A Relational Concept of Machian Relativity

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Abstract

Mach’s principle fits into the wider "relational principle", advocating that not only inertia, but also space and time emerge from the interaction of matter. Concepts of a Machian/relational theory are proposed, where inertia and energy are defined as mutual properties between pairs of objects. Due to Berkeley, only radial motion represents kinetic energy between (point) masses, which is the basis of anisotropic inertia, which in turn underlies the relational principle. The Newtonian definition of potential energy is considered a model for Machian inertia, leading to a frame independent definition of Machian kinetic energy, which comprises of the Newtonian terms (relative to the "fixed stars") and small anisotropic Machian energy terms between objects. The latter account for relativistic trajectories, such as the anomalous perihelion precession and Lense-Thirring frame dragging. However, relativistic effects of remote observation (e.g. time dilation) demand an isotropic model. A relational spacetime metric is derived, which provides an isotropic coordinate transform of the anisotropic Machian model, yielding a relational model which matches GR expressions for relativistic trajectories and effects of remote observation. Therefore, the experimental verification of GR in these cases holds automatically for the relational model. The relational model fits the relational principle (including Mach’s principle) and it is argued that it includes GR as a special case. The relational metric provides both contraction and (unbound) expansion as a function of relative potential, i.e. without invoking dark energy.

Keywords: Relational physics, Mach’s principle, GR, perihelion precession, Lense-Thirring frame dragging, accelerated expansion of the universe

1 Introduction

By Mach’s principle the inertia of a mass can only emerge from the presence of other matter (planets, stars,...). This tells us mass inertia is a relative quantity, just like the force of gravity is, and not an absolute property of an object. It is well known that Einstein attempted for a long time to incorporate Mach’s principle into his general theory of relativity (GR), but finally rejected the idea. This is remarkable, even though GR is

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extremely successful in describing relativistic phenomena by modifying space and time, while leaving inertia invariant. This, however, does not necessarily imply that Einstein’s spacetime concept is the only perceivable model of relativity. After all, Mach’s argument remains valid and several authors demonstrated the inherently relativistic aspects of the Machian principle. For instance, Sciama [17] and recently Gogberashvili and Kanatchikov [8] explored relativistic cosmological implications. Others, e.g. Hoffman [9], Schrödinger [16], Barbour and Bertotti [1] considered particle models, as we will do in this paper.

The approach taken here relies on GR, in particular on the Schwarzschild solution, but mainly as a reference. Rather than a geometric field theory, we follow the views of Bishop Berkeley regarding the anisotropy of inertia (section 2) and extend the approach taken by Schrödinger [16] into a fully relational Machian framework (sections 3,4,5), which enables straightforward calculation of relativistic trajectories, such as the anomalous perihelion precession and Lense-Thirring frame-dragging (section 6).

One aspect that is absent in typical particle models is that of remote observation, which is elementary to GR and which we will be adding (section 7) in a generalized form to the Machian model, by which the model also satisfies the wider relational principle. This relational model, i.e. the Machian model subject to a relational metric, is derived from - so is consistent with - the Schwarzschild solution. It therefore matches relativistic effects of remote observation, such as gravitational time dilation and redshift. The model can be applied to arbitrary configurations of N-body systems, i.e. is not limited to specific GR solutions like the Schwarzschild and Kerr metric.

Even though GR is used as a reference throughout this text, the present relational model differs conceptually from Einstein’s theory. In GR, inertia and energy are attributed to an object, while these are mutual properties between objects in the present approach. A striking difference is that the Schwarzschild solution does not appear as a vacuum solution in the Machian equivalent. The "vacuum" solution seems to assume a background of a flat potential equal to $\varphi_o$, the potential in our part of the universe (section 3.1). This is why the Schwarzschild solution appears as a special case of the Machian representation, where the latter applies to any level of background potential. The absence of a true vacuum also explains why GR (without later modifications such as dark energy) does not predict an accelerated expansion of the universe [15]: against a non-zero flat background potential, particles do not completely lose their inertia when receding from a mass kernel, while proper time and length approach asymptotic coordinate time and length. In the relational model, on the contrary, acceleration is explained by vanishing inertia and expanding spacetime (section 7.4).

2 Descartes, Berkeley, Mach and the relational model

Mach’s principle fits into a long tradition of "relational physics", dating back to Descartes [5] or even Aristotle. The relationalists propose, loosely stated, that physically meaningful quantities only emerge from the interaction of matter. This "relational principle"
not only concerns inertia (Mach’s principle), but also space and time need matter to exist. Consequently, a true vacuum does not exist, even though there is emptiness. One way to capture this is by considering matter not as the material object we see, but rather as the (infinite) gravitational field it causes, i.e., the field is the matter.

The relational principle is philosophically motivated and it has proven difficult to turn it into a consistent physical theory. In the present approach, however, we pursue this relational view. It has strong implications. Firstly, laws of nature must be independent of the chosen frame of reference. This requirement is met in GR; the field equations hold in any frame. Secondly, relational parameters (inertia, distance, time) must dilute completely (i.e. vanish) while moving into empty space; they gradually lose their physical meaning\(^1\). In GR, inertia is invariant, yet space and time do dilute. Though, only to a minor degree and certainly not completely in empty space; in GR-vacuum, spacetime is (asymptotically) flat Minkowskian. It is argued in sections 3.1 and 7 that the GR-vacuum is not a true vacuum and that GR represents a special case of relational physics.

Berkeley’s \([3]\) criticism of Newton’s concept of absolute space regards the notion that position or motion of a sole object (point mass \(m_1\)) in otherwise empty space is unobservable; there is no other physical object to relate the position of \(m_1\) to. Any definition of position, velocity or orientation of \(m_1\) is arbitrary and has no physical meaning, i.e. in the relational view these quantities are physically inexistent in a one-body universe. The same applies to the kinetic energy of the single object. So, how could this object exhibit mass inertia? In agreement with Mach’s principle, it can not. This changes when a second (point) mass \(m_2\) appears. Due to the force of gravity, the two bodies will accelerate towards each other and build up kinetic energy. Therefore, according to Mach, \(m_1\) and \(m_2\) must have acquired some inertia due to each others presence. In Berkeley’s view, contrary to Newton, their radial distance (separation) is the only meaningful geometrical parameter among the two objects. Indeed, the orientation of the two-body system in empty space is unobservable and, as pointed out by Berkeley, any circular motion of the two bodies around each other is physically meaningless (our frame of reference could as well be rotating in the opposite direction). Therefore, by this "anisotropic principle" of Berkeley, contrary to radial motion, tangential motion has no inertia and does not represent kinetic energy between the two bodies\(^2\).

Hence, from a Berkeleyan/Machian point of view, the Newtonian kinetic energy attributed to the circular orbit of two bodies in empty space is all virtual. Therefore, the 'real' (Machian) kinetic energy of the revolving system \((m_1,m_2)\), denoted \(T_{12}\), is zero. Machian kinetic energy may be interpreted as the part of the Newtonian kinetic energy that would be dissipated in an inelastic collision. Indeed, "freezing" the two objects together stops any relative radial motion, including spin of the bodies, but it does

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\(^1\)Vanishing of length and time is to be understood as infinite expansion of unit length and infinite increase of clock rate.

\(^2\)Radial motion must be taken quite literally though for finite-size objects: spin of either one of the bodies at constant separation implies radial motion of mass elements of one body relative to the other body. Therefore, spin does represent kinetic energy between the bodies.
not affect the rotation or translation of the total system in empty space. These latter motions are artificial, unobservable, therefore this part of Newtonian kinetic energy is virtual. The above picture changes if we would move \(m_1\) and \(m_2\) from empty space into our universe, which we may represent by a hollow sphere of mass \(m_o\). Then, the same circular orbit of \(m_1\) and \(m_2\), implies (components of) radial motion of \(m_1\) and \(m_2\) relative to \(m_o\), as explained further in section 5. Thus, the orbit involves non-zero Machian kinetic energies \(T_{01}\) and \(T_{02}\) of the subsystems \((m_1,m_o)\) and \((m_2,m_o)\), respectively. (As pointed out in section 5, these two terms actually represent the Newtonian kinetic energies of \(m_1\) and \(m_2\), defined in a CM frame attached to the "fixed stars"). The Machian energy \(T_{12}\) associated with the subsystem \((m_1,m_2)\) remains zero. If, however, \(m_1\) and \(m_2\) were spinning or in a non-circular orbit, this would involve radial motion between these bodies, therefore \(T_{12} > 0\).

Berkeley's anisotropic principle thus entails the concept of both inertia and kinetic energy as a mutual property between objects. This concept is consistent with the epistemological requirement of a frame independent definition of kinetic energy, as employed by Schrödinger [16]. Expressions for Machian kinetic energy will be provided in section 5. Notably, Berkeley’s principle also provides a rationale for the emergence of space and time from the distribution of matter: along with radial inertia, both radial distance and a certain concept of time (radial acceleration) emerge from the advent of the second body in empty space, while the tangential dimensions remain unobservable (i.e. inexistent) until other bodies appear in those directions. However, from an increasing distance, this collection of bodies gradually begins to appear as one single body, finally shrinking to a point mass in otherwise empty space, making space and time gradually dilute at larger scales and ultimately vanish at infinity. Thus, Berkeley’s principle underlies the relational principle, which includes Mach’s principle.

3 Isotropic v.s. anisotropic model

As a common denominator in typical Machian particle models, the inertia \(\mu\) of an object of mass \(m\) at position \(r\) is related to the gravitational potential of the other masses of the universe. Essentially two approaches can be distinguished: the isotropic model and the anisotropic model.

3.1 Isotropic model

The isotropic model is not consistent with Berkeley’s anisotropic principle. It, however, does match relativistic remote observation phenomena as follows. Consider a mass \(m\) having an isotropic inertia defined proportional to local potential

\[
\mu(r) = \frac{\varphi(r)}{\varphi_o} m
\]

where \(\varphi(r)\) is the total potential at \(r\) due to all other masses. The scaling factor \(\varphi_o\) is equal to the background potential in our part of the universe and is used for convenience;
it makes $\mu(x) = m$ wherever $\varphi(x) = \varphi_0$. This isotropic inertia can straightforwardly explain gravitational time dilation and gravitational redshift as follows.

A clock driven by a harmonic oscillator of mass $m$ and unity spring constant is positioned at a distance $r$ from the center of a spherical mass $M$ (planet), which exerts a potential $\varphi_M(r)$ at the position of $m$. Assume a background potential $\varphi_o$, then the isotropic inertia of mass $m$ is

$$\mu(r) = \frac{\varphi_o + \varphi_M(r)}{\varphi_o} m. \quad (2)$$

The differential equation of the oscillator is $x = -\mu(r)\ddot{x}$, having solution: $x(t) = a \sin(\omega t + \phi_o)$. Hence,

$$\mu(r) = -\frac{x}{\ddot{x}} = \frac{1}{\omega^2(r)}. \quad (3)$$

Elapsed clock time $\tau_R(r)$ between two events is proportional to clock frequency $\omega(r)$. If we have two identical clocks at arbitrary positions $r_A$ and $r_B$, then we may compare the elapsed times of the clocks by the relational dilation factor

$$\alpha_R = \alpha_R(r_A, r_B) = \frac{\tau^2_R(r_B)}{\tau^2_R(r_A)} = \frac{\omega^2(r_B)}{\omega^2(r_A)} = \frac{\mu(r_A)}{\mu(r_B)} = \frac{\varphi(r_A)}{\varphi(r_B)} \quad (4)$$

Following (4), the ratio of elapsed times appears inversely proportional to the ratio of the corresponding potentials. In the special case where we compare the clock at $r_B = r$ with a clock at infinity $r_A = \infty$, we have the specific relational dilation factor

$$\hat{\alpha}_R = \alpha_R(\infty, r) = \frac{\tau^2_R(r)}{\tau^2_R(\infty)} = \frac{\varphi(\infty)}{\varphi(r)} = \frac{\varphi_o}{\varphi_o + \varphi_M(r)} \quad (5)$$

In GR, according to the Schwarzschild metric, time dilation relates local time (i.e. stationary proper time) $\tau_S(r)$ to coordinate time $t = \tau_S(\infty)$ as

$$\alpha_S = 1 - \frac{r_s}{r} = \frac{\tau^2_S(r)}{\tau^2_S(\infty)} \quad (6)$$

where the Schwarzschild radius $r_s = 2GM/c^2$. The specific relational dilation factor (5) and the Schwarzschild dilation factor (6) become virtually identical if we assume

$$\varphi_o = -\frac{1}{2}c^2 \quad (7)$$

since then $r_s/r = \varphi_M(r)/\varphi_o$ and substituting this in (4) yields

$$\frac{\tau^2_R(r)}{\tau^2_R(\infty)} = \hat{\alpha}_R = \frac{1}{1 + \frac{r_s}{r}} \approx 1 - \frac{r_s}{r} = \alpha_s = \frac{\tau^2_S(r)}{\tau^2_S(\infty)}. \quad (8)$$

Hence, the time dilation predicted by the isotropic Machian model is virtually identical to GR time dilation. This has a remarkable consequence: from a Machian point of view,

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$^3$Above the surface of earth, the ratio $r_s/r$ is extremely small ($< 10^{-5}$) for any $r$. Therefore, the relative error in approximation (8) is negligible ($< 10^{-10}$).
the vacuum of the Schwarzschild solution doesn’t look empty really, rather implicitly assumes a flat background potential \( \varphi_0 \), which was explicitly assumed in the Machian model. Recall that the Schwarzschild metric is asymptotically Minkowskian such that Newtonian absolute spacetime is recovered at infinity. Hence, gravity appears in GR as a local curvature of a flat spacetime. The Machian model, on the other hand, is purely relational; \( \alpha_R \) is a dilation parameter between two arbitrary potentials at locations \( r_A \) and \( r_B \). Contrary to GR, clock rate is unbound for a clock moving into empty space, which observation is in agreement with the relational principle. GR’s concept of absolute "coordinate time" (as well as "coordinate distance") at infinity in empty space is therefore considered non-Machian. This has been noted already by several authors, e.g. [12]. However, as pointed out above, assuming a flat background potential \( \varphi_0 \) in the GR "vacuum" turns the relational metric into an (asymptotically) flat Minkowski metric, thus giving rise to the thought that GR represents a special case of the relational metric. Further support of this argument is given in section 6.1 on the anomalous perihelion precession. The unbound relational metric is formalized in the relational model of section 7.2.

Like in GR, gravitational redshift is just a different manifestation of gravitational time dilation. Indeed, (4) provides the Machian ratio between emitted and observed frequency, according to

\[
\alpha_R(r_{obs}, r_{emit}) = \frac{\nu^2_{emit}}{\nu^2_{obs}} = \frac{\varphi(r_{obs})}{\varphi(r_{emit})}
\]

which in the Schwarzschild case for a source at distance \( r \) from \( M \) and an observer at infinity (potential \( \varphi_0 \)) becomes

\[
\frac{\nu^2_{emit}(r)}{\nu^2_{obs}(\infty)} = \alpha_R = \frac{\varphi(\infty)}{\varphi(r_{emit})} = \frac{\varphi_0}{\varphi_0 + \varphi_M(r)} \approx \alpha_S,
\]

consistent with gravitational redshift in the Schwarzschild spacetime.

### 3.2 Anisotropic model

As an example we consider the inertia of mass \( m \) relative to a second mass \( M \) at a separation \( r \). Due to Berkeley, both inertia and the associated kinetic energy exclusively concern radial motion of \( m \), i.e. motion in the direction of \( M \), making inertia anisotropic. Schrödinger [16] used an anisotropic model to derive, without invoking the spacetime concept, the well known GR expression for the anomalous perihelion precession of planetary orbits. The key to this result in the paper of Schrödinger is the small additional inertia of the planet into the direction of the sun. Thus, it is precisely the anisotropy of inertia that accounted for this convincing result. Stated otherwise, isotropic models can not explain the anomalous precession (as pointed out in Appendix A). Even so, there seems to exist a common understanding that anisotropic models are ruled out by the famous experiments by Hughes (1960) and Drever (1961) [7], which
yielded extremely tight upper-bounds on the anisotropy of nuclear magnetic resonance, thus indicating inertial isotropy of the universe. Indeed, the anisotropic model (like the one used by Schrödinger) applied to the harmonic oscillator of section 3.1 shows different frequencies for oscillations in the radial and tangential direction: time dilation by this model only occurs in the radial direction of oscillation, in disagreement with the Hughes-Drever experiments and in disagreement with GR. As a consequence we conclude, even without the Hughes-Drever experiments, that the anisotropic model fails to explain (the isotropic) gravitational time dilation and redshift.

3.3 A way out

So a Machian theory has to deal with seemingly contradicting requirements. It must be isotropic and anisotropic at the same time, while anisotropy is considered conflicting with Hughes-Drever experiments. This is essentially where the Machian doctrine got stuck, while GR passed all the tests. Note that GR’s geometry involves two elementary concepts: time dilation and length contraction, where the first is isotropic and the latter is anisotropic, while the typical Machian approach only employs a single concept, i.e. variable inertia, either isotropic or anisotropic. Apparently, the Machian concept of variable inertia must be enhanced with a second concept to become GR compatible. Nevertheless, as shown in section 6, anisotropic inertia alone can correctly model the relativistic trajectory of a system, like the anomalous precession (section 6.1) and frame dragging (section 6.2). Sections 4 and 5 provide the elementary expressions for such a model. It is shown in section 7 that observational effects, like gravitational time dilation and gravitational redshift, may be accounted for by an isotropic coordinate transform of the anisotropic model.

4 Anisotropic Machian inertia and energy

We briefly review the Newtonian definition of potential energy from a relational perspective and consider the problematic Newtonian definition of kinetic energy. From there we arrive at definitions of Machian inertia and energy.

4.1 Machian potential energy = Newtonian potential energy

A single object $m_i$ at an arbitrary position $r_i$ in empty space has zero potential energy: $V_i = m_i\phi(r_i) = 0$. Indeed, the gravitational potential $\phi$ is nil everywhere in empty space. A second object, $m_j$, is required to obtain a non-zero potential energy $V_{ij}$ of the system $(m_i, m_j)$. Let $\phi_j(r_i)$ denote the gravitational potential due to $m_j$ at the position

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4Anisotropy may not be observable in resonance frequencies, it does manifest itself, though, in the exchange of radial and tangential kinetic energy, as in an elliptic orbit. In fact, the anomalous precession may be considered as experimental evidence of the anisotropic universe.
of $m_i$, then
\[ V_{ij} = m_i \varphi_j(r_i) = m_j \varphi_i(r_j) = V_{ji} \quad (11) \]
which equals the energy required to bring $m_i$ from infinity to a distance $r_{ij}$ from $m_j$. Or, equivalently, the energy to bring $m_j$ from infinity to a distance $r_{ij}$ from $m_i$. Thus, the potential energy $V_{ij}$ belongs as much to $m_i$ as to $m_j$. It doesn’t exist without the other mass present, which is a relational viewpoint. Hence, potential energy is a property of the system $(m_i,m_j)$. In general, for a N-body system, we obtain the total potential energy by adding up for all possible pairs (while avoiding double counting, since $V_{ij} = V_{ji}$)
\[ V = \sum_{i,j>i} V_{ij} \quad i,j=1..N. \quad (12) \]
In conclusion, Newtonian potential energy is a mutual property between every pair of objects and its value only varies with the separation between the objects, therefore is frame independent and fits the relational view.

### 4.2 Machian inertia and Machian kinetic energy

**Newtonian kinetic energy**

Contrary to potential energy, classical kinetic energy is a property of an object $m_i$ and its value $T_i = \frac{1}{2}m_i v_i^2$ is dependent on the choice of frame, while the value of the inertia parameter $m_i$ is not related to the presence of other mass. So it expresses the absoluteness of space and inertia and it clearly does not reflect Mach’s views. The problem of absolute space and inertia becomes most apparent in the classical energy balance of an isolated system ($E$ represents the conserved total energy)
\[ E = T + V \quad (13) \]
where $E$ and $V$ are frame independent, while $T (=\Sigma T_i)$ isn’t. Obviously, one can not create or destroy energy by moving a frame, which makes the frame dependency of $T$ unphysical. So, a consistent formulation of energy conservation requires kinetic energy to be defined independently of the frame too. Moreover, (13) strongly suggests kinetic energy, more specifically, the inertia parameter, must be a reflection of potential energy to provide this consistency.

**Machian inertia**

We let the potential energy $V_{ij}$ between two bodies be a model of their mutual Machian inertia $\mu_{ij}$
\[ V_{ij} = \mu_{ij} \varphi_o \quad (14) \]
i.e., inertia is potential energy (the scaling constant is conveniently chosen to be $\varphi_o$, the background potential in our part of the universe). Accordingly, we define the Machian
partial inertia} between objects \( m_i \) and \( m_j \) as

\[
\mu_{ij} = \frac{V_{ij}}{\varphi_o} = \frac{m_i \varphi_j (r_i)}{\varphi_o} = \frac{m_j \varphi_i (r_j)}{\varphi_o} = \mu_{ji}. \tag{15}
\]

By this definition, inertia is a property of the pair \((m_i,m_j)\). So multiple partial inertiae are associated with a single object, exactly like all the partial potential energies associated with the object. Definition (15) is mutual and symmetrical, which implies that two bodies of totally different mass (say the sun and a neutron) exert an equal inertia to one and another, just like they exert an equal force of gravity to one and another. A consequence of the relational nature of inertia is - contrary to Newton’s universe - that in the Machian model we have to account explicitly for the inertia caused by all the masses of the universe. To do so, we adopt the hollow sphere model of the universe, also used by Einstein and Schrödinger, by which we may treat the masses of the universe as a single object, \( m_o \). The potential inside of the hollow sphere is flat and equal to \( \varphi_o \), the background potential in our part of the universe. Therefore, the inertia \( \mu_{oi} \) between an arbitrary object \( m_i \) and the universe is simply \( \mu_{oi} = m_i \). Note that the inertia \( \mu_{ij} \) between two arbitrary objects is, in general, negligible relative to \( \mu_{oi} = m_i \) and \( \mu_{oj} = m_j \), because \( \varphi_o \), the potential due to the universal masses, is extremely large.

**Machian kinetic energy between point particles**

Using the above definition of partial inertia (15) and recalling Berkeley’s argument that only radial motion between (point) masses represents kinetic energy, we arrive at the following expression of Machian kinetic energy between two point particles \( m_i \) and \( m_j \)

\[
T_{ij} = \frac{1}{2} \mu_{ij} \dot{r}_{ij}^2. \tag{16}
\]

The Machian equation of motion for two masses falling to each other in empty space (see appendix B) reads

\[
\ddot{r}_{ij} = \frac{\varphi_o}{r_{ij}} (1 + \frac{\dot{r}_{ij}^2}{2 \varphi_o}). \tag{17}
\]

This differential equation lets \( m_i \) and \( m_j \) accelerate towards each other until \( \dot{r}_{ij} = 0 \), at which point the relative velocity \( |\dot{r}_{ij}| \) attains its maximum value

\[
|\dot{r}_{ij}|_{\text{max}} = \sqrt{-2 \varphi_o}. \tag{18}
\]

According to (7), \( \varphi_o = -\frac{1}{2} c^2 \), hence, \( |\dot{r}_{ij}|_{\text{max}} = c \), the speed of light. This is not a surprise, as we recognize the Lorentz factor \( \gamma \) in Eq.(17):

\[
(1 + \frac{\dot{r}_{ij}^2}{2 \varphi_o})^{-1/2} = (1 - \frac{\dot{r}_{ij}^2}{c^2})^{-1/2} = \gamma. \tag{19}
\]

Appendix B shows that the velocity limit \( c \) also holds in a non-empty space. Ergo, the speed of light is a consequence of the assumed Machian inertia, i.e. is not a postulate in the Machian context.
The total kinetic energy of a system of \( N \) point masses follows by straightforward summation (16) over all different pairs

\[
T = \sum_{i \neq j} T_{ij} \quad i,j=1,...,N
\]  

(20)

The relationship between Machian and Newtonian kinetic energy will be clarified in the next section.

5 Machian kinetic energy between finite size objects

Eq. (16) provides the elementary expression of kinetic energy between two point particles. The kinetic energy between two solid objects of arbitrary shape and mass distribution is essentially an integral of the kinetic energy between infinitesimal small volume elements of the two objects over their respective volumes. For Machian calculations of celestial mechanics we need models of kinetic energy between two massive spheres (both translating and spinning) and between a massive sphere and the hollow sphere (representing the universe). These models are provided in this section.

5.1 Translational kinetic energy

Between an object and the universe

The homogeneity of the universe allows assuming a constant background potential \( \varphi_o \), at least on the scale of the solar system. This makes the hollow sphere model \( (m_o) \) an accurate representation. Constant potential implies that the partial inertia between an object \( m_i \) and the heavenly masses is constant, so does not depend on the position, particular shape or mass density distribution of the object. By definition (15) we obtain \( \mu_{oi} = m_i \), the Newtonian value. With the hollow sphere in place, the universe has become just another object, \( m_o \). Motion of an object \( m_i \) along any trajectory through space implies components of radial velocity relative to the cosmic masses. The direction of the motion in the universe does not matter for the amount of kinetic energy it represents. Just the magnitude \( v_{oi} \) of the velocity. Therefore, in the case of a body \( m_i \) moving through the universe, we have a translational Machian kinetic energy \( T_{voi} \) between \( m_i \) and \( m_o \) equal to the Newtonian kinetic energy \( T_i \) of object \( m_i \)

\[
T_{voi} = \frac{1}{2} \mu_{oi} v_{oi}^2 = \frac{1}{2} m_i v_i^2 = T_i
\]

(21)

provided that the "absolute" velocity \( v_i \) is defined in the CM frame attached to \( m_o \), so that \( v_i = v_{oi} \).
Between two solid spheres

By definition (15), the partial inertia between the bodies is $\mu_{ij} = V_{ij}/\varphi_o$. For conservation of energy, equal potential energies for spheres and point particles implies equal kinetic energies. Hence, translational kinetic energy between solid spheres is the same as between point particles (16), therefore it only has a radial component

$$T_{v_{ij}} = T_{r_{ij}} = \frac{1}{2} \mu_{ij} \dot{r}_{ij}^2 \quad i,j \neq 0 \quad (22)$$

since the tangential component of motion does not represent kinetic energy between the bodies, i.e.

$$T_{\dot{\phi}_{ij}} = T_{\dot{\theta}_{ij}} = 0 \quad i,j \neq 0 \quad (23)$$

where $\phi_{ij}$ and $\theta_{ij}$ denote the perpendicular tangential directions in spherical coordinates relative to the CM of the two bodies.

5.2 Spin energy

Spin of body $m_i$ and/or $m_j$ implies Machian kinetic energy of the subsystem $(m_i,m_j)$, even if the spinning bodies stay at constant radius. Arbitrary volume elements of these two bodies do have a non-zero radial velocity component relative to each other, as their distance varies during spin of any of the two bodies. Therefore, spin of a body relative to another body, represents radial velocity between masses, hence, Machian kinetic energy. The total kinetic energy of the spinning bodies follows from integrating the Machian kinetic energies between volume elements of the respective bodies over the volumes of the bodies. Instead of solving the (rather involved) integrals, one can derive the solutions as well heuristically by formulating the Machian analogy of Newtonian expressions for spinning bodies, which are based on the same integrals.

Spin energy relative to the universe

Consider a solid sphere $m_i$ of radius $R_i$, spinning at an angular velocity $\omega_i$, relative to the CM frame attached to hollow sphere $m_o$. The Newtonian spin energy is

$$T_{\omega_i} = \frac{1}{2} m_i R_i^2 \omega_i^2 \quad (24)$$

In an inelastic collision of $m_i$ and $m_o$, the spin of $m_i$ will vanish completely. Hence, the Newtonian value is all Machian energy, therefore the analogous Machian expression is simply

$$T_{\omega_{oi}} = \frac{1}{2} \mu_{oi} R_i^2 \omega_i^2 = \frac{1}{2} m_i R_i^2 \omega_i^2 \quad (25)$$

If we add an object $m_j$, then we also get (non-Newtonian) Machian terms between $m_i$ and $m_j$: 
Spin energy between two solid spheres

The case of two spinning spheres appears to be very Machian. One has to bear in mind that circular orbit of the spheres at a rate \( \dot{\phi}_{ij} \), even when the spheres are glued to each other, introduces a spin of both bodies relative to the frame of reference, while this spin clearly does not represent spin energy between the two spheres. Using this heuristic, we derive a frame independent expression for the spin energy between two spheres \( m_i \) and \( m_j \). Assume the two spheres are at separation \( r_{ij} \) and have radii \( R_i \) and \( R_j \). Imagine for a moment they take fixed positions in the frame of reference (\( \dot{\phi}_{ij} = 0 \)). Sphere \( m_i \) has a spin \( \omega_i \) relative to the frame, while \( m_j \) is at rest. In this case, the Machian spin energy of \( m_i \) relative to \( m_j \) is obtained by generalization of (25)

\[
T_{\omega_{ij}/i} = \frac{1}{5} \mu_{ij} R_i^2 \omega_i^2 .
\]  

(26)

In fact, we have only replaced universe \( m_o \) in (25) by object \( m_j \). Obviously, for the opposite case, where \( m_j \) is spinning and \( m_i \) is at rest, we have

\[
T_{\omega_{ij}/j} = \frac{1}{5} \mu_{ij} R_j^2 \omega_j^2 .
\]  

(27)

If both objects are orbiting each other at a rate \( \dot{\phi}_{ij} \), we must subtract the contribution of this rotation from \( \omega_i \) and \( \omega_j \) to obtain a frame independent expression for the spin energy of the system \((m_i, m_j)\)

\[
T_{\omega_{ij}} = T_{\omega_{ij}/i} + T_{\omega_{ij}/j} = \frac{1}{5} \mu_{ij} R_i^2 (\omega_i - \dot{\phi}_{ij})^2 + \frac{1}{5} \mu_{ij} R_j^2 (\omega_j - \dot{\phi}_{ij})^2 .
\]  

(28)

So, spins must be considered relative to the common axis which connects the centers of the two spheres. Indeed, the Machian spin energy between \( m_i \) and \( m_j \) is completely gone if we would freeze the spheres together onto this imaginary axis\(^5\).

Like with translational energy, since \( \mu_{ij} \) is typically extremely small, the spin energy between two spheres is negligible in comparison with each sphere’s (Newtonian) spin energy relative to the universe. Yet, it does account for relativistic effects, such as the Lense-Thirring precession, as will be pointed out in section 6.2.

5.3 Example - Machian energy of two orbiting bodies

In universe \( O \) (background potential \( \varphi_o \)), two non-spinning spherical bodies, \( m \) and \( M \), are orbiting each other at angular velocity \( \dot{\phi} \) and radius \( r = r_m + r_M \), where \( r_m \) and \( r_M \) are the distances from \( m \) and \( M \) to their CM. We assume \( m \ll M \). The total energy of the 3-body system \((m, M, O)\) is assembled as follows

\[
E = T + V
\]

\(^5\)In the general case, the objects may also spin in the perpendicular direction \( \theta_{ij} \), giving an additional spin energy \( T_{\psi_{ij}} = T_{\psi_{ij}/i} + T_{\psi_{ij}/j} = \frac{1}{5} \mu_{ij} R_i^2 (\psi_i - \dot{\theta}_{ij})^2 + \frac{1}{5} \mu_{ij} R_j^2 (\psi_j - \dot{\theta}_{ij})^2 . \) For simplicity, we will leave out this component hereafter, as it can be avoided.
\[ T = T_{mO} + T_{MO} + T_{mM} \]
\[ V = V_{mO} + V_{MO} + V_{mM} = (m + M + \mu)\varphi_o \]  
(29)

where the kinetic energy components are (in polar coordinates)

\[ T_{mO} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m r^2 \dot{\phi}^2 \]  
(30)
\[ T_{MO} = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}M r^2 \dot{\phi}^2 \]  
(31)
\[ T_{mM} = \frac{1}{2}\mu \dot{r}^2 \]  
(32)

and where \( \mu = \mu_{mM} = -GMm/\varphi_0 r \). The radius \( r_m = \frac{M}{M+m}r \) and \( r_M = \frac{m}{M+m}r \).

Considering \( m \ll M \), substitution into (30) and (31) gives

\[ T_{mO} \approx \frac{1}{2}m \left( \frac{M}{M+m} \right)^2 \dot{r}^2 + \frac{1}{2}m \left( \frac{M}{M+m} \right)^2 r^2 \dot{\phi}^2 \approx \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m r^2 \dot{\phi}^2 \]  
(33)

and

\[ T_{MO} = \frac{1}{2}M \left( \frac{m}{M+m} \right)^2 \dot{r}^2 + \frac{1}{2}M \left( \frac{m}{M+m} \right)^2 r^2 \dot{\phi}^2 \]  
(34)

So, \( T_{MO} \approx \frac{m}{M}T_{mO} \ll T_{mO} \), therefore \( T_{MO} \) can be neglected. Accordingly, we eliminate the associated constant potential energy \( V_{MO} \) from the equation. In conclusion, we obtain the following simplified expression for the orbital energy of \( m \) relative to \( M \) and \( O \):

\[ E = T_{mO} + T_{mM} + V_{mO} + V_{mM} = \frac{1}{2}(m + \mu)\dot{r}^2 + \frac{1}{2}m r^2 \dot{\phi}^2 + (m + \mu)\varphi_o \]  
(35)

If, in addition, the two bodies \( m \) and \( M \) spin at angular velocities \( \omega_m \) and \( \omega_M \) relative to \( O \), then we must add spin energy

\[ T_\omega = T_{\omega mO} + T_{\omega MO} + T_{\omega mM} = \frac{1}{2}m R_m^2 \omega_m^2 + \frac{1}{2}M R_M^2 \omega_M^2 + \frac{1}{2}\mu \left( R_m^2 (\omega_m - \dot{\phi})^2 + R_M^2 (\omega_M - \dot{\phi})^2 \right) \]  
(36)

If \( m \) is only an infinitesimal small test particle, then \( R_m = 0 \), which simplifies (36) to

\[ T_\omega = \frac{1}{2}M R_M^2 \omega_M^2 + \frac{1}{2}\mu R_M^2 (\omega_M - \dot{\phi})^2 \]  
(37)

The simplified equations (35) and (37) fit well known celestial problems (see section 6).

6 Consistency of the Machian model and GR

Using the Machian model, i.e. the definitions of anisotropic inertia and Machian energy of section 4 and 5, we derive in this section expressions for the anomalous perihelion precession and for Lense-Thirring frame-dragging, both matching the well-known GR expressions. That is, as long as we assume a background potential equal to \( \varphi_o \). Note that the Machian model does not involve the spacetime concept.
6.1 Anomalous precession of Mercury’s perihelion

The anomalous precession is a classical test case of GR. Using the Schwarzschild solution\(^6\), one derives the following relativistic energy equation for a small test particle \(m\) orbiting a gravitating central mass \(M\)

\[
\frac{1}{2}mr^2 + \frac{1}{2} \frac{m h^2}{r^2} - \frac{GmM}{r} - \frac{GmMh^2}{c^2r^3} = \text{const.} \quad (38)
\]

Here, \(m\) represents Mercury’s mass, \(M\) is the solar mass, \(G\) Newton’s constant and \(h = r^2 \dot{\phi}\) is the (constant) momentum of Mercury’s orbit per unit mass, \(r\) is the radius of Mercury’s orbit and \(\dot{\phi}\) the orbital angular velocity. This expression is Newtonian, except for the last term on the left, which is purely GR and accounts completely for the anomalous precession. From (38) one can derive the famous expression of the anomalous precession per revolution

\[
\Delta \phi = \frac{6\pi G^2 M^2}{c^2 h^2} \quad (39)
\]

Next, we will derive exactly the same expression (38) for the energy of Mercury’s orbit via the present Machian approach. The celestial system is modeled as a 3-body problem between Mercury, the sun and the rest of the universe, which we represent by the hollow sphere \(O\) (accounting for the background potential \(\varphi_o\)). According to (35), the total energy \(E_o\) of the 3-body system \((m, M, O)\) is

\[
E_o = T_mO + T_mM + V_mO + V_mM = \frac{1}{2} (m + \mu) \dot{r}^2 + \frac{1}{2} m \dot{r}^2 \dot{\phi}^2 + (m + \mu) \varphi_o \quad (40)
\]

where

\[
\mu = \mu_mM = \frac{GM}{\varphi_or}. \quad (41)
\]

Note that (40) matches Schrödinger’s result in [16]. Via a Lagrangian approach, we obtain from (40) the momentum of the orbit (per unit mass) as a constant of motion

\[
h = r^2 \dot{\phi}. \quad (42)
\]

Substituting this in (40), and multiplying both sides by \(m/(m + \mu)\) gives

\[
\frac{1}{2}mr^2 + \frac{1}{2} \frac{m}{m + \mu} \frac{m h^2}{r^2} + m \varphi_o = \frac{m}{m + \mu} E_o. \quad (43)
\]

Introducing the auxiliary constant \(\epsilon_o\), the constant total energy \(E_o\) may be defined as

\[
E_o = m \varphi_o + \epsilon_o. \quad (44)
\]

\(^6\)Remarkably, in the case of Mercury’s precession, the Schwarzschild solution has been successfully applied to a non-vacuum space of background potential \(\varphi_o\), which supports the earlier comment that the Schwarzschild vacuum does not actually appear empty.
Note that for sub-relativistic speeds \( E_0 \approx m \varphi_o = -\frac{1}{2}mc^2 \) or
\[
\epsilon_o \ll m \varphi_o. \tag{45}
\]
Since \( \mu \ll m \), we may replace in (43) \( m/(m + \mu) \) by \( (m - \mu)/m \). So, using (44), we get
\[
\frac{1}{2}mr^2 + \frac{1}{2}(m - \mu)\frac{h^2}{r^2} + m\varphi_o = \frac{m - \mu}{m}E_o = \epsilon_o + m\varphi_o - \frac{\mu}{m}\epsilon_o - \mu\varphi_o. \tag{46}
\]
Then, moving the constant energy terms to the right side, ignoring \( \frac{\mu}{m}\epsilon_o \) (since (45)) and substituting (41) yields
\[
\frac{1}{2}mr^2 + \frac{1}{2}m\frac{h^2}{r^2} - \frac{GMm}{c^2}r^2 - \frac{GM}{r} = \epsilon_o \tag{47}
\]
by which we have retrieved the relativistic energy equation of Mercury’s orbit (38), from which of course the same expression (39) follows for the anomalous precession. This result supports the view that inertia is anisotropic, in agreement with Berkeley. In addition, it is pointed out in Appendix A that an isotropic model can not explain the anomalous precession.

6.2 Frame-dragging

Perhaps the most convincing test of the Machian approach is its simplicity in the derivation of frame-dragging effects. We consider the simplest form, linear frame-dragging, followed by rotational (Lense-Thirring) frame-dragging.

Linear frame-dragging

We study the behavior of a small test particle of mass \( m \) near a larger mass \( M \) against the background potential \( \varphi_o \) of universe \( O \). If no (net) forces are exerted on the particle, then it will remain still in its local frame, so the particle’s position marks the position of the frame. If we let \( M \) be a long tube, and place the particle \( m \) halfway inside the tube, exactly on the tube’s axis, then the particle will be weightless, i.e., the net force of gravity on the particle is zero. Let \( x_m(0) \) be the initial position of \( m \) and let \( v_m(0) = 0 \) and \( v_M(0) = 0 \) be the initial velocities of \( m \) and \( M \), respectively. At some point, we make the tube move in the axial direction with a uniform velocity \( v_M \neq 0 \). In the Newtonian context, this would not affect the position of \( m \), so \( v_m \) remains zero. Formally: the Newtonian total kinetic energy is \( T_{Newton} = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2 \), and the linear momentum of \( m \): \( p_{Newton} = \partial T_{Newton}/\partial v_m = mv_m \), which tells that if \( v_m \) is constant and zero, then \( p_{Newton} = 0 \) and the net force on \( m \), \( F_m = \dot{p}_m = 0 \) will also remain zero.

In the Machian approach, however, we have the following expression for the total kinetic energy
\[
T_{Mach} = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2 + \frac{1}{2}\mu(v_m-v_M)^2 \tag{48}
\]
where \( \mu = \mu_{mM} = m\varphi_M(x_m)/\varphi_o \). The last term in (48) represents the kinetic energy of the particle relative to the tube, which term is absent in the classical kinetic energy \( T_{\text{Newton}} \). It is this last term that links the particle and the tube, energy-wise. Also in the Machian approach, \( m \) is not acted upon, so we still have zero total momentum associated with the particle’s axial velocity \( v_m \).

\[
p_{\text{Mach}} = \frac{\partial T_{\text{Mach}}}{\partial v_m} = mv_m + \mu(v_m - v_M) = 0 \quad (49)
\]

from which we obtain the particle’s velocity as a function of the tube’s velocity

\[
v_m = \frac{\mu v_M}{m + \mu} = \frac{\varphi_M(x_m)}{\varphi_o + \varphi_M(x_m)} v_M. \quad (50)
\]

In other words: without exerting any force on it (\( p_{\text{Mach}} = 0 \) at any point in time), the particle seems to start moving with a fraction of the tube’s velocity. This is called frame dragging. However, as pointed out, there actually is no dragging force involved as long as \( p_{\text{Mach}} = 0 \). This notion is in full agreement with the following interpretation: the acceleration of the tube from standstill to some finite velocity \( v_M \) causes the universe \( O \), due to the reaction force, to accelerate from standstill to a velocity \( -v_m \) in the opposite direction of \( v_M \). The tube seems to drag the frame, but from the Machian point of view, the frame and the particle remain at rest and the tube’s velocity is actually \( (v_M - v_m) \). Hence, the particle can be considered to not move at all, in agreement with \( p_{\text{Mach}} = 0 \). Furthermore, note that (49) shows that the total linear momentum of \( m \) is the sum of the particle’s partial linear momenta

\[
p_{\text{Mach}} = p_{mO} + p_{mM} = mv_m + \mu(v_m - v_M) = 0. \quad (51)
\]

The other interesting observation about (50) is that, for arbitrary background potential \( \varphi_b \), the ‘dragging fraction’ at position \( x \) is equal to the ratio \( \varphi_M(x)/(\varphi_b + \varphi_M(x)) \). This means that in the limit of \( \varphi_b \to 0 \) (when we get into empty space) the fraction approaches unity, hence, \( v_m \to v_M \). So the particle moves completely along with the tube, which makes perfectly sense from a Machian point of view, because any common velocity of \( m \) and \( M \) in empty space does not represent kinetic energy, is artificial and is equivalent to no motion at all. If, however, \( \varphi_b = \varphi_o \), as near earth, then the ratio is extremely small, i.e., hard to verify experimentally.

**Lense-Thirring frame dragging**

The Lense-Thirring effect ([11] and [10] for a recent review) concerns frame-dragging near a rotating sphere, say planet earth. Let \( M \) and \( \omega_M \) denote the mass and spin angular velocity of the earth. An infinitesimal small particle of mass \( m \) in the vicinity of \( M \) will be "dragged" by \( M \) in a circular orbit around \( M \) with an angular velocity \( \dot{\phi} \), which is a fraction of the angular velocity of the sphere. To prevent the particle from falling to \( M \), we assume the particle is constrained by a circular polar orbit of radius
Thus, the dragging of \( m \) results in a (slow) precession \( \dot{\phi} \) of the plane of the orbit. We position the origin of the polar coordinates onto the CM of the two spheres. Since \( m \) is infinitesimal small, the CM is at the center of \( M \), so \( r_m = r \) and \( r_M = 0 \). The derivation via GR, involving the Kerr-metric, yields the well-known expression \( \dot{\phi}_{GR} \) for the Lense-Thirring precession of particle \( m \)

\[
\dot{\phi}_{GR} = \frac{r_s \lambda r c}{(r^2 + \lambda^2 \cos^2 \theta)(r^2 + \lambda^2) + r_s \lambda^2 r \sin^2 \theta},
\]

(52)

where \( r_s = \frac{2GM}{c^2} \) is the Schwarzschild radius, \( \lambda = J/Mc \) a distance scale and \( J = \frac{1}{5} MR^2_M \omega_M \) the angular momentum of \( M \). In case of planet earth, \( \lambda \ll r \) and \( r_s \ll r \) for any \( r \geq R_M \). Thus, (52) may be simplified to

\[
\dot{\phi}_{GR} = \frac{r_s \lambda c}{r^3} = \frac{2GM\frac{3}{2}R^2_M}{c^2 r^3} \omega_M,
\]

(53)

which can be rewritten as

\[
\dot{\phi}_{GR} = \frac{\varphi_M(r)\frac{3}{2}R^2_M}{\varphi_o r^2} \omega_M.
\]

(54)

Now, we will start from the other end and derive the Machian equivalent of this GR based expression (54). The Machian Lagrangian for the system reads

\[
\mathcal{L}_{Mach} = T_{Mach} - V_{Mach}
\]

(55)

\[
T_{Mach} = T_{mM} + T_{mO} + T_{MO} = (T_{\dot{r}_mM} + T_{\omega_mM}) + (T_{v_mO} + T_{\omega_mO}) + (T_{v_MO} + T_{\omega_MO})
\]

\[
V_{Mach} = V_{mM} + V_{mO} + V_{MO}
\]

In analogy with linear frame dragging, we only have to derive the expression for the total angular momentum of particle \( m \) associated with \( \dot{\phi} \), set this to zero and solve for \( \dot{\phi} \):

\[
J_\dot{} = \frac{\partial \mathcal{L}_{Mach}}{\partial \dot{\phi}_{GR}} = 0.
\]

(56)

For this purpose, we skip all contributions to \( \mathcal{L}_{Mach} \) which are independent of \( \dot{\phi} \) and keep the energy terms associated with \( \dot{\phi} \), i.e., the Machian term \( T_{\omega_mM} \) of the spin energies between \( m \) and \( M \) (between square brackets) and the Newtonian terms \( T_{v_mO} \) and \( T_{v_MO} \) of the precessing motion of \( m \) respectively \( M \) relative to the universe:

\[
\mathcal{L}_\dot{} = \left[ \frac{1}{2} \mu R^2_m (\omega_m - \dot{\phi})^2 + \frac{1}{2} \mu R^2_M (\omega_M - \dot{\phi})^2 \right] + \frac{1}{2} mr^2_m \dot{\phi}^2 + \frac{1}{2} Mr^2_M \dot{\phi}^2
\]

(57)

where \( \mu = -GmM/\varphi_o r \). The first part of the Machian term in (57) is the spin energy of the particle \( m \) relative to \( M \). This part is zero, because the particle’s radius \( R_m \) is infinitesimal small. The second part of the Machian term is the spin energy of \( M \) relative to the particle, which is (relatively) significant. The third term represents the Newtonian energy of the particle’s precession relative to the universe, which energy is
also significant. The last term represents the Newtonian energy of the precession of \( M \) relative to the universe. This term can be omitted, since \( r_M = 0 \). This leaves

\[
\mathcal{L}_\phi = \frac{1}{2} \mu r_M^2 (\omega_M - \phi)^2 + \frac{1}{2} m r^2 \dot{\phi}^2
\]

where we replaced \( r_m \) by \( r \). So the momentum of particle \( m \) associated with angular velocity \( \dot{\phi} \) is

\[
J_\phi = \frac{\partial \mathcal{L}_\text{Mach}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}_\phi}{\partial \dot{\phi}} = -\frac{1}{2} \mu R_M^2 (\omega_M - \phi) + m r^2 \dot{\phi} = 0
\]

from which we obtain (labeling \( \dot{\phi} = \dot{\phi}_\text{Mach} \))

\[
\dot{\phi}_\text{Mach} = \frac{\frac{1}{2} \mu R_M^2 \omega_M}{m r^2} = \frac{\varphi_M(r) \frac{1}{2} R_M^2}{\varphi_\omega r^2 + \varphi_M(r) \frac{1}{2} R_M^2} \omega_M \approx \frac{\varphi_M(r) \frac{1}{2} R_M^2}{\varphi_\omega r^2} \omega_M = \dot{\phi}_\text{GR},
\]

i.e. the Machian expression matches the GR expression (54): the difference is negligible, as \( \varphi_M(r) \ll \varphi_\omega \) and \( R_M < r \). Note that the expression (59) for \( J_\phi \) allows for a Machian interpretation analog to linear frame-dragging: from the point of view of the particle, earth is rotating with an angular velocity \( \omega_M - \phi \), which is compensated by a backward rotation \(-\phi\) of the rest of the universe, in such a way that the total momentum \( J_\phi \) remains zero. This interpretation agrees with [10], where Iorio et al. point out that the gravitomagnetic field has no physical existence, yet causes Lense-Thirring precession as "a pure coordinate artifact".

7 The relational model as an isotropic transform of the anisotropic Machian model

7.1 Remote observation by the isotropic relational metric

The above examples demonstrate the validity of the anisotropic Machian model in predicting relativistic trajectories of dynamical systems. Notably, this did not involve the spacetime concept. The isotropic model, on the other hand, explains relativistic remote observation phenomena such as time dilation, as pointed out in section 3.1. We assume the observer is essentially a local reference frame in spacetime and its presence does not influence the system under study. In curved spacetime, the change of the observer’s position affects his measurements of the system. In the relational view, units vary along with local potential, i.e. the relational metric is a function of potential. We will derive such a relational metric from the Schwarzschild metric and the corresponding Machian energy equation for a test particle \( m \), orbiting a large sphere of mass \( M \) against a background potential \( \varphi_\omega \). This result will then be generalized for arbitrary spacetimes.

The Schwarzschild metric in spherical coordinates \((r, \phi, \theta)\) reads

\[
d\lambda^2 = c^2 d\tau^2 = \alpha_s c^2 dt^2 - \frac{dr^2}{\alpha_s} - r^2 d\theta^2 - r^2 \sin \theta d\phi^2.
\]
The Schwarzschild dilation parameter is defined \( \alpha_s = 1 - r_s / r \), where \( r_s = 2GM/c^2 \) is the Schwarzschild radius. We again assume the system has a background potential \( \varphi_o \) (see the note on the Schwarzschild vacuum in section 3.1). This is not an arbitrary choice, since, according to (7), \( \varphi_o = -\frac{1}{2}c^2 \). The potential represents the speed of light in the Machian context. Or, conversely, the speed of light represents the background potential \( \varphi_o \) in the Schwarzschild metric. This relationship connects both metrics. Choosing axes such that \( \theta = \pi/2 \) in the plane of the orbit simplifies the metric to

\[
c^2 d\tau^2 = \alpha_s c^2 dt^2 - \frac{dr^2}{\alpha_s} - r^2 d\phi^2.
\]  

(62)

Taking on both sides the derivative with respect to coordinate time (\( \frac{d}{dt} \) denoted by the dot) yields the differential equation for the orbit in coordinate time

\[
c^2 \dot{\tau}^2 = \alpha_s c^2 - \frac{\dot{r}^2}{\alpha_s} - \frac{r^2 \dot{\phi}^2}{\alpha_s}.
\]  

(63)

Next, we isolate the constant energy (per unit mass) \( c^2 \) from the first term on the right

\[
c^2 = \frac{\dot{\tau}^2}{\alpha_s} + \frac{\dot{r}^2}{\alpha_s^2} + \frac{r^2 \dot{\phi}^2}{\alpha_s^2}.
\]  

(64)

from which equation we identify the constant of motion

\[
\frac{\dot{\tau}}{\alpha_s} = k = \text{const},
\]  

(65)

by which \( \dot{\tau} \) can be eliminated from (64), yielding the following equation in coordinate time

\[
c^2 = \alpha_s c^2 k^2 + \frac{\dot{r}^2}{\alpha_s^2} + \frac{r^2 \dot{\phi}^2}{\alpha_s^2},
\]  

(66)

which we may rearrange into

\[
-\frac{c^2}{\alpha_s} + \frac{\dot{r}^2}{\alpha_s^3} + \frac{r^2 \dot{\phi}^2}{\alpha_s^3} = -c^2 k^2.
\]  

(67)

Multiplying both sides by \( \frac{1}{2}m \) and using \( \varphi_o = -\frac{1}{2}c^2 \) yields the "Schwarzschild energy equation" for the orbit

\[
\frac{m\varphi_o}{\alpha_s} + \frac{\frac{1}{2}mr^2}{\alpha_s^3} + \frac{\frac{1}{2}mr^2 \dot{\phi}^2}{\alpha_s^3} = -\frac{1}{2}mc^2 k^2 = E_o.
\]  

(68)

Like in section 3.1, we consider the Schwarzschild dilation factor \( \alpha_s = 1 - \frac{r_s}{r} \) as a special case of the more general relational dilation factor \( \alpha_R \) between two arbitrary positions \( x_A \) and \( x_B \)

\[
\alpha_R(x_A, x_B) = \frac{\varphi(x_A)}{\varphi(x_B)}
\]  

(69)
which, in the present case of the Schwarzschild spacetime, i.e. for a particle \(m\) at position \(r\) and an observer at infinity (where we assume a background potential \(\varphi_o\)), takes on the particular value \(\hat{\alpha}_R\)

\[
\hat{\alpha}_R = \alpha_R(\infty, r) = \frac{\varphi_o}{\varphi_o + \varphi_M(r)} = \frac{1}{1 + \frac{r_s}{r}} \approx 1 - \frac{r_s}{r} = \alpha_s
\]  

(70)

Replacing in (68), like in section 3.1, \(\alpha_s\) by \(\hat{\alpha}_R\) and recalling relations \(\mu = \mu_{mM} = -m\varphi_M(r)/\varphi_o\), \(m = \mu_{om}\) and \(m/\hat{\alpha}_R = m + \mu\) yields the following "relational energy equation" in "coordinate" time \(t\)

\[
(m + \mu)\varphi_o + \frac{1}{2}(m + \mu)\dot{r}^2 + \frac{1}{2}m\dot{s}^2 = -\frac{1}{2}mc^2k^2 = E_o
\]  

(71)

This is the Machian energy equation of the orbit (35), except for the common denominator \(\hat{\alpha}_R^2\) of the two kinetic energy terms, which points at an isotropic coordinate transform between the observer’s coordinates at infinity \((r, s, \phi, t)\) and the local coordinates \((\rho, \sigma, \phi, \tau)\) of the test particle (where we let \(s = r\phi\) and \(\sigma = \rho\phi\) represent the tangential coordinate in the two frames). Eq. (71) in \((r, s, \phi, t)\) coordinates reads

\[
(m + \mu)\varphi_o + \frac{1}{2}(m + \mu)\dot{\rho}^2 + \frac{1}{2}m\dot{s}^2 = E_o
\]  

(72)

If we rewrite the Machian energy equation (35) in local coordinates \((\rho, \sigma, \phi, \tau)\) (the derivative \(\frac{d}{d\tau}\) is denoted by the circle \(\circ\)):

\[
(m + \mu)\varphi_o + \frac{1}{2}(m + \mu)\dot{\rho}^2 + \frac{1}{2}m\dot{\sigma}^2 = E_o,
\]  

(73)

then equating (72) and (73) yields for the radial direction

\[
\dot{\rho}^2 = \frac{\dot{r}^2}{\hat{\alpha}_R^2}
\]  

(74)

and identically for the tangential direction

\[
\dot{\sigma}^2 = \frac{\dot{s}^2}{\hat{\alpha}_R^2}
\]  

(75)

Thus, assuming the validity of both the Schwarzschild metric and the corresponding Machian energy equation, remote observation causes a coordinate transform of velocity. From this velocity transform, we infer the underlying relational spacetime metric for the Schwarzschild case as follows: recalling (8), gravitational time dilation of local time v.s. coordinate time is expressed as

\[
d\tau^2 = \hat{\alpha}_R dt^2.
\]  

(76)
Combining (76) with (74) and (75), respectively, yields \(d\rho^2 = \frac{1}{\alpha_R} dr^2\) and, identically, \(d\sigma^2_{\phi} = \frac{1}{\alpha_R} ds^2_{\phi}\). Because of this spatial isotropy, we need not discern between the radial and tangential directions and consider only a single spatial metric, which applies to a spatial coordinate \(s\) in any direction, so to both \(dr\) and \(ds_{\phi} = rd\phi\) according to

\[
d\sigma^2 = \frac{1}{\alpha_R} ds^2
\]

where \(\sigma\) is the local coordinate and \(s\) the coordinate at infinity. Hence, the relational equivalent of the Schwarzschild metric presumes identical length contraction in both radial and tangential direction, i.e. isotropic contraction. Where the Schwarzschild metric has anisotropic length contraction, this anisotropic feature is represented in the Machian energy equation (73), and consequently in the relational energy equation (71), by the additional inertia into the radial direction:

\[
m + \mu = \frac{m}{\alpha_R}
\]

Hence, by mapping the anisotropic Machian model onto the anisotropic Schwarzschild metric we distilled an isotropic relational metric for the Schwarzschild case.

### 7.2 General form of the relational metric

The specific relational metric (76),(77) for the Schwarzschild case expresses the coordinate transform between local coordinates of the particle \(m\) and the coordinates of the observer at infinity. Actually, considering definition \(\alpha_R = \varphi_o/\varphi(r)\), this specific metric expresses the ratio of local units at two specific potentials. This naturally suggests the generalization of the metric to arbitrary potentials of arbitrary spacetimes by taking, instead of \(\alpha_R\), the general form \(\alpha_r\) (69) of the relational dilation factor between two arbitrary potentials. Hence, the coordinates \((\sigma_A, \tau_A)\) and \((\sigma_B, \tau_B)\) at positions \(x_A\) and \(x_B\) relate according to

\[
d\tau_B^2 = \alpha_R d\tau_A^2
\]

\[
d\sigma_B^2 = \frac{1}{\alpha_R} d\sigma_A^2
\]

or, alternatively

\[
\varphi(x_B) d\tau_B^2 = \varphi(x_A) d\tau_A^2
\]

\[
\varphi(x_A) d\sigma_B^2 = \varphi(x_B) d\sigma_A^2
\]

Eq. (81) and (82) present a general relational metric, satisfying the relational principle: suppose, from earth, at position \(x_A\), we observe an infinitesimal small test particle entering empty space, i.e. getting farther away from earth and all other masses. The local potential \(\varphi(x_B)\) at the position of the particle will gradually vanish and, as a result, the local coordinates \((\sigma_B, \tau_B)\) of the particle dilute, meaning unbound increase of the local clock rate and of the unit length, while moving towards vacuum infinity.
7.3 The relational model

The relational model consists of the Machian model subject to the coordinate transform, defined by the relation metric (81),(82). The relational model fits the relational principle (including Mach’s principle) and appears consistent with GR, provided the background potential is equal to $\varphi_0$. Under that condition we retrieve the Schwarzschild dilation factor ($\alpha_R = \hat{\alpha}_R \approx \alpha_s$). Since the relational model matches GR in the Schwarzschild case, it matches major GR results: gravitational time dilation and redshift, deflection of a photon by the sun, anomalous perihelion precession. The relational model, however, applies to general spacetimes, i.e. is not limited to the Schwarzschild spacetime. One example is Lense-Thirring frame-dragging (section 6.2), which regards the Kerr spacetime. The real proof of concept is likely to be found on the galactic or cosmic scale.

7.4 Cosmological implications

When in the Schwarzschild case the background potential gets significantly below $\varphi_0$, the corresponding relational model starts to deviate significantly from the Schwarzschild metric, since at lower potential the relational metric will expand spacetime, while the Schwarzschild metric will remain unchanged. This may shed a light on some questions of remote observation on the galactic and cosmic scale, such as the galaxy rotation curve problem and the question of the accelerated expansion of the universe.

The galaxy rotation problem - This regards the flat rotation velocity curve of stars on the outskirts of a galaxy. Their speed does not decrease at increasing distance from the galactic center as one would expect by Newton’s laws, i.e. stars seem to move too fast. Considering the relational metric, the lower potential away from the central bulge has several implications: the inertia of a star relative to the central bulge mass is less, the local meter near the star is longer and the local time runs faster. All these factors cause a higher value of the observed velocity relative to the expected Newtonian velocity.

Accelerated expansion of the universe - Objects receding from us into areas of weaker potential lose kinetic energy, however gain speed (as observed from earth) due to decreasing inertia and the expanded spacetime at weaker potentials. Consider an infinitesimal small test particle $m$ entering empty space, moving away from the total universal mass, represented by a sphere $M$ of radius $R_M$. We assume the mass density inside of $M$ is spherically symmetric. The particle is at a distance $r > R_M$ from the center of $M$, where $r$ is measured in the local length unit of the particle. The particle is observed from earth, so the observer is situated inside of $M$ at potential $\varphi_0$ and the relational dilation factor for the observer relative to the position of the particle is

$$\alpha_R = \frac{-\varphi_0 r}{GM}$$

which grows unbound for $r \to \infty$. Recalling (93), the total energy of the two body system $(M, m)$ is

$$E = \frac{1}{2} \mu r^2 + \mu \varphi_0 = \frac{1}{2} \mu r^2 - \frac{1}{2} \mu c^2$$
where $\mu \varphi_o \leq E \leq 0$ and $\mu = -GmM/\varphi_or$, from which we obtain the (squared) velocity of the particle as a function of distance $r$

$$r^2 = \frac{2E}{\mu} - 2\varphi_o = c^2 \left( \frac{E}{GmM} \cdot r + 1 \right). \quad (85)$$

For $r > R_m$ the particle will decelerate (since $E$ is negative) and will reach a maximum distance $r_{max}$ at the point where $\dot{r} = 0$, i.e.

$$r_{max} = \frac{-GmM}{E}, \quad (86)$$

and will eventually fall back to $M$. By the metric (79),(80) the speed of the particle, but as observed from earth, is equal to

$$\dot{r}_{\text{obs}}^2 = \alpha_R^2 \dot{r}^2 = \alpha_R^2 c^2 \left( \frac{E}{GmM} \cdot r + 1 \right) = c^2 \left( \frac{\varphi_o}{GM} \right)^2 \left( \frac{E}{GmM} \cdot r + 1 \right). \quad (87)$$

So when $m$ enters empty space, $\alpha_R$ grows and the particle will at first appear to accelerate, as seen from earth. A maximum observed velocity will be reached where

$$\frac{d(\dot{r}_{\text{obs}}^2)}{dr} = c^2 \left( \frac{\varphi_o}{GM} \right)^2 \left( \frac{E}{GmM} \cdot 3r^2 + 2r \right) = 0, \quad (88)$$

which is at a distance

$$\dot{r} = -\frac{2}{3} \frac{GmM}{E} = \frac{4}{3} r_{max}. \quad (89)$$

It is only beyond this point, at $2/3$ of its total outward journey, that also in the eye of the observer the particle will decelerate, until standstill at $r_{max}$ and fall back. Apparently, and to our comfort, the universe is not yet at this stage and is still in accelerating phase.

### 8 Concluding notes

In the tradition of Descartes, Newton, Berkeley, Mach, Einstein and Schrödinger, we discussed a number of concepts of relational physics and demonstrated the feasibility of a relational theory of relativity. To some extend, one can view this as a reinterpretation of GR by a rotation of concepts. However, the relational concepts have different roots and a merit of their own. In summary:

a) The principle of Berkeley: only radial motion between two (point) masses in otherwise empty space represents inertia and kinetic energy between these masses. This is essentially the basis for the anisotropic Machian model. Moreover, it makes the emergence of space and time from the interaction of masses conceivable.
b) The notion that inertia (Mach) and kinetic energy (Schrödinger) are mutual properties between pairs of objects, like potential energy, leading to Machian definitions of inertia and (frame independent) kinetic energy, which necessarily reflect the definition of potential energy, in order to provide conservation of energy under arbitrary choice of frame. This provides also for the definition of the Machian kinetic energy between two spinning bodies.

c) The notion that the speed of light is not a postulate but inherent in Machian inertia.

d) Newtonian kinetic energy in the CM frame (only) reflects the energy relationship between an object and the distant masses of the universe, while the (anisotropic) Machian energy terms add the energy relations between the various objects in the system under study (e.g. Mercury and the sun). These small Machian terms account for relativistic trajectories, like the anomalous precession of Mercury and Lense-Thirring frame dragging. The Machian model applies to arbitrary configurations of N-body systems.

e) The notion that inertia is anisotropic. Isotropic models can not explain relativistic trajectories.

f) Relativistic effects of remote observation, such as time dilation and redshift, require an isotropic model.

g) The relational principle (including Mach’s principle), by which space, time and inertia emerge from the distribution of matter. Simple models are proposed to relate these quantities to gravitational potential, leading to an isotropic relational metric for general spacetimes, which accounts for relativistic effects of observation equivalently to GR.

h) The relational model, i.e. the anisotropic Machian model subject to the isotropic relational metric, accounts for both relativistic trajectories and relativistic effects of observation equivalently to GR.

i) The notion that, from the relational point of view, the GR-vacuum is not empty, but a space with a flat background potential \( \varphi_o \).

j) From the relational point of view, the Schwarzschild dilation factor agrees with a specific case of the more general relational dilation factor. The latter possibly provides a mechanism to explain phenomena at the galactic and cosmic scale, such as the flat galaxy rotation curves and the accelerated expansion of the universe.

It is not necessarily time dilation that makes a clock run slower. So even fundamental concepts may be exchangeable, allowing for different views of the same world. This makes the relational approach viable, both in its own right and for its consistency with
GR. I maintain some reservations though, reminiscent of John Wheeler\(^7\), about the self-referential definitions of gravitational potential and distance; one determines the other and vice versa.

**Appendix A - Isotropic inertia can not explain anomalous perihelion precession**

We represent the isotropic partial inertia between the planet \(m\) and the sun \(M\) by an arbitrary function \(\xi = \xi(r)\), where \(r\) is the separation between \(m\) and \(M\). Instead of the anisotropic energy equation (40), we have the isotropic equation

\[
\frac{1}{2}(m + \xi)\dot{r}^2 + \frac{1}{2}(m + \xi)r^2\dot{\phi}^2 + (m + \mu)\varphi_o = E_o
\]

(90)

(where \(\mu\varphi_o\) is the potential energy between planet and sun, as in the anisotropic case). Note that the isotropic equation is nearly the Newtonian equation

\[
\frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\phi}^2 + (m + \mu)\varphi_o = E_o.
\]

(91)

The isotropic inertia \(\xi\) has an effect on the velocities \(\dot{r}\) and \(\dot{\phi}\). But this effect is equal in both directions, therefore does not affect the trajectory \(r(\phi)\) of the planet in the \(r,\phi\)-plane. The easy way to show this is by replacing in (90) the time parameter \(\tau\) by \(\tau' = \tau m/(m + \xi)\), yielding the Newtonian equation. Hence, the trajectory of the isotropic system is Newtonian, therefore, is elliptic and has zero precession.

**Appendix B - Basic Machian kinematic equation for two bodies**

We shall adopt a Lagrangian approach in deriving the equations of motion for a system of two point particles \(m_1\) and \(m_2\) falling to each other. This is done for two different cases: in empty space and in our universe:

**In empty space (\(\varphi_b = 0\))**

The Lagrangian associated with the energy of the particles is

\[
L = T - V
\]

(92)

where

\[
T = \frac{1}{2}\mu r^2 \quad V = \mu\varphi_o.
\]

(93)

\(^7\)“Spacetime tells matter how to move; matter tells spacetime how to curve.”
The inertia between the two bodies is \( \mu = -\frac{Gm_1m_2}{r_0} \varphi_o \) (the subscripts of \( \mu \) and \( r \) have been dropped for convenience, so \( \mu = \mu_{12} \) and \( r = r_{12} \)). Lagrange's equation for \( r \) is

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \mu \dot{r} + \mu \ddot{r} + \frac{\mu}{r} (\frac{1}{2} \dot{r}^2 - \varphi_o) = 0, \tag{94}
\]

which yields, using \( \dot{\mu} = -\frac{\mu}{r} \dot{r} \), the Machian kinematic equation for two masses falling to each other in empty space:

\[
\ddot{r} = \frac{\varphi_o}{r} (1 + \frac{\dot{r}^2}{2\varphi_o}). \tag{95}
\]

This differential equation lets \( m_1 \) and \( m_2 \) accelerate towards each other until \( \ddot{r} = 0 \), at which point the relative velocity \( |\dot{r}| \) attains its maximum value

\[
|\dot{r}|_{\text{max}} = \sqrt{-2\varphi_o}. \tag{96}
\]

According to (7), \( \varphi_o = -\frac{1}{2} c^2 \), hence, \( |\dot{r}|_{\text{max}} = c \), the speed of light. This is not a surprise, as we recognize the Lorentz factor \( \gamma \) in Eq.(95):

\[
(1 + \frac{\dot{r}^2}{2\varphi_o})^{-1/2} = (1 - \frac{\dot{r}^2}{c^2})^{-1/2} = \gamma. \tag{97}
\]

Fortunately, differential equation (95) is easy to solve

\[
r(t) = \left( \frac{\dot{r}_0^2 - c^2}{4r_0} \right) t^2 + \dot{r}_0 t + r_0 \tag{98}
\]

where \( r_0 \) are \( r_0 \) are initial values of \( r \) and \( \dot{r} \), respectively. If we set \( \dot{r}_0 = 0 \), then (98) reduces to

\[
r(t) = \frac{-c^2}{4r_0} t^2 + r_0 \tag{99}
\]

so,

\[
\dot{r}(t) = \frac{-c^2}{2r_0} t \tag{100}
\]

from which follows that collision \( (r = 0) \) takes place at \( t_c = 2r_0/c \), at which point

\[
|\dot{r}(t_c)| = |\frac{-c^2}{2r_0} t_c| = c \tag{101}
\]

**In our universe \( (\varphi_b = \varphi_o) \).**

If we move \( m_1 \) and \( m_2 \) into our universe, somewhere near earth, where \( \varphi_b = \varphi_o \), then the kinetic energy term in (92) extends to

\[
T = \frac{1}{2} \mu r^2 + \frac{1}{2} m_1 r_1^2 + \frac{1}{2} m_2 r_2^2 \tag{102}
\]
where \( \dot{r} = \dot{r}_1 + \dot{r}_2 \) and where \( \dot{r}_1 \) and \( \dot{r}_2 \) are the velocities of the bodies relative to their CM. We may simplify (102) to

\[
T = \frac{1}{2}(\mu + m_{12})r^2
\]

(103)

where \( m_{12} = \frac{m_1 m_2}{m_1 + m_2} \) represents the additional inertia due to the presence of the masses of the universe. Via the Lagrangian, we obtain the kinematic equation

\[
\ddot{r} = \frac{\mu}{\mu + m_{12}} \frac{r^2}{r^2} (1 + \frac{\dot{r}^2}{2\varphi_o})
\]

(104)

which is identical to the equation for the empty space case (95), except for an additional 'attenuation' factor \( \lambda = \frac{\mu}{\mu + m_{12}} \). Note that in general \( \lambda \ll 1 \). As a result, the acceleration of the particles toward each other is strongly attenuated by the extra inertia \( m_{12} \) caused by the cosmic masses. However, in the limit for \( r \to 0 \), \( \mu \) grows to infinity and the attenuation disappears (\( \lambda \to 1 \)), thus the kinematic equation (104) is asymptotically equal to the kinematic equation (95) for the empty space case. So finally \( \dot{r} \) will still reach the speed of light at collision, in a much steeper acceleration though, as can be verified easily by numerical simulation.

Hence, at collision, \( m_1 \) and \( m_2 \) reach the speed of light towards each other, whatever the initial conditions are and with or without the cosmic masses present. The condition for this to happen, however, is that the distance between the point particles \( m_1 \) and \( m_2 \) must actually get to zero. The question is how realistic such collisions are, considering that particles have finite size and other forces are involved in particle collisions. The mechanism of (104) suggests that the \( r = 0 \) condition may be (almost) met by particles which can get (almost) at the velocity limit \( c \), like photons and neutrino’s. Another question is what happens during and after collision? The extraordinary property of the Machian kinematic equation (104) is that the state of the particles at collision \( (r = 0, \dot{r} = c, \ddot{r} = 0) \) is always the same, i.e. does not depend on the state of the particles at any instance before collision. This implies that also after collision the state of the particles does not depend on pre-collision conditions (meaning history is gone). From particle physics, however, we know that momentum and energy are being conserved in both annihilation and pair production, i.e., state history is conserved in these collisions. This suggests that the \( r = 0 \) condition is not fully met in such collisions. This, in turn, suggests that the particles have not quite reached the velocity limit \( c \).

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