AN EXPONENTIAL INEQUALITY FOR U-STATISTICS OF I.I.D. DATA

DAVIDE GIRAUDE

ABSTRACT. We establish an exponential inequality for degenerated U-statistics of order $r$ of i.i.d. data. This inequality gives a control of the tail of the maxima absolute values of the U-statistic by the sum of two terms: an exponential term and one involving the tail of $h(X_1, \ldots, X_r)$. We also give a version for not necessarily degenerated U-statistics having a symmetric kernel and furnish an application to the convergence rates in the Marcinkiewicz law of large numbers. Application to invariance principle in Hölder spaces is also considered.

1. Exponential inequalities for U-statistics

1.1. Goal of the paper. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(S, \mathcal{S})$ be a measurable space and let $r \geq 1$ be an integer. Let also $h : S^r \to \mathbb{R}$ be a measurable function and $(X_i)_{i \geq 1}$ an i.i.d. sequence where $X_i : \Omega \to S$. The U-statistic (of order $r$) of kernel $h$ and data $(X_i)_{i \geq 1}$ is defined as

$$U_{r,n}(h) := \sum_{i \in I_r^n} h(X_{i_1}, \ldots, X_{i_r}),$$

(1.1)

where

$$I_r^n := \{i \in \mathbb{Z}^r, 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}.$$

(1.2)

When there is no confusion with the kernel or the order, we will simply write $U_n$.

Our goal is to control the tail of $U_n$. More precisely, we would like to give a control on the following quantity

$$\mathbb{P} \left\{ \max_{r \leq n \leq N} |U_n| > N^{r/2}x \right\}, \quad N \geq r, x > 0,$$

(1.3)

independently on $N$, and with the help of a functional of the tail of $h(X_1, \ldots, X_r)$. This type of inequalities has been studied in the case where the kernel $h$ is bounded in $[1,2,4,12,17,18]$. An extension to the case of U-statistics of the form $\sum_{i \in I_n^r} h_i(X_{i_1}, \ldots, X_{i_r})$ has been considered in $[8,13]$. We will then provide applications to the convergence rates in the law of large numbers. Moreover, the established inequality is also a good tool to verify tightness criterion for partial sum processes in Hölder spaces.

1.2. Statement for degenerated U-statistics. We will first assume that this U-statistic is degenerated, in the sense that for all $s_1, \ldots, s_{r-1} \in S$ and for all $q \in \{1, \ldots, r\}$,

$$\mathbb{E} \left[ h(v_q(s_1, \ldots, s_{r-1}, X_0)) \right] = 0,$$

(1.4)
where \( v_q(s_1, \ldots, s_{r-1}, X_0) \) is a vector of elements of \( S^r \) whose \( q-1 \) first components are respectively \( s_1, \ldots, s_{q-1} \), the component \( q \) is \( X_0 \) and the remaining ones are \( s_{q+1}, \ldots, s_{r-1} \). Notice that here, we are not making any symmetry assumption on the kernel \( h \).

**Theorem 1.1.** Let \( r \geq 1 \) be an integer, \((S, S)\) be measurable space, \( h: S^r \rightarrow \mathbb{R} \) be a measurable function (with \( S^r \) induced with the product \( \sigma \)-algebra) and let \((X_i)_{i \geq 1}\) be an i.i.d. sequence of \( S \)-valued random variables. Suppose that (1.4) holds for all \( s_1, \ldots, s_{r-1} \in S \) and for all \( q \in \{1, \ldots, r\} \). Then the following inequality holds for all positive \( x \) and all \( y \) such that \( x/y > 3^r \):

\[
P\left\{ \max_{r \leq n \leq N} |U_n| > N^{r/2}x \right\} \leq A_r \exp \left( -\frac{1}{2} \left( \frac{x}{y} \right)^2 \right) + B_r \int_1^{+\infty} P\left\{ |h(X_1, \ldots, X_r)| > yu^{1/2}C_r \right\} (1 + \ln(u))^q_r du, \tag{1.5}
\]

where

\[
A_r := 4 \left( 1 - 2^{-r} \right) \tag{1.6}
\]

\[
B_r := 2 \prod_{i=1}^{r} 2^{i-1} \left( 1 + \ln \frac{\kappa_{2(i-1)}}{\kappa_{i-1}} \right)^{i-1} \tag{1.7}
\]

where \( \kappa_q \) is defined for a non-negative \( q \) by

\[
\kappa_q = \begin{cases} 
1 & \text{if } q \leq 1 \\
e^{q-1}/q^q & \text{if } q > 1;
\end{cases} \tag{1.8}
\]

\[
C_r := 4^{-r/2} \prod_{i=1}^{r} \kappa_{i-1}^{1/2} \tag{1.9}
\]

and

\[
q_r := \frac{(r-1)r}{2}. \tag{1.10}
\]

Let us give some comments on the result.

**Remark 1.2.** The second term of the right-hand-side of (1.5) is of order \( E \left[ Y^2 \log (1 + Y)^{q} \mathbf{1}\{Y > 1\} \right] \), where \( Y = |h(X_1, \ldots, X_r)|/y \). This implies that Theorem 1.1 is useful only if \( E \left[ Y^2 \log (1 + Y)^{q} \right] \) is finite.

**Remark 1.3.** Proposition 2.3 in [2] gives also an exponential inequality when \( h \) is bounded. Theorem 1.1 gives a similar result up to the involved constants.

**Remark 1.4.** The bigger \( y \) is, the smaller the second term of the right hand side of (1.5) is but the higher the first term of the right hand side is. Therefore, when applying this inequality, we try to use it with an appropriate value of \( y \).

**Remark 1.5.** The right hand side of (1.5) is independent of \( N \). Therefore, when Theorem 1.1 is applied with a stronger normalization than \( N^{r/2} \), we can get a decay in \( N \).
Remark 1.6. However, this inequality does not seem to be sufficient to deduce directly a satisfactory result for the law of the iterated logarithms. Indeed, one would need to control \( \sum_{N \geq r} \mathbb{P} \{ \max_{r \leq n \leq 2^N} |U_n| > 2^{Nr/2} \alpha \sqrt{\log N} \} \), which forces the choice \( x = \alpha \sqrt{\log N} \) for each fixed \( N \) and one should choose \( y < 3^r x \) hence we would need exponential moments for \( |h(X_1, \ldots, X_r)| \), which is of course suboptimal.

1.3. Statement for general \( U \)-statistics with symmetric kernel. We now would like to give a result similar to Theorem 1.1 but without assuming that the \( U \)-statistic is degenerated. To this aim, we use the Hoeffding decomposition. Here we assume that the kernel \( h: S^r \to \mathbb{R} \) is symmetric in the sense that for all \( x_1, \ldots, x_r \in S \) and all permutation \( \sigma: \{1, \ldots, r\} \to \{1, \ldots, r\} \),

\[
    h\left(x_{\sigma(1)}, \ldots, x_{\sigma(r)}\right) = h\left(x_1, \ldots, x_r\right). \tag{1.11}
\]

The \( U \)-statistic involved in Theorem 1.1 are such that

\[
    \mathbb{E} \left[ h\left(X_1, \ldots, X_r\right) \mid \sigma(X_i, 1 \leq i \leq r, i \neq j) \right] = 0 \tag{1.12}
\]

almost surely for all \( j \in \{1, \ldots, r\} \). In order to extend the result to a larger class of \( U \)-statistic, we need to define a more general notion of degenerateness.

Definition 1.7. Let \((X_k)_{k \geq 1}\) be an i.i.d. sequence with values in the measurable space \((S, \mathcal{S})\). Let \( h: S^r \to \mathbb{R} \) be a measurable function such that \( Y := h(X_1, \ldots, X_r) \) is integrable. Denote \( \mathcal{F}_j \) the \( \sigma \)-algebra generated by the random variables \( X_1, \ldots, X_j \). We say that the \( U \)-statistic \( U_{r,n}(h) \) is degenerated of order \( i - 1 \) for some \( i \in \{2, \ldots, r\} \) if \( \mathbb{E}[Y \mid \mathcal{F}_{i-1}] = 0 \) almost surely but \( \mathbb{E}[Y \mid \mathcal{F}_i] \) is not equal to zero almost surely.

We express the \( U \)-statistic associated to this kernel \( h \) as a sum of \( U \)-statistics of order \( k \) with symmetric kernel. Define

\[
    \pi_{k,r} h(x_1, \ldots, x_k) := (\delta_{x_1} - \mathbb{P}_{X_1}) \cdots (\delta_{x_k} - \mathbb{P}_{X_1}) \mathbb{P}_{X_1}^{r-k} h, \tag{1.13}
\]

where \( Q_1 \cdots Q_r h \) is defined as

\[
    Q_1 \cdots Q_r h = \int \cdots \int h(x_1, \ldots, x_r) dQ_1(x_1) \cdots dQ_r(x_r). \tag{1.14}
\]

Then the following equality holds:

\[
    \binom{n}{r} U_{r,n}(h - \theta) = \sum_{k=1}^{r} \binom{r}{k} \binom{n}{k} U_{k,n}(h_k), \tag{1.15}
\]

where \( \theta := \mathbb{E}[h(X_1, \ldots, X_n)] \) and \( U_{k,n}(h_k) \) is a generated \( U \)-statistic of order \( k \).

If \( U_{r,n}(h) \) is degenerated of order \( i - 1 \) for some \( i \in \{2, \ldots, r\} \), then the first \( i - 1 \) terms in (1.15) vanish (and \( \theta = 0 \)) hence

\[
    \binom{n}{r} U_{r,n}(h) = \sum_{k=i}^{r} \binom{r}{k} \binom{n}{k} U_{k,n}(h_k). \tag{1.16}
\]

Therefore, applying Theorem 1.1 to each degenerated \( U \)-statistic \( U_{k,n}(h_k) \) gives the following result.
Corollary 1.8. Let $r \geq 1$ be an integer, $(S, S)$ be measurable space, $h: S^r \to \mathbb{R}$ be a measurable function (with $S^r$ induced with the product $\sigma$-algebra) and let $(X_i)_{i \geq 1}$ be an i.i.d. sequence of $S$-valued random variables. Assume that $h$ is degenerated of order $i - 1$ for some $i \in \{2, \ldots, r\}$. Then for all positive $x$ and $y$ such that $x/y > 3^r$,

$$
\mathbb{P}\left\{ \max_{r \leq n \leq N} |U_n| > N^{r-{\gamma}\over 2}x \right\} \leq A_r \exp \left( -{1 \over 2} \left( {x \over y} \right)^{2\gamma} \right) + B_r \sum_{k=1}^r \int_{1}^{+\infty} \mathbb{P}\left\{ |\mathbb{E}[h(X_1, \ldots, X_r) | X_1, \ldots, X_k]| > yN^{r-{\gamma}\over 2}u^{1/2}C_r \right\} (1 + \ln(u))^{q_k} du,
$$

(1.17)

where $q_k = (k-1)k^2$ and the constants $A_r$, $B_r$ and $C_r$ depend only on $r$.

When $h(X_1, \ldots, X_n)$ has a finite exponential moments, it turns out that a simpler upper bound can be given.

Corollary 1.9. Let $r \geq 1$ be an integer and $\gamma > 0$. There are constants $x_{r, \gamma}$, $A_{r, \gamma}$ and $B_{r, \gamma}$ such that if $(S, S)$ is a measurable space, $h: S^r \to \mathbb{R}$ is a measurable function (with $S^r$ induced with the product $\sigma$-algebra), $(X_i)_{i \geq 1}$ an i.i.d. sequence of $S$-valued random variables and $h$ is degenerated of order $i - 1$ for some $i \in \{2, \ldots, r\}$, then for all $x \geq x_{r, \gamma}$, for all $\gamma > 0$, then for all $x \geq x_{r, \gamma} := 3^r{\gamma\over 2}2^{-1/r}C_r^{-1}$ (with $C_r$ like in Corollary 1.8) and all $N \geq r$,

$$
\mathbb{P}\left\{ \max_{r \leq n \leq N} |U_n| > N^{r-{\gamma}\over 2}x \right\} \leq A_{r, \gamma} \exp \left( -B_{r, \gamma}x^{2\gamma\over 2\gamma+1} \right) \mathbb{E}[\exp (|h(X_1, \ldots, X_r)|^{\gamma})].
$$

(1.18)

Remark 1.10. Corollary 1.2 in [26] gives a result in the same spirit, but with the following differences.

- Our inequality gives a bound on the tail probability of $\max_{r \leq n \leq N} |U_n|$, while [26] gives a control of the tail of $|U_N|$.
- Our inequality show explicitly how the right hand side depends on $\mathbb{E}[\exp (|h(X_1, \ldots, X_r)|^{\gamma})]$.
- It allows in particular to apply the inequality to $R \cdot h$, $R > 0$, instead of $h$ when $\mathbb{E}[\exp (R |h(X_1, \ldots, X_r)|^{\gamma})]$ is finite.
- The case $0 < \gamma < 2$ was addressed in [26], whereas we cover the case $\gamma > 0$.

1.4. Application to convergence rates in the strong law of large numbers. Let $U_n$ be the U-statistic defined by (1.1). Suppose that $\mathbb{E}[|h(X_1, \ldots, X_r)|]$ is finite. Then

$$
{1 \over n^r} \sum_{i \in I_n^r} (h(X_{i_1}, \ldots, X_{i_r}) - \mathbb{E}[h(X_{i_1}, \ldots, X_{i_r})]) \to 0 \ a.s.
$$

(1.19)

If we assume that the $U$-statistic is degenerated of order $i - 1$ and if we impose more restrictive integrability conditions, an other normalization than $n^r$ can be chosen.

Theorem 1.11 (Theorem 1 in [25]). Let $h: \mathbb{R}^r \to \mathbb{R}$ be a symmetric function, $r \geq 2$ and let $(X_i)_{i \geq 1}$ be an i.i.d. sequence of random variables. Suppose that $h$ is degenerated of order $i - 1$ where $i \in \{2, \ldots, r\}$. Let $q \in (1, 2r/ (2r - i))$. Suppose that for all $j \in \{i, \ldots, r\}$,

$$
\mathbb{E}[h(X_1, \ldots, X_r) | X_1, \ldots, X_j] \in L^{q\over (r-2)/r}.
$$

(1.20)
Then \( n^{-r/q}U_n \to 0 \) almost surely.

Information on the convergence rates in the Marcinkiewicz law of large number have been obtained in \([9]\) and results in the spirit of Baum-Katz \([3]\) for partial sums has been obtained in \([14, 16]\), in all the cases under polynomial moment conditions, that is, finiteness of \( \mathbb{E} \left[ |h(X_1, \ldots, X_r)|^q \right] \) for some \( q \).

Our setting would also allow to derive result under similar assumptions but it seems that the inequality we obtain is not the most suitable in this context. However, under finite exponential moments, one can use Corollary 1.9 in order to quantify the convergence.

**Theorem 1.12.** Let \( (S, \mathcal{S}) \) be a measurable space, \( h: S^r \to \mathbb{R} \) be a symmetric function, \( r \geq 2 \) and let \( (X_i)_{i \geq 1} \) be an i.i.d. sequence of random variables with values in \( S \). Suppose that \( h \) is degenerated of order \( i - 1 \) where \( i \in \{2, \ldots, r\} \). Let \( \alpha \in (r - i/2, r) \). If there exists a positive \( \gamma > 0 \) such that \( \mathbb{E} \left[ \exp \left( R \left| h(X_1, \ldots, X_r) \right|^{\gamma} \right) \right] < +\infty \) is finite for all \( R \), then

\[
\forall \varepsilon > 0, \quad \sum_{N \geq 1} \exp \left( 2^{N\alpha-(r-i/2)} \right) \mathbb{P} \left\{ \max_{r \leq n \leq 2N} |U_n| > \varepsilon 2^{N\alpha} \right\} < +\infty. \tag{1.21}
\]

An other way to measure the speed of convergence in the law of large numbers is by bounding the probabilities of large deviation. In the case of partial sums of an i.i.d. centered sequence \( U_r \), studied. In Corollary 1, it is shown that if \( U_r \) converges in distribution in the Skorohod space \( D \), then in all the cases under polynomial moment conditions, that is, finiteness of \( \mathbb{E} \left[ |h(X_1, \ldots, X_r)|^q \right] \). Extension to martingale with bounded moments have been investigated in \([15]\) and \([6]\).

In the context of the \( U \)-statistics, the normalization will depend on the degree of degeneracy.

**Theorem 1.13.** Let \( (S, \mathcal{S}) \) be a measurable space, \( h: S^r \to \mathbb{R} \) be a symmetric function, \( r \geq 2 \) and let \( (X_i)_{i \geq 1} \) be an i.i.d. sequence of random variables with values in \( S \). Suppose that \( h \) is degenerated of order \( i - 1 \) where \( i \in \{2, \ldots, r\} \). If there exists a positive \( \gamma > 0 \) such that \( M := \sup_{t>0} \exp (t^\gamma) \mathbb{P} \left\{ |h(X_1, \ldots, X_r)| > t \right\} \) is finite, then for all \( x \geq 0 \),

\[
\mathbb{P} \left\{ \max_{r \leq n \leq N} |U_n| > N^{r/2} x \right\} \leq K_1 \exp \left( -K_2 N \frac{x^{2r}}{n^{r/2}} \right), \tag{1.22}
\]

where \( K_1 \) and \( K_2 \) depend on \( \gamma \) and \( M \).

1.5. **Weak invariance principle in Hölder spaces.** In this subsection, we will study another limit theorem for \( U \)-statistics: the functional central limit theorem in Hölder spaces. Given a \( U \)-statistic \( U_n \) of order \( r \) and kernel \( h: S^r \to \mathbb{R} \), we define a partial sum process by

\[
\sigma_n(t) := \frac{1}{n^{r/2}} \left( U_{[nt]} - (nt - [nt]) (U_{[nt]+1} - U_{[nt]}) \right), \tag{1.23}
\]

where for \( x \in \mathbb{R}, [x] \) is the unique integer satisfying \( [x] \leq x < [x] + 1 \). In other words, \( \sigma_n(k/n) = U_k \) and the random function \( t \to \sigma_n(t) \) is affine on the intervals \([k/n, (k + 1)/n]\). In \([19]\), the convergence in distribution in the Skorohod space \( D[0,1] \) of the process \( (n^{-r/2}U_{[nt]})_{n \geq r} \) is studied. In Corollary 1, it is shown that if \( U_n \) is degenerated of order \( i - 1 \), \( i \in \{2, \ldots, r\} \), then \((n^{-r/2}U_{[nt]})_{n \geq r}\) converges in distribution to a process \( I_i(h_i) \) symbolically defined as

\[
I_i(h_i)(t) = \int \cdots \int h_i(x_1, \ldots, x_i) 1_{[0,t]}(u_1) \cdots 1_{[0,t]}(u_i) W(dx_1, du_1) \cdots W(dx_i, du_i), \tag{1.24}
\]

where \( W \) denotes the Gaussian measure (see the Appendix A.1 and A.2 of the paper \([19]\)). For \( i = 2 \), the limiting process admits the expression \( \sum_{j=1}^{+\infty} \lambda_j (B_j^2(t) - t) \), where \( (B_j(\cdot))_{j \geq 1} \)
are independent standard Brownian motions and \( \sum_{j=1}^{+\infty} \lambda_j^2 \) is finite. In particular, such a process has path in Hölder spaces and would also be the limiting process for \( (\sigma_n)_{n \geq 1} \) when \( r = 2 \). Therefore, the study of the limiting behavior of \( (\sigma_n (t))_{n \geq r} \) in Hölder spaces can be considered.

This question has been considered in the context of partial sum processes built on strictly stationary sequences of random variables, that is, of the form

\[
W_n (t) := \frac{1}{a_n} \left( \sum_{i=1}^{[nt]} X_i + (nt - [nt]) X_{[nt]+1} \right),
\]

(1.25)

where \( (X_j)_{j \geq 1} \) is a strictly stationary centered sequence and \( a_n \to +\infty \). The asymptotic behavior of such partial processes in \( D[0,1] \) under dependence has concentrated a lot of effort; see for instance [20] for a survey on the main results.

Define \( \mathcal{R}_i \) as the class of the real-valued functions \( \rho \) defined on \( [0,1] \) which can be expressed as \( \rho (t) = t^\alpha L (1/t) \), where \( L : [1, +\infty) \to \mathbb{R} \) is normalized slowly varying at infinity, positive and continuous, \( \rho \) is increasing on \( [0,1] \) and

\[
\lim_{\delta \to 0} \rho (\delta) \delta^{-1/2} (\ln (\delta^{-1}))^{-i/2} = +\infty.
\]

(1.26)

Notice that this implies that \( 0 < \alpha \leq 1/2 \) and for \( \alpha = 1/2 \), the constraint reads

\[
\lim_{\delta \to 0} L (1/\delta) (\ln (\delta^{-1}))^{-i/2} = +\infty.
\]

(1.27)

For example, if \( c \) is such that the function \( t \mapsto t^{1/2} (\ln (c/t))^\beta \) is increasing, then the latter constraint forces \( \beta > i/2 \).

For \( \rho \in \mathcal{R}_i \), we denote by \( \mathcal{H}_\rho \) the Hölder space associated to the modulus of regularity \( \rho \), that is, the set of function \( x : [0,1] \to \mathbb{R} \) such that \( \|x\|_\rho := \sup_{0 \leq s < t \leq 1} |x(t) - x(s)|/\rho (t-s) + |x(0)| \) is finite. Instead of dealing with the convergence in \( \mathcal{H}_\rho \), we will work with a subspace which is more adapted to the study of convergence in distribution. Let

\[
\mathcal{H}_\rho^0 := \left\{ x : [0,1] \to \mathbb{R} \mid \lim_{\delta \to 0} \sup_{0 < t - s < \delta} \frac{|x(t) - x(s)|}{\rho (t-s)} = 0 \right\}.
\]

(1.28)

The convergence of partial sum processes of the form (1.25) when \( (X_i)_{i \geq 1} \) is i.i.d. has been studied in [21, 22]. The convergence of \( (W_n (\cdot))_{n \geq 1} \) in \( \mathcal{H}_\rho^0 \) for \( \rho \in \mathcal{R}_1 \) holds if and only if

\[
\forall A > 0, \lim_{t \to +\infty} t \mathbb{P} \left\{ |X_1| > At^{1/2} \rho (1/t) \right\} = 0.
\]

(1.29)

Generally, a strategy to prove such results is to establish the convergence on the finite dimensional distribution and prove tightness, which is usually the most difficult part. In Equation (1.3) in [11], a tightness criterion for partial sum processes of the form (1.25) with \( (X_j)_{j \geq 1} \) and \( \rho \) of the form \( t \mapsto t^\alpha \), \( 0 < \alpha < 1/2 \). Its verification is done by using deviation inequalities, see for example [10] or Section 3.3 in [5].

For the purpose of the study of the convergence of \( (\sigma_n (\cdot))_{n \geq 1} \) (defined by (1.23)), we need to extend this criterion in two directions: to partial sum processes like in (1.25) for which the sequence \( (X_j)_{j \geq 1} \) is not necessarily stationary and to the class of modulus of regularity \( \mathcal{R}_1 \).
Proposition 1.14. Let \((X_j)_{j \geq 1}\) be a sequence of random variables. Let \(W_n\) be the partial sum process built on \((X_j)_{j \geq 1}\) defined by (1.25). Let \(\rho \in \mathcal{R}_i\) for some \(i \geq 1\). Suppose that for all positive \(\varepsilon\), the following convergences hold:

\[
\lim_{j \to +\infty} \limsup_{n \to +\infty} \sum_{j=1}^{[\log_2 n]} \sum_{k=0}^{2^j-1} \mathbb{P} \{ |S_{[n(k+1)2^{-j}]} - S_{[nk2^{-j}]}| > a_n \varepsilon (2^{-j}) \} = 0; \tag{1.30}
\]

where \(S_N := \sum_{i=1}^{N} X_i\) and \((a_n)_{n \geq 1}\) is an increasing sequence diverging to infinity and such that \(\sup_{n \geq 1} a_{2n}/a_n\) is finite. Then the partial sum process defined by (1.25) is tight in \(\mathcal{H}_\rho^2\).

Since the previous tightness criterion involves the tails of differences of partial sums, we have to establish a corresponding deviation inequality.

Proposition 1.15. Let \(r \geq 1\) be an integer, \((S, \mathcal{S})\) be measurable space, \(h : S^r \to \mathbb{R}\) be a measurable function (with \(S^r\) induced with the product \(\sigma\)-algebra) and let \((X_i)_{i \geq 1}\) be an i.i.d. sequence of \(S\)-valued random variables. Suppose that (1.4) holds for all \(s_1, \ldots, s_{r-1} \in S\) and for all \(q \in \{1, \ldots, r\}\). Then the following inequality holds for all positive \(x\) and all \(y\) such that \(x/y > 3^r\) and all \(n_2 > n_1 \geq r\):

\[
\mathbb{P} \left\{ \frac{1}{\sqrt{n_2 - n_1 n_2}} |U_{n_2} - U_{n_1}| > x \right\} \leq A_r \exp \left( -\frac{1}{2} \left( \frac{x}{y} \right)^2 \right) + B_r \int_{1}^{+\infty} \mathbb{P} \left\{ h(X_1, \ldots, X_r) > y u^{1/2} C_r \right\} (1 + \ln(u))^{q_r} du, \tag{1.31}
\]

where \(A_r, B_r, C_r\) depend only on \(r\) and \(q_r = r(r - 1)/2\).

By combining this tightness criterion with the obtained deviation inequalities, we get the following function central limit theorem.

Theorem 1.16. Let \(r \geq 1\) be an integer, \((S, \mathcal{S})\) be measurable space, \(h : S^r \to \mathbb{R}\) be a symmetric measurable function (with \(S^r\) induced with the product \(\sigma\)-algebra) and let \((X_i)_{i \geq 1}\) be an i.i.d. sequence of \(S\)-valued random variables. Assume that \(h\) is degenerated of order \(i - 1\) for some \(i \in \{2, \ldots, r\}\) and that \(h\) is symmetric. Let \(\rho \in \mathcal{R}_r\). Assume that

\[
\forall c > 0, \sum_{j=1}^{+\infty} \int_{1}^{+\infty} \mathbb{P} \left\{ |h(X_1, \ldots, X_r)| > c 2^{j/2} \rho (2^{-j}) j^{-r/2} u^{1/2} \right\} (1 + \log(u))^{q_r} du < +\infty. \tag{1.32}
\]

Then

\[
\frac{1}{n^{1/2}} (U_{[nt]} - (nt - [nt]) (U_{[nt]+1} - U_{[nt]})) \to I_1 (h_i) (t) \tag{1.33}
\]

in distribution in \(\mathcal{H}_\rho^0 ([0, 1])\).

In particular, denoting \(Y := |h(X_1, \ldots, X_r)|\), when \(\rho (t) = t^\alpha, 0 < \alpha < 1/2\), the condition

\[
\mathbb{E} \left[ Y^{\alpha + 1/2} (\log Y)^{\alpha/2} \right] < +\infty \tag{1.34}
\]

guarantees (1.33). When \(\rho (t) = t^{1/2} (\log (C/t))^{\beta}, \beta > r/2\), the condition

\[
\forall A > 0, \mathbb{E} \left[ \exp \left( A Y^{\alpha + 1/2} \right) \right] < +\infty \tag{1.35}
\]

guarantees (1.33).
Remark 1.17. When $r = 1$ and $\rho(t) = t^\alpha$, we do not recover exactly the necessary and sufficient condition established in [21], which reads $\lim_{t \to +\infty} t^{1/(1-2-\alpha)}P \{ Y > t \} = 0$. However, when $\rho(t) = t^{1/2} (\log (C/t))^3$, we recover the same condition.

2. Proofs

2.1. Proof of Theorem 1.1. Let us first give the general idea of proof of Theorem 1.1.

(1) We will proceed by an induction argument on $r$. We will use martingale inequality, which helps to control the tail maximum of partial sums of a martingale differences sequences with the help of the tail of the sum of squares and the sum of conditional variances.

(2) Using the notion of convex ordering that will be made explicit later, the sum of squares and the sum of conditional variances satisfy a "convexity domination" hence we will be reduced to tails of $U$-statistics of the previous order.

(3) In order to perform the induction step, we will also need a rearrangement of integrals of the type $\int_1^{+\infty} \int_1^{+\infty} P \{ Y > u/(1 + \ln u)^a v \} du (1 + \ln v)^b dv$, whose treatment will be addressed later.

2.1.1. Martingale inequality. We will formulate the martingale inequality we need to handle the tail of $U$-statistics. First, the following inequality is established in [7].

Theorem 2.1 (Theorem 2.1 in [7]). Let $(D_i)_{i \geq 1}$ be a martingale differences sequence with respect to the filtration $(F_i)_{i \geq 0}$. Suppose that there exists random variables $V_{i-1}$, $1 \leq i \leq n$, which are non-negative, $F_{i-1}$-measurable, non-negative functions $f$ and $g$ such that for some positive $\lambda$ and for all $i \in \{1, \ldots, n\}$,

$$E \left[ \exp (\lambda D_i - g(\lambda) D_i^2) \mathbb{1}_{F_{i-1}} \right] \leq 1 + f(\lambda) V_{i-1}. \quad (2.1)$$

Then for all $x$, $v$, $w > 0$,

$$P \left( \bigcup_{k=1}^{n} \left\{ \sum_{i=1}^{k} D_i \geq x \right\} \cap \left\{ \sum_{i=1}^{k} D_i^2 \leq v^2 \right\} \cap \left\{ \sum_{i=1}^{k} V_{i-1} \leq w \right\} \right) \leq \exp (-\lambda x + g(\lambda) v^2 + f(\lambda) w). \quad (2.2)$$

Notice that the condition (2.1) is satisfied for all $\lambda > 0$ when $f(\lambda) = g(\lambda) = \lambda^2/2$ and $V_{i-1} = E \left[ D_i^2 \mathbb{1}_{F_{i-1}} \right]$. After having optimized over $\lambda$ the right hand side of (2.2) and applying Theorem 2.1 to $D_j$ and then to $-D_j$, we get that

$$P \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} D_i \right| > x \right) \leq 2 \exp \left( -\frac{x^2}{2v^2} \right) + \frac{P \left( \sum_{i=1}^{n} (D_i^2 + E \left[ D_i^2 \mathbb{1}_{F_{i-1}} \right]) > v^2 \right)}{(2.3)}$$

for all $x$ and $v > 0$.

2.1.2. Convex ordering. Let us compare the tails of $Y$ and $E \left[ Y \mathbb{1}_{\mathcal{G}} \right]$. By Markov’s inequality,

$$P \{ E \left[ Y \mathbb{1}_{\mathcal{G}} \right] > x \} \leq \frac{1}{x} E \left[ E \left[ Y \mathbb{1}_{\mathcal{G}} \right] \mathbb{1} \{ E \left[ Y \mathbb{1}_{\mathcal{G}} \right] > x \} \right] = \frac{1}{x} E \left[ Y \mathbb{1} \{ E \left[ Y \mathbb{1}_{\mathcal{G}} \right] > x \} \right]$$

and splitting the last expectation over the sets where $Y \leq x/2$ and $Y > x/2$ gives

$$P \{ E \left[ Y \mathbb{1}_{\mathcal{G}} \right] > x \} \leq \frac{1}{2} P \{ E \left[ Y \mathbb{1}_{\mathcal{G}} \right] > x \} + \frac{1}{x} E \left[ Y \mathbb{1} \{ Y > x/2 \} \right]. \quad (2.4)$$
hence
\[ x \mathbb{P} \{ \mathbb{E} [Y \mid \mathcal{G}] > x \} \leq \mathbb{E} [2Y 1 \{2Y > x\}] \] (2.5)
and writing the last expectation as an integral over \((0, +\infty)\) involving the tail of \(Y\), we get that
\[ x \mathbb{P} \{ \mathbb{E} [Y \mid \mathcal{G}] > x \} \leq x \mathbb{P} \{2Y > x\} + \int_{x}^{+\infty} \mathbb{P} \{2Y > u\} \, du. \] (2.6)
Bounding further \(x \mathbb{P} \{2Y > x\}\) by \(2 \int_{x/2}^{x} \mathbb{P} \{2Y > u\} \, du\) finally gives after the substitution \(v = u/(2x)\) that
\[ \mathbb{P} \{ \mathbb{E} [Y \mid \mathcal{G}] > x \} \leq 4 \int_{1/4}^{+\infty} \mathbb{P} \{Y > xv\} \, dv. \] (2.7)
hence one can control the tails of \(\mathbb{E} [Y \mid \mathcal{G}]\) by a functional of those of \(Y\).

Therefore, if \(Y\) and \(Z\) are two non-negative random variables such that
\[ Z \leq \mathbb{E} [Y \mid Z] \] a.s. (2.8)
then for all \(x > 0\),
\[ \mathbb{P} \{Z > x\} \leq \int_{1}^{+\infty} \mathbb{P} \{Y > xv/4\} \, dv. \] (2.9)

Observe also that if \(Z\) and \(Y\) are two non-negative random variables such that (2.8) holds, then for each convex non-decreasing function \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\),
\[ \mathbb{E} [\varphi (Z)] \leq \mathbb{E} [\varphi (Y)]. \] (2.10)

When (2.10) holds for each convex non-decreasing function, we write \(Z \leq_{\text{conv}} Y\).

By Theorem 6 in [23], part (c), if two random variables \(Y\) and \(Z\) defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfy (2.10), then there exists a probability space \((\Omega', \mathcal{F}', \mathbb{P}')\) and random variables \(Y'\) and \(Z'\) such that
- for all real number \(t\), \(\mathbb{P}' \{Y' \leq t\} = \mathbb{P} \{Y \leq t\}\) and \(\mathbb{P}' \{Z' \leq t\} = \mathbb{P} \{Z \leq t\}\);
- the inequality \(Z' \leq_{\mathbb{P}'} [Y' \mid Z']\) holds almost surely.

Combining this with (2.9) gives the following lemma that will be used in the sequel.

**Lemma 2.2.** Let \(Y\) and \(Z\) be two non-negative random variables such that for all convex non-decreasing function \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+, \mathbb{E} [\varphi (Z)] \leq \mathbb{E} [\varphi (Y)]\). Then for all positive \(x\),
\[ \mathbb{P} \{Z > x\} \leq \int_{1}^{+\infty} \mathbb{P} \{Y > xv/4\} \, dv. \] (2.11)

Using the previous lemma, one can control the tails of the maximum of a martingale whose increments have common majorant for the order \(\leq_{\text{conv}}\).

**Proposition 2.3.** Let \((D_j)_{j \geq 1}\) be a martingale differences sequence with respect to the filtration \((\mathcal{F}_i)_{i \geq 0}\). Suppose that \(\mathbb{E} [D_i^2] \) is finite for all \(i \geq 1\). Suppose that there exists a random variable \(Y\) such that for all \(1 \leq i \leq n\), \(D_i^2 \leq_{\text{conv}} Y^2\). Then for all \(x, y > 0\) and each \(n \geq 1\), the following inequality holds:
\[ \mathbb{P} \left\{ \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} D_i \right| > xn^{1/2} \right\} \leq 2 \exp \left( -\frac{1}{2} \left( \frac{x}{y} \right)^2 \right) + 2 \int_{1}^{+\infty} \mathbb{P} \{Y^2 > y^2 u/4\} \, du. \] (2.12)
Proof. We apply Theorem 2.1 in the following setting: $x$ is replaced by $xn^{1/2}$ and $v = n^{1/2}y$. We obtain that

$$
P \left\{ \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} D_i \right| > x n^{1/2} \right\} \leq 2 \exp \left( - \frac{1}{2} \left( \frac{x}{y} \right)^2 \right)$$

$$+ P \left\{ \sum_{i=1}^{n} D_i^2 > n y^2 \right\} + P \left\{ \sum_{i=1}^{n} E \left[ D_i^2 \mid \mathcal{F}_{i-1} \right] > n y^2 \right\}. \tag{2.13}$$

In order to control the last two terms, we will show that

$$\frac{1}{n} \sum_{i=1}^{n} D_i^2 \leq_{\text{conv}} Y^2 \text{ and } \frac{1}{n} \sum_{i=1}^{n} E \left[ D_i^2 \mid \mathcal{F}_{i-1} \right] \leq_{\text{conv}} Y^2. \tag{2.14}$$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex non-decreasing function. Then by convexity,

$$E \left[ \varphi \left( \frac{1}{n} \sum_{i=1}^{n} D_i^2 \right) \right] \leq E \left[ \frac{1}{n} \sum_{i=1}^{n} \varphi \left( D_i^2 \right) \right] \tag{2.15}$$

and from the fact that $D_i^2 \leq_{\text{conv}} Y^2$, it follows that $E \left[ \varphi \left( D_i^2 \right) \right] \leq E \left[ \varphi \left( Y^2 \right) \right]$ hence

$$E \left[ \varphi \left( \frac{1}{n} \sum_{i=1}^{n} D_i^2 \right) \right] \leq E \left[ \varphi \left( Y^2 \right) \right]. \tag{2.16}$$

Moreover, by convexity of $\varphi$ and Jensen’s inequality,

$$E \left[ \varphi \left( \frac{1}{n} \sum_{i=1}^{n} E \left[ D_i^2 \mid \mathcal{F}_{i-1} \right] \right) \right] \leq E \left[ \frac{1}{n} \sum_{i=1}^{n} \varphi \left( E \left[ D_i^2 \mid \mathcal{F}_{i-1} \right] \right) \right] \leq E \left[ \frac{1}{n} \sum_{i=1}^{n} E \left[ \varphi \left( D_i^2 \right) \mid \mathcal{F}_{i-1} \right] \right]$$

hence

$$E \left[ \varphi \left( \frac{1}{n} \sum_{i=1}^{n} E \left[ D_i^2 \mid \mathcal{F}_{i-1} \right] \right) \right] \leq \frac{1}{n} \sum_{i=1}^{n} E \left[ \varphi \left( D_i^2 \right) \right]. \tag{2.17}$$

By the same argument as before,

$$E \left[ \varphi \left( \frac{1}{n} \sum_{i=1}^{n} E \left[ D_i^2 \mid \mathcal{F}_{i-1} \right] \right) \right] \leq E \left[ \varphi \left( Y^2 \right) \right]$$

which proves (2.14). We conclude by applying Lemma 2.2 twice, first to $Z = n^{-1} \sum_{i=1}^{n} D_i^2$ and then to $Z = n^{-1} \sum_{i=1}^{n} E \left[ D_i^2 \mid \mathcal{F}_{i-1} \right]$. \hfill \qed

2.1.3. Step $r = 1$. As said before, the proof of Theorem 1.1 is done by induction on $r$. In this subsubsection, we treat the case $r = 1$. In this case, $U_n = \sum_{i=1}^{n} h \left( X_i \right)$ and condition (1.4) means that $h \left( X_i \right)$ is centered. Therefore, an application of Proposition 2.3 to $D_j := h \left( X_j \right)$ gives the case $r = 1$ of Theorem 1.1.

2.1.4. The induction step. Let $A \left( r \right)$ be the following assertion: "for each measurable space $(S, \mathcal{S})$, each measurable function $h : S^r \to \mathbb{R}$ (with $S^r$ induced with the product $\sigma$-algebra) and each i.i.d. sequence $(X_i)_{i \geq 1}$ of $S$-valued random variables such that

$$E \left[ h \left( v_q \left( x_1, \ldots, x_{r-1}, X_0 \right) \right) \right] = 0, \tag{2.18}$$


holds for all \( s_1, \ldots, s_{r-1} \in S \) and for all \( q \in \{1, \ldots, r\} \), the following inequality holds for all positive \( x \) and all \( y \) such that \( x/y > 3^r \):

\[
\Pr \left\{ \max_{r \leq n \leq N} |U_n| > N^{r/2}x \right\} \leq A_r \exp \left( -\frac{1}{2} \left( \frac{x}{y} \right)^2 \right) + B_r \int_1^{+\infty} \Pr \left\{ |h(X_1, \ldots, X_r)| > yv^{1/2}C_r^r \right\} (1 + \ln (v)) v^{n} \, dv. \tag{2.19}
\]

We have shown in the previous subsubsection that \( A(1) \) is true.

Now, we assume that \( A(r-1) \) is true for some \( r \geq 2 \) and we will prove \( A(r) \). Let \((S, S)\) be a measurable space, \( h: S^r \to \mathbb{R} \) (with \( S^r \) induced with the product \( \sigma \)-algebra) be a measurable function, \((X_i)_{i \geq 1}\) be an i.i.d. sequence of \( S \)-valued random variables satisfying

\[
\mathbb{E} [h (v_q (x_1, \ldots, x_{r-1}, X_0))] = 0, \tag{2.20}
\]

for all \( s_1, \ldots, s_{r-1} \in S \) and for all \( q \in \{1, \ldots, r\} \), and \( x \) and \( y \) be positive numbers such that \( x/y > 3^r \). We will control \( \Pr \{ \max_{r \leq n \leq N} |U_n| > N^{r/2}x \} \) by a quantity that can be treated by the case of \( U \)-statistics of order \( r - 1 \).

We define

\[
D_j := \frac{1}{N^{r/2}} \sum_{(i_1, \ldots, i_{r-1}) \in I_j^{-1}} h (X_{i_1}, \ldots, X_{i_{r-1}}, X_j), \quad j \geq r, \tag{2.21}
\]

where the notation \( I_j^{-1} \) refers to (1.2) and \( D_j = 0 \) for \( 1 \leq j \leq r - 1 \). Then for all \( n \geq r \),

\[
\frac{1}{N^{r/2}} U_n = \frac{1}{N^{r/2}} \sum_{j=1}^{n} D_j. \quad \text{Moreover, defining } F_j \text{ as the } \sigma \text{-algebra generated by the random variables } X_i, 1 \leq i \leq j \text{ for } j \geq 1 \text{ and } F_0 := \{\emptyset, \Omega\}, \text{ the sequence } (D_j)_{j \geq 1} \text{ is a martingale differences sequence with respect to the filtration } (F_j)_{j \geq 0}. \text{ Let}
\]

\[
Y := \frac{1}{N^{r/2}} \max_{r-1 \leq k \leq N} |U_k'|, \tag{2.22}
\]

where

\[
U_k' := \sum_{(i_1, \ldots, i_{r-1}) \in I_k^{-1}} h (X_{i_1}, \ldots, X_{i_{r-1}}, X) \tag{2.23}
\]

and \( X \) is a random variable which is independent of \((X_i)_{i \geq 1}\) and has the same distribution as \( X_0 \). We will show that for all \( j \geq r \), \( D_j^2 \leq_{\text{conv}} Y^2 \). First observe that since \( X_j \) is independent of the vector \((X_i)_{i \geq 1}^{j-1} \), it follows that \( D_j \) has the same distribution as

\[
D_j' := \frac{1}{N^{r/2}} \sum_{(i_1, \ldots, i_{r-1}) \in I_j^{-1}} h (X_{i_1}, \ldots, X_{i_{r-1}}, X), \quad j \geq r. \tag{2.24}
\]

Let \( \varphi: \mathbb{R}_+ \to \mathbb{R}_+ \) be a convex non-decreasing function. Then

\[
\mathbb{E} [\varphi (D_j^2)] = \mathbb{E} \left[ \varphi \left( D_j'^2 \right) \right]. \tag{2.25}
\]

Since

\[
D_j^2 = \frac{1}{N^{r/2}} U_j^2 \leq Y^2 \tag{2.26}
\]
we get from non-decreasingness of \( \varphi \) that \( \mathbb{E} \left[ \varphi \left( D_j^2 \right) \right] \leq \mathbb{E} \left[ \varphi \left( Y^2 \right) \right] \) hence in view of (2.25), it follows that \( D_j^2 \leq_{\text{conv}} Y^2 \). Writing

\[
P \left\{ \max_{r \leq n \leq N} \left| U_n \right| > N^{r/2} x \right\} = P \left\{ \max_{1 \leq n \leq N} \left| \sum_{j=1}^{n} D_j \right| > N^{1/2} x \right\}, \tag{2.27}
\]

we are in position to apply Proposition 2.3 with \( \bar{x} := x \) and \( \bar{y} := x^{1-1/r} y^{1/r} \). We obtain

\[
P \left\{ \max_{r \leq n \leq N} \left| U_n \right| > N^{r/2} x \right\} \leq 2 \exp \left( -\frac{1}{2} \left( \frac{x}{y} \right)^2 \right) + 2 \int_1^{+\infty} P \left\{ Y > x^{1-1/r} y^{1/r} (u/4)^{1/2} \right\} du. \tag{2.28}
\]

In order to control the last term, we use the fact that \( X \) is independent of the sequence \( (X_i)_{i \geq 1} \) hence

\[
P \left\{ Y > x^{1-1/r} y^{1/r} (u/4)^{1/2} \right\} = \int_S \int_{\max_{r \leq n \leq N}} \left( \sum_{(i_1, \ldots, i_{r-1}) \in I_k^{r-1}} h \left( X_{i_1}, \ldots, X_{i_{r-1}}, s \right) \right) \left| x^{1-1/r} y^{1/r} (u/4)^{1/2} \right) dP_X (s). \tag{2.29}
\]

We bound the probability inside the integral for each fixed \( s \). To this aim, we use the assumption that the assertion \( A(r-1) \) is true. Define for a fixed \( s \in S \) the function \( \bar{h} : S^{r-1} \to \mathbb{R} \) by

\[
\bar{h} (x_1, \ldots, x_{r-1}) = h \left( x_1, \ldots, x_{r-1}, s \right). \tag{2.30}
\]

Then \( \bar{h} \) satisfied the condition (1.4) with \( r \) replaced by \( r - 1 \). Let

\[
\bar{x} := x^{1-1/r} y^{1/r} (u/4)^{1/2}, \quad \bar{y} := y (u/4)^{1/2} (1 + 2 \ln (u))^{-(r-1)^{1/2}}. \tag{2.31}
\]

Since \( x/y \geq 3^r \), it follows that \( \bar{x}/\bar{y} = (x/y)^{1-1/r} (1 + 2 \ln (u))^{(r-1)^{1/2}} \geq (x/y)^{1-1/r} \geq 3^{r-1} \) hence we are in position to apply \( A(r-1) \), which gives

\[
P \left\{ \frac{1}{N^{r-1}} \max_{r-1 \leq k \leq n} \left( \sum_{(i_1, \ldots, i_{r-1}) \in I_k^{r-1}} h \left( X_{i_1}, \ldots, X_{i_{r-1}}, s \right) \right) > x^{1-1/r} y^{1/r} (u/4)^{1/2} \right\}
\begin{align*}
& \leq A_{r-1} \exp \left( -\frac{1}{2} \left( \frac{x}{y} \right)^2 \right) \\
& + B_{r-1} \int_1^{+\infty} P \left\{ \left| \bar{h} (X_1, \ldots, X_{r-1}) \right| > \bar{y} u^{1/2} C_{r-1} \right\} (1 + \ln (u))^{q_{r-1}} du. \tag{2.32}
\end{align*}
\]
Replacing $\tilde{x}$, $\tilde{y}$ and $\tilde{h}$ by their corresponding expression and integrating over $S$ with respect to the law of $X$ gives in view of (2.29) that

$$\mathbb{P}\{Y > x^{1/r}y^{1/r}(u/4)^{1/p}\} \leq A_{r-1} \exp\left(-\frac{1}{2}\left(\frac{x}{y}\right)^2 (1 + \ln (u))\right)$$

$$+ B_{r-1} \int_1^{+\infty} \mathbb{P}\{h(X_1, \ldots, X_{r-1}, X) > y/u^{1/2} (1 + \ln (u))^{-(r-1)/2} v^{1/2}C_{r-1}\} (1 + \ln (v))^{q-1} \mathrm{dv}. \quad (2.33)$$

We derive in view of (2.28) and the fact that $x/y > 3^r$ that

$$\int_1^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{x}{y}\right)^2 (1 + \ln (u))\right) \mathrm{du} \leq \exp\left(-\frac{1}{2}\left(\frac{x}{y}\right)^2 \right) \int_1^{+\infty} t^{-6} \mathrm{dt} \quad (2.34)$$

hence computing the integral,

$$\int_1^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{x}{y}\right)^2 (1 + \ln (u))\right) \mathrm{du} \leq \frac{1}{2} \exp\left(-\frac{1}{2}\left(\frac{x}{y}\right)^2 \right). \quad (2.35)$$

Integrating (2.33) with respect to $u$ on $(1, +\infty)$, we obtain

$$\mathbb{P}\left\{\max_{r \leq n \leq N} |U_n| > N^{r/2} x\right\} \leq (2 + A_{r-1}/2) \exp\left(-\frac{1}{2}\left(\frac{x}{y}\right)^2\right)$$

$$+ B_{r-1} \int_1^{+\infty} \int_1^{+\infty} \mathbb{P}\{h(X_1, \ldots, X_{r-1}, X) > y/u^{1/2} (1 + \ln (u))^{-(r-1)/2} v^{1/2}C_{r-1}\} (1 + \ln (v))^{q-1} \mathrm{dv} \mathrm{du}. \quad (2.36)$$

In order to control the last term, we will make a use of the following lemma.

**Lemma 2.4.** Let $X$ be a non-negative random variable and let $q, q'$ be non-negative numbers. Define

$$\kappa_q = \begin{cases} 1 & \text{if } q \leq 1 \\ e^{q-1}/q^q & \text{if } q > 1. \end{cases} \quad (2.37)$$

Then

$$\int_1^{+\infty} \int_1^{+\infty} \mathbb{P}\{X > u (1 + \ln (u))^{-q} v (1 + \ln (v))^{q'} \} (1 + \ln (v))^{q'} \mathrm{dv} \mathrm{du} \leq \frac{2q}{\kappa q_1} \left(1 + \ln \frac{\kappa q_1}{\kappa q_{q_1}}\right)^{q_1} \int_1^{+\infty} \mathbb{P}\{X > t/\kappa q_1\} (1 + \ln t)^{q_1 + q_2 + 1} \mathrm{dt}. \quad (2.38)$$

**Proof.** Define the function $g_q: u \mapsto u/(1 + \ln u)^{-q}$ for $u \geq 1$. Then $g_q'(u) = (1 + \ln u)^{-q} - uq(1 + \ln u)^{-q-1} \frac{1}{u} = (1 + \ln u)^{-q-1} (1 + \ln u - q)$ hence the function $g_q$ reaches its minimum at $u = e^{q-1}$. In particular,

$$(1 + \ln u)^{q} \leq \kappa_q u, u \geq 1, \quad (2.39)$$

where

$$\kappa_q = \begin{cases} 1 & \text{if } q \leq 1 \\ e^{q-1}/q^q & \text{if } q > 1. \end{cases} \quad (2.40)$$
We do the substitution $w = u (1 + \ln (u))^{-q_1} v$ for a fixed $u$. Then
\[
\int_{1}^{+\infty} \int_{1}^{+\infty} \mathbb{P} \left\{ X > u (1 + \ln (u))^{-q_1} v \right\} (1 + \ln (v))^q dv du
\]
\[
= \int_{1}^{+\infty} \int_{g_{q_1}(u)}^{+\infty} \mathbb{P} \{ X > w \} (1 + \ln (w/g_{q_1}(u)))^{q_2} g_{q_1}(u)^{-1} dw du. \tag{2.41}
\]
Since $g_{q_1}(u)^{-1} \leq \kappa_q$, it follows that
\[
\int_{1}^{+\infty} \int_{1}^{+\infty} \mathbb{P} \left\{ X > u (1 + \ln (u))^{-q_1} v \right\} (1 + \ln (v))^q dv du
\]
\[
\leq \int_{0}^{+\infty} \mathbb{P} \{ X > w \} I(w) (1 + \ln (w\kappa_{q_1}))^{q_2} dw \tag{2.42}
\]
where
\[
I(w) = \int_{1}^{+\infty} 1_{(0,w)} (g_{q_1}(u)) g_{q_1}(u)^{-1} du. \tag{2.43}
\]
Observe that if $w < 1/\kappa_{q_1}$, then $I(w) = 0$. Assume now that $w \geq 1/\kappa_{q_1}$. Using (2.39) with $q = 2q_1$, we get that $g_{2q_1}(u) \geq \kappa_{2q_1}^{-1}$ hence $g_{q_1}(u) \geq \kappa_{2q_1}^{-1} \sqrt{u}$ and it follows that
\[
1_{(0,w)} (g_{q_1}(u)) \leq 1_{(0,\kappa_{2q_1} w^2)} (u) \tag{2.44}
\]
hence
\[
I(w) \leq \frac{1}{q_1 + 1} (1 + 2 \ln (w\kappa_{2q_1}))^{q_1+1} \tag{2.45}
\]
hence by (2.42),
\[
\int_{1}^{+\infty} \int_{1}^{+\infty} \mathbb{P} \left\{ X > u (1 + \ln (u))^{-q_1} v \right\} (1 + \ln (v))^q dv du
\]
\[
\leq \frac{1}{q_1 + 1} \int_{1/\kappa_{q_1}}^{+\infty} \mathbb{P} \{ X > w \} (1 + 2 \ln (w\kappa_{2q_1}))^{q_1+1} (1 + \ln (w\kappa_{q_1}))^{q_2} dw
\]
\[
\leq 2^{q_1} \int_{1/\kappa_{q_1}}^{+\infty} \mathbb{P} \{ X > w \} (1 + \ln (w\kappa_{2q_1}))^{q_1+q_2+1} dw. \tag{2.46}
\]
We get (2.38) after the substitution $t = \kappa_{q_1} w$ and the elementary inequality $(1 + \ln (at))^q \leq (1 + \ln a)^q (1 + \ln t)^q$. This ends the proof of Lemma 2.4. 

To conclude the proof of Theorem 1.1, we apply (2.36) in the following setting:
- $X := 4 |h (X_1, \ldots, X_r)|^2 C_{r-1}^{-2} y^{-2}$;
- $q_1 = r - 1$;
- $q_2 = q_{r-1}$.

2.1.5. Proof of Corollary 1.8. We start from (1.16). We get that for each $r \leq n \leq N$,
\[
\frac{1}{N^{r-i/2}} |U_{r,n} (h)| \leq \sum_{k=1}^{r} \frac{1}{(r-k)!} N^{-k+i/2} |U_{k,n} (h_k)| \tag{2.47}
\]
and taking the maximum over $n \in \{r, \ldots, N\}$ gives
\[
\frac{1}{N^{r-i/2}} \max_{r \leq n \leq N} |U_{r,n} (h)| \leq \sum_{k=1}^{r} \frac{1}{(r-k)!} N^{-k+i/2} \max_{k \leq n \leq N} |U_{k,n} (h_k)|. \tag{2.48}
\]
It follows that
\[
\mathbb{P} \left\{ \max_{r \leq n \leq N} |U_n| > N^r \right\} \leq \sum_{r=1}^{\infty} \mathbb{P} \left\{ \frac{1}{(r-k)!} N^{r+1/2} \max_{k \leq n \leq N} |U_{k,n} (h_k)| > x/r \right\}.
\]

(2.49)
For all \(k \in \{i, \ldots, r\}\), the equality
\[
\mathbb{P} \left\{ \frac{1}{(r-k)!} N^{r+1/2} \max_{k \leq n \leq N} |U_{k,n} (h_k)| > x/r \right\} = \mathbb{P} \left\{ \frac{1}{N^{r+k/2}} \max_{k \leq n \leq N} |U_{k,n} (h_k)| > x \right\}
\]
holds, where \(\bar{x} = N^{(k-i)/2} x (r-k)!/r\).

We can therefore apply Theorem 1.1 in the setting

- \(\bar{x} = N^{(k-i)/2} x (r-k)!/r\);
- \(\bar{y} = N^{(k-i)/2} y (r-k)!/r\) and
- \(\bar{r} = k\)

to get the wanted result.

2.2. Proof of Corollary 1.9. We first use Corollary 1.8 with a \(y < 3^{-r} x\) that will be specified later. Observe that in view of Markov’s inequality,
\[
\int_{1}^{+\infty} \mathbb{P} \left\{ |Y_k| > y N^{(k-i)/2} \right\} (1 + \ln (u))^{q_{r,k}} du \leq I(y) \mathbb{E} \left[ \exp \left( |Y_k|^\gamma \right) \right],
\]

(2.51)
where
\[
I(y) := \int_{1}^{+\infty} \exp \left( - \left( y N^{(k-i)/2} u^{1/2} C_{r,2} \right)^\gamma \right) (1 + \ln (u))^{q_{r,k}} du.
\]

(2.52)
Now, noticing that \(\mathbb{E} \left[ \exp \left( |Y_k|^\gamma \right) \right] \leq c_\gamma \mathbb{E} \left[ \exp \left( |h (X_1, \ldots, X_r)|^\gamma \right) \right]\), it is sufficient to treat the case \(k = i\).

Doing the substitution \(v = u^{1/2} - 1\), we obtain
\[
I(y) = \frac{2}{\gamma} \int_{1}^{+\infty} \exp \left( -C_{r,2} y^{2/3} (v + 1) \right) \left( 1 + \ln \left( (v + 1)^{2/\gamma} \right) \right)^{q_{r,k}} (1 + v)^{\frac{2}{\gamma} - 1} dv.
\]

(2.53)
Assume that \(y\) is such that
\[
y^{2} C_{r,2}^{1/2} \geq 1.
\]

(2.54)
Then
\[
I(y) \leq \exp \left( -RC_{r,2} y^{\gamma} \right) \cdot \frac{2}{\gamma} \int_{1}^{+\infty} \exp \left( -v \right) \left( 1 + \ln \left( (v + 1)^{2/\gamma} \right) \right)^{q_{r,k}} (1 + v)^{\frac{2}{\gamma} - 1} dv.
\]

(2.55)
Now, we choose \(y < 3^{-r} x\) satisfying (2.54) such that
\[
\frac{1}{2} \left( \frac{x}{y} \right)^{2/\gamma} \leq C_{r,2} y^{\gamma},
\]

(2.56)
This leads to the choice
\[
y := x^{2/\gamma} 2^{-\frac{\gamma+2}{\gamma+2}/r} C_{r,2}^{-\frac{\gamma}{\gamma+2}/i}.
\]

(2.57)
Due to the definition of \(x_{r,1}\), the inequalities \(x/y \geq 3^r\) and (2.54) hold. This choice of \(y\) combined with the observation that \(1 \leq \mathbb{E} \left[ \exp \left( |h (X_1, \ldots, X_r)|^\gamma \right) \right] \) gives (1.18), which ends the proof of Corollary 1.9.
2.3. Proof of Theorem 1.12. Let $\varepsilon > 0$. Observe that for all positive $C$,
\[
\mathbb{P} \left\{ \max_{r \in \mathbb{N}} |U_n| > \varepsilon 2^{Na} \right\} = \mathbb{P} \left\{ \max_{r \in \mathbb{N}} \left| \frac{CU_n}{\varepsilon} \right| > 2^{N(r-i/2)} C 2^N(a-(r-i/2)) \right\}. \tag{2.58}
\]
We now choose $C$ such that $\frac{2^{Na}}{C \varepsilon r} B_{r,1} \geq 1$, where $B_{r,1}$ is like in Corollary 1.9. We then apply for Corollary 1.9 in the setting $h := Ch/\varepsilon$, $x = C 2^N(a-(r-i/2))$ and for $N$ such that $C 2^N(a-(r-i/2)) \geq x_{r,i,1}$, with $\tilde{N} = 2^N$.

2.4. Proof of Theorem 1.13. We apply Corollary 1.8 with $x$ replaced by $x N^{i/2}$. This gives
\[
\mathbb{P} \left\{ \max_{r \leq n \leq N} |U_n| > N^r x \right\} \leq A_r \exp \left( -\frac{1}{2} \left( \frac{x N^{i/2}}{y} \right)^{\frac{2}{\gamma}} \right)
+ B_r \sum_{k=1}^r \int_1^{+\infty} \mathbb{P} \left\{ |E (h (X_1, \ldots, X_r) | |X_1, \ldots, X_k)| > y N^{(k-i)} 2^{i/2} C_r \right\} (1 + \ln (u))^{\gamma u} \, du. \tag{2.59}
\]
Using (2.9), we notice that $\sup_{t>0} \exp(t) \mathbb{P} \left\{ |E (h (X_1, \ldots, X_r) | X_1, \ldots, X_k)| > t \right\} \leq \kappa \cdot M$, where $\kappa$ depends only on $\gamma$. Therefore, we will only focus on the term associated to $k = i$ in the right hand side of (2.59). After having used the condition on the tail, we obtain
\[
\mathbb{P} \left\{ \max_{r \leq n \leq N} |U_n| > N^r x \right\} \leq A_r \exp \left( -\frac{1}{2} \left( \frac{x N^{i/2}}{y} \right)^{\frac{2}{\gamma}} \right)
+ B_{r,1} M \int_1^{+\infty} \exp \left( -\left( \frac{1}{2} u^{1/2} C_r \right) \gamma \right) (1 + \ln (u))^{\gamma u} \, du. \tag{2.60}
\]
Then using similar arguments as in the control of $I (y)$ defined by (2.52), we get
\[
\mathbb{P} \left\{ \max_{r \leq n \leq N} |U_n| > N^r x \right\} \leq A_r \exp \left( -\frac{1}{2} \left( \frac{x N^{i/2}}{y} \right)^{\frac{2}{\gamma}} \right)
+ B_{r,1} M \exp \left( -C_{r,1} y^{\gamma} \right). \tag{2.61}
\]
We conclude the proof by choosing $y = N^{2/\gamma} x^{2/\gamma}$.

2.5. Proof of the results of Subsection 1.5.

Proof of Proposition 1.14. First, by using Theorem 3 in [22] (which is a consequence of the Schauder decomposition of $H_\rho^2$ and the tightness criterion given in [24]), the following condition is sufficient for tightness of a sequence of processes $(\xi_n)_{n \geq 1}$ in $H_\rho^2$:
\[
\forall \varepsilon > 0, \lim_{j \to +\infty} \limsup_{n \to +\infty} \mathbb{P} \left\{ \sup \max_{j \geq J} \left| \lambda_{j,t} (\xi_n) \right| / \rho (2^{-j}) > \varepsilon \right\} = 0, \tag{2.62}
\]
where for $j \geq J$, $D_j = \{(k+1) 2^{-j}, 0 \leq k \leq 2^j - 1\}$ and
\[
\lambda_{j,t} (x) = x (t) - \frac{1}{2} \left( x (t+2^{-j}) - x (t-2^{-j}) \right), x \in H_\rho^2, t \in D_j. \tag{2.63}
\]
Since
\[
\left| \lambda_{j,t} (\xi_n) \right| \leq \frac{1}{2} \left| x (t) - x (t+2^{-j}) \right| + \frac{1}{2} \left| x (t) - x (t-2^{-j}) \right|, \tag{2.64}
\]
we infer that
\[
\max_{t \in D_j} \left| \lambda_{j,t} (\xi_n) \right| \leq \max_{1 \leq \ell \leq 2^j} \left| \xi_n \left( (\ell-1) 2^{-j} \right) - \xi_n (\ell 2^{-j}) \right|. \tag{2.65}
\]
and we are therefore reduced to prove that

\[ \forall \varepsilon > 0, \lim_{J \to +\infty} \limsup_{n \to +\infty} \mathbb{P}\left( \sup_{j \geq J} \max_{1 \leq \ell \leq 2^{j}} \left| W_n\left( \ell 2^{-j}\right) - W_n\left( (\ell - 1) 2^{-j}\right) \right| / \rho\left( 2^{-j}\right) > \varepsilon \right) = 0. \quad (2.66) \]

We will now control the differences \( |W_n\left( \ell 2^{-j}\right) - W_n\left( (\ell - 1) 2^{-j}\right)| \) by exploiting the fact that the graph of \( W_n \) is a polygonal line. Fix an integer \( n \) and define the interval \( I_k := [(k-1)/n, k/n], 1 \leq k \leq n \). Define

\[ R_n := \max_{1 \leq k \leq n} |W_n\left( k/n\right) - W_n\left( (k-1)/n\right)|. \quad (2.67) \]

Let \( 0 \leq s < t \leq 1 \).

1. There exists a \( k \in \{1, \ldots, n\} \) such that \( s, t \in I_k \). Since on \( I_k \), \( W_n \) is affine, we derive

\[
|W_n(t) - W_n(s)| \leq n(t - s)|W_n\left( k/n\right) - W_n\left( (k-1)/n\right)|
\leq n(t - s)R_n.
\]

(2.69)

2. There exists a \( k \in \{1, \ldots, n-1\} \) such that \( s \in I_k \) and \( t \in I_{k+1} \). Starting from

\[
|W_n(t) - W_n(s)| \leq |W_n(t) - W_n\left( k/n\right)| + |W_n\left( k/n\right) - W_n\left( (k-1)/n\right)|
\]

and applying the reasoning of the first case to treat the two terms, we get

\[
|W_n(t) - W_n(s)| \leq n(t - s)R_n.
\]

(2.70)

3. There exists a \( k \in \{1, \ldots, n\} \) such that \( s \in I_k \) and \( j \in \{k+2, \ldots, n\} \) such that \( t \in I_j \). We start from

\[
|W_n(t) - W_n(s)| \leq \left| W_n\left( \frac{j-1}{n}\right) - W_n\left( \frac{j-1}{n}\right)\right| + \left| W_n\left( \frac{j-1}{n}\right) - W_n\left( \frac{k}{n}\right)\right| + \left| W_n\left( \frac{k}{n}\right) - W_n(s)\right|.
\]

(2.71)

For the first and third terms of the right hand side, we use the reasoning of case 1 to get that their contribution does not exceed \( 2R_n \). The second term is \( |S_{j-1} - S_k|/a_n \)

hence

\[
|W_n(t) - W_n(s)| \leq 3R_n + \frac{|S_{[nt]} - S_{[ns]}|}{a_n}.
\]

(2.72)

Suppose that \( j \geq \lfloor \log_2 n \rfloor + 1 \) and let \( t = \ell 2^{-j} \) and \( s = (\ell - 1) 2^{-j} \). Then \( t - s = 2^{-j} < 1/n \) hence only the first two cases are possible. Consequently,

\[
\sup_{j \geq \lfloor \log_2 n \rfloor + 1} \max_{1 \leq \ell \leq 2^j} \left| W_n\left( \ell 2^{-j}\right) - W_n\left( (\ell - 1) 2^{-j}\right) \right| / \rho\left( 2^{-j}\right) \leq \sup_{j \geq \lfloor \log_2 n \rfloor + 1} n2^{-j}R_n/\rho\left( 2^{-j}\right).
\]

(2.73)

Now, exploiting the fact that \( \rho\left( u\right) = u^\alpha L\left( 1/u\right) \), we infer that for some constant \( C \) depending only on \( \alpha \) and \( L \),

\[
\sup_{j \geq \lfloor \log_2 n \rfloor + 1} \max_{1 \leq \ell \leq 2^j} \left| W_n\left( \ell 2^{-j}\right) - W_n\left( (\ell - 1) 2^{-j}\right) \right| / \rho\left( 2^{-j}\right) \leq CR_n/\rho\left( 1/n\right).
\]

(2.74)
Let now \( j \in \{J, \ldots, \lfloor \log_2 n \rfloor \} \). This time, with the choices \( t = \ell 2^{-j} \) and \( s = (\ell - 1) 2^{-j} \), the third case applies hence

\[
\max_{J \leq j < \lfloor \log_2 n \rfloor} \max_{1 \leq \ell \leq 2^j} |W_n (\ell 2^{-j}) - W_n ((\ell - 1) 2^{-j})| / \rho (2^{-j})
\leq 3 \max_{J \leq j < \lfloor \log_2 n \rfloor} \rho (2^{-j})^{-1} R_n + \max_{J \leq j < \lfloor \log_2 n \rfloor} \rho (2^{-j})^{-1} \max_{1 \leq \ell \leq 2^j} \frac{1}{a_n} |S_{\lfloor n\ell 2^{-j} \rfloor} - S_{\lfloor n(\ell - 1)2^{-j} \rfloor}|.
\]

(2.76)

In total, we got that for a constant \( C \) depending only on \( \rho \),

\[
\sup_{j \geq J} \max_{1 \leq \ell \leq 2^j} |W_n (\ell 2^{-j}) - W_n ((\ell - 1) 2^{-j})| / \rho (2^{-j})
\leq C \max_{J \leq j < \lfloor \log_2 n \rfloor} \rho (2^{-j})^{-1} R_n + \max_{J \leq j < \lfloor \log_2 n \rfloor} \rho (2^{-j})^{-1} \max_{1 \leq \ell \leq 2^j} \frac{1}{a_n} |S_{\lfloor n\ell 2^{-j} \rfloor} - S_{\lfloor n(\ell - 1)2^{-j} \rfloor}|.
\]

(2.77)

Since (1.30) guarantees that

\[
\lim_{J \to +\infty} \limsup_{n \to +\infty} P \left\{ \max_{J \leq j < \lfloor \log_2 n \rfloor} \rho (2^{-j})^{-1} \max_{1 \leq \ell \leq 2^j} \frac{1}{a_n} |S_{\lfloor n\ell 2^{-j} \rfloor} - S_{\lfloor n(\ell - 1)2^{-j} \rfloor}| > \varepsilon \right\} = 0,
\]

(2.78)

it remains to check that the sequence \( \left( \max_{1 \leq j \leq \lfloor \log_2 n \rfloor} \rho (2^{-j})^{-1} R_n \right)_{n \geq 1} \) converges to 0 in probability. Due to the construction of \( W_n \) and the assumptions on the sequence \( (a_n)_{n \geq 1} \) and \( \rho \), it suffices to check that the convergence in probability of \( (R_{2^N}/\rho (2^{-N}))_{N \geq 1} \). To this aim, notice that (1.30) implies (by considering \( n = 2^N \)) that for each positive \( \varepsilon \),

\[
\lim_{N \to +\infty} \sum_{k=0}^{2^N - 1} P \left\{ |S_{\lfloor 2^N (k+1)2^{-N} \rfloor} - S_{\lfloor 2^N k2^{-N} \rfloor}| > a_{2^N} \varepsilon \rho (2^{-N}) \right\} = 0,
\]

(2.79)

which implies the convergence in probability to 0 of \( R_{2^N}/\rho (2^{-N}) \). This ends the proof of Proposition 1.14.

**Proof of Proposition 1.15.** We start from the equalities

\[
\frac{1}{\sqrt{n_2 - n_1 n_2}} \left( U_{n_2} - U_{n_1} \right) = \frac{1}{\sqrt{n_2 - n_1 n_2}} \sum_{1 \leq i_1 < \cdots < i_r \leq n_2} h (X_{i_1}, \ldots, X_{i_r})
\]

(2.80)

\[
= \frac{1}{\sqrt{n_2 - n_1}} \sum_{j=n_1+1}^{n_2} D_j,
\]

(2.81)

where

\[
D_j = \frac{1}{n_2} \sum_{1 \leq i_1 < \cdots < i_{r-1} < j} h (X_{i_1}, \ldots, X_{i_{r-1}, j}).
\]

(2.82)

Define the random variable \( Y \) as

\[
\frac{1}{n_2^{(r-1)/2}} \max_{r-1 \leq j \leq n_2} \sum_{1 \leq i_1 < \cdots < i_{r-1} < j} h (X_{i_1}, \ldots, X_{i_{r-1}, j}, X).
\]

(2.83)
where \( X \) is independent of the sequence \((X_i)_{i \geq 1}\) and has the same distribution as \( X \). Then in the same way as in the proof of Theorem 1.1, we can prove that \((D_j)_{j=n+1}^{n_2}\) is a martingale differences sequence and that \(D_j^2 \leq \text{conv} Y^2\) for all \( j \in \{n_1 + 1, \ldots, n_2\} \). Applying Proposition 2.3, we derive that

\[
P \left\{ \frac{1}{\sqrt{n_2 - n_1 n_2^{r-1}}} |U_{n_2} - U_{n_1}| > x \right\} \leq 2 \exp \left( -\frac{x^2}{2 y^2} \right) + 2 \int_1^{+\infty} P \left\{ \frac{1}{n_2^{(r-1)/2}} \max_{1 \leq i_1 < \cdots < i_{r-1} < j} \sum_{1 \leq i_1 < \cdots < i_{r-1} < j} h(X_{i_1}, \ldots, X_{i_{r-1}}, X) > y\sqrt{u/2} \right\} du.
\]

(2.84)

Then we treat the last integral in the following way: we integrate with respect to the law of \( X \), apply Theorem 1.1 to the \( U \)-statistics of order \( r - 1 \) and rearrange the integrals. \( \square \)

**Proof of Theorem 1.16.** The convergence of the finite dimensional distributions follows from Corollary 1 in [19] and the fact that for a fixed \( t \), the contribution of \( n^{-i/2} (nt - [nt]) (U_{[nt]+1} - U_{[nt]}) \) is negligible.

It remain to prove tightness in \( \mathcal{H}_\rho^o \). To this aim, we apply the Hoeffding’s decomposition (1.16) and it suffices to show the following.

**Lemma 2.5.** Let \( r \geq 1 \) be an integer, \((S, \mathcal{S})\) be measurable space, \( h: S^r \to \mathbb{R} \) be a symmetric measurable function (with \( S^r \) induced with the product \( \sigma \)-algebra) and let \((X_i)_{i \geq 1}\) be an i.i.d. sequence of \( S \)-valued random variables. Assume that \( h \) is degenerated. Let \( \rho \in \mathcal{R}_r \). Suppose that (1.32) holds. Then

\[
\left( \frac{1}{n^{r/2}} (U_{[nt]} - (nt - [nt]) (U_{[nt]+1} - U_{[nt]}) \right)_{n \geq 1}
\]

is tight in \( \mathcal{H}_\rho^o \).

Let us first show how Lemma 2.5 helps to conclude. After having applied the Hoeffding decomposition, the considered partial sum is a sum of partial sum process defined like in (1.23) but with \( U \)-statistics of lower order and overall degenerated, with a stronger normalization than \( n^{k/2} \) for each term of the sum in (1.16). This shows tightness of the initially considered process and concludes the proof.

**Proof of Lemma 2.5.** By an application of Proposition 1.14, it suffices to show that for all positive \( \varepsilon \),

\[
\lim_{J \to +\infty} \limsup_{n \to +\infty} \sum_{j=J}^{[\log_2 n]} \sum_{k=0}^{2^j - 1} P \left\{ \left| U_{[n(k+1)2^{-j}]} - U_{[nk2^{-j}]} \right| > n^{r/2} \varepsilon \rho \left( 2^{-j} \right) \right\} = 0.
\]

(2.86)

In order to control the involved probabilities, we apply Proposition 1.15 for fixed \( n, J, j \in \{J, \ldots, [\log_2 n]\} \) and \( k \in \{0, \ldots, 2^j - 1\} \) in the following setting: \( n_1 = [nk2^{-j}], n_2 = [n (k+1) 2^{-j}] \) and

\[
x := \frac{n^{r/2} \varepsilon \rho \left( 2^{-j} \right)}{\sqrt{[n (k+1) 2^{-j}] - [nk2^{-j}] [n (k+1) 2^{-j}]}}.
\]

(2.87)
Then we choose $y$ such that
\[
\left( \frac{x}{y} \right)^{2/r} = 3j \ln 2.
\] (2.88)

Observing that $x \geq c2^{j/2} \rho \left( 2^{-j} \right)$, we derive that
\[
P \left\{ \left| U_{\lfloor n(k+1)2^{-j} \rfloor} - U_{\lfloor nk2^{-j} \rfloor} \right| > n^{r/2} \varepsilon \rho \left( 2^{-j} \right) \right\} \leq A_r \rho^{-3/2} + B_r \int_1^{+\infty} P \left\{ \left| h(X_1, \ldots, X_r) \right| > c2^{j/2} \rho \left( 2^{-j} \right) j^{-r/2} u^{1/2} \right\} \left( 1 + \log u \right)^{q_r} du.
\] (2.89)

We conclude by summing over $j \geq J$ and exploiting the convergence of the involved series.

Now, it remains to show that conditions (1.34) (respectively (1.35)) are sufficient in the case $\rho(t) = t^\alpha$ (respectively $\rho(t) = t^{1/2} (\log (c/t))^{\beta}$).

When $\rho(t) = t^\alpha$, we first use the fact that if $a$ and $b$ are two positive real numbers and $Z$ is a non-negative random variable, then
\[
\sum_{j \geq 1} 2^j P \left\{ Z > \frac{2^j a}{(1+j)^b} \right\} \leq C_{a,b} E \left[ Z^{1/a} (1+Z)^b 1 \{ Z > 1 \} \right].
\] (2.90)

This can be seen by cutting the tail probability on a sum of probabilities that $Z$ lies in $(c_j, c_{j+1}]$, where $c_j = \frac{2^j a}{(1+j)^b}$ then switching the sums. Applying this to $Z = Y/u^{1/2}$ for a fixed $v$, $a = 1/2 - \alpha$ and $b = r/2$, bounding the logarithms by $\left( 1 + \log Y \right)^{r/2} = (1 + \log u)^{q_r}$ and accounting $1/(1/2 - \alpha)$ gives (1.32).

When $\rho(t) = t^{1/2} (\log (c/t))^{\beta}$, this follows from similar estimates as in the proof of Corollary 1.9. This concludes the proof of Theorem 1.16.

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