Impossibility of spontaneously breaking local symmetries and the sign problem

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Elitzur’s theorem stating the impossibility of spontaneous breaking of local symmetries in a gauge theory is reexamined. The existing proofs of this theorem rely on gauge invariance as well as positivity of the weight in the Euclidean partition function. We examine the validity of Elitzur’s theorem in gauge theories for which the Euclidean measure of the partition function is not positive definite. We find that Elitzur’s theorem does not follow from gauge invariance alone. We formulate a general criterion under which spontaneous breaking of local symmetries in a gauge theory is excluded. Finally we illustrate the results in an exactly solvable two dimensional abelian gauge theory.

I. INTRODUCTION

A direct method to gain exact and non-perturbative information about gauge theories is available if the weight, $\exp[-S]$, in the Euclidean partition function is positive. Suppose one can find an inequality between two physical quantities which holds for any field configuration. In that case this inequality also holds in the full theory provided that the weight in the Euclidean partition function is positive. In Quantum Chromo Dynamics (QCD) this method has been applied to the propagators of quark bilinears and the resulting QCD/Weingarten inequalities have been successful in explaining aspects of the observed hadron mass spectrum. Another notable example of the use of positivity is the Vafa-Witten theorem [2]. However, as already noted by Vafa and Witten in the original work [2], the assumed positivity of the measure is not just a technical convenience for the proof, it is actually a necessity: Vector-like symmetries can be broken in gauge theories with a non-positive measure. For example the vectorial flavor symmetry in QCD is not protected by the Vafa-Witten theorem [2].

One of the central properties of gauge theories is that local symmetries such as the local gauge symmetry itself can not break spontaneously [3]:

**Elitzur’s Theorem:** In a gauge invariant theory a local quantity with vanishing mean value on its orbit under the action of the gauge group has zero ground state expectation value.

The physical interpretation of the theorem is that spontaneous breaking of the gauge invariance can only occur after having broken the local symmetry explicitly. This is indeed what takes place in the ordinary Higgs mechanism: First one chooses a gauge and in this gauge the remaining global gauge symmetry is spontaneously broken.

In the proof of his theorem Elitzur implicitly assumed that the Euclidean measure of the partition function was positive and explicitly made use of this. In general the proofs [3, 4, 5, 6, 7] of Elitzur’s theorem are all based on the fact that inequalities which hold for any field configuration continue to hold after integrating with respect to a positive measure.

Here we consider the validity of Elitzur’s theorem in gauge theories for which the Euclidean measure is not positive. We will investigate whether the assumed positivity of the measure is just a technical convenience or if it is essential for the theorem. That is, we will examine whether gauge invariance alone is sufficient to protect local order parameters from gaining a non-zero vacuum expectation value.

The heart of the problem at hand is a double limiting process. First the volume is taken to infinity and then the gauge variant source term is taken to zero. The discussion is set in the framework of Euclidean lattice gauge theory. (For an elaborate discussion of the relation between the lattice formulation of Elitzur’s theorem and the continuum perturbative Higgs mechanism see [8].) To be specific, we follow Elitzur and consider the gauged planar spin model in $d$ dimensions. We expect, however, that the statements made generalize beyond this model. In this context we establish that local gauge symmetry alone is not sufficient to protect local symmetries from breaking spontaneously. We then formulate a new criterion under which the local symmetry can not break: If all operators which are bounded, local, as well as gauge invariant have finite vacuum expectation values then Elitzur’s theorem holds. It is then shown that gauge theories with a positive weight automatically satisfy this criterion.

To illustrate how this criterion works we examine a two dimensional pure glue $U(1)$ theory which allows for analytic evaluation. Taking the coupling to scale with the size of the system it is shown that the local gauge invariance break spontaneously. However, in accordance with the criterion we show that the vacuum expectation value of a plaquette in this theory is infinite.
The organization of this paper follows the line of thought above. In section II we define the lattice framework in which we will work. Then in section III we give a specific example which illustrates why Elitzur's theorem does not follow from gauge invariance alone. This example is then generalized in section IV and the general criterion in a pure glue $U(1)$ theory in two dimensions excludes spontaneous breaking of local gauge invariance.

II. THE GAUGED PLANAR SPIN MODEL

In order to address the fate of local gauge invariance we will consider generalized versions of the gauged planar spin model. The spin is parametrized through the angular field

$$\begin{pmatrix} \cos(\phi_i) \\ \sin(\phi_i) \end{pmatrix}. \tag{1}$$

As indicated by the index $i$ the angular field is defined on the sites of a space-time lattice. The gauge field $A_{i,m}$ lives on the link from the site $i$ in the direction $m$ and is also an angular variable. The local gauge transformation of the fields is

$$\phi_i \to \phi_i + C_i$$
$$A_{i,m} \to A_{i,m} + C_i - C_{i+m}. \tag{2}$$

Here $C_i$ is a function taking arbitrary complex values.

The action considered by Elitzur is

$$S = K \sum_{i=1}^{N} \sum_{m=1}^{d} \cos(\phi_i - \phi_{i+m} - A_{i,m}) \tag{3}$$

$$+ \frac{1}{g^2} \sum_{i=1}^{N} \sum_{m=1}^{d} \cos(A_{i,m} + A_{i+m,m} - A_{i,m+m} - A_{i,m}),$$

where $N$ is the volume, $d$ is the number of dimensions, and $K$ and $g$ are constants (for further discussion of this model see [10]). This action is by construction invariant under the local gauge transformation (2). Besides being gauge invariant the action is real and periodic in the fields. Because the action is real the weight $\exp[-S]$ in the partition function

$$Z(N, J = 0) = \int_{-\pi}^{\pi} \prod_{i=1}^{N} d\phi_i \prod_{m=1}^{d} dA_{i,m} e^{-S(\phi,A)} \tag{4}$$

is positive. In this letter we will consider general gauge invariant actions of the angular fields $\phi_i$ and $A_{i,m}$.

The evaluation of a vacuum expectation value involves a double limit. First one introduces a source $J$ for an external field $F(\phi, A)$. The vacuum expectation value of an operator $O(\phi, A)$ is then defined as

$$\langle O(\phi, A) \rangle \equiv \lim_{J \to 0} \lim_{N \to \infty} \frac{\int_{-\pi}^{\pi} \prod_{i=1}^{N} d\phi_i \prod_{m=1}^{d} dA_{i,m} e^{-S(\phi,A)O(\phi, A)} e^{iF(\phi,A)}}{Z(N, J)} \tag{5}$$

We are interested in the vacuum expectation value of a gauge variant local operator whose average over the gauge orbit is zero. Local means that $O(\phi, A)$ only depends on fields at a finite number of sites and links. As a further restriction we shall only consider a source which vanishes upon averaging over the gauge orbit. The external source will be chosen to break the local gauge invariance explicitly. Having an external magnetic field in mind we will for example choose

$$F(\phi, A) = \sum_{i=1}^{N} \sum_{m=1}^{d} \cos(A_{i,m}). \tag{6}$$

With the action (3) and this source Elitzur showed that for a fixed link $(j, \underline{n})$ the vacuum expectation value of $\cos(A_{j,\underline{n}})$ defined by

$$\langle \cos(A_{j,\underline{n}}) \rangle \equiv \lim_{J \to 0} \lim_{N \to \infty} \frac{\int_{-\pi}^{\pi} \prod_{i=1}^{N} d\phi_i \prod_{m=1}^{d} dA_{i,m} e^{-S(\phi,A)\cos(A_{j,\underline{n}})e^{iF(\phi,A)}}}{Z(N, J)} \tag{7}$$

vanishes. The choice of $O(\phi, A) \equiv \cos(A_{j,\underline{n}})$ is not essential. However, it must be a function which vanishes upon average over its gauge orbit.

The original paper by Elitzur is very clear and rather than repeating the proof we encourage the reader to consult Elitzur’s original paper [3]. Below we will give an alternative and general proof of Elitzur’s theorem when the weight is positive.

Assumptions: Here we will consider general actions describing the angular fields $\phi_i$ and $A_{i,m}$. The action is assumed to be gauge invariant but the weight $\exp[-S(\phi, A)]$ is not necessarily real and positive. We will only consider functions of the angular fields $\phi_i$ and $A_{i,m}$ which are periodic on $[-\pi, \pi]$. Moreover, we will only consider actions and sources that act locally, i.e. where the individual terms only connect nearby sites and links. Within these assumptions our main goal is to find a set of constraints under which Elitzur’s theorem holds.

III. SENSITIVITY TO LOCAL PROBES

In this section we consider a specific choice of the external field and the gauge variant local operator. We state the conditions under which the limits $N \to \infty$ and $J \to 0$ in (3) do not commute. The example is generalized in the following section.

We evaluate the vacuum expectation value (vev) of $(j$ is a fixed site and $\underline{n}$ is a fixed direction)

$$\cos(\phi_j - \phi_{j+\underline{n}}) \tag{8}$$
with the source given in (6). That is

$$\langle \cos(\phi_j - \phi_{j+\pi}) \rangle \equiv \lim_{J \to 0} \lim_{N \to \infty} \frac{\int_{-\pi}^{\pi} d\phi d\{A\} e^{-S(\phi, A)} e^{J \sum_{l,m} \cos(A_{l,m}) \cos(\phi_j - \phi_{j+\pi})}}{Z(N, J)} = \langle \cos(\phi_j - \phi_{j+\pi}) \rangle.$$

In order make this evaluation we first change variables from \((\phi_i, A_{i,m})\) to \((\phi_i, l_{i,m})\) where

$$l_{i,m} \equiv \phi_i - \phi_{i+\pi} - A_{i,m}. \quad (10)$$

The action is only a function of \(l_{i,m}\) due to gauge invariance. Furthermore, the Jacobian is field independent and cancels between numerator and denominator when evaluating the vev

$$\langle \cos(\phi_j - \phi_{j+\pi}) \rangle \equiv \lim_{J \to 0} \lim_{N \to \infty} \frac{\int_{-\pi}^{\pi} d\phi d\{l\} e^{-S(l)} \int_{-\pi}^{\pi} d\phi e^{J \sum_{l,m} \cos(l_{i+m} - \phi_i + \phi_{i+\pi}) \cos(\phi_j - \phi_{j+\pi})}}{\int_{-\pi}^{\pi} d\phi e^{-S(l)} \int_{-\pi}^{\pi} d\phi e^{J \sum_{l,m} \cos(l_{i+m} - \phi_i + \phi_{i+\pi})}}. \quad (11)$$

Because of the choice of the source the denominator does not have a term linear in \(J\). Expanding \(\exp[J \sum_{l,m} \cos(l_{i+m} - \phi_i + \phi_{i+\pi})]\) in the numerator we have

$$\int_{-\pi}^{\pi} d\phi d\phi_{j+\pi} e^{J \sum_{l,m} \cos(l_{i+m} - \phi_i + \phi_{i+\pi}) \cos(\phi_j - \phi_{j+\pi})} = J2\pi^2 \cos(l_{j+m}) + O(J^2). \quad (12)$$

Hence for small \(J\) we find

$$\langle \cos(\phi_j - \phi_{j+\pi}) \rangle = \lim_{J \to 0} \lim_{N \to \infty} \frac{J/2 \int_{-\pi}^{\pi} d\phi d\phi_{j+\pi} e^{-S(l)} \cos(l_{j+m})}{J/2 \int_{-\pi}^{\pi} d\phi d\phi_{j+\pi} e^{-S(l)}} + O(J^2). \quad (13)$$

This shows that coherence (ie. gauge invariant modification) in the numerator is possible even if it does not happen in the denominator.

If \(e^{-S(l)}\) is a real and positive function then we can use that \(|\cos(l_{j,m})| \leq 1\) to show that for all \(N\) and \(J\)

$$\frac{J/2 \int_{-\pi}^{\pi} \prod_i d_i \cos(l_{i,m})}{\int_{-\pi}^{\pi} \prod_i d_i e^{-S(l)}} \leq \frac{J/2 Z(N, J)}{Z(N, J)}. \quad (14)$$

The \(N \to \infty\) limit on the right hand side is trivial and the leading term in \(J\) of \(\langle \cos(\phi_j - \phi_{j+\pi}) \rangle\) vanishes linearly with \(J\). The same is true for higher order terms in \(J\), thus confirming Elitzur’s theorem provided that \(e^{-S(l)}\) is positive.

If \(e^{-S(l)}\) is not positive we can not draw such a conclusion. Let us go back to (14) and first perform the integral over all \(l_{i,m}\) with \(i \neq j\) and \(m \neq n\). This leaves

$$\langle \cos(\phi_j - \phi_{j+\pi}) \rangle = \lim_{J \to 0} \lim_{N \to \infty} \frac{J/2 \int_{-\pi}^{\pi} d\phi_{j+\pi} \cos(l_{j,m}) f(l_{j,m}, N)}{\int_{-\pi}^{\pi} d\phi_{j+\pi} f(l_{j,m}, N)} + O(J^2), \quad (15)$$

where

$$f(l_{j,m}, N) \equiv \int_{-\pi}^{\pi} \prod_{i \neq j, m \neq n} d_i e^{-S(l)}. \quad (16)$$

The only requirement we have imposed on the function \(f(l_{j,m}, N)\) is that it is a periodic function in \(l_{j,m}\). Hence gauge invariance and periodicity does not exclude that, say, \(f(l_{j,m}, N) = 1/N + \cos(l_{j,m})\). In this case the \(N \to \infty\) limit in (14) is infinite; the numerator being larger by a factor of \(N\) than the denominator. Therefore, the limits \(N \to \infty\) and \(J \to 0\) do not commute, signaling a possible non-trivial vev of \(\cos(\phi_j - \phi_{j+\pi})\). This example illustrates that gauge invariance alone is not sufficient to prevent local order parameters from obtaining a non-zero vev. The object is now to formulate a constraint on \(e^{-S(l)}\) which is less restrictive than positivity but nevertheless allows us to exclude spontaneous breaking of local symmetries in a gauge theory.

IV. CRITERION UNDER WHICH ELITZUR’S THEOREM HOLDS

We now generalize the example in the previous section and give a general criterion for when Elitzur’s theorem holds. Consider a local, bounded, periodic, and gauge variant function, \(O(\phi, A)\), which vanishes on average over its gauge orbit and consider a general bounded and periodic source \(F(\phi, A)\). Starting from the definition (5) and expanding in \(J\) we get

$$\langle O \rangle = \lim_{J \to 0} \lim_{N \to \infty} \frac{J \int_{-\pi}^{\pi} \prod_{i \neq j, m \neq n} d_i e^{-S(l)} g(l)}{\int_{-\pi}^{\pi} d\{l\} e^{-S(l)}} + \frac{J \int_{-\pi}^{\pi} d\{l\} e^{-S(l)} F(l)}{\int_{-\pi}^{\pi} d\{l\} e^{-S(l)}} + O(J^2) \quad (16)$$
where
\begin{equation}
    g(l) = \int_{-\pi}^{\pi} \prod_{i=1}^{N} d\phi_i F(\phi,l) O(\phi,l),
\end{equation}
\begin{equation}
    F(l) = \int_{-\pi}^{\pi} \prod_{i=1}^{N} d\phi_i F(\phi,l).
\end{equation}

In the numerator the term of order \(J^0\) vanish identically when integrating over \(\phi\) since \(O\) vanish on average over its gauge orbit. For the same reason and because \(F\) acts locally the function \(g(l)\) must be gauge invariant, periodic, bounded, and can only depend on \(i,m\) belonging to a finite part of the lattice.

Now, provided that
\begin{equation}
    \lim_{N \to \infty} \int_{-\pi}^{\pi} \prod_{i,m} d\i_m e^{-S(l)} F(l) = 0,
\end{equation}
we can drop the first term in the denominator of \(\Gamma\). In that case the factors of \(J\) will cancel and the expectation value is given by
\begin{equation}
    \langle O \rangle = \lim_{N \to \infty} \frac{\int_{-\pi}^{\pi} \prod_{i,m} d\i_m e^{-S(l)} g(l)}{\int_{-\pi}^{\pi} \prod_{i,m} d\i_m e^{-S(l)} F(l)}.
\end{equation}

With the factor of \(J\) canceling explicitly the vev is potentially non-zero. In section V we construct an example where \(g\) and \(F\) are identical and the vev is thus unity. In the example where \(F(l) = 0\) given in the previous section we required that
\begin{equation}
    \lim_{N \to \infty} \frac{\int_{-\pi}^{\pi} \prod_{i,m} d\i_m e^{-S(l)} g(l)}{\int_{-\pi}^{\pi} \prod_{i,m} d\i_m e^{-S(l)} F(l)} = 0
\end{equation}
in order to get a non-zero vev. The property \(\Gamma\) or \(\Gamma\), which is needed in order to obtain a non-zero value of \(\langle O \rangle\), has a direct physical meaning: The ratios are by definition the inverse vev’s of \(F\) and \(g\) respectively measured without any external source. Now, since \(g\) and \(F\) are arbitrary, periodic, bounded, and gauge invariant functions we can formulate

**The general criterion:** If all bounded, local, and gauge invariant operators have finite vacuum expectation values when measured without an external source then spontaneous breaking of local symmetries is excluded.

We expect that this criterion is valid beyond the present abelian planar spin models considered here. Proving this, however, is not trivial as soon as the integrations become non-compact.

An alternative way to formulate the criterion is by considering the Fourier expansion of the weight
\begin{equation}
    \exp[-S(l)] = a_0 + \sum_{i,m} a_{i,m}^1 \cos(l_{i,m}) + \sum_{i,m} a_{i,m}^2 \cos(2l_{i,m}) + \ldots
\end{equation}
\begin{equation}
    + \sum_{i,m} b_{i,m}^1 \sin(l_{i,m}) + \sum_{i,m} b_{i,m}^2 \sin(2l_{i,m}) + \ldots
\end{equation}

In the partition function the integration over the field dependent terms vanishes. That is
\begin{equation}
    a_0(N) = Z(N, J = 0)/(2\pi)^N.
\end{equation}

The other Fourier coefficients are the vacuum expectation values of the Fourier modes at zero external source. Therefore, in terms of the Fourier expansion, the criterion for establishing Elitzur’s theorem is:

*If the ratio of all Fourier coefficients of \(\exp[-S]\) and the constant mode is finite for \(N \to \infty\) then the vev of a local quantity which vanishes in average over its gauge orbit is zero.*

In order for the Fourier expansion of \(\exp[-S]\) to be convergent the coefficients must be finite in the \(N \to \infty\) limit. Hence, a necessary requirement for breaking Elitzur’s theorem is that \(Z(N, J = 0)\) is zero in the \(N \to \infty\) limit. This, however, is quite natural; \(Z(N, J = 0)\) will normally be the generating functional for some extensive quantity. Consider for example the baryon density in QCD; there we will expect that \(Z(N, J = 0) \propto \exp[-\mu_B^2 N]\) where \(\mu_B\) is the baryon chemical potential (see eg. \(\Gamma\)).

In the next section we study a two dimensional \(U(1)\) model and show analytically how the criterion excludes spontaneous breaking of local symmetries. First, however, let us show that the criterion is fulfilled automatically if \(\exp[-S]\) is positive.

**Positivity revisited:** Assuming that \(\exp[-S(l)]\) is positive we have for any bounded function with \(\max |f(l)| = \int_{-\pi}^{\pi} \prod_{i,m} d\i_m e^{-S(l)} f(l)\)
\begin{equation}
    \frac{\int_{-\pi}^{\pi} \prod_{i,m} d\i_m e^{-S(l)} g(l)}{\int_{-\pi}^{\pi} \prod_{i,m} d\i_m e^{-S(l)} f(l)} \geq \frac{1}{\int_{-\pi}^{\pi} \prod_{i,m} d\i_m e^{-S(l)} f(l)} > 0.
\end{equation}

Hence by the above criterion spontaneous breaking of local symmetries is excluded.

As for the Fourier coefficients it is trivial to show that the constant mode \(a_0\) is larger than all other modes if \(\exp[-S(l)]\) is positive. That is, the amplitudes of the oscillatory terms as compared to the constant term are restricted by the positivity of the measure. In particular the possibility that the ratio can be infinite for \(N \to \infty\) is excluded and thus \(\langle O(\phi, A) \rangle = 0\). This reestablishes Elitzur’s theorem for a positive weight in a general framework.

V. \(U(1)\) GAUGE THEORY IN 2 DIMENSIONS

In order to make the general discussion from the previous sections more concrete we now look at a pure glue \(U(1)\) gauge theory in 2 dimensions. This theory is analytically solvable even when the action is supplemented by an imaginary term. To be specific we will consider the
weight

\[ e^{-S(A)} = e^{-\beta \sum_{i \in \mathbb{Z}} \cos(Q_i)} e^{-i2 \sum_{i \in L} A_i} \]

(24)

where the plaquette is defined as

\[ \square_{i,m} = A_i + A_{i+m} - A_{i+m+n} - A_{i} \]

and \( L \) defines the contour of a Wilson loop. This complex weight has been used previously \[12\] to discuss the Langevin formulation of Monte Carlo simulations on a complex weight.

We chose to measure the vacuum expectation value of

\[ O(A) \equiv \exp[i2(A_j + A_j)] \]

(26)
in the presence of the source

\[ F(A) \equiv \exp[i2(\square_{j,k} + A_j + A_j)] \]  

(27)
The site \( j \) and the directions \( k \) and \( l \) are fixed so that the plaquette \( \square_{j,k,l} \) lies inside the contour \( L \) of the Wilson loop. Note that, the orientation of this plaquette is chosen opposite of that of the Wilson loop in the weight. We will consider extremely strong coupling, \( \beta \sim \frac{1}{N} \), and show that the local invariance, \( A_{i,m} \to A_{i,m} + C_i - C_{i,m} \), is spontaneously broken. Then we show that in this limit \( \langle \exp[2X] \rangle = \infty \) in accordance with the general criterion formulated above. We proceed as we did in section III by choosing gauge invariant coordinates (now the plaquettes) and integrating over the remaining variables. This is possible because there are twice as many links as there are plaquettes in two dimensions. To be specific we change coordinates according to

\[ \left( \begin{array}{c} A_{i,m} \\ A_{i+m,m} \\ A_{i+m,n} \\ A_{i,n} \end{array} \right) \to \left( \begin{array}{c} A_{i,m} + A_{i+n,m} + A_{i+m,n} + A_{i,n} \\ A_{i,m} + A_{i+n,m} + A_{i+m,n} - A_{i,n} \\ A_{i,m} - A_{i+n,m} - A_{i+m,n} - A_{i,n} \end{array} \right) . \]

Note that, the third coordinate simply is the plaquette. The other three coordinates are not gauge invariant. Due to the periodicity of the integrand the integration range on each of the new coordinates remains the interval \( [-\pi, \pi] \). Expanding in the source \( J \) yields

\[ \left\langle e^{i2(A_{j+k} + A_{j+l})} \right\rangle \equiv \lim_{J \to 0} \lim_{N \to \infty} \frac{\int \mathcal{D}A e^{-S(A) + J \exp[i2(\square_{j,k,l})] e^{i2A_{j+k}}} \left[ \exp[i2(-A_{j+k} - A_{j+l})] e^{-i2A_{j+k} - i2A_{j+l}} \right]}{\int \mathcal{D}A e^{-S(A) + J \exp[i2(\square_{j,k,l})] e^{i2A_{j+k}}} \left[ \exp[i2(-A_{j+k} - A_{j+l})] e^{-i2A_{j+k} - i2A_{j+l}} \right]} \]

(28)

If the second term in the denominator dominates the first then \( \left\langle \exp[i2(A_{j+k} + A_{j+l})] \right\rangle = 1 \). That is, the vev of a local and gauge variant operator which vanishes on average over its gauge orbit is non-zero. Since the gauge group is abelian and we consider two dimensions the two terms in the denominator can be evaluated analytically \[9\]. Using that for abelian theories we have

\[ e^{i \sum_{i \in \mathbb{Z}} A_{i,m}} = e^{i \sum_{i \in \mathbb{Z}} A_{i,m}} \]

(29)
one gets (\( A \) is the total area and \( A_L \) is the area of the Wilson loop \( L \))

\[ \int \mathcal{D}\{e\} e^{-S(\{e\})} e^{i2(\square_{j,k,l})} = \int \mathcal{D}\{e\} e^{-S(\{e\})} + \int \mathcal{D}\{e\} e^{-S(\{e\})} e^{i2(\square_{j,k,l})} \]

(30)

Hence, the ratio is simply a ratio of modified Bessel functions \[4, 12\]

\[ \frac{\int \mathcal{D}\{e\} e^{-S(\{e\})} e^{i2(\square_{j,k,l})}}{\int \mathcal{D}\{e\} e^{-S(\{e\})}} = \frac{\int \mathcal{D}e^{-\beta \cos(\theta)} e^{i2(\square_{j,k,l})}}{\int \mathcal{D}e^{-\beta \cos(\theta)}} = \frac{I_0(\beta)}{I_2(\beta)} \]

(31)

For small values of \( \beta \) this ratio diverges like \( 8/\beta^2 \). Consequently, we can neglect the first term in the denominator of \( \[25\] \) provided that \( \beta^2/J \ll 1 \) in the limits \( N \to \infty \) followed by \( J \to 0 \). For example, with \( \beta = 1/N \) we find that \( \left\langle \exp[i2(A_{j+k} + A_{j+l})] \right\rangle = 1 \). We emphasize that this is possible due to the complex nature of the weight and not just because we allow \( \beta \) to be of order \( 1/N \). (If \( S(\{e\}) \) is real then the first term in the denominator of \( \[25\] \) is larger than the second term for all \( J < 1 \).) In the evaluation above we have only kept track of the leading terms in \( J \). This was done in order to keep the form of the equations as close to those of the previous sections. We stress, however, that it is possible to evaluate
\[ \langle \exp[i2(A_{j, \omega} + A_{j+2, \omega})] \rangle, \text{ as given by the ratio in the first line if (28), for all values of } J \text{ and } \beta. \] The result is
\[ \langle \exp[i2(A_{j, \omega} + A_{j+2, \omega})] \rangle = \lim_{J \to 0} \lim_{N \to \infty} \frac{J \sum_{k=0}^{\infty} (Jk/k!) I_{2k}(-\beta)}{\sum_{k=0}^{\infty} (Jk/k!) I_{2(k-1)}(-\beta)}. \]

(32)

With \( \beta \propto 1/N \) in the limit \( N \to \infty \) the \( k = 0 \) term dominates in the numerator while the \( k = 1 \) term dominates in the denominator. Therefore
\[ \langle \exp[i2(A_{j, \omega} + A_{j+2, \omega})] \rangle = \lim_{J \to 0} \frac{J I_0(0)}{(J^1/1!) I_0(0)} = 1 \]

(33)
in agreement with what we found above.

In order to make the connection to the criterion formulated in the previous section we finally consider the expectation value of \( \exp[i2(D_{j, \omega})] \) on the same weight but with zero external source. This expectation value was evaluated in (31) where we found that \( \langle \exp[i2(D_{j, \omega})] \rangle \) diverges like \( 1/\beta^2 \). From this we conclude: in the case where \( \langle \exp[i2(A_{j, \omega} + A_{j+2, \omega})] \rangle = 1 \) we also have that the vev of a local, bounded, and gauge invariant operator (measured without an external source) is infinite. This example illustrates how the general criterion excludes local order parameters from getting a non-zero vev.

VI. SUMMARY

Elitzur showed that positivity of the measure and gauge invariance is sufficient to protect local symmetries from breaking spontaneously. Here we have considered abelian gauge theories with non-positive measures and have found that gauge invariance alone is not sufficient to prevent a spontaneous breaking of local symmetries. With a non-positive measure the partition function can be dominated by delicate cancellations. We have formulated a general criterion under which Elitzur’s theorem remains valid. It was then shown how in this formulation positivity of the measure implies the vanishing of local order parameters. The restriction in the criterion on the weight \( \exp[-S] \) was formulated in terms of the vacuum expectation values of local gauge invariant operators: If all bounded, local, and gauge invariant operators have finite vacuum expectation values then Elitzur’s theorem holds. Finally, we illustrated analytically how the criterion works in the case of a \( U(1) \) pure glue theory in two dimensions.

Whether the restriction in the criterion is fulfilled for QCD at non-zero baryon chemical potential or other physically relevant field theories with a non-positive Euclidean weight is at present not clear. However, any theory which has infinite expectation values for bounded, local, and gauge invariant operators is likely to be ill-defined. For instance, the infinities of the local variables can imply that also thermodynamic quantities like the baryon density are infinite.

We round off with two remarks: In the generalized gauged planar spin theories considered above the \( U(1) \) invariance allowed us to choose variables such that gauge invariance was manifest. Such a change of variables is in general not trivial to make. However, we expect that also in non-abelian gauge theories one can set up a criterion under which Elitzur’s theorem holds.

Finally, let us mention that Lüscher has constructed a proof of Elitzur’s theorem in the Hamiltonian formulation of lattice \( U(1) \) and \( SU(2) \) pure Yang-Mills theories. Perhaps that line of work can be extended to a gauge theory with dynamical fermions and maybe even to QCD at non-zero baryon chemical potential. Such an extension may cast light on additional physical constraints on lattice gauge theory with a non-positive weight in the partition function.

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