(\tau, k, \cdot)\|^2 d\tau \right)^{1/2} d\Sigma(k) + \mathcal{E}_1(f(t))\mathcal{D}_1(f(t)).$$

Here $\eta > 0$ is a sufficiently small universal constant. Now we give the estimate on the dissipation of $B$ from the Maxwell equations (3.3). Due to $\partial_t \hat{E} - ik \times \hat{B} = \hat{b}$ in (3.3)_{11}, we have
\[ |k \times \hat{B}|^2 = (ik \times \hat{B}, ik \times \hat{B}) = (ik \times \hat{B}, \partial_{x} \hat{E} + \hat{b}) \]
\[ = \partial_{x}(ik \times \hat{B}, \hat{E}) - (ik \times \partial_{x} \hat{B}, \hat{E}) + (ik \times \hat{B}, \hat{b}) \]
\[ = \partial_{x}(ik \times \hat{B}, \hat{E}) + |k \times \hat{E}|^2 + (ik \times \hat{B}, \hat{b}) \]  
\[ (5.4) \]

where \(- (ik \times \partial_{x} \hat{B}, \hat{E}) = -(k \times (k \times \hat{E})), \hat{E}) = |k \times \hat{E}|^2 \) and the dot product \((a, b) = a \cdot \bar{b}\) for any complex vectors \(a, b \in \mathbb{C}^3\) are used. Notice that \(|k|^2 |\hat{B}|^2 = |k \times \hat{B}|^2\) owing to \(k \cdot \hat{B} = 0\) and integrate \((5.4)\) over \([0, t]\) with respect to \(\tau\), we can get

\[ \int_{0}^{t} |k|^2 |\hat{B}(\tau, k)|^2 d\tau \leq (ik \times \hat{B})(t, k) + (ik \times \hat{B}, \hat{E})(0, k) \]
\[ + \int_{0}^{t} |k|^2 |\hat{E}(\tau, k)|^2 d\tau + \eta \int_{0}^{t} |k|^2 |\hat{B}(\tau, k)|^2 d\tau + \frac{1}{4\eta} \int_{0}^{t} |\hat{b}(\tau, k)|^2 d\tau \]
\[ \leq |k|^2 |\hat{B}(t, k)|^2 + |\hat{E}(t, k)|^2 + |k|^2 |\hat{B}(0, k)|^2 + |\hat{E}(0, k)|^2 \]
\[ + \int_{0}^{t} |k|^2 |\hat{E}(\tau, k)|^2 d\tau + \eta \int_{0}^{t} |k|^2 |\hat{B}(\tau, k)|^2 d\tau + \frac{1}{4\eta} \int_{0}^{t} |\hat{b}(\tau, k)|^2 d\tau, \]

where the Young’s inequality are used. Therefore, we have

\[ \int_{\mathbb{Z}^3} \left( \int_{0}^{t} |k|^2 |\hat{B}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \leq \mathcal{E}_1(f(t)) + \mathcal{E}_1(f_0) + \int_{\mathbb{Z}^3} \left( \int_{0}^{t} |k|^2 |\hat{E}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]
\[ + \int_{\mathbb{Z}^3} \left( \int_{0}^{t} |\hat{b}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k). \]
\[ (5.5) \]

for small enough constant \(\eta > 0\). Moreover, since \(\hat{B}(t, k) = \int_{\mathbb{Z}^3} e^{-ik \cdot x} B(t, x) dx, k \in \mathbb{Z}^3\) and the assumption \(\int_{\mathbb{T}^3} B(t, x) dx = 0\) in Theorem 1.2, we have \(\hat{B}(t, 0) = 0\) and the following estiamte

\[ \int_{\mathbb{Z}^3} \left( \int_{0}^{t} \langle k \rangle |\hat{B}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) = \int_{\mathbb{Z}^3} \left( \int_{0}^{t} (1 + |k|^2) |\hat{B}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]
\[ = \int_{\{k \in \mathbb{Z}^3 | |k| = 0\}} \left( \int_{0}^{t} (1 + |k|^2) |\hat{B}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]
\[ + \int_{\{k \in \mathbb{Z}^3 | |k| \geq 1\}} \left( \int_{0}^{t} (1 + |k|^2) |\hat{B}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]
\[ \leq \int_{0}^{t} |\hat{B}(\tau, 0)|^2 d\tau + \sqrt{2} \int_{\{k \in \mathbb{Z}^3 | |k| \geq 1\}} \left( \int_{0}^{t} |k|^2 |\hat{B}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]
\[ \leq \sqrt{2} \int_{\mathbb{Z}^3} \left( \int_{0}^{t} |k|^2 |\hat{B}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]

Combine the above inequality with the estimate of \((5.5)\), one can obtain that

\[ \int_{\mathbb{Z}^3} \left( \int_{0}^{t} \langle k \rangle^2 |\hat{B}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \leq \mathcal{E}_1(f(t)) + \mathcal{E}_1(f_0) + \int_{\mathbb{Z}^3} \left( \int_{0}^{t} |k|^2 |\hat{E}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k) \]
\[ + \int_{\mathbb{Z}^3} \left( \int_{0}^{t} |\hat{b}(\tau, k)|^2 d\tau \right)^{1/2} d\Sigma(k). \]
\[ (5.6) \]

Taking proper linear combination of \((5.2), (5.3)\) and \((5.6)\), we have

\[ \mathcal{E}_1(f(t)) + \mathcal{D}_1(f(t)) \lesssim \mathcal{E}_1(f_0) + \mathcal{E}_1(f(t)) \mathcal{D}_1(f(t)). \]

Due the smallness assumption on \(\mathcal{E}_1(f_0)\), we can get

\[ \mathcal{E}_1(f(t)) + \mathcal{D}_1(f(t)) \lesssim \mathcal{E}_1(f_0), \]
which gives (1.22) in Theorem 1.2. Combing this with the local-in-time existence, the global mild solution and uniqueness follows immediately from the standard continuity argument. This completes the proof of the global existence and the uniform estimate (1.21).

In order to obtain the time decay of the solutions, we let
\[
\hat{f} = e^M \hat{f}, \hat{E} = e^M \hat{E}, \hat{B} = e^M \hat{B}
\]
with \( \lambda > 0 \) which will be chosen later. Since \( \hat{f} \) and \( \hat{E}, \hat{B} \) satisfy the system (3.2) and (3.3), then \( \hat{f} \) and \( \hat{E}, \hat{B} \) satisfy
\[
\partial_t \hat{f} + iv \cdot k \hat{f} + e^{-M} \hat{E} \ast \nabla_v \hat{f} + e^{-M} (v \cdot \hat{B}) \ast \nabla_v \hat{f} - \frac{1}{2} e^{-M} v \cdot \hat{E} \ast \hat{f} - \hat{E} \cdot v \mu^2 = L_{FP} \hat{f} + \lambda \hat{f}.
\]
and
\[
\begin{align*}
\partial_t \hat{E} - ik \times \hat{B} &= -\hat{b} + e^{-M} \hat{E}, \\
\partial_t \hat{B} + ik \times \hat{E} &= e^{-M} \hat{B}, \\
ik \cdot \hat{E} &= \hat{a}, \quad ik \cdot \hat{B} = 0.
\end{align*}
\]
with initial data
\[
\hat{f}(0, k, v) = \hat{f}_0(k, v), \hat{E} = \hat{E}_0(k, v), \hat{B} = \hat{B}_0(k, v).
\]

By virtue of the same method used to derive Lemma 5.1, we have
\[
\mathcal{E}_1(\hat{f}(T)) + \mathcal{D}_1(\hat{f}(T)) \lesssim \mathcal{E}_1(\hat{f}_0) + \sqrt{X} \int_{Z^d_2^k} \left( \int_0^T \| \langle k \rangle \hat{f}(t, k, \cdot) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k)
\]
\[
+ \sqrt{X} \int_{Z^d_2^k} \left( \int_0^T |\langle k \rangle \hat{E}(t, k)|^2 dt \right)^{1/2} d\Sigma(k)
\]
\[
+ \sqrt{X} \int_{Z^d_2^k} \left( \int_0^T |\langle k \rangle \hat{B}(t, k)|^2 dt \right)^{1/2} d\Sigma(k)
\]
for any \( T > 0 \). Since
\[
\int_{Z^d_2^k} \left( \int_0^T \| \langle k \rangle \hat{f}(t, k, \cdot) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) + \int_{Z^d_2^k} \left( \int_0^T \| \langle k \rangle \hat{E}(t, k, \cdot) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k)
\]
\[
+ \int_{Z^d_2^k} \left( \int_0^T \| \langle k \rangle \hat{B}(t, k, \cdot) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \lesssim \mathcal{D}_1(\hat{f}(T))
\]
then take \( \lambda > 0 \) to be small enough in (5.9) yields that
\[
\mathcal{E}_1(\hat{f}(T)) + \mathcal{D}_1(\hat{f}(T)) \lesssim \mathcal{E}_1(\hat{f}_0).
\]
By the definition \( \mathcal{E}_m(\hat{f}(T)) \) in (1.18) and the Minkowski’s inequality \( \| \cdot \|_{L^1} \| L^\infty \| \leq \| \cdot \|_{L^1} \| L^\infty \|_{L^1} \), we can deduce
\[
\int_{Z^d_2^k} \left( \| \langle k \rangle \hat{f}(t, k, \cdot) \|_{L^2_v} + \| \langle k \rangle \hat{E}(t, k) \| + \| \langle k \rangle \hat{B}(t, k) \| \right) d\Sigma(k) \lesssim \mathcal{E}_1(\hat{f}_0).
\]
Recall that \( \hat{f} = e^M \hat{f}, \hat{E} = e^M \hat{E}, \hat{B} = e^M \hat{B} \), then one can obtain
\[
\int_{Z^d_2^k} \left( \| \langle k \rangle \hat{f}(t, k, \cdot) \|_{L^2_v} + \| \langle k \rangle \hat{E}(t, k) \| + \| \langle k \rangle \hat{B}(t, k) \| \right) d\Sigma(k) \lesssim e^{-M} \mathcal{E}_1(\hat{f}_0).
\]
which gives (1.23). Thus we have completed the proof of theorem 1.1.

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