Nonrelativistic Fundamental Quantum and Classical Wave Equations

Z. E. Musielak
Department of Physics, The University of Texas at Arlington, Arlington, TX 76019, USA
E-mail: zmusielak@uta.edu

Abstract. The irreducible representations of the extended Galilean group are used to derive infinite sets of symmetric and asymmetric second-order differential equations with constant coefficients. All derived equations are local and their Lagrangians exist. It is shown that the asymmetric equations are Galilean invariant but the symmetric ones are not. By specifying quantum and classical physical settings, the constants in the equations are determined and the fundamental wave equations, including the Schrödinger, Schrödinger-like and new asymmetric equations, are obtained; the derived wave equation is non-fundamental. Formulation of wave theories based on the fundamental and non-fundamental wave equations is considered, and physical implications of these theories on the wave description are discussed.

1. Introduction

The wave equation is one of the best known equations in classical physics and it describes the propagation of waves in a given background medium (e.g., [1-3]); the equation is called here the classical wave equation. However, the Schrödinger equation (e.g., [4,5]) of quantum mechanics (QM) is called the quantum wave equation. Differences between the classical and quantum wave equations include the physical meaning of the functions representing the waves, and the fact that the former are not Galilean invariant [4] while the latter are [5,6]. The main goal of this paper is to derive Galilean invariant classical and quantum wave equations, and consider their physical implications on the wave description in QM and classical physics.

The fact that the Schrödinger equation is Galilean invariant means that its form remains the same with respect to all transformations of the extended Galilean group $\mathcal{G}_e = [O(3) \otimes sB(3)] \otimes s[T(3+1) \otimes U(1)]$, where $O(3)$ and $B(3)$ are subgroups of rotations and boosts, respectively. In addition, $T(3+1)$ is an invariant Abelian subgroup of combined translations in space and time, and $U(1)$ is a one-parameter unitary subgroup [7,8]. In all inertial frames of reference that exist in the Galilean space and time, the Galilean metric preserves the same form, which means that it is invariant with respect to all transformations of $\mathcal{G}_e$; the observers associated with these inertial frames are called the extended Galilean observers. In other words, Galilean invariance means that the form of
the Schrödinger equation is the same for all extended Galilean observers. Moreover, the presence of $U(1)$ guarantees that the square of the absolute value of the wavefunction is also the same for the observers. This Galilean invariance of the Schrödinger equation is one of the basic requirements for the equation to be the fundamental equation of nonrelativistic QM.

The previous work showed [8] that the Schrödinger equation can be derived from the irreducible representations (irreps) of the invariant subgroup $T(3 + 1)$. Recently, the same approach was used to obtain a new nonrelativistic equation that was found to be also Galilean invariant [9]. Similar to the Schrödinger equation, the new equation is also asymmetric in space and time, and therefore it is called the new asymmetric equation. Nevertheless, these two equations are different and they lead to different quantum theories, which are so different that it was suggested that a new quantum theory based on the new asymmetric equation may instead describe dark matter [9], whose nature and origin still remain unknown (e.g., [10,11]). The Schrödinger equation and new asymmetric equation for the scalar wavefunctions are the only known Galilean invariant equations (see Section 2), and both are the quantum wave equations.

To describe different wave phenomena in classical physics, the classical (symmetric) wave equation is typically used, and this equation has a different form than the (asymmetric) Schrödinger equation of QM. However, it was shown [13-15] that in some special physical settings, a modified Schrödinger equation can be used to represent the time and space evolution of classical waves.

In this paper, the irreps of $T(3 + 1)$ are used to derive infinite sets of symmetric and asymmetric second-order partial differential equations (PDEs) for scalar wavefunctions; the coefficients in these equations are constants whose values are arbitrary but real. Being the second-order PDEs, all derived equations are local and their Lagrangians are given. It is shown that the symmetric PDEs are not Galilean invariant and that all asymmetric equations are. Locality, the existence of Lagrangians and Galilean invariance are required for these equations to be called fundamental (e.g., [16]). Additional requirement imposed in this paper is that among the constants in the symmetric and asymmetric PDEs, only one can be selected in such a way that the resulting equation applies to a given physical setting and describes the time and space evolution of waves in this setting. If, and only if, a given equation satisfies all these requirements, then it is called the fundamental equation.

Thus, the main aim of this paper is to use the infinite sets of PDEs resulting from the irreps of $T(3 + 1)$ to identify the fundamental and non-fundamental wave equations of classical and quantum physics in Galilean Relativity. The obtained results show that there are two fundamental quantum wave equations, two fundamental classical wave equations, and one non-fundamental classical wave equation. The derived equations are compared and formulation of wave theories based on these equations is discussed.

The paper is organized as follows: in Sect. 2, the basic equations, their Lagrangians and Galilean invariance are described; the quantum and classical wave equations are presented in Sect. 3 and 4, respectively; comparison of the wave equations is given in
2. Basic equations, Lagrangians and invariance

2.1. Group theory derivation

As shown in the previous work [8,12], the conditions that the scalar wavefunction \( \phi(t, \mathbf{x}) \), where \( t \) is time and \( \mathbf{x} \) represents 3D Euclidean space \( (x, y, z) \), transforms as one of the irreps of the extended Galilean group \( G_e \) are given by the following eigenvalue equations

\[
i \frac{\partial}{\partial t} \phi(t, \mathbf{x}) = \omega \phi(t, \mathbf{x}) ,
\]

and

\[
- i \nabla \phi(t, \mathbf{x}) = k \phi(t, \mathbf{x}) ,
\]

where \( \omega \) and \( k \) are labels of \( \phi(t, \mathbf{x}) \), which is an eigenfunction of the generators of the invariant Abelian subgroup \( T(3 + 1) \) [9]. These equations can be used to derive all equations of physics for scalar wavefunctions that are allowed to exist in the Galilean space and time. In general, the derived equations can be divided into two separate families, namely, the symmetric equations, with the same order of space and time derivatives, and the asymmetric equations, with different orders of space and time derivatives [9]. In general, the derived equations can be of any order [12] but in this paper only the second-order equations are considered.

The only second-order symmetric equation that can be derived from the eigenvalue equations is given by

\[
\left[ \frac{\partial^2}{\partial t^2} - C_1 \nabla^2 \right] \phi(t, \mathbf{x}) = 0 ,
\]

where \( C_1 = \omega^2/k^2 \) with \( k^{2n} = (\mathbf{k} \cdot \mathbf{k})^n \). Since \( C_1 \) is a real constant coefficient of any value, there is an infinite set of these second-order equations.

Two different asymmetric second-order equations resulting from Eqs. (1) and (2) are

\[
\left[ i \frac{\partial}{\partial t} + C_2 \nabla^2 \right] \phi(t, \mathbf{x}) = 0 ,
\]

and

\[
\left[ \frac{\partial^2}{\partial t^2} - iC_3 \mathbf{k} \cdot \nabla \right] \phi(t, \mathbf{x}) = 0 ,
\]

where \( C_2 = \omega/k^2 \) and \( C_3 = \omega^2/k^2 = C_1 \) are arbitrary constants, which means that there are two infinite sets of the second-order asymmetric equations.

The form of the asymmetric equation given by Eq. (4) is the same as that of the Schrödinger equation [5], except the coefficient \( C_2 \). Therefore, all equations of the same form as Eq. (4) are called the Schrödinger-like equations. However, the equation given by Eq. (5) with different coefficients \( C_3 \) are called the new asymmetric equations.
2.2. Lagrangian formalism

The Lagrangian formalism requires prior knowledge of a Lagrangian, and then it shows how to obtain the resulting dynamical equation from this Lagrangian. Typically, the Lagrangians are presented without explaining their origin because there no methods to derive them from first principle physical theories. Therefore, historically most dynamical equations were established first and only then their Lagrangians were found, often by guessing. Once the Lagrangians are known, the process of finding the resulting dynamical equations is straightforward and it requires substitution of these Lagrangians into the Euler-Lagrange (EL) equation. Despite some progress in deriving Lagrangians for physical systems described by ordinary differential equations (ODEs) (e.g., [17,18]), similar work for PDEs has only limited applications (e.g., [19-22]).

Let \( L(\phi, \partial_t \phi, \nabla \phi) \), where \( \partial_t = \partial/\partial t \), be a Lagrangian that satisfies the EL equation
\[
\frac{\partial L}{\partial \phi} - \partial_t \left( \frac{\partial L}{\partial \partial_t \phi} \right) - \nabla \cdot \left( \frac{\partial L}{\partial \nabla \phi} \right) = 0.
\] (6)
Substituting this Lagrangian into Eq. (6) gives the required dynamical equation if, and only if, the Lagrangian is a priori known. In case the equation is given first, its Lagrangian must be constructed in such a way that when substituted into Eq. (6) the desired dynamical equation is obtained; this is known as the Lagrangian formalism.

Since the symmetric equation given by Eq. (3) is hyperbolic, its Lagrangian can be constructed [21] and the result is
\[
L_s(\partial_t \phi, \nabla \phi) = \frac{1}{2} \left[ C_1^{-1} (\partial_t \phi)^2 - (\nabla \phi)^2 \right].
\] (7)
It is easy to verify that substitution of the Lagrangian \( L = L_s(\partial_t \phi, \nabla \phi) \) into Eq. (6) gives the required symmetric equation.

The asymmetric equations given by Eqs (4) and (5) are parabolic, thus, their Lagrangians must be of a special form that involves both \( \phi \) and its complex conjugate \( \phi^* \). Then, by considering variations with respect to \( \phi^* \), the equation for \( \phi \) is obtained, and alternatively, variations with respect to \( \phi \) give the equation for \( \phi^* \) [22].

For Eq. (4), the Lagrangian can be written as
\[
L_{a1}(\phi, \phi^*, \partial_t \phi, \partial_t \phi^*, \nabla \phi, \nabla \phi^*) = \frac{1}{2} i \left( \phi^* \partial_t \phi - \phi \partial_t \phi^* \right) - C_2 (\nabla \phi^*) \cdot (\nabla \phi).
\] (8)
This Lagrangian gives the first asymmetric equation (see Eq. (4)) after it is substituted into the EL equation in which \( \phi \) is replaced by \( \phi^* \).

For the second asymmetric equation given Eq. (5), the Lagrangian becomes
\[
L_{a2}(\phi, \phi^*, \partial_t \phi, \partial_t \phi^*, \nabla \phi, \nabla \phi^*) = (\partial_t \phi^*) (\partial_t \phi) - \frac{1}{2} i C_3 \left[ \phi (k \cdot \nabla \phi^*) - \phi^* (k \cdot \nabla \phi) \right].
\] (9)
To obtain Eq. (5), this Lagrangian must be substituted into the following EL equation

$$\frac{\partial L_{a2}}{\partial \phi^*} - \partial_t \left( \frac{\partial L_s}{\partial (\partial_t \phi^*)} \right) - (\mathbf{k} \cdot \nabla) \cdot \left( \frac{\partial L_s}{\partial (\mathbf{k} \cdot \nabla \phi^*)} \right) = 0. \quad (10)$$

The symmetric and two asymmetric equations are local, and the above results demonstrate that each equation has its Lagrangian that can be used to derive this equation from the corresponding EL equation. Locality and the existence of Lagrangians are the two requirements for the equations to be called fundamental. The third important requirement is Galilean invariance of these equations that is now investigated.

### 2.3. Galilean invariant equations

Having obtained the three infinite sets of second-order PDEs, their Galilean invariance must be investigated. Galilean invariance requires that Eqs. (3), (4) and (5) are of the same form when transformed to any inertial frame of reference. Let $S$ and $S'$ be two intertial frames moving with respect to each other with the velocity $\mathbf{v} = \text{const}$, which allows writing a boost as $\mathbf{x} = \mathbf{x}' + \mathbf{v}t'$ with $t' = t$. Then, the Galilean metric in space is $ds^2 = d\mathbf{x} \cdot d\mathbf{x} = dx^2 + dy^2 + dz^2$ with $ds'^2 = ds^2$, and in time $dt'^2 = dt^2$. By performing the Galilean transformations given by $G_\epsilon$, which involve translations in space and time, rotations and boosts, Galilean invariance of the metrics can be verified, and Galilean invariance of the derived equations can be determined.

The previously obtained results [12] demonstrated that none of the symmetric equations is Galilean invariant, therefore, these equations cannot be used to formulate any fundamental theory of physics in Galilean Relativity. However, Eq. (4) is Galilean invariant if, and only if, the wave function $\phi(t, \mathbf{x})$ transforms as

$$\phi(t, \mathbf{x}) = \phi'(t', \mathbf{x}') e^{i\eta(t', \mathbf{x}')}, \quad (11)$$

where

$$\eta(t', \mathbf{x}') = \left( \mathbf{v} \cdot \mathbf{x}' + v^2 t'/2 \right)/(2C_2') = \mathbf{v} \cdot \mathbf{r}'/(2C_2'), \quad (12)$$

is the phase factor [5,6], with $\mathbf{r}' = \mathbf{x}' + \mathbf{v}t'/2$, and with the condition $|\phi(t, \mathbf{x})|^2 = |\phi'(t', \mathbf{x}')|^2$ required by the extended Galilean group. Moreover, for Galilean invariance it is also necessary that $C_2 = C_2'$ is valid in all inertial frames. A formal proof of the latter requirement is given in [9], which shows that for $C_2 = C_2'$ to be satisfied, the following condition must be satisfied

$$\left( \frac{\omega}{C_2} - k^2 \right) \left( \frac{v^2}{4C_2} - \mathbf{k} \cdot \mathbf{v} \right) = 0. \quad (13)$$

By assuming that the second term on the LHS of this equation is non-zero, then $C_2 = \omega/k^2$ and the proof that $C_2 = C_2'$ is completed [9]. In general, it is also possible that the second term is also zero, which gives an interesting relationship between the labels $\omega$ and $k$ of the irreps and the velocity $\mathbf{v}$ (see Section 3).
The form of the asymmetric equation given by Eq. (5) is not Galilean invariant. However, it was shown [9] that the equation becomes Galilean invariant after the following transformation of the wavefunction $\phi(x,t) = \phi(r)$, which gives

$$\frac{d}{d(k \cdot r)} \left[ \frac{d\phi}{d(k \cdot r)} - i \frac{4C_3}{v^2} \phi \right] = 0 ,$$  \hspace{1cm} (14)

where $r = x + vt/2$. The transformation allows converting the PDE (see Eq. 5) into the ODE (see Eq. 14). The derived equation is Galilean invariant if, and only if, $C_3 = \omega^2/k^2 = \omega'^2/k'^2C'_3$, which is valid [9] when

$$k \cdot r = k' \cdot r' = \text{const} .$$  \hspace{1cm} (15)

The above results show that there are two infinite sets of equations (see Eqs 4 and 14) that are local, Galilean invariant, and have their Lagrangians. The sets are infinite because the $C_2$ and $C_3$ may have arbitrary real values. Thus, the constants can only be determined once a physical setting to be described by these equations is specified. The physically determined constant allows selecting one equation out of infinite, and this equation is called here the fundamental equation because it is local, posses its Lagrangian and its form remains the same for all Galilean observers. In the following two sections, the fundamental quantum and classical wave equations are obtained for the corresponding microscopic and macroscopic physical systems.

3. Quantum wave equations

3.1. The Schrödinger equation

To describe matter at its microscopic level, the quantum nature of particles or the wave-particle duality must be taken into account [5]. Following [9], it can be shown that the de Broglie relationship gives $C_2 = \hbar/2m$, which means that only one equation is selected from the infinite set of Galilean invariant equations given by Eq. 4. The selected equation is the well-known Schrödinger equation [5,6,8,9]

$$i \frac{\partial}{\partial t} + \frac{\hbar}{2m} \nabla^2 \phi(t, x) = 0 ,$$  \hspace{1cm} (16)

and it is called the fundamental wave equation of QM, or the fundamental quantum wave equation. In this equation, $\phi(t, x)$ represents the probability amplitude, thus, its probability density $|\phi(t, x)|^2$ is needed to make predictions that can be verified experimentally. The condition $|\phi(t, x)|^2 = |\phi'(t', x')|^2$ required by the extended Galilean group guarantees that the probability density is the same for all Galilean observers even if the phase factor (see Eq. 12) is not.

The Schrödinger equation has the well-known Lagrangian [5,21] that can be written as

$$L_{Sch}(\phi, \phi^*, \partial_t \phi, \partial_t \phi^*, \nabla \phi, \nabla \phi^*) = \frac{1}{2} i (\phi^* \partial_t \phi - \phi \partial_t \phi^*)$$

$$\quad - \frac{\hbar}{2m} (\nabla \phi^*) \cdot (\nabla \phi) .$$  \hspace{1cm} (17)
The variations with respect to $\phi^*$ in the EL equation give Eq. (16), and the corresponding equation for the conjugate wavefunction is obtained by performing the variations with respect to $\phi$. The Lagrangian is Galilean invariant [5].

The obtained Schrödinger equation is local, Galilean invariant and has its Galilean invariant Lagrangian, which means that it is the fundamental quantum wave equation.

3.2. The new asymmetric equation

Another possible microscopic description of matter can be given by choosing $C_3 = \varepsilon/2m$, where $\varepsilon$ is a quanta of energy that is independent from frequency [9]. Defining the kinetic energy of a particle as $\varepsilon_v = mv^2/2$, Eq. (14) can be written as

$$\frac{d}{d(k \cdot r)} \left[ \frac{d\phi}{d(k \cdot r)} - \frac{i\varepsilon_o}{\varepsilon_v} \phi \right] = 0 .$$

(18)

It is easy to verify that this equation does not describe any microscopic property of ordinary matter, therefore, it was suggested that this equation may correctly represent the microscopic properties of dark matter, with $\varepsilon$ being a quanta of energy of dark matter [9]. Since neither the physical nature nor origin of dark matter are currently known [23,24], further studies are required to verify this hypothesis.

Since the fundamental new asymmetric wave equation (see Eq. (18)) is the ODE, there are methods to construct its Lagrangian (e.g., [17,18]). Defining $z = (k \cdot r)$, $d_z\phi = d\phi/dz$ and $d_z^2\phi = d^2\phi/dz^2$, Eq. (18) may be written as

$$d_z^2\phi(z) - \frac{i\varepsilon_o}{\varepsilon_v}d_z\phi(z) = 0 ,$$

(19)

for which the following Lagrangian [18] is found

$$L_{asy}(d_z\phi, z) = \frac{1}{2} [d_z\phi(z)]^2 e^{-i\varepsilon_o z/\varepsilon_v} .$$

(20)

By substituting this Lagrangian into the following EL equation

$$\frac{d}{dz} \left( \frac{\partial L_{asy}}{\partial (\partial d_z\phi)} \right) - \frac{\partial L_{asy}}{\partial \phi} = 0 ,$$

(21)

the new asymmetric quantum wave equation given by Eq. (18) is obtained. With $k \cdot r = k' \cdot r' = \text{const}$ and $\phi(r) = \phi'(r')$, the Lagrangian is Galilean invariant.

The obtained new asymmetric equation is local, Galilean invariant and has its Galilean invariant Lagrangian, which means that it is the fundamental quantum wave equation.

4. Classical wave equations

The procedure described in the previous section shows how to use the physical properties of matter to select the fundamental quantum wave equations that describe its microscopic structure. The same procedure can also be used to find the fundamental classical wave equations that describe the propagation of different waves in a uniform background medium. In the following, such wave equations are derived after a brief description of the wave classical equation.
4.1. Non-fundamental wave equation

The symmetric equation given by Eq. (3), with its coefficient $C_1$ being expressed in terms of the eigenvalues (see Eqs 1 and 2), can be written as

$$\left[ \frac{\partial^2}{\partial t^2} - \frac{\omega^2}{k^2} \nabla^2 \right] \phi(t, x) = 0 .$$

(22)

To convert this equation into the wave equation, the eigenvalues $\omega$ and $k$ must be treated as the wave frequency and wavenumber, respectively. In this case, the Fourier transforms of $\phi$ in space and time become the solutions of this equation. Defining the wave characteristic speed as $c_w = \omega/k$, Eq. (22) becomes the well-known wave equation written in its standard form

$$\left[ \frac{\partial^2}{\partial t^2} - c_w^2 \nabla^2 \right] \phi(t, x) = 0 .$$

(23)

The Lagrangian [21] for the wave equation can be written as

$$L_w(\partial_t \phi, \nabla \phi) = \frac{1}{2} \left[ c_w^{-2}(\partial_t \phi)^2 - (\nabla \phi)^2 \right] .$$

(24)

It is easy to verify that substitution of the Lagrangian $L = L_w(\partial_t \phi, \nabla \phi)$ into Eq. (6) gives the classical wave equation (see Eq. (23)).

The obtained wave equation is local and possesses its Lagrangian, but neither the equation nor its Lagrangian is Galilean invariant [4,12] because $c_w$ is a frame dependent quantity. Therefore, the lack of Galilean invariance of this wave equation prevents it from being called fundamental.

4.2. Fundamental Schrödinger-like wave equation

There are infinitely many Galilean invariant Schrödinger-like equations given by Eq. (4) and these equations are local and have their Lagrangians (see Eq. 8). To make one of these equations fundamental classical wave equation, the constant $C_2$ must be evaluated for classical waves. Galilean invariance of Eq. (4) requires that $C_2 = C_2' = \text{const}$, which is valid only when the condition given by Eq. (13) is satisfied. Then, the following three cases can be considered:

(i) \[ \left( \frac{\omega}{C_2} - k^2 \right) = 0 \quad \text{and} \quad \left( \frac{v^2}{4C_2} - \mathbf{k} \cdot \mathbf{v} \right) \neq 0 , \]

which gives $C_2 = \omega/k^2 = c_w^2/\omega$, and Eq. (4) with $C_2 = c_w^2/\omega$ becomes the fundamental classical wave equation;

(ii) \[ \left( \frac{\omega}{C_2} - k^2 \right) \neq 0 \quad \text{and} \quad \left( \frac{v^2}{4C_2} - \mathbf{k} \cdot \mathbf{v} \right) = 0 , \]

with $C_2 = v^2/4(\mathbf{k} \cdot \mathbf{v})$ and $C_2 \neq c_w^2/\omega$, which contradicts the original derivation of Eq. (4), thus, there is no fundamental wave equation with this value of $C_2$; and

(iii) \[ \left( \frac{\omega}{C_2} - k^2 \right) = 0 \quad \text{and} \quad \left( \frac{v^2}{4C_2} - \mathbf{k} \cdot \mathbf{v} \right) = 0 , \]

where $C_2 = \omega/k^2 = c_w^2/\omega$ is substituted into the second relationship to obtain $\omega = k v \cos \theta/4$, with $\theta$ being the angle between $\mathbf{k}$ and $\mathbf{v}$. Since $c_w = v \cos \theta/4$,
Eq. (4) with $c_w^2/\omega = v \cos \theta/4k$ becomes the fundamental classical wave equation. The dependence of $c_w$ exclusively on $v$ and $\theta$ makes this wave unphysical because in typical physical situations $c_w$ depends on physical parameters in the background medium; for this reason, this case will not be further discussed in this paper.

The above results show that the fundamental Schrödinger-like wave equation for classical waves can be written as

$$
\left[ i \frac{\partial}{\partial t} + \frac{c_w^2}{\omega} \nabla^2 \right] \phi(t, \mathbf{x}) = 0 ,
$$

where both $c_w$, the characteristic wave speed, and $\omega$, the wave frequency, are frame-dependent quantities; however, the ratio $c_w^2/\omega$ remains constant in all inertial frames of reference. An interesting result is that $\omega$ appears explicitly in this fundamental wave equation without a Fourier transform in time being made; this is a novel phenomenon in physical theories of waves.

Another new phenomenon is the constant value of the ratio $c_w^2$ to $\omega$, which makes the wave description much easier by the fundamental Schrödinger-like wave equation than that given by the classical wave equation in which there is no constant coefficient. Thus, the coefficient $c_w^2/\omega$ plays similar role for classical waves in Galilean relativity as the speed of light $c$ plays in Special Theory of Relativity (STR) for electromagnetic waves. However, it must be kept in mind that while $c = \text{const}$ is the basic principle of Nature used as the foundation of STR, the ratio $c_w^2/\omega = \text{const}$ is a condition (see Eq. (15)) necessary for Galilean invariance.

The Lagrangian for Eq. (28) is the same as that given by Eq. (17), except the constant coefficient that must be replaced to obtain

$$
L_{Sw}(\phi, \phi^*, \partial_t \phi, \partial_t \phi^*, \nabla \phi, \nabla \phi^*) = \frac{1}{2} i (\phi^* \partial_t \phi - \phi \partial_t \phi^*)
$$

$$
- \frac{c_w^2}{\omega} (\nabla \phi^*) \cdot (\nabla \phi) .
$$

This Lagrangian gives the fundamental Schrödinger-like wave equation when substituted to the EL equation for the variations in $\phi^*$. On the other hand, the variations in $\phi$ lead to the fundamental Schrödinger-like wave equation for $\phi^*$, which becomes important when the probability density $|\phi|^2$ must be calculated. However, in theories of classical waves $|\phi|^2$ does not play any significant role as it does in quantum mechanics.

The fundamental Schrödinger-like wave equation is Galilean invariant. To demonstrate that the Lagrangian given by Eq. (29) is also Galilean invariant, the Galilean transformations are applied and the transformed Lagrangian is obtained. After taking into account $t = t'$, $\nabla = \nabla'$ and $c_w^2/\omega = c_w'^2/\omega' = \text{const}$, the transformed Lagrangian becomes

$$
L_{Sw}'(\phi', \phi^*,', \partial_t \phi', \partial_t \phi^*', \nabla \phi', \nabla \phi'^*) = \frac{1}{2} i (\phi'^* \partial_t \phi' - \phi' \partial_t \phi'^*)
$$

$$
- \frac{c_w'^2}{\omega'} (\nabla \phi'^*) \cdot (\nabla \phi') .
$$
The Lagrangian was obtained by using the condition $\nabla^2 \eta(t', x') = 0$, where
\[
\eta(t', x') = i\omega \left( v \cdot x' + v^2 t'/2 \right) / (2c_w^2). \tag{31}
\]
The fact that Eqs. (29) and (30) are of the same form demonstrates that the Lagrangian for the fundamental Schrödinger-like wave equation is Galilean invariant.

The above results show that Schrödinger-like wave equation has all the characteristics, namely, locality, Galilean invariance and the Galilean invariant Lagrangian that are required for this equation to be called fundamental. A new phenomenon is the existence of the phase factor given by Eq. (31), which is a frame-dependent quantity that is known to all Galilean observers because they already agree on the form of the fundamental Schrödinger-like wave equation. As a result, the observers must extract the phase factors from the solutions in their inertial frame, and the remaining parts of the solutions will be the same for all Galilean observers. In other words, the fundamental theory guarantees that the observers in all inertial frames have identical description of classical waves.

### 4.3. Fundamental new asymmetric wave equation

The new asymmetric equation given by Eq. (14) is already written in its Galilean invariant form. There are infinitely many such equations as the coefficient $C_3$ may be any real number. As shown in Section 3, by selecting $C_3$, one equation may be chosen out of the infinite number of equations, and the selected equation becomes the fundamental quantum wave equation (see Eq. (18), which may describe dark matter. Now, for classical waves, $C_3$ must be selected so that the resulting equation describes these waves. This seems to be straightforward as $C_3 = \omega^2/k^2$, which gives $C'_3 = c_w^2$ (see Section 4.1), and Eq. (14) becomes
\[
\frac{d}{d(k \cdot r)} \left[ \frac{d\phi}{d(k \cdot r)} - \frac{4c_w^2}{v^2} \phi \right] = 0. \tag{32}
\]
Since $\phi(r) = \phi(r')$, $k \cdot r = k' \cdot r'$ = const and $c_w = c'_w = \text{const}$, Eq. (19) is Galilean invariant without introducing any phase factor. Note also that $c_w$ is normalized by the velocity $v$ with which the inertial frames move with respect to each other.

Using Eq. (20), the Lagrangian for Eq. (32) can be written as
\[
L_{aw}(dz\phi, z) = \frac{1}{2} [dz\phi(z)]^2 e^{-4ic_w^2/v^2}, \tag{33}
\]
and its substitution into the EL equation (see Eq. 21) gives the desired wave equation.

The derived Lagrangian depends explicitly on the variable $z$ that involves both $x$ and $t$. In Classical Mechanics, the dependence of Lagrangians on $t$ implies that the total energy of a dynamical system is not conserved and, as a result, the energy function must be calculated [3]. For physical systems with their Lagrangians explicitly time-dependent, the exponentially decaying or increasing terms are present, like in the well-known Caldirola-Kanai Lagrangian [25,26], originally written for the Bateman oscillator [27,28].
However, the Lagrangian given by Eq. (33) is of a different form as its exponential term is periodic in \( z \) (or \( x \) and \( t \)) instead. Despite the presence of the periodic term in \( L_{aw}(d_z \phi, z) \), the resulting fundamental wave equation is independent from this term. Since the first term on the RHS in Eq. (33) represents the wave kinetic energy, the exponential terms shows that this energy is required to oscillate in time and space in the Lagrangian, so that the correct wave equation is obtained. This is a new phenomenon in theories of classical waves described by the fundamental new asymmetric classical wave equation; therefore, it is suggested that the Lagrangian \( L_{aw}(d_z \phi, z) \) forms a separate class among all Lagrangians known in physics.

To demonstrate that \( L_{aw}(d_z \phi, z) \) is Galilean invariant, the Galilean transformations are applied and the following transformed Lagrangian is found

\[
L'_a(d_z' \phi', z') = \frac{1}{2} \left[ (d_z' \phi'(z'))^2 e^{-4ic^2_w/v^2} \right],
\]

which is of the same form as the original Lagrangian given by Eq. (26). The reason is that \( z = z' = \text{const} \) or \( \mathbf{k} \cdot \mathbf{r} = \mathbf{k}' \cdot \mathbf{r}' = \text{const} \) and \( \phi(\mathbf{r}) = \phi'(\mathbf{r}') \). Moreover, \( c^2_w/v^2 = \text{const} \). Therefore, the Lagrangian \( L_{aw}(d_z \phi, z) = \frac{1}{2} [d_z \phi(z)]^2 \exp(-4ic^2_w/v^2) \) is Galilean invariant.

The obtained results show that the new asymmetric wave equation is local, Galilean invariant and it has Galilean invariant Lagrangian, thus, the equation has the required characteristics to be called fundamental.

5. Comparison of wave equations and physical implications

The wave equations derived in this paper can be classified into two separate families: the quantum and classical wave equations. There are two fundamental quantum (Schrödinger and new asymmetric) wave equations. However, among the classical wave equations, one (wave) equation is non-fundamental and two (Schrödinger-like and new asymmetric) equations are fundamental. In the following, the equations within each family are compared and their physical implications on the wave description are discussed.

5.1. Fundamental quantum wave equations

The Schrödinger and new asymmetric quantum wave equations are both fundamental but their specific forms and description of physics at microscopic level are different. Many applications of the Schrödinger equation to atomic physics of ordinary matter are commonly known and well-established [5,22], with the theoretical predictions based on this equation being verified experimentally. However, the new asymmetric quantum wave equation predicts a different structure of the microscopic world, which seems to be not observable in ordinary matter. On the other hand, it is known that the equation is as fundamental as the Schrödinger equation, therefore, it was suggested that it may be used to describe the microscopic structure and evolution of dark matter [9], which
is more abundant in our Universe than ordinary matter [10,11]. Future studies are required to either verify or reject this conjecture.

As pointed out in Section 3, the Schrödinger equation is Galilean invariant only when the frame-dependent phase factor \( \eta(t', x') \) is used. Nevertheless, the frame-dependence does not affect the final results because in QM the condition \( |\phi(t, x)|^2 = |\phi'(t', x')|^2 \) must be satisfied in all inertial frame of references. Now, the new asymmetric quantum wave equation remains Galilean invariant without any phase factor. It must be also noted that both equations were derived from the same irreps of the extended Galilean group, and that they are the only nonrelativistic fundamental quantum wave equations for scalar wavefunctions that exist in Galilean space and time.

Physical implications of the Schrödinger equation on modern physics are well-known and described in most QM textbooks (e.g., [5,22]). However, physical implications of the new asymmetric quantum wave equation on modern physics still remain to be determined. It is clear from the quantum theory developed using this equation that the theory is unlikely to be applicable to ordinary matter at its microscopic level; hence, the suggestion that it may describe dark matter [9]. Another option is that this equation may only be limited to classical waves as it is now discussed.

5.2. Non-fundamental and fundamental classical wave equations

Three different wave equations were derived in this paper for classical waves by using the irreps representations of the extended Galilean group \( \mathcal{G}_c \). In the derived equations, their constants were specified so they describe classical waves. These equations are: the non-fundamental wave equation, the fundamental Schrödinger-like wave equation and the fundamental new asymmetric wave equation. For each wave equation its Lagrangian was found, and it was shown that the Lagrangian for the non-fundamental equation is not Galilean invariant, but both Lagrangians for the fundamental equations are Galilean invariant.

The main physical difference between the non-fundamental and fundamental wave equations is that the latter preserve their forms and the wave description resulting from these two equations for all Galilean observers, while the former is frame-dependent. To demonstrate the differences, the Fourier transforms in space and time are performed for all three wave equations. As expected, the resulting three dispersion relation looks identical and can be written as

\[
\omega^2 = c_w^2 k^2 .
\] (35)

However, the physical meaning of these dispersion relationships is different because for the wave equation the wave frequency \( \omega \), the wavenumber \( k \), and the characteristic wave speed \( c_w \) are frame-dependent. For the fundamental Schrödinger-like wave equation, the resulting dispersion relation is frame-independent since \( c_w^2 / \omega = \text{const} \) in all inertial frames of references. The most interesting is the fundamental new asymmetric wave equation for which both \( \omega^2 / k^2 = \text{const} \) and \( c_w = \text{const} \) are the same for all Galilean observers. Moreover, this is the only wave equation that allows for description of waves
with their characteristic wave speed $c_w$ being constant in all inertial frames of reference in Galilean Relativity. The wave description by the fundamental Schrödinger-like wave equation requires $c_w^2/\omega$ to be constant, and there are no constants in describing waves by the classical wave equation.

In general, Lagrangians possess less symmetry than the dynamical equations resulting from them due to the assumptions on which the Noether theorem is based [29,30]. The best known example is the law of inertia, whose dynamical equation is Galilean invariant but its Lagrangian is not [31,32]. However, a method to restore Galilean invariance of the Lagrangian was developed and applied to the law of inertia [33]. In case of the classical wave equation, neither the equation nor its Lagrangian are Galilean invariant. However, the Lagrangians for both fundamental wave equations are Galilean invariant.

The main physical implication of the above comparison is that the frame-independent theories of waves can only be constructed by using either the fundamental Schrödinger-like wave equation or the fundamental new asymmetric wave equation. Both theories are fundamental, which means that their wave description is the same for all Galilean observers. The two equations that can be used wave theories are now compared.

5.3. Fundamental classical wave equations

The fundamental Schrödinger-like and new asymmetric classical wave equations are of different forms despite the fact that they are both asymmetric. This includes differences in the orders of time and space derivatives but also in their constant coefficients. Specifically, in the fundamental Schrödinger-like wave equation, the ratio $c_w^2/\omega$ remains the same for all inertial observers, which implies that both the wave speed $c_w$ and the wave frequency $\omega$ depend on the frames of reference. This shows that $\omega$ appears explicitly in this equation without performing the Fourier transform in time. However, in case of the fundamental new asymmetric classical wave equation, it is $c_w$ that remains the same for all Galilean observers and $\omega$ is not present this wave equation.

Another main difference between these two fundamental wave equations is the phase factor $\eta(t', \mathbf{x}')$, which guarantees that the fundamental Schrödinger-like classical wave equation is Galilean invariant. Using Eq. (12) and the definition of $C_2 = \omega/k^2 = c_w^2/\omega$, the phase factor can be written as

$$\phi(t, \mathbf{x}) = \phi'(x', t') e^{i(\mathbf{v} \cdot \mathbf{r}' + \omega t'/2)} ,$$

where $\mathbf{r}' = \mathbf{x}' + \mathbf{v} t'/2$.

The presence of the phase factor and its dependence on $\omega$ is a new phenomenon in theories of classical waves. Formally, the extended Galilean group requires that $|\phi(t, \mathbf{x})|^2 = |\phi'(x')|^2$; this condition is only necessary for the fundamental quantum wave equation in which $\phi(t, \mathbf{x})$ represents the probability amplitude and $|\phi(t, \mathbf{x})|^2$ its probability density (see Section 3). However, the condition is not required for classical waves for which full solutions for $\phi(t, \mathbf{x})$ are needed. Since $\mathbf{v} \cdot \mathbf{r} \neq \mathbf{v} \cdot \mathbf{r}'$, the solutions
have different phase factors in different inertial frames of reference. Fortunately, all Galilean observers know the phase factor and therefore they can calculate it in their frame of reference. Then, by eliminating the phase factor in each inertial frame from the solutions, all observers obtain \( \phi(t, x) = \phi(t', x') \), which is required for the same wave description in all their inertial frames of reference.

As compared to the fundamental Schrödinger-like classical wave equation, the fundamental new asymmetric classical wave equation does not require any phase factor and it is valid when the characteristic wave speed \( c_w \) is the same for all Galilean observers. Both facts make a wave theory based on the fundamental new asymmetric classical wave equation to be simpler but yet more powerful than that based on the fundamental Schrödinger-like classical wave equation.

The fact that the characteristic wave speed \( c_w \) in the fundamental new asymmetric classical wave equation is the same for all Galilean observers is a new result, which seems to resemble the speed of light \( c \) being constant in all inertial frames in STR. However, the main difference is that \( c = \text{const} \) in STR is its foundation principle that operates in Nature but the fact that \( c_w = \text{const} \) in Galilean Relativity results from the very special choice of the variables \((k \cdot r)\) describing the wave propagation.

6. Conclusions

This paper deals with derivation of classical and quantum wave equations from the irreps of the extended Galilean group. One infinite set of symmetric (in space and time) and two infinite sets of asymmetric second-order partial differential equations, with constant coefficients of arbitrary real values, are derived. It is shown that the equations are local and their Lagrangians exist. Moreover, all asymmetric equations are Galilean invariant but the symmetric equations are not. Typically, locality, the existence of Lagrangians, and Galilean invariance are required for the equations to be called fundamental. However, in this paper, an equation is called fundamental if, and only if, the value of its constant coefficient is determined by a physical problem, whose evolution in time and space the equation describes.

By specifying the constants in the symmetric and asymmetric equations, the fundamental quantum and classical wave equations are obtained out of the infinite sets of equations. The selected symmetric equation becomes the classical wave equation, which is not fundamental. However, one infinite set of asymmetric equations gives the Schrödinger equation of QM as well as the Schrödinger-like wave equation for classical waves, and both equations are fundamental. Moreover, the other infinite set allows finding two more fundamental equations, namely, the new asymmetric quantum and classical wave equations. By using the fundamental wave equations, fundamental theories of quantum and classical waves can be formulated, with QM based on the Schrödinger equation being well-established and best known. The fundamental new asymmetric quantum wave equation may be used to describe dark matter \([9]\). However, both the fundamental Schrödinger-like and new asymmetric classical wave equations
may be used to formulate fundamental theories of classical waves, with preferences given to the new asymmetric classical wave equation because of its properties described in this paper.

References

[1] W.C. Elmore, M.A. Heald, Physics of Waves, Dover Publ., Inc., New York, 1969
[2] I.G. Main, Vibrations and Waves in Physics, Cambridge Uni. Press, New York, 1993
[3] J.Y. José, E.J. Saletan, Classical Dynamics, A Contemporary Approach, Cambridge Univ. Press, Cambridge, 2002
[4] J.-M. Levy-Leblond, Comm. Math. Phys. 6 (1967) 286
[5] E. Merzbacher, Quantum Mechanics, Wiley & Sons, Inc., New York, 1998
[6] A.B. van Oosten, Apeiron 13 (2006) 449
[7] J.-M. Levy-Leblond, J. Math. Phys. 12 (1969) 64
[8] Z.E. Musielak and J.L. Fry, Ann. Phys. 324 (2009) 296
[9] Z.E. Musielak, Int. J. Mod. Phys. A, 28 (2021) 2150042 (12pp)
[10] K. Freeman, and G. McNamara, In Search of Dark Matter, Springer, Praxis, Chichester, 2006
[11] J.A. Frieman, M.B. Turner, and D. Huterer, Ann. Rev. Astr. Astrophys. 46 (2008) 385
[12] Z.E. Musielak and J.L. Fry, Int. J. Theor. Phys. 48 (2009) 1194
[13] T. Aktosun, R. Weber, Inverse Problems 22 (2006) 89
[14] T. Aktosun, A. Machuca, P. Sacks, Inverse Problems 33 (2017) 115002
[15] H. White, P. Bailey, J. Lawrence, J. George and J. Vera, J., Phys. Open 1 (2019) 100009
[16] L. Susskind, A. Friedman, Special Relativity and Classical Field Theory, Basic Books, New York, 2017
[17] M.C. Nucci and P.G.L. Leach, J. Math. Phys. 48 (2007) 123510
[18] Z.E. Musielak, J. Phys. A Math. Theor. 41 (2008) 055205
[19] J. Lopuszanski, The Inverse Variational Problems in Classical Mechanics, World Scientific, Singapore, 1999
[20] Z.E. Musielak, J. Phys. A Math. Theor. 43 (2010) 425205
[21] G.B. Wignman, Linear and Nonlinear Waves, Wiley & Sons, Inc., New York, 1990
[22] N.A. Daughtey, Lagrangian Interactions, Addison-Wesley Publ. Comp., Inc., Sydney, 1990
[23] J.A. Frieman, M.B. Turner, M.B. and D. Huterer, Ann. Rev. Astr. Astrophys. 46 (2008) 385
[24] K. Sugita, Y. Okamoto and M. Sekine, Int. J. Theor. Phys. 47 (2008) 2875
[25] P. Caldirola, Nuovo Cim. 18 (1941) 393
[26] E. Kanai, Prog. Theor. Phys. 3 (1948) 44
[27] H. Bateman, Phys. Rev. 38 (1931) 815
[28] L.C. Vestal, Z.E. Musielak, Physics 3 (2021) 449
[29] S. Hojman, J. Phys. A: Math. Gen. 17 (1984) 2399
[30] S. Hojman, J. Phys. A: Math. Gen. 27 (1992) L59
[31] L.D. Landau and E.M. Lifshitz, Mechanics, Pergamon Press, Oxford, 1969
[32] J.-M. Levy-Leblond, Comm. Math. Phys. 12 (1969) 64
[33] Z. E. Musielak and T. B. Watson, Phys. Let. A 384 (2020) 126642