Dissipative Floquet Topological Systems

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Motivated by recent pump-probe spectroscopies, we study the effect of phonon dissipation and potential cooling on the nonequilibrium distribution function in a Floquet topological state. To this end, we apply a Floquet-kinetic equation approach to study two dimensional Dirac fermions irradiated by a circularly polarized laser, a system which is predicted to be in a laser induced quantum Hall state. We find that the initial electron distribution shows an anisotropy with momentum dependent spin textures that rely on the switching on protocol of the laser. The phonons then smoothen this out leading to a non-trivial isotropic nonequilibrium distribution distinct from a thermal state. An analytical expression for the distribution at the Dirac point is obtained that is relevant for observing quantized transport.

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Recent years have seen the emergence of topological states of matter which is a new way of characterizing materials by the geometric properties of the underlying band-structure \cite{Hasan2010}. These include time reversal (TR) breaking integer quantum Hall systems, TR preserving spin quantum Hall systems or two-dimensional (2D) topological insulators (TIs), 3D TIs, and their strongly interacting counterparts \cite{Berg2010}. Another intriguing class of systems are those that can show topological behavior only under out of equilibrium conditions, the main candidate being the Floquet TIs which arise under periodic driving \cite{Asboth2014, Zherlitsyn2014}.

Consider a time periodic Hamiltonian $H(t) = H(t+T)$ where the periodicity may be due to an external irradiation by a laser. Then the time-evolution over one period can be written as $U(t+T,t) = e^{-iH_FT}$ where $H_F$ is the Floquet Hamiltonian: an effective time-independent Hamiltonian that captures the stroboscopic time-evolution over one period \cite{Berg2010, Zherlitsyn2014}. Floquet TIs have been mainly described by borrowing concepts from equilibrium where the topological properties are extracted by analyzing the spectrum of $H_F$, with the topological phase showing non-zero Chern numbers and edge-states, though the precise correspondence between the usual equilibrium definition of the Chern number and the number of edge-states does not always work \cite{Fukui2010, Asboth2014}. Experimentally too, Floquet TIs have been realized in a photonic system which is effectively in equilibrium because the periodicity in time is replaced by a periodicity in position \cite{Haug2010}.

However Floquet TIs are manifestly out of equilibrium and so raise a unique set of questions that do not arise in systems in equilibrium, one of them being the issue of the electron distribution function, critical for determining measurable quantities. Obviously the distribution function will depend on how the periodic driving is switched \cite{Oka2013, Oka2014}, where any switching-on protocol breaks time-periodicity. In addition the occupation probability will be very sensitive to any coupling to an external reservoir \cite{Oka2013, Oka2014}. Often reservoir engineering can even produce topological properties absent in the closed system \cite{Oka2013}. The main aim of this work is to understand the electron distribution function of Floquet topological systems by accounting for the initial switching-on protocol of the periodic drive and accounting for coupling to a reservoir of phonons. We will derive and solve a kinetic equation for the electron distribution function, and show that the combined effect of drive and dissipation can stabilize non-trivial steady-states. We will discuss the signature of these states on spin and angle resolved photoemission (ARPES).

We study 2D Dirac fermions coupled to an external circularly polarized laser, and also coupled to a bath of phonons, $H = H_{el} + H_{ph} + H_{ec}$, where

$$H_{el} = \sum_{\vec{k} = [k_x, k_y], \sigma, \sigma' = \uparrow, \downarrow} c_{\vec{k} \sigma}^\dagger \begin{bmatrix} \vec{k} + \vec{A}(t) \end{bmatrix} \cdot \vec{\sigma}_{\sigma \sigma'} c_{\vec{k} \sigma'}. \quad (1)$$

$c_{\vec{k} \sigma}^\dagger, c_{\vec{k} \sigma}$ are creation, annihilation operators for the Dirac fermions whose velocity $v = 1$, $\vec{\sigma} = [\sigma_x, \sigma_y]$ are the Pauli matrices, $\vec{A} = \theta(t) A_0 \cos(\Omega t), -\sin(\Omega t)$ is the circularly polarized electric field which has been suddenly switched...
on at time $t = 0$, we will refer to this switch-on protocol as a quench. This model plays a central role in the study of Floquet topological states where the circularly polarized laser generates a mass term $(m\sigma z)$ in the Floquet Hamiltonian $H_F \equiv \sum_{\omega, q, i} \hbar \omega \hat{b}_q^\dagger \hat{b}_q$ [3, 10], implying a Hall conductivity $\sigma_{xy} = \text{sign}(m) e^2/2h$ provided a zero temperature equilibrium distribution at half-filling is realized. $H_F$ is also the continuum limit of Haldane’s model \[1\] which is an example of a TI (Chern insulator). Experimentally, gapless surface states of a 3D TI, governed by the Dirac Hamiltonian, were recently studied by pump-probe spectroscopy \[26, 27\], while laser induced Hall effect and chiral edge states are being experimentally studied in graphene \[28, 29\].

Dissipation affects the electron distribution and thus the topological signatures. Here we consider coupling to 2D phonons $H_{ph} = \sum_{q,i=x,y} \omega_{q,i} b_{q,i}^\dagger b_{q,i}$ where the electron-phonon coupling is

$$H_c = \sum_{\vec{k}, q, \sigma, \sigma'} c_{\vec{k} \sigma}^\dagger \bar{A}_{ph}(q) \cdot \vec{g}_{\sigma \sigma'} c_{\vec{k} \sigma'},$$

where $\bar{A}_{ph}(q) = [\lambda_{x,q} (b_{x,q}^\dagger + b_{x,q}), \lambda_{y,q} (b_{y,q}^\dagger + b_{y,q})]$. Above we neglect phonon induced scattering between electrons with different momenta. This simplifies the kinetic equation for the electron distribution function, without changing the physics, and is a microscopic way of accounting for a Caldeira-Leggett \[30\] type dissipation. In what follows, we will first discuss the physics in the absence of the phonons $H_c = 0$, but accounting for the sudden switch-on protocol of the AC field, presenting results for the steady-state distribution function and Green’s functions, quantities that are measured in ARPES. We will then address how the results get modified due to coupling to phonons.

**Results for the quench and in the absence of phonons:** Suppose that at $t \leq 0$, the electrons are in the ground-state, i.e., all states below the Dirac point are occupied so that the initial wave-function right before the switching on of the AC field is $|\Psi_{in}(t = 0^-)\rangle = \prod_k |\psi_{in,k}\rangle$, where $|\psi_{in,k}\rangle = (1/\sqrt{2}) \left( e^{-i\theta_k} 1 \right)$, with $\theta_k = k_y/k_x$.

The time-evolution after switching on the AC field is $|\Psi(t > 0)\rangle = U_{cl}(t, t') |\Psi_{in}\rangle$, where $U_{cl}(t, t')$ is the time-evolution operator, $i dU_{cl}(t, t')/dt = H_{cl}(t) U_{cl}(t, t') + U_{cl}(t, t) = 1$. The dynamics is factorizable between different momenta, $U_{cl}(t, t') = \prod_k U_k(t, t')$ so that, $|\Psi(t)\rangle = \prod_k |\psi_{k}(t)\rangle = \prod_k U_k(t, 0) |\psi_{in,k}\rangle$, where $U_k(t, t') = \sum_{\epsilon_n \pm} |\phi_{\epsilon_n,k}(t')\rangle \langle \psi_{\epsilon_n,k}(t)|$ being the exact solution of the Schrödinger equation which may be written in terms of the time-periodic Floquet quasi-modes $|\phi_{\epsilon_n,k}(t + T)\rangle = |\phi_{\epsilon_n,k}(t)|$ and quasi-energies $\epsilon_n$ as follows, $|\psi_{\epsilon_n,k}(t)\rangle = e^{-i\varepsilon_n t} |\phi_{\epsilon_n,k}(t)|$ with $[H_0 - i\varepsilon_n] |\phi_{\epsilon_n,k}\rangle = \epsilon_n |\phi_{\epsilon_n,k}\rangle$. We can now determine the retarded Green’s function

$$g^R_{\sigma \sigma'}(k, t, t') = -i\theta(t - t') \langle \Psi_{in} | c\sigma_k(t) c\sigma'_k(t') \rangle |\Psi_{in}\rangle = -i\theta(t - t') U_{k, \sigma, \sigma'}(t, t') \text{ [31],}$$

and the lesser Green’s function, $g^\less_{\sigma \sigma'}(k, t, t') = -i\langle \Psi_{in} | c\sigma_k(t') c\sigma_k(t) |\Psi_{in}\rangle = -i \sum_{\sigma, \sigma'} \langle \Psi_{in} | c\sigma_{\epsilon_n,k}(t') \rangle \langle \psi_{\epsilon_n,k}(t)| U_{k, \sigma, \sigma'}(0, t) U_{k', \sigma', \sigma'}(t, 0)$. While $g^R$ does not depend on the occupation probability (by not depending on the initial state), $g^\less$ depends on it. We perform a Fourier transformation of the Green’s functions $g(k, t, t')$ with respect to the time-difference $t - t'$ thus moving to the frequency $\omega$ space, and all throughout we present results after time-averaging over the mean time $T_m = (t + t')/2$.

We refer to $ig^\less_{\sigma}(k, \omega)$ as the spin resolved ARPES spectrum, a key quantity in this work that can be directly probed in experiments \[20\]. Note that results for the spectral density $A = \text{Im} [g^R]$ have been discussed elsewhere \[31\], however our results for $g^\less$ even in the absence of phonons are new. We note that number conservation, absence of momentum mixing, and the fact that we are at half-filling imply $\int (d\omega / 2\pi) i \sum_{\sigma} g^\less_{\sigma}(k, \omega) = 1$.

The results for the momentum dependent spin-density $P_z(k) = \int g^\less_{\sigma}(k, \omega) \text{Im} \Omega |\psi_{\epsilon_n,k}(t)|^2 \text{ d}k$ is shown as a contour plot in the left panel of Fig. 1 as well as along the line $k_y = 0$ in Fig. 2. The circularly polarized laser induces a strongly momentum dependent spin density which shows oscillations each time the condition for a photon induced resonance between the Dirac bands $|k| \approx n\Omega/2$, where $n$ is an integer, is obeyed. Further, the density is also anisotropic in momentum space. The spin averaged ARPES spectrum $ig_{\text{tot}}(k, \omega) = i \sum_{\sigma} g^\less_{\sigma}(k, \omega)$ is plotted as an intensity plot in Fig. 3 and its momentum slices in Fig. 4. The spectrum clearly shows the appearance of Floquet bands. Without phonons, the system is free, and the electron distribution is given by the overlap $|\langle \psi_{\epsilon_n,k}(t)| \psi_{\text{in},k}\rangle|^2$. This is a highly non-thermal state that retains memory of the initial state $|\psi_{\text{in},k}\rangle$, and is not expected to thermalize. Just like the spin resolved density, the total density in
Fig. 3 shows a clear asymmetry under $k_x \rightarrow -k_x$, where this particular anisotropy is determined by the phase of the AC field at $t = 0^+$ when $A(t = 0^+) = [A_0, 0]$ is entirely along the $\hat{x}$ direction.

The anisotropy can be understood analytically at $k=0$ \cite{23},

$$P_z(k = 0, \theta_k) = -\frac{2A_0\Delta}{\Delta^2} \cos \theta_k; \Delta = \sqrt{4A_0^2 + \Omega^2} \quad (4)$$

where $\theta_k$ is the angle along which $k = 0$ is approached. This has the same anisotropy as the left panel in Fig.1.

For the lesser Green’s function at $k=0$ we obtain,

$$i\rho_{\sigma\alpha}^{\text{quench}}(k = 0, \theta_k, \omega) = 2\pi \sum_{\alpha = \pm} \rho_{\alpha \alpha}^{\text{quench}} \left[ \frac{\Delta + \sigma\Omega}{2\Delta} \right] \delta \left( \omega + \frac{\sigma(\Delta + \Omega)}{2} \right),$$

$$\rho_{\alpha \alpha}^{\text{quench}} = |\langle \phi_{\alpha \alpha}(0)|\psi_{\text{in},k=0}\rangle|^2 = \frac{1}{2} \left( 1 - \frac{2\alpha A_0}{\Delta} \cos \theta_k \right) \quad (5)$$

above $\sigma = +/-$ for $\uparrow / \downarrow$. The analytic expression for $\rho_{\sigma}$ shows that for $k = 0$, there are exactly four resonances, where the two resonances for spin $\sigma$ occur at $\omega = -\sigma(\Omega \pm \Delta)/2$. Thus the circularly polarized field acts as an effective magnetic field along $\hat{z}$ \cite{10}, splitting the energies of the up and down spin electrons. In particular in the high frequency ($A \ll \Omega$) limit, the lowest energy excitation is $\Delta - \Omega \approx 2A_0^2/\Omega$ and involves flipping a spin from $\downarrow$ to $\uparrow$. The analytic results also show that the weights are far from thermal, where by thermal we imply resonances of the form $\delta(\omega - \epsilon_k)n_F(\omega)$, $n_F$ being the Fermi distribution function. The appearance of only a couple of Floquet bands, and momentum anisotropy is consistent with experimental observations \cite{20}.

**Results in the presence of phonons:** The above results for the time-averaged distribution functions after a quench are exact and will not evolve in time. However if we turn on the electron-phonon coupling, inelastic scattering will cause the distribution functions to relax, we now study how this happens, and what is the resulting steady-state. We first briefly outline the derivation of the kinetic or rate equation in the presence of phonons within the Floquet formalism (see \cite{23} for general discussions). Let $W(t)$ be the density matrix obeying $dW(t)/dt = -i[H, W(t)]$. It is convenient to be in the interaction representation, $W_I(t) = e^{iH_{\text{ph}}U_{\text{el}}(t, 0)W(0)U_{\text{el}}(t, 0)e^{-iH_{\text{ph}}t}}$. To $O(H_{\text{ph}}^2)$, the density matrix obeys the following equation of motion $dW_I(t)/dt = -i[H_{\text{el}}(t), W_I(t)] - \int_{t_0}^{t} dt' [H_{e,I}(t'), W_I(t')]$ where $H_{e,I}$ is in the interaction representation. We assume that at the initial time $t_0$, the electrons and phonons are uncoupled so that $W(t_0) = W_{\text{el}}(t_0) \otimes W_{\text{ph}}(t_0)$, and that initially the electrons are in the state $|\Psi(t)\rangle$ described above, while the phonons are in thermal equilibrium at temperature $T$. Thus, $W_{\text{el}}(t_0) = |\Psi(t)\rangle\langle \Psi(t)| = \prod_k W_{k,0}^{\text{el}}$, where $W_{k,0}^{\text{el}}(t) = \sum_{\alpha, \beta = \pm} e^{-i(\epsilon_{k\alpha} - \epsilon_{k\beta})t}|\phi_{k\alpha}(t)\rangle\langle \phi_{k\beta}(t)|\rho_{k,\alpha\beta}^{\text{quench}}$ with $\rho_{k,\alpha\beta}^{\text{quench}} = |\langle \phi_{\alpha\alpha}(0)|\psi_{\text{in},k}\rangle|^2$. Defining the electron reduced density matrix as $W_{\text{el}} = Tr_{\text{ph}}W$, and noting that $H_e$ being linear in the phonon operators, the trace vanishes, we need to solve, $dW_{\text{el}}(t)/dt = -Tr_{\text{ph}} \int_{t_0}^{t} dt' [H_{e,I}(t'), W_I(t')]$. We assume that the phonons are an ideal reservoir and stay in equilibrium. In that case $W_I(t) = W_{\text{el}}(t) \otimes e^{-H_{\text{ph}}/T} / Tr[e^{-H_{\text{ph}}/T}]$. The most general form of the reduced density matrix for the electrons is

$$W_{\text{el}}(t) = \prod_k \rho_{k,\alpha\beta}(t)|\phi_{k,\alpha}(t)\rangle \langle \phi_{k,\beta}(t)|$$

where in the absence of phonons, $\rho_{k,\alpha\beta}(t) = \rho_{k,\alpha\beta}^{\text{quench}}$ and are time-independent in the interaction representation.

The last remaining assumption is to identify the slow and fast variables, which allows one to make the Markov approximation \cite{33}. Since we would like to treat AC fields that vary rapidly in time, we write $\rho_{k,\alpha\beta}(t) = \sum_{m=\text{int}} e^{im\Omega t} \rho_{k,\alpha\beta}^{(m)}(t)$ where in what follows we assume that $\rho_{k,\alpha\beta}^{(m)}(t)$ are slowly varying on time scales of the period of the AC field and the relevant phonon frequencies. In addition we only study the diagonal components of $\rho_{k,\alpha\alpha}(t)$, which after the Markov approximation, obey the rate equation

$$\left[ \rho_{k,\alpha\alpha}(t) + i\Omega \rho_{k,\alpha\alpha}(t) \right] = -\sum_{m, \beta = \pm} \rho_{k,\beta\beta}^{(m-m')}(t)$$
The initial condition we will consider corresponds to \( \rho^{(m)}_{k, \alpha \alpha}(t = 0) = \delta_{m=0} \rho^{\text{quench}}_{k, \alpha \alpha} \), with the in-scattering and out-scattering rates \( L^{m, m'}_{k, \alpha \beta} \) given in Ref. [32].

Since the rate equation is a weak-coupling quasi-classical approximation in the electron-phonon coupling, the position of the resonances in the spectral density are not modified, and thus even with phonons, \( g^R \) is unchanged. The phonons strongly modify the steady-state lesser Green’s function because the distribution function of the electrons is changed due to inelastic scattering with phonons. In the numerical solutions for the rate equation we assume acoustic phonons with a uniform phonon density of states \( D_{ph} \), and an isotropic electron-phonon coupling \( \lambda_x = \lambda_y = \lambda \). The results can easily be generalized to optical phonons as for frequencies below the optical phonon frequencies, the distribution function will remain unchanged, and will be given by that for the quench.

The time-evolution of the density matrix from a quench-type initial state is shown in Fig. 2, where the time-evolving, the steady-state greater Green’s function in the presence of phonons is given by, \( G_{ss}(k, t, t') = -i \sum_{\alpha=\pm} \rho^{ss}_{k, \alpha \alpha} \langle \phi_{k, \alpha}(t) | c_{\alpha \sigma}^+(t') \phi_{k, \alpha}(0) \rangle \) where \( c_{\alpha \sigma}(t) = \sum_{\sigma'} U_{\alpha \sigma \sigma'}(t, 0) c_{\sigma'}(0) \).

Remarkably, for \( k = 0, L^{m, m'}_{k=0, \alpha \beta} = \delta_{m, m'} L^m_{k=0, \alpha \beta} \), so that again analytic results are possible. Here we find for the spin-density at \( k = 0 \),

\[
P_z(k = 0; H_c \neq 0) = \frac{-2 \Omega (\Delta^2 + \Omega^2) / \Delta}{\sum_{\alpha=\pm} (\Delta - \alpha \Omega)^2 \left( 1 + 2 N (\Delta + \alpha \Omega) \right)}
\]

where \( N(x) \) is the Bose distribution function, while

\[
i G^<_{\sigma \sigma}(k = 0, \omega; H_c \neq 0) = 2 \pi \sum_{\alpha=\pm} \rho^{ss}_{k=0, \alpha \alpha} \times \left[ \left( \frac{\Delta + \alpha \Omega}{2 \Delta} \right) \delta \left( \omega + \sigma \left( \frac{\Delta + \alpha \Omega}{2} \right) \right) \right]
\]

where

\[
\rho^{ss}_{k=0, \alpha \alpha} = \sum_{\beta=\pm} \frac{(\Delta - \beta \Omega)^2 N (\Delta + \beta \Omega)}{\sum_{\alpha=\pm} (\Delta - \alpha \Omega)^2 (1 + 2 N (\Delta + \alpha \Omega))}
\]

with \( \sum_{\alpha=\pm} \rho^{ss}_{k=0, \alpha \alpha} = 1 \). Note that the above results at \( k = 0 \) are isotropic in being independent of the angle \( \theta_k \).

Thus the coupling to phonons makes the electrons lose memory of their initial state, and that reflects in a symmetric distribution of the density in momentum space.

This is also clearly seen in the contour plot of Fig. 3, Fig. 2 shows that the spin-density still retains oscillations at momenta \( k \) for which the photon frequencies are resonant with the Dirac bands, however the magnitude of the oscillations decay with increasing temperature of the phonon bath, with the spin-density \( P_z(k) \) approaching zero as the temperature increases.

The spin averaged ARPES spectrum \( i G^{<}_{tot} \) in the steady-state with phonons is shown as an intensity plot in the middle and lower panels in Fig. 3 and along some momentum slices in Fig. 4. One finds that as the temperature of the phonon bath decreases, the magnitude of the resonances at positive frequencies decrease and the ones at negative frequencies increase, maintaining the sum rule. While this is also the expected result from a simple thermal Green’s function where the weights of the resonances are \( \delta(\omega - \epsilon_k) n_F(\epsilon_k) \), yet note that the precise weights in steady-state are not thermal. This can also be clearly seen in the analytic solution for \( k = 0 \). In particular Eq. (8) implies that in the high frequency limit \( P_z(k = 0; A_0 \ll \Omega) \to \text{tanh} \left( A_{20}^2 / \Omega^2 \right) / \left( 1 + (A_0^2 / \Omega^2) \text{tanh} (A_0^2 / \Omega^2) \text{coth} (\Omega/T) \right) \). In this high-frequency limit, the Floquet Hamiltonian is \( H_F \simeq \sigma_z k_x + \sigma_y k_y + \sigma_z A_0^2 / \Omega \), so a naive guess would be that the thermalized state should have a magnetization of \( \text{tanh} (h_z / 2T) \) where \( h_z = 2A_0^2 / \Omega \). The result for \( P_z \) shows deviations from this guess at \( O(A_0^2 / \Omega^2) \). Thus, the presence of the AC drive causes the electrons to reach a nonequilibrium steady-state even when the phonon reservoir to which the electrons are coupled are themselves always in thermal equilibrium. Fig. 4 also shows that as \( k \) approaches the photon induced resonance condition \( |k| \sim n\Omega/2 \), the effective temperature...
is higher, as more frequencies are excited. This result is clearly reflected in Fig. 3 (central panel) where even when the phonon temperature is very low, the avoided crossings are characterized by a high population density.

In summary we have studied the electron distribution in a Floquet topological system under two circumstances, one is for the closed system, where the distribution function is very sensitive to how the AC field has been switched on, showing highly anisotropic distribution functions, the second is for the open system where the electrons are coupled to a reservoir of phonons. While coupling to phonons causes the system to lose memory of its initial state, yet the presence of the drive gives rise to non-trivial nonequilibrium steady-states observable in ARPES. An important open question is to understand transport phenomena such as Hall conductance. Since the Hall response is dominated by the behavior at $k = 0$ where the Berry curvature is peaked, our results imply that the anisotropic distribution of the closed system will cause significant deviation from the quantum limit. On the other hand coupling to low temperature phonons induces cooling of Floquet states. The cooling works efficiently near the Dirac point, helping the system to approach the quantum limit. However, near resonant points ($\Delta = n\Omega$), the effective temperature stays high due to photo-carriers. **Acknowledgments:** This work was supported by US Department of Energy (DOE-BES) under Award No. DE-SC0010821 (HD and AM), and partially by the Simons Foundation (academic year support for AM).

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Supplementary Material

ANALYTIC SOLUTION AT $k = 0$ FOR THE QUENCH (NO PHONONS)

Let us consider the solution of $H_{el}$ when $k = 0$. In this case, the quasi-modes $|\phi_\alpha\rangle$ obey the equation,

$$H_{el,F}(k = 0)|\phi_\alpha\rangle = \epsilon_\alpha|\phi_\alpha\rangle \quad (11)$$

$$H_{el,F}(k = 0) = \vec{A} \cdot \vec{\sigma} - i\partial_t \quad (12)$$

$$|\phi_\alpha\rangle = \left( \begin{array}{c} \phi_{\uparrow\alpha} \\ \phi_{\downarrow\alpha} \end{array} \right) \quad (13)$$

where $\vec{A} = A_0 (\cos \Omega t, -\sin \Omega t)$, so that $\vec{A} \cdot \vec{\sigma} = A_0 \begin{pmatrix} 0 & e^{-i\Omega t} \\ e^{i\Omega t} & 0 \end{pmatrix}$. Thus, the $\phi_{\uparrow,\downarrow\alpha}$ obey the coupled equation

$$-i\partial_t \phi_{\uparrow\alpha} + A_0 e^{i\Omega t} \phi_{\downarrow\alpha} = \epsilon_\alpha \phi_{\uparrow\alpha} \quad (14)$$

$$-i\partial_t \phi_{\downarrow\alpha} + A_0 e^{-i\Omega t} \phi_{\uparrow\alpha} = \epsilon_\alpha \phi_{\downarrow\alpha} \quad (15)$$

Substituting for

$$\phi_{\downarrow\alpha} = e^{-\frac{i\Omega t}{2}} e^{-i\Omega t} \epsilon_\uparrow \phi_{\uparrow\alpha}$$

into the second equation above gives,

$$\partial_t^2 \phi_{\uparrow\alpha} - i [2\epsilon_\alpha + \Omega] \partial_t \phi_{\uparrow\alpha} + (A_0^2 - \Omega \epsilon_\alpha - \epsilon_\alpha^2) \phi_{\uparrow\alpha} = 0 \quad (17)$$

Writing $\phi_{\uparrow,\downarrow\alpha}(t + T) = \phi_{\uparrow,\downarrow\alpha}(t)$, $\lambda = m\Omega$, where $m$ is an integer. Thus Eq. (16) gives,

$$\phi_{\downarrow\alpha} = d_{\downarrow\alpha} e^{i(m-1)\Omega t} \quad \phi_{\uparrow\alpha} = d_{\uparrow\alpha} e^{im\Omega t} \quad (19)$$

with

$$\epsilon_\pm = \left( \frac{m}{2} - \frac{1}{2} \right) \Omega \pm \frac{\Delta}{2} \quad (20)$$

$$d_{\downarrow\pm} = \frac{-\Omega \pm \Delta}{2A_0} \quad (21)$$

Thus,

$$d_{\uparrow\pm} = \frac{\sqrt{2}A_0}{\sqrt{\Delta(\Delta \mp \Omega)}} \cdot d_{\downarrow\pm} = \pm \frac{1}{\sqrt{2}} \sqrt{1 \mp \frac{\Omega}{\Delta}} \quad (22)$$

$$|\phi_{\pm}(t)\rangle = e^{i\Omega t/2} e^{-i\Omega t/2} d_{\downarrow\pm} \quad (23)$$

Note that while there are infinite possible ways to choose the quasi-modes and the corresponding quasi-energies, where the quasi-energies are related by shifts by integer multiples of the frequency $\Omega$, this degeneracy is absent.
in the wavefunctions corresponding to the exact solutions of the Schrödinger equation, \( |\Psi_n(t)\rangle = e^{-i\epsilon_n t} |\phi_n(t)\rangle \). In particular the wavefunctions are

\[
|\Psi_+(t)\rangle = e^{i\Omega t/2 - i\Delta t/2} \left( \frac{\sqrt{2} A_0}{\sqrt{\Delta(\Delta - \Omega)}} \right) e^{-i\Omega t/2 \sqrt{1 - \Omega^2/\Delta}} |\phi_n(t)\rangle \tag{24}
\]
\[
|\Psi_-(t)\rangle = e^{i\Omega t/2 + i\Delta t/2} \left( -\frac{\sqrt{2} A_0}{\sqrt{\Delta(\Delta + \Omega)}} \right) e^{-i\Omega t/2 \sqrt{1 + \Omega^2/\Delta}} |\phi_n(t)\rangle \tag{25}
\]

One may also construct the time-evolution operator,

\[
U_{k=0}(t, t') = \sum_{\alpha = \pm} e^{-i\epsilon_{n\alpha}(t-t')} |\phi_n(t)\rangle \langle \phi_n(t')| \tag{26}
\]

\[
= \sum_{\alpha = \pm} e^{-i\frac{\Omega + \Delta}{2}(t-t')} \left( d_{\alpha}^\dagger \right) \left( e^{-i\Omega t} d_{\alpha} \right) \tag{27}
\]

If the state just before switching on the AC field is the ground state of the Dirac fermions,

\[
|\psi_{in}\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} e^{-i\theta_k} \\ 1 \end{array} \right) \tag{28}
\]

then the wavefunction after the sudden switch-on of the AC field is given by

\[
|\Psi(t)\rangle = U_{k=0}(t, 0)|\psi_{in}\rangle = \sum_{\alpha = \pm} C_{-\alpha} \sqrt{\Delta(\Delta - \alpha\Omega)} \sqrt{2} A_0 |\Psi_\alpha(t)\rangle \tag{29}
\]

where

\[
C_+ = \frac{[\omega_+ \psi_+(0) + A_0 \psi_2(0)]}{(\omega_+ - \omega_-)} \tag{30}
\]
\[
C_- = \frac{[\omega_+ \psi_-(0) + A_0 \psi_2(0)]}{(\omega_+ - \omega_-)} \tag{31}
\]

with

\[
\omega_\pm = \frac{\Omega \pm \Delta}{2} \tag{32}
\]
\[
\psi_1(0) = -e^{-i\theta_k}/\sqrt{2}; \psi_1(0) = 1/\sqrt{2} \tag{33}
\]

Once the wavefunction \( |\Psi(t)\rangle \) and the time evolution operator \( U(t, t') \) are known, one may compute all the single-time and two-time averages discussed in the main text.

### RATE EQUATIONS FOR GENERAL \( k \) AND EXACT SOLUTION AT \( k = 0 \)

The rate equations after the Markov approximation are found to be (below \( N_q = N(\omega_q) \) is the Bose distribution function)

\[
\rho^{(m)}_{k,\alpha}(t) + im\Omega \rho^{(m)}_{k,\alpha} = -\sum_{q, i=x,y,y'=\pm, n_1,n_2} \pi \lambda_q^2 \left( e^{i\frac{\pi}{4}} \left[ C_{1k\alpha}^{m_1} C_{1k\beta}^{m_2} + C_{2k\beta}^{m_1} C_{2k\alpha}^{m_2} \right] + C_{1k\beta}^{m_1} C_{2k\alpha}^{m_2} + C_{2k\alpha}^{m_1} C_{1k\beta}^{m_2} \right) \times \left[ \left( 1 + N_q \right) \delta(\epsilon_{k\beta} - \epsilon_{k\alpha} + (m - n_1)\Omega + \omega_q) + N_q(\delta(\epsilon_{k\beta} - \epsilon_{k\alpha} + (m - n_1)\Omega - \omega_q) \right) \rho^{(m-n_1-n_2)}_{k,\alpha}(t) \\
- \left( 1 + N_q \right) \delta(\epsilon_{k\beta} - \epsilon_{k\alpha} + (m - n_1)\Omega - \omega_q) + N_q(\delta(\epsilon_{k\beta} - \epsilon_{k\alpha} + (m - n_1)\Omega + \omega_q) \right) \rho^{(m-n_1-n_2)}_{k,\alpha}(t) \\
+ \left( 1 + N_q \right) \delta(\epsilon_{k\beta} - \epsilon_{k\alpha} + (m - n_2)\Omega + \omega_q) + N_q(\delta(\epsilon_{k\beta} - \epsilon_{k\alpha} + (m - n_2)\Omega - \omega_q) \right) \rho^{(m-n_1-n_2)}_{k,\alpha}(t) \\
- \left( 1 + N_q \right) \delta(\epsilon_{k\beta} - \epsilon_{k\alpha} + (m - n_2)\Omega - \omega_q) + N_q(\delta(\epsilon_{k\beta} - \epsilon_{k\alpha} + (m - n_2)\Omega + \omega_q) \right) \rho^{(m-n_1-n_2)}_{k,\alpha}(t) \right] \tag{34}
\]

where \( e^{\hat{x}} = 1, e^{\hat{y}^2} = -1. \)

\[
\langle \phi_{k\alpha}(t) | c^\dagger_{k\beta} c_{k\beta}(t) \rangle = \sum_n e^{in\Omega t} C_{1k\alpha}^{m_n} \tag{35}
\]
\[
\langle \phi_{k\alpha}(t) | c^\dagger_{k\beta} c_{k\beta}(t) \rangle = \sum_n e^{in\Omega t} C_{2k\alpha}^{m_n} \tag{36}
\]
\[
\langle \phi_{k\alpha}(t) | c^\dagger_{k=0,\uparrow} c_{k=0,\uparrow}(t) \rangle = d_{\alpha \uparrow} d_{\beta \uparrow} e^{-i\Omega t} \tag{37}
\]
\[
\langle \phi_{k\alpha}(t) | c^\dagger_{k=0,\downarrow} c_{k=0,\downarrow}(t) \rangle = d_{\alpha \downarrow} d_{\beta \downarrow} e^{i\Omega t} \tag{38}
\]

**Analytic results for the rate equation at \( k = 0 \)**

At \( k = 0 \), the exact expressions for the quasi-modes can be used to show that
Thus, the matrix elements entering the rate equation become,

\[ C_{1++}^{(n)} = \frac{A_0}{\Delta} \delta_{n=-1}; C_{1--}^{(n)} = -\frac{A_0}{\Delta} \delta_{n=1} \]

\[ C_{1+-}^{(n)} = -\frac{1}{2} \left( 1 + \frac{\Omega}{\Delta} \right) \delta_{n=1}; C_{1++}^{(n)} = \frac{1}{2} \left( 1 - \frac{\Omega}{\Delta} \right) \delta_{n=1} \]

\[ C_{2++}^{(n)} = \frac{A_0}{\Delta} \delta_{n=1}; C_{2--}^{(n)} = -\frac{A_0}{\Delta} \delta_{n=1} \]

\[ C_{2+-}^{(n)} = \frac{1}{2} \left( 1 - \frac{\Omega}{\Delta} \right) \delta_{n=1}; C_{2--}^{(n)} = -\frac{1}{2} \left( 1 + \frac{\Omega}{\Delta} \right) \delta_{n=1} \]

Let us assume \( \lambda_{\pi q} = \lambda_{\pi q} \). In this case for \( k = 0 \), \( n_1 + n_2 = 0 \) in the rate equations. So for \( k = 0 \), the rate equations simplify to

\[
\partial_t \begin{pmatrix} \rho_{k=0,++}^{(m)} \\ \rho_{k=0,+-}^{(m)} \end{pmatrix} + i m \Omega \begin{pmatrix} \rho_{k=0,++}^{(m)} \\ \rho_{k=0,+-}^{(m)} \end{pmatrix} = \begin{pmatrix} L_{k=0,++}^{(m)} \\ L_{k=0,+-}^{(m)} \end{pmatrix} \begin{pmatrix} \rho_{k=0,++}^{(m)} \\ \rho_{k=0,+-}^{(m)} \end{pmatrix}
\]

where \( L_{k=0,++}^{(m)} = -L_{k=0,+-}^{(m)} \). We now make the assumption of a uniform phonon density of states \( D_{ph} \) so that the rates are,

\[
L_{k=0,++}^{(m)} = -\pi \lambda^2 D_{ph} \frac{1}{2} \left( 1 + \frac{\Omega}{\Delta} \right)^2 \left[ \{1 + N(\Delta - \Omega - m\Omega)\} \theta(\Delta - \Omega - m\Omega) \\
+ \{1 + N(\Delta - \Omega + m\Omega)\} \theta(\Delta - \Omega + m\Omega) \\
+ N(\Delta - \Omega - m\Omega) \theta(-\Delta + \Omega - m\Omega) \\
+ N(\Delta - \Omega + m\Omega) \theta(-\Delta + \Omega + m\Omega) \right]
\]

(40)

and,

\[
L_{k=0,--}^{(m)} = -\pi \lambda^2 D_{ph} \frac{1}{2} \left( 1 - \frac{\Omega}{\Delta} \right)^2 \left[ \{1 + N(\Delta - \Omega + m\Omega)\} \theta(-\Delta - \Omega - m\Omega) \\
+ \{1 + N(\Delta - \Omega - m\Omega)\} \theta(-\Delta - \Omega + m\Omega) \\
+ N(\Delta - \Omega - m\Omega) \theta(\Delta + \Omega - m\Omega) \\
+ N(\Delta - \Omega + m\Omega) \theta(\Delta + \Omega + m\Omega) \right]
\]

(41)

Above \( \theta \) is the Heaviside step function.