Reply to “A Comment on Singularities in Quantum Cosmology” *

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Abstract

We argue that the reasonings that underlie a recent comment by Gotay and Demaret [1] are defective. We maintain that, contrary to what they assert, our previous papers [2,3,4] are correct and indeed disprove their conjecture that quantum cosmological singularities are predetermined on the classical level by the choice of time.

PACS numbers: 98.80.Hw, 04.20.Dw, 04.60.Gw

CTP # 2535

May 1996

*This work is supported in part by funds provided by the U. S. Department of Energy(D.O.E.) under cooperative research agreement # DF-FC02-94ER40818.

†Supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico(CNPq), Brazil.
I. INTRODUCTION

A significant problem of quantum cosmology is whether classical singularities persist in the quantum domain, and how quantum singularities are connected to the issue of time. In 1983, after classifying the time variable $t$ of a classically singular model as either “slow”, if the classical singularity occurs at a finite value of $t$, or “fast”, if the classical singularity occurs at $t = \pm \infty$, Gotay and Demaret [5] conjectured that (F) self-adjoint quantum dynamics in a fast-time gauge is always singular, whereas (S) self-adjoint quantum dynamics in a slow-time gauge is always nonsingular.

By means of counterexamples, in [2,3,4] we disproved both of these conjectures. In a recent Comment, Gotay and Demaret [1] claim that these papers are incorrect and that their conjectures remain valid for the quantum cosmologies considered in [2,3,4]. In this Reply we argue that their objections are defective, and maintain that our previous results are correct and indeed prove that both of their conjectures are false.

II. FRW MODEL WITH SCALAR FIELD

The model we will be dealing with is a spatially-flat Friedmann-Robertson-Walker universe filled with a massless scalar field. The general model was originally introduced by Blyth and Isham [6], and the special case with $k = 0$ and $m = 0$ was considered in [4], to which the reader is directed for further details. Einstein’s “$G_{00}$ equation” is

$$3 \frac{\dot{R}^2}{R^2} = \frac{1}{4} \dot{\phi}^2,$$  \hspace{1cm} (1)

where $R$ is the scalar factor and $\phi(t)$ is the homogeneous scalar field. In the gauge $t = \phi$ the above equation is equivalent to

$$\dot{R} = \begin{cases} 
R / \sqrt{12} & \text{if } \dot{R} > 0 \\
-R / \sqrt{12} & \text{if } \dot{R} < 0 
\end{cases}.$$  \hspace{1cm} (2)

The field equations allow of expanding or contracting universes, that is, $R(t) = R_0 \exp(\pm t/\sqrt{12})$, where $R_0$ is an arbitrary positive constant. These are mutually exclusive solutions, depending on the initial conditions. The model is singular at $t = -\infty$ in the expanding case or at $t = +\infty$ in the contracting case. Thus $t = \phi$ is a fast time.

The canonical momentum conjugate to $R$ is
whereas the momentum conjugate to $\phi$ is
\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = -\frac{R^3}{N}\dot{\phi}, \tag{4} \]
where $N$ is the lapse. The super-Hamiltonian constraint is
\[ \frac{p_R^2}{24R} - \frac{p_\phi^2}{2R^3} = 0. \tag{6} \]

According to the Arnowitt-Deser-Misner reduction prescription, given the choice $t = \phi$ the Hamiltonian in the reduced phase space is $H = -p_\phi$. Now, solving Eq.(6) for $p_\phi$ and picking up the negative square-root gives rise to the reduced Hamiltonian
\[ H = \frac{1}{\sqrt{12}} R |p_R|. \tag{6} \]

The positive solution for $p_\phi$ was discarded because in the gauge $t = \phi$ it follows from Eq.(4) that $p_\phi < 0$ since $R > 0$ and $N > 0$ by definition. One sees, therefore, that the Hamiltonian (6) is naturally positive. Hamilton’s equation of motion for $R$ in the reduced phase space is
\[ \dot{R} = \frac{\partial H}{\partial p_R} = \begin{cases} \frac{R}{\sqrt{12}} & \text{if } p_R > 0 \\ -\frac{R}{\sqrt{12}} & \text{if } p_R < 0 \end{cases}. \tag{7} \]

Again because $R > 0$ and $N > 0$ by definition, it is a consequence of Eq.(3) that $p_R$ and $\dot{R}$ have the same sign, so that Eqs.(2) and (7) are identical. This completes the verification that the equations of motion generated by the reduced Hamiltonian (6) are completely equivalent to Einstein’s equations for the gravitational field coupled to the scalar field. Such a consistency check is indispensable if minisuperspace quantization is to have any meaning at all.

Gotay and Demaret [1], however, take for Hamiltonian in the reduced phase space
\[ H_{GD} = \frac{1}{\sqrt{12}} R p_R, \tag{8} \]
and with the help of a standard symmetrization procedure write down the corresponding Hamiltonian operator

$$\hat{H}_{GD} = -\frac{i}{\sqrt{12}} \left( R \frac{d}{dR} + \frac{1}{2} \right).$$  \hspace{1cm} (9)

Notice that the Hamiltonian (8) generates only half of Einstein’s equations, that is, it excludes contracting universes. Therefore, it is not the correct reduced Hamiltonian, and the results based on its quantum counterpart (9) that Gotay and Demaret obtain in Section 2 of their Comment are irrelevant because they refer to something different from the model quantized in [4].

It is not difficult to trace their error. The reduced phase space \((R, p_R)\) is the union of the two disjoint sets \((0, \infty) \times (0, \infty)\) and \((0, \infty) \times (-\infty, 0)\). If \(p_R > 0\) the universe expands from a singularity at \(t = -\infty\), whereas if \(p_R < 0\) the universe contracts to a singularity at \(t = +\infty\), and at the classical level these are mutually exclusive situations. Gotay and Demaret consider only the first case, and supposing \(p_R > 0\) they are led to their Hamiltonian (8). The crux of the matter lies in their innocent-looking phrase “we will consider only the first case.” In so doing they in fact discuss a different model, that is, the original model subject to the additional constraint \(p_R > 0\).

The original classical model admits as initial conditions both \(\dot{R} > 0\) and \(\dot{R} < 0\). By assuming from the start that \(p_R > 0\) they have mutilated the original classical model, unduly restricting the allowed set of initial conditions. Simply put, since the set of Hamilton’s equations generated by their Hamiltonian (8) fails to be equivalent to the full set of classical equations of motion, the conclusions based on its quantum version (9) are meaningless.

Contrary to what Gotay and Demaret impute us in their Comment, never do we employ a “modified quantum Hamiltonian” or make “ad hoc modifications to the quantum dynamics.” The Hamiltonian operator considered in [4] is naturally suggested by the form of the correct reduced Hamiltonian (6), and it is as naturally a positive operator as the classical Hamiltonian (6) is a positive function. On the other hand, the operator (9) is simply not the quantum Hamiltonian because it is the quantum counterpart of the incorrect classical Hamiltonian. We maintain, therefore, that our paper [4] is correct and unequivocally disproves Conjecture (F).

III. DUST-FILLED FRW MODELS

The model investigated in [2,3] is a FRW universe filled with dust. The extended phase space is spanned by the canonical coordinates \((R, p_R)\) and \((\varphi, p_\varphi)\), where \(\varphi\) is the only nonvanishing velocity potential for dust, with \(p_\varphi > 0\). The super-Hamiltonian constraint is
\[ p_\phi - \frac{p_R^2}{24R} - 6kR = 0. \] (10)

For \( k = 0 \) or \( k = -1 \), if \( p_R > -12kR \) the model expands from an initial singularity, while if \( p_R < 12kR \) it collapses to a final singularity. In [3] the model is shown to be self-adjoint but nonsingular in the slow-time gauge \( t = p_R \), thus disproving Conjecture (S). Gotay and Demaret [1] object that the choice of time \( t = p_R \) is not permissible quantum mechanically.

It is true that Conjectures (F) and (S) were put forward only for “dynamically admissible” choices of time. In [5] a gauge is said to be dynamically admissible if the variable chosen as time is a priori bounded neither above nor below. For the sake of simplicity, let us take the case \( k = 0 \). As in the previous model containing a scalar field, the reduced phase space of the dust-filled model is disconnected into two components, corresponding to whether \( p_R > 0 \) or \( p_R < 0 \). Then, fixing one of these components, say \( p_R < 0 \), Gotay and Demaret state that “\( p_R \) is a priori bounded above by zero.”

This argument suffers from the same type of deficiency as the one they used regarding the scalar field model. Choices of time such as \( t = R \) or \( t = p_R^2 \) are true examples of dynamically inadmissible gauges, because in these cases \( t \) is a priori bounded below by zero, without the need of unjustifiable additional restrictions on the classical model. The canonical variable \( p_R \) is not a priori bounded above or below. It becomes bounded above or below only after selecting one of the two components of the reduced phase space, and thus again unduly restricting the set of allowed initial conditions, which amounts to a mutilation of the original classical model. Therefore, the choice of time \( t = p_R \) in [3] is dynamically admissible, and the same can be said of the gauge \( t = p_\mu \), with \( \mu = \ln R \), employed in [2]. Thus, we hold that papers [2] and [3] are correct and disprove Conjecture (S).

IV. CONCLUSION

We have discussed quantum singularities on the basis of a singularity criterion that involves the expectation value of an operator \( \hat{f} \) whose classical counterpart \( f \) vanishes at the classical singularity [5], and certain issues raised in [1] pertain more properly to the question whether this is a reasonable criterion. The fact that in the dust model one may have \( \dot{R}(0) = 0 \) without apparently anything catastrophic or pathologic happening to the quantum system is a symptom that this criterion is not physically adequate. It is also doubtful whether it is general enough to be applicable in all gauges. It seems, therefore, that the formulation of a general and physically reasonable singularity criterion remains an open problem in quantum cosmology.
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