ESTIMATES FOR THE POISSON KERNEL AND THE EVOLUTION KERNEL ON NILPOTENT META-ABELIAN GROUPS

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Abstract. Let $S$ be a semi direct product $S = N \rtimes A$ where $N$ is a connected and simply connected, non-abelian, nilpotent meta-abelian Lie group and $A$ is isomorphic with $\mathbb{R}^k$, $k > 1$. We consider a class of second order left-invariant differential operators on $S$ of the form $L_\alpha = L^a + \Delta_\alpha$, where $\alpha \in \mathbb{R}^k$, and for each $a \in \mathbb{R}^k$, $L^a$ is left-invariant second order differential operator on $N$ and $\Delta_\alpha = \Delta - \langle \alpha, \nabla \rangle$, where $\Delta$ is the usual Laplacian on $\mathbb{R}^k$. Using some probabilistic techniques (e.g., skew-product formulas for diffusions on $S$ and $N$ respectively) we obtain an upper bound for the Poisson kernel for $L_\alpha$. We also give an upper estimate for the transition probabilities of the evolution on $N$ generated by $L^{\sigma(t)}$, where $\sigma$ is a continuous function from $[0, \infty)$ to $\mathbb{R}^k$.

1. Introduction

1.1. Poisson kernel on NA groups. Let $S$ be a semi direct product $S = N \rtimes A$ where $N$ is a connected and simply connected, non-abelian, nilpotent Lie group and $A$ is isomorphic with $\mathbb{R}^k$. The dimension $k$ of $A$ is called the rank of $S$. For $s \in S$ we let $x(g) = x$ and $a(g) = a$ denote the components of $s$ in this product so that $g = (x, a)$.

In what follows we identify the group $A$, its Lie algebra $a$, and $a^*$, the space of linear forms on $a$, with the Euclidean space $\mathbb{R}^k$ endowed with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. For the vector $v \in \mathbb{R}^k$ we write $v^2 = v \cdot v = \langle v, v \rangle = \sum_{i=1}^{k} v_i^2$.

We assume that there is a basis $X_1, \ldots, X_d$ for $n$ that diagonalizes the $A$-action. Let $\lambda_1, \ldots, \lambda_d \in a^* = \mathbb{R}^k$ be the corresponding roots, i.e., for
every $H \in \mathfrak{a}$, $[H, X_j] = \lambda_j(H)X_j$, $j = 1, \ldots, d$. As in [4] we assume that there is an element $H_o \in \mathbb{R}^k$ such that $\lambda_j(H_o) > 0$ for $1 \leq j \leq d$.

Let, for $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ and real $d_j$'s,

$$L_\alpha = \sum_{j=1}^{r} (e^{2\lambda_j(\alpha)}X_j^2 + d je^{\lambda_j(\alpha)}X_j) + \Delta_\alpha,$$

where, for $\alpha \in \mathbb{R}^k$,

$$\Delta_\alpha = \sum_{i=1}^{k} (\partial_i^2 - 2\alpha_i \partial_i),$$

and $X_1, \ldots, X_r$ satisfy Hörmander condition, i.e., they generate the Lie algebra $\mathfrak{n}$ of $N$. Then $L_\alpha$ is a left-invariant differential operator on $S$. Define

$$\rho_0 = \sum_{j=1}^{d} \lambda_j$$

and set

$$\chi(g) = \det(\text{Ad}(g)) = e^{\rho_0(\alpha)},$$

where

$$\text{Ad}(g)s = gsg^{-1}, \ s \in S.$$ Let $ds$ be left-invariant Haar measure on $S$. We have

$$\int_S f(sg)ds = \chi(g)^{-1} \int_S f(s)ds.$$ Let

$$A^+ = \text{Int}\{a \in \mathbb{R}^k : \lambda_j(a) \geq 0 \text{ for } 1 \leq j \leq r\}.$$ Remark. It is clear that we could have used all of the roots in defining $A^+$ since from the Hörmander condition the span over $\mathbb{N}$ of $\lambda_j, 1 \leq j \leq r$ contains all of the roots.

If $\alpha \in A^+$ then there exists a Poisson kernel $\nu$ for $L_\alpha$ [4]. That is, there is a $C^\infty$ function $\nu$ on $N$ such that every bounded $L_\alpha$-harmonic function $F$ on $S$ may be written as a Poisson integral against a bounded function $f$ on $S/A = N$,

$$F(g) = \int_{S/A} f(gy)\nu(y)dy = \int_{N} f(y)\tilde{\nu}^\alpha(y^{-1}x)dy,$$

where

$$\tilde{\nu}^\alpha(y) = \nu(a^{-1}y^{-1}a)\chi(a)^{-1}.$$
Conversely the Poisson integral of any $f \in L^\infty(N)$ is a bounded $L^\alpha$-harmonic function.

For $t \in \mathbb{R}^+$ and $\rho \in A^+$, let

$$\delta^\rho_t = \text{Ad}((\log t)\rho)|_N.$$  

Then $t \mapsto \delta^\rho_t$ is a one parameter group of automorphisms of $N$ for which the corresponding eigenvalues on $n$ are all positive. It is known [12] that then $N$ has $\delta^\rho_t$-homogeneous norm: a non-negative continuous function $| \cdot |_\rho$ on $N$ such that $|n|_\rho = 0$ if and only if $n = e$ and

$$|\delta^\rho_t x|_\rho = t|x|_\rho.$$  

For many years the best pointwise estimate in higher rank available in the literature was

$$\nu(x) \leq C_\rho(1 + |x|_\rho)^{-\varepsilon}$$  

for some $\varepsilon > 0$, where $\rho \in A^+$ ([4, 5]). This estimate was significantly improved by the authors in [17, 18]. Assume that the rank (dimension of $A$) is $k > 1$. Let $\nu$ be the Poisson kernel for the operator $L_\alpha$ with $\alpha \in A^+$. A simplified version of [17, Theorem 1.2] says

**Theorem 1.1** ([17, Theorem 1.2]). *For every given $\rho \in A^+$ there exist positive constants $C$ and $c^2$ such that the following estimate holds*

$$\nu(x) \leq C(1 + |x|_\rho)^{-c_\rho(\rho)\gamma(\alpha)},$$  

*where $\gamma(\alpha) = 2 \min_{1 \leq j \leq r} \frac{\lambda_j(\alpha)}{X_j^2}$ and $\rho_0$ is as in (1.3).*

A different estimate, for $\rho = \alpha$ is given by the following

**Theorem 1.2** ([18, Theorem 1.1]). *For all $q > 1$ there exists a constant $C_q = C_{q,\alpha} > 0$ such that for all $x \in N$, the following estimate holds*

$$\nu(x) \leq C_q(1 + |x|_\alpha)^{-\tilde{\gamma}(\alpha)},$$  

*where $\tilde{\gamma}(\alpha) = 2 \min_{1 \leq j \leq r} \frac{\lambda_j(\alpha)^2}{X_j^2}$.  

1.2. **Statements of the main results.** As it is noted in [17, 18] the estimates given in Theorems 1.1 and 1.2 above are not optimal. In this work we consider the case where $N$ is a non-abelian, nilpotent, meta-abelian group, obtaining a more explicit estimate for the Poisson kernel in this setting. Specifically, we assume that

$$N = M \rtimes V$$  

where $M$ and $V$ are abelian Lie groups with the corresponding Lie algebras $\mathfrak{m}$ and $\mathfrak{v}$. Let $\{Y_1, \ldots, Y_m\}$ and $\{X_1, \ldots, X_n\}$ be bases for $\mathfrak{m}$ and $\mathfrak{v}$ respectively.

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2The constant $c$ can be computed explicitly. Moreover, $c \to 0$ as $\rho \to \partial A^+$. 
such that \( \{ Y_1, \ldots, Y_m, X_1, \ldots, X_n \} \) forms an ordered Jordan-Hölder basis for the Lie algebra \( \mathfrak{n} \) of \( N \), ordered so that the matrix of \( \text{ad}_X \) in this basis is strictly lower triangular for all \( X \in \mathfrak{n} \). We use these bases to identify \( M \) and \( V \) with \( \mathbb{R}^m \) and \( \mathbb{R}^n \) respectively. For \( x \in N \) we let \( m(x) = m \) and \( v(x) = v \) denote the components of \( x \) in this product so that \( x = (m, v) \).

We assume also that the \( \{ Y_i \} \) and \( \{ X_j \} \) are eigenvectors for the \( \text{ad}_H \), \( H \in \mathfrak{a} \), action, i.e., there are \( \xi_1, \ldots, \xi_m \in \mathfrak{m}^* \) and \( \vartheta_1, \ldots, \vartheta_n \in \mathfrak{v}^* \) such that for every \( H \in \mathfrak{a} \),

\[
\text{ad}_H Y_i = [H, Y_i] = \xi_i(H)Y_i, \quad 1 \leq i \leq m,
\]

\[
\text{ad}_H X_j = [H, X_j] = \vartheta_j(H)X_j, \quad 1 \leq j \leq n.
\]

We assume that there exists an \( H_o \) such that \( \xi_i(H_o) > 0 \) and \( \vartheta_j(H_o) > 0 \) for \( 1 \leq i \leq m, 1 \leq j \leq n \), i.e,

\[
A^+ \neq \emptyset.
\]

The Jordan-Hölder property implies that in the \( \{ Y_i \} \) basis on \( \mathfrak{m} \), \( \text{ad}_X \) is strictly lower triangular for all \( X \in \mathfrak{v} \).

To simplify our notation we write \( \Lambda \) to denote the set of roots

\[
\Lambda = \{ \xi_1, \ldots, \xi_m, \vartheta_1, \ldots, \vartheta_n \} = \Xi \cup \Theta,
\]

where

\[
\Xi = \{ \xi_1, \ldots, \xi_m \},
\]

\[
\Theta = \{ \vartheta_1, \ldots, \vartheta_n \}.
\]

For \( \Lambda_o \subset \Lambda \) and \( a \in A^+ \) we set

\[
\gamma_{\Lambda_o}(a) = \min_{\lambda \in \Lambda_o} \lambda(a),
\]

\[
\overline{\gamma}_{\Lambda_o}(a) = \min_{\lambda \in \Lambda_o} \frac{\lambda(a)}{\lambda^2}.
\]

For simplicity we assume that all of the \( d_j \) in (1.1) are zero. Hence, in this case, we define

\[
\mathcal{L}_\alpha = \Delta_\alpha + \sum_{j=1}^m e^{2\xi_j(a)}Y_j^2 + \sum_{j=1}^n e^{2\vartheta_j(a)}X_j^2.
\]

In this setting our first main result is the following.

**Theorem 1.3.** Let \( \nu \) be the Poisson kernel for the operator \( \mathcal{L}_\alpha \), defined in (1.6), with \( \alpha \in A^+ \). Under the above assumptions on \( N \), for every \( \rho \in A^+ \) and \( \varepsilon > 0 \) there exists a constant \( C = C_{\rho, \varepsilon} > 0 \) such that

\[
\nu(m, v) \leq C(1 + |(m, v)|_\rho)^{-\gamma},
\]
POISSON KERNEL AND THE EVOLUTION ON META-ABELIAN GROUPS

where
\[
\gamma = \begin{cases}
\frac{1}{2}\gamma(\rho)\Theta(\alpha) + \frac{1}{2}\gamma(\rho)\Lambda(\alpha) & \text{for } \|m\| \geq \varepsilon \text{ and } \|v\| \geq \varepsilon, \\
\gamma(\rho)\Theta(\alpha) & \text{for } \|v\| \geq \varepsilon, \\
\gamma(\rho)\Lambda(\alpha) & \text{for } \|m\| \geq \varepsilon.
\end{cases}
\]

For the estimates for the Poisson kernel and its derivatives on rank-one NA groups, i.e., \(\dim A = 1\), see [9, 8, 22, 3, 7].

The proof of Theorem 1.3 requires both analytic and probabilistic techniques. Some of them were introduced in [4] and used in [6, 9, 8, 17]. In particular, we use the following skew-product formula for the semigroup \(T_t\) generated by \(L_\alpha\), on a general NA group,

\[
(1.7) \quad T_t f(x, a) = E_a U_{\sigma}(0, t) f(x, \sigma_t),
\]

where the expectation is taken with respect to the diffusion \(\sigma_t\) on \(\mathbb{R}^k\) generated by \(\Delta_\alpha\), i.e., the Brownian motion with drift, and \(U_{\sigma}(s, t)\) is the evolution generated by \(L_{\sigma}\) where \(\sigma \in C([0, \infty), \mathbb{R}^k)\) and, for \(a \in \mathbb{R}^k\),

\[
(1.8) \quad L_{\sigma} = \sum_{j=1}^m e^{2\xi_j(a)}Y_j^2 + \sum_{j=1}^n e^{2\varphi_j(a)}X_j^2.
\]

Thus \(U_{\sigma}(s, t)\) is the (unique) family of bounded (on appropriate space of functions on \(N\)) convolution operators \(U_{\sigma}(s, t)f = f * P_{\sigma}(t, s)\), with smooth kernels (transition probabilities) \(P_{\sigma}(t, s)\), which have some properties generalizing semigroup property (see p. 10). The idea of such a decomposition of the diffusion on \(S\) goes back to [15, 16].

In order to get estimates for the Poisson kernel it is necessary to have estimates for \(P_{\sigma}(t, 0)(x)\). The best general result we are aware of in the literature would, when specialized to our current context, be Theorem 1.4 below. See [6, 8] and [17].

Let

\[
(1.9) \quad A_{\sigma}(s, t) = \sum_{\lambda \in \Lambda} \int_s^t e^{2\lambda(\sigma_u)}du.
\]

**Theorem 1.4.** Let \(K \subset N\) be closed and \(e \notin K\). Then there exist constants \(C_1, C_2, \) and \(\nu\) such that for every \(x \in K\) and for every \(t\),

\[
P_{\sigma}(t, 0)(x) \leq C_1 \left( \int_0^t \chi(\sigma_u)^{2/\nu}du \right)^{-\nu/2} \exp \left( \frac{\tau(x)}{4} - \frac{\tau(x)^2}{C_2A_{\sigma}(0, t)} \right)
\]

where \(\tau\) is a subadditive norm which is smooth on \(N \setminus \{e\}\).

It is clear that this estimate is not optimal; it follows from formula (2.7) below, for example, that if \(N\) is abelian, a similar estimate holds without the \(\frac{\tau(x)}{4}\) term. In the rank-one case the presence of this term does not cause
a problem; it is enough to consider $x$ in a compact set. In the higher-rank case this term does create difficulties. Our second main result, Theorem 1.5 below, which plays the crucial role in the proof of Theorem 1.3, is an estimate for $P^{\sigma}(t, 0)$ on $N$ which does not contain such a term. We conjecture that a similar result holds general for nilpotent groups $N$.

In order to state this result let, for a continuous function $\sigma : [0, \infty) \rightarrow A = \mathbb{R}^k$,

$$A^{\sigma}_{M,i}(s, t) = \int_{s}^{t} e^{2\xi_i(\sigma(u))} du, \quad i = 1, \ldots, m,$$

and

$$A^{\sigma}_{V,j}(s, t) = \int_{s}^{t} e^{2\vartheta_j(\sigma(u))} du, \quad j = 1, \ldots, n,$$

and

$$A^{\sigma}_{M,\Sigma}(s, t) = \sum_{i=1}^{m} A^{\sigma}_{M,i}(s, t), \quad A^{\sigma}_{V,\Sigma}(s, t) = \sum_{j=1}^{n} A^{\sigma}_{V,j}(s, t),$$

$$A^{\sigma}_{M,\Pi}(s, t) = \prod_{i=1}^{m} A^{\sigma}_{M,i}(s, t), \quad A^{\sigma}_{V,\Pi}(s, t) = \prod_{j=1}^{n} A^{\sigma}_{V,j}(s, t).$$

We also set

$$A^{\sigma}_{N,\Pi}(0, t) = A^{\sigma}_{M,\Pi}(0, t) A^{\sigma}_{V,\Pi}(0, t),$$

$$A^{\sigma}_{N,\Sigma}(0, t) = A^{\sigma}_{M,\Sigma}(0, t) + A^{\sigma}_{V,\Sigma}(0, t).$$

Finally, for $k \in \mathbb{N}$, we let

$$\phi_k(m) = \left( \frac{\|m\|^{1/k}}{\|m\|^{1/k} + 1} \right)^k, \quad m \in M.$$ 

We also let $k_o$ be the smallest non-negative integer such that

$$\left( \text{ad}_X ight)^{k_o + 1} |_{m} = 0, \quad \forall \ X \in \mathfrak{v}.$$

Note that if $k_o = 0$, then $\mathfrak{v}$ centralizes $m$; hence $N$ is abelian. Thus our hypotheses imply that $k_o > 0$.

**Theorem 1.5.** There are positive constants $C, D$ such that for all $(m, v) \in N$,

$$A^{\sigma}_{N,\Pi}(0, t)^{1/2} P^{\sigma}(0, t)(m, v) \leq C(\|m\|^{1/(2k_o)}) + 1 + A^{\sigma}_{V,\Sigma}(0, t)^{1/2} \times \exp \left( -D \frac{\|v\|^2}{A^{\sigma}_{V,\Sigma}(0, t)} - D \frac{\|m\|^{1/k_o}}{A^{\sigma}_{N,\Sigma}(0, t)} \phi_{2k_o}(m) \right).$$
The proof of Theorem 1.5 is based on our third main result, Corollary 3.5, that allows us to decompose the diffusion defined by $P^\sigma(t,s)$ on a general nilpotent Lie group $N$ into vertical and horizontal components, in much the same way that formula (1.7) decomposes the diffusion defined by $L_\alpha$ on $S$.

1.3. Structure of the paper. The outline of the rest of the paper is as follows:

In §2.1 we recall some basic facts about exponential functionals of Brownian motion and some estimates for the joint distribution of the maximum of the absolute value of the Brownian motion on the time interval $[0,t]$ and its position at time $t$. Next we discuss the skew-product formula for the diffusion generated by $L_\alpha$ and we show how it splits into vertical and horizontal components; that is into a Brownian motion with drift in $\mathbb{R}^k$ and the evolution process in $N$. We give a formula for the evolution kernels in the special case that the nilpotent group is $\mathbb{R}^n$. Finally we recall the construction of the Poisson kernel $\nu$ on $N$ and its extension $\nu^a(x)$ to $N \times \mathbb{R}^k$.

In §3 we give a skew-product formula for split groups. This result is crucial for the proof of Theorem 1.5 in §7. In §5 and §6 we consider diffusions on $M$ and $V$ respectively.

In §8, we prove Theorem 1.3, and finally, in §9 we compare the estimate from Theorem 1.3 with our previous results from [17, 18].

2. Preliminaries

2.1. Exponential functionals of Brownian motion. Let $b_s$, $s \geq 0$, be the Brownian motion on $\mathbb{R}$ starting from $a \in \mathbb{R}$ and normalized so that

\begin{equation}
E_a f(b_s) = \int_{\mathbb{R}} f(x+a) \frac{1}{\sqrt{4\pi s}} e^{-x^2/4s} \, dx.
\end{equation}

Hence $Eb_s = a$ and $\text{Var} b_s = 2s$.

Remark. Our normalization of the Brownian motion $b_s$ is different than that typically used by probabilists who tend to assume that $\text{Var} b_s = s$.

For $d > 0$ and $\mu > 0$ we define the following exponential functional

\begin{equation}
I_{d,\mu} = \int_0^\infty e^{d(b_s - \mu s)} \, ds.
\end{equation}

Such functionals are called perpetual functionals in financial mathematics where they play an important role (see e.g. [11, 23]).

Theorem 2.1 (Dufresne, [11]). Let $b_0 = 0$. Then the functional $I_{2,\mu}$ is distributed as $(4\gamma_{\mu/2})^{-1}$, where $\gamma_{\mu/2}$ denotes a gamma random variable with parameter $\mu/2$, i.e., $\gamma_{\mu/2}$ has a density $(1/\Gamma(\mu/2)) x^{\mu/2 - 1} e^{-x} \mathbb{1}_{[0,\infty)}(x)$. 

The proof of Dufresne’s theorem can be found in many places. See for example [10, 9] or the survey paper [14] and the references therein.

The inverse gamma density on \((0, +\infty)\), with respect to \(dx\), is defined by

\[ h_{\mu, \gamma} = C_{\mu, \gamma} x^{-\mu-1} e^{-\gamma x} 1_{(0, +\infty)}(x). \]

As a corollary of Theorem 2.1, by scaling the Brownian motion and changing the variable, we get the following theorem.

**Theorem 2.2.** Let \(b_0 = a\). Then

\[
E_a f(I_{d, \mu}) = c_{d, \mu} e^{\gamma a} \int_0^\infty f(x) x^{-\mu/d} \exp \left( -\frac{c_{d,a}}{d^2 x} \right) \frac{dx}{x}.
\]

In particular, \(I_{2, \mu}\) has the inverse gamma density \(h_{\mu/2, 1/4}\).

We will also need the following lemma.

**Lemma 2.3.** Let \(\sigma_u = b_u - 2\alpha u\) be the \(k\)-dimensional Brownian motion with a drift, \(d > 0\), and let \(\ell \in (\mathbb{R}^k)^*\) be such that \(\ell(\alpha) > 0\). Then

\[
E_a f \left( \int_0^\infty e^{\ell(\sigma_u) du} \right) = c_{d, \ell, \alpha} e^{\gamma a} \int_0^\infty f(u) u^{-\gamma/d} \exp \left( -\frac{\ell(a)}{2d^2 \ell^2 u} \right) \frac{du}{u},
\]

where \(\gamma = 2\ell(\alpha)/\ell^2\).

In particular, the functional \(\int_0^\infty e^{\ell(b_u - 2\alpha u)} du\) has the inverse gamma density \(h_{2\ell(\alpha)/(d\ell^2), 1/(d\ell^2)}\).

**Proof.** It follows from Theorem 2.2. See [17, Lemma 5.4] for details. \(\square\)

### 2.2. Some probabilistic lemmas

If \(b_t\) is the Brownian motion starting from \(x \in \mathbb{R}\) then the corresponding Wiener measure on the space \(C([0, \infty), \mathbb{R})\) is denoted by \(W_x\). The following lemma follows from formula 1.1.4 on p. 125 in [1].

**Lemma 2.4.** There exists a constant \(c > 0\) such that for all \(x \leq y\),

\[
W_x (\sup_{0 < s < t} |b_s| \geq y) \leq c e^{-(y-x)^2/4t}.
\]

The following two equalities follows easily from the reflection principle for the Brownian motion [13].

**Lemma 2.5.** If \(x > a > 0\), then

\[
W_0 (\sup_{u \in [0, t]} b_u \geq a \text{ and } b_t \leq x) = 2W_0(b_t > a) - W_0(b_t > x),
\]

whereas if \(x < a\) with \(a > 0\), then

\[
W_0 (\sup_{u \in [0, t]} b_u \geq a \text{ and } b_t \leq x) = W_0(b_t > 2a - x).
\]
Let
\[ \Phi(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{x} e^{-u^2/4} du. \]

**Lemma 2.6.** For \( a \geq 0, x, y \in \mathbb{R} \) with \( x < y \), and \( t > 0 \), let
\[ R_1 = \{-a \leq x < y \leq a\}, \quad R_2 = \{x < y < -a\}, \]
\[ R_3 = \{a < x < y\}, \quad R_4 = \{0 < x < a < y\}. \]

Then
\[\int_{W_0} \left( \sup_{u \in [0,t]} |b_u| \geq a \text{ and } b_t \in [x,y] \right) \leq \begin{cases} 2\Phi\left( \frac{a-x}{\sqrt{t}} \right) - 2\Phi\left( \frac{a+y}{\sqrt{t}} \right) + 2\Phi\left( \frac{2a+y}{\sqrt{t}} \right) - 2\Phi\left( \frac{a+x}{\sqrt{t}} \right), & \text{on } R_1, \\
2\Phi\left( \frac{2a-x}{\sqrt{t}} \right) - 2\Phi\left( \frac{2a+y}{\sqrt{t}} \right) + \Phi\left( \frac{-y}{\sqrt{t}} \right) - \Phi\left( \frac{-y}{\sqrt{t}} \right), & \text{on } R_2, \\
\Phi\left( \frac{a}{\sqrt{t}} \right) - \Phi\left( \frac{y}{\sqrt{t}} \right) + 2\Phi\left( \frac{2a+y}{\sqrt{t}} \right) - 2\Phi\left( \frac{2a+x}{\sqrt{t}} \right), & \text{on } R_3, \\
2(1 - \Phi\left( \frac{a}{\sqrt{t}} \right)) - \Phi\left( \frac{y}{\sqrt{t}} \right) - \Phi\left( \frac{2a-x}{\sqrt{t}} \right) + \Phi\left( \frac{2a+x}{\sqrt{t}} \right) - \Phi\left( \frac{2a+y}{\sqrt{t}} \right), & \text{on } R_4. \end{cases}\]

**Proof.** It follows from Lemma 2.5 by an easy calculation. \( \square \)

**Corollary 2.7.** Assume that \( a > |n| + \delta, \delta > 0, \) and \( 0 < \varepsilon/2 < \delta \). Then
\[\varepsilon^{-1} W_0\left( \sup_{u \in [0,t]} |b_u| \geq a \text{ and } b_t \in [n-\varepsilon/2, n+\varepsilon/2] \right) \leq \frac{1}{\sqrt{\pi t}} \left( e^{-(2a-n)^2/(4t)} + e^{-(2a+n)^2/(4t)} \right).\]

**Proof.** Let \( x = n - \frac{\varepsilon}{2} \) and \( y = n + \frac{\varepsilon}{2} \). Our hypotheses imply that \( -a < x < y < a \). In particular
\[0 < (2a - y)/\sqrt{t} < (2a - x)/\sqrt{t} \text{ and } 0 < (2a + x)/\sqrt{t} < (2a + y)/\sqrt{t}.\]

Hence, from Lemma 2.6,
\[\varepsilon^{-1} W_0\left( \sup_{u \in [0,t]} |b_u| \geq a \text{ and } b_t \in [n-\varepsilon/2, n+\varepsilon/2] \right) \leq \frac{1}{\sqrt{\pi t}} \left( e^{-(2a-x)^2/(4t)} + e^{-(2a+y)^2/(4t)} \right)\]
proving the Corollary. \( \square \)

**Corollary 2.8.** Assume that \( a \geq 0 \). Then
\[\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} W_0\left( \sup_{u \in [0,t]} |b_u| \geq a \text{ and } b_t \in [n-\varepsilon/2, n+\varepsilon/2] \right) \leq \begin{cases} \frac{2}{\sqrt{\pi t}} e^{-\frac{(2a-|n|)^2}{4t}} & |n| < a, \\
\frac{2}{\sqrt{\pi t}} e^{-\frac{n^2}{4t}} & 0 \leq a \leq |n|. \end{cases}\]
Proof. The first statement is immediate from Corollary 2.7. For the second statement note that

\[ W_0(b_t \geq a \text{ and } b_t \in [n - \varepsilon/2, n + \varepsilon/2]) \leq W_0(b_t \in [n - \varepsilon/2, n + \varepsilon/2]) \]

\[ = \frac{1}{\sqrt{4\pi t}} \int_{n-\varepsilon/2}^{n+\varepsilon/2} e^{-u^2/(4t)} du \]

from which the lemma follows. \qed

2.3. Disintegration of the diffusion into vertical and horizontal components - skew-product formula.

2.3.1. Vertical component. Let \( L_\alpha \) be defined by (1.6). The process \( \sigma_t \) in \( \mathbb{R}^k \) generated by the operator \( \Delta_\alpha \), i.e., the Brownian motion with drift \(-2\alpha\), is called a vertical component of the diffusion generated by \( L_\alpha \).

2.3.2. Horizontal component. Let \( C_\infty(N) \) be the space of continuous functions \( f \) on \( N \) for which \( \lim_{x \to \infty} f(x) \) exists. For \( X \in n \), we let \( \tilde{X} \) denote the corresponding right-invariant vector field. For a multi-index \( I = (i_1, \ldots, i_m) \), \( i_j \in \mathbb{Z}^+ \) and a basis \( \mathcal{X}_1, \ldots, \mathcal{X}_m \) of the Lie algebra \( n \) we write \( \mathcal{X}^I = \mathcal{X}_{i_1}^{j_1} \cdots \mathcal{X}_{i_m}^{j_m} \). For \( k, l, m, n \) we define

\[ C^{(k,l)}(N) = \{ f : \tilde{X}^I \mathcal{X}^J f \in C_\infty(N) \text{ for every } |I| < k + 1 \text{ and } |J| < l + 1 \} \]

and

\[ \|f\|_{(k,l)} = \sup_{|I| = k, |J| = l} \|\tilde{X}^I \mathcal{X}^J f\|_\infty, \]

(2.4)

\[ \|f\|_{(k,l)} = \sup_{|I| \leq k, |J| \leq l} \|\tilde{X}^I \mathcal{X}^J f\|_\infty. \]

In particular \( C^{(0,k)}(N) \) is a Banach space with the norm \( \|f\|_{0,2} \).

For a continuous function \( \sigma : [0, \infty) \to \mathbb{R}^k \), we consider the operator \( L^\sigma \) where \( L^\sigma \) is as in (1.8). Let \( \{U^\sigma(s, t) : 0 \leq s \leq t\} \) be the (unique) family of bounded operators on \( C_\infty(N) \) which satisfies

1) \( U^\sigma(s, s) = \text{Id} \), for all \( s \geq 0 \),

2) \( \lim_{h \to 0} U^\sigma(s, s + h) f = f \) in \( C_\infty(N) \),

3) \( U^\sigma(s, r)U^\sigma(r, t) = U^\sigma(s, t) \), \( 0 \leq s \leq r \leq t \),

4) \( \partial_s U^\sigma(s, t) f = -L^\sigma U^\sigma(s, t) f \) for every \( f \in C^{(0,2)}(N) \),

5) \( \partial_t U^\sigma(s, t) f = U^\sigma(s, t) L^\sigma f \) for every \( f \in C^{(0,2)}(N) \),

6) \( U^\sigma(s, t) : C^{(0,2)}(N) \to C^{(0,2)}(N) \).

The operator \( U^\sigma(s, t) \) is a convolution operator with a probability measure with a smooth density, i.e., \( U^\sigma(s, t) f = f * P^\sigma(t, s) \). In particular, \( U^\sigma(s, t) \) is left invariant. By iii), \( P^\sigma(t, r) * P^\sigma(r, s) = P^\sigma(t, s) \) for \( t > r > s \). Existence
of $U^\sigma(s,t)$ follows from [20]. Notice that from above properties it follows that
\[ ii) \quad U^{\sigma \circ \theta_u}(s,t) = U^\sigma(s+u, t+u), \]
where $\sigma \circ \theta_u(s) = \sigma_{s+u}$ is the shift operator.

In fact $V(s,t) := U^\sigma(s+u, t+u)$ satisfies i) - vi) with the operator $L^{\sigma_{s+u}}$.
Hence, the result follows from the uniqueness of $U^{\sigma_{s+u}}(s,t)$.

A stochastic process (evolution) in $N$ corresponding to transition probabilities $P^\sigma(t,s)$ is called a horizontal component of the diffusion generated by $L_\alpha$.

2.3.3. Skew-product formula. Let $U^\sigma(s,t)$ and $P^\sigma(t,s)$ be as in §2.3.2. For $f \in C_c(N \times \mathbb{R}^k)$ and $t \geq 0$, we put
\[ (2.5) \quad T_t f(x,a) = \mathbf{E}_a U^\sigma(0,t) f(x, \sigma_t) = \mathbf{E}_a f *_N P^\sigma(t,0)(x, \sigma_t), \]
where the expectation is taken with respect to the distribution of the process $\sigma_t$ (Brownian motion with drift) in $\mathbb{R}^k$ with the generator $\Delta_\alpha$. The operator $U^\sigma(0,t)$ acts on the first variable of the function $f$ (as a convolution operator).

We have the following

**Theorem 2.9.** The family $T_t$ defined in (2.5) is the semigroup of operators generated by $L_\alpha$. That is
\[ \partial_t T_t f = L_\alpha T_t f \]
and
\[ \lim_{t \to 0} T_t f = f. \]

We refer to formula (2.5) as the skew-product formula. By now the proof of the above statement is standard and it goes along the lines of [8] with obvious changes. (In §3 below a more general skew-product formula is proved.)

In fact everything in this section is valid for a general nilpotent Lie groups $N$ which is not assumed to be meta-abelian. Furthermore, Theorem 2.9 is a special case of the following general situation. Consider the product of two manifolds $M \times N$. Let $L_1$ be a differential operator acting on $M$ and let for every $a \in M$, $L_2(a)$ be an operator acting on $N$. Let $L$ be a skew-product of $L_1$ and $L_2(a)$. That is $L$ is the operator on $N \times M$ of the form
\[ L = L_1 + L_2(a). \]
Then it is natural to expect that the semi-group $T_t$ generated by $L$ is given by (2.5), where the expectation is taken with respect to the diffusion $\sigma_t$ on $M$ generated by $L_1$ and $U^\sigma(0,t)$ is the evolution generated by $L_2(\sigma_t)$. The idea of such a decomposition of the diffusion on $N \times M$ goes back to [15, 16] (see also [21]).
2.4. Evolution equation in \( \mathbb{R}^n \). Let

\[
L^t = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(t) \partial_i \partial_j + \sum_{j=1}^{n} b_j(t) \partial_j
\]

be a differential operator on \( C^\infty(\mathbb{R}^n) \), where \( \partial_i = \partial_{x_i} \) and \( a(t) = [a_{ij}(t)] \) is a symmetric, positive definite matrix and the \( a_{ij} \) and \( b_j \) belong to \( C([0, \infty), \mathbb{R}) \). For \( s \leq t \), let \( U(s,t) \) be the unique family of operators on \( C^\infty(\mathbb{R}^n) \) satisfying conditions (i)-(vi) on page 10 where \( L^n \) is replaced by \( L^t \). Our goal in this section is to compute the corresponding convolution kernel \( P(s,t) \).

Let

\[
A_{ij}(s,t) = \int_s^t a_{ij}(u)du \equiv A_{i,j}
\]

\[
B_j(s,t) = \int_s^t b_j(u)du \equiv B_j.
\]

**Proposition 2.10.** Let \( A = [A_{ij}] \) and \( B = (B_1, B_2, \ldots, B_n)^t \). Then

\[
P(t, s)(x) = (2\pi)^{-\frac{n}{2}}(\det A)^{-\frac{1}{2}}e^{-\frac{1}{2}(A^{-1}(x-B):(x-B))}.
\]

**Proof.** For \( f_o \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), we write

\[
f(x, t) = U(s, t)f_o(x) = f_o * P_{t,s}(x).
\]

We note that

\[
\partial_t f(x, t) = L^t f(x, t), \quad t > s,
\]

\[
f(x, s) = f_o(x).
\]

We form the Fourier transform concluding

\[
\partial_t \hat{f}(\xi, t) = \left( -\frac{1}{2} \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j + \sum_{j=1}^{n} iB_j\xi_j \right) \hat{f}(\xi, t).
\]

The above equation is easily solved:

\[
\hat{f}(\xi, t) = \exp \left( -\frac{1}{2} \sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j + \sum_{j=1}^{n} iB_j\xi_j \right) \hat{f}_o(\xi)
\]

Forming the inverse transformation proves the proposition. \( \square \)
2.5. **Poisson kernel.** We return to the context of §1.2. Let \( \mu_t \) be the semigroup of probability measures on \( S = N \times \mathbb{R}^k \) generated by \( L_\alpha \). It is known [5, 9] that

\[
\lim_{t \to \infty} (\pi(\hat{\mu}_t), f) = (\nu, f),
\]

where \( \pi \) denotes the projection from \( S \) onto \( N \) and \((\hat{\mu}, f) = (\mu, \tilde{f}), \tilde{f}(x) = f(x^{-1}). \) Let \( a \in \mathbb{R}^k \) and let \( \mu \) be a measure on \( N \). We define

\[
(\mu^a, f) = (\mu, f \circ \text{Ad}(a)).
\]

For \( a \in \mathbb{R}^k \) we have

\[
\nu^a(x) = \nu(a^{-1}ha)\chi(a)^{-1}, \quad x \in N;
\]

where \( \chi \) is as in (1.4).

It is an easy calculation to check that

\[
\lim_{t \to \infty} (\pi(\hat{\mu}_t)^a, f) = (\nu^a, f).
\]

We need the following

**Lemma 2.11.** We have

\[
(\pi(\hat{\mu}_t)^a, f) = (E_a \tilde{P}^\sigma(t, 0), f).
\]

**Proof.** This equality follows from Theorem 2.9. See [17, Lemma 4.1] for the details. \( \square \)

By (2.9) and Lemma 2.11 it follows that

\[
(\nu^a, f) = \lim_{t \to \infty} (\pi(\hat{\mu}_t)^a, f) = \lim_{t \to \infty} (E_a \tilde{P}^\sigma(t, 0), f).
\]

3. **Diffusions on split groups**

In this section we follow the notation of §1.2 except that we do not assume that either \( M \) or \( V \) is necessarily abelian. We let \( S_0 = V \ltimes S \) considered as a subgroup of \( S \). We denote the general element of \( S \) by

\[
g = (m, v, a) = (m, x), \quad g \in S, \quad m \in M, \quad v \in V, \quad x \in S_0.
\]

Choose \( n_o \) and \( m_o \) so that \( \{Y_1, \ldots, Y_{m_o}\} \) and \( \{X_1, \ldots, X_{n_o}\} \), respectively, generate \( \mathfrak{m} \) and \( \mathfrak{v} \). Let

\[
L_M = \sum_{i=1}^{m_o}(X_i^2 + c_iX_i),
\]

\[
L_V = \sum_{i=1}^{n_o}(Y_i^2 + d_iY_i),
\]

\[
L_N = L_M + L_V,
\]
where \( c_i, d_i \in \mathbb{R} \). We then define
\[
D_o = \Delta_\alpha + LV, \\
D = \Delta_\alpha + LN
\]
considered as elements of the universal enveloping algebra \( \mathfrak{A}(s) \) where \( \Delta_\alpha \) is as in (1.2).

For \( g \in \mathcal{G} \) and \( X \in \mathfrak{A}(s) \) we let
\[
X^g = \text{Ad}(g)X.
\]

We consider the diffusion defined by \( D_o \) on \( S_o \) as the vertical component and
that defined by \( L_M \) on \( M \) as the horizontal. Explicitly, for any topological
space \( X \) we let \( \Omega^r_X = C([r, \infty), X) \) and \( \Omega_X = \Omega^r_X \). Then for \( \tau \in \Omega_{S_o} \) the
operator \( L_M = L^\tau_M \), considered as a left-invariant operator on \( C^\infty(M) \)
produces an operator \( U^\tau_M(s, t) \) on \( C^\infty(M), 0 \leq s \leq t \) as on p. 10. We write
\[
U^\tau_M(s, t) f(x) = \int_M K^M,\tau_t,s(y, x)f(y)dy,
\]
where \( dy \) is Haar measure on \( M \).

The equality
\[
U^\tau_M(s, r)U^\tau_M(r, t) = U^\tau_M(s, t), \quad 0 \leq s < r < t
\]
is equivalent with the Chapman-Kolmogorov equation, [2, (15.8), p. 320],
\[
\int_M K^M,\tau_{t, r}(y, z)K^M,\tau_{r, s}(z, x)dz = K^M,\tau_{t, s}(y, x).
\]

For each \( r \geq 0 \) and \( m \in M \), there is a corresponding Markov process
with state space \( \Omega^r_M \) and a probability measure \( W^M,\tau_{m, r} \). We omit \( r \) from
the notation when it is 0.

In particular, for \( t_n > t_{n-1} > \ldots > t_1 > r \) and the function \( f(\tau) = h(\tau(t_n), \tau(t_{n-1}), \ldots, \tau(t_1)) \),

\[
(3.1) \quad \int_{\Omega^r_M} f(\tau)dW^M,\tau_{m, r}(\tau)
= \int_{M^n} K^M,\tau_{t_n, t_{n-1}}(x_n, x_{n-1}) \ldots K^M,\tau_{t_1, r}(x_1, m)h(x_n, x_{n-1}, \ldots, x_1)dx_n \ldots dx_1.
\]

Similarly, we denote the respective transition kernels for \( D_o, D, \Delta_\alpha \) on \( S_o, S \) and \( R^k \) by \( K^o_{t, s}, K_{t, s}, \) and \( K^A_{t, s} \) respectively. The corresponding operators are
\[
U^o(s, t) = e^{(t-s)D_o}, \quad U(s, t) = e^{(t-s)D}, \quad \text{and} \quad U^A(s, t) = e^{(t-s)\Delta_\alpha}.
\]

We denote the corresponding measures on \( \Omega^r_{S_o}, \Omega^r_S \) and \( \Omega^r_A \) by \( W^S_{x, r}, W^S_{m, x, r} \), and \( W^A_{x, r} \) respectively where \( x = (v, a) \in S_o \) and \( m \in M \).
The following proposition is an extension of Theorem 2.9 to the case where $S_o$ is non-abelian. The proof follows [8].

**Proposition 3.1.** For $f \in C_\infty(S)$,

$$U(0, t)f(m, x) = \int_{\Omega_M} (U^\tau_M(0, t)f)(m, \tau(t))dW_x^{S_o}(\tau) \equiv T_{0,t}f(m, x).$$

In the proof of Proposition 3.1 we will need the following lemma.

**Lemma 3.2.**

$$T_{0,t}f(m, x) = \int_0^t U^\alpha(0, t-u)|_{y}[L^y_{M}|_mT_{0,u}f(m, y)](x)du + (U^\alpha(0, t)f)(m, x),$$

where the subscript indicates the variable on which the operator operates.

**Proof.**

\[
\begin{align*}
(T_{0,t}f - U^\alpha(0, t)f)(m, x) & = \int U^\tau_M(0, t)f(m, \tau(t))dW_x^{S_o}(\tau) - \int U^\tau_M(t, t)f(m, \tau(t))dW_x^{S_o}(\tau) \\
& = \int U^\tau_M(t-u, t)f(m, \tau(t))dW_x^{S_o}(\tau) \\
& = -\int_0^t \partial_u U^\tau_M(t-u, t)f(m, \tau(t))dudW_x^{S_o}(\tau) \\
& = \int_0^t \int L^\tau_{M}(t-u)U^\tau_M(t-u, t)f(m, \tau(t))dW_x^{S_o}(\tau)du \\
& = \int_0^t \int L^{\tau\theta_{t-u}}_{M}(0, u)f(m, \tau\circ\theta_{t-u}(u))dW_x^{S_o}(\tau)du \\
& = \int_0^t \int S_o \int L^y_{M}U^\tau_y(0, u)f(m, \tau(u))dW_x^{S_o}(\tau)K^o_{(t-u)}(x, y)dydu \\
& = \int_0^t U^\alpha(0, t-u)|_{y}[L^y_{M}|_mT_{0,u}f(m, y)](x)du.
\end{align*}
\]

□

**Proof of Proposition 3.1.** From Lemma 3.2

\[
\begin{align*}
\partial_t T_{0,t}(f)(m, x) &= \partial_t \int_0^t U^\alpha(0, t-u)|_{y}[L^y_{M}|_mT_{0,u}f(m, y)](x)du \\
& \quad + \partial_t(U^\alpha(0, t)f)(m, x) \\
& = L^x_{M}|_mT_{0,t}f(m, x) + D^\alpha T_{0,t}f(m, x) - U^\alpha(0, t)f(m, x) \\
& \quad + D^\alpha U^\alpha(0, t)f(m, x) \\
& = L^x_{M}T_{0,t}f(m, x) + D^\alpha T_{0,t}f(m, x),
\end{align*}
\]
where $T_M$ is $L_M$ considered as a left invariant operator on $S$. This proves Proposition 3.1.

We may express $W_{m,x}^S$ in terms of $W_{x}^{S_o}$ and $W_{m}^\tau$. Recall that for $\tau \in \Omega_{S_o}$, $L_M^t = L_{M}^{\tau(t)}$.

Let $f \in C(S)$. Then, by Proposition 3.1, we have

$$
\int_{\Omega_{S}} f(\tau(t))dW_{m,x}^{S}(\tau) = \int_{\Omega_{S_o}} \int_{\Omega_M} \left( \int_{M} K_{l,0}^{M,\eta}(m;l) f(l,\eta(t))dl \right) dW_{x}^{S_o}(\eta) \nonumber
$$

$$
= \int_{\Omega_{S_o}} \int_{\Omega_M} f(\mu(t),\eta(t))dW_{m}^{M,\eta}(\mu)dW_{x}^{S_o}(\eta). \nonumber
$$

(3.2)

Note that $(\mu, \eta) \in \Omega_S$. This suggests the following:

**Theorem 3.3.**

$$
\int_{\Omega_{S}} f(\tau(t))dW_{m,x}^{S}(\tau) = \int_{\Omega_{S_o}} \int_{\Omega_M} f(\mu,\eta)dW_{m}^{M,\eta}(\mu)dW_{x}^{S_o}(\eta). \nonumber
$$

Hence

$$W_{m,x}^{S}(\mu, \eta) = W_{m}^{M,\eta}(\mu)W_{x}^{S_o}(\eta). \nonumber$$

**Proof.** We have the following proposition, where $W_{w;s}$ is the measure corresponding to any general Markov process $\xi(t)$ on $\Omega_{S_X}$ (with $\xi(s) = w$). This is a restatement and generalization of Lemma 4.1.4, p. 189 from [19].

**Proposition 3.4.** Suppose that for $s < t$,

$$f(\tau) = h(\tau|_{[s,t]}, \tau|_{[t,\infty)}). \nonumber$$

Then

$$\int_{\Omega_{S_X}} f(\tau)dW_{w,s}(\tau) = \int_{\Omega_{S_X}} \int_{\Omega_X} h(\tilde{\psi}, \psi)dW_{\tilde{\psi}(t),t}(\psi) dW_{w,s}(\tilde{\psi}). \nonumber$$

Let $g = (m, x)$. The right hand side of (3.3) defines a measure on $\Omega_{S}$ which we temporarily denote $W_{g}^{S}$. The sequence of equalities (3.2) prove that $W_{g}^{S}(f) = W_{g}^{S}(f)$ for $f(\tau) = h(\tau(t))$.

Suppose that

$$f(\tau) = h(\tau(t_1), \tau(t_2)). \nonumber$$

$3dl$ denotes right-invariant Haar measure on $S_o$. Hence $dl dm$ is right-invariant Haar measure on $S$. Expressing densities with respect to right-invariant measure is not a problem as long as we do not write our kernels as convolutions. It has the convenience that the measures split in the semi-direct product decomposition.
Then with \( \tau = (\mu, \eta), w = (m, x) \) and \( 0 < t_1 < t_2 \),

\[
(3.4) \int_{\Omega_S} h(\tau(t_1), \tau(t_2))dW^S_w(\tau) = \int_{\Omega_S} \int_{\Omega_S} h(\bar{\tau}(t_1), \tau(t_2))dW^S_{\bar{\tau}(t_1), t_1}(\tau)dW^S_w(\bar{\tau})
\]

\[
= \int_{\Omega_{\bar{S}}_0} \int_{\Omega_M} \int_{\Omega_{\bar{S}}_0} h(\bar{\mu}(t_1), \bar{\eta}(t_1), \tau(t_2))dW^S_{\bar{\tau}(t_1), t_1}(\tau)dW^{M, \bar{\eta}}_{m}(\bar{\mu})dW^S_w(\bar{\eta})
\]

\[
= \int_{\Omega_{\bar{S}}_0} \int_{\Omega_M} \int_{\Omega_M} h(\bar{\mu}(t_1), \bar{\eta}(t_1), \mu(t_2), \eta(t_2))dW^{M, \bar{\eta}}_{\bar{\mu}(t_1), t_1}(\mu)dW^{M, \eta}_{m}(\bar{\mu})
\]

\[
\cdot dW^{M, \eta}_{\bar{\mu}(t_1), t_1}(\mu)dW^{S_{\eta(t_1), t_1}}_{\bar{\eta}(t_1), t_1}(\eta)dW^{S_{\eta}}_{w}(\bar{\eta}).
\]

We wish to combine (3.4) into a single \( \eta \) integral. We write (3.4) as

\[
\int_{\Omega_S} h(\tau(t_1), \tau(t_2))dW^S_w(\tau) = \int_{\Omega_{\bar{S}}_0} \int_{\Omega_M} H(\bar{\eta}, \eta)dW^{S_{\eta}}_{\bar{\eta}(t_1), t_1}(\eta)dW^{S_{\eta}}_{w}(\bar{\eta}),
\]

where

\[
H(\bar{\eta}, \eta) = \int_{\Omega_M} \int_{\Omega_{\bar{S}}_0} h(\bar{\mu}(t_1), \bar{\eta}(t_1), \mu(t_2), \eta(t_2))dW^{M, \bar{\eta}}_{\bar{\mu}(t_1), t_1}(\mu)dW^{M, \eta}_{m}(\bar{\mu}).
\]

As a function of \( \eta, h \) depends only on \( \eta|_{[t_1, \infty)} \). The \( \bar{\eta} \) dependence is also not a problem since

\[
\int_{\Omega_M} h(\bar{\mu}(t_1), \bar{\eta}(t_1), \mu(t_2), \eta(t_2))dW^{M, \bar{\eta}}_{m}(\bar{\mu})
\]

\[
= \int_{S_0} h(y, \bar{\eta}(t_1), \mu(t_2), \eta(t_2))K^{M, \bar{\eta}}_{t_1, 0}(m, y)dy,
\]

which depends on \( \bar{\eta}|_{[0, t_1]} \). Hence

\[
\int_{\Omega_S} h(\tau(t_1), \tau(t_2))dW^S_w(\tau) = \int_{\Omega_{\bar{S}}_0} \int_{\Omega_M} H(\bar{\eta}, \eta)dW^{S_{\eta}}_{\bar{\eta}(t_1), t_1}(\eta)dW^{S_{\eta}}_{w}(\bar{\eta})
\]

\[
= \int_{\Omega_{\bar{S}}_0} H(\eta, \eta)dW^{S_{\eta}}_{w}(\eta)
\]

\[
= \int_{\Omega_{\bar{S}}_0} \int_{\Omega_M} \int_{\Omega_M} h(\eta(t_1), \bar{\mu}(t_1), \eta(t_2), \mu(t_2))dW^{M, \bar{\eta}}_{\bar{\mu}(t_1), t_1}(\mu)dW^{M, \eta}_{m}(\bar{\mu})dW^{S_{\eta}}_{w}(\eta)
\]

\[
= \int_{\Omega_{\bar{S}}_0} \int_{\Omega_M} h(\eta(t_1), \mu(t_1), \eta(t_2), \mu(t_2))dW^{M, \eta}_{m}(\mu)dW^{S_{\eta}}_{w}(\eta)
\]

as desired. The general case follows similarly. \( \square \)
Corollary 3.5. For a.e. $\sigma$ with respect to $W^A_a$ and $(\mu, \gamma) \in \Omega^M \times \Omega^V$, 
\[ W^{N,\sigma}_{m,x}(\mu, \gamma) = W^{M,\gamma,\sigma}_{m}(\mu) W^V_{x}(\gamma). \]

Proof. Theorem 3.3 implies:
\[ W^S_{m,x,a}(\mu, \gamma, \sigma) = W^{M,\gamma,\sigma}_{m}(\mu) W^S_{x,a}(\gamma, \sigma) = W^{M,\gamma,\sigma}_{m}(\mu) W^V_{x}(\gamma) W^A_{a}(\sigma). \]

On the other hand, Theorem 3.3 also implies
\[ W^S_{m,x,a}(\mu, \gamma, \sigma) = W^{N,\sigma}_{m,x}(\mu, \gamma) W^A_{a}(\sigma) \]
which proves the corollary. \hfill \Box

Corollary 3.6. For a.e. $\sigma$ with respect to $W^A_a$
\[ \int_N K_{i,0}^{N,\sigma}(m, x; m_1, x_1) f(m_1, x_1) \, dm_1 \, dx_1 \]
\[ = \int_M K_{i,0}^{M,\gamma,\sigma}(m, m_1) f(m_1, \gamma_i) \, dm_1 \, dW^V_{x}(\gamma). \]

Proof. This is immediate from Corollary 3.5 and (3.1) with $n = 1$. \hfill \Box

4. Meta-abelian groups

Let notation be as in §1.2. We consider a family of automorphisms $\{\Phi(a)\}_{a \in \mathbb{R}^k}$ of $n$, that leaves $m$ and $v$ invariant. We identify linear transformations on $n$ with $(m+n) \times (m+n)$ matrices, allowing us to write
\[ \Phi(a) = \begin{bmatrix} S(a) & 0 \\ 0 & T(a) \end{bmatrix}, \]
where
\begin{align*}
S(a) &= \text{diag} \left[ e^{\xi_1(a)}, \ldots, e^{\xi_m(a)} \right], \\
T(a) &= \text{diag} \left[ e^{\vartheta_1(a)}, \ldots, e^{\vartheta_n(a)} \right].
\end{align*}
We denote the diagonal entries of $S(a)$ and $T(a)$ by
\begin{align*}
s_i(a) &= e^{\xi_i(a)}, \quad i = 1, \ldots, m, \\
t_j(a) &= e^{\vartheta_j(a)}, \quad j = 1, \ldots, n.
\end{align*}
Let $\sigma$ be a continuous function from $[0, +\infty)$ to $A = \mathbb{R}^k$, and denote
\begin{equation}
\Phi^\sigma(t) = \Phi(\sigma(t)), \quad S^\sigma(t) = S(\sigma(t)), \quad T^\sigma(t) = T(\sigma(t)).
\end{equation}
For $Z \in \mathfrak{n}$ let 

$$Z(t) = \Phi^\sigma(t)Z,$$

and consider

$$\mathcal{L}_M^\sigma(t) = \sum_{j=1}^{m} Y_j^2(t),$$

$$\mathcal{L}_V^\sigma(t) = \sum_{j=1}^{n} X_j^2(t),$$

thought of as left-invariant differential operators on $M$ and $V$ respectively.

For $v \in V$ let

$$\mathcal{L}_M^{\sigma}(t)v = \sum_{j=1}^{m} (\text{Ad}(\Phi^\sigma(t))Y_j)^2.$$  

Then

$$\mathcal{L}^{\sigma(t)}_N f(m, v) = \mathcal{L}_V(t)f(m, v) + \mathcal{L}_M^{\sigma}(t)v f(m, v), \quad t \in \mathbb{R}^+,$$

is a family of left invariant operators on $N$ depending on $t \in \mathbb{R}^+$. Our aim

is to estimate the evolution kernel $P^\sigma(t,s)$ for the time dependent operator

$\mathcal{L}^{\sigma(t)}_N$.

5. Evolution on $M$

We choose coordinates $y_i$ for $M$ for which $Y_i$ corresponds to $\partial_i = \partial_{y_i}$, $1 \leq i \leq m$. Let $\eta \in C([0, \infty), V)$ and consider the evolution on $M$ generated by the time dependent operator

$$\mathcal{L}^{\sigma(t)}_M(t) = \sum_{j=1}^{m} (\text{Ad}(\eta(t))Y_j^2(t)).$$

Then

$$\text{Ad}(\eta(t))Y_j(t) = \text{Ad}(\eta(t))\Phi(t)Y_j = \sum_{k=1}^{m} \psi_{j,k}(t)Y_k,$$

and consequently,

$$\sum_{j=1}^{m} (\text{Ad}(\eta(t))Y_j^2(t))^2 = \sum_{k,l=1}^{m} \sum_{j=1}^{m} \psi_{k,j}(t)\psi_{l,j}(t)Y_kY_l = \sum_{k,l=1}^{m} (\psi(t)\psi(t)^*)_{kl}Y_kY_l,$$
where \( \psi(t) = [\psi_{i,j}(t)] \) is the matrix of \( \Phi^\sigma(t) \)|\( M \). Thus the matrix \([a_{i,j}^\sigma(t)]\) from (2.6) for the operator \( L^\sigma_M(t) \) is

\[
[a_{i,j}^\sigma(t)] = 2[\text{Ad}(\eta(t))S^\sigma(t)][\text{Ad}(\eta(t))S^\sigma(t)]^*,
\]

where the adjoint is in the \( y_j \) coordinates. Let

\[
A_{M}^{\sigma,\eta}(s,t) = \int_{s}^{t} a_{M}^{\sigma,\eta}(u)du.
\]

For a \( d \times d \) invertible matrix \( A \) we set

\[
B(A)(x) = \frac{1}{2} A^{-1} x \cdot x \quad \text{and} \quad D(A) = (2\pi)^{-\frac{d}{2}} (\det A)^{-\frac{1}{2}}.
\]

It follows from Proposition 2.10 that the evolution kernel \( K_{l,s}^{M,\sigma,\eta} \) for the operator \( L^\sigma_M(t) \) is Gaussian, and in our notation, is given by

\[
K_{l,s}^{M,\sigma,\eta}(m_1, m_2) = D(A_{M}^{\sigma,\eta}(t,s)) e^{-B(A_{M}^{\sigma,\eta}(t,s))(m_1^2 - m_2^2)}, \quad m_1, m_2 \in M = \mathbb{R}^m.
\]

We will need the following two lemmas:

**Lemma 5.1.** Let \( A \) be a positive semi-definite symmetric matrix. Then

\[
B(A)(x) \geq \frac{\|x\|^2}{2\|A\|},
\]

where \( \|A\| \) is the \( \ell^2 \rightarrow \ell^2 \)-operator norm.

**Proof.** Let \( A = C^2 \), where \( C^t = C \). Since \( \|A\| = \max_i \lambda_i \) where \( \lambda_i \geq 0 \) are the eigenvalues of \( A \), \( \|C\| = \|A\|^{\frac{1}{2}} \). Then

\[
B(A)(x) = \frac{1}{2} \|C^{-1}x\|^2 = \frac{1}{2} \|C\|^{-2} \|C\|^2 \|C^{-1}x\|^2 \\
\geq \frac{1}{2} \|C\|^{-2} \|x\|^2 = \frac{1}{2} \|A\|^{-1} \|x\|^2.
\]

\( \square \)

**Lemma 5.2.** Let \( M \) and \( D \) be square matrices and let

\[
A = \begin{bmatrix} M & B \\ C & D \end{bmatrix}.
\]

If \( \det M \neq 0 \) then \( \det A = \det M \det(D - CM^{-1}B) \).

**Proof.** See e.g. [24]. \( \square \)

Now we prove an upper bound on \( D(A_{M}^{\sigma,\eta}(s,t)) \) that is independent of \( \eta \). For simplicity of notation we identify \( M, V, \) and, \( N \) with \( \mathbf{m}, \mathbf{v}, \) and \( \mathbf{n} \) using the exponential map.
Lemma 5.3. There is a constant $C > 0$ such that
\[ D(A_M^{\sigma,\eta}(s, t))^{-1/2} \leq C \left( \prod_{i=1}^{m} \int_s^t s_i^\sigma(u)^2 du \right)^{-1/2} = A_{M, M}^{\sigma, \eta}(s, t)^{-1/2}, \]
where $s_i^\sigma(t)$ are the entries of the diagonal matrix $S^\sigma(t)$ defined in (4.1).

Proof. We omit the $t$ and $\sigma$ dependence for the sake of simplicity. From the lower triangularity of the adjoint action of $n$, for $X \in n = N$,
\[
\text{ad}_X = \begin{bmatrix} X_o & 0 \\ v^t & 0 \end{bmatrix}, \quad \text{Ad}_X = e^{\text{ad}_X} = \begin{bmatrix} e^{X_o} & 0 \\ v(X)^t & 1 \end{bmatrix},
\]
where the $X_o$ is an $(m - 1) \times (m - 1)$-matrix and $v$ is an $(m - 1) \times 1$-column vector.

Then
\[ \text{Ad}_X S = e^{\text{ad}_X} \begin{bmatrix} S_0 & 0 \\ 0 & s_m \end{bmatrix} = \begin{bmatrix} e^{X_o}S_0 & 0 \\ v(X)^tS_0 & s_m \end{bmatrix}. \]

Let
\[ F^t = v(X)^tS_0. \]
Thus
\[ \text{Ad}_X S(\text{Ad}_X S)^t = \begin{bmatrix} e^{X_o}S_0S_0^t & e^{X_o}S_0S_0^tG \\ G^t & s_m^2 + |F|^2 \end{bmatrix}, \]
where
\[ G = e^{X_o}S_0F = e^{X_o}S_0S_0^tv(X). \]

Hence,
\[ A^{\sigma,\eta}(s, t) = \begin{bmatrix} A_o & B \\ B^t & A + E \end{bmatrix}, \]
where
\[
A_o = \int_s^t e^{X_o(u)}S_0(u)S_0(u)^t e^{X_o(u)^t} du, \quad B = \int_s^t G(u)du, \\
A = \int_s^t s_m^2(u)du, \quad E = \int_s^t |F(u)|^2 du.
\]

From Lemma 5.2,
\[
\det A_M^{\sigma,\eta}(s, t) = (\det A_o)(A + E - B^tA_o^{-1}B) \\
= (\det A_o)A + (\det A_o)(E - B^tA_o^{-1}B) \\
= (\det A_o)A + \det \begin{bmatrix} A_o & B \\ B^t & E \end{bmatrix}.
\]
The determinant on the right is non-negative since it is the $s_m = 0$ case of formula (5.1). Hence,
\[ \det A_M^{\sigma,\eta}(s, t) \geq A(\det A_o). \]
Our result follows by induction. \[\square\]

Now we estimate the operator norm of the matrix

\[
A_M^\sigma(0, t) = \int_0^t [\text{Ad}(\eta(u))S^\sigma(u)]^t du.
\]

**Lemma 5.4.** Let \(\eta = (\eta(u)) = (\eta_1(u), \ldots, \eta_n(u))\) be a continuous function. Then there exists a constant \(C > 0\) such that

\[
\|A_M^\sigma(0, t)\| \leq C(1 + \Lambda^\eta(0, t)^{2k_0}) \sum_{j=1}^n \int_0^t s_j^\sigma(u)^2 du,
\]

where

\[
\Lambda^\eta(s, t) = \sup_{s \leq u \leq t} \|\eta(u)\|.
\]

**Proof.** We note first that for \(X \in \mathfrak{n}\),

\[
\text{Ad}_X \bigg|_m = \sum_{j=0}^{k_0} \frac{(\text{ad}_X \big|_m)^j}{j!},
\]

\[
\|\text{Ad}_X \big|_m\| \leq C(1 + \|\text{ad}_X\|)^{k_0}
\leq C'(1 + \|X\|)^{k_0}.
\]

Our result follows by bringing the norm inside the integral in (5.2). \(\square\)

### 6. Evolution on \(V\)

Recall that we identified \(V\) with \(\mathbb{R}^n\). The matrix \(T^\sigma(t) = \Phi^\sigma(t)|_V\) is of the form

\[
T^\sigma(t) = \text{diag} \left[ e^{\vartheta_1(\sigma(t))}, \ldots, e^{\vartheta_n(\sigma(t))} \right],
\]

where, \(\vartheta_1, \ldots, \vartheta_n \in (\mathbb{R}^n)^*\). Now we consider the evolution process \(\eta(t)\) on \(V\) generated by

\[
\mathcal{L}_V^\sigma(t) = \sum_{j=1}^n X_j(t)^2 = \sum_{j=1}^n (T^\sigma(t)X_j)^2
\]

(see the notation introduced in (4.2) on p. 19). Thus, since \(X_j = \vartheta_j\),

\[
\mathcal{L}_V^\sigma(t) = \sum_{j=1}^n e^{2\vartheta_j(\sigma(t))} \vartheta_j^2.
\]

The matrix \(a_t = [a_{ij}(t)]\), defined in (2.6), for \(\mathcal{L}_V^\sigma(t)\) is equal to

\[
a_v^\sigma(t) = 2T^\sigma(t)T^\sigma(t)^* = \text{diag} \left[ e^{2\vartheta_1(\sigma(t))}, \ldots, e^{2\vartheta_n(\sigma(t))} \right].
\]

Let \(b_t\) be the 1-dimensional Brownian motion normalized so that

\[
W_x(b_t \in dy) = p_t(x, dy) = \frac{1}{(4\pi t)^{1/2}} e^{-(x-y)^2/4t} dy.
\]
Then, by (2.7),
\[
K_{s,t}^{V,\sigma}(x,dz) = \prod_{1 \leq j \leq n} p_{f_j^\sigma} e^{2\varphi_j(\sigma(u))} du (x_j,dz_j).
\]

Thus the process \( \eta(t) \) generated by \( L_\sigma^V \) has coordinates \( \eta_j(t) \) which are independent Brownian motions with time changed according to the clock governed by \( \sigma \). Let
\[
A_\sigma^V(0,t) = \int_0^t a_\sigma^V(u) du.
\]
Since \( A_\sigma^V(0,t) \) is diagonal we see
\[
(det A_\sigma^V(0,t))^{-1/2} = \left( \prod_{j=1}^n \int_0^t e^{2\varphi_j(\sigma(u))} du \right)^{-1/2},
\]
\[
\|A_\sigma^V(0,t)\| \leq \sum_{j=1}^n \int_0^t e^{2\varphi_j(\sigma(u))} du
= A_{V,\Sigma}(0,t).
\]

7. Estimate for the evolution on \( N \)

In this section we estimate the evolution kernel on \( N = M \rtimes V \). Denote
\[
P^\sigma(0,t)(m,v) := K^\sigma(0,t)(0,0;m,v).
\]
The main result of this section is the following estimate where \( k_0 \) is as in (1.12).

**Theorem 7.1.** There are positive constants \( C, D \) such that
\[
A_{M,\Pi}(0,t)^{1/2} A_{V,\Pi}(0,t)^{1/2} P^\sigma(0,t)(m,v) \leq
C(\|m\|^{1/2} + 1) \exp \left( - D \|v\|^2 - \frac{D\|m\|^2}{A_{V,\Sigma}(0,t)} - \frac{D\|m\|^2}{(\|m\|^{2k_0} + \|v\|^2 + 2k_0 A_{M,\Sigma}(0,t))} \right)
+ C A_{V,\Sigma}(0,t)^{1/2} \exp \left( - D \|v\|^{2k_0} \right) A_{V,\Sigma}(0,t).
\]

**Proof.** We allow the constants \( C \) and \( D \) to change from line to line. By Lemma 5.1 and Lemma 5.3,
\[
K_{s,t}^{M,\sigma,n}(m^1,m^2) = D(A_{M,n}(t,s)) e^{-B(A_{M,n}(s,t)) (m^1-m^2)}
\]
\[
\leq C A_{M,\Pi}(s,t)^{-1/2} e^{-\frac{\|m^1-m^2\|^2}{2\|A_{M,n}(s,t)\|}}.
\]

\[
(7.1)
\]
For \( m^i \in M \) and \( v^1 \in V \),
\[
\int K^\sigma(0, t)(m^1, v^1; m^2, y)\psi(y) \, dy = \int K^M,\sigma,\eta_{t, 0}(m^1, m^2)\psi(\eta(t)) \, dW^V,\sigma_{v^1}(\eta)
\leq CA^\sigma_{M, \Pi}(0, t)^{-1/2} \int \psi(\eta(t)) e^{-\frac{\|m^1 - m^2\|^2}{2A^\sigma_{M, \Pi}(0, t)^2}} \, dW^V,\sigma_{v^1}(\eta).
\]

Then, by Lemma 5.4,
\[
(7.2) \quad A^\sigma_{M, \Pi}(0, t)^{1/2} \int P^\sigma(0, t)(m, v)\psi(v) \, dv
\leq C \int \exp \left( -\frac{D\|m\|^2}{(1 + \Lambda^\sigma(0, t)^{2k_0})A^\sigma_{M, \Sigma}(0, t)} \right) \psi(\eta(t)) \, dW^V,\sigma_{v}(\eta).
\]

For \( v \in \mathbb{R}^n \) given and \( \varepsilon > 0 \), let
\[
\psi_\varepsilon(\cdot) = \varepsilon^{-n}1_{B_\varepsilon(v)}(\cdot),
\]
where
\[
B_\varepsilon(v) = \prod_{j=1}^n B^1_\varepsilon(v_j) \quad \text{and} \quad B^1_\varepsilon(v_j) = [v_j - \varepsilon/2, v_j + \varepsilon/2].
\]
We will estimate (7.2) with \( \psi_\varepsilon \) in place of \( \psi \) as \( \varepsilon \) tend to zero.

Let \( E^\eta_v \) denote expectation with respect to \( dW^V,\sigma_{v}(\eta) \). For \( k = 1, 2, \ldots \), define the sets of paths in \( V \),
\[
\mathcal{A}_k = \{ \eta : k - 1 \leq \Lambda^\eta(0, t) = \sup_{0 \leq u \leq t} \| \eta(u) \|_\infty < k \},
\]
where by \( \| \cdot \|_\infty \) we denote the maximum norm \( \| x \|_\infty = \max_{1 \leq i \leq n} |x_i| \). The integral on the right in (7.2) can be written as an infinite sum and estimated as follows
\[
(7.3) \quad \sum_{k=1}^{\infty} E^\eta_0 \exp \left( -\frac{D\|m\|^2}{(1 + \Lambda^\sigma(0, t)^{2k_0})A^\sigma_{M, \Sigma}(0, t)} \right) \psi_\varepsilon(\eta(t))1_{\mathcal{A}_k}(\eta)
\leq \sum_{k=1}^{\infty} \exp \left( -\frac{D\|m\|^2}{k^{2k_0}A^\sigma_{M, \Sigma}(0, t)} \right) E^\eta_0 \psi_\varepsilon(\eta(t))1_{\mathcal{A}_k}(\eta).
\]
To simplify notation we introduce
\[
c_k = \exp \left( -\frac{D\|m\|^2}{k^{2k_0}A^\sigma_{M, \Sigma}(0, t)} \right),
\]
\[
E_k(\varepsilon) = E^\eta_0 \psi_\varepsilon(\eta(t))1_{\mathcal{A}_k}(\eta) = \varepsilon^{-n}W^V,\sigma_{v}(\eta \in \mathcal{A}_k \text{ and } \eta(t) \in B_\varepsilon(v)).
\]
Lemma 7.2. Assume that $a > \|v\|_{\infty} + \delta$, $\delta > 0$, and $0 < \varepsilon/2 < \delta$. Then
\[
\varepsilon^{-n} W_0^{V,\sigma} \left( \sup_{u \in [0, t]} \|\eta(u)\|_{\infty} \geq a \text{ and } \eta(t) \in B_{\varepsilon}(v) \right) 
\leq A_{V,\Pi}^{\sigma}(0, t)^{-1/2} \sum_{j=1}^{n} \left( e^{-(2a-v_j)^2/2A_{v,j}^{\sigma}(0,t)} + e^{-(2a+v_j)^2/2A_{v,j}^{\sigma}(0,t)} \right).
\]

Proof. Reasoning as in (7.5) we see that the left side of the above inequality is bounded by
\[
C \sum_{j=1}^{n} \left( \prod_{i \neq j} A_{v,j}^{\sigma}(0, t) \right)^{-1/2} \times \varepsilon^{-1} W_0 \left( \sup_{u \in [0, A_{v,j}^{\sigma}(0,t)/2]} |\eta(u)| \geq a \text{ and } \eta_j(A_{v,j}^{\sigma}(0,t)) \in B_{\varepsilon}(v_j) \right).
\]

By our assumption it follows that for every $j$, $a > |v_j| + \delta$. Hence, the result follows by Corollary 2.7. \qed
Let
\[ I = I(\varepsilon) = \sum_{k=1}^{\infty} c_k E_k(\varepsilon). \]

**Lemma 7.3.** We have
\[ A_M^\sigma(0, t)^{1/2} P^\sigma(0, t)(m, v) \leq CI, \]
where
\[ I = \limsup_{\varepsilon \to 0^+} \sum_{k \geq \|v\|_\infty} c_k E_k(\varepsilon). \]
Furthermore, the sum converges uniformly in \( \varepsilon \).

**Proof.** The inequality follows by letting \( \varepsilon \) tend to 0 in (7.3). The uniform convergence follows from Lemma 7.2. \( \square \)

Let \( n_o \) be the smallest natural number such that \( n_o \geq \|v\|_\infty \).

**Lemma 7.4.** We have the following estimates
\[ \limsup_{\varepsilon \to 0^+} E_{n_o}(\varepsilon) \leq CA_{V,\Pi}^\sigma(0, t)^{-1/2} e^{-\|v\|_\infty^2 / 2A_{V,\Sigma}^\sigma(0, t)}, \]
while for \( k \geq n_o + 1 \),
\[ \limsup_{\varepsilon \to 0^+} E_k(\varepsilon) \leq CA_{V,\Pi}^\sigma(0, t)^{-1/2} e^{-\left(\frac{2(k-1) - \|v\|_\infty^2}{2A_{V,\Sigma}^\sigma(0, t)}\right)} \exp\left(-\frac{|v_j|^2}{2A_{V,j}^\sigma(0, t)}\right). \]

**Proof.** Consider \( E_{n_o} \). Let \( j \in \{1, \ldots, n\} \) be fixed. Suppose first that \( |v_j| < n_o - 1 \). Then, using Corollary 2.8, the \( j \)-th term in (7.5) (with \( k = n_o \)) is dominated by a multiple of
\[ A_{V,\Pi}^\sigma(0, t)^{-1/2} e^{-\frac{(2(n_o-1) - |v_j|)^2}{2A_{V,j}^\sigma(0, t)}} \prod_{i \neq j} e^{-\frac{|v_i|^2}{2A_{V,i}^\sigma(0, t)}}. \]
Notice that \( |v_j| \) cannot be equal to \( \|v\|_\infty \). Thus we are done in this case. Now suppose that \( |v_j| \geq n_o - 1 \). Then, using Corollary 2.8 again, we dominate the \( j \)-th term in (7.5) by a multiple of
\[ A_{V,\Pi}^\sigma(0, t)^{-1/2} e^{-\frac{|v_j|^2}{2A_{V,j}^\sigma(0, t)}} \prod_{i \neq j} e^{-\frac{|v_i|^2}{2A_{V,i}^\sigma(0, t)}}. \]
The result for \( E_0 \) follows.

Now we consider \( E_k \). Since \( k \geq n_o + 1 \) it follows that \( k - 1 \geq |v_j| \) for every \( j \). Therefore, by Corollary 2.8 the \( j \)-th term in (7.5) is estimated by
\[ CA_{V,\Pi}^\sigma(0, t)^{-1/2} e^{-\frac{(2(k-1) - |v_j|)^2}{2A_{V,j}^\sigma(0, t)}} \prod_{i \neq j} e^{-\frac{|v_i|^2}{2A_{V,i}^\sigma(0, t)}}. \]
Next we estimate \( I = \limsup_{\varepsilon \to 0^+} \sum_{k \geq \|v\|_\infty} c_k \mathcal{E}_k(\varepsilon) \). From Lemma 7.4

(7.6)

\[
A_{V,\Pi}^\sigma(0, t)^{1/2} I = A_{V,\Pi}^\sigma(0, t)^{1/2} \limsup_{\varepsilon \to 0^+} \left( c_{n_0} \mathcal{E}_{n_0}(\varepsilon) + \sum_{k > n_0} c_k \mathcal{E}_k(\varepsilon) \right)
\]

\[\leq C \exp \left( -\frac{\|v\|_\infty^2}{2A_{V,\Sigma}^\sigma(0, t)} - \frac{D\|m\|^2}{n_o^2 k_o A_{M,\Sigma}^\sigma(0, t)} \right) + \sum_{k = n_o+1}^{\infty} \exp \left( -\frac{D\|m\|^2}{2k_o A_{M,\Sigma}^\sigma(0, t)} - \frac{(2(k-1) - \|v\|_\infty^2)^2}{2A_{V,\Sigma}^\sigma(0, t)} \right).\]

For \( a, b \) non-negative \( a + b \geq \sqrt{a^2 + b^2} \). Also, for \( k \geq n_o + 1 \),

\[(k-1) + (k-1) - \|v\|_\infty \geq n_o + (k-1 - \|v\|_\infty), \]

\[k-1 - \|v\|_\infty \geq n_o - \|v\|_\infty \geq 0.\]

Hence the summation in the last line of (7.6) is bounded by

(7.7)

\[
\sum_{k = n_o+1}^{\infty} \exp \left( -\frac{D\|m\|^2}{k^2 k_o A_{M,\Sigma}^\sigma(0, t)} - \frac{(n_o + (k-1) - \|v\|_\infty^2)}{2A_{V,\Sigma}^\sigma(0, t)} \right) \leq e^{-\frac{n_o^2}{2A_{V,\Sigma}^\sigma(0, t)}} \sum_{k = n_o+1}^{\infty} \exp \left( -\frac{D\|m\|^2}{k^2 k_o A_{M,\Sigma}^\sigma(0, t)} - \frac{(k-1) - \|v\|_\infty^2}{2A_{V,\Sigma}^\sigma(0, t)} \right).
\]

We split the sum in (7.7) into two parts: \( n_o + 1 \leq k \leq n_o + \|m\|^{1/(2k_o)} \) and \( k > n_o + \|m\|^{1/(2k_o)} \), and estimate the corresponding parts by the following two terms.

\[
\|m\|^{1/2k_o} e^{-\frac{n_o^2}{2A_{V,\Sigma}^\sigma(0, t)}} \exp \left( -\frac{D\|m\|^2}{\|m\|^{1/2k_o} + \|v\|_\infty + 2k_o A_{M,\Sigma}^\sigma(0, t)} \right)
\]

and

\[
e^{-\frac{n_o^2}{2A_{V,\Sigma}^\sigma(0, t)}} \sum_{k \geq n_o + \|m\|^{1/2k_o} + 1} \exp \left( -\frac{D\|m\|^2}{k^2 A_{M,\Sigma}^\sigma(0, t)} - \frac{(k-1 - n_o)^2}{2A_{V,\Sigma}^\sigma(0, t)} \right).
\]

The above expression is bounded by

\[
e^{-\frac{n_o^2}{2A_{V,\Sigma}^\sigma(0, t)}} \int_{\|m\|^{1/2k_o}}^{\infty} e^{-\frac{r^2}{2A_{V,\Sigma}^\sigma(0, t)}} \, dr \leq \sqrt{2A_{V,\Sigma}^\sigma(0, t)} e^{-\frac{\|m\|^{2/2k_o}}{2A_{V,\Sigma}^\sigma(0, t)} - \frac{\|m\|^{2/2k_o}}{2A_{V,\Sigma}^\sigma(0, t)}}.
\]

Theorem 7.1 follows. \( \square \)
7.1. Proof of Theorem 1.5. To simplify our notation we set

\[ A_0 = A_{\nu, \Pi}(0, t)^{1/2}, \]
\[ A_1 = A_{\sigma, \Sigma}(0, t), \]
\[ A_2 = A_{\sigma, M}(0, t), \]
\[ A_3 = A_{\nu, \Sigma}(0, t). \]

(7.8)

It follows immediately from Theorem 7.1 that there are positive constants \( C \) and \( D \) such that in the region \( \|v\| \leq \|m\|^{1/2}k_o \),

\[ A_0 P_\sigma(0, t)(m, v) \leq C(\|m\|^{1/(2k_o)} + 1) \exp \left( -D \frac{\|v\|^2}{A_1} - D \frac{\|m\|^2}{A_2} \phi_{2k_o}(m) \right) + CA_1^{1/2} \exp \left( -D \frac{\|m\|^{1/k_o} + \|v\|^2}{A_1} \right) \]

while in the region \( \|v\| \geq \|m\|^{1/2}k_o \),

\[ A_0 P_\sigma(0, t)(m, v) \leq C(\|m\|^{1/(2k_o)} + 1 + A_1^{1/2}) \exp \left( -D \frac{\|m\|^{1/k_o} + \|v\|^2}{A_1} \right). \]

Since \( \phi_{2k_o}(m) \leq 1 \), for all \( m, A_1 \leq A_3 \) and \( A_2 \leq A_3 \), Theorem 1.5 follows.

8. Upper estimate for the Poisson kernel

Let \( \nu(x) = \nu(m, v), m \in \mathbb{R}^m, v \in \mathbb{R}^n \), be the Poisson kernel on \( N \) for the operator \( L_\alpha \) in (1.6). Recall that we assume that

\[ \lambda(\alpha) > 0 \text{ for all } \lambda \in \Lambda. \]

Hence \( \alpha \) belongs to the positive Weyl chamber \( A^+ \). The operator \( \Delta_\alpha \) generates the Brownian motion with drift \( -2\alpha \),

\[ \sigma(u) = b(u) - 2\alpha u, \]

where \( b(u) \) is the \( k \)-dimensional standard Brownian motion normalized so that \( \text{Var} b_u = 2u \).

Let \( \nu^a \) be as in (2.8). We also use the notation introduced in (1.5).

**Theorem 8.1.** For all compact subsets \( K \not
\}

(8.1) \( \nu^{sp}(x) \leq Ce^{-\rho_0(s)p}e^{(s/2)\gamma_\theta(\rho)\gamma_\alpha(\rho)}e^{(s/2)\gamma_\lambda(\rho)\gamma_\lambda(\alpha)} \)

if \( x \in K \cap \{ \phi(m) \geq \varepsilon, \|v\| \geq \varepsilon \} \),

(8.2) \( \nu^{sp}(x) \leq Ce^{-\rho_0(s)p}e^{s\gamma_\theta(\rho)\gamma_\alpha(\rho)} \)

if \( x \in K \cap \{ \|v\| \geq \varepsilon \} \).
and
\begin{equation}
\nu^{sp}(x) \leq C e^{-\rho A(x)} e^{s \gamma \alpha A(x)} \text{ if } x \in K \cap \{ \phi(m) \geq \varepsilon \},
\end{equation}
where \( \phi(m) = \phi_{2k_0}(m) \) is defined in (1.11).

Proof. First we consider elements \( x = (m, v) \) from the set
\[ K_1 = K \cap \{(m, v) : \phi(m) \geq \varepsilon \}. \]
Let \( A_j \) be defined as in (7.8) but with \( t = \infty \). By Theorem 1.5, we have
\begin{equation}
\nu^{sp} \leq C \mathbf{E}_{sp} A_0^{-1} \exp \left( -\frac{D}{A_1} - \frac{D}{A_3} \right) + C \mathbf{E}_{sp} A_0^{-1} A_1^{1/2} \exp \left( -\frac{D}{A_1} - \frac{D}{A_3} \right).
\end{equation}
We estimate the first expectation on the right.
\begin{equation}
\mathbf{E}_{sp} A_0^{-1} \exp \left( -\frac{D}{A_1} - \frac{D}{A_3} \right)
\leq \left( \mathbf{E}_{sp} (A_0^{-1})^2 \right)^{1/2} \left( \mathbf{E}_{sp} \exp \left( -\frac{2D}{A_1} - \frac{2D}{A_3} \right) \right)^{1/2}
\leq \left( \mathbf{E}_{sp} (A_0^{-1})^2 \right)^{1/2} \left( \mathbf{E}_{sp} \exp \left( -\frac{4D}{A_1} \right) \right)^{1/4} \left( \mathbf{E}_{sp} \exp \left( -\frac{4D}{A_3} \right) \right)^{1/4}.
\end{equation}
By the Cauchy-Schwarz inequality we get,
\begin{equation}
\mathbf{E}_{sp} (A_0^{-1})^2 = \mathbf{E}_{sp} (A_{M,1}^{-1})^{-1} (A_{V,1}^{-1})^{-1}
= e^{-2\rho A(x)} \mathbf{E}_0 (A_{M,1}^{-1})^{-1} (A_{V,1}^{-1})^{-1}
\leq e^{-2\rho A(x)} \left( \mathbf{E}_0 (A_{M,1}^{-1})^{-2} \right)^{1/2} \left( \mathbf{E}_0 (A_{V,1}^{-2})^{-2} \right)^{1/2}.
\end{equation}
Since, by Lemma 2.3, the expected values \( \mathbf{E}_0 (A_{M,j}^{-d})^{-d}, j = 1, \ldots, m, \) and \( \mathbf{E}_0 (A_{V,i}^{-d})^{-d}, i = 1, \ldots, n, \) are finite for all \( d > 0 \), we can apply the Cauchy-Schwarz inequality successively to each of the remaining expectation in (8.6) and conclude their finiteness.

Now we consider \( \mathbf{E}_{sp} \exp(-4D_1/A_1) \) and \( \mathbf{E}_{sp} \exp(-4D_2/A_3) \) from (8.5). Clearly,
\begin{equation}
\mathbf{E}_{sp} \exp(-4D_1/A_1) \leq \mathbf{E}_0 \exp(-4D_1/(M(s)A_1))
\end{equation}
where
\[ M(s) = \max_{\phi \in \Theta} e^{2\phi(s)} = e^{2\min_{\phi \in \Theta} \phi(s)} = e^{2s\gamma \phi(s)}. \]
Proceeding exactly in the same way as in the proof of [17, Lemma 6.2] we show that (8.7) is bounded by
\begin{equation}
CM(s) \Phi^{(\alpha)} = C e^{2s\gamma \phi(s) \Phi^{(\alpha)}}.
\end{equation}
The expectation $E_{s\rho} \exp(-4D_2/A_3)$ is similar. Again, in the same way as in the proof of [17, Lemma 6.2] we show that $E_{s\rho} \exp(-4D_2/A_3)$ is bounded by

$$CM_1(s\rho) \theta(\rho),$$

where

$$M_1(s\rho) = \max_{\lambda \in \Lambda} e^{2\lambda(s\rho)} = e^{2s\gamma(\rho)}.$$

Hence,

$$E_{s\rho} \exp(-4D_2/A_3) \leq Ce^{2s\gamma(\rho)\theta(\rho)}.$$

Now we estimate the second expectation on the right in (8.4).

$$E_{s\rho} A_0^{-1} A_1^{1/2} \exp \left(-\frac{D}{A_1} - \frac{D}{A_3} \right)$$

$$\leq \sum_{j=1}^n E_{s\rho} A_0^{-1} A_{V,j}^{1/2} \exp \left(-\frac{D}{A_1} - \frac{D}{A_3} \right)$$

$$= \sum_{j=1}^n E_{s\rho} A_{M,\Pi}^{-1/2} \prod_{k \neq j} A_{V,k}^{1/2} \exp \left(-\frac{D}{A_1} - \frac{D}{A_3} \right)$$

$$= \sum_{j=1}^n e^{-\sum_{\xi \in \Xi} \xi(s\rho)} e^{-\sum_{\theta \neq \theta_j} \theta(s\rho)} E_0 A_{M,\Pi}^{-1/2} \prod_{k \neq j} A_{V,k}^{1/2} \exp \left(-\frac{D}{A_1} - \frac{D}{A_3} \right).$$

Since $s < 0$,

$$e^{-\sum_{\xi \in \Xi} \xi(s\rho)} e^{-\sum_{\theta \neq \theta_j} \theta(s\rho)} \leq e^{-\rho_0(s\rho)}.$$

To estimate

$$E_0 A_{M,\Pi}^{-1/2} \prod_{k \neq j} A_{V,k}^{1/2} \exp \left(-\frac{D}{A_1} - \frac{D}{A_3} \right)$$

we proceed as in (8.5) and (8.6) and get the same estimate. Hence, the estimate (8.1) holds on $K_1$.

Now we have to consider the set

$$K_2 = K \cap \{(m, v) : \|v\| \geq \varepsilon\}.$$

On this set (8.5) simplifies and, using Lemma 2.3, (8.6), (8.7) and (8.8) as above, we get

$$E_{s\rho} A_0^{-1} \exp \left(-\frac{D_1}{A_1} \right) \leq \left( E_{s\rho} (A_0^{-1})^{1/2} \left( E_{s\rho} \exp \left(-\frac{2D_1}{A_1} \right) \right)^{1/2} \right.$$}

$$\leq e^{-\rho_0(s\rho)} e^{s\gamma(\rho)\theta(\alpha)}.$$
As in the previous case the second expectation in (8.4) has the same estimate. Hence, the estimate (8.2) holds on \(K\). Finally, we consider the set
\[ K_3 = K \cap \{(m, v) : \phi(m) \geq \varepsilon\}. \]
Then
\[ E_{s\rho A_0^{-1}} \exp \left( -\frac{D_2}{A_3} \right) \leq \left( E_{s\rho}(A_0^{-1}) \right)^{1/2} \left( E_{s\rho} \exp \left( -\frac{2D_2}{A_3} \right) \right)^{1/2} \]
\[ \leq e^{-\rho_0(s\rho)} e^{s\gamma_\alpha(s\rho)\pi_\alpha}. \]
Again, the second expectation in (8.4) has the same estimate. Thus (8.3) is proved. \(\square\)

8.1. **Proof of Theorem 1.3.** Having Theorem 8.1, we use the standard homogeneity argument as follows.

*Proof of Theorem 1.3.* It is clear that for \(x \in N\) with the norm \(|x|_\rho \leq 1\) we have \(\nu(x) \leq C_\rho\). Let \(\delta_t^\rho = \text{Ad}((\log t)\rho)\). Then \(|\delta_t^\rho x|_\rho = t|x|_\rho\). Let \(x = \delta_\exp(-s)x_0\) with \(|x_0|_\rho = 1\) and \(s < 0\). Then \(|x|_\rho = e^{-s} > 1\). Let \(K = \{x_0 : |x_0|_\rho = 1\}\). By definition (2.8) of \(\nu^\rho\), we get
\[ \nu(x) = \nu(\delta_\exp(-s)x_0) = \nu((s\rho)^{-1}x_0(s\rho)) = e^{\rho_0(s\rho)} \nu^{s\rho}(x_0), \]
where \(\rho_0 = \sum_j \partial_j + \sum_i \xi_i\), and the result follows from Theorem 8.1. \(\square\)

9. **Example**

Consider \(N = \mathcal{H}_n\), the \(2n+1\)-dimensional Heisenberg group, which we realize as \(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\) with the Lie group multiplication given by
\[ (x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 \cdot y_2). \]

The corresponding Lie algebra \(\mathfrak{h}_n\) is then spanned by the left invariant vector fields
\[ X_j = \partial_{x_j}, \quad Y_j = \partial_{y_j} + x_j \partial_z, \quad Z = \partial_z \]
where \(1 \leq j \leq n\). Let \(A = \mathbb{R}^k\) and let \(\xi_{1,j}, \xi_{2,j}, \xi_3 \in (\mathbb{R}^k)^*\), \(1 \leq j \leq n\), be such that
\[ \xi_{1,j} + \xi_{2,j} = \xi_3 \]
independently of \(j\). For \(x \in \mathbb{R}^n\), \(a \in \mathbb{R}^k\), and \(i = 1, 2\), we set
\[ e^{\xi_i(a)} x = (e^{\xi_{i,1}(a)} x_1, e^{\xi_{i,2}(a)} x_2, \ldots, e^{\xi_{i,n}(a)} x_n). \]

We then define an \(A\) action on \(\mathcal{H}_n\) by automorphisms of \(\mathcal{H}_n\) by
\[ a(x, y, z) a^{-1} = (e^{\xi_1(a)} x, e^{\xi_2(a)} y, e^{\xi_3(a)} z), \]
We then let \(S = \mathcal{H}_n \ltimes A\).
Let $X_j$, $Y_j$, and $Z$ be, respectively, left invariant vector fields on $S$. Then

$$X_j = e^{\xi_{1,j}(a)}X_j, \quad Y_j = e^{\xi_{2,j}(a)}Y_j, \quad Z = e^{\xi_a}Z.$$  

We set

$$\mathcal{L}_\alpha = \sum_{j=1}^n \left( X_j^2 + Y_j^2 \right) + Z^2 + \Delta_\alpha$$

(9.1)

$$= \sum_{j=1}^n \left( e^{2\xi_{1,j}(a)}X_j^2 + e^{2\xi_{2,j}(a)}Y_j^2 \right) + e^{2\xi_a}Z^2 + \Delta_\alpha,$$

where $\Delta_\alpha$ is defined in (1.2).

**Example.** Consider the operator $\mathcal{L}_\alpha$ defined in (9.1) on $\mathcal{H}_n \rtimes A$ with $A = \mathbb{R}^2$ and $\xi_{1,j} = (1,0)$, $\xi_{2,j} = (0,1)$. Theorem 1.1 gives

$$\nu(x,y,z) \leq C(1 + |(x,y,z)|\rho)^{-\frac{C_1\rho_0(\rho)\gamma(\alpha)}{4}},$$

where

$$\gamma(\alpha) = 2 \min(\alpha_1, \alpha_2)$$

for some constant $C_1$ which depends on $\rho$ and can be computed. Take $\rho = (1,2)$. We have $\rho_0 = \sum_j \xi_{1,j} + \sum_j \xi_{2,j} + \xi_3$, where $\xi_{i,j}(a) = a_i$, $i = 1, 2$, $j = 1, \ldots, n$. To compute $C_1$ we proceed similarly as in [17, Example 1] and get

$$\nu(x,y,z) \leq C(1 + |(x,y,z)|\rho)^{-\frac{\min(\alpha_1, \alpha_2)}{2}}.$$  

whereas Theorem 1.3 gives, for example for $\phi(y,z) > 1$ and $\|y\| > 1$,

$$\nu(x,y,z) \leq C(1 + |(x,y,z)|\rho)^{-\frac{\alpha_1}{2} - \frac{\min(\alpha_1, \alpha_2)}{2}}.$$  

Similarly, Theorem 1.2 gives

$$\nu(x,y,z) \leq C_q(1 + |(x,y,z)|\alpha)^{-\frac{2}{q}(\min(\alpha_1, \alpha_2))^2},$$

whereas Theorem 1.3 gives,

$$\nu(x,y,z) \leq C(1 + |(x,y,z)|\rho)^{-\frac{\alpha_1}{2} - \frac{\alpha_2}{2}} - \frac{\alpha^2}{2}$$

for $\phi(y,z) > 1$ and $\|y\| > 1$, which is again a better estimate if we take for example an operator with $\alpha_1 \sqrt{\frac{4}{q} - 1} < \alpha_2$.  

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