Trapping Dynamics with Gated Traps: Stochastic Resonance-Like Phenomenon

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We present a simple one-dimensional trapping model prompted by the problem of ion current across biological membranes. The trap is modeled mimicking the ionic channel membrane behaviour. Such voltage-sensitive channels are open or closed depending on the value taken by a potential. Here we have assumed that the external potential has two contributions: a deterministic periodic and a stochastic one. Our model shows a resonant-like maximum when we plot the amplitude of the oscillations in the absorption current vs. noise intensity. The model was solved both numerically and using an analytic approximation and was found to be in good accord with numerical simulations.

During the last decade we have witnessed a growing interest in the phenomenon of stochastic resonance (SR). This phenomenon is related to the enhancement of the response of a nonlinear system due to the interplay between nonlinear oscillations and noise. SR has been studied in different physical, chemical and biological contexts [1]. In particular, it has been found to play a relevant role in several problems in biology: mammalian sensory systems, increment of the tactile capacity, visual perception, low frequency effects and low amplitude electromagnetic fields, etc [2].

Among the many experiments showing SR there is one related to the measurement of the current through voltage-sensitive ion channels in a cell membrane [3]. These channels switch randomly between open and closed states, controlling the ion current. This and other related phenomena have motivated several theoretical studies of the problem of ionic transport through biomembranes. Those studies have used different approaches, as well as different forms of characterizing stochastic resonance in such systems [4].

In this work we want to analyze the effect of the simultaneous action of a deterministic and a stochastic external field on the trapping rate of a gated imperfect trap. Here we do not pretend to make a precise modeling of the behaviour of an ionic channel. We propose a simple model of dynamic trapping. This model is motivated by the above indicated experiment on cell membranes [3]. Our main result is that even a simple model of a gated trapping process can show a SR-like behaviour.

The study of gated trapping processes, i.e. a trapping process where the traps have some kind of internal dynamic has attracted considerable interest [5]. Many authors discussed the way to link the gated trapping processes with the measured behavior of the so called ionic pumps [6]. For example, among other factors, the ion transport depends on membrane electric potential (which plays the role of the barrier height) and can be stimulated by both dc and ac external fields.

We based our study on the so called stochastic model for reactions [7–9], that has been generalized in order to include the internal dynamics of traps. The dynamical process consists of the opening or closing of the traps according to an external field. Such a field has two contributions, one periodic with a small amplitude, and the other stochastic whose intensity will be the tuning parameter.

Our starting model equation is

$$\partial_t \rho(x, t) = D \partial_x^2 \rho(x, t) - \gamma(t) \delta(x) \rho(x, t) + n_u,$$

where $\gamma$ is a stochastic process that represents the absorption probability of an individual trap in our system, $\rho$ is the particle density (particles that have not been yet trapped); for a given realization of $\gamma$, $x$ is the coordinate over the one-dimensional system and $n_u$ is a source term that represents a constant flux of ions. The injection of ions can be at a trap position or at any other position. In this last case the ion can diffuse to the trap position. This diffusion coefficient would represent an effective diffusion through the volume rather a diffusion over the membrane surface.

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Each $\gamma$ is modelled as follows:

$$\gamma(t) = \gamma^* \theta[B \sin(\omega t) + \xi - \xi_c]$$

where $\theta(x)$, the Heaviside function, determines when the trap is open or closed. The trap works in the following manner: if the signal composed of the harmonic part plus $\xi$ (the noise contribution) reaches a threshold $\xi_c$ the trap opens, otherwise it is closed. We are interested in the case where $\xi_c > B$, that is, without noise the trap is always closed. When the trap is open the particles are trapped with a given frequency (probability per unit time) $\gamma^*$.

In other words the open trap is represented by an “imperfect trap”. Finally, in order to complete the model, we must give the statistical properties of the noise $\xi$. We assume that $\xi$ is an uncorrelated Gaussian noise of width $\xi_0$, i.e.

$$\langle \xi(t)\xi(u) \rangle = \delta_{t,u} \xi_0^2$$

Note that this is not a standard white noise due to the Kronecker symbol (instead of Dirac’s delta $\delta(t-u)$).

For future reference we compute $\langle \gamma(t) \rangle$ (brackets mean averages over all realizations of the noise). The result is

$$\langle \gamma(t) \rangle = \frac{\gamma^*}{2} \text{erfc} \left[ \frac{1}{\sqrt{2\xi_0}} (\xi_c - B \sin(\omega t)) \right].$$

Setting $n(x,t) = \langle \rho(x,t) \rangle$ and averaging Eq. \ref{eq:diffusion} we can write

$$n(x,t) = n_u t - \int_0^t du J(u) G(x,t-u),$$

where $G(x,t) = G_0(x,t)$ is the free diffusion propagator, and $J(t) = \langle \gamma_j(t) \rho(jl,t) \rangle$ is the current trough the trap. This current satisfies the relation

$$J(t) = \langle \gamma(t) \rangle \left[ n_u t - \int_0^t du J(u) G(0,t-u) \right]. (6)$$

We solve this equation numerically, using a standard “trapezoidal” integral approach.

The present problem can be extended to a finite lattice with periodic boundary conditions. In this case Eq. \ref{eq:diffusion} is replaced by

$$\partial_t \rho(x,t) = D \partial_x^2 \rho(x,t) - \sum_{j=-\infty}^{\infty} \gamma(t) \delta(x-jl) + n_u,$$ (7)

where, due to the boundary conditions, the sum runs over all the images of the trap. The current satisfies the same Eq. \ref{eq:general_current} but now $G(x,t) = \sum_j G_0(x-jl,t)$. This equation can be solved numerically.

In addition to the numerical solution, as in typical SR studies, we can make some analytic calculations using a perturbation procedure. We assume $B/\xi_0$ to be small and expand $\langle \gamma_j(t) \rangle$ (or $\langle \gamma_j(t) \rangle$) in the following form

$$\langle \gamma(t) \rangle = \langle \gamma_j(t) \rangle = \gamma_0 + \gamma_1 \sin(\omega t) + \mathcal{O}((B/\xi_0)^2). (8)$$

where

$$\gamma_0 = \frac{\gamma^*}{2} \text{erfc} \left( \frac{\xi_c}{\sqrt{2\xi_0}} \right),$$

$$\gamma_1 = \frac{\gamma^* B}{\sqrt{2\pi\xi_0}} \exp \left( -\frac{\xi_c^2}{2\xi_0^2} \right). (10)$$

Now we can transform Eq. \ref{eq:diffusion} to the Laplace domain and solve it iteratively. The expressions obtained are rather complicated and for simplicity we present only the asymptotic behavior. For an infinite lattice the asymptotic current is

$$J(t) \sim 4n_u \sqrt{\frac{Dt}{\pi}} + |C| \cos(\omega t + \phi),$$ (11)

where

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All simulations shown in the figures correspond to averages over 1000 realizations. The reaction times were generated according to the probability density function detailed description of the algorithm used can be found in [9]. The reaction times were generated according to the distribution over the lattice, and are allowed to perform a continuous time random walk (characterized by the jump removed from the system with a given probability distribution characterized by $\langle \gamma \rangle$ at each neighbor site $u$).

There are no particles on the lattice. The particles are injected randomly every $1/L$ units of time, with uniform distribution over the lattice, and are allowed to perform a continuous time random walk (characterized by the jump frequency at each neighbor site $q$). There is no restriction on the number of particles at each site. A particle can be removed from the system with a given probability distribution characterized by $\gamma$ when it reaches the trap site. A detailed description of the algorithm used can be found in [8]. The reaction times were generated according to the following probability density function

$$p(t) = \exp \left( -\int_0^t \langle \gamma(t') \rangle dt' \right).$$

All simulations shown in the figures correspond to averages over 1000 realizations.

The results of both, numerical evolution and simulations, are shown in Figs. 1 to 3. In Fig. 1 we show the temporal behaviour of the absorption current as a function of time. Figure 1a shows the evolution obtained from numerical integrations for a single trap (infinite system) and a periodic array of traps in a time interval such that the periodic array reaches its stationary behavior. It is worth remarking that a system with a single trap does not have a stationary behaviour. However, the figure can induce the false impression that the single trap also presents a stationary behaviour. This is only an artifact of the short time period shown. The insert of Fig. 1a shows, in a much longer time scale, such a non stationary behaviour for the single trap problem. The agreement between both theoretical results makes it difficult to distinguish among them, particularly at short times. Figure 1b depicts the results obtained from the simulations (points) and the corresponding numerical integration of the single trap (solid line) for a fixed value of $\xi_0$. One sees the good agreement between the two curves. When looking at such curves, and for different values of the noise intensity, it is apparent that for intensities smaller or larger than an optimal one, the amplitude is reduced, while in its neighborhood the amplitude becomes a maximum. This is better seen when plotting
\( \Delta J, \) the amplitude of the oscillating part of the absorption current as a function of the noise intensity, as can be seen in Fig. 2 for two different times. Such results make apparent the existence of a maximum as a function of \( \xi_0. \) Here we find good qualitative agreement between the results obtained by numerical integration of Eq. 6 and those from simulations. The insert of Fig. 2 shows the results for the initial slope indicated in the figure by \( h, \) corresponding to the kind of curves shown in the principal part of the figure, as a function of time. That is, the slope of \( \Delta J \) for low noise intensities and short times. Both results, obtained by numerical integration and simulations, show linear dependence with time as predicted in Eq. (15), however the actual values of the slopes are slightly different.

Finally in Fig. 3 we depict the result for \( \phi, \) the phase shift between the input periodic signal and the absorption current (as indicated in Eqs. (12) and (14)). We have compared the results from the numerical integration of Eq. 3 and those from simulations, finding good agreement.

The present results show that even a simple trapping process with an adequate internal dynamic can present a SR like phenomenon. This suggests the possibility that such a kind of phenomenon could be ubiquitous in more complex trapping like processes.

A more elaborate model aimed at describing the experiment in Ref. 3, including a more realistic representation of the ion channels is under study and will be the subject of further work.

Acknowledgments

The authors want to thank to V. Grünfeld for a critical reading of the manuscript; and C. Budde and A. Bruschi for fruitful discussions. Financial support from CONICET and ANPCyT (both Argentine agencies) is acknowledged.

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FIG. 1. Temporal evolution of the absorption current of particles for a gated trapping system. a) Time evolution obtained from numerical integration for a single trap (solid line) and for a finite system with periodic boundary conditions (points). We show the stationary behaviour for the periodic array of traps. The non stationary behaviour of the single trap problem, as discussed in the text, is apparent from the insert. b) Time evolution obtained from simulations (points), and from numerical integration of a single trap system (solid line). The parameters are: \( \omega = 0.1, n_u = 0.01, \xi_c = 1.2, B = 1.0, q_A = 0.1, \gamma^* = 10, \xi_0 = 0.3 \) and \( L = 100. \)

FIG. 2. Variation of the final value of \( \Delta J \) (amplitude of the oscillating part of the absorption current) as a function of \( \xi_0. \) The results obtained from the numerical integration of Eq. (6) are indicated by solid lines, while those from simulations by circles. We have chosen the following time parameters: a) \( t = 517 \) and b) \( t = 895. \) The insert shows the initial slope \( h \) as a function of time. The results obtained from the numerical integration are indicated by triangles and those from simulations by circles. The linear fitting is indicated by continuous and dashed lines respectively. The results were obtained for low noise intensity values (\( \xi_0 \leq 0.1. \))
FIG. 3. Phase shift $\phi$ as a function of $\xi_0$. The results obtained for the numerical integration are indicated by a solid line while those from simulations by circles.
