Amplification limit of weak measurement

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In this work, we solve the open problem of the amplification limit for a weak measurement with post-selection. The pre- and post-selected states of the system and the initial probe state are allowed to be arbitrary. We derive that the maximal output of a weak measurement is the solution of an eigenvalue equation, and reveal a remarkable property that the maximal output is essentially independent of both the observable on the system and the interaction strength, but only dependent on the dimension of the system and the initial state of the probe. As an example, we completely solve the case of dimension two, which is of particular interest for quantum information. We generalize our result to the case where the initial system and probe states are mixed states, and the post-selection is a POVM. Finally, we discuss the application of our result to designing weak measurement experiments.

High precision measurements have been pursued for a long time in classical physics due to their great importance and wide range of practical applications. In recent years, quantum physics has also found to be useful in improving the precision of measurements. For example, when there are $N$ probes, the use of entanglement in the probe states can improve the precision of quantum measurement by a factor of $\sqrt{N}$ over its classical counterpart [1].

Quantum mechanics can also boost the performance of other aspects of measurements. In 1988, a new kind of quantum measurement, the so-called weak measurement, was proposed by Aharonov, Albert, and Vaidman [2]. A significant feature of weak measurement is that when combined with post-selection it can produce outputs beyond the usual range of the eigenvalues of an observable, rather than within the range of the eigenvalues as in a standard quantum projective measurement. This feature has been found useful in amplifying small physical quantities [3].

In a weak measurement, a typical interaction Hamiltonian between the system and the probe has the form

$$gA \otimes \rho(t-t_0),$$

where $A$ is an observable on the system, $p$ is the momentum operator of the probe, and $g$ represents the coupling strength between the system and the probe. In a weak measurement, the coupling strength $g$ is usually very weak (or equivalently the spread of the initial wave packet of the probe is very wide). The shift of the probe in a weak measurement is roughly $gA_w$, where $A_w$ is defined as

$$A_w = \frac{\langle \psi_f | A | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle},$$

and called a weak value. From [1], it is easy to see that when the pre- and post-selections are nearly orthogonal, the weak value $A_w$ can be very large, far beyond the range of the eigenvalues of the observable $A$. So the output of a weak measurement can be perceived an amplification of the small quantity $g$ by a factor $A_w$. If the weak value $A_w$ is complex, the amplification factor is a bit more complicated, but is still basically proportional to $A_w$ [3].

Because of the strange properties of weak measurements with post-selections and their usefulness in application, much research has been done on this subject. This includes weak measurement with an arbitrary probe [3], the general framework of weak measurement [4], weak measurement with orthogonal pre- and post-selections [7], weak measurement with orbital angular momentum pointer states [8], and so on. Repeated weak measurements can implement arbitrary quantum measurements [9]. Experiments have successfully measured weak value [10] and exploited the weak measurement and post-selection to detect small quantities, for example observing the spin Hall effect of light [11], optical beam deflection [12] and optical frequency change [12], among other results. More experimental protocols have also been proposed [14] [10].

Although the weak value formalism exhibits the amplification ability of weak measurement in a concise form, it has an explicit drawback: when the pre-selection and the post-selection are close to being orthogonal, the output of the probe can seemingly become infinitely large. This is obviously impossible in practice, and an upper bound was shown numerically for special cases by [6] [20]. Actually, in the derivation of the weak value, an implicit underlying assumption is that $|gA_w| \ll 1$ [4] [21], so if $\langle \psi_f | \psi_i \rangle \to 0$, this assumption will be violated. Therefore, it is interesting to study the greatest extent that the amplification can reach. This problem has attracted great research interest, and much effort has been devoted to this problem so far. However, to date, only some partial results have been obtained which either made special assumptions about the observable $A$, e.g., $A^2 = I$ [22] [23], or studied special cases like a qubit system with a Gaussian probe [24], or the optimal probe for a given experimental setup [25]. A general bound on the amplifi-
cation ability of a weak measurement is still lacking, even for the simplest case of a two-dimensional system.

In this work, we shall fill this gap for the general case of an arbitrary $d$-dimensional system with an arbitrary continuous probe. We use a variational method to derive a general equation for the maximal shift of the probe in a weak measurement with post-selection, which can be used to obtain a rigorous analytical solution for low dimensional systems (i.e., $d = 2, 3$ or 4), and numerical solutions for arbitrary higher-dimensional systems. The equation also gives a broad family of upper bounds on the maximal output for high dimensional systems with the assistance of matrix norm inequalities. A surprising property of the solution is that the maximal output of a weak measurement is independent of both the system observable $A$ and the coupling strength $g$, but is only determined by the dimension of the system $d$ and the initial state of the probe, up to the order $O(g^d)$.

In addition, we shall give a complete solution to the case of mixed pre-selection, generalized post-selection and mixed probe states. Finally, we shall discuss the application of our result in practical weak measurement experiments.

Suppose the dimension of the system is $d$, and the observable on it is $A$, with $d$ distinct eigenvalues $a_1, \cdots, a_d$, and corresponding eigenstates $|a_i\rangle$, $i = 1, \cdots, d$. The initial state of the system is $|\psi_i\rangle$, the post-selection is $|\psi_f\rangle$, the initial state of the probe is $|\phi\rangle$, and throughout the paper it is assumed without loss of generality that $|\phi\rangle$ is centered on the origin, i.e., $\langle \phi | q | \phi \rangle = 0$. The interaction Hamiltonian between the system and the probe is

$$H_{\text{int}} = g A \otimes \rho \delta(t - t_0).$$

where the $\delta$ function means that the Hamiltonian is an instantaneous action at time $t_0$.

Let $\hbar = 1$. After the evolution under the interaction (2), the final state of the probe is

$$|\phi_f\rangle = \langle \psi_f | \exp(-i g A \otimes p) | \psi_i \rangle | \phi \rangle,$$

where $|\phi_f\rangle$ is unnormalized, and the final average position of the probe is

$$\langle \Delta q \rangle = \frac{\langle \phi_f | q | \phi_f \rangle}{\langle \phi_f | \phi_f \rangle},$$

where $\langle \phi | q | \phi \rangle = 0$ is considered.

When $\langle \Delta q \rangle$ reaches an extremal value $\langle \Delta q \rangle_c$, it should satisfy that $\delta(\Delta q)_c = 0$ where $\delta$ represents variation of $|\psi_i\rangle$ and $|\psi_f\rangle$. By expanding $\delta(\Delta q)_c = 0$, we get

$$\delta(\Delta q)_c = \frac{1}{\langle \phi_f | \phi_f \rangle} ((\delta|\phi_f\rangle)(q|\phi_f\rangle - |\phi_f\rangle)(\Delta q)_c)$$

$$+ (|\phi_f\rangle q - (\Delta q)_c|\phi_f\rangle)(\delta|\phi_f\rangle)) = 0.$$  

(5)

Note that

$$\delta|\phi_f\rangle = (\delta|\psi_f\rangle \exp(-i g A \otimes p)|\psi_i\rangle \phi)$$

$$+ |\psi_f\rangle \exp(-i g A \otimes p)(\delta|\psi_i\rangle)|\phi\rangle,$$

(6)

so Eq. (5) becomes

$$\langle \phi | \psi_i \rangle \exp(i g A \otimes p) (q|\phi_f\rangle - |\phi_f\rangle)(\Delta q)_c (\delta|\psi_f\rangle)$$

$$+ (\delta|\psi_i\rangle \exp(i g A \otimes p)|\psi_f\rangle (q|\phi_f\rangle - |\phi_f\rangle)(\Delta q)_c$$

$$+ \text{H.C.} = 0.$$  

(7)

Since $\delta|\psi_i\rangle$ and $\delta|\psi_f\rangle$ are arbitrary, it follows from (7) that

$$\langle \phi | \exp(i g A \otimes p)|\psi_f\rangle (q - (\Delta q)_c)|\phi_f\rangle = 0,$$

$$\langle \psi_i | \exp(i g A \otimes p)|\psi_f\rangle (q - (\Delta q)_c)|\phi_f\rangle = 0.$$  

(8)

Now expand the first equation of (8) in terms of the eigenstates of $A$. The equation becomes

$$\sum_{i=1}^d \alpha_i \langle a_i | \phi(q - g a_i) (q - (\Delta q)_c)|\phi_f\rangle = 0,$$

(9)

where $\alpha_i = \langle a_i | \psi_f \rangle$ are the coordinates of $|\psi_f\rangle$ in the eigenbasis of $A$.

It is easy to see that, for any $\alpha_i \neq 0$, if $\langle \phi(q - g a_i) | (q - (\Delta q)_c)|\phi_f\rangle \neq 0$, then the left side of (9) will not be zero. Therefore, for all $\alpha_i |\psi_f\rangle \neq 0$ it requires

$$\langle \phi(q - g a_i) | (q - (\Delta q)_c)|\phi_f\rangle = 0.$$  

(10)

Similarly, Eq. (10) also holds true for all $\alpha_i |\psi_f\rangle \neq 0$ according to the second equation of (8). Actually Eq. (10) holds for all $i = 1, \cdots, d$, because $|\psi_i\rangle$ or $|\psi_f\rangle$ can always be chosen as such a state that $\alpha_i |\psi_i\rangle \neq 0 \forall i = 1, \cdots, d$. Let $Q$ be the matrix with elements

$$Q_{ij} = \langle \phi(q - g a_i) | (q - (\Delta q)_c) \langle \phi(q - g a_j) \rangle.$$

As $|\phi_f\rangle$ can be expanded as a linear combination of the $\{|\phi(q - g a_i)\rangle\}_{i=1}^d$, (10) represents a linear equation for the coordinates of $|\phi_f\rangle$ in the nonorthogonal basis $\{|\phi(q - g a_i)\rangle\}_{i=1}^d$. The existence of a non-zero $|\phi_f\rangle$ requires

$$\det Q(\langle \Delta q \rangle_c) = 0,$$

(12)

where the notation $Q(\langle \Delta q \rangle_c)$ explicitly indicates the dependence of $Q$ on $\langle \Delta q \rangle_c$. An extremal value of $\langle \Delta q \rangle$ must be a root of (12), and the largest shift of the probe $|\langle \Delta q \rangle|_{\text{max}}$ is the largest absolute value of these roots.

Now, let us find an orthogonal basis of the subspace spanned by $\{|\phi(q - g a_i)\rangle\}_{i=1}^d$ using the Gram-Schmidt orthogonalization method.

We first consider the case of $d = 2$. Let one state of the orthogonal basis be

$$|e_1\rangle = |\phi(q - g a_1)\rangle,$$

(13)

then the second state in the basis, orthogonal to $|e_1\rangle$, can be constructed as

$$|e_2\rangle = |\phi(q - g a_2)\rangle - |e_1\rangle |\phi(q - g a_2)\rangle$$

(14)

where $|e_2\rangle$ has not been normalized. As a weak measurement is considered here, we assume $g/\lambda(A)|_{\text{max}} \ll 1,$
where $|\lambda(A)|_{\text{max}}$ is the eigenvalue of $A$ with the maximal absolute value. Then $|\phi(q-ga_2)|$ can be expanded as

$$
|\phi(q-ga_2)| = |\phi(q-ga_1)| - g(a_2-a_1)|\phi'(q-ga_1)| + O(q^2),
$$

so (14) can be reduced and normalized to

$$
e_2 = |\phi'(q-ga_1)| - |\phi(q-ga_1)||\phi'(q-ga_1)| + O(q).
$$

From (13) and (14), it can be inferred that the subspace spanned by $|\phi(q-ga_1)|$ and $|\phi(q-ga_2)|$ can be spanned by $|\phi(q-ga_1)|$ and $|\phi'(q-ga_1)|$ to a good approximation.

For dimension $d$, it can be proved by induction that the subspace spanned by $(|\phi(q-ga_1)|, \ldots, |\phi(d-1)(q)|)$ up to $O(g^d)$ as $\lambda(A)|_{\text{max}} \to 0$ (see the Appendix).

Now, let us define the matrix $\Phi$ of which the $k$-th column is $|\phi^{(k)}(q)|$, $k = 1, \ldots, d$. As the subspace spanned by $|\phi(q-ga_1)|, \ldots, |\phi(d-1)(q)|$, equation (12) can be converted to

$$
det(\Phi^\dagger q\Phi - \Delta(q)e_1\Phi^\dagger e_1) = 0,
$$

from which it can be seen that $|\Delta(q)|_{\text{max}}$ is actually the largest absolute value of the eigenvalues of

$$
(\Phi^\dagger s\Phi)^{-\frac{1}{2}}(\Phi^\dagger q\Phi)(\Phi^\dagger s\Phi)^{-\frac{1}{2}}.
$$

From (17) and (18), it can be seen that $|\Delta(q)|_{\text{max}}$ is essentially independent of $g$, and only determined by $|\phi|$ and the dimension of the system $d$. Note that the first $d$ terms in the expansion of any $|\phi(q-ga_j)|$ are linear combinations of $|\phi(q)|, |\phi^{(1)}(q)|, \ldots, |\phi^{(d-1)}(q)|$, so this approximation is accurate up to the order $O(g^d)$. This is quite a surprising and counterintuitive property of the maximal output of a weak measurement. In fact, this feature was once noticed for the special case that the system is two-dimensional and the initial probe state is Gaussian [24]. We now have a rigorous and and more general analytical explanation of that result.

Fig. 1 shows an example of the invariance of $|\Delta(q)|_{\text{max}}$ with different $g$’s.

In quantum information, two-dimensional systems (qubits) are important in many scenarios. We can give a complete solution to this case using the equation we derived.

In the case of $d = 2$, the subspace spanned by $|\phi(q-ga)|$ and $|\phi(q-ga_2)|$ have a basis $|\phi(q)|$ and $|\phi'(q)|$, so it can be worked out from (17) that

$$
|\Delta(q)|_{\text{max}} = \frac{W_q}{2(\Delta p^2)}
$$

where $\langle p^2 \phi | = \langle \phi | p^2 | \phi \rangle^2 = |\Delta p^2|^2$ and

$$
W_q = |\langle \phi | p p q | \phi \rangle - \langle \phi | p | \phi \rangle \langle \phi | q, p \rangle |\phi \rangle|^2 + |\langle \phi | p p q | \phi \rangle - \langle \phi | p | \phi \rangle \langle \phi | q, p \rangle |\phi \rangle|^2
$$

$$
+ 4 |\langle \phi | p q | \phi \rangle |^2 (\Delta p^2)^{\frac{1}{2}}.
$$

When the dimension of the system $d$ is large, an analytical solution for $|\Delta(q)|_{\text{max}}$ does not generally exist, but numerical solutions are easy to obtain from (17). Moreover, $|\Delta(q)|_{\text{max}}$ is the largest absolute value of the eigenvalues of (18), so we can obtain upper bounds on the maximal shift of the probe by noting that the spectral radius of an operator is always bounded by any consistent norm. We discuss this in the Appendix.

In the following, we generalize the previous result to the case that the initial system and probe are mixed states, and the post-selection is a POVM element.

First, note that using a mixed initial state and a POVM post-selection for the system does not change the maximal shift of the probe. This is shown in the Appendix.

So we can suppose the pre- and post-selections are pure
states, and the initial probe state is mixed
\[ \rho_{\phi} = \sum_k \eta_k |\phi_k\rangle \langle \phi_k|. \tag{21} \]

The generalization to this case is mostly straightforward, but one must be cautious to avoid a tricky pitfall. At first glance, as \( \rho_{\phi} \) represents the ensemble \( \{\eta_k, |\phi_k\rangle\} \), it seems that the maximum shift of the probe should be the average maximum shift of the probe over the ensemble, i.e.
\[ |\langle \Delta q \rangle|_{\max} = \sum_k \eta_k |\langle \Delta q \rangle|_{\max}^{\phi_k}, \tag{22} \]
where the superscript \( |\phi_k\rangle \) indicates the dependence of \( \Delta(q)_{\max} \) on \( |\phi_k\rangle \). But the optimal choice of pre- and post-selections to reach the maximum shift depends on the initial probe state, so for different \( |\phi_k\rangle \)'s, the optimal choices of pre- and post-selections are different. One cannot make the optimal choice for all \( |\phi_k\rangle \)'s simultaneously. Thus (22) is not the exact maximal shift for a mixed probe state. (But it is indeed an upper bound of the probe shift).

In fact, for mixed probe states, it can be verified by the previous variational procedure that (17) becomes
\[ \det \left( \sum_k \eta_k \Phi_k^\dagger q \Phi_k - \Delta(q)_{\max} \sum_k \eta_k \Phi_k^\dagger \Phi_k \right) = 0, \tag{23} \]
where \( \Phi_k \) is the matrix with \( |\phi_k(q)\rangle, |\phi_k^{(1)}(q)\rangle, \cdots, |\phi_k^{(d-1)}(q)\rangle \) as its columns, so \( |\langle \Delta q \rangle|_{\max} \) is the largest absolute value of the eigenvalue of
\[ \left( \sum_k \eta_k \Phi_k^\dagger q \Phi_k \right)^{-\frac{1}{2}} \left( \sum_k \eta_k \Phi_k^\dagger q \Phi_k \right) \left( \sum_k \eta_k \Phi_k^\dagger \Phi_k \right)^{-\frac{1}{2}}. \tag{24} \]

The result we derived in this paper not only solved an open problem of weak measurement theory, but also has practical application in weak measurement experiments. When designing a weak measurement scheme to observe a small physical quantity, one must choose proper pre- and post-selections and an initial probe state to amplify the small physical quantity sufficiently. As mentioned before, the weak value decides the amplification extent of a weak measurement. However, since the output of a weak measurement must be bounded, and the numerical evidence has shown that such bounds do exist in weak measurement, it is not straightforward to figure out what pre- and post-selections and initial probe state can achieve the desired amplification without the output “overflowing” the bound of the measurement.

Our result shows that the maximum of the probe shift is determined solely by the initial probe state, so one need only “tune” the initial probe state to make sure the output lies within the bound on the probe shift. For a real measurement, we often have some a priori knowledge of the small quantity \( g \). For example if \( g \) is so small that it cannot be detected by a meter whose minimal output is \( g_0 \), then \( g < g_0 \), so
\[ g A_w < g_0 A_w \leq |\langle \Delta q \rangle|_{\max}. \tag{25} \]
Thus, when the initial state of the probe \( |\phi\rangle \) is given, one can obtain \( |\langle \Delta q \rangle|_{\max} \) from (17), and deduce the largest amplification factor that can be reached from (25) and the optimal choice of the pre- and post-selections correspondingly.

Furthermore, the remarkable property of the maximal output, that it basically has no independence on both the small physical quantity to be observed and the observable of the system, implies that the maximal output of a probe plays a role similar as the range of a meter, which is independent of the observed quantity. So when one fixes the initial state of the probe, he immediately knows the bound on the probe shift in a weak measurement, without having to consider what the small quantity or the observable of the system is.

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Appendix

I. BASIS OF SUBSPACE SPANNED BY
\[ |\phi(q - g_1 a_i)\rangle \quad i = 1, \ldots, d \]
Here, we use mathematical induction to prove that \( |\phi(q)\rangle, |\phi^{(1)}(q)\rangle, \cdots, |\phi^{(d-1)}(q)\rangle \) is a basis (not orthonormalized) of the subspace spanned by \( \{ |\phi(q - g_1 a_i)\rangle \}_{i=1}^d \) up to the order \( O(g^d) \) when \( g(A)_{\max} \rightarrow 0 \).

Previously, we have shown that for the case \( d = 2 \), the space spanned by \( |\phi(q - g_1 a_1)\rangle \) and \( |\phi(q - g_2 a_2)\rangle \) can be alternatively spanned by \( |\phi(q)\rangle, |\phi^{(1)}(q)\rangle, \cdots, |\phi^{(d-1)}(q)\rangle \) up to \( O(g^2) \). Now let us suppose for the dimension \( d \), the subspace can be spanned by \( |\phi(q)\rangle, |\phi^{(1)}(q)\rangle, \cdots, |\phi^{(d-1)}(q)\rangle \) up to \( O(g^d) \), and define the matrix
\[ \Xi = (|\phi(q)\rangle, |\phi^{(1)}(q)\rangle, \cdots, |\phi^{(d-1)}(q)\rangle), \tag{26} \]
then it is straightforward to verify that an orthonormal basis for the dimension \( d \) are the columns of
\[ \Xi(\Xi^\dagger \Xi)^{-\frac{1}{2}}. \tag{27} \]
Then by the Gram-Schmidt orthogonalization procedures, for dimension $d + 1$, the $d + 1$-th basis state can be written as

$$ |e_{d+1}⟩ = |φ(q - ga_{d+1})⟩ - Ξ(Ξ^†Ξ)^{-1}Ξ^†|φ(q - ga_{d+1})⟩ = |φ(q - ga_{d+1})⟩ - Ξ(Ξ^†Ξ)^{-1}ξ|φ(q - ga_{d+1})⟩. \quad (28) $$

Since $g|λ(A)|_{max} ≪ 1$,

$$ |φ(q - ga_{d+1})⟩ = \sum_{k=0}^{d} \frac{(ga_{d+1})^k}{k!}|φ(k)(q)⟩ + O(g^{d+1}) = ΞX + \frac{(ga_{d+1})^d}{d!}|φ(d)(q)⟩ + O(g^{d+1}), \quad (29) $$

where

$$ X = \left(1, ga_{d+1}, \cdots, \frac{(ga_{d+1})^{d-1}}{(d-1)!} \right)^T, \quad (30) $$

therefore, (28) can be simplified to

$$ |e_{d+1}⟩ = ΞX + \frac{(ga_{d+1})^d}{d!}|φ(d)(q)⟩ + O(g^{d+1}) - Ξ(Ξ^†Ξ)^{-1}Ξ^†(ΞX + \frac{(ga_{d+1})^d}{d!}|φ(d)(q)⟩ + O(g^{d+1})) = \frac{(ga_{d+1})^d}{d!}\left(|φ(d)(q)⟩ - Ξ(Ξ^†Ξ)^{-1}Ξ^†|φ(d)(q)⟩\right) + O(g^{d+1}). \quad (31) $$

As the columns of $Ξ$ are $|φ(q)⟩, |φ(1)(q)⟩, \cdots, |φ(d-1)(q)⟩$, it can be seen from (31) that $|e_{d+1}⟩$ is a linear combination of $|φ(q)⟩, |φ(1)(q)⟩, \cdots, |φ(d)(q)⟩$, thus the subspace spanned by $|φ(q - ga_1)⟩, \cdots, |φ(q - ga_{d+1})⟩$ can be spanned by $|φ(q)⟩, |φ(1)(q)⟩, \cdots, |φ(d)(q)⟩$.

In addition, considering that the first $d+1$ terms in the expansion of any $|φ(q - ga_j)⟩$ are linear combinations of $|φ(q)⟩, |φ(1)(q)⟩, \cdots, |φ(d)(q)⟩$, the approximation is actually accurate up to the order $O(g^{d+1})$. This completes the proof.

II. BOUNDS OF $|⟨Δq⟩|_{max}$ FOR LARGE $d$

Recall that if $\| \cdot \|$ is a matrix norm compatible with a vector norm, then for any operator $A$,

$$ \rho(A) ≤ \| A^k \|^\frac{1}{k}, \quad (32) $$

where $\rho(A)$ is the spectral radius of $A$ (i.e. the maximal norm of all eigenvalues of $A$). Therefore, if one chooses different matrix norms, a very wide family of upper bounds can be obtained for $⟨Δq⟩_{max}$ according to (32).

For example, let us choose the Frobenius norm

$$ \| A \|_F = \sqrt{tr(A^†A)} \quad (33) $$

and $k = 1$, then

$$ \rho(A) ≤ \sqrt{tr(A^†A)}. \quad (34) $$

Previously, we have proven that the maximal shift of the probe $|⟨Δq⟩|_{max}$ is the largest absolute value of the eigenvalues of $(Φ^†Φ)^{-\frac{1}{2}}(Φ^†qΦ)(Φ^†Φ)^{-\frac{1}{2}}$, therefore

$$ |⟨Δq⟩|_{max} = \rho((Φ^†Φ)^{-\frac{1}{2}}(Φ^†qΦ)(Φ^†Φ)^{-\frac{1}{2}}) ≤ \sqrt{tr((Φ^†Φ)(Φ^†Φ)^{-1}(Φ^†qΦ)(Φ^†Φ)^{-1})}. \quad (35) $$

Eq. (35) gives an upper bound for $|⟨Δq⟩|_{max}$.

By choosing different matrix norms and different $k$’s, one can derive different upper bounds for $|⟨Δq⟩|_{max}$.

One thing that needs mentioning is that the chosen matrix norm must be a compatible norm, i.e. it must satisfy

$$ \| Ax \| ≤ \| A \| \| x \| \quad (36) $$

for some norm of vectors, where $x$ is any vector, otherwise (32) will be violated.

III. INVARIANCE OF $|⟨Δq⟩|_{max}$ UNDER MIXED PRE-SELECTION AND POVM POST-SELECTION

Suppose the pre- and post-selections are

$$ ρ_i = \sum_k μ_k |ψ^k_i⟩⟨ψ^k_i|, \quad E_f = \sum_k ν_k |ψ^k_f⟩⟨ψ^k_f|, \quad (37) $$

where $E_f$ represents a POVM element, then the shift of the probe is

$$ ⟨Δq⟩ = \frac{tr(qE_f \exp(-igA ⊗ p)ρ_i ⊗ |φ⟩⟨φ| \exp(igA ⊗ p))}{tr(E_f \exp(-igA ⊗ p)ρ_i ⊗ |φ⟩⟨φ| \exp(igA ⊗ p))}, \quad (38) $$

and $δ⟨Δq⟩_c = 0$ implies

$$ δ\left( tr(E_f \exp(-igA ⊗ p)ρ_i ⊗ |φ⟩⟨φ| \exp(igA ⊗ p)) \right) = δ⟨Δq⟩_c δ\left( tr(E_f \exp(-igA ⊗ p)ρ_i ⊗ |φ⟩⟨φ| \exp(igA ⊗ p)) \right). \quad (39) $$

Plug (37) into (39), one can get

$$ \sum_{i,j} μ_i ν_j (δ⟨φ^t_{j}i|q|φ^t_{j}i⟩) - δ⟨Δq⟩_c δ⟨φ^t_{j}i|φ^t_{j}i⟩ = 0, \quad (40) $$

where $|φ^t_{j}i⟩$ denotes $⟨ψ^t_j| \exp(-igA ⊗ p)|ψ^t_i⟩ |φ⟩$.

Note that the $|φ^t_{j}i⟩$ are all independent of one another, so the above equation requires

$$ δ⟨φ^t_{j}i|q|φ^t_{j}i⟩ - δ⟨Δq⟩_c δ⟨φ^t_{j}i|φ^t_{j}i⟩ = 0, \quad ∀i, j. \quad (41) $$

For for any $i, j$, $|φ^t_{j}i⟩$ can vary arbitrarily, thus (11) is equivalent to (5). Therefore, the maximal shift of the probe in this case is the same as that in the case of pure pre- and post-selections.
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