WANDs of the Black Ring

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Abstract

Necessary conditions for various algebraic types of the Weyl tensor are determined. These conditions are then used to find Weyl aligned null directions for the black ring solution. It is shown that the black ring solution is algebraically special, of type I\textsubscript{i}, while locally on the horizon the type is II. One exceptional subclass – the Myers-Perry solution – is of type D.

1 Introduction

Recently a classification of tensors on Lorentzian manifolds of arbitrary dimensions was introduced \cite{1}. When applied to the Weyl tensor \cite{1, 2}, in four dimensions it reduces to the well known Petrov classification, in higher dimensions it leads to a similar, dimensionally independent classification scheme (see Table 1). This classification is based on the existence of preferred null directions – Weyl aligned null directions (WANDs) and corresponding principal null congruences in a given spacetime (see Section 2 for details). In 4 dimensions WANDs correspond to principal null directions of the Weyl tensor.

The Petrov classification in four dimensions was very useful for generating algebraically special exact solutions as well as for physical interpretation of spacetimes. Ultimately, one would like to use the higher dimensional classification for similar purposes as in classical relativity, however, while in higher dimensions some properties related to the classification remain the same as in four dimensions, other differ.

For example, it was proven that for all vacuum spacetimes of types III and N in arbitrary dimension the principal null congruence is geodesic as in four dimensions \cite{3}. This congruence in four dimensions is in addition shear-free but in higher dimensions shear may not vanish (see \cite{3} for additional comments on the Goldberg-Sachs theorem in higher dimensions). Also all spacetimes with vanishing curvature invariants are necessarily of types III and N in arbitrary dimension, including four \cite{4}.

Black holes are another interesting example. First let us point out that the Kerr solution as well as the Myers-Perry solution in five dimensions are of type D with geodesic principal null congruences \cite{3, 5}. More general black holes in four dimensions can be algebraically general, however, if the horizon is isolated, the spacetime is of type II locally on the horizon \cite{5}. Recently, it was proven that locally, on the isolated horizon, the algebraic type is II also in higher dimensions \cite{6}.

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A vacuum, asymptotically flat, stationary black hole solution with a horizon of topology \( S^1 \times S^2 \) - so called black ring - was discovered recently in [10]. There exist black holes of spherical topology and black rings with the same values of the conserved quantities \( M \) and \( J \) and thus black holes in higher dimensions are not uniquely characterized by their mass and angular momenta and uniqueness theorems cannot be (straightforwardly) generalized for higher dimensions.

In this paper it is shown that while the Myers-Perry solution representing a spherical black hole is of type D, black rings, while still algebraically special, belong to the more general class I. In Section 2 the classification of the Weyl tensor in higher dimensions is overviewed and necessary conditions for various classes, which significantly simplify search for WANDs, are determined. In Section 3 we briefly overview the (neutral) black ring solution and analyze some of its properties. In Section 4 we classify static and stationary black rings. In four dimensions it is possible to determine algorithmically an algebraic type of a solution using various invariants and covariants of the metric (see e.g. Chapter 9 in [9]). Since such a method is not developed in higher dimensions, it is necessary to find WANDs explicitly. Components of the Weyl tensor for the black ring are quite complicated and thus corresponding necessary conditions are also complicated even with the use of Maple (however, much simpler than alignment equations). We thus limit ourselves to presenting only their solutions and indicating the procedure how to obtain them. We also explicitly show that locally on the horizon the Weyl tensor is of type II.

2 Algebraic classification of the Weyl tensor in higher dimensions

In this section, algebraic classification of the Weyl tensor in higher dimensions developed in [11, 12] is briefly summarized and at the end necessary conditions for various algebraic types are introduced.

We will work in the frame

\[
\begin{align*}
    m^{(0)} &= n, \quad m^{(1)} = \ell, \quad m^{(i)} = i, j, k = 2 \ldots D - 1, \\
    \ell^a n_a &= 0, \quad a = 0 \ldots D - 1,
\end{align*}
\]

with two null vectors \( n, \ell \)

\[
    \ell^a n_a = 0, \quad a = 0 \ldots D - 1,
\]

and \( D - 2 \) spacelike vectors \( m^{(i)} \)

\[
    m^{(i)a} m_a^{(j)} = \delta_{ij}, \quad m^{(i)a} n_a = 0 = m^{(i)a} n_a, \quad i, j, k = 2 \ldots D - 1,
\]

where the metric has the form

\[
g_{ab} = 2\ell(\ell n_b) + \delta_{ij} m_a^{(i)} m_b^{(j)}.\]

The group of ortochronous Lorentz transformations is generated by null rotations

\[
\hat{\ell} = \ell + z_i m^{(i)} - \frac{1}{2} z^i z_i n, \quad \hat{n} = n, \quad \hat{m}^{(i)} = m^{(i)} - z_i n
\]

(1)

spins and boosts, respectively,

\[
\hat{\ell} = \lambda \ell, \quad \hat{n} = n, \quad \hat{m}^{(i)} = \lambda^{i} m^{(j)}, \quad \hat{\ell} = \lambda \ell, \quad \hat{\ell} = \lambda^{-1} n, \quad \hat{m}^{(i)} = m^{(i)}.
\]

A quantity \( q \) has a boost weight \( b \) if it transforms under a boost according to \( \hat{q} = \lambda^b q \). Boost order of a tensor \( T \) is defined as the maximum boost weight of its frame components and it can be shown that it depends only on the choice of a null direction \( \ell \) (see Proposition III.2 in [1]). For a given tensor \( T \), \( b_{\text{max}} \) denotes the maximum value of \( b(k) \) taken over all null vectors \( k \). Then a null vector \( k \) is aligned with the tensor \( T \) whenever \( b(k) < b_{\text{max}} \) and the integer \( b_{\text{max}} - b(k) - 1 \) is called order of alignment. The classification of tensors [1] is based on the existence of such aligned null vectors of various orders. Namely, for the Weyl tensor the primary alignment type is \( G \) if there are no null vectors aligned with the Weyl tensor and the primary alignment type is 1, 2, 3, 4 if the maximally aligned null vector has order of alignment 0, 1, 2, 3, respectively. Once \( \ell \) is fixed as an aligned null vector of maximal order of alignment, one can search for \( n \) with maximal order of alignment subject to the constraint \( n \cdot \ell = 1 \) and similarly define secondary alignment types.
Let us introduce the operation \{ \} which allows us to construct a basis in the space of Weyl-like tensors by

\[ w_{[a}x_{b}y_{c}z_{d}] = \frac{1}{2}(w_{[a}x_{b}y_{c}z_{d]} + w_{[c}x_{d}y_{a}z_{b]}). \tag{2} \]

Now we can decompose the Weyl tensor in its frame components and sort them by their boost weight (see [1]):

\[
C_{abcd} = 4C_{000j} n_{(a}m_{b}^{(i)}n_{c}m_{d}^{(j)} + 8C_{010i} n_{(a}\ell_{b}n_{c}m_{d}^{(i)} + 4C_{0i0j} n_{(a}m_{b}^{(i)}m_{c}^{(j)}m_{d}^{(k)}
\]

\[+ 4C_{010i} n_{(a}n_{b}m_{c}^{(i)}m_{d}^{(j)} + C_{1i1j} \ell_{(a}m_{b}^{(i)}m_{c}^{(j)}m_{d}^{(k)} + 8C_{0i1j} n_{(a}m_{b}^{(i)}\ell_{c}m_{d}^{(j)} + C_{ij1k} m_{(a}^{(i)}m_{b}^{(j)}m_{c}^{(k)}m_{d}^{(l)}
\]

\[+ 8C_{101i} \ell_{(a}n_{b}m_{c}^{(i)}m_{d}^{(j)} + 4C_{1i1j} \ell_{(a}m_{b}^{(i)}m_{c}^{(j)}m_{d}^{(k)} + 4C_{111j} \ell_{(a}m_{b}^{(i)}\ell_{c}m_{d}^{(j)}]. \]

Additional constraints follow from symmetries of the Weyl tensor

\[
C_{0[i|j]} = 0, \quad C_{0(i|j)} = 0, \quad C_{ij(k|l)} = C_{i(k|l)}, \quad C_{i(j|k)} = 0, \quad C_{01ij} = 2C_{0[i|1]j}, \quad C_{11ij} = 0, \quad C_{1[i|1]j} = 0. \tag{3}
\]

and from its tracelessness

\[
C_{000i} = C_{111i} = 0, \quad C_{010i} = C_{0ij}, \quad C_{101i} = C_{ij1},
\]

\[2C_{01i} = C_{01ij} - C_{ikj}, \quad C_{001} = -\frac{1}{2}C_{ij1}. \tag{4}\]

Now the Weyl tensor is of type I if there exist such \( z_i \) that we can set all components of boost weight 2, \( C_{000j} \), to zero using the transformation [1]. It is of type II if we can set boost order 2 and boost order 1 components to zero and so on. Resulting polynomial equations for \( z_i \) depend on the original choice of the frame \( \ell, n, m^{(i)} \) and can be considerably simplified if the frame is chosen appropriately. However, there is no general method how to make such a choice. For this reason we investigate another method, which involves only the vector \( \ell \) instead of the frame \( \ell, n, m^{(i)} \).

In four dimensions, the following equivalences for principal null directions hold (see e.g. [3])

\[
\ell^{b}\ell^{c}C_{[a}bc|[d}\ell_{f]} = 0 \iff \ell \text{ is PND, at most Petrov type I};
\]

\[
\ell^{b}\ell^{c}C_{a[a}bc|[d}\ell_{c]} = 0 \iff \ell \text{ is PND, at most Petrov type II};
\]

\[
\ell^{b}C_{a[a}bc|[d}\ell_{c]} = 0 \iff \ell \text{ is PND, at most Petrov type III};
\]

\[
\ell^{c}C_{abcd} = 0 \iff \ell \text{ is PND, at most Petrov type N}. \tag{5}\]

By substituting a general form of the Weyl tensor of various primary types in previous equations [4] we can find that in arbitrary dimension

\[
\ell^{b}\ell^{c}C_{[a}bc|[d}\ell_{f]} = 0 \iff \ell \text{ is WAND, at most primary type I};
\]

\[
\ell^{b}\ell^{c}C_{a[a}bc|[d}\ell_{c]} = 0 \iff \ell \text{ is WAND, at most primary type II};
\]

\[
\ell^{b}C_{a[a}bc|[d}\ell_{c]} = 0 \iff \ell \text{ is WAND, at most primary type III};
\]

\[
\ell^{c}C_{abcd} = 0 \iff \ell \text{ is WAND, at most primary type N}. \tag{6}\]

3
In fact for the type I equivalence holds in arbitrary dimension but it is not so for more special types. For example, it can be shown that the most general Weyl tensor satisfying \( \ell^c C_{abcd} = 0 \) has the form
\[
C_{ijkl} m^{(i)}_{a} m^{(j)}_{b} m^{(k)} c m^{(l)}_{d} + 4C_{1ijk} \ell^{(i}_{a} m^{(j)}_{b} m^{(k)} c m^{(l)}_{d}) + 4C_{11ij} \ell^{(i}_{a} m^{(j)}_{b} c m^{(l)}_{d}) = 0.
\]

(7)

Its quadratic curvature invariant is \( C_{ijkl} C^{ijkl} = \sum (C_{ijkl})^2 \), which is non-zero as long as \( C_{ijkl} \) has a non-vanishing component. Note that due to symmetries of \( C_{ijkl} \) it can have a non-vanishing component only for dimensions \( D \geq 6 \). Since the Weyl tensor possesses a non-vanishing invariant, it cannot be of type N or III and thus the equivalence in general does not hold.

Let us conclude with a table comparing the classification of the Weyl tensor in four and higher dimensions.

| D>4 dimensions | 4 dimensions |
|---------------|-------------|
| Petrov type   | alignment type | Petrov type |
| G             | G           | G           |
| I             | (1)         | I           |
| \(I_2\)       | (1,1)       | \(I_2\)     |
| II            | (2)         | D           |
| \(II_2\)      | (2,1)       | II          |
| D             | (2,2)       | D           |
| III           | (3)         | III         |
| \(III_2\)     | (3,1)       | \(III_2\)   |
| N             | (4)         | N           |

Table 1: Comparison of the algebraic classification of the Weyl tensor in four and higher dimensions. Note that in four dimensions alignment type (1) is necessarily equivalent to the type (1,1), (2) to (2,1) and (3) to (3,1) and that type G does not exist.

3 Rotating black ring - overview

The rotating black ring solution was found in [10], here we will use a slightly different form of the metric in coordinates \( \{t, x, y, \phi, \psi\} \) introduced in [11]
\[
d^2 = -\frac{F(x)}{F(y)} \left( dt + R \sqrt{\lambda} \psi (1 + y) dy \right)^2
+ \frac{R^2}{(x-y)^2} \left[ -F(x) \left( G(y) dy \psi^2 \frac{F(y)}{G(y)} G(y) \right) + F(y)^2 \left( \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} \frac{d\phi^2}{d\psi} \right) \right],
\]

(8)

where
\[
F(\xi) = 1 - \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 - \nu \xi).
\]

(9)

Note that only certain regions in the \((x, y)\) plane have signature +3. These regions can be determined by analyzing eigenvalues \( \lambda_1 \ldots \lambda_5 \) of the metric
\[
\begin{align*}
\lambda_2 & = \frac{R^2 F(y)^2}{G(x)(x-y)^2}, \\
\lambda_3 & = -\frac{R^2 F(x) F(y)}{G(y)(x-y)^2}, \\
\lambda_4 & = \frac{R^2 F(y)^2 G(x)}{F(x)(x-y)^2}, \\
\lambda_5 & = -\frac{R^2 F(x)^2 G(y)}{F(y)(x-y)^2}.
\end{align*}
\]
Let us now summarize basic properties of various regions in the black ring solution:

Region $A_1$: Here signs of $\lambda_1 \ldots \lambda_5$ are $++-++$. This region is asymptotically flat, $\partial_{\phi}$ is a timelike Killing vector and thus spacetime is static here. Moreover, the norm of the Killing vector $\partial_{\phi}$ approaches $-1$ at the “flat point” $(x,y)=(-1,-1)$. This region represents an outer part of the black ring solution. Region $A_1$ can be smoothly connected with $A_2$ by identifying $y=-\infty$ with $y=\infty$.

In $A_2$, signs of $\lambda_1 \ldots \lambda_5$ are $+++-+$, both $\partial_{\theta}$ and $\partial_{\phi}$ are spacelike, but there exists a timelike Killing vector as their linear combination. This region represents an ergosphere with a limiting surface of stationarity located at $y=\infty$ and a horizon at $y=1/\nu$.

In $A_3$, signs of $\lambda_1 \ldots \lambda_5$ are $++-++$, spacetime is non-stationary and represents the region below the horizon. Curvature singularity is located at $y=1/\lambda$.

In region $B$, signs of $\lambda_1 \ldots \lambda_5$ are $++-++$. This region is asymptotically flat. In the neighbourhood of the curvature singularity located at $y=1/\lambda$, both $\partial_{\theta}$ and $\partial_{\phi}$ are timelike and thus this region contains closed timelike curves. In the vicinity of the “flat point” $(x,y)=(1,1)$, $\partial_{\psi}$ becomes spacelike and the norm of $\partial_{\theta}$ approaches $-1$. This region represents a spacetime of a spinning naked singularity.

In region $C_1$, signs of $\lambda_1 \ldots \lambda_5$ are $++-++$. Note that $\partial_{\phi}$ is timelike and thus this region admits closed timelike curves. Curvature singularity is located at $x=\infty$. In region $C_2$, signs of $\lambda_1 \ldots \lambda_5$ are $+++$. Curvature singularity is located at $x=1/\lambda$. Regions $C_1$ and $C_2$ are not asymptotically flat and their physical interpretation is unclear.
4 Black ring - algebraic structure

In this section we classify the black ring solution and its various special cases. Our method is to solve the necessity conditions (6) and then check that these solutions indeed represent WANDs by calculating components of the Weyl tensor in an appropriate frame.

4.1 Myers-Perry metric is of type D

By setting $\lambda = 1$ in (8) we obtain the Myers-Perry metric [7] with a single rotation parameter.

It turns out that the second equation in (6) admits two independent solutions 

$$ L_{\pm} = \frac{1}{(x^2 - 1)(-1 + \nu y)} \left( \frac{\nu y x - y + \nu x + 1 - 2 \nu y}{x - y} R \partial_t - \sqrt{\nu} \partial \psi \right) \pm \sqrt{\frac{\nu x - 1}{x - y}} \left( \partial_x + \frac{y^2 - 1}{x^2 - 1} \partial_y \right). \quad (11) $$

When we choose a frame with $\ell \sim L_+$ and $n \sim L_-$ all components of the Weyl tensor with boost weights 2,1,-1,-2 vanish and the spacetime is thus of type D. These two vectors were given in Boyer-Lindquist coordinates in [8] and further discussed in App. D in [3].

4.2 Black ring is of type II on the horizon

The transformation

$$ d\chi = d\psi + \sqrt{-F(y)} \frac{G(y)}{G(x)} dy, \quad \quad (12) $$

$$ dv = dt - R\sqrt{\lambda \nu(1 + y)} \frac{\sqrt{-F(y)}}{G(y)} dy \quad \quad (13) $$

(see [10] for a similar transformation) leads to a metric regular on the horizon $y = 1/\nu$

$$ ds^2 = -\frac{F(x)}{F(y)} (dv + \sqrt{\lambda \nu} R(1 + y) d\chi)^2 $$

$$ + \frac{R^2}{(x - y)^2} \left[ -F(x) \left( G(y)d\chi^2 - 2\sqrt{-F(y)} dxdy \right) + F(y)^2 \left( \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right) \right]. \quad (14) $$

The second equation in (10) admits a solution

$$ L = \partial_v - R\sqrt{\frac{\lambda \nu}{1 + \nu}} \partial_x. $$

One can check that boost order of the Weyl tensor in the frame with $\ell = L$ is 0 and thus the black ring is of the type II on the horizon.

4.3 Black ring is of type I_i

In order to solve the first equation in (10)

$$ I_{a b c d} = \ell^b \ell^c \ell^{[a} C^{n]}_{b c} [d \epsilon f] = 0, \quad (15) $$

we denote

$$ \ell^a = (\alpha, \beta, \gamma, \delta, \epsilon) \quad (16) $$

and from [12] we obtain a set of fourth order polynomial equations in $\alpha \ldots \epsilon$. An additional second order equation follows from $\ell_a \ell^a = 0$. Since components of the Weyl tensor are quite complicated, it is not surprising that these equations are also complicated. However, one can pick some of them which are relatively simple. Let us start with the static case.
4.3.1 The static case

The static case can be obtained by setting \( \nu = 0 \). Two particularly simple equations

\[
I_{tx\psi\phi} = \frac{3\alpha \beta \delta \epsilon \lambda (\lambda^2 - 1)(x - y)^3}{2R^2 (-1 + \lambda x)^2 (-1 + \lambda y)^3} = 0, \tag{17}
\]

\[
I_{xy\psi\phi} = -\frac{3\beta \gamma \delta \epsilon \lambda (\lambda^2 - 1)(x - y)^3}{2R^2 (-1 + \lambda x)^2 (-1 + \lambda y)^3} = 0 \tag{18}
\]

imply that unless \( \lambda \) equals to -1,0 or 1, at least one component of the WAND \( \text{(10)} \) vanishes. More detailed analysis of equations \( \text{(16)} \) shows that the only non-trivial solution is \( \epsilon = 0 \) and

\[
\alpha^2 = \frac{R^2 \gamma (1 + \lambda y)(\gamma (x - 1) + \beta (1 - \lambda y))}{(x - y)^2 (-1 + \lambda x)(y^2 - 1)}, \tag{19}
\]

\[
\delta^2 = \frac{\beta (1 - \lambda x)[\gamma (x^2 - 1) + \beta (1 - y^2)]}{(y^2 - 1)(x^2 - 1)^2}, \tag{20}
\]

with \( \beta, \gamma \) satisfying the quadratic equation

\[
\lambda (x^2 - 1)(-1 + \lambda x)^2 \gamma^2 + \lambda (y^2 - 1)(-1 + \lambda y)(-1 + \lambda x) \beta^2
-\lambda \beta \gamma \{\lambda (x + y)[\lambda (x - 1 + xy) + (1 - x^2)] + (x - y)^2(2 - \lambda^2) + 2(-1 + \lambda x) + 2xy(1 - \lambda y)\} = 0. \tag{21}
\]

Note that a vector as well as its arbitrary multiple represent the same WAND. We can thus set one component of \( \text{(10)} \) equal to a given number. Let us set \( \gamma = \gamma_0 \). Now we can solve \( \text{(21)} \) for \( \beta \). It has in general two real roots, however, only one of them leads to positive values of \( \alpha^2 \) and \( \delta^2 \) as given by \( \text{(16)} \) and \( \text{(20)} \). Let us denote this root by \( \beta_0 \) and corresponding square roots of right hand sides of \( \text{(16)} \) and \( \text{(20)} \) as \( \alpha_0 \) and \( \delta_0 \). Note also that \( \beta = -\beta_0, \gamma = -\gamma_0 \) satisfy \( \text{(21)} \) as well as \( \beta = \beta_0, \gamma = \gamma_0 \) and that this change does not affect values of \( \alpha_0 \) and \( \delta_0 \). We thus arrive at four distinct WANDs \( (\alpha_0, \beta_0, \gamma_0, \pm \delta_0, 0), (\alpha_0, -\beta_0, -\gamma_0, \pm \delta_0, 0) \). The static black ring is thus of principal type I. Furthermore, if we choose a frame with \( \ell = (\alpha_0, \beta_0, \gamma_0, \delta_0, 0) \) and \( n = (\alpha_0, \beta_0, -\gamma_0, -\delta_0, 0) \), we can see that all components with boost weight 2 and -2 vanish and thus the type is \( (1, 1) = I_1 \).

4.3.2 The stationary case

Linear combination

\[
(x - 1)(x + 1)(-1 + \nu x)I_{\psi\psi\phi} - \frac{1}{\lambda}(y - 1)(1 + \lambda)(-1 + \nu y)I_{\psi x\psi\phi} = 0 \tag{22}
\]

leads to

\[
\alpha \delta \left\{ \alpha \gamma \sqrt{\lambda} \nu (x - y)^2 (-1 + \lambda x) + \epsilon \lambda R (y^2 - 1) [\beta (1 - \lambda y) + \gamma (\lambda x - 1)] + \epsilon \nu \lambda R (1 + y) [\beta y (y - 1)(-1 + \lambda y) + \gamma (-2xy + x^2 + y)(-1 + \lambda x)] \right\} = 0. \tag{23}
\]

Assuming \( \alpha \neq 0 \) and \( \delta \neq 0 \), \( \text{(23)} \) is a linear equation for \( \alpha \) and by substituting the result in \( I_{xy\psi\phi} = 0 \) we obtain the quadratic equation for \( \beta \) and \( \gamma \)

\[
\delta \epsilon \left\{ -\beta^2 \lambda (y^2 - 1)(-1 + \lambda y)(-1 + \nu y)(-1 + \lambda x) - \gamma^2 \lambda (x^2 - 1)(-1 + \lambda x)^2 (-1 + \nu x) 
+ \beta \gamma [2(\nu - \lambda)(x - y)^2 + \lambda^2 (1 - \lambda y)[x^2(x + y) + x - y] + 2\lambda (1 - \lambda x)^2 + 2\lambda xy(-1 + \lambda y) + 2\lambda^2 \nu xy(1 - \lambda x) + \lambda \nu (1 - \lambda y)[3x^2(y(1 - \lambda x) + x^3(-1 + \lambda y) - 3x + y) + 2\lambda \nu (x - y)] \right\} = 0. \tag{24}
\]
Figure 2: A numerical example for $\lambda = 3/5, \nu = 1/3$: solutions of equations (23)–(26) are real everywhere in the region $\mathcal{A}_3$ and only to the left from the indicated curve in regions $\mathcal{A}_1, \mathcal{A}_2$.

Now the equation $\ell^a \ell_a = 0$ is linear in $\delta^2$ and using (23) we can express $\delta^2$ in terms of $\beta, \gamma, \epsilon$. Substituting this expression in $I_{x\phi y\psi} = 0$ gives

$$
\left[ \gamma (-\lambda x + 1) - \beta (1 - \lambda y) \right] \left\{ \gamma^3 \nu (-1 + \lambda x)(x - y)^2 - \epsilon^2 \beta \lambda (y^2 - 1)^3 (-1 + \nu y)^3 (-1 + \lambda y) \\
+ \epsilon^2 \gamma (-1 + \lambda x) (y^2 - 1)^2 (-1 + \nu y)^2 \left[ \nu x(1 - \lambda y)(x - 2y) + \lambda (1 - \nu y) + y^2(\nu - \lambda) \right] \right\} = 0,
$$

(25)

which is linear in $\epsilon^2$. Solving this equation for $\epsilon^2$ and substituting to $\ell_a \ell^a = 0$ leads to

$$
\delta^2 = \left\{ \beta^3 \lambda (-1 + \lambda x)(y^2 - 1)(-1 + \lambda y)(-1 + \nu y) + \beta \gamma^2 \lambda (-1 + \lambda x)^2 (x^2 - 1)(-1 + \nu x) \\
- \gamma \beta^2 (-1 + \lambda x) \left[ \nu x(1 - xy)[y(1 - \lambda x) + x(1 - \lambda y)] - \nu (x - y)^2 + \lambda (1 - \lambda y)(x^2 - 1) + \lambda (1 - \lambda x)(y^2 - 1) \right] \right\} \\
/ \left\{ (-1 + \nu x)^2 (x^2 - 1)^2 \left[ \gamma (-1 + \lambda x) \left[ \nu (x - y)^2(-1 + \lambda y) + \lambda (y^2 - 1)(1 - \nu y) \right] \\
+ \beta \lambda (y^2 - 1)(-1 + \lambda y)(-1 + \nu y) \right] \right\}.
$$

(26)

Now we can explicitly express aligned null directions. Since if $\ell$ is aligned, then also any multiple of $\ell$ is aligned, we can set $\beta = 1$ without loss of generality. Then we solve the quadratic equation (24) for $\gamma$ and express $\delta$ and $\epsilon$ from (25) and (26) and finally $\alpha$ from (23).

Since the first condition in (14) is equivalence we have thus obtained expressions for WANDs. In the static limit $\nu \to 0$ the solution corresponds to (13)–(21). Note, however, that for some values of $\lambda, \nu, x, y$ (see Figure 2) the value of $\epsilon^2$ as determined by (25) is negative and thus (23)–(26) do not lead to real WANDs at these points. Thus either spacetime is of type G there or there exists another solution of (15) at these points. This indicates that the corresponding region in the vicinity of the axis $x = 0$ is physically distinct from the rest.

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