General structure of fermion two-point function and its spectral representation in a hot magnetised medium

Aritra Das,*
HENPP Division, Saha Institute of Nuclear Physics, HBNI, 1/AF Bidhan Nagar, Kolkata 700064, India.

Aritra Bandyopadhyay,†
Theory Division, Saha Institute of Nuclear Physics, HBNI, 1/AF Bidhan Nagar, Kolkata 700064, India and
Departamento de Fisica, Universidade Federal de Santa Maria, Santa Maria, RS 97105-900, Brazil.

Pradip K. Roy,‡
HENPP Division, Saha Institute of Nuclear Physics, HBNI, 1/AF Bidhan Nagar, Kolkata 700064, India.

Munshi G. Mustafa§
Theory Division, Saha Institute of Nuclear Physics, HBNI, 1/AF Bidhan Nagar, Kolkata 700064, India.

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We have systematically constructed the general structure of the fermion self-energy and the effective quark propagator in presence of a nontrivial background like hot magnetised medium. This is applicable to both QED and QCD. The hard thermal loop approximation has been used for the heat bath. We have also examined transformation properties of the effective fermion propagator under some of the discrete symmetries of the system. Using the effective fermion propagator we have analysed the fermion dispersion spectra in a hot magnetised medium along with the spinor for each fermion mode obtained by solving the modified Dirac equation. The fermion spectra is found to reflect the discrete symmetries of the two-point functions. We note that for a chirally symmetric theory the degenerate left and right handed chiral modes in vacuum or in a heat bath get separated and become asymmetric in presence of magnetic field without disturbing the chiral invariance. The obtained general structure of the two-point functions is verified by computing the three-point function, which agrees with the existing results in one-loop order. Finally, we have computed explicitly the spectral representation of the two-point functions which would be very important to study the spectral properties of the hot magnetised medium corresponding to QED and QCD with background magnetic field.

* aritra.das@saha.ac.in
† aritra.bandyopadhyay@saha.ac.in
‡ pradipk.roy@saha.ac.in
§ munshigolam.mustafa@saha.ac.in
I. Introduction

In non-central heavy ion collision (HIC) experiments in LHC at CERN and in RHIC at BNL, it is believed that a very strong magnetic field is created in the direction perpendicular to the reaction plane due to the spectator particles that are not participating in the collisions. The experiments conducted by PHENIX Collaboration [1] showed direct-photon anisotropy which has posed a serious challenge to the present theoretical models. It is conjectured that this excess elliptic flow may be due to the excess photons produced by the decay $\rho \to \pi(n)\gamma$ and the branching ratio of which increases in presence of the magnetic field near the critical value where the condensate of $\rho$ is found. The estimated strength of this magnetic field depends on collision energy and impact parameter between the colliding nuclei and is about several times the pion mass squared, $i.e., eB \sim 15m_\pi^2$ at LHC in CERN [2]. Also, a class of neutron star called magnetar exhibits $[3,5]$ a magnetic field of $10^{18} - 10^{20}$ Gauss at the inner core and $10^{12} - 10^{13}$ Gauss at the surface. These observations motivate to study the properties of hot magnetised medium using both phenomenology and quantum field theory.

The presence of a strong magnetic field in HIC influences the QCD phase transitions [6] and particle productions, especially the production of the soft photon [7] and dileptons [8-13], which act as a probe of the medium. Apart from these, there is a large class of other phenomena that take place in presence of background magnetic field like chiral magnetic effects due to axial anomalies [14-16], magnetic catalysis [17, 18], inverse magnetic catalysis [19, 20], superconductivity of the vacuum [21]. It further influences the thermal chiral and deconfining phase transition [22], change of topological charge [23], anomalous transports [24], refractive indices [25, 26] and screening mass [27], decay constant [28] of neutral mesons etc. In addition, efforts were also made to study the bulk properties for a fermi gas [29], low lying hadrons [30] and strongly coupled systems [31], collective excitations in magnetised QED medium [32] using Ritus method and QCD medium [33] using Furry’s picture, neutrino properties [34, 35] and Field theory of the Faraday effects [36, 37].

The magnetic field created in HIC lasts for very short time (~ a few fm). The strength of the field decays rapidly with time after $\tau \sim 1 - 2\, fm/c$. However, the medium effects like electric conductivity can delay the decay and by the time deconfined quarks and gluons equilibrate with QGP medium, the magnetic field strength gets sufficiently weak. At that time the relevant energy scales of the system can be put in this way: $q_f B < m_f^2 \ll T^2$. In this low field limit the properties of the deconfined medium are also affected. So, it becomes important to treat the weak field limit separately. Fermion propagator in presence of a uniform background magnetic field has been derived first by Schwinger [38]. Using this, one loop fermion self-energy and its dispersion property have been studied at zero temperature [42] in weak field approximation and using full propagator at finite temperature [43]. Also a detailed study of the spectral properties of $\rho$ mesons has been performed in presence of magnetic field both at zero [44, 45] and at non-zero temperature [46].

For hot and dense medium (e.g., QED and QCD plasma), it is well know that a bare perturbation theory breaks down due to infrared divergences. A reorganisation of the perturbation theory has been done by performing the expansion around a system of massive quasiparticles [47], where mass is generated through thermal fluctuations. This requires a resummation of certain class of diagrams, known as hard thermal loop (HTL) resummation [48], when the loop momenta are of the order of the temperature. This reorganised perturbation theory, known as HTL perturbation theory (HTLpt), leads to gauge independent results for various physical quantities [49-65]. Within this one-loop HTLpt, the thermomagnetic correction to the quark self-energy [66], quark-gluon three point [66] function at zero chemical potential and four point [67] function at finite chemical potential in weak field limit have been computed. The fermion self-energy has also been extended to the case of non-zero chemical potential and the pressure of a weakly magnetised QCD plasma [68] has also been obtained.

In recent years a huge amount of activity is underway to explore the properties of a hot medium with a background magnetic field using phenomenology as well as using thermal field theory. In a thermal medium the bulk and dynamical properties [48, 69, 70] are characterised by the collective excitations in a time like region and the Landau damping in a space-like domain. The basic quantity associated with these medium properties is the two point correlation function. In this work we construct the general structure of the fermionic two point functions (e.g., self-energy and the effective propagator) in a nontrivial background like a hot magnetised medium. We then analyse its property under the transformation of some discrete symmetries of the system, the collective fermionic spectra, QED like three-point functions and the spectral representation of the two point function and its consequences in a hot magnetised medium. The formulation is applicable equally well to both QED and QCD.

The paper is organised as follows; In section II, the notation and set up are briefly discussed through a fermion propagator in a constant background field using Schwinger formalism. Section III has number of parts in which we obtain the general structure of the self-energy (subsec.III.A), the effective fermion propagator (subsec.III.B), the transformation properties and discrete symmetries of the effective propagator (subsec.III.C), the modified Dirac equations in general and for lowest Landau level (subsec.III.D) and the dispersion properties of the various collective modes (subsec.III.E) in time-like region. In section IV the general structure of the self-energy and the propagator has been verified from one-loop direct calculation. The spectral representation of the effective propagator in space-like domain has been obtained in section V. We have presented some detailed calculations for various sections and subsections in Appendix A-E. Finally, we conclude in section VI.
II. Charged Fermion Propagator in Background Magnetic Field within Schwinger Formalism

In this section we set the notation and briefly outline the fermionic propagator in presence of a background magnetic field following Schwinger formalism [38]. Without any loss of generality, the background magnetic field is chosen along the $z$ direction, $\vec{B} = B\hat{z}$, and the vector potential in a symmetric gauge reads as

$$A^\mu = \left(0, -\frac{yB}{2}, \frac{xB}{2}, 0\right).$$  \hspace{1cm} (1)

Below we also outline the notation we shall be using throughout:

$$a^\mu = (a^0, a^1, a^2, a^3) = (a_0, \vec{a}); \quad a \cdot b \equiv a_0 b_0 - \vec{a} \cdot \vec{b}; \quad g^{\mu\nu} = \text{diag}(1, -1, -1, -1),$$

$$a^\mu = a_\parallel^\mu + a_\perp^\mu; \quad a_\parallel^\mu = (a^0, 0, 0, a^3); \quad a_\perp^\mu = (0, a^1, a^2, 0),$$

$$g^{\mu\nu} = g_\parallel^{\mu\nu} + g_\perp^{\mu\nu}; \quad g_\parallel^{\mu\nu} = \text{diag}(1, 0, 0, -1); \quad g_\perp^{\mu\nu} = \text{diag}(0, -1, 0, 0),$$

$$(a \cdot b) = (a \cdot b)_\parallel - (a \cdot b)_\perp; \quad (a \cdot b)_\parallel = a_0 b_0 - a^3 b^3; \quad (a \cdot b)_\perp = a^1 b^1 + a^2 b^2; \quad \mathbf{s} = \gamma^0 a_0 - \gamma^3 a^3; \quad \mathbf{S} = \gamma^1 a^1 + \gamma^2 a^2$$  \hspace{1cm} (2)

where $\parallel$ and $\perp$ are, respectively, the parallel and perpendicular components, which would be separated out due to the presence of the background magnetic field.

Now, the fermionic two-point function is written as

$$S(x, x') = -i C(x, x') \int_0^\infty \frac{ds}{s \sin(q_f B s)} \exp\left(\frac{im^2 s}{q_f B s}\right)$$

$$\exp\left[-\frac{i}{4s} \left((x - x')^2 - \frac{q_f B s}{\tan(q_f B s)} (x - x')^2_\perp\right)\right]$$

$$\times \left[m_f + \frac{1}{2s} \left((\mathbf{k}_\parallel - \mathbf{k}_\perp) - \frac{q_f B s}{\sin(q_f B s)} \exp\left(-i q_f B s \Sigma_3\right) (\mathbf{k}_\perp - \mathbf{k}_\parallel)\right]\right],$$

where the parameter $s$ is called Schwinger proper time variable [38]. We note that $m_f$ and $q_f$ are mass and absolute charge of the fermion of flavour $f$, respectively. The phase factor, $C(x, x')$, is independent of $s$ but is responsible for breaking of both gauge and translational invariance. Remaining part, denoted as $S(x - x')$, is translationally invariant. However, as shown below, $C(x, x')$ drops out for a gauge invariant calculation. Now $C(x, x')$ reads as

$$C(x, x') = C \exp\left[i q_f \int_{x'}^x dz_\parallel \left(A_\mu(z) + \frac{1}{2} F_{\mu\nu}(z - x')^\nu\right)\right],$$

where $C$ is just a number. The integral in the exponential is independent of the path taken between $x$ and $x'$ and choosing it as a straight line one can write

$$C(x, x') = C \Phi(x, x') = C \exp\left[i q_f \int_{x'}^x d\xi A_\mu(\xi)\right].$$

Using the gauge transformation $A^\mu(\xi) \rightarrow A^\mu(\xi) + \partial^\mu A(\xi)$, and choosing symmetric gauge as given in (1), the phase factor $\Phi(x, x')$ becomes 1, if we take [66]

$$\Lambda(\xi) = B \left(x'_{2\xi_1} - x_{1\xi_2}\right).$$

From equation (3), the momentum space propagator can be obtained as

$$S(K) = \int d^4 x e^{iK x} S(x - x')$$

$$= -i \int_0^\infty ds \exp\left[is \left(K^2 - \frac{\tan(q_f B s)}{q_f B s} K^2_\perp - m_f^2\right)\right]$$

$$\times \left[\left(1 + \gamma_3 q_f \tan(q_f B s)\right) (\mathbf{K}_\parallel + m_f) - \sec^2(q_f B s) \mathbf{K}_\perp\right]$$

$$= \exp\left(-K^2_\perp |q_f B|\right) \sum_{l=0}^\infty (-1)^l \frac{D_l(q_f B, K)}{K^2_\perp - m_f^2 - 2l |q_f B|},$$

where $\mathbf{K}_\parallel = \mathbf{K} - m_f^2$ and $\mathbf{K}_\perp = \mathbf{K} - m_f^2$. The propagator $S(K)$ is then an absolute charge and absolute mass dependent quantity.
where \( k_+^2 = 2|q_j B| \), is quantised with Landau level \( l = 0, 1, \cdots \), and

\[
D_j(q_j B, K) = (K_+^l + m_f) \left[ (1 - i\gamma_5 \gamma_2) L_i \left( \frac{2 K_+^2}{|q_j B|} \right) - (1 + i\gamma_5 \gamma_2) L_{i-1} \left( \frac{2 K_+^2}{|q_j B|} \right) \right] - 4 K_+ L_{i-1}^l \left( \frac{2 K_+^2}{|q_j B|} \right),
\]

(8)

where \( L_i(x) \) is Laguerre polynomial, \( L_{i-1}^l(x) \) is associated Laguerre polynomial with \( L_{i-1}^l(x) = 0 \) and both \( j \) is a non-negative integer.

Below we discuss the structure of the propagator in (7) in presence of background magnetic field. Since fermion propagator is \( 4 \times 4 \) matrix, a new matrix \( \left( \frac{i}{K_+} + \frac{\gamma_5}{u} \right) \) appears in addition to that of the vacuum structure \( (a' K + a' (K^2)) \) is a Lorentz invariant structure function) for a chirally symmetric theory. One can now write the new matrix for a chirally symmetric theory in terms of background electromagnetic field tensor \( F_{\mu \nu} \) as

\[
i\gamma_1 \gamma_2 K_+ B = -\gamma_5 K^\mu F_{\mu \nu} \gamma^\nu,
\]

(9)

where the background dual field tensor reads as

\[
F_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \delta} F^{\rho \delta}.
\]

(10)

The structure of a chirally symmetric free fermion propagator in presence of only magnetic field can be viewed as \( (a' K + \delta' \gamma_5 K^\mu F_{\mu \nu} \gamma^\nu) \), where \( \delta' \) is a new structure constant that appears due to the presence of background magnetic field. When a fermion propagates only in a hot medium, then the vacuum part will be modified only due to the thermal background [69] as \( (a' K + \beta n) \), where \( u \) is the four velocity of the heat bath. When a fermion moves in a nontrivial background like hot magnetised medium then one can write (9) as

\[
i\gamma_1 \gamma_2 K_+ = -\gamma_5 \left[ (K.n) n - (K,u) u \right],
\]

(11)

where

\[
n_\mu = \frac{1}{2B} \epsilon_{\mu \nu \rho \delta} u^\nu F^{\rho \delta} = \frac{1}{B} u^\nu F_{\mu \nu}.
\]

(12)

and the four velocity in the rest frame of the heat bath and the direction of the magnetic field \( B \), respectively, given as

\[
u^\mu = (1, 0, 0, 0), \quad (13a)
\]

\[
n^\mu = (0, 0, 0, 1).
\]

(13b)

One can notice that in a hot magnetised medium both \( u \) and \( n \) are correlated as given in (12) and the contribution due to magnetic field in (9) in presence of heat bath becomes a thermo-magnetic contribution. We also further note that in absence of heat bath, (11) reduces to (9), which is not obvious by inspection but we would see later.

III. General Structure of Fermion Two-point Function in a Hot Magnetised Medium

In previous section the modification of a free propagator has been discussed briefly in presence of a background magnetic field. In this section we would like to obtain the most general structure of a fermion self-energy, the effective fermion propagator and some of its properties in a nontrivial background like hot magnetised medium. We would also discuss the modified Dirac equation and the fermion dispersion spectrum in a hot magnetised medium. For thermal bath we would use HTL approximation and any other approximation required for the purpose will be stated therein.

A. General Structure of the Fermion Self-Energy

The fermionic self-energy is a matrix as well as a Lorentz scalar. However, in presence of nontrivial background, e.g., heat bath and magnetic field, the boost and rotational symmetries of the system are broken. The general structure of fermion self-energy for hot magnetised medium can be written by the following arguments. The self-energy \( \Sigma(P) \) is a \( 4 \times 4 \) matrix which depends, in present case, on the four momentum of the fermion \( P \), the velocity of the heat bath \( u \) and the direction of the magnetic field \( n \).
Now, any $4 \times 4$ matrix can be expanded in terms of 16 basis matrices: $\{ I, \gamma_5, \gamma_\mu, \gamma_\mu' \gamma_5, \sigma_{\mu \nu} \}$, which are the unit matrix, the four $\gamma$-matrices, the six $\sigma_{\mu \nu}$ matrices, and finally $\gamma_5$. So, the general structure can be written as

$$
\Sigma(P) = -a I - b \gamma_5 - a \slashed{\mathbf{P}} - b \slashed{\mathbf{q}} - c \gamma_5 \slashed{\mathbf{P}} - b' \gamma_5 \slashed{\mathbf{q}} - c' \gamma_5 \slashed{\mathbf{q}}$
$$

$$
- h \sigma_{\mu \nu} P^\mu P^\nu - h' \sigma_{\mu \nu} u'^\mu u'^\nu - \kappa \sigma_{\mu \nu} u'^\mu \gamma_5 n'^\nu - d \sigma_{\mu \nu} u'^\mu \gamma_5 n'^\nu - d' \sigma_{\mu \nu} n'^\mu P_\nu - \kappa' \sigma_{\mu \nu} u'^\mu n'^\nu ,
$$

(14)

where various coefficients are known as structure functions. We note that the combinations involving $\sigma_{\mu \nu}$ do not appear due to antisymmetric nature of it in any loop order of self-energy. Also in a chirally invariant theory, the terms $a I$ and $\gamma_5 \slashed{\mathbf{q}}$ will not appear as they would break the chiral symmetry. The term $\gamma_5 \slashed{\mathbf{P}}$ would appear in the self-energy if fermions interact with an axial vector. By dropping those in (14) for chirally symmetric theory, one can now write

$$
\Sigma(P) = -a \slashed{\mathbf{P}} - b \slashed{\mathbf{q}} - c \gamma_5 \slashed{\mathbf{P}} - b' \gamma_5 \slashed{\mathbf{q}} - c' \gamma_5 \slashed{\mathbf{q}}.
$$

(15)

Now we point out that some important information is encoded into the fermion propagator in (7) through (11) for a hot magnetised medium. This suggests that $c \gamma_5$ should not appear in the fermion self-energy and the most general form of the fermion self-energy for a hot magnetised medium becomes

$$
\Sigma(P) = -a \slashed{\mathbf{P}} - b \slashed{\mathbf{q}} - b' \gamma_5 \slashed{\mathbf{q}} - c' \gamma_5 \slashed{\mathbf{q}}.
$$

(16)

When a fermion propagates in a vacuum, then $b = b' = c' = 0$ and $\Sigma(P) = -a \slashed{\mathbf{P}}$. But when it propagates in a background of pure magnetic field without any heat bath, then $a \neq 0$, $b = 0$ and the structure functions, $b'$ and $c'$, will depend only on the background magnetic field as we will see later. When a fermion propagates in a heat bath, then $a \neq 0$, $b \neq 0$ but both $b'$ and $c'$ vanish because there would not be any thermo-magnetic corrections as can also be seen later.

We now write down the right chiral projection operator, $P_+$ and the left chiral projection operator $P_-$, respectively, defined as:

$$
P_+ = \frac{1}{2} \left( I + \gamma_5 \right),
$$

(17a)

$$
P_- = \frac{1}{2} \left( I - \gamma_5 \right),
$$

(17b)

which satisfy the usual properties of projection operator:

$$
P_+^2 = P_+, \quad P_+ P_- = P_- P_+ = 0, \quad P_+ + P_- = I, \quad P_+ - P_- = \gamma_5.
$$

(18)

Using the chirality projection operators, the general structure of the self-energy in (16) can be casted in the following form

$$
\Sigma(P) = - P_+ C \gamma_5 P_- - P_- \slashed{\mathbf{q}} P_+ ,
$$

(19)

where $C$ and $\slashed{\mathbf{q}}$ are defined as

$$
C = a \slashed{\mathbf{P}} + (b + b') \gamma_5 \slashed{\mathbf{q}} + c' \gamma_5 \slashed{\mathbf{q}},
$$

(20a)

$$
\slashed{\mathbf{q}} = a \slashed{\mathbf{P}} + (b - b') \gamma_5 \slashed{\mathbf{q}} - c' \gamma_5 \slashed{\mathbf{q}}.
$$

(20b)

From (16) one obtains the general form of the various structure functions as

$$
a = \frac{1}{4} \text{Tr} \left( \Sigma \slashed{\mathbf{P}} \right) - \left( P.\mathbf{u} \right) \text{Tr} \left( \Sigma \slashed{\mathbf{q}} \right),
$$

(21a)

$$
b = \frac{1}{4} \left( P.\mathbf{u} \right) \text{Tr} \left( \Sigma \slashed{\mathbf{P}} \right) + P^2 \text{Tr} \left( \Sigma \slashed{\mathbf{q}} \right),
$$

(21b)

$$
b' = \frac{1}{4} \text{Tr} \left( \gamma_5 \Sigma \slashed{\mathbf{P}} \right),
$$

(21c)

$$
c' = \frac{1}{4} \text{Tr} \left( \gamma_5 \Sigma \slashed{\mathbf{q}} \right).
$$

(21d)

---

1. The presence of an axial gauge coupling leads to chiral or axial anomaly and a chirally invariant theory does not allow this. Other way, the preservation of both chiral and axial symmetries is impossible, a choice must be made which one should be preserved. For a chirally invariant theory this term drops out. Also the presence of $\gamma_5$ in a Lagrangian violates parity invariance.

2. We have checked that even if one keeps $c \gamma_5$, the coefficient $c$ becomes zero in one-loop order in the weak field approximation.
which are also Lorentz scalars. Beside $T$ and $B$, they would also depend on three Lorentz scalars defined by

\begin{align}
\omega &\equiv P^\mu u_\mu, \\
p^3 &\equiv -P^\mu n_\mu = p_z, \\
p_\perp &\equiv [(P^\mu u_\mu)^2 - (P^\mu n_\mu)^2 - (P^\mu P_\mu)]^{1/2}.
\end{align}

Since $P^2 = \omega^2 - p^2 - p_z^2$, we may interpret $\omega$, $p_\perp$, $p_z$ as Lorentz invariant energy, transverse momentum, longitudinal momentum respectively. All these structure functions for 1-loop order in a weak field and HTL approximations have been computed in Appendix A and quoted here \(^{3}\) as

\begin{align}
a(p_0, |\vec{p}|) &= -\frac{m^2_{\text{th}}}{|\vec{p}|^2} Q_1 \left( \frac{p_0}{|\vec{p}|} \right), \\
b(p_0, |\vec{p}|) &= \frac{m^2_{\text{th}}}{|\vec{p}|} \left[ Q_1 \left( \frac{p_0}{|\vec{p}|} \right) - Q_0 \left( \frac{p_0}{|\vec{p}|} \right) \right], \\
b'(p_0, |\vec{p}|) &= 4C_F g^2 M^2(T, m, q, B) \frac{p_z}{|\vec{p}|^2} Q_1 \left( \frac{p_0}{|\vec{p}|} \right), \\
c'(p_0, |\vec{p}|) &= 4C_F g^2 M^2(T, m, q, B) \frac{1}{|\vec{p}|} Q_0 \left( \frac{p_0}{|\vec{p}|} \right).
\end{align}

We note that the respective vacuum contributions in $a$, $b'$ and $c'$ have been dropped by the choice of the renormalisation prescription, and the general structure of the self-energy, as found in appendix A, agrees with that in (16).

### B. Effective Fermion Propagator

The effective fermion propagator is given by Dyson-Schwinger equation (see Fig. 1) which reads as

\[ S^*(P) = \frac{1}{\slashed{P} - \Sigma(P)} . \]

and the inverse fermion propagator reads as

\[ S^{-1}(P) = \slashed{P} - \Sigma(P) . \]

![Diagram of Dyson-Schwinger equation](image)

**FIG. 1:** Diagramatic representation of the Dyson-Schwinger equation for one-loop effective fermion propagator.

Using (19) the inverse fermion propagator can be written as

\[ S^{-1}(P) = P_+ \left[ (1 + a(p_0, |\vec{p}|)) \slashed{P} + (b(p_0, |\vec{p}|) + b'(p_0, p_\perp, p_z)) \slashed{n} + c'(p_0, |\vec{p}|) \slashed{n} \right] P_- \\
+ P_- \left[ (1 + a(p_0, |\vec{p}|)) \slashed{P} + (b(p_0, |\vec{p}|) - b'(p_0, p_\perp, p_z)) \slashed{n} - c'(p_0, |\vec{p}|) \slashed{n} \right] P_+,
\]

where $\slashed{L}$ and $\slashed{R}$ can be obtained from two four vectors given by

\begin{align}
L^\mu(p_0, p_\perp, p_z) &= A(p_0, |\vec{p}|) P^\mu + B_+(p_0, p_\perp, p_z) u^\mu + c'(p_0, |\vec{p}|) n^\mu, \\
R^\mu(p_0, p_\perp, p_z) &= A(p_0, |\vec{p}|) P^\mu + B_-(p_0, p_\perp, p_z) u^\mu - c'(p_0, |\vec{p}|) n^\mu,
\end{align}

\(^{3}\) In weak field approximation the domain of applicability becomes $m^2_{\text{th}}(\sim \tilde{g}^2 T^2) < q_f B < T^2$ instead of $m^2 < q_f B < T^2$ as discussed in Appendix A.
with

\[ A(p_0, |\vec{p}|) = 1 + a(p_0, |\vec{p}|), \]  
\[ B_\pm(p_0, p_\perp, p_z) = b(p_0, |\vec{p}|) \pm b'(p_0, p_\perp, p_z). \]  

(28a) \hspace{1cm} (28b)

Using (26) in (24), the propagator can now be written as

\[ S^+(P) = P_- \frac{L}{L^2} P_+ + P_+ \frac{R}{R^2} P_- . \]  

(29)

where we have used the properties of the projection operators \( P_\pm \gamma^\mu = \gamma^\mu P_\pm, P_\pm^2 = P_\pm \), and \( P_+ P_- = P_- P_+ = 0 \). It can be checked that \( S^+(P)S^{+\dagger}(P) = P_+ + P_- = 1 \). Also we have

\[ L^2 = L^\mu L_\mu = (A p_0 + B_+)^2 - [ (A p_z + c')^2 + A^2 p_\perp^2 ] = L_0^2 - |\vec{L}|^2, \]  
\[ R^2 = R^\mu R_\mu = (A p_0 + B_-)^2 - [ (A p_z - c')^2 + A^2 p_\perp^2 ] = R_0^2 - |\vec{R}|^2, \]

(30a) \hspace{1cm} (30b)

where we have used \( u^2 = 1, n^2 = -1, P \cdot u = p_0 \), and \( P \cdot n = -p_z \). Note that we have suppressed the functional dependencies of \( L, R, A, B_\pm \) and \( c' \), which would bring them back whenever necessary.

For the lowest Landau Level (LLL), \( l = 0 \Rightarrow p_\perp = 0 \), and these relations reduce to

\[ L^2_{LLL} = (A p_0 + B_+)^2 - (A p_z + c')^2 = L_0^2 - L_z^2, \]  
\[ R^2_{LLL} = (A p_0 + B_-)^2 - (A p_z - c')^2 = R_0^2 - R_z^2. \]  

(31a) \hspace{1cm} (31b)

The poles of the effective propagator, \( L^2 = 0 \) and \( R^2 = 0 \), give rise to quasi-particle dispersion relations in a hot magnetised medium. There will be four collective modes with positive energies: two from \( L^2 = 0 \) and two from \( R^2 = 0 \). Nevertheless, we will discuss dispersion properties later.

C. Transformation Properties of Structure Functions and Propagator

First, we outline some transformation properties of the various structure functions as obtained in (23a), (23b), (23c) and (23d).

1. Under the transformation \( \vec{p} \rightarrow -\vec{p} = (p_\perp, -p_z) \):

\[ a(p_0, |\vec{p}|) = a(p_0, |\vec{p}|), \]  
\[ b(p_0, |\vec{p}|) = b(p_0, |\vec{p}|), \]  
\[ b'(p_0, p_\perp, -p_z) = -b'(p_0, p_\perp, p_z), \]  
\[ c'(p_0, |\vec{p}|) = c'(p_0, |\vec{p}|). \]  

(32a) \hspace{1cm} (32b) \hspace{1cm} (32c) \hspace{1cm} (32d)

2. For \( p_0 \rightarrow -p_0 \):

\[ a(-p_0, |\vec{p}|) = a(p_0, |\vec{p}|), \]  
\[ b(-p_0, |\vec{p}|) = -b(p_0, |\vec{p}|), \]  
\[ b'(-p_0, p_\perp, -p_z) = b'(p_0, p_\perp, p_z), \]  
\[ c'(-p_0, |\vec{p}|) = -c'(p_0, |\vec{p}|). \]  

(33a) \hspace{1cm} (33b) \hspace{1cm} (33c) \hspace{1cm} (33d)

3. For \( P \rightarrow -P = (-p_0, -\vec{p}) \):

\[ a(-p_0, |\vec{p}|) = a(p_0, |\vec{p}|), \]  
\[ b(-p_0, |\vec{p}|) = -b(p_0, |\vec{p}|), \]  
\[ b'(-p_0, p_\perp, -p_z) = b'(p_0, p_\perp, p_z), \]  
\[ c'(-p_0, |\vec{p}|) = -c'(p_0, |\vec{p}|). \]  

(34a) \hspace{1cm} (34b) \hspace{1cm} (34c) \hspace{1cm} (34d)

We have used the fact that \( Q_0(-x) = -Q_0(x) \) and \( Q_1(-x) = Q_1(x) \).
Now based on the above we also note down the transformation properties of those quantities appearing in the propagator: 

1. For $A$:

$$A(p_0, p_L, p_z) \xrightarrow{\tilde{p} \to -\tilde{p}} A(p_0, p_L, p_z).$$ (35a)

$$A(p_0, p_L, p_z) \xrightarrow{p_0 \to -p_0} A(p_0, p_L, p_z).$$ (35b)

$$A(p_0, p_L, p_z) \xrightarrow{\tilde{p} \to -\tilde{p}} A(p_0, p_L, p_z).$$ (35c)

2. For $B_\pm$:

$$B_\pm(p_0, p_L, p_z) \xrightarrow{\tilde{p} \to -\tilde{p}} B_\pm(p_0, p_L, p_z).$$ (36a)

$$B_\pm(p_0, p_L, p_z) \xrightarrow{p_0 \to -p_0} -B_\pm(p_0, p_L, p_z).$$ (36b)

$$B_\pm(p_0, p_L, p_z) \xrightarrow{\tilde{p} \to -\tilde{p}} -B_\pm(p_0, p_L, p_z).$$ (36c)

Using the above transformation properties, it can be shown that $\mathcal{L}$, $\mathcal{R}$, $L^2$ and $R^2$, respectively given in (27a), (27b), (30a) and (30b) transform as

$$\mathcal{L}(p_0, p_L, p_z) \xrightarrow{\tilde{p} \to -\tilde{p}} A(p_0, |\tilde{p}|)(p_0^0 + \tilde{p} \cdot \tilde{\gamma}) + B_+(p_0, p_L, p_z)\gamma^+ - c'(p_0, |\tilde{p}|)\gamma^-, \quad (37a)$$

$$\mathcal{R}(p_0, p_L, p_z) \xrightarrow{\tilde{p} \to -\tilde{p}} -A(p_0, |\tilde{p}|)(p_0^0 + \tilde{p} \cdot \tilde{\gamma}) + B_+(p_0, p_L, p_z)\gamma^+ - c'(p_0, |\tilde{p}|)\gamma^-, \quad (37b)$$

$$L^2(p_0, p_L, p_z) \xrightarrow{\tilde{p} \to -\tilde{p}} R^2(p_0, p_L, p_z), \quad (37c)$$

$$R^2(p_0, p_L, p_z) \xrightarrow{\tilde{p} \to -\tilde{p}} L^2(p_0, p_L, p_z), \quad (37d)$$

and

$$\mathcal{L}(p_0, p_L, p_z) \xrightarrow{p_0 \to -p_0} -L(p_0, p_L, p_z), \quad (38a)$$

$$\mathcal{R}(p_0, p_L, p_z) \xrightarrow{p_0 \to -p_0} -R(p_0, p_L, p_z), \quad (38b)$$

$$L^2(p_0, p_L, p_z) \xrightarrow{p_0 \to -p_0} -L^2(p_0, p_L, p_z), \quad (38c)$$

$$R^2(p_0, p_L, p_z) \xrightarrow{p_0 \to -p_0} -R^2(p_0, p_L, p_z). \quad (38d)$$

Now we are in a position to check the transformation properties of the effective propagator under some of the discrete symmetries:

1. **Chirality**

Under chirality the fermion propagator transform as [71]

$$S(p_0, \tilde{p}) \xrightarrow{\gamma_5} -\gamma_5 S(p_0, \tilde{p}) \gamma_5. \quad (39)$$

The effective propagator, $S^e(p_0, p_L, p_z)$, in (29) transforms under chirality as

$$-\gamma_5 S^e(p_0, p_L, p_z) \gamma_5 = -\gamma_5 S - \gamma_5 \left( L^2(p_0, p_L, p_z) \right) \gamma_5 = \gamma_5 \left( R^2(p_0, p_L, p_z) \right) \gamma_5$$

$$= \gamma_5 \left( L^2(p_0, p_L, p_z) \right) \gamma_5 + \gamma_5 \left( R^2(p_0, p_L, p_z) \right) \gamma_5$$

$$= S^e(p_0, p_L, p_z), \quad (40)$$

which satisfies (39) and indicates that it is chirally invariant.
2. Reflection

Under reflection the fermion propagator transforms \([71]\)

\( S(p_0, \vec{p}) \longrightarrow S(p_0, -\vec{p}). \) \hspace{1cm} (41)

The effective propagator, \( S^*(p_0, p_\perp, p_z) \), in (29) transforms under reflection as

\[
S^*(p_0, p_\perp, -p_z) = \frac{\hat{L}(p_0, \vec{p})}{k(p_0, p_\perp, -p_z)} P_+ + \frac{\hat{R}(p_0, p_\perp, -p_z)}{p_0^2(p_0, p_\perp, p_z)} P_-
\]

\[
= \frac{P_- A(p_0, |\vec{p}|)(p_0^{\gamma_0} + \vec{p} \cdot \vec{\gamma}) + B_-(p_0, p_\perp, p_z) \gamma_0 - c'(p_0, |\vec{p}|) \gamma^3}{p_0^2(p_0, p_\perp, p_z)} P_+ + \frac{P_+ A(p_0, |\vec{p}|)(p_0^{\gamma_0} + \vec{p} \cdot \vec{\gamma}) + B_+(p_0, p_\perp, p_z) \gamma_0 + c'(p_0, |\vec{p}|) \gamma^3}{p_0^2(p_0, p_\perp, p_z)} P_-
\]

\[
\neq S^*(p_0, p_\perp, p_z).
\] \hspace{1cm} (42)

However, now considering the rest frame of the heat bath, \( u^\mu = (1, 0, 0, 0) \), and the background magnetic field along \( z \)-direction, \( n^\mu = (0, 0, 0, 1) \), one can write (42) as

\[
S^*(p_0, p_\perp, -p_z) = \frac{P_- A(p_0, |\vec{p}|)(p_0^{\gamma_0} + \vec{p} \cdot \vec{\gamma}) + B_-(p_0, p_\perp, p_z) \gamma_0 - c'(p_0, |\vec{p}|) \gamma^3}{p_0^2(p_0, p_\perp, p_z)} P_+ + \frac{P_+ A(p_0, |\vec{p}|)(p_0^{\gamma_0} + \vec{p} \cdot \vec{\gamma}) + B_+(p_0, p_\perp, p_z) \gamma_0 + c'(p_0, |\vec{p}|) \gamma^3}{p_0^2(p_0, p_\perp, p_z)} P_-
\]

\[
\neq S^*(p_0, p_\perp, p_z).
\] \hspace{1cm} (43)

As seen in both cases the reflection symmetry is violated as we will see later while discussing the dispersion property of a fermion.

3. Parity

Under parity a fermion propagator transforms \([71]\)

\( S(p_0, \vec{p}) \longrightarrow \gamma_0 S(p_0, -\vec{p}) \gamma_0. \) \hspace{1cm} (44)

The effective propagator, \( S^*(p_0, p_\perp, p_z) \), in (29) under parity transforms as

\[
\gamma_0 S^*(p_0, p_\perp, -p_z) \gamma_0 = \gamma_0 P_- \frac{\hat{L}(p_0, \vec{p})}{k(p_0, p_\perp, -p_z)} P_+ \gamma_0 + \gamma_0 P_+ \frac{\hat{R}(p_0, p_\perp, -p_z)}{p_0^2(p_0, p_\perp, p_z)} P_- \gamma_0
\]

\[
= P_- \gamma_0 \frac{\hat{L}(p_0, \vec{p})}{k(p_0, p_\perp, -p_z)} \gamma_0 P_+ + P_+ \gamma_0 \frac{\hat{R}(p_0, p_\perp, -p_z)}{p_0^2(p_0, p_\perp, p_z)} \gamma_0 P_-
\]

\[
\neq S^*(p_0, p_\perp, p_z).
\] \hspace{1cm} (45)

which does not obey (44), indicating that the effective propagator in general frame of reference is not parity invariant due to the background medium.

However, now considering the rest frame of the heat bath, \( u^\mu = (1, 0, 0, 0) \), and the background magnetic field along \( z \)-direction, \( n^\mu = (0, 0, 0, 1) \), one can write (45) by using (37a), (37b) and \( \gamma_0 \gamma^I = -\gamma^I \gamma_0 \) as

\[
\gamma_0 S^*(p_0, p_\perp, -p_z) \gamma_0 = P_+ \frac{\hat{R}(p_0, p_\perp, p_z)}{k(p_0, p_\perp, p_z)} \gamma_0 P_- + P_- \frac{\hat{R}(p_0, p_\perp, p_z)}{k(p_0, p_\perp, p_z)} P_+ \gamma_0
\]

\[
= S^*(p_0, p_\perp, p_z),
\] \hspace{1cm} (46)

which indicates that the propagator is parity invariant in the rest frame of the magnetised heat bath. We note that other discrete symmetries can also be checked but leave them on the readers.
D. Modified Dirac Equation

1. For General Case

The effective propagator that satisfy the modified Dirac equation with spinor \( U \) is given by

\[
\left( \mathcal{P}_+ \mathcal{P}_- + \mathcal{P}_- \mathcal{K} \mathcal{P}_+ \right) U = 0.
\]  
(47)

Using the chiral basis

\[
\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},
\]  
(48)

one can write (47) as

\[
\begin{pmatrix} 0 & \sigma \cdot R \\ \vec{\sigma} \cdot L & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,
\]  
(49)

where \( \psi_R \) and \( \psi_L \) are two component Dirac spinors with \( \sigma \equiv (1, \vec{\sigma}) \) and \( \vec{\sigma} \equiv (1, -\vec{\sigma}) \), respectively. One can obtain nontrivial solutions with the condition

\[
\det \begin{pmatrix} 0 & \sigma \cdot R \\ \vec{\sigma} \cdot L & 0 \end{pmatrix} = 0
\]
\[
\det[\mathcal{L} \cdot \vec{\sigma}] \det[R \cdot \sigma] = 0
\]
\[
L^2 R^2 = 0.
\]  
(50)

We note that for a given \( p_0 (= \omega) \), either \( L^2 = 0 \), or \( R^2 = 0 \), but not both of them are simultaneously zero. This implies that i) when \( L^2 = 0, \psi_R = 0 \); ii) when \( R^2 = 0, \psi_L = 0 \). These dispersion conditions are same as obtained from the poles of the effective propagator in (29) as obtained in subsec. IIIB.

1. For \( R^2 = 0 \) but \( L^2 \neq 0 \), the right chiral equation is given by

\[(R \cdot \sigma) \psi_R = 0.\]  
(51)

Again \( R^2 = 0 \) \( \Rightarrow \) \( R_0 = \pm |\vec{R}| = \pm \sqrt{R_x^2 + R_y^2 + R_z^2} \) and the corresponding dispersive modes are denoted by \( R^{(\pm)} \). So the solutions of (51) are

\[
\begin{align*}
(i) & \quad R_0 = |\vec{R}|; \quad \text{mode } R^{(+)}; \quad U_{R^{(+)}} = \sqrt{\frac{|\vec{R}| + R_z}{2|\vec{R}|}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \psi_R^{(+) \mp 1} \end{pmatrix}, \\
(ii) & \quad R_0 = -|\vec{R}|; \quad \text{mode } R^{(-)}; \quad U_{R^{(-)}} = \sqrt{\frac{|\vec{R}| + R_z}{2|\vec{R}|}} \begin{pmatrix} 0 & 0 \\ 0 & \frac{R_z - i R_y}{|\vec{R}| + R_z} \end{pmatrix} \begin{pmatrix} 0 \\ \psi_R^{(-)} \end{pmatrix}.
\end{align*}
\]  
(52a-b)

2. For \( L^2 = 0 \) but \( R^2 \neq 0 \), the left chiral equation is given by

\[(L \cdot \vec{\sigma}) \psi_L = 0.\]  
(53)

where \( L^2 = 0 \) implies two conditions; \( L_0 = \pm |\vec{L}| = \pm \sqrt{L_x^2 + L_y^2 + L_z^2} \) and the corresponding dispersive modes are
denoted by $L^{(\pm)}$. The two solutions of (53) are obtained as

$$L_0 = |\vec{L}|; \quad \text{mode } L^{(+)}; \quad U_{L^{(+)}} = -\sqrt{\frac{|\vec{L}| + L_z}{2|\vec{L}|}} \begin{pmatrix} L_x - iL_y \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \psi_L^{(+)} \\ 0 \end{pmatrix}, \quad (54a)$$

$$L_0 = -|\vec{L}|; \quad \text{mode } L^{(-)}; \quad U_{L^{(-)}} = \sqrt{\frac{|\vec{L}| + L_z}{2|\vec{L}|}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \psi_L^{(-)} \\ 0 \end{pmatrix}. \quad (54b)$$

We note here that $\psi_L^{(\pm)}$ and $\psi_R^{(\pm)}$ are only chiral eigenstates but neither the spin nor the helicity eigenstates.

2. For lowest Landau level (LLL)

1. For $R_{LLL}^2 = 0$ in (31b) indicates that $R_0 = \pm R_z$, $R_x = R_y = 0$. The two solutions obtained, respectively, in (B5) and (B6) in Appendix B are given as

$$R_0 = R_z; \quad \text{mode } R^{(+)}; \quad U_{R^{(+)}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix}. \quad (55a)$$

$$R_0 = -R_z; \quad \text{mode } R^{(-)}; \quad U_{R^{(-)}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_- \\ 0 \end{pmatrix}, \quad (55b)$$

where $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2. For LLL, $L_{LLL}^2 = 0$ in (31a) indicates that $L_0 = \pm L_z$, $L_x = L_y = 0$. The two solutions obtained, respectively, in (B7) and (B8) in Appendix B are given as

$$L_0 = L_z; \quad \text{mode } L^{(+)}; \quad U_{L^{(+)}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_- \\ 0 \end{pmatrix}. \quad (56a)$$

$$L_0 = -L_z; \quad \text{mode } L^{(-)}; \quad U_{L^{(-)}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix}. \quad (56b)$$

The spin operator along the $z$ direction is given by

$$\Sigma^3 = \sigma^1 \sigma^2 = \frac{i}{2} \begin{pmatrix} \sigma^1 \sigma^2 \\ \sigma^2 \sigma^1 \end{pmatrix} = i \sigma^1 \sigma^2 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \quad (57)$$

where $\sigma$ with single index denotes Pauli spin matrices whereas that with double indices denote generator of Lorentz group in spinor representation. Now,

$$\Sigma^3 U_{R^{(\pm)}} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \begin{pmatrix} \chi_{\pm} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_{\pm} \end{pmatrix} = \pm \begin{pmatrix} \chi_{\pm} \\ 0 \end{pmatrix} = \pm U_{R^{(\pm)}}, \quad (58)$$

$$\Sigma^3 U_{L^{(\pm)}} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \begin{pmatrix} 0 \\ \chi_{\pm} \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_{\pm} \end{pmatrix} = \mp \begin{pmatrix} 0 \\ \chi_{\pm} \end{pmatrix} = \mp U_{L^{(\pm)}}. \quad (59)$$
So, the modes $L^{-}$ and $R^{+}$ have spins along the direction of magnetic field whereas $L^{+}$ and $R^{-}$ have spins opposite to the direction of magnetic field. Now we discuss the helicity eigenstates of the various modes in LLL. The helicity operator is defined as

$$\mathcal{H}_p = \hat{p} \cdot \vec{\Sigma}. \tag{60}$$

When a particle moves along $+z$ direction, $\hat{p} = \hat{z}$ and when it moves along $-z$ direction, $\hat{p} = -\hat{z}$.

Thus

$$\mathcal{H}_p \ U_R^{(\pm)} = \begin{cases} \pm U_R^{(\pm)}, & \text{for } p_z > 0, \\ \mp U_R^{(\pm)}, & \text{for } p_z < 0. \end{cases} \tag{62}$$

and

$$\mathcal{H}_p \ U_L^{(\pm)} = \begin{cases} \mp U_L^{(\pm)}, & \text{for } p_z > 0, \\ \pm U_L^{(\pm)}, & \text{for } p_z < 0. \end{cases} \tag{63}$$

E. Dispersion

FIG. 2: Dispersion plots for higher Landau level, $l \neq 0$. The energy $\omega$ is scaled with the thermal mass $m_{th}$ for convenience.

In presence of magnetic field, the component of momentum transverse to the magnetic field is Landau quantised and takes discrete values given by $p_z^2 = 2l|q_f B|$, where $l$ is a given Landau levels. In presence of pure background magnetic field and no heat bath ($T = 0$), the Dirac equation gives rise a dispersion relation

$$E^2 = p_z^2 + m_f^2 + (2\nu + 1)q_f |Q| B - q_f Q B \sigma. \tag{64}$$

where $\nu = 0, 1, 2, \ldots$, $Q = \pm 1$, $\sigma = +1$ for spin up and $\sigma = -1$ for spin down. The solutions are classified by energy eigenvalues

$$E_i^2 = p_z^2 + m_f^2 + 2l |q_f B|. \tag{65}$$
where one can define

\[ 2l = (2\nu + 1)|Q| - Q \sigma. \]

Now we discuss the dispersion properties of a fermions in a hot magnetised medium. For general case (for higher LLs, \( l \neq 0 \)) the dispersion curves obtained by solving, \( L^2 = 0 \) and \( R^2 = 0 \) given in (30a) and (30b), numerically. We note that the roots of \( L_0 = \pm |\vec{L}| \Rightarrow L_0 \mp |\vec{L}| = 0 \) are represented by \( L^{(\pm)} \) with energy \( \omega_{L^{(\pm)}} \) whereas those for \( R_0 = \pm |\vec{R}| \Rightarrow R_0 \mp |\vec{R}| = 0 \) by \( R^{(\pm)} \) with energy \( \omega_{R^{(\pm)}} \). The corresponding eigenstates are obtained in (34a), (34b), (32a) and (32b) in subsection III D1. We have chosen \( T = 0.2 \text{ GeV}, \alpha_s = 0.3 \) and \( q_f B = 0.5 m_{\pi} \), where \( m_{\pi} \) is the pion mass. In Fig. 2 the dispersion curves for higher Landau levels are shown where all four modes can propagate for a given choice of \( Q \). This is because the corresponding states for these modes are neither spin nor helicity eigenstates as shown in subsec. III D1. We also note that there will be negative energy modes which are not displayed here but would be discussed in the analysis of the spectral representation of the effective propagator section V.

At LLL \( l = 0 \rightarrow p_z = 0 \) and the roots of \( R_0 = \pm R_\perp \) give rise to two right handed modes \( R^{(\pm)} \) with energy \( \omega_{R^{(\pm)}} \) whereas those for \( L_0 = \pm L_\perp \) produce 4 two left handed modes \( L^{(\pm)} \) with energy \( \omega_{L^{(\pm)}} \). In Appendix D the analytic solutions for the dispersion relations in LLL are presented which show four different modes and the corresponding eigenstates are obtained in subsec. III D2. Now at LLL we discuss two possibilities below:

(i) for positively charged fermion \( Q = 1, \sigma = 1 \) implies \( \nu = 0 \) and \( \sigma = -1 \) implies \( \nu = -1 \). Now we note that \( \nu \) can never be negative. This implies that the modes with \( Q = 1 \) and \( \sigma = -1 \) (spin down) cannot propagate in LLL. Now, the right handed mode \( R^{(+)} \) and the left handed mode \( L^{(-)} \) have spin up as shown in subsec. III D2, will propagate in LLL for \( p_z > 0 \). The \( R^{(+)} \) mode has helicity to chirality ratio \(+1\) is a quasiparticle whereas the mode \( L^{(-)} \) left handed has that of \(-1\) known as plasmino (hole). However, for \( p_z < 0 \), the right handed mode flips to plasmino (hole) as its chirality to helicity ratio becomes -1 whereas the left handed mode becomes particle as its chirality to helicity ratio becomes +1. The dispersion behaviour of the two modes are shown in the left panel of Fig. 3 which begins at mass \( m_{LLL}^{(\pm)} \) as given in (D13).

(ii) for negatively charged fermion \( Q = -1, \sigma = 1 \) implies \( \nu = -1 \) and \( \sigma = -1 \) implies \( \nu = 0 \). Thus, the modes with \( Q = -1 \) and \( \sigma = +1 \) (spin up) cannot propagate in LLL. However, the modes \( L^{(+)} \) and \( R^{(-)} \) have spin down as found in subsec. III D2 will propagate in LLL. Their dispersion are shown in the right panel of Fig. 3 which begin at mass \( m_{LLL}^{(\pm)} \) as given in (D13). For \( p_z > 0 \) the mode \( L^{(+)} \) has helicity to chirality ratio +1 whereas \( R^{(-)} \) has that of -1 and vice-versa for \( p_z < 0 \).

---

4 We make a general note here for left handed modes at LLL. At small \( p_z \), \( L_\perp \) itself is negative for LLL and becomes positive after a moderate value of \( p_z \). This makes the left handed modes \( L^{(+)} \) and \( L^{(-)} \) to flip in LLL than those in higher Landau levels. For details see Appendix D.
In the absence of the background magnetic field ($B = 0$), the two modes, the left handed $L^{(+)}$ and the right handed $R^{(+)}$ fermions, merge together whereas the other two modes, the left handed $L^{(-)}$ and the right handed $R^{(-)}$ fermions, also merge together. This leads to degenerate (chirally symmetric) modes for which the dispersion plots start at $m_{th}$ and one gets back the usual HTL result [49] with quasiparticle and plasmino modes in presence of heat bath as shown in Fig. 4.

As evident from the dispersion plots (Figs. 2 and 3) both left and right handed modes are also degenerate at $p_z = 0$ in presence of magnetic field but at non-zero $|p_z|$ both left and right handed modes get separated from each others, causing a chiral asymmetry without disturbing the chiral invariance (subsec. III C 1) in the system. Also in subsec. III C 2 it was shown that the fermion propagator does not obey the reflection symmetry in presence of medium, which is now clearly evident from all dispersion plots as displayed above.

### IV. Three Point Function

The $(N + 1)$-point functions are related to the $N$-point functions through Ward-Takahashi (WT) identity. The 3-point function is related to the 2-point function as

$$ Q = S^{-1}(P) - S^{-1}(K) = \mathbf{P} - \mathbf{K} - \Sigma(P) + \Sigma(K) $$

Free

$$ = \mathbf{P} - \mathbf{K} - (\Sigma^{B=0}(P, T) - \Sigma^{B=0}(K, T)) $$

Thermal or HTL correction

Thermo-magnetic correction

$$ = Q + a(p_0, |\mathbf{p}|) \mathbf{P} + b(p_0, |\mathbf{p}|) \mathbf{K} - a(k_0, |\mathbf{k}|) \mathbf{P} - b(k_0, |\mathbf{k}|) \mathbf{K} + \mathbf{b}'(p_0, p_\perp, p_z) \gamma_5 \mathbf{P} + \mathbf{c}'(p_0, p_\perp, p_z) \gamma_5 \mathbf{K} - \mathbf{b}'(k_0, k_\perp, k_z) \gamma_5 \mathbf{P} - \mathbf{c}'(k_0, k_\perp, k_z) \gamma_5 \mathbf{K} $$

(67)

where $Q = P - K$. We note that recently the general form of the thermo-magnetic corrections for 3-point [66, 67] and 4-point [67] functions have been given in terms of the involved angular integrals, which satisfy WT identities. Nevertheless, to validate the general structure of the self-energy in (16) vis-a-vis the inverse propagator in (25), we obtain below the temporal component of the 3-point function at $q_0 = 0, \, \mathbf{p} = \mathbf{k}$ and $p = k$. 

FIG. 4: The dispersion plots corresponding to HTL propagator in absence of magnetic field, i.e., $B = 0$. 

Using (23a), (23b), (23c) and (23d), we can obtain

$$\Gamma^0(P, K; Q)\bigg|_{\vec{q} = 0} = \gamma_0 - \frac{m^2_{th}}{p q_0} \delta Q_0 \gamma^0 + \frac{m^2_{th}}{p q_0} \delta Q_1 (\hat{p} \cdot \vec{q})$$

Thermal or HTL correction

$$- \frac{M^2}{p q_0} \left[ \delta Q_0 \gamma_5 + \frac{p_z}{p} \delta Q_1 (i \gamma^1 \gamma^2) \right] \gamma^3$$

Thermo-magnetic correction

$$= \gamma^0 + \delta \Gamma^0_{HTL}(P, K; Q) + \delta \Gamma^0_{TM}(P, K; Q),$$

(68)

with

$$\gamma_5 \gamma^0 = -i \gamma^1 \gamma^2 \gamma^3,$$

$$M^2 = 4 C_F g^2 M^2 (T, m, q_f, B),$$

$$\delta Q_j = Q_j \left( \frac{p_0}{p} \right) - Q_j \left( \frac{k_0}{p} \right).$$

(69)

where $Q_j$ are the Legendre functions of the second kind given in (A7a) and (A7b). Important to note that the thermo-magnetic (TM) correction $\delta \Gamma^0_{TM}$ matches exactly with that from direct calculation in (C5) in Appendix C. The result also agrees with the HTL 3-point function [66, 67] in absence of background magnetic field by setting $B = 0 \Rightarrow M' = 0$ as

$$\Gamma^0_{HTL}(P, K; Q)\bigg|_{\vec{q} = 0} = \left[ 1 - \frac{m^2_{th}}{p q_0} \delta Q_0 \right] \gamma^0 + \frac{m^2_{th}}{p q_0} \delta Q_1 (\hat{p} \cdot \vec{q})$$

(70)

where all components, i.e., $(0, 1, 2, 3)$, are relevant for pure thermal background.

Now in absence of heat bath, setting $T = 0 \Rightarrow m_{th} = 0$ and $M'^2 = 4 C_F g^2 M^2 (T = 0, m, q_f, B)$, the temporal 3-point function in (68) reduces to

$$\Gamma^0_B(P, K; Q)\bigg|_{\vec{q} = 0} = \gamma^0 - \frac{M^2}{p q_0} \left[ \delta Q_0 \gamma_5 + \frac{p_z}{p} \delta Q_1 (i \gamma^1 \gamma^2) \right] \gamma^3$$

Pure magnetic correction

$$= \gamma^0 + \delta \Gamma^0_{TM}(P, K; Q).$$

(71)

(72)

We now note that this is the three-point function with pure background magnetic field but no heat bath. The gauge boson is oriented along the field direction and there is no polarisation in the transverse direction. Thus, only the longitudinal components (i.e., $(0, 3)$-components) of the 3-point function would be relevant for pure background magnetic field in contrast to that of (70) for pure thermal background.

V. Spectral Representation of the Effective Propagator

In this section we obtain the spectral representation of the effective propagator in a hot magnetised medium. This quantity is of immense interest for studying the various spectral properties, real and virtual photon production, damping rates and various transport coefficients etc. of the hot magnetised medium, in particular, for hot magnetised QCD medium.

A. General Case

The effective propagator as obtained in (29) is given by

$$S^+ = P - \frac{1}{L^2} P_+ + P + \frac{R}{R^2} P_-, \quad (73)$$
where $\hat{L}$ and $\hat{R}$ can be written in the rest frame of the heat bath and the magnetic field in the $z$-direction following (27a) and (27b), respectively, as

\[
\hat{L} = [(1 + (a(p_0, p))p_0 + b(p_0, p) + b'(p_0, p_\perp, p_2)] \gamma^0 - [(1 + a(p_0, p))p_2 + c'(p_0, p_\perp, p_2)] \gamma^3
\]
\[
-(1 + a(p_0, p))(\gamma \cdot p_\perp)
\]
\[
= [(1 + a(p_0, p))p_0 + b(p_0, p) + b'(p_0, p_\perp, p_2)] \gamma^0 - [p(1 + a(p_0, p)) (\gamma \cdot \hat{p}) - c'(p_0, p_\perp, p_2)] \gamma^3
\]
\[
= g_L^0(p_0, p_\perp, p_2) \gamma^0 - g_L^3(p_0, p_\perp, p_2)(\gamma \cdot \hat{p}) - g_L^3(p_0, p_\perp, p_2) \gamma^3,
\]

(74)

\[
\hat{R} = [(1 + a(p_0, p))p_0 + b(p_0, p) - b'(p_0, p_\perp, p_2)] \gamma^0 - [(1 + a(p_0, p))p_2 - c'(p_0, p_\perp, p_2)] \gamma^3
\]
\[
-(1 + a(p_0, p))(\gamma \cdot p_\perp)
\]
\[
= [(1 + a(p_0, p))p_0 + b(p_0, p) - b'(p_0, p_\perp, p_2)] \gamma^0 - [p(1 + a(p_0, p))(\gamma \cdot \hat{p}) + c'(p_0, p_\perp, p_2)] \gamma^3
\]
\[
= g_R^0(p_0, p_\perp, p_2) \gamma^0 - g_R^3(p_0, p_\perp, p_2)(\gamma \cdot \hat{p}) + g_R^3(p_0, p_\perp, p_2) \gamma^3,
\]

(75)

where $\hat{p} = p/|p|$, $p = |p|$ and, $p_z$ and $p_\perp$ are given, respectively, in (22b) and (22c). We also note that though $g_L^2 = g_R^2$, $g_L^3 = g_R^3$, but they are treated separately for the sake of notations that we would be using, for convenience, as $g_L^i$ and $g_R^i$. One can decompose the effective propagator into six parts by separating out the $\gamma$ matrices as

\[
S^* = P_+ \gamma^0 P_+ \frac{g_L^i(p_0, p_\perp, p_2)}{L^2} - P_- \gamma^0 P_+ \frac{g_L^0(p_0, p_\perp, p_2)}{L^2} - P_- \gamma^3 P_+ \frac{g_L^3(p_0, p_\perp, p_2)}{L^2} - P_- \gamma^3 P_+ \frac{g_L^3(p_0, p_\perp, p_2)}{L^2}
\]
\[
+ P_+ \gamma^0 P_- \frac{g_R^i(p_0, p_\perp, p_2)}{R^2} - P_- \gamma^0 P_- \frac{g_R^0(p_0, p_\perp, p_2)}{R^2} - P_- \gamma^3 P_- \frac{g_R^3(p_0, p_\perp, p_2)}{R^2} + P_+ \gamma^3 P_- \frac{g_R^3(p_0, p_\perp, p_2)}{R^2}.
\]

(76)

In subsection III E we have discussed that $L^2 = 0$ yields four poles, leading to four modes with both positive and negative energy as $\pm \omega_{L_{(+)}}(p_\perp, p_2)$ and $\pm \omega_{L_{(-)}}(p_\perp, p_2)$. Similarly, $R^2 = 0$ also yields four poles, namely $\pm \omega_{R_{(+)}}(p_\perp, p_2)$ and $\pm \omega_{R_{(-)}}(p_\perp, p_2)$.

With this information one can obtain the spectral representation [49, 72–74] of the effective propagator in (76) as

\[
\rho = (P_- \gamma^0 P_+) \rho_L^1 - (P_- \gamma^0 P_+) \rho_L^3 + (P_- \gamma^3 P_+) \rho_L^3 + (P_- \gamma^3 P_-) \rho_R^3 + (P_+ \gamma^3 P_-) \rho_R^3.
\]

(77)

where the spectral function corresponding to each of the term can be written as

\[
\rho_L^i = \frac{1}{\pi} \text{Im} \left( \frac{g_L^i}{L^2} \right)
\]
\[
= Z_{L_{(+)}}^i(p_\perp, p_2) \delta(p_0 - \omega_{L_{(+)}}(p_\perp, p_2)) + Z_{L_{(-)}}^i(p_\perp, p_2) \delta(p_0 + \omega_{L_{(-)}}(p_\perp, p_2))
\]
\[
+ Z_{L_{(+)}}^i(p_\perp, p_2) \delta(p_0 - \omega_{L_{(+)}}(p_\perp, p_2)) + Z_{L_{(-)}}^i(p_\perp, p_2) \delta(p_0 + \omega_{L_{(-)}}(p_\perp, p_2)) + \beta_{L}^i,
\]

(78)

\[
\rho_R^i = \frac{1}{\pi} \text{Im} \left( \frac{g_R^i}{R^2} \right)
\]
\[
= Z_{R_{(+)}}^i(p_\perp, p_2) \delta(p_0 - \omega_{R_{(+)}}(p_\perp, p_2)) + Z_{R_{(-)}}^i(p_\perp, p_2) \delta(p_0 + \omega_{R_{(-)}}(p_\perp, p_2))
\]
\[
+ Z_{R_{(+)}}^i(p_\perp, p_2) \delta(p_0 - \omega_{R_{(+)}}(p_\perp, p_2)) + Z_{R_{(-)}}^i(p_\perp, p_2) \delta(p_0 + \omega_{R_{(-)}}(p_\perp, p_2)) + \beta_{R}^i,
\]

(79)

where $i = 1, 2, 3$. We note that the delta-functions are associated with pole parts originating from the time like domain ($p_0^2 > p^2$) whereas the cut parts $\beta_{L(R)}^i$ are associated with the Landau damping arises from the space-like domain, $p_0^2 < p^2$, of the propagator.

The residues $Z_{L(R)}^i$ are determined at the various poles as

\[
Z_{L(R)}^i \text{ sgn of pole } (p_\perp, p_2) = g_L^i(p_0, p) \left. \frac{\partial L^2(R^2)}{\partial p_0} \right|_{p_0 = \text{pole}}^{-1}
\]

(80)

As a demonstration, we present analytical expressions of three residues corresponding to the pole $p_0 = +\omega_{L_{(+)}}$ as

\[
Z_{L_{(+)}}^{1+} = \frac{p}{2} \left( \omega_{L_{(+)}}^2 + M^2 \right) \left[ m_{th}^2 \log \left( \frac{\omega_{L_{(+)}} + p}{\omega_{L_{(+)}} - p} \right) - 2p \omega_{L_{(+)}} + M^2 p_2 \left( 2p - \omega_{L_{(+)}} \log \left( \frac{\omega_{L_{(+)}} + p}{\omega_{L_{(+)}} - p} \right) \right) \right]
\]
\[
\times \left[ m_{th}^2 8p^4 \left( \omega_{L_{(+)}} + M^2 / m_{th}^2 p_2 \right) + \log \left( \frac{\omega_{L_{(+)}} + p}{\omega_{L_{(+)}} - p} \right) \right].
\]
\[
Z_{L(+)}^{2+} = p^2 \left( p^2 - \omega_{L(+)}^2 \right) \left[ 2p \left( m_{th}^2 + p^2 \right) - m_{th}^2 \alpha_{L(+)} \frac{\partial}{\partial \alpha_{L(+)}} \log \left( \frac{\alpha_{L(+)} + \beta}{\alpha_{L(+)} - \beta} \right) \right],
\]
\[
m_{th}^2 \left[ 8p^4 \left( \omega_{L(+)} + M^2 / m_{th}^2 p_z \right) + \log \left( \frac{\alpha_{L(+)} + \beta}{\alpha_{L(+)} - \beta} \right) \right],
\]
\[
Z_{L(+)}^{3+} = m_{th}^4 \left[ 8p^4 \left( \omega_{L(+)} + M^2 / m_{th}^2 p_z \right) + \log \left( \frac{\alpha_{L(+)} + \beta}{\alpha_{L(+)} - \beta} \right) \right].
\]

where \( X = 2p^3(M^2 - 2m_{th}^2) + 2M^2 \beta_{p_z}^2 + M^2 \omega_{L(+)} \beta_{p_z}^2 \log \left( \frac{\alpha_{L(+)} + \beta}{\alpha_{L(+)} - \beta} \right). \) The other poles of \( L^2 = 0 \) can trivially be found out by replacing \( \alpha_{L(+)} \) in the above expressions. The expressions for the residues for \( R \) parts can similarly be expressed as the \( L \) parts, but we do not show them.

Below in Fig. 5 we present the residues corresponding to the first Landau level where all the terms are present. We take the value of the magnetic field as \( m_z^2 / 2 \) and temperature to be 200 MeV.

FIG. 5: Different Residues for the first LL \((l = 1)\) are plotted with scaled momentum along the magnetic field direction.

Now, the expressions for the cut parts \( \beta_{L,R} \) are given below:

\[
\beta_L^i = \frac{1}{\pi} \Theta(p^2 - p_0^2) \frac{\text{Im}(g_L^i) \text{Re}(L^2) - \text{Im}(L^2) \text{Re}(g_L^i)}{(\text{Re}(L^2))^2 + (\text{Im}(L^2))^2},
\]
\[
\beta_R^i = \frac{1}{\pi} \Theta(p^2 - p_0^2) \frac{\text{Im}(g_R^i) \text{Re}(R^2) - \text{Im}(R^2) \text{Re}(g_R^i)}{(\text{Re}(R^2))^2 + (\text{Im}(R^2))^2}.
\]
where

\[ \text{Re}(g^1_L) = p_0 - M'^2 p_z \frac{p^2}{p^2} - \frac{m_{th}^2}{p} \left( 1 - \frac{M'^2 p_z p_0}{m_{th}^2} \right) Q_0 \left( \left| \frac{p_0}{p} \right| \right), \]  
\[ \text{Im}(g^1_L) = \frac{\pi m_{th}^2}{2} \left( 1 - \frac{M'^2 p_z p_0}{m_{th}^2} \right), \]  
\[ \text{Re}(g^2_L) = \text{Re}(g^2_R) = p + \frac{m_{th}^2}{p} \left[ 1 - \frac{p_0}{p} Q_0 \left( \left| \frac{p_0}{p} \right| \right) \right], \]  
\[ \text{Im}(g^2_L) = \text{Im}(g^2_R) = \pi m_{th}^2 \frac{p_0}{2 p^2}, \]  
\[ \text{Re}(g^3_L) = \text{Re}(g^3_R) = \frac{M'^2}{p} Q_0 \left( \left| \frac{p_0}{p} \right| \right), \]  
\[ \text{Im}(g^3_L) = \text{Im}(g^3_R) = -\pi M'^2 \frac{p}{2p}, \]  

and

\[ \text{Re}(g^4_L) = p_0 + M'^2 p_z \frac{p^2}{p^2} - \frac{m_{th}^2}{p} \left( 1 + \frac{M'^2 p_z p_0}{m_{th}^2} \right) Q_0 \left( \left| \frac{p_0}{p} \right| \right), \]  
\[ \text{Im}(g^4_L) = \frac{\pi m_{th}^2}{2} \left( 1 + \frac{M'^2 p_z p_0}{m_{th}^2} \right). \]

Also we obtain

\[ \text{Re}(L^2) = A_L + B_L Q_0 \left( \left| \frac{p_0}{p} \right| \right) + C \left( Q^2_0 \left( \left| \frac{p_0}{p} \right| \right) - \frac{\pi^2}{4} \right), \]  
\[ \text{Im}(L^2) = -\frac{\pi B_L}{2} - \pi Q_0 \left( \left| \frac{p_0}{p} \right| \right) C, \]  
\[ \text{Re}(R^2) = A_R + B_R Q_0 \left( \left| \frac{p_0}{p} \right| \right) + C \left( Q^2_0 \left( \left| \frac{p_0}{p} \right| \right) - \frac{\pi^2}{4} \right), \]  
\[ \text{Im}(R^2) = -\frac{\pi B_R}{2} - \pi Q_0 \left( \left| \frac{p_0}{p} \right| \right) C, \]

where

\[ A_L = p_0^2 - p^2 - 2m_{th}^2 - \frac{m_{th}^4}{p^2} - \frac{2M'^2 p_0 p_z}{p^2} + \frac{M'^4 p_0^2}{p^4}, \]  
\[ A_R = p_0^2 - p^2 - 2m_{th}^2 - \frac{m_{th}^4}{p^2} + \frac{2M'^2 p_0 p_z}{p^2} + \frac{M'^4 p_0^2}{p^4}, \]  
\[ B_L = \frac{2m_{th}^4 p_0}{p^3} - \frac{2M'^2 p_z}{p} + \frac{2M'^2 p_0 p_z}{p^3} - \frac{2M'^4 p_0^2}{p^5}, \]  
\[ B_R = \frac{2m_{th}^4 p_0}{p^3} + \frac{2M'^2 p_z}{p} - \frac{2M'^2 p_0 p_z}{p^3} - \frac{2M'^4 p_0^2}{p^5}, \]  
\[ C = \frac{m_{th}^4 - M'^4}{p^2} - \frac{p_0^2 m_{th}^4}{p^4} + \frac{M'^4 p_0^2 p_z^2}{p^6}. \]
B. LLL Case

For LLL, as \( p_\perp = 0 \), so \( g_{L(R)}^2 \) and \( g_{L(R)}^3 \) in (74) and (75) can now be merged as

\[
\begin{align*}
g_{L}^{2+3} &= \left( 1 + a(p_0, p) \right) p_z + c'(p_0, p) \gamma^3, \\
g_{R}^{2+3} &= \left( 1 + a(p_0, p) \right) p_z - c'(p_0, p) \gamma^3.
\end{align*}
\]

FIG. 6: Different Residues for the LLL (\( l = 0 \)) are plotted with scaled momentum along the magnetic field direction.

The spectral function corresponding to LLL reads as

\[
\begin{align*}
\rho_{LLL} &= \left( P_- \gamma^0 P_+ \right) \rho_L^1 - \left( P_- \gamma^3 P_+ \right) \rho_L^{2+3} \\
&\quad + \left( P_+ \gamma^0 P_- \right) \rho_R^1 - \left( P_+ \gamma^3 P_- \right) \rho_R^{2+3},
\end{align*}
\]

where one needs to determine

\[
\rho_{L(R)}^{2+3} = \frac{1}{\pi} \text{Im} \left( \frac{g_{L(R)}^{2+3}}{L^2(R^2)} \right),
\]

which can again be represented in terms of different residues corresponding to different poles of \( L^2(R^2) = 0 \) as in Eq.(79). In Fig. 6, the variation of the residues for the lowest Landau level are shown.

In appendix E we have demonstrated how one gets back the HTL spectral functions when magnetic field is withdrawn from the thermal medium.
VI. Conclusions

In this article the general structure of fermionic self-energy for a chirally invariant theory has been formulated for a hot and magnetised medium. Using this we have obtained a closed form of the general structure of the effective fermion propagator. The collective excitations in such a nontrivial background have been obtained for a time-like momenta in the weak field and HTL approximation in the domain \( m_f^2 \sim g^2 T^2 < |q_f B| < T^2 \). We found that the left and right handed modes get separated and become asymmetric in presence of magnetic field which were degenerate and symmetric otherwise. The transformation of the effective propagator in a hot magnetised medium under some of the discrete symmetries have been studied and its consequences are also reflected in the collective fermion modes in the Landau levels. We have also obtained the Dirac spinors of the various collective modes by solving the Dirac equation with the effective two-point function. Further, we checked the general structure of the two-point function by obtaining the three-point function using the Ward-Takahashi identity, which agrees with the direct calculation of one-loop order in weak field approximation. We also found that only the longitudinal component of the vertex would be relevant when there is only background magnetic field. The spectral function corresponding to the effective propagator is explicitly obtained for a hot magnetised medium which will be extremely useful for studying the spectral properties, e.g., photon/dilepton production, damping rate, transport coefficients for a hot magnetised medium. This has pole contribution due to the various collective modes originating from the time-like domain and a Landau cut contribution appearing from the space-like domain. It has explicitly been shown that the spectral function reduces to that obtained for thermal medium in absence of the magnetic field. Our formulation is in general applicable to both QED and QCD with nontrivial background like hot magnetised medium.

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A. Computations of structure functions in one-loop in a weak field approximation for hot magnetised QCD medium:

Here, we present the computations of the various structure functions in (21a) to (21d) in 1-loop order (Fig.7) in a weak field and HTL approximations following the imaginary time formalism. In Fig.7 the modified quark propagator (bold line) due to background magnetic field is given in (A3). Since gluons are chargeless, their propagators do not change in presence of magnetic field. The gluon propagator in Feynman gauge, is given as\(^{[41]}\)

\[
D_{ab}^{\mu \nu}(Q) = -i\delta_{ab}\frac{g^{\mu \nu}}{Q^2}.
\]

We note that we would like to explore the fermion spectrum in a hot magnetised background in the limit \( m_f^2 < q_f B < T^2 \). In this domain the fermion propagator is obtained by expanding the sum over all Landau levels in powers of \( q_f B \) in (7) and keeping upto \( \mathcal{O}((q_f B)^2) \), it reads as

\[
S(K) = \frac{\tilde{\mathcal{K}} + m_f}{K^2 - m_f^2} + i\gamma_5 \frac{\tilde{\mathcal{K}}_s + m_f}{(K^2 - m_f^2)^{\eta}} q_f B + 2 \left[ \frac{\mathcal{K} + m_f}{K^2 - m_f^2} \tilde{\mathcal{K}} - \frac{K^2 - m_f^2}{(K^2 - m_f^2)^{\eta}} \tilde{\mathcal{K}}_s \right] (q_f B)\]

\[
= \frac{\tilde{\mathcal{K}} + m_f}{K^2 - m_f^2} + i\gamma_5 \frac{\tilde{\mathcal{K}}_s + m_f}{(K^2 - m_f^2)^{\eta}} q_f B + \mathcal{O}((q_f B)^2). \tag{A2}
\]
The first term is the free propagator and the second one is $\mathcal{O}(q_f B)$ correction to it. Now combining (A2) and (11) the fermion propagator in background magnetic field reads as

$$S(K) = i \frac{\slashed{k}}{K^2 - m_f^2} - \frac{\gamma_5}{(K^2 - m_f^2)^2} [q_f B] + \mathcal{O}(q_f B^2)$$

$$= S_1^{B=0}(K) + S_2^{B=0}(K) + \mathcal{O}((q_f B)^2), \quad (A3)$$

where the fermion mass in the numerator has been neglected in the weak field domain, $m_f^2 < (q_f B) < T^2$.

The one-loop quark self-energy up to $\mathcal{O}(q_f B)$ can be written as

$$\Sigma(P) \equiv g^2 C_F T \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu \left( \frac{\slashed{k}}{K^2 - m_f^2} - \frac{\gamma_5}{(K^2 - m_f^2)^2} q_f B \right) \gamma^\mu \frac{1}{(P - K)^2}$$

$$\approx \Sigma^{B=0}(P, T) + \Sigma^{B\neq0}(P, T) \equiv \Sigma^0 + \Sigma^B. \quad (A4)$$

where $g$ is the QCD coupling constant, $C_F = 4/3$ is the Casimir invariant of $SU(3)$ group, $T$ is the temperature of the system. The first term is the thermal bath contribution in absence of magnetic field ($B = 0$) whereas the second one is from the magnetised thermal bath.

Using (A4) in (21a) and (21b), the structure functions $a$ and $b$, respectively, become

$$a(p_0, |\vec{p}|) = \frac{1}{4} \left( \frac{1}{(p_0)^2} - \frac{1}{(p_0)^2 - (P_u)^2} \right), \quad (A5a)$$

$$b(p_0, |\vec{p}|) = \frac{1}{4} \left( \frac{1}{(p_0)^2} - \frac{1}{(p_0)^2 - (P_u)^2} \right), \quad (A5b)$$

where the contributions coming from $\Sigma^B$ vanish due to the trace of odd number of $\gamma$-matrices. Following the well known results in Ref. [69], one can write

$$a(p_0, |\vec{p}|) = -\frac{m_{th}^2}{p_0} \left( 1 - \frac{Q_0}{p_0} \right), \quad (A6a)$$

$$b(p_0, |\vec{p}|) = \frac{m_{th}^2}{|\vec{p}|} \left( 1 - \frac{Q_0}{p_0} \right), \quad (A6b)$$

where the Legendre functions of the second kind read as

$$Q_0(x) = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right), \quad (A7a)$$

$$Q_1(x) = x Q_0(x) - 1 = \frac{x}{2} \ln \left( \frac{x + 1}{x - 1} \right) - 1, \quad (A7b)$$

and the thermal mass [69, 72] of the quark is given as

$$m_{th}^2 = C_F \frac{g^2 T^2}{8}. \quad (A8)$$

The thermal part of the self-energy in (A4) becomes

$$\Sigma^{B=0}(P, T) \equiv \Sigma^0(P, T) = g^2 C_F T \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu \frac{\slashed{k}}{K^2 - m_f^2} \gamma^\mu \frac{1}{(P - K)^2}$$

$$= -a(p_0, |\vec{p}|) \slashed{P} - b(p_0, |\vec{p}|) \slashed{B}. \quad (A9)$$

Again using (A4) in (21c) and (21d), the structure functions $b'$ and $c'$, respectively, become

$$b' = -\frac{1}{4} \left( \frac{1}{(p_0)^2} - \frac{1}{(p_0)^2 - (P_u)^2} \right), \quad (A10)$$

$$c' = \frac{1}{4} \left( \frac{1}{(p_0)^2} - \frac{1}{(p_0)^2 - (P_u)^2} \right), \quad (A11)$$
where the contributions coming from $\Sigma^0$ vanish due to the trace of odd number of $\gamma$-matrices. For computing the above thermo-magnetic structure functions, one needs to use the following two traces:

$$\text{Tr} \left[ \gamma_5 \gamma_{\mu} \gamma_{5} \left[ (K.n)\not{\!\gamma} - (K.u)\not{\!\gamma} \right] \gamma^\mu \right] = 8 (K.n), \quad (A12)$$

$$\text{Tr} \left[ \gamma_5 \gamma_{\mu} \gamma_{5} \left[ (K.n)\not{\!\gamma} - (K.u)\not{\!\gamma} \right] \gamma^\mu \right] = 8 (K.u). \quad (A13)$$

With this one can obtain

$$b' = 2 g^2 C_F T q_f B \sum_{\{K\}}^f (K.n) \Delta_B^2 (K) \Delta_B (P - K), \quad (A14)$$

$$c' = -2 g^2 C_F T q_f B \sum_{\{K\}}^f (K.u) \Delta_B^2 (K) \Delta_B (P - K), \quad (A15)$$

where the boson propagator in Saclay representation is given by

$$\Delta_B (K) = \int_0^\beta \! d\tau e^{k_0 \tau} \Delta_B (\tau, k)$$

and

$$\tilde{\Delta}_B (\tau, k) = \sum_{k_0} e^{-k_0 \tau} \Delta_B (K)$$

$$= \frac{1}{2\omega_k} \left\{ \left[ 1 + n_B (\omega_k) \right] e^{-\omega_k \tau} + n_B (\omega_k) e^{\omega_k \tau} \right\}$$

where the sum is over $k_0 = 2\pi n T$ and $\omega_k^2 = k^2 + m_f^2$. Also the fermion propagator in Saclay representation reads

$$\Delta_F (K) = \int_0^\beta \! d\tau e^{k_0 \tau} \Delta_F (\tau, k)$$

and

$$\tilde{\Delta}_F (\tau, k) = \sum_{k_0} e^{-k_0 \tau} \Delta_F (K)$$

$$= \frac{1}{2\omega_k} \left\{ \left[ 1 - n_F (\omega_k) \right] e^{-\omega_k \tau} - n_F (\omega_k) e^{\omega_k \tau} \right\}$$

where the sum above is over $k_0 = (2n + 1)\pi T$. Now following HTL approximation in presence of magnetic field [66, 68] the (A14) and (A15) are simplified as

$$b' = -4 g^2 C_F M^2 (T, m_f, q_f B) \int \frac{d\Omega}{4\pi} \frac{\hat{K} \cdot n}{P \cdot \hat{K}},$$

$$c' = 4 g^2 C_F M^2 (T, m_f, q_f B) \int \frac{d\Omega}{4\pi} \frac{\hat{K} \cdot u}{P \cdot \hat{K}}.$$

Using the results of the HTL angular integrations [67]

$$\int \frac{d\Omega}{4\pi} \frac{\hat{K} \cdot u}{P \cdot \hat{K}} = \frac{1}{|\vec{p}|} Q_0 \left( \frac{p^0}{|\vec{p}|} \right), \quad (A16)$$

$$\int \frac{d\Omega}{4\pi} \frac{\hat{K} \cdot n}{P \cdot \hat{K}} = -\frac{p^3}{|\vec{p}|^2} Q_1 \left( \frac{p^0}{|\vec{p}|} \right), \quad (A17)$$

the thermo-magnetic structures functions become

$$b' = 4 g^2 C_F M^2 (T, m_f, q_f B) \frac{p^3}{|\vec{p}|^2} Q_1 \left( \frac{p^0}{|\vec{p}|} \right), \quad (A18)$$

$$c' = 4 g^2 C_F M^2 (T, m_f, q_f B) \frac{1}{|\vec{p}|} Q_0 \left( \frac{p^0}{|\vec{p}|} \right). \quad (A19)$$
with the magnetic mass is obtained as

$$M^2(T, m_f, q_f B) = \frac{q_f B}{16\pi^2} \left[ \ln(2) - \frac{T}{m_f} \right].$$ (A20)

We note here that for \( m_f \to 0 \), the magnetic mass diverges but it can be regulated by the thermal mass \( m_{th} \) in (A8) as is done in Refs. [66, 67]. Then the domain of applicability becomes \( m_{th}^2 (\sim g^2 T^2) < q_f B < T^2 \) instead of \( m_f^2 < q_f B < T^2 \).

The thermo-magnetic part of the self-energy in (A4) becomes

$$\Sigma^{B\neq 0}(P, T) \equiv \Sigma^B(P, T) = -g^2 C_F T q_f B \sum_{\mu} \gamma_\mu \left( (K_n)\gamma - (K_u)\gamma \right) \frac{1}{(K^2 - m_f^2)^2} \frac{1}{(P - K)^2}$$

$$= -b'(p_0, |\vec{p}|)\gamma_5 \gamma^\mu - c'(p_0, |\vec{p}|)\gamma_5 \gamma^\mu. \quad (A21)$$

Now combining (A9), (A21) and (A4), the general structure of quark self-energy in hot magnetised QCD becomes

$$\Sigma(p_0, |\vec{p}|) = -a(p_0, |\vec{p}|) P - b(p_0, |\vec{p}|) \mu - \gamma_5 b'(p_0, |\vec{p}|) \mu - \gamma_5 c'(p_0, |\vec{p}|) \mu.$$ (A22)

which agrees quite well with the general structure as discussed in (16) and also with results directly calculated in Refs. [66–68].

### B. Solution of the Modified Dirac equation at Lowest Landau Level (LLL)

At LLL, \( l \to 0 \Rightarrow r = 0 \) and the effective Dirac equation becomes

$$\begin{align*}
\left( p_+ \Lambda + p_- \tilde{\Lambda} \right) U &= 0 \\
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 R_z \\
0 & 0 & 0 & 0 \\
0 & L_0 - L_z & 0 & 0
\end{pmatrix}
\begin{pmatrix}
R_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & L_0 - L_z & 0 & 0
\end{pmatrix}
U &= 0,
\end{align*} \quad (B1)$$

where \( U = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \) with \( \psi_{L(R)} \) are \( 2 \times 1 \) blocks. Now, the condition for the non-trivial solution to exist is given as

$$\det \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
R_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= 0$$

or, \( R_0 = \pm R_z, L_0 = \pm L_z \), \quad (B2)

- Case-I: For \( R_0 = R_z \) one can write (B1) as

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & L_0 + L_z & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi_L^{(1)} \\ \psi_L^{(2)} \\ \psi_R^{(1)} \\ \psi_R^{(2)}
\end{pmatrix}
= 0, \quad (B3)$$

which leads to the following conditions:

$$\begin{align*}
2 R_z \psi_R^{(2)} &= 0, \\
(L_0 + L_z) \psi_L^{(1)} &= 0, \\
(L_0 - L_z) \psi_L^{(2)} &= 0, \\
\psi_R^{(1)} &= \text{Arbitrary}.
\end{align*} \quad (B4)$$

For normalisation, we choose only non-zero component, \( \psi_R^{(1)} = 1 \) which leads to

$$U_R^{(+)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (B5)$$
Now, for $R_0 = -R_z$, similarly one can obtain as
\[
U_R^{(-)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\] (B6)

- Case-II: For $L_0 = L_z$, one gets
\[
U_L^{(+) } = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\] (B7)

whereas for $L_0 = -L_z$, one finds
\[
U_L^{(-)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\] (B8)

C. Verification of the Three Point Function from Direct Calculation

In this appendix we would verify the general structure of the temporal 3-point function as obtained in sec. IV using the general structure of the self-energy.

We begin with the one-loop level 3-point function in a hot magnetised medium in [67] within HTL approximation [48, 75] as
\[
\Gamma(P, K; Q) = \gamma^\mu + \delta\Gamma^\mu_{\text{HTL}}(P, K) + \delta\Gamma^\mu_{\text{TM}}(P, K),
\] (C1)

where the external four-momentum $Q = P - K$. The HTL correction part [49, 74, 75] is given as
\[
\delta\Gamma^\mu_{\text{HTL}}(P, K) = m^2_{\text{th}} G^{\mu\nu} \gamma^\nu = m^2_{\text{th}} \int \frac{d\Omega}{4\pi} \frac{\hat{Y}^\mu \hat{Y}^\nu}{(P \cdot \hat{Y})(K \cdot \hat{Y})} \gamma^\nu = \delta\Gamma^\mu_{\text{HTL}}(-P, -K),
\] (C2)

where $\hat{Y}_\mu = (1, \hat{y})$ is a light like four vector and the thermo-magnetic (TM) correction part [66, 67] is given
\[
\delta\Gamma^\mu_{\text{TM}}(P, K) = 4\gamma^2 S^2 C_F M^2 \int \frac{d\Omega}{4\pi} \frac{1}{(P \cdot \hat{Y})(K \cdot \hat{Y})} [(\hat{Y} \cdot n)\# - (\hat{Y} \cdot u)\#] \hat{Y}^\mu.
\] (C3)

Now, choosing the temporal component of the thermo-magnetic correction part of the 3-point function and external three momentum $\vec{q} = 0$, we get
\[
\delta\Gamma^0_{\text{TM}}(P, K) \bigg|_{\vec{q} = 0} = \gamma_s M^2 \int \frac{d\Omega}{4\pi} \frac{1}{(P \cdot \hat{Y})(K \cdot \hat{Y})} [(\hat{Y} \cdot n)\# - (\hat{Y} \cdot u)\#]
\]
\[
= \gamma_s M^2 \int \frac{d\Omega}{4\pi} \frac{1}{(P \cdot \hat{Y})(K \cdot \hat{Y})} [(\hat{Y} \cdot n)y_0 + (\hat{Y} \cdot u)y^3]
\] (C4)

Along with this following identity:
\[
\left( \frac{1}{K \cdot \hat{Y}} - \frac{1}{P \cdot \hat{Y}} \right) = \frac{Q \cdot \hat{Y}}{(P \cdot \hat{Y})(K \cdot \hat{Y})} = \frac{q_0}{(P \cdot \hat{Y})(K \cdot \hat{Y})},
\]

and , (A16) and (A17), we one finally obtain
\[
\delta\Gamma^0_{\text{TM}}(P, K) \bigg|_{\vec{q} = 0} = \frac{M^2 p_z}{p^2 q_0} \delta Q_1 y_0^3 - \frac{M^2}{pq_0} \delta Q_0 y_3
\]
\[
= - \frac{M^2}{pq_0} \left[ \delta Q_0 y_3 + \frac{p_z}{p} \delta Q_1 (iy^1 y^2) \right] y^3,
\] (C5)

where $\delta Q_n = Q_n \left( \frac{p_0}{p} \right) - Q_n \left( \frac{k_0}{p} \right)$. We note that this expression matches exactly with the expression obtained in (72) from the general structure of fermion self-energy.
D. Analytical Solution of the Dispersion Relations and the Effective Mass in LLL

The dispersion relations at LLL can be written the equations (31a) and (31b) as

\[ L_{LLL}^2 = (A p_0 + B_+)^2 - (A p_z + c')^2 = L_0^2 - L_z^2 = 0, \]  
(D1a)

\[ R_{LLL}^2 = (A p_0 + B_-)^2 - (A p_z - c')^2 = R_0^2 - R_z^2 = 0, \]  
(D1b)
each of which leads to two modes, respectively, as

\[ L_0 = \pm L_z \]
\[ A p_0 + B_+ = \pm (A p_z + c') , \]  
(D2a)

and

\[ R_0 = \pm R_z \]
\[ A p_0 + B_- = \pm (A p_z - c') . \]  
(D3a)

Below we try to get approximate analytical solution of these equations at small and high \( p_z \) limits.

1. Low \( p_z \) limit

In the low \( p_z \) region, one needs to expand \( a(p_0, |p_z|), b(p_0, |p_z|), b'(p_0, 0, p_z), \) and \( c'(p_0, |p_z|) \) defined in (23a), (23b), (23c) and (23d), respectively, which depend on Legendre function of second kind \( Q_0(x) \) and \( Q_1(x) \) as given in equations (A7a) and (A7b), respectively. The Legendre function \( Q_0 \) and structure coefficients are expanded in powers of \( \frac{|p_z|}{p_0} \) as

\[ Q_0 \left( \frac{p_0}{|p_z|} \right) = \frac{|p_z|}{p_0} + \frac{1}{3} \frac{|p_z|^3}{p_0^3} + \frac{1}{5} \frac{|p_z|^5}{p_0^5} + \cdots \]  
(D4)

\[ a(p_0, |p_z|) = -\frac{m_{th}^2}{p_0^2} \left( \frac{1}{3} + \frac{1}{5} \frac{|p_z|^2}{p_0^2} + \cdots \right) , \]  
(D5)

\[ b(p_0, |p_z|) = -2 \frac{m_{th}^2}{p_0} \left( \frac{1}{3} + \frac{1}{15} \frac{|p_z|^2}{p_0^2} + \cdots \right) , \]  
(D6)

\[ b'(p_0, 0, p_z) = 4 g^2 C_F M^2(T, m, qB) p_z \left( \frac{1}{3} \frac{p_0^2}{p_0^2} + \frac{|p_z|^2}{5 p_0^4} + \cdots \right), \]  
(D7)

\[ c'(p_0, |p_z|) = 4 g^2 C_F M^2(T, m, qB) \left( \frac{1}{p_0} + \frac{|p_z|^2}{3 p_0^3} + \cdots \right). \]  
(D8)

Now retaining the terms that are \textit{upto the order} of \( p_z \) in (D5), (D6), (D7), (D8), we obtain the following expressions for the dispersion relation of various modes:

1. \( L_0 = L_z \) leads to a mode \( L^{(+)} \) as

\[ \omega_{L^{(+)}}(p_z) = m^{++}_L + \frac{1}{3} p_z . \]  
(D9)

2. \( L_0 = -L_z \) leads to a mode \( L^{(-)} \) as

\[ \omega_{L^{(-)}}(p_z) = m^{--}_L - \frac{1}{3} p_z . \]  
(D10)

3. \( R_0 = R_z \) leads to a mode \( R^{(+)} \) as

\[ \omega_{R^{(+)}}(p_z) = m^{+-}_L + \frac{1}{3} p_z . \]  
(D11)
4. $R_0 = -R_z$ leads to a mode $R^(-)$ as

$$\omega_{R^-}(p_z) = m_{L,L,L}^{z+} - \frac{1}{3} p_z,$$

where the effective masses of various modes are given as

$$m_{L,L,L}^{z\pm} = \begin{cases} 
\sqrt{m_{th}^2 + 4g^2C_F M^2(T, M, q, B)}, & \text{for } L^{(+) \& R^{-}}, \\
\sqrt{m_{th}^2 - 4g^2C_F M^2(T, M, q, B)}, & \text{for } R^{(+)} \& L^{(-)}. 
\end{cases}$$

2. High $p_z$ limit

We note that $p_z$ can be written as

$$p_z = \begin{cases} 
|p_z|, & \text{for } p_z > 0 \\
-|p_z|, & \text{for } p_z < 0 
\end{cases}$$

In high $p_z$ limit, we obtain

(i)

$$[1 + a(p_0, |p_z|)](p_0 - p_z) + b(p_0, |p_z|) = \begin{cases} 
p_0 - |p_z| - \frac{m_{th}^2}{|p_z|}, & \text{for } p_z > 0 \\
2 |p_z| + \frac{m_{th}^2}{|p_z|} \ln \left( \frac{2 |p_z|}{p_0 - |p_z|} \right), & \text{for } p_z < 0 
\end{cases}$$ \hspace{1cm} (D14)

(ii)

$$[1 + a(p_0, |p_z|)](p_0 + p_z) + b(p_0, |p_z|) = \begin{cases} 
2 |p_z| + \frac{m_{th}^2}{|p_z|} - \frac{m_{th}^2}{|p_z|} \ln \left( \frac{2 |p_z|}{p_0 - |p_z|} \right), & \text{for } p_z > 0 \\
p_0 - |p_z| - \frac{m_{th}^2}{|p_z|}, & \text{for } p_z < 0 
\end{cases}$$ \hspace{1cm} (D15)

(iii)

$$b'(p_0, 0, p_z) + c'(p_0, |p_z|) = \begin{cases} 
\frac{4g^2C_F M^2}{|p_z|} \ln \left( \frac{2 |p_z|}{p_0 - |p_z|} \right) - \frac{4g^2C_F M^2}{|p_z|}, & \text{for } p_z > 0 \\
\frac{4g^2C_F M^2}{|p_z|} \ln \left( \frac{2 |p_z|}{p_0 - |p_z|} \right), & \text{for } p_z < 0 
\end{cases}$$ \hspace{1cm} (D16)

(iv)

$$b'(p_0, 0, p_z) - c'(p_0, |p_z|) = \begin{cases} 
- \frac{4g^2C_F M^2}{|p_z|} \ln \left( \frac{2 |p_z|}{p_0 - |p_z|} \right) + \frac{4g^2C_F M^2}{|p_z|}, & \text{for } p_z > 0 \\
\frac{4g^2C_F M^2}{|p_z|} \ln \left( \frac{2 |p_z|}{p_0 - |p_z|} \right), & \text{for } p_z < 0 
\end{cases}$$ \hspace{1cm} (D17)

1. $L_0 = L_z$ leads to a mode $L^{(+)}$:

For $p_z > 0$,

$$\omega_{L^{(+)}}(p_z) = |p_z| + \frac{(m_{L,L,L}^{z+})^2}{|p_z|}.$$ \hspace{1cm} (D18)

For $p_z < 0$,

$$\omega_{L^{(+)}}(p_z) = |p_z| + \frac{2 |p_z|}{e} \exp \left( -\frac{2 p_z^2}{(m_{L,L,L}^{z+})^2} \right).$$ \hspace{1cm} (D19)
2. \( L_0 = -L_z \) leads to a mode \( L^{(-)} \):
For \( p_z > 0 \),
\[
\omega_{L^{(-)}}(p_z) = |p_z| + \frac{2 |p_z|}{e} \exp \left( -\frac{2 p_z^2}{(m^{+}_{LLL})^2} \right).
\]  
(D20)
For \( p_z < 0 \),
\[
\omega_{L^{(-)}}(p_z) = |p_z| + \frac{(m^{+}_{LLL})^2}{|p_z|}.
\]  
(D21)

3. \( R_0 = R_z \) leads to a mode \( R^{(+)} \):
For \( p_z > 0 \),
\[
\omega_{R^{(+)}}(p_z) = |p_z| + \frac{2 |p_z|}{e} \exp \left( -\frac{2 p_z^2}{(m^{+}_{LLL})^2} \right).
\]  
(D22)
For \( p_z < 0 \),
\[
\omega_{R^{(+)}}(p_z) = |p_z| + \frac{(m^{+}_{LLL})^2}{|p_z|}.
\]  
(D23)

4. \( R_0 = -R_z \) leads to a mode \( R^{(-)} \):
For \( p_z > 0 \),
\[
\omega_{R^{(-)}}(p_z) = |p_z| + \frac{2 |p_z|}{e} \exp \left( -\frac{2 p_z^2}{(m^{+}_{LLL})^2} \right).
\]  
(D24)
For \( p_z < 0 \),
\[
\omega_{R^{(-)}}(p_z) = |p_z| + \frac{(m^{+}_{LLL})^2}{|p_z|}.
\]  
(D25)

Note that in the high momentum limit the above dispersion relations are given in terms of absolute values of \( p_z \), i.e. \( |p_z| \).

We further note that the above dispersion relations in the absence of the magnetic field reduce to HTL results, where left and right handed are degenerate.

E. Recovering HTL Spectral Function

One can easily get back to the HTL thermal spectral function from (77) by turning off the magnetic field, \( i.e., B = 0 \Rightarrow b' = e' = 0 \) and one gets the following simplifications:
\[
g^1_L \big|_{B=0} = g^1_R \big|_{B=0} = g^1; \quad g^2_L \big|_{B=0} = g^2_R \big|_{B=0} = g^2; \quad g^3_L \big|_{B=0} = g^3_R \big|_{B=0} = 0,  \tag{E1}
\]
\[
L^2 \big|_{B=0} = R^2 \big|_{B=0} = H^2; \quad \omega_{L^{(+)}} \big|_{B=0} = \omega_{R^{(+)}} \big|_{B=0} = \omega_{\pm}, \tag{E2}
\]
\[
\rho^1_L \big|_{B=0} = \rho^1_R \big|_{B=0} = \rho^1; \quad \rho^2_L \big|_{B=0} = \rho^2_R \big|_{B=0} = \rho^2, \quad \rho^3 \big|_{B=0} = 0. \tag{E3}
\]
These implies that the spectral function can be written as
\[
\rho \big|_{B=0} = \gamma^0 \rho^1 - (\gamma \cdot \hat{p}) \rho^2. \tag{E4}
\]

Now the HTL spectral function \([49, 73]\) is given by
\[
\rho_{HTL} = \frac{1}{2} (\gamma^0 - \gamma \cdot \hat{p}) \rho_+ + \frac{1}{2} (\gamma^0 + \gamma \cdot \hat{p}) \rho_-
= \frac{1}{2} \gamma^0 (\rho_+ + \rho_-) - \frac{1}{2} (\gamma \cdot \hat{p}) (\rho_+ - \rho_-), \tag{E5}
\]
where $\rho_\pm$ are the HTL spectral function. Since the spectral has both pole and cut part, comparing (E4) and (E5) one gets for the pole parts as

$$\rho_{B=0}^{\text{pole}} = \frac{1}{2} \left[ \rho_+^{\text{HTL}} + \rho_-^{\text{HTL}} \right]$$ \hspace{1cm} (E6)

$$\rho_{B=0}^{\text{pole}} = \frac{1}{2} \left[ \rho_+^{\text{HTL}} - \rho_-^{\text{HTL}} \right].$$ \hspace{1cm} (E7)

and for the cut parts as

$$\beta_{B=0}^{\text{pole}} = \frac{1}{2} \left[ \beta_+^{\text{HTL}} + \beta_-^{\text{HTL}} \right]$$ \hspace{1cm} (E8)

$$\beta_{B=0}^{\text{pole}} = \frac{1}{2} \left[ \beta_+^{\text{HTL}} - \beta_-^{\text{HTL}} \right].$$ \hspace{1cm} (E9)

Now one can obtain either (78) or (79)

$$\rho_{B=0}^{\text{pole}} = [Z_+^{1+} \delta(p_0 - \omega_+) + Z_-^{1-} \delta(p_0 + \omega_-)] + [Z_+^{1+} \delta(p_0 - \omega_-) + Z_-^{1-} \delta(p_0 + \omega_-)] \hspace{1cm} (E10)$$

When the magnetic field is turned off, the different residues can be read from their analytical expressions as given in sec. V as

$$Z_+^{1+} = Z_+^{1-} = \frac{\omega_+^2 - p^2}{4m_{th}^2} = \frac{1}{2} Z_+,$$ \hspace{1cm} (E11)

$$Z_-^{1+} = Z_-^{1-} = \frac{\omega_-^2 - p^2}{4m_{th}^2} = \frac{1}{2} Z_-,$$ \hspace{1cm} (E12)

where, $Z_\pm = (\omega_\pm^2 - p^2)/2m_{th}^2$, are the residues corresponding to the modes $\omega_\pm$. Using these one can write (E10) as

$$\rho_{B=0}^{\text{pole}} = \frac{1}{2} \left[ Z_+ \delta(p_0 - \omega_+) + Z_- \delta(p_0 + \omega_-) \right] + \frac{1}{2} \left[ Z_- \delta(p_0 - \omega_-) + Z_+ \delta(p_0 + \omega_-) \right]
\hspace{1cm}
= \frac{1}{2} \left[ Z_+ \delta(p_0 - \omega_+) + Z_- \delta(p_0 + \omega_-) \right] + \frac{1}{2} \left[ Z_+ \delta(p_0 + \omega_-) + Z_- \delta(p_0 + \omega_-) \right]
\hspace{1cm}
= \frac{1}{2} \left[ \rho_+^{\text{HTL}} + \rho_-^{\text{HTL}} \right],$$ \hspace{1cm} (E13)

which agree with (E6).

Now the other non-zero component of the spectral function can be reduced as

$$\rho_{B=0}^{\text{pole}} = [Z_+^{2+} \delta(p_0 - \omega_+) + Z_-^{2-} \delta(p_0 + \omega_-)] + [Z_+^{2-} \delta(p_0 - \omega_-) + Z_-^{2+} \delta(p_0 + \omega_-)] \hspace{1cm} (E14)$$

along with the remaining non-zero residues as

$$Z_+^{2+} = Z_-^{2-} = \frac{\omega_+^2 - p^2}{4m_{th}^2} \times \frac{\omega_+ m_{th}^2 \log \left( \frac{\omega_+ p}{\omega_+ - p} \right) - 2p m_{th}^2 + p^2}{2p^2 \omega_+ - p m_{th}^2 \log \left( \frac{\omega_+ + p}{\omega_+ - p} \right)}$$

$$= \frac{\omega_+^2 - p^2}{4m_{th}^2} = \frac{1}{2} Z_+,$$ \hspace{1cm} (E15)

and

$$Z_-^{2+} = Z_+^{2-} = \frac{\omega_-^2 - p^2}{4m_{th}^2} \times \frac{\omega_- m_{th}^2 \log \left( \frac{\omega_- p}{\omega_- - p} \right) - 2p m_{th}^2 + p^2}{2p^2 \omega_- - p m_{th}^2 \log \left( \frac{\omega_- + p}{\omega_- - p} \right)}$$

$$= \frac{\omega_-^2 - p^2}{4m_{th}^2} = \frac{1}{2} Z_-.$$ \hspace{1cm} (E16)
Note that we have used the respective dispersion relations coming from $H^2 = 0$, in the last line of (E15) and (E16) for further simplifications. Now (E14) can be rewritten as

$$\rho^2\big|_{\text{pole}} = \frac{1}{2} \left[ Z_+ \delta(p_0 - \omega_+) - Z_+ \delta(p_0 + \omega_+) \right] - \frac{1}{2} \left[ Z_- \delta(p_0 - \omega_-) - Z_- \delta(p_0 + \omega_-) \right]$$

where following the same convention as before

$$\rho^2\big|_{\text{pole}} = \frac{1}{2} \left[ Z_+ \delta(p_0 - \omega_+) + Z_- \delta(p_0 + \omega_-) \right] - \frac{1}{2} \left[ Z_- \delta(p_0 - \omega_-) + Z_+ \delta(p_0 + \omega_+) \right]$$

which agree with (E7).

In absence of magnetic field one write the cut parts from the general expression of $\beta^i_L$ in (82) as

$$\beta^1 = \frac{1}{\pi} \Theta(p^2 - p_0^2) \frac{\text{Im}(g^1) \text{Re}(H^2) - \text{Im}(H^2) \text{Re}(g^1)}{|H^2|^2}$$

$$\beta^2 = \frac{1}{\pi} \Theta(p^2 - p_0^2) \frac{\text{Im}(g^2) \text{Re}(H^2) - \text{Im}(H^2) \text{Re}(g^2)}{|H^2|^2}$$

where for zero magnetic field case

$$H^2 \big|_{B=0} = R^2 \big|_{B=0} = (g^1 + g^2)(g^1 - g^2) = H_- H_+,$$

where following the same convention as before $H_- = g^1 + g^2$ and $H_+ = g^1 - g^2$.

The real and imaginary parts of $H^2$ can be written in terms of $H_- \text{ and } H_+$ as

$$\text{Re}(H^2) + i \text{ Im}(H^2) = \text{Re}(H_-) \text{Re}(H_+) + i \text{ Im}(H_-) \text{Im}(H_+)$$

$$= \left[ \text{Re}(H_-) \text{Re}(H_+) - \text{Im}(H_-) \text{Im}(H_+) \right]$$

$$+ i \left[ \text{Re}(H_-) \text{Im}(H_+) + \text{Re}(H_+) \text{Im}(H_-) \right].$$

Now we can write down

$$\beta^1 + \beta^2 = \frac{1}{\pi} \Theta(p^2 - p_0^2) \frac{(\text{Im}(g^1) + \text{Im}(g^2)) \text{Re}(H^2) - \text{Im}(H^2)(\text{Re}(g^1) + \text{Re}(g^2))}{|H_-|^2|H_+|^2}$$

$$= \frac{1}{\pi} \Theta(p^2 - p_0^2) \frac{\text{Im}(H_-) \text{Re}(H^2) - \text{Im}(H^2) \text{Re}(H_-)}{|H_-|^2|H_+|^2}$$

$$= - \frac{1}{\pi} \Theta(p^2 - p_0^2) \frac{\text{Im}(H_+)}{|H_+|^2}$$

$$= - \frac{1}{2p} m_h^2 \left( 1 - \frac{p_0}{p} \right)$$

$$\left\{ p_0 - p + \frac{m_h^2}{2p} \left( 1 - \frac{p_0}{p} \right) \right\} \frac{\text{log} \left| \frac{p_0 + p}{p_0 - p} \right| + 2}{p_0 - p}$$

$$\frac{\text{log} \left| \frac{p_0 + p}{p_0 - p} \right| + 2}{p_0 - p}$$

$$= \beta_+ \text{.}$$

(E22)

and similarly

$$\beta^1 - \beta^2 = - \frac{1}{\pi} \Theta(p^2 - p_0^2) \frac{\text{Im}(H_-)}{|H_-|^2},$$

$$= - \frac{1}{2p} m_h^2 \left( 1 + \frac{p_0}{p} \right)$$

$$\left\{ p_0 + p - \frac{m_h^2}{2p} \left( 1 + \frac{p_0}{p} \right) \right\} \frac{\text{log} \left| \frac{p_0 + p}{p_0 - p} \right| - 2}{p_0 - p}$$

$$\frac{\text{log} \left| \frac{p_0 + p}{p_0 - p} \right| - 2}{p_0 - p}$$

$$= \beta_- \text{.}$$

(E23)

where both (E22) and (E23) agree with the HTL cut parts [49, 73].

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