Efficient Dominating and Edge Dominating Sets for Graphs and Hypergraphs

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Abstract. Let $G = (V, E)$ be a graph. A vertex dominates itself and all its neighbors, i.e., every vertex $v \in V$ dominates its closed neighborhood $N[v]$. A vertex set $D$ in $G$ is an efficient dominating (e.d.) set for $G$ if for every vertex $v \in V$, there is exactly one $d \in D$ dominating $v$. An edge set $M \subseteq E$ is an efficient edge dominating (e.e.d.) set for $G$ if it is an efficient dominating set in the line graph $L(G)$ of $G$. The ED problem (EED problem, respectively) asks for the existence of an e.d. set (e.e.d. set, respectively) in the given graph.

We give a unified framework for investigating the complexity of these problems on various classes of graphs. In particular, we solve some open problems and give linear time algorithms for ED and EED on dual chordal graphs.

We extend the two problems to hypergraphs and show that ED remains $\text{NP}$-complete on $\alpha$-acyclic hypergraphs, and is solvable in polynomial time on hypertrees, while EED is polynomial on $\alpha$-acyclic hypergraphs and $\text{NP}$-complete on hypertrees.

Keywords: efficient domination; efficient edge domination; graphs and hypergraphs; polynomial time algorithms

1 Introduction and Basic Notions

Packing and covering problems in graphs and their relationships belong to the most fundamental topics in combinatorics and graph algorithms and have a wide spectrum of applications in computer science, operations research and many other fields. Recently, there has been an increasing interest in problems combining packing and covering properties. Among them, there are the following variants of domination problems:

Let $G$ be a finite simple undirected graph with vertex set $V$ and edge set $E$. A vertex dominates itself and all its neighbors, i.e., every vertex $v \in V$ dominates its closed neighborhood $N[v] = \{ u \mid u = v \text{ or } uv \in E \}$. A vertex set $D$ in $G$ is an efficient dominating (e.d.) set for $G$ if for every vertex $v \in V$, there is exactly one $d \in D$ dominating $v$ (sometimes called independent perfect dominating set) \textsuperscript{12}. 

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An edge set $M \subseteq E$ is an efficient edge dominating (e.e.d.) set for $G$ if it is an efficient dominating set in the line graph $L(G)$ of $G$ (sometimes called dominating induced matching). The ED problem (EED problem, respectively) asks for the existence of an e.d. set (e.e.d. set, respectively) in the given graph. Note that both problems are $\mathbb{NP}$-complete. The complexity of ED (EED, respectively) (and their variants) with respect to special graph classes was studied in various papers; see e.g. [2, 20, 34, 35, 38, 39, 40, 41] for ED and [9, 11, 17, 27, 36, 37] for EED. The main contributions of our paper are:

(i) a unified framework for the ED and EED problems solving some open questions,
(ii) linear time algorithms for ED and EED on dually chordal graphs, and
(iii) an extension of the two problems to hypergraphs, in particular to $\alpha$-acyclic hypergraphs and hypertrees: We show that ED remains $\mathbb{NP}$-complete on $\alpha$-acyclic hypergraphs, and is solvable in polynomial time on hypertrees, while EED is polynomial on $\alpha$-acyclic hypergraphs and $\mathbb{NP}$-complete on hypertrees.

Our approach has the advantage that it unifies the proofs of various results obtained in numerous papers. Our proofs are typically very short since we extensively use some theoretical background on the relations of the considered graph and hypergraph classes, and in particular closure properties of graph classes with respect to squares of their graphs, and polynomial time algorithms for Maximum Weight Independent Set and Minimum Weight Dominating Set on some graph classes. The consequences are some new cases where the corresponding problems can be efficiently solved.

2 Further Basic Notions

2.1 Basic notions and properties of graphs

Let $G$ be a finite undirected graph without loops and multiple edges. Let $V$ denote its vertex (or node) set and $E$ its edge set; let $|V| = n$ and $|E| = m$. A vertex $v$ is universal in $G$ if it is adjacent to all other vertices of $G$. A chordless path $P_k$ (chordless cycle $C_k$, respectively) has $k$ vertices, say $v_1, \ldots, v_k$, and edges $v_i v_{i+1}, 1 \leq i \leq k - 1$ (and $v_k v_1$, respectively). A hole is a chordless cycle $C_k$ for $k \geq 5$. $G$ is chordal if no induced subgraph of $G$ is isomorphic to $C_k$, $k \geq 4$. See e.g. [10] for the many facets of chordal graphs. A vertex set $U \subseteq V$ is independent if for all $x, y \in U$, $xy \notin E$ holds. For a graph $G$ and a vertex weight function on $G$, let the Maximum Weight Independent Set (MWIS) problem be the task of finding an independent vertex set of maximum weight.

Let $K_i$ denote the clique with $i$ vertices. Let $K_4 - e$ or diamond be the graph with four vertices and five edges, say vertices $a, b, c, d$ and edges $ab, ac, bc, bd, cd$; its mid-edge is the edge $bc$. Let gem denote the graph consisting of five vertices, four of which induce a $P_4$, and the fifth is universal. Let $W_4$ denote the graph with five vertices consisting of a $C_4$ and a universal vertex.
For $U \subseteq V$, let $G[U]$ denote the subgraph induced by $U$. For a set $\mathcal{F}$ of graphs, a graph $G$ is called $\mathcal{F}$-free if $G$ contains no induced subgraph from $\mathcal{F}$. Thus, it is hole-free if it contains no induced subgraph isomorphic to a hole. A graph $G$ is weakly chordal if $G$ and its complement graph are hole-free. Three pairwise non-adjacent distinct vertices form an asteroidal triple (AT) in $G$ if for each choice of two of them, there is a path between the two avoiding the neighborhood of the third. A graph $G$ is AT-free if $G$ contains no AT.

2.2 Basic notions and properties of hypergraphs

Throughout this paper, a hypergraph $H = (V, E)$ has a finite vertex set $V$ and for all $e \in E$, $e \subseteq V$ ($E$ possibly being a multiset). For a graph $G$, let $\mathcal{N}(G)$ denote the closed neighborhood hypergraph, i.e., $\mathcal{N}(G) = (V, \{N[v] \mid v \in V\})$, and let $\mathcal{C}(G)$ denote the clique hypergraph consisting of the inclusion-maximal cliques of $G$.

A subset of edges $E' \subseteq E$ is an exact cover of $H$ if for all $e, f \in E'$ with $e \neq f$, $e \cap f = \emptyset$ and $\bigcup E' = V$. The Exact Cover problem asks for the existence of an exact cover in a given hypergraph $H$. It is well known that this problem is NP-complete even for 3-elementary hyperedges (problem X3C [SP2] in [24]). Thus, the ED problem on a graph $G$ is the same as the Exact Cover problem on $\mathcal{N}(G)$.

For defining the class of dually chordal graphs, whose properties will be contrasted with those of chordal graphs, as well as for extending the ED and the EED problems to hypergraphs, we need some basic definitions: For a hypergraph $H = (V, E)$, let $2\text{sec}(H)$ denote its 2-section (also called representative or primal) graph, i.e., the graph having the same vertex set $V$ in which two vertices are adjacent if they are in a common hyperedge. Let $L(H)$ denote the line graph of $H$, i.e., the graph with the hyperedges $E$ as its vertex set in which two hyperedges are adjacent in $L(H)$ if they intersect each other.

A hypergraph $H = (V, E)$ has the Helly property if the total intersection of every pairwise intersecting family of hyperedges of $E$ is nonempty. $H$ is conformal if every clique of the 2-section graph $2\text{sec}(H)$ is contained in a hyperedge of $E$ (see e.g. [5,22]).

The notion of $\alpha$-acyclicity [22] is one of the most important and most frequently studied hypergraph notions. Among the many equivalent conditions describing $\alpha$-acyclic hypergraphs, we take the following: For a hypergraph $H = (V, E)$, a tree $T$ with node set $\mathcal{E}$ and edge set $E_T$ is a join tree of $H$ if for all vertices $v \in V$, the set of hyperedges containing $v$ induces a subtree of $T$. $H$ is $\alpha$-acyclic if it has a join tree. For a hypergraph $H = (V, E)$ and vertex $v \in V$, let $\mathcal{E}_v := \{e \in E \mid v \in e\}$. Let $H^* := (\mathcal{E}, \{\mathcal{E}_v \mid v \in V\})$ be the dual hypergraph of $H$. $H = (V, E)$ is a hypertree if there is a tree $T$ with vertex set $V$ such that for all $e \in E$, $T[e]$ is connected.

**Theorem 1 (Duchet, Flament, Slater, see [10]).** $H$ is a hypertree if and only if $H$ has the Helly property and its line graph $L(H)$ is chordal.

The following facts are well known:
Lemma 1. Let $H$ be a hypergraph.

(i) $H$ is conformal if and only if $H^*$ has the Helly property.
(ii) $L(H)$ is isomorphic to $2\sec(H^*)$.

Thus:

Corollary 1. $H$ is $\alpha$-acyclic if and only if $H$ is conformal and its 2-section graph is chordal.

It is easy to see that the dual $N(G)^*$ of $N(G)$ is $N(G)$ itself, and for any graph $G$:

$$G^2 \text{ is isomorphic to } L(N(G)).$$ (1)

In [8], the notion of dually chordal graphs was introduced: For a graph $G = (V, E)$ and a vertex $v \in V$, a vertex $u \in N[v]$ is a maximum neighbor of $v$ if for all $w \in N[v]$, $N[w] \subseteq N[u]$ holds. (Note that by this definition, a vertex can be its own maximum neighbor.) Let $\sigma = (v_1, \ldots, v_n)$ be a vertex ordering of $V$. Such an ordering $\sigma$ is a maximum neighborhood ordering of $G$ if for every $i \in \{1, 2, \ldots, n\}$, $v_i$ has a maximum neighbor in $G_i := G[\{v_1, \ldots, v_n\}]$. A graph is dually chordal if it has a maximum neighborhood ordering. The following is known:

Theorem 2 ([7,8,21]). Let $G$ be a graph. Then the following are equivalent:

(i) $G$ is a dually chordal graph.
(ii) $N(G)$ is a hypertree.
(iii) $C(G)$ is a hypertree.
(iv) $G$ is the 2-section graph of some hypertree.

Thus, Theorems 1 and 2 together with (1) and the duality properties in Lemma 1 imply:

Corollary 2 ([7,8,21]). Let $G$ be a graph and $H$ be a hypergraph.

(i) $G$ is dually chordal if and only if $G^2$ is chordal and $N(G)$ has the Helly property.
(ii) If $H$ is $\alpha$-acyclic then its line graph $L(H)$ is dually chordal.
(iii) If $H$ is a hypertree then its 2-section graph $2\sec(H)$ is dually chordal.

3 Efficient (Edge) Domination in General

Recall that a subset $D \subseteq V$ of vertices is an efficient dominating set if for all $v \in V$, there is exactly one $d \in D$ such that $v \in N[d]$. Also a subset $M \subseteq E$ of edges is an efficient edge dominating set in $G$ if for all $e \in E$, there is exactly one $e' \in M$ intersecting the edge $e$.

Both definitions can be extended to hypergraphs: A subset $D \subseteq V$ is an efficient dominating set for a hypergraph $H$ if it is an efficient dominating set for its 2-section graph $2\sec(H)$. A subset $M \subseteq E$ of hyperedges is an efficient edge dominating set for $H$ if for all $e \in E$, there is exactly one $e' \in M$ intersecting the edge $e$. 

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Corollary 3. A vertex set $D$ is an efficient dominating set in $H$ if and only if $D$ is an efficient edge dominating set in $H^*$. 

The following approach developed in [33] and [39] gives a tool for showing that for various classes of graphs, the ED problem can be solved in polynomial time. For a graph $G = (V, E)$, we define the following vertex weight function: Let $\omega(v) := |N_G[v]|$ (i.e., $\omega(v) := \text{deg}(v) + 1$), and for $D \subseteq V$, let $\omega(D) := \Sigma_{d \in D} \omega(d)$. Obviously, the following holds:

Proposition 1. Let $G = (V, E)$ be a graph and $D \subseteq V$.

(i) If $D$ is a dominating vertex set in $G$ then $\omega(D) \geq |V|$.
(ii) If $D$ is an independent vertex set in $G$ then $\omega(D) \leq |V|$.

Lemma 2. Let $G = (V, E)$ be a graph and $\omega(v) := |N[v]|$ a vertex weight function for $G$. Then the following are equivalent for any subset $D \subseteq V$:

(i) $D$ is an efficient dominating set in $G$
(ii) $D$ is a minimum weight dominating vertex set in $G$ with $\omega(D) = |V|$.
(iii) $D$ is a maximum weight independent vertex set in $G^2$ with $\omega(D) = |V|$.

Proof. (i) $\rightarrow$ (ii): If $D$ is an efficient dominating set in $G$ then, by definition, it is a dominating vertex set where every vertex $v \in V$ is in exactly one closed neighborhood of elements of $D$. Thus, the closed neighborhoods $N[d], d \in D$, give a partition of $V$, and thus, $\omega(D) = |V|$ holds. Also, by Proposition 1 there is no $D'$ with $\omega(D') < \omega(D)$.

(ii) $\rightarrow$ (iii): If $D$ is a dominating set in $G$ with $\omega(D) = |V|$ then the closed neighborhoods $N[d], d \in D$, give a partition of $V$, and in particular, $D$ is a maximum independent vertex set in $G^2$ with $\omega(D) = |V|$.

(iii) $\rightarrow$ (i): If $D$ is an independent vertex set in $G^2$ with $\omega(D) = |V|$ then the closed neighborhoods $N[d], d \in D$, give a partition of $V$, and thus, $D$ is an efficient dominating set in $G$. \hfill $\square$

Note that $D$ is not any independent (dominating) set, but a maximum (minimum) weight one. This implies:

Corollary 4. For every graph class $C$ for which the MWIS problem is solvable in polynomial time on squares of graphs from $C$, the ED problem for $C$ is solvable in polynomial time.

Corollary 5. Let $H$ be a hypergraph, $L(H) = (V, E)$ its line graph and $\omega(v) := |N[v]|$ a vertex weight function for $L(H)$ as above. Then the following are equivalent for any subset $D \subseteq V$:

(i) $D$ is an efficient edge dominating set in $H$
(ii) $D$ is an efficient dominating set in $L(H)$.
(iii) $D$ is a minimum weight dominating vertex set in $L(H)$ with $\omega(D) = |V|$.
(iv) $D$ is a maximum weight independent vertex set in $L(H)^2$ with $\omega(D) = |V|$.
4 Efficient Domination in Graphs

This section presents results for the ED problem on some graph classes.

**Theorem 3 ([38,41]).** The ED problem is NP-complete for bipartite graphs, for chordal graphs as well as for chordal bipartite graphs.

By Corollary 2 (i), the square of a dually chordal graph is chordal. Thus, based on Lemma 2, ED for dually chordal graphs can be solved in polynomial time by solving the MWIS problem on chordal graphs. However, the MWIS problem is solvable in linear time for chordal graphs with the following algorithm:

**Algorithm 1 ([23])**

**Input:** A chordal graph $G = (V, E)$ with $|V| = n$ and a vertex weight function $\omega$.

**Output:** A maximum weight independent set $I$ of $G$.

1. Find a perfect elimination ordering $(v_1, \ldots, v_n)$ and set $I := \emptyset$.
2. For $i := 1$ To $n$
   - If $\omega(v_i) > 0$, mark $v_i$ and set $\omega(u) := \max(\omega(u) - \omega(v_i), 0)$ for all vertices $u \in N(v_i)$.
3. For $i := n$ DownTo 1
   - If $v_i$ is marked, set $I := I \cup \{v_i\}$ and unmark all $u \in N(v_i)$.

By using the following lemmas, the algorithm can be modified in such a way, that it solves the ED problem for dually chordal graphs in linear time.

**Lemma 3 ([7]).** A maximum neighborhood ordering of $G$ which simultaneously is a perfect elimination ordering of $G^2$ can be found in linear time.

The algorithm given in [7] not only finds a maximum neighborhood ordering $(v_1, \ldots, v_n)$. It also computes the maximum neighbors $m_i$ for each vertex $v_i$ with the property that for all $i < n$ no vertex $v_i$ is its own maximum neighbor ($v_i \neq m_i$). This is necessary for the following lemma.

**Lemma 4.** Let $G = (V, E)$ be a graph with $G^2 = (V, E^2)$ and a maximum neighborhood ordering $(v_1, \ldots, v_n)$ where $m_i$ is the maximum neighbor of $v_i$ with $v_i \neq m_i$ and $1 \leq i < j \leq n$. Then: $v_iv_j \in E^2 \iff m_iv_j \in E$.

**Proof.** $\Leftarrow$: $v_j$ is connected with $m_i$ ($m_iv_j \in E$). Thus, the distance between $v_i$ and $v_j$ is at most 2. So $v_i$ and $v_j$ are also connected in $G^2$ ($v_iv_j \in E^2$).

$\Rightarrow$: $v_i$ and $v_j$ are connected in $G^2$ ($v_iv_j \in E^2$). If $v_iv_j \in E$, then $v_iv_j \in E^2$. Now assume, that $v_iv_j \notin E$. Then there is a vertex $v_k$ with $v_kv_k \in E$ and $v_kv_j \in E$. We can distinguish between two cases:

(i) $i < k$. By definition $m_i$ is connected with all neighbors of $v_k$. This also includes $v_j$.

(ii) $k < i$. In this case $v_i$ has a maximum neighbor $m_k$ with $v_im_k \in E$ and $m_kv_j \in E$. Thus, we can repeat the distinction with $v_k := m_k$ until $i < k$. \qed
This allows to modify Algorithm 1 in a way, that it is no longer necessary to compute the square of the given dually chordal graph $G$. Instead, a maximum weight independent set of $G^2$ can be computed on $G$ in linear time.

**Algorithm 2** ([33])

**Input:** A dually chordal graph $G = (V,E)$.

**Output:** An efficient dominating set $D$ (if existing).

1. $D = \emptyset$.
2. For All $v \in V$
   
   Set $\omega(v) := |N(v)|$ and $\omega_p(v) := 0$. $v$ is unmarked and not blocked.

3. Find a maximum neighborhood ordering $(v_1, \ldots, v_n)$ with the corresponding maximum neighbors $(m_1, \ldots, m_n)$ where $v_i \neq m_i$ for $1 \leq i < n$.

4. For $i := 1$ To $n$
   
   For all $u \in N[v_i]$ set $\omega(v_i) := \omega(v_i) - \omega_p(u)$. If $\omega(v_i) > 0$, mark $v_i$ and set $\omega_p(m_i) := \omega_p(m_i) + \omega(v_i)$.

5. For $i := n$ DownTo 1
   
   If $v_i$ is marked and $m_i$ is not blocked, set $D := D \cup \{v_i\}$ and block all $u \in N(v_i)$.

6. $D$ is an efficient dominating set if and only if $\sum_{v \in D} |N[v]| = |V|$.

**Theorem 4.** Algorithm 2 works correctly and runs in linear time.

**Proof.** Algorithm 2 is a modification of Algorithm 1. It computes a maximum weight independent set $D$ of the square of the given graph $G$ and checks if this set is an efficient dominating set.

Based on Lemma 3, line (3) of Algorithm 2 computes a perfect elimination ordering of $G^2$.

Let $v_i$ and $v_j$ be adjacent in $G^2$ and $i < j$. Now Lemma 4 allows to modify the two loops. For the first loop (line (4)), there is an extra vertex weight $\omega_p$. Instead of decrementing the weights $\omega$ of the neighbors of $v_i$ (and their neighbors), $\omega_p$ of the maximum neighbor $m_i$ is incremented by $\omega(v_i)$. Now before comparing $\omega(v_j)$ to 0, $\omega(v_j)$ is decremented by $\omega_p(u)$ for all $u \in N[v_j]$. Because $m_i$ is connected with $v_j$ (Lemma 4), this ensures that each time the weight $\omega(v_j)$ is compared to 0, it has the same value as it would have in Algorithm 1.

For the second loop (line (5)) the argumentation works quite equally. After selecting a vertex $v_j$ (i.e. $v_j \in D$), all its neighbors are blocked. Thus by Lemma 4 if a vertex $v_i$ is adjacent in $G^2$ to a selected vertex, then the maximum neighbor $m_i$ is blocked.

By Lemma 2 it can be checked if $D$ is an efficient dominating set by counting the number of neighbors.

Each line of Algorithm 2 runs in linear time. For line (3) the algorithm given in [7] can be used. The lines (4) to (6) are bounded by the number of nodes and their neighbors (Recall, $\sum_{v \in V} |N(v)| = 2|E|$).

Note that strongly chordal graphs are dually chordal [8]. In [33] one of the open problems is the complexity of (weighted) ED for strongly chordal graphs which is solved by Theorem 4 (for the weighted case see [33]).
Theorem 5. For AT-free graphs, the ED problem is solvable in polynomial time.

Proof. In [19], it is shown that the square of any AT-free graph is a co-comparability graph (which is AT-free). In [12], the MWIS problem for AT-free graphs is solved in polynomial time. This and Corollary [4] implies the result. □

This partially extends the result of [20] showing that the (weighted) ED problem for co-comparability graphs is solvable in polynomial time.

In [38], one of the open problems is the complexity of ED for convex bipartite graphs. This class of graphs is contained in interval bigraphs, and a result of [32] shows that the boolean width of interval bigraphs is at most $2 \log n$, based on a corresponding result for interval graphs [3]. By a result of [4], this leads to a polynomial time algorithm for Minimum Weight Domination on interval bigraphs.

Corollary 6. For interval bigraphs, the ED problem is solvable in polynomial time.

This solves the open question from [38] for convex bipartite graphs.

5 Efficient Edge Domination in Graphs

Lemma 5 ([9,11]). Let $G$ be a graph that has an e.e.d. set $M$.

(i) $M$ contains exactly one edge of every triangle of $G$.
(ii) $G$ is $K_4$-free.
(iii) If $xy$ is the mid-edge of an induced diamond in $G$ then $M$ necessarily contains $xy$. Thus, in particular, $G$ is $W_4$-free and gem-free.

In [36], it was shown that the EED problem is solvable in linear time on chordal graphs. This allows us to solve the EED problem for dually chordal graphs using the following lemma:

Lemma 6. Let $G$ be a graph that has an e.e.d. set. Then $G$ is chordal if and only if $G$ is dually chordal.

Proof. ⇒: $G$ is chordal. Since $G$ has an e.e.d. set there is no gem in $G$ (Lemma [5]). Thus, $G$ is sun-free chordal (strongly chordal respectively). Every strongly chordal graph is dually chordal [8]. So $G$ is dually chordal.

⇐: $G$ is dually chordal. Let $\sigma = (v_1, \ldots, v_n)$ be a maximum neighborhood ordering of $G$. If $G$ is not chordal then $G$ contains an induced subgraph $C$ isomorphic to $C_k$ for some $k \geq 4$. Let $w = v_1$ be the leftmost vertex of $C$ in $\sigma$. Vertex $w$ has a maximum neighbor $u$ in $G_i = G[\{v_1, \ldots, v_n\}]$. Note that $w \neq u$ since $C$ is chordless. Now, if $C$ is a $C_4$, $u$ is adjacent to all vertices of $C$, and if $C$ is a $C_k$ for $k \geq 5$ then $u$ is adjacent to all vertices of a $P_4$ in $C$ and thus contains a gem — in each case it follows by Lemma [5] that $G$ has no e.e.d. set. □
Corollary 7. The EED problem can be solved in linear time for dually chordal graphs.

Efficient edge dominating sets are closely related to maximum induced matchings; it is not hard to see that every efficient edge dominating set is a maximum induced matching but of course not vice versa. However, when the graph has an efficient edge dominating set and is regular then every maximum induced matching is an efficient edge dominating set [10]. On the other hand, the complexity of the two problems differs on some classes such as claw-free graphs where the Maximum Induced Matching (MIM) problem is NP-complete [31] (even on line graphs) while the EED problem is solvable in polynomial time [17]. While every graph has a maximum induced matching, this is not the case for efficient edge dominating sets. Thus, if the graph $G$ has an efficient edge dominating set, this gives also a maximum induced matching but in the other case, the MIM problem is hard for claw-free graphs.

For the MIM problem, there is a long list of results of the following type: If a graph $G$ is in a graph class $C$ then also $L(G)^2$ is in the same class (see e.g. [14,15]), and if the MWIS problem is solvable in polynomial time for the same class, this leads to polynomial algorithms for the MIM problem on the class $C$. A very large class of this type are interval-filament graphs [25] which include co-comparability graphs and polygon-circle graphs; the latter include circle graphs, circular-arc graphs, chordal graphs, and outerplanar graphs. AT-free graphs include co-comparability graphs, permutation graphs and trapezoid graphs (see [10]).

Theorem 6 ([14,18]). Let $G = (V, E)$ be a graph, $L(G)$ its line graph, and $L(G)^2$ its square. Then the following conditions hold:

(i) If $G$ is an interval-filament graph then $L(G)^2$ is an interval-filament graph.
(ii) If $G$ is an AT-free graph then $L(G)^2$ is an AT-free graph.

The MWIS problem for interval-filament graphs is solvable in polynomial time [25]; as a consequence, it is mentioned in [14] that the MIM problem is efficiently solvable for interval-filament graphs. In [12] it is shown that the MWIS problem is solvable for AT-free graphs in time $O(n^4)$. For the EED problem, it follows:

Corollary 8. The EED problem is solvable in polynomial time for interval-filament graphs and for AT-free graphs.

This generalizes the corresponding result for bipartite permutation graphs in [30]; bipartite permutation graphs are AT-free. It also generalizes a corresponding result for MIM on trapezoid graphs [26] (which are AT-free).

In [17], the complexity of the EED problem for weakly chordal graphs was mentioned as an open problem; in [9], however, it was shown that the EED problem (DIM problem, respectively) is solvable in polynomial time for weakly chordal graphs. It is easy to see that long antiholes (i.e., complements of $C_k$,
$k \geq 6)$ have no e.e.d. set, i.e., a hole-free graph having an e.e.d. set is weakly chordal. Thus, for hole-free graphs, the EED problem is solvable in polynomial time \[9\]. Corollary \[6\] leads to an easier way of solving the EED problem on weakly chordal graphs:

**Corollary 9.** The EED problem is solvable in polynomial time for weakly chordal graphs.

**Proof.** In \[15\], it is shown that for a weakly chordal graph $G$, $L(G)^2$ is weakly chordal. For weakly chordal graphs, MWIS is solvable in time $O(n^4)$ \[28,29\]. Thus, by Corollary \[6\] the claim holds. $\square$

The next result contrasts to the fact that the MIM and the EED problem are solvable in polynomial time on chordal graphs:

**Proposition 2.** The MIM problem is $\mathsf{NP}$-complete for dually chordal graphs.

**Proof.** Let $G$ be any graph. We add a new universal vertex, say $u$, to $G$ and denote this graph by $G'$. It is easy to see that $G'$ is dually chordal, and if $G$ is nontrivial, a maximum induced matching of $G'$ cannot contain $u$ in its vertex set. This and the fact that the MIM problem is $\mathsf{NP}$-complete in general \[13\] shows Proposition \[2\] $\square$

### 6 Some Results for Hypergraphs

Theorem \[8\] and the fact that every chordal graph is the 2-section graph of an $\alpha$-acyclic hypergraph (namely, of its clique hypergraph $C(G)$) implies:

**Corollary 10.** The ED problem is $\mathsf{NP}$-complete for $\alpha$-acyclic hypergraphs.

This situation is better for hypertrees:

**Corollary 11.** For hypertrees, the ED problem is solvable in polynomial time.

**Proof.** Let $H$ be a hypertree. Then by Theorem\[2\](iv), $2\sec(H)$ is dually chordal. Thus, ED is solvable in polynomial time using Algorithm \[2\] $\square$

Based on Corollary \[3\] and the duality of hypertrees and $\alpha$-acyclic hypergraphs it follows:

**Corollary 12.** The EED problem for hypertrees is $\mathsf{NP}$-complete.

**Corollary 13.** For $\alpha$-acyclic hypergraphs, the EED problem is solvable in polynomial time.

For the MIM problem we show:

**Theorem 7.** The MIM problem is solvable in polynomial time for $\alpha$-acyclic hypergraphs.
Proof. Let \( H = (V, \mathcal{E}) \) be \( \alpha \)-acyclic. Then by Corollary 2(ii), \( L(H) \) is dually chordal. By Corollary 2(i), the square of a dually chordal graph is chordal, i.e., \( L(H)^2 \) is chordal. So the MIM problem for \( \alpha \)-acyclic hypergraphs can be solved by solving the MWIS problem for chordal graphs.

Theorem 8. The MIM problem for hypertrees is \( \mathbb{NP} \)-complete.

Proof. Let \( H = (V, \mathcal{E}) \) be a hypertree. By definition, \( M \) is an induced matching for \( H \) if it is an independent node set for \( L(H)^2 \). By Theorem 1, for hypertrees \( H \), \( L(H) \) is chordal, and by the same argument as in the proof of Theorem 12, for every chordal graph \( G \), there is a hypertree \( H \) such that \( G \) is isomorphic to \( L(H) \).

Thus, the MIM problem on hypertrees corresponds to the Maximum Independent Set problem on squares of chordal graphs. This problem, however, is \( \mathbb{NP} \)-complete as the following reduction shows: Let \( G = (V, \mathcal{E}) \) be any graph, and let \( F = (V \cup E \cup \{f, g\}, \mathcal{E}_F) \) be the following graph: Its node set consists of all vertices and edges of \( G \) and additionally two new nodes \( f \) and \( g \). Its edge set \( \mathcal{E}_F \) consists of the edges of a clique \( E \cup \{f\} \) and a single edge \( fg \), and all edges between any edge \( e = xy \in \mathcal{E} \) and its vertices \( x, y \) respectively. Thus, \( F \) is a split graph and therefore, \( F \) is chordal. In \( G' = F^2 \), the vertex set \( V \) of \( G \) induces a subgraph isomorphic to \( G \). Let \( \alpha(G) \) denote the maximum size of an independent vertex set in \( G \). We claim:

\[ \alpha(G') = \alpha(G) + 1. \tag{2} \]

Proof of \( \alpha(G') = \alpha(G) + 1 \) is as follows. First assume that \( U \) is an independent vertex set in \( G \) with \( |U| = \alpha(G) \). Then obviously, \( U' = U \cup \{g\} \) is an independent node set in \( G' \) with \( |U'| = \alpha(G) + 1 \). Moreover, \( \alpha(G') \leq \alpha(G) + 1 \) since \( E \cup \{f, g\} \) is a clique in \( G' \).

Conversely assume that \( U' \) is an independent node set in \( G' \) with \( |U'| = \alpha(G) + 1 \). Since \( E \cup \{f, g\} \) is a clique in \( G' \), \( U' \) can contain at most one node of \( E \cup \{f, g\} \), i.e., \( |U' \cap V| = \alpha(G) \), and obviously, \( U' \cap V \) is an independent vertex set in \( G \) which shows the claim.

This reduction shows Theorem 8.

The proof of Theorem 8 also shows that the Maximum Independent Set problem is \( \mathbb{NP} \)-complete for squares of chordal and split graphs. For squares of chordal graphs this was already mentioned without proof in [30].

Theorem 9. The Exact Cover problem is \( \mathbb{NP} \)-complete for \( \alpha \)-acyclic hypergraphs and solvable in polynomial time for hypertrees.

Proof. Exact Cover for \( \alpha \)-acyclic hypergraphs. Let \( H = (V, \mathcal{E}) \) be an arbitrary hypergraph. We construct \( H' = (V', \mathcal{E}') \) as follows: \( V' = V \cup \{u, v\} \) and \( \mathcal{E}' = \mathcal{E} \cup \{g, k\} \) with \( g = V \cup \{u\} \) and \( k = \{u, v\} \). \( H' \) is \( \alpha \)-acyclic. Now we show that \( H \) has an exact cover if and only if \( H' \) has one.

\( \Rightarrow \): Let \( C \subseteq \mathcal{E} \) be an exact cover for \( H \). Then \( C' = C \cup \{k\} \) is an exact cover for \( H' \).
Let $C' \subseteq \mathcal{E}'$ be an exact cover for $H'$. The vertex $v$ is only included in $k$. Therefore $k \notin C'$, so $g$ can not be in $C'$ (otherwise $u$ is covered twice). Thus, $C = C' \setminus \{k\}$ is an exact cover for $H$.

**Exact Cover for hypertrees.** For a hypertree $H = (V, \mathcal{E})$, let $L(H) = (\mathcal{E}, E)$ its line graph and $\omega$ a weight function for $\mathcal{E}$ with $\omega(e) = |e|$. It is easy to see, that $C \subseteq \mathcal{E}$ is an exact cover for $H$ if and only if $C$ is a maximum weight independent set for $L(H)$ with $\sum_{e \in C} \omega(e) = |V|$. Because $L(H)$ is chordal, the Exact Cover problem can be solved for hypertrees in polynomial time by using Algorithm 1.

Because for each hypergraph $H$ an edge set $C \subseteq \mathcal{E}$ is an exact cover for $H$ if and only if $C$ is a maximum weight independent set for $L(H)$, the Exact Cover problem can be solved in in polynomial time if the MWIS problem is polynomial time solvable for the line graph of $H$.

### 7 Conclusion

The subsequent scheme summarizes some of our results; NP-c. means NP-complete, pol. (linear) means polynomial-time (linear-time) solvable, and XC means the Exact Cover problem.

|          | chordal gr. | dually chordal gr | α-acyclic hypergr. | hypertrees |
|----------|-------------|-------------------|--------------------|------------|
| ED       | NP-c. [41]  | linear            | NP-c.              | pol.       |
| EED      | linear [30] | linear            | pol.               | NP-c.      |
| MIM      | pol. [13]   | linear            | pol.               | NP-c.      |
| XC       |             |                   | NP-c.              | pol.       |

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