SIMPLE WEIGHT MODULES WITH FINITE WEIGHT MULTIPLECTIES OVER THE LIE ALGEBRA OF POLYNOMIAL VECTOR FIELDS

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Abstract. Let $W_n$ be the Lie algebra of polynomial vector fields. We classify simple weight $W_n$-modules $M$ with finite weight multiplicities. We prove that every such nontrivial module $M$ is either a tensor module or the unique simple submodule in a tensor module associated with the de Rham complex on $\mathbb{C}^n$.

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1. Introduction

Lie algebras of vector fields have been studied since the fundamental works of S. Lie and E. Cartan in the late 19th century and the early 20th century. A classical example of such Lie algebra is the Lie algebra $W_n$ consisting of the derivations of the polynomial algebra $\mathbb{C}[x_1, ..., x_n]$, or, equivalently, the Lie algebra of polynomial vector fields on $\mathbb{C}^n$. The first classification results concerning representations of $W_n$ and other Cartan type Lie algebras were obtained by A. Rudakov in 1974-1975, [15], [16]. These results address the classification of a class of irreducible $W_n$-representations that satisfy some natural topological conditions. The modules of Rudakov are a particular class of the so-called tensor modules.

General tensor modules $T(P, V)$ are introduced by Shen and Larson, [17], [10], and are defined for a $D_n$-module $P$ and $\mathfrak{gl}(n)$-module $V$, where $D_n$ is the algebra of polynomial differential operators on $\mathbb{C}^n$ (see §2.8 for details). The modules $T(P, V)$ have nice geometric interpretations. If $V$ is finite dimensional, then we have a natural map from $W_n$ to the algebra of differential operators in the section of a trivial vector bundle on $\mathbb{C}^n$ with fiber $V$. This map is a specialization of a Lie algebra homomorphism $W_n \rightarrow D_n \otimes U(\mathfrak{gl}(n))$. The tensor module $T(P, V)$ is nothing but the pull back of the $D_n \otimes U(\mathfrak{gl}(n))$-module $P \otimes V$.

Tensor $W_1$-modules and their extensions were studied extensively in the 1970’s and in the 1980’s by B. Feigin, D. Fuks, I. Gelfand, and others, see for example, [4], [6].
Important results on general tensor modules $T(P, V)$ have been recently established by G. Liu, R. Lu, Y. Xue, K. Zhao, and others, see [18] and the references therein.

In this paper we focus on the category of weight representations of $\mathcal{W}_n$, namely those that decompose as direct sums of weight spaces relative to the subalgebra $\mathfrak{h}$ of $\mathcal{W}_n$ spanned by the derivations $x_1\partial_1, ..., x_n\partial_n$. The study of weight representations of Lie algebras of vector fields is a subject of interest by both mathematicians and theoretical physicists in the last 30 years. Two particular cases in this study have attracted special attention - the cases of $\mathcal{W}_n$ and of the Witt algebra $\text{Witt}_n$. Recall that $\text{Witt}_n$ is the Lie algebra of the derivations of the Laurent polynomial algebra $\mathbb{C}[x_1^\pm, ..., x_n^\pm]$, or, equivalently, the Lie algebra of polynomial vector fields on the $n$-dimensional complex torus. In particular, $\text{Witt}_1$ is the centerless Virasoro algebra. The classification of all simple weight representations with finite weight multiplicities of $\mathcal{W}_1$ and $\text{Witt}_1$ (and hence of the Virasoro algebra) was obtained by O. Mathieu in 1992, [12]. Following a sequence of works of S. Berman, Y. Billig, C. Conley, X. Guo, C. Martin, O. Mathieu, V. Mazorchuk, V. Kac, G. Liu, R. Lu, A. Piard, S. Eswara Rao, Y. Su, K. Zhao, recently, Y. Billig and V. Futorny managed to extend Mathieu’s classification result to $\text{Witt}_n$ for arbitrary $n \geq 1$ (see [1] and the references therein).

The classification of simple bounded (i.e. with a bounded set of weight multiplicities) modules of $\mathcal{W}_n$ was completed in [18]. The result in [18] states that every simple bounded module is a tensor module $T(P, V)$ or a submodule of a tensor module. In order $T(P, V)$ to be bounded, $P$ must be a weight $D_n$-module and $V$ must be a finite-dimensional module.

In this paper we classify all simple weight $\mathcal{W}_n$-modules $M$ with finite weight multiplicities. The main result is surprisingly easy to formulate - every such nontrivial module $M$ is either a tensor module $T(P, V)$ or the unique simple submodule of $T(P, \mathfrak{h}^k \mathcal{O}_m)$ for $k = 1, ..., n$. The necessary and sufficient condition for $P$ and $V$ so that $T(P, V)$ has finite weight multiplicities is given in Theorem 3.5. This condition is expressed in terms of the subsets of roots $\mathcal{W}_n$ and $\mathfrak{gl}(n)$ that act locally finitely or injectively on $P$ and $V$, respectively. For our classification result, we first use a theorem of [14] stating that $M$ is parabolically induced from a bounded simple module $N$ over a subalgebra $\mathfrak{g} = \mathcal{W}_m \ltimes (\mathfrak{t} \otimes \mathcal{O}_m)$ of $\mathcal{W}_n$. This subalgebra $\mathfrak{g}$ plays the role of a Levi subalgebra of a parabolic subalgebra of $\mathcal{W}_n$. The classification of simple bounded $\mathfrak{g}$-modules is one of the most difficult parts of the proof. By introducing the so-called $(\mathfrak{g}, \mathcal{O}_m)$-modules, we prove that $N$ is either the unique submodule of a tensor module, or it is a special generalized tensor module $\mathcal{F}(T(P, V), S)$, see Theorem 5.16. The essential tool for proving this theorem is the twisted localization functor introduced in [13]. For the main theorem we show that the parabolic induction functor maps $\mathcal{F}(T(P, V), S)$ to a tensor module.

Note that the Witt algebra $\text{Witt}_n$ is denoted by $\mathcal{W}_n$ in [1].
The content of the paper is as follows. In Section 2 we collect some important definitions and preliminary results on weight modules, twisted localization, parabolic induction, and tensor modules. In Section 3 we prove the necessary and sufficient condition for the tensor module $T(P, V)$ to be a weight module with finite weight multiplicities. We also show that $T(P, V)$ has a unique simple submodule and explain how the restricted duality functor acts on the tensor modules. The main theorem of this paper is also stated in Section 3. Section 4 is devoted to a few results concerning the parabolic induction theorem. The study of bounded $g$-modules and the classification of all possible $g$-modules $N$ that appear in the parabolic induction theorem are included in Section 5. In Section 6 we complete the proof of the main theorem by showing that the application of the parabolic induction functor on all possible $N$ described in the previous section leads to modules $M$ that are either tensor modules or the unique simple submodules of $T(P, \wedge^k \mathbb{C}^n)$ for $k = 1, ..., n$.

2. Preliminaries

2.1. Notation and convention. Throughout the paper the ground field is $\mathbb{C}$. All vector spaces, algebras, and tensor products are assumed to be over $\mathbb{C}$ unless otherwise stated.

2.2. Weight modules in general setting. Let $\mathcal{U}$ be an associative unital algebra and $\mathcal{H} \subset \mathcal{U}$ be a commutative subalgebra. We assume in addition that $\mathcal{H}$ is a polynomial algebra identified with the symmetric algebra of a vector space $\mathfrak{h}$, and that we have a decomposition

$$\mathcal{U} = \bigoplus_{\mu \in \mathfrak{h}^*} \mathcal{U}^\mu,$$

where

$$\mathcal{U}^\mu = \{ x \in \mathcal{U} | [h, x] = \mu(h)x, \forall h \in \mathfrak{h} \}.$$

Let $Q_\mathcal{U} = \mathbb{Z}\Delta_{\mathcal{U}} = \Delta_{\mathcal{U}} \cup (-\Delta_{\mathcal{U}})$ be the $\mathbb{Z}$-lattice in $\mathfrak{h}^*$ generated by $\Delta_{\mathcal{U}} = \{ \mu \in \mathfrak{h}^* | \mathcal{U}^\mu \neq 0 \}$. We obviously have $\mathcal{U}^\mu \mathcal{U}^\nu \subset \mathcal{U}^{\mu+\nu}$.

We call a $\mathcal{U}$-module $M$ a weight module, or a $(\mathcal{U}, \mathcal{H})$-module, if $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$, where

$$M^\lambda = \{ m \in M | hm = \lambda(h)m \text{ for all } h \in \mathfrak{h} \}.$$  

We call $M^\lambda$ the weight space of $M$, $\dim M^\lambda$ the $\lambda$-weight multiplicity of $M$, and $\text{supp } M = \{ \lambda \in \mathfrak{h}^* | M^\lambda \neq 0 \}$ the support of $M$. Note that

$$\mathcal{U}^\mu M^\lambda \subset M^{\mu+\lambda},$$

for every weight module $M$.

We will call a weight $\mathcal{U}$-module bounded if its set of weight multiplicities is a bounded set. For a bounded $\mathcal{U}$-module $M$, the degree $d(M)$ is the maximal weight multiplicity of $M$. A weight $\mathcal{U}$-module $M$ with finite weight multiplicities is cuspidal if all nonzero elements of $\mathcal{U}^\mu$ act injectively on $M$. If $\Delta_{\mathcal{U}} = -\Delta_{\mathcal{U}}$, then every cuspidal...
$\mathcal{U}$-module is bounded. We use this notion in the case when $\mathcal{U}$ is the Weyl algebra or the universal enveloping algebra of a reductive Lie algebra where the latter property holds.

In the particular case when $\mathcal{U} = U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ and $\mathcal{H} = S(\mathfrak{h})$ for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we have that a weight $\mathcal{U}$-module is a weight $\mathfrak{g}$-module.

2.3. Twisted localization in general setting. We retain the notation of the previous subsection.

Let $a$ be an ad-nilpotent element of $\mathcal{U}$. Then the set $\langle a \rangle = \{a^n \mid n \geq 0\}$ is an Ore subset of $\mathcal{U}$ which allows us to define the $\langle a \rangle$-localization $D_{\langle a \rangle}\mathcal{U}$ of $\mathcal{U}$. For a $\mathcal{U}$-module $M$ by $D_{\langle a \rangle}M = D_{\langle a \rangle}\mathcal{U} \otimes_{\mathcal{U}} M$ we denote the $\langle a \rangle$-localization of $M$. Note that if $a$ is injective on $M$, then $M$ is isomorphic to a submodule of $D_{\langle a \rangle}M$. In the latter case we will identify $M$ with that submodule.

We next recall the definition of the generalized conjugation of $D_{\langle a \rangle}\mathcal{U}$ relative to $x \in \mathbb{C}$. This is the automorphism $\phi_x : D_{\langle a \rangle}\mathcal{U} \to D_{\langle a \rangle}\mathcal{U}$ defined by the formula

$$\phi_x(u) = \sum_{i \geq 0} \binom{x}{i} \text{ad}(a)^i(u) a^{-i}.$$ 

If $x \in \mathbb{Z}$, then $\phi_x(u) = a^x u a^{-x}$. With the aid of $\phi_x$ we define the twisted module $\Phi_x(M) = M^{\phi_x}$ of any $D_{\langle a \rangle}\mathcal{U}$-module $M$. Finally, we set $D_{\langle a \rangle}^x M = \Phi_x D_{\langle a \rangle}M$ for any $\mathcal{U}$-module $M$ and call it the twisted localization of $M$ relative to $a$ and $x$. We will use the notation $a^x \cdot m$ (or simply $a^x m$) for the element in $D_{\langle a \rangle}^x M$ corresponding to $m \in D_{\langle a \rangle}M$. In particular, the following formula holds in $D_{\langle a \rangle}^x M$:

$$u(a^x m) = a^x \left( \sum_{i \geq 0} \binom{-x}{i} \text{ad}(a)^i(u) a^{-i} m \right)$$

for $u \in \mathcal{U}$, $m \in D_{\langle a \rangle}M$.

If $a_1, ..., a_k$ are commuting ad-nilpotent elements in $\mathcal{U}$ and $c = (c_1, ..., c_k)$ is in $\mathbb{C}^k$, then we set $D_{\langle a_1, ..., a_k \rangle} M = \prod_{i=1}^k D_{\langle a_i \rangle} M$ and $D_{\langle a_1, ..., a_k \rangle}^c M = \prod_{i=1}^k D_{\langle a_i \rangle}^c M$. Note that the products $\prod_{i=1}^k D_{\langle a_i \rangle}$ and $\prod_{i=1}^k D_{\langle a_i \rangle}^c$ are well defined because the functors involved pairwise commute.

If $a \in \mathcal{U}$ is an ad-nilpotent weight element and $M$ is a weight module then $D_{\langle a \rangle}^x M$ is again a weight module.

**Lemma 2.1.** Let $a \in \mathcal{U}$ be an ad-nilpotent weight element in $\mathcal{U}$, $M$ be a simple $a$-injective weight $\mathcal{U}$-module, and $z \in \mathbb{C}$. If $N$ is a simple nontrivial submodule of $D_{\langle a \rangle}^z M$, then $D_{\langle a \rangle}M \simeq D_{\langle a \rangle}^{-z} N$. In particular, if $a$ acts bijectively on $M$, $M \simeq D_{\langle a \rangle}^{-z} N$.

**Proof.** We use the fact that if $M$ is a simple weight $\mathcal{U}$-module, then $D_{\langle a \rangle}M$ and $D_{\langle a \rangle}^z M$ are simple $D_{\langle a \rangle}\mathcal{U}$-modules. Since $N$ is submodule of $D_{\langle a \rangle}^z M$, $D_{\langle a \rangle}N$ is a submodule of $D_{\langle a \rangle}^z M$. The simplicity of $N$ implies $D_{\langle a \rangle}N \simeq D_{\langle a \rangle}^z M$ and $D_{\langle a \rangle}^{-z} N \simeq D_{\langle a \rangle}M$. If $a$ acts bijectively, then $M \simeq D_{\langle a \rangle}M$. \hfill $\square$
We will also consider the following particular case of the twisted localization functor for \( \mathcal{U} = U(\mathfrak{g}) \), where \( \mathfrak{g} = \mathcal{W}_m \otimes \mathcal{O}_m \). Let \( a_i \in \mathfrak{h}^{\alpha_i}, \ i = 1, \ldots, \ell \), and \( \Gamma = \{ \alpha_1, \ldots, \alpha_\ell \} \) is a set of commuting roots of \( \mathfrak{h} \) that is linearly independent in \( \mathbb{Z}\Delta_\mathfrak{h} \). Let also \( \lambda \in \mathfrak{h}^* \) be such that \( \lambda = \sum_{i=1}^\ell \varepsilon_i \alpha_i. \) We set \( D^\lambda_\mathfrak{h} = D_1^{\alpha_1} \cdots D_\ell^{\alpha_\ell} \). If \( \bar{M} \simeq D^\lambda_\mathfrak{h} \bar{M} \) we will say that \( \bar{M} \) is obtained by a twisted localization from \( \bar{M} \). If \( \bar{M} \) is bounded, then \( \bar{M} \) is bounded, \cite{13} Lemma 4.4.

### 2.4. The algebras \( \mathcal{O}_n, \mathcal{D}_n, \) and \( \mathcal{W}_n \). In what follows, \( \mathcal{O}_n = \mathbb{C}[x_1, \ldots, x_n] \) and \( \mathcal{D}_n \) will stand for the associative algebra of differential operators in \( \mathcal{O}_n \). In other words, \( \mathcal{D}_n \) is the \( n \)-th Weyl algebra. We will often use the fact that \( \mathcal{D}_n \simeq \mathcal{D}_1 \otimes \ldots \otimes \mathcal{D}_1 \) (\( n \) copies).

Also, \( \mathcal{W}_n \) will stand for the Lie algebra of vector fields on \( \mathbb{C}^n \), i.e. \( \mathcal{W}_n = \text{Der}(\mathcal{O}_n) \).

Henceforth, we fix \( \mathfrak{h} = \text{Span}\{t_1\partial_1, \ldots, t_n\partial_n\} \). Note that \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathcal{W}_n \) and \( \mathcal{H} = \mathbb{C}[t_1\partial_1, \ldots, t_n\partial_n] \) is a maximal commutative subalgebra of \( \mathcal{D}_n \). We will use the setting of \cite{2,23} both for \( \mathcal{U} = U(\mathcal{W}_n) \) and \( \mathcal{U} = \mathcal{D}_n \), and in both cases \( \mathfrak{h} \) is the one that we fixed above. The set of roots of \( \mathcal{W}_n \) is:

\[
\Delta = \left\{ \sum_{j=1}^n m_j \varepsilon_j - \varepsilon_i + \sum_{j \neq i} m_j \varepsilon_j \mid m_j \in \mathbb{Z}_{\geq 0}, \ i = 1, \ldots, n \right\},
\]

where \( \varepsilon_i(t_j\partial_j) = \delta_{ij} \). By \( \Delta' \) we denote the set of all invertible roots of \( \mathcal{W}_n \). One can see that \( \Delta' \) is a root system of type \( A_n \).

A \( \mathcal{W}_n \)-module \( M \) is a \( (\mathcal{W}_n, \mathcal{O}_n) \)-module if \( M \) is an \( \mathcal{O}_n \)-module satisfying

\[
X(fv) = fX(v) + X(f)v, \quad \forall v \in M, f \in \mathcal{O}_n, X \in \mathcal{W}_n.
\]

If \( M \) is a weight \( \mathcal{W}_n \)-module with finite weight multiplicities, then the restricted dual \( M^* \) of \( M \) is by definition the maximal semisimple \( \mathfrak{h} \)-submodule of \( M^* \). The following properties of the restricted dual functor are straightforward.

**Lemma 2.2.** Let \( M \) be a weight \( \mathcal{W}_n \)-module with finite weight multiplicities. Then

1. \( \text{supp } M^* = -\text{supp } M; \)
2. \( \dim M^* = \dim M^{-\mu}; \)
3. \( M \) is simple if and only if \( M^* \) is simple.

Consider the embedding \( \mathbb{C}^n \to \mathbb{C}P^n \). The Lie algebra of vector fields on \( \mathbb{C}P^n \) is isomorphic to \( \mathfrak{sl}(n+1) \) and is a Lie subalgebra of \( \mathcal{W}_n \). In other words we have a canonical embedding \( \mathfrak{sl}(n+1) \subset \mathcal{W}_n \) of Lie algebras.

**Lemma 2.3.** Let \( M \) be a bounded weight \( \mathcal{W}_n \)-module such that \( \text{supp } M \subset \lambda + \mathbb{Z}\Delta_{\mathcal{W}_n} \) for some weight \( \lambda \). Then \( M \) has finite length.

**Proof.** The result holds for \( \mathfrak{sl}(n+1) \)-modules, see Lemma 3.3 in \cite{13}, and hence it holds for \( \mathcal{W}_n \) by using the natural embedding of \( \mathfrak{sl}(n+1) \) in \( \mathcal{W}_n \). \( \square \)
2.5. **Simple weight $D_n$-modules.** According to §2.2, a $D_n$-module $M$ is a weight module if

$$M = \bigoplus_{\lambda \in \mathbb{C}^n} M^\lambda,$$

where $M^\lambda = \{m \in M \mid x_i \partial_i m = \lambda_i m, \text{ for } i = 1, \ldots, n\}$. Below we recall the classification of the simple weight $D_n$-modules.

We will use the automorphism $\sigma : D_n \to D_n$ defined by $\sigma(t_i) = \partial_i, \sigma(\partial_i) = -t_i$ for all $i$. We call $\sigma$ the (full) Fourier transform of $D_n$. If $M$ is a $D_n$-module, by $M^F$ we denote the module $M$ twisted by $\sigma$.

The following gives the classification of all simple weight $D_n$-modules, see for example Corollary 2.9 in [8]

**Proposition 2.4.** (i) Every simple weight module of $D_1$ is isomorphic to one of the following: $O_1 = \mathbb{C}[x], O_1^x, x^\lambda \mathbb{C}[x^\pm 1], \lambda \in \mathbb{C} \setminus \mathbb{Z}$.

(ii) Every simple weight module of $D_n$ is isomorphic to $P_1 \otimes \cdots \otimes P_n$ where $P_i$ is a simple weight $D_1$-module.

We note also that every simple nontrivial weight $D_n$-module $M$ has degree 1, i.e. all its weight multiplicities equal 1. Moreover, for every $i$, $x_i$ (respectively, $\partial_i$) acts either injectively, or locally nilpotently on $M$. Let $I^+(M)$ denote the subset of indices in $\{1, \ldots, n\}$ such that $\partial_i$ acts locally nilpotently on $M$ and $x_i$ acts injectively on $M$, $I^-(M)$ the subset of indices such that $x_i$ acts locally nilpotently on $M$ and $\partial_i$ acts injectively on $M$, and $I^0(M)$ the subset of indices such that both $x_i$ and $\partial_i$ act injectively on $M$. Note that $\{1, \ldots, n\} = I^-(M) \sqcup I^0(M) \sqcup I^+(M)$. Furthermore, there exists $\lambda \in \text{supp } M$ such that

$$\text{supp } M = \lambda + \sum_{i \in I^+(M)} \mathbb{Z}_{\geq 0} \varepsilon_i + \sum_{j \in I^0(M)} \mathbb{Z} \varepsilon_j + \sum_{k \in I^-(M)} \mathbb{Z}_{\leq 0} \varepsilon_k.$$

2.6. **Parabolic induction in general.** Let $\mathfrak{g}$ be any Lie algebra with Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. Let $\gamma \in \mathfrak{h}^*$. Then the subalgebra

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\text{Re}(\gamma, \alpha) \geq 0} \mathfrak{g}_\alpha$$

is called the *parabolic subalgebra* of $\mathfrak{g}$ corresponding to $\gamma$. The Levi subalgebra of $\mathfrak{p}$ is

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\text{Re}(\gamma, \alpha) = 0} \mathfrak{g}_\alpha,$$

and the nilradical of $\mathfrak{p}$ is

$$\mathfrak{n} = \bigoplus_{\text{Re}(\gamma, \alpha) > 0} \mathfrak{g}_\alpha,$$

We are going to use extensively the following standard result.
Proposition 2.5. (a) Let \( N \) be a simple \( l \)-module, considered also as simple \( p \)-modules by letting \( n \) act trivially on \( N \). Then the \( g \)-module \( U(g) \otimes_{U(p)} N \) has a unique simple quotient.

(b) If \( L \) is a simple \( g \)-module such that \( L^n \neq 0 \), then \( L^n \) is a simple \( l \)-module.

(c) If \( L \) and \( M \) are simple \( g \)-modules such that \( M^n \simeq L^n \) as \( l \)-modules, then \( M \simeq L \) as \( g \)-modules.

Remark 2.6. If \( M \) is a simple weight \( g \)-module then \( M^n = \bigoplus_{\lambda \in S} M^\lambda \) where \( S \) is the subset of \( \text{supp} \ M \) such that \( \lambda + \alpha \notin \text{supp} \ M \) for any \( \alpha \in \Delta(n) \). For an arbitrary weight module \( M \) we call \( \bigoplus_{\lambda \in S} M^\lambda \) the \( p \)-top of \( M \) and denote it by \( M^{\text{top}} \).

2.7. Parabolic induction for \( W_n \). In this subsection we recall one of the main results in [14]. Recall the definitions of \( \Delta \) and \( \Delta' \) from §2.4. Let \( \gamma = a_1 \varepsilon_1 + \cdots + a_n \varepsilon_n \) for some \( a_i \in \mathbb{R} \). Set

\[
\Delta_0 = \{ \alpha \in \Delta \mid (\gamma, \alpha) = 0 \}, \quad \Delta_\pm = \{ \alpha \in \Delta \mid (\gamma, \alpha) > (0) \},
\]

\[
\Delta'_0 = \Delta_0 \cap \Delta', \quad \Delta'_\pm = \Delta_\pm \cap \Delta'.
\]

Let

\[
p = h + \bigoplus_{\alpha \in \Delta_0 \cup \Delta_\pm} (W_n)_\alpha, \quad g = h + \bigoplus_{\alpha \in \Delta_0} (W_n)_\alpha.
\]

Theorem 2.7. Let \( M \) be a simple weight \( W_n \)-module.

(a) There exists a weight \( \lambda \in \text{supp} \ M \) and \( \gamma \) such that

\[
\text{supp} \ M \subset \lambda + \mathbb{Z}_{\geq 0}(\Delta'_- \cup \Delta'_0).
\]

(b) One can choose \( \gamma \) in such a way that \( \mathbb{Z}\Delta'_0 = \mathbb{Z}\Delta_0 \) and

\[
\lambda + \mathbb{Z}\Delta_0 \subset \text{supp} \ M.
\]

(c) \( M \) is a unique simple quotient of the parabolically induced module \( U(W_n) \otimes_{U(p)} M_0 \) for some simple weight \( g \)-module \( M_0 \) that is extended in the natural way to a simple \( p \)-module.

2.8. Tensor modules over \( W_n \). Let \( V \) be a \( gl(n) \)-module and \( \tilde{V} := \mathcal{O}_n \otimes V \). One can look at \( \tilde{V} \) as the space of sections of the \( gl(n) \)-bundle on \( \mathbb{C}^n \) with fiber \( V \). Thus, \( \tilde{V} \) has the natural structure of a \( (W_n, \mathcal{O}_n) \)-module.

For a \( D_n \)-module \( P \) and a \( gl(n) \)-module \( V \), we define the tensor \( (W_n, \mathcal{O}_n) \)-module by

\[
T(P, V) := P \otimes_{\mathcal{O}_n} \tilde{V}
\]

and call it the tensor \( W_n \)-module relative to \( P \) and \( V \). If \( P \) is a weight \( D_n \)-module and \( V \) is a weight \( gl(n) \)-module then \( P \otimes_{\mathcal{O}_n} \tilde{V} \) is a weight module and

\[
\text{supp} (P \otimes_{\mathcal{O}_n} \tilde{V}) = \text{supp} P + \text{supp} V.
\]
Alternatively, we can define $T(P, V)$ as follows. Consider $T(P, V)$ as the vector space $T(P, V) = P \otimes_{\mathbb{C}} V$ and define $\mathcal{W}_n$-action and $\mathcal{O}_n$-action by the formulas
\[
x^\alpha \partial_j \cdot (f \otimes v) = x^\alpha \partial_j f \otimes v + \sum_{i=1}^n \partial_i(x^\alpha) f \otimes E_{ij} v,
\]
and define
\[
x^\alpha \cdot (f \otimes v) = x^\alpha f \otimes v,
\]
for $f \in P$, $v \in V$.

In what follows, the $k$-th exterior power $\wedge^k \mathbb{C}^n$ of the natural representation of $\mathfrak{gl}(n)$ will be called the $k$-th fundamental representation. We have the following result from [11] (Theorem 3.1 and Lemma 3.7):

**Proposition 2.8.** (i) Let $P$ be a simple $\mathcal{D}_n$-module and $V$ be a simple $\mathfrak{gl}(n)$-module that is not isomorphic to a fundamental representation. Then $T(P, V)$ is a simple $\mathcal{W}_n$-module.

(ii) Let $P_1$ and $P_2$ be simple $\mathcal{D}_n$-modules and let $V_1$ and $V_2$ be simple $\mathfrak{gl}(n)$-modules such that neither of them is isomorphic to a fundamental representation. Then $T(P_1, V_1) \simeq T(P_2, V_2)$ if and only if $P_1 \simeq P_2$ and $V_1 \simeq V_2$.

We next consider tensor modules $T(P, V)$ for which $V$ is a fundamental representation. For any $\mathcal{D}_n$-module $P$, the differential map
\[
d : T(P, \wedge^k \mathbb{C}^n) \to T(P, \wedge^{k+1} \mathbb{C}^n),
\]
is defined by $d(f \otimes v) = \sum_{i=1}^n (\partial_i f) \otimes (e_i \wedge v)$, where $(e_1, ..., e_n)$ is the standard basis of $\mathbb{C}^n$ associated to the coordinates $x_1, ..., x_n$ of $\mathbb{C}^n$. The map $d$ is a homomorphism of $\mathcal{W}_n$-modules but not $\mathcal{O}_n$-modules. One readily sees that $d^2 = 0$. As a result we have the following generalization of de Rham complex:
\[
0 \xrightarrow{d} T(P, 0 \mathbb{C}^n) \xrightarrow{d} T(P, 1 \mathbb{C}^n) \xrightarrow{d} \cdots \xrightarrow{d} T(P, n \mathbb{C}^n) \xrightarrow{d} 0.
\]

By Theorem 3.5 in [11] we have the following.

**Proposition 2.9.** Let $P$ be a simple $\mathcal{D}_n$-module.

(i) If $k = 0, ..., n - 1$, then the module $T(P, \wedge^k \mathbb{C}^n)$ has a simple quotient isomorphic to $dT(P, \wedge^{k+1} \mathbb{C}^n)$.

(ii) The module $T(P, 0 \mathbb{C}^n)$ is simple if and only if $P$ is not isomorphic to $\mathcal{O}_n$. If $P \simeq \mathcal{O}_n$ then $T(P, 0 \mathbb{C}^n)$ contains a trivial $\mathcal{W}_n$-submodule $\mathbb{C}$ and $dT(P, 0 \mathbb{C}^n) \simeq T(P, 0 \mathbb{C}^n)/\mathbb{C}$.

(iii) The module $T(P, n \mathbb{C}^n)$ is simple if and only if $\sum_i \partial_i P \neq P$.

We finish this subsection by stating the main result from [18] concerning the classification of the simple bounded $\mathcal{W}_n$-modules.

**Theorem 2.10.** Let $M$ be a nontrivial simple bounded $\mathcal{W}_n$-module. Then $M$ is isomorphic to one of the following:
(a) the module $T(P, V)$, where $P$ is a simple weight $\mathcal{D}_n$ module and $V$ is a simple finite-dimensional $\mathfrak{gl}(n)$-module that is not isomorphic to a fundamental representation;
(b) a simple submodule of $T(P, \wedge^k \mathbb{C}^n)$, where $k \in \{1, 2, \ldots, n\}$, and $P$ is a simple weight $\mathcal{D}_n$ module.

Remark 2.11. Proposition [3,7] implies that $T(P, \wedge^k \mathbb{C}^n)$ has a unique simple submodule.

3. Tensor modules with finite weight multiplicities

3.1. Tensor product of weight $\mathfrak{gl}(n)$-modules. Let $M$ be a simple weight $\mathfrak{gl}(n)$-module with finite weight multiplicities. Recall from [3] that $M$ has the following shadow decomposition:

$$\Delta(\mathfrak{gl}(n)) = \Delta^F_M \sqcup \Delta^I_M \sqcup \Delta^+_M \sqcup \Delta^-_M,$$

such that the $\alpha$-root vectors $X_\alpha$ act locally nilpotently on $M$ for all roots $\alpha \in \Delta^+_M \sqcup \Delta^F_M$ and injectively for all roots $\alpha \in \Delta^-_M \sqcup \Delta^I_M$. Moreover, $\Delta^I_M \sqcup \Delta^F_M$ and $\Delta^+_M$ are the roots of the Levi subalgebra $\mathfrak{g}^l + \mathfrak{g}^F$ and the nilradical $\mathfrak{g}^+$, respectively, of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{gl}(n)$, and $M$ is a quotient a parabolically induced module $\text{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)}(M^F \otimes M^I)$, for some cuspidal simple $\mathfrak{g}^l$-module $M^I$ and some finite-dimensional simple $\mathfrak{g}^F$-module $M^F$.

Lemma 3.1. Let $\mathfrak{l}$ be the Levi subalgebra of some parabolic $\mathfrak{p}$ in $\mathfrak{gl}(n)$. Assume that $M'$ and $N'$ are weight $\mathfrak{l}$-modules and that $M' \otimes N'$ has finite weight multiplicities. Then $(\text{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} M') \otimes (\text{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} N')$ has finite weight multiplicities.

Proof. Let $\mathfrak{m}$ denote the nilradical of the opposite to $\mathfrak{p}$ parabolic subalgebra $\mathfrak{p}^-$, and let $U = U(\mathfrak{m})$. Then $U$ has a $\mathbb{Z}_{\geq 0}$-grading $U = \bigoplus_{p \geq 0} U_p$ such that $U_0 = \mathbb{C}$ and each $U_p$ is a finite-dimensional $\mathfrak{l}$-module. This grading induces $\mathbb{Z}_{\geq 0}$-gradings on both $M = \text{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} M'$ and $N = \text{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} N'$ so that $M_p = M' \otimes U_p$ and $N_p = N' \otimes U_p$. Then $M \otimes N$ is also graded and its $m$th graded component is

$$(M \otimes N)_m = \bigoplus_{p+q=m} M' \otimes N' \otimes U_p \otimes U_q.$$

Hence, $M \otimes N$ has finite weight multiplicities. \qed

Lemma 3.2. Let $M$ and $N$ be simple weight $\mathfrak{gl}(n)$-modules. Then $M \otimes N$ has finite weight multiplicities if and only if $(\Delta^I_M \sqcup \Delta^-_M) \subset (\Delta^F_M \sqcup \Delta^-_N)$ or, equivalently, $(\Delta^I_M \sqcup \Delta^-_M) \cap (\Delta^I_N \sqcup \Delta^-_N) = \emptyset$.

Proof. First assume that the condition is not true. There exists a root $\alpha \in \Delta^I_M \sqcup \Delta^-_M$ such that $-\alpha \in \Delta^I_N \sqcup \Delta^-_N$. If $\mu \in \text{supp} M$ and $\nu \in \text{supp} N$ then $\mu + \mathbb{Z}_{\geq 0} \alpha \subset \text{supp} M$ and $\nu - \mathbb{Z}_{\geq 0} \alpha \in \text{supp} N$. Hence $\mu + \nu$ has infinite multiplicity in $M \otimes N$. 

Next assume that the condition holds. Then \( \Delta_M^I \subset \Delta_N^F \), \( \Delta_N^I \subset \Delta_M^F \) and \( \Delta_M^+ \subset (\Delta_N^F \cup \Delta_N^-) \). Choose \( \gamma_M \in \mathbb{Q}\Delta \) such that \( (\gamma_M, \alpha) = 0 \) for all \( \alpha \in \Delta_M^I \cup \Delta_N^F \) and \( (\gamma_M, \alpha) < 0 \) for all \( \alpha \in \Delta_M^- \). Similarly choose \( \gamma_N \), and let \( \gamma = \gamma_M + \gamma_N \). Then 
\[
(\gamma, \Delta_M^I) = (\gamma, \Delta_N^I) = 0 \quad \text{and} \quad (\gamma, \alpha) < 0 \quad \text{for any} \quad \alpha \in \Delta_M^- \cup \Delta_N^-.
\]
Let \( \mathfrak{p} \) be the parabolic defined by \( \gamma \). Then both \( M \) and \( N \) are quotients of the parabolically induced modules \( \text{Ind}^{\mathfrak{g}(n)}_{\mathfrak{p}} M' \) and \( \text{Ind}^{\mathfrak{g}(n)}_{\mathfrak{p}} N' \), respectively. The Levi subalgebra \( I \) of \( \mathfrak{p} \) is isomorphic to \( \mathfrak{g}_M^I \oplus \mathfrak{g}_N^I \oplus (\mathfrak{g}_M^F \cap \mathfrak{g}_N^F) \). Furthermore, \( M' = M^i \otimes M^f \) where \( M^i \) is a simple cuspidal \( \mathfrak{g}_M^I \)-module and \( M^f \) is some finite-dimensional \( \mathfrak{g}_N^F \oplus (\mathfrak{g}_M^F \cap \mathfrak{g}_N^F) \)-module. Similarly, \( N' = N^i \otimes N^f \) where \( N^i \) is a simple cuspidal \( \mathfrak{g}_N^I \)-module and \( N^f \) is some finite-dimensional \( \mathfrak{g}_M^F \oplus (\mathfrak{g}_M^F \cap \mathfrak{g}_N^F) \)-module. Therefore \( M' \otimes N' \) has finite weight multiplicities and the statement follows from Lemma 3.1. \( \square \)

3.2. Weight tensor modules.

**Lemma 3.3.** Let \( P \) be a simple weight \( \mathcal{D}_n \)-module. Then \( P = \bigoplus \kappa P_\kappa \), where \( P_\kappa \) is the eigenspace of \( \sum_{i=1}^n x_i \partial_i \) with eigenvalue \( \kappa \). Furthermore, every nonzero \( P_\kappa \) is a simple \( \mathfrak{g}(n) \)-module and all nonzero \( P_\kappa \) have the same shadow.

**Proof.** The first assertion is obvious. Since the adjoint action of \( \mathfrak{g}(n) \) on \( \mathcal{D}_n \) is locally finite, every root vector \( X_\kappa \in \mathfrak{g}(n) \) either acts locally nilpotently or injectively on all nonzero vectors of \( P \). Therefore all \( P_\kappa \) have the same shadow. By the classification of simple \( \mathcal{D}_n \)-modules, every \( P_\kappa \) is multiplicity free and \( \text{supp} P_\kappa \subset \lambda + \mathbb{Z}\Delta(\mathfrak{g}(n)) \) for any weight \( \lambda \in \text{supp} P_\kappa \). Using these and the fact that \( U(\mathfrak{g}(n)) P_\kappa^\lambda = P_\kappa \), we obtain that \( P_\kappa \) is simple. \( \square \)

**Remark 3.4.** Lemma 3.3 implies that every simple \( \mathcal{D}_n \)-module \( P \) has a well-defined \( \mathfrak{g}(n) \)-shadow. Below we give an explicit description of this shadow in terms of the subsets \( I^+(P) \), \( I^0(P) \) of \( \{1, \ldots, n\} \) defined in 2.3:
\[
\Delta_I^+ = \{ \varepsilon_i - \varepsilon_j \mid i, j \in I^0(P) \}, \quad \Delta_I^- = \{ \varepsilon_i - \varepsilon_j \mid i, j \in I^+(P) \text{ or } i, j \in I^-(P) \},
\]
\[
\Delta_F = \{ \varepsilon_i - \varepsilon_j \mid i \in I^+(P), j \notin I^+(P) \text{ or } i \notin I^+(P), j \in I^+(P) \}, \quad \Delta_F^- = -\Delta_F.
\]

**Theorem 3.5.** Let \( P \) be a simple weight \( \mathcal{D}_n \)-module and \( V \) be a simple weight \( \mathfrak{g}(n) \)-module. Then the \( \mathcal{W}_n \)-module \( T(P, V) \) has finite weight multiplicities if and only if \((\Delta_I^+ \cup \Delta_F^-) \subset (\Delta_I^+ \cup \Delta_F^-)\).

**Proof.** For every semisimple \( \mathfrak{h} \)-module \( X \) we denote by \( X_\kappa \) the eigenspace of \( \sum x_i \partial_i \) with eigenvalue \( \kappa \). By Lemma 3.3 \( P = \bigoplus_{\tau \in \mathbb{N}_0 + \mathbb{Z}} P_\tau \) for some \( \tau_0 \in \mathbb{C} \). Then
\[
T(P, V) = \bigoplus_{\tau \in \mathbb{N}_0 + \mathbb{Z}} P_\tau \otimes V,
\]
and the statement follows from Lemma 3.2. \( \square \)

**Example 3.6.** Consider a simple highest weight module \( \mathfrak{g}(4) \)-module \( V \) such that
\[
\Delta_I^+ \cup \Delta_I^- = \{ \varepsilon_i - \varepsilon_j \mid i = 1, 2; j = 3, 4 \}.
\]
Let $P$ be a simple weight $\mathcal{D}_n$-module $P$ on which $x_1, x_3, \partial_2, \partial_4$ act injectively and $\partial_1, \partial_3, x_2, x_4$ act locally nilpotently. Then by Remark 3.4 and Theorem 3.5, $T(P, V)$ has infinite weight multiplicities as $\varepsilon_1 - \varepsilon_4 \in \Delta^- \cap \Delta^+$. On the other hand, if $P'$ is a simple $\mathcal{D}_n$-module on which $x_1, \partial_2, \partial_3, \partial_4$ act locally nilpotently and $\partial_1, x_2, x_3, x_4$ act injectively, then $T(P', V)$ has finite weight multiplicities.

**Proposition 3.7.** For any simple weight $\mathcal{D}_n$-module $P$ and any simple weight $\mathfrak{gl}(n)$-module $V$, the $\mathcal{W}_n$-module $T(P, V)$ has a unique simple submodule.

**Proof.** If $V$ is not a fundamental representation the statement follows from Proposition 2.8(i). Now let $V = \bigwedge^k \mathbb{C}^n$. It is shown in [9] that if $P$ is cuspidal, i.e., $I^+(P) = I^-(P) = \emptyset$, then $T(P, \bigwedge^k \mathbb{C}^n)$ is simple for $k = 0, n$ and an indecomposable $\mathfrak{sl}(n+1)$-module of length two for $k = 1, \ldots, n-1$. This implies the statement for a cuspidal module $P$. For a general module $P$, consider

$$
\gamma = s \sum_{i \in I^-(P)} \varepsilon_i - \sum_{j \in I^+(P)} \varepsilon_j
$$

for some irrational $s > 1$. Let $\mathfrak{p}$ be the corresponding parabolic subalgebra of $\mathcal{W}_n$ and $\mathfrak{n}$ be the nilradical of $\mathfrak{p}$. The Levi subalgebra $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}(p) \oplus \mathfrak{gl}(q) \oplus \mathcal{W}_m$ where $p = |I^-(P)|$, $q = |I^+(P)|$, and $m = |I^0(P)|$. Note that

$$
P \simeq O_p \otimes O_q \otimes P_m
$$

for some cuspidal $\mathcal{D}_m$-module $P_m$. Since $V$ is finite dimensional and simple, $V^{\mathfrak{w} \oplus \mathfrak{g}(n)} \simeq V_p \otimes V_q \otimes V_m$ is a simple module over $\mathfrak{gl}(p) \oplus \mathfrak{gl}(q) \oplus \mathfrak{gl}(m)$. It is easy to compute that

$$
T(P, V)^n \simeq V_p \otimes V_q \otimes T(P_m, V_m).
$$

Since $P_m$ is cuspidal, $T(P_m, V_m)$ has a unique simple $\mathcal{W}_m$-submodule and hence $T(P, V)^n$ has a unique simple $\mathfrak{g}$-submodule $N$. If $M$ is a simple $\mathcal{W}_n$ submodule of $T(P, V)$ then $M^n \neq 0$ and hence $N \subset M$. That implies the uniqueness of $M$. \qed

3.3. **Duality for tensor modules.**

**Lemma 3.8.** Let $P$ be a simple weight $\mathcal{D}_n$-module. Consider $P$ as a $\mathcal{W}_n$-module via the natural homomorphism $\mathcal{W}_n \to \mathcal{D}_n$. Then $P_* \simeq T(P^F, \Lambda^n \mathbb{C}^n)$.

**Proof.** Recall the definition of $I^\pm(P)$, $I^0(P)$. As a vector space

$$
P = \prod_{i \in I^0(P)} x_i^{\lambda_i} \otimes \mathbb{C}[x_j]_{j \in I^+(P)} \otimes \mathbb{C}[\partial_k]_{k \in I^-(P)} \otimes \mathbb{C}[x_k^{\pm 1}]_{k \in I^+(P)}.
$$

where $\lambda_i$ are nonintegral for all $i \in I^0(P)$. We denote the monomial basis of $P$ by $e(\mu)$ where $\mu_i \in \lambda_i + \mathbb{Z}$ for $i \in I^0(P)$, $\mu_i \in \mathbb{Z}_{\geq 0}$ for $i \in I^+(P) \cup I^-(P)$. We have

$$
x_i e(\mu) = \begin{cases} 
e(\mu + \varepsilon_i) & \text{if } i \in I^0(P) \cup I^+(P) \\ -\mu_i e(\mu + \varepsilon_i) & \text{if } i \in I^-(P) \end{cases}
$$
\[
\partial_i e(\mu) = \begin{cases} 
\mu_i e(\mu - \varepsilon_i) & \text{if } i \in I^0(P) \cup I^+(P), \\
e(\mu + \varepsilon_i) & \text{if } i \in I^-(P)
\end{cases}
\]

Denote the corresponding basis of \(P^F\) by \(f(\mu)\) where \(\mu\) runs over the same set as \(e(\mu)\). Using identification of \(P\) and \(P^F\) as vector spaces, we have that if \(X(e(\mu)) = ce(\nu)\) then \(\sigma_F(X)f(\mu) = cf(\nu)\). This observation allows us to write the action of generators in the basis \(f(\mu)\):

\[
\partial_i f(\mu) = \begin{cases} 
-f(\mu + \varepsilon_i) & \text{if } i \in I^0(P) \cup I^+(P), \\
\mu_i f(\mu - \varepsilon_i) & \text{if } i \in I^-(P),
\end{cases}
\]

\[
x_i f(\mu) = \begin{cases} 
\mu_i f(\mu - \varepsilon_i) & \text{if } i \in I^0(P) \cup I^+(P), \\
f(\mu + \varepsilon_i) & \text{if } i \in I^-(P).
\end{cases}
\]

Let \(\varphi(\mu)\) be a function satisfying

\[
\varphi(\mu + \varepsilon_i) = \begin{cases} 
(\mu_i + 1)\varphi(\mu) & \text{if } i \in I^0(P) \cup I^+(P), \\
-(\mu_i + 1)\varphi(\mu) & \text{if } i \in I^-(P).
\end{cases}
\]

Define a pairing \(P \times P^F \to \mathbb{C}\) by setting \(\langle e(\mu), f(\nu) \rangle = \varphi(\mu)\delta_{\mu,\nu}\). Then we have

\[
\langle \partial_i e(\mu), f(\nu) \rangle = -\langle e(\mu), \partial_i f(\nu) \rangle, \quad \langle x_i e(\mu), f(\nu) \rangle = \langle e(\mu), x_i f(\nu) \rangle.
\]

Hence

\[
\langle g(x)\partial_i e(\mu), f(\nu) \rangle = -\langle e(\mu), g(x)\partial_i f(\nu) \rangle.
\]

Using that \(\partial_i g(x) = g(x)\partial_i + \partial_i (g(x))\) and choosing nonzero \(\omega \in \wedge^n \mathbb{C}^n\), we obtain

\[
g(x)\partial_i (f(\nu) \otimes \omega) = (\partial_i g(x)f(\nu)) \otimes \omega.
\]

This leads to a nondegenerate \(\mathcal{W}_n\)-invariant pairing \(P \times T(P^F, \wedge^n \mathbb{C}^n) \to \mathbb{C}\). \(\square\)

**Lemma 3.9.** Let \(V\) and \(P\) be such that \(T(P, V)\) has finite weight multiplicities, and let \(V^*_r\) be the restricted dual of \(V\). Then \(T(P^F, V^*_r \otimes \wedge^n \mathbb{C}^n)\) and \(T(P, V)\) are restricted dual to each other in the category of weight \(\mathcal{W}_n\)-modules.

**Proof.** We define a pairing

\[
T(P^F, V^*_r \otimes \wedge^n \mathbb{C}^n) \times T(P, V) \to \mathbb{C}
\]

by the formula

\[
\langle f \otimes v, g \otimes w \rangle = \langle f, g \rangle \langle v, w \rangle, \quad v \in V, w \in V^*_r, f \in P, g \in T(P^F, \wedge^n \mathbb{C}^n).
\]

Then we have

\[
\langle x^\alpha \partial_j(f) \otimes v + \sum_i \partial_i(x^\alpha) f \otimes E_{ij} v, g \otimes w \rangle + \langle f \otimes v, x^\alpha \partial_j(g) \otimes w + \sum_i \partial_i(x^\alpha) \otimes E_{ij} w \rangle =
\]

\[
\langle x^\alpha \partial_j(f), g \rangle \langle v, w \rangle + \langle f, x^\alpha \partial_j(g) \rangle \langle v, w \rangle +
\]

\[
\sum_j \langle \partial_j(x^\alpha)f, g \rangle \langle E_{ij}v, w \rangle + \langle f, \partial_j(x^\alpha)g \rangle \langle v, E_{ij}w \rangle = 0,
\]

because of
\[
\langle x^\alpha \partial_j(f), g \rangle + \langle f, x^\alpha \partial_j(f) \rangle = 0,
\]
and
\[
\langle \partial_j(x^\alpha)f, g \rangle = \langle f, \partial_j(x^\alpha)g \rangle
\]
and
\[
\langle E_{ij}v, w \rangle + \langle v, E_{ij}w \rangle = 0.
\]

3.4. **Statement of Main Result.** In this subsection we state and prove the main result in the paper. Some of the results used in the proof will be established in the next three sections.

**Theorem 3.10.** Let \( M \) be a simple weight \( \mathcal{W}_n \)-module with finite weight multiplicities. Then \( M \) is the unique submodule of some tensor module \( T(P, V) \) with finite weight multiplicities. More precisely, exactly one of the following holds:

(i) \( M \) is isomorphic to \( T(P, V) \) for a simple weight \( \mathfrak{d}_n \)-module \( P \) and a simple weight \( \mathfrak{gl}(n) \)-module \( V \) with finite weight multiplicities, such that \( (\Delta_p \cup \Delta_p^-) \subset (\Delta_\gamma \cup \Delta_\gamma^-) \) and such that \( V \) is not isomorphic to a fundamental representation.

(ii) \( M \) is isomorphic to \( dT(P, \wedge^k \mathbb{C}^n) \) for some \( k = 0, 1, ..., n-1 \), and a simple weight \( \mathfrak{d}_n \)-module \( P \).

(iii) \( M \cong \mathbb{C} \), which is the unique simple submodule of \( T(P, \wedge^0 \mathbb{C}^n) \).

**Proof.** Let \( M \) be a simple weight \( \mathcal{W}_n \)-module with finite weight multiplicities. By Theorem 2.7 and Proposition 4.1, \( M \) is a quotient of the parabolically induced module \( \text{Ind}^p_k N \) where \( N \) is a simple bounded \( \mathfrak{g} \)-module over the Levi subalgebra \( \mathfrak{g} \) of \( \mathfrak{p} \). Moreover, by Corollary 5.6 \( N \) satisfies the additional conditions (1) and (2) of Section 5. Theorem 5.16 provides a classification of such \( N \). Finally, Lemma 6.2 and Lemma 6.3 ensure that \( M \) is one of the modules listed in the statement. \( \square \)

4. **Applications of the parabolic induction**

Recall that \( \Delta \) stands for the set of roots of \( \mathcal{W}_n \). We use the setting of §2.7.

In what follows we always assume that \( M \) is a simple weight \( \mathcal{W}_n \)-module that has finite weight multiplicities. We will use that \( M \) is the unique simple quotient of a parabolically induced module \( U(\mathcal{W}_n) \otimes U(\mathfrak{p}) M_0 \), as stated in Theorem 2.7. Let \( \mathfrak{p} \) be the parabolic subalgebra associated with \( \gamma = \sum_{i=1}^n a_i \varepsilon_i \). We assume without loss of generality that
\[
a_1 \geq \cdots \geq a_p > 0 = a_{p+1} = \cdots = a_{p+m} > a_{p+m+1} \geq \cdots \geq a_n.
\]
Henceforth we fix \( \mathfrak{p} \) and denote by \( \mathfrak{g} \) the Levi subalgebra of \( \mathfrak{p} \). Then \( \mathfrak{g} \cong \mathcal{W}_m \ltimes (\mathfrak{f} \otimes \mathcal{O}_m) \) where \( \mathfrak{f} \) is a Levi subalgebra in \( \mathfrak{gl}(p) \oplus \mathfrak{gl}(n-m-p) \). Under this assumptions we have the following
Proposition 4.1. The simple \( \mathfrak{g} \)-module \( M_0 \) is bounded.

Proof. First we prove three preliminary results.

Lemma 4.2. Let \( \alpha = -\varepsilon_i \) or \( \alpha \in \Delta(\mathfrak{t}) \). Then

1. \( \dim \mathfrak{g}_\alpha = 1 \) and any nonzero \( X_\alpha \in \mathfrak{g}_\alpha \) can be included in the \( \mathfrak{sl}_2 \) triple;
2. Either \( \mathfrak{g}_\alpha \) acts locally nilpotently on \( M_0 \) or \( \mathfrak{g}_\alpha : M_0 \to M_0 \) is injective.

Proof. The first assertion is obvious. The second follows from the fact that \( \text{ad} \mathfrak{g}_\alpha \) is locally nilpotent.

Lemma 4.3. Let \( \alpha = -\varepsilon_i \) for \( p < i \leq p + m \) or \( \alpha \in \Delta(\mathfrak{k}) \). Then \( \mathfrak{g}_\alpha \) acts injectively on \( M_0 \).

Proof. Suppose that \( \mathfrak{g}_\alpha \) is locally nilpotent on \( M_0 \). Let \( h \) be the Cartan element in the \( \mathfrak{sl}_2 \)-triple containing \( X_\alpha \in \mathfrak{g}_\alpha \setminus \{0\} \). In particular, \( \alpha(h) = 2 \). Let \( \mu \in \text{supp} M \). Then \( \mu + Z \alpha \subset \text{supp} M \). Furthermore, for any \( n > 0 \) there exist \( k \geq n \) and \( v \in M_{\mu+k\alpha} \) such that \( \mathfrak{g}_\alpha v = 0 \). Let \( M_k \) denote the \( \mathfrak{sl}(2) \)-submodule of \( M \) generated by \( v \). For all sufficiently large \( k \) we have \( \mu \in \text{supp} M_k \). Therefore \( \dim M_\mu = \infty \). A contradiction.

Corollary 4.4. Let \( \alpha = -\varepsilon_i \) or \( \alpha \in \Delta(\mathfrak{t}) \). For any \( \lambda \in \text{supp} M \) and \( X \in \mathfrak{g}_\alpha \setminus \{0\} \) the map \( X : M_\lambda \to M_{\lambda + \alpha} \) is an isomorphism.

Proof. From the previous lemma we know that \( X : M_\lambda \to M_{\lambda + \alpha} \) is injective. Applying the same lemma to \( M_* \) we obtain \( X : M_*^{-\lambda - \alpha} \to M_*^{-\lambda} \) is injective. Hence \( X : M_\lambda \to M_{\lambda + \alpha} \) is surjective.

We are now ready to complete the proof of Proposition 4.1. Corollary 4.4 implies \( \dim M_\mu = \dim M_{\mu+\gamma} \) for any \( \gamma \in \mathbb{Z}\Delta(\mathfrak{W}_m) + \mathbb{Z}\Delta(\mathfrak{t}) \) and \( \mu \in \text{supp} M \). The statement follows.

5. Bounded simple \( \mathfrak{g} \)-modules

5.1. Generalization of tensor modules for the Levi subalgebra \( \mathfrak{g} \) of \( \mathfrak{p} \). We retain the notation of the previous section. In this section we assume that \( m > 0 \). Recall that \( \mathfrak{g} = \mathfrak{W}_m \ltimes (\mathfrak{t} \otimes \mathfrak{O}_m) \). Without loss of generality we may assume \( \mathfrak{O}_m = \mathbb{C}[x_1, \ldots, x_m] \). In this section we will classify simple bounded \( \mathfrak{g} \)-modules \( N \) satisfying the additional properies:

1. \( \text{supp} N = \lambda + \mathbb{Z}\Delta(\mathfrak{g}) \) for any \( \lambda \in \text{supp} N \).
2. All weight spaces of \( N \) have the same dimension \( d \).

It follows from the proof of Proposition 4.1 that \( \mathfrak{g}_\alpha \) acts injectively on \( N \) if \( \alpha = -\varepsilon_i \) or \( \alpha \in \Delta(\mathfrak{t}) \).

First, we generalize the notion of a \((\mathfrak{W}_m, \mathfrak{O}_m)\)-module to that of a \((\mathfrak{g}, \mathfrak{O}_m)\)-module.
Definition 5.1. A $\mathfrak{g}$-module $N$ is a $(\mathfrak{g}, \mathcal{O}_m)$-module if $N$ is a $\mathcal{O}_m$-module satisfying
\begin{equation}
X(fv) = fX(v) + X(f)v \quad \forall v \in N, f \in \mathcal{O}_m, X \in \mathcal{W}_m,
\end{equation}

\begin{equation}
(h \otimes Y)(fv) = (hf)Yv \quad \forall v \in N, f, h \in \mathcal{O}_m, Y \in \mathfrak{t}.
\end{equation}

From now on we assume that all $(\mathcal{W}_m, \mathcal{O}_m)$-modules and all $(\mathfrak{g}, \mathcal{O}_m)$-modules are weight modules.

Remark 5.2. Consider the algebra $\mathcal{A}(m)$ generated by $\mathcal{W}_m \otimes 1$ and $1 \otimes \mathcal{O}_m$ with relations
\begin{equation}
(x \otimes 1)(y \otimes 1) - (y \otimes 1)(x \otimes 1) = [x, y] \otimes 1,
\end{equation}
\begin{equation}
(1 \otimes f)(1 \otimes g) = 1 \otimes fg,
\end{equation}
\begin{equation}
(x \otimes 1)(1 \otimes f) - (1 \otimes f)(x \otimes 1) = 1 \otimes x(f)
\end{equation}
for $x, y \in \mathcal{W}_m$ and $f, g \in \mathcal{O}_m$. Any $(\mathcal{W}_m, \mathcal{O}_m)$ is an $\mathcal{A}(m)$-module, and conversely, any $\mathcal{A}(m)$-module is a $(\mathcal{W}_m, \mathcal{O}_m)$-module. Furthermore, $\mathcal{A}(m)$ is isomorphic to $U(\mathcal{W}_m) \otimes \mathcal{O}_m$ as a vector space by the correspondence $(X \otimes 1)(1 \otimes f) \mapsto X \otimes f$ for all $X \in U(\mathcal{W}_m)$ and $f \in \mathcal{O}_m$. Let $\mathcal{B} := \mathcal{A}(m) \otimes U(\mathfrak{t})$. Then any $(\mathfrak{g}, \mathcal{O}_m)$-module is a $\mathcal{B}$-module.

Example 5.3. Let $S$ be a $\mathfrak{t}$-module. We define a $(\mathfrak{g}, \mathcal{O}_m)$-module structure on the vector space $\mathcal{O}_m \otimes S$ by setting
\begin{equation}
f(h \otimes s) = fh \otimes s, \quad (f \otimes Y)(h \otimes s) = fh \otimes Ys, \quad X(h \otimes s) = X(h) \otimes s
\end{equation}
for all $f, h \in \mathcal{O}_m, Y \in \mathfrak{t}, X \in \mathcal{W}_m$ and $s \in S$. One can easily verify that $\tilde{S} := \mathcal{O}_m \otimes S$ is a $(\mathfrak{g}, \mathcal{O}_m)$-module. Moreover, if $R$ is a $(\mathcal{W}_m, \mathcal{O}_m)$-module then $\mathcal{F}(R, S) := R \otimes_{\mathcal{O}_m} \tilde{S}$ is a $(\mathfrak{g}, \mathcal{O}_m)$-module.

Remark 5.4. A simple weight $(\mathcal{W}_m, \mathcal{O}_m)$-module $R$ with finite weight multiplicities is a tensor module $T(P, V)$ for some simple weight $\mathcal{D}_m$-module $P$ and some simple weight $\mathfrak{gl}(m)$-module $V$, see Theorem 3.7 in [18].

Lemma 5.5. If $R$ is a simple $(\mathcal{W}_m, \mathcal{O}_m)$-module and $S$ is a simple weight $\mathfrak{t}$-module then $\mathcal{F}(R, S)$ is a simple $(\mathfrak{g}, \mathcal{O}_m)$-module, in the sense that it does not contain proper nontrivial $(\mathfrak{g}, \mathcal{O}_m)$-submodules.

Proof. Observe that $\mathcal{F}(R, S)$ is isomorphic to $R \otimes S$ as a $\mathcal{B}$-module. Hence it is a simple $\mathcal{B}$-module. This implies the statement. $\square$

Lemma 5.6. If $N = \mathcal{F}(R, S)$ satisfies conditions (1) and (2), then $S$ is a simple cuspidal $\mathfrak{t}$-module, and $R = T(P, V)$ for some simple cuspidal $\mathcal{D}_m$-module $P$ a simple finite-dimensional $\mathfrak{gl}(m)$-module $V$. For any $\lambda \in \text{supp} \ N$ we have that $\dim N^\lambda = (\dim V)d(S)$, where $d(S)$ is the degree of the cuspidal module $S$.

Proof. The lemma follows from the isomorphism of $\mathfrak{h}$-modules $\mathcal{F}(R, S) \simeq R \otimes S$. $\square$

Lemma 5.7. Let $S$ be a simple nontrivial weight $\mathfrak{t}$-module, $P$ be a simple weight $\mathcal{D}_m$-module, and $V$ be a simple finite-dimensional $\mathfrak{gl}(m)$-module. Then $\mathcal{F}(T(P, V), S)$ is a simple $\mathfrak{g}$-module.
Proof. Choose a regular \( u \in (\mathfrak{t} \cap \mathfrak{h})^* \) that acts nontrivially on \( S \), and denote by \( \mathcal{F}(T(P, V), S)^a \) the eigenspace of \( u \) with eigenvalue \( a \). Let \( M \) be a proper nonzero submodule of \( \mathcal{F}(T(P, V), S) \). Then \( M^a = M \cap \mathcal{F}(T(P, V), S)^a \) is \( \mathcal{O}_m \)-invariant for any \( a \neq 0 \). Using the action of the root elements of \( \mathfrak{t} \), we obtain that \( M^a \) is \( \mathcal{O}_m \)-invariant for \( a = 0 \) as well. Hence \( M \) is a \( (\mathfrak{g}, \mathcal{O}_m) \)-submodule of \( \mathcal{F}(T(P, V), S) \) and we reach a contradiction. \( \square \)

**Lemma 5.8.** Let \( N \) be a simple \( (\mathfrak{g}, \mathcal{O}_m) \)-module satisfying (1) and (2). Then \( N \) is isomorphic to \( \mathcal{F}(R, S) \) for some \( (\mathcal{W}_m, \mathcal{O}_m) \)-module \( R \) and some simple cuspidal \( \mathfrak{t} \)-module \( S \).

**Proof.** Recall the definition of \( \mathcal{B} \) from Remark 5.2. Consider \( N \) as a \( \mathcal{B} \)-module. By definition, for any vector \( v \in N \) we have \( \mathcal{B}v = U(\mathfrak{g})v \) (this follows from the relation \( (f \otimes Y)v = f(Yv) \)). Hence, \( N \) is a simple \( \mathcal{B} \)-module. For a simple \( \mathfrak{t} \)-module \( S' \), the subspace \( \text{Hom}_k(S', N) \otimes S' \) of \( N \) is \( \mathcal{B} \)-stable. Hence, there is a unique up to isomorphism \( S' \), such that \( \text{Hom}_k(S', N) \neq 0 \). The existence of such \( S' \) follows from condition on the support of \( N \). Let \( S \) be such module. We have that \( R = \text{Hom}_k(S, N) \) is a simple \( \mathcal{A}(m) \)-module. Therefore, \( N \simeq R \otimes S \) as a \( \mathcal{B} \)-module. The condition (5.2) ensures that \( N \simeq \mathcal{F}(R, S) \). \( \square \)

Recall that \( \hat{V} \) stands for \( (\mathcal{W}_m, \mathcal{O}_m) \)-module \( \mathcal{O}_m \otimes V \).

**Lemma 5.9.** Let \( N \) be a simple weight \( \mathfrak{g} \)-module, such that \( \partial_i \) acts locally nilpotently for all \( i = 1, \ldots, m \). Then \( N \) is isomorphic to a simple submodule of \( \mathcal{F}(\hat{V}, S) \) for some simple \( \mathfrak{t} \)-module \( S \) and a simple \( \mathfrak{gl}(m) \)-module \( V \).

**Proof.** Let \( N_0 \) be the space of invariants of \( \partial_1, \ldots, \partial_m \). If \( \mathfrak{q} \) is the parabolic subalgebra associated to \( \gamma = -(\varepsilon_1 + \cdots + \varepsilon_m) \), then \( N \) is the unique simple quotient of \( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} N_0 \). Thus, \( N_0 \) is a simple \( \mathfrak{gl}(m) \oplus \mathfrak{t} \)-module, so \( N_0 = V \otimes S \) for some simple modules \( V \) and \( S \). Then we have a natural homomorphism \( \varphi : N_0 \rightarrow \mathcal{F}(\hat{V}, S) \) of \( \mathfrak{gl}(m) \oplus \mathfrak{t} \)-modules, hence, also of \( \mathfrak{q} \)-modules. The homomorphism \( \varphi \) induces a homomorphism \( \Phi : U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} N_0 \rightarrow \mathcal{F}(\hat{V}, S) \) of \( \mathfrak{g} \)-modules. The image of \( \Phi \) is isomorphic to \( N \). \( \square \)

**Lemma 5.10.** Let \( N \) be a simple \( \mathfrak{g} \)-module satisfying (1) and (2). Assume that one can define an \( \mathcal{O}_m \)-module structure on \( N \) in such a way that it satisfies (5.1) and

\[
f(g \otimes Y)v = (g \otimes Y)fv, \quad \forall v \in N, f, g \in \mathcal{O}_m, Y \in \mathfrak{t}.
\]

Then \( N \) is a \( (\mathfrak{g}, \mathcal{O}_m) \)-module.

**Proof.** We need to verify (5.2). First, we claim that verifying (5.2) is equivalent to checking

\[
x_1(1 \otimes Y) = (x_1 \otimes Y), \quad \forall Y \in \mathfrak{t}.
\]

Indeed, let \( f \in \mathcal{O}_m \) and \( X = f \partial_1 \). Then

\[
[X, x_1](1 \otimes Y) = f(1 \otimes Y) = [X, x_1 \otimes Y] = f \otimes Y.
\]
For $f, g \in \mathcal{O}_m$ we have

$$f(g \otimes Y) = fg(1 \otimes Y).$$

Next, observe that (1) and (2) ensure that $\partial$ and $x_i$ act injectively on $N$. Therefore we can localize $N$ with respect $x_i$ and define a $(\mathcal{W}_m, \tilde{\mathcal{O}}_m)$-module structure on $N$, where $\tilde{\mathcal{O}}_m = \mathbb{C}[x_1^{\pm1}, x_2^{\pm1}, \ldots, x_m^{\pm1}]$. Consider the twisted localization $D_{(x_1, \ldots, x_n)}^c N$ of $N$ with $\mathcal{U} = \mathcal{B}$, and some $c \in \mathbb{C}^m$. We can choose $c$ so that $D_{(x_1, \ldots, x_n)}^c N$ has a nonzero weight vector annihilated by all $\partial_i$. Among all such vectors choose one of weight $\mu$ with maximal possible $|\mu|_r := \text{Re} \sum_{i=1}^m \mu_i$. Let $N'$ be a $\mathfrak{g}$-submodule of $D_{(x_1, \ldots, x_n)}^c N$ generated by $u$. Note that for any $\nu \in \text{supp} N'$ we have $|\mu|_r \geq |
u|_r$. Let $v$ have weight $\nu$ with $|\nu|_r = |\mu|_r$. Then $\partial_i v = 0$ and

$$\partial_i (x_1(1 \otimes Y) - x_1 \otimes Y)v = 0,$$

hence $u = (x_1(1 \otimes Y) - x_1 \otimes Y)v$ is annihilated by all $\partial_i$. On the other hand, the weight $\eta$ of $u$ satisfies $|\eta|_r = |\mu|_r + 1$ hence $u = 0$. Let $w \in N'$ be a weight vector of weight $\lambda$ with minimal $|\lambda|$ such that for some $Y \in \mathfrak{k}$

$$(x_1(1 \otimes Y) - x_1 \otimes Y)w \neq 0.$$  

We have

$$\partial_i (x_1(1 \otimes Y) - x_1 \otimes Y)w = (x_1(1 \otimes Y) - x_1 \otimes Y)\partial_i w = 0,$$

which leads to a contradiction. Next we note that $D_{(x_1, \ldots, x_n)}^c N = \tilde{\mathcal{O}}_m \cdot N'$. Since $x_1(1 \otimes Y) - x_1 \otimes Y$ commutes with $\tilde{\mathcal{O}}_m$ we have $(x_1(1 \otimes Y) - x_1 \otimes Y)N(c) = 0$. Then $(x_1(1 \otimes Y) - x_1 \otimes Y)N = 0$. This completes the proof. \hfill $\Box$

**Lemma 5.11.** Assume that $N$ is a simple $\mathfrak{g}$-module satisfying (1) and (2). Let $z$ be a central element of $\mathfrak{k}$ which does not act trivially on $N$. Then $N$ is a $(\mathfrak{g}, \mathcal{O}_m)$-module and hence is isomorphic to a module $\mathcal{F}(R, S)$.

**Proof.** Without loss of generality we may assume that $z$ acts as identity on $N$. Define an $\mathcal{O}_m$-module structure on $N$ by setting $x_i v := (x_i \otimes z)v$. Then $N$ satisfies the assumptions of Lemma 5.10. The statement follows. \hfill $\Box$

**Lemma 5.12.** Let $\mathfrak{k}$ be abelian and $N$ be a simple $\mathfrak{g}$-module satisfying (1) and (2). If $\mathfrak{k}N = 0$, then $(\mathcal{O}_m \otimes \mathfrak{k})N = 0$.

**Proof.** We will show that $(f \otimes h)N = 0$ for any $f \in \mathcal{O}_m$, $h \in \mathfrak{k}$. Note that

$$[\partial_1, x_1 \otimes h]N = hN = 0.$$  

Therefore we have the following identities on $N$:

$$[\partial_1(x_1 \otimes h), \partial_1^2(x_1^2 \otimes h)] = 2\partial_1^2(x_1 \otimes h)^2 = 2(\partial_1(x_1 \otimes h))^2,$$

$$[(\partial_1(x_1 \otimes h))^k, \partial_1^k(x_1^k \otimes h)] = 2k(\partial_1(x_1 \otimes h))^{k+1}.$$  

This implies that, on each weight space $N^\lambda$ of $N$, $\text{tr}_{\mathfrak{n}_L}(\partial_1(x_1 \otimes h))^k = 0$ for all $k > 2$. Since $N$ is bounded this implies nilpotency of $\partial_1(x_1 \otimes h)$ on $N$. Since $\partial_1$ is invertible on $N$ we obtain that $x_1 \otimes h$ is nilpotent on $N$. Let $p$ be the nilpotency degree of
There exists $v \in N$ such that $w := (x_1 \otimes h)^{p-1}v \neq 0$. Then for $f \in \mathcal{O}_m$ we have
$$0 = f\partial_1(x_1 \otimes h)^{p-1}v = p(f \otimes h)(x_1 \otimes h)^{p-1}v.$$ In other words, $w$ is annihilated by $\mathcal{O}_m \otimes h$. The subspace $N'$ of all vectors annihilated by $\mathcal{O}_m \otimes h$ is $\mathfrak{g}$-invariant, but we just proved that $N' \neq 0$. By the irreducibility of $N$, we have $N = N'$. Thus $(\mathcal{O}_m \otimes h)N = 0$. □

**Proposition 5.13.** Let $N$ be a $\mathfrak{g}$-module satisfying (1) and (2). Then for any Borel subalgebra $\mathfrak{b} \subset \mathfrak{k}$ there exists a simple bounded $\mathfrak{g}$-module $\tilde{N}$ satisfying the following two conditions:

(i) There exists a weight $\lambda \in \mathfrak{h}^*$ such that $\text{supp } \tilde{N} \subset \lambda + \sum_{i=1}^m \mathbb{Z}\varepsilon_i - \mathbb{Z}_{\geq 0}\Delta(\mathfrak{b})$ and $\lambda(\mathfrak{h} \cap \mathfrak{b}) \neq 0$.

(ii) The module $N$ is obtained from $\tilde{N}$ by a twisted localization with respect to some set of commuting roots $\Gamma \subset -\Delta(\mathfrak{b})$.

**Proof.** Let $\hat{\mathfrak{k}} = \mathfrak{k} + \mathfrak{h}$, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Then $N$ is a bounded weight $\hat{\mathfrak{k}}$-module, hence, any cyclic $\hat{\mathfrak{k}}$-submodule of $N$ has finite length (see Lemma 3.3 in [13]). Let $N_0$ be a simple $\hat{\mathfrak{k}}$-submodule of $N$. Note that $N_0$ is a cuspidal $\hat{\mathfrak{k}}$-module. By Proposition 4.8 in [13], there exist $\mu \in (\mathfrak{h} \cap \mathfrak{k})^*$ and $\Gamma \subset -\Delta(\mathfrak{b})$ such that $N_0 \simeq D_\mu^\mathfrak{k}M_0$ for some simple bounded $\mathfrak{b}$-highest weight $\hat{\mathfrak{k}}$-module $M_0$. Since $D_\mu^\mathfrak{k}$ is well defined for $\mathfrak{g}$-modules and commutes with the restriction functor $\text{Res}_\hat{\mathfrak{k}}^\mathfrak{k}$, $M := D_\mu^\mathfrak{k}N$ contains an $\mathfrak{n}$-primitive weight vector $v \in M_0$, while $\partial_i$ act injectively on $M$ for all $i = 1, \ldots, m$. Since $U(\hat{\mathfrak{k}})v$ is bounded it has finite $\hat{\mathfrak{k}}$-length and hence there is $\lambda' \in \text{supp } U(\hat{\mathfrak{k}})v$ such that $\lambda' + \alpha \notin \text{supp } U(\hat{\mathfrak{k}})v$ for all $\alpha \in \Delta(\mathfrak{b})$. The injectivity of the action of $\partial_i$ implies that
$$(\lambda' + \alpha + \sum_{i=1}^m \mathbb{Z}_{\geq 0}\varepsilon_i) \cap \text{supp } U(\mathfrak{g})v = \emptyset.$$ If $w$ is a nonzero vector of weight $\lambda'$ then
$$\text{supp } U(\mathfrak{g})w \subset \lambda' + \sum_{i=1}^m \mathbb{Z}\varepsilon_i - \mathbb{Z}_{\geq 0}\Delta(\mathfrak{b}).$$

This implies $\dim(U(\mathcal{O}_m \otimes n))u < \infty$ for any $u \in U(\mathfrak{g})w$. By Lemma 2.3, the boundedness of $U(\mathfrak{g})w$ implies that $U(\mathfrak{g})w$ has finite length. Let $\tilde{N}$ be a simple submodule of $U(\mathfrak{g})w$. Then there is a nonzero weight vector $u \in \tilde{N}$ annihilated by $\mathcal{O}_m \otimes \mathfrak{n}$. Then $\tilde{N}$ satisfies (i) with $\lambda$ being the weight of $u$, while (ii) follows from the simplicity of $N$.

It remains to show that $\lambda(\mathfrak{h} \cap \mathfrak{k}) \neq 0$. For the sake of contradiction, assume that the opposite holds. Take a simple root $\alpha \in \Delta(\mathfrak{b})$. Then a simple computation shows that $\mathfrak{g}_{\alpha}u$ is annihilated by $\mathcal{O}_m \otimes \mathfrak{n}$. The simplicity of $\tilde{N}$ hence implies that $\mathfrak{g}_{\alpha}u = 0$ for all simple roots $\alpha$ and thus $M$ contains a trivial $\mathfrak{k}$-submodule. But the roots of $\Gamma$ act injectively on $N$ and hence on $M$. This leads to a contradiction. □
For a weight $\mu \in (\mathfrak{h} \cap \mathfrak{k})^*$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{t}$, by $L_\mathfrak{b}(\mu)$ (or simply by $L(\mu)$) we denote the simple $\mathfrak{b}$-highest weight $\mathfrak{k}$-module of highest weight $\mu$.

**Lemma 5.14.** The module $\bar{N}$ constructed in Proposition 5.13 is isomorphic to $\mathcal{F}(T(P, V), L(\bar{\lambda}))$ for a cuspidal simple $\mathcal{D}_m$-module $P$, a simple finite-dimensional $\mathfrak{gl}(m)$-module $V$, and a simple highest weight $\mathfrak{k}$-module $L(\bar{\lambda})$, where $\bar{\lambda}$ is the restriction of $\lambda$ to $\mathfrak{h} \cap \mathfrak{t}$.

**Proof.** Consider $\gamma \in (\mathfrak{t} \cap \mathfrak{h})^*$ which determines the Borel subalgebra $\mathfrak{b}$. Then $\gamma$ determines also a parabolic subalgebra $\mathfrak{q}$ in $\mathfrak{g}$. The $\mathfrak{q}$-top of the module $\bar{N}$ is a simple $(\mathfrak{W}_m \oplus \mathfrak{h})$-module. Since $\bar{\lambda} \neq 0$, this module is isomorphic to $T(P, V) \otimes \mathbb{C}_{\bar{\lambda}}$ by Lemma 5.11. A simple computation shows that it is also isomorphic to the top of $\mathcal{F}(T(P, V), L(\bar{\lambda}))$. Hence the statement follows from Proposition 2.5(c).

**Corollary 5.15.** If $\mathfrak{t}$ is not abelian then $N$ is isomorphic to $\mathcal{F}(T(P, V), S)$ for some cuspidal simple $\mathcal{D}_m$-module $P$, a simple finite-dimensional $\mathfrak{gl}(m)$-module $V$, and a simple cuspidal $\mathfrak{k}$-module $S$.

**Proof.** The result follows immediately from Proposition 5.13, Lemma 5.14, and the isomorphism of $\mathfrak{g}$-modules

$$D_\mathfrak{t}^{-\mu}\mathcal{F}(T(P, V), L(\bar{\lambda})) \simeq \mathcal{F}(T(P, V), D_\mathfrak{t}^{-\mu}L(\bar{\lambda})).$$

**Theorem 5.16.** Let $N$ be a simple bounded $\mathfrak{g}$-module satisfying (1) and (2). Then we have one of the following two mutually exclusive statements.

(a) $\mathcal{O}_m \otimes \mathfrak{t}$ acts trivially on $N$ and $N$ is a unique simple submodule of $T(P, V)$ for some cuspidal simple $\mathcal{D}_m$-module $P$ and a simple finite-dimensional $\mathfrak{gl}(m)$-module $V$. In this case $\mathfrak{t}$ must be abelian.

(b) $N$ is isomorphic to $\mathcal{F}(T(P, V), S)$ for some cuspidal simple $\mathcal{D}_m$-module $P$, a simple finite-dimensional $\mathfrak{gl}(m)$-module $V$ and a simple nontrivial cuspidal $\mathfrak{t}$-module $S$.

**Proof.** If $\mathfrak{t}$ is not abelian the statement follows from Corollary 5.15. If $\mathfrak{t}$ acts nontrivially on $N$, the statement follows from Lemma 5.11. If $\mathfrak{t}$ acts trivially on $N$, then by Lemma 5.12 $(\mathcal{O}_m \otimes \mathfrak{t})N = 0$. Then $N$ is a simple bounded $\mathfrak{W}_m$-module and the statement is a consequence of Theorem 1.1 in [18].

6. Back to tensor modules via parabolic induction

6.1. The case of infinite-dimensional $\mathfrak{g}$. We retain the notation of Section 4 and assume again that $M$ is a simple weight $\mathfrak{W}_n$-module that is also the unique simple quotient of the parabolically induced module $U(\mathfrak{W}_n) \otimes_{U(\mathfrak{g})} N$, where $N$ is a simple bounded $\mathfrak{g}$-module satisfying (1) and (2). We will use the properties of $N$ listed in Theorem 5.16.
Recall that $p$ and $m$ are fixed and defined in §4. Let $p' = p \cap \mathfrak{gl}(n)$. The Levi subalgebra of $p'$ is isomorphic to $\mathfrak{t} \oplus \mathfrak{gl}(m)$. Consider a $\mathfrak{g}$-module $F(T(P,V),S)$ where $V$ is a finite-dimensional $\mathfrak{gl}(m)$-module, $P$ is a simple cuspidal $\mathcal{D}_m$-module and $S$ be a simple cuspidal $\mathfrak{t}$-module. Note that $S$ might be a trivial $\mathfrak{t}$-module in the case when $\mathfrak{t}$ is abelian. Let $U$ be the one-dimensional $\mathfrak{t}$-module of weight $\sum_{i=1}^p \varepsilon_i$ and $S^U = S \otimes U$. Finally, let $\hat{S}$ be the unique simple quotient of $U(\mathfrak{gl}(n)) \otimes_{U(p')} (S^U \otimes V)$.

Using the isomorphism

$$\mathcal{D}_n \simeq \mathcal{D}_p \otimes \mathcal{D}_m \otimes \mathcal{D}_{n-p-m},$$

define a $\mathcal{D}_n$-module $\tilde{P}$ by

$$\tilde{P} = \mathbb{C}[x_1, \ldots, x_p]^F \otimes P \otimes \mathbb{C}[x_{p+m+1}, \ldots, x_n],$$

(recall that $X^F$ is the full Fourier transform of $X$).

**Lemma 6.1.** The $p$-top of $T(\tilde{P}, \hat{S})$ is isomorphic to $F(T(P,V),S)$.

**Proof.** The statement follows by comparing the supports of the two modules. Let $p = \mathfrak{g} \oplus \mathfrak{n}$.

$$\text{supp} \tilde{P} = \sum_{i=1}^p \mathbb{Z}_{<0} \varepsilon_i + \text{supp} P + \sum_{i=p+m+1}^n \mathbb{Z}_{\geq 0} \varepsilon_i,$$

$$\text{supp} \hat{S} \subset \text{supp} S + \sum_{i=1}^p \varepsilon_i + \text{supp} V - \text{supp} U(n'),$$

where $n' = \mathfrak{n} \cap \mathfrak{gl}(n)$. Then we have that

$$\text{supp} F(T(P,V),S) \subset \text{supp} T(\tilde{P}, \hat{S}) \subset \text{supp} F(T(P,V),S) + \text{supp} U(n^-),$$

where $n^-$ is the nilradical of the opposite parabolic. Moreover, the multiplicity of any $\mu \in \text{supp} F(T(P,V),S)$ is the same as its multiplicity in $T(\tilde{P}, \hat{S})$. \qed

**Lemma 6.2.** Let $N = F(T(P,V),S)$ be a simple $\mathfrak{g}$-module. Then the unique simple quotient $M$ of $U(\mathcal{W}_n) \otimes_{U(p)} N$ is isomorphic to unique simple submodule of $T(\tilde{P}, \hat{S})$.

**Proof.** The isomorphism of $\mathfrak{p}$-modules $N \rightarrow T(\tilde{P}, \hat{V})^{\text{top}}$ induces a nonzero homomorphism of $\mathcal{W}_m$-modules $U(\mathcal{W}_n) \otimes_{U(p)} N \rightarrow T(\tilde{P}, \hat{V})$. The image of this homomorphism is simple since $T(\tilde{P}, \hat{V})$ has a unique simple submodule. Thus, this submodule is isomorphic to $M$. \qed

Now assume that $F(T(P,V),S)$ is not simple. This is only possible if $\mathfrak{t}$ is abelian, $S$ is trivial, and $V = \Lambda^k \mathbb{C}^m$.

**Lemma 6.3.** Assume that $N$ is the simple submodule $T(P, \Lambda^k \mathbb{C}^m)$ for some $k = 0, \ldots, m - 1$. Then the unique simple quotient $M$ of $U(\mathcal{W}_n) \otimes_{U(p)} N$ is isomorphic to the unique simple submodule of $T(\tilde{P}, \Lambda^{p+k} \mathbb{C}^m)$.
Proof. We consider the monomorphism of \( p \)-modules \( N \rightarrow T(\tilde{P}, \bigwedge^{p+k} \mathbb{C}^n)^{\text{top}} \), and the induced map
\[
U(\mathcal{W}_n) \otimes_{U(p)} N \rightarrow T(\tilde{P}, \bigwedge^{p+k} \mathbb{C}^n).
\]
To complete the proof, we use the same reasoning as the one in the proof of the previous lemma.

6.2. The case of finite-dimensional \( g \). In this case we have \( m = 0 \) and \( g \) is a Lie subalgebra of \( \mathfrak{gl}(n) \). Using arguments similar to the ones used in the previous subsection, one can show that the unique simple quotient of \( U(\mathcal{W}_n) \otimes_{U(p)} S \) is isomorphic to the unique simple submodule of \( T(\tilde{P}, \hat{S}) \).

6.3. The case \( g = \mathcal{W}_m \). This case follows from Theorem 2.10.

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