A note on tight projective 2-designs

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Abstract
We study tight projective 2-designs in three different settings. In the complex setting, Zauner’s conjecture predicts the existence of a tight projective 2-design in every dimension. Pandey, Paulsen, Prakash, and Rahaman recently proposed an approach to make quantitative progress on this conjecture in terms of the entanglement breaking rank of a certain quantum channel. We show that this quantity is equal to the size of the smallest weighted projective 2-design. Next, in the finite field setting, we introduce a notion of projective 2-designs, we characterize when such projective 2-designs are tight, and we provide a construction of such objects. Finally, in the quaternionic setting, we show that every tight projective 2-design for $d$ determines an equi-isoclinic tight fusion frame of $d(2d - 1)$ subspaces of $\mathbb{R}^{d(2d+1)}$ of dimension 3.

KEYWORDS
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1 INTRODUCTION

The vertices of the tetrahedron provide a famously nice arrangement of points on the unit sphere in $\mathbb{R}^3$. This highly symmetric configuration was constructed thousands of years ago in Euclid’s Elements, and it has since emerged as an optimal arrangement for several fundamental problems in metric geometry [3,18]. For example, it solves the $n = 4$ case of Tamme’s inimical dictators problem [19], which asks to maximize the minimum distance between $n$ dictators on the sphere. It also solves the $n = 4$ case of Thomson’s energy minimization problem [50], and more generally, it minimizes every completely monotonic potential [12], which explains the tetrahedral shape exhibited by the methane molecule [30].
If we interpret the sphere as the Bloch sphere, that is, the complex projective space $\mathbb{CP}^1$, then each of the four points corresponds to a line in $\mathbb{C}^2$. For example, the following unit vectors span lines that correspond to vertices of a tetrahedron inscribed in the Bloch sphere:

$$x_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \quad x_3 := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \sqrt{2} \omega \end{bmatrix}, \quad x_4 := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \sqrt{2} \omega^2 \end{bmatrix},$$

where $\omega := \exp(2\pi i/3)$. This arrangement of lines inherits some notions of niceness from the tetrahedron. For example, the lines are equiangular since $\langle x_k, x_\ell \rangle^2 = \frac{1}{3}$ whenever $k \neq \ell$. It inherits much stronger properties as well, and in fact it forms a projective 2-design, as defined below. This paper is devoted to the study of similar objects.

To elaborate, let $S(\mathbb{C}^d)$ denote the sphere of $x \in \mathbb{C}^d$ with $\|x\|_2 = 1$, and consider its uniform probability measure $\sigma$. We let $\text{Hom}_d(t)$ denote the complex vector space spanned by all monomial functions $\mathbb{C}^d \to \mathbb{C}$ that map $z = (z_1, \ldots, z_d)$ to $z_1^{a_1} \cdots z_d^{a_d} x_1^{\beta_1} \cdots x_d^{\beta_d}$ with $t = \sum_j a_j = \sum_j \beta_j$. Next, we take $\Pi_d^{(t)}$ to denote orthogonal projection onto the symmetric subspace $(\mathbb{C}^d)^{\otimes t}_{\text{sym}}$ of $(\mathbb{C}^d)^{\otimes t}$. Finally, put $[n] := \{1, \ldots, n\}$. Having established the necessary notation, a projective $t$-design for $\mathbb{C}^d$ is defined to be any $\{x_k\}_{k \in [n]}$ in $S(\mathbb{C}^d)$ that satisfies the following equivalent properties:

**Proposition 1** (see Waldron [52]). Given $\{x_k\}_{k \in [n]}$ in $S(\mathbb{C}^d)$ and $t \in \mathbb{N}$, the following are equivalent:

(a) $\frac{1}{n} \sum_{k \in [n]} p(x_k) = \int_{S(\mathbb{C}^d)} p(x) d\sigma(x)$ for every $p \in \text{Hom}_d(t)$.

(b) $\frac{1}{n} \sum_{k \in [n]} \langle x_k^{\otimes t} | x_k^{\otimes t} \rangle^* = \binom{d + t - 1}{t}^{-1} \cdot \Pi_d^{(t)}$.

(c) $\frac{1}{n^t} \sum_{k \in [n]} \sum_{\ell \in [n]} \langle x_k, x_\ell \rangle^{2t} = \binom{d + t - 1}{t}^{-1}$.

Returning to the example in Equation (1), we have $d = 2$ and $n = 4$, and one may verify that Proposition 1(c) holds for each $t \in \{1, 2\}$. Thus, these vectors form a projective 1-design and a projective 2-design for $\mathbb{C}^2$.

In words, the cubature rule Proposition 1(a) says that, for the purposes of integration over the sphere $S(\mathbb{C}^d)$, a projective $t$-design “fools” every $p \in \text{Hom}_d(t)$ by mimicking the entire sphere. Note that one may “trace out” one of the $t$ subsystems in Proposition 1(b) to show that every projective $t$-design is also a projective $(t - 1)$-design. Proposition 1(b) says that $\{x_k^{\otimes t}\}_{k \in [n]}$ forms what frame theorists would call a unit norm tight frame [10] for the $\binom{d + t - 1}{t}$-dimensional complex Hilbert space $(\mathbb{C}^d)^{\otimes t}_{\text{sym}}$. With this perspective, Proposition 1(c) corresponds to the frame potential [6] of $\{x_k^{\otimes t}\}_{k \in [n]}$. One may generalize Proposition 1(c) to obtain notions of projective $t$-designs for real, quaternionic, and octonionic spaces [42]. The complex case with $t = 2$ is particularly relevant in quantum state tomography [47], where it is desirable to take $n$ as small as possible. This motivates the following result, which refers to $\{x_k\}_{k \in [n]}$ in $S(\mathbb{C}^d)$ as equiangular if

$$\{|\langle x_k, x_\ell \rangle|^2 : k, \ell \in [n], k \neq \ell\| = 1.$$
**Proposition 2** (Special case of Proposition 1.1 in [4]). Consider $X = \{x_k\}_{k \in [n]}$ in $S(\mathbb{C}^d)$.

(a) If $X$ is a projective 2-design, then $n \geq d^2$ with equality precisely when $X$ is equiangular.

(b) If $X$ is equiangular, then $n \leq d^2$ with equality precisely when $X$ is a projective 2-design.

In the case of equality $n = d^2$, $\{x_k\}_{k \in [n]}$ is known as a tight projective 2-design for $\mathbb{C}^d$, which corresponds to an object in quantum physics known as a symmetric, informationally complete positive operator–valued measure [11]. For example, Equation (1) gives a tight projective 2-design for $\mathbb{C}^2$. In his Ph.D. thesis [57], Zauner conjectured that for every $d > 1$, there exists a tight projective 2-design for $\mathbb{C}^d$ of a particular form. To date, there are only finitely many $d \in \mathbb{N}$ for which a tight projective 2-design is known to exist [24,26,23], and the conjecture is apparently related to the Stark conjectures in algebraic number theory [37]. A solution to Zauner’s conjecture will be rewarded with a 2021 EUR prize from the National Quantum Information Centre in Poland [36].

Pandey et al. [44] recently proposed a new approach to Zauner’s conjecture. They identify an explicit quantum channel $\mathcal{Z}_d : \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d}$ whose so-called entanglement breaking rank is $\text{ebr}(\mathcal{Z}_d) \geq d^2$, and furthermore, equality holds if and only if there exists a tight projective 2-design for $\mathbb{C}^d$. As a consequence, any new upper bound on this entanglement breaking rank represents quantitative progress towards Zauner’s conjecture. In addition, Pandey et al. consider various analytic approaches to obtain such bounds in small dimensions.

As a consequence of Proposition 2, every tight projective 2-design for $\mathbb{C}^d$ is necessarily an equiangular tight frame (ETF), that is, a unit norm tight frame that is also equiangular. ETFs correspond to optimal codes in projective space that achieve equality in the Welch bound [55] (also known as the simplex bound [14]). By virtue of this optimality, ETFs find applications in wireless communication [49], compressed sensing [2], and digital fingerprinting [41]. Motivated by these applications, many ETFs were recently constructed using various mixtures of algebra and combinatorics [49,56,16,22,20,8,32,31,33,34]; see [21] for a survey. Despite this flurry of work, several problems involving ETFs (such as Zauner’s conjecture) remain open, and a finite field model was recently proposed to help study these remaining problems [27,28].

Notice that if $\{x_k\}_{k \in [n]}$ is a tight projective 2-design for $\mathbb{C}^d$, then Propositions 1 and 2 together imply that $\{x_k^{\otimes 2}\}_{k \in [n]}$ is an ETF for $(\mathbb{C}^d)^{\otimes 2}_{\text{sym}}$. This suggests another approach to Zauner’s conjecture [1] in which one seeks ETFs of $d^2$ vectors in the $\binom{d+1}{2}$-dimensional complex Hilbert space $(\mathbb{C}^d)^{\otimes 2}_{\text{sym}}$. There are several known constructions of such ETFs [21,8,32,31], but to correspond to a tight projective 2-design, the ETF must consist of rank-1 symmetric tensors.

We note that one may leverage linear programming bounds to obtain analogous results to Proposition 2 that relate projective $t$-designs over different spaces to different sized angle sets [4]. In the real case, tight projective 2-designs have size $\binom{d+1}{2}$ and are only known to exist for $d \in \{2, 3, 7, 23\}$, with $d = 119$ being the smallest dimension for which existence is currently unknown; see [38,40,5,43,25]. In the quaternion case, tight projective 2-designs have size $2d(d - 1)$; they are only known to exist for $d \in \{2, 3\}$ [13,17], and there is numerical evidence that they do not exist for $d \in \{4, 5\}$ [13]. The octonions are only capable of supporting a projective space for $d \in \{2, 3\}$, and tight projective 2-designs exist in both cases [13].

In this paper, we study tight projective 2-designs in three different settings. In Section 2, we consider the complex setting, specifically, the new quantitative approach of Pandey et al. [44].
Here, we show that \( \text{ebr}(3_d) \) is precisely the size of the smallest weighted projective 2-design for \( \mathbb{C}^d \). This identification allows us to find new upper bounds on \( \text{ebr}(3_d) \). Next, in Section 3, we use Proposition 1(b) to find an analog of projective 2-designs in a finite field setting. This continues the line of inquiry from [27,28] of tackling hard problems from frame theory in a finite field model. In this setting, we obtain an analog of Proposition 2, and then we construct a family of tight projective 2-designs. Finally in Section 4, we consider the quaternionic setting, where we take inspiration from the fact that a tight projective 2-design for \( \mathbb{C}^d \) can be used to produce an ETF of \( d^2 \) vectors in \( (\mathbb{C}^d)^{\otimes 2}_{\text{sym}} \). In particular, we show how a tight projective 2-design for \( \mathbb{H}^d \) can be used to produce an equi-isoclinic tight fusion frame of \( 2d(d-1) \) different three-dimensional subspaces of the \( d(2d+1) \)-dimensional real Hilbert space of \( d \times d \) quaternionic anti-Hermitian matrices.

2 | THE COMPLEX SETTING

A linear map \( \Phi : \mathbb{C}^{d \times d} \to \mathbb{C}^{m \times m} \) is said to be entanglement breaking if it admits an entanglement breaking decomposition:

\[
\Phi(X) = \sum_{k \in [n]} R_k X R_k^*, \quad \sum_{k \in [n]} R_k^* R_k = I_d, \quad \text{rank } R_k = 1 \text{ for every } k \in [n].
\] (2)

The entanglement breaking rank of \( \Phi \), denoted by \( \text{ebr}(\Phi) \), is the smallest \( n \) for which there exists \( \{R_k\}_{k \in [n]} \) in \( \mathbb{C}^{m \times d} \) satisfying (2). Let \( \{e_i\}_{i \in [d]} \) denote the standard basis in \( \mathbb{C}^d \). The Choi matrix of \( \Phi \) is given by

\[
C_\Phi := \sum_{i \in [d]} \sum_{j \in [d]} e_i e_j^* \otimes \Phi(e_i e_j^*) \in \mathbb{C}^{d \times d} \otimes \mathbb{C}^{m \times m}.
\]

In words, \( C_\Phi \) is a \( d \times d \) block array whose \( (i, j) \)th block is \( \Phi(e_i e_j^*) \in \mathbb{C}^{m \times m} \). One may use \( C_\Phi \) to discern useful properties about \( \Phi \). For example, \( \Phi \) is completely positive when \( C_\Phi \) is positive semidefinite; see Theorem 2.22 in [54].

We are interested in the quantum depolarizing channel \( 3_d : \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d} \) defined by

\[
3_d(X) := \frac{1}{d+1} (X + \text{tr} X \cdot I_d).
\]

Remark 3. One may verify that \( 3_d \) is entanglement breaking as a consequence of its scaled Choi matrix \( \frac{1}{d} C_{3_d} \) being a separable bipartite state (specifically, the isotropic state with \( \lambda = 1/d \) from Example 7.25 in [54]).

Pandey et al. [44] pointed to this quantum channel as an opportunity for quantitative progress on Zauner’s conjecture:

Proposition 4 (cf. Corollary III.3 and Theorem V.3 in [44]).

(a) \( \text{ebr}(3_d) \geq d^2 \), with equality if and only if there exists a tight projective 2-design for \( \mathbb{C}^d \).
(b) \( \text{ebr}(3_d) \leq d^2 + d \) whenever \( d \) is a prime power.
We say unit vectors \( \{x_k\}_{k \in [n]} \) in \( \mathbb{C}^d \) form a weighted projective \( t \)-design if there exist weights \( \{w_k\}_{k \in [n]} \) such that

\[
\sum_{k \in [n]} w_k (x_k^{\otimes t}) (x_k^{\otimes t})^* = \left( \frac{d + t - 1}{t} \right)^{-1} \cdot \Pi_d^{(t)}, \quad \sum_{k \in [n]} w_k = 1, \quad w_k \geq 0, \quad k \in [n].
\]

Notice that every projective \( t \)-design is a weighted projective \( t \)-design with weights \( w_k = \frac{1}{n} \). In addition, it is known that every weighted projective \( 2 \)-design for \( \mathbb{C}^d \) has size at least \( d^2 \), and if equality holds, then the weights are all \( \frac{1}{n} \); see Theorem 4 in [47]. What follows is the main result of this section, of which Proposition 4 is a corollary:

**Theorem 5.** The smallest weighted projective \( 2 \)-design for \( \mathbb{C}^d \) has size \( \text{ebr}(3_d) \).

In fact, one may use Theorem 5 to improve upon Proposition 4(b) by collecting various weighted 2-designs from the literature. Specifically, Theorem 4.1, Proposition 4.2, and Corollary 4.2 in [46], and Corollaries 4.4 and 4.6 in [7] give the following:

**Corollary 6.**

(a) \( \text{ebr}(3_d) \leq kd^2 + 2d \) whenever \( kd + 1 \) is a prime power with \( k \in \mathbb{N} \).

(b) \( \text{ebr}(3_d) \leq d^2 + (p + 1)d \) whenever \( d + 1 = p^k \) with \( p \) prime and \( k \in \mathbb{N} \).

(c) \( \text{ebr}(3_d) \leq d^2 + 1 \) whenever \( d - 1 \) is a prime power.

(d) \( \text{ebr}(3_d) \leq d^2 + d - 1 \) whenever \( d \) is a prime power.

Since \( 3_d \) is entanglement breaking, Theorem 5 also implies the existence of weighted projective 2-designs; we note that this also follows from the main result in [48].

**Corollary 7.** For each \( d \in \mathbb{N} \), there is a weighted projective 2-design for \( \mathbb{C}^d \) of size \( \left( \frac{d+1}{2} \right)^2 \).

**Proof.** Since \( 3_d \) is entanglement breaking by Remark 3, Theorem 5 promises a weighted projective 2-design \( \{x_k\}_{k \in [n]} \) for \( \mathbb{C}^d \). Then \( \Pi_d^{(2)} \) resides in the conic hull of \( \{(x_k^{\otimes 2})(x_k^{\otimes 2})^*\}_{k \in [n]} \), which in turn is contained in the \( \left( \frac{d+1}{2} \right)^2 \)-dimensional real vector space of Hermitian operators over \( (\mathbb{C}^d)^{\otimes 2} \). By Carathéodory’s theorem, there exists \( S \subseteq [n] \) with \( |S| \leq \left( \frac{d+1}{2} \right)^2 \) and weights \( w_k \geq 0 \) for \( k \in S \) such that

\[
\sum_{k \in S} w_k (x_k^{\otimes 2}) (x_k^{\otimes 2})^* = \left( \frac{d + 1}{2} \right)^{-1} \cdot \Pi_d^{(2)}.
\]

Furthermore, taking the trace of both sides reveals that \( \sum_{k \in S} w_k = 1 \). As such, \( \{x_k\}_{k \in S} \) is a weighted projective 2-design. \( \square \)

The remainder of this section proves Theorem 5. We first collect a few helpful lemmas. Let \( T : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{d \times d} \) denote the transposition operator defined by \( T(X) := X^T \).
Lemma 8 (cf. Proposition III.5 in [44]). It holds that 
\[ \Phi(X) = \sum_{k \in [n]} x_k y_k^T X \begin{pmatrix} x_k y_k^T \end{pmatrix}^* \iff (T \circ \Phi)(X) = \sum_{k \in [n]} \bar{x}_k \bar{y}_k^T X \begin{pmatrix} \bar{x}_k \bar{y}_k^T \end{pmatrix}^*. \]

Proof. The claim follows from the following manipulation:

\[
\left( \sum_{k \in [n]} x_k y_k^T X \bar{y}_k \bar{x}_k \right)^T = \sum_{k \in [n]} \bar{x}_k \left( y_k^T X^T y_k \right) x_k^T = \sum_{k \in [n]} \bar{x}_k \left( y_k^T X^T y_k \right) x_k^T \\
= \sum_{k \in [n]} \bar{x}_k \bar{y}_k^T X \bar{y}_k \bar{x}_k^T,
\]

where the second step takes the transpose of a scalar. Indeed, we have both 
\((x_k y_k^T)^* = y_k \bar{x}_k^T\) and \((\bar{x}_k \bar{y}_k^T)^* = y_k \bar{x}_k^T\). \(\square\)

Both of the following lemmas were implicitly used in the proof of Corollary III.7 in [44]. To prove them, we will repeatedly use the following identity, which is valid for any linear \(\Phi : \mathbb{C}^{d \times d} \to \mathbb{C}^{m \times m}\), and any \(w, y, x, z \in \mathbb{C}^d\) and \(x, z \in \mathbb{C}^m\):

\[
(w \otimes x)^T C_\Phi(y \otimes z) = \sum_{i \in [d]} \sum_{j \in [d]} w_i y_j \cdot x^T \Phi(e_i e_j^T) z \\
= x^T \Phi \left( \sum_{i \in [d]} \sum_{j \in [d]} w_i y_j \cdot e_i e_j^T \right) z = x^T \Phi(wy^T) z. \tag{3}
\]

Lemma 9. An entanglement breaking map \(\Phi\) has entanglement breaking decomposition

\[ \Phi(X) = \sum_{k \in [n]} a_k b_k^T X \begin{pmatrix} a_k b_k^T \end{pmatrix}^*. \]

if and only if \(\Phi\) has Choi matrix

\[ C_\Phi = \sum_{k \in [n]} (b_k \otimes a_k)(b_k \otimes a_k)^*. \]

Proof. \((\Rightarrow)\) For any \(w, x, y, z\), we may apply (3) to get

\[
(w \otimes x)^T C_\Phi(y \otimes z) = x^T \left( \sum_{k \in [n]} a_k b_k^T (wy^T) \bar{b}_k \bar{a}_k^T \right) z \\
= \sum_{k \in [n]} \left( x^T a_k b_k^T w \right) \left( y^T \bar{b}_k \bar{a}_k^T z \right) \\
= \sum_{k \in [n]} (w \otimes x)^T (b_k \otimes a_k)(b_k \otimes a_k)^T (y \otimes z) \\
= (w \otimes x)^T \left( \sum_{k \in [n]} (b_k \otimes a_k)(b_k \otimes a_k)^* \right) (y \otimes z).
\]
Since $w, x, y, z$ are arbitrary, the result follows.

($\Leftarrow$) For any $w, x, y, z$, we may similarly apply (3) to get

$$x^T \Phi(wy^T)z = (w \otimes x)^T C_\Phi(y \otimes z)$$

$$= (w \otimes x)^T \left( \sum_{k \in [n]} (b_k \otimes a_k)(b_k \otimes a_k)^* \right)(y \otimes z)$$

$$= x^T \left( \sum_{k \in [n]} a_k b_k^T (wy^T) b_k^* \right) z.$$  

Since $w, x, y, z$ are arbitrary, the result follows.

**Lemma 10.** $C_{T^*3d} = \frac{2}{d+1} \Pi_d^{(2)}$.

**Proof.** For any $w, x, y, z$, we may apply (3) to get

$$(w \otimes x)^T C_{T^*3d}(y \otimes z) = x^T [T \circ 3d](wy^T)]z$$

$$= x^T \frac{1}{d+1} (yw^T + \text{tr} wy^T \cdot I_d)z$$

$$= \frac{1}{d+1} (w^T z \cdot x^T y + w^T y \cdot x^T z)$$

$$= (w \otimes x)^T \frac{1}{d+1} (z \otimes y + y \otimes z).$$

Since $w, x$ are arbitrary, it follows that

$$C_{T^*3d}(y \otimes z) = \frac{2}{d+1} \cdot \frac{1}{2} (z \otimes y + y \otimes z) = \frac{2}{d+1} \Pi_d^{(2)}(y \otimes z).$$

Since $y, z$ are arbitrary, the result follows.

We are now ready to prove the main result of this section.

**Proof of Theorem 5.** Suppose $\{x_k\}_{k \in [n]}$ is a weighted projective 2-design for $C^d$ with weights $\{w_k\}_{k \in [n]}$. We claim that $\text{ebr}(3_d) \leq n$. To see this, first recall that

$$\sum_{k \in [n]} w_k(x_k \otimes x_k)(x_k \otimes x_k)^* = \frac{2}{d(d+1)} \cdot \Pi_d^{(2)}.$$

Taking $a_k := x_k$ and $b_k := \sqrt{d w_k} x_k$ then gives

$$\sum_{k \in [n]} (b_k \otimes a_k)(b_k \otimes a_k)^* = d \sum_{k \in [n]} w_k(x_k \otimes x_k)(x_k \otimes x_k)^* = \frac{2}{d+1} \cdot \Pi_d^{(2)} = C_{T^*3d},$$

where the last step applies Lemma 10. Lemmas 8 and 9 then imply that
Next, suppose \( n = \text{ebr}(\mathcal{A}_d) = \text{ebr}(T \circ \mathcal{A}_d) \) and consider the entanglement breaking decomposition

\[
(T \circ \Phi)(X) = \sum_{k \in [n]} a_k b_k^\top X (a_k b_k^\top)^*.
\]

Then Lemmas 9 and 10 together imply

\[
\sum_{k \in [n]} (b_k \otimes a_k)(b_k \otimes a_k)^* = \frac{2}{d + 1} \cdot \Pi_d^{(2)}.
\]

Notice that for every antisymmetric \( x \in (\mathbb{C}^d)^{\otimes 2} \), it holds that

\[
\sum_{k \in [n]} |(b_k \otimes a_k, x)|^2 = x^* \left( \sum_{k \in [n]} (b_k \otimes a_k)(b_k \otimes a_k)^* \right) x = x^* \frac{2}{d + 1} \Pi_d^{(2)} x = 0.
\]

It follows that each \( b_k \otimes a_k \) is necessarily symmetric, and therefore takes the form \( \sqrt{d} w_k x_k \otimes x_k \) for some unit vector \( x_k \) and scalar \( w_k \geq 0 \). Then

\[
\sum_{k \in [n]} w_k (x_k \otimes x_k)(x_k \otimes x_k)^* = \frac{2}{d(d + 1)} \cdot \Pi_d^{(2)}.
\]

The fact that \( \sum_{k \in [n]} w_k = 1 \) follows from taking the trace of both sides. Overall, there exists a weighted projective 2-design for \( \mathbb{C}^d \) of size \( n = \text{ebr}(\mathcal{A}_d) \), as claimed. \( \square \)

## 3  THE FINITE FIELD SETTING

In this section, we introduce a notion of projective 2-designs in a finite field setting. Here, we will find an analog to Proposition 2 in which tight projective 2-designs are identified as maximal systems of equiangular lines, and then we will provide several examples. We start by reviewing some preliminaries; the reader is encouraged to see [27] for more information.

Let \( q \) be a prime power. Given \( a \in \mathbb{F}_q \), we abbreviate \( \bar{a} = a^q \) for its image under the Frobenius automorphism fixing \( \mathbb{F}_q \leq \mathbb{F}_q^2 \). The conjugate transpose of a matrix \( A \) is denoted by \( A^* \). We consider \( \mathbb{F}_q^d \) under the nondegenerate Hermitian form \( \langle x, y \rangle = x^* y \), which is notably conjugate-linear in the first variable. A subspace \( V \leq \mathbb{F}_q^d \) is called nondegenerate if \( V \cap V^\perp = \{0\} \), where

\[
V^\perp := \{ x \in \mathbb{F}_q^d : \langle x, y \rangle = 0 \quad \text{for every} \quad y \in V \}.
\]

In that case, every \( x \in \mathbb{F}_q^d \) can be written uniquely as \( x = Px + Qx \) with \( Px \in V \) and \( Qx \in V^\perp \), where \( P : \mathbb{F}_q^d \to \mathbb{F}_q^d \) is orthogonal projection onto \( V \).
Definition 11. Let $V \leq \mathbb{F}_q^d$ be nondegenerate. We say $\{x_k\}_{k \in [n]}$ in $V$ is a $c$-tight frame for $V$ with constant $c \in \mathbb{F}_q$ if

(i) $\text{span}\{x_k\}_{k \in [n]} = V$, and 
(ii) $\sum_{k \in [n]} \langle x_k, y \rangle x_k = cy$, for every $y \in V$.

We refer to a $c$-tight frame as a tight frame when the constant $c$ is not important. For $a \in \mathbb{F}_q$, a $c$-tight frame is an equal-norm tight frame, or $(a, c)$-NTF, if

(iii) $\langle x_k, x_k \rangle = a$ for every $k \in [n]$.

For $b \in \mathbb{F}_q$, an $(a, c)$-NTF is an equiangular tight frame, or $(a, b, c)$-ETF, if

(iv) $\langle x_k, x_\ell \rangle \langle x_\ell, x_k \rangle = b$ for every $k, \ell \in [n]$ with $k \neq \ell$.

Meanwhile, an $(a, b)$-equiangular system in $V$ satisfies (iii) and (iv), but not necessarily (i) or (ii).

Notice that (ii) implies (i) if $c \neq 0$. Furthermore, if $P$ is orthogonal projection onto $V$, then (ii) is equivalent to

(ii') $\sum_{k \in [n]} x_k x_k^* = cP$.

We will repeatedly make use of the following basic results from [27]:

Proposition 12 (Corollary 3.8 in [27]). If $V \leq \mathbb{F}_q^d$ is nondegenerate and $\{x_k\}_{k \in [n]}$ is a tight frame for $V$ with constant $c = 0$, then $n \geq 2 \dim V$.

Proposition 13 (Equation (3.2) and Proposition 4.7 in [27]).

(a) If $\{x_k\}_{k \in [n]}$ is an $(a, c)$-NTF for $V$, then $na = c \dim V$.
(b) If $\{x_k\}_{k \in [n]}$ is an $(a, b, c)$-ETF for $V$, then $a(c-a) = (n-1)b$.

Proposition 14 (Gerzon’s bound, see Theorem 4.1 in [27] and its proof). If $\{x_k\}_{k \in [n]}$ is an $(a, b)$-equiangular system in $\mathbb{F}_q^d$, and $a^2 \neq b$, then $n \leq d^2$. If equality holds and $a \neq 0$, then $\{x_k x_k^*\}_{k \in [n]}$ is a basis for the $\mathbb{F}_q$-linear space

$$\left\{X \in \mathbb{F}_q^{d \times d} : X = X^* \right\}.$$ 

If equality holds and $a = 0$, then the $\mathbb{F}_q$-span of $\{x_k x_k^*\}_{k \in [n]}$ is the subspace

$$\left\{X \in \mathbb{F}_q^{d \times d} : X = X^*, \text{tr}X = 0 \right\},$$

and $\sum_{k \in [n]} x_k x_k^* = 0$ is the unique $\mathbb{F}_q$-linear dependency of $\{x_k x_k^*\}_{k \in [n]}$ up to a scalar.
3.1 Projective 2-designs

Throughout this section, we assume $q$ is odd. Let $e_1, ..., e_d \in \mathbb{F}_q^d$ denote the standard basis. Then $\{e_i \otimes e_j, i,j \in [d]\}$ is a basis for $(\mathbb{F}_q^d)^\otimes$.

We write

$$(\mathbb{F}_q^d)^\otimes_{\text{sym}} := \left\{ \sum_{i \in [d]} \sum_{j \in [d]} c_{ij} (e_i \otimes e_j) : c_{ij} = c_{ji} \text{ for every } i,j \in [d] \right\}$$

for the subspace of symmetric tensors, and we define

$$\Pi_d^{(2)} := \frac{1}{2} \sum_{i \in [d]} \sum_{j \in [d]} \left( e_i e_j^* \otimes e_j e_j^* + e_j e_j^* \otimes e_i e_i^* \right) \in (\mathbb{F}_q^d)^\otimes_{\text{sym}}.$$

Lemma 15. $(\mathbb{F}_q^d)^\otimes_{\text{sym}} \leq (\mathbb{F}_q^d)^\otimes$ is nondegenerate, and $\Pi_d^{(2)}$ is its orthogonal projection.

Proof. For nondegeneracy, let $y = \sum_{i \in [d]} \sum_{j \in [d]} a_{ij} (e_i \otimes e_j) \in (\mathbb{F}_q^d)^\otimes_{\text{sym}}$ be nonzero. Then there exist $k, \ell \in [d]$ such that $a_{k\ell} = a_{\ell k} \neq 0$. It follows that

$$(y, e_k \otimes e_\ell + e_\ell \otimes e_k) = \sum_{i \in [d]} \sum_{j \in [d]} \overline{a}_{ij} (e_i \otimes e_j, e_k \otimes e_\ell) + \sum_{i \in [d]} \sum_{j \in [d]} \overline{a}_{ij} (e_i \otimes e_j, e_\ell \otimes e_k) = 2\overline{a}_{k\ell},$$

which is nonzero by assumption. Thus, $y$ is not orthogonal to $(\mathbb{F}_q^d)^\otimes_{\text{sym}}$.

Next, we show that $\Pi_d^{(2)}$ projects orthogonally onto $(\mathbb{F}_q^d)^\otimes_{\text{sym}}$. To this end, choose any vector $x = \sum_{k \in [d]} \sum_{\ell \in [d]} c_{k\ell} (e_k \otimes e_\ell)$ and compute

$$\Pi_d^{(2)} x = \frac{1}{2} \sum_{i \in [d]} \sum_{j \in [d]} \sum_{k \in [d]} \sum_{\ell \in [d]} c_{k\ell} \left( (e_i e_i^* \otimes e_j e_j^*) (e_k \otimes e_\ell) + (e_j e_j^* \otimes e_i e_i^*) (e_k \otimes e_\ell) \right)$$

$$= \frac{1}{2} \sum_{i \in [d]} \sum_{j \in [d]} \sum_{k \in [d]} \sum_{\ell \in [d]} c_{k\ell} \left( (e_i e_i^* \otimes e_j e_j^*)(e_k \otimes e_\ell) + (e_j e_j^* \otimes e_i e_i^*) (e_k \otimes e_\ell) \right)$$

$$= \frac{1}{2} \sum_{k \in [d]} \sum_{\ell \in [d]} c_{k\ell} \left( \sum_{i \in [d]} \sum_{j \in [d]} (e_i e_i^*) (e_k \otimes (e_j e_j^*) (e_\ell) + \sum_{i \in [d]} \sum_{j \in [d]} (e_j e_j^*) (e_k \otimes (e_i e_i^*) (e_\ell) \right)$$

$$= \frac{1}{2} \sum_{k \in [d]} \sum_{\ell \in [d]} c_{k\ell} (e_k \otimes e_\ell + e_\ell \otimes e_k),$$

which belongs to $(\mathbb{F}_q^d)^\otimes_{\text{sym}}$. Then

$$x - \Pi_d^{(2)} x = \frac{1}{2} \sum_{k \in [d]} \sum_{\ell \in [d]} c_{k\ell} (e_k \otimes e_\ell - e_\ell \otimes e_k),$$

and it is straightforward to check this is orthogonal to every $y \in (\mathbb{F}_q^d)^\otimes_{\text{sym}}$. As such, $x = (\Pi_d^{(2)} x) + (x - \Pi_d^{(2)} x)$ gives the desired decomposition of $x$. □
As a consequence of Lemma 15, we may consider tight frames over $(\mathbb{F}_q^d)^\otimes_2^{sym}$. This allows us to define the following analog of Proposition 1(b):

**Definition 16.** \( \{x_k\}_{k \in [n]} \) in \( \mathbb{F}_q^d \) is an \((a, c_1, c_2)\)-projective 2-design if \( a, c_1, c_2 \in \mathbb{F}_q \) and

(i) \( \langle x_k, x_k \rangle = a \) for every \( k \in [n] \),

(ii) \( \{x_k\}_{k \in [n]} \) is a \( c_1 \)-tight frame for \( \mathbb{F}_q^d \), and

(iii) \( \{x_k^\otimes_2\}_{k \in [n]} \) is a \( c_2 \)-tight frame for \( (\mathbb{F}_q^d)^\otimes_2^{sym} \).

With this definition, the finite field setting enjoys an analogy to Proposition 2:

**Theorem 17.** If \( \{x_k\}_{k \in [n]} \) in \( \mathbb{F}_q^d \) is a projective 2-design, then \( n \geq d^2 \).

To prove this theorem, we will repeatedly make use of the following:

**Lemma 18.** If \( \{x_k\}_{k \in [n]} \) in \( \mathbb{F}_q^d \) is an \((a, c_1, c_2)\)-projective 2-design with \( c_2 \neq 0 \), then for every \( A \in \mathbb{F}_q^{d \times d} \), it holds that

\[
A = \frac{2}{c_2} \sum_{k \in [n]} x_k^* x_k A x_k x_k^* - \text{tr} A \cdot I_d.
\]

**Proof.** To prove the result, we multiply both sides of the identity \( \sum_{k \in [n]} (x_k^\otimes_2^*)(x_k^\otimes_2^*)^* = c_2 \cdot \Pi_d^{(2)} \) by \( A \otimes I_d \) and then “trace out” the first subsystem. Explicitly, we define the partial trace \( \text{tr}_1 : \mathbb{F}_q^{d \times d} \otimes \mathbb{F}_q^{d \times d} \to \mathbb{F}_q^{d \times d} \) by taking

\[
\text{tr}_1(A \otimes B) := \text{tr}(A) \cdot B
\]

and extending linearly. Since \( \{x_k\}_{k \in [n]} \) is a projective 2-design, we have

\[
\text{tr}_1 \left( \sum_{k \in [n]} (x_k^\otimes_2^*)(x_k^\otimes_2^*)^* (A \otimes I_d) \right) = \text{tr}_1 \left( c_2 \cdot \Pi_d^{(2)} \cdot (A \otimes I_d) \right). \tag{4}
\]

We cycle the trace to simplify the left-hand side of (4):

\[
\text{tr}_1 \left( \sum_{k \in [n]} (x_k^\otimes_2^*)(x_k^\otimes_2^*)^* (A \otimes I_d) \right) = \text{tr}_1 \left( \sum_{k \in [n]} (x_k x_k^*)^\otimes_2^* (A \otimes I_d) \right)
\]

\[
= \text{tr}_1 \left( \sum_{k \in [n]} x_k x_k^* A \otimes x_k x_k^* \right)
\]

\[
= \sum_{k \in [n]} \text{tr}(x_k x_k^* A) \cdot x_k x_k^* = \sum_{k \in [n]} x_k (x_k^* A x_k) x_k^*.
\]
For the right-hand side of (4), we apply the definition of $\Pi_d^{(2)}$:

$$
\text{tr}_1\left( c_2 \cdot \Pi_d^{(2)} \cdot (A \otimes I_d) \right) = \text{tr}_1\left( \frac{c_2}{2} \sum_{i,j \in [d]} \left( e_i e_i^* \otimes e_j e_j^* + e_j e_j^* \otimes e_i e_i^* \right) (A \otimes I_d) \right)
$$

$$
= \frac{c_2}{2} \sum_{i,j \in [d]} \left( \text{tr}(e_i e_i^* A) \cdot e_j e_j^* + \text{tr}(e_j e_j^* A) \cdot e_i e_i^* \right)
$$

$$
= \frac{c_2}{2} \left( \text{tr}A \cdot I_d + A \right).
$$

(6)

The result follows by equating (5) to (6) and rearranging.

Proof of Theorem 17. Suppose $\{x_k\}_{k \in [n]}$ in $\mathbb{F}_{q^2}^d$ is an $(a, c_1, c_2)$-projective 2-design.

Case I: $c_2 = 0$. Then $\{x_k x_k^*\}_{k \in [n]}$ is an $(a^2, 0)$-NTF for $(\mathbb{F}_{q^2}^d)^{\otimes 2}$. By Proposition 12,

$$
n \geq 2 \cdot \dim\left( \left( \mathbb{F}_{q^2}^d \right)^{\otimes 2} \right)_{\text{sym}} = d^2 + d > d^2.
$$

Case II: $c_1 \neq 0$ and $c_2 \neq 0$. Apply Lemma 18 and the identity $x_k x_k^* = c_1 \cdot I_d$:

$$
A = \frac{2}{c_2} \sum_{k \in [n]} x_k x_k^* A x_k x_k^* - \text{tr}A \cdot I_d
$$

$$
= \frac{2}{c_2} \sum_{k \in [n]} x_k x_k^* A x_k x_k^* - \text{tr}A \cdot \frac{1}{c_1} \sum_{k \in [n]} x_k x_k^*
$$

$$
= \sum_{k \in [n]} \left( \frac{2}{c_2} x_k^* A x_k - \frac{1}{c_1} \text{tr}A \right) x_k x_k^*
$$

for every $A \in \mathbb{F}_{q^2}^{d \times d}$.

It follows that the $\mathbb{F}_{q^2}$-span of $\{x_k x_k^*\}_{k \in [n]}$ is $\mathbb{F}_{q^2}^{d \times d}$, and so $n \geq d^2$.

Case III: $c_1 = 0$ and $c_2 \neq 0$. Lemma 18 implies that the $\mathbb{F}_{q^2}$-span of $\{x_k x_k^*\}_{k \in [n]} \cup \{I_d\}$ is $\mathbb{F}_{q^2}^{d \times d}$, while the identity $x_k x_k^* = c_1 \cdot I_d = 0$ implies that $\{x_k x_k^*\}_{k \in [n]}$ is linearly dependent. Since $\{x_k x_k^*\}_{k \in [n]} \cup \{I_d\}$ is a linearly dependent spanning set of size $n + 1$, it follows that $n + 1 \geq d^2 + 1$, that is, $n \geq d^2$.

Theorem 19. Any two of the following statements together imply the third statement:

(a) $\{x_k\}_{k \in [n]}$ in $\mathbb{F}_{q^2}^d$ is a projective 2-design.
(b) $n = d^2$.
(c) There exist $a, b, c_1 \in \mathbb{F}_q$ such that
   (i) $a^2 \neq b$. 
(ii) \( a^2 - b = \frac{bc_1}{a} \) if \( a \neq 0 \),

(iii) \( d \equiv -1 \mod p \) if \( a = 0 \), and

(iv) \( \{x_k\}_{k \in [n]} \) in \( \mathbb{F}^d_{q^2} \) is an \( (a, b, c_1) \)-equiangular tight frame.

When (a), (b), and (c) hold, \( \{x_k\}_{k \in [n]} \) is an \( (a, c_1, c_2) \)-projective 2-design with \( c_2 = 2(a^2 - b) \).

To prove Theorem 19, we need a method of demonstrating that a collection of vectors forms a projective 2-design. For this, we will apply the following:

**Lemma 20.** Take \( \mathbb{F} \) to be \( \mathbb{C} \) or \( \mathbb{F}_{q^2} \). Given \( \{x_k\}_{k \in [n]} \) in \( \mathbb{F}^d \), define \( \Psi : \mathbb{F}^{d \times d} \to \mathbb{F}^{d \times d} \) by

\[
\Psi(A) := \sum_{k \in [n]} x_k x_k^* A x_k x_k^*.
\]

Then

\[
\sum_{k \in [n]} \left(x_k^{\otimes 2}\right)^* \left(x_k^{\otimes 2}\right) = \sum_{i \in [d]} \sum_{j \in [d]} e_i e_j^* \otimes \Psi(e_i e_j^*).
\]

To prove Lemma 20, we will use lemmas from the previous section, with the appropriate interpretation of conjugation in the case \( \mathbb{F} = \mathbb{F}_{q^2} \); indeed, the proofs of these results are valid under this interpretation.

**Proof of Lemma 20.** Consider the linear map \( \Psi : \mathbb{F}^{d \times d} \to \mathbb{F}^{d \times d} \) defined by \( \Psi(A) := \overline{\Psi(A)} \). Then

\[
(T \circ \Psi)(A) = \Psi(A)^* = \sum_{k \in [n]} x_k x_k^* A x_k x_k^* = \sum_{k \in [n]} x_k x_k^* A \left(x_k x_k^T \right)^*.
\]

Lemma 8 then gives

\[
\Psi(A) = \sum_{k \in [n]} x_k x_k^T A \left(x_k x_k^T \right)^*.
\]

Finally, we apply Lemma 9 to get

\[
\sum_{k \in [n]} \left(x_k^{\otimes 2}\right)^* \left(x_k^{\otimes 2}\right) = \sum_{i \in [d]} \sum_{j \in [d]} e_i e_j^* \otimes \Psi(e_i e_j^*),
\]

and we take conjugates of both sides to obtain the result. \( \square \)

**Proof of Theorem 19.** First, \( (a) \land (c) \Rightarrow (b) \) follows from Theorem 17 and Proposition 14. Next, we demonstrate \( (a) \land (b) \Rightarrow (c) \) by considering each case in the proof of Theorem 17.
Case I: \( c_2 = 0 \). This case does not occur since \( n \geq d^2 + d \) implies \( n \neq d^2 \). For the remaining cases, we have \( c_2 \neq 0 \). For these cases, we will use the fact that \( a = 0 \) if and only if \( c_1 = 0 \). To see this, apply Lemma 18 with \( A = I_d \) and the identity 
\[
\sum_{k \in [n]} x_k A x_k^* = c_1 \cdot I_d
\]
We have
\[
I_d = \frac{2}{c_2} \sum_{k \in [n]} x_k x_k^* I_d x_k x_k^* - \text{tr} I_d \cdot I_d = \frac{2a}{c_2} \sum_{k \in [n]} x_k x_k^* - d \cdot I_d
\]
To see this, apply Lemma 18 with \( A = I_d \) and the identity 
\[
x_k x_k^* = d \cdot I_d
\]
If \( 0 \in \{a, c_1\} \), then the above identity implies \( d \equiv -1 \mod p \) and \( n = d^2 \equiv 1 \mod p \). Furthermore, since \( \{x_k\}_{k \in [n]} \) is an \((a, c_1)\)-NTF, Proposition 13(a) gives that \( na = dc_1 \). If \( 0 \in \{a, c_1\} \), then squaring both sides gives
\[
a^2 = n^2 a^2 = d^2 c_1^2 = c_1^2.
\]
It follows that \( a = 0 \) if and only if \( c_1 = 0 \), as claimed. Furthermore, \( d \equiv -1 \mod p \) if \( a = 0 \).

Case II: \( c_1 \neq 0 \) and \( c_2 \neq 0 \). For every \( A \in \mathbb{F}_{q^2}^{d \times d} \), Lemma 18 and the identity
\[
\sum_{k \in [n]} x_k A x_k^* = c_1 \cdot I_d
\]
together imply
\[
A = \sum_{k \in [n]} \left( \frac{2}{c_2} x_k A x_k - \frac{1}{c_1} \text{tr} A \right) x_k x_k^*. 
\tag{7}
\]
Since \( \{x_k x_k^*\}_{k \in [n]} \) is a spanning set of \( \mathbb{F}_{q^2}^{d \times d} \) of size \( n = d^2 = \dim(\mathbb{F}_{q^2}^{d \times d}) \), it is also a basis. Then the decomposition (7) is unique. For \( A := x_\ell x_\ell^* \), this implies
\[
\frac{2}{c_2} \langle x_k, x_\ell \rangle^{q+1} - \frac{a}{c_1} = \frac{2}{c_2} x_k A x_\ell - \frac{1}{c_1} \text{tr} A = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell. \end{cases} 
\tag{8}
\]
It follows that
\[
\langle x_k, x_\ell \rangle^{q+1} = \frac{ac_2}{2c_1} =: b
\]
whenever \( k \neq \ell \), that is, \( \{x_k\}_{k \in [n]} \) is an \((a, b, c_1)\)-ETF. Then (8) gives
\[
\frac{2}{c_2} a^2 - \frac{a}{c_1} = 1, 
\tag{9}
\]
\[
\frac{2}{c_2} b - \frac{a}{c_1} = 0. 
\tag{10}
\]
Subtract (10) from (9) and rearrange to get
\[
a^2 - b = \frac{c_2}{2} \neq 0. 
\tag{11}
\]
Finally, since \( a \neq 0 \), we may rearrange (10) to get \( \frac{c_2}{2} = \frac{b c_1}{a} \), which combined with (11) gives \( a^2 - b = \frac{b c_1}{a} \), as claimed.

**Case III:** \( c_1 = 0 \) and \( c_2 \neq 0 \). We claim that the \( \mathbb{F}_q^2 \)-span of \( \{x_k x_k^* \}_{k \in [n]} \) equals the \( (d^2 - 1) \)-dimensional subspace \( \{X \in \mathbb{F}_q^{d \times d} : \text{tr} X = 0 \} \). Indeed, the inclusion \( \subseteq \) follows from the fact that \( \text{tr}(x_k x_k^*) = a = 0 \) for every \( k \in [n] \). The reverse inclusion follows from a dimension count, since Lemma 18 implies that the \( \mathbb{F}_q^2 \)-span of \( \{x_k x_k^* \}_{k \in [n]} \cup \{I_d\} \) is all of \( \mathbb{F}_q^{d \times d} \). Next, since \( n = (d^2 - 1) + 1 \), it follows that the identity \( \sum_{k \in [n]} x_k x_k^* = c_1 \cdot I_d \) gives the only linear dependency of \( \{x_k x_k^* \}_{k \in [n]} \) up to scalar multiplication, namely, \( \sum_{k \in [n]} x_k x_k^* = 0 \). Taking \( A := x_{\ell} x_{\ell}^* \) in Lemma 18 gives

\[
x_{\ell} x_{\ell}^* = \frac{2}{c_2} \sum_{k \in [n]} x_k x_k^* x_{\ell} x_{\ell}^* x_k x_k^* - \text{tr} x_{\ell} x_{\ell}^* \cdot I_d
\]

and rearranging gives

\[
\sum_{k \in [n]} z_{k \ell} x_k x_k^* = 0,
\]

where

\[
z_{k \ell} := \begin{cases} 
\frac{2}{c_2} \langle x_k, x_{\ell} \rangle^{q+1} & \text{if } k \neq \ell \\
\frac{2}{c_2} & \text{if } k = \ell 
\end{cases}
\]

Since \( \sum_{k \in [n]} x_k x_k^* = 0 \) is the unique dependency up to scaling, the dependency (12) requires \( z_{k \ell} = -1 \) for every \( k \neq \ell \), that is,

\[
\langle x_k, x_{\ell} \rangle^{q+1} = \frac{c_2}{2} = b.
\]

As such, \( \{x_k\}_{k \in [n]} \) is an \((a, b, c_1)\)-ETF. Since \( c_2 \neq 0 \), we have \( b \neq 0 = a^2 \), as claimed.

Finally, we demonstrate \((b) \Rightarrow (c) \Rightarrow (a)\) with the help of Lemma 20. To this end, we consider the linear map \( \Psi^* : A \mapsto \Psi(A)^* \). Since \( \{x_k\}_{k \in [n]} \) is an \((a, b, c_1)\)-ETF, then
\[ \Psi^* (x_\ell x_\ell^*) = \sum_{k \in [n]} x_k x_k^* x_\ell x_\ell^* x_k x_k^* = \sum_{k \in [n]} (x_k, x_\ell)^{q+1} x_k x_k^* \]
\[ = a^2 x_\ell x_\ell^* + \sum_{\substack{k \in [n] \\k \neq \ell}} b x_k x_k^* = (a^2 - b) x_\ell x_\ell^* + b c_1 \cdot I_d. \]

This expression obfuscates the linearity of \( \Psi^* \), which we elucidate in two separate cases.

**Case I:** \( a \neq 0 \). Since \( \text{tr}(x_\ell x_\ell^*) = a \), we may continue (13):
\[ \Psi^* (x_\ell x_\ell^*) = (a^2 - b) x_\ell x_\ell^* + \frac{b c_1}{a} \cdot \text{tr}(x_\ell x_\ell^*) \cdot I_d. \]

Since \( 0 \neq a^2 \neq b \), it follows from Proposition 14 that the \( \mathbb{F}_q \)-span of \( \{x_k x_k^* \}_{k \in [n]} \) has dimension \( \dim_{\mathbb{F}_q} \{X \in \mathbb{F}_q^{d \times d} : X = X^* \} = d^2 \), and so it equals \( \mathbb{F}_q^{d \times d} \). Thus, we may linearly extend the above identity to get
\[ \Psi^*(A) = (a^2 - b) A + \frac{b c_1}{a} \cdot \text{tr}A \cdot I_d \]
for every \( A \in \mathbb{F}_q^{d \times d} \). In particular, we have
\[ \Psi(e_i e_j^*) = \left( \Psi^*(e_i e_j^*) \right)^* = \left( (a^2 - b) e_i e_j^* + \frac{b c_1}{a} \cdot \text{tr}(e_i e_j^*) \cdot I_d \right)^* \]
\[ = (a^2 - b) e_i e_j^* + \frac{b c_1}{a} \cdot \delta_{ij} \cdot I_d = (a^2 - b) \cdot \left( e_i e_i^* + \delta_{ij} \cdot I_d \right), \]
where the last step applies our assumption that \( a^2 - b = \frac{bc_1}{a} \). Then Lemma 20 gives
\[ \sum_{k \in [n]} (x_k \otimes^2 \! x_k^* \otimes \Psi(e_i e_j^*) \sum_{i \in [d]} \sum_{j \in [d]} e_i e_j^* \otimes \Psi(e_i e_j^*) \]
\[ = 2(a^2 - b) \sum_{i \in [d]} \sum_{j \in [d]} e_i e_j^* \otimes \frac{1}{2} \left( e_i e_i^* + \delta_{ij} \cdot I_d \right) = 2(a^2 - b) \cdot \Pi_d^2. \]

Since \( c_2 := 2(a^2 - b) \neq 0 \) by assumption, it follows that \( \{x_k \}_{k \in [n]} \) is an \((a, c_1, c_2)\)-projective 2-design, as desired.

**Case II:** \( a = 0 \). Then \( d \equiv -1 \mod p \) by assumption. Furthermore, since \( \{x_k \}_{k \in [n]} \) is an \((a, c_1)\)-NTF, Proposition 14(a) gives that \( 0 = na = dc_1 = -c_1 \), that is, \( c_1 = 0 \). With this, we continue (13):
\[ \Psi^* (x_\ell x_\ell^*) = (a^2 - b) x_\ell x_\ell^* + b c_1 \cdot I_d = -b x_\ell x_\ell^*. \]
By equality in Gerzon’s bound, Proposition 14 implies that the $\mathbb{F}_q$-span of $\{x_kx_k^*\}_{k \in [n]}$ has dimension $\dim_{\mathbb{F}_q}[X \in \mathbb{F}_q^{d \times d} : X = X^*, \text{tr}X = 0] = d^2 - 1$, and therefore equals $\{X \in \mathbb{F}_q^{d \times d} : \text{tr}X = 0\}$. By linearity, (14) implies $\Psi^\ast(A) = -bA$ for every $A \in \mathbb{F}_q^{d \times d}$ with $\text{tr}A = 0$. To determine $\Psi^\ast(A)$ for all $A$, we also consider

$$\Psi^\ast(I_d) = \sum_{k \in [n]} x_kx_k^*I_d x_kx_k^* = a \sum_{k \in [n]} x_kx_k^* = 0.$$ 

Since $\{x_kx_k^*\}_{k \in [n]} \cup \{I_d\}$ spans $\mathbb{F}_q^{d \times d}$, we may obtain a formula for $\Psi^\ast(A)$ by extending linearly. To this end, denote

$$A := A + \text{tr}A \cdot I_d = A - \frac{\text{tr}A}{d} \cdot I_d \in \{X \in \mathbb{F}_q^{d \times d} : \text{tr}X = 0\}.$$ 

Then

$$\Psi^\ast(A) = \Psi^\ast(A) - \text{tr}A \cdot I_d = \Psi^\ast(A) - \text{tr}A \cdot \Psi^\ast(I_d) = -bA$$ 

$$= -b(A + \text{tr}A \cdot I_d).$$

In particular, we have

$$\Psi\left(e_i e_j^*\right) = \left(\Psi^\ast\left(e_i e_j^*\right)\right)^\ast = \left(-b\left(e_i e_j^* + \text{tr}\left(e_i e_j^*\right) \cdot I_d\right)\right)^\ast = -b\left(e_i e_j^* + \delta_{ij} \cdot I_d\right).$$

Then Lemma 20 gives

$$\sum_{k \in [n]} (x_k^\otimes 2)(x_k^\otimes 2)^* = \sum_{i \in [d]} \sum_{j \in [d]} e_i e_j^* \otimes \Psi\left(e_i e_j^*\right)$$

$$= -2b \sum_{i \in [d]} \sum_{j \in [d]} e_i e_j^* \otimes \frac{1}{2}\left(e_i e_j^* + \delta_{ij} \cdot I_d\right) = -2b \cdot \Pi_d^{(2)}.$$ 

Since $c_2 := -2b \neq -2a^2 = 0$ by assumption, it follows that $\{x_k\}_{k \in [n]}$ is an $(a, c_1, c_2)$-projective 2-design, as desired. 

\[\square\]

### 3.2 A construction for gerzon equality

Theorem 19 allows one to easily identify projective 2-designs over $\mathbb{F}_d$. For example, [27] constructs a $(0, 1, 0)$-ETF of $d^2$ vectors in $\mathbb{F}_3^{d \times d}$ for every $d = 2^{2\ell+1}$ with $\ell \in \mathbb{N}$. Since $2^{2\ell+1} \equiv -1 \mod 3$, Theorem 19 implies that each of these systems of vectors forms a $(0, 0, 1)$-projective 2-design for $\mathbb{F}_3^{d \times d}$. The following result constructs additional examples:

**Theorem 21.** Select any prime $p$, positive integer $k$, and prime power $r$ such that $p$ divides $r - 1$ and $r^2 + r + 1$ divides $p^k + 1$. Put $q := p^k$ and $d := r^2 + r + 1$. Let
\( D \subseteq \mathbb{Z}/d\mathbb{Z} \) denote the Singer difference set, select a primitive element \( \alpha \in \mathbb{F}_q^\times \), put \( \omega := \alpha^{(q^2 - 1)/d} \), and define translation and modulation operators by

\[
(Tf)(x) := f(x - 1), \quad (Mf)(x) := \omega^x \cdot f(x), \quad f : \mathbb{Z}/d\mathbb{Z} \to \mathbb{F}_q^d.
\]

Then \( \{M^sT^t1_D\}_{s,t} \in \mathbb{Z}/d\mathbb{Z} \) is a \((2, 1, 2d)\)-equiangular tight frame of \( d^2 \) vectors in \( \mathbb{F}_q^d \).

The ETF construction in Theorem 21 is a finite field analog of a biangular Gabor frame that was suggested in \([9, 29]\). In the finite field setting, one might view this as a Steiner ETF \([22]\) in which a harmonic ETF \([51, 49, 56, 16]\) plays the role of a “flat” simplex. Empirically, there are many \((p, k, r) \in \mathbb{N}^3\) that satisfy the constraints that \( p \) is prime, \( r \) is a prime power, \( p \) divides \( r - 1 \), and \( r^2 + r + 1 \) divides \( p^k + 1 \). In fact, there are infinitely many, conditioned on the Lenstra–Pomerance–Wagstaff conjecture of the infinitude of Mersenne primes: whenever there exists a Mersenne prime \( r = 2^m - 1 \), we may take \( p = 2 \) and \( k = 3m \). We claim that the ETF construction in Theorem 21 forms a projective 2-design for \( \mathbb{F}_q^d \) whenever \( p > 3 \). First, we have \( a^2 = 4 \neq 1 = b \) and \( a = 2 \neq 0 \), and so by Theorem 19, it suffices to verify that \( a^2 - b = \frac{bc_1}{a} \).

Indeed, \( a^2 - b = 4 - 1 = 3 \) and \( \frac{bc_1}{a} = d \), and since \( p \) divides \( r - 1 \) by assumption, we have \( d = r^2 + r + 1 \equiv 1 + 1 + 1 \equiv 3 \mod p \), as desired. The following table lists the smallest dimensions for which Theorem 21 offers a construction, with gray columns indicating projective 2-designs.

| \( d \) | 13 | 57 | 73 | 307 | 757 | 993 | 1723 | 2173 | 72257 | 2257 | 2257 | 2451 | 3541 | 3541 | 5113 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( p \) | 2 | 2 | 7 | 2 | 2 | 2 | 2 | 2 | 5 | 2 | 23 | 2 | 2 | 29 | 2 |
| \( k \) | 6 | 9 | 12 | 12 | 51 | 378 | 15 | 287 | 287 | 90 | 30 | 63 | 118 | 590 | 213 |
| \( r \) | 3 | 7 | 8 | 17 | 27 | 31 | 41 | 41 | 47 | 47 | 49 | 59 | 59 | 71 |

**Proof of Theorem 21.** Proposition 6.1 in [27] gives that \( \{M^ST^V1_D\}_{s,t} \in \mathbb{Z}/d\mathbb{Z} \) is an \((a, da)\)-NTF with

\[
a = \langle 1_D, 1_D \rangle = |D| \equiv 2 \mod p,
\]

where the last step applies the fact that \(|D| = r + 1 \equiv 2 \mod p\). It remains to verify equiangularity with parameter \( b = 1 \). To this end, we compute

\[
\langle M^ST^V1_D, M^{su}T^v1_D \rangle^{q+1}
\]

for every \((s, t) \neq (u, v)\) in two cases.

**Case I:** \( t = v \). Consider the discrete Fourier transform matrix \( \mathcal{F} = [\omega^{ij}]_{k,l} \in \mathbb{F}_q^{d \times d} \), and observe that \( \{M^ST^V1_D\}_{s,t} \in \mathbb{Z}/d\mathbb{Z} \) equals the columns of \( \text{diag}(1_D) \mathcal{F} \), which in turn is a zero-padded version of the \( D \times d \) submatrix \( \mathcal{F}_{D+t} \). Since \( D + t \) is a difference set with parameter \( \lambda = 1 \), (the proof of) Theorem 5.7 in [27] gives that \( \mathcal{F}_{D+t} \) is a \((|D|, |D| - \lambda, d)\)-ETF. It follows that

\[
\langle M^ST^V1_D, M^{su}T^v1_D \rangle^{q+1} = |D| - \lambda \equiv 1 \mod p
\]
whenever \( s \neq u \).

**Case II:** \( t \neq v \). We exploit the well-known fact that the so-called *development*

\[
\{ D + z : z \in G \}
\]

of a difference set \( D \subseteq G \) with parameter \( \lambda \) gives a symmetric block design in which every pair of blocks intersects in exactly \( \lambda \) points (see Theorem 18.6 in [35]). Since the Singer difference set has parameter \( \lambda = 1 \), we have

\[
\langle M^T 1_D, M^T 1_D \rangle = \sum_{x \in \mathbb{Z}/d\mathbb{Z}} \langle M^T 1_D(x) \rangle^\ell (M^T 1_D)(x) = \sum_{x \in \mathbb{Z}/d\mathbb{Z}} (\log^T 1_D(x - t))(\log^T 1_D(x - v)) = \sum_{x \in \mathbb{Z}/d\mathbb{Z}} \log^T 1_D(x - t) \cdot 1_D(x - v) = \log^T 1_D(x_0),
\]

where \( x_0 \) is the unique member of the block intersection \((D + t) \cap (D + v)\). Then

\[
\langle M^T 1_D, M^T 1_D \rangle^{q + 1} = \left( \log^T 1_D(x_0) \right)^{q + 1} = 1,
\]

as desired. \( \square \)

## 4 | THE QUATERNIONIC SETTING

In this section, we follow the lead of [13,17,53] to consider designs in quaternionic spaces. Consider the following generalization of Proposition 1(c) for \( F \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \} \); see [42,53]. Put \( m \coloneqq \frac{1}{2} \cdot [F : \mathbb{R}] \) and \( N = md \), and given \( A, B \in F^{d \times d} \), define \( \langle A, B \rangle \coloneqq \text{Retr}(A^*B) \). We say unit vectors \( \{x_k\}_{k \in [n]} \) in \( F^d \) form a projective 2-design if

\[
\frac{1}{n^2} \sum_{k \in [n]} \sum_{\ell \in [n]} \langle x_k x_k^*, x_{\ell x_{\ell}^*} \rangle^\ell \frac{m}{N}, \quad \frac{1}{n^2} \sum_{k \in [n]} \sum_{\ell \in [n]} \langle x_k x_k^*, x_{\ell x_{\ell}^*} \rangle^\ell \frac{m}{N(N + 1)}.
\]

We say a projective 2-design \( \{x_k\}_{k \in [n]} \) for \( F^d \) is tight if it is also equiangular, which occurs precisely when equality is achieved in the lower bound \( n \geq d + m(d^2 - d) \) [4]. In this section, we use tight projective 2-designs in quaternionic spaces to form nice arrangements of real subspaces. Given \( r \)-dimensional subspaces \( \{S_k\}_{k \in [n]} \) of a real vector space \( V \), select an orthonormal basis \( \{x_{kl}\}_{l \in [r]} \) for each \( S_k \) and compute the cross-Gramians \( G_{k\ell} \coloneqq \langle x_{kl}, x_{l\ell} \rangle_{l \in [r]} \). We say \( \{S_k\}_{k \in [n]} \) is equi-isoclinic if there exists \( \alpha \geq 0 \) such that \( G_{k\ell}^* G_{k\ell} = \alpha I_r \) for every \( k, \ell \in [n] \) with \( k \neq \ell \); see Theorem 2.3 in [39] for equivalent formulations of equi-isoclinism. We say \( \{S_k\}_{k \in [n]} \) forms a tight fusion frame if

\[
\frac{1}{n^2} \sum_{k \in [n]} \sum_{\ell \in [n]} \|G_{k\ell}\|^2_F = \frac{r^2}{\dim V}.
\]
An equi-isoclinic tight fusion frame is an optimal code in the Grassmannian \( \text{Gr}(r, V) \) under the \textit{chordal distance}, as it achieves equality in the \textit{simplex bound} \[14\]. It is also an optimal packing in terms of the \textit{spectral distance}, where we view each \( S_k \) as a subset of projective space and seek to maximize the minimum distance between these subsets of this metric space \[15\]. One may rightly view such objects as analogs of equiangular tight frames.

Since quaternion multiplication is noncommutative (e.g., \( ij = -ji \)), we start by carefully verifying a few simple facts that may be unfamiliar to the reader:

**Lemma 22.** \( \mathbb{H}^{d \times d} \) is a real Hilbert space with inner product \( \langle A, B \rangle = \text{Retr}(A^*B) \).

**Proof.** First, given \( u, v \in \mathbb{H} \), we observe that \( \text{Re}(uv) \) equals the dot product between the corresponding real coordinate vectors:

\[
\text{Re}((a - bi - cj - dk)(e + fi + gj + hk)) = ae + bf + cg + dh = (a, b, c, d) \cdot (e, f, g, h).
\]

Next, given \( A \in \mathbb{H}^{d \times d} \), let \( \text{vec}(A) \in \mathbb{R}^{4d^2} \) denote the vector of real coordinates of the entries of \( A \); explicitly, one may index the entries of \( \text{vec}(A) \) by \( [1, i, j, k] \times [d] \times [d] \) so that the entry with index \( (r, s, t) \) is given by \( \text{Re}(FA_{rst}) \). Then the above observation gives

\[
\text{Retr}(A^*B) = \text{Re} \sum_{j \in [d]} (A^*B)_{ij} = \text{Re} \sum_{j \in [d]} \sum_{i \in [d]} (A^*)_{ij}B_{ij} = \sum_{i \in [d]} \sum_{j \in [d]} \text{Re}A_{ij}B_{ij} = \text{vec}(A) \cdot \text{vec}(B).
\]

The result follows. \[\square\]

**Lemma 23.** Suppose \( A \in \mathbb{H}^{m \times n} \) and \( B \in \mathbb{H}^{n \times m} \). Then \( \text{Retr}(AB) = \text{Retr}(BA) \).

**Proof.** Consider the algebra homomorphism \( f : \mathbb{H} \to \mathbb{C}^{2 \times 2} \) defined by

\[
f(a + bi + cj + dk) = \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix}.
\]

Observe that \( \text{trf}(z) = 2\, \text{Re}z \). Given \( M \in \mathbb{H}^{n \times n} \), one may apply \( f \) entrywise to obtain \( f(M) \in \mathbb{C}^{2n \times 2n} \), and then \( \text{Retr}M = \frac{1}{2}\text{trf} (M) \). Applying this to \( AB \) and \( BA \) gives

\[
\text{Retr}(AB) = \frac{1}{2}\text{trf}(AB) = \frac{1}{2}\text{trf}(f(AB)) = \frac{1}{2}\text{trf}(f(A)f(B)) = \frac{1}{2}\text{trf}(f(B)f(A)) = \frac{1}{2}\text{trf}(f(BA)) = \text{Retr}(BA),
\]

as claimed. \[\square\]

**Lemma 24.** Suppose \( A, B \in \mathbb{H}^{d \times d} \) satisfy \( A^* = A \) and \( B^* = -B \). Then \( \langle A, B \rangle = 0 \).

**Proof.** Symmetry of the real inner product gives
\( \langle A, B \rangle = \langle B, A \rangle = \text{Retr}(B^*A) = \text{Retr}(-BA^*) = -\text{Retr}(A^*B) = -\langle A, B \rangle. \)

Rearrange to get the result. \(\square\)

In fact, the real subspace of anti-Hermitian matrices is the orthogonal complement of the real subspace of Hermitian matrices. This can be seen by dimension counting: Hermitian matrices have dimension \(d + 4\binom{d}{2}\), while anti-Hermitian matrices have dimension \(3d + 4\binom{d}{2}\), and the sum of these is \(4d^2\), that is, the dimension of \(\mathbb{H}^{d \times d}\).

Given \(x \in \mathbb{H}^d \setminus \{0\}\), let \(S(x)\) denote the three-dimensional real subspace of \(\mathbb{H}^{d \times d}\) defined by

\[
S(x) := \{xz^* : z \in \mathbb{H}, \text{Re} z = 0\}.
\]

For each \(A \in S(x)\), it holds that

\[
A^* = (xz^*)^* = x\overline{z} x^* = -xz^* = -A.
\]

As such, \(S(x)\) is actually a subspace of the anti-Hermitian matrices. Observe that for each \(u, v \in \{i, j, k\}\), it holds that

\[
\langle xux^*, xvx^* \rangle = \text{Retr}((xux^*)^*xvx^*) = \text{Retr}(xu^*x^*xv^*x^*) = \text{Retr}(\overline{u} x^*xv^*x) = \|x\|^4 \cdot \text{Re}(\overline{u} v) = \begin{cases} \|x\|^4 & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}
\]

Thus, \(\{xix^*, xjx^*, xkx^*\}\) is an orthogonal basis for \(S(x)\). With this, we prove the following:

**Theorem 25.** Consider unit vectors \(\{x_k\}_{k \in [n]}\) in \(\mathbb{H}^d\).

(a) If \(\{x_k\}_{k \in [n]}\) is equiangular, then \(\{S(x_k)\}_{k \in [n]}\) is equi-isoclinic.

(b) If \(\{x_k\}_{k \in [n]}\) is a projective 2-design, then \(\{S(x_k)\}_{k \in [n]}\) is tight in anti-Hermitian space.

**Proof.** Given unit vectors \(x, y \in \mathbb{H}^d\), we are interested in the cross-Gramian \(G_{xy}\) between \(\{xix^*, xjx^*, xkx^*\}\) and \(\{yiy^*, yjy^*, yky^*\}\). The entry indexed by \((u, v) \in \{i, j, k\}^2\) is given by

\[
\langle xux^*, yvy^* \rangle = \text{Retr}((xux^*)^*yvy^*) = \text{Retr}(xu^*x^*yvy^*) = \text{Retr}(\overline{u} x^*yvy^*x) = \text{Re}(\overline{u} x^*yvy^*x).
\]

If \(x^*y = 0\), then \(G_{xy} = 0\). Otherwise, define \(z := \frac{x^*y}{|x^*y|}\) and continue (15):

\[
\langle xux^*, yvy^* \rangle = \text{Re}(\overline{u} x^*yvy^*x) = |x^*y|^2 \cdot \text{Re}(\overline{u} zvz^{-1}).
\]

It follows that \(G_{xy}\) equals \(|x^*y|^2\) times a matrix representation of the special orthogonal map \(q \mapsto qzqz^{-1}\) over imaginary \(q \in \mathbb{H}\). This implies (a).

For (b), we apply the facts that \(G_{xy} = |x^*y|^2 \cdot Q\) for some \(Q \in \text{SO}(3)\) and
$|x^*y|^2 = \overline{x^*y}x^*y = y^*xx^*y = \text{Retr}(y^*xx^*y) = \text{Retr}(xx^*yy^*) = \langle xx^*, yy^* \rangle$

In order to compute the frame potential of $\{S(x_k)\}_{k \in [n]}$:

\[
\frac{1}{n^2} \sum_{k \in [n]} \sum_{\ell \in [n]} \|G_{x_kx_\ell}\|_F^2 = \frac{1}{n^2} \sum_{k \in [n]} \sum_{\ell \in [n]} 3 \left| x_k^* x_\ell \right|^4
\]

\[
= \frac{3}{n^2} \sum_{k \in [n]} \sum_{\ell \in [n]} \left\langle x_k^* x_\ell^*, x_\ell x_k^* \right\rangle^2 = 3 \cdot \frac{m(m + 1)}{N(N + 1)} = \frac{9}{d(2d + 1)}.
\]

It follows that $\{S(x_k)\}_{k \in [n]}$ forms a tight fusion frame in the $d(2d + 1)$-dimensional real vector space of $d \times d$ quaternionic anti-Hermitian matrices. □

Theorem 25 implies that every tight projective 2-design for $\mathbb{H}^d$ corresponds to an equi-isoclinic tight fusion frame of $d(2d - 1)$ subspaces of $\mathbb{R}^{d(2d+1)}$ of dimension 3. To date, such designs are only known to exist for $d \in [2, 3]$. First, $\mathbb{H}P^1$ is isometric to $S^4$, and so the six vertices of a five-dimensional regular simplex easily deliver a tight projective 2-design for $\mathbb{H}^2$; this in turn determines six subspaces of $\mathbb{R}^{10}$ of dimension 3. The $d = 3$ case is resolved by Theorem 4.12 in [13], which uses a variant of the Newton–Kantorovich theorem to obtain a computer-assisted proof of the existence of a 15-point simplex in $\mathbb{H}P^2$; this determines 15 subspaces of $\mathbb{R}^{21}$ of dimension 3. The authors are not aware of any other constructions of equi-isoclinic tight fusion frames with these parameters. Also, it is open whether tight projective 2-designs exist for $\mathbb{H}^d$ with $d > 3$.

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**REFERENCES**

1. M. Appleby, I. Bengtsson, S. Flammia, and D. Goyeneche, *Tight frames, Hadamard matrices and Zauner’s conjecture*. J. Phys. A. **52** (2019), 295301.

2. A. S. Bandeira, M. Fickus, D. G. Mixon, and P. Wong, *The road to deterministic matrices with the restricted isometry property*. J. Fourier Anal. Appl. **19** (2013), 1123–1149.

3. E. Bannai, E. Bannai, T. Ito, and R. Tanaka, *Algebraic combinatorics*. De Gruyter, 2021.

4. E. Bannai, and S. G. Hoggar, *On tight t-designs in compact symmetric spaces of rank one*, Proc. Japan Acad. Ser. A. **61** (1985), 78–82.

5. E. Bannai, A. Munemasa, and B. Venkov, *The nonexistence of certain tight spherical designs*. St. Petersburg Math. J. **16** (2005), 609–625.

6. J. J. Benedetto, and M. Fickus, *Finite normalized tight frames*. Adv. Comput. Math. **18** (2003), 357–385.

7. B. G. Bodmann, and J. Haas, *Achieving the orthoplex bound and constructing weighted complex projective 2-designs with Singer sets*. Linear Algebra Appl. **511** (2016), 54–71.
8. B. G. Bodmann, and E. J. King, *Optimal arrangements of classical and quantum states with limited purity*, Mathematics 101 (2020), 393–431.
9. I. Bojarovska, and V. Paternostro, *Gabor fusion frames generated by difference sets*. Wavelets Sparsity XVI. 9597 (2015), 95970D.
10. P. G. Casazza, and G. Kutyniok, eds. *Finite frames: Theory and applications*, Springer, 2012.
11. C. M. Caves, Symmetric informationally complete POVMs UNM Information Physics Group Internal Report, 1999. http://info.phys.unm.edu/caves/reports/infopovm.ps
12. H. Cohn, and A. Kumar, *Universally optimal distribution of points on spheres*. J. Amer. Math. Soc. 20 (2007), 99–148.
13. H. Cohn, and A. Kumar, G. Minton, *Optimal simplices and codes in projective spaces*. Geom. Topol. 20 (2016), 1289–1357.
14. J. H. Conway, R. H. Hardin, and N. J. A. Sloane, *Packing lines, planes, etc.: Packings in Grassmannian spaces*. Exp. Math. 5 (1996), 139–159.
15. I. S. Dhillon, R. W. Heath, T. Strohmer, and J. A. Tropp, *Constructing packings in Grassmannian manifolds via alternating projection*. Exp. Math. 17 (2008), 9–35.
16. C. Ding, and T. Feng, A generic construction of complex codebooks meeting the Welch bound. IEEE Trans. Inform. Theory. 53 (2007), 4245–4250.
17. B. Et-Taoui, *Quaternionic equiangular lines*. Adv. Geom. 20 (2020), 273–284.
18. L. FejesTóth, *Regular Figures*, Pergamon, 1964.
19. L. FejesTóth, Über die Abschätzung des kürzesten Abstandes zweier Punkte eines äuf einer Kugelfläche liegenden Punktsystems, Jber. Deutch. Math. Verein. 53 (1943), 66–68.
20. M. Fickus, J. Jasper, D. G. Mixon, and J. Peterson, *Tremain equiangular tight frames*. J. Combin. Theory A. 153 (2018), 54–66.
21. M. Fickus, and D. G. Mixon, Tables of the existence of equiangular tight frames. arXiv. https://arxiv.org/abs/1504.00253
22. M. Fickus, D. G. Mixon, and J. C. Tremain, *Steiner equiangular tight frames*. Linear Algebra Appl. 436 (2012), 1014–1027.
23. S. T. Flammia, Exact SIC fiducial vectors. http://www.physics.usyd.edu.au/sflammia/SIC/
24. C. A. Fuchs, M. C. Hoang, and B. C. Stacey, *The SIC question: History and state of play*. Axioms. 6 (2017), 21.
25. N. I. Gillespie, Equiangular lines, incoherent sets and quasi-symmetric designs. arXiv. https://arxiv.org/abs/1809.05739
26. M. Grassl, SIC-POVMs. http://sicpovm.markus-grassl.de/
27. G. R. W. Greaves, J. W. Iverson, J. Jasper, and D. G. Mixon, Frames over finite fields: Basic theory and equiangular lines in unitary geometry. arXiv. https://arxiv.org/abs/2012.12977
28. G. R. W. Greaves, J. W. Iverson, J. Jasper, and D. G. Mixon, Frames over finite fields: Equiangular lines in orthogonal geometry. arXiv. https://arxiv.org/abs/2012.13642
29. J. I. Haas, J. Cahill, J. Tremain, and P. G. Casazza, Constructions of biangular tight frames and their relationships with equiangular tight frames. arXiv. https://arxiv.org/abs/1703.01786
30. I. Hargittai, and B. Chamberland, *The VSEPR model of molecular geometry*. Comput. Math. Appl. 12 (1986), 1021–1038.
31. J. W. Iverson, J. Jasper, and D. G. Mixon, Optimal line packings from finite group actions, *Forum Math. Sigma*, 8 (2020). https://doi.org/10.1017/fms.2019.48
32. J. W. Iverson, J. Jasper, and D. G. Mixon, *Optimal line packings from nonabelian groups*. Discrete Comput. Geom. 63 (2020), 731–763.
33. J. W. Iverson, and D. G. Mixon, Doubly transitive lines I: Higman pairs and roux. arXiv. https://arxiv.org/abs/1806.09037
34. J. W. Iverson, and D. G. Mixon, Doubly transitive lines II: Almost simple symmetries. arXiv. https://arxiv.org/abs/1905.06859
35. D. Jungnickel, A. Pott, and K. W. Smith, *Difference Sets, Handbook of Combinatorial Designs*, Chapman and Hall/CRC, 2006, pp. 445–461.
36. KCIK, Award on Quantum Information of Polish National Quantum Information Centre (KCIK). https://kcik.ug.edu.pl/post.php?id=1981
37. G. S. Kopp, SIC-POVMs and the Stark conjectures. Int. Math. Res. Not. (2019).
38. P. W. H. Lemmens, and J. J. Seidel, Equiangular lines. J. Algebra. 24 (1973), 494–512.
39. P. W. H. Lemmens, and J. J. Seidel, Equi-isoclinic subspaces of Euclidean spaces. Nederl. Akad. Wetensch. Proc. Ser. A. 76 (1973), 98–107.
40. A. A. Makhnev, On the nonexistence of strongly regular graphs with the parameters (486, 165, 36, 66). Ukr. Math. J. 54 (2002), 1137–1146.
41. D. G. Mixon, C. J. Quinn, N. Kiyavash, and M. Fickus, Fingerprinting with equiangular tight frames. IEEE Trans. Inform. Theory. 59 (2013), 1855–1865.
42. A. Munemasa, Spherical designs. Handbook of Combinatorial Designs, (2007), 637–643.
43. G. Nebe, and B. Venkov, On tight spherical designs. St. Petersburg Math. J. 24 (2013), 485–491.
44. S. K. Pandey, V. I. Paulsen, J. Prakash, and M. Rahaman, Entanglement breaking rank and the existence of SIC POVMs. J. Math. Phys. 61 (2020), 042203.
45. V. Paulsen, Entanglement Breaking Maps and Zauner’s Conjecture, Codes and Expansions online seminar. https://www.youtube.com/watch?v=VpVwb_i7s0I
46. A. Roy, and A. J. Scott, Weighted complex projective 2-designs from bases: Optimal state determination by orthogonal measurements. J. Math. Phys. 48 (2007), 072110.
47. A. J. Scott, Tight informationally complete quantum measurements. J. Phys. A. 39 (2006), 13507.
48. P. D. Seymour, and T. Zaslavsky, Averaging sets: A generalization of mean values and spherical designs. Adv. Math. 52 (1984), 213–240.
49. T. Strohmer, and R. W. Heath, Grassmannian frames with applications to coding and communication. Appl. Comput. Harmon. Anal. 14 (2003), 257–275.
50. J. J. Thomson, XXIV. On the structure of the atom: An investigation of the stability and periods of oscillation of a number of corpuscles arranged at equal intervals around the circumference of a circle; with application of the results to the theory of atomic structure. London Edinburgh Dublin Philos. Mag. J. Sci. 7 (1904), 237–265.
51. R. J. Turyn, Character sums and difference sets. Pacific J. Math. 15 (1965), 319–346.
52. S. Waldron, A sharpening of the Welch bounds and the existence of real and complex spherical t-designs. IEEE Trans. Inform. Theory. 63 (2017), 6849–6857.
53. S. Waldron, A variational characterisation of projective spherical designs over the quaternions. arXiv. https://arxiv.org/abs/2011.08439
54. J. Watrous, The Theory of Quantum Information, Cambridge University Press, 2018. https://doi.org/10.1017/9781108383595
55. L. Welch, Lower bounds on the maximum cross correlation of signals. IEEE Trans. Inform. Theory. 20 (1974), 397–399.
56. P. Xia, S. Zhou, and G. B. Giannakis, Achieving the Welch bound with difference sets. IEEE Trans. Inform. Theory. 51 (2005), 1900–1907.
57. G. Zauner, Quantum designs—Foundations of a non-commutative theory of designs, Ph.D. thesis, University of Vienna, 1999.

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