The Vacuum Polarization: Power Corrections beyond OPE?

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We compute the vacuum polarization on the lattice using non-perturbatively $O(a)$ improved Wilson fermions. The result is compared with the operator product expansion (OPE).

1. INTRODUCTION

The vacuum polarization is given by

$$\Pi_{\mu\nu}(q) = i \int d^4x \, e^{iqx}(0) T J_\mu(x) J_\nu(0) |0\rangle = (q_{\mu} q_{\nu} - q^2 g_{\mu\nu}) \Pi(q^2),$$

with the vector current $J_\mu(x) = \bar{\psi}(x) \gamma_\mu \psi(x)$. In the following we work at the Euclidean momenta $Q^2 = -q^2$. Because the polarization function $\Pi(-Q^2)$ has logarithmic divergences, it is customary to study the Adler function [1]

$$D(Q^2) = -12\pi^2 Q^2 \frac{d\Pi(-Q^2)}{dQ^2}.$$  (2)

The Adler function provides a way of comparing theoretical predictions from QCD with available time-like experimental data for the $e^+e^-$ total cross section via

$$D(Q^2) = Q^2 \int_{4m^2}^{\infty} \frac{R(s)}{(s + Q^2)^2} ds,$$  (3)

where

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$  (4)

is the ratio of the hadronic to the leptonic cross section at center of mass energy squared $s$. Lattice calculations will give valuable information for the Adler function in the region shown in Figure 1.

The standard operator product expansion (OPE) of the Adler function is of the form

$$D(Q^2) = D_{\text{pert}}(Q^2) + D_{\text{NP}}(Q^2).$$  (5)

The perturbative part is available to three loops in the form of a large-$Q^2$ expansion [2, 3] and reads

$$D_{\text{pert}}(Q^2) = 3 \sum_q Q^2_q \left\{ -6 \frac{m^2_q}{Q^4} - 12 \frac{m^4_q}{Q^6} \ln \frac{m^2_q}{Q^2} + 24 \frac{m^6_q}{Q^8} \left( \ln \frac{m^2_q}{Q^2} + 1 \right) \right. \right.$$ \right.

$$\left. + \frac{\alpha_s(m^2)}{\pi} \left[ 1 - 12 \frac{m^2_q}{Q^2} \left( 1 + \ln \frac{m^2_q}{Q^2} \right) \right. \right.$$ \right.

$$\left. + 16 \frac{m^4_q}{Q^4} \left( \frac{17}{24} + 2\zeta(3) - \frac{3}{4} \ln \frac{m^2_q}{Q^2} - \frac{3}{2} \ln^2 \frac{m^2_q}{Q^2} \right) \right.$$ \right.

$$\left. - 64 \frac{m^6_q}{Q^6} \left( \frac{139}{108} - \frac{19}{9} \ln \frac{m^2_q}{Q^2} - \frac{29}{24} \ln^2 \frac{m^2_q}{Q^2} \right) \right\}. \right.$$  (6)

To leading order in $1/Q^2$ the non-perturbative part is given by

$$D_{\text{NP}}(Q^2) = 3 \sum_q Q^2_q 8\pi^2$$  (7)

$$\left\{ w_1 \frac{\langle \pi^2 GG \rangle}{Q^4} + w_2 \frac{\langle m_g \bar{q}q \rangle}{Q^4} \right.$$ \right.

$$\left. + w_3 \sum_{q'} \frac{\langle m_g \bar{q}'q' \rangle}{Q^4} + \text{higher condensates} \right\}. \right.$$
The last sum in eq. (8) is over $q'$ dynamical quark flavors which will not concern us here since we are working in the quenched approximation. The Wilson coefficients are given by

$$w_1 = \frac{1}{12} \left( 1 - \frac{11}{18} \frac{\alpha_s(\mu^2)}{\pi} \right) + O(\alpha_s^3)$$

$$w_2 = 2 + \frac{2 \alpha_s(\mu^2)}{3 \pi} \left( \frac{47}{4} - \frac{3}{2} \ln \frac{Q^2}{\mu^2} \right) \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^2 + O(\alpha_s^3)$$

$$w_3 = \frac{4}{27} \frac{\alpha_s(\mu^2)}{\pi}$$

$$+ \left( \frac{4}{3} \xi_3 - \frac{88}{243} \right) \frac{1}{3} \ln \frac{Q^2}{\mu^2} \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^2 + O(\alpha_s^3).$$

The polarization tensor splits into two parts: $
\Pi = \Pi^{(a)} + \Pi^{(b)}.$

The first part is given by

$$\Pi^{(a)}_\mu(q) = \sum_x e^{iqx+iq\mu a/2-iq\bar{a}/2} \times \langle \bar{J}^{(1)}_\mu(x)|J^{(1)}_\mu(0)|0 \rangle,$$  

while the second part corresponds to the tadpole diagram

$$\Pi^{(b)}_\mu(q) = \langle 0|J^{(2)}_\mu(0)|0 \rangle \delta_{\mu
u}.$$  

For the current in eq. (8) we take the conserved vector current (CVC)

$$J^{(1)}_\mu(x) = \frac{1}{2} (\bar{\psi}(x + a \hat{\mu}) (1 + \gamma_\mu) U^\dagger_\mu(x) \psi(x) - \bar{\psi}(x) (1 - \gamma_\mu) U_\mu(x) \psi(x + a \hat{\mu})),$$  

and the current in eq. (13) is given by

$$J^{(2)}_\mu(x) = \frac{1}{2} (\bar{\psi}(x + a \hat{\mu}) (1 + \gamma_\mu) U^\dagger_\mu(x) \psi(x) + \bar{\psi}(x) (1 - \gamma_\mu) U_\mu(x) \psi(x + a \hat{\mu})).$$  

We shall take the following ansatz for the polarization tensor:

$$\Pi_\mu = (\hat{q}_\mu \hat{q}_\nu - \delta_\mu\nu) \Pi(-\hat{q}^2).$$

The calculations are done for non-perturbatively $O(a)$ improved Wilson fermions. Besides improving the fermionic action we have
to improve the current as well. The on-shell improved conserved vector current is
\[
J^{(1)}_{\mu}^{\text{imp}}(x) = J^{(1)}_{\mu}(x) + \frac{c_{\text{cvc}}}{2} \alpha s \partial_\lambda \{ \bar{\psi}(x) \sigma_{\mu \lambda} \psi(x) \}. \tag{19}
\]

The value of the improvement coefficient \(c_{\text{cvc}}\) is not known beyond tree level. We take the tree level value \(c_{\text{cvc}} = 1\). The derivative in eq. (19) must be defined such that the Ward identity, eq. (12), is fulfilled. We take
\[
\partial_\lambda f(x) = \frac{1}{4a} \left( f(x + a \lambda) - f(x - a \lambda) + f(x + a \bar{\mu} + a \lambda) - f(x + a \bar{\mu} - a \lambda) \right). \tag{20}
\]

In the momentum space this gives a factor \(-i q_\lambda \cos(a q_\lambda / 2) \cos(a q_\bar{\mu} / 2)\) which results in large \(O(a^2)\) corrections, even in the free case. In the following we omit the \(O(a^2)\) terms, thus leaving us with the factor \(-i q_\lambda\). This keeps the Ward identity fulfilled.

Figure 2. The lattice data for \(\beta = 6.0, \kappa = 0.1345\) on a \(32^4\) lattice. The line is the perturbative contribution. The scale is set using the \(r_0\)-parameter [14].

Figure 3. The coefficient \(P_1\) of the power correction \(1/Q^2\). The fit interval is chosen from approx. 1 GeV to 5 GeV for all lattices.

3. PRELIMINARY RESULTS

We have performed simulations at the \(\beta\) and \(\kappa\) values shown in Table 1. Because the Adler function involves a derivative with respect to \(Q^2\) we prefer to work with the polarization function. The polarization function is thus written as
\[
-12\pi^2 \Pi(-Q^2) = c_0 + c_1 \alpha_s(\mu^2) + c_2 \alpha_s^2(\mu^2) + \left\{ \frac{P_1}{Q^2} + \frac{P_2}{Q^4} + P_3 a^2 Q^2 + P_4 \right\}. \tag{21}
\]

We have included an additive constant \(P_4\) to account for the logarithmically divergent contribution [4,10]. This depends on the renormalization scheme. For the perturbative contribution we take the three-loop result [10] renormalized in the \(\overline{\text{MS}}\) scheme. The first coefficient in eq. (21) reads for a single quark with charge \(Q_q = 1\)
\[
c_0 = \frac{9}{4} \left[ \frac{20}{9} - \frac{4}{3} \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{8 m^2}{Q^2} \right].
\]
Table 1
The values of $\beta$ and $\kappa$ used in the simulations.

| $\beta$ | $\kappa$ | $V$ |
|---------|---------|-----|
| 6.0     | 0.1336  | 16  |
| 6.0     | 0.1339  | 16  |
| 6.0     | 0.1342  | 16  |
| 6.0     | 0.1345  | 16  |
| 6.0     | 0.1345  | 32  |
| 6.2     | 0.1344  | 24  |
| 6.2     | 0.1349  | 24  |
| 6.2     | 0.1352  | 24  |
| 6.4     | 0.1343  | 32  |
| 6.4     | 0.1346  | 32  |
| 6.4     | 0.1350  | 32  |
| 6.4     | 0.1352  | 32  |

The values of $\beta$ and $\kappa$ used in the simulations.

$$+ \left( \frac{4m^2}{Q^2} \right)^2 \left( \frac{1}{4} + \frac{1}{2} \ln \left( \frac{Q^2}{m^2} \right) \right) \ldots$$ (22)

and is related to the leading part in eq. (21) via a derivative with respect to $\ln(Q^2)$. For $\alpha_s(\mu^2)$ we take the four-loop result [11,12] with $\Lambda_{\overline{MS}} = 238(19)$MeV [13]. The quark mass renormalizations are taken from [14]. Thus the perturbative part is completely known. We also allow for a $1/Q^2$ contribution. Residual $O(a^2Q^2)$ corrections are accounted for by $P_3$. In Figure 3 we show the lattice data for $\beta = 6.0$ and $\kappa = 0.1345$. This is compared with the perturbative part of eq. (21) including a constant term $P_4$. There we have also set $\mu = 1/a$. However the result was found not to depend significantly on the exact choice of $\mu$. For values of $Q^2 > 2$GeV$^2$ we find very good agreement between the lattice data and three-loop perturbation theory. Alternatively fitting eq. (21), our data in $1\text{GeV} \lesssim Q \lesssim 5\text{GeV}$ gives the result shown in Figure 3. It turns out that $P_1$ is consistent with zero. (Note that $r_2^2 \approx 6/\text{GeV}^2$.) At present we are not able to quote a reliable number for $P_2$.

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