BUZANO, KREĬN AND CAUCHY-SCHWARZ INEQUALITIES

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ABSTRACT. The Cauchy-Schwarz, Buzano and Kreĭn inequalities are three inequalities about inner product. The main goal of this article is to present refinements of Buzano and Cauchy-Schawartz inequalities, and to present a new proof of a refined version of a Kreĭn-type inequality. Applications that include Buzano-type inequalities for contractions, operator norm and numerical radius inequalities of Hilbert space operators will be presented.

1. Introduction

Let \( H \) be a given complex Hilbert space, with inner product \( \langle \cdot, \cdot \rangle \). The celebrated Cauchy-Schwarz inequality states that

\[
| \langle x, y \rangle | \leq \| x \| \| y \|,
\]

for any vectors \( x, y \in H \). When \( x \) and \( y \) are non-zero vectors, (1.1) implies \( 0 \leq \frac{|\langle x, y \rangle|}{\|x\| \|y\|} \leq 1 \).

This motivates defining the angle between the vectors \( x, y \) by \( \psi_{x,y} \) where

\[
\cos \psi_{x,y} = \frac{| \langle x, y \rangle |}{\| x \| \| y \|}, \quad 0 \leq \psi_{x,y} \leq \frac{\pi}{2}.
\]

Another possible definition for the angle is \( \varphi_{x,y} \) defined as

\[
\cos \varphi_{x,y} = \frac{\text{Re} \langle x, y \rangle}{\| x \| \| y \|}, \quad 0 \leq \varphi_{x,y} \leq \pi.
\]

We refer the reader to [13] for these definitions and some details.

In [10], Kreĭn obtained the following inequality for angles between two vectors

\[
\varphi_{x,z} \leq \varphi_{x,y} + \varphi_{y,z},
\]

for any \( x, y, z \in \mathbb{C}^n \).

In [11], Lin showed that the following triangle inequality

\[
\psi_{x,y} \leq \psi_{x,z} + \psi_{y,z},
\]

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holds for any \( x, y, z \in H \setminus \{0\} \). Lin’s proof used the representation

\[
\psi_{x,y} = \inf_{\alpha, \beta \in \mathbb{C} \setminus \{0\}} \varphi_{\alpha x, \beta y} = \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \varphi_{\alpha x, y} = \inf_{\beta \in \mathbb{C} \setminus \{0\}} \varphi_{x, \beta y}
\]

and inequality (1.3)

Thus, both \( \varphi_{x,y} \) and \( \psi_{x,y} \) satisfy the triangle inequality. Our first target in this article is to present a new proof of (1.4). This new proof will follow from some inner product inequalities, that we present while refining the celebrated Buzano inequality [3], which states

\[
|x\langle y, z \rangle| | \langle z, z \rangle| \leq \| z \| \left( |\langle x, y \rangle| + \| x \| \| y \| \right)
\]

for any \( x, y, z \in H \). It is important to note that Buzano inequality gives a better bound than applying the Cauchy-Schwarz inequality twice on the left side. That is, (1.1) implies

\[
|x\langle y, z \rangle| | \langle z, z \rangle| \leq \| z \| | x \| \| y \|. \tag{1.6}
\]

At the same time, (1.1) implies

\[
\| z \|^2 \left( |\langle x, y \rangle| + \| x \| \| y \| \right) \leq \| z \|^2 | x \| \| y \|.
\]

Consequently, (1.5) provides a refinement of (1.6).

After discussing the Buzano and Krein inequalities, we present some new refinements of the Cauchy-Schwarz inequality (1.1) via contractions.

As further and interesting applications of the obtained inequalities, we present some inequalities for the numerical radius and operator norm of Hilbert space operators. In this context, let \( \mathbb{B}(H) \) denote the algebra of all bounded linear operators acting on a Hilbert space \( H \). An operator \( A \in \mathbb{B}(H) \) is said to be a contraction if \( A \) is a positive and \( A \leq I \), where \( I \) is the identity operator on \( H \). In \( \mathbb{B}(H) \), an operator \( A \) is said to be positive, and is denoted as \( A \geq 0 \), if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in H \). The partial ordering relation “\( \leq \)” is defined among self adjoint operators as

\[
A \leq B \Leftrightarrow B - A \geq 0.
\]

We recall here that the operator norm and the numerical radius of an operator \( T \in \mathbb{B}(H) \) are defined respectively by

\[
\omega(T) = \sup_{\| x \|=1} |\langle Tx, x \rangle| \quad \text{and} \quad \|T\| = \sup_{\| x \|=1} \|Tx\|.
\]

It is well known that \( \frac{1}{2} \| T \| \leq \omega(T) \leq \| T \| \), for \( T \in \mathbb{B}(H) \) (see e.g., [8, Theorem 1.3-1]). Our applications below include refinements of the second inequality above and some other consequences. Among many results, we retrieve the well known inequality [5, Theorem 1]:

\[
\omega(ST) \leq \frac{1}{2} \| S^* \|^2 + |T^2| \|, S, T \in \mathbb{B}(H).
\]
Another interesting application of our results is a new bound for the numerical radius of the product of two operators. In particular, we show that when $B$ is positive, then

$$\omega(AB) \leq \frac{3}{2} \|B\| \omega(A), \ A \in \mathbb{B}(\mathcal{H}).$$

The significance of this inequality is due to the fact that the known bound for $\omega(AB)$ when $B$ is positive is as follows:

$$\omega(AB) \leq \|AB\| \leq \|B\| \|A\| \leq 2 \|B\| \omega(A).$$

Consequently, our new bound provides a considerable refinement of this latter bound. See Remark 3.1 and Corollary 3.3 below for the details.

Before proceeding to the main results, we present the following observation about projections and Buzano inequality, which can be considered as a new proof of the main result in [7].

**Remark 1.1.** Let $P$ be any orthogonal projection in $\mathbb{B}(\mathcal{H})$. Put $z = Px$ in (1.5), and use the fact that $P^2 = P = P^*$, for any projection, then for $x, y \in \mathbb{B}(\mathcal{H})$,

$$\|Px\|^2 |\langle y, Px \rangle| = |\langle Px, Px \rangle| |\langle y, Px \rangle|$$

$$= |\langle x, P^2x \rangle| |\langle y, Px \rangle|$$

$$= |\langle x, Px \rangle| |\langle y, Px \rangle|$$

$$\leq \frac{\|Px\|^2}{2} (|\langle x, y \rangle| + \|x\| \|y\|).$$

Thus, we have shown that

$$|\langle Px, y \rangle| \leq \frac{1}{2} (|\langle x, y \rangle| + \|x\| \|y\|),$$

for any orthogonal projection $P$. It is interesting that contractions satisfy Buzano inequalities, as we show in Corollary 3.2 below.

## 2. Buzano and Kreĭn inequalities

In this section, we present our refinements of the Buzano inequality, then we present a new proof of the Kreĭn-Lin inequality (1.4), with a refinement. First, the Buzano inequality refinement.
Theorem 2.1. For any \( x, y, z \in \mathcal{H} \),

\[
|\langle x, z \rangle \langle z, y \rangle| \\
\leq \frac{1}{2} \left[ |\langle x, z \rangle \langle y, z \rangle| + |\langle x, y \rangle| \|z\|^2 + |\langle x, y \rangle \|z\|^2 - \langle x, z \rangle \langle z, y \rangle | \right] \\
\leq \frac{1}{2} \left[ |\langle x, z \rangle \langle y, z \rangle| + |\langle x, y \rangle| \|z\|^2 + \sqrt{|\|x\|^2\|z\|^2 - \langle x, z \rangle |} \sqrt{|\|y\|^2\|z\|^2 - \langle y, z \rangle |} \right] \\
\leq \frac{\|z\|^2}{2} (\|x\| \|y\| + |\langle x, y \rangle|) .
\]

Proof. We notice that if any of the vectors \( x, y, z \) is the zero vector, then the result follows trivially. Let \( x, y, z \in \mathcal{H} \) be any non-zero vector. Then

\[
|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \sqrt{|\|x\|^2 - |\langle x, e \rangle|^2 \sqrt{|\|y\|^2 - |\langle y, e \rangle|^2}} \\
\leq \|x\| \|y\| - |\langle x, e \rangle \langle y, e \rangle|,
\]

for any unit vector \( e \in \mathcal{H} \). Replacing \( e \) by \( \frac{z}{\|z\|} \), we get

\[
|\langle x, y \rangle - \langle x, \frac{z}{\|z\|} \rangle \langle \frac{z}{\|z\|}, y \rangle| \leq \sqrt{|\|x\|^2 - |\langle x, \frac{z}{\|z\|} \rangle|^2 \sqrt{|\|y\|^2 - |\langle y, \frac{z}{\|z\|} \rangle|^2}} \\
\leq \|x\| \|y\| - \left| \left< x, \frac{z}{\|z\|} \right> \right| \left| \left< y, \frac{z}{\|z\|} \right> \right| .
\]

Multiplying by \( \|z\|^2 \), we infer that

\[
|\langle x, y \rangle \|z\|^2 - \langle x, z \rangle \langle z, y \rangle| \leq \sqrt{|\|x\|^2\|z\|^2 - |\langle x, z \rangle|^2 \sqrt{|\|y\|^2\|z\|^2 - |\langle y, z \rangle|^2}} \\
\leq \|z\|^2 \|x\| \|y\| - |\langle x, z \rangle \langle y, z \rangle| .
\]

Now, from the triangle inequality

\[
|\langle x, z \rangle \langle z, y \rangle| \leq |\langle x, y \rangle| \|z\|^2 \leq |\langle x, y \rangle \|z\|^2 - \langle x, z \rangle \langle z, y \rangle| \\
(2.1) \\
\leq \sqrt{|\|x\|^2\|z\|^2 - |\langle x, z \rangle|^2 \sqrt{|\|y\|^2\|z\|^2 - |\langle y, z \rangle|^2}} \\
\leq \|z\|^2 \|x\| \|y\| - |\langle x, z \rangle \langle y, z \rangle| .
\]

Thus,

\[
|\langle x, z \rangle \langle z, y \rangle| \\
\leq \frac{1}{2} \left[ |\langle x, z \rangle \langle y, z \rangle| + |\langle x, y \rangle| \|z\|^2 + |\langle x, y \rangle \|z\|^2 - \langle x, z \rangle \langle z, y \rangle | \right] \\
\leq \frac{1}{2} \left[ |\langle x, z \rangle \langle y, z \rangle| + |\langle x, y \rangle| \|z\|^2 + \sqrt{|\|x\|^2\|z\|^2 - \langle x, z \rangle |} \sqrt{|\|y\|^2\|z\|^2 - |\langle y, z \rangle|^2}} \right] \\
\leq \frac{\|z\|^2}{2} (\|x\| \|y\| + |\langle x, y \rangle|) .
\]

This completes the proof. \( \square \)
It is interesting that the computations in the proof of Theorem 2.1 imply another proof of Lin’s inequality (1.4), with a refinement. This is our next result.

**Corollary 2.1.** Let $x, y, z \in \mathcal{H}$ be any vectors. Then

\[
\psi_{x,y} \leq \cos^{-1} \left( \cos \psi_{x,y} + |\cos \psi_{x,y} - \cos \psi_{x,z} \cos \psi_{y,z} - \sin \psi_{x,z} \sin \psi_{y,z}| \right) \\
\leq \psi_{x,z} + \psi_{y,z}.
\]

**Proof.** From (2.1), we obtain

\[
|\langle x, z \rangle| |\langle z, y \rangle| \\
\leq |\langle x, y \rangle| \|z\|^2 + |\langle x, z \rangle| |\langle z, y \rangle| - |\langle x, z \rangle| |\langle z, y \rangle| \\
\leq |\langle x, y \rangle| \|z\|^2 + \sqrt{|\langle x, z \rangle|^2 \|z\|^2 - |\langle x, y \rangle|^2 \|y\|^2 \|y\|^2 - |\langle x, z \rangle| |\langle z, y \rangle|}.
\]

If we multiply (2.2), by $0 < \frac{1}{\|x\| \|y\| \|z\|^2}$, we get

\[
\frac{|\langle x, z \rangle| |\langle z, y \rangle|}{\|x\| \|z\| \|y\| \|y\|} \leq \frac{|\langle x, y \rangle|}{\|x\| \|y\|} + \frac{|\langle x, z \rangle|}{\|x\| \|z\|} - \frac{|\langle x, y \rangle|}{\|x\| \|y\|} - \frac{|\langle x, z \rangle|}{\|x\| \|z\|} - \frac{|\langle z, y \rangle|}{\|y\|^2 \|z\|^2} \\
\leq \frac{|\langle x, y \rangle|}{\|x\| \|y\|} + \sqrt{1 - \frac{|\langle x, z \rangle|^2}{\|x\|^2 \|z\|^2} \sqrt{1 - \frac{|\langle z, y \rangle|^2}{\|y\|^2 \|z\|^2}},}
\]

which is equivalent to

\[
\cos \psi_{x,z} \cos \psi_{y,z} \leq \cos \psi_{x,y} + |\cos \psi_{x,y} - \cos \psi_{x,z} \cos \psi_{y,z}| \\
\leq \cos \psi_{x,y} + \sqrt{1 - \cos^2 \psi_{x,z} \sqrt{1 - \cos^2 \psi_{y,z}}} \\
= \cos \psi_{x,y} + \sin \psi_{x,z} \sin \psi_{y,z}
\]

by (1.2). This implies

\[
\cos (\psi_{x,z} + \psi_{y,z}) \\
\leq \cos \psi_{x,y} + |\cos \psi_{x,y} - \cos \psi_{x,z} \cos \psi_{y,z}| - \sin \psi_{x,z} \sin \psi_{y,z} \\
\leq \cos \psi_{x,y}.
\]

Now, since $\cos$ is a decreasing function on $[0, \pi]$ and since $0 \leq \psi_{x,z} + \psi_{y,z} \leq \pi$, the desired inequalities follow.

\[
\square
\]

**Corollary 2.2.** For any $x, y, z \in \mathcal{H}$,

\[
|\langle x, z \rangle| |\langle y, z \rangle| + \sqrt{|\langle x, z \rangle|^2 \|z\|^2 - |\langle x, z \rangle|^2 \sqrt{|\langle y, z \rangle|^2 \|z\|^2 - |\langle y, z \rangle|^2} \leq \|z\|^2 \|x\| \|y\|,
\]

holds.

Another refinement of the Cauchy-Schwarz inequality (1.1) can be stated as follows.
Corollary 2.3. For any \( x, y, z \in \mathcal{H} \),

\[
|\langle x, y \rangle| |z|^2 \leq |\langle x, z \rangle| |\langle y, z \rangle| + \sqrt{|\langle x \rangle|^2 |z|^2 - |\langle x, z \rangle|^2} \sqrt{|\langle y \rangle|^2 |z|^2 - |\langle y, z \rangle|^2}
\]

holds. In particular,

\[
|x, y| \leq |\langle x, e \rangle| |\langle e, y \rangle| + \sqrt{|\langle x \rangle|^2 - |\langle x, e \rangle|^2} \sqrt{|\langle y \rangle|^2 - |\langle y, e \rangle|^2} \leq \|x\| \|y\|,
\]

where \( e \in \mathcal{H} \) is a unit vector.

Proof. It follows from (2.1)

\[
\sqrt{|\langle x \rangle|^2 |z|^2 - |\langle x, z \rangle|^2} \sqrt{|\langle y \rangle|^2 |z|^2 - |\langle y, z \rangle|^2} \geq |\langle x, y \rangle| |z|^2 - |\langle x \rangle \langle z, y \rangle|.
\]

By Corollary 2.2 and the inequality (2.3), we get

\[
|x, y| |z|^2 \leq |\langle x, z \rangle| |\langle z, y \rangle| + \sqrt{|\langle x \rangle|^2 |z|^2 - |\langle x, z \rangle|^2} \sqrt{|\langle y \rangle|^2 |z|^2 - |\langle y, z \rangle|^2}
\]

\[
\leq \|z\|^2 \|x\| \|y\|
\]

as desired. \( \square \)

3. Refinements of the Cauchy-Schwarz inequality via contractions with applications on the numerical radius

The main results in this section include refinements of (1.1) via contractions. These refinements will lead to interesting applications including numerical radius and operator norm inequalities.

Theorem 3.1. Let \( A \in \mathbb{B}(\mathcal{H}) \) be a contraction and let \( x, y \in \mathcal{H} \). Then

\[
0 \leq \sqrt{\langle (A - A^2) x, x \rangle \langle (A - A^2) y, y \rangle} - |\langle (A - A^2) x, y \rangle| \leq \frac{|\|x\| \|y\| - \langle x, y \rangle|}{4}.
\]

Proof. Observe that

\[
|\langle x - Ax, y - Ay \rangle| = |\langle x, y \rangle - \langle x, Ay \rangle - \langle Ax, y \rangle + \langle Ax, Ay \rangle|
\]

\[
= |\langle x, y \rangle - \langle Ax, y \rangle - \langle Ax, y \rangle + \langle A^2 x, y \rangle|
\]

\[
= |\langle x, y \rangle - \langle (2A - A^2) x, y \rangle|
\]

\[
\geq |\langle x, y \rangle| - |\langle (2A - A^2) x, y \rangle|.
\]

(3.2)
On the other hand, by the Schwarz inequality
\[ |\langle x - Ax, y - Ay \rangle| \leq \| x - Ax \| \| y - Ay \| \]
(3.3)
\[ = (\| x \|^2 - \langle (2A - A^2) x, x \rangle)^{\frac{1}{2}} (\| y \|^2 - \langle (2A - A^2) y, y \rangle)^{\frac{1}{2}} \]
\[ \leq \| x \| \| y \| - \sqrt{\langle (2A - A^2) x, x \rangle \langle (2A - A^2) y, y \rangle}, \]
where the second inequality follows from \((a^2 - b^2)(c^2 - d^2) \leq (ac - bd)^2\) \((a, b, c, d \in \mathbb{R}^+)\). Notice that for this to be true, we must have \(A \leq 2I\).

Combining inequality (3.2) by inequality (3.3) gives
\[ |\langle x, y \rangle| - |\langle (2A - A^2) x, y \rangle| \leq \| x \| \| y \| - \sqrt{\langle (2A - A^2) x, x \rangle \langle (2A - A^2) y, y \rangle}, \]
when \(A \leq 2I\). Whence,
\[ \sqrt{\langle (2A - A^2) x, x \rangle \langle (2A - A^2) y, y \rangle} - |\langle (2A - A^2) x, y \rangle| \leq \| x \| \| y \| - |\langle x, y \rangle|, \]
when \(A \leq 2I\). Replacing \(A\) by \(2A\), we get
\[ \sqrt{\langle (A - A^2) x, x \rangle \langle (A - A^2) y, y \rangle} - |\langle (A - A^2) x, y \rangle| \leq \frac{\| x \| \| y \| - |\langle x, y \rangle|}{4}, \]
when \(A \leq I\). This proves the second desired inequality. For the first inequality, since \(A\) is a contraction, \(A - A^2\) is positive. Then by the Cauchy-Schwarz inequality for positive operators, we get
\[ 0 \leq \sqrt{\langle (A - A^2) x, x \rangle \langle (A - A^2) y, y \rangle} - |\langle (A - A^2) x, y \rangle|, \]
which proves the first inequality in (3.1), and completes the proof of the theorem. \(\Box\)

In fact, Theorem 3.1 may be used to obtain the following easier form; as a refinement of the Cauchy-Schwarz inequality.

**Corollary 3.1.** Let \(A \in \mathbb{B}(\mathcal{H})\) be a contraction. Then for \(x, y \in \mathcal{H}\),
\[ |\langle x, y \rangle| + \sqrt{\langle Ax, x \rangle \langle Ay, y \rangle} - |\langle Ax, y \rangle| \leq \| x \| \| y \|. \]
In particular, if \(A \in \mathbb{B}(\mathcal{H})\) is any positive invertible operator, then
\[ |\langle x, y \rangle| + \frac{1}{\| A \|} \left( \sqrt{\langle Ax, x \rangle \langle Ay, y \rangle} - |\langle Ax, y \rangle| \right) \leq \| x \| \| y \|. \]
**Proof.** In Theorem 3.1, we have shown that
(3.4) \[ 0 \leq \sqrt{\langle (A - A^2) x, x \rangle \langle (A - A^2) y, y \rangle} - |\langle (A - A^2) x, y \rangle| \leq \frac{\| x \| \| y \| - |\langle x, y \rangle|}{4}, \]
for the contraction \(B \in \mathbb{B}(\mathcal{H})\) and \(x, y \in \mathcal{H}\). Now, let \(A \in \mathbb{B}(\mathcal{H})\) be such that \(0 \leq A \leq \frac{1}{4} I\), and define
\[ B = \frac{1}{2} \left( I + (I - 4A)^{\frac{1}{2}} \right). \]
Then $B$ is a contraction. Indeed, since the mapping $t \mapsto t^{\frac{1}{2}}$ is operator monotone on $[0, \infty)$ [1, Proposition V.1.8], we have
\[ 0 \leq A \leq \frac{1}{4}I \Leftrightarrow 0 \leq I - 4A \leq I \]
\[ \Rightarrow 0 \leq (I - 4A)^{\frac{1}{2}} \leq I \]
\[ \Rightarrow 0 \leq \frac{1}{2} \left( I + (I - 4A)^{\frac{1}{2}} \right) \leq I. \]
Since $B$ is contractive, (3.4) applies for $B$, and we have
\[ 0 \leq \sqrt{\langle (B - B^2) x, x \rangle \langle (B - B^2) y, y \rangle - \langle (B - B^2) x, y \rangle |\langle x, y \rangle|} \leq \frac{\|x\| \|y\| - |\langle x, y \rangle|}{4}. \]
But by definition of $B$, we have $B - B^2 = A$. This implies
\[ 0 \leq \sqrt{\langle Ax, x \rangle \langle Ay, y \rangle - |\langle Ax, y \rangle|} \leq \frac{\|x\| \|y\| - |\langle x, y \rangle|}{4}, \]
when $0 \leq A \leq \frac{1}{4}I$. Now, if $A$ is an arbitrary contraction, replace $A$ by $\frac{1}{4}A$ in the above inequality. This implies the desired inequality and completes the proof. □

The following result is a Cauchy-Schwarz type inequality for contractions. Following this result, we explain how this extends (1.1).

**Corollary 3.2.** Let $A \in \mathcal{B}(\mathcal{H})$ be a contraction. Then for $x, y \in \mathcal{H}$,
\[ |\langle Ax, y \rangle| \leq \frac{\|x\| \|y\| + |\langle x, y \rangle|}{2}. \]

**Proof.** If $A \leq 2I$, it follows from (3.2) and (3.3) that
\[ |\langle x, y \rangle - \langle (2A - A^2) x, y \rangle| \leq \|x\| \|y\| - \sqrt{\langle (2A - A^2) x, x \rangle \langle (2A - A^2) y, y \rangle}. \]
This implies
\[ |\langle x, y \rangle - \langle (2A - A^2) x, y \rangle| + \sqrt{\langle (2A - A^2) x, x \rangle \langle (2A - A^2) y, y \rangle} \leq \|x\| \|y\|. \]
On the other hand, by the Schwarz inequality,
\[ |\langle x, y \rangle - \langle (2A - A^2) x, y \rangle| + |\langle (2A - A^2) x, y \rangle| \leq \|x\| \|y\|. \]
This implies
\[ -|\langle x, y \rangle| + |\langle (2A - A^2) x, y \rangle| + |\langle (2A - A^2) x, y \rangle| \leq \|x\| \|y\|, \]
when $A \leq 2I$. Thus,
\[ |\langle (2A - A^2) x, y \rangle| \leq \frac{\|x\| \|y\| + |\langle x, y \rangle|}{2}, \]
when $A \leq 2I$. Replacing $A$ by $2A$, we get
\[ |\langle (A - A^2) x, y \rangle| \leq \frac{\|x\| \|y\| + |\langle x, y \rangle|}{8}. \]
whenever $A \leq I$. Applying the same procedure as in the proof Corollary 3.1, we reach
\[
|\langle Ax, y \rangle| \leq \frac{\|x\| \|y\| + |\langle x, y \rangle|}{2},
\]
as desired. □

Notice that when $A = I$, Corollary 3.2 implies (1.1). Therefore, the above corollary provides an extension of (1.1).

**Remark 3.1.** Notice that when $A \in \mathcal{B}(\mathcal{H})$ is a given positive operator, replacing $A$ by $\frac{1}{\|A\|}A$ (when $A \neq 0$) in Corollary 3.2 implies
\[
|\langle Ax, y \rangle| \leq \frac{\|A\|}{2} (|\langle x, y \rangle| + \|x\| \|y\|), x, y \in \mathcal{H}.
\]
At this stage it is natural to ask if this inequality holds for any operator $A \in \mathcal{B}(\mathcal{H})$. That is: if $A \in \mathcal{B}(\mathcal{H})$, does it follow that
\[
|\langle Ax, y \rangle| \leq \frac{\|A\|}{2} (|\langle x, y \rangle| + \|x\| \|y\|), x, y \in \mathcal{H}? \tag{3.5}
\]
It is interesting that this inequality is not true in general. Indeed, if (3.5) is true, then we would have for $A, B \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$ with $\|x\| = 1$,
\[
|\langle ABx, x \rangle| \leq \frac{\|A\|}{2} (|\langle Bx, x \rangle| + \|Bx\| \|x\|)
\leq w(A) (w(B) + \|B\|)
\leq w(A) (w(B) + 2w(B))
= 3w(A)w(B).
\]
Thus, we have shown that if (3.5) is true, then for any $A, B \in \mathcal{B}(\mathcal{H})$, we have $w(AB) \leq 3w(A)w(B)$. Unfortunately, the latter inequality is wrong, as we have
\[
w(AB) \leq 4w(A)w(B), \tag{3.6}
\]
with the constant 4 best possible (see e.g.,[8, Theorem 2.5-2]). This shows that the inequality (3.5) is wrong, in general. However, using the polar decomposition $A = U|A|$, one can see that the inequality
\[
|\langle Ax, y \rangle| \leq \frac{\|A\|}{2} (|\langle Ux, y \rangle| + \|x\| \|U^*y\|)
\leq \frac{\|A\|}{2} (|\langle Ux, y \rangle| + \|x\| \|y\|)
\]
holds for $x, y \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$ with polar decomposition $A = U|T|$. We notice here that $U$ is a partial isometry, and hence $\|U\| = \|U^*\| \leq 1$. This justifies the second inequality above.
It is worth mentioning that, in the case when $B$ is a positive operator, the constant 4 in the inequality (3.6) can be reduced to $3/2$ as shown in the following way:

$$\omega(AB) \leq \frac{\|B\|}{2} (\omega(A) + \|A\|) \leq \frac{3}{2} \omega(A) \|B\| = \frac{3}{2} \omega(A) \omega(B).$$

In particular, if $B$ is a contraction, then

$$\omega(AB) \leq \frac{3}{2} \omega(A).$$

As a conclusion of the above remark, and due to the importance of its finding, we summarize as follows.

**Corollary 3.3.** Let $A, B \in \mathbb{B}(\mathcal{H})$ be such that $B$ is positive. Then

$$\omega(AB) \leq \frac{3}{2} \|B\| \omega(A).$$

As another application of Corollary 3.2, we present the following numerical radius and operator norm applications. In the sequel, the notation $|X|$ will be used to denote $(X^*X)^{1/2}$, for $X \in \mathbb{B}(\mathcal{H})$.

**Corollary 3.4.** Let $A, S, T \in \mathbb{B}(\mathcal{H})$ be such that $A$ is a contraction. Then

$$(3.7) \quad \omega(SAT) \leq \frac{1}{4} \|T\|^2 + |S^*|^2 + \frac{1}{2} \omega(ST).$$

Moreover,

$$(3.8) \quad \|SAT\| \leq \frac{\|T\| \|S\| + \|ST\|}{2}.$$ 

In particular,

$$\omega(ST) \leq \frac{1}{2} \|T\|^2 + |S^*|^2,$$

and

$$\|ST\| \leq \|S\| \|T\|.$$

**Proof.** Replacing $x$ by $Tx$ and $y$ by $S^*x$ in Corollary 3.2, we get

$$|\langle SATx, x \rangle| \leq \frac{\|Tx\| \|S^*x\| + |\langle Tx, S^*x \rangle|}{2}.$$
Thus, by the arithmetic-geometric mean inequality, we obtain
\[
|\langle SATx, x \rangle| \leq \frac{\|Tx\| \|S^*x\| + |\langle STx, x \rangle|}{2}
\]
\[
= \frac{1}{2} \left( \sqrt{\langle |T|^2 x, x \rangle \langle |S^*|^2 x, x \rangle + |\langle STx, x \rangle|} \right)
\]
\[
\leq \frac{1}{2} \left( \frac{1}{2} \sqrt{\langle |T|^2 x, x \rangle + \langle |S^*|^2 x, x \rangle + \frac{1}{2} |\langle STx, x \rangle|} \right)
\]
\[
= \frac{1}{2} \left( \frac{1}{2} \left( \langle |T|^2 + |S^*|^2 \rangle x, x \rangle + \frac{1}{2} |\langle STx, x \rangle| \right) \right).
\]
Therefore,
\[
|\langle SATx, x \rangle| \leq \frac{1}{2} \left( \frac{1}{2} \left( \langle |T|^2 + |S^*|^2 \rangle x, x \rangle + \frac{1}{2} |\langle STx, x \rangle| \right) \right).
\]
Now, by taking supremum over all unit vector \( x \in \mathcal{H} \), we get the inequality (3.7).

To prove (3.8), letting \( x = Tx \) and \( y = S^*y \) in (3.2), then we have
\[
|\langle SATx, y \rangle| \leq \frac{\|Tx\| \|S^*y\| + |\langle Tx, S^*y \rangle|}{2}
\]
\[
= \frac{\|Tx\| \|S^*y\| + |\langle STx, y \rangle|}{2}.
\]
Now, the desired inequality (3.8) follows by taking supremum over \( x, y \in \mathcal{H} \) with \( \|x\| = \|y\| = 1 \).

In dealing with numerical radius inequalities, we are interested in power inequalities. We refer the reader to \([5, 6, 9, 12]\) as a sample of such inequalities. In the following result, we use Corollary 3.4 to obtain a power inequality for the numerical radius.

**Corollary 3.5.** Let \( A, S, T \in \mathbb{B}(\mathcal{H}) \) be such that \( A \) is a contraction. Then
\[
\omega^r(SAT) \leq \frac{1}{4} \| |T|^{2r} + |S^*|^{|2r}| \| + \frac{1}{2} \omega^r(ST),
\]
for \( r \geq 1 \).

**Proof.** This follows from Corollary 3.4 and the facts that \( t \mapsto t^r, r \geq 1 \) is a convex increasing function on \( [0, \infty) \) and that
\[
\left\| f \left( \frac{A + B}{2} \right) \right\| \leq \frac{1}{2} \| f(A) + f(B) \|,
\]
for any increasing convex function \( f : [0, \infty) \to [0, \infty) \) and positive operators \( A, B, [2, \text{Corollary 2.2}] \).

Next, we use Corollary 3.4 to obtain a refinement of the inequality \( \omega(T) \leq \|T\| \).
Corollary 3.6. Let $T \in (\mathcal{H})$ be a given operator with the polar decomposition $T = U|T|$. Then
\[
\omega(T) \leq \frac{1}{2} \left( \|T\| + \|T\|^{\frac{1}{2}} \omega\left(U|T|^{\frac{1}{2}}\right) \right)
\leq \frac{1}{2} \left( \|T\| + \|T\|^{\frac{1}{2}} \|U|T|^{\frac{1}{2}}\| \right)
\leq \frac{1}{2} \left( \|T\| + \|T\|^{\frac{1}{2}} \|U\| \|T\|^{\frac{1}{2}} \right)
\leq \|T\|.
\]

Proof. We prove the first inequality, from which the other inequalities follow immediately. Let $T = U|T|$ be the polar decomposition of $T$. Then, for any vectors $x, y \in \mathcal{H}$,
\[
|\langle Tx, y \rangle| = |\langle U|T|x, y \rangle| = \left| \langle T^{\frac{1}{2}}x, |T|^{\frac{1}{2}}U^*y \rangle \right|.
\]
Let $A$ be any positive operator. Corollary 3.4 implies
\[
|\langle Ax, y \rangle| \leq \frac{\|A\|}{2} (|\langle x, y \rangle| + \|x\| \|y\|).
\]
This together with (3.10) imply
\[
|\langle Tx, x \rangle| = \left| \langle T^{\frac{1}{2}}x, |T|^{\frac{1}{2}}U^*x \rangle \right|
\leq \frac{\|T|^{\frac{1}{2}}\|}{2} \left( \|x, \langle T|^{\frac{1}{2}}U^*x \rangle\| + \|x\| \|T|^{\frac{1}{2}}U^*x\| \right)
= \frac{\|T\|^{\frac{1}{2}}}{2} \left( \|\langle U|T|^{\frac{1}{2}}x, x \rangle\| + \|x\|^{2} \|T|^{\frac{1}{2}}U^*\| \right).
\]
Noting that $\|U^*\| = \|U\| = 1$ and taking the supremum over all unit vectors $x \in \mathcal{H}$, we obtain
\[
\omega(T) \leq \frac{\|T\|^{\frac{1}{2}}}{2} \left( \omega\left(U|T|^{\frac{1}{2}}\right) + \|T\|^{\frac{1}{2}} \right).
\]
This completes the proof of the first inequality, as desired.

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