Towards a closed differential aging formula in special relativity

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Abstract.
It is well known that the Lorentzian length of a timelike curve in Minkowski spacetime is smaller than the Lorentzian length of the geodesic connecting its initial and final endpoints. The difference is known as the differential aging and its calculation in terms of the proper acceleration history of the timelike curve would provide an important tool for the autonomous spacetime navigation of non-inertial observers. I give a solution in 3+1 dimensions which holds whenever the acceleration is decomposed with respect to a lightlike transported frame (lightlike transport will be defined), the analogous and more natural problem for a Fermi-Walker decomposition being still open.

1. Introduction
In this work I consider the problem of determining the spacetime position, with respect to an inertial frame $K$, of an arbitrarily moving observer $O$ in Minkowski spacetime $M$. The restriction to Minkowski spacetime means that we are considering travellers $O$ which move on a region of spacetime which can be considered approximately flat.

Usually this problem is solved by assuming to be known the velocity of the non-inertial observer $O$, $v(t)$, with respect to the inertial frame time $t$. In this way also his position $x(t)$ is known and hence his spacetime position $(t, x(t))$ parametrized with respect to $t$. The non-inertial observer asking his position at proper time $\tau$ may obtain it by direct communication with the inertial observer. Indeed, through the inversion of the relation

$$\tau(t) = \tau_0 + \int_{\tau_0}^{\tau} \sqrt{1 - v^2(t)} \, dt,$$

the inertial observer can obtain his spacetime position as a function of proper time.

However, this procedure has some important drawbacks. For instance, since the needed observables ($v(t)$) are measured by the inertial observer, it requires a communication between the inertial and the non-inertial observers which becomes more and more unlikely as the distance between them increases.

Fortunately, the non-inertial observer may obtain the spacetime position without referring to external observers through measures performed only in the local comoving laboratory. The idea is to use the acceleration history of the observer and to reconstruct the timelike trajectory on spacetime from that information.
Thus, imagine the non-inertial observer transporting an accelerometer and $n$ mutually orthogonal gyroscopes which together with the covariant velocity define a Fermi-Walker transported frame (the spacetime has $n+1$ dimensions). Through the accelerometer he can measure his own acceleration in intensity and direction, in particular he can read the components $\tilde{a}^i(\tau)$, $i = 1, \ldots, n$, with respect to the directions defined by the gyroscopes. The problem is that of recovering the spacetime position $x^\mu(\tau)$ from the historical proper acceleration data $\tilde{a}^i(\tau)$. Since the inertial frame is not fixed by the data $\{\tilde{a}^i(\tau)\}$ the answer will be unique only up to a Poincaré transformation. In particular we may fix the origin of the inertial frame so that it coincides with the event $\tau = 0$ on $O$'s curve $x(\tau)$, and the final answer $x^\mu(\tau)$ will be determined only up to a Lorentz transformation.

In this work the timelike convention is used, $\eta_{00} = 1$, and units are such that $c = 1$. In boldface we denote $n$-vectors and the scalar product between boldfaced vectors is the usual Euclidean one.

![Figure 1](image.png)

**Figure 1.** Fermi-Walker transport in 2+1 dimensional Minkowski spacetime. The problem is to determine the spacetime position in terms of the components of the proper acceleration. A first step is to calculate $T^2(\tau) = x^\mu(\tau)\eta_{\mu\nu}x^\nu(\tau)$.

### 2. The differential aging

Fix a value of $\tau > 0$ and consider the timelike geodesic $\eta$ connecting $x(0)$ with $x(\tau)$. It can be interpreted as the worldline of an inertial observer $\bar{O}$ which leaves $O$ at time $\bar{\tau} = \tau = 0$ and meets again $O$ at time $\tau = T(\tau)$ where $T$ is given by $T^2(\tau) = x^\mu(\tau)\eta_{\mu\nu}x^\nu(\tau)$. In the usual twin examples, $O$ and $\bar{O}$ are twins and $\Delta(\tau) = T(\tau) - \tau > 0$ gives the difference in aging at the second meeting event $x(\tau)$.

The calculation of $T(\tau)$ and hence of the differential aging in terms of the acceleration would represent an important first step in the solution of the navigation problem. Moreover, it would provide a sort of inertial clock $T(\tau)$ which would give to the non-inertial observer the information about his difference in age with respect to an imaginary twin, all this without any use of the concept of distant simultaneity (see the discussion in [2]). In particular, by varying $\tau$ we are actually considering the differences in age with respect to a 1-parameter family of imaginary free falling twins. This problem has been solved at least in 1+1 dimensions [2], the solution being given by

$$T^2(\tau) = \left[\int_0^\tau e^{\int_0^\sigma \tilde{a}(\tau') d\tau'} d\tau\right] \left[\int_0^\tau e^{-\int_0^\sigma \tilde{a}(\tau') d\tau'} d\tau\right].$$  \hspace{1cm} (2)

where $\tilde{a}$ is the only component of the acceleration as measured by a coming accelerometer. In this work I will obtain a generalization to $n+1$ dimensions.
3. The navigation problem
Let us find the equations of spacetime navigation in terms of the acceleration.

The definition of covariant \((n+1)\)-velocity and covariant \((n+1)\)-acceleration is:

\[
\frac{dx^\mu}{d\tau} = u^\mu, \quad \frac{du^\mu}{d\tau} = a^\mu, \tag{3}
\]

where to be rigorous the velocity and acceleration are the vectors \(u^\mu \partial/\partial x^\mu\), \(a^\mu \partial/\partial x^\mu\), i.e. the above \(a^\mu\), \(u^\mu\), are just the components with respect to the tetrad \(\partial/\partial x^\mu\) of the inertial observer at rest in \(K\). The non-inertial observer \(O\) has a different tetrad \(e^\mu_\nu\) which we can identify with its 4-velocity \(e^0 = u^\mu \partial/\partial x^\mu\), and the directions of \(n\) gyroscopes \(e_i\). These last directions can be decomposed with respect to \(\partial/\partial x^\mu\)

\[
e_i = e^\mu_i \frac{\partial}{\partial x^\mu}. \tag{5}
\]

The components \(\ddot{a}^i(\tau)\), of the acceleration with respect to the gyroscopes \(e_i\), are given, so that

\[
a^\mu \frac{\partial}{\partial x^\mu} = \ddot{a}^i(\tau) e^\mu_i \tag{6}
\]

or

\[
a^\mu(\tau) = \ddot{a}^i(\tau) e^\mu_i(\tau), \tag{7}
\]

and our task is the calculation of \(T = \sqrt{x^\mu(\tau)\eta_{\mu\nu}x^\nu(\tau)}\), under the assumption \(x^\mu(0) = 0\), \(u^i(0) = 0\), that is, the non-inertial observer starts at rest and from the origin of the inertial frame \(K\). First, we have to find \(e^\mu_i(\tau)\). The evolution of the tetrad is given by the equation of the Fermi-Walker transport [4]

\[
\frac{de^\mu_i}{d\tau} = (a^\mu u^\nu - u^\mu a^\nu)e^\mu_i. \tag{8}
\]

Using the orthogonality of the comoving tetrad \(\{e_0 = u, e_i\}\), \((e_i \cdot e_j = -\delta_{ij}, e_i \cdot e_0 = 0, e_0 \cdot e_0 = 1\) it reads

\[
\frac{de^\mu_i}{d\tau} = u^\mu \ddot{a}^i(\tau), \quad i = 1, \ldots, n; \quad \mu = 0, \ldots, n. \tag{9}
\]

Summarizing, the navigation problem of recovering the timelike worldline in terms of the components of the proper acceleration taken with respect to a Fermi-Walker transported frame takes the form (I have made explicit the summation over the indexes)

\[
\frac{dx^\mu}{d\tau} = u^\mu, \quad \mu = 0, \ldots, n, \tag{10}
\]

\[
\frac{du^\mu}{d\tau} = \sum_{i=1}^{n} \ddot{a}^i(\tau) e^\mu_i(\tau), \quad \mu = 0, \ldots, n, \tag{11}
\]

\[
\frac{de^\mu_i}{d\tau} = u^\mu \ddot{a}^i(\tau), \quad i = 1, \ldots, n; \quad \mu = 0, \ldots, n. \tag{12}
\]

In particular if the inertial frame \(K\) is chosen such that \(O\) is at rest and oriented as \(K\) at time \(\tau = 0\) then the initial conditions are \(u^i(0) = 0\), \(u^0(0) = 1\), \(x^\mu(0) = 0\), \(e^\mu_i(0) = \delta^\mu_i\). Through these \((n+1)(n+2)\) equations one finds \(x^\mu(\tau)\) and hence \(T(\tau)\) which represents the inertial time dilation with respect to an inertial observer that would see the journey of \(O\) till time \(\tau\) as closed.

No closed analytical solution of these equations is known, nevertheless the problem of obtaining an analytical solution for \(T(\tau)\) may admit a solution. The reason is that \(T\) being a Lorentz invariant is independent of the initial conditions and the problem may therefore simplify. The following section brings evidence in favor of this conclusion.
4. A closed formula for the differential aging

The previous system of differential equations can be decomposed on \( n + 1 \) systems, one for each value of \( \mu \). In order to find \( T^2(\tau) = x^\mu x_\mu \) we define

\[
y_0 = x_\mu e^\mu_0, \quad y_i = x_\mu e^\mu_i,
\]

so that

\[
T^2 = y_\alpha y^\alpha = y_0^2 - y^2. \tag{14}
\]

The reader may convince him or herself that \( y^\alpha \) is the difference of the inertial coordinates of \( x(\tau) \) and \( x(0) \) with respect to an inertial frame centered at \( x(\tau) \) and oriented as the frame \( e_i(\tau) \).

From the previous system of differential equations we obtain

\[
\frac{dT^2}{d\tau} = 2 y_0, \tag{15}
\]

\[
\frac{dy_0}{d\tau} = 1 + \tilde{\alpha} \cdot \hat{y}, \tag{16}
\]

\[
\frac{dy}{d\tau} = y_0 \tilde{\alpha}, \tag{17}
\]

with the initial conditions \( y_\alpha(0) = 0, \quad T(0) = 0 \). The first equation follows also from the last two.

Define \( y = |y|, \hat{y} = y/y, \) \((-\hat{y} \) represents the direction of \( x(0) \) with respect to the mentioned frame centered at \( x(\tau) \)) and multiply Eq. \((17)\) by \( 2y \)

\[
\frac{dy}{d\tau} = y_0 (\tilde{\alpha} \cdot \hat{y}), \tag{18}
\]

to be considered in system with the equation

\[
\frac{dy_0}{d\tau} = 1 + y (\tilde{\alpha} \cdot \hat{y}). \tag{19}
\]

Sum and subtract both equations to obtain

\[
\frac{d(y_0 + y)}{d\tau} = 1 + (\tilde{\alpha} \cdot \hat{y})(y_0 + y), \tag{20}
\]

\[
\frac{d(y_0 - y)}{d\tau} = 1 - (\tilde{\alpha} \cdot \hat{y})(y_0 - y). \tag{21}
\]

Hence multiplying by \( \exp(- \int_0^\tau (\tilde{\alpha} \cdot \hat{y})d\tau) \) the former and by \( \exp(\int_0^\tau (\tilde{\alpha} \cdot \hat{y})d\tau) \) the latter

\[
\frac{d}{d\tau} \{ (y_0 + y) \exp(- \int_0^\tau (\tilde{\alpha} \cdot \hat{y})d\tau) \} = \exp(- \int_0^\tau (\tilde{\alpha} \cdot \hat{y})d\tau), \tag{22}
\]

\[
\frac{d}{d\tau} \{ (y_0 - y) \exp(\int_0^\tau (\tilde{\alpha} \cdot \hat{y})d\tau) \} = \exp(\int_0^\tau (\tilde{\alpha} \cdot \hat{y})d\tau), \tag{23}
\]

and finally, using \( T^2 = (y_0 + y)(y_0 - y) \),

\[
T^2 = \{ \int_0^\tau e^{- \int_0^\tau (\tilde{\alpha} \cdot \hat{y})d\tau'}d\tau' \} \{ \int_0^\tau e^{\int_0^\tau (\tilde{\alpha} \cdot \hat{y})d\tau'}d\tau' \}. \tag{24}
\]

Thus, the differential aging \( \Delta(\tau) \) depends only on the component \( (\tilde{\alpha} \cdot \hat{y}) \) of the acceleration. Note that this formula makes sense only if for all \( \tau', \ y(\tau') \neq 0 \), otherwise \( \hat{y} \) is not defined.
Unfortunately, in $n+1$ dimensions it is not easy to find how the direction $\hat{y}(\tau)$ depends on the acceleration history so that this formula, while giving some insights, does not solve the problem.

Its physical meaning can be understood through the following reasoning. Imagine that at event $x(0)$ an explosion takes place so that the remnants of this explosion begin moving in all directions at a constant velocity with respect to the inertial frame $K$. The free falling observer $\bar{O}$ can be identified with one of this remnants. Now, the non-inertial observer observes at any time $\tau$ a flow of remnants of direction $\hat{y}(\tau)$ passing nearby his comoving laboratory. The above formula states that in order to calculate the differential aging, $O$ does not need to record all the components of the proper acceleration but only the one of direction $\hat{y}(\tau)$, i.e. the one along the remnants’ flow. While interesting this result has limited applicability as we can not always assume, in practice, that an explosion has taken place at event $x(0)$ so that the direction of the remnants’ velocity becomes locally observable.

Note that in 1+1 dimensions since there is only one component of the acceleration ($\tilde{a} \cdot \hat{y} = \pm \tilde{a}$ whenever $y$ does not change sign so that in this domain the formula reduces to (2).

5. The parallel transport which keeps a lightlike direction fixed

A different strategy was followed in [3]. The idea is to replace the Fermi-Walker transport with a different kind of transport. It is well known that software guided telescopes keep their orientation towards a given star under observation despite the fact that the earth rotates. The orientation of the telescope does not change according to the Fermi-Walker transport but according to a transport introduced in [3], that for short I termed lightlike parallel transport. It must not be confused with the generalization of the Fermi-Walker transport above a lightlike curve as studied, for instance, in [5, 1]. Rather, it is a minimal modification of the Fermi-Walker transport over a timelike curve which keeps the direction of a covariantly constant lightlike vector $n$ (interpreted as the light coming from a star) unchanged with respect to a lightlike transported frame.

A telescope fixed with respect to a lightlike transported frame would always be pointing towards the same star, moreover this transport depends on the star (i.e. null direction $n$) chosen.

If $v$ is a vector field over $x(\tau)$, then the lightlike transport is defined through the equation

$$\nabla u v_\mu - \nabla^L u v_\mu = \Omega^L_{\mu\nu} v^\nu,$$

(25)

where $\nabla$ is the usual Levi-Civita connection, and where

$$\Omega^L_{\mu\nu} = \frac{1}{u^\beta n_\beta} [a_\mu n_\nu - a_\nu n_\mu].$$

(26)

As a consequence the Fermi-Walker frame and the lightlike transported frame rotate one with respect to the other but the relative angular velocity is usually very small. This rotation is required in order to correct for the effect of aberration of light that would change the night sky position of the selected star.

The crucial observation is that the lightlike parallel transport can be obtained in practice by suitably correcting, time by time, the orientation of the frame so that the selected star stays always in the same position in the night sky. In fact, that this transport is feasible is proved by the mentioned examples of telescopes.

Let $e^L_i = e^L_{i\mu} \partial / \partial x^\mu$ be the lightlike transported frame. The acceleration measured by the comoving accelerometer can now be projected with respect to the axes of the lightlike transported frame $a^\mu = \tilde{a}(\tau) e^L_{i\mu}(\tau)$ and the problem becomes now that of obtaining the trajectory on spacetime, $x(\tau)$, staring from the data $\{\tilde{a}(\tau)\}$.

Remarkably, this problem can be solved completely, a solution being given in [3]. Here we give only the solution for $T(\tau)$. Note that since the position of the selected star does not change,
we can always orient the lightlike frame so that the light from the star goes on direction $e^L_\nu$. The components of the acceleration can then be decomposed in acceleration along the direction of light, $\vec{a}_\parallel$, and acceleration perpendicular to it, $\vec{a}_\perp$, where this time the boldface denotes an $(n-1)$-vector. The solution is then

$$T^2(\tau) = \left[ \int_0^\tau e^{-\int_0^{\tau'} \vec{a}_\parallel \, d\tau'} \left( \int_0^{\tau'} e^{\int_0^{\tau''} \vec{a}_\parallel \, d\tau''} \vec{a}_\perp \, d\tau'' \right)^2 \, d\tau' \right]$$

$$-\left[ \int_0^\tau e^{-\int_0^{\tau'} \vec{a}_\parallel \, d\tau'} \left( \int_0^{\tau'} e^{\int_0^{\tau''} \vec{a}_\parallel \, d\tau''} \vec{a}_\perp \, d\tau'' \right) \, d\tau' \right]^2 + \left[ \int_0^\tau e^{-\int_0^{\tau'} \vec{a}_\parallel \, d\tau'} \left( \int_0^{\tau'} e^{\int_0^{\tau''} \vec{a}_\parallel \, d\tau''} \vec{a}_\perp \, d\tau'' \right) \, d\tau' \right],$$

and using the Cauchy-Schwarz inequality one can show that $T(\tau) > \tau$ as expected, unless $\vec{a}_\parallel = 0$ and $\vec{a}_\perp = 0$, in which case $T = \tau$.

6. Conclusions
I have considered the problem of reconstructing the trajectory of a non-inertial observer starting from the knowledge of the proper acceleration. This problem is hard to solve if the data is given in terms of the components of the acceleration with respect to a Fermi-Walker transported frame while it can be solved if the newly introduced lightlike parallel transport is used. The offered solution is satisfactory as it can be implemented operationally; nevertheless, it would be nice to solve the problem of finding a closed differential aging formula even in the case of a Fermi-Walker decomposition of the acceleration. This problem remains still open and represents an interesting challenge for future investigations.

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