HIGGS PHYSICS AND THE EQUIVALENCE THEOREM

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ABSTRACT

The equivalence theorem is an extremely useful tool to calculate heavy Higgs and top-quark effects for processes that have center-of-mass-energies (much) larger than the W boson mass. After an explanation of the renormalization procedures involved, the results for one- and two-loop radiative corrections to the fermionic Higgs decay, \( H \rightarrow f \bar{f} \), are given and discussed. Finally, the renormalization scheme dependence is examined, and the reliability of the perturbative series is investigated.

1. Introduction

At LEP I and LEP II, heavy Higgs mass effects are suppressed according to Veltman’s screening theorem.\[1\] However, machines like the LHC and possibly NLC will investigate processes, in which the presence of a Higgs with sufficiently large mass \( M_H \) could cause large nonperturbative effects. The reason is the proportionality of the Higgs quartic coupling \( \lambda \) to \( M_H^2 \), which in perturbative treatments leads to radiative corrections that contain powers of \( M_H \) rather than a logarithmic mass dependence.

It is of interest to study the apparent breakdown of perturbation theory, and to put upper limits on the mass of a weakly interacting Higgs boson. Beyond such an upper mass limit, the perturbative cross sections for, e.g., LHC processes like \( W^+W^- \) scattering become rapidly unreliable.

The present article describes the systematics of calculating heavy-Higgs-mass effects by using the Goldstone boson equivalence theorem.\[3\] The usual Lagrangian of the symmetry-breaking sector is used to calculate the heavy-Higgs-mass corrections in the limit of \( M_H \gg M_W \). Yukawa couplings are kept without violating the Goldstone theorem.\[4\] We obtain a Lagrangian which implements both heavy-Higgs-mass effects and large top-quark-mass corrections. Using a one-loop calculation we show that this Lagrangian reproduces the full Standard-Model electroweak corrections to the decay \( H \rightarrow tt \) in extremely good approximation. Finally, we discuss the leading two-loop correction to \( H \rightarrow ff \), and we conclude with remarks on effects due to the use of different renormalization schemes.

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2. The Goldstone Boson Equivalence Theorem (EQT)

The equivalence theorem (EQT) is usually discussed in the context of scattering processes involving the weak gauge bosons $W^+, W^-, Z$. We will outline the EQT along these lines, with the end of this section being devoted to the application of the EQT to the process $H \to f\bar{f}$.

In the case of scattering processes with gauge bosons, the scattering amplitude of the longitudinally polarized gauge bosons $W^{\pm}_L, Z_L$ and the Higgs boson $H$ are enhanced by factors of $M_H^2/M_W^2 \propto \lambda/g^2$ relative to those which involve transversely polarized gauge bosons and the small electroweak gauge couplings $g$. Next we observe that in momentum space the longitudinal component of the vector boson fields is related to the Goldstone boson fields by

$$W^{\pm}_L(k) = e^\mu_L(k)W^\pm_\mu = w^\pm(k) + O\left(\frac{M_W}{k_0}\right),$$

where $e^\mu$ is the polarization vector, and $k_0$ is the energy component of the four-momentum $k$.

The Goldstone boson equivalence theorem $^2,^3,^6$–$^10$ states that in the limit of a large center-of-mass energy, $\sqrt{s} \gg M_W$, the scattering amplitudes for $n$ longitudinally polarized vector bosons $W^{\pm}_L, Z_L$ and any number of other external particles (including Higgs particles) are related to the corresponding scattering amplitudes for the scalar Goldstone bosons $w^\pm, z$ (to which $W^{\pm}_L, Z_L$ reduce for vanishing electroweak gauge couplings $g$) by

$$T(W^{\pm}_L, Z_L, H, \ldots) = (iC)^nT(w^\pm, z, H, \ldots) + O(M_W/\sqrt{s}).$$

The constant $C$ depends on the renormalization scheme used in the calculation $^8,^9,^10$,

$$C = \frac{M_W^0}{M_W} \left(\frac{Z_W}{Z_w}\right)^{1/2} \left[1 + O(g^2)\right],$$

where the $Z$'s are the wavefunction renormalization constants for the physical fields $W^{\pm}$ and the scalar fields $w^{\pm}$. $C$ is equal to unity for electroweak couplings $g \to 0$ in schemes in which the renormalization constants are defined at mass scales $m \ll M_H$.

We choose to renormalize the $w^{\pm}, z$ fields at $p^2 = m^2 = 0$, a choice which corresponds to massless Goldstone bosons. Then $^8,^9,^10$,

$$C = 1 + O(g^2).$$

In the limit of a heavy Higgs boson, $M_H \gg M_W$, the coupling $g$ is much smaller than the Higgs coupling $\lambda$: $M_W^2/M_H^2 \propto g^2/\lambda \ll 1$, and the gauge couplings can be
neglected. In this approximation, the constant $C$ is equal to unity. Since we started with the assumption $\sqrt{s} \gg M_W$, we obtain the result
\[
\sqrt{s}, M_H \gg M_W \quad \approx \quad T(w^\pm, z, H, \ldots),
\]
where the amplitude on the right-hand-side only depends on the quartic Higgs coupling $\lambda$ and the Yukawa couplings $g_f$, and only involves scalar and fermion fields.\(^a\) (We neglect the QCD sector of the Standard Model throughout this paper.) This makes the use of the equivalence theorem an excellent and easy-to-use approximation.

The equivalence theorem can also be applied in processes that have no external electroweak gauge bosons but receive leading radiative corrections through loops involving $W^+, W^-, Z, or H$. Again, the EQT amplitudes will be a good approximation as long as $\sqrt{s} \gg M_W$ and $M_H \gg M_W$. E.g., in the case of the decay of the Higgs particle, the center-of-mass energy is identical to $M_H$. Therefore, in the limit of $M_H \gg M_W$, the EQT is expected to be an excellent approximation.

Quantitatively we find that a Higgs mass of 400 GeV is sufficiently heavy. This result is based on a comparison of the one-loop result for the Standard-Model decay $H \to t\bar{t}$ based on our equivalence-theorem calculation including Yukawa couplings and the corresponding full electroweak one-loop calculation. For $M_H > 400$ GeV the two results agree to better than 96% for $m_t = 174$ GeV.

3. The Lagrangian consistent with the EQT

All the physics connected with the Higgs particle is determined by the Lagrangian of the Standard Model, the starting point of our EQT calculations. We begin by defining the full Lagrangian for the symmetry-breaking sector of the Standard Model. It is given by
\[
\mathcal{L}_{SB} = \frac{1}{2}(D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{\lambda}{4} (\Phi^\dagger \Phi)^2 + \frac{\mu^2}{2} (\Phi^\dagger \Phi),
\]
where a positive value of $\mu^2$ shifts the minimum of the potential to a non-zero vacuum expectation value of the field $\Phi$. The covariant derivative is defined as $D^\mu = \partial^\mu + igW^\mu \cdot T + \frac{1}{2}g' B^\mu$. The complex field $\Phi$ is written in terms of four real scalar fields,
\[
\Phi = \begin{pmatrix} w_1 + iw_2 \\ h + iw_3 \end{pmatrix}.
\]

The gauge couplings $g$ and $g'$ allow for the interaction of the Higgs sector with the electroweak gauge sector of the Standard Model. The use of the Goldstone boson equivalence theorem corresponds to calculating the physical observables of interest in the limit of $g, g' \to 0$. This reduces the above Lagrangian to a SO(4)-symmetric Lagrangian involving only the scalar fields $h$ and $w = (w_1, w_2, w_3)$.

\(^a\) Note that in the limit of zero gauge couplings the internal electroweak gauge bosons are also replaced by massless scalar Goldstone bosons.
The field $h$ is taken as usual as the component of $\Phi$ which acquires a vacuum expectation value, $v$. This spontaneously breaks the $SO(4) \simeq SU(2) \times SU(2)$ symmetry of the doublet $\Phi$. By virtue of the Goldstone theorem, the spontaneous breaking of the $SO(4)$ symmetry leads to three massless Goldstone bosons. We write $h$ as $h = v + H$ where $\langle \Omega | H | \Omega \rangle = 0$ with respect to the physical vacuum $| \Omega \rangle$. This allows for a perturbative calculation by expanding the Higgs field $H$ and the Goldstone boson fields $w_1, w_2, w_3$ around zero field strength. We obtain the new Lagrangian

\[
\mathcal{L}_H = \frac{1}{2} (\partial_\mu H)^\dagger (\partial^\mu H) + \frac{1}{2} (\partial_\mu w)^\dagger (\partial^\mu w) - \frac{\lambda}{4} (w^4 + 2w^2H^2 + H^4) - \lambda v (w^2H + H^3) - \frac{1}{2} H^2 (3\lambda v^2 - \mu^2) - \frac{1}{2} w^2 (\lambda v^2 - \mu^2) - Hv (\lambda v^2 - \mu^2). \tag{8}
\]

As usual, the tree level relationship $\mu^2 = \lambda v^2$ guarantees vanishing tadpole contributions and massless Goldstone bosons.

In addition to the above Lagrangian, the doublet $\Phi$ also interacts with the left- and right-handed fermion fields, $\psi_{f}^{L,R}$, of the Standard Model, with the strength of the interactions defined by the Yukawa couplings of the theory. This gives the second contribution to our EQT Lagrangian. Choosing the top-bottom-quark generation as an example, the Lagrangian $\mathcal{L}_F$ governing the interactions between the doublet $\Phi$ and the fermion fields is

\[
\mathcal{L}_F^{(t,b)} = - \left( \frac{gb}{\sqrt{2}} \right) \left( \begin{array}{c} \psi_t^L \\ \psi_b^L \end{array} \right) \Phi \psi_b^R + \text{h.c.} - \left( \frac{gt}{\sqrt{2}} \right) \left( \begin{array}{c} \psi_t^L \\ \psi_b^L \end{array} \right) i\tau_2 \Phi^* \psi_t^R + \text{h.c.} \right). \tag{9}
\]

Here $g_t$, $g_b$ are the top- and bottom-quark Yukawa couplings, and $\tau_2$ is the complex Pauli matrix such that $i\tau_2 \Phi^*$ is the charge conjugate of $\Phi$. The complete EQT Lagrangian is now defined as

\[
\mathcal{L}_{\text{EQT}} = \mathcal{L}_H + \mathcal{L}_F. \tag{10}
\]

For a zero vacuum expectation value $v$ of the doublet $\Phi$, the presence of the fermion fields leads to a $SU(2) \times U(1)$ symmetry of the complete Lagrangian $\mathcal{L}_{\text{EQT}}$, $(g_b \neq g_t)$, or respects the chiral $SU(2) \times SU(2)$ symmetry $(g_b = g_t)$. The appearance of a non-zero vacuum expectation value spontaneously breaks the symmetry, leading to the presence of three massless Goldstone bosons (Goldstone’s theorem) even for non-zero Yukawa couplings. Each fermion receives a mass that is proportional to the product of its Yukawa coupling and the vacuum expectation value, $m_f = g_f v / \sqrt{2}$. Depending on the value of the Yukawa couplings, the interaction of the three Goldstone bosons with fermions may either be $SO(3)$-symmetric $(g_t = g_b)$, or

\[b\text{It is only the inclusion of the gauge sector that gives masses to the } W \text{ and } Z \text{ bosons.}\]
it might be broken \((g_t \neq g_b)\). The latter case features a residual symmetry of the charge-conjugate fields \(w^+ = (w_1 - i w_2)/\sqrt{2}\) and \(w^- = (w_1 + i w_2)/\sqrt{2}\).

The Lagrangian \(\mathcal{L}_F\) for any other quark doublet \((q, q')\) of the Standard Model can be obtained by making the substitution \((t, b) \leftrightarrow (q, q')\) in the above expression for \(\mathcal{L}_F\). For a lepton doublet \((\nu, l)\), one also can make the substitution \((t, b) \leftrightarrow (\nu, l)\); however, one has to keep in mind that the Standard Model doesn’t provide for right-handed neutrino fields: \(\psi_R^\nu = 0\) for all lepton flavors.

4. Renormalization of the Lagrangian

The Lagrangian \(\mathcal{L}_{\text{EQT}} = \mathcal{L}_H + \mathcal{L}_F\) provides all the information necessary to calculate Green’s functions and \(S\)-matrix elements. To include quantum corrections, we start by renormalizing \(\mathcal{L}_H\), introducing all possible counterterms that respect the unbroken \(\text{SO}(4)\) symmetry of the Lagrangian, see Eq. (8). This leads to the introduction of the \(\text{SO}(4)\)-symmetric wavefunction renormalization \(Z_\phi\), and the counterterms \(\delta \lambda\) and \(\delta \mu^2\). These quantities are sufficient to guarantee a finite theory even in the broken phase, Eq. (8). In addition, we allow for finite field renormalization constants, \(\tilde{Z}_H\), \(\tilde{Z}_z\), and \(\tilde{Z}_w\), to properly normalize the physical fields of the broken phase (OMS renormalization). The renormalization can be summarized as

\[
\begin{align*}
\lambda &\to \lambda + \frac{\delta \lambda}{Z_\phi^2}, \\
\mu^2 &\to \frac{\mu^2 - \delta \mu^2}{Z_\phi}, \\
v &\to \tilde{Z}_\phi^{1/2}v, \\
H &\to \tilde{Z}_H^{1/2}Z_\phi^{1/2}H, \\
z &\to \tilde{Z}_z^{1/2}Z_\phi^{1/2}z, \\
w^\pm &\to \tilde{Z}_w^{1/2}Z_\phi^{1/2}w^\pm,
\end{align*}
\]

The renormalized Lagrangian in terms of the physical fields is

\[
\mathcal{L}_{H,\text{ren}} = -\frac{1}{2} \tilde{Z}_H H^2 \left(3(\lambda + \delta \lambda)v^2 - (\mu^2 - \delta \mu^2)\right) \\
- \frac{1}{2} (\tilde{Z}_w w^+ w^- + \tilde{Z}_z z^2) \left((\lambda + \delta \lambda)v^2 - (\mu^2 - \delta \mu^2)\right) \\
- \tilde{Z}_H^{1/2} H v \left((\lambda + \delta \lambda)v^2 - (\mu^2 - \delta \mu^2)\right) \\
+ \text{interaction terms},
\]

The coefficient of the term linear in the field \(H\) is fixed as to cancel tadpole contributions to the Higgs one-point function order by order in perturbation theory. This fixes \(v\) to be the vacuum expectation value to all orders, \(i.e.\ \langle \Omega | H | \Omega \rangle = 0\) to all orders. At tree level, we require \(\lambda v^2 = \mu^2\).

It should be noted that both \(w^\pm\) and \(z\) fields have the same mass coefficient and mass counterterm, and the counterterm structure is identical to the coefficient of the linear Higgs term. In the presence of Yukawa interactions the self-energies of the Goldstone bosons yield \(\Pi_z(p^2) \neq \Pi_{w^\pm}(p^2)\) for arbitrary values of \(p^2\). It seems
impossible to cancel tadpole contributions and simultaneously keep all Goldstone fields massless. However, an explicit calculation shows that \( \Pi_z(0) = \Pi_{w^\pm}(0) = T/v \), where \( T \) is the tadpole term. Hence, the OMS renormalization can be used without violating the validity of the Goldstone theorem, i.e., \( \langle |H| \rangle = 0 \) while the Goldstone bosons remain massless at higher orders in perturbation theory.

Next we fix the mass term of the Higgs field. At tree level, using \( \lambda v^2 = \mu^2 \), we find the Higgs mass to be \( M_H^2 = 2\lambda v^2 \). Conversely, this equation defines the Higgs coupling \( \lambda \) in terms of the physical mass \( M_H \) and the physical vacuum expectation value \( v \). At higher orders, the counterterm \( \delta \lambda \) is fixed as to preserve this identity, with the renormalization point at the physical mass value, \( p^2 = M_H^2 \), rather than \( p^2 = 0 \).

Finally, we need to fix the field renormalization constants. In the OMS renormalization, the propagators of the fields are renormalized as to have unit residue at the location of the pole. In the absence of fermion interactions, only one finite field renormalization, \( \bar{Z}_H \), is needed to keep the kinetic terms and free propagators in standard form. In this case, the renormalization constant \( Z_\phi \) is defined such that the propagators of the fields \( w^\pm \) and \( z \) have unit residue at the location of the pole, and \( \bar{Z}_H \) corrects the Higgs propagator. Including fermion interactions, we need to introduce a second finite renormalization constant. In Eq. (13), we have intentionally introduced the finite Goldstone boson renormalization constants, \( \bar{Z}_z \) and \( \bar{Z}_w \), in a symmetric way. However, one of these two quantities is redundant, namely \( \bar{Z}_w = 1 \). This is connected to the fact, that the vacuum expectation value, renormalized according to Eq. (12), is related to the muon decay constant. The decay of the muon, however, is mediated by the \( W \) boson rather than the \( Z \) boson. Hence, both the vacuum expectation value and the fields \( w^\pm \) are renormalized with the same renormalization constant, whereas the field \( z \) obtains an extra finite renormalization in the presence of Yukawa interactions.

In summary, the counterterms and renormalization constants contained in the renormalized Lagrangian \( \mathcal{L}_H \) of Eq. (14) have been fixed as to satisfy the following conditions: (1) \( \langle |H| \rangle = 0 \) to all orders, simultaneously fixing the pole of the Goldstone boson propagators to be at \( p^2 = 0 \); (2) the real part of the pole of the Higgs propagator is located at its physical mass value \( M_H \), fixing the quartic Higgs coupling as \( \lambda = M_H^2/(2v^2) \) to all orders in perturbation theory; (3) the real parts of the residues of all propagators are equal to one at the pole location.

The expressions for the wavefunction renormalization constants in terms of the self-energies are:

\[
Z_z \equiv \bar{Z}_z Z_\phi = 1 + \frac{\partial \Pi_z(0)}{\partial p^2}, \tag{15}
\]

\[
Z_w \equiv Z_\phi = 1 + \frac{\partial \Pi_w(0)}{\partial p^2}, \tag{16}
\]

\[
Z_H \equiv \bar{Z}_H Z_\phi = 1 + \frac{\partial \text{Re}\Pi_H(M_H^2)}{\partial p^2}. \tag{17}
\]
To illustrate the breaking of the SO(3) symmetry of the Goldstone bosons due to the presence of Yukawa interactions, we give the explicit expression for the finite field renormalization $\tilde{Z}_z$ at one loop:

$$\tilde{Z}_z = 1 + \frac{\partial \Pi_z(0)}{\partial p^2} - \frac{\partial \Pi_w(0)}{\partial p^2} \approx 1 - \frac{3g_t^2}{32\pi^2}, \quad (18)$$

where all Yukawa couplings except $g_t$ have been neglected. (Note: for the hypothetical case $g_t = g_b (\neq 0)$ the result is $\tilde{Z}_z = 1$ — the SO(3) symmetry of the Goldstone bosons would persist.)

Because we want to calculate higher order quantum corrections including Yukawa interactions, we also need to renormalize the Lagrangian $L_F$. As in the case of the Lagrangian $L_H$, we use multiplicative field renormalization constants and counterterms for the couplings. Regarding the Yukawa couplings rather than the fermion masses as the fundamental parameters of the theory, we need to renormalize the fields $\psi_f^L$ and $\psi_f^R$ as well as the coupling $g_f$. We introduce the replacements

$$g_f \rightarrow \frac{g_f}{Z_L^{1/2}}(1 + \frac{\delta g_f}{g_f}), \quad (19)$$

$$\psi_f^{R,L} \rightarrow (Z_f^{R,L})^{1/2} \psi_f^{R,L}. \quad (20)$$

In analogy to the OMS renormalization conditions for the Lagrangian $L_H$ of the Higgs sector, the quantities $Z_f^{R,L}$ and $\delta g_f$ are fixed by requiring that the fermion propagator has the real part of its pole equal to the physical mass value, and that the residue of the propagator at the pole is equal to one. This concludes the complete OMS renormalization of $L_{EQT}$.

5. Applications

The classical use of the equivalence theorem has been the investigation of vector boson scattering. The scattering of longitudinally polarized gauge bosons, $W_L$ and $Z_L$, has been studied by a number of authors at tree level and higher orders.

A different application of the equivalence theorem is the calculation of the leading corrections to the $\rho$ parameter. This quantity is — from the point of view of the EQT — a low-energy quantity and defined in the gauge sector of the Standard Model. However, neglecting the gauge couplings, the $\rho$ parameter can be written as

$$\rho = \frac{Z_w}{Z_z} = \tilde{Z}_z^{-1}, \quad (21)$$

with the one-loop result given in Eq. (18).

Another example of a “low-energy” application is the two-loop heavy-top-quark contribution to the $Z \rightarrow b\bar{b}$ coupling which has been calculated using massless Goldstone bosons and arbitrary values of $M_H$. The validity of the equivalence theorem
was explicitly verified using Ward–Takahashi identities. The results have been confirmed in an independent calculation.

A different application of the EQT is the decay $H \to t \bar{t}$ which we will discuss here in detail. This decay process features no external gauge bosons. Yet the EQT is an excellent tool to calculate the radiative corrections in the couplings $g_t$ and $\lambda$.

5.1. One-loop electroweak radiative corrections to $H \to t \bar{t}$

Because of the high mass of the top-quark, we keep the top-quark Yukawa coupling, $g_t$, but set all other Yukawa couplings to zero. In this approximation, the Lagrangian $\mathcal{L}_{\text{EQT}}$ is used to calculate the one-loop corrections to $\Gamma (H \to t \bar{t})$.

The starting point of our analysis is the term of $\mathcal{L}_{\text{EQT}}$ which describes the Higgs-fermion interaction. At tree level we have

$$\mathcal{L}_{\text{Yuk}}^f = -g_f \sqrt{2} \bar{\psi}_f H \psi_f^L + \text{h.c.}.$$  \hspace{1cm} (22)

The Born result for the decay width is given by

$$\Gamma_B (H \to f \bar{f}) = \frac{N_c^f M_H}{16 \pi} \left(1 - \frac{4m_f^2}{M_H^2}\right)^{3/2} g_f^2.$$ \hspace{1cm} (23)

Here $N_c^f = 1$ (3) is the color factor for lepton (quark) flavors.

The renormalized form of Eq. (22) is

$$\mathcal{L}_{\text{Yuk}}^f = -g_f (1 + \delta g_f) \sqrt{2} Z_{w}^{1/2} \bar{\psi}_f^R (Z_f^R)^{1/2} H (Z_f^L)^{1/2} \psi_f^L + \text{h.c.}.$$ \hspace{1cm} (24)

Writing $Z_i = 1 + \delta Z_i$ we obtain the Feynman rule for the $Hf\bar{f}$ coupling at higher order:

$$-i \frac{g_f}{\sqrt{2}} \left(1 + \frac{1}{2} \delta Z_H - \frac{1}{2} \delta Z_w + \frac{1}{2} \delta Z_f^L + \frac{1}{2} \delta Z_f^R + \frac{\delta g_f}{g_f} + \text{O}(\delta^2)\right).$$ \hspace{1cm} (25)

For the one-loop calculation, we neglect the terms of $\text{O}(\delta^2)$.

The radiatively corrected fermionic decay rate of the Higgs boson can now be calculated using the new Feynman rule for the Yukawa coupling and taking into account the one-particle irreducible Feynman diagrams using $\mathcal{L}_{\text{EQT}}$. At one loop, there are six triangular diagrams, [internal lines ($HHt$), ($ttH$), ($zst$), ($ttz$), ($wwb$), and ($bbw$)], which contribute. The corrected fermionic Higgs decay width is defined by

$$\Gamma (H \to f \bar{f}) = (1 + \Delta \Gamma) \Gamma_B (H \to f \bar{f}) + \text{O}(\delta^2).$$ \hspace{1cm} (26)

The explicit result for $\Delta \Gamma$ is given in Ref. (17). At one loop, $\Delta \Gamma$ consists of terms $\text{O}(\lambda = G_F M_H^2/\sqrt{2})$, and $\text{O}(g_t^2 = \sqrt{2} G_F m_t^2)$. In Fig. [1], we show the size of these corrections as a function of $M_H$, and compare them with the full one-loop electroweak
Fig. 1. Comparison of the one-loop results for the ratio $1 + \Delta \Gamma \equiv \Gamma (H \to t\bar{t}) / \Gamma_B (H \to t\bar{t})$ obtained in various approximations with the full one-loop electroweak result ($g_1, g_2, g_t, g_b \neq 0$). The solid curve (EQT) gives the result obtained using the equivalence theorem ($g_1, g_2 = 0$) and a nonzero top-quark Yukawa coupling $g_t$ corresponding to $m_t = 174$ GeV. The dot-dashed curve shows the $O(\lambda) = O(G_F M_H^2)$ correction, and it is equivalent to an EQT curve with $g_t = 0$.

correction including electroweak gauge-couplings $g_1, g_2$ as well as all Yukawa couplings. The full correction was evaluated in the on-shell renormalization scheme using $m_t = 174$ GeV. We see that the $O(\lambda)$ term underestimates the full one-loop electroweak correction term by 32% (24%) at $M_H = 500$ GeV (1 TeV). However, the complete EQT result including the top-quark Yukawa coupling reproduces the full one-loop electroweak result very well. The result obtained using the equivalence theorem with $g_t \neq 0$ is only 3.9% (1.8%) larger than the full electroweak one-loop term at $M_H = 500$ GeV (1 TeV) for $m_t = 174$ GeV. The use of the equivalence theorem therefore gives a quite accurate approximation to the full theory, even for the rather low values of $M_H$ with which we are concerned. The small residual differences away from the decay threshold can be accounted for by the transverse gauge couplings, the nonzero masses of the $W$ and $Z$ bosons, and the finite masses and Yukawa couplings for the remaining fermions. The extra structure of the full electroweak correction close to the threshold, $M_H = 2m_t$, is the result of virtual-photon exchange in QED. This generates a Coulomb singularity and a correction that behaves near threshold as $1 + \alpha_{em} Q_t^2 [\pi/(2\beta) + O(1)]$, where $Q_t$ and $\beta$ are the top-quark electric charge and velocity; see left end of the dashed line in Fig. 1.
5.2. Two-loop radiative corrections to $H \rightarrow f \bar{f}$: $O(\lambda^2)$

We now describe the calculation of the two-loop correction, $O(\lambda^2)$. For $M_H > m_t$ (which always should be satisfied in the EQT limit $M_H \gg M_W$), it is the dominant correction to the decay of the Higgs into any fermion pair $f \bar{f}$. All subleading two-loop electroweak corrections, those of $O(g^2 f \lambda)$ and $O(g_f^4)$, are neglected. It should be noted, that the dominant correction is flavor-independent, whereas the subleading corrections depend on the fermionic decay channel considered.

Since the dominant correction is independent of $g_f$, we need to identify the renormalization pieces that are independent of the Yukawa couplings. Looking at Eq. (24) we find that $Z_H$ and $Z_w$ are the only quantities that obtain pure Higgs coupling corrections, i.e., terms of order $O(\lambda^n)$. All other quantities, including the Feynman diagrams for the vertex corrections, receive contributions proportional to $g_f^2$ or higher powers. Therefore, we obtain the general result for the leading corrections to all orders in $\lambda$ to be

$$1 + \Delta \Gamma(\lambda) = \left. \frac{Z_H}{Z_w} \right|_{g_f=0}$$

(27)

The wave-function renormalization constants $Z_H$ and $Z_w$ were calculated to two loops, $O(\lambda^2)$, in Ref. (12) using dimensional regularization and OMS renormalization. Calculating $\Delta \Gamma$, the divergent pieces cancel, and the $O(\lambda^2)$ electroweak corrections to the fermionic decay rates emerge naturally as

$$1 + \Delta \Gamma = \frac{Z_H}{Z_w} = \frac{1 + a_w \lambda + b_w \lambda^2}{1 + a_H \lambda + b_H \lambda^2}.$$  

(28)

The coefficients in the expansion above have been given analytically and have been confirmed. The numerical values are:

$$a_w = 1, \quad b_w \approx 6.098, \quad a_H \approx -1.12, \quad b_H \approx 41.12.$$  

(29)

The one-loop coefficients $a_H$ and $a_w$ are similar in magnitude, but the two-loop coefficients $b_H$ and $b_w$ differ in magnitude by roughly a factor of 7, despite the fact that almost the same number of diagrams, with similar structures and magnitudes, contribute. It is also interesting that the coefficients in $Z_H^{-1}$ alternate in sign; those in $Z_w^{-1}$ do not.

The above expression for $\Delta \Gamma$ automatically resums one-particle-reducible Higgs-boson self-energy diagrams. However, it is clear that the resummation contains only limited information on higher-order terms. Since we actually have no control of terms beyond $O(\lambda^2)$, and are not aware of a physical principle which would select this as an optimum resummation scheme, we expand Eq. (28) and discard terms beyond
Fig. 2. Complete $O(\lambda)$ and $O(\lambda^2)$ correction factors for $\Gamma(H \to f\bar{f})$ for $100 \text{ GeV} \leq M_H \leq 1700 \text{ GeV}$. These corrections are universal, i.e., they are independent of the flavor of the final-state fermions. In each order, the expanded result given in Eq. (30) is compared to the calculation where the one-particle-reducible Higgs-boson self-energy diagrams are resummed as shown in Eq. (28). The two-loop correction cancels the one-loop correction at $M_H = 1114 \text{ GeV}$ and is twice as large as the latter, with an opposite sign, at $M_H = 1575 \text{ GeV}$.

$O(\hat{\lambda}^2) = O(G_F^2 M_H^4)$. This gives the alternative representation

$$1 + \Delta \Gamma = \frac{Z_H}{Z_w} = 1 + (a_w - a_H)\hat{\lambda} + (b_w - b_H - a_w a_H + a_H^2)\hat{\lambda}^2$$

$$\approx 1 + 2.12\hat{\lambda} - 32.66\hat{\lambda}^2$$

$$\approx 1 + 11.1\% \left(\frac{M_H}{1 \text{ TeV}}\right)^2 - 8.9\% \left(\frac{M_H}{1 \text{ TeV}}\right)^4.$$}

The result agrees at $O(\hat{\lambda})$ with the known one-loop result. \[1,2\]

We are now in a position to explore the phenomenological implications of our results. In Fig. 2, we show the leading electroweak corrections to $\Gamma(H \to f\bar{f})$ in the one- and two-loop approximations with and without resummation of one-particle-reducible higher-order terms plotted as functions of $M_H$. We will concentrate first on the expanded results given in Eq. (30). While the $O(\lambda)$ term (upper solid line in Fig. 2) gives a modest increase of the rates, e.g., by $11\%$ at $M_H = 1 \text{ TeV}$, the situation changes when the two-loop term is included. The importance of this term, which grows as $M_H^4$, increases with $M_H$ in such a way that it cancels the one-loop term completely for $M_H = 1114 \text{ GeV}$, and is twice the size of the one-loop term, with
the opposite sign, for $M_H = 1575$ GeV. The total two-loop correction, shown by the lower solid line in Fig. 2, is then negative and has the same magnitude as the one-loop correction alone. The perturbation series for the corrections to $\Gamma (H \to f \bar{f})$ clearly ceases to converge usefully, if at all, for $M_H \approx 1100$ GeV, or equivalently, for $\lambda \approx 10$. A Higgs boson with a mass larger than about 1100 GeV effectively becomes a strongly interacting particle. Conversely, $M_H$ must not exceed approximately 1100 GeV if the standard electroweak perturbation theory is to be predictive for the decays $H \to f \bar{f}$. Note that one cannot use the usual unitarization schemes invoked in studies of $W_L^\pm, Z_L, H$ scattering to restore the predictiveness for the heavy-Higgs width, as no unitarity violation is involved.

One might expect to improve the perturbative result in the upper range of $M_H$ somewhat by resumming the one-particle-reducible contributions to the Higgs-boson wave-function renormalization by using Eq. (28) rather than Eq. (30). This leads to an increase of the one-loop correction (upper dotted line in Fig. 2), while the negative effect of the two-loop correction is lessened (lower dotted line) for large values of $M_H$. However, in the mass range below $M_H = 1400$ GeV, this effect is too small to change our conclusions concerning the breakdown of perturbation theory. Moreover, the resummed expression for the one-loop terms in the perturbation expansion, when reexpanded to $O \left( \frac{G_F^2 M_H^4}{\lambda^2} \right)$, does not yield a proper estimate for the size of the two-loop terms. There is consequently no reason to favor this approach to the present problem.

The subleading two-loop electroweak corrections, those of $O \left( g_f^2 \lambda \right)$ and $O \left( g_f^4 \right)$, are still unknown, but one may estimate their likely importance by comparing the top-quark Yukawa-coupling correction to the Higgs-coupling correction at one loop.

6. Scheme-dependence of the $O(\lambda^2)$ radiative corrections: OMS versus $\overline{\text{MS}}$ scheme

So far we have carried out the renormalization of $\mathcal{L}_{\text{EQT}}$ using OMS. We found the dominant correction to $H \to f \bar{f}$ to be (Eq. (31))

$$1 + \Delta \Gamma_{\text{OMS}} \approx 1 + 2.12 \hat{\lambda}_{\text{OMS}} - 32.66 \hat{\lambda}_{\text{OMS}}^2 + O \left( \hat{\lambda}_{\text{OMS}}^3 \right), \quad (33)$$

where $16\pi^2 \hat{\lambda}_{\text{OMS}} = M_H^2/(2v^2)$. It is interesting to check whether the convergence of the perturbative series can be improved when using $\overline{\text{MS}}$ renormalization. Since the tree-level result of the fermionic Higgs decay, Eq. (23), is independent of the coupling $\lambda$, we only need the one-loop relation between $\lambda_{\text{OMS}}$ and $\lambda_{\overline{\text{MS}}}$ to convert the two-loop OMS result into $\overline{\text{MS}}$. It is

$$\hat{\lambda}_{\overline{\text{MS}}} = \hat{\lambda}_{\text{OMS}} \left[ 1 + \left( 25 - 3\pi \sqrt{3} + 12 \ln(\mu^2/M_H^2) \right) \right] \hat{\lambda}_{\text{OMS}} + O \left( \hat{\lambda}_{\text{OMS}}^2 \right), \quad (34)$$

where $M_H$ is the physical Higgs mass, and $\mu$ is the mass scale introduced in dimensional regularization. We see that the OMS and the $\overline{\text{MS}}$ couplings are equal for
Fig. 3. The two-loop correction $\Delta \Gamma_{\text{MS}}$ as a function of $M_H$. The curves show the results when either keeping $\mu$ fixed at the value of 200 GeV, or keeping the ratio $\mu/M_H$ fixed at the values indicated. For $\mu \approx 0.7 M_H$ the two-loop OMS result of Fig. 2 is reproduced.

$\mu \approx 0.697 M_H$. Combining the two previous equations, we obtain the correction to the fermionic Higgs decay in $\overline{\text{MS}}$ quantities:

$$1 + \Delta \Gamma_{\overline{\text{MS}}} \approx 1 + 2.12 \hat{\lambda}_{\overline{\text{MS}}}^2 - \left(51.03 - 25.41 \ln\left(M_H^2/\mu^2\right)\right) \hat{\lambda}_{\overline{\text{MS}}}^2 + O\left(\hat{\lambda}_{\overline{\text{MS}}}^3\right). \quad (35)$$

The OMS correction given in Eq. (33) and the $\overline{\text{MS}}$ result are also identical for $\mu \approx 0.697 M_H$.

Truncating the series at two loops leaves a residual $\mu$ dependence which indicates the significance of the $O\left(\hat{\lambda}_{\overline{\text{MS}}}^3\right)$ terms. In Fig. 3 we show the $\overline{\text{MS}}$ correction as a function of $M_H$, keeping $\mu$ or the ratio $\mu/M_H$ fixed at different values. Choosing $\mu \approx 0.697 M_H$ the two-loop OMS result of Fig. 2 is reproduced.

It is interesting to note that for fixed $M_H$ a value $\mu < 0.697 M_H$ improves the convergence of the perturbative $\overline{\text{MS}}$ series twofold: on one hand the value of $\hat{\lambda}_{\overline{\text{MS}}}^2$ decreases as $\mu$ becomes smaller (see Eq. (34)), on the other hand the two-loop coefficient of the $\overline{\text{MS}}$ correction also decreases in magnitude for decreasing $\mu$, vanishing for $\mu = 0.366 M_H$ (see Eq. (33)). For values $\mu > 0.697 M_H$ the opposite is true: the convergence of the series, as indicated by terms up to two loops, gets worse in a twofold way as $\mu$ increases. It seems as if the naive choice of $\mu = M_H$ is not necessarily well motivated.

Varying the scale $\mu$ in the range $M_H/2 < \mu < 2 M_H$ we already find indications for significant three-loop contributions (needed to reduce the $\mu$ dependence) for $M_H >$
650 GeV. Explicitly, for $\mu = M_H (2M_H)$ we find that the two-loop correction $\Delta \Gamma^\text{MS}$ is in magnitude equal to the OMS result, with the opposite sign, for values of $M_H = 870 (650)$ GeV. However, the size of the two-loop correction is still small (about 3–4%) for such values of $M_H$.

7. Summary

We have reviewed the equivalence theorem and the approximations involved. The Lagrangian corresponding to the EQT approximations $\sqrt{s}, M_H \gg M_W$ was formulated and renormalized using OMS conditions. This Lagrangian is the basis for calculating top-quark and heavy-Higgs corrections to many physical observables. We have explicitly discussed the calculation of corrections to the decay $H \rightarrow f \bar{f}$. At one loop we find that the EQT calculation approximates the full electroweak correction very well. Calculating the dominant two-loop corrections we observe the breakdown of perturbation theory for values of $M_H$ in the TeV-range. However, already for values of $M_H > 650$ GeV we find a significant renormalization scheme dependence of the $\overline{\text{MS}}$ result, indicating the unreliability of the perturbative result despite the smallness of the two-loop correction.

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9. References

1. M. Veltman, Acta Phys. Pol. B8, 475 (1977).
2. J.M. Cornwall, D.N. Levin, and G. Tiktopoulos, Phys. Rev. D 10, 1145 (1974); (E) 11, 972 (1975).
3. C.E. Vayonakis, Lett. Nuovo Cim. 17, 383 (1976); M.S. Chanowitz and M.K. Gaillard, Nucl. Phys. B261, 379 (1985); G.J. Gounaris, R. Kögerler, and H. Neufeld, Phys. Rev. D 34, 3257 (1986); Y.-P. Yao and C.-P. Yuan, ibid. 38, 2237 (1988).
4. J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).
5. See for example: J.F. Donoghue, E. Golowich, and B.R. Holstein, Dynamics of the Standard Model, (Cambridge University Press, 1992), pp. 424-426.
6. B.W. Lee, C. Quigg, and H.B. Thacker, Phys. Rev. Lett. 38, 883 (1977); Phys. Rev. D 16, 1519 (1977).
7. M.S. Chanowitz and M.K. Gaillard, Nucl. Phys. B261, 379 (1985).
8. J. Bagger and C. Schmidt, Phys. Rev. D 41, 264 (1990).
9. H. Veltman, Phys. Rev. D 41, 2294 (1990).
10. H.-J. He, Y.-P. Kuang, and X. Li, Phys. Rev. Lett. 69, 2619 (1992).
11. R.S. Lytel, Phys. Rev. D 22, 505 (1980).
12. P.N. Maher, L. Durand, and K. Riesselmann, Phys. Rev. D 48, 1061 (1993); erratum (to appear).
13. For the OMS renormalization of the complete electroweak Lagrangian to one loop see: M. Böhm, H. Spiesberger, and W. Hollik, Fort. Phys. 34, 687 (1986).
14. See, e.g., L. Durand, P.N. Maher, and K. Riesselmann, Phys. Rev. D 48, 1084 (1993), and references therein.
15. R. Barbieri, M. Beccaria, P. Ciafaloni, G. Curci, and A. Vicere Phys. Lett. B 288, 95 (1992); (E) 312, 511 (1993); Nucl. Phys. B409, 105 (1993).
16. J. Fleischer, O.V. Tarasov, and F. Jegerlehner, Phys. Lett. B 319, 249 (1993); Phys. Rev. D 51, 3820 (1995).
17. K. Riesselmann, Ph.D. thesis, Univ. of Wisconsin–Madison, 1994 (unpublished).
18. D.Yu. Bardin, B.M. Vilenski, P.Kh. Khristov, Yad. Fiz. 53, 240 (1991) [Sov. J. Nucl. Phys. 53, 152 (1991)]; B.A. Kniehl, Nucl. Phys. B376, 3 (1992); A. Dabelstein and W. Hollik, Z. Phys. C 53, 507 (1992).
19. L. Durand, B.A. Kniehl, and K. Riesselmann, Phys. Rev. Lett. 72, 2534 (1994); (E) 74, 1699 (1995); Tech. Univ. Munich preprint TUM-HEP-200/94, hep-ph/9412311 (to appear in Phys. Rev. D).
20. A. Ghinculov, Phys. Lett. B 337, 137 (1994); (E) 346, 426 (1994).
21. W.J. Marciano and S.S.D. Willenbrock, Phys. Rev. D 37, 2509 (1988).
22. W.W. Repko and C.J. Suchyta, Phys. Rev. Lett. 62, 859 (1989); D.A. Dicus and W.W. Repko, Phys. Lett. B 228, 503 (1989); Phys. Rev. D 42, 3660 (1990); G. Valencia and S. Willenbrock, Phys. Rev. D 42, 853 (1990); H. Veltman and M. Veltman, Acta Phys. Pol. B22, 669 (1991); K. Hikasa and K. Igi, Phys. Lett. B 261, 285 (1991).
23. A. Sirlin and R. Zucchini, Nucl. Phys. B266, 389 (1986).
24. A.I. Bochkarev and R.S. Willey, Phys. Rev. D 51, 2049 (1995); in Eq. (17) of this paper incorrect numbers are used.
\[ \frac{\Gamma(H \to t\bar{t})}{\Gamma_B} \rightarrow \text{one loop} \]

\[ m_t = 174 \text{ GeV} \]

\[ \Gamma(H \to t\bar{t}) / \Gamma_B(H \to t\bar{t}) \]

- EQT \((m_b = 0)\)
- full electroweak
- \(O(\lambda)\)

\[ M_H \text{ (GeV)} \]

\[ 0.95 \rightarrow 1.00 \rightarrow 1.05 \rightarrow 1.10 \rightarrow 1.15 \]
$\Gamma(H \rightarrow ff) / \Gamma_B(H \rightarrow ff)$

- 1-loop
- 2-loop
- Tree level

$O(\lambda), O(\lambda^2)$

- Resummed
