Degree reduction of disk rational Bézier curves

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Abstract

This paper presents an algorithm for optimal multi-degree reduction of rational disk Bézier curve in the $L^2$ norm. We start by introducing a novel disk rational Bézier based on parallel projection, whose properties are also discussed. Then we transform multi-degree reduction of the error radius curve and the weight curve of the disk rational Bézier curve into solving a constrained quadratic programming (QP) problem. Finally, applying weighted least squares, we provide the optimal multi-degree reduced polynomial approximation of the center curve of the original disk rational Bézier curve. Also this paper gives error estimation for this algorithm, and shows some numerical examples to illustrate the correctness and the validity of theoretical reasoning.

Keywords: Disk rational Bézier curve; Multi-degree reduction; Weighted least squares; Constrained quadratic programming

1. Introduction

Because operations of geometric objects in current state-of-the-art Computer Aided Design systems are based on floating point arithmetic, representations of geometric objects are inaccurate and geometrical computations are...
approximate. In order to deal with the problem of lack of robustness, interval
arithmetic is used in the fields. In 1992, Sederberg and Farouki[1] formally
introduced the concept of interval Bézier curves that can transfer a complete
description of approximation errors along with the curves to applications in
other systems. Based on Sederberg’s work, the algorithms for curve/curve
or surface/surface intersection, solid modeling, and stability of visualization
were studied by Hu et al.[2, 3, 4]. Chen and Lou [5] discussed the problem of
bounding interval Bézier curve with lower degree interval Bézier curve. Lin
et al. [6] provided an efficient and reliable algorithm for representing and
evaluating the boundary of the interval Bézier curves in 2- and 3-D. However, as Chen pointed out [5], interval curves possess some blemish. One is
that interval generally enlarge rapidly in a computational process, another
is that rectangular intervals are not rotationally symmetric. To overcome
these shortcoming, Lin and Rokne [7] applied disk as control points, and
the corresponding Bézier curves are called disk Bézier curves. Since interval
Bézier curves can’t represent conic precisely, Hu et al. [3, 4] introduced inter-
val non-uniform rational B-splines (INURBS) curves and surfaces. In 2011,
using parallel projection, the first author of [8] defined a novel disk rational
Bézier curves, which differ from classic disk rational Bézier in that its error
radii are Bézier polynomials functions.

One of the important theme for rational Bézier curves is degree reduction.
This problem exists because some CAD/CAM systems limit the maximum
polynomial degree so that they can deal with; the other useful application
is the need of data compression and data comparison [9]. Compared with
many methods of degree reduction of Bézier curves, very little research has
been done for rational Bézier curves. Farin [10] described a degree reduction
method for rational Bézier curves for interactive interpolation and approxi-
Sederberg and Chang \[11\] achieved one degree reduction based on perturbing the numerator and denominator polynomials of a rational curve that get a common linear divisor. However, their method can not satisfy interpolations at end points. Chen \[12\] applied the algorithm of shifted Chebyshev polynomials to agree with the degree reduction of rational Bézier with $C^0$-continuity at both end points. The purpose of current paper is to deal with the same problem for disk rational Bézier curves using weighted least-squares (WLS) and constrained quadratic programming (CQP).

This paper has the following structure: To ensure the structural integrity of this paper, in section 2, we review the definition of disk rational Bézier curves and corresponding properties. In section 3, we propose an efficient algorithm to the problem of degree reduction of rational disk Bézier curves. In section 4, bounding errors for degree reduction are analyzed, and some examples are provided.

2. Disk rational Bézier curves

2.1. Disk rational arithmetic

A disk in the plane is defined to be the set

$$(q) = (x_0, y_0)_r = \{x \in \mathbb{R}^2 \mid \|x - q\| \leq r, r \in \mathbb{R}^+ \},$$

whose centric point is $q$ and radius is $r$.

For any two disks $(q_i) = (x_i, y_i)_r$, $i = 1, 2$, the two operations are defined as follow

$$k(q_i) = (k q_i) = (kx_i, ky_i)_{|k| r}, \forall k \in \mathbb{R}, i = 1, 2,$$  \hspace{1cm} (1)
\[(q_1) + (q_2) = (x_1, y_1, r_1) + (x_2, y_2, r_2) = (x_1 + x_2, y_1 + y_2, r_1 + r_2). \]  \hfill (2)

Equations (1) and (2) can be generalized as

\[
\sum_{i=0}^{n} k_i (q_i) = \left( \sum_{i=0}^{n} k_i x_i, \sum_{i=0}^{n} k_i y_i \right) \frac{1}{\sum_{i=0}^{n} |k_i| r_i}. \hfill (3)
\]

In homogeneous coordinates, the disk can be described by

\[
(P^\omega) = (X_0, Y_0, \omega)_R = (\omega x_0, \omega y_0, \omega)_r = \{x^\omega = (\omega x, \omega y, \omega) \in \mathbb{R}^3 \mid ||x^\omega - P^\omega|| \leq r\}. \hfill (4)
\]

Applying the perspective projection \(H(\cdot)\) to the disk \((P^\omega)\) yields a corresponding rational disk in plane \(\omega = 1\). That is

\[
(q) = H((P^\omega)) = H(X_0, Y_0, \omega, R) = \left( \frac{X_0}{\omega}, \frac{Y_0}{\omega} \right)_r = (x_0, y_0, r). \hfill (5)
\]

In addition, the disk can also be represented by the homogeneous coordinates

\[
(P^\omega) = (X_0, Y_0, \omega)_R = (\omega x_0, \omega y_0, \omega)_r = \{x^\omega = (\omega x, \omega y, \omega) \in \mathbb{R}^3 \mid ||x^\omega - P^\omega|| \leq r\}. \hfill (6)
\]

Applying the oblique projection \(I(\cdot)\) to the disk \((P^\omega)\) yields another corresponding rational disk in plane \(\omega = 1\). That is

\[
(q) = I((P^\omega)) = I(X_0, Y_0, \omega, r) = \left( \frac{X_0}{\omega}, \frac{Y_0}{\omega} \right)_r = (x_0, y_0, r). \hfill (7)
\]

Both equations (5) and (7) obviously agree with operation (3). Based on
equations (3) and (5), it can derive a classic disk rational Bézier curve which has been researched in [3][4]. However, using oblique projection $I(\cdot)$, we can define a novel kind of disk rational curve and its properties, such as end interpolation, affine invariant etc., are similar to the classic disk rational Bézier curve.

2.2. Disk rational Bézier curves

A disk rational Bézier curve of degree $n$ with control disk points $(p_i) = (x_i, y_i), r_i$, and corresponding weights $\omega_i \in \mathbb{R}^+, i = 0, ..., n$ is defined by

$$(p)(t) = [p(t); r(t)] = \left[ \sum_{i=0}^{n} p_i \omega_i B_i^n(t) / \sum_{i=0}^{n} \omega_i B_i^n(t); \sum_{i=0}^{n} r_i B_i^n(t) \right], 0 \leq t \leq 1,$$  

(8)

or be written in the basis form

$$(p)(t) = [p(t); r(t)] = \left[ \sum_{i=0}^{n} p_i R_i^n(t); \sum_{i=0}^{n} r_i B_i^n(t) \right], 0 \leq t \leq 1,$$  

(9)

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, i = 0, ..., n$, are Bernstein polynomials, $R_i^n(t) = \frac{\omega_i B_i^n(t)}{\sum_{j=0}^{n} \omega_j B_j^n(t)}, i = 0, 1, ..., n$. $p(t)$ and $r(t)$ are respectively called the center curve and the radius of the disk rational Bézier curve $(p)(t)$. Compare with the classical disk rational Bézier curve whose radius function $r(t)$ is a rational polynomial with weights, the radius function $r(t)$ in equation (8) or (9) is a positive polynomial.

2.3. Properties of disk rational Bézier curves

A disk rational Bézier curve defined by equation (8) satisfies the following properties.

• **End interpolation:** $(p)(0) = (p_0)$ and $(p)(1) = (p_n)$. 


• **Affine invariant:** Let $\mathcal{A}$ be an affine transformation (for example, a rotation, reflection, translation, or scaling), then

$$
\frac{\sum_{i=0}^{n} \omega_i(p_i) B_i^n(t)}{\sum_{i=0}^{n} \omega_i B_i^n(t)} = \frac{\sum_{i=0}^{n} \omega_i A(p_i) B_i^n(t)}{\sum_{i=0}^{n} \omega_i B_i^n(t)}
$$

• **Convex hull:** The disk rational Bézier curve lies in the convex hull of the control disks.

**Proof.** Since the convex hull of control disks $(p_i), i = 0, 1, ..., n,$ are the set of all convex combinations $\sum_{i=0}^{n} \alpha_i(p_i),$ where $\alpha_i = R_i^n(t)$ and $\sum_{i=0}^{n} R_i^n(t) = 1,$ and the proof is complete.

• **Non-uniform convergence:** The disk rational Bézier curve converges non-uniformly on the interval $t \in [0, 1],$ as $\omega_i \to +\infty.$

**Proof.** Since

$$
\lim_{\omega_i \to +\infty} R_i^n(u) = \begin{cases} 
0 & u = 0, i \neq 0, \\
1 & u = 0, i = 0, \\
1 & 0 < u < 1, i = 0, ..., n, \\
1 & u = 1, i = n, \\
0 & u = 1, i \neq n,
\end{cases} \tag{10}
$$

the center curve $p(t)$ in disk rational Bézier curve $(p)(t)$ is non-uniform convergent and the property is desired.

• **De Casteljau algorithm:** For any $t \in [0, 1],$ $(p)(t)$ can be computed as follows:

$$
\begin{align*}
\omega_i^j &= (1 - t) \omega_i^{j-1} + t \omega_i^{j+1}, \\
r_i^j &= (1 - t) r_i^{j-1} + t r_i^{j+1},
\end{align*}
$$

$$
\begin{align*}
p_i^j &= (1 - t) \frac{\omega_i^{j-1}}{\omega_i^j} p_i^{j-1} + t \frac{\omega_i^{j+1}}{\omega_i^j} p_i^{j+1},
\end{align*}
$$

$$
\begin{align*}
r_i^j &= (1 - t) r_i^{j-1} + t r_i^{j+1},
\end{align*}
$$

\(6\)
for \( j = 1, \ldots, n \) and \( i = 0, \ldots, n - j \).

- **Subdivision:** Let \( c \in (0, 1) \) be a real number. Then \((p)(t)\) can be subdivided into two segments:

\[
(p)(t) = \begin{cases} 
\left[ \sum_{i=0}^{n} p_i(c) \omega_i B_i^n(t) \right] ; \sum_{i=0}^{n} r_i(c) B_i^n(t) , & 0 \leq t \leq c, \\
\left[ \sum_{i=0}^{n} p_n^{-i}(c) \omega_n^{-i}(c) B_i^n\left(\frac{t-c}{1-c}\right) \right] ; \sum_{i=0}^{n} r_n^{-i}(c) B_i^n\left(\frac{t-c}{1-c}\right) , & c \leq t \leq 1, 
\end{cases} \tag{12}
\]

- **Degree elevation:** A disk rational Bézier curve \((p)(t)\) of degree \( m \) can be represented as a disk rational Bézier curve of degree \( m + s \) as follows

\[
(p)(t) = \frac{\sum_{i=0}^{m} (p_i) \omega_i B_i^m(t)}{\sum_{i=0}^{m} \omega_i B_i^m(t)} = \frac{\sum_{i=0}^{m+s} (\hat{p}_i) \hat{\omega}_i B_i^{m+s}(t)}{\sum_{i=0}^{m+s} \hat{\omega}_i B_i^{m+s}(t)} , 0 \leq t \leq 1, \tag{13}
\]

where

\[
\hat{\omega}_i = \sum_{j=\max(0,i-s)}^{\min(m,i)} \omega_j \binom{m}{j} \binom{s}{i-j} \binom{i-j}{m-s} , \quad \hat{p}_i = \frac{\sum_{j=\max(0,i-s)}^{\min(m,i)} p_j \omega_j \binom{m}{j} \binom{s}{i-j} \binom{i-j}{m-s}}{\hat{\omega}_i} ,
\]

and

\[
\hat{r}_i = \sum_{j=\max(0,i-s)}^{\min(m,i)} r_j \binom{m}{j} \binom{s}{i-j} \binom{i-j}{m-s} , \quad i = 0, 1, \ldots, m + s. \tag{14}
\]

- **Exact degree reduction:**

Given a degree \( n \) disk rational Bézier curve \((p)(t)\), it represents exactly a degree \( m \) \((m < n)\) disk rational Bézier curve \((\hat{p})(t)\) with control disks \((\hat{p}_i)\) and weights \(\hat{\omega}_i \in \mathbb{R}^+ \), \(i = 0, 1, \ldots, m\), if only if the following equations are
satisfied
\[
\sum_{j=\max(0,i-n)}^{\min(m,i)} \binom{n}{i-j} \omega_j \omega_{i-j} \mathbf{P}_{i-j} = \sum_{j=\max(0,i-n)}^{\min(m,i)} \binom{n}{i-j} \omega_j \omega_{i-j} \mathbf{\hat{P}}_j, \tag{15}
\]
\[i = 0, 1, \ldots, n + m,\]
and
\[
r_k = \sum_{j=\max(0,k-n+m)}^{\min(m,k)} \tilde{\omega}_j \binom{m}{j} \binom{n-m}{k-j} \omega_i B_i^m(t), \quad k = 0, 1, \ldots, n. \tag{16}
\]

**Proof.** For the center curve, by
\[
\mathbf{p}(t) = \mathbf{\hat{p}}(t),
\]
we have
\[
\sum_{i=0}^{n} \mathbf{p}_i \omega_i B_i^m(t) \sum_{i=0}^{m} \omega_i B_i^m(t) = \sum_{i=0}^{n} \mathbf{\hat{p}}_i \omega_i B_i^m(t) \sum_{i=0}^{m} \omega_i B_i^m(t),
\]
which, after some rearrangement of the equation, gives
\[
\sum_{i=0}^{m+n} \sum_{j=\max(0,i-m)}^{\min(m,i)} \binom{m}{j} \binom{n}{i-j} \omega_j \omega_{i-j} \mathbf{P}_{i-j} = \sum_{i=0}^{m+n} \sum_{j=\max(0,i-m)}^{\min(m,i)} \binom{m}{j} \binom{n}{i-j} \omega_j \omega_{i-j} \mathbf{\hat{P}}_j. \tag{15}
\]

Comparing coefficients of like terms on both sides of the equation, this establishes the equation (15).

The proof for the radius (16) is straightforward by equation (14).

3. Degree reduction of disk rational Bézier curves

In this section, we discuss the problem of degree reduction of disk rational Bézier curves using weighted least squares method.
3.1. Description of approximation problem

The problem of degree reduction of disk Rational Béziers curve can be stated as follows:

**Problem** Given a degree \( n \) disk rational Bézier curve \((\mathbf{p})(t)\), find a degree \( m < n \) disk rational Bézier curve \((\mathbf{\hat{p}})(t)\) such that \((\mathbf{\hat{p}})(t)\) is the closure of \((\mathbf{p})(t)\).

The above problem can be decomposed into the following two parts in this paper:

1) For the center curve \( \mathbf{p}(t) \) of the disk rational curve \((\mathbf{p})(t)\), find an optimal \( n - m \) degree reduced approximating function \( \mathbf{\hat{p}}(t) \) to be the center curve of the disk rational curve \((\mathbf{\hat{p}})(t)\) using weighted least squares.

2) For the error radius curve \( r(t) \) of the disk rational Bézier curve \( \mathbf{p}(t) \), find an \( n - m \) optimal degree reduced approximating function \( \mathbf{\hat{r}}(t) \) to be the error radius curve of the disk rational Bézier curve \((\mathbf{q})(t)\), such that \((\mathbf{\hat{p}})(t)\) can bound \((\mathbf{p})(t)\) as tight as possible in \( L_2 \) norm.

In parallel, the problem can be summarized as the following mathematical formulae

\[
\begin{align*}
& (A) \quad \min \| \mathbf{p}(t) - \mathbf{\hat{p}}(t) \|_{\rho(t)} \\
& s.t. \quad \hat{\omega}_i > 0, \; i = 0, \ldots, m \\
& (B) \quad \min_{\hat{r}(t) \geq r(t) + \text{dist}(\mathbf{p}(t), \mathbf{\hat{p}}(t))} \| \hat{r}(t) - r(t) \|
\end{align*}
\]  

(17)

where \( \rho(t) > 0 \) and \( \text{dist}(\mathbf{p}(t), \mathbf{\hat{p}}(t)), \; t \in [0, 1] \) is the Hausdorff distance between the curve \( \mathbf{\hat{p}}(t) \) and the curve \( \mathbf{p}(t) \).

3.2. Degree reduction approximation of center curve

Since the weights of a rational Bézier curve must be positive, we require two steps to achieve degree reduction process for the center curve \( \mathbf{p}(t) \) as
well. First, processing degree reduction of denominator \( \omega(t) \) to obtain degree-reduced weights \( \hat{\omega}_i \) \( (i = 0, ..., m) \) by applying quadratic programming; second, substituting \( \hat{\omega}_i \) into \( \hat{p}(t) \) and then using weighted least squares to accomplish the degree reduction of the center curve \( p(t) \).

3.2.1. Solving weights \( \hat{\omega}_i \)

Let \( \omega(t) = \sum_{i=0}^{n} \omega_i B_i^n(t) \) and \( \hat{\omega}(t) = \sum_{i=0}^{m} \hat{\omega}_i B_i^m(t) \). The distance between \( \omega(t) \) and \( \hat{\omega}(t) \) in \( L_2 \) norm can be expressed as

\[
d(\omega(t), \hat{\omega}(t)) = \| \omega(t) - \hat{\omega}(t) \|^2 = \int_0^1 (\hat{\omega}(t) - \omega(t))^2 dt
\]

\[
= \int_0^1 \left( \sum_{i=0}^{m} \hat{\omega}_i B_i^m(t) \right)^2 dt - 2 \int_0^1 \sum_{i=0}^{m} \hat{\omega}_i (t) \sum_{j=0}^{n} \omega_j (t) dt + \int_0^1 \left( \sum_{i=0}^{n} \omega_i B_i^n(t) \right)^2 dt
\]

\[
= \sum_{i=0}^{m} \sum_{j=0}^{m} \hat{\omega}_i \hat{\omega}_j H_{ij} - 2 \sum_{i=0}^{m} \sum_{j=0}^{n} \hat{\omega}_i \omega_j S_{ij} + \sum_{i=0}^{n} \sum_{j=0}^{n} \omega_i \omega_j G_{ij},
\]

(18)

where

\[
H_{ij} = \int_0^1 B_i^m(t) B_j^m(t) dt = \frac{\binom{n}{i} \binom{n}{j}}{(2m+1) \binom{2n}{i+j}}, \quad S_{ij} = \int_0^1 B_i^m(t) B_j^n(t) dt = \frac{\binom{n}{i} \binom{n}{j}}{(m+n+1) \binom{m+n}{i+j}},
\]

and \( G_{ij} = \int_0^1 B_i^n(t) B_j^n(t) dt = \frac{\binom{n}{i} \binom{n}{j}}{(2n+1) \binom{2n}{i+j}} \).

Note that the term \( \sum_{i=0}^{n} \sum_{j=0}^{n} \omega_i \omega_j G_{ij} \) in equation (18) is constant so we remove this term from (18) and have a constrained quadratic programming as follows

\[
\begin{cases}
\min \sum_{i=0}^{m} \sum_{j=0}^{m} \hat{\omega}_i \hat{\omega}_j H_{ij} - 2 \sum_{i=0}^{m} \sum_{j=0}^{n} \hat{\omega}_i \omega_j S_{ij} \\
\text{s.t.} \quad \hat{\omega}_i > 0, \ i = 0, 1, ..., m.
\end{cases}
\]

(19)

3.2.2. Solving control points \( \hat{p}_i \)

After obtaining the weights \( \hat{\omega}_i \), we apply the weighted least squares to obtain the control points \( \hat{p}_i \) with the contact order \( (k, h) \) \( (k, h = 0, 1, \text{respectively}) \).
of continuity at both endpoints of the rational Bézier curves. That is, given a weight function \( \rho(t) > 0 \),

\[
d_{\rho}(p(t), \tilde{p}(t)) = \int_{0}^{1} \rho(t) (p(t) - \tilde{p}(t))^{2} dt,
\]

the controls points \( \tilde{p}_{l} (l = k + 1, k + 2, \ldots, m - h - 1) \) of \( \tilde{p}(t) \) satisfy

\[
\frac{\partial d_{\rho}(p(t), \tilde{p}(t))}{\partial \tilde{p}_{l}} = -\int_{0}^{1} \frac{\rho(t) \tilde{\omega}_{l} B_{m l}(t)}{\omega(t) \tilde{\omega}^{2}(t)} (x(t) \tilde{\omega}(t) - \tilde{x}(t) \omega(t)) dt = 0, \tag{20}
\]

where \( x(t) = \sum_{i=0}^{n} p_{i} B_{n i}(t) \) and \( \tilde{x}(t) = \sum_{j=0}^{m} \tilde{p}_{j} B_{m j}(t) \).

For simplicity, let \( \rho(t) = \omega(t) \tilde{\omega}(t) \) and equation (20) can be written as

\[
\int_{0}^{1} B_{m l}(t) \tilde{x}(t) \omega(t) dt = \int_{0}^{1} B_{m l}(t) x(t) \tilde{\omega}(t) dt. \tag{21}
\]

Substituting equations \( \omega(t), \tilde{\omega}(t), x(t) \) and \( \tilde{x}(t) \) into (21), we have

\[
\sum_{j=0}^{m} \sum_{i=0}^{n} \frac{(m)}{(i+j+l)} \omega_{i} \tilde{\omega}_{j} \tilde{p}_{j} \int_{0}^{1} B_{2m+n+1+j+l}(t) dt = \sum_{j=0}^{m} \sum_{i=0}^{n} \frac{(m)}{(i+j+l)} \tilde{\omega}_{j} \omega_{i} p_{i} \int_{0}^{1} B_{2m+n+1+i+j+l}(t) dt.
\]

By \( \int_{0}^{1} B_{i}(t) dt = \frac{1}{i+1} \), it yields

\[
\sum_{j=0}^{m} \sum_{i=0}^{n} \frac{(m)}{(i+j+l)} \omega_{i} \tilde{\omega}_{j} \tilde{p}_{j} = \sum_{j=0}^{m} \sum_{i=0}^{n} \frac{(m)}{(i+j+l)} \tilde{\omega}_{j} \omega_{i} p_{i}, l = k + 1, k + 2, \ldots, m - h - 1. \tag{22}
\]

On the other hand, to keep the two rational curves \( p(t) \) and \( \tilde{p}(t) \) have \( (k, h) \) \((k, h = 0, 1)\) order continuous at the endpoints \( t = 0, 1 \), respectively,
the equation (22) can be rewritten as
\[
\sum_{j=0}^{k} \sum_{i=0}^{n} \frac{m_j}{(i+j+l)} \omega_j \dot{\varphi}_j + \sum_{j=k+1}^{m-h-1} \sum_{i=0}^{n} \frac{m_j}{(2m+n)} \omega_i \dot{\varphi}_j + \sum_{j=m-h}^{m} \sum_{i=0}^{n} \frac{m_j}{(i+j+l)} \omega_i \dot{\varphi}_j
\]
\[
= \sum_{j=0}^{m} \sum_{i=0}^{n} \frac{m_j}{(i+j+l)} \omega_j \omega_i \dot{p}_i, \quad l = k + 1, k + 2, ..., m - h - 1,
\]

or equivalently,
\[
\sum_{j=k+1}^{m-h-1} \sum_{i=0}^{n} \frac{m_j}{(2m+n)} \omega_j \dot{\varphi}_j
\]
\[
= \sum_{j=0}^{m} \sum_{i=0}^{n} \frac{m_j}{(i+j+l)} \omega_j \omega_i \dot{p}_i - \sum_{j=0}^{k} \sum_{i=0}^{n} \frac{m_j}{(2m+n)} \omega_i \dot{\varphi}_j - \sum_{j=m-h}^{m} \sum_{i=0}^{n} \frac{m_j}{(i+j+l)} \omega_i \dot{\varphi}_j,
\]
\[
= \sum_{j=k+1}^{m-h-1} \sum_{i=0}^{n} \frac{m_j}{(i+j+l)} \omega_j \dot{\varphi}_j,
\]
\[
l = k + 1, k + 2, ..., m - h - 1,
\]

which is a system of \(m - h - k - 1\) equations in the \(m - h - k - 1\) unknowns \(\dot{\varphi}_{k+1}, \dot{\varphi}_{k+2}, \ldots, \dot{\varphi}_{m-h-1}\).

When \(k, h = 0, 1\), respectively,
\[
\dot{\varphi}_0 = \varphi_0, \dot{\varphi}_m = \varphi_n,
\]
\[
\dot{\varphi}_1 = \dot{\varphi}_0 + \frac{n \omega_0 \omega_1}{m \omega_0 \omega_1} (\varphi_1 - \varphi_0),
\]
and
\[
\dot{\varphi}_{m-1} = \dot{\varphi}_m - \frac{n \omega_{m-1} \omega_m}{m \omega_m \omega_{m-1}} (\varphi_m - \varphi_{m-1}).
\]
3.3. Degree reduction approximation of error radius curve

Elevating the degree of the error radius curve \( \hat{\tau}(t) \) from \( m \) to \( n \) by equation (16), we have

\[
\hat{\tau}(t) = \sum_{i=0}^{m} \hat{\tau}_i B_i^m(t) = \sum_{i=0}^{n} \hat{\tau}_i B_i^n(t),
\]

(25)

where

\[
\hat{\tau}_i = \sum_{j=\max(0,i-n+m)}^{\min(m,i)} \tilde{\tau}_j \binom{m}{j} \binom{n-m}{i-j}.
\]

(26)

Then one of a sufficient condition to satisfy the formula (B) in equation (17) can be stated as

**Theorem 1.** A sufficient condition for (B) in equation (17) is

\[
\hat{\tau}_i > r_i + d, \quad i = 0, 1, \ldots, n,
\]

(27)

where \( \hat{\tau}_i \) is given in equation (26) and

\[
d = \sum_{j=0}^{M} \left( (x(t_j) - \hat{x}(t_j))^2 + y(t_j) - \hat{y}(t_j))^2 \right)^{\frac{1}{2}},
\]

(28)

where \( t_j, j = 0, 1, \cdots, M \) are discrete sample points on curves \( p(t) \) and \( \hat{p}(t) \).

With the analogous method to solving weights \( \hat{\omega}_i \), we have

\[
\|r(t) - \hat{r}(t)\|_2^2 = \int_0^1 \left( \sum_{i=0}^{n} r_i B_i^n(t) - \sum_{j=0}^{m} \hat{\tau}_j B_j^m(t) \right)^2 dt
\]

\[
= \int_0^1 (\sum_{j=0}^{m} \hat{\tau}_j B_j^m(t))^2 dt - 2 \int_0^1 \sum_{i=0}^{n} r_i B_i^n(t) \sum_{j=0}^{m} \hat{\tau}_j B_j^m(t) dt + \int_0^1 (\sum_{i=0}^{n} r_i B_i^n(t))^2 dt
\]

\[
= \sum_{i=0}^{m} \sum_{j=0}^{m} \hat{\tau}_i \hat{\tau}_j H_{ij} - 2 \sum_{i=0}^{m} \sum_{j=0}^{n} \hat{\tau}_i \hat{\tau}_j S_{ij} + \sum_{i=0}^{n} \sum_{j=0}^{n} r_i r_j G_{ij},
\]

where \( H_{ij}, S_{ij} \) and \( G_{ij} \) were given in equation (18), and the third term \( \sum_{i=0}^{n} \sum_{j=0}^{n} r_i r_j G_{ij} \) is constant. So the problem of degree reduction of er-
ror radius function can be transformed to find the optimal solution of the following problem:

\[
\begin{align*}
\min & \quad \sum_{i=0}^{m} \sum_{j=0}^{m} \hat{r}_i \hat{r}_j H_{ij} - 2 \sum_{i=0}^{m} \sum_{j=0}^{n} \hat{r}_i \hat{r}_j S_{ij} \\
\text{s.t.} & \quad \hat{r}_i \geq r_i + d, \quad i = 0, 1, \ldots, n, \\
& \quad \hat{r}_j > 0, \quad j = 0, 1, \ldots, m.
\end{align*}
\]

(29)

4. Errors and Examples

Without loss of generality, we only consider

\[
\text{err}(r(t), \hat{r}(t)) = \|r(t) - \hat{r}(t)\|_\infty,
\]

and

\[
\text{err}(p(t), \hat{p}(t)) = \|p(t) - \hat{p}(t)\|_\infty,
\]

where \(t_j, j = 0, 1, \ldots, M\) are discrete sample points on curves \(r(t), \hat{r}(t), p(t)\) and \(\hat{p}(t)\). Above error equations can be easily generalized to bounding error and relative bounding error[9].

**Example 1.** Given a disk rational Bézier \((p)(t)\) of five degree with control disks \((96, 141)_1, (104, 271)_{10}, (178, 363)_{15}, (331, 378)_{15}, (486, 285)_{10}, (486, 140)_6\) and associated weights \(2, 1, 1, 2, 1, 2\). The best 1-degree reduction curve satisfying \(C^{(0,0)}\)-continuity with the given curve has control disks \((96.0000, 141.0000)_{5.4759}, (23.0187, 356.9572)_{16.7259}, (264.3962, 378.1378)_{22.1426}, (466.7490, 365.0673)_{15.4759}, (486.0000, 140.0000)_{10.4759}\) and associated weights \(2.0344, 0.5115, 1.9717, 1.0274, 1.9563\) (See Fig.1a). The maximum error distances of center curve is 4.4759 (See Fig.1b), which is less than corresponding result in \([8]\). The maximum error distance of error radius curve is 4.6487 (See Fig.1c), which is larger than corresponding in \([8]\).
Figure 1: (a) A degree 5 disk rational Bézier curve (with black) and its one-degree reduced curve (with blue) with $C^{(0,0)}$ continuity at two endpoints; (b) Error distance of center curve; (c) Error distance of error radius curve
Example 2. Given a disk rational Bézier \((p)(t)\) of eight degree with control disks \((60, 149)_{10}, (86, 250)_{4}, (203, 300)_{10}, (350, 310)_{15}, (402, 250)_{20}, (375, 115)_{18}, (472, 81)_{8}, (651, 112)_{10}, (715, 250)_{5}\) and associated weights 10, 4, 10, 15, 20, 18, 8, 10, 5. The best 3-degree reduction curve satisfying \(C^{(1,1)}\)-continuity with the given curve has control disks \((60.0000, 149.0000)_{16.2568}, (103.9233, 319.6251)_{6.6568}, (443.2454, 396.2295)_{32.5068}, (384.0581, 129.1892)_{24.4401}, (482.0062, -252.3928)_{19.2568}, (715.0000, 250.0000)_{11.2568}\) and associated weights 1.8791, 1.5904, 1.4672, 2.7527, 0.4761, 1.9058. The maximum error distances of center curve and error radius curve are 6.2568 and 8.1939, respectively. (See Fig. 2)

5. Summary

In this paper, we discussed the problem of degree reduction of disk rational Bézier curves and proposed an efficient method to solve the problem. Theoretic results and experiments show that the proposed algorithm produces is very effective. The idea presented in this paper can be easily generalized to solve the degree reduction problem of disk rational Bézier surfaces and NURBS curves and surfaces.

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Figure 2: (a) A degree 8 disk rational Bézier curve (with black) and its three-degree reduced curve (with blue) with $C^{(1,1)}$ continuity at two endpoints; (b) Error distance of center curve; (c) Error distance of error radius curve
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