COMPLETE GENTLE ALGEBRAS ARE $g$-TAME

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Abstract. We study a fan consisting of all $g$-vector cones associated with two-term presilting complexes of a complete gentle algebra. We show that any complete gentle algebra is $g$-tame, by definition, the closure of a geometric realization of its fan is the entire ambient vector space. Our main ingredients are their geometric descriptions and their asymptotic behavior under Dehn twists. As a consequence, we also get the $g$-tameness of a class of special biserial algebras containing Brauer graph algebras.

1. Introduction

Gentle algebras, introduced in 1980’s, form an important class of biserial algebras and their representation theory has been studied by many authors (e.g. [AH81, AS87, BR87]). Moreover, the derived categories of gentle algebras are related to various subjects, such as discrete derived categories [Voß01], numerical derived invariants [AAG08, APS19], and Fukaya categories of surfaces [HKK17, LP20].

An aim of this paper is to study two-term silting theory for gentle algebras. In this paper, we don’t assume that gentle algebras are finite dimensional. For our purpose, we consider the complete gentle algebras $\hat{kQ}/\bar{I}$ associated with gentle bound quivers $(Q,I)$, where $\hat{kQ}$ is the complete path algebra of $Q$ over an algebraically closed field $k$, and $\bar{I}$ is the closure of $I$ with respect to topology given by the arrow ideal (see Definition 7.9). They are module-finite over $R$ (i.e., finitely generated as an $R$-module), where $R = k[[t]]$ is the formal power series ring of one value over $k$. In particular, finite dimensional gentle algebras are complete gentle algebras.

We discuss two-term silting theory over module-finite $R$-algebra $A$, see Section 7 (cf. [AIR14, DJ19]). Let $\text{proj} A$ be the category of finitely generated projective right $A$-modules. We denote by $2\text{-presilt} A$ (resp., $2\text{-silt} A$) the set of isomorphism classes of basic two-term presilting (resp., silting) complexes for $A$. Each $T \in 2\text{-presilt} A$ has a numerical invariant $g_A(T) \in \mathbb{Z}^n$, called the $g$-vector of $T$, where $n$ is the number of non-isomorphic indecomposable direct summands of $A$. Then, one can define a cone in $\mathbb{R}^n$, called the $g$-vector cone of $T$, by

$$C_A(T) := \left\{ \sum_X a_X g_A(X) \mid a_X \in \mathbb{R}_{\geq 0} \right\},$$

where $X$ runs over all indecomposable direct summands of $T$. We denote by $\mathcal{F}(A)$ a collection of $g$-vector cones of all basic two-term presilting complexes for $A$, by $|\mathcal{F}(A)|$ its geometric realization. It is shown in [DJ19] that $\mathcal{F}(A)$ is a simplicial fan (i.e., every cone is a simplex) and its maximal faces correspond to basic two-term silting complexes for $A$. Namely,

$$|\mathcal{F}(A)| = \bigcup_{C \in \mathcal{F}(A)} C = \bigcup_{T \in 2\text{-silt} A} C_A(T).$$

Such a fan plays an important role in the study of stability scattering diagrams and their wall-chamber structures (see e.g. [Asa19, Bri17, BST19, Yur18]). The following result is well-known. Now, we say that $A$ is $\tau$-tilting finite if $2\text{-silt} A$ is finite.

Theorem 1.1. [DJ19] If $A$ is $\tau$-tilting finite, then $\mathcal{F}(A)$ is complete, that is,

$$|\mathcal{F}(A)| = \mathbb{R}^n.$$

This result naturally leads the following definition in a general setting.

Definition 1.2. We say that the algebra $A$ is $g$-tame if it satisfies

$$|\mathcal{F}(A)| = \mathbb{R}^n,$$
where \(\overline{\cdot}\) is the closure with respect to the natural topology on \(\mathbb{R}^n\).

Note that a similar notion, called \(\tau\)-tilting tame, was given in [BST19]. In [Yur], it was showed that Jacobian algebras associated with triangulated surfaces are \(g\)-tame. Our main result is the \(g\)-tameness of complete gentle algebras.

**Theorem 1.3.** Any complete gentle algebra is \(g\)-tame.

By a reduction theorem [BZJ18, Kim], we also show the \(g\)-tameness of a special class of special biserial algebras containing Brauer graph algebras (Corollary 7.28). In particular, this means that \(g\)-vector cones are dense in the stability scattering diagram for a finite dimensional gentle algebra or a Brauer graph algebra.

In order to prove Theorem 1.3 our main ingredient is a geometric description of complete gentle algebras (see e.g. [APS19, OPS18, PPP19]). A similar construction has been developed in several areas, such as [AAC18, KS02, OPS18]. For each dissection \(D\) of a \(\bullet\)-marked surface \((S, M)\), one can define a gentle bound quiver \((Q(D), I(D))\) and a complete gentle algebra \(A(D) := kQ(D)/I(D)\). Conversely, any gentle bound quiver arises in this way (see Sections 2.1 and 7.2 for the details). Note that the cardinality \(n\) of \(D\) is completely determined by \((S, M)\) (Remark 2.3).

For a given dissection \(D\) of \((S, M)\), we observe a certain class of non-self-intersecting curves of \(S\), called \(D\)-laminates, and finite multi-set of pairwise non-intersecting \(D\)-laminates, called \(D\)-laminations. Notice that we take account of closed curves here. To each \(D\)-lamine \(\gamma\), we associate an integer vector \(g(\gamma) \in \mathbb{Z}^n\), called \(g\)-vector, whose entries are intersection numbers of \(\gamma\) and \(d \in D\). The next result is an analog of [FT18 Theorems 12.3, 13.6] and a generalization of [PPP19 Proposition 6.14] to an arbitrary dissection.

**Theorem 1.4.** (Theorem 4.1) The map \(\mathcal{X} \mapsto \sum_{\gamma \in \mathcal{X}} g(\gamma)\) gives a bijection between the set of \(D\)-laminates and \(\mathbb{Z}^n\).

We especially consider certain \(D\)-laminates. A \(D\)-lamine \(\mathcal{X}\) is said to be reduced if it consists of pairwise distinct non-closed \(D\)-laminates, and complete if it is reduced and maximal as a set. We denote by \(\mathcal{F}(D)\) a collection of \(C(\mathcal{X})\) of all reduced \(D\)-laminates \(\mathcal{X}\), where \(C(\mathcal{X})\) is a cone in \(\mathbb{R}^n\) spanned by \(g(\gamma)\) for all \(\gamma \in \mathcal{X}\). In particular, \(\mathcal{F}(D)\) is a simplicial fan whose maximal faces correspond to complete \(D\)-laminates (Proposition 2.8).

In Section 7 by giving a geometric model of two-term silting theory for complete gentle algebras, we show that the fans \(\mathcal{F}(D)\) and \(\mathcal{F}(A(D))\) coincide.

**Theorem 1.5** (Theorem 7.25). Let \(D\) be a dissection of a \(\bullet\)-marked surface \((S, M)\) and \(A(D)\) the complete gentle algebra associated with \(D\). There are bijections

\[ T(\cdot) : \{\text{reduced } D\text{-laminates}\} \to \text{2-presilt } A(D) \quad \text{and} \quad \{\text{complete } D\text{-laminates}\} \to \text{2-silt } A(D) \]

such that \(C(\mathcal{X}) = C(T(\mathcal{X}))\). In particular, we have \(\mathcal{F}(A(D)) = \mathcal{F}(D)\) and hence \(|\mathcal{F}(A(D))| = |\mathcal{F}(D)|\).

Therefore, a proof of Theorem 1.3 is completed by the following claim.

**Theorem 1.6.** For a dissection \(D\) of a \(\bullet\)-marked surface \((S, M)\), we have

\[ |\mathcal{F}(D)| = \mathbb{R}^n. \]

A main ingredient of our proof of Theorem 1.6 is the asymptotic behavior of \(g\)-vectors under Dehn twists. This proof is inspired from the proof of [Yur, Theorem 1.5]. In the forthcoming paper [Aok], this method plays a key role for analyzing the polytope associated with the fan \(\mathcal{F}(A(D))\).

This paper is organized as follows. Through to Section 8 we study the geometric and combinatorial aspects of our results. In Section 2 we recall the notions and results of [APS19, PPP19] in terms of our notations. Before proving our results, we give some examples in Section 8. By using the examples, we prove Theorem 1.3 in Section 9. In Sections 5 and 6 we study \(g\)-vectors of \(D\)-laminates and their asymptotic behavior under Dehn twists, and prove Theorem 1.6.

In Section 7 we study the algebraic aspects of our results. First, we recall two-term silting theory over module-finite algebras, in particular, they include complete gentle algebras. Second, we give a geometric description of silting theory over complete gentle algebras, and prove Theorem 1.5. Finally, we prove Theorem 1.3 and also give \(g\)-tameness of a special class of special biserial algebras containing Brauer graph algebras (see Section 7.4). These examples are given in Section 8.
2. Preliminary

In this section, we recall the notions and results of [APS19, PPP19] (see also [OPS18]). Our notations are slightly different from theirs for the convenience of our purpose.

2.1. ◦•-marked surfaces.

Definition 2.1. A ◦•-marked surface is the pair $(S, M)$ consisting of the following data:

(a) $S$ is a connected compact oriented Riemann surface with (possibly empty) boundary $\partial S$.

(b) $M = M_\circ \sqcup M_\bullet$ is a non-empty finite set of marked points on $S$ such that

– both $M_\circ$ and $M_\bullet$ are not empty;

– each component of $\partial S$ has at least one marked point;

– the points of $M_\circ$ and $M_\bullet$ alternate on each boundary component.

Any marked point in the interior of $S$ is called a puncture.

Let $(S, M)$ be a ◦•-marked surface.

Definition 2.2. (1) A ◦-arc (resp., •-arc) $\gamma$ of $(S, M)$ is a curve in $S$ with endpoints in $M_\circ$ (resp., $M_\bullet$), considered up to isotopy, such that the following conditions are satisfied:

• $\gamma$ does not intersect itself except at its endpoints;

• $\gamma$ is disjoint from $M$ and $\partial S$ except at its endpoints;

• $\gamma$ does not cut out a monogon without punctures.

(2) A ◦-dissection (resp., •-dissection) is a maximal set of pairwise non-intersecting ◦-arcs (resp., •-arcs) on $(S, M)$ which does not cut out a subsurface without marked points in $M_\bullet$ (resp., $M_\circ$).

Remark 2.3. Let $g$ be the genus of $S$, $b$ be the number of boundary components and $p_\circ$ (resp., $p_\bullet$) be the number of punctures in $M_\circ$ (resp., $M_\bullet$). By [APS19, Proposition 1.11], a ◦-dissection (resp., •-dissection) of $(S, M)$ consists of $|M_\circ| + p_\bullet + b + 2g - 2 = |M_\bullet| + p_\circ + b + 2g - 2$ ◦-arcs (resp., •-arcs).

By symmetry, the claims in this paper hold if we permute the symbols ◦ and •. Thus we state only one side of each claim. A dissection divides $(S, M)$ into polygons with exactly one marked point.

Proposition 2.4. [APS19, Proposition 1.12] For a •-dissection $D$ of $(S, M)$, each connected component of $S \setminus D$ is homeomorphic to one of the following:

• an open disk with precisely one marked point in $M_\circ \cap \partial S$;

• an open disk with precisely one marked point in $M_\circ$, but not in $\partial S$.

For a •-dissection $D$ of $(S, M)$, the closure of a connected component of $S \setminus D$ is called a polygon of $D$. Proposition 2.4 implies that any polygon of $D$ has exactly one marked point in $M_\circ$. We denote by $\triangle_v$ the polygon with marked point $v \in M_\circ$ (see Figure 1).

![Figure 1. Polygon $\triangle_v$ for a marked point $v \in M_\circ$](image)

Definition-Proposition 2.5. [PPP19, Proposition 3.6] For a •-dissection $D$ of $(S, M)$, there is a unique ◦-dissection $D^*$ whose each ◦-arc intersects exactly one •-arc of $D$. We have $D^{**} = D$. We call $D^*$ the dual dissection of $D$. For $d \in D$, we write the corresponding ◦-arc by $d^* \in D^*$.

2.2. $g$-vectors of $D$-laminates and $D$-laminations. We fix a •-dissection $D$ of $(S, M)$.

Definition 2.6. (1) A ◦-lamine of $(S, M)$ is a curve $\gamma$ in $S$, considered up to isotopy relative to $M$, that is either

• a closed curve, or

[...]
A $D$-laminate is called a closed $D$-laminate if it is a closed curve. Remark that non-closed $D$-laminates coincide with $D$-slaloms in [PPP19]. Now, we treat a certain collection of $D$-laminates, that is central in this paper.

**Definition 2.7.** We say that two $D$-laminates are compatible if they don’t intersect. A finite multi-set of pairwise compatible $D$-laminates is called a $D$-lamination. A $D$-lamination is said to be

- **reduced** if it consists of pairwise distinct non-closed $D$-laminates, and
- **complete** if it is reduced and is the maximum as a set.

Let $\gamma$ be a $D$-laminate. Using the notations in the condition (*), let $p$ be an intersection point of $\gamma$ and $d$ such that $\gamma$ leaves $\triangle_v$ to enter $\triangle_{v'}$ via $p$. Then $p$ is said to be positive (resp., negative) if $v$ is to its right (resp., left), or equivalently, $v'$ is to its left (resp., right). See Figure 3. For $d \in D$, we define an integer

\[
(2.1) \quad g(\gamma)_d := \# \{ \text{positive intersection points of } \gamma \text{ and } d \} - \# \{ \text{negative intersection points of } \gamma \text{ and } d \}.
\]

The $g$-vector $g(\gamma)$ of $\gamma$ is given by $(g(\gamma)_d)_{d \in D} \in \mathbb{Z}^{|D|}$, where $|D|$ is the number of $\bullet$-arcs of $D$. Remark that if $\gamma$ and $d$ intersect twice, then their intersection points are either positive or negative simultaneously. Thus, the absolute value of $g(\gamma)_d$ just counts the number of intersection points of $\gamma$ and $d$. For a $D$-lamination $X$, we denote by $C(X)$ a cone in $\mathbb{R}^{|D|}$ spanned by $g(\gamma)$ for all $\gamma \in X$ and call it the $g$-vector cone of $X$. We denote by $F(D)$ a collection of all $g$-vector cones of reduced $D$-laminations.

**Theorem 2.8.** [PPP19, Theorems 5.12 and 6.12]

1. $F(D)$ is a simplicial fan whose maximal faces correspond to complete $D$-laminations.
2. A reduced $D$-lamination is complete if and only if it has precisely $|D|$ elements.

**Theorem 2.9.** [PPP19, Theorem 6.14] If $F(D)$ is finite, then all $D$-laminates are non-closed. In this case, $F(D)$ is complete.
3. Examples

In this section, we examine our notions defined in the previous section.

(1) Let \((S, M)\) be a disk with \(|M| = 8\) such that all marked points lie on \(\partial S\). For a \(\bullet\)-dissection of \((S, M)\)

\[
D = \begin{array}{c}
1 \\
2 \\
3
\end{array}
\]

all \(D\)-laminates and the corresponding \(g\)-vectors are given as follows:

\[
\begin{array}{cccccccc}
(1, 0, 0) & (-1, 0, 0) & (0, 1, 0) & (0, -1, 0) & (0, 0, 1) & (0, 0, -1) & (1, -1, 0) & (0, 1, -1) & (1, 0, -1)
\end{array}
\]

There are 14 complete \(D\)-laminates. The corresponding fan \(\mathcal{F}(D)\) of \(g\)-vector cones for \(D\) is given as in the left diagram of Figure 4.

(2) Let \((S, M)\) be a disk with \(|M| = 7\) such that one marked point in \(M_\circ\) is a puncture and the others lie on \(\partial S\). For a \(\bullet\)-dissection of \((S, M)\)

\[
D = \begin{array}{c}
1 \\
2 \\
3
\end{array}
\]

all \(D\)-laminates and the corresponding \(g\)-vectors are given as follows:

\[
\begin{array}{cccccccc}
(1, 0, 0) & (-1, 0, 0) & (0, 1, 0) & (0, -1, 0) & (0, 0, 1) & (0, 0, -1) & (1, -1, 0) & (0, 1, -1) & (1, 0, -1)
\end{array}
\]

(1, -1, 0) & (-1, 1, 0) & (0, 1, -1) & (0, -1, 1) & (1, 0, -1) & (-1, 0, 1)

There are 20 complete \(D\)-laminates. The fan \(\mathcal{F}(D)\) is given as in the center diagram of Figure 4.

(3) Consider a torus \(S = T^2\) with \(\partial S = \emptyset\) and \(|M| = 2\) (i.e., both marked points are punctures). Let \(D\) be a \(\bullet\)-dissection of \((S, M)\) given by

\[
\begin{array}{c}
1 \\
2 \\
0
\end{array}
\]
where we identify the opposite sides of the square in the same direction. All $D$-laminates and the corresponding $g$-vectors are given as follows:

\[
\ell \quad \gamma^{-2} \quad \gamma^{-1} \quad \gamma^{0} \quad \gamma^{1} \quad \gamma^{2} \ldots
\]

\[
\ell' \quad \gamma'^{-2} \quad \gamma'^{-1} \quad \gamma'^{0} \quad \gamma'^{1} \quad \gamma'^{2} \ldots
\]

where $\ell, \ell'$ are closed $D$-laminates and $\gamma_m, \gamma'_m$ are non-closed $D$-laminates for all $m \in \mathbb{Z}$. We find that the set $\{\{\gamma_m, \gamma_{m+1}\}, \{\gamma'_m, \gamma'_{m+1}\} | m \in \mathbb{Z}\}$ provides all complete $D$-laminations. The fan $\mathcal{F}(D)$ is given as in the right diagram of Figure 4.

For the closed $D$-lamine $\ell$, its $g$-vector $g(\ell) = (1, -1) \in \mathbb{Z}^2$ does not contained in $|\mathcal{F}(D)|$. It will be approximated by using the Dehn twist $T_\ell$ along $\ell$ (we refer to Section 5 for the details). In fact, we have $T_\ell(\gamma_i) = \gamma_{i+1}$ for any $i \in \mathbb{Z}_{>0}$ and hence

\[
g(\ell) = (1, -1) \in \bigcup_{m \geq 0} C(T^m_\ell(\{\gamma_1\})).
\]

\[\text{Figure 4. A fan } \mathcal{F}(D) \text{ of } g\text{-vector cones for Examples (1)-(3)}\]

4. $g$-VECTORS AND LATTICE POINTS

The aim of this section is to prove the following result.

**Theorem 4.1.** Let $D$ be a $\bullet$-dissection of $(S,M)$. Then there is a bijection

\[
\{D\text{-laminations}\} \to \mathbb{Z}^{|D|}
\]

given by the map $\mathcal{X} \mapsto g(\mathcal{X}) := \sum_{\gamma \in \mathcal{X}} g(\gamma)$, where $g(\emptyset) := 0$.

To prove Theorem 4.1 we first consider the following two cases:
Then we can glue the curves of regard
(5.1. The proof of Theorem 1.6 appears in the next section. Let $D_1$ be
a •-dissection of $(S_1, M_1)$ as in the left diagram of Figure 5.
(b) Let $(S_2, M_2)$ be a disk with $|M_2| = 2n + 1$ such that one marked point in $(M_2)_o$ is a puncture
and the others lie on $\partial S_2$. Let $D_2$ be a •-dissection of $(S_2, M_2)$ as in the right diagram of
Figure 5.
In both cases, we have $|D_i| = n$.\n
\textbf{Figure 5.} Special cases (a) and (b)

\textbf{Proposition 4.2.} For $i \in \{1, 2\}$, $\mathcal{F}(D_i)$ is a complete simplicial fan.

\textbf{Proof.} In the same way as (1) and (2) in Section 3, one can check that the number of $D_1$-laminates is
equal to $\frac{1}{2}n(n + 3)$ and the number of $D_2$-laminates is equal to $n(n + 1)$, in particular, they are finite.
Then the assertion follows from Theorem 2.3.\n
\textbf{Corollary 4.3.} Theorem 4.2 holds for $D = D_1$ or $D = D_2$.

\textbf{Proof.} For $i \in \{1, 2\}$, $\mathcal{F}(D_i)$ is a complete simplicial fan by Proposition 4.2. This implies that the map
$\mathcal{X} \mapsto g(\mathcal{X})$ provides a one-to-one correspondence between the set of $D_i$-laminations consisting only of
non-closed $D$-laminates and $\mathbb{Z}^{|D_i|}$. More precisely, for any $g \in \mathbb{Z}^{|D_i|}$, there is exactly one reduced $D_i$-
lamination $\mathcal{X}$ such that $g(\mathcal{X})$ is contained in the interior of $C(\mathcal{X}^o)$. Since $C(\mathcal{X}^o)$ is simplicial, $g$ is uniquely
written by $g = \sum_{\gamma \in \mathcal{X}^o} a_\gamma g(\gamma)$ for $a_\gamma \in \mathbb{Z}_{>0}$. Then a $D_i$-lamination $\mathcal{X}$ consisting of $a_\gamma$ elements $\gamma \in \mathcal{X}^o$
is a unique one such that $g(\mathcal{X}) = g$.\n
\textbf{Proof of Theorem 4.7.} Let $D$ be a •-dissection of $(S, M)$ and $g = (g_d)_{d \in D}$ any integer vector in $\mathbb{Z}^{|D|}$.
In the following, we construct a $D$-lamination $\mathcal{X}$ such that $g = g(\mathcal{X})$.

Recall that $(S, M)$ is divided into polygons $\triangle_v$ for all $v \in M_o$. For $v \in M_o$, we can naturally embed
$\triangle_v$ into the above •-dissection $D_i$ of $(S_i, M_i)$ for $i = 1$ or 2. More precisely, $(S_i, M_i)$ is obtained from
$\triangle_v$ by gluing a digon with one •-marked point on each •-arc of $D \cap \triangle_v$, where $D \cap \triangle_v$ form $D_i$ in
$(S_i, M_i)$. By Corollary 4.3, there is a unique $D_i$-lamination $\mathcal{X}_v$ such that $g(\mathcal{X}_v) = (g_d)_{d \in D \cap \triangle_v}$. We
regard $\mathcal{X}_v \cap \triangle_v$ as a set of pairwise non-intersecting curves in $(S, M)$ with $|d|$ endpoints on $d \in D \cap \triangle_v$.
Then we can glue the curves of $\mathcal{X}_v \cap \triangle_v$ for all $v \in M_o$ at their endpoints on $D$. As a result, we obtain a set $\mathcal{X}$ of pairwise non-intersecting •-laminates of $(S, M)$. From our construction, every •-laminate
of $\mathcal{X}$ is a $D$-laminate, and hence $\mathcal{X}$ forms a $D$-lamination such that $g(\mathcal{X}) = g$ as desired.

On the other hand, the uniqueness of $\mathcal{X}$ follows from one of $\mathcal{X}_v$ for any $v \in M_o$.\n
\section{Positive position and Dehn twists}

In this section, we fix a •-dissection $D$ of $(S, M)$ and make preparations for proving Theorem 1.6. The proof of Theorem 1.6 appears in the next section.

\subsection{Dehn twist along a closed $D$-laminate.}

We denote by $T_\ell$ the Dehn twist along a closed curve $\ell$ with the orientation defined as follows:

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{dehn_twist.png}
\end{array}
\]

In general, $T_\ell(\gamma)$ is not a $D$-laminate for a given $D$-laminate $\gamma$. We will give a condition that Dehn twists work well.
Let $\gamma$ and $\delta$ be $D$-laminates. For each intersection point $p$ of $\gamma$ and $\delta$, we can assume that $p$ lies in $S \setminus D$, thus $p \in \triangle_v$ for some $v \in M_\delta$. We set orientations of the segments of $\gamma$ and $\delta$ in $\triangle_v$ such that $v$ lies on the right to them. We say that $\gamma$ is in positive position for $\delta$ if $\gamma$ and $\delta$ don’t intersect or $\gamma$ intersects $\delta$ from right to left at each intersection point (see Figure 6).

**Figure 6. A $D$-lamine $\gamma$ is in positive position for a $D$-lamine $\delta$.**

**Lemma 5.1.** Let $\ell$ be a closed $D$-lamine and $\gamma$ a non-closed $D$-lamine which is in positive position for $\ell$. Then

1. $T_\ell(\gamma)$ is a non-closed $D$-lamine;
2. $g(T_\ell(\gamma)) = g(\gamma) + \#(\gamma \cap \ell)g(\ell)$;
3. if a $D$-lamine $\gamma'$ does not intersect $\ell$, then
   \[\#(\gamma' \cap \gamma) = \#(\gamma' \cap T_\ell(\gamma)).\]

**Proof.** The assertions immediately follow from the assumption. □

In the situation of Lemma 5.1, we can repeat the Dehn twist $T_\ell$. Moreover, Lemma 5.1 is generalized for $D$-laminates. For closed curves $\ell_1, \ldots, \ell_k$ and $m_1, \ldots, m_k \in \mathbb{Z}_{\geq 0}$, we write

\[T^{(m_1, \ldots, m_k)} := T^{m_1}_{\ell_1} \cdots T^{m_k}_{\ell_k}.\]

Note that if $\ell_1, \ldots, \ell_k$ are pairwise non-intersecting, then all $T_{\ell_i}$ are commutative.

**Proposition 5.2.** Let $Y = \{\ell_1, \ldots, \ell_k\}$ be a $D$-lamine consisting only of closed $D$-laminates and $X = \{\gamma_1, \ldots, \gamma_h\}$ a $D$-lamine consisting only of non-closed $D$-laminates which are in positive position for any $\ell_i$. Then for any $m_1, \ldots, m_k \in \mathbb{Z}_{\geq 0}$ and $T := T^{(m_1, \ldots, m_k)}_{(\ell_1, \ldots, \ell_k)}$,

1. $\{T(\gamma_1), \ldots, T(\gamma_h)\}$ is a $D$-lamine consisting only of non-closed $D$-laminates;
2. we have the equality
   \[\sum_{i=1}^h g(T(\gamma_i)) = \sum_{i=1}^h g(\gamma_i) + \sum_{i=1}^h \sum_{j=1}^k m_j \#(\gamma_i \cap \ell_j)g(\ell_j).\]

**Proof.** (1) For any $\gamma \in X$ and $\ell \in Y$, by Lemma 5.1, $T_\ell(\gamma)$ is a non-closed $D$-lamine. Lemma 5.1(3) says that $T_\ell(\gamma) \cap \ell'$ is naturally identified with $\gamma \cap \ell'$ for any $\ell' \in Y$. In particular, $T_\ell(\gamma)$ is also in positive position for $\ell'$, thus $T_{\ell'} T_\ell(\gamma)$ is a non-closed $D$-lamine. Repeating this process, $T(\gamma)$ is a non-closed $D$-lamine. Since $T(\gamma)$ and $T(\gamma')$ don’t intersect for any $\gamma, \gamma' \in X$, the assertion holds.

(2) The equality is calculated from Lemma 5.1(2) since Lemma 5.1(3) says that the number of all intersection points of $X$ and $Y$ is not changed by the Dehn twists. □

### 5.2. Non-closed $D$-lamine $\ell^d$ for a closed $D$-lamine $\ell$

Let $X$ be a $D$-lamine consisting only of non-closed $D$-laminates. We assume that there is a closed $D$-lamine $\ell$ such that $X \cup \{\ell\}$ is a $D$-lamine. By the definition of $D$-laminates, there exists $d \in D$ such that $g(\ell)_d > 0$. From now, we construct a non-closed $D$-lamine $\ell^d$ such that

(a) $\ell^d$ is a non-closed $D$-lamine which is compatible with any $D$-lamine of $X$;
(b) $\ell^d$ intersects with $\ell$ so that $\ell^d$ is in positive position for $\ell$.

It plays an important role to prove Theorem 4.6 in the next section.

First, for $d \in D$, we define a $D$-lamine $d^*_+$ (resp., $d^*_-$) as follows (see Figure 7):

- $d^*_+$ (resp., $d^*_-$) is a laminate running along $d^*$ in a small neighborhood of it;
- If $d^*$ has an endpoint $v \in M_\delta$ on a component $C$ of $\partial S$, then the corresponding endpoint of $d^*_+$ (resp., $d^*_-$) is located near $v$ on $C$ in the counterclockwise (resp., clockwise) direction;
• If \(d^*\) has an endpoint at a puncture \(p \in M_\circ\), then the corresponding end of \(d^*_+\) (resp., \(d^-\)) is a spiral around \(p\) counterclockwise (resp., clockwise).

\[
\begin{array}{c}
d^* \\
\cup
\end{array}
\begin{array}{c}
d^*_+ \\
\cup
\end{array}
\begin{array}{c}
d^- \\
\cup
\end{array}
\]

**Figure 7.** Two \(D\)-laminates \(d^*_+\) and \(d^-\).

That is, \(g(d^*_+)_e = \delta_{ed}\) (resp., \(g(d^-)_e = -\delta_{ed}\)) for \(e \in D\), where \(\delta\) is the Kronecker delta.

On this notation, \(g(\ell)_d > 0\) implies \(\ell \cap d^*_+ \neq \emptyset\) and \(d^*_+\) is in positive position for \(\ell\). Without loss of generality, we can assume that \(p \in \ell \cap d^*_+\) lie on \(d\) as in the left diagram of Figure 8.

Second, for each endpoint \(v\) of \(d^*\), we define a curve \(\ell_v\) of \(S\) as follows: Consider the segment \(\alpha := d^*_+ \cap \emptyset\).

• If \(\alpha\) intersects none of \(X\), then let \(\ell_v := \alpha\) (see the center diagram of Figure 8);
• Otherwise, let \(p_v\) be the nearest intersection point of \(\alpha\) and \(X\) from \(p\), where \(p_v \in \alpha \cap \emptyset\) for \(\gamma \in \emptyset\). We denote by \(q\) an endpoint of a connected segment in \(\gamma \cap \emptyset\) containing \(p_v\) such that the intersection point \(q \in \gamma \cap \emptyset\) is negative. Then \(\ell_v\) is a curve obtained by gluing the following two curves at \(p_v\) (see the right diagram of Figure 8):

(i) a segment of \(\alpha\) between \(p\) and \(p_v\);
(ii) a segment of \(\gamma\) obtained by cutting \(\gamma\) at \(p_v\), that contains \(q\).

\[
\begin{array}{c}
\ell \\
\cup
\end{array}
\begin{array}{c}
d^* \\
\cup
\end{array}
\begin{array}{c}
\ell_v = \alpha \\
\cup
\end{array}
\begin{array}{c}
\ell_v \\
\cup
\end{array}
\begin{array}{c}
\ell_v \\
\cup
\end{array}
\begin{array}{c}
q \\
\cup
\end{array}
\begin{array}{c}
\alpha \gamma \\
\cup
\end{array}
\begin{array}{c}
p \end{array}
\]

**Figure 8.** A closed \(D\)-lamine \(d\) and \(d^*_+\) with \(g(d^*_+)_d > 0\) (left), constructions of a curve \(\ell_v\) (center, right)

Finally, we define \(\ell^d\) as a curve obtained by gluing \(\ell_v\) and \(\ell_{v'}\) at \(p\) for endpoints \(v\) and \(v'\) of \(d^*\). A segment of \(\ell^d\) between \(p_v\) and \(p_{v'}\) is called its center segment, where \(p_v\) is a point on \(\ell_v\) sufficiently close to \(v\) if \(\ell_v = \alpha\). It follows from the construction that \(\ell^d\) satisfies (a) and (b) above. Moreover, (b) is generalized as follows.

**Lemma 5.3.** In the above situations, if a \(D\)-laminate \(\gamma\) is compatible with \(X\), then \(\ell^d\) is in positive position for \(\gamma\).

**Proof.** If \(\gamma\) intersects \(\ell^d\), then all the intersection points lie on the center segment of \(\ell^d\), thus the assertion holds.

\[\Box\]

6. **Proof of Theorem 1.6**

In this section, we prove Theorem 1.6. The idea of its proof comes from [Yur]. Fix a \(\bullet\)-dissection \(D\) of \((S,M)\). Let \(g \in \mathbb{Z}^{|D|}\). By Theorem 4.1, there is a \(D\)-laminate \(X\) such that \(g = g(X) = \sum_{\gamma \in X} g(\gamma)\).

It is sufficient to find \(D\)-laminations \(\{X_m\}_{m \in \mathbb{Z}_{\geq 0}}\) consisting only of non-closed \(D\)-laminates such that

\[
X_m^{nc} \subseteq X_m \quad \text{and} \quad g \in \bigcup_{m \in \mathbb{Z}_{\geq 0}} C(X_m).
\]
where \( \mathcal{X} = \mathcal{X}^{nc} \cup \mathcal{X}^{cl} \) is a decomposition such that \( \mathcal{X}^{nc} \) (resp., \( \mathcal{X}^{cl} \)) consists of all non-closed \( D \)-laminates (resp., closed \( D \)-laminates) in \( \mathcal{X} \).

If \( \mathcal{X}^{cl} = \emptyset \), then a family of \( \mathcal{X}_m := \mathcal{X} \) for all \( m \in \mathbb{Z}_{\geq 0} \) is the desired one. Assume that \( \mathcal{X}^{cl} \) is non-empty. For \( \ell_1 \in \mathcal{X}^{cl} \) and \( d_1 \in D \) with \( g(\ell_1,d_1) > 0 \), we obtain a non-closed \( D \)-lamine \( \ell_1^{d_1} \) by the construction of Section 5.2 for \( \mathcal{X}^{nc} \). By Lemma 5.3, \( \ell_1^{d_1} \) is in positive position for every \( \ell \in \mathcal{X}^{cl} \).

If the set \( \{ \ell \in \mathcal{X}^{cl} \mid \ell \cap \ell_1^{d_1} = \emptyset \} \) is non-empty, then we take \( \ell_2 \in \{ \ell \in \mathcal{X}^{cl} \mid \ell \cap \ell_1^{d_1} = \emptyset \} \) and \( d_2 \in D \) with \( g(\ell_2,d_2) > 0 \). By the construction of Section 5.2 for \( \mathcal{X}^{nc} \cup \{ \ell_1^{d_1} \} \), we obtain a non-closed \( D \)-lamine \( \ell_2^{d_2} \). Notice that \( \ell_2^{d_2} \) consists of some of segments of \( D \)-laminates in \( \mathcal{X}^{nc} \), the center segment of \( \ell_2^{d_2} \) and one of \( \ell_1^{d_1} \), where the third type may not appear. In the same way as Lemma 5.3, we can show that \( \ell_2^{d_2} \) is in positive position for every \( \ell \in \mathcal{X}^{cl} \). Repeating this process, we finally get an integer \( h \in \{1, \ldots, k = |\mathcal{X}^{cl}|\} \) and non-closed \( D \)-laminates \( \ell_1^{d_1}, \ldots, \ell_h^{d_h} \) such that

\[ \{ \ell \in \mathcal{X}^{cl} \mid \ell \cap \ell_1^{d_1} = \cdots = \ell \cap \ell_h^{d_h} = \emptyset \} = \emptyset. \]

Moreover, our construction provides the following properties:

- \( \mathcal{X}^{nc} \cup \{ \ell_1^{d_1}, \ldots, \ell_h^{d_h} \} \) is a \( \mathcal{D} \)-laminate consisting only of non-closed \( \mathcal{D} \)-laminates;
- For \( i \in \{1, \ldots, h\} \), \( \ell_i^{d_i} \) is in positive position for every \( \ell \in \mathcal{X}^{cl} \).

We set \( \mathcal{X}^{cl} = \{ \ell_1, \ldots, \ell_h \} \cup \{ \ell_{h+1}, \ldots, \ell_k \} \) and fix the notations \( n_j^{(i)} := \#(\ell_i^{d_i} \cap \ell_j) \), \( n_j := \sum_{i=1}^h n_j^{(i)} \), and \( N := n_1 \cdots n_k \). Set

\[ T := \mathcal{T}(\frac{\ell_1}{\ell_2}, \ldots, \frac{\ell_k}{\ell_1}). \]

By Proposition 5.2, \( \mathcal{T}^m(\ell_i^{d_i}) \) are non-closed \( D \)-laminates for all \( m \in \mathbb{Z}_{\geq 0} \) and \( i \in \{1, \ldots, h\} \), and we get the equalities

\[ \sum_{i=1}^h g(\mathcal{T}^m(\ell_i^{d_i})) = \sum_{i=1}^h g(\ell_i^{d_i}) + m \sum_{i=1}^h \sum_{j=1}^k \sum_{n_j^{(i)}} g(\ell_j) = \sum_{i=1}^h g(\ell_i^{d_i}) + mN \sum_{j=1}^k g(\ell_j) = \sum_{i=1}^h g(\ell_i^{d_i}) + mN g(\mathcal{X}^{cl}). \]

It gives

\[ \lim_{m \to \infty} \frac{\sum_{i=1}^h g(\mathcal{T}^m(\ell_i^{d_i}))}{m} = N g(\mathcal{X}^{cl}). \]

Then the \( \mathcal{D} \)-laminate

\[ \mathcal{X}_m := \mathcal{X}^{nc} \cup \{ \mathcal{T}^m(\ell_i^{d_i}) \}_{i=1}^h \]

is the desired one because

\[ g = g(\mathcal{X}^{nc}) + g(\mathcal{X}^{cl}) \in C(\mathcal{X}^{nc}) + \bigcup_{m \in \mathbb{Z}_{\geq 0}} C(\{ \mathcal{T}^m(\ell_i^{d_i}) \}_{i=1}^h) \subseteq \bigcup_{m \in \mathbb{Z}_{\geq 0}} C(\mathcal{X}_m). \]

7. Representation theory

In this section, we study the algebraic aspects of our results. We can see their examples in the next section.

7.1. Two-term silting complexes for module-finite algebras. Let \( R = k[[t]] \) be the formal power series ring of one value over \( k \). Let \( A \) be a basic \( R \)-algebra which is module-finite (i.e., \( A \) is finitely generated as an \( R \)-module). We denote by \( \text{proj} A \) the category of finitely generated projective right \( A \)-modules, by \( K^b(\text{proj} A) \) the homotopy category of bounded complexes of \( \text{proj} A \). In particular, \( K^b(\text{proj} A) \) is an \( R \)-linear category and \( \text{Hom}_{K^b(\text{proj} A)}(X, Y) \) is a finitely generated \( R \)-module for any \( X, Y \in K^b(\text{proj} A) \).

We begin with the following observation.

**Proposition 7.1.** The category \( K^b(\text{proj} A) \) is a Krull-Schmidt triangulated category.
Proof. For any $X \in \mathcal{K}^b(\text{proj } A)$, $E = \text{End}_{\mathcal{K}^b(\text{proj } A)}(X)$ is a module-finite algebra over the complete local noetherian ring $R$. Therefore, $E$ is semiperfect by [CR62, p.132] and hence $\mathcal{K}^b(\text{proj } A)$ is Krull-Schmidt. □

Now, we study two-term silting theory for a module-finite $R$-algebra $A$. We refer to [AIR14, Aih13, DJ19] for two-term silting theory of finite dimensional algebras, and to [Kim] for ones of module-finite algebras.

**Definition 7.2.** Let $P = (P^i, f^i)$ be a complex in $\mathcal{K}^b(\text{proj } A)$.

1. We say that $P$ is two-term if $P^i = 0$ for any integer $i \neq 0, -1$.
2. We say that $P$ is presilting if $\text{Hom}_{\mathcal{K}^b(\text{proj } A)}(P, P[m]) = 0$ for any positive integer $m$.
3. We say that $P$ is silting if it is presilting and thick $P = \mathcal{K}^b(\text{proj } A)$, where thick $P$ is the smallest triangulated subcategory of $\mathcal{K}^b(\text{proj } A)$ which contains $P$ and is closed under taking direct summands.

We denote by $2$-ips $A$ (resp., $2$-presilt $A$, $2$-silt $A$) the set of isomorphism classes of indecomposable two-term presilting (resp., basic two-term presilting, basic two-term silting) complexes for $A$. Here, we say that a complex $P$ is basic if all indecomposable direct summands of $P$ are pairwise non-isomorphic.

We denote by $|P|$ the number of non-isomorphic indecomposable direct summands of $P$ and by $\text{add } P$ the category of all direct summands of finite direct sums of copies of $P$.

Let $A = \bigoplus_{i=1}^{[A]} P_i$ be a decomposition of $A$, where $P_i$ is an indecomposable projective $A$-module.

**Definition 7.3.** Let $P = [P^{-1} \rightarrow P^0]$ be a two-term complex in $\mathcal{K}^b(\text{proj } A)$.

1. The $g$-vector of $P$ is defined by
   \[g_A(P) := (m_1 - n_1, \ldots, m_{[A]} - n_{[A]}) \in \mathbb{Z}^{[A]}\]
   where $m_i$ (resp., $n_i$) is the multiplicity of $P_i$ as indecomposable direct summands of $P^0$ (resp., $P^{-1}$).
2. The $g$-vector cone $C_A(P)$ is defined to be a cone in $\mathbb{R}^{[A]}$ spanned by $g$-vectors of all indecomposable direct summands of $P$.

We denote by $\mathcal{F}(A)$ a collection of $g$-vector cones of all basic two-term presilting complexes for $A$.

The following are basic properties of two-term presilting complexes.

**Proposition 7.4.** [Kim] Let $P = [P^{-1} \rightarrow P^0] \in 2$-presilt $A$. Then the following hold:

1. $P$ is a direct summand of some basic two-term silting complex for $A$.
2. $P$ is silting if and only if $|P| = |A|$.
3. $\text{add } P^0 \cap \text{add } P^{-1} = 0$.

**Proposition 7.5.** [Kim] The collection $\mathcal{F}(A)$ is a simplicial fan whose maximal faces correspond to basic two-term silting complexes for $A$.

Let $(-) := - \otimes_R R/tR$. In particular, we have $\bar{A} = A/(t)$ and a decomposition $\bar{A} = \bigoplus_{i=1}^{[A]} \bar{P}_i$, where $\bar{P}_i$ are indecomposable projective $\bar{A}$-modules. We identify $\mathbb{Z}^{[A]}$ and $\mathbb{Z}^{[\bar{A}]}$ by this correspondence. For a two-term complex $P = [P^{-1} \rightarrow P^0]$ in $\mathcal{K}^b(\text{proj } A)$, let $\bar{P} := ([P^{-1} \rightarrow P^0])$ be in $\mathcal{K}^b(\text{proj } \bar{A})$.

**Proposition 7.6.** [Kim] Let $P$ be a two-term complex in $\mathcal{K}^b(\text{proj } A)$. Then, we have the following.

1. $P$ is presilting if and only if $\bar{P}$ is presilting.
2. $|P| = |\bar{P}|$.

**Proposition 7.7.** [Kim] The map $(-)$ induces bijections

\[2\text{-presilt } A \rightarrow 2\text{-presilt } \bar{A} \quad \text{and} \quad 2\text{-silt } A \rightarrow 2\text{-silt } \bar{A}\]

such that $C_A(P) = C_{\bar{A}}(\bar{P})$. In particular, we have $\mathcal{F}(A) = \mathcal{F}(\bar{A})$. 

7.2. Gentle algebras from dissections. We recall the definition of gentle algebras. For arrows $\alpha$ and $\beta$, we denote by $s(\alpha)$ and $t(\alpha)$ the starting point and the terminal point of $\alpha$, respectively. Also we write $\alpha\beta$ for the path from $s(\alpha)$ to $t(\beta)$.

**Definition 7.8.** Let $Q$ be a finite quiver and $I$ an ideal in the path algebra $kQ$ satisfying the following conditions:

(i) For each vertex $i$ of $Q$, there are at most two arrows starting at $i$ and there are at most two arrows ending at $i$.

(ii) For every arrow $\alpha$ in $Q$, there exists at most one arrow $\beta$ such that $t(\alpha) = s(\beta)$ and $\alpha\beta \notin I$ and there exists at most one arrow $\gamma$ such that $s(\alpha) = t(\gamma)$ and $\gamma\alpha \notin I$.

(iii) For every arrow $\alpha$ in $Q$, there exists at most one arrow $\beta$ such that $t(\alpha) = s(\beta)$ and $\alpha\beta \in I$ and there exists at most one arrow $\gamma$ such that $s(\alpha) = t(\gamma)$ and $\gamma\alpha \in I$.

(iv) The ideal $I$ is generated by paths of length 2.

In this case, we call the pair $(Q, I)$ gentle bound quiver. We call $kQ/I$ gentle algebra and $\kappa Q/\bar I$ complete gentle algebra, where $\kappa Q$ is a complete path algebra (that is, the completion of $kQ$ with respect to $kQ_+$-adic topology, where $kQ_+$ is the arrow ideal) and $\bar I$ is the closure of $I$.

Notice that we don’t assume that gentle algebras are finite dimensional. It is known in [PPP19, Theorem 4.10] that gentle bound quivers are precisely obtained by the following construction.

**Definition 7.9.** For a $\bullet$-dissection $D$ of $(S,M)$, we define a quiver $Q(D)$ and an ideal $I(D)$ of $kQ(D)$ as follows:

- The set of vertices of $Q(D)$ corresponds bijectively with $D$;
- The set of arrows of $Q(D)$ is a disjoint union of sets of arrows in $C_v$ for all $v \in M_\bullet$ defined as follows (see Figure 9):
  - If $v$ is a puncture and $d_1, \ldots, d_m \in D$ are sides of $\triangle_v$ in counterclockwise order, then there is a cycle $C_v: d_1 \overset{a_1}{\rightarrow} d_2 \overset{a_2}{\rightarrow} \cdots \overset{a_{m-1}}{\rightarrow} d_m \overset{a_m}{\rightarrow} d_1$ in $Q(D)$, that is uniquely determined up to cyclic permutation.
  - If $v$ lies on a boundary segment $b$, and $d_1, \ldots, d_m \in D$ are sides of $\triangle_v$ in counterclockwise order, then there is a path $C_v: d_1 \overset{a_1}{\rightarrow} d_2 \overset{a_2}{\rightarrow} \cdots \overset{a_{m-1}}{\rightarrow} d_m$ in $Q(D)$.
- $I(D)$ is generated by all paths of length 2 which are not a sub-path of any $C_v$.

![Figure 9](image)

**Proposition 7.10.** [PPP19, Theorem 4.10] For a $\bullet$-dissection $D$ of $(S,M)$, the pair $(Q(D), I(D))$ is a gentle bound quiver, and any gentle bound quiver arises in this way.

We prepare a few terminology.

**Definition 7.11.** Let $Q(D)$ be the quiver in Definition 7.9.

- For a puncture $v \in M_\bullet$, a cycle $C_v$ is called a special cycle at $v$. If it is a representative of its cyclic permutation class starting and ending at $d \in D$, then we call it a special $d$-cycle at $v$.
- For $v \in M_\bullet$, every non-constant sub-path of $C_v$ is called a short path.
7.3. **Two-term silting complexes for** \(A(D)\) **via** \(D\)-laminates. Let \(D\) be a \(\bullet\)-dissection of \((S, M)\) and \(A(D) := kQ(D)/I(D)\) the complete gentle algebra associated with \(D\). In this subsection, we establish a geometric model of two-term silting theory for \(A(D)\) and prove Theorem \[3\].

Let \(P_d\) be an indecomposable projective \(A(D)\)-module corresponding to \(d \in D\). For \(d, e \in D\), every short path in \(Q(D)\) from \(d\) to \(e\) provides a non-vanishing homomorphism \(P_e \to P_d\) in proj \(A(D)\), that we call **short map**. Let \(R := k[[t]]\), where \(t\) is a sum of all special cycles in \(Q(D)\). Notice that \(t\) is in the center of \(A(D)\).

**Proposition 7.12.** The complete gentle algebra \(A(D)\) is a module-finite \(R\)-algebra.

**Proof.** It follows from the fact that \(A(D)\) is generated by all short paths and constant paths as an \(R\)-module.

We would discuss a class of complexes in \(K^b(\text{proj } A(D))\) obtained from \(D\)-laminates.

**Definition 7.13.** An indecomposable two-term complex in \(K^b(\text{proj } A(D))\) is called a **two-term string complex** if it can be written as one of the following forms:

\[
\begin{array}{c}
P_{d_1} \xrightarrow{f_{21}} P_{d_2} \\
P_{d_3} \xrightarrow{f_{32}} P_{d_4} \\
\vdots \\
P_{d_{m-1}} \xrightarrow{f_{m-1,m}} P_{d_m} \\
P_{d_m} \xrightarrow{f_{m-1,m}} P_{d_{m-1}}
\end{array}
\]

where each \(f_{ij}\) is of the form \(f_{ij} = t^m f'\) for a short map \(f' : P_j \to P_i\) and a non-negative integer \(m\). It is called **two-term short string complex** if all \(f_{ij}\) are short maps.

We denote by \(2\text{-scx } A(D)\) the set of isomorphism classes of two-term short string complexes such that \(\text{add} T^0 \cap \text{add} T^{-1} = 0\), and indecomposable stalk complexes concentrated in degree 0 or \(-1\). To prove the following proposition, we use \(\tau\)-tilting theory (see [AIR14] for details).

**Proposition 7.14.** Any indecomposable two-term presilting complex in \(K^b(\text{proj } A)\) is a two-term short string complex, that is, \(2\text{-ips } A(D) \subseteq 2\text{-scx } A(D)\).

**Proof.** Since \(\Lambda := A(D)/(t)\) is a finite dimensional special biserial algebra, every indecomposable non-projective \(\Lambda\)-module is either a string module or a band module (see [BR87, WWS85]). A band module \(M\) satisfies \(M = \tau M\) in particular. \(\text{Hom}_\Lambda (M, \tau M) \neq 0\) where \(\tau\) is the Auslander-Reiten translation for \(\Lambda\). By [AIR14] Lemma 3.4, if \(P \in 2\text{-ips } \Lambda\) is a non-stalk complex, then it is a minimal projective presentation of a string module and hence a two-term string complex. In addition, \(P\) must be short since \(t = 0\) on \(\Lambda\). Therefore, Proposition \(\ref{7.3}\) gives \(P \in 2\text{-scx } \Lambda\). By Proposition \(\ref{7.7}\), so is any complex in \(2\text{-ips } A(D)\) because the functor \(- \otimes_R R/tR\) preserves the property being short.

First, we give a geometric model of short maps in proj \(A(D)\).

**Definition 7.15.** A **\(D\)-segment** is a non-self-intersecting curve, considered up to isotopy relative to \(M\), in a polygon \(\triangle_v\) of \(D\) for some \(v \in M_0\) whose ends are unmarked points on sides of \(\triangle_v\) or spirals around \(v\).

Let \(\eta\) be a \(D\)-segment in \(\triangle_v\) whose endpoints lie on \(d, e \in \triangle_v \cap D\). We orient it to satisfy that \(\eta\) is to its right and its starting point lies on \(d\). Then it corresponds to a short path \(d_1 \rightarrow \cdots \rightarrow d_s\) in \(Q(D)\), where \(d_1, \ldots, d_s \in D\) are sides of \(\triangle_v\) in counterclockwise order with \(d_1 = e\) and \(d_s = d\). It induces a short map \(\sigma(\eta) : P_d \to P_e\) in proj \(A(D)\).

**Proposition 7.16.** The map \(\sigma\) induces a bijection

\[
\sigma : \{D\text{-segments whose endpoints lie on } D\} \to \{\text{short maps in proj } A(D)\}.
\]
Proof. The assertion immediately follows from the definition of $\sigma$.

Next, we give a geometric model of two-term short string complexes in $K^b(\text{proj} A(D))$.

**Definition 7.17.** A **generalized D-laminate** is a $\circ$-laminate $\gamma$ intersecting at least one $\bullet$-arc of $D$ such that the condition $(\ast)$ in Definition 2.6(2) and the following conditions are satisfied:

- Each connected component of $\gamma$ in $\triangle_v$ does not intersect itself for any $v \in M_\circ$;
- For any $d \in D$, all intersection points of $\gamma$ and $d$ are either positive or negative simultaneously.

Note that a $D$-laminate is precisely a non-self-intersecting generalized $D$-laminate.

A non-closed generalized (NCG, for short) $D$-laminate $\gamma$ is decomposed into $D$-segments $\gamma_0, \ldots, \gamma_m$ in polygons such that $\gamma_{i-1}$ and $\gamma_i$ have a common endpoint $p_i$ on $d_i \in D$ for every $i \in \{1, \ldots, m\}$. In particular, an end of $\gamma$ is a two-term complex $\big[ T^1, T^0 \big]$ given by

$$\gamma = \bigoplus_{p_j: \text{negative}} T^1_j, \quad \gamma = \bigoplus_{p_i: \text{positive}} T^0_i,$$

where $f = (f_{ij})_{i,j \in \{1, \ldots, m\} \cup \{0\}}$, and $f_{ij} := \begin{cases} \sigma(\gamma_{j-1}) & \text{if } i = j - 1, \\ \sigma(\gamma_j) & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$

From our construction, we have the equality $g(\gamma) = g(\gamma)$ under the identification of $\mathbb{Z}^{(|D|)}$ and $\mathbb{Z}^{(|A|)}$ via the map $d \mapsto P_d$, where the vector $g(\gamma) \in \mathbb{Z}^{(|D|)}$ is defined by the equality 2.14.

**Lemma 7.18.** Suppose that two NCG $D$-laminates $\gamma$ and $\gamma'$ are decomposed into $D$-segments $\gamma_0, \ldots, \gamma_m$ and $\gamma'_0, \ldots, \gamma'_m$, as above, respectively. If $m = m' > 1$ and $\gamma_i = \gamma'_i$ for $i \in \{1, \ldots, m-1\}$, then $\gamma' = \gamma$.

Proof. The $D$-segment $\gamma_0$ (resp., $\gamma_m$) only depends on the sign of $p_1$ (resp., $p_m$), that is uniquely determined by $\gamma_1$ (resp., $\gamma_{m-1}$). Thus we have $\gamma_0 = \gamma'_0$ and $\gamma_m = \gamma'_m$.

**Proposition 7.19.** The map $T_{(-)} : \gamma \mapsto T_\gamma$ induces a bijection

$$T_{(-)} : \{ \text{NCG } D\text{-laminates} \} \to 2\text{-scx } A(D)$$

such that $g(\gamma) = g(T_\gamma)$ for any NCG $D$-laminate $\gamma$.

Proof. By our construction, $T_\gamma$ is a stalk complex or a two-term short string complex such that $\operatorname{add} T^0_\gamma \cap \operatorname{add} T^{-1}_\gamma = 0$, that is, $T_\gamma \in 2\text{-scx } A(D)$. By Lemma 7.16 this map is injective.

In order to prove surjectivity of the map, we give the inverse map. If $P \in 2\text{-scx } A(D)$ is a stalk complex $P_d$ with $d \in D$ concentrated in degree 0 (resp., -1), then we just take $\gamma = d^*_\ast$ (resp., $\gamma = d^{-}_\ast$). Next, let $P \in 2\text{-scx } A(D)$ be a short string complex which is one of (1)-(3) in Definition 7.13. We only consider the form (1) since the others can be proved similarly. By Proposition 7.16 $\gamma_1 := \sigma^{-1}(f_{21}), \gamma_2 := \sigma^{-1}(f_{23}), \ldots, \gamma_{m-1} := \sigma^{-1}(f_{m-1,m})$ are $D$-segments, and $\gamma_{i-1}$ and $\gamma_i$ have a common endpoint on $d_i$ for $i \in \{2, \ldots, m-1\}$. Then, by Lemma 7.18 there are two $D$-segments $\gamma_0$ and $\gamma_m$ such that the curve $\gamma$ obtained by gluing $\gamma_0, \ldots, \gamma_m$ one by one is an NCG $D$-laminate. From our construction, we have $P = T_\gamma$.

Finally, we give a geometric model of two-term presilting/silting complexes in $K^b(\text{proj} A(D))$. To do it, we need some preparations. First of all, for $T, T' \in 2\text{-scx } A(D)$, we consider two kinds of morphisms from $T$ to $T'[1]$ in $K^b(\text{proj} A(D))$. 

**Definition 7.20.** For $T, T' \in \text{2-scx} A(D)$, a morphism $f \in \text{Hom}_{K^b(\text{proj} A(D))}(T, T'[1])$ is called a **singleton single map** if it is induced by a short map $p$ as one of the following forms:

(a) \hspace{1cm} (b) 

\[
\begin{array}{c}
\begin{tikzpicture}
\node (T) at (0,0) {$T$};
\node (T') at (2,0) {$T'$};
\node (p) at (1,1) {$p$};
\node (q) at (0,1) {$q$};
\node (q') at (2,1) {$q'$};
\draw[->] (T) to (q);
\draw[->] (q) to (p);
\draw[->] (p) to (T');
\end{tikzpicture}
\end{array}
\]

(c) \hspace{1cm} (d) 

\[
\begin{array}{c}
\begin{tikzpicture}
\node (T) at (0,0) {$T$};
\node (T') at (2,0) {$T'$};
\node (p) at (1,1) {$p$};
\node (q) at (0,1) {$q$};
\node (q') at (2,1) {$q'$};
\node (p') at (1,2) {$p'$};
\node (p'') at (1,0) {$p''$};
\draw[->] (T) to (q);
\draw[->] (q) to (p');
\draw[->] (p') to (T');
\draw[->] (T) to (p'');
\draw[->] (p'') to (T');
\draw[->] (T) to (q');
\draw[->] (q') to (p'');
\end{tikzpicture}
\end{array}
\]

where $p$ and $q$ (resp., $q'$) have no common arrows as paths, and $p'$ and $p''$ are not constant.

**Definition 7.21.** For $T, T' \in \text{2-scx} A(D)$, a morphism $f \in \text{Hom}_{K^b(\text{proj} A(D))}(T, T')$ is called a **quasi-graph map** if it is induced by the following form:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (T) at (0,0) {$T$};
\node (T') at (2,0) {$T'$};
\node (U) at (1,1) {$U$};
\node (Γ) at (1,0) {$Γ$};
\draw[->] (T) to (U);
\draw[->] (U) to (Γ);
\draw[->] (Γ) to (T');
\end{tikzpicture}
\end{array}
\]

where $U$ and $Γ$ are

(e) \hspace{1cm} (f) 

\[
\begin{array}{c}
\begin{tikzpicture}
\node (T) at (0,0) {$T$};
\node (T') at (2,0) {$T'$};
\node (p) at (1,1) {$p$};
\node (q) at (0,1) {$q$};
\node (q') at (2,1) {$q'$};
\node (P) at (1,2) {$P$};
\node (P') at (1,0) {$P'$};
\draw[->] (T) to (q);
\draw[->] (q) to (P);
\draw[->] (P) to (T');
\draw[->] (T) to (q');
\draw[->] (q') to (P');
\draw[->] (P') to (T');
\end{tikzpicture}
\end{array}
\]

and there is no $p \in \text{Hom}_{A(D)}(P, P')$ such that $pq = q'$ or $r = r'p$. Note that it implies $q' \neq 0$ and $r \neq 0$. A quasi-graph map in $\text{Hom}_{K^b(\text{proj} A(D))}(T, T')$ naturally induces a unique morphism in $\text{Hom}_{K^b(\text{proj} A(D))}(T, T'[1])$, called a quasi-graph map representative.

Regarding two-term short string complexes as homotopy strings defined as in [ALP16] (see also [BM03]), we can obtain the following result from [ALP16].

**Proposition 7.22.** [ALP16] Propositions 4.1 and 4.8 For $T, T' \in \text{2-scx} A(D)$, singleton single maps and quasi-graph map representatives give a basis of $\text{Hom}_{K^b(\text{proj} A(D))}(T, T'[1])$.

Let $γ$ and $δ$ be NCG $D$-laminates. By Propositions 7.19 and 7.22, $\text{Hom}_{K^b(\text{proj} A(D))}(T_γ, T_δ[1])$ has a basis consisting of singleton single maps and quasi-graph map representatives. It follows from the definition that a singleton single map in $\text{Hom}_{K^b(\text{proj} A(D))}(T_γ, T_δ[1])$ is given by one of the following local figures:

(a) \hspace{1cm} (b) \hspace{1cm} (c) \hspace{1cm} (d) 

\[
\begin{array}{c}
\begin{tikzpicture}
\node (p) at (0,0) {$p$};
\node (γ) at (-0.5,0) {$γ$};
\node (δ) at (0.5,0) {$δ$};
\draw[->] (p) to (γ);
\draw[->] (p) to (δ);
\end{tikzpicture}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (p) at (0,0) {$p$};
\node (γ) at (-0.5,0) {$γ$};
\node (δ) at (0.5,0) {$δ$};
\draw[->] (p) to (γ);
\draw[->] (p) to (δ);
\end{tikzpicture}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (p) at (0,0) {$p$};
\node (γ) at (-0.5,0) {$γ$};
\node (δ) at (0.5,0) {$δ$};
\draw[->] (p) to (γ);
\draw[->] (p) to (δ);
\end{tikzpicture}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (p) at (0,0) {$p$};
\node (γ) at (-0.5,0) {$γ$};
\node (δ) at (0.5,0) {$δ$};
\draw[->] (p) to (γ);
\draw[->] (p) to (δ);
\end{tikzpicture}
\end{array}
\]
where $p$ is the associated short map. For a quasi-graph map representative in $\text{Hom}_{\mathcal{K}^\flat(\text{proj}(A|D))}(T_\gamma, T_\delta[1])$, each of $\square$ and $\blacksquare$ as in Definition 7.21 is given by one of the following local figures:

\begin{center}
\begin{tikzpicture}[scale=0.5]
  \node (a) at (0,0) {$\delta$};
  \node (b) at (1.5,0) {$\gamma$};
  \node (c) at (3,0) {$\delta$};

  \draw (a) -- (b);
  \draw (b) -- (c);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}[scale=0.5]
  \node (a) at (0,0) {$\delta$};
  \node (b) at (1.5,0) {$\gamma$};
  \node (c) at (3,0) {$\delta$};

  \draw (a) -- (b);
  \draw (b) -- (c);
\end{tikzpicture}
\end{center}

where the left figure of $(e)$ (resp., $(f)$) is in the case of $q = 0$ (resp., $r' = 0$).

**Proposition 7.23.** The following conditions are equivalent for two NCG $D$-laminates $\gamma$ and $\delta$:

1. $\text{Hom}_{\mathcal{K}^\flat(\text{proj}(A|D))}(T_\gamma, T_\delta[1]) = 0$;
2. $\gamma$ is in positive position for $\delta$.

**Proof.** The assertion follows from Proposition 7.22 and the above observations. $\square$

We are ready to state our results.

**Proposition 7.24.** The following hold.

1. The map $T_{(-)}$ in Proposition 7.14 restricts to a bijection
   \[
   \{\text{non-closed $D$-laminates}\} \to \text{2-ips} A.
   \]
2. Two non-closed $D$-laminates $\gamma$ and $\eta$ are compatible if and only if $T_\gamma \oplus T_\eta$ is presilting.

**Proof.** In general, two NCG $D$-laminates $\gamma$ and $\eta$ are compatible if and only if they are in positive position each by Proposition 7.23. This is equivalent to the condition that $\text{Hom}(T_\gamma, T_\delta[1]) = 0$. Since a non-closed $D$-lamine is precisely a non-self-intersecting NCG $D$-lamine, we get (1) and (2). $\square$

**Theorem 7.25.** The map $X \mapsto T_X := \bigoplus_{\gamma \in X} T_\gamma$ gives bijections
\[
\{\text{reduced $D$-laminates}\} \to \text{2-presilt} A(D) \quad \text{and} \quad \{\text{complete $D$-laminates}\} \to \text{2-silt} A(D)
\]
such that $C(X) = C_A(T_X)$ for all reduced $D$-laminates $X$. In particular, we have
\[
F(D) = F(A(D)) \quad \text{and} \quad |F(D)| = |F(A(D))|.
\]

**Proof.** It follows from Proposition 7.24. $\square$

Now, we prove Theorem 1.3.

**Proof of Theorem 1.3.** By Proposition 7.10 any complete gentle algebra is given as $A(D)$ for some $\bullet$-dissection $D$ of $\circ$-marked surfaces. Thus the assertion follows from Theorems 1.6 and 7.25. $\square$

### 7.4. Application to finite dimensional $k$-algebras.

In this subsection, we study a class of finite dimensional factor algebras containing finite dimensional gentle algebras and Brauer graph algebras.

Let $D$ be a $\bullet$-dissection of $(S, M)$. For a given function $m: M_S \setminus S \to \mathbb{Z}_{>0}$, we define a factor algebra $A(D; m) := kQ(D)/(I(D) \cup J(D; m))$, where $kQ(D)$ and $I(D)$ are in Definition 7.9 and $J(D; m)$ is an ideal generated by
\[
C^m(u) - C^m(v)
\]
for all $d \in D$ and for all special $d$-cycles $C_u$ and $C_v$ at $u$ and $v$ respectively.

**Proposition 7.26.** In the above, $A(D; m)$ is a finite dimensional special biserial algebra.

**Proof.** The assertion follows from the definition. $\square$

Notice that if $M_S$ has no punctures, then $J(D; m) = 0$ and hence $A(D; m)$ is precisely a finite dimensional gentle algebra. On the other hand, if $\partial S = \emptyset$, or equivalently all marked points are punctures, our definition is equivalent to one of Brauer graph algebras. More precisely, one can regard the dual dissection $D^*$ of $D$ together with $m$ as a Brauer graph in the following way:

- The set of vertices corresponds bijectively with $M_S$;
- We draw an edge between two vertices whenever there is a $\circ$-arcs of $D^*$ connecting them.
• A cyclic ordering of edges around a vertex is induced from the orientation of $S$;
• A multiplicity of $v$ is $m(v)$ for every $v \in M$.

Then $A(D;m)$ coincides with the Brauer graph algebra associated with this Brauer graph (see [Sch18 Section 2.4]). Conversely, it is shown in [Lab13 Lemma 2.2.4] that every Brauer graph (and hence Brauer graph algebra) arises in this way.

Several factor algebras have a common geometric model of two-term silting theory as $A(D)$. In the following, let $\Lambda_0 := A(D)/I$, where $I$ is an ideal generated by all special cycles in $Q(D)$. Let $\Lambda_1 := A(D)/(t)$ and $\Lambda_2 := A(D;m)$ for an arbitrary function $m$.

**Theorem 7.27.** For $i \in \{0, 1, 2\}$, there are bijections

$$\varphi_i: \{\text{reduced } D\text{-laminations}\} \to 2\text{-presilt } \Lambda_i$$

and

$$\{\text{complete } D\text{-laminations}\} \to 2\text{-silt } \Lambda_i$$

such that $C(\mathcal{X}) = C_{\Lambda_i}(\varphi_i(\mathcal{X}))$ for all reduced $D$-laminations $\mathcal{X}$. In particular, we have $F(D) = F(\Lambda_i)$.

**Proof.** Remember that $A(D)$ is module-finite over $R$ by Proposition 7.12. Then we have a bijection

$$\varphi_1: \{\text{reduced } D\text{-laminates}\} \xrightarrow{T_{(-)}} 2\text{-presilt } A(D) \xrightarrow{T_{(-)}} 2\text{-presilt } \Lambda_1$$

such that $C(\mathcal{X}) = C_{\Lambda_1}(\varphi_1(\mathcal{X}))$ for all reduced $D$-laminations $\mathcal{X}$, where $T_{(-)}$ and $T_{(-)}$ are bijections in Theorems 7.7 and 7.25 respectively. On the other hand, for $i \in \{1, 2\}$, there is an ideal $J_i$ such that $\Lambda_0 \cong \Lambda_i/J_i$ and $J_i$ is generated by central elements and in the Jacobson radical. In this situation, by [EJR18 Theorem 1.1], we have a canonical bijection between 2-presilt $\Lambda_0$ and 2-presilt $\Lambda_i$ that preserves their $g$-vectors. For $i \in \{0, 2\}$, let $\psi_i$ be the bijection $\psi_i: 2\text{-presilt } \Lambda_1 \to 2\text{-presilt } \Lambda_i$. Then a composition $\varphi_i := \psi_i \circ \varphi_1$ is the desired one for $i \in \{0, 2\}$.

The latter assertion is clear. □

**Corollary 7.28.** For $i \in \{0, 1, 2\}$, the algebra $\Lambda_i$ is $g$-tame. In particular, any Brauer graph algebra is $g$-tame.

**Proof.** It follows from Theorems 1.6 and 7.27. □

8. **Examples for representation theory**

(1) Let $(S, M)$ be a disk with $|M| = 10$ such that one marked point in $M_\circ$ (resp., $M_\ast$) is a puncture and the others lie on $\partial S$. For a $\bullet$-dissection of $(S, M)$

the associated gentle bound quiver $(Q(D), I(D))$ is given by

$$Q(D) = \begin{array}{c}
\circ \\
1 \quad d \quad 4 \quad e \quad 5 \\
\circ \\
\circ \\
2 \quad c \quad f \\
\circ \\
\circ \\
\circ \\
3 \\
\circ
\end{array}$$

and $I(D) = (cf, fc, ed)$. 
We consider an NCG $D$-laminate $\gamma$, but not a $D$-laminate, that is decomposed into $D$-segments $\gamma_0, \ldots, \gamma_5$ as follows:

Then the corresponding two-term string complex $T_\gamma$ is not presilting. In fact, there is a nonzero quasi-graph map representative in $\text{Hom}_{\text{K}^b(\text{proj } A)}(T_\gamma, T_\gamma[1])$ induced by the form

$$
\begin{align*}
T_\gamma & \quad \sigma(\gamma_1) \rightarrow P_2 \\
& \quad \sigma(\gamma_2) \rightarrow P_4 \\
& \quad \sigma(\gamma_3) \rightarrow P_5 \\
& \quad \sigma(\gamma_4) \rightarrow P_5
\end{align*}
$$

where $\sigma(\gamma_1)$ (resp., $\sigma(\gamma_2)$, $\sigma(\gamma_3)$, $\sigma(\gamma_4)$) is the short map in $\text{proj } A(D)$ induced by a short path $b$ (resp., $f, dab, ef$) in $Q(D)$. There is no short map $p$ from $P_4$ to $P_2$ (resp., from $P_5$ to $P_4$) such that $\sigma(\gamma_1) = p\sigma(\gamma_3)$ (resp., $\sigma(\gamma_2) = p\sigma(\gamma_4)$). Therefore, $T_\gamma$ is not presilting.

We consider two $D$-laminates $\delta$ and $\delta'$ as follows:

Then $\delta'$ is a positive position for $\delta$, but $\delta$ is not a positive position for $\delta'$. We observe whether $T_\delta \oplus T_{\delta'}$ is presilting. It is easy to see that $T_\delta$ and $T_{\delta'}$ are presilting respectively. In addition, we have $\text{Hom}_{\text{K}^b(\text{proj } A)}(T_{\delta'}, T_\delta[1]) = 0$. However, there is a nonzero singleton single map from $T_\delta$ to $T_{\delta'}[1]$ induced by a short path $b$, as (d) in Definition 7.20. Thus $T_\delta \oplus T_{\delta'}$ is not presilting.
(2) We consider the \(\bullet\)-marked surface \((S, M)\) and the \(\bullet\)-dissection \(D = \{1, 2\}\) in Section 3(3). Then the associated gentle bound quiver \((Q(D), I(D))\) is given by

\[
Q(D) = \begin{array}{c}
\overset{a_2}{\bullet} \\
\overset{b_2}{\bullet} \\
\overset{a_1}{\bullet} \\
\overset{b_1}{\bullet}
\end{array}
\quad \text{and} \quad
I(D) = \langle a_1 b_1, b_1 a_2, a_2 b_2, b_2 a_1 \rangle.
\]

Consider a function \(m\) identically 1, then

\[
J(D; 1) = \langle a_1 b_1 a_2 b_2 - a_2 b_2 a_1 b_1, b_1 a_2 b_2 a_1 - b_2 a_1 b_1 a_2 \rangle
\]
and \(A(D; 1)\) is a Brauer graph algebra associated with the Brauer graph

\[
\begin{array}{c}
\circ \quad \circ
\end{array}
\]

with multiplicity 1 on the vertex \(\circ\). In Section 3(3), we gave the complete lists of \(D\)-laminates and complete \(D\)-laminations. For \(i \in \mathbb{Z}_{>0}\) and the non-closed \(D\)-lamine \(\gamma_i\), the corresponding two-term string complex \(T_{\gamma_i}\) is given by

\[
\begin{array}{c}
P_1 \\
\sigma(a_2) \\
\sigma(a_1) \\
P_2
\end{array}
\quad \begin{array}{c}
P_1 \\
\sigma(a_2) \\
\sigma(a_1) \\
P_2
\end{array}
\quad \begin{array}{c}
P_1 \\
\sigma(a_2) \\
\sigma(a_1) \\
P_2
\end{array}
\]

where \(\sigma(a_k)\) is a short map induced by \(a_k\) and \(P_1\) only appears \(i\) times in degree 0 (resp., \(P_2\) only appears \(i - 1\) times in degree \(-1\)). For \(i, j \in \mathbb{Z}_{>0}\), it is easy to show that all nonzero maps between \(T_{\gamma_i}\) and \(T_{\gamma_j}[1]\) are quasi-graph map representatives induced by the form

\[
\begin{array}{c}
P_1 \\
\sigma(a_2) \\
\sigma(a_1) \\
P_2
\end{array}
\quad \begin{array}{c}
P_1 \\
\sigma(a_2) \\
\sigma(a_1) \\
P_2
\end{array}
\quad \begin{array}{c}
P_1 \\
\sigma(a_2) \\
\sigma(a_1) \\
P_2
\end{array}
\]

that is, \(i > j + 1\). Therefore, \(T_{\gamma_i} \oplus T_{\gamma_j}\) is two-term silting for \(j = i, i \pm 1\); otherwise it’s not.

**Acknowledgements.** The authors would like to thank their supervisor Osamu Iyama for his guidance and helpful comments. They are Research Fellows of Society for the Promotion of Science (JSPS). This work was supported by JSPS KAKENHI Grant Numbers JP17J04270 and JP19J11408.
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