QUILLEN HOMOLOGY OF SPECTRAL LIE ALGEBRAS
WITH APPLICATION TO MOD P HOMOLOGY OF LABELED CONFIGURATION SPACES

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ABSTRACT. We provide a general method computing the mod $p$ Quillen homology of algebras over a monad that parametrizes the structure of mod $p$ homology of spectral Lie algebras. This is the $E^2$-page of the bar spectral sequence converging to the mod $p$ topological Quillen homology of spectral Lie algebras. The computation of the Quillen homology of the trivial algebra allows us to deduce that the $F_p$-linear spectral Lie operad is not formal. As an application, we study the mod $p$ homology of the labeled configuration space $B_p(M;X)$ in a manifold $M$ with labels in a spectrum $X$, which is the mod $p$ topological Quillen homology of a certain spectral Lie algebra by a result of Knudsen. We obtain general upper bounds for the mod $p$ homology of $B_p(M;X)$, as well as explicit computations for small $k$. When $p$ is odd, we observe that the mod $p$ homology of $B_p(M^r;S^r)$ for small $k$ depends on and only on the cohomology ring of the one-point compactification of $M$ when $n+r$ is even. This supplements and contrasts with the result of Bödigheimer-Cohen-Taylor when $n+r$ is odd.

1. INTRODUCTION

Spectral Lie algebras generalize the notion of Lie algebras over a field $k$ to the $(\infty, \infty)$-category of spectra. They are parametrized by the spectral Lie operad $s\mathcal{L}'$, whose underlying symmetric sequence $\{\partial_n(\text{Id})\}_{n}$ is given by the Goodwillie derivatives of the identity functor on the category of pointed spaces. The spectral Lie operad is Koszul dual to the nonunital $E\infty$-operad $\E_{\infty}$ of Behrens, Antolín-Camarena [Chi05]. The homology operad $\{H_\ast(\partial_n(\text{Id});k)\}_{n}$ of the spectral Lie operad recovers the ordinary Lie operad over $k$ Lie algebras is parametrized by a monad Lie $E_1$ [Kri93] and Basterra [Bas99]. One approach is to use the classical bar spectral sequence

$$E^2_{s,i} = \pi_s \text{Bar}_\ast(\text{id}, s\mathcal{L}', L \otimes F_p) \Rightarrow \text{TQ}^\mathcal{L}_{s+i}(L; F_p),$$

obtained by skeletal filtration of the geometric realization.

To compute the $E^2$-page, it is necessary to understand the structure of the mod $p$ homology of spectral Lie algebras. In [Beh12], Behrens constructed Dyer-Lashof-type unary operations $\tilde{Q}^j$ of degree $j - 1$ on the mod 2 homology of spectral Lie algebras and determined the relations among these operations. Building on the work of Behrens, Antolín-Camarena [AC20] showed that the structure of the mod 2 homology of spectral Lie algebras is parametrized by a monad $\text{Lie}_\mathcal{R}$. An algebra over $\text{Lie}_\mathcal{R}$ is an unstable module over the algebra $\mathcal{R}$ of Behrens’ operations, along with a shifted Lie algebra structure such that brackets of operations always vanish and the self-bracket on an element $x$ is identified with the bottom nonvanishing operation $\tilde{Q}_0 := \tilde{Q}^{|x|}$ on $x$. Following their approach, Kjaer [Kja18] constructed Dyer-Lashof-type unary operations $\beta^r\mathcal{Q}^j$ on the mod $p$ homology of spectral Lie algebras for $p > 2$ and proved that brackets of operations always vanish. However, he did not identify the relations among the unary operations, so the structure map of the monad $\text{Lie}_\mathcal{R}$ parametrizing operations on the mod $p$ homology of spectral Lie algebras is not fully understood.

Hence the $E^2$-page of the bar spectral sequence (1) associated to a spectral Lie algebra $L$ is equivalent to the Quillen homology

$$\text{HQ}^\text{Lie}_\mathcal{R}(H_\ast(L; F_p)) := \pi_s \left(\text{Bar}_\ast(\text{id}, \text{Lie}_\mathcal{R}, H_\ast(L; F_p))\right).$$
of the Lie$_R^e$-algebra $H_*(L;F_p)$ when $p = 2$. Alternatively, this is the total left derived functor $\pi_*(\mathcal{L}Q^\text{Lie}_R^e_{\text{Mod}}(L))$ of the indecomposable functor $\mathcal{L}Q^\text{Lie}_R^e_{\text{Mod}}$ of the Lie$_R^e$-algebra structure.

The main challenge in computing the Quillen homology of Lie$_R^e$-algebras when $p = 2$ arises from the identification of the self-adjoint with the bottom operation $Q$, which precludes a factorization of the free Lie$_R^e$-algebra functor as a composition of the free Lie$_F^e$-algebra functor followed by the free $\mathcal{R}$-algebra functor. Furthermore, since the category of Lie$_F^e$-algebras is nonabelian, we cannot resort to a Grothendieck spectral sequence.

The method we use involves bounding the Quillen homology of Lie$_R^e$-algebras by the Quillen homology of variants of Lie$_R^s$-algebras whose the unary and binary operations are disentangled. We obtain an upper bound is obtained by constructing a May spectral sequence with respect to the $\mathcal{R}$-module structure in Proposition 3.5, whose $E^1$-page is bounded above by the Quillen homology of a variant of Lie$_R^s$-algebras (Definition 3.1). A lower bound can be produced by mapping into the Quillen homology of another variant of Lie$_R^s$-algebras (Definition 3.15). Then we adopt the approach by Brantner-Hahn-Knudsen [BHK19, Section 4.4] in that we replace the bar construction computing the Quillen homology of these variant algebras with a smaller complex, obtained as the total complex of a double complex in Lemma 3.7 and Lemma 3.18.

To compute the homotopy groups of these total complexes, we utilize the machinery of Koszul duality for additive Koszul algebras [Pri70] and Lie algebras [BHK19][CE48][May66A][Pri70], as well as explicit understanding of the Bousfield-Cartan-Dwyer operations

$$\gamma: \pi_{h+r}(\Lambda(V)) \to \pi_{2h+1+r,2r-1}(\Lambda^{2h+1}(V)), 1 \leq i \leq r$$

on the homotopy group of the free simplicial shifted graded exterior algebra $\Lambda(V)$ on a simplicial $\mathcal{F}_2$-module $V$. Thus we obtain general upper bounds for the Quillen homology of Lie$_R^s$-algebras via Proposition 3.5 and Corollary 3.13, precise formulae in low weights in Corollary 3.25, as well as explicit computations in universal cases.

**Theorem 1.1 (Theorem 3.23).** The Quillen homology

$$\text{HQ}^\text{Lie}_R^e_{\text{Mod}}(\Omega^n\text{Free}_{\text{Mod}}(\mathcal{F}_2), \Omega^n\text{Free}_{\text{Mod}}(\mathcal{F}_2); 1 \leq n \leq \infty)$$

of the Lie$_R^e$-algebra $\Omega^n\text{Free}_{\text{Mod}}(\mathcal{F}_2)$, $1 \leq n \leq \infty$ is the shifted graded exterior algebra on generators $\gamma J(x_i)$ satisfying the following conditions:

1. $I = (i_1, \ldots, i_m)$ satisfies $i_j \geq 2i_{j+1}$ for $l < m, i_m \geq 2$, and $i_1 - i_2 - \cdots - i_m \leq r$;
2. $J = (j_1, \ldots, j_r)$ satisfies $0 \leq j_i \leq j_{i+1} + 1$ for $l < r, 0 \leq j_r < n$, and if $j_1 = 0$ then either $r = 1$ or $i_m = 2$.

In particular, the computation of the Quillen homology of the trivial Lie$_R^e$-algebra $\mathcal{F}_2$ allows us to identify all unary operations on the mod 2 Quillen homology of spectral Lie algebras in Remark 3.24 and deduce that the $\mathcal{F}_2$-linear spectral Lie operad is not formal in Corollary 3.26.

**Application to labeled configuration spaces.** The second half of the paper makes use of the computation of the Quillen homology of Lie$_R^e$-algebras to study the mod $p$ homology of labeled configuration spectrum

$$B_k(M, X) := \Sigma_k^\infty \text{Conf}_k(M) \otimes X^{\otimes k}$$

of $k$ points in a parallelizable manifold $M$ with labels in a spectrum $X$.

The study of labeled configuration spaces dates back to as early as Segal [Seg73] and McDuff [McD75] as generalizations of the unordered configuration space $\Sigma_k B_k(M) = B_k(M; S^0)$ of $k$ points in $M$. The rational homology groups of labeled configuration spaces are relatively well understood in cases of interests via classical methods, see for instance [BC88][BCT89][Kri94][Tot96][FT00].

Nonetheless, the mod $p$ homology groups of these objects have remained mostly intractable. Classically, the only known cases are the following:
\( M = \mathbb{R}^n \) with arbitrary labeling spectra by May [May72] and McClure [BMMS88, IX], and \( M = \mathbb{R}^n \) by F. Cohen [CLM76, III]. Then \( \bigoplus_{k \geq 0} B_k(M; X) \) is the free \( E_n \)-algebra on \( X \). Its mod \( p \) homology is captured by Dyer-Lashof operations and Browder brackets as a functor of \( H_*(X; \mathbb{F}_p) \).

- Arbitrary manifold \( M \) with labeling spectrum \( X = \Sigma^\infty S^r \), where either \( p = 2 \) or \( p > 2 \) and \( n + r \) is odd [BCT89][ML88][BCM93]. In these cases, there is a homology decomposition

\[
H_*( \bigoplus_{k \geq 0} B_k(M; \Sigma^r) ) \cong \bigotimes_i H_*(\Omega^i \Sigma^{n+r}) \otimes \dim H_*(M).
\]

In particular, the homology depends only on the \( \mathbb{F}_p \)-module \( H_*(M; \mathbb{F}_p) \).

The most recent developments in the computation of the homology of labeled configuration spaces originate from a result of Knudsen [Knu18]. Using the machinery of factorization homology, he established an equivalence of spectra

\[
\bigoplus_{k \geq 1} B_k(M; V) \simeq |\text{Bar}_* (id, s\mathcal{L}, \text{Free}_{s\mathcal{L}}(\Sigma^r V)^{M^+})|.
\]

Here \( M \) is a parallelizable \( n \)-manifold, \( s\mathcal{L} \) is the monad associated to the free spectral Lie algebra functor \( \text{Free}_{s\mathcal{L}} \), and \((-)^{M^+}\) the cotensor with the one-point compactification of \( M \) in the \( \infty \)-category of spectral Lie algebras. This equivalence can be thought of as a "deformed Koszul duality".

Knudsen's result opens up a path for extracting information about the homology of labeled configuration spaces. In [Knu17], Knudsen provided a general formula for the Betti numbers of unordered configuration spaces by observing that the bar spectral sequence with rational coefficients for the bar construction (3) collapses at the \( E^2 \)-page. Building on Knudsen's work, Drummond-Cole and Knudsen [DCK17] computed the Betti numbers of unordered configuration spaces of surfaces. In [BHK19], Brantner, Hahn, and Knudsen studied Knudsen's spectral sequence with coefficients in Morava \( E \)-theory at an odd prime using Brantner's results on the structure of the Morava \( E \)-theory of spectral Lie algebras [Bra17]. They computed the weight \( p \) part of the labeled configuration spaces in \( \mathbb{R}^n \) and punctured genus \( g \) surfaces \( \Sigma_{g,1} \) for \( g \geq 1 \) with coefficient in a sphere. By letting the height go to infinity, they observed that the integral homology of \( B_p(\Sigma_{g,1}) \) is \( p \)-power-torsion free for any odd prime \( p \).

In this paper, we adapt their approach and study the mod \( p \) homology of \( B_k(M, X) \) for \( M \) a parallelizable \( n \)-manifold and \( X \) any spectrum by examining the mod \( p \) homology of Knudsen's spectral sequence, i.e., the bar spectral sequence (1) with coefficients in \( \mathbb{F}_p \), applied to the bar construction (3).

When \( p = 2 \), our general understanding of the \( E^2 \)-page, i.e., the Quillen homology of \( \text{Lie}_{s\mathcal{L}} \)-algebras, allows us to obtain an upper bound for \( H_*(B_k(M, X); \mathbb{F}_2) \) in Theorem 5.5 for arbitrary parallelizable manifold \( M \) and spectrum \( X \). In the universal case \( M = \mathbb{R}^\infty \) and \( X = \mathbb{S}^r \), the bar spectral sequence has \( E^2 \)-page given by Theorem 3.23. Comparing with the computation of the homology of free \( E_\infty \)-algebras by May [May66A] and McClure [BMMS88], we see that there are infinitely many higher differentials and observe the following universal pattern:

**Conjecture 1.2 (Conjecture 4.5).** Each page of the spectral sequence

\[
E^2_{s,t} = \text{HQ}_{s,t}^\text{Lie}_{s\mathcal{L}}(\Sigma^k \mathbb{S}^r) \Rightarrow \pi_{s+t} \text{Bar}_* (id, s\mathcal{L}, \Sigma^k \mathbb{S}^r)
\]

is an exterior algebra. The higher differentials act on the exterior generators of the \( E^2 \)-page as follows, see Figure 4.1:

1. For an exterior generator \( \alpha = \bar{Q}_{j_1} \cdots \bar{Q}_{j_m} (x_k) \) on the \( E^2 \)-page, we have
   \[
d^{r+1} \gamma_{r+1} (\alpha) = \bar{Q}_{s} \alpha
\]
   for \( r < m \) and \( r \leq j_1 + 1 \).
2. For an exterior generator \( \beta = \gamma_{m+1} \bar{Q}_{j_1} \cdots \bar{Q}_{j_m} (x_k) \) on the \( E^2 \)-page, we have
   (a) \( d^{n+1} (\beta) = \bar{Q}_s \beta \bar{Q}_{j_1} \cdots \bar{Q}_{j_m} (x_k) \),
   (b) \( d^{n+1} d^{m+1} (\beta) = d^{m+1} (\beta) \otimes \beta \),
   (c) \( \gamma_{l-2} d^{n+1} (\beta) = d^{n+1} \gamma_{l+l+1} (\beta) \) for \( n + 1 < l < m \).
These generate all higher differentials under further applications of the \( \eta \) operations in accordance with (2).(b) and (2).(c), as well as the exterior product.

While the pattern of universal differentials is similar to classical ones studied by Dwyer [Dwy80b] and Turner [Tur98], the operations \( \bar{Q}_j \) on coalgebras over the comonad \( HQ_{\pi_+^*}(-) = \pi_+^* \text{Bar}_*^{\pi_+^*} \) increase filtration and hence cannot be constructed using classical methods, see Remark 4.6. In joint work in progress with Andrew Senger, we use a suitable deformation of the comonad associated to the bar construction \( \text{Bar}_*^{\pi_+^*}(\pi_+^* \mathcal{L}, -) \) to the \( \infty \)-category of Beilinson-connective filtered \( \mathbb{F}_2 \)-modules, which allows us to detect the higher differentials.

Using sparsity arguments, we show that the weight \( k \) part of Knudsen’s spectral sequence with \( \mathbb{F}_2 \) coefficients always collapses on the \( E^2 \)-page for small \( k \). Thus we obtain bases of \( H_\ast(B_k(M;X);\mathbb{F}_2) \) for any parallelizable manifold \( M \) and spectrum \( X \) when \( k = 2 \) in Corollary 5.6, and for closed parallelizable \( M \) when \( k = 3 \) in Corollary 5.8. In particular, we observe that the \( \mathbb{F}_2 \)-module \( H_\ast(B_k(M;X)) \) depends on and only on the cohomology ring \( H^\ast(M^+;\mathbb{F}_2) \) when \( H_\ast(X;\mathbb{F}_2) \) has at least two generators. This is in contrast to the case when \( X = S^r \), in that the equivalence (2) depends only on the \( \mathbb{F}_2 \)-module \( H^\ast(M^+;\mathbb{F}_2) \) [BCT89]. As examples, we produce explicit bases for \( H_\ast(B_k(M;X);\mathbb{F}_2) \), \( k = 2,3 \) when \( X \) is an arbitrary spectrum, and \( M \) is a (punctured) genus \( g \) surface in Section 5.3 or the (punctured) real projective space \( \mathbb{R}P^3 \) in Section 5.4.

When \( p > 2 \), we compute the weight \( k \leq p \) part of the \( E^2 \)-page of Knudsen’s spectral sequence with \( \mathbb{F}_p \) coefficients in terms of \( \text{Lie}_{\pi_+^*} \)-algebra homology in Proposition 6.5. We deduce the existence of a single \( d_{p-2} \)-differential in Knudsen’s spectral sequence when \( M = \mathbb{R}^n \) with \( n \geq 1 \), \( X = S^l \) and \( k = p \geq 5 \) in Proposition 6.7. Similar to the case \( p = 2 \), we observe that the \( \mathbb{F}_p \)-linear spectral Lie operad is not formal. Then we show that the mod \( p \) Knudsen’s spectral sequence collapses when \( k = 2 \) or \( k = 3 \) and \( p \geq 5 \).

**Corollary 1.3 (Corollary 6.9).** Let \( M^n \) be a parallelizable manifold and \( X \) any spectrum. Let \( \mathfrak{g} \) be the \( \text{Lie}_{\pi_+^*} \)-algebra \( H^\ast(M^+;\mathbb{F}_p) \otimes \text{Lie}_{\pi_+^*}^\ast(\mathbb{E}^\ast H^\ast(X;\mathbb{F}_p)) \) with brackets given by \( [y \otimes x, y' \otimes x'] := (y \cup y') \otimes [x,x'] \), and \( \text{CE}(\mathfrak{g}) \) the shifted Chevalley-Eilenberg complex (Definition 6.4).

1. For all \( i \), there is an isomorphism of \( \mathbb{F}_p \)-modules
   \[
   H_i(B_2(M;X);\mathbb{F}_p) \cong \bigoplus_{s+t=i} \text{wt}_2(H_{s+t}(\text{CE}(\mathfrak{g}))).
   \]

2. If \( p \geq 5 \), then for all \( i \)
   \[
   H_i(B_3(M;X);\mathbb{F}_p) \cong \bigoplus_{s+t=i} \text{wt}_3(H_{s+t}(\text{CE}(\mathfrak{g}))).
   \]

**Remark 1.4.** When \( X = S^r \) and \( k = 2,3 \), we observe that the \( \mathbb{F}_p \)-module \( H_\ast(B_k(M;S^r);\mathbb{F}_p) \) depends on and only on the cohomology ring \( H^\ast(M^+;\mathbb{F}_p) \) when \( r + l \) is even, see Remark 6.10. This is again in contrast to the case when \( r + l \) is odd in the equivalence (2) [BCT89].

In forthcoming work with Matthew Chen [CZ22], we build on the odd primary method in this paper and Drummond-Cole-Knudsen’s computation of the rational homology of the unordered configurations space \( B_k(\Sigma^s\mathbb{R}^n;\mathbb{S}) \) to identify the higher differentials in Knudsen’s spectral sequence for \( H_\ast(B_k(\Sigma^s\mathbb{S};\mathbb{S});\mathbb{F}_p) \). As a result, we show that the integral homology of \( B_k(\Sigma^s\mathbb{S};\mathbb{S}) \) is \( p \)-power torsion-free for \( p \geq 3 \) when \( k \leq p \) and \( g \geq 1 \).

### 1.1. Outline

In Section 2, we recall the definition of spectral Lie algebras and the structure of their mod 2 homology as algebras over the monad \( \text{Lie}_{\pi_+^*} \), as well as extending the known results to nonconnective objects. Then we define the Quillen homology of \( \text{Lie}_{\pi_+^*} \)-algebras and the mod \( p \) topological Quillen homology of spectral Lie algebras. The two are related by a bar spectral sequence.

In Section 3, we provide general upper bounds for the Quillen homology of \( \text{Lie}_{\pi_+^*} \)-algebras and precise formula in low weights by comparing with the Quillen homology of two variant algebras when \( p = 2 \). Then we explicitly compute the Quillen homology of the \( \text{Lie}_{\pi_+^*} \)-algebras \( \Sigma^k\mathbb{F}_2 \) and \( \Omega^\ast \text{Free}_{\text{mod}_2}^\ast(\Sigma^{n+k}\mathbb{F}_2) \).

In Section 4, we review Knudsen’s result that expresses labeled configuration spaces in parallelizable manifolds as topological Quillen objects of certain spectral Lie algebras. In the universal case \( M = \mathbb{R}^n \), we conjecture patterns of higher differentials.
In Section 5, we apply our understanding of the Quillen homology of Lie^s^R-algebras to extract explicit information about the mod 2 homology of labeled configuration spaces, including general upper bounds and low weight computations. Then we extend the methods to \( p > 2 \) to study the odd primary homology of labeled configuration spaces in Section 6.

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1.3. Conventions. We assume that every object is graded and weighted whenever it makes sense. For instance, \( \text{Mod}_{p}^{\ast} \) stands for the ordinary category of weighted graded \( \mathbb{F}_{p} \)-modules. A weighted graded \( \mathbb{F}_{2} \)-module \( M \) is an \( \mathbb{N} \)-indexed collection of \( \mathbb{Z} \)-graded \( \mathbb{F}_{p} \)-modules \( \{ M(w) \}_{w \in \mathbb{N}} \). The weight grading of an element \( x \in M(w) \) is \( w \). Morphisms are weight preserving morphisms of graded \( \mathbb{F}_{p} \)-modules. The Day convolution \( \otimes \) makes \( \text{Mod}_{p}^{\ast} \) a symmetric monoidal category. The Koszul sign rule \( x \otimes y = (-1)^{|x||y|} x \otimes y \) for the symmetric monoidal product \( \otimes \) depends only on the internal grading and not the weight grading.

Similarly, a shifted Lie algebra \( L \) over \( \mathbb{F}_{p} \) is a weighted graded \( \mathbb{F}_{p} \)-module equipped with a shifted Lie bracket \( [-,-]: L_{m} \otimes L_{n} \to L_{m+n-1} \) that adds weights, as well as satisfying graded commutativity \( [x,y] = (-1)^{|x||y|}[y,x] \) and the graded Jacobi identity
\[
(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||z|}[y,[z,x]] + (-1)^{|z||x|}[z,[x,y]] = 0.
\]

Denote by \( \text{Lie}_{p}^{\ast} \) the category of shifted weighted graded Lie algebras over \( \mathbb{F}_{p} \), as well as the monad associated to the free \( \text{Lie}_{p}^{\ast} \)-algebra functor. When \( p = 2 \), we use the abbreviation \( \text{Lie}^{s} = \text{Lie}_{2}^{\ast} \). We further consider the category \( \text{Lie}^{\ast} \) of totally-isotropic \( \text{Lie}^{s} \)-algebras, i.e., \( \text{Lie}^{s} \)-algebras that have vanishing self-brackets. We use the notation \((\cdot,\cdot)\) exclusively for \( \text{Lie}^{\ast} \) brackets.

We use \( \mathbb{F}_{p}^{\ast} \) for both the field \( \mathbb{F}_{p} \) and its Eilenberg-Maclane spectrum. The coefficients for the homology group \( H_{n}(-) \) is \( \mathbb{F}_{2} \) unless specifically stated.

We use \( \pi_{n}(-) \) to denote the following functors: the functor taking the \( n \)th homotopy group of a spectrum, an \( \mathbb{F}_{p} \)-module spectrum, or a simplicial \( \mathbb{F}_{p} \)-module, as well as the functor taking the \( n \)th homology group of a chain complex over \( \mathbb{F}_{p} \).

We use \( \pi_{n,\ast}(-) \) to denote the functor taking the bigraded homotopy group of a (weighted graded) bisimplicial \( \mathbb{F}_{p} \)-module, which is equivalent to taking the homology of the total complex of the associated double complex via the generalized Eilenberg-Zilber theorem. The bidegree \((s,t)\) is given by the pair (simplicial degree, internal degree).

2. Preliminaries

2.1. Spectral Lie operad. We begin with a brief review of the spectral Lie operad. Ching \[\text{Chi05}\] and Salvatore showed that the Goowillie derivatives \( \partial_{n}(\text{Id}) \) of the identity functor \( \text{Id}: \text{Top} \to \text{Top} \) form an operad \( s\mathcal{L} := \{ \partial_{n}(\text{Id}) \}_{n} \) in Spectra. This operad is Koszul dual to the nonunital commutative operad \( \mathbb{E}^{nu} \) via the operadic bar construction
\[
s\mathcal{L} \simeq \mathbb{D}\text{Bar}(1,\mathbb{E}^{nu},1).
\]

For a description of the operadic bar construction, see \[\text{Chi05}\] for a topological model using trees and \[\text{Bra17}, \text{Appendix D}\] for an \( \infty \)-categorical construction along with a comparison with the topological model.

The \( \ast \)-derivative \( \partial_{n}(\text{Id}) \) admits an explicit description due to Arone and Mahowald \[\text{AM99}\], following the work of Johnson \[\text{Joh95}\]. Let \( \mathcal{P}_{n} \) be the poset of partitions of the set \( \underline{n} = \{1,2,\ldots,n\} \) ordered by refinements, equipped with a \( \Sigma_{n} \)-action induced from that on \( \underline{n} \). Denote by \( \emptyset \) the discrete partition and \( 1 \) the partition \( \{n\} \). Set \( \mathcal{P}_{n} = \mathcal{P}_{n} - \{0,1\} \). Regarding a poset \( \mathcal{P} \) as a category, we obtain via the nerve construction a simplicial set \( N_{\ast}(\mathcal{P}) \). The partition complex \( \Sigma \mathcal{P}_{n} \), the reduced-unreduced suspension of the realization \( |\mathcal{P}_{n}| \), is modeled by the simplicial set
\[
N_{\ast}(\mathcal{P}_{n})/(N_{\ast}(\mathcal{P}_{n} - \emptyset) \cup N_{\ast}((\mathcal{P}_{n} - \{0,1\})).
\]
for \( n \geq 2 \) and the simplicial 0-circle \( S^0 \) for \( n = 1 \). Then there is an equivalence
\[
\partial_\ast(\text{Id}) \simeq \mathbb{D}([\Sigma_\ast]\text{Id}_n]\).
of spectra with \( \Sigma_n \)-action, where \( \mathbb{D} \) denotes the Spanier-Whitehead dual of a spectrum.

2.2. Operations on the mod 2 homology of spectral Lie algebras. Next, we describe the structure on the mod 2 homology of an algebra \( L \) over the spectral Lie operad. It consists of a Lie\(^{\ast}\)-algebra structure along with Dyer-Lashof like unary operations.

The second structure map of a spectral Lie algebra \( L \) is given by
\[
\xi : \partial_2(\text{Id}) \otimes L^{\otimes 2} \simeq \partial_2(\text{Id}) \otimes L^{\otimes 2}_{hS^2} \simeq S^{-1} \otimes L^{\otimes 2}_{hS^2} \rightarrow L.
\]
At the level of homology, this gives rise to a shifted Lie bracket
\[
[-,-] : H_m(L) \otimes H_0(L) \rightarrow H_{m-2-1}(L),
\]
making \( H_\ast(L) \) a graded shifted Lie algebra [AC20, Proposition 5.2].

For \( L \) a connected spectral Lie algebra and \( j \geq d \), Behrens defined unary operations of weight 2
\[
\tilde{Q}^j : H_d(L) \rightarrow H_{d+j-1}(L)
\]
on the mod 2 homology of \( L \) via \( x \mapsto \bar{\xi}_j \sigma^{-1} Q^j(x) \), where \( Q^j : H_d(L) \rightarrow H_{d+j}(L^{\otimes 3}_{hS^2}) \) is an extended power operation \( x \mapsto e_{j-1} \otimes x \otimes x \), \( \sigma^{-1} : H_{d+j}(L^{\otimes 3}_{hS^2}) \rightarrow H_{d+j-1}(\partial_2(\text{Id}) \otimes L^{\otimes 2}_{hS^2}) \) is the desuspension isomorphism, and \( \bar{\xi} \) is the second structure map [Beh12, Section 1.5][AC20, Definition 5.4]. Furthermore, Behrens showed that the quadratic relations
\[
Q^s Q^t = \sum_{s-l \leq r \leq t} \binom{r-l-1}{s-1} Q^{r+s-l} Q^r
\]
for \( s < r \leq 2s \) generate all the relations among the \( \tilde{Q} \) operations [Beh12, Theorem 1.5]. By definition, for \( x \) a homogeneous class \( \tilde{Q}^j(x) = 0 \) for all \( i < |x| \). Hence the relations (4) in lower indexing \( \tilde{Q}^j(x) = \tilde{Q}^{(|x|)}(x) \) become
\[
\tilde{Q}_a \tilde{Q}_b(x) = \sum_{0 \leq c < (a+2b-1)/3} \binom{a+b-2c-2}{b-c} \tilde{Q}_{a+2b-2c} \tilde{Q}_c(x)
\]
for \( 0 \leq a \leq b+1 \) on a homogeneous class \( x \).

Since the Dyer-Lashof operations can be extended to the mod 2 homology of nonconnective spectra, the operations \( \tilde{Q}^i \) for all \( i \in \mathbb{Z} \) can be defined on the mod 2 homology of any spectral Lie algebra \( L \) with \( \tilde{Q}^j(x) = 0 \) for any homogeneous class \( x \in H_\ast(L) \) and \( i < |x| \). Furthermore, the lower-indexed Dyer-Lashof operations are stable under (de)suspension, while in lower-indexing the relations (5) do not depend on the degree of the class and hence are stable under (de)suspension. Switching back to upper-indexing, we therefore can extend the operations \( \tilde{Q}^i \) along with the relations (4) to all integers \( s < r \leq 2s \).

Let \( \tilde{\mathcal{R}} \) be the \( \mathbb{F}_2 \)-algebra obtained by taking the free algebra on the set of symbols \( \{ \tilde{Q}^j \}_{j \in \mathbb{Z}} \) and imposing Behrens’ relations (4). Each \( \tilde{Q}^j \) has internal degree \( j-1 \) and weight two. Therefore \( \tilde{\mathcal{R}} \) has a basis consisting of \( \tilde{Q}^j \) with \( J = \{ j_1, \ldots, j_l \} \) satisfying \( j_i > 2j_{i+1} \) for all \( i \).

**Definition 2.1.** An \( \mathbb{F}_2 \)-module \( M_\ast \) over \( \tilde{\mathcal{R}} \) is **allowable** if for any homogeneous element \( x \in M_\ast \) we have \( \tilde{Q}^{j_1} \tilde{Q}^{j_2} \cdots \tilde{Q}^{j_m} x = 0 \) whenever \( j_1 < j_2 + \cdots + j_m + |x| \).

**Definition 2.2.** [AC20, Definition 6.1] An \( \text{Lie}^{\ast}_{\tilde{\mathcal{R}}} \)-algebra is an \( \mathbb{F}_2 \)-module \( L_\ast \) with a shifted Lie bracket and an allowable \( \tilde{\mathcal{R}} \)-module structure on \( L_\ast \) such that
1. \( \tilde{Q}_0(x) = \tilde{Q}^1(x) = [x,x] \) if \( x \in L_4 \), and
2. \([x, \tilde{Q}^j(y)] = 0 \) for all \( x,y \in L \).

Denote by \( \text{Lie}^{\ast}_{\tilde{\mathcal{R}}} \) the category of \( \text{Lie}^{\ast}_{\tilde{\mathcal{R}}} \)-algebras. To describe the free \( \text{Lie}^{\ast}_{\tilde{\mathcal{R}}} \)-algebra functor, we recall the construction of **Lyndon words** on a set \( S \), which provides a basis for the free \( \text{Lie}^{\ast}_{\tilde{\mathcal{R}}} \)-algebra on an \( \mathbb{F}_2 \)-module with \( \mathbb{F}_2 \)-basis \( S \).
**Construction 2.3.** [Hal50] The Lyndon words on a set $S$ is defined recursively as follows: The elements of $S$ are Lyndon words of length one and given an arbitrary fixed total ordering. Suppose that we have defined Lyndon words of length less than $k$ with a total ordering. Then a Lyndon word of length $k$ is a formal bracket $(w_1, w_2)$ such that

1. $w_1, w_2$ are Lyndon words whose length add up to $k$;
2. $w_1 < w_2$ in the order defined thus far;
3. To take into account the Jacobi identity, if $w_2 = \langle w_3, w_4 \rangle$ for some Lyndon words $w_3, w_4$, then we require $w_3 \leq w_1$.

To extend the total order to Lyndon words of weight at most $k$, we first impose an arbitrary total ordering on Lyndon words of length $k$ and then declare that they are greater than all Lyndon words of lower weights.

The free Lie$_R^s$-algebra functor can thus be computed explicitly as follows:

**Proposition 2.4.** [AC20, Proposition 7.4] Let $V_\bullet$ be an $\mathbb{F}_2$-module with an ordered basis $B$ of $V_\bullet$. First take the free totally isotropic Lie-algebra with $\langle , \rangle$ the free Lie$_{R^s}$ bracket. Denote by $B'$ the set of Lyndon words on the letters $B$, which is an $\mathbb{F}_2$-basis of Free$_{\text{Mod}_2}$($V_\bullet$). Then we take the free $\mathcal{R}$-module on the underlying $\mathbb{F}_2$-module of Free$_{\text{Mod}_2}$($V_\bullet$) and obtain a basis consisting of elements of the form $\mathcal{Q}^i w$ with $w \in B'$. Equip the free $\mathcal{R}$-module Free$_{\text{Mod}_2}$($\langle \text{Lie}_s^R \langle V_\bullet \rangle \rangle$) with a Lie$^s$ bracket $[-,-]$ defined on the induced basis by requiring $[\mathcal{Q}^i w_1, \mathcal{Q}^j w_2] = 0$ if $i \neq 0$ or $j \neq 0$, and setting recursively along the ordering on $B'$

1. If $\langle w_1, w_2 \rangle$ is a Lyndon word, then $\langle w_1, w_2 \rangle = \langle w_1, w_2 \rangle$;
2. $[w_1, w_2] := \mathcal{Q}^{w_1} w_2$;
3. $[w_1, w_2] := [w_1, w_2]$ if $w_1 > w_2$;
4. $[w_1, w_2] := [w_3, w_1, w_4] + [w_4, w_1, w_3]$ if $w_1 < w_2$ and $w_2 = [w_3, w_4]$ with $w_1 < w_3$.

**Remark 2.5.** There is an alternative description of the free Lie$_R^s$-algebra on $V_\bullet$. First take the free $\mathcal{R}$-module over Lie$^s(M)$ and equip Free$_{\text{Mod}_2}$($\langle \text{Lie}_s^R \langle V_\bullet \rangle \rangle$) with a Lie$^s$-structure given by $[x, \mathcal{Q}^i(y)] = 0$ for all $x, y$. Then we take the quotient of Free$_{\text{Mod}_2}$($\langle \text{Lie}_s^R \langle V_\bullet \rangle \rangle$) by the sub $\mathcal{R}$-module generated by $\mathcal{Q}^i(x) + [x, y]$ for all $x \in V_\bullet$. In particular, there is a two-step filtration on Lie$^s_s(M)$ induced by the augmentation of the $\mathcal{R}$-module structure. The associated graded is then $\mathcal{A}_R \circ \text{Lie}_s^R (V_\bullet)$, since the relation $\mathcal{Q}^0(x) + [x, x] = 0$ becomes $[x, x] = 0$.

Antolín-Camarena showed that the monad Lie$^s_R$ parametrizes the mod 2 homology of connected spectral Lie algebras. We note below that the connectivity assumption can be removed. Denote by Free$_{\text{Mod}_2}$ the free spectral Lie algebra functor on Spectra $X \mapsto \bigoplus_{n \geq 1} \partial_n (\text{Id} \otimes X^\otimes n)$.

**Theorem 2.6.** [AC20, Theorem 7.1] There is a natural isomorphism $H_*(\text{Free}_L^s(X); \mathbb{F}_2) \cong \text{Free}_{\text{Mod}_2}(H_*(X; \mathbb{F}_2))$ of Lie$^s_R$-algebras for any spectrum $X$.

**Proof.** Antolín-Camarena proved the isomorphism for $X$ the suspension spectrum of a connected space. To extend the theorem to all spectra, we make use of the fiber sequence $\partial_m (\text{Id} \otimes \mathbb{S}^n) \otimes \mathbb{S}^m \xrightarrow{E} \Sigma^{-1} \partial_m (\text{Id} \otimes \mathbb{S}^n) \otimes \mathbb{S}^{m+1} \otimes \mathbb{S}^m \xrightarrow{H} \Sigma^{-1} \partial_m (\text{Id} \otimes \mathbb{S}^n) \otimes \mathbb{S}^{2m+1} \otimes \mathbb{S}^m$, which was obtained by Behrens via differentiating the EHP sequence [Beh12, Corollary 2.1.4]. In the associated long exact sequence of mod 2 homology, the induced map $H_*$ is surjective, sending the class $\mathcal{Q}^i \mathcal{Q}^j(x_{n+1}) = \mathcal{Q}^i \mathcal{Q}^j(x_{n+1})$ to $\mathcal{Q}^i \mathcal{Q}^j(x_{2n+1})$ for all $J$. This allows us to extend Arone and Mahowald’s computation [AM99, Theorem 3.16] of the mod 2 homology of $\partial_m (\text{Id} \otimes \mathbb{S}^n) \otimes \mathbb{S}^m$ to negative spheres and conclude that $H_*(\text{Free}_L^s(\mathbb{S}^m); \mathbb{F}_2) \cong \text{Free}_{\text{Mod}_2}(H_*(\mathbb{S}^m; \mathbb{F}_2))$ is an isomorphism of Lie$^s_R$-algebras for all $m \in \mathbb{Z}$.
To extend to a finite wedge of spheres, we make use of a result of Arone and Kankaarinta that applies Goodwillie calculus to the Hilton-Milnor Theorem [AK98, Theorem 0.1]. To extend to all spectra, note that \( X \otimes F_2 \) can be written as a filtered colimit of finite wedges of \( S^m \otimes F_2 \) in the category of \( F_2 \)-module spectra. Since free functors and homology commute with filtered colimits, the desired extension follows.

2.3. Quillen homology of spectral Lie algebras. Now we can introduce the main object of interest. The inclusion of trivial \( \text{Lie}_R^s \)-algebras admits a left adjoint \( Q_{\text{Mod}_c}^{\text{Lie}_R^s} \) called the indecomposable functor, i.e. we have an adjunction

\[
\text{Mod}_c \overset{\text{Lie}_R^s}{\underset{Q_{\text{Mod}_c}}{\rightleftarrows}} \text{Lie}_R^s.
\]

Denote again by \( \text{Lie}_R^s \) the monad associated to the free \( \text{Lie}_R^s \)-algebra functor. Any \( \text{Lie}_R^s \)-algebra \( g \) has a free (cofibrant) resolution \( \text{Bar}^\bullet(\text{Free}^{\text{Lie}_R^s}_{\text{Mod}_c}, \text{Lie}_R^s, g) \) in \( \text{Lie}_R^s \). The left derived functor of \( Q_{\text{Mod}_c}^{\text{Lie}_R^s} \) is thus computed by

\[
\llbracket Q_{\text{Mod}_c}^{\text{Lie}_R^s}(g) \rrbracket \cong Q_{\text{Mod}_c}^{\text{Lie}_R^s} \text{Bar}^\bullet(\text{Free}^{\text{Lie}_R^s}_{\text{Mod}_c}, \text{Lie}_R^s, g) \cong \text{Bar}^\bullet(\text{id}, \text{Lie}_R^s, g),
\]

where \( \text{id} : \text{Mod}_c \rightarrow \text{Mod}_c \) is the identity functor considered as the trivial right module over the monad \( \text{Lie}_R^s \) with structure map the augmentation.

**Definition 2.7.** The Quillen homology of a \( \text{Lie}_R^s \)-algebra \( g \), denoted by \( HQ_{s}^{\text{Lie}_R^s}(g) \), is the total left derived functor

\[
HQ_{s}^{\text{Lie}_R^s}(g) := H_* \llbracket Q_{\text{Mod}_c}^{\text{Lie}_R^s}(g) \rrbracket \cong \pi_* \text{Bar}^\bullet(\text{id}, \text{Lie}_R^s, g).
\]

We are interested in computing the Quillen homology of \( \text{Lie}_R^s \)-algebras, since it helps to understand the spectral Lie analog of the mod \( p \) topological André-Quillen homology of nonunital \( E_\infty \)-algebras introduced by Kriz [Kri93] and Basterra [Bas99].

**Definition 2.8.** For \( L \) a spectral Lie algebra, its topological Quillen object is the bar construction

\[
TQ^L_\infty(L) := |\text{Bar}^\bullet(\text{id}, sL, L)|.
\]

We define its mod \( p \) topological Quillen homology to be

\[
TQ^L_\infty(L; \mathbb{F}_p) := \pi_* (|\text{Bar}^\bullet(\text{id}, sL, L)| \otimes \mathbb{F}_p).
\]

Using the skeletal filtration of the geometric realization of the bar construction, we obtain a bar spectral sequence

\[
E^2_{s,t} = \pi_* \pi_0 \text{Bar}^\bullet(\text{id}, sL, L \otimes \mathbb{F}_p) \Rightarrow TQ^L_\infty(L; \mathbb{F}_p)
\]

converging to the mod \( p \) topological Quillen homology. When \( p = 2 \), we can apply Theorem 2.6 repeatedly and deduce that:

**Proposition 2.9.** There is a bar spectral sequence

\[
E^2_{s,t} = \pi_* \pi_0 \text{Bar}^\bullet(\text{id}, s\text{Lie}_R^s, \text{Lie}_R^s(L \otimes F_2)) \cong HQ_{s+t}(H_* (L; F_2)) \Rightarrow TQ^L_{s+t}(L; F_2).
\]

2.4. The derived indecomposable functor. Before diving into computations, we briefly recall without proof the homotopy theory of augmented monads on the category of weighted graded \( F_2 \)-modules and especially the two-sided bar construction for simplicial objects. We mainly follow Sections 3.1 and 3.2 of [JN14].

Let \( T \) be an augmented monad on the category \( \text{Mod}_c \) of weighted graded \( F_2 \)-modules. Denote by \( \text{Alg}_T(\text{Mod}_c) \) the category of \( T \)-algebras. The forgetful functor \( U : \text{Alg}_T(\text{Mod}_c) \rightarrow \text{Mod}_c \) admits a left adjoint, the free functor \( \text{Free}^T : \text{Mod}_c \rightarrow \text{Alg}_T(\text{Mod}_c) \).

Denote by \( s\text{Mod}_c \), the category of simplicial weighted graded \( F_2 \)-modules. Levelwise application of the adjunction \( \text{Free}^T \dashv U : \text{Alg}_T(s\text{Mod}_c) \rightarrow s\text{Mod}_c \),
as well as a monad $T$ on $s\text{Mod}_{\mathbb{F}_2}$. We equip $s\text{Mod}_{\mathbb{F}_2}$ with the standard cofibrantly generated model structure. Then this adjunction induces a right transferred model structure on the category of simplicial $T$-algebras, with weak equivalences and fibrations defined on the underlying simplicial weighted graded $\mathbb{F}_2$-modules.

Denote by $T: \text{Mod}_{\mathbb{F}_2} = \text{Alg}_{\mathcal{U}}(\text{Mod}_{\mathbb{F}_2}) \rightarrow \text{Alg}_T(\text{Mod}_{\mathbb{F}_2})$ the inclusion of trivial $T$-algebras, which is induced by the augmentation. It has a left adjoint $Q^T: \text{Alg}_T(\text{Mod}_{\mathbb{F}_2}) \rightarrow \text{Mod}_{\mathbb{F}_2}$, the indecomposable functor with respect to the $T$-algebra structure. Applying this adjunction levelwise to the corresponding categories of simplicial objects, we obtain a Quillen adjunction

$$Q^T \dashv T: \text{Alg}_T(s\text{Mod}_{\mathbb{F}_2}) \rightarrow s\text{Mod}_{\mathbb{F}_2}.$$ 

The total left derived functor $LQ^T$ of $Q^T$ can be computed by the following standard recipe.

**Construction 2.10.** Given a right module $R: \text{Mod}_{\mathbb{F}_2} \rightarrow \mathcal{D}$ over $T$, and a simplicial object $A$ in $\text{Alg}_T(\text{Mod}_{\mathbb{F}_2})$, one can apply the two-sided bar construction $\text{Bar}_\bullet(R, T, -)$ levelwise to $A$. The diagonal of the resulting bisimplicial complex is a simplicial object in $\mathcal{D}$, denoted by $\text{Bar}_\bullet(R, T, A)$.

In particular, if we regard a $T$-algebra $A$ as the constant simplicial object on $U(A)$ equipped with a simplicial $T$-algebra structure, denoted also as $A$ by abuse of notation, then $\text{Bar}_\bullet(R, T, A)$ agrees with the usual two-sided bar construction.

Since the free resolution $\text{Bar}_\bullet(\text{Free}^T, T, A)$ is a cofibrant replacement of $A$ in the category of simplicial $T$-algebras, the left derived functor of a functor $F$ can be computed by applying $F$ levelwise to a cofibrant replacement, so

$$LQ^T(A) \simeq Q^T\text{Bar}_\bullet(\text{Free}^T, T, A) = \text{Bar}_\bullet(\text{id}, T, A).$$

### 3. Computing the Quillen Homology of Spectral Lie Algebras

In this section, we study the Quillen homology of Lie$_s^R$-algebras when $p = 2$ via comparison with two smaller double complexes that are easy to compute via Koszul duality arguments.

#### 3.1. An upper bound

First we find an upper bound for $\pi_\bullet, \text{Bar}_\bullet(\text{id}, \text{Lie}_s^R, g)$ by constructing a May spectral sequence. The dimensions of its $E^1$-page is bounded above by the homotopy groups of bar construction of a variant of $\text{Lie}_s^R$-algebras whose unary and binary operations do not intertwine. We thank Haynes Miller for suggesting the use of this spectral sequence.

Motivated by Remark 2.5, we would like to first filter away the identification $Q_0(x) = [x, x]$.

**Definition 3.1.** Define a Lie$_s^{\tilde{R}}$-algebra to be an $\mathbb{F}_2$-module $L$ with an allowable $\tilde{R}$-module and a Lie$_s^{\tilde{R}}$-bracket $\langle, \rangle$ such that $\langle x, \tilde{Q}(y) \rangle = 0$ for all $x, y \in L$. Denote by $\text{Lie}_s^{\tilde{R}}$ the category of $\text{Lie}_s^{\tilde{R}}$-algebras and the monad associated to the free $\text{Lie}_s^{\tilde{R}}$-algebra functor.

The underlying $\mathbb{F}_2$-module of the free $\text{Lie}_s^{\tilde{R}}$-algebra on on $\mathbb{F}_2$-module $V$ is given by that of $\mathcal{A}_R \circ \text{Lie}_s^{\tilde{R}}(V)$.

Hence $\text{Lie}_s^{\tilde{R}}$ admits an alternative description as the category of algebras over the composite monad $\mathcal{A}_R \circ \text{Lie}_s^{\tilde{R}}$, with distributive law the natural transformation $\text{Lie}_s^{\tilde{R}} \circ \mathcal{A}_R \Rightarrow \mathcal{A}_R \circ \text{Lie}_s^{\tilde{R}}$ determined by $\langle - , \tilde{Q}(\cdot) \rangle = 0$ for all $i$, cf. [Bec69, Section 1].

**Remark 3.2.** In light of Remark 2.5, we see that the underlying $\tilde{R}$-modules of the free $\text{Lie}_s^R$ and $\text{Lie}_s^{\tilde{R}}$-algebra on any $\mathbb{F}_2$-module agree. The only difference between the two free functors is that in the latter we do not upgrade the $\text{Lie}_s^{\tilde{R}}$-algebra to a $\text{Lie}^s$-algebra via the identification $\tilde{Q}_0(x) = [x, x]$.

Given an arbitrary $\text{Lie}_s^R$-algebra $g$, we would like to equip $g$ with the structure of a $\text{Lie}_s^{\tilde{R}}$-algebra. This boils down to producing a method that equips any $\text{Lie}^s$-algebra with a $\text{Lie}_s^{\tilde{R}}$-algebra structure that is essentially unique.
Construction 3.3. (Lie\textsuperscript{x,ti}-structure on Lie\textsuperscript{x}-algebras.) There is an inclusion $\tau_{\text{Lie}^x}^\text{Lie}_x : \text{Lie}^x \to \text{Lie}^x$ with left adjoint $Q_{\text{Lie}^x}^\text{Lie}_x$ taking the indecomposable of self brackets. Let $\mathfrak{g}$ be a Lie\textsuperscript{x}-algebra with Lie\textsuperscript{t}-bracket $[-,-]$. Hence any Lie\textsuperscript{x,ti}-bracket $\mathfrak{g}$ is equal to a sum of self-brackets and the Lie\textsuperscript{t}-bracket. Let $\mathfrak{g}$ be a Lie\textsuperscript{x}-algebra with Lie\textsuperscript{t}-bracket $[-,-]$. Since $[[x,x],y] = 0$ for all $x,y \in \mathfrak{g}$ by the Jacobi identity, there is a split short exact sequence of $\text{Mod}_{F_2}$-modules

$$V' \to \mathfrak{g} \xrightarrow{\rho} V = Q_{\text{Lie}^x}^\text{Lie}_x(\mathfrak{g}),$$

where $V'$ is the ideal of self-brackets in $\mathfrak{g}$. Note that this is not split as a short exact sequence of Lie\textsuperscript{x}-algebras. Fix an isomorphism $\mathfrak{g} \cong V' \oplus W$ of $F_2$-modules such that $W$ is isomorphic to $\mathfrak{g}$ under the restriction of $\rho$. Denote by $\langle -,- \rangle$ the canonical Lie\textsuperscript{x,ti}-bracket on $V = Q_{\text{Lie}^x}^\text{Lie}_x(\mathfrak{g})$ and consider $V'$ as a trivial Lie\textsuperscript{x,ti}-algebra.

Thus we obtain a Lie\textsuperscript{x,ti}-structure on the underlying $F_2$-module of $\mathfrak{g}$ as the product of $V$ and $V'$ with the above Lie\textsuperscript{x,ti}-structures. Denote by $\tilde{\mathfrak{g}}$ the resulting Lie\textsuperscript{x,ti}-algebra with $\langle -,- \rangle$ the Lie\textsuperscript{x,ti}-bracket.

Therefore, any Lie\textsuperscript{x}-algebra $\mathfrak{g}$ admits a Lie\textsuperscript{x,ti}-structure that is unique up to a change of basis of the underlying $F_2$-module. Denote by $\tilde{\mathfrak{g}}$ the resulting Lie\textsuperscript{x,ti}-algebra, which has the same underlying $\tilde{R}$-module structure as $\mathfrak{g}$, cf. Remark 3.2.

Remark 3.4. If we fix a choice of basis for the underlying $F_2$-module of $\mathfrak{g}$, then any Lie\textsuperscript{t}-bracket $[x,y]$ in $\mathfrak{g}$ is equal to a sum of self-brackets and the Lie\textsuperscript{ti}-bracket $(x,y)$ in $\tilde{\mathfrak{g}}$.

Proposition 3.5. Let $\mathfrak{g}$ be a Lie\textsuperscript{x,ti}-algebra and $\tilde{\mathfrak{g}}$ the associated Lie\textsuperscript{x,ti}-algebra via Construction 3.3. Then there is a May spectral sequence with respect to the $\tilde{R}$-module structure converging to $\pi_q \text{Bar}_{\bullet}(\text{id}, \text{Lie}_x, \tilde{\mathfrak{g}})$. The $E^1$-page $E^1_{p,q} = \oplus_{q \geq 0} E^1_{p,q}$ of this spectral sequence has dimensions bounded above by $\pi_q \text{Bar}_{\bullet}(\text{id}, \text{Lie}_x, \tilde{\mathfrak{g}})$. The $E^2$-page $E^2_{p,q} = \oplus_{q \geq 0} E^1_{p,q}$ of this spectral sequence has dimensions bounded above by $\pi_q \text{Bar}_{\bullet}(\text{id}, \text{Lie}_x, \tilde{\mathfrak{g}})$.

Proof. Let $I_R$ be functor sending an $\tilde{R}$-module $M$ to the kernel of the augmentation $A_R(M) \xrightarrow{\pi_q} M$. Since $I_R(M)$ has an induced $\tilde{R}$-module structure, iterative application yields a filtration

$$\cdots \to (I^\infty_R)^{q+1}(M) \to (I^\infty_R)^{q}(M) \to \cdots \to I^\infty_R(M) \to A_R(M).$$

Hence any $\text{Lie}_x$-algebra $\mathfrak{g}$ admits an increasing filtration

$$F^q(\mathfrak{g}) = \text{coker}((I^\infty_R)^{q-1}(\mathfrak{g}) \to A_R(\mathfrak{g}) \xrightarrow{\text{ev}} \mathfrak{g})$$

where $\text{ev}$ is the $\tilde{R}$-module structure map, with associated graded

$$\text{Gr}^q(\mathfrak{g}) = \text{im}((I^\infty_R)^{q-1}(\mathfrak{g}) \to A_R(\mathfrak{g}) \xrightarrow{\text{ev}} \mathfrak{g}).$$

If $\mathfrak{g}$ is a free Lie\textsuperscript{x,ti}-algebra, then the filtration on $\mathfrak{g}$ is determined by the number of symbols $\tilde{Q}^j$ in each element. Hence we obtain a filtration on $\text{Bar}_{\bullet}(\text{id}, \text{Lie}_x, \tilde{\mathfrak{g}})$, with $F^q \left((\text{Lie}_x^x)^{\infty}(\mathfrak{g})\right)$ the collection of elements $\alpha|x$ in simplicial degree $n$ satisfying the following condition: if we rewrite $\alpha|x$ as an element in $(\text{Lie}_x^x)^{\infty}(\mathfrak{g})$ via Remark 3.2 and Remark 3.4, so any Lie\textsuperscript{t}-bracket in $\alpha|x$ is written as a linear combination of Lie\textsuperscript{ti}-brackets and $\tilde{Q}^0$ applies to other elements, then the sum of the filtration degree of $\alpha|x$ times the number of times $x$ appears and the number of symbols $\tilde{Q}^j$ in any term of $\alpha|x$ coming from applications of the monad $\text{Lie}_x^x$ is at most $q$. Since $\tilde{R}$ is a homogeneous quadratic algebra and evaluation of brackets do not increase the number of $\tilde{Q}^j$’s in the expression, the structure map $\text{Lie}_x^x(\mathfrak{g}) \to \mathfrak{g}$ is compatible with this filtration, and so are the face maps and degeneracy maps in $\text{Bar}_{\bullet}(\text{id}, \text{Lie}_x, \mathfrak{g})$. The induced filtration on the normalized complex of $\text{Bar}_{\bullet}(\text{id}, \text{Lie}_x, \mathfrak{g})$ gives rise to a May spectral sequence

$$\bigoplus_q E^1_{q,s} = \bigoplus_q \pi_q \text{Bar}_{\bullet}(\text{id}, \text{Lie}_x, \mathfrak{g}) \Rightarrow \pi_q \text{Bar}_{\bullet}(\text{id}, \text{Lie}_x, \mathfrak{g}).$$

It follows inductively from Remark 2.5 that for each $n$ the associated graded of the $n$th simplicial level $\bigoplus_q \text{Gr}^q \left((\text{Lie}_x^x)^{\infty}(\mathfrak{g})\right)$ has underlying $F_2$-module $(\text{Lie}_x^x)^{\infty}(\mathfrak{g})$. Again since $\tilde{R}$ is a homogeneous quadratic, the face maps

$$\text{Gr}^q \text{Bar}_n(\text{id}, \text{Lie}_x, \mathfrak{g}) \to \text{Gr}^q \text{Bar}_{n-1}(\text{id}, \text{Lie}_x, \mathfrak{g})$$
induced by filtering the Lie$^\text{tl}_R$-algebra structure map assembles to the Lie$^\text{tl}_R$-algebra structure maps $\text{Lie}^\text{tl}_R(\tilde{\mathfrak{g}}) \to \tilde{\mathfrak{g}}$ when $q \neq 2$. When $q = 2$, the total differential $\partial$ of the normalized complex of $\text{Bar}_s(\text{id}, \text{Lie}^\text{tl}_R, \tilde{\mathfrak{g}})$ sends $[\tilde{Q}^i | x, 1 \tilde{Q}^i | x] \mapsto [\tilde{Q}^i | x, \tilde{Q}^i | x] + [\tilde{Q}^i | x, 1 \tilde{Q}^i | x] = \tilde{Q}^0 | x + \tilde{Q}^i | x, 1 \tilde{Q}^i | x]$ for any $i$ and $x \in \mathfrak{g}$, so the self-bracket is still present in the associated graded $E^0_{\delta,q}$. Whereas when $p > 2$, the self-brackets in the target of such differentials are not visible in the associated graded because the number of $\tilde{Q}^i$‘s in the term decrease after we rewrite in terms of $\tilde{Q}_n$. In comparison, the total differential $\partial'$ in the normalized complex of $\text{Bar}_s(\text{id}, \text{Lie}^\text{tl}_R, \tilde{\mathfrak{g}})$ sends $[\tilde{Q}^i | x, 1 \tilde{Q}^i | x]$ to $[\tilde{Q}^i | x, \tilde{Q}^i | x]$ by $\tilde{Q}^i | x + \tilde{Q}^i | x, 1 \tilde{Q}^i | x]$. Hence $[\tilde{Q}^i | x, 1 \tilde{Q}^i | x]$ can be completed to a cycle on the $E^1$-page only if it does in $\text{Bar}_s(\text{id}, \text{Lie}^\text{tl}_R, \tilde{\mathfrak{g}})$. Therefore the dimension of $\bigoplus_2 E^1_{\delta,q,2}$ is bounded above by $\pi_{\text{Bar}_s(\text{id}, \text{Lie}^\text{tl}_R, \tilde{\mathfrak{g}})}$.

To compute the homotopy groups of $\text{Bar}_s(\text{id}, \text{Lie}^\text{tl}_R, L)$ for $L$ a Lie$^\text{tl}_R$-algebra, we will factor the bar construction into a smaller double complex that is amenable to Koszul duality arguments. We follow the strategy in [BHK19, Proposition 4.19]. We compute the composite of adjunctions

$$\begin{align*}
\text{Mod}_{\mathcal{F}_2} & \xleftarrow{\mathcal{F}_{\text{Mod}_{\mathcal{F}_2}}} \text{Lie}^\text{tl}_R & \text{Lie}^\text{tl}_R & \xrightarrow{\mathcal{F}_{\text{Lie}^\text{tl}_R}} \text{Mod}_{\mathcal{F}_2}
\end{align*}$$

we obtain an isomorphism $Q_{\text{Lie}^\text{tl}_R} \cong Q_{\text{Lie}^\text{tl}_R} \circ Q_{\text{Lie}^\text{tl}_R}$. 

**Construction 3.6.** For $L$ a Lie$^\text{tl}_R$-algebra with Lie$^\text{tl}_R$-bracket $(-, -)$, denote by $\text{AR}_s(L)$ the bar construction $\text{Bar}_s(\text{id}, \mathcal{A}_R, L)$ equipped with a Lie$^\text{tl}_R$-bracket $(-, -)$ given levelwise by

$$\langle \alpha_1 | \alpha_2 | \cdots | \alpha_n | x, \beta_1 | \beta_2 | \cdots | \beta_n | y \rangle = \begin{cases} 1 | \cdots | 1 | x, y \rangle & \text{if } \alpha_i = \beta_i = 1, 1 \leq i \leq n \text{, otherwise} \\ 0 & \end{cases}$$

where $\alpha_i, \beta_j \in R$ and $x, y \in L$. Here we use $L$ to mean the underlying $R$-module $U_{\text{Lie}^\text{tl}_R}(L)$.

**Lemma 3.7.** For $L$ a Lie$^\text{tl}_R$-algebra with Lie$^\text{tl}_R$-bracket $(-, -)$, there is an equivalence of bigraded homotopy groups

$$\pi_{\text{Bar}_s}(\text{id}, \text{Lie}^\text{tl}_R, L) \cong \pi_{\text{Bar}_s}(\text{id}, \text{Lie}^\text{tl}_R, \text{AR}_s(L)).$$

**Proof.** The Lie$^\text{tl}_R$ structure map of $L$ factors as $\text{Lie}^\text{tl}_R(L) \to \mathcal{A}_R(L) \to L$, where the first map evaluates the Lie$^\text{tl}_R$-bracket in $L$. Equipping $\mathcal{A}_R(L)$ with the Lie$^\text{tl}_R$ structure given by

$$\langle \alpha | x, \beta | y \rangle = \begin{cases} 1 | \langle x, y \rangle \rangle & \text{if } \alpha = \beta = 1, 0 & \text{otherwise} \end{cases},$$

we can upgrade $\text{Lie}^\text{tl}_R(L) \to \mathcal{A}_R(L)$ to a map of Lie$^\text{tl}_R$-algebras. Iterating this construction, we obtain a map of simplicial Lie$^\text{tl}_R$-algebras

$$\psi: \text{Bar}_s(\text{Free}_{\text{Mod}_{\mathcal{F}_2}}, \text{Lie}^\text{tl}_R, L) \to \text{Bar}_s(\text{Free}_{\text{Mod}_{\mathcal{F}_2}}, \mathcal{A}_R, L).$$

This map is a weak equivalence, since weak equivalences are detected by underlying simplicial $\mathbb{F}_2$-modules and both sides are free resolutions of $L$. We want to show that $Q_{\text{Lie}^\text{tl}_R}$ preserves this weak equivalence.

Note that there is an isomorphism

$$U_{\text{Lie}^\text{tl}_R} \circ Q_{\text{Lie}^\text{tl}_R} \cong Q_{\text{Mod}_{\mathcal{F}_2}} \circ U_{\text{Lie}^\text{tl}_R}$$
and $\mathcal{O}^\text{Lie}_R^\text{ti}$ is a left Quillen functor. Hence there is a weak equivalence $U^\text{Lie}_R^\text{ti} \circ \mathcal{O}^\text{Lie}_R^\text{ti} \circ \psi$:

$$
U^\text{Lie}_R^\text{ti} \circ \mathcal{O}^\text{Lie}_R^\text{ti} \circ \psi (\text{FreeMod}_2^\text{li}, \text{Lie}_R^\text{ti}, L) \rightarrow U^\text{Lie}_R^\text{ti} \circ \mathcal{O}^\text{Lie}_R^\text{ti} \circ \psi (\text{FreeMod}_2^\text{li}, A, L)
$$

$$
\simeq \mathcal{O}^\text{Mod}_2 \circ U^\text{Lie}_R^\text{ti} \circ \psi (\text{Bar}_2, \text{FreeMod}_2^\text{li}, A, L)
$$

$$
\simeq \text{Bar}_2 (\text{id}, A, L).
$$

Therefore, the homotopy group of

$$
\mathbb{L}^\text{Lie}_R^\text{ti}_2 (L) \simeq \mathcal{O}^\text{Lie}_R^\text{ti} \circ \mathcal{O}^\text{Lie}_R^\text{ti} \circ \psi (\text{FreeMod}_2^\text{li}, \text{Lie}_R^\text{ti}, L)
$$

is given by the homotopy group of the bisimplicial $F_2$-module $\text{Bar}_2 (\text{id}, \text{Lie}_R^\text{ti}, \text{Bar}_2 (\text{id}, \text{id}, A, L))$. 

### 3.2. Homology groups of simplicial Lie$^\text{ti}$-algebras.

The homotopy group of $\text{Bar}_2 (\text{id}, \text{Lie}_R^\text{ti}, V)$ for $V$ a simplicial Lie$^\text{ti}$-algebra can be computed via a shifted version of the classical Chevalley-Eilenberg complex.

Recall from [CE48], [May66A, Section 5] and [Pri70] that given a Lie$^\text{ti}$-algebra $L$, i.e., an unshifted totally isotropic Lie algebra over $F_2$, its Lie$^\text{ti}$-algebra homology is computed by

$$
H^\text{Lie}_L (L) := H^*_s (\mathbb{L}^\text{Lie}_R^\text{ti}_2 (L)[1] \oplus F_2) = H_s (CE(L)).
$$

Here $CE(L)$ is the standard Chevalley-Eilenberg complex, defined to be the exterior algebra on $L[1]$ with differential $\delta$ given by

$$
\delta (\sigma x_1 \otimes \cdots \otimes \sigma x_n) = \sum_{1 \leq i < j \leq n} [\sigma x_i, \sigma x_j] \otimes \sigma x_1 \otimes \cdots \otimes \sigma x_i \otimes \cdots \otimes \sigma x_j \otimes \cdots \otimes \sigma x_n,
$$

Since we are working with shifted, graded totally-isotropic Lie algebras, we use a modified version for ease of notation. First we note that given a Lie$^\text{ti}$-algebra $L$, there are weak equivalences

$$
N (\text{Bar}_2 (\text{id}, \text{Lie}_R^\text{ti}, L)) \simeq N (\Sigma \text{Bar}_2 (\text{id}, \text{Lie}_R^\text{ti}, \Sigma^{-1} L)) \simeq \Sigma CE (\Sigma^{-1} L[1])[-1],
$$

where $CE$ is the reduced complex.

**Definition 3.8.** The Chevalley-Eilenberg complex for a Lie$^\text{ti}$-algebra $L$ is $CE(L) = (\Lambda^* (L), \delta)$, where the shifted graded exterior algebra $\Lambda^* (L)$ has a shifted graded exterior product $\otimes = \Sigma^{-1} \otimes [1]$ that increases homological degree by one and decreases internal degree by one, reflecting the behavior of shifted graded Lie brackets. The differential $\delta$ is given by

$$
\delta (x_1 \otimes \cdots \otimes x_n) = \sum_{1 \leq i < j \leq n} [x_i, x_j] \otimes x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_n.
$$

Then the Lie$^\text{ti}$-algebra homology of $L$ is given by

$$
H^\text{Lie}_L^\text{ti} (L) := \pi_* (\mathbb{L}^\text{Lie}_R^\text{ti}_2 (L) \oplus F_2) \cong H_* (N (\text{Bar}_2 (\text{id}, \text{Lie}_R^\text{ti}, L) \oplus F_2) \cong H_* (CE(L)),
$$

where the last isomorphism follows from rearranging the right hand side in (6).

In the case where $L$ is a simplicial Lie$^\text{ti}$-algebra, its Chevalley-Eilenberg complex $CE(L)$ is the simplicial chain complex obtained by applying the Chevalley-Eilenberg complex levelwise. Then Dold-Kan correspondence says that the homotopy group of $CE(L)$ is isomorphic to the homology of its total complex. A simplicial version of May’s result is recorded in [BHK19, Section 3]. Here we state the shifted version.

**Theorem 3.9.** [BHK19, Theorem 3.13] Let $L$ be a simplicial Lie$^\text{ti}$-algebra. Then there is a natural isomorphism

$$
H_*^\text{Lie}_L (L) := \pi_* (\mathbb{L}^\text{Lie}_R^\text{ti}_2 (L) \oplus F_2) \cong H_* (CE(L)).
$$
In the total complex of CE(L), the differential in the homological direction is given by \( \delta \) in Definition 3.8. The differential \( d \) in the simplicial direction is obtained by applying the shifted graded exterior algebra functor \( \Lambda^* \) to each simplicial differential \( d_i \) of \( L \) and taking the alternating sum, i.e.

\[
d = d_0 \otimes d_0 \otimes \cdots \otimes d_0 + \cdots + d_r \otimes \cdots \otimes d_r.
\]

Both differentials preserve weights.

If the Lie\(^{sl}\) bracket on a simplicial Lie\(^{sl}\) algebra \( L \) is trivial, then the differential \( \delta \) in the homological direction vanishes and \( H_n^{sl}(CE(L)) \cong \pi_n(\Lambda^*(L)) \). The natural operations on the homotopy groups of simplicial exterior algebras are well-understood by the work of Cartan, Bousfield, and Dwyer. We only state Theorem 3.10.

Theorem 3.10. [Dwy80a, Theorem 2.1, Remark 4.4][Bou68][Car54][HM16, Theorem 3.9] Let \( V_* \) be a simplicial graded \( \mathbb{F}_2 \)-module. There are natural operations

\[
\gamma : \pi_{n,r}(\Lambda^b(V_*)) \rightarrow \pi_{2h+1, r+1, 2r-1}(\Lambda^{2h+1}(V_*)), \quad 1 \leq i \leq r
\]

for all \( r \geq 1 \), satisfying the relations

\[
\gamma \gamma_j(x) = \sum_{(i+1)/2 \leq 2i+3/(2j)} \binom{j-i+l-1}{j-l} \gamma_{i-l} \gamma_j(x) \quad \text{for all } i < 2j.
\]

Here in the trigrading \((h,r,t)\) records the number of exterior products \( h \), the simplicial degree \( r \) in \( V_* \), and the internal degree \( t \).

Furthermore, they computed the homotopy group of the free exterior algebra on a simplicial \( \mathbb{F}_2 \)-module.

Definition 3.11. A sequence \( I = (i_1, \ldots, i_m) \) is \( \gamma \)-admissible if \( i_l \geq 2i_{l+1} \) for \( 1 \leq l \leq m-1 \). The excess of \( I \) is

\[
e(I) = i_1 - i_2 - \cdots - i_m.
\]

Theorem 3.12. [Bou68, Theorem 8.6][HM16, Theorem 3.19] Let \( A \) be a graded \( \mathbb{F}_2 \)-basis for \( \pi_*(V_*) \). Then \( \pi_*(\Lambda^*(V_*)) \) is the (shifted graded) exterior algebra on generators \( \gamma(\alpha) \), where \( \alpha \in A \) and \( I = (i_1, \ldots, i_m) \) is \( \gamma \)-admissible with \( e(I) \leq s(\alpha) \), where \( s(\alpha) \) is the simplicial degree of the basis element \( \alpha \).

The following is immediate by combining Theorem 3.9 and Theorem 3.12.

Corollary 3.13. Suppose that \( L \) is a Lie\(^{sl}\) algebra with trivial Lie brackets. Then the homotopy group of \( \text{Bar}_*(\text{id}, \text{Lie}^{sl}, \text{AR}_*(L)) \) is the (shifted graded) exterior algebra on generators \( \gamma(\alpha) \), where \( \alpha \) is a basis element of \( \pi_*(\text{AR}_*(L)) \) (cf. Construction 3.6) and \( I \) is \( \gamma \)-admissible with \( e(I) \leq r \).

We will often omit the adjectives shifted graded for the exterior algebra.

Lemma 3.14. (1). Suppose that \( L = \Sigma^k \mathbb{F}_2 \) is a trivial Lie\(^{sl}\)-algebra. Then \( \pi_*(\text{Bar}_*(\text{id}, \text{Lie}^{sl}, \text{AR}_*(L)) \) is the exterior algebra on generators \( \gamma \tilde{Q}_J(x_k) \), where \( x_k \) is the generator of \( \pi_*(L) \), \( J = (j_1, \ldots, j_r) \) satisfies

\[
j_{i+1} + \cdots + j_r + k - (r-l) \leq j_l \leq 2j_{i+1}
\]

for \( 1 \leq l < r \) and \( j_r > k \), and \( I \) is \( \gamma \)-admissible with \( e(I) \leq r \). In lower indexing, the generators are \( \gamma \tilde{Q}_J(x_k) \), where \( J = (j_1, \ldots, j_r) \) satisfies \( 0 \leq j_l \leq j_{l+1} + 1 \) for all \( l \), and \( I \) is \( \gamma \)-admissible with \( e(I) \leq r \).

(2). Let \( L \) be the Lie\(^{sl}\) algebra with underlying \( \text{AR}_*(\Sigma^k \mathbb{F}_2) \)-module \( \Omega^r \text{Free}_{\Sigma^k \mathbb{F}_2}^\text{Mod}_{\mathbb{F}_2} \), \( n \geq 1 \) and trivial Lie brackets. Then \( \pi_*(\text{Bar}_*(\text{id}, \text{Lie}^{sl}, \text{AR}_*(L)) \) is the exterior algebra on generators \( \gamma \tilde{Q}_J(x_k) \), where \( J = (j_1, \ldots, j_r) \) satisfies \( 0 \leq j_l \leq j_{l+1} + 1 \) for all \( l \) and \( I \) is \( \gamma \)-admissible with \( e(I) \leq r \).

Proof. (1). In light of Corollary 3.13, it suffices to compute

\[
\pi_*(\text{AR}_*(L)) = \pi_*(\text{Bar}_*(\text{id}, \text{AR}_*, \Sigma^k \mathbb{F}_2)),
\]

where the right hand side is the unstable Tor groups \( \text{UnTor}_{n}^\mathbb{F}_2(\mathbb{F}_2, \Sigma^k \mathbb{F}_2) \), cf. [BC70, Section 3].
The quadratic algebra $\tilde{R}$ is a homogeneous Koszul algebra, since the canonical basis $\{\tilde{Q}^i \cdots \tilde{Q}^j, j_i > 2j_{i+1}\}$ of $\tilde{R}$ is a Poincaré-Birkhoff-Witt basis in the sense of Priddy [Pri70, Theorem 5.3]. In particular, it follows from Priddy’s machinery [Pri70, Theorem 2.5, 3.8] that the Tor group $\text{Tor}_R^{\infty}(F_2, F_2)$ has a basis consisting of cycles indexed by $\tilde{Q}^j \cdots \tilde{Q}^i$, where $j_i \leq 2j_{i+1}$ for all $i$.

To compute the unstable Tor groups on a class $x_k$ of internal degree $k$, we need to impose the unstability condition $\tilde{Q}^j(x) = 0$ for $j < |x|$, then the basis of $\text{UnTor}_R^{\infty}(F_2, F_2(x_k))$ consists of basis elements of $\text{Tor}_R^{\infty}(F_2, F_2)$ satisfying $j_i > j_{i-1} - 1 + j_{i-2} - 1 + \cdots + j_r - 1 + |x|$ for all $i < r$ and $j_r \geq k$, or equivalently sequences $\tilde{Q}_{j_1} \cdots \tilde{Q}_{j_k}(x_k)$, where $0 \leq j_i \leq j_{i+1} + 1$ for all $i$.

(2). There is a canonical map of $\tilde{R}$-modules via stabilization, which gives rise to a surjective map of $\text{Lie}_{\tilde{R}}^{s, ti}$-algebras with trivial brackets. The underlying $F_2$-module of $L$ has basis $\tilde{Q}^j x_k$, where $J = (j_1, \ldots, j_r)$ is a basis element of $\tilde{R}$ satisfying $j_r \geq n + k$. Suppose that $\alpha \in A_{\tilde{R}}(L)$ is the cycle completion of an element $\tilde{Q}^j \cdots \tilde{Q}^i x_k$ with $k \leq j_r < n + k$ and $j_{r-1} - 1 + \cdots + j_r - 1 + 1 \leq j_i \leq 2j_{i+1}$ for $i < r$. Since cycle completion via Behrens’ relations cannot increase the index of the right most operation, the differentials supported by $\alpha$ are the same as those supported by its image in $A_{\tilde{R}}(F_2(x_k))$, so $\alpha$ is a nontrivial cycle. Otherwise, all but the rightmost face maps send $\alpha$ to zero, while the rightmost face map from at least one (distinct) term of $\alpha$ is nonzero, so it is impossible to complete the cycle. Switching to lower-indexing yields the desired answer. □

3.3. Quillen homology of $\text{Lie}_{\tilde{R}}^{s, ti}$-algebras with trivial brackets. In order to find a lower bound on the dimensions of the homotopy groups of $\text{Bar}_{\bullet}(\text{id}, \text{Lie}_{\tilde{R}}^{s, ti}, g)$, we map $\text{Bar}_{\bullet}(\text{id}, \text{Lie}_{\tilde{R}}^{s, ti}, g)$ into the bar construction of another variant of $\text{Lie}_{\tilde{R}}^{s, ti}$-algebras.

**Definition 3.15.** Let $\text{Mod}_{\tilde{R}}^{\leq 0} \subset \text{Mod}_{\tilde{R}}$ be the subcategory of $\tilde{R}$-modules $M$ such that $\tilde{Q}_0(x) = 0$ for all $x \in M$. Denote by $\text{Free}_{\text{Mod}_{\tilde{R}}^{\leq 0}}$ the free $\tilde{R}^{>0}$-module functor, and $\text{A}_{\tilde{R}}^{\geq 0}$ the additive monad associated to the free functor. Similarly, denote by $\text{Lie}_{\tilde{R}}^{s, ti \geq 0}$ the subcategory of $\text{Lie}_{\tilde{R}}^{s, ti}$-algebras $L$ satisfying the condition that $\tilde{Q}_0(x) = [x, x] = 0$ for all $x \in L$. Denote by $\text{Lie}_{\tilde{R}}^{s, ti}$ the monad associated to the free $\text{Lie}_{\tilde{R}}^{s, ti}$-algebra functor.

The inclusion $\text{Lie}_{\tilde{R}}^{s, ti} \subset \text{Lie}_{\tilde{R}}^{s, ti \geq 0}$ of subcategory admits a left adjoint $\text{Q}_{\text{Mod}_{\tilde{R}}^{\leq 0}}^{\text{Lie}_{\tilde{R}}^{s, ti \geq 0}}(g)$ that takes the quotient by the ideal of self-brackets closed under the action of the $\tilde{R}$-algebra. When $g$ is a $\text{Lie}_{\tilde{R}}^{s, ti}$-algebra with trivial Lie brackets, $\text{Q}_{\text{Mod}_{\tilde{R}}^{\leq 0}}^{\text{Lie}_{\tilde{R}}^{s, ti \geq 0}}(g)$ is given by equipping the $\tilde{R}^{>0}$-module $\text{Q}_{\text{Mod}_{\tilde{R}}^{\leq 0}}^{\text{Lie}_{\tilde{R}}^{s, ti \geq 0}}(g)$ with trivial $\text{Lie}_{\tilde{R}}^{s, ti}$ brackets.

**Lemma 3.16.** Let $g$ be an $\text{Lie}_{\tilde{R}}^{s, ti}$-algebra. There is a surjective map of simplicial $F_2$-modules

$$\varphi : \text{Bar}_{\bullet}(\text{id}, \text{Lie}_{\tilde{R}}^{s, ti}, g) \rightarrow \text{Bar}_{\bullet}(\text{id}, \text{Lie}_{\tilde{R}}^{s, ti \geq 0}, \text{Q}_{\text{Mod}_{\tilde{R}}^{\leq 0}}^{\text{Lie}_{\tilde{R}}^{s, ti \geq 0}}(g)).$$

**Proof.** By Proposition 2.4, there is an equivalence of monads $\text{Lie}_{\tilde{R}}^{s, ti \geq 0} \simeq \text{A}_{\tilde{R}}^{\geq 0} \circ \text{Lie}_{\tilde{R}}^{s, ti}$. Hence there is a map of monads $\text{Lie}_{\tilde{R}}^{s, ti} \rightarrow \text{Lie}_{\tilde{R}}^{s, ti \geq 0}$ that sends the symbol $\tilde{Q}_0$ to 0, and this induces the map of bar constructions in question. □

The homotopy group of $\text{Bar}_{\bullet}(\text{id}, \text{Lie}_{\tilde{R}}^{s, ti \geq 0}, L)$ is computed in the same way as for $\text{Bar}_{\bullet}(\text{id}, \text{Lie}_{\tilde{R}}^{s, ti}, L)$ with minimal modification to Lemma 3.7 (we use the factorization $\text{Q}_{\text{Mod}_{\tilde{R}}^{\leq 0}}^{\text{Lie}_{\tilde{R}}^{s, ti \geq 0}} \cong \text{Q}_{\text{Mod}_{\tilde{R}}^{\leq 0}} \circ \text{Q}_{\text{Lie}_{\tilde{R}}^{s, ti \geq 0}}$) and Lemma 3.14 (the $\tilde{Q}_0$ operation no longer appears in the generators).
Construction 3.17. For $L$ a Lie$^{\infty}_{\mathcal{R}_{>0}}$-algebra with Lie$^{\infty}$-bracket $\langle \cdot, \cdot \rangle$, denote by $AR^{>0}_{\bullet}(L)$ the bar construction $\text{Bar}_{\bullet}(\text{id}, \mathcal{A}_{\mathcal{R}_{>0}}, L)$ equipped with the simplicial Lie$^{\infty}_{\mathcal{R}_{>0}}$-algebra structure given levelwise by
\[
\langle \alpha_1 | d_2 | \ldots | \alpha_n | x, \beta_1 | f_2 | \ldots | \beta_m \rangle = \begin{cases} 
1 \cdots 1 \langle x, y \rangle & \text{if } \alpha_i = \beta_j = 1, 1 \leq i \leq n, \\
0 & \text{otherwise} 
\end{cases}
\]

Lemma 3.18. (1) There is an isomorphism
\[
\pi_{\ast, s} \text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, \Sigma^{k} \mathbb{F}_2) \cong \pi_{\ast, s} \text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}, \text{Bar}_{\bullet}(\text{id}, \mathcal{A}_{\mathcal{R}_{>0}}, \Sigma^{k} \mathbb{F}_2))
\]
\[
\cong \pi_{\ast, s} \Lambda^s(\text{UnTor}_{\mathcal{R}_{>0}}(\mathbb{F}_2, \mathbb{F}_2(x_i))).
\]

Hence $\pi_{\ast, s} \text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, \Sigma^{k} \mathbb{F}_2)$ is the exterior algebra on generators $\gamma \mathcal{Q}_J(x_i)$, where $J = (j_1, \ldots, j_r)$ satisfies $1 \leq j_i \leq j_i + 1$ for all $l$ and $I$ is $\gamma$-admissible with $e(I) \leq r$.

(2) The homotopy group of $\text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, \Omega^{\infty}_{\text{Free}} \text{Mod}_{\Sigma^{k} \mathbb{F}_2}^{\mathcal{R}_{>0}}(\mathbb{F}_2)$) is the exterior algebra on generators $\gamma \mathcal{Q}_J(x_i)$, where $J = (j_1, \ldots, j_r)$ satisfies $1 \leq j_i < n$ and $1 \leq j_i \leq j_i + 1$ for $l < r$, while $I$ is $\gamma$-admissible with $e(I) \leq r$.

In order to identify the image of the comparison map
\[
\varphi_{\ast, s} : \pi_{\ast, s} \text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, \mathbb{g}) \to \pi_{\ast, s} \text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, \mathbb{g})
\]
in Lemma 3.16 when $L = \Sigma^{k} \mathbb{F}_2$ or $\Omega^{\infty}_{\text{Free}} \text{Mod}_{\Sigma^{k} \mathbb{F}_2}^{\mathcal{R}_{>0}}(\mathbb{F}_2)$ with trivial Lie$^{\infty}$ brackets, which provides a lower bound for $\pi_{\ast, s} \text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, \mathbb{g})$, we want to lift the Bousfield-Cartan-Dwyer operations $\gamma$ on the target to $\pi_{\ast, s} \text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, \mathbb{g})$ for $i \geq 2$. We recall explicit combinatorial formulae of $\gamma$ by Bökstedt and Ottosen. The grading conventions are modified to suit our context.

For $r, i \in \mathbb{N}$ with $1 \leq i \leq r$, let $U(r, i)$ be the set of pairs $(A, B)$ of ordered sequences $a_1 < \cdots < a_i, b_1 < \cdots < b_i$ such that the disjoint union $\{a_1, \ldots, a_i\} \cup \{b_1, \ldots, b_i\}$ is $\{r - i, r - i + 1, \ldots, r + i - 1\}$. Let $V_r(i) \subset U(r, i)$ be the subset with $a_1 = r - i$.

Proposition 3.19. [BO06, Theorem 1.3, Lemma 3.1] Suppose that $V_\ast$ is a simplicial shifted graded commutative algebra with face maps $d_j$. Then $z$ be a representative of a class $[z] \in \pi_{\ast, s}(V_\ast)$ in the normalized complex $N(V_\ast)$. For $2 \leq i \leq s$, define
\[
\gamma(z) = \sum_{(A, B) \in V(s, i)} s_{a_1} \cdots s_{a_i} s_{a_1}(z) \otimes s_{b_1} \cdots s_{b_i} s_{b_1}(z) \in \Lambda^s(z)
\]

Then $d_j(\gamma(z)) = 0$ for $0 \leq j \leq i + 1$, and the induced operation $\gamma : \pi_{\ast, s}(V_\ast) \to \pi_{\ast + i + 2, s - 1}(\Lambda^s(z))$ are exactly the Dwyer-Bousfield operations in Theorem 3.12.

Remark 3.20. If in addition $V_\ast$ is exterior, then the formula above for $i = 1$ induces the operation $\gamma$ on $\pi_{\ast, s}(V_\ast)$. The operation $\gamma$ is not well-defined when there is some element $a$ in the simplicial commutative Lie algebra $V_\ast$ such that $a \otimes a \neq 0$. This is because in $N(V_\ast)$ the differential sends $\gamma_1(a)$ to $a \otimes a$, cf. [Dwy80a, Remark 4.3, 4.4][BO06, Remark 3.2].

Hence we obtain natural operations $\gamma$ for $1 \leq i \leq s$ on
\[
\pi_{\ast, s}(\text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, AR_{\bullet}^{>0}(\Sigma^{k} \mathbb{F}_2))) \cong \pi_{\ast, s}(\Lambda^s(\text{Bar}_{\bullet}(\text{id}, \mathcal{A}_{\mathcal{R}_{>0}}, \Sigma^{k} \mathbb{F}_2))),
\]

and similarly
\[
\pi_{\ast, s}(\text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, AR_{\bullet}(\Sigma^{k} \mathbb{F}_2))) \cong \pi_{\ast, s}(\Lambda^s(\text{Bar}_{\bullet}(\text{id}, \mathcal{A}_{\mathcal{R}_{>0}}, \Sigma^{k} \mathbb{F}_2))).
\]

Suppose that $\xi$ is a cycle in $AR_{\bullet}^{>0}(\Sigma^{k} \mathbb{F}_2)$. In the total complex of $\text{Bar}_{\bullet}(\text{id}, \text{Lie}^{\infty}_{\mathcal{R}_{>0}}, AR_{\bullet}^{>0}(\Sigma^{k} \mathbb{F}_2))$, a representative for the homotopy class $\gamma([\xi])$ is
\[
\gamma(\xi) = \sum_{(A, B) \in V(s, i)} \langle s_{a_1} \cdots s_{a_i} s_{a_1}(\xi), s_{b_1} s_{b_2} \cdots s_{b_i}(\xi) \rangle \in \text{Lie}^{\infty}_{\mathcal{R}_{>0}}(\mathcal{A}_{\mathcal{R}_{>0}})^{\otimes (s + i)}(\Sigma^{k} \mathbb{F}_2).
\]

When we iterate the $\gamma$ operations, the formula is harder to write down explicitly.
Notation 3.21. Suppose that $V_\bullet$ is a simplicial $F_2$-module as a trivial simplicial Lie$^{\ast,\mathbb{n}}$-algebra. For distinct classes $[\xi_1], \ldots, [\xi_n] \in \pi_*(V_\bullet)$, denote by $B(\xi_1, \ldots, \xi_n)$ the cycle in the normalized complex of $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_\bullet, V_\bullet)$ that represents the class $[\xi_1] \otimes \cdots \otimes [\xi_n] \in \pi_*(\text{CE}(V_\bullet)) \cong \pi_*, \text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_\bullet, V_\bullet)$, which is obtained by cycle completion via the Jacobi identity.

Therefore a homotopy class $[\xi_1] \otimes \cdots \otimes [\xi_l]$ with $l > 1$ in $\pi_*(\Lambda^*(\text{Bar}_\bullet(\text{id}, \Lambda_{R>0}^s, \Sigma^k F_2)))$ is represented by an element $B(\xi_1, \ldots, \xi_l)$ in the summand $(\text{Lie}^{\ast,\mathbb{n}}_\bullet)^{l-1}(\Lambda_{R>0}^s)^{l-1}(\Sigma^k F_2)$ of the total complex of $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_\bullet, \Lambda_{R>0}^s(\Sigma^k F_2))$. Since a representative for the homotopy class $\gamma^1_{i}(\xi)$ in the total complex of $\Lambda^*(\text{Bar}_\bullet(\text{id}, \text{Free}_{\Sigma^k F_2}^{\ast,\mathbb{n}}, \Sigma^k F_2)))$ is given by

$$
\gamma^1_{i}(\xi) = \sum_{(C,D) \in V(s+i+j, (A,B) \in V(s+j+1))} s_C(\sigma_A(\xi) \otimes s_B(\xi)) \otimes s_D(\sigma_A(\xi) \otimes s_B(\xi)),
$$

a representative for $\gamma^1_{i}(\xi)$ in the total complex of $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_\bullet, \Lambda_{R>0}^s(\Sigma^k F_2))$ is given by the sum of over all $(A,B) \in V(s,j), (C,D) \in V(s+i+1, j)$ of $B(s_C(\sigma_A(\xi)), s_C(\sigma_B(\xi)), s_D(\sigma_A(\xi)), s_D(\sigma_B(\xi)))$, with the three brackets coming from distinct simplicial filtrations.

Lemma 3.22. Suppose that $L$ is the Lie$^{\ast,\mathbb{n}}_\bullet$ algebra $\Omega^\ast \text{Free}_{\Sigma^k F_2}^{\ast,\mathbb{n}}$, with $\Sigma^k F_2$ the limiting case $n = \infty$. Then the cokernel of $\phi_* : \pi_*, \text{Bar}_\bullet(\text{id}, \text{Lie}^s_\bullet, L) \to \pi_*, \text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_\bullet, L)$ is the ideal generated by all $\gamma^1_{i}(\xi_i)$, where $J = (j_1, \ldots, j_r)$ satisfies $1 \leq j_i \leq j_{i+1} + 1$ for $r < l$ and $1 \leq j_r \leq n$, while $\gamma^1_{i}$ is $\gamma$-admissible with $e(1,1) \leq r$.

Proof. We focus on the case $L = \Sigma^k F_2$, since in the cases $n < \infty$ the only difference is an extra condition on the rightmost operation in basis elements, so the same argument applies with no change. We want to identify all cycles in $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)_{\Sigma^k F_2}$ whose preimage is the source of a differential to an element that is in the kernel of $\phi$. Since $\phi$ is surjective, this is equivalent to finding all $\alpha$ that are cycles in $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$ precisely because the differential $\partial'$ in the normalized complex of $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$ sends $\alpha$ to a linear combination of elements that contain self-brackets or $0$. We start with the generators of the exterior algebra, cf. Lemma 3.14. Let $\alpha = \tilde{Q}_{j_1} \tilde{Q}_{j_2} \cdots \tilde{Q}_{j_k}(x_k)$ be a basis element of $\pi_\Sigma \text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$, represented by a cycle $\alpha = \tilde{Q}_{j_1} \cdots \tilde{Q}_{j_k} | x_k + \tilde{Q}_{j_1} \cdots \tilde{Q}_{j_k} | x_k$ in $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$. The terms in the summand come from standard cycle completion via Behrens’ relations, with the condition that any term containing $\tilde{Q}_0$ is 0. It has preimage $\tilde{\alpha}$ the cycle completion of $\tilde{Q}_{j_1} \cdots \tilde{Q}_{j_k} | x_k$ in $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$ via Behrens’ relations, which is the sum of $\alpha$ and terms $\tilde{Q}_{j_1} \cdots \tilde{Q}_{j_k} | x_k$ such that at least one of the $\tilde{Q}_{j_i}, i > 1$ is equal to $\tilde{Q}_0$. By [BO06, Lemma 3.1], the differential $\partial'$ in the normalized complex of $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$ sends $\gamma^1_{i} \alpha, i \geq 2$ to zero because the terms are either zero or cancel out in pairs due to simplicial identities of face and degeneracy maps. Hence its preimage $\gamma^1_{i} \alpha$ is also a cycle in the normalized complex of $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$. Similarly, for any $\gamma$-admissible sequence $I = (i_1, \ldots, i_m)$ with $i_m \geq 2$, $\gamma^1_{i} \alpha$ lifts to a cycle $\gamma^1_{i} \tilde{\alpha}$ in $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$.

On the other hand, the differential $\partial'$ sends $\gamma^1_{i} \alpha$ to $\langle \alpha, \alpha \rangle = 0$ in $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$, whereas its preimage $\gamma^1_{i} \tilde{\alpha}$ maps to $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = \tilde{Q}_0 \tilde{\alpha}$ under the differential in $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$, which is in the kernel of $\phi$. Similarly, for any $\gamma$-admissible sequence $I = (i_1, \ldots, i_m)$ with $i_m \geq 2$, $\gamma^1_{i} \alpha$ is a cycle in $\text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$ because $\partial' \gamma^1_{i} \alpha = \gamma^1_{i}(\gamma^1_{i} \alpha)$ if the simplicial degree of $\alpha$ is $r > 1$ and $\partial' \gamma^1_{i} \alpha = \partial^1(\gamma^1_{i} \alpha) \otimes x_{i_1} \cdots \otimes x_{i_m-1} \gamma^1_{i}(\alpha)$ if $r = 1$. If $r = 1$, cf. [HM16, 3.9.1]. On the other hand, its preimage $\gamma^1_{i} \tilde{\alpha}$ is mapped by the differential to $\gamma^1(\tilde{\alpha}, \tilde{\alpha}) = \gamma^1(\tilde{Q}_0 \tilde{\alpha})$ with $I' = (i_1 + 1, \ldots, i_m + 1)$ when $r > 1$ and if $r = 1$ to the cycle completion $B(\tilde{Q}_0 \tilde{\alpha}, \gamma^1(\tilde{\alpha}), \gamma^1(\tilde{\alpha}), \gamma^1(\tilde{\alpha}))$ (cf. Notation 3.21). There is a shift by one in the indexing of the $\gamma$ operations because by construction the self-brackets appearing in the same bracket term live in distinct filtrations when more $\gamma$s are applied. Hence all the generators $\gamma^1_{i} \alpha$ of the exterior algebra $\pi_\Sigma \text{Bar}_\bullet(\text{id}, \text{Lie}^{\ast,\mathbb{n}}_{R>0} \Sigma^k F_2)$ are in the cokernel of $\phi$. 


In general, suppose $[\alpha]$ is a basis element of $\pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \text{AR}_\ast^{\geq 0}(\Sigma^k F_2))$ that is the exterior product of generators $\gamma_1(\langle \alpha_1 \rangle), \ldots, \gamma_n(\langle \alpha_n \rangle)$. It is represented by a cycle $\alpha = B(\gamma_1(\langle \alpha_1 \rangle), \ldots, \gamma_n(\langle \alpha_n \rangle))$ in the total complex of $\text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \text{AR}_\ast^{\geq 0}(\Sigma^k F_2))$, cf. Notation 3.21, since $d_j(\gamma_j(\langle \alpha \rangle)) = 0$ for all $j, l$ by Proposition 3.19. The preimage of $\alpha$ is again $B(\gamma_1(\langle \alpha_1 \rangle), \ldots, \gamma_n(\langle \alpha_n \rangle))$ in $\text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$. Hence $[\alpha]$ is in the cokernel of $\phi_\ast$ if and only if at least one of the $\gamma$-admissible sequences $I$ is of the form $I_l = (i_1, \ldots, i_m, 1)$.

**Theorem 3.23.** The Quillen homology

$$H^{\text{Lie}_k}_\ast(\Omega^e_{\text{Free}_{\text{Mod}_{2}}}(\Sigma^{n+k} F_2)) \cong \pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Omega^e_{\text{Free}_{\text{Mod}_{2}}}(\Sigma^{n+k} F_2))$$

of the Lie$^e_k$-algebra $\Omega^e_{\text{Free}_{\text{Mod}_{2}}}(\Sigma^{n+k} F_2), 1 \leq n \leq \infty$ is the (shifted graded) exterior algebra on generators $\gamma_\ast \mathcal{O}_j(x_k)$, where $I = (i_1, \ldots, i_m)$ is $\gamma$-admissible with $e(I) \leq r$ and $i_m \geq 2$, whereas $J = (j_1, \ldots, j_l)$ satisfies $0 \leq j_l \leq j_{l+1} - 1$ for $l < r$, $0 \leq j_r < n$ and if $j_1 = 0$ then either $r = 1$ or $i_m = 2$.

**Proof.** Again we focus on the limiting case $L = \lim_{n \to \infty} \Omega^e_{\text{Free}_{\text{Mod}_{2}}}(\Sigma^{n+k} F_2) \cong \Sigma^k F_2$, since in the cases $n < \infty$ the only difference is an extra condition on the rightmost operation and thus follow from the same considerations.

Consider the subcomplex $\text{Bar}_{\ast}(\text{id}, \text{A}_{\ast}, \Sigma^k F_2)$ of $\text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$. We showed in Lemma 3.14 that a basis for

$$\pi_\ast \text{Bar}_{\ast}(\text{id}, \text{A}_{\ast}, \Sigma^k F_2) \cong \text{UnTor}_{r_\ast}(F_2, F_2)$$

is given by $\mathcal{O}_j(x_k)$ such that $0 \leq j_l \leq j_{l+1} - 1$ for all $l$. Let $\alpha$ be the cycle representing $\mathcal{O}_j(x_k)$ obtained by cycle completion with $r \geq 1$. Since the construction of $\gamma$ is functorial, by Lemma 3.22 the differential in the normalized complex of $\text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$ sends $\gamma_\ast(\alpha)$ to $\mathcal{O}_0(\alpha)$ and $\gamma_\ast \gamma_\ast(\alpha)$ to $\gamma_{j_1+1} \ldots \gamma_n(\mathcal{O}_0(\alpha))$ for $(i_1, \ldots, i_m)$ a $\gamma$-admissible sequence. In comparison, no element in the complement of the subcomplex $\text{Bar}_{\ast}(\text{id}, \text{A}_{\ast}, \Sigma^k F_2)$ hits any of the terms in the cycle representing $\mathcal{O}_j(x_k)$ when $j_0 > 0$, since cycle completion via Behrens’ relations cannot decrease the indexing of the leading operation. Hence $\gamma_{j_1} \ldots \gamma_{j_m}(\mathcal{O}_j(x_k))$ is a cycle for $i_m \geq 2$ and $j_0 > 0$, or $j_0 = 0$ and either $r = 1$ or $i_m = 2$.

In general, suppose we have distinct cycles $\gamma_i(\langle \alpha_1 \rangle), \ldots, \gamma_n(\langle \alpha_n \rangle)$ in $\text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$ that either is of the form above and in the kernel of $\phi$, or maps to exterior generators of $\pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \text{AR}_\ast^{\geq 0}(\Sigma^k F_2))$. The element $B(\gamma_i(\langle \alpha_i \rangle), \ldots, \gamma_n(\langle \alpha_n \rangle))$ gives rise to a nonzero cycle in $\text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \text{Bar}_{\ast}(\text{id}, \text{A}_{\ast}, \Sigma^k F_2))$, since any face map sends $\gamma_i(\langle \alpha_i \rangle)$ to either zero or elements that pair up by Proposition 3.19. Hence $B(\gamma_i(\langle \alpha_i \rangle), \ldots, \gamma_n(\langle \alpha_n \rangle))$ is also a cycle in $\text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$.

As a result, the kernel of the comparison map

$$\phi_\ast : \pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2) \to \pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$$

contains the product of the image of $\phi_\ast$ with the (nonunital) exterior algebra on generators $\gamma(\mathcal{O}_j)$, where $J = (j_1, \ldots, j_r)$ satisfies $0 \leq j_l \leq j_{l+1} - 1$ for all $l$ and $I = (i_1, \ldots, i_m)$ is $\gamma$-admissible with $e(I) \leq r$ and $i_m \geq 2$, with the condition that if $j_1 = 0$ then either $i_m = 2$ or $r = 1$. Combining with Lemma 3.22, we see that $\pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$ contains the exterior algebra on generators $\gamma(\mathcal{O}_j(x_k))$, where $I = (i_1, \ldots, i_m)$ is $\gamma$-admissible with $e(I) \leq r$ and $i_m \geq 2$, whereas $J = (j_1, \ldots, j_r)$ satisfies $0 \leq j_l \leq j_{l+1} - 1$ for all $l$ and if $j_1 = 0$ then either $r = 1$ or $i_m = 2$. By Proposition 3.5, the upper bound on the dimensions of $\pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$ is given by those of $\pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$. Comparing with the basis given in Lemma 3.18, we conclude that we have identified all the homotopy classes of $\text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$.

**Remark 3.24.** Note that $\pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, \Sigma^k F_2)$ is the cofree coalgebra on $\Sigma^k F_2$ over the comonad $\text{Bar}_{\ast}(\text{id}, \text{Lie}^e_\ast, -) = \pi_\ast \text{Bar}_{\ast}(\text{id}, \text{Free}_{\text{Lie}^e_\ast}, -)$.
on \( \text{Mod}_{\mathbb{F}_2} \). The coalgebra structure map is given by

\[
|\text{Bar}_\bullet(\text{id}, \text{Lie}_R^s, \Sigma^k \mathbb{F}_2)| \xleftarrow{\sim} |\text{Bar}_\bullet(\text{id}, \text{Lie}_R^s, |\text{Bar}_\bullet(\text{Free}_{\text{Mod}_{\mathbb{F}_2}}^\text{Lie}_R^s, \Sigma^k \mathbb{F}_2)|)| \\
\rightarrow |\text{Bar}_\bullet(\text{id}, \text{Lie}_R^s, |\text{Bar}_\bullet(\text{id}, \text{Lie}_R^s, \Sigma^k \mathbb{F}_2)|)|,
\]

where the last map makes use of the augmentation \( \text{Free}_{\text{Mod}_{\mathbb{F}_2}}^\text{Lie}_R^s \rightarrow \text{id} \), cf. \([\text{Bra17}, \text{Appendix D}]\). In particular, \( \pi_\ast, \text{Bar}_\bullet(\text{id}, \text{Lie}_R^s, \Sigma^k \mathbb{F}_2) \) records all natural unary operations on a degree \( k \) class in the mod 2 Quillen homology of spectral Lie algebras.

Since differentials preserve weights and the \( g_1 \) operation on \( \text{Bar}_\bullet(\text{id}, \text{Lie}_R^s, L) \) appears in weight at least four, we immediately deduce the following from Proposition 3.5.

**Corollary 3.25.** For any \( \text{Lie}_R^s \)-algebra \( \mathfrak{g} \), the homotopy groups of \( \text{Bar}_\bullet(\text{id}, \text{Lie}_R^s, \mathfrak{g}) \) and \( \text{Bar}_\bullet(\text{id}, \text{Lie}_R^s, \mathfrak{g}) \) are isomorphic in weight less than four.

Furthermore, Theorem 3.23 allows us to deduce a non-formality result for the \( \mathbb{F}_2 \)-linear spectral Lie operad.

**Corollary 3.26.** The spectral Lie operad \( s\mathcal{L} = \partial_\ast(\text{Id}) \) in the category of \( \mathbb{F}_2 \)-module spectra is not formal, i.e. \( C_\ast(\partial_\ast(\text{Id}); \mathbb{F}_2) \) is not quasi-isomorphic to \( H_\ast(\partial_\ast(\text{Id}); \mathbb{F}_2) \) as \( \mathbb{F}_2 \)-linear symmetric operads.

**Proof.** The \( E^\infty \)-page of the spectral sequence (8) is the homology of

\[
C_\ast(|\text{Bar}_\bullet(\text{Id}, \partial_\ast(\text{Id}); \Sigma^k \mathbb{F}_2)|; \mathbb{F}_2) \simeq |\text{Bar}_\bullet(\text{Id}, C_\ast(\partial_\ast(\text{Id}); \mathbb{F}_2), \Sigma^k \mathbb{F}_2)|.
\]

On the right hand side, the bar construction is the operadic bar construction in the category of symmetric sequences in chain complexes over \( \mathbb{F}_2 \), with product given by the circ product \( \circ \) and \( \text{Id} \) the unit. Note that

\[
|\text{Bar}_\bullet(\text{Id}, C_\ast(\partial_\ast(\text{Id}); \mathbb{F}_2), \Sigma^k \mathbb{F}_2)| \simeq C_\ast(\partial_\ast(\text{Id}); \mathbb{F}_2) \circ \Sigma^k \mathbb{F}_2.
\]

Suppose that \( C_\ast(\partial_\ast(\text{Id}); \mathbb{F}_2) \) is quasi-isomorphic to \( H_\ast(\partial_\ast(\text{Id}); \mathbb{F}_2) \) as symmetric sequences. Then the spectral sequence (8) is isomorphic to the spectral sequence obtained by the skeletal filtration of the geometric realization \( |\text{Bar}_\bullet(\text{Id}, H_\ast(\partial_\ast(\text{Id}); \mathbb{F}_2), \Sigma^k \mathbb{F}_2)| \). But \( H_\ast(\partial_\ast(\text{Id}); \mathbb{F}_2) \) is the ordinary shifted Lie operad over \( \mathbb{F}_2 \), cf. \([\text{Fre04}, \text{Section 6}][\text{Chi05}, \text{Example 9.50}]\). The weight two part of the \( E^2 \)-page has only one generator \( [x_k, x_k] \in E^2_{1,2k-1} \), which is much smaller than the weight two part of \( HQ_\ast(\text{Lie}_R^s, \Sigma^k \mathbb{F}_2) \), cf. Theorem 3.23. \( \square \)

In comparison, the spectral Lie operad is formal over \( \mathbb{Q} \) since the bar spectral sequence with rational coefficients collapses on the \( E^2 \)-page, cf. \([\text{Knu17}]\).

### 4. Application to Knudsen’s Spectral Sequence

The rest of the paper is devoted to studying the mod \( p \) homology of labeled configuration spaces using the computation of Quillen homology of spectral Lie algebras. The coefficients for homology is \( \mathbb{F}_2 \) unless otherwise specified.

Let \( M \) be a manifold of dimension \( n \) and \( X \) a spectrum. The configuration space of \( k \) points in \( M \) labeled by \( X \) is the spectrum

\[
B_k(M; X) = \text{Conf}_k(M) \otimes \Sigma^k X^\otimes k,
\]

considered as a weighted spectra of weight \( k \). Denote by \( s\mathcal{L} \) the monad associated to the free spectral Lie algebra functor \( \text{Free}^\mathcal{L} \). The \( \infty \)-category of spectral Lie algebras is cotensored in Spaces, and we write \( (-)^M \) for the cotensor with the one-point compactification of \( M \) in this category. In \([\text{Knu18}]\), Knudsen established the following equivalence using factorization homology.
Theorem 4.1. [Knu18, Section 3.4] [cf. [BHK19, Theorem 5.1]] Let $M$ be a parallelizable $n$-manifold and $X$ a spectrum. Consider $X$ as a weighted spectrum of weight one. Then there is an equivalence of weighted spectra

$$
\bigoplus_{k \geq 1} B_k(M; X) \simeq | \text{Bar}_\ast (\text{id}, s, \mathcal{L}_\ast, \text{Free}^{\mathcal{L}_\ast} (\Sigma^n X)^{M^+}) | .
$$

The left hand side is weighted by the index $k$ and right hand side induced by the weight on $X$.

Applying the bar spectral sequence (Proposition 2.9) to the bar construction on the right hand side, we obtain the following:

**Proposition 4.2.** There is a weighted spectral sequence

$$(7)
E^2_{s,t} = H^{\text{Lie}_{\mathcal{R}_*}}_{s,t} (H_\ast (\text{Free}^{\mathcal{L}_*} (\Sigma^n X)^{M^+})) \Rightarrow \bigoplus_{k \geq 1} H_{s+t}(B_k(M; X)).
$$

The Lie$_{\mathcal{R}_*}$-algebra structure on the cotensor

$$H_\ast (\text{Free}^{\mathcal{L}_*} (\Sigma^n X)^{M^+}) \cong \bar{H}_\ast (M^+) \otimes H_\ast (\text{Free}^{\mathcal{L}_*} (\Sigma^n X)) \cong \bar{H}_\ast (M^+) \otimes \text{Free}_{\text{Mod}_{\mathcal{R}}^*} (H_\ast (\Sigma^n X))
$$

has an explicit description.

**Proposition 4.3.** [BHK19, Proposition 5.9] Let $\mathfrak{g}$ be a spectral Lie algebra. Then there is a spectral Lie algebra structure on the cotensor $\mathfrak{g}^{M^+}$ in the category of spectra. The weight two structural map factors as

$$
\partial_2(1\text{d}) \otimes \mathbb{D}(M^+) \otimes \mathfrak{g}^{\otimes 2}_{\mathcal{R}_*} \rightarrow \mathbb{D}(M^+) \otimes ^2 \mathfrak{g}^{\otimes 2}_{\mathcal{R}_*} \otimes (\partial_2(1\text{d}) \otimes \mathfrak{g}^{\otimes 2}_{\mathcal{R}_*}) \xrightarrow{\text{D}(\delta^s) \otimes \xi^s} \mathbb{D}(M^+) \otimes \mathfrak{g},
$$

where $\mathbb{D}$ is the Spanier-Whitehead dual and $\delta$ the diagonal embedding.

As a result, the shifted Lie bracket on $\bar{H}_\ast (M^+) \otimes H_\ast (\mathfrak{g})$ is given by

$$[y \otimes x] \in \mathfrak{g} \otimes \mathfrak{g} \mapsto [y \cup x] \in \mathfrak{g} \otimes \mathfrak{g}.
$$

Applying the Cartan formula $Q^j(y \otimes x) = \sum \sigma^{j-i} Q^j(y) \otimes Q^i(x)$ for the extended Dyer-Lashof operations $Q^j : x \mapsto e_j \cdot x \otimes x$ and the identification $Q^{-i} = Sq^i$ [May70] to the definition of the $\tilde{Q}^j$ operations, we deduce that the Steenrod operations on $H_\ast (M^+)$ induces a twisted $\mathcal{R}_*$ module structure in the cotensor.

**Proposition 4.4.** The operations $\tilde{Q}^j$ act on $\bar{H}_\ast (M^+) \otimes H_\ast (\mathfrak{g})$ by

$$\tilde{Q}^j(y \otimes x) = \sum_i Sq^{-i} \tilde{Q}^j(y) \otimes \tilde{Q}^i(x).
$$

4.1. The universal case. Now we apply Theorem 3.23 to the case where $M$ is the Euclidean space. While the homology for $B_k(\mathbb{R}^n; X)$ is well-understood [BMMS88][CLM76][May72], we observe interesting patterns of higher differentials in the associated Knudsen’s spectral sequence. Furthermore, the computation of the $E^2$-page in these cases will be useful in deducing the $E^2$-page for a general $M$.

Since $\bar{H}_\ast (\mathbb{R}^n)$ is concentrated in one dimension, the only nonzero Steenrod operation is $Sq^0 = \text{id}$, so the $\mathcal{R}_*$-module structure on $\bar{H}_\ast (\mathbb{R}^n) \otimes H_\ast (\mathfrak{g})$ is given by

$$\tilde{Q}^j(t \otimes x) = \sigma^{-n} \tilde{Q}^j(x) = \tilde{Q}^j(\sigma^{-n}x), x \in \mathfrak{g}.
$$

In the limiting case $M = \mathbb{R}^\infty = \lim_{n \rightarrow \infty} \mathbb{R}^n$, we have the stabilization

$$\lim_{n \rightarrow \infty} \Omega^n \text{Free}^{\mathcal{L}_*} (\Sigma^n X) \simeq X,$

and the spectral sequence (7) becomes

$$(8)
E^2_{s,t} = H^{\text{Lie}_{\mathcal{R}_*}}_{s,t} (\Sigma^k \mathbb{R}^2) \Rightarrow H_{s+t}(\text{Free}^{\mathcal{L}_*} (\mathbb{R}^k)).
$$

The $E^2$-page is computed in Theorem 3.23. Namely, it is the exterior algebra generated by one class $x_k$ and two types of operations on coalgebras over the comonad $\pi_{s,t}, \text{Bar}_\ast (\text{id, Lie}_{\mathcal{R}_*}^\ast, -)$

$$\tilde{Q}^j : E^2_{s,t} \rightarrow E^2_{s+t+j-1, t}, \ j \geq t.$$
under a further splitting of the filtration degree into a sum of homological degree \( h \) counting the number of brackets and simplicial degree \( s \) counting the number of \( \bar{Q}^i \)’s.

Comparing with the computation of \( H_\ast(\text{Free}^Z_\langle \mathbb{Z}^2 \rangle) \) [May72][BMMS88], which is the \( E^\infty \)-page, we can immediately deduce that the \( E^2 \)-page is much larger. Using sparsity arguments, we can identify higher differentials in low degrees, which allows us to make the following conjecture.

**Conjecture 4.5.** Each page of the spectral sequence

\[
E^2_{s,t} = \text{Hom}_{\mathfrak{L}^s_k}(\mathbb{Z}^s,\mathbb{Z}^t) \Rightarrow \pi_{s+t} \text{Bar}_\ast(\text{id},s\mathfrak{L},\Sigma^k\mathbb{F}_2) \cong H_{s+t}(\text{Free}^Z_\langle \mathbb{Z}^2 \rangle)
\]

is an exterior algebra. The higher differentials on the exterior generators of the \( E^2 \)-page are given as follows:

1. For an exterior generator \( \alpha = \bar{Q}_{j_1} \cdots \bar{Q}_{j_m}(x_k) \) on the \( E^2 \)-page, we have
   \[
   d^{r+1} \gamma_{r+1}(\alpha) = \bar{Q}_r(\alpha)
   \]
   for \( r < m \) and \( r \leq j_1 + 1 \).

2. For an exterior generator \( \beta = \gamma_{n+1} \bar{Q}_{j_1} \cdots \bar{Q}_{j_m}(x_k) \) on the \( E^2 \)-page, we have
   a) \( d^{n+1}(\beta) = \bar{Q}_{n+1} \bar{Q}_{j_1} \cdots \bar{Q}_{j_m}(x_k) \),
   b) \( d^{n+1} \mu_{m+1}(\beta) = d^{n+1}(\beta) \otimes \beta \),
   c) \( \gamma_{n+2} \mu_{m+1}(\beta) = d^{n+1}(\mu_{m+1}(\beta)) \) for \( n+1 < l < m \).

These generate all higher differentials under further applications of \( \gamma \) operations in accordance with (2),(b) and (2),(c), as well as the exterior product.

The figure below is an illustration of the higher differentials in homological Adams grading \((s+t,s)\) for \( \beta = \gamma_{n+1} \bar{Q}_{j_1} \cdots \bar{Q}_{j_m}(x_k) \) and \( \alpha = \bar{Q}_{n} \bar{Q}_{j_1} \cdots \bar{Q}_{j_m}(x_k) \) with internal degree \( b \). Set \( a = 2b + m + 1 \). Along the horizontal line \( s = m+1 \) we have generators \( \bar{Q}_1(\alpha), \ldots, \bar{Q}_{n+1}(\alpha) \), each receiving a blue differential via Conjecture 4.5.(1). Along the top slope we have, for each \( i \) with \( n+1 < l < m \), a cyan arrow \( d^{n+1}(\gamma_{n+1}(\beta)) = \gamma_{n+1}(\alpha) \), which correspond to the differentials in Conjecture 4.5.(2),(c). Finally we have a gray arrow \( d^{n+1}(\gamma_{n+1}(\beta)) = \beta \otimes \alpha \), corresponding to Conjecture 4.5.(2),(b).

**Remark 4.6.** The pattern in the universal case is similar to the pattern of universal higher differentials in [Dwy80b, Proposition 2.6] and [Tur98], where divided squares kills off Steenrod operations that are not admissible. Here, the Dyer-Lashof operations \( \bar{Q}^i \) on the \( E^\infty \)-page should be represented by the surviving \( \bar{Q}^i \) operations. On the \( E^2 \)-page, the admissibility condition for \( \bar{Q}^i \) allows for more admissible sequences than
the Dyer-Lashof algebra. The \( \gamma \) operations eliminate the \( \tilde{Q}^j \) operations that do not satisfy the admissibility condition for Dyer-Lashof operations via higher differentials.

One major difference is that while Steenrod operations can be defined on the spectral sequence filtration-wise in [Dwy80b] and [Tur98], the operations \( \tilde{Q}^j \) increase filtration by one and hence the classical methods of constructing operations on spectral sequences no longer apply.

In joint work in progress with Andrew Senger, we use a suitable deformation of the comonad associated to the bar construction \(|\text{Bar}_*(\text{id, } s.\mathcal{L}^r, -)|\) to the \( \infty \)-category of Beilinson-connective filtered \( F_2 \)-modules, which allows us to detect the higher differentials in Conjecture 4.5.

**Remark 4.7.** The spectral sequence we study here is analogous to the bar spectral sequence

\[
E_2^{ij} = \pi_i \pi_j \text{Bar}_*(\text{id, } E^n \otimes F_p, \pi_j(A)) \Rightarrow \pi_{i+j} \text{Bar}_*(\text{id, } E^n \otimes F_p, A)
\]

and its dual. The later was used to identify operations on homotopy groups of spectral partition Lie algebras and mod \( p \) TAQ cohomology operations of nonunital \( E_\infty - F_p \)-algebras in [Zha22], which subsumes unpublished work of Kriz, Basterra and Mandell. The \( E^2 \)-page of this spectral sequence is the André-Quillen homology of \( \text{Poly}_{\mathbb{A}} \)-algebras, i.e., graded polynomial algebras equipped with Dyer-Lashof operations satisfying the Cartan formula. In contrast to Conjecture 4.5, this spectral sequence collapses on the \( E^2 \)-page.

Heuristically, the phenomenon here arises from the nonadditivity of the free \( \text{Lie}^r \)-algebra functor and the order of the factorization \( \text{Lie}^r_{\text{Mod}_{F_2}} = Q_{\text{Mod}_{F_2}} \circ Q_{\text{Lie}^r_{\text{Mod}_{F_2}}} \), which results in simplicial homotopy operations. Whereas the Dyer-Lashof operations are additive away from the bottom operations on even degree classes, so the factorization \( Q_{\text{Mod}_{F_2}} = Q_{\text{Mod}_{F_2}} \circ Q_{\text{Poly}_{\mathbb{A}}} \) does not introduce simplicial homotopy operations.

**4.2. With coefficients.** Next, we take up a slightly more complicated case, where \( M = \mathbb{R}^n \) with labels in an arbitrary spectrum \( X \). Then \( H_*(\text{Free}^{\mathcal{L}^r}(\Sigma^n X)^{F_0^+}) \cong \Omega^n \text{Free}^{\text{Lie}^r_{\text{Mod}_{F_2}}}(\Sigma^n H_*(X)) \) and the spectral sequence (7) becomes

\[
E_2^{ij} = H_{Q_{\text{Lie}_{\text{Mod}_{F_2}}}}^j(\Omega^n \text{Free}^{\text{Lie}_{\text{Mod}_{F_2}}}(\Sigma^n H_*(X))) \Rightarrow H_{n+i}(\text{Free}^{\mathcal{L}^r}(X)).
\]

When \( X = S^1 \), the \( E^2 \)-page \( H_{Q_{\text{Lie}_{\text{Mod}_{F_2}}}}^j(\Omega^n \text{Free}^{\text{Lie}_{\text{Mod}_{F_2}}}(\Sigma^n + \mathcal{L}^r)) \) is computed in Theorem 3.23.

Write \( H_*(X) \cong \bigoplus_{k,l} \mathbb{F}_2 \{ x_{k,l} \} \), where \( \{ x_{k,l} \} \) is an \( \mathbb{F}_2 \)-basis of \( H_k(X) \) for each \( k \). Then

\[
g = H_*(\text{Free}^{\mathcal{L}^r}(\Sigma^n V)) \cong \mathbb{F}_2 \{ t_n \} \otimes H_*(\text{Free}^{\mathcal{L}^r}(\Sigma^n V))
\]

\[
\cong \mathbb{F}_2 \{ t_n \} \otimes \left( \bigoplus_{w \in W} \mathbb{F}_2 \{ \tilde{Q}^j w, J \in \mathcal{R}(d(w)) \} \right)
\]

by [AC20, Proposition 7.3]. Here \( \mathcal{R}(n) \) is the quotient of \( \mathcal{R} \) by the relations \( \tilde{Q}^j \cdots \tilde{Q}^j = 0 \) if \( j_1 < j_2 + \cdots + j_k + n \), and \( W \) is the set of Lyndon words on the set of letters \( \{ \sigma^\alpha x_{k,l} \}_{k,l} \), which is a basis for the free \( \text{Lie}^{\mathcal{L}^r} \)-algebra on generators \( \{ \sigma^\alpha x_{k,l} \}_{k,l} \).

We define the degree of a word \( w \in W \) to be \( d(w) = 1 + \sum_{k,l} m_{k,l}(w)(n + k - 1) \), where \( m_{k,l}(w) \) counts the number of times the letter \( \sigma^\alpha x_{k,l} \) appears in \( w \). Set

\[
g_w = \mathbb{F}_2 \{ t_n \} \otimes \mathbb{F}_2 \{ \tilde{Q}^j w, J \in \mathcal{R}(n + |w|) \}.
\]

Then \( g \cong \bigoplus_{w \in W} g_w \) with trivial brackets. Note that this splitting is induced by an equivalence of \( s.\mathcal{L}^r \)-algebras over \( \mathbb{F}_2 \)-module spectra

\[
\left( \text{Free}^{\mathcal{L}^r}(\Sigma^n X) \right)^{F_2} \cong \mathbb{F}_2 \{ t_n \} \otimes \text{Free}^{\mathcal{L}^r}(\Sigma^n X \otimes \mathbb{F}_2)
\]

\[
\cong \mathbb{F}_2 \{ t_n \} \otimes \text{Free}^{\mathcal{L}^r}(\Sigma^{n+\mathcal{L}^r} \mathbb{F}_2)
\]

\[
\cong \bigoplus_{w \in W} \left( \text{Free}^{\mathcal{L}^r}(\Sigma^d w) \otimes \mathbb{F}_2 \right)^{F_2}.
\]
The last equivalence makes use of Corollary 5.13 in [AB21]. This equivalence would only be that of $F_2$-module spectra if we did not kill the brackets by cotensoring with $(-)^{\wedge}$ . Therefore we deduce the following:

**Proposition 4.8.** The spectral sequence $E^2_{s,t} = HQ_{s,t}^R(\Omega^n \text{Free}_{\text{Mod}_2}^R (\Sigma^n H_\ast(X))) \Rightarrow H_{s+t}(\text{Free}_{\mathbb{F}_2}^R (X))$ splits as

$$E^2_{s,t} \cong \bigoplus_{w \in W} HQ_{s,t}^R(\mathfrak{g}_w) \Rightarrow \bigoplus_{w \in W} \pi_{s+t} \text{Bar}_s(\text{id}, \text{Lie}_R^\ast \mathfrak{g}) \ast \Omega^n \Sigma^d(\mathfrak{w} - n\mathbb{F}_2)).$$

**Remark 4.9.** The canonical map of spectral Lie algebras

$$\Omega^n \text{Free}^{\underline{\cdot}, \mathcal{L}} (\Sigma^k \mathfrak{g}_{\ast}) \to \Omega^n \text{Free}^{\underline{\cdot}, \mathcal{L}} (\Sigma^\ast \mathfrak{g}_{\ast})$$

via stabilization induces an embedding of the $E^2$-pages

$$HQ_{s,t}^R(\Omega^n \text{Free}_{\text{Mod}_2}^R (\Sigma^{n+k} \mathbb{F}_2)) \to HQ_{s,t}^R (\Sigma^k \mathbb{F}_2).$$

We expect that the higher differentials in the source (Conjecture 4.5) pull back to higher differentials in the target. Indeed, combinatorially this will yield the computation of the free $E_n$-algebra on a single generator. If $H(X)$ has multiple generators, then the splitting of the spectral sequence above via Lyndon words corresponds precisely the Browder bracket on the free $E_n$-algebra on those generators, cf. [CLM76, III].

5. Upper bounds and low weight computations

For a general parallelizable manifold $M$ of dimension $n$, the Lie$_R^\ast$-algebra

$$\mathfrak{g} = \widetilde{H}^\ast(M^+) \otimes \text{Free}_{\text{Mod}_2}^R (\Sigma^n H_\ast(X))$$

has non trivial Lie$^\ast$-brackets and the precise image of the comparison map $\varphi_\ast$ in Lemma 3.16 becomes much harder to pin down. Nonetheless, Proposition 3.5 and Lemma 3.7 allow us to obtain a formula for a relatively sharp upper bound of $\pi_{s+t} \text{Bar}_s(\text{id}, \text{Lie}_R^\ast \mathfrak{g})$ by

$$\pi_{s+t} \text{Bar}_s(\text{id}, \text{Lie}_R^\ast \mathfrak{g}) \cong \pi_{s+t} (\text{CE}(\text{AR}_\ast(\mathfrak{g}))).$$

Here $\mathfrak{g} = \widetilde{H}^\ast(M^+) \otimes \text{Free}_{\text{Mod}_2}^R (H_\ast(X))$ is the associated Lie$^\ast$-algebra, and by abuse of notation $\widetilde{H}^\ast(M^+)$ the associated Lie$^\ast$-algebra of the Lie$^\ast$-algebra $\widetilde{H}^\ast(M^+)$ with its usual cup product, cf. Construction 3.3. In particular, it follows from Corollary 3.25 that in weight less than four, the two homotopy groups are isomorphic.

5.1. General upper bounds. We will see that $\pi_{s+t} (\text{CE}(\text{AR}_\ast(\mathfrak{g})))$ admits a description in terms of the Lie$^\ast$-algebra homology of $\mathfrak{g}$. The key observation is that for $\mathfrak{g} = \widetilde{H}^\ast(M^+) \otimes \text{Free}_{\text{Mod}_2}^R (H_\ast(X))$, $\text{AR}_\ast(\mathfrak{g})$ has trivial Lie$^\ast$-structure away from simplicial degree 0 and its degeneracies, cf. Construction 3.6, and the Lie$^\ast$-algebra bracket on $\mathfrak{g}$ vanishes on elements that involve $\overline{Q}$ operations.

**Definition 5.1.** For a Lie$^\ast$-algebra $\mathfrak{g}$, we say that its Lie$^\ast$-structure is **supported entirely** by a sub-Lie$^\ast$-algebra $\mathfrak{g}'$ if the Lie$^\ast$-algebra $\mathfrak{g}$ is isomorphic to the product Lie$^\ast$-algebra $N \oplus \mathfrak{g}'$, where the Lie$^\ast$-algebra bracket vanishes on the complement $N \subset \mathfrak{g}$.

**Lemma 5.2.** Let $\mathfrak{g} = L \otimes \text{Free}_{\text{Mod}_2}^R (V)$ be a product Lie$^\ast$-algebra. Then

$$\pi_{s+t} (\text{CE}(\text{AR}_\ast(\mathfrak{g}))) \cong \Lambda \{ \gamma(\alpha), \alpha \in A \} \otimes H_{s+t}^\text{Lie}_R^\ast (\mathfrak{g}),$$

where $\alpha \in A$ is an element of an $F_2$-basis for $\pi_{s+t} (\text{AR}_\ast(\mathfrak{g}))$ with simplicial degree $s(\alpha)$, and $I$ is $\gamma$-admissible with $e(I) \leq s(\alpha)$. 
Proof. Since brackets of operations are zero, the Lie\textsuperscript{\text{n}ln}-algebra \( \tilde{g} \) is supported entirely by the sub-Lie\textsuperscript{\text{n}ln}-algebra \( g_0^* = L \otimes \text{Free}_{\text{Mod}_{\tilde{g}}^\text{Lie}}(V) \). Furthermore, for all \( m \geq 1 \), the Lie\textsuperscript{\text{n}ln}-algebra \( AR_m(\tilde{g}) \) is supported entirely by the degeneracies coming from \( g_0^* \) by Construction 3.6. Hence each simplicial level \( AR_m(\tilde{g}) \) is isomorphic to the product Lie\textsuperscript{\text{n}ln}-algebra \( T_m \otimes g_m^* \), where \( L_m \) is the sub-Lie\textsuperscript{\text{n}ln}\textsuperscript{\text{Lie}}-algebra consisting of degeneracies of \( g_0^* \) and \( T_m \) a trivial Lie\textsuperscript{\text{n}ln}-algebra. Since the splittings respect the simplicial Lie\textsuperscript{\text{n}ln}-algebra structure of \( AR_\ast(\tilde{g}) \), we deduce that \( AR_\ast(\tilde{g}) \cong T_\ast \otimes g_\ast^* \) as simplicial Lie\textsuperscript{\text{n}ln}-algebras. This induces a splitting of chain complexes

\[ \text{CE}(AR_\ast(\tilde{g})) \cong \text{CE}(T_\ast) \otimes \text{CE}(g_\ast), \]

where \( T_\ast \) is a trivial simplicial Lie\textsuperscript{\text{n}ln}-algebra and \( g_\ast^* \) the constant simplicial object on \( g_0^* \). The lemma then follows from Theorem 3.12, noting that \( H^I_{\text{Lie}\textsuperscript{\text{n}ln}}(\tilde{g}) \cong H^I_{\text{Lie}\textsuperscript{\text{n}ln}}(g_0^*) \). \( \square \)

It remains to compute \( \pi_{\ast,\ast}(\text{AR}_\ast(\tilde{g})) \) for \( \tilde{g} = \tilde{H}^I(M^+) \otimes \text{Free}_{\text{Lie}}(H_\ast(X)) \). Since \( g \) and \( \tilde{g} \) are isomorphic as \( \tilde{R} \)-modules (cf. Remark 3.2), we will not distinguish the two. Recall from Proposition 4.4 that the \( \tilde{R} \)-module structure on \( g \) is twisted by the Steenrod operations in the sense that

\[ \tilde{Q}^I(y \otimes \alpha) = \sum_{0 \leq s \leq n} Sq^{j+s}(y) \otimes \tilde{Q}^I(\alpha). \]

Notation 5.3. Let \( H \cup \{ z \} \) be an \( \mathbb{F}_2 \)-basis of the cohomology ring \( H^\ast(M^+) \), where \( z \) corresponds to the added point in the one-point compactification and \( H \) is a basis for \( H^\ast(M^+) \). For \( y \in H \), denote by \( |y| \) the cohomological degree of \( y \).

Let \( \bar{B} = \{ x_a \}_a \) be a totally ordered basis for \( V = H_\ast(X) \) and \( B = \{ \sigma^ax_a \}_a \) with the induced ordering. Denote by \( W \) the set of basic products on the set \( B \). Then

\[ g = \tilde{H}^I(M^+) \otimes H_\ast(\text{Free}_{\mathbf{F}_2}(\Sigma^n X)) \cong \bigoplus_{y \in W, y \in H} \mathbf{F}_2\{y \} \otimes \tilde{Q}^I y, J \in \tilde{R}(|y|) \}. \]

Proposition 5.4. The bigraded homotopy group \( \pi_{\ast,\ast}(\text{AR}_\ast(g)) \) is isomorphic to \( \pi_{\ast,\ast}(\text{AR}_\ast(g_{\text{inv}})) \), where the the untwisted \( \tilde{R} \)-module \( g_{\text{inv}} \) has the same underlying \( \mathbf{F}_2 \)-module as \( g \) and the \( \tilde{R} \)-module structure is given by \( \tilde{Q}^I(y \otimes x) = y \otimes \tilde{Q}^I(x) \).

Proof. We make use of a spectral sequence to filter away the twisting by the action of the Steenrod operations. We abuse notation here and denote again by \( AR_\ast(g) \) the associated chain complex of \( AR_\ast(g) \). Filter \( g \) by

\[ F_p(g) = \tilde{H}^{\geq p}(M^+) \otimes \text{Free}_{\text{Mod}_{\tilde{g}}^\text{Lie}}(V) \cong \bigoplus_{y \in W, y \in H, |y| \geq p} \mathbf{F}_2\{y \} \otimes \tilde{Q}^I y, J \in \tilde{R}(|y|) \} \]

with associated graded pieces given by

\[ G_p(g) = F_p(g)/F_{p-1}(g) \cong \bigoplus_{y \in W, y \in H, |y| = p} \mathbf{F}_2\{y \} \otimes \tilde{Q}^I y, J \in \tilde{R}(|y|) \}. \]

Since action by Steenrod operations does not decrease cohomological degree, the induced filtration

\[ F_p(AR_\ast(g)) := AR_\ast(F_p(g)) \]

makes \( AR_\ast(g) \) a filtered chain complex. The associated graded pieces are

\[ G_p(AR_\ast(g)) = AR_\ast(G_p(g)) = \bigoplus_{w \in W, y \in H, |y| = p} AR_\ast(\mathbf{F}_2\{y \} \otimes \tilde{Q}^I y, w, J \in \tilde{R}(|y|) \}) \]

and the induced differential preserves direct summands.

Using the case \( M = \mathbb{R}^n \) in Proposition 4.8, we deduce that

\[ E_{-p,q} = H_{-p+q}(G_p(AR_\ast(g))) \cong \bigoplus_{w \in W, y \in H, |y| = p} \pi_{\ast}(AR_\ast(\mathbf{F}_2\{y \} \otimes \tilde{Q}^I y, w, J \in \tilde{R}(|y|) \}) \]

\[ \cong \bigoplus_{w \in W, y \in H, |y| = p} \mathbf{F}_2\{\tilde{Q}^{j_1} \cdots \tilde{Q}^{j_m}(y \otimes w), (j_1, \ldots, j_m) \in \tilde{R}(p, |y|) \}. \]
where $\tilde{R}(p, |w|)$ is the set of sequences $(j_1, \ldots, j_m)$ such that

1. $j_l \leq 2j_{l+1}$ for $1 \leq l < m$ and $|w| - p \leq j_m < |w|$;
2. If $m \geq 2$ then $j_l \geq j_{l+1} + \cdots + j_m + |w| - (m - l)$ for $2 \leq l \leq m - 1$ and $j_1 > j_2 + \cdots + j_m + |w| - (m - 1)$.

We claim that every class on the $E^1$-page survives to a class on the $E^\infty$-page by induction along decreasing cohomological degree on $H^\ast(M^+)$. 

For $y \in \tilde{H}^p(M^+)$ a top cohomology class, there are no nonzero Steenrod action on $y$ other than $Sq^0$, so the differential on $\beta$ in $AR_\ast(g)$ is the same as the differential in $G_n(AR_\ast(g))$, i.e. $\beta$ survives to a nontrivial cycle on the $E^\infty$-page.

Suppose that for all $y' \in \tilde{H}^p(M^+)$ with $q \geq p + 1$, any basis element $\beta' = \hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y' \otimes w')$ of the $E^1$-page is a permanent cycle. Let $[\beta] = \hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y \otimes w)$ be a basis element on the $E^1$-page, with $y \in \tilde{H}^p(M^+)$. A cycle representing this class in $AR_\ast(G_p(g))$ is a finite sum $\beta = \hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y \otimes w) + \sum_i \hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y \otimes w)$ obtained by cycle completion via Behrens’ relations. Note that $l_m \leq j_m < |w|$ for all $l$. Let $d_m$ be the rightmost face map. Then in $AR_\ast(g)$

$$\partial \beta = \partial \left( \hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y \otimes w) + \sum_i \hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y \otimes w) \right) = 0 + d_m \left( \hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y \otimes w) + \sum_i \hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y \otimes w) \right) = \sum_{s \geq 0} \hat{Q}^{j_1} \cdots \hat{Q}^{j_{m-1}} Sq^s(y) \otimes \hat{Q}^{j_{m+s}}(w) + \sum_{s \geq 0} \sum_i \hat{Q}^{j_1} \cdots \hat{Q}^{j_{m-1}} Sq^s(y) \otimes \hat{Q}^{j_{m+s}}(w).$$

Note that each $\theta = \hat{Q}^{j_1} \cdots \hat{Q}^{j_m-1} Sq^s(y) \otimes \hat{Q}^{j_{m+s}}(w)$ or $\hat{Q}^{j_1} \cdots \hat{Q}^{j_{m-1}} Sq^s(y) \otimes \hat{Q}^{j_{m+s}}(w)$ is either 0 or boundary in $AR_\ast(g)$ for all $s \geq 0$:

1. If $l_m + s < |w|$ then $\theta = 0$;
2. If $l_m + s \geq |w|$, then $s \geq 1$, since $l_m \leq j_m < |w|$. By the inductive hypothesis $\theta$ is not a cycle on the $E^\infty$-page, i.e. it is killed by a finite sum of classes of the form $\hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y' \otimes w)$ with $|y'| \geq 1 + s$ in $AR_\ast(g)$. Denote by $\xi$ this finite sum in $AR_\ast(g)$. Note that $\xi$ is not a boundary because it is maximally nondegenerate.

Hence $\partial (\sum (\alpha + \xi)) = 0$ in $AR_\ast(g)$, i.e. $\sum (\alpha + \xi)$ is a cycle in $AR_\ast(g)$ corresponding to the basis element $\beta = \hat{Q}^{j_1} \cdots \hat{Q}^{j_m}(y \otimes w)$ on the $E^1$-page. Therefore no differential can happen in the spectral sequence. \qed

Combing Lemma 5.2, Proposition 5.4 and Corollary 3.25, we deduce the following general upper bound and low weight computation of the $E^2$-page of Knudsen’s spectral sequence.

**Theorem 5.5.** Let $M$ be a parallelizable manifold of dimension $n$ and $X$ any spectrum. Let $g$ denote the Lie$_R^\ast$-algebra $\tilde{H}^\ast(M^+) \otimes \text{Lie}_{\text{Mod}_2}^\ast \left( \Sigma^p BH(X) \right)$ with $\mathbb{F}_2$-basis $B$, and $\tilde{g}$ the associated Lie$_R^\ast,\mathbb{F}_2$-algebra. An upper bound for the $E^2$-page of the weighted spectral sequence

$$E^2_{s, f} = H\text{Lie}_{\text{Mod}_2}^\ast (\tilde{g}) \Rightarrow \bigoplus_{k \geq 1} H_{s+t}(B_k(M; X))$$

is given by

$$\pi_{s+t}(CE(AR_\ast(g))) \cong \Lambda \{ y \hat{Q}^{j_1} \otimes w, y \otimes w \in H \otimes B \} \otimes H_{s+t}^\ast(\tilde{g}),$$

where $y \hat{Q}^{j_1} (y \otimes w)$ satisfies the conditions that

1. $J = (j_1, \ldots, j_m)$ with $m \geq 1$, $0 \leq j_l \leq j_{l+1}$ for $1 \leq l < m$, and $0 \leq j_m < |y|$;
2. $I$ is $\gamma$-admissible with $e(I) \leq m$.

Furthermore, in weight less than four equality is achieved.
5.2. Low weight computations. Theorem 5.5 allows us to deduce the degeneration of the spectral sequence at weight two and three using sparsity arguments. Denote by \(wt_k(M)\) the weight \(k\) part of a weighted (bi)graded \(\mathbb{F}_2\)-module \(M\) and set \(E^r(k) = wt_k(E^r)\).

**Corollary 5.6.** Let \(g, \tilde{g}\) be the same as in Theorem 5.5 and \(B, H\) bases given in Notation 5.3. The weight two part of the spectral sequence (10)

\[
E^2_{s,t}(2) = \text{wt}_2(H^s_{\text{Lie}^n}(\tilde{g})) \Rightarrow H_{s+t}(B_2(M;X))
\]
collapses on the \(E^2\)-page, and hence

\[
E^\infty(2) \cong E^2(2) \cong \text{wt}_2(H^s_{\text{Lie}^n}(\tilde{g})) \oplus \bigoplus_{x \in B, y \in H} \{\tilde{Q}_j(y \otimes x), 0 \leq j < |y|\}.
\]

**Proof.** Since classes in the tensor factor

\[
\Lambda \{ y(\tilde{Q}_j(y \otimes w)), y \otimes w \in H \otimes B\}
\]
of Theorem 5.5 have weight at least two, classes of weight two lie in exactly one of the two tensor components \(A\) and \(H^s_{\text{Lie}^n}(\tilde{g})\). The weight two classes in \(A\) are of the form \(\tilde{Q}_j(y \otimes w)\) where \(w\) has weight one, i.e. \(w\) is an element of the \(\mathbb{F}_2\)-basis \(B = H_1(X)\), cf. Notation 5.3. The weight two classes in \(H^s_{\text{Lie}^n}(\tilde{g})\) are of the form \(y \otimes (x_a, x_b)\) and \((y \otimes x_a) \otimes (y \otimes x_b)\). Hence the weight two part of the spectral sequence has \(E^2\)-page concentrated in simplicial degrees 0, 1 and thus cannot admit higher differentials. \(\square\)

In particular, this demonstrates that for a parallelizable \(M\), the \(\mathbb{F}_2\)-module \(H_*(B_2(M;X))\) depends on and only on the cohomology ring \(H^*(M^+)\) when \(H_*(X)\) has at least two generators.

**Remark 5.7.** This is in contrast to the case where \(X = S^r\) has only one generator in its homology: Bödigheimer-Cohen-Taylor showed that for any closed \(n\)-manifold \(M\),

\[
\bigoplus_{k \geq 1} H_*(B_k(M;S^r)) \cong \bigotimes_{i=0}^n H_*(\Omega^{n-i}S^{r+i}) \otimes \dim H_i(M)
\]
depends only on \(H^*(M)\) as an \(\mathbb{F}_2\)-module [BCT89]. On the other hand, the homology of \(\text{Conf}_2(M)\), the space of ordered configurations of two points in \(M\), also depends only on the cup product structure of \(H^*(M)\) as discussed in [Pet20, Section 1.1].

**Corollary 5.8.** If in addition \(M\) is a closed manifold, then the weight three part of the spectral sequence (10) collapses on the \(E^2\)-page, and a basis for \(H_*(B_3(M;X))\) is given by

\[
E^\infty(3) \cong E^2(3) \cong \bigoplus_{x, x' \in H} \mathbb{F}_2\{\tilde{Q}_j(y \otimes x) \otimes (y' \otimes x'), 0 \leq j < |y|\}
\]

\[
\oplus \text{wt}_3(H^s_{\text{Lie}^n}(\tilde{g}))
\]

**Proof.** Let \(d\) denote the generator for \(H^0(M^+) \cong H^0(M)\). Then any element that is a sum of \(y \otimes \langle x_1, x_2, x_3 \rangle \in H \otimes B\) is killed by a sum \(\langle y \otimes x_1, x_2 \rangle \otimes (d \otimes x_3)\). Since classes in \(A\) have weights positive powers of two, weight three classes on the \(E^2\)-page either live in \(\text{wt}_3(H^s_{\text{Lie}^n}(\tilde{g}))\) with simplicial degree one or two, or have the form

\[
\langle \tilde{Q}_j(y \otimes x) \rangle \otimes (y' \otimes x') \in \text{wt}_3(A) \otimes \text{wt}_3(H^s_{\text{Lie}^n}(\tilde{g}))
\]

with simplicial degree two. Hence \(E^2(3)\) is concentrated in simplicial degree 1 and 2, so there cannot be any higher differentials. \(\square\)

At weight four part we can no longer deduce that the spectral sequence (10) collapses on the \(E^2\)-page using sparsity arguments. An upper bound for the bigraded \(\mathbb{F}_2\)-module \(E^2(4)\) is given by the weight four part of \(A \otimes H^4_{\text{Lie}^n}(\tilde{g})\), which consists of:

1. \(\tilde{Q}_j(y \otimes (x, x'))\) in simplicial degree one,
2. \(\tilde{Q}_j \tilde{Q}_j(y \otimes x)\) and \(\tilde{Q}_j y (\otimes (x_1, x_2))\) in simplicial degree two,
(3) $\tilde{Q}_0(y \otimes x) \otimes \tilde{Q}_1(y' \otimes x')$ and $\tilde{Q}_0(y \otimes x) \otimes (y_1 \otimes x_1) \otimes (y_2 \otimes x_2)$ in simplicial degree three,
(4) Weight four part of $H^{\text{Lie}}(\tilde{g})$.

There could well be a $d^2$-differential from degree considerations.

We close this section by two example computations, the (punctured) genus $g$ surfaces with $g \geq 1$ and the (punctured) real projective space $\mathbb{R}P^3$.

5.3. Example computations: genus $g$ surfaces. Let $\Sigma_{g,1}$ be a once-punctured surface of genus $g \geq 1$. Let $\overline{B} = \{x_i\}$ be a totally ordered basis for $H_*(X)$ and $B = \{\sigma^i x_i\}$ with the induced ordering. Then

$$\overline{H}^*(\Sigma_{g,1}^+) = \begin{cases} \mathbb{F}_2 \{a_i \oplus b_i, i = 1, \ldots, g\} & * = 1 \\ \mathbb{F}_2 \{c\} & * = 2 \\ 0 & \text{otherwise} \end{cases}$$

with nonzero cup products $a_i \cup b_i = c$ for all $i$ and no nontrivial Steenrod operations.

For the closed surface $\Sigma_g$, we have

$$\overline{H}^*(\Sigma_g^+) \cong H^*(\Sigma_g) = \begin{cases} \mathbb{F}_2 \{d\} & * = 0 \\ \mathbb{F}_2 \{a_i \oplus b_i, i = 1, \ldots, g\} & * = 1 \\ \mathbb{F}_2 \{c\} & * = 2 \\ 0 & \text{otherwise} \end{cases}$$

with nonzero cup products $a_i \cup b_i = c$ for all $i$ and $d \cup y = y$ for all $y \in \overline{H}^*(\Sigma_g^+)$.

5.3.1. Weight two. For $M = \Sigma_{g,1}$, the weight two classes supporting CE differentials are

$$\delta(a_i \otimes x_1, b_i \otimes x_2) = c \otimes (x_1, x_2)$$

for $x_1 \neq x_2 \in B$. Denote by $H^1$ the set of generators $\{a_i, b_i, i = 1, \ldots, g\}$ for $\overline{H}^1(\Sigma_{g,1}^+)$. By Corollary 5.6, a basis for $H_*(B_2(\Sigma_{g,1}; V))$ is given by

$$E^2(2) = E^2(2) \cong \bigoplus_{x \in B} \mathbb{F}_2 \{\tilde{Q}_0(y \otimes x), y \in H^1; \tilde{Q}_0(c \otimes x), \tilde{Q}_1(c \otimes x)\}$$

$$+ \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2 \{(y \otimes (x_1, x_2), (y \otimes x_1) \otimes (y \otimes x_2)), y \in H^1; (c \otimes x_1) \otimes (c \otimes x_2)\}$$

$$+ \bigoplus_{x_1, x_2 \in B} \mathbb{F}_2 \{(y \otimes x_1) \otimes (c \otimes x_2), y \in H^1\}$$

$$+ \bigoplus_{x \in B} \mathbb{F}_2 \{(y \otimes x) \otimes (y' \otimes x), y \neq y' \in H^1 \cup \{c\}\}$$

$$+ \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2 \{(y \otimes x_1) \otimes (y' \otimes x_2) + (a_1 \otimes x_1) \otimes (b_1 \otimes x_2), y \neq y' \in H^1, (y, y') \neq (a_1, b_1)\}.$$ 

For $M = \Sigma_g$, the weight two classes supporting CE differentials are

$$\delta((a_i \otimes x_2) \otimes (b_i \otimes x_2)) = c \otimes (x_1, x_2)$$

and

$$\delta((d \otimes x_1) \otimes (y \otimes x_2)) = y \otimes (x_1, x_2)$$
for \(x_1 \neq x_2 \in B\) and \(y \in \tilde{H}^*(\Sigma_+^2)\). By Corollary 5.6, a basis for \(H_\ast(B_2(\Sigma_g; X))\) is given by

\[
E^m(2) = E^2(2) \cong \bigoplus_{x \in B} F_2\{(\tilde{Q}_0(y \otimes x), y \in H^1; \tilde{Q}_0(c \otimes x), \tilde{Q}_1(c \otimes x)\}
\]

\[
\oplus \bigoplus_{x_1 < x_2 \in B} F_2\{(y \otimes x_1) \otimes (y \otimes x_2), y \in H^1; (z \otimes x_1) \otimes (z \otimes x_2)\}
\]

\[
\oplus \bigoplus_{x_1, x_2 \in B} F_2\{(y \otimes x_1) \otimes (z \otimes x_2), y \in H^1\}
\]

\[
\oplus \bigoplus_{x \in B} F_2\{(y \otimes x) \otimes (y' \otimes x), y \neq y' \in H^1, \{y, y'\} \neq \{a_i, b_i\}\}
\]

\[
\oplus \bigoplus_{x_1, x_2 \in B} F_2\{(y \otimes x_1) \otimes (y' \otimes x_2) + (d \otimes x_1) \otimes (c \otimes x_2), \{y, y'\} = \{a_i, b_i\}\text{ or } \{y, y'\} = \{c, d\}\}.
\]

5.3.2. **Weight three.** Classes in \(A = A(y_1(\tilde{Q}u_1) \cdots \tilde{Q}u_m)(y \otimes w), m \geq 1\) have weights positive powers of 2. Hence weight three classes in \(E^2(3)\) either live in \(wt_3(H_{\ast,s}^{\text{Lie},\text{ht}}(\bar{g}))\) or has the form

\[
(\tilde{Q}_j(y \otimes x)) \otimes (y' \otimes x') \in A \otimes H_{\ast,s}^{\text{Lie},\text{ht}}(\bar{g})
\]

with \(x, x' \in B\).

Let \(H\) be the set of generators for \(\tilde{H}^*(\Sigma_{g,1}^+) \cong \tilde{H}^*(\Sigma_g)\) and \(H^1\) the set of generators for \(\tilde{H}^1(\Sigma_{g,1}^+)\). Recall that \(\bar{g} = \tilde{H}^*(\Sigma_g) \otimes \text{Free}_{\text{Mod}_{F_2}}^\text{Lie,s} \Sigma^n H_\ast(X)\). Then we have

\[
E^2(3) \cong \bigoplus_{x_1, x_2 \in B} F_2\{(\tilde{Q}_0(y \otimes x_1)) \otimes (y' \otimes x_2), y \in H^1, y' \in H\}
\]

\[
\oplus \bigoplus_{x_1, x_2 \in B} F_2\{(\tilde{Q}_0(c \otimes x_1)) \otimes (y \otimes x_2), (\tilde{Q}_1(c \otimes x)) \otimes (y \otimes x_2), y \in H\}
\]

\[
\oplus wt_3(H_{\ast,s}^{\text{Lie},\text{ht}}(\bar{g})).
\]

A complete list of an \(F_2\)-basis of \(wt_3(H_{\ast,s}^{\text{Lie},\text{ht}}(\bar{g}))\) can be written down in a straightforward way.

The \(E^2\)-page is concentrated in simplicial degree 0, 1, 2. We need to investigate all classes in \(E^2_{\ast,s}(3)\) to see if they support nontrivial \(d^2\)-differentials to \(E^3_{\ast,s+1}(3)\). Note that all classes in \(E^2_{\ast,s}(3)\) are of the form \(y \otimes (x_1, x_2, x_3)\) for \(y \in H^1\). Since \(E^2(3)\) is natural in \(H_\ast(V)\), we can assume \(x_1, x_2, x_3 \in B\) have internal degree \(k\) respectively. There are two cases:

1. The class \((\tilde{Q}_j(y_1 \otimes x_1)) \otimes (y_2 \otimes x_2) \in E^2_{\ast,s}(3)\) has internal degree at most \(3k - 5\) for all \(y_1, y_2 \in H\), while the class \(y \otimes (x_1, x_2, x_3)\) has internal degree \(3k - 3\) for all \(y \in H^1\). Hence they do not support \(d^2\)-differentials.
2. The other type of classes in filtration 2 are of the form \((y_1 \otimes x_1) \otimes (y_2 \otimes x_2) \otimes (y_3 \otimes x_3)\) with internal degrees at most \(3k - 5\), while the class \(y \otimes (x_1, x_2, x_3)\) has internal degree \(3k - 3\). Hence these classes do not support \(d^2\)-differentials either.

Therefore the weight three part of the spectral sequence collapses at the \(E^2\)-page, and we obtain a basis for \(H_\ast(B_3(\Sigma_g; X))\).
For the closed surface $\Sigma_g$, $\tilde{g} = H^*(\Sigma_g) \otimes \text{Free}_{\text{Lie}_n^\mathbb{R}}^\text{Mod}_2 (\Sigma^gH_*(X))$ and Corollary 5.8 says that

$$E^\infty(3) = E^2(3) \cong \bigoplus_{x_1,x_2\in B} \mathbb{F}_2\{(\tilde{Q}_0(y\otimes x_1)) \otimes (y' \otimes x_2), y \in H^1, y' \in H \cup \{d\}\}
\oplus \bigoplus_{x_1,x_2\in B} \mathbb{F}_2\{(\tilde{Q}_0(c \otimes x_1)) \otimes (y \otimes x_2), (\tilde{Q}_1(c \otimes x)) \otimes (y \otimes x_2), y \in H \cup \{d\}\}
\oplus \text{wt}_3(H_{s,\tilde{g}}^{\text{Lie}_n^\mathbb{R}}(\tilde{g})).$$

We do not list the $\mathbb{F}_2$-basis of $\text{wt}_3(H_{s,\tilde{g}}^{\text{Lie}_n^\mathbb{R}}(\tilde{g}))$ for simplicity.

5.4. Example computations: (punctured) real projective space. The simplest examples of parallelizable manifolds admitting nontrivial Steenrod actions other than $S^0$ are the real projective space $\mathbb{RP}^3$ and the once-punctured real projective space $\tilde{\mathbb{RP}}^3$.

Let $y$ be a generator for $H^1(\mathbb{RP}^3)$. Then

$$\tilde{H}^*(\mathbb{RP}^3) \cong H^*(\mathbb{RP}^3) = \mathbb{F}_2[y]/(y^4), \tilde{H}^*(\tilde{\mathbb{RP}}^3) = \tilde{H}^*(\mathbb{RP}^3) = \mathbb{F}_2\{y, y^2, y^3\}$$

with the obvious cup products and one nontrivial Steenrod operation $S^1(y) = y^2$.

5.4.1. Weight two. We deduce $H_*(B_2(\mathbb{RP}^3; X))$ and $H_*(B_2(\tilde{\mathbb{RP}}^3; X))$ from Corollary 5.6. For $M = \mathbb{RP}^3$, there is only one nontrivial cup product $y \cup y^2 = y^3$, so

$$E^\infty(2) = E^2(2) = \bigoplus_{x \in B, a = 1, 2, 3} \mathbb{F}_2\{\tilde{Q}_j(y^a \otimes x), 0 \leq j < a\}
\oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y \otimes (x_1, x_2)) \otimes (y^a \otimes (x_2)), a = 2, 3\}
\oplus \bigoplus_{x_1, x_2 \in B} \mathbb{F}_2\{(y^a \otimes x_1) \otimes (y^3 \otimes x_2), a = 1, 2\}
\oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y^j \otimes (x_1, x_2)) \otimes (y^2 \otimes (x_2)) + (y^3 \otimes x_1) \otimes (y \otimes (x_2))\}.$$

For $M = \tilde{\mathbb{RP}}^3$, the nonzero cup products are $y \cup y = y^2, y \cup y^2 = y^3$ and $1 \cup y^a = y^a$ for $0 \leq a \leq 3$, so

$$E^\infty(2) = E^2(2) = \bigoplus_{x \in B, a = 1, 2, 3} \mathbb{F}_2\{\tilde{Q}_j(y^a \otimes x), 0 \leq j < a\}
\oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y^a \otimes x_1) \otimes (y^3 \otimes x_2), a = 1, 2\}
\oplus \bigoplus_{x_1, x_2 \in B} \mathbb{F}_2\{(y^a \otimes x_1) \otimes (y^b \otimes x_2) + (y^3 \otimes x_1) \otimes (1 \otimes x_2), (a, b) \neq (3, 0)\}
\oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y^a \otimes x_1) \otimes (1 \otimes x_2) + (1 \otimes x_1) \otimes (y^a \otimes x_2), a = 1, 2, 3\}.$$

5.4.2. Weight three. For the closed manifold $\mathbb{RP}^3$ and $\tilde{g} = H^*(\mathbb{RP}^3) \otimes \text{Free}_{\text{Lie}_n^\mathbb{R}}^\text{Mod}_2 (\Sigma^gH_*(X))$, it follows from Corollary 5.8 that

$$E^\infty(3) = E^2(3) = \text{wt}_3(H_{s,\tilde{g}}^{\text{Lie}_n^\mathbb{R}}(\tilde{g})) \oplus \bigoplus_{x_1, x_2 \in B, 1 \leq a \leq 3, 0 \leq b \leq 3} \mathbb{F}_2\{(\tilde{Q}_j(y^a \otimes x_1)) \otimes (y^b \otimes x_2), 0 \leq j < a\}.$$
For the punctured real projective space $\mathbb{RP}^3$ and $\hat{g} = H^*(\mathbb{RP}^3) \otimes \text{Free}_{\text{Mod}_{\mathbb{F}_2}}(\Sigma^n H_*(X))$, weight three classes in $E^2(3)$ either live in $wt_3(H_{s,s}^{\text{Lie},n}(\hat{g}))$ or has the form

$$(\tilde{Q}_j(y^a \otimes x)) \otimes (y^b \otimes x') \in A \otimes H_{s,s}^{\text{Lie},n}(\hat{g})$$

with $x, x' \in B$ and $1 \leq a, b \leq 3$. Therefore

$E^2(3) = wt_3(H_{s,s}^{\text{Lie},n}(\hat{g})) \oplus \bigoplus_{x_1, x_2 \in B, 1 \leq a, b \leq 3} \mathbb{F}_2\{ (\tilde{Q}_j(y^a \otimes x_1)) \otimes (y^b \otimes x_2), |x_1| - a \leq j \leq |x_1| \}$

A complete list of an $\mathbb{F}_2$-basis for $wt_3(H_{s,s}^{\text{Lie},n}(\hat{g}))$ is given by

1. $y \otimes \langle (x_1, x_2), x_3 \rangle$ for $x_1, x_2, x_3 \in B, x_1 < x_2, x_1 < x_3$ in simplicial degree 0;
2. $(y^a \otimes (x_1, x_2)) \otimes (y^b \otimes x_3) + (y^1 \otimes (x_1, x_3)) \otimes (y^b \otimes x_2) + (y^3 \otimes (x_2, x_3)) \otimes (y^b \otimes x_1)$ for $b = 1, 2$ and $(y \otimes (x_1, x_2)) \otimes (y^2 \otimes x_3) + (y \otimes (x_1, x_3)) \otimes (y^2 \otimes x_2) + (y \otimes (x_2, x_3)) \otimes (y^2 \otimes x_1)$ for distinct $x_i \in B$ in simplicial degree 1;
3. $(y^a \otimes x_1) \otimes (y^b \otimes x_2) \otimes (y^2 \otimes x_3)$ for $\{1, 2\}, \{1, 1\} \not\subseteq \{a, b, c\}$ and $x_i \in B$;

$\sum_{i,j,k=1, \emptyset, j \neq k} (y \otimes x_i) \otimes (y \otimes x_j) \otimes (y \otimes x_k)$,

$\sum_{i,j,k=1, \emptyset, j \neq k} (y \otimes x_i) \otimes (y \otimes x_j) \otimes (y \otimes x_k)$ for distinct $x_1, x_2, x_3 \in B$ in simplicial degree 2.

Again the $E^2$-page is concentrated in simplicial degrees 0, 1, 2, and we use sparsity to rule out higher differentials. Suppose that $x_1, x_2, x_3$ have internal degree $k$. We examine the two cases that could potentially support a $d^2$-differential.

1. The class $(\tilde{Q}_j(y^a \otimes x_1)) \otimes (y^b \otimes x_2) \in E^2_s(3)$ has internal degree at most $3k - 5$ for all $1 \leq a, b \leq 3$; while the class $y \otimes \langle (x_1, x_2), x_3 \rangle$ has internal degree $3k - 3$. Hence they do not support $d^2$-differentials.
2. The other type of classes in simplicial degree 2 are of the form $(y^a \otimes x_1) \otimes (y^b \otimes x_2) \otimes (y^2 \otimes x_3)$ with internal degrees at most $3k - 5$, while the class $y \otimes \langle (x_1, x_2), x_3 \rangle$ has internal degree $3k - 3$. Hence these classes do not support $d^2$-differentials either.

Therefore the weight three part of the spectral sequence collapses on the $E^2$-page, and we obtain a basis for $H_s(B_3(\mathbb{RP}^3; X))$.

6. ODD PRIMARY HOMOLOGY

In the last section, we apply the same methods to study the mod $p$ homology of $B_k(M; X)$ for $p > 2$ via Knudsen’s spectral sequence with $\mathbb{F}_p$ coefficient.

6.1. Odd primary Knudsen’s spectral sequence. We start by recalling partial progress in understanding the unary operations on the mod $p$ homology of spectral Lie algebras by Kjaer [Kja18]. He constructed weight $p$ Dyer-Lashof-type operations in analogy to Behrens’ construction of $\tilde{Q}^i$.

**Proposition 6.1.** [Kja18, Definition 3.2] Let $L$ be a spectral Lie algebra. Then $H_*(L; \mathbb{F}_p)$ admits unary operations $\bar{\beta}^e\beta^j : H_*(L; \mathbb{F}_p) \rightarrow H_{*+2(p-1)e-1}(L; \mathbb{F}_p)$, $e \in \{0, 1\}, j \in \mathbb{Z}$.

On a class $x \in H_*(L; \mathbb{F}_p)$, the class $\bar{\beta}^e\beta^j(x)$ is given by $\xi_*(\sigma^{-1}\beta^e\beta^j(x))$, where $\beta^e\beta^j$ is a mod $p$ Dyer-Lashof operation, $\sigma^{-1}$ the desuspension isomorphism, and $\xi : \partial_p(\text{Id}) \otimes \Sigma_p L^\otimes p \rightarrow L$ the $p$th structure map of the spectral Lie algebra $L$.

It follows from the instability condition of Dyer-Lashof operations that $\bar{\beta}^e\beta^j(x) = 0$ if $j < \frac{|x|}{2}$. Analogous to the case $p = 2$, brackets of unary operations always vanish.

**Proposition 6.2.** [Kja18, Proposition 3.7] For $L$ a spectral Lie algebra, $[\bar{\beta}^e\beta^j(x), y] = 0$ for any $e, j$ and $x, y \in H_*(L; \mathbb{F}_p)$.
Define a functor $\text{Lie}_{\mathcal{F}}^t : \text{Mod}_{\mathbb{F}_p} \to \text{Mod}_{\mathbb{F}_p}$ as follows. For $M \in \text{Mod}_{\mathbb{F}_p}$, let $A$ be an $\mathbb{F}_p$-basis for the free shifted Lie algebra $\text{Free}_{\text{Mod}_{\mathbb{F}_p}}(M)$. The graded $\mathbb{F}_p$-module $\text{Lie}_{\mathcal{F}}^t(M)$ has basis
\[
\{ \beta_k^i Q^{j_1} \cdots \beta_k^i Q^{j_n} | x_1, \ldots, x_n \text{ are } A, j_k \geq \frac{|x_k|}{2}, j_i \geq j_{i+1} - \epsilon_{i+1} \forall i \}. 
\]
The Lie$_{\mathcal{F}}^t$-structure on $\text{Free}_{\text{Mod}_{\mathbb{F}_p}}(M)$ can be extended to one on $\text{Lie}_{\mathcal{F}}^t(M)$ via Proposition 6.2.

**Theorem 6.3.** [Kja18, Theorem 5.2] For $X$ a spectrum, there is an isomorphism of Lie$_{\mathcal{F}}^t$-algebras
\[
\text{Lie}_{\mathcal{F}}^t(H_*(X; \mathbb{F}_p)) \to \text{H}_*(\text{Free}_{\mathbb{F}_p}(X); \mathbb{F}_p).
\]

Now we turn to the odd primary Knudsen’s spectral sequence
\[
E_2^{s,t}(k) = H_{s+t}(B_k(M; X; \mathbb{F}_p)) \Rightarrow H_{s+t}(M; X; \mathbb{F}_p).
\]

By repeatedly applying Theorem 6.3, we see that the $E^2$-page is the homotopy group of a simplicial $\mathbb{F}_p$-module $V_*$ with $(\text{Lie}_{\mathcal{F}}^t)^i(g)$ as the $i$th simplicial level for all $i$, where $g = \text{H}_*(\text{Free}_{\mathbb{F}_p}(\Sigma^n X)^{M_+}; \mathbb{F}_p) \cong \tilde{H}^*(M_+; \mathbb{F}_p) \otimes \text{Lie}_{\mathcal{F}}^t(\Sigma^n H_*(X; \mathbb{F}_p))$.

Furthermore, $g$ has a Lie$_{\mathcal{F}}^t$-structure defined by Proposition 4.3, i.e.,
\[
[y_1 \otimes x_1, y_2 \otimes x_2] := (y_1 \cup y_2) \otimes [x_1, x_2].
\]

At the time of this work, there is no published result on the relations among operations on the odd primary homology of a free spectral Lie algebra.1 While we do not have full knowledge of the behavior of the face maps on concatenations of unary operations in $V_*$, we know how unary operations and Lie$_{\mathcal{F}}^t$-brackets commute by Proposition 6.2 and the face maps on Lie$_{\mathcal{F}}^t$-brackets are simply Lie$_{\mathcal{F}}^t$-algebra structure maps. This allows us to compute the $E^2$-page of the spectral sequence (11) in small weight in terms of Lie$_{\mathcal{F}}^t$-algebra homology.

**Definition 6.4.** [CE48][May66A] For a shifted Lie algebra $L_\mathcal{F}$ over $\mathbb{F}_p$, let $L_{\text{even}}$ and $L_{\text{odd}}$ denote the elements in $L$ with even and odd degree, respectively. The Chevalley-Eilenberg complex of $L$ is the chain complex over $\mathbb{F}_p$
\[
\text{CE}(L) = (\Gamma^*(L_{\text{even}}) \otimes \Lambda^*(L_{\text{odd}}), \partial),
\]
where $\Gamma^*$ and $\Lambda^*$ are respectively the graded, shifted divided power and exterior algebra functors, and the differential $\partial$ on a general element
\[
\gamma_1(x_1) \gamma_2(x_2) \cdots \gamma_m(x_m) \langle y_1, y_2, \ldots, y_n \rangle \in \Gamma^*(L_{\text{even}}) \otimes \Lambda^*(L_{\text{odd}})
\]
is given by
\[
\sum_{1 \leq i < j \leq m} \gamma_1(x_1) \cdots \gamma_{i-1}(x_i) \cdots \gamma_j(x_j) \cdots \gamma_m(x_m) \langle [x_i, x_j], y_1, \ldots, y_n \rangle \\
+ \sum_{1 \leq i < j \leq m} (-1)^{i+j-1} \gamma_1(x_1) \cdots \gamma_{i-1}(x_i) \cdots \gamma_j(x_j) \cdots \gamma_m(x_m) \langle [y_i, y_j], y_1, \ldots, y_n \rangle \\
+ \sum_{j=1}^{m} \gamma_j(x_1) \cdots \gamma_{j-2}(x_i) \cdots \gamma_m(x_m) \langle [x_i, x_i], y_1, \ldots, y_n \rangle \\
+ \sum_{j=1}^{m} \sum_{i=1}^{n} (-1)^{i-j-1} \gamma_1(x_1) \cdots \gamma_{i-1}(x_i) \cdots \gamma_m(x_m) \langle y_1, \ldots, y_j, y_1, \ldots, y_n \rangle.
\]

**Proposition 6.5.** Let $M^n$ be a parallelizable manifold and $X$ any spectrum.

---

1Through private communication, we were informed that Nikolai Konvalov has forthcoming work computing the odd primary relations via Goodwillie calculus.
(1) For $k < p$, the weight $k$ part of the spectral sequence
\[ E^2_{s,t}(k) = \pi_s \pi_t \left( \text{Bar}_* \left( \text{id}, s \mathcal{Z}, \text{Free}^{s \mathcal{Z}} \left( \Sigma^n X \right)^{M^+} \right) \otimes \mathbb{F}_p \right)(k) \Rightarrow H_{s+t}(B_k(M; X); \mathbb{F}_p) \]

has $E^2$-page given by
\[ \text{wt}_s(H_{s+\epsilon}(\text{CE}(g))) \oplus \bigoplus_{y \in H \in B} \mathbb{F}_p \left\{ \overline{\mathcal{B}^s Q^l}_{y \otimes x} : \frac{|x| - |y|}{2} \leq j < \frac{|x|}{2} \right\}, \]

where $H$ is an $\mathbb{F}_p$-basis of $\overline{H}^*(M^+; \mathbb{F}_p)$ and $B$ an $\mathbb{F}_p$-basis of $H_*(X; \mathbb{F}_p)$.

Proof. For $k < p$, all elements in the weight $k$ part of the $E^2$-page of the spectral sequence do not contain unary operations $\overline{\mathcal{B}^s Q^l}$. When $k = p$, nondegenerate elements of weight $p$ on the $E^2$-page are either of the form $\overline{\mathcal{B}^s Q^l y \otimes x} \in \text{Lie}_{\mathbb{R}}^p(g)$, $\overline{\mathcal{B}^s Q^l y \otimes x} \in g$, or a bracket of weight $p$. When $p \geq 5$, the unary operation $\overline{\mathcal{B}^s Q^l}$ cannot be an iteration of brackets on a single element, since $[[x, x], x] = 0$ for any $x$ by the Jacobi identity. Hence there is no $d_1$-differential from a weight $p$ bracket to $\overline{\mathcal{B}^s Q^l y \otimes x}$ or $y \otimes \overline{\mathcal{B}^s Q^l}(x)$. The same argument in Proposition 5.4 implies that the twisting of the action of $\overline{\mathcal{B}^s Q^l}$ by Steenrod operations can be ignored when computing a basis for the $E^2$-page.

Remark 6.6. When $p = 3$, there has to be an identity $\overline{\mathcal{B}^s Q^l}(x) = [[x, x]], x$ when $x$ is an even class in the mod 3 homology of a spectral Lie algebra $L$. To see this, consider the spectral sequence (11) when $M = \mathbb{R}^n$ and $X = S^2$ with $n > 2$, so $g = \mathbb{F}_p(y_n) \otimes \text{Lie}_{\mathbb{R}}^p(\mathbb{F}_p(\sigma^n(x_2)))$ with $y$ in internal degree $-n$ and $x_2l$ in degree $2l$. Set $x = y_n \otimes \sigma^n(x_2)$. Suppose that $[[x, x], x] = 0$. Then the weight $p$ part of the $E^2$-page has basis
\[ \overline{\mathcal{B}^s Q^l}_{x, l \leq j < 2l + n} ; \gamma_l(x) \].

Comparing with the weight three part of the $E^\infty$-page, which is the weight three part of the mod $p$ homology of the free $\mathbb{R}$-algebra on the $S^2$, we see that there are two classes that do not survive to the $E^\infty$-page, i.e., $\gamma_l(x)$ in bigrade $(2, 6l - 2)$ and $\overline{\mathcal{B}^s Q^l}x$ in bigrade $(1, 6l - 2)$ (cf. [CLM76, III]). Hence there has to be a $d_3$-differential from $\gamma_l(x)$ to $\overline{\mathcal{B}^s Q^l}x$. But $\gamma_l(x)$ is represented by the element $[[x, x], x] \in \text{Lie}_{\mathbb{R}}^p \circ \text{Lie}_{\mathbb{R}}^p(g) \subset \text{Lie}_{\mathbb{R}}^p \circ \text{Lie}_{\mathbb{R}}^p(g)$, which is a cycle because the differential sends it to $[[x, x], x] \in \text{Lie}_{\mathbb{R}}^p(g)$ with the two brackets both from the application of $\text{Lie}_{\mathbb{R}}^p$, a contradiction.

When $p \geq 5$, the same analysis applies to show that:

Proposition 6.7. For $p \geq 5$, the only higher differential in the weight $p$ part of the spectral sequence (11) for $M = \mathbb{R}^n, 2 \leq n \leq \infty$, which converges to $H_*(B_p(\mathbb{R}^n; S^2); \mathbb{F}_p)$, is a $d_{p-2}$-differential $\gamma_p(x) \mapsto \overline{\mathcal{B}^s Q^l}y_n \otimes \sigma^n(x)$.

Heuristically, this is because the bottom non-vanishing mod $p$ Dyer-Lashof operation on a class $x$ of degree $2l$ in the mod $p$ homology of an $\mathbb{R}$-algebra is given by $x \otimes x$, so $\gamma_p(x)$ is redundant.

Remark 6.8. Using the same argument in Corollary 3.26, we deduce from Proposition 6.5 that the $\mathbb{F}_p$-linear spectral Lie operad is not formal, since the weight $p$ part of the $E^2$-page
\[ E^2_{s,t} = \pi_s \pi_t \text{Bar}_* \left( \text{id}, s \mathcal{Z}, \text{Free}^{s \mathcal{Z}}(\Sigma^n \mathbb{F}_p) \right) \Rightarrow H_{s+t}(\text{Free}^{\mathbb{R}^{\text{op}}}(\Sigma^n \mathbb{F}_p); \mathbb{F}_p) \]
is larger than the weight $p$ of $H_{s+\epsilon}(\text{CE}(\Sigma^n \mathbb{F}_p))$.

As an immediate corollary to Proposition 6.5, we see that the weight two part of the spectral sequence (11) collapses on the $E^2$-page, since the $E^2$-page is concentrated in simplicial degree 0 and 1. When $p > 3$, weight three elements on the $E^2$-page are in simplicial degree 1 or 2 since $[[x, x], x] = 0$ by the Jacobi identity. Hence the weight three part of the spectral sequence (11) also collapses on the $E^2$-page.

Corollary 6.9. Let $M^n$ be a parallelizable manifold and $X$ any spectrum. Let $g$ be the $\text{Lie}_{\mathbb{F}_p}$-algebra $\overline{H}^*(M^+; \mathbb{F}_p) \otimes \text{Lie}_{\mathbb{F}_p}(\Sigma^n H_*(X; \mathbb{F}_p))$.
(1) For all \( i \), there is an isomorphism of \( \mathbb{F}_p \)-modules
\[
H_i(B_2(M;X);\mathbb{F}_p) \cong \bigoplus_{s+t=i} \text{wt}_2(H_s(\text{CE}(g))).
\]

(2) If \( p \geq 5 \), then for all \( i \)
\[
H_i(B_3(M;X);\mathbb{F}_p) \cong \bigoplus_{s+t=i} \text{wt}_3(H_s(\text{CE}(g))).
\]

Remark 6.10. For \( M \) a connected \( n \)-manifold, Bödigheimer-Cohen-Taylor showed that
\[
\bigoplus_{k \geq 1} H_*(B_k(M;S^1);\mathbb{F}_p) \cong \bigoplus_{i=0}^{n} H_*(\Omega^{n-i}S^{n+r};\mathbb{F}_p) \otimes \dim H_*(M;\mathbb{F}_p)
\]
for \( r + n \) odd and \( r \geq 0 \) [BCT89]. Their proof does not work in the case where \( r + n \) is even due to the existence of nontrivial self-brackets in \( H_*(\Omega^{n}S^n;\mathbb{F}_p) \) when \( l \) is even. Roughly speaking, their inductive proof relies on the canonical map
\[
H_*(\Omega^{\infty}S^1;\mathbb{F}_p) \to H_*(\Omega^{\infty}S^1;\mathbb{F}_p)
\]
being an injection, which is only true when \( l \) is odd. Corollary 6.9 shows that when \( l \) is even, the mod \( p \) homology of \( B_k(M;S^1), k = 2, 3 \) depends on the cup product structure on \( H^*(M^+;\mathbb{F}_p) \): if \( a \cup b = c \) in \( H^*(M^+;\mathbb{F}_p) \), then the \( d_1 \)-differential sends \((a \otimes x) \otimes (b \otimes x)\) to
\[
c \otimes [x,x] \in g = H^*(M^+;\mathbb{F}_p) \otimes \text{Lie}_g^1(\mathbb{F}_p \{x\}),
\]
which is not zero since \( x \) has internal degree \( l \).

At higher weights, there generally will be higher differentials in the odd primary Knudsen’s spectral sequence (11). In forthcoming joint work with Matthew Chen, we make use of Proposition 6.5 and Drummond-Cole-Knudsen’s computation of the rational homology of the unordered configuration spaces \( B_k(\Sigma_g) \). As a result, we show that the integral homology of \( B_k(\Sigma_g) \) is \( p \)-torsion-free for \( k \leq p \) and \( g \geq 1 \).

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