Spectral analysis for linear semi-infinite mass-spring systems

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Abstract

We study how the spectrum of a Jacobi operator changes when this operator is modified by a certain finite rank perturbation. The operator corresponds to an infinite mass-spring system and the perturbation is obtained by modifying one interior mass and one spring of this system. In particular, there are detailed results of what happens in the spectral gaps and which eigenvalues do not move under the modifications considered. These results were obtained by a new technique of comparative spectral analysis and they generalize and include previous results for finite and infinite Jacobi matrices.

Mathematics Subject Classification(2010): 47A75, 47B36,
Keywords: Jacobi matrices; Spectrum; Infinite mass-spring system
*Research partially supported by UNAM-DGAPA-PAPIIT IN105414
1. Introduction

Denote by $l_{\text{fin}}(\mathbb{N})$ the linear space of complex sequences having a finite number of non-zero elements. In the Hilbert space $l^2(\mathbb{N})$, let $J_0$ be the operator with $\text{dom}(J_0) = l_{\text{fin}}(\mathbb{N})$ such that, for every $f = \{f_k\}_{k=1}^\infty$ in $l_{\text{fin}}(\mathbb{N})$,

\[
(J_0 f)_1 := q_1 f_1 + b_1 f_2, \quad (J_0 f)_k := b_{k-1} f_{k-1} + q_k f_k + b_k f_{k+1}, \quad k \in \mathbb{N} \setminus \{1\},
\]

where $q_n \in \mathbb{R}$ and $b_n > 0$ for any $n \in \mathbb{N}$. The operator $J_0$ is symmetric and therefore one can consider the operator $\overline{J_0}$ being its closure. For the symmetric operator $\overline{J_0}$, one of the following two possibilities for the deficiency indices holds [1, Chap. 4, Sec. 1.2]:

\[
\begin{align*}
\text{n}_+ (\overline{J_0}) &= \text{n}_- (\overline{J_0}) = 1, \quad (1.2a) \\
\text{n}_+ (\overline{J_0}) &= \text{n}_- (\overline{J_0}) = 0. \quad (1.2b)
\end{align*}
\]

Fix a self-adjoint extension of $J_0$ and denote it by $J$. Thus, in view of the possible values of the deficiency indices, the von Neumann extension theory tells us that either $J$ is a proper closed symmetric extension of $\overline{J_0}$ or $J = \overline{J_0}$. According to the definition of the matrix representation for an unbounded symmetric operator [2, Sec. 47], $\overline{J_0}$ is the operator whose matrix representation with respect to the canonical basis $\{\delta_n\}_{n=1}^\infty$ in $l^2(\mathbb{N})$ is

\[
\begin{pmatrix}
q_1 & b_1 & 0 & 0 & \cdots \\
b_1 & q_2 & b_2 & 0 & \cdots \\
0 & b_2 & q_3 & b_3 & \cdots \\
0 & 0 & b_3 & q_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

The $k$-th entry of $\delta_n$ is 1 if $k = n$ and 0 if $k \neq n$. Fix $n \in \mathbb{N}$ and consider, along with the self-adjoint operator $J$, the operator

\[
\tilde{J}_n = J + [g_n(\theta^2 - 1) + \theta^2 h] \langle \delta_n, \cdot \rangle \delta_n + b_n(\theta - 1) \langle \delta_n, \cdot \rangle \delta_{n+1} + \langle \delta_{n+1}, \cdot \rangle \delta_n + b_{n-1}(\theta - 1) \langle \delta_{n-1}, \cdot \rangle \delta_n + \langle \delta_n, \cdot \rangle \delta_{n-1}), \quad \theta > 0, \quad h \in \mathbb{R},
\]

where it has been assumed that $b_0 = 0$. Clearly, $\tilde{J}_n$ is a self-adjoint extension of the operator whose matrix representation with respect to the canonical basis in $l^2(\mathbb{N})$ is a Jacobi matrix obtained from (1.3) by modifying the entries $b_{n-1}, g_n, b_n$. For instance, if $n > 2$, $\tilde{J}_n$ is a selfadjoint extension (possibly not proper) of the
operator whose matrix representation is

\[
\begin{pmatrix}
q_1 & b_1 & 0 & 0 & 0 & 0 & \ldots \\
b_1 & \ddots & \ddots & 0 & 0 & 0 & \ldots \\
0 & \ddots & q_{n-1} & \theta b_{n-1} & 0 & 0 & \ldots \\
0 & 0 & \theta b_{n-1} & \theta^2(q_n + h) & \theta b_n & 0 & \ldots \\
0 & 0 & 0 & \theta b_n & q_{n+1} & \theta b_{n+1} & \ddots \\
0 & 0 & 0 & 0 & b_{n+1} & q_{n+2} & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

(1.5)

Note that \( \tilde{J}_n \) is obtained from \( J \) by a rank-three perturbation when \( n > 1 \), and by a rank-two perturbation when \( n = 1 \).

The particular kind of perturbation given in (1.4) arises in the analysis of semi-infinite mass-spring systems. It is known \([5, 6]\) that, within the regime of validity of the Hooke law, the system in Fig. 1, with masses \( \{m_j\}_{j=1}^{\infty} \) and spring constants \( \{k_j\}_{j=1}^{\infty} \), is modeled by the Jacobi operator \( J \) such that

\[
q_j = -\frac{k_{j+1} + k_j}{m_j}, \quad b_j = \frac{k_{j+1}}{\sqrt{m_j m_{j+1}}}, \quad j \in \mathbb{N}
\]

(see \([8, 9]\) for an explanation of the deduction of these formulae in the finite case). Alternatively, the system in Fig. 1 can be interpreted as a one dimensional harmonic crystal \([13, \text{Sec. 1.5}]\). The modified mass-spring system corresponding to the perturbed operator \( \tilde{J}_n \) is obtained by adding \( \Delta m = m_n(\theta^2 - 1) \) to the \( n \)-th mass and \( \Delta k = -hm_n \) to the \( n \)-th spring constant (see Fig. 2).

Figure 1: Semi-infinite mass-spring system

constants \( \{k_j\}_{j=1}^{\infty} \), is modeled by the Jacobi operator \( J \) such that

\[
q_j = -\frac{k_{j+1} + k_j}{m_j}, \quad b_j = \frac{k_{j+1}}{\sqrt{m_j m_{j+1}}}, \quad j \in \mathbb{N}
\]

(see \([8, 9]\) for an explanation of the deduction of these formulae in the finite case). Alternatively, the system in Fig. 1 can be interpreted as a one dimensional harmonic crystal \([13, \text{Sec. 1.5}]\). The modified mass-spring system corresponding to the perturbed operator \( \tilde{J}_n \) is obtained by adding \( \Delta m = m_n(\theta^2 - 1) \) to the \( n \)-th mass and \( \Delta k = -hm_n \) to the \( n \)-th spring constant (see Fig. 2).

Figure 2: Perturbed semi-infinite mass-spring system (\( n \geq 2 \))
This work carries out a comparative spectral analysis of the operators $J$ and $\tilde{J}_n$. This analysis has various distinctive features related to the kind of perturbation under consideration (1.4). As mentioned above, the perturbation has a physical motivation and could be of interest in some applications. An interesting aspect of the perturbation considered here is that the comparative spectral analysis of $J$ and $\tilde{J}_n$ is susceptible of being treated by a method that involves the use of quotients of Green functions (see (3.1)) for deriving a master equation (see (3.3)). This method yields results that cannot be obtained by classical perturbation methods and, remarkably, there is no need of any general assumption on the spectrum of $J$. In particular, there is no need to assume that $J$ has discrete spectrum.

It is worth remarking that the perturbation given in (1.4) has not been studied for the case of semi-infinite Jacobi matrices. The modification of the spectrum of a Jacobi operator as a result of a rank-one perturbation is well understood and has been amply studied (see [11] and references therein), however there is scarce literature treating other kinds of finite rank perturbations.

The main results of this note (Theorems 3.1, 3.2, and 3.3) contain as a particular case all previously known results on the matter ([4, Thm. 2] and [5, Props. 3.1, 3.2]). We point out that the techniques and ideas developed in this work allow to tackle the corresponding generalizations of the inverse spectral analysis carried out in [4] and [5]. This is the subject of a forthcoming paper.

2. Green functions for Jacobi operators

Let us consider the following system of difference equations

\begin{align*}
q_1f_1 + b_1f_2 &= zf_1 \quad (2.1a) \\
b_{k-1}f_{k-1} + q_kf_k + b_kf_{k+1} &= zf_k \quad k \in \mathbb{N} \setminus \{1\}, \quad (2.1b)
\end{align*}

Clearly, by setting $f_1 = 1$, the solution of (2.1) can be found uniquely by recurrence. This solution is an infinite sequence that will be denoted by $\pi(z)$. Note that $\pi_k(z)$ is a polynomial of degree $k - 1$. Alongside this sequence, we define the sequence $\theta(z)$ as the solution of (2.1b) after setting $f_1 = 0$ and $f_2 = b_1^{-1}$. Thus, $\theta_k(z)$ is a polynomial of degree $k - 2$. The elements of the sequence $\pi(z)$, respectively $\theta(z)$, are referred to as the polynomials of the first, respectively second, kind associated with the matrix (1.3). By comparing (1.1) with (2.1), one concludes that $\pi(z) \in \ker(J_0^* - zI)$ if and only if $\pi(z)$ is an element of $l_2(\mathbb{N})$. Of course, in particular, $\pi(z) \in \ker(J - zI)$, if and only if $\pi(z) \in \text{dom}(J)$.

It is easy to verify, directly from the definition of the operator $J$ (see (1.1)),
that
\[ \delta_k = \pi_k(J)\delta_1 \quad \forall k \in \mathbb{N}. \] (2.2)

This implies that \( J \) is simple and \( \delta_1 \) is a cyclic vector (see [2, Sec. 69]). Therefore, if one defines the spectral function as
\[ \rho(t) := \langle \delta_1, E(t)\delta_1 \rangle, \quad t \in \mathbb{R}, \] (2.3)
where \( E \) is the resolution of the identity given by the spectral theorem, then, by [2 Sec. 69, Thm. 2]), one has a unitary map \( \Phi : L_2(\mathbb{R}, \rho) \to l_2(\mathbb{N}) \) such that \( \Phi^{-1}J\Phi \) is the multiplication by the independent variable defined in its maximal domain. This is the canonical representation of \( J \). We note that, on the basis of [2 Sec. 69, Thm. 2]), it follows from (2.2) that \( \pi_k \in L_2(\mathbb{R}, \rho) \) for all \( k \in \mathbb{N} \), that is, all moments of \( \rho \) exists (see also [1, Thm. 4.1.3]), and
\[ \Phi\pi_k = \delta_k \quad \forall k \in \mathbb{N}. \]

In what follows, \( \sigma(J) \), \( \sigma_p(J) \), and \( \sigma_{\text{ess}}(J) \) denote the spectrum, the point spectrum (eigenvalues), and the essential spectrum (in this case, accumulation points of \( \sigma(J) \)) of \( J \), respectively.

Now, consider the Weyl \( m \)-function, given by
\[ m(z) := \langle \delta_1, (J - zI)^{-1}\delta_1 \rangle, \quad z \notin \sigma(J). \] (2.4)

By using the canonical representation, it immediately follows from the definition that
\[ m(z) = \int_{\mathbb{R}} \frac{d\rho(t)}{t - z}. \]
Thus, by the Nevanlinna representation theorem (see [10 Thm. 5.3]), \( m(z) \) is a Herglotz function.

Due to the inverse Stieltjes transform, one uniquely recovers \( \rho \) from \( m \), so \( \rho \) and \( m \) are in one-to-one correspondence.

For every \( z \in \mathbb{C} \setminus \sigma(J) \), let us consider the element \( \psi(z) \) in \( l_2(\mathbb{N}) \) defined by
\[ \psi(z) := (J - zI)^{-1}\delta_1. \] (2.5)

It is known [3 Chap. 7 Eq. 1.39] that for every \( z \in \mathbb{C} \setminus \sigma(J) \) there exists a unique complex number \( m(z) \) such that
\[ \psi(z) = m(z)\pi(z) + \theta(z). \] (2.6)

The overlap with (2.4) in the notation is not an accident, the number \( m(z) \) is actually the value of the Weyl \( m \)-function at \( z \).
Definition 1. For a subspace $G \subset l^2(\mathbb{N})$, let $P_G$ be the orthogonal projection onto $G$. Also, define $G^\perp := \{ \phi \in l^2(\mathbb{N}) : \langle \phi, \psi \rangle = 0 \ \forall \psi \in G \}$ and the subspace $F_n := \text{span}\{\delta_k\}_{k=1}^n$. For the operator $J$ given in the Introduction, consider the operators $J_n^+ := P_{F_n^\perp} J |_{F_n^\perp}$, $J_n^- := P_{F_n} - I J |_{F_n}$. Here, we have used the notation $J |_G$ for the restriction of $J$ to the set $G$, that is, $\text{dom}(J |_G) = \text{dom}(J) \cap G$. Consider also the corresponding $m$-Weyl functions

$$m_n^+(z) := \langle \delta_{n+1}, (J_n^+ - zI)^{-1}\delta_{n+1} \rangle, \quad m_n^-(z) := \langle \delta_{n-1}, (J_n^- - zI)^{-1}\delta_{n-1} \rangle.$$

Note that $J_n^+$ is a selfadjoint extension of the operator whose matrix representation with respect to the basis $\{\delta_k\}_{k=n+1}^\infty$ of the space $F_n^\perp$ is the matrix (1.3) with the first $n$ rows and $n$ columns removed. Moreover, when $J_0$ is not essentially selfadjoint, $J_n^+$ has the same boundary conditions at infinity as the operator $J$. Note that $J_n^-$ is an operator in an $n - 1$-dimensional space whose matrix representation consists of the first $n - 1$ columns and $n - 1$ rows of the matrix (1.3).

The following result can be found in [7, Eqs. 2.10, 2.16]. An alternative proof is provided below.

Proposition 2.1. For any $n \in \mathbb{N} \setminus \{1\}$, one has

$$m_n^+(z) = -\frac{\psi_{n+1}(z)}{b_n \psi_n(z)}, \quad m_n^-(z) = -\frac{\pi_{n-1}(z)}{b_{n-1} \pi_n(z)}.$$

Proof. Define

$$\psi^{(n)}(z) := (J_n^+ - zI)^{-1}\delta_{n+1}.$$

Then one verifies that

$$\psi_k^{(n)} = -\frac{\psi_{k+1}^{(n)}}{b_n \psi_1^{(n)}},$$

and, after further computations, that

$$\psi_k^{(n)} = -\frac{\psi_{k+1}^{(n-1)}}{b_n \psi_1^{(n-1)}}.$$

Therefore,

$$\psi_k^{(n)} = -\frac{\psi_{k+1}^{(n-1)}}{b_n \psi_1^{(n-1)}} = -\frac{\psi_{k+2}^{(n-2)}}{b_n \psi_2^{(n-2)}} = \cdots = \frac{\psi_{k+n}}{b_n \psi_n}.$$

From this, since $m_n^+(z) = \psi_1^{(n)}$, the first identity of the assertion follows. The second formula is the statement of [4, Lem. 2] and a proof for this is provided.
there.

It immediately follows from the previous proposition that the following holds

**Corollary 2.1.** Fix $n \in \mathbb{N} \setminus \{1\}$. The real number $x$ is a zero of the polynomial $\pi_n(\cdot)$ if and only if $x$ is an eigenvalue of $J_n$.

There are various formulae for the matrix entries of the matrix representation of $(J - z)^{-1}$ with respect to the canonical basis. The one provided below is suitable for us (cf. [7, Eq. 2.8]).

**Proposition 2.2.** For any $z \in \mathbb{C} \setminus \sigma(J)$ and $j, k \in \mathbb{N}$, the following holds.

$$\langle \delta_j, (J - zI)^{-1} \delta_k \rangle = \begin{cases} \pi_j(z)\psi_k(z) & \text{if } j \leq k, \\ \psi_j(z)\pi_k(z) & \text{otherwise.} \end{cases} \quad (2.7)$$

**Proof.** Fix any $n \in \mathbb{N}$ and consider the sequence

$$\eta(n, z) := \{\pi_j(z)\psi_j(z)\}_{j=1}^n \cup \{\psi_j(z)\pi_j(z)\}_{j=n+1}^\infty,$$

which clearly is in $l_2(\mathbb{N})$. By the definition of $\pi(z)$ and $\psi(z)$, one verifies that

$$(J - zI)\eta(n, z) = \delta_n$$

This completes the proof. \qed

Let us use the following notation

$$G(z, n) := \langle \delta_n, (J - zI)^{-1} \delta_n \rangle, \quad z \in \mathbb{C} \setminus \sigma(J) \quad (2.8)$$

Note that $m(z) = G(z, 1)$ and, in view of (2.2) and (2.8), one has

$$G(z, n) = \int_\mathbb{R} \frac{d\rho_n(t)}{t - z}, \quad (2.9)$$

where

$$d\rho_n(t) := \pi_n^2(t)d\rho(t). \quad (2.10)$$

The next assertion is found in [7, Thm. 2.8]. We provide a simple proof in which the objects defined above are used.

**Proposition 2.3.** For any $n \in \mathbb{N} \setminus \{1\}$

$$G(z, n) = -\frac{1}{b_n^2 m_n^+(z) + b_{n-1}^2 m_n^-(z) + z - q_n} \quad (2.11)$$
Proof. Consider the (modified) Wronskian of the difference equation \((2.1b)\) for the sequences \(\pi(z)\) and \(\psi(z)\):

\[ W_n[\pi(z), \psi(z)] := b_n \left( \pi_n(z)\psi_{n+1}(z) - \pi_{n+1}(z)\psi_n(z) \right). \]

Since \(\pi(z)\) and \(\psi(z)\) are solutions of \((2.1b)\), one has that, for all \(n \in \mathbb{N}\) and \(z \in \mathbb{C}\),

\[ W_n[\pi(z), \psi(z)] = 1. \]

Using this and Proposition \(2.2\) one writes

\[ G(z,n) = \pi_n \psi_n W_n[\pi, \psi] = \frac{1}{b_n \frac{\psi_{n+1}}{\psi_n} - b_n \frac{\pi_{n+1}}{\pi_n}}. \]

Thus, Proposition \(2.1\) implies

\[ G(z,n) = \frac{-1}{b_n^2 m_n^+(z) - \frac{1}{m_{n+1}(z)}}. \tag{2.12} \]

Finally, by means of the formula

\[ b_{n-1}^2 m_n^-(z) + \frac{1}{m_{n+1}(z)} - q_n + z = 0, \]

which follows from Proposition \(2.1\) and the definition of \(\pi(z)\), one can rewrite \((2.12)\) as \((2.11)\).

Lemma 2.1. Fix \(n \in \mathbb{N}\) and let \(x \not\in \sigma_{ess}(J)\). Then

\[ \lim_{z \to x} (x - z)G(z,n) = \pi_n^2(x)\rho(\{x\}) \tag{2.13} \]

Proof. The proof follows from the integral representation of the function \(G(z,n)\). Indeed,

\[ G(z,n) = \int_{\mathbb{R}} \frac{d\rho_n(t)}{t - z} = \frac{\pi_n^2(x)\rho(\{x\})}{x - z} + C(z), \tag{2.14} \]

where \(C(z)\) is uniformly bounded inside a closed disk intersecting \(\sigma(J)\) only at \(x\). Thus,

\[ (x - z)G(z,n) = \pi_n^2(x)\rho(\{x\}) + (x - z)C(z). \]

Remark 1. If \(x \in \sigma_p(J)\), Lemma \(2.1\) is a special case of \([13, \text{Eq. 2.36}]\). This result amounts to the well-known fact that the residue of the resolvent at an
isolated eigenvalue is equal to the kernel of the projection onto the eigenspace. Indeed, when \( x \) is an eigenvalue, taking into account that \( P(x) := \frac{\langle \pi(x), \cdot \rangle}{\|\pi(x)\|^2} \pi(x) \) is a projection onto the corresponding eigenspace, one verifies that the r.h.s. of (2.13) is the \( n \)-th diagonal element of the matrix representation of \( P(x) \) with respect to the canonical basis.

**Lemma 2.2.** Fix \( n \in \mathbb{N} \) and let \( x \) be an isolated eigenvalue of \( J \). Then, \( x \) is a zero of the polynomial \( \pi_n(\cdot) \) if and only if \( x \) is a zero of \( G(\cdot, n) \). Also, \( x \) is not a zero of \( \pi_n(\cdot) \), if and only if \( x \) is a pole of \( G(\cdot, n) \).

**Proof.** First we prove that \( \pi_n(x) = 0 \) implies \( G(x, n) = 0 \). According to (2.7) and (2.6), one has

\[
G(z, n) = \pi_n(z) \psi_n(z) = m(z) \pi_n^2(z) + \theta_n(z) \pi_n(z).
\]

Since, as in the proof of Lemma 2.1

\[
m(z) = \int_{\mathbb{R}} \frac{d\rho(t)}{t-z} = \frac{\rho(\{x\})}{x-z} + C(z),
\]

where \( C(z) \) is uniformly bounded inside a closed disk intersecting \( \sigma(J) \) only at \( x \), it holds that

\[
G(z, n) = \left( \frac{\rho(\{x\})}{x-z} + C(z) \right) \pi_n^2(z) + \theta_n(z) \pi_n(z)
\]

\[
= \left( \frac{\rho(\{x\})}{x-z} + C(z) \right) (x-z)^2 h^2(z) + \theta_n(z) \pi_n(z).
\]

Here we have written \( \pi_n(z) = (x-z)h(z) \). The assertion follows by noticing that

\[
\lim_{z \to x} \left[ (x-z)\rho(\{x\}) h^2(z) + C(z)(x-z)^2 h^2(z) + \theta_n(z) \pi_n(z) \right] = 0.
\]

Now let us show that \( \pi_n(x) \neq 0 \) implies that \( \lim_{z \to x} G(z, n) = \infty \). Since \( \rho(\{x\}) \neq 0 \), it follows from Lemma 2.1 that

\[
\pi_n(x) \neq 0 \Rightarrow \lim_{z \to x} (x-z)G(z, n) \neq 0.
\]

The remaining converse implications follow from the ones just proven.
The next assertion is reminiscent of Corollary 2.1.

**Lemma 2.3.** Let \( n \in \mathbb{N} \setminus \{1\} \). If \( x \) is an eigenvalue of \( J \) and a zero of the polynomial \( \pi_n(\cdot) \), then \( x \) is an eigenvalue of \( J_n^+ \). On the other hand, if \( x \) is an isolated eigenvalue of \( J \) and \( J_n^+ \), then \( x \) is a zero of \( \pi_n(\cdot) \).

**Proof.** The first assertion is proven by noting that the sequence \( \{\pi_k(x)\}_{k=n+1}^{\infty} \) is an eigenvector of \( J_n^+ \). The second assertion is proven by *reductio ad absurdum*. Indeed, assume that \( x \) is an isolated eigenvalue of \( J \) and \( J_n^+ \), and is not a zero of \( \pi_n(\cdot) \). Then, by Lemma 2.2, \( x \) is a pole of \( G(\cdot, n) \) and therefore, since \( x \) is a pole of \( m^+_n(\cdot) \), Proposition 2.3 implies that \( x \) is also a pole of \( m^+_n(\cdot) \), that is, an eigenvalue of \( J^-_n \). Corollary 2.1 shows that this contradicts our assumptions. \( \square \)

**Remark 2.** By means of Corollary 2.1 one rephrases the previous lemma as follows. For any \( n \in \mathbb{N} \setminus \{1\} \),

\[
\sigma_p(J) \cap \sigma(J^-_n) \subset \sigma_p(J^+_n) \\
\sigma_{disc}(J) \cap \sigma_{disc}(J^+_n) \subset \sigma(J^-_n),
\]

where the notation \( \sigma_{disc}(\cdot) := \sigma(\cdot) \setminus \sigma_{ess}(\cdot) \) has been used.

### 3. Comparative spectral analysis of \( J \) and \( \tilde{J}_n \)

Let \( \tilde{\rho}(t) \) be the spectral function (see (2.3)) corresponding to the operator \( \tilde{J}_n \) defined in (1.4). Also, let \( \tilde{G}(z, n) \) be the function given by (2.8) with \( \tilde{J}_n \) instead of \( J \). We emphasize the fact that the value of the subscript of \( \tilde{J}_n \) coincides with the value of the second argument of \( \tilde{G}(z, n) \), that is, for any \( n \in \mathbb{N} \),

\[
\tilde{G}(z, n) = \langle \delta_n, (\tilde{J}_n - zI)^{-1}\delta_n \rangle.
\]

Define, for \( n \in \mathbb{N} \), the function

\[
\mathcal{M}_n(z) := \frac{G(z, n)}{\tilde{G}(z, n)} \tag{3.1}
\]

and the constant

\[
\gamma := \frac{\theta^2 h}{1 - \theta^2}. \tag{3.2}
\]

**Lemma 3.1.** For any \( n \in \mathbb{N} \),

\[
\mathcal{M}_n(z) = \theta^2 + (1 - \theta^2)(\gamma - z)G(z, n) \tag{3.3}
\]
and
\[
\frac{1}{\mathcal{M}_n(z)} = \frac{1}{\theta^2} + \left(1 - \frac{1}{\theta^2}\right)(\gamma - z)\tilde{G}(z, n)
\] (3.4)

Proof. It follows from (2.11) and (3.1) that
\[
\mathcal{M}_n(z) = \frac{\theta^2(b_n m_n^+(z) + b_{n-1} m_n^-(z) + z\theta^{-2} - q_n - h)}{b_n m_n^+(z) + b_{n-1} m_n^-(z) + z - q_n}
\]
from which one verifies (3.3) for \(n \in \mathbb{N} \setminus \{1\}\). For \(n = 1\), (3.3) follows from the Riccati equation (see [7, Eq. 2.15]) and [12, Eq. 2.23])
\[
b_n m_1^+(z) = q_n - z - \frac{1}{m(z)}
\]
after noticing that, in this case,
\[
P_{F_1} \tilde{J}_1 |_{F_1} = P_{F_1} J |_{F_1} \quad (F_1 = \text{span } \{\delta_1\}).
\]
The proof of (3.4) is completely analogous.

Equation (3.3) for the case \(n = 1\) is [5, Eq. 18]. As in [5], this equation, now for \(n \in \mathbb{N}\), is an important ingredient of the method used for the comparative spectral analysis of \(J\) and \(\tilde{J}_n\). The first immediate consequence of (3.3) and (3.4) is the following assertion

**Corollary 3.1.** For any \(n \in \mathbb{N}\), when \(z \neq \gamma\),
\[
G(z, n) = 0 \iff \tilde{G}(z, n) = 0
\]

**Proposition 3.1.** For any \(n \in \mathbb{N} \setminus \{1\}\),
\[
\sigma_p(J_n^+) \cap \sigma(J_n^-) \subset \sigma_p(J) \cap \sigma_p(\tilde{J}_n)
\]

Proof. Denote by \(\pi^+(z)\) the sequence of polynomials of the first kind associated with the Jacobi operator \(J_n^+\). Let \(\lambda \in \sigma(J_n^-)\), then \(\pi_n(\lambda) = 0\) by Corollary 2.1. This implies that the sequences \(\{\pi_j(\lambda)\}_{j=n+1}^\infty\) and \(\{\pi_j^+(\lambda)\}_{j=1}^\infty\) satisfy the same recurrence relation, including the initial condition
\[
q_{n+1} f_1 + b_{n+1} f_2 = z f_1.
\]
Since the system of recurrence equations with the initial condition is uniquely solvable modulo a multiplying constant, one has
\[
\pi_j(\lambda) = C \pi_{j-n}^+(\lambda), \quad j > n.
\] (3.5)
The constant $C$ can be found by noting that

$$\pi_{n+1}(\lambda) = -b_{n-1}b_{n}^{-1}\pi_{n-1}(\lambda) \text{ and } \pi_{1}^+(\lambda) = 1.$$ 

Therefore $C = -b_{n-1}b_{n}^{-1}\pi_{n-1}(\lambda)$. Since $\lambda \in \sigma_p(J^+_n)$, the vector $\pi^+(\lambda)$ is in $\text{dom}(J^+_n)$. Now, by Definition 2.2, equation (3.3) implies that $\pi(\lambda)$ is in $\text{dom}(J)$, that is, $\pi(\lambda) \in \ker(J - \lambda I)$. Finally, observe that $J^+_n$ and $J^-_n$ do not depend on the perturbation so the result just proven holds for $J_n$. 

**Proposition 3.2.** For any $n \in \mathbb{N}$

$$\gamma \in \sigma(J) \iff \gamma \in \sigma(J_n)$$

**Proof.** Let us prove that $\gamma \in \sigma(J) \Rightarrow \gamma \in \sigma(J_n)$. Since $\sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(\tilde{J})$, it is sufficient to verify that $\gamma \in \sigma_{\text{disc}}(J)$ implies $\gamma \in \sigma_{\text{disc}}(\tilde{J}_n)$ (recall the notation introduced in Remark 2). Since $\gamma$ is an isolated eigenvalue of $J$, $\gamma$ is either a zero or a pole of $G(z, n)$ as a consequence of Lemma 2.2. If $G(\gamma, n) = 0$, then Lemma 2.2 implies that $\pi_n(\gamma) = 0$. Therefore, using Lemma 2.3 one concludes that $\gamma$ is an isolated eigenvalue of $J^+_n$. Thus, taking into account Corollary 2.1 and Proposition 3.1, $\gamma$ is an eigenvalue of $\tilde{J}_n$. If $\gamma$ is a pole of $G(z, n)$, then using Lemma 2.1 and (3.3) one has

$$\mathcal{M}_n(\gamma) = \theta^2 + (1 - \theta^2)\pi^2_n(\gamma)\rho(\{\gamma\}).$$

(3.6)

Thus, since $0 < \pi^2_n(\gamma)\rho(\{\gamma\}) < 1$, the equality (3.6) implies that $\mathcal{M}_n(\gamma) \neq 0$. Then, by (3.1), $\tilde{G}(z, n)$ should have a pole in $\gamma$ which, in turn, implies the assertion. 

**Lemma 3.2.** If $r \in \sigma_p(J) \cap \sigma_p(\tilde{J}_n) \setminus (\sigma_{\text{ess}}(J) \cup \{\gamma\})$, then $G(r, n) = 0$

**Proof.** Note that $r$ is an isolated common eigenvalue. If $\tilde{G}(r, n) = 0$, then $G(r, n) = 0$ by Corollary 3.1. If $\tilde{G}(r, n) \neq 0$, then $\mathcal{M}(r) \in \mathbb{R}$. Indeed, due to Lemma 2.1, one has

$$\lim_{z \to r} \mathcal{M}(z) = \frac{\pi^2_n(r)\rho(\{r\})}{\pi^2_n(r)\rho(\{r\})},$$

(3.7)

where it has been used that the $n$-th polynomial of the first kind for $\tilde{J}_n$ coincides with the one for $J$. By Lemma 2.2, $\pi_n(r) \neq 0$ and $r \in \sigma_p(\tilde{J}_n)$ implies $\tilde{\rho}(\{r\}) \neq 0$. Therefore, the denominator in the r.h.s. of (3.7) is different from zero. Now, since $\mathcal{M}(r)$ is finite and $r \neq \gamma$, (3.3) implies that $G(r, n)$ is finite. Finally, we recur to Lemma 2.2 to conclude that $G(r, n) = 0$. 

**Lemma 3.3.** Let $r \neq \gamma$ be an isolated eigenvalue of $J$. Then $\rho_n(\{r\}) = 0$ if and only if $r \in \sigma(J)$.
Proof. Suppose that $\rho_n(\{r\}) = 0$ and $r \in \sigma_p(J)$. Then $\pi_n(r) = 0$ (see (2.10). By Remark 2, one has that $r \in \sigma_p(J_n^+)$. Now, Proposition 3.1 implies that $r \in \sigma_p(\tilde{J})$. Let us show that if $r \in \sigma_p(\tilde{J}_n)$ then $\rho_n(\{r\}) = 0$. Since $r$ is an isolated common eigenvalue, then by Lemma 3.2 $G(r, n) = 0$, Therefore, by Lemma 2.2 $\pi_n(r) = 0$ which in turn implies $\rho_n(\{r\}) = 0$.

**Lemma 3.4.** Assume $r \notin \sigma_{ess}(J)$. Then, $r \in \sigma(\tilde{J}_n) \setminus \sigma(J)$ implies $\mathcal{M}_n(r) = 0$. Conversely, if $r \neq \gamma$ and $\mathcal{M}_n(r) = 0$, then $r \in \sigma(\tilde{J}_n) \setminus \sigma(J)$.

**Proof.** We begin by proving the first part. By hypothesis $\rho(\{r\}) = 0$, hence $\pi_n(r)\rho(\{r\}) = 0$. Let us show that $\pi_n^2(r)\tilde{\rho}(\{r\}) \neq 0$. Indeed, by Lemma 2.3, $\pi_n(r) \neq 0$, and $\tilde{\rho}(\{r\}) \neq 0$ since $r$ is an isolated element in $\sigma(\tilde{J}_n)$. Thus, using Lemma 2.1 one obtains

$$\mathcal{M}_n(r) = \frac{\rho(\{r\})}{\tilde{\rho}(\{r\})} = 0.$$ 

Let us prove the second part. If $r \neq \gamma$ and $\mathcal{M}_n(r) = 0$, then, by (3.3), $G(r, n)$ cannot be 0 nor $\infty$ and, therefore, due to Proposition 2.2 $r \notin \sigma(J)$. Now, according to (3.1), $\mathcal{M}_n(r) = 0$ and $G(r, n) \in \mathbb{R} \setminus \{0\}$ implies that $\tilde{G}(r, n) = \infty$. Finally, since $r \notin \sigma_{ess}(\tilde{J}_n)$, one uses (2.14) to verify that $r \in \sigma(\tilde{J}_n)$. \hfill \Box

The next theorem describes what happens to the spectrum of $J$ when we perturb it as indicated in (1.4). It states roughly, that between $\gamma$ (see (3.2)) and an eigenvalue of $J$ there is exactly one eigenvalue of the perturbation $\tilde{J}_n$ and there may be at most one common eigenvalue of $J$ and $\tilde{J}_n$. This joint eigenvalue is closer to $\gamma$ than the eigenvalue of $\tilde{J}_n$ which is not shared by $J$. Under the perturbation (1.4), the point $\gamma$ acts as an “attractor” of eigenvalues. The precise statement is as follows:

**Theorem 3.1.** Fix an arbitrary $n \in \mathbb{N}$. Let $\theta < 1$ and $a, b$ be in $\sigma_p(J) \setminus \sigma_p(\tilde{J}_n)$. Define $\mathcal{A} := (-\infty, \gamma) \cap (a, b)$ with $\gamma > a$, where $\gamma$ is defined in (3.2). Assume

(i) $(a, b) \cap \sigma_{ess}(J) = \emptyset$

(ii) $(a, b) \cap \sigma_p(J) \setminus \sigma_p(\tilde{J}_n) = \emptyset$

Then there exists a unique $\mu \in \mathcal{A} \cap \sigma_p(\tilde{J}_n) \setminus \sigma_p(J)$ and, if $\mathcal{A} \cap \sigma_p(J) \cap \sigma_p(\tilde{J}_n) \neq \emptyset$, there exists at most one $\eta \in \mathcal{A} \cap \sigma_p(J) \cap \sigma_p(\tilde{J}_n)$. Moreover

$$|\eta - \gamma| < |\mu - \gamma| .$$

(3.8)

The analogous assertion holds for $\mathcal{B} := (a, b) \cap (\gamma, \infty)$, with $\gamma < b$, instead of $\mathcal{A}$.

**Remark 3.** Observe that the theorem requires that $a, b$ are in $\sigma_p(J)$, but not necessarily in $\sigma_{disc}(J)$. 

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Proof. Step 1. Non common eigenvalues.

It follows from (2.9) and (3.3) that

\[
\frac{M_n(z)}{(1 - \theta^2)(\gamma - z)} = \int_{\mathbb{R}} \frac{d\rho_n(t)}{t - z} + \frac{\theta^2}{(1 - \theta^2)(\gamma - z)} = \int_{\mathbb{R}} \frac{d\omega(t)}{t - z},
\]

(3.9)

where

\[
d\omega(t) := d\rho_n(t) + \frac{\theta^2}{1 - \theta^2}dh(t) \quad h(t) := \begin{cases} 
0 & t < \gamma \\
1 & t \geq \gamma 
\end{cases}
\]

By Lemma 3.3 one has

\[
\rho_n(A) = \langle \delta_n, E_J(A)\delta_n \rangle = 0
\]

(3.10)

which implies that \( \omega(A) = 0 \). Now, since \( a \in \sigma_p(J) \setminus \sigma_p(\tilde{J}_n) \), one concludes that \( \rho(\{a\}) \neq 0 \) and, using again Lemma 3.3 that \( \pi_n^2(a) \neq 0 \). Therefore, taking into account

\[
\rho_n(\{a\}) = \int_{\{a\}} \pi_n^2(t)d\rho(t) = \pi_n^2(a)\rho(\{a\}) \neq 0,
\]

(3.11)

one obtains \( \omega(\{a\}) \neq 0 \). Analogously, \( \omega(\{b\}) \neq 0 \). On the other hand \( \omega(\{\gamma\}) \neq 0 \) by the definition of \( \omega \).

Then, according to Corollary A.2, the function

\[
\int_{\mathbb{R}} \frac{d\omega(t)}{t - x}, \quad x \in \mathbb{R}
\]

has precisely one zero in \( A \), that is, \( M_n(z) \) has one zero in \( A \). Thus, from Lemma 3.4 it follows that there is exactly one eigenvalue of \( \tilde{J}_n \) in \( A \) which is not an eigenvalue of \( J \).

Step 2. Common eigenvalues.

If \( \gamma > b \), that is if \( A = (a, b) \) or if \( \gamma \in (a, b) \) but \( \gamma \notin \sigma_p(J) \) then (3.10) holds. In the first case (3.10) follows by Lemma 3.3 as above and in the second it follows taking into account that

\[
\rho_n(\{\gamma\}) = \int_{\{\gamma\}} \pi_n^2(t)d\rho(t) = \pi_n^2(\gamma)\rho(\{\gamma\}) = 0.
\]

As in (3.11) we have \( \rho_n(\{a\}) \neq 0 \) and \( \rho_n(\{b\}) \neq 0 \).

Then using Corollary A.2 we obtain that

\[
G(x, n) = \int_{\mathbb{R}} \frac{\pi_n^2(t)d\rho(t)}{t - x}
\]

has exactly one zero \( \eta \in (a, b) \). By Lemma 3.2 this is the only point which may
be a common eigenvalue of $J$ and $\tilde{J}$, different from $\gamma$. Therefore there is at most one point in $\mathcal{A} \cap \sigma_p(J) \cap \sigma_p(\tilde{J}_n)$.

If $\gamma \in (a, b) \cap \sigma(J)$ then $\rho(\{\gamma\}) \neq 0$ and $\rho_n(\{\gamma\}) = 0$ if and only if $\pi_n(\gamma) = 0$ and there is only one root of $G(x, n)$ in $(a, b)$, so at most one common eigenvalue of $J$ and $\tilde{J}$ in $(a, b)$. If $\pi_n(\gamma) \neq 0$ then $\rho_n(\{\gamma\}) \neq 0$. Since $\rho_n((a, \gamma) = 0 = \rho_n(\gamma, b)$ applying Corollary A.2 we get one zero of $G(., n)$ in each interval $(a, \gamma)$ and $(\gamma, b)$. These zeros are, as before, the only possibilities of common eigenvalues in these intervals.

Step 3. Proof of inequality (3.8)

Let $\mu, \eta \in \mathcal{A}$ be such that $\mu$ is eigenvalue of $\tilde{J}$ and not eigenvalue of $J$ and $\eta$ is a common eigenvalue.

According to (3.9) $\mu$ satisfies

$$G(\mu, n) = \int_\mathbb{R} \frac{d\rho_n(t)}{t - \mu} = \frac{\theta^2}{(\theta^2 - 1)(\gamma - \mu)} < 0$$ (3.12)

and by Lemma 3.2 we know

$$G(\eta, n) = 0$$ (3.13)

Since $G(., n)$ is strictly increasing in $\mathcal{A}$ we get $\mu < \eta$ and therefore

$$|\mu - \gamma| > |\eta - \gamma|$$

A similar proof works for $\mu, \eta \in \mathcal{B}$. 

The next theorem considers the situation when $\gamma$ is below the bottom of the spectrum of $J$. It happens that between $\gamma$ and an eigenvalue of $J$ there is exactly one eigenvalue of the perturbed operator $\tilde{J}_n$. In this interval there are no common eigenvalues if $\gamma$ is not an eigenvalue. To the left of $\gamma$ there is no spectrum.

**Theorem 3.2.** Fix an arbitrary $n \in \mathbb{N}$. Let $\theta < 1$, $a = -\infty$, and $b$ be in $\sigma_p(J) \setminus \sigma_p(\tilde{J}_n)$. Consider $\mathcal{A, B}$ as were defined in Theorem 3.1 and assume the conditions (i), (ii) of that theorem. Then there is no eigenvalue of $\tilde{J}_n$ in $\mathcal{A}$. If $\mathcal{B} \neq \emptyset$, then $\mathcal{B} \cap \sigma_p(\tilde{J}_n) \setminus \sigma_p(J)$ has precisely one element. If $\gamma \notin \sigma_p(J)$, then the set $\sigma_p(\tilde{J}_n) \cap \sigma_p(J) \cap (-\infty, b)$ is empty. A similar result holds when $a \in \sigma_p(J) \setminus \sigma_p(\tilde{J}_n) \setminus a = \infty$ just interchanging $\mathcal{A}$ and $\mathcal{B}$.

**Proof.** The proof is carried out for the case $\gamma < b$. By (3.10) one has that $\rho_n(-\infty, \gamma) = 0$. Hence

$$G(\lambda, n) = \int_\gamma^\infty \frac{d\rho_n(t)}{t - \lambda}.$$
For any \( \lambda \in \mathcal{A} \), and \( t > \gamma \), we have \( t - \lambda > 0 \), therefore \( G(\lambda, n) > 0 \) since \( \rho_n(\gamma, \infty) \neq 0 \) as a consequence of Lemma 3.3. This contradicts (3.12) with \( \mu = \lambda \). Thus, there is no eigenvalue of \( \tilde{J}_n \) which is not an eigenvalue of \( J \) in \( \mathcal{A} \). Moreover, \( G(\lambda, n) > 0 \) also contradicts (3.13) with \( \eta = \lambda \), hence there are no common eigenvalues in \( \mathcal{A} \).

Now, let us see what happens in \( \mathcal{B} = (\gamma, b) \). We first treat the case when \( \rho_n(\{\gamma\}) = 0 \). Then, Lemma 3.3 and (ii) implies that \( \rho_n(-\infty, b) = 0 \). By Theorem A.1, \( G(\lambda, n) \) is continuous and strictly increasing. Moreover, reasoning as at the beginning of this proof, \( G(\lambda, n) \) is positive. On the other hand, Theorem A.1 implies \( \lim_{\lambda \uparrow b} G(\lambda, n) = +\infty \).

Define

\[
 f(\lambda) := \frac{\theta^2}{(\theta^2 - 1)(\gamma - \lambda)}.
\]

By our assumption on \( \theta \), \( f(\lambda) > 0 \) for \( \lambda > \gamma \). Also,

\[
 \lim_{\lambda \downarrow \gamma} f(\lambda) = +\infty
\]

and \( f \) is decreasing in \( (\gamma, b) \). Thus, for \( \lambda \) close to \( \gamma \), one verifies that \( G(\lambda, n) < f(\lambda) \) and for \( \lambda \) close to \( b \), that \( G(\lambda, n) > f(\lambda) \). Therefore, there exists a unique \( \lambda_0 \in (\gamma, b) \) such that \( G(\lambda_0, n) = f(\lambda_0) \). This \( \lambda_0 \) is an eigenvalue of \( \tilde{J}_n \) which is not an eigenvalue of \( J \). In \( (-\infty, b) \) there are no common eigenvalues since \( G(\lambda, n) > 0 \) in this interval.

Suppose now that \( \rho_n(\{\gamma\}) \neq 0 \). By hypothesis, it follows from Lemma 3.3 that \( \rho_n(\{\gamma\}) \neq 0 \). Thus, it holds that \( \rho_n(\{\gamma\}) \neq 0 \), \( \rho_n(\{b\}) \neq 0 \), and \( \rho_n(\gamma, b) = 0 \). By Theorem A.1, the function \( G(\lambda, n) \) is continuous, strictly increasing in \( (\gamma, b) \) and

\[
 \lim_{\lambda_\downarrow \gamma} G(\lambda, n) = -\infty \quad \lim_{\lambda_\uparrow b} G(\lambda, n) = +\infty.
\]

Hence there is a unique \( \lambda_0 \in (\gamma, b) \) such that \( G(\lambda_0, n) = f(\lambda_0) \). This \( \lambda_0 \) is an eigenvalue of \( \tilde{J}_n \) which is not an eigenvalue of \( J \). Since there is also exactly one \( \lambda_1 \) in \( (\gamma, b) \) such that \( G(\lambda_1, n) = 0 \), one may have a common eigenvalue in this interval.

\[ \square \]

\textbf{Remark 4.} When \( \theta > 1 \), the assertions of Theorems 3.1 and 3.2, modified by interchanging \( J \) and \( \tilde{J}_n \), hold true. The proof is carried out in the same way, but using (3.4) instead of (3.3) and \( \tilde{G}(z, n) \) instead of \( G(z, n) \) instead of (3.13).

\textbf{Remark 5.} If \( n = 1 \) and \( \sigma_{ess}(J) = \emptyset \), then Theorems 3.1, 3.2, and Proposition 3.2 are Propositions 3.1 and 3.2 of [5]. Thus, in this case, a complete description of the interplay of the spectra is obtained.
Remark 6. Note that the validity of our results includes the case of finite dimensional Jacobi matrices. In this particular case, the results of this work coincide with the corresponding ones in [4].

Appendix

We give a simple proof of the following known results for Nevanlinna functions.

**Theorem A.1.** Let \( \rho \) be a positive measure on \( \mathbb{R} \) such that

(i) \( \rho(\mathbb{R}) < \infty \)

(ii) \( \rho(a, b) = 0 \) for an open interval \((a, b)\)

(iii) If \( a \neq -\infty \), then \( \rho(\{a\}) \neq 0 \), and if \( b \neq +\infty \), then \( \rho(\{b\}) \neq 0 \).

Define, for \( \lambda \in \mathbb{R} \),

\[
F(\lambda) := \int_{\mathbb{R}} \frac{d\rho(t)}{t - \lambda}.
\]

Then \( F \) has the following properties

(I) \( F \) is continuous in \((a, b)\)

(II) \( F \) is strictly increasing in \((a, b)\)

(III) If \( a \neq -\infty \), then \( \lim_{\lambda \downarrow a} F(\lambda) = -\infty \)

If \( b \neq +\infty \), then \( \lim_{\lambda \uparrow b} F(\lambda) = +\infty \)

**Proof.** By the definition of \( F \), one has

\[
\frac{F(x + h) - F(x)}{h} = \int_{\mathbb{R}} \frac{1}{h} \left( \frac{1}{t - (x + h)} - \frac{1}{t - x} \right) d\rho(t)
= \int_{\mathbb{R} \setminus (a,b)} \frac{d\rho(t)}{(t - (x + h))(t - x)}.
\]

Let \( x \in \mathcal{E} \subset (a, b) \), where \( \mathcal{E} \) is a closed interval. Then, for \( t \in \mathbb{R} \setminus (a, b) \), there exists \( d > 0 \) such that \( |t - x| > d > 0 \). Choose \( h \in \mathbb{R} \) such that \(|h| < \frac{d}{2}\). Then

\[
|t - xh| > |t - x| - |h| \geq \frac{d}{2} > 0.
\]
Therefore

\[ \left| \frac{1}{(t - x - h)(t - x)} \right| \leq \frac{2}{d^2} \in L^1(\mathbb{R}, d\rho) \]

since \( \rho \) is finite. Thus, one can apply the dominated convergence theorem to obtain that

\[ \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \int_{\mathbb{R} \setminus (a,b)} \frac{d\rho(t)}{(t - x)^2} > 0. \]

This proves (I) and (II).

Now, let \( \{b_n\}_{n=1}^{\infty} \) be a nondecreasing real sequence such that \( b_n \to b \).

Then, for \( n \) sufficiently large,

\[ F(b_n) = \left( \int_{(a, \infty]} + \int_{[b_n, +\infty)} \right) \frac{d\rho(t)}{t - b_n}. \]

For the last term of the r.h.s., one has

\[ \int_{[b_n, +\infty)} \frac{d\rho(t)}{t - b_n} \to \frac{\rho(\{b\})}{b - b_n} + \int_{(b, +\infty)} \frac{d\rho(t)}{t - b_n} \quad n \to \infty. \]  \hspace{1cm} (A.14)

On the other hand, for \( n \) sufficiently large

\[ \left| \int_{(a, \infty]} \frac{d\rho(t)}{t - b_n} \right| \leq \int_{(a, \infty]} \frac{d\rho(t)}{|t - b_n|} \leq C \int_{\mathbb{R}} d\rho(t) < \infty \]  \hspace{1cm} (A.15)

since \( \{b_n\}_{n=1}^{\infty} \) accumulates at \( b \) and \( t \in (a, \infty] \).

It follows from (A.14) and (A.15) that \( F(b_n) \to \infty \) and therefore \( F(\lambda) \) tends to \(+\infty\) whenever \( \lambda \to b \). A similar argument proves that \( F(\lambda) \to -\infty \) if \( \lambda \to a \).

\[ \Box \]

**Remark 7.** \( F \) is not only continuous but holomorphic away of the support \( \rho \) (see the paragraph after [13, Lem. B.4]).

**Corollary A.2.** Let \( F \) be as in Theorem A.1. If \( a \neq -\infty \) and \( b \neq +\infty \), then there exists exactly one point \( p \in (a, b) \) such that \( F(p) = 0 \).

**Proof.** The proof follows directly from (I), (II), (III) of Theorem A.1. \[ \Box \]

**Acknowledgments.** We thank the anonymous referees for useful suggestions and comments that led in particular to Remarks [1] and [7].
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