Field Holography in General Background and Boundary Effective Action from AdS to dS

Sylvain Fichet

ICTP South American Institute for Fundamental Research & IFT-UNESP, R. Dr. Bento Teobaldo Ferraz 271, São Paulo, Brazil

Centro de Ciencias Naturais e Humanas, Universidade Federal do ABC, Santo Andre, 09210-580 SP, Brazil

Abstract

We study quantum fields on an arbitrary, rigid background with boundary. We derive the action for a scalar in the holographic basis that separates the boundary and bulk degrees of freedom. From this holographic action, a relation between Dirichlet and Neumann propagators valid for any background is obtained. As an application in a warped background, we derive an exact formula for the flux of bulk modes emitted from the boundary. We also derive the holographic action in the presence of two boundaries. Integrating out free bulk modes, we derive a formula for the Casimir pressure on a $(d-1)$-brane depending only on the boundary-to-bulk propagators. In AdS$_2$ we find that the quantum force pushes a point particle toward the AdS$_2$ boundary. In higher dimensional AdS$_{d+1}$ the quantum pressure amounts to a detuning of the brane tension, which gets renormalized for even $d$.

We evaluate the one-loop boundary effective action in the presence of interactions by means of the heat kernel expansion. We integrate out a heavy scalar fluctuation with scalar interactions in AdS$_{d+1}$, obtaining the long-distance effective Lagrangian encoding loop-generated boundary-localized operators. We integrate out nonabelian vector fluctuations in AdS$_{4,5,6}$ and obtain the associated holographic Yang-Mills $\beta$ functions. Turning to the expanding patch of dS, following recent proposals, we provide a boundary effective action generating the perturbative cosmological correlators by analytically continuing from dS to EAdS. We obtain the “cosmological” heat kernel coefficients in the scalar case and work out the divergent part of the dS$_4$ effective action which renormalizes the cosmological correlators. More developments are needed to extract all one-loop information from the cosmological effective action.
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1 Introduction and Summary

Imagine a quantum field theory (QFT) supported on a background manifold with boundary. What can an observer standing on the boundary learn about the QFT living in the interior of the manifold?

While this is a simple problem at the classical level, evaluating boundary observables at the quantum level is more challenging, since it involves integrating over the quantum fluctuations occurring in the bulk of the system. Integrating out the bulk quantum fluctuations can be done through the quantum effective action. The latter is then a functional of the boundary degrees of freedom, i.e. a “boundary effective action”.

Such a setup — a QFT seen from the boundary — is common in physics, and is typically referred to as “holographic”. Holography actually refers to a variety of similar-but-not-equivalent concepts that we briefly review further below. Among all examples of QFTs on a background manifold with boundary, we can single out two cases of paramount importance: the Anti-de Sitter (AdS) and de Sitter (dS) spacetimes. These are the maximally symmetric curved spacetimes.

In AdS space, an observer standing on the AdS boundary sees a strongly-coupled CFT, with large number of colors $N$ if the bulk QFT is weakly coupled [1–4]. Therefore holography in AdS leads to a profound connection between gravity and strongly-coupled gauge theories, opening new possibilities to better understand both. The holographic view of AdS was formulated two decades ago [5, 6], AdS/CFT is now studied at loop level. In this work our application to AdS holography will be focused on systematic one-loop computations.

The notion of holography in dS space is even more concrete: Cosmological observations suggest that the chronology of the Universe has at least two phases with approximate dS geometry: the current expanding epoch and the inflationary epoch. When we look at the sky and measure galactic redshifts or the CMB, we are actually observers standing on the late-time boundary of the expanding patch of dS, probing the dS interior with telescopes. The inflationary phase is of great interest because it probes the highest accessible energies and the earliest period of the Universe. Remarkably, taking a glimpse into the quantum fluctuations of the Early Universe is possible by analyzing the cosmological correlators of the CMB. Classical and quantum calculations of cosmological correlators are much less advanced than those in AdS. In this work, we focus on establishing a boundary quantum effective action that generates these correlators, with the goal of extracting loop-level information from it.

We can refer to the field-theoretical setup above as “field holography”. We emphasize that our focus is on “holography” itself, and not on the “holographic principle” which posits a dual $d$-dimensional description of the boundary effective action [7–9].¹ That is, we

¹One concrete incarnation of the holographic principle is the emergence of dual $d$–dimensional theories from $d + 1$ Chern-Simons theories, with for example the 3d CS/WZW correspondence [10]. Another incarnation is the AdS/CFT correspondence. The holographic principle goes beyond weakly coupled QFT, exploring non-perturbative aspects of quantum gravity, black holes, and entanglement entropy (see e.g. [11]). Our work is not about these aspects.
are not concerned with the existence and specification of possible dual boundary theories, but with the content and properties of the boundary effective action itself.

The first goal in this work is to derive the holographic action of a QFT on an arbitrary background. Apart from providing a formalism computing holographic quantities beyond AdS, this approach brings a somewhat different viewpoint on well-known AdS holography. In AdS holography, one may wonder whether a given feature is either a manifestation of AdS/CFT or a more general property of the holographic formalism itself. If the latter is true, the feature under consideration is valid beyond AdS. Thus the boundary action on arbitrary background can, in this sense, be used to shed light on AdS/CFT features.

Using the established holographic formalism, we then investigate the effect of quantum bulk fluctuations on the boundary. Among these effects we can distinguish i) the bulk vacuum bubbles (i.e. 0pt diagrams) which cause a quantum pressure on the boundary and ii) \((n > 0)\)pt connected bulk diagrams which are responsible for correcting/renormalizing the boundary theory. We investigate aspects from both i) and ii).

The crux of the holographic formalism is to single out the boundary-localized degree of freedom in the set of all degrees of freedom of a quantum field. Such a splitting is done by requiring a Dirichlet boundary condition (BC) on the bulk degrees of freedom, such that the one remaining on the boundary is encapsulated into a separate variable. The fluctuation of the field at a given point \(x^M\) in the bulk is completely described by the Dirichlet bulk modes plus the boundary fluctuation. The boundary fluctuation contributes remotely to \(\Phi(x)\), thus a propagator must be involved — we will see in Sec. 2 that it is the so-called boundary-to-bulk propagator \(K\). Such a “holographic” decomposition of the quantum field is illustrated in Fig. 1. A boundary observer does not probe the Dirichlet modes, these can be completely integrated out to give rise to a boundary effective action. Here we will perform this operation perturbatively at one-loop level.

From a technical viewpoint, the integration of quantum fluctuations at one-loop on manifolds with boundary is encoded into the heat kernel coefficients of the one-loop effective action \([12–15]\). The first heat kernel coefficients have been gradually computed along the past decades (see \([16]\) and references therein). To study the boundary correlators, we introduce these results into the framework of the holographic action. Another theme of this work is thus the application of the heat kernel formalism to the evaluation of one-loop
boundary correlators (e.g. one-loop Witten diagrams in AdS).

Here below we review the literature related to our developments.

Vacuum bubbles and quantum pressure: We are unaware of a work about evaluating Casimir pressure or energy in the presence of a $\mathbb{R}^d$ boundary beyond the much studied case of Minkowski background. In our application we are interested in a difference of vacuum pressure between each side of the boundary, hence our setup has little connection to the formal calculations of Casimir energy in AdS$_{d+1}$ from [17] and subsequent references.

Boundary correlators in AdS: There has been a lot of activity about computing and studying loop-level correlators [18–62]. However to the best of our knowledge, such studies are always focused on specific diagrams, and not on the one-loop effective action. It seems that the AdS one-loop effective action has been used only in the very specific case of the one-loop scalar potential (namely, for constant scalar field with dimension $\Delta = d$) [42, 63–68]. The one-loop boundary effective action, through the heat kernel coefficients, contains much more information on the (bulk and boundary) divergences and on the long-distance EFT.

Boundary correlators in dS: There has been a lot of activity about computing cosmological correlators and understanding their structure in terms of conformal symmetry and singularities [69–85]. A bootstrap program analogous to flat space amplitudes techniques has also been developed, often at the level of the dS wavefunction coefficients, with e.g. cutting rules, dispersion relations and positivity bounds, see for example [56, 71, 88–102]. In this work we build on recent developments for computing the cosmological correlators via analytical continuation from dS to EAdS [97, 103, 104], see also [105–108] for earlier works. We build on a proposal from [109] to define an EAdS effective action that generates the perturbative cosmological correlators.

1.1 Summary of Results

- We derived the action of a scalar field in the holographic basis in a general background. The action is diagonal and the Neumann-Dirichlet identity “$G_N = G_D + KG_0K$” immediately follows, proving that such a relation is independent of the background geometry. This relation provides a trivial understanding of the effect of boundary-localized bilinear operators on the propagator.

- We evaluate the holographic action in a generic warped background, including also a dilaton background, and find that holography in this class of background is essentially as simple as in AdS. We obtained a general formula for the flux of bulk modes emitted from the boundary. In the AdS limit we find $G_D = G_{\Delta=\Delta} = G_{\Delta=\Delta}$, which, together with $G_N = G_D + KG_0K$, reproduces a known identity for AdS propagators. We derive the “double” holographic action in the presence of two boundaries, as a functional of the two boundary variables (App. C).

- Integrating out the free bulk modes, we obtain a simple formula for the quantum pressure induced by a bulk scalar fluctuation on a $(d-1)$-brane, expressed only as a

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2See also [83, 86–91] for extension to models without invariance under special conformal transformations.
function of boundary-to-bulk propagators. We recover the scalar Casimir pressure of $(d+1)$-dimensional flat space then study the quantum pressure on a brane in AdS$_{d+1}$ (or a point particle if $d = 1$). For AdS$_2$ we find that the point particle is attracted towards the AdS$_2$ boundary. For a conformally massless scalar, the field effectively sees a flat half-plane and the known 2d Casimir force is recovered. For higher dimensions the pressure can be understood as a contribution to the brane tension. For example, if the brane is static at the classical level, at quantum level it acquires a velocity driven by the quantum pressure. Logarithmic divergences renormalize the brane tension for even $d$.

- We turn to interactions and connected correlators. Integrating out a heavy bulk field gives rise to a long-distance boundary effective action encoding the effect of the heavy field into a series of “holographic vertices” in one-to-one correspondence with contact Witten diagrams. This is the holographic version of a low-energy EFT. We provide the covariant formula for the large mass expansion of the propagator in terms of powers of the Laplacian acting on the Dirac delta, which is the main formula needed to compute the long-distance holographic EFT at classical level. We provide the one-loop holographic effective action using the heat kernel formalism for a spin-0 and spin-1 fluctuations. The heat kernel coefficients give access to the one-loop long-distance holographic EFT and one-loop divergences.

- We argue that AdS boundary correlators generated by the boundary effective action have $\Delta_-$ internal lines. AdS Witten diagrams with $\Delta_-$ internal lines decompose as subdiagrams with $\Delta_+$ internal lines connected to each other by boundary lines. We note that this overall structure of the diagrams amounts to using a $\frac{1}{N}$ perturbative expansion directly in the holographic CFT. The corresponding large-$N$ CFT diagrams are built from vertices made from the amputated $n > 2$pt CFT correlators at leading order in $\frac{1}{N}$, connected to each other by the mean field (i.e. $N = \infty$) 2pt CFT correlator. An analogous diagrammatic approach to large-$N$ CFT was introduced in [110], here we found out how it arises from the AdS side.

- In AdS$_{d+1}$ space, we integrate out at one-loop a heavy scalar fluctuation with scalar interactions. In the regime of distances larger than the AdS radius, the coefficients combine such that only the sum of conformal dimensions appear, in accordance with the dual conformal symmetry. The local piece of the AdS boundary-to-bulk propagator is needed to enforce this feature. Among the generated boundary-localized operators, there are local operators such as mass, kinetic and interaction terms, whose existence is known from the extradimension literature but which had, to the best our knowledge, never been computed exactly. Non-local operators built from combinations of 2pt functions are also generated at one-loop. These contributions renormalize the OPE data of the holographic CFT.

- In AdS$_{4,5,6}$, we integrate out nonabelian vector fluctuations and deduce the holographic $\beta$ function of the gauge coupling. For any of these dimensions the boundary
gauge coupling has a logarithmic running, which either comes from bulk or boundary
heat kernel coefficients depending on spacetime dimension. For \( d = 6 \) the logarithmic
running of the dimensionful gauge coupling is a consequence of the nonzero curvature.

- Turning to the expanding patch of dS, we evaluate the dS 2pt functions directly in
momentum space and perform the analytical continuation from dS to EAdS also used
in [97, 103, 104, 109]. Following a proposal from [109] we establish a boundary effec-
tive action which is the generating functional of amputated cosmological correlators.
Working in a simple scalar case, we obtain the “cosmological” one-loop effective ac-
tion and work out its divergent part. From a simple \( \varphi^4 \) example we show that there is
“wavefunction renormalization” of the cosmological correlators as well as renormal-
ization of the mass/conformal dimension of the field. More developments are needed
to extract all one-loop information from the cosmological heat kernel coefficients.

1.2 Definitions and Conventions

We consider a \( d + 1 \)-dimensional manifold \( \mathcal{M} \) with boundary \( \partial \mathcal{M} \). The bulk and boundary
are taken to be sufficiently smooth such that Green’s identities apply. The metric is either
Euclidian or Lorentzian with \((-++\ldots)\) signature. Latin indices \( M, N, \ldots = \{0, 1, \ldots d\} \)
index the bulk coordinates, denoted by \( x^M \). The bulk metric \( g_{MN} \) is defined by \( ds^2 = g_{MN} dx^M dx^N \). A point belonging to the boundary is labelled as \( x^M_0 \equiv x^M|_{\partial \mathcal{M}} \) in the
bulk coordinates. Greek indices \( \mu, \nu, \ldots = \{0, 1 \ldots d-1\} \) index boundary coordinates,
denoted by \( y^\mu \). The boundary is described by the embedding \( x^M = x^M_0(y^\mu) \). Defining \( E^M_\mu = \frac{\partial x^M}{\partial y^\mu} \), the induced metric is defined by \( \bar{g}_{\mu\nu} = E^M_\mu E^N_\nu g_{MN} \). Let \( e^M_\perp(y^\mu) \) be the unit vector normal to the boundary at the point \( y^\mu \) and outward-pointing. We refer to
the contraction \( \partial_\perp \equiv e^M_\perp \partial_M \) as the normal derivative. The \( \partial_\mu \equiv E^M_\mu \partial_M \) derivatives are
referred to as transverse.\(^3\)

We consider a QFT on the \( \mathcal{M} \) spacetime background. The boundary value of fields, e.g.
\( \Phi \), is denoted by \( \Phi_0 = \Phi|_{\partial \mathcal{M}} \). We will routinely switch between Euclidian and Lorentzian
metric via Wick rotation \( x^0_E = ix^0_L \). We remind the convention for the actions, \( S_E = -iS_L \).
Propagators are related by \( G_E = iG_L \).

We define the bulk inner product as
\[
p \star q = \int_{\mathcal{M}} d^{d+1}x \sqrt{g} p(x) q(x).
\]
We define the boundary inner product as
\[
p \circ q = \int_{\partial \mathcal{M}} d^d y \sqrt{\bar{g}} p(y_0) q(y_0).
\]

\(^3\)The unit normal vector satisfies \( e^M_\perp e^N_\perp g_{MN} = 1 \) and \( e^M_\perp E^N_\mu = 0 \) by definition. Refs.[16, 111] used
orthonormal vielbeins. In our notations, the orthonormal vielbein in \( \partial \mathcal{M} \) is defined by \( \bar{g}_{\mu\nu} = e^M_\perp e^N_\mu \eta^{ab} \). The
orthonormal vielbein in \( \mathcal{M} \) is \( (e^M_\perp, E^M_\mu) \) — the \( (e^M_\perp, E^M_\mu) \) basis forms a non-orthonormal vielbein in \( \mathcal{M} \).
Here we do not need to use these vielbeins explicitly apart from the normal vector. See e.g. [112] for details
on hypersurface geometry.
This is useful to manipulate Green’s identities. For example Green’s first identity is
\( \partial_M \Psi \ast \partial^M \Phi = -\Psi \ast \Box \Phi + \Psi \circ \partial_\perp \Phi \) in this notation.

Outline

In section 2 we establish the holographic action for a scalar in a general background. In
section 3 we apply the results to a warped metric. This is essentially a review with some
scattered new results. In section 4 we integrate out the bulk modes and obtain the quan-
tum pressure on \((d-1)\)-brane in flat and AdS space with arbitrary dimension. In section 5
we give general observations on boundary connected correlators in the holographic formal-
ism and on the long-distance holographic EFT. In section 6 we introduce the heat kernel
coefficients in the holographic formalism. In section 7 we apply it to a scalar fluctuation
with scalar interactions and to a massless nonabelian vector in AdS. We obtain the AdS
one-loop boundary effective action for these cases. In section 7 we derive dS propagators in
the expanding patch, analytically continue from dS to EAdS, and establish a EAdS bound-
ary effective action which generates the perturbative cosmological correlators. Appendices
contain a proof of the discontinuity equation (A), details and checks on propagators in
warped background (B), a derivation of the double holographic action (C), basics of the
heat kernel method and an elementary check from an AdS bubble (D).

2 The Holographic Action in a General Background

We derive the action in the holographic basis for an arbitrary, smooth manifold with
boundary. We assume Euclidian signature. The analogous result in Lorentzian signature
can be obtained via analytical calculation of the Euclidian result. It is sufficient to focus
on a scalar field \( \Phi \) to avoid spin-related technicalities. This section is somewhat technical,
the reader willing to skip the details can go to the main result Eq. (2.22).

Action and Green’s functions

The fundamental action is denoted \( S[\Phi] \). The general partition function with a generic
bulk source \( J \) is \( Z[J] = \int \mathcal{D}\Phi \exp[-S[\Phi] - \int dx^{d+1} \sqrt{g} J \Phi] \). The quadratic part of the
action takes the form

\[
S[\Phi] = \frac{1}{2} \int_M d^{d+1}x \sqrt{g} (\partial_M \Phi \partial^M \Phi + m^2 \Phi^2) + \frac{1}{2} \int_{\partial M} d^d x d^d y' \sqrt{|g|} \sqrt{|\tilde{g}|} y' y \Phi_0 \mathcal{B} \Phi_0 + \text{int.} \tag{2.1}
\]

In the boundary term we have included the most general form of the bilinear operator \( \mathcal{B} \).
It can contain transverse derivatives and can be non-local. Using the product notations of

\[4\]The identity elements for the bulk and boundary products are respectively \( \frac{1}{\sqrt{g}} \delta^{d+1}(x - x') \) and
\( \sqrt{g} \delta^d(y - y') \).

\[5\]In Lorentzian signature the derivation of the holographic action for a timelike boundary is identical
to the Euclidian case. The case of a spacelike boundary is more subtle because time-ordering in the 2pt
functions matters. It is not treated in this section. Holographic calculations in dS space (which has spacelike
boundary) are done in section 8, relying on analytical continuation to Euclidian space.
Eqs. (1.1), (1.2), we have
\[ S[\Phi] = \frac{1}{2} \left( \partial_M \Phi \star \partial_M \Phi + m^2 \Phi \star \Phi + \Phi_0 \circ B \circ \Phi_0 \right) + \text{int.} \quad (2.2) \]

We can identify the wave operator by applying Green’s first identity to the bulk term, giving \( \partial_M \Phi \star \partial_M \Phi + m^2 \Phi \star \Phi = \Phi \star (-\Box + m^2)\Phi + \text{boundary terms} \), with \( \Box \) the Laplacian on \( \mathcal{M} \).

Finally, the inverse of the \(-\Box + m^2\) operator is the propagator \( G \),
\[ (-\Box + m^2)G(x, x') = \frac{1}{\sqrt{g_x}} \delta^{d+1}(x - x'). \quad (2.3) \]

We denote by \( G_D \) the propagator with boundary condition
\[ G_D(x_0, x) = 0 \quad \forall x_0 \in \partial \mathcal{M}, x \in \mathcal{M} \] (Dirichlet) \quad (2.4)

The Neumann boundary condition is a bit more subtle. We denote by \( G_N \) the propagator with the boundary condition
\[ \partial_{x_0}^\perp G_N(x_0, x) = c \quad \forall x_0 \in \partial \mathcal{M}, x \in \mathcal{M} \] (Neumann) \quad (2.5)

where
\[ c = \begin{cases} 0 & \text{if } m \neq 0 \\ \frac{1}{\text{vol}(\partial \mathcal{M})} & \text{if } m = 0. \end{cases} \quad (2.6) \]

Such a distinction is required for consistency of the boundary condition with the bulk equation of motion. For \( m \neq 0 \), the volume integral of Eq. (2.3) gives, upon use of the divergence theorem, \( \int_{\partial \mathcal{M}} d^d y \sqrt{g_y} \partial_{x_0}^\perp G(x_0, x) = 0 \). For \( m = 0 \), one has instead \( \int_{\mathcal{M}} d^d y \sqrt{g_y} \partial_{x_0}^\perp G(x_0, x) = -1 \). This is Gauss’s law on \( \mathcal{M} \). We will show that the subtlety about the Neumann BC has no consequence for physical results.

Finally, we compute the discontinuity in the normal derivative of the propagator with endpoints on the boundary. A derivation is given in App. A, the result is
\[ \partial_{x_0}^\perp G(x_0^-, x_0') - \partial_{x_0}^\perp G(x_0^+, x_0') = \frac{1}{\sqrt{g_y}} \delta^d (y - y'). \] (Discontinuity) \quad (2.7)

For a Neumann propagator the second term of the left-hand side is set by the Neumann boundary condition, giving
\[ \partial_{x_0}^\perp G_N(x_0^-, x_0') = \frac{1}{\sqrt{g_y}} \delta^d (y - y') + c. \] \quad (2.8)

This discontinuity equation is needed for holographic calculations.

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6One uses \( \int_{\mathcal{M}} d^d y \sqrt{g_y} G(x, x') = G \star 1_\mathcal{M} = \langle \Phi \rangle_{1_\mathcal{M}} \). Applying the EOM to this equation gives \( m^2 \langle \Phi \rangle_{1_\mathcal{M}} = 1_\mathcal{M} \), thus \( \int_{\partial \mathcal{M}} d^d y \sqrt{g_y} G(x, x') = \frac{1}{m} 1_\mathcal{M} \). This identity is then used in the integral of Eq. (2.3), resulting in \( c = 0 \) for \( m \neq 0 \).

7The value of \( c \) for \( m = 0 \) could more generally be any unit-normalized distribution on \( \partial \mathcal{M} \). The subsequent results hold in the general case, here we use constant \( c \) for simplicity.
Holographic Basis

The starting point of holography is to split the field variable into boundary and bulk degrees of freedom,

$$\int D\Phi = \int D\Phi_0 \int_{\partial_\mathcal{M} = \Phi_0} D\Phi_{\text{bulk}}. \quad (2.9)$$

The bulk degrees of freedom on the r.h.s describe the set of fluctuations leaving the boundary value unchanged. These bulk modes satisfy thus Dirichlet condition on the boundary. Denoting Dirichlet modes as $\Phi_D$, we have

$$\int_{\Phi|_{\partial_\mathcal{M} = \Phi_0}} D\Phi_{\text{bulk}} = \int D\Phi_D. \quad (2.10)$$

How is $\Phi$ decomposed into the “holographic basis” $(\Phi_0, \Phi_D)$? Let us write $\Phi$ in a general form $\Phi(x) = \Phi_0 \circ K(x) + a\Phi_D(x)$ and determine the $a$ and $K$ functions. To proceed we consider the classical value of $\Phi$, $\langle \Phi \rangle_J(x) = \langle \Phi_0 \rangle_J \circ K(x) + a\langle \Phi_D \rangle_J(x)$ sourced by a generic source $J$, and satisfying $(\Phi)_J(x) = G \ast J(x)$.

If we choose a current $J_D$ that vanishes on the boundary, there is no contribution from the boundary, $(\Phi_0)_J = 0$. We get $\langle \Phi \rangle_J = \langle \Phi_D \rangle_J$ and thus $a = 1$. If, conversely, we choose a boundary-localized current $J_0(x)|_{\partial_\mathcal{M}} = J_0(x_0)$, the Dirichlet component does not contribute, $(\Phi_D)_J = 0$. In that case the field is purely sourced from the boundary, $(\Phi)_J = (\Phi_0)_J \circ K(x)$.

To obtain the $K$ function, we need to use Green’s third identity \(^8\) in the case of a Neumann problem. Using the Neumann boundary condition Eq. (2.5), we get

$$\langle \Phi \rangle_{J_0}(x) = c' \Phi_0 + \partial_\perp \langle \Phi_0 \rangle_{J_0} \circ G_N(x) \quad (2.11)$$

with $c' = -\text{Vol}(\partial_\mathcal{M}) c$. The $\Phi_0$ constant is the average value of the field over the boundary [113] and will drop from the calculations. To proceed, we define the boundary-to-boundary propagator

$$G_0(x_0, x'_0) = G_N(x, x')|_{x, x' \in \partial_\mathcal{M}}. \quad (2.12)$$

We also introduce its inverse $G^{-1}_0$ as

$$G^{-1}_0 \circ G_0(x_0, x'_0) = \frac{1}{\sqrt{\text{Vol} g}} \delta^d(y - y'). \quad (2.13)$$

We evaluate Eq. (2.11) on the boundary, which gives the value $\langle \Phi \rangle_{J_0}(x_0) = \langle \Phi_0 \rangle_{J_0}$ we are interested in. Using the expression of $\langle \Phi_0 \rangle_{J_0}$ we substitute the quantity $\partial_\perp \langle \Phi_0 \rangle$ in Eq. (2.11) by making use of the inverse boundary propagator. The $\Phi_0$ constant cancel. \(^9\) The result is

$$\langle \Phi \rangle_{J_0}(x) = \langle \Phi_0 \rangle_{J_0} \circ G^{-1}_0 \circ G_N(x). \quad (2.14)$$

\(^8\)Here Green’s third identity takes the form $\langle \Phi \rangle(x) = -\langle \Phi_0 \rangle \circ \partial_\perp G(x) + \partial_\perp \langle \Phi_0 \rangle \circ G(x)$.

\(^9\)To see this, first note that $\Phi_0$ can be written as $\Phi_0 = G_N \circ J_0$ (where $J_0$ is an averaged boundary source), then use that $\Phi_0$ is constant in the bulk, which implies $\Phi_0 = G_N \circ J_0 = G_0 \circ J_0$. It follows that $\Phi_0 \circ G^{-1}_0 \circ G_N = \Phi_0$, such that $\Phi_0$ cancels throughout the evaluation and does not appear in Eq. (2.14).
It follows that the $K$ function is

$$K(x_0, x) = G_0^{-1} \circ G_N(x_0, x) \quad (2.15)$$

That is, $K$ is the bulk propagator with an endpoint on the boundary, and amputated by a boundary-to-boundary propagator. It satisfies $\Phi_0 \circ K|_{\partial M} = \Phi_0$. The quantity $K$ is itself typically called the “boundary-to-bulk propagator”.

Summarizing, we have determined $K$ and $a$ by considering the expectation value of $\Phi$. The expression of a bulk field $\Phi$ in the holographic basis is found to be

$$\Phi(x) = \Phi_0 \circ K(x) + \Phi_D(x) \quad (2.16)$$

with $K = G_0^{-1} \circ G_N$.

**Action in the Holographic Basis**

We plug the obtained expression of $\Phi$ in the $(\Phi_0, \Phi_D)$ basis into the partition function. This is

$$Z[J] = \int \mathcal{D}\Phi_0 \mathcal{D}\Phi_D \exp \left[ -S[\Phi_0 \circ K + \Phi_D] - (\Phi_0 \circ K + \Phi_D) \star J \right]. \quad (2.17)$$

The action Eq. (2.1) takes the form

$$S[\Phi] = \frac{1}{2} \left( \partial_M(\Phi_0 \circ K + \Phi_D) \star \partial_M(K \circ \Phi_0 + \Phi_D) + \Phi_0 \circ B \circ \Phi_0 \right) + \text{interactions}. \quad (2.18)$$

We apply Green’s first identity to each of the three bulk terms

$$\partial_M(\Phi_0 \circ K) \star \partial_M(K \circ \Phi_0) + 2\partial_M \Phi_D \star \partial_M(K \circ \Phi_0) + \partial_M \Phi_D \star \partial_M \Phi_D \quad (2.19)$$

For the middle one, one applies the identity such that the obtained Laplacian acts on $K \circ \Phi_0$. The action becomes

$$S[\Phi] = \frac{1}{2} \left( \Phi_0 \circ K \star (\Box + m^2)K \circ \Phi_0 + 2\Phi_D \star (\Box + m^2)K \circ \Phi_0 + \Phi_D \star (-\Box + m^2)\Phi_D \right)$$

$$+ \frac{1}{2} \left( \Phi_0 \circ K \circ \partial_\perp K \circ \Phi_0 + 2\Phi_D \circ \partial_\perp K \circ \Phi_0 + \Phi_D \circ \partial_\perp \Phi_D \right) + \text{int.} \quad (2.20)$$

The first two terms of the first line vanish because $(\Box + m^2)K = 0$ in the bulk. The last two terms of the second line vanish because of the Dirichlet condition, $\Phi_D|_{\partial M} = 0$. We also used that $\partial_\perp \Phi_0 = 0$ and $\Phi_0 \circ K|_{\partial M} = \Phi_0$.

The remaining terms are $\Phi_D \star (\Box + m^2)\Phi_D + \Phi_0 \circ \partial_\perp K \circ \Phi_0$. To evaluate the term with $\partial_\perp K = \partial_\perp G \circ G_0^{-1}$ we note that the point of $G$ on which the derivative does not act belongs to the boundary. Therefore this a derivative of the $\partial_{x_0}^\perp G(x_0^-, x_0)$ form, which requires the use of the discontinuity equation Eq. (2.8). We obtain

$$\Phi_0 \circ \partial_\perp K \circ \Phi_0 = \Phi_0 \circ G_0^{-1} \circ \Phi_0 + c' \tilde{J}_0 \circ \Phi_0. \quad (2.21)$$

The extra term $\tilde{J}_0 \circ \Phi_0$ (with $\tilde{\Phi}_0 = G_0 \circ \tilde{J}_0$) present in the massless case amounts to a shift of the boundary value of the generic $J$ source and can thus be absorbed by a redefinition of $J$. This explicitly shows that this term has no physical relevance.
Combining all the pieces, we find that the partition function of a scalar field supported on an arbitrary background with boundary is

\[
Z[J] = \int \mathcal{D} \Phi_0 \mathcal{D} \Phi_D \exp \left[ -\frac{1}{2} \left( \Phi_D \ast (-\Box + m^2) \Phi_D + \Phi_0 \circ (G_0^{-1} + \mathcal{B}) \circ \Phi_0 \right) \right] - (\Phi_0 \circ K + \Phi_D) \ast J + \text{interactions} \quad (2.22)
\]

The holographic action in Eq. (2.22) is diagonal. The Dirichlet modes have canonical kinetic term. We will often refer to them as “bulk modes”. On the other hand, the boundary degree of freedom has a nontrivial, generally nonlocal self-energy. While the \( \mathcal{B} \) piece is a generic surface term, the \( G_{0}^{-1} \) piece of this holographic self-energy reflects the fact that the boundary degree of freedom knows about the bulk modes.

**Propagator**

Perturbative amplitudes are obtained by taking \( J \) derivatives of Eq. (2.22) or of related quantities such as the generator of connected correlators. In particular, we can compute the propagator

\[
\langle \Phi(x) \Phi(x') \rangle = \frac{1}{\sqrt{g_x} \sqrt{g_{x'}}} \frac{\delta^2}{\delta J(x) \delta J(x')} \log Z[J]. \quad (2.23)
\]

We denote it as \( \langle \Phi(x) \Phi(x') \rangle = G_B^N(x, x') \) because, as shown below, it is the Neumann propagator in the presence of the boundary term \( \mathcal{B} \). Defining the inverse of the boundary operator as

\[
[G_0^{-1} + \mathcal{B}]^{-1} (G_0^{-1} + \mathcal{B}) (x_0, x'_0) = \frac{1}{\sqrt{g_{x_0}}} \delta^d (y - y'), \quad (2.24)
\]

we obtain

\[
G_B^N(x, x') = G_D(x, x') + K \circ [G_0^{-1} + \mathcal{B}]^{-1} \circ K(x, x') \quad (2.25)
\]

Let us verify that \( G_B^N \) satisfies a Neumann boundary condition. To do so we act with \( \partial_\perp \) and convolute with a boundary field \( \hat{\Phi}_0 \) i.e. we act on Eq. (2.25) with \( \hat{\Phi}_0 \cdot \partial_\perp \). The action of \( \partial_\perp \) on the first term is evaluated using the discontinuity equation Eq. (2.8). The second term is evaluated using Green’s third identity, giving \( \hat{\Phi}_0 \circ \partial_\perp G_D(x) = -\hat{\Phi}(x) \). One gets that \( \hat{\Phi}_0 \circ (\partial_\perp + \mathcal{B}) G_N^B(x_0, x) = 0 \). This is true for any \( \hat{\Phi}_0 \), therefore

\[
(\partial_\perp + \mathcal{B}) G_N(x_0, x) = 0 \quad \forall \, x_0 \in \partial \mathcal{M}, x \in \mathcal{M} \quad (2.26)
\]

which is the Neumann boundary condition in the presence of the boundary term.

We have obtained a “holographic” representation of the bulk Neumann propagator in terms of Dirichlet and boundary-to-bulk propagators, valid in any background. We sometimes refer to it as the “Neumann-Dirichlet” identity. The two terms on the r.h.s of Eq. (2.25) correspond to the propagation of Dirichlet modes and to the propagation of the boundary degree of freedom — connected to the bulk endpoints by a boundary-to-bulk
propagator. We can notice that the boundary term takes the form of a Dyson resummation of the boundary operator $B$. The effect of the boundary action in $G_B^N$ can be recovered by dressing $G_N$ with $B$ boundary insertions. We can also see that whenever $B$ becomes infinite, the $G_B^N$ propagator reduces to the Dirichlet one, i.e.

$$G_B^N \big|_{B \to \infty} = G_D. \quad (2.27)$$

## 3 Review: Holography in a Warped Background

We discuss aspects of the holographic formalism for a scalar field in the case of a generic warped background with flat boundary. Since this setup has been studied to death for two decades, this section can be considered as mostly review with bits of less known results. This section also provides sanity checks of the general results from Sec. 2.

### 3.1 Warped Background

We consider a $d+1$-dimensional Lorentzian conformally-flat background with $d$-dimensional Poincaré symmetry along the constant-$z$ slices. The background metric takes the form

$$ds^2_{\text{warped}} = g_{MN} dx^M dx^N = \frac{1}{\rho^2(z)} \left( \eta_{\mu\nu} dy^\mu dy^\nu + dz^2 \right) \quad (3.1)$$

with a boundary at $z = z_0$ constraining $z \geq z_0$.

The quadratic action in Lorentzian signature is

$$S = -\int_M d^{d+1}x \sqrt{|g|} e^{-\varphi} \left( \frac{1}{2} \partial_M \Phi \partial^M \Phi + \frac{1}{2} m^2 \Phi^2 \right) + \frac{1}{2} \int d^d x \sqrt{g} e^{-\varphi_0} \Phi_0 B [\partial_\mu \partial^\mu] \Phi_0 + \cdots \quad (3.2)$$

where $\varphi(z)$ is a dilaton background. This action covers many cases considered in the literature (for a sample, see e.g. [114–119]). For $\rho(z) = k z$, $z_0 = 0$, $\varphi = 0$, the background reduces to the AdS$_{d+1}$ Poincaré patch with curvature $k$.

We Fourier transform along the $z$ slices, using $\Phi(x^M) = \int \frac{d^dp}{(2\pi)^d} e^{ip^\mu y_\mu} \Phi_p(z)$. The wave operator takes the form

$$\mathcal{D} = -\Box + m^2 = -\rho^{d+1} e^\varphi \partial_z \left( \rho^{1-d} e^{-\varphi} \partial_z \right) + \rho^2 p^2 + m^2 \quad (3.3)$$

with $p = \sqrt{\eta_{\mu\nu} p^\mu p^\nu}$.

The physical $p^2$ takes both signs in Lorentzian signature. With the mostly plus metric, we have $p^2 > 0$ for spacelike momentum, $p^2 < 0$ for timelike momentum. In the free theory, $p^2$ is made slightly complex to resolve the non-analyticities arising for timelike momentum. This corresponds to the inclusion of an infinitesimal imaginary shift $p^2 + i \epsilon$, $\epsilon \to 0$. $\epsilon > 0$ is consistent with causality and defines the Feynman propagator. The $i \epsilon$ shift will often be left implicit in our notations.

---

10The normal, outward pointing component of the vielbein obtained from the metric is $e^M_\perp = -\rho(z) \delta^M_z \hat{a}_z$. Hence the normal derivative is $\partial_\perp = -\rho_0 \partial_z$. 

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The homogeneous solutions of $D$ are denoted $f$, $h$ with $Df = 0$, $Dh = 0$. The Wronskian of these solutions, $W = fh' - f'h$, satisfies

$$W(z) = C\rho^{d-1}e^{\varphi}$$  \hspace{1cm} (3.4)

where the only unknown is the overall constant $C$, which depends on the choice of $f$, $h$ solutions.

**Regularity condition**

We specify a regularity condition on a hypersurface away from the boundary, at $z = z_1 > z_0$. On this surface we assume that the $h$ solution blows up while $f$ is the regular solution. The precise condition at $z_1$ is

$$\frac{f(z)}{h(z)} \underset{z \to z_1}{\sim} 0, \quad \frac{f'(z)}{h'(z)} \underset{z \to z_1}{\sim} 0.$$  \hspace{1cm} (3.5)

For timelike momentum, taking $f$ as an outgoing wave and $h$ as an ingoing wave, the condition Eq. (3.5) amounts to a outgoing wave condition. This condition is assumed throughout this section.

**Asymptotics**

In any region where the condition

$$|p^2| \gg m^2 + (\partial_z \log W)^2 + \partial_z^2 \log W$$  \hspace{1cm} (3.6)

is verified, the solutions of the EOM admit the asymptotic behaviour

$$\sqrt{W(z)}e^{\sqrt{p^2}z}, \quad \sqrt{W(z)}e^{-\sqrt{p^2}z}.$$  \hspace{1cm} (3.7)

This can be shown by performing a field redefinition $\Phi = \tilde{\Phi}\rho^{d-1}e^{\varphi/2}$ in the action.

**3.1.1 Propagators**

All the propagators satisfy the bulk equation of motion

$$DG(x, x') = -i\rho^{d+1}e^{\varphi}\delta^{d+1}(x - x').$$  \hspace{1cm} (3.8)

We define $f(z_0) = f_0$, $\partial_z f|_{z_0} = f'_0$ and similarly for $h$, $\rho$ and $\varphi$. The propagators take the following form.

**Neumann propagator:**

$$G^B_N(p; z, z') = \frac{i}{C} \left( \frac{\rho_0h' + Bh_0}{\rho_0f'_0 + Bf_0} f(z_<) - h(z_<) \right) f(z_>)$$  \hspace{1cm} (3.9)

**Boundary-to-boundary propagator:**

$$G^B_N(p; z_0, z_0) = i\rho_0^{d+1}e^{\varphi_0} \frac{f_0}{\rho_0f'_0 + Bf_0}$$  \hspace{1cm} (3.10)
(Amputated) Boundary-to-bulk propagator:

\[ K(p, z) = \rho_0^d e^{\varphi_0} \frac{f(z)}{f_0} \]  

(3.11)

Dirichlet propagator:

\[ G_D(p; z, z') = i \frac{C}{h_0} \left( \frac{h_0}{f_0} f(z_<) - h(z_<) \right) f(z_>) \]  

(3.12)

In some of the above expressions, the Wronskian at \( z_0 \) (Eq. (3.4)) has been used.

In App. B.1 we show that these propagators satisfy the general relations derived in Sec. 2. For example one gets by direct calculation the relation

\[ G_B^R(p; z, z') - G_D(p; z, z') = K(p, z) K(p, z') \left( G_N^{-1}(p; z_0, z_0) - i \rho_0^{-d} e^{-\varphi_0} B \right)^{-1} \]  

(3.13)

which verifies Eq. (2.25) upon translating to Euclidian conventions.

3.1.2 Holographic Action

The holographic basis is

\[ \Phi(p, z) = \frac{f(p, z)}{f_0(p, z_0)} \Phi_0(p, z_0) + \Phi_D(p, z) . \]  

(3.14)

We remind that \( f \) is the regular solution of the EOM away from the boundary, see Eq. (3.5).

We then derive the holographic action. We introduce the spectral representation of the Dirichlet component,

\[ \Phi_D(p; z) = \sum_\lambda \phi_D^\lambda(p) f_D^\lambda(z) \]  

(3.15)

where each Dirichlet mode \( f_D^\lambda \) satisfies the EOM \( D f_D^\lambda(z) = 0 \) with \( p^2 = -m_\lambda^2 \). The Dirichlet modes satisfy the orthogonality and completeness relations

\[ \int dz \rho_0^{-d} e^{-\varphi} f_D^\lambda(z) f_D^{\lambda'}(z) = \delta_{\lambda \lambda'}, \quad \sum_\lambda f_D^\lambda(z) f_D^{\lambda'}(z') = \rho_0^{-d} e^\varphi \delta(z - z') . \]  

(3.16)

Using orthonogonality of the Dirichlet modes and the other properties of the holographic variables previously derived, we get the partition function

\[ Z[J] = \int \mathcal{D} \Phi_0 \mathcal{D} \Phi_D \exp \left[ \frac{i}{2} \int \frac{d^d p}{(2\pi)^d} \left( - \sum \lambda \phi_D^\lambda(p^2 + m_\lambda^2) \phi_D^\lambda + \sqrt{\rho} e^{-\varphi} \Phi_0 \left( \frac{\rho_0}{f_0} \Phi_0 + B \right) \Phi_0 \right) \right. \]

\[ - i \int \frac{d^d p}{(2\pi)^d} \int dz \sqrt{|g|} e^\varphi \left( \frac{f(z)}{f_0} \Phi_0 + \Phi_D(z) \right) J(z) + \text{interactions} \]  

(3.17)

This is a consistent with the main formula Eq. (2.22) upon translating into Euclidian conventions. The \( p \)-dependence of the fields is left implicit, it is \( \Phi_\varphi = \Phi(p) \varphi(-p) \) for each monomial of the quadratic action.
3.1.3 Holographic Mixing

In the class of warped backgrounds considered here, the holographic self-energy may feature a pole indicating the existence of a $d$-dimensional free field, here denoted $\Phi_0$. This is the mode satisfying the bulk EOM with $p^2 = m_0^2 \ll m^2$, which can always exist upon appropriate tuning of the boundary terms in $B$ (see e.g. discussion in [120]). In the presence of this mode, a variant of the holographic basis is to let $\Phi(z; p) = \tilde{\Phi}_0(p)K(p; z) + \Phi_D(z; p)$ (see Refs. [121, 122] for the original proposal in the case of a slice of AdS). In this basis all the degrees of freedom are free fields but the resulting holographic action is nondiagonal. The action contains a cross term between $\tilde{\Phi}_0$ and $\Phi_D$, taking the form

$$\int dz \rho^{1-d} e^{-\epsilon} \tilde{\Phi}_0(p)(p^2 - m_0^2)K(p; z) \sum_{\lambda} f(z) \phi_\lambda^D(-p)$$ (3.18)

upon integration by parts. This term induces both kinetic and mass mixing between the $\tilde{\Phi}_0$ and $\phi_\lambda^D$ fields — this is how the boundary degree of freedom knows about the bulk modes in this basis. The interpretation of this “holographic mixing” is discussed in details in [121, 122]. In our present work the holographic action is instead exactly diagonal, to the price of having a nontrivial self-energy for the boundary degree of freedom.

3.2 Boundary Flux and Unitarity

We present an elementary application of the formalism of Sec.3.1.

Consider a $2 \rightarrow 2$ timelike process where the $\Phi$ is produced from a boundary-localized interaction, e.g. a $\bar{\Psi}\Psi\Phi_{|_{\partial M}}$ coupling. The boundary scattering amplitude is denoted by $iA_{\bar{\Psi}\Psi \rightarrow \bar{\Psi}\Psi}$. This amplitude is proportional to the boundary-to-boundary propagator given in Eq. (3.10),

$$iA_{\bar{\Psi}\Psi \rightarrow \bar{\Psi}\Psi} = -\alpha G_N^B(p; z_0, z_0)$$ (3.19)

The $\alpha$ is a positive coefficient encoding couplings and other overall constants.

The production rate of bulk modes, $\sigma_{\bar{\Psi}\Psi \rightarrow \Phi}$, can be obtained from a unitary cut of the $iA$ amplitude. The unitarity cut amounts to take the imaginary part of $A$, one gets

$$\sigma_{\bar{\Psi}\Psi \rightarrow \Phi} = 2 \text{Im}A = 2\alpha \text{Im}[iG_N^B(p; z_0, z_0)] = 2\alpha \rho_0^d e^{\rho_0} \frac{\rho_0}{\rho_0 f_0 + B f_0} \text{Im}[f_0 f_0^\dagger]$$ (3.20)

We then evaluate $\text{Im}[f_0 f_0^\dagger]$. By definition, $f$ is the outgoing wave solution. The conjugate $f^\dagger$ must contain the ingoing wave solution, $f^\dagger = h + \ldots$ where we chose a unit coefficient for $h$ without loss of generality. It follows that the imaginary part takes the form $\text{Im}[f_0 f_0^\dagger] = \frac{1}{\rho_0}(f_0^\dagger h_0 - f_0 h_0^\dagger)$ and is thus computed by the Wronskian Eq. (3.4). Because of $f^\dagger = h + \ldots$ one has $W = -W^\dagger$, thus the Wronskian is imaginary. Thus the Wronskian can be written as $W = -ic\rho^{d-1}e^\epsilon$ with $c \in \mathbb{R}$.

What is the sign of $c$? Assuming that the EOM is regular everywhere, the Wronskian cannot be zero. Thus $\text{Im}W$ has a definite sign everywhere, encoded into $c$. To determine the sign of $c$ we consider the large $p$ asymptotic limit obtained in Eq. (3.7), with $f \sim e^{ipz}$, $g \sim e^{-ipz}$. The corresponding asymptotic Wronskian is $-2ip$, therefore $c > 0$. 

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Putting the pieces together we obtain that the imaginary part is given by $\text{Im} \left[ f_0 f_0^\dagger \right] = \frac{1}{2} \rho_0^2 e^{-2\varphi_0}$. As a result the production rate of bulk modes is given by

$$\sigma_{\Psi \to \Phi} = \alpha c \frac{\rho_0^2 e^{2\varphi_0}}{|\rho_0 f_0 + B f_0|^2}. \tag{3.21}$$

This is a very simple, exact formula for the flux emitted from the boundary. We see that only the regular solution near the boundary is needed to determine it. We can also notice that the $c > 0$ condition derived above ensures that unitarity is respected.

### 3.3 Properties of AdS Propagators

We review the propagators in AdS background (with regularized boundary) from the viewpoint of our formalism.

In AdS we have $\rho(z) = k z$ with $k$ the AdS curvature. We take $g_{MN} \equiv g_{\text{AdS}}_{MN}$ throughout this subsection. The scalar bulk mass is written as

$$m^2 = \Delta (\Delta - d) k^2. \tag{3.22}$$

Defining $\Delta_+ = d - \Delta_+ > \Delta_-$, the solutions to the wave equation $D \Phi = 0$ near the AdS boundary ($z = 0$) are

$$\Phi(x, z) = z^{\Delta_+} (A_+(x) + O(z^2)) + z^{\Delta_-} (A_-(x) + O(z^2)) \tag{3.23}$$

For $m^2 > -\frac{d^2}{4} + 1$, the $-$ modes are non-normalizable. For $-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1$, both $+$ and $-$ modes are normalizable. The modes can be selected by imposing appropriate condition on the AdS boundary,

$$(z \partial_z - \Delta_+) \Phi|_{z=0} = 0 \tag{3.24}$$

However in AdS calculations (e.g. for Witten diagrams and AdS/CFT), it is often necessary to regulate the AdS boundary, which amounts to truncate spacetime such that $z > z_0$ with $z_0 \neq 0$. Even if $z_0$ is infinitesimal, the propagators reflect on the boundary and take the form given in section (3.1.1) instead of being simply $-i C^{-1} g(z_\prec) f(z_\succ)$. Moreover, in our formalism we have standard Neumann and Dirichlet BC on the regulated boundary. How does this language compare to the BC Eq. (3.24)? To understand how our regulated formalism matches onto the unregulated one, we start from our holographic basis in AdS (given in Eq. (3.14)) and take the near-boundary asymptotic values for $K \propto \frac{1}{f_0}$ and $f_0^\Lambda$. The result is

$$\Phi = \Phi_0 \left( \frac{z}{z_0} \right)^{\Delta_-} \left( 1 + O(z^2) \right) + z^{\Delta_+} \left( A_+(x) + O(z^2) \right) \tag{3.25}$$

for any $\Delta$. The first term, which has Neumann BC, always has $z^{\Delta_-}$ scaling. The second term, which has Dirichlet BC, always has $z^{\Delta_+}$ scaling. Hence the $-$ modes correspond

\[1\] In the notation of Sec. 3.1.1, the solutions satisfying the regularity condition Eq. (3.5) are $f(z) = z^{d/2} K_{\Delta_+ - d/2} (\sqrt{p^2} z)$, $h(z) = z^{d/2} I_{\Delta_+ - d/2} (\sqrt{p^2} z)$. $f$ and $h$ match respectively onto the $-$ and $+$ asymptotics discussed here.
to Neumann and the + modes correspond to Dirichlet. In the context of the boundary effective action, the Dirichlet term in Eq. (3.25) amounts to bulk fluctuations, while the Neumann term can be treated as background. This matches the usual identification of source and fluctuation deduced from AdS asymptotic solutions (see e.g. [123–125]).

3.3.1 AdS Bulk Propagators

The Neumann/Dirichlet identification implies that the regulated bulk propagators satisfy

\[ G_{\Delta^+} = G_N, \quad G_{\Delta^-} = G_D \]

(3.26)

when \( z_0 \to 0 \). Only the second propagator exists if \( \frac{m^2}{k^2} > -\frac{d}{4} + 1 \) because \( - \) modes are not normalizable in that case.

We can readily apply the general results of section 2. The general relation between \( G_D \) and \( G_N \) obtained in Eq. (2.25) becomes here

\[ G_{\Delta^-} (p ; z, z') = G_{\Delta^+} (p ; z, z') + K(p, z) K(p, z') G_{\Delta^-} (p ; z_0, z_0) \bigg|_{z_0 \to 0} \]

(3.27)

in Fourier space. The \( K \) are computed from \( G_N \), and thus here from the \( G_{\Delta^+} \) propagator. The Neumann-Dirichlet identity Eq. (3.27) has been known in the AdS literature see e.g. [67, 126, 127], but was not given a deeper explanation. Our approach shows that such a relation is intrinsic to the holographic formalism and exists for any background geometry.

We can also comment on the conformal spectral representation of AdS propagators (see [128–130], and [41, 42, 52, 57, 62, 131] for some applications). The \( \Delta^+ \) propagator can be written as

\[ G_{\Delta^+} = k^{d-1} \int_{\mathbb{R}} d\nu P(\nu, \Delta) \Omega_\nu \]

(3.28)

where \( \Omega_\nu \) is a known harmonic kernel and \( P(\nu, \Delta) = \frac{1}{\nu^2 + (\Delta - d/2)^2} \). In contrast [42], the \( \Delta^- \) propagator takes the form

\[ G_{\Delta^-} = k^{d-1} \int_{\mathbb{R} + C_u + C_d} d\nu P(\nu, \Delta) \Omega_\nu \]

(3.29)

where \( C_u, d \) are contours wrapping the \( \nu = \pm i(\Delta - d/2) \) clockwise and counterclockwise respectively. We verified explicitly using Ref. [57] that the harmonic kernel satisfies

\[ i \frac{2\pi}{\nu k} \Omega(p ; z, z') = K(p, z) K(p, z') G_{\Delta^-} (p ; z_0, z_0) \bigg|_{z_0 \to 0} \].

(3.30)

This implies that Eq. (3.29) is equivalent to Eq. (3.27). The \( \mathbb{R} \) integral gives \( G_D \), while the \( C_u, d \) integrals give the \( KKG \) term.

3.3.2 AdS Boundary-to-Bulk Propagators

The AdS propagators need to properly encode the regulated AdS boundary in order to obtain the correct CFT correlators, see e.g. [125, 132]. Such a regularization is built-in in our approach, since all propagators are evaluated in the presence of the regulated boundary without any approximation.
Let us work out the CFT 2pt function in our formalism. We start from our general definition $K(x) = G_N \circ [G_0]^{-1}$. One can easily evaluate $\partial_\perp K$ using the discontinuity equation (as in section 2) giving $\partial_\perp K = [G_0]^{-1}$. The subsequent evaluations are best done in momentum space because the final result may have contact terms, that we will need to take into account in subsequent calculations. We have

$$\partial_\perp K = -\frac{1}{\sqrt{g}} \int \frac{dz}{(2\pi)^d} \int \frac{dz}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} K_{\Delta - \frac{d}{2}}(p, z) K_{\Delta - \frac{d}{2}}(p, z_0)$$

(3.31)

which matches the usual result [132]. This expression is exact and is invariant under $\Delta_+ \leftrightarrow \Delta_-$. We then focus, as customary, on the large chordal distance limit $\zeta(x_0, x'_0) \gg \frac{1}{k^2}$ (see App. B.1 for position space expressions), which amounts to the $(y - y')^2 \gg z_0^2$ limit, and thus to $p^2 z z_0 \ll 1$ in momentum space. For any value of $\Delta$, the leading term scales as $p^2 \Delta - d$, where $\Delta_+ > \Delta_-$. Evaluating the Fourier transform and plugging the result into the Euclidian boundary action $S_E = \frac{1}{2} \Phi_0 \circ \partial_\perp K \circ \Phi_0 + \ldots$ gives the asymptotic result

$$S_{2pt}^E = \frac{1}{2} \int d^d y d^d y' \sqrt{g} \left[ \frac{\mathcal{C}_\Delta \eta_\Delta}{(y - y')^{2\Delta_+}} + \Delta_+ k \delta^d(y - y') + O(\zeta^{-\Delta_+ - 1}) \right] \Phi_0(y) \Phi_0(y')$$

(3.32)

with

$$\mathcal{C}_\Delta = \frac{(2\Delta - d) \Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - \frac{d}{2})}, \quad \eta_\Delta = k z_0^{2\Delta - d}$$

(3.33)

The nonlocal part of $S_{2pt}^E$ gives rise to the CFT 2pt function with the correct normalization factor $\mathcal{C}_\Delta$. The remaining dimensionful factor $\eta_\Delta$ (or $\sqrt{g} \eta_\Delta$) can be absorbed into the normalization of the CFT operators. We see that, in addition to the standard nonlocal part, there is a contact term.

If one evaluates $[G_0]^{-1}$ in position space by taking the $\zeta \gg \frac{1}{k^2}$ limit and using a conformal integral to perform the inversion, one gets precisely the nonlocal part of Eq. (3.32) but not the contact term. One could also try to evaluate the whole $G_N \circ [G_0]^{-1}$ expression in position space, by taking the $\zeta \gg \frac{1}{k^2}$ limit for $G_N$ and $G_0$. Such an approximation gives a divergent result for $z \to z_0$. We trace back these discrepancies to the fact that the $\zeta \gg \frac{1}{k^2}$ limit, which requires large $|y - y'|$, does not commute in general with the convolutions in $y$-space—which involve integrals over arbitrary values of $y$. We conclude that the complete result is Eq. (3.32). In the following we will see that the contact term in Eq. (3.32) is needed to ensure consistency of results with conformal symmetry.

### 3.4 Summary

While this section is mostly a review, there are lessons to take away. We have shown that for the warped background Eq. (3.1), even in the presence of a dilaton background, the holographic action and the propagators take simple, explicit expressions, making holography in this class of background essentially as simple as in the Poincaré patch of AdS. We used this background to perform checks of the general formalism of Sec. 2. We also derived
a simple general formula for the flux of bulk modes emitted from the boundary by using a unitarity cut on a boundary $2 \to 2$ exchange diagram.

Going back to AdS, we have cast a new light on an elementary aspect of the propagators, the Neumann-Dirichlet identity Eq. (3.27), which is understood to be an intrinsic property of the holographic formalism, and not as a specificity of AdS. We have also shown that our approach gives the CFT 2pt function with correct normalization and we have taken into account the 2pt contact term.

4 The Quantum Pressure on the Boundary

Throughout the rest of this paper, our focus is essentially on integrating out the bulk modes. In the present section our interest is on free bulk modes and their effect on the boundary.

4.1 Preliminary Observations

We consider a free scalar supported on the class of warped background discussed in section 3. Our aim is to investigate the pressure on the boundary induced by the quantum fluctuations of the field. To obtain this quantum pressure we vary the energy of the quantum vacuum with respect to the position of the boundary.

The energy of the QFT vacuum is a divergent quantity. However, if one varies it with respect to a physical parameter (such as, say, the distance between two plates), the resulting variation is a physical observable and thus must be finite. If divergences still appear in the expression of the vacuum pressure, they have a physical meaning and have to be treated in the framework of renormalization.

To obtain an observable, finite pressure, we need to assume that spacetime does not end on the other side of the boundary. Alternatively one can view the region beyond the boundary as an extended source strictly prohibiting the field to propagate inside it. Assuming that the extended source ends in some region far away from the boundary, the far region provides a contribution that renders the pressure finite. The latter viewpoint was used in [133], where we recovered e.g. the standard Casimir pressure between plates, without the need of any extra regularization method.

Here we adopt the former viewpoint: there is spacetime with a propagating field on each side of the boundary. In that view the boundary can be referred to as a “brane” on which the bulk modes vanish. We refer to the $z < z_0$ and $z > z_0$ regions as “left” and “right”. The physical parameter that we vary is $z_0$, the location of the boundary.

In the presence of curvature and/or dilaton background, the variation of the boundary can be tricky to evaluate. It is useful to rescale the fields in the action as follows

$$\Phi_D = \beta^{d-1} e^{\hat{\Phi}}$$

(4.1)

to absorb the geometric prefactors into the fields. With this field redefinition, the prefactors are moved into an effective position-dependent mass term for $\hat{\Phi}_D$, that leads to no difficulties in the calculation of the variation. This trick simplifies the calculations and, as we will
see, renders manifest the small-distance limit for which the flat metric is asymptotically recovered.

Starting from the partition function with no sources, $Z(z_0)$, the vacuum energy is given by

$$E_{\text{vac}}(z_0) = \frac{W(z_0)}{T}, \quad W(z_0) = i \log Z(z_0).$$

The leading order contribution from the bulk modes is the one-loop determinant,

$$E_{\text{vac}}(z_0) = -\frac{i}{2} \text{Tr} \log (D) = -\frac{i}{2} \sum_\lambda \int A_{d-1} \frac{d^d p}{(2\pi)^d} \log (p^2 + m_\lambda^2)$$

where $A_{d-1}$ is the boundary spatial volume. Taking the $z_0$ variation leads to the definition of the vacuum pressure on the $d$-dimensional boundary:

$$-\frac{F}{A_{d-1}} = \frac{\delta E_{\text{vac}}(z_0)}{\delta z_0} = \frac{1}{2} \sum_\lambda \int d^d p_E \frac{1}{p_E^2 + m_\lambda^2} \frac{\delta m_\lambda^2}{\delta z_0}.$$  \hfill (4.3)

We have performed a Wick rotation to integrate in the Euclidian momentum $p_E$. A positive (negative) value of the pressure means that the boundary is pushed towards positive (negative) $z$.

We could have started from a theory in Euclidian space — the metric signature is irrelevant for the quantity of interest. We work in Lorentzian signature for which the brane has $(d - 1)$ spatial dimensions.

For $d = 1$ the 0-brane is a point particle. For $d > 1$ the brane is an extended object and can have a localized energy density (i.e. brane tension)

$$S \supset \int dx^d \sqrt{g} \sigma.$$  \hfill (4.4)

Depending on the value of $\sigma$ the brane may be static or have velocity in the bulk — the solutions of Einstein’s equation for this system have been classified in [134]. Here we assume that $\sigma$ is tuned such that the brane is static, as in the RS2 model.

### 4.2 Vacuum Pressure from Bulk Modes

What is the mass variation $\delta m_\lambda^2/\delta z_0$ appearing in Eq. (4.3)? To determine it, we consider a piece of the action for one given Dirichlet mode,\(^1\)

$$S_\lambda = -\frac{1}{2} \int_{z_0}^{\infty} dz \left[ \partial_M (f^\lambda (z) \phi_D^\lambda) \partial^M (f^\lambda (z) \phi_D^\lambda) + m^2 (f^\lambda (z) \phi_D^\lambda)^2 \right]$$

We can evaluate the variation $\delta S_\lambda/\delta z_0$ in two ways.

First, one can integrate by part and use the orthogonality relation of the modes, which gives

$$S_\lambda = -\frac{1}{2} (p^2 + m_\lambda^2) (\phi_D^\lambda)^2$$

The variation in $z_0$ is then found to be

$$\frac{\delta S_\lambda}{\delta z_0} = -\frac{1}{2} \frac{\delta m_\lambda^2}{\delta z_0} (\phi_D^\lambda)^2$$

\(^1\)In this section we use the rescaled field $\hat{\Phi}$ everywhere (see Eq. (4.1)), thus the profiles and propagators are understood as $\hat{f}_\lambda, \hat{G}$. The hats will be omitted throughout the section.
Second, we can instead apply the variation directly to Eq. (4.5). The only contribution is from the change of integration domain. This gives
\[
\frac{\delta S_\lambda}{\delta z_0} = \frac{1}{2} \left( \phi_\lambda^D \right)^2 \left. \partial_z f^\lambda(z) \partial_{z'} f^\lambda(z') \right|_{z,z' = z_0}
\] (4.8)

We thus obtain the mass variation\(^\text{13}\)
\[
\frac{\delta m_\lambda^2}{\delta z_0} = - \left. \partial_z f^\lambda(z) \partial_{z'} f^\lambda(z') \right|_{z,z' = z_0}.
\] (4.9)

Plugging Eq. (4.9) into Eq. (4.3), we recognize the spectral representation of the Dirichlet propagator with Euclidian momentum. We obtain the contribution to the pressure from the right region \((z > z_0)\)
\[
\frac{F_R}{A_{d-1}} = i \int \frac{d^d p_E}{(2\pi)^d} \partial_z \partial_{z'} G_D(p_E; z, z') \left|_{z,z' = z_0} \right.
\] (4.10)

We proceed similarly with the contribution from the left region \((z < z_0)\). We obtain the total pressure
\[
\frac{F}{A_{d-1}} = i \int \frac{d^d p_E}{(2\pi)^d} \left[ \partial_z \partial_{z'} G_D^R(p_E; z, z') - \partial_z \partial_{z'} G_D^L(p_E; z, z') \right]_{z,z' = z_0}
\] (4.11)

Moreover we can use the explicit expression for the Dirichlet propagator given in Eq. (3.12) —applied to the rescaled field \(\hat{\Phi}_D\). The regular solution in the left and right regions are denoted by \(f_L, f_R\). Using spherical coordinates and defining \(p = |p_E|\), the final result for the quantum vacuum pressure is found to be\(^\text{14}\)
\[
\frac{F}{A_{d-1}} = - \frac{1}{2^d \pi^{d/2} \Gamma(d/2)} \int dpp^{-d-1} \left( \frac{f'_L(z_0)}{f_L(z_0)} + \frac{f'_R(z_0)}{f_R(z_0)} \right)
\]
\[
= - \frac{1}{2^d \pi^{d/2} \Gamma(d/2)} \int dpp^{-d-1} \partial_z \log \left( K_L(p, z) K_R(p, z) \right) \bigg|_{z = z_0}
\] (4.12)

where \(K_{L,R}\) are the boundary-to-bulk propagators in the left and right regions. Moreover, comparing to Eq. (3.17), we see that the quantum pressure is proportional to the sum of the holographic self-energies from each side of the boundary.

Some sanity checks can be done. In the limit of large Euclidian momentum, when spacetime curvature and mass are negligible, the regular solutions are asymptotically exponentials as dictated by Eq. (3.7). In this limit we have \(f_R(z) \sim e^{p z}, f_L(z) \sim e^{-p z}\). As a result the integrand of Eq. (4.12) vanishes asymptotically in this limit. This is the expected flat space behaviour. Note that this does not imply that the integral is automatically finite. Depending on how fast the metric becomes flat at small distance, \(i.e.\) depending on the large \(p\) behaviour, there can be divergences from the \(p\) integral, that we will treat using standard dimensional regularization.

\(^{13}\)I am grateful to E. Ponton for providing insights on this trick in an early unpublished work [135].

\(^{14}\)We notice a resemblance with the result from the \(D\)-sphere presented in [136]. Our result, however, does not have any unwanted divergences apart from those renormalizing the brane tension.
Second, we recover the Casimir pressure in a Minkowski interval \[137\]. We assume the presence of a second brane in the left region, at \(z = z_0 - \ell < z_0\). We have \(f_R = e^{-pz}\), while \(f_L = \sin(p(z - z_0 + \ell))\). We obtain \(\partial_z \log K_L(z) = p \coth(p\ell)\), \(\log \partial_z K_R(p, z) = -p\), giving exactly the scalar Casimir pressure between two plates of spatial dimension \(d - 1\),

\[
\frac{F_{\text{flat}}(\ell)}{A_{d-1}} = -\frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \int dp \frac{p^d}{e^{2p\ell} - 1} = -\frac{d \Gamma\left(\frac{d+1}{2}\right) \zeta(d + 1)}{2^{d+1} \pi(d+1)/2 \ell^{d+1}}. \tag{4.13}
\]

In particular for \(d = 3\) we recover the well-known result \(F/A_2 = -\pi^2/480\ell^4\).

The fact that the pressure on the \(z_0\) brane is negative means that it is attracted towards negative \(z\), consistent with the fact that the second brane is placed at the left of \(z_0\).

### 4.3 Vacuum Pressure in AdS

We turn to a single brane in AdS metric. We consider the fluctuations from a bulk scalar field with mass \(m^2 = \Delta(\Delta - d)k^2 = (\alpha^2 - \frac{d^2}{4})k^2\) (with \(\alpha \geq 0\)), existing in both left \((z < z_0)\) and right \((z > z_0)\) regions. The solutions to the bulk EOM for the rescaled field are \(\sqrt{z} I_\alpha(\sqrt{p^2 z})\), \(\sqrt{z} K_\alpha(\sqrt{p^2 z})\) in any dimension.

Using the general formula Eq. (4.12), the pressure induced by the scalar fluctuations is

\[
\frac{F_{\text{AdS}}(\alpha)}{A_{d-1}} = -\frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \int dp p^{d-1} \partial_z \log (zI_\alpha(pz)K_\alpha(pz)) \bigg|_{z=z_0} \tag{4.14}
\]

This expression can be evaluated analytically only in particular cases and limits. However, general features can already be deduced. By dimensional analysis the pressure must scale as

\[
\frac{F_{\text{AdS}}(\alpha)}{A_{d-1}} \propto \frac{1}{z^d}. \tag{4.15}
\]

Regarding the sign of the pressure, a qualitative statement can be done when the integral is finite such that no regularization is needed. We notice that the integrand of Eq. (4.14) is positive for \(\alpha \geq \frac{1}{2}\). Under these restrictions we know that the pressure is negative, i.e. the quantum pressure pushes the brane towards the AdS boundary.

#### 4.3.1 Conformally Massless Scalar

For \(\alpha = \frac{1}{2}\) in any \(d\), the scalar is conformally massless upon a Weyl transformation to flat space. For this value of \(\alpha\), the field in AdS effectively behaves just as in flat half-space with boundary at \(z = 0\). Accordingly, the vacuum pressure reduces to the flat space result, for a separation \(z_0\) corresponding to the distance between the \(z = 0\) boundary and the brane \((z = z_0)\),

\[
\left.\frac{F_{\text{AdS}}(\alpha = \frac{1}{2})}{A_{d-1}}\right|_{\alpha = \frac{1}{2}} = \frac{F_{\text{flat}}(\ell)}{A_{d-1}}. \tag{4.16}
\]

#### 4.3.2 Uniform Expansion and the Flat Space Limit

In the large \(pz\) limit and for \(\alpha \neq \frac{1}{2}\) the asymptotic behaviour of the Bessel functions leads to an asymptotically vanishing integrand in Eq. (4.14). However, if one uses this expansion
Figure 2. Force from the quantum vacuum felt by a point particle in AdS$_2$ in the presence of a scalar field with mass $m^2 = (\alpha^2 - \frac{1}{4})k^2$. For a conformally massless scalar $\alpha = \frac{1}{2}$, the flat space Casimir force between the particle and the boundary is recovered. For any $\alpha$ the point particle is attracted towards the AdS boundary. The dotted line shows the result from large mass expansion, Eq. (4.18).

to approximate the $pz > 1$ region of the integral, highly inexact results occur. This can be traced back to the fact that the bulk mass is neglected in the usual large $pz$ asymptotics.

The limit of large $p$ at fixed $z$ amounts to the small distance regime, and thus should asymptotically recover flat space. In flat space, masses play a role in the evaluation of finite loop integrals. Thus we need an approximation where the bulk mass is taken into account in the flat space limit, so that massive (as opposed to massless) flat space QFT appears. This limit is realized by the “uniform expansion” of Bessel functions, [138] giving

$$I_\alpha(\alpha y)K_\alpha(\alpha y) = \frac{u}{2\alpha} \left( 1 + \frac{u^2 - 6u^4 - 5u^6}{8\alpha^2} + O\left(\frac{1}{\alpha^4}\right) \right) \quad \text{for} \quad \alpha \to \infty \quad (4.17)$$

with $u = (1 + y^2)^{-\frac{1}{2}}$.  

4.3.3 AdS$_2$

For $d = 1$, the “brane” is a point particle, thus we talk about force instead of pressure. The force computed by Eq. (4.14) is finite. We evaluate the force at large $\alpha$ using the expansion Eq. (4.17).

$$F_{\text{AdS}_2} \bigg|_{\alpha \gg 1} = -\left( \frac{\alpha}{4} - \frac{1}{256\pi\alpha} + O(\alpha^{-3}) \right) \frac{1}{z_0^2} \quad (4.18)$$

We compute numerically the pressure and find excellent agreement down to $\alpha \sim \frac{1}{2}$. The force is shown in Fig. 2. At $\alpha = \frac{1}{2}$ the flat space Casimir pressure $F_2 = -\frac{\pi}{24z_0^2}$ is recovered.

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We emphasize that the uniform expansion could also be of relevance for other loop calculations in AdS and dS.
The force is found to be negative for any value of the bulk mass: The point particle is attracted towards the AdS boundary.\footnote{The quantum force induces a correction to the classical trajectory of the point particle. What is the classical trajectory? For $\alpha = \frac{1}{2}$ the point particle follows a null geodesic, which is just a straight line along $z$, orthogonal to the boundary (notice that for $\alpha = \frac{1}{2}$ the particle effectively sees flat space thus its geodesic must be a straight line). For $\alpha \neq \frac{1}{2}$ the geodesics are half circles ending on the boundary.}

4.3.4 AdS$_{>2}$

For $\alpha \neq \frac{1}{2}$ and $d \geq 2$, divergences appear in the integral Eq. (4.14). These divergences can be seen using the uniform expansion Eq. (4.17), which gives

$$\partial_z \log \left( z I_\alpha(p z) K_\alpha(p z) \right) \bigg|_{z = z_0} = \frac{1}{2} \frac{a^2}{z^{a^2 + z^2 p^2}} - \frac{4 \alpha^2 z p^2 - 10 \alpha^2 z^3 p^4 + z^5 p^6}{4 (\alpha^2 + z^2 p^2)^4} + \ldots$$  \hspace{1cm} (4.19)

The degree of divergence for each term under the $\int dp p^{d-1}$ integral is then obvious. Through Eq. (4.19) we also see that the integrand takes the same form as in massive flat space, with terms of the form $1/(p^2 + \Delta)^n$. The integrals can thus be treated using textbook dimensional regularization. That is, power-law divergences are automatically removed and only logarithmic divergences remain. There are log divergences for even $d \geq 2$ only.

The result for general $d$ is

$$\frac{F_{\text{AdS}_{d+1}}}{A_{d-1}} \bigg|_{\alpha \gg 1} = -\left( \frac{\alpha^d}{2 d^2 + 1} \right)^{\Gamma \left( 1 - d/2 \right) + O \left( \alpha^{-2} \right)} \frac{1}{z_0^{d+1}}$$  \hspace{1cm} (4.20)

The divergences at even $d \geq 2$ appear via the Gamma function, as customary from dimensional continuation.

Since the Gamma function with negative argument can take both signs, the expression can take either sign depending on the dimension. For odd $d$, keeping the leading term, the expression gives

$$\frac{F_{\text{AdS}_4}}{A_2} \bigg|_{\alpha \gg 1} \sim \frac{\alpha^3}{8 \pi z_0^3}, \quad \frac{F_{\text{AdS}_6}}{A_4} \bigg|_{\alpha \gg 1} \sim -\frac{\alpha^5}{48 \pi^2 z_0^5} \ldots$$  \hspace{1cm} (4.21)

What quantity is being renormalized for even $d$? As can be seen by comparing with AdS$_2$, the divergences are tied to the brane being an extended object, hence we can expect that the quantity being renormalized is brane-localized, and the only candidate is the brane tension. The brane tension takes the form

$$S \supset \int dx^d z_0^{-d} (\sigma + \delta \sigma)$$  \hspace{1cm} (4.22)

where $\delta \sigma$ is the $z_0$-independent counterterm. We introduce $d = n - \epsilon$ with $n$ a positive even integer. Varying in $z_0$, the $\delta \sigma$ counterterm cancels the divergence for

$$d \delta \sigma = \frac{(-1)^{d/2} \alpha^d}{2 \pi^{d/2} \Gamma(d/2) \epsilon} + \text{finite}$$  \hspace{1cm} (4.23)

The finite quantum contribution to the pressure is then proportional to $\log(z_0 \mu) z_0^{-d-1}$, where $\mu$ is the renormalization scale. The finite part in the counterterm Eq. (4.23) is
absorbed in the definition of the renormalization scale. The final result for the quantum pressure felt by the brane is

$$\frac{F_{\text{AdS}}}{A_{d-1}} \sim \frac{(-1)^{d/2} \alpha^d}{2^{d+1} \pi^{d/2} \Gamma(d/2)} \log(z_0 \mu) \frac{\log(z_0 \mu)}{z_0^{d/2} \Gamma(d/2)}$$

for even \(d \geq 2\) (4.24)

at leading order in the \(1/\alpha\) expansion.

The quantum contributions arising in \(\text{AdS}_{d+1}\) would dominate the motion of the brane only if the bare tension is tuned such that the brane is static at classical level (see [134] for brane motion as a function of brane tension.)

5 Interactions and Boundary Correlators: Overall Picture

In this section and the following ones we turn on the interactions. Interactions are easily written in terms of holographic variables in the general action Eq. (2.22). For example the coupling \(\Phi^n(x)\) takes the form

$$\Phi_0 \circ K + \Phi_D$$

(5.1)

This leads to monomials of the form \((\Phi_0 \circ K)^k \Phi_D^{n-k}\). Each of these vertices contribute to the correlators. In this work we are interested in boundary observables, which are defined by insertions of boundary-localized sources \(J_0 = J|_{\partial M}\) in the partition function \(Z[J_0]\).

A \(J_0\) source probes only the boundary degrees of freedom and not the Dirichlet modes. Let us first integrate over the Dirichlet modes in \(Z[J_0]\). This gives

$$Z[J_0] = \int D\Phi_0 e^{-S_D[\Phi_0] - \Phi_0 \circ J_0}$$

(5.2)

where we introduced a nonlocal boundary action \(S_D\) that we refer to as the “Dirichlet action”. It is a functional of the boundary degree of freedom \(\Phi_0\). At perturbative level, the Dirichlet action encodes the diagrams with only internal bulk lines \(G_D\), that we refer to as Dirichlet diagrams. These are for example diagrams \(i)\) to \(iii)\) in Fig. 3. Since \(\Phi_0\) is a dynamical field, \(S_D\) can be seen as a fundamental action in \(\Phi_0\) — encoding very nonlocal operators. In AdS the Dirichlet action encodes the Witten diagrams with \(\Delta\) propagators in internal lines.

Among all the operators in \(S_D\) it is useful to single out the class of operators

$$\int_M d^{d+1}x \sqrt{|g|} \prod_{i=1}^n K_i \circ \Phi_{i,0}(x)$$

(5.3)

These are the operators generated by a single bulk vertex, they are in this sense the most local operators. We define the associated “holographic vertices”

$$\Lambda_n(x_0,i) = \int_M d^{d+1}x \sqrt{|g|} \prod_{i=1}^n K(x_{0,i},x)$$

(5.4)

A 3pt holographic vertex is shown as diagram \(i)\) in Fig. 3. The \(\Lambda_n\) are in one-to-one correspondence with bulk vertices and are proportional to contact Witten diagrams.
Let us turn to the definition of boundary correlators generated by $Z[J_0]$. They are given by

$$\langle \Phi_0(x_{0,a}), \Phi_0(x_{0,b}) \ldots \rangle.$$  
(5.5)

In particular, the connected boundary correlators are generated by taking the derivatives of $W[J_0] = \log Z[J_0]$ (this is the convention for Euclidian signature, there is an extra $i$ for Lorentzian signature). Given the definition of the Dirichlet action Eq. (5.2), the boundary connected diagrams are built from boundary subdiagrams with only Dirichlet lines (i.e. Dirichlet subdiagrams) connected to each other by boundary lines. Examples of such connected correlators are shown as diagrams iv) to vi) in Fig. 3.

**Figure 3.** Sample of boundary connected correlators. Orange and blue points are respectively integrated over $M$ and $\partial M$.

We can further define the boundary effective action, i.e. the generating functional of 1PI boundary connected correlators via the Legendre transform

$$\Gamma[\Phi_{0,cl}] = -W[J_0] + J_0 \circ \Phi_{0,cl}.$$  
(5.6)

Since the Legendre transform is done with respect to the boundary source, it amputates boundary-to-boundary propagators on each leg of the correlators. Likewise, the notion

---

\footnote{We remind that in our notation, the boundary coordinates $y^\mu$ appear through the $x_0^M(y^\mu)$ embedding.}
of one-particle irreducibility (1PI) is here meant with respect to boundary-to-boundary
lines only. Thus in Fig. 3 only diagram \textit{iv}) is 1PR, all the other ones are 1PI since they
cannot be split by cutting a single line on the boundary. Since boundary-to-boundary
correlators are amputated on each external leg, the 1PI correlators obtained by taking
derivatives of \(\Gamma[\Phi_{0,c}]\) have amputated boundary-to-bulk propagators as external legs (see
definition Eq. (2.15)). These objects are the generalization of Witten diagrams to arbitrary
background. In the following we refer to Witten diagrams as “boundary correlators” when
no ambiguity is possible.

5.1 On Holographic CFT Correlators

We have seen that a generic contribution to a given \(n\)pt boundary correlator is made of
Dirichlet subdiagrams (i.e. with only \(G_D\) in internal lines) connected to each other by
boundary lines. This perturbative substructure of the diagrams formally appears when
singling out the Dirichlet action in the partition function, Eq. (5.2). We also know that
when the background is AdS, the boundary correlators are identified as correlators of a
strongly coupled large-\(N\) CFT. We can thus wonder: How does the substructure of a
diagram in terms of Dirichlet subdiagrams translate in the holographic CFT?

The answer certainly involves the notion of large-\(N\) corrections. First we stress that
the Dirichlet subdiagrams can be of arbitrary loop order which in the holographic CFT
is understood as \(1/N\) corrections. Hence the Dirichlet subdiagrams themselves encode \(1/N\)
corrections. Our focus here is rather on the structure of the entire diagram. In order to
figure out the CFT meaning of an entire diagram, let us take an example.

We consider \textit{iv}) in Fig. 3 which is induced by quartic couplings and amounts to \(\Lambda_4 \circ \Lambda_3 \circ G_0\).\(^{18}\) In the CFT language this bubble diagram corresponds to \(^{19}\)

\[
\langle O(y_a)O(y_b)O(u)O(v) \rangle \left[ \langle O(u)O(u') \rangle \right]^{-1} \left[ \langle O(v)O(v') \rangle \right]^{-1} \langle O(u')O(v')O(y_c)O(y_d) \rangle
\]

(5.7)

Here the inverse and the \(\circ\) convolution in spacetime coordinates are written in matrix
notation. We can see that Eq.(5.7) is built solely from other CFT correlators, evaluated at
leading order in the large-\(N\) expansion. The large-\(N\) scaling is \(\sim 1/N^2\) for the 4pt correlators
and \(\sim 1\) for the 2pt correlator.

We can recognize the structure of a perturbative expansion at the level of the large-\(N\)
CFT. The expansion is in the small parameter \(1/N\). To better recognize the perturbative
structure, we define the amputated 4pt correlator,

\[
\langle O(y_a)O(y_b)O(y_c)O(y_d) \rangle_{\text{amp}} =
\left[ \langle O(y_a)O(y_d') \rangle \right]^{-1} \cdots \left[ \langle O(y_d)O(y_d') \rangle \right]^{-1} \langle O(y_d')O(y_d')O(y_d')O(y_d') \rangle
\]

\(^{18}\)The simplest example would be \(\Lambda_3 \circ G_0 \circ \Lambda_3\) but it is 1PR, we rather focus on a 1PI diagram.
\(^{19}\)Strictly speaking the 4pt CFT correlator also contains contribution from \(\Lambda_3 \circ G_0 \circ \Lambda_4\) at leading order
in large-\(N\), corresponding to including box and triangle diagrams in the example. For our discussion we
can safely focus on the bubble diagram only.
This amounts to a 4pt “vertex” in the diagrammatic language. This vertex scales as $\sim \frac{1}{N^2}$.

The expression Eq. (5.7) then becomes

$$\langle O(y_a) O(y'_a) \rangle \langle O(y'_b) O(u) O(v) \rangle_{\text{amp}}$$

$$\times \langle O(u) O(u') \rangle \langle O(v) O(v') \rangle$$

$$\times \langle O(u') O(v') O(y'_c) O(y'_d) \rangle_{\text{amp}} \langle O(y'_c) O(y_c) \rangle \langle O(y'_d) O(y_d) \rangle$$

We recognize the structure of a (non-amputated) bubble diagram — the second line contains the two internal lines of the bubble.

We can reproduce the same steps for more complicated topologies built from Dirichlet subdiagrams, with same outcome. One can also prove the perturbative structure at all order by working at the level of the path integral, using the Dirichlet action as a generator of the Dirichlet subdiagrams.

The conclusion is that AdS diagrams involving internal boundary lines (e.g. iv-vi) in Fig. 3) correspond in the holographic CFT to diagrams made of amputated $n > 2$pt CFT correlators connected by mean field 2pt correlators. Such expressions can, at least in principle, be directly computed in the CFT. Interestingly, an analogous diagrammatic approach to large-$N$ CFT has been introduced in [110] in the context of evaluating a 4pt CFT correlator. Here we have found how it appears from the AdS side.

An implication of the above observations is that, when calculating an AdS Witten diagrams with $\Delta_+$ propagators in internal lines, one actually computes operators of the Dirichlet action $S_D$. If one wants to obtain the complete boundary correlator generated by the boundary action $\Gamma[\Phi_0]$ at a given order in $\frac{1}{N}$, it is necessary to include the extra contributions taking the form of large-$N$ CFT diagrams described above. Examples from Fig. 3 are ii) + iv), iii) + iv) + v). Equivalently, we can say that $\Delta_-$ propagators have to be used in internal lines instead of $\Delta_+$ propagators.

### 5.2 Long-distance Holographic EFT

A field propagating in internal lines may have mass $m$ much higher than the inverse distance scale $\zeta^{-1}$ involved in the correlators, $m \gg \zeta^{-1}$. In that case one can integrate the heavy field out using a large-mass expansion. This produces a series of effective operators depending only on the light degrees of freedom that encodes the effect of the heavy field in the long-distance regime.

In the large $m$ expansion, bulk diagrams are expanded as a series of local bulk interactions suppressed by powers of $m$. From the viewpoint of the boundary effective action, these bulk vertices map one-to-one onto contributions to the holographic vertices $\Lambda_n$. In analogy with flat space low-energy EFT, the contributions from the heavy field to the boundary effective action can be cast into a long-distance effective Lagrangian. Of course, unlike the familiar EFT Lagrangians from flat space, the holographic long-distance EFT is nonlocal in $\partial M$ — even though the bulk vertices are local, nonlocality arises from convolution with the boundary-to-bulk propagators.
Denoting the heavy field by \( \Phi^{(h)} \) and the light field by \( \Phi^{(t)} \), the long-distance Dirichlet action \( S_{D,\text{eff}} + S_{\partial D,\text{eff}} \) generated by integrating \( \Phi^{(h)} \) takes the schematic structure

\[
S_{D,\text{eff}}[\Phi^{(t)}_0, \Phi^{(h)}_0] = S_D[\Phi^{(t)}_0, \Phi^{(h)}_0] + \sum_{n,\alpha} \frac{a_{n,\alpha}}{m^n_{\alpha}} \left( \Lambda_n \right)_{y_1, y_2, \ldots, y_n} \prod_{i=1}^n \Phi^{(h)}_0(y_i) + \ldots \tag{5.10}
\]

\[
S_{\partial D,\text{eff}}[\Phi^{(t)}_0, \Phi^{(h)}_0] = S_{\partial D}^{(0)}[\Phi^{(t)}_0, \Phi^{(h)}_0] + \sum_{n,\alpha} \frac{b_{n,\alpha}}{m^n_{\alpha}} \left( \mathcal{B}_n \right)_{y_1, y_2, \ldots, y_n} \prod_{i=1}^n \Phi^{(h)}_0(y_i) \tag{5.11}
\]

where \( a_{n,\alpha}, b_{n,\alpha} \) encode products of bulk couplings and \( c_{n,\alpha}, d_{n,\alpha} > 0 \). The boundary degree of freedom of the heavy field is not integrated out and thus remains in the Dirichlet action. The \( S^{(0)}_D \) are the fundamental bulk and boundary actions with renormalized constants. The ellipses in \( S_{D,\text{eff}} \) denote Dirichlet subdiagrams other than the holographic vertices \( \Lambda_n \). The \( \mathcal{B}_n \) are —possibly nonlocal— boundary terms generated when integrating out \( \Phi^{(h)}_D \).

Integrating out the \( \Phi^{(h)}_0 \) degree of freedom requires some care because the holographic self-energy can in principle contain a light degree of freedom (see also discussion in section 3.1.3). When integrating out \( \Phi^{(h)}_0 \) except for a possible light mode, additional contributions to the boundary operators \( \mathcal{B}_n \) in \( S_{\partial D,\text{eff}}^{(0)} \) are generated, while the bulk terms for \( \Phi^{(t)}_0 \) in \( S_{D,\text{eff}} \) remain unchanged. This because the boundary-to-bulk propagators of \( \Phi^{(h)} \) get shrunk to the boundary and expands into a series of boundary-localized local terms. As a result, the heavy \( \Phi^{(h)}_0 \) only contributes to the boundary effective operators.

Computing the long-distance EFT arising from integrating out a heavy bulk field is fairly simple at tree-level. The key ingredient is the large-mass expansion of the propagator. This expansion is obtained by inverting the bulk EOM, for example by multiplying by \( \sum_{k=0}^{\infty} \frac{\Box^k}{m^{2(k+1)}} \) and solving order by order. The propagator in the large mass limit is expressed as a sum of derivatives of the delta function

\[
G_D(x, x') = \frac{1}{\sqrt{|g|}} \sum_{k=0}^{\infty} \frac{\Box^k}{m^{2(k+1)}} \delta^{d+1}(x, x') \tag{5.12}
\]

giving rise to local bulk vertices and thus to the holographic vertices. Integrating out a bulk field at loop level is much more technical, this is where the holographic effective action becomes powerful.

6 The Boundary One-loop Effective Action

In this section we integrate out the bulk modes at one-loop in the presence of interactions.

6.1 Preliminary Observations

Our aim is to compute one-loop entries of either the Dirichlet action or of the boundary effective action. To this end we use a background field method that separates fields into
background and fluctuation, $\Phi = \Phi_{\text{bg}} + \Phi_{\text{fl}}$. If we identify $\Phi_{\text{bg}} = \Phi_0 \circ K$, $\Phi_{\text{fl}} = \Phi_D$, the fluctuation has Dirichlet BC and we compute a piece of the Dirichlet action. In AdS this corresponds to a fluctuation with $\Delta_+ \text{ BC}$. If we also let the boundary degree of freedom fluctuate, i.e. $\Phi_0 = \Phi_{0,\text{bg}} + \Phi_{0,\text{fl}}$, the fluctuation has Neumann BC, i.e. $\Phi_{\text{fl}} = \Phi_{0,\text{fl}} \circ K + \Phi_D$. This computes entries of the boundary effective action. In AdS this corresponds to a fluctuation with $\Delta_- \text{ BC}$.

Our focus in the following is on boundary correlators and thus on the boundary effective action. The statements below are also true for Dirichlet action when taking Dirichlet BC. Integrating out Neumann quantum fluctuations at one-loop gives rise to the one-loop effective action,

$$\Gamma = \Gamma_{\text{cl}} + \Gamma_{1-\text{loop}} + \ldots \quad (6.1)$$

In the presence of background fields, $\Gamma_{1-\text{loop}}$ is best evaluated via the heat kernel method, which gives rise to the Gilkey-de Witt “heat kernel coefficients” [12–15]. The heat kernel coefficients are local covariant quantities built from geometric invariants of the background. In the framework of the holographic action, the geometric invariants are expressed in terms of the holographic variables.

The heat kernel coefficients from a light field provide one-loop divergences. The heat kernel coefficients from a heavy field provide both one-loop divergences and the long distance EFT. The Dirichlet component of the fluctuation contributes to the bulk and boundary heat kernel coefficients. The boundary component of the fluctuation contributes solely to the boundary heat kernel coefficients, in accordance with the observations in Sec. 5.2. In all cases the geometric invariants are built from the light field background and thus depend on $\Phi^{(\ell)}_0, \Phi^{(\ell)}_D$. Upon integration in $\Phi^{(\ell)}_D$ these contributions take the general form shown in Eqs. (5.10), (5.11). For the bulk heat kernel coefficients we focus only the holographic vertices throughout this work.

There has been a plethora of studies of perturbative amplitudes in AdS background. AdS loop diagrams and their properties have been investigated in Refs. [18–62]. The one-loop effective action and its applications, however, seem fairly underrepresented in the literature. To the best of our knowledge, it has mostly been used to compute the one-loop effective potential, see [42, 63–68]. The one-loop potential is only a particular case that we briefly discuss below. The full one-loop effective action contains much more information on the long-distance limit of amplitudes and on their divergences.

### 6.2 The One-loop Effective Potential

There is a special case for which the one-loop effective action can be computed exactly. For general spacetime, this special case is defined as follows. It is the case when the one-loop effective action encodes a scalar background with constant value on the boundary (i.e. constant in the $y^\mu$ coordinates) and with appropriate scaling in the bulk such that the background-dependent mass amounts exactly to a constant bulk mass for the fluctuation. In the notation of section 6.3 the condition is

$$X[\Phi^{(\ell)}_0](z) = \text{cste} \quad (6.2)$$
Under this condition, if the free propagator is known, then the one-loop effective potential is automatically known since its derivative is given by the propagator evaluated at coincident endpoints.

For example, in the case of a scalar fluctuation $\Phi^{(h)}_D$ in AdS, and assuming that the action contains
\[
\int d^{d+1}x \sqrt{|g|} \frac{1}{2} (\Phi^{(h)}_D)^2 U[\Phi^{(f)}_0],
\]
the effective potential can be computed if the $U[\Phi^{(f)}_0]$ background is constant in $z$. Notice that $U[\Phi^{(f)}_0]$ is in general a composite whose elements are not necessarily separately constant in $z$. In AdS this becomes a condition on the $z$ scaling of the elements of $U[\Phi^{(f)}_0]$. Assuming $U[\Phi^{(f)}_0] = \prod_{i=1}^r \Phi^{(f)}_0 K_i(z)$ with $K_i(z) \propto z^{a_i}$, the condition means that the scalings of the elements in $U$ have to satisfy
\[
\sum_{i=1}^r a_i = 0 \quad (6.4)
\]
The above condition seems to be usually left implicit in the literature.

What is the CFT dual of the $U$ vev? From the CFT viewpoint, the $U$ vev matches onto the vev of an exactly marginal operator $O_U$ with dimension $\Delta_U = d$. Using AdS/CFT on the separate components of $U$, possibly using both the $\Delta_+$ and $\Delta_-$ branches, one gets that $U$ is in general a product of operators and sources, $O_U = \prod_{i=1}^l O_i \prod_{j=l+1}^r J_j$. In the large $N$ limit associated to the perturbative regime in AdS, the dimension of $O_U$ is the sum of the individual dimensions of the $O_i$ and $J_j$ up to $N$-suppressed corrections,
\[
\sum_{i=1}^l \Delta_{O,i} + \sum_{j=l+1}^r (d - \Delta_{O,j}) + O\left(\frac{1}{N^2}\right) = d. \quad (6.5)
\]
If for example $r = 2$, one has $a_1 = -a_2$. The scaling of the fields in AdS is identified as $a_1 = \Delta$, $a_2 = d - \Delta$, one has $O_U = OJ$ in the CFT, and Eq. (6.5) is verified. One may require that $O_U$ be made of operators only, without sources. In this case, because of the unitarity bound on a scalar primary $\Delta \geq \frac{d-2}{2}$, the number of operators making up the $U$ composite is bounded. For $d = 3, 4, 5, 6$ and $\geq 7$, the allowed values are $r \leq 6$, $r \leq 4$, $r \leq 3$, and $r \leq 2$ operators respectively.

When departing from the specific case of constant background field discussed here, a derivative expansion in the slowly-varying background can be used. This line of reasoning leads to the background field method and to the heat kernel expansion.

### 6.3 The Heat Kernel Coefficients

We use Lorentzian signature. The one-loop effective action takes the form
\[
\Gamma_{1\text{-loop}} = (-)^F \frac{i}{2} \text{Tr} \log \left[ \left(-\Box + m^2 + X(\Phi^{(f)})\right)_{ij} \right] \quad (6.6)
\]
with $\Box = g_{MN} D^M D^N$ the Laplacian built from background-covariant derivatives. The covariant derivatives give rise to a background-dependent field strength $\Omega_{MN} = [D_M, D_N]$,
encoding both gauge and curvature connections. It takes the general form

\[
\Omega_{MN} = -i F^a_{MN} t_a - \frac{i}{2} R_{MN}^{PQ} J_{PQ}
\]

(6.7)

where \( t_a \) and \( J_{PQ} \) are the generators of the gauge and spin representation of the quantum fluctuation. \( X \) is the “field-dependent mass matrix” of the quantum fluctuations, it is a local background-dependent quantity. The effective field strength \( \Omega_{MN} \) and the effective mass \( X \) are, together with the curvature tensor, the building blocks of the heat kernel coefficients.

Using the heat kernel method reviewed in App. D, \( \Gamma_{-\text{loop}} \) takes the form

\[
\Gamma_{-\text{loop}} = (-)^F \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \int_M d^{d+1}x \sqrt{g} \sum_{r=0}^{\infty} \frac{\Gamma(r - \frac{d+1}{2})}{m^{2r-d-1}} \text{tr} b_{2r}(x)
\]

\[
+ (-)^F \frac{1}{2} \int_{\partial M} d^d x \sqrt{g} \sum_{r=0}^{\infty} \left( \frac{\Gamma(r - \frac{d+1}{2})}{(4\pi)^{d/2}} \text{tr} b^0_{2r}(x) + \frac{\Gamma(r - \frac{d}{2})}{(4\pi)^{d/2}} \text{tr} b^0_{2r+1}(x) \right)
\]

(6.8)

with tr the trace over internal (non-spacetime) indexes. Analytical continuation in \( d \) has been used, the expression is valid for any dimension. The local quantities \( b_{2r} \) and \( b^0_{2r} \) are referred to as the bulk and boundary heat kernel coefficients.

For odd bulk dimension all the bulk terms in Eq. (6.8) are finite. There are log divergences from the \( b^0_{2r+1} \) terms for \( r \leq \frac{d}{2} \), which renormalize the boundary-localized fundamental action. For even bulk dimension there are log-divergences from both bulk terms and from the \( b^0_{2r} \) terms for \( r \leq \frac{d+1}{2} \). These log divergences renormalize both the bulk and boundary fundamental actions.

The terms with negative powers of masses in Eq. (6.8) are finite. They amount to an expansion for large \( m \) and give rise to a long-distance effective action, \( S_{\text{eff}} = \int d^{d+1}x \sqrt{|g|} \mathcal{L}_{\text{eff}} + \int d^d x \sqrt{|g|} \mathcal{L}^0_{\text{eff}} \) with

\[
\mathcal{L}_{\text{eff}} = (-)^F \frac{1}{2} \frac{1}{(4\pi)^{d+1/2}} \sum_{r=0}^{\infty} \frac{\Gamma(r - \frac{d+1}{2})}{m^{2r-d-1}} \text{tr} b_{2r}(x)
\]

\[
(6.9)
\]

\[
\mathcal{L}^0_{\text{eff}} = (-)^F \frac{1}{2} \left( \sum_{r=0}^{\infty} \frac{\Gamma(r - \frac{d+1}{2})}{(4\pi)^{d+1/2}} \text{tr} b^0_{2r}(x) + \sum_{r=0}^{\infty} \frac{\Gamma(r - \frac{d}{2})}{(4\pi)^{d/2}} \text{tr} b^0_{2r+1}(x) \right)
\]

(6.10)

Only the first heat kernel coefficients are explicitly known, we use up to \( b_6 \) and \( b^0_5 \) respectively for bulk and boundary coefficients.

### 6.3.1 Bulk Contributions

The general expressions for the bulk coefficients are [16]

\[
b_0 = I
\]

\[
b_2 = \frac{1}{6} RI - X
\]

\[
b_4 = \frac{1}{360} \left( 12 \Box R + 5 R^2 - 2 R_{MN} R^{MN} + 2 R_{MN} R_{NPQ} R^{MN} R^{NPQ} \right) I
\]

\[
- \frac{1}{6} \Box X - \frac{1}{6} RX + \frac{1}{2} X^2 + \frac{1}{12} \Omega_{MN} \Omega^{MN}
\]

(6.11)
\[ b_6 = \frac{1}{360} \left(8D_P \Omega_{MN} D_P \Omega^{MN} + 2D^M \Omega_{MN} D_P \Omega^{PN} + 12\Omega_{MN} \Box \Omega^{MN} - 12\Omega_{MN} \Omega^{NP} \Omega_P^M \\
- 6R_{MN} \Omega_{PQ} \Omega^{PN} \Omega^{PQ} - 4R^N_M \Omega^{MP} \Omega_{NP} + 5R \Omega_{MN} \Omega^{MN} \\
- 6\Box^2 X + 60X \Box X + 30D_M X D^M X - 60X^3 \\
- 30X \Omega_{MN} \Omega^{MN} - 10R \Box X - 4R_{MN} D^N D^M X - 12D_M R D^M X + 30X X R \\
- 12X \Box R - 5XR^2 + 2XR_{MN} R^{MN} - 2XR_{MNPQ} R^{MNPQ} \right) + O(R^3) \]

(6.12)

with \( I \) the identity matrix for internal indexes. Here we give only the part of \( b_6 \) relevant for our applications, the full \( b_6 \) coefficient is given in appendix, Eq. (D.6).

The invariants are built from background fields expressed in holographic variables, including the boundary components such as \( \Phi_0 \circ K \). The boundary-to-bulk propagator satisfies the bulk EOM, hence the terms involving Laplacians such as those arising from \( \Box X \) can be evaluated using the bulk EOMs.

### 6.3.2 Boundary Contributions

More invariants are involved in the boundary contributions to the heat kernel coefficients. Using the conventions from Sec. 1.2, we introduce the extrinsic curvature \( L = \Gamma^\mu_{\nu\mu} \), with \( L_{\mu\nu} = \Gamma^n_{\mu\nu} \) where the normal \( n^M \) vector is outward-pointing from the boundary.\(^{21}\) One also needs to characterize the boundary conditions of the fluctuation, which can either be Dirichlet or Neumann. This is done in general via the projectors \( \Pi_\pm \), satisfying \( (\Pi_\pm)^2 = \Pi_\pm \) and \( \Pi_+ + \Pi_- = I \). The \( \Pi_- (\Pi_+) \) selects the components of the fluctuation satisfying Dirichlet (Neumann) boundary conditions. One also introduces

\[ \chi = \Pi_+ - \Pi_- \tag{6.13} \]

Finally \( S \) is the boundary term appearing in the Neumann BC of the fluctuation as \( (D_\perp - S) \), e.g. \( (\partial_\perp - S) \Phi_{|\perp M} = 0 \).

The boundary heat kernel coefficients are known for arbitrary boundary condition of the fluctuation up to \( b_6 \) [111]. The definitions of the \( \Pi_\pm \) are set accordingly. In the case of the holographic basis, fields are split into a boundary component with Neumann BC and a bulk component with Dirichlet BC.

Focusing on the terms involving \( X \) and \( \Omega \), the boundary heat kernel coefficients are given by [111]

\[ b_3^\theta = \frac{1}{4} \chi X \]

\[ b_4^\theta = \left( \frac{2}{3} \Pi_+ - \frac{1}{3} \Pi_- \right) \partial_\perp X + \frac{1}{3} LX - 2SX \tag{6.14} \]

\(^{21}\)In Ref. [111] the \( n_M \) vector is inward pointing. The sign of \( \partial_\perp, L_{ab} \), and consequently \( D_\perp \), is flipped in the boundary heat kernel coefficients under this change of convention.
\[ b^0_\perp = \frac{1}{5760} \left( -360 \chi D_{\perp} \partial_{\perp} X + 1440 S \partial_{\perp} X + 720 \chi X^2 \right. \]
\[ \left. - 240 \chi \partial_{\mu} \partial_{\mu} X - 240 R \chi X - 2880 S^2 X - (90 \Pi_+ + 450 \Pi_-) L \partial_{\perp} X \right. \]
\[ + 1440 L S X - (195 \Pi_+ - 105 \Pi_-) L L X - (30 \Pi_+ + 25 \Pi_-) L_{\mu \nu} L^{\mu \nu} X \]
\[ - 180 X^2 + 180 \chi X \partial_{\perp} X - \frac{105}{4} \Omega_{\mu \nu} \Omega^{\mu \nu} + 120 \chi \Omega_{\mu \nu} \Omega^{\mu \nu} + \frac{105}{4} \chi \Omega_{\mu \nu} \chi \Omega^{\mu \nu} \]
\[ - 45 \Omega_{\mu \perp} \Omega^{\mu \perp} + 180 \chi \Omega_{\mu \perp} \Omega^{\mu \perp} - 45 \chi \Omega_{\mu \perp} \chi \Omega^{\mu \perp} \right) + \ldots \] (6.15)

The ellipses represent pure curvature terms which are irrelevant for our applications.

The invariants contain the boundary components of the background such as \( \Phi_0 \circ K \).

When a normal derivative acts on such a term, it can be evaluated using the discontinuity equation on the boundary, Eq. (2.8), just like in the evaluation of the free part of the holographic action (see Sec. 2).

### 6.4 Scalar Fluctuation

The action for a set of massive scalar fields \( \Phi \) with mass \( m^2 \) is

\[
S[\Phi] = -\int_M d^{d+1}x \sqrt{g} \left( \frac{1}{2} D_M \Phi^a D^M \Phi^a + \frac{1}{2} m^2 \Phi^a \Phi^a + V(\Phi, \Phi^{(\ell)}) + \frac{1}{2} \xi R \Phi^a \Phi^a \right) + \frac{1}{2} \int_{\partial M} d^d y \sqrt{\bar{g}} \Phi^a_0 B \Phi^a_0
\] (6.16)

We have included the \( \xi \) coupling to the scalar curvature. The potential can depend on other fields denoted collectively as \( \Phi^{(\ell)} \). We split the quantum field as \( \Phi = \Phi|_{\text{bg}} + \Phi|_{\text{fl}} \).

The wave operator for the bulk fluctuation \( \Phi|_{\text{fl}} \) is

\[
\mathcal{D}^{ab} = -(D_M D^M)^{ab} + (V'')^{ab} + \xi \delta^{ab} + m^2 \delta^{ab}
\] (6.17)

with \( V'' = \frac{\delta^2}{\delta \Phi^a \delta \Phi^a} V \).

The canonical invariants needed to evaluate the heat kernel coefficients are thus

\[
X = (V''(\Phi^{(\ell)})|_{\text{bg}})^{ab} + \xi \delta^{ab}
\] (6.18)

\[
\Omega_{MN} = -i F_{MN}|_{\text{bg}}
\] (6.19)

Here \( F_{MN} = F_{MNa} t^a_3 \) where \( t^a_3 \) are the generators of the gauge representation of \( \Phi \).

Regarding boundary conditions of the fluctuation, the \( \Pi_{\pm} \) projectors associated to the BC of the fluctuation are \( \Pi_+ = 1, \Pi_- = (\text{Neumann}) \) or \( \Pi_+ = 0, \Pi_- = 1 \) (Dirichlet). The BC for gauge background is treated in details in next section. If we choose Neumann BC for the vector part \( A_{\mu}|_{\text{bg}} = A_{\mu,0} \circ K \), the normal component of the background must be Dirichlet, \( A_{\perp}|_{\text{bg}} = A_{\perp,D} \). In this case the background field strength invariant is

\[
\Omega_{\mu \nu} = -i F_{\mu \nu,0} \circ K, \quad \Omega_{\mu \perp} = -i(\partial_{\mu} A_{\perp,D} - \partial_{\perp} K \circ A_{\mu,0})
\] (6.20)

The \( \Omega_{\mu \perp} \) component vanishes on the boundary as a consequence of the BCs of \( A_M|_{\text{bg}} \).
6.5 Vector Fluctuation

The YM action is
\[ S[A] = -\frac{1}{4g_{YM}^2} \int d^{d+1}x \sqrt{g} F_{MN} F^{MN} \]  
(6.21)

Splitting the field as \( A_M = A_M|_{\text{bg}} + A_M|_{\text{fl}} \), the bilinear action for the fluctuation is
\[ -\frac{1}{4g_{YM}^2} \int d^{d+1}x \sqrt{g} \left( A_M^a|_{\text{fl}} \left[ -\square^a g_{MN} + R_{MN} \delta^{ab} + 2f^{acb} F_{cMN} \right] A_N^b|_{\text{fl}} - (D^M A_M^n|_{\text{fl}})^2 \right) + \text{bdry term} \]  
(6.22)

where we have used \( D_M A^n_M D^n_M - f^{abc} F_{bMN} - R_{MN} \delta^{ab} \). The terms in bracket in Eq. (6.22) is the wave operator. The subsequent canonical invariants are
\[ (X)_{MN}^{ab} = R_{MN} \delta^{ab} + 2f^{acb} F_{cMN} |_{\text{bg}} \]  
(6.23)
\[ (\Omega_{MN})^P Q_{ab} = -R^P Q_{MN} \delta^{ab} + \delta^P Q f^{acb} F_{cMN} |_{\text{bg}} \]  
(6.24)

6.5.1 Boundary Conditions and Holographic Basis

Near the boundary, the gauge field decomposes as \( A_N = (A_\perp, A_\mu) \). Gauge invariance requires the boundary condition to either be Neumann for \( A_\mu \) and Dirichlet for \( A_\perp \) or the converse (see e.g. [16]). These two BCs are sometimes respectively referred to as “absolute” and “relative”. Explicitly, the two possible BCs are
\[ A_\perp |_{\partial M} = 0, \quad \partial_\perp A_\mu |_{\partial M} = 0 \]  
(absolute)
\[ (D_\perp - L) A_\perp |_{\partial M} = 0, \quad A_\mu |_{\partial M} = 0 \]  
(relative)

When the field is split as \( A_M = A_M|_{\text{bg}} + A_M|_{\text{fl}} \), the background and fluctuation can satisfy different sets of BCs. Picking opposite BCs for background and fluctuation amounts to compute a contribution to the Dirichlet action. Picking the same BCs for background and fluctuation amounts to compute a contribution to the effective holographic action in the chosen background.

If, for example, we choose \( A_\mu \) as the background field and integrate over the \( A_\mu \) fluctuation in both bulk and boundary, we pick the BC Eq. (6.25) for both background and fluctuation, with \( A_\mu |_{\text{bg}} = A_{\mu 0} \circ K \). With this choice of BC the \( F_{\mu \perp} |_{\text{bg}} \) component vanishes on the boundary. The \( \Pi_{\pm} \) projectors associated to the fluctuation \( A_M|_{\text{fl}} \) are
\[ \Pi_+ = \delta_{MN} - \delta_M \perp \delta_N \perp \quad \Pi_- = \delta_M \perp \delta_N \perp \]  
(6.27)

(see [16]).

---

\(^{22}\)Definitions: \( F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + f^{abc} A_M^b A_N^c \), \( D_M A_N^a = D_M A_N^a - D_N A_M^a + f^{abc} A_M^b A_N^c \), \( D_M = D_M^{(R)} - iA_M^a \xi_a, \) \( (\xi_a)_ab = if^{abc} \) with \( D_M^{(R)} \) the Riemann covariant derivative.
6.5.2 Gauge Fixing

We fix the gauge redundancy of the fluctuation using the Faddeev-Popov procedure. As usual in background field calculations (see e.g. [16, 139–141]), we pick the background version of the Feynman-'t Hooft gauge

$$S[A]_{FP} = -\frac{1}{4g_{YM}^2} \int_M d^{d+1}x \sqrt{g} (D^M A_M^a |_b)^2$$

which cancels the corresponding term in Eq. (6.22). The quadratic Lagrangian of the ghost is simply $\bar{c}^a \Box^{ab} c^b$. The corresponding canonical invariants are those of a scalar fluctuation, Eqs. (6.18), (6.19) with $V = 0, \xi = 0$. If the gauge field satisfies the BC Eq. (6.25) [resp. Eq. (6.26)], the ghost has Neumann BC $\partial_\perp c^a |_{\partial M} = 0$ [resp. Dirichlet BC $c^a |_{\partial M} = 0$]. Since ghosts anticommute ($F = \text{odd}$) the total, gauge-invariant heat kernel coefficients from the YM fluctuation are

$$b^{(0)}_{\text{tot}} = b^{(0)}_{A} - 2b^{(0)}_{gh}.$$  

(6.29)

7 Application: Loops in AdS

Using the formalism established in Sec. 6, we evaluate the boundary one-loop effective action for QFTs in $\text{AdS}_{d+1}$ with boundary truncated at $z = z_0$. The canonical invariants are expressed in terms of the boundary variables and boundary-to-bulk propagators $K(x, x_0)$. We work in the long distance regime $(y - y')^2 \gg zz_0$ for which $\zeta(x, x') \gg \frac{1}{k^2}$. The $K|_{\zeta \gg 1}$ approximation satisfies the EOM to $O(\frac{1}{\zeta})$ accuracy. The normal derivative on the boundary is $\partial_\perp = -kz_0 \partial_z$. The extrinsic curvature is $L_\perp \mu \nu = -kg_{\mu \nu}$, $L = -kd$. We focus on on-shell background fields.

7.1 Application 1: Scalar Loops in AdS

We consider a massive scalar $\Phi$ interacting with light scalars $\varphi_i$ via the coupling

$$V = \frac{1}{2} \Phi^2 \prod_{i=1}^n \varphi_i.$$  

(7.1)

Each scalar has a distinct mass given by $m_i^2 = \Delta_i(\Delta_i - d)k^2$. We choose $\Delta_i = \Delta_i^+$. We are interested in the scalar background. We set the coupling to curvature $\xi = 0$ to zero (the generalization to nonzero $\xi$ is straightforward).

To integrate out the bulk modes of $\Phi$, the relevant term of $V$ is $\frac{1}{2} \Phi^2 \prod_{i=1}^n \varphi_i \circ K_i(x)$ with $K_i(x) = K_{\Delta_i}(x)$. It follows that the canonical background-dependent mass is

$$X = \prod_{i=1}^n \varphi_i \circ K_i(x)$$

(7.2)

The field strength $\Omega_{MN}$ vanishes.

The diagrammatic contributions to the heat kernel coefficients computed here are shown in Fig. 4.
Fig. 4. $\text{AdS}_{d+1}$ Witten diagrams contributing to the first heat kernel coefficients. Internal lines can either have Dirichlet ($\Delta_+$) or Neumann ($\Delta_-$) boundary condition. In the latter case each internal line can be further decomposed into a Dirichlet line and a boundary line.

**Bulk Coefficients**

For the bulk heat kernel coefficients it is convenient to introduce the integrated coefficient

$$\tilde{b}_r = \int_{\text{AdS}} d^{d+1} x \sqrt{|g|} b_r(x). \quad (7.3)$$

The nonlocal holographic vertices discussed in section 5 appear in the bulk coefficients. We introduce the vertices

$$\Lambda_{n,r}(x_{0,i}) = \int_0^\infty \frac{dz}{(kz)^{d+1}} \left( \prod_{i=1}^n K_i(x_{0,i},x) \right)^r \quad (7.4)$$

We define the shortcut notations

$$X_0(x) = \prod_{i=1}^n \varphi_{i,0}(x)$$

and

$$\int_0^\infty \frac{dz}{(kz)^{d+1}} \left( \prod_{i=1}^n K_i \circ \varphi_{i,0}(x) \right)^r = \Lambda_{n,r} \circ \left[ \prod_{i=1}^n \varphi_{i,0} \right]^r(x) = \Lambda_n \circ [X_0]^r(x) \quad (7.5)$$

where each of the endpoints of $\Lambda_n$ is convoluted using the $\circ$ product with the relevant boundary variable $\varphi_{i,0}$. Our bulk results will be expressed in terms of these quantities. We remind that $\Lambda_n$ generates a $n$pt contact Witten diagram upon differentiation of the action in the boundary fields $\varphi_{i,0}$.

To evaluate the bulk heat kernel coefficients we need to evaluate $\Box X$. When both derivatives hit the same field inside $X$, the EOM can be used and gives a $\sum_{i=1}^n \Delta_i (\Delta_i - d) k^2$ contribution. When each derivative hits a different field inside $X$, one needs to evaluate the cross terms $\partial_M K_i \partial^M K_j$. The exact answer is complicated. However, in the $\zeta \gg \frac{1}{k^2}$ limit, these cross terms simplifies to $\partial_M K_i \partial^M K_j \sim \Delta_i \Delta_j k^2$. The cross terms combine with the square term to give

$$\Box X |_{\zeta \gg 1} = \left( \sum_{i=1}^n \Delta_i \right) \left( \sum_{i=1}^n \Delta_i - d \right) k^2 X |_{\zeta \gg 1} \quad (7.6)$$

We see that the $\Delta_i$ add up. This is the expected result from $\text{AdS}/\text{CFT}$: The dimension of the $X$ composite is $\Delta_X = \sum_{i=1}^n \Delta_i + O(\frac{1}{N^2})$ at leading order in the large $N$ expansion. The coefficient showing up in Eq. (7.6) is the quadratic Casimir of $X$. Beyond the $\zeta \gg \frac{1}{k^2}$
regime, i.e. for distances shorter that $|y - y'| \sim z_0$, the relation Eq. (7.6) does not hold. This is a way to see that AdS/CFT breaks down at distances shorter than $z_0$.

The contributions to the bulk coefficients from the scalar interactions encoded in $X$ are found to be

$$
\text{tr} \, \tilde{b}_2 = -\Lambda_n \circ X_0
$$

(7.7)

$$
\text{tr} \, \tilde{b}_4 = \frac{1}{2} \Lambda_{n,2} \circ [X_0]^2 + \frac{1}{6} (-\Delta_X (\Delta_X - d) + d(d+1)) k^2 \Lambda_n \circ X_0
$$

(7.8)

$$
\text{tr} \, \tilde{b}_6 = -\frac{1}{6} \Lambda_{n,3} \circ [X_0]^3(x) + \frac{1}{12} (\Delta_X (\Delta_X - d) - d(d+1)) k^2 \Lambda_{n,2} \circ [X_0]^2(x)
$$

$$
- \frac{1}{360} \left( 8d^3 + 5d^4 + 6\Delta_x^2 (\Delta_X - d)^2 + d^2 (7 - 22\Delta_X (\Delta_X - d)) + d(4 - 26\Delta_X (\Delta_X - d)) \right) k^4 \Lambda_n \circ X_0(x)
$$

(7.9)

In the $\hat{b}_6$ coefficients, we performed integration by parts on certain terms i.e. we used Green’s first identity. The generated boundary term contributes to the boundary coefficient $\hat{b}_6^\perp$. In this work we stopped at $\hat{b}_6^\perp$ — the general $\hat{b}_6^\perp$ has apparently not been computed in the heat kernel literature — hence the boundary term arising from integration by parts in $b_6$ is neglected.

**Boundary Coefficients**

We need other identities to evaluate the boundary coefficients. Normal derivatives evaluated on the boundary (such as $\partial_\perp K$) appear. It is convenient to display $\partial_\perp K$ in terms of a CFT 2pt function. We have (see section 3.3.2)

$$
\partial_\perp K_i = -\frac{1}{\sqrt{g}} \left( \frac{C_{\Delta_i, \eta_{\Delta_i}}}{(y - y')^{2\Delta_i}} + \Delta_i^- k \delta^d(y - y') \right)
$$

$$
= -\frac{1}{\sqrt{g}} \left( \langle O_i(y)O_i(y') \rangle - \Delta_i^- k \delta^d(y - y') \right)
$$

(7.10)

where in the second line we have introduced the scalar primary $O_i$ with dimension $\Delta_i$, normalized such that the $C_{\Delta_i, \eta\Delta_i}$ coefficients are absorbed.

We then evaluate $\partial_\perp X$. The contact terms combine and make the dimension $\Delta^-_X = \sum_{i=1}^n \Delta^-_i = \sum_{i=1}^n (d - \Delta_i)$ appear. The $\partial_\perp X$ derivative takes the form

$$
\partial_\perp X = -\Delta^-_X kX_0 - \frac{1}{\sqrt{g}} \sum_{i=1}^n \langle O_i O_i \rangle \circ X_0
$$

(7.11)

where one has introduced a shortcut notation for the $\circ$ convolutions: the correlator for $O_i$ is convoluted with the corresponding $\varphi_{i,0}$ inside $X$.

We turn to the evaluation of $D^\perp \partial_\perp X$. We evaluate $D^\perp \partial_\perp K$ in momentum space and get only a contact term $D^\perp \partial_\perp K = (\Delta (d - \Delta)) k^2 - k^2 z_H^2 \partial_\mu \partial^\mu \delta^d(y - y')$. The cross terms $\partial_\perp K_i \partial^\perp K_j$ are evaluated using Eq. (7.11). The contact terms from the cross terms combine with those from the diagonal terms, leading to a coefficient $\Delta^-_X (d - \Delta^-_X)$ for the
total contact term. We see that the $\Delta_X$ add up, which is consistent with the underlying conformal symmetry. Again, this combination happens from the $|y - y'| \gg z_0$ regime of the boundary-to-bulk propagators. Beyond this regime no such simplification occurs, illustrating the breakdown of AdS/CFT.

The full term reads

\[
D^\perp \partial_\perp X = \Delta_X (d - \Delta_X) k^2 X_0 - k^2 z_0^2 \sum_{i=1}^n (\partial_\mu \partial^\mu)_i X_0
\]

\[+ \frac{2}{\sqrt{g}} \Delta_X k \sum_{i=1}^n \langle O_i O_i \rangle \circ X_0 + \frac{1}{\sqrt{g}} \frac{1}{\sqrt{g}} \sum_{i \neq j} \langle (O_i O_i) \circ [O_j O_j] \circ \rangle X_0.
\]

Again we use a shortcut notation for the convolutions (correlators with label $i$ contract with the $\varphi_{i,0}$ fields inside $X_0$).

We choose Neumann BC for the fluctuation (i.e. we integrate out both $\Phi_D$ and $\Phi_0$). Using that $\Pi_+ = 1$, $\Pi_- = 0$, $\chi = 1$, $S = 0$, the boundary heat kernel coefficients are found to be

\[
\begin{align*}
\text{tr} \, b^\partial_3 &= -\frac{1}{4} X_0 \\
\text{tr} \, b^\partial_4 &= -\frac{2}{3} \frac{1}{\sqrt{g}} \sum_{i=1}^n \langle O_i O_i \rangle \circ X_0 - \frac{1}{3} (2\Delta_X + d) k X_0
\end{align*}
\]

\[
\begin{align*}
\text{tr} \, b^\partial_5 &= \frac{1}{5760} \left( 720 X_0^2 + (-360 \Delta_X (d - \Delta_X) k^2 + 90 d \Delta_X k^2 + 15 d(14 + 3d) k^2) X_0 \\
&+ 360 k^2 z_0^2 \sum_{i=1}^n (\partial_\mu \partial^\mu)_i X_0 - 240 k^2 z_0^2 \partial_\mu \partial^\mu X_0 \\
&+ (90 d - 720 \Delta_X) k \frac{1}{\sqrt{g}} \sum_{i=1}^n \langle O_i O_i \rangle \circ X_0 \\
&- 360 \frac{1}{\sqrt{g}} \frac{1}{\sqrt{g}} \sum_{i \neq j} \langle (O_i O_i) \circ [O_j O_j] \circ \rangle X_0 \right)
\end{align*}
\]

The result from Dirichlet BC (i.e. from integrating out $\Phi_D$ and not $\Phi_0$) is shown in App. D.1 for completeness.

### 7.1.1 $\Phi^2 \varphi^2$ in AdS$_5$

We apply these results to a $L_I = -\frac{\lambda}{4} \Phi^2 \varphi^2$ interaction in AdS$_5$ where the light field $\varphi$ has mass $\Delta(\Delta - d) k^2$. The canonical invariant is $X = \frac{\lambda}{2} \varphi^2$. One has $\Delta_X = 8 - 2\Delta$. The boundary heat kernel coefficients are

\[
\begin{align*}
\text{tr} \, b^\partial_3 &= -\frac{1}{8} \lambda \varphi_0^2 \\
\text{tr} \, b^\partial_4 &= -\frac{2\lambda}{3} \frac{1}{\sqrt{g}} \langle (O O) \circ \varphi_0 \rangle \varphi_0 + \frac{\lambda}{3} (2\Delta - 10) k \varphi_0^2
\end{align*}
\]
\[
\text{tr } b_3^0 = \frac{1}{5760} \left( 120\lambda k^2 z_0^2 \varphi_0 \partial_\mu \partial^\mu \varphi_0 + 60 (12\Delta^2 - 78\Delta + 133) \lambda k^2 \varphi_0^2 + 180\lambda^2 \varphi_0^4 \\
+ 360(4\Delta - 15)\lambda k \frac{1}{\sqrt{g}} ((\mathcal{O}\mathcal{O}) \circ \varphi_0) \varphi_0 - 360\lambda \frac{1}{\sqrt{g}} \frac{1}{\sqrt{g}} ((\mathcal{O}\mathcal{O}) \circ \varphi_0)^2 \right)
\]

These coefficients enter in the holographic effective action as

\[
\Gamma_{1-\text{loop}}^{\partial, \text{fin}} = \frac{1}{2} \int_{\partial\text{AdS}} d^4x \sqrt{g} \left( -\frac{m^2}{16\pi^2} \text{tr } b_4^0(x) \right)
\]

\[
\Gamma_{1-\text{loop}}^{\partial, \text{div}} = \frac{1}{2} \int_{\partial\text{AdS}} d^4x \sqrt{g} \left( -\frac{m^2}{8\pi^2} \text{tr } b_3^0(x) + \frac{1}{8\pi^2} \text{tr } b_5^0(x) \right) \log \mu_r
\]

where $\mu_r$ is the renormalization scale.

**7.1.2 Discussion**

The structure of the boundary one-loop effective action can be read from the heat kernel coefficients. Upon taking $\varphi_0$ derivatives of the boundary action, we see that the bulk heat kernel coefficients generate contact Witten diagrams, which are encoded into the $\Lambda_n$ vertices. These contributions can be intuitively guessed by shrinking loops of bulk fields into bulk points.

In contrast, the boundary heat kernel coefficients lead to boundary contact operators, with possibly some CFT 2pt correlators appended to them as a result of normal derivatives acting on $X$. In the $b_3^0$ coefficients that we have computed, a term with one 2pt correlator appears in $b_4^0$, and a term with two 2pt correlators appears in $b_5^0$.

The boundary heat kernel coefficients determine which boundary-localized operators are radiatively generated. Focusing on the $V(\Phi) = \frac{1}{4}\Phi^2\varphi^2$ example given in section 7.1.1, we can explicitly see that a boundary-localized mass term is generated in $b_3^0$, a boundary-localized kinetic term is generated in $b_5^0$, a boundary quartic is generated in $b_5^0$. In AdS$_5$ all the contributions from $b_3^0, b_5^0$ are log divergences, as encoded in $\Gamma_{1-\text{loop}}^{\partial, \text{div}}$. All these local contributions have been qualitatively discussed in the extradimensional literature, here we have computed them exactly.

Contributions that are nonlocal in $y$ are also generated. From the $b_4^0, b_5^0$ coefficients, corrections to the CFT 2pt function are generated. These corrections contribute to the overall normalization of $\langle \mathcal{O}\mathcal{O} \rangle$ as follows:

\[
C_{\Delta} \left( 1 - \frac{\lambda m_\Phi}{24\pi^2} - \frac{4\Delta - 15}{128\pi^2} \lambda k \log \mu_r \right)
\]

where the first term is the tree-level contribution, the second is from $b_4^0$, the third from $b_5^0$.

From the viewpoint of the boundary CFT, such a “wavefunction renormalization” of the CFT operator contributes to renormalizing the OPE coefficient upon unit-normalization of the 2pt function. Still from the CFT viewpoint, these corrections are understood as leading large-$N$ corrections, with $\lambda k \sim \frac{1}{N^2}$. The heavy mass $m_\Phi^2$ is identified with $\Delta_{\text{gap}}(\Delta_{\text{gap}} - d)k^2$. 


Finally, the \((\langle\mathcal{O}\mathcal{O}\rangle \circ \varphi_0)^2\) term reduces to the 2pt function of another CFT operator, \(\mathcal{O}^2\), which has dimension \(2\Delta\) at this order in perturbation theory. Using the \(\int d^dy\) integral from the action we have

\[
\int d^dy'\langle\mathcal{O}\mathcal{O}\rangle(y, y')\langle\mathcal{O}\mathcal{O}\rangle(y', y'') = i\pi \frac{\Gamma^2\left(\frac{d}{2} - \Delta\right)\Gamma(2\Delta)}{\Gamma^2\left(\Delta\right)\Gamma\left(\frac{d - 2\Delta}{2}\right)}\langle\mathcal{O}^2\rangle^2(y, y''). \tag{7.22}
\]

The \(i\) factor comes from the Lorentzian metric. This explicitly shows how a double trace operator emerges on the AdS boundary as a result of loop corrections.

All the coefficients are consistent with the conformal symmetry of the boundary in the sense that the conformal dimensions of the components of \(X\) appear only via the total dimensions \(\Delta_X, \Delta_{\bar{X}}\), and never in other combinations nor individually. This occurs via nontrivial combinations of coefficients which are valid only in the \(|y - y'| \gg z_0\) regime, illustrating that AdS/CFT is restricted to that energy range.

### 7.2 Application 2: Vector Loops and Yang-Mills Beta Functions

We evaluate the heat kernel coefficients generated by integrating out the spin-1 fluctuation from a non-abelian gauge field.

Our focus is on the gauge background field \(F_{\mu\nu}|_{\text{bg}}\). We thus use the set of BCs Eq. (6.25) for both background and fluctuation. With this choice \(F_{\mu\perp}|_{\text{bg}}|_{\partial M} = 0\), leaving \(F_{\mu\nu}|_{\text{bg}}\) as the non-vanishing component of the field strength on the boundary. The BC of the \(A_M\) fluctuation enforces the \(\Pi^\pm\) projectors to be \(\Pi^+_\perp = \delta_{MN} - \delta_{M\perp}\delta_{N\perp}, \Pi^- = \delta_{M\perp}\delta_{N\perp}\).

We assume the gauge background is on-shell thus \(D_M F_{MN} = 0\). Fixing the gauge of the background to \(D_M A_M = 0\), the EOM reduces to \(\Box A_M = 0\). The homogeneous solutions are \(z^{\frac{d}{2}-1}I_{\frac{d}{2}-1}(pz), z^{\frac{d}{2}-1}K_{\frac{d}{2}-1}(pz)\).

To evaluate the \(F_{MN}\Box F_{MN}\) terms, we use Jacobi’s identity and the commutator \([D_M, D_N]F_{PQ}^a = [F_{MN}, F_{PQ}]^a\). The result is

\[
\text{tr} \left[ F_{MN}\Box F_{MN} \right] = \text{tr} \left[ 4F_{MN} F^{NP} F_{P M} - 2(D_M F_{MN})^2 \right]. \tag{7.23}
\]

The gauge background contributes to the bulk coefficients \(b_4, b_6\) and to the boundary coefficient \(b_5^\partial\). Subtracting the ghost contributions, we find the total coefficients \(b_{\text{tot}} = b_A - 2b_{\text{gh}}\) to be

\[
\begin{align*}
\text{tr} b_{4,\text{tot}} &= -\frac{25 - d}{12}C_2(G)F_{MN}^a F^{a,MN}|_{\text{bg}} \\
\text{tr} b_{5,\text{tot}}^\partial &= \frac{27 - d}{48}C_2(G)F_{\mu\nu}^a F^{a,\mu\nu}|_{\text{bg}} \\
\text{tr} b_{6,\text{tot}} &= \frac{5d^3 - 154d^2 + 557d + 12}{360}C_2(G)k^2 F_{MN}^a F^{a,MN}|_{\text{bg}} \\
&\quad + \frac{d - 1}{90}\text{tr} F_{MN} F^{NP} F_{P M}|_{\text{bg}} \tag{7.24}
\end{align*}
\]

where \(C_2(G)\) is the quadratic Casimir of the gauge representation, \(f^{abc} f^{adb} = C_2(G)\delta^{cd}\).
7.2.1 $\beta$ functions

Both bulk and boundary gauge couplings can be renormalized by the logarithmic divergences appearing in the one-loop effective action. Which term of $\Gamma_{1-\text{loop}}$ diverges depends on the dimension of spacetime. Unlike flat space where a running gauge coupling can only happen in $d = 4$, we will see that in AdS space this can also occur in higher dimensions as a consequence of the curvature. The $\beta$ function of the gauge couplings can be evaluated by putting together the one-loop effective action Eq. (6.8) and the classical part. The classical part is

$$S = -\frac{1}{4g^2} \int d^{d+1}x \sqrt{|g|} (F_{MN}^a)^2 - \frac{1}{4g^2_\theta} \int d^d y \sqrt{|\bar{g}|} (F_{\mu\nu}^a)^2 \bigg|_{z=0}$$

where a boundary-localized kinetic term has been introduced. Bulk and boundary divergences respectively renormalize the $g$ and $g_\theta$ parameters.

For AdS$_4$ ($d = 3$), the log divergence comes from the $\text{tr} b_{4,\text{tot}}$ coefficient. The overall factor is $-\frac{11}{24\pi^2}$. This reproduces the well-known 4d YM $\beta$ function for the bulk coupling, $\beta_{\frac{1}{g^2}} = \frac{11}{24\pi^2} C_2(G)$.

For AdS$_5$ ($d = 4$), the log divergence comes from the $\text{tr} b_{5,\text{tot}}$ coefficient. The $\beta$ function for the boundary gauge coupling is found to be

$$\beta_{\frac{1}{g^2}} \bigg|_{\text{AdS}_5} = -\frac{23}{192\pi^2} C_2(G).$$

(7.26)

For AdS$_6$ ($d = 5$), the log divergence comes from the $\text{tr} b_{6,\text{tot}}$ coefficient. The gauge coupling has canonical dimension $[g] = -1$. The coefficient in Eq. (7.24) reduces to $\frac{107}{1440\pi^3} k^2$. The $\beta$ function for the bulk coupling is found to be

$$\beta_{\frac{1}{g^2}} \bigg|_{\text{AdS}_6} = \frac{107}{1440\pi^3} k^2 C_2(G).$$

(7.27)

Here we can see that the logarithmic running of this dimensionful gauge coupling is a consequence of the AdS curvature.

7.2.2 Boundary Yang-Mills Effective Action

Using the above results, let us write the bilinear boundary action at one-loop. We work in momentum space. Irrespective of the gauge, the $A_z$ background can be eliminated (i.e. integrated out at classical level) using the $D_\mu F^{\mu z} = 0$ component of the EOM, giving the identity

$$F_{MN} F^{MN} = F_{\mu\nu} F^{\mu\nu} + 4 \left( \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) (\partial_z A^\mu)(\partial_z A^\nu)$$

(7.28)

The second term contributes to the boundary action, generating the 2pt function of a conserved CFT current (see e.g. [5]). The boundary action reads

$$S = -\int \frac{d^d p}{(2\pi)^d} \sqrt{\bar{g}} \left( \frac{p^2}{2g_\theta^2} + \frac{1}{g^2} \partial_\perp K \right) A_0^\mu \Pi_{\mu\nu} A_0^\nu \bigg|_{z=z_0}$$

(7.29)

with $\Pi_{\mu\nu} = \left( \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$.
The EOM for $A_\mu$ is simplified by the $D_M A^M = 0$ gauge. Using the same calculations as in section 3.3.2, the exact boundary-to-bulk propagator for $A_\mu$ is

$$K = \frac{1}{\sqrt{\gamma}} \int \frac{d^d p}{(2\pi)^d} \frac{z_0^{d-1} K_{d-1}(pz)}{z_0^{d-1} K_{d-1}(pz_0)}.$$  \hspace{1cm} (7.30)$$

We then take the $pz_0 \ll 1$ limit. The $\partial_\perp K$ derivative goes as $\propto \log(p) - 1/p$, $p^2 \log p$ and $\propto p^2$ for $d = 2, 3, 4$ and $d \geq 5$ respectively. We also integrate the $\beta$ functions and identify the renormalization scale with $p$. The reference scale is denoted $p_0$.

The resulting boundary one-loop effective actions for $d = 3, 4, 5$ are

$$\Gamma[A_0] \bigg|_{\text{AdS}_4} = -\int \frac{d^3 p}{(2\pi)^3} \sqrt{\gamma} \left( \frac{p^2}{2g_b} + \left( \frac{1}{g_0^2} + \frac{11C_2(G)}{24\pi^2} \log \frac{p}{p_0} \right) kp z_0 \right) A_0^\mu \Pi_{\mu\nu} A_0^\nu$$ \hspace{1cm} (7.31)$$

$$\Gamma[A_0] \bigg|_{\text{AdS}_5} = -\int \frac{d^4 p}{(2\pi)^4} \sqrt{\gamma} \left( \frac{1}{2g_{b,0}^2} - \left( \frac{kz_0^2}{g^2} + \frac{23C_2(G)}{384\pi^2} \right) \log \frac{p}{p_0} \right) p^2 A_0^\mu \Pi_{\mu\nu} A_0^\nu$$ \hspace{1cm} (7.32)$$

$$\Gamma[A_0] \bigg|_{\text{AdS}_6} = -\int \frac{d^5 p}{(2\pi)^5} \sqrt{\gamma} \left( \frac{1}{2g_b^2} + \left( \frac{1}{g_0^2} + \frac{107C_2(G)k^2}{1440\pi^3} \log \frac{p}{p_0} \right) k z_0^2 \right) p^2 A_0^\mu \Pi_{\mu\nu} A_0^\nu$$ \hspace{1cm} (7.33)$$

In $\Gamma[A_0] \bigg|_{\text{AdS}_d}$ we have absorbed a constant $\frac{2k^2 z_0^2}{g^2} \left( \gamma + \log \left( \frac{p z_0}{2} \right) \right)$ into the reference value of $\frac{1}{g_{b,0}}$. We see that for AdS$_{4,6}$ the holographic coupling grows in the IR as a consequence of bulk one-loop divergences. For AdS$_5$, both the classical action and the boundary one-loop divergence contribute with negative sign. Thus the holographic coupling grows in the UV in AdS$_5$.

8 A Boundary Effective Action in dS

We turn to an application in de Sitter space dS$_{d+1}$. We will compute one-loop contributions to late-time cosmological correlators.

We consider the flat slicing

$$ds^2_{\text{dS}} = g^\text{dS}_{MN} x^M x^N = \ell^2 \left( -d\eta^2 + (dy)^2 \right)$$ \hspace{1cm} (8.1)$$

with $\eta \in [-\infty, \eta_0]$, $\eta_0 < 0$. $\eta$ is the conformal time, related to the proper time by $d\eta = e^{-\frac{t}{\ell}} dt$. The $\eta_0 = 0$ slice corresponds to future boundary. This coordinate patch describes the expanding de Sitter universe with Hubble radius $\ell$. It covers half of the global dS space, the other half amounts to positive $\eta$. We are interested in correlators with endpoints on the $\eta = \eta_0$ time slice.

In this section we deal with various kinds of two-point function (time-ordered, anti-time-ordered, Wightman), hence we explicitly write the time (anti-)ordering operators $T(\bar{T})$. 

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8.1 Cosmological Correlators and In-in Formalism

The cosmological correlators are conveniently computed in the in-in (or Keldish) formalism [106, 142–144, 144–146]. A in-in correlator is evaluated by performing a time-ordered integral from the initial time to the time of interest \( \eta \), and then performing an anti-time-ordered integral back to the initial time. This can be implemented by slightly shifting the time integral domain away from the real axis, such that the \(-\eta\) variable takes values in either \( \mathbb{R}_+(1+i\epsilon) \) or \( \mathbb{R}_+(1-i\epsilon) \). In the interaction picture, for a given interaction Hamiltonian \( H_I \), one has

\[
\langle \Phi(\eta_0, \vec{y}_1) \cdots \Phi(\eta_n, \vec{y}_n) \rangle = \frac{\langle 0| \hat{T} \left( e^{i \int_{-\infty}^{\eta_0} d\eta H_I} \right) \Phi(\eta_0, \vec{y}_1) \cdots \Phi(\eta_n, \vec{y}_n) \hat{T} \left( e^{-i \int_{-\infty}^{\eta_0} d\eta H_I} \right) |0\rangle}{\langle 0| \hat{T} \left( e^{i \int_{-\infty}^{\eta} d\eta H_I} \right) T \left( e^{-i \int_{-\infty}^{\eta} d\eta H_I} \right) |0\rangle}
\]

(8.2)

with \( \infty_\pm = \infty(1 \pm i\epsilon) \). Here \( |0\rangle \) is the free Bunch-Davis vacuum, for which Minkowski spacetime is recovered at early times \( (t, t' \ll t) \) or short distances \( |t-t'|^2 + |x-x'|^2 \ll \ell^2 \eta^2 \).

The in-in correlators are obtained perturbatively by expanding the left and right exponentials in Eq. (8.2) and performing Wick contractions (see e.g. [142]). Contractions between two right vertices or a right vertex and an external field are done via a time-ordered propagator \( \langle T \Phi(x) \Phi(x') \rangle = G_{-+}(x, x') \). Contractions between two left vertices are done via an anti-time-ordered propagator \( \langle \bar{T} \Phi(x) \Phi(x') \rangle = G_{++}(x, x') \). Contractions between a left vertex and a right vertex or an external field are done via Wightman functions \( \langle \Phi(x) \Phi(x') \rangle = G_{+-}(x, x') \), \( \langle \Phi(x') \Phi(x) \rangle = G_{-+}(x, x') = G^*_{++}(x, x') \).

8.2 dS Propagators

We derive the dS propagators in Fourier space. We will next see that Fourier space renders the analytical continuation from dS to EAdS straightforward. Fourier space is also usually the preferred space to study cosmological correlators.

Particles in dS space are classified following the unitary irreps of \( SO(1, d+1) \) [148, 149]. Their masses are conveniently parametrized as

\[
\ell^2 m^2 = \Delta(d-\Delta) = \frac{d^2}{4} + \nu^2
\]

(8.3)

In the nontachyonic representation one distinguishes the series of “heavy” states with mass \( m > \frac{d}{2} \ell^{-1} \) labelled by \( \Delta_{\mu} = \frac{d}{2} + i\nu \) with \( \nu \in \mathbb{R} \), and the series of “light” states with mass \( 0 < m < \frac{d}{2} \ell^{-1} \) labelled by \( \Delta_{-\mu} = \frac{d}{2} - \mu \) with \( \mu \in [0, \frac{d}{2}] \). More generally the \( \Delta_{\mu} \) function is analytically continued into the whole complex plane.

The (anti-)time-ordered propagators for a scalar particle satisfy the EOM

\[
\mathcal{D}_{dS} G_{-(++)}(x, x') = \mp i \frac{1}{\sqrt{|g|}_{dS}} \delta^{d+1}(x - x')
\]

(8.4)

with \( \mathcal{D}_{dS} = -\Box_{dS} + m^2 \) the scalar Laplacian.

\[23\]We use the convention \( \langle \Phi(x) \Phi(x') \rangle = \langle 0|\Phi(t+i\epsilon, y)\Phi(t'-i\epsilon, y)|0\rangle_{\epsilon \rightarrow 0} \). See e.g. [147].
Since the EOM applies independently on each endpoint, the propagators must have a factorized structure. Explicitly it is

\begin{align}
G_{--}(p; \eta, \eta') &= \frac{i}{C} F_{<}(\eta_{<}) F_{>}(\eta_{>}), \quad G_{++}(p; \eta, \eta') = -\frac{i}{C} \tilde{F}_{<}(\eta_{<}) \tilde{F}_{>}(\eta_{>}) \\
G_{+-}(p; \eta, \eta') &= \frac{i}{C} F_{<}(\eta_{<}) F_{>}(\eta') = -\frac{i}{C} \tilde{F}_{<}(\eta_{<}) \tilde{F}_{>}(\eta) \tag{8.5}
\end{align}

where we have defined $-\eta^{<} = \min(-\eta, -\eta')$, $-\eta^{>} = \max(-\eta, -\eta')$. The $C$ constant is determined by the Wronskian $W = F_{>}^{2} F_{<} - F_{<}^{2} F_{>}$ with $C = W \frac{d^{\ell-1}}{d\eta^{\ell}}$ as in Eq. (3.4), and similarly for $\tilde{C}$. A basis of solutions to the homogeneous EOM in Fourier space is $z^{\frac{d}{2}} H^{(1,2)}_{i\nu}(-p\eta)$.

The condition implementing Bunch-Davies vacuum in the (anti-)time-ordered propagators $G_{--}$, $G_{++}$ is the following [150–154]. As a result of the symmetries of dS, apart from the singularity at coincident endpoints, there can be another singularity when the endpoints are at antipodal positions, $x = x'$, $\eta = -\eta'$. To implement the Bunch-Davies vacuum we require the propagator to be regular in this antipodal configuration (the remaining singularity ensures flat space behaviour at short distance). The antipodal configuration lies outside the expanding Poincaré patch—in fact the two endpoints are separated by a cosmological horizon. The regularity condition is implemented by extending the Poincaré patch into the $\eta > 0$ half of dS. In Fourier space the antipodal configuration amounts to taking the $(p \rightarrow \infty, \eta' \rightarrow -\eta)$ limit. Therefore the Bunch-Davies vacuum is implemented by

\[
\lim_{p \rightarrow \infty, \eta' \rightarrow -\eta} G_{--,++}(\eta, \eta'; p) = 0. \tag{8.6}
\]

Taking into account the $i\epsilon$ shifts, we find that the de Sitter propagators in Fourier space are

\begin{align}
G_{--}^{dS}(p; \eta, \eta') &= -\frac{\pi}{4} \left( \frac{\eta \eta'}{t^{2}} \right)^{\frac{d}{2}} H^{(1)}_{i\nu}(-p\eta^{<}) H^{(2)}_{i\nu}(-p\eta^{>}) \\
G_{++}^{dS}(p; \eta, \eta') &= -\frac{\pi}{4} \left( \frac{\eta \eta'}{t^{2}} \right)^{\frac{d}{2}} H^{(2)}_{i\nu}(-p\eta^{<}) H^{(1)}_{i\nu}(-p\eta^{>}) \\
G_{+-}^{dS}(p; \eta, \eta') &= -\frac{\pi}{4} \left( \frac{\eta \eta'}{t^{2}} \right)^{\frac{d}{2}} H^{(1)}_{i\nu}(-p\eta^{<}) H^{(2)}_{i\nu}(-p\eta^{>}) \tag{8.7}
\end{align}

One can explicitly verify that $G_{++} = (G_{--})^*$ and that the Wightman function $G_{+-}$ is Hermitian whenever $\nu$ is either purely real or imaginary, which corresponds to the two possible representations of states in de Sitter space.

The amputated boundary-to-bulk propagators from the (anti-)time-ordered propagators are given by

\begin{align}
\sqrt{g_{dS}} K_{--}^{dS}(p; \eta, \eta') &= \left( \frac{\eta}{\eta_0} \right)^{\frac{d}{2}} \frac{H^{(2)}_{i\nu}(-p\eta_{0})}{H^{(2)}_{i\nu}(-p\eta_{0})} \tag{8.8} \\
\sqrt{g_{dS}} K_{+-}^{dS}(p; \eta, \eta') &= \left( \frac{\eta}{\eta_0} \right)^{\frac{d}{2}} \frac{H^{(1)}_{i\nu}(-p\eta_{0})}{H^{(1)}_{i\nu}(-p\eta_{0})}. \tag{8.9}
\end{align}
The amputated boundary-to-bulk Wightman function equals either of those depending on which endpoint is put on the boundary.

8.3 From dS to EAdS

We use analytical continuation to express the dS correlators in Euclidian AdS (EAdS). Similar analytical continuation has been discussed in e.g. [105–108] in the (A)dS/CFT context, and has more recently been used in [97, 103, 104] to study the dS correlators in the Mellin-Barnes representation (see also [100, 109, 155, 156]). An advantage of this approach is that much is known about perturbative and spectral techniques in EAdS; all this knowledge is readily transferred to dS via analytical continuation. In the context of the present work, there are additional reasons to work in EAdS. The usual heat kernel formalism needs the fluctuation to have a diagonal Feynman propagator in order to be applicable. In the in-in formalism from the original dS spacetime, the various 2pt functions are simultaneously involved hence we do not know how to straightforwardly proceed. Upon rotating to EAdS one obtains a Euclidian propagator matrix, which is trivially diagonalized. We can thus evaluate the one-loop effective action directly in the Euclidian space. These steps are realized further below.

The Euclidian AdS metric $g_{EAdS}$ is given by

$$ds^2_{EAdS} = \frac{L^2}{z^2} \left( dz^2 + (d\vec{y})^2 \right)$$

with $z \in [z_0, \infty]$. The scalar propagator for a particle with mass $m^2_{EAdS} = (\alpha^2 - \frac{d^2}{4}) \frac{1}{L^2}$ is

$$G_\alpha^{EAdS}(p; z, z') = L \left( \frac{zz'}{L^2} \right)^{\frac{d}{2}} I_\alpha(pz_<)K_\alpha(pz_>) .$$

The boundary-to-bulk propagator is

$$\sqrt{|g|}_{EAdS}K^EAdS_\alpha(p; z) = \left( \frac{z}{z_0} \right)^{\frac{d}{2}} \frac{K_\alpha(pz)}{K_\alpha(pz_0)}$$

The transformation from dS to EAdS is implemented by closing the following contours in the $\eta$ plane

$$\operatorname{Re}(-\eta)$$

which corresponds to the Wick rotations

$$(-\eta)_\pm = ze^{\pm i\left(\frac{\pi}{2} - \epsilon\right)} .$$
Equivalent Wick rotations have been used in [97, 103, 104, 109]. One can readily see that this converts the dS metric to the EAdS one upon identification of the dS and EAdS radii \( L^2 = -\ell^2 \). For our purposes, however, we will use only the \( \ell \) radius. This implies that the EAdS metric resulting from the Wick rotation has only-minus signature, which will imply the presence of \( i^d \), \((-i)^d\) factors in the expressions.

We denote by an arrow the replacement Eq. (8.14) in a function, \( f(\eta_-,\eta_+) \rightarrow f(ze^{-i(\frac{\eta}{2} - \ell)}, ze^{i(\frac{\eta}{2} - \ell)}) \). We plug the Wick rotation into Eq. (8.7). The time coordinates of \( G_{--}, G_{++}, G_{+-} \) are rotated as \((\eta_-,\eta_+), (\eta_+,-\eta_-), (\eta_+,\eta_-)\) respectively. \(^{24}\) For the \( G_{++} \) function, both the \( F_< \) and \( F_> \) pieces are rotated to a Bessel K function with positive argument. For \( G_{--}, G_{++} \), the \( F_>(\eta_+) \) piece turns into a Bessel K function \( \propto K(pz_>) \) with positive argument. In contrast, the Wick rotation in the \( F_<(\eta_<) \) piece gives a Bessel K function with negative argument, which requires careful treatment since \( K \) has a branch cut along the negative axis. To evaluate this piece, the following identity and its complex conjugate are useful:

\[
H_{\nu}^{(1)}(pz_<e^{-i(\frac{\eta}{2} - \ell)}) = \frac{i}{\sin \pi \nu} \left( e^{-i\frac{\eta}{2}\nu}I_{\nu}(pz_<) - e^{i\frac{\eta}{2}\nu}I_{-\nu}(pz_<) \right). \tag{8.15}
\]

We find that the dS propagators take the form

\[
G^{dS}_{--}(p;\eta,\eta') \rightarrow \frac{1}{2\sin(\pi \nu)} \left( e^{-i\nu \Delta_{dS}} G^{EAdS}_{dS}(p; z, z') - e^{-i\nu \Delta_{dS}} G^{EAdS}_{dS}(p; z, z') \right) \tag{8.16}
\]

\[
G^{dS}_{++}(p;\eta,\eta') \rightarrow \frac{1}{2\sin(\pi \nu)} \left( e^{i\nu \Delta_{dS}} G^{EAdS}_{dS}(p; z, z') - e^{i\nu \Delta_{dS}} G^{EAdS}_{dS}(p; z, z') \right) \tag{8.17}
\]

\[
G^{dS}_{+-}(p;\eta,\eta') \rightarrow \frac{1}{2\sin(\pi \nu)} \left( G^{EAdS}_{dS}(p; z, z') - G^{EAdS}_{dS}(p; z, z') \right) \tag{8.18}
\]

where the EAdS propagators are given by Eq. (8.11) with the identification \( L \equiv \ell \). We have introduced

\[
\Delta_{dS,\nu} = \frac{d}{2} \pm i\nu. \tag{8.19}
\]

Our result matches exactly the form of the propagators obtained in [109] up to an overall sign.

For the Wick rotation of the boundary-to-bulk propagators the general formula is

\[
\sqrt{g}_{dS} K^dS_{\pm}(p;\eta) \rightarrow N_\pm(p)\sqrt{g}_{EAdS} K^{EAdS}(p; z) \tag{8.20}
\]

with

\[
N_\pm(p) = \mp i \frac{\rho \Delta_{-\nu}}{\pi} \frac{K_\nu(\rho \eta_0)}{H^{(2)}_{\nu}(-\rho \eta_0)} \tag{8.21}
\]

The same is true for the boundary-to-bulk Wightman function. Let us consider the case of a light mode, which is typically the relevant case for cosmological correlators. We take the small \( p_{20} \) limit and assume real \( i\nu \) with \( |i\nu| = \mu \). For any sign of \( i\nu \) we find \( N_\pm(p) \approx e^{\pm i\frac{\eta}{2} \Delta_{-\nu}} \). Thus

\[
\sqrt{g}_{dS} K^dS_{\pm}(p;\eta) \bigg|_{\text{light, } p_{20} < 1} \rightarrow e^{\pm i\frac{\eta}{2} \Delta_{-\nu}} \sqrt{g}_{EAdS} K^{EAdS}(p; z) \tag{8.22}
\]

\(^{24}\)These are in fact the only possibilities for closing the contours with vanishing contribution from the arcs at \( |\eta| \rightarrow \infty \), as required to perform Wick rotation.
We have used $K_{\mu}^{\text{EAdS}} = K_{-\mu}^{\text{EAdS}}$. We can see that the dimension of the physical operator $\Delta_{-\mu} = \frac{d}{2} - \mu$ always appear. The phases in Eq. (8.22) are consistent with the ones obtained in [109].

Finally, the integral measures are Wick rotated as

$$
\begin{align*}
  i \int_{-i\infty}^{i\infty} \frac{d\eta_+}{(-\eta_+ + 1)^{d+1}} &\to (-i)^{d-1} \int_{z_0}^{\infty} \frac{dz}{(z\ell^{-1})^{d+1}} \\
  -i \int_{-i\infty}^{i\infty} \frac{d\eta_-}{(-\eta_- + 1)^{d+1}} &\to (i)^{d-1} \int_{z_0}^{\infty} \frac{dz}{(z\ell^{-1})^{d+1}}
\end{align*}
$$

(8.23) (8.24)

8.4 The Generator of Cosmological Correlators

We follow, revisit and expand a proposal from Ref. [109]. The analytical continuation from dS to EAdS is used to define a functional $Z_{\text{pert}}^{\text{dS}}[J]$, expressed in EAdS, that generates the cosmological correlators at any order in perturbation theory.

We first realize that, upon the two types of Wick rotation in Eq. (8.13), the dS field with values on $\mathbb{R}(1 \pm i\epsilon)$ is analytically continued as either a holomorphic or an antiholomorphic function of $\eta, \Phi_{\text{dS}}(\pm \eta \pm e^{\pm i\pi/2})$. This implies that there are two distinct fields in $Z_{\text{pert}}^{\text{dS}}$, related to each other by complex conjugation. We define the two fields $\Phi_{\pm}$ as a function of $z$, with $\Phi_{\pm}(z) = \Phi_{\pm}(\pm \eta_{\pm} e^{\mp i\pi/2})$. They satisfy $\Phi^*_{\pm} = \Phi_{\mp}$. In this subsection it is enough to focus on a single pair of EAdS fields $\Phi_{\pm}$, i.e. a single dS field only. The $\Phi_-$ field comes from the time-ordered contour, the $\Phi_+$ field comes from the anti-time-ordered contour. The $\Phi_-$ is used for external legs.

We introduce a boundary source $J$ coupled to the $\Phi_-$ fields. The source is localized on the boundary of EAdS. The source can either be understood as a function of the original dS coordinates or of the EAdS coordinates.

Given these definitions and conventions, we introduce the generating functional

$$
Z_{\text{pert}}^{\text{dS}}[J] = \int D\Phi_{\pm} \exp \left[ -\frac{1}{2} (\Phi_-, \Phi_+) * \hat{G}^{-1}_{\text{EAdS}} * (\Phi_-, \Phi_+)^t - J * \Phi_{-0} \\
- \int_{\text{EAdS}} \sqrt{|g|} dz dx \left( i^{d+1} \mathcal{L}_i[\Phi_-] + (i)^{d+1} \mathcal{L}_i[\Phi_+] \right) \right]
$$

(8.25)

where $J$ is boundary localized. We remind the definition of the convolution products, here in EAdS

$$
A * B = \int_{\text{EAdS}} \sqrt{|g|} dx \int_{\text{EAdS}} A(x) B(x), \quad A \circ B = \int_{\partial\text{EAdS}} \sqrt{|g|} dy A(y) A(y)
$$

(8.26)

with the EAdS radius set to $\ell$. We have introduced the propagator matrix

$$
\hat{G}_{\text{EAdS}} = \frac{1}{2 \sin(\pi i\nu)} \begin{pmatrix}
  e^{-i\Delta_{\nu}} G_{\nu\nu}^{\text{EAdS}} - e^{-i\Delta_{-\nu}} G_{-\nu\nu}^{\text{EAdS}} & e^{-i\Delta_{\nu}} G_{\nu\nu}^{\text{EAdS}} - e^{-i\Delta_{-\nu}} G_{-\nu\nu}^{\text{EAdS}} \\
  e^{i\Delta_{\nu}} G_{\nu\nu}^{\text{EAdS}} - e^{i\Delta_{-\nu}} G_{-\nu\nu}^{\text{EAdS}} & e^{i\Delta_{\nu}} G_{\nu\nu}^{\text{EAdS}} - e^{i\Delta_{-\nu}} G_{-\nu\nu}^{\text{EAdS}}
\end{pmatrix}
$$

(8.27)

Using $\Phi^*_+ = \Phi_-$ one can verify that the fundamental action (i.e. the kinetic term and the interaction term) is real.
The connected perturbative cosmological correlators originally defined by Eq. (8.2) are then given by derivatives of the generating functional

\[ W_{dS}^{pert} [J] = \log(Z_{dS}^{pert} [J]) . \]  

Moreover, as discussed in section 5, taking the Legendre transform \( \Gamma[\Phi_{\pm,0}] = -W[J] + J \circ \Phi_{-0} \) gives the effective action in the boundary variables \( \Phi_{\pm,0} \). This effective action \( \Gamma[\Phi_{-0}] \) generates the 1P-irreducible boundary diagrams, where irreducible means with respect to boundary-to-boundary lines. These diagrams have boundary-to-bulk propagators as external legs — they are the dS version of Witten diagrams.

The reason why the \( \Gamma_{dS}^{pert}[\Phi_{\pm,0}] \) effective action depends on both \( \Phi_{-0} \) and \( \Phi_{+0} \) is that the propagators amputated by the Legendre transform are non-diagonal — both \( \Phi_{+0} \) and \( \Phi_{-0} \) show up when using the chain rule on \( W_{dS}^{pert} [J] \). Thus the operation of amputation requires some care. In order to evaluate the boundary action that generates the amputated cosmological correlators, we take an alternative route by first performing a field redefinition.

**Canonical Normalization**

We can see that the propagator matrix \( G_{EAdS} \) contains linear combinations of \( G_{i\nu} \) and \( G_{-i\nu} \). Thus we introduce a pair of EAdS fields \( \Phi, \tilde{\Phi} \) satisfying

\[ \langle \Phi \Phi \rangle \propto G_{-i\nu}, \quad \langle \tilde{\Phi} \tilde{\Phi} \rangle \propto G_{i\nu}, \quad \langle \Phi \tilde{\Phi} \rangle = 0. \]

We introduce the complex coefficients

\[ c_{\pm i\nu} = \frac{e^{\pm i\frac{\pi}{2}\Delta_{\pm i\nu}}}{\sqrt{2 \sin \pi i\nu}} . \]  

(8.29)

We find that the transformation

\[ \begin{pmatrix} \Phi_- \\ \Phi_+ \end{pmatrix} = U \begin{pmatrix} \tilde{\Phi} \\ \Phi \end{pmatrix}, \quad U = \begin{pmatrix} c_{-i\nu} & i c_{-i\nu} \\ i c_{+i\nu} & c_{+i\nu} \end{pmatrix} \]  

(8.30)

diagonalizes and canonically normalizes the matrix of propagators.

Plugging it into Eq. (8.25) gives canonically normalized kinetic terms. The generating functional reads

\[ Z_{dS}^{pert} [J] = \int D\Phi D\tilde{\Phi} \exp \left[ -\frac{1}{2} (\Phi \ast G_{-i\nu} \ast \Phi + \tilde{\Phi} \ast G_{-i\nu} \ast \tilde{\Phi}) - \mathcal{O}[\Phi_0, \tilde{\Phi}_0] \circ J \right. \]

\[ - \int_{EAdS} \sqrt{|g|} d^{d+1}x \left[ i^{d-1} \mathcal{L}_I [ic_{-i\nu} \Phi + c_{-i\nu} \tilde{\Phi}] + (-i)^{d-1} \mathcal{L}_I [ic_{+i\nu} \Phi + c_{+i\nu} \tilde{\Phi}] \right] \]  

(8.31)

For a heavy dS mass \( (\nu \in \mathbb{R}) \), the fields satisfy \( \Phi^* = -\Phi \). The propagators satisfy \( G_{-i\nu} = G^*_{i\nu} \), and therefore the action is real. For a light dS mass \( (i\nu \in [-\frac{d}{2}, \frac{d}{2}]) \), the fields either satisfy \( \Phi^* = -\Phi, \tilde{\Phi}^* = \tilde{\Phi} \) or \( \Phi^* = \Phi, \tilde{\Phi}^* = -\Phi \) depending on the value of \( i\nu \). The \( G_{\pm i\nu} \) propagators are real, and it follows that again the action is real.
Let us specify the source for a light field. We assume \( i\nu = \mu \in \left[0, \frac{d}{2}\right] \). The kinetic term in Eq. (8.31) indicates that the \( \Phi \) field has Neumann and \( \tilde{\Phi} \) field has Dirichlet BC. Therefore the fields decompose in the holographic basis as

\[
\Phi = \Phi_0 \circ K + \Phi_D, \quad \tilde{\Phi} = \tilde{\Phi}_D.
\] (8.32)

The boundary source term depends only on \( \Phi \). The normalization is fixed by reproducing the cosmological correlators. Using Eq. (8.22) we find

\[
\mathcal{O}[\Phi_0, \tilde{\Phi}_0]_{\text{light}} = \frac{1}{\sqrt{2 \sin \pi \mu}} \Phi_0.
\] (8.33)

Putting the pieces together, the amputated cosmological correlators with light fields in external legs are generated by derivatives of the effective action \( \Gamma_{dS}^{\text{pert}}[\Phi_0] = -W_{dS}^{\text{pert}} + J \circ \mathcal{O}[\Phi_0] \) as follows,

\[
-(2 \sin \pi \nu)^{\frac{n}{2}} \frac{\delta^n \Gamma_{dS}^{\text{pert}}[\Phi_0]}{\delta \Phi_0(\vec{y}_1) \ldots \delta \Phi_0(\vec{y}_n)} = \langle \Phi_0(\vec{y}_1) \ldots \Phi_0(\vec{y}_1) \rangle_{\text{1PI}}
\] (8.34)

for \( i\nu > 0 \). Equivalently, for \( i\nu < 0 \) the derivatives are taken in \( \tilde{\Phi} \). We emphasize that, despite the complex factors present in the action, the resulting cosmological correlators are real.

We remind that, since \( \Gamma_{dS}^{\text{pert}}[\Phi_0] \) is defined by Legendre transform in the boundary source, 1P-irreducibility is here meant with respect to boundary-to-boundary lines (see also Sec. 5). The 4pt exchange diagram with Dirichlet bulk field exchange, for example, is generated by Eq. (8.34) as a 1PI diagram. As a sanity check one can verify that the \( dS \) exchange diagram with cubic vertices and with light states in external legs is correctly reproduced by Eq. (8.34).

### 8.5 Towards the Cosmological One-loop Effective Action

Integrating out a \( dS \) bulk fluctuation at tree-level in \( dS \) is a fairly simple task. One can use, in particular, the covariant large mass expansion of the propagator given in Eq. (5.12). Integrating out a \( dS \) bulk fluctuation at loop-level in \( dS \) is more challenging. However, thanks to the analytical continuation to \( \text{EAdS} \), we can readily use the \( \text{AdS} \) heat kernel coefficients evaluated in the previous section to get the heat kernel expansion of the cosmological one-loop effective action, \( \text{i.e.} \) the one-loop piece of \( \Gamma_{dS}^{\text{pert}} = \Gamma_{dS}^{\text{pert}} + \Gamma_{dS}^{\text{pert}, 1-\text{loop}} + \ldots \).

We focus on integrating out bulk fluctuations of a scalar field \( \Phi \). The field can either be heavy or light in the \( dS \) sense. Depending on cases, either Dirichlet or Neumann BC can be chosen. The interaction Lagrangian in terms of \( dS \) fields is written as

\[
\mathcal{L}_{dS}^{\text{eff}} = -\frac{1}{2} \Phi^2 V''(\varphi_{dS}^{\text{eff}})
\] (8.35)
where the $\varphi_{i}^{dS}$ are light background fields with masses parametrized by $\mu_{i} \in [0, 1/2]$. Upon analytical continuation to EAdS, the interactions are $V''_{\pm} = V''(e^{\pm i\pi/2} \Delta_{\mu_{i}}, K_{\mu_{i}} \circ \varphi_{i,0})$ where the background boundary fields $\varphi_{i,0}$ are real.

We consider the quadratic piece of the fundamental action in $\Phi$,

$$ S_{\text{EAdS}}^{\text{quad}} = \frac{1}{2} (\Phi \ast G_{-i\nu} \ast \Phi + \tilde{\Phi} \ast G_{-i\nu}^{-1} \ast \tilde{\Phi}) $$

$$ + \int_{\text{EAdS}} \sqrt{|g|} d^{d+1}x \left( \frac{i^{d-1}}{2} (ic_{-i\nu} \Phi + c_{i\nu} \tilde{\Phi}) V''_{-} + \frac{(-i)^{d-1}}{2} (ic_{-i\nu} \Phi + c_{i\nu} \tilde{\Phi})^2 V''_{+} \right) $$

with the shortcut notation

$$ V''_{\pm} = V''(e^{\pm i\pi/2} \Delta_{\mu_{i}}, K_{\mu_{i}} \circ \varphi_{i,0}) . $$

The background-field-dependent mass matrix of the fluctuations can be read from Eq. (8.36),

$$ X = i^{d} \left( \begin{array}{cc} (ic_{-i\nu})^2 & c_{i\nu} c_{-i\nu} \\ c_{i\nu} c_{-i\nu} & -i(c_{-i\nu})^2 \end{array} \right) V''_{-} - (-i)^{d} \left( \begin{array}{cc} (ic_{-i\nu})^2 & c_{i\nu} c_{-i\nu} \\ c_{i\nu} c_{-i\nu} & -i(c_{-i\nu})^2 \end{array} \right) V''_{+} $$

$$ = i \frac{1}{2 \sin(\pi \nu)} \left( e^{i\pi \nu} V''_{-} - e^{-i\pi \nu} V''_{+} \begin{array}{c} i(V''_{+} - V''_{-}) \\ i(V''_{+} - V''_{-}) e^{i\pi \nu} V''_{-} - e^{-i\pi \nu} V''_{+} \end{array} \right) $$

For a light fluctuation, $X$ is self-conjugate. For heavy fields, $X$ is not self-conjugate because conjugation swaps the $\Phi$ and $\tilde{\Phi}$ components and thus the entries of $X$, however the physical quantities are the traces $\text{tr}(X^{r})$, which are self-conjugate.

Finally, we evaluate the trace of arbitrary powers of $X$ and find they take the simple form

$$ \text{tr}(X^{r}) = (-1)^{r} \left( (V''_{-})^{r} + (V''_{+})^{r} \right) . $$

This canonical invariant is the only ingredient needed to evaluate the heat kernel coefficients in the case of scalar interactions considered here.

Evaluating the finite, mass-suppressed parts of the one-loop action, i.e. the long-distance EFT, could be interesting. However it may require some more developments due to the fact that one deals with masses which are tachyonic from the EAdS viewpoint. A careful analytic continuation from nontachyonic to tachyonic EAdS masses might be needed. This is left to future work. Here below we work out the divergent part of the one-loop cosmological action in a simple case.

### 8.5.1 Renormalization from Scalar Loops in dS$_{4}$

In dS$_{4}$, one-loop divergences appear from the $b_{4}$ and $b_{4}^{0}$ heat kernel coefficients. The assumed scalar interactions give $V(\varphi_{i}) = \frac{\Delta_{\mu_{i}}}{2} \prod_{i=1}^{n} \varphi_{i}$. The diagrammatic contributions to the $b_{4}, b_{4}^{0}$ coefficients are shown in Fig. 5. The EAdS scalar curvature is $R = -\frac{12}{r^2}$. The $\Box \prod_{i=1}^{n} \varphi_{i}$ and $\partial_{\perp} \prod_{i=1}^{n} \varphi_{i}$ are evaluated like in the AdS case, see Eqs. (7.6), (7.11). In the late time limit, the $\Delta_{i}$ combine to appear only in the $\Delta_{X} = \sum_{i=1}^{n} \Delta_{i}, \Delta_{\overline{X}} = \sum_{i=1}^{n} (3 - \Delta_{i})$ combinations, in accordance with the underlying conformal symmetry.
Analytically-continued $dS^4$ diagrams contributing to bulk and boundary divergences in the one-loop effective action. Internal lines are elements of the $\hat{G}_{EAdS^4}$ propagator matrix.

We take a Neumann BC for the heavy field, accordingly to discussion in Sec. (6). Using Eq. (8.40) with $r = 1, 2$, we find the heat kernel coefficients

$$\text{tr} \, \bar{b}_4 = \cos \left( \pi \Delta_X \right) \Lambda_{n,2} \circ \left[ \prod_{i=1}^{n} \varphi_{i,0} \right]^2 - \frac{\cos \left( \frac{\pi}{2} \Delta_X \right)}{3\ell^2} \left( -\Delta_X (\Delta_X - 3) + 12 \right) \Lambda_n \circ \prod_{i=1}^{n} \varphi_{i,0}$$

$$\text{tr} \, b_4^2 = \frac{4}{3} \cos \left( \frac{\pi}{2} \Delta_X \right) \left( \frac{1}{\sqrt{g}} \sum_{i=1}^{n} (O_i O_i) \circ \prod_{i=1}^{n} \varphi_{i,0} + \frac{1}{\ell} \left( \Delta_X + \frac{d}{2} \right) \prod_{i=1}^{n} \varphi_{i,0} \right)$$

These enter in the cosmological one-loop effective action as

$$\Gamma_{\text{1-loop}}^{dS,\text{div}} [\varphi_i] = -\frac{1}{32\pi^2} \left( \text{tr} \, \bar{b}_4 + \int_{\partial \mathcal{M}} d^3 x \sqrt{\bar{g}} \, \text{tr} \, b_4^2 (x) \right) \log \mu_r$$

The bulk term contains holographic vertices with $n$ and $2n$ legs. The boundary term contains a $n$pt contact operator. It also contains a nonlocal contribution which amounts to attach a 2pt CFT correlator to that local operator. The logarithms obtained here via dimensional regularization are similar to those discussed in [145].

### 8.5.2 Inflationary $\varphi^4$ Example

For inflationary correlators the light fields typically have vanishing mass, which corresponds to $\mu = \frac{d}{2}$ and $\Delta - \mu = 0$. We assume a quartic interaction $V(\varphi) = \frac{1}{4!} \varphi^4$, giving $X = \frac{1}{2} \varphi^2$.  

\[\text{The calculation is here in Euclidian signature. There is a relative \(-\) sign compared to the equivalent expression in Lorentzian signature, due to the usual convention for the actions. See App. D for details.}\]
The total dimensions are $\Delta X = 0$, $\Delta \bar{X} = 6$. We use Neumann BC. We get

$$
\begin{align*}
\text{tr} \bar{b}_4 &= \frac{\lambda^2}{4} \Lambda_4 \circ [\phi_0]^4 - \frac{2\lambda}{\ell^2} \phi_0 \circ \Lambda_2 \circ \phi_0 \\
\text{tr} b_4^0 &= \frac{4\lambda}{3} \frac{1}{\sqrt{g}} (\langle OO \rangle \circ \phi_0) \phi_0 + \frac{5}{\ell} \phi_0^2.
\end{align*}
$$

In the bulk contribution, the first and second terms respectively amount to the generation of a bulk quartic and a bulk mass for $\phi$. Using Eq. (8.43), the $\beta$ function for the mass is

$$
\frac{\partial}{\partial \log \mu_r} m_\phi^2 = \frac{\lambda}{8\pi^2 \ell^2}.
$$

This can also be translated into a correction to the conformal dimension of $\phi$ via $\frac{\partial}{\partial \log \mu_r} \mu^2 = \ell^2 \frac{\partial}{\partial \log \mu_r} m_\phi^2$. The fact that the mass receives a logarithmic correction is a consequence of the nonzero curvature.

In the boundary contribution, a 2pt CFT operator and a boundary-localized mass term are generated. These terms are corrections to the nonlocal and local parts of the 2pt cosmological correlator. This is similar to the log corrections to 2pt correlators discussed in [145]. Like in the AdS case, upon canonical normalization of $O$, the “wavefunction” renormalization of $O$ feeds into the renormalization of the nonlocal part of the $(n > 2)$pt cosmological correlators.

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**A The Discontinuity Equation**

Here we give a proof of Eq. (2.7).

We denote by $\partial M^-$ the surface at an infinitesimal normal distance of the actual boundary $M$ in the inward direction. We define an infinitesimal shell $\sigma_{x_0}$ containing the boundary point $x_0$. The outward part of the boundary of $\sigma_{x_0}$ is in $\partial M$, and the inward part of the boundary of $\sigma_{x_0}$ is in $\partial M^-$. A point belonging to $\partial M^-$ is denoted $x_0^-$. We integrate the bulk equation of motion Eq. (2.3) over the $\sigma_{x_0}$ volume and apply the divergence theorem. The delta function of the EOM is picked by the integral if the other endpoint of the propagator is inside the $\sigma_{x_0}$ shell. Taking the infinitesimal limit for $\sigma_{x_0}$ and using that $M$ is locally Euclidian, we get the discontinuity of the normal derivative evaluated at the boundary. This is Eq. (2.7).
B Details and Checks of Propagators in Warped Background

B.1 Generic Background

Here we show that the propagators given in Sec. 3 satisfy the general properties derived in Sec. 2.

We first define the boundary inner product. In Fourier coordinates it is given by a simple multiplication,

$$
\Phi_0 \circ \Psi_0 = \rho_0^de^{-\phi_0} \Phi_0 \Psi_0 .
$$

(B.1)

Since the inverses are defined using this product, inverses involve a factor:

$$
[a]^{-1} = \rho_0^de^{\phi_0}. 
$$

We will check the general relation

$$
G_N\circ [G_0^{-1} + B]^{-1} \circ K + G_D .
$$

We evaluate

$$
G_N\circ [G_0^{-1} + B]^{-1} \circ K + G_D \circ (z_0, z_0) \ (B.2)
$$

where in the intermediate step one has used the Wronskian in the numerator and we identified the boundary-to-boundary propagator, Eq. (3.10). Since

$$
G_N\circ [G_0^{-1} + B]^{-1} \circ K + G_D \circ (z_0, z_0) \ (B.3)
$$

This reproduces the dressing of the boundary-to-boundary propagator by the boundary-localized bilinear insertion $B$.

It is also instructive to explicitly verify the discontinuity formula. We get

$$
(\rho_0 \partial + B f_0) G_N_B (p; z, z_0) \big|_{z \to z_0} = i \rho_0^d e^{-\phi_0} B
$$

(B.4)

where again one has used the Wronskian at $z_0$. This result verifies Eq. (2.8) upon translation from Lorentzian to Euclidian conventions.

B.2 AdS Propagator in Position Space

In Euclidian AdS the propagator satisfying the equation of motion

$$
(-\Box_{AdS} + \Delta (\Delta - d) k^2 ) G_\Delta (x, x') = \frac{1}{\sqrt{g}} \delta^{d+1}(x - x')
$$

is [157]

$$
G_\Delta (x, x') = \frac{\Gamma(\Delta) k^{d-1-2\Delta} 2F1 \left( \Delta, \Delta - \frac{d}{2}, 2\Delta - d + 1, -\frac{4}{k^2\zeta(x, x')} \right)}{2\pi^{d/2} \Gamma(\Delta - \frac{d}{2} + 1) \zeta(x, x')^2 \delta^{d+1}(x - x')}
$$

(B.5)
with
\[ \zeta(x, x') = \frac{1}{k^2} (z - z')^2 + (x - x')^2 \]

the chordal distance in Poincaré coordinates. Taking one endpoint towards the boundary, the propagator is asymptotically
\[ G_\Delta(x, x')|_{z \to 0} = \frac{\Gamma(\Delta)}{2\pi^{d/2}\Gamma(\Delta - \frac{d}{2} + 1)} \frac{k^{d-1-2\Delta}}{\zeta(x, x')^\Delta} + O\left(\frac{1}{\zeta^{\Delta+1}}\right) \]
for any \( \Delta \).

## C Holography with Two Boundaries

In this appendix we work out the holographic formalism for a \( z_0 < z < z_1 \) slice of the warped background introduced in Eq. (3.1), with both boundaries treated holographically.

Similarly to Sec. 2, our starting point is to write the holographic basis in the presence of two boundaries,
\[ \Phi = \Phi_0 \circ L_0 + \Phi_1 \circ L_1 + \Phi_D \]  
(C.1)

where \( \Phi_{0,1} \) are the field values on each boundary, which will be the variables of the holographic action. The \( L_0, L_1 \) boundary-to-bulk propagators are determined below. We work in Fourier space along the transverse Poincaré slice, hence the coordinates are \((p, z)\). We consider the classical value \( \langle \Phi \rangle \) which satisfies the bulk EOM \( D\langle \Phi \rangle = 0 \). It takes in general the form
\[ \langle \Phi \rangle (z) = af(z) + bh(z). \]  
(C.2)

The associated Wronskian is
\[ W(z) = f(z)h'(z) - f'(z)h(z) = C\rho^{d-1}e^\phi. \]  
(C.3)

The \( f, h \) solutions and the \( a, b \) constants depend on the \( d \)-momentum. We also define
\[ f(z_0) = f_0, \quad f(z_1) = f_1, \quad h(z_0) = h_0, \quad h(z_1) = h_1. \]  
(C.4)

Using the definitions Eqs. (C.2), (C.4), the \( a, b \) constants can be translated into the holographic variables,
\[ a = \frac{\Phi_0 h_1 - \Phi_1 h_0}{f_0 h_1 - f_1 h_0}, \quad b = -\frac{\Phi_0 f_1 - \Phi_1 f_0}{f_0 h_1 - f_1 h_0}. \]  
(C.5)

The holographic basis in Fourier space is therefore
\[ \Phi(p, z) = \Phi_0 \frac{h_1 f(z) - f_1 h(z)}{f_0 h_1 - f_1 h_0} + \Phi_1 \frac{f_0 h(z) - h_0 f(z)}{f_0 h_1 - f_1 h_0} + \Phi_D. \]  
(C.6)

We plug the solution Eq. (C.6) into the quadratic action, giving
\[ S[\Phi_0, \Phi_1] = \int \frac{d^dp}{(2\pi)^d} \sqrt{|g|} e^{-\phi} \frac{1}{2} (\Phi_0 (\rho_0 \partial_z - b_0) \Phi|_{z_0} - \Phi_1 (\rho_1 \partial_z - b_1) \Phi|_{z_1}) + S_D \]  
(C.7)
where $b_0$, $b_1$ encode the boundary actions. We introduce

$$\hat{f}_i = \rho_i f_i(z_i) - b_i f_i(z_i)$$

and similarly for $\hat{g}_i$. Evaluating $\partial_z \Phi_{z_0}$ and $\partial_z \Phi_{z_1}$ using Eq. (C.6) we obtain the holographic self-energies

$$S[\Phi_0, \Phi_1] = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \left( \Phi_0 \Pi_0 \Phi_0 + 2 \Phi_0 \Pi_0 \Phi_1 + \Phi_1 \Phi_1 \right),$$

$$\Pi_0 = \rho_0 e^{-\varphi_0} \frac{\hat{f}_0 h_1 - \hat{h}_0 f_1}{f_0 h_1 - h_0 f_1}, \quad \Pi_1 = \rho_1 e^{-\varphi_1} \frac{\hat{f}_1 h_0 - \hat{h}_1 f_0}{f_0 h_1 - h_0 f_1}, \quad \Pi_{01} = \frac{C}{f_0 h_1 - h_0 f_1}$$

where we have used the Wronskian at $z_0$ and $z_1$ to get the mixed self-energy $\Pi_{01}$.

We can re-express these self-energies using propagators. The propagator in the presence of two boundaries is

$$G^{\pm \pm}(y, y') = \frac{i}{C} \left( \hat{f}_0 h_1(z_0) - \hat{h}_0 f_1(z_0) \right) \left( \hat{f}_1 h_0(z_1) - \hat{h}_1 f_0(z_1) \right)$$

where $\hat{f}_i = f_i$ for a Dirichlet boundary condition (−) and $\hat{f}_i = \hat{f}_i$ for a Neumann boundary condition (+). For the $\Pi_0$, $\Pi_1$ self-energies we find

$$\Pi_0 = \frac{i}{G^{++}_p(z_0, z_0)}, \quad \Pi_1 = \frac{i}{G^{+-}_p(z_1, z_1)}.$$

The $\Pi_{01}$ self-energy can be expressed as

$$\Pi_{01} = \frac{C}{f_0 h_1 - h_0 f_1} = \frac{1}{G^{++}_p(z_0, z_1)} \frac{\hat{f}_0 h_1 - \hat{h}_0 f_1}{f_0 h_1 - h_0 f_1} \frac{\hat{f}_1 h_0 - \hat{h}_1 f_0}{f_0 h_1 - h_0 f_1}$$

$$= \frac{i}{G^{++}_p(z_0, z_1) G^{++}_p(z_0, z_0)} = \frac{i}{G^{++}_p(z_0, z_1) G^{++}_p(z_1, z_1)}.$$

We can recognize (amputated) boundary-to-bulk propagators in the last expressions. We have

$$\Pi_{01} = K_0(z_1) \Pi_1 = K_1(z_1) \Pi_0$$

(C.14)

where $K_0(z)$, $K_1(z)$ denote the amputated boundary-to-bulk propagators from the $z_0$ and $z_1$ boundaries respectively.

If we let the $\Phi_1$ variable be dynamical i.e. integrate it out in the path integral, we should recover our standard one-boundary holographic action. For a Dirichlet boundary condition on $z_1$, we have $\Phi_1 = 0$. It follows trivially that

$$S[\Phi_0] = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{G^{++}_p(z_0, z_0)} \Phi_0^2 + S_D$$

(C.15)

which is the expected result for the holographic action on boundary 0 with Dirichlet boundary condition on boundary 1. For a Neumann boundary condition on $z_1$, we have $(\rho_1 \partial_z - b_1)\Phi_{z_1} = 0$, which implies

$$\Phi_1 = \Phi_0 \frac{W}{h_0 f_1 - f_0 h_1}.$$

(C.16)
Substituting $\Phi_1$ with this relation in the two-boundary holographic action gives
\[ S[\Phi_0] = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{G^++(z_0,z_0)} \Phi_0^2 + S_D \] (C.17)
which is again the expected result for the holographic action on boundary 0 with Neumann boundary condition on boundary 1.

## D Elements of the Heat Kernel Formalism

We work in Euclidian metric and convert to Lorentzian conventions at the end of the calculation. The Euclidian effective action generates the 1PI Euclidian correlators following
\[ -\frac{\delta^n\Gamma_E}{\delta \Phi(x_1)\ldots \delta \Phi(x_n)} = \langle \Phi(x_1)\ldots \Phi(x_n) \rangle. \] (D.1)

The one-loop effective action is put in the form of the heat kernel integral
\[ \Gamma_E^{1-\text{loop}} = -\frac{1}{2} \int \frac{dt}{t} \text{Tr} e^{-tD_E}. \] (D.2)

with $D_E = -\Box_E + m^2 + X$. Tr is the trace over all indexes including spacetime coordinates. The heat kernel trace is expanded as
\[ \text{Tr} e^{-t(-\Box_E + m^2 + X)} = (4\pi t)^{D/2} e^{-tm^2} \int d^Dx \sqrt{g} \sum_{r=0}^{\infty} \text{tr} b^E_{2r}(x)t^r \] (D.3)

where the $b^E_{2r}$ are the Euclidian heat kernel coefficients. Using this expansion in Eq. (D.2), integrating in $t$ and analytically continuing in $D$ gives
\[ \Gamma_E^{1-\text{loop}} = -\frac{1}{2} \frac{\Gamma(r - D/2)}{(4\pi)^{D/2}} \int d^Dx \sqrt{g} \sum_{r=0}^{\infty} \frac{\text{tr} b^E_{2r}(x)}{m^{2r-D}} \] (D.4)
for any spacetime dimension. Converting to Lorentzian metric gives\footnote{Use $x_E^0 = ix^0$, $\Gamma_{E}^{1-\text{loop}} = -i\Gamma_{1-\text{loop}}$.}
\[ \Gamma_{1-\text{loop}} = \frac{1}{2} \frac{\Gamma(r - D/2)}{(4\pi)^{D/2}} \int d^Dx \sqrt{g} \sum_{r=0}^{\infty} \frac{\text{tr} b_{2r}(x)}{m^{2r-D}} \] (D.5)

with tr the trace over internal (non-spacetime) indexes.
The exact $b_0$, $b_2$, $b_4$ coefficients are given in Eq. (6.11). The exact $b_6$ coefficient is

$$b_6 = \frac{1}{360} \left( 8D_P \Omega_{MN} D_P \Omega^{MN} + 2D^2 \Omega_{MN} D_P \Omega^{PN} + 12\Omega_{MN} \Box \Omega^{MN} - 12\Omega_{MN} \Omega^{NP} \Omega_P^M \\
- 6R_{MN} \Omega^{MN} \Omega^{PQ} - 4R^N_M \Omega_M^{MP} \Omega^{NP} + 5R \Omega_M \Omega_M^{MN} \\
- 6 \Box^2 X + 60X \Box X + 30D_M X \Box^2 M X - 60X^3 \\
- 30X \Omega_M \Omega_M^{MN} - 10R \Box X - 4R_M \Omega_M D^N D^M X - 12D_M RD^M X + 30X \Box R \\
- 12X \Box R - 5X \Box^2 R - 2X R_{MN} R^{MN} - 2X R_{MN} \Omega^{PQ} R^{MN} \Omega^{PQ} \Box \\
+ \frac{1}{7!} \left( 18 \Box^2 R + 17D_M RD^M R - 2D_P R_{MN} D^P R^{MN} - 4D_P R_{MN} D^M R^{PN} \\
+ 9D_P R_{MN} Q \Omega^{NL} + 28R \Box R - 8R_{MN} \Box R^{MN} \\
+ 24R_M \Omega_M D^N R^{MP} + 12R_{MN} \Omega^{MN} \Omega^{PQ} + 35/9R^3 \\
- 14/3R_{MN} R^{MN} + 14/3R_{MN} Q \Omega^{MN} R^{PQ} - 208/9R_{MN} R^{MP} R^{NP} \\
- 64/3R_{MN} R_{NP} Q \Omega^{PQ} - 16/3R_{MN} \Omega^{MN} \Omega^{PQ} \Omega^{PQ} \\
- 44/9R_{MN} \Omega^{MN} \Omega^{NPQ} R_{PQAB} - 80/9R_{MN} R_{NP} R^{MN} \Omega^{PQ} R_{PQAB} \right) \right) \right) I \quad (D.6)

D.1 Dirichlet $b_6^\theta$ for Scalar Interaction

In Sec. 7.1 we have used Neumann BC for the fluctuation. If instead one takes Dirichlet BC, using that $\Pi_+ = 0$, $\Pi_- = 1$, $\chi = -1$, $S = 0$, the boundary heat kernel coefficients are found to be

$$\text{tr} b_6^\theta = \frac{1}{4} X_0 \quad (D.7)$$

$$\text{tr} b_4^\theta = \frac{1}{3} \frac{1}{\sqrt{g}} \sum_{i=1}^n (\partial_i \partial^\mu) \xi X_0 + \frac{1}{3} \frac{1}{\sqrt{g}} (\partial_X - d) k X_0 \quad (D.8)$$

$$\text{tr} b_0^\theta = \frac{1}{5760} \left( -720X_0^2 + (360 \Delta_X (d - \Delta_X) k^2 - 450d \Delta_X k^2 - 5d(53 + 27d)k^2) X_0 \\
- 360k^2 \frac{1}{\sqrt{g}} \sum_{i=1}^n (\partial_i \partial^\mu) \xi X_0 + 240k^2 \frac{1}{\sqrt{g}} \partial^\mu \partial^\nu X_0 \\
+ (720 \Delta_X - 450d)k \frac{1}{\sqrt{g}} \sum_{i=1}^n (\partial_i \partial^\mu) \xi X_0 \\
+ 360 \frac{1}{\sqrt{g}} \sum_{i,j=1}^n [(\partial_i \partial^\mu) \xi] [(\partial_j \partial^\nu) \xi] X_0 \right) \quad (D.9)$$

D.2 Elementary Checks from $\mathcal{M}_3$ and EAdS$_3$ Bubble

$\mathcal{M}_3$ Bubble

We check a contribution to the Lorentzian heat kernel coefficient $b_4$ from the scalar bubble diagram in $\mathcal{M}_3$. In $\mathcal{M}_3$, working in momentum space as in particle physics-style calcula-
tions, the amputated bubble at large $m$ is given by

$$B_{M_3}\bigg|_{m\to\infty} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{-p^2 - m^2} \frac{1}{-(p+k)^2 - m^2} \bigg|_{m\to\infty} = \frac{i}{16\pi m}$$ (D.10)

Consider the fundamental interaction $\frac{1}{2} \Phi^2 \mathcal{O}$. The bubble diagram would be generated by two $\mathcal{O}$ derivatives of the Lorentzian effective action $\frac{1}{32\pi m} \mathcal{O}^2$. This matches exactly the $X^2$ term coming from the $b_4$ coefficient (obtained from Eq. (6.11)) in Eq. (D.5).

**EAdS$_3$ Bubble**

We check a contribution to the Euclidian heat kernel coefficient $b_E^4$ from the scalar bubble diagram in EAdS.

Consider the bubble diagram of a real scalar in EAdS,

$$B(x, x') = \frac{1}{2} G(X, Y)^2 = k^2 \int_{\mathbb{R}} d\nu B(\nu, \Delta) \Omega_\nu(x, x')$$ (D.11)

In AdS$_3$, Ref. [42] finds

$$B(\nu, \Delta) = i \frac{\psi\left(\Delta - \frac{1+i\nu}{2}\right) - \psi\left(\Delta - \frac{1-i\nu}{2}\right)}{16\pi \nu}$$ (D.12)

where $\Psi$ is the Digamma function. We expand the functions for large $|\Delta|$ at fixed $\nu$. The leading order result is

$$B(\nu, \Delta)|_{\Delta\to\infty} = \frac{1}{16\pi \Delta} + O\left(\frac{1}{|\Delta|^2}\right).$$ (D.13)

In position space, using that $\int \Omega_\nu(x, x') = \frac{1}{k^{d+1}} \sqrt{|g|} \delta^{d+1}(x, x')$, we obtain the local operator

$$B(x, x')|_{\Delta\to\infty} = \frac{1}{\sqrt{|g|}} \frac{1}{16\pi \Delta k} \delta^3(x, x').$$ (D.14)

Consider the fundamental interaction $\frac{1}{2} \Phi^2 \mathcal{O}$. Using Eq. (D.1), the bubble diagram would be generated by two $\mathcal{O}$ derivatives of the Euclidian effective action

$$- \frac{1}{2} \int dx^{d+1} \int dx'^{d+1} \sqrt{|g|} \sqrt{|g|} \mathcal{O}(x) B(x, x') \mathcal{O}(x').$$ (D.15)

Plugging the limit of Eq. (D.14) into Eq. (D.15) gives the local operator

$$- \frac{1}{32\pi \Delta k} \int dx^{d+1} \sqrt{|g|} \mathcal{O}^2(x)$$ (D.16)

in the Euclidian effective action. This matches exactly the $X^2$ term coming from the $b_4^E$ coefficient (obtained from Eq. (6.11)) in Eq. (D.4).
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