Three-dimensional Noncommutative Gravity

M. Bañados, O. Chandía
Facultad de Física, P. Universidad Católica de Chile
Casilla 306, Santiago 22, Chile

N. Grandi*, F.A. Schaposnik† and G.A. Silva*
Departamento de Física, Universidad Nacional de La Plata
C.C. 67, 1900 La Plata, Argentina

Abstract

We formulate noncommutative three-dimensional (3d) gravity by making use of its connection with 3d Chern-Simons theory. In the Euclidean sector, we consider the particular example of topology $T^2 \times R$ and show that the 3d black hole solves the noncommutative equations. We then consider the black hole on a constant U(1) background and show that the black hole charges (mass and angular momentum) are modified by the presence of this background.

*CONICET
†Associated with CICBA
1 Introduction

Recently, field theories in noncommutative spaces have attracted much attention, partly in connection with string theory. More specifically, it has been shown that noncommutative $U(N)$ gauge theory emerges in a certain low energy limit of a system of $Dp$ branes in a constant Neveu-Schwarz $B$ field background \cite{1}-\cite{3}. In general, gauge theories can be formulated in noncommutative spaces starting from Lagrangians written in terms of ordinary fields multiplied using the Moyal $\ast$ product. It should be noted that consistency requires that the gauge group has to be $U(N)$ (or certain subgroups of $U(N)$ \cite{4}-\cite{6}).

It is then natural to analyse whether noncommutative extensions can be also constructed for gravity. There have been several investigations on this issue, that basically start by gauging, instead of the $SO(d)$ Lorentz group, the $U(1,d−1)$ \cite{7}-\cite{12} (or some orthogonal and symplectic subalgebras of unitary groups \cite{4}-\cite{6}) and then define the theory in terms of vielbeins and spin connection to be multiplied using the $\ast$ product.

It is well-known that in three-dimensional space-time, (ordinary) gravity can be formulated as a Chern-Simons theory \cite{14},\cite{15}. Many aspects, both at the classical and quantum levels, have been understood using this connection since, through field redefinitions, it simplifies the equations and introduces a rich mathematical structure. The construction of a black hole in $2 + 1$ space-time with negative cosmological constant (the so called BTZ blackhole \cite{16}-\cite{17}) also enhanced the interest in 3d gravity, particularly in view of the role it plays in string theory\cite{19}.

The goal of this work is to use the Chern-Simons formulation of three-dimensional (3d) gravity to give a definition for 3d noncommutative gravity. We will take profit on the fact that many classical and quantum aspects of noncommutative Chern-Simons theory are well understood \cite{20}-\cite{31}, to define the noncommutative 3d gravity action in terms of the corresponding noncommutative Chern-Simons action (NCCS)\footnote{There is another kind of noncommutative field theories, namely the so called q-deformed theories. In this context, a q-deformed 3d gravity theory has been discussed using the CS connection \cite{32}.}.

The paper is organized as follows. We start by describing in section 2 the NCCS theory for the group $GL(2,C)$, the one that will be relevant for the formulation of noncommutative 3d gravity. Then, in section 3 we establish the connection between gauge fields and gravitational variables (triad
and spin connection) so that the noncommutative “Einstein equations”, and their corresponding action, can be obtained. We also work out the metric formulation of the equations. In section 4 we study gravitational solutions for the particular topology \(M_3 = T^2 \times \mathbb{R}\). After showing the chiral character of these solutions, we construct the corresponding metric and explore its conformal properties and relate it to the corresponding commutative solutions. In section 5 we couple the chiral solution to a constant Abelian field and discuss how noncommutative effects determine the properties of the resulting blackhole solution.

## 2 Noncommutative Chern-Simons theory

Noncommutative Chern-Simons theory can be defined by the equations of motion,

\[
\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \ast A_\nu - A_\nu \ast A_\mu = 0, \tag{1}
\]

which are invariant under the noncommutative gauge transformations

\[
A'_\mu = U^{-1} \ast A_\mu \ast U + U^{-1} \ast \partial_\mu U. \tag{2}
\]

Here the \(\ast\) product of two functions \(f(x)\) and \(g(x)\) is defined as

\[
(f \ast g)(x) = \exp \left( \frac{i}{2} \theta_{\mu\nu} \partial_{x_\mu} \partial_{y_\nu} \right) f(x)g(y) \bigg|_{y=x} \tag{3}
\]

with \(\theta_{\mu\nu}\) a constant antisymmetric matrix.

The Seiberg-Witten \(\mathbb{N}\) map provides a powerful method to find solutions to (1). In fact, the key property of this map is that \(\hat{A} + \delta \hat{A} = \hat{A}(A + \delta A)\) (with \(A\) and \(\hat{A}\) gauge fields for spaces with different values of \(\theta_{\mu\nu}\)). Thus, if \(A\) is a solution to the commutative equation \(dA + A A = 0\), it follows that \(\hat{A}\) is a solution to the noncommutative equation.

Euclidean gravity, which will be our main interest here, can be formulated as a Chern-Simons theory for the group \(SL(2, \mathbb{C})\). It is well known, however, that in the noncommutative case this group is not closed with respect to the Moyal product and thus we are forced to consider \(GL(2, \mathbb{C})\). The gauge field \(A \in GL(2, \mathbb{C})\) can be expanded in the basis \(\{J_a, i\}\),

\[
A_\mu = A^a_\mu J_a + b_\mu i \tag{4}
\]

3
where \( J_1 = (i/2)\sigma_1, J_2 = -(i/2)\sigma_2, J_3 = (i/2)\sigma_3 \) are anti-Hermitian (\( \sigma_a \) are the Pauli matrices). Since \( A^a \) and \( b \) are complex, we define a second field

\[
\tilde{A}_{\mu} = A^a_{\mu} J_a + \bar{b}_{\mu} i
\]

which satisfies the Chern-Simons equations as well. It is conventional to use the same basis \( \{J_a, i\} \) for both fields and thus \( \tilde{A}_{\mu} \) is not the complex conjugate of \( A_{\mu} \).

The Abelian field \( b \) can be set equal to zero in the commutative case because it decouples from \( A^a \). This is no longer true in the noncommutative theory, although solutions with \( b = 0 \) do exist.

The full set of equations for \( A \) is,

\[
F^a[A] = -i(A^a * b + b * A^a)
\]
\[
db = -i(b * b + (1/4)A^a * A_a)
\]

with \( F^a[A] = dA^a + (1/2)\epsilon^{abc}A^b * A^c \). The right hand side terms are zero at \( \theta = 0 \) showing that \( A^a \) and \( b \) are decoupled in the commutative limit. For future reference, we mention that “flat” solutions with \( F^a = 0 \) exist provided,

\[
A^a * b + b * A^a = 0.
\]

Analogous equations can be written for \( \tilde{A} \).

3 Three-dimensional noncommutative gravity

3.1 Connection representation

Consider a GL(2,C) gauge field \( A \), satisfying two copies of (1)

\[
dA + A * A = 0,
\]
\[
d\tilde{A} + \tilde{A} * \tilde{A} = 0.
\]

(Here, the wedge symbol has been omitted). Now we define the combinations

\[
e = \frac{l}{2i}(A - \tilde{A})
\]
\[
w = \frac{1}{2}(A + \tilde{A})
\]

4
where \( e = e^a J_a + e^4 i \) and \( w = w^a J_a + w^4 i \). These relations are the natural noncommutative generalization of

\[
e^a = \frac{i}{2} (A^a - \bar{A}^a),
\]
\[
w^a = \frac{1}{2} (A^a + \bar{A}^a),
\]

(11)

Adding and subtracting the Chern-Simons equations, it is direct to prove that \( e \) and \( w \) satisfy the noncommutative “Einstein Equations”,

\[
dw + w * w - \frac{1}{l^2} e * e = 0,
\]
\[
de + w * e + e * w = 0
\]

(12)

These equations can be derived from the noncommutative “Einstein-Hilbert” action,

\[
I[e, w] = \int \text{Tr} \left( R * e - \frac{1}{3l^2} e * e * e \right),
\]

(14)

where \( R = dw + w * w \). The variation with respect to the triad yields (12) while the variation with respect to \( w \) yields the noncommutative torsion condition (13). In deriving the eqs. of motion from eq. (14) one has to take into account surface terms which arise in handling Moyal products (and are absent in the ordinary commutative case). This terms vanish for the choice of \( \theta_{\mu\nu} \) that will be done below (see section 4).

Despite the similarities between the action (14) and the usual Einstein-Hilbert action, it should be kept in mind that, in the former, the Abelian fields \( b \) and \( \bar{b} \) are coupled to \( e^a \) and \( w^a \) in a non-trivial way. The full action (14) depends on all fields,

\[
I = I[e^a, w^a, b, \bar{b}].
\]

(15)

The couplings between \( b \) and the gravitational variables are proportional to \( \theta \). We define noncommutative three-dimensional gravity by this action.

If we set the Abelian fields equal to zero, Eqns. (12) and (13) become,

\[
R^a - \frac{1}{l^2} \epsilon^{a}_{bc} e^b * e^c = 0,
\]
\[
de^a + (1/2) \epsilon^{a}_{bc} w^b * e^c + (1/2) \epsilon^{a}_{bc} e^b * w^c = 0.
\]

(16)

(17)
where $R^a = dw^a + (1/2)e^a_{bc}w^b \ast w^c$. The first equation can be regarded as a noncommutative constant curvature condition, written in terms of connections. The second equation is the analogous to a torsion condition. This equation, however, does not imply that the affine connection is symmetric.

The equations (16) and (17) are valid provided the equations for the Abelian field are satisfied with $b = \bar{b} = 0$. This implies,

\[
\frac{1}{l^2} e^a \ast e_a - w^a \ast w_a = 0 \tag{18}
\]
\[
e^a \ast w_a + w_a \ast e^a = 0 \tag{19}
\]

(which are identically satisfied at $\theta = 0$). We shall display below explicit solutions fulfilling these conditions.

### 3.2 Metric representation

Eqns. (16) and (17) have the same form of Einstein equations in the triad formalism, where all products of functions have been replaced by the $\ast$ product. It is now natural to ask whether there exists a metric formulation for them.

We shall assume that the constraints (19) are satisfied and try to write (16) in terms of the metric and affine connection. (See [3]-[11] for other approaches to this problem in four dimensions.)

We define the metric and affine connection as

\[
g_{\mu\nu} = e^a_{\mu} \ast e^b_{\nu} \eta_{ab}, \tag{20}
\]
\[
\Gamma^\mu_{\lambda\rho} = \epsilon_{abc}e^a_{\mu} \ast w^b_{\rho} \ast e^c_{\lambda} + e^a_{\mu} \ast \partial_{\rho}e^a_{\lambda}. \tag{21}
\]

In other words, $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ represent, as usual, the metric and connection in the coordinate basis. Given $e^a$ and $w^a$, the above formulae completely determines $g$ and $\Gamma$. If $e^a$ and $w^a$ satisfy the Chern-Simons equations, we would like to find the differential equation satisfied by $g$ and $\Gamma$.

\[\text{2}\] The definition of the affine connection can be motivated by the gauge invariance of the action. Under gauge transformations the spin connection transforms as $w \rightarrow w' = U^{-1} \ast w \ast U + U^{-1} \ast dU$. Let $w' = \Gamma^\rho_{\lambda\sigma}$ be the connection in a coordinate basis related to the tangent basis via the matrix $U = e^a_{\mu}$. The new connection $\Gamma^\rho_{\lambda\sigma}$ becomes (21). This equation can also be expressed as $\partial_{\rho}e^a_{\lambda} + e^a_{bc}w^b_{\rho} \ast e^c_{\lambda} - e^a_{\mu} \ast \Gamma^\mu_{\lambda\rho} = 0$, i.e., the full covariant derivative of $e^a_{\mu}$ is zero.
The curvature in the coordinate basis is,

\[ R^\mu_{\nu} = d\Gamma^\mu_{\nu} + \Gamma^\mu_{\sigma} \Gamma^\sigma_{\nu} \]  

\[ (\Gamma^\mu_{\nu} = \Gamma^\mu_{\nu\sigma} dx^\sigma) \]  

and it is related to \( R^a \) by the formula,

\[ R^\mu_{\nu} = \epsilon_{abc} e^{\mu a} * R^b * e^c_{\nu}. \]  

(23)

This follows by direct replacement of (21) into (22), and it expresses the fact that the curvature is a tensor. Since \( R^a \) satisfies (16) we find the “Einstein” equation,

\[ R^\mu_{\nu \alpha \beta} = -\frac{1}{l^2} (\delta^\mu_{\alpha} g_{\beta \nu} - \delta^\mu_{\beta} g_{\alpha \nu}) + E^\mu_{\nu \alpha \beta} \]  

(24)

where \( g_{\mu \nu} \) is defined in (20), and

\[ E^\mu_{\nu \alpha \beta} = \frac{1}{2l^2} e^{\mu a} * (e^a_{[\alpha} * e^b_{\beta]} - e^b_{[\beta} * e^a_{\alpha]}) * e^c_{\nu}. \]  

(25)

The first term in (24) is the usual contribution from the cosmological constant to the Einstein equations. Recall, however, that in this theory the metric is not symmetric. The second term \( (E) \) is a purely noncommutative effect, depending on the commutator of triads with respect to the Moyal product, and cannot be expressed in terms of the metric only.

To summarise, given \( e^a \) and \( w^a \) satisfying the Chern-Simons equations of motion then the metric (20) and affine connection (21) satisfy the “Einstein” equation (24). We shall exhibit below a family of solutions satisfying these equations.

## 4 Solutions

Before discussing the gravitational solutions, we shall make some general remarks on the solutions to the Chern-Simons equations.

All solutions considered here live on the topology \( M_3 = T^2 \times \mathbb{R} \). The local coordinates on \( T^2 \) are \( \{z, \bar{z}\} \) and \( \rho \in \mathbb{R} \). The components of the gauge field are then \( \mathcal{A}_\mu = \{A_z, A_{\bar{z}}, A_\rho\} \). We shall take \( \theta_{\rho z} = \theta_{\rho \bar{z}} = 0 \) while the noncommutative coordinates satisfy,

\[ [z, \bar{z}] = \theta. \]  

(26)
This means that, to first order in \( \theta \),

\[
    f \ast g = fg + \frac{\theta}{2}(\partial f \bar{\partial} g - \bar{\partial} f \partial g) + O(\theta^2) \tag{27}
\]

with \( \partial = \partial/\partial z \), \( \bar{\partial} = \partial/\partial \bar{z} \). In particular, we find the Moyal representation of \( \theta^\mu \), \( z \ast \bar{z} - \bar{z} \ast z = \theta \).

The choice of manifold \( M_3 \) and non-trivial component of \( \theta_{\mu\nu} \) ensures that when varying the CS action one can use the cyclic property of the \( \ast \) product without worrying about surface terms. The boundary condition \( A_{\bar{z}} = 0 \) is required in order to have well defined functional derivatives of the CS action.

It should be clear that the 3d black hole [16]-[17] is a solution to the full noncommutative equations simply because this field has two Killing vectors, \( \partial_z \) and \( \partial_{\bar{z}} \), which effectively reduce the Moyal product to the usual one.

In order to explore the noncommutative structure, we need to look at more general solutions. We shall start by looking at solutions to the noncommutative Chern-Simons equations.

### 4.1 The chiral solution

Let us rewrite the first of eqns.(6) in the form

\[
    F_a^{\rho z} [A] + i[b_{\rho}, A_a^z] + i[A_{\rho}^a, b_z] = 0 \\
    F_a^{\rho \bar{z}} [A] + i[b_{\rho}, A_a^{\bar{z}}] + i[A_{\rho}^a, b_{\bar{z}}] = 0 \\
    F_a^{z \bar{z}} [A] + i[b_z, A_a^{\bar{z}}] + i[A_a^z, b_{\bar{z}}] = 0 \tag{28}
\]

where \([A, b] = A \ast b - b \ast A \). Now, fixing the gauge to

\[
    A_{\rho} = iJ^3, \quad b_{\rho} = 0 \tag{29}
\]

the first two equations (28) become

\[
    \partial_{\rho} A_{\bar{z}}^a + i\delta^3_{\rho b} \epsilon^{ab} c A_c^\bar{z} = 0 \\
    \partial_{\rho} A_{z}^a + i\delta^3_{\rho b} \epsilon^{ab} c A_c^z = 0 \tag{30}
\]

with solution

\[
    A_z = d^{-1} \bar{A}_z(z, \bar{z}) d \\
    A_{\bar{z}} = d^{-1} A_{\bar{z}}(z, \bar{z}) d \tag{31}
\]

8
where
\[ d = e^{i\rho J^3} \]  

(32)

Now, the boundary condition \( A_z|_{\partial M} = 0 \) implies \( \tilde{A}_z = 0 \), this resulting in \( A_z = 0 \). Finally, replacing this solution in the last equation in (28), we obtain
\[ \partial_z A_z^a + i[b_z, A_z^a] = D_z[b]A_z^a = 0 \]  

(33)

Let us now study the last equation in (6)
\[ \partial_{\rho}b_z - \partial_z b_{\rho} + i[b_{\rho}, b_z] + \frac{i}{4}[A_{\rho}, A_{za}] = 0 \]  
\[ \partial_{\rho}b_{\bar{z}} - \partial_{\bar{z}} b_{\rho} + i[b_{\rho}, b_{\bar{z}}] + i\frac{i}{4}[A_{\rho}, A_{z\bar{a}}] = 0 \]  
\[ \partial_z b_{\bar{z}} - \partial_z b_{\bar{z}} + i[b_z, b_{\bar{z}}] + i\frac{i}{4}[A_z, A_{\bar{a}a}] = 0 \]  

(34)

Using \( A_z = 0 \) and the gauge condition (29), eq.(34) reads
\[ \partial_{\rho}b_z = 0 \]  
\[ \partial_{\rho}b_{\bar{z}} = 0 \]  
\[ \partial_z b_{\bar{z}} - \partial_z b_{\bar{z}} + i[b_z, b_{\bar{z}}] = 0 \]  

(35)

One then sees that \( b_z, b_{\bar{z}} \) must be independent of \( \rho \). Being the boundary condition \( b_z|_{\partial M} = 0 \), this implies that \( b_{\bar{z}} = 0 \) everywhere. The remaining equation is
\[ \partial_z b_{\bar{z}} = 0 \]  

(36)

and then \( b_z = b_z(z) \). With this solution for the \( U(1) \) field, the eq.(33) simplifies to
\[ \partial_z A_z^a = 0 \]  

(37)

this implying \( A_z = A_z(z) \).

Then, the general solution to eqns.(3) with boundary conditions \( A_z|_{\partial M} = b_{\bar{z}}|_{\partial M} = 0 \), closely related to the 3d black hole, is chiral,
\[ A_z = d^{-1}\tilde{A}_z(z)d \]  
\[ A_{\bar{z}} = 0 \]  
\[ A_{\rho} = iJ_3 = d^{-1}\partial_{\rho}d \]  
\[ b_z = b_z(z) \]  
\[ b_{\bar{z}} = b_{\bar{z}} = 0 \]  

(38)
with $\tilde{A}_z(z), b_z(z)$ arbitrary Lie algebra-valued functions of $z$. This configuration solves both, the commutative and noncommutative equations. It can also be checked that it is a fixed point under the Seiberg-Witten map [3]. A similar analysis can be done for the second complex field $\bar{A}$ leading to a solution analogous to (38) but with $A_z(z) \to \tilde{A}_z(\bar{z}), b_z(z) \to \tilde{b}_z(\bar{z})$ and $d \to d^{-1}$.

A gauge transformation (with group element $d^{-1}$) brings the solution to the simpler form

\begin{align}
A_z &= A_z(z) \\
A_{\bar{z}} &= A_{\rho} = 0 \\
b_z &= b_z(z) \\
b_{\bar{z}} &= b_{\rho} = 0
\end{align}

(39)

An important property of (39) is its Kac-Moody symmetry under holomorphic gauge transformations. To see this, let us specialize to the case $b_z = 0$ and note that the configuration (39) is form-invariant under gauge transformations which only depend on $z$. Let $\lambda = \lambda(z)$. We act with the noncommutative transformation (2) and find,

\begin{align}
\delta A_z &= \partial_z \lambda + A_z * \lambda - \lambda * A_z = \partial_z \lambda + A_z \lambda - \lambda A_z, \\
\delta A_{\bar{z}} &= 0, \\
\delta A_{\rho} &= 0
\end{align}

(40, 41, 42)

The $*$ product has been eliminated because the whole solution only depends on $z$. This symmetry of the space of solutions (39) is generated by a Kac-Moody algebra and play an important role in various approaches to understand the 3d black hole entropy as well as the Brown-Henneaux conformal symmetry.

4.2 The metric

Let us construct the metric corresponding to the solution found above. We start from eq.(20) with the vierbeins $e_\mu$ contracted according to eq.(10) which, for the affine solution takes the form

\begin{align}
e^a_z J_a &= \frac{l}{2i} d^{-1} \tilde{A}(z) d \\
e^a_{\bar{z}} J_a &= -\frac{l}{2i} d \tilde{A}(\bar{z}) d^{-1} \\
e^a_{\rho} J_a &= l J_3
\end{align}

(43)
Defining
\[ \tilde{A} = \frac{i}{2} \begin{pmatrix} A^3 & A^- \\ A^- & -A^3 \end{pmatrix} \quad \tilde{\bar{A}} = \frac{i}{2} \begin{pmatrix} \bar{A}^3 & \bar{A}^- \\ \bar{A}^- & -\bar{A}^3 \end{pmatrix} \] (44)

then, the symmetric (arc length) part of the associated metric is
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \]
\[ ds^2 = l^2 d\rho^2 - \frac{l^2}{4} \left( A^3 A^+ + \bar{A}^3 \bar{A}^- \right) d\bar{z}^2 - \frac{l^2}{4} \left( \bar{A}^3 A^- + \bar{A}^3 A^- \right) d\bar{z}^2 \\
+ \frac{l^2}{8} \left( \{A^3, \bar{A}^3\}_+ + \{A^-, \bar{A}^-\}_+ e^{2\rho} + \{A^+, \bar{A}^-\}_+ e^{2\rho} \right) dzd\bar{z} \\
+ il^2 \bar{A}^3 dzd\rho - il^2 A^3 dzd\rho \] (45)

At this point, we are interested in determining the conditions to be imposed on the gauge fields in order to have an asymptotically AdS metric. To this end, we follow [33] extended to the noncommutative case. The non-diagonal components should be absent. This can be achieved taking \( A^3 = \bar{A}^3 = 0 \), conditions that extend to the noncommutative case the first Polyakov reduction condition. The resulting metric is
\[ ds^2 = l^2 d\rho^2 - \frac{l^2}{4} A^+ A^- dz^2 - \frac{l^2}{4} \bar{A}^+ \bar{A}^- d\bar{z}^2 \\
+ \frac{l^2}{8} \left( \{\bar{A}^+, A^-\}_+ e^{-2\rho} + \{A^+, \bar{A}^-\}_+ e^{2\rho} \right) dzd\bar{z} \] (46)

which has an asymptotic (\( \rho \to \infty \)) form
\[ ds^2 = l^2 d\rho^2 + \frac{l^2}{8} \{A^+, \bar{A}^-\}_+ e^{2\rho} dzd\bar{z} \] (47)

Then, to match with the AdS form we need to impose the condition
\[ \{A^+, \bar{A}^-\}_+ = 8 \] (48)

Taking the derivatives with respect to \( z \) and \( \bar{z} \) we obtain the relations (remember that \( A^+ \) is holomorphic and \( \bar{A}^- \) is antiholomorphic)
\[ \{\partial_z A^+, \bar{A}^-\}_+ = 0 \quad \{A^+, \partial_{\bar{z}} \bar{A}^-\}_+ = 0 \] (49)
In the usual commutative case these relations will imply constants $A^+, \tilde{A}^-$. To test this in the noncommutative case, let us first observe that following [36], one can write

$$f(z) * g(\tilde{z}) = e^{\frac{\theta}{2} \partial \bar{\partial}} f(z) g(\bar{z})$$
$$g(\bar{z}) * f(z) = e^{-\frac{\theta}{2} \partial \bar{\partial}} f(z) g(\bar{z})$$

which implies

$$\frac{1}{2} \{ f, g \} = \frac{1}{2} \left( e^{\frac{\theta}{2} \partial \bar{\partial}} + e^{-\frac{\theta}{2} \partial \bar{\partial}} \right) f(z) g(\bar{z}) = \cosh \left( \frac{\theta}{2} \partial \bar{\partial} \right) f(z) g(\bar{z})$$

Using this, eqs.(50) can be rewritten as

$$\cosh \left( \frac{\theta}{2} \partial \bar{\partial} \right) (\partial_z A^+ \tilde{A}^-) = 0 \quad \cosh \left( \frac{\theta}{2} \partial \bar{\partial} \right) (A^+ \partial_{\bar{z}} \tilde{A}^-) = 0 \quad (52)$$

Calling $\psi_\lambda$ and $\lambda$ the eigenfunctions and eigenvalues of $\partial \bar{\partial}$ and assuming that $\{ \psi_\lambda \}$ is complete, one can write $\cosh((\theta/2)\partial \bar{\partial}) = \sum_\lambda \cosh((\theta/2)\lambda)|\psi_\lambda\rangle \langle \psi_\lambda|$ This ensures that $\cosh((\theta/2)\partial \bar{\partial})$ has no zero modes and then one has, from (52)

$$\partial_z A^+ \tilde{A}^- = 0 \quad A^+ \partial_{\bar{z}} \tilde{A}^- = 0 \quad (53)$$

this implies that $A^+ , \tilde{A}^-$ should be constants. Then we have found the second reduction condition

$$A^+ = 2 \quad \tilde{A}^- = 2 \quad (54)$$

We conclude that in order to have an asymptotic AdS form in the noncommutative case, one needs to impose just the usual Polyakov reduction conditions, previously discussed in [33]. In this case, eqns(14) take the form

$$A_z = i \begin{pmatrix} 0 & e^\rho \\ \frac{1}{2} T(z) e^{-\rho} & 0 \end{pmatrix} \quad (55)$$
$$\bar{A}_\bar{z} = i \begin{pmatrix} 0 & e^\rho \\ e^\rho & 0 \end{pmatrix} \frac{1}{2} T(\bar{z}) e^{-\rho} \quad (56)$$
$$A_\rho = -\bar{A}_\rho = iJ_3$$
$$A_\bar{z} = \bar{A}_\bar{z} = b = \bar{b} = 0 \quad (57)$$
With this, the symmetric metric as defined in (45) becomes
\[ ds^2 = l^2 d\rho^2 - \frac{1}{2} T dz^2 - \frac{1}{2} T d\bar{z}^2 + \frac{1}{8} \left( \{T, T\}_+ e^{-2\rho} + 8l^2 e^{2\rho} \right) dz d\bar{z} \] (58)

We see that the only component of the symmetric metric affected by noncommutativity is \( g^S_{zz} \). Using (50), this component can be written as
\[ g^S_{zz} = \cosh(\frac{\theta}{2} \partial \bar{\partial}) \tilde{g}_{zz} \] (59)

being \( \tilde{g} \) the metric constructed in [18] for the commutative case. The operator \( \cosh((\theta/2)\partial \bar{\partial}) \) acts like the identity when applied to the other components of the metric (all derivative terms vanishes),
\[ g^S_{zz} = \cosh(\frac{\theta}{2} \partial \bar{\partial}) \tilde{g}_{zz} \]
\[ g^S_{\bar{z}\bar{z}} = \cosh(\frac{\theta}{2} \partial \bar{\partial}) \tilde{g}_{\bar{z}\bar{z}} \] (60)

so that the relation between the commutative and the (symmetric) noncommutative solutions can be compactly written as
\[ g^S_{\mu\nu} = \cosh \left( \frac{\theta}{2} \partial \bar{\partial} \right) \tilde{g}_{\mu\nu}. \] (61)

The full metric \( g_{\mu\nu} = g^S_{\mu\nu} + g^A_{\mu\nu} \), where \( g^A_{\mu\nu} \) is the anti-symmetric part, satisfies the "Einstein" equation (24). Note that \( g^A_{\mu\nu} \) is in fact non-zero. Its non-zero contributions come from
\[ g_{zz} = \exp \left( \frac{\theta}{2} \partial \bar{\partial} \right) \tilde{g}_{zz}, \quad g_{\bar{z}\bar{z}} = \exp \left( -\frac{\theta}{2} \partial \bar{\partial} \right) \tilde{g}_{\bar{z}\bar{z}} \] (62)

which imply
\[ g^A_{zz} = \sinh \left( \frac{\theta}{2} \partial \bar{\partial} \right) \tilde{g}_{zz} \] (63)

Recall that the deviation of (24) from the ordinary Einstein equations is encoded in the combination \( E^\nu_{\nu\alpha\beta} \) which depends on the commutator \([e^a_\alpha, e^b_\beta]\). In the present case the only non-vanishing contribution to this commutator is the \((\alpha = z, \beta = \bar{z})\) component, and it is proportional to the commutator \([T, T] = 2 \sinh \left( \frac{\theta}{2} \partial \bar{\partial} \right) T(z) \bar{T}(\bar{z})\).
For future use, let us end this section rewriting the solution (55)-(56) in the $A_\rho = 0$ gauge

$$A_z = i \left( \begin{array}{cc} 0 & 1 \\ \frac{1}{2\pi} T(z) & 0 \end{array} \right)$$  \hspace{1cm} (64)$$

$$\bar{A}_\bar{z} = i \left( \begin{array}{cc} 0 & \frac{1}{2\pi} \bar{T}(\bar{z}) \\ 1 & 0 \end{array} \right)$$  \hspace{1cm} (65)$$

5 Constant Abelian background

We consider in this section the chiral solution considered in the last section coupled to a constant Abelian field of magnitude $F_{z\bar{z}} = i\alpha$. We shall see that the black hole field with constant values of $T$ and $\bar{T}$ will feel the Abelian field due to noncommutative effects.

In order to fix the value of the Abelian field we add to the action the term $-2i \int \text{Tr}(\alpha A)$ where $\alpha$ is a fixed 2-form $\alpha = adz \wedge d\bar{z}$. This is a term of the kind introduced in [25]. The equations of motion (1) are replaced by

$$\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \ast A_\nu - A_\nu \ast A_\mu = \alpha_{\mu\nu} i.$$ \hspace{1cm} (66)$$

$\alpha$ is a number and it contributes only to the Abelian curvature $^3$.

The generalization of the chiral solution satisfying (66) in the $A_\rho = 0$ gauge is simply,

$$A_z = A(z) - i \alpha \bar{z}$$

$$\bar{A}_\bar{z} = 0$$

$$A_\rho = 0.$$ \hspace{1cm} (67)$$

Since the extra term only contributes to the Abelian field, one could naively conclude that the black hole solution has not changed. However, this field depends on both coordinates and noncommutative effects do take place.

The point is that, the noncommutative structure of the gauge transformations changes the affine algebra and, as a result, Polyakov’s reduction conditions needs to be modified. Let $\lambda = \lambda(z)$ and compute the noncommutative gauge transformation (2) acting on (67). The components $A_\rho$ and $A_z$ are left invariant while the transformation for $A_\bar{z}$ yields,

$$\delta A_z = \partial_z \lambda + (A_z - i\alpha \bar{z}) \ast \lambda - \lambda \ast (A_z - i\alpha \bar{z}),$$

$$= (1 + i\theta \alpha) \partial_z \lambda + A_z \lambda - \lambda A_z.$$ \hspace{1cm} (68)$$

$^3$A constant noncommutative Abelian field has been studied in detail in [24].
The extra term proportional to \( \theta \) comes from the Moyal formula \( \bar{z} * f - f * \bar{z} = -\theta \partial f \). The solution (67) still has an affine holomorphic Kac-Moody symmetry but its form has changed.

Even though the extra term \( \alpha \theta \) in (68) does not affect the gauge symmetries in any significant way, it does change the definition of global charges. We shall see that the mass and angular momentum of the black hole are modified by the presence of \( \alpha \).

The point is that under the transformation (68), the reduction condition \( A_z^+ = 2 \) is not consistent, and does not yield the Virasoro algebra. The correct reduction conditions are

\[
A_z^3 = 0, \quad A_z^+ = 2(1 + i\alpha \theta), \quad (69)
\]

and the Virasoro charge is \( T(z) = A^- / (2 + 2i\alpha \theta) \). The reduced field is then

\[
A_z = i(1 + i\alpha \theta) \begin{pmatrix} 0 & 1 \\ \frac{T(z)}{2i} & 0 \end{pmatrix} \quad (70)
\]

In order to match the boundary conditions (keeping the periodicity of the torus fixed) with the solution (64) we perform a constant gauge transformation on \( A_z \) with a group element \( g = e^{iaJ_3} \) and \( a = \text{log}(1 + i\alpha \theta) \). The field (70) is transformed into

\[
A_z = i \begin{pmatrix} 0 & 1 \\ \frac{(1 + i\alpha \theta)^2 T(z)}{2i} & 0 \end{pmatrix} \quad (71)
\]

which is of the form (64). The antiholomorphic field can be constructed in a similar way and one finds,

\[
\bar{A}_z = i \begin{pmatrix} 0 & (1 - i\alpha \theta)^2 \bar{T}(\bar{z}) \\ 1 & 0 \end{pmatrix} \quad (72)
\]

For constant values of \( T \) and \( \bar{T} \) this field represent a black hole. However the relation between the mass and angular momentum and the Virasoro charges have changed,

\[
\begin{align*}
Ml & = (1 + i\alpha \theta)^2 T + (1 - i\alpha \theta)^2 \bar{T}, \\
iJ & = (1 + i\alpha \theta)^2 T - (1 - i\alpha \theta)^2 \bar{T},
\end{align*} \quad (73, 74)
\]

\[\text{In fact one could define } A = (1 + \alpha \theta)A' \text{ and } A' \text{ would transform in the usual way. This corresponds to the Seiberg-Witten map} \]

15
It is instructive to expand these relations to first order in $\theta$,

\[ lM = lM_0 + 2\theta \alpha J_0 \quad (75) \]
\[ J = J_0 - 2\theta \alpha lM_0, \quad (76) \]

where $M_0$ and $J_0$ are the values of $M$ and $J$ at $\alpha = 0$. For example one can start at $\alpha = 0$ with a non-rotating black hole ($J_0 = 0$). Then we turn on the Abelian field with $\alpha \neq 0$ and find that the corresponding black hole will have a non-zero angular momentum.

Acknowledgements: M.B. is supported by Grant # 1000744 (FONDECYT, Chile) and O.C. by Grant # 300026 (FONDECYT, Chile). N.G., F.A.S and G.A.S. are partially supported by UNLP, CICBA, CONICET (PIP 4330/96), ANPCYT (PICT 03-05179), Argentina. M.B. would like to thank Marc Henneaux for a useful conversation on noncommutative field theories.

References

[1] A. Connes, M. R. Douglas and A. Schwarz, JHEP 9802, 003 (1998), hep-th/9711162.
[2] M. R. Douglas and C. Hull, JHEP 9802, 008 (1998), hep-th/9711165.
[3] N. Seiberg and E. Witten, JHEP 9909, 032 (1999), hep-th/9908142.
[4] A. Armoni, Nucl. Phys. B593, 229 (2001), hep-th/0005208.
[5] L. Bonora, M. Schnabl, M. M. Sheikh-Jabbari and A. Tomasiello, Nucl. Phys. B589, 461 (2000), hep-th/0006091.
[6] B. Jurco, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C17 (2000) 521, hep-th/0006246.
[7] Pei-Ming Ho, Int. J. Mod. Phys. A12, 923 (1997), q-alg/9505021.
[8] J. W. Moffat, Phys. Lett. B491, 345 (2000), hep-th/0007181.
[9] J. W. Moffat, Phys. Lett. B493, 142 (2000), hep-th/0008089.
[10] A. H. Chamseddine, Phys. Lett. B504, 33 (2001), hep-th/0009153.
[11] A. H. Chamseddine, Int. J. Mod. Phys. B16, 759 (2001) hep-th/0010268.
[12] A. H. Chamseddine and M. S. Volkov, hep-th/0101202.
[13] G. Bimonte, R. Musto, A. Stern and P. Vitale, Int. J. Mod. Phys. A13, 4023 (1998), hep-th/9706190.
[14] A. Achucarro and P. K. Townsend, Phys. Lett. B180, 89 (1986).
[15] E. Witten, Nucl. Phys. B311, 46 (1988).
[16] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).
[17] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48, 1506 (1993).
[18] M. Bañados, in “Trends in Theoretical Physics II”, Buenos Aires (1998) H. Falomir, R. Gamboa Saraví and F.A. Schaposnik eds. AIP, hep-th/9901148.
[19] A. W. Peet, hep-th/0008241.
[20] A. A. Bichl, J. M. Grimstrup, V. Putz and M. Schweda, JHEP 0007, 046 (2000), hep-th/0004071.
[21] C. Chu, Nucl. Phys. B580, 352 (2000), hep-th/0003007.
[22] G. Chen and Y. Wu, Nucl. Phys. B593, 562 (2001), hep-th/0006114.
[23] F. T. Brandt, A. Das and J. Frenkel, Phys. Lett. B494, 339 (2000), hep-th/0009236; Phys. Rev. D63, 085015 (2001) hep-th/0012087.
[24] N. Grandi and G. A. Silva, hep-th/0010113.
[25] A. P. Polychronakos, JHEP 0011, 008 (2000) hep-th/0010264.
[26] A. Khare and M. B. Paranjape, JHEP 0104, 002 (2001), hep-th/0102016.
[27] G. S. Lozano, E. F. Moreno and F. A. Schaposnik, JHEP 0102, 036 (2001), hep-th/0012260.
[28] D. Bak, S. K. Kim, K. Soh and J. H. Yee, hep-th/0102137.
[29] M. M. Sheikh-Jabbari, hep-th/0102092.
[30] V. P. Nair and A. P. Polychronakos, hep-th/0102181.
[31] D. Bak, K. Lee and J. Park, hep-th/0102188.
[32] A.M. Polyakov, Int. J. Mod. Phys. A5, 833 (1990).
[33] O. Coussaert, M. Henneaux, P. van Driel Class. Quant. Grav. 12, 2961 (1995).
[34] L. Alvarez-Gaume and J. L. Barbon, hep-th/0006209.
[35] J.D. Brown and M. Henneaux, Commun. Math. Phys. 104, 207 (1986).
[36] E. F. Moreno and F. A. Schaposnik, Nucl. Phys. B 596, 439 (2001), hep-th/0008113.