Initial data and first evolutions of dust clouds in bimetric relativity

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Abstract

We present a method for solving the constraint equations in the Hassan–Rosen bimetric theory to determine the initial data for the gravitational collapse of spherically symmetric dust. The setup leads to equations similar to those for a polytropic fluid in general relativity, here called Lane–Emden-like equations. Using a numerical code which solves the evolution equations in the standard $3 + 1$ form, we also obtain a short-term development of the initial data for these bimetric spherical clouds. The evolution highlights some important features of the bimetric theory such as the interwoven and oscillating null cones representing the essential nonbidiagonality in the dynamics of the two metrics. The simulations are in the strong-field regime and show that, at least at an early stage, if the bimetric initial data are close to those for general relativity, the bimetric evolution stays close to the evolution in general relativity as well, and with no instabilities, albeit with small oscillations in the metric fields. In addition, we determine initial data and first evolution for vacuum bimetric spherically symmetric nonstationary solutions, providing generic counterexamples to a statement analogous to Jebsen–Birkhoff theorem in bimetric relativity.

Keywords: modified gravity, ghost-free bimetric theory, bigravity, numerical bimetric relativity

(Some figures may appear in colour only in the online journal)
1. Introduction

In comparison to general relativity (GR), the phenomenology of the Hassan–Rosen (HR) bimetric relativity [1–4] is at an early stage, and the full extent of its physical features is still unknown. At present, bimetric theory is compatible with Solar System and cosmological data [5]. In GR, numerical relativity plays an important role in theoretical astrophysics to clarify structure formation and growth, or formation processes of black holes. Only by means of numerical simulation and experiments, it is possible to get a theoretical understanding of such phenomena occurring in nature [6–10]. In particular, numerical relativity is essential in analyzing strong-field gravitational systems to interpret gravitational wave detections [11].

Similar remarks also hold for the HR theory. In order to perform bimetric numerical simulations (and discriminate bimetric from GR predictions), many questions need to be addressed; for example [12]:

(a) Which initial data solve the bimetric constraints for gravitational collapse?
(b) Which formalism to use for stable bimetric numerical evolution?
(c) What are appropriate gauge conditions?
(d) What algorithms provide numerical stability?
(e) What are suitable boundary conditions for numerical grids?
(f) What is a good method for finding the apparent horizon?

To obtain reliable numerical results, all of these issues have to be resolved.

This work belongs to a series of papers in which we aim to investigate the above issues. Earlier papers in this series establish the bimetric equations in the standard 3 + 1 form [13], and calculate the ratio of lapse functions for the case of spherical symmetry [14]. In the current paper, we present a method for solving the bimetric constraints to determine the initial data for the gravitational collapse of dust in the HR theory. Using a numerical code based on the equations of motion in the standard 3 + 1 form, we then obtain a short term development of the initial data. Obtaining a long term bimetric development is the subject of ongoing work. In order to achieve a more stable numerical evolution, [15] establishes the covariant Baumgarte–Shapiro–Shibata–Nakamura (cBSSN) formalism [16–20] for bimetric relativity. A class of gauge conditions specific to the HR theory are treated in [21].

This paper is structured as follows. The rest of this section summarizes the results by examples; it also reviews the basic equations. Section 2 poses and solves the bimetric constraint equations. Section 3 highlights the properties of the evolution of the initial data. The paper ends with a brief discussion and outlook.

1.1. Summary of results

We consider the case where the two metric sectors are spherically symmetric with common SO(3) Killing vector fields [22]. In GR, with a pressureless dust perfect fluid and spherical symmetry, the Lemaître–Tolman–Bondi spacetime is an analytical solution [23–25]. In bimetric theory, there are no analog solutions, as shown in [26].

In GR, the kinematical and dynamical parts of a metric field can be separated using the 3 + 1 formalism [27, 28]. The same procedure can be applied to bimetric theory using a suitable parametrization based on the geometric mean metric [13]. In this context, the general form of the two metrics in spherical polar coordinates reads,
Figure 1. Initial data for the reference GR solution where the density profile of dust is Gaussian at \( t = 0 \). The radial coordinate is in units of the black hole mass (formed after the collapse).

\[
g = -\alpha^2 dt^2 + A^2 (dr + \beta dt)^2 + B^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{1.1a}
\]

\[
f = -\tilde{\alpha}^2 dt^2 + \tilde{A}^2 (dr + \tilde{\beta} dt)^2 + \tilde{B}^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{1.1b}
\]

where \( \alpha \) and \( \tilde{\alpha} \) denote the lapse functions, \( \beta \) and \( \tilde{\beta} \) are the radial components of the shift vectors, and \((A, B, \tilde{A}, \tilde{B})\) denote the nontrivial components of the spatial vielbeins. In addition, we also evolve the nonzero components of the extrinsic curvature,

\[
K_1 = K^r_r, \quad K_2 = K^\theta_\theta = K^\phi_\phi, \quad \tilde{K}_1 = \tilde{K}^r_r, \quad \tilde{K}_2 = \tilde{K}^\theta_\theta = \tilde{K}^\phi_\phi. \tag{1.2}
\]

The radial components of the shift vectors are conveniently redefined in terms of the mean shift \( q \) and the separation parameter \( p \) [13],

\[
\beta := q + \alpha A^{-1} v, \quad \tilde{\beta} := q - \tilde{\alpha} \tilde{A}^{-1} v, \quad v := p \lambda^{-1}, \quad \lambda := (1 + p^2)^{1/2}. \tag{1.3}
\]

In this parametrization, the lapses \( \alpha, \tilde{\alpha} \), and the mean shift \( q \) do not appear in the constraint equations. All the variables in (1.1a)–(1.3) are functions of \((t, r)\).

We start from the development of existing GR initial data for \( g \), hereinafter called the reference GR solution. At this stage we decouple \( f \); the bimetric theory is reduced to two copies of GR where we only consider the evolution in the \( g \)-sector containing the collapsing matter (letting \( f \) evolve in parallel, for instance, a Minkowski solution). The bimetric interaction will be engaged in a second stage, treating the bimetric collapse after knowing the behavior of the reference GR solution.

As a reference GR solution, we consider a spherical dust cloud with arbitrary density profile at \( t = 0 \). The initial data can be constructed using the time symmetric initial condition where the extrinsic curvature is vanishing, assuming the conformally flat spatial metric. The initial values for the density with a Gaussian profile and the corresponding metric field \( A \) are shown in figure 1. The development of this initial data forms a black hole [30].

After constructing the reference GR solution, we engage the two sectors through the bimetric interaction and solve the bimetric constraint equations for the metric fields, keeping the same matter profile coupled to \( g \). This results in a system of coupled differential equations.

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2 Before coupling matter, the bimetric field equations exhibit duality under the exchange of metrics. The metric sector where the standard model fields are coupled define the physical metric, in the paper selected to be \( g \). Of course, one can also couple another independent matter to \( f \), but this matter will only interact indirectly to the physical matter. Coupling the same matter to both metrics reintroduces the ghost modes (see, e.g., [29]).
whose solution depends on the parameters of the HR theory. The construction of the bimetric initial data is the topic of section 2. Typical initial values for the metric fields $A$ and $\tilde{A}$ which are far from the GR limit are shown in figure 2. These fields carry physical content for the observers in the respective metric sector; they roughly correspond to gravitational potentials in the Newtonian limit.

Finally, as described in section 3, we numerically evolve the initial data. The simulations are not long enough to shed light on the end point of gravitational collapse. Nevertheless, the bimetric collapse of the dust cloud stays very close to the reference GR solution, if the initial data is close to the GR initial data, as illustrated in figure 3. More details about the gravitational collapse in the GR limit are given in section 3.3 (see also figures 5 and 18).

To highlight the bimetric features in the physical content, we employ causal diagrams that display metric configurations according to the method given in [31], an overview of which is given in appendix A. From these diagrams one can read-off the oscillation frequencies of the metric fields in space and time. The causal diagrams are based on the fact that the possible configurations of two metrics can be visualized in terms of the null cone intersections [3], as shown in figure 4(a). Coordinate transformations only deform the null cones, keeping the nature of their intersections. This gives an invariant picture of the bimetric spacetime.
Figure 4. Causal diagram for the evolution of bimetric initial data. These causal types are referring to alignments of the nullcones. Geometrically, they are related how many null directions are shared [3]. The white regions are type I (bidiagonal), the shaded type IIb, and the edges type IIa. Type III does not appear in spherical symmetry because of the dimensional reduction. The dotted lines are isohypses (surface levels) of the $l^2$-norm of the scalar constraint violations (the upper line is at $10^{-2}$, the lower at $10^{-3}$, with no violations at $t = 0$; see equation (3.1)). The plot displays the oscillations of the metric fields in space and time.

The causal diagram for the evolution of the initial data from figure 2 is plotted in figure 4(b), showing the leading frequencies in the dynamics of the metrics fields. Figure 4 also displays how the metric configurations are interwoven in space and time. Note that the metrics have been constructed to be simultaneously diagonalizable at the initial hypersurface. Such ‘bidiagonal slices’ may locally reoccur during the evolution (horizontally slicing the white rhomboid-like patches in some regions in figure 4). Notwithstanding, the two metrics are in general not simultaneously diagonalizable, and the solution lacks a timelike Killing vector field (so it cannot be made stationary by any coordinate transformation). Finally, the dotted lines in figure 4 indicate the regions where the constraints get more violated. They represent surface levels (isohypses) in the contour plot of the $l^2$-norm of the constraint violations\(^3\). These are the artifacts of the numerical simulation and indicate a departure from the bimetric solution. More elaborate comments on the evolution and numerical simulations are found in section 3.

1.2. Basic equations

The action of the Hassan–Rosen bimetric theory reads, [33–35].

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa_g} R_g + \mathcal{L}^m_g \right] + \int d^4x \sqrt{-f} \left[ \frac{1}{2\kappa_f} R_f + \mathcal{L}^m_f \right] + \int d^4x \sqrt{-g} V(S).
\]

\(1.4\)

\(^3\)The $l^2$-norm is used to measure distance in the function space of solutions (see, e.g., [32]).
The metrics $g$ and $f$ are specifically coupled through the ghost-free bimetric potential $V(S)$,

$$V(S) := -\ell^{-2} \sum_{n=0}^{4} \beta_n e_n(S), \quad S := (g^{-1} f)^{1/2}. \quad (1.5)$$

Here, $(g^{-1} f)^{1/2}$ denotes the principal square root of the $(1, 1)$ tensor field $g^{\mu\nu} f_{\mu\nu}$, and $e_n(S)$ are the elementary symmetric polynomials (the principal scalar invariants of $S$). The potential is parametrized by dimensionless real constants $\beta_n$ with an overall length scale $\ell^{-2}$ in the geometrized units $c = G = 1$. The particular form of the potential is dictated by the necessary condition for the absence of ghosts [36], where the dynamics of each metric is given by a separate Einstein–Hilbert term in the action [1]. The principal branch of the square root ensures an unambiguous definition of the theory [3].

The resulting bimetric field equations are (here stated in the operator form),

$$G_g = \kappa_g (V_g + T_g), \quad V_g := -\ell^{-2} \sum_{n=0}^{4} \beta_n Y_n(S), \quad (1.6a)$$

$$G_f = \kappa_f (V_f + T_f), \quad V_f := -\ell^{-2} \sum_{n=0}^{4} \beta_n Y_{4-n}(S^{-1}), \quad (1.6b)$$

where $G_g$ and $G_f$ are the Einstein tensors of the two metrics, $\kappa_g$ and $\kappa_f$ are Einstein’s gravitational constants, $T_g$ and $T_f$ are the stress–energy tensors of the matter fields each minimally coupled to a different metric sector, and $V_g$ and $V_f$ are the contributions of the bimetric potential (1.5), also called the bimetric stress–energy tensors. The functions $Y_n(S)$ in (1.6) encapsulates the variation of the bimetric potential with respect to the metrics,

$$Y_n(S) := \sum_{k=0}^{n} (-1)^{n+k} e_k(S) S^{n-k} = \frac{\partial e_{n+1}(S)}{\partial S}, \quad Y_{n<0}(S) = Y_{n\geq 4}(S) \equiv 0. \quad (1.7)$$

Note that $Y_4 \equiv 0$ truncates the sums in (1.6) accordingly.

The bimetric stress–energy tensors satisfy the following identities [37, 38],

$$\sqrt{-g} V_g^{\mu \nu} + \sqrt{-f} V_f^{\mu \nu} - \sqrt{-g} V^0_{\mu \nu} = 0, \quad (1.8a)$$

$$\sqrt{-g} \nabla_\mu V_g^{\mu \nu} + \sqrt{-f} \tilde{\nabla}_\mu V_f^{\mu \nu} = 0, \quad (1.8b)$$

where $\nabla_\mu$ and $\tilde{\nabla}_\mu$ are the covariant derivatives compatible with $g$ and $f$, respectively. Assuming general covariance of the matter action, the matter conservation laws hold, $\nabla_\mu T_g^{\mu \nu} = 0$ and $\tilde{\nabla}_\mu T_f^{\mu \nu} = 0$, and the field equation (1.6) imply the bimetric conservation law,

$$\nabla_\mu V_g^{\mu \nu} = 0, \quad \tilde{\nabla}_\mu V_f^{\mu \nu} = 0. \quad (1.9)$$

The two equations in (1.9) are not independent according to the differential identity (1.8b).

The $3 + 1$ decomposition. The structure of the bimetric field equations is equivalent to having two copies of GR with the additional stress–energy contributions $V_g$ and $V_f$ coupled through (1.8a); hence, their $3 + 1$ split is straightforward, provided that one knows the projections of $V_g$ and $V_f$ [13]. The $3 + 1$ decomposition of (1.6) results in two sets of the evolution and constraint equations formally analog to those in GR [6–10].
The constraint equations do not depend on the lapse functions and one shift vector because of the particular form of the potential (1.5). The projection of the bimetric conservation law (1.9) gives one additional constraint [13], equivalent to the additional constraint obtained from the Hamiltonian analysis [4] (the so-called secondary constraint). Moreover, as shown in [4, 39], the preservation of this additional constraint relates the two lapses through a ratio, calculated for the case of spherical symmetry in [14].

Since our focus is not on the evolution equations, their standard 3 + 1 form is given in appendix B. The constraint equations are treated in the following section in more detail.

2. Initial data

We let matter (if any) be coupled to only one of the metric sectors, here $g$. The initial values for evolution are subject to certain constraints. In spherical symmetry, the dynamical variables are $(A, B, K_1, K_2)$ and $(\bar{A}, \bar{B}, \bar{K}_1, \bar{K}_2)$. Also, the separation parameter $p$ defines the relative shift between the two metrics (characterizing the relative ‘off-diagonality’ of the metrics). The two lapse functions $\alpha, \bar{\alpha}$, and the mean shift $q$ are kinematical variables and do not appear in the constraint equations. Hence, we need to specify the following metric functions of $r$ for $r = 0$: $A, \bar{A}, B, \bar{B}, K_1, K_2$, and $p$. These functions are constrained by five equations, which leaves a lot of freedom in choosing their initial values. In the rest of this section, we state the constraint equations, solve for a reference GR solution, and construct the related bimetric initial data.

2.1. Constraint equations

To write down the constraint equations, we need the 3 + 1 decomposition of the stress–energy tensors. Let $\rho, j^a, j^b, \bar{\rho}, \bar{j}^a$, and $\bar{j}^b$ denote the normal, tangential and spatial projections of the total stress–energy tensors $V + T$ in the respective sector. These projections sum up the contribution coming from the bimetric potential, denoted by the upper label ‘b’, and the matter contribution, denoted by the upper label ‘m’; for example, $\rho = \rho^b + \rho^m$.

2.1.1. Bimetric sources. The nonzero components of the projections of $V_\alpha$ are [13],

\[ \rho^b = -\left[ \langle R \rangle^2_0 + \lambda \frac{\bar{A}}{A} \langle R \rangle_1^2 \right], \quad \bar{\rho}^b = -p\alpha \langle R \rangle_1^2, \]  

\[ J^b_\alpha = \langle R \rangle^2_0 + \left[ \frac{1}{\lambda} \left( \frac{\tilde{\alpha}}{\alpha} + \frac{\bar{\alpha}}{\bar{\alpha}} \right) - \lambda \frac{\bar{A}}{A} \right] \langle R \rangle_1^2, \]  

\[ J^b_\beta = \langle R \rangle^2_0 + \frac{\tilde{\alpha}}{\alpha A} \langle R \rangle_1^2 + \frac{1}{\lambda} \left( \frac{\tilde{\alpha}}{\alpha} + \frac{\bar{\alpha}}{\bar{\alpha}} \right) \langle R \rangle_1^2, \]

where $J_1 := J^a_\alpha, J_2 := J^b_\alpha = J^b_\theta, J = J_1 + 2J_2$.

Similarly for $V_f$ we have,

\[ \bar{\rho}^b = -\left[ \langle R \rangle^2_0 + \lambda \frac{\bar{A}}{A} \langle R \rangle_1^2 \right] \frac{1}{R^2}, \quad \bar{\rho}_r^b = p\alpha \langle R \rangle_1^2 \frac{1}{R^2}, \]

\[ \bar{J}^b_\alpha = \left\{ \frac{\tilde{\alpha}}{\alpha A} \langle R \rangle_1^2 + \left[ \frac{1}{\lambda} \left( \frac{\tilde{\alpha}}{\alpha} + \frac{\bar{\alpha}}{\bar{\alpha}} \right) - \lambda \frac{\bar{A}}{A} \right] \langle R \rangle_1^2 \right\} \frac{1}{R^2}. \]
where \( J_1 := \tilde{J}' \), \( J_2 := \tilde{J}'_\phi \), and \( \tilde{J} = J_1 + 2J_2 \). To simplify equations, we have defined \( R := \tilde{B}/B \) and,

\[
\langle R \rangle^0_k := - \ell^{-2} \sum_{i=0}^{n} \left( \frac{n}{i} \right) \beta_{(i+k)} R^i, \quad \langle R \rangle^n_k = \langle R \rangle^{n-1}_{k+1} + R \langle R \rangle^{n-1}_{k+1}, \quad \langle R \rangle^0_k = - \ell^{-2} \beta_{(k)}.
\]

The function \( \langle \gamma \rangle^0 \) encapsulates the \( \beta_{(i)} \)-parameters that do not appear elsewhere. The tangential components satisfy the following identity coming from (1.8a),

\[
\sqrt{\gamma} \beta \gamma + \sqrt{\varphi} \beta \varphi = 0, \quad \text{where } \sqrt{\gamma} := AB^2 \text{ and } \sqrt{\varphi} := AB^2.
\]

### 2.1.2. Matter sources

We assume that matter is present only in the \( g \)-sector and let \( T := T_g \). In particular, we consider a perfect fluid with the stress-energy tensor [7, 40],

\[
T_{\mu\nu} = \rho_0 u_{\mu} u_{\nu} + P g_{\mu\nu}, \quad h := 1 + \epsilon + P/\rho_0, \quad g_{\mu\nu} u^\mu u^\nu = -1,
\]

where \( \rho_0 \) is the rest-mass density, \( h \) is the specific enthalpy, \( \epsilon \) is the specific energy, \( P \) is the pressure, and \( u^\mu \) is the four-velocity of the fluid. The general relativistic hydrodynamical equations consist of the conservation law for \( T_{\mu\nu} \), the conservation of the baryon number, and the equation of state for the fluid, respectively given by,

\[
\nabla_{\mu} T^{\mu\nu} = 0, \quad \nabla_{\mu} (\rho_0 u^\mu) = 0, \quad P(\rho_0, \epsilon) = 0.
\]

Following the \( 3 + 1 \) ‘Valencia’ formulation [40, 41], we rewrite (2.6) in the first order flux—conservative form. In spherical coordinates, the resulting equations are expressed in terms of the following conserved variables (here densitized),

\[
\hat{D} := \sqrt{\gamma} \rho_0 w, \quad \hat{S}_r := \sqrt{\gamma} \rho_0 h w A^2 \hat{v}, \quad \hat{\tau} := \sqrt{\gamma} \left( \rho h w^2 - P \right) - \hat{D},
\]

where \( \sqrt{\gamma} \) is defined in (2.4), \( \hat{v} \) is the radial component of the Eulerian three-velocity of the fluid, and \( w \) is the corresponding Lorentz factor,

\[
\hat{v} := (u^\mu / u^\nu + \beta) / \alpha, \quad w := \alpha u^\nu = 1/\sqrt{1 - A^2 \hat{v}^2}.
\]

The flux—conservative evolution equations for \( (\hat{D}, \hat{S}_r, \hat{\tau}) \) are relegated to the ancillary file.

For a pressureless fluid (dust), we have \( P = 0 \), \( \epsilon = 0 \), \( h = 1 \), and the conversion from the conserved to the primitive variables reads,

\[
\rho_0 = \frac{\hat{D}}{w \sqrt{\gamma}}, \quad \hat{v} = \frac{S A^{-1}}{\hat{\tau} + \hat{D}}, \quad w = \frac{\hat{\tau} + \hat{D}}{\sqrt{(\hat{\tau} + \hat{D})^2 - S \hat{D}^2}} = \sqrt{1 + \frac{S A^{-2}}{\hat{D}^2}}.
\]

Finally, the calculation in [40] yields the components of the matter stress—energy tensor,

\[
\rho^m = \rho_0 w^2 = (\hat{\tau} + \hat{D}) / \sqrt{\gamma}.
\]
\[ f_r^m = \rho_0 w^2 A^2 \dot{v} = \dot{S}_r / \sqrt{\gamma}, \quad (2.10b) \]

\[ J_1^m = \dot{v} \dot{S}_r, \quad J_2^m = 0. \quad (2.10c) \]

2.1.3. Bimetric constraints. The scalar and vector constraint equations are obtained as the normal and tangential projections of the bimetric field equation (1.6) on the initial hypersurface. The scalar (or Hamiltonian) constraint equations are,

\[ C_1 := (2K_1 + K_2)K_2 + \frac{1}{A^2} \left( \frac{A^2}{B^2} + 2 \frac{\partial_r A}{A} \frac{\partial_r A}{B} - \frac{(\partial_r B)^2}{B^2} - 2 \frac{\partial_r^2 B}{B} \right) - \kappa_4 (\rho^b + \rho^m) = 0, \quad (2.11a) \]

\[ C_2 := (2\tilde{K}_1 + \tilde{K}_2)\tilde{K}_2 + \frac{1}{A^2} \left( \frac{\tilde{A}^2}{\tilde{B}^2} + 2 \frac{\partial_r \tilde{A}}{\tilde{A}} \frac{\partial_r \tilde{B}}{\tilde{B}} - \frac{(\partial_r \tilde{B})^2}{\tilde{B}^2} - 2 \frac{\partial_r^2 \tilde{B}}{\tilde{B}} \right) - \kappa_4 (\tilde{\rho}^b + \tilde{\rho}^m) = 0. \quad (2.11b) \]

The vector (or momentum) constraint equations are,

\[ C_3 := (K_1 - K_2) \frac{\partial_r B}{B} - \partial_t K_2 - \frac{1}{2} \kappa_4 (f_r^b + f_r^m) = 0. \quad (2.11c) \]

\[ C_4 := (\tilde{K}_1 - \tilde{K}_2) \frac{\partial_r \tilde{B}}{\tilde{B}} - \partial_t \tilde{K}_2 - \frac{1}{2} \kappa_4 (\tilde{f}_r^b + \tilde{f}_r^m) = 0. \quad (2.11d) \]

When compared to the full set of constraint equations in four dimensions [13], the angular components of the momentum constraints become trivial. The last two equations can be combined using the identity (2.4),

\[ \kappa_4 \tilde{A} B \left( \tilde{K}_1 - \tilde{K}_2 \frac{\partial_r \tilde{B}}{\tilde{B}} - \partial_t \tilde{K}_2 - \frac{1}{2} \kappa_4 \tilde{f}_r^m \right) + \kappa_4 \tilde{A} B \left( \tilde{K}_1 - \tilde{K}_2 \frac{\partial_r \tilde{B}}{\tilde{B}} - \partial_t \tilde{K}_2 - \frac{1}{2} \kappa_4 \tilde{f}_r^m \right) = 0. \quad (2.12) \]

The separation parameter \( p \) can be determined either from (2.11c) or (2.11d).

The bimetric scalar and vector constraints (2.11) are the same as in GR with the addition of the bimetric sources. What is specific to the HR theory is the additional constraint obtained from the projection of the bimetric conservation law,

\[ C_{\text{bim}} := \tilde{A} \left( \tilde{K}_1 \langle R \rangle^1_1 + 2 \tilde{K}_2 \langle R \rangle^1_2 \right) + 2 \lambda \langle K \rangle^1_2 - A \left( \langle K \rangle^1_1 + 2 \langle K \rangle^1_2 \right) - 2 \lambda \langle K \rangle^1_2 + 2 p \left( \langle R \rangle^1_1 \frac{\partial_r \tilde{B}}{A} + \langle R \rangle^1_2 \frac{\tilde{A} \partial_r B}{A} \right) + \lambda^{-1} \langle R \rangle^1_2 \partial_r p = 0. \quad (2.13) \]

This constraint eliminates the ghost degrees of freedom. It is obtained using the 3 + 1 projection of the bimetric conservation law, with a general form given in [13] (which is compatible with [4]). Requiring that it stays preserved in time (i.e., \( \partial_t C_{\text{bim}} = 0 \)), yields the ratio of lapses [14].
As an example of the bimetric initial data with no matter sources, consider the initial values
\( p = 0, A = \tilde A = 1, B = \tilde B = r, \) and \( K_1 = K_2 = \tilde K_1 = \tilde K_2 = 0 \) where \( \langle \cdot \rangle_0 \equiv 0, \) hence (2.13) identically vanishes at all times. The \( f \)-sector is set up to develop the Minkowski solution from the initial data \( \tilde A = 1, \tilde B = r, \) and \( \tilde K_1 = \tilde K_2 = 0. \)

The initial data for \( g \) are constructed using the time symmetric initial condition,
\[
K_1 = K_2 = 0, \quad \tilde r = \tilde v = 0 \Rightarrow f_0^a = \tilde f_a = 0, \tag{2.14}
\]
also assuming a conformally flat spatial metric at \( t = 0 \) with the conformal factor \( \psi(r), \)
\[
A = \psi^2, \quad B = \psi^2 r, \quad \Omega_g^2 = \psi^4 \left( \text{dr}^2 + r^2 \text{d}\theta^2 + r^2 \sin^2 \theta \text{d}\phi^2 \right). \tag{2.15}
\]
At this point all the bimetric constraints except (2.11a) are satisfied. Hence, \( \psi \) and \( \tilde D \) must satisfy,
\[
4\psi \Delta \psi + \kappa_g \tilde D = 0, \quad \Delta \psi := \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \psi \right), \tag{2.16}
\]
where \( \rho^m = \tilde D \psi^{-6} \) and \( \Delta \) denotes the spherical Laplacian. Consider the initial density,
\[
\tilde D(r) = c_0^3 \left[ 1 + \frac{1}{2r} \text{erf}(c_0 r \sqrt{\pi}) \right] e^{-r^2 \psi^2}, \quad \text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} \text{d}x. \tag{2.17}
\]
Here, \( c_0 > 0 \) is a free parameter and \( \text{erf}(z) \) is the integral of the Gaussian distribution. This density is the same as in [30] for \( \kappa_g = 8\pi. \) The solution to (2.16) becomes,
\[
\psi(r) = 1 + \frac{1}{2r} \text{erf}(c_0 r \sqrt{\pi}), \quad \lim_{r \to 0} \psi(r) = 1 + c_0, \quad \lim_{r \to \infty} \psi(r) = 1. \tag{2.18}
\]
An example of the initial data for \( c_0 = (3\sqrt{2\pi})^{-1} \approx 0.133 \) is shown in figure 5; the evolution of these data is treated in section 3, with the results shown in the right panel of figure 5.

2.3. Bimetric spherical cloud

We now engage the bimetric interaction, keeping the same initial density of dust as in the reference GR solution. We again use the time symmetric initial condition with the conformally flat spatial metrics in each sectors,
\[
A = \psi_g^2, \quad B = \psi_g^2 r, \quad \Omega_g^2 = \psi_g^4 \left( \text{dr}^2 + r^2 \text{d}\theta^2 + r^2 \sin^2 \theta \text{d}\phi^2 \right), \tag{2.19a}
\]
\[
\tilde A = \psi_f^2, \quad \tilde B = \psi_f^2 r, \quad \Omega_f^2 = \psi_f^4 \left( \text{dr}^2 + r^2 \text{d}\theta^2 + r^2 \sin^2 \theta \text{d}\phi^2 \right). \tag{2.19b}
\]
We set \( p = 0 \) but allow for arbitrary \( \beta_{(\alpha)} \)-parameters. The metric configuration is therefore of type I at the initial hypersurface with simultaneously diagonal \( g \) and \( f. \)

At the moment of time symmetry where \( K_1 = K_2 = \tilde K_1 = \tilde K_2 = 0, \) \( \tilde v = 0, \) and \( p = 0, \) the constraint equations (2.11c), (2.11d), and (2.13) are automatically satisfied. Thus, we only have
to consider the scalar constraints (2.11a) and (2.11b); these form a system of Lane–Emden-like equations,

\[ 4\psi_g \Delta \psi_g + \kappa g \ell^2 (\beta(0)\psi_g^6 + 3\beta(1)\psi_f^2\psi_g^4 + 3\beta(2)\psi_f^4\psi_g^2 + \beta(3)\psi_f^6) + \kappa g \hat{D} = 0, \]

\[ 4\psi_f \Delta \psi_f + \kappa f \ell^2 (\beta(1)\psi_g^6 + 3\beta(2)\psi_f^2\psi_g^4 + 3\beta(3)\psi_f^4\psi_g^2 + \beta(4)\psi_f^6) = 0. \]

(2.20a)

(2.20b)

In astrophysics [42], the Lane–Emden equation represents a self-gravitating spherically symmetric fluid with the polytropic equation of state \( P \propto \rho^{1+1/n} \); in that context, \( \rho \propto \vartheta^n \) satisfies the Lane–Emden equation \( \Delta \vartheta + \vartheta^n = 0 \). Now, interpreting the conformal factors as gravitational potentials in the Newtonian limit, the metric fields obey deformed equations of states (where the polytropic index becomes definite for a specific \( \beta(n) \)-model in vacuum, \( \hat{D} = 0 \)). In other words, engaging the second metric in an existing GR solution introduces a fluid with nontrivial features in both sectors; this gives the name ‘bimetric spherical cloud’ (referred to as ‘pure’ when \( \hat{D} \equiv 0 \)). The departure from GR is controlled by the overall factors \( \kappa g \ell^2 \) and \( \kappa f \ell^2 \).

Given \( \hat{D}(r) \), the system (2.20) should be solved for \( \psi_g(r) \) and \( \psi_f(r) \). We choose asymptotic boundary conditions where \( \hat{D} \to 0 \), \( \psi_g \to \text{const.} \), and \( \psi_f^2/\psi_g^2 \to R_\infty \) at infinity. This imposes a necessary condition \( \langle R_\infty \rangle^3 = \langle R_\infty \rangle^3 = 0 \) for the \( \beta(n) \)-parameters; for example, requiring \( R_\infty = 1 \) implies,

\[ \beta(0) + 3\beta(1) + 3\beta(2) + \beta(3) = 0, \quad \beta(1) + 3\beta(2) + 3\beta(3) + \beta(4) = 0, \]

(2.21)
Figure 6. Bimetric spherical cloud examples: (a) decoupled sectors; (b) $\kappa_f \ll \kappa_g$ and $\beta_{(2)} = -0.04$; (c) and (d) lower frequency modes for $\beta_{(2)} = -4 \times 10^{-4}$; (e) and (f) initial data diverging at a finite radius for $\beta_{(2)} = 2 \times 10^{-4}$; (g) $\kappa_f \gg \kappa_g$, $\beta_{(1)} = 4 \times 10^{-8}$ and $\beta_{(2)} = -2 \times 10^{-7}$; (h) $\kappa_f \gg \kappa_g$, $\beta_{(2)} = -8 \times 10^{-7}$ and $\beta_{(1)} = 2 \times 10^{-7}$; (i) $\beta_{(2)} = 0.004$ and $\beta_{(2)} = -0.02$ for $\kappa_f = \kappa_g$; (j) $\beta_{(2)} = -0.08$ and $\beta_{(1)} = 0.04$ for $\kappa_f = \kappa_g$. All solutions have $\kappa_f = 8 \pi$. The gravitational constants are $\kappa_f \ll \kappa_g$ in (a)–(f), $\kappa_f \gg \kappa_g$ in (g) and (h), and $\kappa_g = \kappa_f$ in (i) and (j). The solved initial data may diverge at finite $r$ for some parameters, as in (e). Oscillation frequencies are higher for large $\beta_{(1)}$-parameters. Note that $A \equiv 1$ in (a), but $A \neq 1$ in (b)–(f). The GR limit is (g) and (h) where $\kappa_f/\kappa_g \to 0$ and $A \approx A_{GR}$.
which fixes $\beta(0)$ and $\beta(4)$ in terms of the other parameters. Moreover, in spherical symmetry, the metric components must be even functions of $r$ [43]. Therefore, we use the Neumann boundary conditions at $r = 0$ requiring $\partial_r \psi_g = 0$ and $\partial_r \psi_f = 0$. Note that we may still choose some arbitrary positive values for $\psi_g$ and $\psi_f$ at $r = 0$.

To summarize: beside $\kappa_g$, $\kappa_f$, $\beta(1)$, $\beta(2)$, $\beta(3)$, $\ell^{-2}$, and the density $\tilde{D}$, we are free to choose $\psi_g$ and $\psi_f$ at $r = 0$, and their ratio at $r \to \infty$. Note that the departure from the reference GR solution is a smooth function of the free parameters (as found by inspection). The GR limit of the HR theory is given by $\kappa_g/\kappa_f \to 0$ [29, 44–46]. The boundary value problem was solved using a shooting method, (i.e., as the initial value problem starting from $r = 0$).

Examples of initial data for different parameter values are shown in figure 6. From the plots we observe higher oscillation frequencies for large $\beta(n)$-parameters, which is compliant with the properties of the usual Lane–Emden equation [42, 47]. The leading frequencies for the pure bimetric spherical clouds are shown in figure 7.

The preservation of the additional constraint (2.13) relates the two lapses through $\alpha W_g + \tilde{\alpha} W_f = 0$ where $W_g$ and $W_f$ are functions of dynamical variables [4, 39]. For the case of spherical symmetry, these functions are established in [14]. The relation holds at all times, including $t = 0$ where it depends on the initial data as shown in figures 8 and 9.

Symmetries in the shapes of $W_g$ and $W_f$ at $t = 0$ may be used as a guideline to what kind of gauge conditions are preferable for the time slicing, e.g., singling out (c) from figure 9.
3. Dynamical evolution of the initial data

Here we present the solutions obtained from the bimetric initial data. We use a free evolution scheme where the constraint equations are solved at the initial surface and the initial data developed by the bimetric evolution equations. The constraint equations are used as error estimators during the evolution. To evolve the equations, we have written a numerical relativity code package for the bimetric theory, \texttt{bim-solver} [48]. The package is written in C++ and provides a framework that combines the evolution equations generated in mathematica into a functioning bimetric numerical relativity code. The code is OpenMP parallelized [49] and supports MPI [50]. The implementation details for \texttt{bim-solver} will be given elsewhere.

3.1. Formalism and numerical methods

The evolution equations are of the standard 3 + 1 form, given in appendix B. Because of the coordinate singularity at $r = 0$, the equations of motion have to be regularized; for this, we use the regularization procedure described in [9, 43, 51, 52]. The regularized equations are relegated to an ancillary file (which can be found on arXiv [53]).

The equations are solved using the finite difference approximation in a uniform grid. The implemented spatial difference scheme is centered in space and with arbitrary finite difference order (we have used up to the sixth order). For the time evolution, we employ the method of lines (MoL) using Runge–Kutta (RK) and iterated Crank–Nicholson (ICN) integration. Moreover, Kreiss–Oliger (KO) dissipation has been added for stability [54, 55]. The form of the Kreiss–Oliger operator is taken from [40]. As diagnostics, we monitor the $l^2$-norm of the scalar constraints. The $l^2$-norm of a quantity $q$ is defined by [56],

$$
|q|_2 = \left( \frac{1}{N} \sum_{i=1}^{N} q_i^2 \right)^{1/2},
$$

where the sum goes over all grid points with valid data. As mentioned, the inner boundaries are fixed by mirroring the grid cells around $r = 0$. The outer boundaries are handled by imposing open boundary conditions. Hence, the ghost cells at $r > r_{\text{max}}$ are updated by extrapolating the interior solution, assuming continuity at the boundary. The extrapolation of the fields is to first order; hence, small unphysical errors are generated in numerical simulations. To delay their propagation, we set the right grid boundary at large radii.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9}
\caption{The lapses at $t = 0$ depending on the gauge (a) $\tilde{\alpha} = 1$, (b) $\alpha = 1$, and (c) $\alpha = \lambda$.}
\end{figure}
The time variation of the radius of each Lagrange shell for the GR simulation. The inset/overlay shows datapoints extracted from [30] for comparison.

The gauge variables are \(q\), \(\alpha\), and \(\tilde{\alpha}\) where the lapses related through \(\alpha W_g + \tilde{\alpha} W_f = 0\). The simplest gauge choices are the algebraic: \(\alpha = 1\), \(\tilde{\alpha} = 1\), or \(\alpha \tilde{\alpha} = \lambda\). The last choice fixes to one the lapse function of the geometric mean metric [13] (other gauges specific to the HR theory are discussed in [21]). Besides using the algebraic gauge conditions, the numerical code also implements the maximal slicing, in this paper used with respect to \(g\).

To detect black hole formation, the code implements an apparent horizon finder in both sectors. In spherical symmetry, a marginally outer trapped surface (MOTS) is given by [6, 57, 58],

\[
\zeta(r) := \partial_t (\log B) - AK_2 = 0.
\]  

The apparent horizon is the outermost MOTS at which (3.2) is satisfied. The solution is obtained using a root-finding algorithm. Alternatively, the function \(\zeta(r)\) can be plotted, and the apparent horizons determined graphically. Note that MOTS is a geometrical concept of a metric and therefore is also valid in bimetric relativity for each metric separately. The outermost MOTS can be found and plotted for each metric separately; however, their implications on the concept of ‘bimetric event horizon’ are still unknown.
3.2. Simulations

Here we give the numerical details for the simulations and show representative results. We have performed two kinds of simulations:

(a) Benchmark (diagnostics) simulation for the reference GR initial data,
(b) Simulations with the engaged bimetric interactions.

For the GR simulation, the grid spacing is $\Delta r = 0.01$ with the outer boundary at $r = 80$. The spatial difference scheme is of the fourth order. The integration employs the method of lines with the third order Runge–Kutta and the Courant–Lax factor (CFL) of $\Delta t / \Delta r = 0.5$. The added Kreiss–Oliger dissipation is of the fourth order with the coefficient 0.03. The lapse is evolved using maximal slicing. The evolution is stable and goes beyond $t = 60$.

The results of the GR simulation are shown in figure 5, which are compatible with [30]. For comparison, figure 10 shows the solution from figure 5 with an inset that contains datapoints which have been extracted from [30].

After the control benchmark, we consider the development of the bimetric initial data constructed in section 2.3. Here we use the grid spacing $\Delta r = 0.04$ with the outer boundary pushed to $r = 300$ and the CFL decreased to 0.25. The code is validated using convergence tests varying $\Delta r$ by factors of two. Since exact solutions are unknown, we employ a Cauchy-type convergence test using the results obtained at three different resolutions. Denote with $f_{(1)}$,
Figure 13. The time variation of $A$ and $\tilde{A}$ at $r = 2$, and the shift vectors at $t = 2$ and $t = 4.5$.

Figure 14. Time slicing represented by the proper time of the Eulerian observers for $g$ and $f$.

Figure 15. Causal diagram for the bimetric spherical cloud near $r = 0$. 
Figure 16. Top: causal diagram for the bimetric spherical cloud at large radii. Bottom: the vertical slice of the constraint violations at $r = 30$.

$f_2$ and $f_4$ the solutions computed using radial grid spacings $\Delta r$, $\Delta r/2$ and $\Delta r/4$, respectively; then the convergence factor is given by [59],

$$\frac{f_2 - f_4}{f_2} = 2^n + O(\Delta r^{n+1}),$$

(3.3)

where $n$ is the order of the sequence of approximations $f_2$, $f_2$, $f_4$. The convergence tests for a solution whose initial data is close to GR are shown in figure 11. The factor remains stable, hence the algorithm converges in the selected range of $\Delta r$.

Next, we examine the evolution of the metric components to the initial conditions shown in figure 12(a). The metric components have a tendency to become bi-proportional, $A \approx \tilde{A}$ and $B \approx \tilde{B}$, at least over a short time period as in figure 13. Note that the null cones oscillate over the time; the same figure shows the radial components of the shift vectors at $t = 2$ and $t = 4.5$.

The lapses are evolved using the maximal slicing with respect to $g$; the proper times of the Eulerian observers for $g$ and $f$ are shown in figure 14. Time is slowed down near the coordinate origin because of the failed regularization. In contrast to the GR simulations that run beyond $t > 60$, the bimetric evolutions fail within $t < 15$, mostly because of the violated boundary conditions. The main cause of the instability are irregularities coming from
round-off errors when calculating the lapse ratio and the corresponding spatial derivatives near $r = 0$.

The causal diagram for the solution near the coordinate origin is shown in figure 15. As noted before, the white regions are type I (bidiagonal) and the shaded type Ib. The edges are type IIa, either with the common left- or right-null directions. The intersections of two left–right type IIa edges are of type I. The irregularities are visible close to the origin. The dotted lines in the causal diagrams again indicate surface levels of the $L^2$-norm of the scalar constraint violations with separation $10^{-3}$. Note that the scalar constraint violations are dimensionful; following [30], they are considered too large when exceeding 10% of the quantity $\kappa_g (\rho_b + \rho_m)$. The causal diagram and the constraint violations at larger radii are given in figure 16. The plot displays regular oscillations of the metric fields in space and time.

Finally, we have performed several thousands of simulations for different parameter values, and the earlier shown examples are typical. All the solutions show the time-dependent nonbidagonality in the dynamics of the two metrics represented by the interwoven and oscillating null cones. An important observation is that the pure bimetric spherical clouds (vacuum solutions) share the same property: they are generically nonbidiagonal and nonstationary; an example is shown in figure 17. The pure bimetric spherical clouds contradict the bimetric analogue of Birkhoff’s theorem [60–62], which is compliant with [63].

### 3.3. Gravitational collapse

The simulations are not long enough to shed light on the end point of gravitational collapse. Nevertheless, the present numerical experiments show that the collapse of the dust cloud follows the pattern from the reference GR solution, showing no instabilities. A physically realistic solution is illustrated in figure 18, where the initial data are close to GR and $\kappa_f \gg \kappa_g = 8\pi$. The time variation of the Lagrange shells is shown in panel (b); the plot is almost the same as in figure 5 (the plots can be compared since the evolution of the time gauge are similar for the two solutions for $t < 7$).

The stress–energy contributions coming from the matter, $\rho^m$, and the bimetric potential, $\rho^b$, are shown in figures 19(a) and (b). These contributions enter the constraint equations (2.11a) and (2.20a). In the GR limit, $\rho^b$ is more than $10^3$ times smaller than $\rho^m$, due to small $\beta_m$–parameters. For comparison, figure 19(c) shows a large contribution from the bimetric potential when the initial data are far from GR (as in figure 2).
Figure 18. (a) A bimetric spherical cloud where the initial data are close to the reference GR solution. (b) The gravitational collapse follows the same pattern as in figure 5. The vertical dotted lines indicate the initial positions of the Lagrange shells. (c) Although $A \approx A_{GR}$, the causal diagram shows the oscillations in the metric fields. The solution is trustworthy for $t < 7$ and $r > 5$.

Figure 19. The initial stress–energy contributions coming from: (a) the matter $\rho^m$, (b) the bimetric potential $\rho^b$ in the GR limit for $\beta_{(1)} = -10^{-4}, \kappa_f = 10^4$, and (c) the bimetric potential $\rho^b$ when the initial data are far from GR, having $\beta_{(1)} = -1$ and $\kappa_f = 8\pi$. 
4. Discussion and outlook

In this work, we have:

(a) Presented a method for solving the constraint equations in the Hassan–Rosen theory to determine bimetric initial data by deforming the existing GR initial data,
(b) Obtained the Lane–Emden-like equations for the bimetric initial data specifically assuming the conformally flat spatial metrics at the moment of time symmetry,
(c) Solved for and analyzed the obtained initial data,
(d) Evolved the initial data using a new numerical code for bimetric relativity.

The results once again point out the importance of nonbidiagonality in the dynamics of the two metrics. The authors in [64] showed that to have dynamically stable solutions, nondiagonal metric elements are needed. Moreover, [65] concludes that considering static black hole solutions in the HR theory is insufficient and that a full dynamical treatment needs to be performed in order to investigate the end point of gravitational collapse. Finally, even in the absence of external matter sources, the bimetric solutions are essentially nonbidiagonal and nonstationary; hence, our results disprove the analogue version of Birkhoff’s theorem for bimetric theory, which is compatible with the findings in [63].

The initial data for bimetric spherical clouds requires negative $\beta_{(0)}$-parameters because of the boundary conditions that are imposed on the constraint equation (2.20a) (see figure 7). This would correspond to an imaginary Fierz–Pauli mass obtained from the spectrum of linearized mass eigenstates on the proportional backgrounds [5, 29, 66]. However, the bimetric spherical clouds are nonproportional solutions in a strong-field regime with the initial conditions that assume conformal flatness. In addition, we are more restrictive on the choice of $\beta_{(0)}$ and $\beta_{(4)}$ in (2.21) than needed to define the Fierz–Pauli mass, which could affect the conclusions on its sign. This suggests that the properties of the bimetric parameter space should be further investigated.

Similarly to GR, the PDE system of evolution equations is ill-posed in general since it is in standard $3+1$ form. A future goal is to obtain a long term development of the initial data. For this purpose, the covariant BSSN formalism is established in [15], and various gauge conditions specific to bimetric relativity are investigated in [21]. The hyperbolicity of the reduced bimetric field equations is discussed in more detail in [15]. Work in progress deals with the implementation of BSSN in bim-solver to obtain stable numerical simulations in spherical symmetry.

Finally, we comment on a possibility to extend the present method to construct initial data for non-spherically symmetric spacetimes. In spherical symmetry, the spacetime dimension is effectively $1+1$, so the square root is parametrized by a scalar $p$. In the case of axial symmetry, the effective dimension is $1+2$, which results in a complicated expression for a two-dimensional vector $p$. Moreover, the closed form of the ratio of lapses is not known for this case, so one of the lapses must be numerically solved during the evolution.

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Table A.1. Causal type discrimination based on $L_\Delta$ and $R_\Delta$.

| $L_\Delta$ | $R_\Delta$ | $L_\Delta R_\Delta$ | Causal type | Comment |
|------------|------------|----------------------|-------------|---------|
| $< 0$      | $< 0$      | $> 0$                | Type IIb    | $g$ left of $f$ |
| $< 0$      | $= 0$      | $= 0$                | Type IIa (R)| right null direction in common |
| $< 0$      | $> 0$      | $< 0$                | Type I      | $f$ inside $g$ |
| $= 0$      | $< 0$      | $= 0$                | Type IIa (L)| left null direction in common |
| $= 0$      | $= 0$      | $= 0$                | Type I      | double null direction in common |
| $= 0$      | $> 0$      | $= 0$                | Type IIa (L)| left null direction in common |
| $> 0$      | $< 0$      | $< 0$                | Type I      | $g$ inside $f$ |
| $> 0$      | $= 0$      | $= 0$                | Type IIa (R)| right null direction in common |
| $> 0$      | $> 0$      | $> 0$                | Type IIb    | $f$ left of $g$ |

Figure 20. Discriminant $L_\Delta R_\Delta$ for the bimetric solution similar to the one in figure 4.

Appendix A. Bimetric causal diagrams

Here we give an overview of the procedure [31] how to plot a bimetric causal diagram in spherical symmetry. Consider two metrics given in (1.1). Let $r_L$ and $r_R$ be the coordinates of the $g$ null cone, and similarly $\tilde{r}_L$ and $\tilde{r}_R$ the coordinates of the $f$ null cone, both cut at $\partial/\partial t$ in the tangent space (these coordinates can be obtained by setting $g, f, d\Omega = 0, dt = 1$, and then solving for $dr$),

$$r_L := -\beta - \alpha/A, \quad r_R := -\beta + \alpha/A, \quad r_L < r_R,$$

$$\tilde{r}_L := -\tilde{\beta} - \tilde{\alpha}/\tilde{A}, \quad \tilde{r}_R := -\tilde{\beta} + \tilde{\alpha}/\tilde{A}, \quad \tilde{r}_L < \tilde{r}_R.$$

Then, the causal type can be determined from the following two distances using table A.1,

$$L_\Delta := r_L - \tilde{r}_L, \quad R_\Delta := r_R - \tilde{r}_R.$$

The causal regions can be shown by plotting the product $L_\Delta R_\Delta$, in which case type I regions have $L_\Delta R_\Delta < 0$, type IIb regions $L_\Delta R_\Delta > 0$, and type IIa regions have $L_\Delta R_\Delta = 0$ (type I if it is a double zero). An example for the solution from figure 4 is shown in figure 20.
Note that $L_\Delta$ and $R_\Delta$ are smooth functions, smoothly deformed by the coordinate transformations. Hence, topological features (e.g., which regions are adjacent to each other) can not be removed. Finally, the causal types can be tracked along any curve (or surface) as shown in figure 21.

**Appendix B. Evolution equations**

The evolution equations for the spatial metrics are,

\[
\partial_t A = - \alpha A K_1 + \partial_r (q A + \alpha v), \quad \partial_t B = - \alpha B K_2 + (q + \alpha A^{-1} v) \partial_r B, \tag{B.1a}
\]

\[
\partial_t \tilde{A} = - \tilde{\alpha} \tilde{A} \tilde{K}_1 + \partial_r (q \tilde{A} - \tilde{\alpha} v), \quad \partial_t \tilde{B} = - \tilde{\alpha} \tilde{B} \tilde{K}_2 + (q - \tilde{\alpha} A^{-1} v) \partial_r \tilde{B}. \tag{B.1b}
\]

The evolution equations for the extrinsic curvatures are,

\[
\partial_t K_1 = \left( q + \alpha A^{-1} v \right) \partial_r K_1 + \alpha K_1 (K_1 + 2K_2) - \alpha \kappa_x \left\{ J_1 - \frac{1}{2} (J - \rho) \right\} + \left( \frac{\partial \alpha A}{A^2} \right)_r - \frac{\partial^2 \alpha}{A^2} + 2 \frac{\partial \alpha A}{A^2} \frac{\partial B}{B} - 2 \frac{\alpha \partial^2 B}{A^2} \right), \tag{B.2a}
\]

\[
\partial_t \tilde{K}_1 = \left( q - \tilde{\alpha} \tilde{A}^{-1} v \right) \partial_r \tilde{K}_1 + \tilde{\alpha} \tilde{K}_1 \left( \tilde{K}_1 + 2\tilde{K}_2 \right) - \tilde{\alpha} \kappa_{\tilde{x}} \left\{ \tilde{J}_1 - \frac{1}{2} (\tilde{J} - \tilde{\rho}) \right\} + \left( \frac{\partial \tilde{\alpha} \tilde{A}}{\tilde{A}^2} \right)_r - \frac{\partial^2 \tilde{\alpha}}{\tilde{A}^2} + 2 \frac{\partial \tilde{\alpha} \tilde{A}}{\tilde{A}^2} \frac{\partial \tilde{B}}{\tilde{B}} - 2 \frac{\tilde{\alpha} \partial^2 \tilde{B}}{\tilde{A}^2} \right), \tag{B.2b}
\]

\[
\partial_t K_2 = \left( q + \alpha A^{-1} v \right) \partial_r K_2 + \alpha K_2 (K_1 + 2K_2) - \alpha \kappa_x \left\{ J_2 - \frac{1}{2} (J - \rho) \right\} + \left( \frac{\alpha B}{B^2} \right)_r - \frac{\partial \alpha B}{B^2} + 2 \frac{\partial \alpha A}{A^2} \frac{\partial B}{B} - 2 \frac{\alpha \partial^2 B}{A^2} \right), \tag{B.2c}
\]
\[
\partial_t \tilde{K}_2 = \left( q - \tilde{\alpha} \Lambda^{-1} \nu \right) \partial_r \tilde{K}_2 + \tilde{\alpha} \tilde{K}_2 \left( \tilde{K}_1 + 2 \tilde{K}_2 \right) - \tilde{\alpha}_K \left\{ \tilde{J}_2 - \frac{1}{2} \left( \tilde{J} - \tilde{\rho} \right) \right\ }
+ \left( \tilde{\alpha} - \frac{\partial_r \tilde{\alpha}}{\tilde{A}^2} B - \frac{\partial_r \tilde{A}}{\tilde{A}^2} A \right) \left( \frac{\partial_r \tilde{B}}{\tilde{B}^2} + \frac{\partial_r \tilde{\alpha}}{\tilde{A}^2} \frac{\partial_r \tilde{B}}{\tilde{B}} - \tilde{\alpha} \frac{\partial_r^2 \tilde{B}}{\tilde{B}^2} \right). \tag{B.2d}
\]

The above set of equations is in each sector equivalent to the GR case [6].

These equations are subject to the regularization procedure described in [9, 43, 51, 52]. There are two types of regularity conditions that must be satisfied. First, the parity which comes from symmetry considerations. Spherical symmetry implies that a reflection in the radial coordinate leaves metrics unchanged. After reflecting \( r \rightarrow -r \), we see that the lapses, the spatial metric components, and the corresponding components of the extrinsic curvature must be even functions of \( r \), while the shift vector must be odd. Beside the parity, the extra regularity conditions require that the manifold must be locally flat at the origin. The regularization is done in several steps (done in both sectors):

(a) Redefine \( B \rightarrow Br \) (and similarly \( \tilde{B} \rightarrow \tilde{Br} \)),
(b) Introduce:
\[
D_\alpha := \partial_r \log \alpha, D_A := \partial_r \log A, D_B := \partial_r \log B,
\]
\[
\sigma = (A - B)/r, \quad \text{and} \quad K_\Delta = (K_1 - K_2)/r,
\]
(c) Find the evolution equation for these variables,
(d) Evolve these variables imposing the odd parity conditions.

The regularized equations are too long to write down here; they are relegated to an ancillary file, which can be found on arXiv [53].

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