Generalized $q$-Onsager algebras and dynamical $K$-matrices

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Abstract
A procedure to construct $K$-matrices from the generalized $q$-Onsager algebra $O_q(\hat{g})$ is proposed. This procedure extends the intertwiner techniques used to obtain scalar ($c$-number) solutions of the reflection equation to dynamical (non-$c$-number) solutions. It shows the relation between soliton non-preserving reflection equations or twisted reflection equations and the generalized $q$-Onsager algebras. These dynamical $K$-matrices are important to quantum integrable models with extra degrees of freedom located at the boundaries; for instance, in the quantum affine Toda field theories on the half-line, they yield the boundary amplitudes. As examples, the cases of $O_q(a_2^{(2)})$ and $O_q(a_2^{(1)})$ are treated in detail.

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1. Introduction

The mathematical structure behind the Yang–Baxter equation (YBE), namely quantum groups, gives a simple way to obtain the $R$-matrix as a solution of the intertwiner equation. For instance, the quantum affine algebras $\mathcal{U}_q(\hat{g})$, formulated by Drinfel’d and Jimbo [D1, J1], allow constructing trigonometric $R$-matrices [J], solutions of the Yang–Baxter equation. These algebras admit different formulations: (1) $q$-Serre–Chevalley formulation$^3$ [D1, J1]; (2) FRT formulation [FaddReshT, RS], which appeared in the context of the quantum inverse scattering method for quantum integrable systems; (3) and Drinfel’d second realization [D2]. Each of these formulations is relevant in different physical and mathematical problems: the $q$-Serre–Chevalley formulation of quantum algebra leads to the construction of the $R$-matrices [J] and its realization in terms of field operators corresponds to non-local conserved currents of the affine Toda field theories (ATFT) [BL]; the FRT formulation gives a simple construction

$^3$ In the limit $q \to 1$, one recovers the Serre–Chevalley formulation of the affine Lie algebras.
of Abelian-conserved quantities of bulk integrable models and in many cases it allows the diagonalization of the transfer matrix using the so-called algebraic Bethe ansatz technique [STF]; the Drinfel’d second realization leads to infinite-dimensional representations of the algebra [FrenJin], the calculation of correlation functions using vertex operators techniques [JMMNDf] or the calculation of scalar products of Bethe vectors [BPR].

In the case of quantum integrable models with boundaries, the reflection equations that ensure the integrability of these models have been known since the seminal works of Cherednik and Sklyanin [Cher, Skly]. The related algebraic structures are given by coideal subalgebras of the quantum algebras. These subalgebras are called reflection algebras (RAs) and the commutation relations are encoded by the reflection equations [MRS]. The definition of the RAs can be constructed from the FRT formulation of the quantum algebra \( U_q(\hat{g}) \) using two different automorphisms. The first one is related to the inverse of the monodromy matrix and we call it the RA. The second one is related to the transposition of the monodromy matrix and we call it the twisted reflection algebra (TRA). In both cases the algebras are given in a FRT-type formulation and the question of a corresponding \( q \)-Serre–Chevalley or Drinfel’d second realization-type formulation is mostly open. For the finite Lie algebra \( g \), the \( q \)-Serre–Chevalley-type formulation of the reflection equation is known and appears in the context of the quantum symmetric pairs and related \( q \)-orthogonal polynomials [NS, Letz]. For special choices of \( g \) it corresponds to special cases of the Zhedanov or Askey–Wilson algebra [Zhed] related to the Askey–Wilson polynomials. Moreover, in the case of \( U_q(a_1^{(1)}) \), an isomorphism of coideal subalgebra has been shown between the RA and the \( q \)-Onsager algebra \( O_q(a_1^{(1)}) \) (which corresponds to a \( q \)-Serre–Chevalley formulation of the reflection equation) [BS, BB2].

In a recent work [BB1], the generalized \( q \)-Onsager algebra \( O_q(\hat{g}) \) was introduced as a coideal subalgebra of the quantum algebra \( U_q(\hat{g}) \). Similar to the calculation of the \( R \)-matrices using the \( q \)-Serre–Chevalley formulation of \( U_q(\hat{g}) \) and the intertwiner equation [J1], the generalized \( q \)-Onsager algebra \( O_q(\hat{g}) \) allows the calculation of the \( K \)-matrices, solutions of the twisted reflection equation. These \( O_q(\hat{g}) \) algebras emerge as the closure relations of the non-local charges of the ATFT on the half-line with soliton non-preserving boundary conditions. Note that these non-local charges have been previously constructed from the \( U_q(\hat{g}) \) generators and were sufficient to construct scalar (\( c \)-number) boundary scattering amplitudes from the intertwiner equation [MN, DM, DG], although their closure relations were unknown. However, the identification of the \( O_q(\hat{g}) \) algebra as a non-Abelian symmetry of the ATFT on the half-line allows us to proceed further; it gives the complete description of the admissible soliton non-preserving boundary conditions: (1) scalar (\( c \)-number) ones, which reproduce the known results [CorDRS] and (2) dynamical (non-\( c \)-number) ones [BB1]. As a consequence, the knowledge of \( O_q(\hat{g}) \) allows us to extend the intertwiner technique to dynamical (non-\( c \)-number) \( K \)-matrices in a systematic way.

From a physical point of view, the dynamical boundary conditions correspond to quantum integrable models with degrees of freedom at the boundary. Some examples are given by conformal models perturbed by a dynamical boundary, such as the massless boundary Sine–Gordon model related to the quantum impurity problem [LesSalCSia], or the generalization to the massless boundary ATFT \( a_2^{(1)} \) [BazHK]. Extensions to the massive boundary Sine–Gordon model have also been considered [BassLeC, BD, BK]. For one of them [BK], a dynamical \( K \)-matrix has been obtained using the intertwiner technique. Among the properties of this model, it is important to note that for real coupling the boundary amplitudes are self-dual [BK], as for the bulk case [BCorDS], contrary to the case with scalar boundary conditions [G, Cor]. There are also examples of dynamical boundary conditions in quantum integrable

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4 Corresponding to the twisted reflection equation.
spin chains, where the dynamical $K$-matrices are obtained as ‘dressing’ of scalar ($c$-number) boundary matrices [ZGLG1, ZGLG2, FraSlav].

In this paper, we present dynamical (non-$c$-number) $K$-matrices obtained using the intertwiner equation associated with the generalized $q$-Onsager algebras for two cases: $O_q(a_2^{(2)})$ and $O_q(a_2^{(1)})$. These two examples show an explicit connection between the generalized $q$-Onsager algebras and the RAs. In section 2 we recall some basics about the quantum algebras, RAs, intertwiner equations and explain the construction of the dynamical $K$-matrices. In section 3 we give the $K$-matrices for $O_q(a_2^{(2)})$ and $O_q(a_2^{(1)})$ realized in terms of the Zhedanov algebra [Zhed]. The mapping to scalar ($c$-number) matrices is described in section 4, recovering known results. Extensions to other affine Lie algebras, applications in physics and further investigations are discussed in section 5.

2. Affine quantum algebras, reflection equations and generalized $q$-Onsager algebras

In this section we briefly recall basic notions on the affine quantum algebras and their coideals. We emphasize the similarities between the FRT formalism (RLL relation) and intertwiner equation (quasi-triangular Hopf structure) for quantum algebras [CP] and the Yang–Baxter equation and intertwiner equation for coideals of quantum algebras [MN, DM, N].

We point out that generalized $q$-Onsager algebras are realized as coideal subalgebras of quantum algebras and the knowledge of these algebras both with the intertwiner equation provides a simple method for the derivation of solutions of the reflection equation. At the end of the section we give a strategy for the construction of dynamical (non-$c$-number or operator-valued) $K$-matrices.

2.1. Hopf algebra and the Yang–Baxter equation

In mathematics, the affine quantum algebras $\mathcal{U}_q(\widehat{\mathfrak{g}})$ are quasi-triangular Hopf algebras (see [CP] for details). Let us consider the coproduct map $\Delta : \mathcal{U}_q(\widehat{\mathfrak{g}}) \to \mathcal{U}_q(\widehat{\mathfrak{g}}) \otimes \mathcal{U}_q(\widehat{\mathfrak{g}})$ and the universal $R$-matrix $R \in \mathcal{U}_q(\widehat{\mathfrak{g}}) \otimes \mathcal{U}_q(\widehat{\mathfrak{g}})$. The universal $R$-matrix is an invertible element such that

$$R \Delta (x) = \sigma \circ \Delta (x) R,$$

with $x \in \mathcal{U}_q(\widehat{\mathfrak{g}})$ and $\sigma$ being the permutation map defined by $\sigma (a \otimes b) = b \otimes a$. As a consequence of the quasi-triangular properties, the universal $R$-matrix is a solution of the universal Yang–Baxter equation:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$  \hspace{1cm} (2.2)

Using a finite-dimensional representation $\pi_u : \mathcal{U}_q(\widehat{\mathfrak{g}}) \to \text{End}(V)$, with $V$ being a finite-dimensional vector space, one maps the universal object to a matrix. It follows that the $R$-matrix satisfies the intertwiner equation

$$R_{12}(u/v)(\pi_u \otimes \pi_v)[\Delta (x)] = (\pi_u \otimes \pi_v)[\sigma \circ \Delta (x)] R_{12}(u/v), \quad x \in \mathcal{U}_q(\widehat{\mathfrak{g}})$$  \hspace{1cm} (2.3)

and the Yang–Baxter equation

$$R_{12}(u/v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u/v),$$  \hspace{1cm} (2.4)

where

$$R_{12}(u/v) = (\pi_u \otimes \pi_v) R_{12}.$$  \hspace{1cm} (2.5)

For formal parameters $u$ and $v$, the tensor product of representations $\pi_u \otimes \pi_v$ is an irreducible representation, which implies that the solution of (2.3) is unique up to a scalar function and
also a solution of \((2.4)\) [J]. These properties allow the construction of the scalar \(R\)-matrices solution of the Yang–Baxter equation from the representation theory and the coproduct of \(\mathcal{U}_q(\hat{\mathfrak{g}})\).

If one evaluates only the first space of the universal \(R\)-matrix with \(\pi_a\), one obtains the affine monodromy matrix
\[ L_1(u) = (\pi_a \otimes 1) R_{12}^a, \]
(2.6)
Evaluating the spaces 1 and 2 of the universal Yang–Baxter equation one obtains the FRT
\[ R_{12}(u/v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u/v). \]
(2.7)
When \(\hat{\mathfrak{g}} = \mathfrak{gl}_{n+1}\), an evaluation homomorphism \(\Pi : \mathcal{U}_q(\mathfrak{gl}_{n+1}) \to \mathcal{U}_q(gln)\) has been constructed [J1] and the corresponding finite monodromy matrix obtained from the intertwiner equation:
\[ L^\Pi(u)(\pi_a \otimes \Pi) \Delta(x) = (\pi_a \otimes \Pi)(\sigma \circ \Delta(x)) L^\Pi(u). \]
(2.8)
The aim of this paper is to provide the construction of a similar object in the case of the RAs.

2.2. Reflection algebras and coideal subalgebras

The reflection equations are representations of the RAs, which can be realized as coideal subalgebras of the quantum affine algebras \(\mathcal{U}_q(\hat{\mathfrak{g}})\). Construction of these RAs is related to automorphisms of the FRT algebra [Skly, MRS]. There are two automorphisms of the FRT algebra leading to two different RAs.

– The first family, the so-called RA, noted \(\mathcal{B}(\hat{\mathfrak{g}})\), is generated by \(K(u) \in \text{End}(V) \otimes \mathcal{B}(\hat{\mathfrak{g}})[[u, u^{-1}]]\) with the commutation relation
\[ R_{12}(u/v)K_1(u)R_{21}(uv)K_2(v) = K_2(v)R_{12}(uv)K_1(u)R_{21}(u/v). \]
(2.9)

The coideal properties follow from the existence of a coaction map \(\delta : \mathcal{B}(\hat{\mathfrak{g}}) \to \mathcal{U}_q(\hat{\mathfrak{g}}) \otimes \mathcal{B}(\hat{\mathfrak{g}})\) given by
\[ \delta(K(u)) = L(u)K(u)L(u^{-1})^{-1} \]
(2.10)
and a counit \(\epsilon : \mathcal{B}(\hat{\mathfrak{g}}) \to \mathbb{C}\) given by
\[ \epsilon(K(u)) = K(u), \]
(2.11)
where the matrix \(K(u)\) is a scalar solution of \((2.9)\). These coideal properties imply that the RA can be constructed as a subalgebra of \(\mathcal{U}_q(\hat{\mathfrak{g}})\) from the homomorphism
\[ (1 \otimes \epsilon) \circ \delta \equiv \psi : \mathcal{B}(\hat{\mathfrak{g}}) \to \mathcal{U}_q(\hat{\mathfrak{g}}); \]
\[ \psi(K(u)) = L(u)K(u)L(u^{-1})^{-1}. \]
(2.12)
The map \(\psi(u) : L(u) \to L(u^{-1})^{-1}\) is an automorphism of the FRT algebra.

Although a clear algebraic framework for these coideal subalgebras and an ‘universal’ \(K\)-matrix is still an open question, at the representation level an intertwiner equation can also be defined for the \(K\)-matrix [MN, DM, N]. Assuming that a \(q\)-Serre–Chevalley formulation of \(\mathcal{B}(\hat{\mathfrak{g}})\) is known, we can define the equation
\[ K(u)(\pi_a \otimes \Pi)\delta(y) = (\pi_{a^{-1}} \otimes \Pi)\delta(y)K(u), \quad y \in \mathcal{B}(\hat{\mathfrak{g}}), \]
(2.13)
with \(\Pi\) being an evaluation homomorphism to a finite algebra. Using the commuting diagram technique [DM], one can show that the solution of this equation will satisfy equation \((2.9)\).

\[ \text{For simplicity, we consider only one family of monodromy matrices, a complete description of the algebra needs two families, see [RS] for details.} \]
The second family, the so-called TRA, noted $B^*(\tilde{g})$, is generated by $K^*(u) \in \text{End}(V) \otimes B^*(\tilde{g})[[u, u^{-1}]]$ with the commutation relation

$$R_{12}(u/v)K^*_1(u)R_{12}^*(1/uv)K^*_2(v) = K^*_2(v)R_{12}^*(1/uv)K^*_1(u)R_{12}(u/v)$$  \hspace{1cm} (2.14)

The coideal properties follow from the existence of a coaction map $\delta : B^*(\tilde{g}) \rightarrow \mathcal{U}_q(\tilde{g}) \otimes B^*(\tilde{g})$ given by

$$\delta(K^*(u)) = L(u)K^*(u)L(u^{-1})^t$$ \hspace{1cm} (2.15)

and a counit $\epsilon : B(\tilde{g}) \rightarrow \mathbb{C}$ given by

$$\epsilon(K^*(u)) = K^*(u),$$ \hspace{1cm} (2.16)

where the matrix $K^*(u)$ is a scalar solution of (2.14). These coideal properties imply that the RA can be constructed as a subalgebra of $\mathcal{U}_q(\tilde{g})$ from the homomorphism $(1 \otimes \epsilon) \circ \delta = \psi : B^*(\tilde{g}) \rightarrow \mathcal{U}_q(\tilde{g})$:

$$\psi(K^*(u)) = L(u)K^*(u)L(u^{-1})^t.$$ \hspace{1cm} (2.17)

The map $w_2 : L(u) \rightarrow L(u^{-1})^t$ is an automorphism of the FRT algebra.

As for the RA case, assuming that a $q$-Serre–Chevalley formulation of $B^*(\tilde{g})$ is known, we can define the intertwiner equation:

$$K^*(u)(\pi_u \otimes \Pi)\delta(y) = (\tilde{\pi}_{u^{-1}} \otimes \Pi)\delta(y)K^*(u), \hspace{1cm} y \in B^*(\tilde{g}),$$ \hspace{1cm} (2.18)

with $\Pi$ being an evaluation homomorphism to a finite algebra and $\tilde{\pi}$ a finite-dimensional conjugate representation of $\mathcal{U}_q(\tilde{g})$ of the same dimension as $\pi$. Using commuting diagrams [DM], one can show that the solution of this equation will satisfy equation (2.14).

Let us mention that if the $R$-matrix of $\mathcal{U}_q(\tilde{g})$ satisfies the crossing symmetry relation

$$R_{12}(u) = M_2R_{12}^*(u^{-1})M_2 \quad \text{with} \quad M^2 = 1 \quad \text{and} \quad M_1M_2R_{12}(u) = R_{23}(u)M_1M_2,$$ \hspace{1cm} (2.19)

then two families of RA are equivalent and

$$K^*(u) = K(u^{1/2})M.$$ \hspace{1cm} (2.20)

### 2.3. The generalized $q$-Onsager algebra

To solve the intertwiner equations (2.13) or (2.18) one has to identify the $q$-Serre–Chevalley-type formulation of the RAs $B(\tilde{g})$ or $B^*(\tilde{g})$ respectively.

The generators made up of $\mathcal{U}_q(\tilde{g})$:

$$a_i = c_i e_i q_i^{b/2} + \bar{c}_i f_i q_i^{b/2} + w_i q_i^b, \quad [c_i, \bar{c}_i, w_i] \in \mathbb{C} \quad \text{and} \quad i = 0, 1, \ldots, \text{rank}(g)$$ \hspace{1cm} (2.21)

correspond to non-local conserved charges of the ATFT with non-preserving boundary conditions (parametrized by $w_i$), where $c_i, \bar{c}_i$ depend on the coupling constant of the theory [MN, DM]. These charges close on the generalized $q$-Onsager algebras $\mathcal{O}_q(\tilde{g})$ upon certain restrictions on $w_i$; see below and [BB1]. These algebras are generated by $\{A_i\}$ with $i = 0, 1, \ldots, \text{rank}(g)$ (see [BB1] for commutation relations) and have coideal properties given by the coaction

$$\delta(A_i) = (c_i e_i q_i^{b/2} + \bar{c}_i f_i q_i^{b/2}) \otimes 1 + q_i^b \otimes A_i$$ \hspace{1cm} (2.22)

and the counit

$$\epsilon(A_i) = w_i.$$ \hspace{1cm} (2.23)
These coideal properties are necessary conditions for a \( q \)-Serre–Chevalley-type formulation of the \( R \)-\( \hat{A} \) \( B(\hat{g}) \) or \( B^*(\hat{g}) \) and appear in the definition of the intertwiner equations \((2.13) \) and \((2.18) \). Moreover, the coideal properties \( \delta, \epsilon \) fix uniquely the homomorphism \( \psi: O_\hat{q}(\hat{g}) \to U_\hat{q}(\hat{g}) \) from the relation \((1 \otimes \epsilon) \circ \delta = \psi \), which implies that \( \psi(\hat{A}_i) = a_i \).

In the case of \( \hat{g} = a_{12}^{(1)} \) or \( \hat{g} = d_{12}^{(1)} \) these generators \((2.21) \) give, from the intertwiner equation, the scalar solutions for the RA \((2.13) \) and in the case \( \hat{g} = a_{12}^{(1)}, n > 1 \), the scalar solutions for the \( R \)-\( \hat{A} \) \( (2.18) \) \([DM, DG]\). For the two first cases, the \( R \)-matrices satisfy the crossing symmetry \((2.19) \) and the solution of \((2.13) \) can be mapped on to the solution of \((2.18) \) using \((2.20) \). For the last case, the intertwiner equation \((2.13) \) with generators \((2.21) \) does not have non-trivial solutions. In view of these results we can assume that the generalized \( q \)-Onsager algebras \( O_q(\hat{g}) \) correspond to a \( q \)-Serre–Chevalley-type formulation of the \( R \)-\( \hat{A} \). For the case of \( a_1^{(1)} \), the dynamical \( K \)-matrices have been obtained in terms of the \( q \)-Onsager algebra \( O_q(a_1^{(1)}) \) \([BS, BB2]\) and in terms of the Zhedanov or Askey–Wilson algebra \( AW \) \([BK]\).

In this paper, we consider dynamical \( K \)-matrices with entries in a finite algebra that gives a realization of \( O_q(\hat{g}) \). The procedure to obtain these dynamical \( K \)-matrices, solutions of the reflection equation, is as follows.

1. Identify from the definition of the \( q \)-Onsager algebra \( O_q(\hat{g}) \) its realization by a finite algebra\(^6\). In the cases considered in this paper the corresponding finite algebra is given by special cases of the Zhedanov or Askey–Wilson algebra \([Zhed]\).
2. Construct an irreducible basis of monomials of this algebra and define a ‘minimal’ solution of the \( K \)-matrix in terms of elements of the irreducible basis. The irreducibility of the basis ensures that the solution is unique up to a scalar factor.
3. Solve the intertwiner equation \((2.13) \) or \((2.18) \) using the commutation relations of the finite algebra.

3. Dynamical \( K \)-matrices for \( O_q(a_2^{(2)}) \) and \( O_q(a_2^{(1)}) \)

In this section we give dynamical (non-\( c \)-number or operator-valued) \( K \)-matrices solutions of the intertwiner equations \((2.13) \) and \((2.18) \). We consider the cases corresponding to the generalized \( q \)-Onsager algebras \( O_q(a_2^{(2)}) \) and \( O_q(a_2^{(1)}) \). As a first step, we introduce the Zhedanov or Askey–Wilson algebra and give a general ansatz for the \( K \)-matrix in terms of generators of this algebra. As a second step, we consider case-by-case the generalized \( q \)-Onsager algebras \( O_q(a_2^{(2)}) \) and \( O_q(a_2^{(1)}) \). We recall their definitions (infinite algebra) \([BB1]\), we give an evaluation homomorphism to the Zhedanov algebra (finite algebra) and, finally, we present the solution of the intertwiner equation—the dynamical \( K \)-matrix.

The Zhedanov algebra or Askey–Wilson algebra \( AW(a, b, c, d, \bar{c}, \bar{d}, q) \) is given by

**Definition 3.1** \([Zhed]\). The Zhedanov algebra or Askey–Wilson algebra \( AW(a, b, c, d, \bar{c}, \bar{d}, q) \) is an associative algebra generated by the elements \( \{W_0, W_1, W_2\} \) and scalars \( \{a, b, c, d, \bar{c}, \bar{d}, q\} \in \mathbb{C} \) with \( q \) (not root of unity), subject to the relations\(^7\):

\[
[W_0, W_1]_q = aW_2, \quad (3.1)
\]

\[
[W_1, W_2]_q = bW_1 + cW_0 + d, \quad (3.2)
\]

\(^6\) The Zhedanov or Askey–Wilson algebra gives a realization by a finite algebra of \( O_q(a_1^{(1)}) \).

\(^7\) In most cases the finite algebra is given by a finite generalized \( q \)-Onsager algebra \( O_q(g) \) isomorphic to a coideal subalgebra of \( U_q(g) \) \([Letz]\), see also remark 1.

\(^8\) We defined \( [A, B]_q = AB - q^{-1}BA \).
The algebra is spanned by the monomials:

\[ W_0^{i_0} W_1^{i_1} W_2^{i_2} \]  

and has the Casimir:

\[ Q = cq^2 W_0^2 + \tilde{c} q^{-2} W_1^2 + a q^2 W_2^2 + b (W_0 W_1 + W_1 W_0) 
- q(q^2 - q^{-2}) W_0 W_1 W_2 + dq(q + q^{-1}) W_0 + \tilde{d} q^{-1} (q + q^{-1}) W_1 \]  

Remark 1. For the generalized \( q \)-Onsager algebra \( \mathcal{O}_q (\tilde{g}) \), a realization in terms of the coideal subalgebra of the quantum algebra \( \mathcal{U}_q (g) \), with \( g \) a simple Lie algebra, can be found. These coideal subalgebras have been classified by Letzter [Letz]. Among them one finds the Zhedanov algebra [Zhed] and the Klimyk–Gavrilik algebra [Klim, GI].

We look for a solution of the intertwiner equations ((2.13) and (2.18)) of the form

\[ K(u) = \sum_{i,j} E_{ij} \sum_{i_1, i_2, i_3} k_{ij}^{i_1i_2i_3} (a, b, c, d, \tilde{c}, \tilde{d}, q, Q) W_0^{i_1} W_1^{i_2} W_2^{i_3} \in \text{End}(\mathbb{C}^3) \otimes \mathcal{W}, \]  

with positive integers \( i_1 + i_2 + i_3 < p \), \( k_{ij}^{i_1i_2i_3} (a, b, c, d, \tilde{c}, \tilde{d}, q, Q) \) some unknown functions and \( \mathcal{W} \) the vector space spanned by the monomials (3.4). We choose \( p = 4 \), similarly to the order of the Casimir (3.5). The irreducibility of the monomials\(^9\) (3.4) ensures that the solution \( K(u) \) is unique in each case and satisfies the corresponding reflection equation.

3.1 The \( \mathcal{O}_q (a_2^{(2)}) \) case

The affine Lie algebra \( a_2^{(2)} \) is the simplest example of the twisted Kac–Moody algebras. We consider this example for its simplicity and its connection with the quantum integrable models as Bullough–Dodd, Mikhailov–Zhiber–Shabat or Izergin–Korepin models [Smir, IK]. In this case, there exists the crossing symmetry relation for the \( R \)-matrix of \( \mathcal{U}_q (a_2^{(2)}) \) of the form (A.3). Therefore, we can consider the intertwiner equation (2.13) to construct solutions of the reflection equation (2.9). The solution for twisted reflection equation follows from the map (2.20).

The generalized \( q \)-Onsager algebra \( \mathcal{O}_q (a_2^{(2)}) \) is defined by

**Definition 3.2 ([BB1]).** The generalized \( q \)-Onsager algebra \( \mathcal{O}_q (a_2^{(2)}) \) is an associative algebra with unit 1, generated by the elements \( \{ A_i \} \), \( i \in \{ 0, 1 \} \) and scalars \( \{ \rho, \tilde{\rho}, \tilde{\rho} \} \in \mathbb{C} \) subject to the relations

\[ [A_0, [A_0, A_1]_{q^4}]_{q^{-4}} - \rho A_1 = 0, \]

\[ [A_1, [A_1, [A_1, A_0]_{q^4}]_{q^{-2}}]_{q^{-2}} - \tilde{\rho} (A_1 A_1 A_0 - w A_1 A_0 A_1 + A_0 A_1 A_1) - \tilde{\rho} A_0] = 0, \]  

\[ w = \frac{(q - 1 + q^{-1}) (q^4 + 2q^2 + 4 + 2q^{-2} + q^{-4})}{q^4 + 3 + q^{-4}}. \]  

\(^9\) Proof of the irreducibility of the monomials can be found in [T].
It is a coideal subalgebra with counit $\epsilon : \mathcal{O}_q(\alpha^2) \to \mathbb{C}$ given by

$$\epsilon(A_0) = w_0, \quad \epsilon(A_1) = w_1 \quad \text{with} \quad \left( w_0^2 + \frac{\rho}{(q^2 - q^{-2})^2} \right) w_1 = 0.$$  \hspace{1cm} (3.10)

The coaction $\delta : \mathcal{O}_q(\alpha^2) \to \mathcal{U}_q(\alpha^2) \otimes \mathcal{O}_q(\alpha^2)$ is given by

$$\delta(A_i) = (c_i q_i^\frac{2}{q} + \tau_i q_i^\frac{2}{q}) \otimes I + q_i^h \otimes A_i,$$  \hspace{1cm} (3.11)

with

$$\rho = c_0 \bar{c}_0, \quad \bar{\rho} = (q + q^{-1})^2 (q^4 + 3 + q^{-4}) c_1 \bar{c}_1 \quad \text{and} \quad \bar{\rho} = \left( \frac{(q^2 + q^{-2})}{(q^4 + 3 + q^{-4})} \right)^2.$$  \hspace{1cm} (3.12)

and we have the evaluation homomorphism $\Pi$.

**Proposition 3.1.** There is an algebra homomorphism $\Pi : \mathcal{O}_q(\alpha^2) \to \mathcal{A}W(1, 0, -q^{-4} \bar{\mu}, C, -q^{-4} \mu, 0, q^3)$ such that

$$\Pi(A_i) = W_i \quad \text{for} \quad i \in \{0, 1\} \quad \text{and} \quad \Pi(\rho) = \mu, \quad \Pi(\bar{\rho}) = \left( \frac{q^4 + 3 + q^{-4}}{q^4 + q^{-2}} \right)^2.$$  \hspace{1cm} (3.13)

For convenience, let us use the following new parameters for $\mathcal{A}W$ and the matrix representation of $\mathcal{U}_q(\alpha^2)$:

$$\alpha = \sqrt{\mu}, \quad \beta = \frac{q^4}{\sqrt{\mu(q + q^{-1})}}, \quad s_0 = q^2 \sqrt{\frac{c_0}{c_1}}, \quad s_1 = \sqrt{\frac{q(q + q^{-1}) c_1}{c_1}} \quad \text{and} \quad c_i, \bar{c}_i \in \mathbb{R}^+$$  \hspace{1cm} (3.14)

From straightforward calculations of the intertwiner equation (2.13) we obtain the dynamical $K$-matrix:

$$K(u) = K^{(1)}(u) + K^{(2)}(u),$$  \hspace{1cm} (3.15)

with

$$K^{(1)}(u) = \begin{pmatrix} h_1(u, C, Q) & 0 & h(u, C) \\ 0 & h_2(u, C, Q) & 0 \\ h(u, C) & 0 & h_3(u, C, Q) \end{pmatrix},$$

$$K^{(2)}(u) = \begin{pmatrix} \alpha \beta \bar{u} W_1 + q^\frac{q}{\rho(q + q^{-1})} W_0 & q^\frac{q}{\rho(q + q^{-1})} [W_1, W_0]_{q^4} + q^2 \bar{u} W_1 & \beta [[W_0, W_1]_{q^4}, W_1]_{q^4} \\ q^{-\frac{q}{\rho(q + q^{-1})}} [W_1, W_0]_{q^4} + q^2 \bar{u} W_1 & -q^\frac{q}{\rho(q + q^{-1})} W_0 & q^\frac{q}{\rho(q + q^{-1})} [W_0, W_1]_{q^4} + q^{-\frac{q}{\rho(q + q^{-1})}} W_1 \\ \beta [W_1, [W_1, W_0]_{q^4}]_{q^4} & q^\frac{q}{\rho(q + q^{-1})} [W_1, W_0]_{q^4} + q^{-\frac{q}{\rho(q + q^{-1})}} W_1 & -\alpha \beta \frac{q}{\rho(q + q^{-1})} W_1 + q^\frac{q}{\rho(q + q^{-1})} \bar{u} W_1 \end{pmatrix}$$

and

$$h(u, C) = \frac{\alpha q^3 (q^3 u - q^{-3} u^{-1})}{\beta(q - q^{-1})(q^4 - q^{-4})} - \beta C$$

$$h_1(u, C, Q) = \frac{\beta Q (q^4 - q^{-4})}{\alpha(u - u^{-1})} + \frac{u \beta C (q - q^{-1})(q^2 + q^{-2})}{(u - u^{-1})}$$

$$+ \frac{q^2 \bar{u} a (q^3 u - q^{-3} u^{-1})}{\beta(u - u^{-1})(q + q^{-1})^2 (q^2 + q^{-2})}.$$
The definition and main properties of the \( R \)-K two families of the reflection equations are different. Moreover, the evaluation of \( \text{in a finite-dimensional algebra is known for the case} \ a \)

\[ h_2(u, C, Q) = \frac{\beta Q(q^4 - q^{-4})}{\alpha(u - u^{-1})} + \frac{\beta C(uq - u^{-1}q^{-1})}{(u - u^{-1})} \]

\[ + \frac{q^4\alpha(qu - q^{-1}u^{-1})(q^3u - q^{-3}u^{-1})}{(u - u^{-1})} \]

\[ h_3(u, C, Q) = \frac{\beta Q(q^4 - q^{-4})}{\alpha(u - u^{-1})} + \frac{\beta C(q - q^{-1})}{u(u - u^{-1})} \]

\[ + \frac{q^4\alpha(q^3u - q^{-3}u^{-1})}{u(u - u^{-1})} \]

\( g = (q^4 - q^{-4}). \)

The direct consequences of the intertwiner equation (2.13) are that the inverse of \( K(u) \) is proportional to \( K(u^{-1}) \), the K-matrix is unique (up to a scalar function) and is a solution of the reflection equation:

\[ K_{12}^* (uv) K_{12} (uv) K_{21} (v) = K_{21} (v) K_{12}^* (uv) K_{12} (uv) K_{21}^* (uv) . \]

with

\[ K_{12}^* (u) = K_{21} (u, q) . \]

The definition and main properties of the \( R \)-matrix are given in the appendix.

### 3.2 The \( \mathcal{O}_q(\alpha^{(1)}_2) \) case

The affine Lie algebra \( \alpha^{(1)}_2 \) is the simplest example of the Kac–Moody algebra of rank 3 and, as in the previous case, is connected with quantum ATFT [Gan]. The \( R \)-matrix of its quantum deformation—\( \mathcal{U}_q(\alpha^{(1)}_2) \)—does not have the crossing symmetry (2.19), and it means that the two families of the reflection equations are different. Moreover, the evaluation of \( K^*(a) \) \(^{10} \)

in a finite-dimensional algebra is known for the case \( \alpha^{(1)}_2 \) [MRS] and supports the fact that \( \mathcal{O}_q(\alpha^{(1)}_2) \) is a q-Serre–Chevalley formulation of TRA.

The generalized q-Onsager algebra \( \mathcal{O}_q(\alpha^{(1)}_2) \) is given by the following definition.

**Definition 3.3 ([BB1]).** The generalized q-Onsager algebra \( \mathcal{O}_q(\alpha^{(1)}_2) \) is an associative algebra with unit 1, generated by the elements \( \{ A_i \} \) and scalars \( \rho_i \in \mathbb{C} \) with \( i = 0, 1, 2 \) subject to the relations

\[ [A_i, A_{i\pm 1}]_{q^2} = \rho_i A_{i\pm 1} , \]

for \( i = 0, 1, 2 \) \( (\equiv 0, -1 \equiv 2) \).

It is a coideal subalgebra with counit, \( \epsilon : \mathcal{O}_q(\alpha^{(1)}_2) \rightarrow \mathbb{C} \) given by

\[ \epsilon(A_i) = w_i , \]

with \( \left( w_i^2 + \frac{\rho_i}{q + q^{-1} - 2} \right) w_j = 0 \) for any \( i, j \in \{ 0, 1, 2 \} \)

and the coaction \( \delta : \mathcal{O}_q(\alpha^{(1)}_2) \rightarrow \mathcal{U}_q(\alpha^{(1)}_2) \otimes \mathcal{O}_q(\alpha^{(1)}_2) \) is given by

\[ \delta(A_i) = (c_i e_q + \tau_i f_q) \otimes 1 + q^{2i} \otimes A_i , \]

with

\[ \{ c_i, \tau_i \} \in \mathbb{C} \]

and we have the evaluation homomorphism \( \Pi . \)

\(^{10}\) The symbol \( * \) here indicates that the \( K \)-matrix is a solution of the twisted reflection equation (2.14).
Proposition 3.2. There is an algebra homomorphism $\Pi : \mathcal{O}_q(\mathfrak{su}_2) \to \mathbb{A}V(\mu_0, 0, -\frac{\mu_1}{\mu_0}, 0, -\frac{\mu_2}{\mu_0}, 0, q)$ such that

$$
\begin{align*}
\Pi(A_i) &= W_i & & j \in \{0, 1, 2\}, \\
\Pi(\rho_0) &= \mu_1, \\
\Pi(\rho_2) &= \mu_2 & \text{and} & \Pi(\rho_0) = q^{-1}\frac{\mu_1\mu_2}{(\mu_0\mu_2)^2}.
\end{align*}
$$

(3.22)

In this case, we consider the intertwiner equation (2.18) for the $K$-matrices $K^*(u)$ transforming the representation $\pi$ into $\tilde{\pi}$ and also the intertwiner

$$
\tilde{K}^*(u)(\pi_{u} \otimes \Pi)\delta(y) = (\pi_{u^{-1}} \otimes \Pi)\delta(y)\tilde{K}^*(u),
$$

$y \in B^*(\hat{g})$ (3.23)

transforming the representation $\pi$ into $\pi$. The two intertwiners are related by the fact that $\tilde{K}^*(u^{-1})$ is proportional to the inverse of $K^*(u)$. For convenience, let us use the following new parameters for $\mathbb{A}V$ and the matrix representation of $U_q(\mathfrak{su}_2)$:

$$
s_1 = \sqrt{q^2/c_i}, \quad a_0 = \sqrt{\mu_0}, \quad a_1 = i\sqrt{\frac{\mu_1}{q\mu_0}},
$$

(3.24)

$$
\alpha_2 = i\sqrt{\frac{\mu_2}{q\mu_0}}, \quad \alpha = \alpha_1\alpha_2\alpha_3 \quad \text{and} \quad c_i, c_i \in \mathbb{R}^+.
$$

The solution of the first intertwiner equation (2.18), is given by

$$
K^*(u) = \begin{pmatrix}
\alpha \kappa(u) & -q^{-\frac{1}{2}}u^{-1}\alpha_1W_1 & iu^{-1}\alpha_0W_0 \\
\alpha_1W_1 & \alpha \kappa(u) & -q^{-\frac{1}{2}}u^{-1}\alpha_2W_2 \\
-iu\alpha_0W_0 & -i\alpha_2W_2 & \alpha \kappa(u)
\end{pmatrix},
$$

(3.25)

where $\kappa(u) = \frac{2\sqrt{u^{-1}+1}}{q^{-1}-1}$. We remark that there is another solution given by the transformation $i \rightarrow -i$ leaving the homomorphism of proposition 3.2 invariant.

The solution for the second intertwiner equation (3.23), $\tilde{K}^*(u)$, is given by

$$
\tilde{K}^*(u) = \tilde{K}^*_{(1)}(u) + \frac{i}{q-q^{-1}}\tilde{K}^*_{(2)}(u),
$$

(3.26)

with

$$
\tilde{K}^*_{(1)}(u) = \begin{pmatrix}
\alpha \tilde{\kappa}(u)^2 - \frac{w^2}{\alpha} (\alpha_2W_2)^2 & 0 & 0 \\
0 & \alpha \tilde{\kappa}(u)^2 - \frac{w^2}{\alpha} (\alpha_0W_0)^2 & 0 \\
0 & 0 & \alpha \tilde{\kappa}(u)^2 - \frac{w^2}{\alpha} (\alpha_1W_1)^2
\end{pmatrix},
$$

$$
\tilde{K}^*_{(2)}(u) = \begin{pmatrix}
0 & q^{\frac{1}{2}u^2}\alpha_1W_1 - \frac{1}{\alpha_1}[W_0, W_1_{q^{-1}}] & q^{-1/2}\alpha_0W_0 - \frac{wu}{\alpha_0}[W_1, W_2]_{q^{-1}} \\
q^{-\frac{1}{2}}u^{-2}\alpha_1W_1 - \frac{1}{\alpha_1}[W_0, W_2]_{q^{-1}} & 0 & q^{\frac{1}{2}u^2}\alpha_2W_2 - \frac{wu}{\alpha_2}[W_1, W_0]_{q^{-1}} \\
q^2\alpha_0W_0 - \frac{wu}{\alpha_0}[W_2, W_1]_{q^{-1}} & q^{\frac{1}{2}u^{-2}}\alpha_2W_2 - \frac{wu}{\alpha_2}[W_0, W_1]_{q^{-1}} & 0
\end{pmatrix}
$$

and $\tilde{\kappa}(u) = \frac{2\sqrt{u^{-1}+1}}{q^{-1}-1}$. Similarly, there is another solution which is also given by the transformation $i \rightarrow -i$.

Finally, these solutions, $K^*(u)$ and $\tilde{K}^*(u)$, are unique (up to a scalar function) and satisfy the twisted reflection equations (2.14) \footnote{The second twisted reflection equation corresponds to the inverse of the first one (up to a scalar function) and $K^*(u)\tilde{K}^*(1/u) = \tilde{K}^*(1/u)K^*(u) \propto 1.$}:

$$
R_{12}(u/v)K^*_1(u)R_{12}^\dagger(u/v) = K^*_2(v)R_{21}^\dagger(v)K^*_1(u)R_{12}(u/v)
$$

(3.27)
The cases obtained from the intertwiner equations \((2)\) are the same as the ones found that our solution \((3)\) coincides with the result obtained in [MRS] for the orthogonal case.

### 4. Scalar limit of the dynamical \(K\)-matrices

Starting from our dynamical solutions \((3.15),(3.25)\) and \((3.26))\), it is straightforward to derive the already known scalar solutions of the reflection equation \((2.9)\) and twisted reflection equations \((2.14)\). These solutions have been derived directly from the reflection equation [PHLSY, K, BFKZ] or from matrix intertwiner operator [N]. We obtain the scalar solutions \((c\text{-number }K\text{-matrices})\) from the dynamical solutions \((\text{non-}\(c\text{-number }K\text{-matrices})\) taking the trivial, or one-dimensional representation of \(A_{\text{W}}\).

**Remark 2.** Using the invariance of the twisted reflection equation by the map \(K^* (u) \rightarrow MK^* (u) M\), with \(M\) being an arbitrary diagonal matrix, and rescaling the generators \(\{W_i\}\) we found that our solution \((3.25)\) coincides with the result obtained in [MRS] for the orthogonal case.

**Remark 3.** The scalar solutions obtained from the dynamical ones are the same as the ones obtained from the intertwiner equations \((2.13)\) and \((2.18)\) replacing \(\Pi\) by \(\epsilon\).

For the case \(a^{(2)}\), the scalar representation of \(W_0\) and \(W_1\) must be a solution of equation \((3.10)\). It follows two types of scalar \(K\)-matrices.

- The case \(w_1 = 0, w_0\) arbitrary and

\[
\frac{q^4w_0}{\beta^2(q + q^{-1})}, \quad Q = -\frac{q^4w_0^2}{\beta^2(q + q^{-1})},
\]

leads to the Type II scalar solution in \([K, N, L-S]\)

\[
K^\text{scal}_{w_0,0} (u) \propto \frac{\alpha}{w_0} + \frac{\alpha}{(q + q^{-1})(q^2 + q^{-2})} \begin{pmatrix}
 u & 0 & \frac{u - u^{-1}}{q - q^{-1}} \\
 0 & \frac{qu - q^{-1}u^{-1}}{q - q^{-1}} & 0 \\
 \frac{u - u^{-1}}{q - q^{-1}} & 0 & u^{-1}
\end{pmatrix}.
\]

The additional boundary parameters in \([K, N]\) are related to the invariance of the reflection equation by conjugation with a diagonal matrix of the form \(M = \text{Diag} (a_1, \{a_1 a_2\}^{1/2}, a_2)\). The trivial solution proportional to unit can be recovered from this solution taking \(\alpha = 0\).

- The cases \(w_0 = \pm i\frac{q^2 - q^{-2}}{q - q^{-1}}, w_1\) arbitrary,

\[
w_2 = \pm i w_1 q^2, \quad C = \pm i a \left( w_1^2 q^4 (q^2 - q^{-2}) + q^4 \frac{1}{\beta^2 (q + q^{-1})(q^2 - q^{-2})} \right),
\]

\[
Q = a^2 \left( (q^4 + q^{-4}) w_1^2 + \frac{q^4}{\beta^2 (q + q^{-1})(q^2 - q^{-2})} \right)
\]

lead to the scalar solutions with all non-zero entries or Type I in \([N]\)

\[
K^\text{scal}_{\pm i\frac{q^2 - q^{-2}}{q - q^{-1}}, w_1} (u) \propto K^{(1)} (u) + \frac{q^7}{(\beta w_1)^2 [4]_q^2 (q - q^{-1})(q^2 u \pm i)} K^{(2)} (u)
\]
Let $12$ known scalar solutions. The dynamical reflection equations. Taking the trivial representation of these subalgebras we recover the $q_K$ equations. These $\{q \}^5$-matrices have entries in the finite subalgebra of the related generalized $q$-Onsager algebra (some special cases of Zhedanov algebra $[\text{Zhed}]$). But the invariance of the homomorphism of proposition 3.2 by the transformation $i \rightarrow -i$ leads to the same number of scalar solutions as for $O_q(a_2^{(1)})$ trivial representations.

- Let $e_i = \pm 1$ for $i = 1, 2$ or $e_1 = e_2 = 0$, then the scalar representation of $O_q(a_2)$ is given by

$$w_0 = e_1 e_2 q^{\frac{\mu_1 \mu_2}{q-1}}, \quad w_0 = e_1 q^{\frac{\mu_0 \mu_2}{q-1}} \quad \text{and} \quad w_0 = e_2 q^{\frac{\mu_0 \mu_1}{q-1}}. \quad (4.1)$$

Considering both solutions, i.e. $i \rightarrow e_0 | i \rightarrow e_0$ with $e_0 = \pm 1$, we recover the scalar solutions $[\text{Gan, DM}]$

$$K_{e_0, e_1, e_2}^{\text{scal}}(u) \propto \begin{pmatrix} \frac{u^{2} w - i q u}{q^{3} - q^{-3}} & -u^{-1} e_1 & i q^{2} w^{-1} e_1 e_2 \\ i q^{2} u e_0 e_1 & \frac{u^{2} w - i q u}{q^{3} - q^{-3}} & -u^{-1} e_2 \\ -u e_1 e_2 & i q^{2} u e_0 e_2 & \frac{u^{2} w - i q u}{q^{3} - q^{-3}} \end{pmatrix}. \quad (4.2)$$

5. Comments

The dynamical $K$-matrices, solutions of the RA and the TRA, have been constructed for the cases $a_1^{(2)}$ and $a_1^{(1)}$ respectively using the generalized $q$-Onsager algebras and the intertwiner equations. These $K$-matrices have entries in the finite subalgebra of the related generalized $q$-Onsager algebra (some special cases of Zhedanov algebra $[\text{Zhed}]$) and are solutions of the reflection equations. Taking the trivial representation of these subalgebras we recover the known scalar solutions.

This procedure can clearly be applied to other generalized $q$-Onsager algebras related to $\hat{g}$ and corresponding dynamical $K$-matrices can be obtained. The main problem is to identify a realization of the generalized $q$-Onsager $O_q(\hat{g})$ in terms of a finite algebra and construct an irreducible basis for the vector space spanned by monomials of generators of the algebra. As we mentioned, the case of $O_q(a_2^{(1)})$ is already known starting directly from TRA $[\text{MRS}]$ and the corresponding finite algebra is the Klimyk–Gavriliuk algebra $[\text{Klim, GI}]$. It corresponds to the orthogonal case of the TRA (noted $Y_q^{\text{w}}(\sigma e)$ in $[\text{MRS}]$). This fact suggests that $O_q(a_2^{(1)})$ corresponds to a $q$-Serre–Chevalley formulation of $Y_q^{\text{w}}(\sigma e)$ (quotient of $Y_q^{\text{w}}(\sigma e)$ by its centre).
For the case $Y^\mu\nu(s_{\mu\nu})$, also considered in [MRS], the $q$-Serre–Chevalley formulation still remains, to our knowledge, an open problem and will be considered elsewhere.

In the case of a finite Lie algebra $g$, the associated RAs have been studied in the context of the quantization of the symmetric spaces and related $q$-orthogonal polynomial [NS, Letz]. Two remarkable facts, to us, appear in these works: there is a one-to-one correspondence between scalar solutions of the reflection equations and the quantum symmetric spaces [NS]; the classification of the coideal subalgebras of $U_q(g)$ is obtained by the classification of the irreducibles pair $(g, \theta)$, with $\theta$ being an involution of $g$, and the $q$-Serre–Chevalley formulation follows from the involution and the coideal properties [Letz]. The extension of this classification for the affine case is still an open problem.

As for physical applications in quantum ATFT with boundary degree of freedom, the dynamical solutions obtained in this paper give the algebraic part of the boundary scattering amplitudes for fundamental particles in the $a_2^{(2)}$ and $a_2^{(1)}$ theories with imaginary coupling. The next step is to consider the fusion procedure and obtain the boundary scattering amplitudes for fused particles. As an application, they lead to scattering amplitudes of the fundamental particles of the real coupling theory. In the case of the $a_2^{(1)}$ boundary ATFT (called the sinh–Gordon model) and dynamical boundary conditions [BK], a remarkable weak–strong coupling duality property of boundary amplitudes is observed. By analogy, weak–strong coupling duality properties could be investigated for $a_2^{(2)}$ and $a_2^{(1)}$ cases using the results presented here. Moreover, considering the infinite-dimensional representation of the $A_1$ algebra, it could be interesting to compare our dynamical solutions with the ones obtained from scalar solutions dressed by equations (2.12) and (2.17) with bulk amplitudes of impurities [CorZam]. These questions will be considered elsewhere.

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Appendix. $U_q(a_2^{(2)})$ and $U_q(a_2^{(1)})$ quantum algebras

In this appendix, we give the definitions of the $U_q(a_2^{(2)})$ and $U_q(a_2^{(1)})$ algebras, their fundamental representations and the associated $R$-matrices used in this paper. The generalized Cartan matrix $A = (a_{ij})_{i,j=0,\ldots,n}$ and associated coprime positive integers $d_0, \ldots, d_n$ are given, respectively, for $a_2^{(2)}$ and $a_2^{(1)}$ by

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \quad (d_i) = \{4, 1\},$$

and

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad (d_i) = \{1, 1, 1\}.$$
\[ [e_i, f_j] = \delta_{ij} \frac{q_i^h - q_i^{-h}}{q_i - q_i^{-1}}, \]

\[ \sum_{r=0}^{1-a_{ij}} (-1)^r \frac{1 - a_{ij}}{r} e_i e_j f_j f_i' = 0. \]

with \( q_i = q^d \). The Hopf algebra structure is ensured by the existence of a comultiplication \( \Delta : U_q(\mathfrak{g}) \mapsto U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \), an antipode \( S : U_q(\mathfrak{g}) \mapsto U_q(\mathfrak{g}) \) and a counit \( \varepsilon : U_q(\mathfrak{g}) \mapsto \mathbb{C} \):

\[ \Delta(e_i) = e_i \otimes q_i^{-h/2} + q_i^{h/2} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q_i^{-h/2} + q_i^{h/2} \otimes f_i, \]

\[ \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \]

\[ S(e_i) = -q_i^{-1} e_i, \quad S(f_i) = -q_i f_i, \quad S(h_i) = -h_i, \quad S(1) = 1, \]

\[ \varepsilon(e_i) = \varepsilon(f_i) = \varepsilon(h_i) = 0, \quad \varepsilon(1) = 1. \]

For both cases, we use three-dimensional representations constructed with the unit matrices \( E_{ij} \), with 1 being at the intersection of the line \( i \) and the column \( j \) and zero elsewhere.

- For the \( U_q(a_2^{(1)}) \) case, the fundamental representation \( \pi_u : U_q(a_2^{(1)}) \mapsto \mathbb{C}^3[[u, u^{-1}]] \) is given by [FRS]

\[
\pi_u(e_1) = s_1(E_{12} + E_{23}), \quad \pi_u(e_0) = u x_0 E_{31}, \\
\pi_u(f_1) = (q + q^{-1}) s_1^{-1} (E_{21} + E_{32}), \quad \pi_u(f_0) = u^{-1} x_0^{-1} E_{13}, \\
\pi_u(q_1^n) = q E_{11} + E_{22} + q^{-1} E_{33}, \quad \pi_u(q_0^n) = q^{-2} E_{11} + E_{22} + q^2 E_{33}.
\]

The parameters \( s_i \) correspond to an automorphism\(^{11} \) of \( U_q(a_2^{(1)}) \) irrelevant for the \( R \)-matrix but relevant for the \( K \)-matrix. Following the results of Jimbo [1], we can derive, up to a scalar function, the \( R \)-matrix associated with this representation from the intertwining property. The solution is given by

\[ R_{12}(u) = (u - 1) q^3 R_{12}^{(q)} + (1 - u^{-1}) q^{-3} R_{12}^{(q^{-1})} + q^{-5} (q^4 - 1) (q^6 + 1) P, \quad (A.1) \]

where \( R_{12}^{(q)} \) is the spin one or three-dimensional \( R \)-matrix associated with the \( U_q(a_1) \) algebra given by

\[
R_{12}^{(q)} = \left( \begin{array}{cccccccc}
q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & q^2 - q^{-2} & 0 & 0 & 0 & 0 & 0 \\
0 & -q^{-2} & 0 & q(q^2 - q^{-2}) & 0 & q(q^2 - q^{-2})(q - q^{-1}) & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & q(q^2 - q^{-2}) & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & q^{-2} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 \\
\end{array} \right).
\]

It follows that this \( R \)-matrix is a solution of the Yang–Baxter equation (2.4). It also satisfies the unitarity condition

\[ R_{12}(u) R_{21}(u^{-1}) = q^2 (1 - q^4 u)(1 + q^{-6} u)(1 - q^4 u^{-1})(1 + q^{-6} u^{-1}) I \quad (A.2) \]

\(^{11} \) Given by \( e_i \rightarrow s_i e_i, f_i \rightarrow \frac{1}{u} f_i \) and \( h_i \rightarrow h_i \).
and the crossing symmetry
\[ R_{12}(u) = M_1 R_{21}^u (-q^6 u^{-1}) M_1 \quad \text{with} \quad M = -qE_{31} + E_{22} - q^{-1}E_{13}. \] (A.3)

- For the \( U_q(a_2^{(1)}) \) case, there are two fundamental representations \( \pi_u, \bar{\pi}_v : U_q(a_2^{(1)}) \mapsto \mathbb{C}^3[[u, u^{-1}]] \) given by
  \[ \pi_u(e_0) = u^3 s_0 E_{31}, \quad \pi_u(f_0) = u^{-2} s_0^{-1} E_{13}, \quad \pi_u(q^h) = q^{-1} E_{11} + E_{22} + qE_{33}, \] (A.4)
  \[ \pi_u(e_1) = s_1 E_{12}, \quad \pi_u(f_1) = s_1^{-1} E_{21}, \quad \pi_u(q^h) = qE_{11} + q^{-1} E_{22} + E_{33}, \] (A.5)
  \[ \pi_u(e_2) = s_2 E_{23}, \quad \pi_u(f_2) = s_2^{-1} E_{32}, \quad \pi_u(q^h) = E_{11} + qE_{22} + q^{-1} E_{33}. \] (A.6)
and
  \[ \bar{\pi}_v(g) = (\pi_u(g))^t |_{q^2 = 1} \] for any \( g \in U_q(a_2^{(1)}) \),
where \( t \) means the transposition. The \( R \)-matrix defined from the intertwiner equation with two \( \pi \) representations is given by
\[ R_{12}(u) = \sum_i E_{ii} \otimes E_{ii} + \frac{u - u^{-1}}{qu - q^{-1} u^{-1}} \sum_{i \neq j} E_{ii} \otimes E_{jj} + \frac{q - q^{-1}}{qu - q^{-1} u^{-1}} \sum_{i \neq j} u^{\text{sign}(i-j)} E_{ij} \otimes E_{ji}. \] (A.8)

This \( R \)-matrix is a solution of the Yang–Baxter equation (2.4) and also satisfies the unitarity condition
\[ R_{12}(u) R_{21}(1/u) = I. \] (A.9)

The \( R \)-matrix defined from the intertwiner equation with two \( \bar{\pi} \) representations is given by
\[ \bar{R}_{12}^u(u) = R_{21}(u) \] and is also a solution of (2.4). The remaining cases \( \pi_a \otimes \bar{\pi}_v \) and \( \bar{\pi}_a \otimes \pi_v \) are given, respectively, by
\[ V_1^{-1} R_{21}^q(iq^{-3/2} q^{-1}) V_1 \] and
\[ V_1 R_{21}^q(iq^{-3/2} q^{-1}) V_1^{-1} \] with \( V = -q^{-1} E_{11} + E_{22} - qE_{33} \). (A.10)

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