Resolvable 3-star designs

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Abstract

Let $K_v$ be the complete graph of order $v$ and $F$ be a set of 1-factors of $K_v$. In this article we study the existence of a resolvable decomposition of $K_v - F$ into 3-stars when $F$ has the minimum number of 1-factors. We completely solve the case in which $F$ has the minimum number of 1-factors, with the possible exception of $v \in \{40, 44, 52, 76, 92, 100, 280, 284, 328, 332, 428, 472, 476, 572\}$.

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1 Introduction

Given a collection of graphs $\mathcal{H}$, an $\mathcal{H}$-decomposition of a graph $G$ (also called $\mathcal{H}$-design) is a decomposition of the edges of $G$ into isomorphic copies of graphs from $\mathcal{H}$; the copies of $H \in \mathcal{H}$ in the decomposition are called blocks. Such a decomposition is called resolvable if it is possible to partition the blocks into classes $\mathcal{P}_i$ such that every point of $G$ appears exactly once in some block of each $\mathcal{P}_i$.

A resolvable $\mathcal{H}$-decomposition of $G$ is sometimes also referred to as a $\mathcal{H}$-factorization of $G$; a class can be called a $\mathcal{H}$-factor of $G$. The case where $\mathcal{H}$ is a single edge ($K_2$) is known as a 1-factorization of $G$ and it is well known to exist for $G = K_v$ if and only if $v$ is even. A single class of a 1-factorization, a pairing of all points, is also known as a 1-factor or a perfect matching.

In many cases we wish to impose further constraints on the classes of an $\mathcal{H}$-decomposition. For example, a class is called uniform if every block of the class is isomorphic to the same graph from $\mathcal{H}$. Of particular note is the result of Rees ([9]) which finds necessary and sufficient conditions for the existence of uniform $\{K_2, K_3\}$-decompositions of $K_v$. Uniformly resolvable decompositions of $K_v$ have also been studied in [2], [4], [5], [6], [7], [8], [10], [11], [12] and [13]. Moreover, recently in the case $\mathcal{H} = \{G_1, G_2\}$ the question of the existence of a uniformly resolvable decomposition of $K_v$ into $r$ classes of $G_1$ and $s$ classes of $G_2$ have been studied in the case in which the number $s$ of $G_2$-factors is maximum. Dinitz, Ling and Danziger ([3]) have solved the case $\mathcal{H} = \{K_2, K_4\}$ and Kucukcifci, Milici and Tuza ([5]) the case $\mathcal{H} = \{K_3, K_{1,3}\}$.

In what follows, we will denote by $(a_1; a_2, a_3, a_4)$ the 3-star, $K_{1,3}$ having vertex set $\{a_1, a_2, a_3, a_4\}$ and edge set $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}\}$. We will use the notation $(K_2, K_{1,3})$-URD($v; r, s$) to denote a uniformly resolvable decomposition of $K_v$ into $r$ classes containing only copies of $K_2$ (i.e. 1-factors) and $s$ classes containing only copies of 3-stars.

In this paper, the main purpose is to investigate the existence of a $(K_2, K_{1,3})$-URD($v; r, s$) in the case in which $s > 0$ and $r$ is minimum. In particular, we will prove the following result:

**Main Theorem.** For each $v \equiv 0 \pmod{4}$, there exists a $(K_2, K_{1,3})$-URD($v; r(v), \frac{2(v-1-r(v))}{3}$), with $r(v)$ as in the Table 1 and with the possible exception of $v \in \{40, 44, 52, 76, 92, 100, 280, 284, 328, 332, 428, 472, 476, 572\}$.
Table 1: The set \( r(v) \).

| \( v \) \( (\text{mod} \ 12) \) | \( r(v) \) |
|---|---|
| 0 | 5 |
| 4 | 3 |
| 8 | 1 |

2 Necessary conditions

In this section we will give necessary conditions for the existence of a uniformly resolvable decomposition of \( K_v \) into \( r \) 1-factors and \( s \) classes of 3-stars, \( s > 0 \).

Lemma 2.1. If there exists a \((K_2, K_{1,3})\)-URD\((v; r, s)\), \( s > 0 \), then \( v \equiv 0 \) (mod 4) and \( s \equiv 0 \) (mod 4).

Proof. Assume that there exists a \((K_2, K_{1,3})\)-URD\((v; r, s)\) \( D \), \( s > 0 \). By resolvability it follows that \( v \equiv 0 \) (mod 4). Counting the edges of \( K_v \) that appear in \( D \) we obtain

\[
\frac{rv}{2} + \frac{3sv}{4} = \frac{v(v - 1)}{2}
\]

and hence

\[
2r + 3s = 2(v - 1). \tag{1}
\]

Denote by \( B \) the set of \( s \) parallel classes of 3-stars and by \( R \) the set of \( r \) parallel classes of \( K_2 \). Since the classes of \( R \) are regular of degree 1, we have that every vertex \( x \) of \( K_v \) is incident with \( r \) edges in \( R \) and \( (v - 1) - r \) edges in \( B \). Assume that the vertex \( x \) appears in \( a \) classes with degree 3 and in \( b \) classes with degree 1 in \( B \). Since

\[
a + b = s \quad \text{and} \quad 3a + b = v - 1 - r,
\]

the equality (1) implies that

\[
2(v - 1 - 3a - b) + 3(a + b) = 2(v - 1)
\]

and hence

\[
b = 3a.
\]

This completes the proof. \( \square \)

Lemma 2.2. A \((K_2, K_{1,3})\)-URD\((v; 0, s)\) does not exist for any \( v \geq 4 \).

Proof. Suppose that there exists a uniformly resolvable decomposition \( D \) of \( K_v \) into \( s \) classes containing only copies of 3-stars with \( s > 0 \). Counting the
edges of $K_v$ that appear in $D$ we obtain

\[ s = \frac{2(v - 1)}{3}. \]

Since, by Lemma 2.1, $s = 4t$ it follows

\[ 2(v - 1) = 12t, \]

which is a contradiction, because $v - 1$ cannot be even for any $v \geq 4$. \hfill \Box

Given $v \equiv 0 \pmod{4}$, define $J(v)$ according to the following table:

| $v$          | $J(v)$                                                                 |
|--------------|-------------------------------------------------------------------------|
| 0 (mod 12)   | \{(v - 1 - 6x, 4x), x = 0, 1, \ldots, \frac{v-1}{6}\}                  |
| 4 (mod 12)   | \{(v - 1 - 6x, 4x), x = 0, 1, \ldots, \frac{v-1}{6}\}                  |
| 8 (mod 12)   | \{(v - 1 - 6x, 4x), x = 0, 1, \ldots, \frac{v-1}{6}\}                  |

Table 2: The set $J(v)$.

**Lemma 2.3.** If there exists a $(K_2, K_{1,3})$-URD$(v; r, s)$ then $(r, s) \in J(v)$.

**Proof.** Let $D$ be a $(K_2, K_{1,3})$-URD$(v; r, s)$. Lemma 2.1 gives $s \equiv 0 \pmod{4}$, Equation (1) $r \equiv (v - 1) \pmod{3}$ and so

- if $v \equiv 0 \pmod{12}$, then $r \equiv 2 \pmod{3}$,
- if $v \equiv 4 \pmod{12}$, then $r \equiv 0 \pmod{3}$,
- if $v \equiv 8 \pmod{12}$, then $r \equiv 1 \pmod{3}$.

Letting $s = 4x$ in the Equation (1), we have $r = (v - 1) - 6x$; since $r$ and $s$ cannot be negative, and $x$ is an integer, the value of $x$ has to be in the range as given in the definition of $J(v)$. \hfill \Box

## 3 Constructions and related structures

In this section we will introduce some useful definitions, results and discuss constructions we will use in proving the main result. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [1] and its online updates. For some results below, we also cite this handbook instead of the original papers. A (resolvable) $H$-decomposition of the complete multipartite graph with $u$ parts each of size $g$ is known as a (resolvable) group divisible design $H$-(R)GDD of type $g^u$, the parts of size $g$ are called the groups of the design. When $H = K_n$ we will call it an $n$-(R)GDD.
A \((K_2, K_{1,3})\)-URGDD \((r, s)\) of type \(g^u\) is a uniformly resolvable decomposition of the complete multipartite graph with \(u\) parts each of size \(g\) into \(r\) 1-factors and \(s\) classes containing only copies of 3-stars.

If the blocks of an \(\mathcal{H}\)-GDD of type \(g^u\) can be partitioned into partial parallel classes, each of which contain all points except those of one group, we refer to the decomposition as a frame. When \(\mathcal{H} = K_n\) we will call it an \(n\)-frame and it is easy to deduce that the number of partial parallel classes missing a specified group \(G\) is \(\frac{|G|}{n-1}\).

An incomplete resolvable \((K_2, K_{1,3})\)-decomposition of \(K_{v+h}\) with a hole of size \(h\) is a \((K_2, K_{1,3})\)-decomposition of \(K_{v+h} - K_h\) in which there are two types of classes, partial classes which cover every point except those in the hole (the points of \(K_h\) are referred to as the hole) and full classes which cover every point of \(K_{v+h}\). Specifically a \((K_2, K_{1,3})\)-URD\((v + h, h; [r_1, s_1], [\bar{r}_1, \bar{s}_1])\) is a uniformly resolvable \((K_2, K_{1,3})\)-decomposition of \(K_{v+h} - K_h\) with \(r_1\) 1-factors and \(s_1\) classes of 3-stars which cover only the points not in the hole, \(\bar{r}_1\) 1-factors and \(\bar{s}_1\) classes of 3-stars which cover every point of \(K_{v+h}\).

We now recall the existence of some 4-RGDDs and 4-frames we will need in the proof.

**Lemma 3.1.** ([1], [13]) There exists a 4-RGDD of type

- \(4^t\) for each \(t \equiv 1 \pmod{3}\), \(t \geq 4\);
- \(3^t\) for each \(t \equiv 0 \pmod{4}\), \(t \geq 4\);
- \(2^{22}, 2^{106}\) and \(2^{142}\).

**Lemma 3.2.** ([13]) There exists a 4-frame of type \(6^t\) for each \(t \equiv 1 \pmod{2}\), \(t \geq 5\) with the possible exception of \(t \in \{7, 23, 27, 35, 39, 47\}\).

We also need the following definitions. Let \((s_1, t_1)\) and \((s_2, t_2)\) be two pairs of non-negative integers. Define \((s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)\). If \(X\) and \(Y\) are two sets of pairs of non-negative integers, then \(X + Y\) denotes the set \(\{(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y\}\). If \(X\) is a set of pairs of non-negative integers and \(h\) is a positive integer, then \(h \ast X\) denotes the set of all pairs of non-negative integers which can be obtained by adding any \(h\) elements of \(X\) together (repetitions of elements of \(X\) are allowed).

**Theorem 3.3.** Let \(v, g, t\) and \(u\) be non-negative integers such that \(v = gtu\). If there exists

1. a 4-RGDD of type \(g^u\);
2. a \((K_2, K_{1,3})\)-URGDD \((r_1, s_1)\) of type \(t^4\) with \((r_1, s_1) \in J_1\);
3. a \((K_2, K_{1,3})\)-URD\((gt; r_2, s_2)\), with \((r_2, s_2) \in J_2\);
then there exists a \((K_2, K_{1,3})\)-URD\((v; r, s)\) for each \((r, s) \in J_2 + h * J_1\), where 
\[ h = \frac{g(u-1)}{3} \]

\(\text{is the number of parallel classes of the 4-RGDD of type } g^u.\)

**Proof.** Let \(G\) be a 4-RGDD of type \(g^u\), with \(u\) groups \(G_i, i = 1, 2, \ldots, u\), of size \(g\); let \(R_1, R_2, \ldots, R_h\), 
\(h = \frac{g(u-1)}{3}\), be the parallel classes of this 4-RGDD. Expand each point \(t\) times and for each block \(b\) of a given resolution class of \(G\) place on \(b \times \{1, 2, \ldots, t\}\) a copy of a \((K_2, K_{1,3})\)-URGDD\((r_1, s_1)\) of type \(t^4\) with \((r_1, s_1) \in J_1\). For each \(i = 1, 2, \ldots, u\), place on \(G_i \times \{1, 2, \ldots, t\}\) a copy of a \((K_2, K_{1,3})\)-URD\((gt; r_2, s_2)\) with \((r_2, s_2) \in J_2\). The result is a \((K_2, K_{1,3})\)-URD\((v; r, s)\) with \((r, s) \in \{J_2 + h * J_1\}. \)

\[ \square \]

**Theorem 3.4.** Let \(v, g, t, h\) and \(u\) be non-negative integers such that \(v = gtu + h\). If there exists

\begin{enumerate}
  \item[(1)] a 4-frame \(F\) of type \(g^u;\)
  \item[(2)] a \((K_2, K_{1,3})\)-URD\((h; r_1, s_1)\) with \((r_1, s_1) \in J_1;\)
  \item[(3)] a \((K_2, K_{1,3})\)-URGDD\((r_2, s_2)\) of type \(t^4\) with \((r_2, s_2) \in J_2;\)
  \item[(4)] a \((K_2, K_{1,3})\)-IURD\((gt+h, h; [r_1, s_1], [r_3, s_3])\) with \((r_1, s_1) \in J_1\) and \((r_3, s_3) \in J_3 = \frac{g}{3} * J_2;\)
\end{enumerate}

then exists a \((K_2, K_{1,3})\)-URD\((v+h; r, s)\) for each \((r, s) \in J_1 + u * J_3.\)

**Proof.** Let \(F\) be a 4-frame of type \(g^u\) with groups \(G_i, i = 1, 2, \ldots, u\); expand each point \(t\) times and add a set \(H = \{a_1, a_2, \ldots, a_h\}\). For \(j = 1, 2, \ldots, \frac{g}{3}\), let \(p_{i,j}\) be the \(j\)-th partial parallel class which miss the group \(G_i\); for each \(b \in p_{i,j}\), place on \(b \times \{1, 2, \ldots, t\}\) a copy \(D^b_{i,j}\) of a \((K_2, K_{1,3})\)-URGDD\((r_2, s_2)\) of type \(t^4\), with \((r_2, s_2) \in J_2\); place on \(G_i \times \{1, 2, \ldots, t\} \cup H\) a copy \(D_i\) of a \((K_2, K_{1,3})\)-IURD\((gt+h, h; [r_1, s_1], [r_3, s_3])\) with \(H\) as hole, \((r_1, s_1) \in J_1\) and \((r_3, s_3) \in J_3 = \frac{g}{3} * J_2.\) Now combine all together the parallel classes of \(D^b_{i,j}, b \in p_{i,j}\), along with the full classes of \(D_i\) so to obtain \(r_3\) 1-factors and \(s_3\) classes of 3-stars, \((r_3, s_3) \in J_3,\) on \(\bigcup^u_{i=1} G_i \times \{1, 2, \ldots, t\} \cup H.\) Fill the hole \(H\) with a copy \(D\) of \((K_2, K_{1,3})\)-URD\((h; r_1, s_1)\) with \((r_1, s_1) \in J_1\) and combine the classes of \(D\) with the partial classes of \(D_i\) so to obtain \(r_1\) 1-factors and \(s_1\) classes of 3-stars on \(\bigcup^u_{i=1} G_i \times \{1, 2, \ldots, t\} \cup H.\) The result is a \((K_2, K_{1,3})\)-URD\((v+h; r, s)\) for each \((r, s) \in J_1 + u * J_3.\)

\[ \square \]

**4 Small cases**

**Lemma 4.1.** There exists a \((K_2, K_{1,3})\)-URGDD\((0, 4)\) of type \(2^4.\)
Proof. Take the groups to be \( \{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\} \) and the classes as listed below:
\[
\{(0; 2, 4, 6), (1; 3, 5, 7)\}, \{(2; 4, 1, 6), (3; 5, 0, 7)\}, \{(5; 2, 0, 7), (4; 1, 3, 6)\},
\{(6; 1, 3, 5), (7; 0, 4, 2)\}.
\]

**Lemma 4.2.** There exists a \((K_2, K_{1,3})\)-URD(8; 1, 4).

Proof. The assertion follows by Lemma [4.1]

**Lemma 4.3.** There exists a \((K_2, K_{1,3})\)-URD(12; 5, 4).

Proof. Let \( V(K_{12})=\mathbb{Z}_{12} \), and the classes as listed below:
\[
\{(0; 4, 5, 6), (7; 8, 9, 10), (11; 1, 2, 3)\}, \{(1; 5, 6, 7), (4; 9, 10, 11), (8; 0, 2, 3)\},
\{(2; 4, 6, 7), (5; 8, 10, 11), (9; 0, 1, 3)\}, \{(3; 4, 5, 7), (6; 8, 9, 11), (10; 0, 1, 2)\},
\{(0; 7), (1; 4), (2; 5), (3; 6), (8; 11), (9; 10)\},
\{(0, 1), (3, 10), (2, 9), (4, 8), (5, 6), (7, 11)\},
\{(0, 11), (1, 8), (2, 3), (4, 7), (6, 10), (5, 9)\},
\{(0, 2), (1, 3), (0, 4), (5, 6), (8, 10), (9, 11)\},
\{(0, 3), (1, 2), (5, 7), (4, 6), (8, 9), (10, 11)\}.
\]

**Lemma 4.4.** There exists a \((K_2, K_{1,3})\)-URGDD(4, 8) of type \(8^3\).

Proof. Let \( \{a_0, a_1, \ldots, a_7\} \), \( \{b_0, b_1, \ldots, b_7\} \) and \( \{c_0, c_1, \ldots, c_7\} \) be the groups and the classes as listed below:
\[
\{(a_0; b_1, b_2, b_3), \ (b_0; c_0, c_2, c_6), \ (c_4; a_1, a_2, a_3), \ (b_7; c_1, c_5, c_7), \ (c_3; a_4, a_5, a_6),
(a_7; b_4, b_5, b_6)\},
\{(a_1; b_0, b_2, b_3), \ (b_1; c_1, c_3, c_7), \ (c_5; a_0, a_2, a_3), \ (b_4; c_2, c_4, c_6), \ (c_0; a_7, a_5, a_6),
(a_3; b_4, b_5, b_6)\},
\{(a_2; b_1, b_0, b_3), \ (b_2; c_0, c_2, c_4), \ (c_6; a_1, a_0, a_3), \ (b_5; c_3, c_5, c_7), \ (c_1; a_4, a_7, a_6),
(a_5; b_4, b_7, b_6)\},
\{(a_3; b_1, b_2, b_0), \ (b_3; c_1, c_3, c_5), \ (c_7; a_1, a_2, a_0), \ (b_6; c_0, c_4, c_6), \ (c_2; a_4, a_5, a_7),
(a_6; b_4, b_5, b_7)\},
\{(a_0; b_5, b_4, c_1), \ (b_7; c_0, c_6, a_1), \ (c_2; b_6, a_2, a_3), \ (b_0; c_3, c_5, a_6), \ (c_7; a_4, a_7, b_3),
(a_5; b_1, b_2, c_1)\},
\{(a_1; b_5, b_6, c_5), \ (b_4; c_1, a_2, c_7), \ (c_3; a_3, a_0, b_7), \ (b_1; a_7, c_0, c_6), \ (c_4; a_4, a_5, b_0),
(a_2; b_5, b_3, c_2)\},
\{(a_2; b_0, b_7, c_6), \ (b_5; a_3, c_2, c_4), \ (c_0; a_1, a_0, b_4), \ (b_2; a_4, c_1, c_7), \ (c_5; b_1, a_5, a_6),
(a_7; b_0, b_3, c_3)\},
\{(a_3; b_1, b_7, c_7), \ (b_6; a_0, c_3, c_5), \ (c_1; a_1, a_2, b_5), \ (b_3; c_2, c_4, a_5), \ (c_6; b_2, a_6, a_7),
(a_4; c_0, b_0, b_1)\},
\{(a_0; b_0), \ (a_1, b_1), \ (a_2, b_2), \ (a_3, b_3), \ (a_4, c_3), \ (a_5, c_6), \ (a_6, c_7), \ (a_7, c_4), \ (b_4, c_3),
(b_5, c_0), \ (b_6, c_1), \ (b_7, c_2)\},
\{(a_0, c_1), \ (a_1, c_2), \ (a_2, c_3), \ (a_3, c_0), \ (a_4, b_3), \ (a_5, b_0), \ (a_6, b_1), \ (a_7, b_2), \ (b_4, c_3),
(b_5, c_6), \ (b_6, c_7), \ (b_7, c_4)\},
\{(a_0, c_2), \ (a_1, c_3), \ (a_2, c_0), \ (a_3, c_1), \ (a_4, b_1), \ (a_5, b_5), \ (a_6, b_6), \ (a_7, b_7), \ (b_0, c_7),
...
Lemma 4.5. There exists a \((K_2, K_{1,3})\)-URD(24; 5, 12).

Proof. Take a \((K_2, K_{1,3})\)-URGDD(4, 8) of type \(8^3\), which exists by Lemma 4.3. Place on each of the groups a copy of a \((K_2, K_{1,3})\)-URD(8; 1, 4) which exists by Lemma 4.2. This completes the proof. \(\square\)

Lemma 4.6. There exists \((K_2, K_{1,3})\)-URGDD(0, 8) of type \(4^4\).

Proof. Take the groups to be \(\{x_1, x_2, x_3, x_4\}\), \(\{a_1, a_2, a_3, a_4\}\), \(\{b_1, b_2, b_3, b_4\}\) and \(\{c_1, c_2, c_3, c_4\}\) and the classes are obtained by reducing subscripts modulo 4 the following base blocks:

\[
\begin{align*}
\{(a_1; b_2, c_3, x_2)\}, \{(b_1; c_3, c_3, x_3)\}, \{(c_1; b_2, a_2, x_3)\}, \{(x_1; b_2, a_3, c_2)\}, \{(a_1; b_1, c_1, x_1)\}, \{(b_1; a_2, c_1, x_1)\}, \{(c_1; b_4, a_4, x_1)\}, \{(x_1; b_4, a_2, c_4)\}.
\end{align*}
\]

\(\square\)

Lemma 4.7. There exists a \((K_2, K_{1,3})\)-IURD(16, 4; [3, 0], [0, 8]).

Proof. Start from the \((K_2, K_{1,3})\)-URGDD(0, 8) of type \(4^4\) of Lemma 4.6 and fill in the groups \(\{a_1, a_2, a_3, a_4\}\), \(\{b_1, b_2, b_3, b_4\}\) and \(\{c_1, c_2, c_3, c_4\}\) with a copy of a \((K_2, K_{1,3})\)-URD(4, 3, 0) to obtain a \((K_2, K_{1,3})\)-IURD(16, 4; [3, 0], [0, 8]) with \(\{x_1, x_2, x_3, x_4\}\) as hole. \(\square\)

Lemma 4.8. There exists a \((K_2, K_{1,3})\)-URD(16; 3, 8).

Proof. The assertion follows by Lemma 4.7. \(\square\)

Lemma 4.9. There exists a \((K_2, K_{1,3})\)-IURD(28, 4; [3, 0], [0, 16]).

Proof. Let the point set be \(V = \{x, a, b, c, d, f, g\} \times \{1, 2, 3, 4\}\) and let \(\{x_1, x_2, x_3, x_4\}\) be the hole.

- Take 16 classes of 3-stars on \(V\) listed below:
  \[
\begin{align*}
\{(x_1; a_{i+3}, b_{i+3}, d_{i+3}), (a_i; a_{i+1}, c_{i+3}, g_{i+2}), (b_i; b_{i+1}, c_{i+2}, f_{i+3}), (d_i; d_{i+1}, g_{i+3}, f_{i+2}), (c_i; c_{i+1}, a_{i+2}, x_{i+3}), (f_i; f_{i+1}, b_{i+2}, x_{i+1}), (g_i; g_{i+1}, x_{i+2}, d_{i+2}), & i \in Z_4\}, \\
\{(x_1; c_{i+2}, f_{i+1}, g_{i+1}), (a_{i+3}; c_i, x_{i+3}, b_i), (b_{i+2}; x_{i+2}, d_{i+2}, f_{i+2}), (d_{i+1}; x_{i+1}, b_{i+3}, a_{i+2}), (c_{i+1}; a_{i+1}, g_{i+2}, d_{i+3}), (f_{i+1}; c_{i+3}, a_i, g_{i+3}), (g_{i+2}; f_{i+1}, b_{i+2}, d_{i+2}), & i \in Z_4\}, \\
\{(x_i; a_{i+1}, b_{i+1}, d_{i+1}), (f_{i+3}; a_{i+2}, x_{i+1}, c_i), (c_{i+2}; x_{i+3}, d_{i+3}, f_i), (g_{i+1}; x_{i+2}, d_{i+1}, a_i), (a_{i+3}; g_{i+2}, b_{i+2}, g_{i+3}), (b_i; g_i, c_{i+1}, f_{i+1}), (d_{i+2}; b_{i+3}, f_{i+2}, c_{i+3}), & i \in Z_4\}, \\
\{(x_i; f_i, c_i, g_i), (a_{i+1}; d_{i+1}, x_{i+3}, b_{i+3}), (b_i; x_{i+2}, c_{i+3}, a_{i+1}), (d_{i+3}; x_{i+1}, f_{i+2}, a_{i+2}), (g_{i+3}; b_{i+1}, b_{i+2}, f_{i+3}), (f_{i+1}; a_{i+3}, d_i, c_{i+1}), (c_{i+2}; d_{i+2}, g_{i+1}, g_{i+2}), i \in Z_4\}.
\end{align*}
\]
• Take three 1-factors on \( \{a, b, c, d, f, g\} \times \{1, 2, 3, 4\} \):

\[
\{a_i, f_{i+3}, \} ; \{b_i, d_{i+1}, \} ; \{c_i, g_{i+2}, \} , i \in \mathbb{Z}_4, \{a_i, d_{i+2}, \} ; \{b_i, c_i, \} ; \{f_i, g_{i+2}, \} , i \in \mathbb{Z}_4, \{a_1, a_3, \} ; \{a_2, a_4, \} ; \{b_1, b_3, \} ; \{b_2, b_4, \} ; \{c_1, c_3, \} ; \{c_2, c_4, \} ; \{d_1, d_3, \} ; \{d_2, d_4, \} ; \{f_1, f_3, \} ; \{f_2, f_4, \} ; \{g_1, g_3, \} ; \{g_2, g_4, \}. 
\]

\(\Box\)

**Lemma 4.10.** There exists a \((K_2, K_{1,3})\)-URD\((28; 3, 16)\).

**Proof.** The assertion follows by Lemma 4.9. \(\Box\)

**Lemma 4.11.** There exists a \((K_2, K_{1,3})\)-IURD\((20, 8; [1, 4], [0, 8])\).

**Proof.** Let the point set be \( V = \mathbb{Z}_{20} \) and let \( H = \{0, 1, \ldots, 7\} \) be the hole.

• Take the 4 classes of 3-stars on \( V \) listed below:

\[
\{0; 8, 9, 10\}, \{1; 11, 12, 13\}, \{2; 14, 15, 16\}, \{17; 3, 4, 5\}, \{18; 6, 7, 19\}, \{0; 11, 12, 13\}, \{1; 8, 9, 10\}, \{2; 17, 18, 19\}, \{14; 3, 4, 5\}, \{15; 6, 7, 16\}, \{3; 8, 9, 10\}, \{4; 11, 12, 15\}, \{5; 16, 18, 19\}, \{13; 2, 6, 7\}, \{14; 0, 1, 17\}, \{3; 11, 12, 13\}, \{4; 8, 9, 16\}, \{6; 14, 17, 19\}, \{10; 2, 5, 7\}, \{15; 0, 1, 18\}, \{5; 8, 9, 11\}, \{7; 14, 16, 17\}, \{10; 4, 6, 19\}, \{12; 2, 13, 15\}, \{18; 0, 1, 3\}, \{6; 8, 9, 16\}, \{11; 2, 7, 10\}, \{15; 3, 5, 13\}, \{17; 0, 1, 18\}, \{19; 4, 12, 14\}, \{7; 8, 12, 19\}, \{9; 2, 10, 14\}, \{11; 6, 15, 18\}, \{13; 4, 5, 17\}, \{16; 0, 1, 3\}, \{8; 2, 10, 15\}, \{9; 7, 11, 13\}, \{12; 5, 6, 17\}, \{18; 4, 14, 16\}, \{19; 0, 1, 3\}. 
\]

• Take the 4 partial classes of 3-stars on \( V \) listed below:

\[
\{8; 9, 12, 18\}, \{11; 13, 14, 19\}, \{17; 10, 15, 16\}, \{8; 11, 13, 17\}, \{10; 12, 15, 18\}, \{16; 9, 14, 19\}, \{9; 17, 18, 19\}, \{14; 10, 12, 15\}, \{16; 8, 11, 13\}, \{(12; 9, 11, 16), \{13; 10, 14, 18\}, \{19; 8, 15, 17\}. 
\]

• Take the partial 1-factors on \( V \):

\[
\{8, 14\}, \{9, 15\}, \{10, 16\}, \{11, 17\}, \{12, 18\}, \{13, 19\}. 
\]

\(\Box\)

**Lemma 4.12.** There exists a \((K_2, K_{1,3})\)-URD\((20; 1, 12)\).

**Proof.** The assertion follows by Lemmas 4.11 and 4.12. \(\Box\)

**Lemma 4.13.** There exists a \((K_2, K_{1,3})\)-URD\((v; 3, \frac{2(v-4)}{3})\) for \( v = 88, 424, 568 \).

**Proof.** Start from a 4-RGDD \( G \) of type \( 2^4 \) which exists for \( v = 88, 424, 568 \) \(\Box\). Give weight 2 to every point of \( G \) and for each block of a given resolution class of \( G \) place a copy of a \((K_2, K_{1,3})\)-RGDD\((0, 4)\) of type \( 2^4 \) which exists by Lemma 4.11. Fill each group of size 4 with a copy of a \((K_2, K_{1,3})\)-URD\((4; 3, 0)\). Applying Theorem 3.3 with \( g = t = 2 \) and \( u = \frac{v}{4} \), we obtain a \((K_2, K_{1,3})\)-URD\((v; 3, \frac{2(v-4)}{3})\). \(\Box\)

9
5 Main results

Lemma 5.1. For every $v \equiv 0 \pmod{24}$ there exists a $(K_2, K_{1,3})$-URD $(v; 5, \frac{2(v-6)}{3})$.

Proof. Let $v = 24s$. The case $s = 1$ corresponds to a $(K_2, K_{1,3})$-URD $(24; 5, 12)$ which exists by Lemma 4.5. For $s > 1$, start from a 4-RGDD of type $3^{1s}$ (1) and apply Theorem 3.3 with $t = 2$ to obtain a $(K_2, K_{1,3})$-URD $(v; 5, \frac{2(v-6)}{3})$ (the input designs are a $(K_2, K_{1,3})$-URGDD(0, 4) of type $2^4$ by Lemma 4.4 and a $(K_2, K_{1,3})$-URD(6; 5, 0)). □

Lemma 5.2. For every $v \equiv 12 \pmod{24}$ there exists a $(K_2, K_{1,3})$-URD $(v; 5, \frac{2(v-6)}{3})$.

Proof. Let $v = 12(2s + 1)$, $s \geq 0$. The case $s = 0$ corresponds to a $(K_2, K_{1,3})$-URD $(12; 5, 4)$ which exists by Lemma 4.3. For $s \geq 1$, start from a $(K_2, K_{1,3})$-URGDD $(0, \frac{2(2s-12)}{3})$ of type $12^{1+2s}$, which exists by Lemma 5.4 of [6]. Filling each group of size 12 with a copy of a $(K_2, K_{1,3})$-URD(12; 5, 4), which exists by Lemma 4.3, gives a $(K_2, K_{1,3})$-URD $(v; 5, \frac{2(v-6)}{3})$. □

Lemma 5.3. For every $v \equiv 8 \pmod{24}$ there exists a $(K_2, K_{1,3})$-URD $(v; 1, \frac{2(v-2)}{3})$.

Proof. Let $v = 8 + 24s$. The case $s = 0$ corresponds to a $(K_2, K_{1,3})$-URD $(8; 1, 4)$ which exists by Lemma 4.2. For $s \geq 1$, start from a 4-RGDD of type $4^{1+3s}$ (1) and apply Theorem 3.3 with $t = 2$ to obtain a $(K_2, K_{1,3})$-URD $(v; 1, \frac{2(v-2)}{3})$ (the input designs are a $(K_2, K_{1,3})$-URGDD(0, 4) of type $2^4$ from Lemma 4.1 and a $(K_2, K_{1,3})$-URD(8; 1, 4) from Lemma 4.2). □

Lemma 5.4. For every $v \equiv 16 \pmod{24}$, with the possible exception of $v \in \{40, 280, 328, 472\}$, there exists a $(K_2, K_{1,3})$-URD $(v; 3, \frac{2(v-4)}{3})$.

Proof. Let $v = 16 + 24s$. The cases $v = 16, 88, 424, 568$ are covered by Lemmas 4.3 and 4.4. For $v > 40$ and $v \neq 280, 328, 472$, start from a 4-frame of type $6^{1+2s}$ (1)2 and apply Theorem 3.3 with $t = 2$ and $h = 4$ to obtain a $(K_2, K_{1,3})$-URD $(v; 3, \frac{2(v-4)}{3})$ (the input designs are a $(K_2, K_{1,3})$-URD(4; 3, 0), a $(K_2, K_{1,3})$-URGDD(0, 4) of type $2^4$ from Lemma 4.1 and a $(K_2, K_{1,3})$-URD(16, 4; [3, 0], [8]) from Lemma 4.7). □

Lemma 5.5. For every $v \equiv 4 \pmod{24}$, with the possible exception of $v \in \{52, 76, 100\}$, there exists a $(K_2, K_{1,3})$-URD $(v; 3, \frac{2(v-4)}{3})$.

Proof. Let $v = 4 + 24s$. The case $v = 28$ follows by Lemma 4.10. For $v > 100$ start from a 4-frame of type $12^s$ (13) and apply Theorem 3.3 with $t = 2$ and $h = 4$ to obtain a $(K_2, K_{1,3})$-URD $(v; 3, \frac{2(v-4)}{3})$ (the input designs are a $(K_2, K_{1,3})$-URD(4; 3, 0), a $(K_2, K_{1,3})$-URGDD(0, 4) of type $2^4$ from Lemma 4.1 and a $(K_2, K_{1,3})$-URD(28, 4; [3, 0], [16]) from Lemma 4.9). □
Lemma 5.6. For every \( v \equiv 20 \pmod{24} \), with the possible exception of \( v \in \{44, 92, 284, 332, 428, 476, 572\} \), there exists a \((K_2, K_{1,3})\)-URD \((v; 1, \frac{2(v-2)}{3})\).

Proof. Let \( v = 20 + 24s \). The case \( v = 20 \) follows by Lemma 4.12. For \( v > 44 \) and \( v \neq 92, 284, 332, 428, 476, 572 \) start from a 4-frame of type \( 6^{1+2s} \) and apply Theorem 3.4 with \( t = 2 \) and \( h = 8 \) to obtain a \((K_2, K_{1,3})\)-URD \((v; 1, \frac{2(v-2)}{3})\) (the input designs are a \((K_2, K_{1,3})\)-URD \(8; 1, 4\) from Lemma 4.2, a \((K_2, K_{1,3})\).URGDD(0, 4) of type 2 from Lemma 4.1 and a \((K_2, K_{1,3})\)-IURD(20, 8; [1,4], [0,8]) from Lemma 4.11).

Combining Lemmas 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6 we obtain the main theorem of this article.

Theorem 5.7. For each \( v \equiv 0 \pmod{4} \), there exists a \((K_2, K_{1,3})\)-URD\((v; r(v), \frac{2(v-1-r(v))}{2})\), with \( r(v) \) as in the Table 1 and with the possible exception of \( v \in \{40, 44, 52, 76, 92, 100, 280, 284, 328, 332, 428, 472, 476, 572\} \).

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