INFRARED SINGULARITIES IN THE
RENORMALIZATION GROUP FLOW OF
YANG-MILLS THEORIES IN THE AXIAL GAUGE

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Abstract

It is shown, by explicit calculation in the axial gauge, that the renormalization group flow
for the Wilson loop in perturbation theory does exhibit singularities and consequently it can
not eventually reproduce the gauge invariant result, when the infrared cut off is removed.

1. Since the appearance of Wilson’s seminal ideas [1] and subsequent Polchinski’s devel-
opments [2], the Renormalization Group Flow (RGF) approach to the quantization and
renormalization of field theories has become more and more popular, both from perturba-
tive and non-perturbative points of view. A truly remarkable amount of work has been
done in the last few years along that line [3] and, in particular, the RGF approach to the
setting up of perturbative Yang-Mills theories in covariant gauges has been thoroughly
discussed in [4]. There, it is neatly pointed out that, in the presence of some continuous
local symmetry such as the non-abelian gauge symmetry, the main problem is the solution
of the so called fine tuning functional equation, which corresponds to the restoration of the
symmetry at the quantum level, after removal of the ultraviolet and infrared regulators Λ
and Λ respectively. The solution of the above functional equation turns out to be either
rather burdensome at the perturbative level (technically extremely heavy beyond one loop)
or insofar unknown at the non-perturbative level. Later on it has been argued [5,6] that
the above mentioned difficulty could be circumvented after transition to the axial gauge.
As a matter of fact, it turns out that (i) regularized Green’s function in the axial gauge
fulfill, in the limit Λ ↓ 0, Lee-Ward-Takahashi identities much simpler than Slavnov-Taylor
identities owing to decoupling of Faddeev-Popov ghosts; (ii) the presence of the infrared
cut off does provide a natural way to screen the non-covariant singularities of the Green’s
functions and, specifically, of the free propagator. Quite remarkably, it has been recently
proved [6] that a modified but simple form of the Lee-Ward-Takahashi identities in the
axial gauge can be maintained to all scales Λ ≠ 0, order by order in perturbation theory,
leading thereby to a mild breaking of the non-abelian gauge symmetry of the RGF. In or-
der to obtain that result, the infrared cut off Λ was suitably identified with a vector boson
mass. Actually, according to universality of the renormalized theory in the limit Λ ↓ 0, any
form of the intermediate infrared regulator must eventually lead, once removed, to the very same renormalized theory in which the gauge symmetry has to be fully restored. Now, it turns out that, at least in the framework of perturbation theory, some gauge fixing Action involving a mass term of the vector boson really represents the simplest tool to investigate the infrared structure of the regulated theory and to check, order by order, the consistency of the RGF procedure, what has been done [5,6] concerning the one loop $\beta$-function.

In spite of the welcome benefit to have simple modified Lee-Ward-Takahashi identities to any order in perturbation theory, the axial gauge RGF approach has been suspected to exhibit singularities [7] in the physical limit $\Lambda \downarrow 0$. In this note we actually prove with an explicit perturbative calculation that the usual expression of the euclidean Wilson loop in the infrared cut off formulation does not flow smoothly to the gauge invariant result, after removal of the infrared cut off itself. More specifically, we shall see that the Wilson loop, up to the fourth order in the non-abelian coupling, does not admit a smooth limit when $\Lambda \downarrow 0$, i.e., infrared singularities do not cancel even for a quantity that is formally gauge invariant at the physical limit.

2. In the spirit of the RGF formulation, the euclidean non-abelian gauge theory can be generally defined in perturbation theory by the usual gauge invariant Action - to our purposes we can neglect matter fields - supplemented by some suitable massive term involving the necessary infrared cut off, together with some gauge fixing Action which allows to study the physical limit of vanishing infrared cut off. In the present investigation we shall consider the axial gauge fixing which is specified, in the euclidean formulation, by a fixed vector $n_\mu$, in such a way that our starting euclidean Action becomes [6]

$$\mathcal{A}[A_\mu] = \int d^4x \, \left\{ \frac{1}{2} \text{tr} [F_{\mu\nu}(x)F_{\mu\nu}(x)] + \Lambda^2 \text{tr} [A_\mu(x)A_\mu(x)] + n_\mu \text{tr} [A_\mu(x)\lambda(x)] \right\}, \quad (1)$$

where trace is over products of $SU(N)$ Lie algebra valued euclidean vector potentials, field strengths and the auxiliary field which enforces the axial gauge choice $n_\mu A_\mu = 0$. Notice that the Faddeev-Popov ghosts can here be disregarded, even in the non-abelian case, thanks to their decoupling within the axial gauge choice. The momentum space euclidean free vector propagator is readily obtained to be $(n^2 \equiv n_\mu n_\mu)$

$$\tilde{D}_{\mu\nu}(k; \Lambda, n) = \frac{1}{k^2 + \Lambda^2} \times$$

$$\left\{ \delta_{\mu\nu} - (n \cdot k) \frac{n_\mu k_\nu + n_\nu k_\mu}{(n \cdot k)^2 + n^2 \Lambda^2} + n^2 \frac{k_\mu k_\nu}{(n \cdot k)^2 + n^2 \Lambda^2} - \Lambda^2 \frac{n_\mu n_\nu}{(n \cdot k)^2 + n^2 \Lambda^2} \right\}, \quad (2)$$

which manifestly enjoys

$$n_\mu \tilde{D}_{\mu\nu}(k; \Lambda, n) = 0. \quad (3)$$

It can be easily checked that, owing to the presence of the infrared cut off $\Lambda$, the above euclidean propagator corresponds to the presence of two independent polarizations and is apparently free from small momenta singularities. Furthermore, since the three and four point elementary non-abelian vertices are the usual ones and it can be shown that the
propagator (2) does fulfill naïve power counting \cite{8}, it turns out that the present model is power counting renormalizable. Of course, this can be obtained at the price of breaking the euclidean $O(4)$ symmetry and, moreover, it can be explicitly verified that the above propagator (2) does not admit a well defined limit when $\Lambda \downarrow 0$, even in the weak topology of the tempered distributions. This means that, on the one hand, the introduction of the Wilson infrared cut off together with the axial gauge subsidiary condition allow for the setting up in perturbation theory of a well defined set of renormalized Schwinger’s function which, however, do break $O(4)$ symmetry and do not admit a smooth limit at the physical gauge invariant point $\Lambda = 0$, even within the weak topology of the tempered distributions. Nonetheless, one can rather reasonably expect that, at least for some suitable renormalized quantities which formally become gauge invariant at the physical point $\Lambda = 0$, the corresponding renormalization group flow is smooth in the limit of vanishing infrared cut off and does reproduce the gauge invariant result, up to renormalization prescriptions. In the non-abelian case such a kind of behaviour could be exhibited by geometrical path-ordered phase factors, because $S$-matrix elements among fundamental fields are affected by severe infrared divergences (generally unmanageable in perturbation theory).

In the present paper we shall analyze the perturbative expansion of the euclidean average path-ordered phase factor

\[ W_\Gamma(\Lambda) = \frac{1}{N} \left\langle \text{tr} \left\{ P \exp \left[ ig \oint_\Gamma dx_\nu A_\nu(x) \right] \right\} \right\rangle \]

\[ \equiv N^{-1} \int [dA_\nu] \exp\{-A[A_\nu]\} \frac{1}{N} \text{tr} \left\{ P \exp \left[ ig \oint_\Gamma dx_\nu A_\nu(x) \right] \right\}, \]

where $\Gamma$ is a closed rectangular path lying on the $Ox_3x_4$ plane, centered at the origin and with sides of lengths $2L$ and $2T$ along the $Ox_3$ and $Ox_4$ axes respectively, whilst we have set $N \equiv \int [dA_\nu] \exp\{-A[A_\nu]\}$. In eq. (4) the vector potential is supposed to belong to a fundamental representation of the $SU(N)$ Lie algebra, i.e. we set $A_\mu(x) = A_\mu^A(x)T_F^A$. The formal quantity in eq. (4) will be referred to as the infrared cut off Wilson loop in the axial gauge. The perturbative expansion of the quantity (4) is well defined in terms of the propagator (2) and of the usual euclidean Feynman’s rules for the three and four point vertices of the pure Yang-Mills theory. This is true provided some regularization is introduced in order to deal with ultraviolet infinities. In this note we shall adopt dimensional regularization and the dimensionally regularized quantity corresponding to eq. (4), order by order in perturbation theory, will be denoted by $\text{reg}W_\Gamma(\Lambda)$. This regularized quantity crucially depends upon the infrared cut off $\Lambda$, at least in perturbation theory. In fact, as already mentioned, the free propagator (2) does not admit a well defined limit $\Lambda \downarrow 0$ in the weak topology of the tempered distributions. This means that in general many individual graphs, which contribute to the quantity $\text{reg}W_\Gamma(\Lambda)$ at a given order, will diverge in the limit $\Lambda \downarrow 0$. Nonetheless, one might conjecture that the full quantity $\text{reg}W_\Gamma(\Lambda)$ keeps a finite value in the physical limit $\Lambda \downarrow 0$, order by order in perturbation theory. Furthermore, if this were true, it might also happen that the possible finite value of the quantity $\text{reg}W_\Gamma(0)$ could also be independent from the specific choice of the axial gauge fixing. In this case, the gauge invariant value of the dimensionally regularized euclidean Wilson loop could eventually be obtained, starting from the perturbative RGF approach in the axial
where \( C \) and \( SU(\alpha) \) are described. Let us first check what happens at the lowest order in perturbation theory. As already mentioned, we shall regulate the ultraviolet divergences by means of dimensional regularization, in such a way that the regularized Green’s functions actually fulfill the modified Lee-Ward-Takahashi identities [7]. To start with, we recall that in the case of euclidean massless \( SU(N) \) gauge theories, the covariant gauge result \( O(g^2) \) in 2\( \omega \)-dimension reads

\[
\text{reg} W_T(\Lambda) = -g^2 C(\omega) \{ I_\omega(L) + I_\omega(T) - J_\omega(L, T) - J_\omega(T, L) \} ,
\]

\[
C(\omega) \equiv 4C_2(F) \mu^{4-2\omega}(2\pi)^{-2\omega},
\]

where \( C_2(F) \) is the quadratic Casimir operator of the fundamental representations of \( SU(N) \) whilst

\[
I_\omega(L) = \int d^{2\omega-1}k \int_0^{+\infty} \frac{\sin^2(k^3 L)}{k^3 L^2} \frac{dk_3}{k^2 + k^2} ,
\]

\[
I_\omega(T) = \int d^{2\omega-1}k \int_0^{+\infty} \frac{\sin^2(k^4 T)}{k^4 T^2} \frac{dk_4}{k^2 + k^2} ,
\]

\[
J_\omega(L, T) = \int_0^{+\infty} dk_3 \int_0^{+\infty} dk_4 \int d^{2\omega-2}k_\perp \frac{\sin^2(k^3 L)}{k_\perp^2} \frac{\cos(2k^4 T)}{k^2 + k^2} .
\]

After setting \( \sigma \equiv 4LT \) - the area of the rectangular contour - and \( \beta \equiv (L/T) \), we find

\[
I_\omega(L) = \pi^{\omega} L^{4-2\omega} \frac{\Gamma(\omega - 2)}{2\omega - 3} ,
\]

\[
J_\omega(L, T) = \pi^{\omega} T^{4-2\omega} \sum_{n=1}^{\infty} (-)^{n+1} \frac{\Gamma(n + \omega - 2)}{(2n - 1)n!} \beta^{2n} , \quad \beta^2 \leq 1 .
\]

Notice that, in the limit \( \omega \uparrow 2 \) and after analytic continuation, one eventually finds for arbitrary \( \beta^2 > 0 \)

\[
\lim_{\omega \uparrow 2} J_\omega(L, T) \equiv J_2(\beta) = 2\pi^2 \left( \beta \arctan \beta - \ln \sqrt{1 + \beta^2} \right) ,
\]

where we understand the inverse trigonometric functions to be their principal branches. It follows that, in the massless case, the \( O(g^2) \) result around \( \omega = 2 \) can be written as

\[
\text{reg} W_2(L, T) \overset{\omega \downarrow 2}{\sim} W_2^{\text{div}} + W_2^{\text{fin}}(L, T) ,
\]

\[
W_2^{\text{div}} = 2g^2 C_2(F) \frac{1}{\epsilon} , \quad \epsilon \equiv 2 - \omega , \quad \hat{g} \equiv \frac{g}{2\pi} ,
\]

\[
W_2^{\text{fin}}(L, T) = 2\hat{g}^2 C_2(F) \times
\]

\[
\left\{ \gamma_E + 1 + \ln \pi + 2\ln \left( \frac{\sigma \mu}{d} \right) + \beta \arctan \beta + \frac{1}{\beta} \arccot \beta \right\} ;
\]

\[
d \equiv 2 \sqrt{L^2 + T^2} .
\]
From the above expressions it follows that, if we add the $O(g^2)$ counterterm
\[ W^{\text{ct.}}_2 = -2g^2C_2(F) \left( \frac{1}{\epsilon} + \gamma_E + 1 + \ln \pi \right) , \tag{10} \]
we eventually obtain the renormalized euclidean Wilson loop in its minimal form, up to
the order $g^2$: namely,
\[ W_{\text{ren}}^2(\beta, \sigma) = 2g^2C_2(F) \left\{ 2 \ln \left( \frac{\sigma \mu}{d} \right) + \beta \arctan \beta + \frac{1}{\beta} \arccot \beta \right\} \]
\[ \beta \ll 1 \sim C_2(F) g^2 T_4 \pi L = -TV_2(L), \quad V_2(L) = -g^2 C_2(F) \frac{4}{4\pi L} . \tag{11} \]
Notice that the quantity $V_2(L)$ just corresponds to the lowest order Coulomb potential of
the non-abelian gauge theory [9].

Let us now turn to the same calculation in the massive case using the euclidean axial
gauge $n_\mu A_\mu = 0$. From the general expression (2) and after choosing, e.g.,
$n_1 = n_2 = n_3 = 0, n_4 = 1$ we find that the only non-vanishing and relevant component of the infrared cut
off vector propagator is
\[ \tilde{D}_{33}(k; \Lambda, n) = \frac{1}{k^2 + \Lambda^2} \left\{ 1 + \frac{k_3^2}{k_1^2 + \Lambda^2} \right\} . \tag{12} \]
As a consequence, we can write the dimensionally regularized and infrared cut off $O(g^2)$
Wilson loop in the axial gauge according to
\[ \text{reg} \tilde{W}_2(\Lambda; L, T) = -g^2C(\omega) \left\{ I_\omega(\Lambda; L) + \hat{I}_\omega(\Lambda, T) - J_\omega(\Lambda; L, T) - \hat{J}_\omega(\Lambda; T, L) \right\} , \tag{13} \]
in which
\[ I_\omega(\Lambda; L) = \int d^{2\omega-1}k \int_{-\infty}^{+\infty} dk_3 \frac{\sin^2(k_3 L)}{k^2 + \Lambda^2} \left( \frac{1}{k_3^2} - \frac{1}{k_3^2 + k^2 + \Lambda^2} \right) , \tag{14a} \]
\[ \hat{I}_\omega(\Lambda; T) = \int d^{2\omega-1}k \int_{-\infty}^{+\infty} dk_4 \frac{\sin^2(k_4 T)}{k^2} \left( \frac{1}{k_4^2 + \Lambda^2} - \frac{1}{k_4^2 + k_4^2 + \Lambda^2} \right) , \tag{14b} \]
\[ J_\omega(\Lambda; L, T) = \int_{-\infty}^{+\infty} dk_3 \int_{-\infty}^{+\infty} dk_4 \int d^{2\omega-2}k_\perp \frac{\sin^2(k_3 L)}{k_3^2} \frac{\cos(2k_4 T)}{k_4^2 + k_4^2 + k_4^2 + \Lambda^2} , \tag{14c} \]
\[ \hat{J}_\omega(\Lambda; T, L) = \int_{-\infty}^{+\infty} dk_3 \int_{-\infty}^{+\infty} dk_4 \int d^{2\omega-2}k_\perp \frac{\sin^2(k_4 T)}{k_4^2 + \Lambda^2} \frac{\cos(2k_3 L)}{k_3^2 + k_3^2 + k_3^2 + \Lambda^2} . \tag{14d} \]
Explicit calculation yields [10]
\[ I_\omega(\Lambda; L) = (L\Lambda) \Lambda^{2\omega-4} \pi^\omega \sqrt{\pi} \Gamma \left( \frac{3}{2} - \omega \right) - \Lambda^{2\omega-4} \pi^\omega \Gamma(2 - \omega) \]
\[ + \Lambda^{2\omega-4} \frac{2\pi^{\omega}}{2\omega-3} (\Lambda L)^{2-\omega} K_{2-\omega}(2\Lambda L) \]
\[- \Lambda^{2\omega-4} \frac{\pi^{\omega} \sqrt{\pi}}{2\Gamma(\omega - \frac{1}{2})} \int_0^\infty \frac{dx}{\sqrt{x}} \frac{x^{\omega-2}}{(1+x)^{3/2}} \exp\{-2\Lambda L\sqrt{1+x}\}, \quad (15a)\]
\[ \hat{I}_\omega(\Lambda; T) = \Lambda^{2\omega-4} \frac{\pi^{\omega}}{2\omega-3} \left\{ 2(\Lambda T)^{2-\omega} K_{2-\omega}(2\Lambda T) - \Gamma(2-\omega) \right\}, \quad (15b)\]
\[ J_\omega(\Lambda; L, T) = \frac{\pi^{\omega}}{\sqrt{\pi TL}} \left( \frac{T}{\Lambda} \right)^{2-\omega} \sum_{n=1}^\infty (-)^{n+1} \frac{(2\Lambda L)^{2n}}{(2n)!} \right. \]
\[ \times \int_0^\infty dx \, x^{n-3/2} (\sqrt{1+x})^{\omega-3/2} K_{\omega-3/2}(2\Lambda T\sqrt{1+x}), \quad (15c)\]
\[ \hat{J}_\omega(\Lambda; L, T) = \frac{\pi^{\omega}}{\sqrt{\pi L\Lambda}} \left( \frac{L}{\Lambda} \right)^{2-\omega} \sum_{n=1}^\infty (-)^{n+1} \frac{(2\Lambda T)^{2n}}{(2n)!} \right. \]
\[ \times \int_0^\infty dx \, x^{n-1/2} (\sqrt{1+x})^{\omega-7/2} K_{\omega-7/2}(2\Lambda L\sqrt{1+x}), \quad (15d)\]

where \( K_\nu(z) \) denotes the Bessel function of imaginary argument [10]. The \( \mathcal{O}(g^2) \) result around \( \omega = 2 \) in this case can be rewritten as

\[ \text{reg} \hat{W}_2(\Lambda; L, T) \sim \hat{W}_2^{\text{div}}(\Lambda; L, T) + \hat{W}_2^{\text{fin}}(\Lambda; L, T), \quad (16)\]
\[ \hat{W}_2^{\text{fin}}(\Lambda; L, T) = 2g^2C_2(F) \left\{ \pi \Lambda L - \gamma_E - \ln \left( \frac{\Lambda^2}{4\pi \mu^2} \right) \right. \]
\[ - K_0(2\Lambda L) - K_0(2\Lambda T) + \frac{1}{2} I_0(2\Lambda L) + \frac{1}{2} \]
\[ + \sum_{n=1}^\infty \frac{(-)^{n+1}}{2(2n)!} \left[ (2\Lambda L)^{2n} I_n(2\Lambda T) + (2\Lambda T)^{2n} \hat{I}_n(2\Lambda L) \right], \quad (17)\]

where we have set

\[ I_0(2\Lambda L) \equiv \int_0^\infty \frac{dx}{\sqrt{x}} (1+x)^{-3/2} \exp\{-2\Lambda L\sqrt{1+x}\}, \quad (18a)\]
\[ I_n(2\Lambda L) \equiv \frac{1}{2\Lambda L} \int_0^\infty dx \, x^{n-3/2} \exp\{-2\Lambda L\sqrt{1+x}\}, \quad n \in \mathbb{N}, \quad (18b)\]
\[ \hat{I}_n(2\Lambda L) \equiv \frac{1}{2\Lambda L} \int_0^\infty dx \, \frac{x^{n-1/2}}{1+x} \exp\{-2\Lambda L\sqrt{1+x}\}, \quad n \in \mathbb{N}. \quad (18c)\]

Again, after summing up the mass independent counterterm

\[ \hat{W}_2^{\text{c.t.}} = -2g^2C_2(F) \left( \frac{1}{\epsilon} + \gamma_E + \frac{3}{2} + \ln \pi \right), \quad (19)\]
we finally get the minimal form of the $O(g^2)$ ultraviolet renormalized and infrared cut off Wilson loop in the axial gauge: namely,

$$\hat{W}_2^{\text{ren}}(\Lambda; L, T) = 2g^2 C_2(F) \left\{ \pi \Lambda L - 1 - \ln \left( \frac{\Lambda^2}{4\mu^2} \right) 
- K_0(2\Lambda L) - K_0(2\Lambda T) + \frac{1}{2} I_0(2\Lambda L) 
+ \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{2(2n)!} \left[ (2\Lambda L)^{2n} I_n(2\Lambda T) + (2\Lambda T)^{2n} \hat{I}_n(2\Lambda L) \right] \right\} ,$$

in such a way that the $O(g^2)$ renormalized quantities in their minimal forms fulfill

$$\lim_{\Lambda \to 0} \hat{W}_2^{\text{ren}}(\Lambda; L, T) = W_2^{\text{ren}}(\beta, \sigma) .$$

(21)

The content of the lowest order result (21) is what could be reasonably expected: the renormalized infrared cut off Wilson loop in the axial gauge flows smoothly to the corresponding gauge invariant value, once the infrared cut off is removed and up to renormalization prescriptions. It is important to stress that the limits $\omega \uparrow 2$ and $\Lambda \downarrow 0$ do not commute and, consequently, in order to obtain the finite and consistent result (21) one has to first subtract the ultraviolet infinities, then take the limit $\omega \uparrow 2$ and finally the limit $\Lambda \downarrow 0$. A further observation is that by performing the very same calculation for the massive case in a general linear covariant gauge, the same result (21) is obtained once again, up to some slightly different renormalization prescription.

4. Let us now turn our attention to the highly non-trivial $O(g^4)$ calculation. It is known [11] that the verification of the large $T$-limit exponential behaviour of the euclidean Wilson loop in perturbation theory does already provide a crucial consistency test of temporal and axial gauges quantization schemes for non-abelian massless gauge theories. In the previous section we have shown that, at the lowest order $O(g^2)$, the renormalization group flow of the infrared cut off Wilson loop in the axial gauge converges smoothly, after subtraction of ultraviolet infinities, to the gauge invariant result, in the limit of the removal of the infrared cut off $\Lambda$ - see eq. (21). In the present section we want to check whether the very same conclusion is true at the next order $O(g^4)$. If this were true, then the RFG approach to the quantization of non-abelian gauge theories, such as proposed in ref.s [5,6], would be extremely appealing. Otherwise, it is ruled out. The gauge invariance test consists of two parts: first, one has to check that all the infrared singularities do cancel and, second, in such a case, one has to verify that the ultraviolet and infrared finite part does coincide, up to renormalization prescriptions, with the corresponding expression as computed, e.g., in the Feynman’s gauge.

\footnote{We stress once again that bad infrared divergences do definitely appear in individual graphs, as already manifest in the $O(g^2)$ calculation, because the free propagator (2) does not admit a limit $\Lambda \downarrow 0$ in the tempered distributions weak topology.}
In order to extract the infrared singular part of any individual graph, the following Lemma is fairly useful. We start from the identity

\[
\frac{1}{x^2 + \Lambda^2} = -\frac{d}{dx} \left( \frac{x}{x^2 + \Lambda^2} \right) + \frac{2\Lambda^2}{(x^2 + \Lambda^2)^2},
\]

whence we obtain that, for any continuous test function \( F(x; \Lambda), \ x \in \mathbb{R}, \ \Lambda \geq 0 \), the following relationship holds true: namely,

\[
\int_{-\infty}^{+\infty} dx \ F(x; \Lambda) \frac{x}{x^2 + \Lambda^2} = \int_{-\infty}^{+\infty} dx \ F(x; \Lambda) \left( -\frac{d}{dx} \frac{x}{x^2 + \Lambda^2} \right)
\]

\[
+ \frac{1}{\Lambda} \int_{-\infty}^{+\infty} dx \ F(x; 0) \frac{2\Lambda^3}{(x^2 + \Lambda^2)^2}
\]

\[
+ \int_{-\infty}^{+\infty} dx \ [F(x; \Lambda) - F(x; 0)] \frac{2\Lambda^2}{(x^2 + \Lambda^2)^2}.
\]

Now, according to the theory of distributions [12] we have

\[
S' \lim_{\Lambda \downarrow 0} \frac{x}{x^2 + \Lambda^2} = \text{CPV} \left( \frac{1}{x} \right) \equiv \frac{1}{[x]},
\]

\[
S' \lim_{\Lambda \downarrow 0} \left( -\frac{d}{dx} \right) \frac{x}{x^2 + \Lambda^2} = \left( -\frac{d}{dx} \right) \text{CPV} \left( \frac{1}{x} \right) \equiv \frac{1}{[x^2]},
\]

\[
S' \lim_{\Lambda \downarrow 0} \frac{2\Lambda^3}{\pi(x^2 + \Lambda^2)^2} = S' \lim_{\Lambda \downarrow 0} \frac{\Lambda}{\pi(x^2 + \Lambda^2)} = \delta(x),
\]

where \( S' \) lim means limit in the tempered distributions topology and CPV indicates the Cauchy Principal Value. Therefrom, under the assumption that

\[
F(x; \Lambda) - F(x; 0) = \Lambda^2 f(x; \Lambda), \quad x \in \mathbb{R},
\]

with \( f(x; \Lambda) \) analytic when \( \Lambda \downarrow 0 \), one finds the following small \( \Lambda \) behaviour

\[
\int_{-\infty}^{+\infty} dx \ F(x; \Lambda) \frac{x}{x^2 + \Lambda^2} \xrightarrow{\Lambda \downarrow 0} \frac{\pi}{\Lambda} F(0; 0) + \int_{-\infty}^{+\infty} dx \ \frac{F(x; 0)}{[x^2]} + \mathcal{O}(\Lambda).
\]

The above formula (26) is the basic tool to single out the infrared divergences of the different graphs contributing to the \( \mathcal{O}(g^4) \) expression of the infrared cut off Wilson loop in the axial gauge. The whole set of diagrams contributing to the genuine non-abelian part of the dimensionally regularized infrared cut off Wilson loop in the axial gauge - which are proportional to the product of the quadratic Casimir operators \( C_2(A)C_2(F) \) of the adjoint and fundamental representations of \( SU(N) \) - can be suitably grouped into three classes we shall call: (a) the self-energy diagrams (fig.1); (b) the crossed propagators diagrams (fig.2); (c) the three point diagrams (fig.3). Let us start by checking whether the strongest\(^2\)

\(^2\) Actually, it is not difficult to verify that all the terms potentially leading to \( \mathcal{O}(\Lambda^{-3}) \) infrared singularities do identically vanish.
infrared singularity $\mathcal{O}(\Lambda^{-2})$ of the non-abelian part of the infrared cut off Wilson loop does cancel or not in the limit $\Lambda \downarrow 0$.

(a) The contribution of the self-energy diagrams reads

\[
\text{reg} \overline{W}_4^{(a)}(\Lambda; T, L) = -\frac{g^4}{8\pi^4}C_2(A)C_2(F) \int_p \int_k \frac{\sin^2(p_3L)}{p_3^2} \frac{1 - \cos(2p_4T)}{(p^2 + \Lambda^2)^2} \times \left( \frac{p_1p_3}{p_4^2 + \Lambda^2} \right) \left( \frac{p_2p_3}{p_4^2 + \Lambda^2} \right) \left( \frac{1}{(k^2 + \Lambda^2)[(p - k)^2 + \Lambda^2]} \right)
\]

\[
\times \left[ \delta_{mn} + \frac{k_mk_n}{k_r^2 + \Lambda^2} \right] \left[ \delta_{ir} + \frac{(p - k)_i(p - k)_r}{(p_4 - k_4)^2 + \Lambda^2} \right] V_{irm} V_{jtn},
\]

where latin indices take values 1,2,3 (remember that the gauge vector is here supposed to have components $n_i = 0, n_4 = 1$) and we have set for later convenience

\[
\int_p \equiv \int \frac{d^2\omega p}{(2\pi\mu)^{2\omega - 4}},
\]

whereas

\[
V_{irm} \equiv (p - 2k)_i \delta_{mr} + (k - 2p)_m \delta_{ir} + (p + k)_r \delta_{im}.
\]

It can be readily checked that

\[
p_{i}(p - k)_r k_{m} V_{irm} = 0, \quad p_{j}(p - k)_i k_{n} V_{jtn} = 0,
\]

whence it follows that the infrared most singular part of the self-energy diagrams turns out to be

\[
\text{reg} \overline{W}_4^{(a)}(\Lambda; T, L) \bigg|_{\text{sing}} = -\frac{g^4}{2\pi^4}C_2(A)C_2(F) \left[ I_1^{(a)}(\Lambda; L, T) + I_2^{(a)}(\Lambda; L, T) \right],
\]

where

\[
I_1^{(a)}(\Lambda; L, T) \equiv \int_p \frac{\sin^2(p_4T)}{(p_4^2 + \Lambda^2)^2} \frac{\sin^2(p_3L)}{(p_3^2 + \Lambda^2)^2} \times \int_k \frac{(p - k)^2[p^2k^2 - (p \cdot k)^2]}{(k_4^2 + \Lambda^2)(k^2 + \Lambda^2)[(p - k)^2 + \Lambda^2]},
\]

\[
I_2^{(a)}(\Lambda; L, T) \equiv \int_p \frac{\sin^2(p_4T)}{(p_4^2 + \Lambda^2)^2} \frac{\sin^2(p_3L)}{2p_3^2} \times \int_k \frac{(p^2)^2k_3^2 + (p \cdot k)^2p_3^2 - 2p_3k_3(p \cdot k)p^2}{(k^2 + \Lambda^2)[(p - k)^2 + \Lambda^2][k_r^2 + \Lambda^2][(p_4 - k_4)^2 + \Lambda^2]},
\]

A direct inspection of the integrals (31) shows, taking the previously quoted Lemma carefully into account, that the self-energy diagrams do not produce any $\mathcal{O}(\Lambda^{-2})$ infrared singularities, or equivalently

\[
\lim_{\Lambda \downarrow 0} (\Lambda^2) \text{reg} \overline{W}_4^{(a)}(\Lambda; T, L) = 0,
\]
as the sum of the self-energy diagrams contains at most $O(\Lambda^{-1})$ infrared singularities.

(b) The contribution of the crossed propagators diagrams reads

\[
\text{reg}\overline{W}^{(b)}_4(\Lambda; T, L) = \frac{\hat{g}^4}{16\pi^4} C_2(A)C_2(F) \int_p \int_q \tilde{D}_{33}(p; \Lambda, n) \tilde{D}_{33}(q; \Lambda, n)
\times \left\{ \begin{array}{l}
2 \exp\{-2iT(p_4 + q_4)\} \left[ \frac{\sin[(p_3 + q_3)L]}{p_3(p_3 + q_3)} - \exp\{-ip_3L\frac{\sin(q_3L)}{p_3q_3}\} \right]^2 \\
-2 \exp\{-2i\omega_4 T\} \frac{\sin(q_3L)}{q_3} \mathcal{F}(p_3, q_3; L)
\end{array} \right. \\
+ 2 \exp\{2i\omega_4 T\} \frac{\sin(q_3L)}{q_3} \mathcal{F}(p_3, q_3; -L)
\]

\[
\mathcal{F}(p_3, q_3; L) \equiv \frac{\sin(q_3L)}{p_3q_3(p_3 + q_3)} - \frac{\sin[(p_3 + q_3)L]}{p_3q_3(p_3 - q_3)} + \exp\{-ip_3 + q_3\}L \frac{\sin(p_3L)}{p_3q_3(p_3 + q_3)} .
\]

A straightforward evaluation, taking again the Lemma into account, actually shows that the expression in eq. (33) does not produce any $O(\Lambda^{-2})$ singularity: namely,

\[
\lim_{\Lambda \to 0}(\Lambda^2)\text{reg}\overline{W}^{(b)}_4(\Lambda; T, L) = 0 .
\]

(c) The contribution of the three point diagrams reads

\[
\text{reg}\overline{W}^{(c)}_4(\Lambda; T, L) = \frac{4i(ig)^3}{N(2\pi)^8} \text{tr}(T_F^AT_F^BT_F^C) \int_p \int_q \int_r \tilde{G}^{ABC}_{333}(p, q, r) \delta(2\omega)(p + q + r)
\times \left\{ (2\pi)^{2\omega - 4} \right. \\
\left. \left\{ 2 \cos(2p_4 T) \frac{\sin^2(p_3L)}{p_3^2q_3} + \frac{\sin^2(p_3L)}{p_3^2q_3} - \frac{\sin^2(r_3L)}{r_3^2q_3} \right\} \right. ,
\]

where $\tilde{G}^{ABC}_{\mu\nu\rho}(p, q, r)$ denotes the three-point Schwinger’s function in momentum space. A straightforward calculation eventually drives to the identification of the infrared most singular part of the expression in eq. (36): namely,

\[
\text{reg}\overline{W}^{(c)}_4(\Lambda; T, L)_{\text{sing}} = \frac{\hat{g}^4}{(4\pi\Lambda)^2} C_2(A)C_2(F) \int_q \int_r (q^2 r^2)^{-1}
\times \left\{ \sin^2[(q_3 + r_3)L] \frac{(q_3 - r_3)^2}{(q_3 + r_3)^2} - 4 \frac{q_3\sin^2(r_3L)}{r_3^2} \right\} ,
\]

with

\[
\int_p \equiv \int d^{2\omega - 1}p (2\pi\mu)^{4 - 2\omega} .
\]
From the values of the basic dimensionally regularized integrals

\[
\lim_{\omega \uparrow 2} \int_p \frac{\sin^2(p_3 L)}{p^2} = -\frac{\pi^2}{2L},
\]
\[
\lim_{\omega \uparrow 2} \int_p \int_q \frac{p_3 q_3 \sin^2[(p_3 + q_3) L]}{p^2 q^2 (p_3 + q_3)^2} = -\frac{\pi^4}{12L^2},
\]
\[
\Phi_\omega(\mu L) \equiv \frac{L^2}{\pi^4} \int_p \int_q \frac{(p_3 + q_3) \sin[(p_3 + q_3) L] \sin[(p_3 - q_3) L]}{p^2 q^2 (p_3 - q_3)}
= \Gamma(2 - \omega) \frac{(\pi/2)^{2\omega-4}(2\pi\mu L)^{8-4\omega}\Gamma\left(\frac{5}{2} - \omega\right)}{(3 - 2\omega)\Gamma(\omega - 1)^2\Gamma\left(\frac{7}{2} - 2\omega\right)},
\]

it follows that

\[
\text{reg} W_4^{(c)}(\Lambda; T, L) \xrightarrow{\Lambda \downarrow 0} \frac{g^4}{(16\pi \Lambda L)^2} C_2(A) C_2(F) \left\{ \Phi_\omega(\mu L) - \frac{1}{6} \right\}.
\]

Summing up the results of eq.s (32),(35),(39) we definitely find the following behaviour of the non-abelian part of the \(\mathcal{O}(g^4)\) dimensionally regularized and infrared cut off Wilson loop in the axial gauge, i.e.,

\[
\text{reg} W_4(\Lambda; T, L) \equiv \text{reg} W_4^{(a)}(\Lambda; T, L) + \text{reg} W_4^{(b)}(\Lambda; T, L) + \text{reg} W_4^{(c)}(\Lambda; T, L)
\]
\[
\xrightarrow{\Lambda \downarrow 0} \frac{g^4}{(16\pi \Lambda L)^2} C_2(A) C_2(F) \left\{ \Phi_\omega(\mu L) - \frac{1}{6} \right\}.
\]

5. The result of eq. (40) shows that the \(\mathcal{O}(g^4)\) dimensionally regularized and infrared cut off Wilson loop in the axial gauge does not converge to some infrared finite result at the physical point \(\Lambda = 0\). This means, in turn, that the Renormalization Group Flow in the axial gauge is discontinuous at the physical point, even for a formally gauge invariant quantity such as the euclidean Wilson loop (this conclusion is \textit{a fortiori} manifestly also true for the Schwinger’s functions). Eq. (40) also shows that the gauge invariance breaking term is extremely bad, since it corresponds to the product of infrared and ultraviolet singularities, the RHS of eq. (40) being \(\mathcal{O}(\Lambda^{-2}(2 - \omega)^{-1})\).

The example we have worked out in this note clearly indicates that the RGF approach to the non-abelian Yang-Mills theories in the axial gauge is absolutely unreliable and, in turn, the proposals suggested in ref.s [5,7] are definitely ruled out at least within perturbation theory. To this concern, it is worthwhile to stress once again that the RGF approach to Yang-Mills theories in Minkowski space-time and in the axial gauge manifestly fulfills, for any \(\Lambda^2 > 0\), power counting renormalizability, unitarity - there are only two field polarizations with positive definite metric - and minimal breaking of gauge invariance - very simple modified Lee-Ward-Takahashi identities hold true to all scales. The whole set of those basic requirements, which make perturbation theory perfectly well defined to all orders, is achieved at the price of breaking simultaneously both gauge invariance - owing
to the presence of the infrared cut off $\Lambda$ - and Lorentz invariance - owing to the presence of the gauge four-vector $n_\mu$. The meaning of the calculation we have developed in the present note is that the RGF approach to non-abelian gauge theories in the axial gauge is physically inconsistent in perturbation theory, as gauge and Lorentz invariances can not be generally recovered in the physical limit, owing to the presence of bad infrared singularities. The wild singularity at the physical point of a formally gauge invariant quantity in the axial gauge also suggests, in our opinion, that the assumption of the regularity at the physical point $\Lambda \downarrow 0$ of the RGF for observable physical quantities in a non-abelian gauge theory is a highly non-trivial requirement, which should be better investigated and understood, in general, also within covariant and light-cone gauge choices [7].

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