Detecting quantum phase transitions in nonintegrable and long-range Ising chains using spatially minimal measurements

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In a recent work [Dağ, Uhrich, and Halimeh, arXiv:2105.05986], single-site observables have been introduced as a versatile tool for the detection of equilibrium and dynamical criticality in short-range near-integrable many-body models. Here, we extend the potential of single-site observables as probes of quantum phase transitions to strongly nonintegrable models with long-range power-law and next-nearest-neighbor interactions. Our t-DMRG calculations verify the results of the mean-field theory for near-integrable transverse-field Ising chains (TFIC) both with finite-size and finite-time scaling analyses. Furthermore, we find that both finite-size and finite-time analyses suggest a dynamical critical point for a strongly nonintegrable and locally connected TFIC. We demonstrate the presence of a quasi-stationary temporal regime in the power-law interacting TFIC, and extract local-order profiles for TFIC in the long-range Ising universality class with algebraic light cones. Finally, we discuss the robustness of the out-of-equilibrium critical exponent $\beta \sim 4/3$ found for the integrable and near-integrable TFICs.

CONTENTS

I. Introduction 1

II. Nonintegrable TFIC with next nearest neighbor terms
   A. Mean-field theory (MFT) analysis and the MFT results of the near-integrable model 3
   B. t-DMRG calculations of near-integrable and strongly-nonintegrable models 7

III. The quasi-stationary regime in long-range interacting nonintegrable TFIC 9

IV. Robustness tests
   A. Independence of the results for different temporal cutoffs 12
   B. Independence of the results from the initial states 12
   C. Analytical expression for the edge magnetization in the vicinity of the transition 12
   D. Nonequilibrium response in the close vicinity of the transition 15
   E. Numerical evidence on quasi-stationary regime not originating from the strong-zero modes 15
   F. Changing the boundary conditions 17

V. Conclusions 17

Acknowledgments 18

REFERENCES

I. INTRODUCTION

Phase transitions are among the most prevalent physical processes in nature, during which certain properties of a medium, such as its specific heat or susceptibility, can change abruptly and discontinuously [1, 2]. The concepts of universality and scaling allow the description of different models by means of universality classes, whereby models in the same class exhibit identical critical behavior independently of their microscopic details [3–5]. In the far-from-equilibrium regime, universality and scaling are well-established in classical systems [6]. However, universality in far-from-equilibrium quantum many-body systems is not fully developed, despite the introduction and exploration of various concepts of dynamical quantum phase transitions and nonequilibrium criticality [7–37], and the impressive experimental observation of out-of-equilibrium critical behavior in various quantum synthetic matter (QSM) setups [38–47].

With the advent of QSM experiments enjoying a high level of precision and control [48], including single-site addressing techniques [49], the prospect of probing equilibrium and dynamical quantum phase transitions following quantum quenches has become a realistic and appealing one. Although it may sound counterintuitive, extracting equilibrium criticality through quench dynamics can be an experimentally more viable and simpler scheme than the challenging procedure of cooling a quantum many-body system to its ground state. Instead, the system is initialized in an easily accessible product state, and subsequently quenched through a control parameter. This method has enabled the extraction of quantum critical
points and universal scaling laws in various models [16–20, 22, 25, 27, 31, 38, 43, 50].

In this vein of probing quantum phase transitions and criticality through quench dynamics, some of us proposed in Ref. [51] to use single-site observables close to an edge in open-boundary and short-range many-body models. In this scenario, an onset of a quasi-stationary temporal regime appears, thus interrupting an otherwise exponential decay to zero in time. Specifically, Ref. [51] demonstrated that single-site observables at locations $r \ll N/2$, where $N$ is the chain size, regardless of their exact location, predict the same quantum critical point (QCP) both in integrable and near-integrable 1D transverse-field Ising chains (TFIC). This was shown by means of exact calculations based on a Pfaffian formalism as well as mean-field theory (MFT). While the extracted QCP at the singularity coincides with the equilibrium one in the integrable model, it is found to be slightly shifted from the equilibrium point in the near-integrable model and hence, following the literature [20], this critical point out of equilibrium is called dynamical critical point (DCP). Importantly, Ref. [51] also showed the presence of a dynamical critical exponent $\beta \sim 4/3$ emerging in the close vicinity of the DCP where the relaxation times to a quasi-stationary value diverge and a critical long-lived relaxation regime appears. The extracted critical exponent was shown to be robust to weak integrability breaking and the location of the measurement site. In other words, Ref. [51] reveals how the microscopic details are effectively washed away in the close vicinity of a predicted DCP due to diverging scales, and hence it does not matter where one performs this spatially minimal measurement to extract the dynamical criticality.

In this long paper, we detail the aforementioned Pfaffian formalism behind the exact results on single-site observables in open-boundary TFICs, and how MFT is implemented in quench dynamics. An analytical form of the latter was first employed in Ref. [20], this critical point out of equilibrium is called dynamical critical point (DCP). Importantly, Ref. [51] also showed the presence of a dynamical critical exponent $\beta \sim 4/3$ emerging in the close vicinity of the DCP where the relaxation times to a quasi-stationary value diverge and a critical long-lived relaxation regime appears. The extracted critical exponent was shown to be robust to weak integrability breaking and the location of the measurement site. In other words, Ref. [51] reveals how the microscopic details are effectively washed away in the close vicinity of a predicted DCP due to diverging scales, and hence it does not matter where one performs this spatially minimal measurement to extract the dynamical criticality.

Finally, we show the presence of a quasi-stationary temporal regime in TFIC with power-law decaying interactions, whose equilibrium criticality properties belong either to short- or long-range Ising universality class with, respectively, $\alpha \geq 3$ and $\alpha < 3$, where $\alpha$ is the power-law exponent of the interaction strength. We reveal that single-site observables still reflect a difference in the quench dynamics between the sites near the edges and in the middle when the model is long-range interacting $\alpha < 3$, up until $\alpha = 2$ where there is no longer an onset of a quasi-stationary regime. Upon closer study of the model with $\alpha = 2.5$, which is long-range with algebraic light cones [52], we reveal that the local order profiles at different $r$ are consistent with each other and suggest a crossover.

Our work demonstrates that the quasi-stationary regime does not originate from integrability. We show explicitly instances of quasi-stationary behaviour appearing in strongly nonintegrable many-body models and its direct application in probing QCP. The origin of the quasi-stationary regime, as explained in Ref. [51], is due to breaking translational symmetry and making an asymmetric measurement with respect to the symmetry center of the chain, e.g., the middle of the chain. This differentiates the two chain edges as either near or far from the perspective of quasi-particles in the integrable models, or of the wavefronts in operator spreading in the nonintegrable models. This is why the presence, and the onset, of the quasi-stationary regime does not depend on integrability. Rather, it depends on the locality of the underlying system, and the linear or sublinear light cones that are hosted in the dynamics of operator spread. Accordingly, we observe that the quasi-stationary regime does not occur when the model becomes long-range with...
logarithmic light cones [53]. To further substantiate our claim that the onset of the quasi-stationary regime, as a boundary-induced phenomenon, is of geometric origin, we demonstrate a lack of conclusive evidence for strong zero modes [54] in the dynamics. Additionally, we find the presence of a quasi-stationary regime even when the hard edges are smoothed out. This implies that our theoretical predictions could in principle be tested even in imperfect conditions where hard boundaries might not be present, e.g., in trapped atomic and ionic gases.

Although we focus on the dynamics of single-site observables, since we aim to dynamically probe ferromagnetic order in our paper, we emphasize that our method could be easily adjusted to probe other quantum orders, e.g., anti-ferromagnetic order, by increasing the unit cell of the underlying lattice. Therefore, our work and Ref. [51] are the first examples of spatially minimal measurements that could probe quantum phase transitions and nonequilibrium criticality.

There are two relevant temporal cutoffs in our study: i) ultraviolet (short-time, short-distance) cutoff $t^*$ and ii) infrared (long-time, long-distance) cutoff $t_l$ [25]. When we utilize the cluster theorem, we set the infrared cutoff as a parametric cutoff in the transverse field $t_l(h)$. For $t$-DMRG calculations, we use infrared cutoffs either parametric in the system size $t_l(N)$ (in finite-size analysis) or set to a fixed time $t_l$ (infinite-time analysis). In Sec. II we exhibit the formalism behind our calculations, express the cluster theorem, and systematically discuss the $t$-DMRG results on our locally-connected nonintegrable models. Then, in Sec. III, we focus on the quench dynamics of single-site observables, and local order profiles in the power-law interacting TFIC. Finally, we test the robustness of the dynamical critical exponent $\beta \sim 4/3$ and provide further evidence on the origin of the quasi-stationary regime in Sec. IV. We conclude in Sec. V.

II. NONINTEGRABLE TFIC WITH NEXT NEAREST NEIGHBOR TERMS

The TFIC with NNN interaction strength $\Delta$ reads

$$H = -J \sum_{r=1}^{N-1} \sigma^z_r \sigma^z_{r+1} - \Delta \sum_{r=1}^{N-2} \sigma^z_r \sigma^z_{r+2} + h \sum_{r=1}^{N} \sigma^x_r, \quad (1)$$

where $\sigma^x_{r,z}$ are the Pauli spin matrices on site $r$, $h$ is the transverse-field strength, $N$ is the length of the chain, and we fix the nearest-neighbor (NN) coupling strength $J = 1$ as the energy scale. While the Ref. [51] focused on either noninteracting or weakly coupled TFIC by invoking MFT, in this section we systematically analyze the single-site quench dynamics of locally connected nonintegrable TFIC with NNN couplings. The following subsection lays out the MFT formalism that has been applied to obtain the results in Ref. [51] on weakly interacting TFIC. Subsequently, we present the $t$-DMRG calculations of both weakly and strongly interacting TFIC.

A. Mean-field theory (MFT) analysis and the MFT results of the near-integrable model

We first review the quench dynamics in the noninteracting fermion picture, and then introduce the MFT formalism in the following discussion.

We map the integrable TFIC to a noninteracting fermionic model in 1D via the transformation [5],

$$\sigma^x_r = -\prod_{s<r} (1 - 2c^\dagger_s c_s) \left( c_r + c_r^\dagger \right), \quad (2)$$

$$\sigma^y_r = 1 - 2c^\dagger_r c_r,$n

$$\sigma^z_r = -i \prod_{s<r} (1 - 2c^\dagger_s c_s) \left( c_r - c_r^\dagger \right),$$

to obtain the noninteracting Hamiltonian

$$H = \sum_r \left[ -J(c^\dagger_{r+1} c_{r+1} + c^\dagger_r c_r + H.c.) + 2hc^\dagger_r c_r \right]. \quad (3)$$

Expressing the local magnetization of a bulk spin ($\sigma^z_r(t)$) in this noninteracting picture brings a string of operators, and hence is not tractable. Instead, one could utilize the Wick theorem and Pfaffian formalism for two-point correlators to encode the string of operators into an anti-symmetric matrix [5, 13]. This direction of solution requires rewriting a single-site observable in terms of two-point correlators [25] and we therefore invoke the cluster theorem [13],

$$\langle \sigma^z_r(t)\sigma^z_{N-r-1}(t) \rangle \sim \langle \sigma^z_r(t) \rangle \langle \sigma^z_{N-r-1}(t) \rangle, \quad (4)$$

where $r \ll N/2$ is close to the left boundary. The cluster theorem breaks down outside of the lightcone, and the corresponding breakdown time $t_l$ can be understood in the context of operator spread between two well separated sites $r$ and $N - r + 1$ that eventually get correlated with each other. This time can be estimated based on the maximum quasiparticle velocities $v_q$, as $t_l = \Delta x/(2v_q)$ where $\Delta x = N - 2r + 1$ is the distance between two spins that are equidistant from the symmetry center of an open-boundary chain in Eq. (4), which is the middle of the chain. For such symmetrically placed sites, the non-equilibrium response is the same, and one can thus write

$$\langle \sigma^z_r(t) \rangle = \sqrt{\langle \sigma^z_r(t) \sigma^z_{N-r+1}(t) \rangle} \equiv C_r(t). \quad (5)$$

Therefore, we can extract the dynamical evolution of a spin at site $r$ from the equal-time two-point correlators of sites $r$ and $N - r + 1$. The latter can be written in terms of auxiliary operators $\phi^\pm = c^\pm + c^\mp$ as

$$\langle \sigma^z_r(t) \sigma^z_{N-r+1}(t) \rangle = \left\langle \phi^r(t) \prod_{s=r+1}^{N-r} \phi^+(s) \phi^-(s) \right\rangle \times \phi^+_{N-r+1}(t). \quad (6)$$

This is, in fact, the expectation value of the so-called string order parameter (SOP) [55]. It can be calculated by invoking Wick’s theorem, which allows one to
re-express the above expectation value as a sum over products of elementary contractions, which in turn is the Pfaffian of an appropriately constructed antisymmetric matrix \( T(t) \) i.e. \((C_r(t))^2 = \text{Pf}(T(t))\) \cite{5, 13}. Although this is in general a complex number, it follows from Eq. (4) that for \( t < t_1 \) we actually have \((C_r(t))^2 \in \mathbb{R} \) and positive, so that we may compute \( C_r(t) = |\text{Pf}(T(t))|^{1/2} = |\sqrt{\det(T(t))}|^{1/2} \). This is advantageous since it is numerically more efficient to calculate determinants as compared to Pfaffians.

To construct the matrix \( T(t) \) we only need to calculate all possible elementary two-point contractions \( \langle \phi^q_p(t) \phi^r_s(t) \rangle \) where \( p, q = \pm \) and \( r \leq a \leq b \leq N-r+1 \). Now we briefly review how to implement a sudden quench in this picture by following Ref. \[22\].

Quenching from an initial Hamiltonian \( H_i \) with transverse field \( h_i \), we first solve

\[
H_i = \sum_k E_k^i \alpha_k^\dagger \alpha_k, \tag{7}
\]

where \( E_k^i \) are the single particle energies, \( \alpha_k \) and \( \alpha_k^\dagger \) are the new annihilation and creation operators, respectively, after diagonalization where the ground state \( |\psi_0\rangle \) satisfies \( \alpha_k |\psi_0\rangle = 0 \). The solution, in a general form, follows as \cite{56}

\[
\begin{pmatrix}
\alpha \\
\alpha^\dagger
\end{pmatrix} = \begin{pmatrix}
G_i & F_i \\
F_i & G_i
\end{pmatrix} \begin{pmatrix}
c_i \\
c_i^\dagger
\end{pmatrix}, \tag{8}
\]

where \( c_i = (c_1, c_2, \cdots, c_N)^T \) and similarly for the creation operator \( c_i^\dagger \). The expressions for the \( G \) and \( F \) block matrices will follow shortly. By solving the eigensystem of

\[
[(A_i - B_i)(A_i + B_i)] |\Phi_k^i\rangle = (E_k^i)^2 |\Phi_k^i\rangle, \tag{9}
\]

we obtain the eigenenergies \( E_k^i \) and eigenvectors \( |\Phi_k^i\rangle \).

Note that \( A_i \) and \( B_i \) are the nearest neighbor hopping and pairing terms of the Hamiltonian, respectively. Then the Hamiltonian could be written as,

\[
H_i = \begin{pmatrix}
A_i & B_i \\
B_i^\dagger & -A_i
\end{pmatrix}, \tag{10}
\]

in the \((\epsilon \epsilon')^T\) basis. Next we use the eigensystem \((E_k^i, |\Phi_k^i\rangle)\) to obtain

\[
|\Psi_k^i\rangle = \frac{1}{E_k^i} \left( |\Phi_k^i\rangle (A_i - B_i) \right)^T. \tag{11}
\]

Let us emphasize that the expression above would not work for zero edge modes where \( E_k^i = 0 \), and handling the case with zero mode is crucial for us. This is because we are primarily interested in the quenches from and to the ordered phase in an open-boundary TFIC. In this case, we set \(|\Psi_k^f\rangle = -|\Psi_k^f\rangle\) \cite{56} and make sure that the resulting state is linearly independent from the other modes. Finally we calculate the matrices \( G_i \) and \( F_i \) in terms of \(|\Phi_k^i\rangle\) and \(|\Psi_k\rangle\). Defining matrices

\[
\Phi_i = \begin{pmatrix}
|\Phi_1^i\rangle & |\Phi_2^i\rangle & \cdots & |\Phi_N^i\rangle
\end{pmatrix}, \\
\Psi_i = \begin{pmatrix}
|\Psi_1^i\rangle & |\Psi_2^i\rangle & \cdots & |\Psi_N^i\rangle
\end{pmatrix},
\]

the block matrices follow

\[
G_i = \frac{1}{2} \left( \Phi_i^T + \Psi_i^T \right), \\
F_i = \frac{1}{2} \left( \Phi_i^T - \Psi_i^T \right). \tag{12}
\]

Similarly for the final Hamiltonian \( H_f \) with transverse field \( h_f \), one could write

\[
\begin{pmatrix}
\beta^i \\
\beta^i^\dagger
\end{pmatrix} = \begin{pmatrix}
G_f & F_f \\
F_f & G_f
\end{pmatrix} \begin{pmatrix}
c_f \\
c_f^\dagger
\end{pmatrix}, \tag{13}
\]

together with corresponding \( \Phi_f \) and \( \Psi_f \). We calculate the transfer matrices with the help of these block matrices,

\[
T_1 = G_f G_i^T + F_f F_i^T, \\
T_2 = G_f F_i^T + F_f G_i^T.
\]
The matrix elements $s_{ks}$ for $k,s$ to utilize in the following contractions, based on this formulation, we construct matrices $M$ would like to make use of basis we have the states and the operators. Because we transfer matrices to write $\phi_T$, $\text{tric}$ matrix These transfer matrices are used to finally calculate the accessible, before showing the effects of finite bond dimension. $\chi$ bond dimensions, $/J$.

Now we can construct the matrix $T(t)$ at time $t$ with the matrix elements $T_{ks}(t) = \langle \phi^+_a(t)|\phi^+_b(t)\rangle$), where $1 \leq k < s \leq 2\Delta$, $p = +(-)$ for $k$ even(odd) and $q = +(-)$ for $s$ even(odd). The relation between parameters $a,b$ and $k,s$ reads $a = r + [k/2]$ and $b = r + [s/2]$, because $r \leq a \leq b \leq N - r + 1$. Having constructed $T(t)$, one can then extract $C_r(t) = |\sqrt{\det T(t)}|^{1/2}$, as discussed below Eq. (6). Let us note that this quench formalism for non-interacting fermions was recently employed in Refs. [22, 25].

In order to incorporate the MFT formalism to this calculation, we need to express the NN term in the TFC Hamiltonian in the fermionic picture. This expression reads,

$$-\Delta \sum_r (c_r - c_r^\dagger) \left( 1 - 2c_{r+1}^\dagger c_{r+1} \right) (c_{r+2} + c_{r+2}^\dagger),$$

$$= \Delta \sum_r (c_r^\dagger - c_r) \left( 1 - 2c_{r+1}^\dagger c_{r+1} \right) (c_{r+2} + c_{r+2}^\dagger),$$

$$= \Delta \sum_r \phi_r^- \phi_{r+1}^+ \phi_{r+1}^- \phi_{r+2}^{+},$$

(16)

where $\Delta > 0$ and $\phi^\pm$ stand for the auxiliary fermions.

In Hartree-Fock expansion, we assume $|\Delta| \ll |J|$, and write Eq. (16) as

$$= \Delta \sum_r \left[ \langle \phi_{r}^{-}\phi_{r+1}^{+}\rangle_{t \rightarrow \infty} \phi_{r+1}^{-}\phi_{r+2}^{+} \right.$$  

$$+ \phi_{r}^{-}\phi_{r+1}^{+}\langle \phi_{r+1}^{-}\phi_{r+2}^{+}\rangle_{t \rightarrow \infty} - \phi_{r}^{-}\langle \phi_{r+1}^{+}\phi_{r+1}^{-}\rangle_{t \rightarrow \infty} \phi_{r+2}^{+}$$

$$- \langle \phi_{r}^{+}\phi_{r+2}^{-}\rangle_{t \rightarrow \infty} \phi_{r+1}^{-}\phi_{r+1}^{+} \right].$$

(17)

Here the $\langle \cdot \rangle_{t \rightarrow \infty}$ means that we calculate the free fermion problem and obtain the correlators with respect to the steady-state in the infinite time limit, instead of the ground state which would be used to calculate the equilibrium QCP. An analytical quench MFT formalism was first introduced in Ref. [20] to calculate a dynamical order parameter, based on two-point correlators in a periodic chain, by utilizing the momentum space representation where the infinite time limit can indeed be taken. In our numerics for an open-boundary chain, we treat the largest time point allowed by the cluster theorem, $t_{f} = \Delta \gamma/(2v_g)$, as the asymptotically infinite time limit. Since the relaxation time of the system observables critically diverges as we approach the QCP (Sec. IV D), it is likely that an infrared cutoff at $t_f$ might not capture a possible steady-state solution in the $t \to \infty$ limit in the close vicinity of the transition. This could explain why our observed DCP at $h_{dc} = 1.1437 \pm 0.0001$ [51] for $\Delta = 0.1$ is slightly smaller than the DCP found in Ref. [20], which is 1.149. Note that for an open-boundary chain one needs to carefully take the edges of the chain into account when calculating Eq. (17). Using the above expansion, we obtain an effective mean field Hamiltonian which has slightly stronger NN coupling compared to the free problem, as well as new NNN couplings. Further, the effective chemical potential slightly decreases, which is reasonable when we think about how the critical point shifts to favor order, e.g., for $\Delta = 0.1$, $h_{c} \sim 1.16$ [20].

When applying the cluster theorem to the MFT of the weakly-interacting nonintegrable TFC, one needs to estimate the lightcone (correlation) velocity $v_q$ of the
model. While for the integrable TFIC this velocity is analytically known, this is not true when we introduce nonintegrability to the model. Here we approximate a quasiparticle velocity based on the analytical prediction of the integrable TFIC: \( v_q = 2h \) for \( h \leq h_c \) and \( v_q = 2h_c \) for \( h > h_c \) [13]. Since this is only an approximation, we sometimes exceed the time when the cluster theorem really breaks down. This means that the distant sites of the chain had already become correlated with one another. This time can be observed with a recurrence attempt in the figures both in the Ref. [51] and in this paper, which is also a sign of finite-size effects.

Fig. 1 shows comparisons between MFT, t-DMRG and exact diagonalization (ED) results at \( h = 0.5, h = 1.1 \) and \( h = 1.2 \), where the QCP is \( h_c \sim 1.16 \). We observe that the MFT analysis can even capture the correct frequency of the oscillations in early times and the general trend of the nonequilibrium response successfully, although it does not totally match with the exact methods, which is expected due to the fact that it is an approximate method that averages out the interactions. In Fig. 1a, both the ED and t-DMRG responses show finite-size effects in the form of quantum revivals (solid-blue and red-dashed-dotted). The finite-size effects of these methods can also be clearly seen in Figs. 1b and 1c. As we approach the QCP and in the dynamically-disordered phase, the match between the exact methods and the MFT does not survive past \( tJ = 10 \). This is either because of the finite-size effects, as apparent in ED response, or the finite maximum bond dimensions set in the t-DMRG algorithm, see Sec. II B for more details. In order to demonstrate the effects of finite bond dimension, we compare the nonequilibrium responses calculated with t-DMRG for different maximum bond dimensions. Fig. 2 shows these responses at \( h/J = 1.3 \) in subfigure (a) and at \( h/J = 1 \) in subfigure (b). We observe that the response with bigger maximum bond dimension, \( \chi_m = 200 \) departs from the response with \( \chi_m = 100 \) at around \( tJ \sim 10 \), exhibiting a downward envelope and hence agreeing with the MFT nonequilibrium responses in the dynamically disordered phase, see Fig. 2a. This feature is shown with a grey-dotted arrow. As time increases, the nonequilibrium response with \( \chi_m = 200 \) starts to exhibit features similar to those of the response with \( \chi_m = 100 \), e.g., recurrences, as pointed out with a black-solid arrow. We note that these are effects of finite maximum bond dimension set in our t-DMRG algorithm. One could predict that as the maximum bond dimension \( \chi_m \) increases, such effects will occur later in time, and instead the response will follow closely to that of MFT response with an oscillatory downward trend. However, there is a trade-off between the maximum bond dimension and the maximum accessible simulation time, and simulating longer times while achieving a satisfactory precision for the response is simply out of our computational reach. Fig. 2b shows the same comparison between different \( \chi_m \) alongside with the MFT result at \( h/J = 1 \), which is a point in the dynamically-ordered phase. The t-DMRG result with \( \chi_m = 100 \) shows an upward trend as time increases, in contrast to the MFT result which reaches a plateau, as expected from our theory based on noninteracting fermions. We also plot the response with \( \chi_m = 200 \) and observe that the nonequilibrium response of greater maximum bond dimension approaches that of the MFT results, departing from the t-DMRG result with \( \chi_m = 100 \) for a small interval of time. Additionally we notice that the response with \( \chi_m = 200 \) (red-sold) also demonstrates effects of finite maximum bond dimension in later times with a recurrence similar to what is observed in the response with smaller bond dimension \( \chi_m = 100 \) (blue-star). Therefore, we argue that as \( \chi_m \) increases, the t-DMRG result should converge to

\[
\begin{array}{|c|c|}
\hline
\text{observable} & \beta \\
\hline
\sigma_1 & 1.3378 \pm 0.0043 \\
\sigma_2 & 1.3568 \pm 0.0039 \\
\sigma_3 & 1.3573 \pm 0.0038 \\
\sigma_4 & 1.3647 \pm 0.0023 \\
\sigma_{12} & 1.3618 \pm 0.0132 \\
\hline
\end{array}
\]

TABLE I. Fit parameters for the scaling law of the near-integrable model in the vicinity of the transition.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{cutoff} & \sigma_1,b & \sigma_2,b & \sigma_3,b \\
\hline
 t^* = 10 & 0.6842 \pm 0.0437 & 0.6973 \pm 0.052 & 0.7003 \pm 0.0505 \\
 t^* = 15 & 0.6957 \pm 0.0449 & 0.6865 \pm 0.0492 & 0.6710 \pm 0.0448 \\
\hline
\end{array}
\]

TABLE II. Fit parameters for the system-size scaling of single-site observables at \( r = 1, 3, 6 \) at the DCP for different ultraviolet cutoffs.

FIG. 3. (Color online). Order profiles calculated with t-DMRG for weakly nonintegrable TFIC with \( \Delta = 0.1J \), (a) for different system sizes \( N = 48, 72, 96 \) at site \( r = 3 \) when a (infrared) temporal cutoff of \( t_0 = N/2 \) is applied. The order profile is tested against different infrared and ultraviolet temporal cutoffs and shown to be robust (see text). (b) Finite-time scaling analysis at a system size of \( N = 96 \) for different infrared cutoffs, shown in the legend, with a fixed ultraviolet cutoff of \( t^* = 0 \). The upward and downward arrows highlight the \( h/J \) points where the order grows or diminishes, respectively with increasing (a) system size and (b) simulation time.
the MFT result. This observation reveals an interesting feature where the approximate MFT result can be obtained in full quantum dynamics only with sufficiently large maximum bond dimensions.

In the Ref. [51], we found the dynamical scaling exponent $\beta \sim 4/3$ for $C_r(h) - C_r(h_c) \equiv C_r(h_n) \propto h_n^p$, where $h_n = (h_c - h)/h_c$. Here, we show that the dynamical scaling exponent is robust to changes in the location $r$ of the observation. Table I presents the fitting parameters for the dynamical scaling law in the vicinity of the transition for the near-integrable model for different sites $r \ll N/2$. For all sites we find a similar exponent of $\beta \sim 4/3$, which matches the scaling exponent found for the integrable model (see Sec. IV and Ref. [51]). We also study the system size scaling at the DCP $h_{dc} = 1.1437 \pm 0.0001$ with different ultraviolet cutoffs $t^*$, all of which give similar exponents $C_r(h_c) \propto aN^{-b}$ where $b \sim 0.7$ and $a$ is nonuniversal. The exponents for different $t^*$ and sites $r$ are summarised in Table II.

B. t-DMRG calculations of near-integrable and strongly-nonintegrable models

We utilize the ITensor environment [57] to construct our matrix product states (MPS) and Trotter decomposition for the time evolution of the MPS. We set a maximum bond dimension $\chi_m$ for the resulting compressed MPS, and set the initial truncation error cutoff for the compression of the MPS as $\epsilon \sim 10^{-8}$. The truncation error cutoff is adaptive: As the maximum bond dimension is reached for the resulting MPS, the error cutoff increases systematically up until a hard error threshold of $\epsilon \sim 10^{-5}$ to be able to access longer times. Setting a maximum allowed bond dimension thus introduces an error which grows with time. Consequently, we are confined to early times for which the above interval of the error thresholds is satisfied.

We study two nonintegrable models with t-DMRG: (i) Weakly interacting near-integrable model, $\Delta = 0.1$, whose MFT results are presented in Ref. [51] and compared to the results obtained from the exact numerics in this paper. (ii) A strongly nonintegrable model, $\Delta = 1$ where MFT is inadequate, and we therefore employ t-DMRG only. The latter is numerically exact, but within a given fidelity threshold the accessible evolution times are limited and naturally shorter than those achievable for the (near-)integrable model.

Figs. 3a and 3b show the finite-size and finite-time scaling analysis, respectively for the weakly interacting TFIC at $r = 3$. In Fig. 3a, we observe a dynamical order profile and a crossing point for system sizes between $N = 48$ and $N = 96$. The crossing point is found to be in the
interval of \( h_{dc}/J \in (1.05, 1.1) \) where the dynamical order builds up for \( h/J \leq 1.05 \) as the system size increases and it vanishes for \( h/J > 1.1 \). We use \( t^* = 10 \) and \( t_l = N/2 \) as ultraviolet and infrared cutoffs in these figures, however the phase diagram is robust to changes in the temporal cutoffs, as we tested with values in the intervals \( t^* \in [8, 12] \) and \( t_l \in [N/3, N/2] \). In Fig. 3b, we employ a form of finite-time scaling analysis \([20]\) where we see a crossing point between order profiles with different infrared cutoffs at a fixed ultraviolet cutoff \( t^* = 0 \). The crossing point resides in the interval of \( h_{dc}/J \in (1.05, 1.1) \) agreeing with the finite-size analysis. For \( h/J < 1.05 \) and \( h/J > 1.1 \), the dynamical order grows or diminishes with longer simulation times, respectively. Given that the system sizes are constrained in \( t\)-DMRG compared to MFT analysis, we observe a smaller DCP than what we have found with MFT analysis. We repeat the same calculation for \( r = 6 \), however with slightly larger system sizes. Figs. 4a and 4b demonstrate the finite-size and finite-time scaling analysis, respectively for this parameter set. In both cases, we find the crossing interval to be \( h_{dc}/J \in (1.1, 1.15) \). Finally, we zoom on the dynamical phase diagram of the MFT analysis in Fig. 4c to demonstrate that the crossing is present in this method too, however for small system sizes, e.g., \( N = 96 \) shown with black arrows around \( h/J \sim 1.14 \). On the contrary, all system sizes \( N \geq 480 \) collapse on a singular point (see Ref. \([51]\)) suggesting that these finite but large system sizes effectively simulate the thermodynamic limit. Therefore, we conclude that the \( t\)-DMRG method of the simulated system sizes \( N \leq 120 \) supports the presence of a dynamical critical point, verifying the results of the MFT method presented in Ref. \([51]\). Whether the \( t\)-DMRG-predicted DCP matches with the QCP better than the MFT-predicted DCP is an interesting question to explore in the future.

Next, we consider the strongly nonintegrable TFIC with \( \Delta = 1 \), whose QCP lies at \( h_c \sim 2.46 \) \([25]\). Figure 5a and 5b show the local order profiles for \( r = 3 \) and \( r = 6 \), respectively. As the system becomes strongly interacting, the maximum system size and the evolution times become more limited. In this set of calculations, we study \( N = 42 \) and \( N = 48 \) with maximum evolution times of \( tJ \sim 20 \). Hence, we apply an infrared cutoff of \( t_l = N/3 \) and set an ultraviolet cutoff \( t^* = 0 \) to increase the temporal range of data to average over. Both figures exhibit a crossing point similar to that of the discussion in the previous paragraph, in the interval of \( h_{dc}/J \in (2.3, 2.35) < h_c \) where for \( h/J < 2.3 \) the dynamical order increase with increasing system size, and for \( h/J > 2.35 \) decreases. Therefore, we conclude that (i) the crossing point seems to be independent of the measurement site, and (ii) the presence of a crossing point in \( t\)-DMRG data hints at the presence of a DCP. We note that more data with larger system sizes and longer simulations times is required to test the robustness of these results, determine the exact location of the DCP and further look for a critical dynamical scaling in this strongly nonintegrable model. Nevertheless, single-site observables close to an edge seem to be a probe of criticality generally in short-range models, not limited to non-interacting or weakly interacting systems.

Finally, we apply the finite-time scaling analysis on the strongly nonintegrable TFIC with \( \Delta = 1 \) shown in Figs. 6a and 6b for \( r = 3 \) and \( r = 6 \), respectively. Although there is a well-defined finite-time crossing point for this set of parameters, the crossing suggests \( h_{dc}/J \sim 2.2 \) for \( r = 3 \) and \( h_{dc}/J \in (2.25, 2.3) \) for \( r = 6 \) all of which is less than the result determined by the finite-size analysis. To check whether the results depend on the maximum bond dimension, we set \( \chi_m = 200 \) for \( r = 3 \), repeat the calculation on the relevant region in Fig. 6c, and observe that the crossing point does not change. Although more data is necessary for conclusive results, we note that the mismatch between finite-size and finite-time analyses as well as the discrepancy between different sites in finite-time analysis might point to a change in the light cone structure, e.g., from lin-
ear to power-law, as the transverse field increases and approaches the equilibrium QPT in finite-size systems.

III. THE QUASI-STATIONARY REGIME IN LONG-RANGE INTERACTING NONINTEGRABLE TFIC

The quasi-stationary temporal regime also emerges in the long-range hard-boundary TFIC with power-law decaying interactions, as was previously noted in Ref. [58] in the context of prethermalization. The Hamiltonian for the one-dimensional long-range TFIC reads,

$$H = -\sum_{r,r'} J(r,r') \sigma^x_r \sigma^x_{r'} + h \sum_r \sigma^z_r; \quad (18)$$

where $J(r,r') = J/|r-r'|^\alpha$. In the limit where $\alpha = 0$, the model becomes integrable with all-to-all interactions, e.g. LMG model; whereas in the limit of $\alpha \to \infty$ the model reduces to short-range NN TFIC. For $\alpha \geq 3$, the model belongs to the short-range Ising universality class [59, 60], and it has algebraic light cones for $\alpha > 2$ that approach linear cones as $\alpha \to \infty$ [52]. In this section, we provide numerical evidence on the presence of a quasi-stationary regime in TFIC with various $\alpha$, and determine when the quasi-stationary regime breaks down. Furthermore, we calculate the local order profiles at $\alpha = 2.5$, which is a long-range model with algebraic light cones.

All data presented in this section is obtained from the time-dependent DMRG method [61–65] with Krylov time evolution [66]. We find convergence for a time-step of $\Delta t = 0.01J$ and a fidelity threshold between $10^{-8} - 10^{-6}$ for our most stringent calculations. Power-law profiles are approximated as a sum of five exponentials fitted over the length of the chain, as is optimal for a matrix product operator formulation [67].

Fig. 7 shows the nonequilibrium responses of single-site observables near the edge, and in the middle of the chain, as well as the total magnetization evolving under Hamiltonian Eq. (18) with $\alpha > 3$ and $h = 0.5\hat{h}_c$ quenched from a polarized state. The unit notation $\hat{h}$ operator formulation [67].

Despite the approximations as a sum of five exponentials fitted over the length of the chain, as is optimal for a matrix product operator formulation [67].

**Fig. 7.** Nonequilibrium responses of $\sigma^x_{3/2}$ (spin in the middle of the chain), $\sigma^z_{c}$ (spin close to the boundary) and the total magnetization, when (a) $\alpha = 10$, (b) $\alpha = 4$ (short-range power-law decaying interactions), at transverse field $h = 0.5\hat{h}_c$. will be explained shortly. We observe that the single-site observable close to the boundary (red) develops a quasi-stationary regime whereas the observable in the middle of the chain (blue) exhibits decay. This behavior, consistently, is similar to what we have observed in the NN TFIC model, c.f. Sec. II and [51]. Although the decay of total magnetization seems to be slowing down, the data is not conclusive to determine its long-time behavior. We notice the oscillatory quasi-stationary regime for $\alpha \lesssim 4$ for the single-site observable near the edge, which was also observed for nonintegrable NNN TFIC [25, 51].

Fig. 8 demonstrates the nonequilibrium responses for long-range TFIC with $\alpha \lesssim 3$. Although the oscillations grow as we decrease $\alpha$, one could still observe the onset of a quasi-stationary regime in the observable near the edge of the chain for $\alpha = 3$, compared to the observable in the middle of the chain (Fig. 8a). The difference in the nonequilibrium responses of spins near the edge and in the middle of the chain decreases as we keep decreasing $\alpha$. Fig. 8b, for $\alpha = 2.5$, demonstrates for the first time an onset of a quasi-stationary regime for a spin in the middle of the chain. However, the onset of the quasi-stationary regime is delayed in the middle of the chain compared to the edge, and this observation points to the locality of the underlying Hamiltonian that is still preserved to an extent. In contrast, there is no boundary effect observed at $\alpha = 2$ (Fig. 8c), where we do not see any difference in the general trend of the nonequilibrium responses. This is physically intuitive, because the model possesses algebraic light cones for $\alpha > 2$ instead of a logarithmic cone, once more pointing to the importance of the locality in the formation of a quasi-stationary regime. As discussed in Ref. [51], the origin of the quasi-stationary regime could be explained in the context of the asymmetric motion of quasi-particles in opposite directions, and the reflection from the nearest hard boundary. When the model is no longer integrable, even though quasi-particles do not exist one could still refer to correlation speeds measured through linear or sublinear light cones. In conclusion, the presence of a quasi-stationary regime does not depend on the integrability, but rather depends on the locality of the underlying Hamiltonian. In this sense, this boundary induced temporal regime acts as a signature of linear, nearly-linear or sublinear light cones.

To demonstrate that the quasi-stationary regime is not a finite-size effect, we show in Fig. 9 the nonequilibrium responses of a spin close to the boundary at $r = 6$ for different system sizes and different $\alpha$. For a given $\alpha$, one can determine the time at which finite-size effects kick in by observing when the data for different system sizes $N = 32, 64$ no longer overlap. Note that for $\alpha = 10$ and $\alpha = 4$, the quasi-stationary regime develops before the finite-size effects appear.

Finally, we set $\alpha = 2.5$, which yields a long-range model [58] with algebraic light cones [52], and study the resulting local order profiles due to the quasi-stationary regime. In all of our $t$-DMRG calculations on long-range TFIC, we Kac-normalize the interaction term by dividing...
FIG. 8. Nonequilibrium responses of $\sigma_{N/2}$ (spin in the middle of the chain), $\sigma_6$ (spin close to the boundary) and the total magnetization, when (a) $\alpha = 3$, (b) $\alpha = 2.5$ and (c) $\alpha = 2$ (long-range power-law decaying interactions), at transverse field $h = 0.5h_c$.

FIG. 9. The nonequilibrium responses of $C_r(t)$ for different $\alpha = 2, 4, 10$ and different system sizes $N = 32, 64$ at transverse field $h = 0.5h_c$.

FIG. 10. Nonequilibrium responses of the order parameter observable, total magnetization $\sum_{r=1}^{N} \sigma_r^z / N$, quenched from a polarized state to $h/h_c = 1.04, 1.05, 1.06, 1.07, 1.08, 1.09, 1.1$, from top to bottom. The dashed line-style indicates curves which cross the time axis. The last response that does not cross the time axis is $h/h_c = 1.06$, meaning that the DCP should lie in the interval of $(1.06, 1.07)h_c$.

FIG. 11. (Color online). (a) Order profiles calculated with $t$-DMRG for the long-range TFIC, with $\alpha = 2.5$ and system size $N = 32$, at different sites $r = 1, 3, 6$ when an ultraviolet temporal cutoff of $t^*J = 2$ and an infrared temporal cutoff of $t_lJ = 6$ are applied. (b) Finite-time analysis curves for $r = 3$ are calculated with $t^* = 0$ and $t_l = 2, 3, 4, 5, 6$, from top to bottom. Since there is no crossing point, finite-time analysis fails.

It by $\sum_{r=1}^{L} 1/r^\alpha$, and hence approximate the equilibrium critical point as $h_c = J$, which in the thermodynamic limit is a good approximation when $\alpha > 1$ [68]. This is performed because the equilibrium critical point is actually not well-defined for a finite system, and finite-size fluctuations are expected to shift the “critical point” to smaller values. Therefore, we first determine the dynamical critical point based on the total magnetization [26]. Fig. 10 shows nonequilibrium responses of the total magnetization for various transverse field strengths. One sees that the response crosses the time axis within the interval $h_{dc}/h_c \in (1.06, 1.07)$, which pins down the dynamical critical point of the model quenched from a polarized state [26]. The observation that $h_{dc} > h_c$ points to the fact that the equilibrium critical point is initially underestimated.

Since the accessible simulation times are significantly limited, we set the infrared and ultraviolet cutoffs as $t_lJ = 6$ and $t^*J = 2$ (Fig. 11a). Remarkably, we find that the order profiles at different sites dip at the same $h/h_c$ value, suggesting a crossover in the interval
of $h_c/\tilde{h}_c \in (1.15, 1.2)$. We note that unlike in previous sections and Ref. [51] where we observe $|C_{r=1}(t)| > |C_{r=3}(t)| > |C_{r=6}(t)| > \ldots$ in the dynamically-ordered phase, here for the long-range model we do not observe such an order. This is because of the presence of algebraic, instead of linear, light cones for $\alpha = 2.5$, which wash away the differences in the nonequilibrium responses of sites that are close to each other, and hence make the system less locally connected. Strengthening this argument, we also find that the finite-time analysis fails for this model. Fig. 11b shows nonequilibrium responses at $r = 3$ between an ultraviolet cutoff $t^* = 0$ and various infrared cutoffs $t_1 = 2, 3, 4, 5, 6$, from top to bottom. The curves do not intersect at any point, unlike what we observed for NNN TFIC models.

The crossover region might be probing the actual QCP $h_c$, as it appears above the DCP determined with total magnetization at $h_c/\tilde{h}_c \in (1.06, 1.07)$, and this is expected in long-range models for quenches from the ferromagnetic phase [18, 69, 70]. To test this idea, we apply finite-size analysis with $N = 16$ and $N = 24$. This analysis accentuates the shortness of accessible times as the system size increases: A system size-dependent infrared cutoff $t_1(N) = 6R_N$, where $R_N = N/32$, corresponds to $t_1 = 4.5$ and $t_1 = 3$ for $N = 24$ and $N = 16$, respectively, throughout the entire set of transverse-field values. An additional challenge in the long-range models is the oscillations with large amplitudes [71] as seen in Fig. 8b. This feature particularly becomes significant as we approach the boundary. Fig. 12a demonstrates how the order profile at $h/\tilde{h}_c = 0.5$ would change with different infrared cutoffs $t_1(N)/R_N$ and system sizes. We observe that the order profiles are affected by the oscillations, in particular for early infrared cutoffs, e.g., $t_1(N) = 6R_N$. As the averaging interval increases with increasing $t_1$, we notice that the oscillations phase out and the dependence on the cutoff becomes less pronounced. Again due to oscillations, comparison between the order profiles of different system sizes becomes less reliable, as can also be seen in Fig. 12a where the ordering between different system sizes keeps changing as we change $t_1(N)/R_N$. While the trend of the data suggests that the oscillations continue phasing out and the ordering possibly converges as we increase the infrared cutoff, we cannot confirm this conclusion due to limited simulation times. Therefore, both to alleviate the effects of oscillations and to utilize the accessible times as much as possible, we adopt an infrared cutoff that is a function of both system size and the transverse field $t_1(N, h) = R_N t_{\text{max}}(N = 32, h)$ where $t_{\text{max}}(N = 32, h)$ is the longest time accessible in our t-DMRG calculations for the largest considered system size $N = 32$ at each $h$. While this adaptive approach helps to decrease the effect of oscillations on the order profiles, in particular deep in the ferromagnetic phase, it does not completely eliminate the effect, which is possible only by increasing the simulation times. Figs. 12b and 12c show the order profiles of different sizes at $r = 1$ and $r = 3$, respectively. We observe that the order profile of $N = 32$ tends to decrease at much smaller $h$ than where the crossover resides. This is exactly because of the limited simulation times and hence the application of very early infrared cutoffs as we increase the transverse field. Also worth noting, finite-size scaling for models with power-law interaction profiles faces the additional challenge that the profile tails will look different for different system sizes, and this becomes particularly nontrivial near criticality. As a result of the above, whether the observed crossover profile is actually a transition cannot be answered conclusively.

IV. ROBUSTNESS TESTS

In this section, we mainly focus on the integrable TFIC with $\Delta = 0$ and test the robustness of the dynamical scaling exponent for different temporal cutoffs, initial states and boundary conditions. Later, we visit the analytical expression of the edge magnetization to demonstrate that the observed dynamical scaling exponent is indeed in the critical relaxation regime instead of the quasi-
stationary regime. This is complemented with more numerical evidence on the critical relaxation regime. Finally, we strengthen our argument on the origin of the quasi-stationary regime being related to the geometry and locality of the underlying system.

A. Independency of the results for different temporal cutoffs

We first test whether our results depend on the choice of ultraviolet cutoff $t^*$. The results in Ref. [51] are produced with a fixed ultraviolet (UV) cutoff of $t^* = 10$, for all $h$. However, none of our results depend on the choice of ultraviolet cutoff: Subfigures (a-k) in Fig. 13 all show the same qualitative behavior for single-site dynamical phase diagrams for various choices of ultraviolet cutoff. Figs. 13a - 13c complement the $t^* = 10$ data in Ref. [51] by showing the single-site phase diagrams of the observables at $r = 3, 9, 12$. Figs. 13d - 13g exhibit another fixed temporal cutoff of $t^* = 20$, whereas Figs. 13h - 13k demonstrate the results of a parametric temporal cutoff for all studied sites. The parametric UV cutoff is determined as follows: We roughly estimate the onset of the quasi-stationary regime as the time required for the quasiparticles to reflect back from the edge closest to the observation site. Therefore, the estimate can be mathematically stated as $t^* = 2\alpha \Delta x / v_\alpha$, where the distance $\Delta x = r - 1$ is the distance between the observation site, $r = 3, 9, 12$ and the closest edge site, $r^* = 1$, in our case. The parameter $\alpha$ is a tuning parameter, as our analytical formula is only an estimate. In fact we find that $\alpha = 2$ presents phase diagrams qualitatively the same as those of other $t^*$, for all $r$ that we studied. Additionally, in all cases the single-site magnetization at the critical point $h_{dc}$ scales with similar exponents in a power-law fashion, $N^{-\beta}$ where $b \sim 1$. These fits can be seen in Figs. 14a, 14b and 14c. Although the choice of cutoff slightly affects this exponent, it does not change the fact that there is a decreasing trend of the magnetization at $h_{dc}$ with system size. This is numerical evidence for the presence of a DCP.

Finally, we test the presence of the universal scaling $C'_r(h_0) \propto h_0^{\beta}$ in the vicinity of the transition when we change the ultraviolet cutoff. Table III provides all fit parameters for the exponent $\beta$ in the integrable TFIC performed with different temporal cutoffs, either fixed or parametric, with the latter denoted as $\text{par}$ in the table. All exponents are very similar and around $\beta \sim 4/3$.

B. Independency of the results from the initial states

In this subsection, we change the initial state to the ground state of an initial Hamiltonian with $h_i = 0.1$, and test whether any of our results depend on the initial state. Figs. 13l - 13o show the single-site dynamical phase diagrams computed with this initial state. We do not observe a change in the qualitative behavior. The single-site magnetization at the critical point still decreases with increasing system size in Fig. 14d by exhibiting $b \sim 1$.

We also test whether the scaling in the vicinity of the transition changes in Fig. 15a. The fit parameters for some observables in the vicinity of the transition are given in Table IV, all of which demonstrate a dynamical critical exponent of $\beta \sim 4/3$. Based on this observation, we demonstrate in Fig. 15b the collapse of the nonequilibrium responses of the system at $r = 6$ for different initial states $h_i = 0$ and $h_i = 0.1$. This collapse is achieved by utilizing the nonuniversal fit parameters in the scaling functions: $C_{r,h_i}(h_n) = a_{r,h_i,h_0}^{n}$ and $C'_{r,h_i}(h_n) = a_{r',h_i}^{n}$ and hence the scaling factor reads $\bar{C}'_{r,h_i}(h_n)/C'_{r',h_i}^{n}(h_n) = a_{r,h_i}/a_{r',h_i}$, i.e., $C_r(t) = C_r(t)a_{r,h_i}/a_{r',h_i}$ in Fig. 15b.

C. Analytical expression for the edge magnetization in the vicinity of the transition

We show that the quasi-stationary value of the edge magnetization does not change its dynamical scaling in the vicinity of the transition. We remind the reader that this scaling is different from the dynamical universal scaling that we numerically observed in the vicinity of the transition. This is because of the divergent relaxation times which means that the times accessible to computation and experiment are still within the relaxation regime when we are in the close vicinity of the transition. The analytical expression for the quasi-stationary value of the edge magnetization in the dynamically-ordered phase is,

$$C_{r=1}^{qs}(h, h_i) = \frac{(1-h_n)(1-h_i)^{1/2}}{1-h_i},$$

for $h, h_i < 1$. Let us rewrite it in terms of the reduced control parameter $h_n = (h_i - h)/h_i$ as

$$C_{r=1}^{qs}(h_n, h_i) = \frac{(2-h_n)h_n(1-h_i)^{1/2}}{1+(h_n-1)h_i}.$$  

In the vicinity of the transition, $h_n \to 0$, we can expand this expression and find up to the third order in $h_n$ and $h_i$,

$$C_{r=1}^{qs}(h_n \to 0, h_i) = a_1(h_i)h_n + a_2(h_i)h_n^2 + a_3(h_i)h_n^3 + \cdots,$$

$$a_1(h_i) = 2 + h_i + \frac{3}{4}h_i^2 + \frac{5}{8}h_i^3 + \cdots,$$

$$a_2(h_i) = -1 - \frac{5}{2}h_i - \frac{7}{8}h_i^2 - \frac{65}{16}h_i^3 + \cdots,$$

$$a_3(h_i) = h_i + \frac{7}{2}h_i^2 + \frac{55}{8}h_i^3 + \cdots.$$ 

Therefore, one can see that $C_{r=1}^{qs}(h_n) \propto h_n$ in the vicinity of the transition, $h_n \to 0$, regardless of the choice of initial state. The initial state only changes the coefficient in front of $h_n$, which is known to be nonuniversal. Based on the observation that the edge magnetization also exhibits
FIG. 13. The single-site dynamical phase diagrams with an ultraviolet temporal cutoff of $t^\star = 10$ for (a) $C_{r=3}(t)$, (b) $C_{r=9}(t)$ and (c) $C_{r=12}(t)$; with an ultraviolet temporal cutoff of $t^\star = 20$ for (d) $C_{r=3}(t)$, (e) $C_{r=9}(t)$, (f) $C_{r=9}(t)$ and (g) $C_{r=12}(t)$; with an ultraviolet temporal cutoff of $t^\star = 2\alpha \Delta x/v$ where $\Delta x \equiv \tau - 1$ with $\tau$ being the single-site observable location and $\alpha = 2$ is a tuning parameter for (h) $C_{r=3}(t)$, (i) $C_{r=9}(t)$, (j) $C_{r=9}(t)$ and (k) $C_{r=12}(t)$. (l-o) The single-site dynamical phase diagrams with an initial state as the ground state of an initial Hamiltonian with $h_i = 0.1$ and an ultraviolet temporal cutoff of $t^\star = 10$ for (l) $C_{r=3}(t)$, (m) $C_{r=6}(t)$, (n) $C_{r=9}(t)$ and (o) $C_{r=12}(t)$. The behavior is qualitatively the same as the results of $h_i = 0$. 
FIG. 14. The system size scaling of different single-site magnetization at the critical point $h_c$ with an initial state at $h_i = 0$ and cutoff (a) $t^*_{\text{c}} = 10$, (b) $t^*_{\text{c}} = 20$, (c) $t^*_{\text{c}} = 2\alpha\Delta x/v_q$ and $\alpha = 2$, and (d) for an initial state at $h_i = 0.1$ and an ultraviolet temporal cutoff of $t^* = 10$. A downward trend can be seen with $aN^{-b}$ where $b \sim 1$ in all curves. In each subfigure, the curves from up to bottom represent the single sites $r = 3, 6, 9, 12$, respectively, as seen in the legend.

FIG. 15. (a) The scaling behavior in the vicinity of the transition for $r = 1, r = 6$ and $r = 12$, in descending order, when the initial state is the ground state of $h_i = 0.1$. All scaling exponents are $\beta \sim 4/3$. (b) $\tilde{C}_r(t)$ for quenches from $h_i = 0$ (blue-solid) or $h_i = 0.1$ (dashed-red) to $h = 0.999$ where the latter is rescaled to collapse on the other according to the fitted power-laws discussed in Ref. [51]. Rescaled quantity is denoted by tilde.
Table III. Fit parameters for the universal scaling law with different ultraviolet temporal cutoffs.

| observable | β | αparm, α = 1 | αparm, α = 2 |
|------------|---|--------------|--------------|
| σ_1^r | 1.3092 ± 0.0059 | 1.3011 ± 0.0016 | 1.3024 ± 0.0016 |
| σ_0^r | 1.3257 ± 0.0079 | 1.3154 ± 0.0047 | 1.3201 ± 0.0046 |
| σ_1^r | 1.3184 ± 0.0067 | 1.3455 ± 0.007 | 1.3572 ± 0.0067 |

Table IV. Fit parameters for the universal scaling law with a different initial state.

| observable | β |
|------------|---|
| σ_0^r | 1.2972 ± 0.0018 |
| σ_0^r | 1.3006 ± 0.0017 |
| σ_0^r | 1.3201 ± 0.0046 |
| σ_0^r | 1.3478 ± 0.0069 |
| σ_0^r | 1.3331 ± 0.0056 |

β ~ 4/3 in its relaxation regime to the quasi-stationary value, it is possible that the rest of the single-site observables close enough to the boundary will also experience a similar scaling ~ 1 in their quasi-stationary regime. However, this is beyond the scope of our work.

We note that the form of the fit function for the quasi-stationary regime of the edge magnetization in the near-integrable model is the same as Eq. (19), and hence the scaling in the vicinity of the transition is the same as well. Let us show this briefly,

\[ C_{qs}^{qs}(h_n) = \frac{\alpha(h_{dc}^3 - (h_{dc} - h_n)^2)}{\beta}, \]

\[ \lim_{h_n \to 0} C_{qs}^{qs}(h_n) = \alpha h_{dc}^{-1} h_n - \frac{1}{2} \alpha (\beta - 1) h_{dc}^{-2} h_n^2 + \cdots. \] (22)

We note that for a different initial state the general trend will remain the same, but that the coefficients α and β might change. In such a case, the scaling in the vicinity of the transition should remain the same as well.

D. Nonequilibrium response in the close vicinity of the transition

Now we plot the nonequilibrium response in the close vicinity of the transition, to demonstrate that the dynamics slow down critically so that the onset of a quasi-stationary regime diverges as we move closer to the critical point. This naturally implies that the dynamical critical exponents that we probe in the vicinity of the transition are of nonequilibrium type, instead of equilibrium which could have been so if we were probing the quasi-stationary regime in the close vicinity. Fig. 16a and 16b show the nonequilibrium responses of edge magnetization and bulk magnetization at r = 6, respectively.

(i) The first observation is that the nonequilibrium responses in the vicinity of the transition exhibit a very similar trend for both cases up to different time-average values (solid-black lines). The collapse of nonequilibrium responses of different sites are already presented in Ref. [51]. (ii) As we move closer to the critical point, we observe that the onset of the quasi-stationary regime diverges: This effect starts to be visible on and after h = 0.99. Therefore, the time-average of the signal no longer matches perfectly with the magnetization value of the quasi-stationary regime, which is beyond accessible simulation times in these figures. We note that it is also around this value of h where we observe the onset of universal scaling appearing with β ~ 4/3 (see for instance Fig. 3a in Ref. [51]). We do not study the dynamical scaling closer than h_n = 10^{-4} (purple), as it is not completely clear whether the dynamics relax to a quasi-stationary regime past this point due to constrained simulation time for a system size of N = 1440. Nevertheless, the allowed parameter regime still demonstrates a robust exponent of β ~ 4/3. In the future, one can extend the simulation time by increasing the system size further and thus test the dynamical critical exponent closer to the critical point.

In this section, we also plot the nonequilibrium response of the integrable TFIC at the DCP which coincides with the QCP h_{dc} = 1 (Fig. 17a), as well as in the dynamically-disordered phase at h = 1.1 (Fig. 17b). One notices the power-law decay of the envelope of the nonequilibrium response, which suggests that there is no quasi-stationary regime appearing in the accessible times. Furthermore, we plot the nonequilibrium MFT responses of the near-integrable model with Δ = 0.1 for h past the DCP in Fig. 17c. One can observe the emergence of long wavelength oscillations as we pass the dynamical critical point, which is helpful in determining the DCP. This is because, such behavior of long wavelength oscillations is a characteristic of the disordered dynamical phase, as seen in Figs. 17a and 17b. One can again notice that the envelope of the nonequilibrium response in the dynamically-disordered phase decays in a power-law fashion, suggesting the absence of a quasi-stationary regime.

E. Numerical evidence on quasi-stationary regime not originating from the strong-zero modes

Next we discuss the coherence times of the edge magnetization (Fig. 18c), as well as the local magnetization...
FIG. 16. The nonequilibrium response of integrable TFIC in the vicinity of the transition (see legend) for (a) the edge magnetization and (b) bulk $r = 6$ magnetization at system size $N = 1440$. The solid-black lines are the time-average with a fixed UV temporal cutoff of $t^* = 10$.

FIG. 17. The nonequilibrium response of integrable TFIC (a) at DCP $h = 1$ and (b) in the dynamically-disordered phase $h = 1.1$ for $r = 6$ magnetization at various system sizes (see legends). (c) Mean-field nonequilibrium responses of the near-integrable model with $\Delta = 0.1$ past the DCP (see legend) at a system size of $N = 1440$.

FIG. 18. Exact diagonalization results for the coherence time of (a-b) $C_{r=3}(t)$ and (c) $C_{r=1}(t)$ at a system size of $N = 14$. (a) and (c) depict certain nonintegrable models (see legend), whereas (b) gives a two-dimensional color plot of the long-time value of the quasi-stationary regime with respect to external field $h$ and the interaction strength $\Delta$. There is no non-monotonic behavior in the plot, demonstrating that the quasi-stationary regime is not caused by strong-zero modes.
of a bulk spin $C_{r=3}(t)$ (Fig. 18a), with respect to time for different interaction strengths $\Delta$. An important evidence of strong zero modes is the presence of resonances, which would result in a non-monotonous trend of the steady-state value with respect to $\Delta$ [54]. The absence of such a behavior can be seen in Figs. 18a and 18c. Additionally, we provide a 2D plot of the long-time steady state value of $C_{r=3}(t)$ with respect to the external field strength $h$ and the interaction strength $\Delta$. The behavior is monotonous everywhere between $0 < h < 1$ and $0 < \Delta < 2$, which allows us to exclude the physics of strong zero modes as a possible explanation of the quasi-stationary temporal regime observed in the open-boundary chains.

We also compare the spatial profiles of the single-site magnetization at a fixed time — determined according to the breakdown of the cluster theorem $t_1$ — with that of a Majorana edge mode $\gamma_1 = \phi^+_1$ [72]. For $h$ in the ordered phase and $r \ll N/2$, where $N = 480$ is set for concreteness, this time corresponds to the quasi-stationary regime, and hence the single-site order parameter value is compared with the probability of the edge mode being found at $r$. As is evident from Fig. 19, the spatial profiles for the magnetization decay exponentially in space, which is expected [10]. However, we observe that the decay rate of an edge mode and of the magnetization at the same $h$ differ by at least two orders of magnitude (red-diamonds vs. blue-circles, respectively). Their spatial profiles coincide perfectly at $r = 1$, which is the boundary of the chain, and start to differ as $r$ increases. This is further evidence that the leakage of a zero mode into the bulk of the chain, alone, cannot explain the presence of a quasi-stationary temporal regime of bulk single-site observables. Fig. 19 also compares the spatial profiles of magnetization at different $h$, showing them to decay faster for external fields close to the QCP.

F. Changing the boundary conditions

Finally, we demonstrate that the quasi-stationary temporal regime emerges not only when we introduce hard boundaries [73], but also for smooth boundaries. A smooth boundary condition can be applied by smoothly turning off the Hamiltonian parameters towards the edges of the chain [74]. Fig. 20 shows the single-site nonequilibrium responses of the integrable TFIC with hard boundaries (red-diamonds), smooth boundaries (green-solid) and periodic boundary condition (blue-dotted). As shown before [25], the middle of a hard-boundary chain (yellow triangles) acts like an arbitrary site in a periodic chain.

We thus conclude that the quasi-stationary regime is robust against altering the boundary conditions, so long as they remain open. This provides further evidence that the stationary regime is a result of the geometry of the chain rather than of the zero modes.

V. CONCLUSIONS

We performed $t$-DMRG calculations to obtain the order profiles of single-site observables at various sites in open-boundary nonintegrable TFICs. The models that we studied range from locally connected with next nearest neighbor interactions to power-law decaying interactions. Our results for the near-integrable model confirm the location of the DCP found with mean-field theory in Ref. [51]. Finite-size analysis for the strongly nonintegrable model determined the DCP to be in the interval of $h_{dc}^{N=1} \in (2.3, 2.35)$, independent of the measurement sites, and to be shifted from the QPT at $h_c \sim 2.46$. Whether this shift is related to finite-size effects, or due to the quench protocol, is a question to answer in the
future. We find that the finite-time analysis with single-site observables matches well with the finite-size analysis in the near-integrable model, whereas there is a mismatch between the two analyses in the strongly nonintegrable model. We argue that this is likely because of a change in the light cone structure as the transverse field increases, which is yet to be explored. We demonstrated the presence of a quasi-stationary regime even in long-range power-law interacting TFIC for $\alpha = 3$ and $\alpha = 5$, where in the former one could still observe a significant difference in the nonequilibrium responses of the spin in the middle and close to an edge, pointing to nearly-linear light cones hosted in the system. Increasing the range of interactions to $\alpha = 2.5$ decreases the differences between sites, emphasizing the long-range nature of the model and expectantly the finite-time analysis fails. The order profiles at $\alpha = 2.5$ for different sites dip at the same transverse field $h_{cr}/h_c < 1.2$ suggesting a crossover, which is found to be larger than the calculated DCP based on total magnetization found to be at $h_{dc}/h_c = (1.06, 1.07)$. Given that the crossover dip appears after $h_{dc}/h_c$, and since the calculations were based on an estimate of QCP $h_c$, it is an interesting direction to check whether the crossover point $h_{cr}$ probes the actual QPT. Our data is significantly limited to early times, especially as we approach the transition boundary, and hence is currently inconclusive to answer this question.

In the second part of our paper, we performed multiple robustness tests for the results found in Ref. [51], and numerically showed that the dynamical critical exponent $\beta = 4/3$ found both for noninteracting and weakly interacting TFIC is robust to the choice of temporal cutoff, measurement site and initial state. We provided additional evidence, both numerically and analytically, that the dynamical critical exponent $\beta = 4/3$ is indeed of nonequilibrium nature, because it arises in a critical relaxation temporal regime in the close vicinity of the DCP where the onset of the quasi-stationary regime diverges. Finally, we demonstrated that the origin of the quasi-stationary regime is not related to the presence of strong-zero modes, but rather — as was argued in Ref. [51] — is due to the locality of the underlying Hamiltonian which defines the geometry of the chain. The observation that the quasi-stationary regime also appears when the hard boundaries are smoothed out is yet further evidence of this argument.

Our work significantly expands Ref. [51] in demonstrating the use of single-site observables to probe quantum phase transitions by (i) confirming the MFT-predicted dynamical critical point, (ii) predicting a dynamical critical point for a strongly nonintegrable model at $\Delta = 1$, (iii) showing that single-site observables could even be useful in long-range interacting TFIC, (iv) further testing the potential of finite-time analysis in the probe of QPTs and (v) performing multiple robustness tests on the results of the noninteracting model.

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