GLOBAL ATTRACTORS
OF IMPULSIVE PARABOLIC INCLUSIONS

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Abstract. In this work we consider an impulsive multi-valued dynamical system generated by a parabolic inclusion with upper semicontinuous right-hand side $\varepsilon F(y)$ and with impulsive multi-valued perturbations. Moments of impulses are not fixed and defined by moments of intersection of solutions with some subset of the phase space. We prove that for sufficiently small value of the parameter $\varepsilon > 0$ this system has a global attractor.

1. Introduction. An autonomous evolution system is called impulsive (or discontinuous) dynamical system (DS) if its trajectories have jumps at moments of intersection with certain surface of the phase space. Unlike systems with impulses at fixed moments of time [23], the behavior of impulsive DS is far from complete understanding. Some aspects of the qualitative behavior of impulsive DS have been studied by many authors in recent years [23], [22], [18], [19], [7], [1], [3], [21], [9]. Focus of investigation was on stability of solutions and on properties of $\omega$-limit sets for discontinuous DS generated by impulsive systems of ordinary differential equations. Recently in [4], [5] several concepts of global attractor were proposed and an application to a scalar reaction-diffusion equation with impulsive perturbation was given. However, it should be noted that in all papers the authors used detailed information about the nature of intersection of a given set by trajectories of the system. These conditions are formulated in an abstract form and cannot be effectively tested without explicit formulas of solutions. In this paper we propose a different concept of global attractor. This concept is based on the definition of uniform attractor for non-autonomous DS [6], [15]. In particular, this definition of attractor is used for systems with impulsive perturbation at fixed moments of time [11]. From this point of view we consider a multi-valued situation when there exists more than one solution for a given initial data. The theory of global attractors of multi-valued DS has many interesting results (see [26] and references there in). Its applications to...
parabolic inclusions and parabolic equations without uniqueness firstly appeared in the works of V.S. Melnik and his pupils [20], [12], [13], [17], [16].

Using the theory of global attractors we construct an abstract multi-valued impulsive DS and prove a result about existence of global attractors for it. We apply the obtained results to describe asymptotic dynamics of a wide class of dissipative infinite-dimensional multi-valued DS, generated by parabolic inclusion with impulsive perturbations at non-fixed moments of time. In particular, we give effective sufficient conditions for existence of global attractors.

The paper is organized in the following way. In Section 2 we give some abstract results concerning the existence and properties of global attractors of multi-valued impulsive dynamical systems. Main results are contained in Section 3 where impulsive parabolic inclusions are considered. The key moment is the presence of a small parameter in the right-hand part of the inclusion. This allows us to use the corresponding linear problem and, as a result, prove the theorem about existence of global attractor for the original nonlinear problem.

2. Multi-valued impulsive dynamical systems. In this section we introduce a concept of impulsive multi-valued dynamical system (impulsive MDS for short). We also discuss a notion of global attractor for such systems.

Let \((X, \rho)\) be a metric space, \(\mathcal{P}(X) = \beta(X)\) be a set of all nonempty (nonempty bounded) subsets of \(X\), for \(A, B \subseteq X\)

\[
\text{dist}_X(A, B) := \sup_{y \in A} \inf_{z \in B} \rho(y, z).
\]

**Definition 2.1.** [20] The multi-valued map \(G : \mathbb{R}_+ \times X \to \mathcal{P}(X)\) is called a multi-valued DS (MDS) if

1) \(\forall x \in X \quad G(0, x) = x;\)
2) \(\forall x \in X \quad \forall t, s \geq 0 \quad G(t + s, x) \subseteq G(t, G(s, x)).\)

**Definition 2.2.** [11], [26] A subset \(\Theta \subset X\) is called a global attractor of MDS \(G\) if

1) \(\Theta\) is compact;
2) \(\Theta\) is uniformly attracting set, i.e.,

\[
\forall B \in \beta(X) \quad \text{dist}_X(G(t, B), \Theta) \to 0, \quad t \to \infty;
\]
3) \(\Theta\) is minimal among closed uniformly attracting sets.

**Remark 1.** In Definition of MDS we assume no conditions of continuity for the map \(x \to G(t, x)\). If the MDS \(G\) also has global attractor in the classical sense [20], i.e. if there exists a compact uniformly attracting set \(\Theta_1 \subset X\) and \(\forall t \geq 0 \quad \Theta_1 \subset G(t, \Theta_1)\), then \(\Theta = \Theta_1\).

The following result provides a criterion of existence of global attractors for dissipative MDS.

**Lemma 2.3.** [11], [26] Suppose the MDS \(G\) satisfies the dissipativity condition

\[
\exists B_0 \in \beta(X) \quad \forall B \in \beta(X) \quad \exists T = T(B) > 0 \quad \forall t \geq T \quad G(t, B) \subset B_0,
\]

then the following conditions are equivalent:

1) \(MDS ~G\) has a global attractor \(\Theta;\)
2) \(MDS ~G\) is asymptotically compact, i.e.,

\[
\forall t_n \not\to \infty \quad \forall B \in \beta(X) \quad \forall \xi_n \in G(t_n, B) \text{ the sequence } \{\xi_n\} \text{ is precompact.}
\]
Moreover, under condition (1)

$$\Theta = \omega(B_0) := \bigcap_{\tau > 0} \bigcup_{t \geq \tau} G(t, B).$$

(3)

Now we will define impulsive MDS. To do this we introduce a set $K$ of continuous maps $\phi : [0, +\infty) \to X$ satisfying the following properties:

K1) $\forall x \in X \ \exists \phi \in K : \phi(0) = x$;

K2) $\forall \phi \in K \ \forall s \geq 0 \ \phi(\cdot + s) \in K$.

We define

$$K_x = \{\phi \in K | \phi(0) = x\}, \ G(t, x) = \{\phi(t) | \phi \in K_x\}.$$

It is easy to prove that $G$ is an MDS [10].

An impulsive MDS $G$ consists of the set $K$, a closed set $M \subset X$ and a multi-valued map $I : M \to P(X)$. The set $M$ is called the impulsive set and the map $I$ is called the impulsive map.

For constructing impulsive trajectories we use arguments with slight modifications from [11, 13]. We assume the following conditions hold:

$$M \cap I(M) = \emptyset;$$

(4)

$$\forall x \in M \ \forall \phi \in K_x \ \exists t = \tau(\phi) > 0 \ \forall t \in (0, \tau) \ \phi(t) \notin M.$$  

(5)

We define

$$M^+(\phi) = \bigcup_{t > 0} \phi(t) \cap M.$$

It follows from the continuity of $\phi$ and [13] that for every $\phi \in K$ if $M^+(\phi) \neq \emptyset$, then there exists $s := \Delta(\phi) > 0$ such that

$$\begin{cases}
\phi(t) \notin M \ \forall t \in (0, s); \\
\phi(s) \in M.
\end{cases}$$

Now we are ready to construct an impulsive trajectory $\tilde{\phi}$ with initial point $x_0 \in X$.

Consider $\phi_0 \in K_{x_0}$.

If $M^+(\phi_0) = \emptyset$, then $\tilde{\phi}(t) = \phi_0(t) \ \forall t \geq 0$.

If $M^+(\phi_0) \neq \emptyset$, then for $s_0 := \Delta(\phi_0) > 0$, $x_1 := \phi_0(s_0) \in M$ and for arbitrary $x_1^+ \in Ix_1$ we define $\tilde{\phi}$ on $[0, s_0]$ by

$$\tilde{\phi}(t) = \begin{cases}
\phi_0(t), & t \in [0, s_0) \\
x_1^+, & t = s_0.
\end{cases}$$

Consider $\phi_1 \in K_{x_1^+}$.

If $M^+(\phi_1) = \emptyset$, then $\tilde{\phi}(t) = \phi_1(t - s_0) \ \forall t \geq s_0$.

If $M^+(\phi_1) \neq \emptyset$, then for $s_1 := \Delta(\phi_1) > 0$, $x_2 := \phi_1(s_1) \in M$ and for arbitrary $x_2^+ \in Ix_2$ we define $\tilde{\phi}$ on $[s_0, s_0 + s_1]$ by

$$\tilde{\phi}(t) = \begin{cases}
\phi_1(t - s_0), & t \in [s_0, s_0 + s_1) \\
x_2^+, & t = s_0 + s_1.
\end{cases}$$

Arguing inductively we obtain an impulsive trajectory $\tilde{\phi}$ with finite or infinite number of impulsive points $\{x_n^+\}_{n \geq 1} \subset X$, the corresponding moments of time $\{s_n\}_{n \geq 0} \subset (0, +\infty)$ and the functions $\{\phi_n\}_{n \geq 0} \subset K$. We shall denote it by

$$\tilde{\phi} = \tilde{\phi}([x_n^+], \{s_n\}, \{\phi_n\})$$
We also put
\[ t_0 := 0, \quad t_{n+1} := \sum_{k=0}^{n} s_k, \quad \tilde{\varphi}(t_{n+1} - 0) := \varphi_n(t_{n+1} - t_n) \in M, \quad n \geq 0. \]

If \( \tilde{\varphi} \) has infinite number of impulses, then \( \forall n \geq 0 \quad \forall t \in [t_n, t_{n+1}] \)
\[
\tilde{\varphi}(t) = \begin{cases} 
\varphi_n(t - t_n), & t \in [t_n, t_{n+1}); \\
x_{n+1}^+, & t = t_{n+1}.
\end{cases}
\]

By \( \tilde{K}_x \) denote the set of all impulsive trajectories with initial point \( x \in X \). We assume that all impulsive trajectories are defined on \([0, +\infty)\), i.e., the following global existence condition holds:
\[
\forall x \in X \quad \text{every} \quad \tilde{\varphi} \in \tilde{K}_x \quad \text{is defined on} \quad [0, +\infty). \tag{6}
\]

The condition (6) means that for every impulsive trajectory the number of its impulsive points is either no more than finite or \( \sum_{k=0}^{\infty} s_k = \infty \).

We define impulsive MDS as a map \( \tilde{G} : \mathbb{R}_+ \times X \to P(X) \) such that
\[
\forall x \in X \quad \forall t \geq 0 \quad \tilde{G}(t, x) = \{ \tilde{\varphi}(t) \mid \tilde{\varphi} \in \tilde{K}_x \} \tag{7}
\]

Note that in the single-valued case (7) defines a classical semigroup [3].

**Lemma 2.4.** Formula (7) defines a MDS.

**Proof.** If \( \varphi \in K_x \) and \( \forall t \in (0, T) \quad \varphi(t) \notin M \), then \( \forall t \in (0, T) \quad \varphi(t) \in \tilde{G}(t, x) \).

Suppose \( \xi \in \tilde{G}(t + s, x), \quad t, s \geq 0 \). Then there exists \( \hat{\varphi} = \tilde{\varphi}(\{x_n^+\}, \{s_n\}, \{\varphi_n\}) \in \tilde{K}_x \) such that \( \xi = \tilde{\varphi}(t + s) \). We consider only the case \( \sum_{n=0}^{\infty} s_n = \infty \). The proof in case of finite number of impulsive points is the same.

Let \( t \in [t_n, t_{n+1}), \quad n \geq 0 \), where \( t_n \) is defined as previously. We consider two cases: \( t + s < t_{n+1} \) or \( t + s \geq t_{n+1} \).

If \( t + s < t_{n+1} \), then \( \xi = \tilde{\varphi}(t+s) = \varphi_n(t+s-t_n) \). Consider the following function
\[
\psi(p) = \varphi_n(p + t - t_n), \quad p \geq 0.
\]

Then \( \psi \in K_{\tilde{\varphi}(t)} \) and \( \forall p \in (0, t_{n+1} - t) \quad \psi(p) \notin M \). So
\[
\xi = \psi(s) \in \tilde{G}(s, \tilde{\varphi}(t)) \subset \tilde{G}(s, \tilde{G}(t, x)).
\]

If \( t + s \geq t_{n+1} \), then for some \( k \geq 1 \)
\[
t_{n+k} \leq t + s < t_{n+k+1}.
\]

Consider the following function
\[
\tilde{\psi}(p) = \begin{cases} 
\varphi_n(p + t - t_n), & p \in [0, t_{n+1} - t), \\
\varphi_{n+1}(p + t - t_{n+1}), & p \in [t_{n+1} - t, t_{n+2} - t), \\
\vdots \\
\varphi_{n+k}(p + t - t_{n+k}), & p \in [t_{n+k} - t, t_{n+k+1} - t), \\
\end{cases}
\]

Then \( \tilde{\psi} \in K_{\tilde{\varphi}(t)} \) and \( \tilde{\psi}(s) = \xi \in \tilde{G}(s, \tilde{\varphi}(t)) \). The Lemma is proved. \( \square \)
3. **Application to impulsive parabolic inclusions.** In this section we consider impulsive problem consisting of differential inclusion with a small parameter in the right-hand part and multi-valued impulsive perturbation. We prove that for sufficiently small value of parameter this impulsive problem generates an impulsive MDS which has a global attractor.

Consider a triplet \( V \subset H \subset V^* \) of Hilbert spaces with compact and dense embedding. Let \( \langle \cdot, \cdot \rangle \) be a canonical duality between \( V \) and \( V^* \). Denote by \( \| \cdot \| \) and \( (\cdot, \cdot) \), respectively, the norm and scalar product in \( H \). Let \( \| \cdot \|_V \) be a norm in \( V \) and

\[
\exists \alpha > 0 \ \forall u \in V \quad \| u \|^2 \leq \alpha \| u \|_V^2. \tag{8}
\]

Consider a linear continuous self-adjoint operator \( A : V \rightarrow V^* \) such that

\[
\exists \beta > 0 \ \forall u \in V \quad (Au, u) \geq \beta \| u \|^2_V. \tag{9}
\]

Then there exists a complete orthonormal in \( H \) family \( \{\psi_i\}_{i=1}^{\infty} \subset V \) such that

\[
\forall i \geq 1 \quad A\psi_i = \lambda_i \psi_i, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_i \rightarrow \infty, \quad i \rightarrow \infty.
\]

Assume that a multi-valued perturbation \( F : H \rightarrow P(H) \) satisfies the following conditions:

\[
\forall y \in H \quad F(y) \text{ is closed and convex in } H, \quad \tag{10}
\]

\[
\exists c_1 > 0 \ \forall y \in H \quad \|F(y)\|_+ := \sup_{z \in F(y)} \|z\| \leq c_1, \tag{11}
\]

\[
\forall y_0 \in H \quad \text{dist}_H(F(y), F(y_0)) \rightarrow 0, \ y \rightarrow y_0. \tag{12}
\]

Now we are ready to formulate our impulsive problem. The first part of this problem is the following parabolic inclusion

\[
\frac{dy}{dt} + Ay \in \varepsilon \cdot F(y), \ t > 0. \tag{13}
\]

We are interested in mild solutions of (13) in the sense of the following definition.

**Definition 3.1.** A pair \( \{y, f\} \in L^2_{loc}(0, +\infty; V) \times L^2_{loc}(0, +\infty; V^*) \) is called a mild solution of (13) if \( \frac{dy}{dt} \in L^2_{loc}(0, +\infty; V^*) \) and for almost all (a.a.) \( t > 0 \)

\[
\begin{cases}
\frac{dy}{dt} + Ay = \varepsilon f(t), \\
f(t) \in F(y(t)).
\end{cases}
\]

The first component \( y \) of the pair \( \{y, f\} \) is also called solution of (13).

It is well known that under conditions (9) - (12) for every \( y_0 \in H, \varepsilon > 0 \) there exists at least one solution \( \{y, f\} \) of (13) with \( y(0) = y_0 \).

The second part of our problem is an impulsive perturbation, which is characterized by the following parameters:

for fixed \( p \geq 1, \ \{\alpha_i\}_{i=1}^{p} \subset (0, +\infty), \ a > 0, \ mu > 0, \)

\[
M = \{y = \sum_{i=1}^{\infty} c_i \psi_i \in H : \ \forall i = 1, p, c_i \geq 0, \ \sum_{i=1}^{p} \alpha_i c_i = a\}, \tag{14}
\]

\[
I : M \rightarrow P(H),
\]

for \( y = \sum_{i=1}^{\infty} c_i \psi_i \in M \)

\[
Iy = \sum_{i=1}^{p} c'_i \psi_i + \sum_{i=p+1}^{\infty} c_i \psi_i : \ \forall i = 1, p, c'_i \geq 0, \ \sum_{i=1}^{p} \alpha_i c'_i = a(1 + \mu). \tag{15}
\]
As an example we consider the following simple situation:

\[ p = 1, \quad I : M \to H, \quad c_1 = (1 + \mu) c_1, \quad \alpha_1 = 1. \]

Then

\[ M = \{ y \in H \mid (y, \psi) = a \}, \quad I y = (1 + \mu) c_1 \psi_1 + \sum_{i=2}^{\infty} c_i \psi_i. \]

**Remark 2.** In the particular case \( H = L^2(\Omega), V = H_0^1(\Omega), A = -\Delta, \)

\[ \forall y \in L^2(\Omega) \quad F(y) = \{ z \in L^2(\Omega) \mid f_1(y(x)) \leq z(x) \leq f_2(y(x)) \} \text{ for a.a. } x \in \Omega, \]

\[ f_1, f_2 : \mathbb{R} \to \mathbb{R} \text{ are continuous bounded functions,} \]

we have a standard parabolic inclusion of reaction-diffusion type \[16\]. But the proposed scheme can also be useful in other situations \[2, 8\].

The main result of the paper is the following theorem.

**Theorem 3.2.** Under conditions \[9 - 12\] for sufficiently small \( \varepsilon > 0 \) the problem \[13 - 15\] generates an impulsive MDS \( G_\varepsilon \) which has a global attractor.

**Proof.** If we denote by \( K^\varepsilon \) the set of solutions of \[13\] then properties \[K1, K2\] are satisfied. Moreover, for sufficiently small \( \varepsilon > 0 \) the corresponding (non-impulsive) MDS \( G_\varepsilon : \mathbb{R}_+ \times H \to P(H) \)

\[ G_\varepsilon(t, y_0) = \{ y(t) \mid y \in K^\varepsilon, \quad y(0) = y_0 \} \]

has a global attractor \[12\]. We are going to prove that this property remains true under impulsive perturbation \[14, 15\].

First of all, we must verify conditions \[4 - 6\]. Let us consider some properties of solutions of \[13\]. For every mild solution \( \{ y, f \} \) and for a.a. \( t > 0 \) we have

\[ \frac{1}{2} \frac{d}{dt} \| y(t) \|^2 + \langle Ay(t), y(t) \rangle = \varepsilon (f(t), y(t)). \]  

(16)

Using \[6, 9, 11\], we get

\[ \frac{1}{2} \frac{d}{dt} \| y(t) \|^2 + \frac{\beta}{\alpha} \| y(t) \|^2 \leq \varepsilon c_1 \| y(t) \|. \]

Then for \( \varepsilon \in (0, \frac{\beta}{\alpha}) \) we have

\[ \forall t \geq s \geq 0 \quad \| y(t) \|^2 \leq \| y(s) \|^2 e^{-\frac{\beta}{\alpha}(t-s)} + 1. \]  

(17)

Every mild solution \( \{ y, f \} \) also satisfies the following equality: \( \forall i \geq 1 \quad \forall t \geq 0 \)

\[ (y(t), \psi_i) = e^{-\lambda_i t} (y_0, \psi_i) + \varepsilon \int_0^t e^{-\lambda_i (t-\tau)} (f(\tau), \psi_i) d\tau. \]  

(18)

From the definition of the set \( M \) and the map \( I \) we immediately obtain \[4\]. To verify \[5\] we take an arbitrary \( y_0 \in M \), an arbitrary mild solution \( \{ y, f \} \) of \[13\] with \( y(0) = y_0 \) and consider the function

\[ g_\varepsilon(t) = \sum_{i=1}^{p} \alpha_i e^{-\lambda_i t} (y_0, \psi_i) + \varepsilon \int_0^t \sum_{i=1}^{p} \alpha_i e^{-\lambda_i (t-\tau)} (f(\tau), \psi_i) d\tau. \]

Then from \[18\] for sufficiently small \( \varepsilon > 0 \) we need to prove the following:

\[ \exists \tau = \tau(\varepsilon, y_0) \quad \forall t \in (0, \tau) \quad g_\varepsilon(t) \neq a. \]  

(19)
Let us consider a function
\[ g(t) = \sum_{i=1}^{p} \alpha_i e^{-\lambda_i t} (y_0, \psi_i). \]
As \( g(0) = a \) and \( \forall i = 1, \ldots, p \), \( \alpha_i > 0 \), \( (y_0, \psi_i) \geq 0 \), we deduce
\[ g'(0) = -\sum_{i=1}^{p} \alpha_i \lambda_i (y_0, \psi_i) \leq -\lambda_1 a. \] \hspace{1cm} (20)
So for some \( \tau_0 = \tau_0(y_0) > 0 \) we have
\[ \forall t \in (0, \tau_0) \quad g(t) < a - \frac{a \lambda_1}{2} t. \]
Therefore, for \( t \in (0, \tau_0) \)
\[ g_\varepsilon(t) < a - \frac{a \lambda_1}{2} t + \varepsilon c_1 \sum_{i=1}^{p} \alpha_i t. \] \hspace{1cm} (21)
Using (21), we get (19) for \( \varepsilon \in (0, \frac{a}{c_1 \sum_{i=1}^{p} \alpha_i}). \)

Let us prove property (6). It is obvious if \( y \notin K \varepsilon \) does not intersect \( M \). To investigate the other situation we take arbitrary \( y_0 \in IM \), \( y \in K \varepsilon \) with \( y(0) = y_0 \) and the function \( g_\varepsilon(t) \) for \( \varepsilon \in (0, \frac{a}{c_1 \sum_{i=1}^{p} \alpha_i}). \) If \( y \) intersects \( M \) then from (4),(5) there exists \( s_\varepsilon = s_\varepsilon(y) > 0 \) such that
\[ \forall t \in (0, s_\varepsilon) \quad y(t) \notin M, \ y(s_\varepsilon) \in M. \]
Let us give an estimation for \( s_\varepsilon \). Using (18), we get
\[ a = \sum_{i=1}^{p} \alpha_i e^{-\lambda_i s_\varepsilon} (y_0, \psi_i) + \varepsilon \int_{0}^{s_\varepsilon} \sum_{i=1}^{p} \alpha_i e^{-\lambda_1 (s_\varepsilon - \tau)} (f(\tau), \psi_i) d\tau \leq \sum_{i=1}^{p} \alpha_i \|y_0\| \cdot e^{-\lambda_1 s_\varepsilon} + \varepsilon c_1 \sum_{i=1}^{p} \alpha_i. \]
So for \( \varepsilon \in (0, \frac{a}{c_1 \sum_{i=1}^{p} \alpha_i}) \) we have
\[ e^{-\lambda_1 s_\varepsilon} \geq \frac{1}{2} \sum_{i=1}^{p} \alpha_i \|y_0\|, \]
\[ s_\varepsilon \leq \frac{1}{\lambda_1} \ln \left( \frac{a}{2 \|y_0\| \sum_{i=1}^{p} \alpha_i} \right). \] \hspace{1cm} (22)
Using again (18), we obtain for \( \varepsilon \in (0, \frac{a}{c_1 \sum_{i=1}^{p} \alpha_i}) \)
\[ a = \sum_{i=1}^{p} \alpha_i e^{-\lambda_i s_\varepsilon} (y_0, \psi_i) + \varepsilon \int_{0}^{s_\varepsilon} \sum_{i=1}^{p} \alpha_i e^{-\lambda_1 (s_\varepsilon - \tau)} (f(\tau), \psi_i) d\tau \geq e^{-\lambda_1 s_\varepsilon} a (1 + \mu) - \frac{a \mu}{2}. \]
Therefore
\[ e^{-\lambda_1 \varepsilon_1} \leq \frac{1}{1 + \mu'}, \quad \varepsilon_1 \geq \frac{1}{\lambda_1} \ln(1 + \mu') \] (23)
where \( \mu' = \frac{\mu}{2\pi^2 p} \). From (23) we get the property (6).

Let us consider impulsive trajectories starting from \( IM \) more accurately.

Firstly, we claim that for
\[ y_0 = \sum_{i=1}^{p} \frac{a(1+\mu)}{\alpha_i p} \psi_i + \sum_{i=p+1}^{\infty} c_i \psi \in IM \]
every \( y \in K^\varepsilon \), \( y(0) = y_0 \) intersects \( M \). Indeed, from (18) \( \forall \ i = \overline{1,p} \)
\[ (y(t), \psi_i) = e^{-\lambda_i t} \frac{a(1+\mu)}{\alpha_i p} + \varepsilon \int_0^t e^{-\lambda_i(t-\tau)} (f(\tau), \psi_i) d\tau \geq \]
\[ e^{-\lambda_i t} \frac{a(1+\mu)}{\alpha_i p} - \varepsilon \frac{c_i}{\lambda_i}. \]
Therefore
\[ \forall \ t \in [0, \frac{1}{\lambda_i} \ln \left( \frac{a(1+\mu)\lambda_i}{\varepsilon \alpha_i c_1} \right)] \ (y(t), \psi_i) \geq 0. \]

On the other hand, for \( \varepsilon \in (0, \frac{1}{\lambda_1} \sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i} \) we have
\[ g_\varepsilon(t) = \frac{a(1+\mu)}{p} \sum_{i=1}^{p} e^{-\lambda_i t} + \varepsilon \int_0^t \sum_{i=1}^{p} \alpha_i e^{-\lambda_i(t-\tau)} (f(\tau), \psi_i) d\tau, \]
\[ g_\varepsilon(0) = a(1+\mu) > a, \quad g_\varepsilon(t) \leq \frac{a(1+\mu)}{p} \sum_{i=1}^{p} e^{-\lambda_i t} + \frac{a}{2}. \]
Then for sufficiently large \( t > 0 \) we have \( g_\varepsilon(t) < a \) and continuity of \( g_\varepsilon \) implies that \( y \) intersects \( M \) at some point \( s_\varepsilon > 0 \). Moreover, from the inequality
\[ a = g_\varepsilon(s_\varepsilon) < a(1+\mu)e^{-\lambda_1 s_\varepsilon} + \frac{a}{2} \]
we obtain
\[ s_\varepsilon \leq \frac{1}{\lambda_1} \ln 2(1 + \mu). \]

Secondly, choosing every time impulsive points with \( c_i = \frac{a(1+\mu)}{\alpha_i p}, \ i = \overline{1,p} \) it is easy to see that if \( y \in K^\varepsilon \), \( y(0) = y_0 \) intersects \( M \) at least at one point then there is an impulsive trajectory \( \tilde{y}, \tilde{y}(0) = y_0 \) which intersects \( M \) infinitely many times.

As a result, we have a nonempty class of impulsive trajectories which intersects \( M \) and we have a nonempty class of impulsive trajectories which intersects \( M \) at an infinite number of points.

Properties (4)-(6) guarantee that for \( \varepsilon \in (0, \varepsilon_1) \),
\[ \varepsilon_1 = \min \left\{ \frac{\beta}{\alpha c_1}, \frac{1}{4} \frac{a\lambda_1}{c_1 \sum_{i=1}^{p} \alpha_i}, \frac{1}{2} \frac{a}{c_1 \sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i}}, \frac{1}{2} \frac{a \mu}{c_1 \sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i}} \right\}, \]
formula
\[ \tilde{G}_\varepsilon(t,y_0) = \{ y(t) \mid y(\cdot) \in \tilde{K}_y^\varepsilon \} \] (24)
generates an (impulsive) MDS, where \( \tilde{K}_y^\varepsilon \) is a set of all solutions of (13)-(15) with initial point \( y_0 \).
Let us prove the dissipativity condition for impulsive MDS (24). If \( y \in \tilde{K}^{\varepsilon}_{y_0} \), 
\[ \|y_0\| \leq R \] 
does not intersect \( M \) then from (17) 
\[ \|y(t)\| \leq \sqrt{2} \quad \forall t \geq T = \frac{2\alpha}{\beta} \ln R. \]  
(25)

If for some \( \tau > 0 \) \( y(t) \notin M \) \( \forall t \in (0, \tau) \), \( y(\tau) \in M \) then from (22) follows 
\[ \tau \leq \frac{1}{\lambda_1} \frac{2R}{\lambda_1} \sum_{i=1}^{p} c_i \]  
(26)

Thus it is enough to prove the following property for sufficiently small \( \varepsilon \in (0, \varepsilon_1) \): 
\[ \exists R_0 > 0 \quad \forall R > 0 \quad \exists T = T(R) > 0 \quad \forall y_0 \in IM, \quad \|y_0\| \leq R, \]  
\[ \forall y \in \tilde{K}^{\varepsilon}_{y_0} \quad \forall t \geq T \quad \|y(t)\| \leq R_0. \]  
(27)

Without loss of generality in all further arguments we assume that if \( y_0 \in IM \) then \( y \in \tilde{K}^{\varepsilon}_{y_0} \) has an infinite number of impulsive points.

So for given \( y \in \tilde{K}^{\varepsilon}_{y_0} \) with \( y_0 \in IM, \quad \|y_0\| \leq R \) from (23) there are \( \{s_i\}_{i=0}^{\infty} \) such that \( y(\cdot) \) has jumps at the moments \( \{s_0, s_0 + s_1, \ldots\} \) with impulsive points \( \{y_i^+\}_{i=1}^{\infty} \) and \( \forall i \geq 0 \quad s_i \geq \frac{1}{\lambda_1} \ln(1 + \mu') \).

Let
\[ y(s_0 - 0) = \sum_{i=1}^{p} c_i \psi_i + \sum_{i=p+1}^{\infty} c_i \psi_i, \]
\[ \|y(s_0 - 0)\|^2 = \sum_{i=1}^{\infty} c_i^2 \leq \|y_0\|^2 e^{-\delta s_0} + 1, \quad \delta = \frac{\beta}{\alpha} > 0, \]
\[ y(s_0) = y_1^+ = \sum_{i=1}^{p} c_i^+ \psi_i + \sum_{i=p+1}^{\infty} c_i \psi_i. \]

Using inequality
\[ \forall i = 1, p \quad c_i^+ \leq \frac{a(1 + \mu)}{\lambda}, \quad \lambda := \min_{1 \leq i \leq p} \alpha_i > 0, \]
we get
\[ \|y_i^+\|^2 = \sum_{i=1}^{p} (c_i^+)^2 + \sum_{i=p+1}^{\infty} c_i^2 \leq \sum_{i=1}^{p} \frac{(1 + \mu)^2}{\lambda^2} a^2 + \|y_0\|^2 e^{-\delta s_0} + 1, \]
\[ \|y(s_0 + s_1 - 0)\|^2 \leq e^{-\delta s_1} \|y_1^+\|^2 + 1 \leq \sum_{i=1}^{p} \frac{(1 + \mu)^2}{\lambda^2} a^2 e^{-\delta s_1} + \|y_0\|^2 e^{-\delta(s_0 + s_1)} + e^{-\delta s_1} + 1 \]
\[ \|y_2^+\|^2 \leq \sum_{i=1}^{p} \frac{(1 + \mu)^2}{\lambda^2} a^2 + \sum_{i=p+1}^{\infty} \frac{(1 + \mu)^2}{\lambda^2} a^2 e^{-\delta s_1} + \|y_0\|^2 e^{-\delta(s_0 + s_1)} + e^{-\delta s_1} + 1 \]

After \( k \) steps we obtain
\[ \|y(\sum_{i=0}^{k} s_i - 0)\|^2 \leq \|y_0\|^2 e^{-\delta \sum_{i=0}^{k} s_i} + \sum_{i=1}^{p} \frac{(1 + \mu)^2}{\lambda^2} a^2 (e^{-\delta s_k} + \ldots + e^{-\delta(s_k + \ldots + s_1)}) + \]
\[ e^{-\delta s_k} + \ldots + e^{-\delta(s_k + \ldots + s_1)} + 1 \]
\[ \|y_{k+1}^+\|^2 \leq \|y_0\|^2 e^{-\delta \sum_{i=1}^{k} s_i} + \sum_{i=1}^{p} \frac{(1 + \mu)^2}{\lambda^2} a^2 + 1 (e^{-\delta s_k} + \ldots + e^{-\delta(s_k + \ldots + s_1)} + 1) \]  
(29)
Using (23), from (28), (29) we get
\[ \exists T = T(R) \forall t \geq T \quad \|y(t)\|^2 \leq 1 + \frac{(\frac{(1+\mu)^2}{\sigma^2}a^2 + 1)}{1 - (1 + \mu') - \sigma p} := R_0 \] (30)

Finally, let us prove that \( \hat{G}_\varepsilon \) is asymptotically compact. Let \( \{y_0^{(n)}\} \) be an arbitrary bounded sequence of initial data, \( \|y_0^{(n)}\| \leq R \), \( \xi_n \in \hat{G}_\varepsilon(t_n, y_0^{(n)}), \, t_n \nearrow +\infty \). Then \( \xi_n = y_n(t_n) \), where \( y_n \in K_{y_0^{(n)}} \). If \( y_n \) does not intersect \( M \) then \( y_n \in K^\varepsilon \) and from (K2) \( \xi_n(\cdot + t_n - 1) \) also belongs to \( K^\varepsilon \) and does not intersect \( M \). So \( \xi_n = y_n(t_n) = z_n(1), \, z_n(0) = y_n(t_n - 1) \).

From (17) we obtain
\[ \|y_n(t_n - 1)\| \leq \sqrt{2} \quad \forall n \geq N(R) \]

Therefore from well-known regularity results [12], [10] the sequence \( \{\xi_n = z_n(1)\} \) is precompact in \( H \). If the function \( y_n \) intersects \( M \) at the first time at a point \( \tau_n \) then from (22) sequence \( \tau_n \) is bounded and \( \{y_n(\tau_n - 0)\} \) is also bounded in \( H \). So from the inequality
\[ \forall y \in M \quad \|Iy\|^2 \leq pa^2 \frac{(1 + \mu)^2}{\sigma^2} + \|y\|^2, \] (31)
it will be enough to prove the precompactness of the sequence \( \{\xi_n\} \subset H \), where
\[ \xi_n \in \hat{G}_\varepsilon(t_n, z_n), \, t_n \nearrow \infty, \, z_n \in IM, \quad \|z_n\| \leq R. \]

Let \( \xi_n = y_n(t_n), \, y_n \in K^\varepsilon_{z_n}, \, \{T^{(n)}_{i+1} = \sum_{i=0}^{\infty} s^{(n)}_{i+1}\} \) be the impulse moments for \( y_n(\cdot), \, \{\eta^{(n)}_{i+1}\} \) be the corresponding impulsive points.

Firstly, we want to prove the precompactness of \( \{\eta^{(n)}_{i+1}\} \). For this aim we consider a bilinear continuous form \( a : V \times V \to \mathbb{R} \)
\[ \forall u, v \in V \quad a(u, v) = \langle Au, v \rangle. \]

For every mild solution \( \{y, f\} \) of the problem [13] with \( y(0) = y_0 \) we have that \( y \) is the unique solution of the following Cauchy problem
\[ \begin{cases} \frac{d}{dt}y + Ay = \varepsilon \cdot f(t); \\ y(0) = y_0. \end{cases} \] (32)

Then for \( \varepsilon \in (0, 1) \) from (24) we get that the function \( t \to a(y(t), y(t)) \) is absolutely continuous on every \( [a, b] \subset (0, +\infty) \) and for a.a. \( t > 0 \)
\[ \frac{d}{dt}a(y(t), y(t)) \leq \|f(t)\|^2 \leq c_1^2, \] (33)
\[ \frac{d}{dt}\|y(t)\|^2 + a(y(t), y(t)) \leq \frac{\alpha}{\beta}c_1^2. \] (34)

Thus from the Uniform Gronwall Lemma [24] and [9] \( \forall r > 0 \)
\[ \beta\|y(r)\|^2 \leq a(y(r), y(r)) \leq \frac{\|y(0)\|^2 + 1}{r} + c_1^2 + \frac{\alpha}{\beta}c_1^2. \] (35)

For the sequence \( y_n \) from (17) and (30) we deduce that there exists \( c = c(R) > 0 \) such that
\[ \forall t \geq 0 \quad \forall n \geq 1 \quad \|y_n(t)\| \leq c(R). \] (36)
Using the inequality
\[ \forall y \in M \cap V \forall y^+ \in Iy \; \|y^+\|_V^2 \leq p\lambda p a^2 \frac{(1 + \mu)^2}{x^2} + \|y\|_V^2, \]
from (22), (23) and (35) we deduce \( \forall i \geq 1 \forall n \geq 1 \)
\[ \|\eta_i^{(n)+}\|_V^2 = \|y_n(T_i^n)\|_V^2 \leq p\lambda p a^2 \frac{(1 + \mu)^2}{x^2} + \|y_n(T_i^n - 0)\|_V^2 \leq \]
\[ p\lambda p a^2 \frac{(1 + \mu)^2}{x^2} + \frac{1}{\beta} \left( c^2(R) + 1 \right) \lambda p + c_1^2 \leq \frac{2c(R) \sum_{i=1}^{p} \alpha_i}{\lambda_1} + \frac{\alpha c_i^2}{3}. \]  
(37)

As the embedding \( V \subset H \) is compact, we obtain the required precompactness of \( \{\eta_i^{(n)+}\} \) in \( H \). Now we are ready to prove the precompactness of \( \{\xi_n = y_n(t_n)\} \).

For every sufficiently large \( n \) there exists a number \( i = i(n) \geq 1, i(n) \to \infty, \; n \to \infty \) such that
\[ t_n \in [T_i^{(n)}(T_i^{(n)}), T_i^{(n)}(T_i^{(n)} + 1)] \]
Let \( y_n = y_n((\eta_i^{(n)+}), \{s_i^{(n)}, \{\varphi_i^{(n)}\}) \). Then \( \xi_n = \varphi_i^{(n)}(t_n - T_i^{(n)}), \; \varphi_i^{(n)}(0) = \eta_i^{(n)+} \).

From (22), (36)
\[ 0 \leq t_n - T_i^{(n)}(T_i^{(n)} - T_i^{(n)}) = s_i^{(n)}(T_i^{(n)}) \leq T := \frac{2c(R) \sum_{i=1}^{p} \alpha_i}{\lambda_1} \frac{a}{\beta}. \]

From (37) we can assume that \( \eta_i^{(n)+} \to \eta \) in \( H \). Using again the regularity result (26), (12) for solutions of (13), we get that
\[ \{\varphi_i^{(n)}(\cdot)\} \text{ is precompact in } C([0, T]; H). \]

Then the sequence \( \xi_n = \varphi_i^{(n)}(t_n - T_i^{(n)}) \) is precompact and Lemma 2.3 finishes the proof.

\[ \square \]

**Remark 3.** It is possible to give an explicit formula for the global attractor \( \Theta_\varepsilon \) of impulsive problem (13)–(15) when \( \varepsilon = 0 \). It is also possible to prove the limit equality
\[ \text{dist}_H(\Theta_\varepsilon, \Theta_0) \to 0, \; \varepsilon \to 0. \]

It will be done in our forthcoming paper.

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