HERON TRIANGLE AND RHOMBUS PAIRS WITH A COMMON AREA AND A COMMON PERIMETER

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Abstract. By Fermat’s method, we show that there are infinitely many Heron triangle and $\theta$-integral rhombus pairs with a common area and a common perimeter. Moreover, we prove that there does not exist any integral isosceles triangle and $\theta$-integral rhombus pairs with a common area and a common perimeter.

1. Introduction

We say that a Heron (resp. rational) triangle is a triangle with integral (resp. rational) sides and integral (resp. rational) area. And a rhombus is $\theta$-integral (resp. $\theta$-rational) if it has integral (resp. rational) sides, and both $\sin \theta$ and $\cos \theta$ are rational numbers.

In 1995, R. K. Guy [5] introduced a problem of Bill Sands, that asked for examples of an integral right triangle and an integral rectangle with a common area and a common perimeter, but there are no non-degenerate such. In the same paper, R. K. Guy showed that there are infinitely many such integral isosceles triangle and rectangle pairs. In 2006, A. Bremner and R. K. Guy [1] proved that there are infinitely many such Heron triangle and rectangle pairs. In 2016, Y. Zhang [6] proved that there are infinitely many integral right triangle and parallelogram pairs with a common area and a common perimeter. At the same year, S. Chern [2] proved that there are infinitely many integral right triangle and $\theta$-integral rhombus pairs. In a recent paper, P. Das, A. Juyal and D. Moody [3] proved that there are infinitely many integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs with a common area and a common perimeter.

By Fermat’s method [4, p. 639], we can give a simple proof of the following result, which is a corollary of Theorem 2.1 in [3].

Theorem 1.1. There are infinitely many Heron triangle and $\theta$-integral rhombus pairs with a common area and a common perimeter.

But for integral isosceles triangle and $\theta$-integral rhombus pair, we have

Theorem 1.2. There does not exist any integral isosceles triangle and $\theta$-integral rhombus pairs with a common area and a common perimeter.

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2. Proofs of the theorems

Proof of Theorem 1.1. Suppose that the Heron triangle has sides \((a, b, c)\), and the \(\theta\)-integral rhombus has side \(p\) and intersection angle \(\theta\) with \(0 < \theta \leq \pi/2\). By Brahmagupta’s formula, all Heron triangles have sides
\[(a, b, c) = ((v + w)(u^2 - vw), v(u^2 + w^2), w(u^2 + v^2)),\]
for positive integers \(u, v, w\), where \(u^2 > vw\).

Noting that the homogeneity of these sides, we can set \(w = 1\), and \(u, v, p\) be positive rational numbers. Now we only need to study the rational triangle and \(\theta\)-rational rhombus pairs with a common area and a common perimeter, then we have

\[
\begin{align*}
uv(v + 1)(u^2 - v) &= p^2 \sin \theta, \\
2u^2(v + 1) &= 4p.
\end{align*}
\]

Since both \(\sin \theta\) and \(\cos \theta\) are rational numbers, we may set
\[
\sin \theta = \frac{2t}{t^2 + 1}, \quad \cos \theta = \frac{t^2 - 1}{t^2 + 1},
\]
where \(t \geq 1\) is a rational number. For \(t = 1\), \(\theta = \pi/2\), this is the case studied by R. K. Guy [5]. Thus it only needs to consider the case \(t > 1\).

Eliminating \(p\) in Eq. (2.1), we have
\[
\frac{u(v + 1)(2t^2u^2v - tu^3v - 2t^2v^2 - tu^3 + 2u^2v - 2v^2)}{2(t^2 + 1)} = 0.
\]

Let us study the rational solutions of the following equation
\[
2t^2u^2v - tu^3v - 2t^2v^2 - tu^3 + 2u^2v - 2v^2 = 0,
\]
and solve it for \(v\), we get
\[
v = \frac{(2t^2u - tu^2 + 2u \pm \sqrt{g(t)})u}{4(t^2 + 1)},
\]
where
\[
g(t) = 4u^2t^4 - 4u(u^2 + 2)t^3 + u^2(u^2 + 8)t^2 - 4u(u^2 + 2)t + 4u^2.
\]

In view of \(v\) is a positive rational number, then \(g(t)\) should be a rational perfect square. So we need to consider the rational points on the curve
\[
C_1 : s^2 = g(t).
\]

The curve \(C_1\) is a quartic curve with a rational point \(P = (0, 2u)\). By Fermat’s method [4, p. 639], using the point \(P\) we can produce another point \(P' = (t_1, s_1)\), which satisfies the condition \(t_1s_1 \neq 0\). In order to construct a such point \(P'\), we put
\[
s = rt^2 + qt + 2u,
\]
where \(r, q\) are indeterminate variables. Then
\[
s^2 - g(t) = \sum_{i=1}^{4} A_i t^i,
\]
where the quantities $A_i = A_i(r, q)$ are given by

\[

t_1 = \frac{2u(u^2 + 2)}{3u^2 - 1},
\]

\[

\begin{align*}
A_1 &= 4u^3 + 4qu + 8u, \\
A_2 &= -4u^2 + 4ru + q^2 - 8u^2, \\
A_3 &= 4u^3 + 2rq + 8u, \\
A_4 &= r^2 - 4u^2.
\end{align*}
\]

The system of equations $A_3 = A_4 = 0$ in $r, q$ has a solution given by

\[
r = -2u, q = u^2 + 2.
\]

This implies that the equation

\[
s^2 - g(t) = \sum_{i=1}^{4} A_i t^i = 0
\]

has the rational roots $t = 0$ and

\[
t = \frac{2u(u^2 + 2)}{3u^2 - 1}.
\]

Then we have the point $P'(t_1, s_1)$ with

\[
\begin{align*}
t_1 &= \frac{2u(u^2 + 2)}{3u^2 - 1}, \\
s_1 &= -\frac{2u(u^6 - 4u^4 + 14u^2 + 3)}{(3u^2 - 1)^2}.
\end{align*}
\]

Put $t_1$ into Eq. (2.2), we get

\[
v = \frac{u^2(u^2 + 2)}{4u^4 + 1}.
\]

Hence, the rational triangle has rational sides $(a, b, c) = \left(\frac{u^2(3u^2 - 1)(u^4 + 6u^2 + 1)}{(4u^2 + 1)^2}, \frac{u^2(u^2 + 2)(u^2 + 1)}{4u^2 + 1}, \frac{u^2(u^6 + 20u^4 + 12u^2 + 1)}{(4u^2 + 1)^2}\right)$.

From the equation $2u^2(v + 1) = 4p$ and $\sin \theta = \frac{2u}{r+1}$, we obtain the corresponding rhombus with side

\[
p = \frac{(u^4 + 6a^2 + 1)u^2}{2(4u^2 + 1)},
\]

and the intersection angle

\[
\theta = \arcsin \frac{4u(u^2 + 2)(3u^2 - 1)}{(4u^2 + 1)(u^4 + 6u^2 + 1)}.
\]

In view of $u, v, p$ are positive rational numbers, $0 < \sin \theta < 1$, $u^2 > v$ and $t_1 > 1$, we get the condition

\[
u > \frac{\sqrt{3}}{3}.
\]

Then for positive rational number $u > \frac{\sqrt{3}}{3}$, there are infinitely many rational triangle and $\theta$-rational rhombus pairs with a common area and a common perimeter. Therefore, there are infinitely many such Heron triangle and $\theta$-integral rhombus pairs. \qed
Example 2.1. (1) If \( u = 1 \), we have a Heron triangle with sides \((8, 15, 17)\), and a \( \theta \)-integral rhombus with side 10 and the smaller intersection angle \( \arcsin(3/5) \), which have a common area 60 and a common perimeter 40.

(2) If \( u = 2 \), we have a Heron triangle with sides \((1804, 2040, 1732)\) and a \( \theta \)-integral rhombus with side 1394 and the smaller intersection angle \( \arcsin(528/697) \), which have a common area 1472064 and a common perimeter 5576.

Proof of Theorem 1.2. As before, we only need to consider the rational isosceles triangle and \( \theta \)-rational rhombus pairs. As in [3], we may take the equal legs of the isosceles triangle to have length \( u^2 + v^2 \), with the base being \( 2(u^2 - v^2) \) and the altitude \( 2uv \), for some rational \( u, v \). The area of the isosceles triangle is \( 2uv(u^2 - v^2) \), with an perimeter of \( 4u^2 \).

Let \( p \) be the length of the side of the rhombus, and \( \theta \) its smallest interior angle. For \( \theta \)-rational rhombus, we have the perimeter \( 4p \) and area \( p^2 \sin \theta \), where \( \sin \theta = 2t/(t^2 + 1) \), for some \( t \geq 1 \).

If the rational isosceles triangle and \( \theta \)-rational rhombus have the same area and perimeter, then

\[
\begin{align*}
2uv(u^2 - v^2) &= p^2 \sin \theta, \\
4u^2 &= 4p.
\end{align*}
\]

From Eq. (2.3), we obtain

\[
\frac{2u(v(u-v)(u+v) - u^3t + v(u-v)(u+v))}{t^2 + 1} = 0.
\]

It only needs to consider \( v(u-v)(u+v) = 0 \). If this quadratic equation has rational solutions \( t \), then its discriminant should be a rational perfect square, i.e.,

\[
u^6 - 4u^4v^2 + 8u^2v^4 - 4v^6 = w^2.
\]

Let \( U = u/v, W = w/v^3 \), we have

\[
W^2 = U^6 - 4U^4 + 8U^2 - 4.
\]

This is a hyperelliptic sextic curve of genus 2. The rank of the Jacobian variety is 1, and Magma’s Chabauty routines determine the only finite rational points are

\[(U, W) = (\pm 1; \pm 1),\]

which lead to

\[(u, w) = (\pm v, \pm v^3),\]

then we get

\[u^3t = 0.\]

So Eq. (2.3) does not have nonzero rational solutions, which means that there does not exist any integral isosceles triangle and \( \theta \)-integral rhombus pairs with a common area and a common perimeter. □
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