ASYMPTOTICS FOR EXPONENTIAL FUNCTIONALS OF RANDOM WALKS

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Abstract

This paper provides a detailed description for the asymptotics of exponential functionals of random walks with light/heavy tails. We give the convergence rate based on the key observation that the asymptotics depends on the sample paths with either slowly decreasing local minimum or final value below a low level. Also, our thoughtful analysis of the interrelationship between the local minimum and the final value provides the exact expression for the limiting coefficients in terms of some transformations of the random walk.

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1 Introduction

Exponential functionals of random walks have been studied deeply in the past decades because of their wide and important applications in various fields such as mathematical finance, physics and population evolution. In the general setting, for a one-dimension random walk $S$ generated by a sequence of independent copies $X_1, X_2, \cdots$ of a random variable $X$, its exponential functional is usually defined by

$$I_n := \sum_{k=1}^{n} e^{-S_k}, \quad n = 1, 2, \cdots$$

with the convention $I_0 = 0$. It increases to the limit $I_{\infty} \in (0, \infty]$ almost surely as $n \to \infty$. According to the Blumenthal zero-one law, the probability that $I_{\infty}$ is finite, $P(I_{\infty} < \infty)$, necessarily equals to 0 or 1. Furthermore, the long-term behavior of $S$ shows this probability equals to 0 if and only if $S$ does not drift to infinity. In this case, much attention has been drawn to the speed at which $I_n$ increases to infinity, especially the decay rate of the following expectation

$$E[F(I_n)] = E[F\left(\sum_{k=1}^{n} e^{-S_k}\right)] \quad (1.1)$$

with $F$ be a positive function on $(0, \infty)$ that vanishes at infinity. The main aim of this work is to provide a detailed description for the decay rate of this expectation as $n \to \infty$.

The study of the expectation $E[F(I_n)]$ has been initiated to explore the asymptotic properties of stochastic systems in random environment. For instance, let $\tau$ be the extinction time of a

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linear fractional Galton-Watson process in an i.i.d. random environment with geometric offspring distribution\footnote{The asymptotic results in the following references are established for Galton-Watson processes in i.i.d. random environment with general branching mechanism and the linear fractional case is considered as a typical example.}. The survival probability of the population at time $n$ can be written as

$$P(\tau > n) = \mathbb{E}[(1 + I_n)^{-1}]$$

in which the random walk $S$ is generated by the offspring distribution that varies randomly as time goes; see Application 1 in [25]. Under the assumption that $\mathbb{E}[X] = 0$ and $\mathbb{E}[|X|^2] < \infty$, Kozlov [26] first showed that $P(\tau > n) \sim C \cdot n^{-1/2}$ as $n \to \infty$ for some $C > 0$. When $\mathbb{E}[X] < 0$, Liu [28] proved a rough asymptotic result

$$\left(P(\tau > n)\right)^{1/n} \sim C \cdot \zeta \quad \text{with} \quad \zeta := \inf_{\lambda \in [0,1]} \mathbb{E}[e^{\lambda X}].$$

More precise results were provided later in [3, 5, 22], i.e., $P(\tau > n) \sim C \cdot \zeta^n, C \cdot n^{-1/2} \zeta^n$ or $C \cdot n^{-3/2} \zeta^n$ if $\mathbb{E}[|X|^2] < \infty$ and $\mathbb{E}[Xe^X] < 0$, $0 \leq 0 > 0$. As the second example, we write $M$ for the maximum of a sample transient random walk moving in an i.i.d. random environment. The tail probability of $M$ can be given by

$$P(M \geq n) = \mathbb{E}[\hat{I}_n / \hat{I}_n + I_n]^{-1}, \quad n = 1, 2, \cdots,$$

where $\hat{I}_n := 1 + \sum_{i=0}^{n-1} e^{S_i}$ and the two random walks $\{S'_k : k = -1, -2, \cdots\}$ and $\{S_k : k = 1, 2, \cdots\}$ are generated by the random environment at negative and positive integer points respectively; see (7) in [1]. The asymptotics of $P(M \geq n)$ was considered in [1] with $\mathbb{E}[X] < 0$, i.e., it is asymptotically equivalent to $C \cdot \zeta^n, C \cdot n^{-1/2} \zeta^n$ or $C \cdot n^{-3/2} \zeta^n$ when $\mathbb{E}[Xe^X] < 0$, $0 \leq 0 > 0$ respectively.

For general function $F$, to our best knowledge the asymptotics of the expectation (1.1) has only been studied in Hirano [25] under the assumption that $\mathbb{E}[Xe^{\alpha X}] = 0$ and $F$ is completely monotone satisfying $F(x) \leq C x^{-\beta}$ for two constants $\beta > \alpha > 0$. His result shows that $\mathbb{E}[F(I_n)] \sim C \cdot n^{-3/2}(\mathbb{E}[e^{\alpha X}])^n$. In this work, we provide under natural assumptions, a more complete and accurate description for the asymptotic behavior of the expectation (1.1) with general function $F$ and random walk $S$ whose generic step $X$ owns light- or heavy-tailed distribution. Our results state that beside of the optimal polynomial decay rate of $F$, i.e., $\theta_F := \sup\{\theta > 0 : \sup_{x > 0} x^\theta F(x) < \infty\}$, the long-term properties of the expectation (1.1) also heavily reply on the speed of the random walk $S$ decreasing to $-\infty$, which can be quantified by

$$\varrho := \inf_{\lambda \in [0, \theta_F]} \mathbb{E}[e^{\lambda X}], \quad \Lambda := \arg \inf_{\lambda \in [0, \theta_F]} \mathbb{E}[e^{\lambda X}]$$

and the behavior of $S_n$ under $\mathbb{P}(\Lambda)$ as $n \to \infty$. Here arg inf stands for argument of the infimum and the probability law $\mathbb{P}(\Lambda)$ is the Esscher transform of $\mathbb{P}$ associated to the martingale $\{e^{-\varrho n \cdot e^{\Lambda S_n}} : n = 0, 2, \cdots\}$. Consequently, we see that the convergence rate of the expectation (1.1) changes dramatically in the following six disjoint cases:

| $S$ oscillates under $\mathbb{P}(\Lambda)$ | $S$ drifts to $-\infty$ under $\mathbb{P}(\Lambda)$ |
|----------------------------------------|----------------------------------------|
| $0 = \Lambda < \theta_F$ | $0 = \Lambda < \theta_F$ |
| $0 < \Lambda < \theta_F$ | $0 < \Lambda < \theta_F$ |
| $0 < \Lambda = \theta_F$ | $0 < \Lambda = \theta_F$ |

Case I  Case II  Case III  Case IV  Case V  Case VI

Figure 1(a)  Figure 1(b)  Figure 1(c)  Figure 1(d)  Figure 1(e)  Figure 1(f)
The feature of this work is that both the convergence rate and the explicit expression of the limiting coefficient in each regime are given. Roughly speaking, the expectation (1.1) decays at some exponential rate with different regularly varying modifying factor, i.e.,

$$E[F(I_n)] \sim \psi_n \cdot \rho^n,$$

where $\psi_n$ is a regularly varying sequence vanishing at infinity and determined by the fluctuation of the random walk $S$ under the probability law $P^{(A)}$. Particularly, when $0 \leq \Lambda < \theta_F$ the comparison with the asymptotics of the first entrance time $\tau_0^-$ of the random walk $S$ in $(-\infty, 0)$ shows that for some constant $C > 0$,

$$E[F(I_n)] \sim C \cdot P(\tau_0^- > n).$$

Our methodology is based on the fluctuation theory for random walks. There are two major challenges, which are also two key steps, in the asymptotic analysis of the expectation (1.1). The first one is to find out the sample paths of random walk that make the main contribution to the expectation (1.1). Enlightened by the asymptotic analysis of survival probabilities of Galton-Watson processes in an i.i.d. random environment; see [4, 2, 3, 9, 32], we observe that the characteristics of these sample paths vary dramatically in different cases. More precisely, sample paths with slowly decreasing local minimum make the main contribution in Case I and III; the expectation (1.1) in Case II mainly relies on sample paths with either the local minimum attained at the beginning of the time interval or the final value below a low level. In Case IV, the main contribution is made by sample paths with early large step. The key sample paths in Case V not only have early large step and low final value but also decay slowly before the large step. Different to other cases, the contribution of any sample path can not be asymptotically ignored in Case VI. The second challenge is to provide an exact expression of the limiting coefficients. In order to achieve it, we decompose each key sample path into several parts at its local minimum and then seek out the subdivisions that make the main contribution to the expectation (1.1). For instance, the contribution of the key sample paths after large time $K$ is negligible in Case I, III and IV, i.e., the impact of $I_n$ can be well approximated by that of $I_K$. However, in Case II and V the key sample paths make the main contribution to the expectation (1.1) at both the beginning and the end of the time interval, i.e., the impact of $I_n$ can be well approximated by that of $I_K + (I_n - I_{n-K})$ for large $K$. Finally, we give the explicit representations for the limiting coefficients in terms of some transformations of these subdivisions.

The second purpose of this work is to offer assistance in our future study of the asymptotic behavior of exponential functional of a Lévy process $\{\xi_t : t \geq 0\}$ defined by

$$I_t(\xi) := \int_0^t e^{-\xi_s} ds, \quad t \geq 0.$$

Similarly, we have $I_t(\xi) \to I_\infty(\xi) \in (0, \infty]$ a.s. as $t \to \infty$ and $I_\infty(\xi) < \infty$ a.s. if and only if $\xi$ drifts to infinity. Readers may refer to [10, 15, 30, 31, 33] and references therein for many interesting results related to $I_t(\xi)$ and $I_\infty(\xi)$. In the case $I_\infty(\xi) = \infty$ a.s., we are usually interested in the decay rate of the expectation $E[F(I_t(\xi))]$ defined as in (1.1), because of its close connection to the long-term properties of random processes in random environment, e.g. continuous-state branching processes in Lévy random environment and diffusion in Lévy random environment; see [7, 24, 27, 29]. Under the assumption that $F(x) \leq Cx^{\theta_F}$ and $\theta_F < \lambda_* := \sup\{\lambda \geq 0 : E[e^{\lambda \xi}] < \infty\}$, four different regimes for the convergence rate of the expectation (1.2) were provided in [7, 29, 27]. When $\lambda_* = 0$, Patie and Savov [31] and Xu [34] proved the polynomial decay rate for $E[F(I_t(\xi))]$ with $\xi$ oscillating and
satisfying the Spitzer’s condition or \( E[\xi_1] < 0 \) and \( P(\xi_1 > x) \) regularly varying at infinity. For the case \( 0 < \lambda_* \leq \theta_F \), to the best of our knowledge, the asymptotics of \( E[F(I_1(\xi))] \) is still an open problem. Notice that \( \xi \) can be well approximated by the random walk \( \xi^\delta := \{\xi^\delta_n : n = 0, 1, \cdots\} \) with \( \xi_n^\delta := \xi_{n\delta} \) for small \( \delta > 0 \). Let \( I^\delta \) be the exponential functional of \( \xi^\delta \), we may conjecture that there exists a positive, regularly varying function \( \Psi \) on \( \mathbb{R}_+ \) satisfying \( \Psi(n) \sim \psi_n \) as \( n \to \infty \) and

\[
E[F(I_1(\xi))] \sim E[F(\delta \cdot I^\delta_{[\delta/n]})] \sim \Psi(t) \cdot |\varrho_\xi|^t \quad \text{and} \quad \varrho_\xi := \inf_{\lambda \in [0, \theta_F]} E[e^{\lambda \xi_1}] .
\]

The remainder of this paper is organized as follows. In Section 2, we recall some basic elements of fluctuation theory for random walks and then provide the accurate asymptotic results for the expectation (1.1). We give in Section 3 some auxiliary asymptotic results for random walks, which will be used in the proofs for our main results. The asymptotic results for the expectation (1.1) with \( S \) oscillating or drifting to \(-\infty\) under \( P^{(\Lambda)} \) are proved separately in Section 4 and 5.

## 2 Preliminaries and main results

In this section we first introduce some basic notation and elements of fluctuation theory for random walks. We then provide the main results in this paper about the asymptotic behavior of exponential functionals of random walks.

### 2.1 Random walks

Suppose that the random walk \( S \) is defined on a complete probability space \( (\Omega, \mathcal{F}, P) \). Let \( \mathcal{F}^S \) and \((\mathcal{F}^S_n)_{n \geq 0}\) denote the \( \sigma \)-algebra and the filtration generated by \( S \), i.e., \( \mathcal{F}^S := \sigma(S_k : k = 0, 1, \cdots) \) and \( \mathcal{F}^S_n := \sigma(S_k : k = 0, 1, \cdots, n) \). For any probability measure \( \mu \) on \( \mathbb{R} \), we denote by \( P_\mu \) and \( E_\mu \) the law and expectation of the random walk \( S \) with initial state \( S_0 \) distributed as \( \mu \). When \( \mu = \delta_x \) is a Dirac measure at point \( x \in \mathbb{R} \), we write \( P_x \) for \( P_{\delta_x} \) and \( E_x \) for \( E_{\delta_x} \). For simplicity, we also write \( P \) for \( P_0 \) and \( E \) for \( E_0 \).

According to its behavior as \( n \to \infty \), the random walk \( S \) can be exactly classified into three types: (i) drifts to \( \infty \): \( S_n \to \infty \) a.s.; (ii) drifts to \( -\infty \): \( S_n \to -\infty \) a.s.; (iii) oscillation: \( \lim sup_{n \to \infty} S_n = -\lim inf_{n \to \infty} S_n = \infty \) a.s. We write \( M := \{M_n : n = 1, 2, \cdots\} \) and \( L := \{L_n : n = 1, 2, \cdots\} \) for the running maximum and minimum processes respectively,

\[
M_n := \max_{1 \leq i \leq n} S_i \quad \text{and} \quad L_n := \min_{1 \leq i \leq n} S_i .
\]

In addition to \( M_n \) and \( L_n \), we will also use the following two random times:

\[
\sigma^+_n := \inf \left\{ 0 \leq i \leq n : S_i = S_0 \lor M_n \right\} \quad \text{and} \quad \sigma^-_n := \inf \left\{ 0 \leq i \leq n : S_i = S_0 \land L_n \right\}
\]

the first times up to time \( n \) that the maximum and minimum are attained. For \( x \in \mathbb{R} \), denote by \( \tau^+_x \) and \( \tau^-_x \) the first entrance times of \( S \) in \((x, \infty)\) and \((\infty, x)\) respectively, i.e.,

\[
\tau^+_x := \inf \{n > 0 : S_n > x\} \quad \text{and} \quad \tau^-_x := \inf \{n > 0 : S_n < x\} .
\]

In the sequel, we always denote by \( \tilde{S} \) an independent copy of \( S \). For the quantities introduced to \( S \), the corresponding ones for \( \tilde{S} \) are denoted by tildes, for instance, \( \tilde{I} \) and \( \tilde{\tau}^-_x \).
The renewal function $V$ associated with the strict descending ladder height process is defined by $V(x) = 0$ if $x < 0$ and

$$V(x) := 1 + \sum_{n=1}^{\infty} P(S_{\gamma_n^-} \geq -x), \quad x \geq 0,$$

where $\{\gamma_i^- : i = 0, 1, \cdots\}$ are strict descending ladder epochs of $S$ with $\gamma_0^- = 0$ and

$$\gamma_i^- := \inf \{n > \gamma_{i-1}^- : S_n < S_{\gamma_{i-1}^-}\}, \quad i \geq 1.$$

It is obvious that $V$ is a non-decreasing right-continuous function with $V(0) = 1$. Using the duality lemma, we also have

$$V(x) = 1 + \sum_{n=1}^{\infty} P(S_n \geq -x, \sigma^-_n = n) = 1 + \sum_{n=1}^{\infty} P(S_n \geq -x, M_n < 0).$$

When the random walk $S$ drifts to $-\infty$, we have $E_x[\tau_0^-] < \infty$ and $V(x) = E_x[\tau_0^-]/E[\tau_0^-]$ for any $x \geq 0$.

The dual process of $S$ is denoted by $\hat{S}$, that is $\hat{S} = -S$ when the starting point is 0. Let $\hat{P}_x$ be the law of $\hat{S}$ under $P_{-x}$. For the quantities introduced to $S$, the corresponding ones for $\hat{S}$ are denoted by hats, for instance $\hat{I}$, $\hat{\tau}_x^{-}$ and so on. Specially, the renewal process $\hat{V}$ associated with the strict descending ladder height process of $\hat{S}$ is equal to the renewal process associated with the strict ascending ladder height process of $S$, i.e.

$$\hat{V}(x) = 1 + \sum_{n=1}^{\infty} P(S_n \leq x, \sigma^+_n = n) = 1 + \sum_{n=1}^{\infty} P(S_n \leq x, L_n > 0).$$

When $S$ drifts to $\infty$, we also have $E_x[\tau_0^+] < \infty$ and $\hat{V}(x) = \hat{E}_x[\tau_0^+]/\hat{E}[\tau_0^+]$ for any $x \geq 0$.

If the random walk $S$ does not drift to $-\infty$, the process $\{V(S_n)1_{\{\tau_0^- > n\}} : n = 0, 1, \cdots\}$ is a $P_x$-martingale for any $x \geq 0$. In this case, we introduce a probability $P^\uparrow_x$ on $(\Omega, \mathcal{F}, \mathbb{P}^S)$ defined by

$$P^\uparrow_x(A) := \int_A V(S_n)1_{\{\tau_0^- > n\}} \frac{dP_x}{V(x)}, \quad A \in \mathbb{F}_n^S, n \geq 0.$$

It is usually well-known as Doob’s $h$-transform of $P_x$. Particularly, under $P^\uparrow$ the process $S$ turns to be a homogeneous Markov process on $[0, \infty)$ with transition function

$$p^\uparrow(x, dy) := \frac{V(y)}{V(x)} P(x + X \in dy), \quad x, y \geq 0.$$

Similarly, in the case that $\hat{S}$ does not drift to $-\infty$, for any $x \geq 0$ we can also introduce Doob’s $h$-transform of $\hat{P}_x$ on $(\Omega, \mathcal{F}, \mathbb{P}^\hat{S})$ defined by the renewal function $\hat{V}$, i.e.,

$$P^\downarrow_x(A) := \int_A \hat{V}(S_n)1_{\{\tau_0^+ > n\}} \frac{d\hat{P}_x}{\hat{V}(x)}, \quad A \in \mathbb{F}_n^\hat{S}, n \geq 0.$$

Under $P^\downarrow$ the process $\hat{S}$ turns to be a homogeneous Markov process on $[0, \infty)$ with transition function

$$p^\downarrow(x, dy) := \frac{\hat{V}(y)}{\hat{V}(x)} P(x - X \in dy), \quad x, y \geq 0.$$
In order to make our following statements much easier to be understood, we write \( S^\uparrow \) and \( S^\downarrow \) for the two independent Markov processes with transition functions \( p^\uparrow \) and \( p^\downarrow \) respectively. Their exponential functionals are denoted as \( I^\uparrow \) and \( I^\downarrow \). Moreover, we also write \( \hat{S}^\uparrow \) and \( \hat{S}^\downarrow \) for the dual processes of \( S^\uparrow \) and \( S^\downarrow \) respectively and also write \( \hat{I}^\uparrow \) and \( \hat{I}^\downarrow \) for their exponential functionals. Repeating the preceding argument, we see that \( \hat{S}^\uparrow \) and \( \hat{S}^\downarrow \) are two homogeneous Markov processes with transition functions

\[
\hat{p}^\uparrow(x, dy) := \frac{\hat{V}(-y)}{\hat{V}(-x)} P(x + X \in dy) \quad \text{and} \quad \hat{p}^\downarrow(x, dy) := \frac{V(-y)}{V(-x)} P(x - X \in dy), \quad x, y \leq 0.
\]

As we have mentioned before, the asymptotics of the tail distribution of generic step \( X \) plays a crucial role in the following classification and asymptotic analysis of exponential functionals of random walks. Thus we need the Laplace transform of \( X \)

\[
\mathcal{L}_X(\lambda) := E[e^{\lambda X}], \quad \lambda \in \mathbb{R}.
\]

Certainly, it may happen that \( \mathcal{L}_X(\lambda) = \infty \) for some \( \lambda \in \mathbb{R} \). Let \( \mathcal{D}_{\mathcal{L}_X} := \{ \lambda \in \mathbb{R} : \mathcal{L}_X(\lambda) < \infty \} \), \( \mathcal{D}^-_{\mathcal{L}_X} := \mathcal{D}_{\mathcal{L}_X} \cap [0, \infty) \) and \( \mathcal{D}^+_{\mathcal{L}_X} := \mathcal{D}_{\mathcal{L}_X} \cap (-\infty, 0] \). For each \( \lambda_0 \in \mathcal{D}_{\mathcal{L}_X} \), the process \( \{\exp\{\lambda_0 S_n\} \cdot |\mathcal{L}_X(\lambda_0)|^{-n} : n = 0, 1, \ldots\} \) is a \( P \) martingale, which allows us to define a probability measure \( P^{(\lambda_0)} \) on \( (\Omega, \mathcal{F}, \mathcal{F}^S) \)

\[
P^{(\lambda_0)}(A) := \int_A \exp\{\lambda_0 S_n\} \cdot |\mathcal{L}_X(\lambda_0)|^{-n} dP, \quad A \in \mathcal{F}_n^S, n \geq 0. \tag{2.1}
\]

It is known that the process \( S \) under \( P^{(\lambda_0)} \) is still a random walk and the generic step \( X \) has Laplace transform

\[
E^{(\lambda_0)}[e^{\lambda X}] = \mathcal{L}_X(\lambda_0 + \lambda)/\mathcal{L}_X(\lambda_0), \quad \lambda \in \mathbb{R}.
\]

Let \( V^{(\lambda_0)} \) and \( \hat{V}^{(\lambda_0)} \) be the renewal functions associated with the strict descending ladder height processes of \( S \) and \( \hat{S} \) under \( P^{(\lambda_0)} \).

### 2.2 Main results

We now provide the asymptotic results for the expectation \( E[F(I_n)] \) in which \( F \) is a positive, bounded function on \((0, \infty)\) and always satisfies the following assumption:

**Assumption 2.1** The set \( \mathcal{D}_F := \{ \theta > 0 : \sup_{x > 0} x^\theta F(x) < \infty \} \) is not null and the supremum is denoted as \( \theta_F \in (0, \infty) \).

To simplify the representation of our main results, let us list the following conditions:

**Condition 2.2** For each \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that \( |F(x) - F(y)| \leq C_\delta |x - y| \) for any \( x, y \geq \delta \).

**Condition 2.3** There exist a constant \( K_0 > 0 \) such that \( F(x) \sim K_0 x^{-\theta_F} \) as \( x \to \infty \).

It is known that when \( S \) does not drift to infinity, with probability one it visits the negative half-line infinite times and hence \( I_\infty = \infty \) a.s. For the converse, when \( S \) drifts to \( \infty \), Erickson’s theorem in [20] shows that \( \lim_{n \to \infty} S_n/n \in (0, \infty) \) a.s. and hence \( I_\infty < \infty \) a.s. The first asymptotic result for exponential functionals of random walks is summarized as follows.
Lemma 2.4 The following assertions are equivalent:
(i) \( I_\infty < \infty \) a.s.;  
(ii) \( P(I_\infty < \infty) > 0 \);  
(iii) \( S \) drifts to \( \infty \), i.e., \( S_n \to \infty \) a.s. as \( n \to \infty \).

In this paper, we are mainly interested in the case \( I_\infty = \infty \) a.s., that is the random walk \( S \) will not drift to \( \infty \). As we have mentioned before, beside of the decay rate of \( F \) the long-term behavior of the expectation \( E[F(I_n)] \) also heavily depends on the speed at which the random walk \( S \) approaches to \( -\infty \). The fluctuation of \( S \) is closely related to the following crucial quantity:

\[
\varphi := \inf_{\lambda \in [0, \theta_F]} \mathcal{L}_X(\lambda).
\]

It is obvious that \( \varphi \leq 1 \) and the equality holds if and only if \( D^+_{\mathcal{L}_X} = \{0\} \) or \( E[X] = 0 \); see Figure 1. In the sequel of this paper we always make the following assumption:

Assumption 2.5 The infimum of \( \mathcal{L}_X(\lambda) \) over \( [0, \theta_F] \) can be attained, i.e. \( \Lambda \in D_{\mathcal{L}_X} \) in which \( \Lambda \) is defined in (1.2) and satisfies \( \mathcal{L}_X(\Lambda) = \varphi \).

According to the location of \( \Lambda \) in \( [0, \theta_F] \) and the asymptotic behavior of random walk \( S \) under \( P^{(\Lambda)} \), six regimes arise for the expectation \( E[F(I_n)] \); see Figure 1. In the next two subsections, we provide an explicit description for the asymptotic behavior of \( E[F(I_n)] \) in each regime.

2.2.1 The oscillating cases

We now provide the asymptotic results for the expectation \( E[F(I_n)] \) with the random walk \( S \) being oscillating under \( P^{(\Lambda)} \). We first consider the case in which \( \Lambda = 0 \) and the well-known Spitzer’s condition holds for \( S \); see Figure 1(a).
**Condition 2.6** There exists a constant \( \rho \in (0, 1) \) such that as \( n \to \infty \),

\[
\frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(S_k > 0) \to \rho.
\]

Spitzer’s condition is of key importance in fluctuation theory of random walks and it is equivalent to the convergence \( \mathbb{P}(S_n > 0) \to \rho \); see Theorem 1 in [18]. Specially, all symmetric random walks satisfy Spitzer’s condition with \( \rho = 1/2 \). To show the exact decay rate of the expectation \( \mathbb{E}[F(I_n)] \), we need the positive, slowly varying function \( \ell_1 \) defined by

\[
\ell_1(x) := \frac{1}{\Gamma(\rho)} \exp \left\{ \sum_{k=1}^{\infty} \frac{(1 - 1/x)^k}{k} \left( \mathbb{P}(S_k \geq 0) - \rho \right) \right\}, \quad x \geq 1,
\]

where \( \Gamma \) is the Gamma function.

**Theorem 2.7** If \( \theta_F > \Lambda = 0 \), under Condition 2.2 and 2.6, we have as \( n \to \infty \),

\[
\mathbb{E}[F(I_n)] \sim C_{F,1} \cdot \mathbb{P}(\tau_0^- > n) \sim C_{F,1} \cdot n^{\rho - 1} \ell_1(n),
\]

where the limit coefficient \( C_{F,1} \in (0, \infty) \) is given by

\[
C_{F,1} = \sum_{k=0}^{\infty} \mathbb{E}[F(I_k + e^{-S_k} \cdot I_{\infty}^-); \sigma_k^- = k].
\]

**Remark 2.8** By the inequality above (2.22) in [4] with \( \delta = \rho/2 > 0 \), we have \( S_n^\uparrow \geq Cn^{\rho/2} \) for some \( C > 0 \) and any \( n \geq 1 \), which directly induces \( I_{\infty}^\uparrow < \infty \) a.s. and then the positiveness of \( C_{F,1} \). Similarly, the other three random variables \( I_{\infty}^\downarrow, \hat{I}_{\infty}^\uparrow, \hat{I}_{\infty}^\downarrow \) are also finite almost surely.

We now turn to consider the asymptotics of the expectation \( \mathbb{E}[F(I_n)] \) with \( 0 < \Lambda < \theta_F \); see Figure 1(b). Let \( \mathbb{P}^{(A)} \) be a probability measure defined in (2.1) with \( \lambda_0 = \Lambda \). Using the change of measure, we have

\[
\mathbb{E}[F(I_n)] = e^{\rho \Lambda} \cdot \mathbb{E}^{(A)}[e^{-\Lambda S_n} F(I_n)],
\]

which indicates that different to the previous case, the long-term behavior of the expectation \( \mathbb{E}[F(I_n)] \) depend on not only the sample paths with slowly decreasing local minimum but also the sample paths with final value \( S_n \) below a low level. In order to get an exact description for the distribution of \( S_n \), we need the following condition.

**Condition 2.9** Under \( \mathbb{P}^{(A)} \) the random walk \( S \) is in the domain of attraction of a strictly stable law without centering with index \( \alpha \in (0, 2] \) and positivity parameter \( \rho \in (0, 1) \), we write \( S \in \mathcal{D}^{(A)}_{\alpha, \rho} \).

This condition is usually stronger than Spitzer’s condition. Indeed, if \( S \in \mathcal{D}^{(A)}_{\alpha, \rho} \) we have \( \mathbb{P}^{(A)}(S_n > 0) \to \rho \). For the converse, Spitzer’s condition usually does not imply a domain of attraction; readers may refer to [12, p.380] for more details. In particular, \( S \in \mathcal{D}^{(A)}_{\alpha, \rho} \) with \( \alpha \in (1, 2] \) if and only if Spitzer’s condition holds for \( S \) under \( \mathbb{P}^{(A)} \), and in this case we always have \( \rho = 1 - 1/\alpha \); see
Proposition 8.9.16 in [12, p.384]. Moreover, Condition 2.9 holds if and only if there exists a function $\ell_2$ that is slowly varying at $\infty$ such that under $P^{(\Lambda)},$

$$\frac{S_n}{n^{1/\ell_2(n)}} \to Y_\alpha$$

in distribution, where $Y_\alpha$ is a strictly stable random variable of parameter $\alpha$ and positivity parameter $\rho$. To simplify the following statements, we define

$$A_n := n^{-1-1/\ell_2(n)}, \quad n = 1, 2, \cdots. \tag{2.6}$$

The following quantities are necessary to give an exact decay rate for the expectation $E[F(I_n)]$. We write $\{g_\alpha(x) : x \in \mathbb{R}\}$ for the probability density function of $Y_\alpha$. The fact that $V^{(\Lambda)}(x) = O(x)$ and $\hat{V}^{(\Lambda)}(x) = O(x)$ as $x \to \infty$ induces that the following two integrals are finite:

$$L^{(\Lambda)}_{V^{(\Lambda)}}(y) := \int_0^y e^{-\lambda z} V^{(\Lambda)}(z) dz \quad \text{and} \quad L^{(\Lambda)}_{\hat{V}^{(\Lambda)}}(y) := \int_0^y e^{-\lambda z} \hat{V}^{(\Lambda)}(z) dz, \quad \lambda, y > 0.$$

Let $\mu^{(\Lambda)}_{V^{(\Lambda)}}(dx)$ and $\mu^{(\Lambda)}_{\hat{V}^{(\Lambda)}}(dx)$ be two probability measures on $\mathbb{R}_+$ defined by

$$\mu^{(\Lambda)}_{V^{(\Lambda)}}(dx) := \frac{e^{-\lambda x} V^{(\Lambda)}(x)}{L^{(\Lambda)}_{V^{(\Lambda)}}(\infty)} dx \quad \text{and} \quad \mu^{(\Lambda)}_{\hat{V}^{(\Lambda)}}(dx) := \frac{e^{-\lambda x} \hat{V}^{(\Lambda)}(x)}{L^{(\Lambda)}_{\hat{V}^{(\Lambda)}}(\infty)} dx. \tag{2.7}$$

**Theorem 2.10** If $\theta_F > \Lambda > 0$, under Condition 2.2 and 2.9, we have as $n \to \infty$,

$$E[F(I_n)] \sim C_{F,2} \cdot P(\tau_0^- > n) \sim C_{F,2} \cdot g_\alpha(0) L^{(\Lambda)}_{\hat{V}^{(\Lambda)}}(\infty) \cdot \sigma^- \cdot A_n, \tag{2.8}$$

where the limit coefficient$^2$ $C_{F,2} \in (0, \infty)$ is given by$^3$

$$C_{F,2} = \sum_{k=0}^{\infty} \left[ \int_0^\infty \mathbf{E}^{(\lambda)}_{(0,0,y)}[e^{-\Delta S_k} F(I_k + e^{-S_k}(I^\lambda_{\infty} + e^{-y} + I^\lambda_{\infty}))]; \sigma_k = k] \mu^{(\Lambda)}_{V^{(\Lambda)}}(dy) + \int_0^\infty \mathbf{E}^{(\lambda)}_{(0,0,z)}[e^{-\Delta S_k} F(e^z(1 + I_k + I^\lambda_{\infty}) + I^\lambda_{\infty})]; \tau_0^+ > k] \frac{e^{\Delta z} V^{(\Lambda)}(z)}{L^{(\Lambda)}_{\hat{V}^{(\Lambda)}}(\infty)} dz \right].$$

We now continue to consider the asymptotics of the expectation $E[F(I_n)]$ with $\theta_F = \Lambda > 0$; see Figure 1(c). In this case, we find that the impact of sample paths with final value below a low level on the expectation on the right side of (2.5) is more complicated than that in the preceding case. In precise, if the random walk ends up below a low level at time $n$, we observe that its local minimum over the time interval $[0, n]$ is not only approximately equal to the final value but also prefers be attained at the end of the time interval, and consequently its contribution to the expectation $E[F(I_n)]$ can be roughly represented as

$$e^{-\Delta S_n} F(I_n) = O(e^{-\Delta S_n} (I_n - I_{\sigma_n}^-)^{-\Lambda}) = O((n - \sigma_n^-)^{-\Lambda}).$$

These make it difficult to provide an exact description for the decay rate of the expectation $E[F(I_n)]$. To keep away from these difficulties, we consider the asymptotics of the expectation $E[F(I_n)]$ with $F$ satisfying Condition 2.3. Let $\hat{\ell}_1^{(\Lambda)}$ be a positive and slowly varying function defined by

$$\hat{\ell}_1^{(\Lambda)}(x) = \frac{1}{\Gamma(1 - \rho)} \exp \left\{ -\sum_{k=1}^{\infty} \frac{(1 - 1/x)^k}{k} \left( P^{(\Lambda)}(S_k > 0) - \rho \right) \right\}, \quad x \geq 1. \tag{2.9}$$

$^2$The positiveness of $C_{F,2}$ is a direct consequence of the finiteness of $I^\lambda_{\infty}$ and $I^\lambda_{\infty}$; see Remark 2.8.

$^3$.$E^{(\lambda)}_{(x,y,z)}$ is the expectation of $(S, S^1, S^2)$ under $P^{(\Lambda)}$ with initial state $(x, y, z)$. 

Theorem 2.11 If \( \theta_F = \Lambda > 0 \), under Condition 2.3 and 2.9, we have as \( n \to \infty \),

\[ \mathbb{E}[F(I_n)] \sim K_0 \cdot C_{F,3} \cdot q^n \cdot \mathbb{P}^{(\Lambda)}(\tau_0^+ > n) \sim K_0 \cdot C_{F,3} \cdot q^n \cdot n^{-\rho} \ell_1^{(\Lambda)}(n), \]

where the coefficient \( C_{F,3} \in (0, \infty) \) is given by

\[ C_{F,3} = \sum_{k=0}^{\infty} \mathbb{E}^{(\Lambda)}[(1 + \hat{I}_k + e^{-\hat{S}_k} \cdot \hat{I}_\infty^+) - \Lambda; \hat{\sigma}_k = k]. \]

2.2.2 The negative drift cases

We now provide the asymptotic results for the expectation \( \mathbb{E}[F(I_n)] \) in which under \( \mathbb{P}^{(\Lambda)} \) the random walk \( S \) has negative drift, i.e.,

\[ a := -\mathbb{E}^{(\Lambda)}[X] > 0. \tag{2.10} \]

We first consider the case with \( \Lambda = \theta_F > 0 \) and \( F \) satisfying Condition 2.3; see Figure 1(f). We notice that the expectation \( \mathbb{E}[F(I_n)] \) is asymptotically equivalent to \( K_0 \cdot \mathbb{E}[I_n^{\theta_F}] \). Using the change of measure and then the duality lemma, we have

\[ \mathbb{E}[I_n^{\Lambda}] \sim \mathbb{E}^{(\Lambda)}[(1 + \hat{I}_{n-1})^{-\Lambda}] \cdot q^n. \]

By Lemma 2.4, as \( n \to \infty \) we have \( \hat{I}_{n-1} \to \hat{I}_\infty < \infty \) a.s. under \( \mathbb{P}^{(\Lambda)} \) and hence \( \mathbb{E}^{(\Lambda)}[(1 + \hat{I}_\infty)^{-\Lambda}] > 0 \).

Theorem 2.12 If \( \Lambda = \theta_F > 0 \), under Condition 2.3, we have as \( n \to \infty \),

\[ \mathbb{E}[F(I_n)] \sim K_0 \cdot \mathbb{E}^{(\Lambda)}[(1 + \hat{I}_\infty)^{-\Lambda}] \cdot q^n. \]

We now provide the asymptotic results for the expectation \( \mathbb{E}[F(I_n)] \) with \( \theta_F > \Lambda > 0 \) and the distribution of generic step \( X \) always satisfying the following two regular variation assumptions:

Assumption 2.13 For some constant \( \beta > 1 \), the tail-probability \( \mathbb{P}^{(\Lambda)}(X > x) \) is regularly varying with index \( -\beta \), i.e., there exists a slowly varying function \( \ell_3(x) \) at \( \infty \) such that \( \mathbb{P}^{(\Lambda)}(X > x) \sim x^{-\beta} \ell_3(x) \) as \( x \to \infty \),

Assumption 2.14 For every \( \delta > 0 \), we have \( \mathbb{P}^{(\Lambda)}(X \in (x, x + \delta)) \sim \beta x^{-1 - \beta} \ell_3(x) \cdot \delta \) as \( x \to \infty \).

When \( \Lambda = 0 \); see Figure 1(d), we first observe that the random walk prefers to attain the local minimum around the first large step. Because of the negative drift, its local minimum will decrease slowly if there is a large step occurring at the beginning, otherwise it drifts to \( -\infty \) very fast and its contribution to the expectation \( \mathbb{E}[F(I_n)] \) can be asymptotically ignored. By analyzing the contribution of sample paths before and after the first large step separately, we give in the next theorem, the exact decay rate of the expectation \( \mathbb{E}[F(I_n)] \).

Theorem 2.15 If \( \theta_F > \Lambda = 0 \), assume \( F \) is non-increasing\(^4\) and satisfies Condition 2.2, we have as \( n \to \infty \),

\[ \mathbb{E}[F(I_n)] \sim \frac{C_{F,4}}{\mathbb{E}[\tau_0^{-}]} \cdot \mathbb{P}(\tau_0^{-} > n) \sim C_{F,4} \cdot \mathbb{P}(X \geq an), \tag{2.11} \]

where the coefficient \( C_{F,4} := \sum_{k=1}^{\infty} \mathbb{E}^{(\Lambda)}[C_{F,4}(k)] \in (0, \infty) \) with

\[ C_{F,4}(k) := \lim_{n \to \infty} \mathbb{E}\left[F(I_{k-1} + e^{-S_{k-1} - X \hat{I}_n}) \mid X \geq an, \mathcal{F}^S\right]. \]

\(^4\)Theorem 2.15 still holds if \( F := F^+ - F^- \) with \( F^+, F^- \) being bounded, non-increasing and satisfying Condition 2.2.
We now start to consider the case $\theta_F > \Lambda > 0$; see Figure 1(e). Similarly as in the asymptotic analysis for the previous case, we see that the slow decreasing of its local minimum also stems from the early arrival of a large step. Moreover, (2.5) shows that the expectation $\mathbf{E}[F(I_n)]$ is also heavily effected by the final value $S_n$. However, we observe that the random walk would stay above a high level for a long time after the large step and hence its final value is more likely above a high level, which gives rise to its meager contribution to the expectation $\mathbf{E}[F(I_n)]$. Consequently, the main contribution to the expectation $\mathbf{E}[F(I_n)]$ is made by the sample paths with not only an early large step but also small final value. In order to get an exact relationship between the local minimum and the final value, we need the following additional technic condition.

**Condition 2.16** Assume that $\beta \neq 2$ and $\mathbf{E}[|X|^\kappa] < \infty$ for some $\kappa \in (1, 2)$.

Under this condition, we have $n^{-1/\kappa}(S_n + an)$ converges to 0 in distribution as $n \to \infty$ by the Kolmogorov-Marcinkiewicz-Zygmund law of large numbers; see Theorem 10.3 in [23, p.311]. These tell that the final value $S_n$ falls into a bounded interval around 0 if and only if the size of the early large step is about $an$. In order to simplify the notation, we define the following sequence

$$B_n = \frac{\beta}{an} \mathbf{P}^{(A)}(X \geq an), \quad n = 1, 2, \ldots.$$  

(2.12)

**Theorem 2.17** If $\theta_F > \Lambda > 0$, under Condition 2.2 and 2.16, we have as $n \to \infty$,

$$\mathbf{E}[F(I_n)] \sim \frac{C_{F:5}}{L^{(A)}(\Lambda)} \mathbf{P}(\tau_0^- > n) \sim C_{F:5} \cdot q^n \cdot B_n,$$

where the coefficient $C_{F:5} := \sum_{k=1}^{\infty} C_{F:5}(k) \in (0, \infty)$ with

$$C_{F:5}(k) := \int_{-\infty}^{\infty} \mathbf{E}^{(A)}[e^{-\Lambda(S_k - \tilde{I}_\infty)}] \mathbf{F}^{(I_k + e^{-S_k - z}(1 + \tilde{I}_\infty))} dz$$

and $\tilde{I}_\infty$ being an independent copy of $\tilde{I}_\infty$.

3 Auxiliary results for random walks

In this section we list as well as generalize some well-known asymptotic results for random walks under conditions and assumptions introduced in Section 2 with $\Lambda = 0$, e.g., asymptotics of the first passage times, local probabilities conditioned to stay positive and conditional limit theorems.

**Remark 3.1** The right-continuity of $V$ and $\hat{V}$ yields that all the following claims still hold with $\tau_0^0 > n$ and $\tau_0^+ > n$ replaced by $L_n > 0$ and $M_n < 0$, respectively.

We first recall a useful asymptotic result for regularly varying sequences; readers can find it in [6].

**Lemma 3.2** Let $\{b_n\}$ be a regularly varying sequence. Consider two summable sequences $\{f_n : n = 0, 1, \cdots\}$ and $\{g_n : n = 0, 1, \cdots\}$ satisfying that $f_n \sim c_1 \cdot b_n$ and $g_n \sim c_2 \cdot b_n$ with $c_1, c_2 \geq 0$. We have as $n \to \infty$,

$$\sum_{k=0}^{n} f_{n-k} g_k \sim \left( \sum_{k=0}^{\infty} c_1 g_k + \sum_{k=0}^{\infty} c_2 f_k \right) \cdot b_n.$$
3.1 Oscillating random walk

The fluctuation theory for oscillating random walks satisfying Spitzer’s condition has been well developed and abundant results have been gotten. Here we list a part of them that will be used in our following proofs. Recall the two slowly varying functions \( \ell_1 \) and \( \hat{\ell}_1 \) defined in (2.2) and (2.9) with \( \Lambda = 0 \). A simple calculation shows that \( \ell_1(x)\hat{\ell}_1(x) \) converges to a positive constant as \( x \to \infty \).

The next lemma comes from Theorem 8.9.12 in [12, p.381] and Lemma 2.1 in [4].

**Lemma 3.3** Under Condition 2.6, for every \( x \geq 0 \) we have as \( n \to \infty \),

\[
\mathbb{P}_x(\tau_0^- > n) \sim V(x)n^{\theta-1}\ell_1(n) \quad \text{and} \quad \mathbb{P}_{-x}(\tau_0^+ > n) \sim \hat{V}(x)n^{-\rho}\hat{\ell}_1(n).
\]

Moreover, there exists a constant \( C > 0 \) such that for all \( x \geq 0 \) and \( n \geq 0 \),

\[
\mathbb{P}_x(\tau_0^- > n) \leq CV(x)n^{\theta-1}\ell_1(n) \quad \text{and} \quad \mathbb{P}_{-x}(\tau_0^+ > n) \leq C\hat{V}(x)n^{-\rho}\hat{\ell}_1(n).
\]

The following result is a direct consequence of the proof for Lemma 2.2 in [4] with \( u(x) = e^{-\lambda x} \).

**Lemma 3.4** Under Condition 2.6, we have for every \( \lambda > 0 \), both of the two sequences

\[
\mathbb{E}[e^{\lambda S_n} : \sigma_n^- = n] = \mathbb{E}[e^{\lambda S_n} : M_n < 0] \quad \text{and} \quad \mathbb{E}[e^{-\lambda S_n} : \sigma_n^+ = n] = \mathbb{E}[e^{-\lambda S_n} : L_n > 0]
\]

are summable and can be uniformly bounded by \( c_0/n \) for some constant \( c_0 > 0 \).

The first conditional limit theorem for random walks in the next lemma was proved by Bertoin and Doney [11] and the second one can be gotten immediately by using the duality lemma.

**Lemma 3.5** For \( k \geq 1 \), let \( f \) be a bounded function on \( \mathbb{R}^{k+1} \) and \( f(S) := f(S_0, S_1, \ldots, S_k) \). Under Condition 2.6, for any \( x \geq 0 \) we have as \( n \to \infty \),

\[
\mathbb{E}_x[f(S) \mid \tau_0^- > n] \to \mathbb{E}_x[f(S^\uparrow)] \quad \text{and} \quad \mathbb{E}_{-x}[f(S) \mid \tau_0^+ > n] \to \mathbb{E}_{-x}[f(\hat{S}^\uparrow)],
\]

where \( f(S^\uparrow) := f(S_0^\uparrow, S_1^\uparrow, \ldots, S_k^\uparrow) \) and \( f(\hat{S}^\uparrow) := f(\hat{S}_0^\uparrow, \hat{S}_1^\uparrow, \ldots, \hat{S}_k^\uparrow) \).

By Remark 2.8, we see that conditioned to stay positive the random walk \( S \) drifts to \( \infty \) a.s. To meet the needs of the following proofs for our main theorems, we provide in the following lemma, some large deviation estimates and uniform upper estimates for the final value of \( S \) conditioned to stay positive. It can be proved by slightly extending and modifying the proofs for Proposition 2.1 and Corollary 2.4 in [2], who considers the case with \( \alpha \in (1, 2] \).

**Lemma 3.6** Suppose \( S \in \mathcal{D}_{\alpha, \rho} \) with \( \alpha \in (0, 2] \) and \( \rho \in (0, 1) \). For any \( \lambda > 0 \), \( x \geq 0 \) and \( y \in [0, \infty] \), we have as \( n \to \infty \),

\[
\mathbb{E}_x[e^{\lambda S_n} : S_n \leq y, \tau_0^- > n] \sim g_\alpha(0)V(x)\mathcal{L}_V^{(\lambda)}(y) \cdot A_n,
\]

\[
\mathbb{E}_{-x}[e^{\lambda S_n} : S_n \geq -y, \tau_0^+ > n] \sim g_\alpha(0)\hat{V}(x)\mathcal{L}_V^{(\lambda)}(y) \cdot A_n.
\]

Moreover, there exists a constant \( C > 0 \) such that uniformly in \( x, y \geq 0 \) and \( n \geq 1 \),

\[
\mathbb{E}_x[e^{\lambda S_n} : S_n \leq y, \tau_0^- > n] \leq C \cdot V(x)\mathcal{L}_V^{(\lambda)}(y) \cdot A_n,
\]

\[
\mathbb{E}_{-x}[e^{\lambda S_n} : S_n \geq -y, \tau_0^+ > n] \leq C \cdot \hat{V}(x)\mathcal{L}_V^{(\lambda)}(y) \cdot A_n.
\]
Let $C_0(\mathbb{R}^2)$ be the space of continuous functions on $\mathbb{R}^2$ vanishing at infinity. The following lemma generalizes the conditional limit theorem from Lemma 10 in [25].

**Lemma 3.7** Suppose $S \in \mathcal{D}_{\alpha, \rho}$ with $\alpha \in (0, 2]$ and $\rho \in (0, 1)$. Let $G \in C_0(\mathbb{R}^2)$ and $f, g$ be two bounded, continuous functions on $\mathbb{R}^{k+1}$ and $\mathbb{R}^{m+1}$ respectively for some $k, m \geq 1$. Let $f(S) := f(S_0, S_1, \cdots, S_k)$ and $g_n(S) := g(S_n, S_{n-1}, \cdots, S_{n-m})$ for $n \geq m$. For every $x \geq 0$ and $\lambda > 0$ we have as $n \to \infty$,

$$\frac{E_x[G(f(S), g_n(S))e^{-\lambda S_n}; \tau_0 > n]}{E_x[e^{-\lambda S_n}; \tau_0 > n]} \to \int_0^{\infty} E_{(x, y)}[G(f(S^\uparrow), g(S^\uparrow))]\mu_\lambda(dy),$$

where $g(S^\uparrow) := g(S_0^\uparrow, S_1^\uparrow, \cdots, S_m^\uparrow)$, $g(\hat{S}^\uparrow) := g(S_0^\uparrow, \hat{S}_1^\uparrow, \cdots, \hat{S}_m^\uparrow)$, $f(S^\uparrow)$, $f(\hat{S}^\uparrow)$ are defined as in Lemma 3.5 and $E_{(x, y)}$ is the expectations of $(S^\uparrow, \hat{S}^\uparrow)$ or $(\hat{S}^\uparrow, S^\uparrow)$ with initial state $(x, y)$.

**Proof.** If $G(f(S), g_n(S)) = f(S)g_n(S)$, the desired results can be gotten immediately by repeating the proof for Lemma 10 in [25]. The general results follow by the Stone-Weierstrass theorem. \(\square\)

### 3.2 Random walk with negative drift

We now list the asymptotic results for random walks with negative drift satisfying Assumption 2.13 and 2.14. For every $x \geq 0$, we need the following important stopping time

$$T^x := \inf\{k \geq 1 : X_k > x\}$$

the first arrival of step with size larger than $x$. For each $k \geq 1$, a simple calculation together with Assumption 2.13 induces that the two events $T^x = k$ and $X_k \geq x$ are asymptotically equivalent as $x \to \infty$, i.e.,

$$P(T^x = k|X_k \geq x) \sim P(X_k \geq x|T^x = k) \to 1,$$

and hence $P(T^x \leq k) \sim kP(X \geq x)$. The following well-known asymptotic results for the tail-probabilities of the final value $S_n$ and the first passage time $\tau_0^-$ can be found in many literature, e.g., Theorem 5 in [6] and Theorem 2.2 in [16].

**Lemma 3.8** For every $x \geq 0$, we have as $n \to \infty$,

$$P(S_n \geq x) \sim nP(X \geq an) \quad \text{and} \quad P_x(\tau_0^- \geq n) \sim E_x[\tau_0^-]P(X \geq an).$$

By this lemma and the Markov property, for each integer $k \geq 0$ we have as $n \to \infty$,

$$P(\sigma_n^- = k) = P(\sigma_k^- = k)P(\tau_0^- > n - k) \sim P(\sigma_k^- = k)E[\tau_0^-]P(X \geq an).$$

The next lemma comes from Remark 3.7 in [34], which extends Theorem 3.2 in [19]. It shows that an early large step is necessary to keep the random walk staying positive for a long time.

**Lemma 3.9** For every integer $k \geq 1$, we have as $n \to \infty$,

$$P(T^{an} > n|\tau_0^- > n) \to 0 \quad \text{and} \quad P(T^{an} \leq k|\tau_0^- > n) \to E[\tau_0^- \land k]/E[\tau_0^-].$$
Moreover, we also observe that the random walk will stay above a high level for a long time after the first large step. This implies that the local minimum of the random walk should be attained near the early large step; see the next lemma and it can be proved by slightly modifying the proof for Lemma 4.6 in [34].

**Lemma 3.10** Let \( b \in (0, a] \) and integer \( k \geq 0 \). For every \( \epsilon > 0 \), there exist two integers \( n_0, t_0 \geq 1 \) such that for any \( T \geq t_0 \) and \( n \geq n_0 \),

\[
P(T^{bn} \geq T, \sigma_n^− = k) \leq \epsilon \cdot P(X \geq an).
\]

**Proof.** By the duality lemma, for \( n > T > k \) we have

\[
P(T^{bn} \geq T, \sigma_n^− = k) \leq P(T^{an} \geq T - k, \tau_0^− = n - k) \cdot P(\sigma_n^− = k) \\
\leq P(T^{an} \geq T - k | \tau_0^− = n - k) \cdot P(\tau_0^− = n - k).
\]

Using Lemma 3.8 and then Lemma 3.9, we have for some \( C > 0 \),

\[
\limsup_{n \to \infty} \frac{P(T^{bn} \geq T, \sigma_n^− = k)}{P(X \geq an)} \leq C \cdot (1 - E[\tau_0^− \land (T - k)]/E[\tau_0^−]),
\]

which goes to 0 as \( T \to \infty \). \( \square \)

Recall the sequence \( B_n \) defined in (2.12). The next lemma provides some asymptotic results for joint local probabilities of the local maximum/minimum \((M_n/L_n)\) and the final value \((S_n)\). Its proof is similar to that of Lemma 3.13 and 3.14 in [34].

**Lemma 3.11** For any \( \lambda_1, \lambda_2 > 0 \) and \( x, y \in [0, \infty] \), we have as \( n \to \infty \),

\[
E[e^{-\lambda_1 M_n - \lambda_2 (M_n - S_n)}; M_n \leq x, M_n - S_n \leq y] \sim \mathcal{L}^{(\lambda_1)}_{\mathcal{V}}(x) \mathcal{L}^{(\lambda_2)}_{\mathcal{Y}}(y) \cdot B_n,
\]

\[
E[e^{\lambda_1 L_n + \lambda_2 (L_n - S_n)}; -L_n \leq x, S_n - L_n \leq y] \sim \mathcal{L}^{(\lambda_1)}_{\mathcal{V}}(x) \mathcal{L}^{(\lambda_2)}_{\mathcal{Y}}(y) \cdot B_n.
\]

**Proof.** Here we just provide a brief proof for the first desired result. Similarly as in the proof for (12) in [32], we have as \( n \to \infty \),

\[
E[e^{-\lambda_1 S_n}; S_n > 0] \sim E[e^{-\lambda_2 S_n}; S_n \leq 0] \sim \frac{1}{a} \cdot P(X > an).
\]

By the factorization identity of random walk; see Theorem 8.9.1 and 8.9.3 in [12, p.376-377],

\[
1 + \sum_{n=1}^{\infty} s^n E[e^{-\lambda_1 M_n - \lambda_2 (M_n - S_n)}] = \exp \left\{ \sum_{n=1}^{\infty} \frac{s^n}{n} (E[e^{-\lambda_1 S_n}; S_n > 0] + E[e^{\lambda_2 S_n}; S_n \leq 0]) \right\}
\]

for \( s \in (0, 1) \) and

\[
\sum_{n=0}^{\infty} E[e^{-\lambda_1 M_n - \lambda_2 (M_n - S_n)}] = \sum_{n=0}^{\infty} E[e^{-\lambda_1 S_n}; \sigma_n^− = n] \cdot \sum_{n=0}^{\infty} E[e^{-\lambda_2 S_n}; \sigma_n^− = 0] = \mathcal{L}^{(\lambda_1)}_{\mathcal{V}}(\infty) \mathcal{L}^{(\lambda_2)}_{\mathcal{Y}}(\infty).
\]

Applying Lemma 2.2.(2) in [17] to the foregoing equations, we have as \( n \to \infty \),

\[
E[e^{-\lambda_1 M_n - \lambda_2 (M_n - S_n)}] \sim \sum_{k=0}^{\infty} E[e^{-\lambda_1 M_k - \lambda_2 (M_k - S_k)}] \cdot B_n \sim \mathcal{L}^{(\lambda_1)}_{\mathcal{V}}(\infty) \mathcal{L}^{(\lambda_2)}_{\mathcal{Y}}(\infty) \cdot B_n.
\]
which induces that for any \( x, y \in (0, \infty) \),
\[
P(M_n \leq x, M_n - S_n \leq y) \sim \int_0^x V(z_1)dz_1 \int_0^y \hat{V}(z_2)dz_2 \cdot B_n
\]
as \( n \to \infty \) and the desired result follows. \( \square \)

The next lemma gives several large deviation estimates for the random walk \( S \) conditioned to stay positive/negative. It comes from Remark 3.16 in [34] and can be proved like the previous lemma with the help of the Baxter identity
\[
1 + \sum_{n=1}^{\infty} s^n \mathbb{E}[e^{\lambda S_n}; \tau_0^+ > n] = \exp \left\{ \sum_{n=1}^{\infty} \frac{s^n}{n} \mathbb{E}[e^{\lambda S_n}; S_n < 0] \right\}, \quad s \in (0, 1),
\]
which can be found in Chapter XVIII.3 in [21] or Chapter 8.9 in [12]. Here we omit the detailed proof.

**Lemma 3.12** For any \( x, \lambda > 0 \) and \( y \in [0, \infty] \), we have as \( n \to \infty \),
\[
\mathbb{E}_x[e^{-\lambda S_n}; S_n \leq y, \tau_n^- > n] \sim V(x)\mathcal{L}_V^{(\lambda)}(y) \cdot B_n,
\]
\[
\mathbb{E}_{-x}[e^{\lambda S_n}; S_n \geq -y, \tau_n^+ > n] \sim \hat{V}(x)\mathcal{L}_V^{(\lambda)}(y) \cdot B_n.
\]

4 Proof for Theorems 2.7-2.11

In this section we prove the asymptotic results for the expectation \( \mathbb{E}[F(I_n)] \) with the random walk \( S \) being oscillating under \( P^{(\Lambda)} \). Although the technical difficulties in the following proofs vary in different cases, one may find that it is the common key point to identify that sample paths with slowly decreasing local infimum make the main contribution to the expectation \( \mathbb{E}[F(I_n)] \). For simplicity, we may always assume \( \theta_F \in \mathcal{D}_F \). All the following proofs still work with \( \theta_F \) replaced by any \( \theta \in \mathcal{D}_F \cap (\Lambda, \infty) \).

4.1 Proof for Theorem 2.7

Under Condition 2.6, we first notice that sample paths with slowly decreasing local infimum would attain the local minimum at the beginning of the time interval. And then, we prove in the next proposition, that the contribution of sample paths with the local minimum being late attained to the expectation \( \mathbb{E}[F(I_n)] \) can be asymptotically ignored.

**Proposition 4.1** For every \( \epsilon > 0 \), there exist two integers \( k_0, n_0 \geq 1 \) such that for any \( K \geq k_0 \) and \( n \geq n_0 \),
\[
\mathbb{E}[F(I_n); \sigma_n^- \geq K] \leq \epsilon \cdot \mathbb{P}(\tau_0^- > n).
\]
Moreover, there exists a constant \( C > 0 \) such that for any \( n \geq 0 \),
\[
\mathbb{E}[F(I_n)] \leq C \cdot \mathbb{P}(\tau_0^- > n).
\]
Proof. By Assumption 2.1, we first have $F(I_n) \leq C \cdot \exp\{\theta_F L_n\}$ for some $C > 0$. Using the Markov property of $S$, we have

$$E[F(I_n); \sigma_n^- \geq K] \leq C \sum_{k=K}^{n} E[e^{\theta_F S_k}; \sigma_k^- = k] \cdot P(\tau_0^- > n - k).$$

Applying Lemma 3.2 together with Lemma 3.3 and 3.4 to the foregoing partial sum, we have for large $n$,

$$E[F(I_n); \sigma_n^- \geq K] \leq C \sum_{k=K}^{\infty} E[e^{\theta_F S_k}; \sigma_k^- = k] \cdot P(\tau_0^- > n).$$

The desired two claims follow directly from the summability of the sequence $E[e^{\theta_F S_k}; \sigma_k^- = k]$; see Lemma 3.4, and the fact that $E[F(I_n); \sigma_n^- = 0] \leq C P(\sigma_n^- = 0) \sim C P(\tau_0^- > n)$. □

We now consider the contribution of sample paths with the local minimum attained at the beginning of the time interval, i.e., $E[F(I_n); \sigma_n^- = k]$ for each fixed $k \geq 0$. Using the Markov property of $S$ and then the duality lemma, we have

$$E[F(I_n); \sigma_n^- = k] = E[E[F(I_k + e^{-S_k} \cdot \tilde{I}_{n-k}); \tau_0^- > n - k | \mathcal{F}^S]; \sigma_k^- = k]. \quad (4.1)$$

Thus it is a crucial step to analyze the asymptotics of the foregoing conditional expectation.

**Proposition 4.2** Let $H$ be a bounded and Lipschitz continuous function on $(0, \infty)$. For every $\epsilon > 0$, there exist two integers $k_0, n_0 \geq 1$ such that for any $K \geq k_0$ and $n \geq n_0$,

$$E[|H(I_n) - H(I_K)|; \tau_0^- > n] = E[|H(I_n) - H(I_K)|; L_n \geq 0] \leq \epsilon \cdot P(\tau_0^- > n).$$

Proof. By Remark 3.1, the boundedness of $H$ induces that $E[|H(I_n) - H(I_K)|; L_n = 0] = o(P(\tau_0^- > n))$ uniformly in $K$ as $n \to \infty$ and hence it suffices to prove

$$E[|H(I_n) - H(I_K)|; L_n > 0] \leq \epsilon \cdot P(\tau_0^- > n).$$

By the Lipschitz continuity of $H$, there exists a constant $C > 0$ such that

$$E[|H(I_n) - H(I_K)|; L_n > 0] \leq C \sum_{k=K+1}^{n} E[e^{-S_k}; L_n > 0]. \quad (4.2)$$

By the Markov property of $S$, we see that $E[e^{-S_k}; L_n > 0]$ can be bounded by

$$E[e^{-S_k} P_{S_k}(\tilde{L}_{n-k} > 0); L_k > 0].$$

By the second result in Lemma 3.3, we have $P_{S_k}(\tilde{L}_{n-k} > 0) \leq CV(S_k) P(L_{n-k} > 0)$ and hence

$$E[e^{-S_k}; L_n > 0] \leq CE[e^{-S_k} V(S_k); L_k > 0] \cdot P(L_{n-k} > 0). \quad (4.3)$$

The fact that $V(x) = O(x)$ as $x \to \infty$ shows $V(x)e^{-x} \leq C e^{-x/2}$ for all $x \geq 0$. From this and Lemma 3.4, we have for large $k$,

$$E[e^{-S_k} V(S_k); L_k > 0] = o(P(\tau_0^- > k)).$$
Taking this and (4.3) back into (4.2) and then using Lemma 3.2, we have for large $n$,
\[
\mathbb{E}[|H(I_n) - H(I_K)|; L_n > 0] \leq C \sum_{k=K}^{\infty} \mathbb{E}[e^{-\theta F S_k/2}; \sigma_k^- = k] \cdot \mathbb{P}(\tau_0^- > n),
\]
Here the summation above vanishes as $K \to \infty$; see Lemma 3.4, and the desired result follows immediately.

**Proposition 4.3** Let $H$ be a bounded function on $(0, \infty)$. For $K \geq 1$, we have
\[
\lim_{n \to \infty} \mathbb{E}[H(I_K) \mid \tau_0^- > n] = \mathbb{E}[H(I_K^\uparrow)] > 0.
\]
Moreover, the sequence $\mathbb{E}[H(I_K^\uparrow)]$ converges to $\mathbb{E}[H(I_K^\infty)] > 0$ as $K \to \infty$.

**Proof.** The first claim follows from Lemma 3.5. The second one follows from the dominated convergence theorem and the fact that $I_K^\infty$ increases to $I_K^\infty < \infty$ a.s. as $K \to \infty$; see Remark 2.8. □

**Proof for Theorem 2.7.** We first prove this theorem with $F$ being globally Lipschitz continuous on $(0, \infty)$. By Proposition 4.1,
\[
\lim_{n \to \infty} \frac{\mathbb{E}[F(I_n)]}{\mathbb{P}(\tau_0^- \geq n)} = \lim_{K \to \infty} \lim_{n \to \infty} \frac{\mathbb{E}[F(I_n); \sigma_n^- \leq K]}{\mathbb{P}(\tau_0^- \geq n)} = \sum_{k=0}^{\infty} \lim_{n \to \infty} \frac{\mathbb{E}[F(I_n); \sigma_n^- = k]}{\mathbb{P}(\tau_0^- \geq n)}. \tag{4.4}
\]
Applying Proposition 4.2 to the conditional expectation on the right side of (4.1) and then using the dominated convergence theorem, we have
\[
\lim_{n \to \infty} \frac{\mathbb{E}[F(I_n); \sigma_n^- = k]}{\mathbb{P}(\tau_0^- \geq n)} = \lim_{K \to \infty} \lim_{n \to \infty} \frac{\mathbb{E}[F(I_k + e^{-S_k} \cdot \tilde{I}_K); \tilde{\tau}_0^- > n - k \mid \mathcal{F}^S]; \sigma_k^- = k]}{\mathbb{P}(\tau_0^- \geq n)}
\]
\[
= \lim_{n \to \infty} \frac{\mathbb{P}(\tilde{\tau}_0^- \geq n-k)}{\mathbb{P}(\tau_0^- \geq n)} \cdot \mathbb{E}\left[\lim_{K \to \infty} \lim_{n \to \infty} \frac{\mathbb{E}[F(I_k + e^{-S_k} \cdot \tilde{I}_K); \tilde{\tau}_0^- > n-k, \mathcal{F}^S]; \sigma_k^- = k]}{\mathbb{P}(\tau_0^- \geq n)}\right]. \tag{4.5}
\]
From Lemma 3.3, we see the first limit on the right side of the second equality equals to 1. Applying Proposition 4.3 with $H(x) = F(I_k + e^{-S_k} \cdot x)$ to the second limit, we also have
\[
\lim_{K \to \infty} \lim_{n \to \infty} \mathbb{E}[F(I_k + e^{-S_k} \cdot \tilde{I}_K); \tilde{\tau}_0^- > n-k, \mathcal{F}^S] = \mathbb{E}[F(I_k + e^{-S_k} \cdot I_k^\uparrow) \mid \mathcal{F}^S].
\]
Taking this back into (4.5) and then (4.4), we have
\[
\lim_{n \to \infty} \frac{\mathbb{E}[F(I_n); \sigma_n^- = k]}{\mathbb{P}(\tau_0^- \geq n)} = \mathbb{E}[F(I_k + e^{-S_k} \cdot I_k^\infty) \mid \mathcal{F}^S]; \sigma_k^- = k > 0.
\]
Hence the desired asymptotic equivalences in (2.3) hold and the limit coefficient $C_{F,1}$ is finite because of Proposition 4.1. For general $F$ satisfying Condition 2.2 and $\delta > 0$, we define $F_\delta(x) = F(x \vee \delta)$, which is globally Lipschitz continuous. By Chebyshev’s inequality,
\[
\mathbb{E}[|F(I_n) - F_\delta(I_n)|] \leq 2C \cdot \mathbb{P}(I_n \leq \delta) \leq 2C \cdot \mathbb{P}(\sigma_n^- = 0) + 2C \delta \cdot \mathbb{E}[I_n^{-1}; \sigma_n^- \geq 1].
\]
By the duality lemma, we have $\mathbb{P}(\sigma_n^- = 0) \sim \mathbb{P}(\tau_0^- \geq n)$ for large $n$. Similarly as in Proposition 4.1, we also can prove $\mathbb{E}[I_n^{-1}; \sigma_n^- \geq 1] \leq C \cdot \mathbb{P}(\tau_0^- \geq n)$ for any $n \geq 1$ and hence
\[
\lim_{n \to \infty} \frac{\mathbb{E}[F(I_n)]}{\mathbb{P}(\tau_0^- \geq n)} = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\mathbb{E}[F_\delta(I_n)]}{\mathbb{P}(\tau_0^- \geq n)}.
\]
The preceding result shows that $\mathbb{E}[F_\delta(I_n)]/\mathbb{P}(\tau_0^- \geq n) \to C_{F_\delta,1} \in (0, \infty)$ as $n \to \infty$, where $C_{F_\delta,1}$ can be represented as the summation in (2.4) with $F$ replaced by $F_\delta$. Specially, when $F(x) = Cx^{-\theta F}$ for any $C > 0$, the monotone convergence theorem induces that $C_{F_\delta,1} \to C_{F,1}$ as $\delta \to 0^+$. This can be extended to the general $F$ satisfying Assumption 2.1 by the dominated convergence theorem. □
4.2 Proof for Theorem 2.10

By (2.5), we observe that the asymptotics of the expectation \(E^{(A)}[F(I_n)e^{-\Lambda S_n}]\) is mainly determined by sample paths with either the local minimum attained at the beginning of the time interval or the final value below a low level. Moreover, we also observe that if the random walk ends up at time \(n\) below a low level, it will tend to attain the local minimum at the end of the time interval \([0, n]\). Precisely, the next proposition proves that the contribution of sample paths with the local minimum attained at neither the beginning nor the end of the time interval \([0, n]\) to the expectation \(E[F(I_n)]\) can be asymptotically ignored.

**Proposition 4.4** For every \(\epsilon > 0\), there exist two integers \(k_0, n_0 \geq 1\) such that for any \(K \geq k_0\) and \(n \geq n_0\),

\[
E[F(I_n); \sigma_n^* \in [K, n - K]] \leq \epsilon \cdot P(\tau_0^- > n).
\]

Moreover, there exists a constant \(C > 0\) such that for any \(n \geq 1\),

\[
E[F(I_n)] \leq C \cdot P(\tau_0^- > n).
\]

**Proof.** By the change of measure given in (2.5), it suffices to prove

\[
E^{(A)}[F(I_n)e^{-\Lambda S_n}; \sigma_n^* \in [K, n - K]] \leq \epsilon \cdot E^{(A)}[e^{-\Lambda S_n}; \tau_0^- > n].
\]

By Assumption 2.1 we have \(E^{(A)}[F(I_n)e^{-\Lambda S_n}; \sigma_n^* = k] \leq C \cdot E^{(A)}[e^{\theta_F S_k - \Lambda S_n}; \sigma_n^- = k]\) for any \(k \geq 1\). Using the Markov property, the independent increments of \(S\) and then the duality lemma, we have

\[
E^{(A)}[e^{\theta_F S_k - \Lambda S_n}; \sigma_n^- = k] = E^{(A)}[e^{\theta_F S_k}; \sigma_k^- = k] \cdot E^{(A)}[e^{-\Lambda S_{n-k}}; \tau_0^- > n - k]
\]

and hence \(E^{(A)}[F(I_n)e^{-\Lambda S_n}; \sigma_n^- \in [K, n - K]]\) can be bounded by

\[
C \sum_{k=K}^{n-K} E^{(A)}[e^{\theta_F S_k}; \tau_0^+ > k] \cdot E^{(A)}[e^{-\Lambda S_{n-k}}; \tau_0^- > n - k].
\]

Recall the sequence \(A_n\) defined in (2.6). Using the first asymptotic equivalence in Lemma 3.6 with \(\lambda = \Lambda, y = \infty\), the second one with \(\lambda = \theta_F - \Lambda, y = \infty\) and then Lemma 3.2 to the foregoing summation, we have for some \(C > 0\) and large \(n\),

\[
E^{(A)}[F(I_n)e^{-\Lambda S_n}; \sigma_n \in [K, n - K]] \leq C \cdot A_n \sum_{k=K}^{\infty} \left( E^{(A)}[e^{\theta_F S_k}; \tau_0^+ > k] + E^{(A)}[e^{-\Lambda S_k}; \tau_0^- > k] \right).
\]

Using Lemma 3.6 again, the last summation vanishes as \(K \to \infty\) and then the first claim follows. The second claim follows from the first one and the fact that \(E^{(A)}[F(I_n)e^{-\Lambda S_n}; \sigma_n^- = 0] \leq CE^{(A)}[e^{-\Lambda S_n}; \tau_0^- > n]\).

We now start to consider the contribution of sample paths with the local minimum attained early to the expectation \(E[F(I_n)]\), i.e. \(E[F(I_n); \sigma_n^* = k]\) for \(k \geq 0\). As in the proof for Theorem 2.7, we first consider the conditional expectation on the right side of (4.1). Different to Proposition 4.2, the next proposition shows that conditioned on \(\tau_0^- > n\) for large \(n\), the exponential functional \(I_n\) can be well approximated by the sum of \(I_J\) and \(I_{n-J,n} := I_n - I_{n-J}\) for large \(J < n/2\).
Proposition 4.5 Let $H$ be a bounded and Lipschitz continuous function. For every $\epsilon > 0$, there exist two integers $j_0, n_0 \geq 0$ such that for any $J \geq j_0$ and $n \geq n_0$,
\[
E[|H(I_n) - H(I_J + I_{n-J,n})|; \tau_0^- > n] \leq \epsilon \cdot P(\tau_0^- > n).
\]

Proof. By the Lipschitz continuity of $H$, we first have
\[
E[|H(I_n) - H(I_J + I_{n-J,n})| | \tau_0^- > n] \leq \sum_{j=J+1}^{n-J} E[e^{-S_j | \tau_0^- > n}].
\]
(4.6)

By the change of measure, we have
\[
E[e^{-S_j | \tau_0^- > n}] = \frac{E[e^{-S_j; \tau_0^- > n}]}{P(\tau_0^- > n)} = \frac{E^{(A)}[e^{-S_j - \Lambda S_n; \tau_0^- > n}]}{E^{(A)}[e^{-\Lambda S_n; \tau_0^- > n}]}.
\]
(4.7)

Using the Markov property and the independent increments of $S$ to the numerator of the last fraction, we have
\[
E^{(A)}[e^{-S_j - \Lambda S_n; \tau_0^- > n}] = E^{(A)}[e^{-S_j}; \tilde{\tau}_0^- > n-j; \tau_0^- > j].
\]

Together with the fact that $V(x) = O(x)$ as $x \to \infty$, the second claim in Lemma 3.6 implies that there exists a constant $C > 0$ such that
\[
E^{(A)}_{S_j}[e^{-\Lambda S_n-j; \tilde{\tau}_0^- > n-j}] \leq C \cdot S_j \cdot A_{n-j}
\]
and hence by the fact that $xe^{-x} \leq Ce^{-x/2}$ for some $C > 0$ and any $x \geq 0$,
\[
E^{(A)}[e^{-S_j - \Lambda S_n; \tau_0^- > n}] \leq C E^{(A)}[e^{-S_j/2; \tau_0^- > j}] \cdot A_{n-j} \leq C \cdot A_j A_{n-j}.
\]

Taking this back into (4.7) and then using the first claim in Lemma 3.6 with $y = \infty$, we have for large $n$,
\[
E[e^{-S_j | \tau_0^- > n}] \leq \frac{C \cdot A_j A_{n-j}}{A_n}.
\]

Taking this back into (4.6) and then using Lemma 3.2, we have for some $C > 0$ and large $n$,
\[
E[|H(I_n) - H(I_J + I_{n-J,n})| | \tau_0^- > n] \leq C \sum_{j=J+1}^{\infty} A_j,
\]
which goes to 0 as $J \to \infty$ and the desired result follows. □

Proposition 4.6 Recall $\mu^{(A)}_{\hat{V}(\lambda)}(dy)$ defined in (2.7) with $\lambda = \Lambda$. Let $H$ be a bounded and continuous function on $(0, \infty)$ vanishing at infinity. For each $J \geq 1$, we have as $n \to \infty$,
\[
E[H(I_J + I_{n-J,n}) | \tau_0^- > n] \to \int_0^{\infty} E^{(A)}_{(0,y)}[H(I_J^+ + e^{-y} + I_{J-1}^+)] \mu^{(A)}_{\hat{V}(\lambda)}(dy).
\]

Moreover, the limit coefficient converges as $J \to \infty$ to a finite limit given by
\[
\int_0^{\infty} E^{(A)}_{(0,y)}[H(I_{\infty}^+ + e^{-y} + I_{\infty}^+)] \mu^{(A)}_{\hat{V}(\lambda)}(dy).
\]
Proof. By the change of measure
\[
E[H(I_J + I_{n-J,n}) \mid \tau_0^- > n] = \frac{E^{(\Lambda)}[H(I_J + I_{n-J,n})e^{-\Lambda S_\nu}; \tau_0^- > n]}{E^{(\Lambda)}[e^{-\Lambda S_\nu}; \tau_0^- > n]}.
\]
Applying Lemma 3.7 to the foregoing expectations with \(G(x_1, x_2) = H(x_1 + x_2)\), \(f(S) = I_J\) and \(g_n(S) = I_{n-J,n}\), we can get the desired convergence immediately. The second claim follows by the dominated convergence theorem and the fact that \(I_{\infty}^k, I_{\infty}^k < \infty\) a.s. under \(P^{(\Lambda)}\); see Remark 2.8. \(\square\)

**Corollary 4.7** For each \(k \geq 0\), we have as \(n \to \infty\),
\[
\frac{E[F(I_n) ; \sigma_n^- = k]}{P(\tau_0^- > n)} \to \int_0^\infty E^{(\Lambda)}_{(0,0,y)}[e^{-\Lambda S_k} F(I_k + e^{-S_k} (I_{\infty}^k + e^{-y} + I_{\infty}^k)); \sigma_k^- = k] \mu^{(\Lambda)}_{\hat{V}^{(\Lambda)}}(dy).
\]

Proof. Applying Proposition 4.5 to the conditional expectation on the right side of (4.1), we have
\[
\lim_{n \to \infty} \frac{E[F(I_k + e^{-S_k} \cdot \tilde{I}_{n-k}) ; \tilde{\tau}_0^- > n - k \mid \mathcal{F}^S]}{P(\tau_0^- > n)} = \lim_{n \to \infty} \frac{E[F(I_k + e^{-S_k} \cdot (\tilde{I}_J + \tilde{I}_{n-k-J,n-k}) ; \tilde{\tau}_0^- > n - k \mid \mathcal{F}^S]}{P(\tau_0^- > n)}
\]
By the change of measure and Lemma 3.6 with \(y = \infty\), the first limit on the right side of the last equality equals to \(\varrho^k\). By Proposition 4.6 with \(H(x) = F(I_k + e^{-S_k}x)\), we have the second limit equals to
\[
\int_0^\infty E^{(\Lambda)}_{(0,0,y)}[F(I_k + e^{-S_k} (I_{\infty}^k + e^{-y} + I_{\infty}^k)) \mid \mathcal{F}^S] \mu^{(\Lambda)}_{\hat{V}^{(\Lambda)}}(dy).
\]
Taking these back into (4.1) and then using the change of measure, we have
\[
\lim_{n \to \infty} \frac{E[F(I_n) ; \sigma_n^- = k]}{P(\tau_0^- > n)} = \varrho^k \int_0^\infty E^{(\Lambda)}_{(0,0,y)}[F(I_k + e^{-S_k} (I_{\infty}^k + e^{-y} + I_{\infty}^k)) \mid \mathcal{F}^S] \mu^{(\Lambda)}_{\hat{V}^{(\Lambda)}}(dy) ; \sigma_k^- = k\]
and the desired result follows by Fubini’s theorem. \(\square\)

We now start to consider the impact of sample paths with the local minimum attained at the end of the time interval on the expectation \(E[F(I_n)]\). Applying the Markov property to \(E[F(I_n) ; \sigma_n^- = n - k]\) for \(n > k \geq 0\), we see it equals to
\[
E[E[F(I_{n-k} + e^{-S_{n-k}} \tilde{I}_k) ; \sigma_{n-k}^- = n - k \mid \mathcal{F}^\tilde{S}] ; \tilde{\tau}_0^- > k].
\]
Like the previous argument, we first need to consider the asymptotics of the conditional expectation. In the next proposition, we show that it is rarely contributed by sample paths with final value below a very low level.
Applying the duality lemma to the expectation on the left side of this inequality, we see it equals to

$$E[F(I_n + xe^{-S_n}); S_n < -y, \sigma_n^- = n] \leq \epsilon \cdot P(\tau_0^- > n).$$

Proof. By the change of measure, we see that the desired inequality holds if and only if

$$E^{(A)}[e^{-\Lambda S_n} F(I_n + xe^{-S_n}); S_n < -y, \sigma_n^- = n] \leq \epsilon \cdot E^{(A)}[e^{-\Lambda S_n}; \tau_0^- > n].$$

Applying the duality lemma to the expectation on the left side of this inequality, we see it equals to

$$E^{(A)}[e^{-\Lambda S_n} F(e^{-S_n}(1 + \hat{I}_{n-1} + x)); S_n < -y, M_n < 0],$$

which can be bounded by $CE^{(A)}[e^{(\theta_p - \Lambda)S_n}; S_n < -y, M_n < 0]$ because of the assumption that $F(x) \leq C x^{\theta_p}$. From Lemma 3.6, there exists a constant $C > 0$ such for any $n \geq 1$,

$$\lim_{n \to \infty} \frac{E^{(A)}[e^{(\theta_p - \Lambda)S_n}; S_n < -y, M_n < 0]}{E^{(A)}[e^{-\Lambda S_n}; \tau_0^- > n]} \leq C \cdot |\mathcal{L}_V^{(\theta_p - \Lambda)}(\infty) - \mathcal{L}_V^{(\theta_p - \Lambda)}(y)|,$$

which goes to 0 as $y \to \infty$ and hence the desired result follows.

Proposition 4.9 Suppose $F$ is globally Lipschitz continuous on $(0, \infty)$. For each $x, y \geq 0$ and $\epsilon > 0$, there exist two integers $j_0, n_0 > 1$ such that for any $J \geq j_0$ and $n \geq n_0$,

$$E[[F(I_n + xe^{-S_n}) - F(I_J + I_{n-J,n} + xe^{-S_n})]; S_n \geq -y, \sigma_n^- = n] \leq \epsilon \cdot P(\tau_0^- > n).$$

Proof. By the Lipschitz continuity of $F$, the expectation on the left side of (4.9) can be bounded by

$$CE[I_{J+1,n-J}; S_n \geq -y, \sigma_n^- = n] = C \sum_{j=J+1}^{n-J} E[e^{-S_j}; S_n \geq -y, \sigma_n^- = n].$$

Applying the duality lemma and then the change of measure to $E[e^{-S_j}; S_n \geq -y, \tau_0^n > n]$, we see that it equals to

$$E[e^{\nu S_n - \gamma S_n}; S_n \geq -y, M_n < 0] = E^{(A)}[e^{\nu S_n - (1+\Lambda)S_n}; S_n \geq -y, M_n < 0] \cdot \rho^n,$$

which can be bounded by $e^{(\kappa+1+\Lambda)\nu}E^{(A)}[e^{\nu S_n - \gamma S_n}; M_n < 0] \cdot \rho^n$ for any $\kappa > 0$. By the Markov property and the second claim in Lemma 3.6, there exists a constant $C > 0$ such that

$$E^{(A)}[e^{\nu S_n - \gamma S_n}; M_n < 0] = E^{(A)}[e^{\nu S_n - \gamma S_n}; M_n < 0] \cdot \tilde{M}_J < 0] \cdot \rho^n \leq CE^{(A)}[e^{\nu S_n - \gamma S_n}; M_n < 0] \cdot A_j \leq CA_j A_{n-J}.$$

Taking this and (4.11) back into (4.10) and then using Lemma 3.6 with $\lambda = 0$ and $y = \infty$, we have for large $n$,

$$\frac{E[I_{J+1,n-J}; S_n \geq -y, \sigma_n^- = n]}{P(\tau_0^- > n)} \leq \frac{C}{A_n} \sum_{j=J+1}^{n-J} A_j A_{n-j},$$

which is asymptotically equivalent to $2C \sum_{j=J}^{\infty} A_j$ as $n \to \infty$; see Lemma 3.2. Hence the desired result follows as $J \to \infty$ because of the summability of the sequence $A_n$. 

□
Proposition 4.10  For any \(x, y \geq 0\) and \(J \geq 1\), we have as \(n \to \infty\),
\[
\mathbb{E}[F(I_J + I_{n-J,n} + xe^{-S_n}); S_n \geq -y, \sigma^-_n = n] \\
\to \int_0^y \mathbb{E}^{(A)}_{(0,z)}[F(e^{z(1 + x + I_{J-1} + I_{J-1}^\dagger})_n)] \frac{e^{\lambda z V(\Lambda)(z)}}{L(\Lambda)(\infty)} dz.
\]

Moreover, the limit coefficient converges as \(J \to \infty\) and then \(y \to \infty\) to
\[
\int_0^\infty \mathbb{E}^{(A)}_{(0,z)}[F(e^{z(1 + x + I_{\infty}^\dagger + I_{\infty}^\dagger^\dagger})_n)] \frac{e^{\lambda z V(\Lambda)(z)}}{L(\Lambda)(\infty)} dz.
\]

Proof. For \(\eta \in (0, \theta_F - \Lambda)\), by the change of measure we have
\[
\mathbb{E}[F(I_J + I_{n-J,n} + xe^{-S_n}); S_n \geq -y, \sigma^-_n = n] \\
= \frac{\mathbb{E}^{(A)}[F(I_J + I_{n-J,n} + xe^{-S_n})e^{-\Lambda S_n}; S_n \geq -y, \sigma^-_n = n]}{\mathbb{E}^{(A)}[e^{-\eta S_n}; \tau^-_n > n]} \\
= \frac{\mathbb{E}^{(A)}[e^{-\eta S_n}; \tau^-_n > n]}{\mathbb{E}^{(A)}[e^{-\Lambda S_n}; \tau^-_n > n]} \mathbb{E}^{(A)}[F(I_J + I_{n-J,n} + xe^{-S_n})e^{-\Lambda S_n}; S_n \geq -y, \sigma^-_n = n].
\]

By Lemma 3.6 with \(x = 0, y = \infty\) and \(\lambda = \Lambda\) or \(\eta\), the first fraction on the right side of second equality in (4.12) converges to \(\mathcal{L}^{(n)}(\Lambda)(\infty)/\mathcal{L}^{(A)}(\Lambda)(\infty)\) as \(n \to \infty\). For the numerator of the second fraction, by the duality lemma we have
\[
\mathbb{E}^{(A)}\left[F(I_J + I_{n-J,n} + xe^{-S_n})e^{-\Lambda S_n}; S_n \geq -y, \sigma^-_n = n\right] \\
= \mathbb{E}^{(A)}\left[F\left(e^{-S_n\left(1 + x + \sum_{j=1}^{J-1} e^{S_j} + \sum_{i=n-J}^{n-1} e^{S_i}\right)}e^{-\Lambda S_n}; S_n \geq -y, M_n < 0\right]\right] \\
= \mathbb{E}^{(A)}\left[e^{-(\eta + \Lambda) S_n} F\left(e^{-S_n\left(1 + x + \sum_{j=1}^{J-1} e^{S_j} + \sum_{i=n-J}^{n-1} e^{S_i}\right)}\right) 1_{\{S_n \geq -y\}} \cdot e^{\eta S_n}, M_n < 0\right].
\]

Applying Lemma 3.7 with \(\lambda = \eta\), \(f(S) = \sum_{j=1}^{J-1} e^{S_j}, g_n(S) = (e^{-(\eta + \Lambda) S_n}, e^{-S_n}, \sum_{i=n-J}^{n-1} e^{S_i}, 1_{\{S_n \geq -y\}}\) and \(G(y, z_1, z_2, z_3, z_4) = z_1 \cdot F(z_2(1 + x + y + z_3) \cdot z_4)\), we have as \(n \to \infty\),
\[
\mathbb{E}^{(A)}\left[F(I_J + I_{n-J,n} + xe^{-S_n})e^{-\Lambda S_n}; S_n \geq -y, \sigma^-_n = n\right] \\
\to \int_0^y \mathbb{E}^{(A)}_{(0,z)}\left[e^{(\eta + \Lambda) z} F\left(e^{z\left(1 + x + \sum_{j=1}^{J-1} e^{S_j} + \sum_{i=1}^{J} e^{S_i}\right)}\right)\right] \mu^{(n)}_{\mathcal{V}(\Lambda)}(dz) \\
= \int_0^y e^{(\eta + \Lambda) z} \mathbb{E}^{(A)}_{(0,z)}\left[F\left(e^{z\left(1 + x + I_{J-1}^\dagger + I_{J-1}^\dagger^\dagger\right)}\right)\right] \mu^{(n)}_{\mathcal{V}(\Lambda)}(dz).
\]

Here the last equality follows from the fact that \(\hat{S}^\dagger = -S^\dagger\) and \(\hat{S}^\dagger = -S^\dagger\). The first claim follows directly by taking these two results back into (4.12) and using the definition of \(\mu^{(n)}_{\mathcal{V}(\Lambda)}(dz)\). The second one can be proved by using the dominated convergence theorem as \(J \to \infty\) and then the monotone convergence theorem as \(y \to \infty\).

Applying the preceding two propositions to (4.8), we can get the following corollary immediately.
Corollary 4.11 For each $k \geq 0$, we have as $n \to \infty$

$$\mathbb{E}[F(I_n); \sigma_n^-=n-k] \xrightarrow{\mathbb{P}(\tau_0^- > n)^\infty} \int_0^\infty \mathbb{E}^{(A)}_{(0,0,z)}[e^{-\Lambda S_k}F(e^{z(1+I_k+I_{\infty}^+)})]; \tau_n^- > k] \frac{e^{AzV^{(A)}(z)}}{\mathcal{L}^{(A)}(\infty)}dz.$$ 

Proof for Theorem 2.10. Here we just prove this theorem with $F$ being globally Lipschitz continuous. Proceeding as in the proof of Theorem 2.7, we can prove this theorem for general $F$. By Proposition 4.4, we have

$$\lim_{n \to \infty} \frac{\mathbb{E}[F(I_n)]}{\mathbb{P}(\tau_0^- > n)} = \sum_{k=0}^\infty \lim_{n \to \infty} \frac{\mathbb{E}[F(I_n); \sigma_n^- = k]}{\mathbb{P}(\tau_0^- > n)} + \sum_{k=0}^\infty \lim_{n \to \infty} \frac{\mathbb{E}[F(I_n); \sigma_n^- = n-k]}{\mathbb{P}(\tau_0^- > n)}.$$ 

The desired result (2.8) follows directly from Corollary 4.7 and 4.11. The finiteness of the limit coefficient $C_{F,2}$ can be gotten from Proposition 4.4.

4.3 Proof for Theorem 2.11

Condition 2.3 and the boundedness of $F$ tell us that for every $\epsilon > 0$, there exists a constant $x_0 > 0$ such that $|F(x) - K_0 \cdot (1 + x)^{-\Lambda}| < \epsilon \cdot (1 + x)^{-\Lambda}$ for any $x \geq x_0$. Thus we need to consider the asymptotic behavior of $\mathbb{E}[(1+I_n)^{-\Lambda}]$ at first. By the change of measure and then duality lemma,

$$\mathbb{E}[(1+I_n)^{-\Lambda}] = \varrho^n \cdot \mathbb{E}^{(A)}[(1+I_n)^{-\Lambda}e^{-\Lambda S_n}] = \varrho^n \cdot \mathbb{E}^{(A)}[(1+\tilde{I}_n)^{-\Lambda}].$$

According to our previous argument, the dual process $\tilde{S}$ satisfies Spitzer’s condition with positivity parameter $1 - \rho$. Using Theorem 2.7 with $F(x) = (1 + x)^{-\Lambda}$, we have as $n \to \infty$,

$$\mathbb{E}^{(A)}[(1+\tilde{I}_n)^{-\Lambda}] \sim C_{F,3} \cdot \mathbb{P}^{(A)}(\tilde{\tau}_0^- > n) = C_{F,3} \cdot \mathbb{P}^{(A)}(\tau_0^- > n).$$

It remains to prove that $\mathbb{E}^{(A)}[(F(I_n) - K_0(1+I_n)^{-\Lambda}e^{-\Lambda S_n}; I_n > x_0)] \leq \epsilon \cdot \mathbb{E}^{(A)}[(1+I_n)^{-\Lambda}e^{-\Lambda S_n}] \leq \epsilon \cdot C \cdot \mathbb{P}^{(A)}(\tau_0^- > n).$

On the other hand, the boundedness of $F$ implies that $F(x) \leq C(1+x)^{-\Lambda}$ for some $C > 0$ and $x \geq 0$. By Chebyshev’s inequality we also have

$$\mathbb{E}^{(A)}[|F(I_n) - K_0(1+I_n)^{-\Lambda}e^{-\Lambda S_n}; I_n \leq x_0)] \leq C(1 + x_0) \mathbb{E}^{(A)}[(1+I_n)^{-\Lambda}e^{-\Lambda S_n}].$$

For each $k \geq 0$, by the Markov property we have

$$\mathbb{E}^{(A)}[(1+I_n)^{-\Lambda}e^{-\Lambda S_n}; \sigma_n^- = k] \leq \mathbb{E}^{(A)}[e^{(A+1)S_k}\Lambda S_n; \sigma_n^- = k] = \mathbb{E}^{(A)}[e^{S_k}; \sigma_k^- = k] \mathbb{E}^{(A)}[e^{-\Lambda S_n-k}; \tau_0^- > n-k].$$

Applying Lemma 3.2 together with Lemma 3.6, we have

$$\mathbb{E}^{(A)}[(1+I_n)^{-\Lambda}e^{-\Lambda S_n}] = \sum_{k=0}^n \mathbb{E}^{(A)}[(1+I_n)^{-\Lambda}e^{-\Lambda S_n}; \sigma_n^- = k] \leq \sum_{k=0}^n \mathbb{E}^{(A)}[e^{S_k}; \sigma_k^- = k] \mathbb{E}^{(A)}[e^{-\Lambda S_n-k}; \tau_0^- > n-k],$$

which can be bounded by $C \cdot A_n = o(\mathbb{P}^{(A)}(\tau_0^- > n))$. Putting all estimates above together, we can get the desired result immediately. \qed
5 Proof for Theorems 2.12-2.17

In this section we prove the asymptotic results for the expectation $E[F(I_n)]$ with $S$ drifting to $-\infty$ under $P^{(A)}$. Different to the oscillating cases, we observe that the slow decreasing of the local minimum of $S$ usually results from an early large step. For simplicity, we again assume $\theta_F \in D_F$.

5.1 Proof for Theorem 2.12

It suffices to prove $E^{(A)}[e^{-\Lambda S_n}|F(I_n) - K_0 I_n^{-A}] \to 0$ as $n \to \infty$. By Condition 2.3, for any $\epsilon > 0$ there exists a constant $x_0 > 0$ such that for any $n \geq 1$ and $x \geq x_0$

$$E^{(A)}[e^{-\Lambda S_n}|F(I_n) - K_0 I_n^{-A}; I_n \geq x] \leq \epsilon \cdot E^{(A)}[e^{-\Lambda S_n} I_n^{-A}] \leq \epsilon.$$

On the other hand, by Assumption 2.1, we have $e^{-\Lambda S_n}|F(I_n) - K_0 I_n^{-A}| \leq Ce^{-\Lambda S_n} I_n^{-A} \leq C$ for some $C > 0$ and hence for any $K > 0$,

$$E^{(A)}[e^{-\Lambda S_n}|F(I_n) - K_0 I_n^{-A}; I_n < x] \leq CP^{(A)}(S_n \geq -K) + CP^{(A)}(S_n < -K; I_n < x).$$

By the duality lemma, the probability on the right side of the above inequality vanishes as $n \to \infty$. by Chebyshev’s inequality, the second one can be bounded by

$$xE^{(A)}[I_n^{-1}; S_n < -K] \leq xE^{(A)}[e^{S_n}; S_n < -K] \leq xe^{-K},$$

which vanishes as $K \to \infty$ and the desired result follows by putting all estimates above together.

5.2 Proof for Theorem 2.15

As we have mentioned before, since the slow decreasing of the local minimum of $S$ usually results from an early large step, the contribution of sample paths with late arrival of the first large step, $E[F(I_n); T^{an} > T]$ for large $T$, can be asymptotically ignored. By Lemma 3.10, we have the local minimum is usually not far from the first large step. Thus for any $\epsilon > 0$ and $K \geq 1$, there exist two integers $t_0, n_0 > 0$ such that for any $T > t_0$ and $n \geq n_0$,

$$E[F(I_n); T^{an} > T, \sigma_n^{-} \leq K] \leq C \cdot P(T^{an} > T, \sigma_n^{-} \leq K) \leq \epsilon \cdot P(\tau_0^- > n). \quad \text{(5.1)}$$

On the other hand, we prove in the next proposition, that the contribution of sample paths with the local minimum attained late, $E[F(I_n); T^{an} > T, \sigma_n^{-} > K]$ $\leq E[F(I_n); \sigma_n^{-} > K]$ for large $K$, also can be asymptotically ignored.

**Proposition 5.1** For every $\epsilon > 0$, there exist two integers $k_0, n_0 \geq 1$ such that for any $K \geq k_0$ and $n \geq n_0$,

$$E[F(I_n); \sigma_n^{-} \geq K] \leq \epsilon \cdot P(\tau_0^- > n).$$

Moreover, there exists a constant $C > 0$ such that for any $n \geq 1$,

$$E[F(I_n)] \leq C \cdot P(\tau_0^- > n).$$
Proof. By Assumption 2.1, we first have $F(I_n) \leq C \exp\{\theta_F L_n\}$ for some $C > 0$ and hence
\[
\mathbb{E}[F(I_n); \sigma_n \geq K] \leq C \sum_{k=K}^{n} \mathbb{E}[e^{\theta_F S_k}; \sigma_n = k] = C \sum_{k=K}^{n} \mathbb{E}[e^{\theta_F S_k}; \sigma_k^- = k] \mathbb{P}(\tau_0^- > n - k).
\]
Here the equality above follows from the Markov property of $S$ and the duality lemma. Applying Lemma 3.2 together with Lemma 3.12 with $y = \infty$ to the last sum, we have as $n \to \infty$ it is asymptotically equivalent to
\[
\mathbb{P}(\tau_0^- > n) \cdot \sum_{k=K}^{\infty} \mathbb{E}[e^{\theta_F S_k}; \sigma_k^- = k] \leq \mathbb{P}(\tau_0^- > n) \cdot \sum_{k=K}^{\infty} \frac{1}{k} \mathbb{P}(X \geq ak).
\]
Since the sequence $k^{-1} \mathbb{P}(X \geq ak)$ is $(-\alpha - 1)$-regularly varying and summable, the two desired inequalities follow immediately.  

We now start to consider the contribution of sample paths with early large step, i.e. $\mathbb{E}[F(I_n); T^a_n = k]$ for each $k \geq 1$. By (3.1), we have for large $n$,
\[
\frac{\mathbb{E}[F(I_n); T^a_n = k]}{\mathbb{P}(\tau_0^- > n)} \sim \frac{\mathbb{E}[F(I_n); X_k \geq an]}{\mathbb{P}(\tau_0^- > n)}.
\]
Applying the strong Markov property to $\mathbb{E}[F(I_n); X_k \geq an]$, we see it is equal to
\[
\mathbb{E}[\mathbb{E}[F(I_{k-1} + e^{-S_k - X_0 \tau_{k-1}})|X \geq an, \mathcal{F}^S] \cdot \mathbb{P}(X \geq an).
\]
It is obvious that we need to consider the asymptotics of the conditional expectation at first.

**Proposition 5.2** Let $H$ be a bounded and Lipschitz continuous function on $(0, \infty)$. For every $\epsilon > 0$, there exist two integers $k_0, n_0 \geq 1$ such that for any $K \geq k_0$ and $n \geq n_0$,
\[
\mathbb{E}[[H(e^{-X} I_n) - H(e^{-X} I_K)]; X \geq an] \leq \epsilon \cdot \mathbb{P}(X \geq an).
\]

**Proof.** Let $I_{K,n} := I_n - I_K$ for $0 \leq K \leq n$. The boundedness and Lipschitz continuity of $H$ induces that there exists a constant $C > 0$ such that
\[
\mathbb{E}[[H(e^{-X} I_n) - H(e^{-X} I_K)]; X \geq an] \leq C \mathbb{E}[1 \wedge (e^{-X} I_{K,n}); X \geq an].
\]
For any $b > a$, it is obvious that the foregoing quantities can bounded by the sum of
\[
\epsilon(b, n) := C \mathbb{P}(X \in [an, bn]) \quad \text{and} \quad \epsilon(b, K, n) := C \mathbb{E}[1 \wedge (e^{-X} I_{K,n}); X \geq bn].
\]
Condition 2.13 induces that $\epsilon(b, n)/\mathbb{P}(X \geq an) \to 0$ as $n \to \infty$ and then $b \to a^+$. Let $S_{n,b} := S_n + nb$, which drifts to $\infty$ as $n \to \infty$. By the independence between $X$ and $I_{K,n}(S)$,
\[
\epsilon(b, K, n) \leq C \mathbb{E}[1 \wedge (e^{-bn} I_{K,n})] \cdot \mathbb{P}\{X \geq bn\} \leq C \mathbb{E}[1 \wedge I_{K,n}^b] \cdot \mathbb{P}\{X \geq bn\},
\]
where $I_{K,n}^b := I_{n,b} - I_K^b$ and $I^b$ is the exponential functional of $S^b$. By Lemma 2.4, we have $I_{K,n}^b < \infty$ a.s. and hence $I_{K,n}^b \leq I_{K,\infty}^b \to 0$ a.s. as $K \to \infty$. Applying the dominated convergence theorem, we immediately get $\epsilon(b, K, n)/\mathbb{P}(X \geq an) \to 0$ as $n \to \infty$ and then $K \to \infty$. The desired result following by putting all preceding estimates together. \(\square\)
Proposition 5.3 Let \( H \) be a bounded and non-increasing function on \((0, \infty)\). For any \( K > 0 \), we have as \( n \to \infty \),
\[
\mathbb{E}[H(e^{-X}I_K) \mid X \geq an] \to C_{H,K} > 0.
\]
Moreover, the sequence \( C_{H,K} \) decreases to a limit \( C_H > 0 \) as \( K \to \infty \) and
\[
\lim_{n \to \infty} \mathbb{E}[H(e^{-X}I_n) \mid X \geq an] = C_H.
\]

Proof. It is easy to see that the first result follows directly from our observation that the monotonicity of \( H \) results in \( \mathbb{E}[H(e^{-X}I_K) \mid X \geq an] \) being non-decreasing in \( n \), i.e., for any \( z > 0 \) a sample calculation show that the expectation \( \mathbb{E}[H(ze^{-X}) \mid X \geq an] \) can written as the sum of \( \mathbb{E}[H(ze^{-X}) \mid X \geq a(n+1)] \) and
\[
\frac{\mathbb{P}\{X \geq [an,a(n+1)]\}}{\mathbb{P}\{X \geq an\}} \left( \mathbb{E}[H(ze^{-X}) \mid X \geq [an,a(n+1)]) - \mathbb{E}[H(ze^{-X}) \mid X \geq a(n+1)] \right).
\]
It is obvious that the foregoing quantity is non-positive and hence \( \mathbb{E}[H(e^{-X}I_K) \mid X \geq an] \geq \mathbb{E}[H(e^{-X}I_K) \mid X \geq a(n+1)] \). We now turn to prove the second claim. Because of the monotonicity of \( H \) and \( I_K \), the sequence \( C_{H,K} \) is non-increasing and converges to a limit \( C_H \in [0, \infty) \) as \( K \to \infty \). The second desired convergence follows directly from Proposition 5.2. It remains to prove \( C_H > 0 \). Recall the modified random walk \( S_n^b := S_n + bn \) for \( b > a \). We have
\[
\mathbb{E}[H(e^{-X}I_n) \mid X \geq an] \geq \frac{\mathbb{E}[H(e^{-X}I_n) \mid X \geq bn]}{\mathbb{P}\{X \geq an\}} \geq \mathbb{E}[H(I_n^b)] \frac{\mathbb{P}\{X \geq bn\}}{\mathbb{P}\{X \geq an\}}.
\]
From Lemma 2.4 and the dominated convergence theorem, we have \( \mathbb{E}[H(I_n^b)] \to \mathbb{E}[H(I_{\infty}^b)] > 0 \) as \( n \to \infty \) and hence \( C_H = \liminf_{n \to \infty} \mathbb{E}[H(e^{-X}I_n) \mid X \geq an] > 0 \).

Corollary 5.4 For each \( k \geq 1 \), we have \( \mathbb{E}[F(I_n); X_k \geq an]/\mathbb{P}(X \geq an) \to \mathbb{E}[C_{F,A}(k)] > 0 \) as \( n \to \infty \).

Proof. Applying Proposition 5.2 and 5.3 with \( H(x) := \mathbb{E}[F(I_{k-1} + xe^{-S_{k-1}}) \mid \mathcal{F}^S] \) to the conditional expectation in (5.3), we have as \( n \to \infty \) it converges to the limit \( C_{F,A}(k) > 0 \) a.s. Taking this back into (5.3) we can get the desired result directly.

Proof for Theorem 2.15. By (5.1), Proposition 5.1 and then (5.2) together, we first have
\[
\lim_{n \to \infty} \frac{\mathbb{E}[F(I_n)]}{\mathbb{P}(X \geq an)} = \lim_{T \to \infty} \lim_{n \to \infty} \frac{\mathbb{E}[F(I_n) ; T^{an} \leq T]}{\mathbb{P}(X \geq an)} = \sum_{k=1}^{\infty} \lim_{n \to \infty} \frac{\mathbb{E}[F(I_n) ; T^{an} = k]}{\mathbb{P}(X \geq an)} = \sum_{k=1}^{\infty} \lim_{n \to \infty} \frac{\mathbb{E}[F(I_n) ; X_k \geq an]}{\mathbb{P}(X \geq an)}.
\]
The asymptotic result (2.11) follows directly from Corollary 5.4. The finiteness of the coefficient \( C_{F,A} \) follows from Proposition 5.1.
5.3 Proof for Theorem 2.17

Recall the sequence $B_n$ defined in (2.12). By (2.5) we first show in the next proposition that the impact of sample paths with fast decreasing local minimum on the expectation $E^{(A)}[e^{-\lambda S_n} F(I_n)]$ can be asymptotically ignored.

Proposition 5.5 For every $\epsilon > 0$, there exist two integers $y_0, n_0 \geq 1$ such that for any $Y \geq y_0$ and $n \geq n_0$,

$$E^{(A)}[F(I_n)e^{-\lambda S_n}; L_n \leq -Y] \leq \epsilon \cdot B_n.$$  

Proof. Assumption 2.1 induces that $F(I_n) \leq C e^{\theta_F L_n}$. By Lemma 3.11, for large $n$ we have

$$E^{(A)}[F(I_n)e^{-\lambda S_n}; L_n \leq -Y] \leq C \cdot e^{-\theta_F Y/2} \cdot E^{(A)}[e^{\theta_F L_n/2-\lambda S_n}] \leq C e^{-\theta_F Y/2} \cdot B_n$$

and hence the desired result follows as $Y \to \infty$. □

Proposition 5.6 For every $\epsilon > 0$, there exist two integers $k_0, n_0 \geq 1$ such that for any $K \geq k_0$ and $n \geq n_0$,

$$E^{(A)}[e^{-\lambda S_n} F(I_n); \sigma_n^- \in [K, n-K]] \leq \epsilon \cdot B_n.$$  

Moreover, there exists a constant $C > 0$ such that for any $n \geq 1$,

$$E^{(A)}[e^{-\lambda S_n} F(I_n)] \leq C \cdot B_n.$$  

Proof. By $F(I_n) \leq C \exp\{\theta_F L_n\}$ and the independent increments of $S$, we have for any $k \geq 1$,

$$E^{(A)}[e^{-\lambda S_n} F(I_n); \sigma_n^- = k] \leq C \cdot E^{(A)}[e^{\theta_F S_n-\lambda S_n}; \sigma_n^- = k]$$

$$= C \cdot E^{(A)}[e^{(\theta_F-\lambda)S_k}; \sigma_k^- = k] \cdot E^{(A)}[e^{-\lambda S_n-k}; \tau_0^- > n-k].$$

Using Lemma 3.2 together with Lemma 3.12, we have for large $n$,

$$E^{(A)}[e^{-\lambda S_n} F(I_n); \sigma_n^- \in [K, n-K]] \leq C \sum_{k=K}^{n-K} E^{(A)}[e^{(\theta_F-\lambda)S_k}; \sigma_k^- = k] E^{(A)}[e^{-\lambda S_n-k}; \tau_0^- > n-k]$$

$$\sim C \sum_{k=K}^{\infty} \left(E^{(A)}[e^{(\theta_F-\lambda)S_k}; \sigma_k^- = k] + E^{(A)}[e^{-\lambda S_k}; \tau_0^- > k]\right) \cdot B_n.$$

The two desired inequalities follows from $E^{(A)}[e^{-\lambda S_n} F(I_n); \sigma_n^- = 0] \leq C \cdot B_n$ and the fact that the two sequences $E^{(A)}[e^{(\theta_F-\lambda)S_k}; \sigma_k^- = k]$ and $E^{(A)}[e^{-\lambda S_k}; \tau_0^- > k]$ are summable; see Lemma 3.12. □

As we mentioned before, the contribution of sample paths with large final value on the expectation $E[F(I_n)]$ also can be ignored; see the following two propositions.

Proposition 5.7 For every $\epsilon > 0$ and integer $k \geq 0$, there exist two integers $n_0, y_0 \geq 1$ such that for any $n \geq n_0$ and $Y \geq y_0$,

$$E^{(A)}[e^{-\lambda S_n} F(I_n); S_n \geq Y; \sigma_n^- = k] \leq \epsilon \cdot B_n.$$
Proof. When \( k = 0 \), by Lemma 3.12 we have for some \( C > 0 \),
\[
\mathbb{E}^{(A)}[e^{-\Lambda S_n} F(I_n); S_n \geq Y; \sigma_n^- = 0] \leq C \mathbb{E}^{(A)}[e^{-\Lambda S_n}; S_n \geq Y; \tau_0^- > n] \leq C |\mathcal{L}^{(A)}_\psi(\infty) - \mathcal{L}^{(A)}_\psi(Y)| \cdot B_n,
\]
which is \( o(B_n) \) for large \( Y \). For \( k \geq 1 \), by Assumption 2.1 we have
\[
\mathbb{E}^{(A)}[e^{-\Lambda S_n} F(I_n); S_n \geq Y; \sigma_n^- = k] \leq C \mathbb{E}^{(A)}[e^{\theta_F S_k - \Lambda S_n}; S_n \geq Y; \sigma_n^- = k].
\]
Conditioned on \( \sigma_n^- = k \), we notice that \( \{S_n \geq Y\} \subset \{S_n - S_k \geq Y\} \) and hence
\[
\mathbb{E}^{(A)}[e^{-\Lambda S_n} F(I_n); S_n \geq Y; \sigma_n^- = k] \leq C \mathbb{E}^{(A)}[e^{-\Lambda S_n + \theta_F S_k}; S_n - S_k \geq Y; \sigma_n^- = k].
\]
The independent increments of \( S \) induces that the last expectation equals to
\[
\mathbb{E}^{(A)}[e^{(\theta_F - \Lambda)S_k}; \sigma_k^- = k] \cdot \mathbb{E}^{(A)}[e^{-\Lambda S_{n-k}}; S_{n-k} \geq Y; \tau_0^- > n - k].
\]
Using Lemma 3.12, there exists a constant \( C > 0 \) such that for some \( C > 0 \),
\[
\mathbb{E}^{(A)}[e^{-\Lambda S_n} F(I_n); S_n \geq Y; \sigma_n^- = k] \leq C \cdot \mathbb{E}^{(A)}[e^{-\Lambda S_{n-k}}; S_{n-k} \geq Y; \tau_0^- > n - k],
\]
which can be bounded by \( C |\mathcal{L}^{(A)}_\psi(\infty) - \mathcal{L}^{(A)}_\psi(Y)| \cdot B_n = o(B_n) \) for large \( Y \).
\[\square\]

Proposition 5.8 For every \( \epsilon > 0 \) and integer \( k \geq 0 \), there exist two integers \( n_0, y_0 \geq 1 \) such that for any \( n \geq n_0 \) and \( Y \geq y_0 \),
\[
\mathbb{E}^{(A)}[e^{-\Lambda S_n} F(I_n); S_n \geq Y; \sigma_n^- = n - k] \leq \epsilon \cdot B_n.
\]

Proof. An argument similar to that in the proof for Proposition 5.7 shows that the expectation in the above inequality can be bounded by
\[
C \mathbb{E}^{(A)}[e^{(\theta_F - \Lambda)S_{n-k}}; \sigma_{n-k}^- = n - k] \cdot \mathbb{E}^{(A)}[e^{-\Lambda S_k}; S_k \geq Y; \tau_0^- > k].
\]
Dividing it by \( B_n \) and then using Lemma 3.11, for large \( n \) we see that it can be bounded by \( C \cdot \mathbb{E}^{(A)}[e^{-\Lambda S_k}; S_k \geq Y; \tau_0^- > k] \), which vanishes as \( Y \to \infty \).
\[\square\]

Recall the drift parameter \( a \) defined in (2.10). We now turn to consider the impact of sample paths with late arrival of the first large step, slowly decreasing local minimum and the final value below some fixed level, i.e. for \( b \in (0, a) \) and \( Y, T \geq 1 \),
\[
\mathbb{E}^{(A)}[e^{-\Lambda S_n} F(I_n); -Y \leq L_n \leq S_n \leq Y; T^{bn} \geq T].
\]
The next two propositions show that the late arrival of the first large jump will cause the random walk to drift to a low level. It extends Lemma 9 in [8], who considered the case \( \beta > 2 \).

Proposition 5.9 Let \( r > \frac{\beta + 1}{\beta - 1} \), \( b \in (0, a/r) \) and \( Y \geq 0 \). For any \( \epsilon > 0 \), there exists an integer \( n_0 \geq 1 \) such that for any \( n \geq n_0 \),
\[
\mathbb{P}^{(A)}(S_n \geq -Y, T^{bn} \geq n) \leq \epsilon \cdot B_n.
\]
Proof. Here we just prove this proposition for $\beta \in (1, 2)$. For $\beta > 2$, it can be proved in the same way with the help of Theorem 4.1.2(i) in [13, p.183]. We first assume that there exists a constant $C > 0$ such that

$$P(\Lambda)(X \leq -x) \leq C \cdot P(\Lambda)(X \geq x), \quad x \geq 0.$$  \hfill(5.4)$$

By Lemma 3.1 in [14] and Assumption 2.13, there exists a constant $C > 0$ such that for large $n$,

$$P(\Lambda)(S_n + an > rb, T^{bn-a} > n) \leq C \cdot |nP(\Lambda)(X \geq bn - a)|^\beta \leq C n^{\beta(1-\beta)\ell_3(n)}.$$  

From this and the assumption on $br < a$ and $r(\beta - 1) > \beta + 1$, we have for large $n$,

$$P(\Lambda)(S_n \geq -Y, T^{bn} > n) \leq C n^{\beta(1-\beta)\ell_3(n)} = o(B_n)$$

We consider the general case with $X$ satisfying Assumption 2.13. For some $K < 0$ we define $X^K := X \lor K$ and $X^K_i = X_i \lor K$ such that $a_K := -E(\Lambda)[X^K] \in (rb, a)$. It is obvious that the random variable $X^K$ satisfies the inequality (5.4). Let $S^K_n$ be the random walk generated by the sequence $\{X^K_i : i = 1, 2, \cdots \}$. It is obvious that $P(\Lambda)(S_n \geq -Y, T^{bn} \geq n) \leq P(\Lambda)(S^K_n \geq -Y, T^{bn} \geq n)$ and our previous result induces that it is $o(B_n)$.

Proposition 5.10 Let $b > 0$ and $Y \geq 0$. For any $\epsilon > 0$, there exist two integers $t_0, n_0 > 1$ such that for any $T \geq t_0$ and $n \geq n_0$,

$$P(\Lambda)(-Y \leq L_n \leq S_n \leq Y, T \leq T^{bn} \leq n) \leq \epsilon \cdot B_n.$$  

Proof. For any $2 \leq k \leq n$, we have $L_{k-1} \geq L_n$ and by the Markov property

$$P(\Lambda)(-Y \leq L_n \leq S_n \leq Y, X_k \geq bn)$$

$$\leq \int_{-Y}^\infty P(\Lambda)(L_{k-1} \geq -Y, S_{k-1} \in [x, \infty)) \cdot \int_x^{\infty} P(\Lambda)(x + y + \tilde{S}_{n-k} \in [-Y, Y]) P(\Lambda)(X_k \in dy)$$

$$\leq P(\Lambda)(L_{k-1} \geq -Y) \cdot \sup_{x \geq -Y} \int_{bn}^\infty P(\Lambda)(x + y + \tilde{S}_{n-k} \in [-Y, Y]) P(\Lambda)(X_k \in dy).$$  \hfill(5.5)$$

By Assumption 2.14, we have $P(\Lambda)(X_k \in [bn + i, bn + i + 1)) \leq 2B_n$ for any $i \geq 1$ and large $n$. Thus the last integral in (5.5) can be bounded by

$$2B_n \sum_{i=0}^\infty P(\Lambda)(x + bn + i + \tilde{S}_{n-k} \in [-Y - 1, Y + 1])$$

$$\leq 2B_n \int_{bn}^\infty P(\Lambda)(x + y + \tilde{S}_{n-k} \in [-Y - 2, Y + 2]) dy.$$  

Applying the integration by parts to this integral, we have it can be bounded by

$$\int_{-\infty}^\infty \int_{-x-y-2}^{-x-y+2} P(\Lambda)(\tilde{S}_{n-k} \in dz) dy = \int_{-\infty}^\infty P(\Lambda)(\tilde{S}_{n-k} \in dy) \int_{-x-y-2}^{-x-y+2} dz \leq 2Y + 4.$$  

Taking these back into (5.5), we have

$$P(\Lambda)(-Y \leq L_n \leq S_n \leq Y, X_k \geq bn) \leq 4(Y + 2) P(\Lambda)(L_{k-1} \geq -Y) \cdot B_n.$$  

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Hence by Lemma 3.8, we have \( P^{(A)}(L_{k-1} \geq -Y) = P^{(A)}_Y(\tau_0^- > k - 1) \leq C \cdot P^{(A)}(X \geq a(k - 1)) \) for large \( k \) and hence
\[
\frac{1}{B_n} P^{(A)}(-Y \leq L_n \leq S_n \leq Y, T \leq T_{bn} \leq n) \leq C \sum_{k=T}^{\infty} P^{(A)}(X \geq a(k - 1)),
\]
which vanishes as \( T \to \infty \) and the desired result follows.

Putting all preceding estimates together, we see that the main contribution to the expectation \( E^{(A)}[F(I_n)e^{-\Lambda S_n}] \) is made by the sample paths with an early large step, i.e. as \( n \to \infty \),
\[
E^{(A)}[F(I_n)e^{-\Lambda S_n}] \sim \sum_{k=1}^{\infty} E^{(A)}[F(I_n)e^{-\Lambda S_n}; T_{bn} = k],
\]
where \( b \in (0, a/r) \) and \( r > \frac{\beta + 1}{\beta - 1} \). By (3.1), for each \( k \geq 1 \) we have
\[
E^{(A)}[F(I_n)e^{-\Lambda S_n}; T_{bn} = k] \sim E^{(A)}[F(I_n)e^{-\Lambda S_n}; X_k \geq bn]. \tag{5.6}
\]
Using Proposition 5.5 again, for large \( Y \) it is also asymptotically equivalent to
\[
E^{(A)}[F(I_n)e^{-\Lambda S_n}; S_{k-1} \geq -Y, X_k \geq bn]. \tag{5.7}
\]
Moreover, as we have mentioned before, the local minimum is usually attained around the early large step and hence the impact of sample paths that stay above a high level before the early large step on the expectation \( E^{(A)}[F(I_n)e^{-\Lambda S_n}] \) can be ignored; see the following proposition.

**Proposition 5.11** Let \( b \in (0, a] \) and integer \( k \geq 1 \). For each \( \epsilon > 0 \), there exist two integers \( n_0, y_0 \geq 1 \) such that for any \( Y \geq y_0 \) and \( n \geq n_0 \),
\[
E^{(A)}[F(I_n)e^{-\Lambda S_n}; S_{k-1} \geq Y, X_k \geq bn] \leq \epsilon \cdot B_n.
\]

**Proof.** Let \( \gamma \in (1/\beta, 1) \). By Assumption 2.1, we see the foregoing expectation can be bounded by
\[
CE^{(A)}[e^{\theta_F L_n - \Lambda S_n}; S_{k-1} \geq n^\gamma, X_k \geq bn] + CE^{(A)}[e^{\theta_F L_n - \Lambda S_n}; Y \leq S_{k-1} \leq n^\gamma, X_k \geq bn]. \tag{5.8}
\]
Applying Hölder’s inequality to the first expectation, for large \( n \) we have it can be bounded by
\[
\left( E^{(A)}[e^{2\theta_F L_n - 2\Lambda S_n}] \cdot P^{(A)}(S_{k-1} \geq n^\gamma) \cdot P^{(A)}(X_k \geq bn) \right)^{1/2}.
\]
Using Lemma 3.11 and then Lemma 3.8, we see that the foregoing quantity can be bounded by \( C \cdot B_n(n \cdot P^{(A)}(S_{k-1} \geq n^\gamma))^{1/2} \), which is \( o(B_n) \) since \( P^{(A)}(S_{k-1} \geq n^\gamma) \sim (k - 1)P^{(A)}(X \geq n^\gamma) = o(1/n) \); see Assumption 2.13. On the other hand, by the Markov property the second expectation in (5.8) can be bounded by
\[
\int_{Y}^{n^\gamma} P^{(A)}(S_{k-1} \in dy) \int_{bn}^{\infty} E^{(A)}_y[e^{\theta_F L_n - k - \Lambda S_{n-k}}]P^{(A)}(X_k \in dx)
= \int_{Y}^{n^\gamma} P^{(A)}(S_{k-1} \in dy) \int_{bn+y}^{\infty} E^{(A)}_y[e^{\theta_F L_n - k - \Lambda S_{n-k}}]P^{(A)}(X_k + y \in dx). \tag{5.9}
\]
By Assumption 2.13 and 2.14, there exists a constant \( C > 0 \) such that for large \( n \),
\[
P^{(A)}(X_k + y \in [bn + i, bn + i + 1]) \leq C \cdot P^{(A)}(X \in [bn + i, bn + i + 1])
\]
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uniformly in $i \geq 0$ and $y \in [Y, n^\gamma]$. There exists a constant $C > 0$ independent of $n$ and $y$ such that the inner integral in the double integral on the right-hand side of (5.9) can be bounded by

$$\sum_{i=0}^{\infty} \mathbf{E}^{(A)}_{bn+i}[e^{\theta_F(L_{n-k+1} - \Lambda(\tilde{S}_{n-k} - 1))} \cdot \mathbf{P}^{(A)}(X_k + y \in [bn + i, bn + i + 1])$$

$$\leq C \sum_{i=0}^{\infty} \mathbf{E}^{(A)}_{bn+i}[e^{\theta_F \tilde{L}_{n-k} - \Lambda \tilde{S}_{n-k}}] \cdot \mathbf{P}^{(A)}(X \in [bn + i, bn + i + 1])$$

$$\leq C \int_{bn}^{\infty} \mathbf{E}^{(A)}_x[e^{\theta_F \tilde{L}_{n-k} - \Lambda \tilde{S}_{n-k}}] \cdot \mathbf{P}^{(A)}(X \in dx)$$

$$= C \cdot \mathbf{E}^{(A)}[e^{\theta_F \tilde{L}_{n-k+1} - \Lambda \tilde{S}_{n-k+1}}; \tilde{X}_1 \geq bn] \leq C \mathbf{E}^{(A)}[e^{\theta_F \tilde{L}_{n-k+1} - \Lambda \tilde{S}_{n-k+1}}] \leq C \cdot B_n.$$

Here the last inequality follows from Lemma 3.11. Taking this back into (5.9), we have for large $n$ and $Y$,

$$\mathbf{E}^{(A)}[e^{\theta_F L_{n-k} - \Lambda S_k}; Y \leq S_{k-1} \leq n^\gamma, X_k \geq bn] \leq C \cdot B_n \cdot \mathbf{P}^{(A)}(S_{k-1} \geq Y) = o(B_n).$$

The desired result follows by putting all estimates above together. \hfill \square

Taking the estimates in (5.7) and Proposition 5.11 back into (5.6), we have for large $n$ and $Y$,

$$\mathbf{E}^{(A)}[F(I_n)e^{-\Lambda S_n}; T^{bn} = k] \sim \mathbf{E}^{(A)}[e^{-\Lambda S_{k-1}} \cdot F(I_n) \cdot e^{-\Lambda(S_{n-S_{k-1}})}; |S_{k-1}| \leq Y, X_k \geq bn].$$

The next proposition shows that for large $Z > 0$, the contribution of sample path with $|S_n - S_{k-1}| > Z$ to the forgoing expectations can be asymptotically ignored.

**Proposition 5.12** Let $Y \geq 0$ and integer $k \geq 1$. For any $\epsilon > 0$, there exist two integers $z_0, n_0 \geq 1$ such that for any $Z \geq z_0$ and $n \geq n_0$,

$$\mathbf{E}^{(A)}[F(I_n)e^{-\Lambda S_{k-1}}; |S_{k-1}| \leq Y, |S_n - S_{k-1}| \geq Z] \leq \epsilon \cdot B_n.$$

**Proof.** By Assumption 2.1, it suffices to prove this equality with $F(I_n)$ replaced by $e^{\theta_F L_n}$. We first notice that $L_n \leq S_{k-1} + L_{k,n}$ with $L_{k,n} := \min\{S_i - S_{k-1}; i = k - 1, \cdots, n\}$. By the Markov property,

$$\mathbf{E}^{(A)}[e^{\theta_F L_n - \Lambda S_{k-1}}; |S_{k-1}| \leq Y, |S_n - S_{k-1}| \geq Z]$$

$$\leq \mathbf{E}^{(A)}[e^{\theta_F(S_{k-1} + L_{k,n}) - \Lambda S_{k-1}}; |S_{k-1}| \leq Y, |S_n - S_{k-1}| \geq Z]$$

$$= \mathbf{E}^{(A)}[e^{\theta_F(\Lambda S_{k-1})}; |S_{k-1}| \leq Y] \cdot \mathbf{E}^{(A)}[e^{\theta_F L_{n-k+1} - \Lambda S_{n-k+1}}; |S_{n-k+1}| \geq Z].$$

It is obvious that the first expectation on the right side of the last inequality can be bounded by $e^{(\theta_F - \Lambda)Y}$. On the other hand, we notice that $\{S_{n-k+1} \leq -Z\} \subset \{L_{n-k+1} \leq -Z\}$ and $\{S_{n-k+1} \geq Z\} \subset \{S_{n-k+1} - L_{n-k+1} \leq -Z\}$ and the second expectation can be bounded by

$$\mathbf{E}^{(A)}[e^{\theta_F L_{n-k+1} - \Lambda S_{n-k+1}}; L_{n-k+1} \leq -Z] + \mathbf{E}^{(A)}[e^{\theta_F L_{n-k+1} - \Lambda S_{n-k+1}}; S_{n-k+1} - L_{n-k+1} \geq Z]. \quad (5.10)$$

We write the first term as

$$\mathbf{E}^{(A)}[e^{\theta_F L_{n-k+1} - \Lambda S_{n-k+1}}] - \mathbf{E}^{(A)}[e^{\theta_F L_{n-k+1} - \Lambda S_{n-k+1}}; -L_{n-k+1} \leq Z]$$

Applying Lemma 3.11 to the first expectation with $\lambda_1 = \theta_F - \Lambda$, $\lambda_2 = \Lambda$, $\gamma = (\infty, \infty)$ and then to the second expectation $(x, y) = (Z, \infty)$, we have the first expectation in (5.10) can be bounded by

$$C \int_{Z}^{\infty} e^{-(\theta_F - \Lambda)z} (V(dz) + V(z)dz) \cdot B_n,$$
Proof. An argument similar to that in the proof of Proposition 5.11 shows that for large $n$, Similarly, the second expectation in (5.10) also can be bounded by
\[
C \int_Z^\infty e^{-\Delta z} (\hat{V}(dz) + \hat{V}(z)dz) \cdot B_n.
\]
Both of these two integrals vanish as $Z \to \infty$, then the desired result follows.

Let $r > \frac{\beta + 1}{\beta - 1}$ and $b \in (0, a/r)$. Combining all estimates above together, we have
\[
E^{(A)}[e^{-\Delta S_n} F(I_n)] \sim \sum_{k=1}^\infty E^{(A)}[e^{-\Delta S_n} F(I_n); |S_{k-1}| \leq Y, |S_n - S_{k-1}| \leq Z, X_k \geq bn]
\]
for large $Y$, $Z$ and $n$. By the Markov property, the summand can be written into
\[
E^{(A)}[e^{-\Delta S_{k-1}} F^b Z (k, n); |S_{k-1}| \leq Y]
\]
with
\[
F^b Z (k, n) := E^{(A)}[e^{-\Delta S_{k-1}} F(I_{k-1} + e^{-\Delta S_{k-1}} \tilde{I}_{n-k+1}); |\tilde{S}_{n-k+1}| \leq Z, \tilde{X}_1 \geq bn].
\]
It is obvious that $F^b Z (k, n)$ plays the main role in the asymptotic analysis of the expectation $E[F(I_n)]$. Recall the constant $\kappa \in (1, 2)$ defined in Condition 2.16. The next proposition states that \{|\tilde{S}_{n-k+1}| \leq Z, \tilde{X}_1 - an| \geq n^{1/\kappa}\}.

**Proposition 5.13** For any $b \in (0, a)$, $Z \geq 1$ and $\epsilon > 0$, there exists an integer $n_0$ such that for any $n \geq n_0$,

\[
P^{(A)}(X_1 \in [bn, an - n^{1/\kappa}] \cup [an + n^{1/\kappa}, \infty); |S_n| \leq Z) \leq \epsilon \cdot B_n.
\]

**Proof.** An argument similar to that in the proof of Proposition 5.11 shows that for large $n$,

\[
P^{(A)}(X_1 \in [bn, an - n^{1/\kappa}); |S_n| \leq Z) \leq C \cdot B_n \cdot \int_{bn}^{an-n^{1/\kappa}} P^{(A)}(|x + S_{n-1}| \leq Z + 2) dx
\]

\[
\leq C \cdot B_n \cdot \int_{bn}^{an-n^{1/\kappa}} \int_{Z-2}^{Z+2} P^{(A)}(S_{n-1} \in dz) dx
\]

\[
\leq C(Z + 2) \cdot B_n \cdot \int_{Z-2}^{Z+2} P^{(A)}(S_{n-1} \in dz)
\]

\[
\leq 4NC \cdot B_n \cdot P^{(A)}\left(\frac{S_{n-1} + a(n - 1)}{n^{1/\kappa}} \geq 1/2\right).
\]

The statements after Condition 2.16 shows that the last probability goes to 0 as $n \to \infty$. Similarly, we also have for large $n$,

\[
P^{(A)}(X_1 > an + n^{1/\kappa}; |S_n| \leq N) \leq C \cdot B_n \cdot \int_{an+n^{1/\kappa}}^\infty P^{(A)}(|x + S_{n-1}| \leq 2N) dx
\]

\[
\leq C \cdot B_n \cdot P^{(A)}(S_{n-1} \leq 2N - an - n^{1/\kappa})
\]

\[
\leq C \cdot B_n \cdot P^{(A)}\left(\frac{S_{n-1} + a(n - 1)}{n^{1/\kappa}} \leq -1/2\right)
\]

and the last probability also goes to 0 as $n \to \infty$. \qed
By this proposition, as \( n \to \infty \) we see that \( F_{\mathscr{F}^S}(k,n) \) can be well approximated by

\[
E^{(A)}[e^{-\Lambda \hat{S}_n-k+1}F(I_{k-1}+e^{-S_{k-1}}\hat{I}_{n-k+1});|\hat{S}_{n-k+1}| \leq Z,|X_1-an| \leq n^{1/\kappa} |\mathscr{F}^S]}
\]

Recall \( \hat{I}_{k,n} = \hat{I}_n - \hat{I}_k \) for \( 0 \leq k \leq n \). We in the next proposition prove that this conditional expectation will not change to much with \( \hat{I}_{n-k+1} \) replaced by \( \hat{I}_{n-k+1-W,n-k+1} \) for large \( W \).

**Proposition 5.14** Let \( H \) be a bounded and Lipschitz continuous function on \((0, \infty)\). For every \( Z, \epsilon > 0 \), there exist two integers \( w_0, n_0 \leq 1 \) such that for any \( W \geq w_0 \) and \( n \geq n_0 \),

\[
E^{(A)}[e^{-\Lambda S_n}|H(I_n) - H(I_{n-w_0,n})|;|S_n| \leq Z,|X_1-an| \leq n^{1/\kappa}] \leq \epsilon \cdot B_n.
\]

**Proof.** Using the Lipschitz continuity of \( H \) and then the duality lemma, the foregoing expectation can be bounded by

\[
Ce^{\Lambda Z} E^{(A)}[(I_{n-w_0} \cap 1);|S_n| \leq Z,|X_1-an| \leq n^{1/\kappa}]
= Ce^{\Lambda Z} E^{(A)}[(e^{-S_n} \hat{I}_{W-1,n-1} \cap 1);|S_n| \leq Z,|X_n-an| \leq n^{1/\kappa}].
\]

As argument similar to that in the proof of Proposition 5.10 shows that there exists a constant \( C > 0 \) such that for large \( n \), the foregoing quantity can be bounded by

\[
Ce^{\Lambda Z} B_n \cdot \int_{an\cdot n^{1/\kappa}}^{an\cdot n^{1/\kappa}} E^{(A)}[(e^{\Lambda Z} \hat{I}_{W-1,n-1} \cap 1);|S_{n-1}+x_n| \leq Z]dx_n.
\]

Changing the variables it turns to be

\[
CB_n \cdot \int_{-Z}^{Z} E^{(A)}[(e^{\Lambda Z} \hat{I}_{W-1,n-1} \cap 1);|S_{n-1}+an-z| \leq n^{1/\kappa}]dz,
\]

which can be bounded by \( 2CZe^{\Lambda Z} \cdot B_n \cdot E^{(A)}[(e^{\Lambda Z} \hat{I}_{W-1,n-1} \cap 1)] \). By the dominated convergence theorem and the fact that \( \hat{I}_n < \infty \) a.s. under \( P^{(A)} \), the desired result follows as \( W \to \infty \). \( \square \)

**Proposition 5.15** Let \( H \) be a nonnegative, bounded and continuous function on \((0, \infty)\). For any two integers \( W, Z > 0 \), we have as \( n \to \infty \),

\[
B_n^{-1} \cdot E^{(A)}[e^{-\Lambda S_n} H(I_{n-W,n});|S_n| \leq Z,|X_1-an| \leq n^{1/\kappa}] \to \int_{-Z}^{Z} E^{(A)}[e^{-\Lambda z} H(e^{-z}(1 + \hat{I}_{W-1}))]dz.
\]

Moreover, the limit coefficient converges as \( W \to \infty \) and then \( Z \to \infty \) to

\[
\int_{-\infty}^{\infty} E^{(A)}[e^{-\Lambda z} H(e^{-z}(1 + \hat{I}_{\infty}))]dz.
\]

**Proof.** By the duality lemma, we first have

\[
E^{(A)}[e^{-\Lambda S_n} H(I_{n-W,n});|S_n| \leq Z,|X_1-an| \leq n^{1/\kappa}]
= E^{(A)}[e^{-\Lambda S_n} H(e^{-S_n}(1 + \hat{I}_{W-1}));|S_n| \leq Z,|X_n-an| \leq n^{1/\kappa}]
\]

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\[
\int_{an-1/n^{1/\kappa}}^{an+1/n^{1/\kappa}} E^{(A)} \left[ e^{-\Lambda(S_{n-1}+x)} H(e^{-S_{n-1}-x}(1+\hat{I}_{W-1})); |S_{n-1}+x| \leq Z \right] P^{(A)}(X \in dx).
\]

By Assumption 2.13 and 2.14, for any \(\delta, \epsilon > 0\) there exists an integer \(n_0 > 1\) such that for any \(n \geq n_0\) and \(|y| \leq n^{1/\kappa}\),

\[
(1 - \epsilon)B_n \cdot \delta \leq P^{(A)}(X \in [an + y, an + y + \delta]) \leq (1 + \epsilon)B_n \cdot \delta.
\]

For each \(i \in \mathbb{Z}\) and \(y \in \mathbb{R}\), we define

\[
H_{n, \delta}(i) := \sup_{\delta \leq x \leq (i+1)\delta} H(e^{-S_{n-1} - an - x}(1 + \hat{I}_{W-1})) \quad \text{and} \quad H_\delta(y) := \sup_{y - \delta \leq x \leq y + \delta} H(e^{-x}(1 + \hat{I}_{W-1})).
\]

A sample calculation shows that

\[
B_n^{-1} E^{(A)} \left[ e^{-\Lambda S_n} H(I_{n-W,n}); |S_n| \leq Z, |X_1 - an| \leq n^{1/\kappa} \right]
\leq (1 + \epsilon) \sum_{i = [n^{1/\kappa}/\delta] - 1}^{[n^{1/\kappa}/\delta]} E^{(A)} \left[ e^{-\Lambda(S_{n-1} + an + i\delta)} H_{n, \delta}(i); |S_{n-1} + an + i\delta| \leq Z + \delta \right] \cdot \delta
\leq (1 + \epsilon) e^{\Lambda \delta} \int_{-Z - 2\delta}^{Z + 2\delta} E^{(A)} \left[ e^{-\Lambda z} H_\delta(z); |S_{n-1} + an - z| \leq n^{1/\kappa} + \delta \right] dz.
\]

Here the last equality following by changing the variables. The statements below Condition 2.16 together with the arbitrariness of \(\epsilon\) yields that as \(n \to \infty\) the last integral converges to

\[
e^{\Lambda \delta} \int_{-Z - 2\delta}^{Z + 2\delta} E^{(A)} \left[ e^{-\Lambda z} H_\delta(z) \right] dz.
\]

Letting \(\delta \to 0^+\), we have \(H_\delta(z) \to H(e^{-z}(1 + \hat{I}_{W-1}))\) for any \(z \in \mathbb{R}\) and then by the dominated convergence theorem,

\[
\limsup_{n \to \infty} B_n^{-1} E^{(A)} \left[ e^{-\Lambda S_n} H(I_{n-W,n}); |S_n| \leq Z, |X_1 - an| \leq n^{1/\kappa} \right] \leq \int_{-Z}^{Z} E^{(A)} \left[ e^{-\Lambda z} H(e^{-z}(1 + \hat{I}_{W-1})) \right] dz.
\]

On the other hand, a similar argument also can yields that

\[
\liminf_{n \to \infty} B_n^{-1} E^{(A)} \left[ e^{-\Lambda S_n} H(I_{n-W,n}); |S_n| \leq Z, |X_1 - an| \leq n^{1/\kappa} \right] \geq \int_{-Z}^{Z} E^{(A)} \left[ e^{-\Lambda z} H(e^{-z}(1 + \hat{I}_{W-1})) \right] dz.
\]

Thus the first claim follows. The second one can be gotten by using dominated convergence theorem as \(W \to \infty\) and then the monotone convergence theorem as \(Z \to \infty\). □

Applying these two propositions to \(F^{b, Z}_{S, S}(k, n)\), we can get the following corollary immediately.

**Corollary 5.16** For every \(b \in (0, 1)\) and \(k, Z \geq 1\), we have as \(n \to \infty\)

\[
B_n^{-1} F^{b, Z}_{S, S}(k, n) \to \int_{-Z}^{Z} E^{(A)} \left[ e^{-\Lambda z} F(I_{k-1} + e^{-S_{k-1} - z}(1 + \tilde{I}_\infty)) \right] \mathcal{F}^S \right] dz.
\]
Proof for Theorem 2.17. By all arguments and proposition above, we have

$$\lim_{n \to \infty} B_n^{-1} E^{(A)}[e^{-\Lambda S_n} F(I_n)] = \sum_{k=1}^{\infty} \lim_{Y \to \infty} \lim_{Z \to \infty} \lim_{n \to \infty} B_n^{-1} E^{(A)}[e^{-\Lambda S_{k-1}} F^{b,Z}_{\neq S}(k,n); |S_{k-1}| \leq Y].$$

Using Proposition 5.14 and 5.14, we have $B_n^{-1} E^{(A)}[e^{-\Lambda S_{k-1}} F^{b,Z}_{\neq S}(k,n); |S_{k-1}| \leq Y]$ converges as $n \to \infty$ to

$$\int_{-Z}^{Z} E^{(A)}[e^{-\Lambda S_{k-1} - \Lambda z} F(I_{k-1} + e^{-S_{k-1-z}(1 + \tilde{I}_\infty)}; |S_{k-1}| \leq Y)] dz,$$

which converges as $Z \to \infty$ and then $Y \to \infty$ to $C_{F,S}(k)$ by the monotone convergence theorem and the dominated convergence theorem.

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