An Elliptic Algebra $U_{q,p}(\hat{sl}_2)$
and
the Fusion RSOS Model

Hitoshi KONNO

Department of Mathematics, Faculty of Integrated Arts and Sciences,
Hiroshima University, Higashi-Hiroshima 739, Japan

†E-mail: konno@mis.hiroshima-u.ac.jp

ABSTRACT

We introduce an elliptic algebra $U_{q,p}(\hat{sl}_2)$ with $p = q^{2r}$ ($r \in \mathbb{R}_{>0}$) and present its free boson representation at generic level $k$. We show that this algebra governs a structure of the space of states in the $k$–fusion RSOS model specified by a pair of positive integers $(r, k)$, or equivalently a $q$–deformation of the coset conformal field theory $SU(2)_k \times SU(2)_{r-k-2}/SU(2)_{r-2}$. Extending the work by Lukyanov and Pugai corresponding to the case $k = 1$, we give a full set of screening operators for $k > 1$. The algebra $U_{q,p}(\hat{sl}_2)$ has two interesting degeneration limits, $p \to 0$ and $p \to 1$. The former limit yields the quantum affine algebra $U_q(\hat{sl}_2)$ whereas the latter yields the algebra $A_{\hbar,\eta}(\hat{sl}_2)$, the scaling limit of the elliptic algebra $A_{q,p}(\hat{sl}_2)$. Using this correspondence, we also obtain the highest component of two types of vertex operators which can be regarded as $q$–deformations of the primary fields in the coset conformal field theory.
1 Introduction

In studying exactly solvable models, especially in calculating correlation functions, the algebraic analysis method has proved to be extremely powerful\[1\]. The method is based on the infinite dimensional quantum group symmetry possessed by a model and its representation theory. In particular, the intertwining operators between the infinite dimensional representation spaces play an important role. There are two types of intertwiners called type I and type II. Remarkably, in solvable lattice models, the type I intertwining operator can be identified with a certain composition of the Boltzmann weights so that its product behaves as a local operator acting on the lattice. One can thus combine this into the Baxter’s corner transfer matrix (CTM) method\[2\]. As a consequence, the type I intertwiner allows us to identify the infinite dimensional irreducible representation with the space of states of the model. Furthermore, the properties of the Boltzmann weights such as the Yang-Baxter equation, the inversion relation and the crossing symmetry yield some universal relations among the type I intertwiners and the CTM. Based on these relations, one can derive $q$–difference equations for correlation functions of local operators.

It is usually a difficult problem to solve such equations. The advantage of the algebraic analysis is that it allows us to derive the correlation functions directly. Correlation functions are formulated as traces of the product of type I intertwiners over irreducible representation space. Especially, in many cases, the infinite dimensional quantum group symmetries admit free boson realization. This enables us to construct the infinite dimensional representations as well as the intertwining operators. Then the calculation of the correlation functions i.e. the traces of the intertwiners is a straightforward task. It is needless to say that the same spirit was already applied to the two dimensional conformal field theory (CFT)\[3, 4\].

In \[1, 5, 6\], the $XXZ$ model, or equivalently, the six vertex model, in the antiferromagnetic regime was solved by applying the representation theory of the quantum affine algebra $U_q(\hat{sl}_2)$. Following this work, its higher spin extension\[7, 8, 9, 10\] and a higher rank extension\[11\] were discussed. The $XYZ$ model, or equivalently, the eight vertex model in the principal regime, was also treated in this approach\[12, 13\]. There the elliptic algebra $A_{q,p}(\hat{sl}_2)$\[14\] was proposed as the basic symmetry of the model. However its free field realizations still remain to be obtained. It should also be remarked that the central extension of the Yangian double $DY(\hat{sl}_2)$, the rational limit of $U_q(\hat{sl}_2)$, was properly defined\[15, 16\]. Its free field realization and application to physical problems were discussed\[15, 16, 17\].

In the recent work\[18\], we discussed two degeneration limits of the elliptic algebra $A_{q,p}(\hat{sl}_2)$, that are the limits $p \to 0$ and $p \to 1$. The first limit gives $U_q(\hat{sl}_2)$, whereas the second limit yields a new algebra. The new algebra turned out to be a relevant symmetry for the $XXZ$ model in the gap-less regime\[18\] as well as for the sine-Gordon theory\[19, 20\]. This new algebra was later reformulated by using the Drinfeld currents and called $\mathcal{A}_{\hbar,\eta}(\hat{sl}_2)$\[21\].

---

1 Here we assume the model to have an infinite number of degrees of freedom.
On the other hand, it is known as vertex-face correspondence that there exists an interaction-round-a-face model corresponding to a vertex model. The eight vertex model and the corresponding eight vertex solid-on-solid (SOS) model, or called the Andrew-Baxter-Forrester (ABF) model, are well-known examples\cite{22}. The higher spin analogues of the ABF model were constructed by a fusion procedure\cite{23}. The $k$-fusion SOS model is obtained by fusing the Boltzmann weights of the ABF model $k$ times in both horizontal and vertical directions.

We are interested in their restricted versions, i.e. the $k$-fusion restricted SOS (RSOS) models\cite{22, 23}. The model is labeled by a pair of positive integers $(r, k)$. At each lattice site $a$, one places a random variable (local height) $m_a$ taking values in the set $S = \{1, 2, ..., r - 1\}$. One further imposes a restriction that for all adjacent sites $a$ and $b$, the local heights $m_a$ and $m_b$ satisfy the admissible conditions

$$m_a - m_b = -k, -k + 2, ..., k, \quad k + 1 \leq m_a + m_b \leq 2r - k - 1.$$

The Boltzmann weight $W(m_a \ m_b \ m_c \ m_d \ | u)$ of the model is attached to each configuration $(m_a, m_b, m_c, m_d)$ on the NW,NE,SE,SW corners of an elementary face with $u$ being the spectral parameter.

The following two facts shown in\cite{23} for the regime III, which is treated through this paper, are fundamental.

1. The one point local height probability (LHP), i.e. a probability in which the center site has a given value of the local height, is given by the so-called branching coefficient appearing in the decomposition of the product of the two irreducible characters of level $k$ and $r - k - 2$ of the affine Lie algebra $A^{(1)}_1$ into level $r - 2$ irreducible character of the diagonal $A^{(1)}_1$.

2. The critical behavior is described by the conformal field theory with the Virasoro central charge $c = \frac{3k}{k+2} \left(1 - \frac{2(k+2)}{r(r-k)}\right)$ and the primary fields of conformal dimensions $h_{J,J',n}$ (2.8).

The corresponding CFT is known as the coset minimal model $SU(2)_k \times SU(2)_{r-k-2}/SU(2)_{r-2}$. The cases $k = 1, 2$ were known before the fusion RSOS model\cite{24}, whereas the $k > 2$ cases were realized inspired by the model\cite{25, 26, 27}. The coset minimal model possesses the extended Virasoro algebra symmetry generated by the Virasoro generator and some extra generators. The super Virasoro algebra is contained as the case $k = 2$. The irreducible characters of the extended Virasoro algebras are given by the same branching coefficients as the LHP in the above.

The first attempt to applying the algebraic analysis to the fusion RSOS model was carried out in\cite{28}. There, in the regime III, the space of states was described based on the representation of $U_q(sl_2)$ and the creation and annihilation operators of the quasi-particles were obtained as a tensor product of certain type I and type II intertwiners in $U_q(sl_2)$. Later, the type I vertex operator in the restricted ABF model in the regime
III was recognized as a lattice operator and a $q$–difference equation for the correlation function was derived [29].

Recently, Lukyanov and Pugai have succeeded to realize this type I vertex operator by using a free boson [30]. This enables us to construct a solution of the above $q$–difference equation exactly. However the most important contribution by this work is not this but a discovery of a symmetry of the restricted ABF model. That is the symmetry generated by the $q$–deformation of the Virasoro algebra [31]. In the same way as the Virasoro algebra, the $q$–Virasoro algebra admits a singular representation corresponding to the minimal series [3]. In such representation, screening operators play an essential role. Constructing screening operators and $q$–deformation of the primary fields, Lukyanov and Pugai discovered that their $q$–primary fields are nothing but the above type I vertex operator in the restricted ABF model.

The purpose of this paper is to extend their result to the $k$–fusion RSOS model in the regime III. Since the Virasoro algebra is realized as the case $k = 1$ in the coset CFT, we expect that a certain $q$–deformation of the extended Virasoro algebra corresponding to the coset $SU(2)_k \times SU(2)_{r-k-2}/SU(2)_{r-2}$ ($k > 1$) exists and it provides a basic symmetry of the $k$–fusion RSOS models [32].

Our strategy is based on the following observation. The screening currents found by Lukyanov and Pugai satisfy an elliptic deformation of the quantum affine algebra $U_q(\widehat{sl}_2)$ at level one. This elliptic algebra has another degeneration limit to $A_{h,\eta}(\widehat{sl}_2)$ at level one [18, 31]. Picking up this algebraic nature, we carry out the extension in the following two steps.

1. We extend the elliptic algebra of the screening currents to generic level $k$. We call this algebra as $U_{q,p}(\widehat{sl}_2)$ [32]. Realizing it by using free bosons, we show that the conformal limit of the generating currents for $U_{q,p}(\widehat{sl}_2)$ coincide with those known in the above coset CFT. Hence these currents give a full set of screening currents for the $q$–deformed coset theory for arbitrary $k$.

2. The elliptic algebra $U_{q,p}(\widehat{sl}_2)$ has two desired degeneration limits, $U_q(\widehat{sl}_2)$ and $A_{h,\eta}(\widehat{sl}_2)$. The Hopf algebra (like) structures are known both in $U_q(\widehat{sl}_2)$ and $A_{h,\eta}(\widehat{sl}_2)$. Using this knowledge, we obtain a free field realization of, at least, the highest component of the type I and type II vertex operators.

The free field realization of the screening currents and the type I, II intertwiners allows us to make a characterization of the $q$–deformation of the coset CFT $SU(2)_k \times SU(2)_{r-k-2}/SU(2)_{r-2}$.

In order to identify our $q$–deformed coset theory with the $k$–fusion RSOS model, we investigate the following two things. The first one is a partition function per site. We show that the correct partition function per site is obtained from the commutation of the two type I vertex operators. This allows us to regard the type I vertex operator as a proper lattice operator for the $k$–fusion RSOS model. The second one is a structure of the Fock modules for the $q$–deformed coset theory. The Fock modules are reducible due to the existence of singular vectors which can be constructed by the screening operators
on the modules. We consider a resolution of the modules and obtain a space which can be regarded as the irreducible highest weight representation of the conjectural $q$–deformation of the extended Virasoro algebra. The character of the space coincides with the desired branching coefficient. Hence one can identify the space with the space of states for the $k$–fusion RSOS model.

This paper is organized as follows. In the next section, we briefly review the free field representation of the coset minimal model $SU(2)_k \times SU(2)_{r-k-2}/SU(2)_{r-2}$. The formulae summarized in this section should be compared with those of the $q$–deformed ones obtained in Sec.4 and 5. In Sec.3, we introduce the elliptic algebra $U_{q,p}(\hat{sl}_2)$ and discuss its properties. In Sec.4, we present a free field representation of $U_{q,p}(\hat{sl}_2)$. As a corollary, the free field representation of the algebra $A_{\bar{h},\eta}(\hat{sl}_2)$ for arbitrary level is obtained. We also derive the type I and type II vertex operators and their commutation relations for the highest components. We argue that the correct partition function per site is obtained from these relations. In Sec.5, based on these results, we propose a $q$–deformation of the coset minimal model. We show that the algebra $U_{q,p}(\hat{sl}_2)$ provides a full set of screening operators which are sufficient for making a resolution of the Fock modules. Then a characterization of the irreducible highest weight modules of the conjectural $q$–deformed extended Virasoro algebras is obtained. The evaluation of the character of the space supports the identification of the space with the space of states for the $k$–fusion RSOS model. The final section is devoted to discussions on the results and some future problems.

2 Coset Conformal Field Theory

In this section we briefly review the coset minimal model $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$, $k, l \in \mathbb{Z}_{>0}$ [24, 25, 26, 27].

The symmetry of the theory is an extended Virasoro algebra generated by the Energy-Momentum(EM) tensor $T(z)$ and the extra generators $A^j_k(z)$ ($j = 1, 2, ..$). The number of extra generators is depend on the value $k$. For example, the case $k = 1$ and $l \in \mathbb{Z}_{>0}$, the theory is the Virasoro minimal model and there are no extra generators [3, 24]. For $k = 2$ with $l \in \mathbb{Z}_{>0}$, the theory is the $N = 1$ super Virasoro minimal model [24, 25, 33, 34]. There are one extra generator of conformal dimension $3/2$, which is nothing but the super generator $A_2(z) = G(z)$. The $k = 4$ case with $l \in \mathbb{Z}_{>0}$ is known as the $S_3$ symmetric model [35]. There are two extra generators $A^4_4(z)$ and $\bar{A}^4_4(z)$ of conformal dimension $4/3$.

The extended Virasoro algebra is defined by the following operator product expansions (OPE) [27].

\[ T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{z-w} + O(1) \]  
\[ T(z)A^j_k(w) = \frac{\sigma A^j_k(w)}{(z-w)^2} + \frac{\partial A^j_k(w)}{z-w} + O(1) \]  
\[ A^j_k(z)A^i_l(w) = \alpha^{ij}(z-w)^{-2\sigma} \left\{ \frac{c}{\sigma} + 2T(w)(z-w)^2 + O((z-w)^3) \right\} \]

[24, 25, 26, 27, 33, 34]
\[ + \beta^{ijm}(z - w)^{-\sigma}\{A_k^m(w) + \frac{1}{2}(z-w)\partial A_k^m(w) + O((z-w)^2)\}, \]  
\tag{2.4}

where \( \sigma = \frac{k+1}{2} \) is the conformal dimension of \( A_k^m(z) \), and \( \alpha^{ij}, \beta^{ijm} \) are the structure constants. The central charge \( c \) of the Virasoro algebra is given by
\[ c = \frac{3k}{k+2} \left( 1 - \frac{2(k+2)}{(l+2)(l+k+2)} \right). \]  
\tag{2.5}

Let \( r = l + k + 2, 1 \leq n \leq r - k - 1, 1 \leq n' \leq r - 1 \) and \( J = |n' - n(\text{mod}2k)|, \ 0 \leq J \leq k \). Define the Virasoro generators \( L_m \) by \( T(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2} \). The Virasoro highest weight state \( |J; n', n\rangle \) is the state satisfying
\[ L_0 |J; n', n\rangle = h_{J, n', n} |J; n', n\rangle, \]  
\tag{2.6}
\[ L_n |J; n', n\rangle = 0, \quad \text{for} \quad n \geq 1. \]  
\tag{2.7}

The highest weights are given by
\[ h_{J, n', n} = \frac{J(k - J)}{2k(k + 2)} + \frac{(nr - n'(r - k))^2 - k^2}{4kr(r - k)}. \]  
\tag{2.8}

One can realize the theory in terms of three boson fields \([25, 26, 36]\). Let us introduce the three independent free boson fields \( \phi_0(z), \phi_1(z) \) and \( \phi_2(z) \) satisfying the OPE \( \langle \phi_0(z)\phi_0(w) \rangle = \langle \phi_1(z)\phi_1(w) \rangle = \log(z-w) = - \langle \phi_2(z)\phi_2(w) \rangle \). Then the EM tensor of the coset theory is realized as
\[ T(z) = T_{Z_k}(z) + \frac{1}{2}(\partial \phi_0(z))^2 + \sqrt{2\alpha_0} \partial^2 \phi_0(z), \]  
\tag{2.9}
\[ T_{Z_k}(z) = \frac{1}{2}(\partial \phi_1(z))^2 - \frac{1}{2} \sqrt{\frac{2}{k+2}} \partial^2 \phi_1(z) - \frac{1}{2}(\partial \phi_2(z))^2. \]  
\tag{2.10}

Here \( 2\alpha_0 = \sqrt{\frac{k}{r(r-k)}} \), and \( T_{Z_k}(z) \) being the EM tensor of the \( Z_k \) parafermion theory \([35]\). The realization of the extra generators can be found in \([25, 26, 27]\).

The expression (2.9) indicates that the coset theory is realized as a composition of the \( Z_k \) parafermion theory and the boson theory \( \phi_0(z) \). Indeed, the primary field of the coset theory is realized as
\[ \Psi_{J, n', n}(z) = \Psi_{J, M, n', n}(z)|_{J = M(\text{mod} \ 2k)}, \]  
\tag{2.11}
\[ \Psi_{J, M, n', n}(z) = \phi_{J, M}(z) : \exp \sqrt{2\alpha_{n', n}} \phi_0(z) : ; \]  
\tag{2.12}
\[ \phi_{J, M}(z) = : \exp \left\{ \frac{J}{\sqrt{2k+2}} \phi_1(z) + \frac{M}{\sqrt{2k}} \phi_2(z) \right\} ; \]  
\tag{2.13}

where \( M = -J, -J + 2, \ldots, J \ (\text{mod} \ 2k), \) \( \alpha_{n', n} = \frac{1-n'}{2} \alpha_- + \frac{1-n}{2} \alpha_+ \), \( \alpha_+ = \sqrt{\frac{r}{k(r-k)}} \), \( \alpha_- = -\sqrt{\frac{r}{kr}} \). Note \( 2\alpha_0 = \alpha_+ + \alpha_- \). Using \( \Psi_{J, n', n}(z) \), the highest weight state \( |J; n', n\rangle \) is obtained as
\[ |J; n', n\rangle = \lim_{z \to 0} \Psi_{J, n', n}(z)|0\rangle, \]  
\tag{2.14}
where $|0\rangle$ denotes the $SL(2, \mathbb{C})$ invariant vacuum state.

The highest weight representation space is then given by the Fock module $\mathcal{F}_{J,n',n}$ constructed by the action of the creation operators of the fields $\phi_j(z)$, $j = 0, 1, 2$ on $|J; n', n\rangle$. These modules are reducible due to the existence of singular vectors. The singular vectors can be constructed by using the screening operators on some highest weight states. In the minimal coset theory, the screening currents, which contour integrals yield the screening operators, are given by

$$S_+(z) = \Psi(z) : \exp\left\{ \sqrt{2} \alpha_+ \phi_0(z) \right\} :,$$

$$S_-(z) = \Psi^\dagger(z) : \exp\left\{ \sqrt{2} \alpha_- \phi_0(z) \right\} :,$$

$$S(z) = - : \left( \sqrt{\frac{k+2}{2}} \partial_1(z) + \sqrt{\frac{k}{2}} \partial_2(z) \right) \exp\left\{ - \sqrt{\frac{2}{k+2}} \phi_1(z) \right\} :,$$

$$\eta(z) = : \exp\left\{ \sqrt{\frac{k+2}{2}} \phi_1(z) + \sqrt{\frac{k}{2}} \phi_2(z) \right\} :,$$

where $\Psi$ and $\Psi^\dagger$ are the parafermion currents given by

$$\Psi(z) = : \left( \sqrt{\frac{k+2}{k}} \partial_1(z) + \partial_2(z) \right) \exp\left\{ \sqrt{\frac{2}{k}} \phi_2(z) \right\} :,$$

$$\Psi^\dagger(z) = : \left( \sqrt{\frac{k+2}{k}} \partial_1(z) - \partial_2(z) \right) \exp\left\{ - \sqrt{\frac{2}{k}} \phi_2(z) \right\} :.$$

These currents are characterized by the properties (i) the conformal dimension is one, (ii) the contour integrals of them commute with the extended Virasoro algebra.

These screening currents defines the nilpotent operators called the BRST charges. One can use these charges to make a resolution of the Fock modules[37, 38]. One should note that the screening currents $S(z)$ and $\eta(z)$ act only on the $\mathbb{Z}_k$ parafermion theory and that the screening operators obtained from $S_\pm(z)$ commute with those from $S(z)$ and $\eta(z)$. Therefore, in order to make a resolution of the Fock modules of the coset theory, one may take the following two steps. First make a resolution of the $\mathbb{Z}_k$ parafermion theory and then consider the coset theory[39]. In section 5, we discuss a $q$–analogue of this resolution.

### 3 The Elliptic Algebra $U_{q,p}(\widehat{sl}_2)$

In this section we define a new elliptic algebra $U_{q,p}(\widehat{sl}_2)$[32] and discuss its relation to known algebras $U_q(\widehat{sl}_2)$ and $A_{h,\eta}(\widehat{sl}_2)$.

---

2 The boson fields $\Phi(z), \varphi(z), \chi(z)$ used in[39] are related to $\phi_0(z), \phi_1(z), \phi_2(z)$ as follows.

$$\Phi(z) = \phi_0(z), \quad \varphi(z) = -\sqrt{\frac{k}{2}} \phi_1(z) - \sqrt{\frac{k+2}{2}} \phi_2(z), \quad \chi(z) = -\sqrt{\frac{k+2}{2}} \phi_1(z) - \sqrt{\frac{k}{2}} \phi_2(z).$$
3.1 Definition

Let \( r \in \mathbb{R}_{>0} \) and \( q \in \mathbb{C}, \ |q| < 1 \). We set \( p = q^{2r} \) and \( p^* = pq^{-2r} \).

**Definition 3.1** The associative algebra \( U_{q,p}(\hat{sl}_2) \) is generated by the operator valued functions (currents) \( k(z), \ E(z), \ F(z) \) of complex variable \( z \) and the central element \( c \) with the following relations\(^3\).

\[
\begin{align*}
    k(z)k(w) &= \left( \frac{z}{w} \right)^{\frac{1}{2}(c-\tau)} \frac{\xi(w/z; p, q) \xi(z/w; p^*, q)}{\xi(w/z; p^*, q) \xi(z/w; p, q)} k(w)k(z), \\
    k(z)E(w) &= \left( \frac{z}{w} \right)^{\frac{1}{2}} \frac{\Theta_p(q^{-1}p^{\frac{3}{2}}w/z)}{\Theta_p(qp^{\frac{1}{2}}w/z)} E(w)k(z), \\
    k(z)F(w) &= \left( \frac{z}{w} \right)^{\frac{1}{2}} \frac{\Theta_p(qp^{\frac{1}{2}}w/z)}{\Theta_p(q^{-1}p^{\frac{3}{2}}w/z)} F(w)k(z), \\
    E(z)E(w) &= q^{2c} \left( \frac{z}{w} \right)^{\frac{1}{2}} \frac{\Theta_p(q^{-1}p^{\frac{3}{2}}w/z)}{\Theta_p(q^{2}w/z)} E(w)E(z), \\
    F(z)F(w) &= q^{-2c} \left( \frac{z}{w} \right)^{\frac{1}{2}} \frac{\Theta_p(q^{2}w/z)}{\Theta_p(q^{-1}p^{\frac{3}{2}}w/z)} F(w)F(z), \\
    [E(z), F(w)] &= \frac{1}{q - q^{-1}} \left\{ \delta(q^{-c}z/w)H^+(q^{-\frac{3}{2}}z) - \delta(q^{-c}z/w)H^-(q^{-\frac{3}{2}}w) \right\},
\end{align*}
\]

where

\[
\begin{align*}
    H^\pm(z) &= \kappa(z)k(q^{\pm(r-\frac{3}{2})+1}z)k(q^{\pm(r-\frac{3}{2})-1}z), \\
    \kappa(z) &= \left( q^{\pm(r-\frac{3}{2})+1}z \right)^{-\frac{1}{2}(r-\tau)} \frac{\xi(x; p^*, q)}{\xi(x; p, q)}|_{x = q^{-2}}, \\
    \xi(z; p, q) &= \frac{(q^{2}z; p, q^4)^{\infty}(pq^{2}z; p, q^4)^{\infty}}{(q^4z; p, q^4)^{\infty}(pz; p, q^4)^{\infty}}.
\end{align*}
\]

and \([A, B] = AB - BA\), \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \), \( \Theta_\ast(z) = (z; s)^\infty(s/z; s)_\infty(s; s)_\infty \), \( (z; s)_\infty = \prod_{n=0}^{\infty}(1 - z s^n) \).

**Remark 3.1.** Let us set \( p = e^{-2\pi i / r}, \ p^* = e^{-2\pi i / r^*}, \ z = q^{2u} \) and denote \( E(z), \ F(z) \) and \( k(z) \) by the same letters \( E(u), \ F(u) \) and \( k(u) \). Then the relations in (3.2)-(3.3) are rewritten in more compact form.

\[
k(u)E(v) = \frac{\vartheta_1\left(\frac{u-v+\frac{1}{2}}{r-c}\right) - \frac{1}{2}r^*}{\vartheta_1\left(\frac{u-v+\frac{1}{2}}{r-c}\right) - \frac{1}{2}r^*} E(v)k(u),
\]

\(^3\)We are indebted to Jimbo for introducing the generator \( k(z) \) and giving its relation to \( H^\pm(z) \) in (3.7).
\[ k(u)F(v) = \frac{\vartheta_1\left(\frac{u-v}{r} - \frac{1}{2}\mid \tau\right)}{\vartheta_1\left(\frac{u-v+1}{r} - \frac{1}{2}\mid \tau\right)} F(v)k(u), \]

(3.11)

\[ E(u)E(v) = \frac{\vartheta_1\left(\frac{u-v+1}{r-c}\mid \tau^*\right)}{\vartheta_1\left(\frac{u-v}{r-c}\mid \tau^*\right)} E(v)E(u), \]

(3.12)

\[ F(u)F(v) = \frac{\vartheta_1\left(\frac{u-v}{r}\mid \tau\right)}{\vartheta_1\left(\frac{u-v+1}{r}\mid \tau\right)} F(v)F(u), \]

(3.13)

\[ H^+(u)H^-(v) = \frac{\vartheta_1\left(\frac{u-v-(1+2)}{r}\mid \tau\right)}{\vartheta_1\left(\frac{u-v+1+2}{r-c}\mid \tau^*\right)} \vartheta_1\left(\frac{u-v+1+2}{r-c}\mid \tau^*\right) H^-(v)H^+(u), \]

(3.14)

\[ H^\pm(u)H^\pm(v) = \frac{\vartheta_1\left(\frac{u-v+1}{r}\mid \tau\right)}{\vartheta_1\left(\frac{u-v+1}{r}\mid \tau\right)} \vartheta_1\left(\frac{u-v-1}{r}\mid \tau^*\right) H^\pm(v)H^\pm(u), \]

(3.15)

where \( \vartheta_1(u|\tau) \) is the Jacobi elliptic theta function

\[ \vartheta_1(u|\tau) = i \sum_{n \in \mathbb{Z}} (-)^n e^{\pi i (n-1/2)^2 \tau} e^{2\pi i (n-1/2)u}. \]

These expressions suggest that the algebra \( U_{q,p}(sl_2) \) is related to some elliptic curves in the similar way to the theory of Enriquez and Felder [40].

### 3.2 Degeneration limit

There are two interesting degeneration limits: \( p \rightarrow 0 \) and \( p \rightarrow 1 \).

The limit \( p \rightarrow 0 \) is taken by letting \( r \rightarrow \infty \). Then the relations (3.11)-(3.15) are reduced to those of the Drinfeld currents in the quantum affine algebra \( U_q(sl_2) \) (see for example, [11]). In the later section (Sec.5), we will make the identification that \( U_{q,p}(sl_2) \) is the algebra of the screening currents in the \( q \)–deformation of the coset theory \( SU(2)_k \times SU(2)_{r-k-2}/SU(2)_{r-2} \). The limit \( r \rightarrow \infty \) to \( U_0(sl_2) \) is then consistent with the well-known fact in CFT and the perturbation of the coset CFT [12], i.e.

\[ \lim_{r \rightarrow \infty} SU(2)_k \times SU(2)_{r-k-2}/SU(2)_{r-2} \text{ theory} \rightarrow SU(2)_k \text{ Wess – Zumino – Witten model}. \]

The second limit \( p \rightarrow 1 \) is taken by setting \( q = e^{\frac{2\pi i}{h}} \) and \( z = e^{-i\alpha} \), \( w = e^{-i\beta} \) and letting \( \varepsilon \rightarrow 0 \). In this limit, the relations in Definition 3.1 tend to those of the currents in the algebra \( A_{h,q}(sl_2) \) [21] (see Appendix), i.e. the degeneration (or scaling) limit of the elliptic algebra \( A_{q,p}(sl_2) \) [4], under the identification \( 1/\eta = \hbar r, 1/\eta' = \hbar (r-k) \) and the interchanges \( H^+ \leftrightarrow H^-, E \leftrightarrow F \). This limit is also consistent with the known fact that the scaling limit of the RSOS model gives the restricted sine-Gordon model and the latter model is obtained by an integrable perturbation of the coset minimal model (see, for example, [12]).

Our algebra \( U_{q,p}(sl_2) \) hence has the same degeneration limits as the elliptic algebra \( A_{q,p}(sl_2) \) [18]. The direct relationship between \( U_{q,p}(sl_2) \) and \( A_{q,p}(sl_2) \) is unknown at this
stage\textsuperscript{4}. However, see \cite{18} Sec.3 where one can find a discussion which suggests the deep relation between the $q$–Virasoro algebra and the elliptic algebra $\mathcal{A}_{q,p}(\hat{sl}_2)$ at level one.

## 4 Free Field Realization of $U_{q,p}(\hat{sl}_2)$ at Level k

We here consider a realization of the algebra $U_{q,p}(\hat{sl}_2)$ at arbitrary level $k \neq 0, -2$ in terms of three bosonic fields\textsuperscript{5}.

### 4.1 Bosonization of $U_{q,p}(\hat{sl}_2)$

Let $a_{j,m}$ ($m \in \mathbb{Z} \neq 0$ $j = 0, 1, 2$) be bosons satisfying the commutation relations

\begin{align}
[a_{0,m}, a_{0,n}] &= \frac{[2m][km]}{m} [rm] \delta_{m+n,0}, \\
[a_{1,m}, a_{1,n}] &= \frac{[2m][k+2m]}{m} [k+2m] \delta_{m+n,0}, \\
[a_{2,m}, a_{2,n}] &= -\frac{[2m][km]}{m} [k+2m] \delta_{m+n,0},
\end{align}

where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$. We also define the primed boson $a'_{0,m}$ and the zero-mode operators $Q_j$ and $P_j$ ($j = 0, 1, 2$) satisfying

\begin{align}
a'_{0,m} &= \frac{[(r-k)m]}{[rm]} a_{0,m}, \\
[P_0, Q_0] &= -i, \\
[P_1, Q_1] &= 2(k+2), \\
[P_2, Q_2] &= -2k.
\end{align}

In order to make the expression of the currents simple, we introduce the following boson fields $\phi_j(A; B, C; z; D)$ ($j = 0, 1, 2; A, B, C, D \in \mathbb{R}$).

\begin{align}
\phi_0(A; B, C; z; D) &= \frac{A}{BC} \sqrt{\frac{2kr}{r-k}} (iQ_0 + P_0 \log z) + \sum_{m \in \mathbb{Z} \neq 0} \frac{[Am]}{[Bm][Cm]} a_{0,m} z^{-m} q^{m|D|}, \\
\phi'_0(A; B, C; z; D) &= \frac{A}{BC} \sqrt{\frac{2k(r-k)}{r}} (iQ_0 + P_0 \log z) + \sum_{m \in \mathbb{Z} \neq 0} \frac{[Am]}{[Bm][Cm]} a'_{0,m} z^{-m} q^{m|D|}, \\
\phi_j(A; B, C; z; D) &= -\frac{A}{BC} (Q_j + P_j \log z) + \sum_{m \in \mathbb{Z} \neq 0} \frac{[Am]}{[Bm][Cm]} a_{j,m} z^{-m} q^{m|D|} \quad (j = 1, 2),
\end{align}

\textsuperscript{4}In the trigonometric limit, the corresponding problem has been discussed by Hou et al.\cite{43}.
\textsuperscript{5}In preparing this paper, we have noticed that Shiraishi has obtained another free field realization for the similar algebra.
\[ \phi_j^{(\pm)}(A; B|z; C) = \frac{P_j}{2} \log q + (q - q^{-1}) \sum_{m \in \mathbb{Z}_{>0}} \frac{[Am]}{[Bm]} a_{j, \pm m} z^{\mp m} q^C \quad (j = 1, 2). \] (4.9)

We often use the abridgment
\[ \phi_j(C|z; D) = \phi_j(A; A, C|z; D), \quad \phi_j(C|z) = \phi_j(C|z; 0). \] (4.10)

We denote by \( \mathbf{::} \) the usual normal ordered product. For example,
\[ \mathbf{::} \exp \{-\phi_0(k|z)\} := \exp \left\{ \sum_{m \in \mathbb{Z}_{>0}} \frac{a_{0, -m}}{km} z^m \right\} \exp \left\{ - \sum_{m \in \mathbb{Z}_{>0}} \frac{a_{0, m}}{km} z^{-m} \right\} \times e^{-i \sqrt{\frac{2r}{k(r-k)}} Q_0 \frac{z}{z} - \sqrt{\frac{2r}{k(r-k)}} P_0}. \] (4.11)

Then we have

**Theorem 4.1** The algebra \( U_{q,p}(\hat{s}l_2) \) has the following free field realization at \( c = k \).
\[ k(z) = \mathbf{::} \exp \left\{ -\phi_0(1; 2, r|z) \right\} \mathbf{::}, \] (4.12)
\[ E(z) = \Psi(z) : \exp \left\{ -\phi_0(k|z) \right\} : \mathbf{::}, \] (4.13)
\[ F(z) = \Psi^\dagger(z) : \exp \left\{ \phi_0'(k|z) \right\} : \mathbf{::}, \] (4.14)

where \( \Psi(z) \) and \( \Psi^\dagger(z) \) are the \( q \)-deformed \( \mathbb{Z}_k \) parafermion currents given by \( \Psi(z) = \Psi^-(z), \Psi^\dagger(z) = \Psi^+(z) \) [44].

\[ \Psi^\pm(z) = \mp \frac{1}{(q - q^{-1})} : \exp \left\{ \pm \phi_2(k|z; \pm \frac{k}{2} \right\} \right\} \times \left\{ \exp \left\{ -\phi_2^{(+)}(1; 2|z; \mp \frac{k}{2} \right) \right\} \pm \phi_1^{(+)}(1; 2|z; \mp \frac{k}{2} \right) \right\} \right\} \times \left\{ \exp \left\{ -\phi_2^{(-)}(1; 2|z; \mp \frac{k}{2} \right) \right\} \mp \phi_1^{(-)}(1; 2|z; \mp \frac{k}{2} \right) \right\} \right\} \mathbf{::}. \] (4.15)

**Proof.** Straightforward calculation. \( \Box \)

**Remark 4.1** In the CFT limit \( q \to 1 \) with \( z \) fixed, the currents \( E(z) \) and \( F(z) \) coincide with the screening currents \( S_+(z) \) [2.13], \( S_-(z) \) [2.16] in the coset CFT.

**Remark 4.2** In the limit \( r \to \infty \), the expressions (1.12) with (3.7) and (4.13)-(4.14) tend to the Matsuo’s bosonization of \( U_q(sl_2) \) [44]. Namely, \( H^+(z) \to K_+(z), H^-(z) \to K_-(z), E(z) \to X^+(z) \) and \( F(z) \to X^-(z) \).

\[ m = a_{1, m}, \alpha_m = a_{2, m}, \beta_0 = P_1, \bar{\alpha}_0 = P_2, 2(k+2)\beta = Q_1, 2k\bar{\alpha} = Q_2. \]
4.2 The $\mathcal{A}_{h,\eta}(\hat{sl}_2)$ limit

Let us consider the $\mathcal{A}_{h,\eta}(\hat{sl}_2)$ limit. The algebra $\mathcal{A}_{h,\eta}(\hat{sl}_2)$ is defined in Appendix. The limit is taken by the following procedure. Setting $q = e^{i \frac{h}{\eta^2}}$, $z = e^{-i \alpha}$, $r = \xi + k$, $m \varepsilon = t \in \mathbb{R}$ and letting $\varepsilon \to 0$. Then we have from (4.1)-(4.3)

$$[a_0(t), a_0(t')] = \frac{1}{\hbar^2} \frac{\sinh \hbar t}{\sinh \frac{\hbar (\xi + k) t}{2}} \delta(t + t'),$$ \hspace{0.5cm} (4.16)

$$[a_1(t), a_1(t')] = \frac{1}{\hbar^2} \frac{\sinh \hbar t}{\sinh \frac{\hbar (k+2) t}{2}} \delta(t + t'),$$ \hspace{0.5cm} (4.17)

$$[a_2(t), a_2(t')] = -\frac{1}{\hbar^2} \frac{\sinh \hbar t}{\sinh \frac{\hbar (k-2) t}{2}} \delta(t + t'),$$ \hspace{0.5cm} (4.18)

and

$$a_0'(t) = \frac{\sinh \frac{\hbar t}{2}}{\sinh \frac{\hbar (\xi + k) t}{2}} a_0(t).$$ \hspace{0.5cm} (4.19)

In this limit, all the zero-mode operators $Q_j$ and $P_j$ $(j = 0, 1, 2)$ are dropped.

The boson fields $\phi_j(A; B, C|\alpha; D)$, $j = 0, 1, 2$ are now reduced to

$$\tilde{\phi}_j(A; B, C|\alpha; D) = \hbar \int_{-\infty}^{\infty} dt \frac{\sinh \frac{\hbar t}{2}}{\sinh \frac{\hbar (\xi + k) t}{2}} a_j(t) e^{i \alpha t + \frac{\hbar D}{2} |t|} \hspace{0.5cm} (j = 0, 1, 2),$$ \hspace{0.5cm} (4.20)

$$\tilde{\phi}_0'(A; B, C|\alpha; D) = \hbar \int_{-\infty}^{\infty} dt \frac{\sinh \frac{\hbar t}{2}}{\sinh \frac{\hbar (k+2) t}{2}} a_0'(t) e^{i \alpha t + \frac{\hbar D}{2} |t|},$$ \hspace{0.5cm} (4.21)

$$\tilde{\phi}_j^{(\pm)}(A; B|\alpha; C) = 2\hbar \int_{0}^{\infty} dt \frac{\sinh \frac{\hbar t}{2}}{\sinh \frac{\hbar (k-2) t}{2}} a_j(\pm t) e^{\pm i \alpha t + \frac{\hbar C}{2} |t|} \hspace{0.5cm} (j = 1, 2).$$ \hspace{0.5cm} (4.22)

Under these notations, we have

**Theorem 4.2** The level $k$ currents in $\mathcal{A}_{h,\eta}(\hat{sl}_2)$ with $1/\eta = \hbar \xi$, $1/\eta' = \hbar (\xi + k)$ is realized as follows.

$$k(\alpha) =: \exp \left\{ -\tilde{\phi}_0(1; 2, \xi + k|\alpha) \right\};$$ \hspace{0.5cm} (4.23)

$$E(\alpha) = \Psi(\alpha) : \exp \left\{ -\tilde{\phi}_0(k|\alpha) \right\};$$ \hspace{0.5cm} (4.24)

$$F(\alpha) = \Psi^\dagger(\alpha) : \exp \left\{ \tilde{\phi}_0'(k|\alpha) \right\};$$ \hspace{0.5cm} (4.25)

where $\Psi(\alpha) = \Psi^-(\alpha)$, $\Psi^\dagger(\alpha) = \Psi^+(\alpha)$ are given by

$$\Psi^\pm(\alpha) = \mp \frac{1}{\hbar} : \exp \left\{ \pm \tilde{\phi}_2(k|\alpha; \pm \frac{k}{2}) \right\}.$$
Although we do not have any results on the Hopf algebra structure of $A_{h,n}(\hat{sl}_2)$, the Hopf algebra like structure and the level zero representation are known\cite{21}. We summarize them in Appendix. From these knowledge, one can construct the intertwining operators between the level $k$ infinite dimensional representations. Let us define the following four vertex operators.

$$\Phi^{(l)}_l(\zeta) = \phi_{l,t}(\zeta) : \exp\left\{ -\tilde{\phi}_0(l; 2, k|\zeta) \right\} : ,$$

$$\Psi^{(l)}_l(\zeta) = \phi_{l,-t}(\zeta) : \exp\left\{ \tilde{\phi}_0(l; 2, k|\zeta) \right\} : ,$$

$$\tilde{\Phi}^{(l)}_l(\zeta) = \phi_{k-t,-(k-t)}(\zeta) : \exp\left\{ -\tilde{\phi}_0(l; 2, k|\zeta) \right\} : ,$$

$$\tilde{\Psi}^{(l)}_l(\zeta) = \phi_{k-t,k-t}(\zeta) : \exp\left\{ \tilde{\phi}_0(l; 2, k|\zeta) \right\} : \quad (l = 0, 1, 2, \ldots, k),$$

where $\phi_{l,\pm t}(\zeta)$ are the analogues of the $Z_k$-parafermion primary fields (2.13) given by:

$$\phi_{l,\pm t}(\zeta) = : \exp\left\{ -\tilde{\phi}_2(\pm l; 2, k|\zeta; \pm \frac{k}{2}) - \tilde{\phi}_1(l; 2, k + 2|\zeta; \pm \frac{k}{2}) \right\} : .$$

Then we have

**Proposition 4.3** The vertex operators $\Phi^{(l)}_l(\zeta)$ and $\Psi^{(l)}_l(\zeta)$ satisfy the intertwining relations of the type I (A.29)-(A.31) and the type II (A.32)-(A.42), respectively, whereas the vertex operators $\tilde{\Phi}^{(l)}_l(\zeta)$ and $\tilde{\Psi}^{(l)}_l(\zeta)$ satisfy the twisted intertwining relations of the type I (A.29), (A.32)-(A.44) and the type II (A.32), (A.44)-(A.42), respectively.

**Proof.** Straightforward. \qed

Using the relations (A.31)-(A.34), one can obtain the other (lower) components of the intertwiners $\Phi^{(l)}_m(\zeta)$, $\Psi^{(l)}_m(\zeta)$, $\tilde{\Phi}^{(l)}_m(\zeta)$, $\tilde{\Psi}^{(l)}_m(\zeta)$ $(m = 0, 1, 2, \ldots, l - 1)$. We omit them here.

### 4.3 The type I and type II vertex operators

Although we do not have any results on the Hopf algebra structure of $U_{q,p}(\hat{sl}_2)$, the $(q, p)$-analogue of the operators (4.27)-(4.30) can be obtained by the following requirements.

1. The procedure taking the limit to $A_{h,n}(\hat{sl}_2)$ from $U_{q,p}(\hat{sl}_2)$ makes the vertex operators reduce to those in (4.27)-(4.30).

2. The zero-modes factors are determined by requiring that the CFT limit ($q \to 1$ with $z$ fixed) of the vertex operators should be expressed as the exponential of the boson fields.
Using the notations in (4.6)-(4.9), we find that the desired vertex operators are given as follows.

\[ \Phi^0_l(z) = \phi_{l,1}(z) : \exp\left\{ -\phi_0(l; 2, k|z) \right\} : \]  
\[ \Psi^0_l(z) = \phi_{l,-1}(z) : \exp\left\{ \phi_0(l; 2, k|z) \right\} : \]  
\[ \tilde{\Phi}^0_l(z) = \phi_{-l,-(k-l)}(z) : \exp\left\{ -\phi_0(l; 2, k|z) \right\} : \]  
\[ \tilde{\Psi}^0_l(z) = \phi_{k-l,-k-l}(z) : \exp\left\{ \phi_0(l; 2, k|z) \right\} : \quad (l = 0, 1, 2, \ldots, k), \]  

where

\[ \phi_{l,\pm 1}(z) = \exp\left\{ -\phi_2(\pm l; 2, k|z; \pm \frac{k}{2}) - \phi_1(l; 2, k + 2|z; \pm \frac{k + 2}{2}) \right\} :. \]  

We also have some conjectural expression for the lower components of these vertex operators. We will discuss them and their commutation relations in a separate paper.

**Remark 4.3** In the \( U_q(s\tilde{sl}_2) \) limit, the type I vertex operator \( \Phi^0_1(z) \) coincides with the result in [14]. On the other hand, in the CFT limit, the same vertex operator coincides with the primary field \( \Psi_{l,1,1}(z) \) in (2.12), whereas the type II vertex operator \( \tilde{\Psi}^0_1(z) \) coincides with \( \phi_{l,-1}(z) : \exp \sqrt{2}\alpha_{l,1} \phi_0(0) : \). Hence one can regard \( \Phi^0_1(z) \) and \( \tilde{\Psi}^0_1(z) \) as the \( q \)-deformation of the primary fields in the coset theory.

**Remark 4.4** At level one, the vertex operator \( \tilde{\Phi}^1_1(z) \) coincides with \( \Psi^+(z) \) obtained by Lukyanov and Pugai in the \( q \)-Virasoro algebra [30]. On the other hand, the degeneration limit (1.29) and (1.30), at level one, coincide with those found in the massless XXZ model [18]. In this way, in all the known cases, the vertex operators relevant for the physical applications obey the twisted intertwining relations.

The vertex operators (4.32)-(4.35) satisfy interesting commutation relations with the currents which allows us to expect the existence of the \( (q,p) \)-analogue of the twisted intertwining relations (4.29)-(4.32) and the existence of the Hopf algebra structure in \( U_{q,p}(s\tilde{sl}_2) \) (see also Sec.6). We here list them only for the vertices \( \tilde{\Phi}^0_i(z) \) and \( \tilde{\Psi}^0_i(z) \).

**Proposition 4.4** The vertex operators \( \tilde{\Phi}^0_i(z) \) and \( \tilde{\Psi}^0_i(z) \) satisfy the following relations.

\[ H^{(\pm)}(w) \tilde{\Phi}^0_i(z) = q^l (z_w \frac{q^{\pm k}}{w})^{-l/r} \theta_p(q^{-l+k/2}z/w) \tilde{\Phi}^0_i(z) H^{(\pm)}(w), \]  
\[ E(w) \tilde{\Phi}^0_i(z) + \tilde{\Phi}^0_i(z) E(w) = 0, \]  
\[ F(w) \tilde{\Phi}^0_i(z) = -q^l (z_w \frac{-l+1}{w})^{-l/r} \theta_p(q^{l-k/2}z/w) \tilde{\Phi}^0_i(z) F(w), \]  
\[ H^{(\pm)}(w) \tilde{\Psi}^0_i(z) = q^{-l} (z_w \frac{q^{\pm k}}{w})^{l/(r-k)} \theta_{p^*}(q^{l+k/2}z/w) \tilde{\Psi}^0_i(z) H^{(\pm)}(w), \]  

14
\[ F(w)\tilde{\Psi}_i^{(l)*}(z) + \tilde{\Psi}_i^{(l)*}(z)F(w) = 0, \]  
\[ E(w)\tilde{\Psi}_i^{(l)*}(z) = -q^{-l}\left(\frac{z}{w}\right)^{l/(r-k)}\frac{\Theta_p^*(q^iz/w)}{\Theta_p^*(q^{-l}z/w)}\tilde{\Psi}_i^{(l)*}(z)E(w). \]

Finally, we present the commutation relations among the type I and type II vertex operators. For application to physics, we are interested in those among \(\tilde{\Phi}_k^{(l)}(z)\) and \(\tilde{\Psi}_1^{(l)}(z)\).

**Proposition 4.5**

\[
\tilde{\Phi}_k^{(l)}(z)\tilde{\Phi}_k^{(l)}(w) = r_k(w/z)\tilde{\Phi}_k^{(l)}(w)\tilde{\Phi}_k^{(l)}(z),
\]

\[
\tilde{\Psi}_1^{(l)}(z)\tilde{\Psi}_1^{(l)}(w) = s_1(z/w)\tilde{\Psi}_1^{(l)}(w)\tilde{\Psi}_1^{(l)}(z),
\]

\[
\tilde{\Phi}_k^{(l)}(z)\tilde{\Psi}_1^{(l)}(w) = \chi(w/z)\tilde{\Psi}_1^{(l)}(w)\tilde{\Phi}_k^{(l)}(z),
\]

where

\[
r_k(z) = z^{-2r} \frac{z^{k(r-k)}(q^{2r-2k+2}/z; q^4, p)\infty(q^{2k+2}/z; q^4, p)\infty(q^{2r+2}/z; q^4, p)\infty(q^{2k+2}/z; q^4, p)\infty}{(q^{2r+2}/z; q^4, p)\infty(q^{2k}/z; q^4, p)\infty(q^{2r-2k+2}/z; q^4, p)\infty(q^{2k+2}/z; q^4, p)\infty},
\]

\[
s_1(z) = z^{-4/(r-k)} \frac{(k)\sigma(w/z)}{\sigma(z/w)},
\]

with

\[
\sigma(z) = \frac{(q^{2(k+1)}/z; q^2, q^{2(k+2)})^2_{\infty}(q^{2(k+2)}/z; q^4, q^{2k})\infty(q^{2k}/z; q^4, q^{2k})\infty(q^{4}/z; q^4, p^*)\infty(z; q^4, p^*)\infty}{(q^{4k}/z; q^2, q^{2(k+2)})^2_{\infty}(q^{4k}/z; q^{2k}, q^{2(k+2)})^2_{\infty}(q^{2(k+1)}/z; q^4, q^{2k})^2_{\infty}(q^{2k}/z; q^4, p^*)^2_{\infty}}(q^{4}/z; q^4, p^*)^2_{\infty},
\]

and

\[
\chi(z) = z^{1/2} \frac{\Theta_q^{1/2}(q/z)}{\Theta_q^{1/2}(q^2z)}.
\]

The function \(r_k(z) \equiv 1/\kappa(u)\) with \(z = q^{2u}\) satisfies the following inversion relations.

\[
\kappa(u)\kappa(-u) = 1,
\]

\[
\kappa(u)\kappa(-2 - u) = \frac{\vartheta_1^{(k+1+u)/(r+u)}\vartheta_1^{(k-1-u)/(r-1-u)}}{\vartheta_1^{(1+u)/(r+1)}\vartheta_1^{(1-u)/(r-1)}}.
\]

Therefore, according to Appendix D in the third reference in [23], one can identify \(\kappa(u)\) with the partition function per site for the \(k\)-fusion RSOS model in the regime III. On the other hand, the logarithmic derivative of the function \(\chi(z)\) gives the excitation energy of the kink[45].
5 \( q \)-Deformation of the Coset Theory

In this section, we discuss a \( q \)-deformation of the coset conformal field theory based on the algebra \( U_{q,p}(\hat{sl}_2) \) and make an identification of it with the \( k \)-fusion RSOS model \( k \in \mathbb{Z}_{>0} \).

5.1 Definition

The Virasoro central charge of the \( \mathbb{Z}_k \) parafermion theory is \( c_{PF} = \frac{2(k-1)}{k+2} \). Hence the level one \( (k = 1) \) parafermion theory has zero central charge and gives a trivial contribution to the coset theory. This should be true in the \( q \)-deformed theory, too. Noting this, one can see that at level one, the currents \( E(z) \) and \( F(z) \) in (4.13) and (4.14) coincides with the screening currents \( \bar{V}(z) \) and \( \bar{U}(z) \) in the \( q \)-Virasoro algebra [30]. In addition, the type I vertex \( \tilde{\Phi}_l(z) \) coincides with the vertex \( \Psi^+(z) \) in [30, 31]. In this sense, the algebra \( U_{q,p}(\hat{sl}_2) \) at level one governs the structure of the \( q \)-Virasoro algebra. In fact, at level one, the \( q \)-Virasoro algebra generator \( T(z) \) is obtained by [31]

\[
T(z) = \Lambda(zq) + \Lambda(zq^{-1}),
\]

\[
\Lambda(z) =: \tilde{\Phi}_1(1)(zq)\Phi_1(1)(zq^{-1}).
\]

For generic level \( k \), as mentioned in Remarks 4.1 and 4.3, one can regard the currents \( E(z) \), \( F(z) \) and the type I vertex \( \tilde{\Phi}_l(z) \) \( (l = 0, 1, \ldots, k) \) as the \( q \)-deformation of the screening currents \( S_+(z) \), \( S_-(z) \) and the primary field \( \Psi_{l;1,1,1} \) in the coset CFT.

These observation lead us to the following characterization of a \( q \)-deformation of the coset theory \( SU(2)_k \times SU(2)_{r-k-2}/SU(2)_{r-2} \), or equivalently a \( q \)-deformation of the extended Virasoro algebra in the free boson realization.

1. The theory is obtained as a composition of the \( q \)-deformed \( \mathbb{Z}_k \) parafermion theory and the \( \phi_0 \) boson theory.

2. The screening currents satisfy the algebra \( U_{q,p}(\hat{sl}_2) \).

3. The \( q \)-deformation of the primary fields are the intertwiners between the infinite dimensional representations of \( U_{q,p}(\hat{sl}_2) \).

We have not yet succeeded to obtain a \( q \)-Virasoro generators with central charge \( c \) [23] and any extra generators. However, the free boson realization of the screening operators and the type I, type II vertex operators enables us to analyze the structure of the highest weight representation of the \( q \)-Virasoro algebras [31]. In the following section, we carry out such analysis for the case \( k > 1 \). The resultant irreducible representation turns out to be identified with the space of states of the \( k \)-fusion RSOS model.
5.2 Fock modules

Let \( J = |n' - n \pmod{2k}| \) and \( M = n' - n \pmod{2k} \). Define the highest weight state \( |J, M; n', n\rangle \) by

\[
|J, M; n', n\rangle = |J, M\rangle_{PF} \otimes |n', n\rangle_0, \tag{5.3}
\]

\[
|J, M\rangle_{PF} = e^{\frac{i}{2}Q_1 + \frac{M}{2k}Q_2}|0\rangle_{PF}, \tag{5.4}
\]

\[
|n', n\rangle = e^{-i\sqrt{2}a_{n', n}Q_0}|0\rangle_0. \tag{5.5}
\]

Here \( |0\rangle_{PF} \) and \( |0\rangle \) denote the \( SL(2, \mathbb{C}) \) invariant vacuum states defined by

\[
a_{j,m}|0\rangle_{PF} = 0 = P_j|0\rangle_{PF} \quad (j = 1, 2 \ m \in \mathbb{Z}_{>0}), \tag{5.6}
\]

\[
a_{0,m}|0\rangle_0 = 0 = P_0|0\rangle_0 \quad (m \in \mathbb{Z}_{>0}). \tag{5.7}
\]

Note that the highest weight states can be obtained by making the vertex operator act on the \( SL(2, \mathbb{C}) \) invariant vacuum state in the same way as (2.14) in CFT.

The Fock modules \( \mathcal{F}_{J,M,n', n} = \mathcal{F}_{J,M}^{PF} \otimes \mathcal{F}_{n', n}^{\phi_0} \) are defined by

\[
\mathcal{F}_{J,M}^{PF} = C[a_{1,-m_1}, a_{2,-m_2} (m_1, m_2 \in \mathbb{Z}_{>0})]|J, M\rangle_{PF}, \tag{5.8}
\]

\[
\mathcal{F}_{n', n}^{\phi_0} = C[a_{0,-m} (m \in \mathbb{Z}_{>0})]|n', n\rangle_0. \tag{5.9}
\]

We also define the degree of the vector in the Fock modules as an eigenvalue of the operator \( L_0 \) given by

\[
L_0 = L_0^{PF} + L_0^{\phi_0}, \tag{5.10}
\]

\[
L_0^{PF} = \sum_{m > 0} \frac{m^2}{[2m][(k+2)m]} a_{1,-m}a_{1,m} + \frac{P_1(P_1 + 2)}{4(k+2)}
\]

\[- \sum_{m > 0} \frac{m^2}{[2m][km]} a_{2,-m}a_{2,m} - \frac{P_2^2}{4k}, \tag{5.11}
\]

\[
L_0^{\phi_0} = \sum_{m > 0} \frac{m^2[(r-k)m]}{[2m][km][rm]} a_{0,-m}a_{0,m} + \frac{1}{2}P_0\left(P_0 - \sqrt{\frac{2k}{r(r-k)}}\right). \tag{5.12}
\]

For a vector \( u \in \mathcal{F}_{J,M,n', n} \)

\[
u = \left( \prod_{i=0,1,2} a_{i,-m_{i,1}a_{i,-m_{i,2}} \cdots a_{i,-m_{i,N_i}}} \right)|J, M; n', n\rangle, \tag{5.13}
\]

its degree \( N \) is given by

\[
L_0u = ( h_{J,M,n', n} + N)u, \quad N = \sum_{i=0,1,2} \sum_{j=1}^{N_i} m_{i,j} \tag{5.14}
\]

where

\[
h_{J,M,n', n} = h_{J,M} + \frac{(nr - n'(r-k))^2 - k^2}{4kr(r-k)}, \tag{5.15}
\]

\[
h_{J,M} = \frac{J(J + 2)}{4(k+2)} - \frac{M^2}{4k}. \tag{5.16}
\]
5.3 Screening currents

In section 5.1, we identified the currents $E(z)$ and $F(z)$ with the $q$-deformation of the screening currents $S_+(z)$ and $S_-(z)$ in the coset CFT. As mentioned in section 2, we need two more screening currents, which governs the structure of the Fock representation space of the $q$-deformed $\mathbb{Z}_k$ parafermion theory. Such screening currents $S(z) : \mathcal{F}_{J,M;n,n'} \to \mathcal{F}_{J-2,M;n,n'}$ and $\eta(z) : \mathcal{F}_{J,M;n,n'} \to \mathcal{F}_{J+k+2,M+k;n,n'}$ were already realized by Matsuo\[44\]. They are given by the following formulae.

$$S(z) = \frac{-1}{(q - q^{-1})} \cdot \exp \left\{ \phi_1 \left( k + 2 \mid z; -\frac{k + 2}{2} \right) \right\} \times \left( \exp \left\{ \phi_2^+(1; 2 \mid z; \frac{k + 2}{2}) + \phi_1^+(1; 2 \mid z; \frac{k}{2}) \right\} - \exp \left\{ -\phi_2^-(1; 2 \mid z; \frac{k + 2}{2}) - \phi_1^-(1; 2 \mid z; \frac{k}{2}) \right\} \right) ; \quad (5.17)$$

$$\eta(z) = \exp \left\{ -\phi_1 \left( 2 \mid z; \frac{k}{2} \right) - \phi_2 \left( 2 \mid z; \frac{k + 2}{2} \right) \right\} \right) ; \quad (5.18)$$

Noting that $S(z)$ and $\eta(z)$ depend only on the boson fields $\phi_1$ and $\phi_2$, the following relations are direct consequences of the Lemma 4.1 and 4.5 in [44].

**Proposition 5.1**

$$E(z)S(w) = S(w)E(z) = O(1), \quad (5.19)$$
$$F(z)S(w) = S(w)F(z) = \left[ k + 2 \right]_{k+2\partial_w} \left[ \frac{1}{z - w} : e^{\Phi_{FS}(z)} :: \right] + O(1) \quad (5.20)$$
$$E(z)\eta(w) = -\eta(w)E(z) = i \partial_w \left[ \frac{1}{z - w} : e^{\Phi_{E\eta}(z)} :: \right] + O(1), \quad (5.21)$$
$$F(z)\eta(w) = -\eta(w)F(z) = O(1), \quad (5.22)$$
$$S(z)\eta(w) = \eta(w)S(z) = i \partial_w \left[ \frac{1}{z - w} : e^{\Phi_{S\eta}(z)} :: \right] + O(1) \quad (5.23)$$

with

$$\Phi_{FS}(z) = \phi_0' (k \mid z) + \phi_1 \left( k + 2 \mid z; \frac{k + 2}{2} \right) + \phi_2 \left( k \mid z; \frac{k}{2} \right),$$

$$\Phi_{E\eta}(z) = -\phi_0 (k \mid z) - \phi_1 \left( 2 \mid z; \frac{k + 2}{2} \right) - \phi_2 \left( k \mid z; -\frac{k}{2} \right) - \phi_2 \left( 2 \mid z; \frac{k + 4}{2} \right),$$

$$\Phi_{S\eta}(z) = \phi_1 \left( k + 2 \mid z; -\frac{k + 2}{2} \right) - \phi_1 \left( 2 \mid z; \frac{k - 2}{2} \right) - \phi_2 \left( 2 \mid z; \frac{k}{2} \right).$$

Here the difference of the function $f(z)$ is defined by

$$a \partial_z f(z) = \frac{f(q^a z) - f(q^{-a} z)}{q - q^{-1}}.$$

In addition, from Theorem 4.1, the following commutation relations hold.
Proposition 5.2

\[
E(z)E(w) = \frac{[u - v + 1]_{r-k}}{[u - v - 1]_{r-k}} E(w)E(z), \quad (5.24)
\]

\[
F(z)F(w) = \frac{[u - v - 1]_{r}}{[u - v + 1]_{r}} F(w)F(z), \quad (5.25)
\]

\[
S(z)S(w) = \frac{[u - v + 1]_{k+2}}{[u - v - 1]_{k+2}} S(w)S(z). \quad (5.26)
\]

Here the notation \([u]_x (x = r, r-k, k+2)\) stands for the theta function

\[
[u]_x = \vartheta_1 \left( \frac{u}{x} \mid \tau_x \right), \quad (5.27)
\]

where we set \(q^{2x} = e^{-2\pi i/\tau_x}\). Hence \(\tau_r = \tau\) and \(\tau_{r-k} = \tau^*\). One should note the following quasi-periodicities.

\[
[u + x]_x = -[u]_x, \quad [u + x\tau_x]_x = -e^{-\pi i (2u + \tau_x)}[u]_x. \quad (5.28)
\]

Now let us define a set of screening operators.

Definition 5.3

\[
Q_+ = \oint \frac{dz}{2\pi i} E(z) \frac{[u - \frac{1}{2} + \hat{\Pi}]_{r-k}}{[u - \frac{1}{2}]_{r-k}}, \quad (5.29)
\]

\[
Q_- = \oint \frac{dz}{2\pi i} F(z) \frac{[u + \frac{1}{2} - \hat{\Pi}']_{r}}{[u + \frac{1}{2}]_{r}}, \quad (5.30)
\]

\[
Q = \oint \frac{dz}{2\pi i} S(z) \frac{[u - \frac{1}{2} + P_1]_{k+2}}{[u - \frac{1}{2}]_{k+2}}, \quad (5.31)
\]

\[
\eta_0 = \oint \frac{dz}{2\pi i} \eta(z), \quad (5.32)
\]

where

\[
\hat{\Pi} = \sqrt{\frac{2r(r-k)}{k}} P_0 - \frac{r-k}{k} P_2, \quad (5.33)
\]

\[
\hat{\Pi}' = \sqrt{\frac{2r(r-k)}{k}} P_0 - \frac{r}{k} P_2. \quad (5.34)
\]

Due to the quasi-periodicity (5.28), the integrands in \(Q_+, Q_-\) and \(Q\) are single valued in \(z\). In addition the integrand in \(\eta_0\) is single valued on \(\mathcal{F}_{J,M;n',n}\). Therefore all the integrations in Definition 5.3 can be taken over a closed contour on \(\mathcal{F}_{J,M;n',n}\).

The following commutation relations hold.
Lemma 5.4

\[ [Q_\pm, Q] = 0, \] \hspace{1cm} (5.35)
\[ [Q_\pm, \eta_0] = 0, \] \hspace{1cm} (5.36)
\[ \{Q, \eta_0\} = 0, \] \hspace{1cm} (5.37)

where \( \{A, B\} = AB + BA \).

Proof of (5.35) From Proposition 5.1 and \([\hat{\Pi}, S(w)] = 0, [P_1, E(z)] = 0\), the relation \([Q_+, Q] = 0\) is trivial. For the commutation \([Q_-, Q]\), we have from Proposition 5.1 and \([\hat{\Pi}', S(w)] = 0, [P_1, F(z)] = 0\)

\[ [Q_-, Q] = -[k + 2] \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} k+2 \partial w \left[ \frac{1}{z - w} : e^{\Phi_{FS}(z)} : \right] \]

\[ \times \frac{[u + \frac{1}{2} - \hat{\Pi}_{1}]_r}{[u + \frac{1}{2}]_r} \frac{[v - \frac{1}{2} + \Pi_1]_{k+2}}{[v - \frac{1}{2}]_{k+2}}. \] \hspace{1cm} (5.38)

Here \( C_z \) denotes a closed contour enclosing the points \( q^{\pm(k+2)} z \). After taking the integral in \( w \), the quasi-periodicity of the theta function \( [\hat{u}]_{k+2} \) makes the right hand side of (5.38) vanish.

The other statements can be proved in the similar way.

\[ \square \]

Remark 5.1 Due to the theta function factor in the Definition 5.3, the screening operators \( Q_+ \) and \( Q_- \) does not commute each other. However, the relation \([Q_n^+, Q_n^-] = 0\) holds on \( \mathcal{F}_{j,M;n',n} \).

Let us set \( Q_n^+ = Q_n^+ \), \( Q_n^- = Q_n^- \), and \( Q_n = Q_n \). Then we claim

\[ \textbf{Theorem 5.5} \text{ The screening operators } Q_+, Q_-, Q \text{ and } \eta_0 \text{ are nilpotent:} \]

\[ Q_n^+ Q_{r-k-n}^+ = Q_{r-k-n}^+ Q_n^+ = Q_{r-k}^+ = 0, \] \hspace{1cm} (5.39)
\[ Q_n^- Q_{r-n}^- = Q_{r-n}^- Q_n^- = Q_r^- = 0, \] \hspace{1cm} (5.40)
\[ Q_n Q_{k+2-n} = Q_{k+2-n} Q_n = Q_{k+2} = 0, \] \hspace{1cm} (5.41)
\[ \eta_0^2 = 0. \] \hspace{1cm} (5.42)

The proof is due to the following lemma\(^7\).

Lemma 5.6

\[ Q_n^+ = \frac{[n]_{r-k}!}{n!} \prod_{j=1}^n \left( \oint \frac{dz_j}{2\pi i z_j} E(z_j) \right) \prod_{i<j} \frac{[u_i - u_j]_{r-k}}{[u_i - u_j - 1]_{r-k}} \prod_{i=1}^n \frac{[u_i + \frac{1}{2} + \hat{\Pi} - n]_{r-k}}{[u_i - \frac{1}{2}]_{r-k}}, \] \hspace{1cm} (5.43)

\(^7\) The screening operator \( S(z) \) is equivalent to the one in \( U_q(\hat{sl}_2) \) discussed in [10]. We hence have proved the nilpotency of the BRST charge in \( U_q(\hat{sl}_2) \) in the improved form as (5.31).
This lemma is proved by using the commutation relations in Proposition 5.2 and the following theta function identity [48].

Lemma 5.7

\[
\frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} [u_{\sigma(i)} - 2i + 2x] \prod_{i \neq j} [u_{\sigma(i)} - u_{\sigma(j)} + 1x] = \prod_{i \neq j} [u_i - u_j + x] \prod_{i=1}^{n} [u_i - n + 1x] \quad (x = r, r - k, k + 2). \quad (5.46)
\]

5.4 Resolution of the Fock modules \( \mathcal{F}_{J,M;n',n} \)

The Fock modules \( \mathcal{F}_{J,M;n',n} = \mathcal{F}_{J,M}^{PF} \otimes \mathcal{F}_{n',n}^{\phi_0} \) are reducible due to the existence of the singular vectors, which are constructed by the screening operators on some highest weight states [10, 31]. In order to obtain irreducible spaces, we consider a resolution of the modules \( \mathcal{F}_{J,M;n',n} \) following the method by Felder [37].

As mentioned in Sec.2, Lemma 5.4 allows us to divide the consideration into the following two steps. First consider the resolution of the \( \mathbb{Z}_k \) parafermion Fock modules \( \mathcal{F}_{J,M}^{PF} \) and get spaces \( \mathcal{H}_{J,M}^{PF} \) as irreducible representations. Then consider the resolution of the modules \( \mathcal{F}_{J,M}^{PF} \otimes \mathcal{F}_{n',n}^{\phi_0} \).

1) Resolution of the \( \mathbb{Z}_k \) q–parafermion Fock modules \( \mathcal{F}_{J,M}^{PF} \)

Let us first remind the reader the following two facts.

1. The \( \mathbb{Z}_k \) parafermion theory is obtained as the coset \( SU(2)_k/U(1) \). In the \( q \)–deformed case, especially in our case, this means that the \( q \)–deformed \( \mathbb{Z}_k \) parafermion theory is obtained from Matsuo’s bosonization of \( U_q(\hat{sl}_2) \) by dropping \{ \( \alpha_m \) \} boson.

2. The screening currents \( S(z) \) and \( \eta(z) \) are independent of the \{ \( \alpha_m \) \} bosons. Therefore these screening currents are common in the \( U_q(\hat{sl}_2) \) theory and the \( q \)–parafermion theory so that the structure of the singular vectors in the \( q \)–parafermion Fock modules are the same as corresponding Fock modules for \( U_q(\hat{sl}_2) \).

The structure of the Fock modules for \( U_q(\hat{sl}_2) \) was investigated in [44, 10]. The modules are given by \( \mathcal{F}_j = \bigoplus_{M \in \mathbb{Z}} \mathcal{F}_{j,M}^{PF} \otimes \mathcal{F}_M^{\phi_0} \). Here \( \mathcal{F}_M^{\phi_0} \) denotes the Fock module of the \{ \( \alpha_m \) \} boson. In [10], we showed, using the equivalent boson realization, that the Fock modules \( \mathcal{F}_j \) are reducible for the highest weight \( \lambda_{a,a'} = J_{a,a'} \Lambda_1 + (k - J_{a,a'}) \Lambda_0 \) due to the existence of
the singular vectors. Here $J_{a,a'} = a - (k+2)a' - 1$, the level $k$ being, in general, a rational number $k + 2 = P/P'$, $P, P' \geq 1$, $\text{GCD}(P, P') = 1$ and $1 \leq a \leq P - 1, 0 \leq a' \leq P' - 1$. The case $P' = 1$ is relevant for the parafermion theory. It was then observed that the Fock module structure, i.e. the degree of the singular vectors and the cosingular vectors as well as the multiplicities in each degree seems to be the same as the CFT case\[38\]. This was checked by calculating the characters of the irreducible representation spaces obtained by a resolution of the Fock modules.

Following the procedure given for $U_q(\hat{sl}_2)$\[14, 10\] and the analysis in CFT\[46, 47\], we make the resolution of the $q$–parafermion Fock modules $\mathcal{F}_{J,M}$ as follows.

We first consider the following restriction.

\[ \tilde{\mathcal{F}}_{J,M} = \ker(\eta : \mathcal{F}_{J,M} \to \mathcal{F}_{J+k+2,M+k}). \] (5.47)

Since $\eta_0^2 = 0$, the complex of the Fock modules associated with the map $\eta_0 : \mathcal{F}_{J,M} \to \mathcal{F}_{J+k+2,M+k}$ has trivial cohomology. We hence have

\[ \text{tr} \left( \tilde{\mathcal{F}}_{J,M} \right) \mathcal{O} = \sum_{u \geq 0} (-)^u \text{tr} \left( \mathcal{F}_{J,M}^{[u]} \right) \mathcal{O} \] (5.48)

and

\[ 0 = \sum_{u \in \mathbb{Z}} (-)^u \text{tr} \left( \mathcal{F}_{J,M}^{[u]} \right) \mathcal{O}, \] (5.49)

where $\mathcal{F}_{J,M}^{[u]} \equiv \mathcal{F}_{J+2u,M+2u}$. On the other hand, the operator $Q$ generates the following complex of the restricted Fock modules $\tilde{\mathcal{F}}_{J,M}$.

\[ \ldots \rightarrow \tilde{\mathcal{F}}_{J,-1} \rightarrow \tilde{\mathcal{F}}_{J,0} \rightarrow \tilde{\mathcal{F}}_{J,1} \rightarrow \tilde{\mathcal{F}}_{J,2} \rightarrow \tilde{\mathcal{F}}_{J,3} \rightarrow \cdots \] (5.50)

Here we introduced the notations $Q_{J+1}^{[2s]} = Q_{J+1}^{[2s+1]} = Q_{J+1}^{[2s+1]}$, $Q_{J+1}^{[2s]} = Q_{J+1}^{[2s+1]}$, $Q_{J+1}^{[2s+1]} = Q_{J+1}^{[2s+1]}$, and $Q_{J+1}^{[2s+1]} = Q_{J+1}^{[2s+1]}$, $Q_{J+1}^{[2s+1]} = Q_{J+1}^{[2s+1]}$, respectively.

The results in CFT\[38, 46, 47, 39\] and the investigation in the quantum affine algebra $U_q(\hat{sl}_2)$\[10\] lead us to the following conjecture

**Conjecture 5.8** The cohomology of the complex (5.50) is given by

\[ \ker Q_{J+1}^{[s]} \mathcal{O} = \begin{cases} \mathcal{H}_{J,M}^{PF} & \text{for } s \neq 0 \\ \mathcal{H}_{J,M}^{PF} & \text{for } s = 0 \end{cases}, \] (5.51)

where $\mathcal{H}_{J,M}^{PF}$ is the irreducible highest weight module of the $q$–deformed $\mathbb{Z}_k$ parafermion theory with the highest weight $h_{J,M}$ (5.16).

As a consequence, we obtain the following trace formula
Corollary 5.9

\[ \text{tr } \mathcal{H}_{J,M}^{PF} \mathcal{O} = \sum_{s \in \mathbb{Z}} (-)^s \text{tr } \mathcal{F}_{J,M}^{PF[s]} \mathcal{O}^{[s]}, \]  

(5.52)

where \( \mathcal{O}^{[s]} \) is an operator on \( \mathcal{F}_{J,M}^{PF} \) obtained by the recursion formula

\[ Q^a_{\alpha} \mathcal{O}^{[s]} = \mathcal{O}^{[s+1]} Q^a_{\alpha} \]  

(5.53)

with \( \mathcal{O}^{[0]} = \mathcal{O} \).

Combining (5.52) and (5.48), we get

\[ \text{tr } \mathcal{H}_{J,M}^{PF} \mathcal{O} = \sum_{s \in \mathbb{Z}} \sum_{u \geq 0} (-)^{s+u} \text{tr } \mathcal{F}_{J,M}^{PF[s,u]} \mathcal{O}^{[s]}, \]  

(5.54)

where

\[
\begin{align*}
\mathcal{F}_{J,M}^{PF[2s,u]} &= \mathcal{F}_{J+(k+2)(u-2s),M+ku}^{PF}, \\
\mathcal{F}_{J,M}^{PF[2s+1,u]} &= \mathcal{F}_{-J-2+(k+2)(u-2s),M+ku}^{PF}.
\end{align*}
\]

To support the conjecture, let us apply this formula to the calculation of the character of the space \( \mathcal{H}_{J,M}^{PF} \). We obtain

\[ \chi_{J,M}(\omega) = \text{tr } \mathcal{H}_{J,M}^{PF} y^{L_0^{PF}-c_{PF}/24} = \eta(\omega) c^J_M(\omega) \]  

(5.55)

with \( y = e^{2\pi i \omega} \). Here the function \( c^J_M(\omega) \) denotes the string function

\[ c^J_M(\omega) = \eta(\omega)^{-3} \sum_{s \in \mathbb{Z}} \sum_{u \geq 0} (-1)^u \left( x_{B_{J,M}^{[s,u]}}^{[s,u]} - x_{B_{-J-2,M}^{[s,u]}}^{[s,u]} \right), \]  

(5.56)

\[ B_{J,M}^{[s,u]} = \frac{(J + 1 - 2(k + 2)(s - u/2))^2}{4(k + 2)} - \frac{(M + ku)^2}{4k} \]  

(5.57)

with \( \eta(\omega) \) being the Dedekind eta function

\[ \eta(\omega) = y^{1/24} \prod_{n=1}^{\infty} (1 - y^n). \]  

(5.58)

The result (5.55) is precisely the irreducible character of the \( \mathbb{Z}_k \) parafermion theory.

2) Resolution of the Fock modules \( \mathcal{F}_{J,M;n',n} \)

Replacing the space \( \mathcal{F}_{J,M}^{PF} \) with \( \mathcal{H}_{J,M}^{PF} \) in \( \mathcal{F}_{J,M;n',n} \), now we consider the Fock modules

\[ \mathcal{F}_{J,M;n',n} \supset \mathcal{H}_{J,M}^{PF} \otimes \mathcal{F}_{n',n}^{2k_0} \]  

with \( J = |n' - n \pmod{2k}|, M = n' - n \pmod{2k}, 1 \leq n \leq r - k - 1, 1 \leq n' \leq r - 1, 0 \leq J \leq k \).
Let us introduce the notations $Q_n^{+[2]} = Q_n^+, Q_n^{+[2t+1]} = Q_{r-k-n}^+, Q_n^{-[2]} = Q_n^−, Q_n^{-[2t+1]} = Q_{r-n'}^− (t \in \mathbb{Z})$ and

\[
\begin{align*}
\tilde{F}_{J,M;n',n}^{[2]} &= \tilde{F}_{J,M+2(r-k)t ; n',n-2(r-k)t}, \\
\tilde{F}_{J,M;n',n}^{[2t+1]} &= \tilde{F}_{J,M+2n+2(r-k)t; n',-n-2(r-k)t}, \\
\tilde{F}_{J,M;n',n}' &= \tilde{F}_{J,M+n; n'} \; ; n'-2rt,n', \\
\tilde{F}_{J,M;n',n}^{[2t+1]}' &= \tilde{F}_{J,M-2n'+2rt ; -n'-2rt,n'}.
\end{align*}
\]

(5.59) (5.60) (5.61) (5.62)

Then, the screening operators $Q_+^+, Q_-^-$ generate the following infinite sequences of the Fock modules $\tilde{F}_{J,M;n',n}$.

\[
\begin{align*}
\cdots Q_n^{+[-2]} \to \tilde{F}_{J,M;n',n}^{[-1]} Q_n^{+[-1]} \to \tilde{F}_{J,M;n',n}^{[0]} Q_n^{+[0]} \to \tilde{F}_{J,M;n',n}^{[1]} Q_n^{+[1]} \to \cdots, \\
\cdots Q_n^{-[-2]} \to \tilde{F}_{J,M;n',n}^{[-2]} Q_n^{-[-1]} \to \tilde{F}_{J,M;n',n}^{[0]} Q_n^{-[0]} \to \tilde{F}_{J,M;n',n}^{[1]} Q_n^{-[1]} \to \cdots.
\end{align*}
\]

(5.63) (5.64)

Due to Theorem 5.5, these are complexes. As in the level one case [30], it is enough to consider one of them. Let us consider the complex (5.63). The following observation suggests the existence of the singular vectors in $\tilde{F}_{J,M;n',n}$ similar to the CFT case [33].

Let $M_{n',n} = n' - n \mod 2k$ and consider the vector

\[
|\chi_{-n',n}⟩ = Q_n^+|J, M_{-n',n}; -n', n⟩ ∈ \tilde{F}_{J,M;-n',n ; -n', n}.
\]

(5.65)

Using Lemma 5.6 and the operator product

\[
\Psi(z_1)\Psi(z_2) = \left(\frac{1}{q - q^{-1}}\right)^2 \frac{z_1^{-2} q (q^{2(k+1)/2} z_2/z_1; q^{2k})}{(q^{-2} z_2/z_1; q^{2k})} × (1 - z_2/z_1) : Ψ_I(z_1)Ψ_I(z_2) : - (1 - q^{-2} z_2/z_1) : Ψ_I(z_1)Ψ_{II}(z_2) : - q^{-2} (1 - q^{2} z_2/z_1) : Ψ_{II}(z_1)Ψ_I(z_2) : + (1 - z_2/z_1) : Ψ_{II}(z_1)Ψ_{II}(z_2) : )
\]

(5.66)

where we set the RHS of (4.13) as $\frac{1}{q - q^{-1}} (Ψ_I(z) - Ψ_{II}(z))$, we have

\[
\begin{align*}
Q_n^+|J, M_{-n',n}; -n', n⟩ &= \frac{[n]_{r-k}^{-1}}{n! [1]_{r-k}^{-1}} \frac{1}{q - q^{-1}} \int \frac{dz_1}{2\pi i z_1} \cdots \int \frac{dz_n}{2\pi i z_n} \prod_{i=1}^n z_i^{2\sqrt{r(r-x)} a_{n',n} - \frac{n}{k} \sum_{i<j} \frac{2}{z_i - z_j} q (2^{k+1} z_j/z_i; q^{2k})}{(q^{-2} z_j/z_i; q^{2k})} \times \text{sum of}\left((\text{polynomials of } z_j/z_i \ (1 \leq i < j \leq n))\right) \times \exp \left\{ \text{polynomials of } a_{0,m}, a_{1,m'}, a_{2,m''}, z_j \ (m, m', m'' \in \mathbb{Z} > 0, j = 1, \ldots, n) \right\} \times |J, M_{-n',n}; -n', n⟩.
\end{align*}
\]

(5.67)

Setting $z_1 = z$, $z_j = zw_j \ (j = 2, \ldots, n)$ and collecting all the $z$-dependence in the integrand, one can factor the integral $\int dz \ z^{-N - \frac{n(n'+n)}{k} - \frac{n}{k} M_{n',n}}$, where $N \in \mathbb{Z}_{\geq 0}$ comes
from the exponent in the fourth line of (5.67). Let us evaluate $N$. From (5.14), the non-vanishing term in (5.67) has degree $N + h_{J,M_n',n'} - h_{n,n}$. On the other hand, the BRST charge $Q_n^+$ commutes with $L_0$. Hence the degree of the same term should be equal to $h_{J,M_n',n'} - h_{n,n}$. Hence we have $N = h_{J,M_n',n'} - h_{n,n} = (4n' + M^2_{n',n'} - M^2_{n',n})/4k$. This value $N$ is consistent with the non-vanishing of the integral $\oint dz$.

We hence conjecture the following statement.

**Conjecture 5.10** The complex (5.63) has the non-trivial cohomology only at $t = 0$, i.e.

$$\operatorname{Ker} Q_n^+[t]/\operatorname{Im} Q_n^+[t-1] = \begin{cases} 0 & \text{for } t \neq 0 \\ H_{J,M_n,n'} & \text{for } t = 0 \end{cases}$$

(5.68)

Here the space $H_{J,M_n,n'}$ is the conjectural irreducible highest weight representation space of the $q-$Virasoro algebra with central charge $c = (2,3)$ with the highest weight $h_{J,M_n,n'} = h_{J,M_n,n'}$.

As a consequence, we can derive a trace formula, which relates the trace over $H_{J,M_n,n'}$ to those over the Fock spaces $\mathcal{F}_{J,M_n'n'}^t$, $t \in \mathbb{Z}$.

**Corollary 5.11**

$$\operatorname{tr} H_{J,M_n,n'} O = \sum_{t \in \mathbb{Z}} (-)^t \operatorname{tr} \mathcal{F}_{J,M_n'n'}^t O^t,$$

(5.69)

where operator $O^t$ on $\mathcal{F}_{J,M_n'n'}^t$ is defined recursively by

$$Q_n^+[t] O^t = O^{t+1} Q_n^+[t]$$

(5.70)

with $O^{[0]} = O$.

Combining (5.54) and (5.69), we finally get the formula

$$\operatorname{tr} H_{J,n',n} O = \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \sum_{u \geq 0} (-)^{s+t+u} \operatorname{tr} \mathcal{F}_{J,M_n'n'}^{s,t,u} \mathcal{F}_{J,M_n',n}^{s,t,u} O^{[s,t]}$$

(5.71)

where the Fock modules $\mathcal{F}_{J,M_n'n'}^{s,t,u}$ denote the following.

$$\mathcal{F}_{J,M_n',n}^{2s,2t,u} = \mathcal{F}_{J-M_n',n}^{PF}[u] \mathcal{F}_{n',n,n}^{0}[u],$$

(5.72)

$$\mathcal{F}_{J,M_n',n}^{2s+1,2t,u} = \mathcal{F}_{J-M_n',n}^{PF}[u] \mathcal{F}_{n',n,n}^{[0]}[u],$$

(5.73)

$$\mathcal{F}_{J,M_n',n}^{2s,2t+1,u} = \mathcal{F}_{J-M_n',n}^{PF}[u] \mathcal{F}_{n',n,n}^{[0]}[u],$$

(5.74)

$$\mathcal{F}_{J,M_n',n}^{2s+1,2t+1,u} = \mathcal{F}_{J-M_n',n}^{PF}[u] \mathcal{F}_{n',n,n}^{[0]}[u],$$

(5.75)

The operator $O^{[s,t]}$ acting on the space $\mathcal{F}_{J,M_n'n'}^{s,t,u}$ is defined recursively by

$$O^{[s,0]} = O^{[s]},$$

(5.76)

$$Q_n^+[t] O^{[s,t]} = O^{[s,t+1]} Q_n^+[t].$$

(5.77)
Applying the formula (5.71), we obtain the character of the space $\mathcal{H}_{J,n',n}$:

$$\chi_{J,n',n}(\omega) = \text{tr}_{\mathcal{H}_{J,n,n}} y^{L_0-c/24}$$

$$= \sum_{\bar{M} = -k+1}^k c_{J,\bar{M}}(\omega) \left( \sum_{t \in \mathbb{Z}} \delta_{\bar{M},M_{n',n-2(r-k)t}} y^{B_{\phi_0}^{[2t]}} - \sum_{t \in \mathbb{Z}} \delta_{\bar{M},M_{n',n-2(r-k)t}} y^{B_{\phi_0}^{[2t+1]}} \right),$$

(5.78)

where $c$ is given by (2.5) and

$$B_{\phi_0}^{[2t]} = \frac{1}{4kr(r-k)} \left( n'(r-k) - nr + 2r(r-k)t \right)^2,$$

(5.79)

$$B_{\phi_0}^{[2t+1]} = \frac{1}{4kr(r-k)} \left( n'(r-k) + nr + 2r(r-k)t \right)^2.$$

(5.80)

The result (5.78) precisely gives the branching coefficient in the formula

$$\chi_{J,0}^{(k)}(\omega)\chi_{n-1}^{(l)}(\omega) = \sum_{n'} \chi_{J,n',n}(\omega)\chi_{n'-1}^{(k+l)}(\omega),$$

(5.81)

where $\chi_{L}^{(k)}(\omega)$ ($L = J, n-1, n'-1$) are the irreducible characters of the affine Lie algebra $\hat{sl}_2$ labeled by the level $k$ and spin $L/2$.

The one point local height probability of the $k-$fusion RSOS model in the regime III is given by the same branching coefficient as (5.78). Hence one may regard the space $\mathcal{H}_{J,n',n}$ as the space of states of the $k-$fusion RSOS model in the regime III, on which Baxter’s CTM acts[2, 23].

6 Discussion

In summary, we have introduced the elliptic algebra $U_{q,p}(\hat{sl}_2)$. Based on this algebra, we have extended the bosonization of the ABF model in [30] to the $k-$fusion RSOS model. We have obtained it as the $q-$deformation of the coset conformal field theory $SU(2)_k \times SU(2)_{r-k-2}/SU(2)_{r-2}$. A full set of screening currents and the highest component of the two types of vertex operators have been derived. We have observed that these operators give a proper characterization of the Fock spaces as the space of states in the $k-$fusion RSOS model.

We have conjectured that there exists a corresponding $q-$deformation of the extended Virasoro algebra in such a way that its screening currents satisfy the algebra $U_{q,p}(\hat{sl}_2)$ and the $q-$deformed primary fields are determined as the intertwiners between the infinite representation space of $U_{q,p}(\hat{sl}_2)$. In order to establish this point, we need to clarify the following two points.

1. The realization of the $q-$Virasoro generator with the central charge (2.3) and the extra generators.
2. The Hopf algebra structure of $U_{q,p}(\hat{sl}_2)$.

Recently, Jimbo has succeeded to derive the algebra $U_{q,p}(\hat{sl}_2)$ by the Gauss decomposition of a central extended dynamical $RLL-$relations with the $R-$matrix introduced by Enriquez and Felder[40]. This result allows us to clarify a Hopf algebra structure of $U_{q,p}(\hat{sl}_2)$. The work along this line is now in progress.

For the completion of the identification of our bosonization with the $k-$fusion RSOS model, we need further to show that the commutation relation among the type I and the type II vertices are given precisely by the $k-$fused RSOS Boltzmann weights[29]. We have checked this point both in the $k = 1$ case and in the case of arbitrary $k$ but for the highest components of the type I and type II vertex operators (Prop. 4.5).

As the scaling limit, we also have obtained a bosonization of the algebra $A_{h,\eta}(\hat{sl}_2)$ at arbitrary level $k$. The level one case has shown to be a relevant symmetry for deriving the form factors in the sine-Gordon theory[20]. In this respect, one may relate the higher level case, especially, the case $k = 2$ to the super sine-Gordon theory. Our bosonization should be useful for deriving the form factors of such theory.

It is also an interesting problem to extend our results to the higher rank case as well as to the case of other types of Lie algebras. For these extensions, the corresponding SOS models are known[49, 50]. We expect that these SOS models could be bosonized based on the corresponding extension of our $U_{q,p}(\hat{sl}_2)$.

7 Acknowledgments

The author would like to thank Michio Jimbo for stimulating discussions and various suggestions. He is also grateful to Kenji Iohara, Atsuo Kuniba, Tetsuji Miwa, Stanislav Pakuliak and Junichi Shiraishi for valuable discussions. This work is supported in part by the Ministry of Education Contract No.09740028.

A The Algebra $A_{h,\eta}(\hat{sl}_2)$

We here give a brief review of the results in [21].

Definition A.1 The algebra $A_{h,\eta}(\hat{sl}_2)$ is generated by the symbols $\hat{e}_\lambda, \hat{f}_\lambda, \hat{t}_\lambda, \lambda \in \mathbb{R}$ and the central element $c$ with the following relations.

$$H^+(\alpha)H^-(\beta) = \frac{\text{sh} \pi \eta(\alpha - \beta - ig(1 - \frac{\pi}{2})) \text{sh} \pi \eta'(\alpha - \beta + ig(1 - \frac{\pi}{2}))}{\text{sh} \pi \eta(\alpha - \beta + ig(1 + \frac{\pi}{2})) \text{sh} \pi \eta'(\alpha - \beta - ig(1 + \frac{\pi}{2}))} H^-(\beta)H^+(\alpha),$$

(A.1)

$$H^\pm(\alpha)H^\pm(\beta) = \frac{\text{sh} \pi \eta(\alpha - \beta - ig) \text{sh} \pi \eta'(\alpha - \beta + ig)}{\text{sh} \pi \eta(\alpha - \beta + ig) \text{sh} \pi \eta'(\alpha - \beta - ig)} H^\pm(\beta)H^\pm(\alpha),$$

(A.2)

$$H^\pm(\alpha)E(\beta) = \frac{\text{sh} \pi \eta(\alpha - \beta - ig(1 + \frac{\pi}{2}))}{\text{sh} \pi \eta(\alpha - \beta + ig(1 + \frac{\pi}{2}))} E(\beta)H^\pm(\alpha),$$

(A.3)
with $\eta$.

The following relations hold.

\begin{align}
H^\pm(\alpha)F(\beta) &= \frac{\sh \pi \eta'(\alpha - \beta + \imath \hbar(1 \mp \frac{\gamma}{4}))}{\sh \pi \eta'(\alpha - \beta - \imath \hbar(1 \pm \frac{\gamma}{4}))} F(\beta)H^\pm(\alpha), \\
E(\alpha)E(\beta) &= \frac{\sh \pi \eta'(\alpha - \beta - \imath \hbar)}{\sh \pi \eta'(\alpha - \beta + \imath \hbar)} E(\alpha)E(\beta), \\
F(\alpha)F(\beta) &= \frac{\sh \pi \eta'(\alpha - \beta + \imath \hbar)}{\sh \pi \eta'(\alpha - \beta - \imath \hbar)} F(\beta)F(\alpha), \\
E(\alpha)F(\beta) &= F(\beta)E(\alpha) = 2 \pi \left( \frac{1}{\alpha - \beta - \frac{i\hbar c}{2}} H^+(\alpha - \frac{i\hbar c}{4}) - \frac{1}{\alpha - \beta + \frac{i\hbar c}{2}} H^-(\beta - \frac{i\hbar c}{4}) \right) + O(1),
\end{align}

where $\hbar$ and $\eta(>0)$ are real parameter and $1/\eta' - 1/\eta = \hbar c > 0$. The generating functions (currents) $E(\alpha), F(\alpha)$ and $H^\pm(\alpha)$ are defined by the following formulae.

\begin{align}
E(\alpha) &= \int_{-\infty}^{\infty} d\lambda e^{\imath \lambda \alpha} \hat{e}_\lambda, \\
F(\alpha) &= \int_{-\infty}^{\infty} d\lambda e^{\imath \lambda \alpha} \hat{f}_\lambda, \\
H^\pm(\alpha) &= -\frac{\hbar}{2} \int_{-\infty}^{\infty} d\lambda e^{\imath \lambda \alpha} \hat{h}_\lambda e^{\mp \lambda/2\eta''}.
\end{align}

with $\eta'' = \frac{2\eta''}{\eta + \eta'}$.

Define another generating functions $e^\pm(\alpha), f^\pm(\alpha)$ and $h^\pm(\alpha)$ by

\begin{align}
\hat{e}^\pm(\alpha) &= \sin \pi \hbar \int_{C} \frac{d\gamma}{2\pi i \sh \pi \eta'(\alpha - \gamma \pm \imath \hbar \frac{\gamma}{4})} E(\alpha), \\
\hat{f}^\pm(\alpha) &= \sin \pi \eta' \int_{C'} \frac{d\gamma}{2\pi i \sh \pi \eta'(\alpha - \gamma \mp \imath \hbar \frac{\gamma}{4})} F(\alpha), \\
\hat{h}^\pm(\alpha) &= \frac{\sin \pi \eta \hbar}{\pi \eta \hbar} H^\pm(\alpha).
\end{align}

The following relations hold.

\begin{align}
e^-(\alpha) &= -e^+(\alpha - i/\eta''), \quad f^-(\alpha) = -f^+(\alpha - i/\eta''), \quad h^-(\alpha) = h^+(\alpha - i/\eta'').
\end{align}

The comultiplication of the algebra $\mathcal{A}_{\hbar, \eta}(\mathbb{S}U(2))$ is given for $e^+(\alpha, \xi) = e^+(\alpha), f^+(\alpha, \xi) = f^+(\alpha), h^+(\alpha, \xi) = h^+(\alpha)$ by the formulae

\begin{align}
\Delta c &= c' + c'' = c \otimes 1 + 1 \otimes c, \\
\Delta e^+(\alpha, \xi) &= e^+(\alpha', \xi) \otimes 1 + \sum_{p=0}^{\infty} (-1)^p (f^+(\alpha' - i\hbar, \xi'))^p h^+(\alpha', \xi') \otimes (e^+(\alpha'', \xi''))^{p+1}, \\
\Delta f^+(\alpha, \xi) &= 1 \otimes f^+(\alpha', \xi),
\end{align}

28
\[ + \sum_{p=0}^{\infty} (-1)^p (f^+(\alpha', \xi'))^{p+1} \otimes \hat{h}^+(\alpha'', \xi'')(e^+(\alpha'' - i\hbar, \xi''))^p, \] (A.17)

\[ \Delta h^+(\alpha, \xi) = h^+(\alpha', \xi') \otimes h^+(\alpha'', \xi'') + \sum_{p=0}^{\infty} (-1)^p[p + 1]_n (f^+(\alpha' - i\hbar, \xi'))^p h^+(\alpha', \xi') \otimes h^+(\alpha'', \xi'')(e^+(\alpha'' - i\hbar, \xi''))^p, \] (A.18)

where \( \xi = 1/\eta = \xi', \xi'' = \xi + \hbar c, \alpha' = \alpha + i\hbar c'/4, \alpha'' = \alpha - i\hbar c'/4 \) and

\[ [p]_n = \frac{\sin \pi \eta \hbar p}{\sin \pi \eta \hbar}, \quad \hat{h}^+(\alpha) = \frac{\eta}{\eta'} \frac{\sin \pi \eta \hbar}{\sin \pi \eta \hbar} h^+(\alpha). \]

Let \( V^{(l)} \) be the \( l + 1 \)-dimensional representation of \( U_\rho(sl_2) \), \( \rho = e^{i\eta \hbar h} \) with a basis \( v_m \ (m = 0, 1, 2, \ldots, l) \). The \( l + 1 \)-dimensional evaluation representation \( V^{(l)}(\xi) = V^{(l)} \otimes \mathbb{C}[e^{i\Lambda \xi}], \lambda \in \mathbb{R}, \xi \in \mathbb{C} \) is given for the base \( v_m(\xi) \in V^{(l)}(\xi) \) by

\[ e^+(\alpha)v_m(\xi) = -\frac{\sin \pi \eta \hbar m}{\sin \pi \eta (\alpha - \xi + i\hbar l - \frac{2m-1}{2})} v_{m-1}(\xi), \] (A.19)

\[ f^+(\alpha)v_m(\xi) = -\frac{\sin \pi \eta \hbar (l - m)}{\sin \pi \eta (\alpha - \xi + i\hbar l - \frac{2m-1}{2})} v_{m+1}(\xi), \] (A.20)

\[ h^+(\alpha)v_m(\xi) = \frac{\sin \pi \eta (\alpha - \xi + i\hbar l + \frac{1}{2})}{\sin \pi \eta (\alpha - \xi + i\hbar l - \frac{2m-1}{2})} v_m(\xi). \] (A.21)

An infinite dimensional highest weight representation of the algebra \( \mathfrak{A}_{h,\eta} (sl_2) \) is constructed as a Fock space \( \mathcal{F} \) of free bosons [18, 21]. See also Sec.6. Here the highest weight property means that

\[ \hat{e}_\lambda |h.w.\rangle = 0, \quad \hat{f}_\lambda |h.w.\rangle = 0 \quad \lambda \in \mathbb{R}. \] (A.22)

There are two types of intertwining operators (vertex operators):

**Type I** \( \Phi^{(l)}(\xi) : \mathcal{F} \rightarrow \mathcal{F} \otimes V^{(l)}(\xi + i\hbar \frac{l}{2}), \) (A.23)

**Type II** \( \Psi^{(l)*}(\xi) : V^{(l)}(\xi + i\hbar \frac{l}{2}) \otimes \mathcal{F} \rightarrow \mathcal{F} \) (A.24)

satisfying

\[ \Phi^{(l)}(\xi) x = \Delta(x) \Phi^{(l)}(\xi), \] (A.25)

\[ \Psi^{(l)*}(\xi) \Delta(x) = x \Psi^{(l)*}(\xi). \] (A.26)

The components of the intertwiners are defined as follows.

\[ \Phi^{(l)}(\xi) u = \sum_{m=0}^{l} \Phi^{(l)}_m(\xi) u \otimes v_m, \] (A.27)

\[ \Psi^{(l)*}(\xi)(v_m \otimes u) = \Psi^{(l)*}_m(\xi) u \quad u \in \mathcal{F}. \] (A.28)
Using the comultiplication (A.15)–(A.18) and the finite dimensional evaluation representation (A.19)–(A.21), the intertwining relations are rewritten as follows. For type I,

\[ \Phi_t^{(l)}(\zeta) h^\pm(\alpha) = \frac{\sinh \pi \eta (\alpha - \zeta + i \hbar (l + \frac{k}{2}))}{\sinh \pi \eta (\alpha - \zeta - i \hbar (l - \frac{k}{2}))} h^\pm(\alpha) \Phi_t^{(l)}(\zeta), \]

(A.29)

\[ [e^\pm(\alpha), \Phi_t^{(l)}(\zeta)] = 0, \]

(A.30)

\[ \sinh i \pi \eta \hbar (l - m + 1) \Phi_m^{(l)}(\zeta) \]

\[ = - \sinh \pi \eta (\alpha - \zeta + \frac{i \hbar}{2} (l - 2m + 2 \mp \frac{k}{2})) [\Phi_m^{(l)}(\zeta) f^\pm(\alpha) \]

\[ - \frac{\sinh \pi \eta (\alpha - \zeta - \frac{i \hbar}{2} (l + 2 \mp \frac{k}{2})) \sinh \pi \eta (\alpha + \zeta + i \hbar (l + k/2))}{\sinh \pi \eta (\alpha + \zeta + \frac{i \hbar}{2} (l - 2m + \frac{k}{2}))} f^\pm(\alpha) \Phi_m^{(l)}(\zeta) \],

(A.31)

and for type II,

\[ h^\pm(\alpha) \Psi_t^{(l)\ast}(\zeta) = \frac{\sinh \pi \eta (\alpha - \zeta + i \hbar (l + \frac{k}{2}))}{\sinh \pi \eta (\alpha - \zeta - i \hbar (l - \frac{k}{2}))} \Psi_t^{(l)\ast}(\zeta) h^\pm(\alpha), \]

(A.32)

\[ [f^\pm(\alpha), \Psi_t^{(l)\ast}(\zeta)] = 0, \]

(A.33)

\[ \sinh i \pi \eta \hbar \Psi_m^{(l)\ast}(\zeta) \]

\[ = - \sinh \pi \eta (\alpha - \zeta + \frac{i \hbar}{2} (l - 2m - 2 \mp \frac{k}{2})) e^\pm(\alpha) \Psi_m^{(l)\ast}(\zeta) \]

\[ + \frac{\sinh \pi \eta (\alpha - \zeta - \frac{i \hbar}{2} (l + 2 \mp \frac{k}{2})) \sinh \pi \eta (\alpha + \zeta + i \hbar (l - k/2))}{\sinh \pi \eta (\alpha + \zeta + \frac{i \hbar}{2} (l - 2m \pm \frac{k}{2}))} \Psi_m^{(l)\ast}(\zeta) e^\pm(\alpha), \]

(A.34)

In some unknown reason, the intertwiners relevant for the physical problems such as the mass less XXZ model[18] and the sine-Gordon theory[19, 20] prefer the twisted intertwining relations. We denote by \( \tilde{\Phi}^{(l)}(\zeta) \) and \( \tilde{\Psi}^{(l)\ast}(\zeta) \) the intertwiners of the same types as (A.23) and (A.24) but obeying the following twisted intertwining relations.

\[ \tilde{\Phi}^{(l)}(\zeta) \iota(x) = \Delta(x) \tilde{\Phi}^{(l)}(\zeta), \]

(A.35)

\[ \tilde{\Psi}^{(l)\ast}(\zeta) \Delta(x) = \iota(x) \tilde{\Psi}^{(l)\ast}(\zeta), \]

(A.36)

where \( \iota \) is the following involution of \( \mathcal{A}_{n, \eta}(s \tilde{t}_2) \)

\[ \iota(e_\lambda) = - e_\lambda, \quad \iota(f_\lambda) = - f_\lambda, \quad \iota(\ell_\lambda) = \ell_\lambda. \]

(A.38)

Then the relations (A.29) and (A.32) remains the same but (A.30), (A.31), (A.33) and (A.34) are replaced with

\[ \{e^\pm(\alpha), \tilde{\Phi}_t^{(l)}(\zeta)\} = 0, \]

(A.39)
\[
\begin{align*}
\text{sh} i\pi \eta h(l - m + 1) \tilde{\Phi}_{m-1}(\zeta) \\
= \text{sh} i\pi\eta' (\alpha - \zeta + i\frac{\eta}{2} (l - 2m + 2 \mp \frac{k}{2})) [\tilde{\Phi}_{m}^{(l)}(\zeta) f^{\pm}(\alpha)] \\
+ \frac{\text{sh} i\pi\eta' (\alpha - \zeta - i\frac{\eta}{2} (l + 2 \pm \frac{k}{2})) \text{sh} i\pi\eta' (\alpha - \zeta - i\eta (l \mp k/2))}{\text{sh} i\pi\eta' (\alpha - \zeta + i\frac{\eta}{2} (l - 2m \mp \frac{k}{2}))} f^{\pm}(\alpha) \tilde{\Phi}_{m}^{(l)}(\zeta),
\end{align*}
\]

\(f^{\pm}(\alpha), \tilde{\Psi}_{m}^{(l)*}(\zeta)\} = 0, \quad \text{(A.40)}

\[
\begin{align*}
\text{sh} i\pi \eta h m \tilde{\Psi}_{m-1}^{(l)*}(\zeta) \\
= \text{sh} i\pi\eta (\alpha - \zeta + i\frac{\eta}{2} (l - 2m - 2 \pm \frac{k}{2})) e^{\pm}(\alpha) \tilde{\Psi}_{m}^{(l)*}(\zeta) \\
+ \frac{\text{sh} i\pi\eta (\alpha - \zeta - i\frac{\eta}{2} (l + 2 \pm \frac{k}{2})) \text{sh} i\pi\eta (\alpha - \zeta + i\frac{\eta}{2} (l \mp k/2))}{\text{sh} i\pi\eta (\alpha - \zeta + i\frac{\eta}{2} (l - 2m \pm \frac{k}{2}))} \tilde{\Psi}_{m}^{(l)*}(\zeta) e^{\pm}(\alpha), \quad \text{(A.42)}
\end{align*}
\]

References

[1] M. Jimbo and T. Miwa. *Algebraic Analysis of Solvable Lattice Models*. CBMS Regional Conference Series in Mathematics vol. 85, AMS, 1994.

[2] R. J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Academic, London, 1982.

[3] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov Infinite conformal symmetry in two dimensional quantum field theory. *Nucl.Phys. B*, 241: 333–380, 1980.

[4] V.S. Dotsenko and V.A. Fateev. Conformal algebra and multipoint correlation functions in two-dimensional statistical models. *Nucl.Phys.B*, 240[FS]: 312–, 1984; Four point correlation functions and the operator algebra in the two-dimensional conformal invariant theories with the central charge \(c < 1\). *Nucl.Phys.B*,251[FS]: 691–,1985.

[5] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki. Diagonalization of the \(XXZ\) Hamiltonian by vertex operators. *Comm.Math.Phys.*,151: 89–153, 1993.

[6] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki. Correlation functions of the \(XXZ\) model for \(\Delta < -1\). *Phys.Lett.A*, 168: 256–263, 1992.

[7] M. Idzumi, K. Iohara, M. Jimbo, T. Miwa, T. Nakashima and T. Tokihiro. Quantum affine symmetry in vertex models *Int.J. Mod. Phys. A*, 8:1479–1511, 1993.

[8] M. Idzumi. Level two irreducible representations of \(U_q(\widehat{sl}_2)\), vertex operators, and their correlations. Int. J. Mod. Phys. A,9:4449–4484,1994.

[9] A.H. Bougourzi and R.A Weston. \(N\) point correlation functions of the spin 1 \(XXZ\) model. *Nucl.Phys.B*,417:439-462,1994.
[10] H. Konno. BRST cohomology in quantum affine algebra $U_q(\widehat{sl}_2)$. *Mod. Phys. Lett. A*, 9: 1253–1265, 1994; Free field representation of the quantum affine algebra $U_q(\widehat{sl}_2)$ and form factors in the higher-spin $XXZ$ model. *Nucl. Phys. B*, 432 [FS]: 457–486, 1994.

[11] Y. Koyama. Staggered polarization of vertex models with $U_q(\widehat{sl}_N)$ symmetry. *Comm. Math. Phys.*, 164: 277–292, 1994.

[12] M. Jimbo, T. Miwa and A. Nakayashiki. Difference equations for the correlation functions of the eight-vertex model. *Jour. of Phys. A*, 26: 2199–2199, 1993.

[13] M. Jimbo, R. Kedem, H. Konno, T. Miwa, and R. Weston. Difference equations in spin chains with a boundary. *Nucl. Phys. B*, 448 [FS]: 429–456, 1995.

[14] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa, and H. Yan. An elliptic quantum algebra for $\widehat{sl}_2$. *Lett. Math. Phys.*, 32: 259–268, 1994; Notes on highest weight modules of the elliptic algebra $A_{q,p}(\widehat{sl}_2)$. *Prog. Theor. Phys., Supplement*, 118: 1–34, 1995.

[15] K. Iohara and M. Kohno. A central extension of $DY_h(gl_2)$ and its vertex representations. *Lett. Math. Phys.*, 37: 319–328, 1996; K. Iohara. Bosonic representations of Yangian double $DY_h(g)$ with $g = gl_n, sl_n$. *J. of Phys. A*, 29: 4593–4621, 1996.

[16] S. Khoroshkin, Central Extension of the Yangian Double. [q-alg/9602031]. S. Khoroshkin, D. Lebedev and S. Pakuliak. Intertwining operators for the central extension of the Yangian double. *Phys. Lett. A*, 222: 381–392, 1996.

[17] H. Konno. Free field representation of level-$k$ Yangian double $DY(\widehat{sl}_2)_k$ and deformation of Wakimoto modules. *Lett. Math. Phys.*, 40: 321–336, 1997.

[18] M. Jimbo and H. Konno and T. Miwa. Massless $XXZ$ model and degeneration of the Elliptic Algebra $A_{q,p}(\widehat{sl}_2)$. Preprint RIMS-1105, 1996, to appear in the proceedings of the conference ”Deformation Theory, Symplectic Geometry and Applications”, Centro Stefano Franscini of ETH Zürich, Monte Verità, Ascona, June 16–22, 1996.

[19] S. Lukyanov. Free field representation for massive integrable models. *Comm. Math. Phys.*, 167: 183–226, 1995.

[20] H. Konno Degeneration of the Elliptic algebra $A_{q,p}(\widehat{sl}_2)$ and form factors in the sine-Gordon theory. Preprint Hiroshima Univ. [hep-th/9701034], to appear in the CRM series in Mathematical Physics, Springer Verlag.

[21] S. Khoroshkin and D. Lebedev and S. Pakuliak. Elliptic algebra $A_{q,p}(\widehat{sl}_2)$ in the scaling limit. Preprint [q-alg/9702002].

[22] G.E. Andrews and R.J. Baxter and P.J. Forrester. Eight vertex SOS model and generalized Rogers-Ramanujan-type identities. *J. Stat. Phys.*, 35: 193–266, 1984.
[23] E. Date and M. Jimbo and T. Miwa and M. Okado. Fusion of the eight vertex SOS model. *Lett. Math. Phys.*, 12: 209–215, 1986; E. Date and M. Jimbo and A. Kuniba and T. Miwa and M. Okado. Exactly solvable SOS models. *Nucl. Phys. B*, 290 [FS20]: 231–273, 1987; Exactly solvable SOS models II. *Adv. Stud. Pure Math.*, 16: 17–122, 1988.

[24] P. Goddard and A. Kent and D. Olive. Virasoro algebras and coset space models *Phys. Lett. B*, 152: 88–1985; Unitary representations of the Virasoro and super-Virasoro algebras *Comm. Math. Phys.*, 103: 105–119, 1986.

[25] D. Kastor and E. Martinec and Z. Qiu. Current algebra and conformal discrete series. *Phys. Lett. B*, 200: 434–440, 1988.

[26] J. Bagger and D. Nemeschansky and S. Yankielowicz. Virasoro algebras with central charge $c > 1$. *Phys. Rev. Lett.*, 60: 389–392, 1988.

[27] F. Ravanini. An infinite class of new conformal field theories with extended algebras *Mod. Phys. Lett. A*, 3: 397–412, 1988.

[28] M. Jimbo, T. Miwa and Y. Ohta. Structure of the space of states in RSOS models. *Int. J. Mod. Phys. A*, 8: 1457–1477, 1993.

[29] O. Foda, M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki. Vertex operators in solvable lattice models. *J. Math. Phys.*, 35: 13–46, 1994.

[30] S. Lukyanov and Y. Pugai. Multi-point Local Height Probabilities in the Integrable RSOS Model”, *Nucl. Phys. B*, 473: 631–658, 1996.

[31] J. Shiraishi, H. Kubo, H. Awata and S. Odake. A Quantum Deformation of the Virasoro Algebra and the Macdonald Symmetric Functions. *Lett. Math. Phys.*, 38: 33–51, 1996; Virasoro-type symmetries in solvable models preprint EFI-96-44, DPSU-96-18, UT-764, [hep-th/9612233](http://arxiv.org/abs/hep-th/9612233).

[32] H. Konno. $q$–deformation of the coset conformal field theory and the fusion RSOS model. Talks given at the XIIth International Congress of Mathematical Physics, 13–19 July, 1997, Brisbane and the International Workshop on “Statistical Mechanics and Integrable Systems”, 28 July–8 August, 1997, ANU, Canberra, Australia.

[33] M.A. Bershadsky and V.G. Knizhnik and M.G. Teitelman. Super conformal symmetry in two dimensions *Phys. Lett. B*, 151: 31–36, 1985.

[34] D. Friedan and Z. Qiu and S. Shenker. Super conformal invariance in two dimensions and the tricritical Ising model. *Phys. Lett. B*, 151: 37–43, 1985.

[35] A.B. Zamolodchikov and V.A. Fateev. Representations of the algebra of “parafermion currents” of spin 4/3 in two-dimensional conformal field theory. Minimal models and the tricritical potts $\mathbb{Z}_3$ model. *Theor. Math. Phys.*, 71: 451–462, 1987.
[36] A. Gerasimov and A. Marshakov and A. Morozov. Free field representation of parafermions and related coset models. *Nucl.Phys.B*, 328: 664, 1989.

[37] G. Felder. BRST approach to minimal model. *Nucl.Phys.B*317:215–236, 1989.

[38] D. Bernard and G. Felder. Fock Representations and BRST cohomology in $SL(2)$ current algebra. *Comm.Math.Phys.*,127:145–168, 1990.

[39] H. Konno. $SU(2)_k \times SU(2)_l/SU(2)_k + l$ coset conformal field theory and topological minimal model on higher genus Riemann surface. *Int. J. Mod. Phys. A*, 8: 5537–5561, 1993.

[40] B. Enriquez and G. Felder. Elliptic quantum groups $E_{\tau,\eta}(sl_2)$ and quasi-Hopf algebras. *q-alg/9703018*.

[41] I. B. Frenkel and N. H. Jing. Vertex Representations of Quantum Affine Algebras. *Proc. Nat’l. Acad. Sci. USA*, 85: 9373–9377, 1988.

[42] C. Ahn, D. Bernard and A. LeClair. Fractional supersymmetries in perturbed coset CFTs and integrable soliton theory. *Nucl.Phys.B*, 346: 409–439, 1990.

[43] B.-Y. Hou and W.-L. Yang. $\hat{h}$ (Yangian) deformed Virasoro algebra as a dynamically twisted $A_{h,\eta}$ algebra. Talk given at the XIIth International Congress of Mathematical Physics, 13-19 July, 1997, Brisbane, Australia.

[44] A. Matsuo. A $q$–deformation of Wakimoto modules, primary fields and screening operators. *Comm. Math. Phys.*, 161: 33–48, 1994.

[45] T. Miwa and R. Weston. Boundary ABF models. *Nucl.Phys.B*486: 517-545, 1997.

[46] J. Distler and Z. Qiu. BRST cohomology and a Feigin-Fuchs representation of Kac-Moody and parafermion theories. *Nucl. Phys. B*, 336: 533–546, 1990.

[47] M. Frau and A. Lerda and J.G. McCarthy and S. Sciuto and J. Sidenius. Free field representation for $\hat{s}l_2$ WZNW models on Riemann surfaces. *Phys. Lett. B*, 245 : 453–464, 1990.

[48] M. Jimbo, M. Lashkevich, T. Miwa and Y. Pugai. Lukyanov’s Screening Operators for the Deformed Virasoro Algebra. preprint RIMS-1087, July 1996, hep-th/9607177

[49] M. Jimbo, T. Miwa and M. Okado. Solvable lattice models whose states are dominant integral weights of $A_{n-1}^{(l)}$. *Lett.Math.Phys.*,14:123-131, 1987.

[50] D. Gepner. On RSOS models associated to Lie algebras and RCFT. *Phys. Lett. B*, 313: 45-54, 1993.