Slow sound in matter-wave dark soliton arrays

Muzzamal I. Shaukat,1,2 Eduardo V. Castro,1,3 and Hugo Terças4

1CeFEMA, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal
2University of Engineering and Technology, Lahore (RCET Campus), Pakistan
3Centro de Física das Universidades do Minho e Porto, Departamento de Física e Astronomia, Faculdade de Ciências, Universidade do Porto, Porto, Portugal
4Instituto de Plasmas e Fusão Nuclear, Lisboa, Portugal

We show the existence of slow propagating phonons in quasi one dimensional Bose-Einstein condensates. The impurities are trapped inside the potential created by dark-soliton due to which the conditions to separate the three levels (qutrit), perfectly, has determined. We compute the phonon-soliton coupling and investigate the decay rates of the three level system. We derive the analytical expression of the linear susceptibility to demonstrate the phenomenon of acoustic induced transparency based on matter wave phononics. The dark-soliton qutrit with unique properties of transmission and dispersion revealing the possibility of slowing down the speed of acoustic pulse. Moreover, the present scheme could be proposed to analyze the diverse applications like quantum gates in the field of quantum information.

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Introduction. Electromagnetically induced transparency (EIT) [1] is a quantum interference effect in which the absorption of a weak probe laser, interacting resonantly with an atomic transition, is reduced in the presence of a coupling laser. EIT is, for instance, crucial in optically controlled slowing of light [2] and optical storage [3], having been extensively investigated in Λ, V and cascade-type three-level atoms [4, 5], and experimentally observed in atoms [6] and semiconductor quantum wells [7]. A major problem in the initial studies of EIT in atomic vapors has to do with the thermal spectral broadening [8, 9], smearing out the EIT window. In order to mitigate this issue, researchers have made use of highly coherent BECs [10–12]. The association of EIT with light-matter coupling can be used to prepare and detect coherent many-body phenomena in ultra-cold quantum gases [13].

Soon after the engineering of photonic crystal structures, the attention has been drawn to the propagation of acoustic waves in periodic media [14, 15]. Many intriguing phenomena, such as the analogue of EIT [16, 17] and Fano resonances [18, 19] have been envisaged in the context of acoustics as well [20, 21]. For example, an isotropic metamaterial consisting of grooves on a square bar traps enhances the acoustic waves due to a strong modulation of wave group velocity [22]: slowing down the speed of sound in sonic crystal waveguides has also been achieved, with a reported group velocity of 26.7 m/s [23]. Soliton propagation and soliton-soliton interaction in EIT media has been studied by Wadati et al. [24], and the formation of solitons via dark-state polaritons has been proposed [25].

Recently, we have shown that a dark-soliton (DS) qubit in a quasi one dimensional (1D) BEC are appealing candidates to store information given their appreciably long lifetimes (∼ 0.01 − 1s) [26]. Moreover, we explored the creation of entanglement between DS qubits, at appreciable large distances of the order of few micrometer, by using the superposition of two maximally entangled states in the dissipative process of spontaneous emission [27, 28]. Dark-soliton qubits thus offer an appealing alternative to quantum optics in solid-state platforms, where information processing involves only phononic degrees of freedom: the quantum excitations on top of the BEC state.

In this Letter, we propose to make use of dark-soliton to achieve a phenomenon with EIT-like characteristics, the acoustic transparency (AT). The active medium is composed of a set of dark-soliton qutrits, i.e. three-level object comprising an impurity trapped at the interior of a dark-soliton potential, in which the BEC acoustic modes propagate (see Fig. 1 for a schematic representation). We start by recalling the conditions under which that qutrit is possible. The qutrit array is shown to be
an open quantum system, where the reservoir is determined by quantum fluctuations (phonons) on top of the BEC state [29]. We compute the linewidth of each of the qutrit transitions by treating the qutrit-phonon interaction within the Born-Markov approximation and analyze the dependence of absorption profile of the AT on the BEC-impurity coupling. We conclude by computing the dispersion relation of a weak envelope of sound waves and show that it propagates at very low speeds (≈ 0.06) mm/s, to the best of our knowledge a record value in acoustics. Our study represents an advance in the direction of 'slow-sound' schemes and the results have potential applications in phononic information processing.

**Dark-soliton qutrit.** We starting by considering a dark soliton in a quasi 1D BEC, with the later being surrounded by a dilute gas of impurities (see Fig. 1). The DS plays the role of a potential for the impurities (considered to be free particles) and the quantum fluctuations (phonons) act like a proper reservoir. At the mean field level, the system is governed by the Gross-Pitaevskii and Schrödinger equations,

\[
\begin{align*}
\frac{i\hbar}{\partial t} \psi_1 &= -\frac{\hbar^2}{2m_1} \frac{\partial^2 \psi_1}{\partial x^2} + g_{11} |\psi_1|^2 \psi_1 + g_{12} |\psi_2|^2 \psi_1, \\
\frac{i\hbar}{\partial t} \psi_2 &= -\frac{\hbar^2}{2m_2} \frac{\partial^2 \psi_2}{\partial x^2} + g_{12} |\psi_1|^2 \psi_2,
\end{align*}
\]

where \( g_{11} \) represents the BEC-inter-particle interaction strength, \( g_{12} = g_{21} \) is the BEC-impurity coupling constant [30] and \( m_1, m_2 \) denotes the BEC particle and impurity masses, respectively. The singular nonlinear solution corresponding to the soliton profile is \( \psi_{\text{sol}}(x) = \sqrt{n_0} \tanh[x/\xi] \) [31, 32], where \( n_0 \) denotes the BEC linear density, \( \xi = \hbar^2 / \sqrt{2m_1 n_0 g} \) is the healing length (of the order \( 0.7-1.0 \) \( \mu \)m, respectively, in a typical 1D BECs, for which the condensate is homogeneous along a trap of size \( l_z \sim 70 \mu \)m [33]). More recent experiments leads eventual trap inhomogeneties to be much less critical by creating much larger traps, \( l_z \sim 100 \mu \)m [34]. Therefore, Eq. (1) can be written as

\[
E'\psi_2 = -\frac{\hbar^2}{2m_2} \frac{\partial^2 \psi_2}{\partial x^2} - g_{21} n_0 \text{sech}^2 \left( \frac{x}{\xi} \right) \psi_2,
\]

where \( E' = E - n_0 g_{21} \). Notice that these results can be easily generalised for the case of gray soliton (travelling with speed \( v \)) by replacing \( \psi_{\text{sol}}(x) = \sqrt{n_0} \tanh[x/\xi] \), where \( \theta = v/c_s \) and \( \gamma = (1 - \theta^2)^{1/2} \). To find the analytical solution of Eq. (2), the potential is casted in the form \( V(x) = -\hbar^2 \nu(1 + \nu) \text{sech}^2(x/\xi) / 2m_2 \xi^2 \) [3] with \( \nu = -1 + \sqrt{1 + 4g_{21} m_2 / g_{11} m_1} \) and the energy spectrum \( E' = -\hbar^2 (\nu - n)^2 / 2m_2 \xi^2 \), where \( n \) is an integer [30]. The number of bound states created by the DS is \( n_{\text{bound}} = \lfloor \nu + 1 + \sqrt{\nu(1 + \nu)} \rfloor \), where the symbol \( \lfloor \cdot \rfloor \) denotes the integer part. As such, for a DS to contain exactly three bound states (i.e. the condition for the qutrit to exist), the parameter \( \nu \) must lie in the range

\[
\frac{4}{5} \leq \nu < \frac{9}{7}.
\]

At \( \nu \geq 9/7 \), the number of bound states increases. However, the effect of the impurity on the profile of the soliton itself becomes more important, and therefore special care must be taken in the choices of the mass ration \( m_2/m_1 \) [30].

**Quantum fluctuations.** The total BEC quantum field includes the DS wave function and quantum fluctuations, \( \psi_1(x) = \psi_{\text{sol}}(x) + \delta \psi(x) \), where \( \delta \psi(x) = \sum_k \eb_k \eb_\dagger_k \) and \( \eb_k \) are the bosonic operators verifying the commutation relation \( [\eb_k, \eb_\dagger_l] = \delta_{k,l} \). The amplitudes \( \eb_k \) and \( \eb_\dagger_k \) satisfy the normalization condition \( \eb_k(x) \eb_\dagger_k(x) + |\eb_k(x)|^2 = 1 \) and are explicitly given in [30]. The total Hamiltonian then reads

\[
H = H_q + H_p + H_{\text{int}},
\]

where \( H_q = h\omega_c \eb_\dagger \eb + h\omega_1 \eb_\dagger \eb + h\omega_2 \eb_\dagger \eb \) is the qubit Hamiltonian, with \( \omega_1 = h(2\nu - 3)/(2m\xi^2) \) and \( \omega_0 = h(2\nu - 1)/(2m\xi^2) \) are the gap energies for \( \eb_\dagger \eb \rightarrow \eb_\dagger \eb \) and \( \eb_\dagger \eb \rightarrow \eb_\dagger \eb \) transitions, respectively. The term \( H_p = \sum_k \epsilon_k \eb_k \eb_\dagger_k \) represents the phonon (reservoir) Hamiltonian, where \( \epsilon_k = m\xi \sqrt{\left( \nu^2 - 2 \right) / 2} \) is the Bogoliubov spectrum with chemical potential \( \mu = gn_0 \). The interaction Hamiltonian is given by

\[
H_{\text{int}} = g_{21} \int dx \eb_\dagger_k \eb_\dagger_\dagger_k \psi_1 \psi_2,
\]

where \( \psi_2(x) = \sum_{l=0}^2 \varphi_l(x) a_l \) describes the qutrit wave function in terms of the bosonic operators \( a_l \), with \( \varphi_0(x) = A_0 \text{Sech}^\alpha(x/\xi) \), \( \varphi_1(x) = A_1 \tanh(x/\xi) \) \( \varphi_2(x) = \sqrt{2} A_2 \left( 1 + 2\alpha \tanh^2(x/\xi) \right) \), where \( A_j (j = 0, 1, 2) \) are the normalization constants and \( \alpha = \sqrt{2g_{21} m_2 / g_{11} m_1} \) [30]. Using the rotating wave approximation (RWA), the first order perturbed Hamiltonian

\[
\begin{align*}
\Delta H &= g_{21} \int dx \eb_\dagger_k \eb_\dagger_\dagger_k \psi_1 \psi_2, \\
&= g_{21} \int dx \eb_\dagger_k \eb_\dagger_\dagger_k \psi_1 \psi_2,
\end{align*}
\]
can be written as

\[ H_{int}^{(1)} = \sum_k \left( g_0^{k+} \sigma_0^+ \right)_k b_k + \left( g_0^{k-} \sigma_0^- \right)_k b_k^\dagger, \]

where \( \sigma_{0,1} = a_{g,e}^{\dagger} a_{g,e}, \sigma_{0,1}^- = a_{g,e}^{\dagger} a_{g,e} \) and the coupling constants \( g_0^{k+} = g_0^k (i = 0, 1) \) are explicitly given in [30]. In our RWA calculation, the counter-rotating terms proportional to \( b_k \sigma_j^- \) and \( b_k^\dagger \sigma_j^+ \) are dropped. The accuracy of such an approximation can be verified post-\textit{eriori}, provided that the emission rates \( \gamma_0 = \gamma_1 \) are much smaller than the qutrit transition frequencies \( \omega_0 \) and \( \omega_1 \), respectively.

\textit{Wigner-Weisskopf spontaneous decay.} We employ the Wigner-Weisskopf theory to find the spontaneous decay rate of the states, by neglecting the effect of temperature and other external perturbations [36]. In this regard, the qutrit is assumed to be initially at the excited state \( |e_2\rangle \) and the phonons to be in the vacuum state \( |0\rangle \). Under such conditions, the wave function of total system (qutrit + phonons) can be described as

\[ |\phi(t)\rangle = a(t)|e_2, 0\rangle + \sum_k b_k(t)|e_1, 1_k\rangle + \sum_{k,p} b_{k,p}(t)|g, 1_k, 1_p\rangle, \]

where \( a(t) \) is the probability amplitude of the excited state \( |e_2\rangle \). The qutrit decays to the state \( |e_1\rangle \) with probability amplitude \( b_k(t) \) by emitting a phonon of wavevector \( k \) and frequency \( \omega_k \). Subsequently, the qutrit de-excites to the ground state \( |g\rangle \) via the emission of \( p \)-phonon of frequency \( \omega_p \) and probability amplitude \( b_{k,p}(t) \). In the interaction picture, these coefficients can be written as

\[ a(t) = e^{-\gamma t/2}, \]

\[ b_k(t) = -ig_0 \frac{e^{i(\omega_k - \omega_1)t - \gamma t/2} - e^{-\gamma t/2}}{i(\omega_k - \omega_1) - \gamma/2}, \]

\[ b_{k,p}(t) = \frac{g_0 g_k^p}{i(\omega_k - \omega_p) - \gamma/2} \left[ e^{i(\omega_p - \omega_1)t - \gamma t/2} - 1 \right] \frac{1 - e^{i(\omega_k + \omega_p - \omega_e)t - \gamma t/2}}{i(\omega_k + \omega_p - \omega_e) - \gamma/2}, \]

where \( \omega_{eg} = \omega_0 + \omega_1 \) and \( \gamma_i \) \( (i = 0, 1) \) is the \( i \)th state decay rate

\[ \gamma_i = \frac{L}{2\hbar} \int d\omega_k \frac{1 + \eta_i}{\eta_i} |g_k^i|^2 \delta(\omega_k - \omega_i), \]

where \( \eta_i = \sqrt{\mu^2 + \hbar^2 c^2} \). Both RWA and Born-Markov approximations can be verified from Fig. 7 and Fig. 3, where it is depicted that the decays rates of both the transitions are much smaller than the respective transition frequencies. It is pertinent to mention here that Feshbach resonances can be used to tune the value of \( g_{12} \) experimentally, allowing for an additional control of the rates \( \gamma_i \). Moreover, dark-soliton quantum diffusion may be the only immediate limitation to the performance of this proposal [4], a feature that has been theoretically predicted but yet not experimentally validated. In any case, quantum evaporation is expected if important trap anisotropies are present, a limitation that we can overcome with the help of box-like or ring potentials [33].

\textit{Acoustic Bloch equations.} To analyze the interference effect of AT, we consider the situation where the qutrit is externally driven by two (probe and control) acoustic fields. The probe field can be excited with a Raman Laser driving the transition \( |g\rangle \leftrightarrow |e_1\rangle \) with frequency \( \omega_p \) and detuning \( \Delta_p = \omega_p - \omega_0 \). Simultaneously, a Raman laser field of frequency \( \omega_c \) and detuning \( \Delta_c = \omega_c - \omega_1 \) couple the states \( |e_1\rangle \) and \( |e_2\rangle \) [38]. Therefore, the qutrit driving can be described, within the RWA approximation, by the following Hamiltonian

\[ H_{\text{drive}} = \frac{\hbar}{2} \left( \Omega_p |e_1\rangle \langle g| + \Omega_c |e_2\rangle \langle e_1| - 2\Delta_p |e_1\rangle \langle e_1| - 2\delta |e_2\rangle \langle e_2| \right) + \text{H.c.}, \]

where \( \delta = \Delta_p + \Delta_c \) and \( \Omega_{p,c} \) denote the Rabi frequency of the probe and control fields, respectively. We obtain the solution for the density matrix \( \rho \) by solving the master equation

\[ \dot{\rho}_q(t) = -\frac{i}{\hbar} [H_q, \rho_q(t)] + \sum_{i=0}^{1} \gamma_i \mathcal{L}_i[\rho], \]

with \( \rho_{ij} = \rho_{ji}^\dagger \) and the Lindblad operator \( \mathcal{L}[\rho] = [\sigma_- \rho(t) \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_- , \rho(t) \}] \). In the limit of the weak-probe approximation, \( \Omega_p \ll \Omega_c \), the steady state-coherences are given by

\[ \rho_{21} = \frac{i \Omega_p}{(\gamma_0 - 2i\Delta_p) + i\Omega_p^2 \gamma_2}, \]

\[ \rho_{31} = \frac{-i \Omega_c}{(\gamma_1 - 2i\delta) \rho_{21}}. \]
In what follows, we assume that a set of solitons (i.e. a soliton gas [39]) of density \( N = 1/d \), with \( d \) denoting the average distance between the solitons. If the solitons are well-separated, \( d \ll \xi \), we can assume the qutrits to be independent. This is not usually the case in one-dimensional systems, unless in the especial comensurability situation, as a consequence of the infinite-range (sinusoidal) character of the collective decay rate [40, 41]. Fortunately in our case, because the solitons locally deplete the BEC, the collective scattering rate vanishes at distances largely exciding the healing length, \( d \gg \xi \) [27, 28]. As such, we can determine the long wavelength behaviour, \( kd \ll 1 \), of the probe field envelope. Using the Heisenberg's relation \( i\hbar \partial (\delta \Psi) / \partial t = \left[ \hat{H}, \delta \Psi \right] \) and the fluctuating field \( \delta \Psi = \phi b e^{iqx} + \psi^* b^e^{iqx} \), where \( \phi \) and \( \psi \) are the Bogoliubov coefficients, we obtained the propagating equation [30]

\[
\frac{\partial \Omega_p}{\partial t} + \frac{\omega_p}{q} \frac{\partial \Omega_p}{\partial x} = -\frac{i}{2\hbar^2} (k_0^0)^2 \rho_{12},
\]

\( \Omega_p = g_0^0 k \delta \Psi / \hbar, \ k_{\text{res}} = 0.9 / \xi \) is the resonant wavevector. By ignoring the time derivative from Eq. (33) (time-independent fluctuating field) and comparing it with \( \partial_x \delta \Psi = ik \chi \delta \Psi / 2 \) [42], we express the susceptibility

\[
\chi = -\frac{i N \xi (g_0^{\text{res}})^2}{\hbar k} \left( \gamma_0 - 2i \Delta_p + \frac{\Omega_c^2}{\gamma_1 - 2\gamma_2} \right),
\]

where \( \rho_{12}^{\text{res}} = N\xi \rho_{12} \) with the total number of solitons per unit length \( N \) in the system and the size of the soliton \( \xi \). The acoustic response of the envelope can be determined by the refractive index \( n = \sqrt{1 + \chi} \). The onset of the AT is demonstrated in Fig. 4. The system reveals initially a normal Lorentzian peak under \( \Omega_c \ll \gamma_1 \) but a dip immediately appears as we increase the control laser power \( \Omega_c \). Moreover, the width of the transparency window increases significantly for \( \Omega_c \gg \gamma_1 \), and carries a signature of Autler-Townes doublet. We expect that the destructive interference between the excitation pathways is reduced due to large value of \( \gamma_1 \). It is important to realize that a change in absorption over a narrow spectral range must be accompanied by a rapid and positive change in refractive index due to which a very low group velocity is produced in AT. Therefore, the group velocity for the acoustic field is given by

\[
v_g = \frac{c_s}{1 + \frac{\Delta p}{2} + \frac{1}{2} (\partial \chi_R / \partial \omega_p)},
\]

where we assume that \( \Omega_p^2 \gg \Gamma_p \Gamma_c \).

**Slow sound in box potentials.**—For the sake of experimental estimates, we consider a one-dimensional BEC loaded in a large box potential. In a typical trap of size \( \sim 100 \mu m \), healing length \( \xi \sim 0.7 \mu m \) and sound speed \( c_s \sim 1 \text{ mm/s} \) [33], we can imagine placing up to 20 well-separated (\( d \sim 3.5 \mu m \)) solitons. Under these conditions, the envelope group velocity can be brought down to a record value of \( \sim 0.06 \text{ mm/s} \), corresponding to the peak appearing in Fig. 5. Indeed, for a wavelength compared to an intersoliton separation \( d \), the estimated group velocity is \( v_g \simeq 5.0 \mu m / s \). The latter is much smaller than that obtained in band-gap arrays [43] and detuned acoustic resonators [44]. In the latter, a sound speed of \( \sim 9.8 \text{ m/s} \) is experimentally reported, which makes our scheme able to achieve a \( 10^{-5} \) smaller speed for an acoustic pulse.

In conclusion, we proposed a scheme for the realization of the acoustic transparency phenomenon with dark-solitons qutrits in a quasi-one dimensional Bose-Einstein
condensates. The qutrits consist of three-level structures formed by impurities trapped by the dark solitons. We investigate the spontaneous decay rates to analyze the interference effect of the acoustic transparency, due to which a narrow absorption window, depending on the BEC-impurity coupling, can be achieved. We show that an acoustic pulse can be slowed down to a record speed of 0.06 mm/s. We believe that the suggested approach opens a promising research avenue in the field of acoustic transport. In general, the present scheme will provide a great feasibility to enable the numerous applications like quantum gates in the field of quantum information processing [45, 46].

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* Electronic address: muzzamalshaukat@gmail.com
† Electronic address: hugo.tercass@tecnico.ulisboa.pt

[1] S. E. Harris, J. E. Field and A. Imamoglu, Phys. Rev. Lett. 64, 1107 (1990).
[2] L. V. Hau, S. E. Harris, Z. Dutton, and C. H. Behroozi, Nature (London) 397, 594 (1999).
[3] D. F. Phillips, A. Fleischhauer, A. Mair, R. L. Walsworth, and M. D. Lukin, Phys. Rev. Lett. 86, 783 (2001).
[4] T. Y. Abi-Salloum, Phys. Rev. A 81, 053836 (2010).
[5] P. M. Anisimov, J. P. Dowling and B. C. Sanders, Phys. Rev. Lett. 107, 163604 (2011).
[6] K. J. Boller, A. Imamoglu and S. E. Harris, Phys. Rev. Lett. 66, 2593 (1990).
[7] G. B. Serapiglia, E. Paspalakis, C. Sirtori, K. L. Vodopyanov and C. C. Philips, Phys. Rev. Lett. 84, 1019 (2000).
[8] E. A. Cornell and C. E. Wieman, Rev. Mod. Phys. 74, 875 (2002).
[9] W. Ketterle, Rev. Mod. Phys. 74, 1131 (2002).
[10] I. Vadeiko, A. V. Prokhorov, A. V. Rybin, and S. M. Arakelyan, Phys. Rev. A 72 013804 (2005).
[11] J. G. Ri, C. K. Kim and K. Nahm, Commun. Theor. Phys. 48, 461464 (2007).
[12] V. Ahufinger, R. Corbalan, F. Cataliotti, S. Burger, F. Minardi, and C. Fort, Opt. Comm. 211, 159 (2002).
[13] J. Ruostekoski, and D. F. Walls, Phys. Rev. A 59, R2571 (1999); ibid Eur. Phys. J. D 5, 335 (1999).
[14] E. Lheurette, Metamaterials and Wave Control (Wiley-ISTE, London) (2013).
[15] R. V. Craster and S. Guenneau, Acoustic Metamaterials: Negative Refraction, Imaging, Lensing and Cloaking (Springer, Berlin), Vol. 166 (2013).
[16] N. Liu, L. Langguth, T. Weiss, J. Kastel, M. Fleischhauer, T. Pfau, and H. Giessen, Nat. Mater. 8, 758–762 (2009).
[17] M. Fleischhauer, A. Imamoglu, and J. P. Marangos, Rev. Mod. Phys. 77, 633 (2005).
[18] B. Lukyanchuk, N. I. Zheludev, S. A. Maier, N. J. Halas, P. Nordlander, H. Giessen, and C. T. Chong, Nat. Mater. 9, 707–715 (2010).
[19] C. Wu, A. B. Khscianikaev, R. Adato, N. Arju, A. A. Yanik, H. Altug, and G. Shvets, Nat. Mater. 11, 69–75 (2011).
[20] A. Santillan and S. I. Bozhevolnyi, Phys. Rev. B 84, 064304 (2011).
[21] M. Amin, A. Elayouch, M. Farhat, M. Addouche, A. Kheiff, and H. Bagci, J. App. Phys. 118, 164901 (2015).
[22] J. Zhu, Y. Chen, X. Zhu, F. J. Garcia-Vidal, X. Yin, W. Zhang, and X. Zhang, Sci. Rep. 3, 1728 (2013).
[23] A. Cicik, O. A. Kaya, M. Yilmaz, and B. Ulug, J. Appl. Phys. 111, 013522 (2012).
[24] M. Wadati, Eur. Phys. J. Special Topics 173, 223 (2009).
[25] X. J. Liu, H. Jing and M. L. Ge, Phys Rev. A 70, 055802 (2004).
[26] M. I. Shaukat, E. V. Castro and H. Terças, Phys. Rev. A 95, 053618 (2017).
[27] M. I. Shaukat, E. V. Castro and H. Terças, arXiv:1801.08169 (2018).
[28] M. I. Shaukat, E. V. Castro and H. Terças, arXiv:1801.08894 (2018).
[29] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation, Clarendon, Oxford (2003).
[30] See Supplemental Material for details on the calculation of the spontaneous decay rates and on the equation governing the probe envelope dynamics.
[31] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP 34, 62 (1972); ibid 37, 823 (1973).
[32] G. Huang, J. Szeftel, and S. Zhu, Phys. Rev. A 65, 053605 (2002).
[33] A. L. Gaunt, T. F. Schmidutz, I. Gotlibovych, R. P. Smith, and Z. Hadzibabic, Phys Rev. Lett. 110, 200406 (2013).
[34] P. Krüger, S. Hofferberth, I. E. Mazets, I. Lesanovsky, and J. Schmiedmayer, Phys. Rev. Lett. 105, 265302 (2010).
[35] J. Lekner, Am. J. Phys. 75, 1151 (2007).
[36] M. Scully and M. Zubairy, Quantum Optics, Cambridge University Press (1997).
[37] J. Dziarmaga, Phys. Rev. A 70, 063616 (2004).
[38] E. Compagno, G. D. Chiara, D. G. Angelakis and G. M. Palma, Sci. Reports 7, 2355 (2017).
[39] H. Terças, D. D. Solnyshkov and G. Malpuech, Phys. Rev. Lett. 110, 035302 (2013); ibid 113, 036403 (2014).
[40] A. Gonzalez-Tudela, D. Martin-Can, E. Moreno, L. Martin- Moreno, C. Tejedor, and F. J. Garcia-Vidal, Phys. Rev. Lett. 106, 020501 (2011).
[41] T. Ramos, H. Pichler, A. J. Daley, P. Zoller, Phys. Rev. Lett. 113, 237203 (2014).
[42] P. Lambropoulos and D. Petroyan, "Fundamentals of quantum optics and and quantum information" (Berlin, Springer) (2007).
[43] W. M. Robertson, C. Baker and C. B. Bennett, Am. J. Phys. 72, 255 (2004).
[44] A. Santillan and S. I. Bozhevolnyi, Phys. Rev. B 89, 184301 (2014).
[45] H. S. Borges and C. J. Villas-Boas, Phys. Rev. A 94, 023337 (2016).
[46] O. Lahad and O. Fristenberg, Phys. Rev. Lett. 119, 113601 (2017).
Supplemental Material

Dark solitons in Bose-Einstein Condensates.— We consider a dark soliton in a quasi 1D BEC, which in turn is surrounded by a dilute set of impurities (see Fig. 1 of the main manuscript). The BEC and the impurity particles are described by the wave functions $\psi_1(x,t)$ and $\psi_2(x,t)$, respectively. At the mean field level, the system is governed by the Gross Pitaevskii and Schrödinger equation, respectively,

$$i\hbar \frac{\partial \psi_1}{\partial t} = -\frac{\hbar^2}{2m_1} \frac{\partial^2 \psi_1}{\partial x^2} + g_{11} |\psi_1|^2 \psi_1 + g_{12} |\psi_2|^2 \psi_1,$$

$$i\hbar \frac{\partial \psi_2}{\partial t} = -\frac{\hbar^2}{2m_2} \frac{\partial^2 \psi_2}{\partial x^2} + g_{21} |\psi_1|^2 \psi_2,$$  \hspace{1cm} (15)

Here, the discussion is restricted to repulsive interactions ($g_{11} > 0$) where the dark solitons are assumed to be not significantly disturbed by the presence of impurities, which we consider to be fermionic in order to avoid condensation at the bottom of the potential and $g_{12} = g_{21}$. To achieve this, the impurity gas is chosen to be sufficiently dilute, i.e. $|\psi_1|^2 \gg |\psi_2|^2$, and much massive than the BEC particles. Such a situation can be produced, for example, choosing $^{133}$Cs impurities in a $^{85}$Rb BEC [1]. Therefore, the impurities can be regarded as free particles that feel the soliton as a potential

$$i\hbar \frac{\partial \psi_2}{\partial t} = -\frac{\hbar^2}{2m_2} \frac{\partial^2 \psi_2}{\partial x^2} + g_{21} |\psi_{\text{sol}}|^2 \psi_2,$$  \hspace{1cm} (16)

where the singular nonlinear solution corresponding to the soliton profile is $\psi_{\text{sol}}(x) = \sqrt{n_0} \tanh [x/\xi]$. The time-independent version of Eq. (16) reads

$$(E - g_{21}n_0)\psi_2 = -\frac{\hbar^2}{2m_2} \frac{\partial^2 \psi_2}{\partial x^2} - g_{21}n_0 \text{sech}^2 \left(\frac{x}{\xi}\right) \psi_2,$$  \hspace{1cm} (17)

To find the analytical solution of Eq. (17), the potential is casted in the Pöschl-Teller form

$$V(x) = -\frac{\hbar^2}{2m_2} \nu(\nu + 1) \text{sech}^2 \left(\frac{x}{\xi}\right),$$  \hspace{1cm} (18)

with $\nu = \left(-1 + \sqrt{1 + 4g_{21}m_2/g_{11}m_1}\right)/2$. The particular case of $\nu$ being a positive integer belongs to the class of reflectionless potentials [3], for which an incident wave is totally transmitted. For the more general case considered here, the energy spectrum associated to the potential in Eq. (18) reads

$$E_n' = -\frac{\hbar^2}{2m_2} (\nu - n)^2,$$  \hspace{1cm} (19)

where $n$ is an integer. The number of bound states created by the dark soliton is $n_{\text{bound}} = \lfloor \nu + 1 + \sqrt{\nu(1 + \nu)} \rfloor$, where the symbol $\lfloor \cdot \rfloor$ denotes the integer part. As such, the condition for exactly three bound states (i.e. the condition for the qutrit to exist) is obtained if $\nu$ sits in the range

$$\frac{4}{5} \leq \nu < \frac{9}{7},$$  \hspace{1cm} (20)

as discussed in the manuscript. At $\nu \geq 9/7$, the number of bound states increases (see Fig. 6 for a schematic illustration). In Fig. 7, we compare the analytical estimates with the full numerical solution of Eqs. (15), for both the soliton and the qutrit wavefunctions, under experimentally feasible conditions.

Hamiltonian.— The interaction Hamiltonian is given by

$$H_{\text{int}} = g_{12} \int dx \psi_1^* \psi_2 \psi_1 \psi_2,$$  \hspace{1cm} (21)
where $\psi_2(x) = \sum_{l=0}^{2} \varphi_l(x) a_l$ describes the qutrit field in terms of the bosonic operators $a_n$, with $\varphi_0(x) = A_0 \text{sech}^2(x/\xi)$, $\varphi_1(x) = 2A_1 \tanh(x/\xi) \varphi_0(x)$ and $\varphi_2(x) = \sqrt{2}A_2 \left(1 - (1 + 3\alpha) \tanh^2(x/\xi)\right) \varphi_0(x)$, where $A_j (j = 0, 1, 2)$ are the normalization constants, given by

$$A_0 = \left(\frac{\sqrt{\Gamma(1+2\alpha)}}{\Gamma[1+2\alpha]}\right)^{-\frac{1}{2}}$$

$$A_1 = \left(2^{2(1+\alpha)} A_0^2 \left(\frac{2F_1[\alpha,2(1+\alpha),1+\alpha,-1]}{\alpha} - \frac{2F_1[1+\alpha,2(1+\alpha),2+\alpha,-1]}{1+\alpha}\right) + \frac{2F_1[2+\alpha,2(1+\alpha),3+\alpha,-1]}{2+\alpha}\right)^{-\frac{1}{2}},$$

$$A_2 = \left(2^{3+2\alpha} A_0^2 A_1^2 \left(\frac{\alpha_2 F_1[\alpha,2(2+\alpha),1+\alpha,-1]}{2+\alpha} - \frac{4\alpha(1+\alpha) F_1[1+\alpha,2(2+\alpha),2+\alpha,-1]}{1+\alpha}\right) + (4 + 6\alpha^2 + 8\alpha) F_1[2+\alpha,2(2+\alpha),3+\alpha,-1] - \frac{4\alpha(1+\alpha) F_1[3+\alpha,2(2+\alpha),4+\alpha,-1]}{3+\alpha}\right) + \frac{\alpha_2 F_1[4+\alpha,2(2+\alpha),5+\alpha,-1]}{4+\alpha}\right)^{-\frac{1}{2}},$$

where $\Gamma[\alpha]$ and $2F_1$ represents the Gamma and Hypergeometric function, respectively, and $\alpha = \sqrt{2g_{12}m_2/g_{11}m_1}$. The inclusion of quantum fluctuations is performed by writting the BEC field as $\psi_1(x) = \psi_{\text{sol}}(x) + \delta \psi(x)$, where $\delta \psi(x) = \sum_k (u_k(x) b_k + v_k^*(x) b_k^*)$ and $b_k$ are the bosonic operators verifying the commutation relation $[b_k, b_q^*] = \delta_{k,q}$. The amplitudes $u_k(x)$ and $v_k(x)$ satisfy the normalization condition $|u_k(x)|^2 - |v_k(x)|^2 = 1$ and are explicitly given.
by [4],

\[
u_k(x) = \sqrt{\frac{1}{4\pi \xi \epsilon_k}} \times
\left[ \left( k\xi \right)^2 + \frac{2\epsilon_k}{\mu} \left( \frac{k\xi}{2} + i\tanh \left( \frac{x}{\xi} \right) \right) + \frac{k\xi}{\cosh^2 \left( \frac{x}{\xi} \right)} \right],
\]

and

\[
v_k(x) = \sqrt{\frac{1}{4\pi \xi \epsilon_k}} \times
\left[ \left( k\xi \right)^2 - \frac{2\epsilon_k}{\mu} \left( \frac{k\xi}{2} + i\tanh \left( \frac{x}{\xi} \right) \right) + \frac{k\xi}{\cosh^2 \left( \frac{x}{\xi} \right)} \right]
\]

Using the rotating wave approximation (RWA), the first order perturbed Hamiltonian can be written as

\[H_{\text{int}}^{(1)} = \sum_k \left( g^k_0 \sigma^+_0 + g^k_1 \sigma^+_1 \right) b_k + \left( g^{k*}_0 \sigma^{-}_0 + g^{k*}_1 \sigma^{-}_1 \right) b^\dagger_k, \tag{23}\]

where \(\sigma^+_{0,1} = a^+_{g,e_1,e_2} a_{g,e_1} \), \(\sigma^-_{0,1} = a^+_{g,e_1,e_2} a_{g,e_1} \), and the coupling constants \(g^k_{0,1} = g^k_i (i = 0, 1)\) are explicitly given by

\[
g^k_0 = \frac{ig_{12}k^2 \xi^3/2}{80\epsilon_k} \sqrt{\frac{\pi_0 \pi}{6}} \left( 2\mu + 8k^2 \mu \xi^2 + 15\epsilon_k \right) \left( -4 + k^2 \xi^2 \right) \mathrm{csch} \left( \frac{k\pi \xi}{2} \right),
\]

\[
g^k_1 = \frac{ig_{12}k^2 \xi^3/2}{896\epsilon_k} \sqrt{\frac{\pi_0 \pi}{15}} \left[ 28 \left( 2k^4 \xi^4 - 35k^2 \xi^2 + 68 \right) \epsilon_k \right.
\]
\[+ \mu \left( 29k^6 \xi^6 - 504k^4 \xi^4 + 896k^2 \xi^2 + 64 \right) \mathrm{csch} \left( \frac{k\pi \xi}{2} \right). \tag{24}\]

Technically speaking, the RWA approximation here means neglecting the intraband terms in Eq. (23), whose amplitudes are given by the coefficients \(g^k_{0,1}\) illustrated in Fig. 8. This is achieved if we assume that only resonant processes (i.e. phonons with wavevectors \(k\) such that their energies \(\omega_k\) are in resonance with the transitions \(\omega_0\) and \(\omega_1\), promoting excitation–deexcitation of the impurity inside the soliton) participate in the dynamics. As explained in the main text, and as we see below, the validity of our RWA approximation is verified \(a \text{ post}erio\ri\), holding if the corresponding spontaneous emission rates \(\gamma_i (i = 0, 1)\) are much smaller than the qutrit transition frequencies \(\omega_i\).

**Wigner-Weisskopf spontaneous decay.** – We employ the Wigner-Weisskopf theory to find the spontaneous decay rate of the states, by neglecting the effect of temperature and other external perturbations. This is extremely well justified in our case, as BECs can nowadays be routinely produced well below the critical temperature for
condensations. The qutrit is assumed to be initially at the excited state $|e_2\rangle$ and the phonons to be in the vacuum state $|0\rangle$. Under such conditions, the wave function of total system (qutrit + phonons) can be described as

$$|\phi(t)\rangle = a(t)|e_2,0\rangle + \sum_k b_k(t)|e_1,1_k\rangle + \sum_{k,p} b_{k,p}(t)|g,1_k,1_p\rangle,$$

where $a(t)$ is the probability amplitude of the excited state $|e_2\rangle$. The qutrit decays to the state $|e_i\rangle$ with probability amplitude $b_k(t)$ by emitting a phonon of wavevector $k$ and frequency $\omega_k$. Subsequently, the qutrit de-excites to the ground state $|g\rangle$ via the emission of $q$-phonon of frequency $\omega_p$ and probability amplitude $b_{k,p}(t)$. The Wigner-Weisskopf ansatz (25) is then let to evolve under the total Hamiltonian in Eq. (23), for which the corresponding Schrödinger equation yields

$$\dot{a}(t) = -\frac{\gamma_1}{2}a(t),$$
$$\dot{b}_k(t) = -i\frac{g}{\hbar} b_k' e^{i(\omega_k - \omega_1)t - \frac{\gamma_1}{2}t} - \frac{\gamma_0}{2} b_k(t),$$
$$\dot{b}_{k,p}(t) = -i\frac{g}{\hbar} b'_{k,p} e^{i(\omega_p - \omega_0)t},$$

which are simplified by following the procedure of Ref. [2]. Here, $\gamma_i (i = 0, 1)$ is the $i$th state decay rate given by

$$\gamma_i = \frac{L}{\sqrt{2\hbar}\xi} \int d\omega_k \sqrt{1 + \frac{\hbar}{\sqrt{2}\xi}} |g|^2 \delta(\omega_k - \omega_i),$$

with

$$\gamma_0 = \frac{\pi N_0 g^2}{76800\hbar^3 \xi^2 \eta_0 \sqrt{\frac{\mu + \eta_0}{\mu}}} (\mu - \eta_0) (-5\mu + 7\eta_0)^2$$
$$\times \left(8\eta_0 + 3\mu \left(-2 + 5\sqrt{\frac{\hbar^2 \omega_0^2}{\mu^2 \xi^2}} \right) \right) \text{csch}^2 \left(\frac{\pi \sqrt{\mu + \eta_0}}{2\sqrt{\mu}}\right),$$

and

$$\gamma_1 = \frac{\pi N_0 g^2}{2.4 \times 10^4 \hbar^3 \xi^2 \eta_0 \sqrt{\frac{\mu + \eta_0}{\mu}}} (\mu - \eta_1) \left[-1956\mu^3 + \hbar^2 \omega_1^2 \left[-591\mu + 56 \sqrt{\eta^2 - \mu^2} + 29\eta_1\right]ight]$$
$$+ 4\mu^2 \left[505\eta_1 + 7 \sqrt{\frac{\eta^2}{\mu^2} - 1} (107\mu - 39\eta_0)\right] \right)^2 \text{csch}^2 \left(\frac{\pi \sqrt{\mu + \eta_0}}{2\sqrt{\mu}}\right),$$

where $\eta_i = \sqrt{\mu^2 + \hbar^2 \omega^2_1}$. In the long time limit $t \gg \gamma_i$, Eq. (26) can be simplified to obtain

$$a(t) = e^{-\gamma_i t/2},$$
$$b_k(t) = -i\frac{g}{\hbar} \left[e^{i(\omega_k - \omega_1)t - \gamma_i t/2} - e^{-\gamma_i t/2}\right],$$
$$b_{k,p}(t) = \frac{g}{i(\omega_k - \omega_1)} \left[e^{i(\omega_k - \omega_1)t - \gamma_i t/2} - e^{-\gamma_i t/2}\right] + \frac{1}{i(\omega_k - \omega_1 - \omega_p)} \left[e^{i(\omega_k - \omega_1 - \omega_p)t - \gamma_i t/2} - e^{-\gamma_i t/2}\right],$$

where $\omega_{eg} = \omega_0 + \omega_1$, leading to Eq. (7) of the manuscript.

**Heisenberg’s relation and propagation.**— With the aim of studying a dilute array of qutrits affects the propagation of sound waves inside the condensate, we compute the equation of motion for a weak acoustic probe coupling the ground and the first excited state (i.e. driving the lower transition of the qutrits). This is done with the help of Heisenberg’s relation

$$i\hbar \frac{\partial (\delta \Psi)}{\partial t} = \left[\hat{H}, \delta \Psi\right],$$
where $\hat{H} = H_q + H_p + H_{\text{drive}}$ denotes the total Hamiltonian and $\delta \Psi = \phi b_q e^{i q x} + \psi^* b_q^\dagger e^{-i q x}$ is the fluctuating field with the Bogoliubov coefficients $\phi$ and $\psi$. Noticing that the commutation relation with the driving Hamiltonian provides

$$[H_{\text{drive}}, \delta \Psi] = \frac{g_0^k}{2\hbar} \rho_{12},$$

where $\Omega_p = \frac{g_0^k \delta \Psi}{\hbar}$, and proceeding similarly for the commutation with $H_q$ and $H_p$, we obtain the following wave equation

$$\frac{\partial \Omega_p}{\partial t} + \frac{\omega_q}{q} \frac{\partial \Omega_p}{\partial x} = -\frac{i}{2\hbar^2} \left( g_0^k \right)^2 \rho_{12},$$

(33)

corresponding to Eq. (12) of the main text. The quantum interference with the second transition, driven by the coupling field of intensity $\Omega_c \gg \Omega_p$, is contained in the coherence $\rho_{12}$ appearing in the rhs of Eq. (33). The latter can be identified as the acoustic analogue of a dynamical susceptibility.

* Electronic address: muzzamalshaukat@gmail.com
† Electronic address: hugo.tercas@tecnico.ulisboa.pt
[1] M. Hohmann, F. Kindermann, B. Ganger, T. Lausch, D. Mayer, F. Schmidt and A. Widera, EPJ Quantum Technology Phys. Rev. A 2, 23 (2015).
[2] M. I. Shaukat, E. V. Castro and H. Terças, Phys. Rev. A 95, 053618 (2017).
[3] J. Lekner, Am. J. Phys. 75, 1151 (2007).
[4] J. Dziarmaga, Phys. Rev. A 70, 063616 (2004).