Multi-point fractional Brownian bridges and their applications

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We propose and test a method to interpolate sparsely sampled signals by a stochastic process with a broad range of spatial and/or temporal scales. To this end, we extend the notion of a fractional Brownian bridge, defined as fractional Brownian motion with a given scaling (Hurst) exponent $H$ and with prescribed start and end points, to a bridge process with an arbitrary number of prescribed intermediate and non-equidistant points. We demonstrate the validity of our method on a signal from fluid turbulence in a high Reynolds number flow. Furthermore, we discuss possible extensions of the present work to include the non-self-similar character of the signal. The derived method could be instrumental within a variety of fields such as astrophysics, particle tracking, specific tailoring of surrogate data, and spatial planning.

Many non-equilibrium phenomena in physics involve random fluctuations with a wide range of spatial and/or temporal scales [1, 2]. The theory of stochastic processes provides a conceptual framework to describe such phenomena [3, 4]. The most emblematic example is provided by Brownian motion, which results from random uncorrelated collisions acting on a particle, and can be described by a simple Wiener process [5]. By contrast, several complex systems in nature also involve long- or short-range correlations, which require a description in terms of fractional Brownian motion [6, 7]. Examples include velocity fluctuations in turbulence [8–10], magnetic field fluctuations in the solar wind [11], random amoebid motion [12], or heart interbeat fluctuations [13]. In the following, we will restrict ourselves to temporal processes, or more generally to processes depending on only one variable. Fractional Brownian motion (fBm) $X(t)$ is a nonstationary centered Gaussian process, and thus entirely characterized by its covariance

$$\langle X(t)X(t') \rangle = \frac{1}{2} \left( t^{2H} + t'^{2H} - |t-t'|^{2H} \right). \quad (1)$$

which implies that $\langle (X(t) - X(t'))^2 \rangle = |t-t'|^{2H}$. The Hurst exponent $H$ determines negative (antipersistent) correlations for $0 < H < 1/2$, whereas positive (persistent) correlations prevail for $1/2 < H < 1$.

In general, the initial value $X(t=0)$ of a fBm $X(t)$ is fixed and no further conditions can be imposed during the entire evolution, for $0 < t < t_1$, where $t_1$ is the final time. Nevertheless, some applications such as the tracking of animal movement [14], information-based financial models [15], the number of neutrons involved in reactor diffusion [16, 17], or cosmic ray propagation in astrophysical turbulence [18] require that the stochastic process follows certain constraints.

To be more specific, we focus on the important example of cosmic ray propagation in turbulent magnetic fields [19, 20]. In order to overcome the overwhelming problem of resolving the wide range of scales involved in these extremely high Reynolds number astrophysical flows, several methods for generating synthetic turbulent fields were developed in the last decades [21–24]. Such methods are frequently implemented and used in major cosmic ray propagation codes (see e.g. [18]). To capture large anisotropies due to the geometry of galaxies (spiral arms, outflow regions, bow shocks), synthetic turbulent fields must be embedded in large-scale magnetohydrodynamic (MHD) simulations of the turbulent interstellar or intergalactic plasma. Therefore, in this problem we face two challenges: i.) fBm, which is used to generate synthetic fields, must be constrained to match the values on the numerical grid of the MHD simulation and ii.) scaling properties, represented by the Hurst exponent of fluctuations of the coarse-grained MHD simulation must be determined from sparse grid data, in order to allow for an “optimal stochastic interpolation” of sparsely-sampled data. In this letter, we address precisely these two issues.

We first present a method to construct a fBm $X(t)$, which takes specific values $X_i$ at times $t_i$, and discuss its application as an optimal stochastic interpolation of a sparsely-sampled real time signal. Accordingly, we start with the well-known notion of a fractional Brownian bridge [25, 26], which is defined as fBm starting from $0$ at $t = 0$, ending at $X_1$ at $t = t_1$, and possessing the same statistical (including scaling (1)) properties as $X(t)$. Such a fractional Brownian bridge (fBb) can be constructed from $X(t)$ according to

$$X^B(t) = X(t) - (X(t_1) - X_1) \frac{\langle X(t)X(t_1) \rangle}{\langle X(t_1)^2 \rangle}. \quad (2)$$

It is possible to generalize this ordinary fBb to an arbitrary number of prescribed intermediate grid points in the following manner: First, we consider the $n$-times con-
ditional moments
\[ \langle X(t)|\{X_i,t_i\}\rangle = \frac{\langle X(t) \prod_{i=1}^n \delta(X(t_i) - X_i) \rangle}{\prod_{i=1}^n \langle \delta(X(t_i) - X_i) \rangle} \], \quad (3)
\[ \langle X(t)X(t')|\{X_i,t_i\}\rangle = \frac{\langle X(t)X(t') \prod_{i=1}^n \delta(X(t_i) - X_i) \rangle}{\prod_{i=1}^n \langle \delta(X(t_i) - X_i) \rangle} \]. \quad (4)

We then demand that our bridge process \( X^B(t) \) is conditional on \( X_i \) at \( t_i \) for \( i = 1, \ldots, n \), which is equivalent to the process possessing the conditional moments (3-4). For a Gaussian process with zero mean and covariance \( \langle X(t)X(t')\rangle \), the conditional moments read
\[ \langle X(t)|\{X_i,t_i\}\rangle = \langle X(t)X(t_i) \rangle \sigma^{-1}_{ij} X_j \], \quad (5)
and
\[ \langle X(t)X(t')|\{X_i,t_i\}\rangle = \langle X(t)X(t') \rangle - \langle X(t)X(t_i) \rangle \left[ \sigma^{-1}_{ij} - \sigma^{-1}_{ik} \delta_{ki} \sigma^{-1}_{jl} \right] \langle X(t)_iX(t_j) \rangle \], \quad (6)
where we implied summation over equal indices and where \( \delta_{ij} = \langle X(t_i)X(t_j) \rangle \) denotes the covariance matrix.

As shown in [27], the multi-point fractional Brownian bridge
\[ X^B(t) = X(t) - \langle X(t_i) - X_i \rangle \sigma^{-1}_{ij} \langle X(t)X(t_i) \rangle \], \quad (7)
possesses one- and two-point moments which are identical to (5-6) and we thus conclude that \( X^B(t) \) is the stochastic process \( X(t) \) conditioned on points \( X_i \) at times \( t_i \). We indeed obtain \( X^B(t_k) = X(t_k) - \langle X(t_i) - X_i \rangle \sigma^{-1}_{ij} \delta_{ij} = X(t_k) - \langle X(t_i) - X_i \rangle \delta_{ij} = X_k \).

In order to check that the simulated bridge processes possess the desired properties (1) and (5-6), we have carried out numerical calculations of the second order structure functions \( S_2(\tau) = \langle (X^B(t+\tau) - X^B(t))^2 \rangle \) for three different Hurst exponents \( H = 0.33, 0.5, 0.66 \) with a total number of \( N = 32768 \) total grid points. From these we prescribed \( N = 32 \) equidistant points generated from fBm (1) with a Hurst exponent \( \tilde{H} = 0.5 \).

The results are shown in Fig. 1(b) and are in agreement with the prediction \( S_2(\tau) = |\tau|^{2H} \) (dashed black lines) for small \( \tau \). For such a case, where the prescribed points follow fBm with Hurst exponent \( \tilde{H} \), we can obtain an explicit formula for \( S_2(\tau) \) from Eq. (7), namely
\[ S_2(\tau) = |\tau|^{2H} \] \quad (8)
\[ - \langle \delta_{ij}X(t)X(t_i) \rangle \left[ \sigma^{-1}_{ij} - \sigma^{-1}_{ik} \delta_{ki} \sigma^{-1}_{jl} \right] \langle \delta_{ij}X(t)X(t_j) \rangle \],
where \( \delta_{ij}X(t) = X(t+\tau) - X(t) \) and where \( \sigma_{ij} = \langle X_iX_j \rangle \) denotes the covariance matrix of the prescribed points with \( \tilde{H} \). Consequently, \( \tilde{H} = H \) implies \( \sigma = \tilde{\sigma} \), which yields \( S_2(\tau) = |\tau|^{2H} \). In other words, given a certain time series \( \{X_i,t_i\} \) that possess a self-similar part governed by \( \tilde{H} \), the bridge with \( H = \tilde{H} \) can be considered as the optimal stochastic interpolation of this time series.

Therefore, as already highlighted in the introduction, we are now in the position to describe an optimization procedure that allows us to estimate the Hurst exponents from sparsely sampled time series. The basic idea is to...
embed a given time series \( \{X_i, t_i\} \) into fBbs (7) with varying Hurst exponents \( H \). For each of these bridges we determine the empirical Hurst exponent \( H_{emp} \) as a function of \( H \) by fitting the second order structure function up to the smallest time scale of the time series \( dt \), (i.e., fitting only the left part up to \( dt_{coarse} \) in Fig. 1(b)). This procedure ensures that we only measure deviations from the scaling \( |\tau|^{2H} \) in the interpolated region (grey lines in Fig. 1(b)) and are not directly contaminated by correlations contained in \( \{X_i, t_i\} \). We have tested our optimization for three different samples of fBm with Hurst exponents \( H = 0.33, 0.5, 0.66 \) and \( N = 128 \) grid points. Each of the samples was embedded in fBbs with varying Hurst exponents \( H \) and \( N = 4096 \) grid points. Fig. 1(c) depicts the minimization of \( H_{emp}(H) \) for the three samples. It can be clearly seen that optimal Hurst exponents \( H_{opt} \) are recovered with high accuracy, although slight variations (\( \Delta H_{opt} \approx 0.015 \) from 20 different samples of the same \( H \)) between different samples can be observed [27]. This effect can be attributed to finite sample sizes, and the corresponding deviations remain rather small, which is quite appealing given the fact that common methods (rescaled range analysis [29] or wavelets [30]) yield erroneous results for such sparsely sampled time series. The proposed method can thus roughly be considered as the extrapolation of self-similar properties of a given time series to finer scales.

In the examples discussed so far, we have systematically chosen the synthetic signal \( X_i \), as well as the process \( X(t) \) used in Eqs. (1,7), to be normalized in the same manner. In order to apply our optimization to real signals, we examine a turbulent velocity time series obtained from hot wire anemometry in the superfluid high Reynolds von Kármán experiment (SHREK) at CEA-Grenoble [31]. The particular experimental setup is a von Kármán cell with two-counter rotating disks (-0.12 Hz on top, +0.18 Hz on bottom) in normal Helium (see [32] for further specifications). The temporal resolution is 50kHz and the attained Taylor-Reynolds number was \( Re_{\lambda} = 2737 \). We applied Taylor’s hypothesis of frozen turbulence [33] to relate single-point velocity measurement at time \( t \) to scales \( x = \langle u \rangle t \), where \( \langle u \rangle \) is the mean velocity. Furthermore, a key prerequisite for the above mentioned optimization procedure is the standardization of the signal by \( \lim_{r \to \infty} \sqrt{\langle (u(x) - \langle u(x) \rangle)^2 \rangle} = \sqrt{2 \langle u(x)^2 \rangle} = \sqrt{2} \sigma \). This standardization ensures the correct large-scale limit of the second-order structure function in Fig. 1(b), which was necessarily fulfilled by the synthetic samples of fBm (1) in the previous study. The blue curve in Fig. 2(a) depicts an extract of the velocity field \( u(x) \) standardized by \( \sigma \) of the total signal for around one integral length scale \( L \).

By contrast to the above optimization procedure for data from synthetic samples, the present analysis is complicated by: i.) the existence of different scaling regimes in the flow, namely a dissipative and integral range of scales, and ii.) non-self similar (intermittent) features in the signal. Turning to point i.), we chose sample sizes of length \( L \) and determined the subset of points in order to guarantee that their grid length lies within the inertial range of scales (here we choose \( \approx 100 \eta \)). As far as point ii.) is concerned, intermittent features manifest themselves in form of strongly varying \( H_{opt} \) for different samples [27]. An example of such an intermittent fluctuation is well visible in the signal shown in

![Figure 2](image-url)

FIG. 2. (a) Turbulent velocity field measurements (blue) in a von Kármán experiment using normal Helium. The number of points \( N = 16384 \) corresponds roughly to one integral length scale \( L \). The corresponding fBb (orange) was constructed from \( N = 64 \) points of the signal (black) and possesses a resolution of 1024 points. The smallest scale of the bridge process therefore corresponds approximately to the Taylor scale of the flow which ensures that the fBb and the velocity field \( u(x) \) possess comparable inertial ranges. Small-scale turbulent fluctuations in the velocity field (blue) cannot be reproduced by the fBb due to its restricting Gaussian properties. The Hurst exponent for the fBb \( H = 0.376 \) was determined from randomized samples of the original turbulent signal such as the one depicted in (b). Due to the self-similarity of these samples, an optimization procedure similar to the one depicted in Fig. 1(c) could be applied. The bridge (orange) is in much better agreement with the self-similar signal (blue) in (b) than with the original signal (blue) in (a).
Fig. 2(a), at $x \approx 0.8L$. In this region our interpolation procedure does not work very well and the bridge process (7) has to be generalized to a non-Gaussian process. In this letter, however, we are solely interested in the self-similar part of the signal and thus perform a randomization of Fourier phases of the turbulent signal. A snapshot of the resulting randomized signal is depicted in Fig. 2(b). Strikingly, and contrary to Fig. 2(a), our interpolation procedure leads to much better results than for the original signal. The randomization procedure has effectively suppressed intermittency, and made the signal essentially Gaussian. We fed several samples of this randomized signal into our optimization routine which reduced fluctuations of the Hurst exponent to an extend comparable to the ones observed in synthetic signals [27]. Moreover, we obtain a Hurst exponent of the randomized signal $H_{opt} = 0.3786 \pm 0.0251$ (evaluated from the optimization of 100 snapshots [27]). To put this result into context, we consider the log-normal model of turbulence [34, 35] which suggests scaling exponents $\zeta_n = n/3 - \mu(n - 3)/18$ for the scaling of velocity structure functions $\langle (u(x + r) - u(x))^n \rangle \sim |r|^{\zeta_n}$, where $\mu$ denotes the intermittency coefficient. Hence, the self-similar part in the model is given by $H_{KGA} = (2 + \mu)/6$ and our analysis suggests that $\mu = 0.2716 \pm 0.1506$ which is comparable to $\mu = 0.2913 \pm 0.0853$ acquired from an analysis of the entire turbulent signal [27].

To conclude, we have presented a generalization of a fractional Brownian bridge to a stochastic process with an arbitrary number of prescribed points. Furthermore, we devised an optimization method which allowed us to estimate the Hurst exponent of a sparsely sampled time series. Our method has proven reliable even in the presence of strong anomalous fluctuations, i.e., non-self-similar features, at small scales. In order to address such features, which are visible but not captured by the multi-point fBb in Fig. 2(a), it will be a task for the future to construct multi-point bridge processes with multiscaling properties (i.e., which are non-self-similar and potentially possess a dissipative scale [36, 37]). A generalization of the bridge process (7) to an arbitrary number of dimensions is straight-forward and might be of potential interest for the construction of various synthetic fields in several physical contexts. In turbulence, the full spatio-temporal (though non-intermittent) Eulerian velocity field $u(x, t)$ can possibly be reconstructed from a set of Lagrangian trajectories $X(y, t)$ where $\dot{X}(y, t) = u(X(y, t), t)$. Latter application could be of considerable interest for particle tracking measurements, which sometimes require a certain knowledge of the flow field in the vicinity of tracer particles [38]. Furthermore, the optimization procedure may help to shed light into the ongoing discussion about the inertial range power spectrum in the solar wind [11, 39]. Last, we mention possible applications to the widely different domain of urban decision making, where our method could be applied to model land price fields [40].

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I. CALCULATION OF CONDITIONAL MOMENTS FOR CENTERED GAUSSIAN PROCESS

Although the conditional moments of a multivariate Gaussian distribution are well known, in what follows, we want to give a brief derivation. To this end, we consider the characteristic functional

\[ \varphi[\alpha] = \left( e^{i \int d\alpha(t) X(t)} \right). \]  

For a centered Gaussian process \( X(t) \), the characteristic functional reduces to (see, for instance [1])

\[ \varphi[\alpha] = e^{-\frac{1}{2} \int d\alpha(t) \int d\alpha(t') \langle X(t)X(t') \rangle \alpha(t')}. \]  

The \( n \)-point probability density function (PDF)

\[ f_n(X_1,t_1; \ldots; X_n,t_n) = \prod_{i=1}^{n} \delta(X(t_i) - X_i), \]  

can thus be re-expressed according to

\[
\begin{align*}
    f_n(X_1,t_1; \ldots; X_n,t_n) &= \int \frac{dk_1}{2\pi} \ldots \frac{dk_n}{2\pi} e^{-i \sum_{i=1}^{n} k_i X_i} \left( e^{i \sum_{i=1}^{n} k_i X(t_i)} \right) \\
    &= \int \frac{dk_1}{2\pi} \ldots \frac{dk_n}{2\pi} e^{-i \sum_{i=1}^{n} k_i X_i} \varphi \left[ \alpha(t) = \sum_{i=1}^{n} k_i \delta(t - t_i) \right] \\
    &= \int \frac{dk_1}{2\pi} \ldots \frac{dk_n}{2\pi} e^{-i \sum_{i=1}^{n} k_i X_i} \exp \left[ -\frac{1}{2} \sum_{i=1}^{n} \sum_{i' \neq i}^{n} k_i k_{i'} \langle X(t_i)X(t_{i'}) \rangle \right].
\end{align*}
\]

Therefore, we obtain a multivariate Gaussian distribution

\[ f_n(X_1,t_1; \ldots; X_n,t_n) = \frac{1}{\sqrt{(2\pi)^n \det(\sigma)}} e^{-\frac{1}{2} X^T \sigma^{-1} X}, \]  

where

\[ X = (X_1, \ldots, X_n)^T, \]  

and where

\[ \sigma_{ij} = \langle X(t_i)X(t_j) \rangle, \]  

denotes the covariance matrix. Next, we can calculate the \( n+1 \)-point quantity,

\[
\begin{align*}
    \prod_{i=1}^{n} \langle X(t) \delta(X(t_i) - X_i) \rangle &= \int \frac{dk_1}{2\pi} \ldots \frac{dk_n}{2\pi} e^{-i \sum_{i=1}^{n} k_i X_i} \langle X(t) e^{i \sum_{i=1}^{n} k_i X(t_i)} \rangle \\
    &= \int \frac{dk_1}{2\pi} \ldots \frac{dk_n}{2\pi} e^{-i \sum_{i=1}^{n} k_i X_i} \left[ \frac{\delta \varphi[\alpha]}{\delta \alpha(t)} \right]_{\alpha(t) = \sum_{i=1}^{n} k_i \delta(t - t_i)} \\
    &= i \int \frac{dk_1}{2\pi} \ldots \frac{dk_n}{2\pi} e^{-i \sum_{i=1}^{n} k_i X_i} \sum_{i=1}^{n} k_i \langle X(t_i)X(t_i) \rangle e^{-\frac{1}{2} \sum_{i=1}^{n} \sum_{i' \neq i}^{n} k_i k_{i'} \langle X(t_i)X(t_{i'}) \rangle} \\
    &= -\sum_{i=1}^{n} \langle X(t)X(t_i) \rangle \frac{d}{dX_i} f_n(X_1,t_1; \ldots; X_n,t_n),
\end{align*}
\]

\[ (7) \]
as well as the $n + 2$-point quantity

$$\prod_{i=1}^{n}(X(t)X(t')\delta(X(t_i) - X_i)) = \int \frac{dk_1}{2\pi} \ldots \frac{dk_n}{2\pi} e^{-i\sum_{i=1}^{n} k_iX_i} \left\langle X(t)X(t')e^{i\sum_{i=1}^{n} k_iX(t_i)} \right\rangle$$

$$= \int \frac{dk_1}{2\pi} \ldots \frac{dk_n}{2\pi} e^{-i\sum_{i=1}^{n} k_iX_i} \left. \frac{\delta^2 \varphi[\alpha]}{\delta \alpha(t)\delta \alpha(t')} \right|_{\alpha(t) = \sum_{i=1}^{n} k_i\delta(t-t_i)}$$

$$= \int \frac{dk_1}{2\pi} \ldots \frac{dk_n}{2\pi} e^{-i\sum_{i=1}^{n} k_iX_i} \left[ \langle X(t)X(t') \rangle - \sum_{i=1}^{n} \sum_{i' = 1}^{n} k_i k_{i'} \langle X(t)X(t_i) \rangle \langle X(t')X(t_{i'}) \rangle \right] e^{-\frac{1}{2} \sum_{i=1}^{n} \sum_{i' = 1}^{n} k_i k_{i'} \langle X(t_i)X(t_{i'}) \rangle}$$

$$= \left[ \langle X(t)X(t') \rangle + \sum_{i=1}^{n} \sum_{i' = 1}^{n} \langle X(t)X(t_i) \rangle \langle X(t')X(t_{i'}) \rangle \frac{d^2}{dX_i dX_{i'}} \right] f_n(X_1, t_1; \ldots; X_n, t_n) , \quad (8)$$

The $n$-times conditional moments

$$\langle X(t)\{X_i, t_i\} \rangle = \frac{\langle X(t) \prod_{i=1}^{n} \delta(X(t_i) - X_i) \rangle}{\prod_{i=1}^{n} \delta(X(t_i) - X_i)} , \quad (9)$$

$$\langle X(t)X(t')\{X_i, t_i\} \rangle = \frac{\langle X(t)X(t') \prod_{i=1}^{n} \delta(X(t_i) - X_i) \rangle}{\prod_{i=1}^{n} \delta(X(t_i) - X_i)} , \quad (10)$$

thus read

$$\langle X(t)\{X_i, t_i\} \rangle = \sigma_i^{-1} \langle X(t)X(t_i) \rangle X_j , \quad (11)$$

and

$$\langle X(t)X(t')\{X_i, t_i\} \rangle = \langle X(t)X(t') \rangle - \left[ \sigma_{ij}^{-1} - \sigma_{ik}^{-1} X_k \sigma_{jl}^{-1} \right] \langle X(t)X(t_i) \rangle \langle X(t')X(t_j) \rangle . \quad (12)$$

where we made use of the symmetry of the covariance matrix and where we imply summation over equal indices.

**II. PROOF FOR MOMENTS OF MULTI-POINT FRACTIONAL BROWNIAN BRIDGE**

In this section, we want to prove that the multi-point fBb

$$X^B(t) = X(t) - (X(t_i) - X_i) \sigma_{ij}^{-1} \langle X(t)X(t_j) \rangle , \quad (13)$$

possesses identical one and two-points moments to the moments fBm process $X(t)$ conditioned on $\{X_i, t_i\}$, i.e., eqs. (11-12). Therefore, we first calculate the mean of the fBb in eq. (13)

$$\langle X^B(t) \rangle = \langle X(t) \rangle - \underbrace{(X(t) - X_i) \sigma_{ij}^{-1} \langle X(t)X(t_j) \rangle}_{=0}$$

$$= X_i \sigma_{jk}^{-1} \langle X(t)X(t_j) \rangle , \quad (14)$$

where we assumed that the fBm possesses zero mean. Next, the correlation function of the generalized fBb reads

$$\langle X^B(t)X^B(t') \rangle = \langle X(t)X(t') \rangle - 2 \langle X(t)X(t_i) \rangle \sigma_{ij}^{-1} \langle X(t)X(t_j) \rangle + \underbrace{(X(t_i)X(t_j)) \sigma_{ik}^{-1} \langle X(t)X(t_i) \rangle \langle X(t')X(t_j) \rangle \sigma_{jl}^{-1}}_{=0}$$

$$+ \sigma_{ik}^{-1} X_k \sigma_{jl}^{-1} X_l \langle X(t)X(t_i) \rangle \langle X(t')X(t_j) \rangle$$

$$= \langle X(t)X(t') \rangle - \sigma_{ij}^{-1} \langle X(t)X(t_i) \rangle \langle X(t')X(t_j) \rangle + \sigma_{ik}^{-1} X_k \sigma_{jl}^{-1} X_l \langle X(t)X(t_i) \rangle \langle X(t')X(t_j) \rangle , \quad (15)$$

where we made use of the identity $\sigma_{ki} \sigma_{ki}^{-1} = \delta_{ki}$. 
III. VARIATIONS OF THE OPTIMAL HURST EXPONENT

An optimization procedure similar to Fig. 1(c) of the main paper for different synthetic samples drawn as fBm with Hurst index $\tilde{H}$ yields slightly different results for the optimal Hurst exponent $H_{\text{opt}}$. In order to compare to the optimization procedure for the turbulent signal later, we choose 100 synthetic samples of fBm with Hurst exponent $\tilde{H} = 0.3786$ with a spatial resolution of $N = 128$ points. Fig. 1(a) depicts the histograms for the optimal Hurst exponent obtained via optimization procedure as discussed in the main paper. The histogram is clearly peaked at the prescribed values $\tilde{H}$ and we obtain $\langle H_{\text{opt}} \rangle = 0.3783 \pm 0.013$. These slight variations are likely due to finite-size effects. Turning next to the turbulent signal in Fig. 2(a) of the main paper, we first perform a Fourier transform of the entire signal and then randomize Fourier phases in order to get rid of strong (intermittent) correlations. The resulting signal possesses Gaussian statistics which can be verified by computing the PDF of velocity increments $\delta u = u(x + r) - u(x)$ at different scales $r$, as it has been done in Fig. 2(a). For comparison, Fig. 2(b) depicts the velocity increment PDF of the original turbulent signal for the same scale separations $r$ as in (a). As to be expected the PDFs develop pronounced tails at small scales, a key signature of small-scale intermittency. Due to the self-similar property of the randomized signal in Fig. 2(a), the corresponding fluctuations can be treated by the same optimization routine as the synthetic fBm. However, as explained in the main text, one has to take care of the existence of a dissipation scale in both the turbulent and randomized signal. To this end, we consider plateaus in log-derivatives of structure functions and determine the small-scale cut-off to be approximately $96\eta$, where $\eta$ denotes the Kolmogorov microscale. We thus perform the optimization procedure for 100 different sub-samples of the randomized signal. The result is shown in Fig. 1(b) and shows many similarities to the one obtained from synthetic samples in Fig. 1(a), although the standard deviation is slightly higher. In fact, we obtain $\langle H_{\text{opt}} \rangle = 0.3786 \pm 0.0251$.

In order to assess this result, we consider the K62 model of turbulence [2, 3], which predicts structure functions

$$
\langle (\delta u)^n \rangle \sim |r|^{\zeta_n} \quad \text{with scaling exponents} \quad \zeta_n = \frac{n}{3} - \frac{\mu}{18} n(n - 3) = an - bn^2 .
$$

(16)
The estimated value $\langle H_{opt} \rangle$ implies that $\mu = 0.2716 \pm 0.1506$. This result, which has been obtained from the optimization procedure of the randomized signal can now be compared to the original turbulent case as follows: First, the velocity increment PDF of the K62 model is accessible via a Mellin transform of the scaling exponents (16), which yields

$$f(\delta r u) = \frac{1}{2\pi \delta r u \sqrt{\ln r}} \int_{-\infty}^{\infty} dx \, e^{-x^2} \exp \left[ -\frac{\left( \ln\frac{\delta r u}{r \sqrt{2\pi}} \right)^2}{4b \ln r} \right],$$

This PDF is now used as a fit for the velocity increment PDFs in Fig. 2(b) after a symmetrization (the PDFs in Fig. 2(b) are strongly skewed, a direct consequence of the turbulent energy transfer). The symmetrization is depicted in Fig. 2(c) and fitting the whole range of scales that were used for the optimization procedure from above, we obtain $\mu = 0.2913 \pm 0.0833$. Examples of the corresponding PDF (17) for different scales $r$ are indicated as the dashed curves in Fig. 2(c). Hence, the intermittency coefficient determined from the optimization procedure of the randomized signal and the intermittency coefficient from the turbulent signal agree fairly well.

Finally, we applied the optimization procedure to the original turbulent signal in keeping the same range of scales and number of sub-samples as for the randomized signal. The result is shown in Fig. 1(c). Compared to (a) and (b) the optimal Hurst exponent exhibits stronger sample-to-sample fluctuations. The mean value, however, agrees very well with the Hurst exponent of the randomized signal and we obtain $\langle H_{opt} \rangle \approx 0.3797 \pm 0.0368$. In order to further categorize these fluctuations, which can be considered as a direct signature of intermittency (i.e., non-self-similarity), we determined the joint PDF of the optimal Hurst parameter and the energy dissipation rate $\varepsilon = 2\nu \left( \frac{\partial u(x)}{\partial x} \right)^2$ averaged over each sample. The result is depicted in Fig. 3 (a). Even with limited statistics, we observe a clear negative correlation between energy dissipation and the determined value of the optimal Hurst exponent.

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