M2-M5 Systems in $\mathcal{N} = 6$ Chern-Simons Theory

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Abstract

We study two aspects of M5-branes in $\mathcal{N} = 6$ $U(N) \times U(N)$ Chern-Simons gauge theory. We first examine multiple M2-branes ending on a M5-brane. We study Basu-Harvey type cubic equations, fuzzy funnel configurations, and derive the M5-brane tension from the $\mathcal{N} = 6$ theory. We also find a limit in which the above M2-M5 system reduces to a D2-D4 system and we recover the Nahm equation from the $\mathcal{N} = 6$ theory. We then examine domain wall configurations in mass-deformed $\mathcal{N} = 6$ theory with a manifest $SU(2) \times SU(2) \times U(1)$ global symmetry. We derive tensions of domain walls connecting between arbitrary M5-brane vacua of the deformed theory and observe their consistency with gravity dual expectations.

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1 Introduction

Multiple M2-brane theory with a manifest $SO(8)$ R-symmetry was shown [1, 2] to be consistent with a totally antisymmetric 3-algebraic description. The only finite dimensional Euclidean 3-algebra assuming total antisymmetry was based on the $so(4)$ 3-algebra with a quantized 4-index structure constant [11, 5]. The corresponding theory can be presented as a $SU(2)_k \times SU(2)_{-k}$ Chern-Simons gauge theory [7, 8] coupled to 8 scalars and 8 fermions in bi-fundamental representations [7]. The theory was shown to arise from two M2-branes moving in an orbifold of transverse $R^8$ space [9], and reduce to a maximally supersymmetric multiple D2-brane theory in a large $k$ and large scalar vev limit [6, 9]. One-loop corrections to the couplings were considered in [41]. Generalizations to include an arbitrary higher rank non-abelian gauge symmetry lead to the Lorentzian 3-algebra [10, 20], but the corresponding theory contains ghost degrees of freedom due to the Lorentzian signature [10]. A ghost-removing procedure turns the theory into that of a dual description of the 3d maximally supersymmetric Yang-Mills theory [16, 17, 18, 19]. Besides, infinite dimensional 3-algebras also exist [40, 21].

An alternative method to include a higher rank gauge symmetry was obtained very recently by considering the $U(N)_k \times U(N)_{-k}$ Chern-Simons gauge theory coupled to four $\mathcal{N} = 2$ superfields in bi-fundamental representations [12]. The Lagrangian of the theory exhibits a manifest $SU(4)$ R-symmetry [13, 14, 15], see also [25, 26, 38, 24], and was proposed to arise from multiple M2-branes moving in a $Z_k$ quotient of the transverse $R^8$ space [12]. The theory was also shown to be consistent with a 3-algebraic description with a less antisymmetric structure constant [14].

The present paper is motivated by trying to understand better the properties of this new $\mathcal{N} = 6$ theory. A nice feature of the previous $\mathcal{N} = 8$ theory is that it admits the Basu-Harvey equation [4, 42] with a $SO(4)$ symmetry, and as a result, there are fuzzy funnel configurations describing multiple M2-branes gradually ending on a M5-brane wrapping a fuzzy 3-sphere. Another nice feature is that the $\mathcal{N} = 8$ theory also admits a mass deformation keeping a $SO(4) \times SO(4)$ global symmetry [34, 35, 36, 37, 22], which has multiple M5-brane vacua charaterized by M5-branes wrapping concentric fuzzy 3-spheres in two possible orthogonal $R^4$ spaces. In this paper, we will study these two aspects in the context of the $\mathcal{N} = 6$ theory.

The organization of this paper is as follows. In section 2.1, we derive Basu-Harvey type equations by the method of forming perfect squares combining the kinetic terms with F-terms or D-terms. Related discussion but with slightly different methods was given in [24, 25]. In section 2.2, we analyze properties of the fuzzy funnel solutions and derive the M5-brane tension from the $\mathcal{N} = 6$ theory. In section 2.3, we show a limit that the above Basu-Harvey equations reduce to Nahm equations describing D2-D4 systems, thus giving another consistency check. In section 3.1, we derive domain wall equations in the mass-deformed $\mathcal{N} = 6$ theory keeping a $SU(2) \times SU(2) \times U(1)$ global symmetry [24]. In section 3.2, we analyze properties of the domain walls and compute their tensions, which are consistent with gravity dual descriptions in terms of M5-brane actions. In section 4, we briefly draw conclusions.
2 Basu-Harvey configurations and M2-M5 system

2.1 Bogomol’nyi completion

We begin by examining the bosonic potential in $\mathcal{N} = 6$ $U(N) \times U(N)$ Chern-Simons theory and expressing it as a sum of several perfect squares. We basically follow the notation of [13], but use a different normalization condition for $U(N)$ generators $\text{tr}(T^a T^b) = (1/2)\delta^{ab}$. In this notation, the potential can be rewritten as

$$V_{\text{scalar}} = V_D + V_F$$

$$= \frac{4\pi^2}{k^2} \text{tr} \left( |Z^A Z^\dagger_A Z^B - Z^B Z^\dagger_A Z^A - W^{\dagger A} W_A Z^B + Z^B W_A W^{\dagger A}|^2 \right.$$  
$$+ |W^{\dagger A} W_A W^{\dagger B} - W^{\dagger B} W_A W^{\dagger A} - Z^A Z^\dagger_A W^{\dagger B} + W^{\dagger B} Z^\dagger_A Z^A|^2 \big)$$

$$+ \frac{16\pi^2}{k^2} \text{tr} \left( |\epsilon^{AC} \epsilon^{BD} W_B Z_C W_D|^2 + |\epsilon^{AC} \epsilon^{BD} Z_B W_C Z_D|^2 \right), \quad (1)$$

where $Z^A, W^{\dagger A}, A = 1, 2$ are the lowest components of four $\mathcal{N} = 2$ superfields respectively, and are all in the $(N, \overline{N})$ representations and have overall $U(1)$ charges +1. The classical vacuum moduli space can be determined by demanding all the squares to be zero simultaneously. In this theory there is an additional residual $Z_k$ symmetry which orbifolds the moduli space.

Next we want to consider Basu-Harvey type BPS equations, which have the dependence of only one of the spatial worldvolume coordinate, say $x^2 = s$. The equations can be obtained by combining the kinetic terms and potential terms in the Hamiltonian and rewriting it as a sum of perfect squares plus some topological terms.

There are two ways to make combinations. If we combine the kinetic terms with F-term potentials, we obtain

$$H = \int dx^1 ds \text{ tr} \left( |\partial_s Z^A|^2 + |\partial_s W^{\dagger A}|^2 + V_{\text{scalar}} \right)$$

$$= \int dx^1 ds \text{ tr} \left( |\partial_s W^{\dagger A} - \frac{4\pi}{k} \epsilon^{AC} \epsilon^{BD} Z_B W_C Z_D|^2 + |\partial_s Z^A - \frac{4\pi}{k} \epsilon^{AC} \epsilon^{BD} W^{\dagger B} Z^\dagger_C W^{\dagger D}|^2 \right.$$  
$$+ \frac{4\pi^2}{k^2} |Z^A Z^\dagger_A Z^B - Z^B Z^\dagger_A Z^A - W^{\dagger A} W_A Z^B + Z^B W_A W^{\dagger A}|^2$$

$$+ \frac{4\pi^2}{k^2} |W^{\dagger A} W_A W^{\dagger B} - W^{\dagger B} W_A W^{\dagger A} - Z^A Z^\dagger_A W^{\dagger B} + W^{\dagger B} Z^\dagger_A Z^A|^2 \big)$$

$$+ \frac{4\pi}{k} \epsilon^{AC} \epsilon^{BD} \int dx^1 \text{ tr} (Z^A W_B Z^D + W^{\dagger A} Z^\dagger_B W^{\dagger C} Z^\dagger_D) \quad (2)$$
or, if the kinetic terms are combined with D-term potentials, we get:

\[
H = \int dx^1 ds \left( |\partial_s W^\dagger A + \frac{2\pi}{k} (W^\dagger B W_B W^\dagger A - W^\dagger A W_B W^\dagger B - Z^B Z_B^\dagger W^\dagger A + W^\dagger A Z_B^\dagger Z^B)|^2 \right.
\]

\[
+ |\partial_s Z^A + \frac{2\pi}{k} (Z^B Z_B^\dagger Z^A - Z^A Z_B^\dagger Z^B - W^\dagger B W_B Z^A + Z^A W_B W^\dagger B)|^2
\]

\[
+ \frac{16\pi^2}{k^2} |\epsilon_{AC}\epsilon_{BD} W_B Z^C W_D|^2 + \frac{16\pi^2}{k^2} |\epsilon_{AC}\epsilon_{BD} Z^B W_C Z^D|^2
\]

\[
+ \frac{\pi}{k} \int dx^1 \text{tr}(W_A W^\dagger A W_B W^\dagger B - W^\dagger A W_A W^\dagger B W_B + 2W^\dagger A W_A Z^B Z^\dagger_B
\]

\[
- 2W_A W^\dagger A Z_B Z^\dagger_B + Z^A Z^\dagger_A Z^B Z^\dagger_B - Z^A Z^\dagger_A Z^B Z^\dagger_B). \tag{3}
\]

In each case, the last term is topological and doesn’t affect the dynamics in the bulk. So we get a set of BPS equations, which minimizes the energy in a given topological sector:

\[
\partial_s W^\dagger A - \frac{4\pi}{k} \epsilon_{AC}\epsilon_{BD} Z^B W_C Z^D = 0 \tag{4}
\]

\[
\partial_s Z^A - \frac{4\pi}{k} \epsilon_{AC}\epsilon_{BD} W^\dagger B Z^\dagger_C W^\dagger_D = 0 \tag{5}
\]

\[
Z^A Z_B^\dagger Z^B - Z^B Z_A^\dagger Z^A - W^\dagger A W_A Z^B + Z^B W_A W^\dagger A = 0 \tag{6}
\]

\[
W^\dagger A W_A W^\dagger B - W^\dagger B W_A W^\dagger A - Z^A Z^\dagger_A W^\dagger B + W^\dagger B Z^\dagger_A Z^A = 0 \tag{7}
\]

for the F-term combination, and

\[
\partial_s W^\dagger A + \frac{2\pi}{k} (W^\dagger B W_B W^\dagger A - W^\dagger A W_B W^\dagger B - Z^B Z_B^\dagger W^\dagger A + W^\dagger A Z_B^\dagger Z^B) = 0 \tag{8}
\]

\[
\partial_s Z^A + \frac{2\pi}{k} (Z^B Z_B^\dagger Z^A - Z^A Z_B^\dagger Z^B - W^\dagger B W_B Z^A + Z^A W_B W^\dagger B) = 0 \tag{9}
\]

\[
\epsilon_{AC}\epsilon_{BD} W_B Z^C W_D = \epsilon_{AC}\epsilon_{BD} Z^B W_C Z^D = 0 \tag{10}
\]

for the D-term combination, respectively. The topological term gives the energy of the configuration when the BPS equations are satisfied.

### 2.2 Fuzzy funnel solution and M5-brane tension

The new Basu-Harvey equation proposed in [24, 25] can be obtained by setting two complex scalars to be zero, and look at the non-trivial equations for the other two complex scalars. For example, we can set $W^\dagger A = 0$, and $Z^A \neq 0$ in (9). The scalar part of the Hamiltonian is given as a square term plus a topological term:

\[
H = \int dx^1 ds \text{tr}(\partial_s Z^A + \frac{2\pi}{k} (Z^B Z_B^\dagger Z^A - Z^A Z_B^\dagger Z^B)|^2)
\]

\[
+ \frac{\pi}{k} \int dx^1 ds \partial_s \text{tr}(Z^\dagger_A Z_B Z^\dagger_B - Z^A Z_B Z^\dagger_B). \tag{11}
\]

3
The first line gives a pair of BPS equations
\[ \partial_s Z^A + \frac{2\pi}{k} (Z^B Z_B^\dagger Z^A - Z^A Z_B^\dagger Z_B) = 0, \tag{12} \]
where \( A, B = 1, 2 \). As opposed to the original Basu-Harvey equation in [4] which has a manifest \( SO(4) \) symmetry, the equation (12) has a manifest \( SU(2) \times U(1) \) symmetry.

As was argued in [25], this equation preserves half of the supersymmetries of the theory. For a configuration on which this equation is satisfied, the energy of the system is given by
\[ E = \frac{\pi}{k} \int dx^1 \text{tr}(Z_A^\dagger Z^A Z_B^\dagger Z^B - Z^A Z_B^\dagger Z_B^\dagger Z^B), \tag{13} \]
\[ = 2 \int ds dx^1 \text{tr}(\partial_s Z_A^\dagger \partial_s Z_A). \tag{14} \]

We used the BPS equation (12) to obtain the second line.

To solve the BPS equation (12), we may separate the \( s \)-dependent and independent part:
\[ Z^A = f(s) G^A, \quad f(s) = \sqrt{\frac{k}{4\pi s}}, \tag{15} \]
where \( G^A \)'s are \( N \times \overline{N} \) matrices satisfying
\[ G^A = G^B G_B^\dagger G^A - G^A G_B^\dagger G^B. \tag{16} \]

This equation is solved in [24] (see also [25]). One can diagonalize \( G_1^\dagger \) using the \( U(N) \times U(N) \) transformations and find that the other matrix \( G_2^\dagger \) must be off-diagonal. The \( G_1^\dagger \)'s have some nice properties: For a \( N \) dimensional irreducible solution,
\[ (G_1^\dagger)_{m,n} = \sqrt{m - 1} \delta_{m,n}, \quad (G_2^\dagger)_{m,n} = \sqrt{N - m} \delta_{m+1,n}, \tag{17} \]
\[ G^1 G_1^\dagger = \text{diag} (0, 1, 2, \ldots, N - 1) = G_1^\dagger G^1, \tag{18} \]
\[ G^2 G_2^\dagger = \text{diag} (N - 1, N - 2, \ldots, 1, 0) \tag{19} \]
\[ G_2^\dagger G^2 = \text{diag} (0, N - 1, N - 2, \ldots, 1) \tag{20} \]
\[ G^A G_A^\dagger = (N - 1) 1_{N \times N}, \quad \text{tr}(G^A G_A^\dagger) = N(N - 1). \tag{21} \]

The eigenvalues of the matrices \( G^1 G_1^\dagger \) and \( G^2 G_2^\dagger \) may be interpreted as the squares of the radial positions of the points on a fuzzy 3-sphere projected onto 2 complex planes, respectively. Since there is a overall \( Z_k \) residual symmetry, the solution would describe a fuzzy \( S^3/Z_k \).

The energy formula (14) is expressed in terms of fields \( Z^A \), which is of mass dimension \( 1/2 \) and does not have the correct mass dimension \( -1 \) as a spatial coordinate. The correct normalization should reproduce the scalar kinetic term of the form,
\[ S_{\text{kinetic}} = -T_2 \int d^3 x \text{tr}(\partial_\mu X_A^\dagger \partial^\mu X^A), \tag{22} \]
where $T_2$ is the M2-brane tension and $X^A$ is the (complexified) spatial coordinate. This implies that we should relate $X^A$ and $Z^A$ by

$$X^A = \sqrt{\frac{1}{T_2}} Z^A. \quad (23)$$

Using this, we can define the radius averaged over each M2-brane as

$$R^2 = \frac{2 \text{tr}(X_A^\dagger X^A)}{N} = \frac{2(N - 1)}{T_2} f^2 \quad (24)$$

$$= \left( \frac{k(N - 1)}{2\pi T_2} \right) \cdot \frac{1}{s} \quad (25)$$

The factor of two in the numerator comes from our normalization condition $\text{tr}(T^a T^b) = (1/2) \delta^{ab}$. The radius vanishes for $N = 1$, and there are non-trivial fuzzy 3-spheres only for $N \geq 2$.

Combining all the above results, after some algebra, we obtain

$$E = T_2^2 \frac{N}{2\pi (N - 1)} \int dx^1 \left( \frac{2\pi^2}{k} \right) R^3 dR \quad (26)$$

$$= T_2^2 \frac{N}{2\pi (N - 1)} \int d^5 x. \quad (27)$$

The factor $k$ in the denominator represents the fact that this M5-brane is divided by the $Z_k$ orbifold action, and $\frac{2\pi^2}{k}$ is the volume of an $S^3/Z_k$ with a unit radius. So the M5-brane wraps an $S^3/Z_k$. The M5-brane tension predicted from the $\mathcal{N} = 6$ theory is

$$T_5 = T_2^2 \frac{N}{2\pi (N - 1)}. \quad (28)$$

The relation between M2-brane and M5-brane tension can also be derived in different ways, by matching the M-theory and type II string theory BPS spectrum [28], or by applying flux and Dirac quantization rules in eleven dimensions [29]:

$$T_5 = \frac{T_2^2}{2\pi}. \quad (29)$$

We see that for large $N$ including the numerical coefficient, (28) exactly agrees with the known result (29). The $1/N$ deviation is due to the fuzziness of the 3-sphere in the finite $N$ regime, and will disappear in the continuum limit for the fuzzy 3-sphere.

### 2.3 Basu-Harvey equations and reduction to Nahm equations

In this section we take a limit in which M2-brane theory reduces to D2-brane theory [30, 31] and show that the Basu-Harvey equation (12) studied in the last section reduces to a Nahm equation, which describes D2-D4 system.
We take a diagonal expectation value in one of the direction, for example, the direction labelled by 3 and expand the fields around the vacuum:

\[ Z^1 = (x^{10} + ix^{20})T^0 + X^1 + iX^2 \]  
\[ Z^2 = ((v + x^{30}) + ix^{40})T^0 + X^3 + iX^4 \]  

(30)

(31)

Here, \( x \)'s represent the \( U(1) \) part and \( T^0 = \frac{1}{\sqrt{2N}} \) 1 for normalization purpose, \( \text{tr}(T^0T^0) = 1/2 \). \( X \)'s take value on \( SU(N) \). We take \( N \) and \( v/k \) finite and fixed, and suppose \( v \) is large, and then we will neglect \( o(1/v) \) terms in the calculation below.

By plugging (30) into the BPS equation (12), we see that

\[ \partial_s Z^2 = \frac{2\pi}{k} (Z^2 Z_1^1 Z^1 - Z^1 Z_2^1 Z^2) \]  
\[ = \frac{2\pi v}{k\sqrt{2N}} [Z_1^1, Z^1] \]  
\[ = \frac{4\pi v}{k\sqrt{2N}} i[Z^1, X^2] \]  

(32)

(33)

(34)

\( U(1) \) part decouples from the equations and we simply set them to zero. \( SU(N) \) part implies

\[ \partial_s X^3 = \frac{4\pi v}{k\sqrt{2N}} i[X^1, X^2], \quad \partial_s X^4 = 0 \]  

(35)

where we compared hermitian and anti-hermitian parts respectively.

In the same way, we can calculate the other component equation

\[ \partial_s Z^1 = \frac{2\pi}{k} (Z_1^1 Z_2^2 Z^2 - Z_2^2 Z_1^1 Z^1) \]  
\[ = \frac{2\pi v}{k\sqrt{2N}} [Z^1, Z_2^2 + Z^2] \]  
\[ = \frac{4\pi v}{k\sqrt{2N}} ([X^1, X^3] + i[X^2, X^3]) \]  

(36)

(37)

(38)

So we get

\[ \partial_s X^1 = \frac{4\pi v}{k\sqrt{2N}} i[X^2, X^3] \]  
\[ \partial_s X^2 = \frac{4\pi v}{k\sqrt{2N}} i[X^3, X^1] \]  

(39)

(40)

Combining the above results, we get

\[ \partial_s X^i = i\frac{1}{2} g_{YM} \epsilon^{ijk}[X^j, X^k] \]  

(41)

where \( i, j, k = 1, 2, 3 \) and \( \epsilon^{ijk} \) is the totally antisymmetric tensor. By using \( g_{YM} = 4\pi v/k\sqrt{2N} \) as in the M2 to D2 reduction [30, 31] for the \( N = 6 \) theory, we get the Nahm equation with the exact coefficient, in the large \( v \) and large \( k \) limit, with \( N \) and \( v/k \) fixed and finite. This describes multiple D2-branes ending on a D4-brane wrapping an \( S^2 \), and the reduction process makes an \( S^2/Z_k \) reducing to an \( S^2 \) that the D4-brane wraps.
3 Domain wall configurations and M2-M5 system

3.1 Domain wall equations

In this section, we turn to the discussion of another aspect of the M5-branes in the $\mathcal{N} = 6$ theory. For the $\mathcal{N} = 8$ M2-brane theory on flat space, we can turn on four fermion mass terms, which preserve at least $\mathcal{N} = 2$ supersymmetry. The most symmetric mass deformation is the one preserving a $SO(4) \times SO(4)$ symmetry \cite{34,35,36} and a $SU(2|2) \times SU(2|2)$ superalgebra. In this case, M5-branes can wrap either of the two geometric $S^3$s in orthogonal $R^4$s.

In the case of $\mathcal{N} = 6$ formulation, the most symmetric mass deformation turns out to preserve a manifest $SU(2) \times SU(2) \times U(1)$ symmetry \cite{24} (see also related discussion \cite{38,39}) and we expect to have a $SU(2|2) \times SU(1|1)$ superalgebra. While, in this case, M5-branes can wrap either of the two possible geometric $(S^3/Z_k)$s, where the $Z_k$ action is due to the residual symmetry, which squash the 3-spheres along their Hopf fiber directions while maintaining a manifest $SU(2) \times U(1)$ symmetry, as in \cite{12}.

We can turn on a D-term deformation corresponding to adding a FI term as found in \cite{24}. In our notation, we have the deformed potential

$$V_{\text{scalar}} = V_D + V_F$$

$$= \frac{4\pi^2}{k^2} \text{tr}(\frac{k}{2\pi} \mu |Z^B + Z^A Z^B_B - Z^B Z^A_B - W^A W_B^B + Z^B W_A W^A|)^2$$

$$+ |\frac{k}{2\pi} \mu W^B + W^A W_B W^B - W^B W_A W^A - Z^A Z^B W^B + W^B Z^A Z^A|^2$$

$$+ \frac{16\pi^2}{k^2} \text{tr}(|\epsilon_{AC} \epsilon^{BD} W_B Z^C W_D|^2 + |\epsilon^{AC} \epsilon_{BD} Z^B W_C Z^D|^2)$$

where $\mu$ is a canonical mass parameter.

We perform the Bogomol’nyi completion combining the kinetic terms and D-terms similar to \cite{3}, and we get

$$H = \int dx^1 d\Sigma (|\partial_s W^A - \mu W^A + \frac{2\pi}{k} (W^B W_B W^A - W^A W_B W^B - Z^B Z^A - W^A W_B Z^B - Z^A Z^B W^B + Z^A W_B W^B)|)^2$$

$$+ |\partial_s Z^A - \mu Z^A + \frac{2\pi}{k} (Z^B Z^A - Z^A Z^B - W^B W_B Z^A + Z^A W_B W^B)|)^2$$

$$+ \frac{16\pi^2}{k^2} |\epsilon_{AC} \epsilon^{BD} W_B Z^C W_D|^2 + \frac{16\pi^2}{k^2} |\epsilon^{AC} \epsilon_{BD} Z^B W_C Z^D|^2)$$

$$+ \frac{\pi}{k} \int dx^1 \text{tr}(W^A W_B W^B - W^A W_B W^B W_B + 2W^A W_A Z^B Z^B - 2W^A W^A Z^B Z^B + Z^A Z^B Z^B - Z^A Z^B Z^B)$$

$$+ \int dx^1 \text{tr}(\mu W^A W_A + \mu Z^A Z^A)$$

New boundary topological terms are produced at the same time when the BPS equations are modified.
The BPS domain wall equations are

\[
\partial_s W^A - \mu W^A + \frac{2\pi}{k}(W^B W^A W^B - W^A W^B W^A - Z_B^A W^A + W^A Z_B^A) = 0 \quad (44)
\]

\[
\partial_s Z^A - \mu Z^A + \frac{2\pi}{k}(Z_B^A Z^A - Z_B^A Z_B^A - W^B W^A + Z^A W^B W^A) = 0 \quad (45)
\]

\[
\epsilon_{AC} \epsilon_{BD} W_B Z_C W_D = \epsilon_{AC} \epsilon_{BD} Z_B W_C W_D = 0. \quad (46)
\]

The equations are modified by just adding the linear terms.

### 3.2 Domain wall solutions and their tensions

In this section we discuss solutions of these domain wall configurations and derive their tensions. Setting \( W^A = 0 \) in equations (44)-(46), we need to solve

\[
\partial_s Z^A - \mu Z^A + \frac{2\pi}{k}(Z_B^A Z^A - Z_B^A Z_B^A - W^B W^A + Z^A W^B W^A) = 0 \quad (47)
\]

We assume the ansatz

\[
Z^A = h(s) G^A, \quad G^A = G_B^B G^A - G^A G_B^B \quad (48)
\]

\[
\partial_s h - \mu h + \frac{2\pi}{k} h^3 = 0 \quad (49)
\]

We then obtain two solutions

\[
h_1(s) = \sqrt{\frac{k\mu}{2\pi (1 - e^{-2\mu s})}} \quad (50)
\]

\[
h_2(s) = \sqrt{\frac{k\mu}{2\pi (1 + e^{-2\mu s})}} \quad (51)
\]

The first solution \( h_1 \) describes a fuzzy funnel where \( s \in (0, \infty) \), and in the \( \mu \to 0 \) limit reproduces (15). The second solution \( h_2 \) is a domain wall solution where \( s \in (-\infty, \infty) \). We have

\[
h_2(-\infty) = 0, \quad h_2(\infty) = \sqrt{\frac{k\mu}{2\pi}} \quad (52)
\]

so this domain wall solution

\[
Z^A = \sqrt{\frac{k\mu}{2\pi (1 + e^{-2\mu s})}} G^A \quad (53)
\]

connects a trivial vacuum with a nontrivial fuzzy sphere vacuum \( \sqrt{\frac{k\mu}{2\pi}} G^A \).
The non-vanishing boundary terms when $W^+ A = 0$ are
\[
H = \int dx^1 ds \partial_s \text{tr}(\mu Z^A Z^A) + \frac{\pi}{k} \int dx^1 ds \partial_s \text{tr}(Z^A_B Z^+ B Z - Z^A Z^+ B Z^A_B) \tag{54}
\]
\[
= \int dx^1 \text{tr} \left( \frac{1}{2} \mu Z^A Z^A \right)_{s=\infty}^{s=-\infty} = 2 \int dx^1 ds \text{tr}(\partial_s Z^A \partial_s Z^A) \tag{55}
\]
\[
= \int dx^1 \frac{k \mu^2}{4\pi} \text{tr}(G^A G^A)_{s=\infty}^{s=-\infty} \tag{56}
\]
\[
= \int dx^1 \frac{k}{4\pi} \mu^2 N(N-1) \tag{57}
\]
where in deriving the second line in (55) we have used the equation of motion (47) to simplify
\[
\frac{\pi}{k} (Z^A Z^B Z^B Z^A Z^A Z^A) = -\frac{1}{2} \mu Z^A Z^A + \frac{1}{2} (\partial_s Z^A) Z^A \tag{58}
\]
and used the fact that $\frac{1}{2} (\partial_s Z^A) Z^A$ vanishes for both $s = -\infty$ and $s = \infty$.

Thereby the tension of this domain wall is
\[
\tau = \frac{k}{4\pi} \mu^2 N(N-1) \tag{59}
\]
It agrees with other results for slightly different theories as discussed in [1], and the second ref. in [22].

Since (58), (55) are the general results for general domain wall solutions, we see that the expression (56) should be a general result for the tension of a domain wall between two arbitrary vacua labelled by integers $\{ N_i |_{s=-\infty}, i = 1, ..., p' \}$, $\{ N_i |_{s=\infty}, i = 1, ..., p \}$, in which the integers label the dimensions of irreducible solutions of the $p'$ and $p$ diagonal-block matrices in $G^A |_{s=-\infty}$ and $G^A |_{s=\infty}$ respectively. The tension of the domain wall between these two arbitrary vacua is therefore
\[
\tau = \frac{k \mu^2}{4\pi} \sum_{i=1}^{p} N_i (N_i - 1) |_{s=\infty} - \frac{k \mu^2}{4\pi} \sum_{i=1}^{p'} N_i' (N_i' - 1) |_{s=-\infty} \tag{60}
\]

The dependence of (59) on mass and $N$ also agrees with the gravity dual analysis in [32] based on computing the action of a M5-brane filling a 4-ball bounded by the 3-sphere on which the M5-brane constructed from M2-branes wraps. The probe M5-brane is also along the $R^{1,1}$ part of the M2-brane worldvolume directions. This computation can also be performed by calculating the action of a M5-brane wrapping a $S^3$ as well as the $x_2$ line-segment across the fermion band at $y = 0$ in the gravity geometry in [36], [37]. In this gravity picture, it is suggestive that if the fermion band is narrow, the M5-brane action is expected to be small.

### 4 Conclusions and discussion

In this paper we have studied two problems of M5-branes in the $\mathcal{N} = 6$ theory. We analyzed the Basu-Harvey type equations and found evidence that the equations describe multiple M2-branes ending on a M5-brane, which wraps on a fuzzy 3-sphere. We
derived the tension of M5-brane and it exactly agrees with the known result in large N limit. We also found that the 3-sphere is orbifolded by a $Z_k$ action as the volume of the M5-brane is suppressed by $1/k$. This is also consistent with the $SU(2) \times U(1)$ symmetry of the equations. We also derived the Nahm equation describing D2-branes ending on a D4-brane wrapping an $S^2$ starting from the above Basu-Harvey type equations and taking a large $k$ limit, providing further evidence for consistency.

We then turned to another situation where M5-branes wrapping on fuzzy 3-sphere emerge as the vacua of the mass-deformed $\mathcal{N} = 6$ theory. We find domain wall solutions and computed their tensions, in agreement with known gravity analysis, thereby adding another evidence for the existence of the M5-branes in the $\mathcal{N} = 6$ theory.

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