Non-commutative geometry of 4-dimensional quantum Hall droplet

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Abstract

We develop the description of non-commutative geometry of the 4-dimensional quantum Hall fluid’s theory proposed recently by Zhang and Hu. The non-commutative structure of fuzzy $S^4$, which is the base of the bundle $S^7$ obtained by the second Hopf fibration, i.e., $S^7/S^3 = S^4$, appears naturally in this theory. The fuzzy monopole harmonics, which are the essential elements in the non-commutative algebra of functions on $S^4$, are explicitly constructed and their obeying the matrix algebra is obtained. This matrix algebra is associative. We also propose a fusion scheme of the fuzzy monopole harmonics of the coupling system from those of the subsystems, and determine the fusion rule in such fusion scheme. By products, we provide some essential ingredients of the theory of $SO(5)$ angular momentum. In particular, the explicit expression of the coupling coefficients, in the theory of $SO(5)$ angular momentum, are given. We also discuss some possible applications of our results to the 4-dimensional quantum Hall system and the matrix brane construction in M-theory.

Keywords: 4-dimensional quantum Hall system, fuzzy monopole harmonics, non-commutative geometry.

1 Introduction

The planar coordinates of quantum particles in the lowest Landau level of a constant magnetic field provide a well-known and natural realization of non-commutative space [1]. The physics of electrons in the lowest Landau level exhibits many fascinating properties. In particular, when the electron density lies in at certain rational fractions of the density corresponding to a fully filled lowest Landau level, the electrons condensed into special incompressible fluid states whose excitations exhibit such unusual phenomena as fractional charge and fractional statistics. For the filling fractions $\frac{1}{m}$, the physics of these states is accurately described by certain wave functions proposed by Laughlin [2], and more general wave functions may be used to describing the various types of excitations about the Laughlin states.

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There has recently appeared an interesting connection between quantum Hall effect and non-commutative field theory. In particular, Susskind [3] proposed that non-commutative Chern-Simons theory on the plane may provide a description of the (fractional) quantum Hall fluid, and specifically of the Laughlin states. Susskind’s non-commutative Chern-Simons theory on the plane describes a spatially infinite quantum Hall system. It, i.e., does the Laughlin states at filling fractions \( \nu \) for a system of an infinite number of electrons confined in the lowest Landau level. The fields of this theory are infinite matrices which act on an infinite Hilbert space, appropriate to account for an infinite number of electrons. Subsequently, Polychronakos [4] proposed a matrix regularized version of Susskind’s non-commutative Chern-Simons theory in an effort to describe finite systems with a finite number of electrons in the limited spatial extent. This matrix model was shown to reproduce the basic properties of the quantum Hall droplets and two special types of excitations of them. The first type of excitations are arbitrary area-preserving boundary excitations of the droplet. The another type of excitations are the analogs of quasi-particle and quasi-hole states. These quasi-particle and quasi-hole states can be regarded as non-perturbative boundary excitations of the droplet. Furthermore, it was shown that there exists a complete minimal basis of exact wave functions for the matrix regularized version of non-commutative Chern-Simons theory at arbitrary level \( \nu^{-1} \) and rank \( N \), and that those are one to one correspondence with Laughlin wave functions describing excitations of a quantum Hall droplet composed of \( N \) electrons at filling fraction \( \nu \). It is believed that the matrix regularized version of non-commutative Chern-Simons theory is precisely equivalent to the theory of composite fermions in the lowest Landau level, and should provide an accurate description of fractional quantum Hall state. However, it does appear an interesting conclusion that they are agreement on the long distance behavior, but the short distance behavior is different [5].

In the matrix regularized version of non-commutative Chern-Simons theory, a confining harmonic potential must be added to the action of this matrix model to keep the particles near the origin. In fact, there has been a translationally invariant version of Laughlin quantum Hall fluid for the two-dimensional electron gas in which it is not necessary to add any confining potential. Such model is Haldane’ that of fractional quantum Hall effect based on the spherical geometry [6]. Haldane’s model is set up by a two-dimensional electron gas of \( N \) particles on a spherical surface in radial monopole magnetic field. A Dirac’s monopole is at the center of two-dimensional sphere. This compact sphere space can be mapped to the flat Euclidean space by standard stereographical mapping. In fixed limit, the connection between this model and non-commutative Chern-Simons theory can be exhibited clearly. Exactly, the non-commutative property of particle’s coordinates in Haldane’s model should be described in terms of fuzzy two-sphere.

In this paper, we do not plan to discuss such description of Haldane’s model in detail, but want to exhibit the character of non-commutative geometry of 4-dimensional generalization of Haldane’s model, proposed recently by Zhang and Hu [7]. The 4-dimensional generalization of the quantum Hall system is composed of many particles moving in four dimensional space under a \( SU(2) \) gauge field. Instead of the two-sphere geometry in Haldane’s model, Zhang and Hu considered particles on a four-sphere surface in radial Yang’s \( SU(2) \) monopole gauge field [8], which replaces the Dirac’s monopole field of Haldane’s model. This Yang’s \( SU(2) \) monopole gauge potential defined on four-sphere can be transformed to the instanton potential of the \( SU(2) \) Yang-Mills theory upon a conformal transformation from four-sphere to the 4-dimensional Euclidean space. Zhang and Hu had shown that at appropriate integer and fractional filling fractions the generalization of system forms an incompressible quantum fluid. They [9] also investigated collective excitations at the boundary of the 4-dimensional quantum Hall droplet proposed by them. In their discussion, an non-commutative algebraic relation between the coordinates of particle moving on four-sphere plays the key role. According to our un-
derstanding about Haldane’s and Zhang et al works, we think that fuzzy sphere structures in their models is the geometrical origin of non-commutative algebraic relations of the particle coordinates. We shall clarify this idea in this paper.

Non-commutative spheres have found a variety of physical applications [14, 15, 16, 17, 13, 19, 20]. The description of fuzzy two-sphere [14] was discovered in early attempts to quantize the super-membrane [14]. The fuzzy four-sphere appeared in [15, 17]. The connection of non-commutative second Hopf bundle with the fuzzy four-sphere has been investigated [13] from quantum group. The fuzzy four-sphere was used [21] in the context of the matrix theory of BFSS [22] to described time-dependent 4-brane solutions constructed from zero-brane degrees of freedom. Furthermore, the non-commutative descriptions of spheres also arise in various contexts in the physics of D-branes. The descriptions of them, e.g., were used to exhibit the non-commutative properties and dielectric effects of D-branes [23]. Recently, Ho and Ramgoolam [24, 25] had studied the matrix descriptions of higher dimensional fuzzy spherical branes in the matrix theory. They have found that the finite matrix algebras associated with the various fuzzy spheres have a natural basis which falls in correspondence with tensor constructions of irreducible representations of the corresponding orthogonal groups. In their formalism, they gave the connection between various fuzzy spheres and matrix algebras by introducing a projection from matrix algebra to fuzzy spherical harmonics. Their fuzzy spheres obey non-associative algebras because of the non-associativity induced by the projected multiplication. The complication of projection makes their constructions of fuzzy spherical harmonics formal.

The goal of this paper is to explore the character of non-commutative geometry of 4-dimensional quantum Hall system proposed recently by Zhang and Hu. Recently, Fabinger [26] had pointed that there exists a connection of the fuzzy $S^4$ with Zhang and Hu’s quantum Hall model of $S^4$. The string theory and brane matrix theory related with such non-commutative structure of fuzzy $S^4$ are discussed by [26, 27, 28]. However, the structure of non-commutative algebra of functions on $S^4$ is not still clear. It is known that the key idea of non-commutative geometry is in replacement of commutative algebra of functions on a smooth manifold by a non-commutative deformation of it [20]. We shall explore the structure the character of non-commutative geometry of 4-dimensional quantum Hall system to find non-commutative algebra of functions on $S^4$.

We should emphasize that the non-commutative structure of fuzzy $S^4$ of Zhang and Hu’s quantum Hall model is different with those commonly considered by people. In fact, the $S^4$ of Zhang and Hu’s quantum Hall model is the base of the Hopf bundle $S^7$ obtained by the second Hopf fibration, i.e., $S^7/S^3 = S^4$, from the connection between the second Hopf map and Yang’s $SU(2)$ [29, 30]. Equivalently, $SU(2) \times SU(2) = SU(2) \times SU(2) = S^4$ since $S^7 = SU(2)/SU(1)$. The bundle $S^7 = SO(5)$ can be parameterized by $\theta, \alpha, \beta, \gamma, \alpha_I, \beta_I, \gamma_I$ (see section 2 in detail). By further smearing out the common $U(1)$ gauge symmetry parameterized by $\gamma_I$, we can obtain the bundle $SU(2) \times U(1) = SO(5)/U(1) = S^6$ and its fibre $U(1)/U(1) = S^3 = S^2$. Furthermore, we can find the sections of the bundle $S^7$ or $S^6$ by solving the eigenfunctions of Zhang and Hu’s model. The eigenfunctions of the LLL consist of the space on which non-commutative algebra of functions on $S^4$ act.

Since at special filling factors, the quantum disordered ground state of 4-dimensional quantum Hall effect is separated from all excited states by a finite energy gap, the lowest energy excitations are quasi-particle or quasi-hole excitations near the lowest Landau level. The quantum disordered ground state of 4-dimensional quantum Hall effect is the state composed coupling by particles lying in the lowest Landau level state. At appropriate integer and fractional filling fractions, the system forms an incompressible quantum liquid, which is called as a 4-dimensional quantum Hall droplet. In fact, the spherical harmonic operators for fuzzy four-sphere are related with quasi-particle’s or quasi-hole’s
creators of 4-dimensional quantum Hall effect. These operators is composed of a complete set of the matrices with the fixed dimensionality. Focusing on the space of particle’s lowest Landau level state, we shall construct explicitly these fuzzy spherical harmonics, also called the fuzzy monopole harmonics, and discuss the nontrivial algebraic relation between them. Furthermore, we shall clarify the physical implications of them.

This paper is organized as follows. Section two introduces the 4-dimensional quantum Hall model proposed by Zhang and Hu, and analyzes the property of Hilbert space and the symmetrical structures of this 4-dimensional quantum Hall system. We shall emphasize the intrinsic properties of the Yang’s SU(2) monopole included in this system, and give the explicit forms of normalized wave functions of this system, which is given by (8) in our paper. The wave functions corresponding to the irreducible representation \( R = [r_1 = I, r_2 = I] \) of SO(5) are those in the LLL. They are the sections of the bundle \( S^7 = SO(5)/SU(2) \) over \( S^4 \) with fibre \( SU(2) = S^3 \) parametered by \( \alpha_I, \beta_I, \gamma_I \). These degeneracy wave functions consist of the LLL Hilbert space which the non-commutative algebra of functions on \( S^4 \) acts on. The elements of this algebra are constructed in the following section. Section three describes the elements of non-commutative algebra of functions on \( S^4 \) which is related with the fuzzy four-sphere from the geometrical and symmetrical structures of 4-dimensional quantum Hall droplet. We find the matrix forms and the symbols of the elements, given by (15) and (13) respectively. We give the explicit construction of complete set of this matrix algebra, determined by (19). In section four, we find the system of algebraic equations satisfied by the generators of the matrix algebra. The results are given by the equations (30) and (31). For the matrix forms of the elements, this non-commutative algebra should be understood as the algebraic relation of matrix multiplication, and for the symbols of the elements, it should be done as that of Moyal product. It can be seen from these results and the complete set (19) that the non-commutative algebra is closed. The associativity of this algebra is shown by the relation (33). Furthermore, a fusion scheme of the fuzzy monopole harmonics of the coupling system from those of the subsystems, and its fusion rule are established in this section, which are given by the relations (40) and (41). Section five includes discussions about the physical interpretations of the results and remarks on some physical applications of them.

2 The Hilbert space of 4-dimensional quantum Hall system

The 4-dimensional quantum Hall system is composed of many particles moving in four dimensional space under a \( SU(2) \) gauge field. The Hamiltonian of a single particle moving on four-sphere \( S^4 \) is read as

\[
\mathcal{H} = \frac{\hbar^2}{2MR^2} \sum_{a<b} \Lambda_{ab}^2 \tag{1}
\]

where \( M \) is the inertia mass and \( R \) the radius of \( S^4 \). The symmetry group of \( S^4 \) is \( SO(5) \). Because the particle is coupling with a \( SU(2) \) gauge field \( A_a, \Lambda_{ab} \) in Eq.(1) is the dynamical angular momentum given by \( \Lambda_{ab} = -i(x_a D_b - x_b D_a) \). From the covariant derivative \( D_a = \partial_a + A_a \), one can calculate the gauge field strength from the definition \( f_{ab} = [D_a, D_b] \). \( \Lambda_{ab} \) does not satisfy the commutation relations of \( SO(5) \) generators. Similar to in Dirac’s monopole field, the angular momentum of a particle in Yang’s \( SU(2) \) monopole field can be defined as \( L_{ab} = \Lambda_{ab} - if_{ab} \), which indeed obey the \( SO(5) \) commutation relations. Yang proved that \( L_{ab} \) can generate all \( SO(5) \) irreducible representations.

In general, the representations of \( SO(5) \) can be put in one-to-one correspondence with Young diagrams, labelled by the row lengths \( [r_1, r_2] \), which obey the constraints \( 0 \leq r_2 \leq r_1 \). For such a representation, the eigenvalue of Casimir operator is given by \( A(r_1, r_2) = \sum_{a<b} L_{ab}^2 = r_1^2 + r_2^2 + 3r_1 + r_2 \),
and its dimensionality is

\[ D(r_1, r_2) = \frac{1}{6}(1 + r_1 - r_2)(1 + 2r_2)(2 + r_1 + r_2)(3 + 2r_1). \]  

(2)

The SU(2) gauge field is valued in the SU(2) Lie algebra \([I_i, I_j] = i\varepsilon_{ijk}I_k\). The value of this SU(2) Casimir operator \(\sum_i I_i^2 = I(I+1)\) specifies the dimension of the SU(2) representation in the monopole potential. \(I\) is an important parameter of \(L_{ab}\), generating all SO(5) irreducible representations and the Hamiltonian Eq.(1). In fact, for a given \(I\), if one deals with the eigenvalues and eigenfunctions the operator of angular momentum \(\sum_{a<b} L_{ab}^2\), it can be found that the SO(5) irreducible representations, which the eigenfunctions called by Yang \([11]\) as SU(2) monopole harmonics belong to, are restricted. Such SO(5) irreducible representations are labelled by the integers \([r_1, r_2 = I]\), and \(r_1 \geq I\). Based on the expressions of SU(2) monopole potentials given by Yang \([11]\), or by Zhang and Hu \([8]\), one can show that \(\sum_{a<b} A_{ab}^2 = \sum_{a<b} L_{ab}^2 - \sum_i I_i^2\) by straightforwardly evaluating. This implies that the eigenvalues and eigenfunctions of the Hamiltonian Eq.(1) can be read off from those of the operator \(\sum_{a<b} L_{ab}^2\). Hence, for a given \(I\), the energy eigenvalues of the Hamiltonian Eq.(1) are read as

\[ E_{[r_1, r_2 = I]} = \frac{\hbar^2}{2MR^2}[A(r_1, r_2 = I) - 2I(I + 1)]. \]  

(3)

The degeneracy of energy level is given by the dimensionality of the corresponding irreducible representation \(D(r_1, r_2 = I)\).

The ground state of the Hamiltonian Eq.(1) plays a key role in the procedure of construction of many-body wave function and the discussion of incompressibility of 4-dimensional quantum Hall system. This ground state, also called the lowest Landau level (LLL) state, is described by the least admissible irreducible representation of SO(5), i.e., labelled by \([r_1, r_2 = I]\) for a given \(I\). The LLL state is \(D(r_1 = I, r_2 = I) = \frac{1}{6}(2I + 1)(2I + 2)(2I + 3)\) fold degenerate, and its energy eigenvalue is \(\frac{\hbar^2}{2MR^2}2I\). Zhang and Hu \([8]\) found the explicit form of the ground state wave function in the spinor coordinates. This wave function is read as

\[ \langle x_a, n_i | m_1, m_2, m_3, m_4 \rangle = \sqrt{\frac{p!}{m_1!m_2!m_3!m_4!}} \Psi_1^{m_1}\Psi_2^{m_2}\Psi_3^{m_3}\Psi_4^{m_4}, \]  

(4)

with integers \(m_1 + m_2 + m_3 + m_4 = p = 2I\). The orbital coordinate \(x_a\), which is defined by the coordinate point of the 4-dimensional sphere \(X_a = Rx_a\), is related with the spinor coordinates \(\Psi_\alpha\) with \(\alpha = 1, 2, 3, 4\) by the relations \(x_a = \hat{\Psi}_\alpha \Gamma_\alpha \Psi\) and \(\sum_\alpha \hat{\Psi}_\alpha \Psi_\alpha = 1\). The five 4x4 Dirac matrices \(\Gamma_\alpha\) with \(\alpha = 1, 2, 3, 4, 5\) satisfy the Clifford algebra \(\{\Gamma_\alpha, \Gamma_\beta\} = 2\delta_{\alpha\beta}\). The isospin coordinates \(n_i = \bar{u}i\sigma_i u\) with \(i = 1, 2, 3\) are given by an arbitrary two-component complex spinor \((u_1, u_2)\) satisfying \(\sum_\sigma \bar{u}_\sigma u_\sigma = 1\). Zhang and Hu gave the explicit solution of the spinor coordinate with respect to the orbital coordinate as following

\[ \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{\frac{1 + x_5}{2}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \sqrt{\frac{1}{2(1 + x_5)}}(x_4 - ix_5) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \]  

(5)

By computing the geometric connection, one can get a non-Abelian gauge potential \(A_a\), which is just the SU(2) gauge potential of a Yang monopole defined on 4-dimensional sphere \(S^4\) \([8]\). Since we do not need the explicit form of it here, we do not write out that of it.

The description of the 4-dimensional quantum Hall liquid involves the quantum many-body problem of \(N\) particle’s moving on the 4-dimensional sphere \(S^4\) in the Yang’s SU(2) monopole field lying
in center of the sphere $S^4$. The wave functions of many particles can be constructed by the nontrivial product of the single particle wavefunctions, among which every single particle wave function is given by the LLL wavefunction Eq.(4). In the case of integer filling, the many-particle wave function is simply the Slater determinant composed of $N$ single-particle wave functions. For the fractional filling fractions, the many particle wave function cannot be expressed as the Laughlin form of a single product. But the amplitude of the many-particle wave function can also be interpreted as the Boltzmann weight for a classical fluid. One can see that it describes an incompressible liquid by means of plasma analogy. Therefore, at the integer or fractional filling fractions, the 4-dimensional system of the generalizing quantum Hall effect forms an incompressible quantum liquid \[6\]. We shall call this 4-dimensional system as a 4-dimensional quantum Hall droplet.

The space of the degenerate states in the LLL is very important not only for the description of the 4-dimensional quantum Hall droplet but also for that of edge excitations and quasi-particle or quasi-hole excitations of the droplet. In fact, this space of the degenerate states is the space which we shall construct the matrix algebra acting on in the following section. In order to construct the complete set of matrix algebra of fuzzy $S^4$, we need to know the explicit forms of the wave functions associated with all irreducible representations $[r_1, r_2 = I]$ of $SO(5)$. Although Yang had found the wave functions for all the $[r_1, r_2 = I]$ states, the form of his parameterizing the four sphere $S^4$ is not convenient for our purpose. Following Hu and Zhang\[12\], we can parameterize the four sphere $S^4$ by the following coordinate system

\[
\begin{align*}
x_1 & = \sin \theta \sin \frac{\beta}{2} \sin(\alpha - \gamma), \\
x_2 & = -\sin \theta \sin \frac{\beta}{2} \cos(\alpha - \gamma), \\
x_3 & = -\sin \theta \cos \frac{\beta}{2} \sin(\alpha + \gamma), \\
x_4 & = \sin \theta \cos \frac{\beta}{2} \cos(\alpha + \gamma), \\
x_5 & = \cos \theta,
\end{align*}
\]

where $\theta, \beta \in [0, \pi)$ and $\alpha, \gamma \in [0, 2\pi)$. The direction of the isospin is specified by $\alpha_I, \beta_I$ and $\gamma_I$.

As the above explanation, we can get the eigenfunctions of the Hamiltonian Eq.(1) from the eigenfunctions of the operator $\sum_{a<b} L_{ab}^2$. The angular momentum operators $L_{ab}$ consists of an orbital part $L_{\mu\nu}^{(0)} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ with $\mu, \nu = 1, 2, 3, 4$ and an isospin part involving the $SU(2)$ monopole field. The angular momentum operators $L_{\mu\nu}^{(0)}$ generate the rotation in the subspace $(x_1, x_2, x_3, x_4)$, and satisfy the commutation relations of $SO(4)$ generators. They can be decomposed into two $SU(2)$ algebras: $J_1^{(0)} = \frac{1}{2}(\frac{3}{2} \varepsilon_{ijk} L_{jk}^{(0)} + L_{4i})$ and $J_2^{(0)} = \frac{1}{2}(\frac{1}{2} \varepsilon_{ijk} L_{jk}^{(0)} - L_{4i})$. They satisfy the identity of operator $\sum_i J_{1i}^{(0)} = \sum_i J_{2i}^{(0)}$. Therefore, if one would use the operators $J_1^{(0)}$ and $J_2^{(0)}$ to generate the $SO(4)$ blocks of the $SO(5)$ irreducible representations, he can not obtain all irreducible representations of $SO(5)$. However, because of the coupling to the Yang’s $SU(2)$ monopole potential, these orbital $SO(4)$ generators are modified into $L_{\mu\nu}$, which are decomposed into $J_1^{(0)} = J_1^{(0)}$ and $J_2^{(0)} = J_2^{(0)} + I_i$. Indeed, the $SO(4)$ generators $J_1^{(0)}$ and $J_2^{(0)}$ can be used to generate the $SO(4)$ block states of all $SO(5)$ irreducible representations \[11\]. Such $SO(4)$ block states can be labelled by the $SO(4)$ quantum numbers $J \equiv \left( \begin{array}{c} j_1 \\ j_{1z} \\ j_2 \\ j_{2z} \end{array} \right)$, where $\sum_i J_{1i}^2 = j_1(j_1 + 1)$ and $\sum_i J_{2i}^2 = j_2(j_2 + 1)$. The $j_{1z}$ and $j_{2z}$ are the magnetic quantum numbers of two $SU(2)$ algebras.
Applying the $SO(4)$ operators to the quantum states described by the Hamiltonian Eq.(1) in the irreducible representations of $SO(5)$, one can see that the complete set of quantum observables of the system is composed of the operators $\sum_{a<b}L_{ab}^I, J_i^2, J_i^z$ and $\hat{J}_{2i}$. Thus, there exist the simultaneous eigenfunctions of those operators, which are just the wave functions of energy eigenvalue being $E_{[r_1,r_2]=I}$. Noticing that the isospin operators $\hat{I}_i$ are coupling with the operator $\hat{J}_{2i}^{(0)}$ into the angular momentum operators $\hat{J}_{2i}$, we should also introduce the quantum number labelling the isospin parameter $I$, which is written as $I^{(0)} \equiv \begin{pmatrix} 0 & I \\ 0 & I_z \end{pmatrix}$. The parameters of group $(\theta, \alpha, \beta, \gamma, \alpha_I, \beta_I, \gamma_I)$ are abbreviated to $(\Omega)$. The wave functions corresponding to the irreducible representation $R \equiv [r_1, r_2 = I]$ of $SO(5)$ are denoted as $D^{(R)}_{J I(0)}(\Omega)$. By using the equation of parameterizing $S^4$ and the parameters of the isospin direction, and following Yang [11], one can obtain the wave functions $D^{(R)}_{J I(0)}(\Omega)$ obeying the system of equations as following

\begin{align}
J_i^2 D^{(R)}_{J I(0)}(\Omega) &= j_i(j_i + 1)D^{(R)}_{J I(0)}(\Omega), \quad \hat{J}_I D^{(R)}_{J I(0)}(\Omega) = j_{1z}D^{(R)}_{J I(0)}(\Omega), \\
\hat{J}_2^2 D^{(R)}_{J I(0)}(\Omega) &= I(I + 1)D^{(R)}_{J I(0)}(\Omega), \\
\hat{J}_2^2 D^{(R)}_{J I(0)}(\Omega) &= j_2(j_2 + 1)D^{(R)}_{J I(0)}(\Omega), \quad \hat{J}_2 D^{(R)}_{J I(0)}(\Omega) = j_{2z}D^{(R)}_{J I(0)}(\Omega), \\
\left\{ \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} (\sin^3 \theta \frac{\partial}{\partial \theta}) - \frac{4J_1^2}{\sin^2 \theta} - \frac{2(1 - \cos \theta)}{\sin^2 \theta} (J_2^2 - \hat{J}_2^2)
\right. \\
&\quad \left. - \frac{(1 - \cos \theta)(3 + \cos \theta)}{\sin^2 \theta} \hat{J}_2^2 \right\} D^{(R)}_{J I(0)}(\Omega) = -A(r_1, r_2 = I) D^{(R)}_{J I(0)}(\Omega).
\end{align}

The first line and second line of the equations tell us that we can use two $SU(2)$ D-functions to realize the wave function with respect to the dependence of the group parameters $\alpha, \beta, \gamma$ and $\alpha_I, \beta_I, \gamma_I$. Furthermore, we also should consider the coupling relations $\hat{J}_2 = \hat{J}_2^{(0)} + \hat{I}_1, \hat{J}_I = \hat{J}_1^{(0)}$ and $\sum_i \hat{J}_i^{(0)} = \sum_i \hat{J}_i^{(0)}$ to make the wave function obey the third line of the equations. The final line is the equation to determine the $\theta$ dependence of the wave function, which had been solved by Yang [11]. Now, we can write the explicit solution form of the normalized wavefunction as

\begin{align}
D^{(R)}_{J I(0)}(\Omega) &= \sqrt{\frac{2r_1 + 3}{2}} (\sin \theta)^{-1} d^{(r_1+1)}_{j_1-j_2-\Omega}(\theta) \frac{\sqrt{(2I+1)(2j_1+1)}}{8\pi^2} \\
&\quad \times \sum_{j_2, I_z} D_{j_1, j_2, I_z}^I(\alpha, \beta, \gamma)(\theta) D_{j_2, I_z}^{IJ}(\alpha_I, \beta_I, \gamma_I),
\end{align}

where

\begin{align}
d^{(r_1+1)}_{j_1-j_2-\Omega}(\theta) &= \sqrt{\frac{2-2(j_1-j_2)}{(r_1 + j_1 - j_2 + 2)! (r_1 - j_1 + j_2 + 2)!}} \\
&\quad \times \left( 1 - \cos \theta \right)^{\frac{j_1+j_2+1}{2}} \left( 1 + \cos \theta \right)^{-\frac{j_1+j_2+1}{2}} P_n^{(\alpha, \beta)}(\cos \theta),
\end{align}

and $P_n^{(\alpha, \beta)}$ is the Jacobi polynomial.

Although $d^{(r_1+1)}_{j_1-j_2-\Omega}(\theta)$ is the d-function of $SO(3)$ rotation group, it should be emphasized that here the quantum numbers of the subgroup $SO(4)$ of $SO(5)$ have replaced the usual magnetic quantum number of $SO(3)$. Practically, $\sqrt{2\pi j_1 j_2} (\sin \theta)^{-1} d^{(r_1+1)}_{j_1-j_2-\Omega}(\theta)$ is the d-function of $SO(5)$ rotation.
group in the special case. The D-function $D^{Iz}_{j_{1}, j_{2}, j_{3}}(0)(\alpha, \beta, \gamma)$ is the standard $SU(2)$ representation matrix for the Euler angles $\alpha, \beta, \gamma$. They are the $SU(2)$ rotation matrix elements generated rotationally by the operators $\hat{J}_{1z}$. Similarly, $D^{Iz}_{j_{1}, I_{z}}(0)(\alpha, \beta, \gamma, I_{z})$ are those generated by the isospin operators $\hat{I}_{z}$. The coupling coefficients $\langle j_{2}^{(0)} | I_{z} | j_{1}^{(0)} \rangle$ are the Clebsch-Gordon coefficients, show the coupling behavior of the $SU(2)$ angular momentums $\hat{J}_{1z}, \hat{I}_{z}$, and $\hat{J}_{2z}$ by means of $\hat{J}_{2z}^{(0)}$.

The isospin direction can be normally specified by two angles $\alpha$ and $\beta$. However, the wave function $D^{(R)}_{I_{z}}(\Omega)$ depends on the Euler angles $\alpha_{I}$, $\beta_{I}$, and $\gamma_{I}$ of the isospin space. In fact, in the $(\hat{I}_{z}, \hat{I}_{1z})$ picture, the $\gamma_{I}$ dependence of $D^{(R)}_{I_{z}}(\Omega)$ is given by the $U(1)$ phase factor $\exp\{-iI_{z}\gamma_{I}\}$, where $I_{z}$ is simply an $U(1)$ gauge index. Therefore, different values of $I_{z}$ correspond to the same physical state. One can smear the $\gamma_{I}$ dependence of the wave function by the gauge choice. Such gauge choice can be fixed by taking $I_{z} = I$ or $I_{z} = -I$. If such gauge choice is taken at every step in all calculations, we call this choice as taking the physical gauge. Analogous to doing usually in the field theory, there exists another gauge choice, which such gauge choice is taken at the end of calculations. The latter gauge choice are called as taking the covariant gauge. We shall take the covariant gauge in this paper, which this thick was also used in the reference [12].

The wave functions $D^{(R)}_{I_{z}}(\Omega)$ are the $SU(2)$ monopole harmonics. Exactly, they are the spherical harmonics on the coset space $SO(5)/SU(2)$, which is locally isomorphic to the sphere $S^{4} \times S^{2}$. By smearing the $U(1)$ degree of freedom of the $\gamma_{I}$ dependence, i.e., taking the physical gauge, they can be viewed as the spherical harmonics on the coset space $SO(5)/U(2)$, which is locally isomorphic to the sphere $S^{4} \times S^{2}$. Globally, $SO(5)/U(2)$ is a bundle over the $S^{4}$ with fibre $S^{2}$. The wave functions are the cross sections in this nontrivial fibre bundle. This implies that there exists a stabilizer group of the wave function solutions $U(2)$, and the action of $SO(5)$ on the state generates a space of the wave function solutions which is the space of cross sections in $SO(5)/U(2)$. If we take the covariant gauge, the $SU(2)$ monopole harmonics should be regarded as the cross sections in the nontrivial fibre bundle $SO(5)/SU(2)$. Then, the stabilizer group of the wave function solutions is $SU(2)$. $SO(5)$ doing the state generates that of cross sections in $SO(5)/SU(2)$. The procedure of parameterizing the four sphere $S^{4}$ Eq. (6) and building up the isomorphic relation of the $SU(2)$ group manifold and the three sphere $S^{3}$ is just that of smearing the stabilizer subgroup $SU(2)$. Of course, if one want to take the physical gauge, he can smear the $U(2)$ by further smearing the $U(1)$ gauge subgroup. Such globally geometrical structure and symmetrical structure can guide us to develop some techniques which we need in this paper.

Let us introduce the state vector $\left| \begin{array}{c} R \\ J \end{array} \right\rangle$, which belongs to the Hilbert space $\mathcal{H}^{(R)}$ composed of the $SU(2)$ monopole harmonics $D^{(R)}_{I_{z}}(\Omega)$. Of course, $\left| \begin{array}{c} R \\ I_{z} \end{array} \right\rangle$ is an element of the Hilbert space $\mathcal{H}^{(R)}$. In general, taking a fixed vector in the Hilbert space, one can use the unitary irreducible representation of an arbitrary Lie group acting in the Hilbert space to produce the coherent state for this Lie group. Now, we are interesting to the coherent state corresponding to the coset space $SO(5)/SU(2)$. The $SU(2)$ is the isotropic subgroup of $SO(5)$ for the state $\left| \begin{array}{c} R \\ I_{z} \end{array} \right\rangle$ since $SU(2)$ is the stabilizer group of the wave function solutions. If we use the unitary irreducible representations of $SO(5)$ acting on the state $\left| \begin{array}{c} R \\ I_{z} \end{array} \right\rangle$ to produce the coherent states, the coherent state vectors belonging to a left coset class
of $SO(5)$ with respect to the subgroup $SU(2)$ differ only in a phase factor and so determine the same state. Consequently, the coherent state vectors depend only on the group parameters parameterizing the coset space $SO(5)/SU(2)$. Thus, we can now introduce the following coherent state vector

$$|\Omega, R\rangle = \sum_{I,I_z} D^{(R)}_{IJ(0)}(\Omega) \begin{pmatrix} R \\ J \end{pmatrix} |I, I_z\rangle,$$

(10)

where the star stands for the complex conjugate. In order to realize the covariant gauge, we have added the isospin frame $|I, I_z\rangle$ to the monopole harmonics $D^{(R)}_{IJ(0)}(\Omega)$. In fact, we can use the finite rotation $\hat{R}(\Omega) = \exp\{i\alpha\hat{J}_z\} \exp\{i\beta\hat{J}_y\} \exp\{i\gamma\hat{L}_5\} \exp\{i\theta\hat{L}_5\} \exp\{i\gamma\hat{L}_5\} \exp\{i\beta\hat{L}_5\} \exp\{i\gamma\hat{L}_5\}$ acting on the state vector $\begin{pmatrix} R \\ I(0) \end{pmatrix}$ to realize the general finite rotation of $SO(5)$ acting on the state vector $\begin{pmatrix} R \\ I(0) \end{pmatrix}$ up to a phase factor, and to generate the above coherent state vector.

The wave functions in the coherent state picture are given by

$$\langle \Omega, R | \begin{pmatrix} R \\ J \end{pmatrix} \rangle = \sum_{I} \langle I, I_z | D^{(R)}_{IJ(0)}(\Omega) \rangle.$$

(11)

The l.h.s. of Eq.(11) is not with the label $I$ since it naturally appears in the label $R = [r_1, r_2 = 1]$, which corresponds to the $SU(2)$ monopole harmonics. It should be emphasized that the above wave functions become the wave functions in the physical gauge only if one smears the isospin frame of them and projects back to the $\alpha$ and $\beta$ angles. In the sense of the finite rotation of $SO(5)$, the explicit forms of wave function solutions given here are the wave function solutions of the 4-dimensional spherically symmetrical top with the $SU(2)$ self-rotating in the isospin direction. The Yang’s $SU(2)$ monopole harmonics can be interpret as the wave functions in the coherent state picture in the physical gauge.

Based on the orthogonality and completeness of the state vectors $| \begin{pmatrix} R \\ J \end{pmatrix} \rangle$ belonging to an irreducible representation $R = [r_1, r_2 = 1]$ of $SO(5)$, we can give the completeness condition of the coherent state vectors

$$\frac{D(r_1, r_2 = I)}{A(\Omega)} \int d\Omega|\Omega, R\rangle \langle \Omega, R| = 1,$$

(12)

where $A(\Omega) = \text{Area}(S^4 \times S^3) = \frac{1}{12}(8\pi)^2$. Every irreducible representation of $SO(5)$ corresponds to a complete set of the coherent state vectors. The coherent states of the different irreducible representations are orthogonal each other. The coherent state corresponding to the LLL states is very important for the description of non-commutative geometry of 4-dimensional quantum Hall droplet.

In order to avoid the label of the irreducible representation of the LLL states confusing with the parameter $I$ of the model, we denote the irreducible representation $R = [r_1 = I, r_2 = 1]$ as $P \equiv [r_1 = I, r_2 = 1]$. The LLL degeneracy states consist of the Hilbert space $\hat{H}(P)$. Because of the LLL states are the lowest energy states of particle’s living in, for a given $I$, the smallest admissible irreducible representation of $SO(5)$ is $R = [r_1 = I, r_2 = 1] = P$. Therefore, the irreducible representations of $SO(5)$ are truncated since there exists an Yang’s $SU(2)$ monopole. If we focus on the Hilbert space of the LLL states, we can determine the matrix forms of tensor operator for the $SU(2)$ monopole harmonics, which are the $D(r_1 = I, r_2 = 1) \times D(r_1 = I, r_2 = 1)$ matrices. The
coupling relation between the tensor operators provides a truncated parameter for the tensor operator for the $SU(2)$ monopole harmonics. The number of independent operators is $(D(r_1 = I, r_2 = I))^2$, which we shall explain in more detail in the next section. Therefore, we should replace the functions by the $D(r_1 = I, r_2 = I) \times D(r_1 = I, r_2 = I)$ matrices on the fuzzy $S^4$. Exactly, the cross sections in the fibre bundle $SO(5)/SU(2)$, which is a bundle over $S^4$ with fibre $S^3$, are replaced by the $D(r_1 = I, r_2 = I) \times D(r_1 = I, r_2 = I)$ matrices on the fuzzy $S^4$. Thus, the algebra on the fuzzy $S^4$ becomes non-commutative. The direct product of $N$ single-particle Hilbert spaces $H(\hat{\Psi})$ can be used to build up the Hilbert space of 4-dimensional quantum Hall droplet. Hence, the fuzzy $S^4$ appears naturally in the description of particle’s moving on the $S^4$ with the Yang’s $SU(2)$ monopole at the center of the sphere. Consequently, the fuzzy $S^4$ is the description of non-commutative geometry of 4-dimensional quantum Hall droplet.

3 Fuzzy monopole harmonics and Matrix operators of fuzzy $S^4$

The construction of fuzzy $S^4$ is to replace the functions on $S^4$ by the non-commutative algebra taken in the irreducible representations of $SO(5)$. This is a full matrix algebra which is generated by the fuzzy monopole harmonics. These fuzzy monopole harmonics consist of a complete basis of the matrix space. In this section, we shall find the explicit forms of such fuzzy monopole harmonics by means of the expressions of the $SU(2)$ monopole harmonics given in the previous section. Our main task of this section is to construct the operators corresponding to the $SU(2)$ monopole harmonics, and to give the matrix elements of these operators acting on the LLL states.

Although for a given $I$, the $SU(2)$ monopole harmonics of the smallest admisible irreducible representations of $SO(5)$ can be regarded as the single-particle wave functions of the LLL in 4-dimensional quantum Hall system, the $SU(2)$ monopole harmonics smaller than the smallest admissible irreducible representations of $SO(5)$ are useful for us to construct the fuzzy monopole harmonics on $SO(5)/SU(2)$. Such $SU(2)$ monopole harmonics can be read off from the expressions obtained in the previous section by the changing of the parameter $I$. If we replace $I$ by $J$, the $SO(5)$ irreducible representation of the monopole harmonics becomes $R = [r_1, r_2 = J]$. Equivalently, we can use $R = [r_1, r_2]$ and $J^{(0)} \equiv \begin{pmatrix} 0 & r_2 \\ r_2 & 0 \end{pmatrix}$ to label them. Thus, the $SU(2)$ monopole harmonics are generally expressed as $D_{J,I}^{(R)}(\Omega)$, which can be obtained by replacing $I^{(0)}$ with $J^{(0)}$ in the equation (8). We can find their corresponding coherent states by the same replacement.

By using the standard techniques of the generalized coherent state, we can now construct the operator $\hat{\Psi}_{J}^{R}$ corresponding to the $SU(2)$ monopole harmonics $D_{J,I}^{(R)}(\Omega)$. Noticing that this operator is an operator of acting in the LLL Hilbert space and $D_{J,I}^{(R)}(\Omega)$ is regarded as the basic function, we can express it as the following form

$$\hat{\Psi}_{J}^{R} = \frac{D(r_1 = I, r_2 = I)}{A(\Omega)} \int d\Omega \sum_{r_{2z}} \langle r_2, r_{2z}|D_{J,I}^{(R)}(\Omega)|\Omega, \frac{P}{2}\rangle|\Omega, \frac{P}{2}|.$$  

(13)

The matrix elements of this operator in the LLL Hilbert space are read as

$$\left< \frac{P}{K^1} | \hat{\Psi}_{J}^{R} | \frac{P}{K^2} \right> = \frac{D(r_1 = I, r_2 = I)}{A(\Omega)} \times \int d\Omega \sum_{l_2z,r_{2z}} D_{K_1, l_2z}^{(R)}(\Omega)D_{J,I}^{(R)}(\Omega)D_{K_2, l_2z}^{(R)}(\Omega)|I, I_2z, r_{2z}, r_{2z}|I, I_2z>.  $$

(14)
The above integral can be analytically performed by making use of the integral formulae about the Jacobi polynomial and the product of three $SU(2)$ D-functions, and the properties of $SU(2)$ D-function. The result is

$$
\left\langle \frac{P}{K^1} \left| \hat{J}^R \right| \frac{P}{K^2} \right\rangle = \left\langle \frac{P}{K^1} \left| R \right| \frac{P}{K^2} \right\rangle \left\{ \begin{array}{ccc}
\hat{P} & j_1 & k_1^2 \\
I & r_2 & I \\
k_2^1 & j_2 & k_2^2
\end{array} \right\}
$$

$$
\times \left\langle \left( k_1^1, k_1^2 \mid j_1, j_1^2 \right|| k_2^1, k_2^2_2 \mid j_2, j_2^2, k_2^2_2 \right\rangle,
$$

where $\left\langle \frac{P}{K^1} \left| R \right| \frac{P}{K^2} \right\rangle$ is independent of the $SU(2)$ magnetic quantum numbers, and is composed of two parts, i.e., it is equal to $\Phi(k_1^1, k_2^1, k_2^2, j_1, j_2, I, r_2)\Theta(k_1^1, k_2^1, k_2^2, j_1, j_2, I, r_1, r_2)$. The part of $\Phi$ is from the contribution of the integral with respect to the variables $\alpha, \beta, \gamma, \alpha_I, \beta_I$ and $\gamma_I$, which is provided by the normalized coefficients and the arisen factors when we expressed the sum over all $SU(2)$ magnetic quantum numbers of six $SU(2)$ coupling coefficients as the 9-j symbol. Its expression is given by

$$
\Phi = (-1)^{j_1+j_2+j_1+k_1^1+k_1^2} \sqrt{\frac{(2I+1)^2(2k_2^1+1)^2(2k_2^2+1)(2k_1^1+1)(2r_1+1)(2j_2+1)}{(8\pi^2)^2}}.
$$

The other part $\Theta$ is given by the integrated part of $\theta$. It is read as

$$
\Theta = \sqrt{\frac{(2I+1)^2(2r_1+1)}{8}} \int_0^\pi d\theta d^{(I+1)}_{k_1^1-k_2^2,-I-1}(\theta) d^{(I+1)}_{k_2^2-k_1^1,-I-1}(\theta) d^{(r_1+1)}_{j_1-j_2-1}(\theta)
$$

$$
\times \left\{ \frac{(I+k_1^1-k_2^2+1)(I+k_2^2-k_1^1+1)(I+k_2^2-k_1^1+1)!}{(2I+3)(2r_1+3)((2I+2)!)^2} \right\}^{-\frac{1}{2}}
$$

$$
\times \left\{ \frac{(r_1+j_1+j_2+2)!(r_1-j_1-j_2)!}{(r_1+j_1+j_2+2)!(r_1-j_1-j_2)!} \right\}^{-\frac{1}{2}}
$$

$$
\times \frac{1}{A(\Omega)} \left\{ \frac{1}{(r_1+j_1+j_2+2)!(2I+3+j_1-j_2)!(2j_1+2)!(r_1+j_1+j_2+3)} \right\}^{-\frac{1}{2}}
$$

$$
\times 3F_2(-r_1+j_1-j_2-1, r_1+j_1-j_2+2, j_1+k_1^1+k_1^2+2; 2j_1+2, 2I+3+j_1-j_2; 1),
$$

where $3F_2$ is the hyper-geometric function.

Although the 9-j symbol of $SO(3)$ is also independent of the $SU(2)$ magnetic quantum numbers, $\Phi, \Theta$ and it are very important for the matrix form of the operator since they all depend on the $SO(4)$ subgroup quantum numbers of $SO(5)$. Hence, they may not be smeared out by re-scaling. In particular, the 9-j symbol together with two $SU(2)$ coupling coefficients will provide the selection rules of the matrix elements of the operator $\hat{J}^R$. From the coupling relations of 9-j symbol’s elements and the triangle relations of $SU(2)$ angular momentum coupling, we find that $r_2$ can be evaluated in the range zero to $2I$. Furthermore, because of $k_1^1+k_1^2 = k_2^1+k_2^2 = I$, the highest values of these $k$’s is $I$. We know that the maximum of $j_1$ and $j_2$ both are $2I$ from the coupling relations of two $SU(2)$ coupling coefficients. The scheme of the $SO(5)$ irreducible representation containing the $SO(4)$ blocks [1] tells us that the largest admissible value of $r_1$ is $2I$. Acting in the LLL Hilbert space, the operators $\hat{J}^R$ producing the nonzero contributions are those belonging to the irreducible representations of $SO(5)$ of $0 \leq r_2 \leq r_1 \leq 2I$. 

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The number of such operators can be calculated by means of the dimension formula of irreducible representation of $SO(5)$. From the equation (2), we have

$$
\sum_{0 \leq r_2 \leq r_1 \leq 2I} D(r_1, r_2) = \frac{1}{6} \sum_{0 \leq r_2 \leq r_1 \leq 2I} (1 + r_1 - r_2)(1 + 2r_2)(2 + r_1 + r_2)(3 + 2r_1)
$$

$$
= \frac{1}{36}(2I + 1)^2(2I + 2)^2(2I + 3)^2.
$$

This is exactly equal to the square of the dimensionality of the LLL Hilbert space. This show that the operators

$$
\hat{\gamma}^R_J, \quad 0 \leq r_2 \leq r_1 \leq 2I
$$

constitute a complete set of irreducible representations of $SO(5)$ in the matrix algebra with the square of the dimension of the LLL Hilbert space. We also call them as the fuzzy monopole harmonics.

Subsequently, we shall discuss the coupling of the $SO(5)$ angular momentums. It is helpful to explore the Hilbert space structure of many-body system corresponding to the system (1), in especial, of the 4-dimensional quantum Hall system. The Kronecker product of two unitary irreducible representations of a semi-simple group is completely reducible, but it is generally not simply reducible. Hence, a given representation may appear more than once in the decomposition of the Kronecker product. However, in the case of $SO(5)$ angular momentum, every admissible irreducible representation only appears once in the decomposition of the Kronecker product. This conclusion is a corollary of the theorems 2, 4 and 8 in the reference [11]. Based on this fact, we have the following coupling relation of the state vectors

$$
\begin{pmatrix} R \\ J \end{pmatrix} = \sum_{J_1, J_2} \begin{pmatrix} R_1 \\ J_1 \\ R_2 \\ J_2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ J_1 \\ J_2 \end{pmatrix}\begin{pmatrix} R \\ J \end{pmatrix}
$$

Then, for the case of $SO(5)$ angular momentum, the decomposition of the Kronecker product of two $SO(5)$ irreducible representations is accomplished by the $SO(5)$ coupling coefficients

$$
\begin{pmatrix} R_1 \\ R_2 \\ J_1 \\ J_2 \end{pmatrix}. \quad \text{These coupling coefficients obey the usual orthogonality relations. We again emphasize that here all state vectors} \begin{pmatrix} R \\ J \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix} \text{belong to the Hilbert space } \mathcal{H}^{(R)}, \text{which is composed of the monopole harmonics } \mathcal{D}^{(R)}_{J_1, J_2(0)}(\Omega).
$$

In the previous section, we have explained that $\sum_{r_{2z}} \mathcal{D}^{(R)(0)}_{J_1, J_2(0)}(\Omega) |r_2, r_{2z})$ can be regarded as the wave function with the $U(1)$ gauge degree of freedom generated by the finite rotation of $SO(5)$ acting on the fixed state vector in the Hilbert space, of course, $\sum_{r_{2z}} \langle r_2, r_{2z} | \mathcal{D}^{(R)}_{J_1, J_2(0)}(\Omega)$ also can be done. By using this property of the wave functions and the orthogonality relations of the coupling coefficients, we find that

$$
\sum_{R, J, r_{2z}} \langle r_2, r_{2z} | \mathcal{D}^{(R)}_{J_1, J_2(0)}(\Omega) \begin{pmatrix} R_1 \\ R_2 \\ J_1 \\ J_2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ J_1 \\ J_2 \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}
$$

$$
= \sum_{r_{2z}} \langle r_1, r_{2z} | \mathcal{D}^{(R_1)}_{J_1, J_1(0)}(\Omega) \sum_{r_{2z}} \langle r_2, r_{2z} | \mathcal{D}^{(R_2)}_{J_2, J_2(0)}(\Omega),
$$

where $R_i = [r_1, r_2]$, and $J_i(0) = \begin{pmatrix} 0 & r_2 \\ r_1 & 0 \end{pmatrix}$ for $i = 1, 2, 3$ (see below). By means of the orthogonality and normalized condition of the $SU(2)$ monopole harmonic, we can obtain the explicit expression of
the coupling coefficients of the \( SO(5) \) angular momentum

\[
\left\langle \begin{array}{ccc} R_2 & R_3 & R_1 \\ J_2 & J_3 & J_1 \end{array} \right\rangle = \langle R_1 \| R_2 \| R_3 \rangle^{-1} \times \int d\Omega \sum_{r_1^{2s}, r_2^{2s}, r_3^{2s}} \mathcal{D}^{(R_1)}_{J_1 J_2 J_3} (\Omega) \mathcal{D}^{(R_2)}_{J_2 J_2 J_3} (\Omega) \mathcal{D}^{(R_3)}_{J_3 J_3 J_3} (\Omega) \langle r_1^{1s}, r_2^{2s}, r_3^{2s} | r_2^{2s}, r_2^{2s} \rangle \langle r_2^{2s}, r_2^{2s} | r_2^{2s}, r_2^{2s} \rangle \langle r_2^{2s}, r_2^{2s} | r_2^{2s}, r_2^{2s} \rangle . \tag{22}
\]

where \( \langle R_1 \| R_2 \| R_3 \rangle = \sum_{r_1^{2s}, r_2^{2s}, r_3^{2s}} \left\langle \begin{array}{ccc} R_1 & R_2 & R_3 \\ J_1 & J_2 & J_3 \end{array} \right\rangle \) is invariant under the rotation transformation of \( SO(5) \), hence is a pure scale factor.

Performing the calculation of the above integral, we get

\[
\left\langle \begin{array}{ccc} R_2 & R_3 & R_1 \\ J_2 & J_3 & J_1 \end{array} \right\rangle = \frac{\langle R_1 \| R_2 \| R_3 \rangle}{\langle R_1 \| R_2 \| R_3 \rangle} \times \langle j_1^{1s}, j_2^{2s} | j_1^{1s}, j_1^{2s}, j_3^{2s}, \rangle \langle j_2^{1s}, j_2^{2s} | j_1^{2s}, j_2^{2s}, j_3^{2s}, \rangle . \tag{23}
\]

Similar to the matrix elements of the fuzzy monopole harmonics, \( \left\langle \begin{array}{ccc} R_1 & R_2 & R_3 \\ J_1 & J_2 & J_3 \end{array} \right\rangle = \hat{\Theta} \hat{\Phi} \). \( \hat{\Phi} \) and \( \hat{\Phi} \) are simply similar to \( \Theta \) and \( \Phi \) respectively. In \( \hat{\Phi} \), it is only different of the replacing the quantum numbers in \( \Phi \) with the corresponding quantum number at present. The expression of \( \hat{\Phi} \) is given by

\[
\hat{\Phi} = (-1)^{j_1^2 - j_3^2 + j_2^2 - r_2^2 + r_2^2} \sqrt{\frac{(2r_1^2 + 1)(2j_1^2 + 1)}{(8\pi^2)^3}} \prod_{i=1}^{3} (-1)^{j_i^2 - r_i^2} \sqrt{(2r_i^2 + 1)(2j_i^2 + 1)(2j_i^2 + 1)} . \tag{24}
\]

The expression of \( \hat{\Theta} \) is given by the following integration

\[
\hat{\Theta} = \int_{-1}^{1} dx (1 - x^2)^{-\frac{1}{2}} \left[ \sum_{i=1}^{3} \frac{2j_i^2 - 2j_i^2 + 1 (r_i^2 + j_i^2 + j_i^2 + 2) (r_i^2 - j_i^2 + j_i^2 + 1) (r_i^2 + j_i^2 + j_i^2 + 1) (r_i^2 + j_i^2 + j_i^2 + 1)}{(2r_i^2 + 3) (2j_i^2 + 1) (2j_i^2 + 1)} \right]^{-\frac{1}{2}} \times (1 - x)^{2j_i^2 + 1} (1 + x)^{-2j_i^2 - 1} P_{r_i^2 - j_i^2 + j_i^2 + 1}^{(2j_i^2 + 1, -2j_i^2 - 1)} (x) . \tag{25}
\]

Performing the analysis parallel to the selection rules of the matrix elements, we find that the coupling coefficient vanishes unless the \( r_i^2 \) and \( r_i^2 \) satisfy the generalized triangular conditions as following

\[
|r_1^2 - r_2^2| \leq r_3^2 \leq r_1^2 + r_2^2 , \tag{26}
\]

etc. The selection rules of the \( SO(5) \) coupling coefficients about the \( SO(4) \) subgroup quantum numbers are provided by the 9-j symbol and two \( SU(2) \) coupling coefficients.

Usually, the evaluation of a coupling coefficient involving a set of basis states labeled by the irreducible representations of a chain of nested subgroup for a semi-simple group. In this approach, the most important ingredient is Racah’s factorization lemma, which ensure the coupling coefficient to factorize into the coupling coefficients of the subgroup. Then, the evaluation of a coupling coefficient becomes the calculations of the re-coupling coefficients, also called the isoscalar factors. In fact, the calculation of the isoscalar factor is very difficult. The advantage of this method is that it can
overcome the trouble with a semi-simple group being not generally simply reducible. However, for the case of $SO(5)$ angular momentum, the group $SO(5)$ is simply reducible in the $SO(5)$ level, i.e., every irreducible representation of $SO(5)$ only appears once in the decomposition of the direct product of two $SO(5)$ irreducible representations. Our deriving the explicit expression of the coupling coefficients of $SO(5)$ is just based on this fact. Since the expression of the coupling coefficients given here is analytically exact, it is not necessary further to calculate the isoscalar factors appearing in the factorization of the coupling coefficients of the chain of nested subgroup.

In fact, $\tilde{\Theta}$ and $\tilde{\Phi}$ together with the 9-j symbol of $SO(3)$ in the Eq.(23) provide the isoscalar factor for $SO(5) \supset SU(2) \times SU(2)$ in the level of $SO(5)$ angular momentum. Here, $SU(2) \times SU(2)$ is the relatively direct product, of which the rotation transformation is generated by one part $\hat{J}^{(0)}_1$ and another relative part $\hat{J}^{(0)}_2 = \hat{I}_1$ by means of $\sum_i \hat{J}^{(0)}_i = \sum_i \hat{J}^{(0)}_2$. Furthermore, we give here the analytically exact expressions of all isoscalar factors of $SO(5)$ in the $SO(5)$ angular momentum level. It is emphasized worthily that all irreducible representations of $SO(5)$ discussed by us here are those belonging to the $SO(5)$ angular momentum, exactly, those corresponding to the Hilbert spaces which are composed of the $SU(2)$ monopole harmonics.

We can construct many-particle’s states from the single-particle states by using of the expression of the coupling coefficients. In the procedure of many-particle state construction, the re-coupling coefficients can be obtained by performing summation over products of the coupling coefficients. Therefore, the expression of the coupling coefficients given by us here is important for the study of the 4-dimensional quantum Hall system and the physical system with the rotation symmetry of $SO(5)$. In fact, the calculation of the matrix elements of fuzzy monopole harmonics is that of the coupling coefficients.

On the other hand, the matrix operators $\hat{Y}^{R}_{r_2 \leq r_1 \leq 2I}$ are the single-body operators acting in the LLL Hilbert space since the matrix forms of them are provided by the matrix elements between the LLL states of single particle system. By determining the ground states, i.e., the LLL states, which can be obtained by the linear combinations of these operators acting on the vacuum state, we can use these operators to generate all possible single particle states including the quasi-particle and quasi-hole states near the LLL states. In this sence, the fuzzy monopole harmonics given by us are the generators of the general wave functions of single particle system. In order to explore the detail structure of this system’s Hilbert space and the properties of quantum states and operators in the 4-diemnsional quantum Hall droplet, we need further to discuss the algebrical structure of these operators, i.e., the commutation relations between them.

4 Matrix algebra of fuzzy $S^4$ and non-commutative geometry from the system

Now, we turn to the discussion of the matrix algebra of the fuzzy monopole harmonics. It is clear that the matrix operators $\hat{Y}^{R}_{r_2 \leq r_1 \leq 2I}$ belong to the irreducible representations of $SO(5)$. All operators belong to the irreducible representations of $SO(5)$ must satisfy the following relation of the tensor products of the operators

$$\hat{T}^{R_1}_{J_1} \hat{T}^{R_2}_{J_2} = \sum_{R,J} \left( R \begin{array}{c|cc} R_1 & R_2 \\ J_1 & J_2 \end{array} \right) \hat{T}^{R}.$$  \hspace{1cm} (27)

Another useful relation is the Wigner-Eckart theorem for the theory of $SO(5)$ angular momentum.
It is read as
\[
\begin{pmatrix}
R_1 & \hat{R}_J \\
J_1 & \hat{J}_J
\end{pmatrix}
\begin{pmatrix}
R_2 \\
J_2
\end{pmatrix}
= \begin{pmatrix}
R_1 & R \\
J_1 & J
\end{pmatrix}
\begin{pmatrix}
R_2 \\
J_2
\end{pmatrix}
\langle R_1 \| \hat{R}_J \| R_2 \rangle,
\]
(28)
where \( \langle R_1 \| \hat{R}_J \| R_2 \rangle \) is independent of the subgroup \( SO(4) \) quantum numbers of \( SO(5) \), which is called as the reduced matrix elements of the operator \( \hat{R}_J \). This theorem can be derived by using of the standard method of the group representation \( [2] \) and the fact that every irreducible representation of \( SO(5) \) appears once in the decomposition of the Kronecker product of two irreducible representations of \( SO(5) \) in the case of \( SO(5) \) angular momentum.

We can obtain the relation between the reduced matrix elements by calculating the matrix elements of the tensor product’s relation of the operators. Furthermore, we scale the operators \( \hat{Y}_J \) into
\[
\hat{Y}_J = \frac{\hat{Y}_J}{\langle \hat{Y}_J \| \hat{Y}_J \rangle}.
\]
(29)
By means of the relation between the reduced matrix elements, we can now read off the matrix algebraic relation of the operators \( \hat{Y}_J \) from the operator product relation (27). This relation is
\[
\hat{Y}_{J_1} \hat{Y}_{J_2} = \sum_{R,J} \begin{pmatrix}
R & R_1 & R_2 \\
J & J_1 & J_2
\end{pmatrix}
\begin{pmatrix}
R_1 & R_2 & \frac{P}{2} \\
J_1 & J_2 & \frac{K_1}{2}
\end{pmatrix}
[D(\frac{P}{2})]^{-1} \hat{Y}_J
\]
(30)
where we have define that \( D(R) = D(r_1, r_2) \) and \( D(\frac{P}{2}) = D(r_1 = I, r_2 = I) \). The operator of the l.h.s. of the Eq.(30) is given by the matrix multiplication between the operators \( \hat{Y}_{J_1} \) and \( \hat{Y}_{J_2} \), which act in the LLL Hilbert space. The 6-J symbol of \( SO(5) \) is given by
\[
\begin{pmatrix}
R_1 & R_2 & \frac{P}{2} \\
J_1 & J_2 & \frac{K_1}{2}
\end{pmatrix}
= \sum_{J,J_1,J_2,K,K_1, K_2} \begin{pmatrix}
R & R_1 & R_2 \\
J & J_1 & J_2
\end{pmatrix}
\begin{pmatrix}
R_1 & R_2 & \frac{P}{2} \\
J_1 & J_2 & \frac{K_1}{2}
\end{pmatrix}
\begin{pmatrix}
R & \frac{P}{2} \\
J & \frac{K_1}{2}
\end{pmatrix}
\begin{pmatrix}
R & R_2 \\
J & J_2
\end{pmatrix}
\begin{pmatrix}
R_1 & \frac{P}{2} \\
J_1 & K_1
\end{pmatrix}
\begin{pmatrix}
R_2 & \frac{P}{2} \\
J_2 & K_2
\end{pmatrix}
\]
(31)
It can be seen easily that the 6-J symbol of \( SO(5) \) is possessed of the properties very similar to the 6-j symbol of \( SO(3) \).

The matrix algebra (30) of the fuzzy \( S^4 \) is formally analogous to that of the fuzzy \( S^2 \) \([33, 34, 35]\). Because of the operators \( \hat{Y}_J, 0 \leq r_2 \leq r_1 \leq 2I \) consist of a complete basis of the \( D(\frac{P}{2}) \times D(\frac{P}{2}) \) matrix algebra, any operator acting in the LLL Hilbert space can be expressed as a linear combination of the matrices \( \hat{Y}_J, 0 \leq r_2 \leq r_1 \leq 2I \). The product of such operators again becomes the linear combination of the matrices due to the matrix algebraic relation (30). In the other words, the matrix algebra is a fundamental relation to determine the algebraic relations of all operators acting in the LLL Hilbert space. The origin of the matrix operators and their algebra appearing in the 4-dimensional quantum Hall system is due to the existence of the Yang’s \( SU(2) \) monopole in the system. Its appearance makes the coordinate space \( S^4 \), which the particles live in, become non-commutative. The description of this non-commutative geometry can be obtained by replacing the algebra of the functions with the matrix operators, i.e., the fuzzy monopole harmonics. The procedure of our finding these matrix operators and their matrix algebra above is just establishing the description of this non-commutative geometry. In this sense, the matrix algebra given here is the algebra of fuzzy \( S^4 \), and describes the
non-commutativity of the coordinate space \(S^4\). When the strength of the Yang’s \(SU(2)\) monopole vanishes, i.e., \(I = 0\), the dimension of the matrix becomes one dimensional and trivial, and then the matrix algebra does the algebra of the functions. Consequently, the fuzzy \(S^4\) becomes a classical \(S^4\) when \(I = 0\).

The matrix algebra given by us is an associative algebra. The simple interpretation of the associativity of the matrix algebra is that since these matrices are the operators acting in the LLL Hilbert space, they can be expressed as

\[
\tilde{Y}_J = \sum_{K_1,K_2} \left( \frac{P}{K_1} \left| \tilde{Y}_{J_1} \right| \frac{P}{K_2} \left| \tilde{Y}_{J_2} \right| \right) \left( \frac{P}{K_1} \left| \tilde{Y}_{J_3} \right| \frac{P}{K_2} \right).
\]

(32)

Thus, the products of three operators become those of three matrices. The associativity of the matrix algebra is equivalently described by the identity of the matrices

\[
\sum_{K',K''} \left( \frac{P}{K_1} \left| \tilde{Y}_{J_1} \right| \frac{P}{K_2} \left| \tilde{Y}_{J_2} \right| \right) \left( \frac{P}{K_1} \left| \tilde{Y}_{J_3} \right| \frac{P}{K_2} \right) = \sum_{K',K''} \left( \frac{P}{K_1} \left| \tilde{Y}_{J_1} \right| \frac{P}{K_2} \left| \tilde{Y}_{J_2} \right| \right) \left( \frac{P}{K_1} \left| \tilde{Y}_{J_3} \right| \frac{P}{K_2} \right).
\]

(33)

Obviously, the results of the above summations are same because there does not exist any singularity to make the summarizing sequences change the results of the summations. Of course, one can straightforwardly show the associativity of the operator algebra of the fuzzy \(S^4\) in the similar manner of the proof of the associativity for the case of fuzzy \(S^2\) \[33\]. This is a more complicate and more technical procedure in which the generalized Biedenharn-Elliott relation for the 6-J symbols of \(SO(5)\) should be established. Although this generalized relation is very useful, we do not discuss it here.

From the above discussions, one can see that the matrix algebra (30) is the multiplicative relation of the matrices produced by the fuzzy monopole harmonics acting in the LLL Hilbert space of single particle system. These matrices consist of a complete set of all operators belonging to the LLL Hilbert space of single particle system. It is well known that the Hilbert space of the many particles can be constructed from the single particle’s Hilbert spaces by the coupling of some few of single-particles. The most simplest way of fuzzy monopole harmonics’ construction of the system of \(N\) particles is to construct first the most elementary fuzzy monopole harmonics of the system of \(N\) particles by making of the symmetrical direct sum of \(N\) single particle fuzzy monopole harmonics. Then, one can produce all fuzzy monopole harmonics of the system of \(N\) particles by using of the matrix algebra (30). This way can not provide the generally truncated rule of the irreducible representations of \(SO(5)\) corresponding to the fuzzy monopole harmonics, which are composed of the LLL Hilbert space of the system of \(N\) particles. On the other hand, it is not obvious that the connection between the construction of the fuzzy monopole harmonics in this way and the wave functions of quasi-particle or quasi-hole excitations in the Laughlin’s and Haldane’s forms. We shall give another scheme of the construction of fuzzy monopole harmonics of the system of \(N\) particles in the rest of this section, and the most simplest way is a special case of the following scheme.

Subsequently, we shall discuss how the fuzzy monopole harmonics of the coupling system are constructed from the subsystems composed of the coupling system. Since the particles spread over the four-sphere \(S^4\), this study is very important for the description of non-commutative geometry of the 4-dimensional quantum Hall droplet. Our starting point is the tensor product relation of two tensor operators belonging to the irreducible representations of \(SO(5)\) which correspond to the Yang’s
SU(2) monopole harmonics. The tensor operators $\hat{T}_j^{R_1}(1)$ and $\hat{T}_j^{R_2}(2)$ are supposed to work on the subsystem 1 and the subsystem 2 respectively of a system in order to distinguish them labelled by adding to 1, 2. Because of these operators belonging to the irreducible representations of $SO(5)$, they must satisfy the tensor product relation of the operators

$$\hat{T}_j^R = \sum_{j_1,j_2} \left\langle R^1_{j_1} \ R^2_{j_2} \mid R \right| \hat{T}_j^{R_1}(1) \hat{T}_j^{R_2}(2) ,$$

(34)

where $\hat{T}_j^R$ is the tensor operator of the coupling system. In order to calculate the matrix elements, we must choose the coupling scheme of the subsystem in the system. Suppose that we want to calculate the operator matrix elements between the left vectors belonging to the irreducible representation of $R^1$ and $R^2$ coupling into $R'$ and the right vectors doing that of $R'^1$ and $R'^2$ coupling into $R''$.

We can first make use the tensor product relation (34) and the Wigner-Eckart theorem (28) to obtain an expression for the reduced matrix element in such coupling scheme. The result is

$$\langle R' \mid \hat{T}^R \mid R'' \rangle = [D(R'')^{-1}]^{-1} \left\langle R'^1 \ R^1 \ R'^2 \mid R^1 \ R^2 \mid R'' \right\rangle \langle R^1 \mid \hat{T}^R(1) \mid R'^1 \rangle \langle R^2 \mid \hat{T}^R(2) \mid R'^2 \rangle ,$$

(35)

where we have defined the 9-J symbol of the $SO(5)$ angular momentum, which is given by

$$\left\langle \begin{array}{cccc} R' & R & R'' \\ R^1 & R^1 & R'^1 \\ R^2 & R^2 & R'^2 \end{array} \right\rangle = \sum_{\text{all } J's} \left\langle \begin{array}{cccc} R^1 & R^2 \\ R & J \end{array} \right\rangle \left\langle \begin{array}{cccc} R'^1 & R'^2 \\ J'^1 & J'^2 \end{array} \right\rangle \left\langle \begin{array}{cccc} R'' & R' \\ J'' & J' \end{array} \right\rangle \left\langle \begin{array}{cccc} R^1 & R^1 \\ J^1 & J^1 \end{array} \right\rangle \left\langle \begin{array}{cccc} R^2 & R^2 \\ J^2 & J^2 \end{array} \right\rangle \times$$

$$\left\langle \begin{array}{cccc} R'^2 & R'^2 \\ J'^2 & J'^2 \end{array} \right\rangle \left\langle \begin{array}{cccc} R'' & R' \\ J'' & J' \end{array} \right\rangle \left\langle \begin{array}{cccc} R'^1 & R'^1 \\ J'^1 & J'^1 \end{array} \right\rangle .$$

(36)

Then, we scale the tensor operator $\hat{T}_j^R$ as

$$\tilde{T}_j^R = \frac{\hat{T}_j^R}{\langle R' \mid \hat{T}^R \mid R'' \rangle} ,$$

(37)

Finally, from the tensor product relation of operators we can read off a fundamental formula of the coupling system’s operators constructed by means of the tensor product of the subsystem’s operators, which is

$$\tilde{T}_j^R = \sum_{R',R'',J_1,J_2} [D(R'')D(R'')]^{-1} \left\langle \begin{array}{cccc} R^1 & R^2 \\ J^1 & J^2 \end{array} \right\rangle \left\langle \begin{array}{cccc} R^1 & R^1 \\ R' & R'^2 \end{array} \right\rangle \left\langle \begin{array}{cccc} R & R'' \\ R'' & R'' \end{array} \right\rangle \tilde{T}_j^{R_1}(1) \tilde{T}_j^{R_2}(2) .$$

(38)

It should be pointed that in the procedure of our obtaining the above formula, we have used the orthogonality relation of the 9-J symbol of the $SO(5)$ angular momentum. In fact, the inverse relation of the above formula also is important since it can be regarded as the operator product expansion of two operators. In order to make the transfer form one to another among them become convenient, here we write out this orthogonality relation

$$\sum_{R',R''} [D(R'')]^{-1} \left\langle \begin{array}{cccc} R^1 & R^2 & R & R'' \\ R'^1 & R'^2 & R' & R'' \end{array} \right\rangle \left\langle \begin{array}{cccc} R & R'' \\ R'' & R'' \end{array} \right\rangle = D(R'')D(R'') .$$

(39)
Matrix operators act in the LLL Hilbert space of our considering system. The description of non-commutative geometry of the 4-dimensional quantum Hall droplet since these

Hence, the matrix algebra is the description of the coordinate non-commutativity of particle’s moving.

The present goal is to establish a scheme that the generators of matrix algebra corresponding to the collective excitations are constructed from the generators of the lowest energy excitations of the single particle.

Because of the existence of the finite energy gap, all matrix operators act in their LLL Hilbert spaces. Generally, for the system including an Yang’s SU(2) monopole, its LLL Hilbert space is described by the irreducible representation of SO(5) $R = [r_1, r_2]$. This implies that for our considering system all irreducible representations labelling the matrix elements of the operators should belong to such irreducible representations of SO(5) as $R = [r_1, r_2]$, e.g., $R', R^{1'}, R^{2'}, R''$, $R^{1''}$ and $R^{2''}$ in the equation (38) should be this kind of the irreducible representations of SO(5). Furthermore, we have that $R' = R''$, $R^{1'} = R^{1''}$ and $R^{2'} = R^{2''}$ because the operators are realized by the matrices.

Now, we can establish a fusion scheme of the matrices of the coupling system from the matrices of the subsystems based on the fundamental formula (38), which is

$$\tilde{\mathcal{Y}}_J = \sum_{R^{1},J^{1},J^{2}} [D(R^{2})D(R^{1''})]^{-1} \left\{ \begin{array}{c} R^{1} \\ J^{1} \\ J^{2} \end{array} \right| R \right\} \tilde{\mathcal{Y}}_{J^{1}}(1)\tilde{\mathcal{Y}}_{J^{2}}(2),$$

where $R' = [r_1', r_2']$, $R^{1'} = [r_1'^{1}, r_2'^{1}]$ and $R^{2'} = [r_1'^{2}, r_2'^{2}]$.

By using of the generalized triangular relation of the SO(5) coupling coefficients and the property of the 9-J symbol of SO(5), a fusion rule of this fusion scheme can be read off as following

$$|r_2^{1'} - r_2^{2'}| \leq r_2 \leq r_1 \leq r_1^{1'} + r_1^{2'}, \quad 0 \leq r_2 \leq r_1 \leq 2r_2^{1'},$$

$$|r_2^{1'} - r_2^{2'}| \leq r_2 \leq r_1^{1'} \leq r_2^{1'} + r_2^{2'}, \quad 0 \leq r_2 \leq r_1 \leq 2r_2^{1'},$$

$$0 \leq r_2 \leq r_1 \leq 2r_2^{1'}).$$

For the 4-dimensional quantum Hall system composed of $N$ particles, one can repeatedly use the fusion formula and its fusion rule for $N - 1$ times to obtain the elements of the matrix algebra of this system. However, the matrix operators obtained in the finals are certainly those of the fuzzy monopole harmonics of the system. That is, if the LLL Hilbert space is the space that corresponds to the irreducible representation of SO(5) $Q = [q_1, q_2]$, the dimension of this Hilbert space is $D(Q)$ and the operators obtained in the finals are the $D(Q) \times D(Q)$ matrices. These matrix operators consist of the complete set of irreducible representations of SO(5) in $D(Q) \times D(Q)$ matrices, and their number is $D(Q) \times D(Q)$. The non-commutativity of these operators is described by the matrix algebra (30) of replacing $\mathbb{P}_E$ with $Q$. This matrix algebra is universal for the 4-dimensional quantum Hall fluids. Hence, the matrix algebra is the description of the coordinate non-commutativity of particle’s moving on $S^4$ in the 4-dimensional quantum Hall system. Exactly, this matrix algebra should be viewed as the description of non-commutative geometry of the 4-dimensional quantum Hall droplet since these matrix operators act in the LLL Hilbert space of our considering system.
Although both the equations (30) and (40) describe the operator product’s relations of the fuzzy monopole harmonics, the significances of them are different. The former should be view as the matrix multiplicater relation of the fuzzy monopole harmonics of the same system, and the latter as the operator tensor product relation of the fuzzy monopole harmonics of the different subsystems. Because the 4-dimensional quantum Hall droplet is a system of many-body, they determine all operators, which act in the LLL Hilbert space of the 4-dimensional quantum Hall droplet, to obey the rules. As our explaining in the section 3, the fuzzy monopole harmonics can be considered as the generators of the general wave functions of single particle system. Hence, the matrix algebra (30) and the operator tensor product (40) can be used the construction of the wave functions of the LLL and its collective excitations of the 4-dimensional quantum Hall system. We shall discuss this construction elsewhere.

5 Summary and outlook

Similar to particle’s motion on a plane in a constant magnetic field, the existence of Yang’s $SU(2)$ monopole in the 4-dimensional quantum Hall system make the coordinates of particle’s moving on the four-sphere become non-commutative. The appearance of such monopole also results in the irreducible representations of $SO(5)$ belonging to the Hilbert space of the system to be truncated. The similar phenomenon occurs in the description of fuzzy two-sphere. This clues us to the description of non-commutative geometry of the 4-dimensional quantum Hall system. Here we found that the fuzzy $S^4$ describes the non-commutative geometry of the 4-dimensional quantum Hall droplet. By determining the explicit forms of fuzzy monopole harmonics and their matrix algebra, we established the description of non-commutative geometry of the 4-dimensional quantum Hall droplet.

The $SU(2)$ monopole harmonics with the isospin rotating frame can be interpreted as the wave functions of a 4-dimensional symmetrical top under the rotation transformation of $SO(5)$. Based on this view, we given the explicit expression of coupling coefficients of the $SO(5)$ angular momentum. The theory of angular momentum of $SO(5)$ is surprisingly simple and excellent, which is very similar to that of $SO(3)$. Many relations, paralleled the relations appearing in the $SO(3)$ theory, can be explicitly written off in $SO(5)$’s that. The expression of coupling coefficients given here is essential to the theory of angular momentum of $SO(5)$. These results are useful for the physics of atoms and molecules with the higher symmetry, i.e., $SO(5)$ symmetry.

Following the proof of equivalence of two-dimensional quantum Hall physics and non-commutative field theory given recently by Hellerman and Raamsdonk 3, we can clarify the physical implication of the fuzzy monopole harmonics here. The second-quantized field theory description of the quantum Hall fluid for various filling fractions should involve some non-commutative field theory. On the 2-dimensional plane, such non-commutative field theory is the regularized matrix version of the non-commutative $U(1)$ Chern-Simons theory. Our constructing the fuzzy monopole harmonics are a complete set of matrix version of non-commutative field theory corresponding to the 4-dimensional quantum Hall fluid. Because the creation and annihilation operators in the second-quantized field theory description of the quantum Hall fluid can be built up by this complete set. In this sense, these fuzzy monopole harmonics can be viewed as the creation and annihilation operators in the second-quantized field theory description of the quantum Hall fluid. The matrix algebra obeyed by these fuzzy monopole harmonics can be interpreted as the non-commutative relations satisfied by the creation and annihilation operators. The construction of the fuzzy monopole harmonics and their matrix algebra given by us is only the first step to establish the second-quantized field theory description of the 4-dimensional quantum Hall fluid, but it is an essential step. The fusion scheme of the fuzzy
monopole harmonics and its inverse relation provide an approach for the calculation of correlation functions in the non-commutative field theory. In fact, the methods and some results present here can be straightforwardly used for the study of the 4-dimensional quantum Hall system.

On the other hand, it is interesting to relate the matrix algebra of fuzzy $S^4$ given here with some applications in D-brane dynamics in string theory and M-theory. The ability to construct the higher dimensional brane configurations using D0-branes is essential for a success of the Matrix theory of BFSS [22], where for example arbitrary membrane configurations in M-theory must be described in terms of the low energy degrees of freedom of the D0-branes. Myers [23] found that D0-branes expand into spherical D2-branes in the constant background RR fields. From the matrix model construction of Kabat and Taylor [36], the non-commutative solution for such spherical D2-brane actually represents the bound state of a spherical D2-brane with some D0-branes. The fuzzy $S^4$ was used in the context of the Matrix theory of BFSS to describe time-dependent 4-brane solutions constructed from the D0-brane degrees of freedom. In this sense, the fuzzy monopole harmonics $\tilde{\mathcal{Y}}^R_{J}$ can be used to describe classical solutions of the corresponding matrix brane model. However, the matrix algebra of fuzzy $S^4$ present here is different with those of fuzzy $S^4$ in the references [21, 37, 24, 25]. The study of the matrix brane construction associated with the matrix algebra of fuzzy $S^4$ present here and the relation between it and the matrix brane constructions of the above references is an interesting topic. The work of this aspect is in progress.

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References

[1] G.V. Dunne, R. Jackiw and C.A. Trugenberger, Phys. Rev. D41, 661 (1990).
[2] R. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[3] L. Susskind, The quantum Hall fluid and non-commutative Chern-Simons theory, hep-th/0101029.
[4] A.P. Polychronakos, Quantum Hall states as matrix Chern-Simons theory, hep-th/0103013.
[5] S. Hellerman and M.V. Raamsdonk, Quantum Hall physics equals non-commutative field theory, hep-th/0103179.
[6] D. Karabali and B. Sakita, Chern-Simons matrix model: coherent states and relation to Laughlin wave functions, hep-th/0106016.
[7] F.D.M. Haldane, Phys. Rev. Lett. 51, 605 (1983).
[8] S.C. Zhang and J.P. Hu, Science 294, 823 (2001).
[9] C.N. Yang, J. Math. Phys. 19, 320 (1978).
[10] A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, Phys. Lett. B59, 85 (1975).
[11] C.N. Yang, J. Math. Phys. 19, 2622 (1978).
[12] J.P. Hu and S.C. Zhang, Collective excitations at the boundary of a 4D quantum Hall droplet, cond-mat/0112432.
[13] A. Connes, Non-commutative geometry, Academic Press, 1994.
[14] J. Madore, Class. Quant. Grav. 9, 69(1992).
[15] H. Grosse, C. Klimcik and P. Presnajder, Commun. Math. Phys. 180, 429(1996).
[16] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B305, 545 (1988).
[17] H. Grosse and A. Strohmaier, Lett. Math. Phys. 48, 163(1999).
[18] A. Connes and G. Landi, Commun. Math. Phys. 221, 141(2001).
[19] F. Bonechi, N. Ciccoli and M. Tarlini, Non-commutative instantons on the 4-sphere from quantum group, math.QA-0012236.
[20] A. Konechny and A. Schwarz, Phys. Rep. 360, 353(2002).
[21] J. Castellino, S. Lee and W. Taylor, Nucl. Phys. B526, 334 (1998).
[22] T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. D55, 5112 (1997).
[23] R. Myers, JHEP 9911, 037 (1999); hep-th/991053.
[24] S. Ramgoolam, On spherical harmonics for fuzzy spheres in diverse dimensions, hep-th/0105006.
[25] P.M. Ho and S. Ramgoolam, Higher dimensional geometries from matrix brane constructions, hep-th/0111278.
[26] M. Fabinger, Higher-dimensional quantum Hall effect in string theory, hep-th/0201010.
[27] D. Karabali and V.P. Nair, Quantum Hall effect in higher dimensions, hep-th/0203264.
[28] Y. Kimura, Non-commutative gauge theory on fuzzy four-sphere and matrix model, hep-th/0204256.
[29] M. Minami, Prog. Theor. Phys. 63, 303(1980).
[30] E. Demler and S.C. Zhang, Ann. Phys. 271, 83(1999).
[31] A.M. Peremolov, Generalized coherent state and their applications, Springer-Verlag, Berlin, 1986.
[32] B.G. Wybourne, Classical groups for physicists, by John Wiley & Sons, 1974.
[33] A. Alekseev, A. Recknagel and V. Schomerus, JHEP 9909, 023 (1999); hep-th/9908040.
[34] C. Chan, C. Chen, F. Lin and H. Yang, CP^n model on fuzzy sphere, hep-th/0105087.
[35] B.Y. Hou, B.Y. Hou and R.H. Yue, Fuzzy sphere bimodule, ABS construction to the exact soliton solution, hep-th/0109091.
[36] D. Kabat and W. Taylor, Adv. Theor. Math. Phys. 2, 181 (1998).
[37] N. Constable, R. Myers and O. Tafjord, JHEP 0106,023(2001); hep-th/0102080.