Some new results on dimension datum

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Abstract
In this paper, we show three new results concerning dimension datum. First, for two subgroups $H_1 \cong U(2n+1)$ and $H_2 \cong \text{Sp}(n) \times \text{SO}(2n+2)$ of $\text{SU}(4n+2)$, we find a family of pairs of irreducible representations $(\tau, \tau') \in \hat{H}_1 \times \hat{H}_2$ such that $\mathcal{D}_{H_1, \tau} = \mathcal{D}_{H_2, \tau'}$. Using these pairs, we construct examples of isospectral hermitian vector bundles. Second, we show that $\tau$-dimension data of one-dimensional representations of a connected compact Lie group $H$ determine the image of homomorphism from $H$ to a given compact Lie group $G$. Third, we show a new compactness result for isospectral sets of normal Riemannian homogeneous spaces $(G/H, m)$ where we allow the bi-invariant Riemannian metrics $m$ on $G$ to vary but assume that $G$ is semisimple.

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INTRODUCTION

Let $G$ be a compact Lie group. Write $\hat{G}$ for the set of isomorphism classes of irreducible complex linear representations of $G$, which is a countable set. The dimension datum of a closed subgroup $H$ in $G$ is defined by

$$\mathcal{D}_H : \hat{G} \to \mathbb{Z}, \quad \rho \mapsto \dim V_\rho^H,$$

where $V_\rho$ is the representation space of an irreducible complex linear representation $\rho$ of $G$ and $V_\rho^H$ means the subspace of $H$ invariant vectors in $V_\rho$. The dimension datum was first studied by Larsen and Pink in their pioneering work [6], with the motivation of helping determine monodromy groups of $\ell$-adic Galois representations. In the beginning of the 21st century, Langlands launched a program of ‘beyond endoscopy’, where he used dimension datum as a key ingredient in his stable trace formula approach to showing general functoriality ([5], [2]). Since then dimension datum catches more attention in the mathematical community. Besides number theory and automorphic form theory, dimension datum also has applications in differential geometry. For example, it is used to construct the first non-homeomorphic isospectral simply-connected closed Riemannian manifolds ([1]), which is based on the generalized Sunada’s method ([8, 10, 11]). In [15], we classified connected closed subgroups of a given compact Lie group with the same dimension datum, and characterized linear relations among different dimension data. In [14], we showed that the space of dimension data of closed subgroups in a given compact Lie group is compact.

In this paper, we show several new results concerning dimension datum after previous works [1, 6, 14, 15]. Let $(\tau, U_\tau)$ be an irreducible complex linear representation of $H$, we define the $\tau$-dimension datum of $H$ in $G$ by

$$\mathcal{D}_{H,\tau} : \hat{G} \to \mathbb{Z}, \quad \rho \mapsto \dim \text{Hom}_H(U_\tau, V_\rho),$$

where $\text{Hom}_H(U_\tau, V_\rho)$ means the space of $H$ equivariant linear maps from $U_\tau$ to $V_\rho$. Like the case of dimension datum, one could consider equalities and linear relations among $\tau$-dimension data. For connected closed subgroups, in Section 1 we reduce the study of these equalities and linear relations to the study of equalities and linear relations among characters associated to subroot systems and weights. Generalizing the treatment in [1] and [15], for two subgroups $H_1 (\cong U(2n + 1))$ and $H_2 (\cong Sp(n) \times SO(2n + 2))$ of $SU(4n + 2)$, we prove that there exists a family of pairs of irreducible representations $(\tau, \tau') \in \hat{H}_1 \times \hat{H}_2$ satisfying that $\mathcal{D}_{H_1,\tau} = \mathcal{D}_{H_2,\tau'}$. This enables us to construct examples of isospectral Hermitian vector bundles. This generalizes examples of isospectral manifolds found in [1] and gives the first examples of isospectral hermitian vector bundles on non-homeomorphic simply connected closed manifolds.

In Section 2, we show that: $\tau$-dimension data of one-dimensional representations of a connected compact Lie group $H$ determine the image of the homomorphism from $H$ to a given compact Lie group $G$ (Theorem 2.1). For two connected semisimple subgroups $H_1$ and $H_2$ of $G$ with the same dimension datum, take $H = H_1 \times H_2$ and let $f_i : H \to G$ be the composition of the projection $H \to H_i$ and the imbedding $H_i \hookrightarrow G$ ($i = 1, 2$). Applying Theorem 2.1, we recover the main theorem of [6].

In Section 3, we show a new compactness result for isospectral sets of normal Riemannian homogeneous spaces $(G/H, m)$ where we allow the bi-invariant Riemannian metrics $m$ on $G$ to
SOME NEW RESULTS ON DIMENSION DATUM

1 \ THE \ \(\tau\)-DIMENSION DATUM OF A CONNECTED SUBGROUP

1.1 \ Root system and character

Let \(T\) be a torus in \(G\). Write \(X^*(T) = \text{Hom}(T, U(1))\) for the weight lattice of \(T\) and write \(X^*(T) = \text{Hom}(U(1), T)\) for the co-weight lattice of \(T\), where \(U(1) = \{z \in \mathbb{C} : |z| = 1\}\). Write

\[
\Gamma^o = N_G(T)/Z_G(T).
\]

Fix a bi-invariant Riemannian metric on \(G\). The restriction of the metric to \(T\) gives a positive definite inner product on the Lie algebra \(\mathfrak{t}_0\) and then induces a positive definite inner product on the dual space \(\mathfrak{t}_0^*\). We have \(X^*(T) \subset \mathfrak{t}_0^*\). Multiplying by \(-1\) and restricting to \(X^*(T)\), we get a positive definite inner product on \(X^*(T)\), which is necessarily \(\Gamma^o\) invariant. Let this inner product on \(X^*(T)\) be denoted by \((\cdot, \cdot)\).

As in [15, Definition 2.2], a root system in the lattice \(X^*(T)\) is a finite subset \(\Phi\) of \(X^*(T)\) satisfying the following conditions.

(i) For any two roots \(\alpha \in \Phi\) and \(\beta \in \Phi\), the element \(\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi\).

(ii) (Strong integrality) For any root \(\alpha \in \Phi\) and any weight \(\lambda \in X^*(T)\), the number \(\frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\) is an integer.

For each root \(\alpha \in \Phi\), the condition (ii) guarantees the existence of a co-weight \(\check{\alpha} \in X_\ast(T)\) such that

\[
\lambda(\check{\alpha}(u)) = u^{\frac{2(\lambda, \alpha)}{(\alpha, \alpha)}}, \quad \forall u \in U(1).
\]

Define the reflection \(s_{\alpha} : T \to T\) associated to \(\alpha\) by

\[
s_{\alpha}(x) = x\check{\alpha}(x)^{-1}, \quad x \in T.
\]

Write \(W_{\Phi}\) for the Weyl group of \(\Phi\), which is a finite group consisting of automorphisms of \(T\) generated by \(s_{\alpha} (\alpha \in \Phi)\). There are induced actions of \(s_{\alpha} (\alpha \in \Phi)\) and \(W_{\Phi}\) on the weight lattice \(X^*(T)\) and on the co-weight lattice \(X_\ast(T)\). As in [15, §3], we set

\[
\Psi_T = \{ 0 \neq \alpha \in X^*(T) : \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}, \forall \lambda \in X^*(T) \}.
\]

It is clear that \(\Psi_T\) is a root system in the lattice \(X^*(T)\) and it contains all root systems in the lattice \(X^*(T)\). Define \(\Psi'_T\) to be the minimal root system in \(X^*(T)\) containing all root systems of connected closed subgroups \(H\) of \(G\) with \(T\) a maximal torus. Then we have \(\Psi'_T \subset \Psi_T\). Both \(\Psi_T\) and \(\Psi'_T\) are clearly \(\Gamma^o\) stable. The following proposition which concerns properties of \(\Psi'_T\) is a combination of Proposition 3.3 and Corollary 3.4 in [15].
**Proposition 1.1.** We have $W_{\psi_T} \subset \Gamma^0$, and $\Psi_T^\prime$ equals to the union of root systems of connected closed subgroups $H$ of $G$ with $T$ a maximal torus.

Choose a positive system $\Psi_T^+$ of $\Psi_T$. For a root system $\Phi$ in the lattice $X^*(T)$, we set

$$\delta_\Phi = \frac{1}{2} \sum_{\alpha \in \Phi \cap \Psi_T^+} \alpha.$$ 

For a root system $\Phi$ in the lattice $X^*(T)$ and a weight $\lambda \in X^*(T)$, we set

$$A_{\Phi,\lambda} = \sum_{w \in W_{\Phi}} \text{sgn}(w)[\lambda + \delta_\Phi - w\delta_\Phi] \in \mathbb{Q}[X^*(T)].$$

For a finite group $W$ between $W_{\Phi}$ and $\Gamma^0$, we set

$$F_{\Phi,\lambda, W} = \frac{1}{|W|} \sum_{\gamma \in W} \gamma(A_{\Phi,\lambda}) \in \mathbb{Q}[X^*(T)].$$

For a weight $\lambda \in X^*(T)$ and a finite subgroup $W$ of $\Gamma^0$, we set

$$\chi^*_{\lambda, W} = \frac{1}{|W|} \sum_{\gamma \in W} [\gamma \lambda] \in \mathbb{Q}[X^*(T)].$$

Then we have

$$F_{\Phi,\lambda, W} = \sum_{w \in W_{\Phi}} \text{sgn}(w) \chi^*_{\lambda + \delta_\Phi - w\delta_\Phi, W}.$$

Note that $\chi^*_{\lambda, W} = \chi^*_{\lambda', W}$ if and only if $W\lambda = W\lambda'$. Choose a set $\Lambda'$ of representatives of $W$ orbits in $X^*(T)$. Then, $\{\chi^*_{\lambda, W} : \lambda \in \Lambda'\}$ form a basis of $\mathbb{Q}[X^*(T)]^W$ (the subspace of $W$ invariant characters on $T$). For any $\chi \in \mathbb{Q}[X^*(T)]^W$, there is a finite linear expansion

$$\chi = \sum_{\lambda \in \Lambda'} a_\lambda \chi^*_{\lambda, W}.$$ 

We define

$$\text{supp} \chi = \{\lambda \in \Lambda' : a_\lambda \neq 0\}$$

and call it the support of $\chi$, which is a finite subset of $\Lambda'$. We call $\chi^*_{\lambda, W}$ a leading term of $\chi$ if $\lambda \in \text{supp} \chi$ and $|\mu| \leq |\lambda|$ for any $\mu \in \text{supp} \chi$; and we call $\chi^*_{\lambda, W}$ a minimal term of $\chi$ if $\lambda \in \text{supp} \chi$ and $|\mu| \geq |\lambda|$ for any $\mu \in \text{supp} \chi$.

**Proposition 1.2.** Let $\tau_1 \in \hat{H}_1$ and $\tau_2 \in \hat{H}_2$. If $\mathcal{D}_{H_1, \tau_1} = \mathcal{D}_{H_2, \tau_2}$, then $H_1$ and $H_2$ have conjugate maximal tori. Assume that $T$ is a maximal torus of both $H_1$ and $H_2$, write $\Phi_i(\subset X^*(T))$ for the root
system of $H_i$ ($i = 1, 2$). Then,

$$D_{H_1, \tau_1} = D_{H_2, \tau_2} \iff F_{\Phi_1, \lambda_1, \Gamma^e} = F_{\Phi_2, \lambda_2, \Gamma^e},$$

where $\lambda_i \in X^*(T)$ is highest weight of $\tau_i$ ($i = 1, 2$).

**Proof.** Let $H$ be a connected closed subgroup of $G$ with $T$ a maximal torus. Write $\Phi \subset X^*(T)$ for the root system of $H$ and set $\Phi^+ = \Phi \cap \Psi_T^+$, which is a positive system of $\Phi$. Let

$$F_{\Phi}(x) = \frac{1}{|W_{\Phi}|} \prod_{\alpha \in \Phi} (1 - \alpha(x))$$

be the Weyl product of $H$ and let $\chi_\lambda(x)$ be the character of an irreducible complex linear representation of $H$ with highest weight $\lambda \in X^*(T)$ with respect to $\Phi^+$. Write $\delta = \delta_{\Phi}$ for simplicity. We calculate $|W_{\Phi}|F_{\Phi}(x)\chi_\lambda(x)$ as follows:

$$|W_{\Phi}|F_{\Phi}(x)\chi_\lambda(x) = \chi_\lambda \prod_{\alpha \in \Phi} (1 - [\alpha]) = \prod_{\alpha \in \Phi^+} \left( \left[ -\frac{\alpha}{2} \right] - \left[ \frac{\alpha}{2} \right] \right) \left( \prod_{\alpha \in \Phi^+} (1 - \left[ \frac{\alpha}{2} \right]) \right) = \left( \sum_{w \in W_{\Phi}} \text{sgn}(w)[-w\delta] \right) \left( \sum_{\gamma \in W_{\Phi}} \text{sgn}(\gamma)[\gamma(\lambda + \delta)] \right) = \sum_{w, \gamma \in W_{\Phi}} \text{sgn}(w)\text{sgn}(\gamma)[-w\delta + \gamma(\lambda + \delta)] = \sum_{\gamma \in W_{\Phi}} \gamma \left( \sum_{w \in W_{\Phi}} \text{sgn}(w)[\lambda + \delta - w\delta] \right) = |W_{\Phi}|F_{\Phi, \lambda, W_{\Phi}}.

In the above computation, we omit the variable $x$ in the other formulas except the first one for simplicity, and we write $[\mu]$ for the complex-valued function on $T$ corresponding to a character $\mu \in X^*(T)$. Then,

$$F_{\Phi}(t)\chi_\lambda(t) = F_{\Phi, \lambda, W_{\Phi}}.$$

Due to $W_{\Phi} \subset \Gamma^0$, we get

$$\frac{1}{|\Gamma^0|} \sum_{\gamma \in \Gamma^0} \gamma \cdot F_{\Phi, \lambda, W_{\Phi}} = F_{\Phi, \lambda, \Gamma^e}.$$

Then, a similar argument as in the proof of [15, Proposition 3.8] shows the conclusion of the proposition. \qed
The following proposition can be shown in the way as for [15, Proposition 3.8].

**Proposition 1.3.** Given a compact Lie group $G$, let $H_1, H_2, ..., H_s \subset G$ be a collection of connected closed subgroups of $G$. For a set of non-zero constants $c_1, ..., c_s$, in order for

$$
\sum_{1 \leq i \leq s} c_i \mathcal{D}_{H_i, \tau_i} = 0
$$

holds it is necessary and sufficient that: for any torus $T$ of $G$,

$$
\sum_{1 \leq i \leq s} c_i F_{\Phi_i, \lambda_i, \Gamma} = 0,
$$

where $\{H_i : i_1 \leq i_2 \leq \cdots \leq i_t\}$ are all subgroups amongst $\{H_i : 1 \leq i \leq s\}$ with maximal tori conjugate to $T$, $\Phi_i$ is the root system of $H_i$ (regarded as a subset of $X^*(T)$ by taking conjugation), and $\lambda_{ij}$ is the highest weight of $\tau_{ij}$ (regarded as an element $X^*(T)$ by taking conjugation).

Motivated by Proposition 1.3, we propose the following two questions which concern the equalities and linear relations among the characters $F_{\Phi, \lambda, W}$.

**Question 1.1.** Given a root system $\Psi$ in a lattice $L$ with rank $\Psi = \text{rank } L$, when does $F_{\Phi_1, \lambda_1, \text{Aut}(\Psi)} = F_{\Phi_2, \lambda_2, \text{Aut}(\Psi)}$ hold for two sub-root systems $\Phi_1, \Phi_2$ of $\Psi$ and two characters $\lambda_1, \lambda_2$ in $L$?

**Question 1.2.** Given a root system $\Psi$ in a lattice $L$ with rank $\Psi = \text{rank } L$, which linear relations are there among the characters $\{F_{\Phi, \lambda, W_\Psi} : \Phi \subset \Psi, \lambda \in L\}$?

One may reduce both Questions 1.1 and 1.2 to the case that $\Psi$ is an irreducible root system. In the next subsection, we discuss Question 1.1 in a special case.

### 1.2 A special case of Question 1.1

There is a nice method in [6] which transfers characters $F_{\Phi,0,W_{BC_n}}$ into polynomials. In [1] and [15], we further find matrix expression for the resulting polynomials. Here, we extend these studies to the characters $F_{\Phi, \lambda, W_{BC_n}}$.

We first recall Larsen–Pink’s method. Set

$$
\mathbb{Z}^n := \mathbb{Z} BC_n = \Lambda_{BC_n} = \text{span}_\mathbb{Z}\{e_1, e_2, ..., e_n\},
$$

$$
W_n := \text{Aut}(BC_n) = W_{BC_n} = \{\pm1\}^n \rtimes S_n,
$$

$$
\mathbb{Z}_n := \mathbb{Q}[\mathbb{Z}^n],
$$

$$
\mathbb{Y}_n := \mathbb{Z}^W_n.
$$

For $m \leq n$, the injection

$$
\mathbb{Z}^n \hookrightarrow \mathbb{Z}^m : (a_1, ..., a_m) \mapsto (a_1, ..., a_m, 0, ..., 0)
$$
extends to an injection \( i_{m,n} : \mathbb{Z}_m \hookrightarrow \mathbb{Z}_n \). Define \( \phi_{m,n} : \mathbb{Z}_m \to \mathbb{Z}_n \) by
\[
\phi_{m,n}(z) = \frac{1}{|W_n|} \sum_{w \in W_n} w(i_{m,n}(z)).
\]
Thus \( \phi_{m,n} \phi_{k,m} = \phi_{k,n} \) for any \( k \leq m \leq n \) and the image of \( \phi_{m,n} \) lies in \( Y_n \). Hence \( \{Y_m : \phi_{m,n}\} \) forms a direct system and we define
\[
Y = \lim_{\to} Y_n.
\]
Define the map \( j_n : \mathbb{Z}_n \to Y \) by composing \( \phi_{n,p} \) with the injection \( Y_p \hookrightarrow Y \). The isomorphism \( \mathbb{Z}^m \oplus \mathbb{Z}^n \to \mathbb{Z}^{m+n} \) gives a canonical isomorphism \( M : \mathbb{Z}_m \otimes \mathbb{Q} \mathbb{Z}_n \to \mathbb{Z}_m \otimes \mathbb{Q} \mathbb{Z}_n \). Given two elements of \( Y \) represented by \( y \in Y_m \) and \( y' \in Y_n \), we define
\[
yy' = j_{m+n}(M(y \otimes y')).
\]
This product is independent of the choice of \( m \) and \( n \) and makes \( Y \) a commutative associative algebra.

The monomials \( [e_1]^{k_1} \cdots [e_n]^{k_n} \) \((k_1, k_2, \ldots, k_n \in \mathbb{Z})\) form a \( \mathbb{Q} \) basis of \( \mathbb{Z}_n \), where \( [e_i]^{k_i} = [k_ie_i] \) is a linear character. Hence \( Y \) has a \( \mathbb{Q} \) basis
\[
e(k_1, k_2, \ldots, k_n) = j_n([e_1]^{k_1} \cdots [e_n]^{k_n})
\]
indexed by \( n \geq 0 \) and \( k_1 \geq k_2 \geq \cdots \geq k_n \geq 0 \). Mapping \( e(k_1, k_2, \ldots, k_n) \) to \( x_{k_1}x_{k_2} \cdots x_{k_n} \), we get a \( \mathbb{Q} \) linear map
\[
E : Y \to \mathbb{Q}[x_0, x_1, \ldots, x_n, \ldots].
\]
This map \( E \) is an algebra isomorphism. Here we remark that we should specify \( x_0 = 1 \) to make \( E \) an algebra isomorphism. We keep the notation \( x_0 \) for the convenience so that resulting polynomials are homogeneous polynomials.

For any \( k_1, \ldots, k_n \in \mathbb{Z} \) with \( k_1 \geq \cdots \geq k_n \geq 0 \), put \( \lambda = k_1e_1 + k_2e_2 + \cdots + k_ne_n \). Then,
\[
J_n(\chi_{\lambda,W_n}^* ) = e(k_1, k_2, \ldots, k_n) \in Y
\]
and
\[
E(J_n(\chi_{\lambda,W_n}^* )) = x_{k_1}x_{k_2} \cdots x_{k_n}.
\]
Given \( f \in \mathbb{Q}[x_0, x_1, \ldots] \), we define
\[
\sigma(f)(x_0, x_1, \ldots, x_{2n}, x_{2n+1}, \ldots) = f(x_0, -x_1, \ldots, x_{2n}, -x_{2n+1}, \ldots).
\]
Then, \( \sigma \) is an involutive automorphism of \( \mathbb{Q}[x_0, x_1, \ldots] \).

Write \( a_n(\lambda), b_n(\lambda), c_n(\lambda), d_n(\lambda) \) for the image of \( j_n(F_{\Phi, \lambda, W_n}) \) under \( E \) for \( \Phi = A_{n-1}, B_n, C_n, D_n \) and a weight \( \lambda \in \mathbb{Z}^n \). We normalize \( a_n(\lambda), b_n(\lambda), c_n(\lambda), d_n(\lambda) \) to be homogeneous polynomials of
degree \( n \). They have integral coefficients. Write \( b_n'(\lambda) = (-1)^{\sum_{1 \leq i \leq n} k_i} \sigma(b_n(\lambda)) \). Define matrices

\[
A_n(\lambda) = (x_{|k_j+i-j|})_{n \times n},
\]

\[
B_n(\lambda) = (x_{|k_j+i-j|} - x_{|k_j+2n+1-i-j|})_{n \times n},
\]

\[
B'(\lambda) = (x_{|k_j+i-j|} + x_{|k_j+2n+1-i-j|})_{n \times n},
\]

\[
C_n(\lambda) = (x_{|k_j+i-j|} - x_{|k_j+2n+2-i-j|})_{n \times n},
\]

\[
D_n(\lambda) = (x_{|k_j+i-j|} + x_{|k_j+2n-i-j|})_{n \times n},
\]

\[
D'(\lambda) = (y_{i,j})_{n \times n},
\]

where \( y_{i,j} = x_{|k_j+i-j|} + x_{|k_j+2n-i-j|} \) if \( i, j \leq n - 1 \), \( y_{n,j} = \sqrt{2} x_{|k_j+n+1-i-j|} \) and \( y_{i,n} = x_{|k_n|} \). The following lemma gives matrix expressions for the polynomials \( a_n(\lambda), b_n(\lambda), c_n(\lambda), d_n(\lambda) \).

**Lemma 1.4.** Let \( \lambda = k_1e_1 + k_2e_2 + \cdots + k_ne_n \), where \( k_1, \ldots, k_n \in \mathbb{Z} \) and \( k_1 \geq \cdots \geq k_n \geq 0 \). Then

\[
\det A_n(\lambda) = a_n(\lambda),
\]

\[
\det B_n(\lambda) = b_n(\lambda),
\]

\[
\det B'(\lambda) = b'_n(\lambda),
\]

\[
\det C_n(\lambda) = c_n(\lambda),
\]

\[
\frac{1}{2} \det D_n(\lambda) = \det D'(\lambda) = d_n(\lambda).
\]

**Proof.** First consider \( \Phi = A_{n-1} \). Then, \( a_n(\lambda) = E(j_n(A_{\Phi,\lambda})) \), where

\[
A_{\Phi,\lambda} = \sum_{w \in S_n} \text{sgn}(w)[\lambda + \delta - w\delta]
\]

with \( \delta = (\frac{n}{2} - \frac{1}{2}, \frac{n}{2} - \frac{3}{2}, \ldots, \frac{n}{2} - \frac{n}{2}) \). For a permutation \( w \in S_n \), one has

\[
E(j_n(\text{sgn}(w)[\lambda + \delta - w\delta])) = \text{sgn}(w) \prod_{1 \leq j \leq n} x_{|k_j+w^{-1}(j)-j|},
\]

which is equal to the term in the expansion of \( \det A_n(\lambda) \) corresponding to the permutation \( w \). Hence, \( \det A_n(\lambda) = a_n(\lambda) \).
Now consider $\Phi = D_n$. Define a new character $\epsilon' : W_n \to \{1\}$ by $\epsilon'|_{W_{D_n}} = \text{sgn}|_{W_{D_n}}$ and $\epsilon'(s_{e_1}) = 1$. Due to $s_{e_n}(\delta_{D_n}) = \delta_{D_n}$, one has

$$F_{D_n,\lambda,W_n} = \frac{1}{2} \sum_{w \in W_n} \epsilon'(w) \chi^\lambda_{\lambda + \delta - w \delta, W_n}$$

where $\delta = (n-1, n-2, \ldots, 0)$. Put $E_n = \langle s_j : 1 \leq j \leq n \rangle \subset W_n$. Then, $W_n = S_n \ltimes E_n$. One can show that for any given $w \in S_n$,

$$\sum_{\gamma \in E_n} \epsilon'(w \gamma) E(j_{n}(\chi^\lambda_{\lambda + \delta - w \gamma \delta, W_n}))$$

is equal to the term in the expansion of $\det D_n(\lambda)$ corresponding to the permutation $w$. Hence, $\frac{1}{2} \det D_n(\lambda) = d_n(\lambda)$.

The proofs for $\det B_n(\lambda) = b_n(\lambda)$ and $\det C_n(\lambda) = c_n(\lambda)$ are similar to that of $\frac{1}{2} \det D_n(\lambda) = d_n(\lambda)$, and are easier since we only need to use the sign function on $W_n$. We get $\det B'_n(\lambda) = b'_n(\lambda)$ by applying the involutive automorphism $\sigma$ to both sides of $\det B_n(\lambda) = b_n(\lambda)$. It is clear that $\det D'_n(\lambda) = \frac{1}{2} \det D_n(\lambda)$. Then, $\det D'_n(\lambda) = d_n(\lambda)$. \hfill \Box

**Proposition 1.5.**

(i) If the integer $n = 2m$ is even and $k_1 \geq k_2 \geq \cdots \geq k_n$ satisfy that $k_{n+1-i} + k_i = 0$ ($1 \leq i \leq m$), then

$$a_{2m}(\lambda) = b_m(\lambda_1)b'_m(\lambda_2),$$

where $\lambda_1 = \lambda_2 = (k_1, \ldots, k_m)$.

(ii) If the integer $n = 2m + 1$ is odd and $k_1 \geq k_2 \geq \cdots \geq k_n$ satisfy that $k_{n+1-i} + k_i = 0$ ($1 \leq i \leq m$), then

$$a_{2m+1}(\lambda) = c_m(\lambda_1)d_{m+1}(\lambda_2),$$

where $\lambda_1 = (k_1, \ldots, k_m), \lambda_2 = (k_1, \ldots, k_{m+1})$.

**Proof.**

(i) For each positive integer $m$, write

$$L_m = (\delta_{i,m+1-j})_{1 \leq i,j \leq m},$$

where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is the Kronecker symbol. Then, $L_m^2 = I$. The matrix $A_{2m}(\lambda)$ is of the form

$$\begin{pmatrix} X & Y \\ L_mYL_m & L_mXL_m \end{pmatrix}.$$
where $X, Y$ are two $m \times m$ matrices. By calculation, we have

$$
\frac{1}{2} \begin{pmatrix}
I & L_m \\
-L_m & I
\end{pmatrix}
\begin{pmatrix}
X & Y \\
L_mYL_m & L_mXL_m
\end{pmatrix}
\begin{pmatrix}
I & -L_m \\
L_m & I
\end{pmatrix}
= \begin{pmatrix}
X + YL_m & 0 \\
0 & L_mXL_m - L_mY
\end{pmatrix}.
$$

Since $X + YL_m$ (respectively, $X - YL_m$) is equal to $B'_m(\lambda_2)$ (respectively, $B_m(\lambda_1)$), we get $a_{2m}(\lambda) = b_m(\lambda_1)b'_{m}(\lambda_2)$ by Lemma 1.4.

(ii) The matrix $A_{2m+1}(\lambda)$ is of the form

$$
\begin{pmatrix}
X & \beta^t & Y \\
\alpha & z & \alpha L_m \\
L_mYL_m & \gamma^t & L_mXL_m
\end{pmatrix},
$$

where $X, Y$ are two $m \times m$ matrices, $\alpha, \beta, \gamma$ are $1 \times m$ vectors. By calculation, we have

$$
\begin{pmatrix}
I & L_m \\
\sqrt{2} & I \\
-L_m & \sqrt{2}
\end{pmatrix}
\begin{pmatrix}
X & \beta^t & Y \\
\alpha & z & \alpha L_m \\
L_mYL_m & \gamma^t & L_mXL_m
\end{pmatrix}
\begin{pmatrix}
I & -L_m \\
L_m & I \\
\sqrt{2} & I
\end{pmatrix}
= 2\begin{pmatrix}
X + YL_m & \frac{\sqrt{2}}{2}(\beta^t + L_m\gamma^t) \\
\sqrt{2}\alpha & z \\
0 & \frac{\sqrt{2}}{2}(L_m\beta^t + \gamma^t) \\
\sqrt{2}\alpha & z & L_mXL_m - L_mY
\end{pmatrix}.
$$

Since the matrix

$$
\begin{pmatrix}
X + YL_m & \frac{\sqrt{2}}{2}(\beta^t + L_m\gamma^t) \\
\sqrt{2}\alpha & z & L_mXL_m - L_mY
\end{pmatrix}
$$

is equal to $D'_{m+1}(\lambda_2)$ and the matrix $X - YL_m$ is equal to $C_m(\lambda_1)$, we get $a_{2m+1}(\lambda) = c_m(\lambda_1)d_m(\lambda_2)$ by Lemma 1.4.

1.3 Isospectral hermitian vector bundles

Let $H$ be a closed subgroup of a connected compact Lie group $G$, and $(\tau, V_\tau)$ be a finite-dimensional irreducible complex linear representation of $H$. Write $E_\tau = G \times_H V_\tau$ for the $G$-equivariant vector bundle on $X = G/H$ induced from $V_\tau$, which consists of equivalence classes in $G \times V_\tau$ of the following equivalence relation:

$$
(g, v) \sim (g', v') \iff \exists x \in H \text{ s.t. } g' = gx \text{ and } v' = x^{-1} \cdot v.
$$
Write \([ (g, v) ] \in E_\tau\) for the equivalence class containing \((g, v)(g \in G, v \in V_\tau)\). The group \(G\) acts on \(G \times V_\tau\) via left translation: \(g' \cdot (g, v) = (g'g, v)(\forall g' \in G)\). It descends to an action on \(E_\tau\) through

\[
g' \cdot [(g, v)] = [(g'g, v)], \forall g' \in G,
\]

still called left translation.

Write \(C^\infty(G, V_\tau)\) for the space of smooth functions \(f : G \to V_\tau\). There are two \(G\) actions \(L\) and \(R\) on \(C^\infty(G, V_\tau)\) through

\[
(L_g f)(g') = f(g^{-1}g')
\]

and

\[
(R_g f)(g') = f(g'g)
\]

(\(\forall f \in C^\infty(G, V_\tau), \forall g, g' \in G\)). Write \(C^\infty(G/H, E_\tau)\) for the space of smooth sections of \(E_\tau\), which could be identified with the subspace of functions \(f : G \to V_\tau\) such that

\[
f(gx) = x^{-1} \cdot f(g), \forall g \in G, \forall x \in H.
\]

The space \(C^\infty(G) \otimes V_\tau\) carries a representation of \(G \times H\) where \(G\) acts through \(L \otimes 1\) and \(H\) acts through \(R \otimes \tau\). This gives a representation of \(G \times H\) on \(C^\infty(G, V_\tau)\) via the identification \(C^\infty(G, V_\tau) = C^\infty(G) \otimes V_\tau\) where the induced action of \(G\) is just \(L\). Then we have

\[
C^\infty(G/H, E_\tau) = (C^\infty(G, V_\tau))^H
\]

and it carries a representation of \(G\), coincides with the action induced from the left translation on \(E_\tau\). By differentiation, we get an action of \(\mathfrak{g} = \text{Lie } G \otimes \mathbb{C}\) on \(C^\infty(G/H, E_\tau)\) and also an action of the universal enveloping algebra \(U(\mathfrak{g})\) on it. Let \(\Delta_\tau\) denote the resulting differential operator on \(C^\infty(G/H, E_\tau)\) from the Casimir element in the center of \(U(\mathfrak{g})\). The action of \(\Delta_\tau\) on \(C^\infty(G/H, E_\tau)\) commutes with the action of \(G\), and it is a second order elliptic differential operator.

Choose an \(H\)-invariant positive definite inner product \((\cdot, \cdot)\) on \(V_\tau\) (which is unique up to scalar). It induces a Hermitian metric on \(E_\tau\) and makes \(E_\tau\) a hermitian vector bundle. Choose a \(G\)-equivariant measure \(d(gH)\) on \(G/H\) with volume 1 and define a Hermitian pairing \((\cdot, \cdot)\) on \(C^\infty(G/H, E_\tau)\) by

\[
(f_1, f_2) = \int_{G/H} (f_1(g), f_2(g)) \, d(gH),
\]

which is clearly \(G\) invariant. In special cases, the Casimir operator \(\Delta_\tau\) is a scalar multiple of the connection Laplacian (also called Bochner Laplacian) \(\nabla^* \nabla\) or other kinds of Laplacians, like Hodge–Laplacian (defined on exterior bundle) and Lichnerowicz Laplacian (defined on symmetric tensor bundle). The author do not know if there is a uniform way to define a kind of Laplacian so that it coincides with Casimir operator \(\Delta_\tau\) in the current setting. The paper [9] is related to this question.
Since $\Delta_\tau$ is an elliptic differential operator, any eigen-function of it in $L^2(G/H, E_\tau)$ is a smooth section. By the Peter–Weyl theorem, we have

$$L^2(G/H, E_\tau) = \bigoplus_{\rho \in \hat{G}} L^2(G/H, E_\tau)_\rho$$

where $L^2(G/H, E_\tau)_\rho \cong V^\oplus m_\rho$ is the $\rho$-isotropic subspace whose multiplicity $m_\rho$ equals $\dim \text{Hom}_H(U_\tau, V_\rho)$ by the Frobenius reciprocity. We know that $\Delta_\tau$ acts on the $\rho$-isotropic component $L^2(G/H, E_\tau)_\rho$ by a scalar determined by $\rho$. Hence, we have the following fact.

**Lemma 1.6.** If $\mathcal{D}_{H_1, \tau_1} = \mathcal{D}_{H_2, \tau_2}$, then the Hermitian vector bundles $E_{\tau_1} = G \times_{H_1} V_{\tau_1}$ (on $G/H_1$) and $E_{\tau_2} = G \times_{H_2} V_{\tau_2}$ (on $G/H_2$) are isospectral with respect to the differential operators $\Delta_{\tau_1}$ and $\Delta_{\tau_2}$.

Put $G = SU(4n + 2)$. Set

$$H_1 = \{(A, \overline{A}) : A \in U(2n + 1)\} \subset G$$

and

$$H_2 = \{(A, B) : A \in \text{Sp}(2n), B \in \text{SO}(2n + 2)\} \subset G.$$ 

Then, $H_1 \cong U(2n + 1), H_2 \cong \text{Sp}(n) \times \text{SO}(2n + 2)$. For a sequence of integers $k_1 \geq k_2 \geq \cdots \geq k_{2n+1}$ with $k_i + k_{2n+2-i} = 0$ for any $i, 1 \leq i \leq n$, write $\lambda = (k_1, k_2, \ldots, k_{2n+1})$ for a weight of $H_1 \cong U(2n + 1)$. Write $\lambda_1 = (k_1, \ldots, k_n)$ for a weight of $\text{Sp}(2n), \lambda_2 = (k_1, \ldots, k_{n+1})$ for a weight of $\text{SO}(2n + 2)$, and $\lambda' = (\lambda_1, \lambda_2)$ for a weight of $H_2$. Write $\tau_\lambda$ (respectively, $\tau_{\lambda'}$) for an irreducible representation of $H_1$ (resp. $H_2$) with highest weight $\lambda$ (respectively, $\lambda'$). By Proposition 1.5 and Lemma 1.6, we have the following theorem.

**Theorem 1.7.** Let $G = SU(4n + 2)$ and let $H_1, H_2 \subset G$ and representations $\tau_\lambda$ (of $H_1$) and $\tau_{\lambda'}$ (of $H_2$) be as above. Then the Hermitian vector bundles $E_{\tau_\lambda} = G \times_{H_1} V_{\tau_\lambda}$ (on $G/H_1$) and $E_{\tau_{\lambda'}} = G \times_{H_2} V_{\tau_{\lambda'}}$ (on $G/H_2$) are isospectral with respect to the Casimir operators $\Delta_\tau$ and $\Delta_{\tau'}$.

Moreover, we can associate a differential operator $D_\tau$ (respectively, $D_{\tau'}$) on $E_\tau$ (respectively, on $E_{\tau'}$) for any element $D \in Z(\mathfrak{g}) = U(\mathfrak{g})^G$. Then, Proposition 1.5 indicates that: $E_{\tau_\lambda}$ and $E_{\tau_{\lambda'}}$ are isospectral with respect to $D_\tau$ and $D_{\tau'}$ for any $D \in Z(\mathfrak{g})$.

## 2 | GENERALIZATION OF A THEOREM OF LARSEN–PINK

A striking theorem of Larsen and Pink ([6, Theorem 1]) says that the dimension datum of a connected compact semisimple subgroup determines the isomorphism class of the subgroup. Fix a connected compact group $H$ (without assuming semi-simplicity) and consider homomorphisms from it to a connected compact Lie group $G$. We show in the following Theorem 2.1 that $\tau$-dimension data for one-dimensional representations of $H$ determine the isomorphism class of the image of a homomorphism. This answers affirmatively a question of Professor Richard Taylor.
Theorem 2.1. Let $G, H$ be connected compact Lie groups, and $f_1, f_2 : H \to G$ be two homomorphisms. If
\[
\dim((\rho \circ f_1) \otimes \chi)^H = \dim((\rho \circ f_2) \otimes \chi)^H
\]
for any $\rho \in \hat{G}$ and any $\chi \in X^*(H) = \text{Hom}(H, U(1))$, then $f_1(H) \cong f_2(H)$.

Proof of Theorem 2.1. The torus case. When $H$ is a torus, we show a stronger conclusion that $f_2 = \text{Ad}(g) \circ f_1$ for some $g \in G$. The method of proof for this motivates the proof in the general case. First, we show $\ker f_1 = \ker f_2$. Suppose this does not hold. Without loss of generality, we assume that $\ker f_1 \subsetneq \ker f_2$. Then, there exists $\chi \in X^*(H)$ such that $\chi|_{\ker f_1} \neq 1$ and $\chi|_{\ker f_2} = 1$. For any $\rho \in \hat{G}$, we have $\rho \circ f_1|_{\ker f_1} = 1$. Hence, $\dim((\rho \circ f_1) \otimes \chi)^H = 0$. As $\chi|_{\ker f_2} = 1$, $\chi$ descends to a linear character $\chi'$ of $f_2(H) \subset G$. Choose some $\rho \in \hat{G}$ such that $\rho \subset \text{Ind}^G_{f_2(H)}(\chi'^*H)$. Then, $\dim((\rho \circ f_2) \otimes \chi')^H > 0$. This is in contradiction with $\dim((\rho \circ f_1) \otimes \chi)^H = \dim((\rho \circ f_2) \otimes \chi)^H$. Thus, $\ker f_1 = \ker f_2$.

By considering $H/\ker f_1$ instead, we may assume that both $f_1$ and $f_2$ are injections. By considering the supports of the Sato–Tate measure of $f_i(H)$ (which is the push-forward to $G^\#$ (the space of $G$-conjugacy classes in $G$) of a normalized Haar measure on $H$ under the map $f_i(H) \hookrightarrow G \to G^\#$), we know that $f_1(H)$ and $f_2(H)$ are conjugate in $G$ ([15, Proposition 3.7]). We may assume that $f_1(H) = f_2(H)$, and denote it by $T$. Write $\Gamma^0 = N_G(T)/Z_G(T)$.

We identify $H$ with $T$ through $f_1$, and regard $f_2$ as an automorphism of $T$, denoted by $\phi$. Then, the condition in the theorem is equivalent to
\[
F_{0, \chi, \Gamma^0} = F_{0, \phi^*(\chi), \Gamma^0}
\]
by Proposition 1.3. This is also equivalent to $\phi^*(\chi) \in \Gamma^0 \cdot \chi$. We show that $\phi = \gamma|_T$ for some $\gamma \in \Gamma^0$. That means $f_2 = \text{Ad}(g) \circ f_1$ for some $g \in G$. Suppose it is not the case. Then,
\[
X_\gamma = \{\chi \in X^*(H) : \phi^*(\chi) = \gamma \cdot \chi\}
\]
is a sublattice of $X^*(H)$ with positive corank for any $\gamma \in \Gamma^0$. Hence,
\[
\bigcup_{\gamma \in \Gamma^0} X_\gamma \neq X^*(H).
\]
This is in contradiction with $\phi^*(\chi) \in \Gamma^0 \cdot \chi$ for any $\chi \in X^*(H)$.

The general case. First we show $H_{\text{der}} \ker f_1 = H_{\text{der}} \ker f_2$, where $H_{\text{der}} = [H, H]$ is the derived subgroup of $H$. Suppose this does not hold. Without loss of generality, we assume that $H_{\text{der}} \ker f_1 \subsetneq H_{\text{der}} \ker f_2$. Then, there exists $\chi \in X^*(H)$ such that $\chi|_{H_{\text{der}} \ker f_1} \neq 1$ and $\chi|_{H_{\text{der}} \ker f_2} = 1$. For any $\rho \in \hat{G}$, we have $\rho \circ f_1|_{\ker f_1} = 1$. Hence, $\dim((\rho \circ f_1) \otimes \chi)^H = 0$. As $\chi|_{H_{\text{der}} \ker f_2} = 1$, $\chi$ descends to a linear character $\chi'$ of $f_2(H) \subset G$. Choose some $\rho \in \hat{G}$ such that $\rho \subset \text{Ind}^G_{f_2(H)}(\chi'^*H)$. Then, $\dim((\rho \circ f_2) \otimes \chi')^H > 0$. This is in contradiction with $\dim((\rho \circ f_1) \otimes \chi')^H = \dim((\rho \circ f_2) \otimes \chi')^H$. Thus, $H_{\text{der}} \ker f_1 = H_{\text{der}} \ker f_2$.

Write $H_i = f_i(H)$. Due to the isomorphism $H/H_{\text{der}} \ker f_i \cong H_i/(H_i)_{\text{der}}$, we have
\[
H_1/(H_1)_{\text{der}} \cong H_2/(H_2)_{\text{der}}.
\]
Choose a maximal torus $T_i$ of $H_i$. Write $(T_i)_s = T_i \cap (H_i)_{\text{der}}$. Then, $(T_i)_s$ is a maximal torus of $(H_i)_{\text{der}}$ and $T_i = Z(H_i)^0 \cdot (T_i)_s$. Due to the isomorphism $T_i/(T_i)_s \cong H_i/(H_i)_{\text{der}}$, we have

$$T_1/(T_1)_s \cong T_2/(T_2)_s.$$ 

By considering the supports of Sato–Tate measures of $H_1$ and $H_2$, we know that $T_1$ and $T_2$ are conjugate in $G$ ([15, Proposition 3.7]). We may assume that $T_1 = T_2$, and denote it by $T$. Write $\Gamma^0 = N_G(T)/Z_G(T)$.

Choose a bi-invariant Riemannian metric on $G$, which induces a $\Gamma^0$ invariant inner product on the Lie algebra of $T$, and also a $\Gamma^0$ invariant inner product on the weight lattice $X^*(T)$. Write $\Phi_1 \subset X^*(T)$ for the root system of $H_i$ and write $X_i = X^*(T_i)/(T_i)_s \subset X^*(T)$.

Then, the isomorphism $T_1/(T_1)_s \cong T_2/(T_2)_s$ gives an isomorphism $\phi : X_1 \to X_2$. For any $\chi_1 \in X_1$, write $\chi_2 = \phi(\chi_1)$. Then, we have

$$F_{\Phi_1,\chi_1,\Gamma_0} = F_{\Phi_2,\chi_2,\Gamma_0}$$

by Proposition 1.3. Due to $\chi_1$ being orthogonal to $\delta_{\Phi_1} - w\delta_{\Phi_1}$ for any $w \in W_\Phi$, $X_{\chi_1,\Gamma_0}$ is the shortest term in the expansion of $F_{\Phi_1,\chi_1,\Gamma_0}$. Thus, $X_\lambda = \gamma \cdot X_1$ for some $\gamma \in \Gamma^0$. Arguing similarly as in the torus case, one shows that $\phi = \gamma|_{X_1}$ for some $\gamma \in \Gamma^0$. Replacing $f_2$ by $\text{Ad}(g)\circ f_2$ for some $g \in N_G(T)$ if necessary, we may assume that $\phi = \text{id}$. Then, $X_1 = X_2$ and $(T_1)_s = (T_2)_s$. As the Lie algebra of $Z(H_i)^0$ is orthogonal to the Lie algebra of $(T_1)_s$, we have $Z(H_1)^0 = Z(H_2)^0$. Write $Z = Z(H_1)^0$, $T_s = (T_1)_s$ and $X = X_1$. Let $G'$ be the centralizer of $Z$ in $G$. Put

$$\Gamma' = N_{G'}(T)/Z_{G'}(T).$$

Then,

$$\Gamma' = \{\gamma \in \Gamma^0 : \gamma|_Z = \text{id}\} = \{\gamma \in \Gamma^0 : \gamma|_X = \text{id}\}.$$ 

If $X$ has rank 0 (that is, $X = 0$), then $H_1$ and $H_2$ are semisimple groups. By [6, Theorem 1], one has $H_1 \cong H_2$. Now we assume that $X$ has positive rank. For any $\gamma \in \Gamma^0 - \Gamma'$,

$$X_\gamma := \{\chi \in X : \gamma \cdot \chi = \chi\}$$

is a sublattice of positive corank. Thus, $\bigcup_{\gamma \in \Gamma^0 - \Gamma'} X_\gamma \neq X$. Choose

$$X_0 \in X - \bigcup_{\gamma \in \Gamma^0 - \Gamma'} X_\gamma.$$ 

Write

$$c = \min\{|\gamma \cdot X_0 - X_0| : \gamma \in \Gamma^0 - \Gamma'| > 0,$$

$$c' = \max\{|\delta_{\Phi_2} - w_2\delta_{\Phi_2}| + |\delta_{\Phi_1} - w_1\delta_{\Phi_1}| : w_1 \in W_{\Phi_1}, w_2 \in W_{\Phi_2}| \geq 0.$$
Take $m \geq 1$ such that $mc > 2c'$. Put $\chi = m\chi_0$. Then, for any $\gamma \in \Gamma^0$ and any $w_j \in W_{\phi_{ij}}$ ($i_j = 1$ or 2),

$$\gamma(\chi + \delta_{\phi_{i_1}} - w_1\delta_{\phi_{i_1}}) = \chi + \delta_{\phi_{i_2}} - w_2\delta_{\phi_{i_2}}$$

if and only if $\gamma \in \Gamma'$ and

$$\gamma(\delta_{\phi_{i_1}} - w_1\delta_{\phi_{i_1}}) = \delta_{\phi_{i_2}} - w_2\delta_{\phi_{i_2}}.$$

Then, the equality $F_{\phi_{1,\chi',\Gamma^0}} = F_{\phi_{2,\chi',\Gamma^0}}$ implies $F_{\phi_{1,0,\Gamma'}} = F_{\phi_{2,0,\Gamma'}}$. Define a root system $\Psi_{T_s}$ as in the Subsection 1.1. Then, $\Gamma' \subset \text{Aut}(\Psi_{T_s})$. Thus, we have

$$F_{\phi_{1,0,\text{Aut}(\Psi_{T_s})}} = F_{\phi_{2,0,\text{Aut}(\Psi_{T_s})}}.$$ 

Hence, results in [15, Section 7] imply that $\Phi_2 = \gamma \cdot \Phi_1$ for some $\gamma \in \text{Aut}(\Psi_{T_s})$. This leads to an isomorphism $\eta : (H_1)_{\text{der}} \to (H_2)_{\text{der}}$ which stabilizes $T_s$ and has $\eta|_{T_s} = \gamma$. Note that $Z \cap (H_1)_{\text{der}} = Z \cap (T_1) = Z \cap T_s \subset T_s \cap Z(G')$. Decompose $\Psi_{T_s}$ into an orthogonal union of irreducible root systems, which leads to a decomposition of $T_s$. Due to the fact that the weight lattice and the root lattice of a root system $BC_n$ coincide, $T_s \cap Z(G')$ is contained in the product of those factors of $T_s$ which correspond to reduced irreducible factors of $\Psi_{T_s}$. The results in [15, Section 7] imply that there exists $\gamma' \in \Gamma'$ such that the action $\gamma$ on reduced irreducible factors of $\Psi_{T_s}$ coincides with that of $\gamma'$. Hence,

$$\eta|_{T_s \cap Z(G')} = \gamma|_{T_s \cap Z(G')} = \gamma'|_{T_s \cap Z(G')} = \text{id}.$$ 

Then, $\eta$ extends to an isomorphism $\eta : H_1 \to H_2$ by letting $\eta|_{Z} = \text{id}$. □

Let $H_1, H_2$ be two connected semisimple subgroups of $G$ with the same dimension datum. Write $i_1, i_2$ for the inclusions of $H_1, H_2$ in $G$, respectively. Let $H = H_1 \times H_2$ and write $p_1, p_2$ for the projection of $H$ to the components $H_1$ and $H_2$, respectively. Set $f_j = i_j \circ p_j$ ($j = 1, 2$). Then, the homomorphisms $f_1, f_2 : H \to G$ satisfy the condition in Theorem 2.1. Applying Theorem 2.1, we get: $H_1 = f_1(H)$ and $H_2 = f_2(H)$ are isomorphic. This recovers [6, Theorem 1].

3 COMPACTNESS OF ISOSPECTRAL SETS

A big conjecture in spectral geometry says that any set of isospectral closed Riemannian manifolds is compact [3, 7]. In [14], we show a result of this flavor for normal Riemannian homogeneous spaces.

**Theorem 3.1** [14, Theorem 3.6]. Let $G$ be a compact Lie group equipped with a bi-invariant Riemannian metric $m_0$ and $H$ be a closed subgroup. Then up to conjugacy, there are only finitely many closed subgroups $H_1, \ldots, H_k$ of $G$ such that the normal Riemannian homogeneous space $(G/H_j, m_0)$ is isospectral to $(G/H, m_0)$. 
It is shown in [1, Theorem 1.2] that the conjugacy class of a closed subgroup \( H \) has only finitely many possibilities if \( D_H \) is given. This confirms an expectation of Langlands. Theorem 3.1 is stronger than [1, Theorem 1.2]. Here we prove a generalization of Theorem 3.1 in case \( G \) is semisimple by allowing the Riemannian metrics to vary.

**Theorem 3.2.** Let \( G \) be compact semisimple Lie group with a bi-invariant Riemannian metric \( m_0 \) and \( H_0 \) be a closed subgroup. Then there are only finitely many conjugacy classes of closed subgroups \( H \) of \( G \) such that there exists a bi-invariant Riemannian metric \( m \) on \( G \) which induces a normal Riemannian homogeneous space \( (G/H, m) \) isospectral to \( (G/H_0, m_0) \).

**Proof.** First we may assume that \( G \) is connected and simply connected. Write \( G = G_1 \times \cdots \times G_s \) for the decomposition of \( G \) into simple factors. For each \( i \), choose a bi-invariant Riemannian metric \( m_{0,i} \) on \( G_i \). By normalization, we may assume that the Laplace operator and the Casimir operator coincide on \( (C^\infty(G_i), m_{0,i}) \) \((1 \leq i \leq s)\).

Suppose that \( \{(G/H_n, m_n) : n \geq 1\} \) is a sequence of normal Riemannian homogeneous spaces such that the Laplace spectrum of each \( (G/H_n, m_n) \) is equal to that of \( (G/H_0, m_0) \), and \( H_n \) \( (n \geq 1) \) are non-conjugate to each other. Write

\[
m_n = \bigoplus_{1 \leq i \leq s} a_i^{(n)} m_{0,i}.
\]

By [14, Theorem 1.1], there exists a closed subgroup \( H \) of \( G \), a subsequence \( \{H_{n,j} : j \geq 1\} \) and a sequence \( \{g_j : j \geq 1, g_j \in G\} \) such that for all \( j \in \mathbb{N} \),

\[ [H^0, H^0] \subset g_j H_{n,j} g_j^{-1} \subset H, \]

and

\[ \lim_{j \to \infty} D_{H_{n,j}} = D_H. \]

Substituting \( \{(G/H_n, m_n) : n \geq 1\} \) by a subsequence if necessary we may assume that: for any \( n \geq 1 \),

\[ [H^0, H^0] \subset H_n \subset H, \]

and

\[ \lim_{n \to \infty} D_{H_n} = D_H. \]

Since \( H_n \) are assumed to be non-conjugate to each other, at most finitely many of them contain \( H^0 \). By removing such exceptions, we may assume that

**Assumption 3.1.** \( \dim H_n < \dim H \) for all \( n \).

We may also assume that each sequence \( \{a_i^{(n)} : n \geq 1\} \) converges. Write

\[ a_i = \lim_{n \to \infty} a_i^{(n)} \in [0, \infty]. \]
Without loss of generality, we assume that

\[ a_1 = \cdots = a_u = 0, \]

\[ 0 < a_{u+1}, \ldots, a_v < \infty, \]

\[ a_{v+1} = \cdots = a_s = \infty, \]

where \( 0 \leq u \leq v \leq s \). Write

\[ G^{(1)} = \prod_{1 \leq i \leq u} G_i, \quad G^{(2)} = \prod_{1 \leq i \leq v} G_i, \quad G^{(3)} = \prod_{v+1 \leq i \leq s} G_i, \]

\[ G' = \prod_{u+1 \leq i \leq v} G_i, \quad H' = G' \cap (HG^{(1)}), \quad m' = \bigoplus_{u+1 \leq i \leq v} a_i m_{0,i}. \]

Write \( \chi_i(\rho) (1 \leq i \leq s) \) for the value of the Casimir operator acting on matrix coefficients of \( \rho \in \hat{G}_i \). We know that: \( \chi_i(\rho) \geq 0 \), and \( \chi_i(\rho) = 0 \) if and only if \( \rho = 1 \). We first show that \( G^{(3)} \subset HG^{(2)} \). Suppose this does not hold. Then, there exists a non-trivial irreducible representation

\[ \rho = \bigotimes_{v+1 \leq i \leq s} \rho_i \]

of \( G^{(3)} \) such that \( V_{\rho}^{G^{(3)} \cap HG^{(2)}} \neq 0 \). Take \( 0 \neq v \in V_{\rho}^{G^{(3)} \cap HG^{(2)}} \) and \( 0 \neq \alpha \in V^* \). Set

\[ f_{v,\alpha}(g_1, \ldots, g_s) = \alpha((g_{v+1}, \ldots, g_s) \cdot v). \]

Then, \( f_{v,\alpha} \in C^\infty(G/H) \subset C^\infty(G/H_n) \) for any \( n \geq 1 \). The Laplace eigenvalue for \( f_{v,\alpha} \in (C^\infty(G/H_n), m_n) \) is equal to

\[ \sum_{v+1 \leq i \leq s} \frac{1}{a_i} \chi_i(\rho_i) > 0. \]

When \( n \to \infty \), this value tends to 0. This is in contradiction with the fact that the Laplace spectrum of each \( G/H_n \) is equal to a given spectrum which is a discrete set in \( \mathbb{R}_{\geq 0} \).

Now we assume \( G^{(3)} \subset HG^{(2)} \). Then, \( H \) is of the form

\[ H = (H \cap G^{(2)}) \times \{ (\phi(x), x) : x \in G^{(3)} \} \]

for some homomorphism \( \phi : G^{(3)} \to G^{(2)} \). Put

\[ G^{(4)} = \{ (\phi(x), x) : x \in G^{(3)} \}. \]

Let \( G^{(5)} \) be the centralizer of \( G^{(4)} \) in \( G \). Then, \( H \cap G^{(2)} \subset G^{(5)} \subset G^{(2)} \). Due to the fact that \([H^0, H^0] \subset H_n \subset H \), each \( H_n \) is of the form

\[ H_n = (H_n \cap G^{(2)}) \times G^{(4)}. \]
Applying [14, Theorem 1.1] to the subgroups \( H_n \cap G^{(2)} \) of \( H \cap G^{(2)} \), we find a subgroup \( \tilde{H} \) of \( H \cap G^{(2)} \) such that \( \lim_{n \to \infty} D_{H_n \cap G^{(2)}} = D_{\tilde{H}} \) as dimension data of subgroups of \( H \cap G^{(2)} \). Put \( \tilde{H}' = \tilde{H} \times G^{(4)} \). Then, \( \lim_{n \to \infty} D_{H_n} = D_{\tilde{H}'} \). Thus, \( \tilde{H}' \subset H \) and \( D_{\tilde{H}'} = D_H \). By [1, Lemma 2.3], we get \( \tilde{H}' = H \). Hence, \( \tilde{H} = H \cap G^{(2)} \). Therefore,

\[
\lim_{n \to \infty} D_{H_n \cap G^{(2)}} = D_{H \cap G^{(2)}}
\]

as dimension data of subgroups of \( G^{(5)} \).

Let \( c \) be a positive real number. Suppose matrix coefficients of

\[
\rho = \bigotimes_{1 \leq i \leq s} \rho_i
\]

contribute to the Laplace spectrum of \((G/H_n, m_n)\) in the eigenvalue scope \([0, c]\). Then,

\[
\sum_{1 \leq i \leq s} \frac{1}{a_i^{(n)}} \chi_i(\rho_i) \leq c
\]

and \( \rho^{G^{(4)}} \neq 0 \). Note that \( a_i^{(n)} \to a_i \), we have when \( n \) is sufficiently large, each \( \rho_i = 1 \) \((1 \leq i \leq u)\) and each \( \rho_i \) \((u + 1 \leq i \leq s)\) lies in a finite set. Due to \( \rho^{G^{(4)}} \neq 0 \), \( \bigotimes_{u+1 \leq i \leq s} \rho_i \) is determined by \( \bigotimes_{1 \leq i \leq u} \rho_i \) up to finitely many possibilities. Then, there are only finitely many \( \rho \) needed to be considered. For each of such \( \rho \), we have

\[
\lim_{n \to \infty} \dim V_{H_n}^{\rho} = \dim V_{\tilde{H}}^{\rho} = \dim V_{H^{G^{(1)}}}^{\rho}
\]

for the invariant dimensions, and

\[
\lim_{n \to \infty} \sum_{1 \leq i \leq s} \frac{1}{a_i^{(n)}} \chi_i(\rho_i) = \sum_{1 \leq i \leq s} \frac{1}{a_i} \chi_i(\rho_i)
\]

for the eigenvalues. Note that

\[
G/HG^{(1)} \cong G^{(2)}/G^{(2)} \cap HG^{(1)} \cong G'/G' \cap HG^{(1)} = G'/H'.
\]

These together imply that: the Laplace spectrum of \((G'/H', m')\) is larger than the Laplace spectrum of \((G/H_0, m_0)\). On the other hand, if matrix coefficients of

\[
\rho = \bigotimes_{1 \leq i \leq s} \rho_i
\]

contribute to the Laplace spectrum of \( G/HG^{(1)} \cong G'/H' \) in the eigenvalue scope \([0, c]\), then we have the same statements for \( \{\rho_i : 1 \leq i \leq s\} \) as above. By the stabilization of invariant dimensions and the convergence of eigenvalues, it follows that the Laplace spectrum of \((G'/H', m')\) is smaller than the Laplace spectrum of \((G/H_0, m_0)\). Therefore, the Laplace spectrum of \((G'/H', m')\) is equal to the Laplace spectrum of \((G/H_0, m_0)\). By the Minakshisundaram–Pleijel asymptotic expansion
formula, Laplace spectrum determines the dimension (cf. [3, Subsection 1.1]). Then,

$$\dim G/H_n = \dim G/H_0 = \dim G'/H' = \dim G/HG^{(1)}$$

for any $n \geq 1$. Hence, $\dim H_n = \dim H G^{(1)}$, which is in contradiction with Assumption (3.1).

Motivated by the compactness conjecture of isospectral sets, we expect the following statement holds.

**Conjecture 3.1.** For any given spectrum, there are only finitely many normal Riemannian homogeneous spaces $(G/H, m)$ up to isometry with Laplace spectrum equal to it.

We remark that in the above conjecture we care about the isometry class of the normal Riemannian homogeneous space $(G/H, m)$, rather than that of the triple $(G, H, m)$. One of the reasons for this is: for a given pair $H \subset G$, there might exist infinitely many different Riemannian metrics $m$ on $G$ which induce the same Riemannian metric on $G/H$ (for example, when $G = K \times K$ and $H = \{(x, x) : x \in K\}$ for $K$ a connected compact Lie group). When $G$ and $m$ are both given, Conjecture 3.1 is confirmed affirmatively by Theorem 3.1.

Any normal Riemannian homogeneous space can be written as the form $M = G/H$ such that the following assumption holds:

**Assumption 3.2.**

(i) $G = T \times G_{\text{der}}$ with $T$ a torus and $G_{\text{der}}$ a connected and simply connected compact semisimple Lie group.

(ii) $H \cap T = 1$.

(iii) $H$ does not contain any simple factor of $G_{\text{der}}$.

Assuming $M = G/H$ is of this form, then $\dim G_{\text{der}}/H \cap G_{\text{der}} \leq \dim M$ and $\dim T \leq \dim M$.

Using the dimension gap of proper subgroups of connected simple Lie groups, one shows that the isomorphism type of $G_{\text{der}}$ has only finitely many possibilities from Assumption 3.2(iii) when $\dim M$ is given. Since $\dim M$ is determined by the Laplace spectrum of $M$, we may assume that the group $G$ is fixed in Conjecture 3.1.

When $G$ is a fixed connected semisimple compact Lie group, there are only finitely many possibilities for the subgroup $H$ by Theorem 3.2. Then, in this case Conjecture 3.1 reduces to the following Question 3.1. Question 3.1 has an affirmative answer when $G/H$ is a compact symmetric space (cf. [4]).

**Question 3.1.** Let $G$ be a given connected semisimple compact Lie group and $H$ be a given closed subgroup satisfying the above assumption. Are there only finitely many normal Riemannian homogeneous spaces $(G/H, m)$ up to isometry with Laplace spectrum equal to a given spectrum?

When $G$ is a fixed torus, then $H = 1$ by Assumption 3.2(ii). In this case, Conjecture 3.1 is the consequence of a theorem of Kneser. A simple proof of Kneser’s theorem is given in [13], which is based on the Mahler compactness theorem for lattices.

For general $G$, the main difficulty is due to the complexity of the restriction on the toric part of the invariant inner product induced by $m$ on the Lie algebra of $G$. Perhaps a sophisticated use
of Mahler compactness theorem coupled with the compactness result for dimension datum ([14, Theorem 1.1]) could overcome this difficulty.

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