LOCAL CURVATURE ESTIMATES FOR THE LAPLACIAN FLOW

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ABSTRACT. In this paper we give local curvature estimates for the Laplacian flow on closed $G_2$-structures under the condition that the Ricci curvature is bounded along the flow. The main ingredient consists of the idea of Kotschwar-Munteanu-Wang [24] who gave local curvature estimates for the Ricci flow on complete manifolds and then provided a new elementary proof of Sesum’s result [36], and the particular structure of the Laplacian flow on closed $G_2$-structures. As an immediate consequence, this estimates give a new proof of Lotay-Wei’s [33] result which is an analogue of Sesum’s theorem.

The second result is about an interesting evolution equation for the scalar curvature of the Laplacian flow of closed $G_2$-structures. Roughly speaking, we can prove that the time derivative of the scalar curvature $R_t$ is equal to the Laplacian of $R_t$, plus an extra term which can be written as the difference of two nonnegative quantities.

1. INTRODUCTION

Let $\mathcal{M}$ be a smooth 7-manifold. The Laplacian flow for closed $G_2$-structures on $\mathcal{M}$ introduced by Bryant [1] is to study the torsion-free $G_2$-structures

\begin{equation}
\partial_t \varphi_t = \Delta_{\varphi_t} \varphi_t, \quad \varphi_0 = \varphi,
\end{equation}

where $\Delta_{\varphi_t} \varphi_t = dd^*_{\varphi_t} \varphi_t + d_{\varphi_t}^* d_{\varphi_t} \varphi_t$ is the Hodge Laplacian of $g_{\varphi_t}$, and $\varphi$ is an initial closed $G_2$-structure. Since $dd^*_{\varphi_t} \varphi_t = d_{\varphi_t} d_{\varphi_t} \varphi_t = 0$, we see that the flow (1.1) preserves the closedness of $\varphi_t$. For more background on $G_2$-structures, see Section 2. When $\mathcal{M}$ is compact, the flow (1.1) can be viewed as the gradient flow for the Hitchin functional introduced by Hitchin [17].

\begin{equation}
\mathcal{H} : [\varphi]_+ \to \mathbb{R}^+, \quad \varphi \mapsto \frac{1}{7} \int_{\mathcal{M}} \varphi \wedge \varphi = \int_{\mathcal{M}} *_{\varphi} 1.
\end{equation}

Here $\overline{\varphi}$ is a closed $G_2$-structure on $\mathcal{M}$ and $[\overline{\varphi}]_+$ is the open subset of the cohomology class $[\overline{\varphi}]$ consisting of $G_2$-structures. Any critical point of $\mathcal{H}$ gives a torsion-free $G_2$-structure.

The study of Laplacian flows on some special 7-manifolds, Laplacian solitons, and other flows on $G_2$-structures can be found in [12, 13, 14, 15, 18, 23, 28, 34, 35, 37, 38].

Recently, Donaldson [6, 7, 8, 9] studied the co-associative Kovalev-Lefschetz fibrations $G_2$-manifolds and $G_2$-manifolds with boundary.

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1.1. Notions and conventions. To state the main results, we fix our notions used throughout this paper. Let $\mathcal{M}$ be as before a smooth 7-manifold. The space of smooth functions and the space of smooth vector fields are denoted respectively by $C^\infty(\mathcal{M})$ and $\mathcal{X}(\mathcal{M})$. The space of $k$-tensors (i.e., $(0,k)$-covariant tensor fields) and $k$-forms on $\mathcal{M}$ are denoted, respectively, by $\otimes^k(\mathcal{M}) = C^\infty(\otimes^k(T^*\mathcal{M}))$ and $\Lambda^k(\mathcal{M}) = C^\infty(\Lambda^k(T^*\mathcal{M}))$. For any $k$-tensor field $T \in \otimes^k(\mathcal{M})$, we locally have the expression $T = T_{i_1\ldots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} =: T_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. A $k$-form $\alpha$ on $\mathcal{M}$ can be written in the standard form as $\alpha = \frac{1}{k!} a_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where $a_{i_1\ldots i_k}$ is fully skew-symmetric in its indices. Using the standard forms, if we take the interior product $X \lrcorner \alpha$ of a $k$-form $\alpha \in \Lambda^k(\mathcal{M})$ with a vector field $X \in \mathcal{X}(\mathcal{M})$, we obtain the $(k-1)$-form $X\lrcorner \alpha = \frac{1}{k!} X_{i_1\ldots i_k} a_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$ which is also in the standard form. In particular, consider the vector space $\otimes^2(\mathcal{M})$ of 2-tensors. For any 2-tensor $A = A_{ij} dx^{i \otimes j}$, define $A^\circ := \frac{1}{2} (A_{ij} + A_{ji}) dx^{i \otimes j} \equiv A^\circ_{ij} dx^{i \otimes j}$ and $A^\wedge := \frac{1}{2} (A_{ij} - A_{ji}) dx^{i \wedge j} \equiv A^\wedge_{ij} dx^{i \wedge j}$. Then $A^\circ$ is an element of $\otimes^2(\mathcal{M})$, the space of symmetric 2-tensors. Since $dx^{i \wedge j} = dx^{i \otimes j} - dx^{j \otimes i}$, it follows that $A^\wedge = \frac{1}{2} A_{ij} dx^{i \wedge j}$. Define $\alpha^A := \frac{1}{2} \alpha_i^A dx^{i \wedge j}$ with $\alpha_i^A := A_{ij}$. Then we see that $\alpha^A = A^\wedge \in \wedge^2(\mathcal{M})$ and $\otimes^2(\mathcal{M}) = \otimes^2(\mathcal{M}) \oplus \wedge^2(\mathcal{M})$.

A given Riemannian metric $g$ on $\mathcal{M}$ determines two isomorphisms between vector fields and 1-forms: $\gamma^g: \mathcal{X}(\mathcal{M}) \rightarrow \Lambda^1(\mathcal{M})$ and $\delta^g: \Lambda^1(\mathcal{M}) \rightarrow \mathcal{X}(\mathcal{M})$, where, for every vector field $X = X^i \frac{\partial}{\partial x^i}$ and 1-form $\alpha = \alpha_i dx^i$, $\gamma^g(X) = X^i g_{ij} dx^j = X_i dx^j$ and $\delta^g(\alpha) = \alpha_i g^{ij} \frac{\partial}{\partial x^j} \equiv \alpha_i \frac{\partial}{\partial x^i}$. Using these two natural maps, we can frequently raise or lower indices on tensors. The metric $g$ also induces a metric on $k$-forms $g \langle dx^{i_1} \wedge \cdots \wedge dx^{i_k}, dx^{j_1} \wedge \cdots \wedge dx^{j_k} \rangle = \det(g)(dx^{i_1}, dx^{j_1}) \cdots g^{j_k i_k} = \sum_{\sigma \in S_k} \text{sgn}(\sigma) g^{j_1 i_1} \cdots g^{j_k i_k}$ where $S_k$ is the group of permutations of seven letters and $\text{sgn}(\sigma)$ denotes the sign $(\pm 1)$ of an element $\sigma$ of $S_k$. The inner product $\langle \cdot, \cdot \rangle_g$ on 2-forms $\alpha, \beta \in \Lambda^2(\mathcal{M})$ now is given by $\langle \alpha, \beta \rangle_g = \frac{1}{2} \alpha_{ij} g^{ij} \beta_{kl} + \frac{1}{2} \beta_{ij} g^{ij} \alpha_{kl}$. Given two 2-tensors $A, B \in \otimes^2(\mathcal{M})$, with the forms $A = A_{ij} dx^{i \otimes j}$ and $B = B_{ij} dx^{i \otimes j}$. Define $\langle \langle A, B \rangle \rangle_g := A_{ij} B_{ij}$. There are two special cases which will be used later:

1. $\alpha = \frac{1}{2} \alpha_{ij} dx^{i \wedge j} \in \wedge^2(\mathcal{M})$ and $B = B_{ij} dx^{i \otimes j} \in \otimes^2(\mathcal{M})$. In this case, $\alpha$ can be written as a 2-tensor $A^\circ = A^\circ_{ij} dx^{i \otimes j}$ with $A^\circ_{ij} = \alpha_{ij}$. Then $\langle \langle \alpha, B \rangle \rangle_g := \langle \langle A^\circ, B \rangle \rangle_g = \alpha_{ij} B_{ij}$.
2. $\alpha = \frac{1}{2} \alpha_{ij} dx^{i \wedge j}$ and $\beta = \frac{1}{2} \beta_{ij} dx^{i \otimes j} \in \wedge^2(\mathcal{M})$. In this case, $\alpha, \beta$ can be both written as 2-tensors $A^\circ = A^\circ_{ij} dx^{i \otimes j}$ and $B^\circ = B^\circ_{ij} dx^{i \otimes j}$ with $A^\circ_{ij} = \alpha_{ij}$ and $B^\circ_{ij} = \beta_{ij}$. Then $\langle \langle \alpha, \beta \rangle \rangle_g := \langle \langle A^\circ, \beta \rangle \rangle_g = \alpha_{ij} \beta_{ij} = 2 \langle \langle A, \beta \rangle \rangle_g$.  

\footnote{In our convention, for any 2-form $\alpha = \frac{1}{2} \alpha_{ij} dx^i \wedge dx^j$, we have $\alpha \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \frac{1}{2} \alpha_{ij} \left( dx^i \otimes dx^j - dx^j \otimes dx^i \right) = \frac{1}{2} \alpha_{ij} \left( \delta_i^l \delta_j^k - \delta_j^l \delta_i^k \right) = \frac{1}{2} \left( \alpha_{kl} - \alpha_{lk} \right) = \alpha_{ik}$, which justifies the notion $\alpha_{ik}$ as $\alpha(\partial/\partial x^k, \partial/\partial x^i)$. In general, for any $k$-form $\alpha = \frac{1}{k!} \alpha_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, we have $\alpha_{i_1\ldots i_k} = \alpha(\partial/\partial x^{i_1}, \ldots, \partial/\partial x^{i_k})$, because $dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \text{sgn}(\sigma) dx^{\sigma(i_1)} \otimes \cdots \otimes dx^{\sigma(i_k)}$.}
The norm of $A \in \otimes^2(M)$ is defined by $||A||^2_g := \langle \langle A, A \rangle \rangle_g = A_{ij}A^{ij}$, while the norm of $\alpha \in \wedge^k(M)$ is $|\alpha|^2_g := \langle \alpha, \alpha \rangle_g = \frac{1}{k!} a_{i_1\cdots i_k}a^{i_1\cdots i_k}$. In particular, $||X||^2_g = X_iX^i = |b_\mathbf{g}(X)|^2_g$ and $||\alpha||^2_g = 2|\alpha|^2_g$, for any vector field $X \in \chi(M)$ and 2-form $\alpha$.

The Levi-Civita connection associated to a given Riemannian metric $g$ is denoted by $\nabla$ or simply $\nabla$. Our convention on Riemann curvature tensor is $\nabla_i\nabla_j - \nabla_j\nabla_i = R_{ijkl}g^{kl}$. The Ricci curvature of $g$ is given by $R_{ij} := R_{ijkl}g^{kl}$. We use $dV_g$ and $*_g$ to denote the volume form and Hodge star operator, respectively, on $M$ associated to a metric $g$ and an orientation.

We use the standard notion $A \ast B$ to denote some linear combination of contractions of the tensor product $A \otimes B$ relative to the metric $g_t$ associated the $\varphi_t$. In Theorem 1.2 and its proof, all universal constants $c, C$ below depend only on the given real number $p$.

1.2. Main results. Applying De Turck’s trick and Hamilton’s Nash-Moser inverse function theorem, Bryant and Xu [2] proved the following local time existence for (1.1).

**Theorem 1.1.** (Bryant-Xu [2]) For a compact 7-manifold $M$, the initial value problem (1.1) has a unique solution for a short time interval $[0, T_\text{max})$ with the maximal time $T_\text{max} \in (0, \infty]$ depending on $\varphi$.

As in the Ricci flow, we can prove following results on the long time existence for the Laplacian flow (1.1).

**Theorem 1.2.** (Lotay-Wei [3]) Let $M$ be a compact 7-manifold and $\varphi_t$, $t \in [0, T)$, where $T < \infty$, be a solution to the flow (1.1) for closed $G_2$-structures with associated metric $g_t = g_{\varphi_t}$ for each $t$.

(a) If the velocity of the flow satisfies

$$\sup_{M \times [0, T)} ||\Delta_t \varphi_t||_t < \infty,$$

then the solution $\varphi_t$ can be extended past time $T$.

(b) If $T = T_{\max}$, then

$$\lim_{t \to T_{\max}} \sup_M \left( ||\nabla_t \varphi_t||^2_t + ||\nabla T_t||^2_t \right) = \infty.$$

Here $T_t$ is the torsion of $\varphi_t$ (see (2.14)).

In this paper, we give a new elementary proof of Theorem 1.2 based on the idea of [2] and the structure of the equation (1.1).

**Theorem 1.3.** Let $M$ be a compact 7-manifold and $\varphi_t$, $t \in [0, T)$, where $T < \infty$, be a solution to the flow (1.1) for closed $G_2$-structures with associated metric $g_t = g_{\varphi_t}$ for each $t$. Suppose that

$$K := \sup_{M \times [0, T)} ||\nabla \varphi_t||_t < \infty, \quad \Lambda := \sup_M ||\nabla T_0||_0.$$
Then
\[ \sup_{\mathcal{M} \times [0, T)} ||Rm_t||_t < \infty, \]
where the bound depends only on \( n, K, T \) and \( \Lambda \).

When \( \mathcal{M} \) is compact, the theorem immediately implies the part (a) in Theorem 1.2. Indeed, we shall show that (see (3.18) and (3.37))
\[ \sup_{\mathcal{M} \times [0, T)} ||\Delta_t \varphi_t||_t < \infty \iff \sup_{\mathcal{M} \times [0, T)} ||\text{Ric}_t||_t < \infty. \]

In the compact case, Theorem 1.3 shows that, if the conclusion in part (a) does not hold, then \( T = T_{\text{max}} \) and \( \sup_{\mathcal{M} \times [0, T_{\text{max}})} ||Rm_t||_t < \infty \), which implies \( \sup_{\mathcal{M} \times [0, T_{\text{max}})} (||Rm_t||_t^2 + ||\nabla_t T_t||_t^2) < \infty \), since the norm \( ||\nabla_t T_t||_t^2 \) can be controlled by \( ||Rm_t||_t^2 \) (see (3.63)). However, by part (b) in Theorem 1.2, it is impossible. Therefore, the conclusion in part (a) is true.

As remarked in [24], to prove Theorem 1.3, it suffices to establish the following integral estimate.

**Theorem 1.4.** Let \( \mathcal{M} \) be a smooth 7-manifold and \( \varphi_t, t \in [0, T) \), where \( T < \infty \), be a solution to the flow \( (1.1) \) for closed G\(_2\)-structures with associated metric \( g_t = g_{\varphi_t} \) for each \( t \). Assume that there exist constants \( A, K > 0 \) and a point \( x_0 \in \mathcal{M} \) such that the geodesic ball \( B_{g_0}(x_0, A/\sqrt{K}) \) is compactly contained in \( \mathcal{M} \) and that
\[ ||\text{Ric}_t||_t \leq K \text{ on } B_{g_0}(x_0, A/\sqrt{K}) \times [0, T]. \]

Then, for any \( p \geq 5 \), there exists \( c = c(p) > 0 \) so that
\[
\int_{B_{g_0}(x_0, A/\sqrt{K})} ||Rm_t||_t^p dV_t \leq c(1 + K)e^{cKT} \int_{B_{g_0}(x_0, A/\sqrt{K})} ||Rm_0||_0^p dV_0
\]
\[ + cK^p \left( 1 + A^{-2p} \right) e^{cKT} \text{vol}_t \left( B_{g_0}(x_0, A/\sqrt{K}) \right), \]
(1.3)
for all \( t \in [0, T] \).

Now by the standard De Giorgi-Nash-Moser iteration (our manifold is compact and the Ricci curvature is uniformly bounded), under the condition in Theorem 1.4, we can prove
\[ ||Rm_T||_T(x_0) \leq d_1(d_2 + \Lambda_0), \]
where \( d_1, d_2 \) are constants depending on \( K, T, A \), and
\[ \Lambda_0 := \sup_{B_{g_0}(x_0, A/\sqrt{K})} ||Rm_0||_0. \]
Actually, this follows from the same argument in [24] by noting that
\[ (\Delta_t - \partial_t)||Rm_t||_t \geq -c||Rm_t||_t^2. \]
(1.5)
To verify (1.5), we use (2.26), (3.61) and (3.65) to deduce that \( ||\nabla_t T_t|| \leq c||Rm_t||_t \) and
\[ ||\nabla_t^2 T_t||_t \leq c||\nabla_t Rm_t||_t + c||Rm_t||_t^{3/2}. \]
Then, by (3.31) and the Cauchy inequality
\[ ||\nabla t Rm_t||^2_t \leq -\frac{1}{2} (\partial_t - \Delta_t) ||Rm_t||^2_t + c ||Rm_t||^{3/2}_t ||\nabla_t Rm_t||_t \]
\[ \leq -\frac{1}{2} (\partial_t - \Delta_t) ||Rm_t||^2_t + c ||Rm_t||^3_t + ||\nabla_t Rm_t||^2_t \]
which implies (1.5). Now the estimate (1.4) yields Theorem 1.3.

The analogue of Theorem 1.2 in the Ricci flow was proved by Hamilton [16] (for part (b)) and Sesum [36] (for part (a)). It is an open question (due to Hamilton, see [3]) that the Ricci flow will exist as long as the scalar curvature remains bounded. For the Kähler-Ricci flow [39] or type-I Ricci flow [10], this question was settled. For the general case, some partial result on Hamilton’s conjecture was carried out in [3].

For the Ricci-harmonic flow introduced by List [29, 30] (see also, [31, 32]), the analogue of Theorem 1.2 was proved in [29, 30] (see also, [31, 32]) and [4] (see [27] for another proof). The author [25, 26] extended Cao’s result [3] to the Ricci-harmonic flow. The same Hamilton’s conjecture was asked by the author in [25, 26].

We can ask the same question for the Laplacian flow on closed $G_2$-structures. In [33] (see Page 171, line -6 to -3, or Open Problem (3) in Page 230), Lotay and Wei asked whether the Laplacian flow on closed $G_2$-structures will exist as long as the torsion tensor or scalar curvature remains bounded. Let $g_t$ be the associated metric of $\varphi_t$. Then the evolution equation for $g_t$ is given by

\[ \partial_t g_{ij} = -2R_{ij} - \frac{4}{3} |T_{ij}|^2 g_{ij} - 4T_{ik}T_{kj}. \]

For the Laplacian flow on closed $G_2$-structures, the torsion $T_i$ is actually a 2-form for each $t$, hence we use the norm $| \cdot |_t$ in (1.6). The standard formula for the scalar curvature $R_t$ gives (see (3.23))

\[ \partial_t R_t = \Delta_t R_t + 2 ||\text{Ric}_t||^2_t - \frac{2}{3} R_t^2 + 4R_{ijk}T^{ik}T^{jl} + 4(\nabla/T^{jk})(\nabla_l T_{jk}). \]

Now the above mentioned open problem states that

Is it true that $\lim_{t \to T_{\max}} R_t = -\infty$?

The “minus infinity” comes from the fact that along the Laplacian flow on closed $G_2$-structures the scalar curvature is always nonpositive (see (2.26)). The following Proposition 1.5 is motivated to solve this problem, and starts from the basic evolution equation (1.7) where the last two terms on the right-hand side do not have good signature. However, using the closedness of $\varphi_t$ (in particular, the identity (3.23)), we can prove the following interesting evolution equation for $R_t$.

Proposition 1.5. Let $\mathcal{M}$ be a smooth 7-manifold and $\varphi_t$, $t \in [0, T)$, where $T \in (0, \infty]$, be a solution to the flow (1.1) for closed $G_2$-structures with associated metric $g_{\varphi_t}$ for
each $t$. Then the scalar curvature $R_t$ satisfies

$$
\partial_t R_t = \Delta_t R_t + \left\{ 2 \left| R_{ij} + \frac{2}{3} \left| T_{ij} \right|_t^2 \right|^2 \right. + \frac{1}{2} \left| R_{ijab} R_{ijmn} - \psi_{abmn} \right|^2_t

+ \frac{1}{2} \left| 2T_{ij} T_{jb} R_{ijmn} - \psi_{abmn} \right|^2_t

+ \frac{1}{2} \left| 2T_{j.tm} \psi_{abmn} \right|^2_t

+ 2 \left( \left| T_{ij} \right|^2_t + 4 \left| \nabla T_{ij} \right|^2_t \right) - \left\{ \left| Rm_t \right|^2_t + \frac{26}{9} \left| \mathcal{R} \right|^2_t + \frac{1}{2} \left| R_{ijab} R_{ijmn} \right|^2_t \right. \\

+ 2 \left| T_{ij} T_{jb} R_{ijmn} \right|^2_t \right\} + 2 \left| \hat{T}_t \right|^4_t + 210.

(1.8)

Here $\hat{T}_{ij} = T_i \hat{T}_{kj}$.}

Observe that the above well-arranged evolution equation can give us a weakly lower bound for $R_t$, which can not prove or disprove the conjecture of Lotay and Wei.

We give an outline of the current paper. We review the basic theory in Section 2 about $G_2$-structures, $G_2$-decompositions of 2-forms and 3-forms, and general flows on $G_2$-structures. In Section 3, we rewrite results in Section 2 for closed $G_2$-structures, and the local curvature estimates will be given in the last subsection.

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2. Basic theory of $G_2$-structures

In this section, we view some basic theory of $G_2$-structures, following [1][9][20][21][22][33]. Let $\{e_1, \ldots, e_7\}$ denote the standard basis of $\mathbb{R}^7$ and let $\{e^1, \ldots, e^7\}$ be its dual basis. Define the 3-form

$$
\phi := e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5 + e^1 \wedge e^6 \wedge e^7 - e^2 \wedge e^4 \wedge e^6 - e^2 \wedge e^5 \wedge e^7 - e^3 \wedge e^4 \wedge e^7 - e^3 \wedge e^5 \wedge e^6,
$$

where $e^{i+j+k} := e^i \wedge e^j \wedge e^k$. The subgroup $G_2$, which fixes $\phi$, of $\text{GL}(7, \mathbb{R})$ is the 14-dimensional Lie subgroup of $\text{SO}(7)$, acts irreducibly on $\mathbb{R}^7$, and preserves the metric and orientation for which $\{e_1, \ldots, e_7\}$ is an oriented orthonormal basis. Note that $G_2$ also preserves the 4-form

$$
* \phi = e^4 \wedge e^5 \wedge e^6 \wedge e^7 + e^2 \wedge e^3 \wedge e^5 \wedge e^7 + e^2 \wedge e^3 \wedge e^4 \wedge e^6 + e^1 \wedge e^3 \wedge e^4 \wedge e^6 - e^1 \wedge e^2 \wedge e^5 \wedge e^6 - e^1 \wedge e^2 \wedge e^5 \wedge e^7 - e^1 \wedge e^2 \wedge e^4 \wedge e^7.
$$

where the Hodge star operator $* \phi$ is determined by the metric and orientation.
For a smooth 7-manifold $\mathcal{M}$ and a point $x \in \mathcal{M}$, define as in \cite{[13]}
\[ \wedge^3_+(T_x^* \mathcal{M}) := \left\{ \varphi_x \in \wedge^3(T_x^* \mathcal{M}) : \text{ for some invertible map } u \in \text{Hom}_R(T_x \mathcal{M}, \mathbb{R}^7) \right\} \]
and the bundle
\[ \wedge^3_+(T^* \mathcal{M}) := \bigsqcup_{x \in \mathcal{M}} \wedge^3_+(T_x^* \mathcal{M}). \]
We call a section $\varphi$ of $\wedge^3_+(T^* \mathcal{M})$ a positive 3-form on $\mathcal{M}$ or a $G_2$-structure on $\mathcal{M}$, and denote the space of positive 3-forms by $\wedge^3_+(\mathcal{M})$. The existence of $G_2$-structures is equivalent to the property that $\mathcal{M}$ is oriented and spin, which is equivalent to the vanishing of the first and second Stiefel-Whitney classes. From the definition of $G_2$-structures, we see that any $\varphi \in \wedge^3_+(\mathcal{M})$ uniquely determines a Riemannian metric $g_\varphi$ and an orientation $dV_\varphi$, hence the Hodge star operator $*_{\varphi}$ and the associated 4-form
\[ \psi := *_{\varphi} \varphi. \]
We also have the isomorphisms $b_\varphi := b_{g_\varphi}$ and $\sharp_\varphi := \sharp_{g_\varphi}$. For a given $G_2$-structure $\varphi \in \wedge^3_+(\mathcal{M})$, we denote by $\langle \cdot, \cdot \rangle_\varphi$, $\langle \langle \cdot, \cdot \rangle \rangle_\varphi$, $| \cdot |_\varphi$, $|| |_\varphi$, the corresponding inner products $\langle \cdot, \cdot \rangle_{g_\varphi}$, $\langle \langle \cdot, \cdot \rangle \rangle_{g_\varphi}$ and norms $| \cdot |_{g_\varphi}$, $|| |_{g_\varphi}$.

Given a $G_2$-structure $\varphi \in \wedge^3_+(\mathcal{M})$. We say that $\varphi$ is torsion-free if $\varphi$ is parallel with respect to the metric $g_\varphi$. Equivalently, $\varphi$ is torsion-free if and only if $\varphi \nabla \varphi = 0$, where $\varphi \nabla$ is the Levi-Civita connection of $g_\varphi$.

**Theorem 2.1. (Fernández-Gray \cite{[13]})** The $G_2$-structure $\varphi$ is torsion-free if and only if $\varphi$ is both closed (i.e., $d\varphi = 0$) and co-closed (i.e., $d^*_{\varphi} \varphi = d\varphi = 0$).

When $\mathcal{M}$ is compact, the above theorem says that a $G_2$-structure $\varphi$ is torsion-free if and only if $\varphi$ is harmonic with respect to the induces metric $g_\varphi$.

We say that a $G_2$-structure $\varphi$ is closed (resp., co-closed) if $d\varphi = 0$ (resp., $d^*_{\varphi} \varphi = 0$). Theorem 2.1 can be restated as that a $G_2$-structure is torsion-free if and only if it is both closed and co-closed.

### 2.1. $G_2$-decompositions of $\wedge^2(\mathcal{M})$ and $\wedge^3(\mathcal{M})$

A $G_2$-structure $\varphi$ induces splittings of the bundles $\wedge^k(T^* \mathcal{M})$, $2 \leq k \leq 5$, into direct summands, which we denote by $\wedge^k(T^* \mathcal{M}, \varphi)$ with $\ell$ being the rank of the bundle. We let the space of sections of $\wedge^k(T^* \mathcal{M}, \varphi)$ by $\wedge^k_+(\mathcal{M}, \varphi)$. Define the natural projections
\[ \pi^k_\ell : \wedge^k(\mathcal{M}) \longrightarrow \wedge^k_+(\mathcal{M}, \varphi), \quad \alpha \longmapsto \pi^k_\ell(\alpha). \]
We mainly focus on the $G_2$–decompositions of $\wedge^2(\mathcal{M})$ and $\wedge^3(\mathcal{M})$. Recall that
\[ \begin{align*}
\wedge^2(\mathcal{M}) &= \wedge^2_+(\mathcal{M}, \varphi) \oplus \wedge^2_{14}(\mathcal{M}, \varphi), \\
\wedge^3(\mathcal{M}) &= \wedge^3_+(\mathcal{M}, \varphi) \oplus \wedge^3_7(\mathcal{M}, \varphi) \oplus \wedge^3_{27}(\mathcal{M}, \varphi).
\end{align*} \]
Here each component is determined by

\[ \wedge^3_2(\mathcal{M}, \varphi) = \{X_\varphi : X \in \mathfrak{X}(\mathcal{M})\} = \{\beta \in \wedge^2(\mathcal{M}) : \ast_\varphi(\varphi \wedge \beta) = 2\beta\}, \]

\[ \wedge^2_{14}(\mathcal{M}, \varphi) = \{\beta \in \wedge^2(\mathcal{M}) : \varphi \wedge \beta = 0\} = \{\beta \in \wedge^2(\mathcal{M}) : \ast_\varphi(\varphi \wedge \beta) = -\beta\}, \]

\[ \wedge^3_1(\mathcal{M}, \varphi) = \{f \varphi : f \in C^\infty(\mathcal{M})\}, \]

\[ \wedge^3_2(\mathcal{M}, \varphi) = \{\ast_\varphi(\varphi \wedge \alpha) : \alpha \in \wedge^1(\mathcal{M})\} = \{X_\varphi : X \in \mathfrak{X}(\mathcal{M})\}, \]

\[ \wedge^3_{27}(\mathcal{M}, \varphi) = \{\eta \in \wedge^3(\mathcal{M}) : \varphi \wedge \eta = \eta \wedge \varphi = 0\}. \]

For any 2-form \( \beta = \frac{1}{2}\beta_{ij}dx^i \wedge dx^j \in \wedge^2(\mathcal{M}) \), its two components \( \pi^2_2(\beta) \) and \( \pi^2_{14}(\beta) \) are determined by

\[ \pi^2_2(\beta) = \frac{\beta + \ast_\varphi(\varphi \wedge \beta)}{3} = \frac{1}{2} \left( \frac{1}{3} \beta_{ab} + \frac{1}{6} \beta^m \psi_{imab} \right) dx^{ab}, \]

\[ \pi^2_{14}(\beta) = \frac{2\beta - \ast_\varphi(\varphi \wedge \beta)}{3} = \frac{1}{2} \left( \frac{2}{3} \beta_{ab} - \frac{1}{6} \beta^m \psi_{imab} \right) dx^{ab}. \]

To decompose 3-forms, recall two maps introduced by Bryant \[ \[ (2.7) \]

\[ i_\varphi : \odot^2(\mathcal{M}) \longrightarrow \wedge^3(\mathcal{M}), \quad j_\varphi : \wedge^3(\mathcal{M}) \longrightarrow \odot^2(\mathcal{M}), \]

where

\[ i_\varphi(h) := h_{ij}g^{ij}dx^i \wedge \left( \frac{\partial}{\partial x^j} \varphi \right) = \frac{1}{2} h_{ij} \varphi^i \psi_{jk} dx^{ijk} \]

\[ = \frac{1}{6} \left( h_{ij} \varphi^i_\ell + h_{ij} \varphi^i_i + h_{kj} \varphi_{kij}^{\ell} \right) dx^{ijk}, \quad h = h_{ij}dx^{ij} \in \odot^2(\mathcal{M}), \]

and

\[ (i_\varphi(\eta))(X,Y) := \ast_\varphi((X_\varphi \wedge (Y_\varphi) \wedge \eta). \]

Then \( i_\varphi \) is injective and is isomorphic onto \( \wedge^3_2(\mathcal{M}, \varphi) \oplus \wedge^3_{27}(\mathcal{M}, \varphi) \), and \( j_\varphi \) is an isomorphism between \( \wedge^3_1(\mathcal{M}, \varphi) \oplus \wedge^3_{27}(\mathcal{M}, \varphi) \) and \( \odot^2(\mathcal{M}) \). Moreover, for any 3-form \( \eta \in \wedge^3(\mathcal{M}) \), we have

\[ (2.10) \]

\[ \eta = i_\varphi(h) + X_\varphi \]

for some symmetric 2-tensor \( h \in \odot^2(\mathcal{M}) \) and vector field \( X \in \mathfrak{X}(\mathcal{M}) \). Then

\[ \eta = h^i_\ell dx^i \wedge \left( \frac{\partial}{\partial x^\ell} \varphi \right) + X^\ell \left( \frac{\partial}{\partial x^\ell} \varphi \right) = \frac{1}{2} h^i_\ell \varphi_{ijk} dx^{ijk} + \frac{1}{6} X^\ell \psi_{ijk} dx^{ijk} \]

\[ = \frac{1}{6} \left( 3h^i_\ell \varphi_{ijk} + X^\ell \psi_{ijk} \right) dx^{ijk} = \frac{1}{6} \ell_{ijk} dx^{ijk}. \]

Write \( h \) as \( h_{ij} = h_{ij} + \frac{1}{2} \text{tr}_\varphi(h)g_{\varphi}, \) where \( \hat{h} \in \odot^2_0(\mathcal{M}) \) is the trace-free part of \( h \), one has

\[ (2.11) \]

\[ \eta = \frac{3}{7} \left( \text{tr}_\varphi(h) \right) \varphi + \frac{1}{2} h^i_\ell \varphi_{ijk} dx^{ijk} + \frac{1}{6} X^\ell \psi_{ijk} dx^{ijk}. \]
2.2. The torsion tensors of a $G_2$-structure. By Hodge duality we obtain the $G_2$-decompositions of 4-forms $\Lambda^4(M) = \Lambda^4_1(M, \phi) \oplus \Lambda^3_2(M, \phi)$ and 5-forms $\Lambda^5(M) = \Lambda^5_1(M, \phi) \oplus \Lambda^5_2(M, \phi)$, respectively. By definition, we can find forms $\tau_0 \in C^\infty(M), \tau_1, \tau_1 \in \Lambda^1(M), \tau_2 \in \Lambda^2_4(M, \phi)$, and $\tau_3 \in \Lambda^3_5(M, \phi)$ such that

$$d\phi = \tau_0 \psi + 3 \tau_1 \wedge \phi + *_\phi \tau_3, \quad d\psi = 4 \tau_1 \wedge \psi - *_{\phi} \tau_2. \tag{2.12}$$

Since $\tau_2 \in \Lambda^2_4(M, \phi)$, it follows that $\tau_2 \wedge \phi = -*_{\phi} \tau_2$. Then (2.12) can be written as in the sense of Bryant [11].

$$d\tau = \tau_0 \psi + 3 \tau_1 \wedge \phi + *_{\phi} \tau_3, \quad d\psi = 4 \tau_1 \wedge \psi + \tau_2 \wedge \phi. \tag{2.13}$$

It can be proved that $\tau_1 = \tilde{\tau}$ (see [22]). We call $\tau_0$ the scalar torsion, $\tau_1$ the vector torsion, $\tau_2$ the Lie algebra torsion, and $\tau_3$ the symmetric traceless torsion. We also call $\tau_{\phi} := \{\tau_0, \tau_1, \tau_2, \tau_3\}$ the intrinsic torsion forms of the $G_2$-structure $\phi$.

Recall that a $G_2$-structure $\phi$ is torsion-free if and only if $d\phi = d\psi = 0$ by Theorem 2.1. From (2.12) we see that $\phi$ is torsion-free if and only if the intrinsic torsion forms $\tau_{\phi} \equiv 0$; that is, $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$.

**Lemma 2.2. (Fernández-Gray, [11])** For any $X \in \mathfrak{X}(M)$, the 3-form $\nabla_X \phi$ lies in the space $\Lambda^3_2(M, \phi)$. Therefore the covariant derivative $\nabla \phi \in \Lambda^1(M) \otimes \Lambda^3_2(M)$.

Consequently, there exists a 2-tensor $T = T_{ij} dx^{i \otimes j}$, called the full torsion tensor, such that

$$\nabla_\ell \phi = T_{\ell ij} \psi_{abc}. \tag{2.14}$$

Equivalently,

$$T_{\ell m} = \frac{1}{24} (\nabla_\ell \phi_{abc}) \psi_{mabc}. \tag{2.15}$$

Write

$$\tau_1 = (\tau_1) dx^i \in \Lambda^1(M), \tag{2.16}$$

$$\tau_2 = \frac{1}{2} (\tau_2)_{ab} dx^{ab} \in \Lambda^2_4(M), \tag{2.17}$$

$$\tau_3 = \frac{1}{2} (\tau_3)_{ijk} dx^{ijk} \in \Lambda^3_5(M, \phi). \tag{2.18}$$

The associated 2-tensor $\tau_3 := (\tau_3)_{ij} dx^{i \otimes j}$ of $\tau_3$ lies in the space $\Lambda^2_5(M)$. With this convenience, the full torsion tensor $T_{\ell m}$ is determined by

$$T_{\ell m} = \frac{\tau_0}{4} s_{\ell m} - (\tau_3)_{\ell m} - (\nabla_\phi (\tau_1) \cdot \phi)_{\ell m} - \frac{1}{2} (\tau_2)_{\ell m} \tag{2.19}$$

or as 2-tensors,

$$T = \frac{\tau_0}{4} s - \tau_3 - \nabla_\phi (\tau_1) \cdot \phi - \frac{1}{2} \tau_2. \tag{2.20}$$

Here the 2-form $\nabla_\phi (\tau_1) \cdot \phi$ is defined by

$$\nabla_\phi (\tau_1) \cdot \phi = \frac{1}{2} (\nabla_\phi (\tau_1) \cdot \phi) dx^{a \wedge b} = \frac{1}{2} ((\tau_1)_{k} q_{ab}) dx^{a \wedge b}. \tag{2.21}$$

As an application, this gives another proof of Theorem 2.1.
For fixed indices $i$ and $j$, set
\begin{equation}
R_{ijk\ell} := R_{ijk\ell} \text{ is skew-symmetric in } k \text{ and } \ell,
\end{equation}
where
\begin{equation}
R_{ij\bullet\bullet} := \frac{1}{2} R_{ijk\ell} dx^k dx^\ell = \frac{1}{2} R_{ijk\ell} dx^k dx^\ell \in \Lambda^2(\mathcal{M}).
\end{equation}
Then, according to (2.5) and (2.6)
\begin{equation}
R_{ijk\ell} = R_{ijk\ell} = \left( \pi_2^2(R_{ij\bullet\bullet}) \right)_{k\ell} + \left( \pi_4^2(R_{ij\bullet\bullet}) \right)_{k\ell},
\end{equation}
where
\begin{align*}
\left( \pi_2^2(R_{ij\bullet\bullet}) \right)_{k\ell} &= \frac{1}{3} R_{ijk\ell} + \frac{1}{6} R_{ijab} \psi^{ab}_{k\ell} = \frac{1}{3} R_{ijk\ell} + \frac{1}{6} R_{ijab} \psi^{ab}_{k\ell}, \\
\left( \pi_4^2(R_{ij\bullet\bullet}) \right)_{k\ell} &= \frac{2}{3} R_{ijk\ell} - \frac{1}{6} R_{ijab} \psi^{ab}_{k\ell} = \frac{1}{3} R_{ijk\ell} - \frac{1}{6} R_{ijab} \psi^{ab}_{k\ell}.
\end{align*}
Karigiannis [22] (see also the equivalent formula obtained by Bryant in [1]) proved that the Ricci curvature is given by
\begin{equation}
R_{jk} = R_{ijk\ell} s^{i\ell} = 3 \left( \pi_2^2(R_{ij\bullet\bullet}) \right)_{k\ell} s^{i\ell} = \frac{3}{2} \left( \pi_4^2(R_{ij\bullet\bullet}) \right)_{k\ell} s^{i\ell}.
\end{equation}
(2.23) \[-(\nabla_i T_{jm} - \nabla_j T_{im}) \psi^{m}_{k} i - T_{i} T_{jk} + (\text{tr}_\psi T) T_{jk} + T_{jb} T_{ia} \psi^{ab}_{jk}, \]
\[-\nabla_i \left( T_{jb} T_{ka} \psi^{ab}_{jk} \right) + \nabla_j \left( T_{ia} \psi^{ab}_{jk} \right) - T_{i} T_{jk} + (\text{tr}_\psi T) T_{jk} - T_{jb} T_{ia} \psi^{ab}_{jk}.
\]
Cleyton and Ivanov [5] also derived a formula for the Ricci tensor for closed $G_2$-structures in terms of $d_\phi^2 \phi$. Taking the trace of (2.23), we obtain Bryant’s formula [1] for the scalar curvature
\begin{equation}
R = -12 \nabla^i (\tau_1) + \frac{21}{8} \tau_0 - ||\tau_3||^2_{\phi} + 9 ||\phi^i (\tau_1) \phi^j||^2_{\phi} - \frac{1}{3} ||\tau_2||^2_{\phi},
\end{equation}
(2.24) \[-12 \nabla^i (\tau_1) + \frac{21}{8} \tau_0 - ||\tau_3||^2_{\phi} + 30 ||\tau_1||^2_{\phi} - \frac{1}{2} ||\tau_2||^2_{\phi}.
\]
For a closed $G_2$-structure, we have $\tau_0 = \tau_1 = \tau_3 = 0$ and then $R = -12 ||\tau_2||^2_{\phi} \leq 0$. On the other hand, we have $(\tau_2)_{ij} = -2 T_{ij}$ by (2.20). Thus the full torsion tensor $T$ is actually a 2-form
\begin{equation}
T = \frac{1}{2} T_{ij} dx^i \in \Lambda^2(\mathcal{M})
\end{equation}
and the scalar curvature can be written in terms of $T$
\begin{equation}
R = -||T||^2_{\phi} = -2|T|^2_{\phi} \leq 0.
\end{equation}
Hence, for closed $G_2$-structures, scalar curvatures are always non-positive.

Finally, we mention a Bianchi type identity
\begin{equation}
\nabla_i T_{j\ell} - \nabla_j T_{i\ell} = -\frac{1}{2} R_{ijab} \psi^{ab}_{k\ell} - T_{ia} T_{jb} \psi^{ab}_{k\ell} = -\left( \frac{1}{2} R_{ijab} + T_{ia} T_{jb} \right) \psi^{ab}_{k\ell}.
\end{equation}
The proof can be found in [22].
2.3. **General flows on** $G_2$-**structures**. For any family $(\varphi_t)_t$ of $G_2$-structures, according to the decomposition (2.10), we can consider the general flow

$$\partial_t \varphi_t = i_{\varphi_t} h_t + X_t \varphi_t$$

where $h_t \in \mathfrak{g}_2(M)$ and $X_t \in \mathfrak{X}(M)$. The general flow (2.28) locally can be written as

$$\partial_t \varphi_{ijk} = h_{\ell} \varphi_{i\ell jk} + h_j \varphi_{i\ell k} + h_k \varphi_{ij\ell} + X_{\ell} \psi_{ij\ell}.$$

We write for $g_t$ and $dV_t$ the metric and volume form associated to $\varphi_t$, respectively.

**Theorem 2.3.** Under the general flow (2.28), we have

$$\begin{align*}
\partial_t g_{ij} &= 2h_{ij}, \\
\partial_t g^{ij} &= -2h^{ij}, \\
\partial_t dV_t &= (\tau_1 h_t) dV_t, \\
\partial_t T_{pq} &= T_{p}^m h_{mq} - T_{p}^m X^k \varphi_{kmq} - (\nabla_k h_t)q \varphi^{ki} + \nabla_p X_q.
\end{align*}$$

These evolution equations can be found in [2].

3. **Laplacian flows on closed** $G_2$-**structures**

We now consider the Laplacian flow for closed $G_2$-structures

$$\partial_t \varphi_t = \Delta \varphi_t = \Delta_t \varphi_t, \quad \varphi_0 = \varphi,$$

where $\Delta \varphi_t = d^* \varphi_t d \varphi_t + d \varphi_t^* d \varphi_t$ is the Hodge Laplacian of $g_{\varphi_t}$ and $\varphi$ is an initial closed $G_2$-structure. The short time existence for (3.1) was proved by Bryant and Xu [2], see also Theorem 1.1.

A criterion for the long time existence for the Laplacian flow on compact manifolds was given in Theorem 1.2. In this section, we give a new elementary proof of Lotay-Wei’s result in compact case.

3.1. **Basic theory of closed** $G_2$-**structures**. Let $\wedge^3_+ (M) \subset \wedge^3_+ (M, \varphi)$ be the set of all closed $G_2$-structures on $M$. If $\varphi \in \wedge^3_+ (M)$ is closed, i.e., $d \varphi = 0$, then $\tau_0, \tau_1, \tau_3$ are all zero, so the only nonzero torsion form is

$$\tau \equiv \tau_2 \equiv \frac{1}{2} \tau_2 = \frac{1}{2} (\tau_2)_{ij} dx^i \wedge dx^j = \frac{1}{2} \tau_{ij} dx^i \wedge dx^j.$$

According to (2.20) and (2.25), we have $T_{ij} = -\frac{1}{2} \tau_{ij}$ so that

$$T \equiv \frac{1}{2} T_{ij} dx^i \wedge dx^j \quad \text{or equivalently} \quad T = -\frac{1}{2} \tau,$$

is a 2-form. Since $d \psi = \tau \wedge \varphi = -d^* \varphi \tau$, we get $d^* \tau = \ast d \varphi \tau = -d^* d^2 \varphi = 0$ which is given in local coordinates by

$$\nabla^i \tau_{ij} = 0.$$

For a closed $G_2$-structure $\varphi$, according to (2.23), the Ricci curvature is given by (in this case $T_{ij}$ is a 2-form)

$$R_{jk} = (\nabla_j T_{im} - \nabla_i T_{jm}) \varphi^m_{\ell k} - T^i_{ij} T_{ik} + T_{j[:k a} \varphi^a_{\ell :b}.$$


Since \( \tau \in \wedge^1_M(M, \varphi) \) and \( T_{ij} = -\frac{1}{2} \tau_{ij} \), it follows from \(^{[33]}\) (see page 179 – 180) that
\[
(\nabla_j T_{ik}) \varphi^m_k = 2T_j^i T_{ik}.
\]
and therefore, for a closed \( G_2 \)-structure \( \varphi \), the Ricci curvature is given by
\[
R_{jk} = - (\nabla_j T_{im}) \varphi^i_k - T_j^i T_{ik}.
\]
Taking the trace of \(^{(3.6)}\) yields \(^{(2.26)}\). Moreover, the factor \( \nabla_j T_{im} \) in \(^{(3.6)}\) can be expressed as (see Proposition 2.4 in \(^{[33]}\))
\[
\nabla_j T_{ik} = - \frac{1}{4} R_{ijklmn} \varphi^m_k - \frac{1}{4} R_{kijnm} \varphi^m_l + \frac{1}{4} R_k_{kijn} \varphi^m_j
\]
\[
- \frac{1}{2} T_{im} T_{jn} \varphi^m_k - \frac{1}{2} T_{kn} T_{jn} \varphi^m_i + \frac{1}{2} T_{im} T_{kn} \varphi^m_j.
\]

If \( \varphi \) is a closed \( G_2 \)-structure, Section 2.2 in \(^{[33]}\) shows that \( \pi_1^2(\Delta \varphi \varphi) = 0 \) and hence, according to \(^{(2.10)}\),
\[
\Delta \varphi \varphi = i_{\varphi}(h) \in \wedge^3(M, \varphi) \oplus \wedge^3(M, \varphi),
\]
where
\[
h_{ij} = \frac{1}{2} \nabla_{\tau_i} \tau_{\varphi j}^m \mu - \frac{1}{6} |\tau|_\varphi^2 \tau_{ij} - \frac{1}{4} \tau^l \tau_{ij} = - R_{ij} - \frac{2}{3} |T|_\varphi^2 g_{ij} - 2T^k \nabla_k T_{ij}.
\]
Here \( |T|_\varphi^2 = x^{T^k \nabla_k T^l} = \frac{1}{2} |T|_\varphi^2 \).

### 3.2. Evolution equations for closed \( G_2 \)-structures

Since the Laplacian flow \(^{(3.1)}\) preserves the closedness of \( \varphi_t \), it follows from \(^{(3.10)}\) that we have
\[
\Delta \varphi_t \varphi_t = i_{\varphi_t}(h_t) \in \wedge^3(M, \varphi_t) \oplus \wedge^3(M, \varphi_t),
\]
where
\[
h_{ij} = - R_{ij} - \frac{2}{3} |T|_t^2 g_{ij} - 2T^k \nabla_k T_{ij}.
\]
From Theorem \(^{[23]}\) we see that the associated metric tensor \( g_t \) evolves by
\[
\partial_t g_{ij} = 2h_{ij} = - 2R_{ij} - \frac{4}{3} |T|_t^2 g_{ij} - 4T^k \nabla_k T_{ij},
\]
and the volume form \( dV_t \) evolves by
\[
\partial_t dV_t = (\text{tr} g_t) dV_t = \left( - R - \frac{14}{3} |T|_t^2 + 4 |T|_t^2 \right) dV_t = \frac{4}{3} |T|_t^2 dV_t.
\]
Hence, along the flow \(^{(3.1)}\), the volume of \( g_t \) is nondecreasing.

Introduce the following notions
\[
\nabla_t := \partial_t - \nabla_t, \quad | \cdot |_t := | \cdot |_{\varphi_t}, \quad \Delta_t := \Delta_{\varphi_t},
\]
where \( \nabla_t := g^{ij} \nabla_i \nabla_j \) is the usual Laplacian of \( g_t \) and \( \Delta_t \) is the Hodge Laplacian of \( g_t \), and also the 2-tenor Sic with components
\[
S_{ij} := R_{ij} + \frac{2}{3} |T|_t^2 g_{ij} + 2T^k \nabla_k T_{ij} = - h_{ij}.
\]
Then the evolution equation \(^{(3.12)}\) can be written as
\[
\partial_t g_{ij} = - 2S_{ij}.
\]
Moreover, the trace of $\text{Sic}_t$ is exactly the scalar curvature, up to a multiplying constant,

$$S_t := \text{tr}_t \text{Sic}_t = R_t + \frac{14}{3} \left| T_{ij} \right|^2 - 4 \left| T_{ij}^i \right|^2 = -\frac{4}{3} \left| T_{ij}^i \right|^2 = \frac{2}{3} R_t.$$  

It was proved in [33] that

$$|\Delta_t \varphi_t|^2 = (\text{tr}_t h_t)^2 + 2 \left| h_t \right|^2 = \frac{16}{9} \left| T_{ij}^i \right|^2 + 2 \left| \text{Sic}_t \right|^2.$$  

This identity together with (2.26) shows that the boundedness of $\Delta_t \varphi_t$ is equivalent to the boundedness of $R \text{Sic}_t$.

The evolution equation (2.33) implies that for the Laplacian flow on closed $G_2$-structures, the torsion $T_{ij}$ evolves by evolves

$$\partial_t T_{ij} = T_{ij}^k h_{kj} - (\nabla_{m} h_{ni}) \varphi_{j}^{mn}.$$  

Furthermore, we can prove

**Proposition 3.1.** Under the flow (3.1), we have

$$\nabla_p T_{qi} \left( T_{pk} \varphi_{j}^{pq} - 2 T_{pq} \varphi_{j}^{pq} \right) = \frac{2}{3} |T_{ij}^i|^2 T_{ij} - 4 T_{ij}^i T_{ki} T_{mj}.$$  

**Proof.** See [33].

For a geometric flow $\partial_t g_{ij} = \eta_{ij}$, for some symmetric 2-tensor $\eta_{ij}$, we have

$$\begin{align*}
\partial_t g_{ijk} &= \frac{1}{2} \Delta^i_{\ell j} \left( \nabla_i \nabla_j \eta_{kp} + \nabla_k \nabla_i \eta_{jp} - \nabla_p \nabla_i \eta_{jk} \\
&\quad - \nabla_j \nabla_k \eta_{ip} + \nabla_j \nabla_p \eta_{ik} \right), \\
\partial_t R_{ij} &= \frac{1}{2} g^{m n} \left( \nabla_q \nabla_i \eta_{kp} + \nabla_k \nabla_q \eta_{ip} - \nabla_q \nabla_i \eta_{jk} + \nabla_j \nabla_q \eta_{kp} \right), \\
\partial_t T_{ij} &= -\Delta_i \text{tr}_t \eta_{ij} + \text{div}_t (\text{div}_t \eta_{ij}) - R_{ij} h_{ij},
\end{align*}$$  

where $\left( \text{div}_t \eta_{ij} \right) = \nabla^i \eta_{ij}$. Applying those evolution equations to $\eta_{ij} = -2 R_{ij} - \frac{4}{3} |T_{ij}^i|^2 + 4 T_{ij}^i T_{kj} = -2 S_{ij}$ we have

$$\begin{align*}
\text{tr}_t \eta_t &= -2 R_t - \frac{28}{3} |T_{ij}^i|^2 + 8 |T_{ij}^i|^2 = \frac{8}{3} |T_{ij}^i|^2, \\
(\text{div}_t \eta_t)_i &= -2 \nabla^i R_{ij} - \frac{4}{3} \nabla_j |T_{ij}^i|^2 - 4 \nabla^i \tilde{T}_{ij} = -\nabla_j R_t - \frac{4}{3} \nabla_j |T_{ij}^i|^2 - 4 \nabla^i \tilde{T}_{ij}, \\
\text{div}_t (\text{div}_t \eta_t) &= \nabla^i (\text{div}_t \eta_t)_i = -\Delta_i R_t - \frac{4}{3} \Delta_i |T_{ij}^i|^2 - 4 \nabla^i \nabla_i \tilde{T}_{ij},
\end{align*}$$  

where the symmetric 2-tensor $\tilde{T}$ is given by

$$\tilde{T}_{ij} := T_{jk}^k T_{ij}.$$  

$$\text{div}_t (\text{div}_t \eta_t) = \nabla^i (\text{div}_t \eta_t)_i = -\Delta_i R_t - \frac{4}{3} \Delta_i |T_{ij}^i|^2 - 4 \nabla^i \nabla_i \tilde{T}_{ij},$$  

where the symmetric 2-tensor $\tilde{T}$ is given by

$$\tilde{T}_{ij} := T_{jk}^k T_{ij}.$$
Plugging those identities into the above evolution equation for $R_t$, we get

$$\partial_t R_t = -4 \nabla^i \nabla^j T_{ij}^2 + 4 \nabla^i \nabla^j \hat{T}_{ij} - R_{ij}^j - 2 R_{ij}^j + \frac{4}{3} |T_1|^2 G_{ij} - 4 \bar{T}_{ij}$$

which implies

$$\Box R_t = 2 |\text{Ric}_t|^2 - \frac{2}{3} R_t^2 - 4 \nabla^i \nabla^j \hat{T}_{ij} + 4 \langle \langle \text{Ric}_t, \hat{T} \rangle \rangle_t.$$  \hfill (3.22)

Observe that the last two terms on the right-hand side of (3.22) are not determined of their signs. In the following, we shall use the identity

$$\nabla^i T_{ij} = 0$$

follows from from (3.3) and (3.4), to simplify those two terms. Using the identity (3.17), we can rewrite (3.24) as

$$\nabla^i \nabla^j T_{ij}^2 = \nabla^j \nabla^i T_{ij}^2 - R_{ij} T_{ij} - R_{ij}^j T_{kl} = R_{ijkl} T_{ij}^k + R_{ik} T_{jk}.$$

On the other hand, from the Ricci identity

$$\nabla^j \nabla^k T_{ij} = \nabla^j \nabla^k T_{ij}^k - R_{ik} T_{ij}^k - R_{ij} T_{ij}^k = R_{ijkl} T_{ij}^k + R_{ik} T_{jk},$$

we see that the evolution equation (3.22) is equivalent to

$$\Box R_t = 2 |\text{Ric}_t|^2 - \frac{2}{3} R_t^2 + 4 R_{ijkl} T_{ik} T_{ij}^j + 4 \langle \langle \nabla^i T_{ik}, \nabla^i T_{jk} \rangle \rangle.$$  \hfill (3.24)

From (3.15) and (3.21) we can rewrite the term $|\text{Ric}_t|^2$ in (3.24) in terms of $\text{Sic}_t$ according to the following relation:

$$|\text{Sic}_t|^2 = \left( R_{ij} + \frac{2}{3} |T_1|^2 G_{ij} + 2 \bar{T}_{ij} \right) \left( R_{ij} + \frac{2}{3} |T_1|^2 G_{ij} + 2 \bar{T}_{ij} \right)$$

$$= |\text{Ric}_t|^2 + \frac{4}{3} |T_1|^2 R_t + 4 \langle \langle \text{Ric}_t, \hat{T}_1 \rangle \rangle_t + \frac{8}{9} |T_1|^2 + \frac{8}{3} |T_1|^2 + 4 \| \hat{T}_1 \|^2$$

$$= |\text{Ric}_t|^2 + \frac{2}{3} R_t^2 + 4 \langle \langle \text{Ric}_t, \hat{T}_1 \rangle \rangle_t + \frac{7}{9} R_t^2 - \frac{4}{3} R_t^2 + 4 |\hat{T}_1|^2$$

$$= |\text{Sic}_t|^2 + 4 |\hat{T}_1|^2 + 4 \langle \langle \text{Ric}_t, \hat{T}_1 \rangle \rangle_t - \frac{11}{9} R_t^2,$$

where we used $\text{tr}_t \hat{T}_1 = g_{ij} T_{ik} T_{kj} = T_{ik} T_{kj} = -2 |T_1|^2$, and $R_t = -2 |T_1|^2$. Replacing $R_t$ by $S_t$ according to the identity (3.17), we can rewrite (3.24) as

$$\Box S_t = \frac{4}{3} |\text{Sic}_t|^2 - \frac{16}{3} |\hat{T}_1|^2 - \frac{16}{3} \langle \langle \text{Ric}_t, \hat{T}_1 \rangle \rangle_t + \frac{32}{27} R_t^2$$

$$+ \frac{8}{3} R_{ijkl} T_{ik} T_{ij}^j + \frac{8}{3} \langle \langle \nabla^i T_{ik}, \nabla^i T_{jk} \rangle \rangle.$$

Similarly, replacing $\langle \langle \text{Ric}_t, \hat{T}_1 \rangle \rangle_t$ by $\langle \langle \text{Sic}_t, \hat{T}_1 \rangle \rangle_t$ with respect to the identity

$$\langle \langle \text{Sic}_t, \hat{T}_1 \rangle \rangle_t = \left( R_{ij} + \frac{2}{3} |T_1|^2 G_{ij} + 2 \bar{T}_{ij} \right) \hat{T}_{ij} = \langle \langle \text{Ric}_t, \hat{T}_1 \rangle \rangle_t - \frac{1}{3} R_t^2 + 2 |\hat{T}_1|^2,$$
we obtain the following evolution equation for $S_t$, \begin{equation}
(3.25) \quad \Box_t S_t = \frac{4}{3} \left[ \left| \nabla T \right| \right]_t^2 - S_t^2 + \frac{8}{3} \left[ R_{ijkl} T^{ik} T^{j\ell} + (\nabla^j T^k)(\nabla T_{jk}) \right].
\end{equation}

Next, we try to deal with the last bracket in (3.25), which contains two terms $R_{ijkl} T^{ik} T^{j\ell}$ and $(\nabla^j T^k)(\nabla T_{jk})$. Using (2.27) and (3.7), the term $(\nabla^j T^k)(\nabla T_{jk})$ is equal to
\[
(\nabla^j T^k)(\nabla T_{jk}) = \left[ \nabla^j T^k + \left( \frac{1}{2} R_{ijkl} T^{ik} T^{j\ell} \right) \right]_{i a} \nabla T_{jk}
= \left| \nabla T_i \right|_t^2 + \frac{1}{2} \left( \frac{1}{2} R_{ijkl} T^{ik} T^{j\ell} \right) \left[ - \frac{1}{2} R_{ijnm} \phi^{mn}_{k} \phi^{kab} - \frac{1}{2} R_{ijnm} \phi^{mn}_{i} \phi^{kab} + \frac{1}{2} R_{ijmn} \phi^{mn}_{j} \phi^{kba} \right]
+ \frac{1}{2} R_{ijmn} \phi^{mn}_{j} \phi^{kba} - T_{km} T_{jn} \phi^{mn}_{k} \phi^{kab} - T_{kn} T_{jm} \phi^{mn}_{i} \phi^{kab} + T_{im} T_{kn} \phi^{mn}_{i} \phi^{kab}
\]

By symmetry the term
\[
\left( \frac{1}{2} R_{ijkl} T^{ik} T^{j\ell} + T^i_a T^j_b \right) \left( - \frac{1}{2} R_{ijmn} \phi^{mn}_{k} \phi^{kab} + \frac{1}{2} R_{ikmn} \phi^{mn}_{i} \phi^{kab} \right)
\]
is equal to, interchanging $i \leftrightarrow j$ and $a \leftrightarrow b$ in the second term,
\[
\left( \frac{1}{2} R_{ijkl} T^{ik} T^{j\ell} + T^i_a T^j_b \right) \left( - \frac{1}{2} R_{ijmn} \phi^{mn}_{k} \phi^{kab} \right) + \left( \frac{1}{2} R_{ijmn} \phi^{mn}_{k} \phi^{kab} \right) \left( \frac{1}{2} R_{ijmn} \phi^{mn}_{i} \phi^{kab} \right)
\]
which is zero. Similarly, we have, by interchanging $m \leftrightarrow n$ and then $i \leftrightarrow j$, $a \leftrightarrow b$ in the first term,
\[
\left( \frac{1}{2} R_{ijkl} T^{ik} T^{j\ell} + T^i_a T^j_b \right) \left( - T_{km} T_{jn} \phi^{mn}_{k} \phi^{kab} + T_{im} T_{kn} \phi^{mn}_{i} \phi^{kab} \right)
\]
\[
= \left( \frac{1}{2} R_{ijkl} T^{ik} T^{j\ell} \right) \left( - T_{km} T_{jn} \phi^{mn}_{k} \phi^{kab} \right) + \left( \frac{1}{2} R_{ijmn} \phi^{mn}_{i} \phi^{kab} \right) \left( \frac{1}{2} R_{ijmn} \phi^{mn}_{j} \phi^{kba} \right)
\]
which is zero. Therefore, using the identity $\phi_{ijk} \phi^{ab} = S_{ia} S_{jb} - S_{ib} S_{ja} + \psi_{ijab}$ (see [22]), we arrive at
\[
(\nabla^j T^k)(\nabla T_{jk}) = ||\nabla T_i||_t^2 - \frac{1}{2} \left( \frac{1}{2} R_{ijkl} T^{ik} T^{j\ell} \right) \left( \frac{1}{2} R_{ijmn} \phi^{mn}_{k} \phi^{kab} + \frac{1}{2} R_{ijmn} \phi^{mn}_{i} \phi^{kab} \right)
\]
\[
= ||\nabla T_i||_t^2 - \frac{1}{8} \left( R_{ijab} + 2 T_{ia} T_{jb} \right) \left( R_{ijab} + 2 T_{ia} T_{jb} \right) \left( R_{ijab} + 2 T_{ia} T_{jb} \right)
\]
\[
\left( R_{ijab} + 2 T_{ia} T_{jb} \right) \left( R_{ijab} + 2 T_{ia} T_{jb} \right) \left( R_{ijab} + 2 T_{ia} T_{jb} \right) = -||\nabla T_i||_t^2 - 4 R_{ijab} T_{ia} T_{jb} + 4 ||\nabla T_i||_t^2.
\]

Since, by our convention,
\[
\left( R_{ijab} + 2 T_{ia} T_{jb} \right) \left( R_{ijab} + 2 T_{ia} T_{jb} \right) = ||\nabla T_i||_t^2 + 4 R_{ijab} T_{ia} T_{jb} + 4 ||\nabla T_i||_t^2,
\]
\[
\left( R_{ijab} + 2 T_{ia} T_{jb} \right) \left( R_{ijab} + 2 T_{ia} T_{jb} \right) = -||\nabla T_i||_t^2 - 4 R_{ijab} T_{ia} T_{jb} + 4 ||\nabla T_i||_t^2,
\]

it follows that

\[(\nabla^j T^k)(\nabla_i T_{jk}) = ||\nabla_i T_t||^2 + \frac{8}{3} ||\nabla T_t||^2 + 8 R_{ijab} T^i_{jb} - 4 ||T_t||^4 + 4 ||\mathring{T}_t||^2 - (R_{ijab} + 2 T_{ia} T_{jb}) (R^{i j mn} + 2 T^{i m} T^{j n}) \psi_{mnab}\]

and (3.25) can be written as

\[\Box S_t = \frac{4}{3} \left(|Sict - 2 \mathring{T}_t|^2 + \frac{8}{3} ||\nabla T_t||^2 + 4 ||T_t||^2 - \frac{2}{3} ||Rm_t||^2 - \frac{13}{3} S_t^2\right)\]

(3.26)

\[-\frac{1}{3} \left(R_{ijab} + 2 T_{ia} T_{jb}\right) \left(R^{i j mn} + 2 T^{i m} T^{j n}\right) \psi_{mnab}.

Finally, we deal with the last term \(J\) on the right-hand side of (3.26). From the identity \(\psi_{j k l} \psi_{j k l} = 168\), we find that

\[J = -\frac{1}{3} \left(R_{ijab} + 2 T_{ia} T_{jb}\right) \left(R^{i j mn} + 2 T^{i m} T^{j n}\right) \psi_{mnab}
= \frac{1}{3} \left(-R_{ij}^{ab} R^{i j mn} \psi_{mnab} - 4 T_i^{a} T_j^{b} R^{i j mn} \psi_{mnab} - 4 T_i^{a} T^{b} T_j^{i} T^{j} \psi_{mnab}\right)
= \frac{1}{3} \left(-R_{ij}^{ab} R^{i j mn} - \frac{1}{2} \psi_{abmn}\right) \left(||R_{ij}^{ab} R^{i j mn}||^2 - 4 ||T_t||^4\right) - 168
+ \left|2 T_i^{a} T_j^{b} R^{i j mn} - \psi_{abmn}\right| \left(||2 T_i^{a} T_j^{b} R^{i j mn}||^2 - 4 ||\mathring{T}_t||^4 - 168\right).
\]

Plugging the expression for \(J\) into (3.26), we obtain

**Proposition 3.2.** The scalar curvature \(R_t\) or \(S_t\) evolves by

\[\Box S_t = \frac{4}{3} \left(|Sict - 2 \mathring{T}_t|^2 + \frac{8}{3} ||\nabla T_t||^2 + \frac{1}{3} ||R_{ijab} R^{i j mn} - \psi_{abmn}||^2 + \frac{4}{3} ||\mathring{T}_t||^2\right)\]

(3.27)

\[+ \frac{1}{3} \left|2 T_{ia} T_{jb} R^{i j mn} - \psi_{ab mn}\right| \left(||2 T_{ia} T_{jb} R^{i j mn}||^2 - \frac{2}{3} ||T_t||^4\right) - \frac{2}{3} ||Rm_t||^2 - \frac{13}{3} S_t^2 - \frac{1}{3} ||R_{ijab} R^{i j mn}||^2 - \frac{4}{3} ||T_{ia} T_{jb} R^{i j mn}||^2 - 126.
\]

Since \(S_t = \frac{2}{3} R_t\), it follows from the above theorem that (1.1) holds true.

Before giving local curvature estimates for Laplacian flow in the next subsection, we derive evolution equations for \(\text{Ric}_t, \text{Rm}_t\), and \(T_t\) in different forms. Using the Lichnerowicz Laplacian

\[\Box L_{t} \eta_{jk} := \Box \eta_{jk} - R_{j}^{lp} \eta_{pk} - R_{k}^{lp} \eta_{jp} + 2 R_{lp kj} h^{lp},\]

we see that the evolution equation for \(R_{ij}\) can be written as

\[\partial_t R_{ij} = -\frac{1}{2} \left[\Box L_{t} \eta_{jk} + \nabla_j \nabla_k \text{tr} \eta_t + \nabla_j (d^i_t \eta_t k + \nabla_k (d^i_t \eta_t j))\right],\]
Moreover, it was proved in [33] that the first term is equal to

\[
\partial_t R_{jk} = \Delta_{L,t} \left( R_{jk} + \frac{2}{3} |T_t| g_{jk} + 2 \tilde{T}_{jk} \right) - \frac{1}{2} \nabla_j \left( \nabla_k R_t + \frac{4}{3} \nabla_k |T_t|^2 + 4 \nabla^i \tilde{T}_{ik} \right) - \frac{4}{3} \nabla_j \nabla_k |T_t|^2 - \frac{1}{2} \nabla_k \left( \nabla_j R_t + \frac{4}{3} \nabla_j |T_t|^2 + 4 \nabla^i \tilde{T}_{ij} \right)
\]

Consequently, the norm of $Ric$ satisfies

\[
\| Ric_t \|^2 = -2 \| \nabla_i Ric_t \|^2 + 4 R_{kij} R^{kij} - \left[ \frac{2}{3} \left( \| T_t \|^2 \right) g_{ij} + 2 \Delta_{L,t} \tilde{T}_{ij} - 2 R_{pjkq} R^{pq} \right]
\]

The general formula for $R^\ell_{ijk}$ gives

\[
\partial_t R^\ell_{ijk} = - \nabla_i \nabla_k R^\ell_j - \nabla_j \nabla_k R^\ell_i + \nabla_i \nabla_j R^\ell_k + \nabla_j \nabla_i R^\ell_k + R_{ijk} R^{q} + R_{ijk} \nabla^q R_{kp} + 2 R_{ikj} \tilde{T}_q^\ell + 2 R_{ijk} \tilde{T}_p^\ell \frac{2}{3} \left( \nabla_i \nabla_k |T_t|^2 \right) g_{pq} - \frac{1}{2} \nabla_i \nabla^q |T_t|^2 \nabla_j \nabla^p |T_t|^2 \frac{2}{3} \left( \nabla_j \nabla^p |T_t|^2 \right) g_{ik}
\]

Hence, the evolution equation for $\| Rm_t \|^2$ is given by

\[
\partial_t \| Rm_t \|^2 = \nabla^2_t Ric_t * Rm_t + Ric_t * Rm_t + Rm_t * Rm_t + Rm_t * T_t + \frac{8}{3} |T_t|^2 \| Rm_t \|^2
\]

Moreover, it was proved in [33] that

\[
\| \nabla_t Rm_t \|^2 \leq - \frac{1}{2} \| Ric_t \|^2 + C_1 \| Rm_t \|^2
\]

\[
+ C_1 \| Rm_t \|^2 |\nabla T_t|^2 + C_1 \| Rm_t \|^2 \| \nabla T_t \|^2
\]
where $C_1$ is some universal constant, and
\[(3.33) \quad \|\nabla T_t\|^2 \leq -\frac{1}{2} \|\nabla T_t\|^2 + C_2 \|\nabla T_t\|^2 + C_2 \|T_t\|^4\]
Squaring (3.33) gives
\[(3.34) \quad \|\nabla T_t\|^2 \leq -\frac{1}{2} \|\nabla T_t\|^2 + C_2 \|\nabla T_t\|^2 + C_2 \|T_t\|^4\]
for another universal constant $C_2$ which may differs from $C_1$. The Cauchy-Schwartz inequality shows $2C_2 \|\nabla T_t\|^2 \leq \|\nabla T_t\|^2 + C_2 \|T_t\|^4$, so that the evolution inequality (3.34) becomes
\[(3.35) \quad \|\nabla T_t\|^2 \leq -\|\nabla T_t\|^2 + C_3 \|\nabla T_t\|^2 + C_3 \|T_t\|^4.
Here $C_3$ is a universal constant.

3.3. Local curvature estimates. In this section, we consider the Laplacian flow (3.1) on $M \times [0, T]$, where $T \in (0, T_{\text{max}})$. From now on we always omit the time subscripts from all considered quantities. From (3.15), (3.29), (3.31), (3.32), and (3.36) we have
\[
\|\nabla Rm\|^2 = -\frac{1}{2} \|\nabla Rm\|^2 + \|\nabla Rm\|^2 + C_2 \|\nabla Rm\|^2 - \frac{2}{3} \|\nabla Rm\|^2 R
+ 2 \langle \nabla Rm, \nabla^2 R \rangle + \frac{2}{3} \langle \nabla Rm, \nabla^2 \nabla R \rangle + \nabla Rm \nabla Rm + \nabla Rm \nabla Rm + \nabla Rm \nabla Rm,
\]
\[
\|\nabla Rm\|^2 \leq -\frac{1}{2} \|\nabla Rm\|^2 + C \|\nabla Rm\|^3 + C \|\nabla Rm\|^3 \|\nabla T\|^2 + C \|\nabla Rm\|^2 \|\nabla T\|^2,
\]
\[
\partial_t \|Rm\|^2 = \nabla^2 Rm + \nabla Rm + \nabla Rm + \nabla Rm + \nabla Rm
+ \nabla Rm + \nabla^2 \|T\|^2 + \nabla Rm + \nabla^2 \|T\|^2 + \nabla Rm + \|T\|^2 \|Rm\|^2,
\]
\[
\|\nabla T\|^2 \leq -\|\nabla T\|^2 + C \|\nabla Rm\|^2 \|T\|^2 + C \|\nabla T\|^4,
\]
\[
\partial_t dV = \frac{2}{3} \|T\|^2 dV, \quad R = -\|T\|^2.
\]
Choose an open domain $\Omega$ of $M$ and assume that
\[(3.36) \quad \|\nabla \phi\| \leq K\]
on $\Omega \times [0, T]$, then the torsion $T$ satisfies $\|T\| \leq K^{1/2}$ and metrics $g_t$ are all equivalent to $g_0$. We also observe from (2.25) and (3.19) that
\[(3.37) \quad \|\nabla \phi\| \leq 1 \iff \Delta \phi \leq 1\]
and the following simple fact
\[(3.38) \quad \partial_t \|A\|^2 = \frac{p}{2} \|A\|^{p-2} \partial_t \|A\|^2\]
for any tensor $A$.

Choose a Lipschitz function $\eta$ with support in $\Omega$ and consider the quantity
\[
\frac{d}{dt} \int \|\nabla Rm\|^{2\eta} \|\nabla T\|^2 dV, \quad : = \int_{M'}
\]
where \( p \geq 5 \). As in [27], we introduce the following “good” quantities

\[
A_1 := \int |Rm|^p \eta^{2p} \, dV, \quad A_2 := \int |Rm|^{p-1} \eta^{2p} \, dV, 
\]

\[
A_3 := \int |Rm|^{p-1} |\nabla \eta|^2 \eta^{2p-1} \, dV, \quad A_4 := \int |Rm|^{p-1} |\nabla \eta|^2 \eta^{2p-2} \, dV 
\]

and also “bad” quantities

\[
B_1 := \frac{1}{K} \int |\nabla \Ric|^2 |Rm|^{p-1} \eta^{2p} \, dV, \quad B_2 := \int |\nabla Rm|^2 |Rm|^{p-3} \eta^{2p} \, dV. 
\]

We split the proof of Theorem 1.4 into four steps.

(a) In the first step, we can show that, see Lemma 3.3,

\[
\frac{d}{dt} A_1 \leq B_1 + cKB_2 + cKA_4 + cKA_1 + cK^2 A_2 + c \int \left(-\frac{1}{2} |T|^2 \right) |Rm|^{p-1} \eta^{2p} \, dV.
\]

(b) In the second step, we can prove that the term

\[
c \int \left(-\frac{1}{2} |T|^2 \right) |Rm|^{p-1} \eta^{2p} \, dV
\]

is bounded from above by (see (3.47))

\[
B_1 + cKB_2 + cK^2 A_2 + cKA_1 - \frac{d}{dt} \left[ \int c(-R) |Rm|^{p-1} \eta^{2p} \, dV \right].
\]

Observe that the above integral is nonnegative, since the scalar curvature \( R \) is nonpositive along the Laplacian flow on closed \( G_2 \)-structures. Hence we obtain from the first step that, see Lemma 3.4

\[
\frac{d}{dt} A_1 \leq 2B_1 + cKB_2 + cKA_4 + cKA_1 + cK^2 A_2
\]

\[
- \frac{d}{dt} \left[ \int c(-R) |Rm|^{p-1} \eta^{2p} \, dV \right].
\]

(c) In the next two steps, we estimate the bad terms \( B_1 \) and \( B_2 \). In the third step, \( B_1 \) is estimated by (see (3.57))

\[
B_1 \leq cKB_2 + cKA_4 + cKA_1 + cK^2 A_2
\]

\[
- \frac{d}{dt} \left[ \frac{1}{K} \int |Rm|^{p-1} |\Ric|^2 \eta^{2p} \, dV + c \int (-R) |Rm|^{p-1} \eta^{2p} \, dV \right].
\]

Then the second step can be simplified as, see Lemma 3.5

\[
\frac{d}{dt} A_1 \leq cKB_2 + cKA_4 + cKA_1 + cK^2 A_2
\]

\[
- \frac{d}{dt} \left[ \frac{1}{K} \int |Rm|^{p-1} |\Ric|^2 \eta^{2p} \, dV + c \int (-R) |Rm|^{p-1} \eta^{2p} \, dV \right].
\]

(d) Finally, we estimate the term \( B_2 \). In this step we shall use the assumption that \( p \geq 5 \). Using the inequality \( |\nabla T| \lesssim |Rm| \) and \( |\nabla^2 T| \lesssim |\nabla Rm| + |Rm||T| + |\nabla T||T| + |T|^3 \), we can prove (see (3.67))

\[
B_2 \leq cA_4 + cA_1 - \frac{d}{dt} \left[ \frac{1}{p-1} \int |Rm|^{p-1} \eta^{2p} \, dV \right].
\]
If we choose a geodesic ball \( \Omega \) := \( B_{x_0}(\rho) \) and a cut-off function \( \eta \) so that \( ||\nabla \phi|| \leq \sqrt{K}e^{KT}/\rho \), then the above inequality gives a proof of Theorem 1.4.

We are going to carry out the above mentioned four steps. From (3.39) and the above evolution equations, we have

\[
\frac{d}{dt} \int ||Rm||^p \eta^{2p} dV = \int (\partial_t ||Rm||^p) \eta^{2p} dV + \int ||Rm||^p \eta^{2p} \partial_t dV
\]

\[
= \frac{p}{2} ||Rm||^{p-2} \left( \partial_t ||Rm||^2 \right) \eta^{2p} dV + \int ||Rm||^p \eta^{2p} \left( -\frac{2}{3} R \right) dV
\]

\[
= \frac{p}{2} ||Rm||^{p-2} \left[ \nabla^2 \text{Ric} \ast Rm + \text{Ric} \ast \nabla^2 \text{Rm} \ast \text{Rm} + \nabla^2 \tilde{T} \ast Rm + \text{Ric} \ast \tilde{T} \ast \nabla^2 T \ast \text{Rm} \right] \eta^{2p} dV
\]

\[
+ \frac{2}{3} \int R ||Rm||^p \eta^{2p} dV
\]

(3.39)

\[
\leq c \int ||Rm||^{p-2} \left[ \nabla^2 \text{Ric} \ast Rm + K ||Rm||^2 + K ||Rm||^2 + \nabla^2 ||T||^2 \ast \text{Ric} + \nabla^2 \tilde{T} \ast Rm \right] \eta^{2p} dV + cK \int ||Rm||^p \eta^{2p} dV
\]

It was proved in [24] that the first integral in (3.39) is bounded by

\[
c \int ||Rm||^{p-2} \left( \nabla^2 \text{Ric} \ast Rm \right) \eta^{2p} dV \leq \frac{1}{K} \int ||\nabla \text{Ric}||^2 ||Rm||^{p-1} \eta^{2p} dV
\]

(3.40)

\[
+ cK \int ||\nabla Rm||^2 ||Rm||^{p-3} \eta^{2p} dV + cK \int ||Rm||^{p-1} ||\nabla \eta||^2 \eta^{2p-2} dV.
\]

Since \( ||T||^2 = -R \), the same inequality holds for the integral

\[
c \int ||Rm||^{p-2} \left( \nabla^2 ||T||^2 \ast \text{Ric} \right) \eta^{2p} dV.
\]

To deal with the last term in the bracket of (3.39), we use the same argument of [24] to conclude

\[
c \int ||Rm||^{p-2} \left( \nabla^2 \tilde{T} \ast Rm \right) \eta^{2p} dV = c \int \left( \nabla ||Rm||^{p-2} \ast \nabla \tilde{T} \ast Rm \right) \eta^{2p} dV
\]

\[
+ c \int \left( ||Rm||^{p-2} \ast \nabla \tilde{T} \ast \nabla Rm \right) \eta^{2p} dV
\]
According to the Cauchy-Schwartz inequality, the first and second integrals are bounded by

\[ c \int \left( ||Rm||^{p-2} \cdot \nabla \hat{T} \cdot Rm \cdot \nabla \eta \right) \eta^{2p-1} dV \leq c \int ||Rm||^{p-2} ||\nabla Rm|| ||\nabla \hat{T}|| \eta^{2p} dV + c \int ||Rm||^{p-2} ||\nabla \hat{T}|| ||\nabla \eta|| \eta^{2p} dV \]

and

\[ c \int ||Rm||^{p-1} ||\nabla \hat{T}|| ||\nabla \eta|| \eta^{2p-1} dV \leq cK \int ||\nabla Rm||^2 ||Rm||^{p-3} \eta^{2p} dV + \frac{1}{K} \int ||\nabla \hat{T}||^2 ||Rm||^{p-1} \eta^{2p} dV \]

Hence we obtain

\[ c \int ||Rm||^{p-2} \left( \nabla^2 \hat{T} \cdot Rm \right) \eta^{2p} dV \leq \frac{1}{K} \int ||\nabla \hat{T}||^2 ||Rm||^{p-1} \eta^{2p} dV \]

(3.41) \[ + cK \int ||\nabla Rm||^2 ||Rm||^{p-3} \eta^{2p} dV + cK \int ||Rm||^{p-1} ||\nabla \eta||^2 \eta^{2p-2} dV. \]

Using \( \hat{T} = T * T \) and \( R = -||T||^2 \) yields

\[ \frac{1}{K} \int ||\nabla \hat{T}||^2 ||Rm||^{p-1} \eta^{2p} dV \]

(3.42) \[ \leq \frac{C}{K} \int ||\nabla T||^2 ||T||^2 ||\nabla \hat{T}||^{p-1} \eta^{2p} dV \leq c \int ||\nabla T||^2 ||Rm||^{p-1} \eta^{2p} dV \]

\[ \leq c \int \left( \frac{1}{4} \|T\|^2 + c ||\nabla \nabla \hat{T}||^{p-1} \eta^{2p} dV \right) \]

\[ = c \int \left( \frac{1}{4} \|T\|^2 \right) ||\nabla \nabla \hat{T}||^{p-1} \eta^{2p} dV \]

\[ + cK \int ||Rm||^p \eta^{2p} dV + cK^2 \int ||Rm||^{p-1} \eta^{2p} dV. \]

Hence, using (3.40), (3.41), and (3.42), we arrive at

Lemma 3.3. One has

\[ A_1' \equiv \frac{d}{dt} A_1 \leq B_1 + cKB_2 + cKA_4 + cKA_1 + cK^2 A_2 \]

(3.43) \[ + c \int \left( \frac{1}{4} \|T\|^2 \right) ||\nabla \nabla \hat{T}||^{p-1} \eta^{2p} dV. \]
In the following computations, we are mainly going to estimate or simplify the bad terms $B_1, B_2$, and also the term involving $-\nabla |T|^2$. Integration by parts on the last integral in (3.43) and using $R = -|T|^2$, we obtain

$$c \int \left( -\nabla |T|^2 \right) ||Rm||^{p-1} \eta^2 dV = c \int \left( (\partial_t - \Delta) R \right) ||Rm||^{p-1} \eta^2 \eta dV$$

$$= c \int (\partial_t R) ||Rm||^{p-1} \eta^2 dV + c \int \left( \nabla R, \nabla \left( ||Rm||^{p-1} \eta^2 \right) \right) dV$$

$$= \frac{d}{dt} \left( c \int R||Rm||^{p-1} \eta^2 dV \right) - c \int R \left( \partial_t ||Rm||^{p-1} \right) \eta^2 dV$$

$$- c \int R||Rm||^{p-1} \eta^2 \eta dV + c \int \left( \nabla R, ||Rm||^{p-3} \nabla \eta \nabla Rm \right) \eta^2 dV$$

$$+ c \int \left( \nabla R, ||Rm||^{p-1} \eta^2 \nabla \eta \right) dV$$

$$\leq c \int ||Rm||^{p-2} \left( \nabla R, \nabla \nabla Rm \right) \eta^2 dV + c \int ||Rm||^{p-1} ||\nabla Rm || ||\nabla \eta || \eta^{p-1} dV$$

$$+ c \int R^2 ||Rm||^{p-1} \eta^2 dV - c \int R \left( \partial_t ||Rm||^{p-1} \right) \eta^2 dV$$

$$+ \frac{d}{dt} \left( c \int R||Rm||^{p-1} \eta^2 dV \right).$$

The first two integrals can be simplified by using the Cauchy-Schwarz inequality as follows:

$$c \int ||Rm||^{p-2} \left( \nabla R, \nabla \nabla Rm \right) \eta^2 dV \leq c \int ||\nabla \nabla Rm || ||Rm||^{p-1} \eta^2 dV$$

$$\leq c \int \left( ||\nabla \nabla Rm || ||Rm|| \frac{p-1}{p} \eta \right) \left( ||\nabla \nabla Rm || ||Rm|| \frac{p-1}{p} \eta \right) dV \leq \frac{1}{50} B_1 + cKB_2$$

and

$$c \int ||Rm||^{p-1} ||\nabla Rm || ||\nabla \eta || \eta^{p-1} dV \leq c \int ||Rm||^{p-1} ||\nabla \nabla Rm || ||\nabla \eta || \eta^{p-1} dV$$

$$\leq c \int \left( ||Rm|| \frac{p-1}{p} ||\nabla \eta || \eta^{p-1} \right) \left( ||Rm|| \frac{p-1}{p} ||\nabla \nabla Rm || dV \right) \leq \frac{1}{50} B_1 + cKA_4.$$

Therefore

$$c \int \left( -\nabla |T|^2 \right) ||Rm||^{p-1} \eta^2 dV \leq \frac{2}{50} B_1 + cKB_2 + cKA_4 + c^2 A_2$$

(3.44)

$$+ \frac{d}{dt} \left( c \int R||Rm||^{p-1} \eta^2 dV \right) - c \int R \left( \partial_t ||Rm||^{p-1} \right) \eta^2 dV.$$

Now, the second integral in (3.44) is equal to

$$- c \int R \left( \partial_t ||Rm||^{p-1} \right) \eta^2 dV = c \int \left( -R \right) ||Rm||^{p-3} \left( \partial_t ||Rm||^2 \right) \eta^2 dV$$

$$= c \int (-R)||Rm||^{p-3} \left[ \nabla^2 \text{Ric} \ast Rm + \text{Ric} \ast \nabla^2 \text{Rm} + \text{Rm} \ast \nabla^2 \text{Rm} \ast \nabla \right.$$

$$\left. + \nabla^2 \text{Rm} \right] \eta^2 dV$$

$$\leq c \int (-R)||Rm||^{p-3} \left[ \nabla^2 \text{Ric} \ast Rm - \nabla \ast \nabla^2 \text{Rm} \ast \nabla \ast \text{Rm} \ast \nabla^2 \text{T} \right] \eta^2 dV + c^2 A_2.$$
Using the identity, where $p \geq 5$,

$$\nabla ||Rm||^{p-3} = \frac{p-3}{2} \left( ||Rm||^2 \right)^{p-4} \nabla ||Rm||^2 \leq ||Rm||^{p-5} Rm * \nabla Rm$$

we obtain

$$c \int (-R)||Rm||^{p-3} \eta^{2p} (\nabla^2 R + Rm) dV = c \int (-R)||Rm||^{p-3} \eta^{2p} (\nabla^2 R + Rm) dV$$

$$+ c \int \{ \nabla \left[ (-R)||Rm||^{p-3} \phi \right] \} * \nabla R + Rm \} dV$$

$$= c \int (-R)||Rm||^{p-3} \eta^{2p} (\nabla^2 R + Rm) dV + c \int ||Rm||^{p-3} \eta^{2p} (\nabla^2 R + Rm) dV$$

$$+ c \int (-R) \eta^{2p} (\nabla ||Rm||^{p-3} \nabla R + Rm) dV$$

$$\leq c \int \left( ||\nabla R|| ||Rm||^{p-1} \eta^p \right) \left( ||\nabla \phi|| ||Rm||^{p-1} \eta^{p-1} \right) dV \leq \frac{1}{50} B_1 + cKB_2 + cKA_4.$$

Similarly, we can prove

$$c \int (-R)||Rm||^{p-3} \left( \nabla^2 R - Rm \right) \eta^{2p} dV \leq \frac{1}{50} B_1 + cKB_2 + cKA_4.$$

Using $\nabla \hat{T} = \nabla T * T \leq c ||\nabla T|| ||T|| \leq cK^{1/2} ||\nabla T||$ yields

$$c \int (-R)||Rm||^{p-3} \eta^{2p} (\nabla^2 \hat{T} + Rm) dV = c \int (-R)||Rm||^{p-3} \eta^{2p} (\nabla^2 \hat{T} + Rm) dV$$

$$+ c \int \{ \nabla \left[ (-R)||Rm||^{p-3} \phi \right] \} * \nabla \hat{T} + Rm \} dV$$

$$= c \int (-R)||Rm||^{p-3} \eta^{2p} (\nabla^2 \hat{T} + Rm) dV + c \int ||Rm||^{p-3} \eta^{2p} (\nabla^2 \hat{T} + Rm) dV$$

$$+ c \int (-R) \eta^{2p} (\nabla ||Rm||^{p-3} \nabla \hat{T} + Rm) dV$$

$$\leq c \int \left( ||\nabla R|| ||Rm||^{p-1} \eta^p \right) \left( ||\nabla \hat{T}|| K^{1/2} ||Rm||^{p-1} \eta^{p-1} \right) dV$$

$$\leq c \int \left( ||\nabla R|| ||Rm||^{p-1} \eta^p \right) \left( ||\nabla \hat{T}|| K^{1/2} ||Rm||^{p-1} \eta^{p-1} \right) dV$$

According to (3.44) we get

$$c \int ||\nabla T||^2 ||Rm||^{p-1} \eta^{2p} dV$$
Choosing $\epsilon$ and $24$ yields

$$
\leq c \int \left( -||T||^2 \right) ||Rm||^{p-1} \eta^2 dV + cKA_1 + cK^2 A_2
$$

$$
\leq \frac{2}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1
$$

$$
\frac{d}{dt} \left( c \int R ||Rm||^{p-1} \eta^2 dV \right) - c \int R \left( \partial_t ||Rm||^{p-1} \right) \eta^2 dV
$$

$$
\leq \frac{2}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1
$$

$$
+ \frac{d}{dt} \left( \int cR ||Rm||^{p-1} \eta^2 dV \right) + c \int (-R) ||Rm||^{p-3} \left( \partial_t ||Rm||^2 \right) \eta^2 dV.
$$

Hence

$$
c \int (-R) ||Rm||^{p-3} \left( \partial_t ||Rm||^2 \right) \eta^2 dV \leq \frac{2}{50} B_1 + cKB_2 + cKA_4 + \frac{1}{\epsilon} B_2 + \frac{1}{\epsilon} A_4
$$

$$
e \left[ \frac{2}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left( \int cR ||Rm||^{p-1} \eta^2 dV \right) \right]
$$

$$
+ ec \int (-R) ||Rm||^{p-3} \left( \partial_t ||Rm||^2 \right) \eta^2 dV.
$$

Choosing $\epsilon = \frac{1}{2}$ yields

$$
\frac{c}{2} \int (-R) ||Rm||^{p-3} \left( \partial_t ||Rm||^2 \right) \eta^2 dV
$$

$$
\leq \frac{3}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left( \int cR ||Rm||^{p-1} \eta^2 dV \right)
$$

and

$$
c \int ||\nabla T||^2 ||Rm||^{p-1} \eta^2 dV
$$

$$
\leq \frac{8}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left( \int 2cR ||Rm||^{p-1} \eta^2 dV \right).
$$

Thus

$$
c \int (-R) ||Rm||^{p-3} \left( \partial_t ||Rm||^2 \right) \eta^2 dV \leq \frac{3}{50} B_1 + cKB_2
$$

(3.45)

$$
+ cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left( \int cR ||Rm||^{p-1} \eta^2 dV \right)
$$

and

$$
c \int ||\nabla T||^2 ||Rm||^{p-1} \eta^2 dV \leq \frac{8}{50} B_1 + cKB_2
$$

(3.46)

$$
+ cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left( \int cR ||Rm||^{p-1} \eta^2 dV \right)
$$

and

$$
c \int \left( -||T||^2 \right) ||Rm||^{p-1} \eta^2 dV \leq \frac{5}{50} B_1 + cKB_2
$$

(3.47)

$$
+ cK^2 A_2 + cKA_1 + \frac{d}{dt} \left( \int cR ||Rm||^{p-1} \eta^2 dV \right).
$$

From (3.43) and (3.47) we arrive at
Lemma 3.4. One has
\[
A_1^t \leq 2B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1 + \frac{d}{dt} \left( \int cR||Rm||^p - \eta^2 dV \right).
\] (3.48)

We next estimate \( B_1 \) and \( B_2 \). Actually, we shall see that \( B_1 \) can be estimated in terms of \( B_2 \). Hence the key step is to estimate \( B_2 \). For \( B_1 \), using
\[
||\nabla Ric||^2 = -\frac{1}{2} ||\nabla Ric||^2 + Ric * Ric * Rm - \frac{1}{3} (\nabla^2 R) T - \frac{2}{3} R ||\nabla Ric||^2 + 2 \langle (Ric, \nabla T) \rangle + \frac{1}{3} \langle (Ric, \nabla^2 R) \rangle + Ric * \hat{T} * Rm + Ric * \hat{\nabla}^2 \hat{T}.
\]

we obtain
\[
B_1 \leq \frac{1}{2K} \int ||Rm||^p - \eta^2 p (\nabla - \nabla t) ||Ric||^2 dV + cKA_1
\] (3.49) + \[
\frac{1}{3K} \int (-R) ||Rm||^p - \eta^2 p |\Delta R| dV + \frac{2}{K} \int \langle (Ric, \nabla T) \rangle ||Rm||^p - \eta^2 p dV
\]
+ \[
\frac{1}{3K} \int \langle (Ric, \nabla^2 R) \rangle ||Rm||^p - \eta^2 p dV + \frac{1}{K} \int ||Rm||^p - \eta^2 p dV.
\]

From the estimates \( \nabla ||Ric||^2 \leq ||Ric|| ||\nabla Ric||, \nabla ||Rm||^p \leq ||Rm||^p - \eta^2 ||\nabla Rm|| \), and \( \partial_t ||Rm||^p - 1 = \frac{p - 1}{2} ||Rm||^p - \eta^2 ||\nabla \partial_t Rm||^2 \), we have
\[
\int ||Rm||^p - \eta^2 p (\nabla - \partial_t) ||Ric||^2 dV
\]
\[
= \int \nabla ||Ric||^2 * \nabla \left( ||Rm||^p - \eta^2 p \right) dV - \int ||Rm||^p - \eta^2 p \left( \partial_t ||Ric||^2 \right) dV
\]
\[
= \int \left( \nabla ||Ric||^2 * \nabla ||Rm||^p - 1 \right) \eta^2 p dV + \int \left( \nabla ||Ric||^2 * \nabla \eta \right) ||Rm||^p - \eta^2 p - 1 dV
\]
\[
- \frac{d}{dt} \left[ \int ||Rm||^p - \eta^2 p ||Ric||^2 dV \right] + \int \left( \partial_t ||Rm||^p - 1 \right) \eta^2 p ||Ric||^2 dV
\]
\[
+ \int ||Rm||^p - \eta^2 p ||Ric||^2 (\partial_t dV)
\]
\[
\leq cK \int ||\nabla Ric|| ||\nabla Rm|| ||Rm||^p - 2 \eta^2 p dV + cK \int ||\nabla Ric|| ||\nabla \eta|| ||Rm||^p - 1 \eta^2 p - 1 dV
\]
+ \[
c \int ||Rm||^p - 3 \left( \partial_t ||Rm||^2 \right) \eta^2 p ||Ric||^2 dV + cK^2 A_1
\]
\[
- \frac{d}{dt} \left[ \int ||Rm||^p - 1 ||Ric||^2 \eta^2 p dV \right]
\]
\[
\leq cK \left( \frac{1}{50c} B_1 + cKB_2 \right) + cK \left( \frac{1}{50c} B_1 + cKA_4 \right) + cK^2 A_1
\]
+ \[
c \int ||Ric||^2 ||Rm||^p - 3 \eta^2 p \left( \partial_t ||Rm||^2 \right) dV - \frac{d}{dt} \left[ \int ||Rm||^p - 1 ||Ric||^2 \eta^2 p dV \right]
\]
\[
\leq \frac{2}{50} KB_1 + cK^2 B_2 + cK^2 A_4 + cK^2 A_1
\]
+ \[
c \int ||Ric||^2 ||Rm||^p - 3 \eta^2 p \left( \partial_t ||Rm||^2 \right) dV - \frac{d}{dt} \left[ \int ||Rm||^p - 1 ||Ric||^2 \eta^2 p dV \right].
\]
Thus

\begin{equation}
(3.50) \quad \int |\text{Rm}|^{p-1}\eta^{2p} ||\text{Ric}||^2 dV \leq \frac{2}{50} KB_1 + cK^2B_2 + cK^2A_4 + cK^2A_1
+ c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \left( \partial_t ||\text{Rm}||^2 \right) dV - \frac{d}{dt} \left[ \int |\text{Rm}|^{p-1} ||\text{Ric}||^2 \eta^{2p} dV \right].
\end{equation}

Consider the term
\[
c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \left( \partial_t ||\text{Rm}||^2 \right) dV = c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \\
\left[ \nabla^2 \text{Ric} \ast \text{Rm} + \text{Ric} \ast \text{Rm} \ast \text{Rm} + \text{Rm} \ast \text{Rm} \ast \hat{T} + \text{Ric} \ast \nabla^2 ||T||^2 + \text{Rm} \ast \nabla^2 \hat{T} \\
+ \frac{4}{3} ||T||^2 |\text{Rm}|^2 \right] dV \leq c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \left[ \nabla^2 \text{Ric} \ast \text{Rm} - \nabla^2 R \ast \text{Ric} \\
+ \nabla^2 \hat{T} \ast \text{Rm} \right] dV + cK^2A_2.
\]

The three terms in the bracket can be estimated as follows. Firstly
\[
c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \left( \nabla^2 \text{Ric} \ast \text{Rm} \right) dV
= c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \left( \nabla \text{Ric} \ast \nabla \text{Rm} \right) dV
+ c \int \left\{ \nabla \left[ |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \right] \ast \nabla \text{Ric} \ast \text{Rm} \right\} dV
= c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \left( \nabla \nabla \text{Ric} \ast \nabla \text{Rm} \right) dV
+ c \int |\text{Rm}|^{p-3}\eta^{2p} \left( \nabla ||\text{Ric}||^2 \ast \nabla \text{Ric} \ast \text{Rm} \right) dV
+ c \int |\text{Ric}|^2\eta^{2p} \left( \nabla ||\text{Rm}|^{p-3} \ast \nabla \text{Ric} \ast \text{Rm} \right) dV
+ c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p-1} \left( \nabla \eta \ast \nabla \text{Ric} \ast \text{Rm} \right) dV
\leq cK \int |\text{Rm}|^{p-3}\eta^{2p} ||\nabla \text{Ric}|| ||\nabla \text{Rm}|| dV + cK \int |\text{Rm}|^{p-1}\eta^{2p-1} ||\nabla \text{Ric}|| ||\eta|| dV
\leq cK \left( eB_1 + \frac{K}{e}B_2 \right) + cK \left( eB_1 + \frac{K}{e}A_4 \right) \leq \frac{1}{50} KB_1 + cK^2B_2 + cK^2A_4.
\]
The same estimate holds for
\[
c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \left( -\nabla^2 R \ast \text{Ric} \right) dV.
\]
Finally,
\[
c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \left( \nabla^2 \hat{T} \ast \text{Rm} \right) dV = c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \\
\left( \nabla \hat{T} \ast \nabla \text{Rm} \right) dV + c \int \left\{ \nabla \left[ |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \right] \ast \nabla \hat{T} \ast \text{Rm} \right\} dV
\leq c \int |\text{Ric}|^2 |\text{Rm}|^{p-3}\eta^{2p} \left( K^{1/2} ||\nabla T|| ||\nabla \text{Rm}|| \right) dV
+ c \int \left( \nabla ||\text{Ric}||^2 \right) |\text{Rm}|^{p-3}\eta^{2p} ||\nabla \hat{T}|| ||\text{Rm}|| dV
+ c \int |\text{Rm}|^2 \left( \nabla ||\text{Rm}||^{p-3} \eta^{2p} ||\nabla \hat{T}|| ||\text{Rm}|| \right) dV.
\]
Therefore and involving the scalar curvature. In the following, we estimate the left four terms in (3.49). We start from terms (3.53) + \( cK \int ||Rm||^{p-1}{\eta}^{2p} dV \)

\[
\leq cK \int ||Rm||^{p-1}{\eta}^{2p} \left( K^{1/2} ||\nabla T|| \right) dV + cK \int ||Rm||^{p-1}{\eta}^{2p-1} \left( K^{1/2} ||\nabla \eta|| \right) dV
\]

\[
\leq K \left[ cKB_2 + \frac{cK}{e} A_4 + ec \int ||\nabla T||^2 ||Rm||^{p-1}{\eta}^{2p} dV \right]
\]

\[
\leq \frac{8}{50} KB_1 + cK^2 B_2 + cK^2 A_4 + cK^3 A_2 + cK^2 A_1 + \frac{d}{dt} \left[ cK \int R ||Rm||^{p-1}{\eta}^{2p} dV \right]
\]

\[
\leq \frac{10}{50} KB_1 + cK^2 B_2 + cK^2 A_4 + cK^3 A_2
\]

In (3.51), we have

\[
\frac{1}{2K} \int ||Rm||^{p-1}{\eta}^{2p} (\nabla \eta - \nabla \eta) ||Ric||^2 dV \leq \frac{6}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1
\]

\[
- \frac{1}{K} \frac{d}{dt} \left[ \int ||Rm||^{p-1} ||Ric||^2{\eta}^{2p} dV \right] + \frac{d}{dt} \left[ \int R ||Rm||^{p-1}{\eta}^{2p} dV \right]
\]

\[
\leq \frac{6}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1
\]

\[
- \frac{d}{dt} \left[ \frac{1}{K} \int ||Rm||^{p-1} ||Ric||^2{\eta}^{2p} dV + c \int (\nabla \eta) ||Rm||^{p-1}{\eta}^{2p} dV \right].
\]

In the following, we estimate the left four terms in (3.49). We start from terms involving the scalar curvature.

\[
\frac{1}{3K} \int (\nabla \eta) ||Rm||^{p-1}{\eta}^{2p} dV = - \frac{1}{3K} \int \nabla R \cdot \nabla \left[ (\nabla \eta) ||Rm||^{p-1}{\eta}^{2p} \right] dV
\]

\[
= - \frac{1}{3K} \int \nabla R \cdot \left[ - \nabla \eta ||Rm||^{p-1}{\eta}^{2p} + (\nabla \eta) ||Rm||^{p-1}{\eta}^{2p} \right]
\]

\[
\leq \frac{2}{3K} \int \nabla R ||Rm||^{p-1}{\eta}^{2p} \nabla \eta dV \leq \frac{2}{3K} \int ||\nabla R||^{2} ||Rm||^{p-1}{\eta}^{2p} dV
\]

\[
+ \frac{c}{K} \int (\nabla \eta) ||Rm||^{p-2} ||\nabla R|| ||\nabla Rm||{\eta}^{2p} dV
\]

\[
+ \frac{c}{K} \int (\nabla \eta) ||Rm||^{p-1}{\eta}^{2p} ||\nabla \eta|| dV
\]

\[
\leq \frac{1}{3K} \int ||\nabla R||^{2} ||Rm||^{p-1}{\eta}^{2p} dV + \frac{1}{3K} \int ||\nabla R||^{2} ||Rm||^{p-1}{\eta}^{2p} dV + cKB_2
\]

\[
+ \frac{1}{3K} \int ||\nabla R||^{2} ||Rm||^{p-1}{\eta}^{2p} dV + cKA_4
\]

\[
\leq \frac{1}{K} \int ||\nabla R||^{2} ||Rm||^{p-1}{\eta}^{2p} dV + cKB_2 + cKA_4.
\]
The another term involving the scalar curvature can be estimated by

\[
\frac{1}{3K} \int \langle (\text{Ric}, \nabla^2 \hat{R} \rangle ||\text{Rm}||^{p-1} \eta^2 dV = \frac{1}{3K} \int (\nabla R \nabla) \left[ R_{ij} ||\text{Rm}||^{p-1} \eta^2 \right] dV
\]

\[
= -\frac{1}{3K} \int \nabla R \left[ \frac{1}{2} \nabla R ||\text{Rm}||^{p-1} \eta^2 + R_{ij} \nabla \eta^2 \right] dV
\]

(3.54) \quad + R_{ij} ||\text{Rm}||^{p-1} 2p \eta^2 \nabla \eta^2 dV \leq \frac{1}{6K} \int ||\nabla R||^2 ||\text{Rm}||^{p-1} \eta^2 dV

+ \frac{c}{K} \int ||\text{Ric}|| ||\nabla R|| ||\text{Rm}||^{p-2} ||\nabla \text{Rm}|| \eta^2 dV

+ \frac{c}{K} \int ||\nabla R|| ||\text{Ric}|| ||\text{Rm}||^{p-1} \eta^2 \nabla \eta^2 dV

\leq -\frac{1}{6K} \int ||\nabla R||^2 ||\text{Rm}||^{p-1} \eta^2 dV + \frac{1}{18K} \int ||\nabla R||^2 ||\text{Rm}||^{p-1} \eta^2 dV + cKB_2

+ \frac{1}{18K} \int ||\nabla R||^2 ||\text{Rm}||^{p-1} \eta^2 dV + cKA_4 \leq cKB_2 + cKA_4.

Using (3.46) we obtain

\[
\frac{2}{K} \int \langle (\text{Ric}, \nabla \hat{\nabla}) \rangle ||\text{Rm}||^{p-1} \eta^2 dV = \frac{1}{K} \int \text{Ric} \ast \nabla \hat{\nabla} ||\text{Rm}||^{p-1} \eta^2 dV
\]

\[
= \frac{1}{K} \int \nabla \text{Ric} \ast \nabla \hat{\nabla} ||\text{Rm}||^{p-1} \eta^2 dV + \frac{1}{K} \int \text{Ric} \ast \nabla \hat{\nabla} \ast \nabla \left[ ||\text{Rm}||^{p-1} \eta^2 \right] dV
\]

\[
\leq \frac{c}{K} \int ||\nabla \text{Ric}|| ||\nabla \hat{\nabla}|| ||\text{Rm}||^{p-1} \eta^2 dV + \frac{c}{K} \int ||\text{Ric}|| ||\nabla \hat{\nabla}|| ||\text{Rm}||^{p-2} ||\nabla \text{Rm}|| \eta^2 dV

+ \frac{c}{K} \int ||\nabla \text{Ric}|| ||\nabla \hat{\nabla}|| ||\text{Rm}||^{p-1} \eta^2 \nabla \eta^2 dV
\]

(3.55) \quad \leq \frac{1}{50} B_1 + c \int ||\nabla \nabla||^2 ||\text{Rm}||^{p-1} \eta^2 dV + cKB_2

+ c \int ||\nabla \nabla||^2 ||\text{Rm}||^{p-1} \eta^2 dV + cKA_4 + c \int ||\nabla \nabla||^2 ||\text{Rm}||^{p-1} \eta^2 dV

\leq \frac{1}{50} B_1 + cKB_2 + cKA_4 + c \int ||\nabla \nabla||^2 ||\text{Rm}||^{p-1} \eta^2 dV

\leq \frac{9}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left[ \int cR ||\text{Rm}||^{p-1} \eta^2 dV \right].

Similarly, we can prove

\[
\frac{1}{K} \int \text{Ric} \ast \nabla^2 \hat{\nabla} ||\text{Rm}||^{p-1} \eta^2 dV = \frac{1}{K} \int \nabla \text{Ric} \ast \nabla \hat{\nabla} ||\text{Rm}||^{p-1} \eta^2 dV
\]

\[
+ \frac{1}{K} \int \text{Ric} \ast \nabla \hat{\nabla} \ast \nabla \left[ ||\text{Rm}||^{p-1} \eta^2 \right] dV \leq \frac{1}{K} \int \nabla \text{Ric} \ast \nabla \hat{\nabla} ||\text{Rm}||^{p-1} \eta^2 dV

+ \frac{c}{K} \int ||\text{Ric}|| ||\nabla \hat{\nabla}|| ||\text{Rm}||^{p-2} ||\nabla \text{Rm}|| \eta^2 dV
\]

(3.56) \quad + \frac{c}{K} \int ||\text{Ric}|| ||\nabla \hat{\nabla}|| ||\text{Rm}||^{p-1} \eta^2 \nabla \eta^2 dV

\leq \frac{c}{K} \int ||\nabla \text{Ric}|| ||\nabla \hat{\nabla}|| ||\text{Rm}||^{p-1} \eta^2 dV + \frac{c}{K} \int ||\text{Ric}|| ||\nabla \hat{\nabla}|| ||\text{Rm}||^{p-2} ||\nabla \text{Rm}|| \eta^2 dV

+ \frac{c}{K} \int ||\text{Ric}|| ||\nabla \hat{\nabla}|| ||\text{Rm}||^{p-1} \eta^2 \nabla \eta^2 dV
we obtain
\[ \frac{9}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left[ \int cR \|Rm\|^p \eta^{2p} dV \right]. \]

Plugging (3.50) and (3.53) − (3.56) into (3.49), and using (3.46) and \( \|\nabla R\|^2 \leq cK\|\nabla T\|^2 \), we obtain
\[ B_1 \leq \frac{6}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 \]
\[ - \frac{d}{dt} \left[ \frac{1}{K} \int \|Rm\|^p \eta^{2p} dV + c \int (-R) \|Rm\|^p \eta^{2p} dV \right] \]
\[ + \frac{1}{K} \int \|\nabla R\|^2 \|Rm\|^p \eta^{2p} dV + \frac{18}{50} B_1 - \frac{d}{dt} \left[ c \int (-R) \|Rm\|^p \eta^{2p} dV \right] \]
\[ \leq \frac{32}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 \]
\[ - \frac{d}{dt} \left[ \frac{1}{K} \int \|Rm\|^p \eta^{2p} dV + c \int (-R) \|Rm\|^p \eta^{2p} dV \right]. \]

Thus (3.57)
\[ B_1 \leq cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 \]
\[ - \frac{d}{dt} \left[ \frac{1}{K} \int \|Rm\|^p \eta^{2p} dV + c \int (-R) \|Rm\|^p \eta^{2p} dV \right] \]

From (3.48) and (3.57), we can conclude that

**Lemma 3.5.** One has
\[ A'_1 \leq cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 \]
\[ - \frac{d}{dt} \left[ \frac{1}{K} \int \|Rm\|^p \eta^{2p} dV + c \int (-R) \|Rm\|^p \eta^{2p} dV \right]. \]

Observe that two terms in the bracket are both nonnegative, since \( R = -\|T\|^2 \leq 0 \).

Finally, we estimate the term \( B_2 \). Using the evolution inequality
\[ \|\nabla Rm\|^2 \leq \left( \frac{1}{2} \right) \|Rm\|^2 + c \|Rm\|^3 + c \|\nabla^2 T\| \|Rm\|^{3/2} + c \|Rm\| \|\nabla T\|^2 \]
we obtain
\[ B_2 = \int \|\nabla Rm\|^2 \|Rm\|^p \eta^{3\eta^{2p}} dV \leq \int \left[ \left( \frac{1}{2} \right) \|Rm\|^2 + c \|Rm\|^3 \right. \]
\[ + c \|\nabla^2 T\| \|Rm\|^{3/2} + c \|Rm\| \|\nabla T\|^2 \left. \right] \|Rm\|^p \eta^{3\eta^{2p}} dV \]
\[ \leq - \frac{1}{2} \int \left( \left( \frac{1}{2} \right) \|Rm\|^2 \right) \|Rm\|^p \eta^{3\eta^{2p}} dV + cA_1 \]
\[ + c \int \|\nabla^2 T\| \|Rm\|^{p-3/2} \eta^{2p} dV + c \int \|\nabla^2 T\|^2 \|Rm\|^{p-2} \eta^{2p} dV. \]

For the first integral one has
\[ - \frac{1}{2} \int \left( \left( \frac{1}{2} \right) \|Rm\|^2 \right) \|Rm\|^p \eta^{3\eta^{2p}} dV = \frac{1}{2} \int \left( \left( \frac{1}{2} \right) \|Rm\|^2 \right) \|Rm\|^p \eta^{3\eta^{2p}} dV \]
To estimate the remainder two integrals, we recall from (3.9) that
\[ \nabla \left( \frac{3.61}{3.62} \nabla \right) \]
In particular, the inequality (3.63) yields
\[ \nabla \left( \frac{3.61}{3.62} \nabla \right) \]
Taking the derivative of (3.61) and using (3.62) we obtain
\[ \nabla^2 T = \nabla Rm \ast \varphi + Rm \ast T \ast \psi + \nabla T \ast T \ast \varphi + T \ast T \ast T \ast \psi. \]
The particular case \[ \nabla^2 T \]
leads to
\[ c \int ||\nabla^2 T||_{Rm} ||^p - 3/2 \eta^2 p || dV \leq c \int \left( ||\nabla Rm|| + ||Rm|| + ||T|| + ||\nabla T|| + ||T||^3 \right) dV \]
In particular, the inequality (3.63) yields
Thus

\[ (3.71) \]

Plugging \((3.69), (3.64), \) and \((3.66)\) into \((3.59)\) we arrive at

\[ (3.67) \]

Together with \((3.58)\) and \((3.67)\) we finally obtain

\[ (3.68) \]

\[ \frac{d}{dt} \left[ \frac{c}{K} \int ||\text{Rm}||^{p-1}||\text{Ric}||^2 \eta^{2p} dV + c \int (-R)||\text{Rm}||^{p-1} \eta^{2p} dV \right] \]

Equivalently,

**Lemma 3.6.** If \(||\text{Ric}|| \leq K \) and \( p \geq 5 \), one has

\[ \frac{d}{dt} \left[ A_1 + cKA_2 + \frac{c}{K} \int ||\text{Rm}||^{p-1}||\text{Ric}||^2 \eta^{2p} dV + c \int (-R)||\text{Rm}||^{p-1} \eta^{2p} dV \right] \]

\[ \leq cK(A_1 + cKA_2) + cKA_4. \]

As in \([24, 27]\), we choose the domain \( \Omega := B_{g_0}(x_0, \rho / \sqrt{K}) \) and the function

\[ \eta = \left( \frac{\rho / \sqrt{K} - d_{g_0}(x_0, \cdot)}{\rho / \sqrt{K}} \right)_+. \]

Then, for all \( t \in [0, T], \)

\[ e^{-cKt}g_0 \leq \eta(t) \leq e^{cKt}g_0, \]

\[ ||\nabla \eta(t)||_{g(t)} \leq e^{cKT}||\nabla \eta||_{g_0} \leq \frac{\sqrt{Ke^{cKT}}}{\rho}. \]

The proof of Theorem 1.4. Define

\[ U := \int ||\text{Rm}||^{p-1}||\text{Ric}||^2 \eta^{2p} dV + cK \int ||\text{Rm}||^{p-1} \eta^{2p} dV \]

\[ + \frac{c}{K} \int ||\text{Rm}||^{p-1}||\text{Ric}||^2 \eta^{2p} dV + c \int (-R)||\text{Rm}||^{p-1} \eta^{2p} dV. \]

Then \((3.69)\) yields

\[ (3.70) \]

\[ U' \leq cKU + cKA_4. \]

For \( A_4, \) using the Young inequality, we have

\[ A_4 = \int ||\text{Rm}||^{p-1}||\nabla \eta||^2 \eta^{2p-2} dV \leq \int_{B_{g_0}(x_0, \rho / \sqrt{K})} ||\text{Rm}||^{p-1} \eta^{2p-2} K \rho^{-2} e^{cKT} dV \]

\[ \leq \int_{B_{g_0}(x_0, \rho / \sqrt{K})} \left[ \left( ||\text{Rm}||^{p-1} \eta^{2p-2} \right)^{p/(p-1)} + \frac{(K \rho^{-2} e^{cKT})^p}{p} \right] dV \]

\[ \leq A_1 + K^p \rho^{-2p} e^{cKT} \text{vol}_{g(t)} \left( B_{g_0} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right) \]

\[ \leq U + cK^p e^{cKT} \rho^{-2p} \text{vol}_{g(t)} \left( B_{g_0} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right). \]

Thus

\[ U' \leq cKU + cK^{p+1} e^{cKT} \rho^{-2p} \text{vol}_{g(t)} \left( B_{g_0} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right). \]
As in the proof of [24], one can easily deduce from above that
\[
\int_{B_{g_0}(x_0,\rho/\sqrt{K})} ||Rm_{g(t)}||^p_{g(t)} dV_{g(t)} \leq c(1+K)e^{cKT} \int_{B_{g_0}(x_0,\rho/\sqrt{K})} ||Rm_{g_0}||^p_{g_0} dV_{g_0}
\]
\[
+ cK^p \left(1+\rho^{-2p}\right)e^{cKT} \vol_{g(t)} \left(B_{g_0}(x_0,\rho/\sqrt{K})\right).
\]

(3.72)

As an immediate consequence of the inequality (3.72) we give another proof of the part (a) in Theorem 1.2.

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