Research article

Algebraic tangles and Jones polynomial ★

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Abstract

We develop a method for computing Kauffman bracket and Jones polynomial for algebraic tangles and their numerator closures. We also introduce the notion of connectivity type of pretzel tangles and give a way of computing it. Several examples are given.

1. Introduction

A two tangle diagram is a region in a knot or link diagram surrounded by a rectangle such that the knot or link diagram crosses the rectangle in four points. These four points are usually thought of as fixed points occurring in the four corners NW, NE, SW and SE. See Fig. 1. Closing the tangle will produce a knot or a link with two components. These operations are called closures of a tangle, and there are two types of closures, the numerator closure and the denominator closure as in Fig. 2. For more details see [1, 2, 3, 4, 5, 6, 7].

In this research we develop a method for computing Kauffman bracket and Jones polynomial for algebraic tangles and their numerator closures. We also introduce the notion of connectivity type of pretzel tangles and give a way of computing it. We give many examples showing the efficiency of our methods and results.

This paper is organized as follows. In Section 2 we introduce concepts and terminologies that we will use in the later sections. In Section 3 we study a binary operation on the bracket polynomial for tangles, we give some properties for this operation and we give formulas for \( \langle n \rangle \) and \( \langle \frac{1}{m} \rangle \); where \( n \in \mathbb{Z} \). In Section 4 we give some formulas to compute the \( X \) polynomial for the numerator closure of rational tangles with two and three components and for pretzel tangles with two components and give some examples to compute the \( X \) polynomial for rational, pretzel and algebraic links. We introduce some properties for a ratio invariant defined by Kauffman in [8]. In Section 5 we construct a method to compute the connectivity type for pretzel tangles, which is an invariant for pretzel tangles.

Fig. 1. Two tangle diagram.

Fig. 2. \( N(T) \) is the numerator closure of \( T \) and \( D(T) \) is the denominator closure of \( T \).

2. Basic concepts and terminology

The most important issue in studying tangles is figuring out if a pair of two tangle diagrams are isotopy equivalent or not. For the following definition see [9].

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Definition 1. Two 2-tangle diagrams are said to be isotopy equivalent if we can get from one of them to the other by a finite sequence of the three Reidemeister moves inside the surrounding rectangle while the four points remain fixed. Reidemeister moves are shown in Figs. 3, 4, 5.

Definition 2. Let $T_1$ and $T_2$ be two 2-tangles. They are added, as in Fig. 6. See [10].

Definition 3. Let $T_1$ and $T_2$ be two 2-tangles. They are multiplied, as in Fig. 7. See [10].

Definition 4. Let $T$ be a 2-tangle. The rotation of $T$ is obtained by rotating $T$ counterclockwise by 90°, as in Fig. 8. See [10].

For a simple example $[0]^r = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = [0]$ and $[\infty]^r = \begin{pmatrix} 1 \\ \infty \end{pmatrix} = [0]$.

Definition 5. Let $T$ be a 2-tangle. The mirror image of a tangle $T$, denoted by $\sim T$, is obtained by switching all crossings. See [8].

Definition 6. Let $T$ be a 2-tangle. The inverse of a tangle $T$, denoted by $T'$ or $T^{-1}$, is defined to be $-T'$ as in Fig. 9. See [8].

Definition 7. A rational tangle is the isotopy class represented by the tangle diagram denoted by $[a_1, a_2, ..., a_n]$ and constructed as follows.

$$T = [a_1, a_2, ..., a_n] = [a_1] + \frac{1}{[a_2]} + \frac{1}{[a_3]} + \cdots + \frac{1}{[a_n]}$$

where $a_2, ..., a_n \in \mathbb{Z} - \{0\}$, $a_1 \in \mathbb{Z}$. Such form of a tangle will be called standard form. See [11].

Rational knots and links are knots and links that are obtained by taking numerator or denominator closure of rational tangles. These have one or two components, they are alternating and they are the easiest knots and links to make. Furthermore all knots and links up to ten crossings are either rational or are obtained by inserting rational tangles into a few simple planar graphs. Rational knots and rational tangles are of fundamental importance in the study of DNA recombination. The rational knots are totally classified. See [12].

Definition 8. A pretzel tangle is a tangle of the form $\frac{1}{[a_1]} + \cdots + \frac{1}{[a_n]}$; where $a_1 \neq 0$, and it is denoted by $(a_1, ..., a_n)$. See Fig. 10 for an example of a pretzel tangle.

A knot (link) is called a pretzel knot (link) if it is a numerator closure of a pretzel tangle.

Definition 9. An algebraic tangle is a tangle obtained by the operations of addition and multiplication of rational tangles.

2.1. Kauffman bracket and Jones polynomial

In this section, we introduce the Kauffman bracket polynomial, the $X$ polynomial and the state sum formula. See [13].

Definition 10. The Kauffman bracket polynomial is a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients in an indeterminate $A$. It maps a diagram $D$ of a link $L$ to $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ and it is characterized by three rules

Rule 1: $\langle \bigcirc \rangle = 1$

Rule 2: $\langle X \rangle = A \langle \rangle + A^{-1} \langle \rangle$

Rule 3: $\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$
The bracket polynomial is an invariant under the second and the third R-moves.

**Theorem 1.** Let $D$ be an oriented link diagram. Then the $X$ polynomial defined by

$$X(D) = (-A^3)^{-\nu(D)}(D)$$

is an invariant of links, where $|D|$ is the diagram without orientation and $\nu(D)$ is the writhe of $D$ that is the sum of the signs of the crossings for a given diagram $D$, where each crossing in the oriented diagram has a sign either $1$ or $-1$ according to the right-hand rule.

The following theorem provides us with a formula for calculating $<D>$ called a state sum formula.

**Theorem 2.**

$$\langle D \rangle = \sum_S A^{x(S)} A^{-y(S)} (-A^2 - A^{-2})^{|S|}$$

where the sum runs over all possible states $S$, which is a choice of how to split all of the $n$ crossings in the projection of $D$, $|S|$ is the total number of circles in the state $S$, $a(S)$ is the number of $A$-splits in $S$ and $b(S)$ is the number of $B$-splits in $S$. The $A$-split and $B$-split are shown in Fig. 11.

**3. Algebraic structures**

In this section we study a binary operation on the bracket polynomial for tangles, and we give some properties for this operation. Let $T$ be any 2-tangle, then the state $S$ consists of circles and the tangle $[0]$ or the tangle $[\infty]$, so the bracket polynomial of $T$ can be defined as

$$\langle T \rangle = \sum_S A^{a(S)} A^{-y(S)} (-A^2 - A^{-2})^{|S|} \varepsilon(S)$$

Where $\varepsilon(S) = [0], \text{or } [\infty]$.

**Lemma 1.** Let $T$ be any 2-tangle and let $\langle T \rangle$ be the formal expansion of the bracket on this tangle. Then there exist elements $a(A)$ and $b(A)$ in $\mathbb{Z}[A, A^{-1}]$, such that

$$\langle T \rangle = a(A) [\infty] + b(A) [0]$$

For simplicity, we write the last formula as

$$\langle T \rangle = a([\infty]) + b([0])$$

Define $M$ to be the free module over $\mathbb{Z}[A, A^{-1}]$ with basis $\{[\infty], [0]\}$. The following operation of multiplication on the module $M$ is defined by Bataineh in [14].

**Definition 11.** Let $\langle T \rangle = a([\infty]) + b([0])$, and $\langle S \rangle = c([\infty]) + d([0])$. Define

$$\langle T \rangle \oplus \langle S \rangle = a([\infty]) + b([0]) \oplus c([\infty]) + d([0])$$

where $\oplus$ is the direct sum. The following four linearity identities hold

$$\langle [\infty] + [\infty] \rangle = \langle [\infty] \rangle \oplus \langle [\infty] \rangle$$

$$\langle [0] + [\infty] \rangle = \langle [0] \rangle \oplus \langle [\infty] \rangle$$

$$\langle [0] + [0]\rangle = \langle [0] \rangle \oplus \langle [0] \rangle$$

**3.1. Algebraic properties**

**Lemma 2.** The following four linearity identities hold

\[
\begin{align*}
\langle [\infty] + [\infty] \rangle &= \langle [\infty] \rangle \oplus \langle [\infty] \rangle, \\
\langle [\infty] + [0] \rangle &= \langle [\infty] \rangle \oplus \langle [0] \rangle, \\
\langle [0] + [\infty] \rangle &= \langle [0] \rangle \oplus \langle [\infty] \rangle, \\
\langle [0] + [0] \rangle &= \langle [0] \rangle \oplus \langle [0] \rangle
\end{align*}
\]

**Proof.**

\[
\begin{align*}
\langle [\infty] + [\infty] \rangle &= \langle [\infty] \rangle \oplus \langle [\infty] \rangle, \\
\langle [\infty] + [0] \rangle &= \langle [\infty] \rangle \oplus \langle [0] \rangle, \\
\langle [0] + [\infty] \rangle &= \langle [0] \rangle \oplus \langle [\infty] \rangle, \\
\langle [0] + [0] \rangle &= \langle [0] \rangle \oplus \langle [0] \rangle
\end{align*}
\]

**Theorem 3.** Let $T$ and $S$ be two 2-tangles, then

$$\langle T + S \rangle = \langle T \rangle \oplus \langle S \rangle$$

**Proof.**

Let $\langle T \rangle = a([\infty]) + b([0])$, and $\langle S \rangle = c([\infty]) + d([0])$, then

$$\langle T + S \rangle = \langle [\infty] \rangle \oplus \langle [0] \rangle$$

where $\oplus$ is the direct sum. The following linearity identities hold

\[
\begin{align*}
\langle [\infty] + [\infty] \rangle &= \langle [\infty] \rangle \oplus \langle [\infty] \rangle, \\
\langle [\infty] + [0] \rangle &= \langle [\infty] \rangle \oplus \langle [0] \rangle, \\
\langle [0] + [\infty] \rangle &= \langle [0] \rangle \oplus \langle [\infty] \rangle, \\
\langle [0] + [0] \rangle &= \langle [0] \rangle \oplus \langle [0] \rangle
\end{align*}
\]

**Lemma 3.** $(M, \oplus)$ is a commutative monoid, that is if $\langle T \rangle, \langle S \rangle, \langle R \rangle \in M$, then

\[
\begin{align*}
1 \ (\langle T \rangle \oplus \langle S \rangle) &\oplus \langle R \rangle = \langle T \rangle \oplus \langle S \rangle \oplus \langle R \rangle \\
2 \ (\langle T \rangle \oplus \langle S \rangle) &\oplus \langle R \rangle = \langle T \rangle \oplus \langle S \rangle \oplus \langle R \rangle \\
3 \ (\langle T \rangle \oplus \langle S \rangle) &\oplus \langle R \rangle = \langle T \rangle \oplus \langle S \rangle \oplus \langle R \rangle
\end{align*}
\]

**Proof.**

(1) Let $\langle T \rangle = a([\infty]) + b([0])$, so

$$\langle T \rangle \oplus \langle S \rangle = \langle [\infty] \rangle \oplus \langle [\infty] \rangle = \langle [\infty] \rangle$$

where $\oplus$ is the direct sum. The following linearity identities hold

\[
\begin{align*}
\langle [\infty] + [\infty] \rangle &= \langle [\infty] \rangle \oplus \langle [\infty] \rangle, \\
\langle [\infty] + [0] \rangle &= \langle [\infty] \rangle \oplus \langle [0] \rangle, \\
\langle [0] + [\infty] \rangle &= \langle [0] \rangle \oplus \langle [\infty] \rangle, \\
\langle [0] + [0] \rangle &= \langle [0] \rangle \oplus \langle [0] \rangle
\end{align*}
\]

And in the same way we get $\langle T \rangle = \langle [\infty] \rangle$.

(2) Let $\langle T \rangle = a([\infty]) + b([0])$, $\langle S \rangle = c([\infty]) + d([0])$ and $\langle R \rangle = e([\infty]) + f([0])$, so we have

\[
\begin{align*}
\langle T \rangle \oplus \langle S \rangle &\oplus \langle R \rangle &= \langle [\infty] \rangle \oplus \langle [\infty] \rangle + \langle [\infty] \rangle \oplus \langle [0] \rangle \\
&\oplus \langle [0] \rangle \oplus \langle [\infty] \rangle + \langle [\infty] \rangle \oplus \langle [0] \rangle \\
&\oplus \langle [0] \rangle \oplus \langle [\infty] \rangle + \langle [\infty] \rangle \oplus \langle [0] \rangle
\end{align*}
\]
\[ T = a e (\langle 0 \rangle) \oplus (\langle 0 \rangle) \oplus (\langle 0 \rangle) + ac f (\langle 0 \rangle) \oplus (\langle 0 \rangle) \oplus (\langle 0 \rangle) + ad f (\langle 0 \rangle) \oplus (\langle 0 \rangle) \oplus (\langle 0 \rangle) + ad f (\langle 0 \rangle) \oplus (\langle 0 \rangle) \oplus (\langle 0 \rangle) + bc f (\langle 0 \rangle) \oplus (\langle 0 \rangle) \oplus (\langle 0 \rangle) + bd f (\langle 0 \rangle) \oplus (\langle 0 \rangle) \oplus (\langle 0 \rangle) + bd f (\langle 0 \rangle) \oplus (\langle 0 \rangle) \oplus (\langle 0 \rangle) + d f (\langle 0 \rangle) \oplus (\langle 0 \rangle) \oplus (\langle 0 \rangle) + d f (\langle 0 \rangle) \oplus (\langle 0 \rangle) \oplus (\langle 0 \rangle) \]

\[ = (a (\langle 0 \rangle) \oplus b (\langle 0 \rangle)) \oplus \left( c (\langle 0 \rangle) + d (\langle 0 \rangle) \right) \oplus (\langle 0 \rangle) \]

Theorem 4. \( (M, \oplus) \) is an associative commutative algebra over the ring \( \mathbb{Z}[A, A^{-1}] \), that is if \( (T), (S), (R) \in M \) and \( p, q \in \mathbb{Z}[A, A^{-1}] \), then

1. \( (T) \oplus (S) + (R) = (T) \oplus (S) + (T) \oplus (R) \)
2. \( (T) + (S) \oplus (R) = (T) \oplus (R) + (S) \oplus (R) \)
3. \( (p(T)) \oplus (q(S)) = pq(T) \oplus (S) \)

Proof. Let \( (T) = a (\langle 0 \rangle) \oplus b (\langle 0 \rangle), (S) = c (\langle 0 \rangle) + d (\langle 0 \rangle) \) and \( (R) = e (\langle 0 \rangle) + f (\langle 0 \rangle) \), so we have

\[ (T) \oplus (S) + (R) = (a (\langle 0 \rangle) + b (\langle 0 \rangle)) \oplus (c + e) (\langle 0 \rangle) \oplus (d + f) (\langle 0 \rangle) \]

Lemma 4. For \( n \in \mathbb{Z} \), we have \( (\langle n \rangle) = \delta^{-1} (\langle -A^{-1} \rangle^{n} - (A^{-n})) (\langle 0 \rangle) + [A^{-n}](\langle 0 \rangle) \)

Proof. The equation clearly holds for \( n = 0 \). For \( n \in \mathbb{Z}^{+} \)

\[ \langle n \rangle = \left( \langle 0 \rangle + \langle 0 \rangle + \cdots + \langle 0 \rangle \right) \]

The third one,

\[ (p(T)) \oplus (q(S)) = (pa (\langle 0 \rangle) + pb (\langle 0 \rangle)) \oplus (qc (\langle 0 \rangle) + qd (\langle 0 \rangle)) \]

On the other hand, for \( n \in \mathbb{Z}^{-} \)

\[ \langle n \rangle = \left( \langle 0 \rangle + \langle 0 \rangle + \cdots + \langle 0 \rangle \right) \]

\[ = \left( \langle 0 \rangle + \langle 0 \rangle + \cdots + \langle 0 \rangle \right) \]

\[ = \left( \langle 0 \rangle + \langle 0 \rangle + \cdots + \langle 0 \rangle \right) \]

\[ = \left( \langle 0 \rangle + \langle 0 \rangle + \cdots + \langle 0 \rangle \right) \]

\[ = \left( \langle 0 \rangle + \langle 0 \rangle + \cdots + \langle 0 \rangle \right) \]
Corollary 1. For $n \in \mathbb{Z}$, we have

$$\frac{1}{[n]} = [A^n](\{\infty\}) + \delta^{-1}(-A^{-3})^n - (A^3)^n(\{0\}) = \frac{1}{([n])}.$$
Figure 12. An oriented knot diagram $D_1$.

\[(c[[n_1, n_2]]) = \left( A^{n_1}[(−A)^{n_2} − (A)^{n_1}] + \delta^{−2}[(−A)^{n_1} − (A)^{n_2}] \right)
\times \left( [−(−A)^{n_2} − A^{n_2}] + (A)^{n_2} \right)\langle(\odot)\right)
\]
\[+ \delta^{-1}A^{n_2}[(−A)^{n_2} − A^{n_2}]\langle(\odot)\right)\]
\[= A^{n_2}[(−A)^{n_2} − (A)^{n_2}] + \delta^{-2}[(−A)^{n_1} − (A)^{n_2}]\]
\[\times \left( [−(−A)^{n_2} − A^{n_2}] + (A)^{n_2} \right)\]
\[= A^{n_2}[(−A)^{n_2} − (A)^{n_2}] + \delta^{-2}[(−A)^{n_1} − (A)^{n_2}]\]
\[\times \left( [−(−A)^{n_2} − A^{n_2}] + (A)^{n_2} \right)\]

Notice that, the X polynomial for the oriented link \((c[[n_1, n_2]])\) as follows:

\[X(c[[n_1, n_2]]) = (−A^3)^{−w(c[[n_1, n_2]])}\left\langle\left( c[[n_1, n_2]]\right)\right\rangle\]

Example 1. Consider the following oriented knot diagram (Fig. 12).

To find the X Polynomial for $D_1$, we must compute the writhe of $D_1$, $w(D_1) = −1 + 1 − 1 + 1 + 1 = −1$. Notice that, this knot is equal to $c([-3, 2])$, so we have

\[X(D_1) = X(c([-3, 2])) = (−A^3)^{−w(c([-3, 2]))}\left\langle\left( c([-3, 2])\right)\right\rangle\]

\[\begin{align*}
&= [−A^3], [−(−A)^3 − A^3] + \delta^{−2}[(−A)^3 − A^3] \left( [−(−A)^3 − A^3] \right) \\
&+ A^3\left( [−(−A)^3 − A^3] \right) \\
&= [−A^3], [−A^−A^3 + \delta^{−2}−A^3 + A^3] \left( [−A^3 − A^3] \right) \\
&= A^3 + \delta^{−2}A^3 + [−A^3 − A^3] \\
&= A^3 + \delta^{−2}A^3 + [−A^3 − A^3] \\
&= A^3 + \delta^{−2}A^3 + [−A^3 − A^3] \\
&= A^3 + \delta^{−2}A^3 + [−A^3 − A^3]
\end{align*}\]

The following proposition gives the bracket of the closure of a pretzel tangle with two components.

Proposition 3. If $n_1, n_2 \in \mathbb{Z}$, then we have $\left( c[[n_1, n_2]]\right) = A^{n_1+2n_2}(\delta − \delta^{-1}) + \delta^{-1}(−A)^{n_1+2n_2}$; where $c[[n_1, n_2]]$ is the numerator closure of $(n_1, n_2)$.
Fig. 14. An oriented knot diagram $D_1$.

\[
\left< \left[3 \right] + \frac{1}{\left[\frac{1}{3} + \frac{1}{3}\right]} \right> = \left< \left[3\right] \right> \oplus \left< \left( \frac{1}{\left[\frac{1}{3} + \frac{1}{3}\right]} \right) \right> = \left[\delta^{-1}[-A^5 - (A^{-3})]|(\infty)| + (A^{-3})|(0)|\right] \\
\left[\delta^{-2}[A^{12} - (A^{-4})]|A^{15} - (A^{-5})|(\infty)\right] \\
+ \left[\delta(A^{9}) + \delta^{-1}[-A^{15} - (A^{-9})] + \delta^{-1}[A^{7} - (A^{-9})]|(0)|\right]
\]

\[
\left< \left[3 \right] + \frac{1}{\left[\frac{1}{3} + \frac{1}{3}\right]} \right> = \left[\delta^{-2}[-A^{9} - (A^{-3})]|A^{12} - (A^{-5})]|-A^{15} - (A^{-5})\right] \\
\left[\delta^{-2}[A^{9} - (A^{-3})]|A^{7} - (A^{-9})\right] \\
+ \delta^{2}[A^{9} - (A^{-7})]|-A^{15} - (A^{-5})\left]\left\langle (\infty)\right]\right\rangle \\
+ \left[\delta^{2}(A^{-12}) + \delta^{-1}[-A^{8} - (A^{-12})] + \delta^{-1}[A^{4} - (A^{-12})]\right]\rangle(0)
\]

$X(D_3) = X\left< \left( 3 \right) + \frac{1}{\left[ \frac{1}{3} + \frac{1}{3} \right]} \right> = \left( -A^3 \right)^n \left< \left( \frac{1}{\left[ \frac{1}{3} + \frac{1}{3} \right]} \right) \right> \left< \left( 3 \right) + \frac{1}{\left[ \frac{1}{3} + \frac{1}{3} \right]} \right>
$


**Definition 14.** Let $T$ be any tangle with $\langle T \rangle = \alpha(\langle \infty \rangle) + \beta(\langle 0 \rangle)$. Define $f(T) = -\frac{n}{\left(\sqrt{\gamma}\right)^n}$.

Recall that, for $n \in \mathbb{Z}$, we have $\langle n \rangle = \left\{ \sum_{i=1}^{n} \left( \frac{\alpha}{\beta} \right)^{\delta^{-1}} \right\} (\infty) + A^{-\gamma}(\langle 0 \rangle)$. See the proof of Lemma 3.

**Lemma 7.** If $T = \langle n \rangle$, then $f(\langle n \rangle) = n$; where $n \in \mathbb{Z}$.

**Proof.** Let $n \in \mathbb{Z}$, then we have

\[
f(\langle n \rangle) = -\frac{n\left(\sqrt{\gamma}\right)^{-n}}{(\sqrt{\gamma})^{-n}} = -\frac{n\left(\sqrt{\gamma}\right)^{-n}}{n\left(\sqrt{\gamma}\right)^{-n}} = n \quad \square
\]

**Lemma 8.** If $T = \frac{1}{\langle n \rangle}$, then $f\left( \frac{1}{\langle n \rangle} \right) = \frac{1}{n}$; where $n \in \mathbb{Z}$.

**Proof.** Let $n \in \mathbb{Z}$ with $\langle \langle n \rangle \rangle = \alpha(\langle \infty \rangle) + \beta(\langle 0 \rangle)$, then $\langle \frac{1}{\langle n \rangle} \rangle = \alpha^{*}(\langle 0 \rangle) + \beta^{*}(\langle \infty \rangle)$.

\[
f\left( \frac{1}{\langle n \rangle} \right) = -\frac{(\sqrt{\gamma})^n}{n(\sqrt{\gamma})^{-n}} = -\frac{(\sqrt{\gamma})^n}{n(\sqrt{\gamma})^{-n}} = \frac{1}{n} \quad \square
\]

**Lemma 9.** Let $T_1$ and $T_2$ be any two tangles, then

\[f(T_1 + T_2) = f(T_1) + f(T_2)\]

**Proof.** Let $T_1$ and $T_2$ be any two tangles with $\langle T_1 \rangle = \alpha_1(\langle \infty \rangle) + \beta_1(\langle 0 \rangle)$ and $\langle T_2 \rangle = \alpha_2(\langle \infty \rangle) + \beta_2(\langle 0 \rangle)$.

\[
\langle T_1 + T_2 \rangle = (T_1 \oplus T_2) = \left( \alpha_1(\langle \infty \rangle) + \beta_1(\langle 0 \rangle) \right) \oplus \left( \alpha_2(\langle \infty \rangle) + \beta_2(\langle 0 \rangle) \right) = \left( \alpha_1 T_1 \right) \oplus \left( \alpha_2 T_2 \right)
\]

\[
f(T_1 + T_2) = -\frac{0 + \alpha_1 \beta_1 + \beta_1 \alpha_2}{\beta_1 \beta_2} A^{\sqrt{\gamma}} \left( \alpha_1 \beta_1 + \beta_1 \alpha_2 \right) A^{\sqrt{\gamma}} = f(T_1) + f(T_2) \quad \square
\]

**Proposition 4.** Let $(n_1, n_2, \ldots, n_m)$ be any pretzel tangle, then

\[
f(n_1, n_2, \ldots, n_m) = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_m}
\]
5. Connectivity type

In this section we construct a method to compute the connectivity type for pretzel tangles, which is an invariant for pretzel tangles. A connectivity type of a tangle $T$ is a tangle $T'$ resulting from $T$ by allowing crossing changes until we get the least number of crossings. Therefore $T'$ has the same number of components and the same endings (NW, NE, SW and SE) as $T$.

Example 4. The connectivity type for $T = [2]$ is $\bigcirc$, the connectivity type for $(2,2) = | \bigcirc |$, also the connectivity type for $T = [1]$ is $[1]$ or $[-1]$.

We will denote the connectivity type of $[1]$ and $[-1]$ by $\bigcirc$.

Lemma 10. Let $(n_1, n_2, \ldots, n_k)$ be a pretzel tangle. If $n_j$ is even and $x_j, x_{j+1}, y_j$ and $y_{j+1}$ are four points as in Fig. 15, then $x_j$ will connect with $y_j$ and $x_{j+1}$ will connect with $y_{j+1}$, and if $n_j$ is odd, then $x_j$ will connect with $y_{j+1}$ and $y_j$ will connect with $x_{j+1}$ as in Fig. 15.

The proof is obvious.

Let $(n_1, n_2, \ldots, n_k)$ be a pretzel tangle, consider the following theorems.

Theorem 6. If $n_j$ is an even number, then a vertical line segment denoted by "|" will appear in the left of the connectivity type, also if $n_j$ is even, then "|" will appear in the right of the connectivity type.

Proof. Let $n_1$ and $n_2$ be two even numbers, so by Lemma 10, $x_1$ will connect with $y_1$ and $x_{k+1}$ will connect with $y_{k+1}$. Therefore "|" will appear in the left and in the right of the connectivity type (Fig. 16).

Theorem 7. If $n_1, \ldots, n_{j-1}$ are odd numbers and $n_j$ is even number, then "|" will appear in the left of the connectivity type, also if $n_j, \ldots, n_k$ are odd numbers and $n_{j-1}$ is even, then "|" will appear in the right of the connectivity type.

Proof. Suppose that $n_1, \ldots, n_{j-1}$ are odd numbers and $n_j$ is even number (Fig. 17).

Fig. 15. How strands link together in the pretzel tangles.

Fig. 16. A pretzel tangle with $n_1$ and $n_2$ which are even.

Fig. 17. A pretzel tangle with $n_1, n_2, \ldots, n_{k-1}$ are odd and $n_k$ is even.

Fig. 18. A pretzel tangle with $n_1, n_2$ are odd and $n_{k-1}$ is even.

Fig. 19. A pretzel tangle with $n_1, \ldots, n_k$ are odd and $k$ is even.

By Lemma 10, if $j$ is even, then $x_1$ connects with $y_j$ connects with $x_{j+1}$ connects with $y_1$, and if $j$ is odd, then $x_1$ connects with $y_{j+1}$ connects with $x_j$. Therefore "|" will appear in the left of the connectivity type, whether $j$ is even or odd. Assume that $n_j, \ldots, n_k$ are odd numbers and $n_{j-1}$ is even number (Fig. 18).

By Lemma 10, if $(k-j)$ is even, then $x_{k+1}$ connects with $y_j$ connects with $x_{j}$ connects with $y_1$, and $x_{j}$ connects with $y_{j+1}$ connects with $y_k$, if $(k-j)$ is odd, then $x_{k+1}$ connects with $y_{j}$ connects with $y_{j+1}$. Hence whether $(k-j)$ is even or odd, "|" will appear in the right of the connectivity type.

Theorem 8. If $n_1, \ldots, n_k$ are odd numbers and $k$ is even, then the connectivity type is $\bigcirc$.

Proof. Suppose that $(n_1, \ldots, n_k)$ is a pretzel tangle and $n_1, \ldots, n_k$ are odd numbers and $k$ is even (Fig. 19).

By Lemma 10, $x_1$ connects with $y_k$, $x_{k+1}$ connects with $y_1$, and $y_1$ connects with $x_2$ connects with $y_2$ connects with $x_3$ connects with $y_{k-1}$ connects with $x_k$, $y_k$ connects with $y_{k+1}$. Hence, the connectivity type is $\bigcirc$.

Theorem 9. If $n_1, \ldots, n_k$ are odd numbers and $k$ is odd, then the connectivity type is $\bigcirc$.
Fig. 20. A pretzel tangle with \(n_1, \ldots, n_k\) are odd and \(k\) is odd.

Fig. 21. A pretzel tangle with \(n_j\) and \(n_{j+1}\) are even.

Fig. 22. A pretzel tangle with \((m-1)\) consecutive even components.

Proof. Suppose that \((n_1, \ldots, n_k)\) is a pretzel tangle and \(n_1, \ldots, n_k\) are odd numbers and also \(k\) is odd (Fig. 20).

By Lemma 10, \(n_1, \ldots, n_k\) is connected with \(y_{j+1}\) from two sides. Therefore \(\bigcirc\) will appear in the connectivity type.

Assume that the statement is true for \((m-1)\) consecutive components all of them are even. Suppose that \(n_j, n_{j+1}, \ldots, n_{j+m}\) are all even (Fig. 22).

Since \(n_{j+1}, n_{j+2}, \ldots, n_{j+m}\) are consecutive and all of them are even, so by our assumption \((m-1)\) of \(\bigcirc\) will appear in the connectivity type, but also notice that \(x_{j+1}\) and \(y_{j+1}\) will be connected from two sides. Therefore \(m\) of \(\bigcirc\) will appear in the connectivity type. □

Theorem 10. If \((n_1, \ldots, n_k)\) is a pretzel tangle and there are \(m\)-consecutive components, all of them are even, then \((m-1)\) of \(\bigcirc\) will appear in the connectivity type.

Proof. Suppose that \(n_j\) and \(n_{j+1}\) are even (Fig. 21).

By Lemma 10, \(n_1, \ldots, n_k\) is connected with \(y_{j+1}\) from two sides. Therefore \(\bigcirc\) will appear in the connectivity type.

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Additional information

No additional information is available for this paper.

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