A discrete curvature on a planar graph

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Summary. Given a planar graph derived from a spherical, euclidean or hyperbolic tessellation, one can define a discrete curvature by combinatorial properties, which after embedding the graph in a compact 2d-manifold, becomes the Gaussian curvature.

1 Introduction

In recent years some approaches to quantum gravity have suggested the hypothesis of a discrete space time [1] as a consequence of the combinatorial properties of spin networks underlying the structure of space [2] and implemented with the discrete time hypothesis of causal sets [3]. We have also presented a philosophical interpretation of a discrete model of space time, based in the relational theory proposed by Leibniz [4].

In this model one starts from a set of elements and the relation among them, without presupposing space and time as a background. We have only at our disposal combinatorial properties of those relations, from which the metrical quantities should be generated.

As Riemann mentioned in his Habilitation Thesis: “The quantitative comparison [of quanta of space] happens for discrete magnitudes through counting, for continuous ones through measurements” [5].

In our model we have to choose the discrete quantities in such a way that in the continuous limit they become the classical ones. To this aim we present a two ways method. The direct way consists on calculating some continuous quantity in a 2d-manifold. For instance, take a tessellation generated by some triangle reflection group. Keeping only the vertices and edges of the tessellation, we obtain a graph (the skeleton) where we can define discrete quantities such as path, distance, curvature.
By the inverse way, we construct an embedding of the graph such that we recover the continuous quantities. In this example, the embedding surface is the original manifold of the tessellation. We can start directly from the graph and find some embedding where the corresponding quantities become analog, like the genus of some $3nj$-symbols graph [6], or the curvature in a triangulated manifold [7] or the area and volume eigenvalues for the gravitational field operator [8]. According to Bombelli one can scatter points in a Lorentzian manifold with uniform density, and then keep the statistical distribution of points from which some discrete quantities can be defined, such as curvature, from combinatorial properties of the set of relations [9].

In section 2 we review algebraic properties of triangle reflection groups in 2d-manifolds. In section 3 we present the spherical, euclidean hyperbolic tessellations generated by Coxeter groups. In section 4 we apply the fundamental properties of Gauss curvature to the continuous tessellations. In section 5 we construct a graph and define on it the curvature such that, when the embedding is performed, we recover the standard one.

These ideas were communicated in the Symposium held in Marseille on loops and spin foams [10].

2 Reflection groups

Let $P$ a finite sided n-dimensional convex polyhedron in a metric space $X$ of finite volume, all of whose dihedral angles are submultiple of $\pi$. Then the group generated by the reflection of $X$ in the sides of $P$ is a discrete reflection group $\Gamma$ with respect to the polyhedron $P$.

In order to construct a presentation for a discrete reflection group, we take all the sides $\{S_i\}$ of $P$ for each pair of indices $i, j$. Let $k_{ij} \equiv \frac{\pi}{\vartheta(S_i, S_j)}$, where $\vartheta(S_i, S_j)$ is the angle between $S_i$ and $S_j$. We call

$$\left\{S_i, (S_iS_j)^{k_{ij}}\right\}$$

(1)

a presentation of the discrete reflection groups $\Gamma$ generated by the reflections on $S_i$. A discrete reflection group is isomorphic to a Coxeter group $G$, that is, an abstract group defined by a group presentation $\left\{S_i, (S_iS_j)^{k_{ij}}\right\}$ where

a) the exponent $k_{ij}$ is a positive integer or infinite,

b) $k_{ij} = k_{ji} > 1$, if $i \neq j$, $k_{ii} = 1$ for each $i, j$,

c) If $k_{ij} = \infty$, the term $(S_iS_j)^{\infty}$ is omitted

The Coxeter graph of $G$ is the labeled graph with vertices $i \in J$ an edges $\{(i, j), k_{ij} > 2\}$. Each edge $i, j$ is labelled by $k_{ij}$. For simplicity the label $k_{ij} = 3$ is omitted.
Let $\Delta$ be an $n$-simplex in $X$ all of whose dihedral angles are submultiple of $\pi$. The group $\Gamma$ generated by the reflections of $X$ in the sides of $\Delta$ is an $n$-simplex discrete reflection group. Notice that $X$ can be $S^n$, $E^n$ or $H^n$.

The classification of all the irreducible $n$-simplex (spherical, euclidean and hyperbolic) reflection groups is complete [11].

Assume that $n = 2$. Then $\Delta$ is a triangle in $X$, whose angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$ are submultiple of $\pi$. If we call $T(l, m, n)$ the group $\Gamma$ generated by the reflections in the sides of $T(l, m, n)$ is call a triangle reflection group. We considere all the cases:

If $X = S^2$ the only spherical triangle reflection groups have the following Coxeter graphs:

- $T(2, 2, 2)$
- $T(2, 3, 4)$
- $T(2, 3, 5)$
- $T(2, 3, 3)$

If $X = E^2$ we have the euclidean triangle reflection groups with Coxeter graphs:

- $T(3, 3, 3)$
- $T(2, 4, 4)$
- $T(2, 3, 6)$

If $X = H^2$ we have the hyperbolic triangle reflection groups with Coxeter graphs

- $T(2, m, n)$ for $m, n \geq 3$
- $T(l, m, n)$ for $l, m, n \geq 3$

Another type of Coxeter groups are generalized (or non-compact) simple reflection groups, that are defined only in $H^n$. A generalized $n$-simplex is an $n$-dimensional polyhedron with $n + 1$ generalized vertices (either a vertex of $P$). the generalized (non compact) hyperbolic n-simple reflection groups exist only for $n \leq 10$ and have been classified completely [12].

The generalized hyperbolic triangle reflection groups have the following Coxeter graph:

- $\infty \infty \infty$ for $l \geq m \geq 3$
- $\infty \infty$ for $n \geq 3$
3 Geometric representation of Coxeter group and tessellations

We have seen in Section 2 that a Coxeter group is isomorphic to a discrete reflection group. Geometrically a reflection can be represented by a linear transformation which fixes an hyperplane point wise and sends some non zero vector to its negative. In the metric space we construct vectors \( \{ \alpha_i \} \) in one to one correspondence to the sides \( \{ S_i \} \) defined before, in such a way that the angle between \( \alpha_i \) and \( \alpha_j \) will be compatible with the values of \( k_{ij} \), namely, 
\[
\theta(\alpha_i, \alpha_j) = \frac{\pi}{k_{ij}}.
\]

In order to construct a reflection with respect to these vectors \( \{ \alpha_i \} \) we define a non-degenerate symmetric bilinear form on \( X \) by the formulas
\[
\langle \alpha_i, \alpha_j \rangle = -\cos \frac{\pi}{k_{ij}} \tag{2}
\]
This expression is interpreted to be \(-1\) for \( k_{ij} = \infty \). Obviously \( \langle \alpha_i, \alpha_i \rangle = 1 \), and \( \langle \alpha_i, \alpha_j \rangle \leq 0 \) for \( i \neq j \). For each vector \( \alpha_i \) we can define a reflection \( S_i \) on \( X \):
\[
S_i \beta = \beta - 2 \langle \alpha_i, \beta \rangle \alpha_i, \quad \beta \in X
\]
clearly \( S_i \alpha_i = -\alpha_i \) and all the vectors \( \gamma \) satisfying \( \langle \alpha_i, \gamma \rangle = 0 \) belong to a plane invariant under \( S_i \). It can be proved that the group of reflections generated by \( \{ S_i \} \) is homomorphic to the Coxeter group \( G \) because they satisfy
\[
S_i^2 = 1, \quad (S_i S_j)^{k_{ij}} = 1 \quad \text{for} \quad i \neq j \tag{3}
\]
we call this homomorphism the geometric representation of \( G \).

The group of reflections generated by \( \{ S_i \} \) is connected with the geometrical property of space called tessellation. This can be seen in an intuitive way by the covering of the, say, two dimensional euclidean space, repeating the same or a finite number of figures without overlapping or holes.

In general, a tessellation of a metric space \( X (= S^n, E^n \text{ or } H^n) \) is a collection \( P \) of \( n \)-dimensional convex polyhedra in \( X \) such that.

a) the interior of the polyhedra in \( P \) are mutually disjoint,

b) the union of the polyhedra in \( P \) is \( X \).

For our purpose we need two more definitions.

A tessellation of \( X \) is exact if and only if each side of a polyhedron \( P \) in \( P \) is a side of exactly two polyhedra \( P \) and \( Q \) in \( P \).

A regular tessellation of \( X \) is an exact tessellation of \( X \) consisting of congruent regular polytopes.

We have shown in section 2 that if \( \Delta \) is an \( n \)-simplex in \( X \) all of whose dihedral angles are submultiple as \( \pi \) then the group \( \Gamma \) generated by the reflection of \( X \) in the sides of \( \Delta \) is a discrete reflection group, the geometrical representation of which was given with the help of the bilinear form (2).
It can be proved that the collection of the polyhedra obtained by the reflections on the side of $\Delta$ is a tessellation of $X$. Since an n-simplex is a regular congruent polytope the tessellation is regular. Therefore all the n-simplex (compact or non-compact) reflection groups lead to regular tessellation of $X$.

We give now some examples of tessellations in $S^2$, $E^2$ and $H^2$ generating by reflecting in the sides of a spherical, euclidean and hyperbolic triangle as defined in section 2.

Notice that the hyperbolic tessellation has been drawn using the conformal disk model [13].

Fig. 1. Tessellation of $S^2$ (in stereographic projection) by reflecting in the sides of the spherical triangle $T(2, 2, 5)$

Fig. 2. Tessellation of $E^2$ generated by reflecting in the sides of the euclidean triangle $T(2, 3, 6)$
4 Gauss curvature of continuous tessellations

Two dimensional tessellations in $X (= S^2, E^2$ or $H^2)$ are generated by 2-simplex (triangle) reflection group. In order to calculate the gaussian curvature, we review some geometrical properties of geodesic triangles.

In $S^2$ the geodesic triangle (that is, triangle whose sides are arcs of geodesics) are spherical. Given a spherical triangle $T(x, y, z)$ with:

$$\begin{align*}
\text{Geodesic sides: } &\quad [y, z], \quad [z, x], \quad [x, y], \\
\text{Length of sides: } &\quad a, \quad b, \quad c, \\
\text{Geodesic arcs: } &\quad f(t) \quad g(t) \quad h(t) \\
&\quad [o, a] \rightarrow S^2, \quad [o, b] \rightarrow S^2, \quad [o, c] \rightarrow S^2
\end{align*}$$
Angle $\alpha$ between $[z, x]$ and $[x, y] = \vartheta (-g'(b), h'(0))$

Angle $\beta$ between $[x, y]$ and $[y, z] = \vartheta (-h'(c), f'(0))$

Angle $\gamma$ between $[y, z]$ and $[z, x] = \vartheta (f'(a), g'(0))$

The excess of the interior angles of a spherical triangle is:

$$\varepsilon = \alpha + \beta + \gamma - \pi$$

It can be proved that this excess is always positive [14].

The area of the triangle $T(x, y, z)$ is [15]

$$\text{Area} \{T(x, y, z)\} = \alpha + \beta + \gamma - \pi = \varepsilon$$

In $E^2$ we have euclidean triangles $T(x, y, z)$

with sides: $[y, z], [z, x], [x, y],$

Length of sides: $a, b, c,$

Angles: $\alpha, \beta, \gamma$

The excess of the interior angles of an euclidean triangle is

$$\varepsilon = \alpha + \beta + \gamma - \pi = 0$$

In $H^2$ the geodesic triangles are hyperbolic. For the hyperbolic triangle $T(x, y, z)$ we have

Geodesic sides: $[y, z], [z, x], [x, y],$

Length of sides: $a, b, c,$

Geodesic arcs: $f(t), g(t), h(t),$ \[ \left[ a, a \right] \rightarrow H^2, \left[ a, b \right] \rightarrow H^2, \left[ a, c \right] \rightarrow H^2 \]

Angle between $[z, x]$ and $[x, y]$: $\alpha = \vartheta (-g'(b), h'(0))$

Angle between $[x, y]$ and $[y, z]$: $\beta = \vartheta (-h'(c), f'(0))$

Angle between $[y, z]$ and $[z, x]$: $\gamma = \vartheta (-f'(a), g'(0))$
A generalized hyperbolic triangle has at least one vertex at infinite. The corresponding angle is zero. An ideal triangle has three vertices at infinite. The excess of the interior angles of an hyperbolic triangle is

$$\epsilon = \alpha + \beta + \gamma - \pi$$  (7)

It can be proved that this excess is always negative [16].

The area of an hyperbolic triangle $T(x, y, z)$ is [17]

$$\text{Area} \ (T) = \pi - (\alpha + \beta + \gamma)$$  (8)

If $T(x, y, z)$ is an generalized hyperbolic triangle

$$\text{Area} \ (T) = \pi - \alpha$$  (9)

If $T(x, y, z)$ is an ideal triangle

$$\text{Area} \ T(x, y, z) = \pi$$  (10)

We now give explicit values for the excess angle and area of the 2-simplex (triangle) reflection groups.

In $S^2$ the spherical triangle are given in Section 2.

The corresponding excess and area are, in units of $\pi$ (remember $\alpha = \frac{\pi}{l}, \beta = \frac{\pi}{m}, \gamma = \frac{\pi}{n}$):

$$\frac{\epsilon}{\pi} = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1$$  (11)

$$\frac{A}{\pi} = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1$$  (12)

We have the following cases

$T(2, 2, 2) : \frac{\epsilon}{\pi} = \frac{4}{\pi} = \frac{1}{2}$

$T(2, 2, n) : \frac{\epsilon}{\pi} = \frac{4}{\pi} = \frac{1}{n}$

$T(2, 3, 3) : \frac{\epsilon}{\pi} = \frac{4}{\pi} = \frac{1}{6}$

$T(2, 3, 4) : \frac{\epsilon}{\pi} = \frac{4}{\pi} = \frac{1}{12}$

$T(2, 3, 5) : \frac{\epsilon}{\pi} = \frac{4}{\pi} = \frac{1}{30}$

In $E^2$ the euclidean triangles are given in Section 2.

The excess angle is $\epsilon = 0$ for

$T(3, 3, 3), \ T(2, 4, 4) \ \text{and} \ T(2, 3, 6)$

In $H^2$ the hyperbolic triangles are given in Section 2.

The corresponding excess and area in units of $\pi$ are (remember $\alpha = \frac{\pi}{l}, \beta = \frac{\pi}{m}, \gamma = \frac{\pi}{n}$):

$$\frac{\epsilon}{\pi} = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1$$  (13)
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\[ \frac{A}{\pi} = 1 - \left( \frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right) \]  
(14)

When some of the vertices are ideal, the corresponding angle is zero (the number \(l, m\) or \(n\) becomes infinite).

For instance, \( T(l, m, \infty) \),

\[ \frac{\epsilon}{\pi} = \frac{1}{l} + \frac{1}{m} - 1 \]  
(15)
\[ \frac{A}{\pi} = 1 - \left( \frac{1}{l} + \frac{1}{m} \right) \]  
(16)

We can now apply these results to the curvature of the surfaces corresponding to the 2-dimensional regular tessellations (spherical, euclidean or hyperbolic). According to Gauss-Bonet theorem [18] the excess angle of some geodesic triangle \( T \) is equal to the integral of the gaussian curvature over \( T \):

\[ \epsilon = \alpha + \beta + \gamma - \pi = \iint_T Kd\sigma \]  
(17)

where \( d\sigma \) is the area element. If \( K=\text{const.} \)

\[ K = \frac{\epsilon}{A} \]  
(18)

Applying this formula to the above results, we have:

- \( K = 1 \) for spherical geodesic triangles
- \( K = 0 \) for euclidean triangles
- \( K = -1 \) for hyperbolic geodesic triangle

We can use another interpretation of Gaussian curvature in terms of parallel transport. If \( \Delta\varphi \) is the change of angle in the parallel transport of a vector along a curve \( C \) the trace of which is the boundary of a region \( R \), containing the point \( p \), then

\[ \Delta\varphi = \iint_R Kd\sigma \]  
(19)

Since \( \Delta\varphi \) does not depend on the choice of \( C \) (but it depends on the enclosed area \( A(R) \))

\[ \lim_{R\to p} \frac{\Delta\varphi}{A(R)} = K(p) \]  
(20)

This formula gives a method to calculate the curvature at a point in terms of the area and the parallel transport along the border.
5 Curvature on planar graphs

A graph is a pair $G = \{V, E\}$ where $V$ is a non-empty set of vertices and $E$ an unordered 2-set of vertices, called edges, in such a way that two vertices are incident to an edge. We exclude loops, which are edges incident twice with the same vertex, and parallel edges, which are pair of edges incident with the same vertices. We are interested in planar sets, that is to say, a graph that can be drawn on a piece of paper, such that its edges intersect only at their common vertices.

A graph can be defined in an abstract way using only combinatorial properties of vertices and edges, or can be obtained from geometrical objects. For instance, given a particular tessellation described in section 3, we keep the edges and vertices of all the triangles and eliminate the embedding manifold (in our case the surface $S^2$, $E^2$ or $H^2$) in such a way that we are left with the points (vertices) and relations among them (edges). In Figures 4, 5, 6, we have drawn the graphs that we have derived by this method from the tessellations given in Figures 1, 2, 3 respectively, where the vertices are represented by points and the edges by arrows.

![Graph obtained from spherical tessellation of Fig. 1.](image)

In a graph one can define such elements as path, circuit, length, distance and other operations on graphs in analogy to the continuous case. For instance, in a given graph one may travel from one vertex to another using several edges. the set of the vertices visited in that journey is called a path. The distance between two vertices of a graph is the length of the shortest path between those two vertices. These definitions coincide with the standard ones when the graph is embedded in some continuous manifold.

Given a planar graph corresponding to Fig. 4, 5, 6 where two adjacent vertices have always the same adjacent third vertex (different from the first two) one can define the excess of this triad of vertices as the quantity, in analogy with (11),

$$\delta = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1$$
Fig. 5. Graph obtained from hyperbolic tessellation of Fig. 2.

Fig. 6. Graph obtained from hyperbolic tessellation of Fig. 3.
where $2l, 2m, 2n$ are the number of edges incident in each of the three vertices, which correspond to $2l$–valued, $2m$–valued or $2n$–valued vertices, respectively. For instance, in Fig. 4, $A, B, C$ are $10$–valued, $4$–valued, $4$–valued vertices, respectively; in Fig. 5, $A, B, C$ are $12$–valued, $4$–valued, $6$–valued vertices, respectively; and in Fig. 6, $A, B, C$ are $16$–valued, $4$–valued, $6$–valued vertices, respectively.

If we define the spherical, euclidean or hyperbolic graph, that is obtained from a spherical, euclidean or hyperbolic tessellation respectively, we can check

- $\delta > 0$, for a spherical graph (Fig. 4)
- $\delta = 0$, for an euclidean graph (Fig. 5)
- $\delta < 0$, for an hyperbolic graph (Fig. 6)

In a similar way, we can define the areas and curvature of a triad in a graph, such that they become the standard quantities when the graph is embedded in a continuous manifold.

Therefore, we define the area of the triad $T(l, m, n)$ in a spherical graph, in analogy with (12),

$$\sigma(T) = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1$$

and the area of the triad $T'(l, m, n)$ in an hyperbolic graph

$$\sigma(T) = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)$$

Similarly, we define the curvature of a triad $T(l, m, n)$

$$K(T) = \frac{\delta}{\sigma} = \begin{cases} 1, & \text{for a spherical graph} \\ 0, & \text{for an euclidean graph} \\ -1, & \text{for an hyperbolic graph} \end{cases}$$

an expression that can be considered the discrete version of the Gauss-Bonet theorem (17). As in the continuous case, the curvature of a graph at a vertex, can be calculated as the parallel transport of a path surrounding the $m$-valued vertex divided by the area of the triads embraced by the path see (20). Obviously

$$K(P) = \frac{2m\delta}{2mA} = K(T)$$

### 6 Some comments

The method outline above can be applied to other more complicated graphs. For instance, in a non regular graph (not derived from a regular tessellation) some discrete Gaussian curvature can be defined in terms of the $l, m$ or $n$-valued vertices of each triad, that leads to positive, zero or negative curvature for that triad.
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Also a more difficult task would be to apply our method to a non-planar graph coming from an n-dimensional tessellation generated by an n-simplex reflection groups. As in the continuous case one can calculate the sectional curvature, that is obtained from some 2-dimensional surface which is the intersection of the space $X(S^n, E^n \lor H^n)$ with the hyperplane perpendicular to the Weyl vector that generates a Coxeter reflection.

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