Maximally extended $\mathfrak{sl}(2|2)$, q-deformed $\mathfrak{d}(2,1;\epsilon)$ and 3D kappa-Poincaré

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Abstract
We show that the maximal extension $\mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3$ of the $\mathfrak{sl}(2|2)$ superalgebra can be obtained as a contraction limit of the semi-simple superalgebra $\mathfrak{d}(2,1;\epsilon) \times \mathfrak{sl}(2)$. We reproduce earlier results on the corresponding q-deformed Hopf algebra and its universal R-matrix by means of contraction. We make the curious observation that the above algebra is related to kappa-Poincaré symmetry. When dropping the graded part $\mathfrak{psl}(2|2)$ we find a novel one-parameter deformation of the 3D kappa-Poincaré algebra. Our construction also provides a concise exact expression for its universal R-matrix.

Keywords: quantum algebra, kappa-Poincaré, integrable models, universal R-matrix, exceptional superalgebra, algebraic contraction

1. Introduction

During the 1970’s and 80’s the Leningrad school led by Ludvig Faddeev developed the quantum inverse scattering method and the algebraic Bethe ansatz to solve large classes of quantum integrable systems [1], see also [2]. These developments laid the foundations for the mathematical formulation of quantum groups and algebras [3]. The theory of quantum algebras based on simple Lie algebras has since been worked out quite exhaustively with numerous results, methods and applications, see e.g. [4]. These quantum algebras display very regular structures which are closely related to their root systems.

In Honour of Petr P Kulish and Ludvig D Faddeev

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Petr Kulish was among the first to formulate quantum algebras based on Lie superalgebras [5]. The generalisation to simple superalgebras largely uses the same regular structures associated to their root systems, and hence many such algebras can be constructed along similar lines. However, there are also a few special cases with peculiar features. For example, there are simple Lie superalgebras such as $\mathfrak{psl}(N|N)$ with vanishing dual Coxeter number. Furthermore, there are the exceptional superalgebras $\mathfrak{d}(2,1;\alpha)$, which form a family depending on the continuous parameter $\alpha$. While similar features do not arise for ordinary simple Lie algebras, they may do for non-simple Lie algebras, with the corresponding quantum algebras sometimes displaying novel structures and applications. Consequently, quantum algebras based on non-simple Lie algebras or Lie superalgebras are explored to a lesser extent, and may still harbour some pleasant surprises.

Examples of such unconventional quantum algebra structures have been found in the one-dimensional Hubbard model and in $\mathcal{N} = 4$ supersymmetric gauge theory in the planar limit (as well as related models in the context of the AdS/CFT correspondence), see [6] and [7] for reviews. In particular, they both possess a peculiar R-matrix that is not of difference form [8–10]. Due to its uncommon structure, this R-matrix escapes the established classification in terms of Yangian and quantum affine algebras based on semi-simple Lie (super)algebras.

A long-standing goal in this regard is to construct the underlying quantum algebra and its universal R-matrix. Several pieces of this puzzle are known. It is clear that the Lie superalgebra $\mathfrak{sl}(2|2)$ and its exceptional central extensions plays a role [9, 11]. Studies of the classical limit have demonstrated that the complete algebra also needs to be extended by derivations [12, 13]. The maximal extension of $\mathfrak{sl}(2|2)$ is a non-simple superalgebra

$$\mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3,$$

which incidentally also serves as a non-standard extended super-Poincaré symmetry in three spacetime dimensions [14]. The classical analysis [13] suggests that the relevant quantum algebra is a peculiar subalgebra, yet to be identified, of the Yangian of the maximal extension of $\mathfrak{sl}(2|2)$.

A complication related to the latter approach is that the underlying Lie algebra is non-simple and the precise form of its algebra relations does not necessarily follow the patterns known from simple Lie (super)algebras. Moreover, the Yangian is a contraction limit of the quantum affine algebra which obscures some of its uniform structure. The $q$-deformation for the maximally extended $\mathfrak{sl}(2|2)$ was explored in [15] and revealed some unconventional terms in the algebra and in the R-matrix. Unfortunately, the form of the result does not make evident how to construct the exact form of q-deformed non-simple Lie (super)algebras, except by applying some amount of trial and error and brute force.

In this paper we revisit the $q$-deformation of maximally extended $\mathfrak{sl}(2|2)$. We will use a different method to construct the algebra and its universal R-matrix. Our idea is based on the connection between extensions of $\mathfrak{psl}(2|2)$ and the exceptional superalgebra $\mathfrak{d}(2,1;\epsilon)$ for $\epsilon = 0$. In fact, there are two ways the limit $\epsilon \to 0$ can be approached, and they yield the superalgebra $\mathfrak{psl}(2|2)$, either with three central extensions $\mathbb{C}^3$ or with an $\mathfrak{sl}(2)$ algebra of derivations [2]. Curiously, both of these extensions can coexist in a consistent Lie superalgebra, which is the maximal extension of $\mathfrak{sl}(2|2)$. However, they cannot both be obtained at the same time from $\mathfrak{d}(2,1;\epsilon)$ alone as the latter lacks three generators. To overcome this shortcoming, we can supply three more generators forming an $\mathfrak{sl}(2)$ algebra, and indeed there is a contraction limit that yields the maximally extended $\mathfrak{sl}(2|2)$

Throughout the paper we will assume algebras to be over the complex numbers. The choice of signature is relevant only to real forms and consequently it will not be of concern to us.

There may be further ways of taking the limit, and attempts have been made to obtain the relevant algebra for the above R-matrix along these lines [16].
\[ \mathfrak{o}(2, 1; \epsilon) \times \mathfrak{sl}(2) \xrightarrow{\epsilon \to 0} \mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3. \] (1.2)

We will use this contraction limit to construct q-deformed maximally extended \( \mathfrak{sl}(2|2) \) based on the (standard) q-deformations of \( \mathfrak{o}(2, 1; \epsilon) \) and \( \mathfrak{sl}(2) \). We will show that this construction yields precisely the algebra relations and the R-matrix obtained in [15]. The \( \mathfrak{o}(2, 1; \epsilon) \) origin of maximally extended \( \mathfrak{sl}(2|2) \) also explains some of the observed peculiarities. For instance, the q-deformed \( \mathfrak{o}(2, 1; \epsilon) \) algebra has three q-deformed \( \mathfrak{sl}(2) \) subalgebras. Importantly, these have deformation parameters \( q, \epsilon \) and \( q^{-1} - \epsilon \), respectively. This implies that in the limit \( \epsilon \to 0 \), some part of the algebra will be (more or less) undeformed (\( q \approx 1 \)) while some other parts remain fully deformed (\( q \not\approx 1 \)).

The fully deformed part of the algebra is the superalgebra \( \mathfrak{psl}(2|2) \) while the \( \mathfrak{sl}(2) \) derivations and charges \( \mathbb{C}^3 \) are only weakly deformed. In fact, one can remove the superalgebra \( \mathfrak{psl}(2|2) \) from the bigger algebra and what remains is a deformation of the 3D Poincaré algebra

\[ \mathfrak{sl}(2) \ltimes \mathbb{C}^3 = \mathfrak{iso}(3). \] (1.3)

Deformations of Poincaré symmetry and associated physical theories have been investigated in their own right (see e.g. the review articles [17]) and this one corresponds to the so-called kappa-Poincaré symmetry. Our algebra turns out to be a novel one-parameter family of deformations of the kappa-Poincaré algebra that is particular to 3D. In this sense, q-deformed maximally extended \( \mathfrak{sl}(2|2) \) is a supersymmetric extension of 3D kappa-Poincaré (with two deformation parameters).

The present paper is organised as follows: We start in section 2 by performing the contraction of the q-deformation of \( \mathfrak{so}(4) \) to obtain a deformation of the 3D Poincaré algebra \( \mathfrak{iso}(3) \). This investigation highlights relevant features in a simplified context, which are then used in the generalisation to superalgebras performed in section 4. In section 3 we compare our deformation of \( \mathfrak{iso}(3) \) to the 3D kappa-Poincaré algebra and show that it is a one-parameter deformation of the latter. We conclude in section 5 and give an outlook.

2. Deformation of 3D Poincaré as a contraction

The maximal extension \( \mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3 \) of the \( \mathfrak{sl}(2|2) \) superalgebra is a non-standard supersymmetric extension [14] of the ordinary 3D Poincaré algebra

\[ \mathfrak{iso}(3) = \mathfrak{sl}(2) \ltimes \mathbb{C}^3. \] (2.1)

The q-deformation of this superalgebra along with its universal R-matrix was constructed in [15], and it was seen to possess a number of unusual features. Most of these unusual features relate to its 3D Poincaré subalgebra, and in fact they remain present in the restriction to it. Therefore it makes sense to study the deformed 3D Poincaré algebra in detail towards understanding the peculiarities in a simpler context.

The 3D Poincaré algebra is well-known to be a contraction of

\[ \mathfrak{so}(4) = \mathfrak{sl}(2) \times \mathfrak{sl}(2). \] (2.2)

Our goal in this section is to lift this contraction to the q-deformed algebras. As we will see, the unusual features alluded to above can all be understood as originating in this limiting procedure. It also transpires, as we will discuss later in section 3, that the q-deformed algebra is closely related to kappa-Poincaré algebras [18–20]. The latter are understood to be constructible as contractions of q-deformed orthogonal algebras [18, 21–23], however, the 3D case
turns out to be particularly tractable and gives rise to some special features. Afterwards, we will return to q-deformed maximally extended $\mathfrak{sl}(2|2)$ in section 4 and show that it originates from a contraction involving the exceptional superalgebra $\mathfrak{o}(2, 1; \alpha)$.

In this paper we will be dealing with several q-deformed (sub)algebras whose (effective) deformation parameter will take different values $q_i$ for some of which we will also take the limit $q_i \to 1$. In order to parameterise the deformations, we will find it convenient to introduce a fixed reference deformation parameter which we denote by $q$ or equivalently by $\hbar q \equiv e^{\hbar}$. (2.3)

The q-deformations of individual (sub)algebras will be specified relative to the reference parameters as $q_{\alpha} = e^{\alpha \hbar}$. This will allow us to tune the relative deformation strength of subalgebras with concrete values, while keeping $q = e^{\hbar}$ fixed to specify the overall strength of the deformation. Furthermore, we shall label q-deformed objects such as q-deformed universal enveloping algebras, q-numbers and q-exponents, by an index ‘$\hbar$’ $U_\hbar(g)$, $[n]_\hbar$, $\exp_\hbar(x)$, ... This helps us specify the relative deformation strength as in $U_{\alpha \hbar}(g)$ in a slightly more legible fashion.

2.1. Hopf algebra

We start by considering two mutually commuting copies of the q-deformed $\mathfrak{sl}(2)$ algebra with different deformation parameters $U_{\epsilon \hbar}(\mathfrak{sl}(2)) \otimes U_{\tilde{\epsilon} \hbar}(\mathfrak{sl}(2))$. (2.5)

For the purpose of taking a contraction limit we introduce the two relative deformation parameters $\epsilon$ and $\tilde{\epsilon}$ which will later be taken to zero. The first copy $U_{\epsilon \hbar}(\mathfrak{sl}(2))$ of the algebra has the following set of defining relations

\[
[H, E] = 2E, \quad \Delta E = E \otimes 1 + q^{-\epsilon H} \otimes E, \quad (2.6)
\]

\[
[H, F] = -2F, \quad \Delta F = F \otimes q^{\epsilon H} + 1 \otimes F, \quad (2.7)
\]

\[
[E, F] = q^{\epsilon H} - q^{-\epsilon H} \over q^{\epsilon} - q^{-\epsilon}, \quad \Delta H = H \otimes 1 + 1 \otimes H. \quad (2.8)
\]

The second copy $U_{\tilde{\epsilon} \hbar}(\mathfrak{sl}(2))$ of the algebra obeys the same set of defining relations (2.6)–(2.8) with the generators $E, F, H$ and parameter $\epsilon$ replaced by $\tilde{E}, \tilde{F}, \tilde{H}$ and $\tilde{\epsilon}$, respectively.

Contraction. We now want to perform the contraction limit $\mathfrak{sl}(2) \times \mathfrak{sl}(2) \to \mathfrak{sl}(2) \ltimes \mathbb{C}^3$. At the same time we also take the limit $\epsilon, \tilde{\epsilon} \to 0$ which ordinarily removes the q-deformation. As usual, the overall limit depends crucially on how the various limits are taken relative to each other, and only for an appropriate fine-tuning of limits we will find the desired algebra which carries some remnants of the q-deformation.

For the contraction limit, it makes sense to introduce the following combinations of generators that we assume to be finite in the limit $\epsilon, \tilde{\epsilon} \to 0$.

\[\text{Note that we use the parameter } \epsilon \text{ not only for performing the contraction but also to specify the relative strength of the q-deformation. This imposes no restriction because there is still the overall deformation parameter } \hbar \text{ which can be adjusted independently.}\]
Furthermore, it is crucial to take the limit $\epsilon, \tilde{\epsilon} \to 0$ in both algebras simultaneously in a coordinated fashion. To understand the requirements, let us inspect some algebra relations in the new basis (2.9)-(2.11). The commutation relation $[E_A, F_A]$ takes the form

$$[E_A, F_A] = \frac{q^{H_C} - q^{-H_C}}{q^{\epsilon} - q^{-\epsilon}} + \frac{q^{2H_C - (\tilde{\epsilon}/\epsilon)H_C} - q^{-2H_C + (\tilde{\epsilon}/\epsilon)H_C}}{q^{\epsilon} - q^{-\epsilon}}$$

$$= \frac{1}{\epsilon} \left[ \frac{q^{H_C} - q^{-H_C}}{2\hbar} + \frac{q^{\beta H_C + \tilde{\epsilon}/\epsilon} - q^{-\beta H_C}}{2\beta \hbar} \right] + \mathcal{O}(\epsilon^0), \quad (2.12)$$

where the expansion in the second line is based on the assumption that $\tilde{\epsilon} \simeq \beta \epsilon$ in the limit $\epsilon \to 0$. We see that the divergent term vanishes for the choices $\beta = \pm 1$. This choice also leads to a well-defined contraction limit for all the other algebra relations. Next we inspect the coalgebra. The coproduct of $E_A$ reads

$$\Delta E_A = E_A \otimes 1 + q^{iH_A + (\tilde{\epsilon}/\epsilon)H_C} \otimes E_A - \frac{1}{\epsilon} \left( q^{-iH_A + (\tilde{\epsilon}/\epsilon)H_C} - q^{-H_C} \right) \otimes E_C$$

$$= -\frac{2\hbar}{\epsilon} \left( q^{\beta H_C} - q^{-H_C} \right) \otimes E_C + \mathcal{O}(\epsilon^0), \quad (2.13)$$

Evidently, we need to set $\beta = -1$ to eliminate the divergent term in this relation. The same choice will eliminate a similar divergence in $\Delta F_A$. Altogether we find that the Hopf algebra has a well-defined contraction limit if $\tilde{\epsilon}/\epsilon \to -1$.

**Limit.** We now write $\tilde{\epsilon}$ as a general expansion in terms of $\epsilon$ subject to the constraint derived above

$$\tilde{\epsilon}(\epsilon) = -\epsilon + \xi \epsilon^2 + \mathcal{O}(\epsilon^3).$$

(14.14)

The parameter $\xi$ can be adjusted freely, and it turns out to survive in the limit. Higher-order terms in the relationship between $\tilde{\epsilon}$ and $\epsilon$ do not contribute in the limit.

The limit of the commutation relations reads

$$[E_A, F_A] = \frac{1}{2} \left( q^{H_C} - q^{-H_C} \right) (H_A + \xi H_C) - \frac{\xi}{2\hbar} \left( q^{H_C} - q^{-H_C} \right), \quad (2.15)$$

as well as

$$[H_A, E_A] = 2E_A, \quad [H_A, F_A] = -2F_A, \quad (2.16)$$

$$[E_A, E_C] = 0, \quad [E_A, F_C] = \frac{q^{H_C} - q^{-H_C}}{2\hbar}, \quad [E_A, H_C] = -2E_C, \quad (2.17)$$

Due to the Hopf algebra isomorphism between $U_q(sl(2))$ we could alternatively assume $i/\epsilon \to +1$ and replace the generators $(E, F, H)$ in the basis (2.9)-(2.11) by $(q^{-i\beta \hbar}E, q^{i\beta \hbar}, -H)$.
\[
[F_A, EC] = -\frac{q^{hc} - q^{-hc}}{2\hbar}, \quad [F_A, FC] = 0, \quad [F_A, HC] = 2FC, \quad (2.18)
\]
\[
[H_A, EC] = 2EC, \quad [H_A, FC] = -2FC, \quad [H_A, HC] = 0, \quad (2.19)
\]
\[
[H_C, EC] = 0, \quad [H_C, FC] = 0, \quad [EC, FC] = 0, \quad (2.20)
\]

while the coproduct relations take the following form
\[
\Delta E_A = E_A \otimes 1 + q^{-hc} \otimes E_A - \hbar (H_A + \xi H_C) q^{-hc} \otimes EC, \quad (2.21)
\]
\[
\Delta F_A = F_A \otimes q^{hc} + 1 \otimes F_A + \hbar FC \otimes q^{hc} (H_A + \xi H_C), \quad (2.22)
\]
\[
\Delta H_A = H_A \otimes 1 + 1 \otimes H_A, \quad (2.23)
\]
\[
\Delta E_C = E_C \otimes 1 + q^{-hc} \otimes EC, \quad (2.24)
\]
\[
\Delta F_C = F_C \otimes q^{hc} + 1 \otimes FC, \quad (2.25)
\]
\[
\Delta H_C = H_C \otimes 1 + 1 \otimes HC. \quad (2.26)
\]

Parameters. One relevant point concerning the above relations is that the parameters \(h\) and \(\xi\) consistently appear as prefactors of the generators \(\{EC, FC, HC\}\). This implies that \(h\) can be eliminated from the algebra and coalgebra by the rescaling
\[
(E_C, FC, HC) \rightarrow h^{-1}(E_C, FC, HC), \quad \xi \rightarrow h\xi. \quad (2.27)
\]
Therefore \(h\) is not a parameter of the Hopf algebra but merely of the presentation given above. Nevertheless we refrain from removing the parameter because it will be useful for later comparisons.

Unlike \(h\), the parameter \(\xi\) cannot be removed from the algebra and coalgebra relations (at the same time) by a redefinition of generators. Note, however, that \(\xi\) can be eliminated from the algebra relations (see [15] in conjunction with section 4) by the redefinition
\[
E_A' = E_A - \xi Y EC, \quad F_A' = F_A - \xi Y FC \quad (2.28)
\]
with
\[
Y := \frac{1}{2} \left( \frac{q^{hc} - q^{-hc}}{2\hbar} \right) HC - 4\hbar X \sqrt{1 + \hbar^2 X \text{arsinh} (\hbar X)} - 1
\]
\[
= \frac{1}{3} \hbar^2 (ECFC + \frac{1}{2} H_C^2) + O(h^4), \quad (2.29)
\]
and the invariant element
\[
X = ECFC + \left( \frac{q^{hc/2} - q^{-hc/2}}{2\hbar^2} \right)^2. \quad (2.30)
\]

While this redefinition removes \(\xi\) from the commutator \([E_A, F_A]\), it does not eliminate it from \(\Delta E_A\) and \(\Delta F_A\); in fact it introduces many additional terms. Therefore \(\xi\) is a non-trivial parameter of the Hopf algebra.

In fact, it is not surprising to find one remaining deformation parameter \(\xi\): The original algebra \(\mathfrak{sl}(2) \times \mathfrak{sl}(2)\) admits two independent \(q\)-deformation parameters \(\epsilon h\) and \(\tilde{\epsilon} h\). One of
them is used up in the contraction limit\(^5\) and it merely appears as the parameter \(\hbar\) of the presentation. The other one survives in the contraction limit as \(\xi\).

The above relations reduce to those of undeformed \(U(\mathfrak{sl}(2) \ltimes \mathbb{C}^3)\) in the further contraction limit \(h \to 0\), \(q = e^{\hbar} \to 1\), where \(\mathfrak{sl}(2)\) and \(\mathbb{C}^3\) are generated by \(\{E_A, F_A, H_A\}\) and \(\{E_C, F_C, H_C\}\), respectively. In this sense the algebra can be viewed as a one-parameter deformation of the 3D Poincaré algebra and we will denote it by

\[
K_\xi(\mathfrak{sl}(2) \ltimes \mathbb{C}^3) = K_\xi(\mathfrak{iso}(3)).
\] (2.31)

2.2. Universal \(R\)-matrix

\(q\)-deformed Hopf algebras based on simple Lie algebras possess a quasi-triangular structure. A natural question in this context is whether the quasi-triangular structure of \(q\)-deformed \(\mathfrak{sl}(2) \ltimes \mathfrak{sl}(2)\) survives the above contraction limit. If it does, we would like to obtain its universal \(R\)-matrix.

The universal \(R\)-matrix of \(U_{ch}(\mathfrak{sl}(2))\) is given by [3]

\[
\mathcal{R}_{\mathfrak{sl}(2)} = \exp_{-2\hbar} \left[ (q^\epsilon - q^{-\epsilon}) E \otimes F \right] \exp \left( \frac{1}{2} \hbar H \otimes H \right),
\] (2.32)

where the \(q\)-exponential is defined via the \(q\)-number and \(q\)-factorial

\[
[n]_h := \frac{1 - q^n}{1 - q}, \quad [n]_h! := \prod_{k=1}^{n} [k]_h, \quad \exp_h[X] := \sum_n \frac{X^n}{[n]_h!}.
\] (2.33)

The \(R\)-matrix of \(U_{ch}(\mathfrak{sl}(2)) \otimes U_{ch}(\mathfrak{sl}(2))\) is the product of the individual \(R\)-matrices\(^6\)

\[
\mathcal{R} = \mathcal{R}_{\mathfrak{sl}(2)} \cdot \mathcal{R}_{\mathfrak{sl}(2)} \\
= \exp_{-2\hbar} \left[ (q^\epsilon - q^{-\epsilon}) E \otimes F \right] \cdot \exp_{-2\hbar} \left[ (q^\epsilon - q^{-\epsilon}) \tilde{E} \otimes \tilde{F} \right] \\
\quad \cdot \exp \left( \frac{1}{2} \chi H \otimes H \right) \exp \left( \frac{1}{2} \tilde{\chi} \tilde{H} \otimes \tilde{H} \right) \\
= \exp_{-2\hbar} \left[ \frac{q^\epsilon - q^{-\epsilon}}{\epsilon} E_C \otimes F_C \right] \\
\quad \cdot \exp_{-2\hbar} \left[ \frac{q^\epsilon - q^{-\epsilon}}{\epsilon^2} \left( E_C - E_A \right) \otimes \left( F_C - F_A \right) \right] \\
\quad \cdot \exp \left( \frac{\hbar}{2\epsilon} H_C \otimes H_C \right) \cdot \exp \left( \frac{\tilde{\chi} H_C - \tilde{\chi} H_A}{2\epsilon^2} \otimes (H_C - H_A) \right),
\] (2.34)

where we have used the transformation (2.9)–(2.11). Each exponential term contains divergences in the limit \(\epsilon \to 0\) paired with a simultaneous removal of the deformation. It therefore requires some work to extract the overall divergences of the terms and determine whether they cancel in the above combination. In order to combine the two \(q\)-exponents into a single

\(^5\)The balance of parameters for contractions can be understood as follows: The contraction requires one parameter which is taken to zero. If the contraction parameter was a genuine parameter of the original algebra, it is eliminated as a parameter of the contracted algebra. If the contraction parameter was merely a parameter of the original algebra’s presentation, the contracted algebra has a new continuous automorphism. The latter can be understood as the limit of the algebra isomorphisms which relate the different presentations of the original algebra.

\(^6\)Another conceivable choice of \(R\)-matrix for the combined algebra is to take the inverse opposite \(R\)-matrix for one of its factors. However, this turns out not to lead to a finite limit.
exponential function we introduce the so-called q-dilogarithm \[24, 25\] as the (ordinary) logarithm of the q-exponent

\[
\log \exp_\hbar [X] = \sum_{n=1}^{\infty} \frac{(1 - q)^{n-1}}{n[n]_\hbar} X^n. \tag{2.35}
\]

We will also require the expansion of the q-dilogarithm close to \(q = 1\) (see corollary 10 of \[25\])

\[
\log \exp_\epsilon \left[ \frac{X}{\epsilon} \right] = -\frac{1}{\epsilon} \Li_2 (-X) + \mathcal{O}(\epsilon). \tag{2.36}
\]

We can now calculate the limit of the R-matrix. The expansion of the logarithm of the first q-exponential in (2.34) is

\[
\log \exp_{-2\hbar} \left[ \frac{q^\epsilon - q^{-\epsilon}}{\epsilon^2} E_C \otimes F_C \right] - \frac{1}{2\hbar} \Li_2 (4\hbar^2 E_C \otimes F_C) + \mathcal{O}(\epsilon). \tag{2.37}
\]

while for the second q-exponential we find

\[
\log \exp_{-2\hbar} \left[ \frac{q^\epsilon - q^{-\epsilon}}{\epsilon^2} (E_C - \epsilon E_A) \otimes (F_C - \epsilon F_A) \right]
\]

\[
= - \left( \frac{1}{2\hbar} + \frac{\xi}{2\hbar} \right) \Li_2 (4\hbar^2 E_C \otimes F_C)
\]

\[
- \frac{1}{2\hbar} (E_C \otimes F_A + E_A \otimes F_C + 2\xi E_C \otimes F_C) \frac{\log (1 - 4\hbar^2 E_C \otimes F_C)}{E_C \otimes F_C} + \mathcal{O}(\epsilon). \tag{2.38}
\]

As the exponents of the two q-exponentials commute (before taking the \(\epsilon \to 0\) limit) the logarithm of their product is simply given by the sum of their logarithms. It immediately follows that the divergences of the two q-exponentials cancel, leaving a finite contribution to the R-matrix. The limit of the exponential functions of the Cartan generators is straight-forward, such that the universal R-matrix takes the following finite form in the limit \(\epsilon \to 0\):

\[
R = \exp \left[ -\frac{\xi}{2\hbar} \Li_2 (4\hbar^2 E_C \otimes F_C) - \frac{\xi}{\hbar} \log (1 - 4\hbar^2 E_C \otimes F_C) \right]
\]

\[
\cdot \exp \left[ - \frac{1}{2\hbar} (E_C \otimes F_A + E_A \otimes F_C) \frac{\log (1 - 4\hbar^2 E_C \otimes F_C)}{E_C \otimes F_C} \right]
\]

\[
\cdot \exp \left[ \frac{1}{2} \hbar (H_C \otimes H_A + H_A \otimes H_C + \xi H_C \otimes H_C) \right]. \tag{2.39}
\]

Let us make two remarks on the form of the resulting R-matrix: First, the R-matrix contains both ordinary dilogarithms \(\Li_2(x)\) and functions \(\log(1 - x)/x = -\Li_2'(x)\) within its exponents. These function may appear unusual at first sight, but given that a q-exponential can be expressed as the exponential of a q-dilogarithm, see (2.35), their appearance is less surprising. Second, the resulting R-matrix is no longer factorised into two constituent R-matrices after taking the limit. The reason for the loss of factorisation is that the constituent R-matrices are divergent on their own, but their combination remains finite in the contraction limit. This behaviour

\[\footnote{Note that in \[25\] the q-exponential and q-dilogarithm functions are defined as \(\exp_\hbar [(1 - q)^{-1}] \) and \(\log \exp_\hbar [(1 - q)^{-1}] \) respectively. The q-dilogarithm function is defined in terms of its series expansion in definition 9 of \[25\], while the relation to the q-exponential function is shown in lemma 8.}
is analogous to the factorisation behaviour of the algebra which starts out as the direct sum 
so(4) = sl(2) × sl(2) and becomes indecomposable in the contraction iso(3) = sl(2) × C².

3. Relation to kappa-Poincaré

Before discussing maximally extended sl(2|2) in section 4, in this section we explore the Hopf algebra and universal R-matrix of section 2 in more detail, including its algebraic structure along with some physical implications. The one-parameter Hopf algebra $K_q(iso(3))$ constructed in the previous section (2.15)–(2.26) as a contraction of $U_q(sl(2)) ⊗ U_q(sl(2))$ is in fact a one-parameter deformation of the well-known 3D kappa-Poincaré algebra, first considered explicitly in [19] as the 3D analog of the 4D kappa-Poincaré algebra of [18, 23].

3.1. Comparison

To compare our one-parameter Hopf algebra with 3D kappa-Poincaré we start by introducing \{L₀, L₁, L₂\} and \{P₀, P₁, P₂\} as the canonical rotation and momentum generators of iso(3) along with the following linear combinations

\begin{align*}
L_\pm &:= \frac{1}{2}(L_1 \pm iL_2), & P_\pm &:= \frac{1}{2}(P_1 \pm iP_2). 
\end{align*}

The new generators are related to those of section 2 as follows

\begin{align*}
H_C &= 2i\text{Ad}_T P_0 & H_A &= 2i\text{Ad}_T L_0 \\
&= 2iP_0 & = 2iL_0, 
E_C &= 2\text{Ad}_T P_+ & E_A &= 2\text{Ad}_T L_+
&= 2q^{-i\phi}P_+ & = 2q^{-i\phi}(L_+ - i\hbar P_+ (L_0 + \xi P_0)), 
F_C &= 2\text{Ad}_T P_- & F_A &= 2\text{Ad}_T L_-
&= 2q^{i\phi}P_- & = 2q^{i\phi}(L_- + i\hbar P_- (L_0 + \xi P_0)),
\end{align*}

with the adjoint action Adₜ and generator $T$ defined as

\begin{align*}
\text{Ad}_T a &:= T a T^{-1}, & T := q^\phi a + \xi \phi \hbar^2/2.
\end{align*}

In this basis the non-vanishing commutation relations (2.15)–(2.20) are given by

\begin{align*}
[L_+, L_-] &= \frac{i}{\hbar}(q^{2i\phi} + q^{-2i\phi})(L_0 + \xi P_0) - \frac{\xi}{8\hbar}(q^{2i\phi} - q^{-2i\phi}), \\
[L_0, L_\pm] &= \mp iL_\pm, \\
[L_\pm, P_\mp] &= \pm\frac{1}{8\hbar}(q^{2i\phi} - q^{-2i\phi}), \\
[L_0, P_\pm] &= [P_0, L_\pm] = \mp iP_\pm,
\end{align*}

such that in the limit $\hbar \to 0$ we recover the algebra iso(3), with the three rotations $L_\mu$ and translations $P_\mu$ satisfying

\begin{align*}
[P_\mu, P_\nu] &= 0, & [L_\mu, L_\nu] &= \epsilon_{\mu\nu\rho}L_\rho, & [L_\mu, P_\nu] &= \epsilon_{\mu\nu\rho}P_\rho.
\end{align*}
Here we have introduced the anti-symmetric tensor $\epsilon_{\mu \nu \rho}$ with $\epsilon^{012} = 1$ and we contract the indices $\mu, \nu, \ldots$ with $\eta_{\mu \nu} = \text{diag} (-1, 1, 1)_{\mu \nu}$. We will denote this algebra as $\mathfrak{iso}(3)$ despite the apparent choice of signature (which is irrelevant in the complexified algebra).

The two quadratic Casimirs of the 3D Poincaré both have generalisations in the deformed algebra. The first is given by (3.30), which in the new basis is

$$X = 4P_+ P_- + \frac{(q^0 - q^{-1})^2}{4\hbar^2},$$

and hence generalises the classical momentum invariant. The second invariant element takes the form

$$\tilde{X} = 4P_+ L_+ + 4P_- L_- + \frac{i}{2\hbar}(q^2 P_0 - q^{-2} P_0)(L_0 + \xi P_0) - \frac{\xi}{2\hbar^2}(q^0 - q^{-1})^2.$$  \hspace{1cm} (3.12)

The classical limit of $\tilde{X}$ is $2P_+ L^\mu$. This scalar is the 3D analogue of the 4D Pauli–Lubański vector, and hence is a measure of the spin.

The adjoint action $\text{Ad}_T$ in the redefinitions (3.2)–(3.4) does not alter the commutation relations, however it does modify the coproduct, which in the new basis is given by

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0,$$

$$\Delta(L_0) = L_0 \otimes 1 + 1 \otimes L_0,$$

$$\Delta(P_\pm) = P_\pm \otimes q^0 P_0 + q^{-1} P_\mp \otimes P_\pm,$$

$$\Delta(L_\pm) = L_\pm \otimes q^{i P_0} + q^{-i P_0} \otimes L_\pm + i\hbar \left[ P_\pm \otimes q^{i P_0} (L_0 + \xi P_0) - (L_0 + \xi P_0) q^{-i P_0} \otimes P_\pm \right].$$

while the R-matrix (2.39) takes the form

$$\mathcal{R} = \text{Ad}_T \left\{ \exp \left[ -\frac{\xi}{2\hbar} L_2 (16\hbar^2 P_+ \otimes P_-) - \frac{\xi}{\hbar} \log(1 - 16\hbar^2 P_+ \otimes P_-) \right] \right\}$$

$$\times \text{Ad}_T \left\{ \exp \left[ \frac{i}{2\hbar} \left( P_+ \otimes L_- + L_+ \otimes P_- \right) \log(1 - 16\hbar^2 P_+ \otimes P_-) \right] \right\}$$

$$\times \exp \left[ -2\hbar (P_0 \otimes L_0 + L_0 \otimes P_0 + \xi P_0 \otimes P_0) \right].$$

(3.17)

Setting $\xi = 0$ we find that the defining relations of $K_0(\mathfrak{iso}(3))$ are equivalent to those of the 3D kappa-Poincaré algebra [19, 22]. To explicitly match the canonical presentation of the latter [20] (usually given in terms of the parameter $\kappa$) one should set $\hbar = \frac{1}{2} \kappa^{-1}$. It then follows that $K_\xi(\mathfrak{iso}(3))$ (3.31) is a one-parameter deformation of the 3D kappa-Poincaré Hopf algebra. As discussed in section 2, in contrast to the parameter $\hbar$ or equivalently $\kappa$, $\xi$ is a genuine parameter of the Hopf algebra, i.e. it cannot be removed by a redefinition of generators [15].

The parameter $\xi$ can however be removed from the algebra relations of $K_\xi(\mathfrak{iso}(3))$ via the transformation (2.29). Furthermore, it was shown in [26] that there is an analogous transformation mapping the algebra relations of $K_0(\mathfrak{iso}(3))$ to those of the undeformed 3D Poincaré algebra. It therefore follows that the algebra relations of $K_\xi(\mathfrak{iso}(3))$ can as well be mapped
to those of $U(\mathfrak{so}(3))$. It is important to note that this does not give a map between the Hopf algebras $K_\xi(\mathfrak{so}(3))$ and $U(\mathfrak{so}(3))$ as the transformation will generate many additional terms in the coproduct depending on $\xi$ and $\hbar$.

In this paper we are working with algebras over the complex numbers, however discussions of kappa-Poincaré and kappa-Euclidean algebras typically focus on particular real forms. Therefore let us briefly comment on the possible real forms [27] of the deformed 3D kappa-Poincaré algebra. A number of the most common real forms can be extended to include the new parameter $\xi$ upon imposing a suitable reality condition. We have checked examples of mixed and definite signature, both with $q \in \mathbb{R}$ and $|q| = 1$. In all cases we find that either $\xi \in \mathbb{R}$ or $\xi \in i\mathbb{R}$ with the former descending from corresponding real forms of $U_{ch}(\mathfrak{sl}(2)) \otimes U_{ch}(\mathfrak{sl}(2))$, and the latter only appearing after taking the limit.

### 3.2. Classical limit

In section 2 the parameter $\xi$ came from an asymmetry (2.14) in the contraction limit of $U_{ch}(\mathfrak{sl}(2)) \otimes U_{ch}(\mathfrak{sl}(2))$. We can further clarify the role of $\xi$ by considering the classical limit of the Hopf algebra. Let us introduce the standard expressions for the cobracket $\delta$ and classical $r$-matrix $R$

$$\Delta(a) - \Delta^\text{cop}(a) := 2\hbar \delta(a) + \mathcal{O}(\hbar^2), \quad R := 1 \otimes 1 + 2\hbar r + \mathcal{O}(\hbar^2),$$

satisfying the coboundary condition

$$\delta(a \otimes 1 + 1 \otimes a, r) = \delta(a).$$

Introducing the anti-symmetrised and symmetrised tensor products, $a \wedge b = a \otimes b - b \otimes a$ and $a \odot b = a \otimes b + b \otimes a$, we expand the coproduct (3.13)–(3.16) to first order in $\hbar$ to find the following coproducts

$$\delta(P_0) = \delta(L_0) = 0,$$

$$\delta(P_{\pm}) = iP_{\pm} \wedge P_0,$$

$$\delta(L_{\pm}) = iL_{\pm} \wedge P_0 + iP_{\pm} \wedge (L_0 + \xi P_0).$$

Similarly expanding (3.17) the classical $r$-matrix takes the form

$$r = 2(P_+ \wedge L_- - L_+ \wedge P_- + \xi P_+ \wedge P_-) + \left( P_\mu \odot L^\mu + \frac{1}{2}\xi P_\mu \odot P^\mu \right).$$

One can check explicitly that the coboundary condition (3.19) is satisfied and further that the classical $r$-matrix (3.23) solves the classical Yang–Baxter equation

$$[r, r] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

The classical $r$-matrix (3.23) takes a form that resembles the Drinfel’d–Jimbo solution for a simple Lie algebra [3]. The symmetric part,

$$x = P_\mu \odot L^\mu + \frac{1}{2}\xi P_\mu \odot P^\mu,$$

is an $\mathfrak{so}(3)$-invariant element of $\mathfrak{so}(3)^{\otimes 2}$, i.e. a quadratic Casimir. In contrast to the situation for simple Lie algebras, the space of quadratic Casimirs of $\mathfrak{so}(3)$ is two dimensional. This follows from the fact that $\mathfrak{so}(3)$ is a contraction of the direct sum algebra $\mathfrak{so}(4) = \mathfrak{sl}(2) \times \mathfrak{sl}(2)$. 


which by definition has a two-dimensional space of quadratic Casimirs. A basis of quadratic Casimirs of $\mathfrak{iso}(3)$ is given by $P_\mu \otimes L^\mu$ and $P_\mu \otimes P^\mu$, and hence $\hbar$ and $\xi$ parameterise an arbitrary element of this space.

Substituting the Casimir (3.25) into the left-hand side of the classical Yang–Baxter equation (3.24) we find

$$[[\mathfrak{x}, \mathfrak{x}]] = -\omega, \quad \omega = -\frac{1}{2} \epsilon^{\mu\nu\rho} \left( P_\mu \wedge P_\nu \wedge L_\rho + \frac{2}{3} \xi P_\mu \wedge P_\nu \wedge P_\rho \right),$$  \hspace{1cm} (3.26)

where the $O(\xi^2)$ term vanishes as the $P_\mu$ commute amongst themselves. By definition $\omega$ is an $\mathfrak{iso}(3)$-invariant element of $\mathfrak{iso}(3)^{\otimes 3}$. This is also a two-dimensional space [28], with $\epsilon^{\mu\nu\rho} P_\mu \wedge P_\nu \wedge L_\rho$ and $\epsilon^{\mu\nu\rho} P_\mu \wedge P_\nu \wedge P_\rho$ forming a basis, such that $\hbar$ and $\xi$ again parameterise an arbitrary element.

It is now the anti-symmetric part of the classical $r$-matrix (3.23) that generates the cobracket in (3.19), and hence generates the deformation of the algebra. This is given by

$$\hat{r} = 2(P_+ \wedge L_- - P_- \wedge L_+ + \xi P_+ \wedge P_-),$$  \hspace{1cm} (3.27)

which solves the modified classical Yang–Baxter equation

$$[[\hat{r}, \hat{r}]] = \omega,$$  \hspace{1cm} (3.28)

such that $\hat{r} + x$ solves the classical Yang–Baxter equation.

It follows from (3.25)–(3.28) that the term $2\xi P_+ \wedge P_-$ in the classical $r$-matrix (3.23) does not correspond to a Drinfel’d twist of the standard 3D kappa-Poincaré algebra. Indeed this would imply that $r - 2\xi P_+ \wedge P_-$ also satisfies the classical Yang–Baxter equation. One can easily see this is not the case as there will no longer be a term on the right-hand side of (3.28) linear in $\xi$ cancelling the corresponding term in (3.26).

### 3.3. Higher-dimensional kappa-Poincaré algebras

In the analysis of the classical limit we have seen that the 3D Poincaré algebra has certain special algebraic features. In order to understand the importance of these, we now consider to the extent that the considerations above can be extended to the kappa-Poincaré algebra in arbitrary dimension [20]. As the $d$-dimensional Poincaré algebra $\mathfrak{iso}(d)$ can be found as a contraction of $\mathfrak{so}(d+1)$, the $d$-dimensional kappa-Poincaré algebra should be found as an analogous contraction of $U_\hbar(\mathfrak{so}(d+1))$. As we have seen this is indeed the case for $d = 3$. It has also been shown explicitly for $d = 2$ [21] and $d = 4$ [23]. However, in these two cases the limit leads to divergences in the universal R-matrix.

The finite limit of the R-matrix for $d = 3$ has its origin in the non-simplicity of $\mathfrak{so}(4)$. In consequence, the R-matrix of $U_\hbar(\mathfrak{so}(d+1))$ factorises for $d = 3$ into commuting $\mathfrak{sl}(2)$ R-matrices $\mathcal{R}_\mathfrak{so}(4) = \mathcal{R}_\mathfrak{sl}(2) \mathcal{R}_\mathfrak{sl}(2)$. Given an R-matrix $\mathcal{R}$ of a quasi-triangular Hopf algebra, taking the inverse and transpose $(\mathcal{R}^{-1})^{\text{cop}}$ also gives an R-matrix of that Hopf algebra. Therefore, for $U_\hbar(\mathfrak{so}(4)) = U_\hbar(\mathfrak{sl}(2)) \otimes U_\hbar(\mathfrak{sl}(2))$ we have the two R-matrices

$$\mathcal{R}_\mathfrak{so}(4) = \mathcal{R}_\mathfrak{sl}(2) \mathcal{R}_\mathfrak{sl}(2), \quad (\mathcal{R}_\mathfrak{so}(4))^{\text{cop}} = (\mathcal{R}_\mathfrak{sl}(2))^{\text{cop}} (\mathcal{R}_\mathfrak{sl}(2))^{\text{cop}},$$  \hspace{1cm} (3.29)

but in addition, since both $\mathfrak{sl}(2)$ parts are independent, there are also another pair of R-matrices of $U_\hbar(\mathfrak{so}(4))$

$$\mathcal{R}'_\mathfrak{so}(4) = \mathcal{R}_\mathfrak{sl}(2) (\mathcal{R}_\mathfrak{sl}(2))^{\text{cop}}, \quad (\mathcal{R}'_\mathfrak{so}(4))^{\text{cop}} = (\mathcal{R}_\mathfrak{sl}(2))^{\text{cop}} \mathcal{R}_\mathfrak{sl}(2),$$  \hspace{1cm} (3.30)
The latter two have a finite contraction limit, while the former two diverge. Indeed the R-matrix (2.34) with \( \xi = 0 \) is of the latter type. To see this let us consider \( \xi = 0 \), in which case the starting point of section 2 is the Hopf algebra \( U_{\text{sl}(2)}(\mathfrak{sl}(2)) \otimes U_{-\text{ch}}(\mathfrak{sl}(2)) \) with an R-matrix of the type \( R_{\text{sl}(2)}(\epsilon h) R_{\text{sl}(2)}(-\epsilon h) \) where we now indicate the dependence of the R-matrices on the deformation parameter. The Hopf algebras \( U_{-\text{ch}}(\mathfrak{sl}(2)) \) and \( U_{\text{ch}}(\mathfrak{sl}(2)) \) are isomorphic, where the isomorphism, however, maps \( R(-h) \to (R^{-1}(h))^{\text{cop}} \). Thus re-expressing the R-matrix (2.34) on \( U_{\text{ch}}(\mathfrak{sl}(2)) \otimes U_{-\text{ch}}(\mathfrak{sl}(2)) \) we indeed find that it is of the type \( R_{\text{sl}(2)}(\tilde{R}_{\text{sl}(2)})^{-1})^{\text{cop}} \).

In order to clarify these results let us take \( d \neq 3 \) and assume we have a contraction of the Hopf algebra \( U_{\text{h}}(\mathfrak{so}(d + 1)) \) to \( K(\mathfrak{iso}(d)) \). We further assume that there is a classical limit, \( h \to 0 \), in which we have a contraction of \( \mathfrak{so}(d + 1) \) to \( \mathfrak{iso}(d) \), and the cobracket generating the deformation \( U_{\text{h}}(\mathfrak{so}(d + 1)) \) contracts to that generating \( K(\mathfrak{iso}(d)) \). Splitting the generators of \( \mathfrak{so}(d + 1) \) into those of an \( \mathfrak{so}(d) \) subalgebra, \( \tilde{L}_{\mu\nu} = \tilde{L}_{[\mu\nu]} \), \( \mu, \nu = 0, \ldots d - 1 \) and the rest, \( P_\mu \), the contraction to \( \mathfrak{iso}(d) \) is given by rescaling \( P_\mu \to \epsilon^{-1} P_\mu \) and taking \( \epsilon \to 0 \). In the limit \( \tilde{L}_{\mu\nu} \) are then the rotations and \( P_\mu \) the translations of the \( d \)-dimensional Poincaré algebra.

The cobracket of kappa-Poincaré takes the form [20]

\[
\delta(P_\mu) = n^\rho P_\rho \land P_\mu, \quad \delta(\tilde{L}_{\mu\nu}) = -n_\rho \tilde{L}_{\rho\mu} \land P_\nu + n_\rho \tilde{L}_{\rho\nu} \land P_\mu, \tag{3.31}
\]

where \( n^\rho \) is a fixed vector. Recalling that the cobracket comes with a power of \( h \) in the expansion of the coproduct (3.18), the expressions (3.31) imply that, in addition to rescaling \( P_\mu \to \epsilon^{-1} P_\mu \), we should also rescale \( h \to h \) for the contraction to be well-defined.

For \( d \neq 3 \) the algebra \( \mathfrak{so}(d + 1) \) is simple and has a single quadratic Casimir. The leading term of this Casimir in the contraction is the quadratic Casimir of \( \mathfrak{iso}(d) \)

\[
\chi_d = P_\mu \otimes P_\mu. \tag{3.32}
\]

Since the symmetric part of the Drinfeld–Jimbo classical R-matrix for a simple Lie algebra is the quadratic Casimir, its leading term in the contraction limit will contain \( \chi_d \) and hence is quadratic in \( P_\mu \). However, as the classical R-matrix comes with a power of \( h \) in the expansion of the R-matrix (3.18), for a finite limit it should be at most linear in \( P_\mu \). Therefore, the classical R-matrix necessarily diverges in the contraction. Returning to the deformed Hopf algebras it follows that taking the contraction limit in the Drinfeld–Jimbo universal R-matrix for \( U_{\text{h}}(\mathfrak{so}(d + 1)) \) is problematic for \( d \neq 3 \).

Taking a different perspective we may instead start from the solution of the modified classical Yang–Baxter equation (3.28) that generates the kappa-Poincaré deformation [20, 29] of the

\[
\hat{r}_d = n_\rho P_\rho \land \tilde{L}^{\mu\nu}, \qquad [\hat{r}_d, \hat{r}_d] = \omega_d = -\frac{1}{2} h^2 P_\mu \land P_\nu \land \tilde{L}^{\mu\nu}, \tag{3.33}
\]

where \( \omega_d \) is an \( \mathfrak{iso}(d) \)-invariant element of \( \mathfrak{iso}(d)^{\otimes 3} \). That is the cobracket for the kappa-Poincaré algebra (3.31) obeys the coboundary condition (3.19) with this classical R-matrix. However, it has been shown for \( d \neq 3 \) that there exists no symmetric term whose sum with \( \hat{r}_d \) (3.33) solves the classical Yang–Baxter equation [20].

Finally returning to \( d = 3 \) the above analysis further clarifies that the finite limit of the universal R-matrix is tied to the existence of a second quadratic Casimir that is linear in \( P_\mu \), which in turn is a consequence of \( \mathfrak{iso}(3) \) being a contraction of the direct sum algebra \( \mathfrak{so}(4) = \mathfrak{sl}(2) \times \mathfrak{sl}(2) \). Indeed, our choice of initial R-matrix when contracting the Hopf algebra \( U_{\text{ch}}(\mathfrak{sl}(2)) \otimes U_{\text{ch}}(\mathfrak{sl}(2)) \) should be such that the leading term in the limit of the symmetrised classical r-matrix is the Casimir linear in \( P_\mu \).

\[\text{In 3D we have } \tilde{L}^{\mu\nu} = \epsilon^{\mu\nu\rho} L_{\rho} \text{ such that (3.33) matches (3.27) and (3.26) if we take } n_0 = -i, n_1 = n_2 = 0 \text{ and } \xi = 0.\]
3.4. R-matrix and 3D scattering problem

Having studied the algebraic structure of the universal R-matrix for kappa-Poincaré symmetry, we conclude this section by asking what purpose it may serve in a physical context. One idea, based on integrable models in 2D, is that the R-matrix describes a two-particle scattering process. Let us therefore discuss some of its implications.

We set up a state $|p\rangle \otimes |q\rangle$ with a pair of well-defined momenta $(p_\mu, q_\mu)$ to describe the two particles and let the R-matrix $\mathcal{R}$ act on it. We will only be interested in the momenta of the particles after the scattering process. To this end we note that the particle momenta are measured as the eigenvalues of the momentum generators $P_\mu$. We therefore compute how the equivalent set of generators $\{E_C, F_C, H_C\}$ commutes past the R-matrix

$$\mathcal{R}^{-1}(E_C \otimes 1)\mathcal{R} = E_C \otimes q^{-\hbar C}, \quad (3.34)$$
$$\mathcal{R}^{-1}(1 \otimes F_C)\mathcal{R} = q^{\hbar C} \otimes F_C, \quad (3.35)$$
$$\mathcal{R}^{-1}(H_C \otimes 1)\mathcal{R} = H_C \otimes 1 - \hbar^{-1} \log \left( 1 - 4\hbar^2 q^{\hbar C} E_C \otimes q^{-\hbar C} F_C \right), \quad (3.36)$$
$$\mathcal{R}^{-1}(1 \otimes H_C)\mathcal{R} = 1 \otimes H_C + \hbar^{-1} \log \left( 1 - 4\hbar^2 q^{\hbar C} E_C \otimes q^{-\hbar C} F_C \right), \quad (3.37)$$
$$\mathcal{R}^{-1}(F_C \otimes 1)\mathcal{R} = F_C \otimes q^{\hbar C} + 1 \otimes F_C - \frac{q^{2\hbar C} \otimes F_C}{1 - 4\hbar^2 q^{\hbar C} E_C \otimes q^{-\hbar C} F_C}, \quad (3.38)$$
$$\mathcal{R}^{-1}(1 \otimes E_C)\mathcal{R} = q^{-\hbar C} \otimes E_C + E_C \otimes 1 - \frac{E_C \otimes q^{-2\hbar C}}{1 - 4\hbar^2 q^{\hbar C} E_C \otimes q^{-\hbar C} F_C}. \quad (3.39)$$

The operators on the right-hand side measure the momenta of the outgoing particles, which is thus completely fixed in terms of the ingoing momenta on the left-hand side. As a result, the outgoing state, curiously, has well-defined momenta

$$\mathcal{R} |p\rangle \otimes |q\rangle \sim |p'\rangle \otimes |q'\rangle. \quad (3.40)$$

This result is in contrast to the intuition that a scattering process in 3D (or any other number of dimensions above 2) produces a linear combination of states with continuously varying momenta.

For concreteness, let us express the relationship between the ingoing momenta $(p_\mu, q_\mu)$ and outgoing momenta $(p'_\mu, q'_\mu)$ in the common basis $P_\mu$ of kappa-Poincaré symmetry. We find

$$p'_+ = e^{-i\eta_\mu/\kappa} r^{-1/2} p_+, \quad (3.41)$$
$$q'_- = e^{i\eta_\mu/\kappa} r^{-1/2} q_-, \quad (3.42)$$
$$p'_0 = p_0 + i\kappa \log r, \quad (3.43)$$
$$q'_0 = q_0 - i\kappa \log r, \quad (3.44)$$
$$p'_- = e^{i\eta_\mu/\kappa} r^{1/2} p_- + e^{i(q_\mu-p_\mu)/2\kappa} r^{1/2} q_- - e^{i(3p_\mu+q_\mu)/2\kappa} r^{-1/2} q_-, \quad (3.45)$$
$$q'_+ = e^{-i\eta_\mu/\kappa} r^{1/2} q_+ + e^{i(q_\mu-p_\mu)/2\kappa} r^{1/2} p_+ - e^{-i(p_\mu+3q_\mu)/2\kappa} r^{-1/2} p_+, \quad (3.46)$$
with
\[ r := 1 - \frac{4}{\kappa^2} e^{i(p_0-\eta_0)/2\kappa} p_+ q_- . \] (3.47)

In fact, five of these six relations are implied by conservation laws, namely the conservation of overall momentum alias quasi-cocommutativity for \{E_C, F_C, H_C\}
\[ p_0' + q_0' = p_0 + q_0 . \] (3.48)
\[ e^{-i\eta_0/2\kappa} p_+' + e^{i\eta_0/2\kappa} q_+' = e^{i\eta_0/2\kappa} p_+ + e^{-i\eta_0/2\kappa} q_+ , \] (3.49)
\[ e^{-i\eta_0/2\kappa} p_-' + e^{i\eta_0/2\kappa} q_-' = e^{i\eta_0/2\kappa} p_- + e^{-i\eta_0/2\kappa} q_- , \] (3.50)
as well as conservation of the mass shell for each particle due the centrality of the element \( X \) in (2.30)
\[ p_+' p_-' - \kappa^2 \sin^2 (q_0'/2\kappa) = p_+ p_- - \kappa^2 \sin^2 (p_0'/2\kappa) , \] (3.51)
\[ q_+' q_-' - \kappa^2 \sin^2 (q_0'/2\kappa) = q_+ q_- - \kappa^2 \sin^2 (q_0'/2\kappa) . \] (3.52)

These five relationships constrain the six outgoing momenta to a one-parameter family. Nevertheless, there is a sixth relationship, which can be expressed in a somewhat symmetric form as\(^{10}\)
\[ e^{i(q_0'-p_0')/2\kappa} p_+ q_- = e^{i(p_0-\eta_0)/2\kappa} p_+ q_- . \] (3.53)

It corresponds to the combination of (3.34) and (3.35)
\[ R^{-1}(E_C \otimes F_C) R = q^{H_C} E_C \otimes q^{-H_C} F_C ; \] (3.54)
in other words it follows from explicit commutation with the R-matrix. The physical origin of this final relationship, for example, how it follows from a hypothetical sixth conserved quantity, and the deeper meaning of the above transformation (3.41)–(3.46) of momenta and whether it can serve within a reasonable particle scattering process, remain to be understood.

Note that a superficially similar scattering process has been discussed in the context of the AdS/CFT correspondence, see appendix B of [30]. More concretely, this is a 2D scattering process with the 2D momenta embedded into a 3D momentum vector whose coproduct is equivalent to the one of \{E_C, F_C, H_C\}. Consequently, all of the above five conservation laws (3.48)–(3.52) are respected by this scattering problem, but the remaining sixth relationship (3.53) is manifestly different. In our basis, it can be expressed as \( p_0' - q_0' = p_0 - q_0 \) implying that the energies of the individual particles are preserved across the scattering. This implies a different transformation for the momenta (3.41)–(3.46).

4. **Maximally extended \( sl(2|2) \) from \( \mathfrak{d}(2,1;\epsilon) \times sl(2) \)**

We now turn to our primary interest, recovering the quasi-triangular Hopf algebra of [15] as a contraction limit of \( \mathcal{U}_h(\mathfrak{d}(2,1;\epsilon)) \otimes \mathcal{U}_h(sl(2)) \). The maximally extended \( sl(2|2) \) Hopf algebra of [15] was constructed as the smallest quasi-triangular Hopf algebra containing the centrally extended \( \mathcal{U}_h(sl(2|2) \otimes \mathbb{C}^2) = \mathcal{U}_h(psU(2) \otimes \mathbb{C}^3) \) as a Hopf subalgebra. The structure of this algebra has the form \( \mathcal{U}_h(sl(2) \otimes psU(2) \otimes \mathbb{C}^3) \), where the \( sl(2) \) factor plays the role of a

\(^{10}\) In fact, this relationship leaves a few other discrete choices but we chose it due to its symmetric form. Alternatively, any one of the relationships (3.41)–(3.46) could be used instead.
continuous outer automorphism for the remainder of the algebra. As discussed at the beginning of section 2 this Hopf algebra possesses a number of unusual features, including the appearance of plain \( h \) factors which are not within exponents \( q = e^{h} \), the existence of an additional free parameter \( \xi \), as well as the non-factorisable form of the universal R-matrix, which involves logarithms and dilogarithms. In section 2 we saw that for the Hopf subalgebra \( U_{h, \xi}(\mathfrak{sl}(2) \times C^{3}) \) these features can be understood by considering a certain contraction of \( U_{h, \xi}(\mathfrak{sl}(2)) \otimes U_{h}(\mathfrak{sl}(2)) \). Therefore, in this section our aim is to recover the full maximally extended \( \mathfrak{sl}(2; 2) \) Hopf algebra in a similar limit. Our starting point for this will be to promote one \( U_{\epsilon}(\mathfrak{sl}(2)) \) factor in the construction of section 2 to \( U_{\epsilon}(\mathfrak{o}(2, 1; \epsilon)) \) while keeping the other factor as \( U_{\epsilon h}(\mathfrak{sl}(2)) \).

### 4.1. Lie superalgebra \( \mathfrak{o}(2, 1; \epsilon) \)

Let us begin by introducing the exceptional Lie superalgebra \( \mathfrak{o}(2, 1; \epsilon) \). This superalgebra depends on the continuous parameter \( \epsilon \) and hence forms a one-parameter family of Lie superalgebras. The even subalgebra consists of three mutually commuting \( \mathfrak{sl}(2) \) algebras, i.e. \( \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2) \). The odd part is spanned by 8 odd generators transforming in the triple Cartan of the even subalgebra. The anti-commutators of two odd generators \( Q \) take the schematic form

\[
\{Q, Q\} \sim s_{1}T_{1} + s_{2}T_{2} + s_{3}T_{3},
\]

where the \( T_{i} \) denote normalised generators of the three \( \mathfrak{sl}(2) \) algebras. The parameters \( s_{i} \) are constrained by the Jacobi identity to satisfy \( s_{1} + s_{2} + s_{3} = 0 \).

The superalgebra with parameters \( s_{i} \) and the one with parameters \( \lambda s_{i} \) \((0 \neq \lambda \in C)\) are isomorphic. The isomorphism is simply given by scaling the odd generators by \( \sqrt{\lambda} \in C \). We can therefore always normalise one parameter and write the parameters \( s_{i} \) in terms of a single parameter \( \epsilon \). There are multiple ways to do this and our choice is

\[
s_{1} = 1, \quad s_{2} = \epsilon, \quad s_{3} = -1 - \epsilon.
\]

In general we will work with the parameters \( s_{i} \) as they preserve the symmetry among the three \( \mathfrak{sl}(2) \) algebras. We will however need to introduce the parameter \( \epsilon \) in order to take the contraction limit in section 4.3.

Considering the superalgebra \( \mathfrak{o}(2, 1; \epsilon) \) by itself there are two ways of taking the \( \epsilon \to 0 \) limit that will be important in our construction. Using the parameterisation (4.2) the first is to directly take \( \epsilon \to 0 \) leading to \( \mathfrak{sl}(2) \times \mathfrak{psl}(2|2) \), that is \( \mathfrak{psl}(2|2) \) together with its \( \mathfrak{sl}(2) \) outer automorphism. The second involves first rescaling \( T(2) \to \epsilon^{-1}T(2) \) and then taking \( \epsilon \to 0 \). This leads to \( \mathfrak{psl}(2|2) \times C^{3} \), that is the triple central extension of \( \mathfrak{psl}(2|2) \).

In order to combine these two limits we will introduce an additional \( \mathfrak{sl}(2) \) algebra in the spirit of section 2 to obtain in the limit \( \mathfrak{sl}(2) \times \mathfrak{psl}(2|2) \times C^{3} \), the maximally extended \( \mathfrak{sl}(2|2) \) algebra. Lifting this limit to the \( q \)-deformed algebras our aim is then to find the Hopf algebra \( U_{h, \xi}(\mathfrak{sl}(2) \times \mathfrak{psl}(2|2) \times C^{3}) \) of [15] as a contraction of \( U_{h}(\mathfrak{o}(2, 1; \epsilon)) \otimes U_{h}(\mathfrak{sl}(2)) \).

### 4.2. q-deformation of \( \mathfrak{o}(2, 1; \epsilon) \)

Let us now define the \( q \)-deformed Hopf algebra \( U_{h}(\mathfrak{o}(2, 1; \epsilon)) \) and its universal R-matrix [31]. The three even \( \mathfrak{sl}(2) \) subalgebras are deformed with \( q^{x} = e^{h x} \).\footnote{A rescaling of the parameters \( s_{i} \) requires an inverse rescaling of \( h \) in addition to the appropriate rescaling to the odd generators for the corresponding Hopf algebras to be isomorphic. In this sense the \( h \) in \( U_{h}(\mathfrak{o}(2, 1; \epsilon)) \) corresponds to the choice (4.2).} The even subalgebra is therefore given by
\[ U_{h_n}(\mathfrak{sl}(2)) \otimes U_{h_0}(\mathfrak{sl}(2)) \otimes U_{h_3}(\mathfrak{sl}(2)). \]  

Note that their coproduct can however contain in addition to the standard coproduct a tail involving the odd generators. Note further that the deformation of the \( \mathfrak{sl}(2) \) corresponding to \( s_2 \) will vanish in the limit \( \epsilon \to 0 \) while the deformation for the other two \( \mathfrak{sl}(2) \) subalgebras will remain. The former \( U_{h_3}(\mathfrak{sl}(2)) \) subalgebra will thus replace the first \( U_{h_n}(\mathfrak{sl}(2)) \) Hopf algebra of section 2. The latter two subalgebras \( U_{h_3}(\mathfrak{sl}(2)) \) and \( U_{h_3}(\mathfrak{sl}(2)) \) will become the two \( \mathfrak{sl}(2) \) subalgebras of \( \mathfrak{psl}(2|2) \) in the limit \( \epsilon \to 0 \).

**Algebra and coalgebra.** We define the q-deformed Hopf algebra \( \mathcal{U}_q(\mathfrak{d}(2,1; \epsilon)) \) in terms of three sets of simple generators \( E_i, F_i \) and their corresponding Cartan generators \( H_i, i = 1, 2, 3 \). The generators \( E_2 \) and \( F_2 \) are odd while \( E_{1,3}, F_{1,3} \) are even. We will make use of the graded q-commutator

\[
[a,b]_\alpha := ab - (-1)^{|a||b|} e^{\alpha} ba,
\]

where the degree is \( |a| = 0 \) for even generators and \( |a| = 1 \) for odd generators. For undeformed commutators we simply write \( [a,b] := [a,b]_0 \). The commutation relations of the simple generators are given by

\[
[H_i, E_j] = a_{ij}E_j, \quad [H_i, F_j] = -a_{ij}F_j, \quad [E_i, F_j] = \delta_{ij} q^d_i - \frac{q^d_i - q^{-d_i}}{q_i - q^{-1_i}},
\]

where the Cartan matrix \( a_{ij} \) and the q-exponents \( d_i \) are given by

\[
a_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ s_1 & 0 & s_3 \\ 0 & -1 & 2 \end{pmatrix}_{ij}, \quad d_i = (s_1 \quad -1 \quad s_3).
\]

The latter are chosen such that they symmetrise the Cartan matrix \( d_{ij} = d_{ji} \). Furthermore, they give rise to the deformation strength of the respective simple generators

\[
q_i := e^{d_i \epsilon}.
\]

To define the non-simple generators and Serre relations we introduce the left and right adjoint action

\[
a \triangleright b := (-1)^{|a||b|} a_{(1)} b S(a_{(2)}),
\]

\[
b \triangleright a := (-1)^{|a||b|} S(a_{(1)}) ba_{(2)},
\]

where we made use of Sweedler’s notation for the coproduct \( \Delta(a) = a_{(1)} \otimes a_{(2)} \), with an implicit sum over all terms. \( S \) denotes the antipode.

We define the six odd non-simple generators

\[
E_{12} := E_1 \triangleright E_2 = [E_1, E_2]_{h_1}, \quad F_{21} := F_2 \triangleright F_1 = [F_2, F_1]_{-h_1},
\]

\[
E_{32} := E_3 \triangleright E_2 = [E_3, E_2]_{h_2}, \quad F_{23} := F_2 \triangleright F_3 = [F_2, F_3]_{-h_2},
\]

\[
E_{132} := (E_1E_3) \triangleright E_2 = [E_1, E_{32}]_{h_3}, \quad F_{213} := F_2 \triangleright (F_1 F_3) = [F_2, F_1]_{-h_3}.
\]
The Serre relations are then given by
\begin{align}
E_1^2 &= 0, & F_1^2 &= 0, \\
E_1 E_3 &= [E_1, E_3] = 0, & F_1 F_3 &= [F_1, F_3] = 0, \\
E_1 \triangleright (E_1 \triangleright E_2) &= [E_1, E_{12}] - \hbar s_1 = 0, & (F_1 \triangleright F_1) \triangleleft F_1 &= [F_{21}, F_1] \hbar s_1 = 0, \\
E_3 \triangleright (E_3 \triangleright E_2) &= [E_3, E_{32}] - \hbar s_3 = 0, & (F_2 \triangleright F_3) \triangleleft F_3 &= [F_{23}, F_3] \hbar s_3 = 0.
\end{align}

Note that by the q-Jacobi identity the non-simple generators \(E_{132}\) and \(F_{213}\) satisfy the identities \([E_3, E_{12}] \hbar s_3 = [E_1, E_{32}] \hbar s_3\) and \([F_{21}, F_3] \hbar s_3 = [F_{23}, F_1] \hbar s_3\) respectively.

The q-deformed coalgebra is defined on the simple generators via the coproduct
\begin{align}
\Delta E_i &= E_i \otimes 1 + q_i^{-\hbar} \otimes E_i, \\
\Delta F_i &= F_i \otimes q_i^\hbar + 1 \otimes F_i, \\
\Delta H_i &= H_i \otimes 1 + 1 \otimes H_i,
\end{align}
where the tensor product is graded in the usual way
\[(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd.\]

**Third \(sl(2)\).** From the expressions above we see that \(\{E_1, F_1, H_1\}\) and \(\{E_3, F_3, H_3\}\) generate the Hopf subalgebras \(U_{\hbar s_1}(sl(2))\) and \(U_{\hbar s_3}(sl(2))\) respectively, deforming two of the \(sl(2)\) subalgebras of \(\hat{\mathfrak{so}}(2, 1; \epsilon)\). The final \(sl(2)\) subalgebra is generated by the following combinations \(\{E_B, F_B, H_B\}\) of the two even non-simple generators and the Cartan generators
\begin{align}
E_B &:= \frac{q - q^{-1}}{q_B^{-1} - q_B} [E_{32}, E_{12}] - \hbar s_1, \\
F_B &:= \frac{q - q^{-1}}{q_B - q_B^{-1}} [F_{21}, F_{23}] \hbar s_2, \\
s_2 H_B &:= s_1 H_1 - 2 H_2 + s_3 H_3,
\end{align}
where we introduced
\[q_B := e^{\epsilon H}.\]

Their commutation relations are
\begin{align}
[H_B, E_B] &= 2E_B, & [H_B, F_B] &= -2F_B, & [E_B, F_B] &= \frac{q_B H_B - q_B^{-1} H_B}{q_B - q_B^{-1}},
\end{align}
which are those of the Hopf algebra \(U_{\hbar s_2}(sl(2))\). However, as \(E_B\) and \(F_B\) are non-simple generators their coproduct has a more complicated form. Indeed, as a consequence of the requirement of compatibility between the coalgebra and algebra, the coproduct for \(\{E_B, F_B, H_B\}\) is
\[ \Delta E_B = E_B \otimes 1 + q_B^{H_B} \otimes E_B \]
\[ - (q - q^{-1})q_B^{-1}E_3q_1^{-1}H_2q_2^{-H_2} \otimes E_{12} \]
\[ + (q - q^{-1})q_B^{-1}(q_1E_{132} + (q_1^2 - 1)E_{32}E_1)q_2^{H_2} \otimes E_2 \]
\[ + (q - q^{-1})q_B^{-1}(q_3^{-2} - 1)E_3q_1^{-H_2}q_2^{-2H_2} \otimes E_2E_{12}, \] (4.26)

\[ \Delta F_B = F_B \otimes q_B^{H_B} + 1 \otimes F_B \]
\[ - (q - q^{-1})q_B F_{21} \otimes q_1^{H_2}F_{23} \]
\[ + (q - q^{-1})q_B F_2 \otimes q_2^{H_2}(q_1^{-1}F_{213} + (q_1^{-2} - 1)F_1F_{23}) \]
\[ + (q - q^{-1})q_B(q_3^{-2} - 1)F_{21}F_2 \otimes q_1^{H_2}q_2^{2H_2}F_3, \] (4.27)

\[ \Delta H_B = H_B \otimes 1 + 1 \otimes H_B, \] (4.28)

where we see that \( \Delta E_B \) and \( \Delta F_B \) pick up a tail involving the odd generators.

It is this deformed \( U_{sl_2}(sl(2)) \) that will replace one \( sl(2) \) of section 2 for the purpose of taking the contraction limit. Therefore, we give the commutation relations of \( \{E_B, F_B, H_B\} \) with the simple roots for convenience

\[ [H_B, E_1] = \delta_{2}E_i, \]
\[ [H_B, F_1] = -\delta_{2}F_i, \] (4.29)

\[ [E_B, E_1] = 0, \]
\[ [E_B, F_1] = 0, \] (4.30)

\[ [E_B, E_2] = (q^{-1} - 1)E_2E_B, \]
\[ [E_B, F_2] = q_1q_B^{-1}(E_{132} + (q_1^{-1}E_{32}E_1)q_2^{H_2}, \] (4.31)

\[ [E_B, E_3] = q_1(q - q^{-1})E_3E_{132}, \]
\[ [E_B, F_3] = q_1(q - q^{-1})E_3E_{12}q_3^{H_3}, \] (4.32)

\[ [F_B, E_1] = 0, \]
\[ [F_B, E_1] = 0, \] (4.33)

\[ [F_B, E_2] = (1 - q_B)F_BF_2, \]
\[ [F_B, E_2] = q_1^{-1}q_B^{-H_2}(F_{213} + (q_1^{-1}F_1F_{23}), \] (4.34)

\[ [F_B, E_3] = -q_1^{-1}(q - q^{-1})F_{213}F_2, \]
\[ [F_B, E_3] = -q_1^{-1}(q - q^{-1})q_3^{-H_2}F_2F_3. \] (4.35)

Notice that these commutation relations as well as the coproduct of \( E_A \) and \( F_A \) do not exhibit a symmetry between the indices 1 and 3. This is an artifact of the choice made in defining the even non-simple generators (4.21)–(4.23). A definition in terms of a symmetric commutator \( [E_{12}, E_{32}] \), however, leads to an inconvenient basis for the purpose of presenting the R-matrix.

**R-matrix.** The R-matrix of \( U_B(sl(2; 1; \epsilon)) \) was explicitly calculated in [31]. The expression for the universal R-matrix depends on a choice of PBW basis for the positive and negative Borel subalgebras. Adapted to our choice of basis for the positive Borel subalgebra \( \{E_2^{ni}E_1^{m}, E_3^{n}, E_1^{m}, E_3^{-m}, H_1^{-m}, H_2^{-m}, H_3^{-m}; n_i, m_j \in \mathbb{N}_0 \} \) the R-matrix takes the form
\[ R = \exp \left[ - (q_2 - q_2^{-1})E_2 \otimes F_2 \right] \cdot \exp \left[ - (q_2 - q_2^{-1})E_{12} \otimes F_{21} \right] \]
\[ \cdot \exp_{-2h \lambda} \left[ (q_{B} - q_{B}^{-1})E_{B} \otimes F_{B} \right] \]
\[ \cdot \exp \left[ - (q_2 - q_2^{-1})E_{32} \otimes F_{32} \right] \cdot \exp \left[ - (q_2 - q_2^{-1})E_{132} \otimes F_{132} \right] \]
\[ \cdot \exp_{-2h \lambda} \left[ (q_3 - q_3^{-1})E_{3} \otimes F_{3} \right] \]
\[ \cdot \exp \left[ \frac{1}{2} \hbar (s_1H_1 \otimes H_1 + s_2H_B \otimes H_B + s_3H_3 \otimes H_3) \right]. \] (4.36)

### 4.3. Contraction limit

We will now apply the contraction limit of section 2 with the role of the generators \( \{E,F,H\}\) of \( U_{sl}(sl(2))\) played by the generators \( \{E_B,F_B,H_B\}\) of \( U_{h}(sl(2,1;\epsilon))\). Indeed, upon using the parameterisation (4.2) we find that the commutation relations of the generators \( \{E_B,F_B,H_B\}\) are exactly those of \( U_{sl}(sl(2))\), while the coproduct for \( E_B\) and \( F_B\) now possesses a tail. Our starting point is therefore the Hopf algebra \( U_{h}(sl(2,1;\epsilon)) \otimes U_{sl}(sl(2))\), where we denote the generators of \( U_{sl}(sl(2))\) by \( \{E,F,H\}\) and where \( \tilde{\epsilon}\) and \( \epsilon\) are related in the same way as in section 2

\[ \tilde{\epsilon}(\epsilon) = - \epsilon + \xi \epsilon^2 + \mathcal{O}(\epsilon^3). \] (4.37)

In particular the minus sign in the linear term is again required to have a well-defined, divergence-free limit. Furthermore, the generators that we keep finite in the limit are directly analogous to those of section 2

\[ E_A := E_B + \tilde{E}, \quad E_C := \epsilon E_B, \] (4.38)
\[ F_A := F_B + \tilde{F}, \quad F_C := \epsilon F_B, \] (4.39)
\[ H_A := H_B + \tilde{H}, \quad H_C := \epsilon H_B. \] (4.40)

The simple generators \( \{E_i,F_i,H_i\}\), and hence also the six odd non-simple generators (4.10)–(4.12), all remain finite in the contraction limit \( \epsilon \to 0\) (\( s_1 \to 1, s_3 \to -1\)). This is consistent with the scaling of \( H_C\), \( E_C\) and \( F_C\) with \( \epsilon\) in (4.38)–(4.40). The former then generate the \( sl(2)\) part of the maximally extended \( sl(2)\) Hopf algebra. In particular the Cartan matrix and \( q\)-exponents all have a finite and non-degenerate \( \epsilon \to 0\) limit, and become those of \( sl(2)\)

\[ a_{ij} = \begin{pmatrix} +2 & -1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & +2 \end{pmatrix}_{ij}, \quad d_i = (1 \ -1 \ -1)_i. \] (4.41)

Furthermore, the Serre relations (4.13)–(4.16) reduce to the standard ones of \( U_h(sl(2))\). On the other hand, as expected, the non-standard Serre elements, \([E_{32},E_{12}]\) and \([F_{21},F_{23}]\), do not vanish in the contraction limit. Instead they become the generators \( E_C\) and \( F_C\) of the extended algebra \( U_h(ps(2) \ltimes \mathbb{C}^3)\). Indeed, in the contracted algebra, the generators \( \{E_C,F_C,H_C\}\) are related to the \( sl(2)\) generators as

\[ E_C = - q - q^{-1} \frac{2 \hbar}{E_{32}}, \] (4.42)
\[ F_C = \frac{q - q^{-1}}{2\hbar} [F_{21}, F_{23}], \quad (4.43) \]

\[ H_C = H_1 - 2H_2 - H_3. \quad (4.44) \]

It now remains to confirm that the commutation relations involving the generators \((4.38)-(4.40)\) have a finite \(\epsilon \to 0\) limit. After taking the contraction limit the commutation relations of the generators \((4.38)-(4.40)\) with themselves are the same as in section 2 and are given in \((2.16)-(2.20)\). The commutation relations of the generators \((4.38)-(4.40)\) with the simple generators are such that \([E_C, F_C, H_C]\) commute with them

\[ [H_C, E_i] = 0, \quad [H_C, F_i] = 0, \quad (4.45) \]

\[ [E_C, H_{1,3}] = [E_C, E_i] = [E_C, F_i] = 0, \quad [F_C, H_{1,3}] = [F_C, E_i] = [F_C, F_i] = 0, \quad (4.46) \]

while the generators \([E_A, F_A, H_A]\) have the following commutators

\[ [H_A E_i] = \delta_2 E_i, \quad [H_A, F_i] = -\delta_2 F_i, \quad (4.47) \]

\[ [E_A, E_i] = [E_A, F_i] = [E_A, H_{1,3}] = 0, \quad [F_A, E_i] = [F_A, F_i] = [F_A, H_{1,3}] = 0, \quad (4.48) \]

\[ [E_A, E_2] = -\hbar E_2 E_C, \quad [E_A, F_2] = q \left( E_{132} + (q - q^{-1}) E_{32} E_1 \right) q_2^{-H_2}, \quad (4.49) \]

\[ [E_A, E_3] = (q - q^{-1}) q E_3 E_{132}, \quad [E_A, F_3] = (q - q^{-1}) q E_3 E_{13} q_2^{H_3}, \quad (4.50) \]

\[ [F_A, F_2] = -\hbar F_C F_2, \quad [F_A, F_3] = q^{-1} q_2^{H_3} (F_{213} - (q - q^{-1}) F_{1} F_{23}), \quad (4.51) \]

\[ [F_A, E_3] = -(q - q^{-1}) q^{-1} F_{213} F_{23}, \quad [F_A, E_3] = -(q - q^{-1}) q^{-1} q_3^{-H_3} F_{21} F_{2}, \quad (4.52) \]

By repeated application of these relations one can then easily find the commutation relations of the generators \((4.38)-(4.40)\) with the non-simple odd generators \((4.10)-(4.12)\).

The contraction limit of the coproduct for \(E_A\) and \(F_A\) is

\[ \Delta E_A = E_A \otimes 1 + q^{-H_C} \otimes E_A - \hbar \left( H_A + \xi H_C \right) q^{-H_C} \otimes E_C \]

\[ - (q - q^{-1}) E_{32} q_1^{-H_1} q_2^{-H_2} \otimes E_{12} \]

\[ + (q - q^{-1}) q \left( E_{132} + (q - q^{-1}) E_{32} E_1 \right) q_2^{-H_2} \otimes E_2 \]

\[ - (q - q^{-1})^2 q^{-1} q_1^{-H_1} q_2^{-H_2} \otimes E_{32} E_{12}, \quad (4.53) \]

\[ \Delta F_A = F_A \otimes q^{H_C} + 1 \otimes F_A + \hbar F_C \otimes q^{H_C} (H_A + \xi H_C) \]

\[ - (q - q^{-1}) F_{21} \otimes q_1^{-H_1} q_2^{H_2} F_{23} \]

\[ + (q - q^{-1}) q^{-1} F_2 \otimes q_2^{H_2} (F_{213} - (q - q^{-1}) F_{1} F_{23}) \]

\[ + (q - q^{-1})^2 q F_{21} F_2 \otimes q_1^{-H_1} q_2^{H_2} F_3, \quad (4.54) \]

while for \(E_C\) and \(F_C\) it is given in \((2.24)-(2.26)\) and is trivial for \(H_A\) and \(H_C\). The coproduct for the remaining generators of \(U_q(\mathfrak{so}(2, 1; \xi))\) does not depend on \(E_B\) or \(F_B\) and hence remains unchanged in the limit up to setting \((s_1, s_2, s_3) = (1, 0, -1)\). Finally it is worth mentioning that the q-deformation of the \(psl(2|2)\)-part is still in place \((q \not\approx 1)\) after the limit, while the
q-deformation of the Poincaré part, as already seen in section 2, is mostly gone \((q \approx 1)\) or reduced to \(h\) for the generators \(\{E_A, H_A, F_A\}\).

**R-matrix.** The R-matrix of \(U_h(\mathfrak{d}(2, 1; \epsilon)) \otimes U_{\ell h}(\mathfrak{sl}(2))\) is given by the product of the individual R-matrices \((4.36)\) and \((2.32)\). The terms involving \(\{E_C, F_C, H_C\}\) and \(\{E_A, F_A, H_A\}\) were already calculated in section 2. The \(\epsilon \to 0\) limit of the remaining terms is straightforward and the complete R-matrix is given by

\[
\mathcal{R} = \exp\left[\left(q - q^{-1}\right)E_2 \otimes F_2\right] \cdot \exp\left[\left(q - q^{-1}\right)E_{12} \otimes F_{21}\right] \\
\cdot \exp\left[-\frac{\xi}{2h} \text{Li}_2\left(4h^2 E_C \otimes F_C\right) - \frac{\xi}{h} \log\left(1 - 4h^2 E_C \otimes F_C\right)\right] \\
\cdot \exp\left[-\frac{1}{2h} \left(E_C \otimes F_A + E_A \otimes F_C\right) \log\left(1 - 4h^2 E_C \otimes F_C\right)\right] \\
\cdot \exp\left[\left(q - q^{-1}\right)E_{32} \otimes F_{32}\right] \cdot \exp\left[\left(q - q^{-1}\right)E_{132} \otimes F_{132}\right] \\
\cdot \exp_{-2\kappa}\left[\left(q - q^{-1}\right)E_1 \otimes F_1\right] \cdot \exp_{2\kappa}\left[\left(q^{-1} - q\right)E_3 \otimes F_3\right] \\
\cdot \exp\left[\frac{1}{2} h \left(H_1 \otimes H_1 - H_A \otimes H_A + H_C \otimes H_A + H_A \otimes H_C + \xi H_C \otimes H_C\right)\right]. \tag{4.55}\]

**Identification with maximally extended \(\mathfrak{sl}(2|2)\).** Comparing the Hopf algebra and R-matrix found in the contraction limit with the results in [15] we see that we recover all relations upon identifying the generators used in [15] as follows

\[
L = E_A + \xi E_C, \quad M = -F_A, \quad H_A = H_N, \tag{4.56}
\]

\[
P = -\frac{2h}{q - q^{-1}} E_C, \quad K = \frac{2h}{q - q^{-1}} F_C, \quad C = \frac{1}{2} H_C, \tag{4.57}
\]

as well as the parameter of [15] as

\[
\kappa = 2\xi. \tag{4.58}
\]

In particular, the R-matrix is in perfect agreement with the appropriate terms in [15] upon using this identification.

Therefore, as claimed, we have recovered the maximally extended \(\mathfrak{sl}(2|2)\) Hopf algebra \(U_{h,\xi}(\mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3)\) as a contraction limit of \(U_h(\mathfrak{d}(2, 1; \epsilon)) \otimes U_{\ell h}(\mathfrak{sl}(2))\).

**Two copies of \(\mathfrak{d}(2, 1; \epsilon)\).** We have seen that we can extend the contraction limit of section 2 to a contraction limit of \(U_h(\mathfrak{d}(2, 1; \epsilon)) \otimes U_{\ell h}(\mathfrak{sl}(2))\) by replacing the \(U_{h,\xi}(\mathfrak{sl}(2))\) of the former contraction by an \(\mathfrak{d}(2)\) subalgebra inside \(\mathfrak{d}(2, 1; \epsilon)\). One may ask if we can also promote the \(U_{\ell h}(\mathfrak{sl}(2))\) factor to \(U_{\ell h}(\mathfrak{d}(2, 1; \epsilon))\). Indeed this is possible without any additional complication. The contraction limit of \(U_h(\mathfrak{d}(2, 1; \epsilon)) \otimes U_h(\mathfrak{d}(2, 1; \epsilon))\) leads to \(U_{h,\xi,\epsilon}(\mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3)\) where we now have two copies of \(\mathfrak{psl}(2|2)\) sharing the generators \(\{E_A, F_A, H_A\}\) and \(\{E_C, F_C, H_C\}\). Note that the resulting algebra has two q-deformed \(\mathfrak{psl}(2|2)\)-parts with independent q-deformation parameters \(q = e^h\) and \(\bar{q} = e^{\bar{h}}\). Together with \(\xi\), this algebra therefore carries 3 deformation parameters.
5. Conclusion

In this paper we have investigated contractions of quasi-triangular Hopf algebras. Our focus was the following three examples

\[ U_{d\hbar}(\mathfrak{sl}(2)) \otimes U_{d\hbar}(\mathfrak{sl}(2)) \rightarrow K_\xi(\mathfrak{iso}(3)), \]  
\[ U_{d\hbar}(\mathfrak{o}(2, 1; \iota)) \otimes U_{d\hbar}(\mathfrak{sl}(2)) \rightarrow U_{d\hbar, \xi}(\mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3), \]  
\[ U_{d\hbar}(\mathfrak{o}(2, 1; \iota)) \otimes U_{d\hbar}(\mathfrak{o}(2, 1; \iota)) \rightarrow U_{d\hbar, \tilde{\iota}}(\mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2)^{\times 2} \ltimes \mathbb{C}^3). \]

In each case the initial algebra is semi-simple and the resulting algebra is non-simple. Exploring the corresponding freedom in the R-matrix it transpired that certain choices had a finite contraction limit. This allowed us to construct universal R-matrices for the contracted Hopf algebras.

The contraction (5.1), discussed in sections 2 and 3, led to a new one-parameter deformation of the 3D kappa-Poincaré algebra. This latter algebra is well-known, underlying the physics on a certain non-commutative version of Minkowski space, and hence it would be interesting to interpret the new parameter $\xi$. We have also obtained explicitly the universal R-matrix for this Hopf algebra.

It would be useful to further explore connections between our results and those in the literature. One example of this would be the infinite boost limit. In the classical analogue of $K_0(\mathfrak{iso}(3))$ we can consider a limit in which the distinguished generator becomes null, $P_0 \rightarrow P_0 + P_1$ and $L_0 \rightarrow L_0 + L_1$ such that $ad_{P_0}^2 + P_1 = ad_{L_0}^2 + L_1 = 0$. In this case the Casimir term in the classical r-matrix is subleading and hence the leading anti-symmetric part solves the classical Yang–Baxter equation in its own right. The resulting classical r-matrix is then of jordanian type [32] and the universal R-matrix is expected to reduce to a twist, the explicit form of which has been constructed [33]. It would be interesting to see if this expression can be recovered on taking the corresponding limit of (2.39).

The contractions (5.2) and (5.3), discussed in section 4, are particularly important as they lead to the maximally extended $\mathfrak{sl}(2|2)$ Hopf algebra and the R-matrix of [15] in a more systematic manner. Indeed lifting these contractions to deformed affine algebras may provide a route to constructing the universal R-matrix for the maximally extended affine $\mathfrak{sl}(2|2)$ Hopf algebra.

To this end it appears convenient to consider the effect of choosing different $\mathfrak{sl}(2)$ subalgebras of $\mathfrak{o}(2, 1; \iota)$ for the contraction of section 2. For example, rather than picking the most non-simple root $E_8$, one may choose one of the simple roots $E_1$ or $E_3$. One may also consider the alternative Dynkin diagram with all fermionic nodes. The resulting Hopf algebras, while appearing different, should be related. For this the isomorphism permuting the three $\mathfrak{sl}(2)$ algebras of $\mathfrak{o}(2, 1; \iota)$, and its extension to the quantum algebra, will be relevant.

As we have seen, the maximally extended $\mathfrak{sl}(2|2)$ superalgebra is a supersymmetric extension of 3D kappa-Poincaré. However, the classical limit of the former is not an ordinary super-Poincaré algebra because the $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$ subalgebra of $\mathfrak{sl}(2|2)$ does not have the canonical form of an R-symmetry. To find such an algebra one can take a further limit

\[ Q \rightarrow \gamma^{-1} Q, \quad (E_C, F_C, H_C) \rightarrow \gamma^{-2} (E_C, F_C, H_C), \quad \gamma \rightarrow 0, \]

where $Q$ represents all odd generators. In this case the $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$ algebra becomes a derivation and thus indeed has the form of an R-symmetry. One might also consider a contraction of $\mathfrak{o}(2, 1; \iota)$ with two of its own $\mathfrak{sl}(2)$ algebras forming the 3D Poincaré algebra in the limit. In this case the R-symmetry would just consist of a single $\mathfrak{sl}(2)$ algebra. It would be interesting
to see whether these limits can be implemented in $U_h(\mathfrak{o}(2, 1; \epsilon))$ and its extensions, and hence if one can find universal R-matrices for 3D super-Poincaré algebras.

Finally, it is well-known that one can consider contraction limits in two-dimensional sigma models with suitable global isometries, for example, the flat space limit of anti-de-Sitter space. Recently such contraction limits were extended to sigma models with q-deformed symmetries [34]. In light of our results it may now be worthwhile exploring these limits in more detail for cases in which the isometry algebra is not simple [35].

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References

[1] Faddeev L D 1963 The inverse problem in the quantum theory of scattering J. Math. Phys. 4 72 (Eng. transl.)
Faddeev L D 1959 The inverse problem in the quantum theory of scattering Usp. Mat. Nauk. 14 57
Faddeev L D, Sklyanin E K and Takhtajan L A 1980 The quantum inverse problem method 1 Theor. Math. Phys. 40 688
[2] Faddeev L D 1995 How algebraic Bethe ansatz works for integrable model Relativistic Gravitation and Gravitational Radiation. Proc., School of Physics (Les Houches, France, 26 September–6 October) pp 149–219
[3] Drinfel’d V G 1985 Hopf algebras and the quantum Yang–Baxter equation Sov. Math. Dokl. 32 254
Drinfel’d V G 1988 Quantum groups J. Math. Sci. 41 898
Jimbo M 1985 A q difference analog of $U(g)$ and the Yang–Baxter equation Lett. Math. Phys. 10 63
Jimbo M 1986 A q-analog of $U(gl(n + 1))$, Hecke algebra and the Yang-Baxter equation Lett. Math. Phys. 11 247
[4] Faddeev L D, Reshetikhin N Y and Takhtajan L A 1990 Quantization of Lie groups and Lie algebras Leningr. Math. J. 1 193
Chari V and Pressley A 1995 A Guide to Quantum Groups (Cambridge: Cambridge University Press)
[5] Kulish P P 1989 Quantum superalgebra osp(2|1) J. Sov. Math. 54 923
Kulish P P and Reshetikhin N Y 1989 Universal R matrix of the quantum superalgebra osp(2|1) Lett. Math. Phys. 18 143
[6] Essler F H L, Frahm H, Göhmann F, Klümper A and Korepin V E 2005 The One-Dimensional Hubbard Model (Cambridge: Cambridge University Press)
[7] Beisert N et al 2012 Review of AdS/CFT integrability: an overview Lett. Math. Phys. 99 3
[8] Shastry B S 1986 Exact integrability of the one-dimensional Hubbard model Phys. Rev. Lett. 56 2453
[9] Beisert N 2008 The su(2|2) dynamic S-matrix Adv. Theor. Math. Phys. 12 945
[10] Beisert N 2007 The analytic Bethe Ansatz for a chain with centrally extended su(2|2) symmetry J. Stat. Mech. P01017
[11] Gómez C and Hernández R 2006 The Magnon kinematics of the AdS/CFT correspondence J. High Energy Phys. JHEP11(2006)021
[12] Torrielli A 2007 Classical R-matrix of the su(2)/2 SYM spin-chain Phys. Rev. D 75 105020
[13] Matsumoto T, Moriyama S and Torrielli A 2007 A secret symmetry of the AdS/CFT S-matrix J. High Energy Phys. JHEP09(2007)099
[14] Beisert N and Spill F 2009 The classical R-matrix of AdS/CFT and its Lie bialgebra structure Commun. Math. Phys. 285 537
[15] Nahm W 1978 Supersymmetries and their representations Nucl. Phys. B 135 149
[16] Beisert N, de Leeuw M and Hecht R 2016 Maximally extended sl(2/2) as a quantum double J. Phys. A: Math. Theor. 49 434005
[17] Matsumoto T and Moriyama S 2008 An exceptional algebraic origin of the AdS/CFT Yangian symmetry J. High Energy Phys. JHEP04(2008)022
[18] Kowalski-Glikman J 2005 Introduction to doubly special relativity Planck Scale Effects in Astrophysics and Cosmology (Lecture Notes in Physics vol 669) pp 131–59
[19] Giller S, Kosinski P, Majewski M, Maslanka P and Kunz J 1992 More about q-deformed Poincaré algebra Phys. Lett. B 286 57
[20] Lukierski J and Ruegg H 1994 Quantum κ-Poincaré in any dimension Phys. Lett. B 329 189
[21] Celeghini E, Giachetti R, Sorace E and Tarlini M 1990 Three dimensional quantum groups from contraction of SU(2)q J. Math. Phys. 31 2548
[22] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 The three-dimensional Euclidean quantum group E(3)q and its R-matrix J. Math. Phys. 32 1159
[23] Lukierski J, Nowicki A and Ruegg H 1992 New quantum Poincaré algebra and κ-deformed field theory Phys. Lett. B 293 344
[24] Faddeev L D and Kashaev R M 1994 Quantum dilogarithm Mod. Phys. Lett. A 9 427
[25] Kirillov A N 1995 Dilogarithm identities Prog. Theor. Phys. Suppl. 118 61
[26] Borowiec A and Pachol A 2010 Classical basis for kappa-Poincaré algebra and doubly special relativity theories J. Phys. A: Math. Theor. 43 045203
[27] Twietmeyer E 1992 Real forms of Uq(g) Lett. Math. Phys. 24 49
[28] Stachura P 1997 Poisson–Lie structures on Poincaré and Euclidean groups in three dimensions J. Phys. A: Math. Gen. 31 4555
[29] Zakrzewski S 1997 Poisson structures on the Poincaré group Commun. Math. Phys. 185 285
[30] Beisert N and Koroteev P 2008 Quantum deformations of the one-dimensional Hubbard model J. Phys. A: Math. Gen. 41 255204
[31] Thys H 2001 R-matrice universelle pour Uq(D(2,1,1)) et invariant d’entrelacs associé (arXiv:math/0104110)
[32] Ogievetsky O 1993 Hopf structures on the Borel subalgebra of sl(2) Suppl. Rend. Circ. Mat. Palermo, II. Ser. 37 185
[33] Lukish P P, Lyakhovsky V D and Mudrov A I 1999 Extended Jordanian twists for Lie algebras J. Math. Phys. 40 4569
[34] Tolstoy V N 2004 Chains of extended Jordanian twists for Lie superalgebras (arXiv:math/0402433)
[35] Borowiec A and Pachol A 2009 kappa-Minkowski spacetime as the result of Jordanian twist deformation Phys. Rev. D 79 045012
[36] Borowiec A and Pachol A 2014 κ-deformations and extended κ-Minkowski spacetimes SIGMA 10 107
[37] Pachol A and van Tongeren S J 2016 Quantum deformations of the flat space superstring Phys. Rev. D 93 026008
[38] Klimčík C 2014 Integrability of the bi-Yang-Baxter sigma-model Lett. Math. Phys. 104 1095
[39] Hoare B 2015 Towards a two-parameter q-deformation of AdS3 × S3 × M4 superstrings Nucl. Phys. B 891 259