Understanding Generalization in Deep Learning via Tensor Methods

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Abstract

Deep neural networks generalize well on unseen data though the number of parameters often far exceeds the number of training examples. Recently proposed complexity measures have provided insights to understanding the generalizability in neural networks from perspectives of PAC-Bayes, robustness, overparametrization, compression and so on. In this work, we advance the understanding of the relations between the network’s architecture and its generalizability from the compression perspective. Using tensor analysis, we propose a series of intuitive, data-dependent and easily-measurable properties that tightly characterize the compressibility and generalizability of neural networks; thus, in practice, our generalization bound outperforms the previous compression-based ones, especially for neural networks using tensors as their weight kernels (e.g. CNNs). Moreover, these intuitive measurements provide further insights into designing neural network architectures with properties favorable for better/guaranteed generalizability. Our experimental results demonstrate that through the proposed measurable properties, our generalization error bound matches the trend of the test error well. Our theoretical analysis further provides justifications for the empirical success and limitations of some widely-used tensor-based compression approaches. We also discover the improvements to the compressibility and robustness of current neural networks when incorporating tensor operations via our proposed layer-wise structure.

1 Introduction

Deep neural networks recently have made major breakthroughs in solving many difficult learning problems, especially in image classification [Simonyan and Zisserman, 2014; Szegedy et al., 2015; He et al., 2016; Zagoruyko and Komodakis, 2016] and object recognition [Krizhevsky et al., 2012; Sermanet et al., 2013; Simonyan and Zisserman, 2014; Zeiler and Fergus, 2014]. The success of deep neural networks depends on the high expressive power and the ability to generalize. The high expressive power has been demonstrated empirically [He et al., 2016; Zagoruyko and Komodakis, 2016] and theoretically [Hornik et al., 1989; Mhaskar and Poggio, 2014]. Yet, fundamental questions on why deep neural networks generalize and what enables their generalizability remain unsettled.

A recent work by [Arora et al., 2018] characterizes the generalizability of a neural network from a compression perspective — the capacity of the network is characterized through its compressed version. The compression algorithm in [Arora et al., 2018] is based on random projection: each weight matrix of the compressed network are represented by a linear combination of basis matrices with entries i.i.d. sampled from $\pm 1$. The effective number of parameters in the weight matrix is the number of coefficients in this linear combination obtained via projection — the inner product between the original weight matrix and these basis matrices. Though the idea of using compression in deriving the generalization bounds is novel, the compression scheme in [Arora et al., 2018] could be made more practical since (1) the cost of forwarding pass in the compressed network still remains the same as the cost in the original one, even though the effective number of parameters to represent the original weight matrices decreases; (2) storing these random projection matrices could require more spaces than storing the original set of parameters. We propose a new theoretical analysis based on a more practical, well-
developed, and principled compression scheme using tensor methods. Besides, we use tensor analysis to derive a much tighter bound for the layer-wise error propagation by exploiting additional structures in the weight tensors of neural networks, which as a result significantly tightens the generalization error bound in Arora et al. (2018).

Our approach aims to characterize the network’s compressibility by measuring the low-rankness of the weight kernels. Existing compression methods in (Jaderberg et al., 2014; Denton et al., 2014; Lebedev et al., 2014; Kim et al., 2015; Garipov et al., 2016; Wang et al., 2018; Su et al., 2018) implement low-rank approximations by performing matrix/tensor decomposition on weight matrices/kernels of well-trained models. However, the layers of SOTA networks, such as VGG (Simonyan and Zisserman, 2014) and WRN (Zagoruyko and Komodakis, 2016), are not necessarily low-rank: we apply CP-tensor decompositions (Kolda and Bader, 2009; Anandkumar et al., 2014b; Huang et al., 2015; Li and Huang, 2018) to the weight tensors of well-trained VGG-16 and WRN-28-10, and the amplitudes of the components from the CP decomposition (a.k.a CP spectrum) are demonstrated by the brown curves in Figure 1, which indicate that the layers of these pre-trained networks are not low-rank. Therefore, a straightforward compression of the network cannot be easily achieved and computationally expensive fine tuning is often needed.

**Figure 1:** CP spectrum comparison (CP-VGG and CP-WRN are neural networks with CP layers).

To overcome this limitation, we propose a layer-wise structure design, CP Layer (CPL), by incorporating the variants of CP decompositions in (Jaderberg et al., 2014; Kossaifi et al., 2017; Howard et al., 2017). CPL re-parametrizes the weight tensors such that a Polyadic form (CP form) (Kolda and Bader, 2009) can be easily learned in an end-to-end fashion.

We demonstrate that empirically, CPL allows the network to learn a low-rank structure more easily, and thus helps with compression. For example, from the pink curves in Figure 1 we see that neural networks with CPL have a spiky CP spectrum, which is an indication of low-rankness. We rigorously prove that this low-rankness in return leads to a tighter generalization bound. Moreover, we are the first to provide theoretical guarantees for the usage of CP decomposition in deep neural networks in terms of compressibility and generalizability.

**Definition 1.1 (Proposed Architecture Layer).** A CP Layer (CPL) with width $R$ consists of $R$ set of parameters $\{\lambda^{(r)}, \{v_j^{(r)}\}^N_{j=1}\}$ where $v_j^{(r)}$ is a vector in $\mathbb{R}^{d_j}$ with unit norm. The weight kernel of this CPL is a $N$-order tensor defined as $K := \sum_{r=1}^R \lambda^{(r)} v_1^{(r)} \otimes \cdots \otimes v_N^{(r)}$, where $\otimes$ denotes the vector outer-product (tensor product) defined in Appendix B.3.9. Note that $K \in \mathbb{R}^{d_1 \times \cdots \times d_N}$.

**Remark.** CPL allows for flexible choices of the structures since the number of components $R$ is a tunable hyper-parameter that controls the number of parameters in CPL. The CP spectrum of this layer is denoted by $\{\lambda^{(r)}\}^R_{r=1}$ in a descending order. The size of the weight kernel is $d_0 \times d_1 \times \cdots \times d_N$, while the number of parameters in CPL is $(d_0 + d_1 + \cdots + d_N + 1) \times R$.

In contrast with existing works which apply CP decomposition to each layer of a reference network, no CP decomposition is needed since the components are explicitly stored as model parameters so that they can be learned from scratch via back-propagation. Moreover, compression in CP layers is natural — simply picking the top $R$ components to retain and pruning out the rest of them. Thus, the compression procedure using CPL does not require any costly fine-tuning while existing works on tensor-based compression may use hundreds of epochs for fine-tuning.

We further propose a series of simple, intuitive, data-dependent and easily-measurable properties to measure the low-rankness in current neural networks. These properties not only guide the selection of the number of components to generate a good compression, but also tighten the bound of the layer-wise error propagation via tensor analysis. The proposed properties:

- characterize the compressibility of the neural network, i.e., how much the original network can be compressed without compromising the performance on a training dataset more than certain range.
- characterize the generalizability of the compressed network, i.e., tell if a neural network is trained using normal data or corrupted data.

In our theoretical analysis, we derive generalization error bounds for neural networks with CP layers, which take both the input distribution and the compressibility of the network into account. We present a rigorous proof showing the connection of our proposed proper-
ties to the generalization error of a network. We will see in experiment section that our proposed bound is very effective at predicting the generalization error.

Notice that, in this paper, the Polyadic form is chosen simply as a demonstration on how tensor methods could be used to improve the analysis of generalization bounds of deep neural networks. Therefore, follow-ups works could potentially analyze the effects of other tensor decomposition methods using our theoretical framework.

Summary of Contributions

1. Better generalization bound of practical use. We verify that our generalization bounds can be used to guide the training of neural networks, since the calculated bound matches the trend of the test error on unseen data during the training process as shown in Figure 2a. Moreover, we demonstrate that our generalization bound is in practice tighter than the bound proposed by (Arora et al., 2018) as shown in Figure 2a and Table 4. Notice that the generalization bound in (Arora et al., 2018) is already orders of magnitude better than previous norm-based or compression based bounds.

2. Intuitive measurements of compressibility and generalizability. We propose a set of properties to characterize the low-rankness in the weight tensors of neural networks in Section 4.2. Our theoretical analysis connects the measured low-rankness with the generalizability of the model, and such connections are verified in Figure 3.

3. First theoretical guarantee on the generalizability and robustness for neural network architectures that allow fast and real time predictions on devices with limited memory (e.g. the architecture designs proposed in (Jaderberg et al., 2014; Kossaifi et al., 2017, Howard et al., 2017), which uses variants of the Polyadic form).

4. Practical improvements. We demonstrate that pruning out the smaller components of CP decomposition in CP layers roughly preserves the test performance without computationally expensive fine tuning (see Section 5.3 and Table 5) as our proposed layer-wise structure is easily compressible. Moreover, we discover that incorporating tensor operations via CPL reduces the generalization error of some well-known neural network architectures, and further improves the robustness of SOTA methods for learning under noisy labels (see Table 2, Table 3, Figure 5 and Figure 6).

2 Related Works

Existing Metrics to Characterizing Generalization. Classical and recent works have analyzed the generalizability of neural networks from different perspective such as VC-dimension (Bartlett et al., 1999; Harvey et al., 2017), sharpness of the solution (Keskar et al., 2016), robustness of the algorithm (Xu and Mannor, 2012), stability and robustness of the model (Hardt et al., 2016; Kuzborskij and Lampert, 2015a), Gonen and Shalev-Shwartz, 2017; Sokolic et al., 2016), and over-parameterization (Neyshabur et al., 2018; Du and Lee, 2018), or using various approaches such as PAC-Bayes theory (McAllester, 1999a,b; Langford and Caruana, 2002; Neyshabur et al., 2015b, 2017b; Dziugaite and Roy, 2017; Golowich et al., 2018), norm-based analysis (Bartlett and Mendelson, 2002; Neyshabur et al., 2015a; Kawaguchi et al., 2017; Golowich et al., 2017), compression based approach (Arora et al., 2018), and combinations of the above approaches (Neyshabur et al., 2017b; Bartlett et al., 2017; Zhou et al., 2018) (see (Jakubovitz et al., 2018) for a complete survey). While these works provide deep theoretical insights to the understanding of the generalizability in neural networks, they did not provide practical techniques to improve generalization.

For the progress on non-vacuous generalization bounds, Dziugaite and Roy (2017) use non-convex optimization and PAC-Bayesian analysis to obtain a non-vacuous sample bound on MNIST, and Zhou et al. (2018) use a PAC-Bayesian compression approach to obtain non-vacuous generalization bounds on both MNIST and ImageNet via smart choices of the prior. While being creative, both bounds are less intuitive and provide little insight into what properties are favorable for networks to have better generalizability. In addition, the tensor-based compression methods are complementary to the compression approach used in (Zhou et al., 2018), which combines pruning, quantization and huffman coding (Han et al., 2015); the tensor-based compression methods can be combined with the approaches used in (Han et al., 2015) to potentially tighten the generalization bound obtained in (Zhou et al., 2018).

Improving generalization in practice. Authors of (Neyshabur et al., 2015a) proposed an optimization method PATH-SGD which improves the generalization performance empirically. While (Neyshabur et al., 2015a) focuses on the optimization approach, we provide a different practical approach that helps the understanding of the relations between the network architecture and its generalization ability.
Comparison with Arora et al. (Arora et al., 2018). Besides practical improvements of generalization error, their work improves the results obtained by (Arora et al., 2018): 1) they provide a tightened layer-wise analysis using tensor methods to directly bound the operator norm of the weight kernel (e.g. Lemma C.5 and Lemma C.8). The interlayer properties introduced by (Arora et al., 2018) are orthogonal to our proposed layer-wise properties and they can be well-combined; 2) in practice, our bound outperforms that of (Arora et al., 2018) in terms of the achieved degree of compression (detailed discussions in Section 5.2 and Section A.2); 3) for fully connected (FC) neural networks, our proposed reshaping factor (definition 4.2) further tightens the generalization bound as long as the inputs to the FC layers have some low-rank structures; 4) we extend our theoretical analysis to neural networks with skip connections, while the theoretical analysis in (Arora et al., 2018) only applies to FC and CNN.

Comparison with existing CP decomposition for network compression. While CP decomposition has been commonly used in neural network compression (Denton et al., 2014; Lebedev et al., 2014; Kossaifi et al., 2017), our proposed compression method is very different from theirs. First, the tensor contraction layer (Kossaifi et al., 2017) is a special case of our CPL for FC layers when we set the number of components to be 1. Second, the number of components in our proposed CPL can be arbitrarily large (as it is a tunable hyper-parameter), while the number of components of layers in (Denton et al., 2014; Lebedev et al., 2014; Kossaifi et al., 2017) are determined by the compression ratio. Third, no tensor decomposition is needed for evaluating the generalizability and compressing neural networks with CP layers as the components from the CP decomposition are already stored as model parameters. Moreover, as the smaller components in CPL are pruned during the compression, the performance of the compressed neural net is often preserved and thus no expensive fine tuning is required (see Table 3). The depthwise-separable convolution used in MobileNet (Howard et al., 2017) is a specific implementation of CPL; thus, our theoretical analysis can provide generalization guarantees for the MobileNet architecture.

3 Notations and Preliminaries

In this paper, we use $S$ to denote the set of training samples drawn from a distribution $D$ with $|S| = m$. Let $n$ denote the number of layers in a given neural network, and superscripts of form $(k)$ denote properties related to the $k^{th}$ layer. We put “CP” in front of a network’s name to denote such a network with CP layers (e.g. CP-VGG denotes a VGG with CP layers). For any positive integer $n$, let $[n] := \{1, 2, ..., n\}$. Let $|a|$ denote the absolute value of a scalar $a$. Given a vector $a \in \mathbb{R}^d$, a matrix $A \in \mathbb{R}^{d \times k}$, and a tensor $A \in \mathbb{R}^{d_1 \times d_2 \times d_3}$, their norms are defined as follows: (1) Vector norm: $\|a\|$ denotes the $\ell_2$ norm. (2) Matrix norms: Let $\|A\|_F$ denote its nuclear norm, $\|A\|_F$ denote its Frobenius norm, and $\|A\|$ denote its operator norm (spectral norm), where $\sigma_i(A)$ denotes the $i^{th}$ largest singular value of $A$. (3) Tensor norms: Let $\|A\| = \max_{x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}, z \in \mathbb{R}^{d_3}} |A(x,y,z)|$ denote its operator norm, and $\|A\|_F$ its Frobenius norm. Moreover, we use $\otimes$ to denote the outer product operator, and $*$ to denote the convolution operator. We use $F_m$ to denote $m$-dimensional discrete Fourier transform, and use tilde symbols to denote tensors after DFT (e.g. $\hat{f} = F_m(f)$). A Polyadic decomposition (CP decomposition) (Kruskal, 1989) of a $N$-order tensor $K \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_N}$ is a linear combination of rank-one tensors that is equal to $K = \sum_{r=1}^{R} \lambda^{(r)} v_1^{(r)} \otimes \cdots \otimes v_N^{(r)}$ where $\forall r \in [R], v_j \in [N], \|v_j^{(r)}\| = 1$. Margin loss (Arora et al., 2018): we use $L_\gamma(M)$ and $\hat{L}_\gamma(M)$ to denote the expected and empirical margin loss of a neural network $M$ with respect to a margin $\gamma \geq 0$. The expected margin loss of a neural network $M$ is defined as $L_\gamma(M) := \mathbb{P}_{(x,y) \in D}[M(x)[y] \leq \gamma + \max_{i \neq y} M(x)[i]]$.

4 CNNs with CPL: Compressibility and Generalization

In this section, we derive the generalization bound for a convolutional neural network (denoted as $M$) using tensor methods and standard Fourier analysis. The complete proof is in Appendix Section D. For simplicity, we assume that there is no pooling layer (e.g. max pooling) in $M$ since adding pooling layer will only lead to a smaller generalization bound (the perturbation error in our analysis decreases with the presence of pooling layers). The derived generalization bound can be directly extended to various neural network architectures (e.g. neural networks with pooling layers, and neural networks with batch normalization). The generalization bounds for fully connected neural networks and neural networks with skip connections are presented in Appendix Section E.4 and E.5 respectively.

4.1 Compression of a CNN with CPL

We first illustrate how to compress any given CNN $M$ by presenting a compression algorithm (Algorithm 1).
We will see that this compression algorithm guarantees a good estimation of the generalization bound for the compressed network $\tilde{M}$.

**Original CNN $\tilde{M}$** is of $n$ layers with ReLU activation, its $k^{th}$ layer weight tensor $M^{(k)}$ is a $4^{th}$ order tensor of size $= \#$ of input channel $s^{(k)} \times \#$ of output channel $o^{(k)} \times \text{kernel height} k_x^{(k)} \times \text{kernel width} k_y^{(k)}$. Let the $3^{rd}$ order tensor $\lambda^{(k)} \in \mathbb{R}^{s^{(k)} \times o^{(k)} \times k_x^{(k)} \times k_y^{(k)}}$ denote the input to the $k^{th}$ layer, and $\gamma^{(k)} \in \mathbb{R}^{o^{(k)} \times o^{(k)} \times k_x^{(k)} \times k_y^{(k)}}$ denote the output of the $k^{th}$ layer before activation. Therefore $\lambda^{(k)} = \text{ReLU} \left( \gamma^{(k-1)} \right)$. We use $i$ to denote the index of input channels, and $j$ to denote the index of output channels. We further use $f$ and $g$ to denote the indices of width and height in the frequency domain.

**Proposition 4.1** (Polyadic Form of original CNN $\tilde{M}$). For each layer $k$, the weight tensor $M^{(k)}$ has a Polyadic form with number of components $R^{(k)} \leq \min \{ s^{(k)} \cdot o^{(k)} \cdot k_x^{(k)} \cdot k_y^{(k)} \}$\cite{Kolda+Bader:09}. $M^{(k)} = \sum_{r=1}^{R^{(k)}} \lambda^{(k)}(r)^{a^{(k)}(r)} \otimes b^{(k)}(r) \otimes C^{(k)}(r)$, where the CP-spectrum is in a descending order, i.e., $\lambda^{(k)}_1 \geq \lambda^{(k)}_2 \geq \cdots \geq \lambda^{(k)}_{R^{(k)}}$. All $a^{(k)}(r)$, $b^{(k)}(r)$ are unit vectors in $\mathbb{R}^{s^{(k)}}$ and $\mathbb{R}^{o^{(k)}}$ respectively, and $C^{(k)}(r)$ is a matrix in $\mathbb{R}^{k_x^{(k)} \times k_y^{(k)}}$ with $\| C^{(k)}(r) \| = 1$. The $R^{(k)}$ is required for the Polyadic Form is called tensor rank.

**Transform original CNN to a CNN with CP layers.** By Proposition 4.1 each weight tensor $M^{(k)}$ in $\tilde{M}$ can be represented in a Polyadic form (CP form) and thus is transformed to a CPL. The total number of parameters in CPL is $R^{(k)} \times (s^{(k)} + o^{(k)} + k_x^{(k)} + k_y^{(k)}) + 1$. Thus, a smaller $R^{(k)}$ leads to fewer number of effective parameters and indicates more compression.

**Compress Original CNN $\tilde{M}$ to $\tilde{M}$**. We illustrate the compression procedure in Algorithm 1. Feeding a CNN $\tilde{M}$ to the compression algorithm, we obtain a compressed CNN $\tilde{M}$, where for each layer $k$, the weight tensor in $\tilde{M}$ is $\tilde{M}^{(k)} = \sum_{r=1}^{R^{(k)}} \lambda^{(k)}(r)^{a^{(k)}(r)} \otimes b^{(k)}(r) \otimes C^{(k)}(r)$ for some $R^{(k)} \leq R^{(k)}$. Similarly, we use $\hat{\lambda}^{(k)}$ to denote the input tensor of the $k^{th}$ layer in $\tilde{M}$ and $\hat{\gamma}^{(k)}$ to denote the output tensor of the $k^{th}$ layer in $\tilde{M}$ before activation. Therefore $\hat{\lambda}^{(k)} = \text{ReLU} \left( \hat{\gamma}^{(k-1)} \right)$. Notice that $\hat{\lambda}^{(k)}$, $\hat{\gamma}^{(k)}$ are of the same shapes as $\lambda^{(k)}$, $\gamma^{(k)}$ respectively and $\lambda^{(1)} = \hat{\lambda}^{(1)}$ since the input data to both networks $\tilde{M}$ and $\tilde{M}$ is the same.

The compression Algorithm 1 is designed to compress any CNN, and therefore requires applying explicit CP decompositions to the weight tensors of traditional CNNs (the step 3 in Algorithm 1). However, for a CNN with CP layers, these CP components are already stored as weight parameters in our CPL structure, and thus are known to the compression algorithm in advance. Therefore, no tensor decomposition is needed when compressing CNNs with CPL as we can prune out the components with smaller amplitudes directly.

**Algorithm 1 Compression of Convolutional Neural Networks**

1. **FBRC** in Appendix B calculates a set of number of components $\{ R^{(k)} \}_{k=1}^n$ for the compressed network such that $\| M(X) - \tilde{M}(X) \|_F \leq \epsilon \| M(X) \|_F$ holds for any given $\epsilon$ and for any input $X \in \text{the training dataset } S$.

2. **CNN-Project** (in Appendix C) takes a given set of number of components $\{ R^{(k)} \}_{k=1}^n$ and returns a compressed network $\tilde{M}$ by pruning out the smaller components in the CP spectrum of the weight tensors of $\tilde{M}$.

More intuitions of the sub-procedures FBRC and CNN-Project are described in Section 4.2 and Appendix C.

**Input:** A CNN $\tilde{M}$ of $n$ layers and a margin $\gamma$

**Output:** A compressed $\tilde{M}$ whose expected error $\tilde{L}_0(\tilde{M}) \leq \tilde{L}_\gamma(\tilde{M}) + \tilde{O} \left( \frac{\sum_{k=1}^{n} R^{(k)}(s^{(k)} + o^{(k)} + k_x^{(k)} + k_y^{(k)})}{m} \right)$

1. Calculate all layer cushions $\{ \zeta^{(k)} \}_{k=1}^n$ based on definition 4.4

2. Pick $R^{(k)} = \min \{ s^{(k)} \cdot o^{(k)} \cdot k_x^{(k)} \cdot k_y^{(k)} \}$ for each layer $k$

3. If $M$ does not have CPL, apply a CP-decomposition to the weight tensor of each layer $k$

4. Set the perturbation parameter $\epsilon := \frac{\gamma}{2 \max_{X} \| M(X) \|_F}$

5. Compute number of components needed for each layer of the compressed network $\left\{ R^{(k)} \right\}_{k=1}^n$ ← \text{FBRC} $\left( \{ M^{(k)} \}_{k=1}^n, \{ R^{(k)} \}_{k=1}^n, \{ \zeta^{(k)} \}_{k=1}^n, \epsilon \right)$

6. $\tilde{M} ← \text{CNN-Project} \left( \tilde{M}, \{ R^{(k)} \}_{k=1}^n \right)$

7. Return the compressed convolutional neural network $\tilde{M}$

**4.2 Characterizing Compressibility of CNN with CPL: Network Properties**

In this section, we propose the following layer-wise properties that can be evaluated based on the training data $S$: tensorization factor (TF), tensor noise bound (TNB), and layer cushion (LC)\cite{Arora et al. 2018}. These proposed properties are very effective at characterizing the compressibility of a neural network. As Algorithm 1’s sub-procedure FBRC selects a set of number of components $\{ R^{(k)} \}_{k=1}^n$ to obtain a compressed network $\tilde{M}$ whose output is similar to that of the original network (i.e., $\| M(X) - \tilde{M}(X) \|_F \leq \epsilon \| M(X) \|_F$ for any input $X \in S$), our proposed properties will assist the selections of $\{ R^{(k)} \}_{k=1}^n$ to guarantee that Algorithm 1 returns a “good” compressed network.
Remark. The tensorization factor characterizes both the generalizability and the expressive power of a given network. For a fixed $j$, a smaller tensorization factor indicates the original network is more compressible and thus has a smaller generalization bound. However, a smaller tensorization factor may also indicate that the given network do not possess enough expressive power. Thus, during the compression of a neural network with good generalizability, we need to find a “good” $j$ that generates a tensorization factor demonstrating the balance between a small generalization gap and high expressive power.

Definition 4.3. [Tensor noise bound $\xi_j^{(k)}$] The tensor noise bound $\{\xi_j^{(k)}\}_{j=1}^{R(k)}$ of the $k$th layer measures the amplitudes of the remaining components after pruning the ones with amplitudes smaller than the $\lambda_j^{(k)}$.

$$
\xi_j^{(k)} := \max_{f,g} \sum_{r=j+1}^{R(k)} |\lambda_r^{(k)}| |\hat{C}_{r}^{(f,g)}| 
$$

Remark. For a fixed $j$, a smaller tensor noise bound indicates the original neural network’s weight tensor is more low-rank and thus more compressible.

Definition 4.4. [Layer cushion $\zeta^{(k)}$] As introduced in (Arora et al., 2018), the layer cushion of the $k$th layer is defined to be the largest value $\zeta^{(k)}$ such that for any $\chi^{(k)} \in S$,

$$
\zeta^{(k)} \left( \left\| \mathcal{M}^{(k)} \right\| F / \sqrt{H(k)W(k)} \right) \left\| \chi^{(k)} \right\| F \leq \left\| \mathcal{M}^{(k+1)} \right\| F 
$$

Suppose $M(k)$ is the smallest $\chi^{(k)}$ such that for any $\chi^{(k)} \in S$,

$$
\zeta^{(k)} \left( \left\| \mathcal{M}^{(k)} \right\| F / \sqrt{H(k)W(k)} \right) \left\| \chi^{(k)} \right\| F \leq \left\| \mathcal{M}^{(k+1)} \right\| F 
$$

Remark. As introduced in (Arora et al., 2018), the layer cushion considers how much the output tensor $\left\| \mathcal{M}^{(k+1)} \right\| F$ grows w.r.t. the weight tensor $\left\| \mathcal{M}^{(k)} \right\| F$ and the input $\left\| \chi^{(k)} \right\| F$.

Definition 4.5. [Main Theorem] For any convolutional neural network $M$ with $n$ layers, Algorithm 1 generates a compressed CNN $\hat{M}$ such that with high probability, the expected error $L_0(M)$ is bounded by the empirical margin loss $\hat{L}_0$ (for any margin $\gamma \geq 0$) and a complexity term defined as follows

$$
L_0(M) \leq \hat{L}_0(M) + \hat{O} \left( \sum_{k=1}^{n} \hat{R}(k) \left( s^{(k)} + \alpha^{(k)} + k_{\xi}^{(k)} + 1 \right) \right) 
$$

where $t_j^{(k)}$, $\xi_j^{(k)}$ and $\zeta^{(k)}$ are data dependent measurable properties — tensorization factor, tensor noise bound, and layer cushion of the $k$th layer in definition 4.2, 4.3 and 4.4 respectively.

Remark. How well the compressed neural network approximates the original network is related to the choice of $\hat{R}(k)$. Inside equation 4.5, $C$ is some value independent of the choice of $j$ in the inequality. Therefore, the number of components for the $k$th layer in the compressed network, $\hat{R}(k)$, is the smallest $j \in [R(k)]$ such that the inequality $\xi_j^{(k)} \Pi_{i=k+1}^{n} t_j^{(i)} \leq C$ holds. Hence, smaller tensorization factors and tensor noise bounds will make the LHS smaller, and larger layer cushions will make the RHS, $C$, larger. As a result, if the above inequality for each layer can be satisfied by a smaller $j$, the obtained generalization bound will be tighter as we can obtain a smaller $\hat{R}(k)$.

Analysis of generalization bounds in Theorem 4.5 This proposed generalization error bound is proportional to the number of components in the CP layers of the compressed neural network. Therefore, when the original neural network is highly compressible or very low-rank, the number of components
Table 1: Comparison of the training and test accuracies between neural networks (NNs) with CPL (CP-VGG-16, CP-WRN-28-10) and traditional NNs (VGG-16, WRN-28-10) on CIFAR10 dataset.

| Dataset | Architect. | VGG-16 | CP-VGG-16 | WRN-28-10 | CP-WRN-28-10 |
|---------|------------|--------|-----------|-----------|--------------|
| CIFAR10 | Training   | 100%   | 100%      | 100%      | 100%         |
|         | Test       | 93.68% | 92.64%    | 95.09%    | 95.83%       |
| CIFAR100| Training   | 100%   | 100%      | 100%      | 100%         |
|         | Test       | 71.9%  | 70.84%    | 76.36%    | 79.5%        |

Table 2: Test accuracy on CIFAR10 with various label corruptions rates (CR).

| Network / CR | 0 | 0.2 | 0.4 | 0.6 | 0.8 |
|--------------|---|-----|-----|-----|-----|
| CIFAR10      | 68.76 | 44.26 | 24.89 | 13.21 |       |
| CIFAR100     | 71.09 | 51.76 | 35.00 | 20.06 |       |

The proof of Theorem 4.5 is in Appendix Section D, and the proof sketch is as follows.

Proof sketch of Theorem 4.5. We first establish that the difference of the outputs between the compressed CNN \( \hat{M} \) and the original CNN \( M \) is bounded by 

\[
\frac{2}{\max \|M(V)\|_F^2}
\]

Then we show the covering number of the compressed network \( \hat{M} \) is \( O(d) \) via Lemma D.7, where \( d \) denotes the total number of parameters in the compressed network. Boundings the covering number of CNNs with CPL to be of order \( O(d) \) is non-trivial as we need careful handlings of the error propagations to avoid a dependence on the product of number of components. After bounding the covering number, the rest of the proof follows from conventional learning theory and Theorem 2.1 in Arora et al. 2018.

5 Experiments

Architecture and optimization setting. The architectures we use in the experiments consist of VGG-16 (Simonyan and Zisserman, 2014), CP-VGG-16, WRN-28-10 (Zagoruyko and Komodakis, 2016) and CP-WRN-28-10 (all with batch normalization). Details of the optimization settings are in A.1.

5.1 Evaluation of Proposed Properties and Generalization Bounds

Tighter Generalization Bound. As shown in Fig 2a, our bound is much tighter than the the state-of-the-art bound achieved in Arora et al. 2018. The effective number of parameters in Arora et al. 2018 is orders of magnitude tighter than other capacity measures, such as \( \ell_1 \) (Bartlett and Mendelson, 2002), Frobenius (Neyshabur et al., 2015b), spec \( \ell_1 \) (Bartlett et al., 2017) and spec-fro (Neyshabur et al., 2017a) as shown in their Figure 4 Left. The use of a more effective and practical compression approach allows us to achieve better compression (detailed discussions are in Appendix Section A.2).

Generalization Bounds Correlated with Test Error. We demonstrate how our generalization bound in Theorem 4.3 is practically useful in characterizing the generalizability during training. In Figure 2b (1) our calculated generalization bound matches well with the trend of the generalization error: after 140 epochs, the training error is almost zero but the test error continues to decrease in later epochs and our computed generalization bound captures these improvements especially well since epoch 150; (2) our calculated bound in Figure 2b for the well-trained CP-VGG-16 at epoch 300 is around 10 while the total number of parameters in this CP-VGG-16 is around 14.7M.

Compressibility of CPL: Property Evaluation. We evaluate and compare our proposed properties measuring compressibility, tensorization factor (TF), tensor noise bound (TNB) and layer cushion (LC), on two different sets of models — well-trained models with small generalization errors (thus expected to obtain small \( \{\hat{R}(k)\}_{k=1}^{n} \) vs. corrupted models with large generalization errors (thus expected to obtain large \( \{\hat{R}(k)\}_{k=1}^{n} \)). In Figure 3(a), the number of components \( \{\hat{R}(k)\}_{k=1}^{n} \) returned by the compression algorithm is much smaller for well-trained models than for corrupted models, which indicates that well-trained models have higher compressibility compared to corrupted ones as expected in our theory. Moreover, in Figure 3(b-d), we can indeed tell if the model is trained

\( \ast \) https://github.com/kuangliu/pytorch-cifar
\( \dagger \) https://github.com/geifmany/cifar-vgg
\( ^{2} \) Zagoruyko and Komodakis (2016)
Table 3: Average test accuracy on MNIST over the last ten epochs. Baseline simply denotes training a neural network on the corrupted training set without further processing. PairFlip denotes that the label mistakes can only happen within very similar classes and Symmetric denotes that the label mistakes may happen across different classes uniformly (Han et al., 2018).

| Task: Rate | Baseline (Han et al., 2018) | F-correction (Han et al., 2018) | MentorNet (Jiang et al., 2017) | CT (Han et al., 2018) | CT + CPL |
|-----------|-----------------------------|-------------------------------|-----------------------------|-------------------|---------|
| PairFlip: 45% | 56.32 ± 0.58 | 0.24 ± 0.03 | 80.88 ± 4.45 | 87.63 ± 0.21 | 92.43 ± 0.01 |
| Symmetric: 50% | 66.05 ± 0.61 | 79.61 ± 1.96 | 90.05 ± 0.30 | 91.32 ± 0.06 | 94.70 ± 0.05 |
| Symmetric: 20% | 94.05 ± 0.16 | 98.80 ± 0.12 | 96.70 ± 0.22 | 97.25 ± 0.03 | 97.91 ± 0.01 |

Using “good” data or corrupted data by evaluating our proposed properties.

Figure 3: Comparison of our proposed properties across layers between well-trained and corrupted CP-VGG-16. The statistics are obtained from 200 models trained under the same optimization settings.

We further apply Algorithm 1 to these well-trained and corrupted models to investigate the consistency between the compression performance of Algorithm 1 and our theoretical results: on average, Algorithm 1 achieves a 31.83% compression rate on the well-trained models, but only an 89.7% compression rate on the corrupted models (lower compression rate is better as it implies a smaller generalization error bound). Clearly, the low-rank structures in well-trained models allow them to be compacted much further, consistent with our theoretical analysis of Algorithm 1.

5.2 Generalization Improvement on Real Data Experiments

Expressive Power of Neural Networks with CP layers. As shown in Table 1, neural networks equipped with CP layers maintain competitive training and test accuracies.

Generalization Improvements under Label Noise. The memorization effect is directly linked to the deteriorated generalization performance of the network (Zhang et al., 2017). Therefore we study how our proposed CPL structure affects the generalizability of a neural network with presence of strong memorization effect — under label noise setting. We assign random labels to a proportion of the training data and train the neural network until convergence. Then we test the network’s performance on the uncorrupted test data. As shown in Table 2, CP-VGG consistently achieves better generalization performance compared to the traditional VGG under various label corruption ratios.

Our CPL, combined with co-teaching (CT) (Han et al., 2018) (the SOTA method for defeating label noise) further improves its performance as shown in Table 3 where we also compare our method CT+CPL against other different label-noise methods (Han et al., 2018). Besides, in Figure 5 our method CT+CPL consistently outperforms the SOTA method (CT) with various choices of number of components.

5.3 CPL Is Natural for Compression

Applying CPL for neural network compression is extensively studied in Su et al. (2018), therefore we focus on explaining why CPL is natural for compression and analyzing the compressibility of CPLs.

Low Rankness in Neural Networks with CPL vs Traditional Neural Networks. The low rankness of a CP-VGG and a traditional VGG is demonstrated by Figure 4 where we display the ratios of the number of components with amplitudes above a given threshold 0.2. We clearly see that VGG with CPL exhibits low rankness consistently for all layers while the traditional VGG is not low-rank. Notice that the CP spectrum in each CPL is normalized by dividing the largest amplitude and the CP components of traditional VGG are obtained via explicit CP decompositions with reconstruction error set to 1e-3.

Figure 4: Comparison of low rankness (compressibility) across layers between neural networks with CPL and standard neural networks.

No Fine-tuning Needed for CPL. Many works using tensor methods for neural network compression require computationally expensive fine-tuning (e.g. 200 epochs end-to-end training on the compressed networks) to recover the compressed network’s test performance. Jaderberg et al. (2014); Denton et al. (2014); Lebedev et al. (2014); Kim et al.
In this work, we derive a practical compression-based generalization bound via the proposed layerwise structure CP layers, and demonstrate the effectiveness of using tensor methods in theoretical analyses of deep neural networks. With a series of benchmark experiments, we show the practical usage of our generalization bound and the effectiveness of our proposed structure CPL in terms of compression and generalization. A possible future direction is studying the effectiveness of other tensor decomposition methods such as Tucker or Tensor Train.

6 Conclusion and Discussion

In this work, we derive a practical compression-based generalization bound via the proposed layerwise structure CP layers, and demonstrate the effectiveness of using tensor methods in theoretical analyses of deep neural networks. With a series of benchmark experiments, we show the practical usage of our generalization bound and the effectiveness of our proposed structure CPL in terms of compression and generalization. A possible future direction is studying the effectiveness of other tensor decomposition methods such as Tucker or Tensor Train.

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A.1 Architecture and optimization setting

We train these four models (VGG-16, CP-VGG-16, WRN-28-10 and CP-WRN-28-10) using standard optimization settings with no dropouts and default initializations provided by PyTorch [Paszke et al., 2017]. We use a SGD optimizer with momentum=0.9, weight decay=5e-4, and initial learning rate=0.05 to start the training process. The learning rate is scheduled to be divided by 2 every 30 epochs for VGG-16 and CP-VGG-16. While for WRN-28-10 and CP-WRN-28-10, the learning rate is scheduled to be divided by 5 at the 60th, 120th and 160th epoch. We run 300 epochs to train each VGG-16 and CP-VGG-16, and we run 200 epochs to train each WRN-28-10 and CP-WRN-28-10.

A.2 Generalization bounds comparison with [Arora et al., 2018]

The generalization bound we calculated for a well-trained CP-VGG-16 (with the same # of parameters as VGG-16) on CIFAR10 dataset is around 12 (thus, of order $10^1$) according to the transformation $f(x) = x/20 - 0.5$ applied in Figure 2b. Our evaluated bound is much better than naive counting of # parameters. Although we may not be able to directly compare our calculated bound with that in [Arora et al., 2018], which is roughly of order $10^5$ as [Arora et al., 2018] uses a VGG-19 to evaluate their generalization bound while our evaluation is done using a CP-VGG-16, we present in Table 4 the effective number of parameters identified by our proposed bound. Compared with the effective number of parameters in [Arora et al., 2018] (Table 1 of [Arora et al., 2018]), we can see that (1) our effective number of parameters is upper bounded by the total number of parameters in original network (thus, the compression ratio is bounded by 1), while the effective number identified by [Arora et al., 2018] could be several times larger than the original number of parameters (e.g. based on Table 1 of [Arora et al., 2018], their effective number of parameters in layer 4 and 6 are more than 4 times of the original number of parameters); (2) the effective number of parameters in [Arora et al., 2018] ignores the dependence on depth, log factors and constants, while our effective number of parameters in Table 4 is exactly the actual number of parameters in the compressed network without these dependences.

A.3 Neural networks with CPL are natural for compression

The compression results in Table 5 are obtained directly without any fine tuning.

A.4 Improved Generalization Achieved by CPL

We provide additional experimental details in the improved generalization ability achieved by CPL under label noise setting. Our CPL combined with co-teaching (CT) [Han et al., 2018] outperforms SOTA method. Co-teaching [Han et al., 2018] is a training procedure for defeating label noise: it avoids overfitting to noisy labels by selecting clean samples out of the noisy ones and using them to update the network. Given the experimental results that neural networks with CPL tend to overfit less to noisy labels (Table 3), we combine Co-teaching to train networks with CPL on three different types of corrupted data (Table 3). The hyperparameters we use in these experiments are the same as the ones in Co-teaching [2].
Table 4: Effective number of parameters identified by our proposed bound in Theorem 4.5.

| layer | original # of params | our effective # of params | our compression ratio | effective # of params in Arora et al. (2018) | compression ratio in Arora et al. (2018) |
|-------|----------------------|---------------------------|----------------------|---------------------------------------------|-------------------------------------------|
| 1     | 1728                 | 1694                      | 0.980324             | 10728.622                                   | 6.208693                                  |
| 2     | 36864                | 36984                     | 1.000255             | 63681.09                                    | 1.727460                                  |
| 3     | 73728                | 73932                     | 1.002767             | 116967.954                                  | 1.586479                                  |
| 4     | 147456               | 147630                    | 1.001180             | 910160.75                                   | 6.172423                                  |
| 5     | 294912               | 295106                    | 1.000658             | 817337.9                                    | 2.771464                                  |
| 6     | 589824               | 590904                    | 1.001831             | 3943927.2                                   | 6.653574                                  |
| 7     | 589824               | 590904                    | 1.001831             | 15346982.0                                  | 26.019596                                 |
| 8     | 1176948              | 1177892                   | 0.998511             | 367775.12                                   | 0.311767                                  |
| 9     | 2359296              | 2288242                   | 0.969883             | 95583.41                                    | 0.040645                                  |
| 10    | 2359296              | 1774344                   | 0.752065             | 87476.836                                   | 0.037078                                  |
| 11    | 2359296              | 350526                    | 0.148572             | 42480.465                                   | 0.018006                                  |
| 12    | 2359296              | 42394                     | 0.017969             | 40184.535                                   | 0.017032                                  |
| 13    | 2359296              | 124080                    | 0.052592             | 137974.52                                   | 0.058481                                  |

Table 5: The compression results of a 28-layer Wide-ResNet equipped with CPL (CP-WRN-28-10) on CIFAR10 dataset. The compression is done via normalizing the CP spectrum and then deleting the components in CPL which have amplitudes smaller than the given cut-off-threshold.

| Cut-off threshold | Compression ratio | # params | Test acc % |
|-------------------|-------------------|----------|------------|
| 0                 | 1 x               | 36.5M    | 95.09      |
| 1e-4              | 0.229 (4 x)       | 8.36M    | 95.08      |
| 1e-3              | 0.164 (5 x)       | 6.90M    | 95.05      |
| 1e-2              | 0.124 (8 x)       | 4.52M    | 94.53      |

As shown in Table 3, we compare our method CT+CPL against various label-noise methods (Han et al., 2018) under standard label noise setting (Han et al., 2018). (1) As shown in Figure 5, our method (CT+CPL) consistently outperforms the SOTA method with various choices of the number of components. (1.1) Specifically, according to Table 3, we see that combining CPL with co-teaching achieves the SOTA results on MNIST for PairFlip\(^3\) with corruption rate 45% and Symmetric\(^4\) with corruption rate 50%. (1.2) We also investigate the learning curve of our method compared with the SOTA (see Figure 6). The models first reach best test accuracy early in the training, and then the test accuracy deteriorates as training goes on due to memorization effect. We see that our method always dominates the vanilla CT method when generalizability of the model starts to deteriorate due to memorization effect. This clearly shows that a neural network with CPL has better generalizability property than the plain neural network under this label noise setting. (2) For the Symmetric-20% in Table 3 as the label corruption rate is low, our method has a low effect in improving the generalization, which is expected.

Remark. The results displayed in Figure 5 and Figure 6 are based on our implementation of the CT method in order to achieve a fair comparison, while the results displayed in Table 3 are based on the reported accuracies by (Han et al., 2018) as we would like to compare our CT+CPL with other different label-noise methods as well.

A.5 Compressibility of CPL: Property Evaluation CPL

Figure 7b displays the CP spectral of a well-trained, a corrupted, and a randomly initialized CP-VGG-16 (at the 13\(^{th}\) convolutional layer). For the unnormalized CP spectra of three models in Figure 7b(a), we can see that the largest amplitude in the CP spectrum of the corrupted CP-VGG-16 is much smaller than that of well-trained and random models. Yet, a smaller leading value in the CP spectrum does not necessarily mean that the corrupted is more low rank. As shown in Figure 7b(b), after normalizing the CP spectrum of each model by its largest amplitude, well-trained CP-VGG-16 still has the most low-rank CP spectrum (the blue curve) than

\(^3\)PairFlip denotes that the label mistakes can only happen within very similar classes (Han et al., 2018)

\(^4\)Symmetric denotes that the label mistakes may happen across different classes uniformly (Han et al., 2018)
Figure 5: Test accuracy vs. different compression ratios

(a) PairFlip-45%

(b) Symmetric-50%

Figure 6: Convergence plots of test accuracy vs. number of epochs on MNIST data

(a) PairFlip-45%

(b) Symmetric-50%

Figure 7: Comparison of the CP spectra of a well-trained, a corrupted, and a randomly initialized CP-VGG-16 at the 13th convolutional layer

(a) CP spectrum with unnormalized amplitudes

(b) CP spectrum with normalized amplitudes
that of corrupted or random models. Notice that the random model has the least low rankness since its weight tensors are the closest to random noise and thus it is hard to compress them.

We also compare our proposed properties among the three different sets of CP-VGG-16: well-trained, corrupted, and randomly initialized. As shown in Figure 8, since random models have the least compressibility as their weight tensors are closest to random noise, properties that focus more on the compressibility of the model are larger on random models (e.g. tensor noise bound), which will lead to larger generalization bounds. In the meantime, properties that focus more on measuring the information loss after compression as well as the expressive power of the models (e.g. Fourier factors) are smaller for random models. The reason why well-trained models have

![Figure 8: Comparison of proposed properties among well-trained, corrupted and randomly-initialized CP-VGG-16 models](image)

the largest tensorization factor is in Figure 7 as the corrupted model usually has a very small leading value in its CP spectrum of later layers; yet as explained before, this does not necessarily indicate that corrupted models have more compressibility or low-rankness. The reason why the CP spectrum of corrupted models tend to have a small leading value is still an interesting question to study and we defer this to future work.

Optimization settings for obtaining the well-trained, corrupted, and randomly initialized models of CP-VGG-16. We obtain well-trained CP-VGG-16 using the same hyperparameter settings as mentioned in Appendix Section A. For corrupted CP-VGG-16, we train the model under 50% of label noise but using the same set of hyperparameters as the well-trained models. For CP-VGG-16 with random initialization, we just train the models for less than 1 epoch. For each set of these models, we obtain 200 instances using different random seeds.

B Common Definitions and Propositions

In this section, we will briefly review three key concepts underlying all analysis in this work, including (multidimensional discrete Fourier transform), CP decomposition and 2D-convolutional layer in neural networks.

B.1 Multidimensional Discrete Fourier Transform (MDFT)

**Definition B.1.** (Multidimensional discrete Fourier transform, MDFT) An $m$-dimensional MDFT $\mathcal{F}_m$ defines a mapping from an $m$-order tensor $\mathcal{X} \in \mathbb{R}^{N_1 \times \cdots \times N_m}$ to another complex $m$-order tensor $\mathcal{X} \in \mathbb{C}^{N_1 \times \cdots \times N_m}$ such
where each column \( \tilde{X}_{f_1, \ldots, f_m} = \left( \prod_{i=1}^{m} N_i \right)^{-\frac{1}{2}} \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} \mathcal{X}_{n_1, \ldots, n_m} \left( \prod_{i=1}^{m} \omega_{N_i}^{f_i n_i} \right) \) (6)

where \( \omega_{N_i} = \exp(-j2\pi/N_i) \) and \( \left( \prod_{i=1}^{m} N_i \right)^{-\frac{1}{2}} \) is the normalization factor that makes \( F_m \) unitary. Through out the paper, we will use symbols with tilde (e.g. \( \tilde{X} \)) to denote tensors after MDFT.

MDFT can also be applied on a subset of the dimensions \( \mathcal{I} \subseteq [m] \), and in this case we denote the mapping as \( F_m^\mathcal{I} \).

\[
\tilde{X}_{i_1, \ldots, i_m} = \left( \prod_{i \in \mathcal{I}} N_i \right)^{-\frac{1}{2}} \sum_{\forall i \in \mathcal{I}} \mathcal{X}_{i_1, \ldots, i_m} \left( \prod_{i \in \mathcal{I}} \omega_{N_i}^{i_i n_i} \right) \quad (7)
\]

where \( i_l = f_l \) if \( l \in \mathcal{I} \) and \( i_l = n_l \) for \( l \notin \mathcal{I} \).

**Fact B.2.** (Separability of MDFT) An \( m \)-dimensional MDFT \( F_m \) is equivalent to a composition of \( m \) unidimensional DFTs, i.e.

\[
F_m = F_m^1 \circ F_m^2 \circ \cdots \circ F_m^m \quad (8)
\]

Similarly, \( F_m^\mathcal{I} \) is identical to a composition of \( |\mathcal{I}| \) unidimensional DFTs over corresponding dimensions.

**Fact B.3.** (MDFT is unitary) For an MDFT \( F \), its adjoint \( F^* \) is equal to its inverse \( F^{-1} \), i.e. \( F^* = F^{-1} \). An immediate corollary of this property is that the operator norm is invariant to MDFT: Given an operator \( A \), its operator norm of \( A \) is equal to \( F^* A F \); i.e. \( \|A\| = \|F^* A F\| \).

### B.2 CP decomposition

**Definition B.4.** (CP decomposition) Given an \( m \)-order tensor \( T \in \mathbb{R}^{N_1 \times \cdots \times N_m} \), a CP decomposition factorizes \( T \) into \( m \) core factors \( \{K^l\}_{l=1}^m \) with \( K^l \in \mathbb{R}^{R \times N_l} \) (with its \( r \)-th column as \( k^l_r \in \mathbb{R}^{N_l} \)) such that

\[
T = \sum_{r=1}^{R} \lambda_r k^1_r \otimes \cdots \otimes k^m_r \quad (9a)
\]

\[
T_{i_1, \ldots, i_m} = \sum_{r=1}^{R} \lambda_r K^1_{r,i_1} \cdots K^m_{r,i_m} \quad (9b)
\]

where each column \( k^l_r \) has unit \( \ell_2 \) norm, i.e. \( \|k^l_r\|_2 = 1 \), \( \forall r \in [R], l \in [m] \). Without loss of generality, we assume the CP eigenvalues are positive and sorted in decreasing order, i.e. \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0 \). If the columns in \( K^l \) are orthogonal, i.e. \( \langle k^l_r, k^l_{r'} \rangle = 1 \) for \( r \neq r' \), the factorization is further named as orthogonal CP decomposition.

**Lemma B.5.** (MDFT of CP decomposition) If an \( m \)-order tensor \( T \in \mathbb{R}^{N_1 \times \cdots \times N_m} \) takes a CP decomposition as in Eq. (9a), its (all-dimensional) MDFT \( \tilde{T} = F_m(T) \in \mathbb{C}^{N_1 \times \cdots \times N_m} \) also takes a CP format as

\[
\tilde{T} = \sum_{r=1}^{R} \lambda_r \tilde{k}^1_r \otimes \cdots \otimes \tilde{k}^m_r \quad (10)
\]

\[
\tilde{T}_{f_1, \ldots, f_m} = \sum_{r=1}^{R} \lambda_r \tilde{K}^1_{r,f_1} \cdots \tilde{K}^m_{r,f_m} \quad (11)
\]

where \( \tilde{K}^l = F_m^\mathcal{I}(K^l), \forall l \in [m] \). The result can be extended to MDFT where a subset of dimensions are transformed.
Proof. (of Lemma B.7) According to the definition of multidimensional discrete Fourier transform, we have

$$\hat{\mathcal{T}}_{n_1, \cdots, n_m} = \left( \prod_{l=1}^{m} N_l \right)^{-\frac{1}{2}} \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} \mathcal{T}_{n_1, \cdots, n_m} \left( \prod_{l=1}^{m} \omega^{\frac{1}{2} n_l} N_l \right)$$

(12)

$$= \left( \prod_{l=1}^{m} N_l \right)^{-\frac{1}{2}} \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} \left( \sum_{r=1}^{R} \lambda_r \hat{K}_{r,n_1}^{1} \cdots \hat{K}_{r,n_m}^{m} \right) \left( \prod_{l=1}^{m} \omega^{\frac{1}{2} n_l} N_l \right)$$

(13)

$$= \sum_{r=1}^{R} \lambda_r \hat{K}_{r,f_1}^{1} \cdots \hat{K}_{r,f_m}^{m}$$

(14)

(15)

which completes the proof. □

B.3 2D-Convolutional Layer in Neural Networks

Definition B.6. (2D-convolutional layer) In CNNs, a 2D-convolutional layer is parametrized by a $4^\text{th}$-order tensor $\mathcal{M} \in \mathbb{R}^{k_x \times k_y \times T \times S}$ (with $k_x \times k_y$ kernels). It defines a mapping from a $3^\text{rd}$-order input tensor $\mathcal{X} \in \mathbb{R}^{H \times W \times S}$ (with $S$ channels) to another $3^\text{rd}$-order output tensor $\mathcal{Y} \in \mathbb{R}^{H \times W \times T}$ (with $T$ channels).

$$\mathcal{Y}_{i,t,t} = \sum_{s=1}^{S} \mathcal{M}_{i,t,t,s} \ast \mathcal{X}_{i,t,s}$$

(16)

$$\mathcal{Y}_{i,j,t} = \sum_{s=1}^{S} \sum_{p,q} \mathcal{M}_{i-p,j-q,t,s} \mathcal{X}_{p,q,s}$$

(17)

where $\ast$ represents a 2D-convolution operator.

Lemma B.7. (Convolutional theorem of 2D-convolutional layer) Suppose $\hat{\mathcal{X}} = \mathcal{F}_{3}^{1,2} (\mathcal{X}) \in \mathbb{C}^{H \times W \times S}$, $\hat{\mathcal{M}} = \mathcal{F}_{4}^{1,2} (\mathcal{M}) \in \mathbb{C}^{H \times W \times T \times S}$ and $\hat{\mathcal{Y}} = \mathcal{F}_{3}^{1,2} (\mathcal{Y}) \in \mathbb{C}^{H \times W \times T}$ are the MDFT of input, weights and outputs tensors $\mathcal{X}$, $\mathcal{M}$ and $\mathcal{Y}$ respectively, then these three tensors satisfy the following equation:

$$\hat{\mathcal{Y}}_{f,g,t} = \sqrt{HW} \sum_{s=1}^{S} \hat{\mathcal{M}}_{f,g,t,s} \hat{\mathcal{X}}_{f,g,s}$$

(18)

Notice that the equation has a constant $\sqrt{HW}$ since we use a normalized MDFT.

Proof. (of Lemma B.7) The theorem can be easily proved by applying MDFT on both sides of Eq. (17).

$$\hat{\mathcal{Y}}_{f,g,t} = \frac{1}{\sqrt{HW}} \sum_{i,j} \mathcal{Y}_{i,j,t} \omega_{H}^{(i-f)} \omega_{W}^{(j-g)}$$

(19)

$$= \frac{1}{\sqrt{HW}} \sum_{i,j} \left( \sum_{s=1}^{S} \sum_{p,q} \mathcal{M}_{i-p,j-q,t,s} \mathcal{X}_{p,q,s} \right) \omega_{H}^{(i-f)} \omega_{W}^{(j-g)}$$

(20)

$$= \sqrt{HW} \sum_{s=1}^{S} \left( \frac{1}{\sqrt{HW}} \sum_{i,j} \mathcal{M}_{i-p,j-q,t,s} \omega_{H}^{(i-t)} \omega_{W}^{(j-q)} \right) \left( \frac{1}{\sqrt{HW}} \sum_{p,q} \mathcal{X}_{p,q,s} \omega_{H}^{(i-f)} \omega_{W}^{(j-g)} \right)$$

(21)

$$= \sqrt{HW} \sum_{s=1}^{S} \hat{\mathcal{M}}_{f,g,t,s} \hat{\mathcal{X}}_{f,g,s}$$

(22)
Lemma B.8. (Operator norm of 2D-convolutional layer) Suppose we rewrite the tensors in matrix/vector form, i.e. \( \hat{X}_{f,g,s} = \hat{x}^{(f,g)}_s \), \( \hat{M}_{f,g,t} = \hat{M}^{(f,g)}_{t,s} \), \( \hat{y}^{(f,g)}_t \), then Eq. (18) can be written using matrix/vector products:

\[
\hat{y}^{(f,g)}_t = \sum_{s=1}^{S} \hat{M}^{(f,g)}_{t,s} \hat{y}^{(f,g)}_s, \quad \forall f,g
\]

The operator norm of \( \hat{M} \), defined as \( \| \hat{M} \| = \| \| \hat{X} \|_{2} = \| Y \|_F \), can be obtained by spectral norms of \( \hat{M}^{(f,g)} \) as:

\[
\| \hat{M} \| = \sqrt{HW} \max_{f,g} \| \hat{M}^{(f,g)} \|_2
\]

Remarks. The bound is first given by Sedghi et al. (2018). In this work, we provide a much simpler proof compared to the original one in Sedghi et al. (2018). In the next section, we show that the bound can be computed without evaluating the spectral norm if the weights tensor \( \hat{M} \) takes a CP format similar to Eq. (23).

Proof. (of Lemma B.8) From Fact B.3 we know that \( \| \hat{M} \| = \| \hat{M} \| \), where \( \| \hat{M} \| = \max_{\| \hat{X} \|_F = 1} \| Y \|_F \). Next, we bound \( \| \hat{Y} \|_F^2 \) (i.e. \( \sum_{f,g} \| \hat{y}^{(f,g)}_f \|_F^2 = 1 \)).

\[
\| \hat{Y} \|_F^2 = \sum_{f,g} \| \hat{y}^{(f,g)}_f \|_2^2
\]

\[
\leq HW \sum_{f,g} \| \hat{M}^{(f,g)} \|_2^2 \| \hat{x}^{(f,g)}_f \|_2^2
\]

\[
\leq HW \max_{f,g} \| \hat{M}^{(f,g)} \|_2^2 \sum_{f,g} \| \hat{x}^{(f,g)}_f \|_2^2
\]

\[
= HW \max_{f,g} \| \hat{M}^{(f,g)} \|_2^2
\]

\[
\| \hat{Y} \|_F \leq \sqrt{HW} \max_{f,g} \| \hat{M}^{(f,g)} \|
\]

We complete the proof by observing all inequalities can achieve equality simultaneously.

\[\square\]

Definition B.9. (Tensor product) For vectors \( a \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \), and \( c \in \mathbb{R}^p \), their tensor product \( a \otimes b \otimes c \) is a 3-way tensor in \( \mathbb{R}^{n \times m \times p} \), with the \((i, j, k)\)th entry being \( a_{i,j}b_{j,k}c_k \). Similarly, for a matrix \( A \in \mathbb{R}^{n \times m} \) and a vector \( c \in \mathbb{R}^p \), their tensor product \( A \otimes c \) is a \( n \times n \times p \) tensor with the \((i, j, k)\)th entry being \( A_{i,j}c_k \).

Definition B.10. (Kronecker product). Let \( A \) be an \( n \times p \) matrix and \( B \) an \( m \times q \) matrix. The \( mn \times pq \) matrix

\[
A \otimes B = \begin{bmatrix}
(a_{1,1}B & a_{1,2}B & \cdots & a_{1,p}B \\
a_{2,1}B & a_{2,2}B & \cdots & a_{2,p}B \\
& \vdots & \ddots & \vdots \\
a_{n,1}B & a_{n,2}B & \cdots & a_{n,p}B
\end{bmatrix}
\]

is called the Kronecker product of \( A \) and \( B \). The outer product is an instance of Kronecker products.

C CP Layers in Tensorial Neural Networks

In this section, we will introduce three types of neural network layers, whose parameters are factorized in CP format as in Eq. (23) (with small variations). For brevity, we omit the layer superscript and denote the input, layer parameters and output as \( X \), \( \mathcal{M} \) and \( \hat{Y} \), and we use \( \hat{Y} = \mathcal{M}(\hat{X}) \) to denote the relations between \( \hat{X} \), \( \mathcal{M} \) and \( \hat{Y} \).
C.1 CP 2D-convolutional Layer

Definition C.1. (CP 2D-convolutional layer) For a given 2D-convolutional layer in Eq. (17), a CP decomposition factorizes the weights tensor \( \mathcal{M} \in \mathbb{R}^{R \times W \times T \times S} \) into three core factors \( \mathcal{C} \in \mathbb{R}^{R \times k_x \times k_y} \), \( \mathbf{U} \in \mathbb{R}^{R \times T} \), \( \mathbf{V} \in \mathbb{R}^{R \times S} \) and a vector of CP eigenvalues \( \lambda \in \mathbb{R}^R \) such that

\[
\mathcal{M} = \sum_{r=1}^{R} \lambda_r \mathbf{C}_r \otimes \mathbf{u}_r \otimes \mathbf{v}_r
\]

(30)

\[
\mathcal{M}_{i,j,t,s} = \sum_{r=1}^{R} \lambda_r \mathbf{C}_{r,i,j} \mathbf{U}_{r,t} \mathbf{V}_{r,s}
\]

(31)

where \( \lambda_r > 0 \), \( ||\mathbf{C}_r||_F = 1 \), \( ||\mathbf{u}_r||_2 = 1 \) and \( ||\mathbf{v}_r||_2 = 1 \) for all \( r \in [R] \).

Lemma C.2. (Operator norm of CP 2D-convolutional layer) For a 2D-convolutional layer whose weights tensor takes a CP format as in Eq. (31), the operator norm of \( \mathcal{M} \) is bound by the CP eigenvalues \( \lambda \) as

\[
||\mathcal{M}|| \leq \sqrt{HW} \sum_{r=1}^{R} |\lambda_r| \max_{f,g} \left| \tilde{C}_{r,f,g} \right|
\]

(32)

Proof. (of Lemma C.2) From Fact B.3, the operator norm of \( \mathcal{M} \) is equal to the one of its MDFT \( \hat{\mathcal{M}} = \mathcal{F}_4^{1,2}(\mathcal{M}) \), i.e. \( ||\mathcal{M}|| = ||\hat{\mathcal{M}}|| \). According to Lemma B.3, it is sufficient to compute the spectral norm for each matrix \( \hat{\mathcal{M}}^{(f,g)} \) individually. Notice that if \( \mathcal{M} \) takes a CP format, each \( \hat{\mathcal{M}}^{(f,g)} \) has a decomposed form as follows

\[
\hat{\mathcal{M}}^{(f,g)} = \sum_{r=1}^{R} \lambda_r \tilde{C}_{r,f,g} \mathbf{u}_r \mathbf{v}_r^T
\]

(33a)

\[
\hat{\mathcal{M}}_{t,s}^{(f,g)} = \sum_{r=1}^{R} \lambda_r \tilde{C}_{r,f,g} \mathbf{U}_{r,t} \mathbf{V}_{r,s}
\]

(33b)

where \( \tilde{C} = \mathcal{F}_3^{2,3}(\mathcal{C}) \) and \( \tilde{C}_{r,f,g} = \tilde{C}_{r,f,g} \). The rest of the proof follows the definition of spectral norm of \( \hat{\mathcal{M}} \), i.e. \( ||\hat{\mathcal{M}}^{(f,g)}||_2 = \max_{||\mathbf{a}||_2=1} ||\hat{\mathcal{M}}^{(f,g)} \mathbf{a}|| \). Let \( \mathbf{b} = \hat{\mathcal{M}}^{(f,g)} \mathbf{a} \), we can bound the \( \ell_2 \) norm of \( \mathbf{b} \):

\[
||\mathbf{b}||_2 = ||\hat{\mathcal{M}}^{(f,g)} \mathbf{a}||_2 = \left\| \sum_{r=1}^{R} \lambda_r \tilde{C}_{r,f,g} \mathbf{u}_r \mathbf{V}_r^T \mathbf{a} \right\|_2
\]

(34)

\[
\leq \sum_{r=1}^{R} \left| \lambda_r \tilde{C}_{r,f,g} \right| \left( \mathbf{u}_r^T \mathbf{a} \right) ||\mathbf{u}_r||_2
\]

(35)

\[
= \sum_{r=1}^{R} \left| \lambda_r \tilde{C}_{r,f,g} \right| \left( \mathbf{v}_r^T \mathbf{a} \right)
\]

(36)

\[
\leq \sum_{r=1}^{R} \left| \lambda_r \right| \left| \tilde{C}_{r,f,g} \right|
\]

(37)

Therefore, \( ||\mathcal{M}|| = ||\hat{\mathcal{M}}|| = \sqrt{HW} \max_{f,g} ||\hat{\mathcal{M}}^{(f,g)}|| \leq \sqrt{HW} \sum_{r=1}^{R} |\lambda_r| \max_{f,g} \left| \tilde{C}_{r,f,g} \right| \).

\[\square\]

C.2 Higher-order CP Fully-connected Layer

Definition C.3. (Higher-order fully-connected layer) The layer is parameterized by a \( 2m \)-th order tensor \( \mathcal{M} \in \mathbb{R}^{T_1 \times \cdots \times T_m \times S_t \times \cdots \times S_m} \). It maps an \( m \)-th order input tensor \( \mathcal{X} \in \mathbb{R}^{S_1 \times \cdots \times S_m} \) to another \( m \)-th order output tensor \( \mathcal{Y} \in \mathbb{R}^{T_1 \times \cdots \times T_m} \) with the following equation:

\[
\mathcal{Y}_{t_1,\ldots,t_m} = \sum_{s_1,\ldots,s_m} \mathcal{M}_{t_1,\ldots,t_m,s_1,\ldots,s_m} \mathcal{X}_{s_1,\ldots,s_m}
\]

(38)
Definition C.4. (Higher-order CP fully-connected layer) Given a higher-order fully-connected layer in Eq. (38), a CP decomposition factorizes the weights tensor $M \in \mathbb{R}^{T_1 \times \cdots \times T_m \times S_1 \times \cdots \times S_m}$ into $m$ core factors $K^m \in \mathbb{R}^{R \times T_{m} \times S_{m}}$.

$$M_{t_1, \ldots, t_m, s_1, \ldots, s_m} = \sum_{r=1}^{R} \lambda_r K^1_{r, t_1, s_1} \cdots K^m_{r, t_m, s_m}$$ (39)

For simplicity, we denote the $r^{th}$ slice of $K^l$ as $K^l_r = K^l_{r, t_1, s_1}$. We assume $K^l_r$ has unit Frobenius norm, i.e. $\|K^l_r\|_F = 1$ and $\lambda_r > 0$ for all $r \in [R]$.

Lemma C.5. (Operator norm of higher-order CP fully-connected layer) For a higher-order fully-connected layer whose weights tensor takes a CP format as in Eq. (45), the operator norm $\|M\|$ is bound by the CP eigenvalues $\lambda$ as

$$\|M\| \leq \sum_{r=1}^{R} |\lambda_r|$$ (40)

Proof. (of Lemma C.5) The proof follows directly the definition of operator norm $\|M\| = \max_{\|X\|_F = 1} \|XM\|_F$.

$$\|Y\|_F \leq \sum_{r=1}^{R} |\lambda_r| \|K^l_r\|_F \cdots \|K^1_r\|_F \|X\|_F$$ (41)

$$\leq \sum_{r=1}^{R} |\lambda_r| \|K^l_r\|_F \cdots \|K^1_r\|_F \|X\|_F$$ (42)

$$= \sum_{r=1}^{R} |\lambda_r| \|X\|_F = \sum_{r=1}^{R} |\lambda_r|$$ (43)

C.3 Higher-order 2D-convolutional layer

Definition C.6. (Higher-order 2D-convolutional layer) The layer is parameterized by a $(2m + 2)^{th}$-order tensor $M \in \mathbb{R}^{k \times k \times T_1 \times \cdots \times T_m \times S_1 \times \cdots \times S_m}$. It maps an $(m + 2)^{th}$-order input tensor $X \in \mathbb{R}^{H \times W \times S_1 \times \cdots \times S_m}$ to another $(m + 2)^{th}$-order output tensor $Y \in \mathbb{R}^{H \times W \times T_1 \times \cdots \times T_m}$ as:

$$Y_{t_1, t_1, \ldots, t_m} = \sum_{i:t_i=1}^{S_1} M_{i, i, \ldots, i, s_1 \ldots, s_m} * X_{i, i, \ldots, i, s_1 \ldots, s_m}$$ (44a)

$$Y_{i, j, t_1, \ldots, t_m} = \sum_{i:t_i=1}^{S_1} \sum_{p,q} M_{i-p, q-j, i, \ldots, i, s_1 \ldots, s_m} X_{p, q, i, \ldots, i, s_1 \ldots, s_m}$$ (44b)

Definition C.7. (CP decomposition of higher-order 2D-convolutional layer) Given a higher-order 2D-convolutional layer in Eq. (38), a CP decomposition factorizes the weights tensor $M \in \mathbb{R}^{H \times W \times T_1 \times \cdots \times T_m \times S_1 \times \cdots \times S_m}$ into $(m + 1)$ core factors $C \in \mathbb{R}^{R \times H \times W}$ and $K^l \in \mathbb{R}^{R \times T_l \times S_l}$, $\forall l \in [m]$.

$$M_{i, j, t_1, \ldots, t_m, s_1 \ldots s_m} = \sum_{r=1}^{R} \lambda_r C_{r, i, j} K^l_{r, t_1, s_1} \cdots K^l_{r, t_m, s_m}$$ (45)

where we we assume $C_r$ and $K^l_r = K^l_{r, t_1, s_1}$ has unit Frobenius norm, i.e. $\|K^l_r\|_F = 1$ and $\|C_r\|_F = 1$

Lemma C.8. (Operator norm of Higher-order CP 2D-convolutional layer) For a higher-order 2D-convolutional layer whose weights tensor takes a CP format as in Eq. (45), the operator norm $\|M\|$ is bound by the CP eigenvalues $\lambda$ as

$$\|M\| \leq \sqrt{HW} \sum_{r=1}^{R} |\lambda_r| \max_{f,g} \left| \tilde{C}^{f,g} \right|$$ (46)
Proof. (of Lemma C.8) The proof is a combination of Lemmas C.2 and C.5. Let $\hat{M} = F_{m}^{1,2}(M)$, we have
\begin{equation}
\|M\| = \|\hat{M}\| = \sqrt{HW} \max_{f,g} \|\hat{F}_{(f,g)}\| (47)
\end{equation}

where $\lambda_r$ is the $r$th largest value in the CP spectrum of $M(k)$.

The operator norm is bounded using Lemma C.6 $\|\hat{F}_{(f,g)}\| \leq \sum_{r=1}^{R} |\lambda_r| \max_{f,g} \|\hat{E}_{r}\|$.

\[ \theta \]

D Convolutional Neural Networks: Compressibility and Generalization

D.1 Complete Proofs of Convolutional Neural Networks

Definition D.1. [tensorization factor $t_j^{(k)}$] The tensorization factors $\{t_j^{(k)}\}_{j=1}^{R(k)}$ of the $k$th layer is defined as
\begin{equation}
t_j^{(k)} := \sum_{r=1}^{j} \lambda_r^{(k)} \max_{f,g} |\hat{C}_{r}^{(f,g)}| (49)
\end{equation}

where $\lambda_r^{(k)}$ is the $r$th largest value in the CP spectrum of $M(k)$.

Definition D.2. [tensor noise bound $\xi_j^{(k)}$] The tensor noise bound $\{\xi_j^{(k)}\}_{j=1}^{R(k)}$ of the $k$th layer measures the amplitudes of the remaining components after pruning the ones with amplitudes smaller than the $\lambda_j^{(k)}$:
\begin{equation}
\xi_j^{(k)} := \sum_{r=j+1}^{R(k)} \lambda_r^{(k)} \max_{f,g} |\hat{C}_{r}^{(f,g)}| (50)
\end{equation}

Definition D.3. [layer cushion $\zeta^{(k)}$] As introduced in Arora et al. (2018), the layer cushion of the $k$th layer is defined to be the largest value $\zeta^{(k)}$ such that for any $X^{(k)} \in S$,
\begin{equation}
\zeta^{(k)} \left\| M^{(k)} \right\|_F \left\| X^{(k)} \right\|_F \leq \left\| M^{(k+1)} \right\|_F (51)
\end{equation}

Following Arora et al. (2018), the layer cushion considers how much smaller the output $\left\| M^{(k+1)} \right\|_F$ of the $k$th layer (after activation) compared with the product between the weight tensor $\left\| M^{(k)} \right\|_F$ and the input $\left\| X^{(k)} \right\|_F$. Note that $H^{(k)}$ and $W^{(k)}$ are constants and will not influence the results of the theorem and the lemmas. For simplicity, we use $H$ and $W$ to denote the maximum $H^{(k)}$ and $W^{(k)}$ over the $n$ layers for the following proofs where upper bounds are desired.

Given these definitions, we can bound the difference of outputs from a given model and its compressed counterpart. The following lemma characterizes the relation between the difference and the factors $t_j^{(k)}$, $\xi_j^{(k)}$, $\zeta^{(k)}$.

Lemma D.4. (Compression bound of convolutional neural networks) Suppose a convolutional neural network $M$ has $n$ layers, and each convolutional layer takes a CP format as in Eq. (31) with rank $R^{(k)}$. If an algorithm generates a compressed network $\tilde{M}$ such that only $\tilde{R}^{(k)}$ components with largest $\lambda_r^{(k)}$’s are retained at the $k$th layer, the difference of their outputs at the $m$th is bounded by $X^{(m+1)}$ as
\begin{equation}
\left\| X^{(m)} - \tilde{X}^{(m)} \right\|_F \leq \left( \prod_{l=1}^{m-1} \zeta^{(l)} \left\| M^{(k)} \right\|_F \prod_{l=k+1}^{m-1} \xi^{(l)} \right) \left\| X^{(m)} \right\|_F (52)
\end{equation}

Therefore for the whole network with $n$ layers, the difference between $M(X)$ and $\tilde{M}(X)$ is bounded by
\begin{equation}
\left\| M(X) - \tilde{M}(X) \right\|_F \leq \left( \prod_{k=1}^{n} \zeta^{(k)} \left\| M^{(k)} \right\|_F \prod_{l=k+1}^{n} \xi^{(l)} \right) \left\| M(X) \right\|_F (53)
\end{equation}
Proof. (of Lemma D.4) We prove this lemma by induction. For $m = 2$, the lemma holds since
\begin{equation}
\|\mathcal{X}^{(2)} - \tilde{\mathcal{X}}^{(2)}\|_F = \|\text{ReLU}(\mathcal{Y}^{(1)}) - \text{ReLU}(\tilde{\mathcal{Y}}^{(1)})\|_F \leq \|\mathcal{Y}^{(1)} - \tilde{\mathcal{Y}}^{(1)}\|_F = \left\|\left(\mathcal{M}^{(1)} - \tilde{\mathcal{M}}^{(1)}\right)\mathcal{X}^{(1)}\right\|_F \leq \sqrt{HW} \xi^{(1)} \|\mathcal{X}^{(1)}\|_F \leq \frac{\xi^{(1)}}{\zeta^{(1)} \|\mathcal{M}^{(1)}\|_F} \|\mathcal{M}^{(2)}\|_F
\end{equation}
where $\mathcal{Y} = \mathcal{M}(\mathcal{X})$ denotes the computation of a convolutional layer. (1) The first inequality follows the Lipschitzness of the ReLU activations; (2) The second inequality uses Lemma C.2 and (3) the last inequality holds by the definition of $\xi^{(1)}$. For $m + 1 > 2$, we assume the lemma already holds for $m$
\begin{equation}
\|\mathcal{X}^{(m+1)} - \tilde{\mathcal{X}}^{(m+1)}\|_F = \|\text{ReLU}(\mathcal{Y}^{(m)}) - \text{ReLU}(\tilde{\mathcal{Y}}^{(m)})\|_F \leq \|\mathcal{Y}^{(m)} - \tilde{\mathcal{Y}}^{(m)}\|_F = \left\|\mathcal{M}^{(m)}\left(\mathcal{X}^{(m)}\right) - \mathcal{M}^{(m)}\left(\tilde{\mathcal{X}}^{(m)}\right)\right\|_F \leq \sqrt{HW} \left(t^{(m)} \|\mathcal{X}^{(m)} - \tilde{\mathcal{X}}^{(m)}\|_F + \xi^{(m)} \|\mathcal{X}^{(m)}\|_F\right) \leq t^{(m)} \sum_{k=1}^{m-1} \frac{\xi^{(k)}}{\zeta^{(k)} \|\mathcal{M}^{(k)}\|_F} \prod_{l=k+1}^{m} \frac{t^{(l)}}{\zeta^{(l)} \|\mathcal{M}^{(l)}\|_F} \|\mathcal{M}^{(m+1)}\|_F
\end{equation}
which completes the induction.

Lemma D.5. For any convolutional neural network $\mathbb{M}$ of $n$ layers satisfying the assumptions in section 3 and any error $0 \leq \epsilon \leq 1$, Algorithm 1 generates a compressed tensorial neural network $\tilde{\mathbb{M}}$ such that for any $\mathcal{X} \in \mathcal{S}$:
\begin{equation}
\|\tilde{\mathbb{M}}(\mathcal{X}) - \mathbb{M}(\mathcal{X})\|_F \leq \epsilon \|\mathbb{M}(\mathcal{X})\|_F
\end{equation}
The compressed convolutional neural network $\tilde{\mathbb{M}}$ has $\sum_{k=1}^{n} \hat{R}^{(k)}(s^{(k)} + o^{(k)} + k_{x}^{(k)} k_{y}^{(k)} + 1)$ total parameters, where each $\hat{R}^{(k)}$ satisfies:
\begin{equation}
\hat{R}^{(k)} = \min \left\{ j \in [R^{(k)}] \left| \sum_{i=k+1}^{n} \xi^{(i)} \Pi_{i=k+1}^{n} \frac{t^{(i)}}{\zeta^{(i)} \|\mathcal{M}^{(i)}\|_F} \leq \epsilon \right. \right\}
\end{equation}
Remark. Equation (64) is slightly different with equation [5] as the margin $\gamma$ is replaced by a perturbation error $\epsilon$. Therefore, how well the compressed tensorial neural network can approximate the original network is related to the choice of $\hat{R}^{(k)}$. Notice that when $\hat{R}^{(k)} = R^{(k)}$, the inequality for the $k^{th}$ layer will be automatically satisfied as $\theta^{(k)} = 0$ in this case by definition.

Proof. (of Lemma D.5) The proof is trivial by observing
\begin{equation}
\frac{\xi^{(k)}}{\zeta^{(k)} \|\mathcal{M}^{(k)}\|_F} \|\mathcal{M}^{(i)}\|_F \leq \frac{\epsilon}{n} \prod_{i=k+1}^{n} \frac{\xi^{(i)} \|\mathcal{M}^{(i)}\|_F}{t^{(i)}},
\end{equation}
\begin{equation}
\Rightarrow \frac{\xi^{(k)}}{\zeta^{(k)} \|\mathcal{M}^{(k)}\|_F} \prod_{i=k+1}^{n} \frac{t^{(i)}}{\xi^{(i)} \|\mathcal{M}^{(i)}\|_F} \leq \frac{\epsilon}{n},
\end{equation}
\begin{equation}
\Rightarrow \sum_{k=1}^{n} \frac{\xi^{(k)}}{\zeta^{(k)} \|\mathcal{M}^{(k)}\|_F} \prod_{i=k+1}^{n} \frac{t^{(i)}}{\xi^{(i)} \|\mathcal{M}^{(i)}\|_F} \leq \epsilon
\end{equation}
Before proving Theorem 1.5, Lemma D.6 (introduced below) is needed.
Lemma D.6. For any convolutional neural network $M$ of $n$ layers satisfying the assumptions in section 3 and any margin $\gamma \geq 0$, $M$ can be compressed to a tensorial convolutional neural network $\hat{M}$ with $\sum_{k=1}^{n} \hat{r}^{(k)}(s^{(k)} + t^{(k)} + k_{x}^{(k)} \times k_{y}^{(k)} + 1)$ total parameters such that for any $X \in S$, $\hat{L}_{0}(\hat{M}) \leq \hat{L}_{\gamma}(M)$. Here, for each layer $k$,

$$\hat{r}^{(k)} = \min \left\{ j \in [\hat{r}^{(k)}] | s_{j}^{(k)} \Pi_{i=k+1}^{n} t_{j}^{(i)} \leq \frac{\epsilon}{n} \Pi_{i=k}^{n} s_{i}^{(i)} \|M^{(i)}\|_{F} \right\}$$

(68)

Proof. (of Lemma D.6)

If $\gamma \geq 2 \max_{X \in S} \|M(X)\|_{F}$, for any pair $(X, y) \in S$, we have

$$|M(X)[y] - \max_{j \neq y} M(X)[j]|^2 \leq (|M(X)[y]| + |\max_{j \neq y} M(X)[j]|)^2 \leq 4 \max_{X \in S} \|M(X)\|_{F} \leq \gamma^2$$

Then the output margin of $M$ cannot be greater than $\gamma$ for any $X \in S$. Thus $\hat{L}_{\gamma}(M) = 1$.

If $\gamma < 2 \max_{X \in S} \|M(X)\|_{F}$, setting

$$\epsilon = \frac{\gamma}{2 \max_{X \in S} \|M(X)\|_{F}}$$

in Lemma D.5, we obtain a compressed fully-connected tensorial neural network $\hat{M}$ with the desired number of parameters and

$$\|M(X) - \hat{M}(X)\|_{F} < \frac{\gamma}{2} \Rightarrow \forall j, |M(X)[j] - \hat{M}(X)[j]| < \frac{\gamma}{2}$$

Then for any pair $(X, y) \in S$, if $M(X)[y] > \gamma + \max_{j \neq y} M(X)[j]$, $\hat{M}$ classifies $X$ correctly as well:

$$\hat{M}(X)[y] > M(X)[y] - \frac{\gamma}{2} > \max_{j \neq y} M(X)[j] + \frac{\gamma}{2} > \max_{j \neq y} \hat{M}(X)[j]$$

Thus, $\hat{L}_{\gamma}(\hat{M}) \leq \hat{L}_{\gamma}(M)$. □

Now we prove the main theorem 4.5 by bounding the covering number given any $\epsilon$.

D.1.1 Covering Number Analysis for Convolutional Neural Network

Proof. (of Theorem 4.5) To be more specific, let us bound the covering number of the compressed network $\hat{M}$ by approximating each parameter with accuracy $\mu$.

Lemma D.7. For any given constant accuracy $\mu$, the covering number of the compressed convolutional network $\hat{M}$ is of order $O(d)$ where $d$ denotes the total number of parameters in $\hat{M}$: $d := \sum_{k=1}^{n} \hat{r}^{(k)}(s^{(k)} + o^{(k)} + k_{x}^{(k)} \times k_{y}^{(k)} + 1)$.

Let $\hat{M}$ denote the network after approximating each parameter in $\hat{M}$ with accuracy $\mu$ (and $\hat{M}(k)$ denote its weight tensor on the $k^{th}$ layer). Based on the given accuracy, we know that $\forall k$, $|\hat{\lambda}_{r}^{(k)} - \hat{\lambda}_{r}^{(k)}| \leq \mu$, $|\hat{a}_{r}^{(k)}(s^{(k)} + o^{(k)} + k_{x}^{(k)} \times k_{y}^{(k)} + 1) \leq \sqrt{O(k)} \mu$, $|\hat{b}_{r}^{(k)} - \hat{b}_{r}^{(k)}| \leq \sqrt{O(k)} \mu$, $|\hat{\hat{B}}^{(k)} - \hat{C}^{(k)}| \leq \sqrt{k_{x}^{(k)} k_{y}^{(k)}} \mu$, where $s$, $o$, $x$, and $y$ are the number of input channels, the number of output channels, the height of the kernel and the width of the kernel, as defined in Section 3. For simplicity, in this proof, let us just use $X^{(k)}, Y^{(k)}, a_{r}^{(k)}, b_{r}^{(k)}, C_{r}^{(k)}$ to denote $\hat{X}^{(k)}, \hat{Y}^{(k)}, \hat{a}_{r}^{(k)}, \hat{b}_{r}^{(k)}, \hat{C}_{r}^{(k)}$. $X^{(k)} \in \mathbb{R}^{H \times W \times x^{(k)}}, Y^{(k)} \in \mathbb{R}^{H \times W \times o^{(k)}}$.

We have

$$\mathcal{F}_{3}^{1,2}(Y^{(k)})_{fg} = \sqrt{H \times W} \sum_{i} |\mathcal{F}_{3}^{1,2}(Y^{(k)})_{fg}^{i}| \sum_{r=1}^{\hat{r}^{(k)}} \hat{\lambda}_{r}^{(k)} \hat{a}_{r}^{(k)}(b_{r}^{(k)} C_{r}^{(k)})_{fg}$$

$$\mathcal{F}_{3}^{1,2} (\hat{Y}^{(k)})_{fg} = \sqrt{H \times W} \sum_{i} |\mathcal{F}_{3}^{1,2} (\hat{Y}^{(k)})_{fg}^{i}| \sum_{r=1}^{\hat{r}^{(k)}} \hat{\lambda}_{r}^{(k)} \hat{a}_{r}^{(k)}(\hat{b}_{r}^{(k)} \hat{C}_{r}^{(k)})_{fg}$$
where $\sqrt{HW}$ is a normalization factor defined in Lemma 3.7

Let $e^{(k)} = \| \hat{Y}^{(k)} - Y^{(k)} \|_F$. Then for each $k$, let $\varphi = \sum_{f,g,i,j} \left( \sum_r \tilde{R}^{(k)} \lambda_r a^{(k)}_{ri} b^{(k)}_{rj} (F_2(c^{(k)}_r)_{fg} - \tilde{F}_2(c^{(k)}_r)_{fg}) \right)^2$

and $\psi = \sum_{f,g,i,j} \left( \sum_r \tilde{R}^{(k)} (\lambda_r a^{(k)}_{ri} b^{(k)}_{rj} - \lambda_r \tilde{a}^{(k)}_{ri} \tilde{b}^{(k)}_{rj}) F_2(\tilde{c}^{(k)}_r)_{fg}) \right)^2$. We first bound $\varphi$ and $\psi$ as follows.

**Bound $\varphi = \sum_{f,g,i,j} \left( \sum_r \tilde{R}^{(k)} (\lambda_r a^{(k)}_{ri} b^{(k)}_{rj} - \lambda_r \tilde{a}^{(k)}_{ri} \tilde{b}^{(k)}_{rj}) F_2(\tilde{c}^{(k)}_r)_{fg}) \right)^2$:** All calculations are based on the $k^{th}$ layer, we remove the layer number $(k)$ for ease of reading. So $a = a^{(k)}$ (the same for $b, c,$ and $R$). Then

$$\sum_{f,g,i,j} \left( \sum_r \tilde{R}^{(k)} (\lambda_r a^{(k)}_{ri} b^{(k)}_{rj} - \lambda_r \tilde{a}^{(k)}_{ri} \tilde{b}^{(k)}_{rj}) F_2(c^{(k)}_r)_{fg}) \right)^2$$

$$\leq \sum_{f,g,i,j} \left( \sum_r (\lambda_r a_{ri} b_{rj} - \lambda_r \tilde{a}_{ri} \tilde{b}_{rj})^2 \sum_r (F_2(c_r)_{fg} - \tilde{F}_2(c_r)_{fg}) \right)^2$$

$$\leq \sum_{r} (\lambda_r^2 \sum_i a_{ri}^2 \sum_j b_{rj}^2) \sum_{f,g} (F_2(c_r)_{fg} - \tilde{F}_2(c_r)_{fg})^2$$

$$\leq \sum_r \lambda_r^2 \tilde{R} \sigma_{kg} \mu^2$$

**Bound $\psi = \sum_{f,g,i,j} \left( \sum_r \tilde{R}^{(k)} (\lambda_r a^{(k)}_{ri} b^{(k)}_{rj} - \lambda_r \tilde{a}^{(k)}_{ri} \tilde{b}^{(k)}_{rj}) F_2(\tilde{c}^{(k)}_r)_{fg}) \right)^2$:** Similarly, we remove the layer number $(k)$ for ease of reading. Then we have

$$\sum_{f,g,i,j} \left( \sum_r \tilde{R}^{(k)} (\lambda_r a^{(k)}_{ri} b^{(k)}_{rj} - \lambda_r \tilde{a}^{(k)}_{ri} \tilde{b}^{(k)}_{rj}) F_2(c^{(k)}_r)_{fg}) \right)^2$$

$$\leq \sum_{f,g,i,j} \left( \sum_r (\lambda_r a_{ri} b_{rj} - \lambda_r \tilde{a}_{ri} \tilde{b}_{rj})^2 \sum_r F_2(c_r)_{fg}) \right)^2$$

$$= \sum_{f,g,i,j} \left( \sum_r (\lambda_r (a_{ri} b_{rj} - \tilde{a}_{ri} \tilde{b}_{rj}) + (\lambda_r - \lambda_r) a_{ri} \tilde{b}_{rj})^2 \sum_r F_2(c_r)_{fg}) \right)^2$$

$$\leq \sum_{f,g,i,j} \left( (2 \sum_r \lambda_r^2 (a_{ri} b_{rj} - \tilde{a}_{ri} \tilde{b}_{rj})^2 + 2 \sum_r (\lambda_r - \lambda_r) a_{ri} \tilde{b}_{rj})^2 \sum_r F_2(c_r)_{fg}) \right)^2$$

$$\leq \sum_{f,g,i,j} \left( 4 \sum_r \lambda_r^2 (a_{ri} b_{rj} - \tilde{a}_{ri} \tilde{b}_{rj})^2 + 2 \sum_r (\lambda_r - \lambda_r) a_{ri} \tilde{b}_{rj})^2 \sum_r F_2(c_r)_{fg}) \right)^2$$

$$= (4 \sum_r \lambda_r^2 (a_{ri}^2 \sum_j (b_{rj} - \tilde{b}_{rj})^2 + \sum_j (a_{ri} - \tilde{a}_{ri})^2 \sum_j \tilde{b}_{rj}^2)^2 + 2 \sum_r (\lambda_r - \lambda_r) a_{ri} \tilde{b}_{rj})^2 \sum_r \sum_{f,g} F_2(c_r)_{fg}) \right)^2$$

$$\leq (4 \sum_r \lambda_r^2 (a_{ri}^2 + s \mu^2) + 2 \tilde{R} \mu^2) \tilde{R}$$

$$= (4 \sum_r \lambda_r^2 (a_{ri} + s \mu) + 2 \tilde{R}^2) \mu^2$$
Bound $\epsilon(k) = \|\tilde{\gamma}(k) - \gamma(k)\|_F^2$: Similarly, we remove the layer number $(k)$. And we let $w_i = \mathcal{F}_3^{1,2}(\chi^{(k)})_{fgi}$, $\hat{w}_i = \mathcal{F}_3^{1,2}(\tilde{\chi}^{(k)})_{fgi}$, $u_i = \sum_r \lambda_r \frac{\partial \hat{w}_i}{\partial r} b_{rj} \mathcal{F}_2(C_r)_{fg}$ and $\hat{u}_i = \sum_r \lambda_r \frac{\partial \tilde{\gamma}(k)}{\partial r} b_{rj} \mathcal{F}_2(\tilde{C}_r)_{fg}$.

$$
\begin{align*}
\|\tilde{\gamma}(k) - \gamma(k)\|_F^2 \\
&= \|\mathcal{F}_3^{1,2}(\tilde{\gamma}(k)) - \mathcal{F}_3^{1,2}(\gamma(k))\|_F^2 \\
&= \sum_{f,g,i,j} \|\mathcal{F}_3^{1,2}(\tilde{\gamma}(k))_{fgi,j} - \mathcal{F}_3^{1,2}(\gamma(k))_{fgi,j}\|^2 \\
&= HW \sum_{f,g,i,j} (w_i - \hat{w}_i)^2 + \sum_{i} (w_i - \hat{w}_i)^2 \\
&\leq 2HW \sum_{f,g,i,j} \left( \sum_{i} (w_i - \hat{w}_i)^2 \right) + \sum_{i} (w_i - \hat{w}_i)^2 \\
&\leq 2HW \sum_{f,g,i,j} \left( \sum_{i} (w_i^2) \sum_{i} (u_i - \hat{u}_i)^2 \right) + \sum_{i} (w_i - \hat{w}_i)^2 \sum_{i} \hat{u}_i^2 \\
&\leq 2HW \sum_{f,g,i,j} \left( \sum_{i} (w_i^2) \sum_{i} (u_i - \hat{u}_i)^2 \right) + \sum_{i} (w_i - \hat{w}_i)^2 \sum_{i} \hat{u}_i^2 \\
&\leq 4HW \left\| \chi^{(k)} \right\|_F^2 \mu^2 \left( \sum_r \lambda_r^2 \hat{R}k_x k_y + 4 \sum_r \lambda_r^2 (o + s) \hat{R} + 2\hat{R}^2 \right) + 2 \sum_{f,g,i,j} (w_i - \hat{w}_i)^2 \sum_{i} \hat{u}_i^2 \\
&\leq 4HW \left\| \chi^{(k)} \right\|_F^2 \mu^2 \left( \sum_r \lambda_r^2 \hat{R}k_x k_y + 4 \sum_r \lambda_r^2 (o + s) \hat{R} + 2\hat{R}^2 \right) + 2 \left\| \chi^{(k)} - \tilde{\chi}^{(k)} \right\|_F^2 \left\| \tilde{\lambda} \right\|_F^2
\end{align*}
$$

When $k = 1$, we know that $\chi^{(1)} = \tilde{\chi}^{(1)}$, so

$$
\begin{align*}
\|\tilde{\gamma}(1) - \gamma(1)\|_F^2 \\
&\leq 4HW \left\| \chi^{(1)} \right\|_F^2 \mu^2 \left( \sum_r \lambda_r^2 \hat{R}k_x k_y + 4 \sum_r \lambda_r^2 (o + s) \hat{R} + 2\hat{R}^2 \right)
\end{align*}
$$

When $k > 1$, we have
Let \( \alpha^{(k)} = 4HW \| \chi^{(k)} \|_F^2 (\sum_r \lambda_r^{(k)} R_x^{(k)} k_x^{(k)} + 4 \sum_r \lambda_r^{(k)} s^{(k)} + 2(\epsilon^{(k)} - 1)) \) and \( \beta^{(k)} = 2 \| \tilde{\mathcal{M}}^{(k)} \|_F^2 \). Then the difference between the final output of the two networks are bounded by:

\[
\| \tilde{\mathbf{M}}(\chi) - \tilde{\mathbf{M}}(\chi) \|_F^2 \\
= \| \text{ReLU} (\tilde{\mathbf{y}}) - \text{ReLU} (\tilde{\mathbf{y}}) \|_F^2 \\
\leq \| \tilde{\mathbf{y}} - \tilde{\mathbf{y}} \|_F^2 \\
\leq \sum_{k=1}^n \alpha^{(k)} \prod_{i=k+1}^n \beta^{(i)}
\]

Since \( \forall k \in [n], \| \chi^{(k)} \| \leq \prod_{i=k}^n \| \chi^{(n+1)} \|_{\mathcal{M}^{(i)}} \), to obtain an \( \epsilon \)-cover of the compressed network, we can first assume \( \beta^{(k)} \geq 1 \forall k \in [n] \). Then \( \mu \) need to satisfy:

\[
\mu \leq \frac{\epsilon}{2 \sqrt{HW n \| \chi^{(n+1)} \|_F^2 R^{(s)} (\frac{\sqrt{2} \| \tilde{\mathcal{M}}^{(i)} \|_{\mathcal{M}^{(i)}}}{\| \tilde{\mathcal{M}}^{(s)} \|_{\mathcal{M}^{(s)}}})^n (\lambda^{(s)})^2 k_x^{(s)} k_y^{(s)} + 4(\lambda^{(s)})^2 (s^{(s)} + o^{(s)}) + 2}
\]

where \( \hat{R}^{(s)} = \max_k r^{(k)} \lambda^{(s)} = \max_r k_x^{(k)}, s^{(s)} = \max_k s^{(k)}, o^{(s)} = \max_k o^{(k)}, k_x^{(s)} = \max_k k_x^{(k)} \). As when \( \mu \) is fixed, the number of networks in our cover will at most be \( (\frac{1}{\mu})^d \) where \( d \) denote the number of parameters in the compressed network. Hence, the covering number w.r.t to a given \( \epsilon \) is \( \hat{O}(n^d) \) (\( n \) is the number of layers in the given neural network). As for practical neural networks, the number of layers \( n \) is usually much less than \( O(\log(d)) \), thus the covering number we obtained w.r.t to a given \( \epsilon \) is just \( \hat{O}(d) \) for practical neural networks.

### E Fully Connected Networks: Compressibility and Generalization

In this section, we derive generalization bounds for fully connected (FC) neural networks (denoted as \( \tilde{\mathcal{M}} \)) using tensor methods.
E.1 Compression of a FC Network with CPL

Original Fully Connected Neural Network: Let $\mathbb{M}$ denote an $n$-layer fully connected network with ReLU activations, where $A^{(k)} \in \mathbb{R}^{h(k) \times h(k+1)}$ denotes the weight matrix of the $k$th layer, $x^{(k)} \in \mathbb{R}^{h(k)}$ denotes the input to $k$th layer, and $y^{(k)} \in \mathbb{R}^{h(k)}$ denotes the output of the $k$th layer before activation in $\mathbb{M}$. Transform original FCN to a CP-FCN: We transform the original fully connected network $\mathbb{M}$ to a network $\tilde{\mathbb{M}}$ with CPLayer. The $k$th layer of $\mathbb{M}$ is denoted by $\mathcal{M}^{(k)} \in \mathbb{R}^{s_1^{(k)} \times s_2^{(k)} \times s_1^{(k+1)} \times s_2^{(k+1)}}$, a 4-dimensional tensor reshaped from $A^{(k)}$ where $s_1^{(k)} \times s_2^{(k)} = h_k, \forall k \in [n].$

Input and Output of $\tilde{\mathbb{M}}$: The original input and output vectors of $\mathbb{M}$ are reshaped into matrices. The input to the $k$th layer of $\mathbb{M}$, denoted by $X^{(k)} \in \mathbb{R}^{s_1^{(k)} \times s_2^{(k)}}$, is a matrix reshaped from the input vector $x^{(k)}$ of the $k$th layer in the original network $\mathbb{M}$. Similarly, the output of the $k$th layer before activation in $\mathbb{M}$, denoted by $Y^{(k)} \in \mathbb{R}^{s_1^{(k)} \times s_2^{(k)}}$, is a matrix reshaped from the output vector $y^{(k)}$ of the $k$th layer in the original network $\mathbb{M}$. For prediction purposes, we reshape the output $Y^{(n)}$ of the last layer in $\tilde{\mathbb{M}}$ back into a vector. So the final outputs of $\mathbb{M}$ and $\tilde{\mathbb{M}}$ are of the same dimension.

Assumption E.1 (Polyadic Form of $\mathbb{M}$): For each layer $k$, assume the weight tensor $\mathcal{M}^{(k)}$ of $\mathbb{M}$ has a Polyadic form with rank $R^{(k)} \leq \min\{s_1^{(k)}, s_2^{(k)}, s_1^{(k+1)}, s_2^{(k+1)}\}$:

$$\mathcal{M}^{(k)} = \sum_{i=1}^{R^{(k)}} \lambda_i^{(k)} a_i^{(k)} \otimes b_i^{(k)} \otimes c_i^{(k)} \otimes d_i^{(k)}$$

(69)

where $\forall i, a_i, b_i, c_i, d_i$ are unit vectors in $\mathbb{R}^{s_1^{(k)}}, \mathbb{R}^{s_2^{(k)}}, \mathbb{R}^{s_1^{(k+1)}}, \mathbb{R}^{s_2^{(k+1)}}$ respectively, and $\forall 1 \leq i \leq R^{(k)}, \langle a_i, a_i \rangle = 1, \langle b_i, b_i \rangle = 1, \langle c_i, c_i \rangle = 1, \langle d_i, d_i \rangle = 1$. Moreover, for each $\mathcal{M}^{(k)}$, $\lambda_i^{(k)} \geq \lambda_{i+1}^{(k)}, \forall i$, and the absolute value of the smallest $|\lambda_{R^{(k)}}|$ can be arbitrarily small.

The total number of parameters in $\mathbb{M}$ is $(s_1^{(1)} + s_2^{(1)}) + (s_1^{(1)} + s_2^{(1)} + 1)R^{(1)}$ and a smaller $R^{(k)}$ renders fewer number of parameters and thus leads to compression. We introduce a compression mechanism that prunes out the smaller components of weight tensor of $\mathbb{M}$, i.e., a low rank approximation of each weight tensor $\mathcal{M}^{(k)}$ of the $k$th layer, and generates a compressed CP-FCN $\tilde{\mathbb{M}}$. The algorithm is depicted in Algorithm 2.

Compression of a FC Network with CPL: In [Li and Huang (2018)], a tensor decomposition algorithm (procedure 1 in [Li and Huang (2018)]) on tensors with asymmetric orthogonal components is guaranteed to recover the top-$r$ components with the largest singular values. To compress $\mathbb{M}$, we apply top-$\tilde{R}^{(k)}$ ($\tilde{R}^{(k)} \leq R^{(k)}$) CP decomposition algorithm on each $\mathcal{M}^{(k)}$, obtaining the components from CP decomposition $(\tilde{\lambda}_i^{(k)}, \tilde{a}_i^{(k)}, \tilde{b}_i^{(k)}, \tilde{c}_i^{(k)}, \tilde{d}_i^{(k)})$, $i \in [\tilde{R}^{(k)}]$. Therefore, we achieve a compressed network $\tilde{\mathbb{M}}$ of $\mathbb{M}$, and the $j$th layer of the compressed network $\tilde{\mathbb{M}}$ has weight tensor as follows

$$\tilde{\mathcal{M}}^{(k)} = \sum_{i=1}^{\tilde{R}^{(k)}} \tilde{\lambda}_i^{(k)} \tilde{a}_i^{(k)} \otimes \tilde{b}_i^{(k)} \otimes \tilde{c}_i^{(k)} \otimes \tilde{d}_i^{(k)}.$$  

(70)

As each $\mathcal{M}^{(k)}$ has a low rank orthogonal CP decomposition by our assumption, the returned results $\{\tilde{\lambda}_i^{(k)}, \tilde{a}_i^{(k)}, \tilde{b}_i^{(k)}, \tilde{c}_i^{(k)}, \tilde{d}_i^{(k)}, i \in [\tilde{R}^{(k)}]\}$ from procedure 1 in [Li and Huang (2018)] are perfect recoveries of $\{\lambda_i^{(k)}, a_i^{(k)}, b_i^{(k)}, c_i^{(k)}, d_i^{(k)}, i \in [R^{(k)}]\}$ according to the robustness theorem in [Li and Huang (2018)]. Our compression procedure is depicted in Algorithm 2.

We denote the input matrix of the $k$th layer in $\tilde{\mathbb{M}}$ as $\tilde{X}^{(k)}$, and the output matrix before activation as $\tilde{Y}^{(k)}$. Note that $X^{(1)} = \tilde{X}^{(1)}$ as the input data is not being modified.

Algorithm 2 is designed for general neural networks. For neural networks with CPLayer, line 3 can be done by pruning out small components from CP decomposition, and only keeping top-$\tilde{R}^{(k)}$ components. For notation simplicity, assume for each layer in $\tilde{\mathbb{M}}$, the width of the $k$th layer is a square of some integer $s^{(k)}$. Then the input to the $k$th layer of $\tilde{\mathbb{M}}$ is a ReLu transformation of the output of the $k - 1$th layer as in equation (71). The output of the $k$th layer of $\tilde{\mathbb{M}}$ is illustrated in equation (72) as the weight tensor which permits a CP forms as in
Algorithm 2 Compression of Fully Connected Neural Networks
$FBR$ (in Appendix B) denotes a sub-procedure which calculates $R^{(k)}$, such that $\|\hat{M}(X) - \hat{M}(X)\|_F \leq \epsilon \|\hat{M}(X)\|_F$ holds for any input $X$ in the training dataset and for any given $\epsilon$.
$TNN-Project$ (in Appendix B) denotes a sub-procedure which returns a compressed network $\hat{M}$ by pruning out the smaller components in the Polyadic form of the weight tensors in the original CNN.

More intuitions of the sub-procedures $FBR$ and $TNN-Project$ are described in Section E.2.

Input: A FCN $M$ of $n$ layers and a margin $\gamma$

Output: A compressed $\hat{M}$ whose expected error $L_0(\hat{M}) \leq \hat{L}_s(M) + \hat{O}\left(\sqrt{\sum_{k=1}^{n} R^{(k)}(2s^{(k)} + 2s^{(k+1)} + 1)}\right)$

1. Calculate all layer cushions $\{s^{(k)}\}_{k=1}^{n}$ based on definition B.4 and $R^{(k)} = \min\{s^{(k)}, s^{(k+1)}\}$ for each layer $k$.
2. If $\hat{M}$ does not have CPL, apply a CP-decomposition to the weight tensor of each layer $k$.
3. Set the perturbation parameter $\epsilon := \frac{2}{2\max_X \|M(X)\|_F}$.
4. Compute number of components needed for each layer of the compressed network $\{\hat{R}^{(k)}\}_{k=1}^{n}$.
5. $\hat{M} \leftarrow \text{TNN-Project}\left(M, \{\hat{R}^{(k)}\}_{k=1}^{n}\right)$
6. Return the compressed convolutional neural network $\hat{M}$.

Equation (69).

$$X^{(k)} = \text{ReLU}\left(Y^{(k-1)}\right)$$

$$Y^{(k)} = \frac{\hat{R}^{(k)}}{\sum_{i=1}^{R^{(k)}} |\lambda_i^{(k)}|^\top X^{(k)} b_i^{(k)} c_i^{(k)} \otimes \hat{d}_i^{(k)} + \phi_i^{(k)}(X^{(k)})}$$  \hspace{1cm} (72)

Where $\phi^{(k)} = \sum_{j=R^{(k)}}^{R^{(k+1)}} |\lambda_j^{(k)}| a_j^{(k)} \otimes b_j^{(k)} \otimes c_j^{(k)} \otimes \hat{d}_j^{(k)}$, $\phi_i^{(k)}(X^{(k)})$ denotes the multilinear operation of the tensor $\phi^{(k)}$ on $X^{(k)}$, i.e., $\{\phi_i^{(k)}(X^{(k)})\}_{i,j} = \sum_{k,l} \phi^{(k)}_{j,k,l} X_{k,l}^{(k-1)}$ and $a_i^{(k)}, b_i^{(k)}, c_i^{(k)}, \hat{d}_i^{(k)} \in \mathbb{R}^{s_i}$. Similarly, the input and output of the $k$th layer of the compressed neural nets $\hat{M}$ satisfy

$$\hat{X}^{(k)} = \text{ReLU}\left(\hat{Y}^{(k-1)}\right)$$

$$\hat{Y}^{(k)} = \frac{\hat{R}^{(k)}}{\sum_{i=1}^{\hat{R}^{(k)}} |\hat{\lambda_i^{(k)}}| \hat{X}^{(k)} b_i^{(k)} c_i^{(k)} \otimes \hat{d}_i^{(k)}}.$$  \hspace{1cm} (74)

### E.2 Characterizing Compressibility of FC Networks with CPL

Now we characterize the compressibility of the fully connected network with CPL $M$ through properties defined in the following, namely reshaping factor, tensorization factor, layer cushion and tensor noise bound.

**Definition E.2.** (reshaping factor) The reshaping factor $\rho^{(k)}$ of layer $k$ is defined to be the smallest value $\rho^{(k)}$ such that for any $x \in S$,

$$\|X^{(k)}\|_G \leq \rho^{(k)} \|X^{(k)}\|_F$$  \hspace{1cm} (75)

The reshaping factor upper bounds the ratio between the spectral norm and Frobenius norm of the reshaped input in the $k$th layer over any data example in the training dataset. Reshaping the vector examples into matrix examples improves the compressibility of the network (i.e., renders smaller $\rho^{(k)}$) as illustrated and empirically verified in [Su et al. 2018]. Note that $\hat{X}^{(k)}$ is the input to the $k$th layer of the compressed network $\hat{M}$, and $\rho^{(k)} \leq 1, \forall k$.

**Definition E.3.** (tensorization factor) The tensorization factor $\{t_j^{(k)}\}_{j=1}^{R^{(k)}}$ of the $k$th layer regarding the network with CPL $M$ and the original network $\hat{M}$ is defined as:

$$t_j^{(k)} = \sum_{r=1}^{j} |\lambda_r^{(k)}|, \forall j.$$  \hspace{1cm} (76)
The tensorization factor measures the amplitudes of the leading components. By Lemma E.5 the tensorization factor is the upper bound of operator norm of the weight tensor.

**Definition E.4.** (layer cushion). Our definition of layer cushion for each layer $k$ is similar to Arora et al. (2018). The layer cushion $\zeta^{(k)}$ of layer $k$ is defined to be the largest value $\zeta^{(k)}$ such that for any $x \in S$, $\zeta^{(k)} \|A^{(k)}\|_F \|x^{(k)}\| \leq \|x^{(k+1)}\|$.

The layer cushion defined in Arora et al. (2018) is slightly larger than ours since our RHS is $\|x^{(k+1)}\| = \text{ReLU} (A^{(k)} x^{(k)})$ while the RHS of the inequality in the definition of layer cushion in Arora et al. (2018) is $A^{(k)} x^{(k)}$. The layer cushion under our settings also considers how much smaller the output $\|x^{(k+1)}\|$ is compared to is compared to the upper bound $\|A^{(k)}\|_F \|x^{(k)}\|$.

**Definition E.5.** (tensor noise bound). The tensor noise bound $\{\zeta^{(k)}\}_{k=1}^n$ of the $k$th layer measures the amplitudes of the remaining components after pruning out the ones with amplitudes smaller than the $j$th component:

$$\zeta_j^{(k)} := \sum_{r=j+1}^{R^{(k)}} |\lambda_r^{(k)}|$$

The tensor noise bound measures the amplitudes of the CP components that are pruned out by the compression algorithm, and the smaller it is, the more low-rank the weight matrix is. We will see that a network equipped with CPL will be much more low-rank than standard networks.

### E.3 Generalization Guarantee of Fully Connected Neural Networks

We have introduced the compression mechanism in Algorithm 2. For a fully connected network with CPL $\hat{M}$ that is characterized by the properties such as reshaping factor, tensorization factor, layer cushion and tensor noise bound, in section E.2, we derive the generalization error bound of a compression network with any chosen ranks $\{\hat{R}^{(k)}\}_{k=1}^n$ as follows.

**Theorem E.6.** For any fully connected network $M$ of $n$ layers satisfying the Assumptions E.1, Algorithm 2 generates a compressed network $\hat{M}$ such that with high probability over the training set, the expected error $L_0(\hat{M})$ is bounded by

$$L_0(\hat{M}) \leq \hat{L}_\gamma(M) + \hat{O}\left(\sqrt{\frac{\sum_{k=1}^{n} \hat{R}^{(k)}(2s^{(k)} + 2s^{(k+1)} + 1)}{m}}\right)$$

(78)

for any margin $\gamma \geq 0$, and the rank of the $k$th layer, $\hat{R}^{(k)}$, satisfies that

$$\hat{R}^{(k)} = \min \left\{ j \in [R^{(k)}] \mid n \rho^{(k)} \zeta_j^{(k)} \Pi_{i=k+1}^{n} t^{(i)} \leq \frac{\gamma}{2 \max_{x \in S} \|M(x)\|_F} \|A^{(k)}\|_F \Pi_{i=k}^{n} \zeta^{(i)}\right\}$$

and $\rho^{(k)}, t^{(k)}, \zeta^{(k)}$ are reshaping factor, tensorization factor, layer cushion and tensor noise bound of the $k$th layer in Definitions E.2, E.3 and E.4 respectively. $\zeta_j^{(k)}$ is defined in the same way with $\zeta^{(k)}$, where $\hat{R}^{(k)}$ is replaced by $j$.

The generalization error of the compressed network $L_0(\hat{M})$ depends on the compressibility of the $M$. The compressibility of the $M$ determines the rank that the compression mechanism should select according to Theorem E.5, which depends on reshaping factor $\rho^{(k)}$, tensorization factor $t^{(k)}$, layer cushion $\zeta^{(k)}$ and tensor noise bound $\zeta_j^{(k)}$.

**Proof sketch of Theorem E.6.** To prove this theorem, we introduce the following Lemma E.7 which reveals that the difference between the output of the original fully connected network $M$ and that of the compressed $M$ is bounded by $\epsilon \|M(x)\|_F$. Then we show the covering number of the compressed network $M$ by approximating each parameter with some certain accuracy is $\hat{O}(d)$ w.r.t to a given $\epsilon$. After bounding the covering number, the rest of the proof follows from conventional learning theory.

**Lemma E.7.** For any fully connected network $M$ of $n$ layers satisfying Assumption E.1, Algorithm 2 generates a compressed Tensorial – FC $\hat{M}$ where for any $x \in S$ and any error $0 \leq \epsilon \leq 1$:

$$\|M(x) - \hat{M}(X)\|_F \leq \epsilon \|M(x)\|_F$$

(79)
The compressed Tensorial – FC \(\hat{M}\) consists of \(\sum_{k=1}^{n} R(k)[2(s(k) + s(k+1)) + 1]\) number of parameters, where each \(R(k)\) is defined as what is stated in Algorithm 4 for each layer \(k \in [n]\),

\[
\hat{R}(k) = \min \left\{ j \in [R(k)] \left| \frac{\rho(k)\xi(k)}{\Pi_{i=k+1}^{n} t(i)} \leq \frac{\epsilon}{n} \left\| A^{(k)} \right\|_{F} \right\}
\]

The complete proofs are in \[E.4\].

### E.4 Complete Proofs of Fully Connected Neural Networks

To prove Lemma \[E.7\], Lemma \[E.8\] (introduced below) is needed.

**Lemma E.8.** For any fully connected network \(M\) of \(n\) layers satisfying the assumptions in section \[3\] given a list of ranks \(\{R(k)\}_{k=1}^{n}\) \(\forall k, R(k) \leq R(k)\), after tensorizing each layer in \(M\) and making \(M\) into \(\hat{M}\), Algorithm \[6\] generates a compressed tensorial neural network \(\hat{M}\) with \(\sum_{k=1}^{n} R(k)[2(s(k) + s(k+1)) + 1]\) total parameters where for any \(x \in S:\]

\[
\left\| M(x) - \hat{M}(x) \right\|_{F} \leq \left( \sum_{k=1}^{n} \frac{\rho(k)\xi(k)}{\Pi_{i=k+1}^{n} t(i)} \right) \left\| M(x) \right\|_{F}
\]

where \(M\) is the matricized version of \(x\), and \(\rho(k),\xi(k)\) are reshaping factor, tensorization factor, layer cushion, and tensor noise bound of the \(k\)th layer in Definitions \[E.2\] \[E.3\] \[E.4\] and \[E.5\] respectively.

**Proof.** (of Lemma \[E.8\]) Based on Algorithm \[2\] since for each layer \(k\) in the compressed network \(\hat{M}\), representing \(\{\hat{\lambda}(k), \hat{a}(k), \hat{b}(k), \hat{c}(k), \hat{d}(k)\}_{k=1}^{R(k)}\) only needs \(\hat{R}(k)[2(s(k) + s(k+1)) + 1]\) parameters, the total number of parameters in \(\hat{M}\) is \(\sum_{k=1}^{n} \hat{R}(k)[2(s(k) + s(k+1)) + 1]\).

Then as for any \(x \in S, M(x) = M(X)\), and by construction, \(M(X) = X^{(n+1)}\) and \(\hat{M}(X) = X^{(n+1)}\), we can prove the lemma by showing \(\left\| X^{(n+1)} - \hat{X}^{(n+1)} \right\|_{F}\) satisfies the above inequality, and we will prove this by induction.

**Induction Hypothesis:** For any layer \(m \geq 0\), \(\left\| X^{(m)} - \hat{X}^{(m)} \right\|_{F} \leq \left( \sum_{k=1}^{m} \frac{s(k)\xi(k)}{\Pi_{i=k+1}^{m} t(i)} \right) \left\| X^{(m)} \right\|_{F}\)

**Base case:** when \(m = 1\), the above inequality hold trivially as \(X^{(1)} = \hat{X}^{(1)}\) as we cannot modify the input, and the RHS is always \(0\).

**Inductive Step:** Now we assume show that the induction hypothesis is true for all \(m\), let us look what happens at layer \(m + 1\). As we assume perfect recovery in each layer, \(\forall k, \{\hat{\lambda}(k), \hat{a}(k), \hat{b}(k), \hat{c}(k), \hat{d}(k)\}_{k=1}^{\hat{R}(k)} = \{\lambda(k), a(k), b(k), c(k), d(k)\}_{k=1}^{R(k)}\).

Let \(\phi(k) := \sum_{i=1}^{\hat{R}(k)} \lambda_{i}^{(m)} (a_{i})^{(m)} \otimes b_{i}^{(m)} \otimes c_{i}^{(m)} \otimes d_{i}^{(m)}\), and note that \(M^{(k)} = \hat{M}^{(k)} + \phi\).

Then we have

\[
\left\| X^{(m+1)} - \hat{X}^{(m+1)} \right\|_{F} = \left\| \text{ReLU} \left( Y^{(m)} \right) - \text{ReLU} \left( \hat{Y}^{(m)} \right) \right\|_{F}
\]

\[
\leq \left\| \sum_{i=1}^{\hat{R}(m)} \lambda_{i}^{(m)} (a_{i})^{(m)} \otimes b_{i}^{(m)} \otimes c_{i}^{(m)} \otimes d_{i}^{(m)} + \phi^{(m)}(X^{(m)}) - \sum_{i=1}^{\hat{R}(m)} \lambda_{i}^{(m)} (a_{i})^{(m)} \otimes b_{i}^{(m)} \otimes c_{i}^{(m)} \otimes d_{i}^{(m)} \right\|_{F}
\]

\[
= \left\| \sum_{i=1}^{\hat{R}(m)} \lambda_{i}^{(m)} (a_{i})^{(m)} \otimes (X^{(m)} - \hat{X}^{(m)}) b_{i}^{(m)} c_{i}^{(m)} \otimes d_{i}^{(m)} + \phi^{(m)}(X^{(m)}) \right\|_{F}
\]

So

\[
\left\| X^{(m+1)} - \hat{X}^{(m+1)} \right\|_{F} \leq \left\| \sum_{i=1}^{\hat{R}(m)} \lambda_{i}^{(m)} (a_{i})^{(m)} \otimes (X^{(m)} - \hat{X}^{(m)}) b_{i}^{(m)} c_{i}^{(m)} \otimes d_{i}^{(m)} \right\|_{F} + \left\| \phi^{(m)}(X^{(m)}) \right\|_{F}
\]
As $\phi^{(m)}(X^{(m)}) = \sum_{i=1}^{R^{(k)}} \lambda_i^{(k)} (a_i^{(k)})^\top X^{(m)} \cdot b_i^{(k)} c_i^{(k)} \otimes d_i^{(k)}$. Since $\{c_i^{(m)}\}$ and are $\{d_i^{(m)}\}$ are sets of orthogonal vectors with unit norms,

$$\|\phi^{(m)}(X^{(m)})\|_F = \sqrt{\sum_{i=R^{(k)}+1}^{R^{(k)}} [\lambda_i^{(k)} (a_i^{(k)})^\top X^{(m)} b_i^{(k)}]^2}$$

$$\leq \sqrt{\sum_{i=R^{(k)}+1}^{R^{(k)}} (\lambda_i^{(k)})^2 \|a_i^{(k)}\|^2 \|X^{(m)} b_i^{(k)}\|^2}$$

$$\leq \sqrt{\sum_{i=R^{(k)}+1}^{R^{(k)}} (\lambda_i^{(k)})^2 \|X^{(m)}\|^2 \|b_i^{(k)}\|^2}$$

$$= \sum_{i=R^{(k)}+1}^{R^{(k)}} (\lambda_i^{(k)})^2 \|X^{(m)}\|$$

$$= \xi^{(m)} \|X^{(m)}\|$$

$$\leq \xi^{(m)} \rho^{(m)} \|X^{(m)}\|_F$$

$$\leq \frac{\rho^{(m)} \xi^{(m)}}{\zeta^{(m)}} \|X^{(m+1)}\|_F$$

Similarly, we can bound $\|\sum_{i=1}^{R^{(m)}} \lambda_i^{(m)} (a_i^{(m)})^\top (X^{(m)} - \hat{X}^{(m)}) b_i^{(m)} c_i^{(m)} \otimes d_i^{(m)}\|_F$ as follows:

$$\|\sum_{i=1}^{R^{(m)}} \lambda_i^{(m)} (a_i^{(m)})^\top (X^{(m)} - \hat{X}^{(m)}) b_i^{(m)} c_i^{(m)} \otimes d_i^{(m)}\|_F$$

$$= \sqrt{\sum_{i=1}^{R^{(m)}} [\lambda_i^{(m)} (a_i^{(m)})^\top (X^{(m)} - \hat{X}^{(m)}) b_i^{(m)}]^2}$$

$$\leq \sqrt{\sum_{i=1}^{R^{(m)}} (\lambda_i^{(m)})^2 \|X^{(m)} - \hat{X}^{(m)}\|^2}$$

$$\leq \sqrt{\sum_{i=1}^{R^{(m)}} (\lambda_i^{(m)})^2 \|X^{(m)} - \hat{X}^{(m)}\|_F}$$

$$= \sqrt{\|A^{(m)}\|_F^2 \|X^{(m)} - \hat{X}^{(m)}\|_F}$$

$$\leq \rho^{(m)} \xi^{(m)} \|A^{(m)}\|_F \|X^{(m)} - \hat{X}^{(m)}\|_F$$

$$\leq \rho^{(m)} \xi^{(m)} \|A^{(m)}\|_F \cdot \frac{\|X^{(m+1)}\|_F}{\|A^{(m)}\|_F} \cdot \prod_{i=k+1}^{m-1} \frac{\rho^{(k)} \xi^{(k)}}{\zeta^{(i)}} \cdot \|X^{(m)}\|_F$$

$$= \left(\sum_{k=1}^{m-1} \rho^{(k)} \xi^{(k)} \|A^{(k)}\|_F \cdot \prod_{i=k+1}^{m-1} \frac{\rho^{(i)} \xi^{(i)}}{\zeta^{(i)}} \right) \|X^{(m+1)}\|_F$$
Combining the above two terms together, we have
\[
\|X^{(m+1)} - \hat{X}^{(m+1)}\|_F \\
\leq \left( \sum_{k=1}^{m-1} \frac{\rho(k) \zeta^{(k)}}{\zeta^{(k)} \|A^{(k)}\|_F} \cdot \Pi_{i=k+1}^{m} t^{(i)} \right) \cdot \|X^{(m+1)}\|_F + \frac{\rho(m) \zeta^{(m)}}{\zeta^{(m)} \|A^{(m)}\|_F} \cdot \|X^{(m+1)}\|_F \\
= \left( \sum_{k=1}^{m-1} \frac{\rho(k) \zeta^{(k)}}{\zeta^{(k)} \|A^{(k)}\|_F} \cdot \Pi_{i=k+1}^{m} t^{(i)} + \frac{\rho(m) \zeta^{(m)}}{\zeta^{(m)} \|A^{(m)}\|_F} \right) \cdot \|X^{(m+1)}\|_F \\
= \left( \sum_{k=1}^{m} \frac{\rho(k) \zeta^{(k)}}{\zeta^{(k)} \|A^{(k)}\|_F} \cdot \Pi_{i=k+1}^{m} t^{(i)} \right) \cdot \|X^{(m+1)}\|_F
\]

Where the second to the last equality is due to the fact that for any \(\alpha, \beta \in \mathbb{R}\), \((\Pi_{i=k+1}^{k} \alpha_i) \times \beta = \beta\).

Then we can proceed to prove Lemma E.7.

**Proof. (of Lemma E.7)** Based on the assumptions of the components from CP decomposition for each \(M^{(k)}\) in section 3, the \(\{\hat{R}^{(k)}\}_{k=1}^{m}\) returned by Algorithm 4 will satisfy:

- \(\forall k, \hat{R}^{(k)} \leq R^{(k)}\)
- \(\rho^{(k)} \zeta^{(k)} \Pi_{i=k+1}^{m} t^{(i)} \leq \frac{\epsilon}{n} \|A^{(k)}\|_F \Pi_{i=k}^{m} \zeta^{(i)}\)

Thus,
\[
\frac{\rho^{(k)} \zeta^{(k)}}{\zeta^{(k)} \|A^{(k)}\|_F} \cdot \Pi_{i=k+1}^{m} t^{(i)} \leq \frac{\epsilon}{n}
\]

Then by Lemma E.8
\[
\|\hat{M}(x) - \hat{M}(X)\|_F \leq \epsilon \|\hat{M}(x)\|_F
\]

Before proving Theorem E.6, Lemma E.9 (introduced below) is needed.

**Lemma E.9.** For any fully connected network \(M\) of \(n\) layers satisfying the assumptions in section 3 and any margin \(\gamma \geq 0\), \(\hat{M}\) can be compressed to a fully-connected tensorial neural network \(\hat{M}\) with \(\sum_{k=1}^{m} \hat{R}^{(k)}[2(s^{(k)} + s^{(k+1)}) + 1]\) total parameters such that for any \(x \in S, \hat{L}_{\gamma}(\hat{M}) \leq \hat{L}_{\gamma}(M)\). Here, for each layer \(k\),
\[
\hat{R}^{(k)} = \min \left\{ j \in [R^{(k)}] \mid \rho^{(k)} \zeta^{(k)} \Pi_{i=k+1}^{m} t^{(i)} \leq \frac{\epsilon}{n} \|A^{(k)}\|_F \Pi_{i=k}^{m} \zeta^{(i)}\right\}
\]

**Proof. (of Lemma E.9)** If \(\gamma \geq 2 \max_{x \in S} \|M(x)\|_F\), for any pair \((x,y) \in S\), we have
\[
|M(x)[y] - \max_{j \neq y} M(x)[j]|^2 \leq (|M(x)[y]| + \max_{j \neq y} M(x)[j])^2 \leq 4 \max_{x \in S} \|M(x)\|_F^2 \leq \gamma^2
\]

Then the output margin of \(M\) cannot be greater than \(\gamma\) for any \(x \in S\). Thus \(\hat{L}_{\gamma}(M) = 1\).

If \(\gamma < 2 \max_{x \in S} \|M(x)\|_F\), setting
\[
\epsilon = \frac{\gamma}{2 \max_{x \in S} \|M(x)\|_F}
\]

in Lemma E.7 we obtain a compressed fully-connected tensorial neural network \(\hat{M}\) with the desired number of parameters and
\[
\|\hat{M}(x) - \hat{M}(X)\|_F < \frac{\gamma}{2} \Rightarrow \forall j, |\hat{M}(x)[j] - \hat{M}(X)[j]| < \frac{\gamma}{2}
\]
Then for any pair \((x, y) \in S\), if \(\tilde{M}(x)[y] > \gamma + \max_{j \neq y} M(x)[j]\), \(\tilde{M}\) classifies \(x\) correctly as well because:

\[
\tilde{M}(X)[y] > M(x)[y] - \frac{\gamma}{2} > \max_{j \neq y} M(x)[j] + \frac{\gamma}{2} > \max_{j \neq y} \tilde{M}(X)[j]
\]

Thus, \(\hat{L}_\alpha(\tilde{M}) \leq \hat{L}_\gamma(M)\).

Now we prove the main theorem [E.6] by bounding the covering number given any \(\epsilon\).

**Proof.** (of Theorem E.6) To be more specific, let us bound the covering number of the compressed network \(\tilde{M}\) by approximating each parameter with accuracy \(\mu\).

**Covering Number Analysis for Fully Connected Neural Network** Let \(\tilde{T}\) denote the network after approximating each parameter in \(M\) with accuracy \(\mu\) (and \(\tilde{T}^{(k)}\) denote its weight tensor on the \(k\)th layer). Based on the given accuracy, we know that \(\forall k\), \(\tilde{\lambda}_i^{(k)} - \lambda_i^{(k)} \leq \mu\) and \(\|\tilde{a}_i^{(k)} - a_i^{(k)}\| \leq \sqrt{s^{(k)}}\mu\) (similar inequalities also hold for \(\tilde{b}_i^{(k)}, \tilde{c}_i^{(k)}, \tilde{d}_i^{(k)}\)). For simplicity, in this proof, let us just use \(a_i^{(k)}, b_i^{(k)}, c_i^{(k)}, d_i^{(k)}\) to denote \(\tilde{a}_i^{(k)}, \tilde{b}_i^{(k)}, \tilde{c}_i^{(k)}, \tilde{d}_i^{(k)}\).

Notice that

\[
Y^{(k)} = \sum_{i=1}^{r} \lambda_i^{(k)} (a_i^{(k)})^T X^{(k)} b_i^{(k)} c_i^{(k)} \otimes d_i^{(k)}
\]

\[
\tilde{Y}^{(k)} = \sum_{i=1}^{r} \hat{\lambda}_i^{(k)} (\tilde{a}_i^{(k)})^T \tilde{X}^{(k)} \tilde{b}_i^{(k)} \hat{c}_i^{(k)} \otimes \hat{d}_i^{(k)}
\]

Let \(\epsilon^{(k)} = \|\tilde{Y}^{(k)} - Y^{(k)}\|_F\). Then for each \(k\), let us first bound \(\|a_i^{(k)}^T X^{(k)} b_i^{(k)} - (\tilde{a}_i^{(k)})^T \tilde{X}^{(k)} \tilde{b}_i^{(k)}\|\) and \(\|c_i^{(k)} \otimes d_i^{(k)} - \hat{c}_i^{(k)} \otimes \hat{d}_i^{(k)}\|_F\) separately.

**Bound** \(\|a_i^{(k)}^T X^{(k)} b_i^{(k)} - (\tilde{a}_i^{(k)})^T \tilde{X}^{(k)} \tilde{b}_i^{(k)}\|\): When \(k = 1\), we know that \(X^{(1)} = \tilde{X}^{(1)}\). Let us first consider the base case where \(k = 1\). For simplicity, let \(a = a_i^{(1)}, \tilde{a} = \tilde{a}_i^{(1)}, b = b_i^{(1)}, \tilde{b} = \tilde{b}_i^{(1)}\), and \(X = X^{(1)}\). Then

\[
\|a_i^{(1)}^T X^{(1)} b_i^{(1)} - (\tilde{a}_i^{(1)})^T \tilde{X}^{(1)} \tilde{b}_i^{(1)}\|
\]

\[
= \|a^T Xb - \tilde{a}^T \tilde{X} \tilde{b}\|
\]

\[
= \|a^T Xb - a^T X \tilde{b} + a^T \tilde{X} \tilde{b} - \tilde{a}^T \tilde{X} \tilde{b}\|
\]

\[
= \|a^T (Xb - \tilde{b}) + (a - \tilde{a})^T \tilde{X} \tilde{b}\|
\]

\[
\leq \|a^T X(b - \tilde{b})\| + \|a - \tilde{a}\| \|X \tilde{b}\|
\]

\[
\leq \|X^T a\| \|b - \tilde{b}\| + \|a - \tilde{a}\| \|X \tilde{b}\|
\]

\[
\leq \mu \sqrt{s^{(1)}} \|X\| (\|a\| + \|\tilde{b}\|)
\]

\[
\leq 2\mu \sqrt{s^{(1)}} \|X\|
\]

The second to the last inequality is because singular values are invariant to matrix transpose.

When \(k \geq 1\), similarly, let \(a = a_i^{(k)}, \tilde{a} = \tilde{a}_i^{(k)}\) (define \(b\) in a similar way), \(X = X^{(k)}\), and \(\tilde{X} = \tilde{X}^{(k)}\). Let
\[ Y = Y^{(k-1)}, \text{ and } \tilde{Y} = \tilde{Y}^{(k-1)} \text{ (basically the output from the } (k-1)\text{th layer before activation). Then} \]

\[
|((a_i^{(k)})^\top X^{(k)} b_i^{(k)} - (\tilde{a_i}^{(k)})^\top \tilde{X}^{(k)} \tilde{b}_i^{(k)})| \\
= |a_i^\top X b - \tilde{a_i}^\top \tilde{X} \tilde{b}| \\
= |a_i^\top X b - \tilde{a_i}^\top \tilde{X} b + \tilde{a_i}^\top \tilde{X} b - \tilde{a_i}^\top \tilde{X} \tilde{b}| \\
\leq |a_i^\top X b - \tilde{a_i}^\top \tilde{X} b| + |\tilde{a_i}^\top X b - \tilde{a_i}^\top \tilde{X} \tilde{b}| \\
\leq 2\mu \sqrt{s(k)} \|X\| + \|X - \tilde{X}\|, \text{ by base case } k = 1 \\
= 2\mu \sqrt{s(k)} \|X\| + \|\text{ReLU}(Y) - \text{ReLU}(\tilde{Y})\| \\
\leq 2\mu \sqrt{s(k)} \|X\| + \|\text{ReLU}(Y) - \text{ReLU}(\tilde{Y})\|_F \\
\leq 2\mu \sqrt{s(k)} \|X\| + \|Y - \tilde{Y}\|_F \\
= 2\mu \sqrt{s(k)} \|X\| + \epsilon^{(k-1)}
\]

Then we can also bound \(|\lambda_i^{(k)} (a_i^{(k)})^\top X^{(k)} b_i^{(k)} - \tilde{\lambda_i}^{(k)} (\tilde{a_i}^{(k)})^\top \tilde{X}^{(k)} \tilde{b}_i^{(k)}|\). For simplicity, let \(\lambda = \lambda_i^{(k)}, \tilde{\lambda} = \tilde{\lambda_i}^{(k)}, x = (a_i^{(k)})^\top X^{(k)} b_i^{(k)}\), and \(\tilde{x} = (\tilde{a_i}^{(k)})^\top \tilde{X}^{(k)} \tilde{b}_i^{(k)}\). Then

\[
|\lambda_i^{(k)} (a_i^{(k)})^\top X^{(k)} b_i^{(k)} - \tilde{\lambda_i}^{(k)} (\tilde{a_i}^{(k)})^\top \tilde{X}^{(k)} \tilde{b}_i^{(k)}| \\
= |\lambda x - \tilde{\lambda} \tilde{x}| \\
\leq |\lambda - \tilde{\lambda}| |x| + |\tilde{\lambda}| |x - \tilde{x}| \\
\leq |\lambda - \tilde{\lambda}| |x| + |\lambda||x - \tilde{x}|, \text{ we can pick } |\tilde{\lambda}| \leq |\lambda| \\
\leq \mu |x| + |\lambda| \times (2\mu \sqrt{s(k)} \|X\| + \epsilon^{(k-1)}) \\
\leq \mu \|X\| + 2\mu \|X\| |\lambda| \sqrt{s(k)} + |\lambda| \epsilon^{(k-1)} \\
= \mu \|X\| (1 + 2|\lambda| \sqrt{s(k)}) + |\lambda| \epsilon^{(k-1)}
\]
Bound $\|c_i^{(k)} \otimes d_i^{(k)} - \tilde{c}_i^{(k)} \otimes \tilde{d}_i^{(k)}\|_F^2$: Similarly let $c = c_i^{(k)}$ and $\tilde{c} = \tilde{c}_i^{(k)}$ (define $d$ and $\tilde{d}$ in a similar way). Then

\[
\|c_i^{(k)} \otimes d_i^{(k)} - \tilde{c}_i^{(k)} \otimes \tilde{d}_i^{(k)}\|_F^2
= \|cd^T - \tilde{c}d\tilde{d}\|_F^2
= \text{Tr}((cd^T - \tilde{c}d\tilde{d})^T (cd^T - \tilde{c}d\tilde{d}))
= \text{Tr}((dc^T - \tilde{c}d\tilde{d}^T)(cd^T - \tilde{c}d\tilde{d}))
= 4\sqrt{s(k+1)\mu}
\]

Bound $\epsilon^{(k)} = \|\tilde{Y}^{(k)} - Y^{(k)}\|_F$: Similarly, for simplicity, let $w_i = \lambda_i^{(k)} (a_i^{(k)})^T X^{(k)} b_i^{(k)}$, $\tilde{w}_i = \tilde{\lambda}_i^{(k)} (\tilde{a}_i^{(k)})^T \tilde{X}^{(k)} \tilde{b}_i^{(k)}$, $U_i = c_i^{(k)} \otimes d_i^{(k)}$, and $\tilde{U}_i = \tilde{c}_i^{(k)} \otimes \tilde{d}_i^{(k)}$. 
Since \( \| \tilde{Y}^{(k)} - Y^{(k)} \|_F = \| \sum_{i=1}^{r^{(k)}} w_i U_i - \sum_{i=1}^{r^{(k)}} \tilde{w}_i \tilde{U}_i \|_F \),

\[
\sum_{i=1}^{r^{(k)}} w_i U_i - \sum_{i=1}^{r^{(k)}} \tilde{w}_i \tilde{U}_i
\]

\[
= \sum_{i=1}^{r^{(k)}} (w_i U_i - \tilde{w}_i \tilde{U}_i)
\]

\[
\leq \sum_{i=1}^{r^{(k)}} \| w_i U_i - \tilde{w}_i \tilde{U}_i \|_F
\]

\[
= \sum_{i=1}^{r^{(k)}} \| w_i U_i - w_i \tilde{U}_i + w_i \tilde{U}_i - \tilde{w}_i \tilde{U}_i \|_F
\]

\[
\leq \sum_{i=1}^{r^{(k)}} \| w_i U_i - w_i \tilde{U}_i \|_F + \| w_i \tilde{U}_i - \tilde{w}_i \tilde{U}_i \|_F
\]

\[
= \sum_{i=1}^{r^{(k)}} \| w_i \| \| U_i - \tilde{U}_i \|_F + \| w_i - \tilde{w}_i \| \| \tilde{U}_i \|_F
\]

\[
\leq \sum_{i=1}^{r^{(k)}} |w_i| \times \sqrt{4 \sqrt{s^{(k+1)}}} \mu + (\mu \| X^{(k)} \| (1 + 2|\lambda_i|\sqrt{s^{(k)}}) + |\lambda_i|\epsilon^{(k-1)}) \times \| \tilde{U}_i \|_F
\]

\[
\leq \sum_{i=1}^{r^{(k)}} \lambda^{(k)} \| X^{(k)} \| \times \sqrt{4 \sqrt{s^{(k+1)}}} \mu + (\mu \| X^{(k)} \| (1 + 2|\lambda_i|\sqrt{s^{(k)}}) + |\lambda_i|\epsilon^{(k-1)}) \times \| \tilde{e}_i^{(k)} \otimes \tilde{d}_i^{(k)} \|_F
\]

\[
= \sum_{i=1}^{r^{(k)}} 2|\lambda_i| \| X^{(k)} \| \times \sqrt{4 \sqrt{s^{(k+1)}}} \mu + \| X^{(k)} \| (1 + 2|\lambda_i|\sqrt{s^{(k)}}) + |\lambda_i|\epsilon^{(k-1)}
\]

\[
\leq \sum_{i=1}^{r^{(k)}} \mu \| X^{(k)} \| (1 + 2|\lambda_i|\sqrt{s^{(k)}}) + |\lambda_i|\epsilon^{(k-1)}
\]

\[
\leq \mu^{(k)} [1 + 2|\lambda_i|\sqrt{s^{(k)}}] \| X^{(k)} \| + r^{(k)}|\lambda_i|\epsilon^{(k-1)}
\]

Let \( \alpha^{(k)} := \mu^{(k)} [1 + 2|\lambda_i|\sqrt{s^{(k)}}] \| X^{(k)} \|, \) and \( \beta^{(k)} = r^{(k)}|\lambda_i|\epsilon^{(k-1)}, \) then by the recurrence relationship in (80) the difference between the final output of the two networks are bounded by:

\[
\| \tilde{M}(X) - \tilde{M}(X) \|_F
\]

\[
= \| \text{ReLU} \left( \tilde{Y}^{(n)} \right) - \text{ReLU} \left( Y^{(n)} \right) \|_F (= X^{(n+1)} - X^{(n+1)})
\]

\[
\leq \| \tilde{Y}^{(n)} - Y^{(n)} \|_F
\]

\[
\leq \sum_{k=1}^{n} \alpha^{(k)} \Pi_{i=k+1}^{n} \beta^{(i)}
\]

Since \( \forall k \in [n], \| X^{(k)} \| \leq \Pi_{i=k}^{n} \| A^{(i)} \|_F \| X^{(n+1)} \|_F, \) to obtain an \( \epsilon \)-cover of the compressed network, we can
first assume $\beta(k) \geq 1 \forall k \in [n]$. Then $\mu$ need to satisfy:

$$
\mu \leq \frac{\epsilon}{(r(s)|\lambda^*|)^n \|X(n+1)\|_F nr(s)(1+4|\lambda^*|\sqrt{s(s)}(\frac{\rho(\cdot)}{\mu(\cdot)}\|A(\cdot)\|_F))^n}
$$

where $r(s) = \max_k r(k) \lambda^*(s) = \max_{i,k} \lambda_i^{(k)}, s(s) = \max_k s(k)$, and $\frac{\rho(\cdot)}{\mu(\cdot)}\|A(\cdot)\|_F = \max_k \frac{\rho(k)}{\mu(k)}\|A(k)\|_F$

As when $\mu$ is fixed, the number of networks in our cover will at most be $(\frac{1}{\mu})^d$ where $d$ denote the number of parameters in the original network. Hence, the covering number w.r.t to a given $\epsilon$ is $\tilde{O}(nd)$ ($n$ is the number of layers in the given neural network). As for practical neural networks, the number of layers $n$ is usually much less than $O(\log(d))$, thus the covering number we obtained w.r.t to a given $\epsilon$ is just $\tilde{O}(d)$ for practical neural networks. 

## F Neural Networks with Skip Connections

### F.1 Problem Setup

For neural nets with skip connections, the current theoretical analyses consider convolutional neural networks with one skip connection used on each layer, since our theoretical results can easily extend to general neural nets with skip connections. Therefore, we used the same the notations for neural nets with skip connections as what we defined for convolutional neural networks.

**Forward pass functions** Under the above assumptions, the only difference that we need to take into account between our analysis of CNN with skip connections and our analysis of standard CNN is the forward pass functions. In neural networks with skip connections, we have

$$
\hat{X}^{(k)} = \text{ReLU} \left( \hat{Y}^{(k-1)} \right)
$$

$$
\hat{Y}^{(k)} = \mathcal{M}^{(k)} \left( \hat{X}^{(k)} \right) + \hat{X}^{(k)}
$$

and

$$
\hat{X}^{(k)} = \text{ReLU} \left( \hat{Y}^{(k-1)} \right)
$$

$$
\hat{Y}^{(k)} = \hat{\mathcal{M}}^{(k)} \left( \hat{X}^{(k)} \right) + \hat{X}^{(k)}
$$

where $\mathcal{M}^{(k)} \left( \hat{X}^{(k)} \right)$ and $\hat{\mathcal{M}}^{(k)} \left( \hat{X}^{(k)} \right)$ compute the outputs of the $k^{th}$ convolutional layer.

Similarly, we use tensorization factor, tensor noise bound and layer cushion as in convolutional neural network defined in 4.2, 4.3 and 4.4. But note that the input $\hat{X}^{(k)}$ in the definition of layer cushion is the input of $k^{th}$ layer after skip connection.

### F.2 Generalization Guarantee of Compressed Network Proposed

**Theorem F.1.** For any convolutional neural network $\mathcal{M}$ of $n$ layers with skip connection satisfying the assumptions in section 3 and any margin $\gamma \geq 0$, Algorithm 1 generates a compressed network $\hat{\mathcal{M}}$ such that with high probability over the training set, the expected error $L_0(\hat{\mathcal{M}})$ is bounded by

$$
\hat{L}_0(\hat{\mathcal{M}}) + \tilde{O} \left( \sqrt{\sum_{k=1}^n \hat{R}^{(k)}(s(k) + o(k) + k_{x}^{(k)} \times k_{y}^{(k)} + 1)} \right)
$$

where

$$
\hat{R}^{(k)} = \min \left\{ j \in [R^{(k)}]|s_j^{(k)} \Pi_{l=k+1}^n (t_j^{(l)} + 1) \leq \frac{\gamma}{2n \max_{X \in S} \|\mathcal{M}(X)\|_F \Pi_{l=k}^n (s_j^{(l)}\|\mathcal{M}(i)\|_F)} \right\}
$$

To prove this theorem, Lemma 2 is needed.
Lemma F.2. For any convolutional neural network $M$ of $n$ layers with skip connection satisfying the assumptions in section 3 and any error $0 \leq \epsilon \leq 1$, Algorithm 1 generates a compressed tensorial neural network $\hat{M}$ such that for any $X \in S$:
\[
\|M(X) - \hat{M}(X)\|_F \leq \epsilon \|M(X)\|_F \tag{83}
\]

The compressed convolutional neural network $\hat{M}$ has with
\[
\sum_{k=1}^{n} \hat{R}(k)(s^{(k)} + o^{(k)} + k_x^{(k)} \times k_y^{(k)} + 1)
\]
total parameters, where each $\hat{R}(k)$ satisfies:
\[
\hat{R}(k) = \min \left\{ j \in [R(k)] | \xi^{(k)} j \prod_{l=k+1}^{n} \zeta^{(l)} \leq \epsilon \right\}
\tag{84}
\]

F.3 Complete Proofs of Neural Networks with Skip Connection

To prove Lemma F.2 the following Lemma F.3 is needed.

Lemma F.3. For any convolutional neural network $M$ of $n$ layers with skip connection satisfying the assumptions in section 3, Algorithm 5 generates a compressed tensorial neural network $\hat{M}$ where for any $X \in S$:
\[
\|M(X) - \hat{M}(X)\|_F \leq \left( \sum_{k=1}^{n} \xi^{(k)} \|M^{(k)}\|_F \prod_{l=k+1}^{n} \zeta^{(l)} \|M^{(l)}\|_F \right) \|M(X)\|_F
\]

where $\xi$, $\zeta$, and $t$ are tensor noise bound, layer cushion, tensorization factor defined in 4.3, 4.4 and 4.2 respectively.

Proof. (of Lemma F.3)

We know by construction, $M(X) = X^{(n+1)}$ and $\hat{M}(X) = \hat{X}^{(n+1)}$, we can just show $\|X^{(n+1)} - \hat{X}^{(n+1)}\|_F$ satisfies the above inequality, and we will prove this by induction. Notice

Induction Hypothesis: For any layer $m > 0$, 
\[
\|X^{(m)} - \hat{X}^{(m)}\|_F \leq \left( \sum_{k=1}^{m} \xi^{(k)} \|M^{(k)}\|_F \prod_{l=k+1}^{m} \zeta^{(l)} \|M^{(l)}\|_F \right) \|X^{(m)}\|_F
\]

Base case: when $m = 1$, the above inequality hold trivially as $X^{(1)} = \hat{X}^{(1)}$ as we cannot modify the input, and the RHS is always $\geq 0$.

Inductive Step: Now we assume show that the induction hypothesis is true for all $m$, then at layer $m + 1$ we have
To prove Theorem F.1, the following lemma is needed.

The proof is similar with the proof of Lemma D.5. The only difference is we replace \( \gamma \).

The proof of Lemma F.3 is then completed by induction.

Now we can prove Lemma F.2

**Proof.** (of Lemma F.2)

The proof is similar with the proof of Lemma D.5. The only difference is we replace \( t^{(l)} \) by \( t^{(l)} + 1 \).

To prove Theorem F.1, the following lemma is needed.

**Lemma F.4.** For any convolutional neural network \( \mathcal{M} \) of \( n \) layers with skip connection satisfying the assumptions in section 3 and any margin \( \gamma \geq 0 \), \( \mathcal{M} \) can be compressed to a tensorial convolutional neural network \( \hat{\mathcal{M}} \) with \( \sum_{k=1}^{n} \hat{R}^{(k)} (s^{(k)} + \ell^{(k)} + k_x^{(k)} \times k_y^{(k)} + 1) \) total parameters such that for any \( \mathcal{X} \in \mathcal{S}, \hat{L}_{\gamma} (\mathcal{M}) \leq \hat{L}_{\gamma} (\mathcal{M}) \). Here, for each layer \( k \),

\[
\hat{R}^{(k)} = \min \left\{ j \in [R^{(k)}] : \frac{1}{2} \sum_{i=k+1}^{n} \left( t^{(i)} + 1 \right) \| M^{(i)} \|_F \right\}
\]

The proof of Lemma F.4 is the same with Lemma E.9. And by setting \( \epsilon = \frac{\gamma}{2 \max_{x \in \mathcal{S}} \xi} \), we get the desired expression of \( \hat{R}^{(k)} \) in the main theorem.

**Proof.** (of Theorem F.1) Similarly, let us bound the covering number of the compressed network \( \hat{\mathcal{M}} \) by approximating each parameter with accuracy \( \mu \).

**Covering Number Analysis for Convolutional Neural Network** Let \( \mathcal{M} \) denote the network after approximating each parameter in \( \hat{\mathcal{M}} \) with accuracy \( \mu \). We use the same assumptions and notations with the proof of Theorem E.5. And we still use \( \mathcal{X}^{(k)}, \mathcal{Y}^{(k)}, \hat{\mathcal{M}}^{(k)} \) to denote \( \mathcal{X}^{(k)}, \mathcal{Y}^{(k)}, \hat{\mathcal{M}}^{(k)} \)

\[
\| \mathcal{X}^{(m+1)} - \hat{\mathcal{X}}^{(m+1)} \|_F = \| \text{ReLU} \left( \mathcal{Y}^{(m)} \right) - \text{ReLU} \left( \hat{\mathcal{Y}}^{(m)} \right) \|_F
\]

\[
\leq \| \mathcal{X}^{(m)} - \hat{\mathcal{X}}^{(m)} \|_F
\]

\[
\leq \left\| \mathcal{M}^{(m)} \left( \hat{\mathcal{X}}^{(m)} \right) + \mathcal{X}^{(m)} - \left( \mathcal{M}^{(m)} \left( \hat{\mathcal{X}}^{(m)} \right) + \hat{\mathcal{X}}^{(m)} \right) \right\|_F
\]

\[
\leq \left\| \mathcal{M}^{(m)} \left( \hat{\mathcal{X}}^{(m)} \right) - \mathcal{M}^{(m)} \left( \hat{\mathcal{X}}^{(m)} \right) \right\|_F + \left\| \mathcal{X}^{(m)} - \hat{\mathcal{X}}^{(m)} \right\|_F
\]

\[
\leq \sqrt{HW} \left( t^{(m)} + 1 \right) \mathcal{X}^{(m)} - \hat{\mathcal{X}}^{(m)} \|_F + \sqrt{HW} s^{(m)} \| \mathcal{X}^{(m)} \|_F
\]

\[
\left( \sum_{k=1}^{m-1} \frac{\xi^{(k)}}{\zeta^{(k)}} \mathcal{M}^{(k)} \|_F \right) \prod_{l=k+1}^{m-1} \frac{t^{(l)} + 1}{\zeta^{(l)}} \mathcal{M}^{(l)} \|_F \mathcal{X}^{(m)} \|_F + \frac{\xi^{(m)}}{\zeta^{(m)}} \mathcal{M}^{(m)} \|_F \mathcal{X}^{(m)} \|_F
\]

\[
\left( \sum_{k=1}^{m} \frac{\xi^{(k)}}{\zeta^{(k)}} \mathcal{M}^{(k)} \|_F \right) \prod_{l=k+1}^{m} \frac{t^{(l)} + 1}{\zeta^{(l)}} \mathcal{M}^{(l)} \|_F \| \mathcal{X}^{(m+1)} \|_F
\]
Since $\alpha$ and $\beta$ need to satisfy:

$\| \tilde{\gamma}^{(k)} - \gamma^{(k)} \|_F \leq 2 \sum_{k=1}^{n} \sum_{t=i}^{k} \alpha^{(i)} \prod_{t=i+1}^{k} \beta^{(t)}$

where $\alpha^{(k)} = 4HW \| \mathcal{X}^{(k)} \|_F^2 (\sum_{\tau} \hat{R}^{(k)} (\lambda^{(k)})^2 \hat{K}^{(k)} k^{(k)} + 4 \sum_{\tau} \hat{R}^{(k)} (\lambda^{(k)})^2 (a^{(k)} + s^{(k)}) \hat{R}^{(k)} + 2(\hat{R}^{(k)})^2) \mu^2$,

and $\beta^{(k)} = 2 \| \hat{M}^{(k)} \|_F^2$.

Since $\forall k \in [n], \| \mathcal{X}^{(k)} \| \leq \Pi_{i=k}^{(n+1)} \frac{\| \mathcal{X}^{(n+1)} \|_F}{\| \hat{M}^{(i)} \|_F}$, to obtain an $\epsilon$-cover of the compressed network, we can first assume $\beta^{(k)} \geq 1 \forall k \in [n]$. Then $\mu$ need to satisfy:

$\mu \leq \frac{\epsilon}{2 \sqrt{HW} n^2 \| \mathcal{X}^{(n+1)} \|_F \hat{R}^{(*)} \sqrt{(\lambda^{(*)})^2 k^{(*)} k^{(*)} + 4(\lambda^{(*)})^2 (a^{(*)} + s^{(*)}) + 2(\hat{R}^{(*)})^2} \mu^{(*)} \| \hat{M}^{(*)} \|_F^n}$

where $\hat{R}^{(*)} = \max_k r^{(k)}$, $\lambda^{(*)} = \max_k \lambda^{(k)}$, $\lambda^{(*)} = \max_k \lambda^{(k)}$, $s^{(*)} = \max_k s^{(k)}$, $a^{(*)} = \max_k a^{(k)}$, $k^{(*)} = \max_k k^{(k)}$, $k^{(*)} = \max_k k^{(k)}$, $k^{(*)} = \max_k k^{(k)}$, and $\mu^{(*)} = \max_k \| \hat{M}^{(*)} \|_F$.

So the skip connections don’t change the limiting behavior of the covering number, which w.r.t to a given $\epsilon$ is $O(nd)$ ($n$ is the number of layers in the given neural network, $d$ is the number of parameters), and $O(d)$ for practical neural networks. Because skip connections don’t need extra parameters, the neural network still has $\sum_{k=1}^{n} \hat{R}^{(k)} (s^{(k)} + r^{(k)} + k^{(*)} k^{(*)} + 1)$ total parameters.

$\square$
G Additional Algorithms and Algorithmic Details

Details of Step 3 in Algorithm 1. We use the alternating least squares (ALS) in the implementation of step 3, which is the ‘parafac’ method of the tensorly library (Kossaifi et al., 2019), to obtain the CP decomposition. Though CP decomposition is in general NP-hard, the ALS method usually converges for most tensors, with polynomial convergence rate w.r.t. the given precision of the allowed reconstruction error (Anandkumar et al., 2015, 2014a,b). In addition, step 3 obtains a CP parametrization of the weight tensor rather than recovers the true components of the weight tensor’s CP decomposition. The rank in the CP decomposition is selected in step 2 and is an upper bound of the true rank of the tensor (Proposition 4.1). Thus, with the chosen rank, we can obtain a CP decomposition with a very low reconstruction error. In practice, for our cases, the CP decomposition method (ALS) used in step 3 always converges within a few iterations, with reasonable run time.

Algorithm 3 Find Best Rank for CNN (FBRC)

Input: A list of weight tensors \( \{M^{(k)}\}_{k=1}^n \), a list of number of components \( \{R^{(k)}\}_{k=1}^n \), a list of layer cushions \( \{\zeta^{(k)}\}_{k=1}^n \), and a perturbation parameter \( \epsilon \) which denotes the maximum error we could tolerate regarding the difference between the output of original network and that of compressed network.

Output: Returns a list of number of components \( \{\hat{R}^{(k)}\}_{k=1}^n \) for the compressed network such that

\[
\|M(\mathcal{X}) - \hat{M}(\mathcal{X})\|_F \leq \epsilon.
\]

Notice that for each \( k \), if the original network does not have skip connections, \( \hat{R}^{(k)} \) satisfies that

\[
\zeta^{(k)} \sum_{i=k+1}^{n} \hat{R}^{(i)} \leq \frac{\epsilon}{n} \sum_{i=k}^{n} \zeta^{(i)} \|M^{(i)}\|_F
\]

or if skip connection is used, \( \hat{R}^{(k)} \) satisfies that

\[
\zeta^{(k)} \sum_{i=k+1}^{n} (\hat{R}^{(i)} + 1) \leq \frac{\epsilon}{n} \sum_{i=k}^{n} \zeta^{(i)} \|M^{(i)}\|_F
\]

1: For each layer \( k \), calculate the following properties: layer cushion \( \zeta^{(k)} \), weight norm \( \|M^{(k)}\|_F \), then calculate the RHS \( \frac{\epsilon}{n} \sum_{i=k}^{n} \zeta^{(i)} \|M^{(i)}\|_F \) for each \( k \)

2: Find the smallest \( \hat{R}^{(n)} \) such that the tensor noise bound for the last layer \( \zeta^{(n)} \) satisfies \( \zeta^{(n)} \leq \frac{\epsilon}{n} \sum_{i=k}^{n} \zeta^{(i)} \|M^{(i)}\|_F \)

3: for \( k = n - 1 \) to 1 do

4: if \( \hat{R} \) does not have skip connections then

5: Calculate the multiplication of tensorization factor for layers upper than \( k \), i.e., \( \Pi_{i=k+1}^{n} \hat{R}^{(i)} \), based on the choices of \( \hat{R}^{(i)} \) for \( k \leq i < n \)

6: Find the smallest \( \hat{R}^{(k)} \) by calculating the largest possible \( \hat{R}^{(k)} \) such that Equation 85 holds.

7: else

8: Calculate the multiplication of tensorization factor for layers upper than \( k \), i.e., \( \Pi_{i=k+1}^{n} (\hat{R}^{(i)} + 1) \), based on the choices of \( \hat{R}^{(i)} \) for \( k \leq i < n \)

9: Find the smallest \( \hat{R}^{(k)} \) by calculating the largest possible \( \hat{R}^{(k)} \) such that Equation 86 holds.

10: Return \( \{\hat{R}^{(k)}\}_{k=1}^n \)

Remark. The FBRC algorithm finds a set of ranks that satisfies inequality 85 (CNNs) or 86 (NNs with skip connections) within polynomial time because of the following guarantees. The total number of possible sets of ranks (say \( T \)), which the FBRC algorithm will at most search through, is equal to the product of the ranks of all layers. The rank of each layer is upper bounded by Proposition 4.1 and thus \( T \) is polynomial w.r.t. the shape of the original weight tensors and the number of layers. Moreover, the search will definitely succeed as the inequalities 85 and 86 automatically hold when \( \hat{R}^{(k)} = \hat{R}^{(k)} \).
Algorithm 4 Find Best Rank (FBR)

Input: A list of tensors \(\{\mathcal{M}^{(k)}\}_{k=1}^{n}\) where each \(\mathcal{M}^{(k)} \in \mathbb{R}^{s_1^{(k)} \times s_2^{(k)} \times s_3^{(k)} \times s_4^{(k)} \times s_5^{(k+1)}}\) is reshaped from a matrix \(A^{(k)}\), a list of number of components \(\{\hat{R}^{(k)}\}_{k=1}^{n}\), a list of layer cushions \(\{\zeta^{(k)}\}_{k=1}^{n}\) of the original network, and a perturbation parameter \(\epsilon\) which denotes the maximum error we could tolerate regarding the difference between the output of original network and that of compressed network.

Output: Returns a list of number of components \(\{\hat{R}^{(k)}\}_{k=1}^{n}\) for the compressed network such that

\[
\left\| M(X) - \hat{M}(X) \right\|_F \leq \epsilon.
\]

1: For each layer \(k\), calculate the following properties: reshaping factor \(\rho^{(k)}\), layer cushion \(\zeta^{(k)}\), weight norm \(\left\| A^{(k)} \right\|_F\), then calculate the RHS \(\frac{\epsilon}{n} \left\| A^{(k)} \right\|_F \Pi_{i=k+1}^{n} \zeta^{(i)}\) for each \(k\).
2: Find the smallest \(\hat{R}^{(n)}\) such that the tensor noise bound for the last layer \(\zeta^{(n)}\) satisfies \(\frac{\rho^{(n)} \zeta^{(n)}}{n} \leq \frac{\epsilon}{n} \left\| A^{(k)} \right\|_F \Pi_{i=k}^{n} \zeta^{(i)}\)
3: for \(k = n - 1\) to 1 do
4: Calculate the multiplication of tensorization factor for layers upper than \(k\), i.e., \(\Pi_{i=k+1}^{n} \zeta^{(i)}\), based on the choices of \(\hat{R}^{(i)}\) for \(k \leq i \leq n\).
5: Find the smallest \(\hat{R}^{(k)}\) by calculating the largest possible \(\zeta^{(k)}\) such that Equation 87 holds.
6: Return \(\{\hat{R}^{(k)}\}_{k=1}^{n}\)

Algorithm 5 CNN-Project

Input: A convolutional neural network \(\mathbb{M}\) of \(n\) layers where its weight tensor \(\mathcal{M}^{(k)}\) of the \(k\)th layer is parametrized by \(\{\lambda_r^{(k)}, a_r^{(k)}, b_r^{(k)}, c_r^{(k)}, d_r^{(k)}\}_{r=1}^{R^{(k)}},\) and a list of ranks \(\{R^{(k)}\}_{k=1}^{n}\).

Output: Returns a compressed network \(\mathbb{M}\) where for each layer \(k\), \(\left\| \mathcal{M}^{(k)} \right\|\) is constructed by the top \(\hat{R}^{(k)}\) components from CP components of \(\mathcal{M}^{(k)}\).

1: for \(k = 1\) to \(n\) do
2: \(\hat{\mathcal{M}}^{(k)} \leftarrow \sum_{r=1}^{R^{(k)}} \lambda_r^{(k)} a_r^{(k)} \otimes b_r^{(k)} \otimes c_r^{(k)} \otimes d_r^{(k)}\)
3: Let \(\hat{\mathcal{M}}^{(k)}\) be the weight tensor of the \(k\)th layer in \(\mathbb{M}\)
4: Return \(\mathbb{M}\)

Algorithm 6 TNN-Project

Input: A fully connected neural network \(\mathbb{M}\) of \(n\) layers where its weight tensor \(\mathcal{M}^{(k)}\) of the \(k\)th layer is parametrized by \(\{\lambda_r^{(k)}, a_r^{(k)}, b_r^{(k)}, c_r^{(k)}, d_r^{(k)}\}_{r=1}^{R^{(k)}},\) and a list of ranks \(\{R^{(k)}\}_{k=1}^{n}\).

Output: Returns a compressed network \(\mathbb{M}\) where for each layer \(k\), \(\left\| \hat{\mathcal{M}}^{(k)} \right\|\) is constructed by the top \(\hat{R}^{(k)}\) components from CP components of \(\mathcal{M}^{(k)}\).

1: for \(k = 1\) to \(n\) do
2: \(\hat{\mathcal{M}}^{(k)} \leftarrow \sum_{r=1}^{R^{(k)}} \lambda_r^{(k)} a_r^{(k)} \otimes b_r^{(k)} \otimes c_r^{(k)} \otimes d_r^{(k)}\)
3: Let \(\hat{\mathcal{M}}^{(k)}\) be the weight tensor of the \(k\)th layer in \(\mathbb{M}\)
4: Return \(\mathbb{M}\)