Explicit analytic solution for the plane elastostatic problem with a rigid inclusion of arbitrary shape subject to arbitrary far-field loadings

Ornella Mattei\(^1\) and Mikyoung Lim\(^2\)

\(^1\)Department of Mathematics, San Francisco State University, CA 94132, USA
\(^2\)Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology, Daejeon 34141, Republic of Korea
Email: mattei@sfsu.edu, mklim@kaist.ac.kr

Abstract

We provide an analytical solution for the elastic fields in a two-dimensional unbounded isotropic body with a rigid isotropic inclusion. Our analysis is based on the boundary integral formulation of the elastostatic problem and geometric function theory. Specifically, we use the coordinate system provided by the exterior conformal mapping of the inclusion to define a density basis functions on the boundary of the inclusion, and we use the Faber polynomials associated with the inclusion for a basis inside the inclusion. The latter, which constitutes the main novelty of our approach, allows us to obtain an explicit series solution for the plane elastostatic problem for an inclusion of arbitrary shape in terms of the given arbitrary far-field loading.

Keywords. Linear elasticity; Lamé system; Transmission problem; Faber polynomials

Contents

1 Introduction 2
2 Preliminary 3
  2.1 Formulation of the problem 3
  2.2 Layer potential technique 4
  2.3 Complex formulation 5
  2.4 Exterior conformal mapping and associated density basis functions 7
  2.5 Faber polynomials 7
3 Series solution for the rigid inclusion problem 8
  3.1 Series expansion of the far-field loading 8
  3.2 Series expansion of the single-layer potential 9
    3.2.1 Series expansion of the single-layer potential inside the inclusion 9
    3.2.2 Series expansion of the single-layer potential outside the inclusion 12
  3.3 Determination of the density function via the transmission condition 13
    3.3.1 Determination of the coefficients \( s_m^{(3)} \) and \( s_m^{(4)} \), \( m \in \mathbb{Z} \) 14
3.3.2 Determination of the coefficients $s_m^{(1)}$ and $s_m^{(2)}$, $m \in \mathbb{Z}$ .................................. 14
3.3.3 Determination of the constant $c_3$ ................................................................. 16
3.4 Solution expansion in the exterior of the inclusion .............................................. 16

4 Examples

4.1 Elliptic inclusion with an arbitrary far-field loading ........................................ 17
4.2 Inclusion of order 2 with an arbitrary far-field loading ...................................... 18
4.3 Inclusion of order 3 with an arbitrary far-field loading ...................................... 19
4.4 Matrix formulation for inclusions of higher order with an arbitrary far-field loading .......................................................... 19

5 Conclusion ......................................................... 20

1 Introduction

The so-called isolated inhomogeneity problem, also known as the Eshelby problem, consists in the determination of the elastic fields in an inclusion embedded in an unbounded medium, given some prescribed elastic fields in the far-field. Indeed, an inclusion with different material parameters from that of the background will induce some perturbation in the hosting medium which will depend on the shape of the inclusion as well as its material parameters. Such a problem has a long history (see, e.g., the review papers [50, 25, 37]), given its fundamental importance in material modelling, especially for its application to the determination of the effective properties of composite materials (see, for instance, the books by Buryachenko [5], Dvorak [13], and Qu and Cherkaoui [40]). It was first addressed for an ellipsoidal inclusion by Poisson [38] within the context of the Newtonian potential problem, then followed by Maxwell [34], who provided some explicit formulas for the electric and magnetic fields inside the ellipsoid.

With specific reference to the elastostatic case, which is the topic of this paper, early results concerned the case of inclusions with simple shapes subject to uniform strains or tractions: just to name a few, see [20, 48] for spherical inclusions, [14] for spheroids, and [11, 45, 46] for ellipsoids. This problem is usually associated with Eshelby as his 1957 paper [16], in which he proved that a homogeneous isotropic ellipsoidal inclusion embedded in an unbounded homogeneous isotropic medium would experience uniform strains and stresses when uniform strains or tractions were applied in the far-field, is one of the most cited papers in Applied Mechanics. Later, Eshelby [17] conjectured that this occurs only for ellipsoidal inclusions, a fact that was proved independently by Kang and Miton [26] and Liu [32] for three-dimensional isotropic media subject to any uniform far-field loading, by Sendeckyi [47] for the two-dimensional case, and by Ru and Schiavone [44] for anti-plane elasticity. Note that such a conjecture has not been proved yet for any uniform far-field loading in the case of anisotropic elastic media [41, 28, 49].

The extension of Eshelby’s work [16] to inclusions of arbitrary shape is not straightforward as it involves either the computation of complicated Green’s functions (e.g., [5, 36]), or solutions only in closed form [42]. In practical applications, on the other hand, such as in metallurgy, in which the goal is to model perturbations of elastic fields due to precipitates, twinnings and martensitic transformations, the inclusions have a more complex
shape. To overcome such a drawback, some authors applied the theory of conformal mapping to the Eshelby problem in plane elastostatics by relying on the complex formulation introduced by Muskhelishvili [35] (see also [15, 33]). This method can be applied to elastic inclusions of arbitrary shape, but in general, it does not provide an explicit solution (see, for instance, [17, 31]), and in those cases in which an explicit solution can be provided, the far-field loading is considered to be uniform [43, 51]. The determination of explicit formulas for the elastic fields in an inclusion of arbitrary shape subject to an arbitrary far-field loading is still, to the best of the authors’ knowledge, an open problem.

We remark that analytic and numerical methods to compute the elastic tensor (often called the Eshelby tensor field) for inclusions of various shapes have been developed [7, 10, 22, 30, 51].

For the conductivity problem (or antiplane elasticity), Jung and Lim [23] found an explicit solution for an inclusion of arbitrary shape based on geometric function theory: the solution inside the inclusion is expanded into a series of Faber polynomials (see [12, 18] for the definition and properties), whereas the solution in the surrounding medium is expanded into a series of harmonic functions expressed in terms of the coordinates of the exterior conformal mapping, supposed to be known. The layer potential operators and the Neumann-Poincaré operator admit expansions into series of the basis functions so that one can find a series solution to the transmission problem by using the layer potential technique (see, for instance, the comprehensive books [1, 2]). Moreover, the two sets of interior and exterior basis functions have explicit relations on the boundary of the inclusion so that the interior and exterior values of the solution can be matched by using the transmission condition on the boundary of the inclusion (see also [10]). This geometric series solution method was successfully applied to the study of inclusion problems in anti-plane elasticity and the spectral property of the Neumann–Poincaré operator [9, 11, 24]. Analogous results for the elastic problem have not been found yet.

In the present paper, we extend the series solution approach for the conductivity transmission problem in [23] to the elastostatic case. In particular, we provide an explicit analytic series solution for the hard inclusion problem in plane elastostatics.

The remainder of this paper is organized as follows. Section 2 mainly describes the complex formulation for the plane elastostatic transmission problem. Section 3 is devoted to the series solution for the rigid inclusion problem. We provide solutions for inclusions of various orders in Section 4. The paper ends with some concluding remarks in Section 5.

2 Preliminary

2.1 Formulation of the problem

Consider an unbounded homogeneous isotropic medium in \( \mathbb{R}^2 \) with an embedded simply connected homogeneous isotropic inclusion \( \Omega \) of arbitrary shape. Let \( \lambda \) and \( \mu \) be the Lamé constants of the system, \( \lambda \) being the bulk modulus and \( \mu \) the shear modulus. Suppose that the displacement field, \( u(x) \), is assigned in the far-field in a quasi-static manner, that is \( u(x) = u_0(x) + O(|x|^{-1}) \) as \( |x| \to \infty \), with \( u_0(x) \) the far-field displacement.

If there were no inclusion, the displacement field \( u(x) \) would be exactly \( u_0(x) \), whereas the strain field \( \varepsilon(x) \) and the stress field \( \sigma(x) \) would be determined uniquely by the following
well-known formulas (assuming the displacements to be small):

\[
\varepsilon = \frac{1}{2} \left( \nabla u + \nabla^T u \right) \quad \text{and} \quad \sigma = C \varepsilon = \lambda \text{tr} \varepsilon + 2 \mu \varepsilon,
\]

where \( \text{tr} \) stands for the trace. If we assume there are no body forces (so that the stress field is divergence-free), then \( u_0(x) \) satisfies the equation \( L_{\lambda,\mu} u_0 = 0 \) in \( \mathbb{R}^2 \), \( L_{\lambda,\mu} \) being the following differential operator

\[
L_{\lambda,\mu} u := \nabla \cdot C \varepsilon(x) = \mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u.
\]

Note \( L_{\lambda,\mu} \) is elliptic under the strong convexity assumption for which \( \mu > 0 \) and \( \lambda + \mu > 0 \) (see, e.g., [29]).

However, due to the presence of the inclusion \( \Omega \), the displacement field in the medium is not \( u_0(x) \). Indeed, under the assumptions that the inclusion is rigid (so that the displacement field on \( \partial \Omega \) is a rigid displacement), and that there are no body forces, \( u(x) \) turns out to be the solution of the system

\[
\begin{align*}
L_{\lambda,\mu} u &= 0 \quad \text{in} \ \mathbb{R}^2 \setminus \overline{\Omega}, \\
\left. u \right|^+ &= \sum_{j=1}^{3} c_j R_j(x) \quad \text{on} \ \partial \Omega, \\
u(x) - u_0(x) &= O(|x|^{-1}) \quad \text{as} \ |x| \to \infty,
\end{align*}
\]

where the superscript \( + \) denotes the limit from outside \( \Omega \), and \( \{R_1, R_2, R_3\} \) is a basis of the space of rigid displacements, say

\[
R_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad R_3 = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}.
\]

2.2 Layer potential technique

A classical way of solving the transmission problem (2.2) is the layer potential technique (see, e.g., [29]), which is based on the ansatz that the displacement field in the medium is the superposition of the far-field loading \( u_0(x) \) (the field that would be in the medium if there were no inclusion), and a perturbation field \( S_{\partial \Omega}[\varphi](x) \) (the term which takes into account the effect of the inclusion):

\[
u(x) = u_0(x) + S_{\partial \Omega}[\varphi](x),
\]
for some density function \( \varphi = (\varphi_1, \varphi_2)^T \) satisfying the equilibrium condition \((2.4)\), i.e.

\[
\int_{\partial \Omega} \varphi \cdot R_j d\sigma = 0, \quad j = 1, 2, 3.
\]

Indeed, \( \varphi(x) \) is given by \( \varphi = \partial_x u^+ \) on \( \partial \Omega \) (see, e.g., [27] Appendix A.2).

The perturbation field \( S_{\partial \Omega}[\varphi](x) \) is called the single-layer potential of the density function \( \varphi(x) \) on \( \partial \Omega \) associated with the Lamé system, and it is defined as

\[
S_{\partial \Omega}[\varphi](x) := \int_{\partial \Omega} \Gamma(x - y)[\varphi](y) d\sigma(y) \quad \text{for} \quad x \in \mathbb{R}^2,
\]

where \( \Gamma = (\Gamma_{ij})_{i,j=1}^{2} \) is the Kelvin matrix of the fundamental solution to the Lamé system in \( \mathbb{R}^2 \) when there is no inclusion, namely,

\[
\Gamma_{ij}(x) = \frac{\alpha_1}{2\pi} \delta_{ij} \ln |x| - \frac{\alpha_2}{2\pi} \frac{x_i x_j}{|x|^2},
\]

\( \delta_{ij} \) being Kronecker’s delta, and

\[
\alpha_1 = \frac{1}{2} \left( \frac{1}{\mu} + \frac{1}{2\mu + \lambda} \right) \quad \text{and} \quad \alpha_2 = \frac{1}{2} \left( \frac{1}{\mu} - \frac{1}{2\mu + \lambda} \right).
\]

Note that, due to \((2.5)\), the problem of finding the displacement field \( u(x) \) satisfying \((2.2)\) is equivalent to finding the single-layer potential \( S_{\partial \Omega}[\varphi](x) \) \((2.7)\) by finding the density function \( \varphi(x) \) satisfying \((2.6)\). The density function \( \varphi(x) \) is associated with the inversion of the operator \( \frac{1}{2}I + K^*_0 \), where \( K^*_0 \) is the so-called elastostatic Neumann–Poincaré operator:

\[
K^*_{\partial \Omega}[\varphi](x) := \text{p.v.} \int_{\partial \Omega} \partial_{\nu_x} \Gamma(x - y) \varphi(y) d\sigma(y) \quad \text{a.e.} \quad x \in \partial \Omega.
\]

The symbol p.v. stands for the Cauchy principal value, \( \partial_{\nu_x} \Gamma(x - y) \) denotes the conormal derivative of the Kelvin matrix with respect to the \( x \)-variable, defined by

\[
\partial_{\nu_x} \Gamma(x - y) b = \partial_{\nu_x} (\Gamma(x - y) b)
\]

for any constant vector \( b \).

### 2.3 Complex formulation

For plane elastostatics, one could identify the coordinates vector \( x = (x_1, x_2)^T \) with the complex variable \( z = x_1 + ix_2 \in \mathbb{C} \). The vector-valued displacement field \( u = (u_1, u_2)^T \) then can be expressed, upon complexification, as the complex function \( u(z) = u_1 + iu_2 \).

Analogously, the density function \( \varphi = (\varphi_1, \varphi_2)^T \) and the single-layer potential \( S_{\partial \Omega}[\varphi](x) \) can be written as \( \varphi(z) = \varphi_1 + i\varphi_2 \) and \( S[\varphi](z) = (S[\varphi])_1 + i(S[\varphi])_2 \), respectively.

Following [35], the equation \( \mathcal{L}_{\lambda, \mu} u = 0 \) in \( \mathbb{C} \setminus \overline{\Omega} \), in which the two unknowns are \( u_1(x) \) and \( u_2(x) \), can then be written as an equation in terms of two complex functions \( \phi(z) \) and \( \psi(z) \), which are analytic in \( \mathbb{C} \setminus \overline{\Omega} \):

\[
2\mu\varphi(z) = \kappa\phi(z) - z\phi'(z) - \psi(z), \quad \kappa = \frac{\lambda + 3\mu}{\lambda + \mu},
\]

\((2.10)\)
where the bar denotes complex conjugation. Similarly, the background solution \( u_0(z) = (u_0)_1 + i(u_0)_2 \) admits the complex representation (dividing the constant \( \mu \) to simplify the formula)

\[
2u_0(z) = \kappa h(z) - \overline{z h'(z)} - \overline{l(z)} \quad \text{in } \Omega \text{ (or in } \mathbb{C} \setminus \Omega),
\]

where \( h(z) \) and \( l(z) \) are analytic functions in \( \Omega \) (or in \( \mathbb{C} \setminus \Omega \)). Recall that \( u_0(x) \) satisfies \( \mathcal{L}_{\lambda,\mu} u_0 = 0 \) in \( \mathbb{R}^2 \), so that the representation (2.11) holds in the whole complex plane. As the single-layer potential \( S_{\partial \Omega}[\phi](x) \) satisfies the Lamé system as well, the following holds

\[
2S[\varphi](z) = \kappa f(z) - z f'(z) - g(z) \quad \text{in } \Omega \text{ (or in } \mathbb{C} \setminus \Omega) \quad \text{(2.12)}
\]

for some complex analytic functions \( f(z) \) and \( g(z) \), which can be expressed as

\[
f(z) = f[\varphi](z) = \alpha_2 \mathcal{L}[\varphi](z), \quad g(z) = g[\varphi](z) = -\alpha_1 \mathcal{L}[\varphi](z) - \alpha_2 \mathcal{C}[\varphi](z)
\]

with the complex integral operators \( \mathcal{L} \) and \( \mathcal{C} \) given by

\[
\mathcal{L}[\psi](z) := \frac{1}{2\pi} \int_{\partial \Omega} \log(z - \zeta) \psi(\zeta) \, d\sigma(\zeta),
\]

\[
\mathcal{C}[\psi](z) := \mathcal{L}[\psi]'(z) = \frac{1}{2\pi} \int_{\partial \Omega} \frac{\psi(\zeta)}{z - \zeta} \, d\sigma(\zeta)
\]

for any complex function \( \psi(z) \) (see [3] for the derivation).

The rigid inclusion transmission condition on \( \partial \Omega \), see (2.2), and the single-layer potential ansatz (2.5) imply that, on the boundary,

\[
S[\varphi](z) = -u_0(z) + c_1 + ic_2 - ic_3 z.
\]

In Section 3 we will develop a series solution method for the transmission problem (2.2) by using the theory of conformal mapping. Therefore, we review some related results in complex analysis in the following subsection.

### 2.4 Exterior conformal mapping and associated density basis functions

From the Riemann mapping theorem, there uniquely exist a real number \( \gamma > 0 \) and a complex function \( \Psi(w) \) that conformally maps the region \( \mathcal{R} = \{ w \in \mathbb{C} : |w| > \gamma \} \) onto \( \mathbb{C} \setminus \Omega \) and satisfies \( \Psi(\infty) = \infty \) and \( \Psi'(\infty) = 1 \). Here, \( \gamma \) is called the capacity of \( \Omega \) and \( \Psi \) is the exterior conformal mapping associated with \( \Omega \). The function \( \Psi(w) \) admits the following Laurent series expansion:

\[
\Psi(w) = w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots = w + \sum_{k=0}^{\infty} a_k w^{-k}
\]

for some complex coefficients \( a_n \), see [39, Chapter 1.2] for the derivation. Being a conformal mapping, \( \Psi(w) \) preserves the angle between two intersecting curves and, hence, it can define an orthogonal curvilinear coordinate system in \( \mathbb{C} \setminus \Omega \) in a simple way. Indeed, we can express \( w \in \mathcal{R} \) in the modified polar coordinates \( (\rho, \theta) \in (\rho_0, \infty) \times [0, 2\pi) \) with \( \rho_0 = \ln \gamma \) as

\[
w = e^{\rho+i\theta}.
\]
Consequently, \((\rho, \theta)\) provides an orthogonal coordinate pair for \(z = \Psi(w) \in \mathbb{C} \setminus \overline{\Omega}\) via the relation
\[
z = \Psi(\rho, \theta) := \Psi(e^{\rho \theta}).
\]

From the Caratheodory extension theorem \[6\], \(\Psi(\rho, \theta)\) admits the continuous extension to the boundary of the domain. We assume that the boundary \(\partial \Omega\) is \(C^{1,\alpha}\) so that, by the Kellogg-Warschawski theorem \[39\], \(\Psi'(\rho, \theta)\) can be continuously extended to the boundary. In particular, the map \(\Psi(\rho, \theta)\) is \(C^1\) on \([\rho_0, \infty) \times [0, 2\pi]\). Therefore, we can use the polar coordinate system \((\rho, \theta)\) associated with (2.18) to define a density basis, that is, a basis for functions defined on \(\partial \Omega\ (\rho = \rho_0)\), as follows
\[
\varphi_m(z) = \frac{e^{im\theta}}{h}, \quad \varphi_{-m}(z) = \frac{e^{-im\theta}}{h}
\]
for each \(m \in \mathbb{N}\), where \(h\) is the scale factor
\[
h(\rho, \theta) = \left|\frac{\partial \Psi}{\partial \rho}\right| = \left|\frac{\partial \Psi}{\partial \theta}\right|.
\]

Later, we will use the basis (2.20) to expand the density function \(\varphi(z)\) of the single-layer potential (2.7). Notice that the length element on \(\partial \Omega\) is then given by
\[
d\sigma(\zeta) = h(\rho_0, \theta) d\theta \quad \text{for} \quad \zeta = \Psi(\rho_0, \theta).
\]

### 2.5 Faber polynomials
In view of the exterior conformal mapping (2.18), the expansion
\[
\frac{w \Psi'(w)}{\Psi(w) - z} = \sum_{m=0}^{\infty} F_m(z) w^{-m}
\]
is valid for sufficiently large \(|w|\), where the function \(F_m(z)\) is an \(m\)-th order monic polynomial, called the \(m\)-th Faber polynomial associated with \(\Psi(w)\) (or \(\Omega\)), that is uniquely determined by the coefficients of \(\Psi(w)\) in (2.18). As an example, the first three polynomials are
\[
F_0(z) = 1, \quad F_1(z) = z - a_0, \quad F_2(z) = z^2 - 2a_0 z + (a_0^2 - 2a_1).
\]

In general, by comparing the \(w^{-m}\) terms in (2.22) (after multiplying both sides by \(\Psi(w) - z\)) one observes the following recursion relation
\[
-m a_m = F_{m+1}(z) + \sum_{s=0}^{m} a_s F_{m-s}(z) - z F_m(z) \quad \text{for each} \quad m = 0, 1, 2, \ldots
\]

The concept of Faber polynomials was first introduced by G. Faber in \[18\] and has been one of the essential elements in geometric function theory (see, e.g., \[12\]). These polynomials satisfy the convenient property that \(F_m(\Psi(w))\) has only one positive term: \(w^m\). In other words,
\[
F_m(\Psi(w)) = w^m + \sum_{k=1}^{\infty} c_{m,k} w^{-k}.
\]
The coefficients $c_{m,k}$ are called the Grunsky coefficients, and they satisfy the following identity: for all $m, k \in \mathbb{N}$, $kc_{m,k} = mc_{k,m}$.

For any $r \geq \gamma$, we denote $\Omega_r$ the region enclosed by $\{ \Psi(w) : |w| = r \}$. By integrating (2.22) with respect to $w$, one can easily find that (see [23, Appendix B] for the derivation)

$$\log(\Psi(w) - z) = \log w - \sum_{m=1}^{\infty} \frac{1}{m} F_m(z) w^{-m}, \quad |w| > r, \quad z \in \Omega_r. \quad (2.25)$$

Note that, thanks to the properties of Faber polynomials, any complex function $v(z)$ analytic in $\Omega_R$, $R > \gamma$, admits the expansion

$$v(z) = \sum_{m=0}^{\infty} d_m F_m(z) \quad \text{in } \Omega \quad (2.26)$$

with the coefficients $d_m$ given by (from the Cauchy integral formula and (2.22))

$$d_m = \frac{1}{2\pi i} \int_{|w|=r} \frac{v(\Psi(w))}{w^{m+1}} \, dw, \quad \gamma < r < R. \quad (2.27)$$

In other words, Faber polynomials form the interior basis: they can be used as an expansion basis on $\Omega$ for complex functions analytic on a domain containing $\Omega$.

## 3 Series solution for the rigid inclusion problem

The goal of this section is to develop a series solution method for the transmission problem (2.2), in order to provide the displacement field in the exterior of the inclusion. Specifically, we will expand the far-field loading in terms of the interior basis and the single-layer potential in terms of the density basis (2.20). By using the transmission condition (2.17) and the properties of Faber polynomials, we will find an explicit expression for the elastic fields in the exterior of the inclusion in terms of the coordinate $\psi_0$ (2.19).

### 3.1 Series expansion of the far-field loading

As shown by equation (2.26), Faber polynomials constitute a basis for analytic functions in $\Omega$. Specifically, we can expand the functions $h(z)$ and $l(z)$ in the complex representation (2.11) of the far-field loading $u_0(z)$ as

$$h(z) = \sum_{m=0}^{\infty} A_m F_m(z), \quad l(z) = \sum_{m=0}^{\infty} B_m F_m(z) \quad (3.1)$$

for some complex coefficients $A_m$ and $B_m$ to be determined by using equation (2.27). Hence, the far-field loading $u_0(z)$ in (2.11) can be written as

$$2u_0(z) = \kappa \sum_{m=0}^{\infty} A_m F_m(z) - z \sum_{m=1}^{\infty} \frac{A_m F'_m(z)}{m} - \sum_{m=0}^{\infty} B_m F_m(z) \quad \text{in } \Omega. \quad (3.2)$$
3.2 Series expansion of the single-layer potential

The single layer potential $S[\varphi](z)$ in (2.12) is written in terms of the analytic functions $f(z)$ and $g(z)$ given by (2.13) and (2.14), respectively. In order to expand the integral operators in (2.13) and (2.14), we need to expand the fundamental solution of the problem (2.2) (see the definition (2.7)), which is related to the Kelvin matrix introduced in (2.8). To do so, let us first write the density function $\varphi$ on $\partial \Omega$ in terms of the density basis (2.20) on $\partial \Omega$:

\[ \varphi(z) = \sum_{m=1}^{\infty} \left( s_m^{(1)} + is_m^{(2)} \right) \varphi_m - \left( s_m^{(3)} + is_m^{(4)} \right) \varphi_m, \]  

(3.3)

where $s_m^{(j)}$, $m \in \mathbb{N}$, $j = 1, 2, 3, 4$, are real coefficients. The constant term $(m = 0)$ is zero due to the equilibrium condition (2.6) when $j = 1, 2$ and the condition with $j = 3$ implies that $\text{Im} \left( \int_{\partial \Omega} \varphi(z) \overline{\varphi}(z) d\sigma(z) \right) = 0$, i.e.,

\[ s_1^{(4)} = -\text{Im} \left( \sum_{m=1}^{\infty} \left( s_m^{(1)} + is_m^{(2)} \right) \overline{a}_m \right). \]  

(3.4)

Notice that, by plugging (3.3) into (2.13) and (2.14), the functions $f(z)$ and $g(z)$ can be expressed in terms of the basis functions (2.20) as follows:

\[ f(z) = \alpha_2 \sum_{m=1}^{\infty} \left( s_m^{(1)} + is_m^{(2)} \right) \mathcal{L}[\varphi_m](z) + \left( s_m^{(3)} + is_m^{(4)} \right) \mathcal{L}[\varphi_m](z) \] 

(3.5)

and

\[ g(z) = -\alpha_1 \sum_{m=1}^{\infty} \left( s_m^{(1)} + is_m^{(2)} \right) \mathcal{L}[\varphi_m](z) + \left( s_m^{(3)} + is_m^{(4)} \right) \mathcal{L}[\varphi_m](z) \]

\[ - \alpha_2 \sum_{m=1}^{\infty} \left( s_m^{(1)} + is_m^{(2)} \right) \mathcal{C}[\zeta \varphi_m](z) + \left( s_m^{(3)} + is_m^{(4)} \right) \mathcal{C}[\zeta \varphi_m](z), \] 

(3.6)

in which the explicit computation of the integral operators $\mathcal{L}[\varphi_{\pm m}](z)$ and $\mathcal{C}[\zeta \varphi_{\pm m}](z)$, $m \in \mathbb{N}$, whose general expression is given by (2.15) and (2.16) depends on whether $z \in \mathbb{C}/\Omega$ or $z \in \overline{\Omega}$, as shown in the following. From now on, we assume $\gamma = 1$ for the sake of simplicity.

3.2.1 Series expansion of the single-layer potential inside the inclusion

Lemma 3.1. We parametrize $\partial \Omega$ by $\zeta = \Psi(\eta) = \Psi(e^{i\theta}) \in \partial \Omega$. For $z \in \Omega$ and $m \in \mathbb{N}$, we have

\[ \mathcal{L}[\varphi_m](z) = c_m - \frac{1}{m} F_m(z), \quad \mathcal{L}[\varphi_{-m}](z) = c_{-m}, \]  

(3.7)

\[ \mathcal{C}[\varphi_m](z) = -\frac{1}{m} F_m'(z), \quad \mathcal{C}[\varphi_{-m}](z) = 0 \]  

(3.8)
and

\[
\mathcal{C}[\zeta \varphi_m](z) = \begin{cases} 
- \sum_{k=-1}^{\infty} \frac{a_k}{k+m} F'_{k+m}(z), & m \geq 2, \\
- \sum_{k=0}^{\infty} \frac{a_k}{k+1} F'_{k+1}(z), & m = 1, 
\end{cases}
\]

(3.9)

\[
\mathcal{C}[\zeta \varphi_{-m}](z) = - \sum_{k=m+1}^{\infty} \frac{a_k}{k-m} F'_{k-m}(z).
\]

Here, \(c_{\pm m}\) are constants given by

\[
c_m = \frac{1}{2\pi} \int_{0}^{2\pi} i \theta e^{i m \theta} d\theta \quad \text{and} \quad c_{-m} = -c_m.
\]

(3.10)

**Proof.** From the definition (2.15), we have

\[
\mathcal{L}[\varphi_m](z) = \frac{1}{2\pi} \int_{0}^{2\pi} \log(z - \Psi(\eta)) \eta^m d\theta
\]

(3.11)

and by using the decomposition of the fundamental solution in terms of Faber polynomials, given by (2.25), we get

\[
\mathcal{L}[\varphi_m](z) = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \log(-1) + \log \eta - \sum_{n=1}^{\infty} \frac{1}{n} F_n(z) \eta^{-n} \right] \eta^m d\theta = c_m - \frac{1}{m} F_m(z).
\]

Analogously,

\[
\mathcal{L}[\varphi_{-m}](z) = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \log(-1) + \log \eta - \sum_{n=1}^{\infty} \frac{1}{n} F_n(z) \eta^{-n} \right] \eta^{-m} d\theta = c_{-m}.
\]

(3.12)

Due to the fact that

\[
\mathcal{C}[\varphi_{\pm m}](z) = \mathcal{L}'[\varphi_{\pm m}](z),
\]

we derive (3.8).

By using the Laurent series expansion (2.18) of \(\Psi\), we get

\[
\mathcal{C}[\zeta \varphi_m](z) = \mathcal{C} \left[ \Psi(\eta) \eta^m \frac{1}{h} \right](z) = \mathcal{C} \left[ \sum_{k=-1}^{\infty} \frac{a_k}{m+k} \eta^{m+k} \frac{1}{h} \right](z) = \sum_{k=-1}^{\infty} \frac{a_k}{m+k} \mathcal{C} \left[ \eta^{m+k} \frac{1}{h} \right](z)
\]

(3.13)

and

\[
\mathcal{C}[\zeta \varphi_{-m}](z) = \mathcal{C} \left[ \Psi(\eta) \eta^{-m} \frac{1}{h} \right](z) = \mathcal{C} \left[ \sum_{k=-1}^{\infty} \frac{a_k}{k-m} \eta^{k-m} \frac{1}{h} \right](z) = \sum_{k=-1}^{\infty} \frac{a_k}{k-m} \mathcal{C} \left[ \eta^{k-m} \frac{1}{h} \right](z)
\]

(3.14)

This completes the proof. □
We just have to plug the relations (3.7), (3.8) and (3.9) back into (3.5) and (3.6) to expand \( f(z) \) and \( g(z) \) in terms of Faber polynomials:

\[
f(z) = \alpha_2 \sum_{m=1}^{\infty} \left[ c_m \left( s_m^{(1)} + is_m^{(2)} \right) + c_m \left( s_m^{(3)} + is_m^{(4)} \right) \right] - \alpha_2 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(3)} + is_m^{(4)} \right) F_m(z)
\]

and

\[
g(z) = -\alpha_1 \sum_{m=1}^{\infty} \left[ c_m \left( s_m^{(1)} + is_m^{(2)} \right) + c_m \left( s_m^{(3)} + is_m^{(4)} \right) \right] + \alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(1)} + is_m^{(2)} \right) F_m(z)
\]

\[
+ \alpha_2 \sum_{m=2}^{\infty} \left( s_m^{(1)} + is_m^{(2)} \right) \sum_{k=m+1}^{\infty} \frac{a_k}{k-m} F_{k-m}(z) + \alpha_2 \left( s_1^{(3)} + is_1^{(4)} \right) \sum_{k=0}^{\infty} \frac{a_k}{1+k} F_{1+k}(z)
\]

\[
+ \alpha_2 \sum_{m=2}^{\infty} \left( s_m^{(3)} + is_m^{(4)} \right) \sum_{k=1}^{\infty} \frac{a_k}{k+m} F_{k+m}(z).
\]

Then, by plugging the expansions of \( f(z) \) and \( g(z) \) into (2.12) and by using the fact that

\[ \kappa \omega_2 = \alpha_1 \]

(see (2.9) and (2.10)), we obtain the following theorem.

**Theorem 3.2.** The single-layer potential \( S[\varphi](z) \) with the density function \( \varphi \) given by (3.3) admits the following expansion, for \( z \in \Omega \):

\[
2S[\varphi](z) = -\alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(3)} + is_m^{(4)} \right) F_m(z) + \alpha_2 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(3)} + is_m^{(4)} \right) F_m'(z)
\]

\[
- \alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(1)} + is_m^{(2)} \right) F_m(z) - \alpha_2 \sum_{m=1}^{\infty} \left( s_m^{(1)} + is_m^{(2)} \right) \sum_{k=m+1}^{\infty} \frac{a_k}{k-m} F_{k-m}(z)
\]

\[
- \alpha_2 \left( s_1^{(3)} + is_1^{(4)} \right) \sum_{k=0}^{\infty} \frac{a_k}{1+k} F_{1+k}(z) - \alpha_2 \sum_{m=2}^{\infty} \left( s_m^{(3)} + is_m^{(4)} \right) \sum_{k=1}^{\infty} \frac{a_k}{k+m} F_{k+m}(z)
\]

(3.15)

The derivative of the \( m \)-th Faber polynomial is a linear combination of the Faber polynomials of lower order, that is

\[
F_m'(z) = \sum_{j=0}^{m-1} \gamma_{m,j} F_j(z),
\]

(3.16)

where the coefficients \( \gamma_{m,j}, j = 0, \ldots, m-1 \), depend on the conformal mapping coefficients \( a_k \) (see (2.18)). In matrix form, (3.16) takes the following expression

\[
F' = \Gamma F + \gamma_0,
\]

(3.17)

where

\[
F := \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}, \quad \Gamma := \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \gamma_{1,1} & 0 & 0 & \cdots \\ \gamma_{3,1} & \gamma_{3,2} & 0 & \cdots \end{bmatrix} \quad \text{and} \quad \gamma_0 := \begin{bmatrix} \gamma_{1,0} \\ \gamma_{2,0} \\ \gamma_{3,0} \end{bmatrix}.
\]

(3.18)
By using (3.16) or, equivalently (3.17), we can express \( S[\varphi](z) \) (3.15) in terms of Faber polynomials only (not their derivatives).

### 3.2.2 Series expansion of the single-layer potential outside the inclusion

**Lemma 3.3.** We parameterize \( \partial \Omega \) by \( \zeta = \Psi(\eta) = \Psi(e^{i\theta}) \in \partial \Omega \). For \( z = \Psi(w) \in \mathbb{C} \setminus \overline{\Omega} \) and \( m \in \mathbb{N} \), we have

\[
L[\varphi_m](w) = -\frac{1}{m} F_m(\Psi(w)) + \frac{w^m}{m}, \quad L[\varphi_{-m}](w) = \frac{w^{-m}}{-m} \tag{3.19}
\]

and

\[
C[\zeta \varphi_m](w) = \sum_{k=-1}^{\infty} \frac{a_k}{\Psi'(w)} w^{k-m-1} - \sum_{k=m+1}^{\infty} \frac{a_k}{k-m} F'_{k-m}(\Psi(w)), \tag{3.20}
\]

\[
C[\zeta \varphi_{-m}](w) = \sum_{k=-1}^{\infty} \frac{a_k}{\Psi'(w)} w^{k+m-1} - \left\{ \sum_{k=-1}^{\infty} \frac{a_k}{k+m} F'_{k+m}(\Psi(w)), \quad m \geq 2, \right. \\
\left. \sum_{k=0}^{\infty} \frac{a_k}{k+1} F'_{k+1}(\Psi(w)), \quad m = 1. \tag{3.21} \right. \]

**Proof.** From (2.24) and (2.25), and the properties of Grunsky coefficients, we have

\[
L[\varphi_m](w) = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \log(w) - \sum_{n=1}^{\infty} \frac{1}{n} F_n(\Psi(\eta)) w^{-n} \right] \eta^m d\theta \]

\[
= -\frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{0}^{2\pi} \frac{1}{n} \left( \eta^n + \sum_{k=1}^{\infty} c_{n,k} \eta^{-k} \right) w^{-n} \eta^m d\theta \]

\[
= -\sum_{n=1}^{\infty} \frac{c_{n,m}}{n} w^{-n} = -\sum_{n=1}^{\infty} \frac{c_{m,n}}{m} w^{-n} = -\frac{1}{m} (F_m(\Psi(w)) - w^m)
\]

and

\[
L[\varphi_{-m}](w) = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \log(w) - \sum_{n=1}^{\infty} \frac{1}{n} F_n(\Psi(\eta)) w^{-n} \right] \eta^{-m} d\theta \]

\[
= -\frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{0}^{2\pi} \frac{1}{n} \left( \eta^n + \sum_{k=1}^{\infty} c_{n,k} \eta^{-k} \right) w^{-n} \eta^{-m} d\theta \]

\[
= -\frac{1}{m} w^{-m}.
\]

By recalling from (3.13) and (3.14) that

\[
C[\zeta \varphi_m](w) = \sum_{k=-1}^{\infty} \frac{a_k}{\Psi'(w)} \left[ \eta^{m+k} \frac{1}{h} \right](w) = \sum_{k=-1}^{\infty} \frac{a_k}{\Psi'(w)} \left( L[\eta^{m+k} \frac{1}{h}](w) \right)'
\]
and
\[
C \left[ \zeta \varphi_m \right](w) = \sum_{k=-1}^{\infty} \alpha_k C \left[ \eta^{k-m} \frac{1}{h} \right](w) = \sum_{k=-1}^{\infty} \alpha_k \left( C \left[ \eta^{k-m} \frac{1}{h} \right](w) \right)',
\]
one can then easily find the the expansions of \( C \left[ \zeta \varphi_m \right](w) \) and \( C \left[ \zeta \varphi_{-m} \right](w) \) in (3.20) and (3.21).

Following the same steps taken for the series expansion of the single layer potential in the interior of the inclusion, one has just to plug (3.19), (3.20), and (3.21) into (2.13) and (2.14) to find the expansion for \( S[\varphi](z) \) in (2.14) with \( z = \Psi(w) \) in the exterior of the inclusion. The result is presented in the following theorem.

**Theorem 3.4.** The single-layer potential \( S[\varphi](z) \) with the density function \( \varphi \) given by (3.3) admits the following expansion, for \( z = \Psi(w) \in \Omega \):

\[
S_{\text{ext}}[\varphi](w) = v_1(w) + v_2(w), \quad (3.22)
\]

where

\[
2v_1(w) = -\alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(3)} + is_m^{(4)} \right) F_m(\Psi(w)) + \alpha_2 \Psi(w) \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(3)} + is_m^{(4)} \right) F_m'(\Psi(w))
\]
\[
- \alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(1)} + is_m^{(2)} \right) F_m(\Psi(w)) - \alpha_2 \sum_{m=1}^{\infty} \left( s_m^{(1)} + is_m^{(2)} \right) \sum_{k=m+1}^{\infty} \frac{a_k}{k-m} F_{k-m}'(\Psi(w))
\]
\[
- \alpha_2 \left( s_1^{(3)} + is_1^{(4)} \right) \sum_{k=0}^{\infty} \frac{a_k}{1+k} F_{1+k}'(\Psi(w)) - \alpha_2 \sum_{m=2}^{\infty} \left( s_m^{(3)} + is_m^{(4)} \right) \sum_{k=-1}^{\infty} \frac{a_k}{k+m} F_{k+m}'(\Psi(w)) \quad (3.23)
\]

and

\[
2v_2(w) = \alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(3)} + is_m^{(4)} \right) \left( w^m - \overline{w^m} \right) + \alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(1)} + is_m^{(2)} \right) \left( \overline{w^m} - w^m \right)
\]
\[
+ \alpha_2 \left( -z + \sum_{k=-1}^{\infty} a_k w^k \right) \sum_{m=1}^{\infty} \left( s_m^{(3)} + is_m^{(4)} \right) \frac{w^m-1}{\Psi'(w)} + \sum_{m=1}^{\infty} \left( s_m^{(1)} + is_m^{(2)} \right) \frac{w^{-m}-1}{\Psi'(w)} \frac{1}{\Psi(w)}
\]
\[
+ \alpha_2 \left( s_1^{(3)} + is_1^{(4)} \right) \sum_{k=1}^{\infty} a_k w^k \frac{1}{\Psi'(w)}. \quad (3.24)
\]

### 3.3 Determination of the density function via the transmission condition

In order to solve the rigid inclusion problem, we have first to find the density function \( \varphi(z) \) that satisfies the transmission condition (2.17), that is, we have to determine the unknown real coefficients \( s_m^{(j)}, m \in \mathbb{Z}, j = 1, 2, 3, 4 \) in the expansion (3.3) of \( \varphi(z) \), by comparing the expansion of the single-layer potential, (3.15), with the expansion of the far-field loading (3.2) via (2.17). We recall that the real coefficients \( c_j, j = 1, 2, 3 \) in (2.17) are implicitly
determined by the equilibrium condition (2.4) for which the solution of the transmission problem has first to be found.

The strategy to determine the coefficients \( s_{m}^{(j)} \), \( m \in \mathbb{Z}, j = 1, 2, 3, 4 \) is the following. We will first determine the coefficients \( s_{m}^{(3)} \) and \( s_{m}^{(4)} \), \( m \in \mathbb{Z} \). Below we will show that, for \( m > 1 \), such coefficients are completely determined by the coefficients of the series expansion (3.2) of the far-field loading \( u_{0}(z) \). The determination \( s_{1}^{(3)} \) and \( s_{1}^{(4)} \), instead, as well as the determination of the coefficients \( s_{m}^{(1)} \) and \( s_{m}^{(2)} \), \( m \in \mathbb{Z} \), requires the knowledge of the constant \( c_{3} \), which we will derive at the end of this section by using the linear dependence of \( s_{m}^{(1)} \) and \( s_{m}^{(2)} \), \( m \in \mathbb{Z} \) on \( c_{3} \). Therefore, the density function will be determined up to the knowledge of the two constants \( c_{1} \) and \( c_{2} \), which will finally be calculated by using the equilibrium condition (2.4) involving the solution \( u(z) \) of the problem that will be provided in Section 3.4.

### 3.3.1 Determination of the coefficients \( s_{m}^{(3)} \) and \( s_{m}^{(4)} \), \( m \in \mathbb{Z} \)

By using the transmission condition (2.17) and comparing the terms depending on \( z \in \partial \Omega \) only (not the conjugate \( \overline{z} \)) in (3.15) and (3.2), we get

\[
\begin{align*}
    s_{1}^{(3)} + is_{1}^{(4)} &= \frac{1}{\alpha_{2}} A_{1} + \frac{1}{\alpha_{1}} 2ic_{3}, \\
    s_{m}^{(3)} + is_{m}^{(4)} &= \frac{1}{\alpha_{2}} mA_{m} \quad \text{for each } m \geq 2.
\end{align*}
\]  

(3.25)

(3.26)

Hence, all the coefficients \( s_{m}^{(3)} \) and \( s_{m}^{(4)} \) for \( m \geq 2 \) are explicitly given by the coefficients \( A_{m} \) of the series expansion (3.2) of the far-field loading \( u_{0}(z) \). The case \( m = 1 \) is more complex, due to the fact that \( c_{3} \) in (3.25) is unknown. However, it will be determined by solving a linear equation as explained in Section 3.3.3.

### 3.3.2 Determination of the coefficients \( s_{m}^{(1)} \) and \( s_{m}^{(2)} \), \( m \in \mathbb{Z} \)

By using the relations (3.25) and (3.26), as well as (3.2) and (3.15), the transmission condition (2.17) on \( \partial \Omega \) turns into

\[
P(z) = c_{3} J_{1}(z) + J_{2}(z) + C,
\]  

(3.27)

where

\[
P(z) = -\alpha_{1} \sum_{m=1}^{\infty} \frac{1}{m} \left( s_{m}^{(1)} + is_{m}^{(2)} \right) F_{m}(z) - \alpha_{2} \sum_{m=1}^{\infty} \left( s_{m}^{(1)} + is_{m}^{(2)} \right) \sum_{k=m+1}^{\infty} \frac{a_{k}}{k-m} F_{k-m}(z),
\]  

(3.28)

\[
J_{1}(z) = \frac{2\alpha_{2}}{\alpha_{1}} \sum_{k=0}^{\infty} \frac{a_{k}}{1+k} F_{1+k}(z),
\]

\[
J_{2}(z) = \sum_{m=1}^{\infty} \overline{B}_{m} F_{m}(z) + A_{1} \sum_{k=0}^{\infty} \frac{a_{k}}{1+k} F_{1+k}(z) + \sum_{m=2}^{\infty} mA_{m} \sum_{k=1}^{\infty} \frac{a_{k}}{k+m} F_{k+m}(z),
\]  

(3.29)

\[
C = -\kappa A_{0} + \overline{B}_{0} + 2c_{1} + 2ic_{2}.
\]  

(3.30)
Note that $\mathcal{P}(z)$ is the only complex function that contains the unknowns $s_m^{(1)}$ and $s_m^{(2)}$, $m \in \mathbb{Z}$, whereas $J_1(z)$ and $J_2(z)$ are known complex functions. Indeed, since $\Omega$ is given, we can compute the Faber polynomials $F_m(z)$ associated with $\Omega$ via (2.22), as well as their derivatives (or, equivalently, the coefficients $\gamma_{m,j}$ in (3.10)). Hence, $J_1(z)$ is determined. Since $u_0(z)$ is given, $J_2(z)$ is also completely determined.

For the sake of clarity, we adopt a matrix notation. Then, let us denote with $s$ the vector containing the unknown coefficients $s_m^{(1)}$ and $s_m^{(2)}$, $m \geq 1$, i.e.,

$$s := \begin{bmatrix} s_1^{(1)} + is_1^{(2)} \\ s_2^{(1)} + is_2^{(2)} \\ \vdots \end{bmatrix}.$$ 

To stress the fact that $\mathcal{P}(z)$ is an operator acting on the unknown vector $s$, let us use the notation $\mathcal{P}[s](z)$. To write $\mathcal{P}[s](z)$ in matrix form, let us start by the term $b_m := \sum_{k=m+1}^{\infty} \frac{a_k}{k-m} F'_{k-m}(z), \quad m \geq 1,$

which, in matrix notation, reads

$$b = D A \overline{F}, \quad (3.31)$$

where $F$ is given by (3.18), and

$$b := \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}, \quad A := \begin{bmatrix} a_2 & a_3 & a_4 & \cdots \\ a_3 & a_4 & a_5 & \cdots \\ a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad D := \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \cdots \\
\vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (3.32)$$

By using (3.17), (3.31) turns into

$$b = D A \left( \overline{F} + \overline{y_0} \right), \quad (3.33)$$

and, consequently, the operator $\mathcal{P}[s](z)$ in (3.23) reads

$$\mathcal{P}[s](z) = -\alpha_2 (\kappa s^T D + s^T D A \overline{\Gamma}) \overline{F} - \alpha_2 s^T D A \overline{y_0}, \quad (3.34)$$

In order to write the right hand side of (3.27) in matrix form, let us introduce the vector $y$, defined as the vector containing the coefficients of the expansion of the function

$$J(z) = c_3 J_1(z) + J_2(z) + C$$

with respect to the basis functions $\overline{F}_m$ (up to the coefficient $-\alpha_2$), that is

$$J(z) = -\alpha_2 y^T \overline{F} - \alpha_2 j_0.$$

Note that the unknown constants $c_1$ and $c_2$ are incorporated in the constant $j_0$, whereas the constant $c_3$ appears linearly both in $y$ and $j_0$, due to the linear dependence of $J(z)$ on $c_3$. Hence, equation (3.27) turns into

$$\left(\kappa s^T D + s^T D A \overline{\Gamma}\right) \overline{F} + s^T D A \overline{y_0} = y^T \overline{F} + j_0 \quad (3.35)$$
and by comparing the coefficients of the series expansion with respect to the Faber polynomials, we get

\[ \kappa s^T D + \bar{s}^T D A \bar{\Gamma} = y^T, \tag{3.36} \]

\[ \bar{s}^T D A \bar{\gamma}_0 = j_0. \tag{3.37} \]

As we will show explicitly in Section 4, for domains of order up to 2, the matrix \( A \) defined in (3.32) is the zero matrix and, therefore, the coefficients \( s^{(1)} \) and \( s^{(2)} \) are easily found in terms of the unknown constant \( c_3 \) as (3.36) reduces to

\[ \kappa D s = y, \]

in which the matrix \( D \) defined in (3.32) is invertible, and the vector \( y \) incorporates \( c_3 \). For domains of higher order, the coefficients \( s^{(1)} \) and \( s^{(2)} \) are found as follows. By taking the transpose of (3.36) and then the complex conjugate of the transpose, we respectively get

\[ \kappa D s + \bar{\Gamma}^T A D \bar{s} = y, \]

\[ \bar{\Gamma}^T A D s + \kappa D \bar{s} = \bar{y}, \tag{3.38} \]

which, in a matrix block form, reads

\[
\begin{bmatrix}
\kappa D & \bar{\Gamma}^T A D \\
\Gamma^T A D & \kappa D
\end{bmatrix}
\begin{bmatrix}
s \\ \bar{s}
\end{bmatrix}
= \begin{bmatrix}
y \\ \bar{y}
\end{bmatrix}.
\tag{3.39}
\]

The invertibility of the block matrix in (3.39) has to be assessed case by case. Specifically, it is invertible if \( \kappa D - 1/\kappa \Gamma^T A A \Gamma^T D \) is invertible. In Section 4 we will show some explicit examples in which it can be proven the matrix is invertible. Again, we stress the fact that the vector \( y \) depends linearly on the constant \( c_3 \) so that the solution \( s \), upon inversion of the block matrix, is

\[ s = c_3 u_1 + u_2 \tag{3.40} \]

3.3.3 Determination of the constant \( c_3 \)

The linear dependence of \( s \) on the constant \( c_3 \) in (3.39) plays a crucial role in the determination of \( c_3 \). Indeed, by combining (3.41) and (3.25) we get

\[ \text{Im} (s \cdot \bar{a}) = \frac{\text{Im}(A_1)}{\alpha_2} - \frac{2c_3}{\alpha_1}, \tag{3.41} \]

where \( a \) is the vector containing the coefficients of the conformal mapping (2.18), that is, \( a^T = [a_1; a_2; a_3; \ldots] \). By using (3.40), we get

\[ c_3 = \frac{\kappa \text{Im}(A_1) - \alpha_2 \text{Im}(u_2 \cdot \bar{a})}{2 + \alpha_1 \text{Im}(u_1 \cdot \bar{a})} \tag{3.42} \]

3.4 Solution expansion in the exterior of the inclusion

Due to the ansatz (2.5), the displacement field outside the inclusion is given by

\[ u(z) = u_0(z) + S_{\text{ext}}[\varphi](z) \quad \text{with} \quad z = \Psi(w) \in \mathbb{C}/\Omega \tag{3.43} \]

where \( u_0(z) \) is the given far-field loading, and \( S_{\text{ext}}[\varphi](z) \) is the expansion (3.22) of the single layer potential in the exterior of the inclusion, in which the coefficients \( s_m^{(j)} \), \( j = 1, 2, 3, 4 \), \( m \in \mathbb{Z} \) are determined as explained in Section 3.3.
4 Examples

In order to provide an explicit expression for the solution (3.43) to the transmission problem, the coefficients \( s^{(j)}_m \), \( j = 1, 2, 3, 4 \) have to be determined explicitly: equation (3.26) allows one to obtain all \( s^{(3)}_m \) and \( s^{(4)}_m \) for \( m > 1 \) explicitly in terms of the coefficients \( A_m \) of the series expansion (3.2) of the far-field loading \( u_0(z) \), whereas the case \( m = 1 \), see equation (3.25), requires the knowledge of the constant \( c_3 \), which can be found when the block matrix in (3.39) is invertible. In such a case, the coefficients \( s^{(1)}_m \) and \( s^{(2)}_m \), \( m \in \mathbb{Z} \) are completely determined as well. The inversion of the matrix in (3.39) is ensured for domains of order up to 3, whereas for domains of higher degree some numerical computations are necessary.

4.1 Elliptic inclusion with an arbitrary far-field loading

Let us start by considering the case in which the inclusion is an ellipse. Consequently, the exterior conformal mapping (2.18) takes the following expression:

\[
\Psi(w) = w + \frac{a}{w}.
\]

For this case, one can obtain a simple formula for the Faber polynomials and their derivatives by applying the recursive formula (2.23), with \( a_{-1} = 1 \), \( a_1 = a \), and \( a_j = 0 \) for all \( j \neq \pm 1 \). Indeed, after some algebra, one gets

\[
F_0(z) = 1, \quad F_m(z) = \frac{1}{2^m m} \left[ (z + \sqrt{z^2 - 4a})^m + (z - \sqrt{z^2 - 4a})^m \right], \quad m = 1, 2, \ldots
\]

Upon derivation one obtains

\[
F'_m(z) = \frac{m}{2^m \sqrt{z^2 - 4a}} \left[ (z + \sqrt{z^2 - 4a})^m - (z - \sqrt{z^2 - 4a})^m \right]
\]

and by using the formula

\[
C^m - D^m = (C - D) \sum_{j=0}^{m-1} C^{m-j-1} D^j
\]

one can derive an explicit formula for \( F'_m(z) \). After some lengthy algebra, we obtain

\[
F'_m(z) = mF_{m-1} + \frac{4am}{2^{m-1}} \begin{cases} \sum_{j=0}^{\frac{m}{2}} 2^{2j}(4a)^{\frac{m-2j}{2}-j} F_{2j} & \text{m odd}, \\ \sum_{j=1}^{\frac{m}{2}-1} 2^{2j-1}(4a)^{\frac{m-2j-1}{2}-j} F_{2j-1} & \text{m even}. \end{cases}
\]

Therefore, the matrix \( \Gamma \) and the vector \( \gamma_0 \) in the matrix equation (3.17) take the following
explicit expression:

\[
\Gamma = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & \cdots \\
a & 0 & 4 & 0 & 0 & 0 & 0 & \cdots \\
0 & 5a/4 & 0 & 5 & 0 & 0 & 0 & \cdots \\
6a^2 & 0 & 3a/2 & 0 & 6 & 0 & 0 & \cdots \\
0 & 7a^2 & 0 & 7a/4 & 0 & 7 & 0 & \cdots \\
32a^3 & 0 & 8a^2 & 0 & 2a & 0 & 8 & \cdots \\
0 & 36a^3 & 0 & 9a^2 & 0 & 9a/4 & 0 & 9 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

and \( \gamma_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3a \\ 0 \\ 5a \\ 0 \\ 7a \\ 0 \\ 7a \\ 0 \\ 9a \\ \vdots \end{bmatrix} \). (4.1)

As already mentioned, the coefficients \( s_m^{(3)} \) and \( s_m^{(4)} \), \( m > 1 \), are explicitly given by (3.26) in terms of the coefficients \( A_m \) of the series expansion (3.2) of the far-field loading \( u_0 \), here supposed to be arbitrary.

For what concerns the coefficients \( s_m^{(1)} \) and \( s_m^{(1)} \), we have that, for this case, the known complex functions in the right-hand side of equations (3.27) take the following expression:

\[
J_1(z) = -i\frac{\alpha_1}{\alpha_2} a F_2'(z),
\]

\[
J_2(z) = \sum_{m=1}^{\infty} B_m F_m(z) + \frac{a}{2} A_1 F_2'(z) + \sum_{m=2}^{\infty} m A_m \left( \frac{F_{m-1}(z)}{m-1} + \frac{a F_{m+1}(z)}{m+1} \right).
\]

whereas the operator \( \mathcal{P}[s](z) \) reads

\[
\mathcal{P}[s](z) = -\alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(1)} + i s_m^{(2)} \right) F_m(z) = -\alpha_1 s^T D F
\]

in which the matrix \( D \), given by (3.32), is invertible. Upon inversion, the coefficients \( s_m^{(1)} \) and \( s_m^{(1)} \) can be written as (3.40), and the constant \( c_3 \) is found by means of equation (3.42).

### 4.2 Inclusion of order 2 with an arbitrary far-field loading

Let us consider an inclusion described by the following exterior conformal mapping:

\[
\Psi(w) = w + \frac{a_1}{w} + \frac{a_2}{w^2}.
\]

In this case, it is not straightforward to determine an analytical expression for the Faber polynomials and their derivatives as it is for the ellipse case. Therefore, one has to use the recursive formula (2.23) to generate the Faber polynomials and, consequently, their derivatives.

Again, the coefficients \( s_m^{(3)} \) and \( s_m^{(4)} \), \( m > 1 \), are explicitly provided by (3.27) in which the operator \( \mathcal{P}[s](z) \) takes the following expression

\[
\mathcal{P}[s](z) = -\alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(1)} + i s_m^{(2)} \right) F_m(z) = -\alpha_1 s^T D F
\]
\[ J_1(z) = -i \frac{\alpha_2}{\alpha_1} \left( \frac{a_1}{2} F_2'(z) + \frac{a_2}{3} F_3'(z) \right), \]
\[ J_2(z) = \sum_{m=1}^{\infty} B_m F_m(z) + A_1 \left( \frac{a_1}{2} F_2'(z) + \frac{a_2}{3} F_3'(z) \right) + \sum_{m=2}^{\infty} mA_m \sum_{k=-1}^{2} \frac{a_k}{m+k} F_{m+k}'(z). \]

Again, thanks to the invertibility of the matrix \( D \), the constant \( c_3 \) and the coefficients \( s_m^{(1)} \) and \( s_m^{(2)} \) can be easily found.

### 4.3 Inclusion of order 3 with an arbitrary far-field loading

Let us consider now the case of an inclusion of order 3, for which the conformal mapping looks like

\[ \Psi(w) = w + \frac{a_1}{w} + \frac{a_2}{w^2} + \frac{a_3}{w^3}. \]

Again, the coefficients \( s_m^{(3)} \) and \( s_m^{(4)} \) are given by (3.26), whereas the coefficients \( s_m^{(1)} \) and \( s_m^{(2)} \) are provided by (3.27) in which

\[ P[s](z) = -\alpha_1 \sum_{m=1}^{\infty} \frac{1}{m} \left( s_m^{(1)} + is_m^{(2)} \right) F_m(z) - \alpha_2 a_3 \left( s_1^{(1)} + is_1^{(2)} \right) F_1(z) \quad (4.5) \]

and

\[ J_1(z) = -i \frac{\alpha_2}{\alpha_1} \sum_{k=1}^{3} \frac{a_k}{1+k} F_{1+k}'(z), \]
\[ J_2(z) = \sum_{m=1}^{\infty} B_m F_m(z) + A_1 \sum_{k=0}^{3} \frac{a_k}{1+k} F_{1+k}'(z) + \sum_{m=2}^{\infty} mA_m \sum_{k=-1}^{3} \frac{a_k}{m+k} F_{m+k}'(z). \]

In this case, one has to invert the block matrix in (3.39), which is invertible, given that the matrix \( \Gamma T A \) has only one non-zero entry, that is, the 11-entry.

### 4.4 Matrix formulation for inclusions of higher order with an arbitrary far-field loading

Suppose that the domain \( \Omega \) has order \( M \); that is, the domain is such that the corresponding conformal mapping (2.18) has a finite number of terms with \( a_M \neq 0 \) and \( a_n = 0 \) for all \( n > M \). Then, the matrix \( A \) defined by (3.32) is an upper anti-triangular matrix and so is \( \Gamma T D A \), given that \( D \) is diagonal (see (3.32)) and \( \Gamma T \) is upper triangular (see (3.18)). In particular, we have that

\[ s_m^{(1)} + is_m^{(2)} = \frac{m \alpha_2}{\alpha_1} \eta_m \quad \text{for all } m \geq M - 1 \quad \text{if } \text{ord}(\Omega) = M. \quad (4.6) \]

Indeed, in such a case, the \( ij \)-th entry of \( A \) is zero for all \( i + j \geq M \) and so is for \( \Gamma T D A \). Therefore, the \( m \)-th entry of \( \Gamma T D A s \) is zero for \( m \geq M - 1 \). As a consequence, the
coefficients $s^{(1)}_m + is^{(2)}_m$ with $m \geq M - 1$ can be determined in a unique way, according to (4.6). All that is left is the computation of the coefficients $s^{(1)}_m + is^{(2)}_m$ when $m \leq M - 2$, for which one has to solve the following equation:

$$\begin{bmatrix}
(\Gamma^T DA)_{M-2} & P_{M-2} \\
P_{M-2} & (\Gamma^T D A)_{M-2}
\end{bmatrix}
\begin{bmatrix}
s_{M-2} \\
s_{M-2}
\end{bmatrix}
= \begin{bmatrix}
y_{M-2} \\
y_{M-2}
\end{bmatrix}, \quad (4.7)
$$

where $v_n = (v_i)_{i=1}^n$ denotes the $n \times 1$ subvector of a given vector $v = (v_i)_{i=1}^\infty$, and $B_n$ denotes the $n \times n$ submatrix $(b_{ij})_{i,j=1}^n$ of a given matrix $B = (b_{ij})_{i,j=1}^{\infty}$. Clearly, to determine the coefficients $s^{(1)}_m + is^{(2)}_m$ with $m \leq M - 2$, one has first to assess the invertibility of the $(2M-4) \times (2M-4)$ matrix in (4.7).

5 Conclusion

The plane elastostatic transmission problem is a classical problem in Applied Mechanics, for which the solution is provided explicitly only when the inclusion has a simple shape, such as in the case of an ellipsoidal inclusion for which one can use elliptic coordinates (see, e.g., [3]). For an inclusion of arbitrary shape such a coordinate system can be defined only locally, thus preventing one from finding an explicit series solution to the transmission problem. To the best of our knowledge, there has been no previous work that provides an explicit series solution in the case of a domain of arbitrary shape with an arbitrary far-field loading.

The key idea of our approach, successfully applied to the conductivity case in [23], consists in the introduction of two sets of bases, one for analytic functions defined outside of the inclusion and one for analytic functions defined inside the inclusion. The exterior basis is based on the coordinate system introduced by the external conformal mapping associated with the inclusion, whereas the interior basis is based on the Faber polynomials associated with the inclusion. The introduction of Faber polynomials allows one to overcome the drawbacks associated with the use of conformal mappings, and it presents a novel method to determine the solution of the elastic transmission problem in an elegant way. Indeed, thanks to the properties of Faber polynomials, the transmission condition at the boundary of the inclusion allows one to derive an explicit formula for the coefficients of the series expansion of the transmission problem in terms of the coefficients of the series expansion of the far-field loading, supposed to be arbitrary. Specifically, in this work, we provide an explicit analytical formula in the case of an arbitrary far-field loading and an algebraic inclusion of order up to 3, whereas, for higher order domains, some numerical computations are required.

This paper represents the first step towards a complete characterization of the plane elastostatic transmission problem. Indeed, for the general case, besides the transmission condition regarding the displacements, one should also consider the one concerning the continuity of tractions at the boundary of the inclusion. This would put forward another research avenue—the solution of the so-called E-inclusion problem for the plane elastostatic case—thus extending the results found in [11] for the conductivity problem. Such a problem involves the determination of the shape of the inclusion that provides uniform
fields inside the inclusion for any or some applied far-field loadings. Note that finding E-
inclusions is an important problem in many practical applications concerning the design
of materials which induce stress fields with small variances in the inclusion phase: these
inclusions, which are tailored to the applied field, are generally less likely to break down
than inclusions with large variances of the stress field. The ultimate goal would be to
solve the elastostatic neutral inclusion problem: some coated inclusions, when placed in
a medium, do not disturb the exterior field, and these are denoted as neutral inclusions.
Once a neutral inclusion has been found, similar inclusions, possibly of different sizes, can
be added to the background matrix without altering the exterior uniform field (e.g., [21]).
In this way it becomes possible to construct a composite, consisting of multiple inclusions
and a background matrix, of which the effective property exactly coincides with that of
the matrix.

Acknowledgments

ML is supported by the Basic Science Research Program through the National Research
Foundation of Korea (NRF) funded by the Ministry of Education (2019R1A6A1A10073887).

References

[1] H. Ammari, J. Garnier, W. Jing, H. Kang, M. Lim, K. Sølna, and H. Wang. Mathematical and
statistical methods for multistatic imaging. Lecture Notes in Mathematics (vol. 2098).
Springer, Cham, 2013.

[2] H. Ammari and H. Kang. Reconstruction of Small Inhomogeneities from Boundary
Measurements, volume 1846 of Lecture Notes in Mathematics. Springer-Verlag, Berlin,
Germany / Heidelberg, Germany / London, UK / etc., 2004.

[3] K. Ando, Y.-G. Ji, H. Kang, K. Kim, and S. Yu. Spectral properties of the Neumann-
Poincaré operator and cloaking by anomalous localized resonance for the elasto-static
system. European J. Appl. Math., 29(2):189–225, 2018.

[4] R. J. Asaro and D. M. Barnett. The non-uniform transformation strain problem for
an anisotropic ellipsoidal inclusion. Journal of the Mechanics and Physics of Solids,
23(1):77–83, Feb. 1975.

[5] G. J. Buryachenko. Micromechanics of Heterogeneous Materials. Springer, New York,
2007.

[6] C. Carathéodory. Über die gegenseitige Beziehung der Ränder bei der konformen Ab-
bildung des Inneren einer Jordanschen Kurve auf einen Kreis. Math. Ann., 73(2):305–
320, 1913.

[7] F. Chen, A. Giraud, I. Sevostianov, and D. Grgic. Numerical evaluation of the eshelby
tensor for a concave superspherical inclusion. Int. J. Eng. Sci., 93:51–58, 2015.

[8] Y. P. Chiu. On the stress field due to initial strains in a cuboid surrounded by an
infinite elastic space. ASME Journal of Applied Mechanics, 44:587–590, 1977.
[9] D. Choi, J. Kim, and M. Lim. Analytical shape recovery of a conductivity inclusion based on Faber polynomials. *To appear in Mathematische Annalen.*

[10] D. Choi, J. Kim, and M. Lim. Geometric multipole expansion and its application to neutral inclusions of general shape. *arXiv preprint arXiv:1808.02446*, 2018.

[11] D. Choi, K. Kim, and M. Lim. An extension of the Eshelby conjecture to domains of general shape in anti-plane elasticity. *arXiv preprint arXiv:1807.09981*, 2018.

[12] P. Duren. *Univalent Functions*. Grundlehren der mathematischen Wissenschaften (vol. 259). Springer-Verlag New York, 1983.

[13] G. J. Dvorak. *Micromechanics of Composite Materials*. Springer, Berlin, 2013.

[14] R. H. Edwards. Stress concentrations around spheroidal inclusions and cavities. *ASME Journal of Applied Mechanics*, 18:19–30, 1951.

[15] A. H. England. *Complex Variable Methods in Elasticity*. Wiley-Interscience, New York, 1971.

[16] J. D. Eshelby. The determination of the elastic field of an ellipsoidal inclusion and related problems. *Proceedings of the Royal Society of London*, 241(1226):376–396, 1957.

[17] J. D. Eshelby. Elastic inclusions and inhomogeneities. In I. N. Sneddon and R. Hill, editors, *Progress in Solid Mechanics*, volume II, pages 87–140, Amsterdam, 1961. North-Holland Publishing Co.

[18] G. Faber. Über polynomische entwicklungen. *Math. Ann.*, 57(3):389–408, 1903.

[19] X.-L. Gao and H. M. Ma. Strain gradient solution for Eshelby’s ellipsoidal inclusion problem. *Proc. R. Soc. Lond. Ser. A*, 466(2120):2425–2446, 2010.

[20] J. N. Goodier. Concentration of stress around spherical and cylindrical inclusions and flaws. *ASME Journal of Applied Mechanics*, 55:39–44, 1933.

[21] Z. Hashin. The elastic moduli of heterogeneous materials. *Journal of Applied Mechanics*, 29(1):143–150, Mar. 1962.

[22] M. Huang, W. Zou, and Q.-S. Zheng. Explicit expression of Eshelby tensor for arbitrary weakly non-circular inclusion in two-dimensional elasticity. *Int. J. Eng. Sci.*, 47(11):1240 – 1250, 2009.

[23] Y. Jung and M. Lim. A new series solution method for the transmission problem. *arXiv preprint arXiv:1803.09458*, 2018.

[24] Y. Jung and M. Lim. A decay estimate for the eigenvalues of the Neumann-Poincaré operator using the grunsky coefficients. *Proc. Amer. Math. Soc.*, 148(2):591–600, 2020.
[25] H. Kang. Conjectures of Pólya–Szegő and Eshelby, and the Newtonian potential problem: A review. Mechanics of Materials: an International Journal, 41(4):405–410, Apr. 2009. Special Issue in Honor of Graeme W. Milton, 2007 Winner of the William Prager Medal of the Society of Engineering Science.

[26] H. Kang and G. W. Milton. Solutions to the Pólya–Szegő conjecture and the Weak Eshelby Conjecture. Archive for Rational Mechanics and Analysis, 188(1):93–116, Apr. 2008.

[27] H. Kang and S. Yu. Quantitative characterization of stress concentration in the presence of closely spaced hard inclusions in two-dimensional linear elasticity. Arch. Ration. Mech. Anal., 232(1):121–196, 2019.

[28] N. Kinoshita and T. Mura. Elastic fields of inclusions in anisotropic media. Physica Status Solidi (a), 5(3):759–768, 1971.

[29] V. D. Kupradze. Potential methods in the theory of elasticity. Israel program for scientific translations, 1965.

[30] Y.-G. Lee and W.-N. Zou. Exterior elastic fields of non-elliptical inclusions characterized by laurent polynomials. Eur. J. Mech. A-Solid, 60:112–121, 2016.

[31] R. D. List and J. P. O. Silberstein. Two-dimensional elastic inclusion problems. Mathematical Proceedings of the Cambridge Philosophical Society, 62(2):303–311, 1966.

[32] L. P. Liu. Solutions to the Eshelby conjectures. Proceedings of the Royal Society A: Mathematical, Physical, & Engineering Sciences, 464(2091):573–594, May 2008.

[33] J.-k. Lu. Complex variable methods in plane elasticity, volume 22. World Scientific, 1995.

[34] J. C. Maxwell. A Treatise on Electricity and Magnetism, volume 1, pages 371–372. Clarendon Press, Oxford, UK, 1873. Article 322.

[35] N. I. Muskhelishvili. Some Basic Problems of the Mathematical Theory of Elasticity: Fundamental Equations, Plane Theory of Elasticity, Torsion, and Bending. P. Noordhoff, Groningen, The Netherlands, 1963.

[36] H. Nozaki and M. Taya. Elastic fields in a polygon-shaped inclusion with uniform eigenstrains. ASME Journal of Applied Mechanics, 64:495–502, 1997.

[37] W. J. Parnell. The Eshelby, Hill, Moment and Concentration Tensors for Ellipsoidal Inhomogeneities in the Newtonian Potential Problem and Linear Elastostatics. Journal of Elasticity, 125(2):231–294, Dec 2016.

[38] S. D. Poisson. Second mémoire sur la théorie de magnétisme. (French) [Second memoir on the theory of magnetism. Mémoires de l’Académie royale des Sciences de l’Institut de France, 5:488–533, 1826.

[39] C. Pommerenke. Boundary behaviour of conformal maps. Grundlehren der mathematischen Wissenschaften (vol. 299). Springer-Verlag, Berlin, Germany, 1992.
[40] J. Qu and M. Cherkaoui. *Fundamentals of Micromechanics of Solids*. Wiley, New York, 2006.

[41] K. Robinson. Elastic energy of an ellipsoidal inclusion in an infinite solid. *Journal of Applied Physics*, 22(8):1045–1054, 1951.

[42] G. J. Rodin. Eshelby’s inclusion problem for polygons and polyhedra. *Journal of the Mechanics and Physics of Solids*, 44:1977–1995, 1996.

[43] C.-Q. Ru. Analytic Solution for Eshelby’s Problem of an Inclusion of Arbitrary Shape in a Plane or Half-Plane. *ASME Journal of Applied Mechanics*, 66(2):315–523, June 1999.

[44] C.-Q. Ru and P. Schiavone. On the elliptic inclusion in anti-plane shear. *Mathematics and Mechanics of Solids : MMS*, 1(3):327–333, Sept. 1996.

[45] M. A. Sadowsky and E. Sternberg. Stress concentration around an ellipsoidal cavity in an infinite body under arbitrary plane stress perpendicular to the axis of revolution of cavity. *ASME Journal of Applied Mechanics*, 14:191–201, 1947.

[46] M. A. Sadowsky and E. Sternberg. Stress concentration around a triaxial ellipsoidal cavity. *ASME Journal of Applied Mechanics*, 16:149–157, 1949.

[47] G. P. Sendeckyj. Elastic inclusion problems in plane elastostatics. *International Journal of Solids and Structures*, 6(12):1535–1543, Dec. 1970.

[48] R. Southwell and H. Gough. VI. On the concentration of stress in the neighbourhood of a small spherical flaw; and on the propagation of fatigue fractures in ”statistically isotropic” materials. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 1(1):71–97, 1926.

[49] L. J. Walpole and R. Hill. The elastic field of an inclusion in an anisotropic medium. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 300(1461):270–289, 1967.

[50] K. Zhou, H. J. Hoh, X. Wang, L. M. Keer, J. H. Pang, B. Song, and Q. J. Wang. A review of recent works on inclusions. *Mechanics of Materials*, 60:144 – 158, 2013.

[51] W. Zou, Q. He, M. Huang, and Q. Zheng. Eshelby’s problem of non-elliptical inclusions. *Journal of the Mechanics and Physics of Solids*, 58(3):346 – 372, 2010.