Courant bracket found out to be T-dual to Roytenberg bracket

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Abstract

Bosonic string moving in coordinate dependent background fields is considered. We calculate the generalized currents Poisson bracket algebra and find that it gives rise to the Courant bracket. Furthermore, we consider the T-dual theory and obtain the dual generalized currents Poisson bracket algebra. It gives rise to the Roytenberg bracket. We conclude that the Courant bracket twisted with the Kalb-Ramond field is T-dual to the Roytenberg bracket, with $Q$ and $R$ flux.

1 Introduction

Non-geometric backgrounds [1, 2, 3] include various dualities. Duality symmetry is a way to show the equivalence between two apparently different theories. Specifically, T-duality [4, 5] is a symmetry between two theories corresponding to different geometries and topologies. It was firstly noticed as the spectrum equivalence of the bosonic closed string with one dimension compactified to a radius $R$, with the bosonic closed string with one dimension compactified to a radius $1/R$.

The Courant bracket [6, 7] is the generalization of the Lie bracket so that it includes both vectors and 1-forms. It is a fundamental structure of the generalized complex geometry. Vectors and 1-forms are treated on equal footing in the generalized complex structures. Moreover, many for string theory relevant geometries, such as complex, symplectic and Kähler geometry, are integrated into the framework of generalized complex structures. Moreover, the generalized complex geometry provides a framework for a unified description of diffeomorphisms and gauge transformations of the Kalb-Ramond field. Hitchin was the first one to introduce the generalized Calabi-Yau manifolds, that unified the concept of a Calabi-Yau manifold with the one of a symplectic manifold [8]. Gualtieri in his PhD thesis contributed further to the mathematical development of generalized complex geometry [9].

In generalized complex geometry, closure under the Courant bracket represents the integrability condition, in a same way that closure under the Lie bracket represents the integrability condition of almost complex structures. Moreover, the Courant bracket governs the gauge transformation in the double field string theory [10].

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The Roytenberg bracket is the generalization of the Courant bracket, so that it includes a bi-vector. It was firstly introduced by Roytenberg [11]. In [12], the \( \sigma \)-model with both 2-form and a bi-vector was considered. The Poisson bracket algebra of the general currents was obtained. It has been observed that, while the current algebra is anomalous, the algebra of charges is closed and gives rise to the Roytenberg bracket. In [13], the Roytenberg bracket was obtained by lifting the topological sector of the first order action for the NS string \( \sigma \)-model to three dimensions. In [14], the higher order Roytenberg bracket is realized, by twisting by a p-vector.

In this paper, we consider the closed bosonic string moving in the coordinate dependent background fields. The generalized currents are defined as linear combinations of worldsheet basis vectors with arbitrary coordinate dependent coefficients, and their Poisson bracket algebra is calculated. We follow the work of [15], that analyzed the most general currents of the general \( \sigma \) model, where it has been shown that the algebra of most general currents gives rise to the Courant bracket. Moreover, we consider the T-duality as a canonical transformation [16], that interchanges momenta with coordinate derivatives. The dual counterparts of the generalized currents are constructed and their algebra obtained. We find that the dual current algebra gives rise to the Roytenberg bracket. Hence, we show that the Courant bracket is T-dual to the Roytenberg one, obtaining the relation that connects the mathematically relevant structures with the T-duality.

## 2 Hamiltonian of the bosonic string

Consider the closed bosonic string in the nontrivial background defined by the metric \( G_{\mu \nu} \) and the Kalb-Ramond antisymmetric tensor \( B_{\mu \nu} \) field, as well as the constant dilaton field \( \Phi = \text{const} \). In the conformal gauge, the propagation is described by the action [17, 18]

\[
S = \int_{\Sigma} d^2 \xi \mathcal{L} = \kappa \int_{\Sigma} d^2 \xi \left[ \frac{1}{2} \eta^{\alpha \beta} G_{\mu \nu}(x) + \epsilon^{\alpha \beta} B_{\mu \nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu,
\]

where integration goes over two-dimensional world-sheet \( \Sigma \) parametrized by \( \xi^\alpha (\xi^0 = \tau, \xi^1 = \sigma) \). Coordinates of the D-dimensional space-time are \( x^\mu (\xi) \), \( \mu = 0, 1, ..., D - 1 \), \( \epsilon^{01} = -1 \) and \( \kappa = \frac{1}{2\pi\alpha'} \).

It is convenient to rewrite the action (2.1) using the light-cone coordinates \( \xi^\pm = \xi^0 \pm \xi^1 \) and derivatives \( \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1) \) as

\[
S = \kappa \int_{\Sigma} d^2 \xi \partial_+ x^\mu \Pi_{+ \mu \nu}(x) \partial_- x^\nu,
\]

where

\[
\Pi_{\pm \mu \nu}(x) = B_{\mu \nu}(x) \pm \frac{1}{2} G_{\mu \nu}(x).
\]

The canonical momenta are given by

\[
\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu \nu}(x) \dot{x}^\nu - 2\kappa B_{\mu \nu}(x) x^\nu.
\]

The Hamiltonian is obtained in a usual way,

\[
\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2\kappa} \pi_\mu (G^{-1})^{\mu \nu} \pi_\nu - 2x^\mu B_{\mu \nu} (G^{-1})^{\nu \rho} \pi_\rho + \frac{\kappa}{2} x^\mu G^{E\mu \nu} x^\nu,
\]
where
\[ G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu} \] (2.6)
is the effective metric.

Energy-momentum tensor components can be written as
\[ T_\pm = \mp \frac{1}{4\kappa} (G^{-1})^{\mu\nu} j_\pm^\mu j_\pm^\nu, \] (2.7)
where the currents \( j_\pm^\mu \) are given by
\[ j_\pm^\mu(x) = \pi^\mu + 2\kappa \Pi_{\pm\mu\nu}(x)x^\nu. \] (2.8)

In terms of the energy-momentum tensor components (2.7), the Hamiltonian is given by
\[ H_C = T_- - T_+ = \frac{1}{4\kappa} (G^{-1})^{\mu\nu} [j_+^\mu j_+^\nu + j_-^\mu j_-^\nu]. \] (2.9)

### 2.1 T-duality as canonical transformation

Let us consider the T-dual transformation given by [16]
\[ x'^\mu \simeq \frac{1}{\kappa} \pi'^\mu, \quad \pi' \simeq \kappa y'_\mu, \] (2.10)
where \( \pi^\mu \) and \( y_\mu \) are the T-dual momenta and coordinates, respectively. The Poisson bracket does not change under the above transformation, therefore the transformation is canonical [16]. This transformation can be obtained by the generating function
\[ F = \kappa \int d\sigma x'^\mu y'_\mu, \] (2.11)
which gives rise to \( \pi_\mu = \frac{\delta F}{\delta x'^\mu} = \kappa y'_\mu \) and \( \pi'^\mu = \frac{\delta F}{\delta y'_\mu} = \kappa x'^\mu. \)

The dual Hamiltonian has the same form as the initial one, with the dual quantities replacing the initial ones
\[ *H_C = \frac{1}{2\kappa} *\pi'^\mu *G^{-1}_{\mu\nu} *\pi'^\nu + \frac{\kappa}{2} y'_\mu *G^{\mu\nu} y'_\nu - 2y'_\mu (*B G^{-1})^\nu_\mu *\pi'^\nu, \] (2.12)
where \( *G^{\mu\nu} \) and \( *B^{\mu\nu} \) are the T-dual metric and Kalb-Ramond field, respectively. The transformations (2.10) does not change the Hamiltonian, since the generating function (2.11) does not depend explicitly on time \( H_C \to H_C + \frac{\partial F}{\partial t} = H_C. \) This is expected, since the T-duality makes the energy spectrum invariant [4]. From this requirement, we read the T-dual transformations of the background fields
\[ *G^{\mu\nu} = (G^{-1})^\mu_\nu, \quad *B^{\mu\nu} = \frac{\kappa}{2} \theta^{\mu\nu}, \] (2.13)
where \( \theta^{\mu\nu} = -\frac{2}{\kappa} (G^{-1}_E B G^{-1})^{\mu\nu} \) is the non-commutativity parameter and \( (G^{-1}_E)^{\mu\nu} \) is the inverse of the effective metric defined in (2.6).
2.2 T-dual Hamiltonian

The dual Hamiltonian can be rewritten as

$$\star H = \frac{1}{4\kappa} \star G^{-1}_{\mu\nu} \left( \star j^\mu_+ j^\nu_+ + \star j^\mu_- j^\nu_- \right), \quad (2.14)$$

where the dual currents are

$$\star j^\mu_\pm = \pi^\mu + 2\kappa \star \Pi^\mu_{\nu} y^\nu. \quad (2.15)$$

We are interested in the transformation of the dual currents (2.15) under the above canonical transformation. Applying (2.10) to (2.15), we get

$$\star j^\mu_\pm \approx \kappa x^\mu + \kappa \theta^\mu_{\nu} \pi^\nu \pm (G^{-1}_E)^{\mu\nu} \pi^\nu \equiv \ell^\mu_\pm, \quad (2.16)$$

where we have marked the result as the new current $\ell^\mu_\pm$. It is worth pointing out that the T-dual transformation of currents $\star j^\mu_\pm$ is not equal to $j^\mu_\pm$, rather the theories in which these currents are defined are T-dual to each other. We can write the Hamiltonian in terms of these currents as

$$\star H = \frac{1}{4\kappa} G^E_{\mu\nu} \left( \ell^\mu_+ \ell^\nu_+ + \ell^\mu_- \ell^\nu_- \right). \quad (2.17)$$

3 Generalized currents in a new basis

In this chapter, we will construct two types of generalized currents. Firstly, we will generalize the currents $j^\pm_{\mu}$. In (2.8) they are defined in the world-sheet basis of coordinate $\sigma-$derivatives and momenta $\{x^\mu, \pi^\mu\}$. The generalized current is defined as an arbitrary vector in a space spanned by $\tau$ and $\sigma$ components of currents, $j_{0\mu}$ and $j_{1\mu}$. In a similar manner, we will generalize the currents $\star j^\mu_\pm$, defined in (2.15).

The convenient components of currents $j_{\pm\mu}$ are

$$j_{0\mu} = \frac{j^+_{\mu} + j^-_{\mu}}{2} = \pi^\mu + 2\kappa B_{\mu\nu}(x)x^{\nu}, \quad j_{1\mu} = \frac{j^+_{\mu} - j^-_{\mu}}{2} = \kappa G_{\mu\nu}(x)x^{\nu}. \quad (3.1)$$

We will mark

$$i^\mu = \pi^\mu + 2\kappa B_{\mu\nu}(x)x^{\nu}, \quad (3.2)$$

as a new, auxiliary current. Therefore, $\{x^\mu, i^\mu\}$ is a new basis on the world-sheet. We can now write currents (2.8) in this new basis as

$$j^\pm_{\mu} = i^\mu \pm \kappa G_{\mu\nu} x^{\nu}. \quad (3.3)$$

In the same way as in [15], we define the generalized currents in the new basis, as the linear combination of both coordinate $\sigma-$derivatives and auxiliary currents

$$J_{C(u,a)} = u^\mu(x)i^\mu + a^\mu(x)x^\mu, \quad (3.4)$$

where $u^\mu(x)$ and $a^\mu(x)$ are the arbitrary coefficients. The charges of these currents are

$$Q_{C(u,a)} = \int d\sigma J_{C(u,a)}. \quad (3.5)$$
If we choose \( a_\mu = \pm \kappa G_{\mu\nu} u^\nu \), we obtain
\[
J_{C(u, \pm \kappa G u)} = u^\mu j_{\pm \mu}. \tag{3.6}
\]
Hence, the currents (2.8) indeed can be obtained from the generalized currents (3.4). On the other hand, for \( a_\mu = -2 \kappa B_{\mu\nu} u^\nu \) we obtain
\[
J_{C(u, -2 \kappa B u)} = u^\mu \pi_\mu, \tag{3.7}
\]
as well as for \( u^\mu = 0 \), we obtain
\[
J_{C(0, a)} = a_\mu x_\nu'. \tag{3.8}
\]
Thus, the starting basis \( \{ x_\mu, \pi_\mu \} \) can be obtained from (3.4) as well.

In an analogous way, we want to generalize the dual currents (2.15). The appropriate basis is given by the \( \tau \) and \( \sigma \) components of the dual currents
\[
* j_0^\mu = * j_+^\mu + * j_-^\mu = \pi^\mu + 2 \kappa * B^{\mu\nu} y_\nu' \equiv \pi^\mu, \quad * j_1^\mu = \frac{ * j_+^\mu - * j_-^\mu }{2} = \kappa * G^{\mu\nu} y_\nu'. \tag{3.9}
\]
The dual currents in the new basis are
\[
* j_{\pm}^\mu = * v^\mu \pm \kappa * G^{\mu\nu} y_\nu'. \tag{3.10}
\]
The generalized currents in the dual theory are given by
\[
* J_{R(v, b)} = v^\mu (x) y_\mu' + b_\mu (x) * j^\mu, \tag{3.11}
\]
where \( v^\mu (x) \) and \( b_\mu (x) \) are arbitrary coefficients. If we put \( v^\mu = \pm \kappa (G_E^{-1})^{\mu\nu} b_\nu \), we obtain
\[
* J_{R(\pm \kappa (G_E^{-1}) b, b)} = b_\mu * j_{\pm}^\mu. \tag{3.12}
\]
For \( b_\mu = 0 \), we obtain
\[
* J_{R(2 \kappa * B b, b)} = b_\mu * \pi^\mu. \tag{3.13}
\]
and for \( v^\mu = 2 \kappa * B^{\mu\nu} b_\nu \), we obtain
\[
* J_{R(2 \kappa * B b, b)} = b_\mu * \pi^\mu. \tag{3.14}
\]

In certain cases the dual background fields are non-local, i.e. not polynomials or functions of the fields or their derivatives evaluated at a single point \[19\]. Therefore, without loss of generality, it is more convenient to work in its dual picture in the initial theory. We will use the basis vectors in the initial theory dual to \( \{ * v^\mu, y_\mu' \} \)
\[
* v^\mu \simeq \kappa x^\mu + \kappa \theta^{\mu\nu} \pi_\nu \equiv k^\mu, \quad y_\mu' \simeq \frac{1}{\kappa} \pi_\mu, \tag{3.15}
\]
where we have defined the new auxiliary current \( k^\mu \).

Therefore, the second set of generalized currents that we will be working with is defined in the basis \( \{ k^\mu, \pi_\mu \} \) by
\[
J_{R(v, b)} = v^\mu (x) \pi_\mu + b_\mu (x) k^\mu, \tag{3.16}
\]
where \( v^\mu (x) \) and \( b_\mu (x) \) are the arbitrary coefficients. Their charges are
\[
Q_{R(v, b)} = \int d\sigma J_{R(v, b)}. \tag{3.17}
\]
If we choose \( v^\mu = \pm (G_E^{-1})^{\mu\nu} b_\nu \), we obtain the currents (2.16)
\[
J_{R(\pm G_E^{-1} b, b)} = b_\mu k_{\pm}^\mu. \tag{3.18}
\]
4 Courant bracket

We are interested in calculating the Poisson bracket algebra of the most general currents $J_{C(u,a)}$, defined in (3.4), as well as of their charges $Q_{C(u,a)}$, defined in (3.5). We will start with the Poisson bracket of the algebra generators, $i_\mu$ and $x'^\nu$, that will be calculated using the standard Poisson bracket relations

$$\{x'^\mu(\sigma, \tau), \pi_\nu(\bar{\sigma}, \tau)\} = \delta^\mu_{\nu}\delta(\sigma - \bar{\sigma}),$$

$$\{x'^\mu(\sigma, \tau), x'^\nu(\sigma, \tau)\} = 0,$$

$$\{\pi_\mu(\sigma, \tau), \pi_\nu(\bar{\sigma}, \tau)\} = 0. \quad (4.1)$$

In the accordance with [15], we will obtain that the algebra of generalized charges (3.5) gives rise to the Courant bracket [6].

We obtain the algebra of generators (3.2)

$$\{i_\mu(\sigma), i_\nu(\bar{\sigma})\} = -2\kappa B_{\mu\nu\rho}x'^\rho\delta(\sigma - \bar{\sigma}), \quad (4.2)$$

where the structural constants are the Kalb-Ramond field strength components, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \quad (4.3)$$

The rest of the generators algebra is given by

$$\{i_\mu(\sigma), x'^\nu(\bar{\sigma})\} = \delta^\nu_\mu \partial_{\sigma}\delta(\sigma - \bar{\sigma}), \quad \{x'^\mu(\sigma), x'^\nu(\bar{\sigma})\} = 0. \quad (4.4)$$

The Poisson bracket of the most general currents (3.4) is obtained using (4.2) and (4.3). It reads

$$\{J_{C(u,a)}(\sigma), J_{C(v,b)}(\bar{\sigma})\} = (v^\nu\partial_\nu u^\mu - u^\nu\partial_\nu v^\mu) i_\mu \delta(\sigma - \bar{\sigma}) - 2\kappa B_{\mu\nu\rho}x'^\rho \delta(\sigma - \bar{\sigma}) - (\partial_\mu a_\nu - \partial_\nu a_\mu) v^\nu - (\partial_\mu b_\nu - \partial_\nu b_\mu) u^\nu \} x'^\mu \delta(\sigma - \bar{\sigma}) + (u^\mu(\sigma) b_\mu(\sigma) + v^\mu(\bar{\sigma}) a_\mu(\bar{\sigma})) \partial_{\sigma} \delta(\sigma - \bar{\sigma}). \quad (4.5)$$

We can modify the anomalous part in the following manner

$$\left(\begin{array}{c} u^\mu(\sigma) b_\mu(\sigma) + v^\mu(\bar{\sigma}) a_\mu(\bar{\sigma}) \end{array}\right) \partial_{\sigma} \delta(\sigma - \bar{\sigma}) =$$

$$= \frac{1}{2} \left( (ub)(\sigma) + (va)(\bar{\sigma}) \right) \partial_{\sigma} \delta(\sigma - \bar{\sigma}) - \frac{1}{2} (ub)(\sigma) \partial_{\sigma} \delta(\sigma - \bar{\sigma}) + \frac{1}{2} (va)(\bar{\sigma}) \partial_{\sigma} \delta(\sigma - \bar{\sigma})$$

$$= \frac{1}{2} \left( (ub)(\sigma) + (ub)(\bar{\sigma}) + va(\sigma) + (va)(\bar{\sigma}) \right) \partial_{\sigma} \delta(\sigma - \bar{\sigma}) + \frac{1}{2} \partial_{\sigma} (va - ub) x'^\mu \delta(\sigma - \bar{\sigma}), \quad (4.6)$$

where we have used the notation $(ub)(\sigma) = u^\mu(\sigma) b_\mu(\sigma)$, and the expression $f(\sigma) \partial_{\sigma} \delta(\sigma - \bar{\sigma}) = f'(\sigma) \delta(\sigma - \bar{\sigma}) + f(\sigma) \partial_{\sigma} \delta(\sigma - \bar{\sigma})$ in the last step. Substituting the previous equation in (4.5) we obtain

$$\{J_{C(u,a)}(\sigma), J_{C(v,b)}(\bar{\sigma})\} = -J_{\bar{C}(\tilde{\omega}, \bar{\epsilon})}(\sigma) \delta(\sigma - \bar{\sigma}) + \frac{1}{2} \left( (ub)(\sigma) + (ub)(\bar{\sigma}) + va(\sigma) + (va)(\bar{\sigma}) \right) \partial_{\sigma} \delta(\sigma - \bar{\sigma}), \quad (4.7)$$

$$\partial_{\sigma} \delta(\sigma - \bar{\sigma}) =$$

$$= \frac{1}{2} \left( (ub)(\sigma) + (ub)(\bar{\sigma}) + va(\sigma) + (va)(\bar{\sigma}) \right) \partial_{\sigma} \delta(\sigma - \bar{\sigma}) + \frac{1}{2} \partial_{\sigma} (va - ub) x'^\mu \delta(\sigma - \bar{\sigma}), \quad (4.6)$$

where we have used the notation $(ub)(\sigma) = u^\mu(\sigma) b_\mu(\sigma)$, and the expression $f(\sigma) \partial_{\sigma} \delta(\sigma - \bar{\sigma}) = f'(\sigma) \delta(\sigma - \bar{\sigma}) + f(\sigma) \partial_{\sigma} \delta(\sigma - \bar{\sigma})$ in the last step. Substituting the previous equation in (4.5) we obtain

$$\{J_{C(u,a)}(\sigma), J_{C(v,b)}(\bar{\sigma})\} = -J_{\bar{C}(\tilde{\omega}, \bar{\epsilon})}(\sigma) \delta(\sigma - \bar{\sigma}) + \frac{1}{2} \left( (ub)(\sigma) + (ub)(\bar{\sigma}) + va(\sigma) + (va)(\bar{\sigma}) \right) \partial_{\sigma} \delta(\sigma - \bar{\sigma}), \quad (4.7)$$
where the coefficients in resulting current are

\[
\bar{w}_\mu = u^\nu \partial_\nu v^\mu - v^\nu \partial_\nu u^\mu,
\]

(4.8) and

\[
\bar{c}_\mu = 2\kappa B_{\mu\nu\rho} u^\nu v^\rho + (\partial_\mu a_\nu - \partial_\nu a_\mu) v^\nu - (\partial_\mu b_\nu - \partial_\nu b_\mu) u^\nu + \frac{1}{2} \partial_\mu (ub - va).
\]

(4.9)

The minus sign in front of the \(J_{\bar{C}(\bar{w},\bar{c})}\) is included for the future convenience. We see that \(\bar{w}_\mu\) does not depend on background fields, while the coefficient \(\bar{c}_\mu\) does, because of the \(H\)-flux term \(B_{\mu\nu\rho}\).

Let us confirm the equivalence between the twisted Courant bracket and the bracket that we have obtained in (4.10). The coordinate free expression for the Courant bracket is given by

\[
[(u, a), (v, b)]_C = (w, c),
\]

(4.10)

where \([u, v]_L\) is the Lie bracket and \(H(u, v, \cdot)\) is a 1-form obtained by contracting a three form. The Lie derivative \(\mathcal{L}_u\) is defined in a usual way \(\mathcal{L}_u = i_u d + di_u\), where \(d\) is the exterior derivative and \(i_u\) the interior derivative. Their action on 1-forms is given by \(da = \partial_\mu a_\nu dx^\mu dx^\nu\) and \(i_u a = u^\mu a_\mu\).

The Lie bracket is given by

\[
[u, v]_L |^\mu = u^\nu \partial_\nu v^\mu - v^\nu \partial_\nu u^\mu.
\]

(4.12)

Using the definition of Lie derivative, we furthermore obtain

\[
\left( \mathcal{L}_u b - \mathcal{L}_v a - \frac{1}{2} d(i_u b - i_v a) \right) |^\mu = u^\nu (\partial_\nu b_\mu - \partial_\mu b_\nu) - v^\nu (\partial_\nu a_\mu - \partial_\mu a_\nu) + \frac{1}{2} \partial_\mu (ub - va).
\]

(4.13)

As for the last term in (4.11), it is given by

\[
H(u, v, \cdot)|^\mu = 2\kappa B_{\mu\nu\rho} u^\nu v^\rho.
\]

(4.14)

The expression for the generalized current corresponding to the Courant bracket is obtained by substituting (4.12), (4.13) and (4.14) in (4.11)

\[
[(u, a), (v, b)]_C = (w, c),
\]

(4.15)

where \(w^\mu\) and \(c_\mu\) are exactly the same as \(\bar{w}^\mu\) and \(\bar{c}_\mu\) defined in (4.8) and (4.9), respectively. Therefore, we see that the bracket defined in (4.10) is indeed the twisted Courant bracket.

Integrating (4.7) over \(\sigma\) and \(\bar{\sigma}\), the anomaly is canceled and one obtains

\[
\{Q_{C(u,a)}, Q_{C(v,b)}\} = -Q_{C[(u,a),(v,b)]_C}.
\]

(4.16)
We see that the algebra of charges is anomaly free. The relation \( (4.16) \) was firstly obtained in [15] for the general case of the Hamiltonian formulation of string \( \sigma \)-model, in which momenta and coordinates satisfy the same Poisson bracket relations as auxiliary currents and coordinates in our theory.

Let us check whether the algebra \( (4.7) \) is consistent with the known results for the Poisson bracket algebra of the currents \( j_{\pm \mu} \) [20]

\[
\{ j_{\pm \mu}(\sigma), j_{\pm \nu}(\tilde{\sigma}) \} = \pm 2\kappa \Gamma_{\mu,\nu,\rho} x^\rho \delta(\sigma - \tilde{\sigma}) - 2\kappa B_{\mu \nu \rho} x^\rho \delta(\sigma - \tilde{\sigma}) \pm 2\kappa G_{\mu \nu} \delta(\sigma - \tilde{\sigma}), \quad (4.17)
\]

\[
\{ j_{\pm \mu}(\sigma), j_{\mp \nu}(\tilde{\sigma}) \} = \pm 2\kappa \Gamma_{\mu,\nu,\rho} x^\rho \delta(\sigma - \tilde{\sigma}) - 2\kappa B_{\mu \nu \rho} x^\rho \delta(\sigma - \tilde{\sigma}),
\]

where \( \Gamma_{\mu,\nu,\rho} = \frac{1}{2}(\partial_\rho G_{\mu \nu} + \partial_\nu G_{\mu \rho} - \partial_\mu G_{\nu \rho}) \) are Christoffel symbols. If we substitute \( a_\mu = \pm \kappa G_{\mu \nu} u^\nu \) and \( b_\mu = \pm \kappa G_{\mu \nu} v^\nu \) for constants \( u^\mu \) and \( v^\mu \) in \( (4.17) \), with the help of \( (3.6) \) we obtain

\[
\{ u^\mu j_{\pm \mu}(\sigma), v^\nu j_{\pm \nu}(\tilde{\sigma}) \} = u^\mu v^\nu \left( -2\kappa B_{\mu \nu \rho} \pm \kappa (\partial_\rho G_{\mu \nu} - G_{\mu \nu}(\tilde{\sigma})) \right) x^\rho \delta(\sigma - \tilde{\sigma}) \pm 2\kappa u^\mu v^\nu G_{\mu \nu}(\sigma) \partial_\sigma \delta(\sigma - \tilde{\sigma}) \quad (4.18)
\]

The consistency with the second relation in \( (4.17) \) is as easily obtained.

## 5 Roytenberg bracket

We use the same procedure as in the previous chapter, in order to calculate the Poisson bracket algebra of the currents \( (3.16) \). Firstly, using \( (4.11) \) we get

\[
\{ k^\mu(\sigma), k^\nu(\tilde{\sigma}) \} = -\kappa \partial_\delta \theta^{\mu \sigma} x^\rho \delta(\sigma - \tilde{\sigma}) - \kappa^2 (\theta^{\mu \sigma} \partial_\sigma \theta^{\nu \rho} - \theta^{\nu \sigma} \partial_\sigma \theta^{\mu \rho}) \pi_\rho \delta(\sigma - \tilde{\sigma}), \quad (5.1)
\]

Secondly, with the help of \( (3.15) \), we express the coordinate in terms of algebra generators and obtain

\[
\{ k^\mu(\sigma), k^\nu(\tilde{\sigma}) \} = -\kappa Q_\rho \pi_\rho k^\mu \delta(\sigma - \tilde{\sigma}) - \kappa^2 R^{\mu \rho \nu} \pi_\rho \delta(\sigma - \tilde{\sigma}), \quad (5.2)
\]

where we expressed the structure constants as fluxes

\[
Q_\rho \pi_\rho = \partial_\rho \theta^{\mu \nu}, \quad R^{\mu \rho \nu} = \theta^{\mu \sigma} \partial_\sigma \theta^{\nu \rho} + \theta^{\nu \sigma} \partial_\sigma \theta^{\mu \rho} + \theta^{\rho \sigma} \partial_\sigma \theta^{\mu \nu}. \quad (5.3)
\]

These are the non-geometric fluxes [21]. They were firstly obtained by applying the Buscher rules [22, 23] on the three-torus with non-trivial Kalb-Ramond field strength [13]. After the T-duality transformations are applied along two isometry directions, one obtains the space that is locally geometric, but globally non-geometric. The flux for this background is \( Q_\rho \pi_\rho \). After the T-duality transformation is applied along all directions, one obtains the space that is neither locally, nor globally geometric, characterized with the \( R^{\mu \rho \nu} \) flux. When considering a generalized T-dualization, the \( R \) flux is obtained when performing T-dualization over the arbitrary coordinate on which the background fields depend [24].
The rest of the generators algebra is calculated in a similar way

\[ \{ k^\mu(\sigma), \pi_\nu(\bar{\sigma}) \} = \kappa \delta^\mu_\nu \partial_\sigma \delta(\sigma - \bar{\sigma}) + \kappa Q^\mu_\rho \pi_\nu \delta(\sigma - \bar{\sigma}) \quad \{ \pi_\mu(\sigma), \pi_\nu(\bar{\sigma}) \} = 0. \quad (5.4) \]

Using (4.6) and (3.15) we can transform the anomaly in the following way

\[ R \]

which is equal to the Roytenberg bracket [11]. In case of only Roytenberg bracket is given by

\[ Q \]

Substituting the last equation in (5.5), we obtain

\[ \{ J_{R(u,a)}(\sigma), J_{R(v,b)}(\bar{\sigma}) \} = (u^\nu \partial_\nu u^\mu - u^\nu \partial_\nu v^\mu) \pi_\mu \delta(\sigma - \bar{\sigma}) - \kappa^2 R^{\mu\nu_\rho} \pi_\rho \delta(\sigma - \bar{\sigma}) - \kappa (\theta^{\mu_\rho} \partial_\rho u^\nu - \nu^\nu \partial_\nu a_\rho - \theta^{\nu_\mu} \partial_\nu v^\mu) \pi_\mu \delta(\sigma - \bar{\sigma}) - \kappa (u^\nu \partial_\nu b_\mu - \nu^\nu \partial_\nu a_\nu - \partial_\nu a_\mu) \pi_\mu \delta(\sigma - \bar{\sigma}) + \kappa (u^\nu (\partial_\nu b_\nu - \partial_\nu b_\mu) - \nu^\nu (\partial_\nu a_\nu - \partial_\nu a_\mu)) \pi_\mu \delta(\sigma - \bar{\sigma}) \quad \{ \rho \} \]

Using (4.6) and (3.15) we can transform the anomaly in the following way

\[ \kappa \left( (ub)(\sigma) + (va)(\bar{\sigma}) \right) \partial_\sigma \delta(\sigma - \bar{\sigma}) = \frac{\kappa}{2} \left( (ub)(\sigma) + (ub)(\bar{\sigma}) + (va)(\sigma) + (va)(\bar{\sigma}) \right) \partial_\sigma \delta(\sigma - \bar{\sigma}) \quad (5.6) \]

Substituting the last equation in (5.5), we obtain

\[ \{ J_{R(u,a)}(\sigma), J_{R(v,b)}(\bar{\sigma}) \} = -J_{\bar{R}(\bar{w},\bar{c})}(\sigma) \delta(\sigma - \bar{\sigma}) + \frac{\kappa}{2} \left( (ub)(\sigma) + (ub)(\bar{\sigma}) + (va)(\sigma) + (va)(\bar{\sigma}) \right) \partial_\sigma \delta(\sigma - \bar{\sigma}), \quad (5.7) \]

where

\[ \bar{w}^\mu = u^\nu \partial_\nu v^\mu - v^\nu \partial_\nu u^\mu + \kappa \theta^{\nu_\rho} \partial_\rho v^\mu a_\nu - \kappa v^\nu \partial_\nu a_\mu \theta^{\nu_\mu} - \kappa Q^\mu_\rho v^\nu a_\rho \quad (5.8) \]

and

\[ \bar{c}_\mu = u^\nu (\partial_\nu a_\nu - \partial_\nu a_\mu) - v^\nu (\partial_\nu b_\nu - \partial_\nu b_\mu) - \frac{1}{2} \partial_\mu (va - ub) + \kappa a_\rho b_\nu Q^\rho_\mu v^\nu b_\mu - \kappa \theta^{\nu_\rho} (\partial_\rho a_\mu b_\nu - \partial_\rho b_\mu a_\nu), \quad (5.9) \]

where we have substituted Q and R fluxes [5.3]. Unlike the currents coefficients in the initial theory, here both coefficients depend on backgrounds, due to presence of fluxes.

As expected, algebra is not closed due to the anomalous part. This Poisson bracket defines a new bracket

\[ [(u, a), (v, b)]_R = (\bar{w}, \bar{c}), \quad (5.10) \]

which is equal to the Roytenberg bracket [11]. In case of only R and Q flux present [5.3], the Roytenberg bracket is given by

\[ [(u, a), (v, b)]_R = \left( [u, v], \mathcal{L}_u b - \mathcal{L}_v a - \frac{1}{2} d(i_u b - i_v a) - \left( \mathcal{L}_v a - \mathcal{L}_u b + \frac{1}{2} d(i_u b - i_v a) \right) \Pi \right) \quad (5.11) \]

\[ = [a, b]_\Pi - [v, a\Pi]_L + [u, b\Pi]_L + \frac{1}{2} [\Pi, \Pi]_L (a, b, \cdot), \]
where $\Pi = \Pi^{\mu\nu}\partial_\mu\partial_\nu$ is the bi-vector. The expression $[\Pi,\Pi]_S(a,b,.)$ represents the Schouten-Nijenhuis bracket \cite{25} contracted with two 1-forms and $[a,b]_\Pi$ is the Koszul bracket \cite{26} given by

$$[a,b]_\Pi = \mathcal{L}_a b - \mathcal{L}_b a + d(\Pi(a,b)).$$  \hfill (5.12)

The Koszul bracket is a generalization of the Lie bracket on the space of differential forms, while the Schouten-Nijenhuis bracket is a generalization of the Lie bracket on the space of multi-vectors.

The terms in (5.11) that we have not calculated yet can be written, using (4.13), as

$$((\mathcal{L}_v a - \mathcal{L}_a b + \frac{1}{2} d(i_v b - i_v a))\Pi)\bigg|^{\mu} = (u^\nu (\partial_\nu b_\rho - \partial_\rho b_\nu) - v^\nu (\partial_\nu a_\rho - \partial_\rho a_\nu) + \frac{1}{2} \partial_\mu (u b - v a))\Pi^{\rho\mu}. \hfill (5.13)$$

The Koszul bracket (5.12) can be further transformed in the following way

$$[a,b]_\Pi|^{\mu} = \Pi^\rho_{\alpha\beta}\partial_\sigma a^\alpha b^\beta + \partial_\mu \Pi^{\rho\sigma} a_\rho b_\sigma, \hfill (5.14)$$

while the remaining terms linear in $\Pi$ become

$$([-v,a]\Pi)_L + [u,b\Pi)_L]^{\mu} = v^\nu (\partial_\nu a_\rho \Pi^{\mu\rho} + a_\rho \partial_\nu \Pi^{\mu\rho}) + a_\rho \Pi^{\mu\nu} \partial_\nu v^\mu - u^\nu (\partial_\nu b_\rho \Pi^{\mu\rho} + b_\rho \partial_\nu \Pi^{\mu\rho}) - b_\rho \Pi^{\mu\nu} \partial_\nu u^\mu. \hfill (5.15)$$

Lastly, we write the expression for the Schouten-Nijenhuis bracket for bi-vectors

$$[\Pi,\Pi]_S|^{\mu\nu} = \epsilon^{\mu\nu}_{\alpha\beta\gamma}\Pi^\sigma a_\sigma \Pi^{\beta\gamma}, \hfill (5.16)$$

where

$$\epsilon^{\mu\nu}_{\alpha\beta\gamma} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\beta & \delta^\rho_\gamma \\ \delta^\nu_\alpha & \delta^\rho_\beta & \delta^\mu_\gamma \\ \delta^\rho_\alpha & \delta^\mu_\beta & \delta^\nu_\gamma \end{vmatrix}. \hfill (5.17)$$

Thus, we get

$$([\Pi,\Pi]_S(a,b,.)|^{\mu} = 2R^{\mu\nu} a_\nu b_\rho, \hfill (5.18)$$

where $R^{\mu\nu}$ is the flux defined in (5.3).

Combining the previously obtained terms, we obtain the expression for the generalized current corresponding to the Roytenberg bracket twisted by the non-commutativity parameter as a bi-vector

$$[(u,a),(v,b)]_R = (w,c), \hfill (5.19)$$

where $w^\mu$ and $c^\mu$ are equal to $\tilde{w}^\mu$ and $\tilde{c}^\mu$, defined in (5.8) and (5.9), respectively, provided that $\Pi^{\mu\nu} = \kappa \theta^{\mu\nu}$.

Integrating the previous equation over $\sigma$ and $\bar{\sigma}$, we see that charges satisfy

$$\{Q_R(u,a),Q_R(v,b)\} = -Q_R([u,a],[v,b])_R. \hfill (5.20)$$

The relation (5.20) was firstly obtained in \cite{12}, where the Hamiltonian approach of the Polyakov 2-dimensional $\sigma$ model coupled to the bi-vector was considered.
What we see is that the basis vectors in mutually dual theories give rise to the Courant and Roytenberg bracket. After the T-dualization, the currents satisfy

\[ J_{C(u,a)} \rightarrow J_{R(v,b)}, \]  

provided that the coefficients satisfy \( u^\mu \rightarrow v^\mu, \ a_\mu \rightarrow b_\mu. \) As a consequence of that, the Courant bracket is dual to the Roytenberg bracket.

6 Conclusion

In this paper, we considered the closed bosonic string propagating in the background composed of a coordinate dependent metric \( G_{\mu\nu}(x) \) and Kalb-Ramond field \( B_{\mu\nu}(x) \), as well as its T-dual theory. T-duality was understood as a canonical transformation interchanging the momenta and coordinate \( \sigma \)-derivatives. The generating function for this transformation does not depend explicitly on time, which means that the Hamiltonians are equal in the initial and dual theory. The duality transformations of all background fields were obtained from this requirement.

The Hamiltonian was expressed in terms of currents \( j^{\pm}_\mu \), as well as in terms of dual currents \( \ast j^{\pm}_\mu \). Currents are functions of momenta and coordinates, that can be rewritten in new, suitable bases. The new basis for the initial description consists of coordinate \( \sigma \)-derivatives \( x'_\mu \) and the auxiliary currents \( i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \), and for the dual description it consists of coordinate \( \sigma \)-derivatives \( y'_\mu \) and dual auxiliary currents \( \ast i_\mu = \ast \pi_\mu + 2\kappa \ast B^{\mu\nu}y'_\nu \).

We considered the most general currents, \( J_{C(u,a)} \) and \( \ast J_{R(v,b)} \), defined as 2-parameter worldsheet vectors in new bases, i.e. the arbitrary linear combination of the coordinate \( \sigma \)-derivatives and auxiliary currents in the initial, and dual coordinate \( \sigma \)-derivatives and dual auxiliary currents in the dual theory. Then, without the loss of generality, we considered the current \( J_{R(v,b)} \) obtained by applying the T-dual relations (3.15) on dual basis vectors, \( \{y'_\mu, \ast i_\mu\} \). By construction, the generalized current \( J_{R(v,b)} \) is function of momenta \( \pi_\mu \) and another set of auxiliary currents \( k_\mu = \kappa x'^\mu + \kappa \theta^{\mu\nu}\pi_\nu \). Therefore, both generalized currents \( J_{C(u,a)} \) and \( J_{R(v,b)} \), are defined in the initial phase space.

It is worth emphasizing that dual currents \( \ast i_\mu \) and \( \ast j^{\pm}_\mu \) cannot be obtained by applying the T-duality canonical transformation laws (2.10) to the currents \( i_\mu \) and \( j^{\pm}_\mu \). Rather than that, when the T-duality transformation relations were applied to the dual currents, the new set of currents defined in the initial phase space, that we have marked as \( k_\mu \) and \( l^{\pm}_\mu \), were obtained. Hence, \( j^{\pm}_\mu \) and \( \ast j^{\pm}_\mu \) represent the same quantity, the current, in mutually dual theories, but do not transform into each other under the duality transformation rules. These results are summarized in the table 1.

The algebra of two types of generalized currents \( J_{C(u,a)} \) and \( J_{R(v,b)} \), both defined in the initial phase space, as well as the algebra of their respective charges \( Q_{C(u,a)} \) and \( Q_{R(v,b)} \), were calculated. The current algebra was found to be anomalous, but the algebra of charges was found to be anomaly free. It has been found that the Poisson bracket algebra of generalized currents gives rise to the Courant bracket in the initial, and to the Roytenberg bracket in the initial description of the T-dual theory. Hence, we concluded that the Courant bracket is dual to the Roytenberg bracket. We find these results important in itself. Both the Courant and the Roytenberg bracket are well understood
mathematical structures. Relations between them and T-duality has a potential to help understand the T-duality better. Moreover, by analyzing characteristics of these brackets we can examine how certain aspects of the mutually T-dual theories relate to each other.

| Initial theory                                                                 | T-dual theory                                                                 | T-dual theory in the initial phase space |
|-------------------------------------------------------------------------------|------------------------------------------------------------------------------|----------------------------------------|
| $H = \frac{1}{2}(G^{-1})^{\mu\nu}(j_{\mu\nu}J_{+\nu} + j_{-\mu}\bar{J}_{\nu})$ | $\Pi H = \frac{1}{2}G^{\nu}_{\mu}(j^{\mu}_{\nu} \ast J + \ast J^{\mu}_{\nu})$ | $H = \frac{1}{2}G^{\nu}_{\mu}(l^{\nu}_{\mu} + l^{\mu}_{\nu})$ |
| Currents                                                                      | Dual currents                                                                | Currents dual to dual currents           |
| $j_{+\mu} = -i\delta_{\mu} + \kappa G_{\mu\nu}x^{\nu}$                      | $\ast j_{\mu} = \ast i\delta_{\mu} + \kappa G^{\mu\nu}y_{\nu}$             | $\Pi j_{\mu} = k^{\mu} + (G^{\nu}_{\mu})^{\nu}_{\mu}y_{\mu}$ |
| $j_{-\mu} = i\delta_{\mu} + 2G_{\mu\nu}B^{\nu}$                            | $j^{\mu}_{\nu} = \ast i\delta_{\mu} + 2\kappa B^{\mu\nu}y_{\nu}$           | $\Pi j^{\mu}_{\nu} = k^{\mu} + \kappa x^{\mu} + \kappa G^{\mu\nu}y_{\nu}$ |
| Generalized currents and charges                                             | Generalized currents and charges                                             | Generalized currents and charges         |
| $J_{C(\nu,a)} = u^{\nu}i_{\mu} + a_{\mu}x^{\nu}$                           | $\ast J_{R(\nu,b)} = v^{\nu}y_{\mu} + b_{\mu}x^{\nu}$                       | $J_{R(\nu,b)} = -J_{C(\nu,a)}$           |
| $Q_{C(\nu,a)} = \int d\sigma J_{C(\nu,a)}$                                 | $\ast Q_{R(\nu,b)} = \int d\sigma \ast J_{R(\nu,b)}$                         | $\{Q_{R(\nu,b)} , Q_{R(\nu,b)} \} = -Q_{R(\nu,b)(u,a)}$ |
| Courant algebra                                                              | Roytenberg algebra                                                           |                                        |
| $\{Q_{C(\nu,a)} , Q_{C(\nu,a)} \} = -Q_{C(\nu,a)(u,a)}$                   |                                              |                                        |

Table 1: Currents in initial and dual theory

It is interesting that both charges $Q_{C(\nu,a)}$ and $Q_{R(\nu,b)}$ can be expressed as the self-dual symmetry generators in the form

$$G = \int d\sigma \left[ \xi^{\mu}{\pi}_{\mu} + \tilde{\Lambda}_{\mu}\kappa x^{\mu} \right].$$

(6.1)

It is easy to show that if we define the new gauge parameter $\Lambda_{\mu} = \tilde{\Lambda}_{\mu} + 2B_{\mu\nu}\xi^{\nu}$, the generators (6.1) are charges $Q_{C(\xi,a)}$; if we define $\tilde{\xi}^{\mu} = \xi^{\mu} + \kappa \theta^{\mu\nu}\tilde{\Lambda}_{\nu}$ the generators are charges $Q_{R(\xi,b)}$. Momenta $\pi_{\mu}$ are generators of general coordinate transformations and $x^{\mu}$ generators of local gauge transformations $\delta_{\Lambda}B_{\mu\nu} = \partial_{\mu}\tilde{\Lambda}_{\nu} - \partial_{\nu}\tilde{\Lambda}_{\mu}$, while $\xi^{\mu}$ and $\tilde{\Lambda}_{\mu}$ are their corresponding parameters. These generators were studied in [20], where it was shown that general coordinate transformations are T-dual to gauge transformations.

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