SYMMETRIES AND SOLUTIONS OF GETZLER’S EQUATION FOR
COXETER AND EXTENDED AFFINE WEYL FROBENIUS MANIFOLDS

IAN A.B. STRACHAN

ABSTRACT. The G-function associated to the semisimple Frobenius manifold $\mathbb{C}^n/W$ (where $W$ is a Coxeter group or an extended affine Weyl group) is studied. The general form of the G-function is given in terms of a logarithmic singularity over caustics in the manifold. The main result in this paper is a universal formula for the G-function corresponding to the Frobenius manifold $\mathbb{C}^n/\tilde{W}^{(k)}(A_{n-1})$, where $\tilde{W}^{(k)}(A_{n-1})$ is a certain extended affine Weyl group (or, equivalently, corresponding to the Hurwitz space $\tilde{M}_{0,k-1,n-k-1}$), together with the general form of the G-function in terms of data on caustics. Symmetries of the G-function are also studied.

1. Introduction

The main result in this paper is the following universal formula, independent of $k$,

$$ G = -\frac{1}{24} t^n $$

for the G-function corresponding to the Frobenius manifold $\mathbb{C}^n/\tilde{W}^{(k)}(A_{n-1})$, where $\tilde{W}^{(k)}(A_{n-1})$ is a certain extended affine Weyl group (or, equivalently, corresponding to the Hurwitz space $\tilde{M}_{0,k-1,n-k-1}$) [DZ1], together with the general form of the G-function in terms of data on caustics. For $n \leq 3$ this result is already known [DZ2], with such solutions being found by directly solving the governing equations for $G$. For arbitrary $k$ and $n$ a different approach is required.

The G-function itself plays a number of important roles within the mathematics and applications of the theory of Frobenius manifolds, all connected, especially in the semisimple case, with the construction of genus one objects from genus zero data. Thus in TQFT it appears in the genus one contribution to the free energy of the field theory; in enumerative geometry it governs genus one Gromov-Witten invariants; in integrable systems it appears in the first order deformation of bi-Hamiltonian structures. The $(n = 2, k = 1)$ case corresponds to the quantum cohomology of $\mathbb{CP}^1$, and the precise form of the G-function is required in establishing, to first-order in the genus expansion, that the underlying integrable system is the Toda lattice. The results in this paper may be used to construct similar conjectures for generalized Toda and Benney hierarchies. It will be assumed in this paper that the reader has an understanding of Frobenius manifolds. In particular, the concepts and notation will follow Dubrovin [D1].

In [DZ2] Dubrovin and Zhang (following from conjectures of Givental [G1]) proved that for semisimple Frobenius manifolds the G-function is given by the formula

$$ G = \log \frac{\tau_I}{J^{1/24}} $$

where $\tau_I$ is the isomonodromic $\tau$-function and $J$ is the Jacobian of the transformation between canonical and flat-coordinates. The governing equations for $G$ itself were obtained by Getzler [Ge] and are the following overdetermined set of linear equations [DZ2]

$$ \sum_{1 \leq a_1, a_2, a_3, a_4 \leq n} z_{a_1} z_{a_2} z_{a_3} z_{a_4} \Delta_{a_1 a_2 a_3 a_4} = 0 $$

Date: 12th December 2002.

1991 Mathematics Subject Classification. 53B25, 53B50.

Key words and phrases. Frobenius manifolds, Coxeter groups, Extended-affine Weyl groups, G-functions.
where
\[ \Delta_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = 3 e^{\mu}_{\alpha_1 \alpha_2} e^{\nu}_{\alpha_3 \alpha_4} \frac{\partial^2 G}{\partial \mu \partial \nu} - 4 e^{\mu}_{\alpha_1 \alpha_2} e^{\nu}_{\alpha_3 \mu} \frac{\partial^2 G}{\partial \alpha_4 \partial \nu} - e^{\mu}_{\alpha_1 \alpha_2} e^{\nu}_{\alpha_3 \alpha_4 \mu} \frac{\partial G}{\partial \nu} + 2 e^{\mu}_{\alpha_1 \alpha_2 \alpha_3} e^{\nu}_{\alpha_4 \mu} \frac{\partial G}{\partial \nu} + \frac{1}{6} e^{\mu}_{\alpha_1 \alpha_2 \alpha_3} e^{\nu}_{\alpha_4 \mu} + \frac{1}{24} e^{\mu}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} e^{\nu}_{\alpha_\mu} - \frac{1}{4} e^{\mu}_{\alpha_1 \alpha_2 \nu} e^{\nu}_{\alpha_3 \alpha_4 \mu}. \]

The proof that (3) satisfies these equations involves rewriting them in terms of canonical coordinates. In particular [DZ3]:

**Theorem 1.** For an arbitrary semisimple Frobenius manifold the system (2) has a unique, up to an additive constant, solution \( G = G(t_1, \ldots, t^n) \) satisfying the quasihomogeneity condition
\[ \mathcal{L}_E G = \gamma \] with a constant \( \gamma \). This solution is given by the formula (3) where \( \tau_1 \) is the isomonodromic tau-function and
\[ J = \det \left( \frac{\partial \tau_1^\alpha}{\partial \tau_1^i} \right) \]
is the Jacobian of the transform from the canonical coordinates to the flat ones. The scaling anomaly \( \gamma \) in (3) is given by the formula
\[ \gamma = -\frac{1}{4} \sum_{\alpha=1}^{n} \mu_\alpha^2 + \frac{n d}{48} \]
where
\[ \mu_\alpha = q_\alpha - \frac{d}{2}, \quad \alpha = 1, \ldots, n. \]

A simple extension of this result appeared in [DZ3]:

**Theorem 2.** The derivatives of the \( G \)-function along the powers of the Euler vector field are given by the following formulae
\[ \partial_{E^k} G = 0, \quad \partial_{E^\nu} G = \frac{n d}{48} - \frac{1}{4} \text{tr} \mu^2, \quad \partial_{E^k} G = -\frac{1}{4} \text{tr} \left( \mu (\mu U^{k-1} + U\mu U^{k-2} + \cdots + U^{k-1} \mu) \right) \]
\[ -\frac{1}{24} \left( \mu U^{k-2} + U\mu U^{k-3} + \cdots + U^{k-2} \mu \right) E - \frac{d}{2} U^{k-2} E, H \]
where
\[ H = c^{\nu}_{\nu} \partial_\nu. \]

For any given Frobenius manifold one may solve the governing equations given in the above theorems to find the \( G \)-function. However, for classes of Frobenius manifold, such an approach is impractical. The idea behind this paper is that it is the singularity structure of these differential equations that drives the solution: the solution may be derived purely in terms of data on the singularities. Geometrically these singularities correspond to caustics in the Frobenius manifold and the singularity data to the \( F \)-manifold structure on the caustics. This idea will be illustrated by finding the \( G \)-function for Frobenius manifolds constructed from Coxeter groups and extended-affine Weyl groups. These caustics also have a number of interesting curvature properties and applications in integrable systems theory [S].

The rest of the paper is laid out as follows. In section 2 the relationship between the multiplication on caustics and the singularities of the \( G \)-function is studied. In section 3 these results will be used to study the \( G \)-function for the Frobenius manifold \( C^n/W \), where \( W \) is a Coxeter group. In section 4 the \( G \)-function will be studied for the Frobenius manifold \( \tilde{C}^n/\tilde{W} \), where \( \tilde{W} \) is an extended affine Weyl group, concentrating in particular on the special case \( \tilde{W} = \tilde{W}^{(k)}(A_{n-1}) \).
manifolds have certain natural symmetries [D1], and such symmetries induce symmetries of the corresponding \( G \)-functions. Such symmetries are studied in section 5.

2. Multiplication on caustics and the \( G \)-function

By definition, a massive Frobenius manifold \( M \) has a semisimple multiplication on the tangent space at generic points of \( M \). The set of points where the multiplication is not semisimple is known as the caustic, and will be denoted \( K \). This is an analytic hypersurface in \( M \), which may consist of a number of components (possibly highly singular),

\[
K = \bigcup_{i=1}^{\# K_i} K_i.
\]

The set of smooth points in \( K \) will be denoted \( K_{\text{reg}} \).

The simplest case case is where the multiplication on the caustic \( K_i \) is of the type \( A_{n-2} \I_2(N_i) \), i.e. the multiplication decomposes into \( n-2 \) one-dimensional algebras and a single two-dimensional algebra based on the Coxeter group \( I_2(N_i) \).

The following theorem studies the behaviour of the \( G \)-function near such caustics.

**Theorem 3.** [H1] Let \((M, \circ, e, E, g)\) be a simply connected massive Frobenius manifold. Suppose that at generic points of the caustic \( K_i \) the germ of the underlying \( \mathcal{F} \)-manifold is of type \( I_2(N_i)A_{n-2} \) for one fixed number \( N_i \geq 3 \).

a) The form \( d\log \tau_I \) has a logarithmic pole along \( K_i \) with residue \( -(N_i - 2)^2/16N_i \) along \( K_i \cap K_{\text{reg}} \).

b) The form \( -\frac{1}{12}d\log J \) has a logarithmic pole along \( K_i \) with residue \( N_i - 2/48 \) along \( K_i \cap K_{\text{reg}} \).

c) The \( G \)-function extends holomorphically over \( K_i \) iff \( N_i = 3 \).

The explicit form of the multiplication used in the proof of this theorem is, in terms of the coordinate fields \( \delta_i = \frac{\partial}{\partial t^i} \), with respect to some not necessarily flat coordinates \( t^i \),

\[
\begin{align*}
\delta_1 \circ \delta_2 &= \delta_2, \\
\delta_2 \circ \delta_2 &= (t^2)^{N-2} \delta_1, \\
\delta_i \circ \delta_j &= \delta_{ij} \delta_j \quad \text{otherwise},
\end{align*}
\]

with caustic \( K = \{ t | t^2 = 0 \} \). The canonical coordinates are, on a simply connected subset of \( M - K \),

\[
\begin{align*}
u_1 &= t^1 + \frac{2}{N} (t^2)^{\frac{N}{2}}, \\
u_2 &= t^1 - \frac{2}{N} (t^2)^{\frac{N}{2}}, \\
u_i &= t^i, \quad i \geq 3,
\end{align*}
\]

the idempotent vector fields are

\[
\begin{align*}
e_1 &= \frac{1}{2} \delta_1 + \frac{1}{2} (t^2)^{\frac{N}{2}} \delta_2, \\
e_2 &= \frac{1}{2} \delta_1 - \frac{1}{2} (t^2)^{\frac{N}{2}} \delta_2, \\
e_i &= \delta_i, \quad i \geq 3
\end{align*}
\]

and the Euler vector field is

\[
E = t^1 \delta_1 + \frac{2}{N} t^2 \delta_2 + \sum_{i=3}^{n} t^i \delta_i.
\]

With these one may directly calculate the forms \( d\log \tau_I \) and \( d\log J \) near the caustic to obtain the above result. However, not all caustics are of this simple kind, as the following example shows.
Example Consider the Frobenius manifold $Q_r(\mathbb{C}P^1)$, related to the quantum cohomology of $\mathbb{C}P^1$, given by the prepotential and Euler field

$$F = \frac{1}{2}(t^1)^2 t^2 + e^{rt^2}, \quad (r > 0),$$

$$E = t^1 \partial_1 + \frac{2}{r} \partial_2.$$

The canonical coordinates are easily calculated:

$$u_1 = t^1 + 2r \frac{1}{r} e^{\frac{1}{2}e^{rt^2}},$$

$$u_2 = t^1 - 2r \frac{1}{r} e^{\frac{1}{2}e^{rt^2}}.$$

Note that $u_1 - u_2$ does not vanish at finite points, but will vanish in the limit $t^2 \to -\infty$. Thus one has a ‘caustic at infinity’ or limiting caustic

$$\mathcal{K}_\infty = \{(t^1, t^2) | t^2 \to -\infty\}.$$

This idea of a ‘caustic at infinity’ may be made more precise by introducing new coordinates

$$\tilde{t}^1 = t^1,$$

$$\tilde{t}^2 = e^{\frac{1}{2}t^2}.$$

In these coordinates the caustic becomes $\mathcal{K}_{\log} = \{(\tilde{t}^1, \tilde{t}^2) | \tilde{t}^2 = 0\}$ and the multiplication takes the form (with $t$ replaced by $\tilde{t}$) given in the first two equations of [7] with $r = N$. Such a caustic will be referred to as a logarithmic caustic. These new variables are no longer flat - the metric is logarithmic along $\mathcal{K}_{\log}$,

$$g = 2 dt^1 dt^2,$$

$$= 2 d\tilde{t}^1 d\tilde{t}^2 \tilde{t}^2,$$

but the $F$-manifold structure (as the next Lemma will show) is easier to understand in these coordinates.

It is illuminating to calculate the $G$-function purely in terms of canonical coordinates, as this will mirror the calculations in the next lemma. The $G$-function is made up of two parts (see [DZ2, DZ3]):

$$d \log \tau_I = \frac{1}{8}(u_1 - u_2) \frac{\eta_2^2}{\eta_1 \eta_2} d(u_1 - u_2),$$

$$= - \frac{r}{16} \frac{dt^2}{t^2}$$

(using the fact that the Egoroff potential $\eta = t^2$) and

$$d \log J = - \frac{r}{2} \frac{d\tilde{t}^2}{\tilde{t}^2}.$$

Hence from (1) $dG = - \frac{r}{2} \frac{dt^2}{t^2}$ and since $G$ is independent of $t^1$, $G = - \frac{r}{2} \log t^2$. Note that $dG$ has a logarithmic pole along $\mathcal{K}_{\log}$ (In fact, for any two-dimensional semisimple Frobenius manifold the scaling anomaly and the equations

$$\mathbf{L}_c G = 0,$$

$$\mathbf{L}_E G = \gamma$$

determine $G$ up to a constant, $G = \gamma \log(u_1 - u_2)$. From this one sees the close relationship between properties of the $G$-function and caustics).

The next result studies the behaviour of the $G$-function near such a limiting caustic. The proof is almost identical to the proof [H1] of theorem 3.
Lemma 4. Let $M \subset \mathbb{C}^n$ be a manifold with coordinates $\tilde{t}^1, ..., \tilde{t}^n$ and $K_{\log} := \{ \tilde{t} | \tilde{t}^2 = 0 \}$ such that $M - K_{\log}$ is a Frobenius manifold $(M - K_{\log}, \circ, e, E, g)$ with the following properties: for some $N_{\log} \geq 1$ the multiplication $\circ$ is given by

$$
\delta_1 \circ \delta_2 = \delta_2,
\delta_2 \circ \delta_2 = (\tilde{t}^2)^{N_{\log} - 2} \delta_1,
\delta_i \circ \delta_j = \delta_{ij} \delta_j \text{ otherwise},
$$

where $\delta_i = \frac{\partial}{\partial t^i}$ (then for $N_{\log} \geq 2$ it extends holomorphically to $K_{\log}$ and for $N_{\log} \geq 3$ $K_{\log}$ is the caustic); the metric is logarithmic along $K_{\log}$, i.e. the matrix of components of the metric $g$ for a base of logarithmic vector fields, with respect to $K_{\log}$, is holomorphic and nondegenerate on the caustic $K_{\log}$. Then near the caustic $K_{\log}$,

$$
d \log J = - \frac{N_{\log}}{16} \frac{d^2}{d\tilde{t}^2} + \text{holomorphic one form in } \tilde{t},
$$

$$
d \log \tau_l = - \frac{N_{\log}}{2} \frac{d^2}{d\tilde{t}^2} + \text{holomorphic one form in } \tilde{t}.
$$

and hence

$$
d G = - \frac{N_{\log}}{24} \frac{d^2}{d\tilde{t}^2} + \text{holomorphic one form in } \tilde{t}.
$$

It will be shown in section 4 that for Frobenius manifolds constructed from extended affine Weyl groups the assumption made in the above lemma holds. The lemma shows that near the caustic $K_{\log}$ the form $dG$ is, in the original flat coordinates, finite. This fact may then be used to exclude the possibility of terms like $e^{-r^2}$ appearing in the $G$-function.

Proof. Note for $N_{\log} \geq 2$ the $F$-manifold structure is of type $I_2(N_{\log})A_n^{n-2}$ (and $I_2(2) = A_2^2$). For $N_{\log} = 1$ the multiplication has a simple pole along $K_{\log}$. The proof is entirely analogous to the proof of Theorem 3. The socle field is

$$
H = 2t^2 \delta_2 + \delta_3 + \ldots + \delta_n,
$$

and now $t^2 \delta_2(\eta)(0) \neq 0$. With this data one can repeat the proof of Theorem 3, as given in [11], to derive the result.

3. The $G$-function for Coxeter groups

The construction of a Frobenius manifold structure on the orbit space $\mathbb{C}^n/W$ (where $W$ is a Coxeter group) is given in [D1]. The only parts of that construction that will be required in this section are the following:

- in flat-coordinates, the prepotential, and hence the structure functions of the Frobenius algebra, are polynomial functions;
- the Euler vector field takes the form

$$
E = \sum_{r=1}^n \frac{d_r}{\hbar} \frac{\partial}{\partial r} \quad d_r > 0
$$

where the $d_r$ are the exponents of the Coxeter group and $\hbar$ is the Coxeter number of $W$.

The components of the caustic $\mathcal{K}$ for such orbit spaces are given in terms of quasihomogeneous polynomials $\kappa_i$ such that $\kappa_i^{-1}(0) = \mathcal{K}_i$. The $F$-manifold structure on these caustics is known, the multiplication is of type $I_2(N_i)A_n^{n-2}$, and this enables Theorem 3 to be used. The data $N_i$ is given in Table 1. It can be extracted with some work from [H2] Theorem 5.22, which builds on [23].

Proposition 5. The $G$-function on $\mathbb{C}^n/W$ takes the form

$$
G = - \frac{1}{24} \frac{(N_1 - 2)(N_1 - 3)}{N_1} \log \kappa_1
$$

and the constant $N_1$, which depends on the Coxeter group $W$, is given in Table 1.
| Coxeter Group $W$ | Number of caustics | Values of $N_i$ |
|------------------|--------------------|----------------|
| $A_n, D_n, E_{6,7,8}$ | 1                  | $N_1 = 3$     |
| $B_n$            | 2                  | $N_1 = 4, N_2 = 3$ |
| $F_4$            | 3                  | $N_1 = 4, N_2 = N_3 = 3$ |
| $H_3$            | 2                  | $N_1 = 5, N_2 = 3$ |
| $H_4$            | 2                  | $N_1 = 5, N_2 = 3$ |
| $I_2(h)$         | 1                  | $N_1 = k$     |

Table 1. Data on the caustics of the Frobenius manifold $\mathbb{C}^n/W$

**Proof** From theorem 2 it follows that the only singularities of $dG$ are on caustics, and it is known that the multiplication on the caustics of a Frobenius manifold obtained from a Coxeter group is of the form where Theorem 3 may be applied (see [Gi2] and [H2] Theorem 5.22).

It follows from the polynomial nature of the structure functions and Theorem 2 that all first derivatives $\partial G/\partial t^\alpha$ are rational functions. Hence, on integrating, $G$ takes the schematic form

$$G(t) = \text{rational function} + \text{logarithmic singularities}$$

By Theorem 3, the only singularities that $G$ has are logarithmic singularities on $K_i$. Thus the rational functions must be polynomial. However, the only polynomial function compatible with the symmetry (3) is a constant (this uses the fact that the exponents of the Coxeter group are all positive). Since $G$ is only defined up to a constant anyway one has:

$$G(t) = \frac{1}{24} \sum_{i=1}^{\#K_i} \frac{(N_i - 2)(N_i - 3)}{N_i} \log \kappa_i$$

(9)

and hence

$$\gamma = -\frac{1}{24} \sum_{i=1}^{\#K_i} \frac{(N_i - 2)(N_i - 3)}{N_i} E(\log \kappa_i).$$

(10)

Using the data in table 1, and in particular that in all cases there is at most one caustic, denoted $K_1$, with $N_i > 3$, the result follows.

The scaling constant $\gamma$ may be calculated purely from the data on the caustics:

$$\gamma = -\frac{1}{24} \frac{(N_1 - 2)(N_1 - 3)}{N_1} E(\log(\kappa_1)).$$

(11)

so in particular for $W = B_n$,

$$E(\kappa_1^{B_n}) = \left(\frac{n - 1}{n}\right)^{B_n} \kappa_1^{B_n}.$$
The $G$-function for extended affine Weyl groups

The construction of a Frobenius manifold structure on the orbit space $\mathbb{C}^n/\tilde{W}$ (where $\tilde{W}$ is an extended affine Weyl group) is given in [DZ1]. The following parts of their construction will be required in this section:

- in flat coordinates, the prepotential, and hence the structure functions of the Frobenius manifold, are polynomial functions in \( \{ t_1, t_2, \ldots, t_{n-1}, e^{t_n} \} \);
- the Euler vector field takes the form
  \[
  E = \sum_{r=1}^{n-1} d_r t_r \partial_t^r + \frac{1}{d_k} \partial_t^{n} 
  \]
  where the $d_r$ are various numbers related to the extended affine Weyl groups, which may be found in Table 2 of [DZ1].

In addition one requires the following properties of caustics and limiting caustics for these manifolds:

**Lemma 6.** The extended affine Weyl group Frobenius manifolds are coverings of Frobenius manifolds

\[
\mathcal{M} - K_{\text{log}} = \mathbb{C}^n - \{ \tilde{t} | \tilde{t}^n = 0 \},
\]

with covering map $\mathbf{t} \mapsto \tilde{\mathbf{t}} = \{ t^1, t^{n-1}, e^{t_n} \}$. At generic points of $K_{\text{log}}$ the assumptions of Lemma 4 are satisfied. The caustics of the extended affine Weyl group Frobenius manifolds are given by the vanishing of certain quasihomogeneous polynomials in the variables \( \{ t^1, t^2, \ldots, t^{n-1}, e^{t_n} \} \).

**Proof** The proof is simpler in flat coordinates $\mathbf{t}$ rather than the $\{ \tilde{t} \}$ coordinates. The canonical coordinates are the roots of the polynomial $\text{poly}(\lambda) = 0$ where

\[
\text{poly}(\lambda) = \det[g^{ij}(t) - \lambda\eta^{ij}(t)].
\]

Using the information contained within the details of Lemma 2.6 in [DZ1] (in particular the equation preceding (2.30)) it is easy to show that

\[
\lim_{t^n \to -\infty} g^{kr} = 0
\]

for $r \neq n$ (recall that for these manifolds the identity element is given by $e = \frac{\partial}{\partial t_n}$ rather than $e = \frac{\partial}{\partial t_r}$). This shows, on expanding the determinant along the $k$th row or column, that in the

| Coxeter Group $W$ | $\gamma_W$ | $G_W$ |
|-------------------|-----------|-------|
| $A_n$             | 0         | 0     |
| $B_n$             | $\frac{(1-n)}{48n}$ | $-\frac{1}{48} \log \kappa_1^{B_n}$ |
| $D_n$             | 0         | 0     |
| $E_{6,7,8}$       | 0         | 0     |
| $F_4$             | $-\frac{1}{48}$ | $-\frac{1}{48} \log[6t_3^2 - 2t_2t_4^2 + t_4^6]$ |
| $H_3$             | $-\frac{3}{100}$ | $-\frac{1}{20} \log[t_2 - t_3^2]$ |
| $H_4$             | $-\frac{1}{25}$ | $-\frac{1}{20} \log[2025t_3^2 - 8100t_2t_4^2 + 630t_3^6 - 16t_4^{12}]$ |
| $I_2(h)$          | $-\frac{1}{12} \frac{(h-2)(h-3)}{h^2}$ | $-\frac{1}{24} \frac{(2-h)(3-h)}{h} \log[t_2]$ |

**Table 2.** The $G$-function on the space $\mathbb{C}^n/W$
limit $t^n \to -\infty$, the polynomial $\text{poly}(\lambda)$ has a repeated root. Hence $\mathcal{K}_\infty$ is a limiting caustic, or equivalently, $\mathcal{K}_{\text{log}}$ is a logarithmic caustic. Consideration of the resultant of $\text{poly}(\lambda)$ also shows that the standard caustics are all given in terms of quasihomogeneous polynomials $\kappa_i$, via $\mathcal{K}_i = \kappa_i^{-1}(0)$, where the $\kappa_i$ are polynomial in the variables $\{\tilde{t}\}$. In addition, the Egoroff potential for these manifolds is given by

$$\eta = t_k = \eta_k t^n = t^n.$$  

Thus $\left(\tilde{t}^n \delta_n\right)|_{\mathcal{K}_{\text{log}}}$ is constant. Thus the assumptions in lemma 4 hold for these manifolds.

**Proposition 7.** Under the assumption that the multiplication on the caustics $\mathcal{K}_i$ of the Frobenius manifold $\mathbb{C}^n/W$ is of the form $A_i^{r-2}I_2(N_i)$ the associated $G$-function is

$$G = -\frac{N_{\text{log}}}{24} t^n - \frac{1}{24} \sum_{i=1}^{\# K_i} \frac{(N_i - 2)(N_i - 3)}{N_i} \log \kappa_i$$  \hspace{1cm} (12)

where

$$\gamma = -\frac{N_{\text{log}}}{24} \delta_k - \frac{1}{24} \sum_{i=1}^{\# K_i} \frac{(N_i - 2)(N_i - 3)}{N_i} E(\log \kappa_i).$$

**Proof** The idea behind this proof is similar to the one used in proposition 5. One is integrating rational functions so the $G$-function has only rational and logarithmic terms. Under the above assumption one can exclude pole singularities and by lemma 4 one can exclude behaviour like $e^{-t^n}$. A scaling argument then forces a polynomial to be a constant.

In more detail, the argument runs as follows. In flat coordinates equations (11),(12) and (13) for $k = 2, \ldots, n$ may be inverted to find all the first derivatives of the $G$-function with respect to the flat coordinates. These must be rational functions with the same denominator, and so may be written

$$\frac{\partial G}{\partial t^i} = \frac{p_i(t)}{\Delta(t)}, \quad i = 1, \ldots, n,$$

(13)

where $p_i$ and $\Delta$ are quasihomogeneous polynomials in $\{t^1, \ldots, t^{n-1}, e^{\tilde{t}^n}\}$. It is useful to introduce a slightly different set of variables, $\{\tilde{t}^\alpha\}$, defined by $\tilde{t}^\alpha = t^\alpha, \alpha = 1, \ldots, n - 1$, and $\tilde{t}^n = e^{\tilde{t}^n}$. In these new variables (13) become

$$\frac{\partial G}{\partial \tilde{t}^i} = \frac{\tilde{p}(\tilde{t})}{\tilde{t}^n \Delta(\tilde{t})}$$

where the extra term in the denominator comes from the chain rule. On integrating the singularities will come from the zeroes of the denominator. Thus on integrating

$$G(\tilde{t}) = \text{rational function} + \text{logarithmic singularities}.$$  

Under the assumption and theorem 3, together with lemma 4 to exclude terms involving $e^{-t^n}$, the rational function must be polynomial and the scaling argument implies that this is then a constant. Thus,

$$G(\tilde{t}) = -\frac{N_{\text{log}}}{24} \log \tilde{t}^n - \frac{1}{24} \sum_{i=1}^{\# K_i} \frac{(N_i - 2)(N_i - 3)}{N_i} \log \kappa_i(\tilde{t})$$

which becomes (12) on converting back to the flat variables. Application of the Euler vector field then gives the final part of the lemma.

Note, this is just the conjectural form of the $G$-function; in order to prove it one requires information on the $F$-manifold structure on the caustics of these manifolds. In the special case of the group $\mathcal{W} = \mathcal{W}^{(k)}(A_{n-1})$ one may easily derive, using ideas from singularity theory, the required $F$-manifold structure and hence the explicit form of the $G$-function.
\[ \begin{array}{c|c}
A_l^{(k)}, D_l, E_{6,7,8} & \gamma_{\tilde{W}} \\
\hline
\tilde{W} & -\frac{1}{24}d_k \\
B_l & -\frac{1}{48}d_k \\
C_l & -\frac{1}{24}d_k \\
F_4 & -\frac{5}{144} \\
G_2 & -\frac{1}{16} \\
\end{array} \]

Table 3. The scaling anomalies of the Frobenius manifolds \( \mathbb{C}^{l+1}/\tilde{W} \)

**Theorem 8.** For the Frobenius manifold \( \mathbb{C}^n/\tilde{W}^{(k)}(A_{n-1}) \) the \( G \)-function takes the universal form

\[ G = -\frac{1}{24} t^n, \]

independent of \( k \).

**Proof** From the above lemma it suffices to show that the manifold \( \mathbb{C}^n/\tilde{W}^{(k)}(A_{n-1}) \) has only one caustic \( K_1 \) (together with the limiting caustic \( K_\infty \)) on which the multiplication is of the form \( A_1^{n-2}I_2(3) \). The scaling anomaly may be derived from the known data (see table 3).

The proof that \( N_1 = 3 \) comes from a standard argument in singularity theory. The manifold \( \mathcal{M} \) is isomorphic to the base space of the unfolding

\[ F_n(x) = x^k + a_1x^{k-1} + \ldots + a_k + \ldots + a_{k+m}x^{-m} \]

with parameters \((a_1, \ldots, a_{k+m}) \in \mathbb{C}^{k+m-1} \times \mathbb{C}^* \) and variable \( x \in \mathbb{C}^* \) (in [DZ1] \( x = e^{i\phi} \)). The total space of the unfolding has a critical space \( \mathcal{C} \subset \mathbb{C}^* \times \mathcal{M} \) (of dimension \( n = k + m \)) and the projection \( \mathcal{C} \to \mathcal{M} \) is a branched covering with the caustic \( \mathcal{K} \) being the image of the critical points of the projection. In this case the critical space is smooth and the set of critical points of the projection \( \mathcal{C} \to \mathcal{M} \) is also smooth: from this it follows that the caustic \( \mathcal{K} \) has only one component and that \( N_1 = 3 \). The same proof may be used to derive the data in the first line of table 1 for the Coxeter group \( A_n \).

A plausible conjecture is that the assumption in proposition 7 is true and also that the data on the caustics of the extended affine Weyl groups is the same as the corresponding Coxeter groups. For \( B_3, C_3, G_2 \) extended affine Weyl groups this may be verified by direct calculation using the explicit formulae for their prepotentials given in [DZ1]. If the conjecture is true then \( G = -1/24 t^n \) for the \( D_l \) and \( E_{6,7,8} \) extended affine Weyl groups as well.

5. **Symmetries of the \( G \)-function**

Symmetries of the WDVV equation are transformations

\[ t^\alpha \mapsto \hat{t}^\alpha, \]
\[ \eta_{\alpha\beta} \mapsto \hat{\eta}_{\alpha\beta}, \]
\[ F \mapsto \hat{F} \]

which preserve the equations. In appendix B in [D1] two types of symmetries were described, a Legendre-type transformation \( S_\kappa \) and an inversion \( I \). Such transformations are defined in terms of flat-coordinates, but in terms of canonical coordinates and isomonodromic data they are very
simple, the details being the content of Lemma 3.13 and Proposition 3.14 in [D1]. Such transformations induce transformations in the isomonodromic $\tau_I$ functions (basically Schlesinger transformations) and the $G$-function:

**Lemma 9.** Under a symmetry $S_\kappa$:

\[ \hat{\tau}_I = \tau_I, \]
\[ \hat{G} = G - \frac{1}{24} \log \det \left( \frac{\partial (\hat{t}^1, \ldots, \hat{t}^n)}{\partial (t^1, \ldots, t^n)} \right), \]
\[ \hat{\gamma} = \gamma - \frac{nq_\kappa}{24}. \]

Under an inversion $I$:

\[ \hat{\tau}_I = \frac{\tau_I}{\sqrt{t^n}}, \]
\[ \hat{G} = G + \left( \frac{n}{24} - \frac{1}{2} \right) \log t^n, \]
\[ \hat{\gamma} = \gamma + \left( \frac{n}{24} - \frac{1}{2} \right) (1 - d). \]

(This second result assumes that the Euler vector field takes the form

\[ E = \sum_{i=1}^{n} d_i v_i \frac{\partial}{\partial t^i} \]

and that $d \neq 1$. The result may be easily extended to cover more general cases).

No proof will be given - it follows immediately from Lemma 3.13 and Proposition 3.14 in [D1] together with the transformation properties of Jacobians. These formulae may be used to define the $G$-function on twisted Frobenius manifolds. Note also the invariant properties of $G$ under inversions if $n = 12$.

**Example** Consider the prepotential corresponding to the extended affine Weyl group $A_2^{(1)}$:

\[ F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 - \frac{1}{24} t_2^4 + t_2 e^{t_3}. \]

The corresponding $G$-function is given by (14):

\[ G = -\frac{1}{24} t_3. \]

Under the symmetry $S_2$ and $S_3$ one obtains, respectively, the prepotentials

\[ \hat{F} = \frac{1}{6} \hat{t}_2^3 + \hat{t}_1 \hat{t}_2 \hat{t}_3 + \frac{1}{6} \hat{t}_1 \hat{t}_3^2 + \frac{1}{2} \hat{t}_1^2 \left( \log \hat{t}_1 - \frac{3}{4} \right), \]
\[ \hat{F} = \frac{1}{2} \hat{t}_1 \hat{t}_3^2 + \frac{1}{2} \hat{t}_2^2 \hat{t}_3 + \frac{1}{2} \hat{t}_1 \log \hat{t}_2. \]

Application of the above lemma then gives the following $G$-functions:

\[ \hat{G} = -\frac{1}{12} \log \hat{t}_1, \]
\[ \hat{G} = -\frac{1}{8} \log \hat{t}_2. \]

The first of these solutions was obtained in [CT] by directly solving Getzler’s equations (4) from the corresponding prepotential. The results of this paper may be used to construct the genus one corrections to the bi-Hamiltonian hierarchy for generalized, multi-component, Toda and Benney hierarchies.
Appendix

In this appendix some of the results of this paper will be extended to the Hurwitz space $H_{1,0;l}$, which coincides with the orbit space $\mathbb{C}^{l+1}/J(A_l)$ where $J(A_l)$ is a Jacobi group - a particular extension of complex crystallographic groups [D1, B]. By direct calculation the $G$-functions for the first two members of this series are

$$G_{J(A_1)} = -\frac{1}{\pi^4} \log \eta(t_0) - \frac{1}{8} \log t_3,$$

$$G_{J(A_2)} = -\frac{1}{\pi^4} \log \eta(t_0) - \frac{1}{6} \log t_4,$$

where $\eta$ is the Dedekind function (for the corresponding prepotentials, see [B]). The scaling anomaly is easily calculated in general,

$$\gamma_{J(A_l)} = -\frac{1}{24} l^2 + 2,$$

which suggests the conjectural form

$$G_{J(A_l)} = -\frac{1}{\pi^4} \log \eta(t_0) - \frac{1}{24} l^2 \log t_{l+2}.$$

Related results have appeared recently in [KK]. It would be interesting to see how these approaches compare. These calculations suggest that the $G$-function for these spaces should have a simple form, though a deeper understanding of the properties of caustics in these spaces will be required in order to apply the methods of this paper.

Acknowledgment: I would like to thank Claus Hertling for explaining certain details of [H1] to me and for his detailed criticism of an earlier version of this paper. Financial support was provided by the EPSRC, grant GR/R05093.

References

[B] Bertola, M. Frobenius manifold structure on orbit space of Jacobi groups; Parts I and II, Diff. Geom. Appl. 13, (2000), 19-41 and 13 (2000), 213-23.

[CT] Chang, Jen-Hsu and Tu, Ming-Hsien, Topological Field Theory approach to the generalized Benney hierarchy, J. Phys A 34, (2001), 251-272.

[D1] Dubrovin, B., Geometry of 2D topological field theories in Integrable Systems and Quantum Groups, ed. Francaviglia, M. and Greco, S.. Springer lecture notes in mathematics, 1620, 120-348.

[D2] Dubrovin, B., Painlevé Transcendents in Two-Dimensional Topological Field Theory in The Painlevé property: One Century Later ed. Conte, R.. CRM series in Mathematical Physics, Springer, 1999, 287-412.

[DZ1] Dubrovin, B. and Zhang, Y., Extended affine Weyl groups and Frobenius manifolds, Compositio Math. 111 (1998), 167-219.

[DZ2] Dubrovin, B. and Zhang, Y., Bihamiltonian hierarchies in the 2D Topological Field Theory at One-Loop Approximation, Commun.Math.Phys. 198 (1998) 311-361.

[DZ3] Dubrovin, B. and Zhang, Y., Frobenius Manifolds and Virasoro Constraints, Selecta Math., New ser.5 (1999) 423-466.

[Ge] Getzler, E., Intersection theory on $\tilde{M}_{1,4}$ and elliptic Gromov-Witten invariants, J. Amer. Math. Soc. 10 (1997), 973-998.

[Gi1] Givental, A.B., Elliptic Gromov-Witten invariants and the generalized mirror conjecture, in Integrable systems and algebraic geometry, Proceedings of the Taniguchi Symposium 1997, ed. Saito, M.H., Shimizu, Y., and Ueno, K., World Scientific (1998), 107-155.

[Gi2] Givental, A.B., Singular Lagrangian manifolds and their Lagrangian maps, J. Sovier Math. 52.4 (1988), 3246-3278.

[H1] Hertling, C., Frobenius manifolds and variance of the spectral numbers, in New developments in singularity theory, ed. D. Siersma et al., Kluwer Acad. Publ. Dordrecht-Boston-London (2001) 235-255.

[H2] Hertling, C., Frobenius manifolds and moduli spaces for singularities, Cambridge Tracts in Mathematics 151, Cambridge University Press (2002).

[KK] Kokotov, A and Korotkin, D., Tau-function on Hurwitz spaces in genus zero and one, math.PH/0202034.

[S] Strachan, I.A.B., Frobenius manifolds: natural submanifolds and induced bi-Hamiltonian structures, math.DG/0201038.

Department of Mathematics, University of Hull, Hull HU6 7RX, U.K.
E-mail address: i.a.strachan@hull.ac.uk