MULTIPLE SIGN-CHANGING SOLUTIONS TO A CLASS OF KIRCHHOFF TYPE PROBLEMS

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Abstract. This paper is concerned with the existence of sign-changing solutions to non local Kirchhoff type problems of the form

\[- \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad (S)\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N = 1, 2, 3)\) with smooth boundary, \( a > 0, \ b > 0, \) and \( f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R} \) is a continuous function. We give a positive answer to a long standing question concerning the existence of more than two sign-changing solutions to \((S)\). More precisely, we show in this paper that if \( f \) is globally 3-superlinear, subcritical and odd with respect to the second variable, then \((S)\) possesses an unbounded sequence of sign-changing solutions.

Our approach is variational and relies on a new sign-changing version of the symmetric mountain pass theorem established in this paper.

1. Introduction

In this paper, we investigate the multiplicity of sign-changing solutions to non local Kirchhoff type problems of the form

\[- \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad (S)\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N = 1, 2, 3)\) with smooth boundary, \( a > 0, \ b \geq 0, \) and \( f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R} \) is a nonlinear function.

This problem is related to the stationary analogue of the hyperbolic equation

\[u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = f(x, u),\]

which is a general version of the equation

\[\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \quad (1.1)\]

proposed by Kirchhoff [10] as an extension of the classical D’Alembert’s wave equation for free vibrations of elastic strings. This model takes into account the changing in length of the string produced by transverse vibrations. In [11], \( L \) is the length of the string, \( h \) is the area of the cross-section, \( E \) is the Young’s modulus of the material, \( \rho \) is the mass density, and \( \rho_0 \) is the initial tension.

When \( b > 0 \), problem \((S)\) is said to be non local. In that case, the first equation in \((S)\) is no longer a pointwise equality. This causes some mathematical difficulties which make the study of such problems particularly interesting. Some early classical
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studies of Kirchhoff type problems can be found in [3, 22]. However, problem (S) received much attention only after the paper of Lions [11], where an abstract framework to attack it was introduced. Some existence and multiplicity results can be found in [20, 8, 4, 12] without any information on the sign of the solutions. Recently, Alves et al. [1], Ma and Rivera [16], and Cheng and Wu [6] obtained one positive solution. In [7], He and Zou obtained infinitely many positive solutions. The existence of sign-changing solutions to (S) was considered by Figuereido and Nascimento [9], Perera and Zhang [21], Mao and Zhang [18], and Mao and Luan [17]. But only one sign-changing solution was found in these papers.

A natural question which seems to still be open is whether (S) has many sign-changing solutions. This is expected by many people, but, to the best of our knowledge, no proof has yet been provided. In this paper, we give a positive answer to that question. Our main result relies on the following standard conditions on the nonlinear term f:

\[ p \int f(x, u) dx \leq c(1 + |u|^{p-1}), \]

where \( p > 4 \) for \( N = 1, 2 \) and \( 4 < p < 6 \) for \( N = 3 \).

\[ f(x, u) = c(|u|), \text{ uniformly in } x \in \Omega, \text{ as } u \to 0. \]

\[ \text{there exists } \mu > 4 \text{ such that } 0 < \mu F(x, u) \leq uf(x, u) \text{ for all } u \neq 0 \text{ and for } a.e x \in \Omega, \]

where \( F(x, u) = \int_0^u f(x, s)ds. \)

\[ f(x, -u) = -f(x, u) \text{ for all } (x, u) \in \Omega \times \mathbb{R}. \]

The following is the main result of the paper.

**Theorem 1.1** (Main result). Let \( a > 0 \) and \( b \geq 0 \). Assume that \( f \) satisfies the conditions (f1,2,3,4). Then (S) possesses a sequence \((u_k)\) of sign-changing solutions such that

\[ \frac{a}{2} \int_{\Omega} |\nabla u_k|^2 dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u_k|^2 dx \right)^2 - \int_{\Omega} F(x, u_k) dx \to +\infty, \text{ as } k \to \infty. \]

If \( b = 0 \), we obtain the following consequence of our main result.

**Corollary 1.2.** Under assumptions (f1,2,3,4), the semilinear problem

\[-\Delta u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]

possesses a sequence \((u_k)\) of sign-changing solutions such that

\[ \frac{1}{2} \int_{\Omega} |\nabla u_k|^2 dx - \int_{\Omega} F(x, u_k) dx \to +\infty, \text{ as } k \to \infty. \]

**Remark 1.3.** Corollary 1.2 was obtained by Qian and Li in [23] by means of the method of invariant sets of descending flow. Earlier proofs were also given in [2, 14] under the stronger assumption that \( f \) is smooth. The arguments of [23, 2, 14] rely on sign-changing critical point theorems built only for functionals of the form

\[ u \in H_0^1(\Omega) \mapsto \frac{1}{2} \|u\|^2 - \Psi(u), \]

where \( \Psi' \) is completely continuous, and cannot then be applied to (S) when \( b > 0 \).

The study of sign-changing solutions is related to several long-standing questions concerning the multiplicity of solutions for elliptic boundary value problems. Compared with positive and negative solutions, sign-changing solutions have more complicated qualitative properties and are more difficult to find. During the last
thirty years, several sophisticated techniques in calculus of variations and in critical point theory were developed to study the multiplicity of sign-changing solutions to nonlinear elliptic partial differential equations. In [2] and [14], the authors established some multiplicity sign-changing critical point theorems in partially ordered Hilbert spaces by using Morse Theory and the method of invariant sets of descending flow respectively. In [25], a parameter-depending sign-changing fountain theorem was established without any Palais-Smale type assumption. More recently, a symmetric mountain pass theorem in the presence of invariant sets of the gradient flows was established in [13]. However, it seems that all these powerful approaches are not directly applicable to find multiple sign-changing solutions to (S).

Our approach in proving Theorem 1.1 relies on a new sign-changing critical point theorem, also established in this paper, which is modelled on the fountain theorem of Bartsch (see Theorem 3.6 in [24]). An essential tool in the proof of this theorem is a deformation lemma, which allows to lower sub-level sets of a functional, away from its critical set. The main ingredient in the proof of the deformation lemma is a suitable negative pseudo-gradient flow, a notion introduced by Palais [19]. Since we are interested in sign-changing critical points, the pseudo-gradient flow must be constructed in such a way that it keeps the positive and negative cones invariant. This invariance property makes the construction of the flow very complicated when the problem contains non local terms. In this paper, we borrow some ideas from recent work by Liu, Liu and Wang [13] on the nonlinear Schrödinger systems and by Liu, Wang and Zhang [15] on the nonlinear Schrödinger-Poisson system, where the pseudo-gradient flows were constructed by using an auxiliary operator. However, the critical point theorem used in [13, 15] cannot be applied to prove Theorem 1.1 because the corresponding auxiliary operator in the case of (S) is not compact. To circumvent this difficulty, we reduce the problem to a finite-dimensional setting by means of a Galerkin approximation. The approximation is incorporated in the statement of our abstract critical point theorem, and this surprisingly makes the proof of Theorem 1.1 simpler than expected.

The rest of the paper is organized as follows. In Section 2 we state and prove the new sign-changing critical point theorem. In Section 3 we provide the proof of Theorem 1.1.

Throughout the paper, we denote by ” $\rightarrow$ ” the strong converge and by ” $\rightharpoonup$ ” the weak convergence.

2. AN ABSTRACT SIGN-CHANGING CRITICAL POINT THEOREM FOR EVEN FUNCTIONALS

In this section, we present a variant of the symmetric mountain pass type theorem which produces a sequence of sign-changing critical points with arbitrary large energy.

Let $\Phi$ be a $C^1$-functional defined on a Hilbert space $X$ of the form

\[ X := \bigoplus_{j=0}^{\infty} X_j, \quad \text{with } \dim X_j < \infty. \tag{2.1} \]

We introduce for $k \geq 2$ and $m > k + 2$ the following notations:

\[ Y_k := \bigoplus_{j=k}^{\infty} X_j, \quad Z_k := \bigoplus_{j=k}^{m-1} X_j, \quad Z_k^m := \bigoplus_{j=k}^{m-1} X_j, \quad B_k := \{ u \in Y_k : \|u\| \leq \rho_k \}, \]

\[ N_k := \{ u \in Z_k : \|u\| = r_k \}, \quad N_k^m := \{ u \in Z_k^m : \|u\| = r_k \}, \] where $0 < r_k < \rho_k$,

\[ \Phi_m := \Phi|_{Y_m}, \quad K_m := \{ u \in Y_m : \Phi_m'(u) = 0 \} \] and $E_m := Y_m \setminus K_m$. 


Let $P_m$ be a closed convex cone of $Y_m$. We set for $\mu_m > 0$
\[ \pm D_m^0 := \{ u \in Y_m \mid \text{dist}(u, \pm P_m) < \mu_m \}, \quad D_m = D_m^0 \cup (-D_m^0), \text{ and } S_m := Y_m \setminus D_m. \]
We will also denote the $\alpha$-neighborhood of $S \subset Y_m$ by
\[ V_\alpha(S) := \{ u \in Y_m \mid \text{dist}(u, S) \leq \alpha \}, \quad \forall \alpha > 0. \]

We have the following version of the symmetric mountain pass theorem.

**Theorem 2.1** (Sign-changing fountain theorem). Let $\Phi \in C^1(X, \mathbb{R})$ be an even functional which maps bounded sets to bounded sets. If, for $k \geq 2$ and $m > k + 2$, there exist $0 < r_k < \rho_k$ and $\mu_m > 0$ such that
(A1) \[ a_k := \max_{u \in B_k} \Phi(u) \leq 0 \text{ and } b_k := \inf_{u \in N_k} \Phi(u) \to +\infty, \text{ as } k \to \infty. \]
(A2) \[ N_k \subset S_m. \]
(A3) There exists an odd locally Lipschitz continuous vector field $B : E_m \to Y_m$ such that:
(i) \[ B((\pm D_m^0) \cap E_m) \subset \pm D_m^0; \]
(ii) there exists a constant $\alpha_1 > 0$ such that \[ \langle \Phi_m'(u), u - B(u) \rangle \geq \alpha_1 \| u - B(u) \|^2, \text{ for any } u \in E_m; \]
(iii) for $a < b$ and $\alpha > 0$, there exists $\beta > 0$ such that $\| u - B(u) \| \geq \beta$ if $u \in Y_m$ is such that $\Phi_m(u) \in [a, b]$ and $|\Phi_m'(u)| \geq \alpha$.
(A4) $\Phi$ satisfies the $(PS)_\alpha$ condition, that is:
- any Palais-Smith sequence of $\Phi_m$ is bounded;
- any sequence $(u_{m_j}) \subset X$ such that $m_j \to \infty, \quad u_{m_j} \in V_{\mu_{m_j}}(S_{m_j}), \quad \sup \Phi(u_{m_j}) < \infty, \quad \Phi_m'(u_{m_j}) = 0$
  has a subsequence converging to a sign-changing critical point of $\Phi$.

Then $\Phi$ has a sequence $(u_k)_k$ of sign-changing critical points in $X$ such that $\Phi(u_k) \to \infty$, as $k \to \infty$.

Condition (A4) in Theorem 2.1 is a version of the usual compactness condition in Critical Point Theory, namely the Palais-Smith condition. We recall that a sequence $(u_n) \subset E$ is a Palais-Smith sequence of a smooth functional $J$ defined on a Banach space $E$ if the sequence $(J(u_n))$ is bounded and $J'(u_n) \to 0$, as $n \to \infty$. If every such sequence possesses a convergent subsequence, then $J$ is said to satisfy the Palais-Smith condition.

We need a special deformation lemma in order to prove the above result. We first recall the following helpful lemma.

**Lemma 2.2** ([25], Lemma 2.2). Let $M$ be a closed convex subset of a Banach space $E$. If $H : M \to E$ is a locally Lipschitz continuous map such that
\[ \lim_{\beta \to 0^+} \frac{\text{dist}(u + \beta H(u), M)}{\beta} = 0, \quad \forall u \in M, \]
then for any $u_0 \in M$, there exists $\delta > 0$ such that the initial value problem
\[ \frac{d\sigma(t, u_0)}{dt} = H(\sigma(t, u_0)), \quad \sigma(0) = u_0, \]
has a unique solution defined on $[0, \delta)$. Moreover, $\sigma(t, u_0) \in M$ for all $t \in [0, \delta)$.

Now we state a quantitative deformation lemma.
Lemma 2.3 (Deformation lemma). Let $\Phi \in C^1(X, \mathbb{R})$ be an even functional which maps bounded sets to bounded sets. Fix $m$ sufficiently large and assume that the condition $(A_3)$ of Theorem 2.3 holds. Let $c \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that

$$
\forall u \in \Phi_m^{-1}([c - 2\varepsilon_0, c + 2\varepsilon_0]) \cap V_m^{-\infty}(S_m) : \|\Phi_m(u)\| > \varepsilon_0.
$$

(2.2)

Then for some $\varepsilon \in ]0, \varepsilon_0[\there exists \eta \in C([0, 1] \times Y_m, Y_m)$ such that:

(i) $\eta(t, u) = u$ for $t = 0$ or $u \notin \Phi_m^{-1}([c - 2\varepsilon, c + 2\varepsilon])$;

(ii) $\eta([1, \Phi_m^{-1}([c - \infty, c + \varepsilon]) \cap S_m) \subset \Phi_m^{-1}([c - \infty, c - \varepsilon])$;

(iii) $\Phi_m(\eta(\cdot, u))$ is not increasing, for any $u$;

(iv) $\eta([0, 1] \times D_m) \subset D_m$;

(v) $\eta(t, \cdot)$ is odd, for any $t \in [0, 1]$.

Proof. Define $V : E_m \to Y_m$ by $V(u) = u - B(u)$, where $B$ is given by $(A_3)$. Then there is $\delta > 0$ such that $V(u) \geq \delta$ for any $u \notin \Phi_m^{-1}([c - 2\varepsilon_0, c + 2\varepsilon_0]) \cap V_m^{-\infty}(S_m)$ (in view $(A_3)$-(iii)). We take $\varepsilon \in ]0, \min(\varepsilon_0, \frac{\delta}{2\alpha_1\mu_m})[\text{ and we define }$}

$$
A_1 := \Phi_m^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap V_m^{-\infty}(S_m), \quad A_2 := \Phi_m^{-1}([c - \varepsilon, c + \varepsilon]) \cap V_m^{-\infty}(S_m),
$$

$$
\chi(u) := \frac{\text{dist}(u, Y_m \setminus A_1)}{\text{dist}(u, Y_m \setminus A_1) + \text{dist}(u, A_2)}, \quad u \in Y_m
$$

so that $\chi = 0$ on $Y_m \setminus A_1$, $\chi = 1$ on $A_2$, and $0 \leq \chi \leq 1$.

Consider the vector field

$$
W(u) := \left\{ \begin{array}{ll}
\chi(u)\|V(u)\|^{-2}V(u), & \text{for } u \in A_1 \\
0, & \text{for } u \in Y_m \setminus A_1.
\end{array} \right.
$$

Clearly $W$ is odd and locally Lipschitz continuous. Moreover, by our choice of $\varepsilon$ above we have

$$
\|W(u)\| \leq \frac{1}{\delta} \leq \frac{\alpha_1\mu_m}{8\varepsilon}, \quad \forall u \in Y_m.
$$

(2.3)

It follows that the Cauchy problem

$$
\frac{d}{dt}\sigma(t, u) = -W(\sigma(t, u)), \quad \sigma(0, u) = u \in Y_m
$$

has a unique solution $\sigma(\cdot, u)$ defined on $\mathbb{R}_+$. Moreover, $\sigma$ is continuous on $\mathbb{R}_+ \times X$.

We have in view of (2.3)

$$
\|\sigma(t, u) - u\| \leq \int_0^t \|W(\sigma(s, u))\|ds \leq \frac{\alpha_1\mu_m}{8\varepsilon},
$$

(2.4)

and by $(A_3)$-(ii)

$$
\frac{d}{dt}\Phi_m(\sigma(t, u)) = -\langle \Phi_m'(\sigma(t, u)), \chi(\sigma(t, u))\|V(\sigma(t, u))\|^{-2}V(\sigma(t, u)) \rangle
\leq -\alpha_1 \chi(\sigma(t, u)).
$$

(2.5)

Define

$$
\eta : [0, 1] \times Y_m \to Y_m, \quad \eta(t, u) := \sigma\left(\frac{2\varepsilon}{\alpha_1}, t, u\right).
$$

Conclusion (i) of the lemma is clearly satisfied and by (2.3) above (iii) is also satisfied. Since $W$ is odd, (v) is a consequence of the uniqueness of the solution to the above Cauchy problem.

We now verify (ii). Let $v \in \eta(1, \Phi_m^{-1}([c - \infty, c + \varepsilon]) \cap S_m)$. Then $v = \eta(1, u) = \sigma\left(\frac{2\varepsilon}{\alpha_1}, u\right)$, where $u \in \Phi_m^{-1}([c - \infty, c + \varepsilon]) \cap S_m$. 


If there exists \( t \in [0, \frac{2\pi}{\alpha_1}] \) such that \( \Phi_m(\sigma(t, u)) < c - \varepsilon \), then by (iii) we have \( \Phi_m(v) < c - \varepsilon \).

Assume now that \( \sigma(t, u) \in \Phi_m^{-1}([c - \varepsilon, c + \varepsilon]) \) for all \( t \in [0, \frac{2\pi}{\alpha_1}] \). By (2.4) we have \( \|\sigma(t, u) - u\| \leq \frac{\mu_m}{4} \), which means, since \( u \in S_m \), that \( \sigma(t, u) \in V_m(S_m) \). Hence \( \sigma(t, u) \in A_2 \) and since \( \chi = 1 \) on \( A_2 \), we deduce from (2.3) that

\[
\Phi_m(\sigma(\frac{2\pi}{\alpha_1}, u)) \leq \Phi_m(u) - \alpha_1 \int_0^{\frac{2\pi}{\alpha_1}} \chi(\sigma(t, u))dt = \Phi_m(u) - 2\varepsilon.
\]

This implies, since \( \Phi_m(u) \leq c + \varepsilon \), that \( \Phi_m(v) = \Phi_m(\sigma(\frac{2\pi}{\alpha_1}, u)) \leq c - \varepsilon \). Hence (ii) is satisfied.

It remains to verify (iv). Since \( \sigma \) is odd in \( u \), it suffices to show that

\[
\sigma([0, +\infty) \times D_m^0) \subset D_m^0.
\]

**Claim:** We have

\[
\sigma([0, +\infty) \times \overline{D_m^0}) \subset \overline{D_m^0}.
\]

Assume by contradiction that (2.10) does not hold. Then there exist \( u_0 \in D_m^0 \) and \( t_0 > 0 \) such that \( \sigma(t_0, u_0) \notin \overline{D_m^0} \). Choose a neighborhood \( N_{u_0} \) of \( u_0 \) such that \( N_{u_0} \subset D_m^0 \). Then there is a neighborhood \( N_0 \) of \( \sigma(t_0, u_0) \) such that \( \sigma(t_0, w) \) \( w \in N_{u_0} \) is a homeomorphism. Since \( \sigma(t_0, u_0) \notin \overline{D_m^0} \), the set \( N_0 \setminus \overline{D_m^0} \) is not empty. Hence there is \( w \in N_{u_0} \) such that \( \sigma(t_0, w) \in N_0 \setminus \overline{D_m^0} \), contradicting (2.7).

We now terminate by giving the proof of our above claim.

By (A3)-(i) we have \( B(D_m^0 \cap E_m) \subset D_m^0 \), which implies that \( B(D_m^0 \cap E_m) \subset \overline{D_m^0} \). Obviously \( \sigma(t, u) = u \) for all \( t \in [0, 1] \) and \( u \in \overline{D_m^0} \cap K_m \).

Assume that \( u \in \overline{D_m^0} \cap E_m \). If there is \( t_1 \in (0, 1) \) such that \( \sigma(t_1, u) \notin \overline{D_m^0} \), then there would be \( s_1 \in [0, t_1) \) such that \( \sigma(s_1, u) \in \partial \overline{D_m^0} \) and \( \sigma(t, u) \notin \overline{D_m^0} \) for all \( t \in (0, t_1] \).

The following Cauchy problem

\[
\frac{d}{dt}u(t, \sigma(s_1, u)) = -W(\mu(t, \sigma(s_1, u))), \quad \mu(0, \sigma(s_1, u)) = \sigma(s_1, u) \in Y_m
\]

has \( \sigma(t, \sigma(s_1, u)) \) as unique solution. Recalling that \( W = 0 \) on \( Y_m \setminus A_1 \), we have \( v - W(v) \in \overline{D_m^0} \cap (Y_m \setminus A_1) \) for any \( v \in \overline{D_m^0} \cap (Y_m \setminus A_1) \).

Assume that \( v \in A_1 \cap \overline{D_m^0} \). Since \( V(u) \geq \delta \), we deduce that \( 1 - \beta \chi(v) \|V(v)\|^{-2} \geq 0 \) for all \( \beta \) such that \( 0 < \beta \leq \delta^2 \). Recalling that \( v \in \overline{D_m^0} \) implies \( dist(v, P_m) \leq \mu_m \), that \( V(v) = v - B(v) \), and that \( aP_m + bP_m \subset P_m \) for all \( a, b \geq 0 \) (because \( P_m \) is a cone), we obtain for any \( \beta \in [0, \delta^2] \)

\[
\text{dist}(v - \beta W(v), P_m) = \text{dist}(v - \beta \chi(v)\|V(v)\|^{-2}V(v), P_m)
\]

\[
= \text{dist}(v - \beta \chi\|V(v)\|^{-2}(v - B(v)), P_m)
\]

\[
= \text{dist}\left( (1 - \beta \chi(v)\|V(v)\|^{-2})v + \beta \chi(v)\|V(v)\|^{-2}B(v), P_m \right)
\]

\[
\leq \text{dist}\left( (1 - \beta \chi(v)\|V(v)\|^{-2})v + \beta \chi(v)\|V(v)\|^{-2}B(v),
\beta \chi(v)\|V(v)\|^{-2}P_m + (1 - \beta \chi(v)\|V(v)\|^{-2})P_m \right)
\]

\[
\leq (1 - \beta \chi(v)\|V(v)\|^{-2}) \text{dist}(v, P_m) + \beta \chi(v)\|V(v)\|^{-2} \text{dist}(B(v), P_m)
\]

\[
\leq (1 - \beta \chi(v)\|V(v)\|^{-2})\mu_m + \beta \chi(v)\|V(v)\|^{-2}\mu_m
\]

\[
= \mu_m.
\]
It follows that \( v - \beta W(v) \in D_m \) for \( 0 < \beta \leq \delta^2 \). This implies that
\[
\lim_{\beta \to 0^+} \frac{\text{dist}(v + \beta(-W(v)), D_m)}{\beta} = 0, \quad \forall u \in D_m.
\]

By Lemma 2.2 there then exists \( \delta_0 > 0 \) such that \( \sigma(t, \sigma(s, u)) \in D_m \) for all \( t \in [0, \delta_0] \). This implies that \( \sigma(t, \sigma(s, u)) = \sigma(t + s, u) \in D_m \) for all \( t \in [0, \delta_0] \), which contradicts the definition of \( s_1 \). This last contradiction assures that \( \sigma([0, +\infty) \times D_m) \subset D_m \).

Proof of Theorem 2.1. (A1) and (A2) imply that \( a_k < b_k \leq \inf_{u \in N_m} \Phi_m(u) \), for \( k \) big enough. Let
\[
\Gamma^m_k := \{ \gamma \in C(B_k, Y_m) \mid \gamma \text{ is odd}, \gamma|_{\partial B_k} = \text{id} \text{ and } \gamma(D_m) \subset D_m \}.
\]
\( \Gamma^m_k \) is clearly non empty and for any \( \gamma \in \Gamma^m_k \) the set \( U := \{ u \in B_k : \| \gamma(u) \| < r_k \} \) is an open bounded and symmetric (i.e. \( -U = U \)) neighborhood of the origin in \( Y_k \).

By the Borsuk-Ulam theorem the continuous odd map \( \Pi_k \circ \gamma : \partial U \to Y_k \to Y_{k-1} \) has a zero, where \( \Pi_k : X \to Y_{k-1} \) is the orthogonal projection. It then follows that \( \gamma(B_k) \cap N^m_k \neq \emptyset \) and, since \( N^m_k \subset S_m \), that \( \gamma(B_k) \cap S_m \neq \emptyset \). This intersection property implies that
\[
k_{m,k} := \inf_{\gamma \in \Gamma^m_k} \max_{u \in \gamma(B_k) \cap S_m} \Phi_m(u) \geq \inf_{u \in N^m_k} \Phi(u) \geq b_k.
\]

We would like to show that for any \( \varepsilon > 0 \), \( \frac{c_{m,k} - a_k}{2} \), there exists \( u \in \Phi_m^{-1}\left( [c_{k,m} - 2\varepsilon_0, c_{k,m} + 2\varepsilon_0] \right) \cap V_{\Phi_m}(S_m) \) such that \( \| \Phi'(u) \| < \varepsilon_0 \).

Arguing by contradiction, we assume that we can find \( \varepsilon_0 \in (0, \frac{c_{m,k} - a_k}{2} ] \) such that
\[
\| \Phi'(u) \| \geq \varepsilon_0, \quad \forall u \in \Phi_m^{-1}\left( [c_{k,m} - 2\varepsilon_0, c_{k,m} + 2\varepsilon_0] \right) \cap V_{\Phi_m}(S_m).
\]

Apply Lemma 2.3 with \( c = c_{k,m} \) and define, using the deformation \( \eta \) obtained, the map
\[
\theta : B_k \to Y_m, \quad \theta(u) := \eta(1, \gamma(u)),
\]
where \( \gamma \in \Gamma^m_k \) satisfies
\[
\max_{u \in \gamma(B_k) \cap S_m} \Phi_m(u) \leq c_{k,m} + \varepsilon,
\]
with \( \varepsilon \) also given by Lemma 2.3.

Using the properties of \( \eta \) (see Lemma 2.3), one can easily verify that \( \theta \in \Gamma^m_k \).

On the other hand, we have
\[
\eta(1, \gamma(B_k)) \cap S_m \subset \eta(1, \Phi_m^{-1}(\left[ -\infty, c_{k,m} + \varepsilon \right]) \cap S_m).
\]

In fact, if \( u \in \eta(1, \gamma(B_k)) \cap S_m \) then \( u = \eta(1, \gamma(v)) \in S_m \) for some \( v \in B_k \). Observe that \( \gamma(v) \in S_m \). Indeed, if this is not true then \( \gamma(v) \in D_m \), and by (iv) of Lemma 2.3 we obtain \( u = \eta(1, \gamma(v)) \in D_m \) which contradicts the fact that \( u \in S_m \). Now \( c_{k,m} \) implies that \( \gamma(v) \in \Phi_m^{-1}(\left[ -\infty, c_{k,m} + \varepsilon \right]) \). It then follows, using (ii) of Lemma 2.3 that \( u = \eta(1, \gamma(v)) \in \eta(1, \left[ -\infty, c_{k,m} + \varepsilon \right]) \cap S_m \). Hence \( \gamma \) holds.
is simple with a positive eigenfunction $e_{\lambda}$. Then each $q$ and norm $\|\cdot\|$ follows from the fact that $b$.

We deduce from $p$ to $\lambda$.

It follows that there is a sequence $\Phi$ changing critical point $u$.

Noting that $\epsilon$, the above contradiction assures that for any $c$.

Throughout this section, we assume that $\Phi$.

Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the distinct eigenvalues of the problem $-\Delta u = \lambda u$ in $\Omega$, $u = 0$ on $\partial \Omega$.

Then each $\lambda_j$ has finite multiplicity. It is well known that the principal eigenvalue $\lambda_1$ is simple with a positive eigenfunction $e_1$, and the eigenfunctions $e_j$ corresponding to $\lambda_j$ ($j \geq 2$) are sign-changing. Let $X_j$ be the eigenspace associated to $\lambda_j$. We set for $k \geq 2$

$$Y_k := \bigoplus_{j=1}^{k} X_j \quad \text{and} \quad Z_k := \bigoplus_{j=k}^{\infty} X_j.$$

**Lemma 3.1.**

(1) For any $u \in Y_k$ we have $\Phi(u) \to -\infty$, as $\|u\| \to \infty$. 

Using (2.9) and (ii) of Lemma 2.3, we obtain

$$\max_{u \in \partial(B_k) \cap S} \Phi_m(u) = \max_{u \in \Omega^{(1)}(B_k) \cap S_m} \Phi_m(u) \leq \max_{u \in \Omega^{(1)}(\{\lambda c_{k,m} + \epsilon\}) \cap S_m} \Phi_m(u) \leq c_{k,m} - \epsilon,$$

contradicting the definition of $c_{k,m}$.

The above contradiction assures that for any $\epsilon_0 \in [0, \frac{c_{k,m} - \epsilon}{2}]$, there exists $u \in \Phi^{-1}(\{c_{k,m} - 2\epsilon_0, c_{k,m} + 2\epsilon_0\}) \cap V_{\Phi_m}(S_m)$ such that $\|\Phi'(u)\| < \epsilon_0$.

It follows that there is a sequence $(u^n_{k,m}) \subset V_{\Phi_m}(S_m)$ such that $\Phi'_m(u^n_{k,m}) \to 0$ and $\Phi_m(u^n_{k,m}) \to c_{k,m}$, as $n \to \infty$.

We deduce from $(A_4)$-(i) that there exists $u_{k,m} \in V_{\Phi_m}(S_m)$ such that $\Phi'_m(u_{k,m}) = 0$ and $\Phi_m(u_{k,m}) = c_{k,m}$.

Noting that $c_{k,m} \leq \max_{u \in B_k} \Phi(u)$, we deduce using $(A_4)$-(ii) that $\Phi$ has a sign-changing critical point $u_k$ such that $b_k \leq \Phi(u_k) \leq \max_{u \in B_k} \Phi(u)$. The conclusion then follows from the fact that $b_k \to \infty$, as $k \to \infty$. 

3. PROOF OF THE MAIN RESULT

Throughout this section, we assume that $(f_{1,3})$ are satisfied. We denote by $|\cdot|_q$, the usual norm of the Lebesgue space $L^q(\Omega)$.

Let $X := H^1_0(\Omega)$ be the usual Sobolev space endowed with the inner product

$$\langle u, v \rangle = \int_{\Omega} \nabla u \nabla v dx$$

and norm $\|u\|^2 = \langle u, u \rangle$, for $u, v \in H^1_0(\Omega)$.

It is well known that solutions of (8) are critical points of the functional

$$\Phi(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \int_{\Omega} F(x, u) dx, \quad u \in X := H^1_0(\Omega). \quad (3.1)$$

By a standard argument, one can easily verify that $\Phi$ is of class $C^1$ and

$$\langle \Phi'(u), v \rangle = (a + b \|u\|^2) \int_{\Omega} \nabla u \nabla v dx - \int_{\Omega} v F(x, u) dx \quad (3.2)$$

Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the distinct eigenvalues of the problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega.$$

Then each $\lambda_j$ has finite multiplicity. It is well known that the principal eigenvalue $\lambda_1$ is simple with a positive eigenfunction $e_1$, and the eigenfunctions $e_j$ corresponding to $\lambda_j$ ($j \geq 2$) are sign-changing. Let $X_j$ be the eigenspace associated to $\lambda_j$. We set for $k \geq 2$

$$Y_k := \bigoplus_{j=1}^{k} X_j \quad \text{and} \quad Z_k := \bigoplus_{j=k}^{\infty} X_j.$$
(2) There exists \( r_k > 0 \) such that
\[
\inf_{u \in Z_k} \Phi(u) \to \infty, \text{ as } k \to \infty.
\]

Proof. (1) It is well known that integrating \((f_3)\) yields the existence of two constants \( c_1, c_2 > 0 \) such that \( F(x, u) \geq c_1 |u|^\mu - c_2 \). This together with the fact that all norms are equivalent in the finite-dimensional subspace \( Y_k \) imply that
\[
\Phi(u) \leq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - c_3 \|u\|^\mu + c_4, \quad \forall u \in Y_k,
\]
where \( c_3, c_4 > 0 \) are constant. Since \( \mu > 4 \), it follows that \( \Phi(u) \to -\infty \), as \( \|u\| \to \infty \).

(2) Using \((f_1)\), we obtain
\[
\Phi(u) \geq \frac{a}{2} \|u\|^2 - c_5 \|u\|^p - c_6, \quad \forall u \in X,
\]
where \( c_5, c_6 > 0 \) are constant. Set
\[
\beta_k := \sup_{\|v\|=1} |v|_p.
\]
Then we obtain
\[
\Phi(u) \geq a \left( \frac{1}{2} - \frac{1}{p} \right) \left( \frac{c_5}{a} \beta_k^p \right)^{\frac{1}{p-2}} - c_6
\]
for every \( u \in Z_k \) such that
\[
\|u\| = r_k := \left( \frac{c_5}{a} \beta_k^p \right)^{\frac{1}{p-2}}.
\]
We know from Lemma 3.8 in [24] that \( \beta_k \to 0 \), as \( k \to \infty \). This implies that \( r_k \to \infty \), as \( k \to \infty \).

Now we fix \( k \) large enough and we set for \( m > k + 2 \)
\[
\Phi_m := \Phi|_{X_m}, \quad K_m := \{ u \in X_m : \Phi_m(u) = 0 \}, \quad E_m := X_m \setminus K_m.
\]
\[
P_m := \{ u \in X_m : u(x) \geq 0 \}, \quad Z_m := \oplus_{j=k}^m X_j, \quad \text{and } N_m := \{ u \in Z_m : \|u\| = r_k \}.
\]
Remark that for all \( u \in P_m \setminus \{0\} \) we have \( \int_X u e_1 dx > 0 \), while for all \( u \in Z_k \), \( \int_X u e_1 dx = 0 \), where \( e_1 \) is the principal eigenfunction of the Laplacian. This implies that \( P_m \cap Z_k = \{0\} \). It then follows, since \( N_m \) is compact, that
\[
\delta_m := dist(N_m \cap L, P_m) > 0. \quad (3.3)
\]
For \( u \in Y_m \) fixed, we consider the functional
\[
I_u(v) = \frac{1}{2} \left( a + b |u|^2 \right) \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} vf(x, u) dx, \quad u \in Y_m.
\]
It is not difficult to see that \( I_u \) is of class \( C^1 \), coercive, bounded below, weakly lower semicontinuous, and strictly convex. Therefore \( I_u \) admits a unique minimizer \( v = Au \in Y_m \), which is the unique solution to the problem
\[
(a + b |u|^2) \Delta v = f(x, u), \quad v \in Y_m.
\]
Clearly, the set of fixed points of \( A \) coincide with \( K_m \). Moreover, the operator \( A : Y_m \to Y_m \) has the following important properties.

Lemma 3.2.

(1) \( A \) is continuous and maps bounded sets to bounded sets.
(2) For any \( u \in Y_m \) we have
\[
\langle \Phi'_m(u), u - Au \rangle \geq a\|u - Au\|^2, \tag{3.5}
\]
\[
\|\Phi'_m(u)\| \leq (a + b)(1 + \|u\|^2)\|u - Au\|. \tag{3.6}
\]

(3) There exists \( \mu_m \in ]0, \delta_m[ \) such that \( \Lambda(\pm D^0_m) \subset \pm D^0_m \), where \( \delta_m \) is defined by (3.3).

Proof. (1) Let \( (u_n) \subset Y_m \) such that \( u_n \to u \). We set \( v_n = Au_n \) and \( v = Au \).

By the definition of \( A \) we have for any \( w \in Y_m \)
\[
(a + b\|u_n\|^2)\int_{\Omega} \nabla v_n \nabla w dx = \int_{\Omega} w f(x, u_n) dx \tag{3.7}
\]
\[
(a + b\|u\|^2)\int_{\Omega} \nabla v \nabla w dx = \int_{\Omega} w f(x, u) dx. \tag{3.8}
\]

Taking \( w = v_n - v \) in (3.7) and in (3.8), and using the Hölder inequality and the Sobolev embedding theorem, we obtain
\[
(a + b\|u_n\|^2)\|v_n - v\|^2 = b(\|u_n\|^2 - \|u\|^2)\int_{\Omega} \nabla v \nabla (v_n - v) dx
\]
\[
+ \int_{\Omega} (v_n - v)(f(u_n) - f(u)) dx \
\leq c_1\|u_n\|^2 - \|u\|^2\|v_n - v\|^2 + c_2\|v_n - v\|\|f(u_n) - f(u)\|_{L^q(\Omega)},
\]
where \( c_1, c_2 > 0 \) are constant. By (f1) and Theorem A.2 in [24], we have \( f(u_n) - f(u) \to 0 \) in \( L^{\frac{2}{p-1}}(\Omega) \). Hence \( \|Au_n - Au\| = \|v_n - v\| \to 0 \), that is, \( A \) is continuous.

On the other hand, for any \( u \in Y_m \) we have, taking \( v = w = Au \) in (3.8)
\[
(a + b\|u_n\|^2)\|Au\|^2 = \int_{\Omega} Au f(x, u) dx.
\]
By using (f1), the Hölder inequality, and the Sobolev embedding theorem, we obtain
\[
a\|Au\| \leq C(1 + \|u\|^p),
\]
where \( C > 0 \) a constant. This shows that \( Au \) is bounded whenever \( u \) is bounded.

(2) Taking \( w = u - Au \) in (3.8), we obtain
\[
(a + b\|u\|^2)\int_{\Omega} \nabla (Au) \nabla (u - Au) dx = \int_{\Omega} (u - Au) f(x, u) dx,
\]
which implies that
\[
\langle \Phi'_m(u), u - Au \rangle = (a + b\|u\|^2)\|u - Au\|^2 \geq a\|u - Au\|^2.
\]

On the other hand, we obtain using (3.8)
\[
\langle \Phi'_m(u), w \rangle = (a + b\|u\|^2)\int_{\Omega} \nabla u \nabla w dx - \int_{\Omega} w f(x, u) dx
\]
\[
= (a + b\|u\|^2)\int_{\Omega} \nabla (u - Au) \nabla w dx, \quad \forall w \in Y_m.
\]
This implies that
\[
\|\Phi'_m(u)\| \leq (a + b\|u\|^2)\|u - Au\|.
\]

(3) It follows from (f1) and (f2) that
\[
\forall \varepsilon > 0, \quad \exists c_{\varepsilon} > 0 : |f(x, t)| \leq \varepsilon |t| + c_{\varepsilon}|t|^{p-1}, \quad \forall t \in \mathbb{R}. \tag{3.9}
\]
Let \( u \in Y_m \) and let \( v = Au \). As usual we denote \( w^\pm = \max\{0, \pm w\} \), for any \( w \in X \).

Taking \( w = v^+ \) in (3.8) and using the Hölder inequality, we obtain
\[
(a + b\|u\|^2)\|v\|^2 = \int_\Omega v^+ f(x, u)dx \leq \varepsilon |u^+|_2|v^+|_2 + c_\varepsilon |u^+|_p^{p-1}|v^+|_p,
\]
which implies that
\[
\|v^+\|^2 \leq \frac{1}{a} \left( \varepsilon |u^+|_2|v^+|_2 + c_\varepsilon |u^+|_p^{p-1}|v^+|_p \right).
\]

On the other hand it is not difficult to see that \( |u^+|_q \leq |u - w|_q \) for all \( w \in -P_m \) and \( 1 \leq q \leq 2^* \). Hence there is a constant \( c_1 = c_1(q) > 0 \) such that \( |u^+|_q \leq c_1 \text{dist}(u, -P_m) \). It is obvious that \( \text{dist}(v, -P_m) \leq \|v^+\| \). So we deduce from (3.10) and the Sobolev embedding theorem that
\[
\text{dist}(v, -P_m)\|v^+\| \leq \|v^+\|^2 \\
\leq c_2 \left( \varepsilon \text{dist}(u, -P_m) + c_\varepsilon \text{dist}(u, -P_m)^{p-1} \right)\|v^+\|,
\]
where \( c_2 > 0 \) is constant. This implies that
\[
\text{dist}(v, -P_m) \leq c_2 \left( \varepsilon \text{dist}(u, -P_m) + c_\varepsilon \text{dist}(u, -P_m)^{p-1} \right).
\]

Similarly on can show that
\[
\text{dist}(v, P_m) \leq c_3 \left( \varepsilon \text{dist}(u, P_m) + c_\varepsilon \text{dist}(u, P_m)^{p-1} \right),
\]
for some constant \( c_3 > 0 \).

Choosing \( \varepsilon \) small enough, we can then find \( \mu_m \in ]0, \delta_m[ \) such that
\[
\text{dist}(v, \pm P_m) \leq \frac{1}{2}\text{dist}(u, \pm P_m)
\]
whenever \( \text{dist}(u, \pm P_m) < \mu_m \). \( \square \)

Using \( \mu_m \) obtained above, we define
\[
\pm D_m^0 := \{ u \in Y_m \mid \text{dist}(u, \pm P_m) < \mu_m \}, \; D_m = D_m^0 \cup (-D_m^0) \\
S_m := Y_m \setminus D_m.
\]

**Remark 3.3.** \( \mu_m < \delta_m \implies N_k^m \subset S_m. \)

The vector field \( A : Y_m \to Y_m \) does not satisfy the assumption \( (A_3) \) of Theorem 2.1 as it is not locally Lipschitz continuous. However, it will be used in the spirit of [3] to construct a vector field which will satisfy the above mentioned condition.

**Lemma 3.4.** There exists an odd locally Lipschitz continuous operator \( B : E_m \to Y_m \) such that

1. \( \langle \Phi'(u), u - B(u) \rangle \geq \frac{1}{2}\|u - A(u)\|^2 \), for any \( u \in E_m \).
2. \( \frac{1}{2}\|u - B(u)\| \leq \|u - A(u)\| \leq 2\|u - B(u)\| \), for any \( u \in E_m \).
3. \( B((\pm D_m^0) \cap E_m) \subset \pm D_m^0 \).

The proof of this lemma follows the lines of [3]. We provide a sketch of the proof here for completeness.
Proof. We define \( \Delta_1, \Delta_2 : E_m \to \mathbb{R} \) as
\[
\Delta_1(u) = \frac{1}{2} \|u - Au\| \quad \text{and} \quad \Delta_2 = \frac{a}{2(a + b)}(1 + \|u\|^2)^{-1}\|u - Au\|. \tag{3.11}
\]
For any \( u \in E_m \) we choose \( \gamma(u) > 0 \) such that
\[
\|A(v) - A(w)\| < \min \{ \Delta_1(v), \Delta_1(w), \Delta_2(v), \Delta_2(w) \} \tag{3.12}
\]
holds for every \( v, w \in N(u) := \{ z \in Y_m ; \|z - u\| < \gamma(u) \} \).
Let \( \mathcal{V} \) be a locally finite open refinement of \( \{ N(u) ; u \in E_m \} \) and define
\[
\mathcal{V}^* := \{ v \in \mathcal{V} ; D_m^0 \cap V \neq \emptyset, -D_m^0 \cap V \neq \emptyset, -D_m^0 \cap D_m^0 \cap V \neq \emptyset \},
\]
\[
\mathcal{U} := \bigcup_{v \in \mathcal{V}^*} \{ v \} \cup \bigcup_{v \in \mathcal{V}^*} \{ v \setminus D_m^0, V \setminus (-D_m^0) \}.
\]
By construction \( \mathcal{U} \) is a locally finite open refinement of \( \{ N(u) ; u \in E_m \} \) and has a property that any \( U \in \mathcal{U} \) is such that
\[
U \cap D_m^0 \neq \emptyset \quad \text{and} \quad U \cap (-D_m^0) \neq \emptyset \implies U \cap D_m^0 \cap (-D_m^0) \neq \emptyset. \tag{3.13}
\]
Let \( \Pi_U : U \in \mathcal{U} \) be the partition of unity subordinated to \( \mathcal{U} \) defined by
\[
\Pi_U(u) := \frac{\alpha_U(u)}{\sum_{v \in \mathcal{U}} \alpha_U(v)}, \quad \text{where} \quad \alpha_U(u) = \text{dist}(u, E_m \setminus U).
\]
For any \( u \in \mathcal{U} \) choose \( \alpha_U \) such that if \( U \cap (\pm D_m^0) \neq \emptyset \) then \( \alpha_U \in U \cap (\pm D_m^0) \) (such an element exists in view of (3.13)). Define \( B : E_m \to Y_m \) by
\[
B(u) := \frac{1}{2}(H(u) - H(-u)), \quad \text{where} \quad H(u) = \sum_{U \in \mathcal{U}} \Pi_U(u)A(a_U).
\]
We then conclude as in [3] by using Lemma 3.2, (3.11), (3.12), and (3.13). \( \Box \)

Remark 3.5. Lemmas 3.2 and 3.4 imply that
\[
\langle \Phi_m'(u), u - B(u) \rangle \geq \frac{1}{8} \|u - B(u)\|^2 \quad \text{and} \quad \|\Phi_m'(u)\| \leq 2(a + b)(1 + \|u\|^2)^{-1}\|u - B(u)\|, \quad \text{for all} \quad u \in E_m.
\]

Lemma 3.6. Let \( c < d \) and \( \alpha > 0 \). For all \( u \in Y_m \) such that \( \Phi_m(u) \in [c, d] \) and \( \|\Phi_m'(u)\| \geq \alpha \), there exists \( \beta > 0 \) such that \( \|u - B(u)\| \geq \beta \).

Proof. By the definition of the operator \( A \), we have for any \( u \in Y_m \)
\[
(a + b\|u\|^2) \int_\Omega \nabla(Au)\nabla u dx = \int_\Omega uf(x, u) dx.
\]
It follows that
\[
\Phi_m(u) - \frac{1}{\mu}(a + b\|u\|^2) \int_\Omega \nabla u \nabla (u - Au) dx = a\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|^2
\]
\[
+ b\left(\frac{1}{4} - \frac{1}{\mu}\right)\|u\|^4 + \int_\Omega \left(\frac{1}{\mu}f(x, u) - F(x, u)\right) dx
\]

which implies, using \( (f_3) \) and Lemma 3.4 (2), that
\[
b(\frac{1}{4} - \frac{1}{\mu})\|u\|^4 \leq |\Phi_m(u)| + \frac{1}{\mu} (a + b|u|^2)\|u\|\|u - Au\|
\]
\[
\leq |\Phi_m(u)| + \frac{2}{\mu} (a + b|u|^2)\|u\|\|u - Bu\|. \tag{3.14}
\]

Suppose that there exists a sequence \((u_n) \subseteq Y_m\) such that \(\Phi_m(u_n) \in [c, d]\), \(\Phi'_m(u_n)\| \geq \alpha\) and \(\|u_n - Bu_n\| \to 0\). By (3.14) we see that \((\|u_n\|)\) is bounded. It follows from Remark 3.5 above that \(\Phi'_m(u_n) \to 0\), which is a contradiction. \(\square\)

Now we verify the compactness condition for \(\Phi\).

**Lemma 3.7.** \(\Phi\) satisfies the \((PS)_{nod}\) condition, that is:
- any Palais-Smale sequence of \(\Phi_m\) is bounded and
- any sequence \((u_{m_j}) \subseteq X\) such that
  \[m_j \to \infty, \quad u_{m_j} \in V_{\sup} (S_{m_j}), \quad \sup \Phi(u_{m_j}) < \infty, \quad \Phi'_m(u_{m_j}) = 0\]
has a subsequence converging to a sign-changing critical point of \(\Phi\).

**Proof.** For any \(u \in Y_m\) we have, in view of \((f_3)\),
\[
\Phi_m(u) - \frac{1}{\mu} \langle \Phi'_m(u), u \rangle = a \left( \frac{1}{2} - \frac{1}{\mu} \right)\|u\|^2 + b \left( \frac{1}{4} - \frac{1}{\mu} \right)\|u\|_4^4
\]
\[\quad + \int_{\Omega} \left( \frac{1}{\mu} u f(x, u) - F(x, u) \right) dx\]
\[\geq a \left( \frac{1}{2} - \frac{1}{\mu} \right)\|u\|^2 + b \left( \frac{1}{4} - \frac{1}{\mu} \right)\|u\|_4^4. \tag{3.15}\]

It then follows that any sequence \((u_n) \subseteq Y_m\) such that \(\sup_n \Phi_m(u_n) < \infty\) and \(\Phi'_m(u_n) \to 0\) is bounded.

Now let \((u_{m_j}) \subseteq X\) be such that
\[m_j \to \infty, \quad u_{m_j} \in V_{\sup} (S_{m_j}), \quad \sup \Phi(u_{m_j}) < \infty, \quad \Phi'_m(u_{m_j}) = 0.\]

In view of (3.14) the sequence \((u_{m_j})\) is bounded. Hence, up to a subsequence, \(u_{m_j} \to u\) in \(X\) and \(u_{m_j} \to u\) in \(L^p(\Omega)\).

Observe that the condition \(\Phi'_m(u_{m_j}) = 0\) is weaker than \(\Phi'(u_{m_j}) = 0\). Therefore, the fact that \((u_{m_j})\) converges strongly, up to a subsequence, to \(u\) in \(X\) does not follow from the usual standard argument.

Let us denote by \(\Pi_{m_j} : X \to Y_{m_j}\) the orthogonal projection. Then it is clear that \(\Pi_{m_j} u \to u\) in \(X\), as \(m_j \to \infty\). We have
\[
\langle \Phi'_{m_j}(u_{m_j}), u_{m_j} - \Pi_{m_j} u \rangle = \langle a + b\|u_{m_j}\|^2 \rangle \langle u_{m_j}, u_{m_j} - \Pi_{m_j} u \rangle
\]
\[\quad - \int_{\Omega} (u_{m_j} - \Pi_{m_j} u) f(x, u_{m_j}) dx. \tag{3.16}\]

Since \((u_{m_j})\) is bounded, we deduce from \((f_1)\) that \((\|f(x, u_{m_j})\|_{p/p-1})\) is bounded. We then obtain by using the Hölder inequality
\[
|\int_{\Omega} (u_{m_j} - \Pi_{m_j} u) f(x, u_{m_j}) dx| \leq \|u_{m_j} - \Pi_{m_j} u\|_p \|f(x, u_{m_j})\|_{p/p-1} \to 0.
\]

Recalling that \(\Phi'_{m_j}(u_{m_j}) = 0\), we deduce from (3.16) that
\[
\langle u_{m_j}, u_{m_j} - \Pi_{m_j} u \rangle = \|u_{m_j}\|^2 - \langle u_{m_j}, u \rangle + \langle u_{m_j}, u - \Pi_{m_j} u \rangle = o(1).
\]
It then follows that $\|u_{m_j}\| \to \|u\|$ which implies, since $X$ is uniformly convex, that $u_{m_j} \to u$ in $X$.

To show that the limit $u$ is sign-changing, we first observe that

$$\langle \Phi'_{m_j}(u_{m_j}), u_{m_j}^\pm \rangle = 0 \iff \left( a + \|u_{m_j}\|^2 \right) \|u_{m_j}^\pm\|^2 = \int_\Omega u_{m_j}^\pm f(x, u_{m_j}) \, dx$$

$$\Rightarrow a \|u_{m_j}^\pm\|^2 \leq \int_\Omega u_{m_j}^\pm f(x, u_{m_j}) \, dx.$$  

By using (3.9) and the Sobolev embedding theorem, we obtain

$$a \|u_{m_j}^\pm\|^2 \leq \int_\Omega u_{m_j}^\pm f(x, u_{m_j}) \, dx \leq c(\varepsilon \|u_{m_j}^\pm\|^2 + c_\varepsilon \|u_{m_j}^\pm\|^p),$$

where $c > 0$ is a constant. Since $u_{m_j}$ is sign-changing, $u_{m_j}^\pm$ are not equal to 0.

Choosing $\varepsilon$ small enough (for instance $\varepsilon < \frac{1}{2c}$), we see that $(\|u_{m_j}^\pm\|)$ are bounded below by strictly positive constants which do not depend on $m_j$. This implies that the limit $u$ of the sequence $(u_{m_j})$ is also sign-changing. \(\square\)

We are now in the position of proving our main result.

**Proof Theorem 1.1.** By Lemmas 3.1, 3.4, 3.6, and 3.7, and Remarks 3.3 and 3.5, the conditions $(A_1)$, $(A_2)$, $(A_3)$ and $(A_4)$ of Theorem 2.1 are satisfied. It then suffices to apply Theorem 2.1 to conclude. \(\square\)

**Conclusion**

In this paper, we presented a simple approach to study multiple sign-changing solutions to a class of Kirchhoff type problems. This seems to be the first paper in literature in that direction. For clarity of our exposition, we limited ourselves to a problem on a bounded domain, but it can be extended to the case of the whole space by working on the space of radially symmetric functions or by introducing a potential which implies a Rellich-Kondrashov compactness type result for the functional space (see e.g [12]). We believe that the abstract critical point theorem established in this paper applies to many other situations. An application to some classes of Schrödinger-Poisson systems will be discussed in a forthcoming paper.

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