Synchronisation and scaling properties of chaotic networks with multiple delays

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received 23 April 2013; accepted in final form 3 July 2013
published online 26 July 2013

PACS 02.30.Ks – Delay and functional equations
PACS 05.45.Xt – Synchronisation; coupled oscillators
PACS 89.75.Da – Systems obeying scaling laws

Abstract – We study chaotic systems with multiple time delays that range over several orders of magnitude. We show that the spectrum of Lyapunov exponents (LEs) in such systems possesses a hierarchical structure, with different parts scaling with the different delays. This leads to different types of chaos, depending on the scaling of the maximal LE. Our results are relevant, in particular, for the synchronisation properties of hierarchical networks (networks of networks) where the nodes of subnetworks are coupled with shorter delays and couplings between different subnetworks are realised with longer delay times. Units within a subnetwork can synchronise if the maximal exponent scales with the shorter delay, long-range synchronisation between different subnetworks is only possible if the maximal exponent scales with the longer delay. The results are illustrated analytically for Bernoulli maps and numerically for tent maps and semiconductor lasers.

Networks of nonlinear units with time-delayed interactions play an important role in various systems, such as coupled semiconductor lasers, predator/prey systems, traffic dynamics, communication networks, genetic circuits, or the brain [1–6]. Delay times may induce high-dimensional chaotic dynamics [7–9], as for example in semiconductor lasers with delayed feedback [10]. A particularly interesting phenomenon in this context is the zero-lag synchronisation of chaotic units, despite the long interaction delays [11–15]. Chaos synchronisation finds applications in encrypted communication [16,17].

Chaos in the network is quantified by the spectrum of Lyapunov exponents (LEs), which measures the sensitivity to initial conditions. For a system with one long delay, chaos can be characterised by the scaling of the maximal LE with increasing delay time \( \tau \). In the region of strong chaos, it approaches a positive constant value whereas for weak chaos the maximal LE decreases as \( 1/\tau \). These scaling properties have consequences for chaos synchronisation: Networks with strong chaos cannot synchronise completely for long delays, whereas for weak chaos synchronisation is possible depending on the value of the maximal LE and the topology of the network [18,19].

Strong and weak chaos have been demonstrated in networks with a single time delay, up to now. However, realistic systems may have different transmission delays for the coupling signals. In a network with a distribution of delays complex behaviour is expected to be suppressed [20–22]. However, if the network has different delay times with special integer ratios, one finds resonances which can either stabilise or rule out chaos synchronisation [23], depending on the ratio.

In this letter we study networks of nonlinear units coupled by multiple delay times which differ by several orders of magnitude. A typical example is a network of networks, with a connection delay \( \tau_1 \) between the nodes within a subnetwork and a much longer connection delay \( \tau_2 \) between the different subnetworks. We explain the scaling of the full spectrum of LEs with increasing delay times and extend the concepts of strong and weak chaos to multiple delay systems: If the maximal LE scales with \( 1/\tau_k \) the system shows \( \tau_k \)-chaos. Finally, we relate the synchronisation
properties of a hierarchical network to the scaling behaviour of the LEs: In the example above, nodes within a subnetwork can synchronise in the $\tau_1$-chaotic regime, while long-range synchronisation between the different subnetworks can only occur in the $\tau_2$-chaotic regime.

**Spectrum of Lyapunov exponents of a Bernoulli map.** – Strong and weak chaos have been found both for time-continuous and discrete systems with delay [8,24]. Since the main results are valid in both cases, we perform our calculations for iterated maps. For networks of Bernoulli maps, even analytical results can be derived. We start with a single chaotic map with $N$ different feedback delays $\tau_1, \ldots, \tau_N$, described by

$$x_{t+1} = (1 - \epsilon)f(x_t) + \epsilon \sum_{k=1}^{N} \kappa_k f(x_{t-\tau_k}),$$

with $x \in \mathbb{R}$. We consider here delays with different orders of magnitudes $1 < \tau_1 < \ldots < \tau_N$. The spectrum of LEs $\Lambda = \{\lambda_1, \ldots, \lambda_{\gamma_N-1}\}$, which describes the evolution of a perturbation $\delta x_t$ along a chaotic trajectory $s_t$, is calculated using the linearised equation

$$\delta x_{t+1} = (1 - \epsilon)f'(s_t)\delta x_t + \epsilon \sum_{k=1}^{N} \kappa_k f'(s_{t-\tau_k})\delta x_{t-\tau_k}.$$  

The coefficients $f'(s_t)$ in general depend on time, since the trajectory $s_t$ is time dependent. However, for a Bernoulli map, given by

$$f(x) = (ax) \mod 1,$$

with $|a| > 1$, we have constant coefficients $f'(x) = a$. The linearised equation (2) reduces then to a polynomial equation for the characteristic multipliers $z$, which characterise the growth of a perturbation $\delta x_t = \delta x_0 z^t$:

$$z = (1 - \epsilon)f' + \epsilon f' \sum_{k=1}^{N} \kappa_k z^{-\tau_k}. \quad (4)$$

The LEs $\lambda \in \Lambda$ are given by $\lambda = \ln|z|$. The calculation of the LEs can thus be performed in the same way as the stability calculation of a steady state. For a single-delay system, it is known that such an equation can have two different types of unstable (with $|z| > 1$) solutions [8,25]: strongly unstable and weakly unstable. The strongly unstable root is approximated by the delay-independent term $z_0 \approx (1 - \epsilon)f'$, provided $|(1 - \epsilon)f'| > 1$, and does not depend on the delays to the leading order. Also the multiple-delay system can have a strongly unstable multiplier.

In analogy with the single-delay system, we assume a scaling behaviour for the next group of multipliers $\ln z = i\omega + \gamma_1/\tau_1$, with $\gamma_1 > 0$. At leading order, by neglecting all terms of order $1/\tau_1$, $e^{-\gamma_1\tau_2/\tau_1}$, and smaller, the characteristic polynomial (4) reads

$$e^{i\omega} = (1 - \epsilon)f' + \epsilon \kappa_1 f' e^{-i\omega\tau_1} - \gamma_1.$$  

In an analogous way, the unstable roots (if they exist) are approximated by a curve $\gamma_2(\omega) = \ln|\epsilon \kappa_2 f'| - \ln|\epsilon^{i\omega} - (1 - \epsilon)f' - e^{-i\omega\tau_1} \epsilon \kappa_1 f'|$. The corresponding LEs form a $\tau_2$-spectrum $\Lambda_2$, which scales inversely with the second delay $\tau_2$. The number of exponents in this spectrum scales linearly with $\tau_2$.

It is possible to calculate unstable spectra $\Lambda_k$ related to each of the delays $\tau_k$; only the $\tau_N$-spectrum, related to the largest delay present in the system, can have a stable part. Figure 1 shows the spectrum of LEs for a Bernoulli map, obtained by solving eq. (4) numerically, and the analytical long-delay approximations $\gamma_{\infty}(\omega)$. Although the different time scales are not so far apart, the analytical curves $\gamma_k(\omega)$ provide a good approximation for the LEs. We can clearly distinguish one strongly unstable multiplier, and the weakly unstable spectra with their respective delay-scaling. Moreover, the spectra show a hierarchical structure: the shorter delay $\tau_1$ appears as a modulation parameter in the $\tau_2$-spectrum, while the $\tau_3$-spectrum shows oscillations with periodicities of both $2\pi/\tau_1$ and $2\pi/\tau_2$. 

![Fig. 1: (Colour on-line) The spectrum of LEs of a Bernoulli map (eq. (4)) (red dots) subject to three feedback delays. The different panels are zooms. The full black lines are the analytic approximations for long delays $\gamma_1(\omega)/\tau_1$ (panel (b)), $\gamma_2(\omega)/\tau_2$ (panel (c)) and $\gamma_3(\omega)/\tau_3$ (panel (d)). Parameters are $a = 3, \epsilon = 0.63, \kappa_1 = 0.3, \kappa_2 = 0.6, \kappa_3 = 0.1, \tau_1 = 40, \tau_2 = 500$ and $\tau_3 = 6000$.](10013-p2)
Synchronisation and scaling properties of chaotic networks with multiple delays

A similar hierarchy of eigenvalues for steady states of time-continuous delay differential equations with multiple delays can be shown applying the same arguments to the corresponding transcendental equation.

**General case.** – The slope of a chaotic map in general depends on the trajectory, so that the linearization (eq. (2)) is time dependent. The Lyapunov spectrum is then evaluated numerically using a Gram-Schmidt orthogonalization procedure according to Farmer [7]. We find that the properties which we derived for the time-independent case are preserved in the presence of fluctuations, so that the complete Lyapunov spectrum \( \lambda \) is a composition of \( \tau_k \)-spectra. The \( \tau_k \)-spectrum is obtained numerically by integrating the evolution of an auxiliary perturbation variable \( \delta x_t^k \), for which the delay terms \( \tau_{k+1} \) to \( \tau_N \) are removed

\[
\begin{align*}
\delta x_{t+1}^0 &= (1 - \epsilon)f'(s_t)\delta x_t^0 \\
&\Rightarrow \Lambda_0 = \{ \lambda_0 \}
\end{align*}
\]

\[
\begin{align*}
\delta x_{t+1}^1 &= (1 - \epsilon)f'(s_t)\delta x_t^1 + \epsilon_1 f'(s_{t-\tau_1})\delta x_{t-\tau_1}^1 \\
&\Rightarrow \Lambda_1 = \{ \lambda_{1,\text{max}}, \ldots, \lambda_{1,n} \} \setminus \Lambda_0
\end{align*}
\]

The first exponent \( \lambda_0 \) is called the sub-LE. If a partial spectrum \( \Lambda_k \) contains positive exponents, they can be said to “survive” the introduction of further time scale separated delay terms, because the contribution of the additional terms becomes exponentially small for the corresponding Lyapunov modes. We exclude these exponents from the definition of the succeeding spectra, so that the partial spectra \( \Lambda_k \) do not overlap, and each spectrum \( \Lambda_k \) scales only inversely with \( \tau_k \).

We demonstrate the composition of the LE spectrum in different partial spectra for a tent map,

\[
f(x) = \begin{cases} 
\frac{a}{1-a}x, & \text{for } 0 \leq x < a, \\
1 - \frac{a}{1-a}x, & \text{for } a \leq x \leq 1, 
\end{cases}
\]

subject to two different delayed feedbacks. Figure 2 compares the sub-LE \( \lambda_0 \), the partial spectrum \( \Lambda_1 \) and the total spectrum \( \Lambda \). We find that the maximal LE \( \lambda_{\text{max}} \) is well approximated by the sub-LE \( \lambda_0 \) for a long enough delay \( \tau_1 \). Since \( \lambda_0 \) depends on the trajectory \( s_t \), it also depends indirectly on the delays \( \tau_k \). Nevertheless, all our numerical observations indicate that this dependence is negligible, as has also been reported for the Lang-Kobayashi model representing a single delayed feedback system [18,19]. The next exponents \( \lambda_2, \ldots, \lambda_1 \) of the full spectrum are approximated by the \( \tau_1 \)-spectrum \( \Lambda_1 \); the full spectrum \( \Lambda \) deviates from \( \Lambda_1 \) as the latter becomes negative. The second LE \( \lambda_2 \), which coincides with \( \lambda_{1,\text{max}} \), decreases with \( \tau_1 \).

The different scaling behaviours of the LE spectrum are depicted in fig. 3. The partial spectrum \( \Lambda_1 \) as a whole (and thus the upper part of the full spectrum) scales inversely with the delay \( \tau_1 \); the number of exponents in the partial spectrum however scales linearly with \( \tau_1 \). These two effects are demonstrated by plotting \( \lambda_k \tau_1 \) vs. \( k/\tau_1 \) (with \( k \) the ranking of the exponent): For different delays \( \tau_1 \) the spectrum converges to a curve for all exponents from the \( \tau_1 \)-spectrum. The curves diverge for smaller exponents \( k/\tau_1 \gtrsim 0.2 \), as these scale with the largest delay \( \tau_2 \). Similar, for varying \( \tau_2 \) the spectrum converges to a curve for all exponents \( \lambda_k \) from \( \Lambda_2 \) when plotting \( \lambda_k \tau_2 \) vs. \( k/\tau_2 \).

The degree of chaos associated with these Lyapunov spectra can be expressed in the Kolmogorov-Sinai entropy. The entropy is inversely proportional to the time interval over which the trajectory can be predicted, and as such, it measures the average loss of information along a trajectory. An upper bound for the entropy is given by the sum of all the positive LEs. From the scaling properties of the LEs it follows that this upper bound does not depend on any of the delay times.

\( \tau_k \)-chaos. – Apart from a hierarchical Lyapunov spectrum as described above, the different time scales can also manifest in the maximal LE, \( \lambda_{\text{max}} \). If \( \lambda_0 \) > 0 we speak of strong chaos; the maximal LE \( \lambda_{\text{max}} \approx \lambda_0 \) and does not vary with any of the delays. In the weakly chaotic regimes \( \lambda_0 < 0 \) holds. If \( \lambda_{1,\text{max}} > 0 \) we speak of \( \tau_1 \)-chaos and \( \lambda_{\text{max}} \propto 1/\tau_1 \). If \( \lambda_{1,\text{max}} < 0 \) the second delay dominates and \( \lambda_{\text{max}} \propto 1/\tau_2 \). Consequently we define \( \tau_k \)-chaos as the scaling of \( \lambda_{\text{max}} \) with \( 1/\tau_k \).

For tent maps and Bernouilli maps transitions from strong to consecutive types of weak chaos are observed.

Fig. 2: (Colour on-line) Panel (a) shows \( \lambda_0 \) and the first 25 exponents of the full and the sub-Lyapunov spectra, \( \lambda \) and \( \lambda_1 \), for a tent map with two feedbacks \( \tau_1 = 60 \) and \( \tau_2 = 500 \) in the strong chaos regime. Panel (b) shows the sub-exponent \( \lambda_0 \) (green “x” symbols), two maximal LEs of the full spectrum \( \lambda \) (red circles), and the maximal exponent of the \( \tau_1 \)-spectrum \( \lambda_{1,\text{max}} \) (blue crosses) in the same setup for different delays \( \tau_1 \). Other parameters are \( a = 0.4, \epsilon = 0.4, \kappa_1 = 0.7, \) and \( \kappa_2 = 0.3 \).

Fig. 3: (Colour on-line) The LE \( \lambda_k, k = 2, \ldots, 100 \) for different values of the feedback delays are shown. Panel (a) shows the scaling of the positive LEs with \( \tau_1 \) for fixed \( \tau_2 = 500 \). Panel (b) shows the \( \tau_2 \)-scaling of the smaller exponents for fixed \( \tau_1 = 30 \). Other parameters are \( a = 0.4, \epsilon = 0.4, \kappa_1 = 0.8 \) and \( \kappa_2 = 0.2 \).
when increasing the total coupling $\epsilon$ (see fig. 6(b)). Similar transitions can as well be found in more complicated systems. To demonstrate the generality of our results, we also consider a semiconductor laser subject to two different delayed feedbacks. The laser dynamics is modelled by the Lang-Kobayashi equations:

$$\dot{E}(t) = \frac{1 + i\alpha}{2} G_N n(t) E(t) + \sigma (\kappa_1 \dot{E}(t - \tau_1) + \kappa_2 \dot{E}(t - \tau_2)), \quad \dot{n}(t) = (p - 1) J_{th} - \gamma n(t) - (\Gamma + G_N n(t)) |E(t)|^2,$$

where $E(t)$ and $n(t)$ denote the complex electric field and the excess carrier density, respectively. The feedback is characterised by the delays $\tau_1$ and $\tau_2$, their relative strengths $\kappa_1$ and $\kappa_2$ and the total feedback strength $\sigma$. A list of the parameters involved can be found in table 1. Figure 4 shows the maximal LE $\lambda_{m}$ and the sub-LEs $\lambda_0$ and $\lambda_{1, \text{max}}$ for varying total feedback strength $\sigma$. We find a region with strong chaos ($\lambda_{m} \approx \lambda_0 > 0$), regions with $\tau_1$-chaos ($\lambda_{m} \approx \lambda_{1, \text{max}} > 0$ and $\lambda_0 < 0$) and regions with $\tau_2$-chaos ($\lambda_{m} > 0$ and $\lambda_0, \lambda_{1, \text{max}} < 0$).

The difference between strong, $\tau_1$-, $\tau_2$-, ..., $\tau_k$-chaotic dynamics can be directly observed in the evolution of a small perturbation. Figure 5 shows the difference between two chaotic tent maps initialised with the same function. We applied a point-like perturbation at $t = 0$ to one of the tent maps. The instantaneous evolution of this perturbation is governed by $\lambda_0$. In the strongly chaotic regime, it thus increases exponentially, as shown in fig. 5(a). In the weakly chaotic regimes the perturbation decays first, to reappear and decay at $t \approx \tau_1$. The perturbation evolves over the consecutive $\tau_1$ intervals according to $\lambda_{1, \text{max}}$. In the $\tau_1$-chaotic regime a perturbation thus spreads on the time scale of $\tau_1$ (illustrated in fig. 5(b)). In the $\tau_2$-chaotic regime it decays over the $\tau_1$ intervals, but it is magnified over the $\tau_2$ intervals. The behaviour is shown in fig. 5(c).

**Synchronisation.** — Multiple delays appear naturally in a network of networks; in this case the interaction delays $\tau_1$ within a subnetwork are much shorter than the connection delays $\tau_2$ between the different subnetworks. Each node has its specific sub-LE $\lambda_0$, while the $\tau_1$-spectra $\Lambda_1$ describe the evolution of small perturbations within a subnetwork. The $\tau_2$-spectra relate to the full network.

**Table 1**: Used constants in the simulation of the Lang-Kobayashi equations. The values are taken from ref. [10].

| Parameter                  | Symbol | Value     |
|----------------------------|--------|-----------|
| Linewidth enhancement factor | $\alpha$ | 5         |
| Differential optical gain   | $G_N$  | $2.142 \times 10^4$ s$^{-1}$ |
| Laser frequency             | $\omega_0$ | $2\pi c/(635 \text{ nm})$ |
| Pump current relative to $J_{th}$ | $p$ | 1.02     |
| Threshold pump current      | $J_{th}$ | $\gamma N_{sol}$     |
| of solitary laser           |        | $0.909 \times 10^9$ s$^{-1}$ |
| Carrier decay rate          | $\gamma$ | $1.707 \times 10^8$    |
| Carrier number of solitary laser | $N_{sol}$ | $3.357 \times 10^{12}$ s$^{-1}$ |
| Cavity decay rate           | $\Gamma$ | $0.357 \times 10^{12}$ s$^{-1}$ |

Fig. 4: (Colour on-line) The maximal LE $\lambda_{m}$ (red, upper curve) and sub-LEs $\lambda_0$ (blue, lower curve) and $\lambda_{1, \text{max}}$ (green, middle curve) for a Lang-Kobayashi laser subject to two different feedback loops. The system undergoes a transition from strong to $\tau_1$-chaos at $\sigma \approx 18$ ms$^{-1}$ and from $\tau_1$- to $\tau_2$-chaos at $\sigma \approx 40$ ms$^{-1}$. Parameters are $\kappa_1 = 0.25$, $\kappa_2 = 0.75$, $\tau_1 = 5$ ms and $\tau_2 = 200$ ms.

Fig. 5: Evolution of a small point-like perturbation for a tent map with two delayed feedbacks. In panel (a) the system exhibits strong chaos ($\epsilon = 0.35$ and $\kappa = 0.8$), the perturbation grows exponentially on a time scale much smaller than the delays. Panel (b) illustrates the evolution in the $\tau_1$-chaotic regime ($\epsilon = 0.7$, $\kappa = 0.8$); the perturbation first decays, but is enhanced after a time $\tau_1$. For $\tau_2$-chaos ($\epsilon = 0.7$, $\kappa = 0.2$), shown in panel (c), the perturbation decays on the time scales $\tau_1$ and $\tau_2$, but grows exponentially after a time $\tau_2$. Note the different time scales in the panels. Other parameters are $\tau_1 = 47$, $\tau_2 = 500$, and $\alpha = 0.4$.

If the nodes are identical, and receive the same amount of input, the network may show identical synchronisation. It is then possible to relate the sign of the sub-LEs to the synchronisation properties. We illustrate this with a hierarchical network of dynamical units, described by

$$x_{t+1} = (1 - \epsilon) f(x_t) + \epsilon k \sum_l A_{jl} f(x^{lm}_t) \quad + \epsilon (1 - k) \sum_{sk} B_{jk}^{(ms)} C_{ms} f(x^{ks}_{t-\tau_2}).$$

The coupling topology within the $m$-th subnetwork is described by the matrix $A^{(ms)}$, the matrix $C$ describes the coupling architecture of the networks, and the matrix $B^{(ms)}$ models the coupling between the $m$-th and the $s$-th subnetwork. We assume that all the elements receive the same amount of input, both within and from outside the subnetwork (i.e. all the matrices have a row sum equal to 1). Moreover, we consider identical rows for the $B$-matrices, so that all the elements of a subnetwork receive the same input from outside.
To determine the stability of a (cluster) synchronised state $s_t^m$, the network model eq. (10) is linearised along the corresponding synchronisation manifold. Evaluating this linear system along the transverse eigendirections, one obtains the master stability function [26]. Within a subnetwork the equation for transverse stability reads

$$\delta x_{t+1} = (1 - \epsilon) f'(s_t^m) \delta x_t + \sigma_A^{(m)} \epsilon f'(s_{t-\tau_1}^m) \delta x_{t-\tau_1}, \quad (11)$$

with $\sigma_A^{(m)}$ the eigenvalues of the connection matrix $A^{(m)}$ corresponding to transverse modes. The explicit dependence on the long-delay connections vanishes, as all the nodes receive the same external input. Such equation resembles the stability equation for a network with a single delay [19,27]; the difference lies within the dynamics of the synchronised state $s_t^m$, which now also depends on the elements outside the subnetwork and the connection delay $\tau_1$. In the strongly chaotic regime, the largest transversal LE is approximately $\lambda_0 > 0$, so that the elements cannot synchronise. In the $\tau_1$-chaotic regime, identical or cluster synchronisation in the $m$-th subnetwork is stable if $|\sigma_A^{(m)}| \leq e^{-\lambda_{max}\tau_1}$; the synchronisation pattern in the subnetwork thus depends on the coupling topology $A^{(m)}$. For $\tau_2$-chaotic behaviour all the nodes of a subnetwork synchronise completely irrespective of the coupling architecture: the elements of the subnetwork show consistent behaviour with respect to the common input from the rest of the network.

The stability of the full network is then governed by the following stability equation:

$$\delta x_{t+1} = (1 - \epsilon) f'(s_t) \delta x_t + \epsilon \kappa f'(s_{t-\tau_1}) \delta x_{t-\tau_1} + \sigma_C \epsilon (1 - \kappa) f'(s_{t-\tau_2}) \delta x_{t-\tau_2}, \quad (12)$$

with $\sigma_C$ the transversal eigenvalues of the intra-network coupling matrix $C$. Synchronisation between subnetworks is only possible in the $\tau_2$-chaotic regime in the limit of long delays, on the condition that $|\sigma_C| \leq e^{-\lambda_{max}t_2}$ holds. It means that in the $\tau_1$-chaotic regime the short-range connections determine the synchronisation pattern in a subnetwork, while there is no synchronisation induced by the long-range connections. In the $\tau_2$-chaotic regime however, the subnetworks act each as one single node, and the synchronisation pattern between these nodes is determined by the long-delay connectivity.

As an example we consider a network of four globally coupled subnetworks, coupled through their mean fields (fig. 6(a)). The subnetworks are bidirectional rings of four elements. As individual dynamics we choose again tent maps (eq. (8)). We increase the total coupling strength $\epsilon$, such that the system undergoes a transition from strong to $\tau_1$- to $\tau_2$-chaos. Figure 6 shows the cross-correlations between several different network elements as a function of $\epsilon$, together with the different sub-LEs $\lambda_0$ and $\lambda_{1,max}$. The sublattice synchronisation between the diagonal elements $B$ and $C$ in a subnetwork is governed by eq. (11), with a transversal eigenvalue $\sigma_A = 0$. The nodes $B$ and $C$ thus synchronise if $\lambda_0 < 0$. The small difference between the two transition points is caused by numerical inaccuracy. The synchronisation between all elements in a subnetwork is governed by the same equation (eq. (11)), but the transversal eigenvalue with maximal magnitude is given by $\sigma_A = -1$. Consequently, the nodes $A$ and $B$ synchronise if $\lambda_1 < 0$. The inset in fig. 6 shows an exact agreement between these two points. We find that the whole network, and thus the nodes $A$ and $D$, synchronises for a slightly higher coupling strength, as soon as $\lambda_{max}$ has decreased sufficiently such that the stability condition $|\sigma_C| = 1/3 < e^{-\lambda_{max}t_2}$ is fulfilled.

Also if the connection delay within a network is small or zero (i.e., the limit $\tau_1 \gg 1$ does not apply), some of these synchronisation properties still hold. In this case one can distinguish between $\tau_2$-chaos and strong chaos, the $\tau_1$-chaotic regime no longer exists [28]. In the strong chaos regime synchronisation between nodes of the same subnetwork is possible in principle. The stability of a synchronised state must then be evaluated using the master stability equation (11), as the simple synchronisation condition $|\sigma_A^{(m)}| < e^{-\lambda_{max}\tau_1}$ is no longer valid. Synchrony between different subnetworks is still impossible in the strongly chaotic regime. In the $\tau_2$-chaotic regime the same conclusions for synchronisation hold as for large $\tau_1$.

**Conclusion.** In conclusion, we showed that a hierarchy of time scales emerges in systems with several delays. These time scales can be characterised by the different components in the spectrum of LEs. Depending on the leading components of the spectrum, one can distinguish strong chaos or $\tau_k$-chaos. In the $\tau_k$-chaotic regime, small
perturbations evolve on the time scale of the time delay $\tau_k$. Although these results are relevant for any systems with different and well-separated time delays, especially interesting is the application to the network of networks, where time delays within a subnetwork are shorter than the corresponding time delays between the different subnetworks. We showed, that in such a case, the units within a subnetwork can only synchronise when strong chaos is absent and the maximal LE scales either with the shorter or with the longer delay. The total synchronisation of all elements involving also all subnetworks is, however, possible only when the whole LE spectrum scales with the longer delay.

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TJ acknowledges support by FEDER (EU) under the project FISICOS (FIS2007-60327). SY acknowledges support by the German Research Foundation in the framework of the Collaborative Research Center SFB 910.

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