TAIL ESTIMATES FOR MARTINGALE UNDER "LLN" NORMING SEQUENCE

E. Ostrovsky, L.Sirota.

Department of Mathematics, Bar-Ilan University, Ramat-Gan, 59200,Israel.
E-mail: galo@list.ru, eugostrovsky@list.ru
Department of Mathematics, Bar-Ilan University, Ramat-Gan, 59200,Israel.
E-mail: sirota@zahav.net.il

Abstract.

In this paper non-asymptotic exponential and moment estimates are derived for tail of distribution for discrete time martingale under norming sequence 1/n, as in the classical Law of Large Numbers (LLN), by means of martingale differences as a rule in the terms of unconditional moments and tails of distributions of summands.

We show also the exactness of obtained estimations.

Key words: Random variables and vectors, martingales, martingale differences, Law of Large Numbers (LLN), lower and upper estimates, great or large deviations, moment, Banach spaces of random variables, tails of distribution, conditional expectation.

Mathematics Subject Classification (2002): primary 60G17; secondary 60E07; 60G70.

1 Introduction. Notations. Statement of problem.

Let \((\Omega, F, P)\) be a probabilistic space, \(\xi(1), \xi(2), \ldots, \xi(n), n < \infty\) being a centered \((E\xi(i) = 0, i = 1, 2, \ldots, n)\) martingale - differences on the basis of the flow of \(\sigma-\) fields (filtration) \(F(i) : F(0) = \{\emptyset, \Omega\}, F(i) \subset F(i+1) \subset F, \xi(0) = 0; E|\xi(i)| < \infty,\) and for every \(i \geq 0, \forall k = 0, 1, \ldots, i - 1 \Rightarrow\)

\[
E\xi(i)/F(k) = 0; E\xi(i)/F(i) = \xi(i) \pmod{P}.
\]

We denote

\[
S(n) = \sum_{i=1}^{n} \xi(i), \quad Q_n(x) = P(S(n)/n > x), \quad x > 0.
\]

Obviously, if the sequence \(\{\xi(i)\}\) satisfies the Weak Law of Large Numbers (WLLN), for instance, the r.v. \(\{\xi(i)\}\) are i.i.d. with finite expectation \(E|\xi(1)| < \infty,\) then
\[
\lim_{n \to \infty} Q_n(x) = 0.
\]

Our aim is to obtain the tail estimates \( Q_n(x) \) for \( S(n)/n \) via the moment and tail estimates (conditional or not) of the sequences \( \{\xi(i)\} \).

Our estimates improve or generalize the well-known inequalities belonging to Fan X., Grama I., and Liu Q [1], Grama I.G. [2], Grama I. and Haeusler E. [3], Lesign E. and Volny D. [6], Li Y. [7], Liu Q. and Watbled F. [9] etc.

Notice that in the articles [6], [9], [17] and many others are described some new applications of these estimates in the theory of dynamical system and in the theory of polymers.

Sometimes, see e.g. [6], [7], [9] etc. the non-asymptotical estimates of probability \( Q_n(x/\sqrt{n}) \) are called ”large deviation” (or ”great deviation”).

By our opinion, there exists a duality in this terminology: in many books and articles [4], [16], [15], [2], [3], [18] and many others the notion ”large deviation” (or ”great deviation”) was used for the asymptotical behavior of the fraction

\[
z := \frac{Q_n(x/\sqrt{n})}{1 - \Phi(x)}, \quad \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-y^2/2) \, dy
\]

as \( x = x(n) \to \infty \).

We denote as usually the \( L(p) \) norm of the r.v. \( \eta \) as follows:

\[
|\eta|_p = \left[ \mathbb{E}[|\eta|^p] \right]^{1/p}.
\]

The paper is organized as follows. The second section contains the main result: estimation of tail-function for normed sum of martingale differences. In the third section we intend to show the exactness of our estimates.

Fourth section contains is devoted to the tail estimates for martingales under more strict conditions; we obtain at the same estimates as in the independent case; following, they are not improvable.

In the last section we consider an inverse statement to our problem.

In all the sections we bring some examples in order to show the exactness of obtained estimates.

## 2 Main result: estimation of tail-function for normed sum.

We need for beginning to introduce some notations. Let \( T(x) \) and \( G(x) \), \( x > 0 \) be a two tail - functions, i.e. such that \( T(0) = G(0) = 1 \), \( T(\cdot), G(\cdot) \) are monotonically decreasing, right continuous and \( T(\infty) = G(\infty) = 0 \). Let us denote

\[
T \vee G(x) = \min(4 \inf_{y>0}(T(y) + G(x/y)), 1).
\]
The function $T \lor G(x)$ has a following sense: if $T(\xi, x) \leq T(x)$, $T(\eta, x) \leq G(x)$, then

$$T(\xi \cdot \eta, x) \leq T \lor G(x).$$

For example, if $\forall x > 0 \ T(\xi, x) \leq \exp(-x^{q(1)})$, $T(\eta, x) \leq \exp(-x^{q(2)})$, $q(1), q(2) = \text{const} > 0$, then

$$T(\xi \cdot \eta, x) \leq \min\left(1, 4 \exp(-C(q(1), q(2))) x^{q(1)q(2)/(q(1)+q(2))}\right).$$

Let $\xi$ be a random variable; its tail-function $T(\xi, x)$, $x \geq 0$ we introduce as follows:

$$T(\xi, x) = \max\left(P(\xi \geq x), P(\xi \leq -x)\right).$$

Further, for the tail-function $T(\cdot)$ we denote the following operator (non-linear)

$$W[T](x) = \min\left(1, \inf_{v>0} \left[\exp(-x^2/(8v^2)) - \int_{v}^{\infty} x^2 dT(x)\right]\right),$$

if there exists the second moment

$$\int_{0}^{\infty} x^2 |dT(x)| < \infty.$$

**Theorem 2.1.** Let again $\xi(i)$ be a sequence of martingale-differences relative to some filtration $\{F(i)\}$ and $T(\xi(i), x) \leq T(x)$, $T(x)$ be some tail-function. Then at $x \geq 2$

$$Q_n(x) \leq W[T](x \sqrt{n}). \quad (2.1)$$

**Proof.** We will use the following fact, see [12], Lemma 1.

Let again $\xi(i)$ be a sequence of martingale-differences relative to some filtration $\{F(i)\}$ and $T(\xi(i), x) \leq T(x)$, $T(x)$ be some tail-function. Then at $x \geq 2$

$$\sup_{b: \sum b(i)=1} T \left(\sum_i b(i)\xi(i), x\right) \leq W[T](x). \quad (2.2)$$

We obtain the proposition (2.1) after choosing $b(i) = 1/\sqrt{n}$ and substituting $x := x \sqrt{n}$.

**Example 2.1.** Assume that $T(\xi(i), x) \leq Y \exp(-(x/K)^q)$, $K, x, q > 0, Y = \text{const} \geq 1$. Denote here

$$\delta = \delta(q) = (\min(q/2, 1))^{-1/4};$$

$$q \in (0, 2] \Rightarrow \beta(q) = \max\left(\frac{1}{q} \Gamma\left(\frac{q}{2}\right), e\left(\frac{2}{eq}\right)\right);$$

$$q > 2 \Rightarrow \beta(q) = \sup_{v \geq 0} \exp(v^q) \int_{v}^{\infty} x \exp(-x^q) \ dx \leq \Gamma(2/q)/(qe);$$

3
\( \Gamma(\cdot) \) is usually Gamma - function. We obtain after some calculations that at \( x \geq 2 \)
\[
Q_n(x) \leq (1 + 2Y \beta(q)) \exp \left( -r^{q/(q+2)} \frac{[x/(K\delta)]^{2q/(q+2)}}{[x/(K\delta)]^{2q/(q+2)}} \right).
\] (2.3)

\textbf{Remark 2.1.} At the value \( q = 1 \) we obtain the main result of article [6]; at the value \( q \in (0, 1) \) the main result of article [1].

\textbf{Example 2.2.} Assume now that
\[
T(\xi(i), x) \leq C_1 \exp \left( -C_2(x/K)^q (\log(F(q, r) + x/K)^r) \right), x > 0,
\]
where by definition
\[
F = F(q, r) = 1, r \leq 0; F(q, r) = \exp(q), r > 0.
\]

Let us introduce the following vector - function \( L(q, r) = \{L(1; q, r), L(2; q, r)\} \)

of a variables \((q, r)\) :
\[
L(1; q, r) = \frac{2q}{q+2}, L(2; q, r) = \frac{2r}{q+2}.
\]

We conclude from theorem 2.1:
\[
T(n^{-1/2} \sum_i \xi(i), x) \leq \exp \left( -C_4(x/K)^{L(1)} (\log(F(L(1), L(2)) + x/K)^{L(2)}) \right), x \leq 2,
\]
where \( L(i) = L(i; q, r), i = 1, 2; \) or equally
\[
Q_n(x) = T(n^{-1} \sum_i \xi(i), x) \leq \exp \left( -G_{q,r}(x/K) \right), x \geq 2.
\]

We consider now moment, more exactly, \( L_p \) estimations for \( S(n)/n \).

\textbf{Theorem 2.2.} Let \( \forall i \xi(i) \in L(p), p \geq 2; x \geq 1. \) Then
\[
Q_n(x) \leq x^{-p} \cdot (p - 1)^p \cdot n^{-p/2} \cdot \left\{ n^{-1} \sum_{i=1}^{n} ||\xi(i)||_p^2 \right\}^{p/2}.
\] (2.4)

\textbf{Proof.} The inequality
\[
|n^{-1/2}S(n)|_p \leq (p - 1) \left\{ n^{-1} \sum_{i=1}^{n} ||\xi(i)||_p^2 \right\}^{1/2}.
\] (2.5)

is proved in [10]; from (2.5) it follows
\[
|n^{-1}S(n)|_p \leq (p - 1) \cdot n^{-1/2} \cdot \left\{ n^{-1} \sum_{i=1}^{n} ||\xi(i)||_p^2 \right\}^{1/2}.
\] (2.6)

It remains to use the Markov’s-Tchebychev’s inequality.

\textbf{Remark 2.2.} The proposition of theorem 2.2 improves the correspondent moment estimate from the article Lesign E., Volny D. [6]. We write the exact value of
constant in the right-hand side, which is less that one in [6]; the non-improperness of the factor \((p - 1)\) is grounded also in [10].

**Remark 2.3.** Assume that for some interval of a values \(p: p \in [2, a)\), where \(a = \text{const} > 2 \forall i |\xi(i)|_p < \infty\). It follows from the inequality (2.4) then

\[
Q_n(x) \leq \inf_{2 \leq p < a} \left[ x^{-p} \cdot (p - 1)^p \cdot n^{-p/2} \cdot \left\{ n^{-1} \sum_{i=1}^{n} |\xi(i)|_p^2 \right\}^{p/2} \right]. \tag{2.7}
\]

\section{Exactness of our estimates.}

We prove in this section that the proposition of theorem 2.1 is non-improvable; we will construct an example of martingales for which in (2.1) is attained asymptotical equality.

Note that the exactness of proposition of theorem 2.2 is proved in [6]. We restrict ourselves only the case when exists a positive value \(q\) such that for all the values \(n = 1, 2, \ldots\) and for all positive values \(x\)

\[
\inf_i T(\xi(i), x) \geq \exp (-x^q), \quad \sup_i T(\xi(i), x) \leq C_1 \exp (-x^q). \tag{3.1}
\]

**Theorem 3.1.** There exists a martingale \((S(n), F(n))\) with correspondent martingale-differences \(\xi(i)\) satisfying the conditions (3.1); and such that there exist finite positive values \(C_2, C_3\) for which

\[
Q_n(1) \geq C_2 \exp \left( -C_3 n^{q/(q+2)} \right). \tag{3.4}
\]

**Proof.** Assume at first \(q < 1\); the case \(q = 1\) is investigated in [6]. Denote \(\beta = 1/(2 + q)\), then \(1 - \beta = (q + 1)/(2 + q)\) and \(\beta \in (0, 1)\).

We will follow the method offered by Lesign E. and Volny D. [6]; namely, we intend to built a very interest martingale of a view

\[
\xi(i) = \eta \cdot \zeta(i), \tag{3.2}
\]

where the r.v. \(\{\zeta(i)\}\), \(\eta\) are symmetrical distributed and total independent:

\[
T(\eta, x) = T(-\eta, x) = \exp (-x^q), \tag{3.3}
\]

the r.v. \(\zeta(i)\) are i.i.d., non-trivial and bounded: \(0 < |\zeta(1)|_\infty < \infty\).

It is easy to verify that the r.v. \(\xi(i) = \eta \cdot \zeta(i)\) satisfy to the conditions (3.1).

Further,

\[
Q_n(1) \geq P \left( \sum_{i=1}^{n} \zeta(i) > C_4 n^{1-\beta} \right) \cdot P \left( \eta > C_5 n^\beta \right) \overset{df}{=} Z_1 \cdot Z_2, \tag{3.4}
\]

where \(C_4, C_5\) are suitable constants depending on \(q\). We get:
\[ Z_2 \geq C_6 \exp \left( -C_7 \left[ n^{q} \right]^q \right) \geq C_8 \exp \left( -C_9 \frac{n^{q/(q+2)}}{2} \right); \quad (3.5) \]

\[ Z_1 = P \left( \sum_{i=1}^{n} \zeta(i) > C_4 n^{(q+1)/(q+2)} \right) = P \left( n^{-1/2} \sum_{i=1}^{n} \zeta(i) > C_4 \frac{n^{q/(2(q+2))}}{2} \right). \quad (3.6) \]

As long as here \( q \in (0, 1) \), \( q/(2(q+2)) \in (0, 1/6) \) and we can apply the classical theory of great deviations [16], [4], [8]:

\[ Z_1 \geq C_{10} \exp \left( -C_{11} \left[ n^{q/(2(q+2))} \right]^2 \right) = C_{10} \exp \left( -C_{11} \frac{n^{q/(q+2)}}{2} \right). \quad (3.7) \]

Multiplying the lower estimates for \( Z_1 \) and \( Z_2 \) we obtain what was required.

Let now \( q > 1 \). Let us define the (unique) integer number \( s = s(q) \) such that \( q - 1 \leq s < q \); then

\[ \frac{s}{2(s+2)} \leq \frac{q}{2(q+2)} \leq \frac{s+1}{2(s+3)}. \quad (3.8) \]

We will again construct a martingale differences by the formula (3.2), and the r.v. \( \eta \) by the formula (3.3); the r.v. \( \zeta(i) \) are again i.i.d., non-trivial, bounded \( 0 < |\zeta(1)|_{\infty} < \infty \).

Moreover, we impose on the distribution \( \zeta(i) \) the following condition:

\[ \mathbf{E} \zeta^m(i) = \mathbf{E} N^m(0, 1), \quad m = 2, 3, 4, \ldots, s+2, \quad (3.9) \]

where \( N(0, 1) \) denotes the standard normal distribution.

We can use on the basis of our assumptions ((3.9) etc.) the theory of large deviations, alike in the case \( 0 < q < 1 \); more exactly, we use theorem of Yu.V.Linnik [8].

After simple calculations we obtain what was required.

### 4 Tail estimates for martingales under more strict conditions.

We impose in this section some strict condition on the martingale differences and obtain an estimate for normed martingale at the same estimate as for sums of independent centered variables.

In order to formulate our result, we need to introduce some addition notations and conditions. Let \( \phi = \phi(\lambda), \lambda \in (-\lambda_0, \lambda_0), \lambda_0 = \text{const} \in (0, \infty] \) be some even strong convex which takes positive values for positive arguments twice continuous differentiable function in the intervals \( |\lambda| \geq 2 \), such that

\[ \phi(0) = 0, \quad \phi^{(i)}(0) \in (0, \infty), \quad \lim_{\lambda \to \lambda_0} \phi(\lambda)/\lambda = \infty. \quad (4.1) \]
We denote the set of all these function as $\Phi$: $\Phi = \{\phi(\cdot)\}$.

We say that the centered random variable (r.v) $\xi = \xi(\omega)$ belongs to the space $B(\phi)$, if there exists some non-negative constant $\tau \geq 0$ such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\lambda \xi) \leq \exp[\phi(\lambda \tau)].$$

(4.2)

The minimal value $\tau$ satisfying (4.2) is called a $B(\phi)$ norm of the variable $\xi$, write

$$||\xi||_{B(\phi)} = \inf \{\tau, \tau > 0 : \forall \lambda \Rightarrow \mathbb{E} \exp(\lambda \xi) \leq \exp(\phi(\lambda \tau))\}.$$  

(4.3)

This spaces are very convenient for the investigation of the r.v. having a exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous of random fields, study of Central Limit Theorem in the Banach space etc.

The space $B(\phi)$ with respect to the norm $|| \cdot ||_{B(\phi)}$ and ordinary operations is a Banach space which is isomorphic to the subspace consisted on all the centered variables of Orlicz’s space $(\Omega, F, \mathbb{P}), N(\cdot)$ with $N$ – function

$$N(u) = \exp(\phi^*(u)) - 1, \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)).$$

The transform $\phi \rightarrow \phi^*$ is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Moraux:

$$\phi^{**} = \phi.$$

The next facts about the $B(\phi)$ spaces are proved in [5], [11], p. 19-27:

1. $\xi \in B(\phi) \Leftrightarrow \mathbb{E} \xi = 0$, and $\exists C = \text{const} > 0,$

$$T(\xi, x) \leq \exp(-\phi^*(Cx)), x \geq 0,$$

and this estimation is in general case asymptotically exact.

2. Denote

$$\overline{\phi}(\lambda) = \sup_n \left[n\phi(\lambda/\sqrt{n})\right].$$

(4.4)

Let $\eta(i)$ be a sequence of i.i.d. random variables belonging to the space $B(\phi)$. We reproduce the following tail inequality for normed sum:

$$T \left(n^{-1/2} \sum_{i=1}^{n} \eta(i), x\right) \leq 2 \exp \left(-\{\overline{\phi}\}^*(x)\right).$$

(4.5)

Let $\phi(\cdot) \in \Phi$ and let $\tilde{F}$ be any sub-sigma algebra of the source sigma-field $F$.

Definition 4.1.
We say that the \textit{centered} random variable (r.v) \( \xi = \xi(\omega) \) belongs to the space \( B(\tilde{F}, \phi) \), if there exists some non-negative \textit{non-random} constant \( \tau \geq 0 \) such that

\[
\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\lambda \xi) / \tilde{F} \leq \exp[\phi(\lambda \tau)].
\]  

(4.6)

The minimal value \( \tau \) satisfying (4.6) is said to be a \( B(\tilde{F}, \phi) \) norm of the variable \( \xi \), write

\[
||\xi||_{B(\tilde{F}, \phi)} = \inf\{\tau, \tau > 0 : \forall \lambda \Rightarrow \mathbb{E} \exp(\lambda \xi) / \tilde{F} \leq \exp(\phi(\lambda \tau))\}.
\]

(4.7)

The space \( B(\tilde{F}, \phi) \) with respect to the norm \( ||\cdot||_{B(\tilde{F}, \phi)} \) and ordinary operations is also a Banach space.

Our definition (4.1) is a direct generalization of a definition in [9], where is considered only the case

\[
\phi(\lambda) = \phi_q(\lambda) := |\lambda|^q, \ |\lambda| > 1; \ \phi(\lambda) = \phi_q(\lambda) = \lambda^2, \ |\lambda| \leq 1,
\]

where \( q = \text{const} \geq 1 \) (in our notations).

\textbf{Theorem 4.1.} Suppose the martingale \((S(n), F(n))\) with correspondent martingale differences \(\{\xi(i)\}\) is such that for some function \(\phi(\cdot) \in \Phi\)

\[
\forall k = 1, 2, \ldots \ \xi(k) \in B(F(k-1), \phi),
\]

and

\[
\sup_k ||\xi(k)||_{B(F(k-1), \phi)} \leq 1 \pmod{\mathbb{P}},
\]

(4.8)

or equally

\[
\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\lambda \xi(k)) / F(k-1) \leq \exp[\phi(\lambda)] \pmod{\mathbb{P}}.
\]

(4.9)

Then

\[
Q_n(x) \leq 2 \exp\left(-\{\overline{\phi}\}^*(x\sqrt{n})\right).
\]

(4.10)

\textbf{Proof.} Let \( \lambda \in (-\lambda_0, \lambda_0) \). We consider an exponential moment of the martingale \( \lambda S(n) \). Indeed: \( \mathbb{E} \exp(\lambda S(n)) = \)

\[
\mathbb{E} \mathbb{E} \exp(\lambda S(n)) / F(n-1) = \mathbb{E} [\lambda S(n-1) \mathbb{E} \exp(\lambda \xi(n)) / F(n-1)] \leq
\]

\[
\mathbb{E}(\lambda S(n-1)) \cdot \exp(\phi(\lambda)) \leq \mathbb{E}(\lambda S(n-2)) \cdot \exp(2\phi(\lambda)) \ldots \leq
\]

\[
\exp(n\phi(\lambda)),
\]

i.e. as in the case of summing of independent identical distributed random variables.

It remains to use the formula (4.5), where instead \( x \) we write \( x\sqrt{n} \).
Example 4.1. Suppose in addition to the conditions of theorem 4.1 as in [9] that
\[ \phi(\lambda) = \phi_q(\lambda), \ q = \text{const} > 0, \]
or equally alike in the article [9]
\[ \exists C \in (0, \infty), \ \sup_{k} \sup_{\Omega} \mathbb{E} \exp(C|\xi(k)|^q) / F(k-1) < \infty. \]
Define the variable \( \gamma(q) \) as follows:
\[ \gamma(q) = \min(2, q), \ q \geq 2; \quad \gamma(q) = 2q/(2 + q), \ q \in (0, 2). \]
We conclude on the basis of theorem 4.1:
\[ Q_n(x) \leq C_1(q) \exp \left(-C_2(q) \frac{x^{\gamma(q)} n^{\gamma(q)/2}}{2} \right). \quad (4.11) \]
Notice that the last estimate is non-improvable still for independent variables \( \xi(i) \).

Remark 4.1 In the book [11], p. 38-43 there are many of exponential tail estimates for sums of independent r.v. They may be generalize on the martingale case satisfying the condition (4.8).

Remark 4.2 The non-asymptotical moment and tail estimates for heavy tailed polynomial martingales are obtained in the article [13].

Notice that in the case of martingales with heave tails the classical normalized sequence \( n^{-1/2} \) in general case may be replaced. For instance, if \( \xi(i) \) are i.i.d. symmetrical r.v. with power tails of a view (symmetrical Pareto distribution)
\[ T(\xi(i), x) = \min(1, x^{-r}), \ x > 0, \ r = \text{const} \in (0, 2), \]
then the normalized sequence \( b(n) \) may be choose as follows: \( b(n) = n^{-1/r} \) (the Stable Limit Theorem).

The case of super-heavy tails, for example, distributions without variance, and when the r.v. \( \{\xi(i)\} \) are independent centered summands was considered in [14] (non-asymptotical approach).

5 Inverse results to our estimates.

Suppose for any martingale \((S(n), F(n))\) the following inequality is true:
\[ Q_n(x) \leq C_1 \exp \left(-C_2 n^{q/(q+2)} x^{2q/(q+2)} \right), \ x \geq 2, \ n \geq 1. \quad (5.1) \]
Moreover, it is sufficient to assume the inequality (5.1) only for some fixed positive values \( x \); say, for the value \( x = 1 \):
\[ Q_n(1) = \mathbb{P}(S(n) > n) \leq C_1 \exp \left(-C_2 n^{q/(q+2)} \right), \ n \geq 1. \quad (5.2) \]
Question: give the possible lower estimate for tail-function \( T(\xi(i), x) \).
**Theorem 5.1.** Assume the martingale difference \( \{\xi(i)\} \) are in addition independent and identically distributed. Suppose also the condition (5.2) is satisfied. Then

\[
\sup_i T(\xi(i), x) \leq C_3 \exp\left(-C_4 x^{2q/(q+2)}\right), \ x \geq 2. \tag{5.3}
\]

**Proof.** We will again follow the method offered by Lesign E. and Volny D. [6]. Namely, we will use the following fact:

\[
\lim_{n \to \infty} \left[ \frac{Q_n(1)}{nP(|\xi(1)| > 2n)} \right] \geq 1. \tag{5.4}
\]

It follows from (5.4) and (5.2) that for all sufficiently greatest values \( n \), say \( n \geq 1 \)

\[
\sup_i T(\xi(i), n) = T(\xi(1), n) \leq C_5 n^{-1} \exp\left(-C_4 n^{2q/(q+2)}\right) \leq C_5 \exp\left(-C_4 n^{2q/(q+2)}\right). \tag{5.5}
\]

This completes the proof of theorem 5.1.

**Remark 5.1.** Assume that for some \( s = \text{const} > 1 \)

\[
Q_n(1) \leq C n^{1-s}, \ n \geq 1. \tag{5.6}
\]

We obtain by means of described method the tail inequality for \( \xi(i) \) in the case when the random variables \( \xi(i) \) are centered and i.i.d. that

\[
\sup_i T(\xi(i), x) \leq C_5(s) x^{-s}, \ x \geq 1. \tag{5.7}
\]

The last inequality implies that the r.v. \( C^{-1} \xi(i) \) belong to the unit ball of the Lorentz space \( L_{s,\infty} = L_{s,\infty}(\Omega) \). Recall that the norm in this space of the r.v. \( \xi \) may be defined as follows:

\[
||\xi||_{L_{s,\infty}} = \sup_{x>0} \left[ x^s T(\xi, x) \right].
\]

**References**

[1] Fan X., Grama I., and Liu Q. Large deviations for martingales with exponential condition. arXiv:1111.1407 [math.PR] 6 Nov 2011.

[2] Grama I.G. On moderate deviations for martingales. Ann. Probab. Volume 25, Number 1 (1997), 152-183.

[3] Grama I. and Haeusler E., 2000. Large deviations for martingales via Cramers method. Stochastic Process. Appl. 85, 279293.
[4] Ibragimov I. A. and Linnik Yu. V. (1965). Independent and Stationary Connected Variables. Nauka, Moscow. (In Russian.) Mathematical Reviews (MathSciNet): MR202176 Zentralblatt MATH: 0154.42201

[5] Kozatchenko Yu. V., Ostrovsky E.I., The Banach Spaces of random Variables of subgaussian type. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43 - 57 (1985).

[6] Lesign E., Volny D. Large deviations for martingales. Stochastic Processes and their Applications, 96, 143 - 159 (2001)

[7] Li Y, 2003. A martingale inequality and large deviations. Statist. Probab. Lett. 62, 317321.

[8] Linnik Yu.V. Limit theorems for sums of independent r.v. taking into account of large deviations. (1961), Theory Probab. Appl., 6, N° 2, p. 131-148; 6, N° 4, p. 363-360.

[9] Liu Q., Watbled F. Exponential inequalities for martingales and asymptotic properties of the free energy of directed polymers in random environment. arXiv:0812.1719v1 [math.PR] 9 Dec 2008.

[10] Ostrovsky E., Sirota L. Moment and tail estimates for martingales and martingale transform, with application to the martingale limit theorem in Banach spaces. arXiv:1206.4964v1 [math.PR] 21 Jun 2012

[11] Ostrovsky E.I., 1999. Exponential estimations for Random Fields and its applications (in Russian). 1999, Obninsk, Russia, OINPE.

[12] Ostrovsky E. Bide-side exponential and moment inequalities for tails of distribution of Polynomial Martingales. arXiv: math.PR/0406532 v.1 Jun. 2004.

[13] Ostrovsky E. and Sirota L. Moment and tail inequalities for polynomial martingales. The case of heavy tails. arXiv:1112.2768v1 [math.PR] 13 Dez 2011

[14] Ostrovsky E. and Sirota L. Non-improved uniform tail estimates for normed sums of independent random variables with heavy tails, with applications. arXiv:1110.4879v1 [math.PR] 21 Oct 2011.

[15] Saulis, L. and Statulevicius, V. (1989). Limit Theorems for Large Deviations. Mokslas, Vilnius. (In Russian.) Mathematical Reviews (MathSciNet): MR93e:60055b

[16] Petrov V.V. Limit theorems of probability theory, volume 4 of Oxford Studies in Probability. The Clarendon Press Oxford University Press, New York, 1995. Sequences of independent random variables, Oxford Science Publications.

[17] Volny D. Approximating martingales and the central limit theorem for strictly stationary processes. Stoch. processes ant their applic., 44, (1993). 41-74.
[18] Ratchkauskas A. Large deviations for martingales with some applications. Acta Applicandae Mathematicae, Volume 38, Number 1 (1995), 109-129, DOI: 10.1007/BF00992617