I. INTRODUCTION

In this paper we demonstrate Dirac’s variation method [1] on a brane. Dirac’s motivation was to describe the electron as a bubble living in a background of electromagnetic field. Since the bubble has one dimension less than the background manifold, it is actually a boundary between the inner and outer parts of the surrounding manifold. A self consistent model is obtainable by the principle of least Action, the action must not change under small variations both in the electromagnetic field and in the location of the bubble.

Dirac claims that the naive way for varying the location of the bubble is wrong. This naive way is: parameterizing the bubble by the coordinates $x^\mu$ and the surrounding manifold by the coordinates $y^A$, the variation in the location of the bubble is naively $\delta y^A(x)$, but this will lead to the wrong equations. The right way to describe this variation is more complicated, this calls for a new coordinate system $z^a$ in the surrounding manifold. In the $z$-system the bubble’s location is fixed, while the variation in the location is done by varying the whole $y$-system with respect to the $z$-system. Therefore the action must be written in terms of the $z$-system, and the canonical fields are the electromagnetic field and the $y$-coordinate system.

The purpose of this paper is to demonstrate Dirac’s method, and to generalize it to include gravity. This paper is built as follows, in section II we give a simple example for this method. This example is Snell’s law of geometrical optics. We show that the naive variation leads to the wrong condition, and only Dirac’s method gives the correct law. In section III we start from the general action in $(N+1)$ dimensions, which includes gravity in $(N+1)$-dimensions and an $N$-dimensional brane. We perform the variation with respect to the $z$-system where the brane is fixed, and derive the equations of motion. The equations of the brane are not new. These are; (i) Israel junction conditions [2], (ii) generalized energy-momentum conservation on the brane, and, (iii) the geodetic equation of the brane [3].

II. A SIMPLE EXAMPLE: SNELL’S LAW

In geometrical optics we describe the light as rays propagating with the speed $v = \frac{c}{n}$, where $n$ is the index of refraction. We start with two media with coefficients $n_1, n_2$, a light source is placed in the first medium while the observer is in the second, see Fig. 1.

![Diagram of light ray propagation](image)

FIG. 1. A light ray propagating from $(0, A)$ to $(L, B)$, crossing the point $(X, 0)$.
The point $X$ where the ray crosses the surface between the two media is the analog of the brane (here it has a zero dimension). The surrounding manifold is the $y$-axis, the brane separates the axis into two parts. The canonical variable here is the height $h(y)$, while the action is the total time, it can be written as an integral over $y$

$$I = \int_{0}^{X} dy \frac{n_1}{c} \sqrt{1 + (dh/dy)^2} + \int_{X}^{L} dy \frac{n_2}{c} \sqrt{1 + (dh/dy)^2}$$  \hspace{1cm} (1)$$

The naive variations are $\delta h(y)$ and $\delta X$. At the end points and on the brane, the variation of $h$ vanishes and therefore the variation of the action is

$$\delta I = \frac{n_1}{c} \int_{0}^{X} dy \delta h \frac{d}{dy} \left( \frac{dh/dy}{\sqrt{1 + (dh/dy)^2}} \right) + \frac{n_1}{c} \delta X \sqrt{1 + (dh/dy)^2} \bigg|_{X^-}$$

$$- \frac{n_2}{c} \int_{X}^{L} dy \delta h \frac{d}{dy} \left( \frac{dh/dy}{\sqrt{1 + (dh/dy)^2}} \right) - \frac{n_2}{c} \delta X \sqrt{1 + (dh/dy)^2} \bigg|_{X^+}$$  \hspace{1cm} (2)$$

The equation of motion within each medium is simply $\frac{dh}{dy} = \text{const.}$, which means that the light propagates in straight lines. The variation with respect to $X$ will lead to the equation

$$n_1 \sqrt{1 + (dh/dy)^2} \bigg|_{X^-} = n_2 \sqrt{1 + (dh/dy)^2} \bigg|_{X^+}$$  \hspace{1cm} (3)$$

The geometrical relations

$$\cot \theta = \frac{dh}{dy}$$  \hspace{1cm} (4)$$

$$\frac{1}{\sin \theta} = \sqrt{1 + (dh/dy)^2}$$  \hspace{1cm} (5)$$

will lead us to the wrong relation

$$\frac{n_1}{\sin \theta_1} = \frac{n_2}{\sin \theta_2}$$  \hspace{1cm} (6)$$

The reason for this wrong relation is that a variation in $X$ will change the path of the ray everywhere and therefore can not be regarded as an independent variation.

Now we will use Dirac’s method for the variation. Take another coordinate for the axis, and call it $z$. In the $z$-axis, the location of the point does not change. The action (1) is written as an integral over $z$

$$I = \int_{0}^{\tilde{X}} dz \frac{n_1}{c} \sqrt{(y')^2 + (h')^2} + \int_{\tilde{X}}^{L} dz \frac{n_2}{c} \sqrt{(y')^2 + (h')^2}$$  \hspace{1cm} (7)$$

The prime denotes differentiation with respect to $z$. The canonical variables are $y(z)$ and $h(z)$, while the limits of integration are fixed. The variation is

$$\delta I = - \frac{n_1}{c} \int_{0}^{\tilde{X}} dz \left[ \delta h \frac{d}{dz} \left( \frac{h'}{\sqrt{(y')^2 + (h')^2}} \right) + \delta y \frac{d}{dz} \left( \frac{y'}{\sqrt{(y')^2 + (h')^2}} \right) \right]$$

$$- \frac{n_2}{c} \int_{\tilde{X}}^{L} dz \left[ \delta h \frac{d}{dz} \left( \frac{h'}{\sqrt{(y')^2 + (h')^2}} \right) + \delta y \frac{d}{dz} \left( \frac{y'}{\sqrt{(y')^2 + (h')^2}} \right) \right]$$

$$+ \frac{n_1}{c} \delta y \left( \frac{y'}{\sqrt{(y')^2 + (h')^2}} \right) \bigg|_{X^-} - \frac{n_2}{c} \delta y \left( \frac{y'}{\sqrt{(y')^2 + (h')^2}} \right) \bigg|_{X^+}$$  \hspace{1cm} (8)$$

\text{\footnotesize 1There is however an option to get Snell's law without Dirac's method. That is to solve the equations for $h(y)$ with the specific boundary conditions, to calculate the total action as a function of $X$ and to impose $\frac{dI}{dX} = 0$.}
The light propagates in straight lines in the \((y, h)\) plane, since \(\frac{dh}{dy} = h' = \text{const}\), while in the \(z\)-plane the motion might look very complicated. The variation \(\delta y\) is continuous over the 'brane', that is \(\delta y^- = \delta y^+\). Using now the geometrical relations (5), the true matching condition is therefore

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \]  

(9)

This simple example demonstrates Dirac’s method for variations on the brane.

III. THE GENERAL ACTION

We would like to apply Dirac’s method to a general \(N\)-dimensional brane living in a background with one extra dimension. The general action for such a brane should look like

\[
I = \int_{V_1} d^{N+1}y \sqrt{-G} \left[ \frac{R}{16\pi G} + \mathcal{L}_1 \right] - \int_{B} d^{N}x \sqrt{-g} \frac{K_1}{8\pi G} \\
+ \int_{B} d^{N}x \sqrt{-g} \mathcal{L}_B \\
+ \int_{V_2} d^{N+1}y \sqrt{-G} \left[ \frac{R}{16\pi G} + \mathcal{L}_2 \right] - \int_{B} d^{N}x \sqrt{-g} \frac{K_2}{8\pi G} 
\]

(10)

- The embedding manifold has been separated into \(V_{1,2}\) by the brane \(B\).
- The \((N + 1)\)-dimensional line element is \(ds_{N+1}^2 = G_{AB} dy^A dy^B\), while the line element on the brane is \(ds_N^2 = g_{\mu \nu} dx^\mu dx^\nu\).
- \(G\) is the bulk gravitational coupling, i.e. Newton’s constant in \((N + 1)\) dimensions.
- The gravitational action in \(V_{1,2}\) includes the \((N + 1)\)-dim integral of the scalar curvature \(R\) over \(V_{1,2}\), plus an \(N\)-dim integral over the brane of the extrinsic curvature of the brane embedded in \(V_{1,2}\) that is \(K_{1,2}\).
- Matter Lagrangians in \(V_{1,2}\) are \(\mathcal{L}_{1,2}\).
- The brane itself is characterized by the integral of the Lagrangian \(L_B\). This Lagrangian might include the \(N\)-dim intrinsic curvature as well as \(N\)-dim matter fields. (The ordinary symbols are used for the brane objects while the script symbols are for the bulk objects.)
- For simplicity we assume here that the extra dimension is space-like. If it is time-like there should be only some sign changes, while if it is null a more careful treatment is needed.

A. Variation in the Rest Frame of the Brane

According to Dirac’s method, we must describe the action in term of a special coordinate system, where the brane is fixed. Let us denote such a system with the coordinates \(z^a\). (Upper case Latin indices are saved for the \(y\) system, while lower case Latin indices are for the \(z\) system. Greek indices are the brane indices). The transformation of tensors from one system to another is done in the usual way with the bi-tensorial object \(y^A_{\alpha} = \frac{\partial y^A}{\partial z^\alpha}\), for example the metric

\[
G_{ab} = y^A_{\alpha} y^B_{\beta} G_{AB} .
\]

(11)

In general one can adopt a different coordinate system for each side of the brane. The following geometrical relations are applicable for each side of the brane separately. The brane we are working with is a hypersurface in the surrounding manifold. The location of the brane is defined by the function \(f(z) = 0\). The normal outward-pointing unit vector is

\[
n_a \propto f_{,a} ; \quad G^{ab} n_a n_b = 1
\]

(12)

The first fundamental form is \(h^\alpha_\beta = \delta^\alpha_\beta - n^a n_a\), it is a projection operator onto the tangent space of the brane, such that
The second fundamental form $K_{ab}$ is the extrinsic curvature of the brane embedded in the external manifold.

$$K_{ab} = \frac{1}{2} h^c_{[a} h^d_{b]} (n_{c,d} + n_{d,c}) .$$ \hspace{1cm} (14)

Two important relations relate the extrinsic curvature of the brane, the intrinsic curvature of the brane and the curvature of the surrounding manifold. These are the Gauss relation

$$R = R - 2 R_{AB} n^A n^B + K^2 - K_{\mu\nu} K^{\mu\nu} ,$$ \hspace{1cm} (15)

and the Codazzi relation

$$(K g^{\mu\nu} - K^{\mu\nu})_{;\nu} = n^A R_{AB} y^{B}_{;\nu} g^{\mu\nu} .$$ \hspace{1cm} (16)

The variation of the unit normal vector (12) is

$$\delta n_a = \frac{1}{2} n_a (n_b n_c \delta G_{bc}) .$$ \hspace{1cm} (17)

Looking first only at the gravitational action on one side of the brane

$$16\pi G I_g = \int_V d^{N+1} z \sqrt{-G} R - 2 \int_B d^N x \sqrt{-g} K ,$$ \hspace{1cm} (18)

the relevant variations are

- The variation of the volume integral

$$\delta (\sqrt{-G} R) = \sqrt{-G} \left[ (R_{ab} - \frac{1}{2} R G_{ab}) \delta G^{ab} + (G^{ab} \delta \Gamma^c_{ab} - G^{ac} \delta \Gamma^b_{ac}) \right] .$$ \hspace{1cm} (19)

- The variation of the brane integral involves the variation of $\sqrt{-g}$

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} = - \frac{1}{2} \sqrt{-g} h_{ab} \delta G^{ab} ,$$ \hspace{1cm} (20)

and the variation of the extrinsic curvature (14)

$$G^{ab} \delta K_{ab} = \frac{1}{2} K n_a n_b \delta G^{ab} + n_c h^{ab} \delta \Gamma^c_{ab} .$$ \hspace{1cm} (21)

where $h_{ab} = G_{ab} - n_a n_b$ is the first fundamental form (13) and we have made use of (17,13,14).

Using Gauss law for an integral over a total divergence

$$\int_V d^{N+1} z \sqrt{-G} V^a_{;a} = \int_B d^N x \sqrt{-g} V^a n_a$$ \hspace{1cm} (22)

the total variation of the gravitational action (18) is simply

$$16\pi G \delta I_g = \int_V d^{N+1} z \sqrt{-G} \left[ (R_{ab} - \frac{1}{2} R G_{ab}) \delta G^{ab} \right]$$
$$+ \int_B d^N x \sqrt{-g} \left[ -2 (K_{ab} - \frac{1}{2} K G_{ab}) \delta G^{ab} + (-2 n_c h^{ab} + n_c G^{ab} - n^a \delta^b_c) \delta \Gamma^c_{ab} \right] .$$ \hspace{1cm} (23)

Using $G_{ab} = h_{ab} + n_a n_b$ (13), expressing $\delta \Gamma^c_{ab}$ in terms of $(\delta G^{ab})_{;c}$, and using the integral relation
the term proportional to $\delta \Gamma_{ab}^c$ in the brane integral in Eq.(23) is turned into $(K_{ab} - K n_a n_b)\delta G^{ab}$. The total variation of the gravitational action on one side (18) is given by

$$16\pi G \delta I_g = \int_{V_1} d^{N+1}x \sqrt{-g} \left[ (R_{ab} - \frac{1}{2} R G_{ab}) \delta G^{ab} \right] + \int_{V_2} d^{N+1}x \sqrt{-g} \left[ (R_{ab} - \frac{1}{2} R G_{ab} - 8\pi G T_{ab}) \delta G^{ab} \right]$$

$$+ \int_B d^{N}x \sqrt{-g} \left[ (K h_{ab} - K_{ab}) \delta G^{ab} \right].$$

The total variation of the action(10) has a contribution from the gravitational action integral on each side of the brane, and matter contributions from the various Lagrangians.

$$16\pi G \delta I = \int_{V_1} d^{N+1}y \sqrt{-G} \left[ \mathcal{E}_{AB} \delta G^{AB} + 2 \left( G_{AB} y^A_a \delta y^B_b \right)_{;b} \right]$$

$$+ \int_{V_2} d^{N+1}y \sqrt{-G} \left[ \mathcal{E}_{AB} \delta G^{AB} + 2 \left( \mathcal{E}^B_A \right)_{;b} \delta y^A \right]$$

$$+ \int_B d^{N}x \sqrt{-g} \left[ \left( (K h_{ab} - K_{ab}) \delta G^{ab} \right)^{(1)} + \left( (K h_{ab} - K_{ab}) \delta G^{ab} \right)^{(2)} - 8\pi G T_{\mu\nu} \delta g^{\mu\nu} \right].$$

Here $T_{ab}^{(1,2)} = -\frac{2}{\sqrt{-G}} \frac{\delta (\sqrt{-G} L_B)}{\delta G^{ab}}$ is the bulk energy- momentum tensor, and $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_B)}{\delta g^{\mu\nu}}$ is the energy-momentum tensor of the brane.

B. The Equations of Motion

The next step is crucial in Dirac’s method. One has to express the variation of the metric in the $z$ system in terms of the variations in the $y$ system. Using Eq.(11)

$$\delta G_{ab} = \delta G_{AB} y^A_a y^B_b + 2 \left( G_{AB} y^A_a \delta y^B_b \right)_{;b}.$$  

The contraction of a general tensor $F_{ab}$ with the variation of the metric $\delta G^{ab}$ is

$$F_{ab} \delta G^{ab} = F_{AB} \delta G^{AB} + 2 (F^{AB})_{;b} G_{AD} \delta y^D - 2 \left( F_{ab} G_{AB} y^A_a \delta y^B_b \right)_{;b}.$$  

After substituting (28) in (26) and using (22) to turn the total divergence into a surface integral, we can transform back to the $y$ system and we are left with

$$16\pi G \delta I = \int_{V_1} d^{N+1}y \sqrt{-G} \left[ \mathcal{E}_{AB} \delta G^{AB} + 2 \left( \mathcal{E}^B_A \right)_{;b} \delta y^A \right]$$

$$+ \int_{V_2} d^{N+1}y \sqrt{-G} \left[ \mathcal{E}_{AB} \delta G^{AB} + 2 \left( \mathcal{E}^B_A \right)_{;b} \delta y^A \right]$$

$$+ \int_B d^{N}x \sqrt{-g} \left[ \left( (K h_{ab} - K_{ab}) \delta G^{ab} \right)^{(1)} + \left( (K h_{ab} - K_{ab}) \delta G^{ab} \right)^{(2)} - 8\pi G T_{\mu\nu} \delta g^{\mu\nu} \right] - 2 \left( n^A \mathcal{E}_{AB} \delta y^B \right)^{(1)} - 2 \left( n^A \mathcal{E}_{AB} \delta y^B \right)^{(2)}.$$  

Where $\mathcal{E}_{AB} = R_{AB} - \frac{1}{2} R G_{AB} - 8\pi G T_{AB}$ is Einstein’s factor in the surrounding manifold.

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2The brane action may include the intrinsic curvature of the brane, this is the Einstein-Hilbert action. In that case, the energy momentum tensor of the brane includes the Einstein tensor

$$T_{\mu\nu} \rightarrow T_{\mu\nu} = \frac{1}{8\pi G_N} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right).$$
The equations of motion in the surrounding manifold are clear. The arbitrariness of $\delta G^{AB}$ leads to Einstein’s equation

$$R_{AB} - \frac{1}{2}RG_{AB} - 8\pi GT_{AB} = 0 .$$

(30)

While the arbitrariness of $\delta y^A$ combined with the Bianchi identity will lead to energy-momentum conservation, which is the conserved Noether current associated with general coordinate transformation

$$(T^{AB})_{;B} = 0 .$$

(31)

The crucial question is what are the exact relations between the variations on the brane. The most important thing is that the brane metric $g_{\mu\nu}$ will be well defined. Therefore the variations of $G^{AB}(y)$ and $y^A(z)$ are not independent, but are constrained in such a way that

$$\delta G^{ab(1)} = \delta G^{ab(2)} = z^a_{,\mu}z^b_{,\nu} \delta g^{\mu\nu}$$

(32)

This will lead to the Israel condition [2]

$$\left( K_{\mu\nu}^{(1)} - K_{\mu\nu}^{(2)} \right) + \left( K_{\mu\nu}^{(2)} - K_{\mu\nu}^{(1)} \right) = 8\pi G T_{\mu\nu}$$

(33)

Notice that in our notation the outward pointing normal vector has a general sign change from one side to the other, and the above expression is actually the difference in extrinsic curvature.

The variation in the location of the brane can be projected tangent and normal to the brane. The tangent variation should be continuous over the brane

$$\left( \delta y^A G^{AB} y^B_{,\mu} \right)^{(1)} = \left( \delta y^A G^{AB} y^B_{,\mu} \right)^{(2)} .$$

(34)

The equation resulting from (29) with the arbitrariness of (34) is

$$\left( n^A (R_{AB} - \frac{1}{2}RG_{AB} - 8\pi GT_{AB}) y^B_{,\mu} \right)^{(1)} + \left( n^A (R_{AB} - \frac{1}{2}RG_{AB} - 8\pi GT_{AB}) y^B_{,\mu} \right)^{(2)} = 0 .$$

(35)

To understand that equation, one should notice that $n_A y^A = 0$, and use Codazzi’s equation (16) and the Israel condition (33) to get the energy momentum conservation on the brane

$$\left( T_{\mu}^{\nu} \right)_{;\nu} = \left( n^A T^{AB} y^B_{,\mu} \right)^{(1)} + \left( n^A T^{AB} y^B_{,\mu} \right)^{(2)} .$$

(36)

The total change in energy-momentum confined to the brane is due to flow of energy momentum in and out of the brane.

The normal variations in the location of the brane are opposite in sign in our notations $(n_A \delta y^A)^{(1)} = -(n_A \delta y^A)^{(2)}$. The arbitrariness of the normal variation in (29) will lead to

$$\left( n^A (R_{AB} - \frac{1}{2}RG_{AB} - 8\pi GT_{AB}) n^B \right)^{(1)} - \left( n^A (R_{AB} - \frac{1}{2}RG_{AB} - 8\pi GT_{AB}) n^B \right)^{(2)} = 0 .$$

(37)

Using Gauss relation (15), and substituting Israel condition (33) multiplied by $(K^{(1)}_{\mu\nu} - K^{(2)}_{\mu\nu})$ to get the non homogenous geodetic brane equation

$$\frac{1}{2}T^{\mu\nu} (K^{(1)}_{\mu\nu} - K^{(2)}_{\mu\nu}) = T_{mn}^{(1)} - T_{mn}^{(2)} .$$

(38)

This is the analog of Newton’s second law. The average value of the extrinsic curvature is the acceleration and the energy momentum of the brane is the analog of mass. The right hand side is the net force (pressure) acting on the brane.

$^3$Let us emphasize that Eq.(38) is a consequence of Eq.(30,33) if the bulk gravitational coupling is strong, but, Eq.(38) remains valid even in the case where the bulk gravitational coupling vanishes.
IV. SUMMARY

The equations of the brane were derived using the variational principle. The action is that of a brane embedded in co-dimension one. These equations are not new and were derived in the past using other methods [2,3]. Dirac’s method is essential in the derivation of Eqs.(36, 38). While Israel condition can be read off Eq.(26), energy momentum conservation on the brane (36) and the geodetic equation of the brane (38) emerge only when using Dirac’s method.

In addition, Israel junction condition is a remnant of Einstein equations in the bulk. It is relevant only when the bulk gravitational coupling is strong. On the other hand, energy momentum conservation on the brane (36) and the geodetic equation of the brane (38) are the equations of the brane. These are valid even if the bulk gravitational coupling vanishes [4].

[1] P.A.M. Dirac, Proc. Roy. Soc. London A 133, 60 (1931).
[2] W. Israel, Nuovo Cimento B44, 1 (1966).
[3] R.A. Battye and B. Carter, Phys. Lett. B509,331-336 (2001), hep-th/0101061; B. Carter Int. J. Theor. Phys. 40,2099-2130 (2001), gr-qc/0012036.
[4] T. Regge and C. Teitelboim, in Proc. Marcel Grossman, p.77 (Trieste, 1975); A. Davidson, Class.Quant.Grav. 16, 653-659 (1999); A. Davidson and D. Karasik, Mod.Phys.Lett. A13, 2187-2192 (1998).