The Banach Poisson geometry of multi-diagonal Toda lattices

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**Abstract**

The Banach Poisson geometry of multi-diagonal Hamiltonian systems having infinitely many integrals in involution is studied. It is shown that these systems can be considered as generalizing the semi-infinite Toda lattice which is an example of a bidiagonal system, a case to which special attention is given. The generic coadjoint orbits of the Banach Lie group of bidiagonal bounded operators are studied. It is shown that the infinite dimensional generalization of the Flaschka map is a momentum map. Action-angle variables for the Toda system are constructed.

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**1 Introduction**

Many important conservative systems have a Hamiltonian formulation in terms of Lie-Poisson brackets. With few notable exceptions, such as the Euler, Poisson-Vlasov, KdV, or sine-Gordon equations, for example, for infinite dimensional systems this Lie-Poisson bracket formulation is mostly formal. It is our belief that these formal approaches can be given a solid functional analytic underpinning. The present paper formulates such an approach for various generalizations of the semi-infinite Toda lattice. It raises fundamental issues about the nature of coadjoint orbits for the Banach Lie groups having only a finite number of non-zero upper diagonals and it poses questions about the integrability of certain generalizations of the Toda lattice in infinite dimensions by providing a functional analytic framework in which these problems can be rigorously formulated. The background of the present work is \([20]\) where the theory of Banach Lie-Poisson spaces was developed.

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The paper is organized as follows. The first two sections develop the theoretical background for the constructions carried out later. Section 2 presents the general theory of induced and coinduced Banach Lie-Poisson structures and derives the analogue of the classical Adler-Kostant-Symes involution theorem \cite{2, 12, 23} from this point of view in the infinite dimensional context. Section 3 introduces the notion of a momentum map for weak symplectic Banach manifolds and presents the abstract symplectic induction method in infinite dimensions.

The next two sections concentrate of the Banach Lie-Poisson geometry of several spaces of trace class operators on a real separable Hilbert space. The general constructions of Section 2 are implemented explicitly on these spaces in Section 4. The multi-diagonal Banach Lie group, its Lie algebra, and its dual are introduced and studied. The naturally induced and coinduced Poisson structures on the preduals of their Banach Lie algebras are presented. Section 5 formulates the equations of motion induced by the Casimir functions of the Banach Lie-Poisson space of trace class operators relative to the various induced and coinduced Poisson brackets discussed previously.

Starting with Section 6 the emphasis is on the important particular case of bidiagonal operators, that is, operators having all entries equal to zero with the possible exception of those on the main and upper \( k \) diagonal. The Banach Lie group of upper bidiagonal bounded operators is studied in detail and the topological and symplectic structure of the generic coadjoint orbit is presented. The Banach space analogue of the Flaschka map is analyzed and its relationship to the coadjoint orbits is pointed out. There are new, typical infinite dimensional, phenomena that appear in this context. For example, the Banach space of trace zero lower bidiagonal trace class operators does not form a single coadjoint orbit and there are non-algebraic invariants for the coadjoint orbits.

Section 7 uses the method of symplectic induction developed Section 3 to derive explicit formulas that are used for the concrete case of the bidiagonal Banach Lie group. A generalization of the Flaschka map introduced in the previous section is presented. This is a Poisson map whose range is the weak symplectic manifold \((\ell^\infty)^k\times(\ell^k)^{k-1}\), endowed with a non-canonical weak symplectic form. Systems with an infinite number of integrals in involution are also introduced in this section. As an example of the theory, the semi-infinite Toda lattice is solved in Section 8 using the method of orthogonal polynomials first introduced, to our knowledge, in \cite{4}. The explicit solution of this system is obtained, both in action-angle as well as in the original variables, thereby extending the formulas in \cite{17} from the finite to the semi-infinite Toda lattice.

Conventions. In this paper all Banach manifolds and Lie groups are real. The definition of the notion of a Banach Lie subgroup follows Bourbaki \cite{5}, that is, a subgroup \( H \) of a Banach Lie group \( G \) is necessarily a submanifold (and not just injectively immersed). In particular, Banach Lie subgroups are necessarily closed.

## 2 Induced and coinduced Banach Lie-Poisson spaces

In this section we quickly review some material from \cite{20} and present some constructions that are necessary for the development of the ideas in the rest of the paper.

Preliminaries. Let us recall how a given Banach Lie-Poisson structure induces and coinduces similar structures on other Banach spaces. All the proofs of the statements below can be found in \cite{20}. Throughout this paper, unless specified otherwise, all objects are over \( \mathbb{R} \).

A Banach Lie algebra \( (g,\cdot,\cdot) \) is a Banach space \( g \) that is also a Lie algebra such that the Lie bracket is a bilinear continuous map \( g \times g \to g \). Thus the adjoint and coadjoint maps \( \text{ad}_x : g \to g \), \( \text{ad}_x y := [x,y] \), and \( \text{ad}_x^* : g^* \to g^* \) are also continuous for each \( x \in g \). Here \( g^* \) denotes the dual of \( g \), that is, the Banach space of all linear continuous functionals on \( g \).

A Banach Lie-Poisson space \( (b,\cdot,\cdot) \) is defined to be a real Poisson manifold such that \( b \) is a Banach space and the dual \( b^* \subset C^\infty(b) \) is a Banach Lie algebra under the Poisson bracket operation. We need to explain what does it mean for a Banach Poisson manifold. The Poisson bracket induces the derivation \( h \to \{\cdot,h\} \) on \( C^\infty(b) \) which defines a map \( X_h : b \to b^{**} \) by \( \langle X_h(b), Df(b) \rangle = \{ f,h \}(b) \) for any \( b \in b \) and \( f \) a smooth real valued function defined in an open subset of \( b \) containing \( b \). Thus, \( X_h(b) \in b^{**} \cong T_b^{**}b \) and therefore \( X_h(b) \) is not a tangent vector to \( b \) at \( b \). The requirement that \( b \) be a Banach Poisson manifold is that \( X_h(b) \in b \cong T_b b \) for all \( b \in b \).

Denote by \( \cdot,\cdot \) the restriction of the Poisson bracket \( \{\cdot,\cdot\} \) from \( C^\infty(b) \) to the Lie subalgebra \( b^* \). The following criterion characterizes the Banach Lie-Poisson structure. The Banach space \( b \) is a Banach Lie-Poisson space \( (b,\cdot,\cdot) \) if and only if its dual \( b^* \) is a Banach Lie algebra \( (b^*,\cdot,\cdot) \) satisfying \( \text{ad}_b^*b \subset b \subset b^{**} \),
for all \( x \in b^* \). Moreover, the Poisson bracket of \( f, h \in C^\infty(b) \) is given by
\[
\{f, h\}(b) = \langle [Df(b), Dh(b)], b \rangle,
\]
where \( b \in b \) and \( Df(b) \in b^* \) denotes the Fréchet derivative of \( f \) at the point \( b \). If \( h \) is a smooth function on \( b \), the associated Hamiltonian vector field is given by
\[
X_h(b) = -\text{ad}^\ast_{Dh(b)} b \in b
\]
for any \( b \in b \). Therefore Hamilton’s equations are
\[
\frac{d}{dt} b(t) = -\text{ad}^\ast_{Dh(b(t))} b(t).
\]

Given two Banach Lie-Poisson spaces \((b_1, \{., .\}_1)\) and \((b_2, \{., .\}_2)\), a smooth map \( \varphi : b_1 \to b_2 \) is said to be canonical or a Poisson map if
\[
\{f, h\}_2 \circ \varphi = \{f \circ \varphi, h \circ \varphi\}_1
\]
for any two smooth locally defined functions \( f \) and \( h \) on \( b_2 \). Like in the finite dimensional case, \( \{., .\} \) is equivalent to
\[
X^\varphi_2 \circ \varphi = T \varphi \circ X^\psi_1
\]
for any smooth locally defined function \( h \) on \( b_2 \). Therefore, the flow of a Hamiltonian vector field is a Poisson map and Hamilton’s equations \( f = \{f, h\}_1 \) in Poisson bracket formulation are valid. If the Poisson map \( \varphi \) is, in addition, linear, then it is called a linear Poisson map.

Given the Banach Lie-Poisson spaces \((b_1, \{., .\}_1)\) and \((b_2, \{., .\}_2)\) there is a unique Banach Poisson structure \{., .\} on the product space \( b_1 \times b_2 \) such that:

(i) the canonical projections \( \pi_1 : b_1 \times b_2 \to b_1 \) and \( \pi_2 : b_1 \times b_2 \to b_2 \) are Poisson maps;

(ii) \( \pi_1^\ast(C^\infty(b_1)) \) and \( \pi_2^\ast(C^\infty(b_2)) \) are Poisson commuting subalgebras of \( C^\infty(b_1 \times b_2) \).

This unique Poisson structure on \( b_1 \times b_2 \) is called the product Poisson structure and its bracket is given by the formula
\[
\{f, g\}_1(b_1, b_2) = \{f_{b_1}, g_{b_2}\}_1(b_1) + \{f_{b_1}, g_{b_2}\}_2(b_2),
\]
where \( f_{b_1}, g_{b_2} \in C^\infty(b_2) \) and \( f_{b_1}, g_{b_2} \in C^\infty(b_1) \) are the partial functions given by \( f_{b_1}(b_2) := f_{b_1}(b_1) := f(b_1, b_2) \) and \( g_{b_2}(b_1) := g(b_2, b_1) := g(b_1, b_2) \). In addition, this formula shows that this unique Banach Poisson structure is Lie-Poisson and that the inclusions \( \iota_1 : b_1 \hookrightarrow b_1 \times b_2, \iota_2 : b_2 \hookrightarrow b_1 \times b_2 \) given by \( \iota_1(b_1) := (b_1, 0) \) and \( \iota_2(b_2) := (0, b_2) \), respectively, are also linear Poisson maps.

**Induced Structures.** Let \( b_1 \) be a Banach space, \((b, \{., .\})\) a Banach Lie-Poisson space, and \( \iota : b_1 \hookrightarrow b \) an injective continuous linear map with closed range. Then \( \ker \iota^\ast \) is an ideal in the Banach Lie algebra \((b^*, \{., .\})\) if and only if \( b_1 \) carries a unique Banach Lie-Poisson bracket \{., .\}\(_{1}^{\text{ind}}\) such that
\[
\{F \circ \iota, G \circ \iota\}_{1}^{\text{ind}} = \{F, G\} \circ \iota
\]
for any \( F, G \in C^\infty(b) \); see Proposition 4.10 in \([20]\). This Poisson structure on \( b_1 \) is said to be induced by the mapping \( \iota \) and it is given by
\[
\{f, g\}_{1}^{\text{ind}}(b_1) = \langle [\iota^\ast((Df(b_1))), [\iota^\ast]^{-1}(Dg(b_1))] \rangle, b_1 \rangle
\]
for any \( f, g \in C^\infty(b_1) \) and \( b_1 \in b_1 \), where \( [\iota^\ast] : b^* \to b_1^* \) is the Banach space isomorphism induced by \( \iota^\ast : b^* \to b_1^* \) and \([., .]\) denotes the Lie bracket on the quotient Lie algebra \( b^* / \ker \iota^\ast \).

Let us assume now that the range \( \iota(b_1) \) is a closed split subspace of \( b \), that is, there exists a projector \( R = R^2 : b \to b \) such that \( \iota(b_1) = R(b) \). Taking in \( \{F, G\} := f \circ \iota^{-1} \circ R, G := g \circ \iota^{-1} \circ R \in C^\infty(b) \) for \( f, g \in C^\infty(b_1) \) and noting that \( \iota^{-1} \circ R \circ \iota = \text{id}_{b_1} \), we get
\[
\{f, g\}_{1}^{\text{ind}}(b_1) = \{f \circ \iota^{-1} \circ R, g \circ \iota^{-1} \circ R\}(\iota(b_1))
\]
\[
= \langle [Df(\iota^{-1} \circ R)(\iota(b_1)), Dg(\iota^{-1} \circ R)(\iota(b_1))] \rangle, \iota(b_1) \rangle.
\]
We shall make use of this formula in \([3]\).

We return now to the general case, that is, we consider an arbitrary quasi-immersion \( \iota : b_1 \hookrightarrow b \) of Banach spaces which means that the range \( \iota(b_1) \) is closed but does not necessarily possess a closed complement.
Proposition 2.1 Let \( \iota : b_1 \hookrightarrow b \) be a quasi-immersion of Banach Lie-Poisson spaces (so range \( \iota \) is a closed subspace of \( b \) and \( \ker \iota^* \) is an ideal in the Banach Lie algebra \( b^* \)). Assume that there is a connected Banach Lie group \( G \) with Banach Lie algebra \( g := b^* \). Then the \( G \)-coadjoint orbit \( O_{\iota(b_1)} := \text{Ad}_{b_1} \iota(b_1) \) is contained in \( \iota(b_1) \) for any \( b_1 \in b_1 \). In addition, if \( N \subset G \) is a closed connected normal Lie subgroup of \( G \) whose Lie algebra is \( \ker \iota^* \), then the \( N \)-coadjoint action restricted to \( \iota(b_1) \) is trivial. Therefore the Banach Lie group \( G/N := \{ g \mid gN \mid g \in G \} \) naturally acts on \( \iota(b_1) \) and the orbit of \( \iota(b_1) \) under this action coincides with \( O_{\iota(b_1)} \) for any \( b_1 \in b_1 \).

**Proof.** Since \( \ker \iota^* \) is an ideal in \( g = b^* \), it follows that \( [x,y] \in \ker \iota^* \) for all \( x \in g \) and \( y \in \ker \iota^* \). Therefore, since \( \ker \iota^* \) is closed in \( g \), it follows that \( \text{Ad}_{\exp_{x,y}} \iota \in \ker \iota^* \) for any \( x \in g \) and \( y \in \ker \iota^* \). This shows that for any \( g \in G \) in an open neighborhood of the identity element of \( G \) we have \( \text{Ad}_g \ker \iota^* \subset \ker \iota^* \). Since \( G \) is connected, it is generated by a neighborhood of the identity and we conclude that \( \text{Ad}_g \ker \iota^* \subset \ker \iota^* \) for any \( g \in G \).

The upper index \( ^\circ \) on a set denotes the annihilator of that set relative to a duality pairing; the annihilator of a set is always a vector subspace. Let \( b_1 \in b_1 \) and \( g \in G \). Since \( \ker \iota^* = \iota(b_1)^0 \), closedness of \( \iota(b_1) \) in \( b \) implies that \( (\ker \iota^*)^0 = \iota(b_1) \). Thus, for any \( g \in G \) and \( x \in \ker \iota^* \), we have

\[
(\text{Ad}_{b_1}^* \iota(b_1), x) = (\iota(b_1), \text{Ad}_g x) = 0
\]

which proves that \( \text{Ad}_G^* \iota(b_1) \subset \iota(b_1) \).

Now let \( N \subset G \) be a closed connected normal Lie subgroup of \( G \) with Banach Lie algebra \( \ker \iota^* \subset g \). For any \( b_1 \in b_1 \), \( x \in g = b^* \), \( y \in \ker \iota^* \), we have

\[
(\text{Ad}_{b_1}^* \iota(b_1), x) = (\iota(b_1), [y,x]) = 0
\]

since \( \ker \iota^* \) is an ideal in \( g \) and \( \ker \iota^* = \iota(b_1)^0 \). Since this is valid for all \( x \in g \), it follows that \( \text{Ad}_{b_1}^* \iota(b_1) = 0 \) for all \( y \in \ker \iota^* \) and \( b_1 \in b_1 \). Using the exponential map, this shows that \( \text{Ad}_{b_1}^* \iota(b_1) = \iota(b_1) \) for any \( n \) in a neighborhood of the identity in \( N \). Since \( N \) is connected, it is generated by a neighborhood of the identity and we conclude that \( \text{Ad}_n^* \iota(b_1) = \iota(b_1) \) for all \( n \in N \).

The quotient \( G/N := \{ g \mid gN \mid g \in G \} \) is a Banach Lie group and the projection \( G \to G/N \) is a smooth surjective submersive Banach Lie group homomorphism (\( 5 \), Chapter III, \( \S 1.6 \)). Since the coadjoint action of \( N \) on \( \iota(b_1) \) is trivial, the Banach Lie group \( G/N \) acts smoothly on \( \iota(b_1) \) by \( [g] \cdot \iota(b_1) := \text{Ad}_g^* \iota(b_1) \). The orbit of a fixed element \( \iota(b_1) \in \iota(b_1) \) by this group action is obviously equal to the \( G \)-orbit \( O_{\iota(b_1)} \).

**Coinduced Structures.** Let \( (b, \{ , \}) \) be a Banach Lie-Poisson space and \( \pi : b \to b_1 \) a continuous linear surjective map onto the Banach space \( b_1 \). Then \( b_1 \) carries a unique Banach Lie-Poisson bracket \( \{ , \}^{\text{coind}}_1 \) making \( \pi \) into a linear Poisson map, that is,

\[
\{ f \circ \pi, g \circ \pi \} = \{ f, g \}^{\text{coind}}_1 \circ \pi \tag{2.10}
\]

for any \( f, g \in C^\infty(b_1) \) if and only if \( \pi^*(b_1^*) \subset b^* \) is closed under the Lie bracket \( [ , ] \) of \( b^* \): see Proposition 4.8 of \( 24 \). This unique Poisson structure on \( b_1 \) is said to be **coinduced** by the Banach Lie-Poisson structure on \( b \) and the linear continuous map \( \pi \). It should be noted that \( \text{im} \pi^* \) is a closed subspace of \( b^* \) since \( \text{im} \pi^* = (\ker \pi)^0 \). To determine the coinduced bracket on \( b_1 \) note that \( \pi^* : b_1^* \to b^* \) is an injective linear continuous map whose closed range is a Banach Lie subalgebra of \( b^* \). Thus, on \( \text{im} \pi^* \) we can invert \( \pi^* \). The coinduced bracket on \( b_1 \) has then the form

\[
\{ f, g \}^{\text{coind}}_1(b_1) = \langle \pi^* \rangle^{-1} \pi^* \left( Df(b_1), \pi^* (Dg(b_1)) \right) \tag{2.11}
\]

for any \( f, g \in C^\infty(b_1) \) and \( b_1 \in b_1 \).

Let us assume that \( \ker \pi \) admits a closed complement. This is equivalent to the existence of a linear continuous injective map \( \iota : b_1 \to b \) with closed range such that \( \pi \circ \iota = \text{id}_{b_1} \). Thus \( 2.10 \) implies that

\[
\{ f, g \}^{\text{coind}}_1 = \{ f \circ \pi, g \circ \pi \} \circ \iota \tag{2.12}
\]

for any \( f, g \in C^\infty(b_1) \).

Assume now that the Banach Lie-Poisson space \( b \) splits into a direct sum \( b = b_1 \oplus b_2 \) of closed Banach subspaces. Denote by \( R_j : b \to b \) the projection onto \( b_j \), for \( j = 1, 2 \). So we have the following relations:
Proof. (i) \( R_1 + R_2 = \text{id}_b, \ R_1^2 = R_1, \ R_2^2 = R_2, \ R_2 R_1 = R_1 R_2 = 0, \ b_1 := \text{im} R_1, \) and \( b_2 := \text{im} R_2. \) Dualizing we get the projectors \( R_1^*, \ R_2^* : b^* \rightarrow b^* \) satisfying \( R_1^* + R_2^* = \text{id}_{b^*}, \ (R_1^*)^2 = R_1^*, \ (R_2^*)^2 = R_2^*, \ R_2^* R_1^* = R_1^* R_2^* = 0. \) The relationship between these spaces is given by

\[
\begin{align*}
\ker R_1 &= \text{im} R_2 = b_2 \quad \text{and} \quad \ker R_2 = \text{im} R_1 = b_1 \\
\ker R_1^* &= \text{im} R_2^* = (\text{im} R_1)^\ast \cong b_2^* \quad \text{and} \quad \ker R_2^* = (\text{im} R_2)^\ast \cong b_1^* \\
b &= b_1 \oplus b_2 \quad \text{and} \quad b^* = b_1^* \oplus b_2^*.
\end{align*}
\]

Let \( \iota_j : b_j \hookrightarrow b \) be the inclusion determined by the splitting \( b = b_1 \oplus b_2 \) for \( j = 1, 2. \) Denote by \( \pi_j : b \rightarrow b_j \) the projection determined by the projector \( R_j : b \rightarrow b, \) that is, \( \iota_j \circ \pi_j = R_j \) and note that \( \pi_j \circ \iota_j = \text{id}_{b_j}. \) We summarize these notations in the following diagram.

\[
\begin{tikzcd}
    & b \\
    b_1 & \pi_1 \ar[ru] & \pi_2 \ar[lu] \\
    b_2 & & 
\end{tikzcd}
\]

From (2.12) we get

\[
\{f, g\}^{\text{coind}}_j = \{f \circ \pi_j, g \circ \pi_j\} \circ \iota_j
\]

or, explicitly

\[
\{f, g\}^{\text{coind}}_j(b_j) = \langle [D(f \circ \pi_j)(\iota_j(b_j)), D(g \circ \pi_j)(\iota_j(b_j))] \rangle, \quad \text{where} \ b_j \in b_j.
\]

The following proposition presents some properties of the induced and coinduced structures on \( b_1 \) and \( b_2. \)

**Proposition 2.2** Assume that \( \text{im} R_1^* \) and \( \text{im} R_2^* \) are Banach Lie subalgebras of \( b^*. \) Then:

(i) \( b_j \) has a Banach Lie-Poisson structure coinduced by \( \pi_j \) and the expression of the coinduced bracket \( \{\ , \}^{\text{coind}}_j \) on \( b_j \) is given by (2.16). The Hamiltonian vector field of \( h \in C^\infty(b_j) \) at \( b_j \in b_j \) is given by

\[
X_h(b_j) = -\pi_j \left( \text{ad}_{\pi_j^* D_h(b_j)} \iota_j(b_j) \right), \quad j = 1, 2,
\]

where \( D_h(b_j) \in b_j^* \) and \( \text{ad}_x \) is the adjoint action of \( x \in b^* \) on \( b^*. \)

(ii) The Banach space isomorphism \( R := \frac{1}{2}(R_1 - R_2) : b \rightarrow b \) defines a new Banach Lie-Poisson structure

\[
\{f, g\}_R(b) := \langle [R^* Df(b), Dg(b)] + [Df(b), R^* Dg(b)], b \rangle
\]

on \( b, f, g \in C^\infty(b), \) that coincides with the product structure on \( b_1 \times b_2, \) where \( b_1 \) carries the coinduced bracket \( \{\ , \}^{\text{coind}}_1 \) and \( b_2 \) denotes \( b_2 \) endowed with the Poisson bracket \( \{\ , \}^{\text{coind}}_2. \)

(iii) The inclusion maps \( \iota_1 : (b_1, \{\ , \}^{\text{coind}}_1) \hookrightarrow (b_1 \times b_2) \) and \( \iota_2 : (b_2, \{\ , \}^{\text{coind}}_2) \hookrightarrow (b_1 \times b_2) \) are linear injective Poisson maps with closed range.

(iv) The map \( \iota_j \) induces from \( (b_1, \{\ , \}^{\text{coind}}_1) \) a Banach Lie-Poisson structure on \( b_j \) which coincides with the coinduced structure described in (i), for \( j = 1, 2. \)

**Proof.** (i) By hypothesis, the range \( \text{im} R_j^* \) of the map \( R_j^* : b^* \rightarrow b^* \) is a Banach Lie subalgebra of \( b^*. \) Thus \( \pi_j \) coinduces a Banach Lie-Poisson structure on \( b_j^*. \) Let \( h \in C^\infty(b_j) \) and note that for any function \( f \in C^\infty(b_j) \) and \( b_j \in b_j \) we have

\[
\langle Df(b_j), X_h(b_j) \rangle = \langle f, h \rangle^{\text{coind}}_j(b_j) = \langle [Df \circ \pi_j)(\iota_j(b_j)), D(h \circ \pi_j)(\iota_j(b_j))] \rangle, \quad \text{where} \ b_j \in b_j.
\]
which proves formula \(2.15\).

(ii) Let \(b = b_1 + b_2 \in b_1 \oplus b_2\). Then \(R_j(b) = b_j\), for \(j = 1, 2\). A direct verification shows then that

\[
\{f, g\}_R(b) = \{R^* Df(b), Dg(b)\} + [D(b), R^* Dg(b)], b
\]

\[
= \frac{1}{2} (\{R_1^* Df(b) + R_2^* Df(b), R_1^* Dg(b) + R_2^* Dg(b)\}, b)
\]

\[
= \frac{1}{2} (\{R_1^* Df(b), R_1^* Dg(b)\}, R_1 b + R_2 b) - \{\{R_2^* Df(b), R_2^* Dg(b)\}, R_1 b + R_2 b\}
\]

\[
= \{R_1^* Df(b), R_1^* Dg(b)\}, R_1 b\) - \{\{R_2^* Df(b), R_2^* Dg(b)\}, R_2 b\}
\]

\[
= \{f_{b_2}, g_{b_2}\}_1^{\text{ind}}(b_1) - \{f_{b_1}, g_{b_1}\}_2^{\text{ind}}(b_2),
\]

where in the third equality we have used the fact that \([R_1^* Df(b), R_1^* Dg(b)] \in \text{im } R_1^* = (\text{im } R_2)^\circ \) and \([R_2^* Df(b), R_2^* Dg(b)] \in \text{im } R_2^* = (\text{im } R_1)^\circ \) and \(b = b_1 + b_2\) with \(b_j \in b_j\). To prove the last equality above it suffices to note that

\[
D_1 f_{b_2}(b_1) \cdot \delta b_1 = D f(b) \cdot \delta b_1 = D f(b) \cdot R_1 \delta b_1 \quad \text{and} \quad D_2 f_{b_1}(b_2) \cdot \delta b_2 = D f(b) \cdot \delta b_2 = D f(b) \cdot R_2 \delta b_2
\]

for any \(\delta b_j \in b_j\), where \(D_j\) is the Fréchet derivative on \(b_j\), for \(j = 1, 2\). The last expression is that of the product Banach Lie-Poisson structure on \(b_1 \times b_2\) (see \(2.6\)).

(iii) This is an immediate consequence of (ii) and the general fact, recalled earlier for products of Banach Lie-Poisson spaces, that these inclusions are Poisson maps with closed range.

(iv) Let \{\cdot, \cdot\}_j^{\text{ind}} and \{\cdot, \cdot\}_j^{\text{coind}} be the induced and coinduced brackets on \(b_j\) from \((b, \{\cdot, \cdot\}_R)\) and \((b, \{\cdot, \cdot\})\), respectively. Therefore,

\[
\{F, G\}_R \circ \iota_j = \{F \circ \iota_j, G \circ \iota_j\}_j^{\text{ind}}
\]

for any \(F, G \in C^\infty(b)\) and, by \(2.16\),

\[
\{f, g\}_j^{\text{coind}} = (-1)^{j-1} \{f \circ \pi_j, g \circ \pi_j\} \circ \iota_j
\]

for any \(f, g \in C^\infty(b_j)\). Apply relation \(2.20\) to the functions \(F := f \circ \pi_j, G := g \circ \pi_j\) and use \(\pi_j \circ R = \frac{1}{2} (-1)^{j-1} \pi_j\), \(\pi_j \circ R = \frac{1}{2} (-1)^{j-1} \pi_j\), and \(2.21\) to get for any \(b_j \in b_j\)

\[
\{f, g\}_j^{\text{ind}}(b_j) = \{f \circ \pi_j, g \circ \pi_j\}_R \circ \iota_j(b_j)
\]

\[
= \{\{R^* D(f \circ \pi_j)(\iota_j(b_j)), D(g \circ \pi_j)(\iota_j(b_j))\}, \iota_j(b_j)\}
\]

\[
+ \{\{R^* D((f \circ \pi_j)(\iota_j(b_j)), R^* D(g \circ \pi_j)(\iota_j(b_j))\}, \iota_j(b_j)\}
\]

\[
= \{\{R^* f_j(b_j), R^* g(b_j)\}, \iota_j(b_j)\}
\]

\[
= (-1)^{j-1} \{\{R^* D(b_j), R^* g(b_j)\}, \iota_j(b_j)\}
\]

\[
= (-1)^{j-1} \{\{D(f \circ \pi_j)(\iota_j(b_j)), D(g \circ \pi_j)(\iota_j(b_j))\}, \iota_j(b_j)\}
\]

\[
= (-1)^{j-1} \{\{f \circ \pi_j, g \circ \pi_j\}_j(b_j)\}
\]

\[
= \{f, g\}_j^{\text{coind}}(b_j).
\]

This proposition implies the following involution theorem.

**Corollary 2.3** In the notations and hypotheses of Proposition 2.3 we have:

(i) The Casimir functions on \((b, \{\cdot, \cdot\})\) are in involution on \((b, \{\cdot, \cdot\}_R)\) and restrict to functions in involution on \(b_j\), \(j = 1, 2\).

(ii) If \(H\) is a Casimir function on \(b\), then its restriction \(H \circ \iota_j\) to \(b_j\) has the Hamiltonian vector field

\[
X_{H\circ \iota_1}(b_1) = \pi_1(\text{ad}^*_{\pi_1^* D(\iota_1(b_1))} \iota_1(b_1))
\]

\[
X_{H\circ \iota_2}(b_2) = \pi_2(\text{ad}^*_{\pi_2^* D(\iota_2(b_2))} \iota_2(b_2))
\]

for any \(b_1 \in b_1\) and \(b_2 \in b_2\), where \(\iota_j : b_j \hookrightarrow b\) is the inclusion, \(j = 1, 2\).
Proof. (i) Let \(F, H \in C^\infty(b)\) be Casimir functions for the Lie-Poisson bracket \(\langle , \rangle\), that is, \(\text{ad}_{DF(b)}^* b = 0\) and \(\text{ad}_{DH(b)}^* b = 0\) for any \(b \in b\). Therefore
\[
\{F, H\}_R(b) = \langle [\text{ad}_{DF(b)}^* b, DH(b)] + [DF(b), \text{ad}_{DH(b)}^* b], b \rangle
= - \langle \text{ad}_{DF(b)}^* b, \text{ad}_{DH(b)}^* b \rangle + \langle \text{ad}_{DF(b)}^* b, \text{ad}_{DF(b)}^* b \rangle = 0
\]
which shows that \(F\) and \(H\) are in involution relative to \(\{ , \}_R\). Then statements (iii) and (iv) of Proposition \ref{prop:involution} show that \(F \circ \iota_j, H \circ \iota_j\) are also in involution on \(b_j, j = 1, 2\).

(ii) Since \(H\) is a Casimir function on \(b\), we have \(\text{ad}_{DH(b)}^* b = 0\) for any \(b \in b\). Therefore, since \(R_1^* + R_2^* = \text{id}_{b^*}\), we get for any \(b_1 \in b_1\)
\[
0 = \text{ad}_{R_1^*DH(b_1)}^* \iota_1(b_1) = \text{ad}_{R_1^*DH(b_1)}^* \iota_1(b_1) + \text{ad}_{R_2^*DH(b_1)}^* \iota_1(b_1).
\]
A similar relation holds for any \(b_2 \in b_2\). So, we have
\[
- \text{ad}_{R_j^*DH(b_j)}^* = \text{ad}_{R_{j+1}^*DH(b_j)}^*, \tag{2.23}
\]
where \(j\) is taken modulo 2.

Since \(\iota_j \circ \pi_j = R_j\), we get
\[
\pi_j^* D(H) \circ \iota_j)(b_j) = D(H \circ \iota_j)(b_j) \circ \pi_j = DH(\iota_j(b_j)) \circ \iota_j \circ \pi_j
= DH(x_{\iota_j(b_j)}) \circ R_j = R_j^* DH(x_{\iota_j(b_j)}),
\]
so (2.18) and (2.23) yield
\[
\iota_j (X_{\iota_j(b_j)}) = -(\iota_j \circ \pi_j) \left( \text{ad}_{\pi_j^* D(H \circ \iota_j)(b_j)} \iota_j(b_j) \right)
= -R_j \left( \text{ad}_{R_j^* DH(x_{\iota_j(b_j)})} \iota_j(b_j) \right)
= R_j \left( \text{ad}_{R_{j+1}^* DH(x_{\iota_j(b_j)})} \iota_j(b_j) \right)
= \text{ad}_{R_{j+1}^* DH(x_{\iota_j(b_j)})} \iota_j(b_j). \tag{2.24}
\]
The last equality follows from the fact that \(\text{ad}_{R_{j+1}^* x} \iota_j(b_j) \in \text{im} R_j = \text{im} \iota_j\) for any \(x \in b^*\) and \(b_j \in b_j\). Indeed, for any \(y \in (\text{im} R_j)^\circ = \text{im} R_{j+1}^*\) we have
\[
\langle \text{ad}_{R_{j+1}^* x} \iota_j(b_j), y \rangle = \langle \iota_j(b_j), [R_{j+1}^* x, y] \rangle = 0
\]
because \([R_{j+1}^* x, y] \in \text{im} R_{j+1}^* = (\text{im} R_j)^\circ\) by hypothesis (the image of \(R_{j+1}^*\) is a Banach Lie subalgebra of \(b^*\)) and \(\iota_j(b_j) \in \text{im} R_j\). Therefore, \(\text{ad}_{R_{j+1}^* x} \iota_j(b_j) \in (\text{im} R_j)^\circ = \text{im} R_{j+1}^* = \text{im} R_j\).

Finally, applying \(\pi_j\) to (2.24) yields (2.22). \(\blacksquare\)

Taken together, Proposition \ref{prop:involution} and Corollary \ref{cor:adjoint} give a version of the Adler-Kostant-Symes Theorem (see [2] [12] [23] [21]) formulated with the necessary additional hypotheses in the context of Banach Lie-Poisson spaces.

**Proposition 2.4** Let \((b, \{ , \})\) be a Banach Lie-Poisson space and let \(R_1, R_3 : b \rightarrow b\) be projectors. Assume that \(\text{im} R_1 = \text{im} R_3 = b_2\), where \(R_{21} := \text{id}_b - R_1, R_{23} := \text{id}_b - R_3, \) and denote \(b_1 := \text{im} R_1, b_3 := \text{im} R_3\). We summarize this situation in the diagram

\[
\begin{array}{ccc}
b & \xrightarrow{\pi_1} & b \\
\iota_1 & & \iota_2 \\
b_1 & \xrightarrow{\pi_21} & b_2 \\
\iota_3 & & \iota_3 \\
b_3 & \xrightarrow{\pi_3} & b_3
\end{array}
\]

where \(\pi_1, \pi_21, \pi_23, \pi_3\) are the projections onto the ranges of \(R_1, R_{21}, R_{23}, \) and \(R_3\) respectively, according to the splittings \(b = b_1 \oplus b_2 = b_2 \oplus b_3, \) and \(\iota_1 : b_1 \hookrightarrow b, \iota_3 : b_3 \hookrightarrow b\) are the inclusions.

Then one has:
(i) If $b^2_3$ is a Banach Lie subalgebra of $b^*$, then $\Phi_{13} := \pi_3 \circ \iota_3 : (b_1, \{ , \}_3^{\text{coind}}) \to (b_3, \{ , \}_3^{\text{coind}})$ and $\Phi_{13} := \pi_1 \circ \iota_3 : (b_3, \{ , \}_3^{\text{coind}}) \to (b_1, \{ , \}_1^{\text{coind}})$ are mutually inverse linear Poisson isomorphisms.

(ii) If $b^2_3$ and $b^3_3$ are Banach Lie subalgebras of $b^*$, then $b_3$ has two coinduced Banach Lie-Poisson brackets $\{ , \}_1^{\text{coind}}$ and $\{ , \}_3^{\text{coind}}$ which are not isomorphic in general.

Proof. (i) Since $b^2_3 = (\im R_{31})^* = \im R^*_{13}$ (see (2.13)) is a Banach Lie subalgebra of $b^*$ it follows that $R_1$ coinduces a Banach Lie-Poisson bracket $\{ , \}_1^{\text{coind}}$ on $b_1$. Similarly, the relation $b^2_3 = (\im R_{23})^* = \im R^*_{31}$ implies that $R_3$ coinduces a Banach Lie-Poisson bracket $\{ , \}_3^{\text{coind}}$ on $b_3$.

Let us notice that $\Phi_{31} \circ \Phi_{13} = \pi_3 \circ \iota_3 \circ \pi_1 \circ \iota_3 = \pi_3 \circ \iota_3 = \pi_3 \circ \iota_3 - \pi_3 \circ R_{21} \circ \iota_3 = \pi_3 \circ \iota_3$ since $\pi_3 \circ R_{21} = 0$. One proves similarly that $\Phi_{13} \circ \Phi_{31} = \id_{b_1}$.

From $\ker \pi_1 = \ker \pi_3 = b_2$ and $b - (\iota_3 \circ \pi_3)(b) \in \ker \pi_3$ for any $b \in b$, it follows that $\pi_1 \circ \iota_3 \circ \pi_3 = \pi_1$. Therefore, if $f, g \in C^\infty(b_1)$ we get from (2.10) and the fact that $\pi_1 : b \to b_1$ is a Poisson map

$$\{ f \circ \pi_1 \circ \iota_3, g \circ \pi_1 \circ \iota_3 \}_3^{\text{coind}} = \{ f \circ \pi_1 \circ \iota_3, g \circ \pi_1 \circ \iota_3 \}_3^{\text{coind}}$$

and the fact that $\pi_1 : b_1 \to b_3$ is a Poisson map.

(ii) By (2.13) we have $b^2_3 = \im R_{23}$ and $b^3_3 = \im R_{31}$ which, by hypothesis, are Banach Lie subalgebras of $b^*$. Therefore, $\pi_{21}$ and $\pi_{23}$ coinduce Poisson brackets $\{ , \}_1^{\text{coind}}$ and $\{ , \}_3^{\text{coind}}$ on $b_2$. ■

3 Symplectic induction

The goal of this section is to present the theory of symplectic induction on weak symplectic Banach manifolds. In the process we shall define the momentum map in this setting, establish some of its elementary properties, and give examples relevant to the subsequent developments in this paper.

Weak symplectic manifolds. In infinite dimensions there are two possible generalizations of the notion of a symplectic manifold.

Definition 3.1 Let $P$ be a Banach manifold and $\omega$ a two-form. Then $\omega$ is said to be weakly nondegenerate if for every $p \in P$ the map $v_p \in T_pP \mapsto \omega(p)(v_p, \cdot) \in T^*_pP$ is injective. If, in addition, this map is also surjective, then the form $\omega$ is called strongly nondegenerate. The form $\omega$ is called a weak or strong symplectic form if, in addition, $d\omega = 0$, where $d$ denotes the exterior differential on forms. The pair $(P, \omega)$ is called a weak or strong symplectic manifold, respectively.

If $P$ is finite dimensional this distinction does not occur since every linear injective map is also surjective. The typical example of an infinite dimensional strongly nondegenerate Banach manifold is a complex Hilbert space endowed with the symplectic form equal to the imaginary part of the Hermitian inner product. Any strong symplectic form is locally constant but weak symplectic forms are not, in general. The usual Hamiltonian formalism extends to the strong symplectic case without any difficulties.

On the other hand, if $(P, \omega)$ is a weak symplectic manifold, the equation $d\hbar = \omega(X_{\hbar}, \cdot)$ that would define the Hamiltonian vector field $X_{\hbar}$ associated to the function $\hbar \in C^\infty(P)$ cannot always be solved for $X_{\hbar}$. But if $X_{\hbar}$ exists, it is necessarily unique. Denote by $C^\omega_\infty(P)$ the vector subspace of smooth functions that admit Hamiltonian vector fields. If $f, h \in C^\infty_\omega(P)$ then Poisson bracket is defined by

$$\{ f, h \}_\omega := \omega(X_f, X_h).$$

(3.1)

In general, it is not true that $C^\omega_\infty(P)$ is a Poisson algebra since $f, h \in C^\infty_\omega(P)$ does not necessarily imply that $\{ f, h \} \in C^\omega_\infty(P)$. However, if $f, g, h \in C^\omega_\infty(P)$ and we assume, in addition, that $\{ f, g \}, \{ g, h \}, \{ h, f \} \in C^\omega_\infty(P)$, $d\omega = 0$ the same proof as in finite dimensions implies the Jacobi identity.

Note that if $f, g \in C^\omega_\infty(P)$ then the product $fg \in C^\omega_\infty(P)$. Indeed, the Hamiltonian vector field defined by $fg$ exists because $X_{fg} = fX_g + gX_f$ as an easy computation shows. Another useful property is that the Poisson bracket $\{ f, g \}(p)$ for $f, g \in C^\omega_\infty(P)$ is completely determined by $df(p)$ and $dg(p)$. Indeed, this follows from the fact that if $df(p) = dg(p)$ then $\omega(p)(X_g(p), \cdot) = df(p) = dg(p) = \omega(p)(X_g(p), \cdot)$ and weak nondegeneracy of $\omega$ implies then that $X_f(p) = X_g(p)$. Using this remark one can recover several standard statements about Hamiltonian vector fields in the weak symplectic case.
The weak symplectic manifold \((\ell^\infty \times \ell^1, \omega)\). In this paper we shall often work with the weak symplectic manifold \((\ell^\infty \times \ell^1, \omega)\), where \(\ell^\infty\) is the Banach space of bounded real sequences whose norm is given by

\[
\|q\|_\infty := \sup_{k=0,1,...} |q_k|, \quad q := \{q_k\}_{k=0}^\infty \in \ell^\infty,
\]

\(\ell^1\) is the Banach space of absolutely convergent real sequences whose norm is given by

\[
\|p\|_1 := \sum_{k=0}^\infty |p_k|, \quad p := \{p_k\}_{k=0}^\infty \in \ell^1,
\]

the strongly nondegenerate duality pairing

\[
\langle q, p \rangle = \sum_{k=0}^\infty q_k p_k, \quad \text{for } q \in \ell^\infty, \quad p \in \ell^1,
\]

establishes the Banach space isomorphism \((\ell^1)^* = \ell^\infty\), and the weak symplectic form \(\omega\) has the expression

\[
\omega((q,p),(q',p')) = \langle q, p' \rangle - \langle q', p \rangle, \quad \text{for } q, q' \in \ell^\infty, \quad p, p' \in \ell^1.
\]

for \(q, q' \in \ell^\infty\) and \(p, p' \in \ell^1\).

The differential form \(\omega\) is conveniently written as

\[
\omega = \sum_{k=0}^\infty dq_k \wedge dp_k.
\]

in the coordinates \(q_k, p_k\). Let us elaborate on the notation used in \((3.3)\). If \(p = \{p_k\}_{k=0}^\infty \in \ell^1\), denote by \(\{\partial/\partial p_k\}_{k=0}^\infty\) the basis of the tangent space \(T_p \ell^1\) corresponding to the standard Schauder basis \(\{|k\}_{k=0}^\infty\) of \(\ell^1\). The same basis in \(\ell^\infty\) has a different meaning: every element \(a := \{a_k\}_{k=0}^\infty \in \ell^\infty\) can be uniquely written as a weakly convergent series \(a = \sum_{k=0}^\infty a_k |k|\). With this notion of basis in \(\ell^\infty\), given \(q \in \ell^\infty\), the sequence \(\{\partial/\partial q_k\}_{k=0}^\infty\) denotes the basis of the tangent space \(T_q \ell^\infty\) corresponding to \(\{|k\}_{k=0}^\infty\). Thus, any smooth vector field \(X\) on \(\ell^\infty \times \ell^1\) is written as

\[
X(q,p) = \sum_{k=0}^\infty \left( A_k(q,p) \frac{\partial}{\partial q_k} + B_k(q,p) \frac{\partial}{\partial p_k} \right),
\]

where \(\{A_k(q,p)\}_{k=0}^\infty \in \ell^\infty\) and \(\{B_k(q,p)\}_{k=0}^\infty \in \ell^1\). If \(Y\) is another vector field whose coefficients are \(\{C_k(q,p)\}_{k=0}^\infty \in \ell^\infty\), \(\{D_k(q,p)\}_{k=0}^\infty \in \ell^1\), employing the usual conventions for the exterior derivatives of coordinate functions to represent elements in the corresponding dual spaces, formula \((3.3)\) gives

\[
\left( \sum_{k=0}^\infty dq_k \wedge dp_k \right)(X,Y)(q,p) = \sum_{k=0}^\infty \left( A_k(q,p) D_k(q,p) - C_k(q,p) B_k(q,p) \right)
\]

which coincides with \((3.3)\). It is in this sense that the writing in \((3.4)\) represents the weak symplectic form \((3.3)\).

In this case we can determine explicitly the space \(C^\omega_\infty(\ell^\infty \times \ell^1)\). To do this, we observe that for any \(h \in C^\infty_\omega(\ell^\infty \times \ell^1)\) its partial derivatives \(\partial h/\partial q \in (\ell^\infty)^*\) and \(\partial h/\partial p \in (\ell^1)^* = \ell^\infty\), respectively. Thus the Hamiltonian vector field \(X_h\) defined by the weak symplectic form \((3.4)\) and the function \(h\) exists if and only if \(\partial h/\partial q \in \ell^1 \subset (\ell^1)^* = (\ell^\infty)^*\). Therefore,

\[
C^\omega_\infty(\ell^\infty \times \ell^1) = \{ f \in C^\infty(\ell^\infty \times \ell^1) \mid \{\partial h/\partial q_k\}_{k=0}^\infty \in \ell^1 \},
\]

and the Hamiltonian vector field defined by \(h \in C^\infty_\omega(\ell^\infty \times \ell^1)\) has the expression

\[
X_h(q,p) = \frac{\partial h}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial h}{\partial q_k} \frac{\partial}{\partial p_k}.
\]

The canonical Poisson bracket of \(f, h \in C^\omega_\infty(\ell^\infty \times \ell^1)\) makes sense and is given by

\[
\{f,g\}_\omega(q,p) = \sum_{k=0}^\infty \left( \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right).
\]
Symplectic induction is a technique that associates to a given Hamiltonian system with a Poisson bracket on functions. We shall give study other momentum maps in subsequent sections. For any \( f \in C_\infty^\infty(\ell^\infty \times \ell^1) \), a direct computation shows that the Poisson bracket of any two functions from the set

\[
\left\{ \begin{array}{l}
f \in C_\infty^\infty(\ell^\infty \times \ell^1) \\
\sum_{j=0}^\infty \frac{\partial^2 f}{\partial q_j \partial q_j^*} (y, z) \in \ell^1,
\end{array} \right.
\]

is again in \( C_\omega^\infty(\ell^\infty \times \ell^1) \).

**Momentum maps on weak symplectic manifolds.** Throughout this section, \( G \) denotes a Banach Lie group and \( \mathfrak{g} \) its Lie algebra. We shall assume that \( \mathfrak{g} \) admits a predual \( \mathfrak{g}_* \) and that the coadjoint action of \( G \) on the dual space \( \mathfrak{g}^* \) leaves \( \mathfrak{g}_* \subset \mathfrak{g}^* \) invariant, that is \( \text{Ad}_{\text{inv}}^* \mathfrak{g}_* \subset \mathfrak{g}_* \), for any \( g \in G \). Recall from [20] that \( \mathfrak{g}_* \) is a Banach Lie-Poisson space (whose bracket is hence given by (2.1)).

**Definition 3.2** Let \( (P, \omega) \) be a weak symplectic manifold and \( G \) a Banach Lie group satisfying the conditions above. A smooth map \( J : P \to \mathfrak{g}_* \) is a *momentum map* if whenever \( \varphi, \psi \) are locally defined smooth functions on \( \mathfrak{g}_* \), such that \( \varphi \circ J, \psi \circ J \) are locally defined elements of \( C_\omega^\infty(P) \), we have \( \{ \varphi \circ J, \psi \circ J \}_\omega = \{ \varphi, \psi \} \circ J \). Here \( \{ \cdot, \cdot \}_\omega \) denotes the Poisson bracket on functions in \( C_\omega^\infty(P) \) and \( \{ \cdot, \cdot \} \) is the Lie-Poisson bracket on \( \mathfrak{g}_* \).

Momentum maps usually appear by the following construction.

**Proposition 3.3** Let \( \Phi : G \times P \to P \) be a smooth symplectic action of the Banach Lie group \( G \) on the weak symplectic Banach manifold \((P, \omega)\). Assume that the smooth map \( J : P \to \mathfrak{g}_* \) is \( G \)-equivariant and is such that for all \( z \in \mathfrak{g} \) we have \( z \circ J \in C_\omega^\infty(P) \) and \( z_P = X_{zJ} \), where \( z_P(p) := \frac{d}{dt} \bigg|_{t=0} \Phi(\exp(tz), p) \) denote the infinitesimal generator of the action. Then \( J \) is a momentum map.

**Proof.** We proceed as in finite dimensions (see, e.g., [13]). First note that if \( \varphi \) is a smooth locally defined function on \( \mathfrak{g}_* \) and \( p \in P \), denoting \( y := d\varphi(J(p)) \in \mathfrak{g} \), we have \( d(\varphi \circ J)(p) = d(y \circ J)(p) \). The Poisson bracket evaluated at \( p \) depends only on the first derivatives of the functions at \( p \) which means that if \( \psi \) is another locally defined function on \( \mathfrak{g}_* \) and \( z := d\psi(J(p)) \in \mathfrak{g} \) we have

\[
\{ \varphi \circ J, \psi \circ J \}_\omega(p) = \{ y \circ J, z \circ J \}_\omega(p).
\]

On the other hand, the derivative at \( g = e \) of the equivariance identity \( J(q \cdot p) = \text{Ad}_{q^{-1}}^* J(p) \) for any \( q \in G \) and \( p \in P \) yields the relation \( T_q J(z_P(p)) = -\text{ad}^*_{qJ(p)}(z) \) for any \( z \in \mathfrak{g} \). Therefore, by (2.1) we get

\[
\{ \varphi, \psi \} \circ J \omega(p) = \{ y \circ J, z \circ J \}_\omega(p)
\]

which shows that \( \{ \varphi \circ J, \psi \circ J \}_\omega = \{ \varphi, \psi \} \circ J \) and hence \( J : P \to \mathfrak{g}_* \) is a momentum map by Definition 3.2.

Note that \( C_\omega^\infty(P) \) is invariant by the \( G \)-action. Indeed, the Hamiltonian vector field of the smooth function \( f \circ \Phi_g \) for \( f \in C_\omega^\infty(P) \) exists and equals \( \Phi_{gJ} X_f \), where \( \Phi_g : P \to P \) denotes the \( G \)-action on \( P \). Similarly, for any \( z \in \mathfrak{g} \), the Hamiltonian vector field of \( df(z_P) \) exists and equals \( [z_P, X_f] \).

Propositions 7.3 and 7.4 in [20] show that if the coadjoint isotropy subgroup of \( \rho \in \mathfrak{g}_* \) is a closed Lie subgroup of \( G \), the coadjoint orbit is a weak symplectic manifold and the inclusion is a momentum map in the sense of Definition 3.2. We shall give study other momentum maps in subsequent sections.

**The symplectic induced space.** Symplectic induction is a technique that associates to a given Hamiltonian \( H \)-space a Hamiltonian \( G \)-space whenever \( H \) is a Lie subgroup of the Lie group \( G \); see [6, 9, 10, 11, 24, 25] for various versions of this construction and several applications. We shall review this method below in the category of Banach manifolds and shall impose also certain splitting assumptions that are satisfied in the example studied later.
Let $G$ be a Banach Lie group with Banach Lie algebra $\mathfrak{g}$. Let $H$ be a closed Banach Lie subgroup of $G$ with Banach Lie algebra $\mathfrak{h}$. Assume that both $\mathfrak{g}$ and $\mathfrak{h}$ admit preduals $\mathfrak{g}_*$ and $\mathfrak{h}_*$, which are invariant under the coadjoint actions of $G$ and $H$, respectively (see [20] for various consequences of this assumption). Throughout this section we shall make the following hypotheses:

- $\mathfrak{h}_* \subset \mathfrak{g}_*$,
- there is an $\mathrm{Ad}_H$-invariant splitting
  \[ \mathfrak{g}_* = \mathfrak{h}_* \oplus \mathfrak{h}_+^\perp, \]  
  (3.8)
  where $\mathfrak{h}_+^\perp$ is a Banach $\mathrm{Ad}_H$-invariant subspace of $\mathfrak{g}_*$, which means that $\mathrm{Ad}_h^* \mathfrak{h}_+^\perp \subset \mathfrak{h}_+^\perp$ for any $h \in H$, where $\mathrm{Ad}^*: G \to \mathrm{Aut}(\mathfrak{g}_*)$ is the $G$-coadjoint action,

- $(\mathfrak{h}_*^\perp)^\circ = \mathfrak{h}$, where $(\mathfrak{h}_*^\perp)^\circ$ is the annihilator of $\mathfrak{h}_+^\perp$,

- the Banach Lie group $H$ acts symplectically on the weak symplectic Banach manifold $(P, \omega)$ and there is a $H$-equivariant map $J^H_P: P \to \mathfrak{h}_*$ satisfying the hypothesis of Proposition 3.3 for the Lie group $H$ and hence $J^H_P$ is a momentum map.

Dualizing the splitting (3.8), we get an $\mathrm{Ad}_H$-invariant splitting
\[ \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}_+, \]  
(3.9)
where $\mathfrak{h}_+ := (\mathfrak{h}_*)^\circ$ is the annihilator of the Banach Lie-Poisson space $\mathfrak{h}_*$.

The induction method produces a Hamiltonian $G$-space by constructing a reduced manifold in the following way. Form the product $P \times G \times \mathfrak{g}_*$ of weak symplectic manifolds, where $G \times \mathfrak{g}_*$ has the weak symplectic form
\[ \begin{aligned}
\omega_L(g, \tilde{p})((u_g, \tilde{\mu}, (v_g, \tilde{\nu})) &= \langle \tilde{\nu}, T_gL_g^{-1}u_g \rangle - \langle \tilde{\mu}, T_gL_g^{-1}v_g \rangle \\
&\quad + \langle \tilde{\rho}, [T_gL_g^{-1}u_g, T_gL_g^{-1}v_g] \rangle,
\end{aligned} \]  
(3.10)
for $g \in G$, $u_g, v_g \in T_gG$, and $\tilde{\rho}, \tilde{\mu}, \tilde{\nu} \in \mathfrak{g}_*$. This formula was introduced in [20] and it looks formally the same as the left trivialized canonical symplectic form on the cotangent bundle of a finite dimensional Lie group (see [1], §4.4, Proposition 4.4.1). From (3.10) it follows that
\[ C_{\omega_L}^\infty(G \times \mathfrak{g}_*) = \{ k \in C^\infty(G \times \mathfrak{g}_*) \mid T^*_eL_gd_1k(g, \tilde{p}) \in \mathfrak{g}_* \}, \]
where $d_1k(g, \rho) \in T^*_eG$ and $d_2k(g, \rho) \in (\mathfrak{g}_*)^* = \mathfrak{g}$ are the first and second partial derivatives of $k$. If $k \in C_{\omega_L}^\infty(G \times \mathfrak{g}_*)$, the Hamiltonian vector field $X_k \in \mathfrak{X}(G \times \mathfrak{g}_*)$ has the expression
\[ X_k(g, \tilde{p}) = \left( T_eL_gd_2k(g, \tilde{p}), \mathrm{ad}_{d_2k(g, \tilde{p})}^* \tilde{p} - T^*_eL_gd_1k(g, \tilde{p}) \right). \]  
(3.11)
Therefore the canonical Poisson bracket of $f, k \in C_{\omega_L}^\infty(G \times \mathfrak{g}_*)$ equals
\[ \{ f, k \}(g, \rho) = \langle d_1f(g, \rho), T_eL_gd_2k(g, \rho) \rangle - \langle d_1k(g, \rho), T_eL_gd_2f(g, \rho) \rangle \\
- \langle \rho, [d_2f(g, \rho), d_2k(g, \rho)] \rangle. \]  
(3.12)
The left $G$-action on $G \times \mathfrak{g}_*$ given by $g' \cdot (g, \rho) := (g'g, \rho)$ induces the momentum map $(g, \rho) \mapsto \mathrm{Ad}_{g^{-1}}^* \rho$ which is $G$-equivariant.

The weak symplectic form $\omega \oplus \omega_L \in \Omega^2(P \times G \times \mathfrak{g}_*)$ is defined by
\[ (\omega \oplus \omega_L)(p, g, \tilde{p})((a_p, T_eL_gx, \tilde{\mu}), (b_p, T_eL_g\tilde{y}, \tilde{\nu})) = \omega(p)(a_p, b_p) + \langle \tilde{\nu}, x \rangle - \langle \tilde{\mu}, \tilde{y} \rangle + \langle \tilde{\rho}, [x, \tilde{y}] \rangle, \]  
(3.13)
where $p \in P, g \in G, \tilde{\rho}, \tilde{\mu}, \tilde{\nu} \in \mathfrak{g}_*, x, \tilde{y} \in \mathfrak{g}$, and $a_p, b_p \in T_pP$.

The Banach Lie group $H$ acts on $P \times G \times \mathfrak{g}_*$ by
\[ h \cdot (p, g, \tilde{p}) := (h \cdot p, gh^{-1}, \mathrm{Ad}_{h^{-1}}^* \tilde{p}). \]  
(3.14)
The infinitesimal generator of this action defined by \( z \in \mathfrak{h} \) equals

\[
z_{P \times G \times \mathfrak{g}_*}(p, g, \tilde{\rho}) = (zp(p), -T_cL_gz, -ad_z^*\tilde{\rho})
\]

which, by (3.11) and the assumption of the existence of a momentum map induced by the action of \( H \) on \( P \), is a Hamiltonian vector field relative to the function \( z \circ (J^H_{P \times G \times \mathfrak{g}_*}(p) - \Pi\tilde{\rho}) \), where \( \Pi : \mathfrak{g}_* \to \mathfrak{h}_* \) is the projection defined by the splitting \( \mathfrak{g}_* = \mathfrak{h}_* \oplus \mathfrak{h}_+^\perp \). Therefore, the action (3.14) admits the equivariant momentum map \( J^H_{P \times G \times \mathfrak{g}_*} : P \times G \times \mathfrak{g}_* \to \mathfrak{h}_* \) given by

\[
J^H_{P \times G \times \mathfrak{g}_*}(p, g, \tilde{\rho}) = J^H_P(p) - \Pi\tilde{\rho}.
\]

The \( H \)-action on \( P \times G \times \mathfrak{g}_* \) is free and proper because \( H \) is a closed Banach Lie subgroup of \( G \). Therefore its restriction to the closed invariant subset \( (J^H_{P \times G \times \mathfrak{g}_*})^{-1}(0) \) is also free and proper. Let us assume at this point that 0 is a regular value and hence that \( (J^H_{P \times G \times \mathfrak{g}_*})^{-1}(0) \) is a submanifold. In concrete applications, such as gravity or Yang-Mills theory, the proof of the regularity of 0 is usually achieved by appealing to elliptic operator theory. With the assumption that 0 is a regular value and that for each \( (p, g, \tilde{\rho}) \in (J^H_{P \times G \times \mathfrak{g}_*})^{-1}(0) \) the map \( h \in H \mapsto h \cdot (p, g, \tilde{\rho}) := (h \cdot p, gh^{-1}, Ad_h^{-1}\tilde{\rho}) \in (J^H_{P \times G \times \mathfrak{g}_*})^{-1}(0) \) is an immersion, it follows that the quotient topological space \( M := (J^H_{P \times G \times \mathfrak{g}_*})^{-1}(0)/H \) carries a unique smooth manifold structure relative to which the quotient projection is a submersion. This underlying manifold topology is that of the quotient topological space and it is Hausdorff (see [5], Chapter III, §1, Proposition 10 for a proof of these statements). Once these topological conditions are satisfied, a technical lemma (stating that the double symplectic orthogonal of a closed subspace in a weak symplectic Banach space is equal to the original subspace) allows one to extend the original proof of the reduction theorem in finite dimensions (see [14]) to the case of weak symplectic Banach manifolds. We shall not dwell here on these technicalities because in the example of interest to us, treated later, the reduction process will be carried out by hand without any appeal to general theorems. Summarizing, we can form the induced space \( (M, \Omega_M) \) which is a smooth Hausdorff weak symplectic Banach manifold, where \( \Omega_M \) is the reduced symplectic form on \( J^H_{P \times G \times \mathfrak{g}_*}^{-1}(0)/H \).

Now note that if we denote \( \tilde{\rho} = \rho + \rho^\perp \in \mathfrak{h}_* \oplus \mathfrak{h}_+^\perp \) we get

\[
(J^H_{P \times G \times \mathfrak{g}_*})^{-1}(0) = \{(p, g, \tilde{\rho}) \in P \times G \times \mathfrak{g}_* \mid J^H_P(p) = \Pi\tilde{\rho}\}
\]

\[
= G \times \{(p, \rho) \in P \times \mathfrak{h}_* \mid J^H_P(p) = \rho\} \times \mathfrak{h}_+^\perp
\]

\[
\cong G \times P \times \mathfrak{h}_+^\perp,
\]

where the \( H \)-equivariant diffeomorphism in the last line is given by

\[
(p, \rho) \in \{(p, \rho) \in P \times \mathfrak{h}_* \mid J^H_P(p) = \rho\} \mapsto p \in P.
\]

Therefore the weak symplectic Banach manifold \( M = (J^H_{P \times G \times \mathfrak{g}_*})^{-1}(0)/H \) is diffeomorphic to the fiber bundle \( G \times_H (P \times \mathfrak{h}_+^\perp) \to G/H \) associated to \( G \to G/H \).

### The weak symplectic form on the induced space.

Let us denote by \( \pi_0 : G \times P \times \mathfrak{h}_+^\perp \to G \times_H (P \times \mathfrak{h}_+^\perp) \) the projection onto the \( H \)-orbit space. The next statement gives the weak symplectic form on \( M \).

**Proposition 3.4** The associated fiber bundle \( G \times_H (P \times \mathfrak{h}_+^\perp) \to G/H \) has a weak symplectic form \( \Omega \) given by

\[
\Omega(\pi_0(g, p, \rho^\perp), (T_{(g, p, \rho^\perp)}\pi_0(T_cL_g(x + x^\perp), a_p, \mu^\perp), T_{(g, p, \rho^\perp)}\pi_0(T_cL_g(y + y^\perp), b_p, \nu^\perp)))
\]

\[
= \omega(p)(a_p, b_p) + \langle T_pJ^H_P(b_p), x \rangle + \langle \nu^\perp, x^\perp \rangle - \langle T_pJ^H_P(a_p), y \rangle - \langle \mu^\perp, y^\perp \rangle
\]

\[
+ \langle J^H_P(p), [x, y] \rangle + \langle \rho^\perp, [x^\perp, y^\perp] \rangle + \langle J^H_P(p), \rho^\perp, [x^\perp, y^\perp] \rangle
\]

\[
= \omega(p)(a_p - x_p(p), b_p - y_p(p)) + \langle T_pJ^H_P(b_p - y_p(p)), 2x \rangle + \langle \nu^\perp + ad_x^\perp\rho^\perp, x^\perp \rangle
\]

\[
- \langle T_pJ^H_P(a_p - x_p(p)), 2y \rangle - \langle \mu^\perp + ad_x^\perp\rho^\perp, y^\perp \rangle + \langle J^H_P(p), [2x, 2y] \rangle
\]

\[
+ \langle \rho^\perp, [x^\perp, 2y] \rangle + \langle [2x, y^\perp], \rho^\perp \rangle + \langle J^H_P(p), \rho^\perp, [x^\perp, y^\perp] \rangle
\]

(3.16)

for \( g \in G, p \in P, \rho^\perp, \mu^\perp, \nu^\perp \in \mathfrak{h}_+^\perp, x, y \in \mathfrak{h}, x^\perp, y^\perp \in \mathfrak{h}^\perp \), and \( a_p, b_p \in T_pP \). The second expression uses only tangent vectors of the form

\[
(a_p - x_p(p), T_cL_g(2x + x^\perp), \mu^\perp + ad_x^\perp\rho^\perp)
\]

(3.17)
which are transversal to the $H$-orbits in the zero level set of the momentum map and hence represent the tangent space $T_{\pi_0(g,p,\rho^\perp)}M$ to the reduced manifold $M$.

**Proof.** We begin with the proof (3.16). Let $\iota_0 : G \times P \times h^\perp \to P \times G \times g$, be the inclusion $\iota_0(g,p,\rho^\perp) := (p, g, J^H_p(p) + \rho^\perp)$. For $p \in P$, $\rho^\perp, \mu^\perp, \nu^\perp \in h^\perp, g \in G$, $\bar{x} = x + x^\perp$, $\bar{y} = y + y^\perp \in g$, $x, y \in h$, $x^\perp, y^\perp \in h^\perp$, and $a_p, b_p \in T_pP$, the reduction theorem and (3.16) give

$$
\Omega(\pi_0(g,p,\rho^\perp))(T_{(g,p,\rho^\perp)}\pi_0(T_c L_g \bar{x}, a_p, \mu^\perp, T_{(g,p,\rho^\perp)}\pi_0(T_c L_g \bar{y}, b_p, \nu^\perp))
= \iota^*_0(\omega \otimes \omega_z)(p, g, \rho^\perp)(a_p, T_c L_g \bar{x}, \mu^\perp, b_p, T_c L_g \bar{y}, \nu^\perp))
= (\omega \otimes \omega_z)(p, g, J^H_p(p) + \rho^\perp)((a_p, T_c L_g \bar{x}, \mu^\perp, b_p, T_c L_g \bar{y}, \nu^\perp))
= \omega(p)(a_p, b_p) + \langle T_pJ^H_p(b_p), \nu^\perp, x + x^\perp - \langle T_pJ^H_p(a_p) + \mu^\perp, y + y^\perp
+ \langle J^H_p(p) + \rho^\perp, [x + x^\perp, y + y^\perp] \rangle

Since $[x + x^\perp, y + y^\perp] = [x, y] + [x^\perp, y] + [x^\perp, y^\perp] + [x, y^\perp]$, $[x, y] \in h = (h^\perp)^\circ$, $[x^\perp, y] \in h^\perp = (h^\perp)^\circ$ (because the splitting $g = h \oplus h^\perp$ is $Ad^*_H$-invariant), $\rho^\perp \in h^\perp$, and $J^H_p(p) \in h_*$, the second term becomes

$$
\langle J^H_p(p) + \rho^\perp, [x + x^\perp, y + y^\perp] \rangle = \langle J^H_p(p), [x, y] \rangle + \langle \rho^\perp, [x^\perp, y] \rangle + \langle \rho^\perp, [x, y^\perp] \rangle
+ \langle J^H_p(p) + \rho^\perp, [x^\perp, y^\perp] \rangle

Since $T_pJ^H_p(b_p) \in h_*$, $\nu^\perp \in h^\perp$, and $x^\perp = h^\perp$, the second term becomes

$$
\langle T_pJ^H_p(b_p) + \nu^\perp, x + x^\perp \rangle = \langle T_pJ^H_p(b_p), x \rangle + \langle \nu^\perp, x \rangle

Similarly, the third term is

$$
\langle T_pJ^H_p(a_p) + \mu^\perp, y + y^\perp \rangle = \langle T_pJ^H_p(a_p), y \rangle + \langle \mu^\perp, y \rangle

Thus we get

$$
\Omega(\pi_0(g,p,\rho^\perp))(T_{(g,p,\rho^\perp)}\pi_0(T_c L_g \bar{x}, a_p, \mu^\perp, T_{(g,p,\rho^\perp)}\pi_0(T_c L_g \bar{y}, b_p, \nu^\perp))
= \omega(p)(a_p, b_p) + \langle T_pJ^H_p(b_p), x \rangle + \langle \nu^\perp, x \rangle - \langle T_pJ^H_p(a_p), y \rangle - \langle \mu^\perp, y \rangle
+ \langle J^H_p(p), [x, y] \rangle + \langle \rho^\perp, [x^\perp, y] \rangle + \langle \rho^\perp, [x, y^\perp] \rangle + \langle J^H_p(p) + \rho^\perp, [x^\perp, y^\perp] \rangle

which proves (3.16).

We want to simplify this expression by taking advantage of the $H$-action on the zero level set of the momentum map. For $x \in h$ we have by $H$-equivariance of $J^H_p$ and the $Ad^*_H$-invariance of the splitting $g_* = h_\ast \oplus h^\perp$

$$
\frac{d}{dt}
\left|_{t=0}
\exp tx \cdot p, g \exp(-tx), Ad^*_{\exp(-tx)}(J^H_p(p) + \rho^\perp)
\right|
= \frac{d}{dt}
\left|_{t=0}
\exp tx \cdot p, g \exp(-tx), J^H_p(exp tx \cdot p) + Ad^*_{\exp(-tx)}(\rho^\perp)
\right|
= (x_P(p), -T_c L_g x, T_P J^H_p(x_P(p)) - ad^*_x \rho^\perp)

Now decompose

$$
(a_p, T_c L_g(x + x^\perp), T_p J^H_p(a_p) + \mu^\perp)
= (x_P(p), -T_c L_g x, T_P J^H_p(x_P(p)) - ad^*_x \rho^\perp)
+ (a_p - x_P(p), T_c L_g(x + x^\perp), T_P J^H_p(a_p - x_P(p)) + \mu^\perp + ad^*_x \rho^\perp)

Since the form $\Omega$ does not depend on the first summand, this means that we can replace everywhere in (3.16) $a_p$ by $a_p - x_P(p)$, $x$ by $2x$, and $\mu^\perp$ by $\mu^\perp + ad^*_x \rho^\perp$. Similarly, we can replace $b_p$ by $b_p - y_P(p)$, $y$ by $2y$, and $\nu^\perp$ by $\nu^\perp + ad^*_y \rho^\perp$. Thus (3.16) becomes

$$
\omega(p)(a_p - x_P(p), b_p - y_P(p)) + \langle T_pJ^H_p(b_p - y_P(p)), 2x \rangle + \langle \nu^\perp + ad^*_y \rho^\perp, x^\perp \rangle
- \langle T_pJ^H_p(a_p - x_P(p)), 2y \rangle + \langle \mu^\perp + ad^*_x \rho^\perp, y^\perp \rangle + \langle J^H_p(p), [2x, 2y] \rangle
+ \langle \rho^\perp, [x^\perp, 2y] \rangle + \langle [2x, y^\perp] \rangle + \langle J^H_p(p) + \rho^\perp, [x^\perp, y^\perp] \rangle

which proves (3.17).
Remark. If \( H = G \), then one can verify directly that the map \( \Psi : G \times_H (P \times \{0\}) \to P \) given by \( \Psi(x_0(p,0)) := g \cdot p \) is a diffeomorphism between the weak symplectic manifolds \((G \times_H (P \times \{0\}), \Omega)\) (the induced space) and \((P, \omega)\) (the original manifold).

The momentum map on the induced space. Now we shall construct a \( G \)-action on the induced space \((G \times_H (P \times H^+_1), \omega)\) and a \( G \)-equivariant momentum map \( J_M^G : G \times_H (P \times H^+_1) \to \mathfrak{g}_* \).

The Banach Lie group \( G \) acts on \( G \times P \times H^+_1 \) by \( g' \cdot (g,p,\rho^+) := (g'g, p, \rho^+) \). This \( G \)-action commutes with the \( H \)-action and so \( G \) acts on the induced space \((G \times_H (P \times H^+_1), \omega)\) by \( g' \cdot [g,p,\rho^+] := [g'g, p, \rho^+] \). It is routine to verify that this action preserves the weak symplectic form \( \Omega \) and that the map

\[
J_M^G([g,p,\rho^+]) = \text{Ad}_g^{-1}(J_H^H(p) + \rho^+) \tag{3.18}
\]

satisfies the hypotheses of Proposition 3.5. We conclude hence the following result.

Proposition 3.5 The map \( J_M^G : G \times_H (P \times H^+_1) \to \mathfrak{g}_* \) given by (3.18) is a \( G \)-equivariant momentum map.

The goal of the induction construction has now been achieved: starting with the Hamiltonian \( H \)-space \((P, \omega)\), where \( H \) is a closed Lie subgroup of a Lie group \( G \), a new Hamiltonian \( G \)-space has been constructed, namely \((G \times_H (P \times H^+_1), \Omega)\).

4 Induction and coinduction from \( L^1(\mathcal{H}) \)

The Banach Lie-Poisson space \( L^1(\mathcal{H}) \). The Banach space of trace class operators \((L^1(\mathcal{H}), \| \cdot \|_1)\) on a separable Hilbert space \( \mathcal{H} \) has a canonical Banach Lie-Poisson bracket defined by

\[
\{f, g\}(\rho) = \text{Tr}(\rho[Df(\rho), Dg(\rho)]), \tag{4.1}
\]

where \( f, g \in C^\infty(L^1(\mathcal{H})) \) and the Fréchet derivatives \( Df(\rho), Dg(\rho) \) are regarded as elements of the Banach Lie algebra \((L^\infty(\mathcal{H}), \| \cdot \|_\infty)\) of bounded operators on \( \mathcal{H} \), identified with the dual of \( L^1(\mathcal{H}) \) by the strongly nondegenerate pairing

\[
\langle \rho, x \rangle = \text{Tr}(\rho x), \quad \text{for} \quad \rho \in L^1(\mathcal{H}), \ x \in L^\infty(\mathcal{H}). \tag{4.2}
\]

Hamilton’s equations defined by the Poisson bracket (4.1) are easily verified to be given in Lax form (see [20] for details)

\[
\frac{d\rho}{dt} = [Dh(\rho), \rho]. \tag{4.3}
\]

The orthonormal basis \( \{|n\}_{n=0}^\infty \) of \( \mathcal{H} \), that is, \( \langle n|m \rangle = \delta_{nm} \) for \( n, m \in \mathbb{N} \cup \{0\} \), induces the Schauder basis \( \{|n\rangle \langle m|\}_{n,m=0}^\infty \) of \( L^1(\mathcal{H}) \) since it is orthonormal in the Hilbert space \( L^2(\mathcal{H}) \) of Hilbert-Schmidt operators and \( L^1(\mathcal{H}) \subset L^2(\mathcal{H}) \). Thus, every trace class operator \( \rho \in L^1(\mathcal{H}) \) can be uniquely expressed as

\[
\rho = \sum_{n,m=0}^\infty \rho_{nm}|n\rangle \langle m|, \tag{4.4}
\]

where the series is convergent in the \( \| \cdot \|_1 \) topology. The coordinates \( \rho_{nm} \in \mathbb{R} \) are given by \( \rho_{nm} = \text{Tr}(\rho|m\rangle \langle n|) \). The rank one projectors \(|l\rangle \langle k|\) thought of as elements of \( L^\infty(\mathcal{H}) \), by giving their values on the Schauder basis of \( L^1(\mathcal{H}) \) as \( \text{Tr}(|l\rangle \langle k| \langle n|m \rangle) = \delta_{nm} \delta_{lm} \) form a biorthogonal family of functionals (see [13]) in \( L^\infty(\mathcal{H}) \) associated to the given Schauder basis \( \{|n\rangle \langle m|\}_{n,m=0}^\infty \) of \( L^1(\mathcal{H}) \). Therefore, each bounded operator \( x \in L^\infty(\mathcal{H}) \) can be uniquely expressed as

\[
x = \sum_{l,k=0}^\infty x_{lk} |l\rangle \langle k|, \tag{4.5}
\]

where the series is convergent in the \( w^* \)-topology. The coordinates \( x_{lk} \in \mathbb{R} \) are also given by \( x_{lk} = \text{Tr}(x|k\rangle \langle l|) \). Recall that \( w^* \)-convergence of the series (4.5) means that the numerical series

\[
\sum_{l,k=0}^\infty x_{lk} \text{Tr}(\rho|l\rangle \langle k|) = \sum_{l,k=0}^\infty x_{lk} \rho_{kl} = \text{Tr}(xp)
\]
is convergent for any \( \rho \in L^1(\mathcal{H}) \).

Since the separable Hilbert space \( \mathcal{H} \) is fixed throughout this paper we shall simplify the notation by writing \( L^1 := L^1(\mathcal{H}) \) and \( L^\infty := L^\infty(\mathcal{H}) \).

**Shift operator notation.** The *shift operator*  

\[
S := \sum_{n=0}^{\infty} |n\rangle\langle n+1|,
\]

and its adjoint  

\[
S^T := \sum_{n=0}^{\infty} |n+1\rangle\langle n|,
\]

turn out to give a very convenient coordinate description of various objects that we shall study in this paper. Note that the matrix of \( S \) has all entries of the upper diagonal equal to one and all other entries equal to zero whereas the matrix of \( S^T \) has all entries of the lower diagonal equal to one and all other entries equal to zero. To facilitate various subsequent computations, we note that  

\[
S^k(S^T)^k = I, \quad (S^T)^kS^k = I - \sum_{i=0}^{k-1} p_i, \quad \text{for} \quad k = 1, 2, \ldots ,
\]

where \( p_i = |i\rangle\langle i| : \mathcal{H} \to \mathcal{H} \) are the orthogonal projectors on \( \mathbb{R}|i\rangle \subset \mathcal{H} \) for any \( i \in \mathbb{N} \cup \{0\} \). Let \( L^\infty_0 \subset L^\infty \) and \( L^1_0 \subset L^1 \) denote the closed subspaces of diagonal operators and define the bounded linear operators \( s, \tilde{s} \) on both \( L^\infty_0 \) and \( L^1_0 \) by  

\[
\begin{align*}
Sx &= s(x)S \quad \text{or} \quad xS^T = S^T s(x) \quad \text{or} \quad xS = S\tilde{s}(x)\end{align*}
\]

for \( x \in L^\infty_0 \) or \( x \in L^1_0 \). The effect of the map \( s \) is that the \( i \)th coordinate of \( s(x) \) equals the \((i+1)\)st coordinate of \( x \), that is, \( s(x_0, x_1, x_2, \ldots , x_n, \ldots) := (x_1, x_2, \ldots , x_n, \ldots) \) for any \( (x_0, x_1, x_2, \ldots , x_n, \ldots) \in \ell^\infty \cong L^\infty_0 \). Similarly, the effect of the map \( \tilde{s} \) is that the \( i \)th coordinate of \( \tilde{s}(x) \) equals the \((i-1)\)st coordinate of \( x \) and the zero coordinate of \( \tilde{s}(x) \) is zero, that is, \( \tilde{s}(0, x_1, x_2, \ldots , x_n, \ldots) := (0, x_0, x_1, x_2, \ldots , x_n, \ldots) \). Thus  

\[
s^k \circ \tilde{s}^k = \text{id} \quad \text{and} \quad \tilde{s}^k \circ s^k = M_{1-\sum_{i=0}^{k-1} p_i}, \quad k = 1, 2, \ldots ,
\]

where \( M_y : L^\infty_0 \to L^\infty_0 \) is defined by \( M_y(x) := yx \) for any \( y \in L^\infty_0 \). The following identities are useful in several computations later on:  

\[
\text{Tr}(\rho s(x)) = \text{Tr}(\tilde{s}(\rho)x) \quad \text{and} \quad \text{Tr}(s(\rho)x) = \text{Tr}(\rho\tilde{s}(x))
\]

for any \( \rho \in L^1_0 \) and \( x \in L^\infty_0 \), which means that \( s \) and \( \tilde{s} \) are mutually adjoint operators.

Any \( x \in L^\infty \) and \( \rho \in L^1 \) can be written as  

\[
x = \sum_{j=1}^{\infty} (S^T)^j x_{-j} + x_0 + \sum_{i=1}^{\infty} x_i S^i,
\]

\[
\rho = \sum_{j=1}^{\infty} (S^T)^j \rho_j + \rho_0 + \sum_{i=1}^{\infty} \rho_{-i} S^i,
\]

where \( x_i, x_0, x_{-j} \in L^\infty \) and \( \rho_j, \rho_0, \rho_{-i} \in L^1 \). *Note the different conventions:* the indices of the lower diagonals for the bounded operators are negative whereas for the trace class operators they are positive. This convention simplifies many formulas later on.

The expressions (4.12) and (4.13) suggest the introduction, for every \( k \in \mathbb{Z} \), of the Banach subspaces  

\[
L^\infty_k := \{ \rho \in L^\infty \mid \rho_{nm} = 0 \text{ for } m \neq n + k \} \subset L^\infty
\]

\[
L^1_k := \{ \rho \in L^1 \mid \rho_{nm} = 0 \text{ for } m \neq n + k \} \subset L^1
\]

consisting of operators whose only non-zero elements lie on the \( k \)th diagonal. We have the following Schauder decompositions  

\[
L^\infty = \bigoplus_{k \in \mathbb{Z}} L^\infty_k \quad \text{and} \quad L^1 = \bigoplus_{k \in \mathbb{Z}} L^1_k.
\]
See [22] Ch. III, §15, namely Definition 15.1 (page 485), Definition 15.3 (page 487), and Theorem 15.1 (page 489) for a detailed discussion of this concept and generalizations. The duality relations between the various spaces $L_1^\infty$ and $L_k^1$ is given by

$$\text{Tr}(\rho_k x_n) = \delta_{kn} \text{Tr}(\rho_k x_k) \quad \text{if} \quad \rho_k \in L_k^1 \quad \text{and} \quad x_n \in L_{-n}^\infty. \quad (4.17)$$

Finally, note that if $k \geq 0$ then $S^k \in L_k^\infty$, $(S^T)^k \in L_k^\infty$, and

$$S^l (S^T)^j = \begin{cases} S^{l-j}, & \text{if} \quad l \geq j \\ (S^T)^{j-l}, & \text{if} \quad l \leq j \end{cases} \quad (4.18)$$

which implies

$$\langle \rho, x \rangle = \sum_{k \in \mathbb{Z}} \text{Tr} \rho_k x_i \quad (4.19)$$

if $\rho$ and $x$ are expressed in the form $[1.14]$ and $[1.12]$.

**Banach subspaces of $L_1^1(\mathcal{H})$ and $L_1^\infty(\mathcal{H})$**. Given the Schauder basis $\{|n\rangle\langle n|\}_{n,m=0}^\infty$ of $L_1^1$ (or biorthogonal family of $L_1^\infty$) inducing the direct sum splitting $[4.16]$, define the **transposition operator** $T : L_1^1 \to L_1^1$ (or $T : L_1^\infty \to L_1^\infty$) by $(\rho^T)_{ij} = \rho_{ji}$, for any $i, j \in \mathbb{N} \cup \{0\}$. We construct the following Banach subspaces of $L_1^1$:

- $L_1^- := \oplus_{k=-\infty}^0 L_k^1$ and $L_1^+ := \oplus_{k=0}^\infty L_k^1$
- $L_S^1 := \{\rho \in L_1^1 \mid \rho = \rho^T\}$ and $L_A^1 := \{\rho \in L_1^1 \mid \rho = -\rho^T\}$
- $L_{-k}^- := \oplus_{i=-k+1}^{i=0} L_i^1$ and $L_{+k}^1 := \oplus_{i=0}^{i=k-1} L_i^1$, for $k \geq 1$
- $L_{-k}^1 := \oplus_{i=-\infty}^{-k} L_i^1$ and $L_{+k}^1 := \oplus_{i=\infty}^{i=k} L_i^1$, for $k \geq 1$
- $L_S^1 := L_S \cap (L_{+k} + L_{-k}^1)$ and $L_A^1 := L_A \cap (L_{+k}^1 + L_{-k}^-)$, for $k \geq 1$.

Relative to operator multiplication, $L_{-k}^1$ is an ideal in $L_1^1$, $L_{+k}^1$ is an ideal in $L_1^1$, but neither is an ideal in $L_1^1$. Therefore, relative to the commutator bracket, the same is true in the associated Banach Lie algebras.

Similarly, using the biorthogonal family of functionals $\{|l\rangle\langle k|\}_{l,k=0}^\infty$ in $L_1^\infty$ inducing the direct sum splitting $[4.16]$, we construct the following Banach subspaces of $L_1^\infty$:

- $L_1^\infty := \oplus_{k=-\infty}^0 L_k^\infty$ and $L_1^\infty := \oplus_{k=0}^\infty L_k^\infty$
- $L_S^\infty := \{x \in L_1^\infty \mid x^T = x\}$ and $L_A^\infty := \{x \in L_1^\infty \mid x^T = -x\}$
- $L_{-k}^\infty := \oplus_{i=-k+1}^{i=0} L_i^\infty$ and $L_{+k}^\infty := \oplus_{i=0}^{i=k-1} L_i^\infty$, for $k \geq 1$
- $L_{-k}^\infty := \oplus_{i=-\infty}^{-k} L_i^\infty$ and $L_{+k}^\infty := \oplus_{i=\infty}^{i=k} L_i^\infty$, for $k \geq 1$
- $L_S^\infty := L_S \cap (L_{+k}^\infty + L_{-k}^-)$ and $L_A^\infty := L_A \cap (L_{+k}^- + L_{-k}^\infty)$, for $k \geq 1$.

The following splittings of Banach spaces of trace class operators

$$L_1^1 = L_1^1 \oplus I_{-1}^1, \quad L_1^1 = L_1^1 \oplus I_{+1}^1, \quad L_1^- = L_{-k}^1 \oplus I_{+k}^1 \quad (4.20)$$

and of bounded operators

$$L_1^\infty = L_1^\infty \oplus I_{-1}^\infty, \quad L_1^\infty = L_1^\infty \oplus L_A^\infty, \quad L_1^\infty = L_{+k}^\infty \oplus L_{+k}^\infty \quad (4.21)$$

will be used below. The strongly nondegenerate pairing $[12]$ relates the splittings $[4.20]$ and $[4.21]$ by

$$\begin{align*}
(L_1^1)^* \cong (I_{+1}^1)^* \cong L_{-1}^\infty, & \quad (L_1^1)^* \cong (I_{-1}^1)^* \cong L_{+1}^\infty, & \quad (L_1^1)^* \cong (I_{+k}^1)^* \cong L_{+k}^\infty, \\
(I_{+1}^1)^* \cong (L_1^1)^* \cong L_{-1}^\infty, & \quad (I_{-1}^1)^* \cong (L_1^1)^* \cong L_{+1}^\infty, & \quad (I_{+k}^1)^* \cong (L_1^1)^* \cong L_{+k}^\infty
\end{align*}$$

where, as usual, $\circ$ denotes the annihilator of the Banach subspace in the dual of the ambient space.
The splittings (4.20) and (4.21) define six projectors of $L^1$ and $L^\infty$, respectively. Let $P^i_0, P^i_1, P^i_\infty : L^1 \to L^1$ be the projectors whose ranges are $I^i_{-1}, L^i_0,$ and $I^i_{+1}$ defined by the splitting $L^1 = I^i_{-1} \oplus L^i_0 \oplus I^i_{+1}$. In particular $P^i_+ + P^i_0 + P^i_\infty = \mathbb{I}$. Let $P^i_{-k} : L^1 \to L^i_{-k}$ be the projector whose range is $L^i_{-k}$ defined by the splitting $L^1_{-k} = L^i_{-k} \oplus I^i_{-k}$. Define the six projectors

$$
\begin{align*}
R_- := P^i_0 + P^i_\infty & , \\
R_+ := P^i_1 & , \\
R_S := P^i_0 + T \circ P^i_1 & , \\
R_{S+} := P^i_1 - T \circ P^i_1 & , \\
R_{-k} := P^i_{-k} & , \\
R_{ik} := R_-|_{L^i_{-k}} & - P_{-k}.
\end{align*}
$$

(4.23)

associated to the splittings (4.20). The order of presentation of these projectors corresponds to the order of the splittings in (4.20).

Similarly, the six projectors associated to the dual splittings (4.21) are given by

$$
\begin{align*}
R^*_- := P^\infty_0 + P^\infty_\infty & , \\
R^*_+ := P^\infty_1 & , \\
R^*_S := P^\infty_0 + P^\infty_\infty + T \circ P^\infty_1 & , \\
R^*_{S+} := P^\infty_1 - T \circ P^\infty_1 & , \\
R^*_{-k} := P^\infty_{-k} & , \\
R^*_{ik} := R^*_-|_{L^\infty_{-k}} & - P^*_{-k}.
\end{align*}
$$

(4.24)

where $P^\infty_0, P^\infty_1, P^\infty_\infty : L^\infty \to L^\infty$ are the projectors whose ranges are $I^\infty_{-1}, L^\infty_0$, $I^\infty_{+1}$ defined by the splitting $L^\infty = I^\infty_{-1} \oplus L^\infty_0 \oplus I^\infty_{+1}$ and $P^\infty_{-k} : L^\infty_{-k} \to L^\infty_{-k}$ is the projector with range $L^\infty_{-k}$ defined by the splitting $L^\infty_{-k} = L^\infty_{-k} \oplus I^\infty_{-k}$.

All Banach spaces appearing in (4.21), with the exception of $L^\infty_k$, are Banach subalgebras of $L^\infty$ or $L^\infty_-$ whereas $I^\infty_k$, for $k \in \mathbb{N}$, are ideals of the Banach algebra $L^\infty$ (but not of $L^\infty$). Therefore, $I^\infty_{-k}$ define a filtration of $L^\infty$ and hence $L^\infty_{-k} \cong L^\infty_0/I^\infty_{-k}$ inherits the structure of an associative Banach algebra. Thus all these associative Banach algebras are naturally Banach Lie algebras. The same considerations apply to the Banach ideals $I^\infty_{-k} \subset L^\infty$.

It will be useful in our subsequent development to distinguish between the projectors defined in (4.20) and (4.24) and the corresponding maps onto their ranges. We shall denote by $\pi_- \circ \pi_+ \circ \pi_S$ and $\pi_{S+}$ the maps on $L^1$ equal to $R_-, R_+, R_S$, and $R_{S+}$ but viewed as taking values in $\text{im} R_-, \text{im} R_+, \text{im} R_S \subseteq L^\infty_0$, and $\text{im} R_{S+} \subseteq I^\infty_{-1}$, respectively. Similarly, denote by $\pi_{-k} \circ \pi_{ik}$ the maps on $L^1_{-k}$ equal to $R_{-k}$ and $R_{ik}$, but viewed as having values in $\text{im} R_{-k} \subseteq L^\infty_{-k}$ and $\text{im} R_{ik} \subseteq I^\infty_{-k}$, respectively. For the projectors on $L^\infty$ we shall denote by $\pi^\infty_-, \pi^\infty_+, \pi^\infty_S$, and $\pi^\infty_{S+}$ the maps equal to $R^*_, R^*+, R^*_S$, and $R^*_{S+}$ viewed as having values in $\text{im} R^*_- \subseteq L^\infty_0$, $\text{im} R^*_+ \subseteq I^\infty_{-1}$, $\text{im} R^*_S \subseteq L^\infty_0$, and $\text{im} R^*_{S+} \subseteq I^\infty_{-1}$, respectively. Finally, let $\pi^\infty_{+k}$ and $\pi^\infty_{ik}$ denote the maps on $L^\infty_{+k}$ equal to $R^*_{+k}$ and $R^*_{ik}$ viewed as having values in $\text{im} R^*_{-k} \subseteq L^\infty_{-k}$ and $\text{im} R^*_{ik} \subseteq I^\infty_{-k}$, respectively.

**Associated Banach Lie groups.** Note that the Banach Lie group

$$GL^\infty := \{ x \in L^\infty \mid x \text{ is invertible} \}$$

(4.25)

has Banach Lie algebra $L^\infty$ and is open in $L^\infty$. Define the closed Banach Lie subgroup of upper triangular operators in $GL^\infty$ by

$$GL^\infty_+ := GL^\infty \cap L^\infty_+.$$  

(4.26)

Since $GL^\infty_+$ is open in $L^\infty_+$, we can conclude that its Banach Lie algebra is $L^\infty_+$. Define the closed Banach Lie subgroup of orthogonal operators in $GL^\infty$ by

$$O^\infty := \{ x \in L^\infty \mid xx^T = x^T x = \mathbb{I} \}.$$  

(4.27)

The Banach Lie algebra $L^\infty_+$ of $O^\infty$ consists of all bounded skew-symmetric operators.

Denote by

$$GL^\infty_{+,k} := (I + I^\infty_{+,k}) \cap GL^\infty_+$$

(4.28)

the open subset of $I + I^\infty_{+,k}$ formed by the group of all bounded invertible upper triangular operators whose strictly upper $(k-1)$-diagonals are identically zero and whose diagonal is the identity. This is a closed normal Banach Lie subgroup of $GL^\infty_+$ whose Lie algebra is the closed ideal $I^\infty_{+,k}$.

**Remark.** Unlike the situation encountered in finite dimensions, the set $I + I^\infty_{+,k}$ does not consist only of invertible bounded linear isomorphisms. An example of an operator in $I + I^\infty_{+,k}$ that is not onto is given by $I - S^2$, where $S$ is the shift operator defined in (4.6), since $\sum_{n=0}^{\infty} \frac{1}{n+1} |n| \notin \text{im}(I - S^2)$.  

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Returning to the general case, define the product
\[ x \circ_k y := \sum_{i=0}^{k-1} \left( \sum_{i=0}^{l} x_i s^i (y_{i-1}) \right) S^l \quad (4.29) \]
of the elements \( x = \sum_{i=0}^{k-1} x_i S^i \) and \( y = \sum_{i=0}^{k-1} y_i S^i \in L_+^\infty,k \), where \( x_i, y_i \) are diagonal operators. Relative to \( \circ_k \), the Banach space \( L_+^\infty,k \) is an associative Banach algebra with unity. It is easy to see that the projection map \( \pi_+^\infty : L_+^\infty \to (L_+^\infty,k, \circ_k) \) is an associative Banach algebra homomorphism whose kernel is \( I_+^\infty,k \). So, it defines a Banach algebra isomorphism \( [\pi_+^\infty] : L_+^\infty/I_+^\infty,k \to (L_+^\infty,k, \circ_k) \) of the factor Banach algebra \( L_+^\infty/I_+^\infty,k \) with \( (L_+^\infty,k, \circ_k) \).

The associative algebra \( L_+^\infty,k \) with the commutator bracket
\[ [x, y]_k := x \circ_k y - y \circ_k x = \sum_{i=0}^{k-1} \sum_{i=0}^{l} (x_i s^i (y_{i-1}) - y_i s^i (x_{i-1})) S^l \quad (4.30) \]
is the Banach Lie algebra of the group
\[ GL_+^\infty,k = \left\{ g = \sum_{i=0}^{k-1} g_i S^i \mid g_i \in L_0^\infty, [g_0] \geq \varepsilon (g_0) \| \text{ for some } \varepsilon (g_0) > 0 \right\} \quad (4.31) \]
of invertible elements in \( (L_+^\infty,k, \circ_k) \).

**Remark.** It is important to note that invertibility in the Banach algebra \( (L_+^\infty,k, \circ_k) \) does not mean invertibility of the operator on \( H \). For example, \( I - S^2 \in GL_+^\infty,3 \), that is, \( I - S^2 \) is an invertible element in \( (L_+^\infty,k,\circ_k) \), but \( I - S^2 \) is not an invertible operator, as noted in the previous remark.

Note that \( (L_+^\infty,k, [\cdot, \cdot]_k) \) is not a Banach Lie subalgebra of \( L_+^\infty \). Since \( \pi_+^\infty : L_+^\infty \to L_+^\infty,k \) is also a Banach Lie algebra homomorphism one has
\[ [x, y]_k = \pi_+^\infty ([x, y]) \quad (4.32) \]

Note that \( \pi_+^\infty (GL_+^\infty) \subset GL_+^\infty,k \), since every invertible operator in \( L_+^\infty \) is mapped by the homomorphism \( \pi_+^\infty \) to an invertible element of \( L_+^\infty,k \). Moreover, if \( x \in \pi_+^\infty (GL_+^\infty) \subset GL_+^\infty,k \), then
\[ \left( \pi_+^\infty (GL_+^\infty) \right)^{-1} (x) = \left\{ g (I + \psi) \mid I + \psi \in GL_+^\infty,k \right\} \quad (4.33) \]
for some \( g \in \pi_+^\infty (GL_+^\infty) \).

Indeed, if \( g' \in \left( \pi_+^\infty (GL_+^\infty) \right)^{-1} (x) \), then there exists some \( g \psi \in I_+^\infty,k \), since \( g \) is invertible, such that \( g^{-1} g' = I + \psi \in GL_+^\infty,k \). The next proposition shows that the restriction of \( \pi_+^\infty \) to \( GL_+^\infty,k \) has range equal to \( GL_+^\infty,k \).

**Proposition 4.1** The Banach Lie group homomorphism \( \pi_+^\infty (GL_+^\infty) : GL_+^\infty \to GL_+^\infty,k \) is surjective and induces a Banach Lie group isomorphism \( \pi_+^\infty : GL_+^\infty,k \to GL_+^\infty,k \) for any \( k = 1, 2, \ldots \).

**Proof.** To show that \( \pi_+^\infty : GL_+^\infty \to GL_+^\infty,k \) is surjective is equivalent to proving that for any \( g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} \in GL_+^\infty,k \) there exists \( \varphi_k \in I_+^\infty,k \) such that
\[ g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \varphi_k \in GL_+^\infty,k \quad (4.34) \]

Assume for the moment that (4.33) holds. We shall draw a consequence from it. By (4.31), \( g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} \in GL_+^\infty,k \) if and only if \( g_0 \) is invertible. Decompose \( \varphi_k = \alpha_k S^k g_0 + \alpha_{k+1} \), where \( \alpha_{k+1} \in I_+^\infty,k+1 \). Choosing \( N \in \mathbb{N} \) large enough so that \( I - \frac{1}{N} \alpha_k S^k \in GL_+^\infty,k \), we obtain
\[ GL_+^\infty \ni \left( I - \frac{1}{N} \alpha_k S^k \right)^N (g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \alpha_k S^k g_0 + \alpha_{k+1}) = g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \varphi_{k+1}, \]
where
\[
\varphi_{k+1} = \left( \sum_{j=2}^{N} \binom{N}{j} (-1)^{j-1} \frac{1}{N^j} (\alpha_k S^k)^j \right) (g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \alpha_k S^k g_0 + \alpha_{k+1}) + \alpha_{k+1} - \alpha_k S^k \left( g_1 S + \cdots + g_{k-1} S^{k-1} + \alpha_k S^k g_0 + \alpha_{k+1} \right) \in I_{+,k+1}^\infty. 
\]

(4.35)

Therefore, if \( g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \varphi \in GL_+^\infty \) for some \( \varphi \in I_{+,k}^\infty \), then there exists some \( \varphi_{k+1} \in I_{+,k+1}^\infty \) such that \( g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \varphi_{k+1} \in GL_+^\infty \).

Now we prove the proposition by induction on \( k \).

If \( k = 1 \), then \( g_0 \in GL_+^\infty \) by definition. Next, let us assume that (4.33) holds. As we just saw, it follows that (4.33) holds. Consider then \( g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + g_k S^k \in GL_+^\infty \) and decompose it in the group \( GL_+^\infty \) as \( g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + g_k S^k = (1 + g_k S^k g_0^{-1}) \circ_k \circ_k \cdots \circ_k \left( g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} \right) \). Let us assume, that \( \|g_k\| < \min(1, \|g_0\|) \) which implies that \( \|g_k S^k g_0^{-1}\| < 1 \) and hence that \( 1 + g_k S^k g_0^{-1} \in GL_+^\infty \). By (4.34) there exists \( \varphi_{k+1} \in I_{+,k+1}^\infty \) such that \( g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \varphi_{k+1} \in GL_+^\infty \). Thus we get

\[
(1 + g_k S^k g_0^{-1})(g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \varphi_{k+1}) = g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + g_k S^k + \psi_{k+1} \in GL_+^\infty
\]

for

\[
\psi_{k+1} = (1 + g_k S^k g_0^{-1})\varphi_{k+1} + g_k S^k g_0^{-1}(g_1 S + \cdots + g_{k-1} S^{k-1}) \in I_{+,k+1}^\infty
\]

which proves the assertion (4.33) for any element in the connected component of \( GL_+^\infty \). Since \( \{1 + g_1 S + \cdots + g_k S^k \mid g_1, \ldots, g_k \text{ diagonal operators in } L^\infty \} \) is a connected Banach Lie subgroup of the connected component of \( GL_+^\infty \), any element of \( GL_+^\infty \) can be written as a product of an element of this group and the Banach Lie subgroup \( GL_+^\infty \) of diagonal operators, it follows that (4.33) holds for any element in \( GL_+^\infty \).

In the Banach Lie group \( (GL_+^\infty, \circ_k) \), the inverse \( g^{-1} = g_0^{-1} + h_1 S + \cdots + h_{k-1} S^{k-1} \) of \( g = g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} \in GL_+^\infty \) is given by

\[
h_p = -g_0^{-1} \left[ \sum_{r=1}^{p-1} \sum_{i=1}^{r} (1 -1)^{-1} g_1 s^{i_1}(g_1^{-1} g_2) \cdots s^{i_r}(g_1^{-1} g_r) \right] s^p(g_0^{-1}),
\]

(4.36)

1 \( \leq p \leq k - 1 \), where the second sum is taken over all indices \( i_1, \ldots, i_r, j_1, \ldots, j_r \) such that \( i_1 + \cdots + i_r = p \) (equality between the \( i_q \) is permitted), \( 0 \leq i_1, \ldots, i_r \leq p, 1 \leq i_1 = j_1 < j_2 < \cdots < j_r = p - i_r \leq p - 1 \). For example, here are the first elements:

\[
\begin{align*}
   h_1 &= -g_0^{-1} g_1 s(g_0^{-1}) \\
   h_2 &= -g_0^{-1} \left[ g_2 - g_1 s(g_0^{-1} g_1) \right] s^2(g_0^{-1}) \\
   h_3 &= -g_0^{-1} \left[ g_3 - g_2 s^2(g_0^{-1} g_1) - g_1 s(g_0^{-1} g_2) + g_1 s(g_0^{-1} g_1) s^2(g_0^{-1} g_1) \right] s^3(g_0^{-1}).
\end{align*}
\]

**Coinduced Banach Lie-Poisson structures.** After these preliminary remarks and notations let us apply the results of the previous section to the Banach Lie-Poisson space \( L^1 \). We shall drop the upper indices “ind” and “coind” on the Poisson brackets because it will be clear from the context which brackets are induced and coinduced on various subspaces.

We start with points (i) of Proposition [2.2] and Proposition [2.4] So let us consider the diagram

![Diagram]

where we recall that \( \pi_S, \pi_{S+}, \pi_+ \) and \( \pi_- \) are the projections onto the ranges of \( R_S, R_{S+}, R_+ \) and \( R_- \) respectively and \( \iota_S, \iota_{S+}, \iota_+ \) and \( \iota_- \) are inclusions. We see from the above that the assumptions in part (i) of Proposition [2.4] are satisfied because \( (L_{+,1}^1)^0 = L_+^\infty \) is a Banach Lie subalgebra of \( (L^1)^* = L^\infty \). Thus we can conclude the following facts.
(i) By Proposition 2.4 (i) it follows that $L^1_{\pm}$ and $L^1_{\pm}$ are isomorphic Banach Lie-Poisson spaces with the Poisson brackets defined by formula (2.17). They are given, respectively, by

$$\{f, g\}_{S}(\sigma) = \text{Tr} (\iota_S(\sigma) [D(f \circ \pi_S)(\iota_S(\sigma)), D(g \circ \pi_S)(\iota_S(\sigma))])$$

(4.37)

for $\sigma \in L^1_S$ and $f, g \in C^\infty(L^1_S)$ and

$$\{f, g\}_{-}(\rho) = \text{Tr} (\iota_{-}(\rho) [D(f \circ \pi_{-})(\iota_{-}(\rho)), D(g \circ \pi_{-})(\iota_{-}(\rho))])$$

(4.38)

for $\rho \in L^1_-$ and $f, g \in C^\infty(L^1_-)$. The coadjoint actions (Ad$^+$)$^{-1}_{\pm}$, $\rho = \pi_{\pm}(g_{\pm}(\rho)g^{-1})$ for $\rho \in L^1_-$

(4.39)

(Ad$^S$)$^{-1}_{\pm}$, $\sigma = \pi_{\pm}(g_{\pm}(\sigma)g^{-1})$ for $\sigma \in L^1_S$

(4.40)

and $g \in GL^\infty$. Differentiating these formulas relative to $g$ at the identity, we get

$$(\text{ad}^+)^*_{\rho} = -\pi_-(\iota_{\pm}(\rho))$$

(4.41)

$$(\text{ad}^S)^*_{\sigma} = -\pi_+(\iota_{\pm}(\sigma))$$

(4.42)

for $x \in L^\infty_\pm$. The isomorphisms $\Phi_{\pm,S} : L^1_{\pm} \to L^1_{\pm}$ and $\Phi_{S,\pm} : L^1_{\pm} \to L^1_S$ are equivariant relative to these coadjoint actions, that is,

$$(\text{Ad}^S)^*_{\pm,\pm} \circ \Phi_{\pm,S} = \Phi_{S,\pm} \circ (\text{Ad}^+)^*_{\pm,\pm}$$

(4.43)

$$(\text{Ad}^+)^*_{\pm,\pm} \circ \Phi_{\pm,S} = \Phi_{\pm,S} \circ (\text{Ad}^S)^*_{\pm,\pm}$$

(4.44)

for any $g \in GL^\infty_\pm$.

(ii) By (4.22), $L^1_{+,1}$ is the predual of the two Banach Lie algebras $I_{-,1}$ and $L^\infty_A$. Thus (4.20) - (4.23), and point (ii) of Proposition 2.4 imply that $I_{+,1}$ carries two different Lie-Poisson brackets, namely by 2.17 we have

$$\{f, g\}_{+}(\rho) = \text{Tr} (\iota_+(\rho) [D(f \circ \pi_+(\iota_+(\rho)), D(g \circ \pi_+(\iota_+(\rho))])$$

(4.45)

and

$$\{f, g\}_{S,+}(\rho) = \text{Tr} (\iota_{S,+}(\rho) [D(f \circ \pi_{S,+}(\iota_{S,+}(\rho)), D(g \circ \pi_{S,+}(\iota_{S,+}(\rho))])$$

(4.46)

where $\rho \in I^1_{+,1}$, $f, g \in C^\infty(I^1_{+,1})$.

The coadjoint actions (Ad$^-)^*_{h}$ and (Ad$^A$)$^*_{g}$ of the groups GI$^\infty_{-,1}$ and O$^\infty$ respectively on $I^1_{+,1}$ are given by

$$(\text{Ad}^-)^*_{h} \rho = \pi_{+}(h_{\pm}(\rho)h^{-1})$$

(4.47)

and

$$(\text{Ad}^A)^*_{g} \rho = \pi_{S,+}(g_{\pm}(\rho)g^{-1})$$

(4.48)

where $\rho \in I^1_{+,1}$. We shall not pursue the investigation of this interesting case in this paper.
Induced Banach Lie-Poisson structures. We begin with the study of the lower triangular case. Denote by \( \iota_{-k} : L^1_{-k} \rightarrow L^1 \) the inclusion and let \( \iota_{-k}^{-1} : \iota_{-k} \left( L^1_{-k} \right) \rightarrow L^1_{-k} \) be its inverse (defined, of course, only on the range of \( \iota_{-k} \)). Then \( \iota_{-k}^{-1} : L^\infty \rightarrow L^\infty_{-k} \). Since \( \ker \iota_{-k}^{-1} = I^\infty_{-k} \) is an ideal in \( L^\infty \), by Proposition 2.1, we have \((\text{Ad}^+)^*_{g^{-1}} \iota_{-k}(L^1_{L^1}) \subset \iota_{-k}(L^1_{-k})\) for any \( g \in GL^\infty_+ \). Therefore there are \( GL^\infty_+ \) and \( L^\infty \) coadjoint actions on \( L^1_{-k} \) defined by

\[
(\text{Ad}^+)^*_{g^{-1}} \rho := \iota_{-k}^{-1} \left( \pi_- \left( g(\iota_{-k} \rho)g^{-1} \right) \right)
\]

\[\text{for } \rho \in L^1_{-k} \quad \text{and } g \in GL^\infty_+ \quad \text{(4.49)}\]

\[
(\text{ad}^+)^* \rho := \iota_{-k}^{-1} \left( \pi_- [\iota_{-k} \rho] \right)
\]

\[\text{for } \rho \in L^1_{-k} \quad \text{and } x \in L^\infty_+ \quad \text{(4.50)}\]

Since the action (4.49) is trivial for all elements of the closed normal Lie subgroup \( GI^{\infty}_{+k} \), it induces the coadjoint action of the group \( GL^\infty_{+k} \equiv GL^\infty_+ / GI^\infty_{+k} \) given by (4.49) that will be also denoted by \((\text{Ad}^+)^*\). Similarly, the Lie algebra action (4.50) is trivial for all elements in the closed ideal \( I^\infty_{+k} \) so it induces the coadjoint action of the Lie algebra \( L^\infty_{+k} \equiv L^\infty / I^\infty_{+k} \) on \( L^1_{-k} \) denoted also by \((\text{ad}^+)^*\).

One can express (4.49) and (4.50) in terms of the expansions \( \rho = \rho_0 + S^T \rho_1 + \cdots + (S^T)^{k-1} \rho_{k-1} \in L^1_{-k} \), \( x = x_0 + x_1 S + \cdots + x_{k-1} S^{k-1} \in L^\infty_{+k} \), and \( g = g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} \in GL^\infty_{+k} \) in the following way

\[
(\text{Ad}^+)^* \rho = \sum_{i,j=0,j \geq i+k}^{k-1} (S^T)^{j-i} \delta^i_j s^j(\delta^i_j g_i) \rho_j k_i h_i
\]

\[\text{where the diagonal operators } h_i \text{ are expressed in terms of the } g_i \text{ in (4.36), and (using (4.18))}\]

\[
(\text{ad}^+)^* \rho = \sum_{i,j=0,j \geq i+k}^{k-1} (S^T)^{j-i} \delta^i_j (\delta^i_j g_i - \rho_i s^j(\delta^i_j g_i)).
\]

By (4.39) and (4.19), the Lie-Poisson bracket on \( L^1_{-k} \) is given by

\[
\{f, g\} := \text{Tr} \left( \rho \left[ Df(\rho), Dg(\rho) \right] \right)
\]

\[= \sum_{i=0}^{k-1} \sum_{l=0}^{k-1} \text{Tr} \left[ \rho_i \left( \frac{\delta f}{\delta \rho_i}(\rho) s^i \left( \frac{\delta g}{\delta \rho_{i-l}}(\rho) - \frac{\delta g}{\delta \rho_i}(\rho) s^i \left( \frac{\delta f}{\delta \rho_{i-l}}(\rho) \right) \right) \right) \quad \text{(4.53)}\]

for \( f, g \in C^\infty_0(L^1_{-k}) \), where \( \frac{\delta f}{\delta \rho_i}(\rho) \) denotes the partial functional derivative of \( f \) relative to \( \rho_i \) defined by

\[Df(\rho) = \frac{\delta f}{\delta \rho_i}(\rho) + \frac{\delta f}{\delta \rho_{i+1}}(\rho) S + \cdots + \frac{\delta f}{\delta \rho_{k-1}}(\rho) S^{k-1} \]

If in the previous formulas we let \( k = \infty \) one obtains the Lie-Poisson bracket on \( L^1 \). Indeed, the Lie-Poisson bracket \( \{ f, g \} \) on \( L^1 \) given by (4.38) expressed in the coordinates \( \{ \rho_i \}_{i=0}^{\infty} \) equals (4.53) for \( k = \infty \).

**Proposition 4.2** The Lie-Poisson bracket (4.53) on \( L^1_{-k} \) coincides with the induced bracket (2.9) determined by the inclusion \( \iota_{-k} : L^1_{-k} \hookrightarrow L^1 \) and the Lie-Poisson bracket (4.38) on \( L^1 \).

**Proof.** We need to prove that the induced bracket (2.9) evaluated on two linear functionals \( x, y \in L^\infty_{+k} \) defined by \( L^\infty_{+k} \subset C^\infty_0(L^1_{-k}) \) coincides with \( \{ x, y \} \). To see this we note that \( D(x \circ \iota_{-k} \circ R_{-k})(\iota_{-k}(\rho)) = \iota_{-k}(x) \in L^\infty \), where \( \iota_{+k} : L^\infty_{+k} \rightarrow L^\infty_+ \) is the inclusion. Then, a direct verification shows that for any \( \rho \in L^1_{-k} \) we have

\[\{ x, y \}^{\text{ind}}(\rho) = \langle [\iota_{+k}x, \iota_{+k}y], \iota_{-k}(\rho) \rangle = \text{Tr} \left( \langle x, y \rangle \rho \right) = \text{Tr} \left( \langle x, y \rangle \rho \right) \quad \text{by (4.30)}. \]

Let us study now the symmetric representation of \( L^1_{-k} \) for \( k \in \mathbb{N} \cup \{ \infty \} \). This will be done by using the Banach Lie-Poisson space isomorphism \( \Phi_{S^\infty} := \pi_{S^\infty} \circ \iota_{-k} : L^1_{-k} \rightarrow L^1_{-k} \). Let \( \pi_{-k} : L^1_{-k} \rightarrow L^1_{-k} \) and \( \pi_{S,k} : L^1_{-k} \rightarrow L^1_{0,k} \) be the projections with the indicated ranges and \( \iota_{S,k} : L^1_{0,k} \rightarrow L^1_{0,k} \) the inclusion. Define \( \Phi_{S^\infty} := \pi_{S^\infty} \circ \Phi_{S^\infty} \circ \iota_{-k} : L^1_{-k} \rightarrow L^1_{-k} \). The following commutative diagram illustrates these maps:
Pushing forward the Poisson bracket \( \{ \cdot, \cdot \}_k \) on \( L_{1,k} \) by the Banach space isomorphism \( \Phi_{S,-} \) endows \( L_{S,k}^1 \) with an isomorphic Poisson structure denoted by \( \{ \cdot, \cdot \}_{S,k} \). From Propositions 2.4 and 4.2, all the maps in the diagram above are linear Poisson maps, with the exception of \( \pi_{-,k} \) and \( \pi_{S,k} \) which are not Poisson.

Recall that \( GL^\infty_+ \) acts on \( L_{1}^- \) and \( L_1^0 \) by \( (4.39) \) and \( (4.40) \) respectively, and that \( GL^\infty_{+,k} \) (and hence \( GL^\infty_{+,k} \)) acts on \( L_{1,k}^- \) by \( (4.49) \). Using the isomorphisms \( \Phi_{S,-} \) and \( \Phi_{S,-,k} \) to push forward these actions to \( L_1^k \) and \( L_1^k \) respectively, all the maps in the diagram above are also \( GL^\infty_+ \)-equivariant. Consequently, one has the \( GL^\infty_+ \)-invariant filtrations

\[
\begin{align*}
\ell_{-1}(L_{1,k}^1) & \hookrightarrow \ell_{-2}(L_{1,k}^2) \hookrightarrow \cdots \hookrightarrow \ell_{-k}(L_{1,k}^1) \hookrightarrow \ell_{-k+1}(L_{1,k}^1) \hookrightarrow \cdots \hookrightarrow L_1^1 \\
\ell_{S,1}(L_{S,k}^1) & \hookrightarrow \ell_{S,2}(L_{S,k}^2) \hookrightarrow \cdots \hookrightarrow \ell_{S,k}(L_{S,k}^1) \hookrightarrow \ell_{S,k+1}(L_{S,k}^1) \hookrightarrow \cdots \hookrightarrow L_1^1
\end{align*}
\]

of Banach Lie-Poisson spaces predual to the sequence

\[
L_1^+ \rightarrow \cdots \rightarrow L_1^{\infty,k} \rightarrow L_1^{\infty,k-1} \rightarrow \cdots \rightarrow L_1^{\infty,2} \rightarrow L_1^{\infty,1}
\]

of Banach Lie algebras in which each arrow is the surjective projector \( \pi_{+,k,k-1}^\infty : L_{+,k}^\infty \rightarrow L_{+,k-1}^\infty \) that maps \( k \)-diagonal upper triangular operators to \((k-1)\)-diagonal upper triangular operators by eliminating the \( k \)th diagonal. We have \( \pi_{+,k,k-1}^\infty \circ \pi_{+,k} = \pi_{+,k-1}^\infty \).

**5 Dynamics generated by Casimirs of \( L^1(\mathcal{H}) \)**

We begin by presenting Hamilton’s equations on \( L_{1}^- \) and \( L_1^0 \) given by arbitrary smooth functions \( h \) and \( f \) defined on the relevant Banach Lie-Poisson spaces. Using formula \((2.18)\) of Proposition \((2.2)\) one obtains Hamilton’s equations

\[
\frac{d}{dt} \rho = \pi_{-,k}(\{D(h \circ \pi_{-})(\ell_{-}(\rho)), \ell_{-}(\rho)\}) \quad \text{for} \quad \rho \in L_{1}^- \quad \text{and} \quad h \in C^\infty(L_{1}^-),
\]

\[
\frac{d}{dt} \sigma = \pi_{S,k}(\{D(f \circ \pi_{S})(\ell_{S}(\sigma)), \ell_{S}(\sigma)\}) \quad \text{for} \quad \sigma \in L_{S}^1 \quad \text{and} \quad f \in C^\infty(L_{S}^1),
\]

on the isomorphic Banach Lie-Poisson spaces \((L_{1}^-,\{\cdot,\cdot\}_-)\) and \((L_{S}^1,\{\cdot,\cdot\}_S)\); from \([11]\) we know that this isomorphism is \( \Phi_{S,-} := \pi_{S} \circ \ell_{-} : (L_{1}^-,\{\cdot,\cdot\}_-) \rightarrow (L_{S}^1,\{\cdot,\cdot\}_S) \). Therefore, if \( f \circ \Phi_{S,-} = h \) then equations \((5.1)\) and \((5.2)\) give the same dynamics. Recall that \( \pi_{-} : L_1^1 \rightarrow L_{1}^- \) and \( \pi_{S} : L_1^1 \rightarrow L_{S}^1 \) are, by definition, the projectors \( P_{1}^1 + P_{0} : L_1^1 \rightarrow L_1^1 \) and \( \pi_{S} := P_{1}^1 + P_{0} + T \circ P_{1}^1 : L_1^1 \rightarrow L_1^1 \) considered as maps on their ranges (see \((3.23)\) and the subsequent comments) and \( \ell_{-} : L_{1}^- \rightarrow L_{1}^- \), \( \ell_{S} : L_{S}^1 \rightarrow L_{S}^1 \) are the inclusions.

Now let us observe that the family of functions \( I_l \in C^\infty(L_1^1) \) defined by

\[
I_l(\rho) := \frac{1}{l} \text{Tr} \rho^l \quad \text{for} \quad l \in \mathbb{N}
\]

are Casimir functions on the Banach Lie-Poisson space \((L_1^1,\{\cdot,\cdot\})\). This follows from \((4.1)\) since one has

\[
DI_l(\rho) = \rho^{l-1} \in L_1^1 \subset L_{1}^\infty \cong (L_1^1)^*.
\]

Restricting \( I_l \) to \( \ell_{-} : L_{1}^- \rightarrow L_{1}^- \) and \( \ell_{S} : L_{S}^1 \rightarrow L_{S}^1 \) we obtain for all \( l \in \mathbb{N} \)

\[
I_l^- (\rho) := I_l(\ell_{-}(\rho)) \quad \text{for} \quad \rho \in L_{1}^- \quad (5.5)
\]

\[
I_l^S(\sigma) := I_l(\ell_{S}(\sigma)) \quad \text{for} \quad \sigma \in L_{S}^1 \quad (5.6)
\]
According to Corollary 2.3(i), (5.5) and (5.6) form two infinite families of integrals in involution
\[ \{I_{-}^{l}, I_{m}\} = 0 \quad \text{and} \quad \{I_{-}^{S}, I_{m}^{S}\} = 0 \quad \text{for} \quad l, m \in \mathbb{N}. \] (5.7)
Since \( I_{-}^{S} \circ \Phi_{S, -} \neq I_{-}^{l} \), the Hamiltonians \( I_{-}^{l} \) and \( I_{-}^{S} \) define on \( (L_{-}^{1}, \{\cdot, \cdot\}_{-}) \) (or \( (L_{-}^{S}, \{\cdot, \cdot\}_{S}) \)) different families of dynamical systems.

Firstly, we shall investigate the systems associated to the Hamiltonians \( I_{-}^{l} \) given by (5.5). As we shall see, the framework of the Banach Lie-Poisson space \( (L_{-}^{1}, \{\cdot, \cdot\}_{-}) \) is more natural in this case. Hence, taking into account Corollary 2.3(ii), substituting \( I_{-}^{l} \) into (5.4), then applying \( \iota_{-} \) to (5.1), and using (5.4), yields the family of Hamilton equations on \( L_{-}^{1} \)
\[ \frac{\partial}{\partial t}\iota_{-}(\rho) = (P_{-}^{1} + P_{0}^{1}) \left[ (P_{+}^{\infty} + P_{0}^{\infty}) \left[ ([\iota_{-}(\rho)]^{l-1} \iota_{-}(\rho) \right] \right. \right. \] (5.8)
or, equivalently, in Lax form
\[ \frac{\partial}{\partial t}\iota_{-}(\rho) = - [P_{-}^{\infty} \left( [\iota_{-}(\rho)]^{l-1} \iota_{-}(\rho) \right], \iota_{-}(\rho)] = [P_{0}^{\infty} \left( [\iota_{-}(\rho)]^{l-1} \iota_{-}(\rho) \right], \iota_{-}(\rho)], \] (5.9)
where \( t_{l} \) denotes the time parameter for the \( l \)th flow.

Equation (5.9) implies that its solution is given by the coadjoint action of the group \( GL_{-}^{\infty} \) on the dual \( L_{-}^{1} \) of its Lie algebra. Hence, there is some smooth curve \( \mathbb{R} \ni t_{l} \mapsto h_{-}(t_{l}) \in GL_{-}^{\infty} \) satisfying \( (Ad_{+})^{*}_{h_{+}(t_{l})} \circ (Ad_{+})^{*}_{h_{+}(t_{l}+s_{l})} = (Ad_{+})^{*}_{h_{+}(t_{l}+s_{l})} \) such that
\[ \iota_{-}(\rho(t_{l})) = (Ad_{+})^{*}_{h_{+}(t_{l})} \rho(0) = (P_{-}^{1} + P_{0}^{1}) \left( h_{+}(t_{l}) \iota_{-}(\rho(0)) \iota_{-}(\rho)(t_{l}) \right) \] (5.10)
is the solution of (5.8) with initial condition \( \rho(0) \) for \( t_{l} = 0 \).

On the other hand, the solution of (5.9) is given by
\[ \iota_{-}(\rho(t_{l})) = h_{-}(t_{l}) \iota_{-}(\rho(0)) h_{-}(t_{l}), \] (5.11)
for a smooth one-parameter subgroup \( \mathbb{R} \ni t_{l} \mapsto h_{-}(t_{l}) \in GL_{-}^{\infty} \) that can be explicitly determined. We shall do this by using the decomposition \( \iota_{-}(\rho) = \rho_{0} + \rho_{-} \), where \( \rho_{-} = \sum_{i=1}^{\infty} (ST)^{i} \rho_{i} \) and \( \rho_{i} \in L_{0}^{1} \) if \( i \in \mathbb{N} \cup \{0\} \). Since \( P_{0}^{\infty} \left( [\iota_{-}(\rho)]^{l-1} \right) = \rho_{0}^{l-1} \), equation (5.9) becomes
\[ \frac{\partial}{\partial t_{l}} \iota_{-}(\rho) = [\rho_{0}^{l-1}, \rho_{0} + \rho_{-}] = [\rho_{0}^{l-1}, \rho_{-}] \] (5.12)
which is equivalent to
\[ \frac{\partial}{\partial t_{l}} \rho_{-} = [\rho_{0}^{l-1}, \rho_{-}] \quad \text{and} \quad \frac{\partial}{\partial t_{l}} \rho_{0} = 0. \] (5.13)
It immediately follows that its solution is given by (5.11) with
\[ h_{-}(t_{l}) = e^{-t_{l} \rho_{0}(0)^{l-1}}, \] (5.14)
where \( \rho(0) = \rho_{0}(0) + \rho_{-}(0) \) is the initial value of \( \rho \) at time \( t_{l} = 0 \).

Note that \( h_{-}(t_{l}) \in GL_{-}^{\infty} \) is in fact a diagonal operator which can also be obtained from the decomposition
\[ e^{t_{l} \iota_{-}(\rho(0))^{l-1}} = h_{-}(t_{l}) h_{-}(t_{l})^{-1}, \] (5.15)
where \( h_{-}(t_{l}) \in GL_{-}^{\infty} \). It follows that we can write the solution also in the form
\[ \iota_{-}(\rho(t_{l})) = k_{-}(t_{l})^{-1} \iota_{-}(\rho(0)) \iota_{-}(t_{l}). \] (5.16)

Finally, note that in (5.10) we can choose \( h_{+}(t_{l}) = h_{-}(t_{l}) \) since also \( h_{-}(t_{l}) \in GL_{+}^{\infty} \).

Let us analyze the system (5.9) in more detail. We begin by noting that there is an isometry between \( \ell_{-}^{\infty} \) and the diagonal bounded linear operators \( L_{0}^{\infty} \subset L_{-}^{\infty} \) and between \( \ell_{-}^{1} \) and the diagonal trace class operators \( L_{0}^{1} \subset L_{-}^{1} \). Fix a strictly lower triangular element
\[ \nu_{-} = \sum_{i=1}^{k-1} (ST)^{i} \nu_{i} \in L_{-}^{1, k} \quad \text{where} \quad k \in \mathbb{N} \cup \{\infty\} \] (5.17)
and define the map \( \mathcal{J}_{\nu_-} : \ell^\infty \times \ell^1 \rightarrow L^1_{\pm} \) by

\[
\mathcal{J}_{\nu_-}(q, p) := p + e^{q_{\nu_-}} e^{-q},
\]

(5.17)

where, on the right hand side, we identify \( p \) and \( q \) with diagonal operators and \( e^q \) is the exponential of \( q \).

It is easy to see that this map is smooth and that \( \mathcal{J}_{\nu_-}(q, p) = \mathcal{J}_{\nu_-}(q + \alpha q, p) \), for any \( \alpha \in \mathbb{R} \). We shall prove in Proposition 6.2 that if \( \nu_- = (S^T_{k-1})_{k=1}^{k-1} \in L^1_{k+1} \), the map \( \mathcal{J}_{\nu_-} : \ell^\infty \times \ell^1 \rightarrow L^1_{0, \pm} \), the space of bidiagonal trace class operators having non-zero entries only on the main and the lower \((k - 1)\)st diagonal, is a momentum map in the sense of Definition 3.2.

We shall argue below, in analogy with the finite dimensional case, that \((q, p)\) can be considered as action-angle coordinates for the Hamiltonian system \((5.9)\). We begin by recalling that the solution of \((5.9)\) is given by \( \nu_- (\rho(t)) = h_- (t_1) \cdots h_- (t_1) \rho_0(t) h_- (t_1) \), where \( h_- (t_1) = e^{-t_1 \rho_0(0)} \), \( \rho_0(0) = \rho_0(0) + \rho_- (0) \in L^1 \), is the initial value of the variable \( \rho \) at \( t_1 = 0 \), \( \rho_0 \in L^0_0 \) a diagonal operator, and \( \rho_- \) a strictly lower triangular operator. Therefore, \( h_- (t_1) h_- (t_m) = h_- (t_m) h_- (t_1) \) for any \( l, m \in \mathbb{N} \) and hence the product

\[
h_- (t) := h_- (t_1, t_2, \ldots) := \prod_{l=1}^{\infty} h_- (t_l)
\]

(5.18)

is independent on the order of the factors and it exists as an invertible bounded operator if we assume that \( t := (t_1, t_2, \ldots) \in \ell^\infty_0 \) which means that \( t \) has only finitely many non-zero elements.

One also has

\[
h_- (t)^{-1} \mathcal{J}_{\nu_-}(q, p) h_- (t) = \mathcal{J}_{\nu_-} \left( q + \sum_{l=1}^{\infty} t_l \rho_0(0)^{l-1}, p \right) \quad \text{for} \quad t \in \ell^\infty_0,
\]

(5.19)

which shows that the flow in the coordinates \((q, p)\) is described by a straight line motion in \( q \) with \( p \) conserved. If this would be a finite dimensional system, since \((q, p)\) are also Darboux coordinates (see (3.3) or (3.4)), we would say that they are action-angle coordinates on \( \mathcal{J}_{\nu_-}(\ell^\infty \times \ell^1) \).

In infinite dimensions, even the definition of action-angle coordinates presents problems. First, if the symplectic form is strong, the Darboux theorem (that is, the symplectic form is locally constant) is valid; see the proof of Theorem 3.2.2 in [1]. Second, if the symplectic form is weak, which is our case, the Darboux theorem fails in general, even if the manifold is a Hilbert space; Marsden’s classical counterexample can be found and discussed in Exercise 3.2H of [1]. Third, even if one could show in a particular case that the Darboux theorem holds, there still is the problem of coordinates. In the case presented above, the action-angle coordinates were constructed explicitly. In general, on Banach weak symplectic manifolds this may well be impossible.

We return now to the systems described by the family of integrals in involution \( I_S^L \) given by \((5.6)\). By Corollary 2.3 ii), substituting \( I_S^L \) into \((5.2)\), and using \((5.4)\), yields the family of Hamilton equations on \( L^1 \)

\[
\frac{\partial I_S(\sigma)}{\partial t_l} = (P_1 + P_0 + T \circ P_1) \left( \left[ (P_0^\infty + P_0^\infty + T \circ P_0^\infty) \left( [I_S(\sigma)]^{l-1}, I_S(\sigma) \right) \right] \right)
\]

(5.20)

or, equivalently, in Lax form

\[
\frac{\partial I_S(\sigma)}{\partial t_l} = - \left[ (P_0^\infty - T \circ P_0^\infty) \left( [I_S(\sigma)]^{l-1}, I_S(\sigma) \right) \right],
\]

(5.21)

where \( t_l \) denotes the time parameter for the \( l \)th flow.

From \((5.20)\) it follows that the solution of this equation can be written in terms of the coadjoint action of the Banach Lie group \( GL^\infty_+ \) on the dual \( L^1 \) of its Lie algebra. More precisely, the solution is necessarily of the form

\[
I_S(\sigma(t_l)) = \left( Ad^{(t_l)}_{g_{+}(t_l)^{-1}} \right) \sigma(0) = \left( P_1 + P_0 + T \circ P_1 \right) \left( g_{+}(t_l) I_S(\sigma(0)) g_{+}(t_l)^{-1} \right)
\]

(5.22)

for some smooth curve \( \mathbb{R} \ni t_l \rightarrow g_{+}(t_l) \in GL^\infty_+ \) and \( \sigma(0) \) the initial condition for \( t_l = 0 \).

On the other hand, the solution of \((5.21)\) is

\[
I_S(\sigma(t_l)) = g_S(t_l)^{T} I_S(\sigma(0)) g_S(t_l),
\]

(5.23)

where \( \mathbb{R} \ni t_l \rightarrow g_S(t_l) \in O^\infty \) is a smooth curve that will be determined in the next proposition by the same method as in the finite dimensional case (see, e.g., [7] [12] [19] [23]).
Proposition 5.1 Assume that we have the decomposition (we set here \( t = t_1 \))
\[
e^{t\sigma} = g(t)g_+(t)
\]
for \( g(t) \in O^\infty \) and \( g_+(t) \in GL_+^\infty \). Then
\[
i_S(t) = g(t)^T[i_S(t)]g(t) = g_+(t)[i_S(t)]g_+(t)^{-1}
\]
is the solution of (5.21) with initial condition \( i_S(\sigma(0)) \).

Proof. To prove the first equality in (5.25), use (5.24) to get
\[
g_+ = g(t)^{-1}g(t)^{-1} = e^{t[i_S(t)]}g(t)^{-1} = e^{t[i_S(t)]}g_+(t)^{-1}
\]
and hence
\[
g_+ = e^{t[i_S(t)]}g_+(t)^{-1} = g_+(t)[i_S(t)]g_+(t)^{-1}
\]
since \( i_S(\sigma(0)) \) commutes with \( e^{t[i_S(t)]} \).

Let \( i_S(\sigma(t)) := g^{-1}[i_S(\sigma(t))]g(t) \). Taking the time derivative of (5.24) and multiplying on the right by \( g(t)^{-1} \) and on the left by \( g(t)^{-1} \) we get
\[
[i_S(t)] = g(t)^{-1}g(t)^{-1} + \dot{g}_+(t)g_+(t)^{-1}
\]
which is equivalent to the equations
\[
g(t)^{-1}\dot{g}_+(t) = (P_\infty^\infty - T \circ P_\infty^\infty) ([i_S(t)]^l)\)
\[
\dot{g}_+(t)g_+(t)^{-1} = (P_\infty^\infty + P_0^\infty + T \circ P_\infty^\infty) ([i_S(t)]^l)\).
\]
(5.26)

(5.27)

Therefore
\[
\frac{dt}{dt}i_S = -g(t)^{-1}\dot{g}_+(t)g(t)^{-1}[i_S(t)]g(t) + g(t)^{-1}[i_S(t)]\dot{g}(t)
\]
\[
= -(P_\infty^\infty - T \circ P_\infty^\infty) ([i_S(t)]^l) i_S(t) + i_S(t) \left( P_\infty^\infty - T \circ P_\infty^\infty \right) ([i_S(t)]^l)
\]
\[
= -\left( P_\infty^\infty - T \circ P_\infty^\infty \right) ([i_S(t)]^l) i_S(t)
\]
which is (5.21). 

This proposition shows that the solution (5.25) of the system (5.21) could be expressed using the analogue of the Iwasawa decomposition \( GL_\infty = O^\infty \cdot GL_0^\infty \cdot GL_+^\infty \) for the Banach Lie group \( GL_\infty \). To our knowledge, there is no proof of this decomposition and there could be technical difficulties that may even render it impossible. However, see the appendix in [15] for the polar decomposition theorem.

Note also that (5.25) produces a smooth curve \( g_+(t) \in GL_+^\infty \) satisfying (5.22) even without the projection operator in that formula. This follows also directly from (5.25) and (4.10).

The previous general considerations involving Proposition 2.1 imply that the families of flows given by (5.1) or (5.2) and, in particular by (5.9) or (5.21), not only preserve the symplectic leaves of \( L_1^1 \) and \( L_1^1 \), but also the filtrations (5.34) and (4.56), respectively. This remark has some important consequences which we discussed below.

We turn now to the study of Hamiltonian systems induced on the filtrations (4.54) and (4.55). A \( k \)-diagonal Hamiltonian system is, by definition, a Hamiltonian system on \( \left( L_{-k}^1, \{\cdot, \cdot \}_k \right) \). Since the map \( \Phi_{-k}: \left( L_{-k}^1, \{\cdot, \cdot \}_k \right) \rightarrow \left( L_{-k}^1, \{\cdot, \cdot \}_k \right) \) introduced at the end of (4) is a Banach Lie-Poisson space isomorphism, we can regard \( k \)-diagonal Hamiltonian systems as being defined also on \( \left( L_{-k}^1, \{\cdot, \cdot \}_k \right) \). From (4.52), Hamilton’s equations on \( \left( L_{-k}^1, \{\cdot, \cdot \}_k \right) \) defined by an arbitrary function \( h_k \in C^\infty(L_{-k}^1) \) are given by
\[
\frac{dt}{dt} h_j = \sum_{l=j}^{k-1} g_l^{-1} \left( \frac{\delta h_k}{\delta p_{l-j}} - p_l \frac{\delta h_k}{\delta p_{l-j}} \right) \]
(5.28)
Note that for all $n > k$ (including $n = \infty$), any $h_k \in C^\infty(L_{-k}^{1,n})$ can be smoothly extended to $h_n := h_k \circ \pi_{kn} \in C^\infty(L_{-n}^{1,n})$, where $\pi_{kn} : L_{-n}^{1,n} \to L_{-k}^{1,k}$ is the projection that eliminates the last lower $n - k$ diagonals of an operator in $L_{-n}^{1,n} := \oplus_{i=-n^+1}^0 L_{i}^{1,n}$. Conversely, any $h_n \in C^\infty(L_{-n}^{1,n})$ gives rise to a smooth function $h_k := h_n \circ \iota_{nk} \in C^\infty(L_{-k}^{1,k})$, where $\iota_{nk} : L_{-k}^{1,k} \to L_{-n}^{1,n}$ is the natural inclusion. Since the flow defined by $h \in C^\infty(L_{-n}^{1,n})$ preserves the filtration (4.3) (see Proposition 2.4), it follows that if the initial condition $\rho(0) \in L_{-k}^{1,k}$, its trajectory is necessarily contained in $L_{-k}^{1,k}$. This means that in order to solve the system (5.28) for a given $k \in \mathbb{N}$, it suffices to solve the Hamiltonian system given by the extension of $h_k$ to $(L_{-1}^{1}, \{\cdot, \cdot\})$ for initial conditions in $L_{-1}^{1,k}$.

Let us now specialize the functions $h_k \in C^\infty(L_{-k}^{1,k})$ and $f_k \in C^\infty(L_{S,k}^{1})$ to

$$I_k^{-,k}(\rho) := I_k^{-,(\iota_{-k} \circ \iota_{-k})(\rho)) \quad \text{for} \quad \rho \in L_{-k}^{1,k} \quad (5.29)$$

$$I_k^{S,k}(\sigma) := I_k^{S,(\iota_{S,k}(\sigma)) \quad \text{for} \quad \sigma \in L_{S,k}^{1} \quad (5.30)$$

respectively, where $\iota_{-k} : L_{-k}^{1,k} \to L_{-k}^{1}$ and $\iota_{S,k} : L_{S,k}^{1} \to L_{S}^{1}$ are the inclusions. Note that since $I_k^{S,k} \circ \Phi_{S,-k} \neq I_k^{-,k}$, the dynamics induced by the functions $I_k^{-,k}$ and $I_k^{S,k}$ are different in spite of the fact that the Poisson structures on $L_{-k}^{1}$ and $L_{S,k}^{1}$ are isomorphic. Therefore, we see that on has the family of Hamiltonian systems indexed by $k \in \mathbb{N}$ which have an infinite number of integrals in involution indexed by $l \in \mathbb{N}$. For $k = 2$ the system is the semi-infinite Toda lattice. Therefore, the $k$-diagonal semi-infinite Toda systems are defined to be the Hamiltonian systems on $L_{S,k}^{1}$ associated to the functions $I_k^{S,k}$, $l \in \mathbb{N}$.

An important consequence of the fact that the Poisson brackets on $L_{-k}^{1,k}$ and $L_{S,k}^{1}$ are induced is that the method of solution of the corresponding Hamilton equations for $I_k^{-,k}$ and $I_k^{S,k}$, respectively, can be obtained by solving these equations on $L_{-1}^{1}$ and $L_{S}^{1}$ respectively. Namely, it suffices to work with the equations of motion (5.29) and (5.21) with initial conditions $\rho(0) \in L_{-k}^{1,k}$ and $\sigma(0) \in L_{S,k}^{1}$, respectively, and use Proposition 5.4. We shall do this in the rest of the paper for a special case related to the semi-infinite Toda system.

6 The bidiagonal case

In this section we shall study in great detail the bidiagonal case consisting of operators that have only two non-zero diagonals: the main one and the lower $k - 1$ diagonal. The results obtained in this section will be used later to give a rigorous functional analytic formulation of the integrability of the semi-infinite Toda lattice.

The coordinate description of the bidiagonal subcase. Due to their usefulness in the study of the Toda lattice, we shall express in coordinates several formulas from §4 adapted to the subalgebra $I^\infty_{+,0,k-1} \subset L^\infty_{+,k}$, $k \geq 2$, consisting of bidiagonal elements

$$x := x_0 + x_{k-1}S^{k-1} = \sum_{i=0}^{\infty} (x_{0,ii}|i\rangle \langle i| + x_{k-1,ii}|i\rangle \langle i + k - 1|), \quad (6.1)$$

where $x_0, x_{k-1}$ are diagonal operators whose entries are given by the sequences $\{x_{0,ii}\}_{i=0}^{\infty}, \{x_{k-1,ii}\}_{i=0}^{\infty} \in \ell^\infty$, respectively. The subalgebra $I^\infty_{+,0,k-1}$ of $L^\infty_{+,k}$ is hence formed by upper triangular bounded operators that have only two non-zero diagonals, namely the main diagonal and the strictly upper $k - 1$ diagonal.

The predual of $I^\infty_{+,0,k-1}$ is $L^1_{+,0,k-1}$ which consists of lower triangular trace class operators having only two non-vanishing diagonals, namely the main one and the strictly lower $k - 1$ diagonal ($k \geq 2$), that is, they are of the type

$$\rho = \rho_0 + (S^{k-1})^T \rho_{k-1} = \sum_{i=0}^{\infty} (\rho_{0,ii}|i\rangle \langle i| + \rho_{k-1,ii}|i + k - 1\rangle \langle i|), \quad (6.2)$$

where $\rho_0$ and $\rho_{k-1}$ are diagonal operators whose entries are given by the sequences $\{\rho_{0,ii}\}_{i=0}^{\infty}, \{\rho_{k-1,ii}\}_{i=0}^{\infty} \in \ell^1$, respectively. The Banach Lie subgroup $GL^\infty_{+,0,k-1}$ of $GL^\infty_{+,k}$ whose Banach Lie algebra is $I^\infty_{+,0,k-1}$ has elements given by

$$g = g_0 + g_{k-1}S^{k-1} = \sum_{i=0}^{\infty} (g_{0,ii}|i\rangle \langle i| + g_{k-1,ii}|i\rangle \langle i + k - 1|), \quad (6.3)$$

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where $g_0$ and $g_{k-1}$ are diagonal operators whose entries are given by the sequences $\{g_{0,i}\}_{i=0}^{\infty}$, $\{g_{k-1,i}\}_{i=0}^{\infty} \in \ell^\infty$, respectively, and the sequence $\{g_{0,i}\}_{i=0}^{\infty}$ is bounded below by a strictly positive number (that depends on $g_0$).

The product of $g, h \in GL^\infty_{+,0,k-1}$ in $GL^\infty_{+,k}$ is given by

$$g \circ_k h = g_0h_0 + (g_0h_{k-1} + g_{k-1}s^{k-1}(h_0))S^{k-1}$$

and the inverse of $g$ in $GL^\infty_{+,k}$ is given by

$$g^{-1} = g_0^{-1}g_{k-1}s^{k-1}(g_0^{-1})S^{k-1} = \sum_{i=0}^{\infty} \frac{1}{g_{0,ii}} |i\rangle \langle i| - \sum_{i=0}^{\infty} \frac{g_{k-1,ii}}{g_{0,ii}g_{0,i+k-1,i+k-1}} |i\rangle \langle i + k - 1|.$$

The Lie bracket of $x, y \in I^\infty_{+,0,k-1}$ has the expression

$$[x, y]_k = (x_{k-1}(s^{k-1}(y_0) - y_0) - y_{k-1}(s^{k-1}(x_0) - x_0))S^{k-1}$$

and are given by

$$\langle x, y \rangle_k = \sum_{i=0}^{\infty} (x_{k-1,ii}(y_{0,i+k-1,i+k-1} - y_{0,ii}) - y_{k-1,ii}(x_{0,i+k-1,i+k-1} - x_{0,ii})) |i\rangle \langle i + k - 1|.$$

The group coadjoint action $\left(Ad^{+,k}\right)^*_g : I^\infty_{+,0,k-1} \rightarrow I^\infty_{+,0,k-1}$ for $g := g_0 + g_{k-1}s^{k-1} \in GL^\infty_{+,0,k-1} \subset GL^\infty_{+,k}$ and Lie algebra coadjoint action $\left(ad^{+,k}\right)^*_x : I^\infty_{+,0,k-1} \rightarrow I^\infty_{+,0,k-1}$, for $x := x_0 + x_{k-1}s^{k-1} \in I^\infty_{+,0,k-1} \subset L^\infty_{+,k}$ are given by

$$\left(Ad^{+,k}\right)^*_g \rho = \rho_0 + g_0^{-1}g_{k-1}\rho_{k-1} - s^{k-1}(g_0^{-1}g_{k-1}\rho_{k-1}) \left(I - \sum_{j=0}^{k-2} p_j \right)$$

and

$$\left(ad^{+,k}\right)^*_x \rho = s^{k-1}(\rho_{k-1}x_{k-1}) - \rho_{k-1}x_{k-1} + (sT)^{k-1}\rho_{k-1}(x_0 - s^{k-1}(x_0))$$

where $\rho := \rho_0 + (sT)^{k-1}\rho_{k-1} \in I^1_{+,0,k-1}$.

Since $\left(I^1_{+,0,k-1}\right)^{*} = I^\infty_{+,0,k-1}$ and the duality pairing is given by the trace of the product, it follows that the Lie-Poisson bracket and its associated Hamiltonian vector field on $I^1_{+,0,k-1}$ are given by

$$\{f, h\}_{0,k-1}(\rho)$$

and

$$\mathcal{L}_\rho h = \sum_{i=0}^{\infty} \frac{\partial f}{\partial \rho_{k-1,ii}} \left(\frac{\partial h}{\partial \rho_{0,i+k-1,i+k-1}} - \frac{\partial h}{\partial \rho_{0,ii}}\right)$$

where

$$\mathcal{L}_\rho h = \sum_{i=0}^{\infty} \frac{\partial f}{\partial \rho_{k-1,ii}} \left(\frac{\partial h}{\partial \rho_{0,i+k-1,i+k-1}} - \frac{\partial h}{\partial \rho_{0,ii}}\right) - \frac{\partial h}{\partial \rho_{k-1,ii}} \left(\frac{\partial f}{\partial \rho_{0,i+k-1,i+k-1}} - \frac{\partial f}{\partial \rho_{0,ii}}\right).$$

(6.9)
and
\[ X_{h}^{0,k-1}(\rho) = \text{Tr} \left[ \rho_{k-1} \left( s^{k-1} \left( \frac{\partial h}{\partial \rho_{0}} - \frac{\partial h}{\partial \rho_{k-1}} \right) - \frac{\partial h}{\partial \rho_{k-1}} \left( s^{k-1} \left( \frac{\partial h}{\partial \rho_{0}} - \frac{\partial h}{\partial \rho_{k-1}} \right) \right) \right) \right] = \sum_{i=0}^{\infty} \rho_{k-1,i} \left[ \left( \frac{\partial h}{\partial \rho_{0,i+k-1,i+k-1}} - \frac{\partial h}{\partial \rho_{0,ii}} \right) \frac{\partial}{\partial \rho_{k-1,ii}} \right] \]
\[ \quad - \frac{\partial h}{\partial \rho_{k-1,ii}} \left( \frac{\partial}{\partial \rho_{0,i+k-1,i+k-1}} - \frac{\partial}{\partial \rho_{0,ii}} \right) \] (6.10)
for \( f, h \in C^{\infty}(I_{-0,k-1}) \). Like in \([3]\) in (6.10) we have used the standard coordinate conventions from finite dimensions to write a vector field. The precise meaning of the symbols \( \partial/\partial \rho \) or, in coordinates, for \( i \) to the Schauder basis \( \{|i| \}_{i=0}^{\infty} \) of \( I_{-0,k-1} \). Thus Hamilton’s equations in terms of diagonal operators are
\[ \frac{d}{dt} \rho_{0} = \rho_{k-1} \frac{\partial h}{\partial \rho_{k-1}} - s^{k-1} \left( \rho_{k-1} \frac{\partial h}{\partial \rho_{k-1}} \right) \] (6.11)
\[ \frac{d}{dt} \rho_{k-1} = \rho_{k-1} \left( s^{k-1} \left( \frac{\partial h}{\partial \rho_{0}} - \frac{\partial h}{\partial \rho_{0}} \right) \right) \] (6.12)
or, in coordinates, for \( i \in \mathbb{N} \cup \{0\}, k \geq 2,
\[ \frac{d}{dt} \rho_{0,i} = \rho_{k-1,i} \frac{\partial h}{\partial \rho_{k-1,i}} - \rho_{k-1,i} \frac{\partial h}{\partial \rho_{k-1,ii}} \] (6.13)
\[ \frac{d}{dt} \rho_{k-1,ii} = \rho_{k-1,ii} \left( \frac{\partial h}{\partial \rho_{0,i+k-1,i+k-1}} - \frac{\partial h}{\partial \rho_{0,ii}} \right) \] (6.14)

**Structure of the generic coadjoint orbit.** By a generic coadjoint orbit we will understand the orbit
\[ \mathcal{O}_{\nu} := \left\{ \left( \text{Ad}^{+}\nu \right)^{g} \nu \mid g \in GI_{+,0,k-1}^{\infty} \right\}, \]
through the element \( \nu = \nu_{0} + (S^{T})^{-1} \nu_{k-1} \in I_{-0,k-1}^{1} \) such that \( \nu_{k-1,ii} \neq 0 \) for \( i = 0, 1, 2, \ldots \).
Let us denote by \( GL_{0}^{\infty,k-1} \) the Banach Lie subgroup of \((k-1)\)-periodic elements of \( GL_{0}^{\infty} \), that is, \( g_{0} \in GL_{0}^{\infty,k-1} \) if and only if \( s^{k-1}(g_{0}) = g_{0} \). Denote by \( L_{0}^{\infty,k-1} \) the Banach Lie algebra of \( GL_{0}^{\infty,k-1} \).

**Proposition 6.1** (i) One has the following equalities
\[ Z(GI_{+,0,k-1}^{\infty}) = (GI_{+,0,k-1}^{\infty})_{\nu} = GL_{0}^{\infty,k-1}, \] (6.15)
where \( Z(GI_{+,0,k-1}^{\infty}) \) is the center of \( GI_{+,0,k-1}^{\infty} \) and \( (GI_{+,0,k-1}^{\infty})_{\nu} \) is the stabilizer of the generic element \( \nu \in I_{-0,k-1}^{1} \).

(ii) The generic orbit
\[ \mathcal{O}_{\nu} := GI_{+,0,k-1}^{\infty}/GL_{0}^{\infty,k-1} \] (6.16)
is a Banach Lie group.

(iii) One has the relation
\[ \mathcal{O}_{\nu} = \nu_{0} + \mathcal{O}_{\left(S^{T}\right)^{-1} \nu_{k-1}} \] (6.17)
between the coadjoint orbits through \( \nu = \nu_{0} + (S^{T})^{-1} \nu_{k-1} \) and through \((S^{T})^{-1} \nu_{k-1} \).

**Proof.** Part (i) follows from a direct verification. Since \( GL_{0}^{\infty,k-1} \) is a normal Banach Lie group of \( GI_{+,0,k-1}^{\infty} \) the quotient \( GI_{+,0,k-1}^{\infty}/GL_{0}^{\infty,k-1} \) is also a Banach Lie group (see [3]). This proves (ii). Part (iii) follows from (6.7). ■

We conclude from (6.17) that to describe any \( \mathcal{O}_{\nu} \) it suffices to study coadjoint orbits through the \((k-1)\)-lower diagonal elements, \( k \geq 2 \).

Since the Banach Lie group \( GI_{+,0,k-1}^{\infty} \) and the generic element \( \nu \in I_{-0,k-1}^{1} \) satisfy all the hypotheses of Theorems 7.3 and 7.4 in [20] we conclude:

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The map $\iota_\nu : GI_{+,0,k-1}^\infty/GI_0^\infty,k-1 \to I_{+,0,k-1}^1$ given by $\iota_\nu([g]) := (\mathrm{Ad}^{+,k})^*_{g^{-1}} \nu$ is a weak injective immersion. This means that its derivative is injective but no conditions on the closedness of its range or the fact that it splits are imposed. The map $\iota_\nu$ is not an immersion as we now show by using Theorem 7.5 in [20].

Since the coadjoint stabilizer Lie algebra $\left( I_{+,0,k-1}^\infty \right)_\nu$ is equal to the center

$$Z(I_{+,0,k-1}^\infty) = \{ x = x_0 + x_k-1 s^{k-1} \in I_{+,0,k-1}^\infty \mid s^{k-1}(x_0) = x_0, x_{k-1} = 0 \}$$

it follows that its annihilator is

$$\left( (I_{+,0,k-1}^\infty)_{\nu}^* \right) = \left\{ \rho = \rho_0 + (S^T)^{k-1} \rho_{k-1} \in I_{-,0,k-1}^1 \mid \text{Tr}(x_0 \rho_0) = 0, \right.$$

$$\text{for all } x_0 \in L_0^\infty \text{ such that } s^{k-1}(x_0) = x_0 \left\} \right..$$

Because

$$\text{Tr} \left( x_0 \left( (\mathrm{ad}^{+,k})^*_x \nu \right)_0 \right) = \text{Tr} \left( x_0 \left( (\mathrm{ad}^{+,k})^*_x \nu \right) \right) = \text{Tr} \left( \left( x_0, x \right)_k \nu \right) = 0$$

for any $x_0 \in Z(I_{+,0,k-1}^\infty)$ and any $x \in I_{+,0,k-1}^\infty$, we have $S_\nu \subset \left( (I_{+,0,k-1}^\infty)_{\nu}^* \right)$, where $S_\nu := \left\{ (\mathrm{ad}^{+,k})^*_x \nu \mid x \in I_{+,0,k-1}^\infty \right\}$ is the characteristic subspace of the Banach Lie-Poisson structure of $I_{+,0,k-1}^1$ at $\nu$. Moreover, the bounded operator $K_\nu : x \in I_{+,0,k-1}^\infty \mapsto (\mathrm{ad}^{+,k})^*_x \nu \in I_{+,0,k-1}^{0}$ has non-closed range in $K_\nu = S_\nu$ and thus the inclusion $S_\nu \subset \left( (I_{+,0,k-1}^\infty)_{\nu}^* \right)$ is strict. To see that the range of $K_\nu$ is not closed, one uses the Banach space isomorphisms $I_{-,0,k-1}^1 \cong \ell^1 \times \ell^1$ and $I_{+,0,k-1}^\infty \cong \ell^\infty \times \ell^\infty$ and shows that the two components of $K_\nu$ are both bounded linear operators with non-closed range. Therefore, since Theorem 7.5 in [20] states that $\iota_\nu$ is an immersion if and only if $S_\nu = \left( (I_{+,0,k-1}^\infty)_{\nu}^* \right)$, this argument shows that $\iota_\nu$ is only a weak immersion.

The quotient space $GI_{+,0,k-1}^\infty/GI_0^\infty,k-1$ is a weak symplectic Banach manifold relative to the closed two-form

$$\omega_\nu([g]) = \text{Tr}(\pi(g \circ_k x), T_g \pi(g \circ_k y)) = \text{Tr}(\nu[x, y]_k)$$

$$= \sum_{i=0}^\infty \nu_{k-1,ii} \left( x_{k-1,ii}(y_{0,i+k-1,i+k-1} - y_{0,ii}) - y_{k-1,ii}(x_{0,i+k-1,i+k-1} - x_{0,ii}) \right),$$

(6.18)

where $x, y \in I_{+,0,k-1}^\infty$, $g \in GI_{+,0,k-1}^\infty$, $[g] := \pi(g)$, $\pi : GI_{+,0,k-1}^\infty \to GI_{+,0,k-1}^\infty/GI_0^\infty,k-1$ is the canonical projection, and $T_g \pi : T_g GI_{+,0,k-1}^\infty \to T_g \left( GI_{+,0,k-1}^\infty/GI_0^\infty,k-1 \right)$ is its derivative at $g$. In this formula we have used the fact that the value at $g$ of the left invariant vector field $\xi_x$ on $GI_{+,0,k-1}^\infty$ generated by $x$ is $g \circ_k x$.

Relative to the Banach manifold structure on $O_\nu$ making $\iota_\nu : GI_{+,0,k-1}^\infty/GI_0^\infty,k-1 \to O_\nu$ into a diffeomorphism, the push forward of the weak symplectic form [0.18] has the expression

$$\omega_{\iota_\nu}(\rho) \left( (\mathrm{ad}^{+,k})^*_x \rho, (\mathrm{ad}^{+,k})^*_y \rho \right) = \text{Tr}(\rho[x, y]_k)$$

$$= \sum_{i=0}^\infty \rho_{k-1,ii} \left( x_{k-1,ii}(y_{0,i+k-1,i+k-1} - y_{0,ii}) - y_{k-1,ii}(x_{0,i+k-1,i+k-1} - x_{0,ii}) \right),$$

(6.19)

where $x, y \in I_{+,0,k-1}^\infty$ and $\rho \in O_\nu$.

We shall express the pull back $\pi^* \omega_\nu$ of the weak symplectic form $\omega_\nu$ in terms of the diagonal operators represented by $\{ g_{0,ii} \}_{i=0}^\infty \in \ell^\infty$ and $\{ g_{k-1,ii} \}_{i=0}^\infty \in \ell^\infty$ defining the element $g \in GI_{+,0,k-1}^\infty$. If $x = x_0 +
\[x_{k-1}S^{k-1}, \ y = y_0 + y_{k-1}S^{k-1} \in I_{+0,k-1}^\infty, \ \text{and} \ \nu = \nu_0 + (ST)^{k-1}\nu_{k-1} \in I_{-0,k-1}^1, \ \text{6.18} \] yields
\[
(\pi^*\omega_\nu)(g) (g \circ_k x, g \circ_k y) = \omega_\nu([g]) (T_g\pi(g \circ_k x), T_g\pi(g \circ_k y)) = \text{Tr}(\nu[x, y])
\]
where \(\nu_{k-1} \) has the diagonal entries \(\{\nu_{k-1,ii}\}_{i=0}^\infty\). The left invariant vector field \(\xi_x\) on \(GI_{+,0,k-1}^\infty\) generated by \(x\) has the expression
\[
\xi_x = \sum_{i=0}^\infty g_{0,ii}x_{0,ii} \frac{\partial}{\partial g_{0,ii}} + \sum_{i=0}^\infty (g_{0,ii}x_{k-1,ii} + g_{k-1,ii}x_{0,ii+k-1,ii+k-1}) \frac{\partial}{\partial g_{k-1,ii}}.
\]
The symbols \(\{\partial/\partial g_{0,ii}, \partial/\partial g_{k-1,ii}\}_{i=0}^\infty\) denote the biorthogonal family in the tangent space \(T_{g}I_{+,0,k-1}^\infty\) corresponding to the standard biorthogonal family \(\{i, i, i, i+k-1\}_{i=0}^\infty\) in \(I_{+,0,k-1}^\infty\). We shall use, as in finite dimensions, the exterior derivative on real valued smooth functions, in particular coordinates, to represent elements in the dual space. With this convention, we have
\[
\pi^*\omega_\nu = \sum_{i=0}^\infty d \log g_{0,ii} \wedge d \left( \nu_{k-1,ii} \frac{g_{k-1,ii}}{g_{0,ii}} - \nu_{k-1,ii} \frac{g_{k-1,ii}}{g_{0,ii+k-1,ii+k-1}} \right), \ \text{6.21}
\]
where, as usual, any element that has a negative index is set equal to zero. To show this, we evaluate the right hand side of \(\text{(6.21)}\) on \(\xi_x\) and \(\xi_y\) and observe that it equals the right hand side of \(\text{(6.20)}\). Note that the computations make sense since \(\nu_{k-1} \in \ell^1\).

The action of the coadjoint isotropy subgroup \(\left(GI_{+,0,k-1}^\infty\right)_\nu = GI_{+,0,k-1}^{\infty,k-1}\) on \(GI_{+,0,k-1}^\infty\) is given by \(g_{0,ii} \mapsto h_{0,ii}g_{0,ii}, g_{k-1,ii} \mapsto h_{0,ii}g_{k-1,ii}\), where \(h_{0,ii} = h_{0,ii+k-1,ii+k-1}\). As expected, the right hand side of \(\text{(6.21)}\) is invariant under this transformation and its interior product with any tangent vector to the orbit of the normal subgroup \(GL_0^{\infty,k-1}\) is zero. This shows, once again, that \(\text{(6.21)}\) naturally descends to the quotient group \(GI_{+,0,k-1}^\infty/GL_0^{\infty,k-1}\).

In order to understand the structure of \(O_\nu\), define the action \(\alpha^k : GI_{+,0,k}^\infty \times L_{1-k+1}^1 \rightarrow L_{1-k+1}^1\) by
\[
\alpha^k ((ST)^{k-1}\nu_{k-1}) := (ST)^{k-1}k^{k-1}(g_0)g_0^{-1}\nu_{k-1}. \ \text{6.22}
\]
The projector \(\delta^k : I_{-0,k-1}^1 \rightarrow L_{1-k+1}^1\) defined by the splitting \(I_{-0,k-1}^1 = L_{1-k+1}^1 \oplus L_0^1\) is a \(GI_{+,0,k-1}^{\infty,k-1}\)-equivariant map relative to the coadjoint and the \(\alpha^k\)-actions of \(GI_{+,0,k-1}^{\infty}\), that is, the diagram
\[
\begin{array}{ccc}
I_{-0,k-1}^1 & \xrightarrow{(\text{Ad}^{-k})^*g^{-1}} & I_{-0,k-1}^1 \\
\downarrow{\delta^k} & & \downarrow{\delta^k} \\
L_{1-k+1}^1 & \xrightarrow{\alpha^k} & L_{1-k+1}^1
\end{array}
\]
commutes for any \(g \in GI_{+,0,k-1}^{\infty}\). We observe that the stabilizer \(GL_0^{\infty,k-1}\) of the \(\alpha^k\)-action does not depend on the choice of the generic element \((ST)^{k-1}\nu_{k-1} \in L_{1-k+1}^1\). The orbits of the coadjoint action of the subgroup \(GL_0^{\infty,k-1}\) on \((\delta^k)^{-1}((ST)^{k-1}\nu_{k-1})\) are of the form
\[
\Delta_{\nu_0,\nu_{k-1}} + (ST)^{k-1}\nu_{k-1} \subset (\delta^k)^{-1}((ST)^{k-1}\nu_{k-1}) \subset I_{-0,k-1}^1,
\]
where
\[
\Delta_{\nu_0,\nu_{k-1}} := \nu_0 + \text{im}N_{\nu_{k-1}} \subset L_0^1 \ \text{6.23}
\]
are affine spaces for each \( ν_0 \in L^1 \) and the linear operator \( N_{ν_{k-1}} : L^∞_0 \to L^1_0 \) is defined by

\[
N_{ν_{k-1}}(g_{k-1}) := ν_{k-1}g_{k-1} + \bar{δ}(ν_{k-1}g_{k-1}) \left( I - \sum_{j=0}^{k-2} p_j \right).
\]

The orbits of the \( α^k \)-action of \( GI^{∞}_{+,0,k-1} \) on \( L^1_{-k+1} \) are

\[
GI^{∞}_{+,0,k-1} \cdot ((S^T)^{k-1}ν_{k-1}) = \{(S^T)^{k-1}s^{k-1}(g_0)g_0^{-1}ν_{k-1} \mid g_0 \in GL^∞_0 \} := \Delta_{ν_{k-1}}.
\]

Note that if \( Δ_{ν_{k-1}} = Δ_{ν’_{k-1}} \), then \( im N_{ν_{k-1}} = im N_{ν’_{k-1}} \) and so \( Δ_{ν_{0},ν_{k-1}} = Δ_{ν_{0},ν’_{k-1}} \). These remarks show that the coadjoint orbit \( O_ν \) is diffeomorphic to the product \( (ν_0 + im N_{ν_{k-1}}) \times Δ_{ν_{k-1}} \) of the affine space \( Δ_{ν_{0},ν_{k-1}} \) with the \( α^k \)-orbit \( Δ_{ν_{k-1}} \). This diffeomorphism does not depend on the choice of \( (S^T)^{k-1}ν_{k-1}’ \in Δ_{ν_{k-1}} \).

Additionally, one identifies the set of generic coadjoint orbits with the total space \( L^1_0 \) of the vector bundle \( L^1_0 \to L^∞_0/α^k(GL^∞_0) \), whose fiber at \( [ν_{k-1}] \) is \( L^1_0/ im N_{ν_{k-1}} \). The vector space \( L^1_0/im N_{ν_{k-1}} \) is not Banach since \( im N_{ν_{k-1}} \) is not closed in \( L^1_0 \) because the operator \( N_{ν_{k-1}} : L^∞_0 \to L^1_0 \) is compact. Consequently, the bundle \( L^1_0 \to L^∞_0/α^k(GL^∞_0) \) does not have the structure of a Banach vector bundle and does not have fixed typical fiber.

The momentum map. Let us now study an important particular case of the map \( J_{ν_{-}} \) by taking in \( ν_{-} = (S^T)^{k-1}ν_{k-1} \in L^1_{-k,k} \). The map \( 5.17 \), denoted in this case \( J_{ν_{k-1}} : \ell^∞ \times \ell^1 \to \ell^1_{-0,k-1} \), becomes

\[
J_{ν_{k-1}}(q,p) = p + (S^T)^{k-1}ν_{k-1}e^{q_{k-1}}(q) - q.
\]

Recall that we identify \( ℓ^1 \) with \( L^1_0 \) and \( ℓ^∞ \) with \( L^∞_0 \). Having fixed \( (S^T)^{k-1}ν_{k-1} \in L^1_{-k+1} \), define the action of \( GI^{∞}_{+,0,k-1} \) on \( \ell^∞ \times \ell^1 \) by

\[
σ^g_{ν_{k-1}}(q,p) := \left( q + log g_0, p + g_{k-1}g_0^{-1}ν_{k-1}e^{q_{k-1}}(q) - q - (g_{k-1}g_0^{-1}ν_{k-1}e^{q_{k-1}}(q) - q) \right).
\]

where \( g := g_0 + g_{k-1}e^{q_{k-1}} \in GI^{∞}_{+,0,k-1} \) and \( (q,p) \in \ell^∞ \times \ell^1 \). The coordinate form of the action \( 6.20 \) is

\[
q'_i = q_i + \log g_{0,i}
\]

\[
p'_i = p_i + \frac{g_{k-1,i}}{g_{0,i}}ν_{k-1,i}e^{q_{k+1}-q_k} - \frac{g_{k-1,i}}{g_{0,k-1,i}}ν_{k-1,i}e^{q_k-q_{k-1}}
\]

for \( i \in \mathbb{N} \cup \{0\} \). Using \( 6.27 \) and \( 6.28 \) one shows that

\[
\sum_{i=0}^{∞} p_i d q_i = \sum_{i=0}^{∞} p_i dq_i - dQ,
\]

where the function \( Q : \ell^∞ \to \mathbb{R} \) is given by

\[
Q(q) := Tr \left( g_0^{-1}g_{k-1}ν_{k-1}e^{q_{k-1}}(q) - q \right) = \sum_{i=0}^{∞} \frac{g_{k-1,i}}{g_{0,i}}ν_{k-1,i}e^{q_{k+1}-q_k}.
\]

Thus we see that \( ω \) is invariant relative to the \( σ^{ν_{k-1}} \)-action, that is \( (σ^{g,ν_{k-1}})^*ω = ω \) for any \( g \in GI^{∞}_{+,0,k-1} \).

**Proposition 6.2** The smooth map \( J_{ν_{k-1}} : \ell^∞ \times \ell^1 \to \ell^1_{-0,k-1} \) given by \( 6.20 \) is constant on the \( σ^{ν_{k-1}} \)-orbits of the subgroup \( GL^∞_{+,0,k-1} \). In addition:

(i) \( J_{ν_{k-1}} \) is a momentum map. More precisely, \( \{ f \circ J_{ν_{k-1}}, g \circ J_{ν_{k-1}} \} ω = \{ f,g \}_{0,k-1} \circ J_{ν_{k-1}} \), for all \( f,g \in \mathcal{C}^∞(I^1_{-0,k-1}) \), where \( \{ ·, · \} ω \) is the canonical Poisson bracket of the weak symplectic Banach space \( (ℓ^∞ \times ℓ^1, ω) \) given by \( 3.7 \) and \( \{ ·, · \}_{0,k-1} \) is the Lie-Poisson bracket on \( I^1_{-0,k-1} \) given by \( 6.9 \).

(ii) \( J_{ν_{k-1}} \) is \( GI^{∞}_{+,0,k-1} \)-equivariant, that is \( J_{ν_{k-1}} \circ g_{ν_{k-1}} = (Ad^{-k})_{g_{ν_{k-1}}} \circ J_{ν_{k-1}} \) for any \( g \in GI^{∞}_{+,0,k-1} \).
(iii) One has \( \mathcal{J}_{\nu_{k-1}}(\ell^\infty \times \ell^1) = (\delta^k)^{-1}(\Delta_{\nu_{k-1}}) \) and \( \mathcal{J}_{\nu_{k-1}}(\ell^\infty \times \{0\}) = \Delta_{\nu_{k-1}} \) and hence \((\ell^\infty \times \ell^1)/\sigma_{\nu_{k-1}}(GL_{0}^{\infty,k-1}) \cong \mathcal{J}_{\nu_{k-1}}(\ell^\infty \times \ell^1) \) consists of those coadjoint orbits which are projected by \( \delta^k \) to the \( \alpha^k \)-orbit \( \Delta_{\nu_{k-1}} \).

**Proof.** To prove (i), let \( f, g \in C_c^\infty(I_{-1,0,k-1}) \) and notice that
\[
\frac{\partial (f \circ \mathcal{J}_{\nu_{k-1}})}{\partial q} \in (L^\infty)^* \quad \text{and} \quad \frac{\partial (f \circ \mathcal{J}_{\nu_{k-1}})}{\partial p} \in (L^1)^* = L^\infty
\]
because \( q \in L^\infty \) and \( p \in L^1 \). However, by (6.25),
\[
\frac{\partial (f \circ \mathcal{J}_{\nu_{k-1}})}{\partial q} = \left( \frac{\partial f}{\partial \rho_{k-1}} \circ \mathcal{J}_{\nu_{k-1}} \right) (q, p) (\rho_{k-1} \circ \mathcal{J}_{\nu_{k-1}})(q, p)(S^{k-1} - 1) \in L^1 \quad (6.30)
\]
since \( (\rho_{k-1} \circ \mathcal{J}_{\nu_{k-1}})(q, p) \in L^1 \) and
\[
\frac{\partial (f \circ \mathcal{J}_{\nu_{k-1}})}{\partial p} = \left( \frac{\partial f}{\partial \rho_0} \circ \mathcal{J}_{\nu_{k-1}} \right) (q, p) \in L^\infty. \quad (6.31)
\]
Note that (6.30) implies that \( f \circ \mathcal{J}_{\nu_{k-1}} \in C_c^\infty(\ell^\infty \times \ell^1) \) for any \( f \in C_c^\infty(I_{-1,0,k-1}) \).

Thus, using the formula for the canonical bracket on the weak symplectic Banach space \((\ell^\infty \times \ell^1, \omega)\) and the fact that the duality pairing \((L^\infty)^* \times L^\infty \to \mathbb{R}\) restricted to \( L^1 \times L^\infty \) equals the trace of the product, we get
\[
\{ f \circ \mathcal{J}_{\nu_{k-1}}, g \circ \mathcal{J}_{\nu_{k-1}} \}_\omega(q, p) = \left( \frac{\partial (f \circ \mathcal{J}_{\nu_{k-1}})}{\partial q}, \frac{\partial (g \circ \mathcal{J}_{\nu_{k-1}})}{\partial q} \right) - \left( \frac{\partial (g \circ \mathcal{J}_{\nu_{k-1}})}{\partial p}, \frac{\partial (f \circ \mathcal{J}_{\nu_{k-1}})}{\partial p} \right)
\]
\[
= \text{Tr} \left[ (\rho_{k-1} \circ \mathcal{J}_{\nu_{k-1}})(q, p) \left( (S^{k-1} - 1) \left( \frac{\partial g}{\partial \rho_0} \circ \mathcal{J}_{\nu_{k-1}} \right) (q, p) \left( \frac{\partial f}{\partial \rho_{k-1}} \circ \mathcal{J}_{\nu_{k-1}} \right) (q, p) \right) \right]
\]
\[
= \{ f, g \}_{\nu_{k-1}} \mathcal{J}_{\nu_{k-1}}(q, p)
\]
by (6.9).

Parts (ii) and (iii) are proved by direct verifications. \( \square \)

Let us define the map \( \Phi^{\nu_{k-1}}(g) : GI_{+0,k-1}^{\infty} \to \ell^\infty \times \ell^1 \) by
\[
\Phi^{\nu_{k-1}}(g) := \sigma_{g}^{\nu_{k-1}}(0, 0), \quad (6.32)
\]
or, in coordinates,
\[
\Phi^{\nu_{k-1}}(g_0, g_{k-1}) = \left( \log g_0, g_{k-1}^{-1}g_0^{-1} - \delta^k(g_{k-1}g_0^{-1}g_{k-1}^{-1}) \right), \quad (6.33)
\]
which shows that \( \Phi^{\nu_{k-1}} \) is smooth and injective.

**Proposition 6.3** *The following diagram*

\[
\begin{array}{cccccc}
1 & \longrightarrow & GL_{0}^{\infty,k-1} & \longrightarrow & GI_{+0,k-1}^{\infty} & \longrightarrow & GI_{+0,k-1}^{\infty}/GL_{0}^{\infty,k-1} & \longrightarrow & 1 \\
\downarrow \Phi^{\nu_{k-1}} & & \downarrow \pi & & \downarrow & & \downarrow t_{\nu_{k-1}} & & \downarrow 0 \\
0 \times \{ p \} & \longrightarrow & L_{0}^{\infty,k-1} \times \{ p \} & \longrightarrow & \ell^\infty \times \ell^1 & \longrightarrow & \mathcal{J}_{\nu_{k-1}} & \longrightarrow & (\delta^k)^{-1}(\Delta_{\nu_{k-1}}) & \longrightarrow & 0
\end{array}
\]

*commutes. The first row is an exact sequence of Banach Lie groups. The second row is also exact in the following sense: the map \( \mathcal{J}_{\nu_{k-1}} \) is onto and its level sets are all of the form \( L_{0}^{\infty,k-1} \times \{ p \} \), where \( p \in L_{0}^{1} \). In addition, \( (\Phi^{\nu_{k-1}})^* \omega = \pi^* \omega_{\nu_{k-1}} \), \( (6.34) \)*
where \( \omega \) and \( \omega_{i_k-1} \) are the weak symplectic forms (5.33) and (6.21) on \( \ell^\infty \times \ell^1 \) and \( GI_{+,0,k-1}/GL_{0,k-1}^\infty \) respectively. We also have

\[
\hat{\Phi}^{\nu_{k-1}} \left( \pi^{-1}(g) \right) = J_{\nu_{k-1}}^{-1} \left( \nu_{k-1}(g) \right)
\]

for any \( g \in GI_{+,0,k-1}^\infty \).

**Proof.** Commutativity is verified using (6.7), (6.25), and (6.32). The identities (6.34) and (6.35) are obtained by direct verifications. \( \blacksquare \)

Remarks. (i) The analysis of the coadjoint orbit \( \mathcal{O}_\nu \cong GI_{+,0,k-1}/GL_{0,k-1}^\infty \) through the generic element \( \nu \in I_{+,0,k-1}^1 \) carried out in this section shows that it is diffeomorphic to \( \Delta_{\nu_0,\nu_{k-1}} \times \Delta_{\nu_{k-1}} \). For an arbitrary \( (\nu_0,\nu_{k-1}) \in \Delta_{\nu_0,\nu_{k-1}} \times \Delta_{\nu_{k-1}} \), the manifolds \( \mathcal{O}_{\nu_{k-1}}^{-1}(\nu_0) \times \Delta_{\nu_{k-1}} \) and \( \mathcal{O}_{\nu_{k-1}}^{-1}(\nu_0,\nu_{k-1}) \) are Lagrangian submanifolds in the sense that their tangent spaces are maximal isotropic.

(ii) If \( k = 2 \) we have \( I_{+,0,1}^1 = L_{+,2}^1 \) and \( GI_{+,0,1}^\infty = GL_{+,2}^\infty \). If, in addition, we consider the finite dimensional case, that is, instead of \( L_{+,2}^1 \) we work with the traceless \( n \times n \) matrices having non-zero entries only on the main and the first lower diagonals, then \( J_\nu \) is a symplectic diffeomorphism of \( \mathbb{R}^{2(n-1)} \), endowed with the canonical symplectic structure, with a single coadjoint orbit of the upper bidiagonal group through a strictly lower diagonal element all of whose entries are non-zero (see [12] or, in triadic form in [22]).

(iii) If \( k = 2 \) and we consider the generic infinite dimensional case, that is, \( \nu_1 \) has all entries different from zero, then the map \( J_\nu \) does not provide a morphism of weak symplectic manifolds between \( \ell^\infty \times \ell^1 \) and a single coadjoint orbit of \( GL_{+,2}^\infty \). The relation between these spaces is more complicated and is explained in the diagram of Proposition (6.3). Each \( GL_{+,2}^\infty \)-coadjoint orbit through a generic element \( S^T \nu_1 \) is only weakly symplectic and Poisson injectively weakly immersed in \( L_{+,2}^1 \) but not equal to it.

(iv) If \( k = 2 \) and we consider the infinite dimensional case with \( \nu_1 \) having also some vanishing entries, the structure of the \( GL_{+,2}^\infty \)-coadjoint orbit through \( S^T \nu_1 \) reduces to the two previous cases as we shall explain below. Let \( i_0 \) be the first index for which the entry \( \nu_{i_0,i_0} = 0 \). Formula (6.7) shows that the first \( i_0 \times i_0 \) block of \( \mathcal{O}_{S^T \nu_1} \) is of that of a finite dimensional orbit of the upper bidiagonal group of matrices of size \( i_0 \times i_0 \) and that the coadjoint action preserves this block. Let \( i_1 \) be the next index for which \( \nu_{i_1,i_1} = 0 \). Again by (6.7) it follows that there is an \( i_1 \times i_1 \) block of \( \mathcal{O}_{S^T \nu_1} \) that is preserved by the coadjoint action and that is equal to a finite dimensional orbit of the upper bidiagonal group of matrices of size \( i_1 \times i_1 \). Continuing in this fashion we arrive either at an infinite sequence of orbits of finite dimensional upper bidiagonal groups (in the case that there is an infinity of indices \( i_s \) such that \( \nu_{i_s,i_s} = 0, s \in N \cup \{0\} \)) or to a generic infinite dimensional orbit of \( GL_{+,2}^\infty \) (if there are only finitely many indices \( i_s, s = 0, 1, \ldots, r \), such that \( \nu_{i_s,i_s} = 0 \)). In the latter case, the last infinite block is preserved by the coadjoint action and we are in the generic case of an orbit of \( GL_{+,2}^\infty \) but on the space complementary to the \( r + 1 \) finite dimensional blocks of sizes \( i_0 \times i_0, \ldots, i_r \times i_r \). Thus, decomposing the orbit as described, the problem of classification of the general \( GL_{+,2}^\infty \)-coadjoint orbit is reduced to the finite dimensional case and to the generic infinite dimensional case.

(v) One can restrict the Hamiltonians \( I_{S^k}^1 \) given by (5.30) to \( I_{+,0,k-1}^1 \) but these functions are not in involution because the inclusion of \( I_{+,0,k-1}^1 \) in \( L_{+,k}^1 \) is not Poisson. Indeed, as recalled in [2], the inclusion would be Poisson if and only if the kernel of its dual map is an ideal in \( L_{+,k}^1 \) which is easily seen to be false unless \( k = 2 \), in which case we have

\[
(I_{S^2}^1 \circ J_\nu)(\mathbf{q}, \mathbf{p}) = \sum_{i=0}^{\infty} p_i
\]

and

\[
H_2(\mathbf{q}, \mathbf{p}) := (I_{S^2}^1 \circ J_\nu)(\mathbf{q}, \mathbf{p}) = \sum_{i=0}^{\infty} p_i^2 + \sum_{i=0}^{\infty} \nu_{i,i}^2 e^{2(q_{i+1} - q_i)}.
\]

The function \( H_2 \) is, up to a renormalization of constants, the Hamiltonian of the semi-infinite Toda lattice. The first integral \( I_{S^2}^1 \circ \hat{J}_0 \) is the total momentum of the system which generates the translation action given by the subgroup \( \mathbb{R}_+^2 \). All integrals \( I_{S^2}^1 \circ \mathcal{J}_\nu, t \in \mathbb{N} \), give the full Toda lattice hierarchy on \( \ell^\infty \times \ell^1 \), see [8].

These considerations justify the name of Flaschka map for the momentum map \( J_\nu : \ell^\infty \times \ell^1 \to L_{+,0,1}^1 = L_{+,2}^1 \). In the next section we will present a momentum map from the weak symplectic manifold \( (\ell^\infty)^{k-1} \times (\ell^1)^{k-1} \), endowed with a weak magnetic symplectic structure, to the Banach Lie-Poisson space \( L_{+,k}^1 \). This momentum map can be considered, as we shall see, as a natural generalization of the Flaschka map to the system of integrals in involution (5.30) for \( k \geq 2 \).
In this section we construct a $GL_{\infty,k}$-equivariant momentum map $J_k : (\ell^\infty)^{k-1} \times (\ell^1)^{k-1} \to L_{1,k}^1$ (see (7.9)) which can be interpreted as a generalization of the Flaschka map (6.25) defined for the bidiagonal case. We also construct a weak symplectic form $\Omega_k$ on $(\ell^\infty)^{k-1} \times (\ell^1)^{k-1}$ (see (7.14)) which has a non-canonical term responsible for the interaction of the Toda system with some kind of an external “field”. We shall illustrate the hierarchy of dynamical systems obtained in this way by studying the special case $k = 3$ in detail (see (7.23)). The simpler case $k = 2$ does not add anything new since one recovers by the symplectic induction method the original semi-infinite Toda system obtained in the previous section.

We shall apply the induction method discussed in [3] to the weak symplectic manifold $(P,\omega) = (\ell^\infty \times \ell^1,\omega)$ with $\omega$ given by (7.3), the Banach Lie group $G := (GL_{\infty,k},\circ_k)$ defined in (4.31), and the Banach Lie subgroup $H := GI_{+,0,k-1}^\infty$ consisting of invertible bidiagonal elements of the form (6.3). As will be seen, the abstract constructions presented in [3] become completely explicit in this case.

We begin by listing the objects involved in this construction. The Banach Lie algebra is $\mathfrak{g} := L_{+,k}^\infty := \oplus_{i=0}^{k-1} L_i^\infty$, the subalgebra is $\mathfrak{h} := I_{+,0,k-1}^\infty = L_0^\infty \oplus L_{-1}^\infty$, and its closed split complement is $\mathfrak{h}^\perp := \oplus_{i=1}^{k-2} L_i^\infty := (I_{+,0,k-1}^\infty)^\perp$. At the level of the preduals we have $\mathfrak{g}_s = L_{1,k}^1 = \oplus_{i=-k+1}^{\infty} L_i$, $\mathfrak{h}_s = I_{+,0,k-1}^1 = L_0^1 \oplus L_{-2}^1$, and its closed split complement $\mathfrak{h}_s^\perp = \oplus_{i=-k+2}^{\infty} L_i^1 := (I_{-,0,k-1}^1)^\perp$. We have hence the Banach space direct sums

$$L_{+,k}^\infty = I_{+,0,k-1}^1 \oplus (I_{+,0,k-1}^1)^\perp \quad (7.1)$$

and

$$L_{1,k}^1 = I_{-,0,k-1}^1 \oplus (I_{-,0,k-1}^1)^\perp \quad (7.2)$$

Thus any $\rho \in L_{1,k}^1$ uniquely decomposes as $\rho = \gamma + \gamma^\perp$, where $\gamma = \rho_0 + (ST)^{k-1} \rho_{k-1} \in I_{+,0,k-1}^1$ and $\gamma^\perp = S^T \rho_1 + \ldots + (S^T)^{k-2} \rho_{k-2} \in (I_{-,0,k-1}^1)^\perp$. Let us show that the splitting (7.2) is invariant relative to the restriction of the adjoint action $Ad^{+k}_h$ of the Banach Lie group $GL_{+,k}^\infty$ to the Lie subgroup $GI_{+,0,k-1}^\infty$. Clearly the factor $I_{+,0,k-1}^\infty$ is preserved because it is the Lie algebra of $GI_{+,0,k-1}^\infty$. To see that the second factor $(I_{-,0,k-1}^1)^\perp$ is also preserved, using (6.5), it suffices to show that for any $h = h_0 + h_{k-1} S^{k-1} \in GI_{+,0,k-1}^\infty$ and any $x_1 S + \ldots + x_{k-2} S^{k-2} \in (I_{+,0,k-1}^1)^\perp$ we have

$$(Ad^{+k}_h)(x_1 S + \ldots + x_{k-2} S^{k-2})$$

$$= (h_0 + h_{k-1} S^{k-1}) \circ_k (x_1 S + \ldots + x_{k-2} S^{k-2}) \circ_k (h_0^{-1} h_{k-1} S^{k-1} (h_0^{-1}) S^{k-1})$$

$$= h_0 s(h_0^{-1}) x_1 S + \ldots + h_0 s^{k-2}(h_0^{-1}) x_{k-2} S^{k-2} \quad (7.3)$$

which is a straightforward verification.

Next we show that the splitting (7.2) is invariant relative to the restriction of the coadjoint action $(Ad^{+k})^*_h$ of $GL_{+,k}^\infty$ to the Lie subgroup $GI_{+,0,k-1}^\infty$. First, by (6.7) the $GI_{+,0,k-1}^\infty$ coadjoint action preserves the predual $L_{1,0,k-1}^1$. Second, to show that the second factor $(I_{+,0,k-1}^1)^\perp$ is also preserved, one verifies directly, using (6.5), that for any $h = h_0 + h_{k-1} S^{k-1} \in GI_{+,0,k-1}^\infty$ and $S^T \rho_1 + \ldots + (S^T)^{k-2} \rho_{k-2} \in (I_{-,0,k-1}^1)^\perp$ we have

$$(Ad^{+k})^*_{h^{-1}} (S^T \rho_1 + \ldots + (S^T)^{k-2} \rho_{k-2})$$

$$= S^T s(h_0) h_0^{-1} \rho_1 + \ldots + (S^T)^{k-2} s(h_0) h_0^{-1} \rho_{k-2} \quad (7.4)$$

According to the general theory we shall take the weak symplectic manifolds $GL_{+,k}^\infty \times L_{1,k}^1$ and $\ell^\infty \times \ell^1$, the canonical action $\sigma^{\nu_k} : GI_{+,0,k-1}^\infty \times (\ell^\infty \times \ell^1) \to \ell^\infty \times \ell^1$ defined in (6.26), and its equivariant momentum map $J_{\nu_k} : \ell^\infty \times \ell^1 \to L_{1,0,k-1}^1$ given by (6.22) (see Proposition 6.2). We fix in all considerations below an element $\nu_{k-1} \in L_0^1$. By (6.14), the Banach Lie group $GI_{+,0,k-1}^\infty$ acts on the product $(\ell^\infty \times \ell^1) \times GL_{+,k}^\infty \times L_{1,k}^1$ by

$$(h, (q,p), (\rho, \gamma)) := \left( (\sigma^{\nu_k} -1)(q,p),\circ_k \gamma h^{-1},(Ad^{+k})_{h^{-1}}^* (\rho) \right),$$

where $h \in GI_{+,0,k-1}^\infty$, $q \in GL_{+,k}^\infty$, $(p,q) \in \ell^\infty \times \ell^1$, and $\rho \in L_{1,k}^1$. This action admits the equivariant momentum map (8.1), which in this case becomes

$$((q,p), (\rho, \gamma)) \in (\ell^\infty \times \ell^1) \times GL_{+,k}^\infty \times L_{1,k}^1 \to J_{\nu_{k-1}} \gamma$$

$$\qquad \quad \quad \quad \quad \rightarrow -J_{\nu_{k-1}} (p,q) - \gamma \in L_{1,0,k-1}^1.$$

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The zero level set of this momentum map is a smooth manifold, $GL^\infty_{+,0,k-1}$-equivariantly diffeomorphic to $GL^\infty_{+,k} \times (\ell^\infty \times \ell^1) \times (I_{0,k-1})^\perp$, the action on the target being

$$h \cdot (g, q, p, \gamma) := \left( g \circ_0 h^{-1}, \sigma^\perp h^{-1}(q), \left( \text{Ad}^+(k) \right)^* h^{-1}_1 \gamma^\perp \right).$$

The symplectically induced bundle is hence the fiber bundle

$$GL^\infty_{+,k} \times GI^\infty_{+,0,k-1} \left( \ell^\infty \times \ell^1 \times (I_{0,k-1})^\perp \right) \rightarrow GL^\infty_{+,k} / GI^\infty_{+,0,k-1}$$

associated to the principal bundle $GI^\infty_{+,k} \rightarrow GL^\infty_{+,k} / GI^\infty_{+,0,k-1}$.

We begin by explicitly determining the base of this bundle. If $g = g_0 + \cdots + g_{k-1}S^{k-1} \in GL^\infty_{+,k}$ and $h = h_0 + h_{k-1}S^{k-1} \in GI^\infty_{+,0,k-1}$ then

$$g \circ_0 h^{-1} = (g_0 + \cdots + g_{k-1}S^{k-1}) \circ_0 (h_0^{-1} - h_{k-1}^{-1}s^{-1}(h_0^{-1})S^{k-1})$$

$$= g_0h_0^{-1} + g_1s(h_0^{-1})S + \cdots + g_{k-2}s^{k-2}(h_0^{-1})S^{k-2} + (g_{k-1}s^{k-1}(h_0^{-1}) - g_0h_0^{-1}h_{k-1}s^{-1}(h_0^{-1}))S^{k-1}.$$

Therefore, the smooth map $GL^\infty_{+,k} \rightarrow (\ell^\infty)^{k-2}$ given by

$$GL^\infty_{+,k} \ni g_0 + \cdots + g_{k-1}S^{k-1} \mapsto (g_0 + \cdots + g_{k-1}S^{k-1}) \ni (g_0^{-1} - g_{k-1}s^{k-1}(h_0^{-1})S^{k-1})$$

$$= \mathbb{I} + g_1s(h_0^{-1})S + \cdots + g_{k-2}s^{k-2}(h_0^{-1})S^{k-2}$$

$$\mapsto (g_1s(h_0^{-1}), \ldots, g_{k-2}S^{k-2}(h_0^{-1})) \in (\ell^\infty)^{k-2}$$

factors through the $GI^\infty_{+,0,k-1}$-action thus inducing a smooth map $GL^\infty_{+,k} / GI^\infty_{+,0,k-1} \rightarrow (\ell^\infty)^{k-2}$. Its inverse is the smooth map

$$(q_1, \ldots, q_{k-2}) \in (\ell^\infty)^{k-2} \mapsto [\mathbb{I} + q_1S + \cdots + q_{k-2}S^{k-2}] \in GL^\infty_{+,k} / GI^\infty_{+,0,k-1}$$

which proves that $GL^\infty_{+,k} / GI^\infty_{+,0,k-1}$ is diffeomorphic to $\ell^\infty)^{k-2}$.

Next, we shall prove that the smooth map

$$\Phi : (\ell^\infty \times \ell^1) \times (\ell^\infty)^{k-2} \times (\ell^1)^{k-2} \rightarrow GL^\infty_{+,k} \times GI^\infty_{+,0,k-1} \left( \ell^\infty \times \ell^1 \times (I_{0,k-1})^\perp \right)$$

given by

$$\Phi((q, p), q_1, \ldots, q_{k-2}, p_1, \ldots, p_{k-2})$$

$$:= \left[ [\mathbb{I} + q_1S + \cdots + q_{k-2}S^{k-2}, (q, p), S^T p_1 + \cdots + (S^T)^{k-2}p_{k-2}] \right]$$

is a diffeomorphism thereby trivializing the associated bundle, which is the reduced space. Indeed, this map has a smooth inverse given by

$$\Phi^{-1}(\{(g_0 + \cdots + g_{k-1}S^{k-1}, (q, p), \gamma^\perp)\})$$

$$= \left( \sigma^\gamma g_0 + g_{k-1}S^{k-1}, (q, p), g_1s(h_0^{-1}), \ldots, g_{k-2}s^{k-2}(g_0^{-1}), \left( \text{Ad}^+(k) \right)^* h_0^{-1}h_{k-1}S^{k-1} \right)^\perp,$$

where, in the third component of the right hand side we have identified $(I_{0,k}^1)^\perp$ with $(\ell^1)^{k-2}$ through the isomorphisms $L_1^k \cong \ell^1$.

The $GL^\infty_{+,k}$-action on the reduced manifold $GL^\infty_{+,k} \times GI^\infty_{+,0,k-1} \left( \ell^\infty \times \ell^1 \times (I_{0,k-1})^\perp \right)$ is given by $g \cdot [g, (q, p), \gamma] = [g \circ_0 g, (q, p), \gamma^\perp]$ for any $g, g' \in GL^\infty_{+,k}$, $(q, p) \in \ell^\infty \times \ell^1$, and $\gamma \in (I_{0,k-1})^\perp$. Via the globally trivializing diffeomorphism $\Phi$, the induced $GL^\infty_{+,k}$-action on $(\ell^\infty \times \ell^1) \times (\ell^\infty)^{k-2} \times (\ell^1)^{k-2}$ has the
expression

\[(g_0 + \cdots + g_{k-1}S^{k-1}) \cdot ((q, p), q_1, \ldots, q_{k-2}, p_1, \ldots, p_{k-2})
\]

\[= \Phi^{-1}((g_0 + \cdots + g_{k-1}S^{k-1}) \cdot \Phi((q, p), (q_1, \ldots, q_{k-2}, p_1, \ldots, p_{k-2}))
\]

\[= \Phi^{-1}((g_0 + \cdots + g_{k-1}S^{k-1}) \cdot (l + q_1S + \cdots + q_{k-2}S^{k-2}, (q, p), S^T p_1 + \cdots + (S^T)^{k-2} p_{k-2}))
\]

\[= \Phi^{-1}((g_0 + \cdots + g_{k-1}S^{k-1}) \circ_k (l + q_1S + \cdots + q_{k-2}S^{k-2}, (q, p), S^T p_1 + \cdots + (S^T)^{k-2} p_{k-2}))
\]

\[= \Phi^{-1}\left(\left[\left(\sum_{i=0}^{k-2} g_{k-1-i}s^{k-1-i}(q_i)\right) S^\ell + \sum_{i=0}^{k-2} g_{k-1-i}s^{k-1-i}(q_i) S^{k-1}, (q, p), S^T p_1 + \cdots + (S^T)^{k-2} p_{k-2}\right]\right)
\]

\[= \left(\sigma_{\nu k-1} g_0 + (\sum_{i=0}^{k-2} g_{k-1-i}s^{k-1-i}(q_i)) s^{k-1}(q, p), s(g_0^{-1})\sum_{i=0}^{k-2} g_{k-1-i}s^{k-1-i}(q_i), \ldots, s(g_0^{-1})\sum_{i=0}^{k-2} g_{k-2-i}s^{k-2-i}(q_i), s(g_0^{-1})s^{k-2}(g_0)g_0^{-1}\right),
\]

where the equality in the last \(k - 2\) components follows from (7.4). Let us summarize the considerations above. Using (6.20) and denoting \(((q', p'), q'_1, \ldots, q'_{k-2}, p'_1, \ldots, p'_{k-2}) := (g_0 + \cdots + g_{k-1}S^{k-1}) \cdot ((q, p), q_1, \ldots, q_{k-2}, p_1, \ldots, p_{k-2})\), we conclude that the \(GL_{k,\ell}^\ell\)-action on the reduced manifold \((\ell^\infty \times \ell^1) \times (\ell^\infty)^{k-2} \times (\ell^1)^{k-2}\) is given by

\[q' = q + \log g_0 \quad (7.5)
\]

\[p' = p + \left(\sum_{i=0}^{k-2} g_{k-1-i}s^{k-1-i}(q_i)\right) g_0^{-1} \nu_{k-1} e^{s^{k-1}(q) - q} - s^{k-1}\left(\sum_{i=0}^{k-2} g_{k-1-i}s^{k-1-i}(q_i)\right) g_0^{-1} \nu_{k-1} e^{s^{k-1}(q) - q} \quad (7.6)
\]

\[q'_l = s(g_0^{-1})\sum_{i=0}^{l} g_{k-1-i}s^{l-i}(q_i) \quad (7.7)
\]

\[p'_l = s'(g_0)g_0^{-1} p_l, \quad l = 1, \ldots, k - 2. \quad (7.8)
\]

All geometric objects described above satisfy the assumptions of Propositions 3.4 and 3.5 and thus one has the weak symplectic form \(\Omega_k\) and the momentum map \(J_k : (\ell^\infty \times \ell^1) \times (\ell^\infty)^{k-2} \times (\ell^1)^{k-2} \rightarrow L_{\nu,\ell}^\ell\) given by (3.16) and (3.18), respectively. By (4.51), \(J_k\) takes the form

\[J_k ((q, p), q_1, \ldots, q_{k-2}, p_1, \ldots, p_{k-2})
\]

\[= (Ad^{+k})^* (l + q_1S + \cdots + q_{k-2}S^{k-2})^{-1} \left(\mathcal{J}_{\nu_{k-1}}(q, p) + S^T p_1 + \cdots + (S^T)^{k-2} p_{k-2}\right)
\]

\[= (Ad^{+k})^* (l + q_1S + \cdots + q_{k-2}S^{k-2})^{-1} \left(p + S^T p_1 + \cdots + (S^T)^{k-2} p_{k-2}\right)
\]

\[+ (S^T)^{k-1} \nu_{k-1} e^{s^{k-1}(q) - q}, \quad (7.9)
\]

where the inverse \((l + q_1S + \cdots + q_{k-2}S^{k-2})^{-1}\) is given by (4.36). We shall call \(J_k\) the generalized Flaschka map.
In order to obtain the explicit expression of the weak symplectic form $\Omega_k$ (see (7.11) on the induced symplectic manifold $(\ell^\infty \times \ell^1) \times (\ell^\infty)^{k-2} \times (\ell^1)^{k-2}$, let us notice that the symplectic form $\omega + \omega_L$ on $(\ell^\infty \times \ell^1) \times GL^\infty_{+k} \times L^1_{-k}$ is given by

$$\omega + \omega_L = -d \left( \text{Tr}(pdq) + \text{Tr}(\rho g^{-1} \circ_k dq) \right),\quad (7.10)$$

where $g^{-1} \circ_k dq$ is the left Maurer-Cartan form on the Banach Lie group $GL^\infty_{+k}$. One has the following decomposition

$$\theta := \text{Tr}(\rho g^{-1} \circ_k dq) = \text{Tr} \left( \sum_{l=0}^{k-1} \rho_l \theta_l \right),\quad (7.11)$$

for $\rho = \rho_0 + S^T \rho_1 + \cdots + (S^T)^{k-1} \rho_{k-1} \in L^1_{-k}$ with

$$\theta_l = \sum_{i=0}^{l} h_i(g) s^{i}(dq_{l-i}), \quad l = 0, 1, \ldots, k - 1.$$ 

The diagonal operators $h_i$ are the components of $g^{-1} = h_0 + h_1 S + \cdots + h_{k-1} S^{k-1}$ given by (4.36). Let $\tilde{\theta}$ be the pull back of $\theta$ to the zero level set of the momentum map (3.15). Next, we pull back the form $\tilde{\theta}$ to $(\ell^\infty \times \ell^1) \times (\ell^\infty)^{k-2} \times (\ell^1)^{k-2}$ by the global section $\Sigma : (\ell^\infty \times \ell^1) \times (\ell^\infty)^{k-2} \times (\ell^1)^{k-2} \to GL^\infty_{+k} \times (\ell^\infty \times \ell^1) \times (L^1_{-0,k-1})^k$ defined by

$$\Sigma((q, p), q_1, \ldots, q_{k-2}, p_1, \ldots, p_{k-2}) := (1 + q_1 S, + \cdots + q_{k-2} S^{k-2}, (q, p), S^T p_1 + \cdots + (S^T)^{k-2} p_{k-2}).$$

Therefore, we get

$$\Sigma^* \tilde{\theta} := \text{Tr}(pdq) + \text{Tr} \left[ (J_{v_{k-1}}(q, p))_{q} \theta_0 \right] + \text{Tr} \left( (J_{v_{k-1}}(q, p))_{k-1} \theta_{k-1} \right) + \text{Tr} \left( \sum_{l=1}^{k-2} p_l \theta_{l} \right)$$

$$= \text{Tr}(pdq) + \text{Tr} \left( \sum_{l=1}^{k-2} p_l \sum_{i=0}^{l-1} h_i(q_1, \ldots, q_i) s^{i}(dq_{l-i}) \right) + \text{Tr} \left( \nu_{k-1} e^{s^{k-1}(q)} - q \sum_{i=1}^{k-2} h_i(q_1, \ldots, q_i) s^{i}(dq_{k-1-i}) \right),\quad (7.12)$$

since $\theta_0 = 0$, where $h_i(q_1, \ldots, q_i)$ is given by (4.36) with $g_0 = (1, 1, \ldots)$, $g_1 = q_1, \ldots, g_{k-2} = q_{k-2}$, $g_{k-1} = (0, 0, \ldots)$. Since $\delta \to \tilde{\delta}^i(\delta)$ for any $\delta \in L^1_0$ and $j \in \mathbb{N}$, by (4.10) the last summand in (7.12) becomes

$$\sum_{i=1}^{k-2} \text{Tr} \left[ \tilde{s}^{i} \left( \nu_{k-1} e^{s^{k-1}(q)} - q h_i(q_1, \ldots, q_i) \right) (I - \sum_{r=0}^{i-1} p_r) dq_{k-1-i} \right]$$

$$= \sum_{i=1}^{k-2} \text{Tr} \left[ \tilde{s}^{i} \left( \nu_{k-1} e^{s^{k-1}(q)} - q h_i(q_1, \ldots, q_i) \right) dq_{k-1-i} \right]$$

because

$$\tilde{s}^{i}(\delta) \sum_{r=0}^{j-1} p_r = 0 \quad \text{for all} \quad \delta \in L^1_0 \quad \text{and} \quad j \in \mathbb{N}.$$ 

Similarly, the second summand in (7.12) equals

$$\sum_{i=1}^{k-2} \sum_{l=0}^{i-1} \text{Tr} \left[ \tilde{s}^{i} (p_l h_i(q_1, \ldots, q_i)) dq_{l-i} \right]$$
so that (7.12) becomes

$$\Sigma^* \tilde{\theta} = \text{Tr}(pdq) + \sum_{i=1}^{k-2} \text{Tr} \left( \sum_{j=0}^{l-1} \tilde{s}^j (p_i h_i(q_1, \ldots, q_l)) dq_{l-i} \right) + \tilde{s}^l \left( \nu_{k-1} e^{s^{k-1}(q)} - q h_i(q_1, \ldots, q_l) \right) dq_{l-1-k} \right)$$

$$= \text{Tr}(pdq) + \sum_{i=1}^{k-2} \left[ \text{Tr} \left( \sum_{j=0}^{l-1} \tilde{s}^j (p_i h_i(q_1, \ldots, q_l)) \right) + \tilde{s}^l \left( \nu_{k-1} e^{s^{k-1}(q)} - q h_i(q_1, \ldots, q_l) \right) \right] dq_i.$$

(7.13)

Then the reduced symplectic form is

$$\Omega_k = -d\Sigma^* \tilde{\theta}. \quad (7.14)$$

Indeed, a straightforward verification shows that $-d\Sigma^* \tilde{\theta}$ satisfies the condition characterizing the reduced symplectic form, so it must be equal to it. Note that the one-form $\Sigma^* \tilde{\theta}$ depends on the chosen section $\Sigma$, but that if $\Sigma$ is any other global section, then $d\Sigma^* \tilde{\theta} = d\Sigma^* \tilde{\theta} = \Omega_k$. In particular, the reduced symplectic form $\Omega_k$ is in this case exact. Note also that the symplectic form $\Omega_k$ is canonical only if $k = 2$ and magnetic only if $k = 3$, a case that we shall analyze in detail below. In general, if $k > 3$, the weak symplectic form $\Omega_k$ is neither canonical nor magnetic due to the presence of the $p_j$-dependent coefficients of $dq$ in the first sum of the second term.

To deal with the Hamiltonian systems defined by the function $I^S_k$ we need to regard the momentum map $J_k$ as having values in $L^{1,k}_S$. This is achieved by defining the equivariant momentum map $J^S_k := \Phi_{S,-k} \circ J^S_k : (L^\infty \times \ell^1) \times (L^\infty \ell^1) \rightarrow L^{1,k}_S$, where $\Phi_{S,-k} : (L^1_{S,-k}, \{\cdot, \cdot\}_{-k}) \rightarrow (L^1_{S,k}, \{\cdot, \cdot\}_S)$ is the isomorphism of Banach Lie-Poisson spaces introduced at the end of §4. Recall that the effect of $\Phi_{S,-k}$ on an element in $L^1_{S,-k}$ is adding to it the transpose of its strictly lower triangular part. Since $J^S_k$ is a Poisson map and the functions $I^S_k$ are in involution on $L^{1,k}_S$, it follows that $I^S_k \circ J^S_k$ are also in involution on the weak symplectic manifold $((L^\infty \times \ell^1) \times (L^\infty \ell^1) \times (L^\infty \ell^1), \Omega_k)$ provided that these functions admit Hamiltonian vector fields.

The case $k = 2$. In this case we have $I^1_{0,1} = L^1_{1,2}$ and $GL^\infty_{1,0,1} = GL^\infty_{1,0,2}$. As we discussed earlier, the induction method yields in this case the original weak symplectic manifold $(L^\infty \times \ell^1, \omega)$. This is the case of the standard semi-infinite Toda lattice.

The case $k = 3$. This is the first situation that goes beyond the Toda lattice. The Banach Lie group $G := (GL^\infty_{1,3}, \cdot)$ consists of bounded operators having only three upper diagonals, while the operators in $GL^\infty_{1,0,2}$ have non-zero entries only on the main and the second strictly upper diagonal. The induced space is now $(L^\infty \times \ell^1) \times (L^\infty \times \ell^1)$. The $GL^\infty_{1,3}$-action on $(L^\infty \times \ell^1) \times (L^\infty \times \ell^1)$ is given, according to (7.9) - (7.8) by

$$q'_l = q_l + \log g_0 \quad (7.15)$$

$$p'_l = p_l + g_2 g_0^{-1} \nu_2 e^{s^2(q)} q_l + g_1 s(q_1) g_0^{-1} \nu_2 e^{s(q)} q_l$$

$$- \tilde{s}^l \left( g_2 g_0^{-1} \nu_2 e^{s^2(q)} q_l + g_1 s(q_1) g_0^{-1} \nu_2 e^{s(q)} q_l \right) \quad (7.16)$$

$$q'_i = s(g_0^{-1})(g_1 + g_0 q_i) \quad (7.17)$$

$$p'_i = s(g_0^{-1}) p_i, \quad l = 1, \ldots, k - 2. \quad (7.18)$$

The reduced symplectic form on $(L^\infty \times \ell^1) \times (L^\infty \times \ell^1)$ is, according to (4.36), (7.13), and (7.14), equal to

$$\Omega_3 = -d \left( \text{Tr}(pdq) + \text{Tr}(p_1 dq_1) - \text{Tr} \left( \nu_2 e^{s^2(q)} - q_1 s(dq_1) \right) \right)$$

$$= -d \left( \text{Tr}(pdq) + \text{Tr} \left( (p_1 - \tilde{s} \left( \nu_2 e^{s^2(q)} q_1 \right) \right) dq_1 \right)$$

$$= -d [\text{Tr}(pdq) + \text{Tr}(p_2 dq_1)], \quad (7.19)$$

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where

\[ \tilde{\mathbf{p}}_1 := \mathbf{p}_1 - \tilde{s} \left( \nu_2 e^{s^2(q)} q \mathbf{q}_1 \right). \]  

(7.20)

We see here exactly the same phenomenon as in classical electrodynamics, where a momentum shift by the magnetic potential transforms the non-canonical magnetic symplectic form to the canonical one.

The equivariant moment map \( (\mathbf{8.3}) \) of this action is by \( (\mathbf{8.1}) \) and \( (\mathbf{8.2}) \) equal to

\[
J_3(q, p, q_1, p_1) = (\text{Ad}^+)^*_{(I + q_1 S)^{-1}} \left( \mathbf{p} + S^T \mathbf{p}_1 + (S^T)^2 \nu_2 e^{s^2(q)} q \right)
\]

\[
= \mathbf{p} + q_1 \mathbf{p}_1 - \tilde{s} \left( q_1 \mathbf{p}_1 + S(q_1) \nu_2 e^{s^2(q)} q \mathbf{q}_1 \right) + s^2 \left( \nu_2 e^{s^2(q)} q \mathbf{q}_1 S(q_1) \right)
\]

\[
+ S^T \left( \mathbf{p}_1 + S(q_1) \nu_2 e^{s^2(q)} q \mathbf{q}_1 - \tilde{s} \left( \nu_2 e^{s^2(q)} q \mathbf{q}_1 \right) \right) + (S^T)^2 \nu_2 e^{s^2(q)} q
\]

\[
= \mathbf{p} + q_1 \mathbf{p}_1 - \tilde{s} \left( q_1 \mathbf{p}_1 \right) - \tilde{s} \left( \nu_2 e^{s^2(q)} q \mathbf{q}_1 \right) q_1 + s^2 \left( \nu_2 e^{s^2(q)} q \mathbf{q}_1 \right) \tilde{s}(q_1)
\]

\[
+ S^T \left( \mathbf{p}_1 + S(q_1) \nu_2 e^{s^2(q)} q - \tilde{s} \left( \nu_2 e^{s^2(q)} q \mathbf{q}_1 \right) \right) + (S^T)^2 \nu_2 e^{s^2(q)} q
\]

\[
= \mathbf{p} + q_1 \mathbf{p}_1 - \tilde{s} (q_1 \mathbf{p}_1) + \tilde{s} \left( \mathbf{p}_1 + S(q_1) \nu_2 e^{s^2(q)} q \right) + (S^T)^2 \nu_2 e^{s^2(q)} q
\]

(7.21)

since the inverse of \( I + q_1 S \) in the Banach Lie group GL_{+,3} is equal to \( (I + q_1 S)^{-1} = I - q_1 S + q_1 s(q_1) S^2 \in GL_{+,3} \).

The Hamiltonians \( I_{S,3} \) given in \( (\mathbf{8.3}) \) are in involution on \( L_{S,3}^1 \) and hence the functions \( I_{S,3} \circ J_3^S \) are in involution on \( (L^\infty \times \ell^1) \times \ell^1, \Omega_3 \), provided that they have Hamiltonian vector fields relative to the weak symplectic form \( \Omega_3 \).

For \( l = 1, 2 \), the Hamiltonians \( H_1 := I_{1}^{S,3} \circ J_3^S \) and \( H_2 := I_{2}^{S,3} \circ J_3^S \) have the expressions

\[
H_1(q, p, q_1, p_1) = \text{Tr}(p)
\]

(7.22)

and

\[
H_2(q, p, q_1, p_1) = \frac{1}{2} \text{Tr} \left[ \mathbf{p} + q_1 \mathbf{p}_1 - \tilde{s} (q_1 \mathbf{p}_1) \right]^2 + \text{Tr} \left( \tilde{s} \left( \mathbf{p}_1 + S(q_1) \nu_2 e^{s^2(q)} q \right) \right)^2
\]

(7.23)

The Hamiltonian system defined by \( H_2 \) describes a semi-infinite family of particles in an external field (given by the magnetic term of the symplectic form \( (\mathbf{7.1}) \) and where the interaction is between every second neighbor. In the case of the Toda lattice (obtained for \( k = 2 \), as discussed above), there is no external field and the interaction is between nearest neighbors. The solution of the semi-infinite Toda lattice will be given in \( \mathbf{8} \). For arbitrary \( k \) there is an external field and interaction of particles is between every \((k - 1)\)st neighbor.

We have given here only the first two Hamiltonians of an infinite family of functions in involution. Involutivity follows because they are obtained from a family of integrals in involution, namely the \( I_{k}^{S,3} \) by pull back with the Poisson map \( J_3^S \).

8 The semi-infinite Toda lattice

In this section we illustrate the theory of the \( k \)-diagonal Hamiltonian systems by the detailed investigation of the semi-infinite Toda lattice which is an example of a bidiagonal system (see Remark (v) at the end of \( \mathbf{8} \)). We shall follow the method of orthogonal polynomials first proposed in \( \mathbf{4} \), as far as we know. We shall extend below the results in \( \mathbf{17} \) for the finite Toda lattice by explicitly solving the semi-infinite Toda lattice both in action-angle variables as well as giving all the flows of the full hierarchy in the original variables.

The family of Hamiltonians \( I_{l}^{S,2} \in C^\infty (L_{S,2}^1) \), \( l \in \mathbb{N} \), leads to the chain of Hamilton equations

\[
\frac{\partial}{\partial t_l} \rho = [\rho, B_l], \quad \text{where} \quad B_l := P^\infty (\rho^l) - (P^\infty (\rho^l))^T,
\]

(8.1)
on the Banach Lie-Poisson space \((L^1_{S,2},\{\cdot,\cdot\}_{S,2})\) (or on the space \((L^1_{L,2},\{\cdot,\cdot\}_{L,2})\) isomorphic to it) induced from \((5.21)\) by the inclusion \(i_{S,2} : L^1_{S,2} \hookrightarrow L^1_S\).

The selfadjoint trace class operator \(\rho \in L^1_{S,2}\) acts on the orthonormal basis \(\{|k\}\}_{k=0}^\infty\) of \(\mathcal{H}\) as follows:

\[
\rho |k\rangle = \rho_{k-1,k} |k-1\rangle + \rho_{kk} |k\rangle + \rho_{k,k+1} |k+1\rangle,
\]

where \(k \in \mathbb{N} \cup \{0\}\) and we set \(\rho_{-1,0} = 0\).

Note that if \(\rho\) is replaced by \(\rho' := \rho \cdot \delta + b \mathbb{I}\), where \(b, c \in \mathbb{R}, c \neq 0\), then the equations \((8.1)\) remain unchanged by rescaling the time \(t' := c^{-1}t\). As will be explained later, the norm \(\|\rho\|_\infty\) and the positivity \(\rho \geq 0\) are preserved by the evolution defined by \((8.1)\). Taking into account the above facts, we can assume, without loss of generality, that \(\|\rho\|_\infty < 1\) and \(\rho \geq 0\). Consequently, from now on we shall work with generic initial conditions \(\rho(0)\) for the Hamiltonian system \((8.1)\), i.e.,

\[
\lambda_m(0) \neq \lambda_n(0), \quad \text{for} \quad n \neq m
\]

\[
\lambda_m(0) > 0 \quad \text{and} \quad \sup_{m \in \mathbb{N} \cup \{0\}} \{\lambda_m(0)\} < 1,
\]

where \(\lambda_m(0)\) are the eigenvalues of \(\rho(0)\). This means that \(\rho(0)\) has simple spectrum, \(\rho(0) \geq 0\), and \(\|\rho(0)\|_\infty < 1\). These hypotheses imply that \(\rho_{k,k+1}(0) > 0\) for all \(k \in \mathbb{N} \cup \{0\}\) and are consistent with the physical interpretation of the semi-infinite Toda system. Let us denote by \(\Omega^1_{S,2} \subset L^1_{S,2}\) the open set consisting of operators satisfying \((8.3)\) and \((8.4)\).

From \((8.2)\), it follows that

\[
|k\rangle = P_k(\rho)|0\rangle,
\]

where the the polynomials \(P_k(\lambda) \in \mathbb{R}[\lambda], k \in \mathbb{N} \cup \{0\}\), are obtained by solving the three term recurrence equation

\[
\lambda P_k(\lambda) = \rho_{k-1,k} P_{k-1}(\lambda) + \rho_{kk} P_k(\lambda) + \rho_{k,k+1} P_{k+1}(\lambda)
\]

with initial condition \(P_0(\lambda) \equiv 1\). Note that the degree of \(P_k(\lambda)\) is \(k\).

We show now that the operator \(\rho \in L^1_{S,2}\) evolving according to \((8.1)\) also has simple spectrum independent of all times \(t\). To do this, we write the spectral resolution

\[
\rho = \sum_{m=0}^\infty \lambda_m \mathbb{P}_m, \quad \mathbb{P}_m \mathbb{P}_n = \delta_{mn} \mathbb{P}_n, \quad \sum_{m=0}^\infty \mathbb{P}_m = \mathbb{I},
\]

where

\[
\mathbb{P}_m := \frac{\langle \lambda_m \rangle}{\langle \lambda_m | \lambda_m \rangle}
\]

are the projectors on the one-dimensional eigenspaces spanned by the eigenvector \(|\lambda_m\rangle\). From \((8.1)\) one obtains

\[
\left( \frac{\partial}{\partial t} \lambda_k \right) \mathbb{P}_n \mathbb{P}_k + (\lambda_n - \lambda_k) \left[ \left( \frac{\partial}{\partial t} \mathbb{P}_n \right) \mathbb{P}_k - \mathbb{P}_n B_l \mathbb{P}_k \right] = 0
\]

for any \(n, k \in \mathbb{N} \cup \{0\}\) and \(l \in \mathbb{N}\). Putting \(n = k\) in \((8.9)\) one finds

\[
\frac{\partial}{\partial t} \lambda_n = 0
\]

for any \(n \in \mathbb{N} \cup \{0\}\) and \(l \in \mathbb{N}\). Thus \(\lambda_m = \lambda_m(0) \neq \lambda_n\) for \(n \neq m\) and we can conclude that the coefficients in

\[
|\lambda_m\rangle = \sum_{l=0}^\infty P_l(\lambda_m)|l\rangle
\]

are the values \(P_l(\lambda_m)\) at the eigenvalue \(\lambda_m\) of the polynomials \(P_l(\lambda)\) which are orthogonal relative to the \(L^2\)-scalar product given by the measure \(\sigma\) in \((8.15)\).

Taking \(n \neq k\) in \((8.9)\) and using properties of orthogonal projectors one obtains

\[
\frac{\partial}{\partial t} \mathbb{P}_n = [\mathbb{P}_n, B_l] \quad \text{for any} \quad n \in \mathbb{N} \cup \{0\} \quad \text{and} \quad l \in \mathbb{N}.
\]
Similarly, for the resolvent
\[ R_\lambda := (\rho - \lambda I)^{-1} = \sum_{m=0}^{\infty} \frac{1}{\lambda_m - \lambda} \mathbb{P}_m \] 
(8.13)
by (8.12) one has
\[ \frac{\partial}{\partial t_i} R_\lambda = \sum_{m=0}^{\infty} \frac{1}{\lambda_m - \lambda} [\mathbb{P}_m, B_i] = [R_\lambda, B_i]. \] 
(8.14)

Note that (8.5) implies that the vector \( |0\rangle \) is cyclic for \( \rho \). Thus, one has a unitary isomorphism of \( \mathcal{H} \) with \( L^2(\mathbb{R}, d\sigma) \), where the measure
\[ d\sigma(\lambda) := d\langle 0|\mathbb{P}\lambda 0\rangle = \sum_{m=0}^{\infty} \mu_m \delta(\lambda - \lambda_m)d\lambda, \] 
(8.15)
is given by the orthogonal resolution of the unity \( \mathbb{P} : \mathbb{R} \ni \lambda \mapsto \mathbb{P}_\lambda \in L^\infty(\mathcal{H}) \) for
\[ \rho = \int_{\mathbb{R}} \lambda d\mathbb{P}_\lambda. \]
The **masses** \( \mu_m \) in (8.15) are given by
\[ \mu_m^{-1} = \langle \lambda_m | \lambda_m \rangle = \sum_{l=0}^{\infty} (P_l(\lambda_m))^2. \] 
(8.16)
Using \( P_m|0\rangle = \mu_m |\lambda_m\rangle \) and \( \mu_m = \langle 0|\mathbb{P}_m 0\rangle \), one obtains from (8.12) the differential equation
\[ \frac{\partial}{\partial t_i} \mu_m = 2\langle \lambda_m | B_i 0\rangle \mu_m = 2 \left( \lambda_m^l - \langle 0|\rho^l 0\rangle \right) \mu_m \] 
(8.17)
for any \( l \in \mathbb{N} \) and \( m \in \mathbb{N} \cup \{0\} \). In order to prove the second equality in (8.17) we notice that
\[ B_i = \rho^l - P_0^\infty (\rho^l) - 2 \left[ P_0^\infty (\rho^l) \right]^T \] 
(8.18)
\[ \left[ P_0^\infty (\rho^l) \right]^T |0\rangle = 0 \] 
(8.19)
\[ P_0^\infty (\rho^l)|0\rangle = \langle 0|P_0^\infty (\rho^l) 0\rangle |0\rangle = \langle 0|\rho^l 0\rangle |0\rangle \] 
(8.20)
which implies
\[ \langle \lambda_m | B_i 0\rangle = \lambda_m^l - \langle 0|\rho^l 0\rangle. \] 
(8.21)
Using (8.17) and noticing that
\[ \sigma_k = \langle 0|\rho^k 0\rangle \] 
(8.22)
one obtains the system of equations
\[ \frac{\partial}{\partial t_i} \sigma_k = 2 \left( \sigma_{k+l} - \sigma_l \sigma_k \right), \] 
(8.23)
where \( \sigma_0 = 1, k \in \mathbb{N} \cup \{0\}, l \in \mathbb{N} \), for the moments
\[ \sigma_k = \int_{\mathbb{R}} \lambda^k d\sigma(\lambda) = \sum_{m=0}^{\infty} \lambda_m^k \mu_m \] 
(8.24)
of the measure (8.15). Let us remark here that in the considered case the moment problem is determined, i.e., the moments \( \sigma_k \) determine the measure (8.15) in a unique way (see, e.g. [3]).

Let us comment on the formulas obtained above. Introduce the diagonal trace class operators \( \lambda, \mu, \sigma \in L_0^1 \) by defining their \( m^{th} \) components to be the eigenvalues \( \lambda_m \), the masses \( \mu_m \), and the moments \( \sigma_m, m \in \mathbb{N} \cup \{0\} \), respectively. On the open subset \( \Omega^1_{1,2} \) one has three naturally defined smooth coordinate systems:

(i) \( \rho \in \Omega^1_{1,2} \),

(ii) \( (\lambda, \mu) \in L_0^1 \times L_0^1 \), where \( \text{Tr} \, \mu = 1 \) and \( \mu > 0 \),

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(iii) \( \sigma \in L_0^1 \) with first component \( \sigma_0 = 1, \sigma > 0, \) and \( d_0 > 0, \)
where
\[
d_0 := \sum_{k=0}^{\infty} d_{0k} |k\rangle \langle k|,
\]
and
\[
d_0k := \det \begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \ldots & \sigma_k \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \ldots & \sigma_{k+1} \\ \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \ldots & \sigma_{k+2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_k & \sigma_{k+1} & \sigma_{k+2} & \sigma_{k+3} & \ldots & \sigma_{2k} \end{bmatrix} > 0, \tag{8.25}
\]
with the convention that \( d_{0,-1} = 1. \) In order to see that \( \sigma \in L_0^1 \) we notice that
\[
\sum_{k=0}^{\infty} \sigma_k = \sum_{k=0}^{\infty} (0|\rho^k|0) \leq \sum_{k=0}^{\infty} \|\rho\|^k \leq \sum_{k=0}^{\infty} \|\rho\|^k = \frac{1}{1 - \|\rho\| \infty} < +\infty.
\]
We also define \( d_1 := \sum_{k=0}^{\infty} d_{1k} |k\rangle \langle k|, \)
where
\[
d_{1k} := \det \begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \ldots & \sigma_{k-1} & \sigma_{k+1} \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \ldots & \sigma_k & \sigma_{k+2} \\ \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \ldots & \sigma_{k+1} & \sigma_{k+3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{k-1} & \sigma_k & \sigma_{k+1} & \sigma_{k+2} & \ldots & \sigma_{2k-1} & \sigma_{2k+1} \end{bmatrix} \tag{8.26}
\]
for \( n \in \mathbb{N} \cup \{0\}. \)

The transformation from \( \rho \)-coordinates to \( \sigma \)-coordinates is given by formula \( 8.22 \). The inverse transformation to \( 8.22 \) has the form
\[
\rho = S^T \rho_1 + \rho_0 + \rho_1 S
\]
\[
= S^T \left[ \tilde{s}(d_0)s(d_0) \right]^{1/2} d_0^{-1} d_0^{-1} d_1 - \tilde{s}(d_0) s(d_0) \right]^{1/2} d_0^{-1} S, \tag{8.27}
\]
or, in components (see, e.g., \[3\]),
\[
\rho_{kk} = d_{0k}^{-1} d_{1k} - d_{0,k-1}^{-1} d_{1,k-1} \quad \text{and} \quad \rho_{k,k+1} = (d_{0,k-1} d_{0,k+1})^{1/2} d_{0k}^{-1} > 0. \tag{8.28}
\]
Formula \( 8.24 \) gives the transformation from \( (\lambda, \mu) \)-coordinates to \( \sigma \)-coordinates. The inverse transformation to \( 8.24 \) is obtained by expanding the so-called Weyl function \( (0|R_\lambda|0) \) in a Laurent series
\[
(0|R_\lambda|0) = \sum_{m=0}^{\infty} \frac{\mu_m}{\lambda_m - \lambda} = - \sum_{k=0}^{\infty} \frac{\sigma_k}{\lambda^{k+1}} \tag{8.29}
\]
for \( |\lambda| > \sup_{m \in \mathbb{N} \cup \{0\}} \{ |\lambda_m| \} = \|\rho\| \infty. \) So, one finds \( (\lambda, \mu) \) by computing the Mittag-Leffler decomposition of the left hand side of \( 8.29. \)

The passage from \( \rho \)-coordinates to \( (\lambda, \mu) \)-coordinates is obtained by composing the previously described transformations. This can also be done directly constructing the spectral resolution for \( \rho. \)

After these remarks we present Hamilton’s equations \( 8.1 \) in the coordinates \( (\lambda, \mu) \)
\[
\frac{\partial}{\partial t_1} \lambda = \{ \lambda, I_1^{S,2} \}_{S,2} = 0 \tag{8.30}
\]
\[
\frac{\partial}{\partial t_1} \mu = \{ \mu, I_1^{S,2} \}_{S,2} = 2 \left( \lambda^t - \text{Tr}(\lambda^t \mu) \right) \mu \tag{8.31}
\]
or, in components,
\[
\frac{\partial}{\partial t_1} \lambda_m = 0 \quad \text{and} \quad \frac{\partial}{\partial t_1} \mu_m = 2 \left( \lambda^t_m - \sum_{n=0}^{\infty} \lambda^n_m \mu_n \right) \mu_m \tag{8.32}
\]
and in the coordinates $\sigma$
\[
\frac{\partial}{\partial t_i} \sigma = \{ \sigma, I_i^{S,2} \} = 2 \left( \sigma^l(\sigma) - \sigma_l \sigma \right)
\]  
(8.33)
whose coordinate expression was already given in (8.23). In deducing equations (8.30), (8.31), and (8.33) we used (8.22) and (8.24).

Let us observe now that (8.23) implies that
\[
\sigma_k = \frac{1}{2} \frac{\partial}{\partial t_k} \log \tau, \quad k \in \mathbb{N}.
\]  
(8.35)
In order to be consistent with the notation assumed in the theory of integrable systems (see, e.g. [16, 19]), we have called this function $\tau$-function.

Substituting (8.35) into (8.23) we obtain the system of linear partial differential equations
\[
\frac{\partial^2 \tau}{\partial t_i \partial t_k} = 2 \frac{\partial \tau}{\partial t_{k+l}}, \quad k, l \in \mathbb{N},
\]  
(8.36)
on the $\tau$-function.

In order to find the explicit form of the $\tau$-function, use (8.21), substitute (8.35) into (8.32), and integrate both sides of the resulting equation to get
\[
\mu_m(t_1, t_2, \ldots, t_{l-1}, t_l, t_{l+1}, \ldots) = \mu_m(t_1, t_2, \ldots, t_{l-1}, 0, t_{l+1}, \ldots) \frac{\tau(t_1, t_2, \ldots, t_{l-1}, t_l, t_{l+1}, \ldots)}{\tau(t_1, t_2, \ldots, t_{l-1}, 0, t_{l+1}, \ldots)} e^{2\sum_{i=1}^{l} \lambda_m t_i}.
\]  
(8.37)
Iterating (8.37) relative to $l \in \mathbb{N}$ yields the final formula for $\mu_m(t_1, t_2, \ldots)$, namely
\[
\mu_m(t_1, t_2, \ldots) = \mu_m(0, 0, \ldots) \frac{\tau(0, 0, \ldots)}{\tau(t_1, t_2, \ldots)} e^{2\sum_{i=1}^{\infty} \lambda_m t_i}.
\]  
(8.38)
Since $\sum_{m=0}^{\infty} \mu_m(t_1, t_2, \ldots) = 1$, we get the following expression for the $\tau$-function
\[
\tau(t_1, t_2, \ldots) = \tau(0, 0, \ldots) \sum_{m=0}^{\infty} \mu_m(0, 0, \ldots) e^{2\sum_{i=1}^{\infty} \lambda_m t_i}
\]  
(8.39)

Let us show that the series in (8.39) is convergent if $\mu(0) \in L^1_{\mathbb{N}} \cong \ell^1$ and $t \in \ell^\infty$. In order to do this we prove that the linear operator defined by
\[
(At)_m := \sum_{l=1}^{\infty} \lambda_m^l t_l
\]
is bounded on $\ell^\infty$. This follows from
\[
\|At\|_\infty = \sup_{m \in \mathbb{N}} \left| \sum_{l=1}^{\infty} \lambda_m^l t_l \right| = \|t\|_\infty \sup_{m \in \mathbb{N}} \left| \sum_{l=1}^{\infty} \lambda_m^l \right| = \|t\|_\infty \sup_{m \in \mathbb{N}} \frac{\lambda_m}{1 - \lambda_m} = \|t\|_\infty \frac{\|\rho\|_\infty}{1 - \|\rho\|_\infty}.
\]
Thus the sequence $\{e^{2\sum_{i=1}^{\infty} \lambda_m t_i}\}_{m \in \mathbb{N}} \in \ell^\infty$. Since $\{\mu_m(0, 0, \ldots)\}_{m \in \mathbb{N}} \in \ell^1$, the series in (8.39) converges.

Summarizing, we see that the substitution of (8.39) into (8.36) and (8.37) gives the $t := (t_1, t_2, \ldots)$-dependence of the moments $\sigma_k(t)$ and the masses $\mu_m(t)$, respectively. The dependence of $\rho_{k,k+1}(t)$ and $\rho_{k,k+1}(t)$ on $t$ is given by (8.21), (8.23), and (8.24) which express these quantities in terms of $\sigma_m(t)$. From the discussion above we see that the conditions (8.3), (8.4) are preserved by the $t$-evolution.
Next, using (8.35), (8.38), and the formula

\[
P_n(\lambda_m) = \frac{1}{\sqrt{d_{0,n-1}d_{0,n}}} \det \begin{bmatrix}
\sigma_0 & \sigma_1 & \sigma_2 & \ldots & \sigma_n \\
\sigma_1 & \sigma_2 & \sigma_3 & \ldots & \sigma_{n+1} \\
\sigma_2 & \sigma_3 & \sigma_4 & \ldots & \sigma_{n+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{n-1} & \sigma_n & \sigma_{n+1} & \ldots & \sigma_{2n-1} \\
1 & \lambda_m & \lambda_2^m & \ldots & \lambda_m^n
\end{bmatrix}
\]  

(8.40)

obtained by orthonormalizing the monomials \(\lambda^n, n \in \mathbb{N} \cup \{0\}\), with respect to the measure \(\sigma\) (see, e.g., [3]), we obtain from \(8.11\) the \(t\)-dependence of the eigenvectors \(|\lambda_m(t)\rangle\) and the corresponding projectors \(P_m(t)\), \(m \in \mathbb{N} \cup \{0\}\).

Formula \(8.11\) defines the operator \(O : \mathcal{H} \to \mathcal{H}\) whose matrix in the basis \(|\langle k\rangle\rangle_{k=0}^{\infty}\) is given by \(O_{kl}(t) := P_l(t)(\lambda_k)\). One has the following identities

\[
\rho(t)O(t) = O(t)\lambda(t) \\
O(t)\mu(t)O(t)^T = I
\]

(8.41) (8.42)

relating the operators \(\rho(t), \lambda(t), \mu(t),\) and \(O(t)\) for any \(t\). Since \(\lambda(t) = \lambda(0)\), where \(0 := (0,0,\ldots)\), we obtain from (8.41) and (8.42)

\[
\rho(t) = O(t)O(0)^{-1}\rho(0)O(t)O(0)^{-1} = Z(t)^T \rho(0)Z(t),
\]

(8.43)

where \(Z(t) := O(0)\mu(0)^{1/2} (O(t)\mu(t))^{1/2} \) is an orthonormal operator, i.e., \(Z(t)^T Z(t) = I\). As shown in [3] and [5] one can express the flows \(t \mapsto \rho(t)\) through the coadjoint action \((\text{Ad}_{S^2}^+) : GL_{\mathbb{R},2} \to \text{Aut}(L_{S,2}^1)\) of the bidual group \(GL_{\mathbb{R},2}^*\) on the Banach Lie-Poisson space \(L_{S,2}^1 \cong L_{-2,2}^1\), i.e.,

\[
\rho(t) = (\text{Ad}_{S^2}^+)_{g(t)^{-1}}(\rho(0))
\]

\[
\quad = S^T s(g_0(t)) g_0(t)^{-1} \rho_1(0) + \rho_0(0) + g_0(t)^{-1}g_1(t)\rho_1(0) - \tilde{s}(g_0(t)^{-1}g_1(t)\rho_1(0))
\]

\[
\quad + s(g_0(t)) g_0(t)^{-1} \rho_1(0) S
\]

\[
\quad = \sum_{i=0}^{\infty} \rho_{i,i+1}(0) \frac{g_{i+1,i+1}(t)}{g_i(t)} |i+1\rangle \langle i|
\]

\[
+ \sum_{i=0}^{\infty} \left( \rho_{i,i+1}(0) + \rho_{i,i+1}(0) \right) \frac{g_{i+1,i+1}(t)}{g_i(t)} - \rho_{i,i+1}(0) \frac{g_{i+1,i+1}(t)}{g_{i+1,i+1}(t)} |i\rangle \langle i|
\]

\[
+ \sum_{i=0}^{\infty} \rho_{i,i+1}(0) \frac{g_{i+1,i+1}(t)}{g_i(t)} |i\rangle \langle i+1|
\]

(8.44)

(the symmetric version of (6.7)), where \(\rho_0 := \text{diag}(\rho_{00}, \rho_{11}, \ldots), \rho_1 := \text{diag}(\rho_{01}, \rho_{12}, \ldots), \rho_0 := (g_{00}, g_{11}, \ldots), \) and \(g_1 := (g_{10}, g_{21}, \ldots) \in L_0^1\).

In order to find the time dependence \(t \mapsto g(t) = g_0(t) + g_1(t)\) for \(g(t) \in GL_{\mathbb{R},2}^\infty\) let us note that from (6.7) and the three term recurrence relation (8.6) it follows that

\[
g_{kk}(t) = g_{00}(t) \frac{\rho_{kk}(0) \cdots \rho_{k,k-1} \cdots \rho_{k-2,k-1} \cdots \rho_{00}(0)}{\rho_{k,k-1} \cdots \rho_{k-2,k-1} \cdots \rho_{00}(0)}
\]

\[
= g_{00}(t) \frac{P_{kk}(0)}{P_{kk}(t)} = g_{00}(t) \sqrt{\frac{d_{0,k-1}(0)d_{0,k}(t)}{d_{0,k}(0)d_{0,k-1}(t)}}
\]

(8.45)

and

\[
g_{k+1,k}(t) = g_{00}(t) \left( \frac{\rho_{00}(0) \cdots \rho_{k,k-1} \cdots \rho_{k-2,k-1} \cdots \rho_{00}(0)}{\rho_{k,k-1} \cdots \rho_{k-2,k-1} \cdots \rho_{00}(0)} \right) \left( \frac{\rho_{00}(t) + \cdots + \rho_{kk}(t) - \rho_{00}(0) - \cdots - \rho_{kk}(0)}{\rho_{kk}(0)} \right)
\]

\[
= g_{00}(t) \frac{P_{k+1,k}(0)P_{k+1,k+1}(t) - P_{k+1,k}(t)P_{k+1,k+1}(0)}{P_{kk}(t)P_{k+1,k+1}(t)}
\]

\[
= g_{00}(t) \sqrt{\frac{d_{kk}(t)d_{0,k+1}(0)}{d_{0,k+1}(0)d_{0,k}(t)d_{0,k-1}(t)d_{0,k-1}(0)}}.
\]

(8.46)
where \( P_n(t) \) are the coefficients of the polynomial \( P_n(t) = P_{n,s}(t)\gamma^s + P_{n,n-1}(t)\gamma^{n-1} + \cdots + P_{n,1}(t)\gamma + P_{n,0}(t) \). The last equalities in (8.43) and (8.46) are obtained using (8.40), (8.25), and (8.26) to get the expressions

\[
P_{kk}(t) = \frac{d_{0,k-1}(t)}{d_{0,k}(t)} \quad \text{and} \quad P_{k+1,k}(t) = \frac{-d_{1,k}(t)}{\sqrt{d_{0,k}(t)d_{0,k+1}(t)}}.
\]

Recall that \( d_0(t) \) and \( d_1(t) \) are given by (8.25) and (8.26), respectively.

Finally, taking in (6.26) (for \( k = 2 \)) \( g_0(t) \) and \( g_1(t) \) given by (8.45) and (8.46), we obtain the explicit expression for the time evolution of the position \( q(t) \) and the momentum \( p(t) \) for all flows in the Toda hierarchy described by the Hamiltonians

\[
H_l(q,p) := \left( I_l^{S,2} \circ \mathcal{J}_{\nu_1} \right) (q,p),
\]

where \( \mathcal{J}_{\nu_1} : \ell^1 \times \ell^\infty \rightarrow L^1_{-2} \cong L^1_{S,2} \) is the Flaschka map given by (6.25) for \( k = 2 \) and \( I_l^{S,2} = I_l^S \circ \iota_{S,2} = I_l \circ \iota_S \circ \iota_{S,2} \) are the restrictions to \( L^1_{S,2} \) of the Casimir functions \( I_l \) of \( L^1 \) (see (5.30)).

Note that the formulas giving the group element \( g(t) \) depend on \( g_{00}(t) \). This first component cannot be determined but it does not matter because \( g_{00}(t)I \) is in the center of \( GL^n_{\infty,2} \) and hence the coadjoint action defined by it is trivial. Also, in terms of the variables \( q \) and \( p \), the action of this group element is a translation in \( q \) and has no effect on \( p \). This corresponds to the flow of \( I_l^{S,2} \).

To solve the Toda system one takes an initial condition \( \rho(0) \) which determines a coadjoint orbit of \( GL^n_{\infty,2} \) in \( L^1_{S,2} \). These coadjoint orbits were studied in detail in [20]. In the generic case, when all entries on the strictly upper (and hence also strictly lower) diagonal of \( \rho(0) \) are strictly positive, the solution of the Toda lattice was given above. If some upper diagonal entries of \( \rho(0) \) vanish, Remark (iv) at the end of [20] describes such orbits as blocks, some of them finite and at most one infinite. Then the Toda lattice equations decouple and we get a smaller Toda system for each block. On the infinite block, the solution is as above. On each finite block one obtains a finite dimensional Toda lattice whose solution is known (see, e.g., [12] [17] [19] [23]).

The method we used above for the semi-infinite case can be also used in the finite case; one works then with measures \( \sigma \) having finite support and uses finite orthogonal polynomials. If one implements the solution method described in this section to this finite dimensional case the results in [17] are reproduced.

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