ADDITION FORMULAS OF LEAF FUNCTIONS AND HYPERBOLIC LEAF FUNCTIONS

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Kazunori Shinohara*
Department of Mechanical Systems Engineering
Daido University
10-3 Takiharu-cho, Minami-ku, Nagoya 457-8530, Japan
shinohara@06.alumni.u-tokyo.ac.jp

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ABSTRACT

Addition formulas exist in trigonometric functions. Double-angle and half-angle formulas can be derived from these formulas. Moreover, the relation equation between the trigonometric function and the hyperbolic function can be derived using an imaginary number. The inverse hyperbolic function
\[ \text{arsinh}(r) = \int_0^r \frac{1}{\sqrt{1+t^2}} dt \]

is similar to the inverse trigonometric function
\[ \text{arcsin}(r) = \int_0^r \frac{1}{\sqrt{1-t^2}} dt , \]

such as the second degree of a polynomial and the constant term 1, except for the sign − and +. Such an analogy holds not only when the degree of the polynomial is 2, but also for higher degrees. As such, a function exists with respect to the leaf function through the imaginary number \( i \), such that the hyperbolic function exists with respect to the trigonometric function through this imaginary number. In this study, we refer to this function as the hyperbolic leaf function. By making such a definition, the relation equation between the leaf function and the hyperbolic leaf function makes it possible to easily derive various formulas, such as addition formulas of hyperbolic leaf functions based on the addition formulas of leaf functions. Using the addition formulas, we can also derive the double angle and half-angle formulas. We then verify the consistency of these formulas by constructing graphs and numerical data.

Keywords Leaf functions · Hyperbolic leaf functions · Lemniscate function · Jacobi elliptic functions · Ordinary differential equation

1 Introduction

1.1 Leaf Functions and Hyperbolic Leaf Functions

An ordinary differential equation consists of both a function raised to the \( 2n - 1 \) power and the second derivative of the function.

\[ \frac{d^2r}{dl^2} = -nr(l)^{2n-1} \] (1)

The preceding equation is the ODE that motivated this study. Although the equation (1) is a simple ordinary differential equation, it has a very important meaning because it generates characteristic waves. By numerically analyzing the solution that satisfies this equation, we can obtain regular and periodic waves (1)(2). The form of these waves differs from the form of the waves based on trigonometric functions. The function that satisfies this ordinary differential equation is called a leaf function, and it describes the features of these functions. Eq. (1) is transformed as follows:

* 10-3 Takiharu-cho, Minami-ku, Nagoya 457-8530, Japan
\[ l = \int_0^r \frac{dt}{\sqrt{1 - t^2}} (= \text{arcsleaf}_n(r)) \]  

(2)

The preceding integral is defined as the inverse function \text{arcsleaf}_n(l) of the leaf function. Another function can be defined as follows:

\[ l = \int_r^1 \frac{dt}{\sqrt{1 - t^2}} (= \text{arcacleaf}_n(r)) \]  

(3)

The preceding integral is also defined as the inverse function \text{arcacleaf}_n(r) of the leaf function with a different integral domain compared to Eq. (2). The variable \( n \) represents a natural number, and it is referred to as the basis. Moreover, the ordinary differential equation that is satisfied by the hyperbolic functions \( r(l) = \sinh(l) \) and \( r(l) = \cosh(l) \) is described as follows.

\[ \frac{d^2r}{dl^2} = r(l) \]  

(4)

Compared to Eq. (1), the difference in Eq. (4) is the positive sign on the right hand side of the equation. The inverse hyperbolic functions \( \text{arsinh}(r) \) and \( \text{arcosh}(r) \) are well known as:

\[ l = \int_0^r \frac{dt}{\sqrt{1 + t^2}} (= \text{asinh}(r)) \]  

(5)

\[ l = \int_r^1 \frac{dt}{\sqrt{t^2 - 1}} (= \text{acosh}(r)) \]  

(6)

The contents of the root in the integrand constitute a polynomial. The polynomial of the inverse hyperbolic function and that of the inverse trigonometric function both have a degree of 2. The magnitude 1 of the constant term in the root is also the same. The difference between the inverse functions of the trigonometric function and the hyperbolic function is the sign of the polynomial in the root. Using Eqs. (5) and (6), it is seen that trigonometric functions and hyperbolic functions have relational equation through imaginary numbers. Based on this relationship, similar functions also could be paired with leaf functions through analogy relation (See Appendix D in detail). These functions are called hyperbolic leaf functions and consist of two functions. One function is defined as follows.

\[ r(l) = \text{sleaf}_n(l)(n = 1, 2, 3 \cdots) \]  

(7)

The limit exists for the function \( \text{sleaf}_n(l) \) (See Appendix F). The domain of the variable \( l \) is defined as follows:

\[ -\zeta_n < l < \zeta_n \]  

(8)

The initial conditions of the preceding equation are defined as follows.

\[ r(0) = \text{sleaf}_n(0) = 0(n = 1, 2, 3 \cdots) \]  

(9)

\[ \frac{dr(0)}{dl} = \frac{d}{dl}\text{sleaf}_n(0) = 1(n = 1, 2, 3 \cdots) \]  

(10)

Next, the another function is defined as follows:

\[ r(l) = \text{cleaf}_n(l)(n = 1, 2, 3 \cdots) \]  

(11)

The limit exists for the function \( \text{cleaf}_n(l) \) (See Appendix G). The domain of the variable \( l \) is as follows:

\[ -\eta_n < l < \eta_n \]  

(12)
The initial conditions of the preceding equation are defined as follows.

\[ r(0) = \text{cleaf}_n(0) = 1(n = 1, 2, 3 \cdots) \]  
\[ \frac{dr(0)}{dt} = \frac{d}{dt}\text{cleaf}_n(0) = 0(n = 1, 2, 3 \cdots) \]  

The ordinary differential equations that are satisfied by the hyperbolic leaf functions that correspond to both equation (7) and equation (11) are as follows.

\[ \frac{d^2r}{dl^2} = nr(l)^{2n-1} \]  

The inverse function of the hyperbolic leaf function is derived as follows:

\[ l = \int_0^r \frac{dt}{\sqrt{1 + t^{2n}}} (= \text{asleaf}_n(r))(n = 1, 2, 3 \cdots) \]  
\[ l = \int_1^r \frac{dt}{\sqrt{t^{2n} - 1}} (= \text{acleaf}_n(r))(n = 1, 2, 3 \cdots) \]  

Here, the prefix \( a \) of both hyperbolic leaf functions \( \text{sleaf}_n(l) \) and \( \text{cleaf}_n(l) \) are defined as the inverse functions.

### 1.2 Comparison of Legacy functions

The leaf functions and the hyperbolic leaf functions based on the basis \( n = 1 \) are as follows:

\[ \text{sleaf}_1(t) = \sin(t) \]  
\[ \text{cleaf}_1(t) = \cos(t) \]  
\[ \text{sleaf}_1(t) = \sinh(t) \]  
\[ \text{cleaf}_1(t) = \cosh(t) \]  

Lemniscate functions were proposed by Gauss [3]. The relation equations between these functions and leaf function are as follows:

\[ \text{sleaf}_2(t) = \text{sl}(t) \]  
\[ \text{cleaf}_2(t) = \text{cl}(t) \]  
\[ \text{sleaf}_2(t) = \text{slh}(t) \]  

The definition of the function \( \text{slh}(t) \) in Eq. (24) can be confirmed based on references [4] [5]. A function corresponding to the hyperbolic leaf function \( \text{cleaf}_2(t) \) is not described in the literature [6][7]. In the case where the basis \( n \geq 3 \), the leaf function or the hyperbolic leaf function cannot be represented by a legacy function such as the lemniscate function.

### 1.3 Originality and Purpose

Fagnano discovered the double formulas of the lemniscate function in the 18th century [8]. Furthermore, based on Fagnano’s formulas, Euler derived the addition formulas of the lemniscate function [9]. Historically, there has been no discussion on the basis \( n = 3 \) in Eq. [21] [5], [16], and [17]. Therefore, the addition formulas of the leaf function based on the basis \( n = 3 \) were investigated [10]. However, the addition formulas of the hyperbolic leaf function based on the basis \( n = 3 \) have not been presented in the case where the basis \( n = 4 \) or more, there is no clear description in the literature about the addition formulas of hyperbolic leaf functions. In the case of hyperbolic leaf functions based on \( n = 1 \), these functions represent the hyperbolic functions \( \sinh(l) \) and \( \cosh(l) \). Therefore, the addition formulas of the hyperbolic leaf function are the same as the addition formulas of the hyperbolic function. The hyperbolic leaf function based on \( n = 2 \) represents the Hyperbolic lemniscate function \( \text{slh}(l) \). There is no clear description in the literature about addition formulas of the function \( \text{slh}(l) \). In the case of a hyperbolic function with the basis \( n = 3 \), as previously discussed, there has not been any historical discussion on this subject. The purpose of this report is to propose addition
formulas for the hyperbolic leaf functions with basis \( n = 2 \) and \( n = 3 \), in addition to establishing both double-angle and the half-angle formulas using addition formulas. A similar analogy exists in the relation between the leaf function and the hyperbolic leaf function such that the relation between the trigonometric function and the hyperbolic function can be derived using imaginary numbers. Using this analogy, the addition formulas of hyperbolic leaf functions based on \( n = 3 \) can be derived from the addition formulas of leaf functions based on \( n = 3 \). Using addition formulas, we present numerical data and curves derived from the hyperbolic leaf function and show that these addition formulas in the section 2 are consistent.

2 Addition formulas

2.1 Addition formulas of leaf function

Let the two variables be \( l_1 \) and \( l_2 \). The addition formulas of the leaf functions \( \text{sleaf}_1(l) \) and \( \text{cleaf}_1(l) \) are as follows.

\[
\text{sleaf}_1(l_1 + l_2) = \text{sleaf}_1(l_1)\text{cleaf}_1(l_2) + \text{cleaf}_1(l_1)\text{sleaf}_1(l_2) \tag{25}
\]

\[
\text{cleaf}_1(l_1 + l_2) = \text{cleaf}_1(l_1)\text{sleaf}_1(l_2) - \text{sleaf}_1(l_1)\text{cleaf}_1(l_2) \tag{26}
\]

The preceding two equations have the same meaning as the addition formulas of trigonometric functions. Next, the addition formulas of the function \( \text{sleaf}_2(l) \) can be stated as follows.

\[
\text{sleaf}_2(l_1 + l_2) = \frac{\text{sleaf}_2(l_1)\frac{\partial\text{sleaf}_2(l_2)}{\partial l_2} + \text{sleaf}_2(l_2)\frac{\partial\text{sleaf}_2(l_1)}{\partial l_1}}{1 + (\text{sleaf}_2(l_1))^2(\text{sleaf}_2(l_2))^2} \tag{27}
\]

Depending on the domain of the variable \( l \) of the leaf function, the signs of both \( \partial\text{sleaf}_2(l_2)/\partial l_2 \) and \( \partial\text{sleaf}_2(l_1)/\partial l_1 \) change. The symbols \( m \) and \( k \) represent integers. Eq. (27) can be summarized according to the domain of variables \( l_1 \) and \( l_2 \).

(i) In case where the domain \( \frac{\pi}{4} (4m - 1) \leq l_1 \leq \frac{\pi}{4} (4m + 1) \), and the domain \( \frac{\pi}{4} (4k - 1) \leq l_2 \leq \frac{\pi}{4} (4k + 1) \). (See Appendix E for the constant \( \pi_2 \), Eq. (27) is as follows:

\[
\text{sleaf}_2(l_1 + l_2) = \frac{\text{sleaf}_2(l_1)\sqrt{1 - [\text{sleaf}_2(l_2)]^2} + \text{sleaf}_2(l_2)\sqrt{1 - [\text{sleaf}_2(l_1)]^2}}{1 + (\text{sleaf}_2(l_1))^2(\text{sleaf}_2(l_2))^2} \tag{28}
\]

(ii) In the case where the domain \( \frac{\pi}{4} (4m - 1) \leq l_1 \leq \frac{\pi}{4} (4m + 1) \), and the domain \( \frac{\pi}{4} (4k + 1) \leq l_2 \leq \frac{\pi}{4} (4k + 3) \). Eq. (27) is as follows:

\[
\text{sleaf}_2(l_1 + l_2) = \frac{-\text{sleaf}_2(l_1)\sqrt{1 - [\text{sleaf}_2(l_2)]^2} + \text{sleaf}_2(l_2)\sqrt{1 - [\text{sleaf}_2(l_1)]^2}}{1 + (\text{sleaf}_2(l_1))^2(\text{sleaf}_2(l_2))^2} \tag{29}
\]

(iii) In the case where the domain \( \frac{\pi}{4} (4m + 1) \leq l_1 \leq \frac{\pi}{4} (4m + 3) \), and the domain \( \frac{\pi}{4} (4k - 1) \leq l_2 \leq \frac{\pi}{4} (4k + 1) \). Eq. (27) is as follows:

\[
\text{sleaf}_2(l_1 + l_2) = \frac{\text{sleaf}_2(l_1)\sqrt{1 - [\text{sleaf}_2(l_2)]^2} - \text{sleaf}_2(l_2)\sqrt{1 - [\text{sleaf}_2(l_1)]^2}}{1 + (\text{sleaf}_2(l_1))^2(\text{sleaf}_2(l_2))^2} \tag{30}
\]

(iii) In the case where the domain \( \frac{\pi}{4} (4m + 1) \leq l_1 \leq \frac{\pi}{4} (4m + 3) \), and the domain \( \frac{\pi}{4} (4k + 1) \leq l_2 \leq \frac{\pi}{4} (4k + 3) \). Eq. (27) is as follows:

\[
\text{sleaf}_2(l_1 + l_2) = \frac{-\text{sleaf}_2(l_1)\sqrt{1 - [\text{sleaf}_2(l_2)]^2} - \text{sleaf}_2(l_2)\sqrt{1 - [\text{sleaf}_2(l_1)]^2}}{1 + (\text{sleaf}_2(l_1))^2(\text{sleaf}_2(l_2))^2} \tag{31}
\]

Next, the addition formulas of \( \text{cleaf}_2(l) \) can be stated as follows:
In the case where both 

\[ \text{cleaf}_2(l_1 + l_2) = \frac{\text{cleaf}_2(l_1) \frac{\partial \text{cleaf}_2(l_2)}{\partial l_2} + \text{cleaf}_2(l_2) \frac{\partial \text{cleaf}_2(l_1)}{\partial l_1}}{1 + (\text{cleaf}_2(l_1))^2(\text{cleaf}_2(l_2))^2} \]  

(iii) In the case where the domain \(2m\pi_2 \leq l_1 \leq (2m + 1)\pi_2\), and the domain \(\frac{n_2}{2} (4k - 3) \leq l_2 \leq \frac{n_2}{2} (4k - 1)\), Eq. (32) is as follows:

\[ \text{cleaf}_2(l_1 + l_2) = \frac{-\text{cleaf}_2(l_1) \sqrt{1 - (\text{cleaf}_2(l_2))^2} - \text{cleaf}_2(l_2) \sqrt{1 - (\text{cleaf}_2(l_1))^2}}{1 + (\text{cleaf}_2(l_1))^2(\text{cleaf}_2(l_2))^2} \]  

(iii) In the case where the domain \(2m\pi_2 \leq l_1 \leq (2m + 1)\pi_2\), and the domain \(\frac{n_2}{2} (4k - 3) \leq l_2 \leq \frac{n_2}{2} (4k - 1)\), Eq. (32) is as follows:

\[ \text{cleaf}_2(l_1 + l_2) = \frac{-\text{cleaf}_2(l_1) \sqrt{1 - (\text{cleaf}_2(l_2))^2} - \text{cleaf}_2(l_2) \sqrt{1 - (\text{cleaf}_2(l_1))^2}}{1 + (\text{cleaf}_2(l_1))^2(\text{cleaf}_2(l_2))^2} \]  

Next, the addition formulas of \(\text{sleaf}_3(l)\) can be described as follows:

\[ (\text{sleaf}_3(l_1 + l_2))^2 = \frac{\left\{\text{sleaf}_3(l_1) \frac{\partial \text{sleaf}_3(l_2)}{\partial l_2} + \text{sleaf}_3(l_2) \frac{\partial \text{sleaf}_3(l_1)}{\partial l_1}\right\}^2}{1 + 4(\text{sleaf}_3(l_1))^2(\text{sleaf}_3(l_2))^2 + 4(\text{sleaf}_3(l_1))^2(\text{sleaf}_3(l_2))^4} \]  

The preceding equation can be summarized as follows according to the domain of the variables \(l_1\) and \(l_2\).  

(i) In the case where both \((4m - 1)\frac{\pi_3}{2} \leq l_1 \leq (4m + 1)\frac{\pi_3}{2}\) and \((4k - 1)\frac{\pi_3}{2} \leq l_2 \leq (4k + 1)\frac{\pi_3}{2}\) or both \((4m + 1)\frac{\pi_3}{2} \leq l_1 \leq (4m + 3)\frac{\pi_3}{2}\) and \((4k + 1)\frac{\pi_3}{2} \leq l_2 \leq (4k + 3)\frac{\pi_3}{2}\), Eq. (37) is defined as follows:

\[ (\text{sleaf}_3(l_1 + l_2))^2 = \frac{\left\{\text{sleaf}_3(l_1) \sqrt{1 - (\text{sleaf}_3(l_2))^2} - \text{sleaf}_3(\text{sleaf}_3(l_2))^2 + 4(\text{sleaf}_3(l_1))^2(\text{sleaf}_3(l_2))^4\right\}^2}{1 + 4(\text{sleaf}_3(l_1))^2(\text{sleaf}_3(l_2))^2 + 4(\text{sleaf}_3(l_1))^2(\text{sleaf}_3(l_2))^4} \]  

The symbol \(\pi_3\) represents a constant (See Appendix E).  

(ii) In the case where both \((4m + 1)\frac{\pi_3}{2} \leq l_1 \leq (4m + 3)\frac{\pi_3}{2}\) and \((4k - 1)\frac{\pi_3}{2} \leq l_2 \leq (4k + 1)\frac{\pi_3}{2}\) or both \((4m - 1)\frac{\pi_3}{2} \leq l_1 \leq (4m + 1)\frac{\pi_3}{2}\) and \((4k + 1)\frac{\pi_3}{2} \leq l_2 \leq (4k + 3)\frac{\pi_3}{2}\), Eq. (37) is defined as follows:
In the case where both $z_1$ and $z_2$ are defined, the preceding equation can be summarized as follows according to the domain of the variables $l$ and the relation between $k$ and $\pi_3$:

\[
(sleaf_3(l_1 + l_2))^2 = \frac{\left\{sleaf_3(l_1)\sqrt{1 - (sleaf_3(l_2))^2} - sleaf_3(l_2)\sqrt{1 - (sleaf_3(l_1))^2}\right\}^2}{1 + 4(sleaf_3(l_1))^2(sleaf_3(l_2))^2 + 4(sleaf_3(l_1))^2(sleaf_3(l_2))^2(sleaf_3(l_1))^2}
\]

$$+ \frac{\left\{sleaf_3(l_1)^3sleaf_3(l_2) - sleaf_3(l_1)(sleaf_3(l_2))^3\right\}^2}{1 + 4(sleaf_3(l_1))^2(sleaf_3(l_2))^2 + 4(sleaf_3(l_1))^2(sleaf_3(l_2))^2(sleaf_3(l_1))^2} \tag{39}$$

Next, the addition formulas of $\text{cleaf}_3(l)$ can be defined as follows:

\[
(sleaf_3(l_1 + l_2))^2 = \frac{\left\{\text{cleaf}_3(l_1)\frac{\partial\text{cleaf}_3(l_2)}{\partial l_2} + \text{cleaf}_3(l_2)\frac{\partial\text{cleaf}_3(l_1)}{\partial l_1}\right\}^2}{1 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2(\text{cleaf}_3(l_1))^4}
\]

$$+ \frac{\left\{(sleaf_3(l_2))^3\text{cleaf}_3(l_1) - sleaf_3(l_2)(\text{cleaf}_3(l_1))^3\right\}^2}{1 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2(\text{cleaf}_3(l_1))^4} \tag{40}$$

The preceding equation can be summarized as follows according to the domain of the variables $l_1$ and $l_2$:

(i) In the case where both $2k\pi_3 \leq l_1 \leq (2k + 1)\pi_3$ and $(4m - 1)\frac{\pi_3}{2} \leq l_2 \leq (4m + 1)\frac{\pi_3}{2}$ or both $2k\pi_3 \leq l_1 \leq (2k + 2)\pi_3$ and $(4m + 1)\frac{\pi_3}{2} \leq l_2 \leq (4m + 3)\frac{\pi_3}{2}$, Eq. (40) can be defined as follows:

\[
(sleaf_3(l_1 + l_2))^2 = \frac{\left\{\text{cleaf}_3(l_1)\sqrt{1 - (sleaf_3(l_2))^2} + \text{cleaf}_3(l_2)\sqrt{1 - (sleaf_3(l_1))^2}\right\}^2}{1 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2(\text{cleaf}_3(l_1))^4}
\]

$$+ \frac{\left\{(sleaf_3(l_2))^3\text{cleaf}_3(l_1) - sleaf_3(l_2)(\text{cleaf}_3(l_1))^3\right\}^2}{1 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2(\text{cleaf}_3(l_1))^4} \tag{41}$$

(ii) In the case where both $2k\pi_3 \leq l_1 \leq (2k + 2)\pi_3$ and $(4m - 1)\frac{\pi_3}{2} \leq l_2 \leq (4m + 1)\frac{\pi_3}{2}$ or both $2k\pi_3 \leq l_1 \leq (2k + 1)\pi_3$ and $(4m + 1)\frac{\pi_3}{2} \leq l_2 \leq (4m + 3)\frac{\pi_3}{2}$, Eq. (40) can be defined as follows:

\[
(sleaf_3(l_1 + l_2))^2 = \frac{\left\{\text{cleaf}_3(l_1)\sqrt{1 - (sleaf_3(l_2))^2} + \text{cleaf}_3(l_2)\sqrt{1 - (sleaf_3(l_1))^2}\right\}^2}{1 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2(\text{cleaf}_3(l_1))^4}
\]

$$+ \frac{\left\{(sleaf_3(l_2))^3\text{cleaf}_3(l_1) - sleaf_3(l_2)(\text{cleaf}_3(l_1))^3\right\}^2}{1 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2 + 4(sleaf_3(l_2))^2(\text{cleaf}_3(l_1))^2(\text{cleaf}_3(l_1))^4} \tag{42}$$

### 2.2 Addition formulas of hyperbolic leaf function

Let the two variables be $l_1$ and $l_2$. Considering the imaginary number $i$, the relation between $\text{sleaf}_1(l)$ and $\text{sleaf}_1(l)$ and the relation between $\text{cleaf}_1(l)$ and $\text{cleaf}_1(l)$ can be obtained as follows (See Appendix D in detail):

\[
\text{sleaf}_1(l_1) = -i \cdot \text{sleaf}_1(i \cdot l)
\]

$$\text{sleaf}_1(l_1) = \text{cleaf}_1(i \cdot l) \tag{43}$$

\[
\text{cleaf}_1(l_1) = \text{cleaf}_1(i \cdot l)
\]

$$\text{cleaf}_1(l_1) = \text{cleaf}_1(i \cdot l) \tag{44}$$

In Eq. (25) and Eq. (26), the variables $l_1$ and $l_2$ are replaced with the variable $i \cdot l_1$ and $i \cdot l_2$, respectively.

\[
\text{sleaf}_1(i \cdot l_1 + i \cdot l_2) = \text{sleaf}_1(i \cdot l_1)\text{cleaf}_1(i \cdot l_2) + \text{cleaf}_1(i \cdot l_1)\text{sleaf}_1(i \cdot l_2) \tag{45}
\]

\[
\text{cleaf}_1(i \cdot l_1 + i \cdot l_2) = \text{cleaf}_1(i \cdot l_1)\text{cleaf}_1(i \cdot l_2) - \text{sleaf}_1(i \cdot l_1)\text{sleaf}_1(i \cdot l_2) \tag{46}
\]
Appendix B), the addition formulas of a hyperbolic function.

As shown in Eq. (49), in the case where \( n \) is an integer, the preceding equation has the same meaning as the addition formulas of a hyperbolic function. The following equation is obtained.

\[
s\text{eaf}_h (l_1 + l_2) = s\text{eaf}_h (l_1) c\text{eaf}_h (l_2) + c\text{eaf}_h (l_1) s\text{eaf}_h (l_2)
\]  

(47)

\[
c\text{eaf}_h (l_1 + l_2) = c\text{eaf}_h (l_1) c\text{eaf}_h (l_2) + s\text{eaf}_h (l_1) s\text{eaf}_h (l_2)
\]  

(48)

The preceding equation has the same meaning as the addition formulas of a hyperbolic function.

Next, let us consider the case of \( n = 2 \). The relation between \( s\text{eaf}_2 (l) \) and \( s\text{eaf}^{\prime}_2 (l) \), and the relation between \( c\text{eaf}_2 (l) \) and \( c\text{eaf}^{\prime}_2 (l) \) are as follows (See Appendix D): 

\[
s\text{eaf}_2 (i \cdot l) = i \cdot s\text{eaf}_2 (l)
\]  

(49)

\[
s\text{eaf}^{\prime}_2 (i \cdot l) = i \cdot s\text{eaf}^{\prime}_2 (l)
\]  

(50)

\[
c\text{eaf}_2 (i \cdot l) = c\text{eaf}_2 (l)
\]  

(51)

As shown in Eq. (49), in the case where \( n = 2 \), the function \( s\text{eaf}_2 (i \cdot l) \) and \( s\text{eaf}^{\prime}_2 (i \cdot l) \) is equal to the functions \( i \cdot s\text{eaf}_2 (l) \) and \( i \cdot s\text{eaf}^{\prime}_2 (l) \), respectively. Therefore, we cannot derive the addition formulas of \( c\text{eaf}_2 (l) \) by replacing \( i \cdot l \) with \( l \) in Eqs. (28) - (31). Using the relation between the function \( s\text{eaf}_2 (l) \) and the function \( s\text{eaf}^{\prime}_2 (l) \)(See Appendix B), the addition formulas of \( s\text{eaf}_2 (l) \) can be obtained. By substituting Eq. (49) into Eqs. (28) - (31), the following equation is obtained.

\[
s\text{eaf}^{\prime}_2 (l_1 + l_2) = 
\frac{s\text{eaf}^{\prime}_2 (l_1) \sqrt{1 + (s\text{eaf}^{\prime}_2 (l_2))^2} + s\text{eaf}_2 (l_2) \sqrt{1 + (s\text{eaf}_2 (l_1))^2}}{1 - (s\text{eaf}_2 (l_1))^2 (s\text{eaf}_2 (l_2))^2}
\]  

(52)

In the Ref. (11), the addition formulas of \( c\text{eaf}^{\prime}_2 (l) \) are obtained using Eq. (98).
The preceding equation can be summarized as follows according to the domain of the variables \( l_1 \) and \( l_2 \).

(i) In the case where both the domains \( 0 \leq l_1 \leq \eta_2 \) and \( 0 \leq l_2 \leq \eta_2 \), or both the domains \( -\eta_2 \leq l_1 \leq 0 \) and \( -\eta_2 \leq l_2 \leq 0 \) (See Appendix C for the constant \( \eta_2 \)), Eq. (53) is defined as follows:

\[
\text{cleafh}_2(l_1 + l_2) = \frac{2\text{cleafh}_2(l_1)\text{cleafh}_2(l_2) + \frac{\partial \text{cleafh}_2(l_1)}{\partial l_1} \frac{\partial \text{cleafh}_2(l_2)}{\partial l_2}}{1 + (\text{cleafh}_2(l_1))^2 + (\text{cleafh}_2(l_2))^2 - (\text{cleafh}_2(l_1))^2(\text{cleafh}_2(l_2))^2}
\]  
(53)

(ii) In the case where both the domains \( 0 \leq l_1 \leq \eta_2 \) and \( -\eta_2 \leq l_2 \leq 0 \), or both the domains \( -\eta_2 \leq l_1 \leq 0 \) and \( 0 \leq l_2 \leq \eta_2 \), Eq. (53) is defined as follows:

\[
\text{cleafh}_2(l_1 + l_2) = \frac{2\text{cleafh}_2(l_1)\text{cleafh}_2(l_2) + \sqrt{\text{cleafh}_2(l_1)^4 - 1} \sqrt{\text{cleafh}_2(l_2)^4 - 1}}{1 + (\text{cleafh}_2(l_1))^2 + (\text{cleafh}_2(l_2))^2 - (\text{cleafh}_2(l_1))^2(\text{cleafh}_2(l_2))^2}
\]  
(54)

Next, let us consider the case of \( n = 3 \). The relation between \( \text{sleaf}_3(l) \) and \( \text{cleafh}_3(l) \), and the relation between \( \text{sleaf}_3(l) \) and \( \text{cleaf}_3(l) \) are as follows (See Appendix D):

\[
\text{cleaf}_3(l) = -i \cdot \text{cleafh}_3(i \cdot l)
\]
(56)

\[
\text{sleaf}_3(l) = \text{cleafh}_3(i \cdot l)
\]
(57)

In Eq. (38) and Eq. (39), the variables \( l_1 \) and \( l_2 \) are replaced with the variables \( i \cdot l_1 \) and \( i \cdot l_2 \), respectively. The addition formulas of \( \text{cleafh}_3(l) \) are defined as follows:

\[
\begin{align*}
\text{(cleafh}_3(l_1 + l_2))^2 &= \left\{ \text{cleafh}_3(l_1) \sqrt{1 + (\text{cleafh}_3(l_2))^6} + \text{cleafh}_3(l_2) \sqrt{1 + (\text{cleafh}_3(l_1))^6} \right\}^2 \\
&\quad - \left\{ \text{cleafh}_3(l_1))^3\text{cleafh}_3(l_2) - \text{cleafh}_3(l_1)(\text{cleafh}_3(l_2))^3 \right\}^2 \\
&\quad - \left\{ \text{cleafh}_3(l_1))^3\text{cleafh}_3(l_2) - \text{cleafh}_3(l_1)(\text{cleafh}_3(l_2))^3 \right\}^2
\end{align*}
\]  
(58)

The addition formulas of \( \text{cleaf}_3(l) \) are defined as follows:

\[
\begin{align*}
\text{(cleaf}_3(l_1 + l_2))^2 &= \left\{ \text{cleaf}_3(l_1) \frac{\partial \text{cleafh}_3(l_2)}{\partial l_1} + \text{cleaf}_3(l_2) \frac{\partial \text{cleafh}_3(l_1)}{\partial l_1} \right\}^2 \\
&\quad - \left\{ \text{cleaf}_3(l_1))^3\text{cleafh}_3(l_2) + \text{cleaf}_3(l_2)(\text{cleafh}_3(l_1))^3 \right\}^2 \\
&\quad - \left\{ \text{cleaf}_3(l_1))^3\text{cleafh}_3(l_2) + \text{cleaf}_3(l_2)(\text{cleafh}_3(l_1))^3 \right\}^2
\end{align*}
\]  
(59)

The preceding equation can be summarized as follows according to the domain of the variables \( l_1 \) and \( l_2 \).
(i) In the case where both the domains $-\eta_3 \leq l_1 \leq 0$ and $-\eta_3 \leq l_2 \leq 0$ or both the domains $0 \leq l_1 \leq \eta_3$ and $0 \leq l_2 \leq \eta_3$ (See Appendix G for the constant $\eta_3$), Eq. (59) is defined as follows:

\[
(c\text{leaf}_3(l_1 + l_2))^2 = \frac{\left\{c\text{leaf}_3(l_1)\sqrt{1 + (s\text{leaf}_3(l_2))^6} + s\text{leaf}_3(l_2)\sqrt{(c\text{leaf}_3(l_1))^6 - 1}\right\}^2}{1 + 4(s\text{leaf}_3(l_2))^4(c\text{leaf}_3(l_1))^2 - 4(s\text{leaf}_3(l_2))^2(c\text{leaf}_3(l_1))^4} - \frac{\left\{(s\text{leaf}_3(l_2))^3c\text{leaf}_3(l_1) + s\text{leaf}_3(l_2)(c\text{leaf}_3(l_1))^3\right\}^2}{1 + 4(s\text{leaf}_3(l_2))^4(c\text{leaf}_3(l_1))^2 - 4(s\text{leaf}_3(l_2))^2(c\text{leaf}_3(l_1))^4} \tag{60}
\]

(ii) In the case where both the domains $-\eta_3 \leq l_1 \leq 0$ and $0 \leq l_2 \leq \eta_3$ or both the domains $0 \leq l_1 \leq \eta_3$ and $-\eta_3 \leq l_2 \leq 0$, Eq. (59) is defined as follows:

\[
(c\text{leaf}_3(l_1 + l_2))^2 = \frac{\left\{c\text{leaf}_3(l_1)\sqrt{1 + (s\text{leaf}_3(l_2))^6} - s\text{leaf}_3(l_2)\sqrt{(c\text{leaf}_3(l_1))^6 - 1}\right\}^2}{1 + 4(s\text{leaf}_3(l_2))^4(c\text{leaf}_3(l_1))^2 - 4(s\text{leaf}_3(l_2))^2(c\text{leaf}_3(l_1))^4} - \frac{\left\{(s\text{leaf}_3(l_2))^3c\text{leaf}_3(l_1) + s\text{leaf}_3(l_2)(c\text{leaf}_3(l_1))^3\right\}^2}{1 + 4(s\text{leaf}_3(l_2))^4(c\text{leaf}_3(l_1))^2 - 4(s\text{leaf}_3(l_2))^2(c\text{leaf}_3(l_1))^4} \tag{61}
\]

3 Double angle formulas and half-angle formulas

3.1 Double angle formulas of leaf function

In the case where the basis $n = 1$, the variables $l_1$ and $l_2$ in Eqs. (25) and (26) are replaced with the variable $l$, and the double angle can be defined as follows:

\[
s\text{leaf}_1(2l) = 2s\text{leaf}_1(l)c\text{leaf}_1(l)\tag{62}
\]

\[
c\text{leaf}_1(2l) = 2(c\text{leaf}_1(l))^2 - 1 = 1 - 2(s\text{leaf}_1(l))^2\tag{63}
\]

In the case where the basis $n = 2$, the variables $l_1$ and $l_2$ in Eq. (27) are replaced with the variable $l$, and the double angle can be expressed as follows:

\[
s\text{leaf}_2(2l) = \frac{2s\text{leaf}_2(l)\frac{d}{dl}s\text{leaf}_2(l)}{1 + (s\text{leaf}_2(l))^4}\tag{64}
\]

The preceding equation can be summarized as follows according to the domain of variable $l$.

(i) In the case where the domain $\frac{\pi}{2} (4m - 1) \leq l \leq \frac{\pi}{2} (4m + 1)$, Eq. (64) is defined as follows:

\[
s\text{leaf}_2(2l) = \frac{2s\text{leaf}_2(l)\sqrt{1 - (s\text{leaf}_2(l))^4}}{1 + (s\text{leaf}_2(l))^4}\tag{65}
\]

(ii) In the case where the domain $\frac{\pi}{2} (4m + 1) \leq l \leq \frac{\pi}{2} (4m + 3)$, Eq. (64) is expressed as follows:

\[
s\text{leaf}_2(2l) = \frac{-2s\text{leaf}_2(l)\sqrt{1 - (s\text{leaf}_2(l))^4}}{1 + (s\text{leaf}_2(l))^4}\tag{66}
\]

The variables $l_1$ and $l_2$ in Eq. (32) are replaced with the variable $l$. The double angle can be defined as follows:
\[
\text{cleaf}_2(2l) = \frac{1 - 2(\text{cleaf}_2(l))^2 - (\text{cleaf}_2(l))^4}{-1 - 2(\text{cleaf}_2(l))^2 + (\text{cleaf}_2(l))^4}
\]  
(67)

In the case where the basis \( n = 3 \), the variables \( l_1 \) and \( l_2 \) of Eq. (37) are replaced with the variable \( l \), and the double angle of the function \( \text{sleaf}_3(2l) \) can be expressed as follows:

\[
\text{sleaf}_3(2l) = \frac{2\text{sleaf}_3(l) \frac{\partial \text{sleaf}_3(l)}{\partial l}}{\sqrt{1 + 8(\text{sleaf}_3(l))^6}}
\]  
(68)

(i) In the case where the domain \( \frac{\pi}{2} (4m - 1) \leq l \leq \frac{\pi}{2} (4m + 1) \) (See Appendix E for the constant \( \pi_3 \), Eq. (68) is defined as follows:

\[
\text{sleaf}_3(2l) = \frac{2\text{sleaf}_3(l) \sqrt{1 - (\text{sleaf}_3(l))^6}}{\sqrt{1 + 8(\text{sleaf}_3(l))^6}}
\]  
(69)

(ii) In the case where the domain \( \frac{\pi}{2} (4m + 1) \leq l \leq \frac{\pi}{2} (4m + 3) \), Eq. (68) is defined as follows:

\[
\text{sleaf}_3(2l) = -\frac{2\text{sleaf}_3(l) \sqrt{1 - (\text{sleaf}_3(l))^6}}{\sqrt{1 + 8(\text{sleaf}_3(l))^6}}
\]  
(70)

In the case where the basis \( n = 3 \), the variable \( l_1 \) and the variable \( l_2 \) of Eq. (42) are replaced with the variable \( l \). The double angle of the function \( \text{cleaf}_3(2l) \) is then expressed as follows:

\[
\text{cleaf}_3(2l) = \frac{2(\text{cleaf}_3(l))^2 + 2(\text{cleaf}_3(l))^4 - 1}{\sqrt{1 + 8(\text{cleaf}_3(l))^2 + 8(\text{cleaf}_3(l))^6 - 8(\text{cleaf}_3(l))^8}}
\]  
(71)

### 3.2 Half-angle formulas of leaf function

In the case where the basis \( n = 1 \), the variables \( l_1 \) and \( l_2 \) in Eq. (26) are replaced with the variable \( l/2 \), and the half angle is defined as follows:

\[
\left( \text{sleaf}_1 \left( \frac{l}{2} \right) \right)^2 = \frac{1 - \text{cleaf}_1(l)}{2}
\]  
(72)

In the case where the basis \( n = 1 \), the variables \( l_1 \) and \( l_2 \) in Eq. (26) are replaced with the variable \( l/2 \), and the half angle can be defined as follows:

\[
\left( \text{cleaf}_1 \left( \frac{l}{2} \right) \right)^2 = \frac{1 + \text{cleaf}_1(l)}{2}
\]  
(73)

In the case where the basis \( n = 2 \), the variables \( l_1 \) and \( l_2 \) in Eqs. (28)-(31) are replaced with the variable \( l/2 \), and the half angle is defined as follows:

(i) In the case where the domain \( \frac{\pi}{4} (4m - 1) \leq l \leq \frac{\pi}{4} (4m + 1) \) (See Appendix E for the constant \( \pi_2 \), the half angle formulas are expressed as follows:

\[
\left( \text{sleaf}_2 \left( \frac{l}{2} \right) \right)^2 = \frac{-1 - \sqrt{1 - (\text{sleaf}_2(l))^2}}{(\text{sleaf}_2(l))^2} + \frac{\sqrt{1 + (\text{sleaf}_2(l))^2}}{(\text{sleaf}_2(l))^2} \sqrt{2 - (\text{sleaf}_2(l))^2 + 2\sqrt{1 - (\text{sleaf}_2(l))^2}}
\]  
(74)

(ii) In the case where the domain \( \frac{\pi}{4} (4m + 1) \leq l \leq \frac{\pi}{4} (4m + 3) \), the half angle formulas are defined as follows:
In the case where the domain \( \pi \leq l \leq \pi \) and the double angle is defined as follows:

\[
\left( \text{sleaf}_2 \left( \frac{l}{2} \right) \right)^2 = -\frac{1}{2} \text{sleaf}_2(l)^2 + \frac{1}{2} \sqrt{1 + (\text{sleaf}_2(l))^2 + (\text{sleaf}_2(l))^4}
\]

\[
-\frac{1}{2} \sqrt{-1 - (\text{sleaf}_2(l))^2 + 2(\text{sleaf}_2(l))^4 + \frac{2 - 2(\text{sleaf}_2(l))^6}{\sqrt{1 + (\text{sleaf}_2(l))^2 + (\text{sleaf}_2(l))^4}}} \quad (75)
\]

In the case where the basis \( n = 2 \), the variables \( l_1 \) and \( l_2 \) in Eqs. (33)-(36) are replaced with the variable \( l/2 \) and the half angle is expressed as follows:

\[
\left( \text{cleaf}_2 \left( \frac{l}{2} \right) \right)^2 = -\frac{1}{2} \text{cleaf}_2(l) + \sqrt{1 + (\text{cleaf}_2(l))^2 + (\text{cleaf}_2(l))^4}
\]

\[
\frac{1}{1 + \text{cleaf}_2(l)} \quad (76)
\]

In the case where the basis \( n = 3 \), the variables \( l_1 \) and \( l_2 \) in Eqs. (38)-(39) are replaced with the variable \( l/2 \) and the half angle of the function \( \text{sleaf}_3(l) \) is defined as follows:

(i) In the case where the domain \( \frac{\pi}{2} (4m - 1) \leq l \leq \frac{\pi}{2} (4m + 1) \) (See Appendix E for the constant \( \pi_3 \)), the half angle is defined as follows:

\[
\left( \text{sleaf}_3 \left( \frac{l}{2} \right) \right)^2 = -\frac{1}{2} \text{sleaf}_3(l)^2 + \frac{1}{2} \sqrt{1 + (\text{sleaf}_3(l))^2 + (\text{sleaf}_3(l))^4}
\]

\[
-\frac{1}{2} \sqrt{-1 - (\text{sleaf}_3(l))^2 + 2(\text{sleaf}_3(l))^4 + \frac{2 - 2(\text{sleaf}_3(l))^6}{\sqrt{1 + (\text{sleaf}_3(l))^2 + (\text{sleaf}_3(l))^4}}} \quad (77)
\]

(ii) In the case where the domain \( \frac{\pi}{2} (4m + 1) \leq l \leq \frac{\pi}{2} (4m + 3) \), the half angle is expressed as follows:

\[
\left( \text{sleaf}_3 \left( \frac{l}{2} \right) \right)^2 = -\frac{1}{2} \text{sleaf}_3(l)^2 + \frac{1}{2} \sqrt{1 + (\text{sleaf}_3(l))^2 + (\text{sleaf}_3(l))^4}
\]

\[
+\frac{1}{2} \sqrt{-1 - (\text{sleaf}_3(l))^2 + 2(\text{sleaf}_3(l))^4 + \frac{2 - 2(\text{sleaf}_3(l))^6}{\sqrt{1 + (\text{sleaf}_3(l))^2 + (\text{sleaf}_3(l))^4}}} \quad (78)
\]

In the case where the basis \( n = 3 \), the variables \( l_1 \) and \( l_2 \) in Eqs. (60)-(61) are replaced with the variable \( l/2 \) and the half angle of the function \( \text{cleaf}_3 \left( \frac{l}{2} \right) \) is defined as follows:

\[
\left( \text{cleaf}_3 \left( \frac{l}{2} \right) \right)^2 = \frac{(\text{cleaf}_3(l))^2 - 1}{4(\text{cleaf}_3(l))^2 + 2} + \frac{\sqrt{3\sqrt{1 + (\text{cleaf}_3(l))^2 + (\text{cleaf}_3(l))^4}}}{2\sqrt{1 + 4(\text{cleaf}_3(l))^2 + 4(\text{cleaf}_3(l))^4}}
\]

\[
\frac{\sqrt{3}\text{cleaf}_3(l)\sqrt{-3 - 6(\text{cleaf}_3(l))^2 + 2\sqrt{3\{1 + 2(\text{cleaf}_3(l))^2\} \sqrt{1 + (\text{cleaf}_3(l))^2 + (\text{cleaf}_3(l))^4}}}}{2\{1 + 2(\text{cleaf}_3(l))^2\}^{\frac{3}{2}}} \quad (79)
\]

### 3.3 Double angle formulas of hyperbolic leaf function

In the case where the basis \( n = 1 \), the variables \( l_1 \) and \( l_2 \) in Eqs. (47) and (48) are replaced with the variable \( l \) and the double angle can be expressed as follows:

\[
\text{sleaf}_1(2l) = 2\text{sleaf}_1(l)\text{cleaf}_1(l) \quad (80)
\]

\[
\text{cleaf}_1(2l) = 2(\text{cleaf}_1(l))^2 - 1 \quad (81)
\]

In the case where the basis \( n = 2 \), the variables \( l_1 \) and \( l_2 \) in Eq. (52) are replaced with the variable \( l \), and the double angle is defined as follows:
The variables $l_1$ and $l_2$ in Eq. (53) are replaced with the variable $l$. The double angle is then defined as follows:

$$\text{cleafh}_2(2l) = \frac{(\text{cleafh}_2(l))^4 + 2(\text{cleafh}_2(l))^2 - 1}{-(\text{cleafh}_2(l))^4 + 2(\text{cleafh}_2(l))^2 + 1}$$  \hspace{1cm} (83)$$

In the case where the basis $n = 3$, the variables $l_1$ and $l_2$ of Eq. (58) are replaced with the variable $l$, and the double angle of the function $\text{sleafh}_3(2l)$ is defined as follows:

$$\text{sleafh}_3(2l) = \frac{2\text{sleafh}_3(l) \sqrt{1 + (\text{sleafh}_3(l))^2}}{\sqrt{1 - 8(\text{sleafh}_3(l))^6}}$$  \hspace{1cm} (84)$$

In the case where the basis $n = 3$, the variables $l_1$ and $l_2$ of Eq. (59) are replaced with the variable $l$, and the double angle of the function $\text{cleafh}_3(2l)$ is defined as follows:

$$\text{cleafh}_3(2l) = \frac{2(\text{cleafh}_3(l))^2 + 2(\text{cleafh}_3(l))^4 - 1}{\sqrt{1 + 8(\text{cleafh}_3(l))^2 + 8(\text{cleafh}_3(l))^6} - 8(\text{cleafh}_3(l))^8}$$  \hspace{1cm} (85)$$

### 3.4 Half-angle formulas of leaf function of hyperbolic leaf function

In the case where the basis $n = 1$, the variables $l_1$ and $l_2$ in Eq. (48) are replaced with the variable $l/2$, and the half angle is defined as follows:

$$\left(\text{sleafh}_1\left(\frac{l}{2}\right)\right)^2 = \frac{\text{cleafh}_1(l) - 1}{2}$$  \hspace{1cm} (86)$$

$$\left(\text{cleafh}_1\left(\frac{l}{2}\right)\right)^2 = \frac{1 + \text{cleafh}_1(l)}{2}$$  \hspace{1cm} (87)$$

In the case where the basis $n = 2$, the variables $l_1$ and $l_2$ in Eq. (52) are replaced with the variable $l/2$, and the half angle is defined as follows:

(i) In the case where the domain $|l| \leq |\zeta_2|$ (See Appendix F for the constant $\zeta_2$) (See Appendix H for the periodicity $n = 2$), the half angle formulas are expressed as follows:

$$\left(\text{sleafh}_2\left(\frac{l}{2}\right)\right)^2 = \frac{1 + \sqrt{1 + (\text{sleafh}_2(l))^2}}{(\text{sleafh}_2(l))^2} - \frac{\sqrt{2}}{\sqrt{1 + (\text{sleafh}_2(l))^2}}$$  \hspace{1cm} (88)$$

(ii) In the case where the domain $|\zeta_2| \leq |l|$, the half angle formulas are defined as follows:

$$\left(\text{sleafh}_2\left(\frac{l}{2}\right)\right)^2 = \frac{1 + \sqrt{1 + (\text{sleafh}_2(l))^2}}{(\text{sleafh}_2(l))^2} + \frac{\sqrt{2}}{\sqrt{1 + (\text{sleafh}_2(l))^2}}$$  \hspace{1cm} (89)$$

In the case where the basis $n = 2$, the variables $l_1$ and $l_2$ in Eq. (53) are replaced with the variable $l/2$, and the half angle can be expressed as follows(See Appendix F for the constant $\zeta_2$) (See Appendix H and the periodicity $n = 2$):

(i) In the case where the domain $|l| \leq |\eta_2|$, the half angle formulas are defined as follows:

$$\left(\text{cleafh}_2\left(\frac{l}{2}\right)\right)^2 = -\frac{1 + \text{cleafh}_2(l) + \sqrt{2}\sqrt{1 + (\text{cleafh}_2(l))^2}}{1 + \text{cleafh}_2(l)}$$  \hspace{1cm} (90)$$

(ii) In the case where the domain $|\eta_2| \leq |l|$, the half angle formulas are defined as follows:
(\text{cleaf}_h(\frac{l}{2}))^2 = -1 + \text{cleaf}_h(l) - \sqrt{2 \sqrt{1 + (\text{cleaf}_h(l))^2}} \over 1 + \text{cleaf}_h(l) \quad (91)

In the case where the basis \( n = 3 \), the variables \( l_1 \) and \( l_2 \) in Eq. (58) are replaced with the variable \( l/2 \) and the half angle of the function \( \text{sleaf}_h(l) \) is defined as follows:

\[
\left( \text{sleaf}_h \left( \frac{l}{2} \right) \right)^2 = -\frac{1}{2} (\text{sleaf}_h(l))^2 - \frac{1}{2} \sqrt{1 - (\text{sleaf}_h(l))^2} + (\text{sleaf}_h(l))^4 + \frac{2 + 2 (\text{sleaf}_h(l))^6}{\sqrt{1 - (\text{sleaf}_h(l))^2} + (\text{sleaf}_h(l))^4} \quad (92)
\]

In the case where the basis \( n = 3 \), the variables \( l_1 \) and \( l_2 \) in Eq. (59) are replaced with the variable \( l/2 \) and the half angle of the function \( \text{cleaf}_h(l) \) is defined as follows:

\[
(\text{cleaf}_h(\frac{l}{2}))^2 = -1 + \text{cleaf}_h(l) + \sqrt{3} \sqrt{1 + \text{cleaf}_h(l)} - (\text{cleaf}_h(l))^4 \quad (93)
\]

\[
\frac{2}{2 + 3(1 + 2(\text{cleaf}_h(l))^2)} \sqrt{1 + (\text{cleaf}_h(l))^2} \quad (93)
\]

4 Numerical analysis

4.1 Numerical analysis of leaf function

In Eq. (1), the graph of the leaf function at the basis \( n = 1 \) is shown in Fig. 2 and Fig. 3. Curves of the functions \( \text{sleaf}_1(l) \) and \( \text{cleaf}_1(l) \) are shown in Fig. 1 and Fig. 2. The horizontal and the vertical axes represent the variables \( l \) and \( r \), respectively. Numerical data for the leaf functions \( \text{sleaf}_1(l) \) and \( \text{cleaf}_1(l) \) are summarized in Table 1. These data are obtained using Eq. (2) and Eq. (3). The curves are the same as those of the trigonometric functions \( r = \sin(l) \) and \( r = \cos(l) \). Using the addition formulas of Eq. (25) and Eq. (26), the curves of the leaf functions \( \text{sleaf}_1(l) \) and \( \text{cleaf}_1(l) \) are translated in the direction of the axis \( l \). Fig. 4 shows graphs of the double angle \( \text{sleaf}_1(2l) \) and the half angle \( \text{sleaf}_1(l/2) \). These data are obtained based on Eq. (62) and Eq. (72). Fig. 5 shows curves for the double angle \( \text{cleaf}_1(2l) \) and the half angle \( \text{cleaf}_1(l/2) \). These data are obtained based on Eq. (63) and (73), respectively. The amplitude of the wave is 1 where one period is \( 2\pi_1(= \pi = 2 \times 3.1415 \cdots) \).

Next, the graph of the leaf function of the basis \( n = 2 \) in Eq. (7) is shown. The curves of the leaf functions \( \text{sleaf}_2(l) \) and \( \text{cleaf}_2(l) \) are shown in Figs. 3 and 4. Numerical data for these two leaf functions are summarized in Table 1. These curves are the same as those of the lemniscate elliptic functions \( r = \text{sl}(l) \) and \( r = \text{cl}(l) \). Using the addition formulas of Eq. (1), the curves of the leaf functions \( \text{sleaf}_2(l) \) and \( \text{cleaf}_2(l) \) are translated in the direction of the axis \( l \). Fig. 8 shows graphs of the double angle \( \text{sleaf}_2(2l) \) and the half angle \( \text{sleaf}_2(l/2) \) obtained using Eqs. (65) and (74)-(75). Fig. 9 shows graphs of the double angle \( \text{cleaf}_2(2l) \) and the half angle \( \text{cleaf}_2(l/2) \) obtained using Eqs. (67) and (76). The amplitude of the wave is 1 and one period of the function \( \text{cleaf}_2(l) \) is \( 2\pi_2(= 2 \times 2.622 \cdots) \). Next, the graph of the leaf function of the basis \( n = 3 \) in Eq. (1) is shown. The curves of the leaf functions \( \text{sleaf}_3(l) \) and \( \text{cleaf}_3(l) \) are shown in Figs. 10 and 11. The horizontal and vertical axes represent the variables \( l \) and \( r \), respectively. The numerical data of the leaf functions \( \text{sleaf}_3(l) \) and \( \text{cleaf}_3(l) \) are summarized in Table 1. Curves of the leaf functions \( \text{sleaf}_3(l) \) and \( \text{cleaf}_3(l) \) are translated in the direction of the axis \( l \). These curves of the leaf functions are obtained using the addition formulas of Eqs. (38) - (39) and the Eqs. (41) - (42). Fig. 12 shows graphs of the double angle \( \text{sleaf}_3(2l) \) and the half angle \( \text{sleaf}_3(l/2) \) obtained using Eqs. (68) and (77) - (78). Fig. 13 shows graphs of the double angle \( \text{cleaf}_3(2l) \) and the half angle \( \text{cleaf}_3(l/2) \) obtained using Eq. (71) and (79). The amplitude of the wave is 1 and one period of the function \( \text{cleaf}_3(l) \) is \( 2\pi_3(= 2 \times 2.429 \cdots) \).

4.2 Numerical analysis of hyperbolic leaf function

The graph of the hyperbolic function obtained using Eq. (4) is shown. Curves of the leaf functions \( \text{sleaf}_h(l) \) and \( \text{cleaf}_h(l) \) are shown in Figs. 14 and 15. The horizontal and vertical axes represent the variables \( l \) and \( r \), respectively. The numerical data of the leaf functions \( \text{sleaf}_h(l) \) and \( \text{cleaf}_h(l) \) obtained using Eq. (4) are summarized in Table 1.
The relation equation between the leaf function and the hyperbolic leaf function are derived using imaginary numbers. These curves are the same curves as those of the hyperbolic functions \( r = \sinh(l) \) and \( r = \cosh(l) \). The curves of the leaf functions \( \text{sleaf}_1(l) \) and \( \text{cleaf}_1(l) \) are translated in the direction of the axis \( l \). These data are obtained using the addition formulas of Eqs. (47) and (48). Fig. 16 shows graphs of the double angle \( \text{sleaf}_1(2l) \) and the half angle \( \text{sleaf}_1(l/2) \) obtained using Eqs. (80) and (86). Fig. 17 shows the graph of the double angle \( \text{cleaf}_1(2l) \) and the half angle \( \text{cleaf}_1(l/2) \) of the leaf function \( \text{cleaf}_1(l) \) obtained using Eqs. (81) and (87).

Next, the curves of the leaf functions \( \text{sleaf}_2(l) \) and \( \text{cleaf}_2(l) \) are shown in Figs. 18 and 19. The horizontal and vertical axes represent the variables \( l \) and \( r \). The numerical data for the leaf functions \( \text{sleaf}_2(l) \) and \( \text{cleaf}_2(l) \) obtained using Eqs. (5) and (6) are summarized in Table 2. Using the addition formulas of Eq. (52) and the Eq. (53), the curves of the leaf functions \( \text{sleaf}_2(l) \) and \( \text{cleaf}_2(l) \) are translated in the direction \( l \). Fig. 20 shows graphs of the double angle \( \text{sleaf}_2(2l) \) and the half angle \( \text{sleaf}_2(l/2) \) obtained using Eqs. (82) and (88). Fig. 21 shows graphs of the double angle \( \text{cleaf}_2(2l) \) and the half angle \( \text{cleaf}_2(l/2) \) obtained using Eqs. (83) and (90). Limits exist for the functions \( \text{sleaf}_2(l) \) and \( \text{cleaf}_2(l) \), respectively. (See Appendix I and Appendix G).

Next, the graph of the hyperbolic function at the basis \( n = 3 \) in Eq. (4) is shown. Curves of the leaf functions \( \text{sleaf}_3(l) \) and \( \text{cleaf}_3(l) \) are shown in Figs. 22 and 23. The horizontal and vertical axes represent the variables \( l \) and \( r \), respectively. The numerical data of the leaf functions \( \text{sleaf}_3(l) \) and \( \text{cleaf}_3(l) \) are summarized in Table 2. Using the addition formulas of Eq. (58) and the Eq. (59), the curves of the leaf functions \( \text{sleaf}_3(l) \) and \( \text{cleaf}_3(l) \) are translated in the direction \( l \). Fig. 24 shows graphs of the double angle \( \text{sleaf}_3(2l) \) and the half angle \( \text{sleaf}_3(l/2) \) obtained using Eq. (82) and Eq. (88). Fig. 24 shows graphs of the double angle \( \text{cleaf}_3(2l) \) and the half angle \( \text{cleaf}_3(l/2) \) obtained using Eqs. (83) and (93). Limits exist in the functions \( \text{sleaf}_3(l) \) and \( \text{cleaf}_3(l) \), respectively. For the function \( \text{sleaf}_3(l) \), the limit exists at \( \pm \zeta_3 \) (See Appendix F for the constant \( \zeta_3 \)). The curve of the function \( \text{sleaf}_3(l) \) monotonically increases in the domain \( -\zeta_3 < l < \zeta_3 \). In the case of the function \( \text{cleaf}_3(l) \), the limit exists at \( \pm \eta_3 \) (See Appendix G for the constant \( \eta_3 \)). The domain of the function \( \text{cleaf}_3(l) \) is \( -\eta_3 < l < \eta_3 \).

5 Conclusion

Based on the analogy between the trigonometric and hyperbolic function, the hyperbolic leaf function paired with the leaf function was defined. The main conclusions can be summarized as follows:

- The relation equation between the leaf function and the hyperbolic leaf function are derived using imaginary numbers.
The addition formulas of the hyperbolic leaf function can be derived by using addition formulas of the leaf function with the basis $n = 1, 2, 3$.

In both the leaf function and hyperbolic leaf function based on the basis $n = 1, 2, 3$, half angle and double angle formulas are derived using addition formulas.

As a future research topic, we will investigate whether the periodicity of the hyperbolic leaf function exists. In the case where the basis is $n = 2$, a limit exists in hyperbolic function. By appropriately setting the initial conditions, the addition formulas $n = 2$ can be applied in all domains over the limit. Although the periodicity of the hyperbolic leaf function $n = 2$ is evident, questions remain concerning the periodicity of the hyperbolic leaf function $n = 3$. In the case where the basis is $n = 3$, a limit also exists for the hyperbolic leaf function. However, the addition formulas of the hyperbolic leaf function cannot be applied outside of its domain. At basis $n = 3$, the periodicity of the hyperbolic leaf function is not observed. Another unaddressed issue is that the addition formulas of the leaf function based on the basis $n = 4$ or more are not known.

### Appendix A

The relation equations based on the basis $n = 1$ are described. The relation equation between the leaf function $sleaf_1(l)$ and the leaf function $cleaf_1(l)$ is as follows:

$$ (sleaf_1(l))^2 + (cleaf_1(l))^2 = 1 $$

(94)

The relation equation between the hyperbolic leaf function $sleahf_1(l)$ and the hyperbolic leaf function $cleahf_1(l)$ is as follows:

$$ (cleahf_1(l))^2 - (sleahf_1(l))^2 = 1 $$

(95)

### Appendix B

The relation equations based on the basis $n = 2$ are described. The relation equation between the leaf function $sleaf_2(l)$ and the leaf function $cleaf_2(l)$ is as follows [2]:

Figure 3: Translation of the curves of the function $cleaf_1(l)$ obtained using the addition formulas based on the basis $n = 1$.
Table 1: Numerical data of the leaf function

| $l$ | sleaf1($l$) | cleaf1($l$) | sleaf2($l$) | cleaf2($l$) | sleaf3($l$) | cleaf3($l$) |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| 0   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.1 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.2 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.3 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.4 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.5 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.6 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.7 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.8 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.9 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |

Table 2: Numerical data of the hyperbolic leaf function

| $l$ | sleaf1h($l$) | cleaf1h($l$) | sleaf2h($l$) | cleaf2h($l$) | sleaf3h($l$) | cleaf3h($l$) |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| 0.0 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.1 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.2 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.3 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.4 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.5 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.6 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.7 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.8 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 0.9 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 1.0 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 1.1 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 1.2 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 1.3 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |
| 1.4 | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   | 0.0000000   | 1.0000000   |

...
Figure 4: Curves of the function $sleaf_1(l)$, $sleaf_1(2l)$ and $sleaf_1(l/2)$ based on the basis $n = 1$

| $n$ | $\pi_n$       |
|-----|-------------|
| 1   | 3.1415926535 ... |
| 2   | 2.6220575542 ... |
| 3   | 2.4286506478 ... |
| ... | ...          |

$(sleaf_2(l))^2 + (cleaf_2(l))^2 + (sleaf_2(l))^2 \cdot (cleaf_2(l))^2 = 1$  \hspace{1cm} (96)

The relation equation between the hyperbolic leaf function $sleaf_{h2}(l)$ and the hyperbolic leaf function $cleaf_{h2}(l)$ is as follows \cite{12} \cite{11}:

$$cleaf_{h2}(\sqrt{2}l) = \frac{1 + (sleaf_{h2}(l))^2}{1 - (sleaf_{h2}(l))^2}$$  \hspace{1cm} (97)

The relation equation between the hyperbolic leaf function $cleaf_2(l)$ and the hyperbolic leaf function $cleaf_{h2}(l)$ is as follows:

Table 4: Limits $\zeta_n$ of the hyperbolic leaf function $sleaf_{h2}(l)$

| $n$ | $\zeta_n$       |
|-----|-------------|
| 1   | Not applicable ... |
| 2   | 1.8540746773 ... |
| 3   | 1.4021821053 ... |
| ... | ...          |
Figure 5: Curves of the functions \( \text{cleaf}_1(l) \), \( \text{cleaf}_1(2l) \) and \( \text{cleaf}_1(l/2) \) based on the basis \( n = 1 \)

Table 5: Limits \( \eta_n \) of the hyperbolic leaf function \( \text{cleaf}_n(l) \)

| \( n \) | \( \eta_n \)         |
|--------|----------------------|
| 1      | Not applicable ⋮     |
| 2      | 1.3110287771 ⋮       |
| 3      | 0.7010910526 ⋮       |
| ⋮      | ⋮                    |

\[ \text{cleaf}_2(l) \cdot \text{cleaf}_2(l) = 1 \] (98)

The relation equation between the hyperbolic leaf function \( \text{sleaf}_2(l) \) and the hyperbolic leaf function \( \text{sleaf}_2(l) \) is as follows:

\[ (\text{sleaf}_2(\sqrt{2}l))^2 = \frac{2(\text{sleaf}_2(l))^2}{1 + (\text{sleaf}_2(l))^4} \] (99)

Appendix C

The relation equations based on the basis \( n = 3 \) are described. The relation equation between the leaf function \( \text{sleaf}_3(l) \) and the leaf function \( \text{cleaf}_3(l) \) is as follows [2]:

\[ (\text{sleaf}_3(l))^2 + (\text{cleaf}_3(l))^2 + 2(\text{sleaf}_3(l))^2 \cdot (\text{cleaf}_3(l))^2 = 1 \] (100)

The relation equation between the hyperbolic leaf function \( \text{sleaf}_3(l) \) and the hyperbolic leaf function \( \text{cleaf}_3(l) \) is as follows [12] [11]:

\[ (\text{cleaf}_3(l))^2 - (\text{sleaf}_3(l))^2 - 2(\text{sleaf}_3(l))^2 \cdot (\text{cleaf}_3(l))^2 = 1 \] (101)
Appendix D

Using the imaginary number, the relations between the leaf function and hyperbolic leaf function are described in section [12] [11]. To derive the relation between these two functions, the following equation is defined.

\[ r = i \cdot u \quad (102) \]

The symbol \( i \) represents the imaginary number. Substituting the preceding equation into Eq. (2) yields the following equation:

\[ l = \int_{0}^{u} \frac{dt}{\sqrt{1 - (i \cdot \xi)^{2n}}} = \text{arcsl}_{n}(i \cdot u) \quad (103) \]

Here, the parameter \( t \) is replaced with \( i \cdot \xi \) \((t = i \cdot \xi)\). In the case where \( t = 0 \), \( \xi \) is zero. In the case where \( t = i \cdot u \), \( \xi \) is \( u \). Thus, the following equation is obtained:

\[ l = \int_{0}^{u} \frac{i \cdot d\xi}{\sqrt{1 - (i \cdot \xi)^{2n}}} = i \cdot \int_{0}^{u} \frac{d\xi}{\sqrt{1 - i^{2n} \cdot \xi^{2n}}} \quad (104) \]

Let \( n \) be an odd number, that is, \( n = 2m - 1 (m = 1, 2, 3, \cdots) \). The following equation is then obtained,

\[ l = i \cdot \int_{0}^{u} \frac{d\xi}{\sqrt{1 - i^{2n} \cdot \xi^{2n}}} = i \cdot \int_{0}^{u} \frac{d\xi}{\sqrt{1 + \xi^{2n}}} = i \cdot \text{asleaf}_{n}(u) \quad (105) \]

The following equation is obtained based on the preceding equation as follows:

\[ \text{sleaf}_{n} \left( \frac{1}{l} \right) = u \quad (106) \]

\[ \text{sleaf}_{n}(-i \cdot l) = u \quad (107) \]
Figure 7: Translation of the curves of the function $\text{cleaf}_2(l)$ obtained using the addition formulas based on the basis $n = 2$

Here, the leaf function $\text{sleaf}_n(l)$ has the following relation [12]:

$$\text{sleaf}_n(-l) = -\text{sleaf}_n(l) \quad (108)$$

Eq. (106) can be defined as follows:

$$-\text{sleaf}_n(i \cdot l) = u \quad (109)$$

The following equation is obtained using Eq. (102) and Eq. (109).

$$\text{sleaf}_n(l) = -i \cdot \text{sleaf}_n(i \cdot l) \quad (110)$$

Next, let us consider the case where $n$ is an even number. In the case where $n = 2m (m = 1, 2, 3 \cdots)$, the following equation is obtained:

$$l = i \cdot \int_0^u \frac{d\xi}{\sqrt{1 - i^{2n} \cdot \xi^{2n}}} = i \cdot \int_0^u \frac{d\xi}{\sqrt{1 - \xi^{2n}}} = i \cdot \text{arcsleaf}_n(u) \quad (111)$$

The following equation is obtained:

$$\text{sleaf}_n\left(\frac{l}{i}\right) = u \quad (112)$$

$$\text{sleaf}_n(-i \cdot l) = u \quad (113)$$

Here, the leaf function $\text{sleaf}_n(l)$ has the following relation [11]:

$$\text{sleaf}_n(-l) = -\text{sleaf}_n(l) \quad (114)$$

Eq. (113) can be expressed as follows:
Figure 8: Translation of the curves of the functions $sleaf_2(l)$, $sleaf_2(2l)$ and $sleaf_2(l/2)$ obtained using the addition formulas based on the basis $n = 2$

\[-sleaf_n(i \cdot l) = u\]  \hspace{1cm} (115)

The following equation is obtained using Eq. (102) and Eq. (115):

$$sleaf_n(l) = -i \cdot sleaf_n(i \cdot l)$$  \hspace{1cm} (116)

In the case where $n$ is an even number, the following equation is also derived:

$$sleafh_n(l) = -i \cdot sleafh_n(i \cdot l)$$  \hspace{1cm} (117)

Next, let us consider Eq. (3). This equation can be transformed as follows:

$$l = \int_1^r \frac{dt}{\sqrt{t^{2n} - 1}} = \int_1^r \frac{dt}{i \sqrt{1 - t^{2n}}} = \frac{1}{i} \cdot \int_1^r \frac{dt}{\sqrt{1 - t^{2n}}} = \frac{1}{i} \arccleaf_n(r)$$  \hspace{1cm} (118)

The following equation is also obtained:

$$r = cleaf_n(i \cdot l)$$  \hspace{1cm} (119)

The following equation is obtained using Eq. (17):

$$r = cleafh_n(l)$$  \hspace{1cm} (120)

The following equation is obtained using Eq. (119) and Eq. (120):

$$cleaf_n(i \cdot l) = cleafh_n(l)$$  \hspace{1cm} (121)

Alternatively, the following equation is obtained by substituting $i \cdot l$ into $l$:  

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Figure 9: Translation of the curves of the functions $\text{cleaf}_2(l)$, $\text{cleaf}_2(2l)$ and $\text{cleaf}_2(l/2)$ obtained using the addition formulas based on the basis $n = 2$

\[ \text{cleaf}_n(l) = \text{cleaf}_n(i \cdot l) \]  

(122)

In the preceding equation, the following equation is applied:

\[ \text{cleaf}_n(l) = \text{cleaf}_n(-l) \]  

(123)

**Appendix E**

The constants $\pi_n$ are defined as follows [1] [2]:

\[ \pi_n = 2 \int_0^1 \frac{1}{\sqrt{1-t^2}} dt (n = 1, 2, 3 \cdots) \]  

(124)

In the case where $n = 1$, the constant $\pi_1$ represents the circular constant $\pi$. The constants $\pi_n (n = 1, 2, 3 \cdots)$ are summarized in table 3.

**Appendix F**

Except for the basis $n = 1$, the limit of the variable $l$ exists in the hyperbolic leaf function $\text{sleafh}_n(l)$ [12]. The limit based on the basis $n$ is defined as $\zeta_n$. The limit $\zeta_n$ is obtained by the following equation:

\[ \zeta_n = \int_0^\infty \frac{1}{\sqrt{1+t^{2n}}} dt (n = 2, 3 \cdots) \]  

(125)

The constants $\zeta_n (n = 2, 3 \cdots)$ are summarized in table 4.
Figure 10: Translation of the curves of the function $sleaf_3(l)$ obtained using the addition formulas based on the basis $n = 3$

Appendix G

Except for the basis $n = 1$, the limit of the variable $l$ exists in the hyperbolic leaf function $cleaf_n(l)$. The limit based on the basis $n$ is defined as $\eta_n$. The limit $\eta_n$ is obtained using the following equation:

$$\eta_n = \int_1^\infty \frac{1}{\sqrt{t^2n - 1}} \, dt (n = 2, 3, \ldots) \quad (126)$$

The constants $\eta_n (n = 2, 3, \ldots)$ are summarized in Table 5.

Appendix H

The function $sleaf_n(l)$ and $cleaf_n(l)$ have limits. The domains of the variable $l$ are defined as Eq. 8 and Eq. 12, respectively. Therefore, the values of hyperbolic leaf function cannot be defined under the domain $|l| > |\zeta_n|$ in the function $sleaf_n(l)$ or $|l| > |\eta_n|$ in the function $cleaf_n(l)$. In the case where $n = 1$, the limits do not exist in the hyperbolic leaf function as $sleaf_1(l)$ and $cleaf_1(l)$ represent sinh($l$) and cosh($l$), respectively. In the case where $n = 2$ ($sleaf_2(l)$ and $cleaf_2(l)$), the initial values of the variables $r(0)$ and $dr(0)/dt$ are defined by Eqs. 9 and 10, or the Eq. 13 and 14. The initial values in the function $sleaf_2(l)$ are redefined as follows:

$$r(2m\zeta_2) = sleaf_2(2m\zeta_2) = 0 \quad (127)$$

$$\frac{dr(2m\zeta_2)}{dl} = \frac{d}{dl} sleaf_2(2m\zeta_2) = 1 \quad (128)$$

The initial values of the function $cleaf_2(l)$ are redefined as follows:

$$r(4m\eta_2) = cleaf_2(4m\eta_2) = 1 \quad (129)$$

$$r((4m - 2)\eta_2) = cleaf_2((4m - 2)\eta_2) = -1 \quad (130)$$
The variable $m$ represents an integer. The graph based on these definitions is shown in Fig. 25($\text{sleaf}_h(2m\zeta_3)$) and Fig. 26($\text{cleaf}_h(2m\zeta_3)$), respectively. Such definitions are consistent for all the formulas such as the addition, double angle, and half angle formulas. These formulas work under all domains. If the case $n = 2$, the hyperbolic leaf functions can be extended for all domains. In the case where $n = 3$ in the hyperbolic leaf function, the addition, double angle, and half angle formulas do not work in the domain $l > |\zeta_2|$ of $l > |\eta_2|$, even if the initial conditions are defined by equations such as $r(2m\zeta_3) = \text{sleaf}_h(2m\zeta_3) = 0$ etc. In the case where $n \geq 3$, the values of $\text{sleaf}_h(l)$ and $\text{cleaf}_h(l)$ are unknown for the domain $l > |\zeta_n|$ of $l > |\eta_n|$.

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Figure 13: Translation of the curves of the functions $\text{cleaf}_3(l)$, $\text{cleaf}_3(2l)$ and $\text{cleaf}_3(l/2)$ obtained using the addition formulas based on the basis $n = 3$.

Figure 14: Translation of the curves of the function $\text{sleaf}_1(l)$ obtained using the addition formulas based on the basis $n = 1$. 

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Figure 15: Translation of the curves of the function \( \text{cleafh}_1(l) \) obtained using the addition formulas based on the basis \( n = 1 \)

Figure 16: Translation of the curves of the functions \( \text{sleafh}_1(l) \), \( \text{sleafh}_1(2l) \) and \( \text{sleafh}_1(l/2) \) obtained using the addition formulas based on the basis \( n = 1 \)
Figure 17: Translation of the curves of the functions $c_{\text{leaf}h_1}(l)$, $c_{\text{leaf}h_1}(l)$ and $c_{\text{leaf}h_1}(l/2)$ obtained using the addition formulas based on the basis $n = 1$.

Figure 18: Translation of the curves of the function $s_{\text{leaf}h_2}(l)$ obtained using the addition formulas based on the basis $n = 2$. 
Figure 19: Translation of the curves of the function $\text{cleaf}_2(l)$ obtained using the addition formulas based on the basis $n = 2$

Figure 20: Translation of the curves of the functions $\text{sleaf}_2(l)$, $\text{sleaf}_2(2l)$ and $\text{sleaf}_2(l/2)$ obtained using the addition formulas based on the basis $n = 2$
Figure 21: Translation of the curves of the functions cleafh₂(l), cleafh₂(2l) and cleafh₂(l/2) obtained using the addition formulas based on the basis \( n = 2 \)

Figure 22: Translation of the curves of the function sleafh₃(l) obtained using the addition formulas based on the basis \( n = 3 \)
Figure 23: Translation of the curves of the functions $s_{\text{leaf}3}(l)$ obtained using the addition formulas based on the basis $n = 3$.

Figure 24: Translation of the curves of the functions $s_{\text{leaf}3}(l)$, $s_{\text{leaf}3}(2l)$ and $s_{\text{leaf}3}(l/2)$ obtained using the addition formulas based on the basis $n = 3$. 
Figure 25: Translation of the curves of the functions $\text{cleafh}_3(l)$, $\text{cleafh}_3(2l)$ and $\text{cleafh}_3(l/2)$ obtained using the addition formulas based on the basis $n = 3$.

Figure 26: Curves of the extended hyperbolic leaf function $\text{sleafh}_2(l)$ for the initial conditions: Eq. (127) and Eq. (128).
Figure 27: Curves of the extended hyperbolic leaf function $c_{\text{leaf2}}(l)$ for the initial conditions: Eq. (129), Eq. (130), and Eq. (131).