Existence of axially symmetric solutions to the Vlasov-Poisson system depending on Jacobi’s integral

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Abstract

We prove the existence of axially symmetric solutions to the Vlasov–Poisson system in a rotating setting for sufficiently small angular velocity. The constructed steady states depend on Jacobi’s integral and the proof relies on an implicit function theorem for operators.

1 Introduction

In stellar dynamics, the evolution of a large ensemble of particles (e.g. stars) which interact only by their self-consistent, self-generated gravitational field, is described by the Vlasov-Poisson system

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \quad (1.1)
\]
\[
\Delta U = 4\pi \rho, \quad (1.2)
\]
\[
\rho(t,x) = \int f(t,x,v) dv. \quad (1.3)
\]

Here \( f = f(t,x,v) \geq 0 \) is the phase-space density, where \( t \in \mathbb{R} \) denotes time, and \( x,v \in \mathbb{R}^3 \) denote position and velocity. \( U = U(t,x) \) is the gravitational potential of the ensemble, and \( \rho = \rho(t,x) \) is its spatial density. We are looking for stationary solutions of (1.1)-(1.3). The ansatz

\[
f_0(x,v) = \Phi(E) = \Phi(\frac{1}{2}v^2 + U(x)) \quad (1.4)
\]

is well known and automatically satisfies the Vlasov equation (1.1), because the particle energy

\[
E(x,v) := \frac{1}{2}v^2 + U(x)
\]
is a conserved quantity along characteristics. But we still have to construct the self-consistent potential. This is done by plugging (1.4) into the Poisson equation, more precisely, we have to solve

$$\Delta U = 4\pi h \Phi(U) = 4\pi \int \Phi(\frac{1}{2}v^2 + U(x))dv$$  \hspace{1cm} (1.5)$$

The ansatz (1.4) only leads to spherically symmetric stationary solutions of (1.1)-(1.3), where $f$ is called spherically symmetric, iff $f(Ax, Av) = f(x, v)$ $\forall A \in O(3)$. Indeed, this is a special case of a more general result of Gidas, Ni and Nirenberg, cf. [2]. If one is interested in stationary solutions with less symmetry, more invariants can be added to (1.4), so that the right-hand side of (1.5) explicitly depends on $x$.

One possibility is to consider a rotating system. If the ensemble is rotating around a given axis, say the $x_3$-axis, we can change to the rotating frame and change coordinates as follows:

$$\zeta := R_t x, \quad \eta := R_t v - \Omega \times (R_t x),$$

where

$$R_t := \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Omega := \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

and the (rotational) velocity $\omega > 0$ is given. The Vlasov-Poisson system then takes the form

$$\partial_t f + \eta \cdot \nabla \zeta f - (\nabla \zeta U + \Omega \times (\Omega \times \zeta) + 2(\Omega \times \eta) \cdot \nabla \eta f = 0, \quad (1.6)$$

$$\Delta \zeta U(t, \zeta) = 4\pi \rho(t, \zeta), \quad (1.7)$$

$$\rho(t, \zeta) = \int f(t, \zeta, \eta)d\eta \quad (1.8)$$

and the characteristic system of the Vlasov equation (1.6) reads

$$\begin{cases} \dot{\zeta} = \eta \\ \dot{\eta} = -\partial_\zeta U(t, \zeta) - 2\Omega \times \eta - \Omega \times (\Omega \times \zeta) \end{cases}$$

which has the following expression as a conserved quantity, if $U$ is time-independent:

$$E_J := \frac{1}{2} \eta^2 + U(\zeta) - \frac{1}{2}\Omega \times \zeta^2,$$

where $E_J$ is also called Jacobi’s integral. A natural ansatz for the construction of stationary solutions of (1.6)-(1.8) is now

$$f(\zeta, \eta) = \varphi(E_J) = \varphi(\frac{1}{2}|\eta|^2 + U(\zeta) - \frac{1}{2}\omega^2 r^2) \quad (1.9)$$
for a suitable function $\varphi: \mathbb{R} \to \mathbb{R}^+$, where $r := r(x) = \sqrt{\zeta_1^2 + \zeta_2^2}$. In the original coordinates $x, v$ one easily verifies that this ansatz leads to

$$g(x, v) := f(\zeta, \eta) = \varphi \left( \frac{1}{2} v^2 + U( \mathbf{R} x) - \omega P \right),$$

where we define $P$ as the third component of the angular momentum, that is $P := x_1 v_2 - x_2 v_1$, which is a conserved quantity of the characteristic system of the Vlasov equation (1.1), if $U$ is axially symmetric with respect to the $x_3$-axis. Obviously, the function $f = f(\zeta, \eta)$ then automatically satisfies (1.6) and one has to solve the Poisson equation, where we relabel $\zeta$ and $\eta$ to $x$ and $v$,

$$\Delta U = \int \varphi \left( \frac{1}{2} v^2 + U( x) - \frac{1}{2} \omega^2 r^2 \right) dv =: \tilde{h}(\omega, r, U(x)). \quad (1.10)$$

So if we construct an axially symmetric $U$ solving (1.10), the corresponding functions $(g, U)$, with $g$ defined as above also will be a stationary solution of (1.1)-(1.3). Clearly, our ansatz for $f$ satisfies (1.6) without any symmetry assumptions on $U$ and this gives hope for the construction of stationary solutions with less symmetry, for example triaxial systems.

Equation (1.10) has been studied, among others, by Vandervoort, cf. [9]. He observed numerically, that if $\varphi$ is of the form

$$\varphi(E_J) = (E_0 - E_J)^{\beta - 3/2}, \quad (1.11)$$

then for $0.5 < \beta \leq 0.808$ there are triaxial solutions to (1.10) for sufficiently large $\omega$. For small $\omega$ or $\beta > 0.808$, all numerically constructed solutions are axially symmetric. Consequently, (1.10) seems to be of particular interest for the construction of ellipsoidal systems, but to our knowledge no self-consistent ellipsoidal systems to (1.1)-(1.3) or (1.6)-(1.8) have been constructed analytically yet.

We will prove that there exist axially symmetric solutions to (1.10) for small $\omega$ under suitable assumptions on $\varphi$, where we treat the case $\beta > 5/2$ in (1.11). For this purpose, we require, that for $\omega = 0$, we have a nontrivial, spherically symmetric solution $(f_0, U_0)$ of (1.10). Note, that in this case the righthand-side of (1.10) only depends on $U_0$. For $\omega \neq 0$, we want to apply an implicit function theorem to get solutions, which arise by deforming $U_0$, where certain symmetries are conserved. The central idea, which makes this approach work is to look for a solution $U^\omega$ as a deformation of $U_0$, i.e., $U^\omega = U_0(g(x))$ for some diffeomorphism $g$ on $\mathbb{R}^3$, and to formulate the problem in terms of finding zeros of a suitable operator $T$ over the space of such deformations instead of the space of the potentials. Whereas the original problem (1.10) had to be solved in $\mathbb{R}^3$, we will only need to know the deformation on a compact neighbourhood of the support of the original solution $(f_0, r_0, U_0)$, and this provides useful compactness properties. Furthermore, finite radius and finite mass of the constructed solutions then are just consequences of the corresponding properties of $(f_0, r_0, U_0)$.

Although the allowed perturbations for the potential $U_0$ only have mirror
symmetry which would match a triaxial system, we have up to now no method
to exclude axial symmetry with respect to the $x_3$-axis for the perturbations
constructed by the implicit function theorem.

The approach described above has been used by Lichtenstein for proving the
existence of slowly rotating Newtonian stars, as described by selfgravitating
fluid balls, cf. [4, 5]. A translation of Lichtenstein’s approach into modern
mathematical language is due to Heilig, cf. [3].

The investigations made there were applied to the Vlasov-Poisson system in
[8], where stationary solutions to (1.1)-(1.3) of the form

$$f(x,v) = \phi(E) \psi(\omega P)$$

were constructed. There, the potential $U$ a-priori was axially symmetric, so
that the expression

$$P = x_1v_2 - x_2v_1$$

is a conserved quantity with respect to
the characteristic system. The procedure described there is the basis of our
approach.

This paper is organized as follows: In the next section we rewrite the problem
in terms of finding zeros of the operator $T$, we then state the main result and
prove it using an implicit function theorem. For this, we need certain properties
of $T$ which can be proved as in [8], except some minor technical modifications
and one lemma, where the symmetry of the allowed perturbations enters in. In
Section 3 we generalize this important lemma dealing with properties of the
operator $\partial_T T(0,0)$ to mirror symmetry.

2 The main result

The mappings, which leave our solutions invariant, are in the set

$$S := \{ \tau_{110} : (x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3), \tau_{101} : (x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3), \\
\tau_{011} : (x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3) \}.$$ 

Now let $B_R := \{ x \in \mathbb{R}^3 | |x| \leq R \}$ and define

$$C_S(B_R) := \{ f \in C(B_R) | f(Ax) = f(x), A \in S, \ x \in B_R \}. \quad (2.12)$$

Then we have

$$\nabla f(0) = 0, \text{ if } f \in C^1(B_R) \cap C_S(B_R).$$

For $\varphi : \mathbb{R} \to [0, \infty]$ we require

(\varphi 1) $\varphi \in C^1(\mathbb{R})$ and there is $E_0 \in \mathbb{R}$ with $\varphi(E_J) = 0$ for $E_J \geq E_0$ and $\varphi(E_J) > 0$ for $E_J < E_0$.

(\varphi 2) $\varphi$ is strictly decreasing in $]-\infty, E_0[$.

(\varphi 3) The ansatz $f_0(x,v) = \varphi(E_J)$ with $\omega = 0$ produces a nontrivial, spherically
symmetric solution $(f_0, \rho_0, U_0)$ of (1.1)-(1.3) with $\rho_0 \in C_0^1(\mathbb{R}^3)$, supp $\rho_0 = B_1$ and $U_0 \in C^2(\mathbb{R}^3)$ with $\lim_{|x| \to \infty} U_0(x) = 0$. 

4
Examples for a functions satisfying (\(\varphi_1\))–(\(\varphi_3\)) are the so-called polytropes

\[
\varphi(E,J) := (E_0 - E_J)^k
\]

for \(k > 1\) and suitable \(E_0 < 0\). Now we can state the main theorem.

**Theorem 2.1.** Let \(r := \sqrt{x_1^2 + x_2^2}\). There exists \(\omega_0 > 0\), such that for all \(\omega \in [-\omega_0, \omega_0]\) there exists a nontrivial solution \((f^\omega, \rho^\omega, U^\omega)\) of (1.6)-(1.8) with

\[
(i) \quad f^\omega(x,v) = \begin{cases} \varphi(\frac{1}{2}v^2 + U^\omega(x) - \frac{1}{2}\omega^2 r^2) & \text{for } |x| < 4 \\ 0 & \text{else} \end{cases}
\]

\[
(ii) \quad (f^0, \rho^0, U^0) = (f_0, \rho_0, U_0) \text{ and for } |\omega| < \omega_0, (f^\omega, \rho^\omega, U^\omega) \text{ has the following symmetry properties: For all } A \in S \text{ we have} \]

\[
f^\omega(Ax, Av) = f^\omega(x,v), \quad \rho^\omega(Ax) = \rho(x), \quad U^\omega(Ax) = U^\omega(x)
\]

and \((f^\omega, \rho^\omega, U^\omega)\) is not spherically symmetric for \(\omega \neq 0\).

\[
(iii) \quad \rho^\omega \in C_1^1(\mathbb{R}^3) \text{ and } U^\omega \in C_6^2(\mathbb{R}^3), \text{ where } \rho^\omega(x) = \int f^\omega(x,v) dv.
\]

\[
(iv) \quad \text{The mappings } [-\omega_0, \omega_0] \ni \omega \mapsto \rho^\omega \text{ and } [-\omega_0, \omega_0] \ni \omega \mapsto U^\omega \text{ are continuous with respect to the norms } \|\cdot\|_{1,\infty} \text{ or } \|\cdot\|_{2,\infty}, \text{ respectively.}
\]

**Remark.** If we add rotations about the \(x_3\)-axis to the set \(S\), the proof of Theorem 2.1 still holds – we can essentially follow the proof given here, and this shows that the constructed solutions in Theorem 2.1 have to be axially symmetric a-posteriori. This follows by the uniqueness of the mapping given by the implicit function theorem, cf. Theorem 4.1.

For the proof of Theorem 2.1 we need some lemmata.

**Lemma 2.2.** The spherically symmetric solution \((f_0, \rho_0, U_0)\) has the following properties.

\[
(a) \quad \text{The potential } U_0 \text{ is given by}
\]

\[
U_0(x) = -\int \frac{\rho_0(y)}{|x-y|} dy = -\frac{4\pi}{|x|} \int_0^{|x|} s^2 \rho_0(s) ds - 4\pi \int_0^\infty s \rho_0(s) ds, \; x \in \mathbb{R}^3.
\]

\[
(b) \quad \rho_0 \text{ is decreasing with } \rho_0(0) > 0, U_0''(0) > 0 \text{ and for every } R > 0 \text{ there exists } C > 0, \text{ such that } U_0'(r) \geq Cr, \quad r \in [0,R], \text{ and } U_0(1) = E_0.
\]

\[
(c) \quad \rho_0' \text{ is Hölder continuous and } U_0' \in C^2(\mathbb{R}^3), \text{ where } \hat{\mathbb{R}}^3 := \mathbb{R}^3 \setminus \{0\}.
\]

**Proof.** The formula

\[
U_0'(r) = \frac{4\pi \int_0^r s^2 \rho_0(s) ds}{r^2}
\]
easily follows from the Poisson equation with spherical symmetry and since we require \( \lim_{|x| \to \infty} U_0(x) = 0 \), the representation for \( U_0 \) holds by uniqueness. As to (b), for \( \omega = 0 \) we have \( f_0(x,v) = f_0(E) = f_0(\frac{1}{2} v^2 + U_0(x)) \) and this implies

\[
\rho_0(x) = \int_{\mathbb{R}^3} f_0(x,v) \, dv = h_0(U_0(x)) := 4\pi \sqrt{2} \int_{U_0(x)}^E \varphi(E) \sqrt{E - U_0(x)} \, dE, \tag{2.13}
\]

where the function \( h \) is continuously differentiable and with \((\varphi 1), (\varphi 2)\) we have \( h'(s) < 0 \) for \( s < E_0 \). Consequently, \( \rho_0 \) is decreasing because \( U_0 \) is increasing and since the steady state \((f_0, U_0)\) is assumed to be nontrivial, we must have \( \rho_0(0) > 0 \). Thus actually \( U_0'(r) > 0, r > 0 \), and since \( U_0''(0) = (4\pi/3) \rho_0(0) > 0 \) this implies the estimate on \( U_0' \) from below. The assertion that \( U_0(1) = E_0 \) follows from (2.13) and the assumption \( \sup \rho_0 = B_1 \). The regularity of \( U_0' \) follows from the formula for \( U_0' \) above and the fact that \( \rho_0 \in C^1_c \), which we deduce again from (2.13). Finally, the Hölder continuity of \( \rho_0 \) will be part of the next Lemma.

**Lemma 2.3.** Let \( E_1 := U_0(2) - E_0 \) and define \( f \) by

\[
f(x,v) = \begin{cases} 
\varphi(\frac{1}{2} v^2 + U(x) - \frac{1}{2} \omega^2 r^2) & \text{for } U(x) < E_0 + E_1, \\
0 & \text{else}
\end{cases}
\]

where \( \varphi \) satisfies \((\varphi 1), (\varphi 2)\) and \( U \in C^2_b(\mathbb{R}^3) \) with \( U(x) > E_0 + E_1 \) for \(|x| > 4\). Then the following holds:

\[
\rho_f(x) := \int_{\mathbb{R}^3} f(x,v) \, dv \\
= \tilde{h}(\omega, r(x), U(x)) \\
= \begin{cases} 
\tilde{h}(U(x) - \frac{1}{2} \omega^2 r^2) & \text{for } U(x) < E_0 + E_1, \\
0 & \text{else}
\end{cases} \tag{2.14}
\]

with

\[
\tilde{h}(s) = 4\pi \sqrt{2} \int_s^{E_0} \sqrt{E - s} \varphi(E) \, dE.
\]

Furthermore, \( \tilde{h} \in C^1(\mathbb{R} \times [0,\infty) \times \mathbb{R}) \) and for every bounded set \( B \subset \mathbb{R} \times [0,\infty) \times \mathbb{R} \) there are constants \( C > 0 \) and \( \mu \in [0,1] \) such that for \((\omega, r, u), (\omega', r, u') \in B \) we have

\[
|\partial_r \tilde{h}(\omega, r, u)| \leq Cr, \\
|\tilde{h}(\omega, r, u) - \tilde{h}(\omega', r, u')| \leq C(|\omega - \omega'| r + |u - u'|), \\
|\partial_u \tilde{h}(\omega, r, u) - \partial_u \tilde{h}(\omega', r, u')| \leq C(|\omega - \omega'| + |u - u'|^{\mu}).
\]

In addition, for \( \omega = 0 \), the function \( \tilde{h}(0,\cdot,\cdot) \) does not depend on \( r(x) \) and we can write \( h_0 := \tilde{h}(0,0,u) \).
Proof. Introducing polar coordinates, we have for $U(x) < E_0 + E_1$

$$
\rho(x) = \int \varphi \left( \frac{1}{2} v^2 + U(x) - \frac{1}{2} \omega^2 r^2 \right) dv
= 4\pi \int_0^\infty t^2 \varphi \left( \frac{1}{2} t^2 + U(x) - \frac{1}{2} \omega^2 r^2 \right) dt
= 4\pi \sqrt{2} \int_{E(x) - \frac{1}{2} \omega^2 r^2}^{E_0} \left( E - U(x) + \frac{1}{2} \omega^2 r^2 \right)^{1/2} \varphi(E) dE,
$$

and (2.14) follows.

We have $h \in C^1(\mathbb{R})$ with

$$
h'(s) = -4\pi \sqrt{2} \int_s^{E_0} \frac{1}{2\sqrt{E-s}} \varphi(E) dE
$$

for $s < E_0$ and $h'(s) = 0$ for $s \geq E_0$ and the first two estimates follow. Next,

$$
h''(s) = -4\pi \sqrt{2} \frac{d}{ds} \int_0^{E_0-s} \frac{1}{2\sqrt{E}} \varphi(E+s) dE
= -4\pi \sqrt{2} \int_0^{E_0-s} \frac{1}{2\sqrt{E}} \varphi'(E+s) dE
= -4\pi \sqrt{2} \int_s^{E_0} \frac{1}{2\sqrt{E-s}} \varphi'(E) dE
$$

yields local Lipschitz continuity of $\partial_u \tilde{h}$ with respect to $\omega$ and $u$ and the proof is complete.

We want to find solutions of the equation

$$
\Delta U = 4\pi \tilde{h}(\omega, r(x), U)
$$

(2.15)

and the main idea is to rewrite problem (2.15) in terms of finding zeros of an operator $T$, which does not act directly on the space of potentials, but on deformations of the given spherically symmetric potential $U_0$. We define Banach spaces, which will serve as domain and range of $T$

$$
X := \{ f \in C_s(B_4) | f(0) = 0, f \in C^1(\bar{B}_4), \exists C > 0 : |\nabla f(x)| \leq C, x \in \bar{B}_4, \\
\forall x \in \partial B_1 : \lim_{t \to 0, t > 0} \nabla f(tx) =: \nabla f(0x) \text{ exists, uniformly in } x \in \partial B_1 \},
$$

where $\partial B_1 := \{ x \in \mathbb{R}^3 \mid |x| = 1 \}$ and $\bar{B}_4 := B_4 \setminus \{0\}$. We equip $X$ with the norm

$$
\| f \|_X := \sup_{x \in \bar{B}_4} |\nabla f(x)|,
$$

and

$$
Y := \{ f \in C_s(B_4) | f(0) = 0, f \in C^1(B_4), \exists C > 0 : |\nabla f(x)| \leq C|x|, x \in B_4, \\
\forall x \in \partial B_1 : \lim_{t \to 0, t > 0} \frac{\nabla f(tx)}{t} =: \nabla f(0x) \text{ exists, uniformly in } x \in \partial B_1 \}
$$

(2.14)
with norm
\[ \|f\|_Y := \sup_{x \in B_4} \frac{\|f(x)\|}{|x|}, \quad f \in Y. \]

To state more precisely, how to use functions in \( X \) to deform the potential \( U_0 \), we need the next lemma.

**Lemma 2.4.** For \( \zeta \in X \) let
\[ g_\zeta : B_4 \to \mathbb{R}^3, \quad g_\zeta(x) := x + \zeta(x) \frac{x}{|x|}, \quad x \in \dot{B}_4, \quad g_\zeta(0) = 0 \]

Then there exists \( r > 0 \), such that for all \( \zeta \in \Omega \), where
\[ \Omega := \{ \zeta \in X \| \zeta \|_X < r \} \]
we have:

(a) \( g_\zeta : B_4 \to B_4, \zeta := g_\zeta(B_4) \) is a homeomorphism, \( g_\zeta : \dot{B}_4 \to \dot{B}_4, \zeta \) is a \( C^1 \)-diffeomorphism, with
\[ |Dg_\zeta(x) - \text{id}| < \frac{1}{2}, \quad x \in \dot{B}_4 \]

and for every \( x \in \partial B_1 \) the mapping
\[ g_\zeta : 0,4x \ni y \mapsto g_\zeta(y) \in 0,|g_\zeta(4x)|x \]
is one-to-one, onto and preserves the natural ordering of points in \( 0,4x \), where we defined \( x_1,x_2 := \{ x_1 + \lambda(x_2 - x_1) \mid \lambda \in [0,1] \} \) for \( x_1,x_2 \in \mathbb{R}^3 \).

(b) \( \frac{1}{2}|x| \leq |g_\zeta(x)| \leq \frac{3}{2}|x|, \quad x \in B_4, \) and \( g_\zeta(B_2) \subset B_3, B_3 \subset g_\zeta(B_4) \subset B_5 \)

(c) \( g_\zeta(Ax) = Ag_\zeta(x), \quad x \in B_4 \) and \( g_\zeta^{-1}(Ax) = Ag_\zeta^{-1}(x), \quad x \in B_4, A \in S \)

(d) \( |Dg_\zeta^{-1}(x) - \text{id}| < \frac{1}{2}, x \in \dot{B}_4, \) and there exists a constant \( C > 0 \), such that for all \( \zeta, \zeta' \in \Omega \):

\[ \frac{1}{|x|} |g_\zeta(x) - g_\zeta'(x)| + |Dg_\zeta(x) - Dg_\zeta'(x)| \leq C \|\zeta - \zeta'\|_X, \quad x \in \dot{B}_4, \]

and
\[ |g_\zeta^{-1}(x) - g_\zeta'^{-1}(x)| \leq C \|\zeta - \zeta'\|_X |x|, \quad x \in B_3 \]

**Proof.** In \( \dot{B}_4 \), we have for \( i,j = 1,2,3 \):
\[ \partial_{x_i} g_{\zeta,j}(x) = \delta_{ij} + \partial_{x_i} \zeta(x) \frac{x_j}{|x|} + \frac{\zeta(x)}{|x|} \left( \delta_{ij} - \frac{x_i x_j}{|x|^2} \right) \quad (2.16) \]

and therefore
\[ |Dg_\zeta(x) - \text{id}| < 3\|\zeta\|_X. \]
With the inverse function theorem the first two assertions in (a) follow. For \( x \in \partial B_1 \),
\[
g_\zeta(tx) = tx + \zeta(tx) x = x(t + \zeta(tx))
\]
and
\[
\frac{d}{dt}(t + \zeta(tx)) = 1 + \nabla\zeta(tx) \cdot x > 0 \quad \text{for } \|\zeta\|_X \text{ small}
\]
and the proof of (a) is complete.

We have \(|\zeta(x)| \leq \|\zeta\|_X |x|\) for \( x \in B_4 \) and this implies (b) for \( r > 0 \) sufficiently small. Assertion (c) is easily verified, too. If we choose \( r \) even smaller we also have the first claim of (d), because
\[
Dg^{-1}_\zeta(x) = (Dg\zeta)^{-1}(g_\zeta^{-1}(x)).
\]
The estimate for \( g_\zeta - g'_\zeta \) follows from the definition of \( g_\zeta \) and the estimate for \( Dg_\zeta - Dg_\zeta' \) follows from (2.10).

For \( x \in \dot{B}_3 \), we have with (b): \( x \in g_\zeta(B_4) \cap g'_\zeta(B_4) \). Consequently, there exists \( y \in B_4 \) mit \( x = g_\zeta'(y) \). Now we have
\[
|g_\zeta^{-1}(x) - g_\zeta'^{-1}(x)| = |g_\zeta^{-1}(g_\zeta(y)) - y|
\]
\[
= |g_\zeta^{-1}(g_\zeta(y)) - g_\zeta^{-1}(g_\zeta(y))|
\]
\[
\leq 2|g_\zeta(y) - g_\zeta(y)| \leq 2\|\zeta - \zeta'\|_X |y|
\]
\[
\leq 4\|\zeta - \zeta'\|_X |x|,
\]
where we used the mean value theorem, the estimate for \( Dg_\zeta^{-1} \) and \( g_\zeta(y), g_\zeta'(y) \subset g_\zeta(\dot{B}_4) \).

We want to find solutions of (2.15) with the following structure
\[
U(x) = U_\zeta(x) := U_0(g_\zeta^{-1}(x)), \quad x \in B_4, \zeta,
\]
with a suitable \( \zeta \in \Omega \). Obviously, we need \( U \) on the whole space \( \mathbb{R}^3 \), but this is only a technical problem. We use the fundamental solution of the Poisson equation to integrate (2.15) and we then have to solve
\[
U_0(x) + \int_{B_4, \zeta} \frac{\tilde{h}(\omega, r(y), U_0(g_\zeta^{-1}(y)))}{|g_\zeta(x) - y|} dy = 0, \quad x \in B_4.
\]
(2.17)

This equation essentially contains the operator we are looking for, but we have to modify things a little and also we want to get rid of the dependence on \( \zeta \) in the integration domain.

**Proof of Theorem 2.7.** For \( \zeta \in \Omega \) and \( \omega \in \mathbb{R} \), we define
\[
T(\omega, \zeta)(x) := U_0(x) + \int_{B_3} \frac{\tilde{h}(\omega, r(y), U_0(g_\zeta^{-1}(y)))}{|g_\zeta(x) - y|} dy
\]
\[
- U_0(0) - \int_{B_3} \frac{\tilde{h}(\omega, r(y), U_0(g_\zeta^{-1}(y)))}{|y|} dy, \quad x \in B_4.
\]
(2.18)
Suppose we already know that this defines a continuous operator
\[ T : \tilde{\omega}, \omega \times \Omega \rightarrow Y \]
for some \( \tilde{\omega} > 0 \) and \( T \) is continuously Fréchet-differentiable with respect to \( \zeta \), where
\[ \partial_\zeta T(0,0) : X \rightarrow Y \]
is an isomorphism – the first two assertion follow from [8], Section 2 and the last assertion will be verified here in Section 3. It is also there that the symmetry of the perturbations plays a crucial role.
The definition of \( Y \) requires \( T(0,0) = 0 \) and therefore we subtracted the constant in (2.18). With assumption \((\phi_3)\), we know \( T(0,0) = 0 \), because \( g_0 = id \) and \( \text{supp}\rho_0 = \text{supp}h_0 \circ U_0 = B_1 \subset B_3 \). The implicit function theorem, cf. [1], Theorem 15.1, also stated in the Appendix as Theorem 4.1, cf. Section 4, now guarantees the existence of \( \omega_1 \in ]0,\tilde{\omega}[ \) and the existence of a continuous mapping
\[ ] - \omega_1, \omega_1[ \ni \omega \mapsto \zeta_\omega \in \Omega \]
such that
\[ T(\omega,\zeta_\omega) = 0, \quad \omega \in ]-\omega_1,\omega_1[ \]
and \( \zeta^0 = 0 \). We also will require that \( \omega^2 r^2 < E_1 \) in \( B_4 \), where \( E_1 \) is defined in Lemma 2.3 and therefore define
\[ \omega_0 := \min \left\{ \omega_1, \sqrt{\frac{|E_1|}{4}} \right\}. \quad (2.19) \]
Now let \( \zeta = \zeta_\omega \), where we choose a fixed \( \omega \in ]-\omega_0,\omega_0[ \) and define
\[ \rho_\zeta(x) := \tilde{h}(\omega, r(x), U_0(g_\zeta^{-1}(x))), \quad x \in B_3. \quad (2.20) \]
Then we have \( \rho_\zeta \in C_S(B_3) \cap C^1(\hat{B}_3) \). By Lemma 2.3, \( \rho_\zeta > 0 \) at most, if \( U_0(g_\zeta^{-1}(x)) < E_0 + E_1 \), which is equivalent to \( |g_\zeta^{-1}(x)| < 2 \) by Lemma 2.2. Consequently,
\[ \text{supp}\rho_\zeta = g_\zeta(B_2) \subset \hat{B}_3. \]
We extend \( \rho_\zeta \) to 0 to all of \( \mathbb{R}^3 \) and we achieve
\[ \rho_\zeta \in C_c(\mathbb{R}^3), \quad \text{supp}\rho_\zeta \subset \hat{B}_3. \]
We want equation (2.20) to hold everywhere, but we have not defined \( g_\zeta \) globally. We can rewrite \( T(\omega,\zeta) = 0 \) as
\[ U_0(x) = - \int_{B_3} \frac{\rho_\zeta(y)}{|g_\zeta(x) - y|} dy + C, \quad x \in B_4, \]
or
\[ U_0(g_\zeta^{-1}(x)) = - \int_{B_3} \frac{\rho_\zeta(y)}{|x - y|} dy + C, \quad x \in B_4, \zeta, \]
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where
\[ C := U_0(0) + \int_{B_3} \frac{\rho(y)}{|y|} \, dy. \]
Now define
\[ U_\zeta(x) := -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x - y|} \, dy + C. \]
Then we have \( U_\zeta \in C^1(\mathbb{R}^3) \) with
\[ U_\zeta(x) = U_0(g_\zeta^{-1}(x)), \quad x \in B_3 \subset B_{4,\zeta} \tag{2.21} \]
and thus \( \rho_\zeta \in C^1_\zeta(\mathbb{R}^3) \) and \( U_\zeta \in C^2_\rho(\mathbb{R}^3) \) with \( \Delta U_\zeta = 4\pi \rho_\zeta \) in \( \mathbb{R}^3 \). Furthermore,
\[ \Delta U_\zeta = 4\pi \tilde{h}(\omega, r(x), U_\zeta(x)), \quad x \in B_3 \subset B_{4,\zeta}. \tag{2.22} \]
The last equation holds even in \( \mathbb{R}^3 \). We have to show
\[ \rho_\zeta(x) = \tilde{h}(\omega, r(x), U_\zeta(x)), \quad x \in \mathbb{R}^3, \]
that is, \( U_\zeta(x) > E_0 + E_1 \) for \( x \in \mathbb{R}^3 \setminus g_\zeta(B_2) \). We know
\[ \Delta U_\zeta(x) = 0, \quad x \in \mathbb{R}^3 \setminus g_\zeta(B_2), \]
\lim_{|x| \to \infty} U_\zeta(x) = C \] and
\[ U_\zeta(x) = E_0 + E_1, \quad x \in \partial g_\zeta(B_2), \]
\[ U_\zeta(x) > E_0 + E_1, \quad x \in B_3 \setminus g_\zeta(B_2). \]
Here we used (2.21) and the monotonicity of \( U_0(|x|) \) with \( U_0(2) = E_0 + E_1 \). If \( C \leq E_0 + E_1 \), we have a contradiction to the maximum principle. Therefore, \( C > E_0 + E_1 \) and again by the maximum principle: \( U_\zeta > E_0 + E_1 \) on \( \mathbb{R}^3 \setminus g_\zeta(B_2) \) and consequently, (2.22) holds in \( \mathbb{R}^3 \).

Now define \( \rho^\omega := \rho_\zeta, \quad U^\omega := U_\zeta \) and
\[ f^\omega(x,v) := \begin{cases} \varphi(\frac{1}{2}v^2 + U^\omega(x) - \frac{1}{2}\omega^2 r^2), & \text{for } U^\omega(x) < E_0 + E_1 \\ 0, & \text{else} \end{cases} \]
\[ = \begin{cases} \varphi(\frac{1}{2}v^2 + U^\omega(x) - \frac{1}{2}\omega^2 r^2), & \text{for } |x| < 4 \\ 0, & \text{else} \end{cases}. \tag{2.23} \]
Now \( f^\omega \) defined by (2.23) solves the Vlasov equation (1.6) because it is constant along characteristics. More precisely, we have \( U_\zeta(x) - \frac{1}{2}\omega^2 r^2 > E_0 \) in a neighbourhood of \( \partial B_4 \), if we choose \( \omega_0 \) sufficiently small as in (2.19). If we then fix \( (x,v) \) with \( E_J(x,v) < E_0 \) and consider a characteristic \( (X,V) \) going through \( (x,v) \) we conclude that if \( x \in B_4 \), we have \( X \in B_4 \) for all time. On the other hand, if \( x \notin B_4 \), we have \( X \notin B_4 \) for all time.
Altogether, assertions (i)-(iii) of the theorem follow, except the non-spherical symmetry in the case $\omega \neq 0$. Choose $x \in \mathbb{R}^3$ with $\rho^\omega(x) > 0$, $x_1 := a \neq 0, x_2 = x_3 = 0$. Then there exists some $\eta \in \mathbb{R}^3$, such that

$$\frac{1}{2} \eta^2 + U^\omega(x) - \frac{1}{2} \omega^2 a^2 < E_0.$$ 

Now if $(f^\omega, U^\omega)$ were spherically symmetric, there would exist a rotation $A$ around the $x_2$-axis such that $(Ax)_1 = (Ax)_2 = 0$ and $f^\omega(Ax, Av) = f^\omega(x, v)$. But the monotonicity of $\varphi$ implies

$$f^\omega(x, v) = \varphi\left(\frac{1}{2} v^2 + U^\omega(x) - \frac{1}{2} \omega^2 a^2\right) = \varphi(E_J(x, v))$$

$$\neq \varphi(E_J(Ax, Av)) = \varphi\left(\frac{1}{2} v^2 + U^\omega(x)\right) = f^\omega(Ax, Av),$$

which contradicts our assumption of spherical symmetry. With a similar argument, one can also show that the constructed solutions cannot be axially symmetric with respect to any axis in $\mathbb{R}^3$ except for the $x_3$-axis. Though our deformations only have mirror symmetry with respect to every coordinate plane, which would match a triaxial system, we would still have to prove that the constructed $\zeta^\omega$ are not axially symmetric with respect to the $x_3$-axis to construct triaxial solutions.

The asserted continuity properties (iv) can be proved as follows: For $x \in B_3$ we have

$$|U^\omega(x) - U^\omega'(x)| \leq \|U_0\|_\infty \|g_\omega^{-1}(x) - g_\omega'^{-1}(x)\| \leq C \|\zeta^\omega - \zeta^\omega'\|_X.$$ 

By the implicit function theorem, $\zeta^\omega$ continuously depends on $\omega$ with respect to the $\|\cdot\|_X$-norm and we have $\rho^\omega(x) = \tilde{h}(\omega, r(x), U^\omega(x))$.

Lemma 2.3 implies that $\rho^\omega$ is continuous in $\omega$ with respect to $\|\cdot\|_\infty$ and

$$U^\omega(x) = -\int_{B_3} \frac{\rho^\omega(y)}{|x-y|} dy + U_0(0) + \int_{B_3} \frac{\rho^\omega(y)}{|y|} dy, \quad x \in \mathbb{R}^3$$

implies the continuity of $U^\omega$ in $\omega$ with respect to $\|\cdot\|_\infty$. Differentiating the above expression for $\rho^\omega$ yields the continuity of $\rho^\omega$ with respect to $\|\cdot\|_1$, and therefore also the continuity of $U^\omega$ in the norm $\|\cdot\|_2$.

3 $\partial\zeta T(0,0)$ is an isomorphism

In this section, we want to establish some of the assumptions needed for the implicit function theorem. We will prove the following result:

Proposition 3.1. The mapping $\partial\zeta T(0,0) : X \to Y$ is a linear isomorphism.
Let $\omega_2 := \sqrt{|E_1|}/4$, where $E_1$ is defined in Lemma 2.3 and let us recall from [8], Proposition 3.1 that the Fréchet-derivative of $T\colon -\omega_2, \omega_2[\times \Omega \rightarrow Y$ is given by

$$[\partial_\zeta T(\omega, \zeta)\Lambda](x) =$$

$$= \int_{B_3} \left( \frac{1}{|g_\zeta(x) - y|} - \frac{1}{|y|} \right) \partial_\zeta \tilde{h}(\omega, r(y), U_\zeta(y)) \nabla U_\zeta(y) \cdot \frac{g_\zeta^{-1}(y)}{|g_\zeta^{-1}(y)|} \Lambda(g_\zeta^{-1}(y)) dy$$

$$- \int_{B_3} \frac{g_\zeta(x) - y}{|g_\zeta(x) - y|^2} \tilde{h}(\omega, r(y), U_\zeta(y)) dy \cdot \frac{x}{|x|} \Lambda(x), \quad x \in B_3,$$

(3.24)

where $\omega \in [-\omega_2, \omega_2[, \zeta \in \Omega, \Lambda \in X$, and $U_\zeta(y) := U_0(g_\zeta^{-1}(y)), y \in B_3$

We abbreviate $L_0 \Lambda := \partial_\zeta T(0,0)\Lambda$ for $\Lambda \in X$. We observe that $g_0 = id$ and therefore the function $U_\zeta$ in (3.24) coincides with the potential $U_0$ of the spherically symmetric steady state we started with, if $\zeta = 0$. We have

$$\rho_0(|x|) = \partial_\zeta \tilde{h}(0, r(x), U_0(|x|)) U_0(|x|)$$

$$= \partial_\zeta \tilde{h}(0, r(x), U_0(|x|)) \nabla U_0(x) \cdot \frac{x}{|x|}, \quad x \in \mathbb{R}^3.$$

This implies

$$(L_0 \Lambda)(x) = - \int_{B_3} \left( \frac{1}{|x - y|} - \frac{1}{|y|} \right) \rho_0(|y|) \Lambda(y) dy - \int_{B_3} \frac{x - y}{|x - y|^2} \rho_0(|y|) dy \cdot \frac{x}{|x|} \Lambda(y)$$

$$= -U_0'(|x|) \Lambda(x) - \int_{B_3} \left( \frac{1}{|x - y|} - \frac{1}{|y|} \right) \rho_0(|y|) \Lambda(y) dy, \quad x \in B_4, \Lambda \in X.$$

Now let

$$(K\Lambda)(x) := - \frac{1}{U_0(|x|)} \int_{B_3} \left( \frac{1}{|x - y|} - \frac{1}{|y|} \right) \rho_0(|y|) \Lambda(y) dy, \quad x \in \mathbb{R}^3, \Lambda \in C_S(B_4).$$

Then we can write

$$(L_0 \Lambda)(x) = -U_0'(|x|)((id - K)\Lambda)(x), \quad x \in B_4, \Lambda \in X. \quad (3.25)$$

In order to prove Proposition 3.1 we need

**Lemma 3.2.** The linear operator $K : C_S(B_4) \rightarrow C_S(B_4)$ is compact, where $C_S(B_4)$ is equipped with the supremum norm $\| \cdot \|_\infty$.

**Proof.** For $\Lambda \in C_S(B_4)$ let

$$V_\Lambda(x) := - \int_{B_3} \frac{1}{|x - y|} \rho_0(|y|) \Lambda(y) dy, \quad x \in \mathbb{R}^3.$$
Then $V_\Lambda \in C^1(\mathbb{R}^3)$, $\nabla V_\Lambda(0) = 0$, and

$$(K\Lambda)(x) = \frac{1}{U_0'(|x|)}(V_\Lambda(x) - V_\Lambda(0)), \quad x \in \hat{B}_4.$$  

Using Lemma 2.2(c), we obtain the estimate

$$| (K\Lambda)(x) | \leq \frac{1}{C|x|} \| \nabla V_\Lambda \|_\infty |x| \leq C \| \Lambda \|_\infty, \quad x \in \hat{B}_4,$$

where the constant $C$ depends on $\rho_0$ and $U_0$, but not on $\Lambda$ or $x$. Thus $K$ maps bounded sets into bounded sets. We next show that $K\Lambda$ is Hölder continuous with exponent 1/2, uniformly on bounded sets in $C_S(B_4)$. Let $M > 0$ and assume $\| \Lambda \|_\infty \leq M$. In the following, constants denoted by $C$ depend on $\rho_0, U_0$ and $M$, but not on $\Lambda$. Obviously, $\rho_0\Lambda \in L^\infty(\mathbb{R}^3)$ and we deduce from Lemma 4.2 the existence of $C > 0$ with

$$| \nabla V_\Lambda(x) - \nabla V_\Lambda(x') | \leq C \| \rho_0\Lambda \|_\infty |x - x'|^{1/2}, \quad x, x' \in B_4$$

Since $\nabla V_\Lambda(0) = 0$, the latter implies

$$| \nabla V_\Lambda(x) | \leq C|x|^{1/2}, \quad x \in B_4.$$  

Now let $x, x' \in \hat{B}_4$ and $|x| \leq |x'|$. Then

$$|(K\Lambda)(x) - (K\Lambda)(x')| \leq \left| \frac{1}{U_0'(|x|)} - \frac{1}{U_0'(|x'|)} \right| |V_\Lambda(x) - V_\Lambda(0)| + \frac{1}{U_0'(|x'|)} |V_\Lambda(x) - V_\Lambda(x')| =: I_1 + I_2$$

and we obtain for some $z \in B_4$ with $|z| \leq |x'|$ the estimates

$$I_1 \leq \frac{|U_0'(|x|) - U_0'(|x'|)|}{|x| |x'|} |\nabla V_\Lambda(z)| |x| \leq C |x - x'|^{1/2} \frac{(|x| + |x'|)^{1/2}}{|x'|} |z|^{1/2} \leq C |x - x'|^{1/2},$$

and

$$I_2 \leq C \frac{|\nabla V_\Lambda(z)| |x - x'|}{|x|} \leq C \frac{|z|^{1/2} |x - x'|}{|x|} \leq C |x - x'|^{1/2},$$

so that

$$|(K\Lambda)(x)K\Lambda)(x')| \leq C |x - x'|^{1/2}, \quad x, x' \in \hat{B}_4$$

and

$$|(K\Lambda)(x)| \leq C |\nabla V_\Lambda(z)| \leq C |x|^{1/2}, \quad x \in \hat{B}_4.$$  

We have shown that $K$ maps bounded sets of $C_S(B_4)$ into bounded and equicontinuous subsets of $C_S(B_4)$. Thus $K$ is compact by the Arzelà-Ascoli theorem and the proof is complete. \qed
Lemma 3.3. $id - K : C_S(B_4) \to C_S(B_4)$ is one-to-one and onto.

Proof. Since $K$ is compact, it suffices to show that $id - K$ is one-to-one. Let $\Lambda \in C_S(B_4)$ with $\Lambda - K\Lambda = 0$. Now $\Lambda = 0$ can be shown by expanding $\Lambda$ into spherical harmonics. For that purpose, let

$$\{S_{n,j}, n \in \mathbb{N}, j = 1, \ldots, 2n + 1\}$$

be the orthonormal set of spherical harmonics introduced in the Appendix, cf. Section 4, where for $n \in \mathbb{N}$, the functions $S_{n,j} : \partial B_1 \to \mathbb{R}$, $j = 1, \ldots, 2n + 1$ are homogeneous polynomials of degree $n$. We define

$$\Lambda_{n,j}(r) := \int_{\partial B_1} S_{n,j}(\xi) \Lambda(\xi) d\omega_\xi = \frac{1}{r^2} \int_{\partial B_r} S_{n,j}(x/r) \Lambda(x) d\omega_x$$

and we use the expansion of the integral kernel $1/|x - y|$ into spherical harmonics, cf. Lemma 4.3 and Lemma 4.4. For that purpose, let $r, s \in \mathbb{R}^+$, $r \neq s$, we have

$$\frac{1}{|x - y|} = \max(r,s)^{-1} \sum_{n=0}^\infty \sum_{j=1}^{2n+1} \frac{4\pi}{2n+1} \left( \frac{\min(r,s)}{\max(r,s)} \right)^n S_{n,j}(\xi) S_{n,j}(\eta).$$

$K\Lambda - \Lambda = 0$ then implies

$$\Lambda_{n,j}(r) = \int_{\partial B_1} \int_{\partial B_1} \left( \frac{1}{|r\xi - y|} - \frac{1}{|y|} \right) S_{n,j}(\xi) d\omega_\xi \rho_0(|y|) \Lambda(y) dy$$

$$= -\frac{4\pi}{2n+1} \frac{1}{U_0'(r)} \int_0^3 s^2 \rho_0(s) \frac{\min(r,s)}{\max(r,s)^n+1} \int_{\partial B_1} S_{n,j}(\eta) \Lambda(\eta) d\omega_\eta ds$$

$$+ \frac{4\pi}{2n+1} \frac{1}{U_0'(r)} \int_0^3 s^2 \rho_0(s) \frac{\min(r,s)}{\max(r,s)^n+1} \int_{\partial B_1} S_{n,j}(\eta) \Lambda(\eta) d\omega_\eta ds$$

$$= -\frac{4\pi}{2n+1} \frac{1}{U_0'(r)} \int_0^3 s^2 \rho_0(s) \left( \frac{\min(r,s)}{\max(r,s)^n+1} - \frac{0^n}{s^{n+1}} \right) \Lambda_{n,j}(s) ds,$$

where we used that the functions $S_{n,j}$ are orthonormal with respect to $\langle \ldots \rangle_{L^2(\partial B_1)}$. We find that

$$\Lambda_{01}(r) = -\frac{4\pi}{r U_0'(r)} \int_0^r \rho_0(s) s(s-r) \Lambda_{01}(s) ds$$

and we obviously have $\lim_{r \to 0} \Lambda_{01}(r) = 0$. Let $R \geq 0$ be maximal such that $\Lambda_{01}(r)$ vanishes on $[0,R]$. Then for $r \in [R,3]$,

$$|\Lambda_{01}(r)| \leq \frac{4\pi}{r U_0'(r)} \|\rho_0\|_\infty \sup_{0 \leq s \leq r} |\Lambda_{01}(s)| \int_0^r s(s-r) ds \leq C(r-R) \sup_{0 \leq s \leq r} |\Lambda_{01}(s)|.$$
where we made the transformation $\xi \mapsto (-\xi_1, \xi_2, \xi_3)$. Analogously,
\[
\int_{\partial B_1} \xi_2 \Lambda(\xi) d\omega_\xi = \int_{\partial B_1} \xi_3 \Lambda(\xi) d\omega_\xi = 0,
\]
and we have $\Lambda_{11} = \Lambda_{12} = \Lambda_{13} \equiv 0$. Let $n \geq 2$. Then
\[
\Lambda_{nj}(r) = -\frac{4\pi}{2n+1} \frac{1}{U_0'(r)} \left( \int_0^r s^2 \rho_0(s) \frac{s^n}{r^{n+1}} \Lambda_{nj}(s) ds + \int_r^3 s^2 \rho_0(s) \frac{s^n}{s^{n+1}} \Lambda_{nj}(s) ds \right),
\]
and
\[
|\Lambda_{nj}(r)| \leq \frac{4\pi}{2n+1} \frac{1}{U_0'(r)} ||\Lambda_{nj}||_\infty \left( \frac{1}{r^2} \int_0^r (-\rho_0'(s)) \frac{s^{n-1}}{r^{n-1}} s^3 ds + r \int_r^3 (-\rho_0'(s)) \frac{s^{n-1}}{s^{n-1}} ds \right)
\]
\[
\leq \frac{4\pi}{2n+1} \frac{1}{U_0'(r)} ||\Lambda_{nj}||_\infty \left( \frac{1}{r^2} \int_0^r (-\rho_0'(s)) s^3 ds + r \int_r^3 (-\rho_0'(s)) ds \right)
\]
\[
= \frac{4\pi}{2n+1} \frac{1}{U_0'(r)} ||\Lambda_{nj}||_\infty \left( \frac{1}{r^2} r^3 (-\rho_0)(r) + \frac{3}{r} \int_0^r s^2 \rho_0(s) ds + r \rho_0(r) \right)
\]
\[
= \frac{3}{2n+1} ||\Lambda_{nj}||_\infty,
\]
where we integrated by parts in the third line and used the fact that $U_0'(r) = \frac{4\pi}{2n+1} \int_0^r s^2 \rho_0(s) ds$ in the last line, also recall from (2.13) that $-\rho_0'(r) \geq 0$.

Now $2n+1 > 3$ for $n \geq 2$ implies that $\Lambda_{nj} \equiv 0$ for $n \geq 2$ as well and the completeness of $\{S_{nj}\}$ induces $\Lambda \equiv 0$. We conclude that $id - K$ is one-to-one as claimed. \[ \square \]

It is now clear that $L_0 : X \to Y$ is one-to-one as well – this follows from Eq. (3.24) and the fact that $U_0'(r) > 0$ for $r > 0$. So once we have proved the next lemma, the proof of Proposition 3.1 will be complete.

**Lemma 3.4.** $L_0 : X \to Y$ is onto.

**Proof.** Let $g \in Y$ and define $q := g/U_0'$. We will show $q \in X$. We have $q \in C^1(B_4) \cap C(S(B_4))$ and
\[
|\nabla q| \leq \frac{|\nabla g(x)|}{U_0'(|x|)} + |g(x)| \left| \frac{U_0''(|x|)}{U_0'(|x|)^2} \frac{x}{|x|} \right| \leq C \left( \frac{|\nabla g(x)|}{|x|} + \frac{|g(x)|}{|x|^2} \right) \leq 2C||g||_Y.
\]

By definition of $Y$ and since $U_0 \in C^2([0, \infty[)$ with $U_0''(0) > 0$ we have that for every $x \in \partial B_4$,
\[
\nabla q(tx) = \frac{\nabla g(tx)}{U_0'(t)} \frac{t}{U_0'(t)} - \frac{g(tx)}{t^2} \frac{U_0''(t)}{U_0'(t)^2} \frac{x}{|x|^2}
\]
\[
= - \frac{\nabla g(0x)}{U_0''(0)} \frac{1}{U_0''(0)} \frac{g(0x)}{0^2} \frac{U_0''(0)}{U_0'(0)^2} \frac{x}{|x|^2},
\]
as $t \to 0+$, uniformly in $x \in \partial B_4$.  

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Since $X \subset C_S(B_4)$, there exists by Lemma 3.3 an element $\Lambda \in C_S(B_4)$ such that
\[
\Lambda - K\Lambda = -q = -\frac{g}{U_0'}.
\]
This implies that $L_0\Lambda = g$ and thus that $L_0$ is onto, provided $\Lambda \in X$. To see the latter we observe that $\Lambda = K\Lambda - q$ is Hölder continuous since $K\Lambda$ is Hölder continuous. If we now define $V_\Lambda$ as above in the proof of Lemma 3.2 we also conclude that $V_\Lambda \in C^2(\mathbb{R}^3)$ and thus $K\Lambda \subset C^1(\mathbb{R}^4)$. Denoting by $H_{V_\Lambda}$ the Hessian of $V_\Lambda$ we obtain for each $x \in B_4$ a point $z \in \mathbb{R}^3$ such that
\[
\begin{aligned}
|\nabla (K\Lambda)(x)| &\leq \left| \frac{U_0''(|x|)}{U_0(|x|)^2} \right| |V_\Lambda(x) - V_\Lambda(0)| + \frac{1}{U_0'(|x|)}|\nabla V_\Lambda(x)| \\
&\leq \frac{C}{|x|^2}|(H_{V_\Lambda}(z)x,x)| + C |\nabla V_\Lambda(x)| \leq C\|D^2V_\Lambda\|_\infty.
\end{aligned}
\]

Finally, for $x \in \partial B_1$, we have
\[
\begin{aligned}
\nabla (K\Lambda)(tx) &= -\frac{U_0''(t)}{U_0(t)^2} x (V_\Lambda(tx) - V_\Lambda(0)) + \frac{1}{U_0(t)} \nabla V_\Lambda(tx) \\
&= -\frac{U_0''(t)}{U_0(t)^2} x 1 \frac{1}{t^2} \frac{1}{2} (H_{V_\Lambda}(tx)x,tx) + \frac{t}{U_0(t)} \nabla V_\Lambda(tx) \\
&= -\frac{U_0''(t)}{2U_0(0)} (H_{V_\Lambda}(0)x,x) x + \frac{1}{U_0(0)} D^2V_\Lambda(0)x,
\end{aligned}
\]
as $t \to 0^+$, uniformly in $x \in \partial B_1$. We have shown that $K\Lambda \in X$ and this implies $\Lambda = K\Lambda + q \in X$ and the proof is complete. $\square$

4 Appendix

In this section, we firstly state the implicit function theorem which is used for the proof of Theorem 2.1. Then we give a regularity result for the Poisson equation and finally introduce spherical harmonics and state two important lemmas: an addition theorem and the expansion of the integral kernel $1/|x-y|$ in spherical harmonics.

**Theorem 4.1.** Let $X,Y,Z$ be Banach spaces, $U \subset X$ and $V \subset Y$ neigbourhoods of $x_0 \in X$ and $y_0 \in Y$ respectively, $F: U \times V \to Z$ continuous and continuously Fréchet-differentiable with respect to the second variable. Suppose also that $F(x_0,y_0) = 0$ and $F_{y_0}^{-1}(x_0,y_0) \in \mathcal{L}(Z,Y)$. Then there exist balls $B_r(x_0) \subset U$, $B_\delta(y_0) \subset V$ and exactly one continuous map $G: B_r(x_0) \to B_\delta(y_0)$ such that $Gx_0 = y_0$ and $F(x,Gx) = 0$ on $B_r(x_0)$.

**Proof.** [1], Theorem 15.1. $\square$

**Lemma 4.2.** Let $n < p \leq \infty$ and let $\rho(x) \in L^p(\mathbb{R}^n)$ with compact support. Define
\[
V_\rho(x) := -\int_{\mathbb{R}^n} \frac{1}{|x-y|} \rho(y) \, dy.
\]
Then for every $0 < \alpha < 1 - n/p$ we have $V_\rho \in C^{1,\alpha}(\mathbb{R}^n)$ and
\[
|\partial_i V_\rho(x) - \partial_i V_\rho(x')| \leq C(n, \alpha, p)|x' - x|^\alpha \|f\|_p\mathcal{L}^n(\text{supp}\{\rho\}) \frac{1}{(1 + |x'|)^{\frac{n}{p} - \frac{1}{p}}}
\]

Proof. \cite{6}, Theorem 10.2.

\[\square\]

Some facts about spherical harmonics

In the following, we use the notation of \cite{7} and we will always consider the case, where the space dimension $q$ is equal to 3. For $n \in \mathbb{N}$, consider a homogeneous polynomial $H_n$ of degree $n$, which satisfies
\[
\Delta H_n(x) = 0.
\]
Then for $\xi \in \partial B_1 := \{x \in \mathbb{R}^3||x| = 1\}$,
\[
S_n(\xi) := H_n(\xi)
\]
is called a spherical harmonic of order $n$. For each $n$, there exist $2n+1$ linearly independent spherical harmonics, which we call $S_{n,j}, \ j = 1, \ldots, 2n+1$, cf. \cite{7}, Lemma 4. We denote by \(\{S_{n,j}, \ n = 0, \ldots, \infty, \ j = 1, \ldots, 2n+1\}\) the orthonormal set of all spherical harmonics, where we orthonormalize with respect to $\langle \ldots \rangle_{L^2(\partial B_1)}$. Then we have the following

Lemma 4.3. For a fixed $n \in \mathbb{N}$ and $\xi, \eta \in \partial B_1$, we have
\[
\sum_{j=1}^{2n+1} S_{n,j}(\xi)S_{n,j}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta),
\]
where $P_n(t)$ is the Legendre Polynomial of degree $n$.

Lemma 4.4. Let $x, y \in \mathbb{R}^3$ with $x = R\xi, \ y = r\eta$, for suitable $\xi, \eta \in \partial B_1$ and $r, R \in \mathbb{R}$. Then we have for $R > r$
\[
\frac{1}{|x-y|} = R^{-1}\sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n P_n(\xi \cdot \eta),
\]
and for $R < r$
\[
\frac{1}{|x-y|} = r^{-1}\sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^n P_n(\xi \cdot \eta),
\]
where $P_n(t)$ is the Legendre Polynomial of degree $n$.

Proofs can be found in \cite{7}, Theorem 2 and Lemma 19.

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