Universal crossover in interacting fermions within the low-energy expansion

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We generalize the Bardeen-Cooper-Schrieffer-Bose-Einstein-condensation (BCS-BEC) crossover of two-component fermions, which is realized by tuning the s-wave scattering length $a$ between the fermions, to the case of an arbitrary effective range $r_e$. By using the Nozières-Schmitt-Rink (NSR) approach, we show another crossover by changing $r_e$ and present several similarities and differences between these two crossovers. The region $(r_e > a/2)$ where the effective range expansion breaks down and the Hamiltonian becomes non-Hermitian is found. Our results are universal for interacting fermions with low-energy constants $a$ and $r_e$ and are directly relevant to ultracold Fermi atomic gases as well as dilute neutron matter.

\textbf{Introduction}—The BCS-BEC crossover, which is realized by tuning the s-wave scattering length $a$ in cold atom experiments \cite{1,2}, has been widely accepted as an important concept to understand strongly correlated quantum systems \cite{3,4}. Indeed, this phenomenon has been discussed in various systems such as superconductors \cite{5,6}, \cite{7,8} and dense quark matter \cite{9,10}. In this regard, thermodynamic properties of strongly interacting ultracold Fermi gases have been experimentally investigated by changing $a$ near the unitarity limit \cite{11,12}. The observed quantities are universal for homogeneous two-component fermions with a dimensionless coupling parameter $1/(k_F a)$ where $k_F$ is the Fermi momentum. The equation of state in this atomic system \cite{13,14,15} shows an excellent agreement with a variational calculation for dilute pure neutron matter (PNM) \cite{16,17,18} which has a relatively large negative scattering length $a = -18.5 \text{ fm}$ \cite{19} and the coupling parameter $1/(k_F a) \simeq -0.04$ at a subnuclear density $\rho \simeq 0.08 \text{ fm}^{-3}$.

On the other hand, there are of course various differences between ultracold Fermi gases and pure neutron matter such as non-locality of the interaction. The most important difference is the magnitude of an effective range $r_e$. While $r_e$ in ultracold Fermi gases near a broad Feshbach resonance is negligible, that in neutron matter given by $r_e = 2.8 \text{ fm}$ \cite{20} largely affects system’s properties even around the subnuclear density \cite{21}. On the other hand, a narrow Feshbach resonance in ultracold atoms gives a large and negative effective range \cite{22}. Since $r_e$ is directly related to the phase shift $\delta (p)$ ($p$ is the momentum), one can expect that the negative (positive) effective range induces a strong (weak) attraction \cite{23,24,25,26}. In this sense, a natural question arises: \textit{How does the superfluid transition behave if one arbitrarily changes the effective range?}

The purpose of this Letter is to answer this question and show that another crossover of the superfluid phase transition from the BCS pairing to the molecular BEC occurs when changing the effective range $r_e$. For this purpose, we use the Nozières-Schmitt-Rink (NSR) theory \cite{27,28} which is a standard approach to describe the BCS-BEC-crossover physics in the presence of strong pairing fluctuations. Figure 1 is a proposed phase diagram of strongly interacting homogeneous spin-1/2 fermions with respect to two dimensionless parameter $1/(k_F a)$ and $r_e/k_F$. While the BCS and the BEC sides are qualitatively separated at $1/(k_F a) = 0$ in the zero-range case, such a crossover boundary is generalized to $k_F \cot \delta (k_F) = -1/(k_F a) + r_e/k_F/2 = 0$. PNM corresponds to the region $r_e/k_F > 0$ and $1/(k_F a) < 0$ where the neutron Fermi momentum $k_F$ can vary from the inner crust to the outer core of a neutron star interior. In addition, we show that the solution for $T_c$ disappears in the large-
positive-effective-range region with finite positive scattering length due to the breakdown of the effective range expansion at $r_c > a/2$. Since an optical control of the scattering length and the effective range can be achieved in ultracold atomic gases\cite{51,52,53,54}, one can expect that our results can be checked by future experiments.

**Formulation**—We consider a homogeneous two-component fermion system with low-energy $s$-wave scattering described by a two-channel Hamiltonian\cite{12,13,55,50}:

\[
H = \sum_{p,\sigma} \xi_p c_{p,\sigma}^\dagger c_{p,\sigma} + \sum_{q} (\varepsilon_q/2 + \nu - 2\mu) b_q^\dagger b_q + \sum_{p,q} (g_p b_q^\dagger c_{p+q/2,\uparrow} c_{-p+q/2,\downarrow} + H.c.),
\]

where $\xi_p = \varepsilon_p - \mu \equiv p^2/2 - \mu$ is the kinetic energy with the momentum $p$ and the fermion mass $m$ measured from the chemical potential $\mu$, $c_{p,\sigma}$ ($b_q$) fermionic (bosonic) annihilation operator (where $\sigma = \uparrow, \downarrow$ is the fermionic spin state), $\nu$ the energy of a diatomic boson, and $g_p = g/\sqrt{1 + (p/p_c)^2}$ the momentum-dependent Feshbach coupling with a cutoff parameter $p_c$ (noting that we use $h = k_B = 1$ and the system volume is taken to be one for simplicity). In this model, the two-body $T$-matrix is obtained as

\[
T(p, p', \omega) = \frac{g_pg_p}{\omega + \nu - \sum_k \omega_k - 2ip_k},
\]

where $\omega_k = \omega + i\delta$ and $\delta$ is an infinitesimal small positive number. Eq. (2) gives an exact relation between the parameters in Eq. (1) and the low-energy phase shift

\[
p\cot\delta(p) = -\frac{4\pi}{m} T^{-1}(p, p, 2\varepsilon_p + i\delta)
\equiv -\frac{1}{a} + 1 + \frac{r_c p^2}{s p^4},
\]

where

\[
a = \left[ -\frac{4\pi \nu}{m g^2} + p_c \right]^{-1},
\]

\[
0 < a \ll 1,
\]

\[
r_c = -\frac{8\pi}{m^2 g^2} + \frac{2}{p_c} \left( 1 - \frac{1}{p_c a} \right),
\]

and

\[
s = -\frac{4\pi}{m^2 g^2 p_c^2}
\]

are the scattering length, the effective range, and the shape parameter\cite{61}, respectively. In this letter, we take $g \to \infty$ ($p_c \to \infty$) when $r_c > 0$ ($< 0$) such that $s = 0$ since we limit ourselves to the effective range expansion. We note that higher-order coefficients $O(p^3)$ are exactly zero in this model. Figure 2 shows the phase shift $\delta(p)$ at various scattering lengths and effective ranges. While the low-momentum region $p|a| \leq 0.5$ of $\delta(p)$ does not depend on $r_c$, the high-momentum region is characterized by $r_c$. Since the positive phase shift indicates how the phase of the wave function is drawn to the short-distance region due to the attraction, one can interpret the magnitude of $\delta(p)$ as a strength of the attractive interaction. From this viewpoint, one can expect that a negative (positive) effective range induces effectively strong (weak) attraction between two fermions.

The condition for obtaining $T_c$ is given by the Thouless criterion\cite{61}:

\[
\frac{mg^2}{4\pi a} + 2\mu = -\sum_p g_p^2 \left[ \frac{1 - 2f(\xi_p)}{2\xi_p} - \frac{m}{p^2} \right],
\]

where $f(\xi) = [e^{\xi/T} + 1]^{-1}$ is the Fermi distribution function. In the NSR scheme, the chemical potential $\mu$ at a given number density $N$ is obtained by solving\cite{12,62}:

\[
N = 2 \sum_p f(\xi_p) + 2 \sum_q b(\varepsilon_q/2 + \nu - 2\mu)
- T \sum_q \frac{\partial}{\partial \mu} \ln \left[ 1 - D_q(i\nu_n)\Pi_q(i\nu_n) \right],
\]

where $D_q(i\nu_n) = [i\nu_n - \varepsilon_q/2 - \nu + 2\mu]^{-1}$ is the thermal Green’s function of a bare molecular boson ($i\nu_n$ is the bosonic Matsubara frequency) and

\[
\Pi_q(i\nu_n) = \sum_p g_p \frac{1 - f(\xi_p + q) - f(\xi_p - q)}{i\nu_n - \xi_p + q - \xi_p - q}
\]

is the pair-correlation function. The third term of Eq. (5) is the many-body correction associated with strong.
The superfluid phase transition temperature $T_c$ and the chemical potential $\mu$ at $T = T_c$ as functions of the inverse scattering length $1/(k_Fa)$ and the effective range $r_{e}k_F$, where $\varepsilon_F$ is the Fermi energy of non-interacting fermions. In each figure, the dashed and dash-dotted curves show results at PNM parameters and $r_e = 0$, respectively. The dotted line in the panel (b) shows the half of the critical binding energy $E_{b,c} = -4/(ma^2)$ at $r_e = a/2$. The effective range expansion breaks down in the region beyond $r_e = a/2$ (BD).

Results—Figure 3 shows the calculated superfluid phase transition temperature $T_c$ and chemical potential $\mu$ at $T = T_c$ in the parameter plane of $1/(k_Fa)$ and $r_{e}k_F$. One can find that the effective-range dependence of $T_c$ at fixed $1/(k_Fa) \leq 0$ is very similar to the ordinary zero-range results shown as the dash-dotted lines. $T_c$ increases with increasing the magnitude of the negative effective range even at the negative scattering length since the effective interaction is enhanced [49, 51, 62]. In the weak-coupling regime where $1/(k_Fa) < 0$ and $r_{e}k_F > 0$, the critical temperature is qualitatively explained by $T_c \sim T_F \exp \left[ -\frac{\pi}{2} \left( -\frac{1}{k_Fa} + \frac{1}{2}r_{e}k_F \right) \right]$ [51, 62], whereas the ordinary BCS prediction with the zero-range potential is given by $T_c^{\text{BCS}} \simeq 0.6117T_F \exp \left( \frac{\mu}{2\kappa a} \right)$. In the large-negative-effective-range limit with any finite scattering lengths, $T_c$ and $\mu$ approach $T_{\text{LNR}} = 0.2047T_F$ and 0, respectively [51, 64], where the system consists of the BEC of diatomic molecules and a small number of thermal excited fermions given by $N_0 = 2 \sum_p f(\varepsilon_p)$ which causes the quantitative difference between $T_{\text{LNR}}$ and the ordinary BEC limit $T_{\text{BCS}} = 0.2187T_F$ [10, 14]. On the other hand, $T_c$ gradually decreases and $\mu$ approaches $\varepsilon_F$ with increasing $r_e$ due to the reduction of the cutoff as [40]

$$p_c = \frac{1 + \sqrt{1 - 2r_e/a}}{r_e} \quad (r_e > 0).$$

In this regard, the diagonal effective interaction $U_{\text{eff}}(p,p) = -g^2_F/(\nu - 2\mu)$ at $p = k_F$ becomes weaker, resulting in the decrease of $T_c$ [40, 14, 62]. The pure neutron matter (PNM) corresponding to the dashed curve in Fig. 3 is also located in this region, with $a = -18.5$ fm and $r_e = 2.8$ fm [14]. PNM gradually approaches the unitarity limit $(1/a = 0)$ with increasing the neutron density $\rho_n = k_F^3/(3\pi^2)$ and finally goes to the large-positive-effective-range region.

In contrast, in the region where $a > 0$ and $r_e > 0$, $r_e$ induces the enlargement of the two-body binding energy $E_b$ and $\mu$ approaches the half of $E_b$, given by

$$E_b = \frac{1}{ma^2} \frac{1}{\left[ 1 - 1/(ap_c) \right]^2 a}$$

$$= \frac{1}{ma^2} \left[ \frac{1 + \sqrt{1 - 2r_e/a}}{1 - r_e/a + \sqrt{1 - 2r_e/a}} \right]^2.$$  

One can find that the real solution of $E_b$ disappears if $r_e$ exceeds $a/2$ since $p_c$ given by Eq. (10) becomes complex. This result indicates that the effective range expansion breaks down to describe such a deep bound state within the two low-energy parameters. In such a situation, the Hamiltonian given by Eq. (11) becomes non-Hermitian and we cannot obtain a thermodynamic equilibrium state in our homogeneous model. We also note that while such a singularity was overlooked in the previous mean-field study at $T = 0$ [50], interestingly a cluster formation has been predicted in a similar region where $r_e > 0.46a$ in trapped Fermi gases [62]. Indeed, $\mu$ approaches the half of a critical binding energy $E_{b,c} = -4/(ma^2)$ at this boundary $r_e = a/2$ ($p_c = r_e^{-1}$) in the strong-coupling side. We note that although the spin-triplet neutron-proton interaction in dilute nuclear matter has a positive scattering length $[a_t = 5.42$ fm$]$ and an effective range $[r_{e,t} = 1.76$ fm$]$ [44], it does not satisfy $r_{e,t} > a_t/2 = 2.71$ fm and the deuteron (neutron-proton pair) binding energy $E_d = -2.2$ MeV is consistent with Eq. (11) [66].

To see the effects of pairing fluctuations on $T_c$ and $\mu$, we compare the results of the NSR approach to the

FIG. 3. (a) The superfluid phase transition temperature $T_c$ and the chemical potential $\mu$ at $T = T_c$ as functions of the inverse scattering length $1/(k_Fa)$ and the effective range $r_{e}k_F$, where $\varepsilon_F$ is the Fermi energy of non-interacting fermions. In each figure, the dashed and dash-dotted curves show results at PNM parameters and $r_e = 0$, respectively. The dotted line in the panel (b) shows the half of the critical binding energy $E_{b,c} = -4/(ma^2)$ at $r_e = a/2$. The effective range expansion breaks down in the region beyond $r_e = a/2$ (BD).
condensed bosons $2 \sum_q b(\varepsilon_q/2 + \nu_r - 2\mu)$ becomes finite. Such a simple approximation becomes exact in the large-negative-effective-range limit \cite{68} and this fact is in a sharp contrast with the strong-coupling BEC side realized at small positive scattering length and zero effective range. We note that this is a direct consequence of the difference between BEC of closed-channel molecules in the two-channel model at $r_c k_F \leq -2$ and that of tightly bound molecules at $1/(k_F a) \gtrsim 1$ with $r_c = 0$ \cite{13}.

\textbf{Conclusions.}— We have addressed the universal crossover within the low-energy expansion involving two interaction parameters, that is, the scattering length $a$ and the effective range $r_c$. By using the Nozières-Schmitt-Rink (NSR) approach to incorporate effects of strong pairing fluctuations, we predict the superfluid phase transition temperature $T_c$ at arbitrary dimensionless parameters $1/(k_F a)$ and $r_c k_F$. The crossover of $T_c$ from BCS superfluid to molecular BEC can be achieved by changing $r_c$ from the positive value to negative one at $1/a \lesssim 0$. In this regime, the positive effective range reduces the effective interaction at the Fermi momentum due to the momentum cutoff $p_F \simeq r_c^{-1}$. In the large-negative-effective-range limit at any finite scattering lengths, $T_c$ approaches $T_{LNR} = 0.204 T_F$ which is equivalent to the BEC temperature of diatomic molecules in the presence of thermal excited fermions. In addition, the chemical potential $\mu$ at $T = T_c$ shifts from the Fermi energy $\varepsilon_F$ to zero by decreasing $r_c$. On the other hand, at $r_c > a/2$ with the positive $a$, the effective range expansion breaks down and the physical bound state vanishes, resulting the disappearance of $T_c$ and a non-Hermiticity of the model Hamiltonian in this regime.

In this Letter, we have not considered the effects of the Hartree shift which is of importance in the positive-effective-range region \cite{40}. In particular, it is reported that the Hartree shift with a large positive effective range makes the system thermodynamically unstable at $1/a = 0$ \cite{68}. Furthermore, the Gor'kov-Melik-Barkhudarov (GMB) corrections \cite{72} on $T_c$ associated with particle-hole fluctuations \cite{64, 73, 77} play a significant role especially in the weak-coupling region. These are left as future problems.

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{(a) The superfluid phase transition temperature $T_c$ and (b) the chemical potential $\mu$ at $T = T_c$ as a function of $r_c k_F$ at $1/a = 0$, within the NSR and MF approaches. MF$^*$ is the mean-field results with the ultraviolet renormalization of $\nu$. The dotted line in the panel (a) shows the large-negative-effective-range limit $T_{LNR} = 0.204 T_F$ \cite{58, 63}.}
\end{figure}

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