Comment on “The Winfree model with non-infinitesimal phase-response curve: Ott-Antonsen theory” [Chaos 30, 073139 (2020)]

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In a recent paper [Chaos 30, 073139 (2020)] we analyzed an extension of the Winfree model with nonlinear interactions. The nonlinear coupling function $Q$ was mistakenly identified with the non-infinitesimal phase-response curve (PRC). Here, we assess to what extent $Q$ and the actual PRC differ in practice. By means of numerical simulations, we compute the PRCs corresponding to the $Q$ functions previously considered. The results confirm a qualitative similarity between the PRC and the coupling function $Q$ in all cases.

In Ref.\textsuperscript{1} we studied this generalization of the Winfree model of globally coupled phase oscillators:

\begin{align}
\dot{\theta}_i &= \omega_i + Q(\theta_i,A), \quad (1a) \\
A &= \epsilon \sum_{j=1}^{N} P(\theta_j). \quad (1b)
\end{align}

Here, $A$ is proportional to the sum over the pulses emitted by the $N$ oscillators of the population. In contrast to the original model\textsuperscript{2,3}, function $Q$ in Eq. (1a) has a nonlinear dependence on the mean field $A$. The motivation for this is the fact that nonlinearity is an unavoidable consequence of applying phase reduction beyond the first order to oscillator ensembles\textsuperscript{4}. Note that a Taylor expansion of $Q$ to $n$th order in $A$ yields up to $(n+1)$-body phase interactions, similarly to Ref.\textsuperscript{5}.

We mistakenly called $Q$ ‘non-infinitesimal phase-response curve’ in Ref.\textsuperscript{1}. Properly speaking, function $Q$ is a non-linear ‘coupling function’\textsuperscript{4}. The aim of this comment is to clarify to what extent the coupling function $Q$ determines the actual phase-response curve (PRC). The PRC quantifies the phase shift gained by an oscillator in response to an external stimulus\textsuperscript{6}. There is no analytic relation between $Q$ and the PRC beyond the small $\epsilon$ limit; in that case $Q(\theta,A) \approx Q(\theta)A$, where $Q$ turns out to be so-called infinitesimal PRC (iPRC). In consequence, we rely here on numerical simulations to compute the PRC empirically.

The family of functions $Q(\theta,A)$ considered in\textsuperscript{1} was:

\begin{equation}
Q(\theta,A) = f_1(A)(1 - \cos \theta) - f_2(A) \sin \theta. \quad (2)
\end{equation}

Four representative pairs of functions $f_{1,2}(A)$ were studied in detail in\textsuperscript{1} and the corresponding coupling functions $Q(\theta,A)$ were depicted in Fig. 2 of Ref.\textsuperscript{1}. With the aim of comparing them, we obtain the PRC for each of the four coupling functions $Q$ considered in\textsuperscript{1}.

The PRC value depends on the timing as well as on the specific shape of the stimulus, which is not necessarily weak or brief\textsuperscript{6}. Numerically, we obtain the PRC measuring the effect on one oscillator’s phase of a pulse generated by another oscillator. This means that the two oscillators are unidirectionally coupled (i.e., a master-slave configuration). We adopt $\omega = 1$ as the natural frequency for both, perturbed and perturbing oscillators, which is the obvious choice as it is the central frequency of the distribution of $\theta$\textsuperscript{1}. Moreover, we follow\textsuperscript{7} and use the same $2\pi$-periodic symmetric unimodal pulse function $P(\theta)$. It vanishes at $\theta = \pm \pi$, and a free parameter $r < 1$ controls the narrowness of $P$: The height of the pulse is $P(0) = 2/(1 - r)$, and $\lim_{\theta \to \mp \pi} P(\theta) = 2\pi \delta(\theta)$. In this comment we consider two different pulse widths: $r = 0.9$ (the value selected in\textsuperscript{1}), and $r = 0.99$ corresponding to an extremely narrow pulse.

The simulation starts at time $t = 0$ with the (slave) oscillator at an initial phase $\theta_2$. Then, we let it to evolve under the influence of the forcing oscillator. The phase of this one grows linearly, such that the input felt by the first oscillator is $A(t) = \epsilon P(t - \pi)$. Parameter $\epsilon$ determines the strength of the stimulus. The simulation runs from $t = 0$ to $t = 2\pi$, since $A$ exactly vanishes at these times. Note that we do not need to run the simulation further since phase oscillators are governed by first-order differential equations. For a given $\epsilon$ value, we measure the phase shift at $t = 2\pi$ such that PRC($\theta_2, \epsilon$) = $\theta(\theta = 2\pi) - (\theta_2 + 2\pi)$. The phase $\theta_2$ in the argument of the PRC is the phase value when $A$ attains its maximum, assuming no input exists: $\theta_2 = \theta_0 + \pi$. The results are shown in Figs. 1 and 2 for a set of $\epsilon$ values; in each panel for one particular coupling function $Q(\theta,A)$ already adopted in\textsuperscript{1}. In all panels, the corresponding iPRC is shown as a reference. Note that the normalization of the $y$-axis in Figs. 1 and 2 includes a $2\pi$ factor —in addition to $\epsilon$— because this is the integral of the pulse over an interval of length $2\pi$. Figures 1 and 2 are quite similar, though for $r = 0.9$ (Fig. 1) the PRCs remain closer to their iPRCs up to a larger $\epsilon$ value.

The comparison of the PRCs in this comment with the corresponding $Q$ functions in Fig. 2 of Ref.\textsuperscript{1} evidences that $Q(\theta,A)$ is not simply the PRC. Indeed, $Q$ in (2) has only the first harmonic in $\theta$, whereas the non-infinitesimal PRCs in the figures display additional Fourier components. In spite of these dissimilarities, simple visual inspection indicates that the PRC strongly resembles the coupling function $Q$ in all four cases. For example, we observe the same loss of non-negativity of the (type-I) iPRC as $A$ increases in panel (a),
or the transition from a synchronizing iPRC to a desynchronizing PRC for large enough $A$ in panel (c). Summarizing, our simulations confirm that the main attributes of the coupling function $Q$ are shared by the non-infinitesimal PRC.

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