ABSTRACT

We study strategic network formation games in which agents attempt to form (costly) links in order to maximize their network centrality. Our model derives from Jackson and Wolinsky’s symmetric connection model, but allows for heterogeneity in agent utilities by replacing decay centrality (implicit in the J-W model) by a variety of classical centrality measures, as well as game-theoretic measures of centrality. We are primarily interested in characterizing the asymptotically pairwise stable networks, i.e., those networks that are pairwise stable for all sufficiently small, positive edge costs. We uncover a rich typology of stability:

- we give an axiomatic approach to network centrality that allows us to predict the stable network for a rich set of combination of centrality utility functions, yielding stable networks with features reminiscent of structural properties such as “core periphery” and “rich club” networks.
- We show that a simple variation on the model renders it universal, i.e., every network may be a stable network.
- We also show that often we can infer a significant amount about agent utilities from the structure of stable networks.

KEYWORDS

Strategic Network Formation; Asymptotic Pairwise Stability; Network Centrality Measures

1 INTRODUCTION

Centrality in social networks is a topic that has seen an overwhelming amount of recent work at the intersection of Social Network Analysis [55], Physics of Complex Systems [45], Economics [36], Theoretical Computer Science and Artificial Intelligence [22]. Many of the existing models of network formation are stochastic. In reality, networks form and evolve as a consequence of agent incentives. Among these incentives centrality maximization is certainly a pervasive one. To give just one too familiar example: the increasing competitive nature of the scientific enterprise, coupled perhaps with an ever more common reliance on quantitative measures.

and rankings of individuals and publishing venues as proxies for research quality, has often resulted in a significant explosion in the number of submissions to venues perceived as “top ones” (e.g., AI conferences such as NeurIPS, IJCAI or AAAI). The decisive factor seems to be, of course, authors’ perception that publication in top venues is a way to increase their papers’ impact, which ultimately increases their own “centrality”, that they want maximized.

Less clear are the strategic consequences of agents’ propensity for competing for central positions. By this we mean understanding the manner in which agents’ preferences for central network positions influence the actual unraveling of network structure. Until recently, models of strategic network formation (e.g. [4, 37]) were unable to reproduce the rich typology of emerging networks uncovered by social network analysis (witnessed e.g. by concepts such as core-periphery structure [10], the rich-club effect [58], or small-world networks [56]). Very interesting recent work [7] showed that strategic network formation models can reproduce the basic characteristics of social networks, low diameter, a power-law degree distribution and high clustering in its equilibrium networks. The model of Biló et al. assumes that the cost of an agent a is a sum of (a). The sum of costs of all adjacent edges we, assumed to be an arbitrary convex function of the distance between u and v, should the edge not be present, and (b). The sum of distances to all other agents. Interesting as this result is, it assumes that agents’ utilities have a quite specific nature, fairly similar for all agents: what all agents attempt to minimize is a (generalized form) of sum of distances to all other networks. Real agents may have objectives that are not distance based (e.g. intermediate communication between various parts of the network). Furthermore, different agents may have different, unrelated, centrality objectives.

The goal of this paper is to contribute to understanding network formation from a game-theoretic perspective that assumes that agents are willing to modify network structure (at a small cost per extra link) in order to improve their centrality. We start from the realization that the most well-known model of strategic network formation, the symmetric connection model [36, 37] can be seen as maximizing agents’ decay centrality [17, 53], subject to constant edge cost. Our model accommodates heterogeneity in centrality objectives: we show via an axiomatic approach to network centrality [9], that salient features of centrality measures can influence the structure of emerging networks in predictable ways.

A key design choice for models of network formation is their handling of tie strength: It is well-known that weak ties play an important role in social dynamics, being extremely useful for information dissemination [29]. Our model restricts agents tie manipulation attempts to weak ties only, whose cost of establishing/maintenance can realistically be assumed to be a tiny positive constant. An
2 PRELIMINARIES

We assume familiarity with basics of graph theory, coalitional game theory (see Chalkiadakis et al. [14]), strategic network formation (e.g. Jackson [36]) and centrality measures in social networks [19, 40]. In particular, given network $g$ and vertex $i$ of $g$, we will denote by $deg(i)$ the degree of $i$ in $g$, by $N(i)$ the set of neighbors of $i$ in $g$, and by $N(i)$ the set $N(i) = \{i\} \cup N(i)$. We also denote by $g + ij$ the network obtained by adding missing edge $ij$, by $d(i, j)$ the distance between $i$ and $j$, and by $Conn(i)$ the connected component of $i$ in $g$. We will write $g_1 + g_2$ for the disjoint union of two networks $g_1, g_2$. An edge is called a bridge edge if its removal disconnects the graph. The neighborhood domination relation is a classical concept in graph theory (e.g. Definition 1.16 in Brandstädt et al. [12]), first formulated (under the name vicinal preorder) in Foldes and Hammer [25] and formalized as follows:

**Definition 1.** Given vertices $x, y$ in graph $g$, we say that $y$ dominates $x$ (and write $x \preceq y$) iff $N(x) \subseteq N(y) \cup \{y\}$.

Given a set of vertices $V$, denote by $\bar{g}_V$ the set of graphs on vertex set $V$. A centrality measure is a function $C : V \times \bar{g}_V \to \mathcal{R}$. We will force notation and write $C_v[g]$ instead of $C(v, g)$. We also review the following special cases:

**Definition 2.** Given node $i$ in network $g$, define:

- The degree centrality of $i$ is $C_{deg}[i] = \frac{1}{n} \cdot deg(i)$.
- The closeness centrality of $i$ is $C_{close}[i] = \frac{1}{\sum_{\text{all } j \neq i} d(i, j)}$.
- The eccentricity centrality of $i$ is defined [3] as $C_{ec}[i] = \min((n - 1)/d(i, j) : j \in Conn(i))$.
- The random walk closeness centrality of $i$ is defined [57] as $C_{close}[i] = \frac{1}{\sum_{j \in Conn(i)} h_{t[i, j]},}$ where $h_{t[i, j]}$ is the expected time for a random walk started at $j$ to first hit $i$.
- The decay centrality of $i$ is defined as $C_{dec}[i] = \sum_{j \in g} \beta^{d(i, j)}$, where $\beta$ is a fixed parameter, $0 < \beta < 1$.
- The harmonic centrality of $i$ is defined as $C_{harm}[i] = \sum_{j \in g} \frac{1}{d(i, j)}$.
- The betweenness centrality of $i$ is defined $C_{between}[i] = \sum_{x \neq z \neq i} \frac{\sigma_{xz}(i)}{\sigma_{xz}}$, which is the sum of percentages of shortest paths between arbitrary vertices $y, z$ that pass through $i$.
- The random walk (a.k.a. current flow) betweenness centrality of $i$ is defined [46] as $C_{RWB}[i] = \sum_{j \neq i} r_{j, k}$, where $r_{j, k}$ is the probability that a random walk starting at node $j$ with absorbing node $k$ passes through $i$.
- The eigenvector centrality of $i$ is defined as $C_{eig}[i] = \lambda[i]$, where $\lambda$ is the eigenvector corresponding to the largest eigenvalue of the adj. matrix of $g$.
- The Katz centrality is defined as $C_{Katz}[i] = \sum_{k=1}^{\infty} \sum_{j=1}^{n} \alpha^{k} (A^{k})_{ji}$, where $\alpha$ is a parameter, $0 < \alpha < 1$.
- PageRank. See e.g. [8] for formal definitions and some properties.

*In many papers eccentricity is defined as $max{d(i, j) : j \in Conn(i)}$. Defining eccentricity centrality like we do has been done before, and has the advantage that bigger values correspond to "more central nodes".*
We also need centralities defined using coalitional games (see also [35]):

Definition 3. The Michalak et al. centrality of i [42] is defined as the Shapley value of node i in the coalitional game $(N, v)$, where $v(S) = |S \cup N(S)|$. It has the formula $C_{\text{GT}}[i] = \sum_{j \in N(i)} \frac{1}{\text{deg}(j)}$. We will also use a variant (to our knowledge first considered here) based on the Banzhaf, rather than the Shapley value:

Definition 4. The Banzhaf-Michalak centrality of i is defined as the Banzhaf value of node i in the coalitional game $(N, v)$, where $v(S) = |S \cup N(S)|$. An easy computation shows that it has the formula $C_{\text{BG}}[i] = \sum_{j \in N(i)} \frac{1}{\text{deg}(j)}$.

Definition 5. Given centrality measure $C = (C_i)$ and threshold $\theta = (\theta_i) \in (\mathbb{R} \cup \{+\infty\})^N$, the $\theta$-truncation of $C$ is the centrality $C_\theta$ defined by

$$C_\theta[g] = \begin{cases} C_i[g], & \text{if } C_i[g] < \theta_i \\ \theta_i, & \text{otherwise.} \end{cases}$$

Note that for $\theta = \infty$ (or just a large integer) we get the original utility. So truncations really extend our previous framework. An agent threshold $\theta_i$ is called individually feasible if $\theta_i = 0$ or $\theta_i > 0$ and there exists some network $h$ on the set of vertices $\{j \in N : \theta_j > 0\}$ such that $C_h[\theta] = \theta_i$. A network $w$ is called feasible iff $C_\theta[w] \leq \theta_i$ for all agents $i$.

3 MODEL AND AXIOMATIC SETTING

Our framework, which extends the Jackson-Wolinsky symmetric connection model, is specified as follows:

Definition 6. The symmetric connection model with generalized centralities $(C_i)$, thresholds $\theta_i$ and edge cost $c$ is defined as follows: the utility of player $i$ on network $g$ is $u_i(g) = C_{i,\theta_i}(g) - c \cdot \text{deg}(i)$.

where $C_{i,\theta_i}$ is the $\theta_i$-truncation of centrality $C_i$. We will occasionally avoid mentioning the family of centralities $(C_i)$ and thresholds $\theta_i$ when they are clear from the context.

An edge flip of a pair of nodes $i, j$ of a network $g$ is the addition of $ij$ to $g$, if $ij \notin g$, or its removal from $g$, if $ij \in g$. The outcome of an edge flip is the resulting network $h$.

Definition 7. An edge flip is a weakly improving move for player $i$ if $u_i(h) \geq u_i(g)$, and a strongly improving move if $u_i(h) > u_i(g)$. An edge flip is an improving move iff:
- it is an edge addition that is strongly improving for at least one endpoint and at least weakly improving for both, or
- it is an edge deletion, strongly improving for some endpoint.

The main model of the emerging network structure employed in the area of strategic network formation, defined in Jackson and Wolinsky [37] is:

Definition 8. Network $g$ is called pairwise stable if no edge flip is an improving move.

In this paper we use a version of pairwise stability that is appropriate to our setting that assumes weak ties only, in which the edge cost is a tiny (but positive) value $\epsilon > 0$. Therefore, the following variant of pairwise stability will be our main notion of interest:

Definition 9. Consider the symmetric connection model with generalized centralities. Network $g$ is called asymptotically pairwise stable (APSN) if there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, $g$ is pairwise stable in the model instantiation with edge cost $\epsilon$.

Since APSN is a version of pairwise stability, results on APSN relate to existing literature. For instance the original Jackson-Wolinsky result can be interpreted as stating that for decay centralities the unique family of APSN consists of complete graphs $K_n$.

3.1 Axioms for network centralities

The first axiom is a simple one and has been discussed before in the literature [9]. It formalizes the intuition that adding edges always improves the centrality of adjacent nodes:

Axiom 1. A centrality measure $C$ is increasing if whenever $ij \notin g$, $C[i, g + ij] > C[i, g]$.

If $C$ is a centrality measure satisfying some axiom then $C' = 1/(1 + C)$ (or even $C' = 1/C$, when $C$ is strictly positive) satisfies a correspondingly modified “dual” axiom. For instance, here’s the dual of Axiom 1:

Axiom 1'. A centrality measure $C$ is decreasing if whenever $ij \notin g$, $C[i, g + ij] < C[i, g]$.

Our next axiom represents a different type of monotonicity: the benefit of extra links only incurs for agents already in the same connected component. It is a more precise version of an axiom due to Boldi et al. [8]:

Axiom 2. A centrality measure $C$ is locally increasing if it satisfies the following conditions: if $ij \notin g$ and $i, j$ are in the same connected component then $C[i, g + ij] > C[i, g]$. If $ij \notin g$ and $i, j$ are not in the same connected component in $g$ then $C[i, g + ij] \leq C[i, g]$.

In the dual scenario agent only benefit when forming bridges between previously disconnected components:

Axiom 2'. A centrality measure $C$ is peripherally decreasing if it satisfies the following conditions: If $ij \notin g$ and $i, j$ are in the same connected component then $C[i, g + ij] \leq C[i, g]$. If $ij \notin g$ and $i, j$ are not in the same connected component in $g$ then $C[i, g + ij] > C[i, g]$.

The setting of Axiom 2 is not vacuous, as we have:

Theorem 1. Closeness centrality and random walk closeness centrality satisfy Axiom 2.

The next axiom has a different flavor, and encode a scenario when agents benefit by connecting only when they were “of the same/different types”. In our particular setting homophily is assessed with respect to agents’ degree:

Axiom 3. A centrality measure $C$ is degree homophilic if the following is true: there exists a strictly increasing function $f$ such that for any network $g$ and edge $ij \notin g$, adding edge $ij$ to $g$ is an improving move for $i$ iff $\text{deg}(i) \leq f(\text{deg}(j))$. 

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| Centrality | Axiom | Reference |
|------------|-------|-----------|
| Degree     | 1, 4, 5 | trivial   |
| Harmonic   | 1     | Boldi and Vigna [9] |
| Katz       | 1     |          |
| Decay      | 1     |          |
| Pagerank   | 1     | Chien et al. [16] Boldi et al. [8] |
| Closeness  | 2     | Theorem 1 |
| r.w. Closeness | 2 | Theorem 1 |
| Michalak et al. | 3 | Theorem 2 |

**Theorem 1.** Theorem 2

**Theorem 1.**

The following axiom introduces a weak version of a fairness axiom that has been considered in previous literature ([43, 47]):

**Axiom 4.** Centrality measure $C$ is called weakly fair if the following are true: whenever $g$ is a network, $S \subseteq V(g)$, $i, j \in S$ such that $ij \notin E(g)$ and $C_i[g + ij] - C_i[g] = \max\{C_k[g + kl] - C_k[g] : k, l \in S, kl \notin E(g)\}$, we have $C_i[g + ij] \geq C_i[g + r]$ for all $r \in S, rj \notin E(g)$. In other words edges that are (global) maximizers of the increase in network centrality are (local) maximizers of the increase in network centrality for both endpoints.

**Axiom 5.** Increasing centrality measure $C$ is called ordered if the following are true: whenever $g, h$ are networks on the same set of vertices such that $E(g) \subseteq E(h)$ and $i, j, k$ are vertices of $g$ (and $h$) such that $ij, ik \notin E(h)$, we have: if $C_i[g + ij] \geq C_i[g + ik]$ then $C_i[h + ij] \geq C_i[h + ik]$. In other words the relative effect of edges is closed under the addition of (unrelated) edges.

**Example 1.** Degree centrality satisfies axioms 1, 4, and 5. So do related variations such as $C_i[g] = \deg_g(i)^2$, weighted versions of the degree, or the following (more interesting) centrality measure: $C^*_v[g] = 0 \text{ if } v \text{ is isolated in } g, C^*_v[g] = \frac{\deg_g(v)}{|V(g)|}$ otherwise.

For space reasons, we refer for some of the missing proofs to the longer version [34]. On the other hand, many of our results (even theoretical ones) arise from implementing our model in Python using the networkx package [31] and performing computational experiments.

### 4 APSN FOR MONOTONE CENTRALITIES

In this section we consider mixtures of agents satisfying axioms 1, 1', and 2, 2'. Our first result deals with the case when all agents thresholds are $\theta_i = \infty$. In this case we can characterize the APSN as those graphs whose connected components are (perhaps single node) cliques plus, maybe, one complex component, which is not a complete graph. This complex component displays an extreme form of core-periphery structure: it consists of a core, a clique of nodes of type 1 and 2, and a periphery consisting of nodes of type 2' attached to core nodes only.

**Theorem 3.** The APSN in the centrality model with agents satisfying one of Axioms 1, 1', 2, 2' are precisely those networks $g$ satisfying the following rules:

- All agents of type 1' are isolated.
- There is a single connected component that contains all agents of type 1. Agents of type 1 and 2 in this component form a clique ("the core"). All agents of type 2' belong to this component, and are pendant vertices attached to vertices of type 1 of the clique ("the periphery").
- All other connected components are complete graphs (including isolated nodes) containing agents of type 2 only.

**Proof.** Consider an APSN. All agents of type 1 must be connected in a clique, since adding an edge is an improving move for all of them. For agents of type 2 belonging to this component it is beneficial to connect to all nodes of type 1 (and among themselves), hence they are also part of the clique core. Agents of type 2' want to stay connected to the component, but only minimally: once they are connected to a node in the component, adding any extra edge is not improving for them. Their contacts must be of type 1: if they were of type 2 they'd benefit from severing the connection.

All other components consist of agents of type 2 only, for which it is beneficial to fully connect.

![Figure 2: APSN for mixtures of monotone centralities.](image)

As a corollary of the previous result, we can infer quite a lot about agent types from the structure of stable networks:

**Corollary 1.** Consider, in the setting of Theorem 3, an asymptotically pairwise stable network $g$. Then:

- a). Isolated nodes are either of type 1 or 2.
- b). An agent $y$ in clique components of size $\geq 2$ may be of type 1 or 2. It is guaranteed to be of type 2 when there exists a complex component in $g$ which doesn't contain $y$, or when some agent $x$ in a different component is known to be of type 1 (Fig. 2).
- c). In the complex component pendant vertices are of type 2' (Fig. 2) and their neighbors are of type 1. All other nodes are of types 1 or 2.
- d). One cannot distinguish between centrality measures of the same type, nor between nodes of different types in the same listing (a)-(c).

As the next result shows, even for monotone measures, adding finite thresholds has a profound effect on the structure of APSN, changing them from complete graphs to all graphs:

**Theorem 4.** The following are true:
Figure 3: Greedy algorithm for finding APSN.

Proof.  

a). For every network \( g \) and centralities \( (C_i)_{i\in V(g)} \) satisfying axiom 1 there exist thresholds \( (\theta_i) \) s.t. \( g \) is an APSN for the truncated centrality game with thresholds \( \theta_i \).

b). If agents’ original centrality measures satisfy axiom 1, then for all families \( (\theta_i) \) of thresholds, the APSN for the truncated centrality games with thresholds \( \theta_i \), if they exist, can be characterized as the graphs with “Pareto optimal centralities”, i.e. graphs \( h \) satisfying:
- for every \( ij \notin E(h), C_i[h] \geq \theta_i \) or \( C_j[h] \geq \theta_j \), and
- for every edge \( ij \in E(h) \), removing \( ij \) from \( h \) would yield a network \( l \) with \( C_i[l] < \theta_i \) and \( C_j[l] < \theta_j \).

c). For all \( \theta_i \geq 0 \), APSN exist in all truncated centrality games with centralities \( (C_i) \) satisfying Axioms 1, 4 and 5.

Let \( \theta_i = C_i[g] \). We claim that \( g \) is an APSN for the truncated centrality game with centralities \( C_i \) and thresholds \( \theta_i \). Indeed, consider an edge \( ij \) of \( g \). Nodes \( i, j \) don’t want to drop edge \( ij \), since their current centrality values are \( \theta_i, \theta_j \), while, by Axiom 1, their centralities decrease below these values if they dropped \( ij \). Let now \( i, j \) be vertices such that \( ij \notin g \). Since their current centrality values are \( \theta_i, \theta_j \) (at the threshold), adding edge \( ij \) would not increase their truncated centralities, while incurring the extra (positive) cost of edge \( ij \). So adding edge \( ij \) is not an improving move.

b. First, by essentially repeating the proof at point a, it is easy to see that graphs with Pareto optimal centralities are APSN. The opposite direction is equally easy: consider an APSN \( h \) and two vertices \( i, j \). If \( ij \notin E(h) \) then adding edge \( ij \) must not be an improving move for at least one of \( i, j \). Since \( C_i, C_j \) are increasing, the only possibility is that \( C_i[h] \geq \theta_i \) (so that adding edge \( ij \) does not increase the truncated centrality of \( i \) and, in fact, decrease its utility, because of the extra cost of edge \( ij \)) or, similarly, \( C_j[h] \geq \theta_j \). Consider now the case when \( ij \in E(h) \). Because centralities are increasing and removing edge \( ij \) is not an improving move, removing \( ij \) must strictly decrease truncated centralities for both nodes \( i, j \). This is only possible if the centralities of \( i, j \) in the resulting network \( l \) satisfy \( C_i[l] < \theta_i \) and \( C_j[l] < \theta_j \).

c. First of all, a comment about the result at point b.: it does not establish the existence of APSN, since it is not clear that the conditions in the characterization are actually feasible. This is what we show next, under the hypothesis that all centralities satisfy axioms 1,4,5.

We will prove the existence of APSN as follows. Consider the algorithm in Figure 3. We claim that its outcome \( g \) is an APSN. Indeed, none of the missing edges could be added to \( g \) if \( ab \notin E(g) \) then \( C_a[g] \geq \theta_a \) or \( C_b[g] \geq \theta_b \) at the moment when edge \( ab \) was considered for inclusion. Since centralities only increase during the algorithm, the condition is valid at the end of the algorithm as well.

On the other hand, consider an edge \( ab \in E(g) \) with \( C_a[g] \geq \theta_a \). In order to apply point (b), we aim to prove that for every edge \( ad \in E(g) \), \( C_a[g - ad] < \theta_a \). Let \( af \) be the last edge adjacent to a added to \( g \) by the algorithm in Figure 3. By the algorithm, adding \( af \) is the first moment the centrality of node \( a \) increases beyond value \( \geq \theta_a \). So \( C_a[g - af] < \theta_a \). We will prove that in fact \( C_a[g - ad] \leq C_a[g - af] < \theta_a \) if \( d = f \) then our claim is true. Otherwise, edge \( ad \) must have been added to \( g \) before edge \( ef \). Consider the moment when adding edge \( ad \). Let \( g_0 \) be the network before the addition. By Axiom 4, adding edge \( ad \) maximized the centralities increases of both nodes \( a, d \). Since \( af \) was a candidate for edge addition, \( C_a[g_0 + ad] < C_a[g_0 + af] \). By the fact that \( C_a \) satisfies Axiom 5, \( C_a[g_1 + af] < C_a[g_1 + ad] \), where \( g_1 \geq g_0 \) is the graph \( g - \{af, ad\} \). But \( g_1 + af = g - ad \) and \( g_1 + ad = g - af \).

5 DEGREE HOMOPHILY YIELDS RICH-CLUB APSN

Next we study centrality games for degree-homophilic centrality measures. The following result shows that APSN in this case have a “rich club” hierarchical structure:

Theorem 5. Let \( h \) be an APSN for the centrality game with upward degree homophilic centralities with function \( f(\cdot) \) satisfying \( f(0) = -1 \) and \( f(x) \geq x \) for every \( x \geq 1 \).

Let \( m \) be the maximum degree of a node in \( h \). Let \( n_r^* = \min \{ k : \text{deg}(k) \geq m \} \) for \( r \geq 2 \), \( n_1^* = \min \{ r : f(r) \geq n_{r-1}^* \} \). Clearly \( n_1^* \geq n_2^* \geq \ldots \) (and one can assume w.l.o.g. by removing multiple copies of the same value, that \( n_1^* > \ldots > n_r^* = 1 \) for some \( r \geq 1 \))

a). If \( \text{deg}(i), \text{deg}(j) \geq n_i^* \), then \( ij \in E(h) \).

b). If \( \text{deg}(i), \text{deg}(j) \in [n_k^*, n_{k-1}^*] \) for some \( k \geq 2 \) then \( ij \in E(h) \) (“alike nodes connect to each other”)

c). If \( k \geq 2 \), \( \text{deg}(i) \leq n_k^*, \text{deg}(j) > n_{k-1}^* \), then \( ij \notin E(h) \).

Proof. We use the definition of degree homophily:

a). Since \( \text{deg}(i) \geq n_k^* \) and \( f(\cdot) \) is monotonically non-decreasing, \( f(\text{deg}(i)) \geq f(n_k^*) \geq m \geq \text{deg}(j) \), and similarly \( f(\text{deg}(j)) \geq \text{deg}(j) \). If \( i, j \) were not connected, then adding \( ij \) would be an improving move for both of them.

b). Similar to (a): as \( \text{deg}(i) \geq n_k^* \) and \( f(\cdot) \) is monotonically non-decreasing, \( f(\text{deg}(i)) \geq f(n_k^*) = \text{deg}(j) \), so \( f(\text{deg}(i)) \geq \text{deg}(j) \), and similarly \( f(\text{deg}(j)) \geq \text{deg}(j) \). If \( i, j \) were not connected, then adding \( ij \) would be an improving move.

c). We have \( \text{deg}(i) \leq n_k^* \) so \( f(\text{deg}(i)) \leq f(n_k^*) < f(n_{k-1}^*) \leq \text{deg}(j) - 1 \). Hence \( \text{deg}(i) - 1 > f(\text{deg}(i) - 1) \), so removing edge \( ij \) is an improving move for \( j \), since in the graph \( h = g - ij \) adding edge \( ij \) is not an improving move for \( j \).

Corollary 2. Let \( a_1 > a_2 > \ldots > a_p > a_{p+1} = 1 \) be a sequence of integers such that \( a_i - 1 > f(a_1k + 1) \) for all \( i = 1, \ldots, p \). Then all graphs of type \( K_{a_1} + K_{a_2} + \ldots + K_{a_p} + K_{\theta} \) (“stratified clique graphs”) are APSN for upward degree homophilic centrality games.
with function \(f\) and conversely, all APSN are unions of cliques with this structure.

**Proof.** We need the following simple

**Lemma 1.** Let \(g\) be a network and \(ij \in E(g)\). Then removing edge \(ij\) from \(g\) is an improving move if \(\deg(j) > f(\deg(i))\) or \(\deg(i) > f(\deg(j))\).

Proof. Removing edge \(ij\) is an improving move iff for at least one of the two nodes \(i, j\), its betweenness centrality stays the same when removing the edge. In this case adding edge \(ij\) to \(g = g - ij\) is not an improving move and vice-versa: if adding edge \(ij\) to \(g\) is not an improving then one of \(i, j\) has the same betweenness centrality in \(g\) as in \(h\), hence removing edge \(ij\) is an improving move in \(g\).

By definition, adding edge \(ij\) to \(g\) is not improving iff \(\deg(i) > f(\deg(i))\) or \(\deg(j) > f(\deg(j))\).

Consider now a sequence \(a_1 > a_2 > \ldots > a_p > a_{p+1} = 1\) be a sequence of integers such that \(a_i - 1 > f(a_{i-1})\) for all \(i = 1, \ldots, p\). We first need to prove that all graphs of type \(K_{a_1} + K_{a_2} + \ldots + K_{a_p} + 1K_0\), \(i \geq 0\), are APSN.

This is easy, by applying points a), b), c) of the theorem: let, indeed, \(y, z\) be nodes in the same clique \(K_{a_i}\), \(1 \leq r \leq p\). We need to show that removing edge \(yz\) is not an improving move. Since they are in the same clique, the degrees of \(y, z\) are both equal to \(a_r - 1\). Since \(a_r - 1 \leq f(a_r - 1)\) (because \(a_r \geq 2\) and \(f(x) \geq x\) for \(x \geq 1\)), the desired conclusion follows by Lemma 1.

Let now \(y, z\) be nodes in different cliques, \(y \in K_{a_i}, z \in K_{a_r}, a_r > a_i\). We have \(\deg(y) = a_r - 1 > f(a_{r-1}) = f(\deg(z))\). By the definition, adding edge \(yz\) is not an improving move.

Since \(0 > f(0) = -1\) connecting any isolated node to any other node is not an improving move. So \(g\) is an APSN.

Conversely, let \(g\) be an APSN. By applying points a) and b) of the Theorem, we get that \(g\) has edges between every two vertices whose degrees are in the same interval \([n_{i-1}^+, n_i^-]\), where by convention \(n_0^- = m\).

To infer the fact that \(g\) has the structure claimed in the corollary we need to prove that no other edges are present. Point c) of the theorem excludes edges between node whose degrees are not in the same interval.

The only potential trouble is that there might be a node \(x\) of degree \(n_i^+\) who is connected with nodes whose degrees are in both intervals \([n_{i-1}^+, n_i^-]\) and \([n_{i-1}^-, n_i^+]\), thus "joining two cliques". We will show that something like this doesn’t happen by induction on \(i\).

**Case \(i = 1\):** Let \(z\) be a node of maximum degree \(m\). Let \(A\) be the set of nodes with degree in the range \([n_i^-, n_i^+]\). Then all the nodes in \(A\) are connected to each other. \(z\) is not connected to any node outside \(A\). If there were some other node \(w\) in \(A\) that is connected to a node outside \(A\) then \(w\) would have degree higher than \(m\), a contradiction. Hence nodes in \(A\) form a connected component that is a clique.

The **induction step:** Assume we have obtained \(l - 1\) connected components that are cliques of size \(a_1 > a_2 > \ldots > a_{l-1}\) satisfying the condition \(a_i - 1 > f(a_{i-1})\) for \(i = 1, \ldots, l - 2\). Applying the reasoning in the induction case \(i = 1\) to the remaining graph we obtain a connected component of size \(a_l\) that is a clique. Furthermore \(a_{l-1} - 1 > f(a_l - 1)\), since nodes in the \(l\)th clique component are not connected to those in the \(l - 1\)st component.

It is possible that the tail of the resulting sequence \(a_1, \ldots, a_l\) is composed of components of size 1, that is isolated nodes. The required condition is satisfied, since \(1 - 1 > f(1 - 1) = f(0) = -1\).

In the previous theorem the condition \(f(x) > x\) is necessary: as the next result shows, without it the structure of APSN is much simpler:

**Theorem 6.** Consider a centrality game with upward degree homophilic centralities with common function \(f(x) = x + 1\). Then no edge addition can be an improving move.

Assume that, additionally, for every agents \(i, j\) such that \(ij \notin g\) and \(\deg(i) \leq \deg(j)\) we have \(C_i [g + ij] \leq C_i [g]\). Then the unique APSN for the centrality game is the empty network \(\varnothing\).

**Proof.** Adding a missing edge \(ij\) can never be an improving move: to be so, one would need, simultaneously that \(\deg(i) > \deg(j)\) and \(\deg(j) \leq \deg(i)\), which is impossible.

For similar reasons, removing an existing edge \(ij\) is always improving for one of the endpoints. Indeed, assume that \(h\) is a network containing edge \(ij\) and, w.l.o.g. \(\deg(i) \leq \deg(j)\). Let \(g = h - ij\). Then \(u_i(g) - u_i(h) = C_i [g] - C_i [h] + c > 0\). So removing edge \(ij\) is an improving move for \(i\).

**Observation 1.** The Banzhaf-Michalak centrality satisfies the conditions of Theorem 6. Indeed, assume \(ij \notin g\) and \(\deg(i) \leq \deg(j)\). Then \(C_i [g + ij] - C_i [g] = \frac{1}{2 \deg(i) + 1} - \frac{1}{2 \deg(j) + 1} - \frac{1}{2 \deg(i) + 1} = \frac{1}{2 \deg(i) + 1} \leq 0\).

6 DOMINATION AND APSN IN BETWEENNESS CENTRALITY GAMES

In this section we completely characterize APSN for betweenness centrality games. First, simple computations provide examples of APSN with components that are not complete graphs: networks \(C_4 + nK_1\), \(n \geq 0\). What about the general structure of APSN? We will show that the domination relation plays a decisive role in their characterization. To accomplish this, we first prove:

**Lemma 2.** The following statements are true:

- Adding any bridge edge \(ij\) weakly increases \(i\)’s betweenness centrality, strictly unless \(i\) was isolated. Consequently adding a bridge edge is improving for \(i\), unless \(i\) was isolated. Conversely, a disconnecting edge removal is improving for \(i\) iff \(i\) was a pendant node.

- Adding any non-bridge edge \(ij\) weakly increases \(i\)’s betweenness centrality.

We now prove the following result, which gives an unexpected (and fairly elegant) algorithmic characterization of APSN for betweenness games using the domination relation:

**Theorem 7.** Graphs \(g\) that are APSN for betweenness centrality games consist of isolated vertices plus at most one connected component \(C\) with at least two vertices which satisfies the following condition: \(\deg(l) \geq 2\) for every \(l \in C, \text{diam}(C) = 2\) and for every \(i \neq j \in C, ij \in E(g)\) if and only if sets \(N(i) \setminus \{j\}\) and \(N(j) \setminus \{i\}\) are incomparable, i.e. if none of \(i, j\) dominates the other.
Proof. First, it is easy to see that the networks that satisfy the condition of Theorem 7 are APSN: indeed, by Lemma 2 isolated vertices have no incentive to connect to anyone else, as their utility would decrease. Consider, on the other hand two vertices $i$, $j$ in a large component $C$.

If $ij \in E(g)$ then, by the condition of the theorem, there exist vertices $k \in N(i) \setminus \{j\}$ and $l \in N(j) \setminus \{i\}$. Since $d(k,j) \leq 2$ it follows that $k-i-j$ is a shortest path between $i$ and $j$ that would disappear if we dropped edge $ij$, decreasing the betweenness centrality of $i$ and ultimately its utility. Similarly, if we dropped edge $ij$ the utility of $j$ would also decrease, hence it is not an improving move.

On the other hand if $ij \notin E(g)$ then $N(i) \setminus \{j\}$ and $N(j) \setminus \{i\}$ are comparable. Assume w.l.o.g. that $N(i) \setminus \{j\} \subseteq N(j) \setminus \{i\}$. Then every shortest path between two vertices $s,t \neq i$ that goes through $i$ stays a shortest path when we add edge $ij$: This is clear when $s,t \neq j$, so assume w.l.o.g. $t = j$. Then $d(s,j) = 1$. Adding edge $ij$ creates new shortest paths, hence it is not an improving move for $i$.

Let us now prove the converse direction, that APSN satisfy the conditions in the theorem. A first statement to prove is that any APSN has at most one component with at least two vertices. Indeed, if we had two components with at least two vertices, then the utility of any vertex in the smaller component would increase as a result of the addition of an edge between two vertices in the larger component, hence it is an improving move.

As for eccentricity centrality, although it seems not to have any monotonicity properties, experimental evidence is consistent with the following conjecture, that seems to situate this measure together with the monotonic ones:

Conjecture 2. The complete graphs $K_n$ are the only asymptotically pairwise stable networks for eccentricity centrality.
8 LEARNING AGENT THRESHOLDS

In spite of the previous result, we can still talk about learning agent utility functions. However, the problem that we will deal with is not that of learning agent centralities (which we will, in fact, assume known), but agent thresholds. In other words, we want to answer the following variant of Q3: Can we learn (something about) agents’ thresholds from the structure of stable networks?

The learning model we will assume is a type of oracle learning [1]. Specifically, oracle queries are pairs $(g, i)$ consisting of a network $g$ and an agent index $i$. Given query $(g, i)$ the oracle will either reply with an APSN $h$ such that $C_i[h] > C_i[g]$, or with ‘NONE’, in the case such an APSN $h$ does not exist.

It is important to realize that thresholds may fail to be fully identifiable simply due to the coarse resolution of centralities: for instance, any values between two consecutive integers (e.g. 2.3 and 2.7) are completely equivalent as thresholds for degree centrality, since degrees in graphs are integral, and jumps in centrality (as a result of an edge flip) have a magnitude at least one. The best we can hope for in such a scenario is to identify the interval $[2, 3]$ as an interval that contains the threshold. The interval corresponds to a single edge flip in a network that decreased the centrality of the given node below the threshold value.

A potential issue with the identification of thresholds is the fact that (consistent with the model in our Corollary 1) we only get APSN as oracle answers. If for all APSN $g, C_i[g] < \theta_i$ then all estimates provided by the oracle on the value of the threshold are too low. If this doesn’t happen, we can prove:

**Theorem 9.** Given an agent $i$, assume that there exists an APSN $h$ with $\theta_i \leq C_i[h]$. Then there exists an algorithm that uses oracle queries and outputs an APSN $g$ and edge $ij$ s.t. $C_i[g-ij] \leq \theta_i \leq C_i[g]$. For linear centralities the algorithm runs in polynomial time.

9 RELATED LITERATURE

The area of network games is quite large, and a comprehensive survey is impossible. We list here two such overviews: the first one, most relevant to our interest is [38]. Another one with an algorithmic bent is [51]. The model that we are concerned with is a variant of the symmetric connection model [37] (see also [36]). Some notable subsequent work includes [21, 28, 39]. Many alternative models have been investigated. A more preeminent one is [4].

Our work owes much to the axiomatic approach to network centralities. For significant work in this area see [5, 8, 9, 47–49, 54].

More related work exists in the theoretical computer science literature: for example, Hopcroft and Sheldon [32] discuss an oriented model in which nodes have control over outgoing edges. There is no cost for changing their links, and their purpose is to increase their Pagerank. They show that the Nash equilibria in this game have a fairly sophisticated structure (see also Chen et al. [15]). Undirected versions of this game have been studied [3]. Recently Kouroupas et al. [41] have studied a model in which the utility of a node is a product of two factors: content quality multiplied by the traffic level. In this model pure Nash equilibria always exist. On the other hand Avin et al. [2] prove that preferential attachment models can be seen as Nash equilibria of some network games.

Other related work comes from the sociology literature [13, 33, 44]. For instance, in the Buskens and Van De Rijt model every node strives to fill "structural holes" (including lack of connectedness) between nodes. This is somewhat analogous to maximizing betweenness, but the precise model (and the results) are different.

Our model allows heterogeneity in agents’ utilities, corresponding to distinct measures of centrality. Heterogeneous network formation models have been studied before, e.g. Galeotti et al. [26].

Finally, several papers (e.g. [6, 18]) have treated the problem of improving the centrality of a node by adding or removing links. Our work is different in several respects: first of all, in our setting all agents aim to improve their respective centralities. Second, in our model maintaining a link has a (small) cost.

10 CONCLUSIONS, POSSIBLE EXTENSIONS

We have shown that our models can accommodate a wide range of agent centrality objectives. Still, we do not see our results as adequate enough yet for the analysis of real-life networks. They have, instead, more of a proof-of-concept nature, and could conceivably be made more realistic in many ways. Some variations (we believe) worth investigating are listed below:

**Probabilistic edge addition/removal:** In real life an edge may only form with some probability even though both agents would benefit from it. Studying such a variation could produce networks with core structures that are dense but not quite complete.

**Strong and weak links, forced links, affiliation models:** In the model we have discussed all the links are weak links. A natural extension allows for both strong and weak links. This would entail using two types of costs: fixed, constant costs for the strong links, small (“ε”) costs for weak links. A second, orthogonal, distinction that could be useful is that of forced versus free links. We assumed implicitly that link formation is completely under the control of the agent. Often this is not so: there are social ties in real life that could be regarded as fixed, since severing them entails a significant cost. Forced links may be a consequence of affiliation: people meet as the result of joining the same clubs. A possibly relevant model is the social effort model of [11]. Another one is the social clubs model of [24]. For centrality in affiliation networks see [23].

**Manipulating link strength:** agents could manipulate link strength, rather than completely severing them.

**Tagged networks:** agents have a tag and care about the tags of their neighbors, like in Schelling’s segregation model.

**Spatial agents:** Agents interaction may result from placement in space. A standard reference for spatial connection models is [28].

**Multilayer networks:** Sometimes (e.g. [20]) link formation may encompass multiple, correlated, link types. E.g. two coworkers may end up being friends as well. It would be interesting to formulate multilayer extensions of the Jackson-Wolinsky model.

**Overlapping communities:** For centrality in such models see [27, 30, 50, 52].

**Dynamic models:** Finally, our concepts of network stability are steady-state concepts. It would be interesting to study the emerging networks in dynamic models of network formation with a similar philosophy.
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