NIL HECKE ALGEBRAS AND WHITTAKER $\mathcal{D}$-MODULES

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To the memory of Bertram Kostant

ABSTRACT. Given a semisimple group $G$, Kostant and Kumar defined a nil Hecke algebra that may be viewed as a degenerate version of the double affine nil Hecke algebra introduced by Cherednik. In this paper, we construct an isomorphism of the spherical subalgebra of the nil Hecke algebra with a Whittaker type quantum Hamiltonian reduction of the algebra of differential operators on $G$. This result has an interpretation in terms of geometric Satake and the Langlands dual group. Specifically, the isomorphism provides a bridge between very differently looking descriptions of equivariant Borel-Moore homology of the affine flag variety (due to Kostant and Kumar) and of the affine Grassmannian (due to Bezrukavnikov and Finkelberg), respectively.

It follows from our result that the category of Whittaker $\mathcal{D}$-modules on $G$, considered by Drinfeld, is equivalent to the category of holonomic modules over the nil Hecke algebra, and it is also equivalent to a certain subcategory of the category of Weyl group equivariant holonomic $\mathcal{D}$-modules on the maximal torus.

CONTENTS

1. Introduction 1
2. Three constructions of the universal centralizer 7
3. Review of quantum Hamiltonian reduction and Kazhdan filtrations 10
4. Proofs of Theorem 1.2.2 and Theorem 1.6.3 15
5. Hamiltonian reduction of $T^*(G/\bar{N})$ and $\mathcal{D}(G/\bar{N})$ 17
6. The Miura bimodule 24
7. Nil-Hecke algebras 26
8. Spherical degenerate nil DAHA via Hamiltonian reduction 30
References 33

1. INTRODUCTION

1.1. Reminder on nil-Hecke algebras. In this paper, we work over $\mathbb{C}$. We use the notation $\text{Sym} \, \mathfrak{k}$, resp. $\mathcal{U}\mathfrak{k}$, for the symmetric, resp. enveloping, algebra of a vector space, resp. Lie algebra, $\mathfrak{k}$. Let $T^*X$, resp. $\mathcal{D}_X$ and $\mathcal{D}(X)$, denote the cotangent bundle, resp. the sheaf and ring of algebraic differential operators, on a smooth algebraic variety $X$. Throughout the paper, we fix a complex connected and simply connected semisimple group $G$ with Lie algebra $\mathfrak{g}$.

Let $\mathfrak{h}$ be a finite-dimensional vector space, $\mathcal{R} \subset \mathfrak{h}^*$ a reduced root system with the set $\Sigma$ of simple roots. An associated Coxeter group $\mathcal{W}$ acts naturally on $\mathfrak{h}$ and it is generated by a set $s_\alpha$, $\alpha \in \Sigma$, of simple reflections. In their work on equivariant cohomology of flag varieties, Kostant and Kumar [KK1]–[KK2] introduced a noncommutative $\mathbb{Z}$-graded algebra $\mathcal{H}(\mathfrak{h}, \mathcal{W})$ called a nil Hecke algebra, cf. Section 7.2 for an overview. The algebra $\mathcal{H}(\mathfrak{h}, \mathcal{W})$ is generated by the vector space $\mathfrak{h}$ and a collection, $\theta_\alpha$, $\alpha \in \Sigma_{\text{aff}}$, of Demazure elements. The elements of $\mathfrak{h}$ pairwise commute and generate
a copy of the algebra \( \text{Sym}\, h \) inside \( \mathcal{H}(h, W) \). Demazure elements satisfy the braid relations. The other defining relations among the generators of \( \mathcal{H}(h, W) \) are as follows, cf. (7.1.3):

\[
(\theta_{\alpha})^2 = 0, \quad \theta_{\alpha} \cdot s_{\alpha}(h) - h \cdot \theta_{\alpha} = \langle \alpha, h \rangle, \quad \forall h \in h, \alpha \in \Sigma.
\]

Let \( T \) be the (abstract) maximal torus of \( G \) and \( t = \text{Lie} \, T \). Let \( X^* \) be the weight lattice and \( W \) the abstract Weyl group. Further, let \( t_{\text{aff}} \) be the affine Cartan algebra, \( W_{\text{aff}} \) the affine Weyl group, and \( \mathcal{H}(t_{\text{aff}}, W_{\text{aff}}) \) the corresponding nil Hecke algebra. It is convenient to enlarge the group \( W_{\text{aff}} \) and consider \( \tilde{W} = W \ltimes X^* \), an extended affine Weyl group. Similarly, there is an enlargement, \( \mathcal{H}(t_{\text{aff}}, \tilde{W}) \), of \( \mathcal{H}(t_{\text{aff}}, W_{\text{aff}}) \). This is a \( \mathbb{Z} \)-graded \( \mathbb{C}[h] \)-algebra that may be viewed as a degeneration of the nil DAHA introduced by Cherednik and studied further by Fegin and Cherednik [CF].

Below, we will mostly be interested in the algebra \( \mathbb{H} := \mathcal{H}(t_{\text{aff}}, \tilde{W})|_{h=1} \), a specialization of \( \mathcal{H}(t_{\text{aff}}, \tilde{W}) \) at \( h = 1 \). The grading on \( \mathcal{H}(t_{\text{aff}}, \tilde{W}) \) induces an ascending \( \mathbb{Z} \)-filtration on \( \mathbb{H} \). By construction, one has \( \text{gr} \, \mathbb{H} = \mathcal{H}(t_{\text{aff}}, \tilde{W})|_{h=0} \), resp. \( \text{gr} \, \mathbb{H}_h = \mathcal{H}(t_{\text{aff}}, \tilde{W}) \), where \( \text{gr} \, A \), resp. \( A_h \), denotes an associated graded, resp. Rees algebra, of a filtered algebra \( A \). The algebra \( \mathbb{H} \) is a kind of (micro-)localization of the cross product \( W \times \mathcal{D}(T) \). In particular, there are algebra embeddings \( \text{Sym} \, t \hookrightarrow \mathcal{D}(T) \hookrightarrow \mathbb{H} \), where \( \text{Sym} \, t \) is identified with the algebra of translation invariant differential operators on the torus \( T \). The filtration on \( \mathbb{H} \) agrees with the natural filtration on \( \mathcal{D}(T) \) by order of the differential operator.

Let \( e = \frac{1}{\pi h} \sum_{w \in W} w \in CW \) be the symmetrizer idempotent. The algebra \( \mathbb{H}^{\text{ph}} = e \mathbb{H} e \), called spherical subalgebra, is a filtered algebra with unit \( e \). The embeddings above restrict to algebra embeddings \( (\text{Sym} \, t)^W \hookrightarrow \mathcal{D}(T)^W \hookrightarrow \mathbb{H}^{\text{ph}} \).

### 1.2. Spherical subalgebra via Hamiltonian reduction.

Let \( K \) be a linear algebraic group and \( X \) a smooth \( K \)-variety. Given a character \( \chi : \mathfrak{k} = \text{Lie} \, K \to \mathbb{C} \), we write \( \mathfrak{k}^\chi \) for the image of \( \mathfrak{k} \) in \( \text{Sym} \, t \), resp. \( \mathcal{D}(X) \), etc., under the map \( k \mapsto k - \chi(k) \).

Throughout the paper, we fix a maximal unipotent subgroup \( N \) of \( G \) and a nondegenerate character \( \psi : n = \text{Lie} \, N \to \mathbb{C} \). Let \( N \times N \) act on \( G \) by \( (n_t, n_r) : g \mapsto n_t^{-1} g n_r \). We write \( N_t \), resp. \( N_r \), for the first, resp. second, factor of \( N \times N \). (We will use subscripts ‘\( t \)’ and ‘\( r \)’ in other similar contexts.) Thus, we get a pair, \( n_t^\psi \) and \( n_r^\psi \), of commuting Lie subalgebras of \( \mathcal{D}(G) \). We define the following quantum Hamiltonian reduction

\[
\mathbb{W} := \left( \mathcal{D}(G)/\mathcal{D}(G)(n_t^\psi + n_r^\psi) \right)_{N_t \times N_r}.
\]

This is an associative algebra that comes equipped with a natural ascending \( \mathbb{Z} \)-filtration, called Kazhdan filtration, cf. Section 3.

We identify \( Z(\mathcal{U}g) \), the center of \( \mathcal{U}g \), with the algebra of \( G \)-bi-invariant differential operators on \( G \). The embedding of the algebra of \( G \)-bi-invariant differential operators into \( \mathcal{D}(G) \) induces an injective homomorphism \( Z(\mathcal{U}g) \to \mathbb{W} \) of filtered algebras.

One of our main results reads as follows:

**Theorem 1.2.1.** There is an isomorphism \( \mathbb{W} \to \mathbb{H}^{\text{ph}} \), of filtered algebras, that maps the subalgebra \( Z(\mathcal{U}g) \subset \mathbb{W} \) to the subalgebra \( (\text{Sym} \, t)^W \subset \mathbb{H}^{\text{ph}} \). The resulting map \( Z(\mathcal{U}g) \to (\text{Sym} \, t)^W \) is the Harish-Chandra isomorphism.

Theorem 1.2.1 has a classical (a.k.a. Poisson) counterpart that involves the moment map \( \mu_{n \times n} : T^* G \to n^* \times n^* \), associated with the Hamiltonian \( N_t \times N_r \)-action on \( T^* G \) induced by the one on \( G \). The variety \( Z = \mu^{-1}(\psi \times \psi) / (N_t \times N_r) \), a classical Hamiltonian reduction, comes equipped with the structure of an integrable system. Specifically, \( Z \) is a smooth symplectic algebraic variety equipped with a natural smooth Lagrangian fibration \( \kappa : Z \to g^* / \text{Ad}^* G \) whose fibers are abelian.
algebraic groups. The resulting group scheme on \(g^* / \text{Ad}^* G\) is known as the universal centralizer, cf. Section 2 for a review. Our second result, to be proved in Section 4.3 reads as follows:

**Theorem 1.2.2.** There is an isomorphism \(gr \mathbb{W} \cong \mathbb{C}[3]\) of graded Poisson algebras, which restricts to an isomorphism \(gr (Z g) \cong \kappa^*(\mathbb{C}[g^* / \text{Ad}^* G])\), of maximal commutative subalgebras.

As a consequence of the two theorems above, one obtains an isomorphism \(gr \mathbb{H}^{sph} \cong \mathbb{C}[3]\), which shows that the algebra \(\mathbb{H}^{sph}\) may be viewed as a quantization of the symplectic variety 3. A slightly modified form of the isomorphism \(gr \mathbb{H}^{sph} \cong \mathbb{C}[3]\) that does not however reveal a connection with nil Hecke algebras has been proved earlier by Bezrukavnikov, Finkelberg, and Mirkovic [BFM, Proposition 2.8].

We remark that Theorem 1.2.2 neither implies, nor is a simple consequence of Theorem 1.2.1 due to the fact that the filtrations and gradings on the algebras involved are not bounded below.

1.3. **Strategy of the proof of Theorem 1.2.1** The theorems above are, in fact, formal consequences of a combination of results (to be recalled in Section 4) of Kostant and Kumar, [KK1], [KK2], on the one hand, and of Bezrukavnikov and Finkelberg, [BF], on the other hand. That approach is, however, rather indirect; it relies on the geometric Satake and involves the Langlands dual group. Thus, one of our primary motivations was to find a more direct geometric approach.

The strategy of our approach to the isomorphism \(\mathbb{H}^{sph} \cong \mathbb{W}\) is as follows. First, we construct an algebra homomorphism \(\mathcal{O}(T)^W \rightarrow \mathbb{W}\). Then, we show that this homomorphism can be extended to a homomorphism \(\mathbb{H}^{sph} \rightarrow \mathbb{W}\). Finally, in Section 5.2 we prove using Theorem 7.1.4 that the latter homomorphism \(\mathbb{H}^{sph} \rightarrow \mathbb{W}\) is an isomorphism.

We should point out that we do not know how to construct the homomorphism \(\mathcal{O}(T)^W \rightarrow \mathbb{W}\) directly. The construction of such a homomorphism is not obvious even quasi-classically, where it amounts to a construction of Beilinson and Kazhdan [BK], to be recalled in Section 2.3. Thus, we use an indirect approach that involves the algebra \(\mathcal{O}(G/\tilde{N})\), where \(\tilde{N}\) is a maximal unipotent subgroup opposite to \(N\). Put \(\bar{n} = \text{Lie } \tilde{N}\). Our construction starts with the following Hamiltonian reduction

\[
A := (\mathcal{O}(G/\tilde{N})/\mathcal{O}(G/\tilde{N})\bar{n}_i^{\psi})^N_i = (\mathcal{O}(G)/\mathcal{O}(G)(\bar{n}_i^{\psi} + \bar{n}_r))^N_i \times N_r.
\]

There is a Weyl group action on \(\mathcal{O}(G/\tilde{N})\) by algebra automorphisms that has been introduced by Gelfand and Graev a long time ago. The Gelfand-Graev \(W\)-action descends to \(A\) so one has an algebra \(W \times A\). The key ingredient of our approach is a \((W \times A, \mathbb{W})\)-bimodule \(\mathbb{M}\), the Miura bimodule defined Section 3, an object closely related to the one introduced by D. Kazhdan and the author in [CK]. We prove (Theorem 5.5.1) that the algebra \(A\) is isomorphic to \(\mathcal{O}(T)\); moreover, the isomorphism intertwines the \(W\)-action on \(A\) and the natural \(W\)-action on \(\mathcal{O}(T)\), see Proposition 5.5.2. Thus, we may view \(\mathbb{M}\) as a \((W \times \mathcal{O}(T), \mathbb{W})\)-bimodule. Further, using a general criterion of Proposition 7.2.4 we deduce that the left action of \(W \times \mathcal{O}(T)\) on \(\mathbb{M}\) can be extended to an action of \(\mathbb{H}\), a larger algebra. Now, the Miura bimodule comes equipped with a canonical generator \(1_M \in \mathbb{M}\). We prove that for any \(a \in \mathbb{H}^{sph} \subset \mathbb{H}\) there exists a uniquely determined element \(a_\mathbb{W} \in \mathbb{W}_r\), such that one has \(a_1_M = 1_M a_\mathbb{W}\). The assignment \(a \mapsto a_\mathbb{W}\) yields the desired homomorphism \(\mathbb{H}^{sph} \rightarrow \mathbb{W}\).

1.4. **Relation to equivariant homology of flag varieties.** Let \(\hat{G}\) be the Langlands dual group of \(G\) and \(\hat{T}\) the maximal torus of \(\hat{G}\). Let \(I \subset \hat{G}((z))\) be an Iwahori subgroup. Let \(B_{\text{aff}} = \hat{G}((z))/I\) be the affine flag variety, resp. \(\text{Gr} = \hat{G}((z))/\hat{G}[[z]]\) the affine Grassmannian.

Given a group \(K\) and a \(K\)-action on a space \(X\), let \(H^K_X\), resp. \(H^K_{\hat{G}}(X)\), denote equivariant cohomology, resp. homology, of \(X\). One has canonical isomorphisms

\[
H^I_{\times G_m}(pt) = H^I_{\times G_m}(pt) = (\text{Sym} t)[h], \quad \text{resp.} \quad H^I_{\hat{G}[z] \times G_m}(pt) = (\text{Sym} t)^W [h].
\]
In the theorems below, the algebra structure on equivariant homology of $B_{\text{aff}}$, resp. $\text{Gr}$, is given by convolution. The multiplicative group $\mathbb{G}_m$ acts on $B_{\text{aff}}$, resp. $\text{Gr}$, by loop rotation.

Kostant and Kumar, [KK1], [KK2], [Ku] proved the following theorem which is a Kac-Moody generalization of a well-known result of Bernstein-Gelfand-Gelfand, [BGG].

**Theorem 1.4.1.** There are graded algebra isomorphisms

$$H^i_{G[[z]] \times \mathbb{G}_m}(B_{\text{aff}}) \cong \mathbb{H}^i_{\mathfrak{g}}, \quad H^i_{G[[z]] \times \mathbb{G}_m}(\text{Gr}) \cong \mathbb{H}^i_{\mathfrak{g}}^{\text{hol}}.$$ 

On the other hand, one has the following result.

**Theorem 1.4.2.** There are graded algebra isomorphisms

$$H^i_{G[[z]]}(\text{Gr}) \cong \text{gr} \mathbb{W}, \quad H^i_{G[[z]]}(\text{Gr}) \cong \mathbb{W}.$$ 

Here, the first isomorphism is due to Bezrukavnikov, Finkelberg, and Mirković [BFM], Theorem 2.1.2(b) and Proposition 2.8(b), and the second isomorphism is due to Bezrukavnikov and Finkelberg [BF], Theorem 3. Thus, our Theorem [1.2.1] is a formal consequence of a combination of the second isomorphism in Theorem [1.4.1] and Theorem [1.4.2] respectively.

**1.5. The Whittaker category.** Let $A$ be a $\mathbb{Z}$-filtered algebra such that $\text{gr} A$ is a finitely generated commutative algebra. Given a finitely generated $A$-module $M$, one can choose a good filtration on $M$ and let $\text{SS}(M) = \text{supp}(\text{gr} M)$. We say that $M$ is holonomic if $\dim \text{SS}(M) \leq \frac{1}{2} \dim \text{Spec}(\text{gr} A)$. (Although this definition is certainly not reasonable in the generality of arbitrary filtered algebras $A$ as above, it is sufficient for our limited purposes.) We write $A$-$\text{hol}$ for the abelian category of holonomic left $A$-modules. In the case of the algebra $\mathbb{H}$, one has $\text{gr} \mathbb{H} = W \times A$, where $A$ is a commutative algebra, cf. Proposition [7.2.7].

We define the notion of holonomicity by replacing $\text{gr} \mathbb{H}$ by $A$ in the previous definition.

Let $K$ be a connected linear algebraic group, $\Omega_K$ the canonical bundle on $K$, and $\chi : \mathfrak{t} = \text{Lie} K \to \mathbb{C}$ a character. We will work with right $\mathfrak{g}$-modules and put $e^z \Omega_K := \mathfrak{g} \mathfrak{k} / \mathfrak{t} \mathfrak{k}$. This is a line bundle on $K$ equipped with a flat connection that does not necessarily have regular singularities, in general. Let $X$ be a smooth variety equipped with a $K$-action $a_K : K \times X \to X$. Let $pr_1$, resp. $pr_2$, denote the first, resp. second, projection $K \times X \to X$. A $(K, \chi)$-Whittaker $\mathcal{D}_X$-module is, by definition, a $\mathcal{D}_X$-module $\mathcal{F}$ equipped with an isomorphism $a_K^* \mathcal{F} \cong e^{\chi} \Omega_K \boxtimes \mathcal{F}$ of $\mathcal{D}_{K \times X}$-modules that satisfies an appropriate cocycle condition (here and elsewhere, we abuse notation and write $\mathcal{F}' \boxtimes \mathcal{F}$ for $pr_1^* \mathcal{F}' \boxtimes \mathcal{F}_{\mathcal{O}_{K \times X}} pr_2^* \mathcal{F}$). There is an abelian category $(\mathcal{D}_X, K, \chi)$-$\text{mod}$ of $(K, \chi)$-Whittaker $\mathcal{D}_X$-modules. In the special case where $\chi = 0$, one gets the category $(\mathcal{D}_X, K)$-$\text{mod}$ of $K$-equivariant $\mathcal{D}_X$-modules. If $X$ is affine, we may (and will) identify $\mathcal{D}_X$-modules with $\mathcal{D}(X)$-modules via the functor of global sections. If, moreover, the group $K$ is unipotent then a $(K, \chi)$-Whittaker structure on $\mathcal{F}$ amounts to the property (rather than an additional structure) that, for any $k \in \mathfrak{t}$, the action of $k_g - \chi(k)$ on $\Gamma(X, \mathcal{F})$ is locally nilpotent, where $k_g$ denotes the action that comes from the $\mathcal{D}(X)$-action.

Following Drinfeld, we consider the category $(\mathcal{D}_G, N_l \times N_r, \psi \times \psi)$-$\text{mod}$ and its full subcategory, to be denoted $\mathcal{W}$ and called the Whittaker category, whose objects are holonomic $\mathcal{D}$-modules. Drinfeld raised a question of finding a description of an $\ell$-adic counterpart of the Whittaker category in terms of $W$-equivariant sheaves on the maximal torus $T$. The following result provides, in particular, an answer to an analogous question in the $\mathcal{D}$-module setting.

**Theorem 1.5.1.** The Whittaker category $\mathcal{W}$ is equivalent to any of the following categories:

1. The category of holonomic $\mathcal{W}$-modules;
2. The category of holonomic $\mathbb{H}$-modules;
Remark 1.5.3. There is a canonical projection $t^*/s_W \to t^*/W$, of a stacky quotient of $t^*$ by $W$ to the categorical quotient. Isomorphism (1.5.2) means that $M$, viewed as a quasi-coherent sheaf on $t^*/s_W$, descends to $t^*/W$.

In the recent work [Lo1], G. Lonergan established an equivalence of categories which is analogous to the equivalence $Wh \cong \cat(T, W)$ in the theorem above. However, the important holonomicity condition is not present in the setting of [Lo1]. The approach in [Lo1] is of a topological nature; it relies on the geometric Satake equivalence (via [BF]) and on some general results of Goresky, Kottwitz, and MacPherson, [GKM], cf. also [LLMSSZ]. That approach is totally different from ours.

Our proof of Theorem 1.5.1 is closely related to the theorem below that may be of independent interest and which is, in a sense, ‘dual’ to the statement of [BBM], Theorem 1.5(1).

Given a map $f : X \to Y$, let $f^j_!$ denote the $i$th derived push-forward functor on $\mathcal{D}$-modules. Let $a_N : \bar{N} \times G \to G$, $(\bar{n}, g) \mapsto g\bar{n}$, be the action.

**Theorem 1.5.4.** For any $M \in (\mathcal{D}_G, N_r, \psi)$-mod, we have $\int^j_! (\Omega^n_N \boxtimes M) = 0$, $\forall j \neq 0$. In particular, the functor $(\mathcal{D}_G, N_r, \psi)$-mod $\to$ $(\mathcal{D}_G, \bar{N}_r)$-mod, $M \mapsto \int^0_! (\Omega^n_N \boxtimes M)$ is exact.

The proof of Theorem 1.5.4 is given in Section 6.1.

1.6. The Whittaker functor. Let $X$ be a smooth $G \times G$-variety, $G \subset G \times G$ the diagonal, and $(\mathcal{D}_X, G)$-mod the corresponding category of $G$-equivariant $\mathcal{D}_X$-modules. Write $N_r = 1 \times N \subset G \times G$. It is easy to check that one has a well-defined functor

$$
\Psi : (\mathcal{D}_X, G)\text{-mod} \to (\mathcal{D}_X, \bar{N}_r \times N_r, \psi \times \psi)\text{-mod}, \quad M \mapsto \int^0_{a_{N_r}} (e^\psi \Omega_N \boxtimes M). \quad (1.6.1)
$$

Assume now that $X$ is affine and form a Hamiltonian reduction $\mathcal{D} := (\mathcal{D}(X) \times n^\psi \mathcal{D}(X))^N_r$. The map $n^\psi \to \mathcal{D}(X)$ descends to a map $n^\psi \to \mathcal{D}$. We let $(\mathcal{D}, n^\psi)$-mod be a full subcategory of the category of right $\mathcal{D}$-modules such that the induced action of the Lie algebra $n^\psi$ on the module is locally nilpotent. One checks that for a right $\mathcal{D}(X)$-module $M$ the space $M/Mn^\psi$, of $n^\psi$-coinvariants, has the natural structure of a right $\mathcal{D}$-module; furthermore, the resulting $n^\psi$-action on $M/Mn^\psi$ is locally nilpotent if $M$ is $G$-equivariant. Also, the space $\mathcal{D}(X)/n^\psi \mathcal{D}(X)$ has the structure of a $(\mathcal{D}, \mathcal{D}(X))$-bimodule. We have the following functors

$$
(\mathcal{D}(X), G)\text{-mod} \xrightarrow{M \mapsto M/Mn^\psi} (\mathcal{D}, n^\psi)\text{-mod} \xrightarrow{- \otimes_\mathcal{D} \mathcal{D}(X)/n^\psi \mathcal{D}(X)} (\mathcal{D}(X), N_r \times N_r, \psi \times \psi)\text{-mod}. \quad (1.6.2)
$$
Theorem 1.6.3. Let $X$ be an affine $G \times G$-variety. Then the composite functor in (1.6.2) is isomorphic to $\Psi$ and the second functor in (1.6.2) is an equivalence. Furthermore, for any $M \in (\mathcal{D}(X), G)$-mod and all $j \neq 0$, we have $H_j(n^\psi, M) = 0$, resp. $\int_{a_N^j}(\Omega_{N_r} \boxtimes M) = 0$. 

Thus, the first functor in diagram (1.6.2) is exact and it takes holonomic modules to holonomic modules.

Below, we are interested in a special case of the above setting where $X = G$ is viewed as a $G \times G$-variety via left and right translations. The diagonal $G$-action on $X$ corresponds to the conjugation action of $G$ on itself, so $(\mathcal{D}(X), G)$-mod is, in this case, the category of $Ad G$-equivariant $\mathcal{D}(G)$-modules. Further, there is an equivalence $(\mathcal{D}(G), N_l \times N_r, \psi \times \psi)$-mod, see Section 3.1 and Lemma 5.4.1 Also, one checks that for any $\mathcal{D}(G)$-module $M$, the subspace $(M/Mn^\psi)^{n_l^\psi} \subset M/Mn^\psi$ of $n_l^\psi$-invariants has the natural structure of a $\mathcal{W}$-module, see Section 3.1 From the above theorem, we will deduce the following result.

Theorem 1.6.4. (i) For any $M \in (\mathcal{D}(G), Ad G)$-mod and all $j \neq 0$, we have $H_j(n^\psi, M) = 0$, resp. $\int_{a_N^j}(\Omega_{N_r} \boxtimes M) = 0$.

(ii) The functor (1.6.1) corresponds, via the equivalence $(\mathcal{D}(G), N_l \times N_r, \psi \times \psi)$-mod $\cong \mathcal{W}$-mod, to the functor $(\mathcal{D}(G), Ad G)$-mod $\rightarrow \mathcal{W}$-mod, $M \mapsto (M/Mn^\psi)^{n_l^\psi}$. Furthermore, the latter functor induces via the equivalence $\text{Wh} \cong \mathcal{W}$-hol, of Theorem 1.5.1 an exact functor $(\mathcal{D}(G), Ad G)$-hol $\rightarrow \text{Wh}$.

Theorem 1.6.3 will be proved in Section 4.2 and Theorem 1.6.4 will be proved in Section 5.4.

Remarks 1.6.5. (i) In the special case where $X = G$ and $M \in (\mathcal{D}(G), Ad G)$-mod is holonomic (not necessarily regular), the vanishing of $\int_{a_N^j}(\Omega_{N_r} \boxtimes M) = 0$ for all $j \neq 0$ can be proved in a different way by adapting the proof of a result of Bezrukavnikov, Braverman, and Mirkovic, [BBM, Theorem 1.5(2)], to the $\mathcal{D}$-module setting.

(ii) One can show that the $\Psi$ takes finitely generated $\mathcal{D}(G)$-modules to finitely generated $\mathcal{W}$-modules. Furthermore, it is likely that this functor respects singular supports in the sense that $SS(\Psi(M))$ is obtained from $SS(M)$ by a suitable classical Hamiltonian reduction.

Observe that group multiplication $G \times G \rightarrow G$ induces a monoidal structure on a suitably defined (bounded) derived category counterpart $D(\mathcal{D}(G), Ad G)$-hol, resp. $\text{DWh}$, of the abelian category $(\mathcal{D}(G), Ad G)$-hol, resp. $\text{Wh}$. The functor $\Psi$ in Theorem 1.6.4 can be upgraded to a monoidal functor $D(\mathcal{D}(G), Ad G)$-hol $\rightarrow \text{DWh}$, cf. Remark 3.1.6 (In the setting of $\infty$-categories for an affine Kac-Moody group an analogous functor has been considered in [Be]). Using that the functor (1.5.2) is exact, it is also possible to define a derived version, $D\hat{\mathcal{A}}t(T, W)$, of the abelian category $\mathcal{A}t(T, W)$. The group $T$ being abelian, multiplication $T \times T \rightarrow T$ gives $D\hat{\mathcal{A}}t(T, W)$ the structure of a symmetric monoidal category. We expect that there is a derived counterpart of Theorem 1.5.1 that provides, in particular, a monoidal equivalence $\text{DWh} \cong D\hat{\mathcal{A}}t(T, W)$. Following Drinfeld, we observe that the existence of such an equivalence would imply that the monoidal structure on $\text{DWh}$ is symmetric. The symmetry of the monoidal structure on $\text{DWh}$ seems to be unknown at the time of writing this paper.

We defer a more detailed discussion of these topics to a separate paper.

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1 was informed by V. Drinfeld that the last equation has also been obtained by Sam Raskin (unpublished).
2. THREE CONSTRUCTIONS OF THE UNIVERSAL CENTRALIZER

Given an algebraic group $K$ with Lie algebra $\mathfrak{k}$, we write $\text{Ad}^*$ for the coadjoint action of $K$ in $\mathfrak{k}^*$. Recall that we have fixed a connected and simply connected semisimple group $G$ with Lie algebra $\mathfrak{g}$. We write $G_x$ for the stabilizer of an element $x \in \mathfrak{g}^*$ under the $\text{Ad}^*$-action and let $\mathfrak{g}_x = \text{Lie } G_x$. We say that $x$ is regular if $\dim \mathfrak{g}_x = \text{rk } \mathfrak{g}$. Let $\mathfrak{g}^\text{reg}$ be the set of regular elements of $\mathfrak{g}^*$.

2.1. The first construction of $\mathfrak{z}$. One can identify the cotangent bundle on $G$ with the variety of triples

$$T^*G = \{(x', x, g) \in \mathfrak{g}^* \times \mathfrak{g}^* \times G \mid x' = \text{Ad}^* g(x)\}. \tag{2.1.1}$$

The group $G$ acts on itself by left and right translation. The induced $G \times G$-action on $T^*G$ is given by $(g_1, g_2) : (x', x, g) \mapsto ((\text{Ad}^* g_1(x'), \text{Ad}^* g_2(x), g_1 g g_2^{-1})$. This action is Hamiltonian with moment map

$$\mu_t \times \mu_r : T^*G \to \mathfrak{g}^* \times \mathfrak{g}^*, \quad \mu_t(x', x, g) = x', \quad \mu_r(x', x, g) = x.$$

Next, let $G$ act on itself by conjugation and act on $\mathfrak{g}^* \times G$ diagonally. We define

$$\mathcal{Z} := \{(x, g) \in \mathfrak{g}^* \times G \mid \text{Ad}^* g(x) = x\}, \quad \mathcal{Z}^\text{reg} := \{(x, g) \in \mathcal{Z} \mid x \in \mathfrak{g}^\text{reg}\}.$$

In terms of (2.1.1), the moment map associated with the $\text{Ad} G$-action on $T^*G$ has the form:

$$\mu_{\text{Ad}}(x', x, g) = x' - x.$$ Writing $\mathfrak{g}^\text{diag}$ for the diagonal copy of $\mathfrak{g}^*$ in $\mathfrak{g}^* \times \mathfrak{g}^*$, one obtains the following identifications:

$$\mathcal{Z} \xrightarrow{(x, g) \mapsto (x, x, g)} (\mu_t \times \mu_r)^{-1} (\mathfrak{g}^\text{diag}) = \mu_{\text{Ad}}^{-1}(0).$$

Let $T^\text{reg}G := \mu_t^{-1} (\mathcal{Z}^\text{reg}) = \mu_r^{-1} (\mathfrak{g}^\text{reg})$. This is a $G \times G$-stable Zariski open subset of $T^*G$, and we have $\mathcal{Z}^\text{reg} = (\mu_t \times \mu_r)^{-1} (\mathfrak{g}^\text{diag}) = \mu_{\text{Ad}}^{-1}(0) \cap T^\text{reg}G$.

The first projection $\mathfrak{g}^* \times G \to \mathfrak{g}^*$ makes $\mathfrak{g}^* \times G$ a $G$-equivariant group scheme on $\mathfrak{g}^*$ with fiber $G$. The fiber of $\mathcal{Z}^\text{reg}$ over $x \in \mathfrak{g}^\text{reg}$ equals $G_x$, which is a not necessarily connected abelian group. This makes $\mathcal{Z}^\text{reg} \to \mathfrak{g}^\text{reg}$ a smooth $G$-equivariant abelian group subscheme of the group scheme $\mathfrak{g}^\text{reg} \times G \to \mathfrak{g}^\text{reg}$.

Let $\theta : \mathfrak{g}^* \to \mathfrak{c} := \mathfrak{g}^* / \text{Ad}^* G$ be the coadjoint quotient map. The fibers of the composite $\mathfrak{g}^\text{reg} \to \mathfrak{g}^* \to \mathfrak{c}$ are the regular coadjoint orbits. It follows that the map $\mathcal{Z}^\text{reg} \to \mathfrak{c}$, $(x, x, g) \mapsto \theta(x)$, is a $G$-torsor; in particular, $\mathcal{Z}^\text{reg}$ is affine. The Kostant slice, see Section 2.2, provides a section of this $G$-torsor.

**Definition.** The **universal centralizer** is defined as $\mathfrak{z} := \mathcal{Z}^\text{reg} / G := \text{Spec}(C[\mathcal{Z}^\text{reg}]^G)$. The map $\mathcal{Z}^\text{reg} \to \mathfrak{c}$ descends to a morphism $\mathfrak{z} \to \mathfrak{c}$, making $\mathfrak{z}$ an abelian group scheme on $\mathfrak{c}$.

We see from the above that the universal centralizer may be identified with a Hamiltonian reduction of $T^\text{reg}G$ over $0 \in \mathfrak{g}^*$:

$$\mathfrak{z} \cong (\mu_{\text{Ad}}^{-1}(0) \cap T^\text{reg}G) / G =: T^\text{reg}G / (\text{Ad } G, 0). \tag{2.1.3}$$

From now on, we fix a principal $\mathfrak{sl}_2$-triple $(e, h, f)$. Let $e + \mathfrak{g}_F$ be the Kostant slice and $S \subset \mathfrak{g}^*$ the image of $e + \mathfrak{g}_F$ under the Killing form $(\cdot, \cdot)$. Further, let $\mathcal{Z}_S = \{(x, g) \in S \times G \mid \text{Ad}^* g(s) = s\}$. Using the identification $\mathcal{Z} = (\mu_t \times \mu_r)^{-1} (\mathfrak{g}^\text{diag})$, one can identify $\mathcal{Z}_S$ with the preimage of $S$ under the map $\mu |_{\mathcal{Z}} = \mu_r |_{\mathcal{Z}} : \mathcal{Z} \to \mathfrak{g}^*$, $(x, x, g) \mapsto x$. According to Kostant [Ko1, Ko3], all elements of $S$ are regular; furthermore, the composite $S \hookrightarrow \mathfrak{g}^\text{reg} \to \mathfrak{c}$ is an isomorphism. We deduce that $\mathcal{Z}_S$
is a smooth closed subvariety of $T^*G$ contained in $Z^\text{reg}$; moreover, the composite $Z_S \hookrightarrow Z^\text{reg} \twoheadrightarrow Z^\text{reg} \sslash G = \mathcal{Z}$ is an isomorphism. It follows that the group scheme $\mathcal{Z}$ is smooth, so the Hamiltonian reduction construction in (2.1.3) provides $\mathcal{Z}$ with the structure of a smooth symplectic variety.

We leave the proof of the following lemma to the interested reader.

**Lemma 2.1.4.** The restriction of the symplectic 2-form on $T^*G$ to the subvariety $Z_S$ is nondegenerate, i.e., $Z_S$ is a symplectic submanifold of $T^*G$. Furthermore, the isomorphism $Z_S \twoheadrightarrow \mathcal{Z}$ is a symplectomorphism.

### 2.2. The second construction of $\mathcal{Z}$

This construction uses a two-sided Whittaker Hamiltonian reduction as follows.

Let $n$ be the unique maximal nilpotent subalgebra of $\mathfrak{g}$ that contains the element $f$ and $N$ the corresponding unipotent subgroup. Let $\psi \in \mathfrak{g}^*$ be defined by $\psi(x) = (e, x)$. We will abuse notation and also write $\psi$ for $\psi|_n$. Thus, we have $\psi \in S \subset pr^{-1}(\psi) = \psi + n^\perp$, where $pr : \mathfrak{g}^* \to n^*$ is the natural projection. According to Kostant, $[\text{Ko3}]$, the action map $N \times S \to \psi + n^\perp$ is an isomorphism. This easily yields the following result, see $[\text{Gi} \text{ Corollary 1.3.8}]$.

**Lemma 2.2.1.** Let $X$ be a $G$-variety equipped with a $G$-equivariant map $f : X \to \mathfrak{g}^*$. Let $O_\psi$ be the localization of $\mathbb{C}[n^*]$ at $\psi \in n^*$. Then, we have:

(i) Let $\mathcal{F}$ be a $G$-equivariant quasi-coherent $O_X$-module. Then, $\mathcal{F}$ is a flat $(pr \circ f)^*O_\psi$-module. In particular, the composite map $X \xrightarrow{f} \mathfrak{g}^* \xrightarrow{pr} n^*$ is flat at $\psi \in n^*$.

(ii) The subscheme $f^{-1}(\psi + n^\perp)$ is $N$-stable. The action of $N$ on $f^{-1}(\psi + n^\perp)$ yields an $N$-equivariant isomorphism $N \times f^{-1}(S) \to f^{-1}(\psi + n^\perp)$.

Let $X$ be a smooth (not necessarily affine) symplectic variety equipped with a Hamiltonian action of a (not necessarily reductive) linear algebraic group $K$, with moment map $\mu : X \to \mathfrak{k}^*$. Let $\chi \in \mathfrak{k}^*$ be a character and assume that $\chi$ is a regular value of $\mu$ and the $K$-action on $\mu^{-1}(\chi)$ admits a slice, that is, a subvariety $S \subset \mu^{-1}(\chi)$ such that the action $K \times S \to \mu^{-1}(\chi)$ is an isomorphism of algebraic varieties. In such a case, the variety $S$ is smooth and inherits a natural symplectic structure. Furthermore, one can show that any two slices are isomorphic, and we write $X \sslash K = S$.

We will especially be interested in the case where $X = T^*G$ is viewed as a $G_l \times G_r$-variety and $f = \mu_l \times \mu_r$ is the moment map $T^*G \to \mathfrak{g}^* \oplus \mathfrak{g}^*$. In this setting, the role of the Lie algebra $n^\psi$ is played by $n^\psi_l \oplus n^\psi_r$.

**Corollary 2.2.2.** One has a natural isomorphism $\mathcal{Z} = T^*G \sslash (N_l \times N_r, \psi \times \psi)$.

**Proof.** Applying Lemma 2.2.1(ii) in the case of the Lie algebra $n^\psi_l \oplus n^\psi_r$, we compute

\[
T^*G \sslash (N_l \times N_r, \psi \times \psi) = (\mu^{-1}_l(\psi) \cap \mu^{-1}_r(\psi))/N_l \times N_r
\]

\[
= (\mu^{-1}_l(\psi + n^\perp) \cap \mu^{-1}_r(\psi + n^\perp))/N_l \times N_r
\]

\[
= \mu^{-1}_l(S) \cap \mu^{-1}_r(S) = \{(x, g) \in S \times G \mid Ad^* g(x) \in S\}
\]

\[
= \{(x, g) \in S \times G \mid Ad^* g(x) = x\} = \mathcal{Z}.
\]

### 2.3. The third construction of $\mathcal{Z}$

This construction, due to $[\text{BFM}]$, is via affine blow-ups.

Let $B$ be the flag variety of all Borel subalgebras of $\mathfrak{g}$. Let $\tilde{g} := \{(x, b) \in \mathfrak{g}^* \times B \mid x \in b^\perp\}$. The first projection $\pi : \tilde{g} \to \mathfrak{g}$, $(x, b) \mapsto x$, is the Grothendieck-Springer resolution. Let $t$ be the universal Cartan, $T$ the universal Cartan torus, $W$ the Weyl group, and $t^* \to t^*/W \cong \mathfrak{g}^* / Ad^* G = \mathcal{E}$ the quotient. It is known that the following map is an isomorphism:

\[
\tilde{g}^\text{reg} := \{(x, b) \in \tilde{g} \mid x \in \mathfrak{g}^\text{reg}\} \twoheadrightarrow \mathfrak{g}^\text{reg} \times \mathfrak{e}, \quad (x, b) \mapsto (x, x \bmod b^\perp).
\]
Beilinson and Kazhdan [BK] constructed a natural map 
\[ \varphi : \mathfrak{z} \times \mathfrak{t}^* \longrightarrow T \times \mathfrak{t}^*, \] 
(2.3.2)
of \(W\)-equivariant group schemes on \(t^*\), where \(W\) acts naturally on \(T\), resp. \(t^*\), diagonally on \(T \times t^*\), and trivially on \(\mathfrak{z}\). The construction is as follows.

First, it was shown by Kostant that for any \((x, b) \in \mathfrak{g}^{\text{reg}}\), one has an inclusion \(G_x \subset B\), where \(B\) stands for the Borel subgroup with Lie algebra \(b\). Therefore, there is a well-defined map 
\[ G_x \hookrightarrow B \twoheadrightarrow B/[B, B] = T, \quad g \mapsto g \mod [B, B]. \]
Hence, the assignment \((g, x, b) \mapsto (g \mod [B, B], x, b)\) gives a morphism \(Z^{\text{reg}} \times \mathfrak{g}^{\text{reg}} \rightarrow T \times \mathfrak{g}^{\text{reg}}\), of group schemes on \(\mathfrak{g}^{\text{reg}}\). Using Proposition 2.3.3, we obtain a morphism 
\[ Z^{\text{reg}} \times \mathfrak{g}^{\text{reg}} \rightarrow Z^{\text{reg}} \times \mathfrak{g}^{\text{reg}} (\mathfrak{g}^{\text{reg}} \times \mathfrak{t}^*) = Z^{\text{reg}} \times \mathfrak{t}^* \rightarrow T \times \mathfrak{g}^{\text{reg}} = T \times (\mathfrak{g}^{\text{reg}} \times \mathfrak{t}^*), \]
of \(W \times G\)-equivariant group schemes on \(\mathfrak{g}^{\text{reg}}\). Taking categorical quotients by \(G\) on each side yields the following map, which is the required map \(\varphi\) in (2.3.2):
\[ 3 \times \mathfrak{t}^* = Z^{\text{reg}} \rightarrow G \times \mathfrak{t}^* = (Z^{\text{reg}} \times \mathfrak{g}^{\text{reg}} \times \mathfrak{t}^*) / G \rightarrow (T \times \mathfrak{g}^{\text{reg}} \times \mathfrak{t}^*) / G = T \times \mathfrak{t}^*. \]

The graph of \(\varphi\) gives the following variety:
\[ \Lambda = \{ (z, x, t) \in 3 \times \mathfrak{t}^* \times T \mid \varpi(z) = x \mod W, \quad \varphi(z, x) = (t, x) \}, \]
where \(\varpi : 3 \rightarrow \mathfrak{t}^*\) is the canonical map. We identify \(\mathfrak{t}^* \times T\) with \(T^* T\).

The proof of the following result is left to the reader.

**Proposition 2.3.3.** (i) \(\Lambda\) is a smooth closed Lagrangian subvariety of \(3 \times T^* T\).

(ii) For any \(t \in T\), every irreducible component of the fiber, \(pr_T^{-1}(t)\), of the map \(pr_T : \Lambda \rightarrow T, (z, x, t) \mapsto t\) is a (possibly singular) Lagrangian subvariety of \(3\).

The variety \(\Lambda\) is a classical counterpart of the Miura bimodule, to be introduced in Section 6.2 below.

Let \(R \subset \mathbb{X}^*\) be the set of roots and the weight lattice of \(G\), respectively.

**Example 2.3.4.** Let \(\Sigma \subset R\) be the set of simple roots and \(S \subset \Sigma\) a subset. Let \(L_S\) be an associated standard Levi subgroup of \(G\) and \(L_S^{\text{der}} = [L_S, L_S]\), the derived group of \(L_S\). Let \(e_S\) be a principal nilpotent of the Levi subalgebra \(\text{Lie} L_S\) and \(Z(L_S^{\text{der}}, e_S)\) its centralizer in \(L_S^{\text{der}}\). Finally, put
\[ \mathfrak{t}_S^{\text{reg}} := \{ \lambda \in \mathfrak{t}^* \mid (\lambda, \tilde{\alpha}) = 0 \& (\lambda, \tilde{\beta}) \neq 0, \forall \alpha \in S, \beta \in \Sigma \setminus S \}. \]
Then, one has a decomposition
\[ pr_T^{-1}(1) \cong \bigcup_{S \subset \Sigma} \mathfrak{t}_S^{\text{reg}} \times Z(L_S^{\text{der}}, e_S), \]
where each piece is isomorphic to a union of irreducible components of \(pr_T^{-1}(1)\) permuted transitively by the center of \(G\).

Next, identify \(\text{Sym} t\) with \(\mathbb{C}[t^*]\). Following [BFM], let \(\mathbb{C}[T \times t^*, \frac{t^{\alpha-1}}{\alpha}, \alpha \in R]\) be an algebra obtained from \(\mathbb{C}[T \times t^*]\) by adjoining all rational functions \(\frac{t^{\alpha-1}}{\alpha}, \alpha \in R\), where \(t^\lambda\) stands for an element \(\lambda \in \mathbb{X}^*\) viewed as a regular function on \(T\).

**Theorem 2.3.5** ([BFM], Proposition 2.8). The algebra map \(\varphi^* : \mathbb{C}[T \times t^*] \rightarrow \mathbb{C}[3 \times t^*]\), induced by the morphism \(\varphi\) in (2.3.2), extends to a \(W\)-equivariant algebra isomorphism
\[ \mathbb{C}[T \times t^*, \frac{t^{\alpha-1}}{\alpha}, \alpha \in R] \cong \mathbb{C}[3 \times t^*]. \]

Thus, we have
\[ 3 \cong \left( \text{Spec} \mathbb{C}[T \times t^*, \frac{t^{\alpha-1}}{\alpha}, \alpha \in R] \right) / W. \]
An important step in the proof of the theorem is played by the following result, see [BFM Section 4].

**Proposition 2.3.6.** The algebra $\mathbb{C}[T \times \mathfrak{t}^*, \frac{t^\alpha - 1}{\alpha}]$, $\alpha \in R^1 W$ is flat over $\mathbb{C}[\mathfrak{t}^*] W$.

A simple proof of this proposition based on nil Hecke algebras will be given in Section 7.1 below.

3. REVIEW OF QUANTUM HAMILTONIAN REDUCTION AND KAZHDAN FILTRATIONS

3.1. **Quantum Hamiltonian reduction.** In this subsection we collect some general, mostly standard results on quantum Hamiltonian reduction.

Given a Lie algebra $\mathfrak{t}$, we write $E/E\mathfrak{t}$, resp. $\mathfrak{t}E\setminus E$ or $E/\mathfrak{t}E$, for the space of coinvariants of a right, resp. left, $\mathfrak{t}$-module $E$. We write $E^\psi$ for the space of invariants.

Let $\mathfrak{b}$, resp. $\bar{\mathfrak{b}}$, be the unique Borel subalgebra of $\mathfrak{g}$ such that $\mathfrak{f} \in \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, resp. $\mathfrak{e} \in \bar{\mathfrak{n}} = [\mathfrak{b}, \bar{\mathfrak{b}}]$. We get a triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \bar{\mathfrak{n}}$, where we identify $\mathfrak{t}$ with $\mathfrak{b} \cap \bar{\mathfrak{b}}$. We have algebra embeddings $U\mathfrak{n} \hookrightarrow U\mathfrak{b} \hookrightarrow U\mathfrak{g}$. It is clear that $\mathfrak{n}(U\mathfrak{b}) = (U\mathfrak{b})\mathfrak{n}$ is a two-sided ideal of $U\mathfrak{b}$ and we have $U\mathfrak{b}/(U\mathfrak{b})\mathfrak{n} \cong U\mathfrak{t}$. Let $Z \mathfrak{g}$ be the center of $U\mathfrak{g}$ and $Q := U\mathfrak{g}/(U\mathfrak{g})\mathfrak{n}^\psi$.

The following result is well-known, cf. [Ko3].

**Proposition 3.1.1.** (i) The embedding $Z \mathfrak{g} \hookrightarrow U\mathfrak{g}$ induces an algebra isomorphism $Z \mathfrak{g} \cong (U\mathfrak{g}/\mathfrak{n}^\psi)^{\text{ad}} \mathfrak{n}$.

(ii) $Q$ is free as a left $(U\mathfrak{n} \otimes Z \mathfrak{g})$-module.

(iii) The embedding $U\mathfrak{t} \hookrightarrow U\mathfrak{g}$ induces an isomorphism $U\mathfrak{t} \cong \mathfrak{n}Q/Q$ of left $U\mathfrak{t}$-modules.

(iv) For all $i > 0$, we have $H^i(\mathfrak{n}^\psi, Q) = 0$; furthermore, $H^0(\mathfrak{n}^\psi, Q) = Q^\psi \cong Z \mathfrak{g}$, by (i).

We will also use

**Proposition 3.1.2.** Let $M$ be an $(U\mathfrak{g}, U\mathfrak{g})$-bimodule such that the adjoint $\mathfrak{g}$-action on $M$ is locally finite. Then, for any $j > 0$, we have $H_j(\mathfrak{n}^\psi, M) = 0$, where we view $M$ as an $\mathfrak{n}^\psi$-module via the right action.

**Proof.** Let $M$ be as in the statement and assume, in addition, that $M$ is finitely generated as a left, equivalently right, $U\mathfrak{g}$-module. In this case, the required statement follows from [Gi] Lemma 4.4.1. More precisely, the lemma was stated under an additional assumption that the center of $U\mathfrak{g}$ acts on $M$ by scalars both on the left and on the right. However, this additional assumption was not used in the proof of the lemma.

In the general case, we can find a family of sub $(U\mathfrak{g}, U\mathfrak{g})$-bimodules $M_\alpha \subseteq M$, such that each $M_\alpha$ is finitely generated as a left $U\mathfrak{g}$-module and we have $M = \lim_{\to} M_\alpha$. The functor $H_*(\mathfrak{n}^\psi, -) = \text{Tor}_*^{U\mathfrak{g}}(\mathfrak{n}^\psi, -)$ commutes with direct limits. Hence, we have $H_*(\mathfrak{n}^\psi, M) = \lim_{\to} H_*(\mathfrak{n}^\psi, M_\alpha)$. This yields the proposition by the first paragraph of the proof.

From now on, we let $A$ be a finitely generated left noetherian associative algebra and $U\mathfrak{g} \rightarrow A$ an algebra homomorphism such that

$$\text{the adjoint action } \text{ad } \chi : a \mapsto xa - ax, \text{ of } \mathfrak{g} \text{ on } A, \text{ is locally finite}. \quad (3.1.3)$$

Let $\mathfrak{t}$ be a nilpotent Lie subalgebra of $U\mathfrak{g}$, such as $\mathfrak{n}, \mathfrak{n}^\psi$, or $\bar{\mathfrak{n}}$. We write $(A, \mathfrak{t})$-mod for a full subcategory of the category of $A$-modules such that the induced $\mathfrak{t}$-action on the module is locally nilpotent. Condition (3.1.3) implies that the adjoint action of $\mathfrak{t}$ on $A$ is locally nilpotent. In particular, one has $A/\mathfrak{t}A \in (A, \mathfrak{t})$-mod.

The space $A/\mathfrak{t} := (A/\mathfrak{t}A)^{\text{ad}} \mathfrak{t}$, resp. $\mathfrak{t} \setminus A := (A/\mathfrak{t}A)^{\text{ad}} \mathfrak{t}$, acquires an algebra structure. One has an algebra isomorphism $(A/\mathfrak{t})^{\text{op}} \cong \mathfrak{t}^{\text{op}} \setminus A^{\text{op}}$. A left, resp. right, $A$-module structure on $E$ induces a natural left, resp. right, $A/\mathfrak{t}$-module structure on $E/\mathfrak{t}$, resp. $E/\mathfrak{t}E$. More generally, for each $j \geq 0$, the cohomology group $H^j(\mathfrak{t}, E)$, resp. homology group $H_j(\mathfrak{t}, E)$, has the structure of a left, resp.
right, $A \mathcal{M}$-module. Similarly, there is a left, resp. right, $A \mathcal{M}$-module structure on $H_\ast \mathcal{M}(\psi, E)$, resp. $H^\ast \mathcal{M}(\psi, E)$. In particular, $A/\mathcal{M}$ is an $A \mathcal{M}$-$A$-bimodule, resp. $A \mathcal{M}$-$A$-bimodule.

In this Section, we will only consider the case $\mathcal{M} = n^\psi$ and put $\mathfrak{A} = A \mathcal{M}$. The composite map $Zg \hookrightarrow Ug \to A$ descends to an algebra homomorphism $Zg \to \mathfrak{A}$. The map $Ug \to A$ induces a morphism $Q \to A/An^\psi$, of $(Ug, Zg)$-bimodules, and also a morphism $Ut = \bar{n}(Ug)\mathcal{M}/Ug)n^\psi \to \bar{n}A/A(\mathfrak{A})$, of $(Ut, Zg)$-bimodules.

Below, we work with left-modules; similar results hold for right modules.

**Proposition 3.1.4.** (i) For any $M \in (A, n^\psi)$-mod and $i \neq 0$, we have $H^i(n^\psi, M) = 0$. Furthermore, the functor $(A, n^\psi)$-mod $\to \mathfrak{A}$-mod, $M \mapsto M^{n^\psi}$ is an equivalence and $L \mapsto A/An^\psi \otimes_{\mathfrak{A}} L$ is its quasi-inverse.

(ii) For $L \in \mathfrak{A}$-mod and all $i \neq 0$, we have $H_i(\bar{n}, A/An^\psi \otimes_{\mathfrak{A}} L) = 0$; furthermore, there are natural isomorphisms $H_0(\bar{n}, A/An^\psi \otimes_{\mathfrak{A}} L) \cong (\bar{n}A/A(\mathfrak{A}) \otimes_{\mathfrak{A}} L \cong Ut \otimes_{Zg} L$.

(iii) The algebra $\mathfrak{A}$ is left noetherian.

(iv) For all $i \neq 0, we have $H_i(n^\psi, A) = 0$, where $n^\psi$ acts on $A$ by right multiplication.

**Proof.** Put $Z = Zg$. Let $L$ be a $Z$-module. View $Q \otimes Z L$ as an $n^\psi$-module and let $C^\ast(n^\psi, Q \otimes Z L)$ be the corresponding cohomological Chevalley-Eilenberg complex. Thus, we have

$$C^\ast(n^\psi, Q \otimes Z L) = (\wedge^\ast n^\psi)^* \otimes (Q \otimes Z L) = ((\wedge^\ast n^\psi)^* \otimes Q) \otimes Z L.$$

Assume first that the $Z$-module $L$ is flat. Then, we find

$$H^i(n^\psi, Q \otimes Z L) = H^i(C^\ast(n^\psi, Q \otimes Z L)) = H^i(((\wedge^\ast n^\psi)^* \otimes Q) \otimes Z L) = H^i(n^\psi, Q) \otimes Z L.$$

Proposition 3.1.1(iv) says that the cohomology group on the right equals $L$ if $i = 0$, resp. vanishes if $i > 0$. Next, let $L$ be an arbitrary, not necessarily flat, $Z$-module and choose a resolution $E \mathcal{M}$ of $L$, by flat $Z$-modules. Since $Q$ is free over $Z$, see Proposition 3.1.1(ii), we deduce that the complex $Q \otimes Z E \mathcal{M}$ is a resolution of $Q \otimes Z L$. Since all $E \mathcal{M}$ are flat, each term of the latter resolution is acyclic with respect to the functor $(-)^{n^\psi}$, by the above. It follows that one has

$$H^i(n^\psi, Q \otimes Z L) = H^i((Q \otimes Z E \mathcal{M})^{n^\psi}) = H^i(E \mathcal{M}^\ast), \text{ where } H^0(E \mathcal{M}^\ast) = L, \ H^i(E \mathcal{M}^\ast) = 0 \ \forall i > 0.$$

This implies part (i) in the special case $A = Ug$, cf. also [GG, Section 6.2].

To complete the proof of (i) in the general case, we view an object $M \in (A, n^\psi)$-mod as an object of $(Ug, n^\psi)$-mod. This yields the cohomology vanishing of part (i) and shows that the functor $M \mapsto M^{n^\psi}$ restricts to an exact functor $(A, n^\psi)$-mod $\to \mathfrak{A}$-mod. Further, applying the equivalence in the special case above to $M = A/An^\psi$, viewed as an object of $(Ug, n^\psi)$-mod, and using that $(A/An^\psi)^{n^\psi} = \mathfrak{A}$, we deduce that the natural map $Q \otimes Z \mathfrak{A} \to A/An^\psi$ is an isomorphism. This yields isomorphisms of functors

$$A/An^\psi \otimes_{\mathfrak{A}} (-) = (Q \otimes Z \mathfrak{A}) \otimes_{\mathfrak{A}} (-) = Q \otimes Z (-). \quad (3.1.5)$$

Using the special case $A = Ug$, we deduce that the map $L \to (Q \otimes Z L)^{n^\psi} = (A/An^\psi \otimes_{\mathfrak{A}} L)^{n^\psi}$ is an isomorphism, for any $L \in \mathfrak{A}$-mod. It follows that the functors in (i) are quasi-inverse to each other. Further, since $Q$ is free over $U\mathfrak{A} \otimes Z$, the isomorphism $Q \otimes Z \mathfrak{A} \cong A/An^\psi$ implies that $A/An^\psi$ is free as a left $(U\mathfrak{A} \otimes \mathfrak{A})$-module. Therefore, one has $H_i(\bar{n}, A/An^\psi \otimes_{\mathfrak{A}} L) = H_i(\bar{n}, Q) \otimes Z L$, for any free $\mathfrak{A}$-module $L$. Moreover, the group on the right vanishes for all $i \neq 0$, and we have $H_0(\bar{n}, Q) \otimes Z L = \bar{n}Q \otimes Z L \cong Ut \otimes Z L$. The case of an arbitrary $\mathfrak{A}$-module $L$ then follows by considering a resolution of $L$ by free $\mathfrak{A}$-modules, similar to the proof of (i). This proves (ii).

Now, let $L$ be a finitely generated left $\mathfrak{A}$-module. Then $A/An^\psi \otimes_{\mathfrak{A}} L$ is a finitely generated $A$-module, hence it is noetherian. The equivalence of part (i) implies that $L$ is also noetherian, proving that $\mathfrak{A}$ is left Noetherian. Finally, view $A$ as a $(Ug, Ug)$-bimodule via left and right multiplication. This bimodule satisfies the assumptions of Proposition 3.1.2 by 3.1.3. Thus, part (iv) follows from Proposition 3.1.2.
Remark 3.1.6. (i) Associated with the homomorphism $n^\psi \to A$, there is a BRST complex $D\mathfrak{A} := \wedge^*(n^\psi \oplus (n^\psi)^*) \otimes A$ that has the natural structure of a dg algebra. Using that the groups $H_i(n^\psi, A)$ and $H^i(n^\psi, A/An^\psi)$ vanish for all $i \neq 0$, see parts (i), (iv) of Proposition 3.1.4 it is immediate to deduce that $H^0(D\mathfrak{A}) \cong \mathfrak{A}$ and $H^i(D\mathfrak{A}) = 0$, for all $i \neq 0$.

(ii) Assume that the algebra $A$ has finite global dimension, i.e., there is an integer $n = n(A) > 0$ such that one has $\text{Ext}_A^{n+1}(M, M') = 0$, for all left $A$-modules $M, M'$. Then, for all left $\mathfrak{A}$-modules $L, L'$, by Proposition 3.1.4 we get $\text{Ext}_A^{n}(L, L') = \text{Ext}_A^{n}(A/An^\psi \otimes_{\mathfrak{A}} L, A/An^\psi \otimes_{\mathfrak{A}} L') = 0$. Hence, the algebra $\mathfrak{A}$ has finite global dimension. This can also be deduced from (i).

Let $D^b(A, n^\psi)$-mod be a full triangulated subcategory of $D^b(A$-mod) whose objects $M^\ast$ have the property that $H^i(M^\ast) \in (A, n^\psi)$-mod, $\forall i$. The proof of Proposition 3.1.4 can be easily upgraded to show that the functor $\wedge^*(n^\psi \oplus (n^\psi)^*) \otimes (-)$ yields a triangulated equivalence of $D^b(A, n^\psi)$-mod and the bounded derived category of dg $D\mathfrak{A}$-modules. The latter category is triangulated equivalent to $D^b(\mathfrak{A}$-mod), by (i).

3.2. Kazhdan filtrations. Several proofs in the paper will involve a ‘filtered-to-graded’ reduction. In this subsection we introduce the necessary definitions and notation.

Let $E$ be a vector space equipped with a $\mathbb{Z}$-grading $E = \bigoplus_{\ell} E(\ell)$ and an ascending $\mathbb{Z}$-filtration $E_\leq$ by graded subspaces $E_{\leq j} = \bigoplus_{\ell} E_{\leq j}(\ell), j \in \mathbb{Z}$. The Kazhdan filtration associated with the given grading and filtration is an ascending $\mathbb{Z}$-filtration $F, E$ defined as follows:

$$F_n E := \bigoplus_{\ell \in \mathbb{Z}} F_n E(\ell), \quad F_n E(\ell) := E_{\leq (n-\ell)/2}(\ell).$$

If $E$ is an algebra and the $\mathbb{Z}$-grading and filtration $E_\leq$ are compatible with the algebra structure, then so is the Kazhdan filtration.

The adjoint action on $\mathfrak{g}$ of the semisimple element $h$, of the fixed $sl_2$-triple, gives a $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}(\ell) \quad \text{where} \quad \mathfrak{g}(\ell) = \{ x \in \mathfrak{g} \mid [h, x] = \ell \cdot x \},$$

by even integers. The $\mathbb{Z}$-grading on $\mathfrak{g}$ induces one on $U\mathfrak{g}$. We equip $U\mathfrak{g}$ with the Kazhdan filtration, $F, U\mathfrak{g}$, associated with that $\mathbb{Z}$-grading and the standard PBW filtration on $U\mathfrak{g}$. The Kazhdan filtration will be our default filtration on $U\mathfrak{g}$. The quotient filtration $F, Q$, on $Q = U\mathfrak{g}/(U\mathfrak{g})n^\psi$ has no negative terms, i.e., $F_{-1}Q = 0$. We write $U\mathfrak{g}, Z\mathfrak{g}, Q_h, \mathfrak{g}_h$, etc., for the corresponding Rees objects. Note that the Kazhdan and PBW filtrations on $U\mathfrak{g}$ restrict to the same filtration on $Z\mathfrak{g}$.

The algebra $U_h\mathfrak{g}$ may be identified, in a natural way with a $\mathbb{C}[h]$-algebra generated by a copy of the vector space $\mathfrak{g}$ such that the space $\mathfrak{g}(\ell) \subset \mathfrak{g}, \ell \in \mathbb{Z}$, is placed in degree $\ell + 2$, with defining relations $x \otimes y - y \otimes x = h[x, y], \forall x, y \in \mathfrak{g}$. With this identification, the space $n^\psi = \{ n - \psi(n), n \in \mathfrak{g} \}$ becomes a graded Lie subalgebra of $U_h\mathfrak{g}$; moreover, one has a natural graded algebra isomorphism $U_hn = U(n^\psi)[h]$. Furthermore, the $U_h\mathfrak{g}$-module $Q_h$ may be identified with $U_h\mathfrak{g}/(U_h\mathfrak{g})n^\psi$.

Proposition 3.2.2. (1) Analogues of the statements (ii)–(iv) of Proposition 3.1.1 hold with $U\mathfrak{g}$, resp. $Z\mathfrak{g}$, and $Q$, being replaced by $gr U\mathfrak{g}$, resp. $gr Z\mathfrak{g}$ and $gr Q$.

(2) The functors $M \mapsto M^{n^\psi}$ and $L \mapsto U_h\mathfrak{g}/(U_h\mathfrak{g})n^\psi \otimes_{Z\mathfrak{g}} L$ give quasi-inverse equivalences $(U_h, n^\psi)$-grmod $\cong (Z\mathfrak{g})$-grmod of the corresponding categories of $\mathbb{Z}$-graded $\mathbb{C}[h]$-modules.

(3) Let $M \in (U\mathfrak{g}, n^\psi)$-mod and let $F, M$ be a $\mathbb{Z}$-filtration which is compatible with the Kazhdan filtration on $U\mathfrak{g}$ and such that each of the spaces $F_i M, i \in \mathbb{Z}$ is $n$-stable. Equip $M^{n^\psi}$ with the filtration induced from the one on $M$ by restriction. Then the natural map $gr Q \otimes_{gr Z\mathfrak{g}} gr(M^{n^\psi}) \to gr M$, resp. $gr(M^{n^\psi}) \to (gr M)^{n^\psi}$, is an isomorphism.

(4) Let $M$ be an $(U\mathfrak{g}, U\mathfrak{g})$-bimodule equipped with a $\mathbb{Z}$-filtration $F, M$ which is compatible with the Kazhdan filtration of $U\mathfrak{g} \otimes U\mathfrak{g}$. Assume that the ad $\mathfrak{g}$-action on $M$ is locally finite and each of the spaces $F_i M$ is ad $\mathfrak{g}$-stable. Let $\mathbb{C}_\psi$ denote a 1-dimensional $U(n^\psi)[h]$-module such that the vector space $n^\psi$ and the
element $h$ kill $\mathbb{C}_\psi$. Then $\text{Tor}_j^{\mathcal{U}(n^\psi)|[h]}(\mathbb{C}_\psi, M_h) = 0$ for all $j > 0$, where the algebra $\mathcal{U}(n^\psi)|[h] = \mathcal{U}_h n \subset \mathcal{U}_h g$ acts on $M_h$ on the right.

Proof. Part (1) is a direct consequence of results of Kostant, [Ko3]. Given (1), the proof of (2) is completely analogous to the proof of Proposition 3.1.4(i) in the case $A = U g$. Now, let $M_h$ be as in (3) and let $M_h$ be the corresponding Rees $U_{h g}$-module. It follows from (2) that the natural map $Q_h \otimes Z_\mathbb{C}_h ((M_h)^{n^\psi}) \rightarrow M_h$ is an isomorphism. The functor $\mathbb{C}[h]/(h) \otimes_{\mathbb{C}[h]} (\mathbb{C}[h]/(h)) = \mathbb{C}$.

As in Proposition 3.1.4(i), we will assume that the Kazhdan filtration is canonically isomorphic as a $\mathbb{C}$-algebra (but not as a graded algebra), to the Rees algebra associated with the filtration $A_{\leq}$. We identify $\mathbb{C}$-algebra (but not as a graded algebra), to the Rees algebra associated with the filtration $A_{\leq}$, one has $\cap_{i \in \mathbb{Z}} h^i A_h = 0$. It follows that the Kazhdan filtration on $A$ is separating.

We identify $n^\psi$ with a graded Lie subalgebra of $\mathcal{U}(n^\psi)|[h] = \mathcal{U}_h n$ as in Proposition 3.1.4(4). It follows from definitions that there is a natural graded space, resp. graded algebra, isomorphism

$$((A_h/A_h n^\psi)|_{h=0} \cong \text{gr}(A)/(\text{gr}(A)n^\psi), \text{ resp. } (A_h/\mathbb{F} n^\psi)|_{h=0} \cong (\text{gr}(A)/\mathbb{F}n^\psi).$$

(3.3.1)

Lemma 3.3.2. Assume that $\text{gr}_{\leq} A$ is a finitely generated commutative algebra. Then, $A_h$ is a finitely generated left noetherian algebra and $A_h/A_h n^\psi$ is flat over $\mathbb{C}[h]$.

Assume, in addition, that the Kazhdan filtration on $A/An^\psi$ is separating. Then, we have:

1. The Kazhdan filtration on $\mathfrak{A}$ is separating and there are natural graded algebra isomorphisms $A_h/\mathbb{F} n^\psi \cong \mathfrak{A}_h$, resp. $\text{gr}(A)/\mathbb{F}n^\psi \cong \text{gr} \mathfrak{A}$.

2. For any finitely generated module $M \in (A, n^\psi)$-mod, one has

$$\dim \text{SS}(M) - \dim \text{SS}(M^{n^\psi}) = \frac{1}{2}(\dim \text{Spec}(\text{gr}(A)) - \dim \text{Spec}(\text{gr} \mathfrak{A})).$$

Proof. The algebra $\text{gr}_{\leq} A$ being finitely generated and commutative, it follows easily that the Rees algebra of $A$ associated with the filtration $A_{\leq}$, hence also $A_h$, is finitely generated and left noetherian. To prove that $A/An^\psi$ is flat over $\mathbb{C}[h]$ we apply Proposition 3.1.4(4) in the case $M = A$. We
deduce that \( \text{Tor}^C_{j}([h]/(h), A_{h}/A_{h}n^\psi) = 0 \) for all \( j > 0 \). It follows that \( A_{h}/A_{h}n^\psi \) is \( h \)-torsion free. Since \( C[h, h^{-1}] \otimes_{C[h]} A_{h}/A_{h}n^\psi \cong \mathbb{C}[h, h^{-1}] \otimes_{\mathbb{C}} A/An^\psi \), we deduce that \( A_{h}/A_{h}n^\psi \) is flat over \( \mathbb{C}[h] \).

To prove statement (1), we let \( Un^\psi \) act on \( A \) by right multiplication. The filtrations on \( A \) and \( n^\psi \) make the corresponding Chevalley-Eilenberg complex \( C_{\ast}(n^\psi, A) \) a filtered complex. An associated graded complex, \( gr C(n^\psi, A) \), may be identified with \( C_{\ast}(n^\psi, gr A) \), the Koszul complex of \( gr A \). Here, we view \( gr A \) as a \( \mathbb{C}[[n^\psi]] \)-module via the map \( gr Un^\psi \to gr A \) induced by the homomorphisms \( Un^\psi \to U_{\ast} \to A_\ast \). There is a standard spectral sequence for homology:

\[
E_1 = H(gr C_{\ast}(n^\psi, A)) \Rightarrow E_\infty = gr H(C_{\ast}(n^\psi, A)). \tag{3.3.3}
\]

The locally finite \( ad_{\mathfrak{g}} \)-action on \( A \) can be exponentiated to a \( G \)-action. Therefore, the map \( gr U_{\mathfrak{g}} \to gr A \), induced by \( U_{\mathfrak{g}} \to A \), gives a \( G \)-equivariant morphism \( \text{Spec gr} A \to \text{Spec}(gr U_{\mathfrak{g}}) = g^\ast \).

From Lemma [2.2.1], we deduce that \( \text{Spec gr} A \) is flat over \( O_{\psi} \). It follows that the homology groups \( H_j(C_{\ast}(n^\psi, gr A)) \) vanish for all \( i \neq 0 \). Hence, our spectral sequence degenerates at the \( E_1 \)-term.

Next, we claim that the above spectral sequence is convergent. This is not immediately clear since the Kazhdan filtration on \( A \) is not necessarily bounded below. To prove convergence, observe first that the action of the algebra \( A \) on itself by left multiplication makes \( C_{\ast}(n^\psi, A) \) a complex of finitely generated left \( A \)-modules. The Chevalley-Eilenberg differential respects the \( A \)-module structure. We deduce, as in the proof of [6], Theorem 4.14(i), that in the spectral sequence (3.3.3), one has \( Z_\infty = \cap_{r \geq 0} Z_r^\ast = 0, \forall p \), i.e., the first equation in formula (4.3.5) from [6] holds. Further, we know that \( H_j(C_{\ast}(n^\psi, gr A)) = 0, \forall j > 0 \), see Proposition 5.1.4(iv). This, combined with the assumption that the filtration on \( A/An^\psi \) is separating implies that the second equation in [6], formula (4.3.5), holds. Thus, the convergence criterion stated in [6] Lemma 4.3.7 applies in our present setting. It follows that the spectral sequence in (3.3.3) does converge. By [6] Lemma 4.3.3, we deduce that the canonical map \( gr A/(gr A)n^\psi \to gr(A/An^\psi) \) is an isomorphism. Explicitly, this means that the following surjective map is an isomorphism:

\[
F_{n}A/\sum_{i}(F_{n-1-i}A)n_{i}^\psi \to (F_{n}A + An^\psi)/(F_{n-1}A + An^\psi), \quad \forall n \in \mathbb{Z}. \tag{3.3.4}
\]

Here, \( n_{i}^\psi \) stands for the \( i \)-th homogeneous component of \( n_{i}^\psi U(n_{i}^\psi)[h] = U_{h}n_{i}^\psi \), i.e., \( n_{i}^\psi \) is a copy of the vector space \( n \cap g(i + 2) \) if \( i < -2 \) and \( n_{0}^\psi = \{ n - \psi(n), n \in n \cap g(-2) \} \).

The isomorphisms in (1) are immediate consequences of (3.3.4) and Proposition [3.2.2](3) applied to \( M = A/An^\psi \). Also, the assumption of the proposition clearly implies that the filtration \( F_{\mathfrak{g}} \mathfrak{f} \) is separating.

Now, let \( M \) be as in (2). Choose a finite-dimensional \( n^\psi \)-stable subspace \( M_0 \subset M \), such that \( M = AM_0 \). The filtration on \( M \) defined by \( F_iM := (F_iA)M_0 \), \( i \in \mathbb{Z} \), satisfies the assumptions of Proposition [3.2.2](3). Hence, we have \( gr M \cong gr Q \otimes_{gr z_{\emptyset}} gr(M^{n^\psi}) \). Using that \( gr Q \cong C[\psi + n^1] \cong C[N \times S] \) and \( S \cong \mathfrak{c} \cong \text{Spec}(gr Z_{\emptyset}) \), we compute

\[
\dim \text{Supp}(gr Q \otimes_{gr z_{\emptyset}} gr(M^{n^\psi})) = \dim \text{Supp} gr Q - \dim \mathfrak{c} + \dim \text{Supp} gr(M^{n^\psi}) = \dim N + \dim \text{Supp} gr(M^{n^\psi}).
\]

In particular, for \( M = A/An^\psi \), we get \( \dim \text{Supp}((gr A/(gr A)n^\psi)) = \dim N + \dim \text{Spec} gr \mathfrak{f} \). On the other hand, since \( \text{Spec gr} A \) is flat over \( O_{\psi} \), using part (ii) of Lemma [2.2.1], we get

\[
\dim \text{Supp}((gr A/(gr A)n^\psi)) = \dim \text{Spec} gr A - \dim n^\psi.
\]

The required equation follows. \( \square \)
4. Proofs of Theorem 1.2.2 and Theorem 1.6.3

4.1. Whittaker coinvariants of \( \mathcal{D} \)-modules. Let \( X \) be a smooth \( G \)-variety. The algebra of differential operators comes equipped with an algebra homomorphism \( \mathcal{U}_G \to \mathcal{D}(X) \). The homomorphism intertwines the PBW filtration on \( \mathcal{U}_G \) with the natural filtration \( \mathcal{D}(X) \) by order of the differential operator. Following Section 3.3, we get a \( \mathbb{Z} \)-grading on \( \mathcal{D}(X) \) and an associated Kazhdan filtration \( F, \mathcal{D}(X) \).

Remark 4.1.1. Let \( \gamma : \mathbb{G}_m \to T \) be a 1-parameter subgroup generated by the element \( h \) of our fixed \( \mathfrak{sl}_2 \)-triple. We let the group \( \mathbb{G}_m \) act on \( X \) by \( z \mapsto \gamma(z)x \). The induced \( \mathbb{G}_m \)-action on \( \mathcal{D}(X) \) by algebra automorphisms gives a \( \mathbb{Z} \)-grading on \( \mathcal{D}(X) \) that agrees with the \( \mathbb{Z} \)-grading on \( \mathcal{D}(X) \) considered above.

Let \( K \) be a connected algebraic subgroup of \( G \) and \( \chi : \mathfrak{t} = \text{Lie} K \to \mathbb{C} \) a character. We have \( (\mathfrak{t}^\chi)^{\text{op}} \cong \mathfrak{t}^{-\chi}. \) The Lie algebra map \( \mathfrak{t}^\chi \to \mathcal{D}(X) \) gives a Lie algebra map \( \mathfrak{t}^{-\chi} \to \mathcal{D}(X)^{\text{op}}. \) Assume, in addition, that \( X \) is affine and the quotient \( X/\mathbb{K} \) is a \( K \)-torsor. Then, there is a canonical isomorphism \( \mathcal{D}(X/\mathbb{K}) \cong \mathcal{D}(X) \otimes \mathcal{D} \). In the special case where \( K \) is unimodular, e.g., unipotent, and the canonical bundle on \( X \) has a \( K \)-invariant trivialization, we get \( \mathcal{D}(X) \cong \mathcal{D}(X)^{\text{op}} \). We deduce an isomorphism \( \mathcal{D}(X) \otimes \mathfrak{t} \cong (\mathcal{D}(X) \otimes \mathfrak{t})^{\text{op}}. \) More generally, with the same assumptions one can construct a natural isomorphism \( \mathcal{D}(X) \otimes \mathfrak{t}^{-\chi} \cong (\mathcal{D}(X) \otimes \mathfrak{t})^{\text{op}}. \)

On the other hand, by general properties of Hamiltonian reduction there is an isomorphism \( \mathcal{D}(X) \otimes \mathfrak{t}^{\text{op}} \cong \mathcal{D}(X)^{\text{op}} \). Thus, under the same assumptions, one also has an isomorphism \( \mathcal{D}(X) \otimes \mathfrak{t}^{\text{op}} \cong \mathfrak{t}^{-\chi} \mathcal{D}(X)^{\text{op}} \).

Now, let \( X \) be a smooth \( G \times G \)-variety. We will abuse notation and write \( G \times G = G_1 \times G_r. \) Thus, we have a natural algebra map \( \mathcal{U}_{G_1} \otimes \mathcal{U}_{G_r} \to \mathcal{D}(X) \), so \( \mathcal{U}_{G_1} \otimes \mathcal{U}_{G_r} \) acts on any (say, right) \( \mathcal{D}(X) \)-module \( M \). Changing a sign of the \( \mathfrak{g} \)-action, we may view \( M \) as an \( \mathcal{U}_G \)-bimodule. Let \( (\mathcal{D}(X), G) \)-mod be the category of right \( \mathcal{D}(X) \)-modules equivariant with respect to the diagonal \( G \)-action. For such a module \( M \), the \( \mathfrak{g} \)-action on \( M \) is locally finite. We conclude that Proposition 4.1.1 yields the following result.

Corollary 4.1.2. For any \( M \in (\mathcal{D}(X), G) \)-mod and \( j \neq 0 \), we have \( H_j(n^\psi, M) = 0. \) In particular, the functor \( (\mathcal{D}(X), G) \)-mod \( \mapsto (\mathcal{D}(X) \otimes n^\psi)_\text{mod} \), \( M \mapsto M/Mn^\psi \), is exact.

4.2. Proof of Theorem 1.6.3 Let \( X \) be a smooth affine \( G \times G \)-variety and \( M \) a right \( \mathcal{D}(X) \)-module. We may view \( M \) as a left \( \mathcal{D}(X)^{\text{op}} \)-module, in particular, an \( n_r^{-\psi} \)-module. The Lie algebra \( \mathfrak{n} \) being unimodular, one has a canonical isomorphism of functors \( H_r(n, -) \cong H^{d-r}(n, -) \), where \( d = \dim N \). It follows that there is a canonical isomorphism \( H^r(n_r^{-\psi}, M) \cong H_{d-r}(n, -). \) By Section 3.1, this space has the natural structure of a left \( \mathcal{D} \)-module, where \( \mathcal{D} = \mathcal{D}(X)^{\text{op}} \otimes n_r^{-\psi}. \) We have the identification \( \mathcal{D}(X)^{\text{op}}/\mathcal{D}(X) \otimes n_r^{-\psi} = \mathcal{D}(X)/n_r^{-\psi} \mathcal{D}(X) \), and it follows from Proposition 3.1.4(i) that the functor \( \mathcal{D}(X)^{\text{op}}/\mathcal{D}(X) \otimes n_r^{-\psi} \otimes \mathcal{D}(-) : \mathcal{D} \)-mod \( \mapsto (\mathcal{D}(X), n_r^{-\psi}) \)-mod is an equivalence. Also, as has been mentioned in Section 1.5, there is a canonical equivalence of \( (\mathcal{D}(X), n_r^{-\psi}) \)-mod \( \cong (\mathcal{D}(X), N_r, \psi) \)-mod. We deduce that the second functor in (1.6.2) is an equivalence. The first functor in (1.6.2) is exact, by Corollary 4.1.2.

It remains to identify the composite functor in (1.6.2) with the functor (1.6.1). It is well-known, cf. [MV], that an averaging functor \( \int_{\mathbb{G}_N} \mathfrak{g}_K \otimes - \) is a right adjoint to the natural full embedding \( \mathcal{D}(X), N_r \)-mod \( \to \mathcal{D}(X) \)-mod (a homological shift by \( d \) is due to a shift in adjunction formulas, e.g., [HTT], Theorem 3.2.14.) Similarly, one shows that the functor \( \int_{\mathbb{N}_N} e^{-\psi} \mathfrak{g}_N \otimes - \) is a right adjoint to the natural full embedding \( \mathcal{D}(X), n_r^{-\psi} \)-mod \( \to \mathcal{D}(X) \)-mod, to be denoted \( \mathbb{I} \).
On the other hand, the functor $I$ can be factored as a composition $(\mathcal{D}(X), n^\psi_r)_{\text{mod}} \to \mathcal{D}_{\text{mod}} \to \mathcal{D}(X)_{\text{mod}}$, where the first functor is the equivalence $M \mapsto M^{n^\psi_r}$, to be denoted $J$, and the second functor is $\mathcal{D}(X)^{op}/\mathcal{D}(X)^{op} n^\psi_r \otimes_D (-)$. From the canonical isomorphisms

$$\text{Hom}_{\mathcal{D}(X)^{op}}(\mathcal{D}(X)^{op}/\mathcal{D}(X)^{op} n^\psi_r \otimes_D M, F) \cong \text{Hom}_D(M, F^{n^\psi_r}) \cong \text{Hom}_D(M, H_d(n^\psi_r, F)),$$

we see that the functor $J^{-1} \circ H_d(n^\psi_r, -)$ is a right adjoint of $I$. Hence, the functor $J^{-1} \circ H_d(n^\psi_r, -)$ must be isomorphic to the averaging functor $\int_{n_r}^{-d}(e^{-\psi} \Omega_N \boxtimes -)$. These two functors are left exact. The corresponding $j$-th right derived functors are $J^{-1} \circ H_d(n^\psi_r, -)$ and $\int_{n_r}^{-d}(e^{-\psi} \Omega_K \boxtimes -)$, respectively. It follows that these derived functors are isomorphic. The composite functor in (1.6.2) is nothing but the functor $J^{-1} \circ H_0(n^\psi_r, -)$, proving the required isomorphism of functors. □

4.3. **Proof of Theorem 1.2.2** Let $\mathbb{X}^{++} \subset \mathbb{X}^*$ be the set of dominant weights. We write $\lambda \leq \mu$ if $\mu - \lambda$ is a linear combination of simple roots with nonnegative integer coefficients.

Let $V_\lambda$ be a simple $G$-representation with highest weight $\lambda \in \mathbb{X}^{++}$, i.e., $V_\lambda$ has a unique, up to a constant factor, nonzero vector $v_\lambda \in V_\lambda$ such that $tv_\lambda = \lambda(t)v_\lambda$, $\forall t \in T$. Let $V^*_\lambda$ be the contragredient representation. We have the Peter-Weyl decomposition $G = \oplus_{\lambda \in \mathbb{X}^{++}} V_\mu \otimes V^*_\mu$. We put $R_\lambda := \oplus_{\mu \leq \lambda} V_\mu \oplus V^*_\mu$. The spaces $R_\lambda$, $\lambda \in \mathbb{X}^{++}$, form an exhaustive multiplicative filtration of the algebra $\mathbb{C}[G]$.

Now, view $X = G$ as a $G \times G$-variety where $G \times G = G_1 \times G_r$ acts on $G$ by left and right translations. Thus, we have an algebra homomorphism $U(g_1 \oplus g_r) = U(g_1) \otimes U(g_r) \to \mathcal{D}(G)$. The diagonal $G$-action on $X$ corresponds to the action of $G$ on itself by conjugation.

We have a natural embedding $\mathbb{C}[G] \to \mathcal{D}(G)$. For $\lambda \in \mathbb{X}^{++}$, let $P_\lambda \mathcal{D}(G) := R_\lambda \cdot U(g_1 \oplus g_r)$ be a right $U(g_1 \oplus g_r)$-submodule of $\mathcal{D}(G)$ generated by the subspace $R_\lambda \subset \mathbb{C}[G]$. We have

$$\mathcal{D}(G) = \mathbb{C}[G] \cdot U(g_1) = \cup_{\lambda \in \mathbb{X}^{++}} R_\lambda \cdot U(g_1) \subset \cup_{\lambda \in \mathbb{X}^{++}} R_\lambda \cdot U(g_1 \oplus g_r) = \cup_{\lambda \in \mathbb{X}^{++}} P_\lambda \mathcal{D}(G).$$

Thus, the spaces $P_\lambda \mathcal{D}(G)$, $\lambda \in \mathbb{X}^{++}$, form an exhaustive filtration of $\mathcal{D}(G)$.

**Remark 4.3.1.** The filtration $P_\lambda \mathcal{D}(G)$, $\lambda \in \mathbb{X}^{++}$ has a natural geometric interpretation in terms of the *wonderful compactification* of $G^{ad}$, the adjoint group. It is immediate from this interpretation that $U(g_1 \oplus g_r) \cdot R_\lambda = R_\lambda \cdot U(g_1 \oplus g_r)$ and each of the spaces $P_\lambda \mathcal{D}(G)$ is stable under the conjugation action of $G$ on itself. Furthermore, for any $\lambda, \mu \in \mathbb{X}^{++}$ one has $P_\lambda \mathcal{D}(G) \cdot P_\mu \mathcal{D}(G) \subset P_{\lambda + \mu} \mathcal{D}(G)$.

The adjoint action of the element $h_1 + h_r \in g_1 \oplus g_r$ gives a $\mathbb{Z}$-grading on $U(g_1 \oplus g_r)$. The corresponding Kazhdan filtration $F_\lambda U(g_1 \oplus g_r)$ is a tensor product of the Kazhdan filtrations on $U(g_1)$ and $U(g_r)$, respectively. Recall that the Kazhdan filtration on $Q = U(g_1/(g_1) n^\psi$ has no components of negative degrees. It follows that the quotient Kazhdan filtration on $U(g_1 \oplus g_r)/U(g_1 \oplus g_r) n^\psi$ has no components of negative degree. Here, we have used simplified notation $n^\psi = n^\psi_1 \oplus n^\psi_r$.

The conjugation action of the $1$-parameter subgroup $\gamma$ on $G$ induces a $\mathbb{Z}$-grading on $\mathcal{D}(G)$. Let $F_\lambda \mathcal{D}(G)$ be the corresponding Kazhdan filtration on $\mathcal{D}(G)$. By definition, one has $(P_\lambda \mathcal{D}(G)) n^\psi = R_\lambda \cdot U(g_1 \oplus g_r) n^\psi$. We deduce from the above that, for every $\lambda$ there exists $m = m(\lambda) \gg 0$ such that the quotient Kazhdan filtration on $(P_\lambda \mathcal{D}(G) + \mathcal{D}(G)) n^\psi / \mathcal{D}(G) n^\psi$ has no components of degrees $\leq -m(\lambda)$. This implies that the Kazhdan filtration on $\mathcal{D}(G)/\mathcal{D}(G) n^\psi$ is separating.

To complete the proof of Theorem 1.2.2 we consider the setting of Section 3 in the case where $A = \mathcal{D}(G)$ and the Lie algebra $g$, resp. $n^\psi$, is replaced by $g_1 \oplus g_r$, resp. $n^\psi$. Thus, Lemma 3.3.2 is applicable. The filtration on $\mathcal{D}(G)/\mathcal{D}(G) n^\psi$, being separating, from the second isomorphism in statement (1) of the lemma we deduce that the natural map

$$(\text{gr}^F \mathcal{D}(G)) \oplus (N_l \times N_r, \psi \times \psi) \to \text{gr}^F (\mathcal{D}(G) / n^\psi)$$
is an isomorphism. Further, we have $\mathbb{C}[T^*G \sslash (N_l \times N_r, \psi \times \psi)] = (\text{gr} F \mathcal{D}(G)) \sslash (N_l \times N_r, \psi \times \psi)$, by Corollary 2.2.2. Theorem 1.2.2 follows.

Corollary 4.3.2. The algebra $\mathcal{W} = \mathcal{D}(G) \sslash (n^\psi_l + n^\psi_r)$ is simple and the Kazhdan filtration on this algebra is separating.

Proof. The filtration on $\mathcal{D}(G) \sslash (n^\psi_l + n^\psi_r)$ is separating by Lemma 3.3.2. Therefore, for a nonzero two-sided ideal $I \subsetneq \mathcal{W}$, we have that $\text{gr } I \subsetneq \text{gr } \mathcal{W}$ is a nonzero Poisson ideal of $\text{gr } \mathcal{W} = \mathbb{C}[3]$, where $\mathbb{C}[3]$ is viewed as a Poisson algebra. But this Poisson algebra contains no nontrivial Poisson ideals, since 3 is a smooth symplectic variety, see Section 2.1.

Remark 4.3.3. Associated with the filtration $\mathcal{R}_l$ on $\mathbb{C}[G]$, one has the corresponding Rees algebra $\oplus_{\lambda \in \mathbb{X}^{++}} R_\lambda$. This $\mathbb{X}^{++}$-graded algebra is the coordinate ring of the Vinberg semigroup associated with $G$. Similarly, the filtration $P_\lambda \mathcal{D}(G)$ on $\mathcal{D}(G)$ induces a quotient filtration on $\mathcal{D}(G)/\mathcal{D}(G) n^\psi_{l \text{r}}$. By restriction, one gets a filtration $P_\lambda \mathcal{W}$ on $\mathcal{D}(G)/\mathcal{D}(G) n^\psi_{l \text{r}}$. The corresponding Rees algebra $\oplus_{\lambda \in \mathbb{X}^{++}} P_\lambda \mathcal{W}$ may be viewed as a quantum counterpart of the coordinate ring of a Vinberg type deformation of the universal centralizer 3. This deformation, as well as an analogue of the wonderful compactification of 3 has been studied in [Ba1]–[Ba2].

5. Hamiltonian reduction of $T^*(G/\tilde{N})$ and $\mathcal{D}(G/\tilde{N})$

5.1. The affine closure of $T^*(G/\tilde{N})$. Let $B$, resp. $\tilde{B}$, be the Borel subgroup of $G$, corresponding to the Borel subalgebra $b$, resp. $\tilde{b}$. Thus, $N$, resp. $\tilde{N}$, is the unipotent radical of $B$, resp. $\tilde{B}$, and $B \cap \tilde{B}$ is a maximal torus. We identify the flag variety $B$ with $G/B$ and let $\tilde{B} := G/\tilde{N}$.

The group $G$ acts on $\tilde{B}$ on the left and the torus $T = \tilde{B}/\tilde{N}$ acts on $\tilde{B}$ on the right. Thus we have a $G_l \times T_r$-action on $\tilde{B}$ and an induced Hamiltonian $G_l \times T_r$-action on $T^*\tilde{B}$, with moment map $\mu_{G_l,T_r,\tilde{B}} : T^*\tilde{B} \to g^* \times t^*$. We have a natural isomorphism $(T^*\tilde{B})/T_r = \tilde{g}$, so the quotient map $\psi : T^*\tilde{B} \to (T^*\tilde{B})/T_r = \tilde{g}$ makes $T^*\tilde{B}$ a $G_l$-equivariant $T$-torsor over $\tilde{g}$. The moment map $\mu_{G_l,T_r,\tilde{B}} : T^*\tilde{B} \to T^*\tilde{B}$ factors as a composition $T^*\tilde{B} \to \tilde{g} \to g^* \times t^*$.

Let $z : \xi \mapsto z : \xi$ denote the standard $\mathbb{G}_m$-action on a cotangent bundle by dilations along the fibers. We define a $\bullet$-action of $\mathbb{G}_m$ on $T^*G$, resp. $\tilde{g}^*$, by $z : \xi \mapsto z : \xi := z^{-2}$. $\text{Ad}^* \gamma(z)$-action on $T^*G$ is induced by $\gamma(z) : g \mapsto \gamma(z) g \gamma(z)^{-1}$. The action on $T^*G$ of the torus $T_l \times T_r \subset G_l \times G_r$ descends to a well-defined action on $T^*\tilde{B}$. Hence, the $\bullet$-action descends to a well-defined action on $T^*\tilde{B}$.

The variety $T^*\tilde{B}$ is quasi-affine; furthermore, it was shown in [GR]. Section 5.5] that the algebra $\mathbb{C}[T^*\tilde{B}]$ is finitely generated. Thus, $(T^*\tilde{B})_{\text{aff}} := \text{Spec } \mathbb{C}[T^*\tilde{B}]$ is an affine algebraic variety that contains $T^*\tilde{B}$ as a Zariski open and dense subvariety. It is clear that any action of a linear algebraic group on $T^*\tilde{B}$ extends to an action on $(T^*\tilde{B})_{\text{aff}}$. The moment map $\mu_{G_l,T_r,\tilde{B}} \times T_{\text{aff}} : (T^*\tilde{B})_{\text{aff}} \to g^* \times t^*$.

The following result, [GR]. Section 5.5], [GK], is a commutative counterpart of the Gelfand-Graev construction mentioned in the introduction.

Proposition 5.1.1. There is a $W$-action on $(T^*\tilde{B})_{\text{aff}}$ such that:

1. The $T_r$-action and the $W$-action combine together to give a $W \rtimes T$-action on $(T^*\tilde{B})_{\text{aff}}$.
2. The $W \rtimes T$-action commutes both with the $G_l$-action and with the $\bullet$-action on $(T^*\tilde{B})_{\text{aff}}$.
3. The map $\mu_{G_l,T_r,\tilde{B}} \times T_{\text{aff}} : (T^*\tilde{B})_{\text{aff}} \to g^* \times t^*$ is $W \rtimes T$-equivariant, where $W$ acts naturally on $t^*$ and $W \rtimes T$ acts trivially on $g^*$.
**Remark 5.1.2.** Let $T_{\text{reg}}\vec{B} := \mu_{G,T\cdot\vec{B}}^{-1}(g^{\text{reg}})$. We have open embeddings $T_{\text{reg}}\vec{B} \subset T^*\vec{B} \subset (T^*\vec{B})_{\text{aff}}$. The set $T_{\text{reg}}\vec{B}$ is stable under the $W$-action, but the set $T^*\vec{B}$ is not.

**Lemma 5.1.3.** The morphism $\mu_{G,T\cdot\vec{B}}|_{T_{\text{reg}}\vec{B}} : T_{\text{reg}}\vec{B} \to g^{\text{reg}}$ is flat and we have $\mu_{G,T\cdot\vec{B}}^{-1}(g^{\text{reg}}) = T_{\text{reg}}\vec{B}$. Further, let $S \subset g^*$ be a closed subvariety such that $S \subset g^{\text{reg}}$, and let $I_Y \subset \mathbb{C}[g^*]$ be the corresponding ideal. Then, we have $\mathbb{C}[\mu_{G,T\cdot\vec{B}}^{-1}(S)] = \mathbb{C}[(T^*\vec{B})/\mathbb{C}[T^*\vec{B}]I_S$.

**Proof.** To simplify the notation, we write $\mu = \mu_{G,T\cdot\vec{B}}$, resp. $\bar{\mu} = \bar{\mu}_{G,T\cdot\vec{B}}$. Let $A := \mu_\ast O_{T\cdot\vec{B}}$. This is a quasi-coherent sheaf of $O_{g^*}$-algebras and we have $\mathbb{C}[(T^*\vec{B})_{\text{aff}}] = \mathbb{C}[T^*\vec{B}] = \Gamma(g^*,A)$. Let $i_x : \{x\} \hookrightarrow g^*$ be a one point embedding and $I_x \subset \mathbb{C}[g^*]$ the ideal of $x$. Then $\mu^{-1}(x)$ is a closed affine subvariety of $(T^*\vec{B})_{\text{aff}}$ and we have

$$\mathbb{C}[\bar{\mu}^{-1}(x)] = \mathbb{C}[(T^*\vec{B})_{\text{aff}}]/\mathbb{C}[(T^*\vec{B})_{\text{aff}}]I_x = \Gamma(g^*,A)/\Gamma(g^*,A)I_x = i_x^\ast A.$$

The map $\mu : T_{\text{reg}}\vec{B} \to g^{\text{reg}}$ factors as $\pi g_{\text{reg}} \circ \pi$, where $\pi$ is a $T$-torsor, $\pi$ is the Grothendieck-Springer map, and $\pi g_{\text{reg}}$ is a finite morphism. Therefore, $T_{\text{reg}}\vec{B} \to g^{\text{reg}}$ is a flat affine morphism. Hence, for $x \in g^{\text{reg}}$, the fiber $\mu^{-1}(x)$ is affine. Writing $i : \mu^{-1}(x) \hookrightarrow T_{\text{reg}}\vec{B}$ for the corresponding embedding, by base change, we get $i_x^\ast A = \mu_\ast i^\ast O_{T\cdot\vec{B}} = \mu_\ast i^\ast O_{T\cdot\vec{B}}$. Thus, from the previous paragraph, we deduce that

$$\mathbb{C}[\bar{\mu}^{-1}(x)] = i_x^\ast A = \Gamma(\mu^{-1}(x), i^\ast O_{T\cdot\vec{B}}) = \mathbb{C}[\mu^{-1}(x)].$$

We conclude that $\mu^{-1}(x) = \bar{\mu}^{-1}(x)$ for any $x \in g^{\text{reg}}$. The proof of the last statement of the lemma is very similar.

The remainder of Section 5.1 will not be used in the paper and is only given for completeness.

Let $\vec{B}_{\text{aff}}$ be the affine closure of $\vec{B}$ and $\bar{\mu} : (T^*\vec{B})_{\text{aff}} \to \vec{B}_{\text{aff}}$ the canonical extension of the projection $p : T^*\vec{B} \to \vec{B}$. For every $w \in W$, let $\bar{\mu}_w : (T^*\vec{B})_{\text{aff}} \to \vec{B}_{\text{aff}}$ be a map defined by $\bar{\mu}_w(y) = \bar{\mu}(w(y))$. Thus, we have $\bar{\mu} = \bar{\mu}_1$.

**Proposition 5.1.4.** For $y \in (T^*\vec{B})_{\text{aff}}$ the following properties are equivalent:

(i) We have $\bar{\mu}_{G,T\cdot\vec{B}}(y) \in g^{\text{reg}}$;

(ii) The $T_r$-orbit, $T_r \cdot y$, of $y$ is a closed orbit in $(T^*\vec{B})_{\text{aff}}$ of maximal dimension, i.e., $\dim(T_r \cdot y) = \dim T$;

(iii) For every $w \in W$ we have $\bar{\mu}_w(y) \in \vec{B}$.

Observe first that the fibers of the maps $\bar{\mu}_{G,T\cdot\vec{B}} \times \bar{\mu}_{T,T\cdot\vec{B}}$ are $T_r$-stable. We need the following standard result.

**Lemma 5.1.5.** Every fiber of the map $\bar{\mu}_{G,T\cdot\vec{B}} \times \bar{\mu}_{T,T\cdot\vec{B}} = \bar{\mu}_{T,T\cdot\vec{B}} \times \bar{\mu}_{G,T\cdot\vec{B}}$ contains exactly one closed $T_r$-orbit; furthermore this orbit is contained in the closure of any other $T_r$-orbit in that fiber.

**Proof.** We have $\mathbb{C}[(T^*\vec{B})_{\text{aff}}]^{T_r} = \mathbb{C}[T^*\vec{B}]^{T_r} = \mathbb{C}[[g]]$. The algebra $\mathbb{C}[[g]]$ may be identified with the pullback of $\mathbb{C}[g^* \times T / W \times T^r]$ via $\bar{\mu}_{G,T\cdot\vec{B}} \times \bar{\mu}_{T,T\cdot\vec{B}}$. Hence, for any $s \in g^* \times T / W \times T^r$, the fiber of the map $(T^*\vec{B})_{\text{aff}} \to \text{Spec} \mathbb{C}[(T^*\vec{B})_{\text{aff}}]^{T_r}$ may be identified with $(\bar{\mu}_{G,T\cdot\vec{B}} \times \bar{\mu}_{T,T\cdot\vec{B}})^{-1}(s)$. The lemma follows since $T$-invariant functions on an affine variety separate closed $T$-orbits.

**Proof of Proposition 5.1.4.** Let $y \in (T^*\vec{B})_{\text{aff}}$ be such that $\bar{\mu}_{G,T\cdot\vec{B}}(y) \in g^{\text{reg}}$. Then, it follows from Lemma 5.1.3 that $y \in T_{\text{reg}}\vec{B}$. Hence, the $T_r$-orbit of $y$ is a closed orbit of maximal dimension. Also, for every $w \in W$, we have $\bar{\mu}_w(y) \in \vec{B}$, since the set $T_{\text{reg}}\vec{B}$ is $W$-stable.

Next, fix $y \in (T^*\vec{B})_{\text{aff}}$ such that the element $\bar{\mu}_{G,T\cdot\vec{B}}(y)$ is not regular. We claim that the $T_r$-orbit of $y$ is either not closed or it does not have maximal dimension. Let $F$ be the fiber of the map
subalgebra that contains the element $e$ in $G$. It follows that there are no closed $T_\circ$-orbits in $F$ of maximal dimension. To prove this, let $F' := F \cap T^\circ \bar{B}$. All $T_\circ$-orbits in $T^\circ \bar{B}$, in particular those in $F'$, have maximal dimension. Further, since $\bar{\mu}_{G,T^\circ \bar{B}}(y)$ is not regular in $g^*$, the Springer fiber over $\bar{\mu}_{G,T^\circ \bar{B}}(y)$ has dimension $> 0$. It follows that $F'$, hence also $F$, contains infinitely many $T_\circ$-orbits of maximal dimension. On the other hand, Lemma 5.1.5 says that $F$ contains exactly one closed $T_\circ$-orbit $O$ and, moreover, $O$ is contained in the closure of any other $T_\circ$-orbit in $F$. Hence, none of the infinitely many $T_\circ$-orbits above is closed and $O$ is contained in the closure of each of those orbits. It follows that the dimension of $O$ is not maximal.

To complete the proof, let $y \in (T^\circ \bar{B})_{\text{aff}} \setminus T^\circ \bar{B}$ be as above and suppose by contradiction that $\bar{p}_w(y) \in \bar{B}$ for every $w \in W$. Since all orbits of the $T_\circ$-action on $\bar{B}$ have maximal dimension, the $T_\circ$-orbit of $y$ has maximal dimension. Therefore, this orbit is not closed in $(T^\circ \bar{B})_{\text{aff}}$, by the paragraph above. Hence, by Hilbert-Mumford, there is a 1-parameter subgroup $\tau : \mathbb{C}^* \to T$ such that the limit $y' := \lim_{z \to 0} \tau(z) \cdot y$ exists and $y' \notin T_\circ \cdot y$. Hence, for every $w \in W$, we have $\lim_{z \to 0} \tau(z) \cdot \bar{p}_w(y) = \bar{p}_w(y')$.

Let $h \in \text{Lie } T = \mathfrak{t}$ be a generator of the 1-parameter subgroup $\tau$. Observe that the existence of a well-defined limit $\lim_{z \to 0} \tau(z) \cdot x \in \bar{B}_{\text{aff}}$ for some $x \in \bar{B}$ implies that $\mu(h) \geq 0$ for every dominant integral weight $\mu \in \mathfrak{t}^*$. Since the limit $\lim_{z \to 0} \tau(z) \cdot \bar{p}_w(y)$ exists for every $w \in W$, it follows that we must have $\mu(h) \geq 0$ for all integral weights $\mu \in \mathfrak{t}^*$. This forces $h = 0$, a contradiction.

5.2. Classical Hamiltonian reduction of $T^\circ(G/\bar{N})$. The following result goes back to Kostant.

**Lemma 5.2.1.** Let $x \in e + b$ and $b_+ \subset B$ a Borel subalgebra such that $x \in b_+$. Then the $N$-orbit through the point $b_+ \in B$ is the (unique) open $N$-orbit in $B$.

**Proof.** Recall the $\mathbb{Z}$-grading (3.2.1) and introduce a $\mathbb{Z}$-filtration $g^{l \leq k} := \oplus_{\ell \leq k} g(\ell), k \in \mathbb{Z}$, on $g$. We have $b = g^{\leq 0}$, resp. $n = g^{\leq -2}$, and $b = \oplus_{\ell \geq 0} g(\ell)$. Also, we have $f \in g(-2)$, resp. $e \in g(2)$.

Now, let $x$ and $b_+$ be as in the statement of the lemma. Conjugating the pair $(x, b_+)$ by an element of $N$ if necessary, we may (cf. Section 2.2) and will assume that $x \in e + g_\ell$. Thus, since $g_\ell \subset g^{\leq -2}$, in $\mathfrak{g}_\mathbb{C}$, we have $\text{gr } x = x \mod g^{< 0} = e$. Observe next that the filtration on $g$ induces a filtration $b_+ \cap g^{\leq k}, k \in \mathbb{Z}$, on the Borel subalgebra $b_+$. It is clear that $\text{gr } b_+$ is a solvable Lie subalgebra of $\text{gr } g$ that contains the element $e = \text{gr } x$. Since $\text{dim gr } b_+ = \text{dim } b_+$ and the Lie algebra $\text{gr } g$ is isomorphic to $g$, we conclude that $\text{gr } b_+$ is a Borel subalgebra of $\text{gr } g$. But the only Borel subalgebra that contains the element $e$ is the algebra $b$. Therefore, we must have that $\text{gr } b_+ = b$. It follows that $\text{gr } b_+ = 0$ for all $i < 0$. We deduce that $b_+ \cap n = b_+ \cap g^{\leq -2} = 0$, and the lemma follows.

**Lemma 5.2.2.** Let $Y := \mu_{G,T^\circ \bar{B}}^{-1}(\psi + n^\perp)$. Then, the image of $Y$ under the projection $p : T^\circ \bar{B} \to \bar{B}$ is contained in the open set $B\bar{N}/\bar{N} \subset G/\bar{N} = \bar{B}$. Furthermore, the map

$$
\mu_{T^\circ \bar{B}} \times p : Y \to t^* \times B\bar{N}/\bar{N} = N \times T \times t^* = N \times T^*T
$$

is an $N$-equivariant isomorphism.

**Proof.** We identify $g^*$ with $g$ and $B$ with $G/\bar{B}$, respectively. Then, Lemma 5.2.1 translates into the first statement of Lemma 5.2.2. Further, by Lemma 2.2.1 one has an $N$-equivariant isomorphism $N \times \mu_{G,T^\circ \bar{B}}^{-1}(S) \to Y$. Since $q : T^\circ \bar{B} \to \bar{g}$ is a $T$-torsor, we deduce that $\mu_{G,T^\circ \bar{B}}^{-1}(S)$ is a $T$-torsor over $\pi^{-1}(S)$, where $\pi$ is the Grothendieck-Springer map. Recall that $S \subset g^{\text{reg}}$ and $\bar{g}^{\text{reg}} \cong g^{\text{reg}} \times T^*$. The map $\pi$ corresponds to the first projection $g^{\text{reg}} \times T^* \to g^{\text{reg}}$. Hence, we obtain, $\pi^{-1}(S) =$
\( \pi^{-1}(S) \cap (g^{\text{reg}} \times t^*) = S \times_{\epsilon} t^* \cong t^* \). We conclude that \( \mu_{G,T-\overline{B}}^{-1}(S) \) is a \( T \)-torsor over \( t^* \). The result follows. \( \square \)

Let \( \mathcal{Y} := T^* \overline{B} / \langle N, \psi \rangle \). This means, according to the definition given in Section 2.2 and Lemma 5.2.2, that the map \( \mu_{T,T-\overline{B}} \times p \) yields an isomorphism \( p_\mathcal{Y} : \mathcal{Y} \rightarrow T^*T \). Further, from Lemma 5.1.3, in the case \( S = \psi + n^\perp \), we find
\[
\mathbb{C}[\mathcal{Y}] = \mathbb{C}[Y]^N = \mathbb{C}[\mu_{G,T-\overline{B}}^{-1}(\psi + n^\perp)]^N = (\mathbb{C}[T^* \overline{B}] / \mathbb{C}[T^* \overline{B}] n^\psi)^N = \mathbb{C}[T^* \overline{B}] / n^\psi.
\]
Thus, we have \( \mathcal{Y} = \text{Spec}(\mathbb{C}[T^* \overline{B}] / n^\psi) \).

The subset \( \psi + n^\perp \subset g^* \) is stable under the \( \cdot \)-action. It follows from Proposition 5.1.1(2) that the \( W \times T_r \)-action, resp. \( \cdot \)-action, on \( \mathbb{C}[T^* \overline{B}] \) by algebra automorphisms descends to an action on the algebra \( \mathbb{C}[T^* \overline{B}] / n^\psi \). Therefore, this action induces a \( W \times T_r \)-action, resp. \( \cdot \)-action, on \( \mathcal{Y} \).

Let \( W \) act on \( T^*T = t^* \times T \) diagonally, resp. \( T \) act by translations along the second factor.

**Lemma 5.2.3.** The map \( p_\mathcal{Y} \) is a \( T \times W \)-equivariant symplectomorphism \( \mathcal{Y} \rightarrow T^*T \). Furthermore, this map intertwines the \( \cdot \)-action on \( Y \) with the \( \mathbb{C}_m \)-action on \( T^*T \) by dilations by \( z^{-2} \) along the factor \( t^* \).

**Proof.** All statements, except for the \( W \)-equivariance, are straightforward and are left for the reader. Compatibility of the \( W \)-actions is a consequence of properties of the \( W \)-action on \( \mathbb{C}[T^* \overline{B}] \), see [GR] Lemma 5.2.5. \( \square \)

### 5.3. Quantum Hamiltonian reduction of \( \mathcal{D}(\overline{B}) \)

One has a chain of algebra isomorphisms
\[
\mathcal{D}(NT \overline{N} \cap N) / n^\psi \cong \mathcal{D}(N \times T) / n^\psi \cong (\mathcal{D}(N) / n^\psi) \otimes \mathcal{D}(T) \cong \mathbb{C} \otimes \mathcal{D}(T) \cong \mathcal{D}(T).
\]

Let \( j : NT \overline{N} \cap N \rightarrow G / \overline{N} = \overline{B} \) be the open embedding.

**Theorem 5.3.1.** (i) The following functors are mutually inverse equivalences:
\[
\begin{align*}
(\mathcal{D}_\overline{B}, N, \psi)\text{-mod} & \quad \overset{\Gamma(\overline{B}, -)}{\underset{\mathcal{D}_\overline{B} \otimes \mathcal{D}(\overline{B})}{\longrightarrow}} (\mathcal{D}(\overline{B}), N, \psi)\text{-mod} \quad \overset{(-) n^\psi}{\longrightarrow} (\mathcal{D}(\overline{B}) / n^\psi)\text{-mod}.
\end{align*}
\]

(ii) The map \( j^* : \mathcal{D}(\overline{B}) / n^\psi \rightarrow \mathcal{D}(NT \overline{N} \cap N) / n^\psi \cong \mathcal{D}(T) \), induced by restriction of differential operators via \( j \), is an algebra isomorphism.

We begin with the following result.

**Lemma 5.3.3.** (i) For any \( F \in (\mathcal{D}_\overline{B}, N, \psi)\)-mod the canonical map \( F \rightarrow j_* j^* F \) is an isomorphism.

(ii) The functor \( \mathcal{D}_T\text{-mod} \rightarrow (\mathcal{D}_\overline{B}, N, \psi)\text{-mod}, M \mapsto j_*(e^\psi \Omega_N \boxtimes M) \), is an equivalence. An inverse is given by the functor \( \mathcal{F} \mapsto (pr \cdot j^* \mathcal{F}) n^\psi \), where \( pr : B \overline{N} / \overline{N} = N \times T \rightarrow T \) is the second projection.

**Proof.** The argument is an adaptation of the argument from [MS] to the setting of not necessarily holonomic \( \mathcal{D} \)-modules. Given a point \( x \in \overline{B} \), let \( i_x : \{x\} \rightarrow \overline{B} \) denote the embedding, \( N_x \) the stabilizer of \( x \) in \( N \), and \( \psi_x \) the restriction of \( \psi \) to the Lie algebra \( n \cap \text{Lie } N_x \). Note that if \( \psi_x \neq 0 \), then the \( \mathcal{D}_{N_x} \)-module \( e^{\psi_x} \Omega_{N_x} \) cannot be obtained as a pullback via the map \( N_x \rightarrow pt \) of a \( \mathcal{D} \)-module on a point. Also, it is clear that for any \( E \in (\mathcal{D}_\overline{B}, N, \psi)\)-mod and \( x \in \overline{B} \), the \( \mathcal{D} \)-module \( R^k i_x^* \mathcal{E} \), on \( \{x\} \), is an \( (N_x, \psi_x) \)-Whittaker \( \mathcal{D} \)-module. It follows that if \( \psi_x \neq 0 \), then \( R^k i_x^* \mathcal{E} = 0 \) for all \( k \in \mathbb{Z} \).

Now let \( E \in (\mathcal{D}_\overline{B}, N, \psi)\)-mod be a nonzero \( \mathcal{D}_\overline{B} \)-module such that \( \text{Supp } E \subset \overline{B} \setminus B \overline{N} / \overline{N} \). Then, for a sufficiently general point \( x \) in the support of \( E \), there exists \( k \) such that \( R^k i_x^* \mathcal{E} \neq 0 \). Write \( x = g \overline{N} / \overline{N} \), so we have \( \text{Lie } N_x = \text{Ad } g(\mathfrak{n}) \). Lemma 5.2.2(ii) implies that \( n \cap \text{Lie } N_x = \text{Ad}^* g(\mathfrak{n}) \cap \).
\((ψ + n^⊥) = ∅\). This means that \(ψ_x \neq 0\). It follows from the above that \(E = 0\), a contradiction. Thus, there are no nonzero objects \(E \in (\mathcal{D}_B, N, ψ)\)-mod such that \(\text{Supp } E \subset B \setminus B \tilde{N} / \tilde{N}\).

To complete the proof of (i) observe that, for any \(F \in (\mathcal{D}_B, N, ψ)\)-mod, the kernel, resp. cokernel, of the map \(F \to j_∗j^∗F\) is an object of \(E \in (\mathcal{D}_B, N, ψ)\)-mod supported on \(B \setminus B \tilde{N} / \tilde{N}\). Hence \(E = 0\), proving (i). Further, it is immediate from definitions that the functors \(e^\psi \Omega_N \boxtimes (-)\) and \((pr_*(-))^\psi\) give mutually inverse equivalences \(\mathcal{D}(T)\)-mod \(\cong (\mathcal{D}_N × T, N, ψ)\)-mod. Therefore, part (ii) follows from (i).

\[\] **Proof of Theorem 5.3.4** First, we apply Proposition 3.1.4 in the case where \(A = \mathcal{D}(\tilde{B})\). It follows from part (i) of the proposition that the functors \((-)^\psi\) and \(\mathcal{D}(\tilde{B})/\mathcal{D}(\tilde{B})n^\psi \otimes_{\mathcal{D}(\tilde{B})n^\psi} (-)\), on the right of diagram (5.3.2), are mutually inverse equivalences.

Next, let \(F \in (\mathcal{D}_B, N, ψ)\)-mod. The variety \(B \tilde{N} / \tilde{N} \cong T\) being affine, from Lemma 5.3.3(i), we deduce \(H^i(\tilde{B}, F) = H^i(\tilde{B}, j^∗j_∗F) = H^i(B \tilde{N} / \tilde{N}, j^∗F) = 0\), for all \(i \neq 0\). It follows that \(Γ(\tilde{B}, -)\) is an exact functor on \((\mathcal{D}_B, N, ψ)\)-mod. Furthermore, we have

\[Γ(\tilde{B}, F)^\psi = Γ(\tilde{B}, j^∗j_∗F)^\psi = Γ(B \tilde{N} / \tilde{N}, j^∗F)^\psi = Γ(T, pr_∗j^∗F)^\psi = Γ(T, (pr_∗j^∗F)^\psi).\] (5.3.4)

We see that the action of the algebra \(\mathcal{D}(\tilde{B})/n^\psi\) on \(Γ(\tilde{B}, F)^\psi\) factors through the restriction map \(j^∗: \mathcal{D}(\tilde{B})/n^\psi \to \mathcal{D}(T)\). Moreover, the resulting functor \(Γ(\tilde{B}, F)^\psi: (\mathcal{D}_B, N, ψ)\)-mod \(→ \mathcal{D}(T)\)-mod is an equivalence, by Lemma 5.3.3(ii), and an inverse equivalence is the functor \(j_∗(e^\psi \Omega_N \boxtimes -)\).

We apply \(Γ(\tilde{B}, -)^\psi\), an exact functor on \((\mathcal{D}_B, N, ψ)\)-mod, to the morphism \(\mathcal{D}_B \otimes n^\psi \to \mathcal{D}_B\) given by right multiplication by \(n^\psi\). The cokernel of this morphism equals \(\mathcal{D}_B/\mathcal{D}_Bn^\psi\). We deduce that \(Γ(\tilde{B}, \mathcal{D}_B/\mathcal{D}_Bn^\psi)^\psi = (\mathcal{D}(\tilde{B})/\mathcal{D}(\tilde{B})n^\psi)^\psi = \mathcal{D}(\tilde{B})/n^\psi\). On the other hand, it is clear that \(j^∗(\mathcal{D}_B/\mathcal{D}_Bn^\psi)^\psi = e^\psi \Omega_N \boxtimes \mathcal{D}(T)\). Hence, \((pr_∗j^∗(\mathcal{D}_B/\mathcal{D}_Bn^\psi))^\psi = (pr_∗j^∗(e^\psi \Omega_N \boxtimes \mathcal{D}(T))^\psi = \mathcal{D}(T)\). Thus, using (5.3.4), we obtain

\[\mathcal{D}(\tilde{B})/n^\psi = Γ(\mathcal{D}_B, \mathcal{D}_B/\mathcal{D}_Bn^\psi)^\psi = Γ(T, pr_∗j^∗(\mathcal{D}_B/\mathcal{D}_Bn^\psi)^\psi) = Γ(T, \mathcal{D}(T)) = \mathcal{D}(T)\].

Thus, we have shown that the restriction map \(j^∗: \mathcal{D}(\tilde{B})/n^\psi \to \mathcal{D}(T)\), in Theorem 5.3.4(ii), is an algebra isomorphism and the functor \(Γ(\tilde{B}, -)^\psi: (\mathcal{D}_B, N, ψ)\)-mod \(→ \mathcal{D}(T)\)-mod is an equivalence. We conclude that both the functor \((-)^\psi\) and the composite functor \(Γ(\tilde{B}, -)^\psi\), in diagram (5.3.2), are equivalences. It follows that the functor \(Γ(\tilde{B}, -)\), in diagram (5.3.2), is also an equivalence. This proves part (i) of Theorem 5.3.4.

5.4. **Proof of Theorem 1.6.4** We begin with the following result.

**Lemma 5.4.1.** The following natural maps are algebra isomorphisms

\[\mathcal{D}(G)/n^\psi_i \xrightarrow{n_r} \mathcal{D}(G)/n^\psi_i + \bar{n}_r \xleftarrow{\mathcal{D}(G)/n^\psi_i \otimes \mathcal{D}(G)/n^\psi_i};\]

\[\mathcal{D}(G)/n^\psi_i \otimes \mathcal{D}(G)/n^\psi_i \xrightarrow{\mathcal{D}(G)/n^\psi_i \otimes \mathcal{D}(G)/n^\psi_i} \mathcal{D}(G)/n^\psi_i \xrightarrow{\mathcal{D}(G)/n^\psi_i \otimes \mathcal{D}(G)/n^\psi_i} \mathcal{D}(G)/n^\psi_i \leftarrow (\bar{n}_r \\mathcal{D}(G)) \mathcal{D}(G)/n^\psi_i\].

**Proof.** Let \(N = N_\ast\), resp. \(N = \bar{N}_\ast\). First, we prove that the map \(\psi\) is an isomorphism. To prove this, consider an open embedding \(j_G: NT \tilde{N}/\tilde{N} \xrightarrow{G/\tilde{N}}\). We have a commutative diagram of algebra
homomorphisms

\[
\mathcal{D}(G/\tilde{N}) \mathcal{D}(n_r^n) \xrightarrow{\nu} \mathcal{D}(G) \mathcal{D}(n_r^n + \bar{n}_r).
\]

By commutativity, it suffices to show that the map \(j^*_G\) on the right is an algebra isomorphism. The latter is proved by mimicking the arguments of Section 5.3.

Next, we prove that the map \(u\) is an isomorphism. We simplify the notation and write \(\mathcal{D} = \mathcal{D}(G)\), resp. \(M = \mathcal{D}/(\mathcal{D} \otimes n_r^n)\), and \(D_l = \mathcal{D}(G) \mathcal{D}(n_r^n) = M^n^n\). The action of \(n_r^n\) on \(\mathcal{D}\) by right multiplication descends to \(M\). Observe that \(M\) is an object of the category \((\mathcal{D}, n_r^n)\)-mod. In that category, one has an exact sequence

\[
M \otimes n_r^n \xrightarrow{\alpha} M \rightarrow M/Mn_r^n \rightarrow 0,
\]

where the map \(\alpha\) is given by the \(n_r^n\)-action on \(M\). We apply the functor \((-)^{n_r^n}\), which is an exact functor on \((\mathcal{D}, n_r^n)\)-mod, to (5.4.2). We have

\[
(M/Mn_r^n)^{n_r^n} = \left(\mathcal{D}(n_r^n)^{\mathcal{D}(n_r^n)}\right)^{n_r^n} = \mathcal{D}(\mathcal{D}(n_r^n + n_r^n)),
\]

\[
(Mn_r^n)^{n_r^n} = \text{Im}(\mathcal{D}(n_r^n) \otimes n_r^n) = \text{Im}(M \otimes n_r^n) \rightarrow M^n^n = (Mn_r^n)^{n_r^n} = D_ln_r^n.
\]

Thus, from the exact sequence (5.4.2), we obtain the following exact sequence in \((\mathcal{D}, n_r^n + n_r^n)\)-mod:

\[
0 \rightarrow D_ln_r^n \rightarrow D_l \rightarrow \mathcal{D}(\mathcal{D}(n_r^n + n_r^n)) \rightarrow 0.
\]

We now apply the functor \((-)^{n_r^n}\), which is an exact functor on \((\mathcal{D}, n_r^n + n_r^n)\)-mod, to (5.4.3). It is immediate to see that the exactness of the resulting sequence implies that the map \(u\) is an isomorphism. All other isomorphisms of the lemma are proved in a similar way. \(\square\)

**Proof of Theorem 1.6.4** Let \(D_r := \mathcal{D}(G) \mathcal{D}(n_r^n)\). The embedding \(n_r^n \rightarrow \mathcal{D}(G)\) induces a map \(n_r^n \rightarrow D_r\).

The functor \((D_r, n_r^n)\)-mod \(\rightarrow (D_r, n_r^n)\)-mod, \(M \rightarrow M^n^n\), is an equivalence, by Proposition 3.1.4.

Further, by Lemma 5.4.1, we have an isomorphism \(D_r \mathcal{D}(n_r^n) \cong \mathcal{W}\). The result now follows from Theorem 1.6.3. \(\square\)

5.5. **The Gelfand-Graev action.** Let \(\rho\) be the half-sum of positive roots. Let \(W\) act on \(\text{Sym} t\) via the dot-action: \(w \cdot a := w(a) + \langle w(a) - a, \rho \rangle\), \(w \in W, a \in t\). We will identify \(ZG\) with \(\text{Sym} t^W\) via the Harish-Chandra isomorphism. The differential of the \(G \times T\)-action on \(\mathcal{B}\) gives an algebra map \(U_g \rightarrow \mathcal{D}(\mathcal{B}), a \mapsto a_t, \text{resp. } Ut = \text{Sym} t \rightarrow \mathcal{D}(\mathcal{B}), a \mapsto a_t\). It is known that the map \(a \otimes a' \mapsto a_t \otimes a'_t\) factors through a homomorphism \(U_g \otimes ZG \text{Sym} t \rightarrow \mathcal{D}(\mathcal{B}), \text{cf. [BB]}\).

Restricting the \(G \times T\)-action on \(\mathcal{B}\) to the subgroup \(T \times T \subset G \times T\) gives a weight decomposition \(\mathbb{C}[\mathcal{B}] = \bigoplus_{\lambda, \mu \in \mathbb{X}^+} \mu \mathbb{C}[\mathcal{B}]^\lambda, \text{resp. } \mathbb{C}[T^* \mathcal{B}] = \bigoplus_{\lambda, \mu \in \mathbb{X}^+} \lambda \mathbb{C}[T^* \mathcal{B}]^\mu\). Further, the \(\mathcal{G}_m\)-action on \(\mathcal{B}\) given by \(\gamma(z) : gN \mapsto \gamma(z)g^{-1}N\) induces a \(\mathbb{Z}\)-grading on \(\mathcal{D}(\mathcal{B})\). This \(\mathbb{Z}\)-grading may be expressed in terms of the weight decomposition as follows:

\[
\mathcal{D}(\mathcal{B}) = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{D}(\mathcal{B})(\ell), \quad \mathcal{D}(\mathcal{B})(\ell) := \bigoplus_{\mu, \lambda \in \mathbb{X}^+ \langle \lambda, h \rangle + \langle \mu, h \rangle = \ell} \mu \mathcal{D}(\mathcal{B})^\lambda.
\]

Let \(F, \mathcal{D}(\mathcal{B})\) be the Kazhdan filtration associated with the \(\mathbb{Z}\)-grading and filtration \(\mathcal{D}_{\leq}(\mathcal{B})\), as in Section 4.1.
We have maps \( n^\psi = n^\psi_t \to UG_t \to \mathcal{D}(\widetilde{B}) \). The right \( T \)-action on \( \mathcal{D}(\widetilde{B}) \) survives the Hamiltonian reduction by \( n^\psi \). We get an \( X^* \)-grading \( \mathcal{D}(\widetilde{B}) / n^\psi = \oplus_{\mu \in X^*} (\mathcal{D}(\widetilde{B}) / n^\psi)^{(\mu)} \). The Kazhdan filtration induces a \( \mathbb{Z} \)-filtration \( F_n(\mathcal{D}(\widetilde{B}) / n^\psi) \) which is compatible with the \( X^* \)-grading. Thus, we have \( F_n(\mathcal{D}(\widetilde{B}) / n^\psi) = \oplus_{\mu \in X^*} F_n(\mathcal{D}(\widetilde{B}) / n^\psi)^{(\mu)} \). The (Kazhdan shifted) principal symbol map \( \text{gr}^F \mathcal{D}(\widetilde{B}) \to \mathbb{C}[T^* \widetilde{B}] / n^\psi \) descends to a map \( \text{gr}^F(\mathcal{D}(\widetilde{B}) / n^\psi) \to \mathbb{C}[T^* \widetilde{B}] / n^\psi \), of \( \mathbb{Z} \times X^* \)-graded algebras.

**Proposition 5.5.1.** The map \( \text{gr}^F \mathcal{D}(\widetilde{B}) / n^\psi \to \mathbb{C}[T^* \widetilde{B}] / n^\psi \) is an isomorphism.

**Proof.** According to [GR Corollary 3.6.1], the principal symbol map \( \text{gr} \mathcal{D}(\widetilde{B}) \to \mathbb{C}[T^* \widetilde{B}] \) is a graded algebra isomorphism, where \( \text{gr}(-) \) is taken with respect to the order filtration on differential operators. This isomorphism respects the \( X \)-gradings. Therefore, it gives a graded algebra isomorphism \( \text{gr}^F \mathcal{D}(\widetilde{B}) \cong \mathbb{C}[T^* \widetilde{B}] \), where the grading on \( \mathbb{C}[T^* \widetilde{B}] \) comes from the \( \bullet \)-action. Further, an argument similar to the one used in Section 4.3 shows that the Kazhdan filtration on \( \mathcal{D}(\widetilde{B}) \) is separating. The statement of the proposition now follows from the isomorphism of Lemma 3.3.2(1) in the case where \( A = \mathcal{D}(\widetilde{B}) \).

In the 1960’s, Gelfand and Graev constructed, using partial Fourier transforms, a Weyl group action on \( \mathcal{D}(\widetilde{B}) \) by algebra automorphisms. The Gelfand-Graev action on \( \mathcal{D}(\widetilde{B}) \) commutes with the \( G \)-action, in particular with the \( N \)-action, by left translations. Therefore, the Gelfand-Graev action descends to a \( W \)-action on \( \mathcal{D}(\widetilde{B}) / n^\psi \) by algebra automorphisms.

We view the function \( t^\nu \) as a zero order differential operator on \( T \), and define a \( W \)-action on the algebra \( \mathcal{D}(T) \) by the formula \( (\mathcal{D}(T)) t^\nu = w(t^\nu \circ w \circ t^{-\rho}) \). Here \( w(-) \) denotes the natural action of \( w \in W \) on \( \mathcal{D}(T) \) induced by the action of \( w \) on \( T \). Note that \( w \cdot u = w(u) \) for any \( u \in \mathbb{C}[T] \subset \mathcal{D}(T) \). Also, the embedding \( \text{Sym} \longrightarrow \mathcal{D}(T) \) intertwines the dot-actions on \( \text{Sym} \) and \( \mathcal{D}(T) \).

It is clear that the map \( j^*: \mathcal{D}(\widetilde{B}) / n^\psi \to \mathcal{D}(BN/\bar{N}) / n^\psi = \mathcal{D}(T) \) takes \( F_n(\mathcal{D}(\widetilde{B}) / n^\psi) \) to \( \mathcal{D}_{\leq n}(T) \) for all \( n \).

**Proposition 5.5.2.** The isomorphism \( j^*: \mathcal{D}(\widetilde{B}) / n^\psi \to \mathcal{D}(T) \) of Theorem 5.3.1 induces an isomorphism \( \text{gr}^F j^*: \text{gr}^F(\mathcal{D}(\widetilde{B}) / n^\psi) \to \mathcal{D}(T) = \mathbb{C}[T^* T] \) of graded algebras. Furthermore, the map \( j^* \) intertwines the \( \bar{W} \)-action on \( \mathcal{D}(\widetilde{B}) / n^\psi \) and the dot-action on \( \mathcal{D}(T) \).

To prove the proposition, recall that \( v_\lambda \in V_\lambda \) denotes a nonzero \( \bar{N} \)-fixed vector of the irreducible representation \( V_\lambda \) such that \( v_\lambda = \lambda(t)v_\lambda \), \( \forall t \in T \). Let \( v^*_\lambda \in V^*_\lambda \) be a nonzero \( N \)-fixed vector and \( f^\lambda(g) := (v^*_\lambda, gv_\lambda) \). This function is right \( \bar{N} \)-invariant, hence it descends to a function \( f^\lambda \) on \( \mathcal{D}(\widetilde{B}) \), such that \( f^\lambda(n \cdot t \cdot \bar{N}/\bar{N}) = \lambda(t) \), for all \( t \in T \), \( n \in \bar{N} \), \( \bar{n} \in \bar{N} \).

To simplify the notation, put \( A = \mathcal{D}(\widetilde{B}) / n^\psi \). The function \( f^\lambda \), viewed as an element of \( \mathcal{D}(\widetilde{B}) \), survives the Hamiltonian reduction giving an element \( f^\lambda_A \in A \). Let \( F_w, w \in W \) denote the Gelfand-Graev automorphisms of the algebra \( \mathcal{D}(\widetilde{B}) \), resp. \( A \).

**Proof of Proposition 5.5.2** The map \( \text{Sym} t \to \mathcal{D}(\widetilde{B}) \), \( u \mapsto u_t \), survives the Hamiltonian reduction by \( n^\psi \subset \mathcal{D}(\widetilde{B}) \), giving an algebra map \( \text{Sym} t \to A \), \( u \mapsto u_A \). It is clear that for any \( u \in \text{Sym} t \), resp. \( f^\lambda_A, X^{++} \), we have \( j^*(u_A) = u \), resp. \( j^*(f^\lambda_A) = t^\lambda \). The elements \( \{u, t^\mu | u \in \text{Sym} t, \mu \in X^{++}\} \) generate the algebra \( \mathcal{D}(T) \). Therefore, the elements \( \{u_A, f^\mu_A | u \in \text{Sym} t, \mu \in X^{++}\} \) generate the algebra \( A \), by Theorem 5.3.1.

Let \( \text{ord} u \) denote the order of a differential operator \( u \in \mathcal{D}(\widetilde{B}) \). Clearly, one has \( f^\mu \in \text{Sym}^{n}(\mathcal{D}(\widetilde{B}))^\mu \). Thus, we have

\[
 f^\mu_A \in F_0 A(\mu), \quad \forall \mu \in X^{++}, \quad \text{resp.} \quad u_A \in F_{2n} A(0), \quad \forall u \in \text{Sym}^n t.
\]
Furthermore, \( \text{gr}_0(f_A^\mu) \neq 0 \), resp. \( \text{gr}_{2n}(u_A) \neq 0 \). Let \( \text{symb} \) be the isomorphism of Proposition \[5.5.1\] and recall the isomorphism \( p_\psi \) from Section \[5.2\]. One has the following diagram of graded algebra maps:

\[
\begin{array}{ccc}
\mathbb{C}[T^*T] & \xrightarrow{p_\psi} & \mathbb{C}[T^*\mathcal{B}] \\
\xrightarrow{\text{symb}} & & \xrightarrow{\text{gr}} \mathbb{C}[T^*T] \xrightarrow{\text{gr}^*} \text{gr}(T) = \mathbb{C}[T^*T].
\end{array}
\] (5.5.3)

Let \( p_A = p_\psi \circ \text{symb} \). It is immediate from the construction of the maps that, for any element \( a \) from our generating set \( \{u_A, f_A^\mu | u \in \text{Sym} t, \mu \in \mathbb{X}^{++} \} \) of \( A \), one has \( p_A(\text{gr} a) = (\text{gr}^*a)(\text{gr} a) \). It follows that the map \( (\text{gr}^*a) \circ p_A^{-1} : \mathbb{C}[T^*T] \to \mathbb{C}[T^*T] \) is the identity. We deduce that the map \( \text{gr}^* \) is a graded algebra isomorphism, proving the first statement of the proposition.

For any, not necessarily dominant, \( \lambda \in \mathbb{X}^* \), we let \( f_A^{(\lambda)} \) be the preimage of \( t^\lambda \) under the isomorphism \( j^* \) of Theorem \[5.5.1\]. This agrees with the notation \( f_A^{(\lambda)} \) for dominant \( \lambda \).

It is known that the map \( \text{Sym} t \to \mathcal{D}(\mathcal{B}) \), \( u \mapsto u_r \), intertwines the dot-action on \( \text{Sym} t \) and the Gelfand-Graev action on \( \mathcal{D}(\mathcal{B}) \), cf. [BBP]. It follows that the map \( u \mapsto u_A \) intertwines the dot-action on \( \text{Sym} t \) and the \( W \)-action on \( A \). Thus, to prove that the map \( j^* \) is \( W \)-equivariant, it remains to show that for any \( \mu \in \mathbb{X}^{++} \) and \( w \in W \), one has \( F_w(f_A^\mu) = f_A^{w(\mu)} \). First, it is immediate from the construction of the action that for any \( \lambda \in \mathbb{X}^* \) and \( w \in W \), the map \( F_w \) yields an isomorphism \( \mathcal{D}(\mathcal{B})^{(\lambda)} \cong \mathcal{D}(\mathcal{B})^{(w(\lambda))} \), cf. [BBP], [GR]. Also, it was shown in the course of the proof of [BBP] Lemma 3.18, that one has

\[
\text{ord } F_w(u) = \text{ord } u + \frac{1}{2}\langle \lambda, h \rangle - \frac{1}{2}\langle w(\lambda), h \rangle, \quad \forall \lambda \in \mathbb{X}^*, \ u \in \mathcal{D}(\mathcal{B})^{(\lambda)}, \ w \in W.
\]

It follows that the automorphisms \( F_w, \) of \( \mathcal{D}(\mathcal{B}) \), respect the Kazhdan filtration. Hence, the induced automorphisms of the algebra \( A \) have similar properties. In particular, the map \( F_w : A^{(\lambda)} \to A^{(w(\lambda))} \) is an isomorphism and the \( W \)-action on \( A \) induces a \( W \)-action on \( \text{gr} F^H W \) by \( \mathbb{Z} \)-graded algebra automorphisms.

Next, we observe that for any \( \lambda \in \mathbb{X}^* \), the space \( \mathcal{D}(T)^{(\lambda)} \) is a rank-one free (say, left) \( \text{Sym} t \)-module with generator \( t^\lambda \). Therefore, the space \( A^{(\lambda)} \) is a rank one free \( \text{Sym} t \)-module with generator \( f_A^\lambda \). On the other hand, we know that the map \( F_w : A^{(\mu)} \to A^{(w(\mu))} \) is an isomorphism and, for all \( u \in \text{Sym} t, \mu \in \mathbb{X}^{++} \), we have \( F_w(f_A^\mu u_A) = F_w(f_A^\mu)(w \cdot u)_A \). It follows that the element \( F_w(f_A^\mu) \) is a generator of \( A^{(w(\mu))} \) as a \( \text{Sym} t \)-module. The generator of a rank-one free \( \text{Sym} t \)-module is determined uniquely up to a nonzero constant factor. We deduce that \( F_w(f_A^\mu) = c_{w,\mu} f_A^{w(\mu)} \), where \( c_{w,\mu} \in \mathbb{C} \) is a nonzero constant. To complete the proof of \( W \)-equivariance of the map \( j^* \), we must show that \( c_{w,\mu} = 1 \) for all \( w \in W \) and \( \mu \in \mathbb{X}^{++} \). We know that the elements \( F_w(f_A^\mu) \) and \( f_A^{w(\mu)} \) both belong to \( F_0 A^{(w(\mu))} \). Hence, in \( \text{gr}_0 A \), we have an equation \( \text{gr}(F_w(f_A^\mu)) = c_{w,\mu} \cdot \text{gr} f_A^{w(\mu)} \). Further, it follows from [GR] Proposition 5.4.1 that the isomorphism \( p_A = p_\psi \circ \text{symb} \), in (5.5.3), respects the \( W \)-actions. Therefore, we compute

\[
p_A(\text{gr}(F_w(f_A^\mu))) = p_A(F_w(\text{gr} f_A^\mu)) = w \cdot p_A(\text{gr} f_A^\mu) = w \cdot f_A^{w(\mu)} = (\text{gr}^* j)(\text{gr} f_A^{w(\mu)}) = p_A(\text{gr} f_A^{w(\mu)}).
\]

It follows that \( \text{gr}(F_w(f_A^\mu)) = f_A^{w(\mu)} \), hence \( c_{w,\mu} = 1 \), and we are done.

\[ \square \]

6. THE MIURA BIMODULE

6.1 Proof of Theorem \[5.5.4\] In this subsection, we consider the actions of \( N \) and \( \bar{N} \) on \( G \) by right translations. Let \( D_r = \mathcal{D}(G) \bigotimes \mathfrak{n}_r^\psi \). The natural projection \( p : G \to G/\bar{N}_r \) is an affine morphism. Therefore, for any open affine \( U \subset G/\bar{N}_r \), the set \( p^{-1}(U) \) is affine. Let \( A_U := \Gamma(p^{-1}(U), \mathcal{D}_G) \). Given a left \( D_r \)-module \( F \), put \( F_U = A_U \bigotimes_{\mathcal{D}_r} F \). We have \( A_U \otimes \mathcal{D}(G) (\mathcal{D}(G)/\mathcal{D}(G)\mathfrak{n}_r^\psi \otimes_{D_r} F) = \).
We have a chain of equivalences of categories of left modules
\[ D_r\text{-mod} = \mathcal{D}(G) / n_r^\psi\text{-mod} \cong (\mathcal{D}(G), N_r, \psi) \cong (\mathcal{D}_G, N_r, \psi)\text{-mod}. \] (6.1.1)

Let \( F \in (\mathcal{D}_G, N_r, \psi)\text{-mod} \) be the object that corresponds to an object \( F \in D_r\text{-mod} \) via the above equivalences. Thus, \( \Gamma(p^{-1}(U), F) = A_U / A_U n_r^\psi \otimes_{A_U n_r^\psi} F_U \). For any \( j \in \mathbb{Z} \), by the definition of the functor
\[
\int^\text{derived}_p : \mathcal{D}_G\text{-mod} \rightarrow \mathcal{D}_G^{\text{op}}\text{-mod} \rightarrow \mathcal{D}_G^{\text{op}}\text{-mod} = \mathcal{B}_G\text{-mod},
\]
one has \( \Gamma(U, \int^\text{derived}_p F) = H_{-j}(\n, \Gamma(p^{-1}(U), F)) \). We deduce that \( \Gamma(U, \int^\text{derived}_p F) = 0 \), and there is an isomorphism \( \Gamma(U, \int^\text{derived}_p F) \cong \mathcal{U} \otimes_{\mathcal{Z}_G} F_U \). In particular, the functor \( \int^\text{derived}_p : D_r\text{-mod} \rightarrow \mathcal{D}_G / N_r\text{-mod} \) is exact.

To complete the proof, we recall that the averaging functor \( F \mapsto \int^\text{derived}_p (\Omega_{N_r} \times F) \) on the derived category of \( \mathcal{D}_G\)-modules is isomorphic to the functor \( p^* \circ \int^\text{derived}_p \), cf. [MV, Section 2.5]. The functor \( p^* : \mathcal{D}_G / N_r\text{-mod} \rightarrow (\mathcal{D}_G, N_r)\text{-mod} \) is well-known to be an equivalence, and the theorem follows.

Let \( \mathcal{M} = \n, \mathcal{D}_G / \mathcal{D}_G n_r^\psi \). This is a sheaf of coinvariants, which is a sheaf of vector spaces on \( G \), to be called the Miura sheaf. Using an isomorphism \( \mathcal{D}_G \cong \mathcal{O}_G \otimes \mathcal{U}_G \) and the PBW theorem for \( \mathcal{U}_G \), it is easy to show that for any affine open \( V \subset G \), there is an isomorphism \( \Gamma(V, \mathcal{M}) = \n, \mathcal{D}(V) / \mathcal{D}(V)n_r^\psi \). According to Section 3.1, the sheaf \( p_* \mathcal{M} \) has the natural structure of a \( (\n, \mathcal{D}_G, \mathcal{D}(G) / n_r^\psi) \)-bimodule. Recall that \( \mathcal{D}(G) / n_r^\psi = D_r \). Also, by Section 3.1, one has algebra isomorphisms \( \n, \mathcal{D}(G) / \n, \mathcal{D}(G) \cong (p, \mathcal{D}_G)^{\text{op}} / \n, \mathcal{D}(G) \cong \n, \mathcal{D}_G \mathcal{B}_G \). Thus, we can view \( p_* \mathcal{M} \) as a \( (\mathcal{D}_G, D_r) \)-bimodule. By Proposition 3.1.4(ii), there is a canonical isomorphism
\[ p_* \mathcal{M} \cong \mathcal{U} \otimes_{\mathcal{Z}_G} (p, \mathcal{D}_G) / n_r^\psi, \] (6.1.2)
of sheaves of \( (\mathcal{U}, D_r) \)-bimodules. Furthermore, the proof of Theorem 1.5.4 shows that on the category \( D_r\text{-mod} \), there is an isomorphism of functors
\[
\int^\text{derived}_p (\mathcal{D}_G / \mathcal{D}_G n_r^\psi \otimes_{D_r} -) \cong p_* \mathcal{M} \otimes_{D_r} (-). \] (6.1.3)

6.2. From \( \mathcal{W} \)-modules to \( \mathcal{D}(T)\)-modules. We are going to mimic formulas (6.1.2)–(6.1.3) in the setting where the ring \( \mathcal{D}(G) \) is replaced by the ring \( D_I := \mathcal{D}(G) / n_I^\psi \).

View \( D_I \) as an \( (\mathcal{U} n_r, \mathcal{U} n_r) \)-bimodule, where the left, resp. right, action is provided by left, resp. right, multiplication inside the algebra \( D_I \) by the elements of \( \n, \mathcal{D}_r \), resp. \( n_r \). The analogue of the Miura sheaf \( \mathcal{M} \) is played by \( \mathcal{M} := \n, D_I \setminus D_I / D_I n_I^\psi \). By Section 3.1, this space of coinvariants has the structure of a \( (\n, D_I, D_I / n_I^\psi) \)-bimodule. By Lemma 5.4.1, we have \( D_I / n_I^\psi \cong \mathcal{W} \), and there is a chain of algebra isomorphisms
\[ \n, D_I = \n, \mathcal{D}_G (\mathcal{D}(G) / n_I^\psi) = (\n, \mathcal{D}_G (\mathcal{D}(G) / \n, \mathcal{D}_G)) / n_I^\psi = \mathcal{D}(G) / n_I^\psi = \mathcal{D}(T), \] where the last isomorphism comes from Theorem 5.3.1(i). We conclude that \( \mathcal{M} \) has the structure of a \( (\mathcal{D}(T), \mathcal{W}) \)-bimodule. Further, according to Proposition 3.1.4(ii), one has natural isomorphisms
\[ \mathcal{M} = \n, D_I \setminus D_I / D_I n_I^\psi \cong \text{Sym}_t \otimes_{Z} (D_I / n_I^\psi) \cong \text{Sym}_t \otimes_{Z} \mathcal{W}. \]
We let the group $W$ act on $\text{Sym} \, t \otimes_{Z\mathfrak{g}} \mathcal{W}$ by $w(a \otimes h) = (w \cdot a) \otimes h$, where $a \mapsto w \cdot a$ is the dot-action. It follows from the construction that the left action $\mathcal{D}(T) \otimes \mathcal{M} \to \mathcal{M}$ is a $W$-equivariant map. Hence, this action can be extended to a left action of $W \times \mathcal{D}(T)$ on $\mathcal{M}$. Thus, $\mathcal{M}$ acquires the structure of a $(W \times \mathcal{D}(T), \mathcal{W})$-bimodule, to be called the Miura bimodule.

For any $\mathcal{W}$-module $F$, the $W$-action on $\mathcal{M}$ gives the $\mathcal{D}(T)$-module $\mathcal{M} \otimes_\mathcal{W} F$ a $W$-equivariant structure. Thus we obtain a functor $\mathcal{W}$-mod $\to W \times \mathcal{D}(T)$-mod, $F \mapsto \mathcal{M} \otimes_\mathcal{W} F$. This functor is exact since $\mathcal{M} \otimes_\mathcal{W} (-) = (\text{Sym} \, t \otimes_{Z\mathfrak{g}} \mathcal{W}) \otimes_\mathcal{W} (-) = \text{Sym} \, t \otimes_{Z\mathfrak{g}} (-)$, and $\text{Sym} \, t$ is free over $Z\mathfrak{g}$.

**Proposition 6.2.1.** The functor $\mathcal{W}$-mod $\to W \times \mathcal{D}(T)$-mod, $F \mapsto \mathcal{M} \otimes_\mathcal{W} F$, takes holonomic $\mathcal{W}$-modules to holonomic $\mathcal{D}(T)$-modules.

**Proof.** Using an analogue of the composition of the chain of equivalences in (6.1.1), we obtain an equivalence $(\mathcal{D}(G),N_l \times N_r,\psi \times \psi) \text{-mod} \to \mathcal{W}$-mod, to be denoted $I$. Further, Theorem 5.3.1(ii) yields an equivalence $(\mathcal{D}(G)/N_r,\psi) \text{-mod} \to \mathcal{D}(T)$-mod, which we denote by $J$. Recall the notation $D_l = \mathcal{D}(G)/\mathfrak{n}_l^\psi$, resp. $D_r = \mathcal{D}(G)/\mathfrak{n}_r^\psi$, and the projection $\pi : G \to \tilde{G} = G/N_r$.

We consider the following diagram of functors where $\mathcal{D}$ stands for $\mathcal{D}(G)$ and horizontal inclusions in the middle of the diagram are the natural full embeddings:

$$
\begin{array}{ccc}
\mathcal{W}\text{-mod} & \xrightarrow{I} & (\mathcal{D},N_l \times N_r,\psi \times \psi)\text{-mod} \\
\mathcal{D}(T)\text{-mod} & \xrightarrow{J} & (\mathcal{D},N_l \times N_r,\psi \times 0)\text{-mod} \\
\end{array}
$$

Using $J$ is an equivalence and formula (6.1.3), it is easy to verify that two composite functors $\mathcal{W}$-mod $\to \mathcal{D}(G)/N_r$-mod, along the perimeter of the diagram, are isomorphic. It follows that the functor $f_p^0 \circ I$ is isomorphic to the functor $J \circ \mathcal{M} \otimes_\mathcal{W} (-)$. This implies the statement of the proposition since the functor $f_p^0$ sends holonomic $\mathcal{D}$-modules to holonomic $\mathcal{D}$-modules. \qed

**Remark 6.2.2.** For $F \in (\mathcal{D}(G),N_l \times N_r,\psi \times \psi)$-mod, the $W$-equivariant structure on the $\mathcal{D}(T)$-module $J(f_p^0 \cdot F)$ is not immediately visible from the definition of the functor $J^{-1} \circ f_p^0$. \end{proof}

The proof of the following result is left for the reader.

**Lemma 6.2.3.** View $\mathcal{M}$ as a $\mathcal{D}(T) \otimes \mathcal{W}^{\text{op}}$-module. Then, we have $\text{SS}(\mathcal{M}) = \Lambda$, the Lagrangian subvariety from Proposition 2.3.3. In particular, $\mathcal{M}$ is a holonomic $\mathcal{D}(T) \otimes \mathcal{W}^{\text{op}}$-module.

7. **Nil-Hecke algebras**

7.1. **Degenerate nil-Hecke algebras.** Let $\Sigma \subset R \subset \mathfrak{h}^*$ be a reduced root system, where $\Sigma$ denotes the set of simple roots. Let $\mathcal{W}$ be the corresponding Coxeter group, $\{s_\alpha, \alpha \in \Sigma\}$ the set of generators of $\mathcal{W}$, and $\ell : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ the length function. For $\alpha, \beta \in \Sigma$, let $m_{\alpha, \beta}$ denote the order of the element $s_\alpha s_\beta \in \mathcal{W}$. The nil-Hecke algebra $\mathcal{H}(\mathcal{W})$ is defined as a $\mathbb{C}$-algebra with generators $t_\alpha$, $\alpha \in \Sigma$, subject to the relations

$$
(t_\alpha)^2 = 0, \quad (t_\alpha t_\beta)^{m_{\alpha, \beta}} = (t_\beta t_\alpha)^{m_{\alpha, \beta}}, \quad \forall \alpha, \beta \in \Sigma.
$$

(7.1.1)

For each $w \in \mathcal{W}$, there is an element $t_w \in \mathcal{W}$ defined as a product $t_{w_1} \cdots t_{w_k}$ for a reduced factorization $w = s_{\alpha_1} \cdots s_{\alpha_k}$ into simple reflections. It is known that $t_w$ is independent of such a factorization and one has $t_w t_y = t_{wy}$ if $\ell(w) + \ell(y) = \ell(wy)$ and $t_w t_y = 0$, otherwise, see [Ku]. Moreover, the set $\{t_w, \ w \in \mathcal{W}\}$ is a $\mathbb{C}$-basis of $\mathcal{H}(\mathcal{W})$.

Let $\mathbb{C}(\mathfrak{h}^*)$ be the field of fractions of the algebra $\text{Sym} \mathfrak{h} = \mathbb{C}[\mathfrak{h}^*]$. For every $\alpha \in R$, define an element $\theta_\alpha = \frac{1}{\ell} (s_\alpha - 1) \in \mathcal{W} \times \mathbb{C}(\mathfrak{h}^*)$. There is a natural algebra map $\mathcal{H}(\mathcal{W}) \hookrightarrow \mathbb{C}(\mathfrak{h}^*) \times \mathcal{W}$ given
on the generators by \( t_\alpha \mapsto \theta_\alpha \), \( \alpha \in \Sigma \). Let \( \mathcal{H}(\mathfrak{h}, W) \) be a free left \( \operatorname{Sym} \mathfrak{h} \)-submodule of \( \mathbb{C}(\mathfrak{h}^*) \rtimes W \) with basis \( \theta_w \), \( w \in W \). It is immediate to check that \( \mathcal{H}(\mathfrak{h}, W) \) is a subalgebra of \( \mathbb{C}(\mathfrak{h}^*) \rtimes W \) and that \( \mathcal{H}(\mathfrak{h}, W) \) is free as a \( \operatorname{Sym} \mathfrak{h} \)-module via right multiplication, [Ku].

**Proposition 7.1.2 ([Ku], Theorem 11.1.2).** The algebra \( \mathcal{H}(\mathfrak{h}, W) \) is generated by the algebras \( \mathcal{H}(W) \) and \( \operatorname{Sym} \mathfrak{h} \) subject to the following commutation relations:

\[
\theta_\alpha \cdot s_\alpha(h) - h \cdot \theta_\alpha = \langle \alpha, h \rangle, \quad \forall h \in \mathfrak{h}, \ \alpha \in \Sigma. \tag{7.1.3}
\]

The subspace \( \operatorname{Sym} \mathfrak{h} \subset \mathbb{C}(\mathfrak{h}^*) \) is stable under the action of the subalgebra \( \mathcal{H}(\mathfrak{h}, W) \subset \mathbb{C}(\mathfrak{h}^*) \rtimes W \) on \( \mathbb{C}(\mathfrak{h}^*) \). Conversely, one has

**Theorem 7.1.4 ([Ku], Section 11.2).** The \( \mathcal{H}(\mathfrak{h}, W) \)-action on \( \operatorname{Sym} \mathfrak{h} \) is faithful, and we have

\[
\mathcal{H}(\mathfrak{h}, W) = \{ u \in \mathbb{C}(\mathfrak{h}^*) \rtimes W \mid u(\operatorname{Sym} \mathfrak{h}) \subseteq \operatorname{Sym} \mathfrak{h} \}.
\]

We equip \( \operatorname{Sym} \mathfrak{h} \) with the natural grading and extend it to a grading on \( \mathcal{H}(\mathfrak{h}, W) \) by placing \( \theta_w \) in degree \( -\ell(w) \). This makes \( \mathcal{H}(\mathfrak{h}, W) \) a \( \mathbb{Z} \)-graded algebra, resp. \( \operatorname{Sym} \mathfrak{h} \) a \( \mathbb{Z}_{\geq 0} \)-graded \( \mathcal{H}(\mathfrak{h}, W) \)-module.

The assignment \( s_\alpha \mapsto \alpha \cdot \theta_\alpha + 1, \ \alpha \in \Sigma \), extends to an algebra embedding \( \mathbb{C}W \hookrightarrow \mathcal{H}(\mathfrak{h}, W) \). We will identify \( \mathbb{C}W \) with its image, which is contained in the degree zero homogeneous component of \( \mathcal{H}(\mathfrak{h}, W) \). It is clear that for any \( w \in W \) and \( \alpha \in \mathcal{R} \), inside \( W \times \mathbb{C}(\mathfrak{h}^*) \), one has \( \theta_w(\alpha) = w \cdot \theta_\alpha \cdot w^{-1} \).

Taking \( \alpha \) to be a simple root, we see that the element \( \theta_{\beta} \) belongs to \( \mathcal{H}(\mathfrak{h}, W) \) for any root \( \beta \).

Given an \( \operatorname{Sym} \mathfrak{h} \rtimes W \)-module \( M \) and \( \alpha \in \Sigma \), let \( M^\pm = \{ m \in M \mid s_\alpha(m) = \pm m \} \). It is clear that the action of \( \alpha \in \mathfrak{h} \subset \operatorname{Sym} \mathfrak{h} \) on \( M \) sends \( M^\pm \) to \( M^\mp \). Observe also that the action of \( \mathcal{H}(\mathfrak{h}, W) \) on any \( \mathcal{H}(\mathfrak{h}, W) \)-module is \( (\operatorname{Sym} \mathfrak{h})^W \)-linear since the center of \( \mathcal{H}(\mathfrak{h}, W) \) equals \( (\operatorname{Sym} \mathfrak{h})^W \).

**Lemma 7.1.5.** Assume that the group \( W \) is finite. Then the \( \mathcal{H}(\mathfrak{h}, W) \)-action on \( \operatorname{Sym} \mathfrak{t} \) yields an algebra isomorphism

\[
\mathcal{H}(\mathfrak{h}, W) \cong \operatorname{End}_{(\operatorname{Sym} \mathfrak{h})^W} \operatorname{Sym} \mathfrak{h}. \tag{7.1.6}
\]

Furthermore, for a \( \operatorname{Sym} \mathfrak{h} \rtimes W \)-module \( M \), the following are equivalent:

(i) The natural map \( \operatorname{Sym} \mathfrak{h} \otimes_{(\operatorname{Sym} \mathfrak{h})^W} M^W \rightarrow M \) is an isomorphism.

(ii) The action of \( W \rtimes \operatorname{Sym} \mathfrak{h} \) on \( M \) admits a (necessarily unique) extension to an \( \mathcal{H}(\mathfrak{h}, W) \)-action on \( M \).

(iii) For every \( \alpha \in \Sigma \), the action map \( \alpha : M^+ \rightarrow M^- \) is a bijection.\(^2\)

**Proof.** Recall that \( \operatorname{Sym} \mathfrak{h} \) is a free \( (\operatorname{Sym} \mathfrak{h})^W \)-module of rank \( \#W \). Isomorphism (7.1.6) is a simple consequence of Theorem 7.1.4. The equivalence of (i) and (ii) now follows from Morita equivalence of the algebras \( (\operatorname{Sym} \mathfrak{h})^W \) and \( \operatorname{End}_{(\operatorname{Sym} \mathfrak{h})^W} \operatorname{Sym} \mathfrak{h} \). The equivalence of (ii) and (iii) easily follows from the formula \( s_\alpha = \alpha \cdot \theta_\alpha + 1 \).

\[\Box\]

### 7.2. Degenerate nil-DHA

Let \( \Sigma \subset R \subset \mathfrak{t}^* \) be a finite reduced root datum, and \( \Sigma_{\text{aff}} = \Sigma \cup \{ \alpha_0 \} \subset R_{\text{aff}} \subset \mathfrak{t}_{\text{aff}}^* \) an associated affine root datum. Let \( \mathbb{X}^* \) and \( Q \) be the weight and root lattice of \( R \), respectively. Let \( \widetilde{W} = W \ltimes \mathbb{X}^* \supset W_{\text{aff}} = W \ltimes Q \) be the extended affine Weyl group. Thus, \( \widetilde{W} = (\mathbb{X}^*/Q) \ltimes W_{\text{aff}} \).

Similar to the case of affine Hecke algebras, we define \( \mathcal{H}(t_{\text{aff}}, \widetilde{W}) := \mathbb{X}^*/Q \mathcal{H}(t_{\text{aff}}, W_{\text{aff}}) \). To avoid confusion, we write elements of \( \mathbb{X}^* \), viewed as a subgroup of \( \widetilde{W} \), in the form \( e^\mu \) and also use the same symbol for the image of that element under the algebra embedding \( \mathbb{C}W \hookrightarrow \mathcal{H}(t_{\text{aff}}, \widetilde{W}) \).

Thus, for any root \( \alpha \in R \) in \( \mathcal{H}(t_{\text{aff}}, \widetilde{W}) \), there is an element \( \theta_{w, \alpha} = (\theta_w \text{ for } w = e^\alpha) \), and also a different element, \( \theta_{\alpha} \), the Demazure element associated with \( \alpha \) viewed as a (not necessarily simple) root in \( R_{\text{aff}} \).

\(^2\)The equivalence of (ii) and (iii) was pointed out to me by Gwyn Bellamy.
Let $h \in \text{aff}$ be the minimal imaginary coroot. We identify $\text{aff}$ with $t \oplus Ch$, resp. $\text{Sym}(\text{aff})$ with $C[t^*][h]$. The natural linear action of $\tilde{W}$ on $\text{aff}$ induces a $\tilde{W}$-action on $C[t^*][h]$, by algebra automorphisms. Explicitly, for $\mu \in X^*$, the action of $e^\mu$ on $C[t^*][h]$ is given by the formula $e^\mu f(x, h) = f(x - h\mu, h)$. The action of the algebra $\mathcal{H}(\text{aff}, \tilde{W})$ on $C[t^*][h]$ extends to a faithful $\mathcal{H}(\text{aff}, \tilde{W})$-action which agrees with the $\tilde{W}$-action, where $\tilde{W}$ is viewed as a subset of $\mathcal{H}(\text{aff}, \tilde{W})$ via the canonical embedding. From the above formula for the action of $e^\mu$, one finds the following commutation relations in $\mathcal{H}(\text{aff}, \tilde{W})$:

$$\xi \cdot e^\mu = e^\mu \cdot (\xi + (\mu, \xi) \cdot h), \quad \mu \in X^*, \xi \in t.$$  \hfill (7.2.1)

Let $\mathcal{D}_h(T)$ be the Rees algebra of the algebra $\mathcal{D}(T)$ of differential operators, equipped with the filtration by order of the differential operator. Recall the notation $t^\mu \in C[T]$ for the function on $T$ associated with $\mu \in X^*$. For $\xi \in t$, let $\partial_\xi$ denote the corresponding translation invariant vector field on $T$. We may view $t^\mu$ and $\partial_\xi$ as elements of $\mathcal{D}_h(T)$ placed in degrees 0 and 1, respectively. The commutation relations (7.2.1) are identical to the commutation relations in the algebra $\mathcal{D}_h(T)$ between the elements $t^\mu$ and $\partial_\xi$. Further, it follows from (7.2.1) that the set $S := (\text{Sym} \text{aff}) \setminus \{0\}$ is an Ore subset of the algebra $\mathcal{D}_h(T)$. The corresponding noncommutative localization $S^{-1} \cdot \mathcal{D}_h(T)$ may be viewed as a kind of microlocalization of $\mathcal{D}_h(T)$. We obtain algebra embeddings

$$W \times \mathcal{D}_h(T) \xleftarrow{\text{w} \rightarrow \text{w}, t^\mu \rightarrow e^\mu, \xi \rightarrow \partial_\xi} \mathbb{H} \leftarrow \tilde{W} \times Q(\text{aff}) = W \times S^{-1} \cdot \mathcal{D}_h(T).$$  \hfill (7.2.2)

Given a $C[h]$-algebra $A$ and $c \in C$, we let $A|_{h=c} := A/(h - c)A$ denote its specialization at $h = c$. We have $\mathcal{D}_h(T)|_{h=1} = \mathcal{D}(T)$, resp. $\mathcal{D}_h(T)|_{h=0} = C[T^*T]$. We define $\mathbb{H} = \mathcal{H}(\text{aff}, \tilde{W})|_{h=1}$. The graded algebra embedding $W \times \mathcal{D}_h(T) \hookrightarrow \mathbb{H}$ induces an embedding $W \times \mathcal{D}(T) \hookrightarrow \mathbb{H}$, resp. $W \times C[T^*T] \hookrightarrow \text{gr} \mathbb{H}$, of specializations at $h = 1$, resp. $h = 0$.

Recall from the introduction, the notation $e = \frac{1}{\#W} \sum_{w \in W} w \in C$ and the spherical algebra $e\mathcal{H}(\text{aff}, \tilde{W})e$, resp. $\mathcal{H}(\text{aff}, \tilde{W})^{\text{sp}} := e\mathbb{H}e$. The space $\mathcal{H}(\text{aff}, \tilde{W})e$, resp. $\mathbb{H}e$, has the natural structure of an $(\mathcal{H}(\text{aff}, \tilde{W}), e\mathcal{H}(\text{aff}, \tilde{W})e)$-bimodule, resp. $(\mathbb{H}, \mathbb{H}^{\text{sp}})$-bimodule. The embedding $W \times \mathcal{D}_h(T) \hookrightarrow \mathbb{H}$ induces an embedding $\mathcal{D}(T)^W \hookrightarrow \mathbb{H}^{\text{sp}}$, resp. $C[T^*T]^W \hookrightarrow \text{gr} \mathbb{H}^{\text{sp}}$.

**Lemma 7.2.3.** Let $c \in C$ and put $\mathcal{H}_c = \mathcal{H}(\text{aff}, \tilde{W})|_{h=c}$. The algebras $\mathcal{H}(\text{aff}, \tilde{W})$ and $e\mathcal{H}(\text{aff}, \tilde{W})e$, resp. $\mathcal{H}_c$ and $e\mathcal{H}_c e$, are Morita equivalent and the action map induces an algebra isomorphism

$$\mathcal{H}(\text{aff}, \tilde{W}) \to \text{End}_{e\mathcal{H}(\text{aff}, \tilde{W})} e\mathcal{H}(\text{aff}, \tilde{W}) e, \quad \text{resp.} \quad \mathcal{H}_c \to \text{End}_{e\mathcal{H}_c} \mathcal{H}_c e.$$  

**Proof.** It is immediate from (7.2.6) that $e\mathcal{H}(W, t)e = (\text{Sym} t)^W$ and, moreover, there exist elements $h'_t, h''_t \in \mathcal{H}(t, W)$ such that one has $1 = \sum_i h'_t \cdot h''_t$. Since $\mathcal{H}(W, t)$ is a subalgebra of $\mathcal{H}(\text{aff}, \tilde{W})$, this equation may be viewed as an equation in $\mathcal{H}(\text{aff}, \tilde{W})$. Thus, we have $\mathcal{H}(\text{aff}, \tilde{W})e\mathcal{H}(\text{aff}, \tilde{W}) = \mathcal{H}(\text{aff}, \tilde{W})$, which is known to imply all statements of the lemma involving $\mathcal{H}(\text{aff}, \tilde{W})$. The case of specializations at $h = c$ is similar. \hfill \Box

**Proposition 7.2.4.** (i) Let $M$ be a $W \times \mathcal{D}_h(T)$-module. The action of $W \times \mathcal{D}_h(T)$ on $M$ can be extended (necessarily uniquely) to an $\mathcal{H}(\text{aff}, \tilde{W})$-action on $M$ if and only if the natural map $C[t^*] \otimes_{C[t^*]} M^W \to M$ is an isomorphism. A similar statement holds in the case of specializations at $h = c$, for any $c \in C$.

(ii) Let $L$ be a $\mathcal{D}_h(T)^W$-module. The action of $\mathcal{D}_h(T)^W$ on $L$ can be extended (necessarily uniquely) to an $\mathbb{H}^{\text{sp}}$-action on $L$ if and only if the map $\text{Sym} t \otimes (\text{Sym} t)^W L \to \mathcal{D}(T)^W \otimes \mathcal{D}(T)^W L$, induced by the inclusion $\text{Sym} t \hookrightarrow T^*$, is an isomorphism.

**Proof.** The simple reflection $s_{0,0} : \text{aff} \to (\text{aff})$ is a reflection with respect to the hyperplane $\delta(x) = h$, where $\delta \in R$ is the highest root. The corresponding Demazure operator acts on $C[t^*][h]$ as follows:

$$s_{0,0}(f)(x, h) = \frac{f(x^0_s(x, h)) - f(x, h)}{x - (x, h) - h} = f(x - (x, h) - h - f(x, h)).$$  \hfill (7.2.5)
For any weight \( \mu \in X^* \), we compute
\[
(e^\mu \circ \theta_\delta \circ e^{-\mu} f)(x, h) = (e^\mu \circ \theta_\delta f)(x + h \mu, h) = e^\mu \left( \frac{f(x - \langle x, \delta \rangle \delta + h \mu, h) - f(x + h \mu, h)}{\langle x, \delta \rangle} \right)
\]
\[
= \frac{f(x - h \mu - \langle x - h \mu, \delta \rangle \delta + h \mu, h) - f(x - h \mu + h \mu, h)}{\langle x - h \mu, \delta \rangle}
\]
\[
= \frac{f(x - \langle x, \delta \rangle - h \langle \mu, \delta \rangle \delta) - f(x, h)}{\langle x, \delta \rangle - h \langle \mu, \delta \rangle}.
\]

(7.2.6)

Now let \( M \) be as in (i). By Lemma 7.1.5 the action of \( W \ltimes \text{Sym} t \) on \( M \) can be extended to an action of the algebra \( \mathcal{H}(t, \tilde{W}) \). Next, we use the action of the subalgebra \( \mathbb{C}[T] \subset \mathcal{D}_{t}(T) \) on \( M \). Specifically, since any root is \( W \)-conjugate to a simple root, there exists a weight \( \mu \in X^* \) such that we have \( \langle \mu, \delta \rangle = 1 \). For such a \( \mu \), formula (7.2.6) shows that in \( \mathcal{H}(t_{\text{aff}}, \tilde{W}) \), one has \( \theta_{\alpha_0} = e^\mu \circ \theta_\delta \circ e^{-\mu} \).

Accordingly, we let \( \theta_{\alpha_0} \) act on \( M \) by the operator \( e^\mu \circ \theta_\delta \circ e^{-\mu} \). We claim that the action of \( \theta_{\alpha_0} \) thus defined and the actions of \( \mathcal{H}(t, W) \) and \( \mathbb{C}[T] \) on \( M \) combine together to give \( M \) the structure of an \( \mathcal{H}(t_{\text{aff}}, \tilde{W}) \)-module. Thus, we must check that the relations (7.1.1) and (7.1.3) hold. The relations which do not involve \( \theta_{\alpha_0} \) hold by construction. The equation \( (\theta_{\alpha_0})^2 = 0 \) is clear. Also, using (7.2.1) and (7.2.6) it is straightforward to check that (7.1.3) holds for \( \alpha = \alpha_0 \).

To prove the braid relations we use [Lo2]. Specifically, let \( \mathcal{I}(t_{\text{aff}}) \) be the algebra associated, as in [Lo2, Section 2.7], with the Coxeter group \( \Gamma := W_{\text{aff}} \) and its reflection representation \( \mathfrak{h} := t_{\text{aff}} \). The relations we have checked so far say that \( M \) has the structure of an \( \mathcal{I}(t_{\text{aff}}) \)-module. The algebra \( \mathcal{H}(t_{\text{aff}}, W_{\text{aff}}) \) is a quotient of \( \mathcal{I}(t_{\text{aff}}) \) by a two-sided ideal. Furthermore, the main result of [Lo2] is essentially equivalent, see [Lo2, Section 2.7], to the statement that the canonical map \( \mathcal{I}(\mathfrak{h}) \to \mathcal{H}(\mathfrak{h}, \Gamma) \) is, in fact, an isomorphism, which is what we want. We remark that in [Lo2] the Coxeter group is assumed to be finite. The braid relation \( (\theta_\alpha \theta_\beta)^{m_{\alpha, \beta}} = (\theta_\beta \theta_\alpha)^{m_{\alpha, \beta}} \) was proved in [Lo2, Section 3] under that assumption. However, in the case of an infinite Coxeter group with two generators, \( t_\alpha, t_\beta \), the element \( t_\alpha t_\beta \) has an infinite order, i.e., one has \( m_{\alpha, \beta} = \infty \). Therefore, the corresponding braid relation is, in that case, vacuous. This proves that the map \( \mathcal{I}(\mathfrak{h}) \to \mathcal{H}(\mathfrak{h}, \Gamma) \) is an isomorphism for any, not necessarily finite, Coxeter group \( \Gamma \).

Thus, we have extended the action of \( W \ltimes \mathcal{D}_{t}(T) \) on \( M \) to an \( \mathcal{H}(t_{\text{aff}}, \tilde{W}) \)-action. Note that if such an extension exists, then it is unique since each of the operators \( \theta_\alpha, \alpha \in \Sigma_{\text{aff}}, \) is defined uniquely due to the Morita equivalence (7.1.6) in the special case of the rank-one nil Hecke algebra associated with the root system \( \{ \alpha, -\alpha \} \). Part (i) of Proposition 7.2.4 follows.

Combining part (i) for the specialization at \( h = 1 \) with Morita equivalence of \( W \ltimes \mathcal{D}(T) \) and \( \mathcal{D}(T)^W \) yields part (ii).

By definition, one has \( \mathcal{H}(t_{\text{aff}}, \tilde{W})|_{h=0} = \mathfrak{h} \), resp. \( e\mathcal{H}(t_{\text{aff}}, \tilde{W})e|_{h=0} = \mathfrak{h}\mathfrak{h}_{\text{sph}} \). Note that the last statement of the proposition below implies Proposition 2.3.6.

**Proposition 7.2.7.** There is an algebra isomorphism
\[
\text{gr } \mathfrak{h} \cong W \ltimes \mathbb{C}[T \times t^*, \frac{t^\alpha - 1}{\alpha}, \alpha \in R], \quad \text{resp. } \text{gr } \mathfrak{h}_{\text{sph}} \cong \mathbb{C}[T \times t^*, \frac{t^\alpha - 1}{\alpha}, \alpha \in R]^W.
\]
Furthermore, we have \( (\text{gr } \mathfrak{h})e \cong \mathbb{C}[T \times t^*, \frac{t^\alpha - 1}{\alpha}, \alpha \in R], \) and each of the above objects is flat over \( \mathbb{C}[t^*]^W \).

**Proof.** It is clear from (7.2.1) that the elements \( e^\mu \in \mathcal{H}(t_{\text{aff}}, \tilde{W})|_{h=0}, \mu \in X^* \), and \( \xi \in t \), generate a copy of the algebra \( \mathbb{C}[T^*T] \) inside \( \mathcal{H}(t_{\text{aff}}, \tilde{W})|_{h=0} \). For \( w \in W \), let \( \theta_w \) denote the image of the

\[\text{This correction of an original, incorrect, construction of } \theta_{\alpha_0} \text{ was suggested to me by Gus Lonergan.}\]
element $\theta_w \in \mathcal{H}(t_\text{aff}, \widetilde{W})$ in $\mathcal{H}(t_\text{aff}, \widetilde{W})|_{h=0}$. Then, the commutation relations $(7.1.3)$ imply easily that the assignment

$$\tilde{\theta}_\alpha \mapsto \frac{1}{\alpha}(s_\alpha - 1), \quad \tilde{\theta}_{\alpha_0} \mapsto \frac{1}{\beta}(s_\delta - 1), \quad \alpha \in \Sigma,$$

extends by multiplicativity to a well-defined algebra homomorphism

$$\Phi : \mathfrak{gr} \mathbb{H} = \mathcal{H}(t_\text{aff}, \widetilde{W})|_{h=0} \to W \ltimes T \times t^*, \frac{\ell}{\alpha}, \alpha \in R].$$

(7.2.8)

For any $\mu \in X^*$, using that $t^s \alpha = t^\mu s = t^{(\mu, \tilde{\alpha})}$, we find that

$$t^\mu \frac{1}{\alpha}(s_\alpha - 1) t^{-\mu} s = \frac{1}{\alpha}(t^{(\mu, \tilde{\alpha})} - s_\alpha) + \frac{1}{\alpha}(s_\alpha - 1) = t^{(\mu, \tilde{\alpha})}.\frac{1}{\alpha}.\alpha \in R].$$

(7.2.9)

Letting $\mu$ run through the set of fundamental weights, we see that the image of the map $\Phi$ contains all elements $\frac{\ell}{\alpha}$, $\alpha \in \Sigma$. The algebra on the right of (7.2.8) is generated by these elements and its subalgebra $\widetilde{W} \ltimes \mathbb{C}[T \times t^*]$ follows that the map $\Phi$ is surjective. Also, it is clear that applying $\mathbb{C}(t^*) \otimes_{\mathbb{C}[t^*]} (-)$, a localization functor, to the map $\Phi$ yields an isomorphism between the corresponding localized $\mathbb{C}(t^*)$-modules. Hence, $\ker \Phi$ is a torsion $\mathbb{C}[t^*]$-module.

Further, the algebra $\mathcal{H}(t_\text{aff}, \widetilde{W})$ is a free $\mathbb{C}[t_\text{aff}]$-module with basis $\theta_w, w \in \widetilde{W}$, by definition. It follows that $\mathcal{H}(t_\text{aff}, \widetilde{W})|_{h=0}$ is a free $\mathbb{C}[t^*]$-module with basis $\tilde{\theta}_w, w \in \tilde{W}$, in particular, this module is torsion free. Therefore, we have $\ker \Phi = 0$, so the map $\Phi$ is an isomorphism. Since $\mathbb{C}[t^*]$ is free over $\mathbb{C}[t^*]$, we deduce that $W \ltimes \mathbb{C}[T \times t^*, \frac{\ell}{\alpha}, \alpha \in R]$ is a free $\mathbb{C}[t^*]$-module.

Finally, the map $\Phi$ being an isomorphism, it follows that its restriction to the spherical subalgebra yields an isomorphism $\ker \mathbb{H}^{\text{aff}}|_{h=0} \to \mathbb{C}[T \times t^*, \frac{\ell}{\alpha}, \alpha \in R][W]$. Viewed as a $\mathbb{C}[t^*]$-module, the algebra $\ker \mathbb{H}^{\text{aff}}$ is a direct summand of $\ker \mathbb{H}$, hence it is flat over $\mathbb{C}[t^*]$. This implies that $\mathbb{C}[T \times t^*, \frac{\ell}{\alpha}, \alpha \in R][W]$ is a flat $\mathbb{C}[t^*]$-module, completing the proof. \qed

8. Spherical degenerate nil DAHA via Hamiltonian reduction

8.1. The action of $\mathbb{H}$ on the Miura bimodule $\mathcal{M}$. The following result provides the link between nil Hecke algebras and Hamiltonian reduction.

**Proposition 8.1.1.** The left action of $W \ltimes \mathcal{D}(T)$ on the Miura bimodule $\mathcal{M}$ can be extended to an $\mathbb{H}$-action, making $\mathcal{M}$ an $(\mathbb{H}, W)$-bimodule.

**Proof.** This follows from Proposition [7.2.4] since $\mathcal{M}[W] = \mathcal{W}$, so the map $\text{Sym} t \otimes_{\text{Sym} t} \mathcal{W} \to \mathcal{M} = \text{Sym} t \otimes_{\text{Sym} t} \mathcal{W}$ is an isomorphism. \qed

Recall that the algebra $\mathcal{W}$ comes equipped with the Kazhdan filtration. We equip $\text{Sym} t$ with the natural filtration induced by the grading and equip $\mathcal{M} = \text{Sym} t \otimes_{\mathcal{W}} \mathcal{W}$ with a tensor product filtration. Let, $\mathcal{M}_h$ be an associated Rees module. It is straightforward to see from the construction that the $(\mathbb{H}, \mathcal{W})$-bimodule structure on $\mathcal{M}$ is compatible with the filtrations. Therefore, the left $\mathbb{H}$-action on the Miura bimodule $\mathcal{M}$ can be lifted to an $\mathcal{H}_h$-action on $\mathcal{M}_h$. Thus, $\mathcal{M}_h$ acquires the structure of a $\mathbb{Z}$-graded $(\mathbb{H}_h, \mathcal{W}_h)$-bimodule. We will identify $\mathcal{M}_h$ with $(\text{Sym} t)[h] \otimes_{\text{Sym} t} \mathcal{W}_h$, resp. $\mathcal{M}[W]_h$ with $\mathcal{W}_h$. Let $1_{\mathcal{M}} = 1 \otimes 1 \in (\text{Sym} t)[h] \otimes_{\text{Sym} t} \mathcal{W}_h$ be the generator. Recall that $\mathbb{H}_h = \mathcal{H}(t_\text{aff}, \widetilde{W})$.

The following theorem and its corollary are an extended version of Theorem [12.1] from the introduction.

**Theorem 8.1.2.** (i) The map $\mathcal{H}(t_\text{aff}, \widetilde{W}) \to \mathcal{M}_h, u \mapsto u(1 \otimes 1)$, induces an isomorphism $\mathcal{H}(t_\text{aff}, \widetilde{W}) e \to \mathcal{M}_h$ of graded left $\mathcal{H}(t_\text{aff}, \widetilde{W})$-modules.
(ii) The map \( e\mathcal{H}(t_{\text{aff}})\mathcal{W} \rightarrow \mathbb{M}_h^W = \mathbb{W}_h \), \( e\mathcal{H}(t_{\text{aff}})\mathcal{W} = e\mathcal{H}(t_{\text{aff}})\mathcal{W} \), yields a graded algebra isomorphism

\[
\begin{array}{c}
\mathcal{H}(t_{\text{aff}})\mathcal{W} \\
\downarrow \\
\mathcal{H}(t_{\text{aff}})\mathcal{W} \\
\downarrow \\
\mathcal{H}(t_{\text{aff}})\mathcal{W} \\
\end{array}
\]

such that the following diagram commutes:

\[
\begin{array}{c}
(Sym t)^W [h] \\
\downarrow \\
\mathbb{Z}_h \mathfrak{g} \\
\downarrow \\
\mathbb{W}_h \\
\end{array}
\]

The action of \( \mathcal{H} \) on \( \mathbb{M}_h \) makes \( \mathcal{H} \) a cyclic left \( \mathcal{H} \)-module with generator \( S_h \). The annihilator of \( 1_{S_h} \) is a left ideal \( J \subset \mathcal{H} \), generated by the elements \( \theta_w \), \( w \in \mathcal{T} \). Since \( e\mathcal{H}(t_{\text{aff}})\mathcal{W} \), \( e\mathcal{H}(t_{\text{aff}})\mathcal{W} \), the map \( \mathcal{H} \rightarrow \mathbb{M}_h \), \( h \mapsto h1_{S_h} \), descends to a surjection \( \mathcal{H} e \rightarrow S_h \) with kernel \( J e \). This gives an \( \mathcal{H}^{sph} \)-module surjection \( \mathcal{H}^{sph} = e\mathcal{H} e \rightarrow S_h^W = eS_h \), with kernel \( J^{sph} := eJ e \), a left ideal of \( \mathcal{H}^{sph} \).

Also, write \( \mathcal{H}^{sph} = e\mathcal{H} e \rightarrow S_h^W = eS_h \), with kernel \( J^{sph} := eJ e \), a left ideal of \( \mathcal{H}^{sph} \).

The action of \( \mathcal{H} \) on \( \mathbb{M}_h \) gives a map \( \tau : e\mathcal{H} e \rightarrow \mathbb{W}_h = \mathbb{M}_h^W \), \( h \mapsto h1_{\mathbb{W}_h} \). The image of \( J^{sph} \subset \mathcal{H}^{sph} \) under the map \( \tau \) is a left \( \mathcal{H}^{sph} \)-submodule \( J^{sph} \subset \mathbb{W}_h \). This submodule is not, a priori, stable under left multiplication by \( \mathbb{W}_h \). We let \( \mathcal{K} := \mathbb{W}_h : \tau(J^{sph}) \) be a left ideal of the algebra \( \mathbb{W}_h \) generated by \( \tau(J^{sph}) \). Let \( \mathcal{K} = \cap_{i \in \mathbb{Z}} (K + h\mathbb{W}_h) \) be the ‘closure’ of \( \mathcal{K} \subset \mathbb{W}_h \) in \( \mathcal{H} \)-adic topology.

**Lemma 8.2.1.** One has a decomposition \( \mathbb{W}_h = Z_h \mathfrak{g} \oplus \mathcal{K} \) as a direct sum of \( Z_h \mathfrak{g} \)-stable graded subspaces.

**Proof.** We first prove an analogue of the statement of the lemma at \( h = 0 \). To this end, let \( \mathcal{H}_0 := \mathcal{H}|_{h=0} \), resp. \( \mathcal{H}_0^{sph} = e\mathcal{H}_0 e \). We have a chain of isomorphisms

\[
(\text{gr} \mathcal{H}) e = \mathcal{H}_0 e \cong C[T \times t^*, \frac{t^\alpha - 1}{\alpha}, \alpha \in R] \cong C[3 \times t] \cong \text{Sym} t \otimes (\text{Sym} t)^W \text{gr} \mathbb{W},
\]

where the first, resp. second, and third, isomorphism is Proposition 7.2.7, resp. Theorem 7.2.3,1 and Theorem 7.2.2.

Let \( J_0 \subset \mathcal{H}_0 e \) be the image of \( J \) in \( \mathcal{H}_0 = \mathcal{H}/h\mathcal{H} \) and let \( I \subset C[T \times t^*, \frac{t^\alpha - 1}{\alpha}, \alpha \in R] \), resp. \( L \subset \text{Sym} t \otimes (\text{Sym} t)^W \text{gr} \mathbb{W} \), be an ideal that corresponds to \( J_0 \) via the first, resp. the composite, isomorphism in (8.2.2). The proof of Proposition 7.2.7 combined with (7.2.8)–(7.2.9), shows that the ideal \( I \) is generated by the elements \( \frac{t^\mu - 1}{\mu} \), \( \mu \in \mathbb{X}^* \). It is clear from this description, cf. also (7.2.9), that one has a direct sum decomposition \( C[T \times t^*, \frac{t^\alpha - 1}{\alpha}, \alpha \in R] \cong C[t] \oplus I \), of graded \( \text{Sym} t \)-stable subspaces. Hence, one has a decomposition \( \text{Sym} t \otimes (\text{Sym} t)^W \text{gr} \mathbb{W} = \text{Sym} t \oplus L \). Taking \( W \)-invariants and writing \( J_0^{sph} = J_0^W \), we deduce the following isomorphism that respects the direct sum decompositions:

\[
\mathcal{H}_0^{sph} = (\text{Sym} t)^W \oplus J_0^{sph} \cong \text{gr} \mathbb{W} = (\text{Sym} t)^W \oplus L^W.
\]

It is straightforward to check that the isomorphism \( \mathcal{H}_0^{sph} \rightarrow \text{gr} \mathbb{W} \), in (8.2.3), agrees with \( \tau_0 : \mathcal{H}_0^{sph} \rightarrow \text{gr} \mathbb{W} \), the specialization of the map \( \tau : \mathcal{H}^{sph} \rightarrow \mathbb{W}_h \) at \( h = 0 \). The image of the ideal \( J_0^{sph} \)
equals $\tau_0(J_0^{\text{sph}})$, by definition. We conclude that $L^W = \tau_0(J_0^{\text{sph}})$. Thus, the decomposition on the right of (8.2.3) reads
\[ \mathbb{W}_h/\mathbb{W}_h = (Z_h\mathfrak{g}/h\cdot Z_h\mathfrak{g}) \oplus \tau_0(J_0^{\text{sph}}). \] (8.2.4)

This is equivalent to a pair of equations
\[ \mathbb{W}_h = Z_h\mathfrak{g} + \tau(J^{\text{sph}}) + h\mathbb{W}_h, \quad Z_h\mathfrak{g} \cap (\tau(J^{\text{sph}}) + h\mathbb{W}_h) = h\mathbb{W}_h. \] (8.2.5)

Remark 8.2.6. The decomposition in (8.2.4) does not necessarily imply that $\mathbb{W}_h = Z_h\mathfrak{g} \oplus \tau(J^{\text{sph}})$, since the $Z$-grading on $\mathbb{W}_h$ is not bounded below.

It is immediate from definitions that $\bar{K}$ is a graded left ideal of $\mathbb{W}_h$. Furthermore, the ideals $\bar{K}$ and $K$ have the same image in $\text{gr} \mathbb{W} = \mathbb{W}_h/h\mathbb{W}_h$. Also, the image of $K$ in $\text{gr} \mathbb{W}$ equals $\text{gr} \mathbb{W} \cdot \tau_0(J_0^{\text{sph}})$, by definition. We have shown that $L^W = \tau_0(J_0^{\text{sph}})$, where $L^W$ is an ideal of $\text{gr} \mathbb{W}$. Therefore, $\tau_0(J_0^{\text{sph}})$ is an ideal. Hence, inside $\text{gr} \mathbb{W}$, we have $J_0^{\text{sph}}|_{\text{gr} \mathbb{W}} = \text{gr} \mathbb{W} \cdot \tau_0(J_0^{\text{sph}})$. Combining these observations together, we deduce
\[ \tau(J^{\text{sph}}) + h\mathbb{W}_h = K + h\mathbb{W}_h = \bar{K} + h\mathbb{W}_h. \] (8.2.7)

Using (8.2.7), we see that equations (8.2.5) take the following form:
\[ \mathbb{W}_h = Z_h\mathfrak{g} + \bar{K} + h\mathbb{W}_h, \quad Z_h\mathfrak{g} \cap (\bar{K} + h\mathbb{W}_h) = h\mathbb{W}_h. \] (8.2.8)

We claim that the equations in (8.2.8) imply the required direct sum decomposition $\mathbb{W}_h = Z_h\mathfrak{g} \oplus \bar{K}$. Indeed, let $a \in \mathbb{W}_h$. Separating homogeneous components, one may assume without loss of generality that $a$ is homogeneous of some degree $n \in \mathbb{Z}$. Note that $Z_h\mathfrak{g}$ has no homogeneous components of negative degree. Hence, in the case $n < 0$, the first equation in (8.2.8) implies that there exist $k_1 \in \bar{K}$ and $a_1 \in \mathbb{W}_h$, of degrees $n$ and $n - 1$, respectively, such that $a = k_1 + h a_1$. Applying the same argument to $a_1$, one gets $a_1 = k_2 + h a_2$, and so on. Thus, for any $m \geq 1$, one can find $k_1, \ldots, k_m \in \bar{K}$ such that one has $a = h^{m+1} a_{m+1} + \sum_{i=1}^{m} h^{-1} k_i$. Since $K$ is an ideal, we have $\sum_{i=1}^{m} h^{-1} k_i \in \bar{K}$ so $a \in h^{m+1}\mathbb{W}_h + K$. We conclude that $a \in \cap_{m \in \mathbb{Z}} (h^{m+1}\mathbb{W}_h + K) = \bar{K}$. In the case $n \geq 0$, one shows by induction on $n$ using a similar argument that $a \in Z_h\mathfrak{g} + K$. Finally, from the second equation in (8.2.8), one deduces that $Z_h\mathfrak{g} \cap \bar{K} = 0$.

Proof of Theorem 8.1.2. We know that $\mathcal{H}^{\text{sph}}$ and $\mathcal{H}e$ are torsion free $S^W_h$-modules. Therefore, the map $\mathcal{H}^{\text{sph}} \to \mathbb{C}(t_{\text{aff}}^*)^W \otimes_{S^W_h} \mathcal{H}^{\text{sph}}$ is injective. Further, the algebra $\mathfrak{g}$ is a flat scheme over $e$, see Section 2.1.

Hence, the algebra $\text{gr} \mathbb{W} \cong \mathbb{C}[3]$ is a torsion free $S^W_h$-module. It follows, since the filtration on $\mathbb{W}$ is separating by Lemma 4.3.2, that the algebra $\mathbb{W}_h$ is torsion free over $S^W_h = Z_h\mathfrak{g}$. We conclude that the map $\mathbb{W}_h \to \mathbb{C}(t_{\text{aff}}^*)^W \otimes_{S^W_h} \mathbb{W}_h$ is injective. It follows that the map $\tau : \mathcal{H}^{\text{sph}} \to \mathbb{W}_h$ is injective.

To prove that $\tau$ is surjective, recall from Section 7.2 that $\mathcal{H}$ is a subalgebra of $\mathcal{W}_{\text{aff}} \ltimes \mathbb{C}(t_{\text{aff}}^*)$. The latter algebra acts faithfully on $\mathbb{C}(t_{\text{aff}}^*)$. We have a decomposition $\mathbb{C}(t_{\text{aff}}^*) = e\mathbb{C}(t_{\text{aff}}^*) \oplus (1 - e)\mathbb{C}(t_{\text{aff}}^*)$. The action of the subalgebra $e(W \ltimes \mathbb{C}(t_{\text{aff}}^*))e$ respects this decomposition, moreover, the action of this subalgebra kills the second direct summand. Further, we have $\mathcal{H}^{\text{sph}} \subset e(W \ltimes \mathbb{C}(t_{\text{aff}}^*))e$, so the action of $\mathcal{H}^{\text{sph}}$ kills $(1 - e)\mathbb{C}(t_{\text{aff}}^*)$ and $S^W_h = eS_h \subset e\mathbb{C}(t_{\text{aff}}^*)$ is an $\mathcal{H}^{\text{sph}}$-stable subspace.

Next, we use the map $\tau$ to pullback the $\mathcal{H}^{\text{sph}}$-action to a $\mathbb{W}_h$-action on $e\mathbb{C}(t_{\text{aff}}^*)^W$. Thanks to the injectivity of $\tau$, the resulting $\mathbb{W}_h$-module is faithful. We claim that the subspace $S^W_h \subset \mathbb{C}(t_{\text{aff}}^*)^W$ is stable under the $\mathbb{W}_h$-action. To see this, we use Lemma 8.2.1. Specifically, it follows from the lemma that the map $\tau$ induces a bijection $\mathcal{H}^{\text{sph}}/J^{\text{sph}} \cong S^W_h \to Z_h\mathfrak{g} \cong \mathbb{W}_h/\bar{K}$. Via this bijection, the action of the algebra $\mathcal{H}^{\text{sph}}$ on $S^W_h$ can be extended to a $\mathbb{W}_h$-action on $S^W_h \cong \mathbb{W}_h/\bar{K}$. Is is immediate from the construction that the action so defined agrees with the above defined $\mathbb{W}_h$-action on $\mathbb{C}(t_{\text{aff}}^*)^W$. The claim follows.
We now apply Theorem 7.1.4 to conclude that inside \( \text{End}_C C(t^0_{ad}) \), one has an inclusion \( \mathbb{W}_h \subseteq \tau(\mathbb{H}^{\text{sph}}) \). Thus \( \tau \) is surjective, hence, it is an isomorphism. All other statements of the theorem easily follow from this. \( \Box \)

8.3. Proof of Theorem 1.5.1 It follows from Proposition 3.1.4 and the isomorphism \( (\mathcal{D}(G)/\mathfrak{h}^\psi)\mathfrak{h}^\psi \) of Lemma 5.4.1 that the functor \( M \mapsto M^n \times_n \psi \) gives an equivalence \( (G, N_l \times N_r, \psi \times \psi)\)-mod \( \cong \mathbb{W}\text{-mod} \). The equation in Lemma 3.3.2(3) shows that this functor induces, by restriction to holonomic modules, an equivalence \( \mathbb{W}h \cong \mathbb{W}\text{-hol} \). Also, by Theorem 1.2.1 we have an equivalence \( \mathbb{W}\text{-hol} \cong \mathbb{H}^{\text{sph}}\text{-hol} \) follows from this by Morita equivalence, see Lemma 7.2.3 since the latter equivalence takes holonomic modules to holonomic modules, by definition.

Let \( \mathcal{C} \) be the category of not necessarily holonomic \( W \times \mathcal{D}(T)\)-modules \( M \) such that (1.5.2) is an isomorphism. Further, let \( \mathcal{C}^{\text{sph}} \) be the category of not necessarily holonomic \( \mathcal{D}(T)^W\)-modules \( L \) such that the map \( \text{Sym} t \otimes \text{(Sym} t)^W L \rightarrow \mathcal{D}(T) \otimes \mathcal{D}(T)^W L \) is an isomorphism. It follows from Morita equivalence of the algebras \( \mathcal{D}(T)^W \) and \( W \times \mathcal{D}(T) \) that the functor \( \mathcal{D}(T) \otimes \mathcal{D}(T)^W (-) \) provides an equivalence \( \mathcal{C}^{\text{sph}} \rightarrow \mathcal{C} \).

Further, we have a functor \( \mathbb{H}\text{-mod} \rightarrow \mathcal{C} \) induced by the algebra embedding \( W \times \mathcal{D}(T) \hookrightarrow \mathbb{H} \). The proof of Proposition 7.2.4 shows that this functor is fully faithful. Thus, this functor is an equivalence. Using Morita equivalences above, we deduce that the functor \( \mathbb{H}^{\text{sph}}\text{-mod} \rightarrow \mathcal{C}^{\text{sph}} \), induced by the algebra embedding \( \mathcal{D}(T)^W \hookrightarrow \mathbb{H}^{\text{sph}} \), is also an equivalence.

To complete the proof of the theorem we must show that an object \( L \in \mathcal{C}^{\text{sph}} \) is holonomic as a \( \mathcal{D}(T)^W\)-module if and only if it is holonomic as an \( \mathbb{H}^{\text{sph}}\text{-module}. \) Assume first that \( L \) is holonomic as an \( \mathbb{H}^{\text{sph}}\text{-module}, \) equivalently, as a \( \mathbb{W}\text{-module.} \) Then, Proposition 6.2.1 implies that \( \mathcal{M} \otimes \mathbb{W} L = \text{Sym} t \otimes \text{(Sym} t)^W L \) is a holonomic \( \mathcal{D}(T)\)-module. Hence, \( L \) is holonomic as a \( \mathcal{D}(T)^W\)-module.

Conversely, assume that \( L \) is holonomic as a \( \mathcal{D}(T)^W\)-module \( L \). Choose a good filtration on \( L \), and equip \( M := \text{Sym} t \otimes \text{(Sym} t)^W L \), resp. \( \mathcal{M} \otimes \mathcal{D}(T)^W L \) and \( \mathcal{W} \otimes \mathcal{D}(T)^W L \), with the tensor product filtrations. The map \( T^*T \rightarrow (T^*T)/W \) being finite, we deduce that \( \text{Supp} \text{gr} M \) is a Lagrangian subvariety of \( T^*T \). We have the following diagram, cf. Proposition 2.3.3

\[
\mathfrak{Z} = \text{Spec} (\text{gr} \mathcal{W}) = \text{Spec} (\text{gr} \mathbb{H}^{\text{sph}}) \xleftarrow{p_3} \Lambda \xrightarrow{p_{T^*T}} T^*T = \text{Spec} (\text{gr} \mathcal{D}(T)).
\]

where \( p_3 \), resp. \( p_{T^*T} \), is the first, resp. second, projection and we have used Theorem 1.2.1 to identify \( \text{gr} \mathcal{W} \) with \( \text{gr} \mathbb{H}^{\text{sph}} \), resp. Theorem 1.2.2 to identify \( \text{Spec} \text{gr} \mathcal{W} \) with \( \mathfrak{Z} \). Thus, we have

\[
\text{Supp} \text{gr} (\mathcal{W} \otimes \mathcal{D}(T)^W L) = \text{Supp} (\text{gr} \mathcal{W} \otimes \text{gr} \mathcal{D}(T)^W \text{gr} L) \subseteq p_3 (p_{T^*T}^{-1} (\text{Supp} \text{gr} M)).
\]

By Proposition 2.3.3 \( \Lambda \) is a Lagrangian subvariety of \( \mathfrak{Z} \times T^*T \). It follows that \( p_3 (p_{T^*T}^{-1} (\text{Supp} \text{gr} M)) \) is an isotropic subvariety of \( \mathfrak{Z} \). We deduce that \( \mathcal{W} \otimes \mathcal{D}(T)^W L \) is a holonomic \( \mathbb{W}\text{-module.} \) On the other hand, the action of the algebra \( \mathbb{H}^{\text{sph}} \cong \mathcal{W} \) on \( L \) gives a canonical surjection \( \mathcal{W} \otimes \mathcal{D}(T)^W L \twoheadrightarrow L \), of \( \mathbb{H}^{\text{sph}}\text{-modules.} \) This implies that \( L \) is holonomic as a \( \mathbb{W}\text{-module.} \) \( \Box \)

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