I study the topology of quantum fluctuations which take place at the earliest stage of high-energy processes. A new exact solution of Yang-Mills equations with fractional topological charge and carrying a single color is found.

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1. The problem of initial data for ultra-relativistic heavy ion collisions has been a sore subject for more than a decade. The roots of the problem penetrate deeply into the least explored areas of QCD like the nature of the QCD vacuum and hadronic structure. The parton picture of a nucleus completely disregards the properties of the vacuum and only partially respects the hadron structure by replacing the true bounded QCD state with an artificial flux of free quarks and gluons. For heavy ion collisions, the evolution of parton distributions is very different from the evolution in the simplest processes like $ep$-DIS or high-energy $pp$-collisions. In a series of papers [1,2] we have studied the causal character of the QCD evolution and found that high-energy processes explore all possible quantum fluctuations that may develop before the collision and are consistent with a given inclusive probe. All these fields propagate forward in time and collapse at the vertex of interaction with the probe. These fluctuations are the snapshots of nuclei frozen by the measurement, and they cannot be arbitrary since the emerging final-state configurations must be consistent with all interactions that are effective on the timescale of the emission process. In other words, we have proved that the QGP as the final state can be created only in a single quantum transition, as an ensemble of collective modes of expanding matter. The scale of the entire evolution process appeared to be set by the physical properties of the final state.

Besides a fair treatment of the final state-interactions, the theory has to rely on a realistic picture of the initial state. It must allow one to treat nuclei as Lorentz-contracted finite-size objects. It has to account for those interactions that keep nuclei intact before the collisions and are responsible for the coherence of the nuclei wave function. It is commonly accepted that the effective interactions responsible for the existence of hadrons are due to topological fluctuations in the QCD vacuum, i.e., instantons. These exist only in Euclidean space, and only spherically symmetric field configurations are used in the theory that describe stable hadrons. They cannot be directly transferred to Minkowsky space.

The solutions of the QCD field equations with non-trivial topology, by their design, rely heavily on the identification the directions in physical (geometric) space with the directions of the internal (tangent) color space so that the local rotations of the geometric coordinates can be compensated by the gauge transformation in color space. It is impossible to match the $SU(2) \times SU(2) \approx O(4)$ local color group with the Lorentz group, $SL(2,C) \approx O(1,3)$, since the first one is a compact Lie group, the second is not. Which of them should be given priority, and what properties have to be sacrificed in order to identify the angular coordinates of internal color and external physical spaces? An empirical answer to this question is already known. One must use the Euclidean metric and construct the self-dual solutions of the Yang-Mills equations (instantons). Acting in this way, we indeed achieve remarkable successes in the description of many properties of stable hadrons [3]. These successes are not by chance and should be considered as important physical inputs. However, this theory is incapable of describing moving hadrons. Motion is possible only in the Minkowsky world (where no self-dual fields exist). Therefore, we may ask if this commonly used prescription is sufficiently motivated physically? It is perfectly clear that would such a motivation exist, it can be only due to the nature of the measurement process: as viewed from Minkowsky world of moving stable hadrons, the Euclidean calculations provide an effective theory in the rest frame of a hadron. If a precise resolution of the color field coordinate takes place, then (since the moment of the measurement) the Euclidean picture is no longer valid.

2. In this note, I show that the solutions of the Yang-Mills equations, which interpolate between the Euclidean and Minkowsky worlds, do exist. Such an interpolation becomes possible because two regimes are separated by the light-cone of the point-like interaction. In the Euclidean domain (before the interaction) the transient topological object has finite action and a fractional winding number. The fields of this object evolve in Euclidean proper time $\tau$, and after collapse at the time $\tau = 0$, they can be continued as the waves propagating in Minkowsky space. [4] Constructing the Euclidean part of the solu-

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1 According to English orthography, the suffix -on in the name of this object (would it deserves a name) seems unavoidable. I would suggest ephemeron (rather than transient) in order not to create an image of a particle and emphasize an ephemeral nature of this field configuration.
tion, I map the $[SU(2) \times SU(2)]_{\text{color}}$ on $[O(4)]_{\text{space}}$ and require that the spin connections of the metric and the gauge field potential (both taken in the same group representation) must be identical. Acting in this way, I give priority to the geometry of the gauge group. Moreover, I insist that, before the interaction has resolved the color field on a sufficiently short scale, the space-time metric is defined by the internal Euclidean geometry of the gauge group.

The only tool which is capable of coping with this view of the relationship between the internal color dynamics and the geometry is the so-called tetrad formalism (see, e.g., [5, 6]). Indeed, the tetrad vector $e_\alpha^\tau$ includes two connections (gauge fields). One of them, the spin connection $\omega_{\alpha\beta}^\tau$, can be found from the condition that the tetrad vector $e_\alpha^\tau$ behaves like Lorentz tensors under the local Lorentz group transformations. The second gauge field, the spin connection $\omega_{\alpha\beta}^\tau$, provides covariance with respect to the local Lorentz rotation.

Let $x^\mu = (\tau, r, \phi, \eta)$ be the contravariant components of the curvilinear coordinates that cover the past of the hyperplane $t = 0$, $z = 0$ of the interaction.

$$x^0 = -\tau \cosh \eta, \quad x^3 = -\tau \sinh \eta, \quad x^1 = r \cos \phi, \quad x^2 = r \sin \phi,$$

where $x^\alpha = (t, x, y, z)$. The flat Minkowsky space is easily done by making the time-like tetrad vector $e_0^\mu$ imaginary, $e_0^\mu \to (e_0^\mu)E = (i, 0, 0, 0)$, $(e_0^\mu)E = (-i, 0, 0, 0)$. Then Eqs. (2) take the form

$$g_{\mu\nu} = g_{\alpha\beta}(e_\alpha^\mu)E(e_\beta^\nu)E = \text{diag}[\pm 1, 1, \tau^2, \tau^2],$$

$$g^{\alpha\beta} = g^{\mu\nu}(e_\alpha^\mu)E(e_\beta^\nu)E = \text{diag}[\mp 1, 1, 1, 1].$$

In the tetrad formalism, the transition to the Euclidean space is easily done by making the time-like tetrad vector $e_0^\mu$ imaginary, $e_0^\mu \to (e_0^\mu)E = (i, 0, 0, 0)$, $(e_0^\mu)E = (-i, 0, 0, 0)$. The only non-vanishing components of the connections are

$$\Gamma_{\eta\eta\tau} = -\Gamma_{\tau\eta\eta} = -\tau, \quad \Gamma_{\phi\phi\tau} = -\Gamma_{\tau\phi\phi} = -r, \quad \omega_\eta^{30} = -\omega_\eta^{03} = 1, \quad \omega_\phi^{12} = -\omega_\phi^{21} = -1.$$  

The spin connection can be found from the condition that the tetrad vectors form a matrix

$$e_\alpha^\mu = \text{diag}(1, 1, r, \tau).$$

These vectors correctly reproduce the curvilinear metric $g_{\mu\nu}$ and the flat Minkowsky metric $g_{\alpha\beta}$, i.e.,

$$g_{\mu\nu} = g_{\alpha\beta}e_\alpha^\mu e_\beta^\nu = \text{diag}[-1, 1, r^2, \tau^2],$$

$$g^{\alpha\beta} = g^{\mu\nu}e_\alpha^\mu e_\beta^\nu = \text{diag}[-1, 1, 1, 1].$$

The spin connection can be found from the condition that the covariant derivatives of the tetrad vectors equal zero [3, 4], i.e.,

$$\omega_\mu^{\alpha\beta} = [\Gamma_\mu^\lambda e_\lambda^\alpha - \partial_\mu e_\alpha^\alpha] e_\beta^\nu.$$  

Indeed, the tetrad vector $e_\alpha^\mu$ is the coordinate vector and the Lorentz vector at the same time. (The Lorentz index $\alpha$ and the coordinate index $\mu$ are moved up and down by the local Minkowsky metric tensor $g_{\alpha\beta}$ and the global metric tensor $g_{\mu\nu}$, respectively.) The only non-vanishing components of the connections are

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This formal step also leads to a set of standard prescriptions for the transition to the Euclidean version of the field theory: $A_\tau = (e_0^\tau)E A^\tau = -iA^\tau$. The same rule holds for the spin connection, $(\omega_{\mu}^{03})_M \to (\omega_{\mu}^{03})_E = -i(\omega_{\mu}^{03})_M$. These formulae indicate that we perform a transition to an imaginary proper time $\tau$. The choice between the signs in Eq. (3) is a subject of an independent analysis.

3. Assuming the Euclidean long-distance behavior, we employ the metric

$$ds^2 = dr^2 + dr^2 + d\phi^2 + \tau^2 d\eta^2$$

with the only non-vanishing components of the spin connection being $\omega_\eta^{03} = -1$, and $\omega_\phi^{12} = -1$. (Christoffel symbols $\Gamma$ remain the same as in Eq. (3).) Overall, we have four domains with the same Euclidean metric, which explains the result (9), (10). Introducing the (iso)vector representation $A_{\mu}^{\alpha\beta}$ of the gauge field of the $O(4)$ group, and insisting on a one-to-one mapping of the color and space directions, we require that

$$A_{\mu}^{\alpha\beta} = h \omega_{\mu}^{\alpha\beta},$$

where the factor $h$ is an arbitrary real number which defines the relationship between the cyclic components of space-time and color coordinates. The gauge fields of $O(4)$ have two projections on its two $SU(2)$-subgroups,

$$(A_{\mu}^{a})_{\pm} = \frac{1}{2} \eta_{\pm}^{a\alpha\beta} A_{\mu}^{\alpha\beta} = \pm A_{\mu}^{0a} \pm \frac{1}{2} e^{a\alpha\beta} A_{\mu}^{\alpha\beta},$$

where $\eta_{\pm}^{a\alpha\beta}$ are the ‘t Hooft symbols [3], and the subscripts $(\pm)$ denote two chiral projections. Thus, we have

$$(A_{\eta}^{a})_{\pm} = \mp h \omega_\eta^{03} = \pm h, \quad (A_{\phi}^{3})_{\pm} = h \omega_\phi^{12} = -h,$$

which is compatible with the gauge condition $A^{\tau} = 0$ that we adopt for both the Euclidean and the Lorentz regimes of the process. One can easily find a representation for this potential which manifests its pure gauge origin,

$$A_{\mu}(x) = (1/2)A^a_{\mu}(x)\sigma^a = S\partial_\mu S^{-1}.$$


Using the decomposition, \( S = i u_0 1 + u_\alpha \sigma^\alpha \), and \( S^{-1} = -iu_0 1 + u_\alpha \sigma^\alpha \) we arrive at
\[
A_\mu(x) = (1/2)A^\nu_\mu(x)\sigma^\nu.
\]
By comparison with \( (10) \), and accounting for the unitarity, \( SS^{-1} = 1 \), we obtain a system of equations,
\[
-2(u_1 \partial_\eta u_2 - u_2 \partial_\eta u_1 + u_0 \partial_\eta u_3 - u_3 \partial_\eta u_0) = \pm \hbar ,
-2(u_1 \partial_\phi u_2 - u_2 \partial_\phi u_1 + u_0 \partial_\phi u_3 - u_3 \partial_\phi u_0) = -\hbar ,
\]
\[
u_0^2 + u_3^2 = 1 \quad (13)
\]
which has a solution
\[
(u_0)_\pm = \mp 2^{-1/2} \cos \hbar \eta , \quad (u_3)_\pm = 2^{-1/2} \sin \hbar \eta ,
(u_1)_\pm = 2^{-1/2} \cos \hbar \phi , \quad (u_2)_\pm = 2^{-1/2} \sin \hbar \phi .
\]
\[
(A^3)_{\pm} = E(\tau, r) i S \eta S^{-1} = \mp E(\tau, r) \to \pm 1 ,
(A^3)_{\phi} = \Phi(\tau, r) i S \Phi^{-1} = -\Phi(\tau, r) \to -1 ,
\]
\[
\quad where the arrows point to the values of potentials at \( \tau \to \infty \), where the field must approach a pure gauge. (For now, we assume that \( h = 1 \).) Since the field has only one color component, the commutator in the definition, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \), vanishes and the components of the field tensor are the same as in the Abelian case where
\[
F_{\tau\eta} = \mp \partial_\tau E , \quad F_{\tau\phi} = -\partial_\tau \Phi , \quad F_{\tau r} = 0 ,
F_{r\eta} = \mp \partial_\tau E , \quad F_{r\phi} = -\partial_\tau \Phi , \quad F_{\eta \phi} = 0 .
\]
The condition for the (anti)self-duality of the field tensor \( F_{\mu\nu} \) reads as
\[
\tilde{F}_{\mu\lambda} = g_{\mu\nu} g_{\lambda\alpha} \epsilon^{\rho\sigma\nu\alpha} F_{\rho\sigma} = \pm F_{\mu\lambda} .
\]
Note that the definition of the dual tensor is different from the familiar definition in flat space. This modification is obvious. Indeed, the co- and contravariant tensor components are even of different dimensions. The requirement of the self-duality of the field \( (15) \) yields a system of equations,
\[
\frac{1}{r} \frac{\partial E}{\partial r} = \frac{1}{r} \frac{\partial E}{\partial r} , \quad \frac{\partial E}{\partial r} = -r \frac{\partial E}{\partial r} .
\]
Two conditions of the self-consistency for this system are
\[
\partial_\tau^2 \Phi - \frac{1}{r} \partial_\tau \Phi = - (\partial^2 \Phi + \frac{1}{r} \partial_\tau \Phi) ,
\partial_\tau^2 E + \frac{1}{r} \partial_\tau E = - (\partial^2 E - \frac{1}{r} \partial_\tau \Phi) .
\]
This system is easily solved by separation of variables. The solution which obeys the original system \( (18) \), the condition for finiteness, the boundary condition of pure gauge at \( \tau \to \infty \), and the condition that \( A_\eta \to 0 \) at \( \tau \to 0 \), is as follows,
\[
\mp A^3_\eta = E(\tau, r) = -\lambda r K_1(\lambda r) J_0(\lambda r) + 1 ,
- A^3_\phi = \Phi(\tau, r) = \lambda r J_1(\lambda r) K_0(\lambda r) + 1 .
\]
From these expressions for the potentials, we may easily find the field strength of the ephemeron,
\[
\mp \tau^{-1} F^3_{\tau\eta} = -r^{-1} F^3_{\tau\phi} = \lambda^2 K_0(\lambda r) J_0(\lambda r) ,
\mp \tau^{-1} F^3_{\tau\phi} = -r^{-1} F^3_{\tau\eta} = \lambda^2 J_1(\lambda r) K_1(\lambda r) .
\]
The geometry of this field is noteworthy. It has the symmetry of a torus. The magnetic field has two components, one along the torus pipe, and the second winding around the pipe. This is a well known configuration of a toroidal magnetic trap. Since \( r < \tau \), both radii of the torus get smaller when \( \tau \to 0 \); the torus collapses at \( \tau = 0 \). The electric fields have two similar components which are created in accordance with the Abelian induction law. Every pipe is mono-colored. The parameter \( \rho = \lambda^{-1} \) clearly plays the role of the “size” of the ephemeron.

4. Starting from the fields given by the Eqs. \( (10) \) we may find the Euclidean action of the ephemeron:
\[
S_E = \int d^4 x \sqrt{g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{2\pi^2}{g^2} \int_0^\infty \tau d\tau \int_0^\infty \frac{dr}{r} \left[ \left( \frac{\partial \Phi}{\partial \tau} \right)^2 + \left( \frac{\partial \Phi}{\partial r} \right)^2 \right] .
\]
In the same way, we compute the topological charge
\[
Q = \int d^4 x \sqrt{g} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \pm \frac{1}{g^2} \int_0^\infty \tau d\tau \int_0^\infty \frac{dr}{r} \left[ \left( \frac{\partial \Phi}{\partial \tau} \right)^2 + \left( \frac{\partial \Phi}{\partial r} \right)^2 \right] .
\]
Thus, we have reproduced a standard relation, between the instanton action and its winding number,
\[
S_E = - \frac{8\pi^2}{g^2} |Q| .
\]
We now have to find the winding number \( Q \).
We shall do it using the representation of topological charge via the divergence of the Chern-Simons current,
\[
Q = \int d\sigma A^\mu ,
\]
where
\[
K^\mu = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} [A^\nu_\rho \partial_\sigma A^\mu_\sigma + \frac{\theta}{3} \epsilon_{\tau\alpha\beta} A^\nu_\rho A^\mu_\alpha A^\sigma_\beta] .
\]
The second term (usually the major one) identically vanishes since the ephemeron field has only one color component. In our geometry, only two components of \( K^\mu \) survive,
\[ K^\tau = \pm\frac{1}{4} \left[ E \frac{\partial \Phi}{\partial \tau} - \Phi \frac{\partial E}{\partial \tau} \right] = \pm\frac{1}{8} \left[ \frac{r}{\tau} \frac{\partial E^2}{\partial \tau} + \frac{\partial \Phi^2}{\partial \tau} \right] \quad (27) \]

and

\[ K^\tau = \pm\frac{1}{4} \left[ E \frac{\partial \Phi}{\partial \tau} - \Phi \frac{\partial E}{\partial \tau} \right] = \pm\frac{1}{8} \left[ \frac{r}{\tau} \frac{\partial E^2}{\partial \tau} + \frac{\partial \Phi^2}{\partial \tau} \right]. \quad (28) \]

Correspondingly, the total flux of the vector \( K^\mu \) over a closed surface can be split into the sum of integrals over three surfaces, \( \tau = \infty, r = 0, \) and \( \tau = r, \)

\[ Q = \int d\tau dr (\tau - r) \left[ \partial_\tau K^\tau + \partial_r K^r \right] = \int_0^\infty dr \left[ K^\tau(\infty, r) - K^\tau(r, r) \right] \]
\[ + \int_0^\infty d\tau \left[ K^\tau(\tau, \tau) - K^\tau(\tau, 0) \right] \quad (29) \]

(The factor \( 4\pi^2 \) has come from two angular integrations.) Straightforward computation of the integrals lead to the following expressions,

\[ \int_0^\infty K^\tau(\infty, r) dr = 0, \quad -\int_0^\infty K^\tau(\tau, 0) d\tau = \frac{1}{4}, \]
\[ \int_0^\infty \left[ K^\tau(\tau, \tau) - K^\tau(\tau, \tau) \right] d\tau = -\frac{1}{8}. \quad (30) \]

Eventually, we find

\[ Q = \frac{1}{4\pi^2} \int d\sigma_\mu K^\mu = \frac{1}{8}. \quad (31) \]

The transient topological gluon configuration carries a fractional topological charge of \( 1/8 \) in that part of the Euclidean space which is an image of the interior of the past light cone of the interaction vertex. In a full chart, we have several domains with similar Euclidean picture, and the sum over all of them gives \( Q_{\text{tot}} = 1. \)

5. The ephemeron field is defined by four parameters. Two of them, \( x_0 \) and \( y_0, \) are obvious and correspond to the preserved translation symmetry in the \( xy- \) plane. To include them explicitly, we must read \( r \) as \( \sqrt{(x-x_0)^2 + (y-y_0)^2}. \) Next, we have the ephemeron “radius” \( \rho = \lambda^{-1}. \) The fourth parameter is the scale factor \( h, \) which has been introduced in Eq. (8) an dropped from the explicit calculations after Eq. (14). In fact, the presence of scale factor \( h \) among the parameters of ephemeron can be recovered from the observation that equations (15) are homogeneous (and thus admit an arbitrary scale factor) and the fact that solutions for the vector potential then have a pure gauge asymptote given by Eqs. (11) - (14). With \( h \) explicitly retained, the topological charge is \( Q = h^2/8 \) and thus becomes a continuous parameter. This is in contrast with the case of spherically symmetric BPST instantons (8) which can have only integer charges. Equations (12) and (14) explain the origin of the non-trivial topology of the ephemeron. The solutions with different values of \( h \) correspond to different field configurations whose asymptotes at \( \tau \to \infty \) are the different gauge transforms of the classical vacuum with \( A(x) = 0. \)

The most impressive feature of the ephemeron is that it is one-dimensional both in physical and color spaces. This is not so surprising from a mathematical point of view since we map the two spaces onto one another. In physical space, the full spherical symmetry is corrupted by the interaction and only two rotations in the \( tz- \) and the \( xy- \) planes survive as an actual symmetry. The high-precision measurement of the coordinate inside an object formed by the strong interaction necessarily resolves a mono-colored field pattern. Topologically, the ephemeron is a collapsing ring before the interaction, and an opened expanding string after it.

Existence of the transient topological field configurations poses several important questions.

(i) How does the toroidal geometry of the ephemeron affect the distribution of the gluons produced in high-energy collisions? Is the proper field of the resolved color charge Coulomb-like or does it carry some remnants of the twisted geometry of the ephemeron fields.

(ii) The geometry of the electric and magnetic field of the ephemeron implies strong spin polarization effects. All known evidences of \( P- \) and \( CP- \) violations come from the study of various decays which are genuinely non-stationary processes. What is the role of the transient topological configurations in these decays? Formally, the ephemeron must be included into the path-integral representation of the point-to-point correlators of hadronic currents on equal footing with the BPST instantons.

Finally, the border between perturbative and non-perturbative QCD is clearly defined by their relation to the non-trivial topological properties of the QCD vacuum. The very existence of the ephemeron solutions of the Yang-Mills equations undoubtedly indicates that there is a hope to bridge the gap.

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