Formation of Arm Spurs/Feathers in Local Simulations of the Wiggle Instability

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ABSTRACT

Gaseous substructures such as feathers and spurs dot the landscape of spiral arms in disk galaxies. One of the candidates to explain their formation is the wiggle instability of galactic spiral shocks. We study the wiggle instability using local 2D isothermal hydrodynamical simulations of non-self gravitating gas flowing in an externally imposed spiral potential. In the first part of the paper we compare the results of simulations with predictions from analytic linear stability analysis, and find good agreement between the two. By repeating the simulations with different types of boundary condition we also demonstrate that a distinct, parasitic Kelvin-Helmholtz instability can develop in addition to the wiggle instability, particularly in systems with small gas sound speed and strong spiral potential. In the second part of the paper we explore the parameter space and study the properties of the substructures generated by the wiggle instability. We find that the predicted separation between spurs/feather is highly sensitive to the sound speed, to the spiral potential strength and to the interarm distance. The feather separation decreases and the growth rate increases with decreasing sound speed, increasing potential strength, and decreasing interarm distance. We compare our results with a sample of 20 galaxies from the HST archival survey of La Vigne et al., and find that the wiggle instability can reproduce the range of typical feather spacing seen in observations. It remains unclear how the wiggle instability relates to competing mechanisms for spurs/feather formation such as the magneto-jeans instability and the stochastic accumulation of gas due to correlated supernova feedback.

Key words: instabilities - shock waves - hydrodynamics - ISM: kinematics and dynamics - galaxies: kinematics and dynamics

1 INTRODUCTION

Gaseous spiral arms in galaxies often exhibit substructure such as spurs and feathers that extend from the arms into the interarm region (e.g. La Vigne et al. 2006; Leroy et al. 2017; Schinnerer et al. 2017; Kreckel et al. 2018; Elmegreen & Elmegreen 2019). The formation of these substructures is not fully understood. Various mechanisms have been proposed, including gravitational instability/amplification of perturbations in the arm crest (Elmegreen 1979; Cowie 1981; Balbus & Cowie 1985; Balbus 1988; Elmegreen 1994), magneto-gravitational instabilities (Kim & Ostriker 2002, 2006; Lee & Shu 2012; Lee 2014), and clustering of gas due to correlated supernova feedback (Kim et al. 2020).

One possible mechanism that does not rely on the gas self-gravity nor on magnetic fields is the wiggle instability (Wada & Koda 2004). The wiggle instability is a purely hydrodynamic instability that occurs as small density perturbations in the interstellar gas are amplified by multiple passages through a spiral arm crest (Kim et al. 2014; Kim et al. 2015; Sormani et al. 2017). The wiggle instability causes the spiral shock front to fragment and break into regularly spaced clumps whose morphology resembles that of observed spurs/feathers in disk galaxies (Wada & Koda 2004; Kim & Kim 2014). However, there is little work on how properties of the structures induced by the wiggle instability depend on the underlying parameters.

In this paper, we perform local idealised 2D hydrodynamical simulations of the wiggle instability. Our aims are (i) to compare the results of the simulations with predictions from previous analytical linear stability analysis; (ii) to explore how the wiggle instability depends on parameters such as the gas sound speed, the spiral potential strength, the background shear and the interarm separation, and formulate empirical trends that can be used to interpret observations.

This paper is structured as follows. In Section 2, we present the formulation of the problem. In Section 3 we describe our numerical setup and our methodology for the shock front analysis. In Section 4 we compare in detail two example simulations with predictions from the linear stability analysis of Sormani et al. (2017) by performing a Fourier decomposition of the unstable shock front as a function of time. In Section 5 we demonstrate that boundary conditions are critical for the development of the wiggle instability, which also proves that a distinct, parasitic Kelvin-Helmholtz instability can develop in addition to the wiggle instability for certain values of the parameters. In Section 6 we study how the properties of the wiggle instability depend on the underlying parameter. In Section 7 we compare our results with HST observations of spurs/feathers in disk galaxies and we discuss the relation of the wiggle instability with other proposed mechanisms of spurs/feather formation. We sum up in Section 8.
2 BASIC EQUATIONS

Our goal is to study the wiggle instability using the simplest possible setup. Following Roberts (1969) we approximate the equations of hydrodynamics in a local Cartesian patch that is corotating with a segment of a spiral arm located at galactocentric radius \( R = R_0 \). We briefly summarise the setup here, and offer a detailed derivation of the equations in Appendix A.

The gas is assumed to flow in an externally imposed gravitational potential that is the sum of an axisymmetric component with circular angular velocity \( \Omega(R) \) plus a spiral perturbation \( \Phi_s \) that rigidly rotates with pattern speed \( \Omega_0 \). We assume an isothermal equation of state \( P = \frac{c^2}{ho} \), where \( P \) is the pressure, \( \rho \) is the surface density and \( c_s \) is the (constant) sound speed, and that the gas is two-dimensional, non self-gravitating and unmagnetised.

We call \((x,y)\) the local Cartesian coordinates in a frame that is sliding along the arm segment with a speed equal to the local circular velocity, where \( x \) is the coordinate perpendicular to the arm and \( y \) the coordinate parallel to the arm. The equations of motion in this frame approximated under the assumption of small pitch angle (\( \sin(i) \ll 1 \)) are (see Appendix A):

\[
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{\rho} - \nabla \Phi_s - 2\Omega_0 \times \mathbf{v} + q\Omega_0 v_y \mathbf{e}_y + F \left(1 - \frac{q}{2}\right) \mathbf{e}_y
\]

\[
\partial_t \rho + \mathbf{v} \cdot (\rho \mathbf{v}) = 0,
\]

where \( \mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y \) is the velocity in the \((x,y)\) frame (in other words, \( v_x \) is the velocity perpendicular to the arm in the frame corotating with the spiral arms, while \( v_y \) is the difference between the velocity parallel to the arm and the local circular velocity), \( \Omega_0 = \Omega(R_0) \), \( \Phi_0 = \Phi(R_0) \),

\[
q = -\left( \frac{d \log \Omega}{d \log R} \right)_{R=R_0},
\]

is the shear parameter calculated at \( R = R_0 \),

\[
F = 2\Omega_0 v_y \sin(i),
\]

is a constant, and \( v_{c0} = (\Omega_0 - \Omega_y)R_0 \) is the circular velocity at \( R_0 \) in the frame corotating with the spiral arms.

We assume that in our local frame the spiral perturbation to the potential has the form:

\[
\Phi_s = \Phi_0 \cos \left( \frac{2\pi x}{L_s} \right),
\]

where \( \Phi_0 \) is constant and \( L_s \ll R_0 \) is the size of our local patch in the \( x \) direction, which is equal to the separation between two consecutive spiral arms at \( R = R_0 \) (see Eq. A34). Note that, since we assume that the spiral potential only depends on \( x \), our system is transitionally invariant in the \( y \) direction.

Equations (1) and (2) are identical to those of a simple 2D fluid that is subject to the following forces:

1. The pressure \(-\nabla P/\rho\).
2. The external spiral potential \(-\nabla \Phi_s\).
3. The Coriolis force \(-2\Omega_0 \times \mathbf{v}\).
4. A constant force \( F(1 - q/2) \mathbf{e}_y\).
5. The “shear” force \( q\Omega_0 v_y \mathbf{e}_y\).

2.1 Parameters

The problem posed by Equations (1) and (2) is completely specified by the six parameters \( \{c_s, \Phi_0, L_s, q, \Omega_0, F\} \). From these, we can derive four dimensionless parameters and two scaling constants. Without loss of generality we choose to use \( \Omega_0 \) and \( F \) as scaling constants. We rescale the others as indicated in Table 1 to obtain four independent dimensionless parameters \( \{\xi_0, \Phi_0, L_s, q, \tilde{q}\} \). In this paper we explore how the properties of the wiggle instability depend on these four. For simplicity of notation we drop the ‘tilde’ superscript hereafter and always refer to the dimensionless parameters unless otherwise specified. The range of values explored in this work is indicated in Table 1. The table indicates both the dimensionless values and the physical values calculated assuming typical galactic values for the two scaling constants. The physical value of \( F \) indicated in the table corresponds to a typical circular velocity in the frame corotating with the spiral arms of \( v_{c0} = 100 \text{km s}^{-1} \) and pitch angle \( \sin(i) = 0.1 \). The explored values of \( q \) include the cases of solid body rotation \( (q = 0) \), a flat rotation curve \( (q = 1) \), and a Keplerian rotation curve \( (q = 1.5) \).

Using the typical values of the scaling constants indicated in Table 1, one unit of dimensionless time corresponds to \( 1/\Omega_0 = 48.9 \text{Myr} \) of physical time, one unit of dimensionless length corresponds to \( F/\Omega_0^2 = 1 \text{kpc} \), and one unit of dimensionless velocity corresponds to \( F/\Omega_0 = 20 \text{km s}^{-1} \).

2.2 Steady state solution

Equations (1) and (2) admit steady state solutions that are periodic in the \( x \) coordinate and do not depend on the \( y \) coordinate. As noted by Roberts (1969) (see also Shu et al. 1973), these steady state solutions must contain a shock if \( \Phi_0 \) is above a critical value (this critical value depends on the values of the other parameters). These shocked steady state solutions constitute the initial conditions for the simulations described below. The aim of this paper is to see under which conditions these solutions are prone to the development of the wiggle instability and to study the properties of the substructures generated by the instability.

Assuming steady state, Equations (1) and (2) reduce to the following system of ordinary differential equations:

\[
\frac{dv_x}{dx} = 2\Omega_0 v_y - \frac{d\Phi}{dx} \left[ \frac{v_x - c_s^2}{v_x} \right]^{-1},
\]

\[
\frac{dv_y}{dx} = (q - 2) \left( \Omega_0 v_x + \frac{F}{2} \right).
\]

The density is given by \( \rho(x) = C/v_x \), where \( C \) is an arbitrary constant which without loss of generality we set to unity. Note that the solution for \( \Phi = 0 \) is \( v_x = -F/(2\Omega_0) = v_{c0} \sin(i) \), \( v_y = 0 \) (\( \xi = 1/2 \), \( \xi = 0 \) in dimensionless units). This solution corresponds to the fact that in absence of the spiral perturbation to the potential, the component of the circular motion perpendicular to the spiral arm is simply \( v_{c0} \sin(i) \). Note that this solution is subsonic if \( \xi_0 > 0.5 \) and supersonic if \( \xi_0 < 0.5 \) (in dimensionless units).

The system of equations (6) and (7) are solved numerically using the shooting method. The numerical procedure is described in more detail in Appendix B.

3 METHODOLOGY

3.1 Simulation setup

We solve the equations of hydrodynamics in the public grid code PLUTO version 4.3 (Mignone et al. 2007). We use a two-dimensional static Cartesian grid with uniform spacing \( \Delta x = \Delta y = 1.25 \times 10^{-3} \) in dimensionless units (corresponding to \( 1.25 \text{pc} \) in the linear regime, with \( J = 1 \times 10^{-2} \) (in dimensionless units).
Table 1. Parameters of the problem.

| Parameter | Brief description | Physical values | Dimensionless formulation | Dimensionless values |
|-----------|------------------|-----------------|--------------------------|---------------------|
| $c_s$     | isothermal sound speed | 4 - 14 km s$^{-1}$ | $\ddot{c}_s = \frac{c_s}{F/\Omega_0}$ | 0.2 - 0.7 |
| $\Phi_0$ | strength of the spiral potential | 10 - 100 (km s$^{-1}$)$^2$ | $\Phi_0 = \frac{\Phi_0}{(F/\Omega_0)^2}$ | 0.025 - 0.25 |
| $L_s$     | spiral arm separation | 0.4 - 2.0 kpc | $\tilde{L}_s = \frac{L_s}{(F/\Omega_0)}$ | 0.4 - 2.0 |
| $q$       | shear factor       | 0 - 1.5 | $\tilde{q} = q$ | 0 - 1.5 |
| $F$       | background Coriolis force $\perp$ to arm | 400 (km s$^{-1}$)$^2$ kpc$^{-1}$ | $\tilde{F} = \frac{F}{F}$ | 1 |
| $\Omega_0$ | local rotation velocity | 20 km s$^{-1}$ kpc$^{-1}$ | $\tilde{\Omega}_0 = \frac{\Omega_0}{\Omega_0}$ | 1 |

physical units). The size of the computational box is $L_x \times L_y \times L_z$, $L_z$ is varied within the range indicated in Table 1. We adopt $L_y = 2$ for most of the simulations reported in this paper, but we experiment with different values in Section 4. These box sizes correspond to $N_x = 320$-$1600$ grid points in the $x$ direction and $N_y = 1600$ grid points in the $y$ direction. We use the following parameters within the PLUTO code: RK2 time-stepping, no dimensional splitting, isothermal equation of state, Roe Riemann Solver, and the default flux limiter. The time-step is determined according to the Courant-Friedrichs-Lewy (CFL) criterion, with a CFL number of 0.4.

The initial conditions are provided by the steady states described in Section 2.2. We let the system evolve till $t_f = 30.0$ in dimensionless units, corresponding to $t_f \simeq 1.5$ Gyr in physical units. We introduce some random seed noise in the initial conditions to accelerate the onset of instability and save significant computational time. The instability would develop at a later time even without this initial noise, and we have tested that the properties of the induced substructure (morphology and feather separation) are unaffected by the introduction of the noise. The way in which noise is introduced is described in detail in Appendix C.

We use two types of boundary conditions in this work. The first is periodic boundary conditions in both the $x$ and $y$ directions. This type of boundary conditions is the most appropriate for galactic spiral shocks because it takes into account the fact that the material leaving one spiral arm will pass through the next spiral arm. The second type of boundary conditions are inflow-outflow boundary conditions, also known as D’yakonov-Kontoroich (DK) boundary conditions after the classic shock front stability analysis of D’yakov (1954) and Kontorovich (1958). With DK boundary conditions, we assume a constant injection of gas at the $x = 0$ boundary at the rate given by the steady state solution (Section 2.2), while gas can freely escape at the $x = L_x$ boundary thanks to standard outflow boundary conditions. The boundary is periodic in the $y$ direction. This second type of boundary conditions is appropriate for most “normal” non-astrophysical circumstances, in which the pre-shock flow should be left unperturbed on account of the fact that signal cannot travel backwards at supersonic velocity (see §90 in Landau & Lifshitz 1987).

3.2 Shock front analysis

The wiggle instability is an instability of the shock front. We therefore track the shock front in the simulations as a function of time, and we analyse its evolution by performing a Fourier decomposition of its shape.

Let us call $f(y,t)$ the $y$ position of the shock front. We detect the shock front in each snapshot using the large density jump that characterises it. We estimate the density gradient $\partial_\rho$ using finite differences along each horizontal slice, and we define the shock position to be the point where $|\partial_\rho|$ is maximum. In this way, we obtain the position of the shock front $f(y = y_n, t)$ at each grid point $y = y_n$ and timestep $t = t_i$. This simple method tracks well the shock position as a function of time (see Figure 1).

We analyze the shape of the shock $f(y, t)$ for fixed $t$ using a discrete Fourier transform:

$$f(y_n, t) = \sum_{m=0}^{N_y-1} B_m(t) \exp(i k_m y_n),$$

where the wavenumber is

$$k_m = \frac{2\pi m}{L_y}, \quad m \in \{0, 1, \ldots, N_y - 1\}$$

and

$$y_n = \frac{n L_y}{N_y}, \quad n \in \{0, 1, \ldots, N_y - 1\}$$

and $B_m = \tilde{B}_{N_y-m}$, where the bar indicates the complex conjugate, because $f$ is real. Note that due to the finite size of the computational box in the $y$ direction, only discrete values of the wavenumber $k_m$ and of the wavelength $\lambda_m = 2\pi/k_m = L_y/m$, where $m$ is a positive integer, are allowed.

To analyse the temporal growth of the amplitudes, we smooth the $B_m(t)$ curves with a moving average of over 13 snapshots (corresponding to a total smoothing interval of $\Delta t = 2.6$ in dimensionless units). This removes small-scale noise. Then we fit the smoothed curves with the following function:

$$B_{m, \text{smooth}}(t) = a e^{-\omega_m t},$$

where $a$ is a constant and $\omega_m$ is real. This approach neglects potential oscillations coming from the imaginary part of $\omega_m$, which cannot be detected reliably due to noise in our numerical setup and are washed out by our smoothing procedure. However, since these oscillations have by definition a zero net time-average, they do not affect our measurement of the long-term growth rates.
4 TWO EXAMPLE SIMULATIONS VS LINEAR STABILITY ANALYSIS

In this section we analyse in detail two example simulations and compare them to predictions from linear stability analysis. The parameters of the two example simulations are chosen to correspond to the cases studied in the linear analysis of Sormani et al. (2017) and are reported in Table 2.

The top row in Figure 4 shows the evolution of the surface density in the two simulations, and the second row zooms-in on the shock front. The third show shows the amplitude of the first 20 modes of the Fourier decomposition of the shock front (see Equation 8) at various times during the simulation. At $t = 0$ the shock front is a straight line in both simulations, and then starts oscillating due to the wiggle instability. It is immediately evident that in simulation A (left) the instability is dominated by a single mode with a large wavelength ($m = 1$), while in simulation B (right) there are multiple unstable modes.

The bottom row in Figure 4 shows the amplitude of the individual modes as a function of time. The black dashed horizontal lines mark the values at which the amplitude becomes larger than the grid resolution, an the green shaded area indicates the region where an instability is detectable, defined as the region where at least one mode is above the horizontal dashed line. Let us first consider in more detail simulation A, which is simpler thanks to the single dominant unstable mode. In simulation A, the mode $m = 1$ is the first to cross the horizontal dashed line at $t \approx 2$. The subsequent evolution of this mode is very well approximated by exponential growth, until at $t \approx 25$ the curve flattens and the growth stops. This is because instability starts to be saturated and we are entering the non-linear regime.

The mode $m = 2$ is the second mode to cross the black dashed line, but it does so at much later times than the $m = 1$ mode ($t \approx 22$). Further modes become cross the line $t > 22$. The $m \geq 2$ modes also appear to grow exponentially after crossing the black dashed line. However, as we discuss below, we believe that the growth of the $m > 1$ modes is driven by non-linear coupling between them and the $m = 1$ mode.

The evolution of the amplitudes in simulation B is more complex. The first mode to cross the black dashed line is $m = 8$, followed shortly by $m = 4,7,11,1,10$. Of these, modes $m = 10$ and 11 saturate very quickly, while modes 8, 4, 7 grow exponentially for some time before saturating.

Figure 2 compares the growth rates of modes in the two simulations, measured by fitting Equation 11 to the amplitudes in Figure 1, with rates predicted by the linear stability analysis of Sormani et al. (2017). In the linear analysis all values of $k_m$ are allowed, while in the simulations only discrete values $k_m = 2\pi m/L_y$ where $m$ is an integer are allowed because of the finite size of the box in the $y$ direction. We measure the growth rate only for modes that start to grow exponentially immediately after the beginning of the green shaded area (which indicates when the amplitude of the first mode crosses the black dashed line). We argue below that modes that start growing later are not genuinely unstable, because their growth is driven by the non-linear coupling between them and the modes who are genuinely unstable. We have verified that the measured growth rates do not depend significantly on the random seed of the initial noise.

The blue dot in the top panel of Figure 1 shows the growth rate of the the dominant $m = 1$ for simulation A. The growth rate of this mode matches very well the one predicted by the linear stability analysis. The red squares indicate the growth rates for an additional simulation that is identical to simulation A except that we doubled the $y$ size of the simulation box, $L_y = 4$, allowing for twice possible values of $k_m$. The growth rate for of the first three modes of this simulation also match very well those predicted by the linear analysis.

As mentioned above, the amplitude of the mode with $k_y = 2\pi$ in simulation A (see $m = 2$ curve in the bottom-left of Figure 1) also grows exponentially at $t > 22$. This growth does not match the linear analysis, which predicts very slow growth (see $k_m = 2\pi$ in the top panel of Fig. 2). To investigate the origin of this mismatch we have run another simulation identical to simulation A except that $L_y = 1$. In this simulation, the smallest possible wavenumber is $k_m = 2\pi$, while the $k_m = \pi$ mode is not possible because the box is not large enough. In this additional simulation, we do not observe any significant growth of the $k_m = 2\pi$ mode. This suggests that the growth of the $k_m = 2\pi$ mode in the simulation with $L_y = 2$ is driven by its coupling to the $k_m = \pi$ mode. We have further confirmed that this is the case by running additional simulations with controlled initial conditions in which we excite only selected modes. We observe that if excite only a certain wavenumber $k_y = k_0$, the growth of the modes whose wavenumber is an integer multiple of $k_0$ is enhanced. This confirms that there is significant non-linear interaction between the modes, even for relatively small values of the amplitudes ($|B_m|/k_m \simeq 1\%$). This coupling is neglected in the linear analysis, which assumes that modes each mode is independent.

The bottom panel in Figure 4 shows that simulation B, unlike simulation A, does not match very well the predictions of the linear analysis. However, the comparison is incomplete because the most unstable mode observed in simulation B corresponds to $k_y = 8\pi$, which lies outside the range studied in the linear analysis of Sormani et al. (2017), which only studied $k_y \lesssim 6\pi$. Only one of the observed unstable modes ($k_y = 4\pi$) is covered by the linear analysis in Sormani et al. (2017). The growth rate of this mode in simulation B does not match very well the one predicted by the linear analysis. This is likely because the growth of the $m = 4$ mode is affected by non-linear coupling to the $m = 8$ mode, since their wavelengths are multiple of each other. Another, less likely possibility is that an unstable modmode has been missed in the linear analysis of Sormani et al. (2017).

We conclude that the linear analysis generally works well in predicting the wavelength and growth rate of the most unstable mode. However, there is significant non-linear coupling between modes that can be significant even when the modes have relatively low amplitudes ($|B_m|/k_m \simeq 1\%$).

| Simulation | $c_s$ | $\Phi_0$ | $L_y$ | $q$ |
|------------|------|--------|------|----|
| A          | 0.7  | 0.25   | 1.0  | 0  |
| B          | 0.3  | 0.025  | 1.0  | 0  |
| C          | 0.3  | 0.25   | 1.0  | 0  |

Table 2. Parameters of the example simulations analysed in Sections 4 and 5. Parameters are given in dimensionless form (see Section 2.1).

5 EFFECTS OF THE BOUNDARY CONDITIONS AND PARASITIC KELVIN-HELMHOLTZ INSTABILITIES

The boundary conditions are critical for the development of the wiggle instability (Kim et al. 2014; Sormani et al. 2017). To understand why, let us briefly discuss the stability of shocks fronts in general. The stability of shocks with respect to the formation of “ripples” and “corrugations” on their surface was studied in the classic work of D’yakov (1954) and Kontorovich (1958) (see also §90 in Landau & Lifshitz 1987). The unanimous conclusion of these works was that shock fronts are essentially always stable, except under exotic cir-
Figure 1. Analysis of the two example simulations that we compare to the linear stability analysis (see Section 4 and Table 2). Left: Simulation A. Right: Simulation B. Top row: time evolution of the surface density. The white line indicates the shock front detected using the method described in Section 3.2. Second row: zoom onto the shock front. Ochre points indicate the shock front detected in the simulation. The coloured lines indicate the shock front reconstructed using only the Fourier modes indicated in the legend. Third row: the amplitude of the first 20 Fourier modes at different times. Simulation A is completely dominated by the single mode with $m = 1$. Simulation B has multiple prominent modes. Bottom row: time evolution of the amplitudes of the individual modes, smoothed with a moving average of width $\Delta t = 2$. The horizontal dashed black line indicates where the amplitude of each mode becomes greater than the grid resolution, $B_m > \Delta y = 1.25 \times 10^{-2}$. The green shaded area indicates the region where an instability is detectable, defined to be the region where the amplitude of at least one mode is above the black dashed line. The $\omega$ indicates the best-fitting growth rate obtained by fitting Equation (11), and the green vertical dashed lines indicate the time range used for the fit. One time unit corresponds to $\approx 48.9\,\text{Myr}$ in physical units, and one spatial unit corresponds to $1\,\text{kpc}$ (see Section 2.1).
performed in the case $L$ to the finite size $L$ studied in Section 4. The linear analysis of Sormani et al. (2017) demonstrated using a linear stability analysis that the same amplification of perturbations through successive shock passages is posed (akin to the classic works above), but becomes unstable with the very high vertical shear $\tau = \partial_z \nu_z$ present immediately after the shock in the steady state solution. Note that, consistent with this interpretation, the instability develops much faster in Simulation C than in Simulations A and B, because the KHI does not need to wait for material to complete one period in the $x$ direction to grow. The KHI is a “local” instability while the wiggle instability is a “global” instability. We will see in Section 6 that the amount of post-shock shear in the steady state solution is indeed a very good predictor of whether the system is subject to a parasitic KHI (i.e., of whether it is unstable with inflow/outflow boundary conditions). The wiggle and the KHI instabilities can be simultaneously present in spiral arms and may contribute to the creation of spurs/feathers.

We take $m_{\text{est}}/\Delta t$ to be the nearest integer to $m_{\text{est}}/\Delta t$, weighted by the amplitude of the perturbation at successive shock passages (or to the accumulation of potential vorticity, see Kim et al. 2014). It is instead a parasitic Kelvin-Helmholtz instability (KHI) caused by the very high vertical shear $\tau = \partial_z \nu_z$ present immediately after the shock in the steady state solution. Note that, consistent with this interpretation, the instability develops much faster in Simulation C than in Simulations A and B, because the KHI does not need to wait for material to complete one period in the $x$ direction to grow. The KHI is a “local” instability while the wiggle instability is a “global” instability. We will see in Section 6 that the amount of post-shock shear in the steady state solution is indeed a very good predictor of whether the system is subject to parasitic KHI (i.e., of whether it is unstable with inflow/outflow boundary conditions). The wiggle and the KHI instabilities can be simultaneously present in spiral arms and may contribute to the creation of spurs/feathers.

6 PARAMETER SPACE SCAN

We explore the parameter space by running three sets of simulations. In the first set we vary the sound speed and the spiral potential strength, in the second we vary the shear factor and the interarm separation, and in the third we look in more detail at the effects of varying the interarm separation. The parameters of all simulations are listed in Table 3.

We quantify the properties of the wiggle instability using two main quantities. The first is the mean average wavenumber of the unstable modes, defined by:

$$\langle m \rangle = \frac{1}{N} \sum_{i=0}^{N} (m)_i \rangle,$$

where

$$\langle m \rangle_i = \frac{\sum_{m=1}^{30} m B_m(t_i)}{\sum_{m=1}^{30} B_m(t_i)}$$

is the average wavenumber at time $t_i$, weighted by the amplitude of the various modes. The sum over $i$ is extended over $t_i = \{t_0, t_0 + i\Delta t, t_0 + 2\Delta t, \ldots, t_0 + N\Delta t\}$ where $\Delta t = 0.02$ in dimensionless units is the time interval between snapshots, and $t_0$ is the earliest time at which one mode (which we call $m_0$) becomes greater than the grid resolution (i.e., the beginning of the green shaded area in Figure 1). We take $N$ to be the nearest integer to $t_{\text{est}}/\Delta t$, where $t_{\text{est}} = 1/\omega_{m_0}$ is an estimate of the typical growth time of the instability obtained using the instantaneous growth rate $\omega_{m_0}$ of the $m_0$ mode at $t = t_0$. Typical values of $t_{\text{est}}$ are in the range 1-10 in dimensionless units.

A quantity closely related to $\langle m \rangle$ is the average wavelength of the unstable modes (see Section 3.2):

$$\langle \lambda \rangle = \frac{L_y}{\langle m \rangle}$$

The value of $\langle \lambda \rangle$ is useful for comparison with observations because...
Figure 3. Comparison between simulations with periodic boundary conditions (left) and inflow/outflow (DK) boundary conditions (right). Top: simulation A. Middle: simulation B. Bottom: simulation C. Under periodic boundary conditions, all simulations are unstable. Under inflow/outflow (DK) boundary conditions, simulations A and B become stable, while simulation C is still unstable due to parasitic Kelvin-Helmholtz instabilities. See discussion in Section 5.
it characterises the average spacing between feathers/spurs generated by the wiggle instability.

The second quantity we define is the average growth rate:

$$\langle \omega \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle \omega \rangle_i,$$

where

$$\langle \omega \rangle_i = \frac{\sum_{m=1}^{30} \omega_m B_m(t_i)}{\sum_{m=1}^{30} B_m(t_i)},$$

and $\omega_m$ is growth rate of mode $m$ obtained by fitting Equation (11) to the smoothed $B_m(t)$ curves in the range $[t_0, t_0 + t_{\text{est}}]$. A related quantity is $\langle T_{\text{grow}} \rangle = 2\pi/\langle \omega \rangle$, which quantifies the timescale over which the instability grows.

The values of $\langle m \rangle$ and $\langle \omega \rangle$ for each simulation under periodic boundary conditions are listed in Table 3.

### 6.1 Sound speed and spiral potential strength

Figures 4 and 5 show the initial conditions for the first set of simulations, in which we vary the sound speed $c_s$ and the spiral potential strength $\Phi_0$ while keeping fixed the other parameters (see Table 3). Figures 6 and 7 show the surface density at a later time for periodic and inflow/outflow (DK) boundary conditions respectively. Most of the simulations with periodic boundary conditions display the wiggle instability (Figs. 6 and 8), and in those who do not the wiggle instability would probably develop if the simulation were continued for longer times. Many of the simulations with high $\Phi_0$/low $c_s$ are also unstable for inflow/outflow boundary conditions (Figs. 7 and 9), which means that a parasitic KHI occurs in addition to the wiggle instability (see Section 5). Table 3 shows that the post-shock shear $\tau$ is a good predictor of whether the simulation will display KHI.

Figures 8 and 9 show the time evolution of the Fourier modes for the case of periodic and inflow/outflow (DK) boundary conditions respectively, while the top part of Figure 11 shows the average wavenumber and growth rates of the unstable modes for the simulations with periodic boundary conditions. We can see the following trends:

(i) Simulations with higher $\Phi_0$ and lower $c_s$ tend to be more unstable.

(ii) The average wavenumber $\langle m \rangle$ of the unstable modes increases for increasing $\Phi_0$ and decreasing $c_s$ (top panels in Fig. 11).

(iii) The average growth rate $\langle \omega \rangle$ increases for increasing $\Phi_0$ and decreasing $c_s$ (bottom panels in Fig. 11).

Finally, it is worth noting that modes within the same simulation tend to saturate to similar values of the normalized amplitudes $B_m/\lambda_m$ (Fig. 8).

### 6.2 Shear factor

In the second set of simulations we analyse the dependence of the instability on the term containing $q$ in Equation 1, which arises due to differential rotation in the galaxy. The value $q = 0$ corresponds to solid-body rotation, $q = 1$ to a flat rotation curve, and $q = 1.5$ to Keplerian rotation. Figure 10 shows the amplitude of the Fourier modes as a function of time for both periodic and inflow/outflow boundary conditions, while bottom part of Figure 11 shows the average wavenumber and growth rate as a function of $q$. From these figures we conclude that the wiggle instability depends only weakly on the shear factor $q$, although decreasing $q$ tends to stabilise the system.

### 6.3 Spiral arms separation

In the third set of simulations we examine the instability as a function of the interarm separation $L_x$. Figure 12 shows the evolution of the Fourier amplitudes for the second and third set of simulations as a function of time, and Figure 13 shows the average wavenumber and growth rate of the unstable modes. We note that:

(i) The average wavenumber $\langle m \rangle$ decreases as we increase $L_x$. This could be related to the fact that with larger interarm separation, disturbances travel a longer way in the $y$ direction during one period.

(ii) The growth rate $\langle \omega \rangle$ tends to decrease for increasing $L_x$. This is expected since for larger $L_x$ the time interval between successive shock passages is larger (disturbances travel longer distances in the $x$ direction to reach the next shock).

(iii) For $L_x \geq 1.5$ the growth is too slow to for the wiggle instability to have any impact on real galaxies ($\langle T_{\text{grow}} \rangle \gtrsim 1.5$ Gyr in physical units).

Finally, note that simulations with large $L_x$ may appear stable because of the finite length $L_x$, which does not allow large wavelengths that may dominate the instability in these cases (see also discussion in Section 4).

### 7 DISCUSSION

#### 7.1 Comparison with observations

We can compare our simulations to observations of spurs/feathers in disc galaxies. Figure 14 shows the spacing between feathers along the arm for a sample of 20 galaxies in the survey of La Vigne et al. (2006). La Vigne et al. (2006) provide values for only two galaxies (NGC 3433 and NGC 5985). To increase the number of data points, we manually measured the feather separation for 20 galaxies directly from the figures in La Vigne et al. (2006) by superimposing a Cartesian grid onto them. We corrected for the effect of inclination, although this has a minor impact on the results since galaxies are selected to have low inclination (see Table 2 in La Vigne et al. 2006). We have checked that for NGC 3433 and NGC 5985 our values are consistent with those provided by La Vigne et al. (2006) within the errors. Our values are also listed in Table 4.

Comparing values of $\langle \lambda \rangle$ from Table 3 to those of $\langle \lambda_{\text{obs}} \rangle$ in Table 4, we can see that the wiggle instability is able to reproduce the range of spacing seen in observations (200-2000 pc). However, it is not straightforward to make a correspondence between parameters of the simulations and observations because (i) there are several degeneracies between parameters, so that for a given $\langle \lambda_{\text{obs}} \rangle$ there is no unique set of parameter that can reproduce it; (ii) as discussed in Section 6, the predicted wavelength of the wiggle instability is extremely sensitive to parameters such as the sound speed and the spiral potential strength.

La Vigne et al. (2006) note that galaxies with weak spiral potential (class Sc and Sd) show decreased feather formation. This is consistent with our results where decreasing $\Phi_0$ decreases the strength of the instability.

Figure 14 shows that there is a clear correlation between feather spacing $\langle \lambda_{\text{obs}} \rangle$ and galactocentric radius $R$ (the spacing increases with radius). Our simulations predict a similar trend, because $\langle \lambda \rangle$ increases with interarm distance $L_x$ (Figure 13), and the latter typically increases with $R$. To compare these two trends in more detail we need to convert the interarm distance $L_x$ into galactocentric radius $R$. For the idealised logarithmic spiral arms used in this paper
the two are related by (see Eq. A26):

\[ L_x = \left( \frac{2\pi \sin(i)}{m_s} \right) R, \]  

(17)

where \( \sin(i) \) is the pitch angle and \( m_s \) is the number of spiral arms (not to be confused with wavenumber). In this comparison we have the freedom to choose \( \sin(i) \) arbitrarily, because specifying all the six parameters that characterise our idealised problem only fixes the product \( v_0 \sin(i) \) (see Section 2.1). The value of \( \sin(i) \) controls the predicted slope of the trend in the \( \lambda - R \) plane. If we assume \( m_s = 2 \) and a typical pitch angle of \( \sin(i) \approx 0.1 \) (e.g. Savchenko & Reshetnikov 2013) then our simulations predict a relation which is too steep (i.e., \( \lambda \) increases too quickly with \( R \)). The simulations can reproduce the observations well only if we assume an unrealistically small value of the pitch angle \( \sin(i) = 0.02 \) (see the black dashed line in Figure 13).

However, the picture is complicated by several factors. First, the extreme sensitivity to the underlying parameters discussed above. It may be possible that the observed trend is better reproduced with a lower value of the sound speed than used in third set of simulations. Second, the very idealised nature of our simulations. We assume that the gas is isothermal with constant sound speed (to which \( \lambda \) is very sensitive, as discussed in Section 6). However, this is a very crude approximation because the interstellar medium is in reality a complex multi-phase medium, and it is not clear what effect this would have on the properties of the wiggle instability. We ne-
Figure 4. Initial conditions for the first set of simulations (see Table 3). These initial conditions are the steady state solutions calculated using Eqs. (6) and (7). Annotated are the Mach number $M$ and the post-shock shear $\tau$ (see caption of Table 3 for definitions).

We conclude that while the wiggle instability appears to be able to reproduce the typical range of feather spacing observed in real galaxies, the extreme sensitivity of the predicted spacing to the underlying parameters and the very idealised nature of our setup makes a detailed comparison and constraining the underlying parameters very difficult tasks.

7.2 Wiggle instability vs other mechanisms for formation of spurs and feathers

There have been numerous mechanism that have been proposed for spurs/feather formation. Kim & Ostriker (2002, 2006) (see also Lee & Shu 2012; Lee 2014) proposed that spurs might form via a magneto-Jeans instability (MJI), in which magnetic fields favour the gravitational fragmentation and collapse of spiral arm crests by removing angular momentum from contracting regions. Kim et al. (2020) proposed that spurs/feathers originate from the stochastic accumulation of gas due to correlated supernova feedback. Dobbs & Bonnell (2006) proposed that spurs/feathers originate from the amplification of pre-existing perturbations at the arm crest, in a way that is reminiscent of the mechanism at the basis of the wiggle instability.

One of the key properties of the wiggle instability is that it does not rely on the presence of magnetic fields, self-gravity and supernova feedback. Thus, feathering due to the wiggle instability should be present even in regions without much stellar feedback (e.g. the outer HI spiral arms in disc galaxies) or where the gas self-gravity is negligible. It might also be relevant in other contexts, such as spiral arms in protoplanetary discs (e.g. Rosotti et al. 2020).

There has been some confusion regarding the relation between the Kelvin-Helmholz instability and the wiggle instability. Some studies do not present the distinction, or have mistaken one for the other. In Section 5 we have presented evidence that they are physically dis-
tinct mechanisms that can both occur simultaneously in real galaxies.

It has been noted that vertical shear, magnetic fields and the presence of supernova feedback all tend to suppress the wiggle instability Kim & Ostriker (2006); Kim et al. (2015); Kim et al. (2020). It is however challenging to determine exactly the contribution of each mechanism in a complex situation. Observations of regions where one can exclude some of these mechanisms (e.g. where self-gravity or supernova feedback are negligible) might help in discriminating between them.

Figure 5. Initial surface density for the first set of simulations (see Table 3). Lines with arrows are instantaneous streamlines, and the color indicates the total velocity $v = (v_x^2 + v_y^2)^{1/2}$. Note that the density jump at the shock becomes larger for increasing $\Phi_0$ and decreasing $c_s$. One unit of $x$ corresponds to 1 kpc in physical units (see Section 2.1).
8 CONCLUSION

We simulated a small patch of a typical spiral galaxy to study the wiggle instability in the simplest possible setup. We found the following results:

(i) We compared in detail the results of simulations with predictions from linear stability analysis by doing a Fourier decomposition of the perturbed shock front. We find that the linear analysis works well in predicting the wavelength and growth rate of the instability, but also that there is a surprisingly strong non-linear coupling between the different modes even when the amplitudes are relatively small (with the ratio between the
displacement of the shock front and the wavelength of the unstable mode being $\sim 1\%$, Section 4).

(ii) A parasitic Kelvin-Helmholtz instability can be present in addition to the wiggle instability when the post-shock shear is large. This occurs preferentially for stronger spiral potentials and smaller sound speeds. That the two instabilities are physically distinct is demonstrated by the fact that for some values of the parameters the system is stabilised by changing the boundary conditions from periodic to inflow-outflow (Section 5).

(iii) The properties of the wiggle instability are very sensitive to the underlying parameters, in particular the gas sound speed.

Figure 7. Same as Fig. 6, but for inflow-outflow boundary conditions (see Sect.3.1). Only simulations in the bottom-right panels are unstable, due to parasitic Kelvin-Helmholtz instability.
Figure 8. Amplitudes of the first 28 Fourier modes as a function of time for the simulations in Figure 6. The average growth rate \( \langle \omega \rangle \) and the average unstable wavenumber \( \langle m \rangle \) is larger for larger \( \Phi_0 \) and smaller \( c_s \). The vertical gray dashed line indicates the time when at least one amplitude becomes larger than the grid resolution. The black dashed line indicates Equation 11 with the average growth rate. Blue-shaded panels indicate simulations that are unstable, while unshaded plots indicate simulations that do not show sign of instability within the simulation time. One time unit corresponds to roughly 48.9 Myr in physical units.

\( c_s \), the spiral potential strength \( \Phi_0 \) and the interarm spacing \( L_x \). The wavelength \( \langle \lambda \rangle \) of the wiggle instability decreases and the growth rate increases for larger \( \Phi_0 \), smaller \( c_s \) and smaller \( L_x \) (Section 6).

(iv) The wiggle instability is able to reproduce the range of spacing between feathers observed in real galaxies. However, the extreme sensitivity of the predicted spacing to the underlying parameters and the very idealised nature of our setup makes constraining the underlying parameters very difficult tasks. Moreover, it is very challenging to disentangle the contribution of the wiggle instability from other mechanisms for substructure formation such as magneto-Jeans instabilities or correlated supernova feedback (Section 7).

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Figure 10. Top: initial conditions for the second set of simulations (see Table 3). Bottom: Fourier amplitudes of the first 28 modes as a function of time under periodic (left) and inflow/outflow (DK, right) boundary conditions.

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APPENDIX A: DERIVATION OF THE BASIC EQUATIONS

Here we provide a detailed derivation of Equations (1) and (2). We start by writing down the equations of fluid dynamics in a rotating frame. Then we introduce Roberts’ spiral coordinate system, and
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Figure 11. The average growth rate $\langle \omega \rangle$ (Eq. 15) and wavenumber $\langle m \rangle$ (Eq. 12) as a function of the various parameters for the first set of simulations (top) and the second set of simulations (bottom). Empty squares correspond to stable simulations, to which we assign $\langle \omega \rangle = \langle m \rangle = 0$.

### A1 Fluid equations in a rotating frame

The Euler and continuity equations in a frame rotating with pattern speed $\Omega_p = \Omega_p \mathbf{e}_z$ are:

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \]  
\[ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla P}{\rho} - \nabla \Phi - 2\Omega_p \times \mathbf{v} - \Omega_p \times (\Omega_p \times \mathbf{R}), \]

where $\mathbf{v}$ is the velocity in the rotating frame, $\rho$ is the surface density, $P$ is the pressure, $\Phi$ is an external gravitational potential, $-2\Omega_p \times \mathbf{v}$ is the Coriolis force, $-\Omega_p \times (\Omega_p \times \mathbf{R})$ is the centrifugal force. The explicit form of the potential $\Phi$ will be specified later.

### A2 Spiral coordinates

Following Roberts (1969), we define the following spiral coordinates:

\[ \eta = \log \left( \frac{R}{R_0} \right) \cos(i) + \theta \sin(i), \]
\[ \xi = -\log \left( \frac{R}{R_0} \right) \sin(i) + \theta \cos(i). \]

The inverse relations are:

\[ \log \left( \frac{R}{R_0} \right) = \eta \cos(i) - \xi \sin(i), \]
\[ \theta = \eta \sin(i) + \xi \cos(i), \]

where $R$, $\theta$ are usual polar coordinates and $R_0$ and $i$ are constants. There are several possible choices for the domain of the $(\eta, \xi)$ coordinates, two of which are shown in Figure A1. In the following, we use the domain corresponding to the orange shaded region in Figure A1:

\[ \eta = \left[ -\pi \sin(i), \pi \sin(i) \right], \quad \xi = [0, \infty). \]

Figure A2 shows lines of constant $\eta$ and $\xi$ for this choice of the domain and $R_0 = 1, \theta = 20^\circ$. The origin ($\eta = 0, \xi = 0$) corresponds to...
the point \((R = 1, \theta = 0)\). As we increase the coordinate \(\eta\) at constant \(\xi\), we move perpendicularly to the spirals. Note that when we cross the value \(\eta = \pi \sin(i)\) (reappearing on the other side at \(\eta = -\pi \sin(i)\)), there is a jump in the coordinate \(\xi\) of \(\Delta \xi = 2\pi \cos(i)\).

Therefore, when working with these coordinates all physical quantities such as for example the density must satisfy the following condition:

\[
\rho(\xi, \eta) = \rho(\eta + 2\pi \sin(i), \xi + 2\pi \cos(i)).
\]  

(A8)

The unit vectors in spiral coordinates are:

\[
\hat{e}_\eta = \cos(i) \hat{e}_R + \sin(i) \hat{e}_\theta,
\]  

(A9)

\[
\hat{e}_\xi = -\sin(i) \hat{e}_R + \cos(i) \hat{e}_\theta.
\]  

(A10)
and the derivatives of the unit vectors are:

$$\nabla = \frac{1}{R} \left( \hat{e}_\eta \frac{\partial}{\partial \eta} + \hat{e}_\xi \frac{\partial}{\partial \xi} \right)$$  \hspace{1cm} (A11)

and the derivatives of the unit vectors are:

$$\frac{\partial \hat{e}_\eta}{\partial \eta} = \sin(i) \hat{e}_\xi, \quad \frac{\partial \hat{e}_\eta}{\partial \xi} = \cos(i) \hat{e}_\zeta, \quad (A12)$$

$$\frac{\partial \hat{e}_\xi}{\partial \eta} = -\sin(i) \hat{e}_\eta, \quad \frac{\partial \hat{e}_\xi}{\partial \xi} = -\cos(i) \hat{e}_\eta. \quad (A13)$$

### A3 Equations of motion in spiral coordinates

Using the relations of the previous subsection it is straightforward to rewrite Equations (A1) and (A2) in spiral coordinates. The continuity equation becomes:

$$\partial_t \rho + \frac{1}{R} \left[ \partial_\eta (\rho \nu_\eta) + \partial_\xi (\rho \nu_\xi) + \rho \left( \nu_\eta \cos(i) - \nu_\xi \sin(i) \right) \right] = 0, \quad (A14)$$

and the Euler equation:

$$\partial_t \nu_\eta + \frac{1}{R} \left[ \nu_\eta \left( \partial_\eta \nu_\eta \right) + \nu_\xi \left( \partial_\xi \nu_\eta \right) - \nu_\eta \nu_\xi \sin(i) - \nu_\xi^2 \cos(i) \right]$$

$$= -\frac{1}{R} \frac{\partial P}{\partial \rho} - \frac{1}{R} \hat{\Phi} + 2 \Omega_\rho v_\eta + \cos(i) \Omega_\rho^2 \hat{\Phi} \quad \text{(A15)}$$

$$\partial_t \nu_\xi + \frac{1}{R} \left[ \nu_\eta \left( \partial_\eta \nu_\xi \right) + \nu_\xi \left( \partial_\xi \nu_\xi \right) + \nu_\eta \nu_\xi \cos(i) + \nu_\xi^2 \sin(i) \right]$$

$$= -\frac{1}{R} \frac{\partial P}{\partial \rho} - \frac{1}{R} \hat{\Phi} - 2 \Omega_\rho v_\xi - \sin(i) \Omega_\rho^2 \hat{\Phi}. \quad (A16)$$

### A4 Split into circular and spiral components

We now write

$$v = v_c + v_s$$

$$\rho = \rho_c + \rho_s \quad (A17)$$

$$P = P_c + P_s$$

$$\Phi = \Phi_c + \Phi_s$$

where the subscript c refers to a steady state axisymmetric solution in the axisymmetric potential $\Phi_c$ and s to a “spiral” departure from the axisymmetric solution. We assume that the axisymmetric solution is of the type

$$\rho_c = \text{constant}$$

$$P_c = \text{constant} \quad (A18)$$

$$v_c = (\Omega(R) - \Omega_\eta) \times \mathbf{R} \quad (A19)$$

where $\Omega = \Omega \hat{e}_\zeta$. In spiral coordinates we can write:

$$\mathbf{R} = R \left( \cos(i) \hat{e}_\eta - \sin(i) \hat{e}_\xi \right). \quad (A20)$$

The circular velocity can therefore be written as:

$$v_{c\eta} = (\Omega(R) - \Omega_\eta) R \sin(i),$$

$$v_{c\xi} = (\Omega(R) - \Omega_\eta) R \cos(i). \quad (A21)$$

### A5 Radial spacing between spiral arms

Starting from the spiral arm defined by the relation $\eta = -\pi \sin(i)$ (black line in Figure A2) and moving along the coordinate $\eta$ at constant $\xi$, we meet the next arm after $\Delta \eta = 2\pi \sin(i)/m$ (and $\Delta \xi = 0$ by definition). Using these in Equations (A5)-(A6), we find the corresponding change in $R$ and $\theta$ as we move from one arm to the next:

$$\log \left( 1 + \frac{L}{R} \right) = \Delta \eta \cos(i) - \Delta \xi \sin(i) = \frac{2 \pi \sin(i) \cos(i)}{m}, \quad (A22)$$

$$\Delta \theta = \Delta \eta \sin(i) + \Delta \xi \cos(i) = \frac{\pi \sin(i)^2}{m}, \quad (A23)$$

where $L$ is defined as the radial separation between two arms. In the limit $\sin(i) \ll 1$ these become

$$L \approx \frac{2 \pi \sin(i)}{m}, \quad (A24)$$

$$\Delta \theta \approx 0. \quad (A25)$$

### A6 Approximations

Following Roberts (1969) (see also Balbus 1988) we now approximate the equations of motion under the following assumptions:

(i) The pitch angle is small,

$$\sin(i) \ll 1. \quad (A26)$$

(ii) The velocity $v_{c\xi} \sim \Omega R$ is much greater than $v_{c\eta}, \nu_\eta$ and $v_{c\eta} \sim \Omega \sin(i) R$. The latter are all comparable in size. Thus

$$v_{c\xi} \sim \Omega R \gg v_{c\eta} \gg v_{c\eta} \sim \Omega \sin(i). \quad (A27)$$

(iii) All quantities with subscript ‘s’ vary much faster in the direction $\hat{e}_\eta$ (with a length-scale $L$, corresponding to $\Delta \eta \approx 2\pi \sin(i)/m$), than in the direction $\hat{e}_\xi$ (length-scale $R$, corresponding to $\Delta \xi \approx 2\pi \cos(i)$). Thus

$$\partial_\eta \sim \frac{1}{\sin(i)}, \quad \partial_\xi \sim 1. \quad (\text{spiral quantities}) \quad (A28)$$

(iv) All quantities with subscript ‘c’ vary with a length-scale $R$ in the direction $R$ (in the other directions they do not vary since they are axisymmetric by definition). According to Equations (A3) and (A4), in order to double $R$ we need to move by $\Delta \xi = \log(2)/\sin(i)$ in the $\xi$ direction (at constant $\eta$) or by $\Delta \eta = \log(2)/\cos(i)$ in the $\eta$ direction (at constant $\xi$). Thus

$$\partial_\eta \sim 1, \quad \partial_\xi \sim \sin(i). \quad (\text{background quantities}) \quad (A29)$$
Figure 14. Separation of feathers/spurs as a function of galactocentric radius $R$ for a sample of 20 galaxies in the survey of La Vigne et al. (2006) (see Table 4). The grey band indicates a linear fit to the data, and the thickness is the standard deviation. The dashed line indicates $\langle \lambda \rangle$ for simulation set 3 (Table 3) assuming $\sin(i) = 0.02$ to convert from interarm separation $L$ to galactocentric radius $R$ (see Eq. A26), and using the scalings discussed in Section 2.1 to convert from dimensionless to physical units. Changing the value of $\sin(i)$ changes the slope of the curve, while changing the scalings enlarges or shrinks the curve without affecting its slope.

Figure A1. Blue shaded: domain of the $(\eta, \xi)$ coordinates if we require that $\theta \in [-\pi, \pi]$. Orange shaded: another possible choice for the domain. This choice is more natural in terms of the $(\eta, \xi)$ coordinates, but corresponds to a more convoluted $\theta$ domain. The two vertical lines that bound the shaded orange region correspond to $\eta = \pm \pi \sin(i)$. The two red dots represent the same physical point and correspond to the single red dot in Figure A2.

(v) The sound speed of the gas is smaller or comparable to the spiral velocities:

$$c_s \lesssim \Omega R \sin(i).$$  \hspace{1cm} (A32)

(vi) The strength of the spiral potential is comparable to the spiral velocities squared:

$$\Phi_s \sim [\Omega R \sin(i)]^2.$$  \hspace{1cm} (A33)

As a consequence of the first assumption, we have that the radial spacing between the spiral arms $L$ is much smaller than $R$ (see Equation A26):

$$\frac{L}{R} \sim \frac{2\pi \sin(i)}{m} \ll 1.$$  \hspace{1cm} (A34)

A7 Local coordinates

We define local $(x, y)$ coordinates centred on a small patch around $R_0$ as follows:

$$x = R_0 \eta, \quad y = R_0 \xi.$$  \hspace{1cm} (A35)

The point $(x = 0, y = 0)$ corresponds to the point $(R = R_0, \theta = 0)$. We require the range of $\eta$ to be from one spiral arm to the next, so $\eta = [-\pi \sin(i)/m, \pi \sin(i)/m]$, and the range of $\xi$ to be comparable. Thus we have

$$|x| \lesssim R_0 \sin(i) \ll R_0, \quad |y| \lesssim R_0 \sin(i) \ll R_0$$  \hspace{1cm} (A37)

Under these approximations, we can rewrite the radius as (see Equation A5):

$$R = R_0 \exp(\eta \cos(i) - \xi \sin(i)), \quad R_0 = R_0(1 + \eta) \quad R = R_0 + x.$$  \hspace{1cm} (A38)
The derivatives with respect to the \( (x,y) \) coordinates are
\[
\begin{align*}
\partial_x &= \frac{1}{R_0} \partial_\eta, \\
\partial_y &= -\frac{1}{R_0} \partial_\xi.
\end{align*}
\] (A41)

The velocities in \( (x,y) \) coordinates are the same as in spiral coordinates:
\[
\begin{align*}
v_x &= v_\eta, \\
v_y &= v_\xi.
\end{align*}
\] (A42)

### A8 Approximating the continuity equation

We now want to approximate the continuity equation (A14) to leading order in \( \sin(i) \). To do this, we first substitute (A17) into (A14) and eliminate all the terms that simplify because the axisymmetric solution (subscript ‘c’) also separately satisfies (A14). This gives:
\[
\begin{align*}
\partial_t \rho_c + \frac{1}{R} [\partial_\eta (\rho_c v_{\eta}^c) + \partial_\xi (\rho_c v_{\xi}^c) + \rho_c (v_\eta \cos(i) - v_\xi \sin(i))] + \frac{1}{R} [\partial_\eta (\rho_c v_{\eta}^c) + \partial_\xi (\rho_c v_{\xi}^c)] = 0.
\end{align*}
\] (A43)

We now have to estimate the order of each term according to the relations listed in Section A6 and eliminate the negligible ones. For the terms in the first row of Equation (A43) we have:
\[
\begin{align*}
\partial_\eta (\rho_c v_{\eta}^c) &\sim \frac{1}{\sin(i)} \rho_c \Omega R \sin(i) \sim \rho_c \Omega R \\
\partial_\xi (\rho_c v_{\xi}^c) &\sim \rho_c \Omega R \sin(i) \\
\rho_c v_{\eta}^c \cos(i) &\sim \rho_c \Omega R \sin(i) \\
\rho_c v_{\xi}^c \sin(i) &\sim \rho_c \Omega R (\sin(i))^2
\end{align*}
\] (A44)

To leading order in \( \sin(i) \) we only need to keep \( \partial_\eta (\rho_c v_{\eta}^c) \) (all the others are negligible compared to this). For the terms in the second row we have (remember \( \partial_\eta \rho_c \sim \rho_c / \sin(i) \), while \( \partial_\eta v_{\eta} \sim v_{\eta} \)):
\[
\begin{align*}
\partial_\eta (\rho_c v_{\eta}^c) &= v_{\eta} \partial_\eta (\rho_c) + \rho_c \partial_\eta (v_{\eta}^c) \\
&\sim \rho_c \Omega R + \rho_c \Omega R \sin(i) \\
\partial_\xi (\rho_c v_{\xi}^c) &= \rho_c \partial_\xi (v_{\xi}^c) + v_{\xi} \partial_\xi (\rho_c) \\
&\sim \rho_c \Omega R \sin(i) + \rho_c \Omega R \\
\rho_c v_{\eta}^c \cos(i) &= \rho_c \Omega R \sin(i) \\
\rho_c v_{\xi}^c \sin(i) &\sim \rho_c \Omega R \sin(i)
\end{align*}
\] (A45)

Thus we need to keep \( \partial_\eta \rho_c \) and \( v_{\xi} \partial_\xi (\rho_c) \). Proceeding similarly for the third row and approximating \( 1/R \approx 1/R_0 \) everywhere in Equation (A43) (which is correct to leading order in \( \sin(i) \)), see Equations A40 and A37 we find that Equation (A43) reduces to:
\[
\begin{align*}
\partial_\eta \rho_c + \frac{1}{R_0} \left[ \partial_\eta (\rho_c v_{\eta}^c) + \partial_\eta (\rho_c v_{\xi}^c) + v_{\eta} \partial_\xi (\rho_c) \right] + O(\sin(i)) = 0.
\end{align*}
\] (A46)

This is the minimal amount of terms that we need to keep. We can however some terms of order \( \sin(i) \) to make the final equation appear more familiar while committing a negligible error of \( O(\sin(i)) \). Going back to Equation (A14), we see that all the terms of order \( O(1) \) that appear in Equation (A53) originate from the first two terms inside the square parentheses. Therefore, we can approximate Equation (A14) as:
\[
\begin{align*}
\partial_\rho \rho + \frac{1}{R} [\partial_\eta (\rho v_\eta) + \partial_\xi (\rho v_\xi)] + O(\sin(i)) = 0.
\end{align*}
\] (A47)

This equation is equivalent to (A53) to order \( O(\sin(i)) \), but is more useful because it looks like the normal continuity equation. Finally, we can use (A41) to re-express (A47) in local coordinates as:
\[
\begin{align*}
\partial_\rho \rho + \partial_\rho (\rho v_\rho) + \partial_\rho (\rho v_\rho) + O(\sin(i)) = 0.
\end{align*}
\] (A48)

This equation coincides with Equation (2.1c) of Balbus (1988) and with Equation (2) of Kim et al. (2014).

### A9 Approximating the Euler equation

We split the Euler equation (A15) and (A16) into a ‘circular’ and ‘spiral’ component and then approximate to leading order in \( \sin(i) \). Substituting (A17) into (A15) and (A16), and eliminating all the terms that simplify because the axisymmetric solution also separately satisfies Equation (A17), we obtain respectively:
\[
\begin{align*}
\partial_t v_{\eta} + \frac{1}{R} [v_{\eta} \partial_\eta (\rho v_{\eta}) + v_{\xi} \partial_\xi (\rho v_{\xi}) - v_{\xi} \rho v_{\xi} \sin(i) - v_{\xi} \chi \cos(i)] \\
&+ \frac{1}{R} [v_{\eta} \partial_\eta (\rho v_{\eta}) + v_{\xi} \partial_\xi (\rho v_{\xi}) - v_{\xi} v_{\xi} \cos(i)] \\
&+ \frac{1}{R} [v_{\eta} \partial_\eta (\rho v_{\eta}) + v_{\xi} \partial_\xi (\rho v_{\xi}) - v_{\xi} v_{\xi} \sin(i) - v_{\xi} v_{\xi} \cos(i)] \\
&= - \frac{1}{R} \partial_\rho \rho + \frac{1}{R} \rho \partial_\rho (\Phi_\rho) + 2 \Omega_\rho v_{\xi}.
\end{align*}
\] (A49)

and
\[
\begin{align*}
\partial_t v_{\xi} + \frac{1}{R} [v_{\eta} \partial_\eta (\rho v_{\eta}) + v_{\xi} \partial_\xi (\rho v_{\xi}) + v_{\xi} \rho v_{\xi} \cos(i) + v_{\xi} \chi \sin(i)] \\
&+ \frac{1}{R} [v_{\eta} \partial_\eta (\rho v_{\eta}) + v_{\xi} \partial_\xi (\rho v_{\xi}) + v_{\xi} v_{\xi} \cos(i) + v_{\xi} v_{\xi} \sin(i)] \\
&+ \frac{1}{R} [v_{\eta} \partial_\eta (\rho v_{\eta}) + v_{\xi} \partial_\xi (\rho v_{\xi}) + v_{\xi} v_{\xi} \cos(i) + v_{\xi} v_{\xi} \sin(i)] \\
&= - \frac{1}{R} \partial_\rho \rho - \frac{1}{R} \partial_\rho (\Phi_\rho) - 2 \Omega_\rho v_{\eta}.
\end{align*}
\] (A50)
So far we have not performed any approximation. We now estimate the order of each term in these equations using the relations in Section A6. We want to keep only the leading order in \( \sin(i) \). For the terms within the square parentheses in Equation (A56) we have:

\[
\begin{align*}
\varphi_n(\partial_\eta \varphi_n) & \sim [\Omega R]^2 \sin(i) \quad (\ast) \quad (A58) \\
\varphi_x(\partial_\xi \varphi_n) & \sim [\Omega R]^2 [\sin(i)]^2 \quad (A59) \\
\varphi_x \varphi_n \sin(i) & \sim [\Omega R]^2 [\sin(i)]^3 \quad (A60) \\
\varphi_x \cos(i) & \sim [\Omega R]^2 [\sin(i)]^2 \quad (A61) \\
\varphi_n(\partial_\eta \varphi_n) & \sim [\Omega R]^2 [\sin(i)]^2 \quad (A62) \\
\varphi_x(\partial_\xi \varphi_n) & \sim [\Omega R]^2 [\sin(i)]^3 \quad (A63) \\
\varphi_x \varphi_n \sin(i) & \sim [\Omega R]^2 [\sin(i)]^3 \quad (A64) \\
\varphi_x \varphi_x \cos(i) & \sim [\Omega R]^2 [\sin(i)] \quad (\ast) \quad (A65) \\
\varphi_n(\partial_\eta \varphi_n) & \sim [\Omega R]^2 [\sin(i)]^2 \quad (\ast) \quad (A66) \\
\varphi_x(\partial_\xi \varphi_n) & \sim [\Omega R]^2 [\sin(i)]^2 \quad (\ast) \quad (A67) \\
\varphi_x \varphi_n \sin(i) & \sim [\Omega R]^2 [\sin(i)]^2 \quad (A68) \\
\varphi_x \varphi_x \cos(i) & \sim [\Omega R]^2 [\sin(i)] \quad (\ast) \quad (A69)
\end{align*}
\]

Thus to leading order in \( \sin(i) \) we need to keep terms marked with (\ast). To the same order we can also approximate \( 1/R \simeq 1/R_0 \) everywhere in Equation (A56) (see Equations A40 and A37). Using the relations given in Section A6 we see that all the terms on the right-hand-side of Equation (A56) are of order \( [\Omega R]^2 \sin(i) \), so we have to keep them. We can put \( \cos(i) \simeq 1 \) in terms (A65) and (A69). To the same order we can also approximate \( \varphi_x \simeq (\Omega_0 - \Omega_\varphi) R_0 = \text{constant} \) (see Equation A23) in terms (A65), (A67) and (A69), where \( \Omega_0 = \Omega(R_0) \). Putting everything together, we can rewrite Equation (A56) as:

\[
\partial_t \varphi_n + \frac{1}{R_0} \left[ \left( \varphi_n + \varphi_x \right) (\partial_\eta \varphi_n) + \varphi_x (\partial_\xi \varphi_n) \right] = -\frac{1}{R} \partial_\rho \varphi_x - \frac{1}{R} \partial_\Phi \varphi_x + 2\Omega_0 \varphi_x + O \left( [\sin(i)]^2 \right). \quad (A70)
\]

Committing a negligible error of order \( O(\sin(i)) \) we can add a term \( \varphi_x (\partial_\xi \varphi_n) \) inside the square parentheses, to make the result look more similar to the usual Euler equation in Cartesian coordinates. Using (A17), (A41) and (A42) we can finally rewrite Equation (A56) as:

\[
\partial_t \varphi_n + \varphi_x (\partial_\xi \varphi_n) + \varphi_y (\partial_\eta \varphi_n) = -\frac{\partial_p \rho}{\rho} - \partial_\Phi \Phi_x + 2\Omega_0 \varphi_x + O \left( [\sin(i)]^2 \right).
\]

(A71a)

Now we repeat similar calculations for Equation (A57). For the various terms within the square parentheses in this equation we have:

\[
\begin{align*}
\varphi_n (\partial_\eta \varphi_n) & \sim [\Omega R]^2 \sin(i) \quad (\ast) \quad (A72) \\
\varphi_x (\partial_\xi \varphi_n) & \sim [\Omega R]^2 \sin(i)^2 \quad (A73) \\
\varphi_n \varphi_x \cos(i) & \sim [\Omega R]^2 [\sin(i)]^2 \quad (A74) \\
\varphi_n \sin(i) & \sim [\Omega R]^2 [\sin(i)]^3 \quad (A75) \\
\varphi_n (\partial_\eta \varphi_n) & \sim [\Omega R]^2 \sin(i) \quad (\ast) \quad (A76) \\
\varphi_x (\partial_\xi \varphi_n) & \sim [\Omega R]^2 \sin(i) \quad (\ast) \quad (A77) \\
\varphi_n \varphi_x \cos(i) & \sim [\Omega R]^2 [\sin(i)]^2 \quad (A78) \\
\varphi_n \sin(i) & \sim [\Omega R]^2 [\sin(i)]^3 \quad (A79) \\
\varphi_n (\partial_\eta \varphi_n) & \sim [\Omega R]^2 \sin(i) \quad (\ast) \quad (A80) \\
\varphi_x (\partial_\xi \varphi_n) & \sim [\Omega R]^2 \sin(i) \quad (\ast) \quad (A81) \\
\varphi_n \varphi_x \cos(i) & \sim [\Omega R]^2 [\sin(i)]^2 \quad (A82) \\
\varphi_n \sin(i) & \sim [\Omega R]^2 [\sin(i)]^3 \quad (A83)
\end{align*}
\]

To leading order in \( \sin(i) \) we need to keep terms that are marked with (\ast). To the same order the derivative in term (A76) can be rewritten as (see Equations A23 and A39):

\[
\partial_\eta \varphi_x \simeq (\Omega_0 - \Omega_\varphi) R_0 + \frac{d\Omega}{dR} \frac{R^2}{R_0}, \quad (A84)
\]

\[
\simeq (\Omega_0 - \Omega_\varphi) R_0 - q \Omega_0 R_0, \quad (A85)
\]

where we introduced the shear parameter

\[
q = -\frac{d\log(\Omega)}{d\log(R)} R_0. \quad (A86)
\]

To the same order term (A78) can be written as:

\[
\varphi_n \varphi_x \cos(i) \simeq \varphi_n (\Omega_0 - \Omega_\varphi) R_0. \quad (A87)
\]

To the same order we can put \( \cos(i) = 1 \) in term (A78) and approximate \( 1/R \simeq 1/R_0 \) everywhere in Equation (A57). The terms with \( \Phi_0 \) on the RHS of Equation (A57) are of order \( [\Omega R]^2 \sin(i)^2 \) and could be neglected, but we keep them, committing a negligible error of order \( O(\sin(i) \sin(i)) \). We also keep terms (A77) and (A81) committing negligible errors. Putting everything together and using (A17), (A41) and (A42) we find:

\[
\partial_t \varphi_n + \varphi_x (\partial_\xi \varphi_n) + \varphi_y (\partial_\eta \varphi_n) = -\frac{\partial_p \rho}{\rho} - \partial_\Phi \Phi_x - 2\Omega_0 \varphi_x + q \Omega_0 \varphi_x + O \left( [\sin(i)]^2 \right).
\]

(A88a)

Equations (A71) and (A88) agree with Equations (2.1a) and (2.1b) of Balbus (1988) and with Equation (3) of Kim et al. (2014). Some remarks:

(i) Despite velocities \( \varphi_x \) and \( \varphi_y \) being in the frame rotating at \( \Omega_\varphi \), the Coriolis term that appears in Equations (A71) and (A88) is calculated using \( \Omega_0 \), not \( \Omega_\varphi \).

(ii) In deriving Equations (A55), (A71), and (A88), we have not expanded to first order in the quantities with subscript ‘s’, as we would have done in a standard linear analysis. Indeed, we have kept
quadratic terms such as $v_{cx}(\partial_y v_{sx})$, which we would not have kept in a linear analysis. Instead, the small parameter in the present expansion is $\sin(i)$.

(iii) In solving Equations (A55), (A71), and (A88), the circular velocity $v_c$ must be specified, because it enters through the terms $v_x = v_{cx} + v_{sx}$ and $v_y = v_{cy} + v_{sy}$. However, in all instances in which $v_c$ appears, it can be considered a constant. This can be shown by considering one by one the various terms that contain it. For example, in Equation (A71) we have the term

$$v_y(\partial_y v_{sx}) = v_{cy}(\partial_y v_{sx}) + v_{sy}(\partial_y v_{sx}).$$

(A89)

The circular velocity can be expanded as (see Equations A23):

$$v_{cx} = v_{c0} + O\left(\sin(i)^2\right),$$

(A90)

$$v_{cy} = v_{c0} + O\left(\sin(i)\right),$$

(A91)

where

$$v_{c0} = (\Omega_0 - \Omega_p) R_0 \sin(i),$$

(A92)

$$v_{c0} = (\Omega_0 - \Omega_p) R_0,$$  

(A93)

are the circular velocities at $R = R_0$. Since according to the relations in Section A6 we have $v_{cx} = (1/R_0) \partial_y v_{sx} \sim \Omega R \sin(i)$, when we substitute (A91) into the first term on the right hand side of (A89) we obtain:

$$v_{cy}(\partial_y v_{sx}) = v_{c0}(\partial_y v_{sx}) + O(|\sin(i)|^2).$$

(A94)

When we put this back into (A89) and then into (A71), we see that we can approximate $v_{cx} \simeq v_{c0}$ to the same level of approximation under which (A71) is valid, i.e. $O(\sin(i)^2)$. Repeating the same argument with all the terms in which $v_{cx}$ and $v_{cy}$ appear in Equations (A55), (A71), and (A88), one sees that we can approximate everywhere $v_{cx} \simeq v_{c0}$ and $v_{cy} \simeq v_{c0}$.

**A10 Final equations**

We now re-express the equations in the form used in the main text. Equation (A55) is already in the same form as (2). To bring Equations (A71) and (A88) in the form (1), note that as specified in item (iii) in Section A9 above we can consider $v_c$ to be a constant everywhere in these equations. So we can replace $v$ with $v$ in all terms containing a derivative, and we can substitute $v_{sx} = v_x - v_{c0}$ and $v_{sy} = v_y - v_{c0}$ (see Equations A92 and A93). We arrive at:

$$\partial_t v + (v \cdot \nabla) v = -\frac{\nabla p}{\rho} - \nabla \Phi_s - 2\Omega_0 \times v + G,$$

(A95)

where $G$ is a constant given by:

$$G = 2\Omega_0(\Omega_0 - \Omega_p) R_0 \left(\sin(i) \left(1 - \frac{q}{2}\right) \hat{e}_y - \hat{e}_x\right).$$

(A96)

Now we perform the following Galilean transformation to put ourselves in a frame that moves along the spiral arm with a speed equal to the circular velocity:

$$x' = x, \quad y' = y - v_{c0} t, \quad t' = t,$$

(A97)

Equation (A55) is invariant under this transformation and remains unchanged. All terms in Equation (A95) are invariant except the last two, which substituting $v_y = v'_y + v_{c0}$ become:

$$-2\Omega_0 \times v + G = -2\Omega_0 \times v' + 2\Omega_0 v_{c0} \sin(i) \left(1 - \frac{q}{2}\right) \hat{e}_y.$$  

(A98)

Therefore the final Euler equation becomes:

$$\partial_t v + (v \cdot \nabla) v = q\Omega_0 v_x \hat{e}_y - \frac{\nabla p}{\rho} - \nabla \Phi_s - 2\Omega_0 \times v + F \left(1 - \frac{q}{2}\right) \hat{e}_y$$

(A99)

where we have dropped the primes for simplicity and $F$ is a constant given by:

$$F = 2\Omega_0(\Omega_0 - \Omega_p) R_0 \sin(i).$$

(A100)

Equation (A99) coincides with Equation (1).

**APPENDIX B: NUMERICAL SOLUTION OF THE STEADY STATE EQUATIONS**

Here we describe the numerical procedure used to solve Equations (6) and (7). We use the ‘shooting’ method (e.g. Press et al. 2007). Our procedure is similar to those of Shu et al. (1973) and Kim et al. (2014).

As mentioned in the main text, we are interested in solutions that contain a shock. These solutions must contain a sonic point $x_s$, defined as the point where $v_x(x_s) = c_s$. We do not know the position of the sonic point a priori, so we take an initial guess for the position of the sonic point. We then determine the velocity $v_x(x_s) = \Phi_s'(x_s)/(2\Omega_0)$ at the sonic point by using the fact that Equation (6) should not be singular at the sonic point.

We use these initial values of $v_x$ and $v_y$ to integrate the differential equation both in the backward and forward directions starting from the sonic point $x_s$ (see Fig. B1). Next, we determine if there is a point where the shock jump conditions are satisfied. There are two jump conditions: (i) the component of the velocity parallel to the shock is continuous at the shock, $v_{y+} = v_{y-}$; (ii) there must be a jump in the perpendicular component such that $v_{x+} + v_{y+} = c_s^2$. Here, $v_\pm$ indicate the velocities just before/after the shock. We first check whether there is a point that quantities, assuming that quantities are periodic with period $L_x$. If there is a point where this condition is satisfied, then we check the second condition. If both conditions are satisfied within a given tolerance ($\sim 10^{-6}$) we stop the procedure and we have found a solution. Otherwise, we change the guess for the sonic point and repeat the procedure until the jump conditions are met.

**APPENDIX C: INITIAL NOISE**

Here we discuss in more details the random noise that we introduce to accelerate the onset of the instability and save computational time. We perturb the initial density according to

$$\rho(x,y,t=0) = \rho_0(x) \left(1 + h(x,y)\right),$$

(C1)

where $\rho_0(x)$ is the density of the steady states described in Section B and $h(x,y)$ is a random noise calculated as follows. We write

$$h(x,y) = \sum_{(k,l)} H_{kl} \exp \left[i2\pi (\frac{xk}{N_x} + \frac{yk}{N_y})\right],$$

(C2)

where $N_x$ and $N_y$ are the number of grid points in $x$ and $y$ directions respectively and $k \in \{0,1,2,\ldots,N_x - 1\}$, $l \in \{0,1,2,\ldots,N_y - 1\}$. We write $H_{kl} = |H_{kl}| \exp(i\theta_{kl})$ and draw the amplitudes $|H_{kl}|$ from a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 0.01$, and the phases $\theta_{kl}$ from a uniform distribution between 0 and $2\pi$. In this way we obtain white noise with an amplitude of roughly $|h(x,y)| \approx 4\%$. The initial noise is visible for example in the initial conditions shown in the top-left corner of Figure 1.
Figure B1. An example numerical solution to Equations (6) and (7). These steady state solutions are used as initial conditions for our simulations. For this example, $q = 0$, $L_x = 1$, $c_s = 0.7$ and $\Phi_0 = 0.25$. The top panels show the final solution as solid lines, and the discarded continuations of the forward and backward integrations as dashed lines. The red dotted line is the sound speed. The bottom panels illustrate the fulfilment of Rankine-Hugoniot jump conditions.