On Properties of Vacuum Axial Symmetric Spacetime of Gravitomagnetic Monopole in Cylindrical Coordinates

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Abstract

We investigate general relativistic effects associated with the gravitomagnetic monopole moment of gravitational source through the analysis of the motion of test particles and electromagnetic fields distribution in the spacetime around nonrotating cylindrical NUT source. We consider the circular motion of test particles in NUT spacetime, their characteristics and the dependence of effective potential on the radial coordinate for the different values of NUT parameter and orbital momentum of test particles. It is shown that the bounds of stability for circular orbits are displaced toward the event horizon with the growth of monopole moment of the NUT object. In addition, we obtain exact analytical solutions of Maxwell equations for magnetized and charged cylindrical NUT stars.

KEY WORDS: Relativistic stars; gravitomagnetic charge; particle motion; electromagnetic fields.
1 Introduction

At present there is no any observational evidence for the existence of gravitomagnetic monopole or so-called NUT charge which may be source of gravitomagnetic field in static case. However there are several attempts and proposals to detect it through astronomical observations in galactic, extragalactic range and in Solar system. For example, the anomalous acceleration of Pioneer satellites \[1\] is attempted to explain through the effect of gravitational field of magnetic mass on motion of test particles \[2, 3\]. In recent papers \[4, 5\], the theoretical effect of magnetic mass in NUT space on the microlensing light curve and the possibility of magnetic mass detection using the gravitational microlensing technique have been studied.

It is not less interesting to study vacuum axial-symmetric solutions of Einstein equations describing the gravitational field of nonrotating NUT source. Nowadays cylindrically symmetric stationary metrics are analyzed due to the fact that they can be used as simple first order approximation model for describing some astrophysical objects. Here we will concentrate in studying properties of NUT sources related to the cylindrical symmetry. The NUT metric components generated by the solution to the Newtonian Laplacian equation corresponding to a Newtonian potential of a rod of length \(2A\) and mass per unit length being equal to \(1/2\) was presented in \[6\]. In the paper \[7\] the metric of cylindrical NUT source in the approximation of weak field has been obtained and studied. In the paper \[8\] we have shown the metric of Nouri-Zonoz describing spacetime outside a line gravitomagnetic field could be obtained from Papapetrou solution of vacuum equations of gravitational field.

Physical interpretation of NUT solution was also investigated in the literature. In particular, it was shown in recent paper \[9\] that the sources of the NUT spacetime are two semi-infinite counter-rotating rods of negative masses and a finite static rod of positive mass. It was also noted that this model is the only possibility for the NUT solution to be stationary and possess zero total angular momentum. Although the metric is not, from the mathematical point of view, cylindrically symmetric, it is physically cylindrically symmetric, which is clear from the gravitoelectric and gravitomagnetic fields which are both explicitly cylindrically symmetric functions.

In our preceding paper \[10\] the electromagnetic fields of conducting shell embedded in the weak gravitational field of cylindrical gravitomagnetic monopole were investigated and analytical solutions to the Maxwell equations in this spacetime were found. Here we extend them to the space time of cylindrical NUT source obtained in \[9\] as exact solution of Einstein equations without making any assumption on the strength of gravitational field. We are interested to study the motion of test particles and electro-
magnetic fields in NUT space with the aim to get tools for studying new important general relativistic effects which are associated with nondiagonal components of the metric tensor and have no Newtonian analogues due to their origin.

The paper is organized as follows. In section 2 we study the motion of test particles in spacetime of cylindrical NUT source. We investigate the effective potential and consider the influence of NUT parameter on characteristics of particles motion. In section 3 we solve Maxwell equations in the exterior metric of cylindrical NUT source around (i) magnetized star with the dipolar magnetic field and (ii) electric charge put on cylindrical gravitomagnetic monopole. We summarize our results in section 4.

Throughout, we use the units \( G = c = 1 \) and the metric signature \((-;+++)\). The round and square brackets on the indices denote symmetrization and antisymmetrization respectively, e.g., \( A_{(\mu\nu)} = (A_{\mu\nu} + A_{\nu\mu}) / 2 \) and \( A_{[\mu\nu]} = (A_{\mu\nu} - A_{\nu\mu}) / 2 \).

# 2 Particles Motion in a Space of Nonrotating Cylindrical NUT Source

The general stationary axially symmetric solution \cite{6, 9} to the vacuum Einstein equations in the standard Weyl-Papapetrou cylindrical coordinates \( \{t, \rho, \varphi, z\} \) is given by

\[
ds^2 = f^{-1} \left[ e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right] - f (dt - \omega d\varphi)^2 ,
\]

where the metric coefficients \( f, \gamma, \omega \) are functions of \( \rho \) and \( z \) only. The explicit expressions for \( f, \gamma, \omega \) have been obtained by Gautreau and Hoffman \cite{6} in the form

\[
f = \frac{A^2(x^2 - 1)}{(Ax + M)^2 + l^2} , \quad e^{2\gamma} = \frac{x^2 - 1}{x^2 - y^2} , \quad \omega = 2ly + C ,
\]

\[
x = \frac{p_+ + p_-}{2A} , \quad y = \frac{p_+ - p_-}{2A} ,
\]

\[
p_{\pm} = \sqrt{\rho^2 + (z \pm A)^2} , \quad A = \sqrt{M^2 + l^2} ,
\]

where \( C \) is an arbitrary real constant.

Following to \cite{9} we will take here \( C = 0 \) (case \( C = 0 \) also corresponds to the general NUT metric

\[
ds^2 = -\alpha [dt - 2I \cos \theta d\varphi]^2 + \sin^2 \theta (r^2 + l^2) d\varphi^2 + \alpha^{-1} dr^2 + (r^2 + l^2) d\theta^2 ,
\]

which can be received from the quadratic form \cite{4} with help of the following
transformations

\[ f = \frac{N^2 - (M^2 + l^2)}{(N + M)^2 + l^2} , \quad f^{-1} e^{2\gamma} = \frac{(N + M)^2 + l^2}{p_+ p_-} , \quad r = N + M , \]

\[ N = \frac{1}{2}(p_+ + p_-) , \quad \cos \theta = \frac{p_+ - p_-}{2A} , \quad \omega = \frac{l}{A}(p_+ - p_-) , \]

\[ \rho = \sqrt{(r - M)^2 - A^2} \sin \theta , \quad z = (r - M) \cos \theta , \quad (4) \]

used in paper [6], parameter \( \alpha = \frac{(r^2 - 2Mr - l^2)}{(r^2 + l^2)} \). Here parameters \( M \) and \( l \) represent gravitoelectric (total mass of the source) and gravitomagnetic charges respectively. The parameter \( l \) is also called NUT parameter or the most commonly known as 'gravitomagnetic monopole'. From astrophysical point of view it is reasonable to assume that the string has no electric charge and consequently in the space-time around the nonspinning NUT string the metric tensor has the following nonvanishing components

\[ g_{00} = -f , \quad g_{11} = g_{33} = f^{-1} e^{2\gamma} , \]

\[ g_{02} = f \omega , \quad g_{22} = f^{-1} \rho^2 - f \omega^2 . \quad (5) \]

The Euler-Lagrange equations

\[ \frac{d}{d\lambda} \left( \frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0 , \quad (6) \]

governing geodesics of the metric [11] can be derived from the Lagrangian

\[ 2L = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} . \quad (7) \]

where a dot stands for differentiation with respect to an affine parameter \( \lambda \).

The integrals of motion

\[ \frac{dp_t}{d\lambda} = \frac{\partial L}{\partial t} = 0 , \quad p_t = -E , \]

\[ \frac{dp_\varphi}{d\lambda} = \frac{\partial L}{\partial \varphi} = 0 , \quad p_\varphi = L , \quad (8) \]

follow from Euler-Lagrange equations (6). Here \( p_\alpha = \partial L / \partial \dot{x}^\alpha \) is canonical momenta and the conserved quantities represent, respectively, the total energy, orbital angular momentum of the test particle [11].

Knowing that \( g_{\alpha\beta} p^\alpha p^\beta = -m^2 \) we can conclude that value of \( L \) in (7) equals to \(-m^2/2\). For spacetime [11] the Lagrangian (7) is

\[ L = -\frac{1}{2} f \dot{t}^2 + f \omega \dot{\varphi} + \frac{1}{2} f^{-1} e^{2\gamma}(\dot{\rho}^2 + \dot{z}^2) + \frac{1}{2} (f^{-1} \rho^2 - f \omega^2) \dot{\varphi}^2 , \quad (9) \]
and the corresponding canonical momenta are
\[ E = f(t - \omega \dot{\phi}) \] \[ L = f \omega t + (f^{-1} \rho^2 - f \omega^2) \dot{\phi} \]. (10)

Our task here is complicated due to the fact that the metric coefficients depend not only on radial coordinate, but also on coordinate \( z \) (the spacetime is axially symmetric). We have to express \( z \) through \( \rho \). In the paper [12] the motion of a free uncharged test particle in the Kerr-Newman-Taub-NUT spacetime was considered. Solving the Hamilton-Jacobi equation which describes the motion of test particle in the vicinity of non rotating source endowed with gravitomagnetic charge it was shown that the orbit will lie on a cone with
\[ \cos \theta = \pm \left| \frac{2lE}{L} \right| \]. (11)

As \( l \to 0 \) such an orbit degenerates to a single orbit on the equatorial plane. Using transformations (11) one can get the expression for \( z \) in cylindrical coordinates
\[ z = \frac{2lE}{L} \sqrt{\frac{L^2(A^2 + \rho^2) - 4l^2 A^2 E^2}{L^2 - 4l^2 E^2}} \]. (12)

It is easy to see that \( z = 0 \) corresponds to the equatorial plane \( \theta = \pi/2 \) in spherical coordinates. In order to investigate the radial dependence of effective potential we restrict ourselves to the study of circular geodesics neglecting possible motion of particles along the \( z \) axis, hence \( \dot{z} = 0 \). This implies that the equation (9) is
\[ -f \dot{t}^2 + 2f \omega \dot{t} + f^{-1} e^{2\gamma} \rho^2 + (f^{-1} \rho^2 - f \omega^2) \dot{\phi}^2 = -m^2. \] (13)

From equations (10) one could have
\[ \dot{t} = \frac{E(\rho^2 - \omega^2 f^2) + L \omega f^2}{f \rho^2} \] \[ \dot{\phi} = \frac{L - E \omega}{f^{-1} \rho^2}. \] (14)

Consider a case when particle has a mass \( m \neq 0 \). Then it is convenient to normalize quantities \( E, L \) to the unit of mass of the particle \( m \) and label them with a bar as \( \bar{E} = E/m, \bar{L} = L/m \). The ‘tilded’ quantities are normalized to the total mass of the source \( M \). Then the equation of circular motion (13) can be rewritten in the form
\[ f^{-1} e^{2\gamma} \left( \frac{1}{m} \frac{d \rho}{d \lambda} \right)^2 = V(\rho, z), \quad V(\rho, z) = -1 + \frac{\bar{E}^2}{f} \] \[ - \frac{f(\bar{L} - \bar{E} \omega)^2}{\rho^2} \]. (15)

The elliptic integrals resulting from the solution of the equations of motion are not particularly informative, but one can get a general picture
of the orbits by considering behaviour of an effective potential. Define an effective potential $V_{\text{eff}}$ as that value of $E$ such that $\dot{\rho} = 0$ at radius $\rho$:

$$-1 + \frac{V_{\text{eff}}^2}{f} - \frac{f(\overline{\mathcal{L}}/M - V_{\text{eff}}}){\rho^2} = 0.$$  \hfill (16)

The solution for effective potential

$$V_{\text{eff}} = -\overline{\omega}f^2(\overline{\mathcal{L}}/M) \pm \sqrt{-f(f^2\overline{\omega}^2 - \overline{\rho}^2)\overline{\rho}^2 + f^2(\overline{\mathcal{L}}/M)^2\overline{\rho}^2}$$  \hfill (17)

should have a sense of energy and a root which corresponds to the asymptotic condition $\lim_{\rho \to \infty} V_{\text{eff}} = 1$ has to be chosen.

Figure 1 illustrates the dependence of the effective potential on radial coordinate $\rho/M$ for a source endowed with mass $M$ and gravitomagnetic mass $l$.

Figure 1: The radial dependence of the effective potential for particles with nonzero rest mass for different values of angular momentum $\overline{\mathcal{L}}$. The left hand side figure is responsible for the case when the NUT parameter $\overline{l} = 0$. For the right hand side figure the NUT parameter $\overline{l} = 0.5$. Maxima in the effective potential indicate unstable circular orbits and minima stable circular orbits. Curves for particles with equal angular momentum and different gravitomagnetic charges have more monotonous behavior with the increase of the value of NUT parameter.

To simplify further calculations we can make the assumptions concerning the value of $\overline{l}$. We select second order approximation in gravitomagnetic monopole moment which can be justified by the fact that at the moment there is no any astrophysical evidence for strong gravitomagnetic mass. For example, in the recent papers [4, 5] the magnetic mass detection using the gravitational microlensing technique has been explored and it has been evaluated that the minimum observable gravitomagnetic mass to be
about 14 meters. The following assumption is made concerning energy $E$ and angular momentum $L$ of the particle. As the quantity $\tilde{l}^2$ is small then for the nonrelativistic particles with small values of energy and angular momentum one can neglect terms being proportional to $E^2\tilde{l}^2$ and $E^2\tilde{l}^2$ and get

$$\tilde{z} = \frac{2\tilde{l}E}{L/M} \sqrt{1 + \tilde{\rho}^2} . \quad (18)$$

The conditions for circular motion of test particle at radius $\rho = \rho_0$ are

$$V(\rho_0) = 0 , \ V'|_{\rho=\rho_0} = 0 , \quad (19)$$

where prime denotes the derivative on radial coordinate. Furthermore, the circular motion is stable if

$$V''|_{\rho=\rho_0} < 0 . \quad (20)$$

From the equation (19) after some lengthly algebra one can obtain the following expression for the energy of the particle

$$\frac{E^2}{\rho^2} \approx \frac{2\tilde{\rho}^2 (X + \tilde{\rho}^2) \left[ 2\tilde{l}^2 (X^2 + \tilde{\rho}^2(1 + \tilde{\rho}^2)) + X\tilde{\rho}^2(X + \tilde{\rho}^2) \right]}{X^3 \left[ 2(1 + \tilde{\rho}^2)(\tilde{\rho}^4 + \tilde{\rho}^2(X - 2) - 4X) + \tilde{l}^2 Y \right]} , \quad (21)$$

where $X = 1 + \sqrt{1 + \tilde{\rho}^2}$ and $Y = 3\tilde{\rho}^4(X - 2) + 10X + 6\tilde{\rho}^2X$. The circular motion is possible provided that the expression for $\frac{E^2}{\rho^2}$ is nonnegative [13].

Circular orbits exist till a limiting case corresponding to the photon orbit ($E = E/m \rightarrow \infty$). When NUT parameter is equal to zero this limit is equal to $\rho^2 = \tilde{\rho}^2 M^2 = 3M^2$. This corresponds to the value $r = 3M$ in spherical symmetric Schwarzschild spacetime. The influence of NUT parameter is such that the limit of existence of circular orbits decreases. For example, for $\tilde{l} = 0.5$ we have $\tilde{\rho}^2 = 2.49$.

The orbit is maximally bound when $\frac{E^2}{\rho^2} = 1$. Equating the expression (21) to unity one can obtain the equation governing the bounds of maximally binding orbits. In the field of a cylindrical source endowed with NUT parameter, taken as $\tilde{l} = 0.1$, radius of maximally binding orbit is $\tilde{\rho}^2 \approx 7.98$. Under the influence of NUT parameter the value of maximal radius of bound orbits decreases. For comparison: when $\tilde{l} = 0$ radius of the orbit is $\tilde{\rho}^2 = 8$ which corresponds to $r = 4M$ in Schwarzschild spacetime.

The orbit situated at an inflection point of the effective radial potential, that is, with

$$V'''|_{\rho=\rho_0} = 0 \quad (22)$$

is the least tightly bound and describes the bounds of stability region. The solution of this equation for $\tilde{l} = 0$ is $\tilde{\rho}^2 = 24$. This corresponds to $r = 6M$.
for spherical symmetric Schwarzschild case. The orbits are stable for $\tilde{\rho} > 2\sqrt{6}$ and unstable for $\tilde{\rho} < 2\sqrt{6}$. For $\tilde{l} = 0.7$ we have for limit of stability $\tilde{\rho}^2 = 34.41$. Therefore, the radius of critical orbit separating stable orbits from unstable for nonzero gravitomagnetic monopole increases.

The influence of the NUT parameter on the motion of a test particle can be seen from the following table

| Gravitomagnetic influence on motion |  |  |  |
|-----------------------------------|---|---|---|
| $|\tilde{l}|$ | $\tilde{\rho}^2_{\text{circular}}$ | $\tilde{\rho}^2_{\text{max.bound}}$ | $\tilde{\rho}^2_{\text{stable}}$ |
| 0 | 3 | 8 | 24 |
| 0.05 | 2.99 | 7.99 | 24.02 |
| 0.1 | 2.98 | 7.98 | 24.08 |
| 0.5 | 2.49 | 7.5 | 27.21 |
| 0.7 | 1.97 | 6.94 | 34.41 |

where $\tilde{\rho}^2_{\text{circular}}$ defines the radius of last circular orbit, $\tilde{\rho}^2_{\text{max.bound}}$ defines the radius of the last bound orbit and $\tilde{\rho}^2_{\text{stable}}$ defines the radius of first stable orbit.

Hereafter for simplicity we will omit widetilded and overlined labels but keep in mind that all quantities are normalized to the mass of the source, energy and angular momentum are also normalized to the mass of the particle. The four-velocity of the proper observer has the following nonvanishing components

$$u^\alpha \equiv \frac{1}{\sqrt{f}} \left\{ 1, 0, 0, 0 \right\}, \quad u_\alpha \equiv \sqrt{f} \left\{ -1, 0, \omega, 0 \right\}.$$ (23)

Components of absolute acceleration $w_\alpha = u_{\alpha;\beta} u^\beta$ of the proper observer in the gravitational field are given by

$$w_\alpha \equiv \frac{1}{2f} \left\{ 0, f, \rho \omega, 0, f, z \right\}.$$ (24)

The nonvanishing components of the relativistic rate of rotation $A_{\beta\alpha} = u_{[\alpha,\beta]} + u_{[\beta]w_{\alpha}}$ are

$$A_{r\varphi} = \frac{1}{2} \sqrt{f} \dot{\omega}_{,\rho}, \quad A_{z\varphi} = \frac{1}{2} \sqrt{f} \dot{\omega}_{,z}.$$ (25)

For the case of zero gravitomagnetic charge all components of relativistic rate of rotation are equal to zero. It means that NUT parameter influences on the observer as a force which drags the observer and makes him rotate around the source.
The geodesic equations $\delta^2 x^\alpha / \delta \sigma^2 = 0$ in spacetime take the form

\[
\ddot{\chi} + \frac{i}{f^2 \rho^2} \left[ \rho^2 \mu_f + f^3 \omega \mu_\omega \right]
+ \frac{\dot{\phi}}{f^2 \rho^2} \left[ 2f \omega \rho \dot{\rho} + 2 \rho^2 \omega \mu_f - f (\rho^2 + f^2 \omega^2) \mu_\omega \right] = 0 ,
\tag{26}
\]

\[
\ddot{\rho} + (\gamma_r - \frac{f \rho}{2f}) (\dot{\rho}^2 - \dot{z}^2) + \frac{e^{-2\gamma}}{2f} \left[ f^2 f_{\rho} \dot{\rho}^2 + (-2 \rho f + \xi_\rho) \dot{\phi}^2 \right] + 2(\gamma_r - \frac{f \rho}{2f}) \rho \dot{z} - e^{-2\gamma} f(\omega f_{\rho} + f \omega_{\rho}) \dot{\phi} \dot{t} = 0 ,
\tag{27}
\]

\[
\ddot{z} + (\gamma_r - \frac{f \rho}{2f}) (\dot{z}^2 - \dot{\rho}^2) + \frac{e^{-2\gamma}}{2f} \left[ \xi_\rho \dot{\phi}^2 + f^2 f_{\rho} \dot{\rho}^2 \right] - e^{-2\gamma} f(\omega f_{\rho} + f \omega_{\rho}) \dot{\phi} \dot{t} + 2(\gamma_r - \frac{f \rho}{2f}) \rho \dot{z} = 0 ,
\tag{28}
\]

\[
\ddot{\phi} + \frac{i f^2}{\rho^2} \mu_\omega + \frac{\dot{\phi}}{f^2 \rho^2} (2f \rho \dot{\rho} - \rho^2 (f_{\rho} \dot{\rho} + f_{\rho} \dot{z}) - f^3 \omega \mu_\omega) = 0 ,
\tag{29}
\]

where $\delta$ denotes the covariant derivative and $\mu_f = f_{\rho} \dot{\rho} + f_{\rho} \dot{z} \dot{\rho} + \omega_{\rho} \dot{\rho} + \omega_{\rho} \dot{z}, \xi_\rho = (\rho^2 + f^2 \omega^2) f_{\rho} + 2 f^3 \omega \omega_{\rho}, \xi_\rho = (\rho^2 + f^2 \omega^2) f_{\rho} + 2 f^3 \omega \omega_{\rho}$.  

For the circular geodesics $\dot{\rho} = \dot{z} = 0$ only radial and $z$ component of the geodesic equations are not vanishing

\[
f_{\rho} \dot{\rho}^2 + \left( -\frac{2 \rho}{f} + f^{-2} \xi_\rho \right) \dot{\phi}^2 - 2(\omega f_{\rho} + f \omega_{\rho}) \dot{\phi} \dot{t} = 0 ,
\tag{30}
\]

\[
f^{-2} \xi_\rho \dot{\phi}^2 + f_{\rho} \dot{t}^2 - 2(\omega f_{\rho} + f \omega_{\rho}) \dot{\phi} \dot{t} = 0 .
\tag{31}
\]

The angular velocity of the test particle is given by $\omega = d\phi/dt = d\phi/d\sigma (dt/d\sigma)^{-1} = \dot{\phi} / \dot{t}$ which becomes, using (30) and (31) as

\[
\omega = f \frac{\kappa \pm \sqrt{\kappa^2 - (f_{\rho} + f_{\rho} \dot{z}) \chi}}{\chi} ,
\tag{32}
\]

where

\[
\kappa = f [f(\omega_{\rho} + \omega_{z}) + \omega(f_{\rho} + f_{\rho} \dot{z})]
\tag{33}
\]

and

\[
\chi = (f_{\rho} + f_{\rho} \dot{z})(\rho^2 + f^2 \omega^2) - 2f \rho + 2f^3 \omega(\omega_{\rho} + \omega_{z}) .
\tag{34}
\]

Equation (32) defines the angular velocity of a zero angular momentum particle. Due to the effect of dragging of inertial frames the particles are dragged by the influence of NUT parameter. This effect weakens with distance as $1/\rho^3$ and it makes the gravitomagnetic charge of source measurable in principle.
3 Electromagnetic Fields of Nonrotating Cylindrical NUT Source

The general form of the first pair of general relativistic Maxwell equations is given by
\[ F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0, \quad (35) \]
where \( F_{\alpha\beta} \) is the electromagnetic field tensor expressing the strict connection between the electric and magnetic four-vector fields \( E^\alpha, B^\alpha \). For an observer with four-velocity \( u^\alpha_{(obs)} \), the covariant components of the electromagnetic tensor are given by
\[ F_{\alpha\beta} \equiv 2u_{(obs)}[\alpha E_\beta] + \eta_{\alpha\beta\gamma\delta} u^\gamma_{(obs)} B^\delta, \quad (36) \]
where \( \eta_{\alpha\beta\gamma\delta} \) is the pseudo-tensorial expression for the Levi-Civita symbol
\[ \eta^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}} \epsilon_{\alpha\beta\gamma\delta}, \quad \eta_{\alpha\beta\gamma\delta} = \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta}, \quad (37) \]
with \( g \equiv \det[g_{\alpha\beta}] = -e^{4\gamma} \rho^2 f^{-2} \) for the metric (II).

A useful class of observers is represented by the “zero angular momentum observers” or ZAMOs [14]. These are observers that are locally stationary (i.e. at fixed values of \( r \) and \( \theta \)) but who are “dragged” into rotation with respect to a reference frame fixed with respect to distant observers. They have four-velocity components given by
\[ (u^\alpha)_{ZAMO} \equiv b^{-1} \left( 1, 0, -\frac{f^2 \omega}{a}, 0 \right); \quad (u^\alpha)_{ZAMO} \equiv b \left( -1, 0, 0, 0 \right), \quad (38) \]
where \( a = \rho^2 - f^2 \omega^2, \ b = \rho \sqrt{f/a} \).

The general form of the second pair of Maxwell equations is given by
\[ F^{\alpha\beta ;} ;_{\beta} = 4\pi J^\alpha, \quad (39) \]
where \( J^\alpha \) is the four-current.

Maxwell equations assume a familiar flat-spacetime form when projected onto a locally orthonormal tetrad. In principle such tetrad is arbitrary, but in the case of a relativistic rotating metric source a “natural” choice is offered by the tetrad carried by the ZAMOs. Using (35) we find that the components of the tetrad \( \{e^\alpha_0\} = (e_0, e_\rho, e_\phi, e_z) \) carried by a ZAMO observer are
\[ e^\alpha_0 = b^{-1} \left( 1, 0, -\frac{f^2 \omega}{a}, 0 \right), \quad (40) \]
\[
e^\alpha_\rho = \sqrt{f} e^{-\gamma} \left( 0, 1, 0, 0 \right), \quad (41)
\]
\[
e^\alpha_\phi = \sqrt{f} \left( 0, 0, 1, 0 \right), \quad (42)
\]
\[
e^\alpha_z = \sqrt{f} e^{-\gamma} \left( 0, 0, 0, 1 \right). \quad (43)
\]

We can now rewrite the Maxwell equations (35) and (39) in the ZAMO reference frame by contracting them with (40)–(43). After some lengthy but straightforward algebra, we obtain Maxwell equations in the more useful form

\[
\frac{e^{2\gamma}}{f} B^\phi_{,\rho} + \left( \frac{e^{\gamma} \sqrt{a}}{f} B^z_{,z} \right)_{,\rho} + \left( \frac{e^{\gamma} \sqrt{a}}{f} B^\rho_{,\phi} \right)_{,\phi} = 0, \quad (44)
\]

\[
\frac{\partial B^\phi}{\partial t} = \frac{f}{a} \left( \rho E^z_{,\rho} + \omega f B^\phi_{,\rho} \right) - \frac{e^{-\gamma} f \rho}{\sqrt{a}} E^\phi_{,z}, \quad (45)
\]

\[
\frac{\partial B^\phi}{\partial t} = \frac{f}{e^{2\gamma}} \left[ \rho \left( \frac{e^{\gamma}}{\sqrt{a}} E^\rho_{,\rho} - \frac{e^{\gamma} \omega f}{\sqrt{a}} B^z_{,z} \right) - \left( \frac{e^{\gamma} \rho}{\sqrt{a}} E^z_{,z} - \frac{e^{\gamma} \omega f}{\sqrt{a}} B^\rho_{,\rho} \right) \right], \quad (46)
\]

\[
\frac{\partial B^z}{\partial t} = e^{-\gamma} f \left( \rho E^\phi_{,\rho} \right)_{,\rho} - \frac{f}{a} \left( \rho E^\phi_{,\phi} - \omega f B^z_{,\rho} \right), \quad (47)
\]

and

\[
\left( \frac{e^{\gamma} \sqrt{a}}{f} E^\phi \right)_{,\rho} + \left( \frac{e^{2\gamma}}{f} E^\phi_{,\rho} \right) + \left( \frac{e^{\gamma} \sqrt{a}}{f} E^z \right)_{,z} = 4\pi \frac{e^{2\gamma}}{f^{3/2}} \sqrt{a} J^z, \quad (48)
\]

\[
\frac{e^{\gamma}}{\sqrt{a}} \left( \omega f E^\rho_{,\phi} - \rho B^z_{,\phi} \right) + \rho B^z_{,z} = \frac{e^{\gamma}}{f} \frac{\partial E^\rho}{\partial t} + 4\pi \frac{e^{\gamma} \rho}{\sqrt{a}} J^\rho, \quad (49)
\]

\[
- \left[ \frac{e^{\gamma}}{\sqrt{a}} (\omega f E^\phi_{,\rho} - \rho B^z_{,\rho}) \right]_{,\rho} - \left[ \frac{e^{\gamma}}{\sqrt{a}} (\omega f E^z_{,z} + \rho B^\rho_{,\rho}) \right]_{,z}
\]
\[
= \frac{e^{2\gamma}}{f} \frac{\partial E^\phi}{\partial t} + 4\pi \frac{e^{2\gamma}}{\sqrt{f a}} (\rho J^\rho - f \omega J^z), \quad (50)
\]
\[ \frac{e^\gamma}{\sqrt{a}} \left( \omega f E^z_{,\phi} + \rho B^\rho_{,\phi} \right) - (\rho B^\rho_{,\phi})_{,\rho} = \frac{e^\gamma}{\sqrt{a}} \frac{\partial E^z}{\partial t} + 4\pi \frac{e^\gamma \rho}{\sqrt{f}} J^z. \] (51)

We will look for stationary and axially symmetric solutions of the Maxwell equations, taking into account that in the vacuum region around the source all components of electric current are equal to zero. We could study the realistic configuration of magnetic fields, for example, the dipolar magnetic field. Due to the possible smallness of the gravitomagnetic mass of the star estimated from some astrophysical observations we may perform calculations to the first order in NUT parameter.

Figure 2: Dependence of general relativistic modification factor \( B^\rho_{GR}/B^\rho_{Newt} \) of magnetic field on the radial coordinate \( \rho/M \) normalized in units of stellar mass. Near to the NUT source the magnetic field will be amplified, then in some intermediate region it will be weakened and in the asymptotically far zone the influence of NUT parameter is negligible, the behavior of field is Newtonian and relation tends to unity. The influence of the NUT parameter is more strong near to the source of the \( z \) axis.

According to the transformations (4) solution for vector potential in
spherical coordinates (see, for example, [15, 16])

\[ A_\phi(r, \theta) = -\frac{3\mu \sin^2 \theta}{2M} \left[ \left( \frac{r}{2M} \right)^2 \ln\left(1 - \frac{2M}{r}\right) + \frac{1}{2} \left( \frac{r}{M} + 1 \right) \right] \]  (52)

can be rewritten in cylindrical ones as

\[ A_\phi(\rho, z) = -\frac{3\mu \rho^2}{8M^3 f} \left[ \ln f + \frac{2M(2M + N)}{(M + N)^2} \right], \]  (53)

where \( \mu \) is the dipolar magnetic moment.

This yields the following exact solutions for the nonvanishing components of magnetic field

\[
B^\rho = -\frac{3\mu \rho e^{-\gamma}}{8M^3 f(M + N)^3} \left\{ 2M f N_z (N + 3M) \\
+ f_z(M + N) \left[ (M + N)^2 \ln f + 3M^2 - N^2 \right] \right\},
\]  (54)

\[
B^z = \frac{3\mu e^{-\gamma}}{8M^3 f(M + N)^3} \left\{ 2M f_\rho N_\rho (N + 3M) \\
+ (M + N) \left[ 2M(N + 2M)(\rho f_\rho - 2f) \\
+ (M + N)^2 [\rho f_\rho(\ln f - 1) - 2f \ln f] \right] \right\}. \]  (55)

Figure 3: The radial dependence of electric field \( E^\rho \) for different values of the gravitomagnetic monopole \( l \). The effect of the NUT parameter on the electric field is becoming important near to the source of the \( z \) axis.

The dependence of relation \( B^\rho_{GR}/B^\rho_{Newt} \) on parameter \( \rho/M \) is shown on Figure 3.
In the limit of flat spacetime when $\rho = r \sin \theta$, $z = r \cos \theta$ and $\lim_{M \to 0} = 0$ the expressions (54)-(55) reduce to the Newtonian ordinary expressions for magnetic field of a magnetic dipole [17]

$$B^\phi = \frac{\mu (2z^2 - \rho^2)}{(z^2 + \rho^2)^{5/2}}, \quad B^z = \frac{3\mu z \rho}{(z^2 + \rho^2)^{5/2}}.$$  \hspace{1cm} (56)

Consider now as a toy model the charged NUT star with the monopolar electric field configuration

$$E^\rho = E^\rho (\rho, z) \neq 0,$$  \hspace{1cm} (57)

$$E^z = E^\phi = 0.$$  \hspace{1cm} (58)

If $Q$ is the electric charge per unit length of the line tube then the solution for radial electric field admitted by Maxwell equations is

$$E^\rho = Q \frac{fe^{-\gamma}}{\sqrt{\alpha}}.$$  \hspace{1cm} (59)

Figure 3 shows that for small values of $z$ and $\rho$ (near to the source) the influence of NUT parameter is noticeable.

## 4 Conclusion

We have investigated the general relativistic effects associated with the possible existence of the gravitomagnetic monopole moment of cylindrical NUT source and in particular studied the influence of the gravitomagnetic monopole moment on the motion of the test particles along circular orbits. These effects could be divided into two parts. First effect is related to the fact that due to the influence of the NUT parameter the particle’s circular orbits will be not kept in the equatorial plane. Our aim was mainly related to the investigation of the second effect connected with the influence of the NUT parameter on the circular orbits of the particles. It is shown that under the influence of the NUT parameter the particle’s orbits are becoming less stable which can be seen from the table presented in the paper.

Analytical general relativistic expressions for the dipolar magnetic field around NUT star are presented. It is shown that in the linear approximation in NUT parameter there is only effect of mass of the star on the value of stationary magnetic field. In the case of charged star there is strong influence of NUT parameter on the value of monopolar electric field. Our results in principle could be combined with analysis of astrophysical data on electromagnetic fields of compact objects in order to get upper limits on the value of NUT parameter.
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