Abstract. Associated with a given graph $G$, typically an infinite tree, and motivated by applications, we introduce two families of operators in a Hilbert space $H_G$ induced by $G$. To realize the Hilbert space, we first develop some representation theory. We obtain the first family of operators on $H_G$ by an extension of the more familiar case of groups: free representations of the group-algebra. Because of their classical counterparts, we call the operators in our first family, graph operators; and the second Toeplitz operators. We focus on the interconnections between the two families. We introduce and study graph operators in two steps: first, starting with a fixed graph $G$, we introduce a groupoid over $G$; and from this, the groupoid von Neumann algebra $M_G$. Our graph operators will then be finitely supported elements of $M_G$.

1. Introduction

Before, starting our problem, we open with a historical comment, and a comparison between the case of groups and graphs. In a number of recent papers there have been a variety of different approaches to introducing algebras of operators in Hilbert space (for a sample, see the papers cited below). A number of these ideas are motivated by what works for groups, i.e., starting with the group algebra, and then build representations of it. Each representation serves some purpose, or is dictated by an application, for example to harmonic analysis or to quantum mechanics. More than half a century ago, von Neumann introduced the ring of operators (now called von Neumann algebras) generated by the free group $F_n$ with $n$-generators, leading to non-hyperfinite factors $L(F_n)$ (See [27]). While the construction is simple enough, the questions are difficult. Now, for $F_n$, the natural Hilbert space is $l^2(F_n)$. Since a group acts on itself, we get operators in $l^2(F_n)$, i.e., regular representation; and $L(F_n)$ is simply the von Neumann algebra generated by the regular representation.

Now, let $G$ be a countable directed graph, i.e., a system of vertices and edges (with direction) subject to simple axioms, details below. It is tempting, in the analysis of graphs, to mimic some of the constructions used for groups. But a glance at the comments above and literature shows that there are difficulties for graphs that do not arise in the case of groups. A key idea we employ is in brief outline this: Starting with a graph $G$, we introduce first an “enveloping” groupoid
G and a groupoid algebra $A_G$. We show that $A_G$ contains a canonical abelian subalgebra $D_G$ (the letter $D$ for diagonal!) and a conditional expectation $E_G$ from $A_G$ onto $D_G$. We are then able to apply Stinespring’s theorem (See [26]) to $E_G$. The resulting representation is acting on the Stinespring Hilbert space $H_G$, and this will be the starting point of our analysis.

For the benefit of the readers, we collect here some references: [26] the paper by Stinespring. While there are several relevant papers about von Neumann’s construction, the following will do for our present purpose [27]. There is a diverse set of approaches to Hilbert space, operators, and operator algebras, and we list here only a sample: [4], [5], [6], [8], [14], [15], [23], [24], and [25], dealing with graph groupoid dynamical systems and corresponding crossed product von Neumann algebras, graph Laplacian Operators, and reproducing kernels, etc. In a different direction, there is a large literature on graph $C^*$-algebras, see for example [20], [21], and [22]. For relevant papers in graph theory proper, we cite [1], [2], [16], [17], [18], and [19]; again just a small sample.

Starting with analysis on countable directed graphs $G$, we introduce Hilbert spaces $H_G$ and a family of weighted operators $T$ on $H_G$. When the weights (which are called coefficients later in context) are chosen, $T$ is called a graph operator. From its weights (or coefficients), we define the support $\text{Supp}(T)$ of $T$.

Let $G$ be a countable directed graph. Then there exists a corresponding algebraic structure $G$, as a form of groupoid (e.g., see [4], [9], and [10]). Such a groupoid $G$ is called the graph groupoid of $G$. By constructing the ($C^*$- or von Neumann) operator algebra $A$, generated by $G$, we can study the elements of $G$ (or $G$) as operators in $A$, under a suitable representation of $G$ (e.g., see [4], [5], and [7]). Interestingly, every operator $x$ on an arbitrary (separable countable dimensional) Hilbert space $H$ can generate the corresponding graph $G$, and the $C^*$-algebra $C^*(x)$, generated by $x$, is characterized by the groupoid crossed product algebra $A_q \times_\alpha G$ induced by the groupoid dynamical system $(A_q, G, \alpha)$, whenever $a$ is polar-decomposed by $aq$, where $a$ is the partial isometry part of $x$, and $q$ is the positive part of $x$ (e.g., see [6]).

The main motivation of our study is the above close connection between directed graphs and operators.

In [8], We defined the graph operators by the “finitely” supported operators in the von Neumann algebra $M_G = C[L(G)]^w \text{w}$, generated by $G$, acting on the Hilbert space $H_G$, where a representation $(H_G, L)$ of $G$ is the canonical representation of $G$, consisting of the Stinespring Hilbert space $H_G$, and the canonical groupoid action $L$ (Also, see Section 2 below).

Self-adjointness, the unitary property, Hyponormality and Normality of graph operators are characterized in [8]. This means that the spectral-property of graph operators are characterized. These operator-theoretic properties are characterized by the combinatorial data on supports and the analytic data on coefficients of graph operators.

In this paper, we find connections between our graph operators and the well-known Toeplitz operators.

A further point motivating our work is from the analysis of general classes of graphs. For general discrete models, there is no obvious group and therefore no Fourier duality available. Graphs typically are not endowed with a group structure
that invites any kind of Fourier duality. As a basis for our harmonic analysis, we
instead introduce a natural groupoid which serves as a substitute.

In the body of our paper, we will be using freely tools from the operators in
Hilbert space, the theory of $C^*$-algebras and von Neumann algebras. The reader
may find the following background references helpful: [35], and [36]. Recent relevant
papers on graph analysis include: [2], [4], [5], [6], [7], [28], [29], [30], [31], [32], [33],
and [34].

1.1. Overview. A graph is a set of objects called vertices (or points or nodes)
connected by links called edges (or lines). In a directed graph, the two directions
are counted as being distinct directed edges (or arcs). A graph is depicted in a
diagrammatic form as a set of dots (for vertices), jointed by curves (for edges).
Similarly, a directed graph is depicted in a diagrammatic form as a set of dots
jointed by arrowed curves, where the arrows point the direction of the directed
edges.

Recently, we have studied the operator-algebraic structures induced by directed
graphs. The key idea to study graph-depending operator algebras is that: every
directed graph $G$ induces its corresponding groupoid $G$, called the graph groupoid
of $G$. By considering this algebraic structure $G$, we can determine the groupoid
actions $\lambda$, acting on Hilbert spaces $H$. i.e., we can have suitable representations
$(H, \lambda)$ for $G$.

And this guarantees the existence of operator algebras $A_G = \mathbb{C}[\lambda(G)]$, generated by $G$ (or induced by $G$), in the operator algebras $B(H)$. Indeed, the
operator algebras $A_G$ are the groupoid topological $(C^*$- or $W^*$-)subalgebras of
$B(H)$.

It is interesting that each edge $e$ of $G$ assigns a partial isometry on $H$;
each vertex $v$ of $G$ assigns a projection on $H$ (under various different types of
representations of $G$). For the continuation of our recent research, we will fix
the canonical representation $(H_G, L)$ of $G$, and construct the corresponding von
Neumann algebra

$$M_G = \mathbb{C}[L(G)]$$

where $H_G$ is the graph Hilbert space $l^2(G)$. This von Neumann algebra $M_G$ is
called the graph von Neumann algebra of $G$.

In this paper, we are interested in certain elements $T$ of $M_G$. Recall that, by the
definition of graph von Neumann algebras, if $T \in M_G$, then

$$T = \sum_{w \in G} t_w L_w$$

with $t_w \in \mathbb{C}$.

Define the support $\text{Supp}(T)$ of $T$ by

$$\text{Supp}(T) = \{w \in G : t_w \neq 0\}.$$
isomorphic to the $l^2$-space $l^2(G)$ of $G$. Then this Hilbert space $H_G$ contains its subspace $H_V = l^2(V)$, where $V$ means the vertex set of $G$. In fact, 

$$H_G = H_V \oplus H_F,$$

for some subspace $H_F$. Remark here that, if the edge set $E$ of $G$ is nonempty, then the orthogonal complement $H_F$ of $H_V$ is nontrivial in $H_G$.

We remark here that if a graph $G$ is the $N$-regular tree $T_N$, for $N \in \mathbb{N}$, then

$$H_V^\text{Hilbert} = l^2(\mathbb{N}^{\oplus N})$$

(See Section 4 below). For example, the 2-regular tree $T_2$ is a tree

Clearly, the 1-regular tree $T_1$ is an infinite linear graph with its root,

$$\cdots \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots$$

In [8], we considered the unitarily equivalent infinite matrix $A_e$ on $l^2(\mathbb{N})$ of a graph operator $L_e$, for $e \in E_1$ of $T_1$, represented on $H_V$. Remark in this case that

$$H_V^\text{Hilbert} = l^2(\mathbb{N}),$$

and

$$A_e = \begin{pmatrix}
0 & 0 & 0 \\
\ddots & \ddots & \ddots \\
0 & 0 & 1 & 1 \\
& 0 & 0 & \ddots \\
0 & \ddots & \ddots & \ddots \\
\end{pmatrix}$$

(Also, See Section 4). This shows that the sum of graph operators

$$\sum_{e \in E_1} L_e$$

is represented by the Toeplitz operator
Also, this new graph \(N\) induced by \(G\) is well-defined on \(G\).

Based on the above observation, we study the relation between graph operators induced by \(N\)-regular trees \(T_N\) and Toeplitz operators.

2. Definitions and Background

In this section, we introduce the concepts precisely, and definitions we will use.

2.1. Graph Groupoids. Let \(G\) be a directed graph with its vertex set \(V(G)\) and its edge set \(E(G)\). Let \(e \in E(G)\) be an edge connecting a vertex \(v_1\) to a vertex \(v_2\). Then we write \(e = v_1 \rightarrow v_2\), for emphasizing the initial vertex \(v_1\) of \(e\) and the terminal vertex \(v_2\) of \(e\).

For a fixed graph \(G\), we can define the oppositely directed graph \(G^{-1}\), with \(V(G^{-1}) = V(G)\) and \(E(G^{-1}) = \{e^{-1} : e \in E(G)\}\), where each element \(e^{-1}\) of \(E(G^{-1})\) satisfies that

\[ e = v_1 \rightarrow v_2 \in E(G), \text{ with } v_1, v_2 \in V(G), \]

if and only if

\[ e^{-1} = v_2 \leftarrow v_1, \text{ in } E(G^{-1}). \]

This opposite directed edge \(e^{-1} \in E(G^{-1})\) of \(e \in E(G)\) is called the shadow of \(e\). Also, this new graph \(G^{-1}\), induced by \(G\), is said to be the shadow of \(G\). It is clear that \((G^{-1})^{-1} = G\).

Define the shadowed graph \(\widehat{G}\) of \(G\) by a directed graph with its vertex set

\[ V(\widehat{G}) = V(G) = V(G^{-1}) \]

and its edge set

\[ E(\widehat{G}) = E(G) \cup E(G^{-1}), \]

where \(G^{-1}\) is the shadow of \(G\).

We say that two edges \(e_1 = v_1 \rightarrow v'_1\) and \(e_2 = v_2 \rightarrow v'_2\) are admissible, if \(v'_1 = v_2\), equivalently, the finite path \(e_1 e_2\) is well-defined on \(\widehat{G}\). Similarly, if \(w_1\) and \(w_2\) are finite paths on \(G\), then we say \(w_1\) and \(w_2\) are admissible, if \(w_1 w_2\) is a well-defined finite path on \(G\), too. Similar to the edge case, if a finite path \(w\) has its initial vertex \(v\) and its terminal vertex \(v'\), then we write \(w = v_1 \rightarrow v_2\). Notice that every admissible finite path is a word in \(E(\widehat{G})\). Denote the set of all finite path by \(FP(\widehat{G})\). Then \(FP(\widehat{G})\) is the subset of the set \(E(\widehat{G})^*\), consisting of all finite words in \(E(\widehat{G})\).

Suppose we take a part

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & & \\
\end{pmatrix}
\]

on \(l^2(N) = H_N\).

\[
(\widehat{G})_{10} = \{e \in E(G) : \exists v, w \in V(G), e = v \rightarrow w\}
\]

\[
(\widehat{G})_{01} = \{e \in E(G) : \exists v, w \in V(G), e = w \rightarrow v\}
\]

\[
(\widehat{G})_{11} = \{e \in E(G) : \exists v, w \in V(G), e = v \rightarrow w\} \cup \{e \in E(G) : \exists v, w \in V(G), e = w \rightarrow v\}
\]

In this section, we introduce the concepts precisely, and definitions we will use.
admissible, since we can obtain a finite path $e_1e_2$, however, the edges $e_1$ and $e_3$ are not admissible, since a finite path $e_1e_3$ is undefined.

We can construct the free semigroupoid $\mathbb{F}^+(\hat{G})$ of the shadowed graph $\hat{G}$, as the union of all vertices in $V(\hat{G}) = V(G) = V(G^{-1})$ and admissible words in $FP(\hat{G})$, equipped with its binary operation, the admissibility. Naturally, we assume that $\mathbb{F}^+(\hat{G})$ contains the empty word $\emptyset$, as the representative of all undefined (or non-admissible) finite words in $E(\hat{G})$.

Remark that some free semigroupoid $\mathbb{F}^+(\hat{G})$ of $\hat{G}$ does not contain the empty word; for instance, if a graph $G$ is a one-vertex-multi-edge graph, then the shadowed graph $\hat{G}$ of $G$ is also a one-vertex-multi-edge graph too, and hence its free semigroupoid $\mathbb{F}^+(\hat{G})$ does not have the empty word. However, in general, if $|V(G)| > 1$, then $\mathbb{F}^+(\hat{G})$ always contain the empty word. Thus, if there is no confusion, we always assume the empty word $\emptyset$ is contained in the free semigroupoid $\mathbb{F}^+(\hat{G})$ of $\hat{G}$.

**Definition 2.1.** By defining the reduction (RR) on $\mathbb{F}^+(\hat{G})$, we define the graph groupoid $\mathbb{G}$ of a given graph $G$, by the subset of $\mathbb{F}^+(\hat{G})$, consisting of all “reduced” finite paths on $\hat{G}$, with the inherited admissibility on $\mathbb{F}^+(\hat{G})$ under (RR), where the reduction (RR) on $\mathbb{G}$ is as follows:

\[
(\text{RR}) \quad w w^{-1} = v \quad \text{and} \quad w^{-1}w = v',
\]

for all $w = v v' \in \mathbb{G}$, with $v, v' \in V(\hat{G})$.

Such a graph groupoid $\mathbb{G}$ is indeed a categorial groupoid with its base $V(\hat{G})$ (See Appendix A).

2.2. Canonical Representation of Graph Groupoids. Let $G$ be a given countable connected directed graph with its graph groupoid $\mathbb{G}$. Then we can define the (pure algebraic) algebra $\mathcal{A}_G$ of $\mathbb{G}$ by a vector space over $\mathbb{C}$, consisting of all linear combinations of elements of $\mathbb{G}$, i.e.,

\[
\mathcal{A}_G \overset{\text{def}}{=} \mathbb{C} \cup \left\{ \sum_{j=1}^{k} t_j w_j \mid w_j \in \mathbb{G}, \ t_j \in \mathbb{C}, \ j = 1, \ldots, k \right\},
\]

under the usual addition ($+$), and the multiplication ($\cdot$), dictated by the admissibility on $\mathbb{G}$. Define now a unary operation ($*$) on $\mathcal{A}_G$ by

\[
\sum_{j=1}^{k} t_j w_j \in \mathcal{A}_G \rightarrow \sum_{j=1}^{k} t_j w_j^{-1} \in \mathcal{A}_G,
\]

where $\overline{z}$ means the conjugate of $z$, for all $z \in \mathbb{C}$, and of course $w^{-1}$ means the shadow of $w$, for all $w \in \mathbb{G}$. We call this unary operation ($*$), the adjoint (or the shadow) on $\mathcal{A}_G$. Then the vector space $\mathcal{A}_G$, equipped with the adjoint ($*$), is a well-defined (algebraic) $*$-algebra.

Now, define a $*$-subalgebra $\mathcal{D}_G$ of $\mathcal{A}_G$ by

\[
\mathcal{D}_G \overset{\text{def}}{=} \mathbb{C} \cup \left\{ \sum_{j=1}^{n} t_j v_j \mid v_j \in V(\hat{G}), \ t_j \in \mathbb{C}, \ j = 1, \ldots, k \right\}.
\]

This $*$-algebra $\mathcal{D}_G$ acts like the diagonal of $\mathcal{A}_G$, so we call $\mathcal{D}_G$, the diagonal ($*$)-subalgebra of $\mathcal{A}_G$.

2.2.1. The Hilbert Space $H_G$. Below, we identify the canonical Hilbert space $H_G$. The algebra $\mathcal{A}_G$ is represented by bounded linear operators acting on $H_G$. The representation is induced by the canonical conditional expectation, via the Stinespring construction (e.g., see [14]).
We can construct a (algebraic *-)conditional expectation

$$E : \mathcal{A}_G \to \mathcal{D}_G$$

by

$$E \left( \sum_{w \in X} t_w w \right) \overset{def}{=} \sum_{v \in X \cap V(\hat{G})} t_v v,$$

for all $$\sum_{w \in X} t_w w \in \mathcal{A}_G$$, where $$X$$ is a finite subset of $$\mathbb{G}$$.

Since the conditional expectation $$F$$ is completely positive under a suitable topology on $$\mathcal{A}_G$$, we may apply the Stinespring’s construction. i.e., the diagonal subalgebra $$\mathcal{D}_G$$ is represented as the $$l^2$$-space, $$l^2(V(\hat{G}))$$, by the concatenation. Then we can obtain the Hilbert space $$H_G$$,

$$H_G \overset{def}{=} \text{the Stinespring space of } \mathcal{A}_G \text{ over } \mathcal{D}_G, \text{ by } F,$$

containing $$l^2(V(\hat{G}))$$, i.e., if $$\pi_{(E, \mathcal{D}_G)}$$ is the Stinespring representation of $$\mathcal{A}_G$$, acting on $$l^2(V(\hat{G}))$$,

$$H_G = \pi_{(E, \mathcal{D}_G)}(\mathcal{A}_G).$$

This Stinespring space $$H_G$$ is the Hilbert space with its inner product $$\langle ., . \rangle$$ satisfying that:

$$\langle h, \pi_{(E, \mathcal{D}_G)}(a) k \rangle_G = \langle h, E(a) k \rangle_{2},$$

for all $$h, k \in l^2(V(\hat{G}))$$, for all $$a \in \mathcal{A}_G$$, where $$\langle . , . \rangle_2$$ is the inner product on $$l^2(V(\hat{G}))$$.

i.e., The Stinespring space $$H_G$$ is the norm closure of $$\mathcal{A}_G$$, by the norm,

$$(2.2.2) \quad \left\| \sum_{j=1}^{n} w_i \otimes h_i \right\|_G^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} \langle h_i, E(w_i^* w_k) h_k \rangle_2,$$

induced by the Stinespring inner product $$\langle ., . \rangle_G$$ on $$\mathcal{A}_G$$, for all $$a_i \in \mathcal{A}_G$$, $$h_i \in l^2(V(\hat{G}))$$, for all $$n \in \mathbb{N}$$.

**Definition 2.2.** We call this Stinespring space $$H_G$$, the graph Hilbert space of $$\mathbb{G}$$ (or of $$\mathbb{G}$$).

Denote the Hilbert space element $$\pi_{(E, \mathcal{D}_G)}(w)$$ by $$\xi_w$$ in the graph Hilbert space $$H_G$$, for all $$w \in \mathbb{G}$$, with the identification,

$$\xi_\emptyset = 0_H_G, \text{ the zero vector in } H_G,$$

where $$\emptyset$$ is the empty word (if exists) of $$\mathbb{G}$$. We can check that the subset $$\{\xi_w : w \in \mathbb{G}\}$$ of $$H_G$$ satisfies the following multiplication rule:

$$\xi_{w_1} \xi_{w_2} = \xi_{w_1 w_2}, \text{ on } H_G,$$

for all $$w_1, w_2 \in \mathbb{G}$$. Thus, we can define the canonical multiplication operators $$L_w$$ on $$H_G$$, satisfying that

$$L_w \xi_{w'} \overset{def}{=} \xi_w \xi_{w'} = \xi_{w w'},$$

for all $$w, w' \in \mathbb{G}$$. The existence of such multiplication operators $$L_w$$ guarantees the existence of a groupoid action $$L$$ of $$\mathbb{G}$$, acting on $$H_G$$:

$$L : w \in \mathbb{G} \mapsto L(w) \overset{def}{=} L_w \in B(H_G).$$

This action $$L$$ of $$\mathbb{G}$$ is called the canonical groupoid action of $$\mathbb{G}$$ on $$H_G$$. 

\[ \text{TOEPLITZ OPERATORS OVER INFINITE GRAPHS 7} \]
2.2.2. The Operators $L_w$. Let $w$ and $w_i$ denote reduced finite paths in $FP_i(\hat{G})$, for $i \in \mathbb{N}$, equivalently, they are the reduced words in the edge set $E(\hat{G})$, under the reduction (RR). Consider
\begin{equation}
L_w \left( \sum_i w_i \otimes h_i \right) = \sum_i w w_i \otimes h_i,
\end{equation}
for $h_i \in l^2(\mathbb{N})$. Here, the element $\sum_i w_i \otimes h_i$ denotes a finite sum of tensors in $A_G$. And $ww_i$ in (2.2.3) means concatenation of finite words. With the conditional expectation $E : A_G \to D_G$ (See (2.2.1) above), we get the Stinespring representation $(H_G, \pi_{(E,D_G)}(w)) : H_G \to H_G$, and the operators $\pi_{(E,D_G)}(w) : H_G \to H_G$ obtained from (2.2.3) by passing to the quotient and completion as in Definition 2.2. To simplify terminology, in the sequel, we will simply write $L_w$ for the operator $\pi_{(E,D_G)}(w)$.

2.2.3. Graph von Neumann Algebras. Let $G$, $\mathbb{G}$, and $H_G$ be given as above. And let $\{L_w : w \in G\}$ the multiplication operators on $H_G$, where $L$ is the canonical groupoid action of $G$.

**Definition 2.3.** Let $G$ be a countable directed graph with its graph groupoid $G$. The pair $(H_G, L)$ of the graph Hilbert space $H_G$ and the canonical groupoid action $L$ of $G$ is called the canonical representation of $G$. The corresponding groupoid von Neumann algebra $M_G \overset{\text{def}}{=} C[\mathbb{G}[G]]^w$, generated by $G$ (equivalently, by $L(G) = \{L_w : w \in G\}$), as a $W^*$-subalgebra of $B(H_G)$, is called the graph von Neumann algebra of $G$.

We can check that the generating operators $L_w$'s of the graph von Neumann algebra $M_G$ of $G$ satisfies that:

$L_w^* = L_{w^{-1}}$, for all $w \in G$,

and

$L_w, L_{w_2} = L_{w_1 w_2}$, for all $w_1, w_2 \in G$.

It is easy to check that if $v$ is a vertex in $G$, then the graph operator $L_v$ is a projection, since

$L_v^* = L_{v^{-1}} = L_v = L_{v^2} = L_v^2$.

Thus, by the reduction (RR) on $G$, we can conclude that if $w$ is a nonempty reduced finite path in $FP_i(\hat{G})$, then the operator $L_w$ is a partial isometry, since

$L_w^* L_w = L_{w^{-1}}$, and

and $w^{-1} w$ is a vertex, and hence $L_w^* L_w$ is a projection on $H_G$.

3. Graph Operators

In this section, we introduce graph operators and summarize the operator-theoretical properties of graph operators obtained in [8]. These results will be applied to characterize the operator-theoretic properties of Toeplitz operators in Section 5.
Let $G$ be a graph with its graph groupoid $G$. Let $M_G = \mathbb{C}[L(G)]^G$ be the graph von Neumann algebra of $G$ in $B(H_G)$, where $(H_G, L)$ is the canonical representation of $G$. Since $M_G$ is a groupoid von Neumann algebra generated by $G$, every element $T$ of $M_G$ satisfies the expansion,

$$T = \sum_{w \in G} t_w L_w,$$

with $t_w \in \mathbb{C}$.

For the given operator $T \in M_G$, having the above expansion, define the subset $\text{Supp}(T)$ of $G$ by

$$\text{Supp}(T) \overset{\text{def}}{=} \{ w \in G : t_w \neq 0 \}.$$

This subset $\text{Supp}(T)$ of $G$ is called the support of $T$. And the constants $t_w$'s, for $w \in \text{Supp}(T)$, are said to be the coefficients of $T$.

**Definition 3.1.** Let $T$ be an element of the graph von Neumann algebra $M_G$ of a given graph $G$, and let $\text{Supp}(T)$ be the support of $T$. If $\text{Supp}(T)$ is finite, then we call the operator $T$, a graph operator. The graph operators $L_w$, generating $M_G$, for all $w \in G \setminus \{\emptyset\}$, are called the generating (graph) operators.

i.e., the graph operators are the finitely supported operators on $H_G$.

In [8], we characterize the spectral-theoretical properties of graph operators, in terms of their supports and coefficients. In this paper, we concentrate on studying the connections between certain graph operators and Toeplitz operators.

4. Background for Main Results

In this section, we consider the fundamental background of the main results of this paper obtained in Sections 5 and 6. In Section 4.1, we discuss about the decomposition of graph Hilbert spaces. We observe that whenever a graph Hilbert space $H_G$ is given, there exists a (closed) subspace $H_V$, induced by all vertices of $G$, such that

$$H_G = H_V \oplus H_V^\perp,$$

and $H_V^\perp$ is induced by all reduced finite paths of $G$. Moreover, we will restrict our interests to the case where a given graph $G$ is a regular tree.

In Section 4.2, we consider generalized Toeplitz algebra $\text{Toep}(H)$ over an arbitrary Hilbert space $H$. In fact, the $C^*$-algebra $\text{Toep}(H)$ is well-known, but we are particularly interested in the anti-$\ast$-isomorphic $C^*$-algebra $\text{Toep}^\ast(H)$ of $\text{Toep}(H)$.

4.1. Graph Operators Induced by Regular Trees. In this section, we restrict our interests to the case where the given graphs are regular trees $T_N$, for $N \in \mathbb{N}$. Notice that the regular trees are simplicial, in the sense that (i) they do not allow loop-edges, and (ii) they do not have multi-edges, equivalently, if there is an edge connecting two vertices, then there is no other edge connecting those vertices. For instance, the following three graphs $G_1$, $G_2$, and $G_3$ are not simplicial, where

$$G_1 = \bullet \rightarrow \bullet,$$

$$G_2 = \bullet \leftrightarrow \bullet \leftrightarrow \bullet,$$

and

$$G_3 = \bullet \rightarrow \bullet \leftrightarrow \bullet.$$
Indeed, the graph $G_1$ has a loop-edge connecting from the vertex $v$ to itself, and hence it is not simplicial; the graph $G_2$ contain two edges connecting the vertex $v_1$ to the vertex $v_2$, and hence it is not simplicial; the graph $G_3$ is not simplicial because it has both loop-edge and multi-edges.

Since the regular trees $T_N$ are simplicial, we can put the suitable name (or indices) for the vertices. For the $N$-regular tree $T_N$, we will put the name 1 for the root of $T_N$, and the $N$-vertices in the 1-st level of $T_N$ have their names $11, 12, ..., 1N$. And the $N^2$-vertices in the 2-nd level of $T_N$ have their names $111, ..., 11N, 121, ..., 12N, ..., 1N1, ..., 1NN$, etc. For instance, the 2-regular tree $T_2$ has its vertices with their indices as follows:

And each edge $e$ of $T_N$ connecting the vertex $v_1$ to the vertex $v_2$ can be denoted by the pair $(v_1, v_2)$, again by the simpliciality of $T_N$. For instance, in $T_2$, the edge $x$ in the above figure is denoted by the pair $(111, 1112)$. Such a pair notation does not fit for arbitrary graph case (in particular, where a graph allows multi-edges). But, for simplicial graphs, this pair notation works well.

Thus a length-$k$ finite path $w$ can be denoted by $(k + 1)$-tuple of passing vertices, for $k \in \mathbb{N}$. For example, if $w = x_1 x_2$ in $T_2$, where $x_1$ and $x_2$ are edges in the above figure, then

$$x_1 = (12, 122), \quad x_2 = (122, 1221),$$

and

$$w = x_1 x_2 = (12, 122, 1221).$$

Clearly, if we have a finite path expressed by the $(k + 1)$-tuple $w = (v_1, v_2, ..., v_{k+1})$, then we can understand $w$ as a length-$k$ finite path $w = e_1 ... e_k$, generated by the admissible edges $e_1, ..., e_k$, where

$$e_j = (v_j, v_{j+1})$$

for all $j = 1, ..., k$.

For the given $N$-regular tree $T_N$ (under the above setting on vertices and edges) we can determine the graph groupoid $G_N = G_{T_N}$, and the corresponding graph von
Neumann algebra $M_N = M_{T_N}$, for $N \in \mathbb{N}$. We are interested in graph operators in $M_N$.

Let $T_N$ be the $N$-regular tree with its graph groupoid $G_N$, and let $H_N$ and $M_N$ be the corresponding graph Hilbert space and the graph von Neumann algebra of $T_N$, respectively, for $N \in \mathbb{N}$. By the Stinespring construction, the graph Hilbert space $H_N$ has its subspace

$$H_V = l^2(V(T_N))$$

where $V(T_N) = V(\hat{T}_N)$ is the vertex set of $T_N$, where

$$\xi_v = \pi(E,D_{T_N})(v), \text{ for all } v \in V(T_N)$$

(See Section 2.2). We call the subspace $H_V$ of $H_N$, the vertex space of $T_N$. For convenience, let’s denote $V(T_N)$ simply by $V_N$, for $N \in \mathbb{N}$. Thus, the Hilbert space $H_N$ is decomposed by

$$H_N = H_V \oplus H_{FP},$$

where

$$H_{FP} = H_N \ominus H_V = l^2(FP_r(T_N)),$$

where $FP_r(T_N)$ is the reduced finite path set of $G_N$.

Now, let’s denote $E^N_k$ be the length-$k$ reduced finite path set, which is the subset of $FP_r(\hat{T}_N)$ consisting of all length-$k$ reduced finite paths on the shadowed graph $\hat{T}_N$ of $T_N$, for all $k \in \mathbb{N}$. Clearly, the edge set $E(\hat{T}_N)$ of $\hat{T}_N$ is the set $E^N_1$, and the set $FP_r(\hat{T}_N)$ is partitioned by

$$FP_r(\hat{T}_N) = \biguplus_{k=1}^{\infty} E^N_k,$$

set-theoretically, where $\biguplus$ means the disjoint union. So, the subspace $H_{FP}$ of $H_N$ is Hilbert-space isomorphic to

$$H_{FP} = \bigoplus_{k=1}^{\infty} \left( \bigoplus_{w \in E_k} \mathbb{C}\xi_w \right),$$

whenever

$$\mathbb{C}\xi_w \ominus \text{Hilbert} = \mathbb{C} \otimes \cdots \otimes \mathbb{C} = \mathbb{C}^{\otimes k},$$

for all $w \in E_k$, for all $k \in \mathbb{N}$, where

$$\xi_w = \pi(E,D_{T_N})(w), \text{ for all } w \in FP_r(\hat{T}_N).$$

The above observation shows that the graph Hilbert space $H_N$ has its orthonormal basis (or its Hilbert basis),

$$\{\xi_w : w \in G_N \setminus \{\emptyset\}\}.$$ 

Therefore, if we define the Hilbert space $l^2(G_N)$ by the $l^2$-space generated by $G_N \setminus \{\emptyset\}$, more precisely,

$$l^2(G_N) \overset{def}{=} \left( \bigoplus_{v \in V(T_N)} \mathbb{C}\eta_v \right) \oplus \left( \bigoplus_{w \in FP_r(\hat{T}_N)} \mathbb{C}\eta_w \right),$$

(4.1.2)
with its Hilbert basis
\[ \{ \eta_w : w \in \mathbb{G}_N \setminus \{ \emptyset \} \}, \]
then the graph Hilbert space \( H_G \) and the Hilbert space \( l^2(\mathbb{G}_N) \) are Hilbert-space isomorphic
\[ H_G \cong l^2(\mathbb{G}_N). \]
So, without loss of generality, we may consider our graph Hilbert space \( H_N \) (the Stinespring space) as \( l^2(\mathbb{G}_N) \).

In the following context, we use \( H_G \) and \( l^2(\mathbb{G}_N) \), alternatively.

Remark that, in [4] and [7], we define the graph Hilbert space \( H_G \) of a given arbitrary countable directed graph \( G \) by \( l^2(\mathbb{G}) \), where \( \mathbb{G} \) is the graph groupoid of \( G \).

Now, consider the graph groupoid \( \mathbb{G}_N \) of the \( N \)-regular tree \( T_N \) more in detail.

Let \((v_1, v_2)\) be an edge of \( T_N \). Then its shadow has its pair notation \((v_2, v_1)\).

So, we can have
\[ (v_1, v_2) (v_2, v_1) = (v_1, v_2, v_1) = v_1, \]
by the reduction (RR) on \( \mathbb{G}_N \). This means that, if we have a “nonempty” element
\[ (v_1, v_2, \ldots, v_n, \ldots, v_{j+1}, \ldots, v_k), \]
in \( \mathcal{FP}_r(\hat{T}_N) \), then it is reduced (and hence identical) to a length-\((k - 1)\) reduced finite path
\[ (v_1, \ldots, v_j, v_{j+1}, \ldots, v_k), \]
in \( \mathcal{FP}_r(\hat{T}_N) \), for \( k \in \mathbb{N} \), where the length-0 reduced finite paths mean the vertices (i.e., where \( k = 1 \)).

By operator theory, we can represent each element \( T \) of the graph von Neumann algebra \( M_N \) on \( H_N \). However, we are interested in the representation of \( T \) on the vertex space \( H_V \).

### 4.2. Toeplitz Algebras \( \text{Toepl}(H) \)

Let \( H \) be an arbitrary Hilbert space, throughout this section. The **Fock space** \( \mathcal{F}_H \) denote \( \mathcal{F}(H) \) over \( H \) is defined by a Hilbert space,
\[ \mathcal{F}_H \overset{\text{def}}{=} \bigoplus_{n=0}^{\infty} H \otimes^n, \quad \text{with } H \otimes^n = C\Omega = C, \]
where \( \Omega \) means the vacuum vector, where the direct sum \( \oplus \) and the tensor product \( \otimes \) are all defined under the Hilbert (product) topology.

Now, fix a Hilbert-space element \( h \) of \( H \), and then define an operator \( l_h \) on \( \mathcal{F}_H \) by an operator satisfying
\[ (4.2.1) \]
\[ l_h(\Omega) = h, \quad \text{and} \]
\[ l_h : \xi_1 \otimes \ldots \otimes \xi_n \mapsto h \otimes \xi_1 \otimes \ldots \otimes \xi_n, \]
for all \( n \in \mathbb{N} \). Then clearly, we can check that the adjoint \( l_h^* \) of \( l_h \) is an operator satisfying that:
\[ (4.2.2) \]
\[ l_h^*(\Omega) = 0_{\mathcal{F}_H}, \quad \text{and} \]
\[ l_h^* : \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n \mapsto <h, \xi_1 >_H \xi_2 \otimes \ldots \otimes \xi_n, \]
for all \( n \in \mathbb{N} \), where \( 0_{\mathcal{F}_H} \) is the zero vector in \( \mathcal{F}_H \), and \( <,>_H \) means the inner product on the Hilbert space \( H \).

**Definition 4.1.** Let \( l_h \) be an operator on the Fock space \( \mathcal{F}_H \) over a Hilbert space \( H \), defined in (4.2.1), for a fixed element \( h \in H \). Then it is called the (left) creation operator induced by \( h \). The adjoint \( l_h^* \) of \( l_h \), satisfying (4.2.2), is called the (left) annihilation operator induced by \( h \).

Remark that the operators \( \{l_h, l_h^* : h \in H\} \) satisfy the relation,
\[
(4.2.3) \quad l_h^* l_h = < h, h >_H 1_{\mathcal{F}_H},
\]
where \( 1_{\mathcal{F}_H} \) is the identity operator on \( \mathcal{F}_H \), for all \( h \in H \).

**Definition 4.2.** The \( C^* \)-subalgebra \( \text{Toep}(H) \) of \( B(\mathcal{F}_H) \), generated by the creation operators,
\[
\{l_h : h \in H\}
\]
is called the Toeplitz operator over \( H \). And the elements of \( \text{Toep}(H) \) are said to be (generalized) Toeplitz operators (over \( H \)). In particular, if \( \dim H = 1 \), then \( \text{Toep}(H) \) is \( * \)-isomorphic to the classical Toeplitz algebra \( \text{Toep} \) in \( B(l^2(\mathbb{N})) \).

Let \( \mathcal{F}_H \) be the Fock space over \( H \) given as above, also let’s fix a Hilbert-space element \( h \in H \). Now, we will define a new operator \( r_h \) induced by \( h \) by an operator on \( \mathcal{F}_H \) satisfying that
\[
(4.2.4) \quad r_h(\Omega) = h, \quad \text{and} \quad r_h : \xi_1 \otimes \ldots \otimes \xi_n \mapsto \xi_1 \otimes \ldots \otimes \xi_n \otimes h,
\]
for all \( n \in \mathbb{N} \). Then, the adjoint \( r_h^* \) of \( r_h \) satisfies that
\[
(4.2.5) \quad r_h^*(\Omega) = 0_{\mathcal{F}_H} \quad \text{and} \quad r_h^* : \xi_1 \otimes \ldots \otimes \xi_n \otimes \xi_{n+1} \mapsto \xi_1 \otimes \ldots \otimes \xi_n < h, \xi_{n+1} >_H,
\]
for all \( n \in \mathbb{N} \).

**Definition 4.3.** The operators \( r_h \), satisfying (4.2.4), on \( \mathcal{F}_H \) is called the right creation operator induced by \( h \), and its adjoint \( r_h^* \), satisfying (4.2.5), is called the right annihilation operator induced by \( h \).

The family of operators
\[
\{r_h, r_h^* : h \in H\}
\]
satisfies the relation,
\[
(4.2.6) \quad r_h^* r_h = < h, h >_H 1_{\mathcal{F}_H},
\]
for all \( h_1, h_2 \in H \).

**Definition 4.4.** The \( C^* \)-subalgebra \( \text{Toep}^R(H) \) of \( B(\mathcal{F}_H) \), generated by the right creation operators,
\[
\{r_h : h \in H\}
\]
is called the right Toeplitz algebra over \( H \). And the elements of \( \text{Toep}^R(H) \) are said to be right (generalized) Toeplitz operators on \( \mathcal{F}_H \).
The following theorem shows the relation between the Toeplitz algebra $\text{Toep}(H)$, and the right Toeplitz algebra $\text{Toep}^R(H)$.

**Theorem 4.1.** The Toeplitz algebra $\text{Toep}(H)$ over $H$, and the right Toeplitz algebra $\text{Toep}^R(H)$ over $H$ are anti-$*$-isomorphic.

**Proof.** Recall that, by definition, $\text{Toep}(H) = C^* \{ l_h : h \in H \}$, and $\text{Toep}^R(H) = C^* \{ r_h : h \in H \}$, as $C^*$-subalgebras of $B(\mathcal{F}_H)$, where $l_h$ and $r_h$ are the left and right creation operators, respectively, for all $h \in H$. Thus, we can define a map $\Phi : \text{Toep}(H) \to \text{Toep}^R(H)$ by a generator-preserving linear transformation, satisfying

\[
\begin{aligned}
\Phi : & \quad 1_L \in \text{Toep}(H) \mapsto 1_R \in \text{Toep}^R(H), \\
& l_h \in \text{Toep}(H) \mapsto r_h^* \in \text{Toep}^R(H), \\
& l_h^* \in \text{Toep}(H) \mapsto r_h \in \text{Toep}^R(H), \\
& l_h^* r_{h_2}^* q_{h_1} \in \text{Toep}(H) \mapsto r_{h_2}^* q_{h_1}^* ,
\end{aligned}
\]

for all $h, h_1, h_2 \in H$, and for all $q_1, q_2 \in \{ 1, * \}$, where $1_L$ is the identity element in $\text{Toep}(H)$, and $1_R$ is the identity element in $\text{Toep}^R(H)$.

Since $\Phi$ is generator-preserving, it is bijective and bounded. Moreover, it is not difficult to check $\Phi$ is isometric.

Observe now that by the 4-th condition of (4.2.7), the linear map $\Phi$ is anti-multiplicative, i.e.,

\[
\Phi \left( l_{h_1}^* r_{h_2}^* q_{h_1} \right) = r_{h_2}^* q_{h_1}^* = \Phi \left( l_{h_2}^* \right) \Phi \left( l_{h_1}^* \right),
\]

for all $h_1, h_2 \in H$, and $q_1, q_2 \in \{ 1, * \}$. Therefore, by (4.2.8), we can have that

\[
\Phi \left( T_1 T_2 \right) = \Phi \left( T_2 \right) \Phi \left( T_1 \right), \quad \text{in} \quad \text{Toep}^R(H),
\]

for all $T_1, T_2 \in \text{Toep}(H)$.

To show this morphism $\Phi$ is an anti-$*$-isomorphic, it suffices to show that $\Phi$ preserves the relation (4.2.3) in $\text{Toep}(H)$ to the relation (4.2.6) in $\text{Toep}^R(H)$:

\[
< h_1, h_2 \succ_H 1_L = \Phi \left( l_{h_1}^* r_{h_2}^* \right) = \Phi \left( l_{h_2}^* \right) \Phi \left( l_{h_1}^* \right)
\]

by (4.2.8) and (4.2.9)

\[
= r_{h_2}^* r_{h_1} = < h_1, h_2 \succ_H 1_R.
\]

This shows that $\Phi$ is a bijective anti-multiplicative isometric linear transformation preserving (4.2.3) to (4.2.6), and hence it is an anti-$*$-isomorphism from $\text{Toep}(H)$ onto $\text{Toep}^R(H)$. □

The above theorem shows that the (left)Toeplitz algebra $\text{Toep}(H)$ and the right Toeplitz algebra $\text{Toep}^R(H)$ are anti-$*$-isomorphic. We will use this results in Section 5.3, later.
5. Representations of $N$-Tree Operators on Vertex Spaces

As in Section 4.1, we restrict our interests to the case where given graphs are $N$-regular trees $\mathcal{T}_N$, for $N \in \mathbb{N}$. Throughout this section, we will use the same notations we used in Section 4.1. We want to represent graph operators $T$ on the vertex space $H_N$. Of course, the vertex spaces $H_V$ would be different whenever $N$ varies. For emphasizing we are working on $N$-regular trees, we call the graph operators of the graph von Neumann algebra $M_N$, the $N$-tree operators (or tree operators).

In the first two following subsections, we consider the special cases where $N = 1$, and $N = 2$, respectively. And then in Subsection 4.2.3, we will consider the general case.

5.1. 1-Tree Operators. In this subsection, we consider the 1-regular tree $\mathcal{T}_1$, and its corresponding Hilbert space $H_1$, and von Neumann algebra $M_1$. We will represent the graph operators of $M_1$ on the vertex space $H_V$ of $H_1$.

When $N = 1$, the vertex space $H_V$ is Hilbert-space isomorphic to the $l^2$-space $l^2(\mathbb{N}) = \mathbb{C}^{\oplus \infty}$, i.e.,

$$H_V^{\text{Hilbert}} = l^2(\mathbb{N}),$$

if $N = 1$. Thus, we will use the isomorphic Hilbert spaces $H_V$ and $l^2(\mathbb{N})$, alternatively.

Now, put the name (or indices) of vertices of $\mathcal{T}_1$ by $\mathbb{N}$, i.e.,

$$\mathcal{T}_1 = \bullet \to 1 \to 2 \to 3 \to 4 \to \cdots.$$  

Then all reduced finite paths $w$ of the graph groupoid $\mathcal{G}_1$ are expressed by

$$(j, j + 1, j + 2, \ldots, j + k)$$

or

$$(j + k, \ldots, j + 2, j + 1, j).$$

Indeed, we can obtain that

$$(5.1.2)$$

$$FP_r\left(\hat{T}_1\right) = FP(\mathcal{T}_1) \sqcup FP(\mathcal{T}_1^{-1}).$$

Remark here that, in general,

$$FP_r(\hat{G}) \supseteq FP(G) \cup FP(G^{-1}),$$

for an arbitrary graph $G$.

Now, let’s denote the inner product of $H_V$ by $\langle \cdot, \cdot \rangle_2$, since $H_V$ is isomorphic to $l^2(\mathbb{N})$. Now, to represent the 1-tree operators $T$ of $M_1$, we can use the Fourier expansion with respect to the inner product $\langle \cdot, \cdot \rangle$. i.e., if $T = \sum_{w \in \text{Supp}(T)} t_w L_w$, then the representation $\alpha_T$ of $T$ on $H_V$ is

$$(5.1.3)$$

$$\alpha_T = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \langle T \xi_k, \xi_l \rangle_2 \alpha_{k,l},$$

where $A_{k,l}$ are the rank-one operators on $l^2(\mathbb{N})$,

$$\alpha_{k,l} = | l > < k |, \text{ for all } k, l \in \mathbb{N}.$$
and

\[ \xi_k = \begin{pmatrix} 0, \ldots, 0, 1, 0, 0, \ldots \end{pmatrix} \in l^2(\mathbb{N}) = H_V, \]

for all \( k \in \mathbb{N} \).

Here, \(| \cdot > < \cdot |\) means the Dirac-operator notation. Here, notice that the inner product \( << \cdot, \cdot >> \) in (5.1.3) means the inner product on \( H_V \), not the inner product on the graph Hilbert space \( H_1 \).

By (5.1.3), we can define an action \( \alpha \) of the graph von Neumann algebra \( M_1 \), acting on the vertex space \( H_V \), satisfying that

\[ \alpha(T) \overset{\text{def}}{=} \alpha_T, \text{ for all } T \in M_N. \]

Then this morphism \( \alpha \) is indeed a well-defined action of \( M_N \), since it is bounded linear, and

\[ \alpha(T_1 T_2) = \alpha_{T_1} \alpha_{T_2} = \alpha(T_1) \circ \alpha(T_2), \]

and

\[ \alpha(T_1^+) = \alpha T_1^* = (\alpha(T_1))^*, \]

for all \( T_1, T_2 \in M_N \), where \( (\circ) \) means the usual composition.

Therefore, we can obtain the following lemma.

**Lemma 5.1.** Let \( e = (j, j+1) \) be an edge of the 1-regular tree \( T_1 \) of (5.1.2), with its shadow \( e^{-1} = (j + 1, j) \), for \( j \in \mathbb{N} \). Then the graph operator \( L_e \) of \( M_1 \) is unitarily equivalent to the operator \( \alpha_{L_e} \in B(H_V) \), where

\[ (5.1.5) \]

\[ \alpha_{L_e} \overset{\text{U.E}}{=} \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix}, \]

with

\[ \alpha_{L_e}^* = \alpha_{L_{e^{-1}}} = \alpha_{L_e^{-1}} \overset{\text{U.E}}{=} \begin{pmatrix} 0 & 0 \\ \vdots & \ddots \\ 0 & 0 & 1 \\ 0 & \vdots & \ddots \\ 1 \end{pmatrix}, \]

where \( \boxed{\text{th position}} \) means the \((j, j)\)-th position, for \( j \in \mathbb{N} \), and where \( \overset{\text{U.E}}{=} \) means “being unitarily equivalent.” □
The proof is straightforward by (5.1.4), and (5.1.3).
Consider now the operator $T_E$,

$$T_E = \sum_{e \in E(T)} L_e$$

in $M_1$, i.e., this operator $T_E$ is the infinite sum of the graph operators $L_e$'s, for all $e \in E(T)$. Then it is represented on $l^2(\mathbb{N}) = H_V$ by the operator, unitarily equivalent to

$$
\begin{pmatrix}
1 & 1 & 0 & \cdots & \cdots \\
0 & 1 & 1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}
$$

Remark that the identity operator $1_{M_1} = \sum_{v \in V(T)} L_v$ of $M_1$ is unitarily equivalent to the diagonal infinite matrix

$$
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
\ddots & \ddots \\
0 & \cdots & \cdots 
\end{pmatrix}
$$
on $H_V$. Thus, we can easily check that (5.1.6)

$$T_E - 1_{M_1} \cong \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & \ddots \\
\ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & 0
\end{pmatrix},$$
on $H_V$, and it is unitarily equivalent to the adjoint $U^*$ of the unilateral shift $U$. 
Recall that the unilateral shift $U$ on $l^2(\mathbb{N})$ is the operator defined by

$$U : (t_1, t_2, t_3, ...) \mapsto (t_2, t_3, ...),$$
for all $(t_n)_{n=1}^\infty \in l^2(\mathbb{N})$. So, the adjoint $U^*$ of $U$ is the operator, satisfying

$$U^* : (t_1, t_2, t_3, ...) \mapsto (0, t_1, t_2, ...),$$
on $l^2(\mathbb{N})$.
Recall also that the classical Toeplitz algebra $U_1$ is the $C^*$-subalgebra $C^*(U)$ of $B (l^2(\mathbb{N}))$, generated by the unilateral shift $U$. Notice that the Toeplitz algebra $U_1$ is also understood as the $C^*$-subalgebra of $B (H^2(T))$, generated by the classical Toeplitz operators $T_\varphi$, with their symbols $\varphi$, contained in the von Neumann algebra $L^\infty(T)$, where $T$ is the unit circle in $\mathbb{C}$. Here, the Hilbert space $H^2(T)$, where $T_\varphi$'s
acting on, is the Hardy space consisting of all analytic functions on $\mathbb{T}$, which is a subspace of the $L^2$-space $L^2(\mathbb{T})$ equipped with the Haar measure.

From the above observation, we can conclude that:

**Theorem 5.2.** Let $\mathcal{U}_1$ be the classical Toeplitz algebra. Then $\mathcal{U}_1$ is a $C^*$-subalgebra of the graph von Neumann algebra $M_1$.

**Proof.** Let $\mathcal{T}_1$ be the 1-regular tree and $M_1$, the corresponding graph von Neumann algebra of $\mathcal{T}_1$ (generated by all 1-tree operators). And let $\mathcal{U}_1$ be the classical Toeplitz algebra $C^*(U)$ acting on $l^2(\mathbb{N})$, where $U$ is the unilateral shift. By (5.1.5), and (5.1.6), the unilateral shift $U$ is unitarily equivalent to

$$U \overset{U.E}{=} T_E - 1_{M_1},$$

where

$$T_E = \sum_{e \in E(\mathcal{T}_1)} L_e \in M_1,$$

and

$$1_{M_1} = \sum_{v \in V(\mathcal{T}_1)} L_v \in M_1.$$

Therefore, the classical Toeplitz algebra $\mathcal{U}_1$ satisfies that

$$\mathcal{U}_1 \overset{\text{def}}{=} C^*(U) \overset{\text{iso}}{=} C^*(\alpha(T_E - 1_{M_1})) \overset{\text{iso}}{=} C^*(\alpha(T_E)),$$

in $B(l^2(\mathbb{N})) \overset{\text{iso}}{=} B(H_V)$, where $\alpha$ is the action of $M_1$ acting on $l^2(\mathbb{N})$, in the sense of (5.1.4). Therefore, by the very definition of $M_1$,

$$M_1 = vN(L(G_1)) \text{ in } B(H_1),$$

the algebra $\mathcal{U}_1$ is a $C^*$-subalgebra of $M_1$. \[\square\]

The above theorem shows that all classical Toeplitz operators are the representations of certain elements of the graph von Neumann algebra $M_1 = M_{\mathcal{T}_1}$. In particular, the generator $U$ of $\mathcal{U}_1$ is the infinite sum of graph operators, by (5.1.5).

Notice now that all (classical) Toeplitz operators can be understood as the certain infinite sum of certain graph operators with pattern.

Let $w = (j, j + 1, j + 2)$ be a length-2 reduced finite path in the graph groupoid $\mathcal{G}_1$ of $\mathcal{T}_1$. Then, by (5.1.3), the graph operator $L_w \in M_1$, induced by $w$, is unitarily equivalent to the infinite matrix $\alpha_{L_w} = \alpha(L_w)$,

$$L_w \overset{U.E}{=} \alpha_{L_w} \overset{U.E}{=} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots \\ \end{array} \right).$$
on $l^2(\mathbb{N}) = H_V$, satisfying

\[
L_w^* = L_{w^{-1}} \overset{U.E}{=} \alpha_{L_w}^{-1} = \alpha_{L_w}^i = \begin{pmatrix}
0 & & & & & & & 0 \\
\vdots & & & & & & & \vdots \\
0 & & & & & & & 0 \\
0 & & & & & & & 0 \\
\vdots & & & & & & & \vdots \\
0 & & & & & & & 0 \\
0 & 0 & 0 & & & & & \vdots \\
1 & 0 & & & & & & \vdots \\
0 & 0 & & & & & & \vdots \\
& & & & & & & \vdots
\end{pmatrix},
\]

on $l^2(\mathbb{N})$, where $\square$ means the $(j, j)$-th entry, for all $j \in \mathbb{N}$.

So, inductively, we obtain that:

**Proposition 5.3.** Let $w = (j, j + 1, \ldots, j + k) \in \mathbb{G}_1$, for $j, k \in \mathbb{N}$. Then the corresponding graph operator $L_w$ of $M_1$ is unitarily equivalent to the operator $\alpha L_w$, (5.1.7)

\[
\begin{pmatrix}
0 & \cdots & \cdots & 0 \\
\vdots & & & \vdots \\
\vdots & & & \vdots \\
0 & \cdots & \cdots & 0 \\
\vdots & & & \vdots \\
\vdots & & & \vdots \\
\vdots & & & \vdots \\
0 & \cdots & \cdots & 0 \\
\end{pmatrix}
\]

on $l^2(\mathbb{N}) = H_V$, i.e., $\alpha L_w$ is represented as an infinite matrix with only nonzero $(j, j)$ and $(j + k, j)$ entries 1’s. □

The proof of the above proposition is straightforward by (5.1.3). Now, Define an element $T_{E(k)}$ in $M_1$ by

\[
T_{E(k)} \overset{def}{=} \sum_{w \in E(+k)} L_w,
\]

where the support $E(+k)$ of $T_{E(k)}$ is defined by the subset (5.1.8)

\[
E(+k) \overset{def}{=} \{ w \in FP(\hat{T}_1) : |w| = k \},
\]

consisting of all length-$k$ (non-reduced) finite paths of the finite path set $FP(\hat{T}_1)$ of $\hat{T}_1$, for all $k \in \mathbb{N}$. Here, remark again that the “reduced” finite path set $FP_r(\hat{T}_1)$ of the shadowed graph $\hat{T}_1$ of $T_1$ is identified with the disjoint union of the “non-reduced” finite path set $FP(\hat{T}_1)$ of $T_1$ and the non-reduced finite path set $FP(\hat{T}_1^{-1})$ of the shadow $T_1^{-1}$ of $T_1$ (See (5.1.2)).
By (5.1.8), we have that
\begin{equation}
FP(T_1) = \bigcup_{k=1}^\infty E(+k).
\end{equation}

Also, by (5.1.2) and (5.1.8), we can obtain the subsets $E(-k)$ of the reduced finite path set $FP_r\left(\widehat{T_1}\right)$, where
\begin{equation}
E(-k) \overset{\text{def}}{=} \{ w \in FP(T_1^{-1}) : |w| = k \},
\end{equation}
for all $k \in \mathbb{N}$, satisfying that
\begin{equation}
FP\left(T_1^{-1}\right) = \bigcup_{k=1}^\infty E(-k),
\end{equation}
and hence
\begin{equation}
FP\left(\widehat{T_1}\right) = \left( \bigcup_{k=1}^\infty E(+k) \right) \cup \left( \bigcup_{k=1}^\infty E(-k) \right),
\end{equation}
by (5.1.2) and (5.1.11). Thanks to (5.1.10) and (5.1.12), we can define an element $T(-k)$ of $M_1$ by
\begin{equation}
T(-k) \overset{\text{def}}{=} \sum_{w \in E(-k)} L_w.
\end{equation}
Then we can easily check
\begin{equation}
T^*_(-k) = T(-k), \text{ for all } k \in \mathbb{N},
\end{equation}
since $L^*_w = L^{-1}_w$, for all $w \in G_1$. Also, the operator $T_E$, defined at the beginning of this section, is nothing but the element $T(+1)$, in $M_1$.

Therefore, by the above discussion, we can obtain the following lemma.

**Lemma 5.4.** For $k \in \mathbb{N}$, let
\begin{equation}
T_{(+k)} \overset{\text{def}}{=} \sum_{w \in E(+k)} L_w, \text{ and } T_{(-k)} \overset{\text{def}}{=} \sum_{x \in E(-k)} L_x,
\end{equation}
in $M_1$, where $E(+k)$, and $E(-k)$ are defined in (5.1.8) and (5.1.10), respectively.
Then
\begin{equation}
T^*_(-k) = T(-k) \text{ in } M_1, \text{ for all } k \in \mathbb{N},
\end{equation}
and
\begin{equation}
T_{(+k)} \overset{U.E}{=} \alpha_{T_{(+k)}} = \alpha_{(+k)} \overset{U.E}{=} \underset{k\text{-times}}{\begin{pmatrix}
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & & & & & & & & 
\end{pmatrix}},
\end{equation}
where $U.E$ denotes the unitary extension of the operator.
on $l^2(\mathbb{N})$, for all $k \in \mathbb{N}$. Hence, the element $T_{(-k)}$ is unitarily equivalent to the operator $\alpha_{k+1}^*$ on $l^2(\mathbb{N})$. i.e.,

$$T_{(-k)} \overset{U,E}{=} \alpha_{T_{(-k)}} = \alpha_{(-k)} = \alpha_{(k+1)}^*.$$

□

By the above lemma, we can obtain the following proposition.

**Proposition 5.5.** Let $T_{(+k)}$ and $T_{(-k)}$ be given as in the above lemma in the graph von Neumann algebra $M_1$ of the 1-regular tree $T_1$, for $k \in \mathbb{N}$. Then the elements $T_{(+k)} - 1_{M_1}$ and $T_{(-k)} - 1_{M_1}$ are unitarily equivalent to $U^*k$, and $U^k$ on $l^2(\mathbb{N}) = H_V$, respectively for all $k \in \mathbb{N}$, where $U$ is the unilateral shift on $l^2(\mathbb{N})$.

**Proof.** By (5.1.14) and (5.1.15), we can have that

$$T_{(+k)} - 1_{M_1} \overset{U,E}{=} \alpha_{(+k)} - I = \begin{pmatrix}
0 & \cdots & 0 & \mathbb{1} & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 \\
0 & \vdots & \ddots & \ddots & \ddots 
\end{pmatrix},$$

on $l^2(\mathbb{N}) = H_V$, where $\mathbb{1}$ is the $(k, 1)$-entry, for all $k \in \mathbb{N}$, and where $I$ means the identity operator on $l^2(\mathbb{N})$. By (5.1.16), we obtain that

$$T_{(-k)} - 1_{M_1} \overset{U,E}{=} \alpha_{(-k)} - I = \begin{pmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 \\
\mathbb{1} & \ddots & \ddots & \ddots \\
0 & \vdots & \ddots & \ddots \\
0 & \cdots & 0 & 1 
\end{pmatrix},$$

on $l^2(\mathbb{N})$, where $\mathbb{1}$ is the $(1, k)$-th entry. Therefore, the element $T_{(+k)} - 1_{M_1}$ (resp., $T_{(-k)} - 1_{M_1}$) is unitarily equivalent to $U^*k$ (resp., $U^k$), for all $k \in \mathbb{N}$, by (5.1.16) (resp., by (5.1.17)), on $l^2(\mathbb{N})$, for all $k \in \mathbb{N}$. □

The above proposition is the generalization of (5.1.5). Under the settings of the above proposition, we can re-write (5.1.5) that: the element $T_{(+1)} - 1_{M_1}$ of $M_1$ is unitarily equivalent to the adjoint of the unilateral shift $U^*$ on $l^2(\mathbb{N})$; and the element $T_{(-1)} - 1_{M_1}$ of $M_1$ is unitarily equivalent to the unilateral shift $U$ on $l^2(\mathbb{N})$.

By the above proposition, we can obtain the following theorem.
Theorem 5.6. Let $S = \sum_{j=-n}^{1} t_j U^{*j} + t_0 I + \sum_{i=1}^{k} t_i U^{i}$ be a Toeplitz operator, with $t_0 \in \mathbb{C}$, for $p = -n, \ldots, -1, 0, 1, \ldots, k$, in the classical Toeplitz algebra $\mathcal{U}_1$ (or equivalently, $S$ is the Toeplitz operator $T_\varphi$ with its trigonometric polynomial symbol $\varphi(z) = \sum_{p=-n}^{k} t_p z^p$ in $L^\infty(\mathbb{T})$, on $H^2(\mathbb{T})$). Let $T_{(\pm k)}$ be the elements of the graph von Neumann algebra $M_1$ of the 1-regular tree $\mathcal{T}_1$, for all $k \in \mathbb{N}$. Then the Toeplitz operator $S$ is unitarily equivalent to the element $S'$ of $M_1$, on $l^2(\mathbb{N}) = H_V$, where (5.1.18)

$$S' = \sum_{j=1}^{n} t_{-j} T_{(+j)} + s_0 1_{M_1} + \sum_{i=1}^{k} t_i T_{(-i)},$$

with

$$s_0 = t_0 - \left(\sum_{j=-n}^{1} t_j\right) - \left(\sum_{i=1}^{k} t_i\right) \in \mathbb{C}.$$

Proof. By the above corollary, the operators $U^k$ and $U^{*l}$ of $B(l^2(\mathbb{N}))$ are unitarily equivalent to the elements $T_{(-k)} - 1_{M_1}$, and $T_{(+k)} - 1_{M_1}$ of $M_1$ on $l^2(\mathbb{N})$, respectively, where $U$ is the unilateral shift on $l^2(\mathbb{N})$. So, the given Toeplitz operator $S$ is unitarily equivalent to the element,$$$S' = \sum_{j=-n}^{1} t_j (T_{(-j)} - 1_{M_1}) + t_0 1_{M_1} + \sum_{i=1}^{k} t_i (T_{(-i)} - 1_{M_1})$$

$$= \left(\sum_{j=-n}^{1} t_j T_{(-j)} - \sum_{j=1}^{1} t_j 1_{M_1}\right) + t_0 1_{M_1}$$

$$+ \left(\sum_{i=-n}^{k} t_i T_{(-i)} - \sum_{i=1}^{k} t_i 1_{M_1}\right)$$

$$= \left(\sum_{j=-n}^{1} t_j T_{(-j)}\right) + \sum_{i=1}^{k} t_i (T_{(-i)} - 1_{M_1})$$

$$- \left(\sum_{j=-n}^{1} t_j\right) 1_{M_1} + t_0 1_{M_1} - \left(\sum_{i=1}^{k} t_i\right) 1_{M_1}$$

$$= \left(\sum_{j=-n}^{1} t_j T_{(-j)}\right) + \sum_{i=1}^{k} t_i (T_{(-i)} - 1_{M_1})$$

$$+ t_0 - \left(\sum_{j=-n}^{1} t_j\right) - \left(\sum_{i=1}^{k} t_i\right) 1_{M_1}.$$ 

The above theorem characterizes the classical Toeplitz operators in terms of 1-tree operators.

Corollary 5.7. The classical Toeplitz algebra $\mathcal{U}_1$ is $*$-isomorphic to the $C^*$-subalgebra

$$C^* \left(\alpha \{T_n \in M_1 : n \in \mathbb{Z}, \text{ with } T_0 \overset{def}{=} 1_{M_1}\}\right)$$

of $B(l^2(\mathbb{N}))$, where $\alpha$ is the action of $M_1$, in the sense of (5.1.4), acting on $H_V = l^2(\mathbb{N})$. □

The above corollary also shows that the classical Toeplitz algebra $\mathcal{U}_1$ is indeed a $C^*$-subalgebra of the graph von Neumann algebra $M_1$ of the 1-regular tree $\mathcal{T}_1$.

5.2. 2-Tree Operators. In this section, we will consider the relation between the generalized Toeplitz operators in $\text{Toep}(\mathbb{C}^{\otimes 2})$ and 2-tree operators which are the graph operators in the graph von Neumann algebra $M_2$ of the 2-regular tree $\mathcal{T}_2$. 
As we discussed in Section 4.1, the corresponding graph Hilbert space $H_2$ of $T_2$ is decomposed by the vertex space $H_V$ and its orthogonal complemented subspace $H_V^\perp$. Like in Section 5.1, we want to represent graph operators in the graph von Neumann algebra $M_2$ as an operator on $H_V$. To do that we first concentrate on characterizing $H_V$.

Now, we denote the process sending the vertex $i_1 i_2 \ldots i_n$ in the $n$-th level of $T_2$ to the vertex $i_1 i_2 \ldots i_n 1$ in the $(n + 1)$-th level of $T_2$ by $\gamma_1$, for all $n \in \mathbb{N}$, where $i_1, \ldots, i_n \in \{1, 2\}$.

Similarly, we denote the process sending $i_1 \ldots i_n$ to $i_1 \ldots i_n 2$ by $\gamma_2$, for all $n \in \mathbb{N}$. i.e., $\gamma_j$ are the function on the vertex set $V(T_2)$ of $T_2$, defined by

\begin{equation}
\gamma_j (i_1 i_2 \ldots i_n) \overset{\text{def}}{=} i_1 i_2 \ldots i_n j,
\end{equation}

for all $j = 1, 2$, for all $n \in \mathbb{N}$. Also, we can understand these functions $\gamma_j$ generate the edges in $T_2$, i.e., the function

\begin{equation}
\gamma_j (i_1 i_2 \ldots i_n)
\end{equation}

can be understood as the edge

\begin{equation}
(i_1 i_2 \ldots i_n, i_1 i_2 \ldots i_n j),
\end{equation}

in $T_2$, for $j = 1, 2$.

Now, let $X_2$ be the set $\{1, 2\}$, and define $X_2^*$, by the union of $X_2^*$ and $\{\emptyset\}$, where $\emptyset$ is the empty word in $X_2$, i.e.,

\begin{equation}
X_2 = \{\emptyset\} \cup X_2^*,
\end{equation}

where $X_2^*$ means the set of all words in $X_2$. So, we can understand the set $X_2$ is the collection of all words in $X_2$ and the empty word. Then we can regard the functions $\gamma_j$ of (5.2.1) on as functions on $X_2$, for all $j = 1, 2$. 
Notation Let $W$ be a finite words in $X^*_2$. Then we denote the compositions
\[ \gamma_{j_1} \circ \gamma_{j_2} \circ \ldots \circ \gamma_{j_k} \text{ on } X_2 \]
simply by
\[ \gamma_{j_1j_2 \ldots j_k}, \text{ or } \gamma_W, \]
whenever $W = j_1j_2 \ldots j_k \in X^*_2$. □

The above notation gives us a motivation for the following proposition.

**Proposition 5.8.** Let $X^*_2$ be the set of all finite words in $X_2 = \{1, 2\}$, and let $X_2$ be the set defined in (5.2.2). Then there exists an (right) action $\gamma$ of $X^*_2$, acting on $X_2$, such that
\[ (5.2.3) \quad \gamma : W \in X^*_2 \mapsto \gamma_W : X_2 \rightarrow X_2, \]
with additional equality
\[ \gamma_W(\emptyset) = W, \text{ for all } W \in X_2. \]

**Proof.** The map $\gamma : X^*_2 \rightarrow F(X_2)$ is well-defined, where $F(X_2)$ is the set of all functions on $X_2$. So, it is enough to show that
\[ \gamma(W_1W_2) = \gamma(W_2) \circ \gamma(W_1), \]
on $X_2$, for all $W_1, W_2 \in X^*_2$. Indeed, we can have that
\[ \gamma(W_1W_2) = \gamma_{W_1W_2} = \gamma_{W_2} \gamma_{W_1} = \gamma(W_2) \circ \gamma(W_1). \]

The above proposition, indicating the existence of the right action $\gamma$ of $X^*_2$ on $X_2$, shows that the vertex set $V(T_2)$ is generated by the action $\gamma$. Therefore, we can obtain the set-equalities
\[ (5.2.4) \quad V(T_2) = X_2 = \{\emptyset\} \cup \{\gamma_W(X_2) : W \in X^*_2\}. \]

Now, consider the vertex space more in detail. Motivated by (5.2.3) and (5.2.4), we may expect that the vertex space
\[ H_V = l^2(X_2) = l^2(X^*_2) = l^2(V(T_2)) \]
is Hilbert-space isomorphic to the (generalized) Fock space over $\mathbb{C}^{\oplus 2}$,
\[ (5.2.5) \quad F_2 = F(\mathbb{C}^{\oplus 2}) = \bigoplus_{k=0}^{\infty} (\mathbb{C}^{\oplus 2})^{\otimes k}, \]
with the identity,
\[ (\mathbb{C}^{\oplus 2})^{\otimes 0} = \mathbb{C}. \]

Let $F_H$ be the Fock space over an arbitrary Hilbert space $H$, like in Section 4.2. If $H$ is an $n$-dimensional Hilbert space $\mathbb{C}^{\oplus n}$, then we denote $F_{\mathbb{C}^{\oplus n}}$ simply by $F_n$, for all $n \in \mathbb{N}$.

**Theorem 5.9.** Let $H_V$ be the vertex space of the graph Hilbert space $H_2$ of $T_2$. Then
\[ (5.2.6) \quad H_V \overset{\text{Hilbert}}{=} l^2(X_2) \overset{\text{Hilbert}}{=} F_2. \]
Proof. First, recall the relation (4.1.1), and (4.1.2). In particular, by (4.1.1), the vertex space $H_V$ of the graph Hilbert space $H_2$ is Hilbert-space isomorphic to

$$(5.2.7) \quad H_V = \bigoplus_{v \in V(T_2)} \mathbb{C}\xi_v = \bigoplus_{W \in \mathcal{X}_2} \mathbb{C}\xi_W,$$

where $B_V = \{ \xi_v : v \in V(T_2) \} = \{ \xi_W : W \in \mathcal{X}_2 \}$ is the Hilbert basis (or the orthonormal basis) of $H_V$. Motivated by (5.2.4), we may determine a linear map, satisfying

$$(5.2.8) \quad \xi_{j_1,j_2,\ldots,j_k} \in B_V \mapsto \eta_{j_1} \otimes \eta_{j_2} \otimes \ldots \otimes \eta_{j_k} \in B_k,$$

where $B_k$ is the Hilbert basis for $(\mathbb{C} \oplus \mathbb{C})^\otimes k$, for all $k \in \mathbb{N}$, where

$$(5.2.9) \quad \eta_{j_i} = \begin{cases} (1, 0) & \text{if } j_i = 1, \\ (0, 1) & \text{if } j_i = 2, \end{cases}$$

for all $i = 1, \ldots, k$, for $k \in \mathbb{N}$. We denote this morphism with (5.2.8) by $\Phi_k$, for all $k \in \mathbb{N}$. Then we define a linear map

$$(5.2.10) \quad \Phi : \bigoplus_{W \in \mathcal{X}_2} \mathbb{C}\xi_W \to \mathcal{F}_2$$

by

$$\Phi = \bigoplus_{n=0}^{\infty} \Phi_n, \text{ with the identity } \Phi_0 = i_d,$$

where $\Phi_n$ is the linear map satisfying (5.2.8), for $n \geq 1$, and where $i_d$ means the identity map on $\mathbb{C}$ (i.e., $i_d(z) = z$, $\forall z \in \mathbb{C}$). Indeed, we can define such a linear map, because

$$(5.2.11) \quad \bigoplus_{W \in \mathcal{X}_2} \mathbb{C}\xi_W = \mathbb{C} \oplus \left( \bigoplus_{n=2}^{\infty} \left( \bigoplus_{W \in \mathcal{X}_2(k)} \mathbb{C}\xi_W \right) \right),$$

where

$$(5.2.12) \quad \mathcal{X}_2(k) \equiv \{ W \in \mathcal{X}_2 : |W| = k \}, \text{ for all } k \in \mathbb{N},$$

where $|W|$ means the length of the word $W$. So, the linear map $\Phi$ of (5.2.10) is a well-defined into $\mathcal{F}_2$, by (5.2.9). i.e., the summands

$$\mathbb{C}\xi_{j_1,j_2,\ldots,j_k}$$

of (5.2.11) corresponds to the subspace

$$p_{j_1} (\mathbb{C} \oplus \mathbb{C}) \otimes p_{j_2} (\mathbb{C} \oplus \mathbb{C}) \otimes \ldots \otimes p_{j_k} (\mathbb{C} \oplus \mathbb{C})$$

of the summand $(\mathbb{C} \oplus \mathbb{C})^\otimes k$ of the Fock space $\mathcal{F}_2$, where $p_1$ and $p_2$ are the natural projections on $\mathbb{C} \oplus \mathbb{C}$.
Therefore, this linear map \( \Phi \) is basis-element preserving, and hence it is bijective. Moreover, it is easy to check that \( \| \Phi \| = 1 \), by the very definition. So, \( \Phi \) is the isometric bijective linear map, preserving Hilbert bases. Therefore, the Hilbert spaces 

\[
\bigoplus_{W \in \mathcal{X}_2} \mathbb{C} \xi_W \quad \text{and} \quad \mathcal{F}_2
\]

are Hilbert-space isomorphic, with its isomorphism \( \Phi \). This shows that 

\[
H_{V_{\text{Hilbert}}} = \bigoplus_{W \in \mathcal{X}_2} \mathbb{C} \xi_W = H_{V}
\]

The above theorem characterize the vertex space \( H_V \) in the graph Hilbert space \( H_2 \) of \( T_2 \) by the Fock space \( \mathcal{F}_2 \) over \( \mathbb{C}^{\otimes 2} \). So, from now on, we use \( H_V \) and \( \mathcal{F}_2 \), alternatively.

Similar to Section 5.1, we will represent graph operators as operators on the vertex space \( H_V = \mathcal{F}_2 \).

**Lemma 5.10.** Let \( (W, W_j) \) be an edge connecting a vertex \( W = j_1 j_2 \ldots j_k \in \mathcal{X}_2 \)

to a vertex \( W_j = j_1 j_2 \ldots j_k \in \mathcal{X}_2 \), for \( j = 1, 2 \), for \( k \in \mathbb{N} \). Then the corresponding 2-tree operator \( L_{(W, W_j)} \) induced by an edge \( (W, W_j) \) is unitarily equivalent to the restriction \( r_{e_j} |_{\mathcal{H}_W} \) of the right creation operator \( r_{e_j} \) on \( \mathcal{F}_2 \) induced by \( e_j \), for \( j = 1, 2 \), more precisely,

\[
L_{(W, W_j)} |_{\mathcal{F}_2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

\( U.E \), for all \( j = 1, 2, \) where

\[
\mathcal{H}_W = p_{j_1} (\mathbb{C}^{\otimes 2}) \otimes \cdots \otimes p_{j_k} (\mathbb{C}^{\otimes 2})
\]

and

\[
\mathcal{H}_W(j) = p_{j_1} (\mathbb{C}^{\otimes 2}) \otimes \cdots \otimes p_{j_k} (\mathbb{C}^{\otimes 2}) \otimes p_j (\mathbb{C}^{\otimes 2})
\]

where \( e_1 \) and \( e_2 \) are the natural basis-vectors of \( \mathbb{C}^{\otimes 2} \) and where \( p_1 \) and \( p_2 \) are the natural projections on \( \mathbb{C}^{\otimes 2} \).

**Proof.** By definition and by (5.2.6), for a fixed vertex \( W = j_1 j_2 \ldots j_k \in \mathcal{X}_2 = V (T_2) \), for \( k \in \mathbb{N} \),

the corresponding 2-tree operator \( L_{(W, W_j)} \), induced by the edge \( (W, W_j) \), is unitarily equivalent to an operator \( T_j \) on the Fock space \( \mathcal{F}_2 \), sending all Hilbert-space elements in \( \mathcal{H}_W \) to the Hilbert-space elements in \( \mathcal{H}_W(j) \), where \( \mathcal{H}_W \) and \( \mathcal{H}_W(j) \) are given as in (5.2.12) above, for \( j = 1, 2 \). If an Hilbert space elements are not contained in \( \mathcal{H}_W \), then \( T_j \) send such elements to the zero vector \( 0_{\mathcal{F}_2} \) of \( \mathcal{F}_2 \), i.e., the 2-tree operator \( L_{(W, W_j)} \) is unitarily equivalent to the operator \( T_j \) on the vertex space \( H_V = \mathcal{F}_2 \), where

\[
(5.2.13)
\]
$$T_j \eta = \begin{cases} r_{e_j} \eta = \eta \otimes e_j & \text{if } \eta \in \mathcal{H}_W \\ 0_{\mathcal{F}_2} & \text{otherwise}, \end{cases}$$

for $j = 1, 2$, where $r_{e_j}$ is the right creation operator induced by $e_j$, where $e_1 = (1, 0)$, and $e_2 = (0, 1)$ in $\mathbb{C}^{\oplus 2}$.

Remark that the operator $T_j$ is nothing but the restriction $r_{e_j} |_{\mathcal{H}_W}$ of the right creation operator $r_{e_j}$ to the subspace $\mathcal{H}_W$ of $(\mathbb{C}^{\oplus 2})^k$ in $\mathcal{F}_2$. Therefore, the 2-tree operator $L_{(W,Wj)}$ is unitarily equivalent to the restriction $r_{e_j} |_{\mathcal{H}_W}$ of $r_{e_j}$ on the vertex space $H_V = \mathcal{F}_2$, for $j = 1, 2$. 

By the previous lemma, we can obtain the following theorem.

**Theorem 5.11.** Let $r_{e_j}$ be the right creation operator on the Fock space $\mathcal{F}_2$ over $\mathbb{C}^{\oplus 2}$, induced by the natural basis-vector $e_j$, for $j = 1, 2$. Then it is unitarily equivalent to the element $R_j$ of the graph von Neumann algebra $M_2$ of the 2-regular tree $\mathcal{T}_2$, where

\begin{equation}
R_j \overset{\text{def}}{=} \sum_{(W,Wj) \in E(\mathcal{T}_2), W \in \mathcal{X}_2} L_{(W,Wj)} \in M_2, \text{ for } j = 1, 2,
\end{equation}

where $L_{(W,Wj)}$ are the 2-tree operators induced by edges $(W,Wj) \in E(\mathcal{T}_2)$. Also, the right annihilation operator $r_{e_j}^*$ is unitarily equivalent to the adjoint $R_j^*$ of $R_j$, and hence

\begin{equation}
r_{e_j}^* \overset{U.E}{=} \sum_{(W,Wj) \in E(\mathcal{T}_2^{-1}), W \in \mathcal{X}_2} L_{(Wj,W)} \in M_2, \text{ for } j = 1, 2,
\end{equation}

where $\mathcal{T}_2^{-1}$ means the shadow of $\mathcal{T}_2$.

**Proof.** By the previous lemma, the 2-tree operators $L_{(W,Wj)}$ induced by the edges $(W,Wj)$ are unitarily equivalent to the restrictions $r_{e_j} |_{\mathcal{H}_k}$ on the vertex space $H_V = \mathcal{F}_2$, whenever $|W| = k$, for $k \in \mathbb{N}$. Therefore, the right creation operators $r_{e_j}$ induced by the natural basis-elements $e_j$ of $\mathbb{C}^{\oplus 2}$ are unitarily equivalent to the elements $R_j$ of the graph von Neumann algebra $M_2$ of the 2-regular tree $\mathcal{T}_2$, in the sense of (5.1.14), on $\mathcal{F}_2$, for all $j = 1, 2$.

So, the adjoint $r_{e_j}^*$, the right annihilation operators induced by $e_j$, are unitarily equivalent to the adjoints $R_j^*$ of $R_j$, in the sense of (5.1.15), in $M_2$. 

The above theorem shows that the products

$r_{e_{j_1}}^{s_1} r_{e_{j_2}}^{s_2} \ldots r_{e_{j_k}}^{s_k}$

of right creation operators and right annihilation operators, for $j_1, \ldots, j_k \in \{1, 2\}$, and for $s_1, \ldots, s_k \in \{1, \ast\}$, is also unitarily equivalent to the elements

$R_{j_1}^{s_1} R_{j_2}^{s_2} \ldots R_{j_k}^{s_k}$

of the graph von Neumann algebra $M_2$ of the 2-regular tree $\mathcal{T}_2$, where $t$ is a certain constant and $R_{j_1}, \ldots, R_{j_k}$ are given as in (5.1.14).

Therefore, by the above theorem and by the very above discussion, we can obtain the following corollary.

**Corollary 5.12.** The right Toeplitz algebra $\text{Toep}^R(\mathbb{C}^{\oplus 2})$, in the sense of Section 4.2, is a $C^*$-subalgebra of the graph von Neumann algebra $M_2$ of the 2-regular tree $\mathcal{T}_2$.  \[\square\]
Thus, by Section 4.2 and by the above corollary, we can obtain the following theorem.

**Theorem 5.13.** Let $H$ be a Hilbert space with its dimension $\dim H = 2$, and let $\text{Toep}(H)$ be the Toeplitz algebra over $H$. Then $\text{Toep}(H)$ is a $C^*$-subalgebra of the graph von Neumann algebra $M_2$ of the 2-regular tree $T_2$.

**Proof.** By the above corollary, the right Toeplitz algebra $\text{Toep}^R(\mathbb{C}^2)$ over $\mathbb{C}^2$ is a $C^*$-subalgebra of $M_2$. In Section 4.2, we showed that, for any arbitrary Hilbert space $H$, the right Toeplitz algebra $\text{Toep}^R(H)$ and the Toeplitz algebra $\text{Toep}(H)$ are anti-$*$-isomorphic from each other.

If a Hilbert space $H$ has its dimension, $\dim H = 2$, then it is Hilbert-space isomorphic to $\mathbb{C}^2$, and moreover, there exists an anti-$*$-isomorphism

$$\Phi^{-1} : \text{Toep}^R(H) \rightarrow \text{Toep}(H),$$

where $\Phi$ is an anti-$*$-isomorphism defined in (4.2.7). Therefore, the right Toeplitz algebra $\text{Toep}^R(\mathbb{C}^2)$ is anti-$*$-isomorphic to the Toeplitz algebra $\text{Toep}(H)$ over $H$, whose dimension $\dim H = 2$. So, $\text{Toep}(H)$ is a $C^*$-subalgebra of $M_2$. $\blacksquare$

### 5.3. $N$-Tree Operators for $N \geq 2$.

Now, we will consider $N$-tree operators, where $N \geq 2$. In Section 5.2, we already observed the case where $N = 2$. So, we need to study for $N > 2$. However, we can obtain the similar results as in Section 5.2 by induction. In this section, we will extend the main results of Section 5.2 to the general case where $N \geq 2$, i.e., we consider the relation between generalized Toeplitz operators on the Fock space $\mathcal{F}_N$ over $\mathbb{C}^{\oplus N}$, and $N$-tree operators in the graph von Neumann algebra $M_N$ of the $N$-regular tree $T_N$. Again, notice that our extension would be simply done by induction.

Now, we denote the process sending the vertex

$$i_1 i_2 \ldots i_n$$

in the $n$-th level of $T_N$ to the vertex

$$i_1 i_2 \ldots i_n j$$

in the $(n + 1)$-th level of $T_N$ by $\gamma_j$, for all $n \in \mathbb{N}$, where $i_1, \ldots, i_n \in \{1, 2, \ldots, N\}$, i.e.,

$$\gamma_j (i_1 i_2 \ldots i_n) \overset{df}{=} i_1 i_2 \ldots i_n j,$$

for all $j = 1, 2, \ldots, N$, for all $n \in \mathbb{N}$. Also, we can understand these functions $\gamma_j$ generates the edges in $T_2$, i.e., the function

$$\gamma_j (i_1 i_2 \ldots i_n)$$

can be understood as the edge

$$(i_1 i_2 \ldots i_n, i_1 i_2 \ldots i_n j),$$

in $T_N$, for $j = 1, 2, \ldots, N$.

Now, let $X_N$ be the set $\{1, 2, \ldots, N\}$, and let $X_N^*$ be the set

$$X_N = \{\emptyset\} \cup X_N^*,$$

where $X_N^*$ is the set of all words in $X_N$, and $\emptyset$ means the empty word in $X_N$.

We can check that

$$V(T_N) = X_N = \{\emptyset\} \cup \{\gamma_W(X_N) : W \in X_N^*\},$$

where

$$\gamma_W = \gamma_1 \circ \gamma_2 \circ \ldots \circ \gamma_k,$$
whenever \( W = j_1 j_2 \ldots j_k \in X_N \). Similar to the case where \( N = 2 \), we can conclude that

\[
H_V = l^2(\mathcal{X}_N) = l^2(V(T_N))
\]

is Hilbert-space isomorphic to the (generalized) Fock space \( \mathcal{F}_N = \mathcal{F}(\mathbb{C}^{\oplus N}) \) over \( \mathbb{C}^{\oplus N} \).

**Theorem 5.14.** Let \( H_V \) be the vertex space of the graph Hilbert space \( H_N \) of \( T_N \). Then

\[
H_V \overset{\text{Hilbert}}{=} l^2(\mathcal{X}_N) \overset{\text{Hilbert}}{=} \mathcal{F}_N,
\]

where \( \mathcal{F}_N \) is the Fock space over \( \mathbb{C}^{\oplus N} \). \( \square \)

Let \( p_1, \ldots, p_N \) be the natural projection on \( \mathbb{C}^{\oplus N} \), defined by

\[
p_j = \begin{pmatrix}
0 & & & \\
& \ddots & & \\
& & 0 & j \text{-th entry} \mathbb{1} \\
& & & \ddots \\
0 & & & 0
\end{pmatrix},
\]

having only nonzero entry \((j, j)\)-entry 1, expressed by \( \mathbb{1} \) above, for all \( j = 1, \ldots, N \).

Let \( e_1, \ldots, e_N \) be the natural basis elements of \( \mathbb{C}^{\oplus N} \), i.e.,

\[
e_j = \begin{pmatrix}
0 & \ldots & 0 & 1 & 0 & \ldots & 0
\end{pmatrix},
\]

for all \( j = 1, \ldots, N \).

The above theorem characterize the vertex space \( H_V \) in the graph Hilbert space \( H_N \) of the \( N \)-regular tree \( T_N \) by the Fock space \( \mathcal{F}_N \) over \( \mathbb{C}^{\oplus N} \). So, from now on, we use \( H_V \) and \( \mathcal{F}_N \), alternatively.

**Lemma 5.15.** Let \((W, W_j)\) be an edge connecting a vertex

\[
W = j_1 j_2 \ldots j_k \in X_N
\]

to a vertex

\[
W_j = j_1 j_2 \ldots j_k j \in X_N,
\]

for \( j = 1, \ldots, N \), for \( k \in \mathbb{N} \). Then the corresponding \( N \)-tree operator \( L_{(W, W_j)} \) induced by an edge \((W, W_j)\) is unitarily equivalent to the restriction \( r_{e_j} |_{H_W} \) of the right creation operator \( r_{e_j} \) on \( \mathcal{F}_N \) induced by \( e_j \), for \( j = 1, \ldots, N \), more precisely,

\[
L_{(W, W_j)} \overset{U.E}{=} r_{e_j} |_{H_W} : H_W \to \mathcal{H}_W(j)
\]
on \( \mathcal{F}_N = H_V \), for all \( j = 1, \ldots, N \), where

\[
\mathcal{H}_W = p_{j_1} (\mathbb{C}^{\oplus 2}) \otimes \ldots \otimes p_{j_k} (\mathbb{C}^{\oplus 2})
\]

and

\[
\mathcal{H}_W(j) = p_{j_1} (\mathbb{C}^{\oplus 2}) \otimes \ldots \otimes p_{j_k} (\mathbb{C}^{\oplus 2}) \otimes p_j (\mathbb{C}^{\oplus 2}).
\]

\( \square \)

By the previous lemma, we can obtain the following theorem.
Theorem 5.16. Let $r_{e_j}$ be the right creation operator on the Fock space $\mathcal{F}_N$ over $\mathbb{C}^{\oplus N}$, induced by the natural basis vector $e_j$, for $j = 1, \ldots, N$. Then it is unitarily equivalent to the element $R_j$ of the graph von Neumann algebra $M_N$ of the $N$-regular tree $\mathcal{T}_N$, where
\[
R_j \overset{\text{def}}{=} \sum_{(W,W_j) \in E(\mathcal{T}_N), W \in X_N} L_{(W,W_j)} \in M_2,
\]
for $j = 1, \ldots, N$, where $L_{(W,W_j)}$ are the $N$-tree operators induced by edges $(W, W_j) \in E(\mathcal{T}_N)$. Also, the right annihilation operator $r_{e_j}^*$ is unitarily equivalent to the adjoint $R_j^*$ of $R_j$, and hence
\[
r_{e_j}^* \overset{\text{def}}{=} \sum_{(W,W_j) \in E(\mathcal{T}_N^{-1}), W \in X_N} L_{(W_j,W)} ,
\]
for $j = 1, \ldots, N$, where $\mathcal{T}_N^{-1}$ means the shadow of $\mathcal{T}_N$. \qed

The above theorem shows that the products
\[
r_{e_{j_1}}^* r_{e_{j_2}}^* \ldots r_{e_{j_k}}^*
\]
of right creation operators and right annihilation operators, for $j_1, \ldots, j_k \in \{1, \ldots, N\}$, and for $s_1, \ldots, s_k \in \{1, \ast\}$, is also unitarily equivalent to the elements
\[
R_{s_{j_1}}^* R_{s_{j_2}}^* \ldots R_{s_{j_k}}^*
\]
of the graph von Neumann algebra $M_N$ of the $N$-regular tree $\mathcal{T}_N$.

Therefore, by the above theorem and by the very above discussion, we can obtain the following corollary.

Corollary 5.17. The right Toeplitz algebra $\text{Toepl}^R(\mathbb{C}^{\oplus N})$, in the sense of Section 4.2, is a $C^*$-subalgebra of the graph von Neumann algebra $M_N$ of the $N$-regular tree $\mathcal{T}_N$, for $N \in \mathbb{N} \setminus \{1\}$. \qed

Thus, by Section 4.2 and by the above corollary, we can obtain the following theorem.

Theorem 5.18. Let $H$ be a Hilbert space with its dimension $\dim H = N$, for $N \in \mathbb{N} \setminus \{1\}$, and let $\text{Toepl}(H)$ be the Toeplitz algebra over $H$. Then $\text{Toepl}(H)$ is a $C^*$-subalgebra of the graph von Neumann algebra $M_N$ of the $N$-regular tree $\mathcal{T}_N$.

Proof. By the above corollary, the right Toeplitz algebra $\text{Toepl}^R(\mathbb{C}^{\oplus N})$ over $\mathbb{C}^{\oplus N}$ is a $C^*$-subalgebra of $M_N$, for $N \in \mathbb{N} \setminus \{1\}$. In Section 4.2, we showed that, for any arbitrary Hilbert space $H$, the right Toeplitz algebra $\text{Toepl}^R(H)$ and the Toeplitz algebra $\text{Toepl}(H)$ are anti-*$*$-isomorphic from each other.

If a Hilbert space $H$ has its dimension, $\dim H = N$, then it is Hilbert-space isomorphic to $\mathbb{C}^{\oplus N}$, and moreover, there exists an anti-*$*$-isomorphism
\[
\Phi^{-1} : \text{Toepl}^R(H) \rightarrow \text{Toepl}(H),
\]
where $\Phi$ is an anti-*$*$-isomorphism defined in (4.2.7). Therefore, the right Toeplitz algebra $\text{Toepl}^R(\mathbb{C}^{\oplus N})$ is anti-*$*$-isomorphic to the Toeplitz algebra $\text{Toepl}(H)$ over $H$. So, $\text{Toepl}(H)$ is a $C^*$-subalgebra of $M_N$. \qed

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