Maximizing the Sum of Radii of Disjoint Balls or Disks

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Abstract

Finding nonoverlapping balls with given centers in any metric space, maximizing the sum of radii of the balls, can be expressed as a linear program. Its dual linear program expresses the problem of finding a minimum-weight set of cycles (allowing 2-cycles) covering all vertices in a complete geometric graph. For points in a Euclidean space of any finite dimension $d$, this graph can be replaced by a sparse subgraph obeying a separator theorem. This graph structure leads to an algorithm for finding the optimum set of balls in time $O(n^{2-1/d})$, improving the $O(n^3)$ time of a naive cycle cover algorithm.

1 Introduction

Researchers of map labeling (the placement of nonoverlapping text labels on maps or other visualizations) have studied various problems of assigning shapes of maximum size to a given set of points in the plane [3,13,10,21]. One simple case of this problem, finding circles centered at the given points that maximize the minimum radius, can be solved by finding the closest pair of points and setting all radii equal to half of this pair’s distance. However, this measure of solution quality penalizes the label sizes even for points far away from the closest pair, where larger radii could be used without overlap. On the other hand, an $L_2$ measure of solution quality (maximizing the sum of disk areas) would also be unsatisfactory: even with only two points to be labeled, the $L_2$ solution would assign zero radius to one point. In this paper, we study the $L_1$ version of this problem, in which we are given as input $n$ points and must maximize the sum of radii of disjoint disks centered at those points. Two points can be optimally labeled by two equal-radius disks, and this criterion has the largest value of $p$ among $L_p$ criteria (maximizing sums of $p$th powers of the radii) that allow this equal-radius solution.

The same problem can be generalized to arbitrary metric spaces, with some care about definitions. We may specify a ball by its center (a point in the space) and radius (a real number); we consider balls to be distinct when they have different centers or radii, even when they contain the same sets of points. For a geodesic metric space (in which every two points can be joined by an isometrically embedded line segment), a pair of balls has a nonempty intersection if and only if their sum of radii equals or exceeds the distance of their centers. In other metric spaces (for instance finite metric spaces) two balls may have larger sum of radii than their center distance without intersecting. Nevertheless we will say that two balls in an arbitrary metric space overlap when their sum of radii exceeds the center distance, and are nonoverlapping otherwise. If the sum of radii equals the center distance, we say that the two balls touch.

Then in any metric space, finding a set of nonoverlapping metric balls with $n$ given center points $p_i$, maximizing the sum of non-negative radii $r_i$, can be expressed as a linear program. The objective is to maximize the linear function $\sum r_i$, subject to constraints that each pair of balls remain nonoverlapping, i.e., that $r_i + r_j \leq d(p_i, p_j)$.

This linear program has two variables per constraint, a well-studied special case of linear programming. But although strongly polynomial algorithms for feasibility with two variables per constraint are known [4,12], they do not extend to optimization, and their running time is higher than might be desired. Therefore, it remains of interest to find a purely combinatorial algorithm for the problem, with as low a running time as possible.

In this paper, we provide such a combinatorial algorithm, running in cubic time for general metrics and subquadratic time for low-dimensional Euclidean spaces.

1.1 New results

We prove the following results:

- Finding metric balls with maximum sum of radii is equivalent under linear programming duality to finding a minimum-length set of cycles (allowing 2-cycles) that cover all vertices of the complete geometric graph on the given centers. The maximum sum of radii equals half of the minimum total cycle length. By reducing cycle covers to weighted matching, the optimal set of balls in any metric space can be constructed in cubic time.

- For points in Euclidean spaces of bounded dimension $d$, the edges of the optimal cycle cover can be found in a subgraph of the complete geometric graph, the intersection graph of nearest neighbor balls of the points. This graph is sparse, having $O(n)$ edges with a constant of proportionality depending singly-exponentially in the dimension. Moreover, it obeys a...
Along with the map labeling research discussed above, we must find non-negative weights for the edges of a complete geometric graph such that each vertex has incident edge weights totaling at least one, and minimizing the weighted sum of edge lengths.

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1.2 Related research

Along with the map labeling research discussed above, researchers have studied other types of geometric optimization problems in which the optimization criterion is a sum of radii of balls or circles. These include finding a set of \( k \) balls with given centers, drawn from a larger set of \( n \) points, that cover all points and minimize the sum of radii of the balls \([10]\), finding a connected set of disks in the plane with given centers that minimize the sum of radii \([2]\), and finding both a collection of disks centered at a subset of input points that covers all input points, and a tour connecting the disk centers, minimizing a combination of disk radii and tour length \([1]\).

2 Equivalence to cycle cover

Let \( p_i \) (\( i = 0, \ldots, n - 1 \)) be a set of points in a metric space, with distances \( d(p_i, p_j) \). The maximum sum of radii problem can be expressed as a linear program:

\[
\text{maximize } \sum r_i
\]

subject to the inequality constraints

\[
\forall i : r_i \geq 0
\]

\[
\forall i, j : r_i + r_j \leq d(p_i, p_j).
\]

By linear programming duality \([20]\) Ch. 12 this has the same value as the dual linear program:

\[
\text{minimize } \sum w_{ij} d(p_i, p_j)
\]

subject to the inequality constraints

\[
\forall i, j : w_{ij} \geq 0
\]

\[
\forall i : \sum_j w_{ij} \geq 1
\]

That is, we must find non-negative weights \( w_{ij} \) for the edges of a complete geometric graph such that each vertex has incident edge weights totaling at least one, and minimizing the weighted sum of edge lengths.

Lemma 1 There is an optimal solution to the dual linear program described above in which the incident edge weights at each vertex total exactly one.

Proof. Define the excess of a vertex to be the total weight of its incident edges minus one. If any edge has weight greater than one, its weight can be reduced to exactly one, improving the quality of the solution without changing its feasibility. Otherwise, if any vertex has positive excess, we can subtract an equal positive weight from two of its incident edges and add the same weight to the edge between the other endpoints of these edges. By the triangle inequality, this change does not worsen the solution, and again it does not change its feasibility. By making such changes we can reduce the total excess, maintaining feasibility, until eventually all vertices have excess zero.

This dual program (either in the form first given above, or with the restriction that the weights at each vertex sum to exactly one according to Lemma 1) is the linear programming relaxation of the problem of finding a minimum weight perfect matching in the complete geometric graph. Such a relaxation is primarily used for bipartite graphs, for which it is exact \([20]\) Ex. 12.7. For non-bipartite graphs such as the complete graph, it has a half-integral optimal solution in which all weights \( w_{ij} \) belong to \( \{0, 1/2, 1\} \) \([20]\) Ex. 14.8. Doubling the weights to make them integers, and interpreting each doubled weight as an edge multiplicity, gives us a combinatorial description of the dual solution as a multiset of edges in which (by Lemma 1) each vertex has degree two. That is, we have the following result:

Theorem 2 The maximum sum of radii of nonoverlapping balls, centered at points \( p_i \) of a metric space, equals half of the minimum total edge length of a collection of vertex-disjoint cycles (allowing 2-cycles) spanning the complete geometric graph on the points \( p_i \), with edge lengths equal to the distances between the edge endpoints.

We call such a collection of cycles a minimum cycle cover. Theorem 2 can be expressed more briefly as stating that half of the minimum cycle cover length

![Figure 1: Nonoverlapping disks that (by Corollary 4) maximize the sum of radii for their centers.](image-url)
equals the maximum sum of radii of nonoverlapping balls. There always exists a minimum cycle cover in which all cycles of more than two edges have odd length, for any long even cycle can be partitioned into two disjoint matchings, and at least one of these matchings gives a covering of the same vertices by 2-cycles that is at least as good. Therefore, from now on we will assume that our cycle covers have no long even cycles.

This result also gives us an easy-to-test optimality condition for the maximum sum of radii problem, that applies to inputs in general position (without extra touching balls beyond the ones required by the solution). In what follows, the touching graph of a family of nonoverlapping balls has a vertex for each ball and an edge connecting each two balls that touch. It may differ from the intersection graph in spaces that are not geodesic.

**Corollary 3** Suppose that no two balls in a given family of balls overlap, and that each connected component of the touching graph of the balls is an odd cycle or an isolated edge. Then this family of balls has the maximum sum of radii of any family with the same centers.

**Proof.** The touching graph of the balls (viewing each isolated edge as a 2-cycle) gives a cycle cover of length half the sum of radii of the given balls. By Theorem 2 no cycle cover can be shorter and no system of balls with the same centers can have a larger sum of radii. \( \square \)

Figure 1 shows a family of disks meeting the conditions of the corollary, together with their touching graph.

### 3 Cycle covers from matchings

The **bipartite double cover** \( 2G \) of a graph \( G \) is the tensor product \( G \times K_2 \), a bipartite graph with two copies of each vertex of \( G \) (one of each color) and two copies of each edge of \( G \) (one for each pair of oppositely-colored copies of the endpoint of the edge). We use the same edge weights in \( G \) and \( 2G \).

**Lemma 4** Every perfect matching in \( 2G \) corresponds (under the mapping that takes each vertex of \( 2G \) to the corresponding vertex in \( G \)) to a cycle vertex cover with equal total length in \( G \). Every cycle vertex cover in \( G \) comes from a perfect matching in \( 2G \) in this way.

See Figure 2 for an example. Each cycle of more than two vertices in \( G \) has two different representations as a set of matched edges in \( 2G \), but this ambiguity is not a problem. We can find a minimum cycle vertex cover in \( G \) by finding a minimum weight perfect matching in \( 2G \).

A minimum weight perfect matching on a bipartite graph of \( n \) vertices and \( m \) edges can be found in \( O(mn + n^2 \log n) \) \( [9] \). (Better times are known when the weights are small integers \( [6] \).) By solving this problem on the doubled complete geometric graph, the optimal sum of radii can be computed in time \( O(n^3) \). However, this is still slower than we would like for Euclidean spaces, and does not yet tell us the radii of the individual balls.

### 4 From cycle covers to balls

Each odd cycle in a minimum cycle cover of the complete geometric graph corresponds to a unique system of balls that maximizes the sum of radii for those points. Let the cycle have vertices \( p_0, p_1, \ldots, p_{k-1} \) and edge lengths \( \ell_i = d(p_i, p_{i+1 \mod k}) \). Then for any \( j \) with \( 0 \leq j < i \) set

\[
r_j = \sum_{i<j} \pm \ell_i / 2,
\]

choosing the signs so that the two edges adjacent to \( p_j \) have positive sign and every other point \( p_i \) is incident to edges of opposite sign.

We omit the proof that this formula gives us a unique solution, because instead we use a more general solution that also provides radii for the points in 2-cycles. The 2-cycles in the cycle cover cause us more trouble, because their radii may be constrained by other nearby points but are not in general uniquely determined. For instance, in Figure 3 each of the three pairs of touching circles must have nearly-equal radii to avoid overlaps with nearby circles.

To describe how we transform a minimum weight perfect matching problem on \( 2K_4 \) into a feasible system of balls with maximum sum of radii, we need to look

![Figure 2: The correspondence between a cycle cover of a graph \( G \) (left, in this case, a complete graph \( K_4 \)) and a matching in \( 2G \) (right).](image1)

![Figure 3: Pairs of disks corresponding to 2-cycles in the minimum cycle cover may constrain the radii of other nearby pairs of disks.](image2)
more deeply into the details of the Hungarian algorithm for minimum weight perfect matching. The version of this algorithm described by Tarjan [17, Secs. 8.4 & 9.1] maintains a matching (initially empty) which it uses to orient the graph. Unmatched edges are directed from one side of the bipartition to the other, and matched edges are directed in the opposite direction. In each iteration the algorithm finds a minimum-cost alternating path between two unmatched vertices; here, the cost of a path is the sum of lengths of unmatched edges, minus the sum of lengths of matched edges. It uses this path to extend the matching by one edge.

Because matched edges are subtracted from the path length, each path search involves negative-weight edges. However, the algorithm also maintains a system of non-negative weights that are equivalent (in the sense of having the same shortest paths), allowing Dijkstra’s algorithm to be used, and adjusts these weights to keep them non-negative after each iteration. Each iteration increases the number of matched edges, so there are $O(n)$ iterations. The time per iteration can be bounded by the time for Dijkstra’s algorithm. Using Fibonacci heaps this gives a total running time of $O\left( n^2 \log n \right)$ [9].

To adjust edge weights we maintain dual variables: a real number for each vertex of the bipartite graph. We subtract these variables from the length of each incident unmatched edge, and add these variables to the (negated) length of each incident matched edge. In order for the adjusted edge weights to be non-negative, the algorithm maintains an invariant that the length of each unmatched edge is at least the sum of the dual variables at its endpoints, and each matched edge length equals the sum of the dual variables at its endpoints.

Recall that the graph $2K_n$ to which we apply this algorithm has two vertices for each input point $p_i$, one of each color (say red and blue). We may visualize the dual variables at these two vertices as two balls (again, one red and one blue) both centered at point $p_i$. Then the invariants maintained by the Hungarian algorithm can be expressed in our terms as stating that pairs of balls of opposite colors cannot overlap unless they have the same center, and that each matched edge comes from a touching pair of oppositely colored balls (Figure 4). However, this visualization may be somewhat misleading, as we allow the dual variables to be negative.

**Lemma 5** If we are given the dual variables found by the Hungarian algorithm for minimum weight perfect matching in the bipartite graph $2K_n$, we can construct from them a system of nonoverlapping balls with maximum sum of radii, in linear time.

**Proof.** Let the two dual variables for point $p_i$ be $a_i$ and $b_i$, and calculate the radius of the ball centered at $p_i$ to be the average $r_i = (a_i + b_i)/2$ of the two dual variables. For each $i$ and $j$, we have (by assumption) $a_i + b_j \leq d(p_i, p_j)$ and $b_i + a_j \leq d(p_i, p_j)$, and averaging these two inequalities gives us that $r_i + r_j \leq d(p_i, p_j)$. That is, the balls with radius $r_i$ are non-overlapping.

Each edge of the cycle cover of $K_n$ has two balls of opposite color at its endpoints whose radii sum to the edge length. For each cycle of the cycle cover of $K_n$, each ball centered at a cycle vertex will contribute to this equality for exactly one of the two incident cycle edges, so the sum of the radii of the blue and red balls centered at the cycle vertices equals the length of the cycle. Therefore, when we average these red and blue radii to give the radii $r_i$ of a single system of balls, the sum of the averaged radii equals half the length of the cycle. Thus, over the whole input, the sum of all the radii $r_i$ equals half the length of the cycle cover. The optimality of this system of nonoverlapping balls then follows from Theorem 2.

Combining Lemma 4, Lemma 5, and the known algorithms for minimum weight perfect matching in bipartite graphs gives us the following result.

**Theorem 6** Given $n$ points in an arbitrary metric space, we can construct a system of nonoverlapping balls centered at those points, maximizing the sum of radii of the balls, in time $O(n^3)$.

## 5 Euclidean speedup

To speed up the search for balls with maximum sum of radii in low-dimensional Euclidean spaces, we replace the complete geometric graph of the previous sections with a sparser graph that contains the optimal cover and behaves the same with respect to testing whether systems of balls are nonoverlapping. Given a system of points $p_i$ in a metric space, let the nearest neighbor distance $\delta_i$ of each point $p_i$ be

$$\delta_i = \min_{j \neq i} d(p_i, p_j).$$

In Euclidean spaces of bounded dimension, these distances can be found for all points in total time
The ply of a system of (overlapping) balls is the maximum number of balls that have a nonempty intersection. The kissing number of a space is the maximum number of unit balls that can touch a single unit ball; it is single-exponential in the dimension for Euclidean spaces. The following lemma is from [18, Lem. 3.2].

Lemma 9 The nearest neighbor balls of a finite set of points in d-dimensional Euclidean space have ply at most the kissing number of the space.

It follows that \( \mathcal{N} \) is sparse, having only \( O(n) \) edges. We can measure this more precisely by its degeneracy, the minimum number \( \Delta \) such that the vertices of the graph can be ordered into a sequence with each vertex having at most \( \Delta \) neighbors that are later than it in the sequence. The following lemma is from [18, Thm. 4.2].

Lemma 10 The intersection graph of any system of balls with bounded ply has bounded degeneracy.

This immediately gives a speedup from \( O(n^3) \) to \( O(n^2 \log n) \), by applying the \( O(m n + n^2 \log n) \) time bound for matching on the sparse bipartite graph \( 2\mathcal{N} \) and using the degeneracy bound to substitute \( m \leq n \Delta = O(n) \). But we can do better using more graph structure. Define a separator of an \( n \)-vertex graph to be a subset of the vertices the removal of which allows the graph to be partitioned into two subgraphs, disconnected from each other and each having at most \( s n \) vertices for a constant \( s < 1 \). The following result is from [8]:

Lemma 11 The intersection graphs of systems of balls with bounded ply in Euclidean spaces of dimension \( d \) have separators of size \( O(n^{1 - 1/d}) \) that can be found deterministically in linear time from the balls. By recursively constructing a hierarchy of separators, the intersection graph of a system of \( n \) balls can be found from the balls in time \( O(n \log n) \).

Another way of expressing the existence of small separators (of size a fractional power of \( n \)) is that the intersection graphs of bounded-ply systems of balls, such as \( \mathcal{N} \), are graphs of polynomial expansion [7,15]. We remark that any hierarchy of separators for \( \mathcal{N} \) can be transformed into a hierarchy of separators for \( 2\mathcal{N} \), simply by including both copies of each vertex of a separator for \( \mathcal{N} \) in the corresponding separator for \( 2\mathcal{N} \).

An efficient algorithm for weighted matching in graphs with small separators is given by [14, Sec. 7]. They directly consider only planar graphs, but their method (if it works) would also work for other graphs that have separator hierarchies. However, they consider a different variant of matching, maximum weight matching on non-bipartite graphs, and are not explicit in how they maintain and update the analogue for that problem of the dual variables that we need. They also do not take
advantage of more recent advances in shortest path algorithms for graphs with separators [11]. In an appendix we describe an algorithm based on similar ideas that maintains the dual variables that we need and is faster by a logarithmic factor, proving the following result:

**Lemma 12** Let $G$ be a bipartite graph given together with a separator hierarchy in which each subgraph of $p$ vertices has a separator of size $O(p^c)$, for some constant $c$ with $0 < c < 1$. Then one can find both a minimum weight perfect matching of $G$, and the dual variables of the matching, in time $O(n^{1+c})$.

Putting the results of this section together, we have our main theorem:

**Theorem 13** Given $n$ points in a Euclidean space of constant dimension $d \geq 2$, we can construct a system of balls centered at the given points, with maximum sum of radii, in time $O(n^{2-1/d})$.

**Proof.** We find the nearest neighbors of all points, construct graph $N$ and its separator hierarchy, apply the matching algorithm of Lemma 12 to $2N$ (using separators formed by doubling the separators in $N$), and average the dual variables from the two copies of each input point to give a single radius for each point. □

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A Weighted matching in separated bipartite graphs

In this appendix we prove Lemma 12 on the existence of an efficient minimum weight perfect matching algorithm for bipartite graphs obeying a separator theorem.

Suppose we are given a weighted graph $G$ with a separator $S$ (part of a separator hierarchy for $G$), such that $G \setminus S$ can be partitioned into two subgraphs $L$ and $R$, disconnected from each other, with $L$ and $R$ both having a number of vertices that is smaller by a constant factor. For such a graph, Lipton and Tarjan [14, Sec. 7] describe a divide-and-conquer algorithm for maximum-weight matching (an equivalent problem to minimum-weight perfect matching) that recursively matches $L$ and $R$, and then adds the separator vertices one at a time to the union of $L$ and $R$. When each separator vertex is added, the maximum matching may change along a single alternating path.

Lipton and Tarjan write that “given a suitable representation”, each alternating path can be found in the time for a single application of Dijkstra’s algorithm. This, in turn, allows the most expensive term in the running time of their algorithm to be bounded by the product of the time for finding each path with the number of separator vertices. The “suitable representation” that they refer to is made more complicated by the possibility that $G$ might not be bipartite, but in the bipartite case it is essentially the system of dual variables that we need. However, their handling of these variables is lacking in detail, so we provide here a more explicit divide and conquer matching algorithm, with the following differences:

- We formulate the problem as finding a minimum-weight maximum-cardinality matching in a graph with non-negative edge weights. This slight generalization of minimum-weight perfect matching allows us to perform recursive calls without needing to guarantee that each subproblem has a perfect matching, and simplifies the definition of the dual variables relative to their definition for maximum-weight matching.

- We recurse on the induced subgraphs $G[S \cup L]$ and $G[S \cup R]$ rather than on $L$ and $R$. This inclusion of the separator vertices in the recursive subproblems results in the calculation of two inconsistent systems of dual variables for these vertices, which must be reconciled when the subproblem solutions are combined.

- We compute shortest paths using an algorithm of Henzinger et al. rather than by Dijkstra’s algorithm. This algorithm takes linear time per path, after a separator hierarchy has already been computed, saving a logarithmic time factor compared to Dijkstra.

- We consider only bipartite graphs rather than the more general case of weighted matching in arbitrary graphs.

Because we recurse on $G[S \cup L]$ and $G[S \cup R]$, we use a separator hierarchy in which these two subgraphs (rather than $L$ and $R$) are the ones that are recursively subdivided. In more detail, our algorithm performs the following steps:

1. Recursively compute a minimum-weight maximum-cardinality matching, and its corresponding system of dual variables, for the two subgraphs $G[S \cup L]$ and $G[S \cup R]$. Recall that the dual variables are numbers associated with each vertex such that, for each unmatched edge, the two numbers at its endpoints sum to at most the edge length, and for each matched edge they sum to exactly the edge length.

2. Combine the two subproblems to give a single matching (with fewer than the maximum number of matched edges) and valid system of dual variables for the whole graph $G$. To do so, for each vertex $s$ in $S$, we choose the dual variable for $s$ in $G$ to be the minimum of the two values computed for $s$ in $G[L \cup S]$ and in $G[R \cup S]$, and we remove the matched edge incident to this vertex in one of the two subproblems, the subproblem that did not supply the minimum dual variable value. (If the two subproblems have equal dual variables at $s$, we choose arbitrarily which of the two matched edges to remove.)

3. Repeat the following steps until no more alternating paths can be found:

   - Modify the original weight of each edge by subtracting the dual variables at its endpoints.

   - Add an artificial source vertex adjacent by zero-length edges to all unmatched vertices on one side of the bipartition. Orient the unmatched edges of the graph from the source side to the opposite side, and the matched edges in the other direction.

   - Use the algorithm of Henzinger et al. starting from this source vertex to find the alternating path of minimum modified weight between two unmatched vertices.

   - Augment the matching using the alternating path, trading matched and unmatched edges along the path.

   - Use the distances from the source vertex found by the path search to update the dual variables so that they remain valid.
To update the dual variables, decrease each dual variable on the source side of the bipartition by its distance from the source, and increase each dual variable on the opposite side by its distance from the source. The sum of dual variables at the endpoints of each matched edge remains unchanged, the sum of variables on any shortest path edge increases to equal the edge length, and the sum of variables on any other edge remains at most equal to the edge length (else that edge would have supplied a shorter path).

The result of this algorithm is a maximum-cardinality matching on $G$, and a feasible system of dual variables, whose existence ensures that the matching has minimum weight (as any alternating cycle would have non-negative modified weight and therefore non-negative total cost).

The union of the matchings in $G[S \cup L]$ and $G[S \cup R]$ is at least as large as the single maximum matching in $G$, so the reconciliation process (in which we remove at most $|S|$ edges from the union of the matchings) produces a single matching that differs in cardinality from the maximum by at most $|S|$. Therefore, the inner loop of the algorithm involves at most $|S|$ linear-time path searches. The time analysis of Lemma 12 follows straightforwardly as a divide-and-conquer recurrence.