A GAMMA CLASS FORMULA FOR OPEN GROMOV-WITTEN CALCULATIONS

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Abstract. For toric Calabi-Yau 3-folds, open Gromov-Witten invariants associated to Riemann surfaces with one boundary component can be written as the product of a disk factor and a closed invariant. Using the Brini-Cavalieri-Ross formalism, these disk factors can often be expressed in terms of gamma classes. When the Lagrangian boundary cycle is preserved by the torus action and can be locally described as the fixed locus of an anti-holomorphic involution, we prove a formula that expresses the disk factor in terms of a gamma class and combinatorial data about the image of the Lagrangian cycle in the moment polytope. We verify that this formula encodes the expected invariants obtained from localization by comparing with several examples. We then examine a novel application of this formula to disk enumeration on the quintic 3-fold. Finally, motivated by large $N$ duality, we show that this formula also unexpectedly applies to Lagrangian cycles on $\mathcal{O}_{\mathbb{P}^3}(-1,-1)$ constructed from torus knots.

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1. Introduction

1.1. Background and motivation. Gromov-Witten theory has a rich history, both in physics and mathematics. Physically, Gromov-Witten invariants appear in type IIA topological string theory as instanton counts associated to interactions between particles. Mathematically, they are invariants associated to symplectic manifolds that, roughly speaking, count pseudoholomorphic curves in the manifold. The relationship between these two perspectives is conceptually straightforward: as a string moves in time, it sweeps out a compact Riemann surface (its 'worldsheet'). The amplitudes in string theory encode counts of maps from Riemann surfaces into a 3-(complex)-dimensional Calabi-Yau manifold, and Gromov-Witten theory assigns invariants to spaces of such maps.

In general, counting holomorphic maps from Riemann surfaces to a given target space is a difficult problem in enumerative geometry. Gromov-Witten theory has famously benefited from its connections with string dualities, first with mirror symmetry [COGP, Wi1], and more recently, large \( N \) duality [GV, OV]. Beginning with [Kon], for toric manifolds, Gromov-Witten invariants associated to maps of closed surfaces have also been systematically computed using localization [GP, CKYZ, KZ]. “Closed” Gromov-Witten theory is a natural mathematical counterpart to closed topological string theory, and, in contrast to the “open” theory (i.e., for maps of Riemann surfaces with boundary), the moduli spaces involved are rigorously defined.

Open Gromov-Witten theory is the subject of this paper. By analogy with the closed case, open Gromov-Witten theory is a mathematical counterpart to open topological string theory: open strings sweep out compact Riemannian surfaces with boundary, and the boundary of the strings are constrained to lie on branes. These boundary constraints are expressed mathematically as Lagrangian submanifolds \( L \subset X \), and the string amplitudes are encoded by counts of holomorphic maps \( f: \Sigma \to X \), with the image of the boundary constrained to lie on \( L: f(\partial \Sigma) \subset L \). However, as observed in [AKV, KL], there are additional subtleties in adapting the methods of the closed theory to the open case. In particular, even for well-behaved Lagrangian boundary cycles, open Gromov-Witten invariants depend on an additional integral parameter (in localization, this parameter corresponds to the weights of the torus action).

In spite of this, the same computational tools of mirror symmetry, large \( N \) duality, and localization can still be used. In fact, through these string dualities, open Gromov-Witten theory can be connected to both classical and homological knot theory [OV, DSV, GJKS, MV, Wi2, Wi3]. Motivated by relationships with the crepant resolution conjecture [Rua], open Gromov-Witten theory can also be generalized to orbifolds [BC]. This setting has driven a more abstract formulation of open Gromov-Witten invariants, which has led to a deeper understanding of the A-model. In particular, as will be discussed in detail below, the open invariants contain gamma classes coming from disk terms [BCR]. In addition, the open Gromov-Witten generating function can be obtained from a modification of Givental’s \( J \) function [BC, Bri].

The primary goal of this paper is to describe a concise and consistent framework for computing open Gromov-Witten invariants directly, via localization. Leveraging the formalism of [BCR], the main tool is a formula for open Gromov-Witten invariants expressed in terms of local combinatorial data and a gamma class. As
expected from [AKMV], the construction depends only on the local geometry near a vertex of the moment polytope of $X$. In the case where the associated moduli space of open maps is rigorously defined ([KL]), this formula is proven to be correct. Most intriguingly, this result is shown to apply in two unexpected contexts: enumerative invariants associated to the quintic 3-fold, and Lagrangian cycles obtained from torus knots appearing in large $N$ duality.

The author hopes that the approach described herein will lead to a more general construction of open Gromov-Witten invariants.

1.2. Organization of the paper. The paper is organized in the following way. Section 2 reviews some general facts about open Gromov-Witten theory, including deformation theory and localization. Most importantly, this section describes how to express an open Gromov-Witten invariant as the product of a “disk term” and an invariant of closed maps. Section 3 contains the proof of the main computational tool of this paper:

**Theorem.** Let $X$ be a Calabi-Yau 3-fold and $L \subset X$ a Lagrangian submanifold. Let $S^1$ act on $X$ such that the $S^1$ action preserves $L$, and $L$ intersects a rigid circle-invariant curve $C$. Suppose that $L$ can be described in a neighborhood of $L \cap C$ as the fixed locus of an anti-holomorphic involution. Let $\gamma \in H^2(X; \mathbb{Q})$. Then, the genus $g$, 1 boundary component, degree $d$, winding $w$ open Gromov-Witten invariant with Lagrangian boundary $L$ is

$$\langle \gamma \rangle^{g,1}_{d,w} = \left( \Delta_{X,L} \circ \langle \gamma, \frac{\phi_p}{z - \psi} \rangle_{g,d} \right) \bigg|_{z = \alpha} ,$$

where $\Delta_{X,L}$ is the disk function

$$\Delta_{X,L}(\gamma) := \frac{\pi}{wz \Gamma_X \sin \left( \frac{\pi \lambda}{2} \right)} \cdot \gamma .$$

Here, $\Gamma_X$ is the homogeneous Iritani gamma class, $\lambda$ is the weight of the $S^1$ action along a normal direction to $C$, $\alpha = c_1(T_0 \Delta)$ is the equivariant Chern class of the induced representation of $S^1$ at the attachment point of the disk, and $\phi_p$ is the equivariant class of the image $p \in X$ of the disk attachment point.

This gamma class formula was encountered previously in [BCR], where the authors study a Lagrangian locally described as the fixed locus of the antiholomorphic involution $\sigma (\xi, x, y) = (1/\xi, y\xi, x\xi)$. As will be seen (lemma 3), this result implies the formula above after a change of coordinates. Section 4 describes how to apply the gamma class formula to several examples where the resulting invariant is already known, and demonstrates that this formula reproduces the expected result. The last two sections study two examples where the assumptions on the local geometry of $L$ are not satisfied. Section 5 examines disk enumeration on the quintic 3-fold and finds that a slight modification of this formula again applies. Finally, Section 6 applies this formula to a novel class of Lagrangian cycles motivated by large $N$ duality. These Lagrangian cycles are obtained from the conormal bundles of torus knots in $S^3$ after the conifold transition, and do not have the same local description required in the above theorem. Nevertheless, the main result of this paper is still found to apply to these cycles. The examples in sections 5 and 6 hint that a version of the theorem may hold for a broader class of Lagrangian cycles.
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2. Preliminaries

2.1. Deformation theory for stable maps. Open Gromov-Witten invariants are enumerative invariants of maps \( f : \Sigma \to X \) from Riemann surfaces with boundary into a Calabi-Yau manifold \( X \) with a chosen Lagrangian submanifold \( L \), such that \( f(\partial \Sigma) \subset L \). By analogy with the definition for closed stable maps appearing in [Kon], [KL] define the open Gromov-Witten invariant in the following way. Fix integers \( g \) (the genus of \( \Sigma \)) and \( h \) (the number of connected components of \( \partial \Sigma \)), and a relative homology class \( d \in H_2(X,L;\mathbb{Z}) \) with \( \partial d = \sum w_i \in H_1(L;\mathbb{Z}) \). Then, the open Gromov-Witten invariant \( GW_{g,h}^{d,w_1,\ldots,w_h} \) is a virtual count of continuous maps \( f : (\Sigma,\partial \Sigma) \to (X,L) \) satisfying:

- \((\Sigma,\partial \Sigma) \) is a Riemann surface of genus \( g \) with boundary \( \partial \Sigma \) consisting of \( h \) oriented circles,
- \( f \) is holomorphic in the interior of \( \Sigma \),
- \( f_* [\Sigma] = d \), and
- \( f_* [\partial \Sigma] = \sum w_i \).

For brevity, \( w_1,\ldots,w_h \) will sometimes be denoted by \( \vec{w} \). In order to define such an invariant, [KL] construct a moduli space \( \overline{M}_{g,h}(X,L;d,\vec{w}) \) of stable maps which compactify the maps described above, and give a local description of an orientation and a virtual fundamental class on this moduli space. In particular, the authors generalize the deformation complex in ordinary Gromov-Witten theory to the open case.

Recall that for smooth, closed \( \Sigma \) in ordinary Gromov-Witten theory, there is a normal bundle exact sequence of vector bundles on \( \Sigma \):

\[
0 \longrightarrow T_\Sigma \longrightarrow f^* T_X \longrightarrow N_{\Sigma/X} \longrightarrow 0.
\]

The corresponding long exact sequence in cohomology is

\[
0 \longrightarrow H^0(\Sigma, T_\Sigma) \longrightarrow H^0(\Sigma, f^* T_X) \longrightarrow H^0(\Sigma, N_{\Sigma/X}) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
H^1(\Sigma, T_\Sigma) \quad H^1(\Sigma, f^* T_X) \quad H^1(\Sigma, N_{\Sigma/X}) \quad 0
\]

The terms in this sequence can be interpreted as infinitesimal automorphisms, deformations, and obstructions to deformations for \( \Sigma \) and \( f \), so this sequence can be re-written as the deformation complex:

\[
0 \quad Aut(\Sigma) \quad \text{Def}(f) \quad \text{Def}(\Sigma, f) \\
\text{Def}(\Sigma) \quad Obs(f) \quad Obs(\Sigma, f) \quad 0.
\]

Suitably interpreted, the same sequence holds for nodal, open curves in open Gromov-Witten theory: Over a smooth point \((\Sigma, f)\) in \( \overline{M}_{g,h}(X,L;d,\vec{w}) \), \( H^k(\Sigma, f^* T_X) \) are the cohomology groups associated to sections \( s \) of \((\Sigma, f^* T_X)\) satisfying \( s|_{\partial \Sigma} \in \Gamma(\partial \Sigma, f^* T_L) \).
The expected (virtual) dimension of $\overline{M}_{g,h} (X,L; d, \vec{w})$ is
\[
\text{vdim } \overline{M}_{g,h} (X,L; d, \vec{w}) = \text{rank Def (} \Sigma, f) - \text{rank Obs (} \Sigma, f) \\
= \mu (f^* T_X, f|_{\partial \Sigma}^* T_L) - (\dim X - 3) \chi (\Sigma),
\]
where $\mu$ denotes the generalized Maslov index of the real subbundle $(f|_{\partial \Sigma})^* T_L \subset f^* T_X$ [KL]. When $X$ is a complex manifold and $L$ is the fixed locus of an anti-holomorphic involution, $\mu (f^* T_X, (f|_{\partial \Sigma})^* T_L) = \int_d c_1 (T_X)$, and if $X$ is a Calabi-Yau threefold, $\text{vdim } \overline{M}_{g,h} (X,L; d, \vec{w}) = 0$. When $\overline{M}_{g,h} (X,L; d, \vec{w})$ has a well-behaved torus action, [KL] give an explicit description for the localization of the virtual fundamental class to the fixed loci of the torus action. In contrast to closed Gromov-Witten theory, the virtual cycle found in [KL] depends on the torus action. Additionally, the invariants defined in [KL] depend on a choice of orientation. This choice is reflected in the overall sign of the invariant, and the invariant formula proposed in this note has an analogous orientation-dependent sign.

2.2. Separating the disk term. Gromov-Witten invariants are, in general, difficult to compute. The primary computational tool is the Atiyah-Bott fixed-point formula [AB]. When applied to computations of Gromov-Witten invariants for toric varieties, this “localizes” integrals over the entire moduli space of stable maps to integrals over only those maps which are fixed by the torus action [Kon, GP, KZ]. The Atiyah-Bott fixed point formula for the integral of a class $\phi$ over a manifold (or more generally, a Deligne-Mumford stack) $M$ is
\[
\int_M \phi = \sum_P \int_P \left( \frac{i_P^* \phi}{e(N_P)} \right),
\]
where the sum is over the fixed point sets $P$, $i_P$ is the embedding of $P$ into $M$, and $e(N_P)$ is the (equivariant) Euler class of the normal bundle of $P$ in $M$.

Following [GP, Kon], a stable map $(\Sigma, f)$ can be naturally described as a decorated graph. The vertices $v$ of the graph correspond to contracted components of the nodal curve $\Sigma$, and are labeled by the genus $g(v)$ of that component. The edges correspond to $\mathbb{P}^1$’s which are not contracted by $f$, and are labeled by the degree $d_e$ of the map $f|_{\mathbb{P}^1}$ associated to the edge. When the fixed stable maps are described as decorated graphs in this way, (2) becomes
\[
\text{GW}^g_d := \int_{[\overline{M}_{g,0}(X,d)]^{vir}} 1 = \sum_{\Gamma} \frac{1}{|A_\Gamma|} \int_{M_\Gamma} \frac{1}{e(N^{vir}_\Gamma)}.
\]

As observed in [GZ], the graph description of stable maps can be extended to the open stable maps defined in [KL] by treating the open disk component as a “leg” of the graph. A crucial consequence of this is that open Gromov-Witten invariants can be expressed as a closed Gromov-Witten invariant multiplied by a “disk term.” For simplicity, restrict attention to surfaces with one boundary component. Let $X$ be a Calabi-Yau manifold equipped with an $S^1$ action that fixes a Lagrangian submanifold $L \subset X$. Suppose that $f : \Sigma \to X$ is a stable map from a genus $g$ Riemann surface with one boundary component such that $f_* [\Sigma] = d \in H_2 (X; \mathbb{Z})$ and $f|_{\partial \Sigma} : \partial \Sigma \to f (\partial \Sigma)$ has winding $w$ as a map between homotopy circles. Let $\overline{M} := \overline{M}_{g,1,0} (X,L; d, w)$ denote the the moduli space of such maps (genus $g$, 1 boundary component, 0 marked points).
The $S^1$ action on $X$ naturally induces an $S^1$ action on $\mathcal{M}$. If $(\Sigma, f) \in \mathcal{M}$ is fixed by the $S^1$ action, then $\Sigma$ must take the form

$$\Sigma = \Sigma_0 \cup_\nu \Delta,$$

where $\Sigma_0$ is a closed genus $g$ Riemann surface, $\Delta$ is a disk, and $\nu$ is a simple node on $\Sigma_0$ at which $\Delta$ is attached. $S^1$ invariance further requires that $(\Sigma_0, f|_{\Sigma_0}, \nu)$ is fixed by the induced action on $\mathcal{M}_{g,1}(X, d)$.

Then, a virtual localization formula analogous to (3) for the genus $g$, degree $d$, winding $w$ open Gromov-Witten invariant would take the form

$$GW_{g,w}^{d,1} = \int_{\mathcal{M}_{g,w}^{vir}} \frac{1}{|A_\Gamma|} \int_{T} e\left( N_{\Gamma}^{vir}\right),$$

where $A_\Gamma = \mathbb{Z}/w\mathbb{Z} \times A_\Gamma'$ ($\Gamma'$ is the graph associated to the closed curve $(\Sigma_0, \nu)$).

Note that $\mathcal{M}_{g,1}(X, d)$ is equipped with a natural map $e_\nu : \mathcal{M}_{g,1}(X, d) \to X$ given by evaluation at the marked point. The conditions on $(\Sigma, f)$ specified above (in particular, that $f(\nu) = p$) imply that the fixed locus $M_\Gamma$ is isomorphic to the fixed subspace $e_\nu^{-1}(p)^{S^1} \subset \mathcal{M}_{g,1}(X, d)^{S^1}$.

As in the closed case, the equivariant normal bundle $e\left(N_{\Gamma}^{vir}\right)$ and the virtual fundamental cycle are determined, respectively, by the moving and fixed parts of the deformation complex (1):

$$0 \to \text{Def}(\Sigma) \to \text{Def}(\Sigma, f) \to \text{Obs}(\Sigma, f) \to 0$$

This gives the following relationship in the representation ring of $S^1$:

$$\text{Obs}(\Sigma, f) - \text{Def}(\Sigma, f) = \text{Aut}(\Sigma) + \text{Obs}(f) - \text{Def}(\Sigma) - \text{Def}(f)$$

with $\text{Obs}(f) = H^1(\Sigma', f^*T_X)$, $\text{Def}(f) = H^0(\Sigma', f^*T_X)$, $\text{Aut}(\Sigma) = \text{Ext}^0 (\Omega_{\Sigma}(D), \Omega_{\Sigma})$, and $\text{Def}(\Sigma) = \text{Ext}^1 (\Omega_{\Sigma}(D), \Omega_{\Sigma})$. (Here, $D$ is the divisor associated to the nodal points of $\Sigma$. When $\Sigma$ is smooth, these spaces are just $H^0(\Sigma, T_{\Sigma})$ and $H^1(\Sigma, T_{\Sigma})$, respectively).

Now, relate the terms in this sequence to the terms concerning $\Sigma_0$ and $\Delta$: Let $f_0 := f|_{\Sigma_0}$ and $f_\Delta := f|_{\Delta}$. Suppose that $\Delta$ is parametrized by $\{|t| \leq 1\}$, with $\nu$ identified with the point $t = 0$. Then, there is an exact sequence

$$0 \to \mathcal{O}_\Sigma \to \mathcal{O}_{\Sigma_0} \oplus \mathcal{O}_\Delta \to \mathcal{O}_\nu \to 0.$$

This becomes the exact sequence on cohomology:

$$0 \to \text{Def}(f) \to H^0(\Delta, T_{(\Delta, f_\Delta)}) \oplus \text{Def}(f_0) \to T_pX$$

which yields the following relations in the representation ring:

$$\text{Obs}(f) - \text{Def}(f) = H^1(\Delta, T_{(\Delta, f_\Delta)}) - H^0(\Delta, T_{(\Delta, f_\Delta)}) + \text{Obs}(f_0) - \text{Def}(f_0).$$
\[ \text{Obs}(f)^m - \text{Def}(f)^m = H^1(\Delta, T_{(\Delta,f\Delta)})^m - H^0(\Delta, T_{(\Delta,f\Delta)})^m \\
+ \text{Obs}(f_0)^m - \text{Def}(f_0)^m + T_pX, \]

where \( p = f(\nu) \in X \) and the \( f, m \) superscripts denote fixed and moving terms with respect to the \( S^1 \) action.

Similarly,

\[
\text{Aut}(\Sigma)^m = \text{Aut}(\Sigma_0, \nu)^m + \text{Aut}(\Delta, 0)^m, \\
\text{Def}(\Sigma)^f = \text{Def}(\Sigma_0, \nu)^f + \text{Aut}(\Delta, 0)^f, \\
\text{Aut}(\Sigma)^f = \text{Aut}(\Sigma_0, \nu)^f, \\
\text{Def}(\Sigma)^m = \text{Def}(\Sigma_0, \nu)^m + T_p\Sigma_0 \otimes T_0\Delta.
\]

Note that \( \text{Aut}(\Delta, 0)^m \) consists of the infinitesimal automorphisms of \( \Delta \) preserving the origin \( t = 0 \), which are generated by the sections \( t\partial_t \) over \( \mathbb{R} \). Therefore, \( \text{Aut}(\Delta, 0)^m \) is trivial, and \( \text{Aut}(\Delta, 0)^f = \mathbb{R} \).

Collecting the above observations,

\[
\text{Obs}(\Sigma, f)^f - \text{Def}(\Sigma, f)^f = H^1(\Delta, T_{(\Delta,f\Delta)})^f - H^0(\Delta, T_{(\Delta,f\Delta)})^f \\
+ \text{Obs}(f_0)^f - \text{Def}(f_0)^f \\
+ \text{Aut}(\Sigma_0, \nu)^f - \text{Def}(\Sigma_0, \nu)^f \\
+ \text{Aut}(\Delta, 0)^f,
\]

and

\[
\text{Obs}(\Sigma, f)^m - \text{Def}(\Sigma, f)^m = H^1(\Delta, T_{(\Delta,f\Delta)})^m - H^0(\Delta, T_{(\Delta,f\Delta)})^m \\
+ \text{Obs}(f_0)^m - \text{Def}(f_0)^m \\
+ \text{Aut}(\Sigma_0, \nu)^m - \text{Def}(\Sigma_0, \nu)^m \\
+ T_pX - T_p\Sigma_0 \otimes T_0\Delta.
\]

The first equation implies that the virtual fundamental cycle of the fixed locus

is the restriction of the natural virtual cycle of the fixed locus \([\overline{\mathcal{M}}_{g,1}(X, d)\mathbb{G}^{vir}]^\nu \)

to the subspace \( e^{-1}(p)^{S^1} \). The second equation yields the following relationship

between the normal bundles \( e(N_{g,1}^{vir}) \) and \( e(N_{g,1}^{vir}) \) (where \( \Gamma^f \) is the graph for the closed curve with the disk “leg” removed, i.e., the stable map associated to \( \Sigma_0 \) with one marked point):

\[
N_{g,1}^{vir} = N_{g,1}^{vir} - T_pX + R\mathbb{L}^{-1} \\
- H^1(\Delta, T_{(\Delta,f\Delta)})^m + H^0(\Delta, T_{(\Delta,f\Delta)})^m.
\]

Here \( R \) is the representation of \( S^1 \) on \( T_0\Delta \cong \mathbb{C} \) induced by the pullback of the \( S^1 \) action on \( f(\Delta) \), and \( \mathbb{L} \) is the tautological cotangent line bundle on \( \overline{\mathcal{M}}_{g,1}(X, d) \) associated to the marked point \( \nu \), i.e., the line bundle whose fiber at the point \( (f_0, \Sigma_0, \nu) \) is \( T_p\Sigma_0 \). \( R\mathbb{L}^{-1} \) is contribution from the term \( T_p\Sigma_0 \otimes T_0\Delta \): \( T_p\Sigma_0 \) is the fiber of \( \mathbb{L} \) and \( T_0\Delta \) is a constant vector space which carries the representation \( R \) by \( S^1 \).

Hence,

\[
(5) \quad \int_{M_{g,1}} \frac{1}{e(N_{g,1}^{vir})} = e_{S^1}(H^1(\Delta, T_{(\Delta,f\Delta)})) e_{S^1}(T_pX) \\
\cdot \int_{M_{g,\nu}} \frac{1}{e(N_{g,\nu}^{vir})(\alpha - \psi)^\nu},
\]
where $\alpha = c_1(R)$ and $\psi = c_1(L)$ (so that $e_{S^1}(R_{\mathbb{C}^*}) = \alpha - \psi$). As before, $e_{S^1}(\cdot)$ denotes the $S^1$-equivariant Euler class of the specified bundle. Denote by $D_{X,L}$ the “disk factor”

$$D_{X,L} := \left(\frac{1}{w}\right) \frac{e_{S^1}(H^1(\Delta, T(\Delta, f_{\Delta})))}{e_{S^1}(H^0(\Delta, T(\Delta, f_{\Delta})))}.$$ 

Then, (4) becomes

$$GW^{g,d,w}_{d,w} = \int_{M_{\text{vir}}} 1 = D_{X,L} \left(\sum_{i' \neq i} \frac{1}{|A_{i'}|} \int_{M_{i'}} \frac{i^*ev^*(\phi_p)}{e\left(N_{\text{vir}}^{i'}\right)(\alpha - \psi)}\right),$$

where $\phi_p$ is the equivariant Thom class of the point $p \in X$, and $i^*$ the pullback to the fixed locus $M_{i'}$. Comparing this formula with (3) shows that the parenthetical quantity is the localization of a closed Gromov-Witten invariant:

$$GW^{g,1}_{d,w} = D_{X,L} \int_{[\overline{M}_{g,1}(X, d)]^{\text{vir}}} \frac{ev^*(\phi_p)}{e\left(N_{\text{vir}}^{i'}\right)(\alpha - \psi)}.$$ 

3. The Gamma Class Formula

3.1. A formula for open Gromov-Witten invariants. The proposed formula for open Gromov-Witten invariants is obtained by composition of a disk function $\Delta_{X,L}$ and a descendant invariant. $\Delta_{X,L}$ is built from combinatorial data about the moment polytope, and a characteristic class. Recall that if $\delta_i$ are the Chern roots of a complex vector bundle $E$, Iritani’s gamma class [Iri] is a characteristic class associated to $E$ defined by

$$\Gamma_E := \prod_{\delta_i} \Gamma(1 + \delta_i).$$

As observed in [BCR], in some cases the disk term $D_{X,L}$ (6) can be expressed using gamma classes. The gamma class also appears in quantum cohomology, and can be regarded as a localization contribution from constant maps in Floer theory [GGI].

The inputs of $\Delta_{X,L}$ are the torus weight $\lambda$ of a normal direction to $f(\partial \Sigma)$, and the homogenized Iritani gamma class $\hat{\Gamma}_X \in H^* (X; \mathbb{Q}) (z)$, defined by

$$\hat{\Gamma}_X := \prod_{\delta_i} \Gamma \left(1 + \frac{\delta_i}{z}\right),$$

where $\delta_i$ are the Chern roots of the tangent bundle $T_X$. (When $\deg z = 2$, $\deg \hat{\Gamma}_X = 0$). With these definitions, the main result is:

**Theorem 1.** Let $X$ be a Calabi-Yau 3-fold and $L \subset X$ a Lagrangian submanifold. Let $S^1$ act on $X$ such that the $S^1$ action preserves $L$, and $L$ intersects a rigid circle-invariant curve $C$. Suppose that $L$ can be described in a neighborhood of $L \cap C$ as the fixed locus of an anti-holomorphic involution. Let $\gamma \in H^2 (X; \mathbb{Q})$. Then, the genus $g$, 1 boundary component, degree $d$, winding $w$ open Gromov-Witten invariant with Lagrangian boundary $L$ is

$$\langle \gamma \rangle^{g,1}_{d,w} = \left(\Delta_{X,L} \circ \left(\frac{\phi_p}{z - \psi}\right)_{g,d}\right)|_{z = \alpha},$$
where $\Delta_{X,L}$ is the disk function

$$\Delta_{X,L}(\gamma) := \frac{\pi}{w z \Gamma_X} \sin \left( \frac{\pi \cdot \gamma}{2} \right) \cdot \gamma.$$  

Here, $\Gamma_X$ is the homogeneous Iritani gamma class, $\lambda$ is the weight of the $S^1$ action along a normal direction to $C$, $\alpha = c_1(T_0 \Delta)$ is the equivariant Chern class of the induced representation of $S^1$ at the attachment point of the disk, and $\phi_p$ is the equivariant class of the image $p \in X$ of the disk attachment point.

Remark. The setup in theorem 1 is depicted in Figure 1. In [BCR], the authors study the orbifold version of this scenario with a Lagrangian obtained by the anti-holomorphic involution $\sigma(\xi,x,y) = (1/\xi, \xi y, \xi x)$, and obtain an analogous result. The disk term (9) only differs from [BCR, equation 28] in notation: here, $z = \alpha$, and the winding $w$ is included to account for automorphisms of the leg.

Although the result above is stated in a more general form, it will be shown below (lemma 3) that after a change of coordinates, $\sigma(\xi,x,y) = (1/\xi, \xi y, \xi x)$ is actually the only $S^1$-equivariant anti-holomorphic involution satisfying these assumptions. In particular, the result proven in this note applies to Aganagic-Vafa branes [AV]. This author is unsure how to prove the result without any assumptions on the local geometry of $L$—in order to apply localization, one requires a description of the torus-fixed disks with boundary on $L$. However, as will be shown in Section 6, this result also applies without modification to a family of Lagrangian cycles which are not obtained from the fixed locus of an anti-holomorphic involution.

Figure 1. A local picture near a vertex in the toric polytope.

A Lagrangian $L$, obtained as the fixed locus of an anti-holomorphic involution, intersects an edge of the toric polytope, labeled by the local coordinate $\xi$. The only torus-fixed disks are the hemispheres $D$ with $\partial D = L \cap \{x = y = 0\} \cong S^1$. Such a map is given locally by $t \mapsto (\xi = t^w, x = 0, y = 0)$. The $x$-$y$ hyperplane is normal to the disk. The normal directions to $C$ are spanned by $\partial_x$ and $\partial_y$, so the weight $\lambda$ appearing in $\Delta_{X,L}(\gamma)$ can be the weight of any $S^1$-invariant line spanned by these vectors (for example, $\lambda_x$ or $\lambda_y$).

In the genus 0 case, the closed Gromov-Witten invariants in formula (7) also appear as terms in Givental’s $J$ function [Giv]. Givental’s $J$ function is the map on quantum cohomology $J_X: H^\ast(X; \mathbb{Q}) \rightarrow H^\ast(X; \mathbb{Q}) (z)$ given by

$$J_X(\gamma) = z + \gamma + \sum_{n=0}^{\infty} \sum_{d \in H_2(X, \mathbb{Z})} \left\langle \gamma^n \frac{T^a}{z - \psi} \right\rangle_{0,d} T^a.$$
where $T^\alpha$ is a basis for the cohomology of $X$, $T^\alpha$ is the dual basis with respect to the Poincaré pairing, and

$$\left\langle \gamma^n, \frac{T^\alpha}{z - \psi} \right\rangle_{0,d} = \sum_{k=0}^{\infty} z^{-(k+1)} \left\langle \gamma^n, \tau_k T^\alpha \right\rangle_{0,d}$$

is a power series of gravitational descendant closed Gromov-Witten invariants. To obtain the genus-$g$ generating function, a higher-genus version of the $J$-function is needed. Define the genus-$g$ modified $J$-function $J^g_X : H^\ast (X; \mathbb{Q}) \to H^\ast (X; \mathbb{Q}) (z)$ to be

$$J^g_X (\gamma) = z + \sum_{n=0}^{\infty} \sum_{d \in H_2 (X; \mathbb{Z})} \frac{q^d}{n!} \left\langle \gamma^n, \frac{T^\alpha}{z - \psi} \right\rangle_{g,d} T^\alpha,$$

where

$$q^d = e^{2\pi i \int \omega}$$

and $\omega$ is the complexified Kähler class of $X$. From (10) and theorem 1, it is easy to write a generating function for open Gromov-Witten invariants. The generating function for the one-boundary-component, winding $w$ open Gromov-Witten invariants $\langle \gamma \rangle_{g,1}^{d,w}$ is the function

$$\Phi_w (\gamma) := \sum_{g \geq 0} \sum_{n \geq 0} \sum_{d \in H_2 (X; \mathbb{Z})} g_s^{2g-1} \frac{q^d}{n!} \langle \gamma^n \rangle_{d,w}^{g,1},$$

where $g_s$ is the string coupling constant, $\gamma \in H^\ast (X; \mathbb{Q})$, and the summation is only over combinations of $g$, $n$, and $d$ where the summands are defined.

**Corollary 2.** Let $X$ and $L$ be as in Theorem 1. Then, a generating function for the winding-$w$ open Gromov-Witten invariants of $\overline{M}_{g,1} (X, L; d, w)$ is given by the formula

$$\Phi_w (\gamma) = \sum_{g \geq 0} g_s^{2g-1} (\Delta_{X,L} \circ J^g_X (\gamma, \phi_p)) |_{z=\alpha}.$$  

**Remark.** Two observations about this generating function merit mention. First, in [BC, Bri], the authors use a similar procedure to obtain a generating function for open invariants from a modification of the $J$ function. The main distinction here is the presentation of the disk term. Second, for $\gamma = 1$, $\Phi_w$ has the expression

$$\Phi_w = \sum_{g \geq 0} \sum_{d \in H_2 (X; \mathbb{Z})} g_s^{2g-1} q^d GW_{d,w}^{g,1}.$$  

As will be discussed in Section 6, [DSV] have found that, for a certain class of Lagrangian cycles originating from torus knots, the expression above encodes the HOMFLY polynomial associated to the original knot.

### 3.2. Proof of the main result

The main content of the proof of theorem 1 is the comparison of the predicted disk term from (9) with an explicit localization calculation of the open Gromov-Witten invariant.
Virtual localization of open Gromov-Witten invariants. First, recall the virtual localization technique: As described in Section 2.2, the open Gromov-Witten invariant $GW^g_{d,w}$ can be expressed as a product of a disk term $D_{X,L}$ and a descendant invariant. The surface $\Sigma$ can be written as $\Sigma_0 \cup_\nu \Delta$, with $\Sigma_0$ is a closed surface, $\Delta$ a disk, and $\nu$ the point of attachment. In terms of the cohomology of sheaves over $\Delta$, $D_{X,L}$ was found to have the following expression:

$$D_{X,L} = \left( \frac{1}{w} \right) e_{S^1} \left( H^1(\Delta, T_{(\Delta, f_\Delta)}) \right) e_{S^1} \left( H^0(\Delta, T_{(\Delta, f_\Delta)}) \right),$$

where $f_\Delta : \Delta \to X$ is the restriction of the map $f : \Sigma \to X$ to the disk $\Delta$, $p = f(\nu)$, and $e_{S^1}(\cdot)$ denotes the $S^1$-equivariant Euler classes of the specified bundles. To compute the disk contribution to $GW^g_{d,w}$, one must compute each of these cohomology groups.

In contrast to the analogous computation of closed invariants, the Lagrangian $L$ imposes boundary conditions on the sections of $T_{(\Delta, f_\Delta)}$. Let $f_\theta$ denote $f_\Delta|_{\partial \Delta}$. Then, $T_{(\Delta, f_\Delta)}$ consists of sections of $f_\Delta^*T_X$ satisfying $s|_{\partial \Delta} \in f_\Delta^*T_L$. To obtain an explicit presentation of the boundary conditions, let $Ann(L) \subset T_X|_L$ be the subbundle of the cotangent bundle $T^*_X$ which annihilates the tangent bundle $T_L \subset T^*_X|_L$. Choose a basis of sections $\alpha_1, \alpha_2, \alpha_3$ of $Ann(L)$ along the boundary $\partial D$ of the disk. (The $\alpha_i$ can be obtained by, for example, linearizing the equations defining $L$.) $T_{(\Delta, f_\Delta)}$ consists of the sheaf of germs of holomorphic sections of the bundle $f_\Delta^*T_X$ satisfying the boundary conditions

$$f_\theta^*(\alpha_j)(s|_{\partial \Delta}) = 0, \quad j = 1, 2, 3.$$  \hspace{1cm} (11)

With this presentation of the boundary conditions, computing $H^1(\Delta, T_{(\Delta, f_\Delta)})$ becomes an exercise in Čech cohomology: Let $U = \{t : 0 < |t| \leq 1\}$, $U' = \{t : 0 \leq |t| < 1\}$ be an open cover of $\Delta$, and let $x, y, \xi$ be local coordinates such that $f(\nu) = p$ is the origin $(x, y, \xi) = (0, 0, 0)$. Then, local sections over $U$ and $U'$ are of the form

$$s = \sum_{k \in \mathbb{Z}} (a_k t^k \partial_x + b_k t^k \partial_y + c_k t^k \partial_\xi),$$

$$s' = \sum_{k \geq 0} (a'_k t^k \partial_x + b'_k t^k \partial_y + c'_k t^k \partial_\xi),$$

and the coefficients $a_k, b_k, c_k$ are subject to boundary conditions imposed by (11).

Finally, to apply localization, let $\rho_\theta : X \to X$ denote the $S^1$ action determined by

$$\rho_\theta(x, y, \xi) = (e^{i\lambda_x \theta} x, e^{i\lambda_y \theta} y, e^{i\lambda_\xi \theta} \xi), \quad \theta \in S^1.$$  \hspace{1cm} (11)

The weights $\lambda_x$, $\lambda_y$, $\lambda_\xi$ are required to satisfy $\lambda_x + \lambda_y + \lambda_\xi = 0$ so that the holomorphic volume form is preserved by the $S^1$ action.

Boundary conditions. Now, suppose that near the intersection $L \cap C$, $L$ is described as the fixed locus of an anti-holomorphic involution $\sigma$. Choose the local coordinates $x, y, \xi$ such that $L \cap C$ is defined by $x = y = 0, |\xi|^2 = 1$.  

Lemma 3. For a Calabi-Yau $S^1$ action on $\mathbb{C}^3$, the only $S^1$-equivariant anti-holomorphic involution defined on $(x, y, \xi) \in \mathbb{C}^2 \times \mathbb{C}^*$ and fixing the circle $x = y = 0$, $|\xi|^2 = 1$ is

$$\sigma(x, y, \xi) = \left(X_0 \overline{x}, Y_0 \overline{x}, \frac{1}{\xi}\right),$$

where $X_0, Y_0 \in \mathbb{C}^*$ are two constants satisfying $X_0 Y_0 = 1$.

Note that the Calabi-Yau condition means that the holomorphic 3-form $dx \wedge dy \wedge d\xi$ is preserved by the $S^1$ action, i.e., if $S^1$ acts with weights $(\lambda_x, \lambda_y, \lambda_\xi)$, then $\lambda_x + \lambda_y + \lambda_\xi = 0$.

Proof. $S^1$-equivariance is the requirement that $\rho_\theta \circ \sigma = \sigma \circ \rho_\theta$, and is a necessary condition to apply localization to $L$. (By assumption, in a neighborhood of $L \cap C$, $p \in L$ must satisfy $\rho_\theta(p) = \sigma(\rho_\theta(p))$. Because $L$ is defined locally by $p = \sigma(p)$, this is equivalent to $\rho_\theta \circ \sigma = \sigma \circ \rho_\theta$.

Generically, $\sigma$ takes the form

$$\sigma(x, y, \xi) = (\sigma_x, \sigma_y, \sigma_\xi),$$

where $\sigma_x$, $\sigma_y$, and $\sigma_\xi$ are Laurent series in the variables $\overline{\pi}, \overline{y}, \overline{\xi}$:

$$\sigma_x = \sum_{j,k,l \in \mathbb{Z}} X_{j,k,l} \overline{\pi}^j \overline{y}^k \overline{\xi}^l,$$
$$\sigma_y = \sum_{j,k,l \in \mathbb{Z}} Y_{j,k,l} \overline{\pi}^j \overline{y}^k \overline{\xi}^l,$$
$$\sigma_\xi = \sum_{j,k,l \in \mathbb{Z}} Z_{j,k,l} \overline{\pi}^j \overline{y}^k \overline{\xi}^l.$$

$S^1$-equivariance imposes restrictions on the coefficients $X_{j,k,l}$, $Y_{j,k,l}$, $Z_{j,k,l}$. For example, $\sigma_\xi$ must satisfy

$$\sigma_\xi(e^{i\lambda_\xi \theta} x, e^{i\lambda_x \theta} y, e^{i\lambda_\xi \theta} \xi) = e^{i\lambda_x \theta} \sigma_\xi(x, y, \xi)$$

for all $\theta \in S^1$. Expanding this, this is

$$\sum_{j,k,l} Z_{j,k,l} e^{-i\theta(j \lambda_x + k \lambda_y + l \lambda_\xi)} \overline{\pi}^j \overline{y}^k \overline{\xi}^l = e^{i\lambda_x \theta} \sum_{j,k,l} Z_{j,k,l} \overline{\pi}^j \overline{y}^k \overline{\xi}^l.$$

Hence, $-(l + 1) \lambda_\xi = j \lambda_x + k \lambda_y$. Recalling that $\lambda_x + \lambda_y = -\lambda_\xi$, this forces $j = k = l + 1$. Similar results hold for $\sigma_x$ and $\sigma_y$. Therefore, anti-holomorphic involutions commuting with the $S^1$ action along $L$ must take the form

$$\sigma_x = (y \xi) \sum_{l \in \mathbb{Z}} X_l \overline{y}^l \xi^l, \quad \sigma_y = (\overline{x} \xi) \sum_{l \in \mathbb{Z}} Y_l \overline{x}^l \xi^l, \quad \sigma_\xi = (\overline{y} \xi) \sum_{l \in \mathbb{Z}} Z_l \overline{y}^l \xi^l.$$

In fact there are further restrictions on $\sigma$. Because $\sigma$ must be defined along $\{x = y = 0, |\xi|^2 = 1\}$, $X_l = Y_l = 0$ for $l < 0$, and $Z_{-1} = 1, Z_l = 0$ for $l < -1$. 
Substituting these relations into $\sigma^2 = 1$, the equation $x = \sigma_x \circ \sigma (x, y, \xi)$ becomes

\[
x = \left( x \sum_{l \geq 0} Y_l (xy\xi)^l + x^2 y \sum_{j, k \geq 0} Y_j Z_k (xy\xi)^{j+k} \right) \times \sum_{l \geq 0} X_l \left( \frac{1}{\xi} \sum_{j, k \geq 0} X_j Y_k (xy\xi)^{j+k+1} + \sum_{i, j, k \geq 0} X_i Y_j Z_k (xy\xi)^{i+j+k+2} \right)^l = x Y_0 X_0 + (x^2 y \xi) g (x, y, \xi),
\]

where $g$ is a power series in $x, y, \xi$. Therefore, $X_0 Y_0 = 1$, and, for degree reasons, $g (x, y, \xi) = 0$. An analogous relation holds for $\sigma_y$. Similarly, $\xi = \sigma_\xi \circ \sigma (x, y, \xi)$ becomes

\[
\xi = \xi - \sum_{k \geq 2} (-1)^k \frac{\xi^k}{k} \left( xy \sum_{l \geq 0} Z_l (xy\xi)^l \right)^k + \left( \xi^2 xy \sum_{j, k \geq 0} X_j Y_k (xy\xi)^j \right) \times \sum_{l \geq 0} \left( \frac{1}{\xi} \sum_{j, k \geq 0} X_j Y_k (xy\xi)^{j+k+1} + \sum_{i, j, k \geq 0} X_i Y_j Z_k (xy\xi)^{i+j+k+2} \right)^l = \xi + (\xi^2 xy) h (x, y, \xi).
\]

As with $\sigma_x$, $h (x, y, \xi) = 0$. A careful examination of the coefficients in the above equations reveals that the only solutions to $\sigma^2 = 1$ are $Z_l = 0$ for $l \geq 0$ and $X_l = Y_l = 0$ for $l \geq 1$. This completes the proof of the lemma. $\square$

Applying the above lemma, the defining equations of $L$ near $L \cap C$ are

\[
x = X_0 y \xi, \quad y = Y_0 x \xi, \quad \xi = \frac{1}{\xi}.
\]

Linearizing these equations yields

\[
dx = X_0 (\xi d\bar{y} + y d\xi), \quad dy = Y_0 (\bar{\xi} d\bar{x} + \bar{x} d\xi), \quad d\xi = -\left( \frac{1}{\xi} \right)^2 d\xi^2,
\]

and at $x = y = 0$, these equations simplify to

\[
dx = X_0 \bar{\xi} d\bar{y}, \quad dy = Y_0 \bar{\xi} d\bar{x}, \quad d\xi = -\left( \frac{1}{\xi} \right)^2 d\xi.
\]

As $\xi = \bar{\xi}^{-1}$ along $|\xi|^2 = 1$, a basis for $Ann (L)$ along $\partial D$ is given by the 1-forms

\[
\alpha_1 = dx - X_0 \bar{\xi} d\bar{y}, \quad \alpha_2 = dy - Y_0 \bar{\xi} d\bar{x}, \quad \alpha_3 = \bar{\xi} d\xi + \xi d\bar{\xi}.
\]

Recall that local sections over $U = \{ t : 0 < |t| \leq 1 \}$ are of the form

\[
s = \sum_{k \in \mathbb{Z}} (a_k t^k \partial_x + b_k t^k \partial_y + c_k t^k \partial_\xi).
\]
Parameterizing $|t| = 1$ by $e^{i\theta} = t$, the map $f_\theta$ takes the form $f_\theta(e^{i\theta}) = (x = 0, y = 0, \xi = e^{iw\theta})$.
The boundary conditions $f_\theta^*(\alpha_j)(s|_{\partial U}) = 0$ impose restrictions on the coefficients $a_k, b_k, c_k$:

$$
f_\theta^*(\alpha_1)(s|_{\partial U}) = (dx - X_0e^{-iw\theta}dy) \left( \sum_{k \in \mathbb{Z}} (a_k e^{ik\theta} \partial_x + b_k e^{ik\theta} \partial_y + c_k e^{ik\theta} \partial_\xi) \right)
= \sum_{k \in \mathbb{Z}} (a_k e^{ik\theta} - X_0 e^{-iw\theta}b_k e^{-ik\theta}),
$$

$$
f_\theta^*(\alpha_2)(s|_{\partial U}) = (dy - Y_0e^{-iw\theta}dx) \left( \sum_{k \in \mathbb{Z}} (a_k e^{ik\theta} \partial_x + b_k e^{ik\theta} \partial_y + c_k e^{ik\theta} \partial_\xi) \right)
= \sum_{k \in \mathbb{Z}} (b_k e^{ik\theta} - Y_0 e^{-iw\theta}a_k e^{-ik\theta}),
$$

$$
f_\theta^*(\alpha_3)(s|_{\partial U}) = (e^{-iw\theta}d\xi + e^{iw\theta}d\bar{\xi}) \left( \sum_{k \in \mathbb{Z}} (a_k e^{ik\theta} \partial_x + b_k e^{ik\theta} \partial_y + c_k e^{ik\theta} \partial_\xi) \right)
= \sum_{k \in \mathbb{Z}} (c_k e^{(k-w)\theta} + \bar{c}_k e^{-(k-w)\theta}).
$$

These yield the following equations for the coefficients $a_k, b_k, c_k$:

$$
a_k - X_0\bar{\alpha}_{-k-w} = 0, \quad b_k - Y_0\bar{\alpha}_{-k-w} = 0, \quad c_k + \bar{\alpha}_{2w-k} = 0.
$$

The first two equations are actually equivalent: after complex conjugation, relabeling of indices, and substituting $X_0Y_0 = 1$, the second equation becomes the first. So, the Lagrangian boundary conditions on sections over $U$ are

$$
(12) \quad a_k = X_0\bar{\alpha}_{-k-w}, \quad c_k = \bar{\alpha}_{2w-k}.
$$

From these boundary conditions, the cohomology groups $H^i(\Delta, T(\Delta, f_\Delta))$ can be computed explicitly.

3.2.3. Computation of cohomology groups and equivariant classes. $H^0(\Delta, T(\Delta, f_\Delta))$ consists of the global sections, i.e., holomorphic sections $s$ on $\Delta$. These take the form

$$
s = \sum_{k \geq 0} (a_k t^k \partial_x + b_k t^k \partial_y + c_k t^k \partial_\xi),
$$

with $a_k, b_k$ and $c_k$ subject to the boundary conditions in (12), and $a_k = b_k = c_k = 0$ for $k < 0$. In particular, the equation $a_k = X_0\bar{\alpha}_{-k-w}$ implies that $a_k = 0$ for all $k$. Shifting indices $k \rightarrow -k - w$, this equation also implies that $b_k = 0$ for all $k$. Finally, from the last boundary equation $c_k = \bar{\alpha}_{2w-k}$, $c_k = 0$ for $k > 2w$. So, $H^0(\Delta, T(\Delta, f_\Delta))$ consists of sections of the form

$$
s = \sum_{k=0}^{w-1} (c_k t^k \partial_\xi + \bar{c}_k t^{2w-k} \partial_\xi) + c_w t^w \partial_\xi,
$$

where $c_w$ is real. As a vector space, $H^0(\Delta, T(\Delta, f_\Delta))$ is isomorphic to

$$
\mathbb{R} \{t^w \partial_\xi\} \oplus \bigoplus_{k=0}^{w-1} \mathbb{C} \{t^k \partial_\xi\}.
$$
The map \( f_\Delta \) takes \( t \mapsto (x = 0, y = 0, \xi = t^w) \), so \( S^1 \) action the section \( t^k \partial_x \) with weight \( \lambda_\xi (k/w - 1) \). \( t^w \partial_x \) is fixed by the \( S^1 \) action, so \( H^0 (\Delta, \mathcal{T} (\Delta, f_\Delta)) \) is just the complex part of this vector space. Hence,

\[
\begin{align*}
\epsilon_{S^1} (H^0 (\Delta, \mathcal{T} (\Delta, f_\Delta))) &= \prod_{k=0}^{w-1} \lambda_\xi \left( \frac{k}{w} - 1 \right).
\end{align*}
\]

\( H^1 (\Delta, \mathcal{T} (\Delta, f_\Delta)) \) consists of the cokernel to the Čech differential. Sections over \( U \cap U' \) can be written as

\[
\delta = \sum_{k \in \mathbb{Z}} (\alpha_k t^k \partial_x + \beta_k t^k \partial_y + \gamma_k t^k \partial_\xi)
\]

\[
= \sum_{k \leq -w} \alpha_k t^k \partial_x + \sum_{k=1-w}^{1} \alpha_k t^k \partial_x + \sum_{k \geq 0} \alpha_k t^k \partial_x
\]

\[
+ \sum_{k < 0} \beta_k t^k \partial_y + \sum_{k \geq 0} \beta_k t^k \partial_y
\]

\[
+ \sum_{k \geq 0} \gamma_k t^k \partial_\xi.
\]

The image of the Čech differential consists of sections \( \delta \) of the form \( \delta = s - s' \). In terms of the coefficients, this is

\[
\alpha_k = \begin{cases} a_k, & k < 0, \\ a_k - a'_k, & k \geq 0, \end{cases}, \quad \beta_k = \begin{cases} b_k, & k < 0, \\ b_k - b'_k, & k \geq 0, \end{cases}, \quad \gamma_k = \begin{cases} c_k, & k < 0, \\ c_k - c'_k, & k \geq 0, \end{cases}
\]

where again, \( a_k, b_k, \) and \( c_k \) are subject to the boundary conditions (12). Solutions always exist for \( \gamma_k \): set \( c_k = \gamma_k \) for \( k < 0 \), and set \( c'_k = c_{2w-k} \) for \( k \geq 0 \). Similarly, because \( a'_k \) and \( b'_k \) are completely free, any \( a_k \) and \( \beta_k \) for \( k \geq 0 \) can be solved for. However, to solve for \( \alpha_k \) and \( \beta_k \) for \( k < 0 \), it must be the case that \( b_k = \beta_k \). The first boundary equation \( a_k = X_0 b_{-w-k} \) then implies that \( a_k \) is fixed for \( -w < k < 0 \). So, there are no solutions if \( a_k \neq X_0 \beta_{-w-k} \) in \( -w < k < 0 \). When \( k \leq -w, -k - w \geq 0 \), so setting \( b_{-k-w} = \alpha_k \) and \( b'_{-k-w} = \alpha_k - \beta_{-k-w} \) will solve these equations. Therefore, the cokernel of the Čech differential is isomorphic to the space of sections \( \delta \) of the form

\[
\delta = \sum_{k=1-w}^{-1} \alpha_k t^k \partial_x.
\]

The induced \( S^1 \) action on \( t^k \partial_x \) has weight \( \frac{k}{w} \lambda_\xi - \lambda_x \). As a vector space, \( H^1 (\Delta, \mathcal{T} (\Delta, f_\Delta)) \) is

\[
\bigoplus_{k=1-w}^{-1} \mathbb{C} \langle t^k \partial_x \rangle,
\]

and

\[
\epsilon_{S^1} (H^1 (\Delta, \mathcal{T} (\Delta, f_\Delta))) = \prod_{k=1-w}^{-1} \left( \frac{k}{w} \lambda_\xi - \lambda_x \right).
\]
3.2.4. Comparison of disk terms. Substituting (13) and (14) in (6) yields

\[ D_{X,L} = \frac{1}{w} \prod_{k=1-w}^{w-1} \left( \frac{k}{w} \lambda_{\xi} - \lambda_x \right) \prod_{k=0}^{w-1} \lambda_{\xi} \left( \frac{k}{w} - 1 \right). \]

The proof will be complete if (15) is equivalent to the claimed expression (9):

\[ \left. \frac{1}{w} \prod_{k=1-w}^{w-1} \left( \frac{k}{w} \lambda_{\xi} - \lambda_x \right) \prod_{k=0}^{w-1} \lambda_{\xi} \left( \frac{k}{w} - 1 \right) \right|_{z=\alpha} = \frac{\pi}{wz \Gamma_X \sin \left( \frac{\lambda_y z}{2} \right)}. \]

First, observe that

\[ \prod_{k=0}^{w-1} \lambda_{\xi} \left( \frac{k}{w} - 1 \right) = \left( -\frac{\lambda_x}{w} \right)^w \Gamma \left( w + 1 \right). \]

Similarly,

\[ \prod_{k=1-w}^{w-1} \left( \frac{k}{w} \lambda_{\xi} - \lambda_x \right) = \left( -\frac{\lambda_{\xi}}{w} \right)^{-w} \prod_{k=1}^{w-1} \left( k + w \frac{\lambda_x}{\lambda_{\xi}} \right) = \left( -\frac{\lambda_{\xi}}{w} \right)^{-w} \Gamma \left( w \frac{\lambda_{\xi}}{\lambda_x} + w \right) \Gamma \left( w \frac{\lambda_{\xi}}{\lambda_x} + 1 \right). \]

Recall that, by assumption, \( \lambda_x + \lambda_y + \lambda_{\xi} = 0 \), so \( \frac{\lambda_x}{\lambda_{\xi}} = -1 - \frac{\lambda_y}{\lambda_{\xi}} \). Therefore,

\[ \Gamma \left( w \frac{\lambda_{\xi}}{\lambda_x} + w \right) = \Gamma \left( -w \frac{\lambda_y}{\lambda_{\xi}} \right). \]

The above manipulations show that

\[ \left. \frac{1}{w} \prod_{k=1-w}^{w-1} \left( \frac{k}{w} \lambda_{\xi} - \lambda_x \right) \prod_{k=0}^{w-1} \lambda_{\xi} \left( \frac{k}{w} - 1 \right) \right|_{z=\alpha} = \left( -\frac{\lambda_x}{w} \right)^w \Gamma \left( -w \frac{\lambda_y}{\lambda_{\xi}} \right) \Gamma \left( w \frac{\lambda_{\xi}}{\lambda_x} + 1 \right). \]

The induced action on \( T_0 \Delta \) carries weight \( \alpha = \frac{\lambda_{\xi}}{w} \). Substitute \( \frac{\lambda_{\xi}}{w} = z \) and apply Euler’s reflection formula to get

\[ D_{X,L} = \left( -\frac{1}{w} \right) \frac{\Gamma \left( -\frac{\lambda_y}{z} \right)}{z \Gamma \left( \frac{\lambda_{\xi}}{z} + 1 \right) \Gamma \left( \frac{\lambda_x}{z} + 1 \right)} \]

\[ = \left( \frac{1}{w} \right) \frac{\pi}{z \Gamma \left( \frac{\lambda_{\xi}}{z} + 1 \right) \Gamma \left( \frac{\lambda_x}{z} + 1 \right) \Gamma \left( \frac{\lambda_x}{z} + 1 \right) \sin \left( \frac{\pi \lambda_y}{2} \right)}{wz \Gamma_X \sin \left( \frac{\pi \lambda_x}{2} \right)}. \]

Generically, there are two \( S^1 \)-invariant normal directions to \( L \cap C \) in \( X \), given by the tangent vectors \( \partial_x \) and \( \partial_y \). In the formula in (9), \( \lambda \) may be the weight of either of these directions, i.e., \( \lambda = \lambda_y \) or \( \lambda_x \). The choice of \( \lambda \) changes the sign of (9) because \( \sin \left( \frac{\pi \lambda_y}{2} \right) = -\sin \left( \frac{\pi \lambda_x}{2} \right) \). This sign ambiguity reflects an overall choice of orientation of \( \overline{M}_{g,1,0} (X, L; d, w) \) [AKV, GZ, KL]. This completes the proof of the main result.
4. Comparison to Known Localization Calculations

In this section, (8) is compared to previous virtual localization calculations of open Gromov-Witten invariants. In the interest of brevity, only the geometric setup and final results are stated below; the reader interested in further localization calculations is referred to the original sources, or the computation appearing in the proof of theorem 1.

4.1. Simple Lagrangians for $\mathcal{O}_p: (-1, -1)$. This situation was first described in [KL], and the gamma class formula for an orbifold generalization of this case was given in [BCR]. For completeness, the scenario will also be described here. Let $X = \mathcal{O}_p: (-1, -1)$. $X$ appears often in Gromov-Witten theory and mirror symmetry: it is the small resolution of the conifold singularity, the normal bundle to a smooth rational line in a Calabi-Yau 3-fold, and it can be obtained from a $U(1)$ gauge theory with 4 chiral fields with charges $(1, 1, -1, -1)$. $X$ can be described symplectically using symplectic reduction on $\mathbb{C}^4$, and in this setting it is easiest to obtain the moment polytope of $X$.

Let $S^1$ act on $\mathbb{C}^4$ with weights $(1, 1, -1, -1)$. Then, the moment map for this action is

$$\mu: \mathbb{C}^4 \rightarrow \mathfrak{g}^1 \cong \mathbb{R}$$

$$(z_1, z_2, z_3, z_4) \mapsto \frac{1}{2} \left( |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 \right),$$

and it can be seen (for example, by choosing appropriate local coordinates and checking transition functions) that

$$X \cong \mu^{-1} \left( \left( \frac{r}{2} \right) \right) / S^1 = \left\{ |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = r \right\} / S^1$$

for $r \in \mathbb{R}_{>0}$ ($r$ determines the symplectic volume of the base $\mathbb{P}^1$). There’s a natural anti-holomorphic involution $\sigma$ on $\mathbb{C}^4$ given by

$$\sigma (z_1, z_2, z_3, z_4) = (\bar{z_2}, \bar{z_1}, \bar{z_3}, \bar{z_4}).$$

The fixed locus of this involution is a Lagrangian submanifold $\bar{L}$ of $\mathbb{C}^4$ defined by the equations

$$|z_1|^2 = |z_2|^2,$$

$$|z_3|^2 = |z_4|^2,$$

$$\bar{z_1} \bar{z_2} \bar{z_3} \bar{z_4} = z_1 z_2 z_3 z_4.$$

Because $\bar{L}$ is preserved by the $S^1$ action, $\mu^{-1} \left( \left( \frac{r}{2} \right) \right) \cap \bar{L} / S^1$ defines a Lagrangian $L \subset X$.

This Lagrangian is easy to visualize in the moment polytope of $X$. The moment polytope is the image of $X$ in $\mathbb{R}^4$ under the projection $z_i \mapsto |z_i|^2$. Then, $L$ is the intersection of the two planes $|z_1|^2 = |z_2|^2$ and $|z_3|^2 = |z_4|^2$ in the polytope. $L$ intersects the zero section $\mathbb{P}^1$ along its equator, so that

$$L \cap \mathbb{P}^1 = \left\{ |z_1|^2 = |z_2|^2, |z_3|^2 = |z_4|^2 = 0, |z_1|^2 + |z_2|^2 = r \right\} \cong S^1,$$

as depicted in Figure 2. There are two unique disks with boundary on $L$, corre-
Figure 2. The toric polytope for $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1)$ with a Lagrangian. This figure depicts the geometry described in Section 4.1. A Lagrangian $L$, obtained as the fixed locus of an anti-holomorphic involution, intersects the zero section of $X$. Local coordinates $(x, y)$ and $(z, w)$ parametrize the fibers of $X$ in a neighborhood of the two vertices of the moment polytope. One possible torus-fixed disk $D$ with boundary $\partial D = L \cap \mathbb{P}^1 \cong S^1$ is depicted.

Corresponding to the two hemispheres of the $\mathbb{P}^1$. In local coordinates $(\xi, x, y)$ defined by

$$
\xi = z_1 z_2, \quad x = z_2 z_3, \quad y = z_2 z_4,
$$

$L$ is defined by the fixed locus of the anti-holomorphic involution

$$
\sigma(\xi, x, y) = \left( \frac{1}{\xi}, \frac{x}{\xi} y, \frac{x}{\xi} x \right).
$$

It is readily checked that at $L \cap \mathbb{P}^1 = \{ |\xi|^2 = 1 \}$, and the disk based at the $z_1 = 0$ pole of the $\mathbb{P}^1$ takes the form $|\xi| \leq 1$. The winding $w$ disk map is

$$
t \mapsto (\xi = t^w, x = 0, y = 0).
$$

In this situation, $[KL]$ computed

$$
\left( \frac{1}{w} \right) e_{S^1} \left( H^1(\Delta, T_{\Delta,f_{\Delta}}) \right) = \left( \frac{1}{w} \right) \prod_{k=1-w}^{1} \lambda_{\xi - \lambda z} \prod_{k=0}^{1} \lambda_{\xi - \lambda - 1}.
$$

Comparing with (15) in Section 3.2.4, this is $i^*(\Delta_{X,L})_{z=\alpha}$, so this result agrees with the proposed formula.

4.2. The canonical bundle of $\mathbb{P}^2$. This situation was described in [GZ]. As in the previous example, $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ can be obtained via symplectic reduction. Let $S^1$ act on $\mathbb{C}^4$ with weights $(1, 1, 1, -3)$. Then,

$$
X \cong \mathbb{P}^1 \left( \frac{r}{2} \right) / S^1 = \left\{ |z_1|^2 + |z_2|^2 + |z_3|^2 - 3 |z_4|^2 = r \right\} / S^1
$$

for $r \in \mathbb{R}_{>0}$. [GZ] consider the Lagrangian submanifold $\tilde{L}_c \subset \mathbb{C}^4$ defined by

$$
|z_1|^2 - |z_3|^2 = c,
|z_2|^2 - |z_4|^2 = 0,
\bar{z}_1 z_2 \bar{z}_3 z_4 = \frac{1}{z_1 z_2 z_3 z_4},
$$
where \(-r < c < r\). Because \(\bar{L}_c\) is preserved by the \(S^1\) action, it descends to a Lagrangian \(L_c \subset X\), as depicted in Figure 3. Note that \(c\) parametrizes the intersection of \(L_c\) with the \(\mathbb{P}^1\) given by the image of \(|z_1|^2 + |z_3|^2 = r\) in the quotient space. At \(c = 0\), \(L_c\) intersects this curve along its equator. For simplicity, restrict attention to \(L = L_0\) (locally, other values of \(c\) can be obtained by a coordinate transformation). In local coordinates

\[
(16) \quad \xi = \frac{z_1}{z_3}, \quad x = \frac{z_2}{z_3}, \quad y = z_3 z_4,
\]

the Lagrangian \(L\) is the fixed locus of the anti-holomorphic involution

\[
\sigma (\xi, x, y) = \left( \frac{1}{\xi}, \xi y, \xi x \right).
\]

The disk is \(|\xi|^2 \leq 1\), and the winding \(w\) disk map is

\[
t \mapsto (\xi = t^w, x = 0, y = 0).
\]

So, locally, the situation computed in [GZ] is identical to [KL]. As seen in Section 4.1, this agrees with theorem 1.

Slightly extending the computation in [GZ], theorem 1 can also be used to compute the invariants associated to a Lagrangian cycle intersecting an external leg of the moment polytope. Let \(\tilde{L}\) be the submanifold of \(\mathbb{C}^4\) defined by

\[
|z_1|^2 - |z_2|^2 = 0,
|z_3|^2 - |z_4|^2 = 0,
\]

\[
z_1 z_2 z_3 z_4 = z_1 z_2 z_3 z_4.\]

Again, these equations are preserved by the \(S^1\) action, so the image of \(\tilde{L}\) in the quotient space \(X\) is a well-defined Lagrangian submanifold \(L\). \(L\) can be equivalently described as the fixed locus of the anti-holomorphic involution \((z_1, z_2, z_3, z_4) \mapsto \)

**Figure 3.** The toric polytope for \(X = O_{\mathbb{P}^2} (-3)\) with a Lagrangian. This figure depicts the geometry described in Section 4.2. A Lagrangian \(L\), obtained as the fixed locus of an anti-holomorphic involution, intersects the zero section of \(X\) along one edge of the moment polytope. One possible torus-fixed disk \(D\) with boundary \(\partial D = L \cap \mathbb{P}^2\) is depicted.
In local coordinates \((\xi, x, y)\) (16), \(L\) is defined by
\[
\begin{align*}
|y|^2 &= 1, \\
|x|^2 &= |\xi|^2, \\
\xi xy &= \xi xy.
\end{align*}
\]
(This Lagrangian cycle is shown in Figure 4). The disk is \(|y|^2 \leq 1\), and the winding

\[L \quad \text{disk map is} \quad t \mapsto (\xi = 0, x = 0, y = t^w). \quad \text{So,} \quad \alpha = \frac{\lambda_y}{w}. \quad \text{Applying theorem 1,}
\]
\[
\left( \frac{1}{w} \right) e^{1 \gamma} \left( H^1(\Delta, T(\Delta, f\Delta)) \right) = \left. \frac{\pi}{w z \Gamma_X \sin \left( \frac{\pi \lambda_y}{w} \right)} \right|_{z = \alpha} = -\frac{\pi \Gamma \left( -w \frac{\lambda_y}{\lambda_x} \right)}{\lambda_y \Gamma \left( \frac{w \lambda_y}{\lambda_x} + 1 \right) \Gamma (w + 1)}
\]
\[
= \frac{1}{w} \prod_{k=1-w}^{w-1} \left( \frac{k}{w} \lambda_y - \lambda_x \right).
\]
Here, the normal direction weight \(\lambda = \lambda_\xi\) has been chosen in (9). Choosing \(\lambda = \lambda_x\) instead changes the sign, reflecting the overall dependence of these counts on the choice of torus weights. This can be seen from the product identity:
\[
\prod_{k=1-w}^{w-1} \left( \frac{k}{w} \lambda_y - \lambda_x \right) = (-1)^{w-1} \prod_{k=1-w}^{w-1} \left( \frac{k}{w} \lambda_y - \lambda_x \right).
\]

5. Disk invariants on the quintic 3-fold

5.1. Real Lagrangian submanifolds of quintic 3-folds. In [PSW], the authors investigate open invariants for disks with boundary on a Lagrangian submanifold \(Q_R\) of a quintic 3-fold \(Q\). \(Q_R\) is obtained as the restriction to \(Q\) of the fixed locus of an anti-holomorphic involution on \(\mathbb{P}^4\); however, the associated invariants are computed by an integral over the moduli of stable disk maps to \(\mathbb{P}^4\), rather than \(Q\). As will be seen in Section 5.2, the result of theorem 1 still applies in this setting. This section reviews the geometric setup of [PSW].
Let $Q \subset \mathbb{P}^4$ be a nonsingular quintic hypersurface with symplectic form $\omega$ obtained from the Fubini-Study metric on $\mathbb{P}^4$. Let $\sigma$ be the anti-holomorphic involution on $\mathbb{P}^4$, described in homogeneous coordinates as

$$
\sigma \left([z_0 : z_1 : z_2 : z_3 : z_4]\right) = [\overline{z_0} : \overline{z_1} : \overline{z_2} : \overline{z_3} : \overline{z_4}].
$$

The fixed locus of $\sigma$ is a real Lagrangian submanifold $\mathbb{P}^4_\mathbb{R} \subset \mathbb{P}^4$, and its restriction to $Q$ is a real Lagrangian submanifold $Q_\mathbb{R} \subset Q$.

For the diagonal $T^5$ action on $\mathbb{P}^4$, there are five torus fixed points:

$$
\zeta_0 = [1 : 0 : 0 : 0 : 0], \quad \ldots, \quad \zeta_4 = [0 : 0 : 0 : 0 : 1].
$$

Only $\zeta_0$ is fixed by both the $T^5$ action and $\sigma$, so it is the unique real $T^5$-fixed point. Now, consider the rank-2 subtorus $T^2 \subset T^5$ acting by

$$
(\theta_1, \theta_2) \cdot [z_0 : z_1 : z_2 : z_3 : z_4] = [z_0 : e^{i\lambda_1} z_1 : e^{-i\lambda_1} z_2 : e^{i\lambda_2} z_3 : e^{-i\lambda_2} z_4].
$$

There are two rational lines invariant with respect to this $T^2$ action: $L$, connecting $\zeta_1$ to $\zeta_2$; and $L'$, connecting $\zeta_3$ to $\zeta_4$.

As in the doubling construction in [KL], a holomorphic degree-$d$ map $\tilde{f} : \mathbb{P}^1 \to L$ (or $L'$) can be halved to obtain two winding-$d$ disk maps corresponding to the two hemispheres of $L$. The boundaries of these disks are the intersections $L \cap \mathbb{P}^4_\mathbb{R}$ and $L' \cap \mathbb{P}^4_\mathbb{R}$. However, when $d$ is even, not every stable disk map can be obtained in this way [PSW], so only odd-winding disk maps will be considered in the remainder of this section.

To enumerate the disk maps $f : (\Delta, \partial \Delta) \to (Q, Q_\mathbb{R})$, [PSW] perform an analogous computation to the closed quintic genus-zero enumeration computation described in [Kon]. Let $\overline{\mathcal{M}}_\Delta (\mathbb{P}^4, \mathbb{P}^4_\mathbb{R}; d)$ denote the moduli space of stable disk maps $f : \Delta \to \mathbb{P}^4$ of winding $d$ satisfying $f(\partial \Delta) \subset \mathbb{P}^4_\mathbb{R}$, as in Section 2.1, and let $\tilde{F}_d$ be the real vector bundle over this moduli space with fiber

$$
\tilde{F}_d\big|_{f : (\Delta, \partial \Delta) \to (\mathbb{P}^4, \mathbb{P}^4_\mathbb{R})} = H^0 \left(C, \tilde{f}^* \mathcal{O}_{\mathbb{P}^4} (5)\right)_\mathbb{R},
$$

where $\tilde{f} : C \to \mathbb{P}^4$ is the stable rational map obtained from the stable disk map via reflection, and the $\mathbb{R}$ subscript denotes real sections. Generically, the boundary of $\overline{\mathcal{M}}_\Delta (\mathbb{P}^4, \mathbb{P}^4_\mathbb{R}; d)$ consists of maps with two boundary components. Because $\tilde{F}_d$ is not trivial near this boundary, the integral $c(\tilde{F}_d)$ over the moduli space is not well-defined.

To remedy this, consider a generic stable disk map $(\Delta, f) \in \partial \overline{\mathcal{M}}_\Delta (\mathbb{P}^4, \mathbb{P}^4_\mathbb{R}; d)$. $\partial \Delta$ has two components, $B_1$ and $B_2$. Modifying $f$ by replacing the image of one of the two components by its image under the involution $\sigma$ (e.g., defining $f'$ such that $f'(B_2) = \sigma \circ f(B_2)$) yields another two-component map $(f', \Delta)$. Let $\tilde{\mathcal{M}}_\Delta (\mathbb{P}^4, \mathbb{P}^4_\mathbb{R}; d)$ denote the quotient of $\overline{\mathcal{M}}_\Delta (\mathbb{P}^4, \mathbb{P}^4_\mathbb{R}; d)$ under the equivalence relation identifying two-component maps of the form $(f, \Delta)$ and $(f', \Delta)$. The vector bundle $\tilde{F}_d$ descends to a bundle $F_d$ over $\tilde{\mathcal{M}}_\Delta (\mathbb{P}^4, \mathbb{P}^4_\mathbb{R}; d)$, and here [PSW] prove that the virtual disk count for maps $f : (\Delta, \partial \Delta) \to (Q, Q_\mathbb{R})$ is given by

$$
N_{disk}^d = \int_{\tilde{\mathcal{M}}_\Delta (\mathbb{P}^4, \mathbb{P}^4_\mathbb{R}; d)} e(F_d)
$$

[PSW, theorem 3]. Localization can be used to compute the right-hand side of this equation.
5.2. Disk enumeration and Theorem 1. There are two rational lines \( L, L' \) fixed by the \( T^2 \) action on \( \mathbb{P}^4 \) (17). Consequently, there are four \( T^2 \)-fixed disks, identified by their incident torus-fixed points \( \zeta_i \). These disk maps are completely characterized by the winding \( p \) and the incident point \( \zeta_i \). In the notation of [PSW], the disk count is related to the contributions from each of these disk maps by the formula

\[
N_d^{\text{disk}} = \sum_{i=1}^{4} \sum_{p \text{ odd}} \text{Cont}(\zeta_i, p) (N_d^{\text{disk}})^i,
\]

where \( \text{Cont}(\zeta_i, p) (N_d^{\text{disk}}) \) denotes the contribution of the disk map \( (\zeta_i, p) \) to the disk count. The intersection disk term \( I(\zeta_i, p) \) of \( \text{Cont}(\zeta_i, p) (N_d^{\text{disk}}) \) is the contribution of the unique \( T^2 \)-fixed map \( f : (\Delta, \partial \Delta) \to (\mathbb{P}^4, \mathbb{P}^2) \) with winding \( p \) and incident point \( \zeta_i \). In [PSW, lemma 6], the terms \( I(\zeta_i, p), i = 1, \ldots, 4 \) are computed explicitly.

In fact, this expression can also be obtained using the result of theorem 1. For simplicity, restrict attention to the disk term associated to \( I(\zeta_1, p) \). Here, using localization, [PSW] compute

\[
I(\zeta_1, p) = \frac{(-1)^{p \frac{1}{2}} 2\lambda}{p} \frac{(5p)!!}{p!!} \left( \frac{\lambda}{2p} \right)^p \frac{\lambda + \lambda'}{\Gamma \left( \frac{5p}{2} + 1 \right) \Gamma \left( \frac{3p}{2} + 1 \right) \Gamma \left( \frac{3p}{2} + 1 \right)},
\]

where \( \lambda \) and \( \lambda' \) are the torus weights from (17).

The vertex of the disk is the point \( \zeta_1 \), so the induced torus representation \( R \) on \( T_0 \Delta \) has weight \( \alpha = c_1 (R) = 2\lambda/p \). Now, re-write the terms of (19) using gamma functions (recalling the identity \( n!! = \pi^{-1/2} 2^{n+1/2} \Gamma \left( \frac{n}{2} + 1 \right) \) for \( n \) odd):

\[
\prod_{i=0}^{(p-1)/2} \left( 1 - \frac{2\lambda'}{p} \right) \lambda - \lambda' = \left( \frac{2\lambda}{p} \right)^{(p+1)/2} \frac{\Gamma \left( 1 + \frac{p}{2} \left( \lambda - \lambda' \right) \right)}{\Gamma \left( \frac{1}{2} - \frac{\lambda' p}{2} \right)},
\]

\[
\prod_{i=0}^{(p-1)/2} \left( 1 - \frac{2\lambda'}{p} \right) \lambda + \lambda' = \left( \frac{2\lambda}{p} \right)^{(p+1)/2} \frac{\Gamma \left( 1 + \frac{p}{2} \left( \lambda + \lambda' \right) \right)}{\Gamma \left( \frac{1}{2} + \frac{\lambda' p}{2} \right)}.
\]

Substituting these expressions into the third fraction of (19) yields

\[
\frac{\Gamma \left( \frac{5p}{2} + 1 \right) \Gamma \left( \frac{3p}{2} + 1 \right) \Gamma \left( \frac{3p}{2} + 1 \right)}{\left( \frac{2\lambda}{p} \right)^{\alpha \sin \left( \frac{\pi}{2} + \frac{\lambda'}{2} \right)} \Gamma \left( 1 + \frac{p}{2} \left( \lambda - \lambda' \right) \right) \Gamma \left( 1 + \frac{p}{2} \left( \lambda + \lambda' \right) \right)}
\]

\[
= \frac{\Gamma \left( \frac{5\lambda}{z} + 1 \right) \Gamma \left( \frac{3\lambda}{z} + 1 \right) \Gamma \left( \frac{3\lambda}{z} + 1 \right)}{\Gamma \left( \frac{2\lambda}{z} + 1 \right) \Gamma \left( \frac{\lambda}{z} + 1 \right) \Gamma \left( \frac{\lambda}{z} + 1 \right) \Gamma \left( \frac{\lambda + \lambda'}{z} + 1 \right) \Gamma \left( \frac{\lambda + \lambda'}{z} + 1 \right)}
\]

after the additional substitution \( z = \alpha = 2\lambda/p \).

This formula now closely resembles (9). By definition, the gamma class is multiplicative. Hence, \( \hat{\Gamma} Q \) can be obtained from the normal bundle exact sequence of sheaves

\[
0 \to T_Q \to T_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(5) \to 0,
\]
yielding
\[ \hat{\Gamma}_Q = \frac{\hat{\Gamma}_{p^4}}{\hat{\Gamma}_{O(5)}}. \]

An additional subtlety is that the above exact sequence is not torus-equivariant for the real quintic. However, from the hard Lefshetz theorem, it suffices to write the (non-equivariant) gamma class as a ratio, and then choose an equivariant lift for each of the factors.

The weights of the tangent space \( T_{p^4} \) at the fixed point \( \zeta_1 \) are \( 2\lambda, \lambda, \lambda + \lambda', \) and \( \lambda - \lambda' \), and the bundle \( O_{p^4} (5) \) has weight \( 5\lambda \) at this fixed point. Therefore, the second term in the product above is the localized form of the inverse of \( \hat{\Gamma}_Q \). Finally, because \( p = 2\lambda/z, (-1)^{(p-1)/2} = \sin \left( \frac{\lambda}{z} \right) \). Combining these observations, (19) can be expressed as
\[ I(\zeta_1, p) = \frac{\pi}{p \hat{\Gamma}_Q \sin \left( \frac{\lambda + \lambda'}{2} \right)}, \]
which has a similar form to (9). The disk is contained in the rational line \( L \). In local coordinates
\[ (x_1, x_2, x_3, x_4) = \left( \frac{z_0}{z_1}, \frac{z_2}{z_1}, \frac{z_3}{z_1}, \frac{z_4}{z_1} \right) \]

near the vertex \( \zeta_1 \), the disk map is \( t \mapsto (0, t^p, 0, 0) \), and \( \lambda + \lambda' \) is the weight associated to a normal direction to the disk (namely, \( \partial_{x_4} \)). The only discrepancy between (20) and (9) is an additional factor of \( z \) in the latter equation. As these are invariants appearing from the localization of an integral (18) over \( \mathcal{M}_\Delta (\mathbb{P}^4, \mathbb{P}^4_\mathbb{R}; d) \), rather than \( \mathcal{M}_\Delta (Q, Q_\mathbb{R}; d) \), there is no a priori reason to expect a form of theorem 1 to apply.

6. Lagrangian Cycles in Large N Duality

6.1. Lagrangian cycles and the conifold transition. In addition to Lagrangians appearing as the fixed loci of anti-holomorphic involutions, there is another family of Lagrangians on \( X = O_{p^4} (-1, -1) \) motivated by large N duality and knot theory. Recent work in this area has yielded many connections between knot theory and Gromov-Witten theory ([BEM, DSV, GJKS]); this section reviews the geometric relationship between knots on \( S^3 \) and open Gromov-Witten theory on \( X \).

Recall that \( X \) can be identified with the resolved conifold—\( X \) is the small resolution of the conifold singularity
\[ xy - zw = 0 \]
in \( \mathbb{C}^4 \). In particular, by blowing up the subspace \( y = z = 0, X \) can be described by the equations
\[ xy - zw = 0, \quad x\lambda = w \rho, \quad y\lambda = z \rho, \]
where \( (x, y, z, w) \in \mathbb{C}^4 \) and \( |\lambda : \rho| \in \mathbb{P}^1 \). The conifold singularity is also the singular limit of the smooth hypersurface threefold \( Y_\mu \subset \mathbb{C}^4 \) defined by
\[ xy - zw = \mu, \]
where \( \mu \in \mathbb{R}_{>0} \). As described in [DSV], \( Y_\mu \) is symplectomorphic to the cotangent bundle \( T^*_S \). The base \( S_\mu \cong S^3 \) is the fixed locus of the anti-holomorphic involution
\[ \sigma (x, y, z, w) = (\overline{x}, -\overline{w}, \overline{z}, -\overline{y}), \]
expressed by the equations \(|x|^2 + |y|^2 = \mu|\).
The large $N$ duality conjecture states that the large $N$ limit of the topological A-model on $Y_\mu$ with $N$ Lagrangian branes wrapping $S_\mu$ is equivalent to the topological A-model on $X$ [GV]. This has been checked in several ways. First, according to [Wi2], the topological A-model on $Y_\mu$ with $N$ Lagrangian branes wrapping $S_\mu$ is equivalent to the $U(N)$ Chern-Simons theory on $S_\mu$. Then, in the large $N$ expansion, the partition function $Z_{CS}(k,N)$ is equivalent to the topological A-model partition function $Z_X(g_s,t)$ [GV]. The parameters determining the A-model theory on $X$ are the string coupling constant $g_s$ and the symplectic area $t$ of the zero section $P_1 \subset X$, which are related to the Chern-Simons parameters $k$ and $N$ by
\[
g_s = \frac{2\pi}{k+N}, \quad t = -\frac{2\pi i N}{k+N}.
\]
Large $N$ duality is extended to incorporate Wilson loops in [OV]. Following [Wi3], Wilson loop observables in the Chern-Simons theory on $S^3$ correspond to colored HOMFLY polynomials of knots $K \subset S^3$. The conormal bundle $N_K^*$ to a knot $K \subset S^3$ is Lagrangian submanifold of $T^*S^3$. The main difficulty in extending large $N$ duality in this manner is determining the corresponding A-model on $X$: $N_K^*$ intersects the zero section $S^3$ in the knot $K$, which becomes contracted after the conifold transition. To remedy this difficulty, the Lagrangian cycle $N_K^*$ must be lifted to a new Lagrangian $\tilde{L}$ disjoint from the zero section before performing the conifold transition [AMV, MV], as depicted in Figure 5.

\[\begin{array}{c}
\text{Figure 5. The conifold transition for lifted Lagrangian cycles.}
\end{array}\]

This figure depicts the conifold transition. The Lagrangian $\tilde{L} \subset Y_\mu \cong T^*S^3$ is constructed by shifting the conormal bundle of a knot $K \subset S^3$ off of the zero section. This lift introduces a holomorphic cylinder $C$ connecting the knot on $S^3$ to its image in $\tilde{L}$. $Y_0$ is the conifold singularity $xz - yw = 0$ in $\mathbb{C}^4$. The map $\phi_\mu : Y_\mu \to Y_0$ is a symplectomorphism away from the zero section, so $\phi_\mu(\tilde{L})$ is a Lagrangian submanifold of $Y_0$. $X \cong O_{\mathbb{P}^1}(-1,-1)$ is the small resolution of the conifold singularity, and $\sigma_\epsilon : X \to Y_0$ is the corresponding natural map. In fact, there are a family of such symplectomorphisms, where $\epsilon$ parametrizes the symplectic form on the zero section $\mathbb{P}^1 \subset X$. Hence, $L := \sigma_\epsilon^{-1} \circ \phi_\mu(\tilde{L})$ is a Lagrangian submanifold of $X$. The holomorphic disk $D$ is the image of $C$ under the conifold transition.
Such a lift is easy to construct: define coordinates \((\vec{u}, \vec{v})\) for \(T^{*}_{S^{3}}\) by
\[
T^{*}_{S^{3}} = \{ (\vec{u}, \vec{v}) \in \mathbb{R}^4 \times \mathbb{R}^4 : |\vec{u}| = 1, \vec{u} \cdot \vec{v} = 0 \}.
\]
Any knot \(K \subset S^{3}\) is given by a parametrization \(\vec{u} = f(\theta)\). Then, the conormal bundle \(N^{*}_{K}\) can be expressed as
\[
N^{*}_{K} = \left\{ (\vec{u}, \vec{v}) \in T^{*} S^{3} : \vec{u} = f(\theta), \frac{df}{d\theta} \cdot \vec{v} = 0 \right\}.
\]
Lifts of \(N^{*}_{K}\) can be specified by maps \(g : S^{1} \rightarrow T^{*}_{f(\theta)} S^{3}\) such that \(\frac{df}{d\theta} \cdot g(\theta) \neq 0\); for such a \(g\), define the lifted conormal bundle \(\tilde{L}\) to be
\[
\tilde{L} := \left\{ (\vec{u}, \vec{v}) \in T^{*} S^{3} : \vec{u} = f(\theta), \frac{df}{d\theta} \cdot (\vec{v} - g(\theta)) = 0 \right\}.
\]
The image of \(\tilde{L}\) under the conifold transition will be a Lagrangian \(L \subset X\), and the open A-model on \(X\) with this Lagrangian boundary can be computed. Shifting \(N^{*}_{K}\) off of the zero section modifies large \(N\) duality in the following ways: The lift of \(N^{*}_{K}\) to \(\tilde{L}\) introduces corrections to the Wilson loop observables in the Chern-Simons theory proportional to the area of the holomorphic cylinder \(C\) connecting the lift of the knot to its image in the zero section [DSV]. Instead of the closed A-model on \(X\), the corresponding theory should be an open A-model with Lagrangian boundary \(L\). This statement of large \(N\) duality is found to be true for torus knots in [DSV], and their construction provides a novel source of Lagrangians.

6.2. Toric Lagrangian cycles and Theorem 1. It is important to note that the Lagrangians considered in [DSV] are not obtained as the fixed loci of anti-holomorphic involutions, so there is no a priori reason to expect that the formula proposed in theorem 1 should apply in this situation. For the \((r, s)\) torus knot, the corresponding Lagrangian \(L\) is found to be fixed under the torus action
\[
\rho_{\theta} \left( (x, y, z, w), [\lambda : \rho] \right) = \left( e^{i s \theta} x, e^{i r \theta} y, e^{-i s \theta} z, e^{-i r \theta} w \right), \left[ e^{-i (r+s) \theta} \lambda : \rho \right].
\]
There is only one holomorphic disk in \(X\) fixed by this \(S^{1}\) action, and it lies entirely in the \(x\)-\(y\) face of the moment polytope, as depicted in Fig. 6. A neighborhood of the disk can be described by local coordinates \(x, y, \xi = \lambda/\rho\). In these coordinates, the disk map is
\[
t \mapsto (\xi = 0, x = b_{1}t^{ws}, y = b_{1}^{*}t^{wr}),
\]
where \(|t| \leq 1\) and \(b_{1} \in \mathbb{R}_{>0}\) is a constant obtained from the geometric construction in [DSV]. After a lengthy localization calculation, [DSV] compute the winding-1 open Gromov-Witten invariants with Lagrangian boundary \(L\). This computation readily generalizes to higher winding [GJKS], and gives the following expression for \(D_{X,L}\):
\[
D_{X,L} = (-1)^{ws} \prod_{k=1}^{ws-1} \frac{(r + s - \frac{k}{w})}{w \prod_{k=0}^{ws-1} (s - \frac{k}{w})}.
\]
This can be re-written in terms of gamma functions in the following way:
\[
\frac{\prod_{k=1}^{ws-1} (r + s - \frac{k}{w})}{\prod_{k=0}^{ws-1} (s - \frac{k}{w})} = \frac{\Gamma(wr + ws)}{(\frac{1}{w}) \Gamma(ws + 1) \Gamma(wr + 1)}.
\]
This figure depicts the geometry described in Section 6.2. The Lagrangian $L$ is the image of a shifted conormal bundle to a knot in $S^3$ under the conifold transition. Local coordinates $(x, y)$ and $(z, w)$ parametrize the fibers of $X$ in a neighborhood of the two vertices of the moment polytope. The boundary of the disk $D$ is symplectomorphic to the torus knot (in the depiction above, the trefoil). $D$ is contained entirely in the $x$-$y$ face of the polytope, and the disk map can be written in local coordinates as $t \mapsto (\xi = 0, x = b_1 t^w, y = b_1 t^w)$.

Locally (Figure 6), the weights of the torus action are $\lambda_\xi = -r - s$, $\lambda_x = s$, $\lambda_y = r$. The induced torus action on $\Delta$ is $t \mapsto e^{i\theta/w} t$, so $\alpha = \frac{1}{w}$. Replacing $\alpha = z$ and substituting these weights into the above formula yields

$$
\frac{\Gamma \left( \frac{-\lambda_\xi}{z} \right)}{wz \Gamma \left( \frac{\lambda_x}{z} + 1 \right) \Gamma \left( \frac{\lambda_y}{z} + 1 \right)} = \frac{\pi}{wz \Gamma_X \sin \left( \pi \frac{\lambda_\xi}{z} \right)}
$$

after Euler’s reflection identity. As remarked above, the Lagrangian $L$ is not the fixed locus of an anti-holomorphic involution. However, the result still applies. Let $C$ be the $S^1$-invariant curve given locally by $(0, t^s, t^r)$ for $t \in C$. Then, $L \cap C = \partial D$ and $\lambda_\xi$ is the weight of an $S^1$-invariant normal direction to $C$ as $C$ is entirely contained in the $x$-$y$ plane of the moment polytope. The author finds it curious that the result of theorem 1 appears to apply in this situation, and hopes that this is evidence that, properly formulated, a more general version of this theorem exists.

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