On a Spectral Representation for Correlation Measures in Configuration Space Analysis

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Abstract

The paper is devoted to the study of configuration space analysis by using the projective spectral theorem. For a manifold \( X \), let \( \Gamma_X \), resp. \( \Gamma_{X,0} \) denote the space of all, resp. finite configurations in \( X \). The so-called \( K \)-transform, introduced by A. Lenard, maps functions on \( \Gamma_{X,0} \) into functions on \( \Gamma_X \) and its adjoint \( K^* \) maps probability measures on \( \Gamma_X \) into \( \sigma \)-finite measures on \( \Gamma_{X,0} \). For a probability measure \( \mu \) on \( \Gamma_X \), \( \rho_\mu := K^* \mu \) is called the correlation measure of \( \mu \). We consider the inverse problem of existence of a probability measure \( \mu \) whose correlation measure \( \rho_\mu \) is equal to a given measure \( \rho \). We introduce an operation of \( \ast \)-convolution of two functions on \( \Gamma_{X,0} \) and suppose that the measure \( \rho \) is \( \ast \)-positive definite, which enables us to introduce the Hilbert space \( \mathcal{H}_\rho \) of functions on \( \Gamma_{X,0} \) with the scalar product \( (G^{(1)}, G^{(2)})_{\mathcal{H}_\rho} = \int_{\Gamma_{X,0}} (G^{(1)} \ast G^{(2)})(\eta) \rho(d\eta) \). Under a condition on the growth of the measure \( \rho \) on the \( n \)-point configuration spaces, we construct the Fourier transform in generalized joint eigenvectors of some special family \( A = (A_\phi)_{\phi \in \mathcal{D}} \), \( \mathcal{D} := C^\infty_0(X) \), of commuting selfadjoint operators in \( \mathcal{H}_\rho \). We show that this Fourier transform is a unitary between \( \mathcal{H}_\rho \) and the \( L^2 \)-space \( L^2(\Gamma_X, d\mu) \), where \( \mu \) is the spectral measure of \( A \). Moreover, this unitary coincides with the \( K \)-transform, while the measure \( \rho \) is the correlation measure of \( \mu \).

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Introduction

The configuration space $\Gamma_X$ over a (non-compact) Riemannian manifold $X$ is defined as the set of all locally finite subsets (configurations) in $X$. Such spaces as well as probability measures on them appear naturally in several topics of mathematics and physics. Let us mention only the theory of point processes [9, 6], classical statistical mechanics [22, 8], and nonrelativistic quantum field theory, e.g., [20, 21] and references therein.

An important tool in the study of configuration space analysis is the so-called $K$-transform. Roughly speaking, this transform maps functions defined on the space $\Gamma_{X,0}$ of finite configurations in $X$ into functions defined on the space $\Gamma_X$ of all configurations. Interpreting the algebra of functions on $\Gamma_X$ as observables of our system, we may consider functions on $\Gamma_{X,0}$ as quasi-observables, from which we are able to reconstruct observables by using the $K$-transform. This special kind of observables is known in physics and called additive type observables, see [5]. A. Lenard was the first to recognize the operator nature of the $K$-transform [14, 15, 16]. Recently, this theory was reanalyzed and further developed in [10, 11, 12, 13], where the reader can find also many applications of this transform.

The adjoint $K^*$ of the $K$-transform, defined by the formula

$$\int_{\Gamma_X} (KG)(\gamma) \mu(d\gamma) = \int_{\Gamma_{X,0}} G(\eta)(K^*\mu)(d\eta),$$

maps probability measures on $\Gamma_X$ into $\sigma$-finite measures on $\Gamma_{X,0}$, and $\rho_\mu := K^*\mu$ is called the correlation measure of $\mu$.

In several applications, a $\sigma$-finite measure $\rho$ on $\Gamma_{X,0}$ appears as a given object and the problem is to show that this $\rho$ can be seen as a correlation measure for a probability measure on $\Gamma_X$. Different types of sufficient conditions were given for this to hold. A. Lenard [15, 16] used essentially a positivity condition for the correlation measure, which allowed him to construct a linear positive functional and apply a version of the Riesz–Krein extension theorem. His conditions were also necessary. O. Macchi [18] (see also [6]) needed an additional condition in order to get an explicit construction of the measure on $\Gamma_X$.

The present paper is also devoted to this problem. As a first step, we utilize the idea of [10, 13], introducing the so-called $*$-convolution on a space of functions on $\Gamma_{X,0}$ and demanding that $\rho$ be $*$-positive definite, that is,

$$\int_{\Gamma_{X,0}} (G*G)(\eta) \rho(d\eta) \geq 0.$$
Unlike the approach of [10, 13], where the authors prove a Bochner type theorem, we use a spectral approach. The condition (1) enables us to introduce in Section 2 the $\star$-convolution Hilbert space $H_\rho$ of functions on $\Gamma_{X,0}$ with the scalar product

$$(G^{(1)}, G^{(2)})_{H_\rho} := \int_{\Gamma_{X,0}} (G^{(1)} \star \overline{G}^{(2)})(\eta) \rho(d\eta).$$

Next, we follow the general strategy of representation of positive definite kernels and functionals by using the projective spectral theorem, see [2, 3, 4, 17]. We consider in the space $H_\rho$ a family $(A_\varphi)_{\varphi \in \mathcal{D}}$ of Hermitian operators defined by the formula

$$(A_\varphi G)(\eta) := (\varphi \star G)(\eta)$$

on an appropriate domain. Here, $\mathcal{D} := C^\infty_0(X)$ is the nuclear space of all $C^\infty$ functions on $X$ with compact support, and each $\varphi \in \mathcal{D}$ is identified with the function on $\Gamma_{X,0}$ given as follows: $\varphi(\eta) := \varphi(x)$ if $\eta = \{x\}$ and $\varphi(\eta) := 0$ if the number of points in $\eta \in \Gamma_{X,0}$ is not equal to one.

Under a rather weak condition on the measure $\rho$, we show that the operators $A_\varphi$ are essentially selfadjoint in $H_\rho$ and their closures $A^\sim_\varphi$ constitute a family of commuting selfadjoint operators in $H_\rho$. Moreover, these operators are shown to satisfy the conditions of the projective spectral theorem, and the Fourier transform in generalized joint eigenvectors of the family $(A^\sim_\varphi)_{\varphi \in \mathcal{D}}$ gives a unitary isomorphism between $H_\rho$ and an $L^2$-space $L^2(\mathcal{D}', d\mu)$, where $\mathcal{D}'$ is the dual of $\mathcal{D}$ and $\mu$ is the spectral measure of the family $(A^\sim_\varphi)_{\varphi \in \mathcal{D}}$. Under this isomorphism, each operator $A^\sim_\varphi$ goes over into the operator of multiplication by the monomial $(\cdot, \varphi)$. The corresponding Parseval inequality gives the required spectral representation of the functional determined by the measure $\rho$.

In Section 3, following an idea in [10], we prove, under an additional, natural condition on $\rho$, that the measure $\mu$ is concentrated actually on $\Gamma_X$. Notice that the configuration space can be considered as a subset of $\mathcal{D}'$ by identifying any configuration from $\Gamma_X$ with a sum of delta functions having support in the points of the configuration. Moreover, the unitary constructed in Section 2 coincides now with the $K$-transform, since $\rho = \rho_{\mu}$ is the correlation measure of $\mu$.

Finally, let us stress that the spectral approach not only gives an alternative way to find sufficient conditions for a measure to be a correlation measure, but also gives a new understanding of the $K$-transform as a unitary operator between the Hilbert spaces $H_\rho$ and $L^2(\Gamma_X, d\mu)$ which has the form of Fourier transform.
2 The $\ast$-convolution Hilbert space and the corresponding Fourier transform

Let $X$ be a connected, oriented $C^\infty$ (non-compact) Riemannian manifold. We denote by $D$ the space $C^\infty_0(X)$ of all real-valued infinite differentiable functions on $X$ with compact support. This space can be naturally endowed with a topology of a nuclear space, see e.g. [7].

Let $F_0(D) := \mathbb{C}$ and $F_n(D) := D^{\mathbb{C}} \hat{\otimes} C, \quad n \in \mathbb{N}$, where $D^{\mathbb{C}}$ denotes the complexification of the real space $D$ and $\hat{\otimes}$ stands for the symmetric tensor product. Notice that $F_n(D)$ is the complexification of the space of all real-valued $C^\infty$ symmetric functions on $X^n$ with compact support. Next, we define $F_{\text{fin}}(D) := \bigoplus_{n=0}^{\infty} F_n(D)$ to be the topological direct sum of the spaces $F_n(D)$, i.e., an arbitrary element $G \in F_{\text{fin}}(D)$ is of the form $G = (G^{(0)}, G^{(1)}, \ldots, G^{(n)}, 0, 0, \ldots)$, where $G^{(i)} \in F_i(D)$, and the convergence in $F_{\text{fin}}(D)$ means the uniform finiteness and the coordinate-wise convergence. In what follows, we will identify a $G^{(n)} \in F_n(D)$ with the element $(0, \ldots, 0, G^{(n)}, 0, 0, \ldots) \in F_{\text{fin}}(D)$.

Next, we define the space $\tilde{\Gamma}_{X,0}$ of multiple finite configurations over $X$:

$$\tilde{\Gamma}_{X,0} := \bigsqcup_{n \in \mathbb{N}_0} \tilde{\Gamma}^{(n)}_X.$$ 

Here, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\tilde{\Gamma}^{(0)}_X := \emptyset$ and $\tilde{\Gamma}^{(n)}_X, \quad n \in \mathbb{N}$, is the factor space

$$\tilde{\Gamma}^{(n)}_X := X^n / S_n$$

with $S_n$ being the group of all permutations of $\{1, \ldots, n\}$, which naturally acts on $X^n$:

$$\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n.$$ (2)

Thus, an $\eta = [x_1, \ldots, x_n] \in \tilde{\Gamma}^{(n)}_X$ is an equivalence class consisting of $n$ elements each of which is a point in $X$ (an $n$-point configuration in $X$ with possibly multiple points).

Each $\tilde{\Gamma}^{(n)}_X$ is equipped with the factor topology generated by the topology on $X^n$, and $\tilde{\Gamma}_{X,0}$ is equipped then by the topology of disjoint union. It follows directly from the construction of $F_{\text{fin}}(D)$ that each $G \in F_{\text{fin}}(D)$ can be considered as the function on $\tilde{\Gamma}_{X,0}$ defined by

$$G(\emptyset) := G^{(0)}.$$
\( G([x_1, \ldots, x_n]) = G^{(n)}(x_1, \ldots, x_n), \quad n \in \mathbb{N}. \) \hspace{1cm} (3)

Notice that in the formula (3) we fixed, in fact, a numeration of the points in \( X \) defining the equivalence class, but the right hand side of (3) is independent of the numeration. Now, we will need a numeration once more to define the notion of summation over partitions of an equivalence class.

So, let \( \eta = [x_1, \ldots, x_n] \) be an equivalence class with a fixed numeration of points. To each nonempty subset \( \xi \) of the set \( \{1, \ldots, n\} \) there corresponds the equivalence class defined by the points \( x_i, \ i \in \xi \). The \( \xi = \emptyset \) as a subset of \( \{1, \ldots, n\} \) corresponds to the \( \emptyset \) as an element of \( \tilde{\Gamma}_X^{(0)} \). Thus, we will preserve the notation \( \xi \) for the corresponding element of \( \Gamma_X, 0 \). For a function \( F: (\Gamma_X, 0)^k \to \mathbb{C} \), we let

\[
\sum_{(\xi_1, \ldots, \xi_k) \in P_3(\eta)} F(\xi_1, \ldots, \xi_k) \quad (4)
\]

denote the summation over all partitions \( (\xi_1, \ldots, \xi_k) \) of \( \{1, \ldots, n\} \). As easily seen, the result of the summation (4) is independent of the numeration.

Now, we define a convolution \( \star \) as the mapping

\[
\star: F_{\text{fin}}(\mathcal{D}) \oplus F_{\text{fin}}(\mathcal{D}) \to F_{\text{fin}}(\mathcal{D}) \quad (5)
\]
given by

\[
(G_1 \star G_2)(\eta) := \sum_{(\xi_1, \xi_2, \xi_3) \in P_3(\eta)} G_1(\xi_1 \cup \xi_2)G_2(\xi_2 \cup \xi_3), \quad (6)
\]

where \( P_3(\eta) \) denotes the set of all partitions \( (\xi_1, \xi_2, \xi_3) \) of \( \eta \) in 3 parts.

**Lemma 1** \( F_{\text{fin}}(\mathcal{D}) \) with the operation \( \star \) is a commutative nuclear algebra.

*Proof.* For a class \( \eta = [x_1, \ldots, x_n] \), let \( |\eta| := n \) Since \( G_1, G_2 \in F_{\text{fin}}(\mathcal{D}) \), there exist \( n_1, n_2 \in \mathbb{N}_0 \) such that \( G_i(\eta) = 0 \) if \( |\eta| > n_i \). Then, (6) implies that \( (G_1 \star G_2)(\eta) = 0 \) if \( |\eta| > n_1 + n_2 \).

Next, we note that, for arbitrary \( G_1^{(n_1+n_2)} \in \mathcal{D}^{\otimes(n_1+n_2)} \) and \( G_2^{(n_2+n_3)} \in \mathcal{D}^{\otimes(n_2+n_3)} \), the function

\[
G_1^{(n_1+n_2)}(x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_1+n_2}) \times \times G_2^{(n_2+n_3)}(x_{n_1+1}, \ldots, x_{n_1+n_2}, x_{n_1+n_2+1}, \ldots, x_{n_1+n_2+n_3})
\]

belongs to \( \mathcal{D}^{\otimes(n_1+n_2+n_3)} \)—the \( (n_1 + n_2 + n_3) \)-th tensor power of \( \mathcal{D} \)—and moreover, it depends continuously on \( G_1 \) and \( G_2 \). Hence, it is easy to see that \( G_1 \star G_2 \) indeed belongs to \( F_{\text{fin}}(\mathcal{D}) \) and that the operation (5) is continuous.
The commutativity of $\star$ follows directly from the definition. Thus, it remains only to show the associativity. It follows from (6) and an easy combinatoric consideration that

$$
((G_1 \star G_2) \star G_3)(\eta) = \sum_{(\xi_1, \xi_2, \xi_3) \in \mathcal{P}_3(\eta)} (G_1 \star G_2)(\xi_1 \cup \xi_2)G_3(\xi_2 \cup \xi_3)
$$

$$
= \sum_{(\xi_1, \xi_2, \xi_3) \in \mathcal{P}_3(\eta)} \sum_{(\psi_1, \psi_2, \psi_3) \in \mathcal{P}_3(\xi_1 \cup \xi_2)} G_1(\psi_1 \cup \psi_2)G_2(\psi_2 \cup \psi_3)G_3(\xi_2 \cup \xi_3)
$$

$$
= \sum_{(\xi_1, \xi_2, \xi_3) \in \mathcal{P}_3(\eta)} \sum_{(\psi_1, \psi_2, \psi_3) \in \mathcal{P}_3(\xi_1)} \sum_{(\psi_2, \psi_{22}, \psi_{23}) \in \mathcal{P}_3(\xi_2)} G_1(\psi_{11} \cup \psi_{12} \cup \psi_{21} \cup \psi_{22}) \times
$$

$$
G_2(\psi_{12} \cup \psi_{13} \cup \psi_{22} \cup \psi_{23})G_3(\psi_{21} \cup \psi_{22} \cup \psi_{23} \cup \xi_3)
$$

$$
= \sum_{(\xi_1, \ldots, \xi_7) \in \mathcal{P}_7(\eta)} G_1(\xi_1 \cup \xi_4 \cup \xi_6 \cup \xi_7)G_2(\xi_2 \cup \xi_4 \cup \xi_5 \cup \xi_7)G_3(\xi_3 \cup \xi_5 \cup \xi_6 \cup \xi_7).
$$

Absolutely analogously, one arrives at the same result when calculating $(G_1 \star (G_2 \star G_3))(\eta)$. ■

We will need now also the (usual) space of finite configurations over $X$—denoted by $\Gamma_{X,0}$—which is defined as a subset of $\hat{\Gamma}_{X,0}$ consisting of $\emptyset$ and all $\eta = [x_1, \ldots, x_n] \in \hat{\Gamma}_{X,0}$ such that $x_i \neq x_j$ if $i \neq j$. Each $\eta = [x_1, \ldots, x_n] \in \Gamma_{X,0}$ can be identified with the set $\{x_1, \ldots, x_n\}$. We have $\Gamma_{X,0} = \bigcup_{n \in \mathbb{N}_0} \Gamma^{(n)}_X$, where $\Gamma^{(n)}_X$ is the space of $n$-point configurations in $X$.

The space $\Gamma_{X,0}$ is endowed with the relative topology as a subset of $\hat{\Gamma}_{X,0}$.

Let $\rho$ be a measure on the Borel $\sigma$-algebra $\mathcal{B}(\Gamma_{X,0})$. Of course, $\rho$ can be considered as a measure on $\mathcal{B}(\hat{\Gamma}_{X,0})$ such that the (measurable) set $\hat{\Gamma}_{X,0} \setminus \Gamma_{X,0}$ is of zero $\rho$ measure. One sees that the restriction of $\rho$ to $\Gamma^{(n)}_X$ is actually a measure on $\mathcal{B}_{\text{sym}}(X^n)$. Here, $\mathcal{B}_{\text{sym}}(X^n)$ denotes that sub-$\sigma$-algebra of the Borel $\sigma$-algebra $\mathcal{B}(X^n)$ consisting of symmetric sets, i.e., sets in $X^n$ which are invariant with respect to the action (2) of the permutation group $S_n$ on $X^n$. For example, for each Borel $\Lambda \in \mathcal{B}(X)$ we have $\Lambda^n \in \mathcal{B}_{\text{sym}}(X^n)$.

We will suppose that $\rho$ satisfies the following assumptions:

(A1) **Normalization:** $\rho(\Gamma^{(0)}_X) = 1$.

(A2) **Local finiteness:** For each $n \in \mathbb{N}$ and each compact subset $\Lambda \subset X$, we have $\rho(\Gamma^{(n)}_\Lambda) < \infty$

(where $\Gamma^{(n)}_\Lambda$ denotes, of course, the $n$-point configuration space over $\Lambda$).
Positive definiteness: For each $G \in \mathcal{F}_{\text{fin}}(\mathcal{D})$

$$
\int_{\Gamma_{X,0}} (G \ast \overline{G})(\eta) \rho(d\eta) \geq 0,
$$

where $\overline{G}$ is the complex conjugate of $G$.

Thus, it follows from (A2) and (A3) that

$$
\mathcal{F}_{\text{fin}}(\mathcal{D}) \oplus \mathcal{F}_{\text{fin}}(\mathcal{D}) \ni (G_1, G_2) \mapsto a_\rho(G_1, G_2) := \int_{\Gamma_{X,0}} (G_1 \ast G_2)(\eta) \rho(d\eta) \in \mathbb{C}
$$

is a bilinear continuous form which is positive definite: $a_\rho(G, \overline{G}) \geq 0$. Therefore, by using the general technique, e.g., [2], Ch. 5, Sect. 5, subsec. 1, we can construct a nuclear factor-space

$$
\hat{\mathcal{F}}_{\text{fin}}(\mathcal{D}) := \mathcal{F}_{\text{fin}}(\mathcal{D}) / \{G' : a_\rho(G', \overline{G'}) = 0\},
$$

consisting of factor classes

$$
\hat{G} = \{G' \in \mathcal{F}_{\text{fin}}(\mathcal{D}) : a_\rho(G - G', \overline{G} - \overline{G'}) = 0\},
$$

and then a Hilbert space $\mathcal{H}_\rho$ as the closure of $\hat{\mathcal{F}}_{\text{fin}}(\mathcal{D})$ with respect to the norm generated by the scalar product $(\hat{G}_1, \hat{G}_2)_{\mathcal{H}_\rho} := a_\rho(G_1, \overline{G}_2)$. Thus, as a result we get a nuclear space $\hat{\mathcal{F}}_{\text{fin}}(\mathcal{D})$ that is topologically, i.e., densely and continuously, embedded into the Hilbert space $\mathcal{H}_\rho$.

Now, for each $\varphi \in \mathcal{D}$, we define an operator $A_\varphi$ acting on $\mathcal{F}_{\text{fin}}(\mathcal{D})$ as

$$
A_\varphi G := \varphi \ast G, \quad G \in \mathcal{F}_{\text{fin}}(\mathcal{D}),
$$

and let $A_\varphi$ be the operator in $\mathcal{H}_\rho$ with domain $\text{Dom } A_\varphi = \hat{\mathcal{F}}_{\text{fin}}(\mathcal{D})$ defined by

$$
A_\varphi \hat{G} := \overline{A_\varphi G} = \overline{\varphi \ast G}, \quad G \in \mathcal{F}_{\text{fin}}(\mathcal{D}).
$$

By Lemma 1,

$$
a_\rho(A_\varphi G_1, \overline{G}_2) = \int_{\Gamma_{X,0}} ((\varphi \ast G_1) \ast \overline{G}_2)(\eta) \rho(d\eta)
$$

$$
= \int_{\Gamma_{X,0}} (G_1 \ast (\overline{\varphi \ast G}_2))(\eta) \rho(d\eta)
$$

$$
= a_\rho(G_1, A_\varphi \overline{G}_2),
$$

and therefore the definition (8) makes sense due to [2], Ch. 5, Sect. 5, subsec. 2, which uses essentially the Cauchy–Schwartz inequality.

We strengthen now the condition (A2) by demanding the following:
(A2') For every compact $\Lambda \subset X$, there exists a constant $C_\Lambda > 0$ such that
\[
\rho(\Gamma(n)_\Lambda) \leq C^n_\Lambda \quad \text{for all } n \in \mathbb{N}.
\] (9)

**Lemma 2** Let (A1), (A2'), and (A3) hold. Then the operators $A_\varphi$, $\varphi \in \mathcal{D}$, with domain $\mathcal{F}_n(D)$ are essentially selfadjoint in $\mathcal{H}_\rho$ and their closures, $A_\varphi^\sim$, constitute a family of commuting selfadjoint operators, where the commutation is understood in the sense of the resolutions of the identity.

**Proof.** Let us show that, for any $G(n) \in F_n(D)$, $\hat{G}(n)$ is an analytical vector of each $A_\varphi$, i.e., the series
\[
\sum_{k=0}^{\infty} \frac{\|A_\varphi^k \hat{G}(n)\|_{\mathcal{H}_\rho}}{k!} |z|^k, \quad z \in \mathbb{C},
\] (10)

has a positive radius of convergence. So, let us fix $\varphi \in D$ and $G(n) \in F_n(D)$ and let $\Lambda$ be a compact set in $X$ such that $\text{supp } \varphi \subset \Lambda$ and $\text{supp } G(n) \subset \Lambda$.

We will say that a measurable function $G$ on $\Gamma_X(0)$ has bounded support if there exists a compact set $\Lambda \subset X$ and $N \in \mathbb{N}$ such that $\text{supp } G \subset \bigcup_{n=0}^{N} \Gamma_{\Lambda}(n)$. The space of all bounded measurable functions with bounded support will be denoted by $B_{bs}(\Gamma_X(0))$. Evidently, the formula (6) can be extended to the case where $G_1, G_2 \in B_{bs}(\Gamma_X(0))$.

Set now
\[
\tilde{\varphi}(\eta) := \sup_{x \in X} |\varphi(x)| \mathbf{1}_\Lambda(\eta), \quad \tilde{G}(n)(\eta) := \sup_{\eta \in \Gamma_X(n)} |G(n)(\eta)| \mathbf{1}_{\Lambda^n}(\eta),
\]
where $\mathbf{1}_Y(\cdot)$ denotes the characteristic function of a set $Y$. Denote by $m$ the volume measure on $X$. Without loss of generality, we can suppose that $m(\Lambda) \geq 1$. Let $\tilde{\rho}_\Lambda$ be the measure on $\tilde{\Gamma}_{X,0}$ defined by
\[
\tilde{\rho}_\Lambda \mid \tilde{\Gamma}_{X,(n)} := C_\Lambda m^{\otimes n},
\]
where $C_\Lambda$ is the constant from (A2') corresponding to the set $\Lambda$. Then, it is easy to see that
\[
\|A_\varphi^k \tilde{G}(n)\|^2_{\mathcal{H}_\rho} = \int_{\Gamma_{X,0}} \left( (\varphi^\ast G(n)) \ast (\varphi^\ast \tilde{G}(n)) \right)(\eta) \rho(d\eta)
\leq \int_{\Gamma_{X,0}} \left( (\tilde{\varphi}^\ast \tilde{G}(n))^2 \right)(\eta) \tilde{\rho}_\Lambda(d\eta)
= \int_{\Gamma_{X,0}} \left( (\tilde{G}(n))^2 \ast \tilde{\varphi}^{2k} \right)(\eta) \tilde{\rho}_\Lambda(d\eta).
\] (11)
For any $R^{(n)}$ and $f$ from $B_{bs}(\tilde{\Gamma}_{X,0})$ which are only not equal to zero on $\tilde{\Gamma}^{(n)}_X$ and $\tilde{\Gamma}^{(1)}_X$, respectively, we have

$$
(R^{(n)} \ast f)([x_1, \ldots, x_k]) = \begin{cases} 
\sum_{i=1}^{n+1} f(x_i) R^{(n)}([x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}]), & \text{if } k = n + 1, \\
\sum_{i=1}^{n} f(x_i) R^{(n)}([x_1, \ldots, x_{n}]), & \text{if } k = n, \\
0, & \text{otherwise.}
\end{cases}
$$

(12)

Here, $\hat{x}_i$ denotes the absence of $x_i$. Therefore, if additionally $R^{(n)} \geq 0$ and $f \geq 0$, then

$$
\int_{\tilde{\Gamma}_{X,0}} (R^{(n)} \ast f)(\eta) \tilde{\rho}_\Lambda(d\eta) \leq C_{\Lambda,f}(2n + 1) \int_{\tilde{\Gamma}_{X,0}} R^{(n)}(\eta) \tilde{\rho}_\Lambda(d\eta),
$$

(13)

where

$$
C_{\Lambda,f} := \max \{ \text{ess sup } f(x), C_{\Lambda} \int_X f(x) m(dx) \},
$$

which yields

$$
\int_{\tilde{\Gamma}_{X,0}} (R \ast f)(\eta) \tilde{\rho}_\Lambda(d\eta) \leq 2C_{\Lambda,f}(n + 1) \int_{\tilde{\Gamma}_{X,0}} R(\eta) \tilde{\rho}_\Lambda(d\eta)
$$

for each $R \in B_{bs}(\tilde{\Gamma}_{X,0})$, $R \geq 0$, satisfying $R \upharpoonright \tilde{\Gamma}^{(k)}_X = 0$ if $k > n$.

Hence, by using (13), we get

$$
\int_{\tilde{\Gamma}_{X,0}} ((\tilde{G}^{(n)})^{*2} \ast \tilde{\varphi}^{*2k})(\eta) \tilde{\rho}_\Lambda(d\eta)
$$

$$
\leq 2C_{\Lambda,\tilde{\varphi}}(2n + 2k) \int_{\tilde{\Gamma}_{X,0}} ((\tilde{G}^{(n)})^{*2} \ast \tilde{\varphi}^{*2k-1})(\eta) \tilde{\rho}_\Lambda(d\eta)
$$

$$
\leq (2C_{\Lambda,\tilde{\varphi}})^2(2n + 2k)(2n + 2k - 1) \int_{\tilde{\Gamma}_{X,0}} ((\tilde{G}^{(n)})^{*2} \ast \tilde{\varphi}^{*2k-2})(\eta) \tilde{\rho}_\Lambda(d\eta)
$$

$$
\leq \cdots \leq (2C_{\Lambda,\tilde{\varphi}})^{2k} \frac{(2n + 2k)!}{(2n)!} \int_{\tilde{\Gamma}_{X,0}} (\tilde{G}^{(n)})^{*2}(\eta) \tilde{\rho}_\Lambda(d\eta).
$$

(14)

Thus, (11) and (14) give

$$
\|A^{k \tilde{G}^{(n)}}\|_{H_{\rho}} \leq (2C_{\Lambda,\tilde{\varphi}})^k \frac{(2n)!}{(2n+k)!} \cdot (\tilde{G}^{(n)})^{*2} \tilde{\rho}_\Lambda(n+k+1) - 1/2 \sum_{k=0}^{\infty} (4C_{\Lambda,\tilde{\varphi}})^k (n+k)! |z|^k < \infty \quad \text{if } |z| < (4C_{\Lambda,\tilde{\varphi}})^{-1},
$$

Since

$$
\sum_{k=0}^{\infty} \frac{(4C_{\Lambda,\tilde{\varphi}})^k (n+k)!}{k!} |z|^k < \infty \quad \text{if } |z| < (4C_{\Lambda,\tilde{\varphi}})^{-1},
$$

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the analyticity of $\hat{G}^{(n)}$ for $A_\varphi$ is proven. By using Nelson’s analytic vector criterium (e.g., [23], Sect. X.6, or [2], Ch. 5, Th. 1.7) we conclude that the operators $A_\varphi$ are essentially selfadjoint on $\hat{F}_\text{fin}(D)$.

Next, by Lemma 1 and (8), the operators $A_{\varphi_1}$ and $A_{\varphi_2}$, $\varphi_1, \varphi_2 \in D$, commute on $\hat{F}_\text{fin}(D)$. Since the operator $A_{\varphi_2}$ is essentially selfadjoint on $\hat{F}_\text{fin}(D)$, the set

$$(A_{\varphi_2}^{-} - z \text{id})\hat{F}_\text{fin}(D), \quad z \in \mathbb{C}, \exists z \neq 0,$$

is dense in $H_\rho$. Next, again using Lemma 1 and (8), we get

$$(A_{\varphi_2}^{-} - z \text{id})\hat{F}_\text{fin}(D) = ((A_{\varphi_2}^{-} - z \text{id})F_\text{fin}(D))^\sim \subset \hat{F}_\text{fin}(D).$$

Therefore, the operators $A_{\varphi_1}^-$, $A_{\varphi_2}^-$, and

$$A_{\varphi_1}^- | (A_{\varphi_2}^- - z \text{id})\hat{F}_\text{fin}(D)$$

have a total set of analytical vectors. Thus, by [2], Ch. 5, Th. 1.15, the operators commute in the sense of the resolutions of the identity. ■

Let $D'$ denote the dual space of $D$ and let $C_\sigma(D')$ be the cylinder $\sigma$-algebra on $D'$ (see e.g. [2], Ch. 2, Sect. 1, subsec. 9).

**Theorem 1** Let a measure $\rho$ on $(\Gamma_{X,0}, B(\Gamma_{X,0}))$ satisfy the assumptions (A1), (A2'), and (A3). Then, there exists a probability measure $\mu$ on $(D', C_\sigma(D'))$ and a unitary isomorphism

$$K: H_\rho \rightarrow L^2(D', C_\sigma(D'), d\mu) := L^2(d\mu)$$

such that the image of each operator $A_\varphi^-$, $\varphi \in D$, under $K$ is the operator of multiplication by the monomial $\langle \varphi, \cdot \rangle$ in $L^2(d\mu)$:

$$KA_\varphi^- K^{-1} = \langle \varphi, \cdot \rangle \cdot.$$ (15)

The unitary $K$ is defined first on the dense set $\hat{F}_\text{fin}(D)$ in $H_\rho$ by the formula

$$\hat{F}_\text{fin}(D) \ni \hat{G} = (\hat{G}^{(n)})_{n=0}^\infty \mapsto K\hat{G} = (K\hat{G})(\omega) = \sum_{n=0}^\infty \langle G^{(n)}, :\omega^{\otimes n}: \rangle$$ (16)

(the series in (16) is actually finite) and then it is extended by continuity to the whole $H_\rho$ space. Here, $G = (G^{(n)})_{n=0}^\infty \in F_\text{fin}(D)$ is an arbitrary representative of $\hat{G} \in \hat{F}_\text{fin}(D)$, and for any $\omega \in D'$, $:\omega^{\otimes n}: \in D'^{\otimes n}$ is the $n$-th Wick power of $\omega$ defined by the recurrence relation

$$:\omega^{\otimes 0}: = 1, \quad :\omega^{\otimes 1}: = \omega, \quad \langle \varphi^{\otimes (n+1)}, :\omega^{\otimes (n+1)}: \rangle = \frac{1}{n+1} \left[ \langle \varphi^{\otimes (n+1)}, :\omega^{\otimes n} \otimes \omega \rangle - n \langle (\varphi^{\otimes 2}) \otimes \varphi^{\otimes (n-1)}, :\omega^{\otimes n}: \rangle \right],$$ (17)

$$\varphi \in D.$$
Remark 1 Let $\mathcal{F}_\text{fin}^*(\mathcal{D})$ stand for the dual of $\mathcal{F}_\text{fin}(\mathcal{D})$. This is the topological direct product of the dual spaces $\mathcal{F}_n(\mathcal{D}') = \mathcal{D}'_{\mathcal{C}}^\otimes n$ of $\mathcal{F}_n(\mathcal{D})$. Thus, an arbitrary element $R$ of $\mathcal{F}_\text{fin}^*(\mathcal{D})$ has the form

$$ R = (R^{(n)})_{n=0}^\infty $$

such that $a_\rho(G, R) = 0$ for each $G \in \mathcal{F}_\text{fin}(\mathcal{D})$. Next, it follows from (7) that the dual $\hat{\mathcal{F}}_\text{fin}^*(\mathcal{D})$ of $\hat{\mathcal{F}}_\text{fin}(\mathcal{D})$ can be identified with the factor-space

$$ \hat{\mathcal{F}}_\text{fin}^*(\mathcal{D}) = \mathcal{F}_\text{fin}^*(\mathcal{D}) / \{ R : \langle G, R \rangle = 0 \text{ for each } G \in \mathcal{F}_\text{fin}(\mathcal{D}) \}. $$

Here, $\langle \cdot, \cdot \rangle$ denotes the dual pairing between the spaces $\mathcal{F}_\text{fin}(\mathcal{D})$ and $\mathcal{F}_\text{fin}^*(\mathcal{D})$ (as well as the pairing between the spaces $\hat{\mathcal{F}}_\text{fin}(\mathcal{D})$ and $\hat{\mathcal{F}}_\text{fin}^*(\mathcal{D})$ below). Thus, each element $R \in \mathcal{F}_\text{fin}^*(\mathcal{D})$ is a representative of some element $\hat{R} \in \hat{\mathcal{F}}_\text{fin}^*(\mathcal{D})$. Define now

$$ R(\omega) := (\omega^{\otimes n})_{n=0}^\infty \in \mathcal{F}_\text{fin}^*(\mathcal{D}), $$

and let $\hat{R}(\omega)$ be the corresponding element of $\hat{\mathcal{F}}_\text{fin}^*(\mathcal{D})$. Then, the formula (16) can be rewritten in the form

$$ \hat{\mathcal{F}}_\text{fin}(\mathcal{D}) \ni \hat{G} \mapsto K\hat{G} = (K\hat{G})(\omega) = \langle \hat{G}, \hat{R}(\omega) \rangle, $$

and hence (15) yields

$$ \langle A_\varphi \hat{G}, \hat{R}(\omega) \rangle = (K(A_\varphi \hat{G}))(\omega) = (\varphi, \omega)(K\hat{G})(\omega) = \langle \varphi, \omega \rangle \langle \hat{G}, \hat{R}(\omega) \rangle, \quad \hat{G} \in \hat{\mathcal{F}}_\text{fin}(\mathcal{D}). $$

So, $\hat{R}(\omega)$ is a generalized joint eigenvector of the family $A_\varphi^\sim$, $\varphi \in \mathcal{D}$, belonging to the joint eigenvalue $\omega \in \mathcal{D}'$, and the unitary $K$, written in the form (18), is the Fourier transform in generalized joint eigenvectors of this family (see [2], Ch. 3, for a detailed exposition of the general theory).

**Proof of Theorem 1.** We will use the standard technique of construction of the Fourier transform in generalized joint eigenvectors of a family of commuting self-adjoint operators [2, 17, 4]. In fact, the existence of a measure and a unitary $K$ satisfying (15) and given by the formula (16) with some kernels $\omega^{\otimes n} : \mathcal{D}'_{\mathcal{C}}^\otimes n$ follows from the following lemma.

**Lemma 3** 1) For each $\varphi \in \mathcal{D}$, $A_\varphi$ is a linear continuous operator on $\hat{\mathcal{F}}_\text{fin}(\mathcal{D})$.

2) For an arbitrary fixed $\hat{G} \in \hat{\mathcal{F}}_\text{fin}(\mathcal{D})$, the mapping

$$ \mathcal{D} \ni \varphi \mapsto A_\varphi \hat{G} \in \hat{\mathcal{F}}_\text{fin}(\mathcal{D}) $$

is linear and continuous.
3) The vacuum \( \hat{\Omega} = (1, 0, 0, \ldots) \in \hat{F}_{\text{fin}}(D) \) is a strong cyclic vector of the family \((A_{\varphi})_{\varphi \in D} \), i.e., the linear span of the set
\[
\{ \hat{\Omega} \} \cup \{ A_{\varphi_1} \cdots A_{\varphi_n} \hat{\Omega} \mid \varphi_i \in D, i = 1, \ldots, n, n \in \mathbb{N} \}
\]
is dense in \( \hat{F}_{\text{fin}}(D) \).

**Proof of Lemma 3.** 1) and 2) Clear by Lemma 1.

3) Denote by \( \Omega = (1, 0, 0, \ldots) \) the vacuum in \( F_{\text{fin}}(D) \). It suffices to show that the set
\[
\{ \Omega \} \cup \{ A_{\varphi_1} \cdots A_{\varphi_n} \Omega \mid \varphi_i \in D, i = 1, \ldots, n, n \in \mathbb{N} \}
\]
is dense in \( F_{\text{fin}}(D) \).

Because of (12), we have on \( F_{\text{fin}}(D) \)
\[
A_{\varphi} = A_{\varphi}^+ + A_{\varphi}^0,
\]
where \( A_{\varphi}^+ \) is a creation operator:
\[
A_{\varphi}^+ \psi \otimes n^\otimes = (n + 1) \varphi \widehat{\otimes} \psi \otimes n,
\]
and \( A_{\varphi}^0 \) is a neutral operator:
\[
A_{\varphi}^0 \psi \otimes n^\otimes = n (\varphi \otimes \psi^\otimes) \otimes n^{\otimes (n-1)}.
\]
Therefore, taking to notice that the \( A_{\varphi}^+ \)'s are the usual creation operators, the cyclicity of \( \Omega \) for the operators \( A_{\varphi} \) follows from the proof of Theorem 2.1 in [17], p. 65. ■

To finish the proof of the theorem, we need only to show that (17) holds. To this end, denote for \( G \in F_{\text{fin}}(D) \) \( KG := K \hat{G} \). Then, upon (15), (16), (19)–(21),
\[
\langle \varphi, \cdot \rangle K(\varphi \otimes n) = KA_{\varphi} \varphi \otimes n = (n + 1)K(\varphi \otimes (n+1)) + nK((\varphi^2) \otimes \varphi \otimes (n-1)),
\]
which implies (17). ■

**Corollary 1** Under the conditions of Theorem 1, we have for each \( G \in \mathcal{H}_\rho \)
\[
\int_{\Gamma_{X,0}} G(\eta) \rho(d\eta) = \int_{\mathcal{D}'} KG(\omega) \mu(d\omega).
\]

**Proof.** Since \( K \) is unitary, we have, for arbitrary \( G_1, G_2 \in \mathcal{H}_\rho \),
\[
\int_{\Gamma_{X,0}} (G_1 \star G_2)(\eta) \rho(d\eta) = \int_{\mathcal{D}'} (KG_1)(\omega)(KG_2)(\omega) \mu(d\omega).
\]
By setting in this formula \( G_1 = G \) and \( G_2 = \hat{\Omega} \) and noting that, from one hand side, the vacuum is the identity element for the \( \star \)-convolution and on the other hand \( K \hat{\Omega} \equiv 1 \), we get the corollary. ■
Remark 2 Let us consider the functional
\[ L(\varphi; \omega) := e^{\langle \log(1+\varphi), \omega \rangle}, \]
which is evidently analytical in \( \varphi \) in a neighborhood of zero in \( D_C \) for each fixed \( \omega \in D' \). Then, by differentiating this functional and by using the recurrence relation (17), one can show that \( L \) is the generating functional of the Wick monomials \( \langle \varphi^\otimes n, \omega^\otimes n \rangle \), i.e.,
\[ L(\varphi, \omega) = \sum_{n=0}^{\infty} \langle \varphi^\otimes n, \omega^\otimes n \rangle \]
for \( \varphi \) from a neighborhood of zero (more exactly, for \( \varphi \in D_C \) such that \( \sup_{x \in X} |\varphi(x)| < 1 \)). Notice that the functional \( L \) is just the character in the generalized translation operator approach to Poisson analysis [1].

3 The measure \( \rho \) as a correlation measure

The configuration space \( \Gamma_X \) over \( X \) is defined as the set of all locally finite subsets (configurations) in \( X \):
\[ \Gamma_X := \{ \gamma \subset X \mid |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset X \}. \]
Here \( |A| \) denotes the cardinality of a set \( A \). One can identify any \( \gamma \in \Gamma_X \) with the positive Radon measure
\[ \sum_{x \in \gamma} \delta_x \in \mathcal{M}(X), \]
where \( \delta_x \) is the Dirac measure with mass in \( x \), \( \sum_{x \in \emptyset} \delta_x := \text{zero measure} \), and \( \mathcal{M}(X) \) stands for the set of all positive Radon measures on \( B(X) \). The space \( \Gamma_X \) can be endowed with the relative topology as a subset of the space \( \mathcal{M}(X) \) with the vague topology, i.e., the weakest topology on \( \Gamma_X \) such that all maps
\[ \Gamma_X \ni \gamma \mapsto \langle f, \gamma \rangle := \int_X f(x) \gamma(dx) = \sum_{x \in \gamma} f(x) \]
are continuous. Here, \( f \in C_0(X) \) (:=the set of all continuous functions in \( X \) with compact support). We will denote by \( \mathcal{B}(\Gamma_X) \) the Borel \( \sigma \)-algebra on \( \Gamma_X \). In fact, \( \Gamma_X \) is a measurable subset of \( D' \) and the trace \( \sigma \)-algebra of \( \mathcal{C}_\sigma(D') \) on \( \Gamma_X \) (i.e., the \( \sigma \)-algebra on \( \Gamma_X \) consisting of intersections of sets from \( \mathcal{C}_\sigma(D') \) with \( \Gamma_X \)) coincides with \( \mathcal{B}(\Gamma_X) \).

The following lemma gives a direct representation of the Wick powers \( :\omega^\otimes n: \) in the case where \( \omega = \gamma \) is a configuration.
Lemma 4 For each $\gamma \in \Gamma_X$, we have

$$\gamma \otimes n := \sum_{\eta \supseteq \gamma, |\eta| = n} \widehat{\delta}_x,$$

where the summation is extended over all $n$-point subconfigurations from $\gamma$.

Proof. For $n = 0$ and $n = 1$ the formula evidently holds, and let us suppose that it holds for all $m \leq n$. Then, upon (17)

$$\langle \varphi \otimes (n+1), \gamma \otimes (n+1) \rangle = \frac{1}{n+1} \left[ \langle \varphi \otimes n, \gamma \otimes n \rangle \langle \varphi, \gamma \rangle - n \langle \varphi^2 \otimes \gamma \otimes (n-1), \gamma \otimes n \rangle \right]$$

$$= \frac{1}{n+1} \left( \sum_{\eta \supseteq \gamma, |\eta| = n} \prod_{y \in \eta} \varphi(y) - \sum_{\eta \supseteq \gamma, |\eta| = n} \sum_{x \in \eta} \varphi^2(x) \prod_{y \in \eta \setminus \{x\}} \varphi(y) \right)$$

$$= \frac{1}{n+1} \sum_{\eta \supseteq \gamma, |\eta| = n} \prod_{x \in \gamma} \varphi(x) \sum_{y \in \gamma \setminus \eta} \varphi(y) = \sum_{\eta \supseteq \gamma, |\eta| = n} \varphi(y). \quad \blacksquare$$

As a direct consequence of Lemma 4 and Corollary 1, we get

Proposition 1 Suppose that, under the assumptions of Theorem 1, the measure $\mu$ has the configuration space $\Gamma_X$ as a set of full measure. Then, the operator $K$ coincides with the $K$-transform between the spaces of functions of finite and infinite configurations, while the measure $\rho$ is the correlation measure of $\mu$ [14, 15, 16, 10].

To restrict the measure $\mu$ to $\Gamma_X$, we need an additional condition on $\rho$, which is also not very restrictive.

(A4) Every compact $\Lambda \subset X$ can be covered by a finite union of open sets $\Lambda_1, \ldots, \Lambda_k$, $k \in \mathbb{N}$, which have compact closures and satisfy the estimate

$$\rho(\Gamma^{(n)}_{\Lambda_i}) \leq (2 + \varepsilon)^{-n} \text{ for all } i = 1, \ldots, k \text{ and } n \in \mathbb{N},$$

where $\varepsilon = \varepsilon(\Lambda) > 0$.

Suppose, for example, that a measure $\rho$ on $\Gamma_{X,0}$ has density $\tilde{\rho}$ with respect to the Lebesgue–Poisson measure

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{\otimes n},$$

and suppose that this density fulfills the estimate

$$\operatorname{ess sup}_{\eta \in \Gamma^{(n)}_{X}} \tilde{\rho}(\eta) \leq n! C^n, \quad n \in \mathbb{N},$$

for some constant $C > 0$. Then $\rho$ satisfies trivially (A2') as well as (A4). (We note that this situation where the measure $\rho$ has density with respect to the Lebesgue–Poisson measure is typical in applications.)
Theorem 2 Let a measure $\rho$ on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$ satisfy the assumptions (A1), (A2'), (A3), (A4), and let $\mu$ be the probability measure on $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'))$ constructed in Theorem 1. Then, $\Gamma_X$ is of full $\mu$ measure.

Proof. The proof is a modification of part of the proof of Theorem 5.5 in [10].

For a function $\varphi \in \mathcal{D}_C$, define a function $e(\varphi, \cdot)$ on $\Gamma_{X,0}$ as follows:

$$\Gamma_{X,0} \ni \eta \mapsto e(\varphi, \eta) := \prod_{x \in \eta} \varphi(x) \in \mathbb{C}.$$  

It follows from Remark 2 that

$$e^{\langle \varphi, \omega \rangle} = \sum_{n=0}^{\infty} \langle (e^\varphi - 1)^{\otimes n}, : \omega^{\otimes n} : \rangle,$$

where $\varphi$ belongs to a neighborhood of zero in $\mathcal{D}_C$, more exactly, if $\sup_{x \in X} |\varphi(x)| < \delta$ for some $\delta > 0$. Therefore,

$$e^{\langle \varphi, \cdot \rangle} = K e(e^\varphi - 1, \cdot). \quad (23)$$

Fix a compact $\Lambda \subset X$. Let $\mathcal{C}_{\sigma, \Lambda}(\mathcal{D}')$ denote the sub-$\sigma$-algebra of $\mathcal{C}_\sigma(\mathcal{D}')$ generated by the functionals of the form

$$\mathcal{D}' \ni \omega \mapsto \langle \varphi, \omega \rangle \in \mathbb{C}, \quad \varphi \in \mathcal{D}(\Lambda),$$

where $\mathcal{D}(\Lambda)$ denote the subspace of $\mathcal{D}$ consisting of those $\varphi$ having support in $\Lambda$. Next, let $\mu_\Lambda$ stand for the restriction of the measure $\mu$ to the sub-$\sigma$-algebra $\mathcal{C}_{\sigma, \Lambda}(\mathcal{D}')$.

Let now $\varphi \in \mathcal{D}_C(\Lambda)$. It follows from (23) that

$$e(e^\varphi - 1, \cdot) \ast e(e^{\overline{\varphi}} - 1, \cdot) = e(e^\varphi + \overline{\varphi} - 1, \cdot).$$

Therefore, by using (A2'), we see that there exists $\delta_\Lambda > 0$ such that $e(e^\varphi - 1, \cdot) \in \mathcal{H}_\rho$ provided $\sup_{x \in X} |\varphi(x)| \leq \delta_\Lambda$. Thus, by Corollary 1,

$$\int_{\mathcal{D}'} e^{\langle \varphi, \omega \rangle} \mu_\Lambda(d\omega) = \int_{\Gamma_{X,0}} e(e^\varphi - 1, \eta) \rho(d\eta), \quad \varphi \in \mathcal{D}_C(\Lambda), \sup_{x \in X} |\varphi(x)| \leq \delta_\Lambda. \quad (24)$$

Thus, the formula (24) gives the analytic extension of the Fourier transform of the measure $\mu_\Lambda$ in a neighborhood of zero.

Let us introduce now a mapping $\mathcal{R}$ which transforms the set of measurable functions on $\Gamma_\Lambda$ into itself as follows:

$$(\mathcal{R}F)(\eta) := \sum_{\xi \in \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_\Lambda.$$
Let now $\Lambda$ satisfy the condition
\[
\rho(\Gamma^{(n)}_\Lambda) \leq (2 + \varepsilon)^n, \quad \varepsilon > 0.
\] (25)
Define on $B(\Gamma_{\Lambda})$ the set function
\[
\tilde{\mu}_\Lambda(A) := \int_{\Gamma_{\Lambda}} (\mathcal{R}1_{\Lambda})(\eta) \rho(d\eta).
\]
Since $\sum_{\xi \subset \eta} 1 = 2^n$ if $|\eta| = n$, we conclude that the bound (25) implies that $\tilde{\mu}_\Lambda$ is a signed measure. Therefore, for $\varphi \in \mathcal{D}_C(\Lambda)$, we have
\[
\int_{\Gamma_{\Lambda}} e^{\langle \varphi, \eta \rangle} \tilde{\mu}_\Lambda(d\eta) = \int_{\Gamma_{\Lambda}} (\mathcal{R}e^{\langle \varphi, \cdot \rangle})(\eta) \rho(d\eta).
\] (26)
Direct calculation shows that
\[
(\mathcal{R}e^{\langle \varphi, \cdot \rangle})(\eta) = e^{(e^{\varphi} - 1, \eta)},
\]
and therefore, we have from (24) and (26)
\[
\int_{\mathcal{D}'} e^{\langle \varphi, \omega \rangle} \mu_\Lambda(d\omega) = \int_{\Gamma_{\Lambda}} e^{\langle \varphi, \eta \rangle} \tilde{\mu}_\Lambda(d\eta), \quad \varphi \in \mathcal{D}_C(\Lambda), \sup_{x \in X} |\varphi(x)| \leq \delta_\Lambda.
\]
Therefore, $\tilde{\mu}_\Lambda$ is a probability measure on $\Gamma_{\Lambda}$, and moreover it coincides with the restriction of the measure $\mu_\Lambda$ to the set $\Gamma_{\Lambda}$ considered as a subset of $\mathcal{D}'$.

Hence
\[
\mu(\tilde{\Gamma}_\Lambda) = 1,
\] (27)
where $\tilde{\Gamma}_\Lambda$ denotes the set of all $\omega \in \mathcal{D}'$ whose restriction to the set $\Lambda$ is a finite sum of delta functions concentrated in $\Lambda$ and having disjoint support.

Now, let $\Lambda$ be an arbitrary compactum in $X$ and let $\Lambda_1, \ldots, \Lambda_k$ be open subsets of $X$ as in (A4) corresponding to $\Lambda$. Since
\[
\tilde{\Gamma} \bigcup_{i=1}^k \Lambda_i = \bigcap_{i=1}^k \tilde{\Gamma}_\Lambda_i,
\]
we conclude that (27) holds for each compact $\Lambda \subset X$. From here, we immediately conclude that $\mu(\Gamma_X) = 1$. ■

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