The Rare Decays $\pi^0 \to e^+e^-$, $\eta \to e^+e^-$ and $\eta \to \mu^+\mu^-$ in Chiral Perturbation Theory

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Abstract

We calculate the decay rates for $\pi^0 \to e^+e^-$, $\eta \to e^+e^-$ and $\eta \to \mu^+\mu^-$ in chiral perturbation theory. The linear combination of counterterms necessary to render these amplitudes finite is fixed by the recently measured branching fraction for $\eta \to \mu^+\mu^-$. We find $\text{Br}(\pi^0 \to e^+e^-) = 7 \pm 1 \times 10^{-8}$ and $\text{Br}(\eta \to e^+e^-) = 5 \pm 1 \times 10^{-9}$.

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1. Introduction

In this letter we use chiral perturbation theory to calculate the electromagnetic decays of the $\pi^0$ and $\eta$ to $\ell^+\ell^-$. These decays proceed through two photon intermediate states containing the anomalous $\pi^0\gamma\gamma$ ($\eta\gamma\gamma$) coupling as well as from a local $\pi^0\ell^+\ell^-$ ($\eta\ell^+\ell^-$) operator which is required as a counterterm to render the one-loop diagram finite. We determine the coefficient of the counterterm by fitting to a recent measurement of $\text{Br}(\eta \to \mu^+\mu^-)$ [1], which then allows us to predict the rates for $\eta \to e^+e^-$ and $\pi^0 \to e^+e^-$. 

Our work differs from previous calculations in which a hadronic form factor is associated with the $\pi^0\gamma\gamma$ and $\eta\gamma\gamma$ vertices [2]-[8]. This makes the one loop integral finite but introduces model dependence into the dispersive piece of amplitude. The absorptive piece of the amplitude is related to the $\pi^0(\eta) \to \gamma\gamma$ width by unitarity and is therefore unambiguous.

A precise theoretical prediction for these decays is interesting not only in itself, but also because the $\eta\mu\mu$ and $\pi^0\mu\mu$ couplings contribute to the “background” for parity violating observables in $K^+ \to \pi^+\mu^+\mu^-$, as has been emphasized recently in refs. [9] and [10]. These couplings also provide a “background” to the $T$ odd observables in this decay which have been investigated in refs. [3] [11]. The $\mu^+$ spin polarisation is a parity violating observable whose magnitude gets a contribution from short distance physics. In the standard model this is dominated by top quark loops and so provides a measurement of the real part of the CKM matrix element $V_{td}$ in the phase convention where $V_{bc}$ is real. The long-distance physics background from the $\eta\mu\mu$ and $\pi^0\mu\mu$ couplings has been studied in detail in [10] where it was found to be significant for small values of $\text{Re}V_{td}$ and small top quark masses. Therefore, it is important to understand these electromagnetic rare decays of the $\pi^0$ and $\eta$ in order to form a reliable estimate of the background to the determination of $\text{Re}V_{td}$ from $\mu^+$ polarisation measurements.

2. $\eta \to \mu^+\mu^-$

The graphs in fig. [1] give the leading contribution to $\eta \to \mu^+\mu^-$ in chiral perturbation theory. The $\eta\gamma\gamma$ and $\pi^0\gamma\gamma$ vertices arise from the Wess-Zumino term [12]

$$
L_{WZ} = \frac{\alpha}{4\pi f} \epsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} \left( \pi^0 / \sqrt{2} + \eta / \sqrt{6} \right) + \ldots \ldots \ , \quad (2.1)
$$

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where $f = 135 \text{ MeV}$. This leads to a decay width
\[
\Gamma(\eta \to \gamma\gamma) = \frac{\alpha^2 m_\eta^3}{96\pi^3 f^2} .
\] (2.2)

The imaginary part of the one-loop graph is finite and related by unitarity to the width (2.2)\[2\]; however, the real part diverges and requires a local counterterm
\[
\mathcal{L}_{\text{c.t.}} = \frac{3i\alpha^2}{32\pi^2} \bar{e}\gamma^\mu\gamma_5 e \left[ \chi_1 Tr(Q^2 \Sigma^\dagger \partial_\mu \Sigma - Q^2 \partial_\mu \Sigma^\dagger \Sigma) 
+ \chi_2 Tr(Q \Sigma^\dagger Q \partial_\mu \Sigma - Q \partial_\mu \Sigma^\dagger Q \Sigma) \right] ,
\] (2.3)

where $\ell = e$ or $\mu$, $Q$ is the electromagnetic charge matrix
\[
Q = \begin{pmatrix}
\frac{2}{3} & 0 & 0 \\
0 & -\frac{1}{3} & 0 \\
0 & 0 & -\frac{1}{3}
\end{pmatrix} .
\] (2.4)

Each term in (2.3) contains two factors of $Q$ because $\mathcal{L}_{\text{c.t.}}$ arises from Feynman diagrams where two photons produce the $\ell^+\ell^-$ pair. The field $\Sigma = \exp(i2M/f)$ is the usual exponentiation of the goldstone boson matrix where
\[
M = \begin{pmatrix}
\pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\
\pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\
K^- & K^0 & -2\eta/\sqrt{6}
\end{pmatrix} .
\] (2.5)

The coefficients $\chi_1$ and $\chi_2$ are renormalisation scheme dependent and subtraction scheme dependent; we use dimensional regularisation with $\overline{MS}$ (the gamma matrix algebra is performed in 4 dimensions) and choose the subtraction point to be $\Lambda = 1 \text{ GeV}$. We find the width for $\eta \to \mu^+\mu^-$ to be
\[
\Gamma(\eta \to \mu^+\mu^-) = \frac{\alpha^2 m_\mu^2 m_\eta}{48\pi f^2} |A(m_\eta^2)|^2 \sqrt{1 - \frac{4m_\mu^2}{m_\eta^2}} ,
\] (2.6)

where
\[
\text{Im} A(s) = \frac{\alpha}{\pi} \frac{1}{\sqrt{1 - \xi^2}} \log \left( \frac{1 + \sqrt{1 - \xi^2}}{\xi - 1} \right) ,
\] (2.7)

and
\[
\text{Re} A(s) = \frac{\alpha}{4\pi^2} \left[ \chi_1(\Lambda) + \chi_2(\Lambda) + 11 - 6 \log \left( \frac{m_\mu^2}{\Lambda^2} \right) 
+ 2\xi^2 - 4\xi^4 + 4\xi^2 \log(4\xi^2) + 8\xi^4 \log(4\xi^2) 
\right. 
- 4 \int_0^1 dx \left[ 3 + \frac{2(\xi^2 - 1)\sqrt{x}}{x + \xi^2(1 - x)} \right] \lambda_+^2 \log |\lambda_+| 
\left. 
- 4 \int_0^1 dx \left[ 3 - \frac{2(\xi^2 - 1)\sqrt{x}}{x + \xi^2(1 - x)} \right] \lambda_-^2 \log |\lambda_-| \right] ,
\] (2.8)
We have defined \( \xi^2 = s/4m^2_\mu \) and \( \lambda_\pm = \sqrt{x\xi^2 \pm \sqrt{x\xi^2 + (1-x)}} \). This amplitude is renormalisation scheme independent. A change in the value of \( \log\left(\frac{m^2_\mu}{\Lambda^2}\right) \) due to a different choice of \( \Lambda \) is compensated by a change in the coefficient \( \chi_1(\Lambda) + \chi_2(\Lambda) \).

The real part of the amplitude agrees with previous computations which introduced a form factor for the \( \eta\gamma\gamma \) vertex when we take the mass associated with the form factor to be large and retain only the leading term.

The branching fraction for \( \eta \to \mu^+\mu^- \) has recently been remeasured at SATURNE, a machine dedicated to \( \eta \) physics, which finds \( \text{Br}(\eta \to \mu^+\mu^-) = (5 \pm 1) \times 10^{-6} \), near the unitary limit of \( 4.3 \times 10^{-6} \) set by \( \text{Im}A(m^2_\eta) \). This fixes the sum of the counterterms \(-40 < \chi_1(\Lambda) + \chi_2(\Lambda) < -13\). Note that the rate is relatively insensitive to the precise value of the counterterms. This is because the one loop amplitude is infrared divergent as \( m_\ell \to 0 \) and so dominates the contribution from the counterterm. Similarly, in phenomenological models it has been found that the predictions for the branching ratio are relatively insensitive to the exact form and scale of the hadronic form factors.

3. Decays to \( e^+e^- \)

Having fixed the sum of counterterms \( \chi_1(\Lambda) + \chi_2(\Lambda) \) we may now unambiguously predict the rates for \( \eta \to e^+e^- \) and \( \pi^0 \to e^+e^- \). It is important that \( \chi_1 \) and \( \chi_2 \) are the same for the cases \( l = e \) and \( l = \mu \). This occurs because both the \( e \) and \( \mu \) masses are small compared with the chiral symmetry breaking scale. From expressions analogous to (2.6)–(2.8) found by substituting \( m_e \) for \( m_\mu \) and evaluating at either \( s = m^2_\eta \) or \( s = m^2_\pi \) (in the \( \pi^0 \) case (2.2) and (2.9) are multiplied by 3) we find

\[
\text{Br}(\pi^0 \to e^+e^-) = 7 \pm 1 \times 10^{-8} \\
\text{Br}(\eta \to e^+e^-) = 5 \pm 1 \times 10^{-9}
\]

(3.1)

compared to the present experimental upper bounds [13]

\[
\text{Br}(\pi^0 \to e^+e^-)_{\text{exp.}} < 1.3 \times 10^{-7} \\
\text{Br}(\eta \to e^+e^-)_{\text{exp.}} < 3 \times 10^{-4}.
\]

(3.2)

For \( \pi^0 \to e^+e^- \), the present upper limit is within a factor of two of the theoretical prediction, and one may hope that in the near future this decay mode will be observed. A precise determination of the branching ratio would test the validity of chiral perturbation
theory for these decays. In contrast, the experimental upper limit for $\eta \rightarrow e^+e^-$ is five orders of magnitude above the theoretical prediction. This upper limit was determined in a bubble chamber experiment performed in 1966 with $\sim 10^4 \eta$’s; hopefully this limit will be dramatically improved at SATURNE where $10^8 \eta$’s are produced per day.

Corrections to our predictions come from higher dimension operators in the chiral Lagrangian which contain more derivatives or more factors of $m_s$. These are suppressed by factors of $m_\eta^2/\Lambda^2_\chi \sim 25\%$.

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Figure Captions

Fig. 1. Leading graphs contributing to $\eta \rightarrow \mu^+\mu^-$. 