Analytical approximation of neutrino distribution function in core-collapse supernova

A Dobrynina¹,², E Koptyaeva¹ and I Ognev¹

¹ Department of Theoretical Physics, P. G. Demidov Yaroslavl State University, Sovetskaya 14, 150003 Yaroslavl, Russia
² II. Institute for Theoretical Physics, University of Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
E-mail: dobrynina@uniyar.ac.ru, koptiaeva2016@yandex.ru, ognev@uniyar.ac.ru

Abstract. Results of the numerical simulations of the neutrino angular distribution in core-collapse supernova were analyzed. As analysis shown, the neutrino angular distribution function can be approximated by a one-parametric Gaussian function. The proposed approximation describes about 90% of all neutrinos in the core-collapse supernova at times \( t \gtrsim 0.2 \) sec after a core bounce and relatively large distances from proto-neutron-star center. Therefore, it could be a good analytical approximation of the numerical data. At small values of time and distance an accuracy of this approximation significantly decreases.

1. Introduction
Core-collapse supernova (SN) remains for a long time an important part of modern astrophysics. Despite a significant progress in numerical simulations of such phenomenon, there is still no final answer about the nature of the successful explosion of collapsing stars. Nevertheless, it has been determined that in most scenarios the neutrino radiation plays a dominant role in this process, and without its participation a successful supernova explosion is impossible. Perhaps, with the exception of the magnetorotational supernova models, where a strong magnetic field is generated in supernova and leads to explosion [1–4]. However, even in these models, neutrino emission can play an essential role.

An important factor is that a matter of collapsing SN is just partially transparent for neutrino radiation. Thus, the description of a neutrino propagation in such objects requires a self-consistent solution of the their transport equation with hydrodynamics. Up to now, this is a very difficult and resource-intensive problem, despite the various simplifications used. Because of that, the calculation of the neutrino propagation in a supernova includes only the main processes of their interaction with the matter. Therefore, many aspects of neutrino physics of supernovae remain outside the scope of explosion modeling. So, it is important to estimate the influence of such non-included effects on the dynamics of a supernova without their insertion in the self-consistent simulation. But such a calculations and estimations for processes involving neutrinos can be carried out if their local nonequilibrium distribution function is known.
2. Neutrino distribution function in supernova
One of the general methods to describe neutrino emission in supernova is an approach based on their one-particle distribution function. Because of supernova matter is partially transparent for neutrinos, neutrino distribution function is nonequilibrium and can be obtained by solving the Boltzmann equation. Under supernova conditions, this equation is solved only numerically, which makes difficulties for use of results obtained to other problems. Thus, it is important to find an analytical approximation of the results of numerical simulations, which reproduce the neutrino distribution in supernova.

In a case of spherically symmetric propagation of neutrinos in supernova, a local nonequilibrium distribution function of neutrinos at each moment of time depends on three parameters. For example, it can be the distance from a center of proto-neutron star (PNS) \( r \), angle between neutrino momentum and radial direction of star \( \theta \) and neutrino energy \( \omega \). Usually in the spherically symmetric case, it is also assumed that the neutrino energy and angle distributions are independent. This assumption is approximately confirmed by numerically calculations of neutrino transport and allows to simplify the analytical calculations. Note, that the neutrino energy spectrum in supernova is more investigated than neutrino angular distribution. In particular, for the neutrino energy distribution was suggested two analytical approximations [5,6].

The study of the neutrino angular distribution in a supernova is presented in the literature insignificantly. An analytical approximation for it was found only in ref. [7]. In this analysis the mass of SN progenitor is equal to 15 \( M_\odot \). An analytical fit of normalized neutrino angular distribution was proposed in [7]

\[
I_{fit}(\theta) = \left[ \frac{0.9994}{(1 + (\theta/0.0029)^{1.5})^2} \right]^5 + \left[ \frac{0.0006}{(1 + (\theta/0.01)^{1.43})^2} \right]^5. \tag{1}
\]

This fit corresponds to the distance \( r = 10^4 \) km from PNS center and the time \( t = 0.28 \) sec after a core bounce. Authors of [7] generalize this fit to another values of distance \( r \) as follows

\[
I(r, \theta) \propto I_{fit}(\theta r/10^4 \text{ km}). \tag{2}
\]

Unfortunately, this fit is obtained only for one mass value of SN progenitor and one time slot, and it is not clear how to employ it for another cases. Moreover, the suggested function is difficult from the viewpoint of further integration. All mentioned weaknesses are reasons of a search of alternative approximation of neutrino angular distribution in supernova.

3. Analytical approximation of neutrino angular distribution function
In our analysis we use the results of neutrino angle-dependent distributions from simulations of explosion of SN progenitors with masses of 11.2, 13.8, 15, 17.8, 20.6 and 25 \( M_\odot \). The simulations were performed with the 1D version of the PROMETHEUS-VERTEX code [8]. In this simulations authors employ a progenitor model from [9]. The nuclear equation of state is taken from [10] with compressibility modulus \( K=220 \) MeV.

As follows from calculations of macroscopic quantities characterized neutrino interaction with supernova matter, it is more convenient with respect to further calculations to look for an approximation of neutrino angular distribution in terms of variable \( y = 1 - \cos \theta \). Performing an analysis of data provided by numerical simulation \( J_{data}(r, y) \), we have obtained that the angular neutrino distribution in terms of \( y \) can be approximated by simple one-parametric Gaussian function

\[
\Phi(r, y) = \exp \left[ - (A(r) y)^2 \right]. \tag{3}
\]

1 [https://wwwmpa.mpa-garching.mpg.de/ccsnarchive/data/Huedepohl2012_1d_accr/index.html](https://wwwmpa.mpa-garching.mpg.de/ccsnarchive/data/Huedepohl2012_1d_accr/index.html)
In order to estimate an applicability of the proposed approximation, we consider a relative deviation of $\Phi(r, y)$ from $J_{\text{data}}(r, y)$ in following form

$$
\delta_{J-\Phi} = \frac{\int_0^2 |J_{\text{data}}(r, y) - \Phi(r, y)| dy}{\int_0^2 J_{\text{data}}(r, y) dy}.
$$

(4)

The relative deviation $\delta_{J-\Phi}$ of results of SN simulations $J_{\text{data}}(r, y)$ of angular distribution of electron neutrinos, electron antineutrinos and all other neutrino types in SN matter from proposed analytical approximation of its $\Phi(r, y)$ in dependence on distance $R$ (in km) from PNS center for several values of time after a bounce ($t = 0.02, 0.17, 0.256$ sec) and four values of progenitor masses ($11.2, 15, 20.6$ and $25\, M_\odot$) are presented in figure 1 and figure 2.

**Figure 1.** The relative deviation $\delta_{J-\Phi}$ of results of SN simulations $J_{\text{data}}(r, y)$ of angular distribution of electron neutrinos (solid lines), electron antineutrinos (dashed lines) and all other neutrino types (dotted lines) in SN matter from proposed analytical approximation of its $\Phi(r, y)$ in dependence on distance $R$ from PNS center for several values of $t$ and different progenitor masses. Masses of progenitor are $11.2\, M_\odot$ (a) and $15\, M_\odot$ (b). Blue lines: $t = 0.02$ sec; green: $t = 0.17$ sec; red: $t = 0.256$ sec.

**Figure 2.** The legend is the same as in figure 1 except for values of progenitor masses. Masses of progenitor are $20.6\, M_\odot$ (a) and $25\, M_\odot$ (b).
As seen from figure [1] and figure [2], the relative deviation $\delta_{J-\Phi}$ is sufficiently large and differs for different types of neutrinos at time $t \sim 0.02$ sec. Apparently, this indicates that the angular distribution has not yet formed. At more late times, the deviation of numerical data from the proposed function $\Phi(r, y)$ decreases, and its behavior for different neutrino types becomes more similar. The proposed approximation of neutrino angular distribution well describes the data provided by numerical simulations at large distances from PNS center. Moreover, the deviation at $t \gtrsim 0.2$ sec is approximately equal to 10%. Therefore, the proposed one-parameter distribution can be considered as good analytical approximation of the neutrino angular distribution in a supernova. Note, that the character of relative deviation for different masses of progenitor has similarities.

4. Conclusions
The analysis of angular distribution of different neutrino types is performed on data provided by 1D supernova simulation with progenitor masses from 11 to 25 $M_\odot$. As it was shown, the neutrino angular distribution can be approximated by one-parameter Gaussian function. The proposed approximation describes the neutrino angular distribution in a supernova with an accuracy of approximately 90% at times $t \gtrsim 0.2$ sec after a core bounce and relatively large distances from proto-neutron-star center. At small distances as well as at small values of time after a core bounce, the proposed function describes the neutrino angular distribution in a supernova much worse. Presented results are practically similar for different values of progenitor mass.

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