SAMPLING AND RESOLUTION CHARACTERISTICS IN REDUCED ORDER MODELS OF SHALLOW WATER EQUATIONS: INTRUSIVE VS NON-INTRUSIVE

Shady E. Ahmed  
School of Mechanical & Aerospace Engineering,  
Oklahoma State University,  
Stillwater, Oklahoma - 74078, USA.  
shady.ahmed@okstate.edu

Omer San  
School of Mechanical & Aerospace Engineering,  
Oklahoma State University,  
Stillwater, Oklahoma - 74078, USA.  
osan@okstate.edu

Diana A. Bistrian  
Department of Electrical Engineering and Industrial Informatics,  
University "Politechnica" of Timisoara,  
331128, Hunedoara, Romania.  
diana.bistrian@fih.upt.ro

Ionel M. Navon  
Department of Scientific Computing,  
Florida State University, Tallahassee,  
Florida 32306 USA.  
inavon@fsu.edu

ABSTRACT

We investigate the sensitivity of reduced order models (ROMs) to training data resolution as well as sampling rate. In particular, we consider proper orthogonal decomposition (POD), coupled with Galerkin projection (POD-GP), as an intrusive reduced order modeling technique. For non-intrusive ROMs, we consider two frameworks. The first is using dynamic mode decomposition (DMD), and the second is based on artificial neural networks (ANNs). For ANN, we utilized a residual deep neural network, and for DMD we have studied two versions of DMD approaches: one with hard thresholding and the other with sorted bases selection. Also, we highlight the differences between mean-subtracting the data (i.e., centering) and using the data without mean-subtraction. We tested these ROMs using a system of 2D shallow water equations for four different numerical experiments, adopting combinations of sampling rates and resolutions. For these cases, we found that the DMD basis obtained with hard thresholding is sensitive to sampling rate, performing worse at very high rates. The sorted DMD algorithm helps to mitigate this problem and yields more stabilized and converging solution. Furthermore, we demonstrate that both DMD approaches with mean subtraction provide significantly more accurate results than DMD with mean-subtracting the data. On the other hand, POD is relatively insensitive to sampling rate and yields better representation of the flow field. Meanwhile, resolution has little effect on both POD and DMD performances. Therefore, we preferred to train the neural network in POD space, since it is more stable and better represents our dataset. Numerical results reveal that an artificial neural network on POD subspace (POD-ANN) performs remarkably better than POD-GP and DMD in capturing system dynamics, even with a small number of modes.

Keywords Reduced order modeling · Proper orthogonal decomposition · Dynamic mode decomposition · Artificial neural network · Resolution · Sampling rate

1 Introduction

Despite the gigantic progress in computational power and numerical simulations accuracy, dealing with complex realistic situations is still prohibitive. This is particularly important in geophysical flows (e.g., atmosphere and oceans) characterized by a wide span of spatio-temporal scales. Resolving all of these scales on a fine grid covering the globe is just impossible, even with the largest supercomputer in the world nowadays [1,2]. Reduced order modeling (ROM) offers an alternative to handle these complex simulations in a more efficient way. In general, ROM presents approximate
substitutes to the original system. Those substitutes should be cheaper to solve and analyze while retaining similar
dynamics to the original system [3,4]. Fortunately, this is feasible given the fact that these complex systems are often
dominated by a few number of underlying manifolds controlling most of the mass, momentum, and energy transfer.
The idea of ROM is simple; instead of ‘watching’ each grid point, we may just follow a small number of features and
still be able to know equivalent amount of information.

In other words, rather than simulating every single point (which represents a degree of freedom in computations), ROM
enables us to detect the major directions over which most variations occur. By superposing the contributions of a
small number of these directions, an approximate ‘numerical twin’ of the original system is obtained. The remaining
directions are cut-off assuming their contributions are minimum. Dealing with that truncated numerical twin saves a
large portion of computational power, time, and storage. This is especially evident when multiple forward simulations
are required. Examples include data assimilation [5–14], optimal control [15–21], parameter identification [22–24], and
uncertainty quantification [25–29].

Model order reduction was introduced to fluid dynamics community many decades ago through the work of Lumley [30]
as a mathematical technique to extract coherent structures from turbulent flow fields. Different ROM techniques vary
in the way they handle these underlying structures, either growing/decaying with a specific rate, oscillating with a
constant frequency or containing a significant amount of kinetic energy. Living in the era of big data, snapshot-based
projection methods have gained the largest popularity, where the response of a system, either recorded from experiments
or high-fidelity numerical simulations given a certain input, is assumed to contain the essential behavior of that system.

Proper orthogonal decomposition (POD), also known as principal component analysis [31], computes a set of orthonor-
mal spatial basis vectors which describe the main directions (modes), by which the given flow field is characterized, in
the $L_2$ sense [32]. Each of these modes contains a proportion of the system’s total energy. The most energetic POD
modes are selected to generate the reduced order twin. Often, POD is coupled with Galerkin projection (GP) to generate
ROMs for linear and nonlinear systems (POD-GP) [33–40]. A recent mathematical discussion on POD-based ROMs is
given in Ref. [41] for nonlinear parametric and time-dependent PDEs. In brief, GP makes use of the orthonormality of
the POD modes to reduce the full order model (FOM) (i.e., governing equations) from partial differential equations
(PDEs) to a simpler set of ordinary differential equations (ODEs) involving the POD temporal coefficients. These ODEs
can be solved with much more simplicity than the original set of PDEs. In practice, ROM approximation can lead to
speedups of several orders of magnitude as well as great reduction in memory requirements [42,43] with minor impact
on accuracy of numerical solution.

The spatiotemporal biorthogonal decomposition, a variant of POD, has been introduced to recover the temporal
information associated with the POD modes [44,45]. A spectral POD (SPOD) has been applied to extract low-
dimensional compression of the full order model (FOM) extending POD to account for temporal dynamics in addition
to energetic optimality [46]. Although the spatial SPOD modes are no longer orthonormal, it was shown that the norm
of the spatial modes gives further insights into the dataset and reduces POD in the limiting case. The name ‘SPOD’ was
also given to the frequency-based POD where standard POD algorithm is applied to a set of realizations of the temporal
Fourier transform of the flow field [4,47,48]. Loiseau et al. [49] have also introduced a manifold modeling approach
as a potentially viable compression model. They showed that a two-dimensional manifold is more accurate than an
expansion using 50 POD modes for the transient cylinder wake. Another variation of POD in linear cases is balanced
proper orthogonal decomposition, which extracts two sets of modes for specified inputs and outputs, corresponding to
the controllable states, and the observable states, respectively. Balanced truncation of linear systems has been presented
by combining POD and balanced realization theory [50]. In ROM, the aim is to retain both the most controllable
modes and the most observable modes [51]. However, for some systems, states that have very small controllability
might have very large observability, and vice versa. Balancing involves determining a coordinate system in which the
most controllable directions in state space are also the most observable directions [52–54]. The modes that are least
controllable/observable are then truncated.

Projection-based techniques (e.g., POD-GP) belong to a family of intrusive reduced order modeling, also called
physics-informed ROM, since they require knowledge of the high-order governing equations, and projecting them onto
the reduced basis to obtain a set of ordinary differential equations (ODEs). This limits their use to systems which we
fully understand and can formulate in a mathematical representation [55]. Also, as the projection depends on governing
equations, the source codes have to be modified for even slightly different cases, making the process inconvenient for
complex nonlinear problems. Moreover, in fluid applications, POD-GP results in a dense system consisting of triadic
interactions due to the quadratic nonlinearity with an order of $O(R^3)$ computational load, where $R$ refers to the retained
number of modes. This poses a constraint on the number of modes to be selected. Several efforts have been devoted to
minimize the number of selected modes through closure modeling to reduce the amount of information lost due
to truncation [56–60]. Stefanescu and Navon [61] employed the discrete empirical interpolation method (DEIM) to
reduce the computational complexity of the POD-GP reduced order model caused by its dependence on the nonlinear
An optimized DMD method which tailors the decomposition to a desired number of modes was introduced by Chen was proposed by Kutz et al. [68] employing one-level separation recursively to separate background (low-rank) and another class for ROM which is gaining popularity in recent years, is the purely data-driven, non-intrusive ROMs. That is a family of methods that solely access datasets to identify system’s dynamics, with little-to-no knowledge of the governing equations, sometimes called physics-agnostic modeling. Dynamic mode decomposition (DMD) is one of the very popular techniques in this area, relying on Koopman theory. Mezic [63] was the first to apply the Koopman theory for the purposes of reduced order modelling. Later, Schmid and Sesterhenn [64] introduced the DMD algorithm which is considered the numerical implementation of the Koopman modal decomposition. In DMD, time-resolved data are decomposed into modes, each having a single characteristic frequency of oscillation and growth/decay rate. There are a number of variants for DMD algorithms. The monograph written by Kutz et al. [65] collects various DMD methods and their implementations. Hemati et al. [66] proposed an online-DMD algorithm that is capable of handling large and streaming datasets. The algorithm reads consecutive pairs of snapshots, whenever available, and updates the DMD modes accordingly, while keeping track of historical datasets with minimal storage requirements. In order to achieve a desirable trade-off between the quality of approximation and the number of modes that are used to approximate the given fields, Jovanović et al. [67] developed a sparsity-promoting DMD. In their algorithm, sparsity is induced by regularizing the least-squares deviation between the data matrix and a linear combination of DMD modes with an additional term that penalizes the $L_1$-norm of the vector of DMD amplitudes. A multiresolution DMD (mrDMD) was proposed by Kutz et al. [68] employing one-level separation recursively to separate background (low-rank) and foreground (sparse) dynamics. This technique has been recently extended to treat a broader variety of multiscale systems and more faithfully reconstruct their isolated components [69].

An optimized DMD method which tailors the decomposition to a desired number of modes was introduced by Chen et al. [70]. Dang et al. [71] proposed a denoising and feature extraction algorithm for multi-component coupled noisy mechanical signals by combining multiscale permutation entropy with sparse optimization based on non-convex regularization. Bistrian and Navon [72, 73] utilized an adaptive randomized DMD to obtain a reduced basis in the offline stage, and the temporal values in the online stage through an interpolation using radial basis functions. Also, they provided an improved selection method for the DMD modes and associated amplitudes and Ritz values, that retains the most influential modes and reduces the size of the model [74, 75]. In their method, modes are arranged in descending order of the energy of the DMD modes weighted by the inverse of the Strouhal number. Insignificant modes are truncated based on the conservation of quadratic integral invariants by the reduced order flow. Higham et al. [76] suggested that DMD is more favorable than POD in situations where that nonlinear dynamics can arise in the transition of underlying structures to a quasi-two-dimensional behaviour. However, Taira et al. [4] referred that DMD often yields less reliable results when the system’s nonlinearity is significant unless more measurements are included in the snapshots. Theoretical work on DMD has centered mainly on exploring connections with other methods, such as Koopman spectral analysis [77–79], POD [80], and Fourier analysis [70]. Lu and Tartakovsky [81] provides a theoretical error estimator of DMD predictability for linear and nonlinear parabolic equations.

There have been many efforts to hybridize and bridge the gap between POD (energy-based decomposition) and DMD (frequency-based decomposition). A single-frequency POD method (F-POD) has been proposed by Zhao et al. [82] to re-decompose the POD modes from the perspective of frequency by utilizing discrete Fourier transform to enable dynamic analysis. The associated modal growth rate information is obtained through a band of adjacent frequencies centered about some isolated peaks. A combination of POD and DMD was proposed by Cammilleri et al. [83] performing DMD on the temporal coefficients of the POD modes to force purely harmonic temporal modes, while keeping the main energy optimality of the POD. Noack et al. [84] used a recursive DMD approach to extract oscillatory modes from the data sequence, a feature of DMD, while ensuring the orthogonality of the modes and minimization of the truncation error, features of POD. First, DMD is employed on the dataset, and a normalized DMD mode is chosen such that it minimizes the averaged error. The following modes are constructed recursively from the orthogonal residual of the Galerkin expansion with the preceding modes. This results in an orthonormal set of modes while retaining the monochromatic feature of DMD. Alla and Kutz [85] proposed the use of DMD for the approximation of nonlinear terms in POD-based projection framework. Mendez et al. [86] proposed a multi-scale POD (mPOD) that combines multi-resolution analysis with POD. The multi-resolution analysis splits the correlation matrix into the contributions of different scales. Then, the temporal mPOD basis functions are computed by setting frequency constraints to the classical POD. Williams et al. [87] proposed a hybrid PDE/ROM integrator to allow for online ROM generation based on a comparison between POD-GP and DMD results. When they agree to some limit, the forward simulation is pursued.
in the reduced space. Otherwise, the original PDE is solved in full space for a specific time interval to update the POD and DMD basis functions.

Recently, machine learning (ML) tools, especially artificial neural networks (ANNs) have been playing a significant role in different fields of science and engineering [88–90]. The merits of machine learning algorithms have been demonstrated for prototypical fluid mechanics applications, including flow modeling, reconstruction, and control [91–95]. This is due to their superior capability to identify the linear and non-linear maps between input and output data as well as extracting underlying patterns [96–99]. For model order reduction purposes, neural networks can be used as an auto-encoder [100–103], composed mainly of two parts; encoder and decoder. For the encoder, the input is the full order flow field, while the output is a reduced order representation of the field (i.e., using a smaller number of neurons). This represents the projection step in ROM paradigm. The decoder does the opposite job, taking the reduced representation and recovering back the flow field (i.e., reconstruction). However, it has been noted that the autoencoder space is not optimal in terms of orthonormality, order, and conserving physical quantities. Alternatively, ANNs capability of identifying maps can be hybridized with POD basis optimality (POD-ANN) [104–106]. San et al. [105] proposed a supervised machine learning framework for the non-intrusive model order reduction of unsteady fluid flows to provide accurate predictions of state variables while varying the control parameter values. Instead of using the flow field as an input, a pre-processing step is conducted by projecting the field onto the POD modes to obtain temporal modal coefficients, reducing memory requirements. Then, ANNs can be exploited to predict the evolution of these coefficients with time. Finally, a post-processing step is employed to reconstruct the flow field. In other words, based on a POD compression, ANN is used to bypass Galerkin projection and solving the resulting ODEs. This allows the use of POD in a non-intrusive ROM framework, eliminating the need to access the governing equations. Also, neural networks can be utilized for closure modeling and correction of ROMs [107–111].

Since data amounts witness tremendous expansion, it is increasingly important to assess which dataset better represents the system. This is particularly important in snapshot-based ROM since these datasets are used to distill the main dynamics and influences further use of the obtained model. Therefore, in the present paper, we investigate ROM sensitivity to the training data, in terms of resolution and sampling rate. We choose the two-dimensional (2D) shallow water equations (SWEs) as our testbed. This should provide more insights and guidelines when formulating ROMs, especially for convective, nonlinear flows. Four numerical experiments were performed using two different resolutions and sampling rates. POD and DMD basis functions are computed for these four experiments, and ROM performance is assessed accordingly. For DMD, we studied two approaches for basis computation and selection. One of them is based on hard thresholding without further sorting and selection, and the other one uses a sorting criteria to select the most influential DMD modes. For the present dataset, hard thresholding DMD was found to suffer from overfitting issues when using a relatively high sampling rate. Sorted DMD algorithm helps in reducing this problem by truncating the modes with low amplitudes and low growth/decay rates. On the other hand, POD does not suffer from this problem and gave almost matching results for different sampling rates. Resolution (i.e., full order 2D field) was found to have negligible effect as long as it carries sufficient information about the flow, although lower resolution is observed to mitigate the overfitting problems resulting from unnecessarily high sampling rate. We investigated the effect of data preparation and centering on ROM performances. We repeated the same four experiments with centered datasets (through mean-subtraction) and uncentered datasets (without mean-subtraction). Although POD-based ROMs gave similar results for both cases, DMD performed significantly better for data without mean-subtraction.

The rest of the paper is organized as follows. In Section 2, our testbed, the two-dimensional shallow water equations, is described. Governing equations as well as initial and boundary conditions are introduced. Also, numerical schemes used to solve these equations are presented briefly. Reduced order modeling techniques are given in Section 3, where POD and DMD basis computations are explained. Also, the use of POD with either Galerkin projection as an intrusive ROM or artificial neural networks as a non-intrusive ROM is discussed. Results are demonstrated in Section 4 by illustrating the geopotential flow field obtained after a one-day prediction using POD-GP, POD-ANN, and DMD, compared to the true projection for different number of modes. Also, time series prediction of POD modal coefficients are presented to compare the ANNs prediction capability with its POD-GP counterpart. Finally, concluding remarks are drawn in Section 5.

2 Shallow Water Equations

Most atmospheric and oceanic flows feature horizontal length scales which are much larger than the vertical length scale (i.e., layer depth) [112]. Therefore, representing those systems using shallow water equations (SWEs) has been a common practice in geophysical fluid dynamics studies. Conventionally, SWEs describe a thin layer of constant density fluid in hydrostatic balance, bounded from below by a rigid surface and from above by a free surface. Above that free surface, another fluid of negligible inertia is assumed to exist.
Conceptually, to derive SWEs, the classic Navier-Stokes equations are integrated, and averaged across the vertical height, and pressure term is substituted by potential height, through the hydrostatic approximation. As for most geophysical flows, Coriolis term is included to account for forces due to Earth’s rotation. This results in a set of simplified one-dimensional equations (in the case of 1D SWEs) or two-dimensional equations (as in 2D SWEs). In the present study, we consider the latter case in inviscid form. SWEs have been used as a testbed for many geophysical fluid dynamics studies, since they provide valuable physical and numerical insights, while avoiding the computational burden of the general primitive equations. For a single-layer inviscid shallow water equations, also called Saint-Venant equations, the momentum and mass conservations can be written as follows, \[74\]

\[
\frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} = -\nabla \eta, \tag{1}
\]

\[
\frac{D\eta}{Dt} + \eta \nabla \cdot \mathbf{u} = 0, \tag{2}
\]

where \(\mathbf{u}\) is the horizontal velocity vector (i.e., \(\mathbf{u} = u\mathbf{i} + v\mathbf{j}\)), \(\eta\) is the geopotential height, \(\eta = gh\), and \(h\) is fluid depth. Here, \(\mathbf{f}\) represents the Coriolis factor, and \(g\) is the gravitational acceleration. The material derivative, \(D/Dt\) is defined as

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{3}
\]

Considering a rectangular 2D domain, \(\Omega = [0, L_x] \times [0, L_y]\), SWE can be rewritten in Cartesian form,

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial \eta}{\partial x} - fu = 0, \tag{4a}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial \eta}{\partial y} + fu = 0, \tag{4b}
\]

\[
\frac{\partial \eta}{\partial t} + \frac{\partial (\eta u)}{\partial x} + \frac{\partial (\eta v)}{\partial y} = 0. \tag{4c}
\]

Adopting a \(\beta\)-plane assumption, the effect of the Earth’s sphericity is modeled by a linear variation in the Coriolis factor in the spanwise direction,

\[
f = f_0 + \frac{\beta}{2}(2y - L_y), \tag{5}
\]

where \(f_0\) and \(\beta\) are constants. The framework used here consists of the nonlinear SWEs in a channel on the rotating earth, associated with periodic boundary conditions in the \(x\)-direction and slip-wall boundary condition in the \(y\)-direction, which can be mathematically formulated as follows,

\[
\left.\frac{\partial u}{\partial y}\right|_{(x,0,t)} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{(x,L_y,t)} = 0, \quad \left.\frac{\partial \eta}{\partial y}\right|_{(x,0,t)} = \text{constant}, \tag{6}
\]

\[
\left.\frac{\partial v}{\partial y}\right|_{(x,0,t)} = 0, \quad \left.\frac{\partial v}{\partial y}\right|_{(x,L_y,t)} = 0, \quad \left.\frac{\partial \eta}{\partial y}\right|_{(x,L_y,t)} = \text{constant}.
\]

In the above expressions, it follows that \(\frac{\partial^2 \eta}{\partial y^2}\bigg|_{(x,0,t)} = 0, \frac{\partial^2 \eta}{\partial y^2}\bigg|_{(x,L_y,t)} = 0\), which allows the following approximation (i.e., also known as linear extrapolation),

\[
\eta_0 = 2\eta_1 - \eta_2, \tag{7}
\]

where \(\eta_0\), \(\eta_1\), and \(\eta_2\) are the geopotential heights defined discretely at cell centers shown in Figure 1.

For initial state, Grammeltvedt [113] introduced several stable initial conditions for geopotential heights, and here we adopt their initial condition I1 given as,

\[
\eta_0(x, y) = gH_0 + gH_1 \tanh \left(\frac{9(L_y/2 - y)}{2L_y}\right) + gH_2 \frac{\sin \left(\frac{2\pi x}{L_x}\right)}{\cosh^2 \left(\frac{9(L_y/2 - y)}{2L_y}\right)}. \tag{8}
\]
Using the geostrophic relation, where the Coriolis force is balanced by the pressure force, the dominant balance at the initial state becomes,

\[ u_0(x, y) = -\frac{1}{f} \frac{\partial \eta_0(x, y)}{\partial y}, \]  
\[ v_0(x, y) = \frac{1}{f} \frac{\partial \eta_0(x, y)}{\partial x}. \]  

(9a) \hspace{1cm} (9b)

Hence, the initial velocity field can be obtained by differentiating Eq. (8). Initial \( u, v, \) and \( \eta \) fields are shown in Figure 2.

2.1 Numerical methods

2.1.1 Spatial discretization for full order model

In order to solve the SWEs represented before, staggered rectangular grid is adopted, specifically Arakawa C-grid \[114\] as shown in Figure 3. Staggered grid is often used in computational fluid dynamics (CFD) as a way of conserving total mass, energy and enstrophy through numerics. Also, it prevents the odd-even decoupling problem that is usually encountered when using collocated grid. Use of a staggered grid necessitates some attention, since interpolations and/or approximations are required to communicate data between different locations.
Figure 3: Staggered grid (Arakawa C-grid) arrangement for mass, energy, and enstrophy conserving discretization

For first-order spatial derivatives, standard second-order-accurate centered-finite-difference scheme is utilized. This yields the following semi-discretized SWEs,

\[
\begin{align}
\frac{\partial u}{\partial t} \bigg|_{i,j+\frac{1}{2}} &= -u_{i,j+\frac{1}{2}} \left( \frac{u_{i+1,j+\frac{1}{2}} - u_{i-1,j+\frac{1}{2}}}{2\Delta x} - \frac{1}{4} \left( v_{i+\frac{1}{2},j} + v_{i-\frac{1}{2},j} + v_{i+\frac{1}{2},j+1} + v_{i-\frac{1}{2},j+1} \right) \right) \frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{2\Delta y} \\
&\quad - \eta_{i+\frac{1}{2},j+\frac{1}{2}} - \eta_{i-\frac{1}{2},j+\frac{1}{2}} \\
&\quad + \frac{f_{i,j+\frac{1}{2}}}{4} \left( v_{i+\frac{1}{2},j} + v_{i-\frac{1}{2},j} + v_{i+\frac{1}{2},j+1} + v_{i-\frac{1}{2},j+1} \right), \\
\frac{\partial v}{\partial t} \bigg|_{i+\frac{1}{2},j} &= -v_{i+\frac{1}{2},j} \left( \frac{u_{i+1,j+\frac{1}{2}} + u_{i+1,j-\frac{1}{2}} + u_{i+1,j+\frac{1}{2}} + u_{i+1,j-\frac{1}{2}}}{2\Delta x} \frac{v_{i+\frac{1}{2},j} - v_{i-\frac{1}{2},j}}{2\Delta y} \\
&\quad - \frac{\eta_{i+\frac{1}{2},j+\frac{1}{2}} - \eta_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} \\
&\quad - \frac{f_{i+\frac{1}{2},j}}{4} \left( u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j} + u_{i+1,j+\frac{1}{2}} + u_{i+1,j-\frac{1}{2}} \right), \\
\frac{\partial \eta}{\partial t} \bigg|_{i+\frac{1}{2},j+\frac{1}{2}} &= -\left( \frac{(\eta_{i+\frac{1}{2},j+\frac{1}{2}} + \eta_{i+\frac{1}{2},j-\frac{1}{2}})u_{i+1,j+\frac{1}{2}} - (\eta_{i+\frac{1}{2},j+\frac{1}{2}} + \eta_{i-\frac{1}{2},j+\frac{1}{2}})u_{i+1,j-\frac{1}{2}}}{2\Delta x} \\
&\quad - \left( \frac{(\eta_{i+\frac{1}{2},j+\frac{1}{2}} + \eta_{i+\frac{1}{2},j-\frac{1}{2}})v_{i+\frac{1}{2},j+1} - (\eta_{i+\frac{1}{2},j+\frac{1}{2}} + \eta_{i+\frac{1}{2},j-\frac{1}{2}})v_{i+\frac{1}{2},j-1}}{2\Delta y} \right). 
\end{align}
\]
The system’s total mass, \( m \), energy, \( E \), and enstrophy, \( \zeta \), can be computed at any time using the following relations,

\[
m(t) = \int_\Omega h(x, y, t) \, dx \, dy, \tag{11}
\]

\[
E(t) = \int_\Omega \frac{1}{2}(u^2(x, y, t) + v^2(x, y, t)) + \frac{1}{2}gh^2(x, y, t) \, dx \, dy, \tag{12}
\]

\[
\zeta(t) = \int_\Omega \left(\frac{\xi(x, y, t) + f(x, y)}{h(x, y, t)}\right)^2 \, dx \, dy, \tag{13}
\]

where \( \xi \) is the vorticity field defined as \( \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \). The time history of these three invariants, with respect to their initial values are given in Figure 4 for the two resolutions studied in this paper. From this figure, we can easily verify that this staggered grid satisfies the conservation of total mass, energy and enstrophy.

![Figure 4: Time histories of SWE system’s total mass, energy, and enstrophy. Above row is zoom-out scale and below is zoom-in scale. Solid line is for fine grid (240 × 176) and dashed line is for coarse grid (60 × 44) ](image)

### 2.1.2 Temporal discretization

For temporal discretization, third-order total variational diminishing (TVD) Runge-Kutta (TVD-RK3) scheme [115] is used. We apply a conservative finite difference formulation for our numerical framework in the current study. We cast the governing equations in the following semi-discretized ordinary differential equations form,

\[
\frac{d\vec{q}}{dt} = \vec{\mathcal{L}},
\]

where

\[
\vec{q} = \begin{bmatrix} u \\ v \\ \eta \end{bmatrix}, \quad \vec{\mathcal{L}} = \begin{bmatrix} -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \frac{\partial \eta}{\partial x} + fu \\ -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - \frac{\partial \eta}{\partial y} - fu \\ -\frac{\partial (\eta u)}{\partial x} - \frac{\partial (\eta v)}{\partial y} \end{bmatrix}.
\]
Then, the TVD-RK3 algorithm consists of the following three steps,

\[ q^{(1)} = q^{(n)} + \Delta t \mathcal{L}^{(n)}, \]
\[ q^{(2)} = \frac{3}{4} q^{(n)} + \frac{1}{4} q^{(1)} + \frac{1}{4} \Delta t \mathcal{L}^{(1)}, \]
\[ q^{(n+1)} = \frac{1}{3} q^{(n)} + \frac{2}{3} q^{(2)} + \frac{2}{3} \Delta t \mathcal{L}^{(2)}. \]  

### 2.1.3 Data storage

Although a staggered grid is used within the solver and solution is calculated at shifted positions, flow fields \((u, v, \text{and } \eta)\) are finally stored at the collocated grid positions, to ensure consistency in the constituted ROMs. To do so, the following simple interpolations are utilized,

\[ u_{i,j} = \frac{1}{2} \left( u_{i,j-1/2} + u_{i,j+1/2} \right), \]
\[ v_{i,j} = \frac{1}{2} \left( u_{i-1/2,j} + u_{i+1/2,j} \right), \]
\[ \eta_{i,j} = \frac{1}{4} \left( \eta_{i-1/2,j-1/2} + \eta_{i+1/2,j-1/2} + \eta_{i+1/2,j+1/2} + \eta_{i-1/2,j+1/2} \right). \]

### 2.2 Numerical integration

Integration over the whole domain might be required for checking conservative quantities. Also, it will be used in proper orthogonal decomposition to calculate the POD modes and their coefficients as will be seen in Section 3.1 as well as Galerkin projection in Section 3.2. For the two-dimensional numerical integration of any quantity \(r(x, y)\) over the domain \(\Omega\), we use the Simpson’s 1/3rd rule [116, 117]. This integration goes as follows

\[ \int_{\Omega} r(x, y) \, dx \, dy = \frac{\Delta y}{3} \sum_{j=1}^{N_y/2} (\hat{r}_{2j-1} + 4\hat{r}_{2j} + \hat{r}_{2j+1}), \]  

where

\[ \hat{r}_j = \frac{\Delta x}{3} \sum_{i=1}^{N_x/2} \left( r_{2i-1,j} + 4r_{2i,j} + 4r_{2i+1,j} \right), \quad \text{for } j = 1, 2, \ldots, N_y + 1. \]  

Here, \(N_x\) and \(N_y\) are the number of grid intervals in both \(x\) and \(y\) directions, respectively. For valid implementation of Simpson’s 1/3rd rule, \(N_x\) and \(N_y\) should be even. Also, uniform interval sizes in each coordinate are assumed in the above formulae.

### 3 Reduced Order Modeling

#### 3.1 Proper orthogonal decomposition

Despite being introduced a few decades ago, POD-based ROM is still the state-of-the-art in model order reduction, especially when coupled with Galerkin projection. In POD, the dominant spatial subspaces are extracted from a given dataset. In other words, POD computes the dominant coherent directions in an infinite space which best represent the spatial evolution of a system. POD-ROM is closely-related to either singular value decomposition or eigenvalue decomposition of snapshot matrix. However, in most fluid flow simulations of interest, the number of degrees of freedom (number of grid points) is usually orders of magnitude larger than number of collected datasets. This results in a tall and skinny matrix, which makes the conventional decomposition inefficient, as well as time and memory consuming. Sirovich [118] proposed an alternative efficient technique for generating the POD basis, called ‘Method of Snapshots’. A number of snapshots, \(N_s\), of the 2D flow field, denoted as \(w(x, y, n) \in \{u(x, y, t_n), v(x, y, t_n), \eta(x, y, t_n)\}\), are stored at consecutive times \(t_n\) for \(n = 1, 2, \ldots, N_s\). The time-averaged field, called ‘base flow’, can be computed as

\[ \bar{w}(x, y) = \frac{1}{N_s} \sum_{n=1}^{N_s} w(x, y, t_n). \]
The mean-subtracted snapshots, also called anomaly or fluctuation fields, are then computed as the difference between the instantaneous field and the mean field

\[ w'(x, y, t_n) = w(x, y, t_n) - \bar{w}(x, y). \]  

This subtraction has been common in ROM community, and it guarantees that ROM solution would satisfy the same boundary conditions as full order model \[70\]. This anomaly field procedure can be also interpreted as a mapping of snapshot data to its origin.

Then, an \( N_s \times N_s \) snapshot data matrix \( A = [a_{ij}] \) is computed from the inner product of mean-subtracted snapshots

\[ a_{ij} = \langle w'(x, y, t_i); w'(x, y, t_j) \rangle, \]  

where the angle-parenthesis denotes the inner product defined as

\[ \langle q_1(x, y); q_2(x, y) \rangle = \int_{\Omega} q_1(x, y)q_2(x, y)dxdy. \]  

Then, an eigen-decomposition of \( A \) is performed,

\[ AV = V\Lambda, \]  

where \( \Lambda \) is a diagonal matrix whose entries are the eigenvalues \( \lambda_k \) of \( A \), and \( V \) is a matrix whose columns \( v_k \) are the corresponding eigenvectors. It should be noted that these eigenvalues need to be arranged in a descending order (i.e., \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N_s} \)), for proper selection of the POD modes. In general, the eigenvalues, \( \lambda_k \), represent the respective POD mode contribution to the total variance. In case of velocity time series, it represents the turbulent kinetic energy. The POD modes \( \phi_k \) are then computed as

\[ \phi_k(x, y) = \frac{1}{\sqrt{\lambda_k}} \sum_{n=1}^{N_s} v^m_k w'(x, y, t_n) \]  

where \( v^m_k \) is the \( n \)-th component of the eigenvector \( v_k \). The scaling factor, \( 1/\sqrt{\lambda_k} \), is to guarantee the orthonormality of POD modes, i.e., \( \langle \phi_i; \phi_j \rangle = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta. The algorithm used to calculate the POD modes can be summarized in Algorithm 1.

**Algorithm 1:** POD algorithm with mean-subtraction

**Result:** POD modes, \( \phi_k \), and eigenvalues, \( \lambda_k \), for \( k = 1, 2, \ldots, R \)

Read snapshots: \( w(x, y, t_n) \in \{u(x, y, t_n), v(x, y, t_n), \eta(x, y, t_n)\} \)

1. \( \bar{w}(x, y) = \frac{1}{N_s} \sum_{n=1}^{N_s} w(x, y, t_n); \) ! calculate mean field
2. \( w'(x, y, t_n) = w(x, y, t_n) - \bar{w}(x, y); \) ! calculate mean-subtracted field
3. \( a_{ij} = \langle w'(x, y, t_i); w'(x, y, t_j) \rangle; \) ! calculate correlation matrix \( AN_s \times N_s \)
4. \( AV_k = \lambda_k v_k; \) ! eigenvalue decomposition
5. \( \phi_k(x, y) = \frac{1}{\sqrt{\lambda_k}} \sum_{n=1}^{N_s} v^m_k w'(x, y, t_n); \) ! calculate POD modes for largest \( R \) eigenvalues

### 3.2 Galerkin projection

As stated in Section 1, POD is usually coupled with Galerkin projection to generate intrusive ROMs. This requires the projection of main governing equations on the truncated POD space. To do so, we first have to rewrite the SWEs Eq. (4) in the following way,

\[
\begin{align*}
\frac{\partial u}{\partial t} & = -u \frac{\partial u}{\partial x} - \frac{\partial \eta}{\partial y} \frac{\partial \eta}{\partial x} + f v, \\
\frac{\partial v}{\partial t} & = -u \frac{\partial v}{\partial x} - \frac{\partial \eta}{\partial y} \frac{\partial \eta}{\partial y} - f u, \\
\frac{\partial \eta}{\partial t} & = \frac{\partial (\eta u)}{\partial x} - \frac{\partial (\eta v)}{\partial y}.
\end{align*}
\]
The flow variables, $u$, $v$, and $\eta$, are approximated using the most energetic $R_u$, $R_v$, and $R_\eta$ POD modes, respectively as,

$$
\begin{align*}
  u(x, y, t) &= \bar{u}(x, y) + \sum_{k=1}^{R_u} a_k(t) \phi_k^u(x, y), \\
  v(x, y, t) &= \bar{v}(x, y) + \sum_{k=1}^{R_v} b_k(t) \phi_k^v(x, y), \\
  \eta(x, y, t) &= \bar{\eta}(x, y) + \sum_{k=1}^{R_\eta} c_k(t) \phi_k^\eta(x, y),
\end{align*}
$$

(29a, 29b, 29c)

where $\bar{u}(x, y)$, $\bar{v}(x, y)$, and $\bar{\eta}(x, y)$ represent the time-averaged flow field variables across the training snapshots, $\phi_k^u(x, y)$, $\phi_k^v(x, y)$, $\phi_k^\eta(x, y)$ are the $k$-th POD modes for the fluctuating part of the flow field variables, and $a_k(t)$, $b_k(t)$, and $c_k(t)$ are the corresponding $k$-th time dependent coefficients. Although the formulae in this section are generally applicable for any number of modes $R_u$, $R_v$, and $R_\eta$, we use equal number of modes in the present study (i.e., $R = R_u = R_v = R_\eta$). Substituting Eq. (29) into Eq. (28), an orthonormal Galerkin projection can be subsequently performed by multiplying Eq. (28) by the spatial POD modes $\phi_k^u(x, y)$, $\phi_k^v(x, y)$, $\phi_k^\eta(x, y)$, then integrating over the entire domain $\Omega$, and making use of the orthonormality of POD modes. The resulting dense dynamical system for $a_k$, $b_k$, and $c_k$ can be written as

$$
\begin{align*}
  \frac{da_k}{dt} &= \mathfrak{B}^{(u)}_k(a_l) + \sum_{i=1}^{R_u} \mathfrak{B}^{(u, u)}_{ik} a_i a_j + \sum_{i=1}^{R_v} \mathfrak{B}^{(u, v)}_{ik} b_i b_j + \sum_{i=1}^{R_\eta} \mathfrak{B}^{(u, \eta)}_{ik} c_i c_j + \sum_{i=1}^{R_u} \sum_{j=1}^{R_v} \sum_{l=1}^{R_\eta} \mathfrak{B}^{(u, u, u)}_{ijkl} a_i a_j a_l, \\
  \frac{db_k}{dt} &= \mathfrak{B}^{(v)}_k(b_l) + \sum_{i=1}^{R_u} \mathfrak{B}^{(v, u)}_{ik} a_i a_j + \sum_{i=1}^{R_v} \mathfrak{B}^{(v, v)}_{ik} b_i b_j + \sum_{i=1}^{R_\eta} \mathfrak{B}^{(v, \eta)}_{ik} c_i c_j + \sum_{i=1}^{R_u} \sum_{j=1}^{R_v} \sum_{l=1}^{R_\eta} \mathfrak{B}^{(v, u, v)}_{ijkl} a_i a_j a_l, \\
  \frac{dc_k}{dt} &= \mathfrak{B}^{(\eta)}_k(c_l) + \sum_{i=1}^{R_u} \mathfrak{B}^{(\eta, u)}_{ik} a_i a_j + \sum_{i=1}^{R_v} \mathfrak{B}^{(\eta, v)}_{ik} b_i b_j + \sum_{i=1}^{R_\eta} \mathfrak{B}^{(\eta, \eta)}_{ik} c_i c_j + \sum_{i=1}^{R_u} \sum_{j=1}^{R_v} \sum_{l=1}^{R_\eta} \mathfrak{B}^{(\eta, u, \eta)}_{ijkl} a_i a_j a_l,
\end{align*}
$$

(30a, 30b, 30c)

where the predetermined model coefficients can be computed by the following numerical integration during an offline stage (i.e., they need to be computed only once)

$$
\begin{align*}
  \mathfrak{B}^{(u)}_k &= \langle -\bar{u} \frac{\partial \bar{u}}{\partial x} - \bar{v} \frac{\partial \bar{u}}{\partial y} - \bar{\eta} \frac{\partial \bar{u}}{\partial x} + f \bar{u}; \phi_k^u \rangle, \\
  \mathfrak{B}^{(v)}_k &= \langle -\bar{u} \frac{\partial \bar{v}}{\partial x} - \bar{v} \frac{\partial \bar{v}}{\partial y} - \bar{\eta} \frac{\partial \bar{v}}{\partial y} - f \bar{u}; \phi_k^v \rangle, \\
  \mathfrak{B}^{(\eta)}_k &= \langle -\bar{u} \frac{\partial \bar{\eta}}{\partial x} - \bar{v} \frac{\partial \bar{\eta}}{\partial y}; \phi_k^\eta \rangle,
\end{align*}
$$

(31a, 31b, 31c)
while the coefficients of linear terms are computed as

\[
\begin{align*}
\Delta_{ik}^{(u,u)} &= \left\langle -\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} - \nu \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} ; \phi^u_k \rightangle, \\
\Delta_{ik}^{(u,v)} &= \left\langle -\phi^u_i \frac{\partial v}{\partial x} + f \phi^u_i ; \phi^v_k \rightangle, \\
\Delta_{ik}^{(u,\eta)} &= \left\langle -\frac{\partial \phi^u_i}{\partial x} \phi^u_k \rightangle, \\
\Delta_{ik}^{(v,u)} &= \left\langle -\phi^v_i \frac{\partial u}{\partial x} - \nu \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} - \phi^v_i \frac{\partial v}{\partial y} ; \phi^u_k \rightangle, \\
\Delta_{ik}^{(v,v)} &= \left\langle -\phi^v_i \frac{\partial v}{\partial x} \phi^v_k \rightangle, \\
\Delta_{ik}^{(v,\eta)} &= \left\langle -\frac{\partial \phi^v_i}{\partial y} \phi^v_k \rightangle, \\
\Delta_{ik}^{(\eta,u)} &= \left\langle -\frac{\partial \phi^\eta_i}{\partial x} \phi^u_k \rightangle, \\
\Delta_{ik}^{(\eta,v)} &= \left\langle -\frac{\partial \phi^\eta_i}{\partial y} \phi^v_k \rightangle, \\
\Delta_{ik}^{(\eta,\eta)} &= \left\langle -\nu \frac{\partial \phi^\eta_i}{\partial x} \frac{\partial \phi^\eta_i}{\partial y} ; \phi^\eta_k \rightangle.
\end{align*}
\]

Finally, the coefficients of nonlinear terms are computed as

\[
\begin{align*}
\gamma_{ijk}^{(u,u,u)} &= \left\langle -\phi^u_i \frac{\partial \phi^u_j}{\partial x} \phi^u_k \rightangle, \\
\gamma_{ijk}^{(u,v,u)} &= \left\langle -\phi^v_i \frac{\partial \phi^u_j}{\partial y} \phi^u_k \rightangle, \\
\gamma_{ijk}^{(v,u,v)} &= \left\langle -\phi^u_i \frac{\partial \phi^v_j}{\partial x} \phi^v_k \rightangle, \\
\gamma_{ijk}^{(v,v,v)} &= \left\langle -\phi^v_i \frac{\partial \phi^v_j}{\partial y} \phi^v_k \rightangle, \\
\gamma_{ijk}^{(\eta,u,u)} &= \left\langle -\phi^\eta_i \frac{\partial \phi^u_j}{\partial x} \phi^u_k \rightangle, \\
\gamma_{ijk}^{(\eta,v,v)} &= \left\langle -\phi^\eta_i \frac{\partial \phi^v_j}{\partial y} \phi^v_k \rightangle.
\end{align*}
\]

To solve the dynamical system given by Eq. (30) using any ODE solver, the initial conditions for \(a_k(t), b_k(t),\) and \(c_k(t)\) have to be determined which can be obtained by the following projection of the initial snapshot on the corresponding POD modes

\[
\begin{align*}
a_k(t_0) &= \left\langle u(x, y, t_0) - \bar{u}(x, y) ; \phi^u_k \rightangle, \\
b_k(t_0) &= \left\langle v(x, y, t_0) - \bar{v}(x, y) ; \phi^v_k \rightangle, \\
c_k(t_0) &= \left\langle \eta(x, y, t_0) - \bar{\eta}(x, y) ; \phi^\eta_k \rightangle.
\end{align*}
\]

### 3.3 Artificial neural network

A neural network is simply a mapping from some input space to an output space. The two common applications of neural networks are regression and classification. Neural networks are powerful in terms of their capability to identify the underlying patterns in data. First, they were inspired by the Nobel prize winning work of Hubel and Wiesel [119] using neuronal networks organized in hierarchical layers of cells for processing visual stimulus. A typical feed forward network consists of a sequence of layers, each having a number of nodes, or neurons. Information is carried from one layer to the next layers through connections between their respective neurons with some weighting and biasing parameters. In mathematical representation, the input \(x_i\) to a neuron in the \(l\)-th layer can be written as

\[
x^l_i = \sum_{k=1}^{N} w^l_{ki} y_{k}^{l-1} + b^l_i,
\]
where \( w_{ki}^{l-1} \) is the weight parameter from the \( k \)-th neuron from the \((l-1)\)-th to the \(i\)-th neuron in the \(l\)-th layer, \( y_{ki}^{l-1} \) is the output of the \(k\)-th neuron in the \((l-1)\)-th layer, and \( n \) is the number of neurons at this layer. \( b_{ki}^{l-1} \) represent the bias that is added as an input to the \(i\)-th neuron in the \(l\)-th layer. Finally, each node, or neuron, acts as a processing unit on its input by applying some activation, or transfer, function \( G \) as follows,

\[
y_{ki}^l = G_l(x_{ki}^l).
\]  

In general, any differentiable function may qualify as a transfer function, however a few number of standard activation functions are used in practice (e.g., Tan-Sigmoid, Log-Sigmoid, Linear, Radial-Basis). In theory, a neural network can represent any mapping function between inputs and targets, given sufficient number of hidden layers and neurons, i.e. deep neural network (DNN) [120]. This is done by tuning the weights and biases through an optimization algorithm using some training data. In brief, starting from arbitrarily random weights and biases, and given input data, the neural network (mapping) output is calculated, and compared with the true target (true labels) to calculate the error at the output layer. This error is then back-propagated through the preceding layers, and the weights and biases are corrected accordingly. This process is repeated until convergence happens. So, in concept, training a network is simply solving a minimization problem in which the objective function represents the root mean square error between the network output and the true target.

Exploiting the mapping and predictability strengths of neural networks, they can offer an effective alternative to predict the temporal coefficients of POD modes, substituting for the Galerkin projection step. More importantly, this eliminates the need to access the governing equations and makes the process purely data-driven. This is crucial when the system in-hand is still not well-understood, or its mathematical representation is prohibitive.

In the present study, we limit our study for 4, 8, and 16 modes. A feed-forward neural network consisting of 4 hidden layers/60 neurons each, is used for number of modes \( R = 4 \) or 8, and 5 layers/100 neurons each for the 16 modes case. The ‘ReLu’ activation function is used for all hidden layers and all tests, defined as

\[
G_l(x_{ki}^l) = \max(0, x_{ki}^l).
\]  

This ANN is trained in POD space, where the training data are obtained from projecting the full-order snapshots on the POD modes, to obtain the POD modal coefficients. Instead of mapping the modal coefficients at time \( n \) to the coefficients at next time, \( n + 1 \), the target is set as the difference between them. In other words, the network takes as input a set of \( a_k^{(n)} \) and gives as output a set of \( a_k^{(n+1)} - a_k^{(n)} \), referring to it as residual network, or ResNet. The residual between two time steps is then applied to update the current state during prediction. The learning of the residual information was observed to result in a more stabilized performance [105,121] and also improves the accuracy of neural network prediction [122]. Also, it is typical in machine learning community to normalize the input and output data. This is particularly effective when dealing with data of different order of magnitudes. In our case, we have chosen to use normalization from \(-1\) to \(+1\) using the following formula,

\[
\tilde{z} = \frac{2z - (z_{\max} + z_{\min})}{z_{\max} - z_{\min}}.
\]  

3.4 Dynamic mode decomposition

While modes in POD are computed based on energy cascade [123] (i.e., the modes that contains the largest amount of energy), their interpretation and usage for feature detection might become unclear in some situations which can occur at largely distinct frequencies, with very similar energy content. On the other hand, frequency-based methods characterize each mode with a specific oscillating frequency and growth/decay rate. They are considered as data-driven adaptations of the discrete Fourier transform (DFT), with the advantage of allowing each mode to grow or decay exponentially. The most popular frequency-based decomposition method is the dynamic mode decomposition (DMD), approximating the modes of the so-called Koopman operator [124]. The Koopman operator is a linear operator that represents nonlinear dynamics without linearization [77,78,125] taking the form \( \mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n \), where \( \mathbf{A} \) is a time-independent linear operator that maps the data from time \( t_n \) to \( t_{n+1} \). The terms ‘DMD mode’ and ‘Koopman mode’ are often used interchangeably in the fluids literature.

The method can be viewed as computing the eigenvalues and eigenvectors (low dimensional modes) of a linear model that approximates the underlying dynamics, even if the dynamics is nonlinear [126]. The DMD gives the growth rates and frequencies associated with each mode. Theorems regarding the existence and uniqueness of DMD modes and eigenvalues can be found in [70]. Assuming the mean-subtracted snapshot matrix to be \( \mathbf{X} = \{\mathbf{w}', \mathbf{w}''_1, \mathbf{w}''_2, \mathbf{w}''_3, \ldots, \mathbf{w}''_{N_{\alpha} - 2}, \mathbf{w}''_{N_{\alpha} - 1}, \mathbf{w}''_{N_{\alpha}}\} \), we construct two matrices, \( \mathbf{X}_1 = \{\mathbf{w}'_1, \mathbf{w}'_2, \mathbf{w}'_3, \ldots, \mathbf{w}'_{N_{\alpha} - 2}, \mathbf{w}'_{N_{\alpha} - 1}\} \) and \( \mathbf{X}_2 = \{\mathbf{w}''_2, \mathbf{w}''_3, \ldots, \mathbf{w}''_{N_{\alpha} - 2}, \mathbf{w}''_{N_{\alpha} - 1}, \mathbf{w}''_{N_{\alpha}}\} \). Here, \( \mathbf{w}'_{\alpha} \) refers to
any field data (i.e., $u$, $v$, or $\eta$) at time $t_n$ that is written as a solution on our 2D grid domain mapped into column vectors. Therefore, using the Koopman operator $A$, the following can be written,

$$X_2 = AX_1.$$  \hfill (39)

So, $A$ is simply $X_2X_1^+$, where $X_1^+$ is the Moore–Penrose pseudo-inverse of $X_1$. It should be noted that the size of $A$ is $N_p \times N_p$ where $N_p$ is the total number of grid points, which makes this computation practically unfeasible in fluid flows problems where $N_p \gg N_s$. Instead, we implement DMD following the method of snapshots by substituting for the SVD of $X_1$ as $X_1 = U\Sigma V^*$, where the asterisk superscript denotes the conjugate transpose,

$$X_2 = AU\Sigma V^*.$$  \hfill (40)

Defining $\tilde{A} = U^*AU$,

$$\tilde{A} = U^*X_2V\Sigma^{-1}.$$  \hfill (41)

Now, computing the eigenvalues and eigenvectors of $\tilde{A}$

$$\tilde{A}v_j = \lambda_j v_j.$$  \hfill (42)

The eigenvalues $\lambda_j$ of $\tilde{A}$ are the same as the eigenvalues of $A$. The DMD modes (eigenvectors of $A$) can be computed as follows,

$$\phi_j = UV_j.$$  \hfill (43)

In literature, there are slight variations from Eq. 43 for computing the DMD modes [65, 126]. Now, defining $\alpha_j = \frac{\ln (\lambda_j)}{\Delta t}$, the reconstructed DMD flow field can be rewritten as,

$$w^{DMD} = \bar{w} + \sum_{j=1}^{R} b_j \phi_j e^{(\alpha_j t)}$$  \hfill (44)

$$= \bar{w} + \Phi \text{diag}(e^{\alpha t}) \ b,$$  \hfill (45)

where $b$ is a vector of the coefficients $b_j$ representing the initial amplitudes of the DMD modes. $\Phi$ is a matrix whose columns $\phi_j$ are the DMD modes, and $\text{diag}(e^{\alpha t})$ is a diagonal matrix whose entries are $(e^{\alpha t})$. The initial amplitudes of DMD modes can be computed from projecting the first snapshot (i.e., at $t = 0$) onto the DMD modes as,

$$b = \Phi^+w'_1,$$  \hfill (46)

where $\Phi^+$ is the pseudo-inverse of $\Phi$, and $w'_1$ is the first snapshot (mean-subtracted). It should be noted here that the amplitudes vector, $b$, is computed here from only the first snapshot. This is expected to be less representative of the true modal amplitudes, especially for integration over a long time interval. Alternatively, these amplitudes can be calculated from any arbitrary snapshot using Eq. 45, or it can be calculated using an optimization algorithm that minimizes the average error between true data and DMD predictions. To summarize, the DMD is a non-intrusive ROM technique which transforms the system state into its future state.

### 3.4.1 DMD mode selection

Selection of DMD modes used in the reduced order flow model constitutes the source of many discussions. Here, we will describe two methods for computing the DMD modes. The first method is based on what is called ‘hard thresholding’, computing a reduced rank matrix, $\tilde{A}_R$. That is instead of retaining all the vectors in $U$ and $V$, we use only the first $R$ of them, $U_R$ and $V_R$, respectively. Also, only the first largest $R$ singular values in $\Sigma$ are kept and the remaining are truncated. This reduces the dimension of $\tilde{A}$ from an $(N_s - 1) \times (N_s - 1)$ to $R \times R$, giving exactly $R$
modes without any further modal selection. This algorithm, which will be denoted as DMD-HT from here onward, can be summarized in Algorithm 2.

Algorithm 2: DMD [Hard Threshold] (DMD-HT) algorithm with mean-subtraction

Result: DMD modes, \( \phi_k \), and eigenvalues, \( \lambda_k \) for \( k = 1, 2, \ldots, R \)

Read snapshots: \( w(x, y, t_n) \in \{w(x, y, t_n), v(x, y, t_n), \eta(x, y, t_n)\} \)

1. \( w(x, y, t_n) = \frac{1}{N_s} \sum_{n=1}^{N_s} w(x, y, t_n) \) ! calculate mean field
2. \( w'(x, y, t_n) = w(x, y, t_n) - \bar{w}(x, y) \) ! calculate anomaly field snapshots
3. \( w_n' := w'(x, y, t_n), \) and \( \bar{w} := \bar{w}(x, y) \) ! stack fields into vector form
4. \( X_1 = \{w_1', w_2', w_3', \ldots, w_{N_s-1}', w_{N_s-1}'\} \), and \( X_2 = \{w_2', w_3', \ldots, w_{N_s-2}', w_{N_s-2}'\} \) ! obtain initial, and future snapshot matrices
5. \( X_1 = UV^* \) ! singular value decomposition, where \( U \in \mathbb{R}^{N_s \times N_s-1} \), \( V \in \mathbb{R}^{N_s-1 \times N_s-1} \), and \( \Sigma \in \mathbb{R}^{N_s-1 \times N_s-1} \)
6. \( U_R, \Sigma_R, V_R \Leftarrow U, \Sigma, V \) ! hard thresholding, where \( U_R \in \mathbb{R}^{N_s \times R} \), \( V_R \in \mathbb{R}^{N_s-1 \times R} \), and \( \Sigma_R \in \mathbb{R}^{R \times R} \)
7. \( \hat{A}_R = U_R^{*}X_2V_R\Sigma_R^{-1} \) ! compute the lower-rank dynamics matrix
8. \( \hat{A}_Rv_k = \lambda_k v_k \) ! eigenvalue decomposition
9. \( \phi_k = U_Rv_k \) ! calculate DMD modes
10. \( \phi_k(x, y) \Leftarrow \phi_k \) ! unstack the vector data into a field form

The second approach introduces some sorting criteria to select the most effective modes. Without any hard thresholding, the eigendecomposition of \( \hat{A} \) is computed, giving \( (N_s - 1) \) eigenvalues and DMD modes through Eq. 42-43, with their initial amplitudes using Eq. 39. Then, we define the contribution or the influence of any DMD mode, \( \phi_j \), to ROM accuracy as

\[
I_j = |b_j| \left( e^{\sigma_j} + e^{-\sigma_j} \right),
\]

where \( \sigma_j \) is the growth rate defined as \( \sigma_j = \text{real}(\alpha_j) \). The reason we choose this criterion and definition is that we can have quickly decaying modes with high amplitude, or rapidly growing modes with lower amplitudes. In both cases, these modes can be influential, having a significant contribution to ROM accuracy. Also, it was previously demonstrated that a selection based solely on either growth/decay rate or the amplitudes is insufficient criterion [127]. Based on this definition of modal contribution, the DMD modes are subsequently arranged in a descending order based on their respective \( I \) values, and the first \( R \) of them are retained in the ROM (i.e., \( I_1 \geq I_2 \geq I_3 \geq \cdots \geq I_R \)). This algorithm can be summarized in Algorithm 3, denoted as DMD-S. It should be noted that a soft thresholding (or regularization) can be also implemented before computing \( \hat{A} \), truncating the vectors in \( U \) and \( V \) corresponding to singular values smaller than a predetermined threshold.

Algorithm 3: DMD [Sorted] (DMD-S) algorithm with mean-subtraction

Result: DMD modes, \( \phi_k \), and eigenvalues, \( \lambda_k \), for \( k = 1, 2, \ldots, R \)

Read snapshots: \( w(x, y, t_n) \in \{w(x, y, t_n), v(x, y, t_n), \eta(x, y, t_n)\} \)

1. \( w(x, y, t_n) = \frac{1}{N_s} \sum_{n=1}^{N_s} w(x, y, t_n) \) ! calculate mean field
2. \( w'(x, y, t_n) = w(x, y, t_n) - \bar{w}(x, y) \) ! calculate anomaly field snapshots
3. \( w_n' := w'(x, y, t_n), \) and \( \bar{w} := \bar{w}(x, y) \) ! stack the field data into a vector form
4. \( X_1 = \{w_1', w_2', w_3', \ldots, w_{N_s-1}', w_{N_s-1}'\} \), and \( X_2 = \{w_2', w_3', \ldots, w_{N_s-2}', w_{N_s-2}'\} \) ! obtain initial, and future snapshot matrices
5. \( X_1 = UV^* \) ! singular value decomposition
6. \( \hat{A} = U^*X_2V\Sigma^{-1} \) ! compute the dynamics matrix
7. \( \hat{A}v_k = \lambda_k v_k \) ! eigenvalue decomposition
8. \( \phi_k = U_Rv_k \) ! calculate DMD modes (unsorted)
9. \( \Phi = \{\phi_1, \phi_2, \ldots, \phi_{N_s-1}\} \) ! generate DMD basis matrix
10. \( b = \Phi^*w_1 \) ! compute modal amplitudes
11. \( I_k = |b_k| \left( e^{\sigma_k} + e^{-\sigma_k} \right) \) ! compute modal contributions using the growth rate
12. \( I_1 \geq I_2 \geq I_3 \geq \cdots \geq I_R \) ! sorting and selecting the most influential modes
13. \( \phi_k(x, y) \Leftarrow \phi_k \) ! unstack the vector data into a field form (after sorting)
4 Results

4.1 Numerical experiments

For data collection, we consider SWE within a rectangular 2D domain of $L_x = 6000$ km, and $L_y = 4400$ km. To define the initial field as in Eq. 8, values of $g = 9.81$ m/s$^2$, $H_0 = 2000$ km, $H_1 = 220$ km, and $H_2 = 133$ km are adopted. Also, Coriolis constants of $f_0 = 10^{-4}$ s$^{-1}$ and $\beta = 1.5 \times 10^{-11}$ m$^{-1}$s$^{-1}$ are used in Eq. 5. Since the primary objective of the current study is to assess and compare the performance of different ROM and field decomposition techniques and their sensitivities to data quality, we study two different resolutions, $\{N_x = 240, N_y = 176\}$ and $\{N_x = 60, N_y = 44\}$, and two sampling rates, $\{N_s = 1440\}$ and $\{N_s = 240\}$, where $N_x$ and $N_y$ are the number of grid spacings in the $x$ and $y$ directions, respectively and $N_s$ is the number of snapshots collected from solving the SWEs full order model (FOM) for 24 hours. This gives us four numerical experiments as shown in Figure 5.

![Figure 5: Four numerical experiments in present study: Experiment 1: High resolution, high sampling rate; Experiment 2: High resolution, low sampling rate; Experiment 3: Low resolution, low sampling rate; Experiment 4: Low resolution, high sampling rate](image)

For all experiments, a time-step of $10$ s is used to guarantee numerical stability through the Courant-Friedrichs-Lewy (CFL) number as well as convergence. We have chosen to focus on geopotential height, $\eta$, predictions being a conservative quantity. Therefore, for DNN and DMD, only $\eta$ fields, their bases, $\phi^\eta$, and corresponding coefficients, $c(t)$, are used for training and prediction. However, for Galerkin projection the $u$ and $v$ fields are also considered since they are coupled with $\eta$ in the ROM ODE system of equations given in Eq. 30.

4.2 Eigenvalue analysis

Eigenvalues decomposition is a principal step in both POD and DMD algorithms. In POD, the modal eigenvalues gives an indication of the amount of information (or variance) contained in this respective mode, which is the main criteria in POD modal selection. In DMD, the eigenvalues are, in general, complex numbers, providing information about the temporal growth rate and frequency of DMD modes. In sorted DMD (Algorithm 3), the analysis of DMD eigenvalues is important and gives a deeper insight onto the influence of DMD modes on ROM dynamical evolution. The POD eigenvalues for Experiments 1-4 are shown in Figure 6, where the largest four eigenvalues (corresponding to the most four energetic POD modes) are denoted with red color. The next largest four eigenvalues are given green color, and the next eight ones (i.e., from $\lambda^{\eta}_9$ to $\lambda^{\eta}_{16}$) are given orange color. The same color code is used for DMD eigenvalues in Figure 7 to denote the selected modes based on the criteria given in Eq. 47.
Figure 6: POD eigenvalues for geopotential fields

In POD, the percentage modal energy is computed using the following relative information content (RIC) formula [128],

\[
P(k) = \left( \frac{\sum_{j=1}^{k} \lambda_j}{\sum_{j=1}^{N} \lambda_j} \right) \times 100. \tag{48}
\]

The Kolmogorov n-width [129, 130] provides a mathematical guideline to quantify the optimal n-dimensional linear subspace and the associated error (i.e., a measure of system’s reducibility). More specifically, it quantifies how a given n-dimensional subspace compares with all other possible n-dimensional subspaces. In POD, it can be considered as a measure of how well a linear superposition of POD modes might represent the underlying dynamics. If the decay of Kolmogorov width is slow, there exists no feasible linear subspace that fits well the data. Therefore, in our context, RIC can be used as a simple quantitative metric to understand the Kolmogorov width of the systems. Table 1 presents the RIC of the flow field variables, \(u\), \(v\), and \(\eta\).

4.3 Geopotential field

Now, we would like to compare reconstructed full-order fields obtained from POD-GP, POD-ANN, and DMD. Here, DMD serves as a comparison with a different example of non-intrusive ROMs. The true (full-order) final field after 24 hours, using the higher resolution is given in Figure 8 for reference. Compared to the initial field from Figure 2, we can easily visualize the convective nature of the flow field. This promotes the 2D SWEs as a challenging and more realistic testing framework than many other simpler prototypical cases characterizing periodic behaviors. In order to make fair judgments of ROMs predictability, fields obtained from true field projection on POD modes are also shown. These are
the fields reconstructed from only the first $R$ modes and their corresponding true coefficients, and they represent the maximum attainable accuracy from POD-based ROMs.

Figures 9-12 show geopotential fields from the four experiments using 4, 8, and 16 modes. It is easily observed that, for the current datasets, both POD-GP and POD-ANN fields converge to the true prediction with increasing the number of modes, irrespective of resolutions or sampling rates. On the other hand, DMD shows unsimilar behavior, especially the one with hard thresholding (DMD-HT). For Experiment 1 (with both high sampling rate and resolution), increasing the number of modes significantly deteriorates the prediction accuracy. However in Experiment 2 and Experiment 3 (with low sampling rates), using more DMD modes helps the predicted field to converge to the true one. For Experiment 4, with high sampling rates and lower resolution, the predicted flow field is slightly worse than that in Experiment 3 which uses the same resolution, but lower sampling rate. This implies that, for the present study, sampling rate plays an important role in DMD performance.

This is, in particular, true for DMD-HT algorithm. That is unnecessarily increasing the number of snapshots, while adding no significant information about the system’s underlying dynamical evolution, results in a very big data matrix making the raw DMD basis computation and selection, Algorithm 2, ill-conditioned. Each couple of consecutive snapshots, in some sense, can be viewed as a constraint in the process of approximating the linear operator $A$ in Eq. 39. Using too many snapshots (constraints) that do not contain additional information can lead to overfitting problems. This is significantly evident in Experiment 1 with both high resolutions and sampling rates. Also, the frequencies resolved by

![Figure 7: DMD eigenvalues for geopotential fields](image-url)
Table 1: Relative information content using different numbers of POD modes

| R  | $P(k^u)$ | $P(k^v)$ | $P(k^\eta)$ |
|----|----------|----------|-------------|
| Experiment 1 |
| 4  | 98.09    | 97.70    | 87.62       |
| 8  | 99.63    | 99.35    | 96.07       |
| 16 | 99.99    | 99.90    | 99.47       |
| Experiment 2 |
| 4  | 98.09    | 97.70    | 87.62       |
| 8  | 99.62    | 99.35    | 96.06       |
| 16 | 99.99    | 99.90    | 99.47       |
| Experiment 3 |
| 4  | 97.91    | 97.40    | 85.56       |
| 8  | 99.46    | 99.18    | 94.57       |
| 16 | 99.88    | 99.84    | 98.65       |
| Experiment 4 |
| 4  | 97.90    | 97.40    | 85.54       |
| 8  | 99.46    | 99.18    | 94.58       |
| 16 | 99.88    | 99.84    | 98.66       |

the DMD are still subjected to the Nyquist sampling theorem [131], implying that the frequencies which are essential to the flow physics should be known in advance to adjust the sampling rate.

Compared with DMD-HT results, the sorted algorithm, DMD-S, is found to give more stable results and be less prone to overfitting due to sampling rate issues. However, for the range of number of modes tested here (4, 8, and 16), this algorithm is found to be less capable of capturing the main convective dynamics. Regarding the effect of resolution, Experiments 2 and 3 infer that data resolution has less influence on DMD predictions as long as the number of snapshots is reasonable. Meanwhile, results from Experiment 4 suggest that using lower resolution can act as a filter to mitigate the overfitting issues, compared to Experiment 1.

Figure 8: True geopotential field at the end of 24 hours

4.4 Time series predictions

Since the ANN is trained in the POD space, it is reasonable to assess its performance in that space compared to Galerkin projection. Figures 13-16 show the performance of POD-GP, DMD and POD-ANN in different experiments using the first four modes along with the full order model (FOM) resulting field projected on these modes using Eq. 34c. Although there is no similar time series coefficients associated with DMD algorithm, we artificially generated the DMD time series. First the flow fields are reconstructed by DMD using Eq. 44 at different times, then they are projected onto
the POD space to get their corresponding modal coefficients, \( c(t) \). For the first 24 hours, while POD-GP is incapable of capturing the main dynamics with only four modes, the ANN significantly outperforms GP predictions, being capable to better approximate the dynamical POD modal evolution. POD-ANN results are almost coincident with the true values for most of the time. As indicated in Section 4.3, resolutions and sampling rates have negligible effects on POD-based predictions, although higher sampling rates offer larger amounts of training datasets, which in turn results in a better ANN model.

Using the first 8 and 16 modes, POD-GP starts to capture the system’s dynamics and follows the true trajectories, as shown in Figures 17-24. Meanwhile, POD-ANN performs perfectly well within the 24-hour time frame for 8 and 16 modes with cheaper cost than POD-GP, leveraging its potential as a non-intrusive ROM technique to bypass both the Galerkin projection and solving the resulting ODEs steps. It should be noted here that many other advanced and evolving machine learning-based time series prediction tools can be exploited in a similar way to the present POD-ANN framework.

### 4.5 Mean subtraction

As mentioned in Section 3.1, it has been customary in ROM community to use mean-subtracted (or anomaly) fields, \( w'(x_i, y_j) \), to construct ROMs. This is the case in almost every POD-based work. However, in DMD, this is not generally accurate. Chen et al. [70] showed that subtracting the mean field strips away important dynamical information, yielding a periodic trajectory, and makes the DMD algorithm incapable of determining modal growth rates. Conversely, in a very recent study, Hirsh et al. [132] demonstrated that centering the data through mean-subtraction is an effective pre-processing step and gives more accurate results for exact DMD spectra which might not be extracted from data without centering, especially when the dynamics matrix is full rank. Therefore, we replicated our four experiments without subtracting the mean field (i.e., using \( w \) instead of \( w' \)) to study the effect of centering data in convective non-periodic flow dynamics. For POD, we found that final results almost matching and no major difference can be noticed, Figures 25-28.

To quantify the results in a more quantitative way, we use the root mean square error (\( RMSE \)) as an error measure, defined as

\[
RMSE = \sqrt{\frac{1}{N_x N_y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (w(x_i, y_j) - w^{FOM}(x_i, y_j))^2}.
\]

Interestingly, the \( RMSE \) resulting from the projection of FOM onto the POD space is slightly smaller when using anomaly fields for decomposition, Table 2. This implies that POD modes resulting from mean-subtracted fields captures more information than those resulting from the flow field without subtraction. On the other hand, the Galerkin projection results are better in case of no mean-subtraction. This might be a result of amplification of noise retained in the mean-subtracted fields, being mapped to their origin.

On the other hand, the DMD results are dramatically affected by this subtraction. This can be easily seen from the reconstructed geopotential height fields, shown in Figures 25-28, compared to Figures 9-12. Also, the \( RMSE \) for cases with and without mean subtraction is shown in Table 4. Similar observations can be obtained from plotting the timeseries prediction, Figures 33-40.

### 5 Concluding Remarks

In this work, we investigate the influences of training data on different ROM frameworks. We tested two popular energy-based and frequency-based decomposition techniques, namely POD and DMD, respectively. We used four datasets spanning a wide range of resolutions (about 2700 and 42, 600 grid points) and sampling rates (240 and 1440 snapshots) generated from solving the inviscid 2D SWEs. Different ROM frameworks were studied; POD-GP as a classical intrusive technique, and DMD and POD-ANN as evolving non-intrusive ROMs. We have compared the final results reconstructed in full order space, as geopotential height field using 4, 8, and 16 modes. Furthermore, we investigated the POD modal time series predictions from these frameworks. For comparison purposes, we artificially generated DMD time series by projecting the DMD field predictions onto POD space.

From the aforementioned experiments, we can conclude that DMD is more sensitive to the training datasets, and more dense information (i.e., lower sampling rates and resolutions) is favored. A very high sampling rate results in overfitting problems in approximating Koopman operator. Resolution has negligible effect on DMD convergence, provided that it carries sufficient amount of information. Also, coarse grid can serve as a filter to mitigate overfitting resulting from high sampling rates, especially in DMD-HT algorithm. On the other hand, POD modes have proved to be more robust than DMD ones, showing a very stable and converging behavior for all experiments.
Moreover, we investigated the effect of centering data through mean-subtraction. We demonstrated that constructing DMD without subtracting the mean geopotential height field provides much better predictions for both DMD-HT and DMD-S algorithms. Therefore, we believe that although centering data in periodic flows might offer a valuable pre-processing step, it might be unfavorable in convective flow situations, like SWE systems. In terms of POD, centering data and applying POD on the anomaly field was found to yield better decomposition of flow fields. However, this could result in slight amplification of noise while solving the ROM from POD-GP.

POD-ANN was shown to provide a valuable non-intrusive ROM, benefiting from the optimality, order, and robustness of POD modes, as well as the mapping and predictive capabilities of artificial neural networks. Unlike DMD, which requires either further treatment and analysis of the system and its invariants to better select the most effective modes or a priori knowledge of essential frequencies to adjust sampling rate, the selection criterion in POD-based frameworks is
Table 4: RMSE for DMD at $t = 24$ hours

| $R$ | DMD [Hard Threshold] | DMD [Sorted] |
|-----|----------------------|--------------|
|     | With Subtraction     | Without Subtraction | With Subtraction | Without Subtraction |
| **Experiment 1** | | | | |
| 4   | 304.80               | 61.83        | 152.40          | 83.56               |
| 8   | 543.08               | 70.84        | 138.01          | 62.10               |
| 16  | 381.83               | 62.75        | 128.53          | 53.98               |
| **Experiment 2** | | | | |
| 4   | 247.22               | 62.69        | 150.09          | 44.20               |
| 8   | 477.84               | 63.82        | 133.08          | 44.05               |
| 16  | 61.50                | 52.06        | 119.75          | 36.34               |
| **Experiment 3** | | | | |
| 4   | 233.21               | 58.70        | 143.68          | 47.62               |
| 8   | 245.70               | 66.36        | 128.04          | 38.86               |
| 16  | 67.27                | 68.31        | 114.69          | 16.32               |
| **Experiment 4** | | | | |
| 4   | 296.31               | 57.79        | 145.92          | 48.22               |
| 8   | 262.01               | 75.38        | 132.81          | 42.40               |
| 16  | 100.68               | 107.44       | 124.07          | 35.46               |

trivial and automated based on the modal eigenvalues. Table 5 gives a summary of the main findings of the present study and provides some insights on sensitivity of decomposition techniques to the training data as well as possible enhancements. Although, we only used ANN in the present study, many other time series prediction tools (e.g., LSTM, Gaussian Processes, etc) can be incorporated in the same framework.

Acknowledgements

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research under Award Number DE-SC0019290. OS gratefully acknowledges their support. Disclaimer: This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

References

[1] Eugenia Kalnay. *Atmospheric modeling, data assimilation and predictability*. Cambridge University Press, New York, 2003.

[2] Jordan G. Powers, Joseph B. Klemp, William C. Skamarock, Christopher A. Davis, Jimy Dudhia, David O. Gill, Janice L. Coen, David J. Gochis, Ravan Ahmadov, Steven E. Peckham, Georg A. Grell, John Michalakes, Samuel Trahan, Stanley G. Benjamin, Curtis R. Alexander, Geoffrey J. Dimego, Wei Wang, Craig S. Schwartz, Glen S. Romine, Zhiquan Liu, Chris Snyder, Fei Chen, Michael J. Barlage, Wei Yu, and Michael G. Duda. The weather research and forecasting model: overview, system efforts, and future directions. *Bulletin of the American Meteorological Society*, 98(8):1717–1737, 2017.

[3] Alfio Quarteroni, Andrea Manzoni, and Federico Negri. *Reduced basis methods for partial differential equations: an introduction*. Springer, New York, 2015.

[4] Kunihiko Taira, Steven L Brunton, Scott TM Dawson, Clarence W Rowley, Tim Colonius, Beverley J McKeon, Oliver T Schmidt, Stanislav Gordeyev, Vassilios Theofilis, and Lawrence S Ukeiley. Modal analysis of fluid flows: an overview. *AIAA Journal*, pages 4013–4041, 2017.
| Strengths | Weaknesses | Mitigation strategies |
|-----------|------------|-----------------------|
| **POD-GP** | • Bases are orthonormal in space, ordered and optimal in $L_2$ sense<br>• Harness the power of physics and often rely on the underlying equations<br>• Arguably first hybrid physics-data driven approach for complex dynamical systems<br>• Less prone to overfitting due to sampling rate | • Lack of conservation<br>• Online prediction cost with $O(R^3)$ for fluid dynamics applications (quadratic nonlinearity)<br>• Intrusive and needs access to the physical processes and subcomponents<br>• Its intrinsic global nature adds modal deformation, especially for convective flows | • Many closure modeling ideas have been developed<br>• Online basis adaption is applicable with increasing computational cost<br>• Frequency domain forms of POD are available |
| **DMD** | • Fully non-intrusive and does not need knowledge of governing equations<br>• DMD modes are temporally orthogonal (i.e., a frequency based approach)<br>• Projection-free approach<br>• Online prediction is very fast | • Basis functions are not orthogonal in space<br>• Usually requires more basis functions compared to POD<br>• Highly dependent on the sampling procedure and decay/growth rate<br>• Intrinsically interpolatory, and hard to generalize to parametric systems<br>• Basis functions are not sorted automatically | • Dynamic streaming approaches are applicable<br>• Additional optimization procedures can be applied to select better representative basis functions<br>• Sparsity promoting, multi-scale, multi-resolution, and recursive approaches are available to improve accuracy and computational efficiency |
| **POD-ANN** | • Fully non-intrusive and does not need knowledge of governing equations<br>• Harnesses the power of emerging machine learning tools<br>• Quite modular, and different time series prediction tools can be applied<br>• Basis functions are orthogonal and optimal<br>• Basis are also sorted trivially<br>• Online prediction is fast | • Prone to overfitting<br>• Intrinsically interpolatory<br>• Deeper architectures are required for larger number of modes<br>• Works well if there is scale separation (i.e., with higher decay rate in eigenspectrum or lower Kolmogorov-width)<br>• Hyper-parameter tuning is usually required<br>• Requires a black-box integrator | • Partitioning or local approaches can be applied<br>• Gaussian processes can be used to estimate uncertainties<br>• Closure models can be developed for the problems with higher Kolmogorov width (i.e., smaller decay rate in eigenspectrum) |

[5] Dacian N Daescu and I Michael Navon. Efficiency of a POD-based reduced second-order adjoint model in 4D-Var data assimilation. *International Journal for Numerical Methods in Fluids*, 53(6):985–1004, 2007.

[6] Ionel M Navon. Data assimilation for numerical weather prediction: a review. In Seon K. Park and Liang Xu, editors, *Data Assimilation for atmospheric, oceanic and hydrologic applications*, pages 21–65. Springer, New York.
York, 2009.

[7] Yanhua Cao, Jiang Zhu, I Michael Navon, and Zhendong Luo. A reduced-order approach to four-dimensional variational data assimilation using proper orthogonal decomposition. *International Journal for Numerical Methods in Fluids*, 53(10):1571–1583, 2007.

[8] Jincong He, Pallav Sarma, and Louis J Durlofsky. Use of reduced-order models for improved data assimilation within an EnKF context. In *SPE Reservoir Simulation Symposium*. Society of Petroleum Engineers, Texas, 2011.

[9] Peter L Houtekamer and Herschel L Mitchell. Data assimilation using an ensemble Kalman filter technique. *Monthly Weather Review*, 126(3):796–811, 1998.

[10] Peter L Houtekamer and Herschel L Mitchell. A sequential ensemble Kalman filter for atmospheric data assimilation. *Monthly Weather Review*, 129(1):123–137, 2001.

[11] Andrew F Bennett. *Inverse modeling of the ocean and atmosphere*. Cambridge University Press, New York, 2005.

[12] Geir Evensen. *Data assimilation: the ensemble Kalman filter*. Springer-Verlag, Berlin, Heidelberg, 2009.

[13] Kody JH Law and Andrew M Stuart. Evaluating data assimilation algorithms. *Monthly Weather Review*, 140(11):3757–3782, 2012.

[14] Vladimir Buljak. *Inverse analyses with model reduction: proper orthogonal decomposition in structural mechanics*. Springer-Verlag, Berlin, Heidelberg, 2011.

[15] Kazufumi Ito and Sivaguru S Ravindran. Reduced basis method for optimal control of unsteady viscous flows. *International Journal of Computational Fluid Dynamics*, 15(2):97–113, 2001.

[16] Antoine McNamara, Adrien Treuille, Zoran Popović, and Jos Stam. Fluid control using the adjoint method. In *ACM Transactions on Graphics*, volume 23, pages 449–456. ACM, 2004.

[17] Michel Bergmann and Laurent Cordier. Optimal control of the cylinder wake in the laminar regime by trust-region methods and POD reduced-order models. *Journal of Computational Physics*, 227(16):7813–7840, 2008.

[18] Sivaguru S Ravindran. A reduced-order approach for optimal control of fluids using proper orthogonal decomposition. *International Journal for Numerical Methods in Fluids*, 34(5):425–448, 2000.

[19] WR Graham, J Peraire, and KY Tang. Optimal control of vortex shedding using low-order models. Part I–open-loop model development. *International Journal for Numerical Methods in Engineering*, 44(7):945–972, 1999.

[20] WR Graham, J Peraire, and KY Tang. Optimal control of vortex shedding using low-order models. Part II–model-based control. *International Journal for Numerical Methods in Engineering*, 44(7):973–990, 1999.

[21] Steven L Brunton and Bernd R Noack. Closed-loop turbulence control: progress and challenges. *Applied Mechanics Reviews*, 67(5):050801, 2015.

[22] Hongfei Fu, Hong Wang, and Zhu Wang. POD/DEIM reduced-order modeling of time-fractional partial differential equations with applications in parameter identification. *Journal of Scientific Computing*, 74(1):220–243, 2018.

[23] Muriel Boulakia, Elisa Schenone, and J-F Gerbeau. Reduced-order modeling for cardiac electrophysiology, application to parameter identification. *International Journal for Numerical Methods in Biomedical Engineering*, 28(6-7):727–744, 2012.

[24] Boris Kramer, Benjamin Peherstorfer, and Karen Willcox. Feedback control for systems with uncertain parameters using online-adaptive reduced models. *SIAM Journal on Applied Dynamical Systems*, 16(3):1563–1586, 2017.

[25] David Galbally, K Fidkowski, K Willcox, and O Ghattas. Non-linear model reduction for uncertainty quantification in large-scale inverse problems. *International Journal for Numerical Methods in Engineering*, 81(12):1581–1608, 2010.

[26] Lorenz Biegler, George Biros, Omar Ghattas, Matthias Heinkenschloss, David Keyes, Bani Mallick, Luis Tenorio, Bart Van Bloemen Waanders, Karen Willcox, and Youssef Marzouk. *Large-scale inverse problems and quantification of uncertainty*. Wiley Online Library, Chichester, 2011.

[27] Lionel Mathelin, M Youssuff Hussaini, and Thomas A Zang. Stochastic approaches to uncertainty quantification in CFD simulations. *Numerical Algorithms*, 38(1-3):209–236, 2005.

[28] Ralph C Smith. *Uncertainty quantification: theory, implementation, and applications*. Siam, Philadelphia, 2014.

[29] Habib N Najm. Uncertainty quantification and polynomial chaos techniques in computational fluid dynamics. *Annual Review of Fluid Mechanics*, 41:35–52, 2009.
[30] John Leask Lumley. The structure of inhomogeneous turbulent flows. In *Atmospheric Turbulence and Radio Wave Propagation*, pages 166–178. Nauka Publishing House, Moscow, 1967.

[31] Karl Pearson. On lines and planes of closest fit to systems of points in space. *Philosophical Magazine*, 2(11):559–572, 1901.

[32] Gal Berkooz, Philip Holmes, and John L Lumley. The proper orthogonal decomposition in the analysis of turbulent flows. *Annual Review of Fluid Mechanics*, 25(1):539–575, 1993.

[33] Kazufumi Ito and SS Ravindran. A reduced-order method for simulation and control of fluid flows. *Journal of Computational Physics*, 143(2):403–425, 1998.

[34] Angelo Iollo, Stéphane Lanteri, and J-A Désidéri. Stability properties of POD–Galerkin approximations for the compressible Navier—Stokes equations. *Theoretical and Computational Fluid Dynamics*, 13(6):377–396, 2000.

[35] Clarence W Rowley, Tim Colonius, and Richard M Murray. Model reduction for compressible flows using POD and Galerkin projection. *Physica D: Nonlinear Phenomena*, 189(1-2):115–129, 2004.

[36] René Pinnau. Model reduction via proper orthogonal decomposition. In Wilhelms H. A. Schilders, A. Henk van der Vorst, and Joost Rommes, editors, *Model order reduction: theory, research aspects and applications*, pages 95–109. Springer, Berlin, Heidelberg, 2008.

[37] Witold Stankiewicz, Marek Morzyński, Bernd R Noack, and Gilead Tadmor. Reduced order Galerkin models of flow around NACA-0012 airfoil. *Mathematical Modelling and Analysis*, 13(1):113–122, 2008.

[38] Ekkehard W Sachs and Stefan Volkwein. POD-Galerkin approximations in PDE-constrained optimization. *GAMM-Mitteilungen*, 33(2):194–208, 2010.

[39] Imran Akhtar, Ali H Nayfeh, and Calvin J Ribbens. On the stability and extension of reduced-order Galerkin models in incompressible flows. *Theoretical and Computational Fluid Dynamics*, 23(3):213–237, 2009.

[40] Matthew F Barone, Irina Kalashnikova, Daniel J Segalman, and Heidi K Thornquist. Stable Galerkin reduced order models for linearized compressible flow. *Journal of Computational Physics*, 228(6):1932–1946, 2009.

[41] Carmen Gräßle, Michael Hinze, and Stefan Volkwein. Model order reduction by proper orthogonal decomposition. *arXiv preprint arXiv:1906.05188*, 2019.

[42] René Milk, Stephan Rave, and Felix Schindler. pyMOR – Generic algorithms and interfaces for model order reduction. *SIAM Journal on Scientific Computing*, 38(5):S194–S216, 2016.

[43] Vladimir Puzyrev, Mehdi Ghommem, and Shiv Meka. pyROM: A computational framework for reduced order modeling. *Journal of Computational Science*, 30:157–173, 2019.

[44] Nadine Aubry, Régis Guyonnet, and Ricardo Lima. Spatiotemporal analysis of complex signals: theory and applications. *Journal of Statistical Physics*, 64(3-4):683–739, 1991.

[45] Nadine Aubry. On the hidden beauty of the proper orthogonal decomposition. *Theoretical and Computational Fluid Dynamics*, 2(5-6):339–352, 1991.

[46] Moritz Sieber, C Oliver Paschereit, and Kilian Oberleithner. Spectral proper orthogonal decomposition. *Journal of Fluid Mechanics*, 792:798–828, 2016.

[47] C Picard and J Delville. Pressure velocity coupling in a subsonic round jet. *International Journal of Heat and Fluid Flow*, 21(3):359–364, 2000.

[48] Aaron Towne, Oliver T Schmidt, and Tim Colonius. Spectral proper orthogonal decomposition and its relationship to dynamic mode decomposition and resolvent analysis. *Journal of Fluid Mechanics*, 847:821–867, 2018.

[49] Jean-Christophe Loiseau, Bernd R Noack, and Steven L Brunton. Sparse reduced-order modelling: sensor-based dynamics to full-state estimation. *Journal of Fluid Mechanics*, 844:459–490, 2018.

[50] Karen Willcox and Jaime Peraire. Balanced model reduction via the proper orthogonal decomposition. *AIAA Journal*, 40(11):2323–2330, 2002.

[51] Bruce Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE transactions on automatic control*, 26(1):17–32, 1981.

[52] Clarence W Rowley. Model reduction for fluids, using balanced proper orthogonal decomposition. *International Journal of Bifurcation and Chaos*, 15(03):997–1013, 2005.

[53] John R Singler and Belinda A Batten. A proper orthogonal decomposition approach to approximate balanced truncation of infinite dimensional linear systems. *International Journal of Computer Mathematics*, 86(2):355–371, 2009.
[54] John R Singler. Balanced POD for model reduction of linear PDE systems: convergence theory. *Numerische Mathematik*, 121(1):127–164, 2012.

[55] Toni Lassila, Andrea Manzoni, Alfio Quarteroni, and Gianluigi Rozza. Model order reduction in fluid dynamics: challenges and perspectives. In *Reduced order methods for modeling and computational reduction*, pages 235–273. Springer, Cham, 2014.

[56] Sk. Mashfiqur Rahman, Shady E. Ahmed, and Omer San. A dynamic closure modeling framework for model order reduction of geophysical fluids. *Journal of Fluid Mechanics*, 747:518–544, 2014.

[57] Omer San and Traian Iliescu. Proper orthogonal decomposition closure models for fluid flows: Burgers equation. *International Journal of Numerical Analysis & Modeling - Series B*, 5:217–237, 2014.

[58] Zhu Wang, Imran Akhtar, Jeff Borggaard, and Traian Iliescu. Proper orthogonal decomposition closure models for turbulent flows: A numerical comparison. *Computer Methods in Applied Mechanics and Engineering*, 237:10–26, 2012.

[59] Jeff Borggaard, Traian Iliescu, and Zhu Wang. Artificial viscosity proper orthogonal decomposition. *Mathematical and Computer Modelling*, 53(1-2):269–279, 2011.

[60] RȘtefănescu and Ionel Michael Navon. POD/DEIM nonlinear model order reduction of an ADI implicit shallow water equations model. *Journal of Computational Physics*, 237:95–114, 2013.

[61] Yuepeng Wang, Ionel M Navon, Xin Yu Wang, and Yue Cheng. 2D Burgers equation with large Reynolds number using POD/DEIM and calibration. *International Journal for Numerical Methods in Fluids*, 82(12):909–931, 2016.

[62] Igor Mezić. Spectral properties of dynamical systems, model reduction and decompositions. *Nonlinear Dynamics*, 41(1-3):309–325, 2005.

[63] Peter Schmid and Joern Sesterhenn. Dynamic mode decomposition of numerical and experimental data. *Bulletin of the American Physical Society*, 53, 2008.

[64] J Nathan Kutz, Steven L Brunton, Bingni W Brunton, and Joshua L Proctor. *Dynamic mode decomposition: data-driven modeling of complex systems*. SIAM, Philadelphia, 2016.

[65] Maziar S Hemati, Matthew O Williams, and Clarence W Rowley. Dynamic mode decomposition for large and streaming datasets. *Physics of Fluids*, 26(11):111701, 2014.

[66] Daniel Dylewsky, Molei Tao, and J Nathan Kutz. Dynamic mode decomposition for multiscale nonlinear physics. *Physical Review E*, 99(6):063311, 2019.

[67] Kevin K Chen, Jonathan H Tu, and Clarence W Rowley. Variants of dynamic mode decomposition: boundary condition, Koopman, and Fourier analyses. *Journal of Nonlinear Science*, 22(6):887–915, 2012.

[68] Ting Zhang, Yong Lv, Youting Li, and Cancan Yi. Optimized dynamic mode decomposition via non-convex regularization and multiscale permutation entropy. *Entropy*, 20(3):152, 2018.

[69] Diana Alina Bistrian and Ionel Michael Navon. Randomized dynamic mode decomposition for nonintrusive reduced order modelling. *International Journal for Numerical Methods in Engineering*, 112(1):3–25, 2017.

[70] Diana Alina Bistrian and Ionel Michael Navon. Efficiency of randomised dynamic mode decomposition for reduced order modelling. *International Journal of Computational Fluid Dynamics*, 32(2-3):88–103, 2018.

[71] Diana Alina Bistrian and Ionel Michael Navon. An improved algorithm for the shallow water equations model reduction: dynamic mode decomposition vs POD. *International Journal for Numerical Methods in Fluids*, 78(9):552–580, 2015.

[72] Diana Alina Bistrian and Ionel Michael Navon. The method of dynamic mode decomposition in shallow water and a swirling flow problem. *International Journal for Numerical Methods in Fluids*, 83(1):73–89, 2017.

[73] JE Higham, W Brevis, and CJ Keylock. Implications of the selection of a particular modal decomposition technique for the analysis of shallow flows. *Journal of Hydraulic Research*, 56(6):796–805, 2018.
[77] Clarence W Rowley, Igor Mezić, Shervin Bagheri, Philipp Schlatter, and Dan S Henningson. Spectral analysis of nonlinear flows. *Journal of Fluid Mechanics*, 641:115–127, 2009.

[78] Igor Mezić. Analysis of fluid flows via spectral properties of the Koopman operator. *Annual Review of Fluid Mechanics*, 45:357–378, 2013.

[79] Shervin Bagheri. Koopman-mode decomposition of the cylinder wake. *Journal of Fluid Mechanics*, 726:596–623, 2013.

[80] Peter J Schmid. Dynamic mode decomposition of numerical and experimental data. *Journal of Fluid Mechanics*, 656:5–28, 2010.

[81] Hannah Lu and Daniel M Tartakovsky. Predictive accuracy of dynamic mode decomposition. *arXiv preprint arXiv:1905.01587*, 2019.

[82] Yijia Zhao, Ming Zhao, Xiaojian Li, Zhengxian Liu, and Juan Du. A modified proper orthogonal decomposition method for flow dynamic analysis. *Computers & Fluids*, 182:28–36, 2019.

[83] Ada Cammilleri, Florimond Guéniat, Johan Carlier, Luc Pastur, Etienne Ménin, François Lusseyran, and Guillermo Artana. POD-spectral decomposition for fluid flow analysis and model reduction. *Theoretical and Computational Fluid Dynamics*, 27(6):787–815, 2013.

[84] Bernd R Noack, Witold Stankiewicz, Marek Morzyński, and Peter J Schmid. Recursive dynamic mode decomposition of transient and post-transient wake flows. *Journal of Fluid Mechanics*, 809:843–872, 2016.

[85] Alessandro Alla and J Nathan Kutz. Nonlinear model order reduction via dynamic mode decomposition. *SIAM Journal on Scientific Computing*, 39(5):B778–B796, 2017.

[86] MA Mendez, M Balabane, and J-M Buchlin. Multi-scale proper orthogonal decomposition of complex fluid flows. *Journal of Fluid Mechanics*, 870:988–1036, 2019.

[87] M. Williams, P. Schmid, and J. Kutz. Hybrid reduced-order integration with proper orthogonal decomposition and dynamic mode decomposition. *Multiscale Modeling & Simulation*, 11(2):522–544, 2013.

[88] Steven L Brunton and J Nathan Kutz. *Data-driven science and engineering: machine learning, dynamical systems, and control*. Cambridge University Press, New York, 2019.

[89] J Nathan Kutz. Deep learning in fluid dynamics. *Journal of Fluid Mechanics*, 814:1–4, 2017.

[90] MS Al-Haik, H Garmentani, and I Michael Navon. Truncated-Newton training algorithm for neurocomputational viscoplastic model. *Computer Methods in Applied Mechanics and Engineering*, 192(19):2249–2267, 2003.

[91] Michele Milano and Petros Koumoutsakos. Neural network modeling for near wall turbulent flow. *Journal of Computational Physics*, 182(1):1–26, 2002.

[92] Kai Fukami, Koji Fukagata, and Kunihiko Taira. Super-resolution reconstruction of turbulent flows with machine learning. *Journal of Fluid Mechanics*, 870:106–120, 2019.

[93] N Benjamin Erichson, Lionel Mathelin, Zhewei Yao, Steven L Brunton, Michael W Mahoney, and J Nathan Kutz. Shallow learning for fluid flow reconstruction with limited sensors and limited data. *arXiv preprint arXiv:1902.07358*, 2019.

[94] Jeremy Morton, Antony Jameson, Mykel J Kochenderfer, and Freddie Witherden. Deep dynamical modeling and control of unsteady fluid flows. In *Advances in Neural Information Processing Systems*, pages 9258–9268, 2018.

[95] Byungsoo Kim, Vinicius C. Azevedo, Nils Thuery, Theodore Kim, Markus Gross, and Barbara Solenthaler. Deep fluids: a generative network for parameterized fluid simulations. *Computer Graphics Forum*, 38(2):59–70, 2019.

[96] Christopher M Bishop. *Pattern recognition and machine learning*. Springer-Verlag, New York, 2006.

[97] Christopher M Bishop. *Neural networks for pattern recognition*. Oxford University Press, Oxford, 1995.

[98] Brian D Ripley and NL Hjort. *Pattern recognition and neural networks*. Cambridge University Press, New York, 1996.

[99] Kumpati S Narendra and Kannan Parthasarathy. Identification and control of dynamical systems using neural networks. *IEEE Transactions on Neural Networks*, 1(1):4–27, 1990.

[100] Pierre Baldi and Kurt Hornik. Neural networks and principal component analysis: learning from examples without local minima. *Neural Networks*, 2(1):53–58, 1989.

[101] N Benjamin Erichson, Michael Muehlebach, and Michael W Mahoney. Physics-informed autoencoders for Lyapunov-stable fluid flow prediction. *arXiv preprint arXiv:1905.10866*, 2019.
[102] Samuel E Otto and Clarence W Rowley. Linearly recurrent autoencoder networks for learning dynamics. *SIAM Journal on Applied Dynamical Systems*, 18(1):558–593, 2019.

[103] Kookjin Lee and Kevin Carlberg. Model reduction of dynamical systems on nonlinear manifolds using deep convolutional autoencoders. *arXiv preprint arXiv:1812.08373*, 2018.

[104] S Narayanan, AI Khibnik, CA Jacobson, Y Kevrekidis, R Rico-Martinez, and K Lust. Low-dimensional models for active control of flow separation. In *Proceedings of the 1999 IEEE International Conference on Control Applications* (Cat. No. 99CH36328), volume 2, pages 1151–1156. IEEE, 1999.

[105] Omer San, Romit Maulik, and Mansoor Ahmed. An artificial neural network framework for reduced order modeling of transient flows. *Communications in Nonlinear Science and Numerical Simulation*, 77:271–287, 2019.

[106] S Pawar, SM Rahman, H Vaddireddy, O San, A Rasheed, and P Vedula. A deep learning enabler for non-intrusive reduced order modeling of fluid flows. *arXiv preprint arXiv:1907.04945*, 2019.

[107] Omer San and Romit Maulik. Neural network closures for nonlinear model order reduction. *Advances in Computational Mathematics*, 44(6):1717–1750, 2018.

[108] Omer San and Romit Maulik. Machine learning closures for model order reduction of thermal fluids. *Applied Mathematical Modelling*, 60:681–710, 2018.

[109] Sk Rahman, Omer San, and Adil Rasheed. A hybrid approach for model order reduction of barotropic quasi-geostrophic turbulence. *Fluids*, 3(4):86, 2018.

[110] Muhammad Mohebujjaman, Leo G Rebolholz, and Traian Iliescu. Physically constrained data-driven correction for reduced-order modeling of fluid flows. *International Journal for Numerical Methods in Fluids*, 89(3):103–122, 2019.

[111] Saddam Hijazi, Giovanni Stabile, Andrea Mola, and Gianluigi Rozza. Data-driven POD-Galerkin reduced order model for turbulent flows. *arXiv preprint arXiv:1907.09909*, 2019.

[112] Geoffrey K Vallis. *Atmospheric and oceanic fluid dynamics: fundamentals and large-scale circulation*. Cambridge University Press, New York, 2006.

[113] Arne Grammeltvedt. A survey of finite-difference schemes for the primitive equations for a barotropic fluid. *Monthly Weather Review*, 97(5):384–404, 1969.

[114] Akio Arakawa. Computational design for long-term numerical integration of the equations of fluid motion: two-dimensional incompressible flow. Part I. *Journal of Computational Physics*, 1(1):119–143, 1966.

[115] Sigal Gottlieb and Chi-Wang Shu. Total variation diminishing Runge-Kutta schemes. *Mathematics of Computation*, 67(221):73–85, 1998.

[116] Parviz Moin. *Fundamentals of engineering numerical analysis*. Cambridge University Press, New York, 2010.

[117] Joe D Hoffman and Steven Frankel. *Numerical methods for engineers and scientists*. CRC Press, New York, 2001.

[118] Lawrence Sirovich. Turbulence and the dynamics of coherent structures. I. Coherent structures. *Quarterly of Applied Mathematics*, 45(3):561–571, 1987.

[119] David H Hubel and Torsten N Wiesel. Receptive fields, binocular interaction and functional architecture in the cat’s visual cortex. *The Journal of Physiology*, 160(1):106–154, 1962.

[120] Kurt Hornik, Maxwell Stinchcombe, and Halbert White. Multilayer feedforward networks are universal approximators. *Neural Networks*, 2(5):359–366, 1989.

[121] Eldad Haber and Lars Ruthotto. Stable architectures for deep neural networks. *Inverse Problems*, 34(1):014004, 2017.

[122] Yiping Lu, Aoxiao Zhong, Quanzheng Li, and Bin Dong. Beyond finite layer neural networks: bridging deep architectures and numerical differential equations. In *International Conference on Machine Learning*, pages 3282–3291, 2018.

[123] Philip Holmes, John L Lumley, Gahl Berkooz, and Clarence W Rowley. *Turbulence, coherent structures, dynamical systems and symmetry*. Cambridge University Press, New York, 2012.

[124] Bernard O Koopman. Hamiltonian systems and transformation in Hilbert space. *Proceedings of the National Academy of Sciences of the United States of America*, 17(5):315, 1931.

[125] AK Alekseev, DA Bistrian, AE Bondarev, and IM Navon. On linear and nonlinear aspects of dynamic mode decomposition. *International Journal for Numerical Methods in Fluids*, 82(6):348–371, 2016.
[126] Jonathan H Tu, Clarence W Rowley, Dirk M Luchtenburg, Steven L Brunton, and J Nathan Kutz. On dynamic mode decomposition: theory and applications. *Journal of Computational Dynamics*, 1:391–421, 2014.

[127] Bernd R Noack, Marek Morzynski, and Gilead Tadmor. *Reduced-order modelling for flow control*. Springer-Verlag, Wien, 2011.

[128] Max D Gunzburger. *Flow control*, volume 68. Springer-Verlag, New York, 2012.

[129] A Kolmogoroff. Über die beste Annäherung von Funktionen einer gegebenen Funktionenklasse. *Annals of Mathematics*, 37(1):107–110, 1936.

[130] Allan Pinkus. *N-widths in Approximation Theory*, volume 7. Springer Science & Business Media, 2012.

[131] Harry Nyquist. Certain topics in telegraph transmission theory. *Transactions of the American Institute of Electrical Engineers*, 47(2):617–644, 1928.

[132] Seth M Hirsh, Kameron Decker Harris, J Nathan Kutz, and Bingni W Brunton. Centering data improves the dynamic mode decomposition. *arXiv preprint arXiv:1906.05973*, 2019.
Figure 9: Geopotential field at 24 hours from Experiment 1
Figure 10: Geopotential field at 24 hours from Experiment 2
Figure 11: Geopotential field at 24 hours from Experiment 3
Figure 12: Geopotential field at 24 hours from Experiment 4
Figure 13: Time series prediction using the first 4 modes from Experiment 1

Figure 14: Time series prediction using the first 4 modes from Experiment 2
Figure 15: Time series prediction using the first 4 modes from Experiment 3

Figure 16: Time series prediction using the first 4 modes from Experiment 4
Figure 17: Time series prediction using the first 8 modes from Experiment 1

Figure 18: Time series prediction using the first 8 modes from Experiment 2
Figure 19: Time series prediction using the first 8 modes from Experiment 3

Figure 20: Time series prediction using the first 8 modes from Experiment 4
Figure 21: Time series prediction using the first 16 modes from Experiment 1

Figure 22: Time series prediction using the first 16 modes from Experiment 2
Figure 23: Time series prediction using the first 16 modes from Experiment 3

Figure 24: Time series prediction using the first 16 modes from Experiment 4
Figure 25: Geopotential field at 24 hours from Experiment 1 - obtained without mean subtraction
Figure 26: Geopotential field at 24 hours from Experiment 2 - obtained without mean subtraction
Figure 27: Geopotential field at 24 hours from Experiment 3 - obtained without mean subtraction
Figure 28: Geopotential field at 24 hours from Experiment 4 - obtained without mean subtraction
Figure 29: Time series prediction using the first 4 modes from Experiment 1 - obtained without mean subtraction

Figure 30: Time series prediction using the first 4 modes from Experiment 2 - obtained without mean subtraction
Figure 31: Time series prediction using the first 4 modes from Experiment 3 - obtained without mean subtraction

Figure 32: Time series prediction using the first 4 modes from Experiment 4 - obtained without mean subtraction
Figure 33: Time series prediction using the first 8 modes from Experiment 1 - obtained without mean subtraction

Figure 34: Time series prediction using the first 8 modes from Experiment 2 - obtained without mean subtraction
Figure 35: Time series prediction using the first 8 modes from Experiment 3 - obtained without mean subtraction

Figure 36: Time series prediction using the first 8 modes from Experiment 4 - obtained without mean subtraction
Figure 37: Time series prediction using the first 16 modes from Experiment 1 - obtained without mean subtraction.

Figure 38: Time series prediction using the first 16 modes from Experiment 2 - obtained without mean subtraction.
Figure 39: Time series prediction using the first 16 modes from Experiment 3 - obtained without mean subtraction

Figure 40: Time series prediction using the first 16 modes from Experiment 4 - obtained without mean subtraction