Information Assisted Dictionary Learning for fMRI data analysis.

Manuel Morante$^{a,b}$, Yannis Kopsinis$^{c,b}$, Sergios Theodoridis$^{d,a,b}$

$^a$Dept. of Informatics and Telecommunications, University of Athens (Greece)
$^b$Computer Technology Institute & Press “Diophantus” (CTI), Patras (Greece)
$^c$LIBRA MLI Ltd, Edinburgh (UK)
$^d$Chinese University of Hong Kong, Shenzhen (China)

Abstract

In this paper, the task-related fMRI problem is treated in its matrix factorization formulation. The focus of the reported work is on the dictionary learning (DL) matrix factorization approach. A major novelty of the paper lies in the incorporation of well-established assumptions associated with the GLM technique, which is currently in use by the neuroscientists. These assumptions are embedded as constraints in the DL formulation. In this way, our approach provides a framework of combining well-established and understood techniques with a more “modern” and powerful tool. Furthermore, this paper offers a way to relax a major drawback associated with DL techniques; that is, the proper tuning of the DL regularization parameter. This parameter plays a critical role in DL-based fMRI analysis since it essentially determines the shape and structures of the estimated functional brain networks. However, in actual fMRI data analysis, the lack of ground truth renders the a priori choice of the regularization parameter a truly challenging task. Indeed, the values of the DL regularization parameter, associated with the $\ell_1$ sparsity promoting norm, do not convey any tangible physical meaning. So it is practically difficult to guess its proper value. In this paper, the DL problem is reformulated around a sparsity-promoting constraint that can directly be related to the minimum amount of voxels that the spatial maps of the functional brain networks occupy. Such information is documented and it is readily available to neuroscientists and experts in the field.

The proposed method is tested against a number of other popular techniques and the obtained performance gains are reported using a number of synthetic fMRI data. Results with real data have also been obtained in the context of a number of experiments and will be soon reported in a different publication.

Keywords: fMRI, semi-blind, Dictionary Learning, sparsity, weighted norms

1. Introduction

Functional Magnetic Resonance Imaging (fMRI) is a powerful non-invasive technique for localizing and analyzing brain activity [1]. Studying the different
areas within the brain that are associated with important tasks such as vision, perception, etc., constitutes a major area of research that demands robust and high precision techniques for the analysis of fMRI data [2], [3], [4].

Performing various human actions lead to the simultaneous activation of a number of functional brain networks (FBN), which are engaged in proper interactions to effectively execute the task. Such networks are usually related to low-level brain functions and they are defined by the number of segregated specialized small brain regions, which are potentially distributed over the whole brain. These regions collaborate in order to coherently perform a specific brain function. For each FBN, the involved segregated brain regions define a spatial map. Moreover, these brain regions, irrespective of their anatomical proximity or remoteness, exhibit a strong functional connectivity. This can be quantified via measuring the underlying correlation between the corresponding activation/deactivation time-patterns, referred to as time courses. Examples of such brain networks are the visual, sensorimotor, auditory, default-mode, dorsal attention, and executive control networks.

Most commonly, fMRI is based on blood oxygenation level-dependent (BOLD) contrast, which essentially measures the local changes in the level of oxygen caused by the neuronal activity. This is achieved by exploiting the different magnetic properties of oxygen-saturated versus oxygen-desaturated hemoglobin. Accordingly, the time courses, rather than being the actual neuronal activation patterns of the involved FBNs, constitute the evoked hemodynamic response of the brain to the corresponding neuronal activity. This process is modeled as a convolution between the actual neuronal activation and a person-dependent impulse response function called the Hemodynamic Response Function (HRF).

The obtained fMRI measurements associated to each voxel, comprise a mixture of the time courses corresponding to all the FBNs, which are active at the specific voxels. Moreover, beyond the aforementioned brain-induced sources that are contributing to the measured mixture, a number of machine-induced interfering sources are also present. The aim of fMRI data analysis is to unmix the measured mixture in order to reveal the brain-induced time courses of interest and the associated spatial maps.

In any experimental fMRI design, there are two common activation schemes: the block paradigm and the event-related paradigm [5]. In the block paradigm, the subject is presented with a number of preselected stimuli, usually two or three, referred to as conditions. Typical examples comprise a set of different images or an image appearing in different places or different colors, patterns, categories, etc. The conditions appear repetitively in a specific order and each one of them occupies specific time frames of fixed duration. Moreover, the subject might be asked to perform an action, e.g., to press a switch to signal a positive or a negative feedback as his/her response. In the event-related experimental design paradigm, the conditions appear in a random order and for random time durations, which are usually shorter than those in the block design. Brain areas of high activation and correlation to the stimulation/response pattern are considered as highly relevant to the specific functional task (visual/motor centers, pain receptors, etc). A challenging fMRI experimental procedure is the
task-free one, referred to as resting-state fMRI. In this case, the brain activity is not related to activation patterns that are induced by external stimuli to the subject; the functional analysis of the brain activity needs to be realized fully blindly.

One way to proceed the analysis of the fMRI data, in the case of block design or event-related experimental design, is to estimate the shape of the time-courses that correspond to the externally imposed experimental activation set up. Such time-courses are hereafter referred to as task-related time courses. Conventional analysis of fMRI data heavily relies on approaches focused on the General Lineal Model (GLM), which is based on the assumption of the prior availability of the task-related time courses [6]. After fitting the model in each voxel independently, statistical testing is performed in order to detect those voxels that are activated by each task-related time course or linear combinations of them. Moreover, visualisations of the voxels’ activation level is achieved via statistical parameter mapping (SPM), where active voxels are color-coded according to their p-values (or other comparable statistics) [7]. The GLM/SPM framework and its variants are implemented in the SPM software [8], which has arguably become the dominant tool to analyse fMRI data. Nevertheless, the aforementioned approach suffers from several limitations. The most prominent among them are: a) It assumes that the HRF function is known; although the HRF varies significantly across different persons as well as across different areas of the same brain [9]. b) The hypothesis testing on whether a voxel is active due to the experimental task or not, requires the proper tuning of a threshold, which ensures statistical significance. However, the setup of this critical threshold is tricky and it is often the source of systematic errors and misinterpretations [10].

An alternative route to fMRI data analysis is via the Blind Source Separation (BSS) method performed via a matrix factorization scheme. In general, matrix factorization methods are flexible approaches to unravel the involved latent factors via, for example, low-rank or sparse modeling techniques. Moreover, such methods are robust enough to accommodate externally imposed sets of constraints and have gained significant attention in several applications beyond fMRI [2]. BSS methods aim at alleviating some of the limitations that the classical GLM/SPM approach suffers from. For example, they need not adopt any assumption regarding the HRF, they inherently model interfering artifacts, such as machine-induced artifacts or uncorrected head-motion residuals that may obscure the brain activity of interest, and they can discover brain-induced sources beyond the task-related ones. Hence, they can also be applied on resting state fMRI experimental tasks and they can support several strategies for joint analysis of multi-sessions and multi-subjects datasets.

However, a blind matrix factorization task does not lead to a unique solution, and the results depend on the specific imposed constraints. Based on such constraints, two are the major techniques adopted for fMRI data analysis: Independent Component Analysis (ICA) and Dictionary Learning (DL).

ICA can be seen as a matrix factorization approach, which imposes the assumption of independence in various ways. This method has been used in fMRI data for solving the unmixing problem for many years [11] and it is still an
active field of research [12]. Nevertheless, although ICA has been widely applied for fMRI analysis, due to its capability to separate spatially or temporally independent components, it has been observed by several authors that there exist certain situations where the independence hypothesis is seriously violated [13], [14]. Moreover, and more important, there is not any physical evidence to justify that sources are necessarily independent; the independence hypothesis has been strongly challenged, in particular in the context of resting state fMRI [15].

On the other hand, DL approaches are built around the concept of sparsity. Although sparsity is a natural assumption underlaying the fMRI data generation, standard sparsity-aware learning methods still have several drawbacks that should be taken into consideration. The most remarkable is that most of the DL methods require the selection of a regularization parameters that controls the “sparsity level” [2], [16], [17], [18]. Such user-defined parameters lack a direct physical interpretation and are “artificially” introduced in the optimization process; this implies that their selection depends on the specific data set and relies on cross-validation arguments. Nevertheless, cross-validation arguments “looses” its practical meaning when working with fMRI real data.

Another less obvious –but decisive– drawback is that in most of the cases sparsity is imposed by the user in a rather “unnatural” way without any connection to the underlying activity in the brain. In this paper, we present a variant of the $\ell_1$-norm constraint, which promotes sparsity in a way that attempts to exploit information available in brain networks and, hence, it is a step towards alleviating the aforementioned drawbacks.

Despite the numerous interesting properties that the matrix factorization approaches have, they are still not the top in popularity for task-fMRI analysis. The reason for that is a critical weakness that they exhibit against GLM/SPM approach; when two or more task-related sources manifest themselves in highly overlapped brain regions, then all BSS methods, to a larger (ICA) or smaller (DL) degree, can fail to discriminate them. This situation is common in numerous experimental designs of interest in which the different experimental conditions can activate the same major functional brain network, e.g., the auditory, the visual, etc. In contrast, SPM offers the option of studying the contrast among different conditions, aiming at detecting brain areas which are explicitly activated due to a specific task.

The enhanced discriminative power of SPM comes from the fact that the time-course related information, corresponding to each experimental condition, is explicitly incorporated into the GLM modeling via the estimated BOLD sequences. Such information is left unexploited in the BSS framework. Hence, in order to fully exploit the power and merits of matrix factorisation-based methods, it is imminent need to be able to effectively enhance the source unmixing by exploiting the aforementioned time-courses; such a reasoning renders the whole decomposition process to be semi-blind rather than fully blind.

Concerning ICA, semi-blind techniques have been used in an attempt to embed prior knowledge into the associated estimation task [19]. Unfortunately, the independence hypothesis is very restrictive and consequently limits the ben-
efits that one expects to get. Semi-blind ICA methods require a special care when task-related time courses are involved, because they may elicit largely overlapping responses with other sources or even other imposed task-related time courses [10]; thus, this can lead to a strong violation of the independence assumption [20]. For this reason, when external information is imposed, it is required to first analyze how the imposed task-related time courses affect the rest of the sources, which, in practice, is impossible to predict in practice, since the sources are a priori unknown.

On the other hand, DL-based methods can adopt, in principle, any prior information in the form of a constraint, since sparsity is the only underlying assumption. In fact, a method called Supervised Dictionary Learning (SDL) [21] allows the incorporation of external information from the task-related time courses similarly to GLM. Thus, SDL paves the way to new directions of research in fMRI, as well as for other applications, like Hyperespectral imaging, where, also, semi-blind ICA-based methods can not be implemented due to the independence hypothesis [22]. However, the SDL method still shares the same drawbacks as the GLM: the imposed time courses are assumed to be known using a predefined functional form for the HRF, which is considered fixed over the whole brain and from subject to subject.

This is a strong and unrealistic hypothesis, which, somehow, sacrifices a major advantage of BSS methods; that is, they are not restricted by HRF assumptions, that are deemed to be wrong in a larger or smaller degree. In this paper, the DL approach has been appropriately reformulated in order to allow the incorporation of weak constraints associated with each one of the time-course; thus, the unmixed sources are not restricted to fully comply with the imposed time-courses but they are only encouraged to be similar.

**Major Contributions**

The major contributions of this paper are summarized as follows:

- We propose a new semi-blind DL approach that incorporates a priori knowledge related to the approximate shape of the task-related time courses. This approach enhances the discrimination power of the DL method, while at the same time accounts for the fact that our knowledge concerning the HRF is deemed to be relatively inaccurate or even dynamically changing.

- We introduce a new sparsity constraint based on the weighted \( \ell_1 \)-norm, which explicitly incorporates sparsity in an interpretable way. Instead, the new constraints are tied with a physical interpretation, which is related to the area occupancy of various functional brain networks. The latter is information, which is documented in available brain atlases [23].

- The new proposed sparsity constraint, also, offers the flexibility of treating sources which are dense. In real fMRI data, some of the sources can be dense, and this is related to certain machine artifacts.
- The new proposed method lends itself as an alternative to other standard DL methods that impose sparsity on a voxel by voxel basis [16]. Instead, the proposed DL method incorporates sparsity on a source by source basis; this is possible by handling the sparsity of each spatial map and it is naturally complying with the physical interpretation of the segregated functional brain networks.

- We provide a realistic synthetic data set, which is used to study the efficiency of the proposed method and to compare its performance with other standard DL methods. Although results with real data have been obtained, in this paper, we will only report simulated experiments. Real data are to be reported in a separate paper.

**Notation**

Scalars are denoted by lower case letters, $x$, a bold capital letter, $X$, denotes a matrix and a bold lower case letter, $x$, denotes a vector with its $i^{th}$ component denoted as $x_i$. The $i^{th}$ row and the $i^{th}$ column of a matrix $X \in \mathbb{R}^{M \times N}$, are denoted as $x^i \in \mathbb{R}^{1 \times N}$ and $x_i \in \mathbb{R}^{M \times 1}$, respectively. Moreover, $x_{ij}$ denotes the element “located” at row $i$ and column $j$ of matrix $X$.

Given an arbitrary vector, $w \in \mathbb{R}^N$, the notation $w \succ a$ stands for $w_i > a \ \forall \ i = 1, 2, \ldots, N$.

The vector $\mathbf{1}_N$ denotes a column vector of size $N$ having all components equal to one and $\mathbf{I}_N$ is the identity matrix of size $N \times N$. Furthermore, $A = \text{diag}(x)$ is a diagonal matrix having $a_{ii} = x_i$.

### 2. Problem Statement

During the course of a task-related fMRI experiment, a sequence of 3D images of the brain are successively acquired along time, while the subject performs a set of predetermined tasks according to the chosen experimental design. Each one of these 3D images is formed by a grid of elementary volume units called voxels. These voxels can be thought of as cubes, which define the smallest measurable volume of a brain tissue. Accordingly, the signal detected at each voxel reflects the degree of activity of the set of neurons contained in the corresponding small volume [1].

Each 3D image is unfolded and stored in a vector, $x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{R}^{1 \times N}$, where $N$ is the total number of voxels per image. The vectors corresponding to the sequence of, say $T$, acquired images are concatenated as rows in the data matrix $X \in \mathbb{R}^{T \times N}$.

Note that, in practice, prior to the formation of the data matrix, $X$, several standardized preprocessing steps are conducted in order to account for a number of detrimental effects related to the fMRI image acquisition process, such as slice timing correction, head motion, realignment normalization, etc.

From a mathematical point of view, the source separation problem can be described as a matrix factorization task of the data matrix, i.e.,

$$ X \approx DS, \quad (1) $$
where, following the dictionary learning jargon, $D \in \mathbb{R}^{T \times K}$ is the dictionary matrix, whose columns represent the time courses, $S \in \mathbb{R}^{K \times N}$, is the coefficient matrix, whose rows represent the spacial maps associated with the corresponding time courses and $K$ is the number of sources.

In general, this matrix factorization can be expressed as a constrained optimization task given by:

$$\arg\min_{D,S} \|X - DS\|_F^2 \quad \text{s.t.} \quad D \in \mathcal{D}, \quad S \in \mathcal{L},$$

where $X$ is the data matrix and $\mathcal{D}$, $\mathcal{L}$ are two sets of admissible matrices, defined via an adopted set of appropriately imposed constraints.

### 2.1. Sparse Approximation

The concept of sparsity refers to the number of zero-values that a discrete signal has. In numerous applications, sparsity is the cornerstone of several systems whose response can be modeled using a small number of non-zero coefficients. In the fMRI framework, sparsity is a natural way to model the segregated nature of the spatial maps. In particular, each row of the coefficient matrix, $S$, should have non-zero values at the entries that correspond to voxels activated by the corresponding time course, only.

No doubt, the best way to measure sparsity is using the $\ell_0$-norm. Formally, given an arbitrary vector $x \in \mathbb{R}^N$, the $\ell_0$-norm (which strictly speaking is not a true norm [2]) can be defined as the cardinality of the support of the vector $x$:

$$\|x\|_0 = \#\text{supp}(x),$$

where supp$(x) = \{i = 1, N \mid x_i \neq 0\}$ denotes the support of the vector $x$. In other words, the $\ell_0$-norm is equal to the number of non-zero components of a vector.

Observe that the $\ell_0$-norm depends explicitly on the number of non-zero entries. However, sometimes, it is more convenient to use the proportion of zero values of a vector, often called level of sparsity [24]. For example, given an arbitrary vector $x \in \mathbb{R}^N$, then its level of sparsity is given by:

$$l = 1 - \frac{\|x\|_0}{N},$$

where $N$ is the total number of elements of the vector $x$, and $l \in [0, 1]$ is the proportion of non-zero values. For this reason, the level of sparsity is more informative to use, since it simultaneously considers the number of non-zero values as well as the size of the vector.

The matrix factorization framework that explicitly exploits sparsity is Dictionary Learning (DL), in which the coefficient matrix is constrained to be sparse. In DL, sparsity can by imposed in two ways: Either each column of $S$ is separately constrained to be sparse, e.g. $\|s_i\|_0 \leq \gamma_i$, where $\gamma_i$ is the maximum number of the non-zeros values of the $i$th column of $S$, or the full coefficient
matrix is constrained to have $\gamma$ non-zeros \[16\], \[21\], \[25\]. In the fMRI context, the column-wise sparsity constraint corresponds to imposing sparsity per voxel; that is, the number of active sources that each source can have is constrained to be sparse. However this piece of information is voxel-dependent and is hard to accommodate in general. On the other hand, enforcing sparsity in the full coefficient matrix is easier to handle, yet it is still restrictive; this is because the available spatial map-level information cannot be exploited on a source by source basis.

In this paper, to the best of our knowledge, it is the first time that the DL framework is extended in order to allow sparsity promotion among rows (or sets of rows) of the coefficient matrix. Indeed, such a modeling perfectly fits in the fMRI framework, but it also can be extended to other fields where DL is adopted as Blind Source Separation (BSS), e.g. in the hyperspectral imaging context.

Going back to the problem in Eq. (2), the $\ell_0$-norm constitutes the best candidate to impose sparsity constraints. Imposing sparsity per rows, one admissible set of coefficient matrices can be defined as:

$$
\mathcal{L}_0 = \{ S \in \mathbb{R}^{K \times N} \mid \| s^i \|_0 \leq \phi_i \ \forall \ i = 1, 2, \ldots, K \}, \quad (5)
$$

where $\phi_i$ is a user-defined parameter, which denotes the maximum number of non-zeros values of the $i$th row of the matrix $S$. As it will be later discussed, such constraints bear a physical meaning that renders $\phi_i$ values easily tunable in practice.

The implementation of the $\ell_0$-norm often results in an NP-hard task. For this reason, the $\ell_1$-norm has been widely adopted as a convex relaxation, and it has been established that under certain conditions both minimization problems are equivalent \[26\].

In this way, after replacing the $\ell_0$-norm with the $\ell_1$-norm the previous constrained set becomes:

$$
\mathcal{L}_1 = \{ S \in \mathbb{R}^{K \times N} \mid \| s^i \|_1 \leq \lambda_i \ \forall \ i = 1, 2, \ldots, K \}, \quad (6)
$$

where $\lambda_i$ are new extra user-defined parameters to implicitly control the sparsity of the rows of the coefficient matrix. In contrast to the parameters $\phi_i$, the new parameters, $\lambda_i$, are not directly related to the level of sparsity, rendering them hard to tune in practice, unless cross validation is a possibility.

In order to face the critical issues introduced by this convex relaxation, the weighted $\ell_1$-norm is proposed as an alternative sparsifying constraint.

### 2.2. The weighted $\ell_1$-norm

Given an arbitrary vector $x \in \mathbb{R}^N$, the weighted $\ell_1$-norm is defined as:

$$
\| x \|_{1,w} = \sum_{i=1}^{N} w_i |x_i|, \quad (7)
$$
where $\mathbf{w} > 0$ is a corresponding vector of weights, associated with. According to [27], [28], the values of the weights are set equal to:

$$
 w_i = \frac{1}{|x_i| + \varepsilon} \quad \forall i = 1, 2, \ldots, N,
$$

where $\varepsilon \in \mathbb{R}^+$ is a real positive number, which is introduced in order to avoid division by zero, providing enhanced numerical stability [2]. Under the above definition of weights, the weighted $\ell_1$-norm constitutes a good approximation to the $\ell_0$-norm [28]. Accordingly, a newly defined constrained set for the coefficient matrix is the following:

$$
 \mathcal{L}_W = \left\{ \mathbf{S} \in \mathbb{R}^{K \times N} \mid \|\mathbf{s}\|_{1,\mathbf{W}} \leq \phi_i \quad \forall i = 1, 2, \ldots, K \right\},
$$

where $\phi_i$ is exactly the same user-defined parameter as it was introduced in Eq. (5) and $\mathbf{w}$ is the vector of weights defined as in Eq. (8). Here it is the key point: $\phi_i$ stands for the maximum number of non-zero values, exactly in the same way as for the $\ell_0$-norm.

Furthermore, the weighted $\ell_1$-norm constrain can be alternatively imposed over the full coefficient matrix as well:

$$
 \mathcal{L}_w = \left\{ \mathbf{S} \in \mathbb{R}^{K \times N} \mid \|\mathbf{s}\|_{1,\mathbf{W}} \leq \phi_i \quad \forall i = 1, 2, \ldots, K \right\},
$$

where $\phi_i$ is exactly the same user-defined parameter as it was introduced in Eq. (5) and $\mathbf{w}$ is the vector of weights defined as in Eq. (8). Here it is the key point: $\phi_i$ stands for the maximum number of non-zero values, exactly in the same way as for the $\ell_0$-norm.

At this point, observe that knowing the values $\phi_i$, it is possible to set the sparsity level of $\mathcal{L}_W$ as $\Phi = \sum_{i=1}^{K} \phi_i$ forcing both constrained sets to lead to the same overall sparsity of $\mathbf{S}$. However, in the case of $\mathcal{L}_w$, the parameters $\phi_i$ impose a specific structure, which restricts the set of admissible coefficient matrices. In fact, $\mathcal{L}_w$ is a subset of $\mathcal{L}_W$. The consequences of this will be analyzed in detail in Section 4.

**General Case**

From a more general perspective, we can look at the sets $\mathcal{L}_W$ and $\mathcal{L}_w$ as two special cases of a more general constrained set. In order to exploit this, let us first define as $\mathcal{B}$ a set containing a subset of $M$ elements of the set $\{1, 2, \ldots, K\}$ and as $\mathbf{S}_B$ a row submatrix of $\mathbf{S}$ comprising the rows of $\mathbf{S}$ listed in the set $\mathcal{B}$. Then, a generalized constraint set is defined as:

$$
 \mathcal{L}_B = \left\{ \mathbf{S} \in \mathbb{R}^{K \times N} \mid \|\mathbf{s}\|_{1,\mathbf{W}_B} \leq \phi_i, \forall i = 1, 2, \ldots, B \right\},
$$
where \( \phi_i \) is the number of non-zero elements of the \( \text{i}^{\text{th}} \) submatrix \( S_{B_i} \), \( W_{B_i} \) is the associated submatrix of weights and \( B \) is the total number of submatrices considered.

Then, the set \( L_w \) corresponds to the case of \( B_i = i \), where, \( i = 1, 2, \ldots, K \), whereas \( L_W \) corresponds to the case where all the row indexes of the matrix are considered within the same set, i.e., \( B_i = B = 1, 2, \ldots, K \). Besides these two cases, the above set covers a plethora of constrained sets that correspond to associated with matrix \( S \). Thus, we can study all these different cases under a single framework, using the general constrained set.

2.3. Supervised Dictionary Learning

As already discussed, a method called Supervised Dictionary Learning (SDL) [21] allows the incorporation of external information from the task-related time courses in the DL framework.

The starting point in the formulation of the SDL lies on splitting the dictionary in two parts:

\[
D = [\Delta, D_F] \in \mathbb{R}^{T \times K}, \tag{13}
\]

where the first part, \( \Delta \in \mathbb{R}^{T \times M} \), is fixed and equal to the imposed task-related time courses. On the contrary, the second part, \( D_F \in \mathbb{R}^{T \times (K-M)} \), is left to vary and it is estimated via DL optimization arguments. In [21], the authors propose to implement SPAMS [29], which is a well-known DL algorithm based on the conventional \( \ell_1 \)-norm regularization.

However, this approach still inherits the major drawbacks associated with GLM; the constrained dictionary atoms (columns of the matrix \( \Delta \)) lead to improvement only if the imposed task-related time courses are sufficiently accurate. If the imposed task-related time courses are miss-modeled, their contributions can introduce detrimental effects, leading to wrong results.

2.4. Assisted Dictionary Learning

In this paper, an alternative approach is presented, which incorporates a modified set of constraints that are associated with the task-related time courses. Specifically, the strong \textit{equality} requirement of the previous approach is relaxed by a looser \textit{similarity} distance-measuring norm constraint. Then, if part of the a priori information is inaccurate, the method is able to adjust the constrained atoms in an optimal way, since they are not enforced to be equal to a preselected shape.

The kick off point lies on splitting the dictionary in two parts:

\[
D = [D_C, D_F] \in \mathbb{R}^{T \times K}, \tag{14}
\]

where, in contrast to the SDL approach, the part, \( D_C \in \mathbb{R}^{T \times M} \) is constrained to be \textit{similar} to the imposed task-related time courses.

Then, the new \textit{convex} set of dictionaries is defined as:

\[
\mathcal{D}_\delta = \left\{ D \in \mathbb{R}^{T \times K} \mid \|d_i - \delta_i\|^2_2 \leq c_\delta \quad i = 1, \ldots, M \right\}, \quad \left\| d_{i} \right\|_2^2 \leq c_d \quad i = M + 1, \ldots, K \right\}, \tag{15}
\]
where ∥·∥_2 denotes the Euclidean norm, d_i is the i-th column of the dictionary D and δ_i is the i-th a priori selected task-related time course. The constant c_δ is a user-defined parameter, which controls the degree of similarity between the constrained atoms and the imposed time courses.

In practice, ill conditioned phenomena may occur associated with this kind of matrix factorization approaches [30]. Hence, in order to prevent degenerated solutions, the free atoms of the dictionary are constrained to be of a bounded norm controlled by c_d, a user-defined parameter, which is usually set equal to 1 for simplicity.

3. Proposed Algorithm

In this section, we present a new algorithm, called Assisted weighted Dictionary Learning (AwDL), to solve Eq. (2), incorporating the new proposed sets of constraints. Put succinctly, the new optimization task is cast as:

\[
(D, S) = \arg\min_{D,S} \|X - DS\|_F^2 \quad \text{s.t.} \quad D \in \mathcal{D}_\delta, \ S \in \mathcal{L}_S,
\]

where \( \mathcal{D}_\delta \) and \( \mathcal{L}_S \) are the new constrained sets proposed before.

However, the simultaneous minimization with respect to D and S is a challenging one. In order to solve it, the Block Majorized Minimization (BMM) algorithm is adopted. This procedure is not new and it provides a powerful framework for such type of optimization tasks, e.g. [31]. Essentially, the BMM algorithm simplifies the optimization task by adopting a two-step alternating minimization, under certain assumptions.

In particular, starting from an arbitrary set of estimates, \( D^{[0]} \) and \( S^{[0]} \), at the k-th iteration, the algorithm comprises the following steps:

I. \( S^{[k+1]} = \min_S \psi_S(S, S^{[k]}) \) s.t. \( S \in \mathcal{L}_S \),

II. \( D^{[k+1]} = \min_D \psi_D(D, D^{[k]}) \) s.t. \( D \in \mathcal{D}_\delta \),

where, according to the assumptions underlying any BMM algorithm, \( \psi_S \) is the so called surrogate function of the first step, given D is fixed to its current estimate \( D^{[k]} \) and \( \psi_D \) is the corresponding surrogate function of the second step, given S to be fixed to its current update \( S^{[k+1]} \).

Hence, for each step, the main loss function is replaced by its surrogate counterpart; the latter is chosen so that to majorize the former and at the same time to be easier to minimize compared to the original. In general, the surrogate function is not unique but it has to satisfy certain specific conditions [31].

3.1. Step I: Coefficient Update

According to Eq. (17), the update step at the k-th iteration is formulated as:

\[
S^{[k+1]} = \min_S \psi_S(S, S^{[k]}) \quad \text{s.t.} \quad S \in \mathcal{L}_S.
\]
Regarding the surrogate function, \( \psi_S \), an admissible candidate, which facilitates the optimization task in this case, can be obtained using a second order Taylor’s expansion, e.g. [31], i.e.:

\[
\psi_S(S, S^{[k]}) = \| X - DS \|_F^2 - \| DS - DS^{[k]} \|_F^2 + c_S \| S - S^{[k]} \|_F^2,
\]

where \( c_S \geq \| D^T D \| \) is a constant and \( \| \cdot \| \) stands for the spectral norm.

Then, in order to solve the current task, the Lagrangian relaxation method is implemented. Therefore, introducing the Lagrange multipliers, the task becomes:

\[
\min_S \psi_S(S, S^{[k]}) + \mathcal{P}_\gamma(S),
\]

where \( \mathcal{P}_\gamma \) is a penalty term that depends on the weighted \( \ell_1 \)-norm constraints and it is defined as:

\[
\mathcal{P}_\gamma(S) = \sum_{i=1}^B \gamma_i \left( \| S_{B_i} \|_1, w_{B_i} - \phi_i \right),
\]

where the introduced parameters \( \gamma_i \geq 0 \), for \( i = 1, 2, \ldots, B \), are the Lagrange multipliers and \( B \) is the number of different submatrices considered.

From the theory of optimization, at the optimal point, the Lagrange multipliers have to satisfy the corresponding KKT conditions, i.e.,

\[
\gamma_i \left( \| S_{B_i} \|_1, w_{B_i} - \phi_i \right) = 0,
\]

with \( i = 1, 2, \ldots, B \).

Observe that the optimization task in Eq. (20) involves a convex loss function. Hence, a global minimum exists at the \( k^{th} \) iteration. This is a nice feature and at the same time is a requirement as part of the sufficient conditions that guarantee the convergence of the BMM algorithm [32].

Although the loss function is convex, it is not differentiable. Nevertheless, the minimum can be obtained at a point with zero subgradient. That is, a minimizer of the function exists if and only if the zero (matrix) belongs to the subdifferential set given by:

\[
\mathbf{0} \in \partial \left( \psi_S(S, S^{[k]}) + \mathcal{P}_\gamma(S) \right).
\]

In this case, the first terms is fully differentiable, and the subdifferential set becomes a singleton given by the standard derivative:

\[
\partial \left( \psi_S(S, S^{[k]}) \right) = \left\{ \nabla \psi_S(S, S^{[k]}) \right\} = \left\{ -2D^T X + 2c_S S - 2c_S S^{[k]} - 2DD^T S^{[k]} \right\},
\]
that is,

\[ \partial \left( \psi_S(S^{[k]}) \right) = \{ 2c_S(S - A) \}, \]

where \( A \) is matrix defined as:

\[ A \triangleq \frac{1}{c_S} \left[ D^T X + (c_S I_K + D^T D) S^{[k]} \right]. \]

In contrast, the second term is not differentiable and its corresponding subdifferential set is given by:

\[ \partial \mathcal{P}_\gamma(s_{ij}) = \begin{cases} \gamma_i w_{ij} \text{sign}(s_{ij}) & \text{if } s_{ij} \neq 0 \\ \gamma_i w_{ij} g_{ij} & \text{if } s_{ij} = 0 \end{cases}. \]

(26)

However, this set depends on the specific selection of the imposed Lagrange multipliers \( \gamma_i \), which are determined via the associated KKT conditions, i.e.,

\[ \gamma_i(\|S_{B_i}\|_1, W_{B_i}, \phi_i) = 0, \quad \forall i = 1, 2, \ldots, B. \]

Therefore, the task now becomes to find the matrix \( S \) in the subdifferential set which simultaneously satisfies

\[ 2c_S(S - A) + \partial \mathcal{P}_\gamma(S) = 0, \]

and its corresponding KKT conditions, where \( \partial \mathcal{P}_\gamma \) is given by Eq. (26).

Thus, following standard arguments and after some algebraic manipulations, it can be shown that the solution is given by:

\[ S_{B_i} = \begin{cases} A_{B_i} & \text{if } \|A_{B_i}\|_1, W_{B_i} \leq \phi_i \\ \mathcal{P}_{B_{\phi_i}}[W_{B_i}, \phi_i](A_{B_i}) & \text{if } \|A_{B_i}\|_1, W_{B_i} > \phi_i \end{cases}, \]

(27)

with \( i = 1, 2, \ldots, B \), where \( S_{B_i} \) and \( A_{B_i} \) are the submatrix of the coefficient matrix and the matrix \( A \) indexed by the \( i \)th tuple, respectively. Matrix \( W_{B_i} \) is the corresponding sub-matrix of weights, and \( \mathcal{P}_{B_{\phi_i}}[W_{B_i}, \phi_i] \) is the projection operator over the weighted \( \ell_1 \)-norm ball, \( B_{\phi_i}[W_{B_i}, \phi_i] \) of weights \( W_{B_i} \) of radius \( \phi_i \).

For the projection over the weighted \( \ell_1 \)-norm ball, many available algorithms can be implemented, e.g., [28]. Nevertheless, the majority of those algorithms are designed for vectors. Then, in order to apply the solution in Eq. (27), each submatrix must be properly vectorized prior to their application. This operation is an isomorphism, which does not disrupt the projection.

With respect to the associated Lagrange multipliers, \( \gamma_i \), the values are given by:

\[ \gamma_i = \max \left\{ 0, \frac{2c_S(\|A_{B_i}\|_1, W_{B_i} - \phi_i)}{\|W_{B_i}\|_F} \right\}. \]

(28)

Finally, the coefficient update can be compactly written by introducing a new operator:

\[ S^{[k+1]} = Q_B(A), \]
where $Q_B(A) : A_{B_i} \rightarrow S_{B_i}^{[k+1]}$, $i = 1, 2, \ldots, B$, controls the update for each sub-matrix and it is defined via the solution provided in Eq. (27); that is,

$$S_{B_i}^{[k+1]} = \begin{cases} A_{B_i} & \text{if } \|A_{B_i}\|_{1,w_{B_i}} \leq \phi_i \\ \mathcal{P}_{B_i}[w_{B_i}, \phi_i](A_{B_i}) & \text{if } \|A_{B_i}\|_{1,w_{B_i}} > \phi_i \end{cases},$$

where the parameters are the same as they were described before.

**Iterative Weight Estimates**

Theoretically, the best possible value for the weights corresponds to the optimal solution of the main optimization task. Of course, this is impossible in practice, since the solution is unknown. However, one can use the currently available estimates according to Eq. (8). Indeed, this iterative update tends to obtain successively better estimates of the weights, and it is smoothly integrated within the optimization process [27], [28].

### 3.2. Step II: Dictionary Update

In the second step of the alternating minimization, i.e. Eq. (18), the optimization task at the $k$th step is:

$$D^{[k+1]} = \min_D \psi_D(D, D^{[k]}) \quad \text{s.t. } D \in \mathcal{D}_\delta.$$

For the surrogate function, $\psi_D$, an admissible choice can be obtained from the second order Taylor’s expansion [31] around $D^k$, that is,

$$\psi_D(D, D^{[k]}) = \|X - DS\|_F^2 - \left\|DS - D^{[k]}S\right\|_F^2 + c_D \left\|D - D^{[k]}\right\|_F^2,$$

where $c_D \geq \|SS^T\|$ and $\|\cdot\|$ stands for the spectral norm.

In order to solve the current task, the BMM algorithm does not impose any restrictions. For simplicity, a Lagrangian relaxation is again implemented. Thus, introducing a new penalty term, the original task can be rewritten as:

$$\min_D \psi_D(D, D^{[k]}) + \mathcal{P}_\gamma(D),$$

where $\mathcal{P}_\gamma$ is defined according to the constraints over the dictionary:

$$\mathcal{P}_\gamma(D) = \sum_{i=1}^{M} \gamma_i \left[ (d_i - \delta_i)^T (d_i - \delta_i) - c_\delta \right] + \sum_{i=M+1}^{K} \gamma_i (d_i^T d_i - c_d),$$

where $\gamma_i$, $i = 1, 2, \ldots, K$, are the respective $K$ Lagrange multipliers.
Note that $P_\gamma(D)$ is formed by two different terms; the first one refers to the first $M$ columns of the dictionary and it “deals” with the similarity constraint. The second relates to the remaining $(K-M)$ columns that impose a penalization over the respective norms.

The associated KKT conditions are given by:

\begin{align}
\gamma_i \left( \|d_i - \delta_i\|^2 - c_d \right) &= 0 \quad i = 1, 2, \ldots, M, \\
\gamma_i \left( \|d_i\|^2 - c_d \right) &= 0 \quad i = M + 1, \ldots, K. 
\end{align}

For simplicity, the term in Eq. (32) can be rewritten in a more compact form by introducing the following matrix notation: Define the mask matrix $M \in \mathbb{R}^{M \times K}$, which is a rectangular matrix ($M < K$) with zero elements everywhere except the elements along its principal diagonal, which are equal to one, i.e.,

\begin{equation}
M \triangleq (m_{ij})_{ij} \rightarrow \begin{cases} 
  m_{ij} = 0 & \text{if } i \neq j \\
  m_{ij} = 1 & \text{if } i = j. 
\end{cases}
\end{equation}

Using this mask matrix and the properties of the trace, we can compactly write:

$$P_T(D) = \text{tr} \left[ \Gamma \left( (D - \Delta M)^T (D - \Delta M) - C \right) \right],$$

where $\Gamma$ is a diagonal matrix $\Gamma = \text{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_K\}$ and $C$ is another diagonal matrix given by $C = \text{diag}\{c_d I_M, c_d I_{K-M}\}$.

Eventually, the minimization task for this step is compactly written as:

$$D^{[k+1]} = \min_D \psi(D, D^{[k]}) + P_T(D).$$

The current loss function is convex, after fixing $S$ to its current estimate $S = S^{[k+1]}$, and a global minimum exists at the $k^{th}$ iteration. Besides, as it was described before, the uniqueness is also a sufficient condition to guarantee the convergence of the BMM algorithm [32].

In this case, the loss function is fully differentiable and the minimum is found as the point with the zero gradient, i.e.:

$$\nabla_D \left( \psi(D, D^{[k]}) + P_T(D) \right) = 0. \quad (36)$$

The derivative of the first term is given by:

$$\nabla_D \psi_D = -2XS^T + 2D^{[k]} SS^T + 2c_D D - 2c_D D^{[k]},$$

and the derivative of the second term is given by:

$$\nabla_D P_T(D) = 2D \Gamma - 2 \Delta M \Gamma,$$

Now, solving with respect to $D$ results to:

$$D = \left[ B + \frac{1}{c_D} \Delta M \Gamma \right] \left( \frac{1}{c_D} \Gamma + I_K \right)^{-1}, \quad (37)$$
where \( B \) is a matrix defined as:

\[
B \triangleq \frac{1}{cD} \left[ XS^T + D [k] \left( cD I_K - SS^T \right) \right],
\]

and the Lagrange multipliers in \( \Gamma \) are obtained so that to satisfy the corresponding KKT conditions.

A careful observation of the previous set of equations reveals that the solution is separable; thus, one can equivalently write \( K \) equations, one for each specific column of \( D \), that is:

\[
d_i = \left[ b_i + \frac{1}{cD} m_i \delta_i \gamma_i \right] \left( \frac{\gamma_i}{cD} + 1 \right)^{-1},
\]

with \( i = 1, 2, \ldots, K \) and \( m_i \) is defined so that \( m_i = 1 \) if \( i \leq M \) and \( m_i = 0 \) if \( i > M \). Defining \( \mu_i = \frac{cD}{cD + \gamma_i} \), the previous expression can be easily written as:

\[
d_i = \mu_i b_i + m_i (1 - \mu_i) \delta_i.
\] (38)

Note that according to the values of \( m_i \), these equations can be partitioned in two different groups:

\[
d_i = \mu_i b_i + (1 - \mu_i) \delta_i, \quad i = 1, 2, \ldots, M, \quad (39)
\]

\[
d_i = \mu_i b_i, \quad i = M + 1, \ldots, K, \quad (40)
\]

that is, one for each constrained part of the dictionary.

**Similarity Constraint**

Eq. (39) stands for the similarity constraints, whose associated KTT conditions are given by \( \gamma_i \left( \| d_i - \delta_i \|^2 - c_\delta \right) = 0 \). Hence, only two solutions are admissible: \( \gamma_i = 0 \) or \( \| d_i - \delta_i \|^2 = c_\delta \).

Then, according to the dictionary constraint, it can be shown, that

\[
\text{if } \| b_i - \delta_i \|^2 \leq c_\delta \iff \gamma_i = 0,
\]

leads to a correct solution, and the update is easily written as \( d_i^{[k+1]} = d_i = b_i \).

On the other hand, if \( \| b_i - \delta_i \|^2 > c_\delta \), the selection of \( \gamma_i = 0 \) is not admissible. However, if \( \gamma_i \neq 0 \), according to the KTT conditions, it necessarily leads to:

\[
\| d_i - \delta_i \|^2 = c_\delta,
\]

where by definition,

\[
\| d_i - \delta_i \|^2 = \mu_i^2 \| d_i - \delta_i \|^2 = c_\delta.
\]

Thus,

\[
\mu_i = \frac{c_\delta^{1/2}}{\| b_i - \delta_i \|} \Rightarrow \gamma_i = cD \left( \frac{1}{\mu_i} - 1 \right).
\]
Finally, the dictionary part for the similarity constraint can be written as:

\[
\mathbf{d}^{[k+1]}_i = \begin{cases} 
\mathbf{b}_i & \text{if } \|\mathbf{b}_i - \delta_i\|^2 \leq c_d \\
\frac{c_d^{1/2}}{\|\mathbf{b}_i - \delta_i\|}(\mathbf{b}_i - \delta_i) + \delta_i & \text{if } \|\mathbf{b}_i - \delta_i\|^2 > c_d
\end{cases}
\]  

(41)

with \(i = 1, 2, \ldots, M\).

**Normalization Constraint**

Concerning the normalization constrain in Eq. (40), the KKT conditions are given by \(\gamma_i \left(\|\mathbf{d}_i\|^2 - c_d\right) = 0\), which has two admissible solutions: \(\gamma_i = 0\) or \(\|\mathbf{d}_i\|^2 = c_d\), for \(i = M + 1, M + 2, \ldots, K\).

Once more, it can be shown that, if \(\|\mathbf{b}_i\|^2 \leq c_d \Leftrightarrow \gamma_i = 0\) is a solution and the update step can be easily written as \(\mathbf{d}^{[k+1]}_i = \mathbf{d}_i = \mathbf{b}_i\).

However, if \(\|\mathbf{b}_i\|^2 > c_d\), the solution \(\gamma_i = 0\) is not admissible anymore; if, \(\gamma_i \neq 0\), then, the KKT conditions impose that,

\[\|\mathbf{d}_i\|^2 = c_d.\]

By definition,

\[\|\mathbf{d}_i\|^2 = \mu_i^2 \|\mathbf{b}_i\|^2 = c_d,\]

that is,

\[\mu_i = \frac{c_d^{1/2}}{\|\mathbf{b}_i\|} \Rightarrow \gamma_i = c_d \left(\frac{1}{\mu_i} - 1\right).\]

Eventually, for the normalization constraint, the update step can be written as:

\[
\mathbf{d}^{[k+1]}_i = \begin{cases} 
\mathbf{b}_i & \text{if } \|\mathbf{b}_i\|^2 \leq c_d \\
\frac{c_d^{1/2}}{\|\mathbf{b}_i\|} \mathbf{b}_i & \text{if } \|\mathbf{b}_i\|^2 > c_d
\end{cases}
\]  

(42)

with \(i = M + 1, M + 2, \ldots, K\).

Thus, the dictionary update can be compactly performed introducing a new operator:

\[\mathbf{D}^{[k+1]} = \mathcal{U}(\mathbf{B}),\]

where \(\mathcal{U}(\mathbf{B}) : \mathbf{b}_i \rightarrow \mathbf{d}_i\) with \(i = 1, 2, \ldots, K\), it controls the two possible alternatives and it is defined as:

\[
\mathbf{d}_i = \begin{cases} 
\mathbf{b}_i & \text{if } \|\mathbf{b}_i\|^2 \leq c_d \\
\frac{c_d^{1/2}}{\|\mathbf{b}_i\|} (\mathbf{b}_i - \delta_i) + \delta_i & \text{otherwise}
\end{cases} 
\]  

(44)
Put succinctly, the new algorithm for the proposed optimization task comprises the following pseudo-code depicted below:

**Proposed Algorithm**

1. **Initialization:** $S^{[0]}, D^{[0]}, \varepsilon > 0$
2. $k = 0$
3. repeat
4. 
5. Step I - Coefficient Update
6. Init: $D = D^{[k]}, c_S \geq \|D^T D\|
7. A = \frac{1}{c_S} [D^T X + (c_S I_K - D^T D) S^{[k]}]$
8. $W \rightarrow w_{ij} = \frac{1}{|a_{ij}| + \varepsilon}$
9. $S^{[k+1]} = Q_B(A)$
10. 
11. Step II - Dictionary Update
12. Init: $S = S^{[k+1]}, c_D \geq \|S^T S\|
13. B = \frac{1}{c_D} [X S^T + D^{[k]} (c_D I_K - S S^T)]$
14. $D^{[k+1]} = U(B)$
15. $k = k + 1$
16. until some stopping criterion is satisfied
17. output: $D^{[k]}, S^{[k]}$

3.3. Initialization

In general, DL related optimization is a non-convex problems, which are characterized by containing a number of potential local minima. Hence, if bad initialization estimates are adopted, any algorithm may likely converge to an undesirable solution, corresponding to a shallow minimum. In fact, this is a common problem for all the matrix factorization-based techniques. For this reason, each specific algorithm considers the initialization that better suits its specific needs.

Accordingly, different initialization procedures have been used and tested, including random initialization. Finally, it turned out that initializing the DL via the ICA solution provided the most consistent results. This initialization has also a Bayesian flavor. The imposed sparsity constraint that acts component-wise can be thought as an implicit independence assumption over a prior. Of course, independence over the prior does not necessarily mean independence over the posterior. Besides, the ICA solution have been sucessfully implemented as initialization for DL algorithms, e.g.,[33].

Furthermore, starting with ICA as initialization of the proposed algorithms turns out to be a nice synergy: the algorithm starts from the solution of ICA and the DL scheme goes beyond the independence hypothesis limitations leading to enhanced performance.
In this paper, ICA was implemented via JADE [34], which identifies independent components computing fourth-order cumulants. In the sequel, the JADE solution is projected over the constrained set of matrices, using the defined weighted $\ell_1$-norm projection operator. The result of the projection is then used as initial estimate of the coefficient matrix.

No doubt, other more recent advanced or novel ICA algorithms can be applied. For example, an alternative ICA methods that simultaneously provides independent-sparse decomposition [35], [36] constitute interesting alternative possibilities.

4. Performance evaluation

This section studies the performance of the newly proposed algorithm based on a synthetic, albeit realistic data set.

4.1. Generation of a Realistic synthetic Data Set

In this paper, we present a new realistic synthetic data set using SimTB, which enables a flexible composition of synthetic fMRI-like data sets and has been widely used in a number of fMRI studies, e.g., [38], [39], [40].

SimTB implements a simple spatiotemporal separable model, characterized by two features: a) the spatial maps are smooth (Gaussian-shaped) and b) the sources are barely overlapped; the overlap only affects small areas with low relative intensity. Both features are non-realistic, because real brain actively often exhibits non-linear effects, leading to the saturation of the brain response, and the overlap often affects extensive areas.

For this reason, we have modified the default sources of SimTB: first, instead of Gaussian shapes, which show activation at just one specific voxel, we rather prefer to have more than one neighbor voxels being activated with the same relative energy. In order to do that, we started from the Gaussian shapes from SimTB, and in the sequel, we reduced the highest activated voxels to have the same relative value. Finally, we rearranged in the space some of the sources to increase their overlap.

After these modifications, the final data set comprises 16 brain-like sources (see Fig. (1)); each source is formed by a single slice of size 100 $\times$ 100 voxels and a time course, which comprises 300 time instances with a difference of two seconds between consecutive acquisition times (TR=2).

Besides the sources that are directly related to the brain activity, true fMRI data contain sources that are either scanner-induced or related to biological processes such as heart-beating, breathing, etc. All these phenomena are collectively referred as artifacts, and some of them exhibit large intensity compared with the brain-induced sources of interest. Accordingly, we included five extra

---

1SimTB is an open MATLAB toolbox for the creation of brain-like fMRI datasets. It can be downloaded from [http://mialab.mrn.org/software](http://mialab.mrn.org/software)
Brain-like Sources

Figure 1: Spatial maps and their corresponding time courses of all the brain-like sources proposed for this synthetic data set.

Artifacts

Figure 2: Artifacts from the data set [37]. The first column depicts the spatial maps and their corresponding time courses for the sources 3, 4 and 5, while the second column shows the sources 7 and 8.
### Brain-like Sources

| Source | SinTB Correspondence | lvl. Sparsity |
|--------|----------------------|---------------|
| #      | Anatomical correspondence [38] |               |
| 1      | Bilateral Visual - more posterior | 95.28 % |
| 2      | Medial Frontal | 95.33 % |
| 3      | Precuneus | 95.53 % |
| 4      | Default Mode Network | 88.25 % |
| 5      | Subcortical nuclei | 93.30 % |
| 6      | Subcortical nuclei - putamen | 97.07 % |
| 7      | White matter tracts (anterior) | 88.07 % |
| 8      | Dorsal Attention Network | 91.82 % |
| 9      | Frontoparietal (Right dominance) | 85.51 % |
| 10     | Subcortical nuclei - thalamus | 92.67 % |
| 11     | SensoriMotor | 77.38 % |
| 12     | Right Auditory | 91.60 % |
| 13     | Left Auditory | 91.53 % |
| 14     | Right Hippocampus | 94.51 % |
| 15     | Left Hippocampus | 94.57 % |
| 16     | White matter tracts (posterior) | 71.95 % |

### Artifacts

| # | Features according to [37] | lvl. Sparsity |
|---|-----------------------------|---------------|
| 17 | sub-Gaussian | 1.00 % |
| 18 | Gaussian | 1.00 % |
| 19 | super-Gaussian | 1.99 % |
| 20 | sub-Gaussian | 86.14 % |
| 21 | super-Gaussian | 71.84 % |

### Mean values of the Levels of Sparsity

|                  | lvl. Sparsity |
|------------------|---------------|
| Whole data set   | 76.49 % |
| Brain-like sources | 90.27 % |
| Artifacts        | 32.39 % |

Table 1: Summary of the main features of the proposed synthetic data set, including their corresponding level of sparsity. Observe that although the mean level of sparsity of the data set is 76.5%, the brain-like sources are clearly dominated by high level of sparsity, whereas the artifacts are mainly dense (see Fig. (1) and Fig. (2)).
sources from the data set in [18]. Precisely, the sources 3, 4, 5, 7 and 8 that represent fMRI-like artifacts (see Fig. (2)).

Table 1 summarizes the main features of the generated data set. Observe that the majority of the brain-like sources are sparse (like the natural brain activity behavior), while the artifacts are mostly dense.

Apart from brain-like sources and artifacts, fMRI data are corrupted by Rician noise [41]. In the next section, we will add Rician noise as an external component, with the aim to carry out a comparative study of the different algorithms under various situations and levels of noise [42].

4.2. Performance Study

In this section, we perform a comparative study between the proposed method and other DL and ICA-based approaches that are often adopted in the fMRI analysis context. In all the experiments that follow, the true number of sources involved in the synthetic fMRI signal is considered to be known. This assembles the best scenario for all the studied methods. Note, however, that all the methods studied appear to be relatively robust to overestimation of the actual number of sources.

**Experiment 1 - Study of the weighted $\ell_1$-norm sparsity constraint**

The first experiment aims to study the advantages gained by using the weighted $\ell_1$-norm constraint rather than the conventional $\ell_1$-norm one. In order to solely focus on the effects of the norm constraint, this experiment is fully blind without time-curse assistance, i.e., we do not impose any extra constraint over the dictionary besides the sparsity-related one.

The results of this study are illustrated in Fig. (3). In particular, we compared the performance of three different DL methods: a) SPAMS algorithm [29] (denoted by a green curve), which is an on-line DL sparse coding algorithm based on the $\ell_1$-norm constraint imposed on the columns of the coefficient matrix; b) a well-known Majorization Minimization-based DL algorithm [25] (depicted by an orange curve), which is again based on the $\ell_1$-norm constraint imposed on the whole coefficient matrix; and c) the proposed DL algorithm, which is based on the weighted $\ell_1$-norm constraint. Two variants of the proposed algorithm were studied; one that constraint the whole coefficient matrix (depicted by a purple line) and one that the constraint is imposed on the rows of the coefficient matrix; that is, on a source by source basis (denoted by a blue curve). In particular, these two cases are defined as follows:

**Case (A):** The sparsity constraint is imposed over the whole coefficient matrix. The set of constraints for this case can be written as:

$$\mathcal{L}_W = \left\{ S \in \mathbb{R}^{K \times N} \mid \| S \|_{1,W} \leq NL \right\},$$

where $N = 10000$ and $T = 300$ are the size of the synthetic data set. The level of sparsity, $L$, in this case is set equal to $70\%$, which is a rough underestimation
of the true mean level of sparsity, 76.49 %, of the whole coefficient matrix (see Table 1).

Case (B): In this case, the sparsity constraint is imposed row by row of the coefficient matrix. Thus, the sparsity constraint is given by:

$$\mathcal{L}_w = \{ S \in \mathbb{R}^{K \times N} \mid \| s^i \|_{1,w_i} \leq N l_i \quad \forall i = 1, 2, \ldots, K \},$$

where $l_i$ are the expected levels of sparsity for each spatial map and $K$ is the specific number of sources. Table 2 shows the imposed levels of sparsity for this case together with the true values.

In practice, the exact values of the levels of sparsity are unknown. For this reason, in order to show that the method is robust to inexact sparsity level assumptions, the imposed sparsity level values diverge with respect to the true ones. A more detailed simulation analysis on the robustness of the proposed technique to sparsity level mistuning is conducted later, in experiment 2.

| True | Imposed (l) |
|------|-------------|
| 97.07| 95          |
| 95.53| 94          |
| 95.33| 93          |
| 95.28| 92          |
| 94.57| 91          |
| 94.51| 90          |
| 93.30| 89          |
| 92.67| 88          |
| 91.82| 87          |
| 91.60| 86          |
| 91.53| 85          |
| 88.25| 85          |
| 88.07| 80          |
| 86.14| 80          |
| 85.51| 75          |
| 77.38| 70          |
| 71.95| 65          |
| 71.84| 60          |
| 1.99 | 0           |
| 1.00 | 0           |
| 1.00 | 0           |

Table 2: Comparison between levels of sparsity

Note that in Table 2, we explicitly sorted the levels of sparsity in order to easily compare the values. However, in practice, ordering the levels of sparsity is optional: the user is free to use any arbitrary order. Observe, also, that three of the levels of sparsity were set to 0%, because we expected that at least three sources are going to be dense.
In particular, Fig. (3) shows the results of this experiment for the different studied methods. Every point represents the mean value for 100 different noise realizations with exactly the same parameters. The horizontal axis assembles different levels of Rician noise. The vertical axis indicates the squared value of the mean Pearson correlation, $R^2$, between the estimated decomposition and the correct one. That is:

$$R^2 = \left( \frac{1}{K} \sum_{i=1}^{K} \rho(F_i, \hat{F}_i) \right)^2,$$

where $K$ is the total number of sources considered, $F_i$ is the $i^{th}$ source of the decomposition, that is, $F_i = d_i s_i$ where $d_i$ and $s_i$ are the $i^{th}$ column of the dictionary, $D$, and the $i^{th}$ row of the coefficient matrix, $S$, whereas $\hat{F}_i$ is the corresponding true source from the synthetic data set.

The quantity $\rho(A, B)$, with $A, B \in \mathbb{R}^{M \times N}$, is defined as the Pearson’s correlation between the matrices $A$ and $B$, which is given by:

$$\rho(A, B) = \frac{1}{M + N - 1} \sum_{i,j=1}^{M,N} \left( \frac{a_{ij} - \mu_A}{\sigma_A} \right) \left( \frac{b_{ij} - \mu_B}{\sigma_B} \right),$$

where $\mu_A$ and $\sigma_A$ are the mean and the standard deviation of $A$, respectively, and $\mu_B$ and $\sigma_B$ are the mean and the standard deviation of $B$.

Comparing the performance of either version of the proposed algorithm with that of MM-DL, it is evident that the incorporation of the weighted $\ell_1$-norm clearly enhances the overall performance of both: in the brain-like sources and in the all-sources case. Moreover, the performance of the proposed approaches is similar to the one obtained by SPAMS. However, the most important advantage associated with the proposed approach is that it does not require any “optimization” w.r. the choice of the regularization parameter. In contrasts, the regularization parameters for SPAMS and MM-DL were properly optimized through cross-validation, which is possible here since the experiment is synthetic and the ground truth is available.

**Experiment 2 - Information Assisted Dictionary Learning**

The aim of this experiment is twofold: first, we study the performance of the proposed method when information related to the task-related time courses is imposed. As a benchmark, the Semi-blind Dictionary Learning (SDL) [21] method is used, since it is the only alternative DL-based approach which incorporates task-related time courses as external information. As it has already been discussed, SDL is essentially modification of the SPAMS algorithm that incorporates Dictionary assistance. Second, we study the sensitivity of the aforementioned algorithms to the tuning of their free parameters; namely, the regularization parameter, $\lambda$, in the case of SDL and the per-row sparsity constraints of our proposed method.

We have chosen 3 different time courses from the synthetic dataset, in particular sources 1, 12 and 13, to be considered as task-related. The specific
sources represent brain areas that are commonly targeted in practice and they exhibit overlap with their surrounding sources. Since one of the scopes of this experiment is to analyze the effect of mis-modeling of the imposed task-related time courses, we have to synthetically generate HRFs, which differ to a varying degree from the canonical HRF (shown in thick red line in Fig. (4)) that was used for the generation of the dataset. Then, for each mis-modeled HRF, the three aforementioned task-related time-courses can be computed and imposed in the studied methods.

Accordingly, we generated five different HRFs (shown with color lines in Fig. (4)) by feeding with randomly generated parameters the two-gamma distribution model [7], taking care, however, to keep the generated parameters within limits that resembles the expected variability and characteristics of real HRFs. To confirm this, 100 HRFs were generated with the adopted randomized method (shown with gray lines in Fig. (4)) and visually compared to human brain HRFs that have been estimated in several studies, e.g. in [43].

In the context of our proposed algorithm, the user now can define both constraints using the task-related time courses as well as the level of sparsity per source. Thus, the constrained set of dictionaries can be written as following:

$$
\mathcal{D}_{\delta} = \left\{ D \in \mathbb{R}^{T \times K} \left| \begin{array}{c}
\|d_i - \delta_i\|^2 \leq c_{\delta} \\
\|d_i\|^2 \leq 1 \\
\|d_i\|^2 \leq 1 \\
i = 1, 2, 3 \\
i = 4, \ldots, K
\end{array} \right. \right\},
$$
Figure 4: Representation of the 100 HRFs (grey) that were randomly generated from the two-gamma distribution model. The red HRF represents the Canonical HRF and the colored HRFs stands for the selected miss-modeled HRFs.

where $\delta_i$ are the considered task-related time courses; thus, in the specific case, the sources 1, 12 and 13 are imposed as $\delta_1$, $\delta_2$ and $\delta_3$ respectively. The similarity parameter was set to $c_\delta = 0.1$, which is large enough to encapsulate the variability introduced by the considered HRFs.

With respect to the sparsity constraint, the sparsity level of the task-related time courses has to be explicitly set because they are now used in the dictionary constraint, $D_\delta$. This requirement is not essentially restrictive since the task-related sources usually correspond to functional brain networks that are depicted in brain atlases, so the specialist can have a good indication regarding to the spatial area that these networks occupy on an average brain. More importantly, the proposed method is relatively insensitive in the mis-tuning of the sources sparsity. In order to demonstrate this, three independent source-sparsity configurations are tested, which are indicated in Table 3 as $(l_1)$, $(l_2)$ and $(l_3)$.

For simplicity, the task-related sources are the first three shown in the Table 3, whereas the sparsity levels of the rest of the sources is shown in descending order. As before, the sparsity level of the non task-related time sources need not be explicitly set in on one-to-one base. Observe that configuration $(l_1)$ is closer to the true one whereas $(l_3)$ is a grossly tuned sparsity level configuration having large discrepancies from the true sparsity levels of both the task-related and the non task-related time sources. Configuration $(l_2)$ is somewhere in between.

In contrast to the proposed method, the SDL requires to tune a single regularization parameter, $\lambda$; however, as it will be seen in practice, it is rather impossible for this single parameter to be tuned effectively since: a) the perfor-
Levels of Sparsity

| True | Imposed (l_1) | Imposed (l_2) | Imposed (l_3) |
|------|--------------|--------------|--------------|
| 95.28| 95           | 95           | 75           |
| 91.60| 90           | 90           | 85           |
| 91.53| 90           | 80           | 85           |
| 97.07| 97           | 94           | 90           |
| 95.53| 95           | 93           | 90           |
| 95.33| 95           | 92           | 90           |
| 94.57| 93           | 91           | 90           |
| 94.51| 92           | 90           | 90           |
| 93.30| 92           | 89           | 85           |
| 92.67| 92           | 88           | 85           |
| 91.82| 90           | 87           | 85           |
| 88.25| 88           | 86           | 85           |
| 88.07| 88           | 85           | 85           |
| 86.14| 85           | 80           | 80           |
| 85.51| 85           | 75           | 75           |
| 77.38| 77           | 70           | 75           |
| 71.95| 70           | 65           | 60           |
| 71.84| 70           | 60           | 20           |
| 1.99 | 2            | 0            | 5            |
| 1.00 | 0            | 0            | 2            |
| 1.00 | 0            | 0            | 1            |

Table 3: Comparison between the different implemented levels of sparsity

Performance achieved is very sensitive to the value of this parameter and b) it does not bear a concise physical interpretation that could serve as a guide for its tuning. Moreover, since in practical scenarios we do not know the true solution, tuning via cross-validation arguments is pointless. In order to study the $\lambda$ mis-tune tolerance that SDL exhibits, four scenarios were tested: first, $\lambda$ was set equal to the value that leads to the best estimate of the three task-related sources, when the canonical HRF was imposed. In our synthetic example, since we know the true solution, this $\lambda$ value can be empirically found through cross-validation. It turned out to be $\lambda_1 = 0.0008$. Then, three other values where tested, namely $\lambda_2 = 0.00008$, $\lambda_3 = 0.008$ and $\lambda_4 = 0.0000008$.

The performance results corresponding to the 5 mis-modeled HRFs as well as the true canonical HRF are shown in Fig. (4)), where the number of sources has been set equal to the correct one ($K = 21$), the level of Rician noise corresponds to SNR = 15 and all the shown performance curves represent the ensemble average of the results obtained from 100 independent noise realizations. Moreover, the left, mid and right subfigures correspond to the average performance of the three task-related sources, the brain-like sources and all the sources, respectively. The performance of AwDL is denoted with filled circled curves and that of SDL with open squared curves.
Concerning the performance of the proposed method, three points are worthy to be stressed: first, it performs always better compared to the performance that is obtained without task-related course assistance, which is depicted with a straight red curve. More specifically, this latter curve represents the best performance achieved in experiment 1. Second, for all three sparsity configurations, the obtained performance for the assisted curves is remarkably stable. Third, the average performance of brain-only sources as well as of all the sources corresponding to the three implemented sparsity level configurations ($l_1, l_2$, and $l_3$) is similar, which is a manifestation of the robustness and the tolerance of AwDL with respect to the imposed sparsity levels.

In contrast to AwDL, SDL exhibits a rather erratic behavior for different $\lambda$ values. Although for $\lambda = 0.0008$ appears to perform in the all-brain and all-sources cases somewhat better than the proposed algorithm, this performance was obtained through intensive fine-tuning and it cannot be secured in practice. Moreover, it is not guaranteed that SDL tends to perform better with relatively small $\lambda$ values. The best performing value appears to be highly problem-specific and depends on the overall sparsity of the sources, the number of sources that can be simultaneously manifested in each voxel and also, to a lessen degree, on the noise level.

Concerning the robustness w.r. to the HRF mis-modeling, AwDL appears much more tolerant compared to SDL, something that can be observed in the relative performance degradation caused in the mis-matched HRFs. SDL tends to degrade faster than AwDL. This should be expected since SDL keeps the imposed time-courses fixed, whereas AwDL allows them to get improved within the limits of the corresponding constraint.

Figure 5: Experiment 2: Analysis of the different performance of the methods against the miss-modeling of the HRFs
Finally, a last but important point, is the fact that SDL is computationally much more complex compared to AwDL. Detailed computational complexity analysis will be shown in a next version of the current paper.

5. Conclusions

This paper presents a new Dictionary Learning method that includes external information in a natural way via two novel convex constraints: i) a sparsity constraint based on the weighted $\ell_1$-norm, which incorporates the true level of sparsity of the spatial maps on a source by source basis and ii) a similarity constraint over the dictionary, which integrates external a priori information available from the experimental scheme.

The proposed sparsity constraint constitutes a more natural alternative to the standard $\ell_1$-norm regularization. This bypasses the problem of selecting the regularization parameters, following cross-validation arguments that have no practical meaning, when real data are involved.

The incorporation of the task-related time courses from the experimental paradigm enhances the performance of all the methods. However, the newly proposed constraint exhibits more tolerance against miss-modeling of the HRF, offering a more robust alternative compared to more restrictive methods.

Extensive simulations over realistic synthetic data sets verified the advantages and the enhanced performance obtained by the new proposed method, which will be reported in a separate paper.

Bibliography

References

[1] R. A. Poldrack, J. A. Mumford, T. E. Nichols, Handbook of functional MRI data analysis, Cambridge University Press, 2011.

[2] Sergios Theodoridis, Machine Learning: A Bayesian and Optimization Perspective, Academic Press, 2015.

[3] A. Protopapas, E. Orfanidou, J. S. H. Taylor, E. Karavasilis, E. C. Kapnoula, G. Panagiotaropoulou, G. Velonakis, L. S. Poulou, N. Smyrnis, D. Kelekis, Evaluating cognitive models of visual word recognition using fMRI: effects of lexical and sublexical variables, Neuroimage 128 (2016) 328–341.

[4] M. G. Bright, K. Murphy, Is fMRI noise really noise? resting state nuisance regressors remove variance with network structure 114 (2015) 158–169.

[5] E. Amaro, G. J. Barker, Study design in fMRI: basic principles 60 (3) (2006) 220–232.
[6] F. I. Karahanoğlu, C. Caballero-Gaudes, F. Lazeyras, D. Van De Ville, Total activation: fMRI deconvolution through spatio-temporal regularization, Neuroimage 73 (2013) 121–134.

[7] W. D. Penny, K. J. Friston, J. T. Ashburner, S. J. Kiebel, T. E. Nichols, Statistical parametric mapping: the analysis of functional brain images, Academic press, 2011.

[8] K. Friston et al., Statistical Parametric Mapping (SPM) (1999). URL http://www.fil.ion.ucl.ac.uk/spm/

[9] G. K. Aguirre, E. Zarahn, M. D'esposito, The variability of human, BOLD hemodynamic responses 8 (4) (1998) 360–369.

[10] A. Protopapas, E. Orfanidou, J. S. H. Taylor, E. Karavasilis, E. C. Kapnoula, G. Panagiotaropoulou, G. Velonakis, L. S. Poulou, N. Smyrnis, D. Kelekis, Evaluating cognitive models of visual word recognition using fMRI: effects of lexical and sublexical variables, Neuroimage 128 (2016) 328–341.

[11] Vince D. Calhoun and Tülay Adalı, Multisubject independent component analysis of fMRI: A decade of intrinsic networks, default mode, and neurodiagnostic discovery, IEEE Reviews in Biomedical Engineering 5 (2012) 60–73.

[12] W. Du, Y. Levin-Schwartz, G.-S. Fu, S. Ma, V. D. Calhoun, T. Adalı, The role of diversity in complex ICA algorithms for fMRI analysis, Journal of Neuroscience Methods 264 (2016) 129–135.

[13] I. Daubechies, E. Roussos, S. Takerkart, M. Benharrosh, C. Golden, K. D’Ardenne, W. Richter, J. D. Cohen, J. Haxby, Independent component analysis for brain fMRI does not select for independence 106 (26) (2009) 10415–10422.

[14] V. D. Calhoun, V. K. Potluru, R. Phlypo, R. F. Silva, B. A. Pearlmutter, A. Caprihan, S. M. Plis, T. Adalı, Independent component analysis for brain fMRI does indeed select for maximal independence 8 (8) (2013) e73309.

[15] Y.-B. Lee, J. Lee, S. Tak, K. Lee, D. L. Na, S. W. Seo, Y. Jeong, J. C. Ye, A. D. N. Initiative, Sparse SPM: Group sparse-dictionary learning in SPM framework for resting-state functional connectivity MRI analysis 125 (2016) 1032–1045.

[16] Yannis Kopsinis et al., fnri unmixing via properly adjusted dictionary learning, in: Signal Processing Conference (EUSIPCO), 2014 Proceedings of the 22nd European, IEEE, 2014, pp. 2075–2079.
[17] G. Zhang, Z. Jiang, L. S. Davis, Online semi-supervised discriminative dictionary learning for sparse representation, in: Asian Conference on Computer Vision, Springer, 2012, pp. 259–273.

[18] M. M. Moreno, Y. Kopsinis, E. Kofidis, C. Chatzichristos, S. Theodoridis, Assisted dictionary learning for FMRI data analysis, in: Acoustics, Speech and Signal Processing (ICASSP), 2017 IEEE International Conference on, IEEE, 2017, pp. 806–810.

[19] Jagath Rajapakse and Wei Lu, Extracting task-related components in functional mri, Submitted to ICA 2001 1 (2001) 120.

[20] V. D. Calhoun, T. Adali, M. C. Stevens, K. A. Kiehl, J. J. Pekar, Semi-blind ICA of fMRI: A method for utilizing hypothesis-derived time courses in a spatial ICA analysis 25 (2) 527–538.

[21] S. Zhao, J. Han, J. Lv, X. Jiang, X. Hu, Y. Zhao, B. Ge, L. Guo, T. Liu, Supervised dictionary learning for inferring concurrent brain networks 34 (10) (2015) 2036–2045.

[22] W.-K. Ma, J. M. Bioucas-Dias, T.-H. Chan, N. Gillis, P. Gader, A. J. Plaza, A. Ambikapathi, C.-Y. Chi, A signal processing perspective on hyperspectral unmixing: Insights from remote sensing 31 (1) 67–81.

[23] R. Salvador, J. Suckling, M. R. Coleman, J. D. Pickard, D. Menon, E. D. Bullmore, Neurophysiological architecture of functional magnetic resonance images of human brain, Cerebral cortex 15 (9) (2005) 1332–1342.

[24] B. Murphy, P. Talukdar, T. Mitchell, Learning effective and interpretable semantic models using non-negative sparse embedding, Proceedings of COLING 2012 (2012) 1933–1950.

[25] M. Yaghoobi, T. Blumensath, M. E. Davies, Dictionary learning for sparse approximations with the majorization method, IEEE Transactions on Signal Processing 57 (6) (2009) 2178–2191.

[26] D. L. Donoho, M. Elad, Optimally sparse representation in general (nonorthogonal) dictionaries via 1 minimization 100 (5) (2003) 2197–2202.

[27] E. J. Candes, M. B. Wakin, S. P. Boyd, Enhancing sparsity by reweighted 1 minimization, Journal of Fourier analysis and applications 14 (5) (2008) 877–905.

[28] Y. Kopsinis, H. Georgiou, S. Theodoridis, fMRI unmixing via properly adjusted dictionary learning (2014) 2075–2079.

[29] J. Mairal, F. Bach, J. Ponce, G. Sapiro, Online learning for matrix factorization and sparse coding, The Journal of Machine Learning Research 11 (2010) 19–60.
[30] G. I. Allen, Sparse and functional principal components analysis.

[31] Y. Sun, P. Babu, D. P. Palomar, Majorization-minimization algorithms in signal processing, communications, and machine learning 65 (3) (2017) 794–816.

[32] M. Razaviyayn, M. Hong, Z.-Q. Luo, A unified convergence analysis of block successive minimization methods for nonsmooth optimization 23 (2) (2013) 1126–1153.

[33] V. Abrol, P. Sharma, S. F. Roohi, A. K. Sao, A. A. Kassim, Fast and robust FMRI unmixing using hierarchical dictionary learning, in: 2016 IEEE International Conference on Image Processing (ICIP), 2016, pp. 714–718.

[34] J. F. Cardoso, A. Souloumiac, Blind beamforming for non-gaussian signals, IEE Proceedings F (Radar and Signal Processing) 140 (6) (1993) 362–370.

[35] A. M. Bronstein, M. M. Bronstein, M. Zibulevsky, Y. Y. Zeevi, Sparse ICA for blind separation of transmitted and reflected images, International Journal of Imaging Systems and Technology 15 (1) (2005) 84–91.

[36] Z. Boukouvalas, Y. Levin-Schwartz, V. D. Calhoun, T. Adalı, Sparsity and independence: Balancing two objectives in optimization for source separation with application to fMRI analysis, Journal of the Franklin Institute 355 (4) (2018) 1873–1887.

[37] N. Correa, T. Adah, Y.-O. Li, V. D. Calhoun, Comparison of blind source separation algorithms for FMRI using a new Matlab toolbox: GIFT, in: Acoustics, Speech, and Signal Processing, 2005. Proceedings. (ICASSP’05). IEEE International Conference on, Vol. 5, IEEE, 2005, pp. v–401.

[38] E. B. Erhardt, E. A. Allen, Y. Wei, T. Eichele, V. D. Calhoun, SimTB, a simulation toolbox for fMRI data under a model of spatiotemporal separability, Neuroimage 59 (4) (2012) 4160–4167.

[39] E. A. Allen, E. B. Erhardt, Y. Wei, T. Eichele, V. D. Calhoun, Capturing inter-subject variability with group independent component analysis of fMRI data: a simulation study, Neuroimage 59 (4) (2012) 4141–4159.

[40] E. Castro, D. Hjelm, S. Plis, L. Dinh, J. Turner, V. Calhoun, Deep independence network analysis of structural brain imaging: A simulation study, in: Machine Learning for Signal Processing (MLSP), 2015 IEEE 25th International Workshop on, IEEE, 2015, pp. 1–6.

[41] H. Gudbjartsson, S. Patz, The Rician distribution of noisy MRI data, Magnetic resonance in medicine 34 (6) (1995) 910–914.

[42] M. Welvaert, Y. Rosseel, On the definition of signal-to-noise ratio and contrast-to-noise ratio for fmri data, PloS one 8 (11) (2013) e77089.
[43] D. A. Handwerker, J. M. Ollinger, M. D’Esposito, Variation of BOLD hemodynamic responses across subjects and brain regions and their effects on statistical analyses, Neuroimage 21 (4) (2004) 1639–1651.