Convergence proof for stochastic gradient descent
in the training of deep neural networks with
ReLU activation for constant target functions

Martin Hutzenthaler\textsuperscript{1}, Arnulf Jentzen\textsuperscript{2,3}, Katharina Pohl\textsuperscript{4},
Adrian Riekert\textsuperscript{5}, and Luca Scarpa\textsuperscript{6}

\textsuperscript{1} Faculty of Mathematics, University of Duisburg-Essen,
Essen, Germany, e-mail: martin.hutzenthaler@uni-due.de
\textsuperscript{2} Applied Mathematics: Institute for Analysis and Numerics,
Faculty of Mathematics and Computer Science, University of Münster,
Münster, Germany, e-mail: ajentzen@uni-muenster.de
\textsuperscript{3} School of Data Science and Shenzhen Research Institute of Big Data,
The Chinese University of Hong Kong, Shenzhen, China, e-mail: ajentzen@cuhk.edu.cn
\textsuperscript{4} Faculty of Mathematics, University of Duisburg-Essen,
Essen, Germany, e-mail: katharina.pohl@uni-due.de
\textsuperscript{5} Applied Mathematics: Institute for Analysis and Numerics,
Faculty of Mathematics and Computer Science, University of Münster,
Münster, Germany, e-mail: ariekert@uni-muenster.de
\textsuperscript{6} Department of Mathematics, Politecnico di Milano,
Milano, Italy, e-mail: luca.scarpa@polimi.it

December 15, 2021

Abstract

In many numerical simulations stochastic gradient descent (SGD) type optimization methods perform very effectively in the training of deep neural networks (DNNs) but till this day it remains an open problem of research to provide a mathematical convergence analysis which rigorously explains the success of SGD type optimization methods in the training of DNNs. In this work we study SGD type optimization methods in the training of fully-connected feedforward DNNs with rectified linear unit (ReLU) activation. We first establish general regularity properties for the risk functions and their generalized gradient functions appearing in the training of such DNNs and, thereafter, we investigate the plain vanilla SGD optimization method in the training of such DNNs under the assumption that the target function under consideration is a constant function. Specifically, we prove under the assumption that the learning rates (the step sizes of the SGD optimization method) are sufficiently small but not $L^1$-summable and under the assumption that the target function is a constant function that the expectation of the risk of the considered SGD process converges in the training of such DNNs to zero as the number of SGD steps increases to infinity.
1 Introduction

In many numerical simulations stochastic gradient descent (SGD) optimization methods perform very effectively in the training of deep artificial neural networks (ANNs) but till this day it remains an open problem of research to provide a mathematical convergence analysis which rigorously explains the success of SGD optimization methods in the training of deep ANNs.

Even though the mathematical analysis of gradient descent (GD) type optimization methods in the training of deep ANNs is still in its infancy, there are several auspicious analysis results for GD type optimization methods in the literature. In the following we mention several of such analysis results for GD type optimization methods in a very brief way and we point to the below mentioned references for further details.

We refer, for instance, to [6,26,40–42,46,49] and the references mentioned therein for abstract convergence results for GD type optimization methods under convexity assumptions on the objective function, by which we mean the function of which one intends to approximatively compute minimum points. The objective functions arising in the training of deep ANNs via GD type optimization methods are in nearly all situations not convex (see Corollary 2.19 in Subsection 2.7 in this article below for a characterization of convexity in the situation deep ANNs with the rectified linear unit (ReLU) activation) and in such scenarios there often exist infinitely many saddle and non-global local minimum points of the considered objective functions (cf., e.g., [14,24,33,34,43,44] and the references mentioned therein).

We also point, for example, to [1,5,10,17,23,31,32,35,37,39,45,48,51] for abstract convergence results for GD type optimization methods without convexity assumptions on the objective function. Moreover,
we refer, for instance, to [5, 16, 32, 35, 51, 52] for convergence results for GD type optimization methods under Kurdyka–Lojasiewicz type assumptions.

We also mention convergence results for GD type optimization methods in the training of ANNs in the so-called overparametrized regime in which, roughly speaking, the architecture of the considered ANNs is chosen large enough so that there are much more ANN parameters than input-output data pairs of the considered supervised learning problem. We refer, for example, to [4, 12, 18, 21, 25, 36, 53] for convergence results in this overparametrized regime in the context of shallow ANNs and to [2, 3, 19, 48, 54] for convergence results in this overparametrized regime in the context of deep ANNs.

We also mention convergence results for GD type optimization methods in the training of shallow ANNs under specific assumptions on the target function under consideration, by which we mean the function describing the relationship between the input and the output data (the factorization of the conditional expectation of the output datum given that the corresponding input datum is known). Specifically, we refer to [29] for the situation of piecewise affine target functions, we refer to [22] for the situation of piecewise polynomial target functions, and we refer to [13, 28] for the situation of constant target functions. In particular, a central contribution of this work is to extend the findings of [28] from shallow ANNs with just one hidden layer to deep ANNs with an arbitrary large number of hidden layers.

Furthermore, we also refer, for instance, to [30] for lower bounds for approximations errors for GD type optimizations methods and we refer, for instance, to [15, 38, 50] for certain divergence results for GD type optimization methods in the training of deep ANNs. Further references and overviews on GD type optimizations methods can, for example, also be found in [9, 11, 20, 23, Section 1.1], [26, Section 1], and [47].

In this work we study the plain vanilla SGD optimization method in the training of deep ANNs under the assumption that the target function under consideration is a constant function. Specifically, one of the main results of this work, Theorem 5.11 in Subsection 5.7 below, proves that the expectation of the risk of the considered SGD process converges in the training of fully-connected feedforward deep ANNs with the rectified linear unit (ReLU) activation to zero as the number of SGD steps increases to infinity. To illustrate the findings of this work in more detail, we present in Theorem 1.1 below a special case of Theorem 5.11 and we now add some further explanatory comments regarding the mathematical objects appearing in Theorem 1.1.

The natural number $L \in \mathbb{N} = \{1, 2, 3, \ldots \}$ in Theorem 1.1 is related to the number of hidden layers of the deep ANNs in Theorem 1.1 and the sequence $\ell_k \in \mathbb{N}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots \}$, of natural numbers in Theorem 1.1 describes the dimensions (the numbers of neurons) of the layers of the deep ANNs in Theorem 1.1. More formally, in Theorem 1.1 we consider fully-connected feedforward deep ANNs with $L - 1$ hidden layers and, including the input and the output layer, $L + 1$ layers overall with $\ell_0$ neurons on the input layer (with an $\ell_0$-dimensional input layer), with $\ell_1$ neurons on the 1st hidden layer (with an $\ell_1$-dimensional 1st hidden layer), with $\ell_2$ neurons on the 2nd hidden layer (with an $\ell_2$-dimensional 2nd hidden layer), $\ldots$, with $\ell_{L-1}$ neurons on the $(L-1)$-th (and last) hidden layer (with an $\ell_{L-1}$-dimensional $(L-1)$-th hidden layer), and with $\ell_L$ neurons on the output layer (with an $\ell_L$-dimensional output layer).

There are thus $\ell_1\ell_0$ real weight parameters and $\ell_1$ real bias parameters to describe the affine linear transformation between the $\ell_0$-dimensional input layer and the $\ell_1$-dimensional 1st hidden layer in the deep ANN, there are thus $\ell_2\ell_1$ real weight parameters and $\ell_2$ real bias parameters to describe the affine linear transformation between the $\ell_1$-dimensional 1st hidden layer and the $\ell_2$-dimensional 2nd hidden layer, $\ldots$, there are thus $\ell_{L-1}\ell_{L-2}$ real weight parameters and $\ell_{L-1}$ real bias parameters to describe the affine linear transformation between the $\ell_{L-2}$-dimensional $(L-2)$-th hidden layer and the $\ell_{L-1}$-dimensional $(L-1)$-th hidden layer, and there are thus $\ell_L\ell_{L-1}$ real weight parameters and $\ell_L$ real bias parameters to describe the affine linear transformation between the $\ell_{L-1}$-dimensional $(L-1)$-th hidden layer and the $\ell_L$-dimensional output layer. Overall in Theorem 1.1 we thus use

$$v = \sum_{k=1}^{L} (\ell_k\ell_{k-1} + \ell_k) = \sum_{k=1}^{L} \ell_k(\ell_{k-1} + 1) \quad (1.1)$$

real numbers to describe the employed deep ANNs. We also refer to Figure 1 for a graphical illustration of the used deep ANN architecture in the case of a simple example deep ANN with 3 hidden layers (corresponding to $L = 4$ affine linear transformations), with 5 neurons on the input layer (corresponding
Figure 1: Graphical illustration of the used deep ANN architecture in the case of a simple example deep ANN with 3 hidden layers (corresponding to $L = 4$ affine linear transformations), with 5 neurons on the input layer (corresponding to $\ell_0 = 5$), 8 neurons on the 1st hidden layer (corresponding to $\ell_1 = 8$), 7 neurons on the 2nd hidden layer (corresponding to $\ell_2 = 7$), 9 neurons on the 3rd hidden layer (corresponding to $\ell_3 = 9$), and 3 neurons on the output layer (corresponding to $\ell_4 = 3$). In this situation we have for every ANN parameter vector $\theta \in \mathbb{R}^3 = \mathbb{R}^{213}$ that the realization function $\mathbb{R}^5 \ni x \mapsto N^4_{\infty, 3}(x)$ of the considered deep ANN maps the 5-dimensional input vector $x = (x_1, x_2, x_3, x_4, x_5) \in [a, b]^5$ to the 3-dimensional output vector $N^4_{\infty, 3}(x) = (N^4_{\infty, 1}(x), N^4_{\infty, 2}(x), N^4_{\infty, 3}(x))$. 
to $\ell_0 = 5$), 8 neurons on the 1st hidden layer (corresponding to $\ell_1 = 8$), 7 neurons on the 2nd hidden layer (corresponding to $\ell_2 = 7$), 9 neurons on the 3rd hidden layer (corresponding to $\ell_3 = 9$), and 3 neurons on the output layer (corresponding to $\ell_4 = 3$).

The matrices $w^{0,k}, k \in \mathbb{N}, \theta \in \mathbb{R}^k$, and the vectors $b^{0,k}, k \in \mathbb{N}, \theta \in \mathbb{R}^k$, in (1.2) in Theorem 1.1 are used to describe the affine linear transformations in the considered deep ANN. More formally, for every ANN parameter vector $\theta \in \mathbb{R}^d$ and every $k \in \{1, \ldots, L\}$ we have that the $(\ell_k \times \ell_{k-1})$-matrix $w^{\theta,k} = (w^{\theta,k}_{i,j})_{(i,j) \in \{1, \ldots, \ell_k\} \times \{1, \ldots, \ell_{k-1}\}}$ describes the linear part in the affine linear transformation between the $k$-th and the $(k+1)$-th layer (the $(k-1)$-th and the $k$-th hidden layer, respectively) and we have that the $\ell_k$-dimensional vector $b^{\theta,k} = (b^{\theta,k}_1, b^{\theta,k}_2, \ldots, b^{\theta,k}_{\ell_k})$ describes the inhomogenous part in the affine linear transformation between the $k$-th and the $(k+1)$-th layer (the $(k-1)$-th and the $k$-th hidden layer, respectively). The functions $A^\theta_{k}, k \in \mathbb{N}, \theta \in \mathbb{R}^d$, in Theorem 5.11 are used to describe the affine linear transformations in the considered deep ANN in the sense that for every ANN parameter vector $\theta \in \mathbb{R}^d$ and every $k \in \mathbb{N}$ we have that the affine linear transformation between the $k$-th and the $(k+1)$-th layer (the $(k-1)$-th and the $k$-th hidden layer, respectively) is given through the function $\mathbb{R}^{\ell_k-1} \ni x \mapsto A^\theta_{k}(x) = b^{\theta,k} + w^{\theta,k}x$ in $\mathbb{R}^{\ell_k}$.

The real numbers $a \in \mathbb{R}$, $b \in (a, \infty)$ in Theorem 1.1 are used to specify the region where the input data of the considered supervised learning problem take values in. Specifically, we assume in Theorem 1.1 that the input data takes values in the $\ell_0$-dimensional cube $[a, b]^{\ell_0} \subseteq \mathbb{R}^{\ell_0}$, the target function in Theorem 1.1 and which describes the relationship between the input data and the output data, is thus a function from the $\ell_0$-dimensional set $[a, b]^{\ell_0}$ to the $\ell_L$-dimensional set $\mathbb{R}^{\ell_L}$.

We assume that the target function is a constant function which is represented by the $\ell_L$-dimensional vector $\xi \in \mathbb{R}^{\ell_L}$ in Theorem 1.1. The target function in Theorem 1.1 is thus given through the constant function $[a,b]^{\ell_0} \ni x \mapsto \xi \in \mathbb{R}^{\ell_L}$.

In Theorem 1.1 we study the training of deep ANNs with ReLU activation. The ReLU activation function $\mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R}$ fails to be differentiable and this lack of differentiability transfers from the activation function to the risk function. To specify the SGD optimization method, one thus need to specify a suitably generalized gradient function. We accomplish this by means of an appropriate approximation procedure in which the ReLU activation function $\mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R}$ is approximated via appropriate continuously differentiable functions which converge pointwise to the ReLU activation function and whose derivative converge pointwise to the left derivative of the ReLU activation function; see (1.3) below for details. Our approximation procedure in (1.3) is partially inspired by the shallow ANN case in [28] (see also [13]) but in the case of deep ANNs such an approximation procedure is a more more delicate issue since the non-differentiable ReLU activation function appears in the argument of the non-differentiable ReLU activation function in the case of risk functions associated to deep ANNs; see Proposition 2.5 below for further details. The mathematical description of the generalized gradient function in (1.3) and (1.5) in Theorem 1.1 corresponds precisely to the standard procedure how such appropriately generalized gradients are computed in a PYTHON implementation in TENSORFLOW (cf., e.g., [28], Section 1 and Subsection 3.7). The functions $M_r : (\cup_{n \in \mathbb{N}}\mathbb{R}^n) \to (\cup_{n \in \mathbb{N}}\mathbb{R}^n), r \in [1, \infty)$, in Theorem 1.1 specify appropriate multidimensional versions of the ReLU activation function and its continuously differentiable approximations.

The function $||| : (\cup_{n \in \mathbb{N}}\mathbb{R}^n) \to \mathbb{R}$ in Theorem 1.1 is used to express the standard norm for tuples of real numbers. In particular, observe that for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we have that the real number $||x||$ is thus given through the standard norm of the $n$-dimensional vector $x$.

The probability space $(\Omega, \mathcal{F}, P)$ in Theorem 1.1 is the probability space on which the input data is defined and the i.i.d. random variables $X^{n,m} : \Omega \to [a, b]^{\ell_0}, n, m \in \mathbb{N}_0$, in Theorem 1.1 describe the input data of the considered supervised learning problem. The sequence of natural numbers $M_n \in \mathbb{N}, n \in \mathbb{N}_0$, in Theorem 1.1 specifies the size of the mini-batches in the SGD optimization method and the stochastic process $\Theta = (\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \times \Omega \to \mathbb{R}^d$ in Theorem 1.1 specifies the SGD process. In particular, we note for every SGD step $n \in \mathbb{N}_0$ that the random variable $\Theta_n$ (the SGD process at time $n$) and the input data random variables $X^{n,1}, X^{n,2}, \ldots, X^{n,M_n}$ are used to compute the random variable $\Theta_{n+1}$ (the SGD process at time $n+1$).

The sequence of non-negative real numbers $\gamma_n \in [0, \infty), n \in \mathbb{N}_0$, in Theorem 1.1 describes the learning rates (the step sizes) for the SGD optimization method. In Theorem 1.1 we assume – in dependence of
the size of the norm of the initial value of the SGD process, of the architecture of the considered deep ANNs, and of the target function – that the learning rates are sufficiently small but also that the learning rates are not summable.

Under the above outlined hypotheses we prove in item (iii) in Theorem 1.1 that the expected risk of the SGD process converges to zero as the number of SGD steps increases to infinity. We now present the precise statement of Theorem 1.1.

Theorem 1.1. Let $L, d \in \mathbb{N}$, $(\ell_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}$, $a \in \mathbb{R}$, $b \in (a, \infty)$, $\xi \in \mathbb{R}^{Ld}$, satisfy $\delta = \sum_{k=1}^L \ell_k (\ell_{k-1} + 1)$, for every $\theta = (\theta_1, \ldots, \theta_\delta) \in \mathbb{R}^\delta$ let $w^{k, \theta} = (w^{k, \theta}_{i,j})_{(i,j) \in \{1, \ldots, \ell_k\} \times \{1, \ldots, \ell_{k-1}\} \in \mathbb{R}^{Ld \times \ell_{k-1}}$, $k \in \mathbb{N}$, and $b^{k, \theta} = (b^{k, \theta}_i)_{i \in \{1, \ldots, \ell_k\} \in \mathbb{R}^{Ld}}$, $k \in \mathbb{N}$, satisfy for all $k \in \{1, \ldots, L\}$, $i \in \{1, \ldots, \ell_k\}$, $j \in \{1, \ldots, \ell_{k-1}\}$ that
\[
w^{k, \theta}_{i,j} = \theta_{\ell_{j-1}+\ell_j+\sum_{s=1}^{k-1} \ell_s \ell_{s-1}+1}
\]
and
\[
b^{k, \theta}_i = \theta_{\ell_{i-1}+i+i+\sum_{s=1}^{k-1} \ell_s \ell_{s-1}+1}.
\]
for every $k \in \mathbb{N}$, $\theta \in \mathbb{R}^k$, $\theta \geq 0$ satisfy for all $x \in \mathbb{R}^{Ld-1}$ that $A^k(x) = b^{k, \theta} + w^{k, \theta} x$, let $\mathcal{R}_r : \mathbb{R} \to \mathbb{R}$, $r \in [1, \infty]$, satisfy for all $r \in [1, \infty]$, $x \in (-\infty, 2^{-1}r^{-1}]$, $y \in \mathbb{R}$, $z \in [r^{-1}, \infty)$ that
\[
\mathcal{R}_r(x) = 0, \quad 0 \leq \mathcal{R}_r(y) \leq \mathcal{R}_\infty(y) = \max\{y, 0\}, \quad \text{and} \quad \mathcal{R}_r(z) = z, \quad (1.3)
\]
assume $\sup_{r \in [1, \infty]} \sup_{x \in \mathbb{R}} \mathcal{R}_r((\mathcal{R}_r)'(x)) < \infty$, let $\|\cdot\| : (\cup_{n \in \mathbb{N}} \mathbb{R}^n) \to \mathbb{R}$ and $\mathbb{M}_n : (\cup_{n \in \mathbb{N}} \mathbb{R}^n) \to (\cup_{n \in \mathbb{N}} \mathbb{R}^n)$, $\mathbb{R} \to \mathbb{R}$, $r \in [1, \infty]$, satisfy for all $r \in [1, \infty]$, $n \in \mathbb{N}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ that $\|x\| = (\sum_{k=1}^n |x_k|^2)^{1/2}$ and $\mathbb{M}_n(x) = (\mathbb{R}_r(x_1), \ldots, \mathbb{R}_r(x_n))$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X^{n,m}, \Omega \to [a, b]^n, n, m \in \mathbb{N}_0,$ be i.i.d. random variables, let $(M_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{N}$, for every $r \in [1, \infty]$ let $N^{k, \theta}_r : \mathbb{R}^r \to \mathbb{R}^k$, $\theta \in \mathbb{R}^k$, $k \in \mathbb{N}$, and $\mathcal{L}_r^\theta : \mathbb{R}^r \times \Omega \to \mathbb{R}, n \in \mathbb{N}_0$, satisfy for all $x \in \mathbb{R}^r$, $\theta \in \mathbb{R}^k$, $k \in \mathbb{N}$, $\omega \in \Omega$ that
\[
\mathcal{N}^{k, \theta}_r(x) = A^k(x), \quad \mathcal{N}^{k+1,-1}_r(x) = A^k_{r+1}(\mathcal{R}_r((\mathcal{N}^{k, \theta}_r(x)))), \quad (1.4)
\]
and $\mathcal{L}_r^\theta(\omega) = \frac{1}{\delta^{M_n}} \sum_{k=1}^M_n \|N^{k, \theta}_r(X(n,m)(\omega)) - \xi\|^2$, let $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ satisfy for all $\theta \in \mathbb{R}^k$ that $\mathcal{L}(\theta) = E[\|N^{k, \theta}_r(X(0,0)) - \xi\|^2]$, for every $n \in \mathbb{N}$ let $\mathcal{G}^0 : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfy for all $\theta \in \mathbb{R}^k$, $\omega \in \Omega : ((\nabla \mathcal{L}^{\mathcal{G}^0}(\theta,\omega))_{r \in [1, \infty]}$ is convergent) that
\[
\mathcal{G}^0(\omega, \theta) = \lim_{r \to \infty} (\nabla \mathcal{L}^{\mathcal{G}^0}(\theta,\omega)), \quad (1.5)
\]
let $\Theta = (\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \times \Omega \to \mathbb{R}$ be a stochastic process, let $(\gamma_n)_{n \in \mathbb{N}_0} \subseteq [0, \infty)$, assume that $\Theta_0$ and $(X(n,m))_{n,m \in \mathbb{N}} \subseteq [0, \infty)$ are independent, and assume for all $n \in \mathbb{N}_0$, $\omega \in \Omega$ that $\Theta_{n+1}(\omega) = \Theta_n(\omega) - \gamma_n \mathcal{G}^0(\Theta_n(\omega), \omega), (1.6)$ and $\max\{a, b, \|\xi\|, 1\}^{2L} \gamma_n \leq (\|\Theta_0\| + 1)^{-2L}$, and $\sum_{k=0}^\infty \gamma_k = \infty$. Then
(i) there exists $C \in \mathbb{R}$ such that $\mathbb{P}(\sup_{n \in \mathbb{N}_0} \|\Theta_n\| \leq C) = 1$,
(ii) it holds that $\mathbb{P}(\lim_{n \to \infty} \mathcal{L}(\Theta_n) = 0) = 1$, and
(iii) it holds that $\lim_{n \to \infty} \mathbb{E}[\mathcal{L}(\Theta_n)] = 0$.

Theorem 1.1 above is a direct consequence of Corollary 5.12 in Subsection 5.7. Corollary 5.12, in turn, follows from Theorem 5.11 (one of the main results of this work).

The remainder of this article is structured in the following way. In Section 2 we establish several fundamental regularity properties and representation results for the risk function and its generalized gradient function. While Theorem 1.1 is restricted to the situation where the target function is a constant function, the findings in Section 2 are proved for general measurable target functions and are – beyond their employment in the proof of Theorem 1.1 – of general use in the mathematical analysis of the training of deep ReLU ANNs. In Section 3 we use the main findings from Section 2 to prove in the case where the target function is a constant function that the risks of gradient flow (GF) processes converge with convergence rate 1 to zero. In Section 4 we employ some of the results from Sections 2 and 3 to establish that the risks of GD processes converge to zero provided that the target function is a constant function and that the learning rates are not $L^1$-summable but sufficiently small. Finally, in Section 5 we extend the findings from Section 4 and prove that the expectations of the risks of SGD processes converge to zero provided that the target function is a constant function and that the learning rates are not $L^1$-summable but sufficiently small. Theorem 1.1 above is a direct consequence of Corollary 5.12 in Section 5.


2 Properties of the risk function associated to deep artificial neural networks (ANNs)

In this section we establish several fundamental regularity properties and representation results for the risk function and its generalized gradient function. While Theorem 1.1 in the introductory section above is restricted to the situation where the target function is a constant function, the findings in this section are proved for general measurable target functions and are – beyond their deployment in the proof of Theorem 1.1 above – of general use in the mathematical analysis of the training of deep ReLU ANNs.

In Setting 2.1 in Subsection 2.1 we present our mathematical framework to introduce, among other things, the number \( L \in \mathbb{N} \) of affine linear transformations in the considered deep ANNs, the dimensions \( \ell_0, \ell_1, \ldots, \ell_L \in \mathbb{N} \) of the layers of the considered deep ANNs (the number of neurons on the layers of the considered deep ANNs), the number \( \theta = \sum_{k=1}^{L} \ell_k(\ell_{k-1} + 1) \in \mathbb{N} \) of real parameters of the considered deep ANNs, the realization functions \([a, b]^{\ell_0} \ni x \mapsto \mathcal{N}_{L, \theta}(x) \in \mathbb{R}^{\ell_L} \), \( \theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{R}^\theta \), of the considered deep ReLU ANNs (see (2.4)), the risk function \( \mathcal{L}_\infty : \mathbb{R}^\theta \to \mathbb{R} \) (see (2.5)), the generalized gradient function \( \mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_\ell) : \mathbb{R}^\ell \to \mathbb{R}^\ell \) associated to the risk function, and the Lyapunov-type function \( V : \mathbb{R}^\ell \to \mathbb{R} \) for the mathematical analysis of GD type optimization methods in the training of deep ReLU ANNs (see (2.6)). In Setting 2.1 we thus consider ReLU ANNs with \( L - 1 \) hidden layers and, including the input and the output layer, with \( L + 1 \) layers overall.

In Theorem 2.9 in Subsection 2.3 below (one of the main results of this section) we establish in (2.61) and (2.62) an explicit representation for the generalized gradient function \( \mathcal{G} : \mathbb{R}^\theta \to \mathbb{R}^\ell \) of the risk function \( \mathcal{L}_\infty : \mathbb{R}^\theta \to \mathbb{R} \). Our proof of Theorem 2.9 employs the approximation result for realization functions of deep ReLU ANNs in Proposition 2.5 in Subsection 2.2, the representation result for gradients of suitable approximations of the realization functions of deep ReLU ANNs in Lemma 2.6 in Subsection 2.2, the a priori bound result in Lemma 2.7 in Subsection 2.3, and the elementary integrability result for the target function \( f : [a, b]^{\ell_0} \to \mathbb{R} \) in Lemma 2.8 in Subsection 2.3.

Our proof of the approximation result in Proposition 2.5, in turn, uses the elementary auxiliary result in Lemma 2.2 in Subsection 2.2 and the elementary convergence rate result for suitable approximations of the ReLU activation function \( \mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R} \) in Lemma 2.3 in Subsection 2.2. In the elementary result in Lemma 2.4 in Subsection 2.2 we also explicitly construct examples for such approximations of the ReLU activation function \( \mathbb{R} \ni x \mapsto \max\{x, 0\} \in \mathbb{R} \) (cf. (2.2) in Setting 2.1).

In Theorem 2.11 in Subsection 2.5 below (another main result of this section) we establish in (2.75) an explicit polynomial growth estimate for the generalized gradient function \( \mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_\ell) : \mathbb{R}^\ell \to \mathbb{R}^\ell \) of the risk function \( \mathcal{L}_\infty : \mathbb{R}^\ell \to \mathbb{R} \). As a consequence of (2.75), we show in Corollary 2.12 in Subsection 2.5 that the generalized gradient function \( \mathcal{G} : \mathbb{R}^\ell \to \mathbb{R}^\ell \) is locally bounded. Our proof of Corollary 2.12 employs, beside (2.75), also the well-known local Lipschitz continuity result in Lemma 2.10 in Subsection 2.4. Our proof of Theorem 2.11, in turn, makes use of the explicit representation of the generalized gradient function \( \mathcal{G} : \mathbb{R}^\ell \to \mathbb{R}^\ell \) in (2.61) and (2.62) in Theorem 2.9.

In Corollary 2.19 in Subsection 2.7 we establish that the risk function \( \mathcal{L}_\infty : \mathbb{R}^\ell \to \mathbb{R} \) in the training of deep ReLU ANNs (see (2.5)) is convex if and only if the product of the total mass \( \mu([a, b]^{\ell_0}) \) and the number \( L - 1 \) of hidden layers vanishes \((L - 1)\mu([a, b]^{\ell_0}) = 0\). Corollary 2.19 hence proves that, except of the degenerate situation where the underlying measure \( \mu : \mathcal{B}([a, b]^{\ell_0}) \to [0, \infty] \) is the zero measure or where the considered ANNs just describe affine linear transformations without any hidden layer, it holds for every arbitrary measurable target function \( f : [a, b]^{\ell_0} \to \mathbb{R}^{\ell_L} \) that the risk function \( \mathcal{L}_\infty : \mathbb{R}^\ell \to \mathbb{R} \) is not convex. Our proof of Corollary 2.19 employs the essentially well-known convexity result in Corollary 2.16 in Subsection 2.6 and the basic non-convexity result in Corollary 2.18 in Subsection 2.7. Our proof of Corollary 2.16 uses the elementary characterization result for affine linear functions in Lemma 2.14 in Subsection 2.6 and the well-known convexity result in Proposition 2.15 in Subsection 2.6.

In Proposition 2.13 in Subsection 2.6 we recall the fact that convex functions from \( \mathbb{R}^\ell \) to \( \mathbb{R} \) do not admit any no-global local minimum point (see (2.94)).

Many of the results in this section extend the results in [28, Section 2] from shallow ReLU ANNs with just one hidden layer to deep ReLU ANNs with an arbitrarily large number of hidden layers. In particular, Proposition 2.5 in Subsection 2.2 extends [28, Proposition 2.2], Theorem 2.9 in Subsection 2.3 extends [28, Proposition 2.3], Lemma 2.10 in Subsection 2.4 extends [28, Lemma 2.4], Theorem 2.11 in Subsection 2.5 extends [28, Lemma 2.5], and Corollary 2.12 in Subsection 2.5 extends [28, Corollary 2.6].
2.1 Mathematical framework for deep ANNs with ReLU activation

**Setting 2.1.** Let \( L, d \in \mathbb{N}, (k_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}, \) \( a, a \in \mathbb{R}, b \in (a, \infty), \mathcal{A} \in (0, \infty), \mathcal{B} \in (\mathcal{A}, \infty) \) satisfy \( \mathcal{D} = \sum_{k=1}^{L} \ell_k (k_k - 1) + 1 \) and \( a = \max \{|a|, |b|, 1| \), for every \( \theta = (\theta_1, \ldots, \theta_{k}) \in \mathbb{R}^k \) let \( m_{k, \theta} = (w_{i,j})_{(i,j) \in \{1, \ldots, k_k\} \times \{1, \ldots, k_{k-1}\}} \in \mathbb{R}^{k_1 \times k_{k-1}}, k \in \mathbb{N}, \) and \( b_{k, \theta} = (b_{1, \theta}^k, \ldots, b_{k_k, \theta}^k) \in \mathbb{R}^{k_1}, k \in \mathbb{N}, \) satisfy for all \( k \in \{1, \ldots, L\}, i \in \{1, \ldots, k_k\}, j \in \{1, \ldots, k_{k-1}\} \)

\[
\begin{align*}
\theta_{1,k-1} = \sum_{i=1}^{j} \ell_i (\ell_{i+1} + 1) \\
b_{i,j} = \theta_{i,k-1} + \sum_{i=1}^{j} \ell_i (\ell_{i+1} + 1),
\end{align*}
\]

(2.1)

for every \( k \in \mathbb{N}, \theta \in \mathbb{R}^k \) let \( A_{k,1}^\theta = (A_{1,1}^\theta, \ldots, A_{r,1}^\theta) : \mathbb{R}^{k_1} \rightarrow \mathbb{R}^{k_k} \) satisfy for all \( x \in \mathbb{R}^{k_1} \) that \( A_{k,1}^\theta(x) = b_{k,\theta} + m_{k,\theta} x \) and \( \mathcal{R}_r : \mathbb{R} \rightarrow \mathbb{R}, r \in [1, \infty], \) satisfy for all \( r \in [1, \infty], x \in (-\infty, \mathcal{A}^{-1}), y \in \mathbb{R}, z \in [\mathcal{B}^{-1}, \infty) \) that

\[
\mathcal{R}_r \in C^1(\mathbb{R}, \mathbb{R}), \quad \mathcal{R}_r(x) = 0, \quad 0 \leq \mathcal{R}_r(y) \leq \mathcal{R}_\infty(y) = \max\{y, 0\}, \quad \text{and} \quad \mathcal{R}_r(z) = z, \quad (2.2)
\]

assume \( \sup_{r \in [1, \infty]} \sup_{x \in \mathbb{R}} (|\mathcal{R}_r'(x)| < \infty, \) let \( \|\cdot\| : (\cup_{n \in \mathbb{N}} \mathbb{R}^n) \rightarrow \mathbb{R}, \langle \cdot, \cdot \rangle : (\cup_{n \in \mathbb{N}} (\mathbb{R}^n \times \mathbb{R}^n)) \rightarrow \mathbb{R}, \) and \( \mathcal{M}_r : (\cup_{n \in \mathbb{N}} \mathbb{R}^n) \rightarrow (\cup_{n \in \mathbb{N}} \mathbb{R}^n), r \in [1, \infty], \) satisfy for all \( r \in [1, \infty], n \in \mathbb{N}, \) \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) that

\[
\begin{align*}
\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{1/2}, \quad \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i, \quad \text{and} \quad \mathcal{M}_r(x) = (\mathcal{R}_r(x_1), \ldots, \mathcal{R}_r(x_n)),
\end{align*}
\]

(2.3)

for every \( \theta \in \mathbb{R}^k \) let \( \mathcal{N}_{r,1}^\theta = (\mathcal{N}_{r,1}^\theta, \ldots, \mathcal{N}_{r,1}^\theta) : \mathbb{R}^{k_1} \rightarrow \mathbb{R}^{k_k}, r \in [1, \infty], k \in \mathbb{N}, \) and \( \mathcal{N}_{r,1}^\theta \subseteq \mathbb{R}^{k_1}, k, i \in \mathbb{N}, \) satisfy for all \( r \in [1, \infty], k \in \mathbb{N}, i \in \{1, \ldots, k_k\}, x \in \mathbb{R}^n \) that

\[
\begin{align*}
\mathcal{N}_{r,1}^\theta(x) &= A_{i,1}^\theta(x), \quad \mathcal{N}_{r,1}^{k+1} \theta(x) = A_{k+1}^\theta(\mathcal{M}_{r+1}^\theta(\mathcal{N}_{r,1}^\theta(x))),
\end{align*}
\]

(2.4)

and \( \mathcal{N}_{r,1}^\theta = \{y \in [a, b]^{\mathbb{N}} : \mathcal{N}_{r,1}^\theta(y) > 0\}, \) \( \mu : \mathcal{B}[a, b]^{\mathbb{N}} \rightarrow [0, \infty] \) be a measure, let \( m \in \mathbb{R} \) satisfy \( m = \mu([a, b]^{\mathbb{N}}), \) \( f = (f_1, \ldots, f_{k_k}) : [a, b]^{\mathbb{N}} \rightarrow \mathbb{R}^{k_1} \) be measurable, for every \( r \in [1, \infty] \) let \( \mathcal{L}_r : \mathbb{R}^k \rightarrow \mathbb{R} \) satisfy for all \( \theta \in \mathbb{R}^k \) that

\[
\mathcal{L}_r(\theta) = \int_{[a, b]^{\mathbb{N}}} \|\mathcal{N}_{r,1}^\theta(x) - f(x)\|^2 \mu(dx),
\]

(2.5)

let \( \mathcal{G} = (G_1, \ldots, G_k) : \mathbb{R}^k \rightarrow \mathbb{R}^k \) satisfy for all \( \theta \in \mathbb{R}^k : ((\nabla \mathcal{L}_r)(\theta))_{r \in [1, \infty]} \) is convergent that \( \mathcal{G}(\theta) = \lim_{r \rightarrow \infty} (\nabla \mathcal{L}_r)(\theta) \), and let \( V : \mathbb{R}^k \rightarrow \mathbb{R} \) satisfy for all \( \theta \in \mathbb{R}^k \) that

\[
V(\theta) = \left[ \sum_{k=1}^{L} \ell_k \|b_{k,\theta}\|^2 + \sum_{j=1}^{k_{k-1}} \sum_{i=1}^{k_k} |w_{i,j,k}^{k,\theta}|^2 \right] - 2L(0, b_{k,\theta}).
\]

(2.6)

2.2 Approximations of the realization functions of the considered deep ANNs

**Lemma 2.2.** Assume Setting 2.1 and let \( \theta \in \mathbb{R}^k, r \in [1, \infty]. \) Then

(i) it holds for all \( k \in \mathbb{N}, i \in \{1, \ldots, k_k\}, x = (x_1, \ldots, x_{k_{k-1}}) \in \mathbb{R}^{k_{k-1}} \) that

\[
A_{k,1}^\theta(x) = b_{i,1}^\theta + \sum_{j=1}^{k_{k-1}} w_{i,j}^{k,\theta} x_j,
\]

(2.7)

(ii) it holds for all \( i \in \{1, \ldots, k_1\}, x = (x_1, \ldots, x_{k_0}) \in \mathbb{R}^{k_0} \) that

\[
\mathcal{N}_{r,1}^\theta(x) = b_{i,1}^\theta + \sum_{j=1}^{k_0} w_{i,j}^{1,\theta} x_j,
\]

(2.8)

and

(iii) it holds for all \( k \in \mathbb{N}, i \in \{1, \ldots, k_{k+1}\}, x \in \mathbb{R}^{k_{k+1}} \) that

\[
\mathcal{N}_{r,1}^{k+1} \theta(x) = b_{i,1}^{k+1,\theta} + \sum_{j=1}^{k_{k+1}} w_{i,j}^{k+1,\theta} \mathcal{R}_{r+k}(\mathcal{N}_{r,1}^\theta(x)).
\]

(2.9)
Proof of Lemma 2.2. Observe that (2.4) and the assumption that for all $k \in \mathbb{N}$, $x \in \mathbb{R}^{d_k-1}$ it holds that $A_k^t(x) = b^t \theta^k + w^{t,k} x$ establish (2.7), (2.8), and (2.9). The proof of Lemma 2.2 is thus complete.

Lemma 2.3 (Approximations of the rectifier function). Assume Setting 2.1. Then

(i) it holds for all $x \in \mathbb{R}$ that

$$\limsup_{r \to \infty} (|\mathcal{R}_r(x) - \mathcal{R}_\infty(x)| + |(\mathcal{R}_r)'(x) - \mathbbm{1}_{(0,\infty)}(x)|) = 0 \quad (2.10)$$

and

(ii) it holds for all $r \in [1, \infty)$, $x \in \mathbb{R}$ that $|\mathcal{R}_r(x) - \mathcal{R}_\infty(x)| \leq \mathcal{B} r^{-1}$.

Proof of Lemma 2.3. Note that (2.2) ensures that for all $r \in [1, \infty)$, $x \in (-\infty, 0]$ it holds that $\mathcal{R}_r(x) = 0 = \max\{x, 0\} = \mathcal{R}_\infty(x)$. Therefore, we obtain for all $r \in [1, \infty)$, $x \in (-\infty, 0]$ that

$$|\mathcal{R}_r(x) - \mathcal{R}_\infty(x)| + |(\mathcal{R}_r)'(x) - \mathbbm{1}_{(0,\infty)}(x)| = 0. \quad (2.11)$$

Furthermore, observe that (2.2) proves that for all $x \in (0, \infty)$ there exists $R \in [1, \infty)$ such that for all $r \in [R, \infty)$, $y \in (\ell^2/2, \infty)$ it holds that $\mathcal{R}_r(y) = \mathcal{R}_\infty(y)$. Hence, we obtain for all $x \in (0, \infty)$ that

$$\limsup_{r \to \infty} (|\mathcal{R}_r(x) - \mathcal{R}_\infty(x)| + |(\mathcal{R}_r)'(x) - (\mathcal{R}_\infty)'(x)|) = 0. \quad (2.12)$$

Combining this with (2.11) establishes item (i). Note that (2.2) shows that for all $r \in [1, \infty)$, $y \in (0, \mathcal{B} r^{-1})$ it holds that

$$|\mathcal{R}_r(y) - \mathcal{R}_\infty(y)| = \mathcal{R}_\infty(y) - \mathcal{R}_r(y) \leq \mathcal{R}_\infty(y) \leq y \leq \mathcal{B} r^{-1}. \quad (2.13)$$

Moreover, observe that (2.2) proves that for all $r \in [1, \infty)$, $x \in (-\infty, 0] \cup [\mathcal{B} r^{-1}, \infty)$ it holds that $\mathcal{R}_r(x) = \mathcal{R}_\infty(x)$. Combining this with (2.13) establishes item (ii). The proof of Lemma 2.3 is thus complete.

Lemma 2.4 (Examples of approximations of the rectifier function). Let $\mathcal{A} \in (0, \infty)$, $\mathcal{B} \in (\mathcal{A}, \infty)$, $\eta \in C^1(\mathbb{R}, \mathbb{R})$ satisfy for all $x \in (-\infty, 0]$, $y \in \mathbb{R}$, $z \in [1, \infty)$ that $\eta(x) = 0 \leq \eta(y) \leq 1 = \eta(z)$ and for every $r \in [1, \infty)$ let $\mathcal{R}_r : \mathbb{R} \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}$ that $\mathcal{R}_r(x) = \max\{x, 0\} = \eta(\frac{x - \mathcal{B} r^{-1}}{\mathcal{A}})$. Then

(i) it holds for all $r \in [1, \infty)$, $x \in (-\infty, \mathcal{A} r^{-1}]$ that $\mathcal{R}_r(x) = 0$,

(ii) it holds for all $r \in [1, \infty)$, $x \in [\mathcal{B} r^{-1}, \infty)$ that $\mathcal{R}_r(x) = x$,

(iii) it holds for all $r \in [1, \infty)$, $x \in \mathbb{R}$ that $0 \leq \mathcal{R}_r(x) \leq \max\{x, 0\}$,

(iv) it holds for all $r \in [1, \infty)$ that $\mathcal{R}_r \in C^1(\mathbb{R}, \mathbb{R})$, and

(v) it holds that $\sup_{r \in [1, \infty)} \sup_{x \in \mathbb{R}} |(\mathcal{R}_r)'(x)| < \infty$.

Proof of Lemma 2.4. Note that the assumption that for all $x \in (-\infty, 0]$ it holds that $\eta(x) = 0$ establishes item (i). Observe that the assumption that for all $x \in [1, \infty)$ it holds that $\eta(x) = 1$ proves item (ii). Note the assumption that for all $x \in \mathbb{R}$ it holds that $0 \leq \eta(x) \leq 1$ establishes item (iii). Observe that item (i), the fact that for all $r \in [1, \infty)$, $x \in (0, \infty)$ it holds that $\mathcal{R}_r(x) = x \eta(\frac{x - \mathcal{B} r^{-1}}{\mathcal{A}})$, and the assumption that $\eta \in C^1(\mathbb{R}, \mathbb{R})$ establish item (iv). Note that the chain rule implies for all $r \in [1, \infty)$, $x \in (0, \mathcal{B} r^{-1}]$ that

$$|(\mathcal{R}_r)'(x)| = \eta'(\frac{x - \mathcal{B} r^{-1}}{\mathcal{A}}) + x \frac{\mathcal{B} r^{-1}}{\mathcal{A}^2} \eta'(\frac{x - \mathcal{B} r^{-1}}{\mathcal{A}}) \leq 1 + \frac{\mathcal{B} r^{-1}}{\mathcal{A}^2} \left[ \sup_{y \in \mathbb{R}} |\eta'(y)| \right]. \quad (2.14)$$

Combining this with items (i) and (ii) proves that for all $r \in [1, \infty)$, $x \in \mathbb{R}$ it holds that

$$|(\mathcal{R}_r)'(x)| \leq \max\{0, 1 + \left[ \frac{\mathcal{B} r^{-1}}{\mathcal{A}^2} \right] \left[ \sup_{y \in \mathbb{R}} |\eta'(y)| \right] \} \leq 1 + \left[ \frac{\mathcal{B} r^{-1}}{\mathcal{A}^2} \right] \left[ \sup_{y \in \mathbb{R}} |\eta'(y)| \right] < \infty. \quad (2.15)$$

This establishes item (v). The proof of Lemma 2.4 is thus complete.

Proposition 2.5 (Approximations of realization functions of deep ReLU ANNs). Assume Setting 2.1, for every $r \in [1, \infty)$, $k \in \mathbb{N}$ let $\mathcal{M}_{r,k} : \mathbb{R}^{d_k} \to \mathbb{R}^{d_k}$ satisfy for all $x \in \mathbb{R}^{d_k}$ that $\mathcal{M}_{r,k}(x) = \mathcal{M}_r(x)$, and let $k \in \{1, \ldots, L\}$, $\theta \in \mathbb{R}^\theta$. Then

9
(i) it holds for all \( r \in [1, \infty] \) that
\[
\mathcal{N}_r^{k, \theta} = \begin{cases} 
A_k^0 & : k = 1 \\
A_k^0 \circ M_{r/(k-1), k-1} \circ A_{k-1}^0 \circ \cdots \circ M_{r/2, 2} \circ A_2^0 \circ M_{r, 1} \circ A_1^0 & : k > 1,
\end{cases}
\]

(ii) it holds that
\[
\sup_{r \in [1, \infty)} \sup_{x \in \mathbb{R}^d_0} \max_{i \in \{1, \ldots, \ell_k\}} \left\{ r^{1/k} |\mathcal{R}_{\ell_k} / (\mathcal{N}_r^{k, \theta}(x) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x)))\right\} \\
\leq \mathcal{R} \left[ \sum_{j=0}^{k-1} \left( \sup_{r \in [1, \infty)} \sup_{x \in \mathbb{R}} |(\mathcal{R}_{\ell_k})'(x)| \right)^j \left( \sum_{\theta=1}^{\ell_k} |\theta \theta|^j \right)^2 \right] < \infty,
\]

(iii) it holds that
\[
\sup_{r \in [1, \infty)} \sup_{x \in \mathbb{R}^d_0} \max_{i \in \{1, \ldots, \ell_k\}} \left\{ r^{1/(\max(k-1,1))} |\mathcal{N}_r^{k, \theta}(x) - \mathcal{N}_{\infty,i}^{k, \theta}(x)\right\} \\
\leq \mathcal{R} \left[ \sum_{j=0}^{k-1} \left( \sup_{r \in [1, \infty)} \sup_{x \in \mathbb{R}} |(\mathcal{R}_{\ell_k})'(x)| \right)^j \left( \sum_{\theta=1}^{\ell_k} |\theta \theta|^j \right)^2 \right] \mathcal{R}(1/(1))(k) < \infty,
\]

(iv) it holds for all \( i \in \{1, \ldots, \ell_k\} \), \( x \in \mathbb{R}^d_0 \) that
\[
\lim_{r \to \infty} \sup_{r \in [1, \infty)} \left\{ |\mathcal{N}_r^{k, \theta}(x) - \mathcal{N}_{\infty,i}^{k, \theta}(x)| + |\mathcal{R}_{\ell_k} / (\mathcal{N}_r^{k, \theta}(x) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x)))| \right\} = 0,
\]and

(v) it holds for all \( i \in \{1, \ldots, \ell_k\} \), \( x \in \mathbb{R}^d_0 \) that
\[
\lim_{r \to \infty} \sup_{r \in [1, \infty)} |(\mathcal{R}_{\ell_k})'(\mathcal{N}_r^{k, \theta}(x)) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x))| = 0.
\]

**Proof of Proposition 2.5.** Throughout this proof let \( L \in \mathbb{R} \) satisfy
\[
L = \sup_{r \in [1, \infty)} \sup_{x \in \mathbb{R}} |(\mathcal{R}_{\ell_k})'(x)|,
\]
let \( c \in \mathbb{R} \) satisfy \( c = L(\sum_{j=1}^{\ell_k} |\theta_j|) \), and for every \( K \in \{1, \ldots, L\} \), \( r \in [1, \infty) \) let \( e_{K,r} \in \mathbb{R} \) satisfy
\[
e_{K,r} = \sup_{x \in \mathbb{R}^d_0} \max_{i \in \{1, \ldots, \ell_K\}} |\mathcal{R}_{\ell_k} / (\mathcal{N}_r^{k, \theta}(x) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x)))|.
\]
Observe that (2.21) and the fundamental theorem of calculus ensure that for all \( r \in [1, \infty) \), \( x, y \in \mathbb{R} \) it holds that
\[
|\mathcal{R}_r(x) - \mathcal{R}_r(y)| \leq L|x - y|.
\]
Furthermore, note that (2.4) establishes item (i). Observe that Lemma 2.3 assures that for all \( r \in [1, \infty) \) it holds that
\[
\sup_{y \in \mathbb{R}} |\mathcal{R}_r(y) - \mathcal{R}_\infty(y)| \leq \mathcal{R}^{-1}.
\]
Item (ii) of Lemma 2.2 therefore ensures that for all \( r \in [1, \infty) \), \( i \in \{1, \ldots, \ell_1\} \), \( x \in \mathbb{R}^d_0 \) it holds that
\[
|\mathcal{R}_r(\mathcal{N}_{\infty,i}^{k, \theta}(x)) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x))| = |\mathcal{R}_r(\mathcal{N}_{\infty,i}^{k, \theta}(x)) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x))| \leq \mathcal{R}^{-1}.
\]
Combining this with (2.22) proves for all \( r \in [1, \infty) \) that
\[
e_{1,r} \leq \mathcal{R}^{-1}.
\]
Moreover, note that item (iii) of Lemma 2.2 and (2.23) ensure that for all \( r \in [1, \infty) \), \( K \in \mathbb{N} \cap [1, L] \), \( i \in \{1, \ldots, \ell_{K+1}\} \), \( x \in \mathbb{R}^d_0 \) it holds that
\[
|\mathcal{R}_{\ell_k} / (\mathcal{N}_{\infty,i}^{k+1, \theta}(x)) - \mathcal{R}_{\ell_k} / (\mathcal{N}_{\infty,i}^{k+1, \theta}(x))| \leq L|\mathcal{N}_r^{k+1, \theta}(x) - \mathcal{N}_{\infty,i}^{k+1, \theta}(x)|
\]
\[
= L\left[ \sum_{j=1}^{\ell_K} \left| w_{l_k}^{K+1, \theta} \left( \mathcal{R}_{\ell_k} / (\mathcal{N}_{\infty,i}^{k, \theta}(x)) \right) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x)) \right| \right]
\]
\[
\leq L\left[ \sum_{j=1}^{\ell_K} \left| w_{l_k}^{K+1, \theta} \left( \mathcal{R}_{\ell_k} / (\mathcal{N}_{\infty,i}^{k, \theta}(x)) \right) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x)) \right| \right]
\]
\[
\leq L\left[ \sum_{j=1}^{\ell_K} \left| \mathcal{N}_{\infty,i}^{k+1, \theta}(x) \right| \max_{i \in \{1, \ldots, \ell_K\}} \left| \mathcal{R}_{\ell_k} / (\mathcal{N}_{\infty,i}^{k, \theta}(x)) \right| - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x)) \right]
\]
\[
\leq L\left[ \sum_{j=1}^{\ell_K} \left| \mathcal{N}_{\infty,i}^{k+1, \theta}(x) \right| \max_{i \in \{1, \ldots, \ell_K\}} \left| \mathcal{R}_{\ell_k} / (\mathcal{N}_{\infty,i}^{k, \theta}(x)) \right| - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{k, \theta}(x)) \right]
\]
Combining this with (2.24) ensures that for all \( r \in [1, \infty) \), \( K \in \mathbb{N} \cap [1, L) \), \( i \in \{1, \ldots, \ell_{K+1}\} \), \( x \in \mathbb{R}^{e_0} \) it holds that
\[
\begin{align*}
|\mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K+1,\theta}(x)) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{K+1,\theta}(x))| \\
\leq |\mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K+1,\theta}(x))| - \mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K+1,\theta}(x))| \\
+ |\mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K+1,\theta}(x))| - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{K+1,\theta}(x))| \\
\leq L \left[ \sum_{j=1}^\theta \|	heta_j\| \right] e_{K,r} + dB^{-1/(K+1)} = e_{K,r} + dB^{-1/(K+1)}. \tag{2.28}
\end{align*}
\]

Hence, we obtain for all \( r \in [1, \infty) \), \( K \in \mathbb{N} \cap [1, L) \) that \( e_{K+1,r} \leq e_{K,r} + dB^{-1/(K+1)} \). This shows for all \( r \in [1, \infty) \), \( K \in \mathbb{N} \cap [1, L) \) that
\[
\begin{align*}
e_{K,r} &\leq c \left[ e_{K-1,r} \right] + dB^{-1/K} \leq c \left[ e_{K-2,r} + dB^{-1/(K-1)} \right] + dB^{-1/K} \\
&= c^2 e_{K-2,r} + \sum_{j=0}^{K-2} dB^{-1/(K-j)} \leq \ldots \leq c^K e_{K,(K-1),r} + \sum_{j=0}^{K-2} dB^{-1/(K-j)} \\
&= c^{K-1} e_{K,1,r} + \sum_{j=0}^{K-2} dB^{-1/(K-j)}. \tag{2.29}
\end{align*}
\]

Combining this with (2.26) demonstrates for all \( r \in [1, \infty) \), \( K \in \mathbb{N} \cap [1, L) \) that
\[
\begin{align*}
e_{K,r} &\leq c^{K-1}e_{1,1,r} + \sum_{j=0}^{K-2} dBc^{-1/(K-j)} \leq c^{K-1}e_{1,1,r} + \sum_{j=0}^{K-2} dBc^{-1/(K-j)} \\
&= \sum_{j=0}^{K-1} DBc^{-1/(K-j)} \leq \left[ \sum_{j=0}^{K-1} DB \right] \max(-1/(K-1),\ldots,\ldots, \ldots, -1) = \left[ \sum_{j=0}^{K-1} DB \right] dB^{-1/K}. \tag{2.30}
\end{align*}
\]

This establishes item (ii). Observe that (2.1), (2.4), and (2.22) ensure that for all \( r \in [1, \infty) \), \( K \in \mathbb{N} \cap [1, L), \ i \in \{1,2,\ldots,\ell_K\} \), \( x \in \mathbb{R}^{e_0} \) it holds that
\[
\begin{align*}
|\mathcal{N}_{r,i}^{K,\theta}(x) - \mathcal{N}_{\infty,i}^{K,\theta}(x)| &\leq \left[ \sum_{j=1}^{K-1} DB \right] \left[ \mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K,\theta}(x)) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{K,\theta}(x)) \right] \\
&\leq \left[ \sum_{j=1}^{K-1} DBc^{-1/(K-j)} \right] \left[ \mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K,\theta}(x)) - \mathcal{R}_\infty(\mathcal{N}_{\infty,i}^{K,\theta}(x)) \right] \\
&\leq \left( \sum_{j=1}^{K-1} DBc^{-1/(K-j)} \right) \max(-1/(K-1),\ldots,\ldots,\ldots, -1) \leq \left( \sum_{j=1}^{K-1} DBc^{-1/(K-j)} \right) dB^{-1/K}. \tag{2.31}
\end{align*}
\]

Therefore, we obtain for all \( K \in \mathbb{N} \cap [1, L) \) that
\[
\begin{align*}
\sup_{r \in [1, \infty)} \sup_{x \in \mathbb{R}^{e_0}} \max_{i \in \{1,2,\ldots,\ell_K\}} DB \left[ \mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K,\theta}(x)) - \mathcal{N}_{\infty,i}^{K,\theta}(x) \right] \\
\leq \left( \sum_{j=1}^{K-1} DB \right) \left[ \sup_{r \in [1, \infty)} DB \left[ \mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K,\theta}(x)) - \mathcal{N}_{\infty,i}^{K,\theta}(x) \right] \right] = \left( \sum_{j=1}^{K-1} DB \right) \left[ \sup_{r \in [1, \infty)} DB \left[ \mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K,\theta}(x)) - \mathcal{N}_{\infty,i}^{K,\theta}(x) \right] \right]. \tag{2.32}
\end{align*}
\]

Combining this with (2.4), (2.22), and item (ii) establishes item (iii). Note that items (ii) and (iii) prove item (iv). It thus remains to prove item (v). For this observe that item (iv) assures for all \( i \in \{1, \ldots, \ell_k\} \), \( x \in \mathbb{R}^{e_0} \) with \( \mathcal{N}_{\infty,i}^{K,\theta}(x) < 0 \) that there exists \( R \in [1, \infty) \) such that for all \( r \in [R, \infty) \) it holds that \( \mathcal{N}_{r,i}^{K,\theta}(x) < 0 \). Combining this with (2.2) demonstrates for all \( i \in \{1, \ldots, \ell_k\} \), \( x \in \mathbb{R}^{e_0} \) with \( \mathcal{N}_{\infty,i}^{K,\theta}(x) < 0 \) that there exists \( R \in [1, \infty) \) such that for all \( r \in [R, \infty) \) it holds that
\[
\begin{align*}
(\mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K,\theta}(x)) = 0 = \mathbb{1}_{(0,\infty)}(\mathcal{N}_{\infty,i}^{K,\theta}(x)) = \mathbb{1}_{X^{K,\theta}}(x). \tag{2.33}
\end{align*}
\]

In addition, note that item (iv) shows for all \( i \in \{1, \ldots, \ell_k\} \), \( x \in \mathbb{R}^{e_0} \) with \( \mathcal{N}_{r,i}^{K,\theta}(x) > 0 \) that there exists \( R \in [1, \infty) \) such that for all \( r \in [R, \infty) \) it holds that \( \mathcal{N}_{r,i}^{K,\theta}(x) > DB^{-1/K} \). Combining this with (2.2) demonstrates for all \( i \in \{1, \ldots, \ell_k\} \), \( x \in \mathbb{R}^{e_0} \) with \( \mathcal{N}_{r,i}^{K,\theta}(x) > 0 \) that there exists \( R \in [1, \infty) \) such that for all \( r \in [R, \infty) \) it holds that
\[
\begin{align*}
(\mathcal{R}_{r,1/(K+1)}(\mathcal{N}_{r,i}^{K,\theta}(x)) = 1 = \mathbb{1}_{(0,\infty)}(\mathcal{N}_{\infty,i}^{K,\theta}(x)) = \mathbb{1}_{X^{K,\theta}}(x). \tag{2.34}
\end{align*}
\]

Furthermore, observe that item (iii) assures that for all \( i \in \{1, 2, \ldots, \ell_k\} \), \( x \in \mathbb{R}^{e_0} \) it holds that
\[
\begin{align*}
\sup_{r \in [1, \infty)} \left( r^{(1+1/(k))}/(k-1/(1,\infty))(K_{r,i}^{K,\theta}(x) - \mathcal{N}_{r,i}^{K,\theta}(x)) \right) \\
= \sup_{r \in [1, \infty)} \left( r^{(1+1/(k))}/(max(k-1,1))(K_{r,i}^{K,\theta}(x) - \mathcal{N}_{r,i}^{K,\theta}(x)) \right) < \infty. \tag{2.35}
\end{align*}
\]
This and the fact that $-(1 + \mathbb{1}_{(1)}(k))/(k - \mathbb{1}_{(1)}(k)) < -1/k$ assure for all $i \in \{1, 2, \ldots, \ell_k\}$, $x \in \mathbb{R}^\ell$ with $\mathcal{N}^k_{\infty, i}(x) = 0$ that there exist $\mathcal{C}, R \in [1, \infty)$ such that for all $r \in [R, \infty)$ it holds that

$$|\mathcal{N}^k_{\infty, i}(x)| = |\mathcal{N}^k_{\infty, i}(x) - \mathcal{N}^k_{\infty, i}(x)| \leq \mathcal{C}\left[r^{-(1+1/(1))}(k-1_{(1)}(k))\right] < \mathcal{A}_r^{-1/k}. \quad (2.36)$$

Moreover, note that (2.2) assures that for all $r \in [1, \infty)$, $y \in \mathbb{R}$ with $|y| < \mathcal{A}_r^{-1/k}$ it holds that $(\mathcal{R}_r^{-1/k})'(y) = 0$. Combining this with (2.36) proves for all $i \in \{1, 2, \ldots, \ell_k\}$, $x \in \mathbb{R}^\ell$ with $\mathcal{N}^k_{\infty, i}(x) = 0$ that there exists $R \in [1, \infty)$ such that for all $r \in [R, \infty)$ it holds that

$$(\mathcal{R}_r^{-1/k})'(\mathcal{N}^k_{\infty, i}(x)) = 0 = \mathbb{1}_{(0, \infty)}(\mathcal{N}^k_{\infty, i}(x)) = \mathbb{1}_{\mathcal{A}_r^{1/k}}(x). \quad (2.37)$$

Combining this, (2.33), and (2.34) establishes item (v). The proof of Proposition 2.5 is thus complete. \(\Box\)

Lemma 2.6. Assume Setting 2.1, for every $k \in \mathbb{N}_0$ let $d_k \in \mathbb{N}_0$ satisfy $d_k = \sum_{n=1}^k \ell_n(\ell_{n-1} + 1)$, and let $x = (x_1, \ldots, x_{d_1}) \in [a, b]^\ell$, $r \in [1, \infty)$. Then

(i) it holds for all $k \in \{1, \ldots, L\}$, $i \in \{1, \ldots, \ell_k\}$ that $\mathbb{R}^\ell \ni \theta \mapsto \mathcal{N}^k_{\infty, i}(x) \in \mathbb{R}^\ell$ is differentiable,

(ii) it holds for all $K \in \{1, \ldots, L\}$, $k \in \{1, \ldots, K\}$, $i \in \{1, \ldots, \ell_k\}$, $j \in \{1, \ldots, \ell_{k-1}\}$, $h \in \{1, \ldots, \ell_K\}$, $\theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{R}^\ell$ that

$$\frac{\partial}{\partial \theta_{(i-1)\ell_k + j + d_k - 1}} (\mathcal{N}^k_{\infty, i}(x)) = \sum_{\nu_k, \nu_{k+1}, \ldots, \nu_{k-1} \in \mathbb{N}, \forall \nu_k \in \mathbb{N}\cap [k, K]: \nu_k \leq \ell_k} \left[ \mathbb{1}_{[i, k)}(\nu_k) \left( \mathbb{1}_{[i, k)}(\nu_k) \prod_{\nu_k = k+1}^K (w_{n, \nu_{k-1}}) \left( [\mathcal{R}_{r/(n-1)}'](\mathcal{N}^k_{\nu_{k-1}}(x)) \right) \right) \right],$$

and

(iii) it holds for all $k \in \{1, \ldots, L\}$, $K \in \{1, \ldots, K\}$, $i \in \{1, \ldots, \ell_k\}$, $h \in \{1, \ldots, \ell_K\}$, $\theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{R}^\ell$ that

$$\frac{\partial}{\partial \theta_{(i-1)\ell_k + j + d_k - 1}} (\mathcal{N}^k_{\infty, i}(x)) = \sum_{\nu_k, \nu_{k+1}, \ldots, \nu_{k-1} \in \mathbb{N}, \forall \nu_k \in \mathbb{N}\cap [k, K]: \nu_k \leq \ell_k} \left[ \mathbb{1}_{[i, k)}(\nu_k) \left( \mathbb{1}_{[i, k)}(\nu_k) \prod_{\nu_k = k+1}^K (w_{n, \nu_{k-1}}) \left( [\mathcal{R}_{r/(n-1)}'](\mathcal{N}^k_{\nu_{k-1}}(x)) \right) \right) \right].$$

Proof of Lemma 2.6. Observe that items (ii) and (iii) of Lemma 2.2 and the assumption that $\mathcal{R}_r \in C^1(\mathbb{R}, \mathbb{R})$ establish item (i). We now prove (2.38) and (2.39) by induction on $K \in \{1, \ldots, L\}$. Note that item (ii) of Lemma 2.2 implies that for all $i \in \{1, \ldots, \ell_1\}$, $j \in \{1, \ldots, \ell_0\}$, $h \in \{1, \ldots, \ell_1\}$, $\theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{R}^\ell$ it holds that

$$\frac{\partial}{\partial \theta_{(i-1)\ell + j}} (\mathcal{N}^1_{\infty, i}(x)) = x_j \mathbb{1}_{[i, h)}(i) \quad \text{and} \quad \frac{\partial}{\partial \theta_{(i-1)\ell + j}} (\mathcal{N}^1_{\infty, i}(x)) = \mathbb{1}_{[h, i)}(i). \quad (2.40)$$

This establishes (2.38) and (2.39) in the base case $K = 1$. For the induction step let $K \in \mathbb{N}\cap [1, L]$ satisfy for all $k \in \{1, \ldots, K\}$, $i \in \{1, \ldots, \ell_k\}$, $j \in \{1, \ldots, \ell_{k-1}\}$, $h \in \{1, \ldots, \ell_K\}$, $\theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{R}^\ell$ that

$$\frac{\partial}{\partial \theta_{(i-1)\ell_k + j + d_k - 1}} (\mathcal{N}^k_{\infty, i}(x)) = \sum_{\nu_k, \nu_{k+1}, \ldots, \nu_{k-1} \in \mathbb{N}, \forall \nu_k \in \mathbb{N}\cap [k, K]: \nu_k \leq \ell_k} \left[ \mathcal{R}_{r/(\ell_{k-1})} (\mathcal{N}^k_{\infty, i}(x)) \left( [\mathcal{R}_{r/(n-1)}'](\mathcal{N}^k_{\nu_{k-1}}(x)) \right) \right],$$

and

$$\frac{\partial}{\partial \theta_{(i-1)\ell_k + j + d_k - 1}} (\mathcal{N}^k_{\infty, i}(x)) = \sum_{\nu_k, \nu_{k+1}, \ldots, \nu_{k-1} \in \mathbb{N}, \forall \nu_k \in \mathbb{N}\cap [k, K]: \nu_k \leq \ell_k} \left[ \mathcal{R}_{r/(\ell_{k-1})} (\mathcal{N}^k_{\infty, i}(x)) \left( [\mathcal{R}_{r/(n-1)}'](\mathcal{N}^k_{\nu_{k-1}}(x)) \right) \right]. \quad (2.41)$$

This completes the proof of Lemma 2.6.
and
\[
\frac{\partial}{\partial \theta_{(i-1)\ell_k+j+\delta_k}} (N^R_{r,h}(x)) = \sum_{v_k, v_{k+1}, \ldots, v_K \in \mathbb{N}, \forall \omega \in \mathbb{N} \cap [k,K]: v_\omega \leq \ell_\omega} [\mathbb{I}(v_k)] [\mathbb{I}(v_K)] \left[ \prod_{n=0}^k \left( w_{n,\theta,\nu_{n-1}} \left[ (\mathcal{R}_{r_1/\nu_{(n-1)}}') (N^\nu_{r_1,\nu_{(n-1)}}(x)) \right] \right) \right].
\]

(2.42)

Observe that item (iii) of Lemma 2.2 and (2.41) demonstrate that for all \( k \in \{1, \ldots, K \}, \ i \in \{1, \ldots, \ell_k \}, \ j \in \{1, \ldots, \ell_{K+1} \}, \ h \in \{1, \ldots, \ell_K \}, \ \theta = (\theta_1, \ldots, \theta_3) \in \mathbb{R}^3 \) it holds that
\[
\frac{\partial}{\partial \theta_{(i-1)\ell_k+j+\delta_k}} (N^R_{r,h}(x)) = \frac{\partial}{\partial \theta_{(i-1)\ell_k+j+\delta_k}} \left( b_h^{K+1,\theta} + \sum_{\ell=1}^{\ell_k} w_{h,\ell}^{K+1,\theta} \mathcal{R}_{r,\ell}(N^R_{r,h}(x)) \right)
\]
\[
= \sum_{\ell=1}^{\ell_k} w_{h,\ell}^{K+1,\theta} \left[ (\mathcal{R}_{r,\ell}(N^R_{r,h}(x))') \left( \frac{\partial}{\partial \theta_{(i-1)\ell_k+j+\delta_k}} (N^R_{r,h}(x)) \right) \right]
\]
\[
= \sum_{\ell=1}^{\ell_k} w_{h,\ell}^{K+1,\theta} \left[ (\mathcal{R}_{r,\ell}(N^R_{r,h}(x))') \left( \frac{\partial}{\partial \theta_{(i-1)\ell_k+j+\delta_k}} (N^R_{r,h}(x)) \right) \right]
\]
\[
\cdot \sum_{v_k, v_{k+1}, \ldots, v_K \in \mathbb{N}, \forall \omega \in \mathbb{N} \cap [k,K]: v_\omega \leq \ell_\omega} \left[ \mathbb{I}(v_k) \right] [\mathbb{I}(v_K)] \left[ \prod_{n=0}^k \left( w_{n,\theta,\nu_{n-1}} \left[ (\mathcal{R}_{r_1/\nu_{(n-1)}}') (N^\nu_{r_1,\nu_{(n-1)}}(x)) \right] \right) \right].
\]

(2.43)

Furthermore, note that item (iii) of Lemma 2.2 ensures that for all \( i \in \{1, \ldots, \ell_{K+1} \}, \ j \in \{1, \ldots, \ell_K \}, \ h \in \{1, \ldots, \ell_K \}, \ \theta = (\theta_1, \ldots, \theta_3) \in \mathbb{R}^3 \) it holds that
\[
\frac{\partial}{\partial \theta_{(i-1)\ell_k+j+\delta_k}} (N^R_{r,h}(x)) = \frac{\partial}{\partial \theta_{(i-1)\ell_k+j+\delta_k}} \left( b_h^{K+1,\theta} + \sum_{\ell=1}^{\ell_k} w_{h,\ell}^{K+1,\theta} \mathcal{R}_{r,\ell}(N^R_{r,h}(x)) \right)
\]
\[
= \mathcal{R}_{r_{i+1}} (N^R_{r,h}(x)) \mathbb{I}(v_0) (i).
\]

(2.44)

Moreover, observe that item (iii) of Lemma 2.2 and (2.42) demonstrate that for all \( k \in \{1, \ldots, K \}, \)
\[ i \in \{1, \ldots, \ell_K\}, \ h \in \{1, \ldots, \ell_{K+1}\}, \ \theta = (\theta_1, \ldots, \theta_\delta) \in \mathbb{R}^\delta \text{ it holds that} \]
\[
\frac{\partial}{\partial \theta_{\ell_h, k+1+i+a_{h,k}}}(N_{r,h}^{K+1,\theta}(x)) = \frac{\partial}{\partial \theta_{\ell_h, k+1+i+a_{h,k}}}(b_h^{K+1,\theta} + \sum_{i=1}^{\ell_K} w_{h,i}^{K+1,\theta} \mathcal{R}_{r,i/K}(N_{r,i}^{K,\theta}(x)))
\]
\[
= \sum_{i=1}^{\ell_K} w_{h,i}^{K+1,\theta} \left( (\mathcal{R}_{r,i/K})'(N_{r,i}^{K,\theta}(x)) \right) \left( \frac{\partial}{\partial \theta_{\ell_h, k+1+i+a_{h,k}}}(N_{r,i}^{K,\theta}(x)) \right)
\]
\[
= \sum_{i=1}^{\ell_K} w_{h,i}^{K+1,\theta} \left( (\mathcal{R}_{r,i/K})'(N_{r,i}^{K,\theta}(x)) \right) \sum_{\forall w \in \mathcal{N}(r,K) : v_h \subseteq \mathcal{C}_w} \left[ \mathbb{I}\{i\}(v_h) \right] \left[ \mathbb{I}\{i\}(v_K) \right]
\]

(2.45)

In addition, note that item (iii) of Lemma 2.2 shows for all \( i \in \{1, \ldots, \ell_{K+1}\}, \ h \in \{1, \ldots, \ell_{K+1}\}, \ \theta = (\theta_1, \ldots, \theta_\delta) \in \mathbb{R}^\delta \)

\[
\frac{\partial}{\partial \theta_{\ell_h, k+1+i+a_{h,k}}}(N_{r,h}^{K+1,\theta}(x)) = \frac{\partial}{\partial \theta_{\ell_h, k+1+i+a_{h,k}}}(b_h^{K+1,\theta} + \sum_{i=1}^{\ell_K} w_{h,i}^{K+1,\theta} \mathcal{R}_{r,i/K}(N_{r,i}^{K,\theta}(x)))
\]

(2.46)

Induction thus establishes (2.38) and (2.39). The proof of Lemma 2.6 is thus complete. \qed

2.3 Explicit representations for the generalized gradients of the risk function

Lemma 2.7. Assume Setting 2.1 and let \( K \subseteq \mathbb{R}^d \) be compact. Then

(i) it holds for all \( x \in [0, \infty) \) that

\[
\sup_{r \in [1, \infty)} \sup_{y \in [-x,x]} \left( (\mathcal{R}_r(y)) + (\mathcal{R}_r(y))' \right) < \infty,
\]

(2.47)

(ii) it holds for all \( k \in \{1, \ldots, L\} \)

\[
\sup_{\theta \in K} \sup_{r \in [1, \infty)} \sup_{x \in [a,b]/0} |\mathcal{N}^{k,\theta}(x)| < \infty,
\]

(2.48)

(iii) it holds for all \( k \in \{1, \ldots, L\} \)

\[
\sup_{\theta \in K} \sup_{r,s \in [1, \infty)} \sup_{x \in [a,b]/0} \left( (\mathcal{R}_s(\mathcal{N}^{k,\theta}(x))) + (\mathcal{R}_s(\mathcal{N}^{k,\theta}(x)))' \right) < \infty,
\]

(2.49)

and

(iv) it holds that

\[
\sup_{\theta \in K} \sup_{r \in [1, \infty)} \sup_{x \in [a,b]/0} \sup_{y \in [a,b]/0} \frac{\partial}{\partial \theta}(\mathcal{N}^{L,\theta}(x)) \right) < \infty.
\]

(2.50)

Proof of Lemma 2.7. Observe that the fundamental theorem of calculus and the assumption that for all \( r \in [1, \infty) \) it holds that \( \mathcal{R}_r \in C^1(\mathbb{R}, \mathbb{R}) \) ensure that for all \( r \in [1, \infty), \ x \in \mathbb{R} \) we have that \( \mathcal{R}_r(x) = \mathcal{R}_r(0) + \int_0^r \mathcal{R}_r(y) dy \). Combining this with the assumption that \( \sup_{r \in [1, \infty)} \sup_{y \in [a,b]/0} |\mathcal{R}_r(y)| < \infty \) shows that for all \( x \in [0, \infty) \) it holds that \( \sup_{r \in [1, \infty)} \sup_{y \in [-x,x]} |\mathcal{R}_r(y)| < \infty \). This establishes item (i).

We now prove (2.48) by induction on \( k \in \{1, \ldots, L\} \). Note that the assumption that \( K \) is compact ensures that there exists \( C \in \mathbb{R} \) which satisfies that

\[
\sup_{\theta \in K} ||\theta|| < C.
\]

(2.51)
Observe that (2.51) and item (ii) of Lemma 2.2 demonstrate for all \( \theta \in K, r \in [1, \infty), i \in \{1, \ldots, \ell_1\}, x \in [a, b]^{\ell_0} \) that

\[
|N_{r,i}^{k,\theta}(x)| \leq |b_1^{1,\theta}| + \sum_{j=1}^{\ell_1} |w_{r,j}^{1,\theta}| |x_j| \leq a(\ell_1 + 1)c. 
\]

This shows (2.48) in the base case \( k = 1 \). For the induction step let \( k \in \mathbb{N} \cap [1, L) \) satisfy that

\[
\sup_{\theta \in K} \sup_{r \in [1, \infty)} \sup_{i \in \{1, \ldots, \ell_k\}} \sup_{x \in [a, b]^{\ell_0}} |N_{r,i}^{k,\theta}(x)| < \infty. \tag{2.53}
\]

Note that (2.51) and item (iii) of Lemma 2.2 imply for all \( \theta \in K, r \in [1, \infty), i \in \{1, \ldots, \ell_{k+1}\}, x \in [a, b]^{\ell_0} \) that

\[
|N_{r,i}^{k+1,\theta}(x)| \leq |b_1^{k+1,\theta}| + \sum_{j=1}^{\ell_k} |w_{r,j}^{k+1,\theta}| r_m(\mathcal{R}_r(N_{r,j}^{k,\theta}(x))) \leq c(1 + \ell_k) \max_{j \in \{1, \ldots, \ell_k\}} |\mathcal{R}_r(N_{r,j}^{k,\theta}(x))|. \tag{2.54}
\]

This, item (i), and (2.53) demonstrate that

\[
\sup_{\theta \in K} \sup_{r \in [1, \infty)} \sup_{i \in \{1, \ldots, \ell_{k+1}\}} \sup_{x \in [a, b]^{\ell_0}} |N_{r,i}^{k+1,\theta}(x)| < \infty. \tag{2.55}
\]

Induction thus establishes (2.48). This completes the proof of item (ii). Observe that items (i) and (ii) prove item (iii). Note that item (iii) and Lemma 2.6 establish item (iv). The proof of Lemma 2.7 is thus complete.

**Lemma 2.8** (Integrability properties of the target function). Assume Setting 2.1. Then

\[
\int_{[a, b]^{\ell_0}} \|f(x)\|^2 \mu(dx) + \int_{[a, b]^{\ell_0}} \|f(x)\| \mu(dx) < \infty. \tag{2.56}
\]

**Proof of Lemma 2.8.** Observe that (2.5), the fact that \( \mathcal{L}_\infty(0) \in \mathbb{R} \), and the fact that for all \( x \in \mathbb{R}^{\ell_0} \) it holds that \( N_{r,i}^{L,\theta}(x) = 0 \) assure that

\[
\int_{[a, b]^{\ell_0}} \|f(x)\|^2 \mu(dx) = \mathcal{L}_\infty(0) < \infty. \tag{2.57}
\]

Hölder’s inequality and the fact that \( \mathfrak{m} = \mu([a, b]^{\ell_0}) \in \mathbb{R} \) hence show that

\[
\int_{[a, b]^{\ell_0}} \|f(x)\| \mu(dx) \leq \left[ \int_{[a, b]^{\ell_0}} 1 \mu(dx) \right]^{1/2} \left[ \int_{[a, b]^{\ell_0}} \|f(x)\|^2 \mu(dx) \right]^{1/2} = \sqrt{\mathfrak{m}} \left[ \int_{[a, b]^{\ell_0}} \|f(x)\|^2 \mu(dx) \right]^{1/2} < \infty. \tag{2.58}
\]

Combining this with (2.57) establishes (2.56). The proof of Lemma 2.8 is thus complete.

**Theorem 2.9.** Assume Setting 2.1, for every \( k \in \mathbb{N}_0 \) let \( d_k \in \mathbb{N}_0 \) satisfy \( d_k = \sum_{n=1}^{k} \ell_n(n-1) + 1 \), and let \( \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^{d} \). Then

(i) it holds for all \( r \in [1, \infty) \) that \( \mathcal{L}_r \in C^1(\mathbb{R}^d, \mathbb{R}) \)

(ii) it holds for all \( r \in [1, \infty), k \in \{1, \ldots, L\}, i \in \{1, \ldots, \ell_k\}, j \in \{1, \ldots, \ell_{k-1}\} \) that

\[
\left( \frac{\partial \mathcal{L}_r}{\partial \theta} (i-1)\ell_{k-1} + j + d_{k-1} \right) (\theta) = \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall w \in \mathbb{N}(i-1), i \leq k} \int_{[a, b]^{\ell_0}} 2 \left[ \mathcal{R}_{r+i-1}^{(i-1)\mathcal{R}_{r,k-1}^{(k-1),\theta}}(N_{r,j}^{\mathcal{R}_{r,k-1}^{(k-1),\theta}}(x)) \right] \|1(1)\| \mu(dx), \tag{2.59}
\]

(iii) it holds for all \( r \in [1, \infty), k \in \{1, \ldots, L\}, i \in \{1, \ldots, \ell_k\} \) that

\[
\left( \frac{\partial \mathcal{L}_r}{\partial \theta} (i-1)\ell_{k-1} + i + d_{k-1} \right) (\theta) = \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall w \in \mathbb{N}(i-1), i \leq k} \int_{[a, b]^{\ell_0}} 2 \left[ \|1(1)\| \right] \mu(dx). \tag{2.60}
\]
(iv) it holds that \( \limsup_{r \to \infty} (|\mathcal{L}_r(\theta) - \mathcal{L}_\infty(\theta)| + \| (\nabla \mathcal{L}_r)(\theta) - \mathcal{G}(\theta) \|) = 0 \),

(v) it holds for all \( k \in \{1, \ldots, L\} \), \( i \in \{1, \ldots, \ell_k\} \), \( j \in \{1, \ldots, \ell_{k-1}\} \) that

\[
\mathcal{G}(i-1)_{k-1+i+j} + \mathcal{G}_{k-1-i+j} = \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}} \int_{[a, b]^{\ell_k}} 2 \left[ \mathcal{R}_r(\mathcal{N}_{\infty,j}^{\max(k-1,1), \theta}(x)) \mathbb{1}_{(1, L]}(k) + x_j \mathbb{1}_{(1)}(k) \right] \]

and

(vi) it holds for all \( k \in \{1, \ldots, L\} \), \( i \in \{1, \ldots, \ell_k\} \) that

\[
\mathcal{G}(i)_{k-1+i} = \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}} \int_{[a, b]^{\ell_k}} 2 \left[ \mathbb{1}_{(1)}(v_k) \right] \left[ \mathcal{N}_{\infty,j,v_L}^{L, \theta}(x) - f_{v_L}(x) \right] \left[ \prod_{n=k+1}^{L} \left( \mathcal{w}_{v_N,v_{n-1}}^n \mathbb{1}_{\mathcal{N}_{v_N-1,s}(x)}(x) \right) \right] \mu(dx),
\]

Proof of Theorem 2.9. Note that Lemma 2.6 and the chain rule show that for all \( r \in [1, \infty) \), \( i \in \{1, \ldots, \ell\} \), \( x \in [a, b]^{\ell_0} \) it holds that \( \mathbb{E} [\| \mathcal{N}_{\infty,j,v_L}^{L, \theta}(x) - f(x) \|^2] \in \mathbb{R} \) is differentiable at \( \theta \) and that

\[
\frac{\partial}{\partial \theta} \left[ \mathcal{N}_{\infty,j,v_L}^{L, \theta}(x) - f(x) \right] = 2 \sum_{j=1}^{\ell_0} \left( \mathcal{N}_{r,j,v_L}^{L, \theta}(x) - f_j(x) \right) \frac{\partial}{\partial \theta} \left[ \mathcal{N}_{r,j,v_L}^{L, \theta}(x) \right].
\]

Combining this with Lemma 2.7, Lemma 2.6, and the dominated convergence theorem establishes items (i), (ii), and (iii). Observe that the dominated convergence theorem and the fact that for all \( x \in [a, b]^{\ell_0} \) it holds that \( \limsup_{r \to \infty} |\mathcal{L}_r(\theta) - f(x) - (\mathcal{N}_{\infty,j,v_L}^{L, \theta}(x) - f(x))| = 0 \) ensure that

\[
\limsup_{r \to \infty} |\mathcal{L}_r(\theta) - \mathcal{L}_\infty(\theta)| = 0.
\]

Furthermore, note that item (iv) and (v) of Proposition 2.5 demonstrate that for all \( k \in \{1, \ldots, L\} \), \( i \in \{1, \ldots, \ell_k\} \), \( j \in \{1, \ldots, \ell_{k-1}\} \), \( x \in [a, b]^{\ell_0} \) it holds that

\[
\lim_{r \to \infty} \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, v_u \in \mathbb{N}[k, L]\colon v_u \leq \ell_u} \left[ \mathcal{R}_r(\mathcal{N}_{\infty,j}^{\max(k-1,1), \theta}(x)) \mathbb{1}_{(1, L]}(k) + x_j \mathbb{1}_{(1)}(k) \right] \right)
\]

and

\[
\lim_{r \to \infty} \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, v_u \in \mathbb{N}[k, L]\colon v_u \leq \ell_u} \left[ \mathbb{1}_{(1)}(v_k) \left[ \mathcal{N}_{\infty,j,v_L}^{L, \theta}(x) - f_{v_L}(x) \right] \left[ \prod_{n=k+1}^{L} \left( \mathcal{w}_{v_N,v_{n-1}}^n \mathbb{1}_{\mathcal{N}_{v_N-1,s}(x)}(x) \right) \right] \right] \right)
\]

Combining Lemma 2.7, Lemma 2.8, (2.64), (2.65), (2.66), and the dominated convergence theorem establishes items (iv), (v), and (vi). The proof of Theorem 2.9 is thus complete.
2.4 Local Lipschitz continuity properties of the risk function

**Lemma 2.10.** Assume Setting 2.1 and let $K \subseteq \mathbb{R}^o$ be compact. Then there exists $\mathcal{L} \in \mathbb{R}$ which satisfies for all $\theta, \vartheta \in K$

\[
|\mathcal{L}_\infty(\theta) - \mathcal{L}_\infty(\vartheta)| + \sup_{x \in [a, b]^{\ell_o}} \|N_{\infty}^{L, \theta}(x) - N_{\infty}^{L, \vartheta}(x)\| \leq \mathcal{L}\|\theta - \vartheta\|. \tag{2.67}
\]

**Proof of Lemma 2.10.** Observe that, e.g., Beck et al. [8, Theorem 2.36] (applied with $d \cap d$, $l = (l_0, l_1, \ldots, l_L)$) in the notation of [8, Theorem 2.36]) implies that for all $\theta, \vartheta \in K$ it holds that

\[
\sup_{x \in [a, b]^{\ell_o}} \|N_{\infty}^{L, \theta}(x) - N_{\infty}^{L, \vartheta}(x)\| \leq L \left(\prod_{p=0}^{l-1}(\ell_p + 1)\right) \sqrt{\mathcal{L}} \left(\max\{1, \|\vartheta\|, \|\theta\|\}\right)\|\theta - \vartheta\|. \tag{2.68}
\]

Furthermore, note that the fact that $K$ is compact ensures that there exists $\kappa \in [1, \infty)$ which satisfies for all $\theta \in K$

\[
\|\theta\| \leq \kappa < \infty. \tag{2.69}
\]

Observe that (2.68) and (2.69) demonstrate that there exists $\mathcal{L} \in \mathbb{R}$ which satisfies for all $\theta, \vartheta \in K$

\[
\sup_{x \in [a, b]^{\ell_o}} \|N_{\infty}^{L, \theta}(x) - N_{\infty}^{L, \vartheta}(x)\| \leq \mathcal{L}\|\theta - \vartheta\|. \tag{2.70}
\]

Combining this with the Cauchy-Schwarz inequality shows for all $\theta, \vartheta \in K$

\[
|\mathcal{L}_\infty(\theta) - \mathcal{L}_\infty(\vartheta)|
= \left[\int_{[a, b]^{\ell_o}} \|N_{\infty}^{L, \theta}(x) - f(x)\|^2 \mu(dx)\right] - \left[\int_{[a, b]^{\ell_o}} \|N_{\infty}^{L, \vartheta}(x) - f(x)\|^2 \mu(dx)\right]
\leq \int_{[a, b]^{\ell_o}} \left[\|N_{\infty}^{L, \theta}(x) - f(x)\|^2 - \|N_{\infty}^{L, \vartheta}(x) - f(x)\|^2\right] \mu(dx)
\leq \int_{[a, b]^{\ell_o}} \left[\|N_{\infty}^{L, \vartheta}(x) - N_{\infty}^{L, \theta}(x)\|^2 + \|N_{\infty}^{L, \theta}(x) - N_{\infty}^{L, \vartheta}(x)\|^2 + 2f(x)\| \mu(dx)
\leq \mathcal{L}\|\theta - \vartheta\| \left[\int_{[a, b]^{\ell_o}} \|N_{\infty}^{L, \vartheta}(x) + N_{\infty}^{L, \theta}(x) - 2f(x)\| \mu(dx)\right]. \tag{2.71}
\]

This, (2.69), (2.70), and the fact that for all $x \in [a, b]^{\ell_o}$ it holds that $N_{\infty}^{L, \theta}(x) = 0$ imply that for all $\theta, \vartheta \in K$ we have that

\[
|\mathcal{L}_\infty(\theta) - \mathcal{L}_\infty(\vartheta)| \leq \mathcal{L}\|\theta - \vartheta\| \left[\int_{[a, b]^{\ell_o}} \|N_{\infty}^{L, \theta}(x) + N_{\infty}^{L, \vartheta}(x)\| + 2f(x)\| \mu(dx)\right]
\leq \mathcal{L}\|\theta - \vartheta\| \left[\left(\mathcal{L}\left(\sup_{y \in [a, b]^{\ell_o}} \|N_{\infty}^{L, \theta}(y)\| + \|N_{\infty}^{L, \vartheta}(y)\|\right)\right) + 2 \int_{[a, b]^{\ell_o}} \|f(x)\| \mu(dx)\right]
= \mathcal{L}\|\theta - \vartheta\| \left[\left[\left(\mathcal{L}\left(\sup_{y \in [a, b]^{\ell_o}} \|N_{\infty}^{L, \theta}(y) - N_{\infty}^{L, \vartheta}(y)\|\right)\right) + 2 \int_{[a, b]^{\ell_o}} \|f(x)\| \mu(dx)\right]
\leq \mathcal{L}\|\theta - \vartheta\| \left[\mathcal{L}\left(\sup_{y \in [a, b]^{\ell_o}} \|N_{\infty}^{L, \theta}(y) - N_{\infty}^{L, \vartheta}(y)\|\right) + 2 \int_{[a, b]^{\ell_o}} \|f(x)\| \mu(dx)\right]
\leq 2\mathcal{L} \left[\kappa \mathcal{L} + \int_{[a, b]^{\ell_o}} \|f(x)\| \mu(dx)\right]\|\theta - \vartheta\|. \tag{2.72}
\]

Combining this, Lemma 2.8, and (2.70) establishes (2.67). The proof of Lemma 2.10 is thus complete. \qed
2.5 Upper estimates for the norm of the generalized gradients of the risk function

Theorem 2.11. Assume Setting 2.1, let $\theta \in \mathbb{R}^p$, for every $k \in \mathbb{N}$, $i \in \{1, \ldots, \ell_k\}$ let $Q_{k,i} \in \mathbb{R}$ satisfy $Q_{k,i} = |b_i^1\theta|^2 + \sum_{j=1}^{\ell_k} |w_{i,j}^k\theta|^2$, and let $Q_k \in \mathbb{R}$, $k \in \mathbb{N}_0$, satisfy for all $k \in \mathbb{N}$ that $Q_0 = 1$ and $Q_k = 1 + \sum_{i=1}^{\ell_k} Q_{k,i}$. Then

(i) it holds for all $k \in \{1, \ldots, L\}$, $i \in \{1, \ldots, \ell_k\}$, $x \in [a, b]^\ell$ that

$$|\mathcal{R}_\infty(N_{\infty,i}^k(x))|^2 \leq |N_{\infty,i}^k(x)|^2 \leq a^2Q_{k,i}\left[\prod_{p=0}^{k-1}((\ell_p + 1)Q_p)\right],$$

(2.73)

(ii) it holds for all $K \in \{1, \ldots, L\}$ that $\prod_{p=0}^{K}Q_p \leq (||\theta||^2 + 1)^K$,

(iii) it holds for all $K \in \{1, \ldots, L\}$, $k \in \{1, \ldots, K\}$, $i \in \{1, \ldots, \ell_k\}$ that

$$\sum_{v_k, v_{k+1}, \ldots, v_{\ell_k} \in \mathbb{N}} \sum_{v_k \in \mathbb{N}\setminus[k,K]}: v_k \leq \ell_k$$

and

(iv) it holds that

$$\|\mathcal{G}(\theta)\|^2 \leq 4L\max[\prod_{k=0}^{L}((\ell_k + 1))][||\theta||^2 + 1]^{L-1}L_\infty(\theta).$$

Proof of Theorem 2.11. We first prove item (i) by induction on $k \in \{1, \ldots, L\}$. Note that (2.4), the fact that for all $x \in \mathbb{R}$ it holds that $|\mathcal{R}_\infty(x)| = |\max(x, 0)| \leq |x|$, and the Cauchy-Schwarz inequality ensure that for all $i \in \{1, \ldots, \ell_1\}$, $x = (x_1, \ldots, x_{\ell_1}) \in [a, b]^\ell$ it holds that

$$|\mathcal{R}_\infty(N_{\infty,i}^1(x))|^2 \leq |N_{\infty,i}^1(x)|^2 \leq a^2(|b_i^1\theta|^2 + \sum_{j=1}^{\ell_1} |w_{i,j}^1\theta|^2)^2$$

$$\leq (\ell_1 + 1)(|b_i^1\theta|^2 + \sum_{j=1}^{\ell_1} |w_{i,j}^1\theta|^2)^2 \leq a^2(\ell_1 + 1)(|b_i^1\theta|^2 + \sum_{j=1}^{\ell_1} |w_{i,j}^1\theta|^2) = a^2(\ell_1 + 1)Q_{1,i}.$$ 

(2.76)

This establishes item (i) in the base case $k = 1$. For the induction step let $k \in \mathbb{N} \cap [1, L]$ satisfy for all $i \in \{1, \ldots, \ell_k\}$, $x \in [a, b]^\ell$ that

$$|\mathcal{R}_\infty(N_{\infty,i}^k(x))|^2 \leq |N_{\infty,i}^k(x)|^2 \leq a^2Q_{k,i}\left[\prod_{p=0}^{k-1}((\ell_p + 1)Q_p)\right].$$

(2.77)

Observe that (2.77), the fact that for all $x \in \mathbb{R}$ it holds that $|\mathcal{R}_\infty(x)| = |\max(x, 0)| \leq |x|$, the fact that for all $p \in \mathbb{N}_0$ it holds that $Q_p \geq 1$, and the Cauchy-Schwarz inequality demonstrate that for all $i \in \{1, \ldots, \ell_{k+1}\}$, $x = (x_1, \ldots, x_{\ell_{k+1}}) \in [a, b]^\ell$ it holds that

$$|\mathcal{R}_\infty(N_{\infty,i}^{k+1}(x))|^2 \leq |N_{\infty,i}^{k+1}(x)|^2 \leq a^2Q_{k+1,i}\left[\prod_{p=0}^{k}((\ell_p + 1)Q_p)\right].$$

(2.78)

(2.79)
establishes item (ii). We now prove item (iii) by induction on $K \in \{1, \ldots, L\}$. Observe that the fact that for all $i \in \{1, \ldots, \ell_k\}$ it holds that $\sum_{v_k}^K \|w_{v_k, v_{k-1}}\|^2 = 1$ establishes (2.74) in the base case $K = 1$. For the induction step let $K \in \mathbb{N} \cap [1, L)$ satisfy for all $k \in \{1, \ldots, K\}$, $i \in \{1, \ldots, \ell_k\}$ that

$$\sum_{v_k, v_{k+1}, \ldots, v_{K+1} \in \mathbb{N}, \ \forall w \in \mathbb{N}[K, K+1]: v_w \leq \ell_w} \left[ \mathbb{I}(i)(v_k) \right] \left[ \prod_{v_{n=K+1}}^{K+1} |w_{v_{n}, v_{n-1}}\|^2 \right] \leq \|\theta\|^2(K-k). \quad (2.80)$$

Note that (2.80) implies that for all $k \in \{1, \ldots, K + 1\}$, $i \in \{1, \ldots, \ell_k\}$ it holds that

$$\sum_{v_k, v_{k+1}, \ldots, v_{K+1} \in \mathbb{N}, \ \forall w \in \mathbb{N}[K, K+1]: v_w \leq \ell_w} \left[ \mathbb{I}(i)(v_k) \right] \left[ \prod_{v_{n=K+1}}^{K+1} |w_{v_{n}, v_{n-1}}\|^2 \right] = \left[ \sum_{v_k, v_{k+1}, \ldots, v_{K+1} \in \mathbb{N}, \ \forall w \in \mathbb{N}[K, K+1]: v_w \leq \ell_w} \left[ \mathbb{I}(i)(v_k) \right] \left[ \prod_{v_{n=K+1}}^{K+1} |w_{v_{n}, v_{n-1}}\|^2 \right] \right] \left[ \prod_{v_{K+1} = 1}^{\ell_{K+1}} |w_{v_{K+1}, v_{K}}\|^2 \right] \quad (2.81)$$

$$\leq \left[ \sum_{v_k, v_{k+1}, \ldots, v_{K+1} \in \mathbb{N}, \ \forall w \in \mathbb{N}[K, K+1]: v_w \leq \ell_w} \left[ \mathbb{I}(i)(v_k) \right] \left[ \prod_{v_{n=K+1}}^{K+1} |w_{v_{n}, v_{n-1}}\|^2 \right] \right] \|\theta\|^2 \leq \|\theta\|^2(K+1-k).$$

Induction thus establishes item (iii). It thus remains to prove item (iv). For this we assume without loss of generality that $m > 0$, for every $k \in \mathbb{N}_0$ let $d_k \in \mathbb{N}_0$ satisfy $d_k = \sum_{n=1}^\infty \ell_n (\ell_n + 1)$, and for every $k \in \{1, \ldots, L\}$, $i \in \{1, \ldots, \ell_k\}$ let $W_{k,i} \in \mathbb{R}$ satisfy

$$W_{k,i} = \sum_{v_k, v_{k+1}, \ldots, v_{L} \in \mathbb{N}, \ \forall w \in \mathbb{N}[K, L]: v_w \leq \ell_w} \left[ \mathbb{I}(i)(v_k) \right] \left[ \prod_{v_{n=K+1}}^{L} |w_{v_{n}, v_{n-1}}\|^2 \right]. \quad (2.82)$$

Observe that item (vi) of Theorem 2.9 proves that for all $k \in \{1, \ldots, L\}$, $i \in \{1, \ldots, \ell_k\}$ it holds that

$$|G_{\ell_k, \ell_{k-1} + 1 + d_{k-1}}(\theta)|^2$$

$$= \left[ \sum_{v_k, v_{k+1}, \ldots, v_{L} \in \mathbb{N}, \ \forall w \in \mathbb{N}[K, L]: v_w \leq \ell_w} \int_{[a,b]^\ell} 2 \left[ \mathbb{I}(i)(v_k) \right] \left[ \mathcal{N}_{\infty, v_k}(x) - f_{v_k}(x) \right]$$

$$\cdot \left[ \prod_{v_{n=K+1}}^{L} \mathbb{I}_{\mathbb{N}_0}^{\ell_n} \mathbb{I}_{\mathbb{N}_{v_{n-1}+1}}(x) \right] \mu(dx) \right]^2 \quad (2.83)$$

$$\leq 4 \left[ \int_{[a,b]^\ell} \|\mathcal{N}_{\infty, v_k}(x) - f_{v_k}(x)\| \sum_{v_k, v_{k+1}, \ldots, v_{L} \in \mathbb{N}, \ \forall w \in \mathbb{N}[K, L]: v_w \leq \ell_w} \left[ \mathbb{I}(i)(v_k) \right] \left[ \prod_{v_{n=K+1}}^{L} |w_{v_{n}, v_{n-1}}\|^2 \right] \mu(dx) \right]^2$$

$$= 4 \left[ W_{k,i} \int_{[a,b]^\ell} \|\mathcal{N}_{\infty, v_k}(x) - f_{v_k}(x)\| \mu(dx) \right]^2.$$
Furthermore, note that item (v) of Theorem 2.9 and Jensen’s inequality imply for all \( k \in \{ 1, \ldots, L \}, i \in \{ 1, \ldots, \ell_k \}, j \in \{ 1, \ldots, \ell_{k-1} \} \) that

\[
|G_{(i-1)\ell_{k-1}+j+d_{k-1}}(\theta)|^2 \\
= \left[ \sum_{v_k, v_{k+1}, \ldots, v_{\ell_k} \in \mathbb{N}} \int_{[a,b]^{\ell_k}} 2 \mathcal{A}_\infty(\mathcal{N}_\infty^{\max(k-1,1)}(x) \mathbb{I}_{(1,L)}(k) + x_j \mathbb{I}_{(1)}(k)) \right]^2 \\
= 4m^2 \left[ W_{k,i} \left[ \frac{1}{m} \int_{[a,b]^{\ell_k}} \left[ \mathcal{A}_\infty(\mathcal{N}_\infty^{\max(k-1,1)}(x) \mathbb{I}_{(1,L)}(k) + x_j \mathbb{I}_{(1)}(k) ) \| \mathcal{N}_\infty^{\ell_j} \mathbb{I}_{(1)}(k) \| \| \mathcal{N}_\infty^{L,\theta}(x) - f(x) \| \mu(dx) \right]^2 \right] \\
\leq 4m^2 \left[ W_{k,i}^2 \left[ \frac{1}{m} \int_{[a,b]^{\ell_k}} \left[ \mathcal{A}_\infty(\mathcal{N}_\infty^{\max(k-1,1)}(x) \mathbb{I}_{(1,L)}(k) + x_j \mathbb{I}_{(1)}(k) ) \| \mathcal{N}_\infty^{\ell_j} \mathbb{I}_{(1)}(k) \| \| \mathcal{N}_\infty^{L,\theta}(x) - f(x) \| \mu(dx) \right]^2 \right] \\
\right].
\] (2.85)

This ensures for all \( i \in \{ 1, \ldots, \ell_1 \}, j \in \{ 1, \ldots, \ell_0 \} \) that

\[
|G_{(i-1)\ell_0+j}(\theta)|^2 \leq 4ma^2(W_{1,i})^2 L_\infty(\theta).
\] (2.86)

Moreover, observe that (2.85) and item (i) demonstrate for all \( k \in \mathbb{N} \cap \{ 1, L \}, i \in \{ 1, \ldots, \ell_k \}, j \in \{ 1, \ldots, \ell_{k-1} \} \) that

\[
|G_{(i-1)\ell_{k-1}+j+d_{k-1}}(\theta)|^2 \\
\leq 4m(W_{k,i})^2 \left[ \mathcal{A}_\infty^{Q_{k-1,j}} \left[ \frac{1}{m} \int_{[a,b]^{\ell_{k-1}}} \mathcal{A}_\infty^{Q_{k-1,j}}(x) \| \mathcal{N}_\infty^{\ell_j} \mathbb{I}_{(1)}(k) \| \| \mathcal{N}_\infty^{L,\theta}(x) - f(x) \| \mu(dx) \right]^2 \right] \\
\leq 4m(W_{k,i})^2 a^2 Q_{k-1,j} \left[ \frac{1}{m} \int_{[a,b]^{\ell_{k-1}}} \mathcal{A}_\infty^{Q_{k-1,j}}(x) \| \mathcal{N}_\infty^{\ell_j} \mathbb{I}_{(1)}(k) \| \| \mathcal{N}_\infty^{L,\theta}(x) - f(x) \| \mu(dx) \right] L_\infty(\theta).
\] (2.87)

In addition, note that the Cauchy-Schwarz inequality and item (iii) imply for all \( k \in \{ 1, \ldots, L \}, i \in \{ 1, \ldots, \ell_k \} \) that

\[
(W_{k,i})^2 \leq \prod_{p=k+1}^L \ell_p \sum_{v_{k}, v_{k+1}, \ldots, v_{\ell_k} \in \mathbb{N}} \left[ \prod_{n=k+1}^L |w_{v_n, v_{n-1}}^n| \right]^2 \left[ \prod_{n=k+1}^L |w_{v_n, v_{n-1}}^n| \right]^2 \leq \prod_{p=k+1}^L \ell_p \| \theta \|^2(L-k).
\] (2.88)

Combining this with (2.84) and (2.86) assures that

\[
\sum_{i=1}^{\ell_k} |G_{(i-1)\ell_0+i}(\theta)|^2 + \sum_{j=1}^{\ell_{k-1}} |G_{(i-1)\ell_0+j}(\theta)|^2 \leq \sum_{i=1}^{\ell_k} \left[ 4m(W_{1,i})^2 + \sum_{j=1}^{\ell_{k-1}} 4ma^2(W_{1,i})^2 \right] L_\infty(\theta) \\
\leq 4m \left[ \prod_{p=0}^L \ell_p \right] \| \theta \|^2(L-1)(\ell_1 + a^2 \ell_1 \ell_0) L_\infty(\theta) \leq 4ma^2 \left[ \prod_{p=0}^L (\ell_p + 1) \right] \| \theta \|^2(L-1) L_\infty(\theta).
\] (2.89)
Furthermore, observe that (2.84), (2.87), and (2.88) prove that
\[
\sum_{k=2}^{L} \left[ |G_{i_k} + \ldots + a_{i_1}|^2 + \ldots + |G_{i_1}|^2 \right]^2 \leq 4\alpha^2 \left[ \sum_{k=2}^{L} \left( \left| \sum_{j=1}^{|G_{i_k}|} (W_{k,i})^2 \right|^2 + \ldots + \left| \sum_{j=1}^{|G_{i_1}|} (W_{k,i})^2 \right|^2 \right) \right] \mathcal{L}_\infty(\theta)
\]
This, item (ii), and the fact that for all \( k \in \mathbb{N} \) it holds that \( 1 \leq \prod_{p=0}^{k-2} Q_p \) show that
\[
\sum_{k=2}^{L} \sum_{j=1}^{|G_{i_k}|} |G_{i_k} + \ldots + a_{i_1}|^2 \leq 4\alpha^2 \left[ \sum_{k=2}^{L} \left( \left| \sum_{j=1}^{|G_{i_k}|} (W_{k,i})^2 \right|^2 + \ldots + \left| \sum_{j=1}^{|G_{i_1}|} (W_{k,i})^2 \right|^2 \right) \right] \mathcal{L}_\infty(\theta)
\]
Combining this with (2.89) ensures that
\[
\|\mathcal{G}(\theta)\|^2 = \sum_{k=1}^{\ell_\theta} \left[ |G_{i_k} + \ldots + a_{i_1}|^2 + \ldots + |G_{i_1}|^2 \right]^2 \leq 4\alpha^2 \left[ \sum_{k=2}^{L} \left( \left| \sum_{j=1}^{|G_{i_k}|} (W_{k,i})^2 \right|^2 + \ldots + \left| \sum_{j=1}^{|G_{i_1}|} (W_{k,i})^2 \right|^2 \right) \right] \mathcal{L}_\infty(\theta)
\]
This establishes item (iv). The proof of Theorem 2.11 is thus complete.  

**Corollary 2.12.** Assume Setting 2.1 and let \( K \subseteq \mathbb{R}^p \) be compact. Then
\[
\sup_{\theta \in K} \|\mathcal{G}(\theta)\| \leq \infty.
\]

**Proof of Corollary 2.12.** Note that Lemma 2.10 and the assumption that \( K \) is compact demonstrate that \( \sup_{\theta \in K} \mathcal{L}_\infty(\theta) < \infty \). Item (iv) of Theorem 2.11 therefore establishes (2.93). The proof of Corollary 2.12 is thus complete.  

### 2.6 Convexity properties of the risk function

**Proposition 2.13** (Non-existence of non-global local minima for convex objective functions). Let \( \vartheta \in \mathbb{N} \) and let \( \mathcal{L} : \mathbb{R}^p \to \mathbb{R} \) satisfy for all \( \theta, \vartheta \in \mathbb{R}^p \), \( \lambda \in [0, 1] \) that \( \mathcal{L}(\lambda \theta + (1 - \lambda)\vartheta) \leq \lambda \mathcal{L}(\theta) + (1 - \lambda)\mathcal{L}(\vartheta) \). Then
\[
\{ \theta \in \mathbb{R}^p : (\exists \varepsilon \in (0, \infty) : \inf_{\vartheta \in \mathbb{R}^p} \mathcal{L}(\vartheta) = \mathcal{L}(\theta) > \inf_{\vartheta \in \mathbb{R}^p} \mathcal{L}(\vartheta) ) \} = \emptyset.
\]
Proof of Proposition 2.13. We prove (2.94) by contradiction. For this let $\theta, \vartheta \in \mathbb{R}^p$, $\varepsilon \in (0, \infty)$ satisfy
\[
\inf_{\varphi \in \{ \varphi \in \mathbb{R}^p : \| \varphi - \vartheta \| \leq \varepsilon \}} \mathcal{L}(\varphi) = \mathcal{L}(\theta) > \mathcal{L}(\vartheta). \tag{2.95}
\]
Observe that (2.95) and the fact that for all $\lambda \in [0, 1]$ it holds that $\mathcal{L}(\lambda \theta + (1-\lambda)\vartheta) \leq \lambda \mathcal{L}(\theta) + (1-\lambda)\mathcal{L}(\vartheta)$ imply that for all $\lambda \in (0, 1)$ it holds that
\[
\mathcal{L}(\lambda \theta + (1-\lambda)\vartheta) \leq \lambda \mathcal{L}(\theta) + (1-\lambda)\mathcal{L}(\vartheta) < \mathcal{L}(\lambda \theta) + (1-\lambda)\mathcal{L}(\vartheta) = \mathcal{L}(\theta). \tag{2.96}
\]
Combining this and (2.95) with the fact that $\inf_{\varphi \in \{ \varphi \in \mathbb{R}^p : \| \varphi - \theta \| \leq \varepsilon \}} \mathcal{L}(\varphi) = \inf_{\varphi \in \{ \varphi \in \mathbb{R}^p : \| \varphi - \vartheta \| \leq \varepsilon \}} \mathcal{L}(\theta + \varphi)$ shows that for all $\lambda \in (0, 1)$ it holds that
\[
\mathcal{L}(\theta + \lambda(\theta - \vartheta)) < \inf_{\varphi \in \{ \varphi \in \mathbb{R}^p : \| \varphi - \theta \| \leq \varepsilon \}} \mathcal{L}(\theta + \varphi). \tag{2.97}
\]
This contradiction establishes (2.94). The proof of Proposition 2.13 is thus complete. □

Lemma 2.14 (Characterization of affine linearity). Let $V$ and $W$ be $\mathbb{R}$-vector spaces and let $\varphi : V \to W$ be a function. Then the following three statements are equivalent:

(i) It holds for all $\lambda \in [0, 1]$, $v, w \in V$ that $\varphi(\lambda v + (1-\lambda)w) = \lambda \varphi(v) + (1-\lambda)\varphi(w)$.

(ii) It holds for all $\lambda \in (0, 1)$, $v, w \in V$ that $\varphi(\lambda v + (1-\lambda)w) = \lambda \varphi(v) + (1-\lambda)\varphi(w)$.

(iii) It holds for all $\lambda \in \mathbb{R}$, $v, w \in V$ that $\varphi(\lambda v + w) - \varphi(0) = \lambda(\varphi(v) - \varphi(0)) + (\varphi(w) - \varphi(0))$.

Proof of Lemma 2.14. Note that the fact that for all $\lambda \in [0, 1]$, $v, w \in V$ it holds that $\varphi(\lambda v + (1-\lambda)w) = \lambda \varphi(v) + (1-\lambda)\varphi(w)$ establishes that (item (i) $\iff$ item (iii)). We now prove that (item (iii) $\implies$ item (i)). Observe that item (iii) ensures that for all $\lambda \in [0, 1]$, $v, w \in V$ it holds that
\[
\varphi(\lambda v + (1-\lambda)w) = \lambda(\varphi(v) - \varphi(0)) + (1-\lambda)(\varphi(w) - \varphi(0)) + \varphi(0) = \lambda \varphi(v) + (1-\lambda)\varphi(w). \tag{2.98}
\]
This establishes that (item (iii) $\implies$ item (i)). We now prove that (item (i) $\implies$ item (iii)). Note that item (i) implies that for all $\lambda \in [0, 1]$, $v, w \in V$ it holds that
\[
\varphi(\lambda v) = \varphi(\lambda v + \lambda v) = \lambda(\varphi(v)) + (1-\lambda)\varphi(v) = \lambda(\varphi(v) - \varphi(0)) + \varphi(0). \tag{2.99}
\]
Furthermore, observe that item (i) shows that for all $\lambda \in (1, \infty)$, $v \in V$ it holds that
\[
\lambda \varphi(v) = \lambda \varphi(\frac{1}{\lambda} \lambda v + (1-\frac{1}{\lambda})0) = \lambda \frac{1}{\lambda} \varphi(\lambda v) + \lambda(1-\frac{1}{\lambda})\varphi(0) = \varphi(\lambda v) - (1-\lambda)\varphi(0). \tag{2.100}
\]
This and (2.99) prove that for all $\lambda \in [0, \infty)$, $v \in V$ it holds that
\[
\varphi(\lambda v) = \lambda(\varphi(v) - \varphi(0)) + \varphi(0). \tag{2.101}
\]
Combining this with item (i) ensures that for all $v \in V$ it holds that
\[
0 = \varphi(0) - \varphi(0) = \varphi(\frac{1}{2}2v - \frac{1}{2}2v) - \varphi(0) = \frac{1}{2}\varphi(2v) + \frac{1}{2}\varphi(-2v) - \varphi(0) = \frac{1}{2}\varphi(v - v) + \frac{1}{2}\varphi(-v - v) + \frac{1}{2}\varphi(0) - \varphi(0) = \varphi(v) + \varphi(-v) - 2\varphi(0). \tag{2.102}
\]
This and (2.101) show that for all $\lambda \in (0, \infty)$, $v \in V$ it holds that
\[
\varphi(\lambda v) - \varphi(0) = -\varphi(-\lambda v) + \varphi(0) = -(\varphi(-\lambda v) - \varphi(0)) = -(-\varphi(v) - \varphi(0) + \varphi(0) - \varphi(0)) = \lambda(\varphi(v) - \varphi(0)). \tag{2.103}
\]
Combining this with (2.101) proves that for all $\lambda \in \mathbb{R}$, $v \in V$ it holds that
\[
\varphi(\lambda v) = \lambda(\varphi(v) - \varphi(0)) + \varphi(0). \tag{2.104}
\]

22
Hence, we obtain for all \(v, w \in V\) that
\[
\varphi(v + w) = \varphi\left(\frac{1}{2}v + \frac{1}{2}w\right) = \frac{1}{2}\varphi(v) + \frac{1}{2}\varphi(w) = \frac{1}{2}(\varphi(v) + \varphi(w)) + \frac{1}{2}(\varphi(v) - \varphi(w)) = \varphi(v) + \varphi(w) - \varphi(0).
\]
This and (2.104) demonstrate that for all \(\lambda \in \mathbb{R}, v, w \in V\) it holds that
\[
\varphi(\lambda v + w) - \varphi(0) = \varphi(\lambda v) + \varphi(w) - 2\varphi(0) = \lambda(\varphi(v) - \varphi(0)) + \varphi(w) + \varphi(w) - 2\varphi(0) = \lambda(\varphi(v) - \varphi(0)) + (\varphi(w) - \varphi(0)).
\]
This establishes that (item (i) \(\rightarrow\) item (iii)). The proof of Lemma 2.14 is thus complete. \(\square\)

**Proposition 2.15** (Convexity with respect to the last affine linear transformation). Assume Setting 2.1, let \(\delta \in \mathbb{N}_0\) satisfy \(\delta = \sum_{k=1}^{L-1} \ell_k(\ell_{k-1} + 1)\), let \(\theta = (\theta_j)_{j \in [N]} : \mathbb{N} \cap [0, \delta] \to \mathbb{R}\) be a function, and let \(L : \mathbb{R}^{\ell_L(\ell_{L-1}+1)} \to \mathbb{R}\) satisfy for all \(v = (v_1, \ldots, v_{\ell_L(\ell_{L-1}+1)}) \in \mathbb{R}^{\ell_L(\ell_{L-1}+1)}\) that
\[
L(v) = \mathcal{L}_\infty(\theta_1, \theta_2, \ldots, \theta_\delta, v_1, v_2, \ldots, v_{\ell_L(\ell_{L-1}+1)}).
\]
Then it holds for all \(v, w \in \mathbb{R}^{\ell_L(\ell_{L-1}+1)}, \lambda \in [0, 1]\) that
\[
L(\lambda v + (1 - \lambda)w) \leq \lambda L(v) + (1 - \lambda)L(w).
\]

**Proof of Proposition 2.15.** Throughout this proof let \(\psi = (\psi_1, \ldots, \psi_\delta) : \mathbb{R}^{\ell_L(\ell_{L-1}+1)} \to \mathbb{R}^a\) satisfy for all \(v = (v_1, \ldots, v_{\ell_L(\ell_{L-1}+1)}) \in \mathbb{R}^{\ell_L(\ell_{L-1}+1)}\) that
\[
\psi(v) = (\theta_1, \theta_2, \ldots, \theta_\delta, v_1, v_2, \ldots, v_{\ell_L(\ell_{L-1}+1)}),
\]
and for every \(v \in \mathbb{R}^{\ell_L(\ell_{L-1}+1)}\) let \(N^a : \mathbb{R}^a \to \mathbb{R}^{\ell_L}\) satisfy for all \(x \in \mathbb{R}^a\) that
\[
N^a(x) = N^a_{L, \psi}(x).
\]
Note that Lemma 2.14, (2.1), and (2.109) ensure that for all \(v, w \in \mathbb{R}^{\ell_L(\ell_{L-1}+1)}, \lambda \in [0, 1], x \in \mathbb{R}^{\ell_L}\) it holds that
\[
\begin{aligned}
b^{L, \psi}(\lambda v + (1 - \lambda)w) + w^{L, \psi}(\lambda v + (1 - \lambda)w)x &= b^{L, \psi}(v) + w^{L, \psi}(w)x + \lambda b^{L, \psi}(v) + w^{L, \psi}(w)x + (1 - \lambda)\lambda b^{L, \psi}(v) + w^{L, \psi}(w)x + (1 - \lambda)w^{L, \psi}(w)x \\
&= \lambda b^{L, \psi}(v) + (1 - \lambda)b^{L, \psi}(w) + \lambda w^{L, \psi}(v)x + (1 - \lambda)w^{L, \psi}(w)x\end{aligned}
\]
Next observe that (2.4) shows that for all \(v \in \mathbb{R}^{\ell_L(\ell_{L-1}+1)}, x \in \mathbb{R}^{\ell_L}\) it holds that
\[
N^a_{L, \psi}(x) = \begin{cases} b^{L, \psi}(v) + w^{L, \psi}(w)x & : L = 1 \\
& b^{L, \psi}(v) + w^{L, \psi}(w)\left(\mathcal{M}_\infty(N^a_{L-1, \psi}(x))\right) & : L > 1.\end{cases}
\]
Therefore, we obtain that for all \(v, w \in \mathbb{R}^{\ell_L(\ell_{L-1}+1)}, \lambda \in [0, 1], x \in \mathbb{R}^{\ell_L}\) it holds that
\[
N^a_{L, \psi}(\lambda v + (1 - \lambda)w) = \begin{cases} b^{L, \psi}(\lambda v + (1 - \lambda)w) + w^{L, \psi}(\lambda v + (1 - \lambda)w)x & : L = 1 \\
& b^{L, \psi}(\lambda v + (1 - \lambda)w) + w^{L, \psi}(\lambda v + (1 - \lambda)w)\left(\mathcal{M}_\infty(N^a_{L-1, \psi}(\lambda v + (1 - \lambda)w)(x))\right) & : L > 1.\end{cases}
\]
Combining this with (2.111) implies that for all \(v, w \in \mathbb{R}^{\ell_L(\ell_{L-1}+1)}, \lambda \in [0, 1], x \in \mathbb{R}^{\ell_L}\) it holds that
\[
\begin{aligned}
N^a_{L, \psi}(\lambda v + (1 - \lambda)w) &= \begin{cases} \lambda b^{L, \psi}(v) + (1 - \lambda)\lambda b^{L, \psi}(w) + w^{L, \psi}(w)x & : L = 1 \\
& \lambda b^{L, \psi}(v) + w^{L, \psi}(w)\left(\mathcal{M}_\infty(N^a_{L-1, \psi}(\lambda v + (1 - \lambda)w)(x))\right) + (1 - \lambda)w^{L, \psi}(w)\left(\mathcal{M}_\infty(N^a_{L-1, \psi}(\lambda v + (1 - \lambda)w)(x))\right) & : L > 1.\end{cases}
\end{aligned}
\]
Item (i) of Proposition 2.5, (2.1), and the fact that for all \( i \in \mathbb{N} \setminus \{1, \sum_{k=1}^{L-1} f_k(\ell_{k-1} + 1)\} \), \( v, w \in \mathbb{R}^{f_L(\ell_{L-1} + 1)} \) it holds that \( \psi_i(v) = \psi_i(w) \) hence imply that for all \( v, w \in \mathbb{R}^{f_L(\ell_{L-1} + 1)} \), \( \lambda \in [0, 1] \), \( x \in \mathbb{R}^{d_0} \) it holds that

\[
\mathcal{N}_L^\psi(\lambda v + (1 - \lambda)w)(x) = \begin{cases} 
\lambda [b^L, \psi(v) + w^L, \psi(x)] + (1 - \lambda) [b^L, \psi(w) + w^L, \psi(x)] & : L = 1 \\
\lambda [b^L, \psi(v) + w^L, \psi(v)] + M_\infty \left( \mathcal{N}_L^\psi(\lambda v + (1 - \lambda)w)(x) \right) & : L > 1.
\end{cases}
\] (2.115)

Moreover, note that (2.112) ensures that for all \( v, w \in \mathbb{R}^{f_L(\ell_{L-1} + 1)} \), \( \lambda \in [0, 1] \), \( x \in \mathbb{R}^{d_0} \) it holds that

\[
\mathcal{N}_L^\psi(\lambda v + (1 - \lambda)w)(x) = \lambda \mathcal{N}_L^\psi(v)(x) + (1 - \lambda) \mathcal{N}_L^\psi(w)(x).
\] (2.117)

This and (2.110) show that for all \( v, w \in \mathbb{R}^{f_L(\ell_{L-1} + 1)} \), \( \lambda \in [0, 1] \), \( x \in \mathbb{R}^{d_0} \) it holds that

\[
\mathcal{N}_L^\psi(\lambda v + (1 - \lambda)w)(x) = \mathcal{N}_L^\psi(v)(x) + (1 - \lambda) \mathcal{N}_L^\psi(w)(x).
\] (2.118)

Next observe that (2.5), (2.107), (2.109), and (2.110) ensure that for all \( v \in \mathbb{R}^{f_L(\ell_{L-1} + 1)} \) it holds that

\[
L(v) = \mathcal{L}_\infty(\psi(v)) = \int_{[a, b]^{\ell_0}} \| \mathcal{N}_L^\psi(v)(x) - f(x) \|^2 \mu(dx) = \int_{[a, b]^{\ell_0}} \| \mathcal{N}_L^\psi(v) - f(x) \|^2 \mu(dx).
\] (2.119)

Combining this, (2.118), and the fact that for all \( \lambda \in [0, 1] \), \( x, y \in \mathbb{R} \) it holds that \((\lambda x + (1 - \lambda)y)^2 \leq \lambda x^2 + (1 - \lambda)y^2\) shows that for all \( v, w \in \mathbb{R}^{f_L(\ell_{L-1} + 1)} \), \( \lambda \in [0, 1] \) it holds that

\[
L(\lambda v + (1 - \lambda)w) = \int_{[a, b]^{\ell_0}} \| \mathcal{N}_L^{\lambda v + (1 - \lambda)w} - f(x) \|^2 \mu(dx)
\]

\[
= \int_{[a, b]^{\ell_0}} \| \mathcal{N}_L^{\lambda v} - f(x) \|^2 + (1 - \lambda) \| \mathcal{N}_L^{\lambda w} - f(x) \|^2 \mu(dx)
\]

\[
\leq \int_{[a, b]^{\ell_0}} \| \mathcal{N}_L^{\lambda v} - f(x) \|^2 + (1 - \lambda) \| \mathcal{N}_L^{\lambda w} - f(x) \|^2 \mu(dx)
\]

\[
\leq \lambda \int_{[a, b]^{\ell_0}} \| \mathcal{N}_L^{\lambda v} - f(x) \|^2 \mu(dx) + (1 - \lambda) \int_{[a, b]^{\ell_0}} \| \mathcal{N}_L^{\lambda w} - f(x) \|^2 \mu(dx)
\]

\[
= \lambda L(v) + (1 - \lambda) L(w).
\]

This establishes (2.108). The proof of Proposition 2.15 is thus complete.

\[\Box\]

**Corollary 2.16 (Convexity of the risk function).** Assume Setting 2.1 and assume \((L - 1)m = 0\). Then it holds for all \( \theta, \vartheta \in \mathbb{R}^d \), \( \lambda \in [0, 1] \) that

\[
\mathcal{L}_\infty(\lambda \theta + (1 - \lambda)\vartheta) \leq \lambda \mathcal{L}_\infty(\theta) + (1 - \lambda) \mathcal{L}_\infty(\vartheta).
\] (2.121)

**Proof of Corollary 2.16.** Throughout this proof we distinguish between the case \( m = 0 \) and the case \( m \neq 0 \). We first prove (2.121) in the case \( m = 0 \).

\[
m = 0.
\] (2.122)
Note that (2.5) and (2.122) ensure that for all $\theta \in \mathbb{R}^9$ it holds that
\[
L_\infty(\theta) = \int_{[a,b]^9} \|N_{\infty}^{\theta_0}(x) - \xi\|^2 \mu(dx) = 0.
\] (2.123)
Therefore, we obtain that for all $\theta, \vartheta \in \mathbb{R}^9$, $\lambda \in [0, 1]$ it holds that
\[
L_\infty(\lambda \theta + (1 - \lambda) \vartheta) = 0 = \lambda L_\infty(\theta) + (1 - \lambda) L_\infty(\vartheta).
\] (2.124)
This establishes (2.121) in the case $m = 0$. Next we prove (2.121) in the case $m \neq 0$.
\[
\frac{m}{16} + \frac{1}{2} \left[ \int_{[a,b]^9} f_1(x) \mu(dx) \right].
\] (2.125)
Observe that (2.125) and the assumption that $(L - 1)m = 0$ imply that $L = 1$. Proposition 2.15 hence demonstrates that for all $\theta, \vartheta \in \mathbb{R}^9$, $\lambda \in [0, 1]$ it holds that
\[
L_\infty(\lambda \theta + (1 - \lambda) \vartheta) \leq \lambda L_\infty(\theta) + (1 - \lambda) L_\infty(\vartheta).
\] (2.126)
This establishes (2.121) in the case $m \neq 0$. The proof of Corollary 2.16 is thus complete. □

2.7 Non-convexity properties of the risk function

**Proposition 2.17** (Arithmetical averages in the argument of the risk function). Assume Setting 2.1, assume $L > 1$, let $\xi = (\xi_1, \ldots, \xi_L) \in \mathbb{R}^L$, let $\{a,b\}^{L+1} \subseteq \mathbb{R}$ satisfy for all $i, j \in \mathbb{N}_0$, $k \in \mathbb{N}$ that $\alpha_{i,j,k} = i\ell_{k-1} + j + \sum_{h=1}^{k-1} \ell_h(\ell_{h-1} + 1)$, and let $\theta = (\theta_1, \ldots, \theta_2), \vartheta = (\vartheta_1, \ldots, \vartheta_2) \in \mathbb{R}^9$ satisfy for all $i \in [1, \ell_L], \varphi \in \mathbb{R}^9$, $j \in [1, \ell_L]$, $\theta_{ik} = \vartheta_{ik}$ that
\[
\begin{align*}
\theta_{ik} &= \vartheta_{ik}, \\
\alpha_{ik} &= \alpha_{ik}, \\
\alpha_{ik} &= \alpha_{ik}.
\end{align*}
\] (2.127)

If $L \in [2, \infty)$ then $L_\infty(\theta) = L_\infty(\vartheta)$ and
\[
L_\infty(\theta + \frac{\vartheta}{2}) = \frac{L_\infty(\theta) + L_\infty(\vartheta)}{2} + \frac{m}{16} + \frac{1}{2} \left[ \int_{[a,b]^9} f_1(x) \mu(dx) \right].
\] (2.128)

**Proof of Proposition 2.17.** Note that items (ii) and (iii) of Lemma 2.2, and (2.127) ensure that for all $x \in [a, b]^{L+1}$ it holds that $N_{\infty}^{\theta_0}(x) = N_{\infty}^{\vartheta_0}(x) = \xi$. This and (2.5) imply that
\[
L_\infty(\theta) = L_\infty(\vartheta) = \int_{[a,b]^9} \|\xi - f(x)\|^2 \mu(dx).
\] (2.129)
Furthermore, observe that items (ii) and (iii) of Lemma 2.2 and (2.127) demonstrate that for all $x \in [a, b]^{L+1}$ it holds that
\[
N_{\infty}^{\theta_0, (\theta + \vartheta)/2}(x) = \frac{1}{2} \max \left\{ \xi_1, \xi_1 + \frac{1}{2} \right\} = \xi_1 = \xi_1 + \frac{1}{2}.
\] (2.130)
Moreover, note that items (ii) and (iii) of Lemma 2.2 and (2.127) ensure that for all $i \in \mathbb{N} \cap (1, \ell_L]$, $x \in [a, b]^{L+1}$ it holds that $N_{\infty}^{\theta_0, (\theta + \vartheta)/2}(x) = \xi_i$. Combining this with (2.129) and (2.130) shows that
\[
L_\infty(\frac{\theta + \vartheta}{2}) = \int_{[a,b]^9} \left[ (\xi_1 + \frac{1}{2} - f_1(x))^2 + \xi_1 - f_1(x))^2 \right] \mu(dx) \]
\[
= \int_{[a,b]^9} \|\xi - f(x)\|^2 \mu(dx) + \int_{[a,b]^9} \frac{1}{16} \mu(dx) + \int_{[a,b]^9} \frac{1}{2} \left[ \xi_1 - f_1(x) \right] \mu(dx) \]
\[
= \frac{L_\infty(\theta) + L_\infty(\vartheta)}{2} + \frac{m}{16} + \frac{1}{2} \left[ \int_{[a,b]^9} f_1(x) \mu(dx) \right].
\] (2.131)
This establishes (2.128). The proof of Proposition 2.17 is thus complete. □
Corollary 2.18 (Non-convexity of the risk function). Assume Setting 2.1, assume \((L-1)m \neq 0\), let \(\xi = (\xi_1, \ldots, \xi_{l_L}) \in \mathbb{R}^{l_L}\) satisfy \(\xi_1 = m^{-1} \int_{[a,b]^*} f_1(x) \mu(dx)\), let \((\alpha_{i,j,k})_{(i,j,k) \in (\mathbb{N}_0)^3} \subseteq \mathbb{R}\) satisfy for all \(i, j \in \mathbb{N}_0, k \in \mathbb{N}\) that \(\alpha_{i,j,k} = \xi_{i-1} + j + \sum_{h=1}^{k-1} \xi_h\), and let \(\theta = (\theta_1, \ldots, \theta_m), \vartheta = (\vartheta_1, \ldots, \vartheta_m) \in \mathbb{R}^m\) satisfy for all \(i \in \{1, \ldots, l_L\}, j \in \{1, \ldots, \vartheta\}\) that
\[
\theta_{\alpha_{i-1-1,i-1}} = \theta_{\alpha_{i-1,i-1}} = 1, \quad \theta_{\alpha_{i,i}} = \theta_{\alpha_{i-1,i}} = \xi_i,
\]
and \(\theta_{\alpha_{1,1}} = \vartheta_{\alpha_{1,1}} = \theta_{\vartheta_{1,1}} = 0\).

Then
\[
\mathcal{L}_\infty^\infty \left( \frac{\theta + \vartheta}{2} \right) = \left[ \mathcal{L}_\infty^\infty(\theta) + \mathcal{L}_\infty^\infty(\vartheta) \right] + m \frac{m}{16} \mathcal{L}_\infty^\infty(\theta) + \mathcal{L}_\infty^\infty(\vartheta).
\]

Proof of Corollary 2.18. Observe that Proposition 2.17 and the assumption that \(\xi_1 = m^{-1} \int_{[a,b]^*} f_1(x) \mu(dx)\) demonstrate that
\[
\mathcal{L}_\infty^\infty \left( \frac{\theta + \vartheta}{2} \right) = \left[ \mathcal{L}_\infty^\infty(\theta) + \mathcal{L}_\infty^\infty(\vartheta) \right] + m \frac{m}{16} \left[ \frac{1}{2} \xi_1 m - \int_{[a,b]^*} f_1(x) \mu(dx) \right]
\]
and the fact that \(m \neq 0\) therefore implies that
\[
\mathcal{L}_\infty^\infty \left( \frac{\theta + \vartheta}{2} \right) > \mathcal{L}_\infty^\infty(\theta) + \mathcal{L}_\infty^\infty(\vartheta).
\]
The proof of Corollary 2.18 is thus complete.

Corollary 2.19 (Characterization of convexity of the risk function). Assume Setting 2.1. Then it holds that \((L-1)m = 0\) if and only if it holds for all \(\theta, \vartheta \in \mathbb{R}^m, \lambda \in [0, 1]\) that
\[
\mathcal{L}_\infty^\infty(\lambda \theta + (1 - \lambda) \vartheta) \leq \lambda \mathcal{L}_\infty^\infty(\theta) + (1 - \lambda) \mathcal{L}_\infty^\infty(\vartheta).
\]

Proof of Corollary 2.19. Note that Corollary 2.16 and Corollary 2.18 establish (2.136). The proof of Corollary 2.19 is thus complete.

3 Gradient flow (GF) processes in the training of deep ANNs

In this section we establish in Theorem 3.8 in Subsection 3.4 below in the training of deep ReLU ANNs, under the assumption that the target function \(f: [a,b]^* \to \mathbb{R}\) in (2.5) is a constant function, that the risk of every solution \(\Theta = (\Theta_\ell)_{\ell \in [0, \infty)}: [0, \infty) \to \mathbb{R}^m\) of the associated GF differential equation converges with rate 1 to zero. Our proof of Theorem 3.8 uses the elementary regularity result for the Lyapunov function \(V: \mathbb{R}^m \to \mathbb{R}\) in Proposition 3.1 in Subsection 3.1 below (cf. (2.6) in Setting 2.1), the weak chain rule for compositions of the Lyapunov function \(V: \mathbb{R}^m \to \mathbb{R}\) and GF solutions in the constant target function case in Corollary 3.5 in Subsection 3.2 below, and the weak chain rule for compositions of the risk function \(\mathcal{L}_\infty^\infty: \mathbb{R}^m \to \mathbb{R}\) and GF solutions in the general target function case in Proposition 3.7 in Subsection 3.3 below (cf. (2.5) in Setting 2.1).

Our proof of the weak chain rule for compositions of \(V: \mathbb{R}^m \to \mathbb{R}\) and GF solutions in the constant target function case in Corollary 3.5, in turn, employs the weak chain rule for compositions of \(V: \mathbb{R}^m \to \mathbb{R}\) and GF solutions in the general target function case in Proposition 3.4 in Subsection 3.2. Our proof of the weak chain rule for compositions of \(V: \mathbb{R}^m \to \mathbb{R}\) and GF solutions in the general target function case in Proposition 3.4 make use of the local boundedness result for the generalized gradient function \(G: \mathbb{R}^m \to \mathbb{R}\) in Corollary 2.12 in Subsection 2.5 above as well as of the identity for the scalar product of \(\mathbb{R}^m \ni \theta \mapsto (\nabla V)(\theta) \in \mathbb{R}^m\) and \(G: \mathbb{R}^m \to \mathbb{R}^m\) in Proposition 3.2 in Subsection 3.1. In Corollary 3.3 in Subsection 3.1 we also specialize Proposition 3.2 to the constant target function case. In particular,
Corollary 3.3 implies that the scalar product of \( \mathbb{R}^3 \ni \theta \mapsto (\nabla V)(\theta) \in \mathbb{R}^3 \) and \( G \) is non-negative and, thereby, proves that \( V \) serves as a Lyapunov function for the considered GF processes.

Our proof of the weak chain rule for compositions of \( L_\infty : \mathbb{R}^3 \to \mathbb{R} \) and GF solutions in the general target function case in Proposition 3.7 employs Theorem 2.9 in Subsection 2.3 above, Corollary 2.12 in Subsection 2.5, and the uniform local boundedness result for the gradients \( \mathbb{R}^3 \ni \theta \mapsto (\nabla L_r)(\theta) \in \mathbb{R}^3, \ r \in [1, \infty) \), of the approximated risk functions \( L_r : \mathbb{R}^3 \to \mathbb{R}, \ r \in [1, \infty) \), in Lemma 3.6 in Subsection 3.3 (cf. (2.5) in Setting 2.1).

The results in this section extend the findings in [28, Subsection 2.6] and [13, Section 3] from shallow ReLU ANNs with just one hidden layer to deep ReLU ANNs with an arbitrarily large number of hidden layers. In particular, Proposition 3.1 in Subsection 3.1 extends [28, Proposition 2.8], Proposition 3.2 in Subsection 3.1 extends [28, Proposition 2.9], Corollary 3.3 in Subsection 3.1 extends [28, Corollary 2.10], Proposition 3.4 in Subsection 3.2 and Corollary 3.5 in Subsection 3.2 extend [13, Lemma 3.2] (cf. also [27, Proposition 4.1]), Lemma 3.6 in Subsection 3.3 extends [13, Lemma 3.4], Proposition 3.7 in Subsection 3.3 extends [13, Lemma 3.5], and Theorem 3.8 in Subsection 3.4 extends [13, Theorem 3.7].

### 3.1 Lyapunov type estimates for the dynamics of GF processes

**Proposition 3.1.** Assume Setting 2.1 and let \( \theta \in \mathbb{R}^3 \). Then

(i) it holds that

\[
V(\theta) = \sum_{k=1}^{L} (k\|b^{k,\theta}\|^2 + \sum_{i=1}^{\ell_k} \sum_{j=1}^{\ell_{k-1}} |w_{i,j}^{k,\theta}|^2) - 2L \langle f(0), b^{L,\theta} \rangle ,
\]

(ii) it holds that

\[
\frac{1}{2}\|\theta\|^2 - 2L^2\|f(0)\|^2 \leq V(\theta) \leq 2L\|\theta\|^2 + L\|f(0)\|^2 ,
\]

(iii) it holds that

\[
(\nabla V)(\theta) = 2(\mathbf{w}^{1,\theta}, b^{1,\theta}, \mathbf{w}^{2,\theta}, b^{2,\theta}, \ldots , \mathbf{w}^{L-1,\theta}, b^{L-1,\theta}, (L-1)b^{L-1,\theta}, \mathbf{w}^{L,\theta}, L(b^{L,\theta} - f(0))).
\]

**Proof of Proposition 3.1.** Observe that (2.6) establishes item (i). Next note that (2.6), the Cauchy-Schwarz inequality, and the Young inequality demonstrate that

\[
V(\theta) = \sum_{k=1}^{L} (k\|b^{k,\theta}\|^2 + \sum_{i=1}^{\ell_k} \sum_{j=1}^{\ell_{k-1}} |w_{i,j}^{k,\theta}|^2) - 2L \langle f(0), b^{L,\theta} \rangle
\]

\[
= \|\theta\|^2 + \sum_{k=1}^{L} (k-1)\|b^{k,\theta}\|^2 - 2L \langle f(0), b^{L,\theta} \rangle \geq \|\theta\|^2 - 2L \langle f(0), b^{L,\theta} \rangle
\]

\[
\geq \|\theta\|^2 - 2L\|f(0)\||b^{L,\theta}| \geq \|\theta\|^2 - 2L^2\|f(0)\|^2 - \frac{1}{2}\|b^{L,\theta}\|^2 \geq \frac{1}{2}\|\theta\|^2 - 2L^2\|f(0)\|^2 .
\]

Furthermore, observe that (2.6) shows that

\[
V(\theta) = \sum_{k=1}^{L} (k\|b^{k,\theta}\|^2 + \sum_{i=1}^{\ell_k} \sum_{j=1}^{\ell_{k-1}} |w_{i,j}^{k,\theta}|^2) - 2L \langle f(0), b^{L,\theta} \rangle
\]

\[
= \|\theta\|^2 + \sum_{k=1}^{L} (k-1)\|b^{k,\theta}\|^2 - 2L \langle f(0), b^{L,\theta} \rangle \leq \|\theta\|^2 + \sum_{k=1}^{L} (k-1)\|b^{k,\theta}\|^2 + |2L \langle f(0), b^{L,\theta} \rangle|.
\]

The Cauchy-Schwarz inequality and the Young inequality hence imply that

\[
V(\theta) \leq \|\theta\|^2 + \sum_{k=1}^{L} (k-1)\|b^{k,\theta}\|^2 + 2L\|f(0)\||b^{L,\theta}|
\]

\[
\leq \|\theta\|^2 + \sum_{k=1}^{L} (k-1)\|b^{k,\theta}\|^2 + L\|f(0)\|^2 + L\|b^{L,\theta}\|^2 \leq 2L\|\theta\|^2 + L\|f(0)\|^2 .
\]

Combining this and (3.3) proves item (ii). Note that item (i) establishes item (iii). The proof of Proposition 3.1 is thus complete.

\[\square\]
Proposition 3.2. Assume Setting 2.1 and let \( \theta \in \mathbb{R}^3 \). Then
\[
\langle (\nabla V)(\theta), G(\theta) \rangle = 4L \int_{[a,b]_0} \left< \mathcal{N}^{L}_{\infty}(x) - f(x), \mathcal{N}_{\infty}^{L,\theta}(x) - f(0) \right> \mu(dx). \tag{3.6}
\]

Proof of Proposition 3.2. Throughout this proof let \( (d_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}_0 \) satisfy for all \( k \in \mathbb{N}_0 \) that \( d_k = \sum_{n=1}^{k} \ell_n(x_{n-1} + 1) \). Observe that item (iii) of Proposition 3.1 demonstrates that
\[
\langle (\nabla V)(\theta), G(\theta) \rangle = \sum_{k=1}^{L} \sum_{i_k=1}^{\ell_k} 2k b^{k,\theta}_{i_k} G_{i_k}^k \epsilon_{k-1} + d_{k-1}(\theta)
+ \sum_{k=1}^{L} \sum_{i_k=1}^{\ell_k-1} \sum_{i_{k-1}=1}^{\ell_{k-1}} 2m^{k,\theta}_{i_k,i_{k-1}} G_{i_k}^k \epsilon_{k-1} + d_{k-1}(\theta) - \sum_{i_k=1}^{\ell_L} 2L f_{L}(0) G_{L} \epsilon_{L} + d_{L}(\theta).
\tag{3.7}
\]

Next we claim that for all \( k \in \{1, \ldots, L\} \) it holds that
\[
\sum_{m=k}^{L} \sum_{i_m=1}^{\ell_m} b^{m,\theta}_{i_m} G_{i_m}^m \epsilon_{m-1} + d_{m-1}(\theta) + \sum_{i_k=1}^{\ell_k-1} \sum_{i_{k-1}=1}^{\ell_{k-1}} m^{k,\theta}_{i_k,i_{k-1}} G_{i_k}^k \epsilon_{k-1} + d_{k-1}(\theta) = 2 \int_{[a,b]_0} \left< \mathcal{N}_{\infty}^{L,\theta}(x) - f(x), \mathcal{N}_{\infty}^{L,\theta}(x) \right> \mu(dx). \tag{3.8}
\]

We prove (3.8) by induction on \( k \in \{1, \ldots, L\} \). Note that items (v) and (vi) of Theorem 2.9 and (2.4) ensure that
\[
\sum_{i=1}^{\ell_L} b^{L,\theta}_{i} G_{i}^L \epsilon_{L-1} + d_{L-1}(\theta) + \sum_{i=1}^{\ell_L} m^{L,\theta}_{i} G_{i}^L \epsilon_{L-1} + d_{L-1}(\theta) = \sum_{i=1}^{\ell_L} 2b^{L,\theta}_{i} \int_{[a,b]_0} \left< \mathcal{N}_{\infty}^{L,\theta}(x) - f_i(x) \right> \mu(dx)
+ \sum_{j=1}^{L} \sum_{j=1}^{\ell_{L-1}} 2m_{i,j}^{L,\theta} \int_{[a,b]_0} \left( R_{\infty}^{L,\theta}(x) \mathbb{1}_{\{L\}}(L) + x_j \mathbb{1}_{\{1\}}(L) \right) \left< \mathcal{N}_{\infty}^{L,\theta}(x) - f_i(x) \right> \mu(dx)
= 2 \int_{[a,b]_0} \left< \mathcal{N}_{\infty}^{L,\theta}(x) - f(x), \mathcal{N}_{\infty}^{L,\theta}(x) \right> \mu(dx). \tag{3.9}
\]

This establishes (3.8) in the base case \( k = L \). For the induction step let \( k \in \mathbb{N} \cap [2, L] \) satisfy
\[
\sum_{m=k}^{L} \sum_{i_m=1}^{\ell_m} b^{m,\theta}_{i_m} G_{i_m}^m \epsilon_{m-1} + d_{m-1}(\theta) + \sum_{i_k=1}^{\ell_k-1} \sum_{i_{k-1}=1}^{\ell_{k-1}} m^{k,\theta}_{i_k,i_{k-1}} G_{i_k}^k \epsilon_{k-1} + d_{k-1}(\theta) = 2 \int_{[a,b]_0} \left< \mathcal{N}_{\infty}^{L,\theta}(x) - f(x), \mathcal{N}_{\infty}^{L,\theta}(x) \right> \mu(dx). \tag{3.10}
\]
Observe that items (v) and (vi) of Theorem 2.9 show that

\[
\sum_{m=k-1}^{L} \sum_{l_m=1}^{\ell_m} b_{m}^{\ell_m} \mathcal{G}_{(m-1)l_m+i_m+d_m-1}(\theta) + \sum_{k_k=1}^{\ell_{k-1}} \sum_{i_{k-2}=1}^{\ell_{k-2}} w_{k_{k-1},i_{k-2}}^{k-1} \mathcal{G}_{(k_{k-1}-1)i_{k-2}+i_{k-2}+d_{k-2}}(\theta) \\
= \sum_{m=k}^{L} \sum_{l_m=1}^{\ell_m} b_{m}^{\ell_m} \mathcal{G}_{m-1+i_m+d_m-1}(\theta) + \sum_{k_k=1}^{\ell_{k-1}} \sum_{i_{k-2}=1}^{\ell_{k-2}} w_{k_{k-1},i_{k-2}}^{k-1} \mathcal{G}_{k_{k-1}-1i_{k-2}+i_{k-2}+d_{k-2}}(\theta) \\
+ 2 \sum_{i_k=1}^{k_{k-1}} \left[ \int_{[a,b]} \mathcal{R}_\infty(\mathcal{N}_{\infty,i_{k-2}^\infty}(k-2,1,\theta)(x)) \mathbb{1}(1,L)(k-1) + x_{i_{k-2}} \mathbb{1}(1)(k-1) \right] \\
\sum_{(m,n)\in\mathbb{N}\times[1,L] : n_m \leq \ell_m} \left( (\mathcal{N}_{\infty,n_k} \mathcal{L}_\infty \mathcal{L}_n) (x) - f_{n_k}(x) \right) \\
\cdot \mathbb{1}(i_{k-1})(v_{k-1}) \left[ \prod_{i=1}^{L} (w_{i_k,v_{i_k-1}}) \left( \mathbb{1}_{\mathcal{V}_{i_k-1}}(x) \right) \right] \mu(dx).}

Therefore, we obtain that

\[
\sum_{m=k-1}^{L} \sum_{l_m=1}^{\ell_m} b_{m}^{\ell_m} \mathcal{G}_{(m-1)l_m+i_m+d_m-1}(\theta) + \sum_{k_k=1}^{\ell_{k-1}} \sum_{i_{k-2}=1}^{\ell_{k-2}} w_{k_{k-1},i_{k-2}}^{k-1} \mathcal{G}_{(k_{k-1}-1)i_{k-2}+i_{k-2}+d_{k-2}}(\theta) \\
= \sum_{m=k}^{L} \sum_{l_m=1}^{\ell_m} b_{m}^{\ell_m} \mathcal{G}_{m-1+i_m+d_m-1}(\theta) + \sum_{k_k=1}^{\ell_{k-1}} \sum_{i_{k-2}=1}^{\ell_{k-2}} w_{k_{k-1},i_{k-2}}^{k-1} \mathcal{G}_{k_{k-1}-1i_{k-2}+i_{k-2}+d_{k-2}}(\theta) \\
+ 2 \sum_{i_k=1}^{k_{k-1}} \left[ \int_{[a,b]} \mathcal{R}_\infty(\mathcal{N}_{\infty,i_{k-2}^\infty}(k-2,1,\theta)(x)) \mathbb{1}(1,L)(k-1) + x_{i_{k-2}} \mathbb{1}(1)(k-1) \right] \\
\sum_{(m,n)\in\mathbb{N}\times[1,L] : n_m \leq \ell_m} \left( (\mathcal{N}_{\infty,n_k} \mathcal{L}_\infty \mathcal{L}_n) (x) - f_{n_k}(x) \right) \\
\cdot \mathbb{1}(i_{k-1})(v_{k-1}) \left[ \prod_{i=1}^{L} (w_{i_k,v_{i_k-1}}) \left( \mathbb{1}_{\mathcal{V}_{i_k-1}}(x) \right) \right] \mu(dx).}
\]
Item (v) of Theorem 2.9, (2.4), and (3.10) hence imply that

\[
\sum_{m=k-1}^L \sum_{i_m=1}^\ell_m b^m_{i_m} \mathcal{G}_{(i_m)} \mathbf{e}_{\ell_m+i_m+d_{m-1}}(\theta) + \sum_{i_{k-1}=1}^{\ell_{k-1}} \sum_{i_{k-2}=1}^{\ell_{k-2}} w^k_{i_{k-1},i_{k-2}} \mathcal{G}_{(i_{k-1}-1)i_{k-2}+i_{k-2}+d_{k-2}}(\theta)
\]

\[
= \sum_{m=k}^L \sum_{i_m=1}^\ell_m b^m_{i_m} \mathcal{G}_{(i_m)} \mathbf{e}_{\ell_m+i_m+d_{m-1}}(\theta) + 2 \sum_{i_{k-1}=1}^{\ell_{k-1}} \int_{[a,b]^0} \sum_{i_{k-1}}^{\ell_{k-1}} w_{i_{k-1}}^{k,\theta} \left[ \mathcal{R}_\infty(\mathcal{N}^{k-1,\theta}_{\infty,i_{k-1}}(x)) \right] 
\]

\[
\cdot \sum_{v_{k-1},v_{k-1+1},\ldots,v_{L} \in \mathbb{N}} \sum_{\forall \omega \in \mathbb{N}[k,L]: \nu_{\omega} \leq \ell_{\omega}} \left[ \mathcal{N}^{L^\theta}_{\infty,v_{k-1}}(x) - f_{v_{k-1}}(x) \right] \mathbb{I}_{i_{k-1}}(v_{k}) \mu(dx) 
\]

\[
= \sum_{m=k}^L \sum_{i_m=1}^\ell_m b^m_{i_m} \mathcal{G}_{(i_m)} \mathbf{e}_{\ell_m+i_m+d_{m-1}}(\theta) + \sum_{k=1}^L \sum_{i_k=1}^{\ell_k} w^k_{i_k} \mathcal{G}_{(i_k-1)f_{i_k}+i_k+d_{k-1}}(\theta) 
\]

\[
= 2\int_{[a,b]^0} \left\langle \mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) - f(x), \mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) \right\rangle \mu(dx). 
\]

Induction thus establishes (3.8). Next that (3.8) ensures that

\[
\sum_{k=1}^L \sum_{i_k=1}^{\ell_k} b^k_{i_k} \mathcal{G}_{(i_k)} \mathbf{e}_{\ell_k+i_k+d_{k}}(\theta) + \sum_{k=1}^L \sum_{i_{k-1}=1}^{\ell_{k-1}} w^k_{i_{k-1}} \mathcal{G}_{(i_{k-1}-1)f_{i_{k-1}}+i_{k-1}+d_{k-1}}(\theta) 
\]

\[
= 2L \int_{[a,b]^0} \left\langle \mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) - f(x), \mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) \right\rangle \mu(dx). 
\]

Combining this and (3.7) demonstrates that

\[
\left\langle (\nabla V(\theta), \mathcal{G}(\theta)) \right\rangle = \sum_{k=1}^L \sum_{i_k=1}^{\ell_k} 2kb^k_{i_k} \mathcal{G}_{(i_k)} \mathbf{e}_{\ell_k+i_k+d_{k}}(\theta) 
\]

\[
+ \sum_{k=1}^L \sum_{i_k=1}^{\ell_k} \sum_{i_{k-1}=1}^{\ell_{k-1}} 2w^k_{i_k,i_{k-1}} \mathcal{G}_{(i_{k-1}-1)f_{i_{k-1}}+i_{k-1}+d_{k-1}}(\theta) 
\]

\[
- \sum_{i_{k-1}=1}^{\ell_{k-1}} 2f_{i_{k-1}}(0) \mathcal{G}_{\ell_{k-1}+i_{k-1}+d_{k-1}}(\theta) 
\]

\[
= 4L \int_{[a,b]^0} \left\langle \mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) - f(x), \mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) \right\rangle \mu(dx) 
\]

\[
- 4L \sum_{i_{k-1}=1}^{\ell_{k-1}} \int_{[a,b]^0} f_{i_{k-1}}(0) \left( \mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) - f_{i_{k-1}}(x) \right) \mu(dx) \right) 
\]

\[
= 4L \int_{[a,b]^0} \left\langle \mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) - f(x), \mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) - f(0) \right\rangle \mu(dx). 
\]

The proof of Proposition 3.2 is thus complete. \(\square\)

**Corollary 3.3.** Assume Setting 2.1, assume for all \(x \in [a,b]^0\) that \(f(x) = f(0)\), and let \(\theta \in \mathbb{R}^\mathbb{N}\). Then

\[
\left\langle (\nabla V(\theta), \mathcal{G}(\theta)) \right\rangle = 4L \mathcal{L}_{\infty}(\theta). 
\]

**Proof of Corollary 3.3.** Observe that Proposition 3.2 and the fact that for all \(r \in [1, \infty]\) it holds that \(\mathcal{L}_{r}(\theta) = \int_{[a,b]^0} |\mathcal{N}^{\mathcal{L},\theta}_{\infty}(x) - f(0)|^2 \mu(dx)\) establish (3.16). The proof of Corollary 3.3 is thus complete. \(\square\)
3.2 Weak chain rule for compositions of Lyapunov functions and GF processes

**Proposition 3.4.** Assume Setting 2.1 and let \( T \in (0, \infty), \Theta \in C([0, T], \mathbb{R}^p) \) satisfy for all \( t \in [0, T] \) that \( \Theta_t = \Theta_0 - \int_0^t \mathcal{G}(\Theta_s) \, ds \). Then it holds for all \( t \in [0, T] \) that

\[
V(\Theta_t) = V(\Theta_0) - 4L \int_0^t \int_{[a, b]} \langle \mathcal{N}_{L, \theta}^L(x) - f(x), \mathcal{N}_{L, \theta}^L(x) - f(0) \rangle \, \mu(dx) \, ds.
\] (3.17)

**Proof of Proposition 3.4.** Note that Corollary 2.12 and the assumption that \( \Theta \in C([0, T], \mathbb{R}^p) \) ensure that \( [0, T] \ni t \mapsto \mathcal{G}(\Theta_t) \in \mathbb{R}^p \) is bounded. Proposition 3.2 and, e.g., Cheridito et al. [13, Lemma 3.1] (applied with \( T \wedge T, n \wedge \delta, F \wedge (\mathbb{R}^p \ni \theta \mapsto V(\theta) \in \mathbb{R}), \delta \wedge ((0, T] \ni t \mapsto \mathcal{G}(\Theta_t) \in \mathbb{R}^p) \) in the notation of [13, Lemma 3.1]) therefore prove that for all \( t \in [0, T] \) it holds that

\[
V(\Theta_t) - V(\Theta_0) = -\int_0^t \langle (\nabla V)(\Theta_s), \mathcal{G}(\Theta_s) \rangle \, ds
\]

\[
= -4L \int_0^t \int_{[a, b]} \langle \mathcal{N}_{L, \theta}^L(x) - f(x), \mathcal{N}_{L, \theta}^L(x) - f(0) \rangle \, \mu(dx) \, ds.
\] (3.18)

The proof of Proposition 3.4 is thus complete. \( \square \)

**Corollary 3.5.** Assume Setting 2.1, assume for all \( x \in [a, b] \) that \( f(x) = f(0) \), and let \( T \in (0, \infty), \Theta \in C([0, T], \mathbb{R}^p) \) satisfy for all \( t \in [0, T] \) that \( \Theta_t = \Theta_0 - \int_0^t \mathcal{G}(\Theta_s) \, ds \). Then it holds for all \( t \in [0, T] \) that

\[
V(\Theta_t) = V(\Theta_0) - 4L \int_0^t \mathcal{L}_\infty(\Theta_s) \, ds.
\] (3.19)

**Proof of Corollary 3.5.** Observe Proposition 3.4 and the fact that \( \mathcal{L}_\infty(\theta) = \int_{[a, b]} \| \mathcal{N}_{L, \theta}^L(x) - f(x) \|^2 \mu(dx) = \int_{[a, b]} \langle \mathcal{N}_{L, \theta}^L(x) - f(x), \mathcal{N}_{L, \theta}^L(x) - f(0) \rangle \, \mu(dx) \) establish (3.19). The proof of Corollary 3.5 is thus complete. \( \square \)

3.3 Weak chain rule for the risk of GF processes

**Lemma 3.6.** Assume Setting 2.1 and let \( K \subseteq \mathbb{R}^p \) be compact. Then

\[
\sup_{\theta \in K} \sup_{r \in [1, \infty)} \| (\nabla \mathcal{L}_r)(\theta) \| < \infty.
\] (3.20)

**Proof of Lemma 3.6.** Throughout this proof assume without loss of generality that \( m > 0 \) and for every \( k \in \mathbb{N}_0 \) let \( d_k \in \mathbb{N}_0 \) satisfy \( d_k = \sum_{n=1}^k \ell_n (\ell_n - 1) + 1 \). Note that Lemma 2.7 ensures that there exists \( \mathfrak{D} \in [1, \infty) \) which satisfies for all \( k \in \{1, \ldots, L\} \) that

\[
\mathfrak{D} \geq a + \sup_{\theta \in K} \sup_{r \in [1, \infty)} \sup_{t \in [1, \ell_k]} \sup_{\ell_{n-1} \leq k} \left( |\mathcal{N}_{r, \theta}^L(x)| + |\mathfrak{R}_s(\mathcal{N}_{r, \theta}^L(x))| + |(\mathfrak{R}_s)'(\mathcal{N}_{r, \theta}^L(x))| \right).
\] (3.21)

Furthermore, observe that item (iii) of Theorem 2.9 proves that for all \( \theta \in K, r \in [1, \infty), k \in \{1, \ldots, L\}, \)
\[ i \in \{1, \ldots, \ell_k \} \text{ it holds that} \]
\[
\left| \frac{\partial L_r}{\partial \theta_{e_k k-1 + i + d_k-1}}(\theta) \right|^2 \\
= \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall w \in \mathbb{N}^n[k, L]: v_w \leq \ell_w} \int_{[a, b]^n} 2 \left[ \mathbb{1}(v_k) \left[ (N_{r, i}^{\ell, \theta})_r(x) - f_{\ell, i}(x) \right] \right] \right)^2 \]
\[
\text{Moreover, note that item (ii) of Theorem 2.9 demonstrates for all } \theta \in K, r \in [1, \infty), k \in \{1, \ldots, L\}, i \in \{1, \ldots, \ell_k\} \text{ that} \]
\[
\left| \frac{\partial L_r}{\partial \theta_{e_k k-1 + i + d_k-1}}(\theta) \right|^2 \\
\leq 4m^2 \mathcal{D}^{2L} \left( \int_{[a, b]^n} \|N_r^{\ell, \theta}(x) - f(x)\| \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall w \in \mathbb{N}^n[k, L]: v_w \leq \ell_w} \mathbb{1}(v_k) \left[ \prod_{n=1}^{L} w_{v_n, v_{n-1}}^{\theta} \right] \right) \mu(dx) \right)^2 \]
\[
= 4m^2 \mathcal{D}^{2L} \left( \int_{[a, b]^n} \|N_r^{\ell, \theta}(x) - f(x)\| \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall w \in \mathbb{N}^n[k, L]: v_w \leq \ell_w} \mathbb{1}(v_k) \left[ \prod_{n=1}^{L} w_{v_n, v_{n-1}}^{\theta} \right] \right) \frac{1}{m} \int_{[a, b]^n} \|N_r^{\ell, \theta}(x) - f(x)\| \mu(dx) \right)^2 \]
\[
\leq 4m^2 \mathcal{D}^{2L} \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall w \in \mathbb{N}^n[k, L]: v_w \leq \ell_w} \mathbb{1}(v_k) \left[ \prod_{n=1}^{L} w_{v_n, v_{n-1}}^{\theta} \right] \right)^2 L_r(\theta). \]
\[
\left| \frac{\partial \mathcal{L}_r}{\partial \theta(i-1)_{k_{i-1}+j+d_{k_i-1}}} (\theta) \right|^2 \\
= \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall v \in \mathbb{N} [k, L], v_w \leq \ell_w} \int_{[a, b]^{\ell_0}} 2 \left[ \mathcal{R}^{1/(\max(k-1, 1))}_r (\mathcal{N}^{\max(k-1, 1)}_{r, j}) (x) \mathcal{P}_{1,L}(x) + x \mathcal{P}_{1,1}(x) \right] \right)^2 \\
\cdot \left[ \mathcal{L}_r^1 \right] ^2 \\
\leq 4 \mathcal{D}_r^2 \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall v \in \mathbb{N} [k, L], v_w \leq \ell_w} \left[ \mathcal{L}_r^1 \right] ^2 \right) \\
= 4 \mathcal{D}_r^2 \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall v \in \mathbb{N} [k, L], v_w \leq \ell_w} \left[ \mathcal{L}_r^1 \right] ^2 \right) \\
\cdot \frac{1}{m} \int_{[a, b]^{\ell_0}} \left( \mathcal{L}_r^1 \right) ^2 \mu(dx).
\]

Jensen’s inequality therefore proves that for all \( \theta \in K, r \in [1, \infty), k \in \{1, \ldots, L\}, i \in \{1, \ldots, \ell_k\}, j \in \{1, \ldots, \ell_j\} \), we have that
\[
\left| \frac{\partial \mathcal{L}_r}{\partial \theta(i-1)_{k_{i-1}+j+d_{k_i-1}}} (\theta) \right|^2 \\
\leq 4 \mathcal{D}_r^2 \mathcal{L}_r^1 \left( \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall v \in \mathbb{N} [k, L], v_w \leq \ell_w} \left[ \mathcal{L}_r^1 \right] ^2 \right) \\
\cdot \frac{1}{m} \int_{[a, b]^{\ell_0}} \left( \mathcal{L}_r^1 \right) ^2 \mu(dx).
\]

This and (3.25) assure for all \( \theta \in K, r \in [1, \infty) \) that
\[
\left\| (\nabla \mathcal{L}_r^1) (\theta) \right\|^2 \\
= \sum_{k=1}^{L} \sum_{i=1}^{\ell_k} \left| \nabla \mathcal{L}_r^1 (\theta) \right|^2 \\
\leq 4 \mathcal{D}_r^2 \left( \sum_{k=1}^{L} \sum_{i=1}^{\ell_k} \left[ \mathcal{L}_r^1 \right] ^2 \right) \\
\cdot \frac{1}{m} \int_{[a, b]^{\ell_0}} \left( \mathcal{L}_r^1 \right) ^2 \mu(dx).
\]

Furthermore, observe that (2.5) implies for all \( \theta \in K, r \in [1, \infty) \) that
\[
\mathcal{L}_r^1 (\theta) = \int_{[a, b]^{\ell_0}} \left\| \mathcal{N}^{L+\theta}_{r} (x) - f(x) \right\|^2 \mu(dx) \leq 2 \int_{[a, b]^{\ell_0}} \left[ \left\| \mathcal{N}^{L+\theta}_{r} (x) \right\|^2 + \left\| f(x) \right\|^2 \right] \mu(dx).
\]

This, Lemma 2.8, and (3.21) prove that \( \sup_{\theta \in K} \sup_{r \in [1, \infty) \mathcal{L}_r^1 (\theta) < \infty \). Combining this with (3.26) and item (iii) of Theorem 2.11 establishes (3.20). The proof of Lemma 3.6 is thus complete.
Proposition 3.7. Assume Setting 2.1 and let $T \in (0, \infty)$, $\Theta \in C([0,T], \mathbb{R}^p)$ satisfy for all $t \in [0,T]$ that $\Theta_t = \Theta_0 - \int_0^t \mathcal{G}(\Theta_s) \, ds$. Then it holds for all $t \in [0,T]$ that

$$
\mathcal{L}_\infty(\Theta_t) = \mathcal{L}_\infty(\Theta_0) - \int_0^t \|\mathcal{G}(\Theta_s)\|^2 \, ds.
$$

(3.28)

Proof of Proposition 3.7. Note that, e.g., Cheridito et al. [13, Lemma 3.1] (applied with $T \cap T$, $n \cap n$, $F \cap (\mathbb{R}^p \in \mathbb{R})$) in the notation of [13, Lemma 3.1] implies that for all $r \in [1, \infty)$, $t \in [0,T]$ we have that

$$
\mathcal{L}_r(\Theta_t) = \mathcal{L}_r(\Theta_0) - \int_0^t (\langle \nabla \mathcal{L}_r(\Theta_s) \rangle, \mathcal{G}(\Theta_s)) \, ds.
$$

(3.29)

Furthermore, observe that item (iv) of Theorem 2.9 ensures that for all $t \in [0,T]$ it holds that $\lim_{r \to \infty} (\mathcal{L}_r(\Theta_t) - \mathcal{L}_r(\Theta_0)) = \mathcal{L}_\infty(\Theta_t) - \mathcal{L}_\infty(\Theta_0)$ and $\lim_{r \to \infty} (\langle \nabla \mathcal{L}_r(\Theta_t) \rangle, \mathcal{G}(\Theta_t)) = \langle (\mathcal{G}(\Theta_t), \mathcal{G}(\Theta_t)) \rangle = \|\mathcal{G}(\Theta_t)\|^2$. Moreover, note that the assumption that $\Theta \in C([0,T], \mathbb{R}^p)$ assures that there exists a compact set $K \subseteq \mathbb{R}^p$ such that for all $t \in [0,T]$ it holds that $\Theta_t \in K$. This, the Cauchy-Schwarz inequality, Corollary 2.12, item (i) of Theorem 2.9, and Lemma 3.6 hence demonstrate that

$$
\sup_{r \in [1, \infty)} \sup_{t \in [0,T]} \|\nabla \mathcal{L}_r(\Theta_t)\| \leq \sup_{r \in [1, \infty)} \sup_{t \in K} \|\nabla \mathcal{L}_r(\Theta_t)\| < \infty.
$$

(3.30)

The dominated convergence theorem and item (iv) of Theorem 2.9 therefore show that for all $t \in [0,T]$ it holds that

$$
\lim_{r \to \infty} \left[ \int_0^t \langle \nabla \mathcal{L}_r(\Theta_s) \rangle, \mathcal{G}(\Theta_s) \rangle \, ds \right] = \int_0^t \left[ \lim_{r \to \infty} \langle \nabla \mathcal{L}_r(\Theta_s) \rangle, \mathcal{G}(\Theta_s) \rangle \right] \, ds = \int_0^t \|\mathcal{G}(\Theta_s)\|^2 \, ds.
$$

(3.31)

Combining this with (3.29) establishes (3.28). The proof of Proposition 3.7 is thus complete. 

\(\square\)

3.4 Convergence analysis for GF processes

Theorem 3.8. Assume Setting 2.1, assume for all $x \in [a,b]$ that $f(x) = f(0)$, and let $\Theta \in C([0, \infty), \mathbb{R}^p)$ satisfy for all $t \in [0, \infty)$ that $\Theta_t = \Theta_0 - \int_0^t \mathcal{G}(\Theta_s) \, ds$. Then

(i) it holds that $\sup_{t \in [0, \infty)} \|\Theta_t\| \leq 2V(\Theta_0) + 4L^2 \|f(0)\|^2 < \infty$,

(ii) it holds for all $t \in (0, \infty)$ that $\mathcal{L}_\infty(\Theta_t) \leq \frac{1}{2L} \|\Theta_0\|^2 + 2L^2 \|f(0)\|^2 < \infty$, and

(iii) it holds that $\lim_{t \to \infty} \mathcal{L}_\infty(\Theta_t) = 0$.

Proof of Theorem 3.8. Observe that item (ii) of Proposition 3.1 implies that for all $t \in [0, \infty)$ it holds that $\|\Theta_t\| \leq 2V(\Theta_t) + 4L^2 \|f(0)\|^2 \leq 2V(\Theta_0) + 4L^2 \|f(0)\|^2 < \infty$. Moreover, note that Corollary 3.5 and the fact that for all $\theta \in \mathbb{R}^p$ it holds that $\mathcal{L}_\infty(\theta) \geq 0$ prove that for all $t \in [0, \infty)$ we have that $V(\Theta_t) \leq V(\Theta_0)$. This establishes item (i). Next observe that Proposition 3.7 implies that $\Theta_t \in \mathbb{R}^p$, $t \to \mathcal{L}_\infty(\Theta_t) \in \mathbb{R}$ is non-increasing. Combining this with Corollary 3.5 and item (ii) of Proposition 3.1 demonstrates that for all $t \in [0, \infty)$ it holds that

$$
\mathcal{L}_\infty(\Theta_t) \leq \frac{1}{2L} \|\Theta_0\|^2 + 2L^2 \|f(0)\|^2.
$$

(3.32)

Hence, we obtain for all $t \in (0, \infty)$ that

$$
\mathcal{L}_\infty(\Theta_t) \leq \frac{1}{2L} \|\Theta_0\|^2 + 2L^2 \|f(0)\|^2.
$$

(3.33)

This establishes items (ii) and (iii). The proof of Theorem 3.8 is thus complete. 

\(\square\)
4 Gradient descent (GD) processes in the training of deep ANNs

In this section we use some of the results from Sections 2 and 3 above to establish in Theorem 4.7 in Subsection 4.3 below that in the training of deep ReLU ANNs we have that the sequence of risks $\mathcal{L}_\infty(\Theta_n)$, $n \in \mathbb{N}_0$, of any time-discrete GD process $\Theta = (\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \to \mathbb{R}^3$ converges to zero provided that the target function $f : [a, b]^k \to \mathbb{R}^k$ is a constant function and provided that the learning rates (the step sizes) $\gamma_n \in [0, \infty)$, $n \in \mathbb{N}_0$, in the GD optimization method are sufficiently small (see (4.18) below for details) but fail to be $L^1$-summable so that $\sum_{n=0}^{\infty} \gamma_n = \infty$. Our proof of Theorem 4.7 employs Lemma 2.10 in Subsection 4.1, Corollary 2.12 in Subsection 2.5 above, Proposition 3.1 in Subsection 3.1 above, as well as the recursive upper estimate for the composition $V(\Theta_n)$, $n \in \mathbb{N}_0$, of the Lyapunov function $V : \mathbb{R}^3 \to \mathbb{R}$ and the time-discrete GD process $\Theta = (\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \to \mathbb{R}^3$ in (4.14) in Lemma 4.6 in Subsection 4.2 below.

Our proof of the time-discrete Lyapunov estimate in (4.14) in Lemma 4.6 is based on induction and applications of the time-discrete Lyapunov estimate in Corollary 4.5 in Subsection 4.2. The time-discrete Lyapunov estimate in Corollary 4.5, in turn, is an immediate consequence of the time-discrete Lyapunov estimate in Corollary 4.4 in Subsection 4.2. While Theorem 4.7, Lemma 4.6, and Corollary 4.5 are restricted to the situation where the target function $f : [a, b]^k \to \mathbb{R}^k$ is a constant function, the Lyapunov estimate in Corollary 4.4 is applicable in the general case of a measurable target function $f : [a, b]^k \to \mathbb{R}^k$.

Our proof of Corollary 4.4 is based on an application of the one-step Lyapunov estimate for the GD method in Corollary 4.2 in Subsection 4.1. Our proof of Corollary 4.2, in turn, uses the one-step Lyapunov estimate for the GD method in Lemma 4.1 in Subsection 4.1. In Corollary 4.2 in Subsection 4.1 we also specialize the one-step Lyapunov estimate for the GD method in Lemma 4.1 to the situation where the target function $f : [a, b]^k \to \mathbb{R}^k$ is a constant function. Corollary 4.2 is employed in our convergence analysis of the SGD method in Section 5 below.

The results in this section extend the findings in [28, Section 2] from shallow ReLU ANNs with just one hidden layer to deep ReLU ANNs with an arbitrarily large number of hidden layers. In particular, Lemma 4.1 in Subsection 4.1 and Corollary 4.2 in Subsection 4.1 extend [28, Lemma 2.12], Corollary 4.3 in Subsection 4.1 extends [28, Corollary 2.13], Corollary 4.4 in Subsection 4.2, Corollary 4.5 in Subsection 4.2, and Lemma 6 in Subsection 4.2 extend [28, Corollary 2.14 and Lemma 2.15], and Theorem 4.7 in Subsection 4.3 extends [28, Theorem 2.16].

4.1 Lyapunov type estimates for the dynamics of GD processes

**Lemma 4.1.** Assume Setting 2.1 and let $\gamma \in \mathbb{R}, \theta \in \mathbb{R}^3$. Then

$$V(\theta - \gamma \mathcal{G}(\theta)) - V(\theta) = \gamma^2 \|\mathcal{G}(\theta)\|^2 + \gamma^2 \sum_{k=1}^{L} \sum_{i=1}^{t_k} (k - 1)! |\mathcal{G}_{\ell_k} \mathcal{G}_{\ell_{k-1}+1+...+\ell_{k-1+1}}(\theta)|^2 \nonumber$$

$$- 4\gamma L \left[ \int_{[a,b]} \langle \mathcal{N}_L^\theta(x) - f(x), \mathcal{N}_L^\theta(x) - f(0) \rangle \, \mu(dx) \right] \nonumber$$

$$\leq \gamma^2 \|\mathcal{G}(\theta)\|^2 - 4\gamma L \left[ \int_{[a,b]} \langle \mathcal{N}_L^\theta(x) - f(x), \mathcal{N}_L^\theta(x) - f(0) \rangle \, \mu(dx) \right]. \tag{4.1}$$

**Proof of Lemma 4.1.** Throughout this proof let $e_1, e_2, \ldots, e_2 \in \mathbb{R}^3$ satisfy $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $e_3 = (0, \ldots, 0, 1)$ and let $g : \mathbb{R} \to \mathbb{R}$ satisfy for all $t \in \mathbb{R}$ that

$$g(t) = V(\theta - t \mathcal{G}(\theta)). \tag{4.2}$$

Note that (4.2) and the fundamental theorem of calculus demonstrate that

$$V(\theta - \gamma \mathcal{G}(\theta)) = g(\gamma) = g(0) + \int_0^\gamma g'(t) \, dt = g(0) + \int_0^\gamma \left( \langle \nabla V(\theta - t \mathcal{G}(\theta)), -\mathcal{G}(\theta) \rangle \right) \, dt \nonumber$$

$$= V(\theta) - \int_0^\gamma \langle \nabla V(\theta - t \mathcal{G}(\theta)), \mathcal{G}(\theta) \rangle \, dt. \tag{4.3}$$
Proposition 3.2 therefore proves that

\[ V(\theta - \gamma G(\theta)) = V(\theta) - 4\gamma L \left[ \int_{[a,b]_0} \langle N^{L,\theta}_\infty(x) - f(x), N^{L,\theta}_\infty(x) - f(0) \rangle \, \mu(dx) \right] + \int_0^\gamma \left( \langle (\nabla V)(\theta) - (\nabla V)(\theta - tG(\theta)), G(\theta) \rangle \right) dt \]

Moreover, observe that item (iii) of Proposition 3.1 implies that for all \( t \in \mathbb{R} \) it holds that

\[ (\nabla V)(\theta) - (\nabla V)(\theta - tG(\theta)) = 2tG(\theta) + 2 \left[ \sum_{k=1}^L \sum_{i=1}^{\ell_k} (k - 1)b_i^{k,\gamma G(\theta)} e_{\ell_k,\ell_{k-1}+i+\sum_{h=1}^{k-1} \ell_h(\ell_h+1)} \right]. \]

Combining this with (4.4) shows that

\[ V(\theta - \gamma G(\theta)) = V(\theta) - 4\gamma L \left[ \int_{[a,b]_0} \langle N^{L,\theta}_\infty(x) - f(x), N^{L,\theta}_\infty(x) - f(0) \rangle \, \mu(dx) \right] + \int_0^\gamma \left( 2tG(\theta) + 2 \left[ \sum_{k=1}^L \sum_{i=1}^{\ell_k} (k - 1)b_i^{k,\gamma G(\theta)} e_{\ell_k,\ell_{k-1}+i+\sum_{h=1}^{k-1} \ell_h(\ell_h+1)} \right], G(\theta) \right) dt \]

Hence, we obtain that

\[ V(\theta - \gamma G(\theta)) = V(\theta) - 4\gamma L \left[ \int_{[a,b]_0} \langle N^{L,\theta}_\infty(x) - f(x), N^{L,\theta}_\infty(x) - f(0) \rangle \, \mu(dx) \right] + \gamma^2 \| G(\theta) \|^2 \]

The proof of Lemma 4.1 is thus complete.
Corollary 4.2. Assume Setting 2.1, assume for all \( x \in [a, b]^I_\nu \) that \( f(x) = f(0) \), and let \( \gamma \in \mathbb{R}, \theta \in \mathbb{R}^p \). Then
\[
V(\theta - \gamma G(\theta)) - V(\theta) = \gamma^2 \|G(\theta)\|^2 + 2 \sum_{k=1}^{L} \sum_{l=1}^{m_k} (k - 1) |G_x(t_k, t_{k-1}+\sum_{j=1}^{l-1} f_x(t_{k-1}+\sum_{j=1}^{l-1} (\nu_j))(\nu_j)|^2 - 4\gamma L \mathcal{L}_\infty(\theta)
\]
(4.8)

Proof of Corollary 4.2. Note that Lemma 4.1, (2.5), and the assumption that for all \( x \in [a, b]^I_\nu \) it holds that \( f(x) = f(0) \) establish (4.8). The proof of Corollary 4.2 is thus complete.

Corollary 4.3. Assume Setting 2.1 and let \( \gamma \in [0, \infty), \theta \in \mathbb{R}^p \). Then
\[
V(\theta - \gamma G(\theta)) - V(\theta) \leq 4\gamma^2 mL^2 a^2 [\prod_{p=0}^{L} (\ell_p + 1)] (2V(\theta)) + 4L^2 \|f(0)\|^2 + 1)^{L-1} \mathcal{L}_\infty(\theta)
\]
(4.9)

Proof of Corollary 4.3. Observe that item (iv) of Theorem 2.11 and item (ii) of Proposition 3.1 demonstrate that
\[
\|G(\theta)\|^2 \leq 4mLa^2 [\prod_{p=0}^{L} (\ell_p + 1)] (\|\theta\|^2 + 1)^{L-1} \mathcal{L}_\infty(\theta)
\]
(4.10)

Combining this with Lemma 4.1 establishes (4.9). The proof of Corollary 4.3 is thus complete.

4.2 Upper estimates for compositions of Lyapunov functions and GD processes

Corollary 4.4. Assume Setting 2.1, let \( (\gamma_n)_{n \in \mathbb{N} \subseteq [0, \infty) \}, let (\Theta_n)_{n \in \mathbb{N} : \mathbb{N} \to \mathbb{R}^p \) satisfy for all \( n \in \mathbb{N} \) that \( \Theta_{n+1} = \Theta_n - \gamma_n \mathcal{G}(\Theta_n) \), and let \( n \in \mathbb{N} \). Then
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq 4(\gamma_n)^2 mL^2 a^2 [\prod_{p=0}^{L} (\ell_p + 1)] (2V(\Theta_n)) + 4L^2 \|f(0)\|^2 + 1)^{L-1} \mathcal{L}_\infty(\Theta_n)
\]
(4.11)

Proof of Corollary 4.4. Note that Corollary 4.3 establishes (4.11). The proof of Corollary 4.4 is thus complete.

Corollary 4.5. Assume Setting 2.1, assume for all \( x \in [a, b]^d \) that \( f(x) = f(0) \), let \( (\gamma_n)_{n \in \mathbb{N} \subseteq [0, \infty) \), let \( (\Theta_n)_{n \in \mathbb{N} : \mathbb{N} \to \mathbb{R}^p \) satisfy for all \( n \in \mathbb{N} \) that \( \Theta_{n+1} = \Theta_n - \gamma_n \mathcal{G}(\Theta_n) \), and let \( n \in \mathbb{N} \). Then
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq 4L((\gamma_n)^2 mL^2 a^2 [\prod_{p=0}^{L} (\ell_p + 1)] (2V(\Theta_n)) + 4L^2 \|f(0)\|^2 + 1)^{L-1} \gamma_n \mathcal{L}_\infty(\Theta_n).
\]
(4.12)

Proof of Corollary 4.5. Observe that Corollary 4.4 and the assumption that for all \( x \in [a, b]^I_\nu \) it holds that \( f(x) = f(0) \) establish (4.12). The proof of Corollary 4.5 is thus complete.

Lemma 4.6. Assume Setting 2.1, assume for all \( x \in [a, b]^d \) that \( f(x) = f(0) \), let \( (\gamma_n)_{n \in \mathbb{N} \subseteq [0, \infty) \), let \( (\Theta_n)_{n \in \mathbb{N} : \mathbb{N} \to \mathbb{R}^p \) satisfy for all \( n \in \mathbb{N} \) that \( \Theta_{n+1} = \Theta_n - \gamma_n \mathcal{G}(\Theta_n) \), and assume
\[
\sup_{n \in \mathbb{N}} (\gamma_n m) \leq (La^2 [\prod_{p=0}^{L} (\ell_p + 1)] (2V(\Theta_n)) + 4L^2 \|f(0)\|^2 + 1)^{L-1}^{-1}.
\]
(4.13)

Then it holds for all \( n \in \mathbb{N} \) that
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq -4L(\gamma_n (1 - \sup_{m \in \mathbb{N}} (\gamma_m)) mL^2 a^2 [\prod_{p=0}^{L} (\ell_p + 1)] (2V(\Theta_n)) + 4L^2 \|f(0)\|^2 + 1)^{L-1} \mathcal{L}_\infty(\Theta_n) \leq 0.
\]
(4.14)
Proof of Lemma 4.6. Throughout this proof let \( g \in \mathbb{R} \) satisfy \( g = \sup_{n \in \mathbb{N}_0}(\gamma_n m) \). We prove (4.14) by induction on \( n \in \mathbb{N}_0 \). Note that Corollary 4.5, (4.13), and the fact that \( \gamma_n m \leq g \) imply that

\[
V(\Theta_1) - V(\Theta_0) \leq 4L(\gamma_0 + \gamma_0 L \mathbb{E}[\ell_p + 1)] (2V(\Theta_0) + 4L^2\|f(0)\|^2 + 1)^{(L-1)}L_\infty(\Theta_0) \]

This establishes (4.14) in the base case \( n = 0 \). For the induction step let \( n \in \mathbb{N} \) satisfy for all \( m \in \{0,1,\ldots,n-1\} \) that

\[
V(\Theta_{m+1}) - V(\Theta_m) \leq -4L\gamma_m (1 - gLm^2[\prod_{p=0}^{L} (\ell_p + 1)] (2V(\Theta_0) + 4L^2\|f(0)\|^2 + 1)^{(L-1)}L_\infty(\Theta_m). \]

Observe that (4.16) and the fact that for all \( \theta \in \mathbb{R}^d \) it holds that \( L_\infty(\theta) \geq 0 \) ensure that \( V(\Theta_n) \leq V(\Theta_{n-1}) \leq \cdots \leq V(\Theta_0) \). Combining this with Corollary 4.5, (4.13), and the fact that \( \gamma_n m \leq g \) demonstrates that

\[
V(\Theta_{n+1}) - V(\Theta_n) \leq 4L(\gamma_n + \gamma_n L^2m^2[\prod_{p=0}^{L} (\ell_p + 1)] (2V(\Theta_0) + 4L^2\|f(0)\|^2 + 1)^{(L-1)}L_\infty(\Theta_n) \]

This establishes (4.14). The proof of Lemma 4.6 is thus complete.

\[ \square \]

4.3 Convergence analysis for GD processes

Theorem 4.7. Assume Setting 2.1, assume for all \( x \in [a,b]^d \) that \( f(x) = f(0) \), let \( (\gamma_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \to \mathbb{R}^d \) satisfy for all \( n \in \mathbb{N}_0 \) that \( \Theta_{n+1} = \Theta_n - \gamma_n \mathcal{G}(\Theta_n) \), and assume

\[
\sup_{n \in \mathbb{N}_0}(\gamma_n m) < (Lm^2[\prod_{p=0}^{L} \ell_p + 1)] (2V(\Theta_0) + 4L^2\|f(0)\|^2 + 1)^{(L-1)} \]

and \( \sum_{n=0}^\infty \gamma_n = \infty \). Then

(i) it holds that \( \sup_{n \in \mathbb{N}_0} \|\Theta_n\| < \|2V(\Theta_0) + 4L^2\|f(0)\|^2\|^{1/2} < \infty \) and

(ii) it holds that \( \limsup_{n \to \infty} L_\infty(\Theta_n) = 0 \).

Proof of Theorem 4.7. Throughout this proof let \( \eta \in (0,\infty) \) satisfy

\[
\eta = 4L(1 - \sup_{m \in \mathbb{N}_0}(\gamma_n m) m^2[\prod_{p=0}^{L} (\ell_p + 1)] (2V(\Theta_0) + 4L^2\|f(0)\|^2 + 1)^{(L-1)} \]

and let \( \varepsilon \in \mathbb{R} \) satisfy \( \varepsilon = (\eta/3)\min\{1, \limsup_{n \to \infty} L_\infty(\Theta_n)\} \). Note that item (ii) of Proposition 3.1 ensures that for all \( n \in \mathbb{N}_0 \) it holds that

\[
\|\Theta_n\| < \|2V(\Theta_0) + 4L^2\|f(0)\|^2\|^{1/2}. \]

Furthermore, observe that Lemma 4.6 implies that for all \( n \in \mathbb{N}_0 \) it holds that \( V(\Theta_n) \leq V(\Theta_{n-1}) \leq \cdots \leq V(\Theta_0) \). This and (4.20) establish item (i). Next note that item (ii) of Proposition 3.1 ensures that for all \( n \in \mathbb{N} \) we have that \( V(\Theta_n) \geq 4L\|\Theta_n\|^2 - 2L^2\|f(0)\|^2 \geq -2L^2\|f(0)\|^2 \). Combining this with Lemma 4.6 and (4.19) implies that for all \( N \in \mathbb{N} \) it holds that

\[
\eta \left[ \sum_{n=0}^{N-1} \gamma_n L_\infty(\Theta_n) \right] \leq \sum_{n=0}^{N-1} (V(\Theta_n) - V(\Theta_{n+1})) = V(\Theta_0) - V(\Theta_N) \leq V(\Theta_0) + 2L^2\|f(0)\|^2. \]
This demonstrates that
\[
\sum_{n=0}^{\infty} [\gamma_n \mathcal{L}_\infty(\Theta_n)] \leq \eta^{-1}(V(\Theta_0) + 2L^2||f(0)||^2) < \infty. \tag{4.22}
\]
Combining this with the assumption that \(\sum_{n=0}^{\infty} \gamma_n = \infty\) ensures that \(\liminf_{n \to \infty} \mathcal{L}_\infty(\Theta_n) = 0\). In the following we prove item (ii) by contradiction. For this assume that
\[
\limsup_{n \to \infty} \mathcal{L}_\infty(\Theta_n) > 0. \tag{4.23}
\]
Observe that (4.23) implies that
\[
0 = \liminf_{n \to \infty} \mathcal{L}_\infty(\Theta_n) < \varepsilon < 2\varepsilon < \limsup_{n \to \infty} \mathcal{L}_\infty(\Theta_n). \tag{4.24}
\]
This ensures that there exist \((m_k, n_k) \in \mathbb{N}^2, k \in \mathbb{N}\), which satisfy for all \(k \in \mathbb{N}\) that \(m_k < n_k < m_{k+1}\), \(\mathcal{L}_\infty(\Theta_{n_k}) > 2\varepsilon\), and \(\mathcal{L}_\infty(\Theta_{n_k}) < \varepsilon < \min_{j \in \mathbb{N} \cap [m_k, n_k]} \mathcal{L}_\infty(\Theta_j)\). Note that (4.22) and the fact that for all \(k \in \mathbb{N}, j \in \mathbb{N} \cap [m_k, n_k]\) it holds that \(1 \leq \frac{1}{\varepsilon} \mathcal{L}_\infty(\Theta_j)\) demonstrate that
\[
\sum_{k=0}^{\infty} \sum_{j=m_k}^{n_k-1} \gamma_j \leq \frac{1}{\varepsilon} \left[ \sum_{j=m_k}^{n_k-1} (\gamma_j \mathcal{L}_\infty(\Theta_j)) \right] \leq \frac{1}{\varepsilon} \left[ \sum_{j=0}^{\infty} (\gamma_j \mathcal{L}_\infty(\Theta_j)) \right] < \infty. \tag{4.25}
\]
Moreover, observe that Corollary 2.12 and item (i) imply that there exists \(\mathcal{C} \in \mathbb{R}\) which satisfies that
\[
\sup_{n \in \mathbb{N}_0} \|\mathcal{G}(\Theta_n)\| \leq \mathcal{C}. \tag{4.26}
\]
Note that (4.26), the triangle inequality, and (4.25) prove that
\[
\sum_{k=1}^{\infty} \|\Theta_{n_k} - \Theta_{m_k}\| \leq \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} \|\mathcal{G}(\Theta_j)\| \leq \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} \gamma_j < \infty. \tag{4.27}
\]
Next observe that Lemma 2.10 and item (i) ensure that there exists \(\mathcal{L} \in \mathbb{R}\) which satisfies for all \(m, n \in \mathbb{N}_0\) that \(|\mathcal{L}(\Theta_m) - \mathcal{L}(\Theta_n)| \leq \mathcal{L} \|\Theta_m - \Theta_n\|\). Combining this with (4.27) shows that
\[
\limsup_{k \to \infty} |\mathcal{L}_\infty(\Theta_{n_k}) - \mathcal{L}_\infty(\Theta_{m_k})| \leq \limsup_{k \to \infty} (\mathcal{L} \|\Theta_{n_k} - \Theta_{m_k}\|) = 0. \tag{4.28}
\]
The fact that for all \(k \in \mathbb{N}_0\) it holds that \(\mathcal{L}_\infty(\Theta_{n_k}) < \varepsilon < 2\varepsilon < \mathcal{L}_\infty(\Theta_{m_k})\) therefore implies that
\[
0 < \varepsilon \leq \inf_{k \in \mathbb{N}} |\mathcal{L}_\infty(\Theta_{n_k}) - \mathcal{L}_\infty(\Theta_{m_k})| \leq \limsup_{k \to \infty} |\mathcal{L}_\infty(\Theta_{n_k}) - \mathcal{L}_\infty(\Theta_{m_k})| = 0. \tag{4.29}
\]
This contradiction establishes item (ii). The proof of Theorem 4.7 is thus complete. \(\blacksquare\)

5 Stochastic gradient descent (SGD) processes in the training of deep ANNs

In this section we use some of the results from Sections 2, 3, and 4 above to establish in Theorem 5.11 in Subsection 5.7 below that in the training of deep ReLU ANNs we have that the sequence of risks \(\mathcal{L}(\Theta_n)\), \(n \in \mathbb{N}_0\), of any time-discrete SGD process \(\Theta = (\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \times \Omega \to \mathbb{R}^d\) converges to zero provided that the target function \(f : [a, b]^d \to \mathbb{R}^d\) is a constant function and provided that the learning rates (the step sizes) \(\gamma_n \in [0, \infty), n \in \mathbb{N}_0\), in the GD optimization method are sufficiently small (see (5.31) in Theorem 5.11 for details but fail to be \(L^1\)-summable so that \(\sum_{n=0}^{\infty} \gamma_n = \infty\).

In Corollary 5.12 in Subsection 5.7 we simplify the statement of Theorem 5.11 by imposing a more restrictive smallness condition on the learning rates \((\gamma_n)_{n \in \mathbb{N}_0} \subseteq [0, \infty)\) (compare the smallness assumption on the learning rates in (5.50) in Corollary 5.12 with the smallness assumption on the learning rates in (5.31) in Theorem 5.11 for details). Theorem 1.1 in the introduction is a direct consequence of Corollary 5.12. In our proof of Corollary 5.12 we employ the elementary upper estimate for the Lyapunov
function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ (see (5.5) in Setting 5.1 in Subsection 5.1 below) in Proposition 3.1 in Subsection 3.1 above to verify that the smallness assumption in (5.31) in Theorem 5.11 is satisfied so that Theorem 5.11 can be applied.

Our proof of Theorem 5.11 employs Proposition 3.1 in Subsection 3.1, the well-known integrability result in Corollary 5.5 in Subsection 5.3 below, the uniform local boundedness result for the generalized gradients $\mathcal{G}^n = (\mathcal{G}^n_1, \ldots, \mathcal{G}^n_d) : \mathbb{R}^d \rightarrow \mathbb{R}$, $n \in \mathbb{N}_0$, (see (5.6) in Setting 5.1) of the empirical risk functions $\mathcal{L}_n^\infty : \mathbb{R}^d \rightarrow \mathbb{R}$, $n \in \mathbb{N}_0$, in Lemma 5.7 in Subsection 5.4 below, the local Lipschitz continuity result for the risk function $\mathcal{L} : \mathbb{R}^d \rightarrow \mathbb{R}$ in Lemma 2.10 in Subsection 2.4 above, and the probabilistic recursive upper estimate for the composition $V(\Theta_n)$, $n \in \mathbb{N}_0$, of the Lyapunov function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ and the time-discrete SGD process $\Theta = (\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \times \Omega \rightarrow \mathbb{R}^d$ in Corollary 5.10 in Subsection 5.6 below.

Our proof of Lemma 5.7 also uses the local Lipschitz continuity result for the risk function in Lemma 2.10 in Subsection 2.4 as well as the explicit pathwise polynomial growth estimates for the generalized gradients $\mathcal{G}^n = (\mathcal{G}^n_1, \ldots, \mathcal{G}^n_d) : \mathbb{R}^d \rightarrow \mathbb{R}$, $n \in \mathbb{N}_0$, of the empirical risk functions in Lemma 5.6 in Subsection 5.4. Lemma 5.6, in turn, is a direct consequence of the explicit polynomial growth estimate for the generalized gradient function $\mathcal{G} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ in Theorem 2.11 in Subsection 2.5 above.

Our proof of Corollary 5.10 is based on induction as well as on the pathwise recursive upper estimate for the composition $V(\Theta_n)$, $n \in \mathbb{N}_0$, of the Lyapunov function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ and the time-discrete SGD process $\Theta = (\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \times \Omega \rightarrow \mathbb{R}^d$ in Lemma 5.9 in Subsection 5.6 below. Our proof of Lemma 5.9 employs Proposition 3.1 in Subsection 3.1, the explicit pathwise polynomial growth estimates for the generalized gradients $\mathcal{G}^n = (\mathcal{G}^n_1, \ldots, \mathcal{G}^n_d) : \mathbb{R}^d \rightarrow \mathbb{R}$, $n \in \mathbb{N}_0$, of the empirical risk functions in Lemma 5.6, and the one-step Lyapunov estimate for the SGD method in Lemma 5.8 in Subsection 5.5 below. Lemma 5.8 is a direct consequence of the one-step Lyapunov estimate for the GD method in Corollary 4.2 in Subsection 4.1 above.

In Corollary 5.5 we demonstrate that for every time point $n \in \mathbb{N}_0$ we have that the expectation $E[\mathcal{L}_n^\infty(\Theta_n)]$ of the empirical risk of the SGD process at time $n$ coincides with the expectation $E[\mathcal{L}(\Theta_n)]$ of the risk of the SGD process at time $n$. Our proof of Corollary 5.5 employs the well-known integrability result in Proposition 5.3 in Subsection 5.3 and the well-known measurability result in Lemma 5.4 in Subsection 5.3. In Proposition 5.3 we assert that the expectations $E[\mathcal{L}_n^\infty(\Theta)]$, $\Theta \in \mathbb{R}^d$, $n \in \mathbb{N}$, of the empirical risk functions $\mathcal{L}_n^\infty : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, coincide with the risk function $\mathbb{R}^d \ni \theta \mapsto \mathcal{L}(\theta) \in \mathbb{R}$.

In Lemma 5.4 we collect well-known measurability and independence properties for the input data, the SGD process, and the generalized gradients of the empirical risk functions. Our proof of Lemma 5.4 makes use of the local Lipschitz continuity result for the risk function in Lemma 2.10 in Subsection 2.4 and of the explicit representation result for the generalized gradients of (approximations of) the empirical risk functions in Proposition 5.2 in Subsection 5.2 below. Proposition 5.2 is a direct consequence of the explicit representation for the generalized gradient function of the risk function in Theorem 2.9 in Subsection 2.3 above.

The findings in this section extend the findings in [28, Section 4] from shallow ReLU ANNs with just one hidden layer to deep ReLU ANNs with an arbitrarily large number of hidden layers. In particular, Proposition 5.2 in Subsection 5.2 extends [28, Proposition 3.2], Proposition 5.3 in Subsection 5.3 generalizes [28, Proposition 3.3], Lemma 5.4 in Subsection 5.3 extends [28, Lemma 3.4], Corollary 5.5 in Subsection 5.3 extends [28, Corollary 5.5], Lemma 5.6 in Subsection 5.4 extends [28, Lemma 3.6], Lemma 5.7 in Subsection 5.4 extends [28, Lemma 3.7], Lemma 5.8 in Subsection 5.5 extends [28, Lemma 3.9], Lemma 5.9 in Subsection 5.6 extends [28, Lemma 3.10], Corollary 5.10 in Subsection 5.6 extends [28, Corollary 3.11], Theorem 5.11 in Subsection 5.7 extends [28, Theorem 3.12], and Corollary 5.12 in Subsection 5.7 extends [28, Corollary 3.13].

### 5.1 Mathematical framework for SGD processes and deep ANNs with ReLU activation

**Setting 5.1.** Let $L, d \in \mathbb{N}$, $(\ell_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}$, $\xi \in \mathbb{R}^{d \times \ell_k}$, $a, b \in \mathbb{R}$, $b \in (a, \infty)$, $\mathcal{A} \in (0, \infty)$, $\mathcal{B} \in (\mathcal{A}, \infty)$ satisfy $\mathcal{A} = \sum_{k=1}^L \ell_k(\ell_k-1)$ and $a = \max\{|a|, |b|, 1\}$, for every $\theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{R}^\ell$ let $w^{k,\theta} = (w_{i,j}^{k,\theta})_{i,j \in \{1, \ldots, \ell_k\} \times \{1, \ldots, \ell_k-1\}} \in \mathbb{R}^{d \times \ell_k-1}$, $k \in \mathbb{N}$, and $b^{k,\theta} = (b_{i,j}^{k,\theta})_{i,j \in \{1, \ldots, \ell_k\} \times \{1, \ldots, \ell_k-1\}} \in \mathbb{R}^{d\times\ell_k}$, $k \in \mathbb{N}$, satisfy for all...
\[ k \in \{1, \ldots, L\}, i \in \{1, \ldots, \ell_k\}, j \in \{1, \ldots, \ell_{k-1}\} \] that
\[ w_{i,j}^{k,\theta} = \theta_{(i-1)\ell_{k-1} + j + \sum_{k'=1}^{k-1} \ell_{k'}(\ell_{k'-1} + 1)} \quad \text{and} \quad v_{i,j}^{k,\theta} = \theta_{(i-1)\ell_{k-2} + j + \sum_{k'=1}^{k-2} \ell_{k'}(\ell_{k'-1} + 1)}, \quad (5.1) \]
for every \( k \in \mathbb{N}, \theta \in \mathbb{R}^n \) let \( A_r^k = (A_r^k, \ldots, A_r^k_k) : \mathbb{R}^{\ell_k-1} \to \mathbb{R}^{\ell_k} \) satisfy for all \( x \in \mathbb{R}^{\ell_k-1} \) that \( A_r^k(x) = b^{k,\theta} + w^{k,\theta} x \), let \( \mathcal{A}_r : \mathbb{R} \to \mathbb{R} \), \( r \in [1, \infty] \), satisfy for all \( r \in [1, \infty) \), \( x \in (-\infty, \mathcal{A}^{-1}_r) \), \( y \in \mathbb{R}, z \in [\mathcal{A}^{-1}_r, \infty) \) that
\[ \mathcal{A}_r \in C^1(\mathbb{R}, \mathbb{R}), \quad \mathcal{A}_r(x) = 0, \quad 0 \leq \mathcal{A}_r(y) \leq \mathcal{A}_\infty(y) = \max\{y, 0\}, \quad \text{and} \quad \mathcal{A}_r(z) = z, \quad (5.2) \]
assume \( \sup_{r \in [1, \infty)} \sup_{x \in \mathbb{R}} \| \mathcal{A}_r'(x) \| < \infty \), let \( \| \cdot \| : (\cup_{n \in \mathbb{N}} \mathbb{R}^n) \to \mathbb{R}, \langle \cdot, \cdot \rangle : (\bigcup_{n \in \mathbb{N}} (\mathbb{R}^n \times \mathbb{R}^n)) \to \mathbb{R} \), and \( \mathcal{M}_r : (\cup_{n \in \mathbb{N}} \mathbb{R}^n) \to (\bigcup_{n \in \mathbb{N}} \mathbb{R}^n) \), \( r \in [1, \infty], n \in \mathbb{N}, x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) that
\[ \| x \| = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2}, \quad \langle x, y \rangle = \sum_{i=1}^n x_i y_i, \quad \text{and} \quad \mathcal{M}_r(x) = (\mathcal{A}_r(x_1), \ldots, \mathcal{A}_r(x_n)), \quad (5.3) \]
for every \( \theta \in \mathbb{R}^n \) let \( N_{r,\theta}^{k,\theta} = (N_{r,\theta}^{k,\theta}, \ldots, N_{r,\theta}^{k,\theta}) : \mathbb{R}^k \to \mathbb{R}^r \), \( r \in [1, \infty], k \in \mathbb{N}, \) and \( \Lambda_{r,\theta}^{k,\theta} \subseteq \mathbb{R}^k, k, i \in \mathbb{N} \), satisfy for all \( r \in [1, \infty) \), \( k \in \mathbb{N}, n \in \mathbb{N}, x = (x_1, \ldots, x_k) \), \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) that
\[ N_{r,\theta}^{k,\theta}(x) = \Lambda_{r}^{k}(x), \quad N_{r,\theta}^{k+1,\theta}(x) = \Lambda_{r+1}^{k+1}(\mathcal{M}_{r+1}^{-1}(N_{r,\theta}^{k,\theta}(x))), \quad (5.4) \]
and \( \Lambda_{r,\theta}^{k,\theta} = \{ y \in [a, b] : \Lambda_{r,\theta}^{k,\theta}(y) > 0 \} \), let \( (\Omega, F, \mathbb{P}) \) be a probability space, \( X_{n,m} = (X_{n,m}^1, \ldots, X_{n,m}^L) : \Omega \to [a, b]^L \), \( n, m \in \mathbb{N}_0 \) be i.i.d. random variables, let \( \mathcal{L} : \mathbb{R}^L \to \mathbb{R} \) and \( V : \mathbb{R}^L \to \mathbb{R} \) satisfy for all \( \theta \in \mathbb{R}^n \) that \( \mathcal{L}(\theta) = \mathbb{E}[\| \mathcal{L}_{\theta}^{\mathbb{R}^n}(X_{n,m}(\theta)) - \| \|^2] \) and
\[ \mathcal{L}(\theta) = \left[ \sum_{k=1}^L (\| b^{k,\theta} \|^2 + \sum_{i=1}^{\ell_k} \sum_{j=1}^{\ell_{k-1}} |w_{i,j}^{k,\theta}|^2 ) \right] - 2L(\xi, b^{L,\theta}), \quad (5.5) \]
let \( (M_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{N} \), for every \( n \in \mathbb{N}_0 \), \( r \in [1, \infty) \) let \( \mathcal{L}_r^n : \mathbb{R}^n \times \Omega \to \mathbb{R} \) satisfy for all \( \theta \in \mathbb{R}^n \), \( \omega \in \Omega \) that \( \mathcal{L}_r^n(\theta, \omega) = \frac{1}{\ell_k} \sum_{k=1}^L \mathcal{L}_{\theta}^{\mathbb{R}^n}(X_{n,m}(\theta)) - \| \|^2 \), for every \( n \in \mathbb{N}_0 \) let \( \mathcal{G}^n = (\mathcal{G}_1^n, \ldots, \mathcal{G}_L^n) : \mathbb{R}^L \times \Omega \to \mathbb{R} \) satisfy for all \( \theta \in \mathbb{R}^n \), \( \omega \in \Omega \) that \( \mathcal{G}^n(\theta, \omega) = \lim_{r \to \infty} \langle \mathcal{L}_r^n(\theta, \omega) \rangle_{r \in [1, \infty)} \) is convergent that
\[ \mathcal{G}^n(\theta, \omega) = \lim_{r \to \infty} \langle \mathcal{L}_r^n(\theta, \omega) \rangle_{r \in [1, \infty)} \quad (5.6) \]
let \( \Theta = (\Theta_n)_{n \in \mathbb{N}_0} : \mathbb{N}_0 \times \Omega \to \mathbb{R} \) be a stochastic process, \( \gamma_n = (\gamma_n)_{n \in \mathbb{N}_0} \subseteq [0, \infty) \), assume that \( \Theta_n \) and \( (X_{n,m})_{n,m \in \mathbb{N}_0} \) are independent, and assume for all \( n \in \mathbb{N}_0 \), \( \omega \in \Omega \) that \( \Theta_{n+1}(\omega) = \Theta_n(\omega) - \gamma_n \mathcal{G}^n(\Theta_n(\omega), \omega), \)

### 5.2 Explicit representations for the generalized gradients of the empirical risk function

**Proposition 5.2. Assume Setting 5.1 and let \( n \in \mathbb{N}_0 \), \( \omega \in \Omega \). Then**

(i) it holds for all \( r \in [1, \infty) \) that \( \mathbb{R}^n \ni \theta \mapsto \mathcal{L}_r^n(\theta, \omega) \in \mathbb{R} \) \( \in C^1(\mathbb{R}^n, \mathbb{R}) \),

(ii) it holds for all \( r \in [1, \infty), k \in \{1, \ldots, L\}, i \in \{1, \ldots, \ell_k\}, j \in \{1, \ldots, \ell_{k-1}\}, \theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^n \) that
\[ \left( \frac{\partial}{\partial \theta_{(i-1)\ell_{k-1} + j + \sum_{k'=1}^{k-1} \ell_{k'}(\ell_{k'-1} + 1)}} \mathcal{L}_r^n(\theta, \omega) \right)(\theta, \omega) \]
\[ = \sum_{v_k, v_{k+1}, \ldots, v_L \in \mathbb{N}, \forall w \in \mathbb{N}[k, L] : v_\leq w} \left[ \mathcal{L}_{r+1}^{\mathbb{R}^n}(X_{n,m}(\theta)) \right] \prod_{q=k+1}^L \left( \mathcal{L}_{r+q-1}^{\mathbb{R}^n}(X_{n,m}(\theta)) \right) \left( \mathcal{G}_q^{n,\theta} \right) \left( v_k, v_{k+1}, \ldots, v_L \right) \left( v_\leq w \right) \]
(iii) it holds for all $r \in [1, \infty)$, $k \in \{1, \ldots, L\}$, $i \in \{1, \ldots, \ell_k\}$, $\theta = (\theta_1, \ldots, \theta_3) \in \mathbb{R}^3$ that
\[
\left(\partial_{\theta_{\ell_k \ell_{k-1}+i+\sum_{k=1}^{k-1} \ell_k \ell_{k-1}+1}} \mathcal{G}^n\right)(\theta, \omega) = \\
\sum_{v_k, v_{k+1}, \ldots, v_L \in N_k, \forall \omega \in N_{[k, L]} : \nu_{\omega} \leq \ell_k} \frac{2}{M_3} \sum_{m=1}^{M_2} \bigg[ \mathbb{I}(i)(v_k) \bigg[ \mathcal{N}_{r, v_k}^{\ell_r, \omega} (X^{n, \cdot m}(\omega)) - \xi_{v_k} \bigg] \\
\cdot \left[ \prod_{q=k+1}^{L} \left( \mathcal{W}_{q, \omega}^{\theta, \gamma} \mathcal{R}_{q+1}^{\gamma, \omega} (X^{n, \cdot m}(\omega)) \right) \right] \bigg],
\]
(iv) it holds for all $\theta = (\theta_1, \ldots, \theta_3) \in \mathbb{R}^3$ that $\limsup_{r \to \infty} \| (\nabla \mathcal{G}^n)(\theta, \omega) - \mathcal{G}^n(\theta, \omega) \| = 0$, 
(v) it holds for all $k \in \{1, \ldots, L\}$, $i \in \{1, \ldots, \ell_k\}$, $j \in \{1, \ldots, \ell_{k-1}\}$, $\theta = (\theta_1, \ldots, \theta_3) \in \mathbb{R}^3$ that
\[
\mathcal{G}^n(\ell_k \ell_{k-1}+i+\sum_{k=1}^{k-1} \ell_k \ell_{k-1}+1)(\theta, \omega) = \\
\sum_{v_k, v_{k+1}, \ldots, v_L \in N_k, \forall \omega \in N_{[k, L]} : \nu_{\omega} \leq \ell_k} \frac{2}{M_3} \sum_{m=1}^{M_2} \bigg[ \mathbb{I}(i)(v_k) \bigg[ \mathcal{N}_{\mathcal{R}_{q+1}^{\gamma, \omega}}^{\ell_r, \omega} (X^{n, \cdot m}(\omega)) - \xi_{v_k} \bigg] \\
\cdot \left[ \prod_{q=k+1}^{L} \left( \mathcal{W}_{q, \omega}^{\theta, \gamma} \mathcal{R}_{q+1}^{\gamma, \omega} (X^{n, \cdot m}(\omega)) \right) \right] \bigg].
\]

Proof of Proposition 5.2. Note that items (i), (ii), (iii), (iv), (v), and (vi) of Theorem 2.9 (applied with $\mu \cap \{B([a, b], \theta) \cap A \to \sum_{m=1}^{M_a} \mathbb{I}(X^{n, \cdot m}(\omega)) \in [0, 1]\}$, $f \cap \{[a, b] \cap A(x) \in \mathbb{R}^{\mathbb{C}}\}$ in the notation of Theorem 2.9) prove items (i), (ii), (iii), (iv), (v), and (vi). The proof of Proposition 5.2 is thus complete. 

5.3 Properties of expectations of the empirical risk functions

Proposition 5.3. Assume Setting 5.1. Then it holds for all $n \in \mathbb{N}_0$, $\theta \in \mathbb{R}^3$ that
\[
\mathbb{E}[\mathcal{G}^n(\theta)] = \mathcal{L}(\theta).
\]

Proof of Proposition 5.3. Observe that the fact that for all $n \in \mathbb{N}_0$, $\theta \in \mathbb{R}^3$, $\omega \in \Omega$ it holds that $\mathcal{G}^n(\theta, \omega) = \frac{1}{M_3} \sum_{m=1}^{M_3} \| \mathcal{N}_{\mathcal{R}_{q+1}^{\gamma, \omega}}^{\ell_r, \omega} (X^{n, \cdot m}(\omega)) - \xi \| ^2$, the fact that for all $n \in \mathbb{N}_0$, $\theta \in \mathbb{R}^3$ it holds that $\mathcal{L}(\theta) = \mathbb{E}(\| \mathcal{N}_{\mathcal{R}_{q+1}^{\gamma, \omega}}^{\ell_r, \omega} (X^{0, \cdot m}) - \xi \| ^2)$, and the assumption that $X^{n, \cdot m}: \Omega \to [a, b]^{\mathbb{C}}$, $n, m \in \mathbb{N}_0$, are i.i.d. random variables imply that for all $n \in \mathbb{N}_0$, $\theta \in \mathbb{R}^3$ we have that
\[
\mathbb{E}[\mathcal{G}^n(\theta)] = \frac{1}{M_3} \sum_{m=1}^{M_3} \mathbb{E}(\| \mathcal{N}_{\mathcal{R}_{q+1}^{\gamma, \omega}}^{\ell_r, \omega} (X^{n, \cdot m}) - \xi \| ^2) = \mathbb{E}(\| \mathcal{N}_{\mathcal{R}_{q+1}^{\gamma, \omega}}^{\ell_r, \omega} (X^{0, \cdot m}) - \xi \| ^2) = \mathcal{L}(\theta).
\]

The proof of Proposition 5.3 is thus complete. 

Lemma 5.4. Assume Setting 5.1 and let $\mathbb{P}_n \subseteq \mathcal{F}$, $n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}$ that $\mathbb{P}_0 = \sigma(\theta_0)$ and $\mathbb{P}_n = \sigma(\theta_n, (X^{n, \cdot m})_{m \in \mathbb{N} \cap [0, n)} \times \mathbb{N}_0)$. Then
(i) it holds for all $n \in \mathbb{N}_0$ that $\mathbb{R}^\theta \ni \Omega \ni (\theta, \omega) \mapsto \mathcal{G}_n^\theta(\theta, \omega) \in \mathbb{R}^\theta$ is $(B(\mathbb{R}^\theta) \otimes \mathcal{F}_{n+1})/B(\mathbb{R}^\theta)$-measurable, 
(ii) it holds for all $n \in \mathbb{N}_0$ that $\Theta_n$ is $\mathcal{F}_n/B(\mathbb{R}^\theta)$-measurable, and 
(iii) it holds for all $m, n \in \mathbb{N}_0$ that $\sigma(X^{n,m})$ and $\mathcal{F}_n$ are independent.

Proof of Lemma 5.4. Note that Lemma 2.10, items (ii) and (iii) of Proposition 5.2, and the assumption that for all $r \in [1, \infty)$ it holds that $\mathcal{S}_r \subset C^1(\mathbb{R}, \mathbb{R})$ ensure that for all $n \in \mathbb{N}_0$, $r \in [1, \infty)$, $\omega \in \Omega$ we have that $\mathbb{R}^\theta \ni \Omega \ni (\theta, \omega) \mapsto (\nabla_{\theta} \mathcal{L}_n^\theta(\theta, \omega)) \in \mathbb{R}^\theta$ is continuous. Furthermore, observe that items (ii) and (iii) of Proposition 5.2 and the assumption that for all $m, n \in \mathbb{N}_0$ it holds that $X^{n,m}$ is $\mathcal{F}_{n+1}/B([a, b]^{(\alpha)})$-measurable imply that for all $n \in \mathbb{N}_0$, $r \in [1, \infty)$, $\theta \in \mathbb{R}^\theta$ it holds that $\Omega \ni (\theta, \omega) \mapsto (\nabla_{\theta} \mathcal{L}_n^\theta(\theta, \omega)) \in \mathbb{R}^\theta$ is $\mathcal{F}_{n+1}/B(\mathbb{R}^\theta)$-measurable. Combining this and, e.g., Beck et al. [7, Lemma 2.4] (applied with $(E, \delta) \subset (\mathbb{R}^\theta, B(\mathbb{R}^\theta))$, $(\Omega, \mathcal{F}) \subset (\Omega, \mathcal{F}_{n+1})$, $X \ni (\mathbb{R}^\theta \times \Omega \ni (\theta, \omega) \mapsto (\nabla_{\theta} \mathcal{L}_n^\theta(\theta, \omega)) \in \mathbb{R}^\theta$) in the notation of [7, Lemma 2.4]) demonstrates that for all $n \in \mathbb{N}_0$, $r \in [1, \infty)$ we have that $\mathbb{R}^\theta \ni \Omega \ni (\theta, \omega) \ni (\nabla_{\theta} \mathcal{L}_n^\theta(\theta, \omega)) \in \mathbb{R}^\theta$ is $(B(\mathbb{R}^\theta) \otimes \mathcal{F}_{n+1})/B(\mathbb{R}^\theta)$-measurable. Item (iv) of Proposition 5.2 hence proves that for all $n \in \mathbb{N}_0$ it holds that $\mathbb{R}^\theta \ni \Omega \ni (\theta, \omega) \ni \mathcal{G}_n^\theta(\theta, \omega) \in \mathbb{R}^\theta$ is $(B(\mathbb{R}^\theta) \otimes \mathcal{F}_{n+1})/B(\mathbb{R}^\theta)$-measurable. This establishes item (i).

Next we prove item (ii) by induction on $n \in \mathbb{N}_0$. The assumption that $F_0 = \sigma(\Theta_0)$ implies that $\Theta_0$ is $\mathcal{F}_0/B(\mathbb{R}^\theta)$-measurable. This establishes item (ii) in the base case $n = 0$. For the induction step let $n \in \mathbb{N}_0$ satisfy that $\Theta_n$ is $\mathcal{F}_n/B(\mathbb{R}^\theta)$-measurable. Note that the fact that $\mathcal{F}_n = \sigma(\Theta_0, (X^{n,m})_{m, n \in \mathbb{N} \cap [0, n]} \times \mathbb{N}_0)$ ensures that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Therefore, we obtain that $\Theta_n$ is $\mathcal{F}_{n+1}/B(\mathbb{R}^\theta)$-measurable. Moreover, observe that the fact that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and item (i) imply that $\Theta_n$ is $\mathcal{F}_{n+1}/B(\mathbb{R}^\theta)$-measurable. Combining this, the fact that $\Theta_n$ is $\mathcal{F}_{n+1}/B(\mathbb{R}^\theta)$-measurable, and the assumption that $\Theta_{n+1} = \Theta_n - \gamma_n \mathcal{G}_n^\theta(\Theta_n)$ proves that $\Theta_{n+1}$ is $\mathcal{F}_{n+1}/B(\mathbb{R}^\theta)$-measurable. Induction thus establishes item (ii).

In addition, note that the assumption that $X^{n,m}$, $n, m \in \mathbb{N}_0$, are independent, the assumption that $\Theta_0$ and $(X^{n,m})_{(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0}$ are independent, and the fact that $\mathcal{F}_n = \sigma(\Theta_0, (X^{n,m})_{n, m \in \mathbb{N} \cap [0, n]} \times \mathbb{N}_0)$ establish item (iii). The proof of Lemma 5.4 is thus complete. □

Corollary 5.5. Assume Setting 5.1. Then it holds for all $n \in \mathbb{N}_0$ that

$$E[\mathcal{L}_n^\theta(\Theta_n)] = E[\mathcal{L}(\Theta_n)].$$

Proof of Corollary 5.5. Throughout this proof let $\mathcal{F}_n \subset \mathcal{F}$, $n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}_0$ that $F_0 = \sigma(\Theta_0)$ and $\mathcal{F}_n = \sigma(\Theta_0, (X^{n,m})_{m, n \in \mathbb{N} \cap [0, n]} \times \mathbb{N}_0)$ and let $L^n : (\{a, b\}^{M_n} \times \mathbb{R}^n) \ni (x, \theta) \mapsto 1_{M_n}^{\sum_{m=1}^{M_n} ||\chi(x, \theta) - \xi||^2}$. 

Observe that (5.14) and the assumption that for all $n \in \mathbb{N}_0$, $\theta \in \mathbb{R}^\theta$, $\omega \in \Omega$ it holds that $\mathcal{G}_n^\theta(\theta, \omega) = \frac{1}{M_n} \sum_{m=1}^{M_n} ||\chi(x, \theta) - \xi||^2$ imply that for all $n \in \mathbb{N}_0$, $\theta \in \mathbb{R}^\theta$, $\omega \in \Omega$ we have that

$$L^n(\chi(x, \theta), \Theta_n) = \frac{1}{M_n} \sum_{m=1}^{M_n} \left|\chi(x, \theta) - \xi\right|^2.$$

Combining this with Proposition 5.3 demonstrates that for all $n \in \mathbb{N}_0$, $\theta \in \mathbb{R}^\theta$ it holds that

$$E[L^n(\chi(x, \theta), \Theta_n)] = E[L^n(\Theta_n)] = \mathcal{L}(\Theta_n).$$

Moreover, note that (5.15) assures that for all $n \in \mathbb{N}_0$ it holds that

$$\mathcal{G}_n(\Theta_n) = \mathcal{G}_n(\Theta_n, \Theta_n).$$

Combining this, (5.16), Lemma 5.4, and, e.g., [26, Lemma 2.8] (applied with $(\Omega, \mathcal{F}, \mathbb{P}) \subset (\Omega, \mathcal{F}, \mathbb{P})$, $G \ni (\chi, \Theta_n) \ni (\{a, b\}^{M_n} \times \mathcal{F}_{n+1})$, $(\chi, \Theta_n) \ni (\mathbb{R}^n, \mathcal{F}_{n+1})$, $X \ni (\Omega \ni (\Theta_n, \omega) \ni X^{n,M_n}(\omega), \Theta_n(\omega) \ni \omega) \ni \mathbb{R}^\theta$) in the notation of [26, Lemma 2.8]) proves that for all $n \in \mathbb{N}_0$ it holds that

$$E[L^n(\chi(x, \theta), \Theta_n) \ni \mathcal{G}_n(\Theta_n)] = E[\mathcal{L}(\Theta_n)].$$

This and (5.15) establish (5.13). The proof of Corollary 5.5 is thus complete. □
5.4 Upper estimates for the norm of the generalized gradients of the empirical risk function

Lemma 5.6. Assume Setting 5.1 and let \( n \in \mathbb{N}_0, \theta \in \mathbb{R}^p, \omega \in \Omega \). Then
\[
\|\tilde{\mathbf{g}}^n(\theta, \omega)\|^2 \leq 4L^2a^2\left[\prod_{k=p}^n(\ell_p + 1)\right]\|\theta\|^{2(L-1)}\mathcal{L}_\infty^n(\theta, \omega).
\]  
(5.19)

Proof of Lemma 5.6. Observe that item (iv) of Theorem 2.11 (applied with \( \mu \wedge (\mathcal{B}[a, b]^n) \ni A \mapsto \frac{1}{M_n}\sum_{m=1}^{M_n} \mathbb{I}_A(X_n^m(\omega)) \in [0, 1], f \wedge ([a, b]_n) \ni x \mapsto \xi \in \mathbb{R}^{L_\xi} \) in the notation of Theorem 2.11) establishes (5.19). The proof of Lemma 5.6 is thus complete. \( \square \)

Lemma 5.7. Assume Setting 5.1 and let \( K \subseteq \mathbb{R}^p \) be compact. Then
\[
\sup_{n \in \mathbb{N}_0}\sup_{\theta \in K}\sup_{\omega \in \Omega}\|\tilde{\mathbf{g}}^n(\theta, \omega)\| < \infty.
\]  
(5.20)

Proof of Lemma 5.7. Note that Lemma 2.10 (applied with \( \mu \wedge (\mathcal{B}[a, b]^n) \ni A \mapsto \frac{1}{M_n}\sum_{m=1}^{M_n} \mathbb{I}_A(X_n^m(\omega)) \in [0, 1], f \wedge ([a, b]_n) \ni x \mapsto \xi \in \mathbb{R}^{L_\xi} \) in the notation of Lemma 2.10) ensures that there exists \( \mathbf{c} \in \mathbb{R}^p \) which satisfies for all \( \theta \in K \) that \( \sup_{x \in [a, b]^n}\|\mathcal{L}^n_{\infty}(\mathbf{x})\| \leq \mathbf{c} \). The fact that for all \( n, m \in \mathbb{N}_0, \theta, \omega \in \Omega \) it holds that \( X_n^m(\omega) \in [a, b]^n \) hence demonstrates that for all \( n \in \mathbb{N}_0, \theta, \omega \in \Omega \) we have that
\[
\mathcal{L}_\infty^n(\theta, \omega) = \frac{1}{M_n}\sum_{m=1}^{M_n}\|\mathcal{L}^n_{\infty}(X_n^m(\omega)) - \xi\|^2 \leq \frac{2}{M_n}\sum_{m=1}^{M_n}\|\mathcal{L}^n_{\infty}(X_n^m(\omega))\|^2 + \|\xi\|^2 \leq 2\mathbf{c}^2 + 2\|\xi\|^2.
\]  
(5.21)

Combining this with Lemma 5.6 establishes (5.20). The proof of Lemma 5.7 is thus complete. \( \square \)

5.5 Lyapunov type estimates for the dynamics of SGD processes

Lemma 5.8. Assume Setting 5.1 and let \( n \in \mathbb{N}_0, \theta \in \mathbb{R}^p, \omega \in \Omega \). Then
\[
V(\theta - \gamma_n\tilde{\mathbf{g}}^n(\theta, \omega) - V(\theta)) = (\gamma_n)^2\|\tilde{\mathbf{g}}^n(\theta, \omega)\|^2 + (\gamma_n)^2\left[\sum_{k=1}^{L} \sum_{i=1}^{L}(k - 1)\mathbf{g}^n_{\theta_{k-1+i+\sum_{h=1}^{k-1}}}(\theta, \omega)^2\right] - 4\gamma_nL\mathcal{L}_\infty^n(\theta, \omega).
\]  
(5.22)

Proof of Lemma 5.8. Observe that Corollary 4.2 (applied with \( \mu \wedge (\mathcal{B}[a, b]^n) \ni A \mapsto \frac{1}{M_n}\sum_{m=1}^{M_n} \mathbb{I}_A(X_n^m(\omega)) \in [0, 1], f(0) \wedge \xi, \gamma \wedge \gamma_n \) in the notation of Corollary 4.2) establishes (5.22). The proof of Lemma 5.8 is thus complete. \( \square \)

5.6 Upper estimates for compositions of Lyapunov functions and SGD processes

Lemma 5.9. Assume Setting 5.1. Then it holds for all \( n \in \mathbb{N}_0 \) that
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq 4L\left((\gamma_n)^2a^2\left[\prod_{k=0}^{L}(\ell_p + 1)\right](2V(\Theta_n) + 4L^2\|\xi\|^2 + 1)^{(L-1)} - \gamma_n\right)\mathcal{L}_\infty^n(\theta_n).
\]  
(5.23)

Proof of Lemma 5.9. Note that Lemma 5.6 and item (ii) of Proposition 3.1 (applied with \( \mu \wedge (\mathcal{B}[a, b]^n) \ni A \mapsto \frac{1}{M_n}\sum_{m=1}^{M_n} \mathbb{I}_A(X_n^m(\omega)) \in [0, 1], f(0) \wedge \xi, \gamma \wedge \gamma_n \) in the notation of Proposition 3.1) demonstrate that for all \( n \in \mathbb{N}_0 \) it holds that
\[
\|\tilde{\mathbf{g}}^n(\Theta_n)\|^2 \leq 4La^2\left[\prod_{k=0}^{L}(\ell_p + 1)\right]\|\Theta_n\|^2 + 1)^{(L-1)}\mathcal{L}_\infty^n(\theta_n)
\]  
(5.24)

Lemma 5.8 therefore implies that
\[
V(\Theta_{n+1}) - V(\Theta_n) \leq 4(\gamma_n)^2L\|\tilde{\mathbf{g}}^n(\Theta_n)\|^2 - 4\gamma_nL\mathcal{L}_\infty^n(\theta_n)
\]  
(5.25)

The proof of Lemma 5.9 is thus complete. \( \square \)
Corollary 5.10. Assume Setting 5.1 and assume
\[ \mathbb{P}(\sup_{n \in \mathbb{N}_0} \gamma_n \leq \left[ La^2 \prod_{p=0}^L (\ell_p + 1) \right] (2V(\Theta_0) + 4L^2\|\xi\|^2 + 1)^{(L-1)})^{-1} = 1. \] (5.26)
Then it holds for all \( n \in \mathbb{N}_0 \) that
\[ \mathbb{P}\left(V(\Theta_{n+1}) - V(\Theta_n) \leq -4\gamma_n L \left[ 1 - \sup_{m \in \mathbb{N}_0} \gamma_m \right] \cdot La^2 \prod_{p=0}^L (\ell_p + 1) \right] (2V(\Theta_0) + 4L^2\|\xi\|^2 + 1)^{(L-1)}) \mathcal{L}_\infty^n(\Theta_n) \leq 0 \right) = 1. \] (5.27)

Proof of Corollary 5.10. Throughout this proof let \( g \in \mathbb{R} \) satisfy that \( g = \sup_{n \in \mathbb{N}_0} \gamma_n \). We prove (5.27) by induction on \( n \in \mathbb{N}_0 \). Observe that Lemma 5.9 and the assumption that \( \gamma_0 \leq g \) ensure that it holds \( \mathbb{P}\)-a.s. that
\[ V(\Theta_{m+1}) - V(\Theta_m) \right. \left. \leq -4\gamma_m L \left[ 1 - gLa^2 \prod_{p=0}^L (\ell_p + 1) \right] (2V(\Theta_0) + 4L^2\|\xi\|^2 + 1)^{(L-1)}) \mathcal{L}_\infty^m(\Theta_m) \leq 0 \right. \] (5.29)
Note that (5.29) implies that it holds \( \mathbb{P}\)-a.s. that \( V(\Theta_n) \leq V(\Theta_{n-1}) \leq \ldots \leq V(\Theta_0) \). Combining Lemma 5.9 with (5.26) and the assumption that \( \gamma_n \leq g \) hence shows that it holds \( \mathbb{P}\)-a.s. that
\[ V(\Theta_{n+1}) - V(\Theta_n) \right. \left. \leq -4\gamma_n L \left[ 1 - gLa^2 \prod_{p=0}^L (\ell_p + 1) \right] (2V(\Theta_0) + 4L^2\|\xi\|^2 + 1)^{(L-1)}) \mathcal{L}_\infty^n(\Theta_n) \leq 0 \right. \] (5.30)
Induction thus establishes (5.27). The proof of Corollary 5.10 is thus complete.

5.7 Convergence analysis for SGD processes

Theorem 5.11. Assume Setting 5.1, assume \( \sum_{n=0}^{\infty} \gamma_n = \infty \), and let \( \delta \in (0,1) \) satisfy
\[ \inf_{n \in \mathbb{N}} \mathbb{P}(\gamma_n La^2 \prod_{p=0}^L (\ell_p + 1) \right. \left. (2V(\Theta_0) + 4L^2\|\xi\|^2 + 1)^{(L-1)} \leq \delta) = 1. \] (5.31)
Then
(i) there exists \( C \in \mathbb{R} \) such that \( \mathbb{P}(\sup_{n \in \mathbb{N}_0} \|\Theta_n\| \leq C) = 1 \),
(ii) it holds that \( \mathbb{P}(\limsup_{n \to \infty} \mathcal{L}(\Theta_n) = 0) = 1 \), and
(iii) it holds that \( \limsup_{n \to \infty} \mathbb{E}[\mathcal{L}(\Theta_n)] = 0 \).
Proof of Theorem 5.11. Throughout this proof let $g \in [0, \infty]$. Observe that (5.31) and the assumption that $\sum_{n=0}^{\infty} \gamma_n = \infty$ ensure that $g \in (0, \infty)$. This and (5.31) imply that there exists $\mathcal{C} \in [1, \infty)$ which satisfies

$$P(V(\Theta_0) \leq \mathcal{C}) = 1.$$  

(5.32)

Note that (5.32) and Corollary 5.10 demonstrates that

$$P(\sup_{n \in \mathbb{N}_0} V(\Theta_n) \leq \mathcal{C}) = 1.$$  

(5.33)

Item (ii) of Proposition 3.1 (applied with $\mu \supseteq (B([a, b]^a) \supseteq A \mapsto \frac{1}{M_n} \sum_{i=1}^{M_n} \mathbb{I}_A(X^{n,m}(\omega)) \in [0, 1])$, $f \supseteq ([a, b]^a \supseteq x \mapsto \xi \in \mathbb{R}^L)$ in the notation of Proposition 3.1) therefore shows that

$$P(\sup_{n \in \mathbb{N}_0} ||\Theta_n|| \leq \mathcal{C}) = 1.$$  

(5.34)

This establishes item (i). Observe that Corollary 5.10 and (5.31) ensure that for all $n \in \mathbb{N}_0$ it holds $P$-a.s. that

$$- (V(\Theta_n) - V(\Theta_{n+1})) \leq -4\gamma_n L(1 - gL) \left[ \prod_{p=0}^{L}(\xi^0 + 1) \right] (2V(\Theta_0) + 4L^2||\xi||^2 + 1)^{L-1}) \mathbb{L}^\infty(\Theta_n).$$  

This and (5.31) prove that for all $n \in \mathbb{N}_0$ it holds $P$-a.s. that

$$\gamma_n \mathbb{L}^\infty(\Theta_n) \leq \frac{V(\Theta_n) - V(\Theta_{n+1})}{4L[1 - gL\left[ \prod_{p=0}^{L}(\xi^0 + 1) \right] (2V(\Theta_0) + 4L^2||\xi||^2 + 1)^{L-1})]} \leq \frac{V(\Theta_n) - V(\Theta_{n+1})}{4L(1 - \delta)}. \tag{5.35} \leq \frac{V(\Theta_n) - V(\Theta_{n+1})}{4L(1 - \delta)} \tag{5.36}$$

Furthermore, note that item (ii) of Proposition 3.1 (applied with $\mu \supseteq (B([a, b]^a) \supseteq A \mapsto \frac{1}{M_n} \sum_{i=1}^{M_n} \mathbb{I}_A(X^{n,m}(\omega)) \in [0, 1])$, $f \supseteq ([a, b]^a \supseteq x \mapsto \xi \in \mathbb{R}^L)$ in the notation of Proposition 3.1) ensures that for all $n \in \mathbb{N}_0$ it holds $P$-a.s. that $V(\Theta_n) \geq \frac{1}{2}||\Theta_n||^2 - 2L^2||f(0)||^2 \geq -2L^2||f(0)||^2$. Combining this with (5.33) and (5.36) implies that for all $n \in \mathbb{N}_0$ it holds $P$-a.s. that

$$\sum_{n=0}^{N-1} \gamma_n \mathbb{L}^\infty(\Theta_n) \leq \frac{\sum_{n=0}^{N-1} (V(\Theta_n) - V(\Theta_{n+1}))}{4L(1 - \delta)} \leq \frac{V(\Theta_n) - V(\Theta_{N})}{4L(1 - \delta)} \leq \frac{V(\Theta_n) + 2L^2||f(0)||^2}{4L(1 - \delta)} \leq \frac{\mathcal{C} + 2L^2||f(0)||^2}{4L(1 - \delta)} < \infty. \tag{5.37}$$

Hence, we obtain that

$$\sum_{n=0}^{\infty} \gamma_n E[\mathbb{L}^\infty(\Theta_n)] = \lim_{N \to \infty} \sum_{n=0}^{N-1} \gamma_n \mathbb{L}^\infty(\Theta_n) \leq \frac{\mathcal{C} + 2L^2||f(0)||^2}{4L(1 - \delta)} < \infty. \tag{5.38}$$

Moreover, observe that Corollary 5.5 proves that for all $n \in \mathbb{N}_0$ it holds that $E[\mathbb{L}^\infty(\Theta_n)] = E[\mathbb{L}(\Theta_n)]$. This and (5.38) show that $\sum_{n=0}^{\infty} \gamma_n E[\mathbb{L}(\Theta_n)] < \infty$. The monotone convergence theorem and the fact that for all $n \in \mathbb{N}_0$, $\omega \in \Omega$ it holds that $\mathbb{L}(\Theta_n(\omega)) \geq 0$ therefore ensure that

$$E[\sum_{n=0}^{\infty} \gamma_n \mathbb{L}(\Theta_n)] = \sum_{n=0}^{\infty} E[\gamma_n \mathbb{L}(\Theta_n)] < \infty. \tag{5.39}$$

Hence, we obtain that

$$P(\sum_{n=0}^{\infty} \gamma_n \mathbb{L}(\Theta_n) < \infty) = 1. \tag{5.40}$$

Next let $A \subseteq \Omega$ satisfy

$$A = \{ \omega \in \Omega : \left( \sum_{n=0}^{\infty} \gamma_n \mathbb{L}(\Theta_n(\omega)) < \infty \right) \& \left( \sup_{n \in \mathbb{N}_0} ||\Theta_n(\omega)|| \leq \mathcal{C} \right) \}. \tag{5.41}$$

Note that (5.34), (5.40), and (5.41) ensure that $A \in \mathcal{F}$ and $P(A) = 1$. In the following let $\omega \in A$ be arbitrary. Observe that (5.41) ensures that

$$\sup_{n \in \mathbb{N}_0} ||\Theta_n(\omega)|| \leq \mathcal{C}. \tag{5.42}$$
In addition, note that the assumption that \( \sum_{n=0}^{\infty} \gamma_n = \infty \) and the fact that \( \sum_{n=0}^{\infty} \gamma_n L(\Theta_n(\omega)) < \infty \) show that \( \liminf_{n \to \infty} L(\Theta_n(\omega)) = 0 \). In the following we prove item (ii) by contradiction. For this assume that
\[
\limsup_{n \to \infty} L(\Theta_n(\omega)) > 0.
\] (5.43)
This implies that there exists \( \varepsilon \in (0, \infty) \) which satisfies
\[
0 = \liminf_{n \to \infty} L(\Theta_n(\omega)) < \varepsilon < 2\varepsilon < \limsup_{n \to \infty} L(\Theta_n(\omega)).
\] (5.44)
Observe that (5.44) assures that there exist \( (m_k, n_k) \in \mathbb{N}^2, k \in \mathbb{N} \), which satisfy that for all \( k \in \mathbb{N} \) it holds that \( m_k < n_k < m_{k+1}, L(\Theta_{m_k}(\omega)) > 2\varepsilon, \) and \( L(\Theta_{n_k}(\omega)) < \varepsilon < \varepsilon < \min_{j \in \mathbb{N} \setminus \{m_k, n_k\}} L(\Theta_j(\omega)) \). Combining this with the fact that \( \sum_{n=0}^{\infty} \gamma_n L(\Theta_n(\omega)) < \infty \) shows that
\[
\sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} \gamma_j \leq \frac{1}{\varepsilon} \left[ \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} (\gamma_j L(\Theta_j(\omega))) \right] \leq \frac{1}{\varepsilon} \left[ \sum_{k=0}^{\infty} (\gamma_k L(\Theta_k(\omega))) \right] < \infty.
\] (5.45)
Furthermore, note that (5.42) and Lemma 5.7 ensure that there exists \( D \in \mathbb{R} \) which satisfies for all \( n \in \mathbb{N}_0 \) that \( \|D^a(\Theta_n(\omega), \omega)\| \leq D \). This, (5.45), and the fact that for all \( n \in \mathbb{N}_0, \omega \in \Omega \) it holds that \( \Theta_{n+1}(\omega) - \Theta_n(\omega) = -\gamma_n D^a(\Theta_n(\omega), \omega) \) demonstrate that
\[
\sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} \gamma_j \leq \frac{1}{\varepsilon} \left[ \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k-1} (\gamma_j L(\Theta_j(\omega))) \right] \leq \frac{1}{\varepsilon} \left[ \sum_{k=0}^{\infty} (\gamma_k L(\Theta_k(\omega))) \right] < \infty.
\] (5.46)
Moreover, observe that Lemma 2.10 (applied with \( \mu \cap (B([a, b]^6) \ni A \mapsto \frac{1}{M_n} \sum_{m=1}^{M_n} 1_A(X^{n,m}(\omega)) \in [0, 1]) \), \( f \cap [a, b]^6 \ni \xi \mapsto \xi \in \mathbb{R} \)) in the notation of Lemma 2.10 and (5.42) prove that there exists \( L \in \mathbb{R} \) which satisfies for all \( m, n \in \mathbb{N}_0 \) that \( |L(\Theta_m(\omega)) - L(\Theta_n(\omega))| \leq L \|\Theta_m(\omega) - \Theta_n(\omega)\| \). Combining this with (5.46) proves that
\[
\limsup_{k \to \infty} (L(\Theta_m(\omega)) - L(\Theta_{m_k}(\omega))) = 0.
\] (5.47)
This and the fact that for all \( k \in \mathbb{N}_0 \) it holds that \( L(\Theta_m(\omega)) < \varepsilon < 2\varepsilon < L(\Theta_{m_k}(\omega)) \) demonstrate that
\[
0 < \varepsilon \leq \inf_{k \in \mathbb{N}} |L(\Theta_m(\omega)) - L(\Theta_{m_k}(\omega))| \leq \limsup_{k \to \infty} |L(\Theta_m(\omega)) - L(\Theta_{m_k}(\omega))| = 0.
\] (5.48)
This contradiction proves that \( \limsup_{n \to \infty} L(\Theta_n(\omega)) = 0 \). Combining this with the fact that \( \mathbb{P}(A) = 1 \) establishes item (ii). Note that item (i) and the fact that \( L \) is continuous show that there exists \( \mathcal{E} \in \mathbb{R} \) which satisfies that
\[
\mathbb{P}(\sup_{n \in \mathbb{N}_0} L(\Theta_n(\omega)) \leq \mathcal{E}) = 1.
\] (5.49)
Observe that item (ii), (5.49), and the dominated convergence theorem establish item (iii). The proof of Theorem 5.11 is thus complete. \( \square \)

**Corollary 5.12.** Assume Setting 5.1, assume \( \sum_{n=0}^{\infty} \gamma_n = \infty \), and assume for all \( n \in \mathbb{N}_0 \) that
\[
\mathbb{P}(4Ld \max\{a, \|\xi\|\})^2L \gamma_n \leq (\|\Theta_0\| + 1)^{-2L} = 1.
\] (5.50)
Then

(i) there exists \( \mathcal{E} \in \mathbb{R} \) such that \( \mathbb{P}(\sup_{n \in \mathbb{N}_0} \|\Theta_n\| \leq \mathcal{E}) = 1 \),

(ii) it holds that \( \mathbb{P}(\limsup_{n \to \infty} L(\Theta_n) = 0) = 1 \), and

(iii) it holds that \( \limsup_{n \to \infty} \mathbb{E}[L(\Theta_n)] = 0 \).
Proof of Corollary 5.12. Throughout this proof let \( B \in \mathbb{R} \) satisfy \( B = \max \{ a, ||\xi|| \} \). Note that item (ii) of Proposition 3.1 (applied with \( \mu \) and \( f([a, b]_0) \)) \( A \mapsto \sum_{m=1}^{M_n} \mathbb{I}_A(X_{n,m}^\omega) \in [0, 1] \), \( f \) \( \mathbb{I}([a, b]_0) \in \mathbb{R}^\ell \) in the notation of Proposition 3.1 and the fact that for all \( x, y \in \mathbb{R}, M \in \mathbb{N} \) it holds that \( (x + y)^M \leq (2^M + 1 - 1)(x^M + y^M) \) ensure that it holds \( \mathbb{P} \)-a.s.

\[
(2V(\Theta_0) + 4L^2||\xi||^2 + 1)^{(L-1)^2} \leq (2^L - 1)^{(L-1)}(4L^2||\Theta_0||^2 + 1)^{(L-1)} + (7B^2L^2)^{(L-1)}
\]

Therefore, we obtain that it holds \( \mathbb{P} \)-a.s. that

\[
(2V(\Theta_0) + 4L^2||\xi||^2 + 1)^{(L-1)^2} \leq (2^L - 1)^{(L-1)}(4L^2||\Theta_0||^2 + 1)^{(L-1)} + (7B^2L^2)^{(L-1)}
\]

This and (5.50) show that for all \( n \in \mathbb{N}_0 \) it holds \( \mathbb{P} \)-a.s.

\[
\gamma_n L a^2 \prod_{i=0}^L (\ell_0 + 1) \leq (2V(\Theta_0) + 4L^2||\xi||^2 + 1)^{(L-1)^2}
\]

Combining this with Theorem 5.11 establishes items (i), (ii), and (iii). The proof of Corollary 5.12 is thus complete. \( \square \)

Acknowledgments

Benno Kuckuck is gratefully acknowledged for several helpful suggestions. This work has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure and under the research project HU1889/7-1.

References

[1] Absil, P.-A., Mahony, R., and Andrews, B. Convergence of the iterates of descent methods for analytic cost functions. SIAM J. Optim. 16, 2 (2005), 531–547.

[2] Allen-Zhu, Z., Li, Y., and Liang, Y. Learning and generalization in overparameterized neural networks, going beyond two layers. In Advances in Neural Information Processing Systems (2019), H. Wallach, H. Larochelle, A. Beygelzimer, F. d Alché-Buc, E. Fox, and R. Garnett, Eds., vol. 32, Curran Associates, Inc.

[3] Allen-Zhu, Z., Li, Y., and Song, Z. A convergence theory for deep learning via overparameterization. In Proceedings of the 36th International Conference on Machine Learning (09–15 Jun 2019), K. Chaudhuri and R. Salakhutdinov, Eds., vol. 97 of Proceedings of Machine Learning Research, PMLR, pp. 242–252.

[4] Arora, S., Du, S., Hu, W., Li, Z., and Wang, R. Fine-grained analysis of optimization and generalization for overparameterized two-layer neural networks. In Proceedings of the 36th International Conference on Machine Learning (Long Beach, California, USA, 6 2019), K. Chaudhuri and R. Salakhutdinov, Eds., vol. 97 of Proceedings of Machine Learning Research, PMLR, pp. 322–332.

[5] Attouch, H., and Bolte, J. On the convergence of the proximal algorithm for nonsmooth functions involving analytic features. Mathematical Programming 116, 1 (May 2007), 5–16.
[6] Bach, F., and Moulines, E. Non-strongly-convex smooth stochastic approximation with convergence rate \( o(1/n) \). In Advances in Neural Information Processing Systems (2013), C. J. C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Q. Weinberger, Eds., vol. 26, Curran Associates, Inc.

[7] Beck, C., Becker, S., Grohs, P., Jaffard, N., and Jentzen, A. Solving the Kolmogorov PDE by means of deep learning. J. Sci. Comput. 88, 3 (2021), Paper No. 73, 28.

[8] Beck, C., Jentzen, A., and Kuckuck, B. Full error analysis for the training of deep neural networks. To appear in Infin. Dimens. Anal. Quantum Probab. Relat. Top., arXiv:1910.00121 (2020), 53 pages.

[9] Bercu, B., and Fort, J.-C. Generic Stochastic Gradient Methods. American Cancer Society, 2013, pp. 1–8.

[10] Bertsekas, D. P., and Tsitsiklis, J. N. Gradient convergence in gradient methods with errors. SIAM J. Optim. 10, 3 (2000), 627–642.

[11] Bottou, L., Curtis, F. E., and Nocedal, J. Optimization methods for large-scale machine learning. SIAM Rev. 60, 2 (2018), 223–311.

[12] Chen, Z., Rotskoff, G., Bruna, J., and Van den-Eijnden, E. A dynamical central limit theorem for shallow neural networks. In Advances in Neural Information Processing Systems (2020), H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin, Eds., vol. 33, Curran Associates, Inc., pp. 22217–22230.

[13] Cheridito, P., Jentzen, A., Riekert, A., and Rossmannek, F. A proof of convergence for gradient descent in the training of artificial neural networks for constant target functions. Revision requested from J. Complexity, arXiv:2102.09924 (2021), 23 pages.

[14] Cheridito, P., Jentzen, A., and Rossmannek, F. Landscape analysis for shallow ReLU neural networks: complete classification of critical points for affine target functions. Minor revision requested from J. Nonlinear Sci., arXiv:2103.10922 (2021), 19 pages.

[15] Cheridito, P., Jentzen, A., and Rossmannek, F. Non-convergence of stochastic gradient descent in the training of deep neural networks. J. Complexity 64 (2021), Paper No. 101540, 10.

[16] Dereich, S., and Kassing, S. Convergence of stochastic gradient descent schemes for Lojasiewicz-landscapes. arXiv:2102.09385 (2021), 24 pages.

[17] Dereich, S., and Müller-Gronbach, T. General multilevel adaptations for stochastic approximation algorithms of Robbins-Monro and Polyak-Ruppert type. Numer. Math. 142, 2 (2019), 279–328.

[18] Du, S., Lee, J., Li, H., Wang, L., and Zhai, X. Gradient descent finds global minima of deep neural networks. In Proceedings of the 36th International Conference on Machine Learning (Long Beach, California, USA, 6 2019), K. Chaudhuri and R. Salakhutdinov, Eds., vol. 97 of Proceedings of Machine Learning Research, PMLR, pp. 1675–1685.

[19] Du, S., Lee, J., Li, H., Wang, L., and Zhai, X. Gradient descent finds global minima of deep neural networks. In Proceedings of the 36th International Conference on Machine Learning (09–15 Jun 2019), K. Chaudhuri and R. Salakhutdinov, Eds., vol. 97 of Proceedings of Machine Learning Research, PMLR, pp. 1675–1685.

[20] E, W., Ma, C., Wojtowytsch, S., and Wu, L. Towards a mathematical understanding of neural network-based machine learning: what we know and what we don’t. arXiv:2009.10713 (2020), 56 pages.

[21] E, W., Ma, C., and Wu, L. A comparative analysis of optimization and generalization properties of two-layer neural network and random feature models under gradient descent dynamics. Sci. China Math. 63, 7 (2020), 1235–1258.
[22] Eberle, S., Jentzen, A., Riekert, A., and Weiss, G. S. Existence, uniqueness, and convergence rates for gradient flows in the training of artificial neural networks with ReLU activation. arXiv:2108.08106 (2021), 30 pages.

[23] Fehrman, B., Gess, B., and Jentzen, A. Convergence rates for the stochastic gradient descent method for non-convex objective functions. J. Mach. Learn. Res. 21 (2020), Paper No. 136, 48.

[24] Ge, R., Huang, F., Jin, C., and Yuan, Y. Escaping from saddle points — online stochastic gradient for tensor decomposition. In Proceedings of The 28th Conference on Learning Theory (Paris, France, 03–06 Jul 2015), P. Grünwald, E. Hazan, and S. Kale, Eds., vol. 40 of Proceedings of Machine Learning Research, PMLR, pp. 797–842.

[25] Jentzen, A., and Kröger, T. Convergence rates for gradient descent in the training of overparameterized artificial neural networks with biases. arXiv:2102.11840 (2021), 38 pages.

[26] Jentzen, A., Kuckuck, B., Neufeld, A., and von Wurstemberger, P. Strong error analysis for stochastic gradient descent optimization algorithms. IMA J. Numer. Anal. 41, 1 (2021), 455–492.

[27] Jentzen, A., and Riekert, A. Convergence analysis for gradient flows in the training of artificial neural networks with ReLU activation. arXiv:2107.04479 (2021), 37 pages.

[28] Jentzen, A., and Riekert, A. A proof of convergence for stochastic gradient descent in the training of artificial neural networks with ReLU activation for constant target functions. arXiv:2104.00277 (2021), 29 pages.

[29] Jentzen, A., and Riekert, A. A proof of convergence for the gradient descent optimization method with random initializations in the training of neural networks with ReLU activation for piecewise linear target functions. arXiv:2108.04620 (2021), 44 pages.

[30] Jentzen, A., and von Wurstemberger, P. Lower error bounds for the stochastic gradient descent optimization algorithm: sharp convergence rates for slowly and fast decaying learning rates. J. Complexity 57 (2020), 101438, 16.

[31] Karimi, B., Miasojedow, B., Moulines, E., and Wai, H.-T. Non-asymptotic analysis of biased stochastic approximation scheme. In Proceedings of the Thirty-Second Conference on Learning Theory (25–28 Jun 2019), A. Beygelzimer and D. Hsu, Eds., vol. 99 of Proceedings of Machine Learning Research, PMLR, pp. 1944–1974.

[32] Karimi, H., Nutini, J., and Schmidt, M. Linear convergence of gradient and proximal-gradient methods under the polyak-Łojasiewicz condition. arXiv:1608.04636 (2020), 25 pages.

[33] Lee, J. D., Panageas, I., Piliouras, G., Simchowitz, M., Jordan, M. I., and Recht, B. First-order methods almost always avoid strict saddle points. Math. Program. 176, 1-2, Ser. B (2019), 311–337.

[34] Lee, J. D., Simchowitz, M., Jordan, M. I., and Recht, B. Gradient descent only converges to minimizers. In 29th Annual Conference on Learning Theory (Columbia University, New York, New York, USA, 23–26 Jun 2016), V. Feldman, A. Rakhlin, and O. Shamir, Eds., vol. 49 of Proceedings of Machine Learning Research, PMLR, pp. 1246–1257.

[35] Lei, Y., Hu, T., Li, G., and Tang, K. Stochastic gradient descent for nonconvex learning without bounded gradient assumptions. IEEE Trans. Neural Netw. Learn. Syst. 31, 10 (2020), 4394–4400.

[36] Li, Y., and Liang, Y. Learning overparameterized neural networks via stochastic gradient descent on structured data. In Advances in Neural Information Processing Systems (2018), S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, Eds., vol. 31, Curran Associates, Inc.
[37] Lovas, A., Lytras, I., Rásonyi, M., and Sabanis, S. Taming neural networks with TUSLA: Non-convex learning via adaptive stochastic gradient Langevin algorithms. arXiv:2006.14514 (2021), 37 pages.

[38] Lu, L., Shin, Y., Su, Y., and Karniadakis, G. E. Dying ReLU and initialization: theory and numerical examples. Commun. Comput. Phys. 28, 5 (2020), 1671–1706.

[39] Ömer Deniz Akyildiz, and Sabanis, S. Nonasymptotic analysis of stochastic gradient hamiltonian monte carlo under local conditions for nonconvex optimization. arXiv:2002.05465 (2021), 26 pages.

[40] Moulines, E., and Bach, F. Non-asymptotic analysis of stochastic approximation algorithms for machine learning. In Advances in Neural Information Processing Systems (2011), J. Shawe-Taylor, R. Zemel, P. Bartlett, F. Pereira, and K. Q. Weinberger, Eds., vol. 24, Curran Associates, Inc.

[41] Nesterov, Y. Introductory lectures on convex optimization, vol. 87 of Applied Optimization. Kluwer Academic Publishers, Boston, MA, 2004. A basic course.

[42] Nesterov, Y. E. A method for solving the convex programming problem with convergence rate $O(1/k^2)$. Dokl. Akad. Nauk SSSR 269, 3 (1983), 543–547.

[43] Panageas, I., and Piliouras, G. Gradient descent only converges to minimizers: non-isolated critical points and invariant regions. In 8th Innovations in Theoretical Computer Science Conference, vol. 67 of LIPIcs. Leibniz Int. Proc. Inform. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2017, pp. Art. No. 2, 12.

[44] Panageas, I., Piliouras, G., and Wang, X. First-order methods almost always avoid saddle points: The case of vanishing step-sizes. In Advances in Neural Information Processing Systems (2019), H. Wallach, H. Larochelle, A. Beygelzimer, F. d Alché-Buc, E. Fox, and R. Garnett, Eds., vol. 32, Curran Associates, Inc.

[45] Patel, V. Stopping criteria for, and strong convergence of, stochastic gradient descent on bottou-curtis-nocedal functions. arXiv:2004.00475 (2021), 44 pages.

[46] Rakhlin, A., Shamir, O., and Sridharan, K. Making gradient descent optimal for strongly convex stochastic optimization. In Proceedings of the 29th International Conference on International Conference on Machine Learning (Madison, WI, USA, 2012), ICML’12, Omnipress, p. 1571–1578.

[47] Ruder, S. An overview of gradient descent optimization algorithms. arXiv:1609.04747 (2017), 14 pages.

[48] Sankararaman, K. A., De, S., Xu, Z., Huang, W. R., and Goldstein, T. The impact of neural network overparameterization on gradient confusion and stochastic gradient descent. arXiv:1904.06963 (2020), 28 pages.

[49] Schmidt, M., and Roux, N. L. Fast convergence of stochastic gradient descent under a strong growth condition. arXiv:1308.6370 (2013), 5 pages.

[50] Shamir, O. Exponential convergence time of gradient descent for one-dimensional deep linear neural networks. In Proceedings of the Thirty-Second Conference on Learning Theory (25–28 Jun 2019), A. Beygelzimer and D. Hsu, Eds., vol. 99 of Proceedings of Machine Learning Research, PMLR, pp. 2691–2713.

[51] Wojtowytsch, S. Stochastic gradient descent with noise of machine learning type. Part I: Discrete time analysis. arXiv:2105.01650 (2021), 34 pages.

[52] Xu, Y., and Yin, W. A block coordinate descent method for regularized multiconvex optimization with applications to nonnegative tensor factorization and completion. SIAM Journal on Imaging Sciences 6, 3 (2013), 1758–1789.
[53] Zhang, G., Martens, J., and Grosse, R. B. Fast convergence of natural gradient descent for over-parameterized neural networks. In Advances in Neural Information Processing Systems 32, H. Wallach, H. Larochelle, A. Beygelzimer, F. d Alché-Buc, E. Fox, and R. Garnett, Eds. Curran Associates, Inc., 2019, pp. 8082–8093.

[54] Zou, D., Cao, Y., Zhou, D., and Gu, Q. Gradient descent optimizes over-parameterized deep ReLU networks. Mach. Learn. 109, 3 (2020), 467–492.