Multistable jittering in oscillators with pulsatile delayed feedback

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Abstract

Oscillatory systems with time-delayed pulsatile feedback appear in various applied and theoretical research areas, and received a growing interest in the last years. For such systems, we report a remarkable scenario of destabilization of a periodic regular spiking regime. In the bifurcation point numerous regimes with non-equal interspike intervals emerge simultaneously. We show that this bifurcation is triggered by the steepness of the oscillator’s phase resetting curve and that the number of the emerging, so-called “jittering” regimes grows exponentially with the delay value. Although this appears as highly degenerate from a dynamical systems viewpoint, the “multi-jitter” bifurcation occurs robustly in a large class of systems. We observe it not only in a paradigmatic phase-reduced model, but also in a simulated Hodgkin-Huxley neuron model and in an experiment with an electronic circuit.

Interaction via pulse-like signals is important in neuron populations, biological, optical and opto-electronic systems. Often, time delays are inevitable in such systems as a consequence of the finite speed of pulse propagation. In this letter we demonstrate that the pulsatile and delayed nature of interactions may lead to novel and unusual phenomena in a large class of systems. In particular, we explore oscillatory systems with pulsatile delayed feedback which exhibit periodic regular spiking (RS). We show that this RS regime may destabilize via a scenario in which a variety of higher-periodical regimes with non-equal interspike intervals (ISIs) emerge simultaneously. The number of the emergent, so-called “jittering” regimes grows exponentially as the delay increases. Therefore we adopt the term “multi-jitter” bifurcation.

Usually, the simultaneous emergence of many different regimes is a sign of degeneracy and it is expected to occur generically only when additional symmetries are present. However, for the class of systems treated here no such symmetry is apparent. Nevertheless, the phenomenon can be reliably observed when just a single parameter, for example the delay, is varied. This means that the observed bifurcation has codimension one. In addition to the theoretical analysis of a simple paradigmatic model, we provide numerical evidence for the occurrence of the multi-jitter bifurcation in a realistic neuronal model, as well as an experimental confirmation in an electronic circuit.

As a universal and simplest oscillatory spiking model in the absence of the feedback, we consider the phase oscillator $d\varphi/dt = \omega$, where $\varphi \in \mathbb{R}$ (mod 1), and $\omega = 1$ without loss of generality. When the oscillator reaches $\varphi = 1$ at some moment $t$, the phase is reset to zero and the oscillator produces a pulse signal. If this signal is sent into a delayed feedback loop [Fig. 1(a)] the emitted pulses affect the oscillator after a delay $\tau$ at the time instant $t^* = t + \tau$. When the pulse is received, the phase of the oscillator undergoes an instantaneous shift by an amount

$$\Delta \varphi = Z(\varphi(t^* - 0)),$$

where $Z(\varphi)$ is the phase resetting curve (PRC). Thus, the dynamics of the oscillator can be described by the following equation:

$$\frac{d\varphi}{dt} = 1 + Z(\varphi) \sum_{t_j} \delta(t - t_j - \tau),$$

where $t_j$ are the instants when the pulses are emitted. Although our analysis is valid for an arbitrary shape of the PRC, for numerical illustrations we use $Z_{ex}(\varphi) :=$
0.1 \sin^6(\pi \varphi), \text{ where } q \text{ controls the steepness of } Z_{\text{ex}}(\varphi) \text{ [see Fig. 1(b)]}. 

In [14] it was proven that a system with pulsatile delayed coupling can be reduced to a finite-dimensional map under quite general conditions. To construct the map for system [1] let us calculate the ISI \( T_{j+1} := t_{j+1} - t_j \). It is easy to see that \( T_{j+1} = 1 - Z(\psi_j) \), where \( \psi_j = \varphi(t_j^\ast - 0) = t_j^\ast - t_j \) is the phase at the moment of the pulse arrival \( t_j^\ast = t_j - p + \tau \) [Fig. 1(c)]. Here, \( P \) is the number of ISIs between the emission time and the arrival time. Substituting \( t_j = t_{j-p} + T_{j-p} + \ldots + T_j \), we obtain the ISI map

\[
T_{j+1} = 1 - Z \left( \tau - \sum_{k=j-P+1}^j T_k \right). \tag{2}
\]

The most basic regime possible in this system is the regular spiking (RS) when the oscillator emits pulses periodically with \( T_j = T \) for all \( j \). Such a regime corresponds to a fixed point of the map (2) and therefore all possible periods \( T \) are given as solutions to

\[
T = 1 - Z (\tau - PT), \tag{3}
\]

where \( P = \lceil \tau/T \rceil \), and hence \( \tau - PT = \tau(\text{mod}T) \). Figure 1(d) shows the period \( T \) as a function of \( \tau \) for \( Z_{\text{ex}}(\varphi) \) and \( q = 5 \).

To analyze the stability of the RS regime, we introduce small perturbations \( \delta_j \) such that \( T_j = T + \delta_j \), and study whether they are damped or amplified with time. The linearization of (2) in \( \delta_j \) is straightforward and leads to the characteristic equation

\[
\lambda^P - \alpha \lambda^{P-1} - \alpha^2 \lambda^{P-2} - \ldots - \alpha^P - \alpha = 0, \tag{4}
\]

where \( \alpha := Z'(\psi) \) is the slope of the PRC at the phase \( \psi = \tau \mod T \) (cf. Fig. 3).

There are two possibilities for the multipliers \( \lambda \) to become critical, i.e. \( |\lambda| = 1 \). The first scenario takes place at \( \alpha = 1/P \) when the multiplier \( \lambda = 1 \) appears, which indicates a saddle-node bifurcation [diamonds in Fig. 1(d)]. In general these folds of the RS-branch lead to the appearance of multistability and hysteresis between different RS regimes [15, 17].

The second scenario is much more remarkable and takes place at \( \alpha = -1 \), where \( P \) critical multipliers \( \lambda_k = e^{i2\pi k/(P+1)} \), \( 1 \leq k \leq P \), appear simultaneously. This feature is quite unusual since in general bifurcations one would not expect more than one real or two complex-conjugate Floquet multipliers become critical at once 9. In the following we study this surprising bifurcation in detail and explain why we call it "multi-jitter".

In order to observe the multi-jitter bifurcation, the PRC \( Z(\varphi) \) must possess points with sufficiently steep negative slope \( Z'(\varphi) = -1 \). For instance, in the case \( Z(\varphi) = Z_{\text{ex}}(\varphi) \), such points exist for \( q > q^* \approx 27 \). For such \( q \), two points \( \psi_A, \psi_B \in (0,1) \) exist where \( Z'_{\text{ex}}(\psi_{A,B}) = -1 \) [see stars in Fig. 1(b)]. This means that for appropriate values of the delay time \( \tau \), such that \( \tau(\text{mod}T) \in \{\psi_A, \psi_B\} \), it holds \( \alpha = -1 \), and the multi-jitter bifurcation takes place. Using Eq. 3 one may determine the corresponding values of \( \tau \) for each possible \( P \geq 1 \):

\[
\tau_{A,B}^P = P(1 - Z(\psi_{A,B})) + \psi_{A,B}. \tag{5}
\]

Figure 2 shows the numerically obtained bifurcation diagram for \( q = 28 \). All values of its ISIs \( T_j \) observed after a transient are plotted by solid lines versus the delay \( \tau \). Black lines correspond to RS regimes, while irregular regimes with distinct ISIs are indicated in red color. Black dashed lines correspond to unstable RS solutions obtained from Eq. 3. For the intervals \( \tau \in (\tau_{A,B}^P,\tau_{A,B}^{P+1}) \), the RS regime destabilizes and several stable irregularly spiking regimes appear.

Let us study in more detail the bifurcation points \( \tau = \tau_{A,B}^P \) for different values of \( P \). For \( P = 1 \), only one multiplier \( \lambda = -1 \) becomes critical. Note that in this case the map (2) is one-dimensional and the corresponding bifurcation is just a supercritical period doubling giving birth to a stable period-2 solution existing in the interval \( \tau \in (\tau_{A,B}^1,\tau_{B}^1) \) [Fig. 2(b)]. For this solution the ISIs \( T_j \) form a periodic sequence \( (\Theta_1,\Theta_2) \) where the periodicity of the sequence is indicated by an overline. It satisfies

\[
\Theta_2 = 1 - Z (\tau - \Theta_1) \quad \text{and} \quad \Theta_1 = 1 - Z (\tau - \Theta_2). \tag{6}
\]

For \( P \geq 2 \), \( P \) multipliers become critical simultaneously at \( \tau = \tau_{A,B}^P \) and the RS solution is unstable for \( \tau \in (\tau_{A,B}^P,\tau_{B}^P) \). Numerical study shows that various irregular spiking regimes appear in this interval, which typically have ISIs sequences of period \( P+1 \). Moreover, they typically exhibit only two different ISIs in varying order [see Fig. 2(b)] bottom. As a result, each solution corresponds to only two, and not \( P+1 \), points in Figs. 2(a), (c)–(e). In the following we call such solutions "bipartite". For larger \( P \), a variety of different bipartite solutions with \( (P+1) \)-periodic ISIs sequences can be observed in \( \tau \in (\tau_{A,B}^P,\tau_{B}^P) \). The stability regions of these solutions alternate and may overlap leading to multistable regimes.

The bipartite structure of the observed solutions can be explained by their peculiar combinatorial origin. Indeed, all bipartite solutions can be constructed from the period-2 solution \( (\Theta_1,\Theta_2) \) existing for \( P = 1 \). Consider an arbitrary \( (P+1) \)-periodic sequence of ISIs \( (T_1, T_2, \ldots, T_{P+1}) \), where each \( T_j \) equals one of the solutions \( \Theta_1, \Theta_2 \) of (6) for some delay \( \tau = \tau_0 \in [\tau_{A,B}^1, \tau_{B}^1] \). Let \( n_1 \geq 1 \) and \( n_2 \geq 1 \) be the number of ISIs equal to \( \Theta_1 \) and to \( \Theta_2 \) respectively. Then it is readily checked that the constructed sequence is a solution of (2) at the feedback delay time

\[
\tau_{n_1,n_2} = \tau_0 + (n_1 - 1) \Theta_1 + (n_2 - 1) \Theta_2. \tag{7}
\]

Red dotted lines in Figs. 2(c), (d), and (e) show the branches of bipartite solutions constructed from (6) with \( P = n_1 + n_2 - 1 = 2, 3, 4 \). Note that these solutions lie
Figure 2: (Wide, color online) (a) ISIs $T_j$ versus the delay $\tau$ for $\mathbf{I}$ with $Z = Z_{ex}$ and $q = 28$. Solid lines correspond to stable, dashed and dotted to unstable solutions. Black color indicates RS regimes, red stands for bipartite solutions (with two different ISIs). Diamonds indicate saddle-node bifurcations and stars multi-jitter bifurcations. (b), (c), (d), and (e) are zooms of (a) for $P = 1, 2, 3, 4$. The lower panels show examples of stable bipartite solutions, for which the corresponding sequences of ISIs are plotted versus time. Solutions within the same dashed frame coexist at a common value of $\tau$, which is also indicated by a vertical blue line in the corresponding zoom.

exactly on the numerical branches which validates the above reasoning. However, some parts of the branches are unstable and not observable. Since each bipartite solution corresponds to a pair of points $(\tau_{n_1, n_2}, T_1)$ and $(\tau_{n_1, n_2}, T_2)$, solutions with identical $n_1$ and $n_2$ correspond to the same points in the bifurcation diagrams in Fig. 2. For instance, when $P = 3$ the branches corresponding to the solutions $(\Theta_1, \Theta_2, \Theta_1, \Theta_2) = (\Theta_1, \Theta_2)$ and $(\Theta_1, \Theta_2, \Theta_2, \Theta_2)$ lie on top of each other.

Let us estimate the number of different bipartite solutions for a given $P \in \mathbb{N}$. The number of different binary sequences of the length $P + 1$ equals $2^{P+1}$. Subtracting the two trivial sequences corresponding to the RS one gets $2^{P+1} - 2$. Disregarding the possible duplicates by periodic shifts (maximally $P + 1$ per sequence) one obtains an estimate for the total number of bipartite solutions for a given value of $P$ as

$$\# \{\text{solutions at } P\} \geq \frac{2^{P+1} - 2}{(P + 1)}.$$ (8)

Notice that all these bipartite solutions exist at the same value of $P$ but for different ranges of the delay $\tau$. Nevertheless, all emerge from the RS solution in the bifurcation points $\tau^P_{A,B}$. To see this, let us consider the limit $\tau_0 \rightarrow \tau^P_A$. In this case the ISIs $\Theta_1(\tau_0)$ and $\Theta_2(\tau_0)$ tend to the same limit $T_A = 1 - Z(\psi_A)$ which is the period of the RS at the bifurcation point. Then, (7) converges to $\tau^P_A + (n_1 + n_2 - 2)T_A = \tau^P_A$, while all bipartite solutions converge to the RS with period $T_A$.

Thus, all bipartite solutions branch off the RS in the bifurcation points $\tau^P_{A,B}$. This finding is clearly recognizable in Figs. 2(c)–(e), where stars indicate the multi-jitter bifurcation points. Numerical simulations show that many of the bipartite solutions stabilize leading to high multistability. In particular, we observe that all bipartite solutions with same values of $n_1$ and $n_2$ exhibit identical stability. This emergence of numerous irregular spiking, or jittering, regimes motivates the choice of the name “multi-jitter bifurcation”.

High multistability is a well-known property of systems with time delays. A common reason is the so-called reappearance of periodic solutions 15. This mechanism may cause multistability of coexisting periodic solutions, whose number is linearly proportional to the delay. Due to multi-jitter bifurcation, multistability can develop much faster, since the number of coexisting solutions grows exponentially with the delay [cf. Eq. (3)]. This suggests that the underlying mechanism is quite different.

Two main features of the feedback in system (1) can be identified as the origin of the multi-jitter bifurcation: (i) the feedback is delayed and (ii) it comes in form of discrete $\delta$-pulses. Firstly, the incorporation of the delay is necessary since fundamental properties of time are responsible for the degeneracy of the multi-jitter bifurcation. To explain this let us consider (2) as a $P$-dimensional mapping $(T_{j-P+1}, ..., T_j) \rightarrow (T_{j-P+2}, ..., T_{j+1})$. Disregarding the calculation of the new ISI $T_{j+1}$ as in (2), all the map does is to move the timeframe by shifting all ISIs one place ahead. The rigid nature of time allows no physically meaningful modification of this part of the map which could unfold the degenerate bifurcation. Moreover, the new ISI $T_{j+1}$ depends exclusively on the sum of the previous ISIs which
than one, a PRC may have slope $Z'$ of discrete solutions with differently ordered ISIs is generated. As a consequence the combinatorial accumulation of coexisting solutions with different orders of ISIs leads to the emergence of a large number ($P + 1$)-periodic bipartite solutions. The insets in the lower part of Fig. 3(a) show two such period-5 bipartite regimes of the FitzHugh-Nagumo oscillator. Note that both of them coexist at $\tau = 126$ ms, which illustrates the multistability of the system.

Bipartite solutions are the basic form of jittering both in phase-reduced models and realistic systems. However, other jittering solutions are possible as well. Beyond the multi-jitter bifurcation the emergent bipartite solutions may undergo subsequent bifurcations. In system (3), we observed higher-periodic, quasiperiodic and chaotic regimes for larger steepnesses of the PRC ($q > 70$). Similar regimes were also found for the Hodgkin-Huxley model. An example showing the onset of chaotic jittering is shown in Fig. 3(b) for $\tau = 59.5$ ms. Jittering may also destabilize as can be seen in Fig. 3(b) at $\tau \approx 62.5$, where the corresponding branch cannot be traced further due to the stability loss.

To conclude, in a phase oscillator with delayed pulsatile feedback (1) we discovered a surprising bifurcation leading to the emergence of a large number ($\sim \exp \tau$) of jittering solutions. We showed that this multi-jitter bifurcation does not only appear in such a simple model, but also in more realistic neuron models and even in physically implemented electronic systems. These findings confirm the generality of our theoretical results and provide motivation for a deeper study of the multi-jitter phenomenon.

The possibility of jittering depends on the steepness of the PRC which is an easily measurable quantity for most oscillatory systems (3). Thus, our findings provide an easy criterion to check for the existence of jittering in a given system. This may prove useful in a variety of research areas, where pulsatile feedback or interactions of oscillating elements takes place. For instance, this might be one of the mechanisms behind the appearance of irregular spiking in neuronal models with delayed feedback (2) and timing jitter in semiconductor laser systems with delayed feedback (25). For applications which exploit complex transient behavior such as liquid state machines (26) the high dimension of the unstable manifold at the bifurcation can be interesting. Furthermore, in view of the possibility of a huge number of coexisting attracting orbits beyond the bifurcation the system can serve as a memory device by associating inputs with the attractors to which they make the system converge.
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