On a Hadamard fractional boundary value problem for $3 < \alpha \leq 4$.

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Abstract. This note deals with Hadamard fractional differential equation of order $3 < \alpha \leq 4$, subject to a mixed boundary conditions on $[1, e]$. The investigation made here involves Hadamard integral operator of a function with a construction of an appropriate Green's function. Using its properties as well as its maximum value, we will be able to get Hartman-Winter and Lyapunov-type inequalities for a class of Hadamard fractional differential equations. Finally, we will illustrate this result by giving an example.

1. Introduction

The first result appeared in a class of fractional differential equations is due to Lyapunov [1]. Assuming that $u$ is a solution of

\[
\begin{cases}
 u''(t) + q(t)u(t) = 0, & a < t < b, \\
 u(a) = u(b) = 0
\end{cases}
\]

where $q$ defined on $[a, b]$ with real values, is supposed to be continuous. It was proved in [1] that $q$ must realize the following integral inequality

\[
\int_a^b |q(s)|ds > \frac{4}{(b-a)^2}, \quad a < t < b,
\]

in order to guarantee that the non-trivial solution exists.

This condition (2) is essential and was extend to more general fractional differential equations with fractional orders. It is known as Lyapunov's inequality. Depending on each type of fractional derivative, for such type of inequalities, a lot of papers are derived for both existence and nonexistence solutions of fractional boundary value problems. Unlikely, for a fractional Hadamard differential equation with different fractional orders, analogous Lyapunov-type inequalities are not shown a lot.

The result that we would like to state and prove in this paper, is the Hadamard fractional differential equation subject to a mixed boundary conditions on $[1, e]$ with fractional order $3 < \alpha \leq 4$. Following [1], Ferreira in [2] improved two different results. The first one focused on existence of $u$ satisfying
\[ (_{a}D^{\alpha}u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2, \]
\[ u(a) = u(b) = 0, \]

where \( a \) and \( b \) are two real constants and \( q \) is a real function supposed to be continuous on \([a, b]\). This fractional boundary value problem admits a non-trivial solution provided that the following inequality
\[ \int_{a}^{b} |q(s)| ds > \Gamma(\alpha) \left( \frac{4}{(b-a)^{\alpha-1}} \right) \]
is realized.

However, in [3], the situation is different for the following fractional problem
\[ (_{a}D^{\alpha}u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2. \]
\[ u(a) = u(b) = 0, \]

Basically, this is due to the variation of the constructed Green’s function for each appropriate boundary value problem.

As condition of existence, Ferreira [3] got the following one
\[ \int_{a}^{b} |q(s)| ds > \Gamma(\alpha) a^{\alpha-1} (b-a)^{\alpha-1}. \]

For a fractional calculus material, we refer the reader to ([4],[5],[6]) and the references therein.

In this paper, we consider the following Hadamard high order fractional boundary value problem
\[ \begin{aligned}
\left( D_{a, \log}^{\alpha}u \right)(t) + q(t)u(t) &= 0, \quad 1 < t < e, \\
u(1) = u'(1) = u''(1) = 0, u'''(e) &= 0
\end{aligned} \tag{3} \]

where \( 3 < \alpha \leq 4 \). In (3), the derivative operator \( D_{a, \log}^{\alpha} \) stands for left Hadamard fractional derivative of order \( \alpha \) and \( q \) is a continuous function on \([1, e]\) to \( \mathbb{R} \).

We aim to get Hartman-Winter and Lyapunov-type inequalities for problem (3).

**Theorem 1.1** Let \( u \) be a non-trivial solution of the following fractional boundary value problem
\[ \begin{aligned}
\left( D_{a, \log}^{\alpha}u \right)(t) + q(t)u(t) &= 0, \quad 1 < t < e, \\
u(1) = u'(1) = u''(1) = 0, u'''(e) &= 0
\end{aligned} \tag{4} \]

where \( 3 < \alpha \leq 4 \), and \( q \) is a real and continuous function on \([1, e]\). Then (4) admits a non-trivial solution \( u \) if
\[ \int_{1}^{e} |q(s)| \frac{ds}{s} > 2\Gamma(\alpha) \tag{5} \]
is satisfied.

As auxiliary results, the following definitions and lemmas are necessary for the proof of Theorem 1.1.

**2. Definitions and lemmas:**

**Definition 2.1** The Hadamard fractional integral of a function \( g \in AC^{n}([a, b]) \) is defined by
\[ l_{a}^{a} \ g(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \frac{g(s)}{(log t - log s)^{1-\alpha}} \frac{ds}{s}, \]
Definition 2.2 Let \( a \geq 0 \), and \( n = [a] + 1 \). If \( f \in AC^n([a, b]) \) then the Hadamard fractional derivative of order \( \alpha \) of \( g \) defined by

\[
\frac{d^\alpha}{dt^\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(\log t - \log s)^{n-1}} \, ds,
\]

exists almost everywhere on \([a, b]\) (\([a]\) is the entire part of \( a \)).

Lemma 2.1 [4] For \( \alpha > 0 \); \( f(t) \in C(0,1) \), the homogeneous fractional differential equation

\[
\left( D_{a,\log}^\alpha u(t) \right) + q(t)u(t) = 0
\]

has a solution

\[
f(t) = c_1(\log(t) - \log(a))^{\alpha - 1} + c_2(\log(t) - \log(a))^{\alpha - 2} + \ldots + c_n(\log(t) - \log(a))^{n-1}
\]

where, \( c_i \in \mathbb{R} \), \( i = 1, \ldots, n \), and \( n = [\alpha] + 1 \), (\( \alpha \) non-integer).

Definition 2.3 [4] Let \( f \in L^1((a, b); \mathbb{R}) \). The fractional Hadamard integral of order \( \alpha > 0 \) of \( f \) is defined by

\[
\int_{a,\log}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(\log(t) - \log(s))^{\alpha-1}} \, ds.
\]

Definition 2.4 [4] Let \( \alpha > 0 \), and \( n \) be the smallest integer greater than or equal to \( \alpha \). Let \( f : [a, b] \rightarrow \mathbb{R} \) be a function such that \( D_{a,\log}^\alpha f \) exists almost everywhere on \([a, b]\). In this case, the Hadamard fractional derivative of order \( \alpha \) of \( f \) is defined by

\[
D_{a,\log}^\alpha f(t) = \frac{d^n}{dt^n} \int_{a,\log}^{\alpha-a} f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t \frac{g^{(n)}(s)}{(\log(t) - \log(s))^{n-1}} \, ds
\]

for \( a.e \ t \in [a, b] \).

3. Estimate of the Green’s function and Results:

In this section, we look to the maximum of \( G \) and therefore improve the two inequalities: Hartman-Winter and -type inequalities.

Lemma 3.1 Let \( 3 < \alpha \leq 4 \), \( q \in C([1, e]) \). Then the unique nontrivial solution of the fractional boundary value problem

\[
\left( D_{1,\log}^\alpha u(t) \right) + q(t)u(t) = 0, \quad 1 < t < e, \quad (6)
\]

\[
u(1) = u(e) = 0 \quad (7)
\]

is given by
\[
u(t) = \frac{1}{\Gamma(\infty)} \int_t^e G(t,s)u(s)q(s) \frac{ds}{s} + \frac{1}{\Gamma(\infty)} \int_t^e G(t,s)u(s)q(s) \frac{ds}{s},
\]

where the Green function \( G(t,s) \) is defined by
\[
\Gamma(\alpha)G(t,s) = \begin{cases} 
(\log(t))^{\alpha-1}(1 - \log(s))^{\alpha-3} - (\log(t) - \log(s))^{\alpha-1} & , 1 \leq s \leq t, \\
(\log(t))^{\alpha-1}(1 - \log(s))^{\alpha-3} & , t \leq s \leq b. 
\end{cases} \tag{8}
\]

For the forthcoming analysis, we bound the Green’s function \( G \) and approximate its value. The next lemma is devoted to solving this issue.

\textbf{Lemma 3.2} Assume that \( u \) is a non-trivial solution of the Hadamard fractional boundary value problem
\[
(D_{1+\log}^\alpha u)(t) + q(t)u(t) = 0, \quad 1 < t < e, \quad 3 < \alpha \leq 4 \tag{9}
\]
\[
u(1) = u'(1) = 0. \tag{10}
\]

Then the Green’s function \( G \) defined in (8) is positive and realizes the following inequality for all \((t, s) \in [1, e] \times [1, e]\)
\[
0 \leq G(t, s) \leq G(e, s),
\]

where \( G(e, s) \) is defined by \( G(e, s) := \frac{(\log(s))^{\alpha-2}}{2} \Gamma(\alpha), \quad t, s \in [1, e]. \)

\textbf{Theorem 3.1} Let \( u \) be a non-trivial solution of
\[
(D_{1+\log}^\alpha u)(t) + q(t)u(t) = 0, \quad 1 < t < e, \\
u(1) = u'(1) = u''(1) = 0, u''(e) = 0, \tag{11}
\]

where \( 3 < \alpha \leq 4, \) and \( q \) is a real and continuous function on \([1, e]\). Then the following inequality is satisfied
\[
\int_1^e (\log(s))\big(1 - \log(s)\big)^{\alpha-3} \frac{(\log(s))^{\alpha-1}}{t} \geq \Gamma(\alpha),
\]

where \( t, s \in [1, e]. \)

One may observe that the conclusion of Theorem 3.1 represents the Hartman-Winter inequality for a Hadamard boundary value problem.

\textbf{Theorem 3.2} Let \( u \) be a non-trivial solution of
\[
(D_{1+\log}^\alpha u)(t) + q(t)u(t) = 0, \quad 1 < t < e, \\
u(1) = u'(1) = u''(1) = 0, u''(e) = 0. \tag{12}
\]

where \( 3 < \alpha \leq 4, \) and \( q \) is a real and continuous function on \([1, e]\) to \( R. \) Then the fractional boundary value problem (12) admits a non-trivial solution if the following inequality is satisfied
\[
\int_1^e \frac{(\log(s))(1 - \log(s))^{\alpha - 3}(\log(s))}{\Gamma(\alpha)} \frac{q(s)}{s} ds \geq \Gamma(\alpha),
\]
where \(t, s \in [1, e]\).

The next theorem expresses the necessary condition of existence of non-trivial solutions of (4), namely the Lyapunov’s inequality.

**Theorem 3.3** Assume \(u\) satisfies the fractional boundary value problem (3), then \(u\) is a non-trivial solution if

\[
\int_1^e |q(s)| \frac{ds}{s} > 2\Gamma(\alpha)
\]

(13)

The proof is straightforward since

\[
\max_{s \in [1, e]} G(e, s) = \frac{(\log e - \log 1)^{\alpha + 1}}{2^\alpha} = \frac{1}{2}
\]

4. **Application:**

This section deals with one example focusing on an eigenvalue fractional boundary value problem. Let us consider

\[
\left( D_{1.5}^{\alpha, 2} \log \right)(t) + \lambda u(t) = 0, \quad 1.5 < t < 2,
\]

\[
u(1.5) = u'(1.5) = u''(1.5) = 0, u''(2) = 0,
\]

(14)

where \(3 < \alpha \leq 4\).

When we argue on \([1.5, 2] \subset [1, e]\), and set \(q(t) = \lambda\), we obtain directly the result by applying Theorem 3.2,

\[
|\lambda| \geq 2\Gamma(\alpha).
\]

(15)

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