Feynman integrals are completely determined by linear algebra

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We find that all Feynman integrals (FIs), with any number of loops, can be determined completely by linear relations between FIs. Therefore, FIs computation is conceptually changed to a linear algebraic problem. Examples up to 5 loops are given to verify this observation. As a byproduct, we get a powerful method to calculate perturbative corrections in quantum field theory.

Introduction. — Feynman integrals (FIs) encode key information of quantum field theories. Study of FIs is important both for exploring mysteries of quantum field theories and for phenomenological application of them. Integrating over some variables is found to be a necessary step to determine FIs in all known systematic methods. This seems to be a reasonable phenomenon, as FIs themselves are defined by integrating over loop momenta. However, because it is usually hard to perform integration in a systematic and efficient way, is it possible to totally bypass integration in determining FIs?

Systematic methods to compute FIs on the market can be divided into direct methods and indirect methods. Direct methods include sector decomposition \([1,13]\), Mellin-Barnes representation \([8,12]\), loop-tree duality \([14,22]\), and so on, where one computes FIs by directly performing integration over some variables. Indirect methods compute FIs indirectly by solving corresponding equations, which include difference equations \([24,27]\) and differential equations \([28,40]\). To uniquely determine the solution, boundary information are needed in these indirect methods. Unfortunately, the only known systematic way to obtain boundary information is to use direct methods to calculate them. Therefore, integration is still necessary in these indirect methods.

The auxiliary mass flow (AMF) method \([38,40]\) is a kind of differential equations method, which computes FIs by setting up and solving differential equations with respect to an auxiliary mass term \(\eta\) (called \(\eta\)-DEs). The virtue of AMF is that its boundary conditions at \(\eta \to \infty\) are simply vacuum bubble integrals, which can be more easily calculated by using other methods \([41,50]\).

The observation in this Letter is following. Boundary information for AMF, which can always be casted to single-mass vacuum FIs, can be related to propagator integrals (p-integrals) with one less loops. Then, p-integrals can again be calculated by using the AMF method, with input of new boundary information having one less loops. By using this strategy iteratively, we eventually do not need any input for boundary information in the AMF framework. It is thus surprising to find that integration is totally bypassed in determining FIs.

As a result of our observation, FIs are completely determined by linear relations between FIs, which are used to decompose all FIs to a small set of bases, called master integrals (MIs), and to set up \(\eta\)-DEs of these MIs. We note that numerically solving ordinary differential equations (like \(\eta\)-DEs) is a well solved mathematical problem \([51]\). Therefore, the problem of integrating over loop momenta is now conceptually changed to an algebraic problem of exploring the linear space of FIs.

In the rest of the Letter, we first review the AMF method and emphasize its input. We then describe our method to compute boundary conditions within the AMF framework, without any unknown information. Some examples are in order to verify this method. Finally, we propose a powerful way to calculate perturbative corrections within dimensional regularization.

Before continuing, let us first give a brief introduction to FIs. A family of FIs are defined by the following integrals with various values of \(\vec{\nu}\),

\[
I_{\vec{\nu}} = \int \left( \prod_{i=1}^{L} \frac{D^{\nu_i}/\nu_i^{D/2}}{D^{\nu_i}} \right) \frac{D_{K+1}^{-\nu_{K+1}} \cdots D_{N}^{-\nu_{N}}}{D_{1}^{\nu_1} \cdots D_{K}^{\nu_{K}}}.
\]

(1)

where \(L\) is the number of loops, \(\nu_i\) are loop momenta, \(D\) is the dimensionality of \(\nu_i\), \(D_1, \ldots, D_K\) are inverse propagators with \(\nu_1, \ldots, \nu_K\) being integers, and \(D_{K+1}, \ldots, D_N\) are irreducible scalar products introduced for completeness with \(\nu_{K+1}, \ldots, \nu_{N}\) being nonpositive integers. It was proved that a family of FIs form a finite-dimensional linear space \([52]\). That is, any FI in a give family can be decomposed into a linear combination of MIs, which is a finite set of bases of the linear space formed by the family of FIs. Coefficients in this decomposition are rational functions of all natural variables, like \(D\), Mandelstam variables, masses, and the \(\eta\) introduced in AMF. Information of the linear space are completely encoded in these decompositions, or linear relations between FIs. Decomposition of FIs is usually called integration-by-parts (IBP) reduction, which have been extensively studied \([24,53,73]\). Having IBP reduction relations, we then only need to study MIs.

Furthermore, because FIs containing linear propagators can be determined by FIs containing only quadratic propagators \([74]\), we will not consider linear propagators anymore.

The auxiliary mass flow method. — To determine \(I_{\vec{\nu}}\) defined in Eq. (1), in the AMF method one in-
produces an auxiliary family of integrals defined by

$$\tilde{I}_\nu(\eta) = \int \left( \prod_{i=1}^{L} \frac{d^D \ell_i}{i\pi^{D/2}} \right) \frac{\tilde{D}_{K+1}^{\nu_{K+1}} \cdots \tilde{D}_{N}^{\nu_N}}{\tilde{D}_1^{\nu_1} \cdots \tilde{D}_K^{\nu_K}}. \quad (2)$$

Without loss of generality, we assume $\nu_1 > 0$ and $D_1 = \ell_1^2 - m^2 + i0^+$. We can then choose the propagator mode (40) to set $D_i = D_1$ for $i > 1$ and modify the mass term for $i = 1$ by

$$\tilde{D}_1 = \ell_1^2 - m^2 - \eta. \quad (3)$$

Original $I_\nu$ can be obtained by taking $\eta \to i0^-$,

$$I_\nu = \lim_{\eta \to i0^-} \tilde{I}_\nu(\eta). \quad (4)$$

Let us denote MIs of the auxiliary family by $\tilde{J}_\nu(\eta)$, and denote its dimension by $n$. Using IBP reduction, $\frac{\partial}{\partial \eta} \tilde{J}_\nu(\eta)$ can be again expressed as linear combinations of $J_\nu(\eta)$, which results in a system of closed $\eta$-DEs,

$$\frac{\partial}{\partial \eta} \tilde{J}_\nu(\eta) = A(\eta)\tilde{J}_\nu(\eta), \quad (5)$$

where $A(\eta)$ is an $n \times n$ matrix with entries rationally depending on $\eta$. Supposing that we already have boundary conditions in hand, we can solve the $\eta$-DEs numerically to obtain $J_\nu(\eta)$ and thus their limit $J(0^-)$. As $\tilde{I}_\nu(\eta)$ can be expressed as linear combinations of $\tilde{J}_\nu(\eta)$ using IBP reduction, all original FIs $I_\nu$ (and certainly also their MIs) are eventually determined.

An advantage of AMF is that boundary conditions at $\eta \to \infty$ can be systematically calculated. In this limit, nonzero contributions only come from integration regions where linear combinations of loop momenta are either of $O(\sqrt{\eta})$ or $O(1)$ [38, 51]. In each of these limited number of regions, a general propagator can be expressed as $1 / (\ell_1 + \ell_S + p)^2 - m^2 - \kappa \eta$, where $\ell_L$ is the $O(\sqrt{\eta})$ part of loop momenta, $\ell_S$ is the $O(1)$ part of loop momenta, $p$ is a linear combination of external momenta, $m$ is the mass, and $\kappa = 0$ or 1. Then, if $\ell_L \neq 0$ or $\kappa \neq 0$, we can simplify the propagator by

$$\frac{1}{(\ell_L + \ell_S + p)^2 - m^2 - \kappa \eta} \sim \frac{1}{\ell_L^2 - \kappa \eta}. \quad (6)$$

Otherwise, the propagator is unchanged. After the above simplification, the resulted new FIs at boundary are either single-mass vacuum FIs or simpler FIs comparing with the original FIs. For the later cases, we can compute them again using AMF, which needs even simpler FIs as input for boundary conditions.

By using AMF iteratively, to determine any $L$-loop FI, we eventually only need single-mass vacuum FIs no more than $L$ loops as additional input besides IBP reductions. Diagrams of some typical single-mass vacuum FIs are shown in Fig. 1.

**Determine single-mass vacuum Feynman integrals.** — Now let us assume that $I_\nu$ defined in Eq. (1) are single-mass vacuum FIs, with $D_1 = \ell_1^2 - m^2 + i0^+$ as the only massive propagator and $\nu_1 > 0$. Without loss of generality, we set $m^2 = 1$ in the rest of this Letter.

Let us define a massless $p$-integral

$$\tilde{I}_{\nu'}(\ell_1^2) = \int \left( \prod_{i=2}^{L} \frac{d^D \ell_i}{i\pi^{D/2}} \right) \frac{D_{K+1}^{\nu_{K+1}} \cdots D_{N}^{\nu_N}}{D_2^{\nu_2} \cdots D_K^{\nu_K}}, \quad (7)$$

with $\nu' = (\nu_2, \cdots, \nu_N)$, where $\ell_1$ presents as its “external momentum” and $\ell_1^2$ is its only mass scale. Based on dimensional counting, we have

$$\tilde{I}_{\nu'}(\ell_1^2) = (-\ell_1^2)^{(i-1)D/2 - \nu + \nu_1} \tilde{I}_{\nu'}(-1), \quad (8)$$

where $\nu = \sum_{i=1}^{N} \nu_i$. The original integral $I_\nu$ is then factorized to two parts and can be evaluated as

$$I_\nu = \int \frac{d^D \ell_1}{i\pi^{D/2}} \frac{(-\ell_1^2)^{(i-1)D/2 - \nu + \nu_1}}{(\ell_1^2 - 1 + i0^+)^{\nu_1}} \tilde{I}_{\nu'}(-1) = \frac{\Gamma(\nu - LD/2)\Gamma(LD/2 - \nu + \nu_1)}{\Gamma(\nu_1)\Gamma(D/2)} \tilde{I}_{\nu'}(-1), \quad (9)$$

which determines a $L$-loop single-mass vacuum FI $I_\nu$ by a $(L - 1)$-loop massless $p$-integral $\tilde{I}_{\nu'}(-1)$, which relation is well-known.

Here comes the key observation: the $(L - 1)$-loop massless $p$-integral $\tilde{I}_{\nu'}(-1)$ can be computed via AMF discussed in the last section, which requires single-mass vacuum FIs no more than $(L - 1)$ loops as additional input besides IBP reductions. Therefore, we find that, with linear algebra provided by IBP reductions, single-mass vacuum FIs with $L$ loops are determined by that with
less than \( L \) loops. This works iteratively until the boundary at \( L = 1 \). Vacuum FIs with \( L = 1 \) are completely determined by the relation (9) by noticing that the value of 0-loop \( p \)-integral is simply 1.

We eventually arrive at a surprising conclusion that all single-mass vacuum FIs are completely determined by linear algebra between different FIs, and therefore, all FIs are completely determined by linear algebra between different FIs. This conclusion is valid for any number of loops \( L \) and arbitrary dimensionality \( D \).

**Examples.** — To better understand the above observation, let us compute some FIs.

One of the simplest examples is the 2-loop single-mass vacuum integral shown in Fig. 4 (b), defined by

\[
I_{(1,1,1)} = \int \frac{\prod_{i=1}^{2} d^{D} \ell_{i}}{i\pi D/2} \frac{1}{(\ell_{1}^{2} - 1)(\ell_{2}^{2} + \ell_{2})^{2}},
\]

(10)

where Feynman prescription \( i0^{+} \) for each denominator is suppressed. The relation (9) gives

\[
I_{(1,1,1)} = \frac{Γ(3 - D)Γ(D - 2)}{-Γ(1)Γ(D/2)} \tilde{I}_{(1,1)}(-1),
\]

with

\[
\tilde{I}_{(1,1)}(-1) = \int \frac{d^{D} \ell_{2}}{i\pi D/2} \frac{1}{\ell_{2}^{2}(\ell_{2} + p)^{2}},
\]

(12)

where \( p^{\mu} \) satisfies \( p^{2} = -1 \).

To calculate the 1-loop \( p \)-integral \( \tilde{I}_{(1,1)}(-1) \) via the AMF method, we introduce auxiliary integrals

\[
\tilde{I}_{(1,0)}(\eta) = \int \frac{d^{D} \ell_{2}}{i\pi D/2} \frac{1}{\ell_{2}^{2} - \eta},
\]

\[
\tilde{I}_{(1,1)}(\eta) = \int \frac{d^{D} \ell_{2}}{i\pi D/2} \frac{1}{(\ell_{2}^{2} - \eta)(\ell_{2} + p)^{2}},
\]

(14)

which are MIs of the corresponding auxiliary family. Denoting \( \mathcal{J} = (\tilde{I}_{(1,0)}, \tilde{I}_{(1,1)})^{T} \), \( \eta \)-DEs can be obtained using IBP reductions,

\[
\frac{\partial \mathcal{J}(\eta)}{\partial \eta} = \begin{pmatrix} \frac{1-\epsilon}{1-\eta} & 0 \\ \frac{1+\epsilon}{1-\eta} & \frac{1-2\epsilon}{1+\eta} \end{pmatrix} \mathcal{J}(\eta).
\]

(15)

As \( \eta \to \infty \), only the integration region \( |\ell_{2}| \sim \mathcal{O}(\sqrt{\eta}) \) gives nonzero contribution. Thus we have

\[
\tilde{I}_{(1,0)}(\eta) = \eta^{D/2-1} \int \frac{d^{D} \ell_{2}}{i\pi D/2} \frac{1}{\ell_{2}^{2} - 1} = \eta^{D/2-1}(-1)Γ(1 - D/2),
\]

(16)

where in the last step the relation (9) has been used, and

\[
\tilde{I}_{(1,1)}(\eta) \sim \eta^{D/2-2} \int \frac{d^{D} \ell_{2}}{i\pi D/2} \frac{1}{(\ell_{2}^{2} - 1)(\ell_{2}^{2} - 1)} = \eta^{D/2-2} \int \frac{d^{D} \ell_{2}}{i\pi D/2} \frac{1}{\ell_{2}^{2} - 1} = I_{(1,0)}(\eta),
\]

(17)

where scaleless integrals are omitted in the third line.

By solving the \( \eta \)-DEs (15) together with boundary conditions at \( \eta \to \infty \) in Eqs. (10) and (17), \( \tilde{I}_{(1,1)}(-1) = \tilde{I}_{(1,1)}(i0^{-}) \) is determined. We thus obtain the desired FI \( I_{(1,1,1)} \) using the relation (11).

Clearly, the same procedure can be used to compute any FI. Let us give the result of another example shown in Fig. 4 (c), which is one of the most complicated 5-loop single-mass vacuum FIs. Following the above described procedure, we can compute all MIs in this family to very high precision, with only input of IBP reductions. The result of the corner integral with 10-digit precision is given by

\[
-2.073855510e^{-2} - 7.812755312e^{-1} - 17.25882864 + 717.6808845e + 8190.87648e^{2} + 78840.29598e^{3}
+ 56664.1116e^{4} + 3901713.802e^{5} + 23702384.71e^{6},
\]

(18)

where we have set \( D = 4 - 2\epsilon \) with only 9 orders in \( \epsilon \) expansion are shown, although more orders and digits can be easily obtained. The first seven terms of the expansion agree with that obtained in Ref. [48], and other terms are new.

**A new method to calculate perturbative corrections.** — An important feature of our strategy is that the FIs we calculate can have arbitrary dimensionality. This on the one hand makes our strategy applicable for a general theory, e.g., nonrelativistic theory with dimensionality equals 3. And on the other hand, by sampling different dimensionality around a fixed value, say \( 4 \), we can fit the Laurent expansion with respect to \( \epsilon \) to any desired order, which is actually the way we obtain the results in Eq. (18).

If we apply the above strategy directly to physical processes, we arrive at a new and powerful method to calculate perturbative corrections. Let us explain this by an example: the next-to-next-to-leading order (NNLO) QCD correction to top-antitop quark pair production cross section in lepton colliders \( e^{+}e^{-} \to \gamma^{*} \to t\bar{t} + X \), which have been previously calculated in Refs. [77–79].

In our method, we calculate bare cross section (before renormalization) with an numerical value of \( \epsilon \), and then renormalize it in the standard \( \overline{\text{MS}} \) scheme, with the same value of \( \epsilon \). To show a numerical result, we choose center-of-mass energy \( s \), renormalization scale \( \mu \) and top quark mass as \( \mu = \sqrt{s} = 1 \) and \( m^{2} = 1/8 \). We ignore contributions from internal top quark loops and that from photon...
interacting with other five type of quarks, because these contributions are very small. Then, if we set \( \epsilon = 0.001 \), the NNLO correction gives

\[
\sigma_{0.001}^{\text{NNLO}} / (\alpha_s^2) = 9.261823090, \quad (19)
\]

where only 10 digits are shown. Because cross section is a physical quantity that is free of divergence, \( \sigma_{0.002}^{\text{NNLO}} \) can give an estimation of total cross section up to \( \mathcal{O}(\epsilon) \) error. Now let us calculate the cross section with another value \( \epsilon = 0.002, \) which gives

\[
\sigma_{0.002}^{\text{NNLO}} / (\alpha_s^2) = 9.269638349. \quad (20)
\]

The fact that \( \sigma_{0.001}^{\text{NNLO}} \) and \( \sigma_{0.002}^{\text{NNLO}} \) has a relative difference at \( \mathcal{O}(1/1000) \) level confirms two things. First, \( \sigma_{\epsilon}^{\text{NNLO}} \) calculated here is free of \( 1/\epsilon^n \) divergence, or else the difference should be at \( \mathcal{O}(1) \) level. Second, \( \sigma_{\epsilon_1}^{\text{NNLO}} = \sigma_{\epsilon_2}^{\text{NNLO}} + \mathcal{O}(\epsilon_1 - \epsilon_2) \) is justified. Therefore, we can fit a linear function of \( \epsilon \) by combining values of \( \sigma_{0.001}^{\text{NNLO}} \) and \( \sigma_{0.002}^{\text{NNLO}} \) to provide a better estimation of \( \sigma_0^{\text{NNLO}} \),

\[
\sigma_0^{\text{NNLO}} / (\alpha_s^2) \approx 9.2540, \quad (21)
\]

which becomes closer to the exact result 9.253454354. By calculating more values of \( \epsilon \), we can further improve the estimation to any desired precision.

In this method, we do not need to manipulate a Laurent expansion of \( \epsilon \) during the intermediate stage of calculation, and thus the computational time can be usually reduced by several times. This improvement of efficiency is very important for cutting-edge problems. Actually, using this method we have successfully calculated the above mentioned \( t\bar{t} \) production to next-to-next-to-next-to-leading order for the first time, which will be presented elsewhere.

**Summary and outlook.** — By combining the recently proposed AMF method and the relation Eq. \( 19 \), we find that all FIs, with any number of loops and arbitrary dimensionality, are completely determined by linear relations between FIs. This interesting observation conceptually changes FIs computation to an algebra problem. This observation has been explicitly verified by some examples up to 5 loops.

For phenomenological purpose, many general FIs need to be calculated. The mainstream method on the market to compute FIs can be divided into two steps. In the first step one reduces all FIs to MIs and in the second step one calculates these MIs. Both of the two steps are found to be very difficult for current cutting-edge problems. With our strategy, IBP reduction becomes the only obstacle for FIs calculation. Our strategy has been implemented in the package AMFFlow \( 18 \), which can automatically calculate general FIs, with any number of loops, to high precision, as far as IBP reduction is successful. These features make our method unique comparing with other methods of FIs computation on the market.

Because FIs with any dimensionality can now be calculated, we can calculate physical processes directly with a given small value of \( \epsilon \), the dimensional regulator. In this way, we can significantly improve the efficiency of perturbative calculation. Furthermore, our method is applicable for a general theory like nonrelativistic theory with dimensionality equals 3.

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