Two-velocity hydrodynamics in fluid mechanics: Part II
Existence of global $\kappa$–entropy solutions to compressible Navier-Stokes systems with degenerate viscosities

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Abstract: This paper addresses the global existence problem of so-called $\kappa$–entropy solutions of Navier–Stokes equations for viscous compressible and barotropic fluids with degenerate viscosities. Such solutions satisfy in a weak sense the mass and momentum conservation equations and also a generalization of the BD–entropy identity called: $\kappa$–entropy. This new entropy involves a mixture parameter $\kappa \in (0, 1)$ between the two velocities $u$ and $u + 2\nabla \varphi(\rho)$ (the latter was introduced by the first two authors in [C. R. Acad. Sci. Paris 2004]), where $u$ is the velocity field and $\varphi$ is a function of the density $\rho$ defined by $\varphi'(s) = \mu'(s)/s$. The assumption $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$ is also required as in previous works by the two first authors to establish the $\kappa$-entropy identity. As a byproduct of the existence proof, a two-velocity hydrodynamical model (in the spirit of works by S.C. Shugrin for instance) is shown to be included in the barotropic compressible flows with degenerate viscosities. It allows for the construction of approximate solutions based on an augmented approximate scheme of the compressible Navier-Stokes equations extending an approach developed for some inviscid compressible systems with capillarity by other authors.

Keywords. Hypocoercivity, Compressible Navier-Stokes, Augmented system, Two-velocity hydrodynamics, $\kappa$-entropy.

1 Introduction

In 2006–2007, the first two authors [6] [7] [8] introduced the concept of global weak solutions of Navier–Stokes equations for compressible barotropic fluids satisfying the energy inequality and an extra math-
mathematical entropy called BD entropy in the case of degenerate viscosities. The BD entropy identity was derived in [4] under the assumption \( \lambda(g) = 2(\mu'(g)g - \mu(g)) \) generalizing the one with \( \mu(g) = g \) and \( \lambda(g) = 0 \) that was introduced by the first two authors and C.K. LIN in [5]. This BD entropy involves an energy related to the velocity \( \mathbf{u} + 2 \nabla \varphi(g) \) where \( \varphi \) is a function of the density \( g \) defined by \( \varphi'(s) = \mu'(s)/s \).

It is interesting to note that the quantity \( 2 \nabla \varphi(g) \) appears in other mathematical works for inviscid systems: see [3] and references therein with \( 2 \nabla \log g \) for example. See also the works by E. NELSON (for instance [27]) related to kinematics of Markovian motion where the presence of two velocities is also discussed: The quantity \( \mu \nabla \varphi \) is described as a velocity required for the particle to counteract osmotic effects (osmotic velocity) and the current velocity.

Using some drag force in the momentum conservation equation or an additional singular pressure terms, stability of global weak solutions (to the barotropic compressible Navier–Stokes equations with density dependent viscosities) satisfying the extra BD entropy was shown in [8], [6] and with D. GÉRARD-VARET in [9] with no need of multiplying the momentum equation by \( g \) as in [5]. Note that without such extra terms, stability of global weak solutions satisfying the extra BD entropy was obtained by A. MELLET and A. VASSEUR in [23] showing an extra estimate on the velocity field but the construction of approximate solutions satisfying the energy, the BD entropy and the Mellet–Vasseur estimate together is still open [4]. Concerning the construction of approximate solutions with singular pressure, drag terms or capillarity terms, the authors gave some hints in [7] for the general setting \( \lambda(g) = 2(\mu'(g)g - \mu(g)) \).

This has been fully developed in [25] and [26] in the case of a linear viscosity \( \mu(g) = g \) based on well chosen but complicated regularized terms: regularization of the mass equation and extra regularized terms in the momentum equation inspired by those given in [7]; and high-order derivative capillarity term, together with regularization for the velocity \( \mathbf{u} + 2 \nabla \varphi(g) \). Note that complete proof of the construction of approximate solutions satisfying the energy and the BD entropy in the general setting \( \lambda(g) = 2(\mu'(g)g - \mu(g)) \) seems not so easy even with the hints given in [7].

In this paper, we propose to come back to the compressible Navier-Stokes equation with degenerate viscosities in the general setting with a very simple construction of approximate solutions and concept of global solutions. We propose a very natural concept of global \( \kappa \)-entropy solutions based on a generalized BD entropy. Through this \( \kappa \)-entropy, we introduce a two-velocity hydrodynamical model in the compressible Navier–Stokes equations with degenerate viscosities. Interestingly enough, it turns out to be linked to [30] and [17] introducing a mixture parameter \( \kappa \in (0, 1) \) involving two velocity fields \( \mathbf{u} \) and \( \mathbf{u} + 2 \nabla \varphi(g) \) sharing the same reference density. The augmented system we will use to construct approximate solution is similar to systems which may be found in [30] with the main velocity \( \mathbf{u} + 2 \kappa \nabla \varphi(g) \), a drift \( 2 \nabla \varphi(g) \) and same reference density \( g \). In our system, regularization of the mass conservation equation will follow from the introduction of the new velocity \( \mathbf{w} \) (no need of mass regularization). The generalized BD entropy (\( \kappa \)-entropy) reflects some non-linear hypercoercivity property of the nonlinear compressible Navier-Stokes equations under the aforementioned relation between \( \lambda \) and \( \mu \). For an introduction to hypercoercivity, the interested reader is referred to the paper by K. BEAUCHARD and E. ZUAZUA [1] and the book by C. VILLANI [32] (and references cited therein) which describe its link to global existence around equilibrium and large-time behavior (see the interesting paper by R. DANCHIN [13] for the application to fluid mechanics). Readers interested on papers related to entropy for nonlinear partial differential equations are referred to [12]. Our generalized BD entropy (\( \kappa \)-entropies) may be seen as a nonlinear version of the identity that was proven by A. MATSUMURA–T. NISHIDA on the linearized compressible system around

\footnote{The first author has been recently informed that A. VASSEUR and C. YU have obtained a very nice result which may be summarized as follows: If we have a global weak solution satisfying the energy and the BD entropy estimates of the compressible Navier–Stokes equations with turbulent drag terms, then it is possible to construct smooth multipliers allowing to get the Mellet–Vasseur estimate (uniformly with respect to the drag coefficient) and therefore to suppress the drag terms letting such drag coefficient tend to zero. This is a real breakthrough which, coupled with our present paper, would give the first result (construction and stability) of global existence to the compressible Navier–Stokes equations with general degenerate viscosities under the relation \( \lambda(g) = 2(\mu'(g)g - \mu(g)) \) without extra terms!}
the equilibrium \((\varrho_{eq}, u_{eq}) = (1, 0)\). For the reader’s convenience, we revisit hypocoercivity on linearized compressible Navier–Stokes systems in the last two sections of the paper (barotropic and heat-conducting case). This will show that \(\kappa\)-entropy estimates on such linearized system with a coefficient \(\kappa\) may be chosen arbitrarily small. The definition of the global \(\kappa\)-entropy solution is strongly associated with two velocity hydrodynamics in the compressible Navier–Stokes equations with degenerate viscosities. The reader interested on the subject is referred for instance to [30] for some discussions around two velocity Hydrodynamics. Remark that a 1/2–entropy solution has been obtained recently by [M. GISCLON and I. VIOLET, Preprint (2014)], for \(\mu(\varrho) = \varrho\) and \(\lambda(\varrho) = 0\), starting from the quantum compressible Navier-Stokes equations studied by A. JÜNGEL in [20] (extended in [14] and [19]) with an additional singular pressure and letting the scaled Planck constant vanish. Their proof strongly relies on the Bohm potential identity and therefore works only for \(\mu(\varrho) = \varrho\) in the multi-dimensional in space case. Remark also that a global weak solution of the compressible Navier-Stokes equations which satisfies the BD entropy is also a \(\kappa\)-entropy solution for all \(0 \leq \kappa \leq 1\). The construction of the approximate solutions is based on a nice augmented approximate scheme of the compressible Navier-Stokes equations which extends an approach developped in inviscid compressible systems with dispersion (see for instance S. BENZONI, R. DANCHIN, S. DESCOMBES and F. BÉHUEL, R. DANCHIN, P. GRAVEJAT, J.–C. SAUT, D. SMETS). Similar approximate schemes have been introduced for zero Mach number systems in Two-Velocity Hydrodynamics in Fluid Mechanics: Part I, see [10]. In some sense, it seems that our result is the first showing clearly the existence of two-velocity hydrodynamics in the barotropic compressible Navier–Stokes equations with degenerate viscosities. The augmented system is closed to systems written in [30] (with a joint density for the two species) with the main velocity given by \(w = u + 2\kappa \nabla \varphi(\varrho)\) and the drift by \(v = 2\nabla \varphi(\varrho)\). The reader is referred to the mathematical paper by E. FEIREISL and A. VASSEUR where they study a compressible system with two-velocities proposed by H. BRENNER. It is also important to note that the introduction of an additional unknown (the gradient of a function of the density) in order to rewrite the system into a hyperbolic system perturbed by a second order skew symmetric term has been used recently by P. NOBLE and J.–P. VILA to study the stability of various approximations of the one-dimension Euler–Korteweg equations (dispersive system). Our scheme is thus relevant to derive numerical schemes with an additional unknown for Navier-Stokes-Korteweg systems (dispersive system). This will be the purpose of a forthcoming paper.

In the last section, we introduce a thermodynamically consistent two velocity model with heat conductivity associated with \(u\) and \(u + 2\nabla \varphi(\varrho)\) in the spirit of the work written by S.M. SHUGRIN in [30], each component being respectively associated with the density \((1 - \kappa)\varrho\) and \(\kappa \varrho\). The main objective is not a mathematical proof of global existence but to show that two-velocity hydrodynamics as in [30] is consistent with the study which has been made in Two-Velocity Hydrodynamics in Fluid Mechanics: Part I namely [10] for some low Mach number system. More precisely, we will first formally show that we do not solve the usual heat-conducting compressible Navier-Stokes equation solving the two-velocity hydrodynamics system with total energy because the \(\kappa\)-temperature is not \textit{a priori} the usual temperature. We also show that formal low-Mach number limit on such system gives the augmented system used in [10] to get the existence result: a kind of consistency between the approaches and the systems.

2 The barotropic Navier-Stokes system

Two compressible fluid models with degenerate viscosity and pressure depending only on the density (barotropic flows) will be considered.

1) Compressible Navier-Stokes with singular pressure. First the compressible Navier–Stokes equations...
for compressible and barotropic fluids write as follows:

\[
\begin{aligned}
&\frac{\partial}{\partial t} \rho + \text{div}(\rho \mathbf{u}) = 0, \\
&\frac{\partial}{\partial t} (\rho \mathbf{u}) + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) - \text{div} (2\mu(\rho)D(\mathbf{u})) - \nabla(\lambda(\rho)\text{div} \mathbf{u}) + \nabla p(\rho) = 0,
\end{aligned}
\]

in \((0,T) \times \Omega \) (1)

where \(D(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla' \mathbf{u})\), \((\rho \mathbf{u} \otimes \mathbf{u})_j = \partial_i(\rho \mathbf{u}_i \mathbf{u}_j)\) and \(\Omega\) is a periodic box \(\Omega = T^3\). The pressure \(p\) is singular close to zero density as previously introduced in [8]: more precisely the pressure \(p(\rho)\) has some stabilizing part close to vacuum as in [25]

\[
p'(\rho) = \begin{cases} 
    c_1 \rho^{-\gamma^+ - 1} & \text{for } \rho \leq \rho^*, \\
    c_2 \rho^{\gamma^- + 1} & \text{for } \rho > \rho^*
\end{cases}
\]

with \(c_1 > 0, \gamma^+ > 1\) and \(\gamma^- > 0\). A more precise estimation of \(\gamma^-\) will be given below. The viscosity coefficients \(\mu(\rho), \lambda(\rho)\) satisfy the Bresch-Desjardins relation introduced in [4]

\[
\lambda(\rho) = 2(\rho \mu'(\rho) - \mu(\rho)).
\]

This system is completed with initial data

\[
\rho|_{t=0} = \rho_0, \quad (\rho \mathbf{u})|_{t=0} = \mathbf{m}_0.
\]

**Remark 1** Note that, with the assumptions on \(p'\), negative pressures are possible: The fluid behaves as a solid for small density. In fact negative power in a neighborhood of vacuum is sufficient for our purposes.

2) **Compressible Navier-Stokes with a turbulent drag term.** We discuss, in Section 4, how to handle the compressible Navier–Stokes equations with turbulent drag terms and standard gamma-type pressure law, namely

\[
\begin{aligned}
&\frac{\partial}{\partial t} \rho + \text{div}(\rho \mathbf{u}) = 0, \\
&\frac{\partial}{\partial t} (\rho \mathbf{u}) + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) - \text{div} (2\mu(\rho)D(\mathbf{u})) - \nabla(\lambda(\rho)\text{div} \mathbf{u}) + r_1 \rho |\mathbf{u}| \mathbf{u} + \nabla p(\rho) = 0,
\end{aligned}
\]

with the usual pressure law \(p(\rho) = a \rho^\gamma\) with \(\gamma > 1\). It is important to cover such situation, see the footnote in the introduction. The assumptions on the viscosity in this section will be similar to the one introduced in [23] with the extra control \(\mu(\rho) \leq C \rho^{2/3 + 1/3\nu}\) with \(\nu \in (1/\gamma, 1)\) when \(\rho \geq 1\). This additional assumption is introduced because the energy and BD entropy are mixed in the \(\kappa\)-entropy (less information is available compared to [9]). For this part, we will only show the extra terms that have to be included in the augmented system and show how the \(\kappa\)-entropy is changed. The existence results and asymptotic limit with respect to smoothing parameters \(\alpha, n\) and \(\delta\) will not be affected when \(\varepsilon\) is fixed. The asymptotic limit when \(\varepsilon\) tends to 0 is then the same as in [9].

**Remark 2** The present work may be extended to dispersive compressible Navier-Stokes system (Korteweg type system) when the additional dispersive term is compatible with the velocity \(\mathbf{u} + 2\kappa \nabla \varphi(\rho)\). For instance, when \(\varphi(\rho) = \log \rho\), the quantum dispersive term \(-\rho \nabla \left( \Delta \sqrt{\rho} / \sqrt{\rho} \right)\) studied in [20]: see recent works by M. Gisclon–I. Violet or A. Jüengel. This will be considered in [11] for numerical purposes.
2.1 Definition of a global $\kappa$-entropy solution of (1)-(4)

**Definition 1** Let $\kappa$ be such that $0 < \kappa < 1$, the couple of functions $(\rho, \mathbf{u})$ is called a global $\kappa$-entropy solution to system (1)-(4) if the following properties are satisfied:

- The mass equation is satisfied in the following sense
  \[ -\int_\Omega \rho \partial_t \xi \, dx - \int_\Omega \rho \mathbf{u} \cdot \nabla \xi \, dx = \int_\Omega \rho^0 \xi(0) \, dx \tag{6} \]
  for all $\xi \in C^\infty_c((0, T) \times \Omega)$.

- The momentum equation is satisfied in the following sense
  \[ -\int_\Omega \rho \mathbf{u} \cdot \partial_t \mathbf{\phi} \, dx - \int_\Omega (\rho \mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{\phi} \, dx + \int_\Omega 2\mu(\rho) D(\mathbf{u}) : \nabla \mathbf{\phi} \, dx + \int_\Omega \lambda(\rho) \text{div} \mathbf{u} \, \text{div} \mathbf{\phi} \, dx - \int_\Omega \rho \phi(0) \, dx \tag{7} \]
  for all $\mathbf{\phi} \in (C^\infty_c((0, T) \times \Omega))^3$.

- Moreover $(\rho, \mathbf{u})$ satisfies, for all $t \in [0, T]$, the following $\kappa$-entropy estimates
  \[ \sup_{t \in [0, T]} \left[ \int_\Omega \rho \left( \frac{|\mathbf{u} + 2\kappa \nabla \varphi(\rho)|^2}{2} + (1 - \kappa)\kappa \frac{|2\nabla \varphi(\rho)|^2}{2} \right) \right] \tag{8} \]
  \[ + 2\kappa \int_0^T \int_\Omega \mu(\rho) |A(\mathbf{u})|^2 \, dx \, ds + 2\kappa \int_0^T \int_\Omega \mu'(\rho) \rho' \frac{\rho}{\eta} \nabla \varphi|^2 \, dx \, ds \]
  \[ + 2(1 - \kappa) \int_0^T \int_\Omega (\mu(\rho) |D(\mathbf{u})|^2 + |\mu'(\rho) \rho - \mu(\rho)| \text{div} \mathbf{u}|^2) \, dx \, ds \]
  \[ \leq \int_\Omega \rho \left( \frac{|\mathbf{u} + 2\kappa \nabla \varphi(\rho)|^2}{2} + (1 - \kappa)\kappa \frac{|2\nabla \varphi(\rho)|^2}{2} \right) \, dx + \int_\Omega \rho_0 \varphi(\rho_0) \, dx \]
  with $\varphi'(s) = \mu'(s)/s$ and where we have introduced the internal energy $e(\rho)$ defined by
  \[ \frac{\rho^2 \text{d}e(\rho)}{\text{d}\rho} = p(\rho). \]

Let us note that such $\kappa$-entropy is a nonlinear generalization of the pseudocoercivity properties shown on linearized compressible Navier-Stokes equations. For reader’s convenience, we provide the calculations in this framework in Section 5 after the proof of existence.

**Remark 3** Note that
  \[ \int_\Omega \rho \left( \frac{|\mathbf{u} + 2\kappa \nabla \varphi(\rho)|^2}{2} + (1 - \kappa)\kappa \frac{|2\nabla \varphi(\rho)|^2}{2} \right) \, dx \]
  \[ = \int_\Omega \rho \left( (1 - \kappa) \frac{|\mathbf{u}|^2}{2} + \kappa \frac{|\mathbf{u} + 2\nabla \varphi(\rho)|^2}{2} \right) \, dx. \tag{9} \]
  That means that the $\kappa$-entropy shows two-velocity hydrodynamics in the usual compressible Navier-Stokes system. See for instance [30] and [17] for unknowns introduction in two-velocity hydrodynamics and thermodynamics. Indeed, using the identity
  \[ \mathbf{u} + 2\kappa \nabla \varphi(\rho) = (1 - \kappa) \mathbf{u} + \kappa (\mathbf{u} + 2\nabla \varphi(\rho)) \]
  we see that (9) is the kinetic energy of a two-fluid mixture having the velocities $\mathbf{u}$ and $\mathbf{u} + 2\nabla \varphi(\rho)$ and $\kappa$ playing the role of the mass fraction. The interested reader is referred to [30] for similar calculations.
Remark 4 Equality \( [8] \) is a generalization of the BD entropy obtained by D. Bresch and B. Desjardins in the case \( \kappa = 1 \). More precisely, they formally derived the following identity

\[
\frac{d}{dt} \int_{\Omega} \varrho \left| \frac{u + 2 \nabla \varphi(\varrho)}{2} \right|^2 \, dx + \frac{d}{dt} \int_{\Omega} \varrho c(\varrho) \, dx + \int_{\Omega} 2\mu(\varrho) |A(w)|^2 \, dx + \int_{\Omega} \frac{2\mu(\varrho)\mu'(\varrho)}{\varrho} \left| \nabla \varphi \right|^2 \, dx = 0. \tag{10}
\]

Remark 5 If a global solution satisfies the BD entropy (10) and the standard energy balance

\[
\frac{d}{dt} \int_{\Omega} \varrho \frac{|u|^2}{2} \, dx + \frac{d}{dt} \int_{\Omega} \varpi(\varrho) \, dx + \int_{\Omega} 2\mu(\varrho)|D(u)|^2 \, dx + \int_{\Omega} \lambda(\varrho) |\text{div} u|^2 \, dx = 0,
\]

it also satisfies the \( \kappa \) entropy for all \( 0 < \kappa < 1 \). It suffices to use the identity

\[
\kappa |u + 2 \nabla \varphi(\varrho)|^2 + (1 - \kappa) |u|^2 = |u + 2\kappa \nabla \varphi(\varrho)|^2 + (1 - \kappa) \kappa |2 \nabla \varphi(\varrho)|^2
\]

and therefore to add \( \kappa \) times the BD entropy to \((1 - \kappa)\) times the energy. A global weak solution satisfying the BD entropy is therefore a \( \kappa \)-entropy solution for all \( 0 < \kappa < 1 \). The converse is not clear because there is no reason for a \( \kappa \)-entropy solution to be a global weak solution which satisfies the BD entropy.

Remark 6 Note that our constructive scheme is relevant for numerical purposes. It has already been used in the inviscid framework by P. Noble and J.-P. Vila in \([29]\) with extra dispersive terms (see also references cited therein).

2.2 Main results

The initial data \([4]\) are assumed to satisfy:

\[
\varrho^0 \geq 0, \quad \varrho^0 \in L^1(\Omega), \quad \varrho^0 c(\varrho^0) \in L^1(\Omega), \quad \nabla \mu(\varrho^0) \in L^2(\Omega), \quad \frac{\nabla \mu(\varrho^0)}{\sqrt{\varrho^0}} \in L^2(\Omega), \tag{11}
\]

\[
\frac{|m^0|^2}{\varrho^0} = 0, \quad \text{a.e. on } \{ x \in \Omega : \varrho^0(x) = 0 \}, \quad \frac{m^0}{\sqrt{\varrho^0}} \in L^2(\Omega). \tag{12}
\]

1. Compressible Navier–Stokes equations with singular pressure. In what follows we will make some assumptions concerning the viscosity coefficients and the pressure. We assume that \( \mu(\cdot), \lambda(\cdot) \) are \( C^1([0, \infty)) \) such that \( \mu'(\varrho) \geq c > c, \mu(0) = 0 \), the following relation is satisfied

\[
\lambda(\varrho) = 2(\mu'(\varrho) \varrho - \mu(\varrho)).
\]

Moreover, there exists positive constants \( c_0, c_1, \varrho^* \),

\[
m > 3/4, \quad 2/3 < n < \frac{\gamma^- + 1}{2}, \tag{13}
\]

such that:

\[
\text{for all } s < \varrho^*, \quad \mu(s) \geq c_0 s^n \quad \text{and} \quad 3\lambda(s) + 2\mu(s) \geq s^n,
\]

\[
\text{for all } s \geq \varrho^*, \quad c_1 s^m \leq \mu(s) \leq \frac{s^m}{c_1} \quad \text{and} \quad c_1 s^m \leq 3\lambda(s) + 2\mu(s) \leq \frac{s^m}{c_1}.
\]

The pressure \([2]\) is a \( C^1([0, \infty)) \) of \( \varrho \) and we assume that:

\[
\gamma^+ > 1, \quad \gamma^- > \frac{2n(3m - 2)}{4m - 3} - 1. \tag{14}
\]
Remark 7 Assumptions (13) and (14) follow from information we need to pass to the limit in the convective term. More precisely one needs to ensure that
\[ \frac{1}{\rho^2} u \in L^r(0, T; L^s(\Omega)) \]
with \( r, s > 2 \): the details of this estimate will be given in at the end of Section 3.4.

Our main results reads as follows:

Theorem 1 Assume that \( 0 < \kappa < 1 \) is fixed. If the initial data, viscosity coefficients and \( \gamma^\pm \) satisfy all the assumptions above with a singular pressure given by (2) then there exists global in time weak \( \kappa \)-entropy solution in the sense of Definition 1.

2. Compressible Navier–Stokes equations with a turbulent drag term. In this part, we assume the following hypotheses on viscosities \( \mu(\rho) \) and \( \lambda(\rho) \) that may be found for instance in [23]: let \( \mu(\cdot), \lambda(\cdot) \) are \( C^1([0, \infty)) \) such that there exists a positive \( \nu \in (0, 1) \) with
\[ \mu'(\rho) \geq \nu, \quad \mu(0) \geq 0, \] (15)
and
\[ |\lambda'(\rho)| \leq \frac{1}{\nu} \mu'(\rho), \quad \nu \mu(\rho) \leq 2\mu(\rho) + 3\lambda(\rho) \leq \frac{1}{\nu} \mu(\rho). \] (16)
The following relation (introduced by the two first authors) is also assumed
\[ \lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)). \] (17)
As stressed in [23], the hypothesis above imply that
\[ C\rho^{2/3+\nu/3} \leq \mu(\rho) \leq C\rho^{2/3+1/(3\nu)} \text{ when } \rho \geq 1, \] (18)
\[ C\rho^{2/3+1/(3\nu)} \leq \mu(\rho) \leq C\rho^{2/3+\nu/3} \text{ when } \rho \leq 1. \] (19)
The interested reader is referred to [23] for more details to see where such hypothesis are used for stability purposes.

Remark 8 It is important to remark that an extra assumption is required to get the Mellet-Vasseur estimate when no turbulent drag term is included in the momentum equation. Namely if \( \gamma \geq 3 \), it is also assumed that
\[ \liminf_{\rho \to +\infty} \frac{\mu(\rho)}{\rho^{\gamma/3+\zeta}} > 0 \]
with \( \zeta > 0 \).

In this part, we add the following hypothesis to deal with the turbulent drag term because, compared to [9], the \( \kappa \)-entropy mix the usual energy estimate and the BD-entropy (less information is available):
\[ \mu(\rho) \leq C\rho^{2/3+1/(3\eta)} \text{, with } \eta \in (1/\gamma, 1) \text{ when } \rho \geq 1. \] (20)

Theorem 2 Assume that \( 0 < \kappa < 1 \) be fixed. Let us assume (15)–(20) and (11)–(12) be satisfied with a pressure \( p(\rho) = a\rho^\gamma \) with \( \gamma > 1 \), then there exists global in time weak \( \kappa \)-entropy solution of (24) in the similar sense of Definition 1 (with the extra term coming from the turbulent drag and the power-law pressure).

Remark 9 Note that the drag term gives the extra information on \( \rho|u|^2 \) needed to pass to the limit in the convective term: It replaces Mellet-Vasseur’s estimate.
2.3 Change of variable and $\kappa$–entropy

We generalize to the compressible framework some ideas developped recently in [10] for low Mach number systems with heat conductivity effects. More precisely, let us define the following velocity field generalizing the one introduced in the BD entropy estimate

$$w = u + 2\kappa \nabla \varphi(\rho)$$  \hspace{1cm} (21)

with

$$\varphi'(\rho) = \frac{\mu'(\rho)}{\rho}.$$  \hspace{1cm} (22)

Note that BD entropy, as introduced in [4] by the first two authors, corresponds to $\kappa = 1$. Assuming that solutions to (1) are smooth enough, it can be shown that $w$ satisfies the following evolution equation

$$\partial_t (\rho w) + \text{div} \left( \rho u \otimes w \right) - 2(1 - \kappa) \text{div}(\mu(\rho) D(w)) - 2\kappa \text{div}(\mu(\rho) A(w)) + 4(1 - \kappa)\mu(\rho) \nabla \varphi(\rho) - \nabla((\lambda(\rho) - 2\kappa(\mu'(\rho) \rho - \mu(\rho))) \text{div} u) + \nabla P(\rho) = 0.$$  \hspace{1cm} (23)

Let us now write the equation satisfied by $(\rho, w, \nabla \varphi(\rho))$. We get the system

$$\begin{aligned}
\partial_t \rho + \text{div} \left( \rho u \right) - 2\kappa \Delta \mu(\rho) &= 0, \\
\partial_t (\rho w) + \text{div} \left( \rho u \otimes w \right) - 2(1 - \kappa) \text{div}(\mu(\rho) \nabla w) - 2\kappa \text{div}(\mu(\rho) A(w)) + 4(1 - \kappa)\mu(\rho) \nabla \varphi(\rho) - \nabla((\lambda(\rho) - 2\kappa(\mu'(\rho) \rho - \mu(\rho))) \text{div} u) + \nabla P(\rho) &= 0, \\
\partial_t (\rho \nabla \varphi(\rho)) + \text{div}(\rho u \otimes \nabla \varphi(\rho)) - 2\kappa \text{div}(\mu(\rho) \nabla \varphi(\rho)) + \nabla(\mu(\rho) \nabla^t w) + \nabla((\mu'(\rho) \rho - \mu(\rho)) \text{div} u) &= 0, \\
w &= u + 2\kappa \nabla \varphi(\rho).
\end{aligned}$$  \hspace{1cm} (24)

Taking the scalar product of the equation satisfied by $w$ with $w$, the scalar product of the equation satisfied by $\nabla \varphi(\rho)$ with $4(1 - \kappa)\kappa \nabla \varphi(\rho)$ and adding the resulting expressions we get the $\kappa$–entropy

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \left( \frac{|w|^2}{2} + \frac{(1 - \kappa)|2\nabla \varphi(\rho)|^2}{2} \right) \, dx + \frac{d}{dt} \int_\Omega \rho c(\rho) \, dx \\
+ 2\kappa \int_\Omega \mu(\rho)|A(w)|^2 \, dx + 2\kappa \int_\Omega \frac{\mu'(\rho)\rho(\rho)}{\rho} |\nabla \varphi|^2 \, dx \\
+ 2(1 - \kappa) \left[ \int_\Omega \mu(\rho)|D(u)|^2 \, dx + \int_\Omega (\mu'(\rho) \rho - \mu(\rho)) |\text{div} u|^2 \, dx \right] = 0
\end{aligned}$$  \hspace{1cm} (25)

where we used assumption [3] to write

$$\int_\Omega (\lambda(\rho) - 2\kappa(\mu'(\rho) \rho - \mu(\rho))) \text{div} u \, dw + 4(1 - \kappa)(\mu'(\rho) \rho - \mu(\rho)) \text{div} u \, \nabla \varphi(\rho) \, dx = 2(1 - \kappa) \int_\Omega (\mu'(\rho) \rho - \mu(\rho)) |\text{div} u|^2 \, dx$$  \hspace{1cm} (26)

recalling that $u = w - 2\kappa \nabla \varphi(\rho)$.

Relation [25] also includes the one obtained in the recent paper of M. GISCLON, I. VIOLET for quantum Navier–Stokes equations who have followed the work by A. J"UENGERL (choose $\kappa = 1/2$ and $\varphi = \log \rho$ in [25] to find the one in their works), however we do not need the Bohm potential formula to conclude as in the paper by M. GISCLON, I. VIOLET. Our definition of global in time $\kappa$–entropy solutions will be linked to [25]. A solution which satisfies the mass and momentum equations in a weak sense and the $\kappa$–entropy estimate.
3 Construction of solution

Following the idea developed recently by D. Bresch, V. Giovangigli and E. Zatorska in [10] for the following low Mach system

\[
\begin{align*}
\partial_t \varrho + \text{div}(\varrho \mathbf{u}) &= 0, \\
\partial_t (\varrho \mathbf{u}) + \text{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \pi &= 2 \text{div}(\mu(\varrho) D(\mathbf{u})) + \nabla(\lambda(\varrho) \text{div} \mathbf{u}), \\
\text{div} \mathbf{u} &= -2\kappa \Delta \varphi(\varrho),
\end{align*}
\]

(27)

with \( \varphi \) an increasing function of \( \varrho \) and \( 0 < \kappa < 1 \) fixed. We construct the approximate solution to system (24) using the augmented approximate system. More precisely, we introduce a new unknown \( \mathbf{v} \), which is not yet known to satisfy \( \mathbf{v} = 2\nabla \varphi(\varrho) \). Our aim will be to find a solution \((\varrho, \mathbf{w}, \mathbf{v})\) of the following system:

\[
\begin{align*}
\partial_t \varrho + \text{div}(\varrho \mathbf{w}) - 2\kappa \Delta \varphi(\varrho) &= 0, \\
\partial_t (\varrho \mathbf{w}) + \text{div}((\varrho \mathbf{w} - 2\kappa \nabla \varphi(\varrho) \otimes \mathbf{w}) - \nabla ((\lambda(\varrho) - 2\kappa(\mu'(\varrho) \varrho - \mu(\varrho))) \text{div}(\mathbf{w} - \kappa \nabla \mathbf{w}))) \\
&- 2(1 - \kappa) \text{div}(\mu(\varrho) D(\mathbf{w})) - 2\kappa \text{div}(\mu(\varrho) A(\mathbf{w})) + \nabla p(\varrho) = -2\kappa(1 - \kappa) \text{div}(\mu(\varrho) \nabla \mathbf{v}), \\
\partial_t (\varrho \mathbf{v}) + \text{div}((\varrho \mathbf{w} - 2\kappa \nabla \varphi(\varrho) \otimes \mathbf{v}) - 2\kappa \text{div}(\mu(\varrho) \nabla \mathbf{v}) + 2\nabla((\mu'(\varrho) \varrho - \mu(\varrho)) \text{div}(\mathbf{w} - \kappa \nabla \mathbf{v})) \\
= -2\kappa \mu(\varrho) \nabla \mathbf{v},
\end{align*}
\]

(28)

Remark 10 Note that such system is close to the one written in [30]. Here the main velocity is \( \mathbf{w} = \mathbf{u} + 2\kappa \nabla \varphi(\varrho) \) and the drift is \( \mathbf{v} = 2\nabla \varphi(\varrho) \).

Our idea of construction of solution is based on the following \( \kappa \)-entropy equality

\[
\begin{align*}
\frac{d}{dt} \int_{\Omega} \varrho \left( \frac{|\mathbf{w}|^2}{2} + (1 - \kappa) \kappa \frac{|\nabla \varrho|^2}{2} \right) \, dx + \frac{d}{dt} \int_{\Omega} \varrho \left( (\lambda(\varrho) - 2\kappa(\mu'(\varrho) \varrho - \mu(\varrho))) \text{div}(\mathbf{w} - \kappa \nabla \mathbf{w}) \right) \, dx \\
+ 2\kappa \int_{\Omega} \mu(\varrho) |A(\mathbf{w})|^2 \, dx + 2(1 - \kappa) \int_{\Omega} (\mu'(\varrho) \varrho - \mu(\varrho)) |\text{div}(\mathbf{w} - \kappa \nabla \mathbf{w})|^2 \, dx \\
+ 2 \int_{\Omega} \frac{\kappa \mu'(\varrho)}{\varrho} |\nabla \varrho|^2 \, dx = 0.
\end{align*}
\]

(29)

that holds for any sufficiently smooth solution of (28). In order to build such a solution for \( \varepsilon > 0 \) given, we need to go through several levels of approximations. For example, to build a solution of the nonlinear parabolic equation for \( \varrho \), some assumptions are required on the coefficients. Two smoothing parameters \( \alpha > 0 \) and \( \delta > 0 \) (denoting standard mollification with respect to \( t \) and \( x \)) are therefore introduced in all the transport terms, so that the approximate system can be rewritten as

\[
\begin{align*}
\partial_t \varrho + \text{div}(\varrho |\mathbf{w}|_\delta) - 2\kappa \text{div} \left( (\mu'(\varrho))_\alpha \nabla \varrho \right) &= 0, \\
\partial_t (\varrho \mathbf{w}) + \text{div}((\varrho |\mathbf{w}|_\delta - 2\kappa(\mu'(\varrho))_\alpha \nabla \varrho) \otimes \mathbf{w}) - \nabla ((\lambda(\varrho) - 2\kappa(\mu'(\varrho) \varrho - \mu(\varrho))) \text{div}(\mathbf{w} - \kappa \nabla \mathbf{w})) \\
&- 2(1 - \kappa) \text{div}(\mu(\varrho) D(\mathbf{w})) - 2\kappa \text{div}(\mu(\varrho) A(\mathbf{w})) + \varepsilon \Delta^2 \mathbf{w} - \varepsilon \text{div}(1 + |\nabla \mathbf{w}|^2) \nabla \mathbf{w} + \nabla p(\varrho) \\
= -2\kappa(1 - \kappa) \text{div}(\mu(\varrho) \nabla \mathbf{v}), \\
\partial_t (\varrho \mathbf{v}) + \text{div}((\varrho |\mathbf{w}|_\delta - 2\kappa(\mu'(\varrho))_\alpha \nabla \varrho) \otimes \mathbf{v}) \\
- 2\kappa \text{div}(\mu(\varrho) \nabla \mathbf{v}) + 2\nabla((\mu'(\varrho) \varrho - \mu(\varrho)) \text{div}(\mathbf{w} - \kappa \nabla \mathbf{w})) \\
= -2\kappa \mu(\varrho) \nabla \mathbf{v},
\end{align*}
\]

(30)

Compared to [10], we have an extra term in the continuity equation because \( \mathbf{w} \) is no longer divergence free. In addition, a smoothing high-order derivative term \( \Delta^2 \mathbf{w} \) depending on small parameter \( \varepsilon > 0 \).
has to be introduced to control large spatial variations of $\mathbf{w}$, because $\text{div} \mathbf{w}$ is not \textit{a-priori} bounded in $L^1(0,T; (L^\infty(\Omega))^3)$. Such bound will be required to be able to have bounds on the density. We will also need to show that $\mathbf{v} = 2\nabla \varphi(\rho)$ at some point of the construction process and the second term in the regularization process will be helpful.

Then, in order to solve \((30)\) for given $\alpha, \delta, \varepsilon > 0$, the equations for the two velocities $(\mathbf{w}, \mathbf{v})$ is projected onto finite dimensional spaces $(X_n, Y_n)$ as is classically done when using Faedo-Galerkin approximation. The first step is to prove global in time existence of solutions when $n, \alpha, \delta$ and $\varepsilon$ are fixed, then to pass to the limit $\alpha \to 0$ and after to let $n \to \infty$. Next, we let $\delta \to 0$ in order to prove that $\mathbf{v} = 2\nabla \varphi(\rho)$. At the end, we combine the equation satisfied by $\mathbf{w}$ and $\mathbf{v}$ to get an equation satisfied by $\mathbf{u}$ (multiply the equation satisfied by $\mathbf{v}$ by $\kappa$ and subtract from the equation for $\mathbf{w}$). The last limit passage $\varepsilon \to 0$ is performed already for the system written in terms of $\mathbf{u}$ which gives the weak formulation of the original system \([1]\). This last step follows the same lines as the proof presented by the two first authors for the heat-conducting case, see \([8]\). It essentially uses the presence of singular pressure, which allows to stabilize the system close to vacuum or the presence of the turbulent drag term to have enough control on the velocity field. More complicated construction in the case of Navier-Stokes type model for compressible mixture with viscosity $\mu(\rho) = \varrho$ can be found in the PhD thesis of the third author \([33]\), see also \([25, 26]\).

### 3.1 Existence of solutions for the full approximation

Below we present the basic level of approximation procedure.

1. The continuity equation is replaced by its regularized version

$$
\partial_t \rho + \text{div}(\rho |\mathbf{w}|_\delta) - \kappa \text{div}(\mu'(\rho)|\mathbf{w}|) + \kappa \nabla \rho = 0, \quad (31)
$$

where $\alpha, \delta$ denote the standard regularizations with respect to time and space. With such smoothing properties, we can use the estimates given by Theorem 1 which are uniform with respect to $\delta$ when far from vacuum. This will be used at different stages: the construction and the proof that $\mathbf{v} = 2\nabla \varphi(\rho)$ before passing to the limit with respect to $\varepsilon$. Note that similar double regularization have been recently used in \([22]\) for instance and discussed in the book by J.L. Vazquez \([31]\).

2. The momentum equation is replaced by its Faedo-Galerkin approximation with additional regularizing term $\varepsilon[\Delta^2 \mathbf{w} - \text{div}((1 + |\nabla \mathbf{w}|^2) \nabla \mathbf{w})]$

$$
\int_\Omega \varrho \mathbf{w}(\tau) \cdot \phi \text{ d}x - \int_0^T \int_\Omega \left(\varrho |\mathbf{w}|_\delta - 2\kappa |\mu'(\rho)| |\nabla \varrho| \otimes \mathbf{w} \right) : \nabla \phi \text{ d}x \text{ d}t + 2(1 - \kappa) \int_0^T \int_\Omega \mu(\rho) D(\mathbf{w}) : \nabla \phi \text{ d}x \text{ d}t + 2\kappa \int_0^T \int_\Omega \mu(\rho) A(\mathbf{w}) : \nabla \phi \text{ d}x \text{ d}t + 2(1 - \kappa) \int_0^T \int_\Omega (\mu'(\rho) \varrho - \mu(\rho)) \text{ div} \mathbf{w} \text{ div} \phi \text{ d}x \text{ d}t
$$

$$
- 2\kappa(1 - \kappa) \int_0^T \int_\Omega \mu(\rho) \nabla \mathbf{v} : \nabla \phi \text{ d}x \text{ d}t - 2\kappa(1 - \kappa) \int_0^T \int_\Omega (\mu'(\rho) \varrho - \mu(\rho)) \text{ div} \mathbf{v} \text{ div} \phi \text{ d}x \text{ d}t
$$

$$
- \int_0^T \int_\Omega p(\rho) \text{ div} \phi \text{ d}x \text{ d}t + \varepsilon \int_0^T \int_\Omega (\Delta^2 \mathbf{w} \cdot \Delta^2 \phi + (1 + |\nabla \mathbf{w}|^2) \nabla \mathbf{w} : \nabla \phi) \text{ d}x \text{ d}t = \int_\Omega (\varrho \mathbf{w})^0 \cdot \phi \text{ d}x,
$$

satisfied for any $\tau \in [0,T]$ and any test function $\phi \in X_n$, where $X_n = \text{span}\{\phi_i\}_{i=1}^n$ and $\{\phi_i\}_{i=1}^\infty$ is an orthonormal basis in $(W^{1,2}(\Omega))^3$ with $\phi_i \in (C^\infty(\Omega))^3$ for all $i \in N$. 


Theorem 3

result: 10.24 from [15], which is a combination of Theorems 7.2, 7.3 and 7.4 from [21]) yields the following
nuity equation, which is now quasi-linear parabolic equation with smooth coefficients. Thus, application
Existence of solutions to the continuity equation.
For fixed \( W \) orthonormal basis in (\( s \)

\[ \varrho \] continuous with values in \( \varrho \) solution

point problem

\[ T \] fixed point argument. More precisely, we will prove that there exists time
the integral equations (32) and (33) possess the unique solution on possibly short time interval via
\( w \) proof performed in [10] for
Local existence of solutions to the Galerkin approximations.

\[ \nu \] \( \nu \) \( \nu \) 3. The Faedo-Galerkin approximation for the artificial equation

\[ \int_\Omega \varrho \tau \cdot \xi \ dx - \int_0^T \int_\Omega \left( (\varrho[w] \delta - 2\kappa[\mu'(\varrho)]_n \nabla \varrho) \otimes \nu \right) : \nabla \xi \ dx \ dt \]

\[ + 2\kappa \int_0^T \int_\Omega \mu(\varrho) \nabla \nu : \nabla \xi \ dx \ dt + 2\kappa \int_0^T \int_\Omega (\mu'(\varrho) \varrho - \mu(\varrho)) \div \nu \div \xi \ dx \ dt \]

\[ - 2 \int_0^T \int_\Omega (\mu'(\varrho) \varrho - \mu(\varrho)) \div\nu \div \xi \ dx \ dt - 2 \int_0^T \int_\Omega \mu(\varrho) \nabla^{t} \nu : \nabla \xi \ dx \ dt \]

\[ = \int_\Omega (\varrho \nu)^0 \cdot \xi \ dx, \] (33)
satisfied for any \( \tau \in [0, T] \) and any test function \( \xi \in Y_n \), where \( Y_n = \text{span}\{\xi_i\}_{i=1}^n \) and \( \{\xi_i\}_{i=1}^\infty \) is an orthonormal basis in \( (W^{1,2}(\Omega))^3 \) with \( \xi_i \in (C^\infty(\Omega))^3 \) for all \( i \in N \).

Existence of solutions to the continuity equation. For fixed \( \varrho \in C([0, T]; X_n) \) we solve the continuiy equation, which is now quasi-linear parabolic equation with smooth coefficients. Thus, application of classical existence theory of Ladyženskaja, Solonnikov and Uralceva [21] (see for example Theorem 10.24 from [15], which is a combination of Theorems 7.2, 7.3 and 7.4 from [21]) yields the following result:

**Theorem 3** Let \( \nu \in (0, 1) \) and suppose that the initial condition \( \varrho^0 \in C^{2+\nu}(\Omega) \) is such that \( 0 < r \leq \varrho^0 \leq R \) and it satisfies the periodic boundary conditions. Then problem (31) possesses a unique classical solution \( \varrho \) from the class

\[ V_{[0,T]} = \left\{ \begin{array}{l}
\varrho \in C([0,T]; C^{2+\nu}(\Omega)) \cap C^1([0,T] \times \Omega), \\
\frac{\partial}{\partial t} \varrho \in C^{\nu/2}([0,T]; C(\Omega))
\end{array} \right\} \] (34)

and satisfying classical maximum principle

\[ 0 < r \leq \varrho(t, x) \leq R. \] (35)

Moreover, the mapping \( \varrho \mapsto \varrho(\varrho) \) maps bounded sets in \( C([0,T]; X_n) \) into bounded sets in \( V_{[0,T]} \) and is continuous with values in \( C([0,T]; C^{2+\nu}(\Omega)) \), \( 0 < \nu' < \nu < 1 \).

Local existence of solutions to the Galerkin approximations. Here we proceed as in the analogous

\[ (\varrho(t), \nu(t)) = \begin{pmatrix}
\mathcal{M}_{\varrho(t)}[P_{X_n}(\varrho \varrho)^0 + \int_0^t \mathcal{K}(\varrho(s)) \mathcal{N}_{\varrho(t)}[P_{X_n}(\varrho \varrho)^0 + \int_0^s \mathcal{L}(\nu(s)) ds] \\
T[\varrho, \nu](t)
\end{pmatrix} \] (36)

where \( (X_n^*) \) is identified with \( X_n \), so the symbol \( \langle \cdot, \cdot \rangle_{(X_n, X_n)} \) denotes the action of a functional from \( X_n^* = X_n \) on the element from \( X_n \), similarly for \( \langle \cdot, \cdot \rangle_{(Y_n, Y_n)} \)

\[ \mathcal{M}_{\varrho(t)} : X_n \to X_n, \quad \int_\Omega \varrho \mathcal{M}_{\varrho(t)}[\phi] \cdot \psi \ dx = \langle \phi, \psi \rangle_{(X_n, X_n)}, \quad \phi, \psi \in X_n, \]
\[ N_{\varrho(t)} : Y_n \to Y_n, \quad \int_{\Omega} \varrho N_{\varrho(t)}[\xi] : \zeta \, dx = \langle \xi, \zeta \rangle_{(Y_n, Y_n)}, \quad \xi, \zeta \in Y_n, \]

\( P_{X_n}, P_{Y_n} \) denote the projections of \( L^2(\Omega) \) onto \( X_n, Y_n \), respectively, and \( K(\varrho), L(\upsilon) \) are two operators defined as follows

\[ K : X_n \to X_n, \]

\[ \langle K(\varrho), \xi \rangle_{(Y_n, Y_n)} = \int_{\Omega} \left( (\varrho|w|_{\beta} - 2\kappa|\mu'(\varrho)|_{\alpha} \nabla \varrho)|w|_{\beta} \right) : \nabla \xi \, dx - 2\kappa \int_{\Omega} \mu(\varrho) \varrho \nabla \varrho : \nabla \xi \, dx \]

\[ - 2(1 - \kappa) \int_{\Omega} \left( \mu'(\varrho) \varrho - \mu(\varrho) \right) \text{div} \, w \text{div} \, \phi \, dx \]

\[ - 2\kappa \int_{\Omega} \mu(\varrho) A(\varrho) : \nabla \phi \, dx + 2\kappa(1 - \kappa) \int_{\Omega} \mu(\varrho) \nabla \upsilon : \nabla \phi \, dx \]

\[ + 2\kappa(1 - \kappa) \int_{\Omega} \left( \mu'(\varrho) \varrho - \mu(\varrho) \right) \text{div} \, w \text{div} \, \phi \, dx + \int_{\Omega} \mu(\varrho) \text{div} \, \phi \, dx \]

\[ - \varepsilon \int_{\Omega} (\Delta^{*} w : \Delta^{*} \phi + (1 + |\nabla w|^{2}) \nabla w : \nabla \phi) \, dx, \]

\[ L : Y_n \to Y_n, \]

\[ \langle L(\upsilon), \xi \rangle_{(Y_n, Y_n)} = \int_{\Omega} \left( (\varrho|w|_{\beta} - 2\kappa|\mu'(\varrho)|_{\alpha} \nabla \varrho)|w|_{\beta} \right) : \nabla \xi \, dx - 2\kappa \int_{\Omega} \mu(\varrho) \nabla \upsilon : \nabla \xi \, dx \]

\[ - 2\kappa \int_{\Omega} \left( \mu'(\varrho) \varrho - \mu(\varrho) \right) \text{div} \, w \text{div} \, \xi \, dx + 2 \int_{\Omega} \left( \mu'(\varrho) \varrho - \mu(\varrho) \right) \text{div} \, \xi \, dx \]

\[ + 2 \int_{\Omega} \mu(\varrho) \nabla^{*} w : \nabla \xi \, dx. \]

Since \( \varrho(t, x) \) is bounded from below by a positive constant, we have

\[ \| M_{\varrho(t)} \|_{L(X_n, X_n)}, \quad \| N_{\varrho(t)} \|_{L(Y_n, Y_n)} \leq \frac{1}{r}. \]

Moreover

\[ \| M_{\varrho(t)} - M_{\varrho(t)} \|_{L(X_n, X_n)} + \| N_{\varrho(t)} - N_{\varrho(t)} \|_{L(Y_n, Y_n)} \leq c(n, r_{1}, r_{2}) \| \varrho^{1} - \varrho^{2} \|_{L^{1}(\Omega)}, \quad (37) \]

and by the equivalence of norms on the finite dimensional space we prove that

\[ \| K(\varrho) \|_{X_n} + \| L(\upsilon) \|_{Y_n} \leq c(r, R, \| \nabla \varrho \|_{L^{2}(\Omega)}, \| w \|_{X_n}, \| \upsilon \|_{Y_n}). \quad (38) \]

Next, we consider a ball \( B \) in the space \( C([0, \tau]; X_n) \times C([0, \tau]; Y_n) \):

\[ B_{M, \tau} = \{ (w, \upsilon) \in C([0, \tau]; X_n) \times C([0, \tau]; Y_n) : \| w \|_{C([0, \tau]; X_n)} + \| \upsilon \|_{C([0, \tau]; Y_n)} \leq M \} \cdot \]

Using estimates \( (37), (38), (34) \) and \( (35) \), one can check that \( T \) is a continuous mapping of the ball \( B_{M, \tau} \) into itself and for sufficiently small \( \tau = T(n) \) it is a contraction. Therefore, it possesses a unique fixed point which is a solution to \( (32) \) and \( (33) \) for \( T = T(n) \).

**Global existence of solutions.** In order to extend the local in-time solution obtained above to the global in time one, we need to find uniform (in time) estimates, so that the above procedure can be iterated. First let us note, that \( w, \upsilon \) obtained in the previous paragraph have better regularity with respect to time. It follows by taking the time derivative of \( (36) \) and using the estimates \( (34), (35), \) that

\[ (w, \upsilon) \in C^{1}([0, \tau]; X_n) \times C^{1}([0, \tau]; Y_n). \]
This is an important feature since now we can take time derivatives of (32) and (33) and use the test functions $\phi = w$ and $\xi = v$, respectively. We then obtain
\[
\begin{align*}
&\frac{d}{dt} \int_{\Omega} \frac{|w|^2}{2} \, dx + \frac{d}{dt} \int_{\Omega} \rho \varepsilon (\rho) \, dx + 2(1 - \kappa) \int_{\Omega} \mu (\rho)|D(w)|^2 \, dx \\
&\quad + 2\kappa \int_{\Omega} \mu (\rho) |A(w)|^2 \, dx + 2(1 - \kappa) \int_{\Omega} (\mu' (\rho) \rho - \mu (\rho)) \text{div} w \, dx \\
&\quad - 2\kappa (1 - \kappa) \int_{\Omega} \mu (\rho) \nabla : \nabla w \, dx - 2\kappa (1 - \kappa) \int_{\Omega} (\mu' (\rho) \rho - \mu (\rho)) \text{div} v \text{div} w \, dx \\
&\quad + 2\kappa \int_{\Omega} \mu' (\rho) \rho' (\rho) |\nabla \rho|^2 \, dx + 2\kappa \int_{\Omega} \mu (\rho) |\nabla \rho|^2 \, dx \\
&\quad + 2\kappa \int_{\Omega} \mu' (\rho) \rho' (\rho) |\nabla \rho|^2 \, dx + \varepsilon \int_{\Omega} (|\Delta^s w|^2 + (1 + |\nabla w|^2) |\nabla w|^2) \, dx = 0,
\end{align*}
\]
and
\[
\begin{align*}
&\frac{d}{dt} \int_{\Omega} \frac{|v|^2}{2} \, dx + 2\kappa \int_{\Omega} \mu (\rho) |\nabla v|^2 \, dx + 2\kappa \int_{\Omega} \mu (\rho) |\nabla w|^2 \, dx \\
&\quad - 2\int_{\Omega} (\mu' (\rho) \rho - \mu (\rho)) \text{div} w \text{div} v \, dx - 2\int_{\Omega} \mu (\rho) \nabla : \nabla w \, dx \, dt = 0.
\end{align*}
\]
Therefore, multiplying (40) by $(1 - \kappa)\kappa$ and adding it to (39), we obtain
\[
\begin{align*}
&\frac{d}{dt} \int_{\Omega} \rho \left( \frac{|w|^2}{2} + (1 - \kappa)\kappa \frac{|v|^2}{2} \right) \, dx + \frac{d}{dt} \int_{\Omega} \rho \varepsilon (\rho) \, dx + 2(1 - \kappa) \int_{\Omega} \mu (\rho)|D(w)|^2 \, dx \\
&\quad + 2(1 - \kappa) \int_{\Omega} (\mu' (\rho) \rho - \mu (\rho)) \text{div} w - \kappa \text{div} v \, dx + 2\kappa \int_{\Omega} \mu (\rho)|A(w)|^2 \, dx \\
&\quad + 2\kappa \int_{\Omega} \mu' (\rho) \rho' (\rho) |\nabla \rho|^2 \, dx + \varepsilon \int_{\Omega} (|\Delta^s w|^2 + (1 + |\nabla w|^2) |\nabla w|^2) \, dx \\
&\quad = - \int_{\Omega} p(\rho) \text{div} ([|w|_\delta - v] \, dx.
\end{align*}
\]
Integrating the above estimate with respect to time, using the Hölder and the Gronwall inequalities, we obtain uniform estimate for $w$ and $v$ necessary to repeat the procedure described in the previous paragraph. Thus, we obtain a global in time unique solution $(\rho, w, v)$ satisfying equations (31) [32] [33].

**Uniform estimates.** Below we present uniform estimates that will allow us to pass to the limit with $\alpha$ and $n$ respectively. First observe that multiplying continuity equation (31) by $\varrho_\alpha$ and integrating by parts with respect to $x$ gives
\[
\begin{align*}
1 \frac{d}{dt} \int_{\Omega} |\varrho_\alpha|^2 \, dx + 2\kappa \int_{\Omega} |\mu' (\varrho_\alpha) | \varrho_\alpha |\nabla \varrho_\alpha|^2 \, dx &= - \int_{\Omega} \text{div} [\varrho_\alpha] \varrho_\alpha |\nabla \varrho_\alpha|^2 \, dx \\
&\leq \|\text{div} [\varrho_\alpha] \|_{L^\infty (\Omega)} \int_{\Omega} |\varrho_\alpha|^2 \, dx.
\end{align*}
\]
Integrating this equality with respect to time and using the uniform bound of $\text{div} [\varrho_\alpha]$ in $L^1 (0, T; L^\infty (\Omega))$ provides the following estimates
\[
\| \varrho_\alpha \|_{L^\infty (0, T; L^2 (\Omega))} + \| \sqrt{\mu' (\varrho_\alpha) | \nabla \varrho_\alpha |} \|_{L^2 (0, T; L^2 (\Omega))} \leq c.
\]
Moreover, the standard maximum principle gives boundedness of $\varrho_\alpha$ from above and below. Indeed, multiplying equation (31) by
\[
\varrho_{\alpha}^+ = \max (0, r - \varrho_\alpha) \quad \text{and} \quad \varrho_{\alpha}^- = \min (0, R - \varrho_\alpha),
\]

13
respectively we obtain also using the bound on \( |w|_\delta \) that
\[
0 < r_\varepsilon \leq g_\alpha(t, x) \leq R_\varepsilon. \tag{43}
\]

**Remark 11** To prove these bounds one needs to know that \( g_\alpha \in L^2(0, T; W^{1,2}(\Omega)) \), which doesn’t follow from \([42]\). This problem could be solved by adding a small viscosity parameter \( \alpha \) and considering \([\mu'(\varrho)]_\alpha = [\mu'(\varrho)]_\alpha + \alpha \) in place of \([\mu'(\varrho)]_\alpha \).

Next, using \([43]\) and integrating \([41]\) with respect to time we see that for \( 0 < \kappa < 1 \) we have
\[
\begin{align*}
|w_\alpha|_{L^\infty(0, T; L^2(\Omega))} + \|w_\alpha\|_{L^2(0, T; H^2(\Omega))} + \|\nabla w_\alpha\|_{L^4(0, T; L^4(\Omega))} + \|\nabla w_\alpha\|_{L^2(0, T; H^1(\Omega))} &
\leq c.
\end{align*}
\]

In the above estimate the constant \( c \) is uniform with respect to all approximation parameters except \( \varepsilon \).

### 3.2 Passage to the limit with respect to \( \alpha \) and with respect to \( n \).

**Passage to the limit \( \alpha \to 0 \).** On the finite dimensional subspace all the norms are equivalent, therefore the space compactness of \( w_\alpha \) and \( v_\alpha \) is automatic. In fact, for \( n \) fixed we also know that \( \partial_t w_\alpha \) is bounded in \( L^2(0, T; X_n) \), thus up to the extraction of a subsequence, \( w_\alpha \to w \) strongly in \( L^2(0, T; X_n) \) and the same can be deduced for \( v_\alpha \). The biggest problem is thus to pass to the limit in the term
\[
[\mu'(\varrho_\alpha)]_\alpha \nabla \varrho_\alpha \otimes w_\alpha 
\text{ when } \alpha \to 0 \tag{45}
\]
which requires the strong convergence of the density and the weak convergence of the gradient of density. Observe that higher order estimates for \( \varrho_\alpha \), cannot be used here due to \( \alpha \)-regularization of the coefficient \( \mu'(\varrho_\alpha) \), however the information from \([43]\) and \([42]\) is enough. Indeed, using \([43]\) we can deduce that there exists \( \alpha_0 \) such that for \( \alpha < \alpha_0 \) one has \( 3R/2|\mu'(\varrho_\alpha)|_\alpha > r/2 \) uniformly with respect to \( \alpha \). Therefore \([42]\) implies that up to a subsequence
\[
\varrho_\alpha \to \varrho \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)) \text{ when } \alpha \to 0,
\]
moreover \( \partial_t \varrho_\alpha \in L^2(0, T; W^{-1,2}(\Omega)) \) and \( \varrho_\alpha \in L^\infty((0, T) \times \Omega) \), thus the Aubin-Lions lemma implies strong convergence of \( \varrho_\alpha \)
\[
\varrho_\alpha \to \varrho \text{ strongly in } L^p((0, T) \times \Omega) \text{ when } \alpha \to 0, \quad p < \infty.
\]

This justifies the passage to the limit in \([56]\). Therefore, one is able to pass to the limit \( \alpha \to 0 \) in both velocity equations \([32]\) and \([33]\). The limit functions \((\varrho, w, v) = (\varrho_n, w_n, v_n)\) satisfy the following system of equations:

- the momentum equation
\[
\begin{align*}
&\langle \partial_t (\varrho_n w_n)(t), \phi \rangle_{X_n^*, X_n} - \int_\Omega ((\varrho_n[w_n]_\delta - 2\kappa \nabla \mu(\varrho_n)) \otimes w_n)(t) : \nabla \phi \, dx
\end{align*}
\]
\[
+ 2(1 - \kappa) \int_\Omega \mu(\varrho_n) D(w_n)(t) : \nabla \phi \, dx + 2\kappa \int_\Omega \mu(\varrho_n) A(w_n)(t) : \nabla \phi \, dx
\]
\[
+ 2(1 - \kappa) \int_\Omega (\mu'(\varrho_n) \varrho_n - \mu(\varrho_n)) \text{ div } w_n(t) \text{ div } \phi \, dx
\]
\[
- 2\kappa(1 - \kappa) \int_\Omega \mu(\varrho_n) \nabla v_n(t) : \nabla \phi \, dx - 2\kappa(1 - \kappa) \int_\Omega (\mu'(\varrho_n) \varrho_n - \mu(\varrho_n)) \text{ div } v_n(t) \text{ div } \phi \, dx
\]
\[
- \int_\Omega p(\varrho_n)(t) \text{ div } \phi \, dx + \varepsilon \int_\Omega (\Delta^a w_n(t) \cdot \Delta^a \phi + (1 + |\nabla w_n|^2) \nabla w_n(t) : \nabla \phi) \, dx = 0
\]

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\[ \langle \partial_t (\varrho_n \mathbf{v}_n) (t), \xi \rangle_{(Y^*_n; Y_n)} - \int_{\Omega} \left( (\varrho_n [w_n]_\delta - 2\kappa \nabla \mu(\varrho_n)) \otimes \mathbf{v}_n(t) \right) : \nabla \xi \, dx \]

\[ + 2\kappa \int_{\Omega} \mu(\varrho_n) \nabla \mathbf{v}_n(t) : \nabla \xi \, dx + 2\kappa \int_{\Omega} (\mu'(\varrho_n) \varrho_n - \mu(\varrho_n)) \nabla \mathbf{v}_n(t) \cdot \nabla \xi \, dx \]

\[ = \int_{\Omega} (\mu'(\varrho_n) \varrho_n - \mu(\varrho_n)) \nabla \mathbf{w}_n(t) : \nabla \xi \, dx - 2 \int_{\Omega} \mu(\varrho_n) \nabla \mathbf{w}_n(t) : \nabla \xi \, dx = 0 \] (47)

is satisfied for all \( \xi \in Y_n \) with \( t \in [0, T] \).

However, so far we only know that the approximate continuity equation \([31]\) is satisfied in the sense of distributions

\[ \int_0^T \langle \partial_t \varrho_n, \phi \rangle_{(W^{1,2}(\Omega), W^{1,2}(\Omega))} \, dt - \int_0^T \int_{\Omega} \varrho_n [w_n]_\delta \cdot \nabla \phi \, dx \, dt + \kappa \int_0^T \int_{\Omega} \nabla \mu(\varrho_n) \cdot \nabla \phi \, dx \, dt = 0 \] (48)

for any test function \( \phi \) from \( L^2(0, T; W^{1,2}(\Omega)) \). But on the other hand, we know that in the sense of distributions

\[ \psi = \partial_t \varrho_n - \kappa \text{div} (\mu'(\varrho_n) \nabla \varrho_n) = -\varrho_n \text{div}[w_n]_\delta - [w_n]_\delta \cdot \nabla \varrho_n \in L^2((0, T) \times \Omega) \]

if \( w_n \in L^\infty(0, T; L^\infty(\Omega)) \). Indeed, note that we have

\[ \| \varrho_n \text{div}[w_n]_\delta \|_{L^2(0,T;L^2(\Omega))} \leq \| \varrho_n \|_{L^\infty(0,T;L^\infty(\Omega))} \| \text{div}[w_n]_\delta \|_{L^2(0,T;L^2(\Omega))} \]

(49)

\[ \|[w_n]_\delta \cdot \nabla \varrho_n\|_{L^2(0,T;L^2(\Omega))} \leq \|[w_n]_\delta \|_{L^\infty(0,T;L^\infty(\Omega))} \| \nabla \varrho_n \|_{L^2(0,T;L^2(\Omega))} \]

(50)

and the r.h.s. of above inequalities is bounded at the level of Galerkin approximations, therefore also

\[ \psi \mu'(\varrho_n) \in L^2((0, T) \times \Omega) \].

Taking the product of \( \psi \) and \( \psi \mu'(\varrho_n) \) we obtain

\[ \int_{\Omega} \mu'(\varrho_n) (\partial_t \varrho_n - \kappa \Delta \mu(\varrho_n))^2 \, dx \leq c(t) \in L^1((0, T)) \]

and the above integral gives rise to estimates

\[ \int_{\Omega} \mu'(\varrho_n) (\partial_t \varrho_n)^2 \, dx + \kappa^2 \int_{\Omega} \mu'(\varrho_n) \Delta \mu(\varrho_n)^2 \, dx - 2\kappa \int_{\Omega} \mu'(\varrho_n) \partial_t \varrho_n \Delta \mu(\varrho_n) \, dx \]

\[ = \int_{\Omega} \mu'(\varrho_n) (\partial_t \varrho_n)^2 \, dx + \kappa^2 \int_{\Omega} \mu'(\varrho_n) \Delta \mu(\varrho_n)^2 \, dx + \kappa \frac{d}{dt} \int_{\Omega} |\nabla \mu(\varrho_n)|^2 \, dx \leq c(t). \] (51)

Note that we have passed to the limit with \( \alpha \) in the previous paragraph.

Note that this estimate asks for \( L^\infty((0, T) \times \Omega) \) bound for \( w_n \), which is possible only at the level of Galerkin approximation. Nevertheless, regularity \([51]\) allows us to first rewrite \([31]\) as

\[ \partial_t \mu(\varrho_n) + \text{div}([w_n]_\delta \mu(\varrho_n)) - 2\kappa \mu'(\varrho_n) \Delta \mu(\varrho_n) + (\mu'(\varrho_n) \varrho_n - \mu(\varrho_n)) \text{div}[w_n]_\delta = 0. \] (52)
Multiplying the above equation by $-\Delta \mu(\varrho)$ and integrating by parts we obtain

$$\frac{d}{dt} \int_{\Omega} |\nabla \mu(\varrho_n)|^2 \, dx + 2\kappa \int_{\Omega} \mu'(\varrho_n) |\Delta \mu(\varrho_n)|^2 \, dx = \int_{\Omega} \mu'(\varrho_n) \varrho_n \text{div}[w_n] \delta \Delta \mu(\varrho_n) \, dx + \int_{\Omega} [w_n] \delta \nabla \mu(\varrho_n) \Delta \mu(\varrho_n) \, dx.$$  

Note that for any $f \in H^2(\Omega) \cap L^\infty(\Omega)$ we have the following realization of the Gagliardo-Nierenberg inequality

$$\|\nabla f\|_{L^4(\Omega)}^2 \leq c \|\Delta f\|_{L^2(\Omega)} \|f\|_{L^\infty(\Omega)},$$  

(53) see for example [28]. Therefore, applying this inequality with $f = \mu(\varrho_n)$ and using the uniform bound for $w_n \in L^2([0,T;H^1(\Omega))]$ and $\varepsilon$-dependent bound in $L^4([0,T;L^4(\Omega)])$ following from (44), we obtain

$$\|\nabla \mu(\varrho_n)\|_{L^\infty([0,T;L^2(\Omega)])} + 2n^{1/2} \|\Delta \mu(\varrho_n)\|_{L^2([0,T] \times \Omega)} \leq c \left( \|\varrho_n^0\|_{H^1(\Omega)}, R, \kappa, \varepsilon \right),$$  

(54) and thus, coming back to (52) and using (43) we easily check the uniform (with respect to $n$ and $\delta$) estimate

$$\|\nabla \varrho_n\|_{L^\infty([0,T;L^2(\Omega)])} + \|\Delta \varrho_n\|_{L^2([0,T] \times \Omega)} \leq c.$$  

(55)

**Passage to the limit** $n \to \infty$. As in the previous paragraph the biggest problem is to identify the limit of the term

$$\mu'(\varrho_n) \nabla \varrho_n \otimes w_n,$$  

(56)

and in the convective term

$$\varrho_n w_n \otimes w_n.$$  

(57)

Having obtained estimate (55) we can estimate the time-derivative of gradient of $\varrho_n$. Indeed taking gradient of (31) we obtain

$$\partial_t \nabla \varrho_n = -\nabla (\varrho_n \text{div}[w_n] \delta) - \nabla ([w_n] \delta \cdot \nabla \varrho_n) - \kappa \nabla \Delta \mu(\varrho_n) \in L^2(0,T;W^{-1,3/2}(\Omega)).$$

Note, however, that unlike in the analogous step of [10] the above estimate is no longer uniform with respect to $\varepsilon$. Now, applying the Aubin-Lions lemma for $\nabla \varrho_n$ we obtain

$$\nabla \varrho_n \to \nabla \varrho \quad \text{strongly in } L^2(0,T;L^2(\Omega)),$$

therefore due to (43) we also have

$$\varrho_n \to \varrho \quad \text{and} \quad \frac{1}{\varrho_n} \to \frac{1}{\varrho} \quad \text{strongly in } L^p(0,T;L^p(\Omega)),$$  

(58)

for $p < \infty$ and

$$\varrho_n w_n \to \varrho w \quad \text{weakly in } L^{p_1}(0,T;L^{q_1}(\Omega)) \cap L^{p_2}(0,T;L^{q_2}(\Omega)),$$  

(59)

where $p_1 < 2$, $q_1 < 6$, $p_2 < \infty$, $q_2 < 2$. These convergences justify the limit passage in (56).
To justify passage to the limit in (57) we first estimate

\[
\|\nabla (\varrho_n \mathbf{w}_n)\|_{L^2(0,T;L^2(\Omega))} \leq \|\nabla \varrho_n\|_{L^\infty(0,T;L^2(\Omega))} \|\mathbf{w}_n\|_{L^2(0,T;L^6(\Omega))} + \|\nabla \mathbf{w}_n\|_{L^2(0,T;L^2(\Omega))} \|\varrho_n\|_{L^\infty(0,T;L^\infty(\Omega))}.
\]  

(60)

We can also estimate the time derivative of momentum, from (46) we obtain

\[
\sup_{\|\phi\| \leq 1} \left\{ \left| \int_0^T \partial_t (\varrho_n \mathbf{w}_n) : \phi \, dx \, dt \right| \right. 
+ 2(1-\kappa) \left| \int_0^T \mu(\varrho_n) D(\mathbf{w}_n) : \nabla \phi \, dx \, dt \right| 
+ 2\kappa \left| \int_0^T \mu(\varrho_n) A(\mathbf{w}_n) : \nabla \phi \, dx \, dt \right| 
+ 2\kappa(1-\kappa) \left| \int_0^T \mu(\varrho_n) \nabla \mathbf{w}_n : \nabla \phi \, dx \, dt \right| 
+ \left. \int_0^T (\mu(\varrho_n) \varrho_n - \mu(\varrho_n)) \text{div} \mathbf{w}_n \text{div} \phi \, dx \, dt \right| 
+ \varepsilon \left| \int_0^T \nabla (\Delta \mathbf{w}_n : \Delta \phi) \, dx \, dt \right| + \varepsilon \left| \int_0^T \nabla (1 + |\nabla \mathbf{w}|^2) \nabla \mathbf{w} : \nabla \phi \, dx \, dt \right| \right\},
\]

(61)

where the norm \(\|\phi\|\) denotes the norm in the space \(W_T := L^2(0,T;H^{2s}(\Omega)) \cap L^4(0,T;W^{1,4}(\Omega))\) with \(s \geq 2\). For the highest order terms we have

\[
\varepsilon \left| \int_0^T \nabla (\Delta^s \mathbf{w}_n : \Delta^s \phi) \, dx \, dt \right| \leq \varepsilon \|\mathbf{w}_n\|_{L^2(0,T;H^{2s}(\Omega))} \|\phi\|_{L^3(0,T;H^{2s}(\Omega))},
\]

and

\[
\varepsilon \left| \int_0^T (1 + |\nabla \mathbf{w}|^2) \nabla \mathbf{w} : \nabla \phi \, dx \, dt \right| \leq \varepsilon \int_0^T \|\nabla \phi\|_{L^4(\Omega)} \left( \|\nabla \mathbf{w}_n\|_{L^4(\Omega)}^3 + \|\nabla \mathbf{w}_n\|_{L^4(\Omega)}^4 \right) \, dt 
\leq \varepsilon \|\nabla \phi\|_{L^1(0,T;L^4(\Omega))} \left( \|\nabla \mathbf{w}\|_{L^4(0,T;L^4(\Omega))}^3 + \|\nabla \mathbf{w}\|_{L^4(0,T;L^4(\Omega))}^4 \right).
\]

The other terms from (61) are less restrictive, therefore

\[
\|\partial_t (\varrho_n \mathbf{w}_n)\|_{W^*} \leq c. 
\]

(62)

Collecting (59), (60), (62) and applying the Aubin-Lions lemma to \(\varrho_n \mathbf{w}_n\), we prove that

\[
\varrho_n \mathbf{w}_n \rightharpoonup \varrho \mathbf{w} \quad \text{strongly in } L^p(0,T;L^p(\Omega))
\]
for some $p > 1$ and therefore thanks to \cite{58} and \cite{44}
\[ \nabla w_n \rightarrow \nabla w \quad \text{strongly in } L^p(0, T; L^p(\Omega)) \]  
for $1 \leq p < 4$. In particular, convergence in \cite{57} is proved.

For future purposes we now estimate the time derivative of $\varrho v$. We use \cite{17} to obtain
\[
\sup_{\|\xi\| \leq 1} \left| \int_0^T \int_{\Omega} \partial_t (\varrho_n v_n) : \xi \, dx \, dt \right|
\]
\[
= \sup_{\|\xi\| \leq 1} \left\{ \int_0^T \int_{\Omega} \left( (\varrho_n[w_n])_\delta - 2\kappa\mu'(\varrho_n)\nabla \varrho_n \right) \otimes v_n : \nabla \xi \, dx \, dt \right. 
+ 2\kappa \int_0^T \int_{\Omega} \mu(\varrho_n) \nabla v_n : \nabla \xi \, dx \, dt 
+ 2\kappa \int_0^T \int_{\Omega} \mu'(\varrho_n) \varrho_n - \mu(\varrho_n)) \div v_n \div \xi \, dx \, dt 
+ 2 \int_0^T \int_{\Omega} \mu'(\varrho_n) \varrho_n - \mu(\varrho_n)) \div w_n \div \xi \, dx \, dt 
+ 2 \int_0^T \int_{\Omega} \mu'(\varrho_n) \varrho_n - \mu(\varrho_n)) \div w_n \div \xi \, dx \, dt \right\}
\]
for $\xi \in L^4(0, T; W^{1,4}(\Omega))$. We will only estimate the convective term since it is most restrictive
\[
\left| \int_0^T \int_{\Omega} \left( (\varrho_n[w_n])_\delta - 2\kappa\mu'(\varrho_n)\nabla \varrho_n \right) \otimes v_n : \nabla \xi \, dx \, dt \right|
\leq \int_0^T \|\nabla \xi\|_{L^4(\Omega)} \left( R \|w_n\|_{L^4(\Omega)} \|v_n\|_{L^2(\Omega)} + c(\kappa, R) \|\nabla \varrho_n\|_{L^1(\Omega)} \|v_n\|_{L^2(\Omega)} \right) \, dt
\leq c(\kappa, R) \|\xi\|_{L^4(0, T; W^{1,4}(\Omega))} \|v_n\|_{L^2(0, T; L^2(\Omega))} \left( \|w_n\|_{L^4(0, T; L^4(\Omega))} + \|\Delta \varrho_n\|_{L^2(0, T; L^2(\Omega))} \right)
\]
thus
\[
\|\partial_t (\varrho_n v_n)\|_{L^4(0, T; W^{-1,4}(\Omega))} \leq c.
\]  
(64)

Hence, the limit functions $(\varrho, w, v) = (\varrho_\delta, w_\delta, v_\delta)$ fulfill

- the continuity equation
\[
\partial_t \varrho_\delta + \div (\varrho_\delta[w_\delta]) - 2\kappa\Delta \mu(\varrho_\delta) = 0
\]
(65)
a.e. in $(0, T) \times \Omega$,

- the momentum equation
\[
\begin{align*}
&\langle \partial_t (\varrho_\delta w_\delta), \phi \rangle_{W_\tau', W_\tau} - \int_0^T \int_{\Omega} ((\varrho_\delta[w_\delta])_\delta - 2\kappa \nabla \mu(\varrho_\delta)) \otimes w_\delta : \nabla \phi \, dx \, dt 
+ 2(1 - \kappa) \int_0^T \int_{\Omega} \mu(\varrho_\delta) D(w_\delta) : \nabla \phi \, dx \, dt 
+ 2\kappa \int_0^T \int_{\Omega} \mu(\varrho_\delta) A(w_\delta) : \nabla \phi \, dx \, dt 
+ 2(1 - \kappa) \int_0^T \int_{\Omega} \mu'(\varrho_\delta) \varrho_\delta - \mu(\varrho_\delta)) \div w_\delta \div \phi \, dx \, dt 
- 2\kappa(1 - \kappa) \int_0^T \int_{\Omega} \mu(\varrho_\delta) \nabla v_\delta : \nabla \phi \, dx \, dt 
- 2\kappa(1 - \kappa) \int_0^T \int_{\Omega} \mu'(\varrho_\delta) \varrho_\delta - \mu(\varrho_\delta)) \div v_\delta \div \phi \, dx \, dt 
- \int_0^T \int_{\Omega} p(\varrho_\delta) \div \phi \, dx \, dt + \varepsilon \int_0^T \int_{\Omega} (\Delta^2 w_\delta \cdot \Delta^2 \phi + (1 + |\nabla w_\delta|^2) \nabla w_\delta : \nabla \phi) \, dx \, dt = 0
\end{align*}
\]  
(66)

for all $\phi \in W_\tau$ with $\tau \in [0, T]$,  

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3.3 Passage to the limit \( \delta \) tends to 0 and identification of \( v \) with \( 2\nabla \varphi(\vartheta) \)

The aim of this paragraph is to let \( \delta \to 0 \) in the equations \( (65), (66) \) and \( (67) \). This limit passage can be performed exactly as \( n \to \infty \) presented above. The only difference is that after this step we may drop the additional equation for \( v \) thanks to identification \( v = 2\nabla \varphi(\vartheta) \). Below we present the details of this reasoning.

Note that the coefficients of the quasi-linear parabolic equation \( (65) \) (i.e. \( [w_\delta]_\delta \) are sufficiently regular and the maximum principle \( (43) \) holds uniformly with respect to all approximation parameters except \( \varepsilon \). Therefore, the classical theory of Ladyženskaja, Solonnikov and Uralceva [21] (Theorems 7.2, 7.3 and 7.4 from [21]) can be applied to show further regularity of \( g_\delta \), we have in particular

\[
\partial_t g_\delta \in C([0,T]; C(\Omega)), \quad g_\delta \in C([0,T]; C^2(\Omega)).
\]

Let us now rewrite \( (65) \) using \( (22) \) as

\[
\partial_t g_\delta + \text{div} (g_\delta ([w_\delta]_\delta - 2\kappa \nabla \varphi(\vartheta_\delta))) = 0
\]

and therefore multiplying the above equation by \( \mu'(\vartheta) \) we obtain

\[
\partial_t \mu(\vartheta_\delta) + \text{div} (\mu(\vartheta_\delta) ([w_\delta]_\delta - 2\kappa \nabla \varphi(\vartheta_\delta))) + (\mu'(\vartheta_\delta) \vartheta_\delta - \mu(\vartheta_\delta)) (\text{div}[w_\delta]_\delta - 2\kappa \Delta \varphi(\vartheta_\delta)) = 0.
\]

Differentiating it with respect to space one gets in the sense of distributions

\[
\partial_t (g_\delta \tilde{v}_\delta) + \text{div}(g_\delta ([w_\delta]_\delta - 2\kappa \nabla \varphi(\vartheta_\delta)) \otimes \tilde{v}_\delta) + 2\nabla \left((\mu'(\vartheta_\delta) \vartheta_\delta - \mu(\vartheta_\delta)) (\text{div}[w_\delta]_\delta - \kappa \text{div} \tilde{v}_\delta)\right)
\]

\[
+ 2\text{div}((\mu(\vartheta_\delta) \nabla [w_\delta]_\delta) - 2\kappa \text{div}(\mu(\vartheta_\delta) \nabla \tilde{v}_\delta) = 0
\]

where by \( \tilde{v}_\delta \) we denoted \( 2\nabla \varphi(\vartheta_\delta) \). Note that due to particular case of the Gagliardo-Nirenberg interpolation inequality \( (53) \) and \( (43) \) we know that \( \nabla g_\delta \) is bounded

\[
\| \nabla g_\delta \|_{L^4(0,T; L^4(\Omega))} \leq c,
\]

uniformly with respect to \( \delta \). One can thus estimate the convective term of \( (69) \) in \( L^2(0,T; W^{1,2}(\Omega)) \) uniformly with respect to \( \delta \). Indeed, we now \( w_\delta \) uniformly bounded in \( L^4(0,T; L^4(\Omega)) \) with respect to \( \delta \) and therefore

\[
\| \Lambda \psi \|_{L^4(0,T; L^4(\Omega))} \leq c(R) \| \nabla \psi \|_{L^2(0,T; L^2(\Omega))} \| \nabla \vartheta_\delta \|_{L^4(0,T; L^4(\Omega))} \| \nabla [w_\delta]_\delta \|_{L^4(0,T; L^4(\Omega))} \| \nabla \vartheta_\delta \|_{L^4(0,T; L^4(\Omega))}
\]

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for $\xi \in L^2(0, T; W^{1,2}(\Omega))$ (uniformly with respect to $\delta$), which justifies that

\[
(\partial_t (g_\delta \tilde{v}_\delta), \xi)_{L^2(0, T; W^{-1,2}(\Omega)), L^2(0, T; W^{1,2}(\Omega))} - \int_0^T \int_\Omega g_\delta ([w_\delta]_\delta - 2\kappa \nabla \varphi(g_\delta)) \otimes \tilde{v}_\delta : \nabla \xi \, dx \, dt
\]

\[
- 2 \int_0^T \int_\Omega (\mu'(g_\delta) g_\delta - \mu(g_\delta)) \text{div}[w_\delta]_\delta \, dx \, dt
\]

\[
+ 2\kappa \int_0^T \int_\Omega (\mu'(g_\delta) g_\delta - \mu(g_\delta)) \text{div}[\tilde{v}_\delta] \, dx \, dt
\]

\[
- 2 \int_0^T \int_\Omega \mu(g_\delta) \nabla^t [w_\delta]_\delta : \nabla \xi \, dx \, dt = 0
\]

is satisfied for any $\xi \in L^2(0, T; W^{1,2}(\Omega))$.

**Remark 12** As we noticed in [63], the regularity of $g_\delta$ is in fact much higher and allows to formulate the equation for $\tilde{v}_\delta$ in much stronger sense than merely (70). This formulation, however, will be used when passing to the limit with respect to $\varepsilon$ after passing to the limit with respect to $\delta$.

We now want show that $\tilde{v}_\delta - v_\delta$ tends to 0 when $\delta$ goes to zero in an appropriate norm. To this purpose let us expand

\[
I = \frac{d}{dt} \int_\Omega \frac{[v_\delta - \tilde{v}_\delta]^2}{2} \, dx + 2\kappa \int_\Omega \mu(g_\delta) |\nabla (v_\delta - \tilde{v}_\delta)|^2 \, dx
\]

\[
+ 2\kappa \int_0^T \int_\Omega (\mu'(g_\delta) g_\delta - \mu(g_\delta)) (\text{div}(v_\delta - \tilde{v}_\delta))^2 \, dx \, dt
\]

\[
\frac{d}{dt} \int_\Omega \frac{|v_\delta|^2}{2} \, dx + 2\kappa \int_\Omega \mu(g_\delta) |\nabla v_\delta|^2 \, dx \, dt + 2\kappa \int_0^T \int_\Omega (\mu'(g_\delta) g_\delta - \mu(g_\delta)) (\text{div} v_\delta)^2 \, dx \, dt
\]

\[
- 2 \int_\Omega (\mu'(g_\delta) g_\delta - \mu(g_\delta)) \text{div} w_\delta \text{div} v_\delta \, dx - 2 \int_0^T \int_\Omega \mu(g_\delta) \nabla^t w_\delta : \nabla v_\delta \, dx \, dt \leq 0.
\]

Now, the last term in (71) can be computed using $\xi = \tilde{v}_\delta$ in (70), we have

\[
\frac{d}{dt} \int_\Omega \frac{[\tilde{v}_\delta]^2}{2} \, dx + 2\kappa \int_\Omega \mu(g_\delta) |\nabla \tilde{v}_\delta|^2 \, dx \, dt + 2\kappa \int_0^T \int_\Omega (\mu'(g_\delta) g_\delta - \mu(g_\delta)) (\text{div} \tilde{v}_\delta)^2 \, dx \, dt
\]

\[
- 2 \int_\Omega \mu(g_\delta) \nabla^t [w_\delta]_\delta : \nabla \tilde{v}_\delta \, dx - 2 \int_0^T \int_\Omega (\mu'(g_\delta) g_\delta - \mu(g_\delta)) \text{div} w \text{div} \tilde{v}_\delta \, dx \, dt = 0.
\]

The middle term in (71) equals

\[
\frac{d}{dt} \int_\Omega g_\delta v_\delta \cdot \tilde{v}_\delta \, dx = \int_\Omega (\partial_t (g_\delta v_\delta) \cdot \tilde{v}_\delta + v_\delta \cdot \partial_t (g_\delta \tilde{v}_\delta) - \partial_t g_\delta v_\delta \cdot \tilde{v}_\delta) \, dx
\]
and the two first terms make sense and can be handled using $\xi = \tilde{\nu}_\delta$ in \(67\) and $\xi = \nu_\delta$ in \(70\). Note that $\tilde{\nu}_\delta$ and $\partial_t \varrho_\delta$ are due to \(68\) regular enough to justify the integrability of the last term in \(73\) and we can write

$$
\int_\Omega \partial_t \varrho_\delta \nu_\delta \cdot \tilde{\nu}_\delta \, dx
$$

$$
= \int_\Omega (\varrho_\delta([w_\delta]_\delta - 2\kappa \nabla \varphi(\varrho_\delta))) \otimes \nu_\delta : \nabla \tilde{\nu}_\delta \, dx + \int_\Omega (\varrho_\delta([w_\delta]_\delta - 2\kappa \nabla \varphi(\varrho_\delta))) \otimes \tilde{\nu}_\delta : \nabla \nu_\delta \, dx.
$$

Therefore, after summing all expressions together and after some manipulation, we can show that

$$
I - 2 \int_0^T \int_\Omega (\mu'(\varrho_\delta)\varrho_\delta - \mu(\varrho_\delta)) \text{div} [w_\delta]_\delta \text{div} \tilde{\nu}_\delta \, dx \, dt + 2 \int_0^T \int_\Omega (\mu'(\varrho_\delta)\varrho_\delta - \mu(\varrho_\delta)) \text{div} w_\delta \text{div} \tilde{\nu}_\delta \, dx \, dt
$$

$$
+ 2 \int_0^T \int_\Omega (\mu'(\varrho_\delta)\varrho_\delta - \mu(\varrho_\delta)) \text{div} [w_\delta]_\delta \text{div} \nu_\delta \, dx \, dt - 2 \int_0^T \int_\Omega (\mu'(\varrho_\delta)\varrho_\delta - \mu(\varrho_\delta)) \text{div} w_\delta \text{div} \nu_\delta \, dx \, dt
$$

$$
- \int_\Omega \mu(\varrho_\delta) \nabla^t [w_\delta]_\delta : \nabla \tilde{\nu}_\delta \, dx - \int_\Omega \mu(\varrho_\delta) \nabla^t [w_\delta]_\delta : \nabla \nu_\delta \, dx
$$

$$
+ \int_\Omega \mu(\varrho_\delta) \nabla^t w_\delta : \nabla \tilde{\nu}_\delta \, dx + \int_\Omega \mu(\varrho_\delta) \nabla^t [w_\delta]_\delta : \nabla \nu_\delta \, dx \leq 0,
$$

in particular

$$
I \leq \int_\Omega \mu(\varrho) \nabla^t ([w_\delta]_\delta - w_\delta) : \nabla (\tilde{\nu}_\delta - \nu_\delta) \, dx
$$

$$
+ 2 \int_0^T \int_\Omega (\mu'(\varrho_\delta)\varrho_\delta - \mu(\varrho_\delta)) \text{div} ([w_\delta]_\delta - w_\delta) \text{div} (\tilde{\nu}_\delta - \nu_\delta) \, dx \, dt.
$$

Note that the r.h.s. of this inequality tends to 0 when $\delta \to 0$. Indeed, we can bound $\nabla (\tilde{\nu}_\delta - \nu_\delta)$ in $L^2(0,T; L^2(\Omega))$ uniformly with respect to $\delta$ and $[w_\delta]_\delta \to w$ strongly in $L^p(0,T; W^{1,p}(\Omega))$ for $p < 4$. Therefore, using \(71\), we conclude that $\nu_\delta - \tilde{\nu}_\delta$ converges to zero in $L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ when $\delta \to 0$. □

The limit functions $(\varrho, w) = (\varrho_e, w_e)$ fulfill

- the continuity equation

$$
\partial_t \varrho_e + \text{div} (\varrho_e w_e) - \kappa \Delta \mu(\varrho_e) = 0 \quad (74)
$$

a.e. in $(0, T) \times \Omega$,

- the momentum equation

\[
\langle \partial_t (\varrho_e w_e), \phi \rangle_{(W,f,W^e)} - \int_0^T \int_\Omega (\varrho_e w_e - 2\kappa \nabla \mu(\varrho_e)) \otimes w_e : \nabla \phi \, dx \, dt
\]

$$
+ 2(1 - \kappa) \int_0^T \int_\Omega \mu(\varrho_e) D(w_e) : \nabla \phi \, dx \, dt + 2\kappa \int_0^T \int_\Omega \mu(\varrho_e) A(w_e) : \nabla \phi \, dx \, dt
$$

$$
+ 2(1 - \kappa) \int_0^T \int_\Omega (\mu'(\varrho_e)\varrho_e - \mu(\varrho_e)) \text{div} w_e \text{div} \phi \, dx \, dt
$$

$$
- 4\kappa(1 - \kappa) \int_0^T \int_\Omega \mu(\varrho_e) \nabla^2 \varphi(\varrho_e) : \nabla \phi \, dx \, dt
$$

$$
- 4\kappa(1 - \kappa) \int_0^T \int_\Omega (\mu'(\varrho_e)\varrho_e - \mu(\varrho_e)) \Delta \varphi(\varrho_e) \text{div} \phi \, dx \, dt
$$

$$
- \int_0^T \int_\Omega p(\varrho_e) \text{div} \phi \, dx \, dt + \varepsilon \int_0^T \int_\Omega (\Delta^e \varrho_e \cdot \Delta^e \phi + (1 + |\nabla w_e|^2) \nabla w_e : \nabla \phi) \, dx \, dt = 0
$$

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for $\phi \in W_\tau, \tau \in [0, T]$,

- the auxiliary equation for $\nabla \varphi(\rho_e)$

\[
\langle \partial_t \nabla \mu(\rho_e), \xi \rangle_{(L^2(0, \tau; W^{-1,2}(\Omega)), L^2(0, \tau; W^{1,2}(\Omega)))}
- \int_0^T \int_\Omega ((\rho_e \dot{w}_e - 2\kappa \nabla \mu(\rho_e) \otimes \nabla \varphi(\rho_e)) : \nabla \xi) \ dx \ dt + 2\kappa \int_0^T \int_\Omega \nabla \mu(\rho_e) : \nabla \xi \ dx \ dt
+ 2\kappa \int_0^T \int_\Omega (\mu'(\rho_e) \rho_e - \mu(\rho_e)) \nabla \varphi(\rho_e) \cdot \nabla \xi \ dx \ dt
- \int_0^T \int_\Omega (\mu'(\rho_e) \rho_e - \mu(\rho_e)) \partial \rho_e \cdot \nabla \xi \ dx \ dt - \int_0^T \int_\Omega \mu(\rho_e) \nabla^2 \rho_e : \nabla \xi \ dx \ dt = 0,
\]

for all $\xi \in L^2(0, \tau; W^{1,2}(\Omega))$ with $\tau \in [0, T]$.

### 3.4 Recovering the global weak $\kappa$-entropy solution

Due to the identification, we may now define a new quantity

\[
u = w - 2\kappa \nabla \varphi(\rho).
\]

(77)

Compared to [10], we cannot pass to the limit in the equations satisfied by $w$ and $v$. We have to combine them to deal with an equation on $u$. More precisely, multiplying (33) by $\kappa$ and substracting to (32) we obtain the weak formulation on $u$

\[
\langle \partial_t u_e \otimes u_e, \phi \rangle_{(W_\tau^2, W_\tau^2)} - \int_0^T \int_\Omega (u_e \otimes u_e) : \nabla \phi \ dx \ dt + 2 \int_0^T \int_\Omega \mu(\rho_e) \nabla^2 u_e : \nabla \phi \ dx \ dt
+ \int_0^T \int_\Omega \Delta^2 u_e \cdot \Delta \phi \ dx \ dt + \epsilon \int_0^T \int_\Omega (1 + |\nabla w_e|^2) \nabla w_e \cdot \nabla \phi \ dx \ dt = 0
\]

satisfied for all $\phi \in W_\tau$ with $\tau \in [0, T]$.

Passing to the limit in (78) with respect to $\epsilon$ follows the lines introduced by [6] using the uniform estimates given by the regularized $\kappa$-entropy. See also [33] for details regarding the barotropic case with chemical species. Now, after passing to the limit in the energy estimate [41] (with all parameters $\alpha \to 0, n \to \infty, \delta \to 0$ in the same way), we obtain

\[
\frac{d}{dt} \int_\Omega \rho_e \left( \frac{|w_e|^2}{2} + (1-\kappa)\kappa \frac{|\nabla \varphi(\rho_e)|^2}{2} \right) \ dx + \frac{d}{dt} \int_\Omega \rho_e \varphi(\rho_e) \ dx + 2(1-\kappa) \int_\Omega \mu(\rho_e) |D(u_e)|^2 \ dx
+ 2\kappa \int_\Omega \mu(\rho_e) |A(w_e)|^2 \ dx + 2(1-\kappa) \int_\Omega (\mu'(\rho_e) \rho_e - \mu(\rho_e)) |\nabla u_e|^2 \ dx
+ 2\kappa(\mu'(\rho_e) \rho_e - \mu(\rho_e)) |\nabla \varphi(\rho_e)|^2 \ dx + \epsilon \int_\Omega |\Delta^2 w_e|^2 \ dx + \epsilon \int_\Omega (1 + |\nabla w_e|^2) |\nabla w_e|^2 \ dx \leq 0,
\]

(79)
where we essentially used the regularization \( \varepsilon [\Delta^{2*} w_\varepsilon - \text{div}(1 + |\nabla w_\varepsilon|^2) \nabla w_\varepsilon] \). We then show that the limit when \( \varepsilon \to 0 \) give a solution \((\varrho, u)\) which satisfies the following energy inequality

\[
\sup_{t \in [0,T]} \left[ \int_\Omega \left( \frac{|w|^2}{2} + (1 - \kappa) \frac{2\nabla \varphi(\varrho)^2}{2} \right) (t) \, dx + \int_\Omega \varrho \varphi(t) \, dx \right] + 2(1 - \kappa) \int_0^T \int_\Omega \mu(\varrho) |D(u)|^2 \, dx
\]

\[
+ 2\kappa \int_0^T \int_\Omega \mu(\varrho) |A(w)|^2 \, dx + 2(1 - \kappa) \int_0^T \int_\Omega (\mu'(\varrho) \varrho - \mu(\varrho))(\text{div} u)^2 \, dx
\]

\[
+ 2\kappa \int_0^T \int_\Omega \frac{\mu'(\varrho)p'(\varrho)}{\varrho} |\nabla \varrho|^2 \, dx \leq \int_\Omega \left( \varrho_0 \left( \frac{|w_0|^2}{2} + (1 - \kappa) \frac{2\nabla \varphi(\varrho_0)^2}{2} \right) + \varrho_0 \epsilon(\varrho_0) \right) \, dx.
\]

(80)

For the limit passage in the corresponding equations we use the density estimates due to the singular pressure, as it was introduced in the work by the first two authors and generalized to chemically reacting mixture case by the third author and collaborators. In particular, one needs to ensure that

\[
\varrho_{\frac{1}{2}} u \in L^s(0, T; L^r(\Omega))
\]

with \( r, s > 2 \). Here assumptions of \( n, m \) and \( \gamma^- \) [13], [14] play an important role. Indeed, assuming for the moment that \( \varrho_{\frac{1}{2}} u \in L^\infty(0, T; L^2(\Omega)) \), \( \varrho \in L^\infty(0, T; L^p(\Omega)) \) and \( u \in L^{q_1}(0, T; L^{q_2}(\Omega)) \), we can write exactly as in the proof of Lemma 6.1. from [8] that

\[
\varrho_{\frac{1}{2}} |u| = \varrho^{\frac{1}{2} - \varrho} |u|^{2\varrho} |u|^{1-2\varrho}
\]

for some \( 0 \leq \varrho \leq \frac{1}{2} \). Therefore

\[
\| \varrho_{\frac{1}{2}} u \|_{L^s(0, T; L^r(\Omega))} \leq \| \varrho \|_{L^\infty(0, T; L^p(\Omega))} \| \varrho_{\frac{1}{2}} u \|_{L^\infty(0, T; L^2(\Omega))} \| u \|_{L^{q_1}(0, T; L^{q_2}(\Omega))}^{1-2\varrho}
\]

with

\[
\frac{1}{s} = 1 - 2\varrho, \quad \frac{1}{r} = \frac{1}{p} \left( \frac{1}{2} - \varrho \right) + \varrho + \frac{1}{q_2}, \quad \frac{1}{q_1} = \frac{1}{p} + \frac{1}{q_2} - \varrho.
\]

(81)

It follows from the \( \kappa \)-entropy estimate [80] that the best \( p \) we can take is equal

\[
p = 6m - 3.
\]

Let us now determine \( q_1, q_2 \), we first check that the \( \kappa \)-entropy estimate [80] gives us uniform bound on

\[
\int_0^T \int_\Omega \frac{\mu'(\varrho)p'(\varrho)}{\varrho} |\nabla \varrho|^2 \, dx \, dt.
\]

In particular, for \( \varrho < \varrho^* \), we obtain

\[
\nabla \left( \varrho^{-\frac{n+1}{2}} \right) \in L^2(0, T; L^2(\Omega)),
\]

which controls the negative powers of \( \varrho \) close to vacuum. We can use this estimate to determine \( q_1 \) and some \( q_3 \), we have

\[
\| \nabla u \|_{L^{q_1}(0, T; L^{q_3}(\Omega))} \leq c \left( 1 + \| \varrho^{-\frac{n}{2}} \|_{L^{2q}(0, T; L^q(\Omega))} \right) \| \varrho_{\frac{n}{2}} \nabla u \|_{L^2(0, T; L^2(\Omega))}
\]

\[
\leq c \left( 1 + \| \nabla \left( \varrho^{-\frac{n+1}{2}} \right) \|_{L^2(0, T; L^2(\Omega))} \right) \| \varrho_{\frac{n}{2}} \nabla u \|_{L^2(0, T; L^2(\Omega))}
\]

(82)

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where
\[ j = \frac{\gamma^- + 1 - n}{n}, \quad q_1 = \frac{2j}{j + 1} = 2 \left( 1 - \frac{n}{\gamma^- + 1} \right), \quad \frac{1}{q_3} = \frac{1}{6j} + \frac{1}{2}. \] (83)

Thus in (81) we can take
\[ q_2 = \frac{3q_3}{3 - q_3} = 3q_1 \]
provided that
\[ q_1 > 1, \quad \text{i.e.} \quad n < \frac{\gamma^- + 1}{2}. \]

Inserting \( q_2 = 3q_1 \) and \( p = 6m - 3 \) to (81) and using (83) we obtain \( r, s > 2 \) provided that in addition we assume that
\[ \gamma^- > \frac{2n(3m - 2)}{4m - 3} - 1, \quad m > \frac{3}{4}, \]
but there is no restriction on \( \gamma^+ > 1 \).

4 Navier–Stokes equations with drag terms

Let us recall the compressible Navier–Stokes system with quadratic turbulent drag force
\[
\begin{aligned}
\partial_t \varrho &+ \text{div}(\varrho \mathbf{u}) = 0, \\
\partial_t (\varrho \mathbf{u}) + \text{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \text{div}(2\mu(\varrho)D(\mathbf{u})) - \nabla(\lambda(\varrho)\text{div} \mathbf{u}) + r_1 \varrho|\mathbf{u}| + \nabla p(\varrho) = 0,
\end{aligned}
\] (84)

with the usual pressure law \( p(\varrho) = a\varrho^\gamma \) with \( \gamma > 1 \). The augmented regularized system reads
\[
\begin{aligned}
\partial_t \varrho &+ \text{div}(\varrho |\mathbf{w}|) - 2\kappa \text{div} \left( (\mu'(\varrho))_\alpha \nabla \varrho \right) = 0, \\
\partial_t (\varrho \mathbf{w}) &+ \text{div}(\varrho |\mathbf{w}| \delta - 2\kappa \mu'(\varrho)_\alpha \nabla \varrho \otimes \mathbf{w}) - \nabla(\lambda(\varrho - 2\kappa \mu'(\varrho) - \mu(\varrho))) \text{div}(\mathbf{w} - \kappa \mathbf{v}) \\
&- 2(1 - \kappa) \text{div}(\mu(\varrho)D(\mathbf{w})) - 2\kappa \text{div}(\mu(\varrho)A(\mathbf{w})) \\
&+ \varepsilon \Delta^2 \varrho \mathbf{w} - \varepsilon \text{div}((1 + |\nabla \mathbf{w}|^2)\nabla \mathbf{w}) + r_1 \varrho|\mathbf{w} - 2\kappa \nabla \varphi(\varrho)|(\mathbf{w} - 2\kappa \nabla \varphi(\varrho)) + \nabla p(\varrho) \\
&= -2\kappa(1 - \kappa) \text{div}(\mu(\varrho)\nabla \mathbf{v}), \\
\partial_t (\varrho \mathbf{v}) &+ \text{div}(\varrho |\mathbf{w}| - 2\kappa(\mu'(\varrho))_\alpha \nabla \varrho \otimes \mathbf{v}) \\
&- 2\kappa \text{div}(\mu(\varrho)\nabla \mathbf{v}) + 2\nabla \left( (\mu'(\varrho)) - \mu(\varrho)) \text{div}(\mathbf{w} - \kappa \mathbf{v}) \right) \\
&= -2\kappa \text{div}(\mu(\varrho)\nabla \mathbf{w}),
\end{aligned}
\] (85)

The additional drag term \( r_1 |\mathbf{w} - 2\kappa \nabla \varphi(\varrho)|(\mathbf{w} - 2\kappa \nabla \varphi(\varrho)) \) will not really affect the \( \kappa \)-entropy. More precisely the contribution to the \( \kappa \)-entropy estimate may be written as
\[
\int_\Omega \varrho |\mathbf{w} - \kappa \nabla \varphi(\varrho)|(\mathbf{w} - \kappa \nabla \varphi(\varrho)) \cdot \mathbf{w} \, dx = \int_\Omega \varrho |\mathbf{w} - \kappa \nabla \varphi(\varrho)|^3 \, dx + \kappa \int_\Omega \varrho |\mathbf{w} - \kappa \nabla \varphi(\varrho)|(\mathbf{w} - \kappa \nabla \varphi(\varrho)) \cdot \nabla \varphi(\varrho) \, dx
\]

Note that the second term of the above right hand side rewrites:
\[
\begin{aligned}
\int_\Omega \varrho |\mathbf{w} - 2\kappa \nabla \varphi(\varrho)|(\mathbf{w} - 2\kappa \nabla \varphi(\varrho)) \cdot \nabla \varphi(\varrho) \\
&= - \int_\Omega \mu(\varrho)|\mathbf{w} - 2\kappa \nabla \varphi(\varrho)| \text{div}(\mathbf{w} - 2\kappa \nabla \varphi(\varrho)) \, dx - \int_\Omega \mu(\varrho) |\mathbf{w} - 2\kappa \nabla \varphi(\varrho)| \cdot (\mathbf{w} - 2\kappa \nabla \varphi(\varrho)) \cdot (\mathbf{w} - 2\kappa \nabla \varphi(\varrho)) \, dx
\end{aligned}
\]
Now we use the hypothesis made on viscosities. We can bound
\[ \int_{\Omega} \mu(\varrho)|\text{div}\, u| \, dx \leq \frac{1}{2} \|\sqrt{\mu(\varrho)} D(u)\|^2_{L^2(\Omega)} + \frac{r_1}{3} \|\varrho|u|^3\|_{L^1(\Omega)} + c(r_1) \int_{\Omega} \frac{\mu(\varrho)}{\varrho^2} \, dx. \]

and using \([19]\) and \([20]\)
\[ \int_{\Omega} \frac{\mu(\varrho)^3}{\varrho^2} \, dx \leq C + \int_{\Omega} \frac{\mu(\varrho)^3}{\varrho^2} 1_{\{(t,x);\varrho(t,x)\geq 1\}} \, dx \leq C + c(r_1) \int_{\Omega} \varrho e(\varrho) \, dx \]
with \(c(r_1) \to 0\) when \(r_1\) tends to zero. Thus the \(\kappa\)-entropy is not perturbed and we get uniform estimate on \(\varrho|u|^3\) with respect to \(\varepsilon\) in \(L^1(0,T; L^1(\Omega))\). It is enough to conclude as in \([9]\) because the extra estimate we get replaces the Mellet-Vasseur quantity involved in \([23]\).

5 Hypocoercivity revisited on linearized compressible Navier-Stokes

In this section, we want to show to readers who are not familiar with compressible Navier-Stokes equations why our \(\kappa\)-entropy equality may be seen as a nonlinear version of the so-called hypocoercivity property which is strongly used in the framework of strong solutions. The interested readers is referred to \([32]\) and \([1]\) for more general discussions around hypocoercivity and to \([13]\) for deep mathematical results on compressible Navier-Stokes equations in critical spaces.

5.1 Linearized barotrophic system

In this subsection we consider the barotropic compressible Navier-Stokes system. Linearization around the state \((\varrho^0, u^0) = (1, 0)\) on the barotropic system gives
\[
\begin{cases}
\partial_t \varrho + \text{div}\, u = 0, \\
\partial_t u + \nabla \varrho - 2\mu \text{div}(D(u)) + \frac{2}{d} \mu \nabla \text{div}\, u = 0,
\end{cases}
\tag{86}
\]
with \(D(u) = (\nabla u + \nabla' u)/2\). The chosen coefficient \(\lambda\) is the borderline case that means the one such that \(\lambda + 2\mu/d = 0\). For the sake of simplicity, we focus on the periodic domain \(\Omega = T^d\). The standard energy (multiplying the first equation of \([86]\) by \(\varrho\) and the second equation by \(u\) and summing them up) reads
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |u|^2 + |\varrho|^2 \right) \, dx + \mu \int_{\Omega} \left| D(u) - \frac{\text{div}\, u}{d} \right|^2 \, dx = 0. \tag{87}
\]
Let \(\kappa\) be a constant that will be determined later one, then \(v = u + 2\kappa \mu \nabla \varrho\) satisfies the equation
\[
\partial_t v + \nabla \varrho - \mu \Delta u - \mu \left( 1 - \frac{2}{d} - 2\kappa \right) \nabla \text{div}\, u = 0
\]
and therefore (multiplying this equation by $v$, the first mass equation by $\varrho$ and summing them up)

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (|v|^2 + |\varrho|^2) \, dx + 2\mu \int_{\Omega} |\nabla v|^2 \, dx + 2\mu \int_{\Omega} |\nabla v|^2 \, dx + 2\mu \int_{\Omega} |A(u)|^2 \, dx \\
+ 2\mu \left(1 - \frac{1}{d} - \kappa\right) \int_{\Omega} \nabla \varrho \cdot \nabla \varrho \, dx = 0.
$$

This equality may be rewritten (using that $\Delta = \nabla \div - \curl \curl$)

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (|v|^2 + |\varrho|^2) \, dx + 2\kappa \mu \int_{\Omega} |\nabla \varrho|^2 \, dx + 2\mu \int_{\Omega} |A(u)|^2 \, dx \\
+ 2\mu \left(1 - \frac{1}{d} - \kappa\right) \int_{\Omega} \nabla \varrho \cdot \nabla \varrho \, dx = 0.
$$

where $A(u) = (\nabla u - (\nabla u)^t)/2$. Testing the gradient of the mass equation by $\nabla \varrho$, we get

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \varrho|^2 \, dx + \int_{\Omega} \nabla \varrho \cdot \nabla \div u \, dx = 0.
$$

Assume now that $0 < \kappa < (d - 1)/d$, then multiplying the last relation by $4\mu^2 \kappa \left(1 - \frac{1}{d} - \kappa\right)$ and adding to (88) we get

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (|v|^2 + |\varrho|^2) \, dx + 2\mu \int_{\Omega} |\nabla v|^2 \, dx \\
+ 2\mu \left(1 - \frac{1}{d} - \kappa\right) \int_{\Omega} \nabla \varrho \cdot \nabla \varrho \, dx = 0.
$$

Remark that this equality is the analog, deleting the term $1/d$, of the $\kappa$-entropy we found in the nonlinear framework: Note that

$$
\int_{\Omega} |A(u)|^2 \, dx + (1 - \kappa) \int_{\Omega} |\div u|^2 \, dx = \kappa \int_{\Omega} |A(u)|^2 \, dx + (1 - \kappa) \int_{\Omega} |D(u)|^2 \, dx
$$

In particular, this $\kappa$-entropy (89) provides the exponential decay in time to $(0,0)$ of $(\varrho, u)$ in the $L^2(\Omega)$ norm if the initial perturbation $(\varrho_0, u_0)$ is in $L^2(\Omega)$.

Let us now look at exponential decay in $H^1$ norm. Taking the curl of the momentum equation in (86) and testing against $\curl u$, we get

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\curl u|^2 \, dx + \mu \int_{\Omega} |\curl \curl u|^2 \, dx = 0.
$$

Taking the div of the momentum equation and testing it against $\div u$

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\div u|^2 \, dx + (2\mu + \lambda) \int_{\Omega} |\nabla \div u|^2 \, dx + \int_{\Omega} \Delta \varrho \div u \, dx = 0.
$$

Adding these equation to the mass equation tested against $\Delta \varrho$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (|\div u|^2 + |\nabla \varrho|^2) \, dx + (2\mu + \lambda) \int_{\Omega} |\nabla \div u|^2 \, dx = 0.
$$

Thus (87), (89), (90), (91) provide an exponential decay in time for $\sqrt{\kappa}||\varrho, u||_{H^1(\Omega)}$ to $0$ assuming initial perturbation $\sqrt{\kappa}(\varrho_0, u_0)$ in $H^1(\Omega)$. Note in particular, that coefficient $\kappa$ can be arbitrary small.
5.2 Linearized heat-conducting compressible Navier–Stokes equations

In this subsection, we consider the linearized compressible Navier–Stokes equation with heat conductivity around $(1, 0, 1)$ namely

$$
\begin{align*}
\partial_t \theta + \text{div} \, u &= 0, \\
\partial_t u + \nabla \theta + \nabla \theta - 2\mu \text{div}(D(u)) + \frac{2}{d} \mu \nabla \text{div} \, u &= 0, \\
\partial_t \theta + \frac{2}{d} \text{div} \, u - K \Delta \theta &= 0,
\end{align*}
$$

with the periodic boundary conditions. Such system may be found for instance in [32] page 51. For this system, we easily check that there exists an energy, defined as

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |u|^2 + |\theta|^2 + \frac{d}{2} |\theta|^2 \right) \, dx + 2\mu \int_{\Omega} \left| D(u) - \frac{\text{div} \, u \cdot \mathbf{1}}{d} \right|^2 \, dx + \frac{K d}{2} \int_{\Omega} |\nabla \theta|^2 \, dx = 0. \tag{93}
$$

Differentiating the mass equation and the temperature equation with respect to space, we find the following equations

$$
\begin{align*}
\partial_t \theta + \text{div} \, u &= 0, \\
\partial_t u + \nabla \theta + \nabla \theta - 2\mu \text{div}(D(u)) + \frac{2}{d} \mu \nabla \text{div} \, u &= 0, \\
\partial_t \theta + \frac{2}{d} \text{div} \, u - \kappa \Delta \theta &= 0, \\
\partial_t \nabla \theta + \nabla \text{div} \, u &= 0, \\
\partial_t \nabla \theta + \frac{2}{d} \nabla \text{div} \, u - \kappa \Delta \nabla \theta &= 0.
\end{align*}
$$

Thus we can write a new equation satisfied by artificial velocity $u + 2\kappa \mu \nabla \theta$

$$
\partial_t (u + 2\kappa \mu \nabla \theta) + \nabla \theta + \nabla \theta - \mu \Delta u - \mu (1 - \frac{2}{d} - 2\kappa) \nabla \text{div} \, u = 0.
$$

Testing it against $u + 2\kappa \mu \nabla \theta$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |u + 2\kappa \mu \nabla \theta|^2 + \theta^2 \right) \, dx + 2\kappa \mu \int_{\Omega} |\nabla \theta|^2 \, dx + \mu \int_{\Omega} |\nabla u|^2 \, dx + \mu (1 - \frac{2}{d} - 2\kappa) \int_{\Omega} |\text{div} \, u|^2 \, dx

- 4\mu^2 \kappa \left( 1 - \frac{1}{d} - \kappa \right) \int_{\Omega} \nabla \text{div} \, u \cdot \nabla \theta \, dx + \int_{\Omega} \nabla \theta \cdot u \, dx + 2s\kappa \mu \int_{\Omega} \nabla \theta \cdot \nabla \theta \, dx = 0. \tag{95}
$$

Thus as in the barotropic case, we have

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |u + 2\kappa \mu \nabla \theta|^2 + \theta^2 \right) \, dx + 2\kappa \mu \int_{\Omega} |\nabla \theta|^2 \, dx + 2\mu \int_{\Omega} |A(u)|^2 \, dx + 2\mu (1 - \frac{1}{d} - \kappa) \int_{\Omega} |\text{div} \, u|^2 \, dx

- 4\mu^2 \kappa \left( 1 - \frac{1}{d} - \kappa \right) \int_{\Omega} \nabla \text{div} \, u \cdot \nabla \theta \, dx + \int_{\Omega} \nabla \theta \cdot u \, dx + 2\kappa \mu \int_{\Omega} \nabla \theta \cdot \nabla \theta \, dx = 0. \tag{96}
$$

Recall that for $\nabla \theta$, we have

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \theta|^2 \, dx + \int_{\Omega} \nabla \text{div} \, u \cdot \nabla \theta \, dx = 0. \tag{97}
$$
Multiplying this relation by $4\mu^2\kappa \left(1 - \frac{1}{d} - \kappa\right)$ and adding to the previous one, we get

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |u + 2\kappa \mu \nabla \varrho|^2 + \varrho^2 + 4\mu^2\kappa \left(1 - \frac{1}{d} - \kappa\right) |\nabla \varrho|^2 \right) dx + 2\kappa \mu \int_{\Omega} |\nabla \varrho|^2 dx + 2\mu \int_{\Omega} |A(u)|^2 dx + 2\mu \left(1 - \frac{1}{d} - \kappa\right) \int_{\Omega} |\text{div} u|^2 dx + \int_{\Omega} \nabla \theta \cdot \nabla \varrho \ dx = 0.
$$

(98)

Recall that

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{d}{2} \theta^2 dx + \frac{Kd}{2} \int_{\Omega} |\nabla \theta|^2 dx + \int_{\Omega} \theta \text{div} u \ dx = 0,
$$

(99)

which, when added to (98), yields

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |u + 2\kappa \mu \nabla \varrho|^2 + \varrho^2 + 4\mu^2\kappa \left(1 - \frac{1}{d} - \kappa\right) |\nabla \varrho|^2 + \frac{d}{2} \theta^2 \right) dx + 2\mu \int_{\Omega} |A(u)|^2 dx + 2\mu \left(1 - \frac{1}{d} - \kappa\right) \int_{\Omega} |\text{div} u|^2 dx + \int_{\Omega} \nabla \theta \cdot \nabla \varrho \ dx + \kappa \mu \int_{\Omega} |\nabla \varrho|^2 dx + \kappa \mu \int_{\Omega} |(\varrho + \theta)|^2 dx = 0.
$$

(100)

Choosing $\kappa$ such that $0 < \kappa \mu < dK/2$ and $0 < \kappa < (d-1)/d$, the $\kappa$-entropy balance (100) gives the exponential decay of $(\nabla \varrho, u, \theta)$ in the $L^2$ norm, note in particular an interesting interplay between conductivity and pressure.

Let us now focus on exponential decay in $H^1$ norms. Let us take the equation satisfied by $\text{div} u$, test it against $\text{div} u$ and integrate, we get

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\text{div} u|^2 dx + \int_{\Omega} \Delta \varrho \text{div} u \ dx + \int_{\Omega} \Delta \theta \text{div} u \ dx + (2\mu + \lambda) \int_{\Omega} |\nabla \text{div} u|^2 \ dx = 0.
$$

(101)

Let us now take the equation satisfied by $\nabla \theta$ and test it against $\nabla \theta$, we get

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \theta|^2 dx - \frac{2}{d} \int_{\Omega} \text{div} u \Delta \theta \ dx + K \int_{\Omega} |\Delta \theta|^2 dx = 0.
$$

(102)

Recall that

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \theta|^2 dx - \int_{\Omega} \text{div} u \Delta \theta \ dx = 0.
$$

(103)

Thus adding (101), (102) to (103) gives

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |\text{div} u|^2 + |\nabla \theta|^2 + \frac{d}{2} |\nabla \theta|^2 \right) dx + \frac{dK}{2} \int_{\Omega} |\Delta \theta|^2 dx + (2\mu + \lambda) \int_{\Omega} |\nabla \text{div} u|^2 \ dx = 0.
$$

(104)

Recalling that the curl $u$ satisfies (90), combining it with (104) and the $\kappa$-entropy equality (100), we get the exponential decay in time of $\|\sqrt{\kappa}(\varrho, u, \theta)\|_{H^1(\Omega)}$ if the initial perturbation $\sqrt{\kappa}(\varrho_0, u_0, \theta_0)$ is uniformly bounded in $H^1(\Omega)$. 

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6 Heat-conducting Navier–Stokes equations with \( \kappa \)-energy.

In this section, we present the equations of motions for the heat-conducting fluid written in terms of the two velocities \( \mathbf{u} \) and \( \mathbf{u} + 2\nabla \varphi(\rho) \) with corresponding densities \((1 - \kappa)\rho \) and \( \kappa \rho \). We do not aim at proving the existence result for such system but on showing that the two-velocity hydrodynamics in the spirit of the work by S.M. Shugrin [30] is consistent with the study performed for the low Mach number system in the first part of this diptych in [10]. More precisely, we will show that the formal low-Mach number limit for the two-velocities system gives the augmented system used in [10] to construct the approximate solution. An important observation is that the system presented below does not coincide with the usual heat-conducting compressible Navier-Stokes equations. Indeed, the two-velocities description of the fluid lead to different energy equation with a generalised temperature, called the \( \kappa \)-temperature. However, this is not \textit{a priori} the usual temperature, unless the system reduces to the angle velocity one (i.e. the density \( \kappa \rho \) is equal to 0). This property was also explained in the works [30] and [17] where the authors discuss the capillary-temperature.

We assume that \( \Omega \) is a periodic box in \( \mathbb{R}^3 \), i.e. \( \Omega = \mathbb{T}^3 \), and we consider the following two-velocity system

\[
\begin{aligned}
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) &= 0 \\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \text{div}(2\mu(\rho)D(\mathbf{u})) - \nabla(\lambda(\rho)\text{div} \mathbf{u}) + \nabla p(\rho, e_\kappa) &= 0 \\
\frac{\partial (\rho(\mathbf{u} + 2\nabla \varphi(\rho)))}{\partial t} + \text{div}(\rho \mathbf{u} \otimes (\mathbf{u} + 2\nabla \varphi(\rho))) - \text{div}(2\mu(\rho)A(\mathbf{u})) + \nabla p(\rho, e_\kappa) &= 0 \\
\frac{\partial \kappa}{\partial t} + \text{div}(\kappa \mathbf{u}) + \text{div}(p(1 - \kappa)\mathbf{u} + \kappa(\mathbf{u} + 2\nabla \varphi(\rho))) + \text{div} Q_\kappa &= 0,
\end{aligned}
\]

where we denoted \( D(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla^t \mathbf{u}) \) and \( A(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} - \nabla^t \mathbf{u}) \).

The viscosity coefficients \( \mu(\rho) \), \( \lambda(\rho) \) satisfy the Bresch-Desjardins relation

\[
\lambda(\rho) = 2\mu'(\rho)\rho - 2\mu(\rho).
\]

The total \( \kappa \)-energy \( E_\kappa \) is defined as follows

\[
E_\kappa = \rho \left( e_\kappa + \frac{(1 - \kappa)}{2} |\mathbf{u}|^2 + \frac{\kappa}{2} |\mathbf{u} + 2\nabla \varphi(\rho)|^2 \right).
\]

Remark 13 Note that \( E_\kappa \mathbf{u} \) is expressed as a sum of two energies

\[
(1 - \kappa)\rho \mathbf{u} \left[ e_\kappa + \frac{|\mathbf{u}|^2}{2} \right] + \kappa \rho \mathbf{u} \left[ e_\kappa + \frac{|\mathbf{u} + 2\nabla \varphi(\rho)|^2}{2} \right]
\]

similarly to energy from [30].

Remark 14 Integrating the total \( \kappa \)-energy equation with respect to space, we obtain

\[
\frac{d}{dt} \int_\Omega E_\kappa \, dx = 0.
\]

Thus (107) and the identity

\[
\frac{(1 - \kappa)}{2} |\mathbf{u}|^2 + \frac{\kappa}{2} |\mathbf{u} + 2\nabla \varphi(\rho)|^2 = \frac{|\mathbf{u} + 2\kappa \nabla \varphi(\rho)|^2}{2} + (1 - \kappa)\kappa \frac{|2\nabla \varphi(\rho)|^2}{2}
\]

(108)
yields the following conservation property
\[ \frac{d}{dt} \int_{\Omega} \rho \left( e_\kappa + \frac{1}{2} |u + \kappa \nabla \varphi(\rho)|^2 + \frac{(2-\kappa)\kappa}{2} |\nabla \varphi(\rho)|^2 \right) \, dx = 0. \]
This quantity may be treated as a generalization of the \( \kappa \)-entropy, found for the barotropic case, to the heat-conducting case.

The viscous tensors \( S_1, S_2 \) are given by
\[ S_1 = 2\mu(\rho)D(u) + \lambda(\rho) \text{div } u I \]
and
\[ S_2 = 2\mu(\rho)A(u). \]
The heat flux \( Q_\kappa \) is given by standerd Fourier’s law, i.e.
\[ Q_\kappa = -K \nabla \theta_\kappa, \]
where \( K \) is the positive heat-conductivity coefficient and \( \theta_\kappa \) denotes the generalized temperature (the \( \kappa \)-temperature). Let us consider an ideal polytropic gas, namely
\[ p(\rho, e_\kappa) = r \rho \theta_\kappa + p_c(\rho), \quad e_\kappa = C_v \theta_\kappa + e_c(\rho), \]
where \( r \) and \( C_v \) are two constant positive coefficients, see for instance [8]. For convenience, we denote \( \gamma = 1 + r/C_v \). Moreover, the additional pressure and internal energy, \( p_c \) and \( e_c \) respectively, are associated to the "zero Kelvin isothermal", which roughly speaking means that
\[ \lim_{\rho \to 0^+} p_c(\rho) = -\infty. \]
Further, we require that \( e_c \) is a \( C^2(0, \infty) \) nonnegative function and that the following constraint is satisfied
\[ p_c(\rho) = \rho^2 \frac{de_c}{d\rho}(\rho). \]
Below we present two different formulations of the internal energy equation which lead to useful bounds on \( \kappa \)-temperature similarly as in [8] for the usual temperature.
The first formulation reads
\[ C_v \left( \partial_t(\rho \theta_\kappa) + \text{div}(\rho u \theta_\kappa) + \Gamma \rho \theta_\kappa \text{div } w \right) = 2(1-\kappa)\mu(\rho)|D(u)|^2 + 2\kappa \mu(\rho)|A(u)|^2 + 2(1-\kappa)(\mu'(\rho)\rho - \mu(\rho))|\text{div } u|^2 + \text{div}(K \nabla \theta_\kappa), \]
with \( \Gamma \) the Gruneisen parameter and where the mixing temperature \( \theta_\kappa \) becomes the usual temperature if \( \kappa = 0 \). Note that for \( 0 \leq \kappa \leq 1 \), the \( \kappa \)-temperature remains non-negative in view of the Maximum Principle.
The second formulation is based on the notion of generalized \( \kappa \)-entropy \( s_\kappa \). It is the usual entropy in which the standard temperature has been replaced by the \( \kappa \)-temperature, i.e.
\[ s_\kappa = C_v \log \theta_\kappa - r \log \rho, \]
Thus, when \( \rho, \theta, \kappa \) is sufficiently regular we can derive the following equation
\[
\partial_t (\rho s) + \text{div} (\rho \mathbf{u} s) - \text{div} (K \nabla \log \theta) = 0
\]
\[
= 2(1 - \kappa) \frac{\mu(\rho)|D(\mathbf{u})|^2}{\theta} + 2\kappa \frac{\mu(\rho)|A(\mathbf{u})|^2}{\theta} + 2(1 - \kappa) (\mu'(\rho) \rho - \mu(\rho))|\text{div} \mathbf{u}|^2
\]
\[
- 2\kappa \Gamma \rho \Delta \varphi(\rho) + K \frac{\nabla \theta^2}{\theta^2}.
\]
Note that the terms on right-hand side, when integrated over space, give nonnegative contribution. Indeed, it suffices to check that for the penultimate term we have
\[
\int_{\Omega} -\Gamma \rho \theta^2 \Delta \varphi(\rho) \frac{\theta}{\kappa} \, dx = \int_{\Omega} \Gamma \varphi'(\rho) |\nabla \varphi|^2 \, dx \geq 0.
\]
Using all this information, it could be possible to prove global existence of \( \kappa \)-entropy solution for the heat–conducting compressible-Navier Stokes system under analogous assumptions on than in [8], replacing the usual temperature by the \( \kappa \)-temperature. The existence of the approximate solution could be then proven by using the augmented system written in terms of \( \mathbf{w} = \mathbf{u} + 2\kappa \nabla \varphi(\rho) \) and \( \mathbf{v} = 2 \nabla \varphi(\rho) \) as it was done in [30], or in Section 3 addressing barotropic flows
\[
\partial_t \varphi + \text{div} (\varphi |\mathbf{w}|^2) - \kappa \text{div} (\varphi |\mathbf{w}|^2) = 0,
\]
\[
\partial_t (\rho \mathbf{w}) + \text{div} ((\rho |\mathbf{w}| - \kappa |\mathbf{w}|) \nabla \varphi \otimes \mathbf{w}) - \nabla ((\lambda(\rho) - \kappa \mu'(\rho) \rho - \mu(\rho)) \text{div} (\mathbf{w} - \kappa \mathbf{v}))
\]
\[
- (2 - \kappa) \text{div} (\mu(\rho)D(\mathbf{w})) - \kappa \text{div} (\mu(\rho)A(\mathbf{w})) + \varepsilon \Delta^2 \mathbf{w} - \varepsilon \text{div} ((1 + |\nabla \mathbf{w}|^2) \nabla \mathbf{w}) + \nabla p(\rho, \kappa \rho)
\]
\[
= -\kappa (2 - \kappa) \text{div} (\mu(\rho) \nabla \mathbf{v}),
\]
with the \( \kappa \)-total energy supplemented by the \( \varepsilon \) correction corresponding to the \( \varepsilon \) regularisation of the momentum
\[
\partial_t (\rho E_\kappa) + \text{div} ((\rho |\mathbf{w}| - \kappa |\mathbf{w}|) \nabla \varphi) E_\kappa + \text{div} (\rho \mathbf{w} + \mathbf{Q})
\]
\[
- \text{div} \left( S_1 \mathbf{w} + (2 - \kappa) \kappa S_2 \mathbf{v} \right) + \varepsilon \Delta^2 \mathbf{w} - \varepsilon \text{div} ((1 + |\nabla \mathbf{w}|^2) |\mathbf{w}|^2 = 0
\]
and the set of initial conditions. Above, the total \( \kappa \)-energy \( E_\kappa \) is defined as
\[
E_\kappa = e_\kappa + \frac{1}{2} |\mathbf{w}|^2 + \frac{\kappa(2 - \kappa)}{2} |\mathbf{v}|^2.
\]
Note, however, this construction would not lead to the usual heat–conducting compressible Navier-Stokes system in the limit \( \varepsilon \to 0 \). Indeed, the difference is again due to \( \kappa \)-temperature that is not the usual one. But, performing a formal low Mach number limit for this system, we would get \( p = 1 \), \text{div} \mathbf{w} = 0 (comparing terms of the same order). In the equation on \( \mathbf{w} \), being now incompressible, the pressure gradient \( \nabla p \) would be replaced by Lagrangian multiplier \( \nabla \pi \). As a result, we would get the augmented system defined in [10] in the part devoted to construction of solution.

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