Multivalued Matrices and Forbidden Configurations

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Abstract

An $r$-matrix is a matrix with symbols in $\{0, 1, \ldots, r-1\}$. A matrix is simple if it has no repeated columns. Let $\mathcal{F}$ be a finite set of $r$-matrices. Let $\text{forb}(m, r, \mathcal{F})$ denote the maximum number of columns possible in a simple $r$-matrix $A$ that has no submatrix which is a row and column permutation of any $F \in \mathcal{F}$. Many investigations have involved $r = 2$. For general $r$, $\text{forb}(m, r, \mathcal{F})$ is polynomial in $m$ if and only if for every pair $i, j \in \{0, 1, \ldots, r-1\}$ there is a matrix in $\mathcal{F}$ whose entries are only $i$ or $j$. Let $\mathcal{T}_\ell(r)$ denote the following $r$-matrices. For a pair $i, j \in \{0, 1, \ldots, r-1\}$ we form four $\ell \times \ell$ matrices namely the matrix with $i$’s on the diagonal and $j$’s off the diagonal and the matrix with $i$’s on and above the diagonal and $j$’s below the diagonal and the two matrices with the roles of $i, j$ reversed. Anstee and Lu determined that $\text{forb}(m, r, \mathcal{T}_\ell(r))$ is a constant. Let $\mathcal{F}$ be a finite set of 2-matrices. We ask if $\text{forb}(m, r, \mathcal{T}_\ell(3) \setminus \mathcal{T}_\ell(2) \cup \mathcal{F})$ is $\Theta(\text{forb}(m, 2, \mathcal{F}))$ and settle this in the affirmative for some cases including most 2-columned $F$.

Keywords: extremal set theory, $(0,1)$-matrices, multivalued matrices, forbidden configurations, trace, Ramsey Theory.

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1 Introduction

We define a matrix to be simple if it has no repeated columns. A (0,1)-matrix that is simple is the matrix analogue of a set system (or simple hypergraph) thinking of the matrix as the element-set incidence matrix. We generalize to allow more entries in our matrices and define an \( r \)-matrix be a matrix whose entries are in \( \{0,1,\ldots,r-1\} \). We can think of this as an \( r \)-coloured matrix. For \( r = 2 \), \( r \)-matrices are (0,1)-matrices and for \( r = 3 \), \( r \)-matrices are (0,1,2)-matrices. We examine extremal problems and let \( \|A\| \) denote the number of columns in \( A \).

We will use the language of matrices in this paper rather than sets. For two matrices \( F \) and \( A \), we write \( F \prec A \), and say that \( A \) has \( F \) as a configuration, if there is a submatrix of \( A \) which is a row and column permutation of \( F \). Row and column order matter to submatrices but not to configurations. Let \( F \) denote a finite set of matrices. Let

\[
\text{Avoid}(m, r, F) = \{ A : A \text{ is } m\text{-rowed and simple } r\text{-matrix, } F \not\prec A \text{ for } F \in F \}.
\]

Our extremal function of interest is

\[
\text{forb}(m, r, F) = \max_A \{ \|A\| : A \in \text{Avoid}(m, r, F) \}.
\]

In the case \( r = 2 \), we are considering (0,1)-matrices and then we drop \( r \) from the notation to write \( \text{Avoid}(m, 2, F) = \text{Avoid}(m, F) \) and \( \text{forb}(m, 2, F) = \text{forb}(m, F) \). We define

\[
\text{forbmax}(m, r, F) = \max_{m' \leq m} \text{forb}(m', r, F).
\]

It has been conjectured by Anstee and Raggi [9] that \( \text{forbmax}(m, 2, F) = \text{forb}(m, 2, F) \) for large \( m \) (which is a type of monotonicity). For many \( F \) this is readily proven.

The following dichotomy between polynomial and exponential bounds is striking. Denote an \((i,j)\)-matrix as a matrix whose entries are \( i \) or \( j \).

**Theorem 1.1** (Füredi and Sali [8]) Let \( F \) be a family of \( r \)-matrices. If for every pair \( i, j \in \{0,1,\ldots,r-1\} \), there is an \((i,j)\)-matrix in \( F \) then for some \( k \), \( \text{forb}(m, r, F) \) is \( O(m^k) \). If there is some pair \( i, j \in \{0,1,\ldots,r-1\} \) so that \( F \) has no \((i,j)\)-matrix then \( \text{forb}(m, r, F) \) is \( \Omega(2^m) \).

It would be of interest to have more examples of forbidden families of configurations where we can determine the asymptotics of \( \text{forb}(m, r, F) \). There are known examples given in [8]. There is a generalization of a result of Balogh and Bollobás [6] for (0,1)-matrices to \( r \)-matrices. Define the generalized identity matrix \( I_\ell(a, b) \) as the \( \ell \times \ell \) \( r \)-matrix with \( a \)'s on the diagonal and \( b \)'s elsewhere. The standard identity matrix is \( I_\ell(1,0) \). Define the generalized triangular matrix \( T_\ell(a, b) \) as the \( \ell \times \ell \) \( r \)-matrix with \( a \)'s below the diagonal and \( b \)'s elsewhere. The standard upper triangular matrix is \( T_\ell(0,1) \). Let

\[
T_\ell(r) = \{ I_\ell(a, b) : a, b \in \{0,1,\ldots,r-1\}, a \neq b \}
\]
By Theorem 1.1, \( \text{forb}(m, r, \mathcal{T}_\ell(r)) \) is bounded by a polynomial but much more is true.

**Theorem 1.2** Given \( r, \ell \), there is a constant \( c(r, \ell) \) so that \( \text{forb}(m, r, \mathcal{T}_\ell(r)) \leq c(r, \ell) \).

We will use the constant \( c(r, \ell) \) repeatedly in this paper. This is a kind of Ramsey Theorem, a particular structured configuration appears in any \( r \)-matrix of a suitably large number of distinct columns. An important result is that \( c(r, \ell) \) is \( O(2^{c_r \ell^2}) \) for some constant \( c_r \). Not unexpectedly, Ramsey Theory shows up in the proof. Section 2 contains a number of proofs using Ramsey theory.

\( \mathcal{T}_\ell(2) \) consists of \((0,1)\)-matrices (i.e. 2-matrices). This paper considers forbidding the matrices \( \mathcal{T}_\ell(r) \cap \mathcal{T}_\ell(2) \). Note that any \((0,1)\)-matrix \( A \in \text{Avoid}(m, r, \mathcal{T}_\ell(r) \cap \mathcal{T}_\ell(2)) \) and so \( \text{forb}(m, r, \mathcal{T}_\ell(r) \cap \mathcal{T}_\ell(2)) = \Omega(2^m) \). Forbidding \( \mathcal{T}_\ell(r) \cap \mathcal{T}_\ell(2) \) may be somewhat like asking the matrices to be \((0,1)\)-matrices.

**Theorem 1.3** Let \( r, \ell \) be given. Then \( \text{forb}(m, r, \mathcal{T}_\ell(r) \cap \mathcal{T}_\ell(2)) \) is \( \Theta(2^m) \).

**Proof:** A construction in \( \text{Avoid}(m, r, \mathcal{T}_\ell(r) \cap \mathcal{T}_\ell(2)) \) is to take all 2-columns on \( m \) rows.

Take any matrix \( A \in \text{Avoid}(m, r, \mathcal{T}_\ell(r) \cap \mathcal{T}_\ell(2)) \) and replace all entries 2, 3, \ldots, \( r-1 \) by 1’s to obtain the 2-matrix \( A' \), not necessarily simple. The number of different columns in \( A' \) is at most \( 2^m \).

Let \( \alpha \) be a column of \( A' \). Let \( B \) denote the submatrix of \( A \) consisting of all columns of \( A \) that map to \( \alpha \) under the replacements. Let \( B' \) be the simple submatrix of \( B \) consisting of the rows of \( B \) where \( \alpha \) has 1’s. Then \( \|B\| = \|B'\| \leq c(r-1, \ell) \) else we have a configuration in \( \mathcal{T}_\ell(r) \cap \mathcal{T}_\ell(2) \) in \( B' \) (using Theorem 1.2 with symbols chosen from \( \{1, 2, \ldots, r-1\} \).

Combining these two observations yields the desired bound. \( \square \)

By the same argument we can show \( \text{forb}(m, r, \mathcal{T}_\ell(r) \cap \mathcal{T}_\ell(s)) \) is \( \Theta(s^m) \) but the focus is on \( s = 2 \) in this paper. In this paper we will also take \( r = 3 \). Note that Lemma 1.6 provides a justification for this restriction. Define the matrices \( T_\ell(a, b, c) \) as the \( \ell \times \ell \) matrix with \( a \)'s below the diagonal, \( b \)'s on the diagonal and \( c \)'s above the diagonal. In our problems we can require \( a \neq b \). These appear in the proof of Theorem 1.2 but, for \( a \neq b \neq c \), are not matrices of just two entries which are referred to in Theorem 1.1. One general result in this direction is the following.

**Theorem 1.4** Let \( \mathcal{F} \) be a finite family of \((0,1)\)-matrices. Then \( \text{forb}(m, 3, \mathcal{T}_\ell(3) \cap \mathcal{T}_\ell(2) \cup T_\ell(0, 2, 1) \cup \mathcal{F}) \) is \( O(\text{forbmax}(m, \mathcal{F})) \).

Another version of Theorem 1.4 with restricted column sums (column sum will refer in this setting to the number of 1’s) is given in Section 2 with the analogous proof. We are not pleased with the inclusion of \( T_\ell(0, 2, 1) \) in Theorem 1.4 and think it can be avoided.
**Problem 1.5** Let $F$ be a $(0,1)$-matrix. Is it true that \( \text{forb}(m, 3, T_\ell(3) \setminus T_\ell(2) \cup F) \) is \( \Theta(\text{forb}_{\max}(m, F)) \)?

Obviously the configuration $T_\ell(0, 2, 1)$ will be problematical. We will let $\ell$ take on large but constant values. Some results given below support a yes answer. For example if we forbid nothing in the $(0,1)$-world then the maximum number of possible distinct $(0,1)$-columns is $2^m$. One could say that “\( \text{forb}_{\max}(m, \emptyset) = 2^m \)” . Now using Theorem 1.3, we see that Problem 1.5 is true in this case.

Given $s = 2$, one can show it suffices to consider $r = 3$ in Problem 1.5. The argument is similar to Theorem 1.3 and uses Ramsey Theory. The proof is given in Section 2.

**Lemma 1.6** Let $r > 2$ and $\ell$ be given. Then there is a constant $bd(\ell)$ so that \( \text{forb}(m, r, T_\ell(r) \setminus T_\ell(2) \cup F) \) is \( O(\text{forb}(m, 3, T_{bd(\ell)}(3) \setminus T_{bd(\ell)}(2) \cup F)) \).

Given the answer ‘yes’ to Problem 1.5, this yields a justification for restricting to $r = 3$. The argument could also be extended to $T_\ell(r) \setminus T_\ell(s)$ but the focus is on $s = 2$.

Many configurations $F$ can be handled by Theorem 1.7 and in particular configurations with more than two columns.

**Theorem 1.7** Let $F \prec T_{\ell/2}(0,1)$. Then \( \text{forb}(m, 3, T_\ell(3) \setminus T_\ell(2) \cup F) \) is \( \Theta(\text{forb}_{\max}(m, F)) \).

**Proof:** Note that $T_{\ell/2}(0,1) \prec T_\ell(0,2,1)$ by considering the submatrix of $T_\ell(0,2,1)$ consisting of the even indexed columns and the odd indexed rows. Thus if $F \not\prec A$, then $T_\ell(0,2,1) \not\prec A$. Apply Theorem 1.4.

One important corollary is the following.

**Corollary 1.8** \( \text{forb}(m, 3, T_\ell(3) \setminus T_\ell(2) \cup [0,1]) \) is \( \Theta(1) \).

**Proof:** $[0,1] \prec T_{\ell/2}(0,1)$ for $\ell \geq 4$.

This paper provides a number more results in this direction mostly involving configurations of two columns. Define $F_{a,b,c,d}$ to be the $(a+b+c+d) \times 2$ configuration with $a$ rows $[1,1]$, $b$ rows $[1,0]$, $c$ rows $[0,1]$, and $d$ rows $[0,0]$. The asymptotics of $\text{forb}(m, F_{a,b,c,d})$ have been completely determined by Anstee and Keevash [1]. Note that we can assume $a \geq d$ since otherwise we can take the $(0,1)$-complement $F_{a,b,c,d}^c = F_{d,c,b,a}$. Also we may assume $b \geq c$ since as configurations $F_{a,b,c,d} = F_{a,c,b,d}$. We note that $\text{forb}(m, F_{a,b,0,0})$ is $\Omega(m^{a+b-1})$ by taking all columns of column sum $a+b$ and a different construction shows $\text{forb}(m, F_{0,b,0,0})$ is $\Omega(m^b)$. The important upper bounds are for $a \geq 1$, $\text{forb}(m, F_{a,b,0,0})$ is $\Theta(m^{a+b-1})$ [1] and $\text{forb}(m, F_{0,b+1,0,0})$ is $\Theta(m^b)$ [1]. Note that $I_2 = F_{0,1,1,0}$. This is the first result not covered by Theorem 1.7.

**Theorem 1.9** \( \text{forb}(m, 3, T_\ell(3) \setminus T_\ell(2) \cup I_2) \) is $\Theta(\text{forb}_{\max}(m, I_2))$. 

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Theorem 1.10 Let $a \geq 0$ and $b \geq 2$ be given. Then $\text{forb}(m, 3, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2) \cup F_{a,b,b,a})$ is $\Theta(\text{forbmax}(m, F_{a,b,b,a}))$.

We give the proofs in Section 4. Note the subtlety that $\text{forbmax}(m, F_{0,b,b,a})$ is $\Theta(m^b)$ whereas, for $a \geq 1$, $\text{forbmax}(m, F_{a,b,b,a})$ is $\Theta(m^{a+b-1})$. The proofs use results for two columned forbidden configurations from [1]. The other critical two columned result concerns $F = F_{0,b+1,b,0}$ for which we don’t know the answer for Problem 1.5.

Define $t \cdot F = [F F \cdots F]$ to be the concatenation of $t$ copies of $F$.

Theorem 1.11 Let $F$ be a given $k \times p$ $(0,1)$-matrix. Then $\text{forb}(m, 3, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2) \cup t \cdot F)$ is $O(\max\{m^k, \text{forb}(m, 3, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2) \cup F)\})$.

Proof: Let $A \in \text{Avoid}(m, 3, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2))$ with $\|A\| > (t-1)p\binom{m}{k} + \text{forb}(m, 3, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2) \cup F) + 1$. Then $F \prec A$. Remove from $A$ the $p$ columns containing a copy of $F$ and repeat. We will generate at least $(t-1)\binom{m}{k} + 1$ copies of $F$ and hence at least $t$ column disjoint copies of $F$ in the same set of $k$ rows and so $t \cdot F \prec A$. 

To apply this, we need to know $\text{forb}(m, 3, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2) \cup F)$. The following is established in Section 4.

Theorem 1.12 Let

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $\text{forb}(m, 3, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2) \cup H)$ is $\Theta(m)$.

Corollary 1.13 Given $H$ in (2), we have $\text{forb}(m, 3, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2) \cup t \cdot H)$ is $\Theta(m^2)$.

Proof: We apply Theorem 1.11 and Theorem 1.12.

Theorem 1.12 and Corollary 1.13 are yes instances of Problem 1.5 since $\text{forb}(m, H)$ is $\Theta(m)$ and $\text{forb}(m, t \cdot H)$ is $\Theta(m^2)$ [5].

Given an $m_1 \times n_1$ matrix $A$ and a $m_2 \times n_2$ matrix $B$, define the product of two matrices $A \times B$ as the $(m_1 + m_2) \times n_1 n_2$ matrix obtained from placing each column of $A$ on top of each column of $B$ for all possible pairs of columns. Let $F$ be given with

$$0 \times 1 \times F = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ F \end{bmatrix}$$

In [5], we establish that $\text{forb}(m, 0 \times 1 \times F)$ is $O(m \cdot \text{forb}(m, F))$. We establish this version of the Problem 1.5 in Section 3.

Theorem 1.14 $\text{forb}(m, 3, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2) \cup 0 \times 1 \times F)$ is $O(m \cdot \text{forb}(m, \mathcal{T}_t(3) \setminus \mathcal{T}_t(2) \cup F))$.

This result extends results for $F_{a,b,b,a}$ to $F_{a+1,b,b,a+1}$ and can be used in other instances such as $H$ above. We finish the paper with some open problems.
2 Results using Ramsey Theory

We apply Ramsey Theory to help us find configurations in \( T_\ell(3) \setminus T_\ell(2) \) etc. We use the \( p \) colour Ramsey number \( R_p(t_1, t_2, \ldots, t_p) \) as the smallest number \( n \) such that for every edge colouring of \( K_n \) with \( p \) colours there is some colour \( i \) so that there is a clique of size \( t_i \) with all edges of colour \( i \). Typical notation is that for \( t_1 = t_2 = \cdots = t_p = t \), we write \( R_p(t_1, t_2, \ldots, t_p) = R_p(t^p) \). While these numbers can be large, we can for example bound \( R_p(t^p) \leq 2^{pt} \).

Let \( r, s \) be given integers with \( r > s \geq 2 \). Let us define a set \( \mathcal{P}_t^x(r) \) of \( t \times t \) matrices by the following template which will have choices \( x, y_1, y_2, \ldots, y_t \in \{1, 2, \ldots, r - 1\} \) where we require \( y_j \neq x \) for \( j \in [t] \). The entries marked * may be given entries in \( \{0, 1, \ldots, r - 1\} \) in any possible way.

\[
\mathcal{P}_t^x: \begin{bmatrix}
y_1 \\
x & y_2 & * \\
x & x & y_3 \\
\vdots \\
x & x & x & \cdots & y_{t-1} \\
x & x & x & \cdots & x & y_t
\end{bmatrix}
\] (3)

**Lemma 2.1** Let \( \ell, r, s \) be given with \( r > s \geq 2 \). Let \( t = (r - 1)(R_r((2\ell)^r) - 1) - 1 \). Assume \( A \) is an \( m \)-rowed simple \( r \)-matrix. Assume there is some \( G \in \mathcal{P}_t^x(r) \) with \( G \prec A \) and such that if \( x \in \{0, 1, \ldots, s - 1\} \) then \( y_j \in \{s, s + 1, \ldots, r - 1\} \) for all \( j \in [t] \). Then there is some \( F \in F \prec G \) and

\[
F \in (T_\ell(r) \setminus T_\ell(s)) \cup \{T_\ell(x, z, u); x, u \in \{0, 1, \ldots, s - 1\}, x \neq u, z \notin \{0, 1, \ldots, s - 1\}\}.
\]

**Proof:** Assume there is some \( G \in \mathcal{P}_t^x(r) \) with \( G \prec A \) and such that if \( x \in \{0, 1, \ldots, s - 1\} \) then \( y_i \in \{s, s + 1, \ldots, r - 1\} \) for all \( i \in [t] \).

First assume \( x \notin \{0, 1, \ldots, s - 1\} \). There are \( r - 1 \) choices for each \( y_j \) and hence there is some choice \( z \in \{0, 1, \ldots, s - 1\} \setminus x \) which appears at least \( R_r((2\ell)^r) \) times on the diagonal. Now form a graph whose vertices are the rows \( i \) with \( y_j = z \) and we colour edge \( a, b \) for \( a < b \) by the entry in the \( a, b \) location of \( G \) (above the diagonal). There will be at least \( R_r((2\ell)^r) \) vertices and there will be at most \( r \) colours and so by the Ramsey number there will be a clique of size \( 2\ell \) of all edges of the same colour, say colour \( u \). If \( u = z \) we have \( T_{2\ell}(x, z) \prec A \). If \( u = x \) we have \( I_{2\ell}(x, z) \prec A \). If \( u \neq x, z \) then we consider the configuration \( T_{2\ell}(x, u, z) \) of size \( 2\ell \) induced by the clique and the even columns and the odd rows to show \( T_{\ell}(x, y) \prec A \). All three cases yield a configuration in \( T_\ell(r) \setminus T_\ell(s) \).

Now assume \( x \in \{0, 1, \ldots, s - 1\} \) then there are \( r - s - 1 \) choices for each \( y_j \) and hence there is some choice \( z \notin \{0, 1, \ldots, s - 1\} \) which appears at least \( R_r((2\ell)^r) \) times.
on the diagonal. Now we proceed as above to obtain a configuration $T_{2t}(x, z, u)$. If
$u \in \{s, s + 1, \ldots, r - 1\}$ then we obtain a configuration in $T_{t}(r \setminus T_{t}(s))$. If $u = x$ we
obtain a configuration $I_{2t}(x, z)$ which is in $T_{t}(r \setminus T_{t}(s))$. If $u \in \{0, 1, \ldots, s - 1\}$ with
$u \neq x$, then we obtain a configuration $T_{2t}(x, z, u)$ with $x, u \in \{0, 1, \ldots, s - 1\}$ and $x \neq u$
(which does not yield a configuration in $T_{t}(r \setminus T_{t}(s))$).

Our application of the Lemma [2.1] to Theorem [1.4] will be in the case $r = 3$ and $s = 2$
and then $\{T_{t}(x, z, u) ; x, u \in \{0, 1, \ldots, s - 1\}, x \neq u, z \notin \{0, 1, \ldots, s - 1\}\}$ is the single
configuration $T_{t}(0, 2, 1)$. We prove in greater generality.

**Proof of Theorem [1.4]** The idea of the proof is to use the induction to generate
configurations corresponding to matrices in $P_{t}$ that enable us to apply the proof of
Lemma [2.1] and obtain matrices in $T_{t}(r \setminus T_{t}(s))$.

We use the following function $f$ in our proof. Let $f$ be determined by the recurrence

$$f(p_{0}, p_{1}, \ldots, p_{r-1}) = \sum_{i=0}^{r-1} f(p_{0}, p_{1}, \ldots, p_{i-1}, \ldots, p_{r-1}), \tag{4}$$

and the base cases that $f(p_{0}, p_{1}, \ldots, p_{r-1}) = 1$ if $p_{i} = 1$ for any $i \in \{0, 1, \ldots, r - 1\}$.
Solving this exactly seems difficult but since $f$ satisfies the same recurrence as the
multinomial coefficients, with smaller base cases, we obtain

$$f(p_{0}, p_{1}, \ldots, p_{r-1}) \leq \frac{(p_{0} + p_{1} + \cdots + p_{r-1} - r)!}{(p_{0} - 1)!(p_{1} - 1)! \cdots (p_{r-1} - 1)!} \tag{5}$$

Let $g(p_{0}, p_{1}, \ldots, p_{r-1}) = f(p_{0}, p_{1}, \ldots, p_{r-1}) \cdot \text{forbmax}(m, F)$.

We will establish for fixed $m$ but by induction on $\sum p_{i}$, that if $A$ is an $n$-rowed simple
$r$-matrix with $n \leq m$ and $\|A\| > g(p_{0}, p_{1}, \ldots, p_{r-1})$ then for some $i \in \{0, 1, \ldots, r - 1\}$,
$A$ will contain configuration $F \in F$ or a configuration in $P_{p_{i}}$ satisfying the condition
that if $i \in \{0, 1, \ldots, s - 1\}$, then $y_{j} \in \{s, s + 1, \ldots, r - 1\}$ for $j \in [P_{i}]$. We use forbmax
so that forbmax$(m, s, F) \geq \text{forb}(n, s, F)$.

If $p_{i} = 1$, then an element of $P_{p_{i}}$ is a $1 \times 1$ matrix. For $i \in \{0, 1, \ldots, s - 1\}$, then
the entry in the $1 \times 1$ matrix must not be in $\{0, 1, \ldots, s - 1\}$ and if $i \notin \{0, 1, \ldots, s - 1\}$,
then the entry in the $1 \times 1$ matrix must not be $i$. In the former case, we require the
matrix to have some entry not in $\{0, 1, \ldots, s - 1\}$ which would only be difficult if $A$
was an $s$-matrix. In that case $\|A\| \leq \text{forb}(n, s, F) \leq \text{forbmax}(m, s, F)$ and we note that
$f(p_{0}, p_{1}, \ldots, p_{r-1}) = 1$ for $p_{i} = 1$. In the latter case we are merely requiring that the
matrix $A$ has at least two different entries which would only not occur for $\|A\| = 1$. In
either case we are able to obtain an instance of $P_{1}$ in $A$ if $\|A\| > g(p_{0}, p_{1}, \ldots, p_{r-1})$.
This establishes the required base cases for the induction.

Assume $p_{i} \geq 2$ or all $i \in \{0, 1, \ldots, r - 1\}$. Consider a matrix $A \in \text{Avoid}(n, r, P_{p_{0}} \cup
P_{p_{1}} \cup \cdots \cup P_{p_{r-1}} \cup F)$ with $n \leq m$ and $\|A\| > g(p_{0}, p_{1}, \ldots, p_{r-1})$. We wish to obtain a
contradiction.
Choose a row $w$ of $A$ which has at least two different entries one of which is not in 
\{0, 1, \ldots, s - 1\}. If there is no such row then either $\|A\| = 1$ or $A$ is an $s$-matrix. In the
latter case, we have $\|A\| > g(p_1, p_2, \ldots, p_{r-1}) \geq \text{forbmax}(m, s, \mathcal{F}) \geq \text{forb}(n, s, \mathcal{F})$ and so 
$F \prec A$, a contradiction. We may assume a row $w$ of $A$, which has at least two different
entries one of which is not in \{0, 1, \ldots, s - 1\}, exists.

Decompose $A$ as follows by permuting rows and columns

\[
A = \begin{array}{c}
\text{row } w \rightarrow \left[
\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
G_0 & 1 & 1 & \cdots \\
G_1 & 2 & 2 & \cdots \\
G_2 & \cdots & \cdots & \cdots \\
G_{r-1} & \cdots & \cdots & \cdots
\end{array}
\right]
\end{array}
\] (6)

Each $G_i$ is simple. Now

\[
\|A\| = \sum_{i=0}^{r-1} \|G_i\| > g(p_0, p_1, \ldots, p_{r-1}) = f(p_0, p_1, \ldots, p_{r-1}) \cdot \text{forbmax}(m, s, \mathcal{F})
\]

\[
= \left(\sum_{i=0}^{r-1} f(p_0, p_1, \ldots, p_i - 1, \ldots, p_{r-1})\right) \cdot \text{forbmax}(m, s, \mathcal{F}).
\]

From the recurrence (4), there is some $i$ with

\[
\|G_i\| > g(p_0, p_1, \ldots, p_i - 1, \ldots, p_{r-2}, p_{r-1}).
\]

Certainly $G_i \prec A$ and $G \in \text{Avoid}(n-1, 3, \mathcal{P}_{p_0}^0 \cup \mathcal{P}_{p_1}^1 \cup \cdots \cup \mathcal{P}_{p_{r-1}}^{r-1} \cup \mathcal{F})$. Then by induction on $\sum_i p_i$, we can assume $G_i$ has a copy of $\mathcal{P}_{p_i-1}$ using the template (5) with $x = i$ and

if $x = i \in \{0, 1, \ldots, s - 1\}$ then $y_j \in \{s, s + 1, \ldots, r - 1\}$ for all $j = 1, 2, \ldots, p_i - 1$. We

can extend to a copy of $\mathcal{P}_{p_i}$ in $A$ by adding row $w$ to extend by a row of $i$’s and then extend by a column from some $G_j$ with $j \neq i$. If $i \in \{0, 1, \ldots, s - 1\}$, then we

can extend to a copy of $\mathcal{P}_{p_i}$ in $A$ by adding row $w$ to extend by a row of $i$’s and then extend by a column from some $G_h$ with $h \in \{s, s + 1, \ldots, r - 1\}$. This is possible since we have assumed that row $w$ has at least two different entries one of which is not in

\{0, 1, \ldots, s - 1\}. Now some matrix $G$ in the family $\mathcal{P}_{p_i}$ has $G \prec A$.

Specializing to $p_0 = p_1 = \cdots = p_{r-1} = (r-1)(R_r((2\ell)^r) - 1$ and applying Lemma 2.1

yields that $G$ contains a configuration in $(\mathcal{T}_l(r) \setminus \mathcal{T}_l(s)) \cup \{T_\ell(x, z, u) : x, u \in \{0, 1, \ldots, s - 1\}, x \neq u, z \notin \{0, 1, \ldots, s - 1\}\}$ and then specializing to $r = 3$ and $s = 2$ yields the
result. $\blacksquare$

It was convenient to consider general $r, s$ but we will focus on $r = 3$ and $s = 2$. The

proof of Theorem 1.4 can be adapted to considering fixed column sum i.e. columns with

a fixed number of 1’s. In the case of 3-matrices, we define the column sum of a 3-column matrix

$\alpha$ to be the number of 1’s present. When there are no 2’s in $\alpha$, this is the usual column sum. Define

\[
\text{forb}_k(m, 3, \mathcal{F}) = \max\{\|A\| : A \in \text{Avoid}(m, 3, \mathcal{F}), \text{ all columns in } A \text{ have } k \text{ 1’s}\},
\]

and define $\text{forbmax}_k$ similarly. There are $\mathcal{F}$ for which we can exploit information about

$\text{forb}_k(m, 3, \mathcal{F})$, deducing some information from $\text{forb}_k(m, \mathcal{F})$. 

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**Theorem 2.2** Let $\mathcal{F}$ be a finite set of $(0,1)$-matrices. Let $\ell$ be given. Then there exists a constant $d$ so that

$$\text{forb}_k(m, 3, \mathcal{T}_\ell(3) \setminus \mathcal{T}_\ell(2) \cup \mathcal{T}_\ell(0, 2, 1) \cup \mathcal{F}) \leq O\left(\sum_{j=k-d}^{k} \text{forbmax}_j(m, \mathcal{F})\right)$$

(7)

**Proof:** We will follow the proof of Theorem 1.4 but note how columns sums are affected. Let $g_k(p_0, p_1, p_2) = f(p_0, p_1, p_2) \cdot \text{forbmax}_k(m, \mathcal{F})$.

Consider a matrix $A \in \text{Avoid}_k(n, 3, \mathcal{P}^0_{p_0} \cup \mathcal{P}^1_{p_1} \cup \mathcal{P}^2_{p_2} \cup \mathcal{F})$ with $n \leq m$ and $k > p + 2R_3(2\ell^3)$ and $\|A\| > g_k(p_0, p_1, p_2)$. We wish to obtain a contradiction.

It is convenient to interpret the proof of Theorem 1.4 as growing a tree where each node is associated with a matrix with three associated parameters $(p, p, p)$ and has some fixed column sum $s$. We begin with a root node corresponding to a matrix $A$ with parameters $(p, p, p)$ where $p = 2R_3(2\ell, 2\ell, 2\ell)$. Then the matrices $G_0, G_1, G_2$ can be viewed as the children. Our recursive growth of the tree begins with a node corresponding matrix $B$ for which we decompose by some row $w$ with at least two entries one of which is 2. If we can’t decompose then either $\|B\| = 1$ or $B$ is an $(0,1)$-matrix.

Assume each column of $B$ has $s$ 1’s. Decompose $B$ as follows by permuting rows and columns

$$B = \begin{array}{c|c|c}
\text{row } w & 0 & 1 \\
\text{H}_0 & 0 & 1 \\
\text{H}_1 & 1 & 2 \\
\text{H}_2 & 2 & 2 \\
\end{array}$$

(8)

Each $H_i$ is simple. Given that each column in $B$ has $s$ 1’s then for each column in $H_0$ and $H_2$ has $s$ 1’s and each column in $H_1$ has $s - 1$ 1’s. Thus the nodes of our tree correspond to matrices with fixed column sum.

We also need to keep track of the current triple $(q_0, q_1, q_2)$ for each node. Thus if $B$ has the triple $(q_0, q_1, q_2)$ then $G_0$ has triple $(q_0 - 1, q_1, q_2)$, $G_1$ has triple $(q_0, q_1 - 1, q_2)$ and $G_2$ has triple $(q_0, q_1, q_2 - 1)$. We do not decompose $B$ if $q_0 = 1$ or $q_1 = 1$ of $q_2 = 1$.

Otherwise the node corresponding to $B$ has children $G_0, G_1, G_2$ with the possibility that $\|G_0\| = 0$ or $\|G_1\| = 0$ in which case $B$ would only have two children.

Given the decomposition (8), then $\|A\|$ is the sum of $\|B\|$ over all leaves $B$ of the tree. The leaves of the tree which cannot be further decomposed correspond to matrices $B$ with $\|B\| = 1$ or $B$ is a $(0,1)$-matrix or $B$ where the three parameters $(q_0, q_1, q_2)$ have either $q_0 = 1$ or $q_1 = 1$ or $q_2 = 1$.

We deduce that the depth of the tree is at most $d = 3p = 6R_3(2\ell, 2\ell, 2\ell)$ with a branching factor of 3 and so there are at most $3^d$ nodes in the tree which is a constant. Also we have that each node corresponds to a matrix with constant column sum $s \in \{k - d, k - d + 1, \ldots, k\}$ which is a constant cardinality set.

Now continue growing the tree until no further growth is possible. If the process generates a node $B$ with $q_0 = 1$ or $q_1 = 1$ or $q_2 = 1$, then by the arguments of Theorem 1.4 there will be some configuration in $\mathcal{T}_\ell(3) \setminus \mathcal{T}_\ell(2) \cup \mathcal{T}_\ell(0, 2, 1)$ in $B$ and hence in $A$. A leaf node is one which corresponds to some $(0,1)$-matrix $B$ with constant column sum $s \in \{k - d, k - d + 1, \ldots, k\}$ for which we deduce that $\|B\| \leq \text{forbmax}_s(m, \mathcal{F})$. 


The bound (7) now follows with the inclusion of some large constants. 

We will apply this result to 2-columned $F$.

**Proof of Lemma 1.6** We readily note that $\text{forb}(m, r, T_{\ell}(r)\setminus T_{\ell}(2) \cup F) \geq \text{forb}(m, 3, T_{\ell}(r)\setminus T_{\ell}(2) \cup F)$ since $\text{Avoid}(m, 3, T_{\ell}(3)\setminus T_{\ell}(2) \cup F) \subseteq \text{Avoid}(m, r, T_{\ell}(r)\setminus T_{\ell}(2) \cup F)$.

Let $bd(\ell) = R_{(r-2)(r-2)}((2\ell)^{(r-2)(r-2)})$ where we assume $bd(\ell) > (r - 2)\ell$. Let $A \in \text{Avoid}(m, r, T_{\ell}(r)\setminus T_{\ell}(2) \cup F)$. Replace all entries 3, 4, . . . , $r - 1$ by 2’s to obtain $A'$. The number of different columns in $A'$ is at most $\text{forb}(m, 3, T_{bd(\ell)}(3)\setminus T_{bd(\ell)}(2) \cup F)$ for the following reason. If $F < A'$, then $F < A$ so we may assume $F \not\approx A'$. Let $A''$ be the matrix obtained from $A'$ by keeping exactly one copy of each column. If $\|A''\| > \text{forb}(m, 3, T_{bd(\ell)}(3)\setminus T_{bd(\ell)}(2) \cup F)$ then there is configuration $G < A''$ with $G \in T_{bd(\ell)}(3)\setminus T_{bd(\ell)}(2)$. There are several cases.

If $G$ is a generalized identity matrix say $I_{bd(\ell)}(1, 2)$, then in $A$, we have a configuration which has entries in $\{2, 3, . . . , r - 1\}$ on the diagonal and 1’s off the diagonal. Then there is some entry $q \in \{2, 3, . . . , r - 1\}$ appearing $[bd(\ell)/(r - 2)] \geq \ell$ times (using $bd(\ell) > (r - 2)\ell$) and we obtain a principal submatrix of $G$ (row and column indices given by the diagonal entries $q$) in $T_{(bd(\ell)/(r-2))}(r)\setminus T_{(bd(\ell)/(r-2))}(2)$ in $A$.

If $G$ is a generalized identity matrix say $I_{bd(\ell)}(2, 1)$, then in $A$, we have a configuration which has entries in $\{2, 3, . . . , r - 1\}$ off the diagonal and 1’s on the diagonal. Now apply Ramsey Theory by colouring a graph on $bd(\ell)$ vertices with the colour of edge $(i, j)$ for $i < j$ being the 2-tuple $a_{i,j}, a_{j,i}$. There are $(r - 2)^2$ colours and so if $bd(\ell) > R_{(r-2)(r-2)}((2\ell)^{(r-2)(r-2)})$, then there is a clique of colour $p, q$ of size $2\ell$ and so $2\ell \times 2\ell$ configuration whose entries on the diagonal are 1’s and above the diagonal are $p$ and whose entries below the diagonal are $q$. If $p = q$, we have a configuration in $T_{2\ell}(r)\setminus T_{2\ell}(2)$. If $p \neq q$, then we form an $\ell \times \ell$ configuration with $p$’s above the diagonal and $q$’s below the diagonal (by taking even indexed columns and odd indexed rows) which is a configuration in $T_{r}(r)\setminus T_{r}(2)$. Similar arguments handle the remaining cases.

To determine the maximum number of columns of $A$ that map into a given $(0, 1, 2)$-column $\alpha$ in $A'$, let $\alpha$ have $t$ 2’s and then the columns mapping into $\alpha$ correspond to a $t$-rowed simple matrix with entries in $\{2, 3, . . . , r - 1\}$. If the number of columns is bigger that $c(r - 2, \ell)$, then those columns contain a configuration in $T_{\ell}(r)$ whose entries are in $\{2, 3, . . . , r - 1\}$ and so the configuration is in $T_{r}(r)\setminus T_{r}(2)$. We now deduce that $\|A\| \leq c(r - 2, \ell) \times \text{forb}(m, 3, T_{bd(\ell)}(3)\setminus T_{bd(\ell)}(2) \cup F)$ yielding our bound.

**3 0 × 1 × F**

Let $A \in \text{Avoid}(m, 3, T_{\ell}(3)\setminus T_{\ell}(2) \cup 0 \times 1 \times F)$. If we can choose a pair of rows $i, j$ so that there are $\text{forb}(m - 2, 3, T_{\ell}(3)\setminus T_{\ell}(2) \cup F) + 1$ columns of $A$ which have 0’s on row $i$ and 1’s in row $j$, then we have $F < A$, a contradiction.

**Lemma 3.1** Let $\epsilon > 0$ be given. Let $A$ be an $m$-rowed simple $3$-matrix with each column
having both a 0 and a 1 and at least \( \epsilon m \) entries either 0 or 1. Assume

\[ \|A\| > 2 \cdot \text{forbmax}(m - 2, 3, T_\ell(3) \cup T_\ell(2) \cup F) \frac{(m)}{e^2 \epsilon - 1}. \]  

(9)

Then \( 0 \times 1 \times F \prec A \).

**Proof:** We note that a column of \( m \) rows that has \( p \) 0’s and \( q \) 1’s will have \( pq \) pairs of rows \( i, j \) containing the configuration \([0 \ 1]\). For a given \( p, q \), the minimum number of configurations \([0 \ 1]\) is \( p + q - 1 \) when for example there is one 1 and \( p + q - 1 \) 0’s. An \( m \)-rowed column with at least one 0 and at least one 1 and at least \( \epsilon m \) entries that are 0 or 1 will have at least \( \epsilon m - 1 \) configurations \([0 \ 1]\). There are \( 2^m \) choices for \( i, j \) when considered as an ordered pair.

If (9) is valid then there will be a pair of rows \( i, j \) with more than 

\[ 2 \cdot \text{forbmax}(m - 2, 3, T_\ell(3) \cup T_\ell(2) \cup F) \] 

columns with the configuration \([0 \ 1]\). Thus there will be a pair of rows \( i, j \) with at least \( \text{forb}(m, T_\ell(3) \cup T_\ell(2) \cup F) + 1 \) columns all with the submatrix \([0 \ 1]\) (or all the reverse). Then we can form an \( (m - 2) \times (\text{forbmax}(m, T_\ell(3) \cup T_\ell(2) \cup F) + 1) \) simple matrix \( A' \) that when extends by a row of 0’s and a row of 1’s is contained in \( A \). Since \( A' \in \text{Avoid}(m, T_\ell(3) \cup T_\ell(2)) \), we deduce that \( F \prec A' \) and then \( 0 \times 1 \times F \prec A \), as desired.

**Proof of Theorem 1.14** If we have many columns with few 0’s and 1’s then we will show we are able to find in \( A \) a \( c \times c \) configuration \( G \) in \( \mathcal{P}_c^2 \) of \( A \) as in (11) and then can use Lemma 2.1.

Let \( A \in \text{Avoid}(m, T_\ell(3) \cup T_\ell(2) \cup F) \). There are at most \( c(2, \ell) \) (0,2)-columns and at most \( c(2, \ell) \) (1,2)-columns. Let \( A' \) be the matrix obtained from \( A \) by deleting (0,2)-columns and (1,2)-columns.

Now each column in \( A' \) has at least one 0 and one 1. Let

\[ \epsilon = \frac{1}{4R(2\ell, 2\ell, 2\ell)}. \]  

(10)

Delete from \( A' \) any rows entirely of 2’s to obtain a simple matrix \( A'' \in \text{Avoid}(t, 3, T_\ell(3) \cup T_\ell(2) \cup F) \) where \( t \leq m \). Let \( A_2 \) denote those columns of \( A'' \) with at most \( \epsilon t \) 0’s and 1’s and let \( A_{01} \) denote those columns of \( A'' \) with more than \( \epsilon t \) 0’s and 1’s.

We select columns of \( A_2 \) in turn to form the pattern \( \mathcal{P}_c^2 \) in (3) with \( c = 2 \cdot R((2\ell)^3) \).

We can begin with a column on \((1 - \epsilon)t \) 2’s. At the \( k \)th stage we have \( k \) columns (selected
in the order displayed) with

$$
\begin{array}{c}
\neq 2 \\
2 \neq 2 \\
2 2 \neq 2 \\
2 2 \cdots \neq 2 \\
2 2 \cdots 2 \neq 2 \\
2 2 \cdots 2 2 \\
2 2 \cdots 2 2 \\
2 2 \cdots 2 2 \\
\end{array}
\geq (1 - k\epsilon)t
$$

(11)

where the final block of 2’s in rows S has $|S| \geq (1 - k\epsilon)t$. Any column of $A_2$ not already chosen has 2’s in at least $(1 - (k + 1)\epsilon)t$ rows of S. To proceed we need that $\|A_2\| \geq c = 2R_3(2\ell, 2\ell, 2\ell)$ and we require that $(1 - k\epsilon)t \geq 1$ for $k + 1 \leq c = 2R(2\ell, 2\ell, 2\ell)$. Our choice of $\epsilon$ (10) ensures this. $A_2$ has no rows of 2’s and so a column with a 0 or 1 in rows S can be used to extend (11) to the situation with $k + 1$ columns. We repeat until we have $c = 2R_3(2\ell, 2\ell, 2\ell)$ columns. Applying Lemma [2,1] we obtain a matrix $F \in T_\ell(3)$ with $F \prec A$ that has 2’s below the diagonal and so we have obtained a configuration in $T_\ell(3) \setminus T_\ell(2)$, a contradiction. Thus

$$
\|A_2\| \leq 2 \cdot R_3(2\ell, 2\ell, 2\ell).
$$

(12)

If

$$
\|A_{01}\| > 2 \cdot \text{forb}_{\text{max}}(m - 2, 3, T_\ell(3) \setminus T_\ell(2) \cup F) \frac{\binom{2}{\ell}}{\ell^t - 1}
$$

then by Lemma [3,1] $0 \times 1 \times F \prec A_{01}$. Thus

$$
\|A_{01}\| \leq 2 \cdot \text{forb}(m - 2, T_\ell(3) \setminus T_\ell(2) \cup F) \frac{\binom{2}{\ell}}{\ell^t - 1} \leq m \cdot \text{forb}_{\text{max}}(m - 2, T_\ell(3) \setminus T_\ell(2) \cup F).
$$

(13)

Using $\|A\| = 2 \cdot c(2, \ell) + \|A_2\| + \|A_{01}\|$, we obtain our desired bound.

4 Two-columned matrices

The main result of this section is the following. The proof is given after Lemma [4,5] and Lemma [4,6].

**Theorem 4.1** $\text{forb}(m, 3, T_\ell(3) \setminus T_\ell(2) \cup F)$ is $O\left( \sum_{k=0}^{m} \text{forb}_{\text{max}}(m, F) \right)$ for all two-columned matrices $F$.

We have some useful results for two-columned $F$. Theorem 1.2 in [1], gives us insight into $F_{0, b, b, 0}$ with a strong stability result. For our purposes we only need the following.
Lemma 4.2 \[\text{[1]}\] Let \( k, b \) be given with \( b \geq 1 \). Then \( \text{forb}_k(m, F_{0,b,b,0}) \) is \( O(m^{b-1}) \).

Lemma 5.4 in \[\text{[1]}\], repeated below, gives us insight into \( F_{0,b,b,1} \) which helps us consider \( F_{1,b,b,1} \) for \( b \geq 2 \).

Lemma 4.3 Suppose \( r \geq 1 \) and \( F \) is a \( k \)-uniform family of subsets of \([m]\), with \( k \geq r + 2 \), so that every pair \( A, B \in F \) is either disjoint or intersects in at least \( k - r \) points, and for every \( A \in F \) we have \( 1 \notin A \). Then \( |F| \) is \( O(m^r) \).

Translated in our language it says \( \text{forb}_k(m, F_{0,r+1,r+1,1}) \) is \( \Theta(m^r) \) for \( k \neq r + 1 \). Note that \( \text{forb}_{r+1}(m, F_{0,r+1,r+1,1}) \) is \( \Theta(m^{r+1}) \) by taking all columns of \( r + 1 \) 1’s. By taking \((0,1)\)-complements where \( F_{0,r+1,r+1,1}^c = F_{1,r+1,r+1,0} \). Thus for \( k \neq m - r - 1 \), \( \text{forb}_k(m, F_{1,r+1,r+1,0}) \) is \( \Theta(m^r) \) and \( \text{forb}_{r+1}(m, F_{1,r+1,r+1,0}) \) is \( \Theta(m^{r+1}) \) by taking all columns of \( r + 1 \) 1’s.

Corollary 4.4 Let \( r \geq 1 \). Let \( m \) be given. For \( k \neq r + 1, r + 2, m - r, m - r - 1 \), we have \( \text{forb}_k(m, F_{1,r+1,r+1,1}) \) is \( \Theta(m^r) \). For \( k = r + 1, r + 2, m - r - 2 \) or \( m - r - 1 \) we have \( \text{forb}_k(m, F_{1,r+1,r+1,1}) \) is \( \Theta(m^{r+1}) \).

Proof: Assume \( k \neq r + 1, r + 2, m - r, m - r - 1 \) and \( \text{forb}_k(m, F_{1,r+1,r+1,0}) \leq cm^r \). Let \( A \in \text{Avoid}(m, F_{1,r+1,r+1,1}) \). Consider row 1. The number of 0’s plus the number of 1’s in row 1 is \( ||A|| \). Let \( B \) be the submatrix of \( A \) formed by the columns with a 1 in row 1 and rows 2, 3, \ldots \( m \). Then \( B \in \text{Avoid}_{k-1}(m - 1, F_{0,r+1,r+1,1}) \) and so \( ||B|| \leq \text{forb}_{k-1}(m - 1, F_{0,r+1,r+1,1}) \). Note that \( k - 1 \neq r + 1 \). Let \( C \) be the submatrix of \( A \) formed by the columns with a 0 in row 1 and rows 2, 3, \ldots \( m \). Then \( C \in \text{Avoid}_k(m - 1, F_{1,r+1,r+1,0}) \).

Now \( F_{1,r+1,r+1,0} \) is the \((0,1)\)-complement of \( F_{0,r+1,r+1,1} \) and so \( \text{forb}_k(m - 1, F_{1,r+1,r+1,0}) = \text{forb}_{m-1}(m - 1, F_{0,r+1,r+1,1}) \). Note that \( k \neq (m - 1) - r \). We deduce that \( ||A|| = ||B|| + ||C|| \leq 2cm^r \) for \( k \neq r + 1, r + 2, m - r - 2, m - r - 1 \). For the remaining cases \( k = r + 1, r + 2, m - r - 2 \) or \( m - r - 1 \), we deduce that \( ||A|| \) is \( O(m^{r+1}) \).

Let a two-columned \( F \) and an \( A \in \text{Avoid}(m, 3, \mathcal{T}_L(3) \setminus \mathcal{T}_L(2) \cup F) \) with fixed column sum \( k \) be given. Note that by Theorem 2.2 given \( L \), there exists a constant \( t \) such that after \( t \left( \sum_{i=k-t}^k \text{forb} [m, F] \right) \) columns, we either find in \( A \) a configuration of \( \mathcal{T}_L(3) \setminus \mathcal{T}_L(2) \), or \( T_L(0, 2, 1) \) or \( F \). If \( L \geq \ell \), the only object on this list not forbidden is \( T_L(0, 2, 1) \), so we may assume we find this configuration. Note that \( T_{L/2}(1, 0) \prec T_L(0, 2, 1) \), so \( T_{L/2}(1, 0) \) must appear. Reorder the columns so that the 1’s are above the diagonal in \( T_L(0, 1) \). Now, using the previous notation for two-columned matrices, let \( F = F_{a,b,c,d} \). Notice that if we delete the first \( a \) columns of \( T_L(0, 1) \), every pair of columns has \( a \) copies of \([11]\); if we delete the last \( d \) columns, every pair of columns has \( d \) copies of \([00]\); and if we take every \( c \)th column of what remains, every pair of columns has \( c \) copies of \([01]\).

Let \( A' \) be the submatrix of \( A \) obtained by taking the selected columns from \( T_L(0, 1) \) and deleting the rows from \( T_L(0, 1) \). Note that in the deleted rows, every pair of columns
has a copies of $[1 \ 1]$, c copies of $[0 \ 1]$, and d copies of $[0 \ 0]$, so if any pair of columns of $A'$ have b copies of $[1 \ 0]$, $A$ contains $F$. Also, since $A$ has fixed column sum, and the column sums of $T_L(0,1)$ increase from left to right, the column sums of $A'$ decrease from left to right. To use these facts we need the following lemma.

**Lemma 4.5** Let $M$ be an $m$-rowed matrix such that:

(i) $M$ does not contain $[1_b \ 0_b]$ as a $b \times 2$ submatrix for some $b$.

(ii) If $i < j$, column $i$ of $M$ has more 1’s than column $j$

(iii) $M$ avoids $T_{\ell}(3) \setminus T_{\ell}(2)$

Then for every $r$, there exists a constant $c_r$ (dependent on $k$) such that if $M$ has more than $c_r$ columns, it has an $r \times r$ configuration in $P_r^2[3]$ with 1’s on the diagonal and 2’s below the diagonal.

**Proof:** We proceed by induction. When $r = 1$, the desired object is just a single 0, so the lemma is trivial. Suppose the lemma holds for $r$. We claim that the lemma holds for $r + 1$ with $c_{r+1} = R_5(\ell, \ell, \ell, c_r, b+1) + b$. Suppose $M$ satisfies the hypotheses of the lemma and has $c_{r+1}$ columns. Define $M'$ to be the restriction of $M$ to the rows with a 1 in the first column. Since the column sums of $M$ strictly decrease from left to right, the $(b+1)$th column of $M$ has at least $b$ fewer 1’s than the first, which implies that there must be at least $b$ non-1 entries in the $(b+1)$th column of $M'$. At most $b-1$ of these entries are 0 by condition (i), so there is at least one 2. Pick one. The $(b+2)$th column of $M'$ has at least two 2’s, at least one of which is in a different row than the one already chosen. Pick one such 2. Similarly the $(b+3)$th column of $M'$ has a 2 in a different row than the 2’s already selected, and so on; continuing in this way, we find a diagonal of 2’s of length $\|M\| - b = R_5(\ell, \ell, \ell, c_r, b+1)$. Let the square submatrix of $A$ induced by the row and column indices of the chosen diagonal be $M''$.

We now produce a colouring of the complete graph on $\|M''\|$ vertices as follows. Given $i < j$, if $M''_{ij}, M''_{ji} \neq 0$, colour edge $\{i, j\}$ with the ordered pair $(M''_{ij}, M''_{ji})$; if $M''_{ij} = 0$ or $M''_{ji} = 0$, colour $\{i, j\}$ with 0. Now there are five colours: $(1,1), (1,2), (2,1), (2,2)$, and 0. By Ramsey Theory, we have a clique of size $\ell$ of colour $(1,1), (1,2)$ or $(2,1)$ or a clique of size $b+1$ of colour 0 or a clique of size $c_r$ of colour $(2,2)$. In the first case, all three colours give rise to a member of $T_{\ell}(3) \setminus T_{\ell}(2)$. In the second case, we have a column with $b$ 0’s opposite the 1’s in the first column of $M$, contradicting condition (i). Hence the only allowed case is the third, which corresponds to a block of 2’s. In particular, there is a row $x$ of $M$ with $c_r$ 2’s and a 1. Look under the 2’s; the resulting matrix has $c_r$ columns and satisfy the hypotheses of the lemma, so by induction there is an $r \times r$ configuration with 1’s on the diagonal and 2’s above. Adding in row $x$ gives an $(r+1) \times (r+1)$ configuration of the desired type. 

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Lemma 4.6 Let $M$ satisfy the hypotheses of Lemma 4.5 with $c_r$ defined there. Then $\|M\| < c_{R_3(2\ell, \ell, \ell)}$.

Proof: We use the notation $c_r$ from the statement of Lemma 4.5. Suppose $\|M\| \geq c_{R_3(2\ell, \ell, \ell)}$. By Lemma 4.5 there exists a configuration $N \prec M$ in $\mathcal{P}_{R_3(2\ell, \ell, \ell)}$ with 1’s on the diagonal and 2’s below the diagonal. We colour a complete graph $K_{R_3(2\ell, \ell, \ell)}$ as follows: for $i < j$, colour edge $(i, j)$ with $N_{ji}$ (note that $N_{ij} = 2$). By the definition of $R_3(2\ell, \ell, \ell)$, there is a monochromatic clique. Three cases are possible. If there is a clique of size $2\ell$ of colour 0, we get $T_{2\ell}(0, 1, 2)$, which contains $T_{\ell}(0, 2)$. If there is a clique of size $\ell$ of colour 1, we have $T_{\ell}(1, 2)$, and a clique of size $\ell$ of colour 2 yields $I_{\ell}(1, 2)$. This contradicts our assumption that $\|M\| \geq c_{R_3(2\ell, \ell, \ell)}$. 

Proof of Theorem 4.1 Let $A \in \text{Avoid}(m, 3, T_3(3) \setminus T_2(2) \cup F)$ be given, with fixed column sum $k$. By Theorem 2.2 there exist constants $C$ and $d$ independent of $k$ such with more than $C(\sum_{i=k-d}^{k}c_{R_3(2\ell, \ell, \ell)})$ columns, we either have one of the forbidden objects or a very large triangular matrix $T_{t}(0, 2, 1)$ with $t = c_{R_3(2\ell, \ell, \ell)}$. This yields a matrix $M$ satisfying the hypotheses of Lemma 4.5 with $\|M\| > c_{R_3(2\ell, \ell, \ell)}$. Then Lemma 4.6 yields a contradiction. Hence, $|A| \leq C(\sum_{i=k-d}^{k}c_{R_3(2\ell, \ell, \ell)})$, so $\text{forb}_{k}(m, 3, T_3(3) \setminus T_2(2) \cup F) \leq C(\sum_{i=k-d}^{k}c_{R_3(2\ell, \ell, \ell)})$. Summing over $k$ gives the desired result. 

This result can be used to give bounds for many 2-columned matrices.

Proof of Theorem 1.10 For $F_{0,b,b,0}$ we use Lemma 4.2 which yields that $\text{forb}_{k}(m, 3, T_3(3) \setminus T_2(2) \cup F_{0,b,b,0})$ is $O(m^{b-1})$. Then Theorem 4.1 yields that $\text{forb}(m, 3, T_3(3) \setminus T_2(2) \cup F_{0,b,b,0})$ is $O(m^{b})$. From [1], $\text{forb}(m, F_{0,b,b,0})$ is $\Theta(m^{b})$.

By Corollary 4.4, $\text{forb}_{k}(m, F_{1,b,b,1})$ is $O(m^{b-1})$ for $b \geq 2$ and $k \neq r+1, r+2, m-r-2, m-r-1$. For $k = r+1, r+2, m-r-2$ or $m-r-1$, the bound is $O(m^{b})$. By Theorem 4.1 $\text{forb}(m, 3, T_3(3) \setminus T_2(2) \cup F_{1,b,b,1})$ is $O(m^{b})$. From Theorem 1.14 we may extend this to obtain $\text{forb}(m, 3, T_3(3) \setminus T_2(2) \cup F_{a,b,b,a})$ is $O(m^{a+b-1})$. This is the correct bound by [1]. 

Proof of Theorem 1.9 Use Lemma 4.2 with $b = 1$ which by Theorem 4.1 yields that $\text{forb}(m, 3, T_3(3) \setminus T_2(2) \cup F_{0,1,1,0})$ is $O(m)$. 

We do not know how to do solve for $F = F_{1,1,1,1}$ for which $\text{forb}(m, F_{1,1,1,1})$ and $\text{forb}_{k}(m, F_{1,1,1,1})$ are both $\Theta(m)$. Similarly, the case $F = F_{a,1,1,1}$ for $a \geq 2$ is not solved. We have $\text{forb}(m, F_{a,1,1,1})$ is $\Omega(m^{a})$. The following results give bounds which must be close to the correct bounds.

Theorem 4.7 $\text{forb}(m, 3, T_3(3) \setminus T_2(2) \cup F_{1,1,1,1})$ is $O(m \log m)$

Proof: Let $A \in \text{Avoid}(m, F_{1,1,1,1})$ with column sum $k$ be given. If $k \leq m/2$, then every pair of columns has a $[00]$. Since the column sum is fixed, every pair of columns has
an $I_2$. Hence there must be no $[11]$ in any pair of columns. This means the $1$'s must all appear on disjoint rows, so there are at most $\frac{m}{k}$ columns. If $k > \frac{m}{2}$, take the 0-1 complement to get a similar result. Applying Theorem 4.1 and summing over $k$ gives the result. ■

Applying Theorem 1.14 gives the following corollary.

**Corollary 4.8** \(\text{forb}(m, 3, T_\ell(3) \setminus T_\ell(2) \cup F_{a,1,1,a}) = O(m^a \log m)\)

Of course, the extra factor of $\log m$ is undesirable. However, given that all known forbidden families have a polynomial bound, this strongly suggests that the actual bound for $F_{a,1,1,a}$ is $O(m^a)$.

## 5 An example with 3 columns

Define the useful notation $A|_S$ to denote the submatrix of $A$ given by the rows $S$. In order to prove Theorem 1.12 for $H$ given in (2), we find the following lemma useful. A standard decomposition applied to 3-matrices considers deleting a row $i$ from a simple 3-matrix $A$. The resulting matrix might not be simple. Let $C_{a,b}(i)$ be the simple 3-matrix that consists of the repeated columns of the matrix that is obtained when deleting row $r$ from $A$ that lie under both symbol $a$ and $b$ in row $i$. In particular $[ab] \times C_{a,b}(i) \not\prec A$.

Let $B(i)$ denote the $(m-1)$-rowed simple 3-matrix obtained from $A$ by deleting row $i$ and any repeats of columns so that

\[ \|A\| \leq \|B(i)\| + \|C_{0,1}(i)\| + \|C_{1,2}(i)\| + \|C_{0,2}(i)\|. \]

The inequality arises from columns that are repeated three times in the matrix obtained from $A$ by deleting row $i$ but get counted four times on the right hand side. This bound on $\|A\|$ is often amenable to induction on the number of rows. If $K_2 = [01] \times [01] \not\prec A$, or in our case $H \not\prec A$, we deduce that $\|C_{0,1}(i)\|$ is $O(1)$, namely the constant bound for $\text{forb}(m, 3, T_\ell(3) \setminus T_\ell(2) \cup [01])$ using Corollary 1.8. The following lemma could help with $\|C_{1,2}(i)\|$ and $\|C_{0,2}(i)\|$.

**Lemma 5.1** Let $A \in \text{Avoid}(m, 3, T_\ell(3) \setminus T_\ell(2))$. Assume for some set of rows $S$ we have $[0 \mid I_{|S|}] \not\prec A|_S$ and for each pair of rows $i, j \in S$, we have no $[1] \mid [1]$ in $A$. If $|S| > 3\ell \cdot c(2, \ell)$, then there is some row $i \in S$ for which $\|C_{1,2}(i)\| = \|C_{0,2}(i)\| = 0$.

**Proof:** We will show that $\|C_{1,2}(i)\| > 0$ for only a few choices $i \in S$ and similarly show that $\|C_{0,2}(i)\| > 0$ for only a few choices $i \in S$. Then for $S$ large enough, there will be some $i \in S$ with $\|C_{1,2}(i)\| = \|C_{0,2}(i)\| = 0$.

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Let $U$ denote the rows $i \in S$ for which $\|C_{1,2}(i)\| > 0$. Assume $|U| \geq \ell \cdot c(2, \ell)$. When $\|C_{1,2}(i)\| > 0$, we have (at least) two columns in $A$ differing only in row $i$, one with a $1$ and one with a $2$. Choose one such pair of columns $\gamma, \delta$ as shown:

$$
\begin{align*}
& \begin{pmatrix} i \\ U \setminus i \\ [m] \setminus U \end{pmatrix} \\
& \begin{pmatrix} 1 & 2 \\ \alpha & \alpha \\ \beta & \beta \end{pmatrix} < A.
\end{align*}
$$

It is possible that for many $i$, the same second column might be chosen. By the property of $A$ that $A$ has no $\begin{pmatrix} 1 \\ \end{pmatrix}$ rows of $S$ and hence $U$, we deduce $\delta|_U = [^2_\alpha]$ is a $(0,2)$-vector. By Theorem 1.2 (in this case due to [6]), we have that there are at most $c(2, \ell)$ choices. Now there are $|U|$ choices for $i$ and so, given our bound on $|U|$, there are $\ell$ choices for $i \in U$ which have the same $[^2_\alpha]$. Now considering the $\ell$ columns $[^1_i]$ yields an $\ell \times \ell$ matrix in $A|_U$ with $1$'s on diagonal and $2$'s off the diagonal namely $I_\ell(2,1) \in T_\ell(3) \setminus T_\ell(2)$, a contradiction. Thus $\|C_{1,2}(i)\| > 0$ for less than $\ell \cdot c(2, \ell)$ choices $i$.

Assume $\|C_{0,2}(i)\| > 0$ for $2\ell \cdot c(2, \ell)$ choices $i$. Denote the choices by $V$. Then we have the following

$$
\begin{align*}
& \begin{pmatrix} i \\ V \setminus i \\ [m] \setminus V \end{pmatrix} \\
& \begin{pmatrix} 0 & 2 \\ \alpha & \alpha \\ \beta & \beta \end{pmatrix} < A
\end{align*}
$$

(14)

This case is a little more complicated because $\alpha$ may have up to one $1$. We choose a subset $W \subseteq V$ of the rows where $\alpha$ has no $1$'s. This can be done as follows. Choose some row $i_1 \in V$ and assume the corresponding choice of columns yields an $\alpha$ with a 1 in row $j_1 \in V$ and if not let $j_1 = i_1$. Now choose a row $i_2 \in V \setminus \{i_1, j_1\}$ and assume the corresponding $\alpha$ has a 1 in row $j_2 \in V$ and if not $j_2 = i_2$. Now choose a row $i_3 \in V \setminus \{i_1, j_1, i_2, j_2\}$ and assume the corresponding $\alpha$ has a 1 in row $j_3 \in V$ and if not $j_3 = i_3$. Continue in this way to form $W = \{i_1, i_2, \ldots, i_{2\ell \cdot c(2, \ell)}\}$ using the fact $|V| \geq 2\ell \cdot c(2, \ell)$, $|W| \geq \ell \cdot c(2, \ell)$ and for each $i \in W$ we have $\|C_{0,2}(i)\| > 0$ where we have on pair of cols in $A$ as in (14) with $\alpha$ having no $1$’s. Now repeat the above argument for the $(1,2)$-case to obtain an $\ell \times \ell$ matrix in $A$ with $0$’s on diagonal and $2$’s off the diagonal, namely $I_\ell(2,0) \in T_\ell(3) \setminus T_\ell(2)$, a contradiction. Thus $\|C_{0,2}(i)\| > 0$ for less than $2\ell \cdot c(2, \ell)$ choices $i$.

We deduce that for $|S| > 3\ell \cdot c(2, \ell)$, there exists a row $i$ with $|C_{1,2}(i)| = |C_{0,2}(i)| = 0$.

**Proof of Theorem 1.12** Let $A \in Avoid(m, 3, T_\ell(3) \setminus T_\ell(2) \cup H)$ be given. If $A$ contains a large identity (or its complement) then by Lemma 5.4 there exists a row $i$ with $C_{1,2}(i) = C_{0,2}(i) = \emptyset$. Note that $\|C_{0,1}(i)\|$ is $O(1)$ by Corollary 1.8 since $C_{0,1}(i)$ avoids $[01]$. Thus, we can delete row $i$ and at most $O(1)$ columns and obtain a simple matrix. Then induction on $m$ would yield the desired $O(m)$ bound. Our goal is to show that a large identity must occur.

Let $A_k$ be the submatrix of $A$ with column sum $k$. If, for any $L$, $\|A_k\| > c(3, L)$ then either $I_L(0,1) \prec A_k$, $I_L(1,0) \prec A_k$, or $T_L(0,1) \prec A_k$. We will take $L$ to be large.
We note that $H \prec I_L(0,1)$ and so the first case does not occur. In the second case $I_L(1,0) \prec A_k$ we have that $A_k$ and indeed $A$ does not have $[0]_0$ on the $L$ rows containing $I_L(1,0)$ else $H \prec A$. Then apply Lemma 5.1 using the (0,1)-complement and note that $\|C_{1,0}(i)\|$ is $O(1)$ by Corollary 1.8 since $H \prec [0 \ 1] \times [0 \ 1]$ and hence $[0 \ 1] \not\prec C_{0,1}(i)$. This yields that $\|A\|$ is $O(m)$ by induction on $m$.

In the third case, with the triangular matrix $T_L(0,1) \prec A_k$, let $A_k'$ be the matrix consisting of the columns from $A_k$ containing $T_L(0,1)$. Assume the columns of $A_k'$ are ordered consistent with $T_L(0,1)$. Let $A_k''$ be the submatrix obtained from $A_k'$ by deleting the $L$ rows containing $T_L(0,1)$. Then the column sums of $A_k''$ are decreasing from left to right. Let $S$ be the rows containing 1’s in the first column of $A_k''$. Every triple of columns in $T_L(0,1)$ has the submatrix $[001]$, so $A_k''$ does not contain any submatrix $[100]$ else $H \prec A_k'' \prec A$. Thus $A_k''|_S$ does not contain $[00]$. Also $A_k''$ has decreasing column sums from left to right. We proceed in a manner similar to the proof of Lemma 4.5. We first find a diagonal of entries either 0 or 2. By the pigeonhole principle, there is a long diagonal of 2’s or a long diagonal of 0’s. If there is a long diagonal of 2’s we apply Ramsey Theory as before. Large cliques involving 0’s are not allowed since $[0 \ 0]$ is forbidden, and hence we are forced to have a block of 2’s. This yields a submatrix $[1 \ 2 \ 2 \ \cdots \ 2]$ and so we proceed, as in proof of Lemma 4.5 considering the columns containing the 2’s. If we continue doing this for enough rows, we find a forbidden object. Hence, there must be a point where no sufficiently long diagonal of 2’s exists, so there is a long diagonal of 0’s. In this case, apply Ramsey Theory again. Recalling that we have no submatrix $[0 \ 0]$, the only configurations that result are either in $T_L(3)\setminus T_L(2)$ or an identity complement $I_L(1,0)$ for some large $t$. Given that $H \not\prec A$, we have that there is no $[0 \ 1]$ on any pair of the $t$ rows. This allows us to use Lemma 5.1.

If $\|A_k\|$ is bounded by a constant for all $k$ then $\|A\|$ is $O(m)$. If $\|A_k\|$ is a big enough constant then we obtain an $I_L(1,0) \prec A_k$ for some appropriately large $t$. By Lemma 5.1 we find some $i \in [m]$ with $\|C_{1,2}(i)\| = \|C_{0,2}(i)\| = 0$. As noted above, $\|C_{0,1}(i)\|$ is $O(1)$. Thus we can delete row $i$ of $A$ and at most $O(1)$ columns from $A$ to obtain a simple matrix in $\text{Avoid}(m-1,3, T_L(3)\setminus T_L(2) \cup H)$ and then apply induction.

6 Open problems

Some small examples of $F$ for which we have not handled $\text{forb}(m,3, T_L(3)\setminus T_L(2) \cup F)$ include:

$$K_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \text{ bound should be } O(m)$$

$$F_{0,2,1,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ bound should be } O(m)$$
\[ F_{1,1,1,0} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \text{ bound should be } O(m) \]

We would particularly like to have a general result that \( \text{forb}(m, 3; \mathcal{T}_e(3) \setminus \mathcal{T}_e(2) \cup ([0, 1] \times F)) \) is \( O(m \times \text{forb}(m, 3; \mathcal{T}_e(3) \setminus \mathcal{T}_e(2) \cup F)) \) matching the standard induction results for \((0, 1)\)-forbidden configurations.

Given a \((0, 1)\)-column \( \alpha \), we might consider a 3-matrix \( A \in \text{Avoid}(m, 3; \mathcal{T}_e(3) \setminus \mathcal{T}_e(2)) \) such that each column of \( A \) arises from \( \alpha \) by setting certain entries to 2. We deduce that \([0, 1] \not\prec A\) and so by Corollary 1.8 we have the interesting fact that \( \|A\| \) is \( O(1) \). In some sense the columns of \( A \) are a 3-matrix replacement for \( \alpha \). We were unable to exploit this for Problem 1.5.

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