Spacing distributions in random matrix ensembles

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1 Introduction

1.1 Motivation and definitions

The topic of spacing distributions in random matrix ensembles is almost as old as the introduction of random matrix theory into nuclear physics. Both events can be traced back to Wigner in the mid 1950’s [37, 38]. Thus Wigner introduced the model of a large real symmetric random matrix, in which the upper triangular elements are independently distributed with zero mean and constant variance, for purposes of reproducing the statistical properties of the highly excited energy levels of heavy nuclei. This was motivated by the gathering of experimental data on the spectrum of isotopes such as $^{238}$U at energy levels beyond neutron threshold. Wigner hypothesized that the statistical properties of the highly excited states of complex nuclei would be the same as those of the eigenvalues of large random real symmetric matrices. For the random matrix model to be of use at a quantitative level, it was necessary to deduce analytic forms of statistics of the eigenvalues which could be compared against statistics determined from experimental data.

What are natural statistics for a sequence of energy levels, and can these statistics be computed for the random matrix model? Regarding the first question, let us think of the sequence as a point process on the line, and suppose for simplicity that the density of points is uniform and has been normalized to unity. For any point process in one dimension a fundamental quantity is the probability density function for the event that given there is a point at the origin, there is a point in the interval $[s, s + ds]$, and further there are $n$ points somewhere in between these points and thus in the interval $(0, s)$. Let us denote the probability density function by $p(n; s)$. In the language of energy levels, this is the spacing distribution between levels $n$ apart.

Another fundamental statistical quantity is the $k$-point distribution function $\rho_k(x_1, \ldots, x_k)$. This can be defined recursively, starting with $\rho_{(1)}(x)$, by the requirement that

$$\rho_k(x_1, \ldots, x_k)/\rho_{(k-1)}(x_1, \ldots, x_{k-1}) = \sum_{n=0}^{\infty} p(n; s).$$

(1.1)
From empirical data of a long energy level sequence, the quantity \( p(n; s) \) for small values of \( n \) at least is readily estimated (the statistical uncertainty gets worse as \( n \) increases). Use of (1.2) then allows for an estimation of \( \rho(2)(0; s) \).

We thus seek the theoretical determination of \( p(n; s) \) for matrix ensembles.

### 1.2 Spacing between primes

Before taking up the problem of determining \( p(n; s) \) for matrix ensembles, which is the theme of these lectures, let us digress a little and follow the line of introduction to spacing distributions given by Porter in the review he wrote as part of the book [31], which collected together the major papers written in the field up to 1965. Porter’s introduction is particularly relevant to the theme of the present school because it uses the prime numbers as an example of a deterministic sequence which, like energy levels of heavy nuclei, nevertheless exhibit pronounced stochastic features.

It turns out the spacing distributions between primes relate to perhaps the simplest example of a point process. This is when the probability that there is a point in the interval \([s, s + ds]\) is equal to \( ds \), independent of the location of the other points. This generates the so called Poisson process with unit density, or in the language of statistical mechanics, a perfect gas. By definition of the process the ratio (1.1) is unity for all \( k \) and thus

\[
\rho(k)(x_1, \ldots, x_k) = 1. \tag{1.3}
\]

To compute \( p(n; s) \), we think of the Poisson process as the \( N \to \infty \) limit of a process in which each unit interval on the line is broken up into \( N \) equal sub-intervals, with the probability of there being a particle in any one of the subintervals equal to \( 1/N \). Thus

\[
p(s; n) = \lim_{N \to \infty} (1 - \frac{1}{N})^{sN - n} N^{-n} \binom{sN}{n} = \frac{s^n}{n!} e^{-s}. \tag{1.4}
\]

In the first equality of (1.4), the first factor is the probability that \( sN - n \) subintervals do not contain a particle, the second factor is the probability that \( n \) subintervals do contain a particle, while the final factor is the number of ways of choosing \( n \) occupied sites amongst \( sN \) sites in total. The probability density in the final equality of (1.4) is the Poisson distribution. Substituting (1.4) in (1.2) gives \( \rho(2)(0, x) = 1 \), as required by (1.3).

The distribution (1.4) ties in with prime numbers through Kramér’s model (see the lectures by Heath-Brown in the present volume). In this approximation, statistically the primes are regarded as forming a Poisson process on the positive integer lattice. The probability of occupation of the \( N \)th site is taken to equal \( 1/\log N \), so as to be consistent with the prime number theorem. Kramér’s model predicts that as an approximation

\[
p^{(N)}(n; s) = \frac{s^n}{n!} e^{-s}, \quad s = t/\log N \tag{1.5}
\]

where \( p^{(N)}(n; s) \) refers to the probability that for primes \( p \) in the neighbourhood of a prime \( N \), there is a prime at \( p + t \), and furthermore there are exactly \( n \) primes between \( p \) and \( p + t \).
Figure 1: Distribution of the spacing $t$ between primes (leftmost graph) and the spacing $t$ between every second prime for 2,000 consecutive primes starting with $N = 10^9 + 7$. The distributions are given in units of $s = t / \log N$. The smooth curves are the Poisson distributions $p(0; s) = e^{-s}$ and $p(1; s) = se^{-s}$.

To compare the prediction against empirical data, we choose a value of $N$, say $10^9$, and for the subsequent $M$ primes (say $M = 2,000$) record the distance to the following prime (in relation to $p^{(N)}(1; s)$) and the distance to the second biggest prime after that (in relation to $p^{(N)}(s; 1)$). We form a histogram, with the scale on the horizontal axis measured in units of $s = t / \log N$, where $t$ is the actual spacing. The natural units for $t$ are multiples of 2, and this provides a width for the bars of the histogram. We see from Figure 1 that the general trend of the histograms do indeed follow the respective Poisson distributions.

1.3 Empirical determination of spacing distributions for matrix ensembles

Wigner’s interest was in the statistical properties of the eigenvalues of large real symmetric random matrices. More particularly, he sought the statistical properties of the eigenvalues in what may be termed the bulk of the spectrum (as opposed to the edge of the spectrum [9]). The eigenvalues in this region are characterized by having a uniform density, which after rescaling (referred to as ‘unfolding’) may be taken as unity. In distinction to the situation with the sequence of primes, for random matrices it is not necessary to study the statistical properties of a large sequence of (unfolded) eigenvalues from a single matrix. Rather the spacing distributions with respect to the middle eigenvalue (this is the eigenvalue most in the bulk) in multiple samples from the class of random matrices in question can be listed, and then this list used to create a histogram. Moreover, to approximate large matrix size behaviour, it is only necessary to consider quite small matrix sizes, say $13 \times 13$.

In Figure 2 we have plotted the empirical determination of $p(0; s)$ and $p(1; s)$ obtained from lists of eigenvalue spacings for realizations of the so called GOE (Gaussian orthogonal ensemble) eigenvalue distribution. As we know from the lectures of Fyodorov in this volume, the GOE
Figure 2: Plot of the distribution of the unfolded spacing between the 6th and 7th, and 7th and 8th eigenvalues (pooled together) for 2,000 samples from the $13 \times 13$ GUE eigenvalue distribution. The smooth curve is the Wigner surmise $p_4(0; s)$. The rightmost graph is the distribution between the 6th and 8th eigenvalues in the same setting, while in the smooth curve in this case is $(1/2)p_4(0; s/2)$ with $p_4$ given by (2.13).

consists of real symmetric random matrices, with each diagonal element chosen from the normal distribution $N[0, 1]$, and each (strictly) upper triangular element chosen from the normal distribution $N[0, 1/\sqrt{2}]$. For such matrices, it is well known that to leading order in the matrix rank $N$, the eigenvalue density is given by the Wigner semi-circle law

$$
\rho_{1}(x) = \frac{\sqrt{2N}}{\pi} \sqrt{1 - \frac{x^2}{2N}}.
$$

Multiplying the eigenvalues at point $x$ by this factor allows us to unfold the sequence giving a mean eigenvalue spacing of unity.

A less well known, and much more recent result relating to GOE matrices is that their spectrum can be realized without having to diagonalize a matrix [4] (see also [13]). Thus one has that the roots of the random polynomial $P_N(\lambda)$, defined recursively by the stochastic three term recurrence

$$
P_k(\lambda) = (\lambda - a_k)P_{k-1}(\lambda) - b_{k-1}^2P_{k-2}(\lambda)
$$

where

$$
a_k \sim N[0, 1], \quad b_k^2 \sim \text{Gamma}[k/2, 1],
$$

have the same distribution as the eigenvalues of GOE matrices (the notation Gamma[$s, \sigma$] denotes the gamma distribution with density proportional to $x^{s-1}e^{-x/\sigma}$). Generating such polynomials and finding their zeros then provides us with a sequence distributed as for GOE eigenvalues, from which we have determined $p(0; s)$ and $p(1; s)$.
2 Eigenvalue product formulas for gap probabilities

2.1 Theory relating to \( p(n; s) \)

Consider a point process consisting of a total of \( N \) points. Let the joint probability density function of the \( N \) points be denoted \( p(x_1, \ldots, x_N) \). A quantity closely related to the spacing distribution \( p(0; s) \) is the gap probability

\[
E_{\text{bulk}}(0; s) := \lim_{N \to \infty} a_N^N \int_{-s/2}^{s/2} \int_{-s/2}^{s/2} \cdots \int_{-s/2}^{s/2} p(a_N x_1, \ldots, a_N x_N)
\]

(2.1)

where \( \tilde{I} = (-\infty, \infty) - (-s/2, s/2) \) and \( a_N \) is the leading large \( N \) form of the local density at the origin (and thus the unfolding factor). Thus it is easy to see that

\[
p(0; s) = \frac{d^2}{ds^2} E_{\text{bulk}}(0; s).
\]

(2.2)

More generally we can define

\[
E_{\text{bulk}}(n; s) := \lim_{N \to \infty} \left( \frac{N}{n} \right) a_N^N \int_{-s/2}^{s/2} \int_{-s/2}^{s/2} \cdots \int_{-s/2}^{s/2} \int_{I} \cdots \int_{I} p(a_N x_1, \ldots, a_N x_N)
\]

(2.3)

These quantities can be calculated from the generating function

\[
E_{\text{bulk}}(s; \xi) := \lim_{N \to \infty} a_N^N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{I} \cdots \int_{I} p(a_N x_1, \ldots, a_N x_N) \prod_{l=1}^{N} (1 - \xi \chi_{J}^{(l)}(-s/2, s/2))
\]

(2.4)

where \( \chi_{J}^{(l)} = 1 \) for \( x^{(l)} \in J \) and \( \chi_{J}^{(l)} = 0 \) otherwise, according to the formula

\[
E_{\text{bulk}}(n; s) = \frac{(-1)^n}{n!} \frac{d^n}{d\xi^n} E_{\text{bulk}}(s; \xi) \bigg|_{\xi=1}.
\]

(2.5)

It follows from the definitions that

\[
p(n; s) = \frac{d^2}{ds^2} E_{\text{bulk}}(n; s) + 2p(n-1; s) - p(n-2; s),
\]

(2.6)

or equivalently

\[
p(n; s) = \frac{d^2}{ds^2} \sum_{j=0}^{n} (n - j + 1) E_{\text{bulk}}(j; s).
\]

(2.7)

Hence knowledge of \( \{ E_{\text{bulk}}(j; s) \}_{j=0, \ldots, n} \) is sufficient for the calculation of \( p(n; s) \).

It is possible to relate \( E_{\text{bulk}}(j; s) \) to the \( k \)-point distribution functions. In the finite system the latter are given by

\[
\rho_{(k)}^{(N)}(x_1, \ldots, x_k) = \frac{N!}{(N-k)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x_1, \ldots, x_N).
\]

(2.8)

With

\[
\rho_{(k)}^{\text{bulk}}(x_1, \ldots, x_k) := \lim_{N \to \infty} a_N^k \rho_{(k)}^{(N)}(a_N x_1, \ldots, a_N x_k),
\]
by expanding (2.4) in a power series in $\xi$ and making use of (2.8) we see that
\[
E_{\text{bulk}}(s; \xi) = 1 + \sum_{k=1}^{\infty} \frac{(-\xi)^k}{k!} \int_{-s/2}^{s/2} dx_1 \cdots \int_{-s/2}^{s/2} dx_k \rho_{(k)}(x_1, \ldots, x_k).
\] (2.9)

For the limiting process to be rigorously justified, because $[-s/2, s/2]$ is a compact interval, it is sufficient that $\rho_{(k)}(x_1, \ldots, x_k)$ be bounded by $M^k$ for some $M > 0$.

With these basic formulas established, we will now proceed to survey some of the main results relating to spacing distributions in the bulk of the various matrix ensembles (orthogonal, unitary and symplectic symmetry classes).

### 2.2 Wigner surmise

For the Poisson process we have seen that $p(0; s) = e^{-s}$. Thus in this case the spacing distribution is actually maximum at zero separation between the points. The opposite feature is expected for $p(0; s)$ in relation to the eigenvalues of random real symmetric matrices, as can be seen by examining the $2 \times 2$ case of matrices of the form

\[
A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.
\]

This matrix is diagonalized by the decomposition $A = R \text{diag}[\lambda_+, \lambda_-]R^T$ where

\[
R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.
\]

Expressing $a, b, c$ in terms of $\lambda_+, \lambda_-, \theta$ it is simple to show

\[
dadbdc = |\lambda_+ - \lambda_-| d\lambda_+ d\lambda_- d\theta.
\] (2.10)

Thus for small separation $s := |\lambda_+ - \lambda_-|$ the probability density function vanishes linearly.

Let $\mu(s)$ denote the small $s$ behaviour of $p(0; s)$. We have seen that for the Poisson process $\mu(s) = 1$, while for the bulk eigenvalues of real symmetric matrices $\mu(s) \propto s$. Wigner hypothesized [38] that as with the Poisson process, $p(0; s)$ for the bulk eigenvalues of random real symmetric matrices could be deduced from the ansatz

\[
p(0; s) = c_1 \mu(s) \exp \left( - c_2 \int_0^s \mu(t) \, dt \right)
\] (2.11)

where the constants $c_1$ and $c_2$ are determined by the normalization requirements

\[
\int_0^\infty p(0; s) \, ds = 1, \quad \int_0^\infty sp(0; s) \, ds = 1
\]

(the second of these says that the mean spacing is unity). Thus one arrives at the so called Wigner surmise

\[
p(0; s) = \frac{\pi}{2} s e^{-\pi s^2/4}
\] (2.12)
for the spacing distribution of the bulk eigenvalues of random real symmetric matrices.

The ansatz (2.11) does not apply if instead of real symmetric matrices one considers complex Hermitian matrices, or Hermitian matrices with real quaternion elements. Examining the $2 \times 2$ case (see the introductory article by Porter in [31]) one sees that in the analogue of (2.10), the factor $|\lambda_+ - \lambda_-|$ should be replaced by $|\lambda_+ - \lambda_-|^\beta$ with $\beta = 2$ (complex elements) or $\beta = 4$ (real quaternion elements). Choosing the elements to be appropriate Gaussians, one can reclaim (2.12) and furthermore obtain

$$p_2(0; s) = \frac{32s^2}{\pi^2}e^{-4s^2/\pi}, \quad p_4(0; s) = \frac{2^{18}s^4}{3^6\pi^3}e^{-64s^2/9\pi}.$$  

(2.13)

as approximations to the spacing distributions in the cases $\beta = 2$ and $\beta = 4$ respectively.

### 2.3 Fredholm determinant evaluations

A unitary invariant matrix ensemble of $N \times N$ random complex Hermitian matrices has as its eigenvalue probability density function

$$\frac{1}{C} \prod_{l=1}^{N} w_2(x_l) \prod_{1 \leq j < k \leq N} (x_k - x_j)^2,$$  

(2.14)

which we will denote by $\text{UE}_N(g)$. We know (see the lectures by Fyodorov in this volume) that the $k$-point distribution function can be expressed in terms of the monic orthogonal polynomials $\{p_k(x)\}_{k=0,1,...}$ associated with the weight function $w_2(x)$,

$$\int_{-\infty}^{\infty} w_2(x)p_j(x)p_k(x) \, dx = h_j \delta_{j,k}.$$  

Thus with

$$K_N(x, y) = (w_2(x)w_2(y))^{1/2} \sum_{k=0}^{N-1} \frac{p_k(x)p_k(y)}{h_k},$$  

$$= (w_2(x)w_2(y))^{1/2} \frac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{x - y}$$  

(2.15)

we have

$$\rho^{(N)}_{(k)}(x_1, \ldots, x_k) = \det \left[ K_N(x_j, x_l) \right]_{j,l=1,\ldots,k}.$$  

(2.16)

This structure is significant for the evaluation of the generating function

$$E_{N,2}(J; \xi; w_2) := \left\langle \prod_{l=1}^{N} (1 - \xi x_l^{(0)}) \right\rangle_{\text{UE}_N(g)}$$  

(2.17)

(the subscript 2 on $E_{N,2}$ indicates the exponent in (2.14)). Expanding (2.17) in a power series analogous to (2.9) we obtain

$$E_{N,2}(J; \xi; w_2) = 1 + \sum_{k=1}^{N} \frac{(-\xi)^k}{k!} \int_J dx_1 \cdots \int_J dx_k \det \left[ K_N(x_j, x_l) \right]_{j,l=1,\ldots,k}.$$  

(2.18)
where use has been made of (2.16). The sum in (2.18) occurs in the theory of Fredholm integral equations [36], and is in fact an expansion of the determinant of an integral operator,

$$E_{N,2}(J; \xi; w_2) = \det(1 - \xi K_J)$$

(2.19)

where $K_J$ is the integral operator on the interval $J$ with kernel $K_N(x, y)$,

$$K_N[f](x) = \int_J K_N(x, y)f(y) dy.$$

It is well known that in the bulk scaling limit, independent of the precise functional form of $w_2(x)$,

$$\lim_{N \to \infty} a_N K_N(a_N x, a_N y) = \frac{\sin \pi(x - y)}{\pi(x - y)} =: K^\text{bulk}(x, y)$$

(2.20)

for a suitable scale factor $a_N$. Thus we have

$$E^\text{bulk}_2(J; \xi) = \det(1 - \xi K^\text{bulk}_J)$$

(2.21)

where $K^\text{bulk}_J$ is the integral operator on the interval $J$ with kernel (2.20) (the so called sine kernel). This is a practical formula for the computation of $E^\text{bulk}_2$ if we can compute the eigenvalues $\{\mu_j\}_{j=0,1,...}$ of $K^\text{bulk}_J$, since we have

$$E^\text{bulk}_2(J; \xi) = \prod_{j=0}^{\infty} (1 - \xi \mu_j).$$

(2.22)

In fact for $J = (-s, s)$ the eigenvalues can be computed [18] by relating $K^\text{bulk}_{-s,s}$ to a differential operator which has the prolate spheroidal functions as its eigenfunctions, and using previously computed properties of this eigensystem.

Wigner’s interest was not in complex Hermitian random matrices, but rather real symmetric random matrices. Orthogonally invariant ensembles of the latter have an eigenvalue probability density function of the form

$$\frac{1}{C} \prod_{l=1}^{N} w_1(x_l) \prod_{1 \leq j < k \leq N} |x_k - x_j|,$$

(2.23)

to be denoted $\text{OE}_N(w_1)$. For such matrix ensembles, the $k$-point distribution function can be written as a quaternion determinant (or equivalently Pfaffian) with an underlying $2 \times 2$ matrix kernel (see e.g. [3] Ch. 5). From this it is possible to show that

$$\left( E^\text{bulk}_1(J; \xi) \right)^2 = \det(1 - \xi K^\text{bulk}_{1,J})$$

(2.24)

where $K^\text{bulk}_{1,J}$ is the integral operator on $J$ with matrix kernel

$$K^\text{bulk}_{1,J}(x, y) = \begin{bmatrix}
\sin \pi(x - y) & \frac{1}{\pi} \int_0^{\pi(x-y)} \frac{\sin t}{t} dt - \frac{1}{2} \text{sgn}(x - y) \\
\frac{\partial}{\partial x} \frac{\sin \pi(x - y)}{\pi(x - y)} & \frac{\sin \pi(x - y)}{\pi(x - y)}
\end{bmatrix}.$$  

(2.25)
However, unlike the result (2.21), this form has not been put to any practical use.

Instead, as discovered by Mehta [26], a tractable formula results from the scaling limit of an inter-relationship between the generating function of an orthogonal symmetry gap probability and a unitary symmetry gap probability. The inter-relationship states

\[ E_{2N,1}((−t, t); \xi; e^{-x^2/2}) \big|_{\xi=1} = E_{N,2}((0, t^2); \xi; y^{-1/2}e^{-y}X_{y>0}) \big|_{\xi=1}, \tag{2.26} \]

and in the scaling limit leads to the result

\[ E_{1,\text{bulk}}^{\text{bulk}}((-s, s); \xi) \big|_{\xi=1} = \det(1 - K_{(-s, s)}^{\text{bulk}+}) \tag{2.27} \]

where \( K_{(-s, s)}^{\text{bulk}+} \) is the integral operator on \((-s, s)\) with kernel

\[ \frac{1}{2} \left( \frac{\sin \pi(x - y)}{\pi(x - y)} + \frac{\sin \pi(x + y)}{\pi(x + y)} \right), \tag{2.28} \]

which we recognize as the even part of the sine kernel (2.20). (For future reference we define \( K_{(-s, s)}^{\text{bulk}−} \) analogously, except that the kernel consists of the difference of the two terms in (2.28), or equivalently the odd part of the sine kernel (2.20).) Because the eigenvalues \( \mu_{2j} \) of the integral operator on \((-s, s)\) with kernel (2.20) correspond to even eigenfunctions, while the eigenvalues \( \mu_{2j+1} \) correspond to odd eigenfunctions, we have that

\[ E_{1,\text{bulk}}^{\text{bulk}}((-s, s); \xi) \big|_{\xi=1} = \prod_{l=0}^{\infty} (1 - \mu_{2l}). \tag{2.29} \]

Gaudin [18] used this formula, together with (2.2), to tabulate \( p_{1,\text{bulk}}(0; s) \) and so test the accuracy of the Wigner surmise (2.12). In fact this confirmed the remarkable precision of the latter, with the discrepancy between it and the exact value no worse than a few percent.

The case of Hermitian matrices with real quaternion elements and having a symplectic symmetry remains. The eigenvalue p.d.f. of the independent eigenvalues (the spectrum is doubly degenerate) is then

\[ \frac{1}{C} \prod_{l=1}^{N} w_4(x_l) \prod_{1 \leq j < k \leq N} (x_k - x_j)^4, \tag{2.30} \]

which we denote by \( \text{SE}_N(w_4) \). The computation of the corresponding bulk gap probability relies on further inter-relationships between matrix ensembles with different underlying symmetries. These apply to the eigenvalue probability density function for Dyson’s circular ensembles,

\[ \frac{1}{C} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta, \]

where \( \beta = 1, 2 \) or 4 according to the underlying symmetry being orthogonal, unitary or symplectic respectively. The corresponding matrix ensembles are referred to as the \( \text{COE}_N, \text{CUE}_N \) and \( \text{CSE}_N \) in order. In the \( N \to \infty \) scaling limit these ensembles correspond with the bulk of the ensembles \( \text{OE}_N(w_1), \text{UE}_N(w_2) \) and \( \text{SE}_N(w_4) \) respectively.
The first of the required inter-relationships was formulated by Dyson [5] and proved by Gunson [19]. It states that

\[ \text{alt}(\text{COE}_N \cup \text{COE}_N) = \text{CUE}_N \] (2.31)

where the operation \( \text{COE}_N \cup \text{COE}_N \) refers to the superposition of two independent realizations of the \( \text{COE}_N \) and alt refers to the operation of observing only every second member of the sequence. The second of the required inter-relationships is due to Dyson and Mehta [7]. It states that

\[ \text{alt} \text{COE}_2 = \text{CSE}_N. \] (2.32)

(For generalizations of (2.31) and (2.32) to the ensembles \( \text{OE}_N(w_1), \text{UE}_N(w_2) \) and \( \text{SE}_N(w_4) \) with particular \( w_1, w_2 \) and \( w_4 \) see [12].) Using (2.31) and (2.32) together one can deduce that in the scaled limit

\[ E_4^{\text{bulk}}(0; (-s/2, s/2)) = \frac{1}{2} \left( \frac{E_1^{\text{bulk}}(0; (-s, s)) + E_2^{\text{bulk}}(0; (-s, s))}{E_1^{\text{bulk}}(0; (-s, s))} \right), \] (2.33)

which upon using (2.22) and (2.29) reads

\[ E_4^{\text{bulk}}(0; (-s/2, s/2)) = \frac{1}{2} \left( \prod_{l=0}^{\infty} (1 - \lambda_{2l}) + \prod_{l=0}^{\infty} (1 - \lambda_{2l+1}) \right). \] (2.34)

Another consequence of (2.32) is that

\[ p_4(0; s) = 2p_1(1; 2s). \] (2.35)

It is this relationship, used together with the approximation for \( p_4(0; s) \) in (2.13), which is used to approximate \( p(1; s) \) as a smooth curve in Figure 2.

In summary, as a consequence of the pioneering work of Mehta, Gaudin and Dyson, computable formula in terms of the eigenvalues of the integral operator on \((-s, s)\) with the sine kernel (2.20) were obtained for

\[ E_2^{\text{bulk}}((-s, s); \xi), \quad E_1^{\text{bulk}}(0; (-s, s)), \quad E_4^{\text{bulk}}(0; (-s/2, s/2)). \]

### 3 Painlevé transcendent evaluations

#### 3.1 The results of Jimbo et al.

An explicit connection between the multiple interval gap probability

\[ E_2^{\text{bulk}} \left( \bigcup_{j=1}^{p} (a_{2j-1}, a_{2j}); \xi \right) \]

and integrable systems theory — specifically the theory of isomonodromic deformations of linear differential equations — was made by Jimbo, Miwa, Mōri and Sato in 1980. Here the endpoints \( a_1, \ldots, a_{2p} \) of the gap free intervals become dynamical time like variables, inducing flows which turn out to be integrable.
As part of this study the quantity
\[
E_2^{\text{bulk}}((-s, s); \xi) = \det(1 - \xi K^{\text{bulk}}_{(-s, s)}) = \prod_{j=0}^{\infty} (1 - \xi \mu_j)
\] (3.1)
was expressed in terms of the solution of a nonlinear equation. In fact knowledge of (3.1) is sufficient to calculate the products appearing in (2.29) and (2.34). Thus with
\[
D_+(s; \xi) := \prod_{j=0}^{\infty} (1 - \xi \mu_{2j}), \quad D_-(s; \xi) := \prod_{j=0}^{\infty} (1 - \xi \mu_{2j+1})
\]
Gaudin (see [28]) has shown
\[
\log D_\pm(s; \xi) = \frac{1}{2} \log E_2^{\text{bulk}}((-s, s); \xi) \pm \frac{1}{2} \int_0^s \sqrt{-\frac{d^2}{dx^2} \log E_2^{\text{bulk}}((-x, x); \xi)} \, dx.
\] (3.2)
The result of [23] is that
\[
E_2^{\text{bulk}}((-s, s); \xi) = \exp \int_0^{\pi s} \frac{\sigma(u; \xi)}{u} \, du
\] (3.3)
where \(\sigma(u; \xi)\) satisfies the nonlinear differential equation
\[
(u\sigma''(u))^2 + 4(u\sigma' - \sigma)(u\sigma' - \sigma + (\sigma')^2) = 0
\] (3.4)
subject to the boundary condition
\[
\sigma(u; \xi) \sim -\frac{\xi u}{\pi}, \quad u \to 0^+.
\]
In fact the equation (3.4) is an example of the so called \(\sigma\) form of a Painlevé V equation. In view of this it is appropriate to give some background into the Painlevé theory, following [21]. First we remark that the Painlevé differential equations are second order nonlinear equations isolated as part of the study of Painlevé and his students into the moveable singularities of the solution of such equations. Earlier Fuchs and Poincaré had studied first order differential equations of the form
\[
P(y', y, t) = 0
\] (3.5)
where \(P\) is a polynomial in \(y', y\) with coefficients meromorphic in \(t\). In contrast to linear differential equations, nonlinear equations have the property that the position of the singularities of the solution will depend in general on the initial condition. The singularities are then said to be moveable. For example
\[
\frac{dy}{dt} = y^2
\] (3.6)
has the general solution \(y = 1/(c - t)\), where \(c\) determines the initial condition, and so exhibits a moveable first order pole. The nonlinear equation
\[
y\frac{dy}{dt} = \frac{1}{2}
\]
has the general solution $y = (t - c)^{1/2}$, which exhibits a moveable branch point (essential singularity). Fuchs and Poincaré sought to classify all equations of the form (3.5) which are free of moveable essential singularities. They were able to show that up to an analytic change of variables, or fractional linear transformation, the only such equations with this property were the differential equation of the Weierstrass $P$-function,

$$
\left( \frac{dy}{dt} \right)^2 = 4y^3 - g_2y - g_3, \quad (3.7)
$$

or the Riccati equation

$$
\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t) \quad (3.8)
$$

where $a, b, c$ are analytic in $t$ (note that (3.6) is of the latter form).

Painlevé then took up the same problem as that addressed by Fuchs and Poincaré, but now with respect to second order differential equations of the form

$$
y'' = R(y', y, t)
$$

where $R$ is a rational function in all arguments. It was found that the only equations of this form and with no moveable essential singularities were either reducible to (3.7) or (3.8), reducible to a linear differential equation, or were one of six new nonlinear differential equations, now known as the Painlevé equations. As an explicit example of the latter, we note the Painlevé V equation reads

$$
y'' = \left( \frac{1}{2y} + \frac{1}{1-y} \right)(y')^2 - \frac{1}{x}y' + \frac{(y-1)^2}{x^2} \left( \alpha y + \frac{\beta}{y} \right) + \frac{\gamma y + \delta(y+1)}{y-1} \quad (3.9)
$$

where $\alpha, \beta, \gamma, \delta$ are parameters.

An immediate question is to how (3.9) relates to (3.4). For this one must develop a Hamiltonian theory of the Painlevé equations. The idea is to present a Hamiltonian $H = H(p, q; t; \vec{v})$, where the components of $\vec{v}$ are parameters, such that after eliminating $p$ in the Hamilton equations

$$
q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad (3.10)
$$

$q'$ and $p'$ denoting derivatives with respect to $t$, the equation in $q$ is the appropriate Painlevé equation (in (3.10) the role of $p$ and $q$ is interchanged relative to their usual meaning of position and momentum in physics; here we are following the convention of Okamoto. Malmquist [25] was the first to present such Hamiltonians, although his motivation was not to further the development of the Painlevé theory itself. This was left to Okamoto in a later era, and it is aspects of his theory we will briefly present here.

The Hamiltonian for the PV equation as presented by Okamoto [29] is

$$
tH_V = q(q-1)^2p^2 - \{(v_1 - v_2)(q-1)^2 - 2(v_1 + v_2)q(q-1) + tq\}p + (v_3 - v_2)(v_4 - v_2)(q-1), \quad (3.11)
$$

where the parameters are constrained by $v_1 + v_2 + v_3 + v_4 = 0$ and are further related to those in (3.9) according to

$$
\alpha = \frac{1}{2}(v_3 - v_4)^2, \quad \beta = \frac{1}{2}(v_1 - v_2)^2, \quad \gamma = v_1 + 2v_2 - 1, \quad \delta = -\frac{1}{2}.
$$

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It turns out that, as a consequence of the Hamilton equations (3.10), $tH_V$ itself satisfies a nonlinear differential equation. It is this differential equation which relates to (3.4). Okamoto made use of this equation for the symmetry it exhibits in the parameters $v_1, \ldots, v_4$.

The equation in question, which is fairly straightforward to derive, is presented for the so-called auxiliary Hamiltonian

$$h_V(t) = tH_V + (v_3 - v_2)(v_4 - v_2) - v_2t - 2v_2^2.$$  

Okamoto showed

$$(th'_V)^2 - (h_V - th'_V + 2(h'_V)^2) + 4 \prod_{k=1}^{4}(h'_V + v_k) = 0.$$  

Setting

$$\sigma_V(t) = h_V(t) + v_2 t + 2v_2^2, \quad \nu_{j-1} = v_j - v_2 \quad (j = 1, \ldots, 4)$$

in this one obtains the so-called Jimbo-Miwa-Okamoto $\sigma$-form of the Painlevé V equation

$$(t\sigma'_V)^2 - \left(\sigma_V - t\sigma'_V + 2(\sigma'_V)^2 + (\nu_0 + \nu_1 + \nu_2 + \nu_3)\sigma'_V\right)^2 + 4(\nu_0 + \sigma'_V)(\nu_1 + \sigma'_V)(\nu_2 + \sigma'_V)(\nu_3 + \sigma'_V) = 0 \quad (3.12)$$

(Jimbo and Miwa [22] arrived at (3.12) in their study of isomonodromic deformations of linear differential equations). We note that (3.4) is an example of this equation with

$$\nu_0 = \nu_1 = \nu_2 = \nu_3 = 0, \quad t \mapsto -2iu.$$

### 3.2 Unveiling more structure

The result of Jimbo et al. relates to the Fredholm determinant of the integral operator with the sine kernel. What is special about the sine kernel that relates it to integrable systems theory? This question was answered by Its, Izergin, Korepin and Slanov [20] who exhibited integrability features of all kernels of the Christoffel-Darboux type (recall (2.15) in relation to the latter terminology)

$$\xi K(x, y) = \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x - y}, \quad (3.13)$$

the sine kernel begin the special case

$$\phi(x) = \sqrt{\xi} \sin x, \quad \psi(y) = \sqrt{\xi} \cos y. \quad (3.14)$$

One of their key results related to the form of the kernel $R(x, y)$ for the so-called resolvent operator

$$R_J := \xi K_J(1 - \xi K_J)^{-1}.$$  

With

$$Q(x) := (1 - \xi K_J)^{-1}\phi, \quad P(x) := (1 - \xi K_J)^{-1}\psi \quad (3.15)$$
they showed

\[ R(x, y) = \frac{Q(x)P(y) - P(x)Q(y)}{x - y}. \quad (3.16) \]

The significance of the resolvent kernel is evident from the general formula

\[ \frac{\partial}{\partial a_j} \log \det(1 - \xi K_{(a_1, a_2)}) = (-1)^{j-1}R(a_j, a_j) \quad (j = 1, 2). \quad (3.17) \]

To derive this formula, one notes that

\[ \log \det(1 - \xi K_{(a_1, a_2)}) = \text{Tr} \log(1 - \xi K_{(a_1, a_2)}) = \int_{-\infty}^{\infty} \log(1 - \xi K(x, x) \chi_{(a_1, a_2)}) \, dx. \]

Thus

\[ \frac{\partial}{\partial a_j} \log \det(1 - \xi K_{(a_1, a_2)}) = (-1)^{j-1}(1 - \xi K(a_j, a_j))^{-1}\xi K(a_j, a_j) \]

as required.

According to (3.16)

\[ R(a_j, a_j) = -Q(x)P'(x) + P(x)Q'(x) \Big|_{x = a_j}, \quad (3.18) \]

so we see from (3.17) that the Fredholm determinant is determined by the quantities (3.16) and their derivatives evaluated at the endpoints of the interval. Indeed a close examination of the workings of [23], undertaken by Mehta [24], Dyson [6] and Tracy and Widom [32], revealed that the former study indeed proceeds via the equations (3.17) and (3.18), and in fact \( \sigma(t) \) in (3.3) is related to the resolvent kernel evaluated at an endpoint by \( \sigma(t) = -tR(t/2, t/2) \). Moreover it was realized that like (3.16), there are other equations contained in the working of [23] which apply to all kernels of the form (3.13). However it was also clear that other equations used in [23] were specific to the form of \( \phi \) and \( \psi \) in (3.14).

Tracy and Widom were able to identify these latter properties, which are that \( \phi \) and \( \psi \) are related by the coupled first order differential equations

\[
\begin{align*}
m(x)\phi'(x) &= A(x)\phi(x) + B(x)\psi(x) \\
m(x)\psi'(x) &= -C(x)\phi(x) - A(x)\psi(x)
\end{align*}
\]

where \( m, A, B, C \) are polynomials. This structure allows the so called universal equations (independent of the specific form of (3.13)) such as (3.18) to be supplemented by a number of case specific equations. For some choices of \( \phi \) and \( \psi \) in addition to that corresponding to sine kernel, the resulting system of equations closes. Examples relevant to spacing distributions at the soft and hard edge of matrix ensembles with unitary symmetry are

\[
\phi(x) = \sqrt{\xi} \text{Ai}(x), \quad \psi(x) = \phi'(x), \quad \phi(x) = \sqrt{\xi} J_0(\sqrt{x}), \quad \psi(x) = x\phi'(x).
\]

In both these cases it was possible to obtain an evaluation of the generating function for the corresponding gap probability in a form analogous to [33, 34].
We will make note of the hard edge result because it, by virtue of Mehta’s inter-relationship \([2.26]\), relates to the gap probability in the bulk in the case of an underlying orthogonal symmetry. First, we define the hard edge gap probability in the case of an underlying orthogonal symmetry as the scaled limit of the ensemble \([2.14]\) with \(w_2(x) = x^a e^{-x} \chi_{x>0}\). Explicitly
\[
E_2^\text{hard}((0, s); \xi) = \lim_{N \to \infty} E_2 \left(0, \frac{s}{4N}; \xi; x^a e^{-x} \chi_{x>0} \right). \tag{3.20}
\]
It was shown in \([9]\) that
\[
E_2^\text{hard}((0, s); \xi) = \det(1 - \xi K^\text{hard}_{(0,s)}) \tag{3.21}
\]
where \(K^\text{hard}_{(0,s)}\) is the integral operator on \((0, s)\) with kernel
\[
K^\text{hard}(x, y) = \frac{J_a(x^{1/2})y^{1/2}J'_0(x^{1/2}) - x^{1/2}J'_a(x^{1/2})J_0(y^{1/2})}{2(x-y)}.
\]
As part of the study \([34]\) the Fredholm determinant \([3.21]\) was given the evaluation
\[
E_2^\text{hard}((0, s); \xi) = \exp \int_0^s u(t; a; \xi) \frac{dt}{t} \tag{3.22}
\]
where \(u\) satisfies the differential equation
\[
(tu'')^2 - a^2(u')^2 - u'(4u' + 1)(u - tu') = 0 \tag{3.23}
\]
subject to the boundary condition
\[
u(t; a; \xi) \sim -\xi t K^\text{hard}(t, t) \quad \text{as} \quad t \to 0^+.
\]
The equation \([3.23]\) is a special case of the \(\sigma\) form of the Painlevé III’ system \([30]\).

It follows from \([2.26], 3.20\) and \(3.22\) that \([10]\)
\[
E_1^\text{bulk}(0; (-s, s)) = E_2^\text{hard}(0; (0, \pi^2 s^2)) \bigg|_{a=-1/2} = \exp \int_0^{(\pi s)^2} u(t; a; \xi) \frac{dt}{t} \bigg|_{a=-1/2} \tag{3.24}
\]
This is an alternative Painlevé transcendent evaluation to that implied by \([2.28], 3.2\) and \(3.3\). Similarly, by noting that
\[
2\sqrt{xy} K^\text{hard}(x^2, y^2) \bigg|_{a=1/2} = \frac{1}{2} \left( \frac{\sin(x-y)}{x-y} - \frac{\sin(x+y)}{x+y} \right)
\]
we see from \([2.31], 3.21\) and \(3.22\) that \([10]\)
\[
E_4^\text{bulk}(0; (-s/2, s/2)) \bigg|_{a=1/2} = \frac{1}{2} \left( \exp \int_0^{(\pi s)^2} u(t; a; \xi) \frac{dt}{t} \bigg|_{a=-1/2} + \exp \int_0^{(\pi s)^2} u(t; a; \xi) \frac{dt}{t} \bigg|_{a=-1/2} \right). \tag{3.25}
\]

In summary, the Fredholm determinants in the expressions for the bulk gap probabilities can each be written in terms of Painlevé transcendent. From a practical viewpoint these expressions are particularly well suited for generating power series expansions, and also allow for a numerical tabulation of each of \(E_2^\text{bulk}(0; (-s, s)), E_1^\text{bulk}(0; (-s, s))\) and \(E_4^\text{bulk}(0; (-s, s))\), as well as \(E_2^\text{bulk}(n; (-s, s))\) for \(n \geq 1\). For the latter quantity, according to \([2.5]\) we must differentiate \(E_2^\text{bulk}((-s, s); \xi)\) with respect to \(\xi\) then set \(\xi = 1\). Doing this in \([3.4]\) gives a coupled system of differential equations for \(\partial^j \sigma(u; \xi)/\partial \xi^j \big|_{\xi=1} (j = 0, \ldots, n)\) which is only numerically stable for small values of \(n\).
3.3 Distribution of bulk right or left nearest neighbour spacings

The spacing distribution refers to the distribution of the distance between consecutive points as we move along the line left to right. Another simple to measure statistic of this type is the distribution of the smallest of the left neighbour spacing and right neighbour spacing for each point. Let us denote this by \( p_{\beta}^{n.n.}(s) \) (the superscript n.n. stands for nearest neighbour, while the subscript \( \beta \) indicates the symmetry class). Let \( E_{\beta}^{n.n.}(0; (-s, s)) \) denote the probability that about a fixed eigenvalue at the origin, there is no eigenvalue at distance \( s \) either side. Analogous to (2.2) it is easy to see that

\[
p_{\beta}^{n.n.}(s) = -\frac{d}{ds} E_{\beta}^{n.n.}(0; (-s, s)). \tag{3.26}
\]

In the case \( \beta = 2 \) (unitary symmetry) the generating function \( E_{\beta}^{n.n.}((-s, s); \xi) \) can be expressed as a Fredholm determinant

\[
E_{\beta}^{n.n.}((-s, s); \xi) = \det(1 - \xi K_{(-s, s)}) \tag{3.27}
\]

where \( K_{(-s, s)} \) is the integral operator on \((-s, s)\) with kernel

\[
K^{n.n.}(x, y) := (\pi x)^{1/2}(\pi y)^{1/2} \frac{J_{a+1/2}(\pi x)J_{a-1/2}(\pi y) - J_{a+1/2}(\pi y)J_{a-1/2}(\pi x)}{2(x-y)} \tag{3.28}
\]
evaluated at \( a = 1 \). Following the strategy which leads to (3.22), the Fredholm determinant (3.27) for general \( a \in \mathbb{Z}_{\geq 0} \) can be characterized as the solution of a nonlinear equation. Explicitly

\[
E_{\beta}^{n.n.}((-s, s); \xi) = \exp \left( \int_0^{2\pi s} \sigma_a(t; \xi) \frac{dt}{t} \right) \tag{3.29}
\]

where \( \sigma_a \) satisfies the nonlinear equation

\[
(s \sigma_a'')^2 + 4(-a^2 + s \sigma_a') - \sigma_a \left( (\sigma_a')^2 - \{a - (a^2 - s \sigma_a' + \sigma_a)'^1/2\}^2 \right) = 0 \tag{3.30}
\]
subject to the boundary condition

\[
\sigma_a(s; \xi) \sim -\frac{\xi 2(s/4)^{2a+1}}{(1/2 + a)\Gamma(3/2 + a)}. \tag{3.31}
\]

In the case \( a = 0 \), (3.28) reduces to the sine kernel and the differential equation (3.30) reduces to (3.4). For general \( a \in \mathbb{Z}_{\geq 0} \) the differential equation (3.30) is satisfied by an auxiliary Hamiltonian for PIII (as distinct from PIII') (39).

Substituting (3.29) in (3.26) gives

\[
p_{2}^{n.n.}(s) = -\frac{\sigma_a(2\pi s; \xi)}{2\pi s} \exp \int_0^{2\pi s} \sigma_a(t; \xi) \frac{dt}{t} \bigg|_{a=\xi=1}. \tag{3.31}
\]

An application of this result can be made to the study of the zeros of the Riemann zeta function on the critical line (Riemann zeros). We recall that the Montgomery-Odlyzko law states that the statistics of the large Riemann zeros coincide with the statistics of bulk eigenvalues of an
Figure 3: Comparison of $nn(t) := p_2^{n.n}(s)$ for the matrix ensembles with unitary symmetry in the bulk (continuous curve) and for $10^6$ consecutive Riemann zeros, starting near zero number 1 (open circles), $10^6$ (asterisks) and $10^{20}$ (filled circles).

ensemble of random matrices with unitary symmetry, where both the zeros and eigenvalues are assumed to be unfolded so as to have mean spacing unity. As a test of this law, in [11] the empirical determination of $p_2^{n.n}(s)$ for large sequences of Riemann zeros, starting at different positions along the critical line, was compared with (3.31). The results, which are consistent with the Montgomery-Odlyzko law, are reproduced in Figure 3. A significant feature is that the empirical determination of $p_2^{n.n}(s)$ for the Riemann zeros is so accurate that it is not possible to compare against an approximate form of $p_2^{n.n}(s)$ for the random matrices. Thus the exact, readily computable, Painlevé evaluation (3.31) is of a practical importance.
4 Gap probabilities from the Okamoto $\tau$-function theory

4.1 Other strategies

The method of Tracy and Widom may be described as being based on function theoretic properties of Fredholm determinants. Alternative methods which also lead to the characterization of gap probabilities in terms of the solution of nonlinear equations have been given by a number of authors. One alternative method is due to Adler and van Moerbeke [35], who base their strategy on the fact that for suitable underlying weight $w_2$, gap probabilities in the case of a unitary symmetry satisfy the KP hierarchy of partial differential equations known from soliton theory. The first member of this hierarchy is then used in conjunction with a set of equations referred to as Virasora constraints, satisfied by the gap probabilities as a function of the endpoints of the gap free regions, to arrive at third order equations for some single interval gap probabilities. These third order equations are reduced to the $\sigma$-form of the Painlevé theory, making use of results of Cosgrove [3, 2]. Borodin and Deift [1] have given a method based on the Riemann-Hilbert formulation of the resolvent kernel [3, 10, 24]. This makes direct contact with the Schlesinger equations from the theory of the isomonodromic deformation of linear differential equations, and is thus closely related to the work of Jimbo et al. [23]. The other approach to be mentioned is due to Forrester and Witte [14]. It is based on Okamoto’s development of the Hamiltonian approach to Painlevé systems, and proceeds by inductively constructing sequences of multi-dimensional integral solutions of the $\sigma$ form of the Painlevé equations, and identifying these solutions with gap probabilities for certain random matrix ensembles with unitary symmetry.

For detailed accounts of all these methods, see [8, Ch. 6&7]. In the remainder of these lectures we will restrict ourselves to results from the work of Forrester and Witte which relate directly to gap probabilities in the bulk.

4.2 Direct calculation of spacing distributions

We have taken as our objective the exact evaluation of the bulk spacing distributions for the three symmetry classes of random matrices. So far exact evaluations have been presented not for the spacing distribution itself, but rather the corresponding gap probability, which is related to the spacing distribution by (2.2). It was realized by Forrester and Witte [15] that in all three cases one of the derivatives could be performed analytically by using theory relating to the $\sigma$ form of the Painlevé transcendents.

As an explicit example, consider the result [3, 2, 24]. It was shown in [15] that

$$\frac{d}{ds} \exp \int_0^{(\pi s)^2} u(t; a; \xi) \frac{dt}{t} \bigg|_{a=-1/2} = -\exp \left( -\int_0^{(\pi s)^2} \tilde{u}(t) \frac{dt}{t} \right)$$

where $\tilde{u}$ satisfies the nonlinear equation

$$s^2(\tilde{u}'')^2 = (4(\tilde{u}')^2 - \tilde{u}'(s\tilde{u}' - \tilde{u})) + \frac{9}{4}(\tilde{u}')^2 - \frac{3}{2} \tilde{u}' + \frac{1}{4}.$$
subject to the boundary condition

\[ \tilde{u}(s) \sim \frac{s}{3} + \frac{s^2}{45} + \frac{8s^{5/2}}{135\pi} \]

Recalling now (2.2) we see that

\[ p_{1bulk}^{ bulk}(0; s) = \frac{2\tilde{u}((\pi s/2)^{2})}{s} \exp \left( - \int_{0}^{(\pi s/2)^{2}} \frac{\tilde{u}(t)}{t} dt \right) \]

(cf. (2.12)).

The identity (4.1) can be understood from the approach to gap probabilities of Forrester and Witte. The key advance from earlier studies is that the generating function (2.4), with \( p \) given by (2.14), can be generalized to the quantity

\[ E_{bulk}(s; \mu; \xi) := \lim_{N \to \infty} a_{N}^{s} \int_{\infty}^{\infty} dx_{1} \cdots \int_{\infty}^{\infty} dx_{N} \prod_{l=1}^{N} (1-\xi \chi_{(-s/2,s/2)})|s/2-a_{N}x_{i}|^{\mu} p(a_{N}x_{1}, \ldots, a_{N}x_{N}) \]

and still be characterized as the solution of a nonlinear equation. This is also true at the hard and soft edges, and in the neighbourhood of a spectrum singularity (before the generalization the latter is controlled by the kernel (3.28)).

It is the generalization in the case of the hard edge which leads to (4.1). The quantity of interest is defined by

\[ E_{2}^{ hard}((0, s); \mu; \xi) = \lim_{N \to \infty} \frac{I_{N}(a)}{I_{N}(a+\mu)} E_{2}\left((0, \frac{s}{4N}); (x-\frac{s}{4N})^{\mu} x a e^{-x} \chi_{x>0}\right) \]

where

\[ I_{N}(a) := \int_{0}^{\infty} dx_{1} \cdots \int_{0}^{\infty} dx_{N} \prod_{l=1}^{N} e^{-x_{l}a} \prod_{1 \leq j < k \leq N} (x_{k} - x_{j})^{2} \]

(the factor \( I_{N}(a)/I_{N}(a+\mu) \), which is readily evaluated in terms of gamma functions, is chosen so that when \( s = 0, \) (4.3) is equal to unity). By using theory from the Okamoto \( \tau \) function approach to the Painlevé systems PV and PIII' it is shown in [16] that

\[ E_{2}^{hard}((0, s); \mu; \xi) = \exp \int_{0}^{s} u^{h}(t; a, \mu; \xi) \frac{dt}{t}, \]

where \( u^{h} \) satisfies the differential equation

\[ (tu'')^{2} - (\mu + a)^{2}(u')^{2} - u'(4u' + 1)(u - tu') - \frac{\mu(a + \mu)}{2} u' - \frac{\mu^{2}}{4} = 0. \]

Thus we have

\[ \frac{d}{ds} \exp \left( \int_{0}^{s} u^{h}(t; a, \mu; \xi)|_{\mu=0} \frac{dt}{t} \right) = \frac{s^{a}}{2^{a+2}\Gamma(a+1)\Gamma(a+2)} \exp \left( \int_{0}^{s} u^{h}(t; a, \mu; \xi)|_{\mu=2} \frac{dt}{t} \right), \]

which in the case \( a = -1/2 \) reduces to (4.1).
We also read off from (4.6) that
\[
\frac{d}{ds} \exp \int_0^{(\pi s)^2} u(t; a; \xi) \frac{dt}{t} \bigg|_{\xi=1/2} = -\frac{2}{3} (\pi s)^2 \exp \left( -\int_0^{(\pi s)^2} \tilde{v}(t) \frac{dt}{t} \right)
\] (4.7)
where \( \tilde{v}(t) = -u^b(t; a, \mu; \xi)|_{a=1/2, \mu=2, \xi=1} \) and thus satisfies (4.5) appropriately specialized. The boundary condition consistent with (4.7) is
\[
\tilde{v}(t) \sim t \rightarrow 0_+ \frac{t}{5} \left( 1 + O(t) \right) + \frac{8t^{7/2}}{3^5 \cdot 5^3 \cdot 7\pi} \left( 1 + O(t) \right).
\] (4.8)
Hence, according to (2.16) and (2.2),
\[
p_4^{\text{bulk}}(0; s) = 2p_1^{\text{bulk}}(0; 2s) + \frac{2\pi^2 s}{3} \left( \tilde{v}(s) - 1 \right) \exp \left( -\int_0^{(\pi s)^2} \tilde{v}(t) \frac{dt}{t} \right).
\] (4.9)

The Okamoto \( \tau \)-function theory of PVI and PV allows (4.3) to be computed for general \( \mu \), and also its generalization in which there is a further factor \( | -s/2 - a_Nx_l |^\alpha \) in the product over \( l \) in the integrand \( \text{[17]} \). These results allow not only the first derivative with respect to \( s \) of (4.3) to be computed by an identity analogous to (4.1), but also the second derivative. In particular, it is found that
\[
p_2^{\text{bulk}}(0; s) = \frac{\pi^2}{3} s^2 \exp \int_0^{2\pi s} v(t; \xi) \frac{dt}{t}
\] (4.10)
where \( v \) satisfies the nonlinear equation (which can be identified in terms of the \( \sigma \) form of the PIII’ equation)
\[
(sv'')^2 + (v - sv')\{v - sv' + 4 - 4(v')^2\} - 16(v')^2 = 0
\]
subject to the boundary condition
\[
v(s; \xi) \sim s \rightarrow 0 - \frac{1}{15} s^2.
\]

The exact evaluations (4.2), (4.9) and (4.10) are perhaps the most compact Painlevé evaluations possible for the bulk spacing distributions. A striking feature of (4.2) and (4.10) is that they are of the functional form \( a(s) \exp(-\int_0^s b(t) \, dt) \) and thus extend the Wigner surmise (2.12) and its \( \beta = 2 \) analogue in (2.13) to exact results.

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