Two-Parameter Generalizations of Cauchy Bi-Orthogonal Polynomials and Integrable Lattices

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Abstract
In this article, we consider the generalised two-parameter Cauchy two-matrix model and the corresponding integrable lattice equation. It is shown that with parameters chosen as \(1/k_i\), \(k_i \in \mathbb{Z}_{>0}\) \((i = 1, 2)\), the average characteristic polynomials admit \((k_1 + k_2 + 2)\)-term recurrence relations, which can be interpreted as spectral problems for integrable lattices. The tau function is then given by the partition function of the generalised Cauchy two-matrix model as well as Gram determinant. The simplest solvable example is given.

Keywords Two-parameter Cauchy two-matrix model · Toda-type lattice · Gram determinant technique

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1 Introduction

The Cauchy two-matrix model was proposed in the study of integrable systems, inspired by the analysis of the peakon solution of the Degasperis–Procesi equation (Lundmark and Szmigielski 2003). Later on, this matrix model has attracted much attention from groups in random matrix theory and many properties have been systematically studied. For example, the limiting behaviour of the Cauchy two-matrix model and the corresponding Riemann–Hilbert problem of the Cauchy bi-orthogonal polynomials were done in Bertola et al. (2009), Bertola et al. (2010) and Bertola et al. (2014); its connections with Bures ensemble studied on the levels of partition functions and correlation functions were demonstrated in Forrester and Kieburg (2016) and Forrester and Li (2019). The joint probability density function (jPDF) of this model has the form

$$\prod_{1 \leq j < k \leq N} (x_k - x_j)^2 (y_k - y_j)^2 \prod_{j, k=1}^{N} \omega_1(x_j) \omega_2(y_k), \quad x_j, y_k \in \mathbb{R}_+$$

with some nonnegative weight functions $\omega_1$ and $\omega_2$. As is known, the development of random matrix model gives insights into orthogonal polynomials theory and classical integrable systems, e.g. the considerations of Hermitian matrix model with unitary invariance are closely related to orthogonal polynomials, and KP/1d-Toda hierarchy (Adler and van Moerbeke 1997; Gerasimov et al. 1991; Tsujimoto and Kondo 2000), and it is interesting to know whether there are also some integrable systems behind the Cauchy two-matrix models. Very recently, the answer was given in Li and Li (2019). It was shown that if $\omega_1 = \omega_2$ and some proper time flows are involved, then the time-dependent partition function of the Cauchy two-matrix model can be regarded as the tau function of the CKP hierarchy as well as the so-called Toda lattice of the CKP type (C-Toda lattice for brevity). Moreover, the average characteristic polynomials of the Cauchy two-matrix model—the Cauchy bi-orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ can act as the wave functions, and the remarkable four-term recurrence relation (with $a_n$, $b_n$, $c_n$ and $d_n$ properly chosen)

$$P_{n+1}(x) + a_n P_n(x)) = P_{n+2}(x) + b_n P_{n+1}(x) + c_n P_n(x) + d_n P_{n-1}(x) \quad (1.1)$$

provides a $3 \times 3$ spectral problem for the integrable system.

Compared with the previous work, there are two main motivations for our present study. One is from the recent work on the integrable system related to a new class of extended affine Weyl group $\tilde{W}^{(k,k+1)}(A_l)$ (Zuo 2020). As is known, the extended affine Weyl group $\tilde{W}^{(k)}(A_l)$ is related to the spectral operator (where $\Lambda$ is the shift operator and $\{a_j\}_{j=1}^{l+1}$ are diagonal matrices)

$$\Lambda^k + a_1 \Lambda^{k-1} + \cdots + a_{l+1} \Lambda^{k-l-1}, \quad 1 \leq k < l, \quad a_{l+1} \neq 0, \quad (1.2)$$

which can be regarded as the superpotential of the bigraded Toda hierarchy (Carlet 2006) and it has intimate connection with Muttalib–Borodin model, which can be
viewed as a θ-deformed model of Hermitian matrix model with unitary invariance. Regarding the extended affine Weyl group \( \tilde{W}^{(k,k+1)}(A_l) \), the superpotential is given by

\[
(\Lambda - a_{l+2})^{-1}(\Lambda^{k+1} + a_1 \Lambda^k + \cdots + a_{l+1} \Lambda^{k-l}), \quad 1 \leq k < l, \quad a_{l+1}a_{l+2} \neq 0, \quad (1.3)
\]

where \( \{a_j\}_{j=1}^{l+2} \) are diagonal matrices and the condition \( a_{l+1}a_{l+2} \neq 0 \) means the multiplication operator is invertible. For details, we give an appendix A about how this superpotential generates from the extended affine Weyl group. Obviously, the four-term recurrence relation (1.1) is a very special example with \( k = 1 \) and \( l = 2 \). The question we want to answer in this paper is whether there are any orthogonal families of polynomials admitting such general recurrence relations and, if the answer is affirmative, what are the corresponding integrable lattices.

Another motivation is from a very recent work on the θ-deformation of the Cauchy two-matrix model (Forrester and Li 2019). The idea to generalise the Cauchy two-matrix model is to consider the jPDF

\[
\prod_{1 \leq j < k \leq N} (x_k - x_j)(y_k - y_j) \prod_{j,k=1}^{N} (x_j - x_k)(y_j - y_k) \prod_{j=1}^{N} \omega_1(x_j) \omega_2(y_j) dx_j dy_j \quad (1.4)
\]

with some nonnegative weight functions \( \omega_1 \) and \( \omega_2 \). It was also shown in Forrester and Li (2019) that if the weight functions are specifically chosen as Laguerre weights, i.e. \( \omega_1(x) = x^a e^{-x} \) and \( \omega_2(y) = y^b e^{-y} \), then the hard edge kernel of this model can be depicted in terms of the Fox H-kernel, generalising the original work about the Meijer G-kernel (Bertola et al. 2014). However, the jPDF in (1.4) is not the most general case of Cauchy two-matrix model. In other words, one can consider a two-parameter generalisation

\[
\prod_{1 \leq j < k \leq N} (x_k - x_j)(y_k - y_j) \prod_{j,k=1}^{N} (x_j - x_k)(y_j - y_k) \prod_{j=1}^{N} \omega_1(x_j) \omega_2(y_j) dx_j dy_j \quad (1.5)
\]

to distinguish the eigenvalues \( \{x_k\}_{k=1}^{N} \) and \( \{y_k\}_{k=1}^{N} \) not only from the weight functions, but the interactions within themselves.

This paper is organised as follows. In Sect. 2, we discuss the generalised Cauchy bi-orthogonal polynomials, which give us general spectral problems related to (1.3). Moreover, with the Laguerre weight, these generalised Cauchy bi-orthogonal polynomials can be connected with bi-orthogonal Jacobi polynomials. Besides, we show that these generalised bi-orthogonal polynomials can be written as a series sum and furthermore as a contour integral. In Sect. 3, we study the time evolutions and consider how to derive the corresponding integrable systems. Without symmetry property of the measures, the derivation becomes rather difficult and the method in this paper is totally different from the ones shown in Li and Li (2019) and Miki and Tsujimoto
A Two-Parameter Generalisation of Cauchy Bi-Orthogonal Polynomials

2.1 Orthogonality and Recurrence Relation

Consider the inner product \( \langle \cdot, \cdot \rangle \) such that

\[
\langle x^i, y^j \rangle = \int_{\mathbb{R}^+ \times \mathbb{R}^+} \frac{x^i y^j}{x + y} d\mu_1(x) d\mu_2(y) := m_{i,j}
\]

with two nonnegative measures \( d\mu_1 \) and \( d\mu_2 \) and positive real number \( i \) and \( j \) \((i, j \in \mathbb{R}^+)\). Then, we can define a family of monic two-parameter Cauchy bi-orthogonal polynomials \( \{P_n, Q_n\}_{n=0}^{\infty} \), satisfying the orthogonal relation

\[
\langle P_n(x^{\theta_1}), Q_m(y^{\theta_2}) \rangle = h_n \delta_{n,m}, \text{ for some } h_n > 0,
\]

where \( \theta_1, \theta_2 \in \mathbb{R}^+ \). Therefore, from the linear system (2.2), we can get a closed form for these polynomials, showing

\[
P_n(x^{\theta_1}) = \frac{1}{\tau_n} m_{0,0} \cdots m_{0,(n-1)\theta_2} 1 \begin{vmatrix} m_{\theta_1,0} & \cdots & m_{\theta_1,(n-1)\theta_2} & x^{\theta_1} \\ \vdots & \vdots & \vdots \\ m_{n\theta_1,0} & \cdots & m_{n\theta_1,(n-1)\theta_2} & x^{n\theta_1} \end{vmatrix},
\]

\[
Q_n(y^{\theta_2}) = \frac{1}{\tau_n} m_{0,0} \cdots m_{0,n\theta_2} 1 \begin{vmatrix} m_{(n-1)\theta_1,0} & \cdots & m_{(n-1)\theta_1,n\theta_2} & y^{\theta_2} \\ \vdots & \vdots & \vdots \\ m_{(n-1)\theta_1,0} & \cdots & m_{(n-1)\theta_1,n\theta_2} & y^{n\theta_2} \end{vmatrix},
\]

where \( \tau_n \) is the normalisation factor \( \det(m_{(k-1)\theta_1,(l-1)\theta_2})_{k,l=1}^{n} \). By direct computations, one can show \( h_n = \tau_{n+1}/\tau_n \). Moreover, the existence and uniqueness of the polynomials defined by the linear system (2.2) are equal to the condition \( \tau_n \neq 0 \), which could be verified by using Andréief formula and shown that

\[
\tau_n = \frac{1}{(n!)^2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \det \left[ \frac{1}{x_j + y_k} \right]_{j,k=1}^{n} \Delta_n(x^{\theta_1}) \Delta_n(y^{\theta_2}) \prod_{j=1}^{n} d\mu_1(x_j) d\mu_2(y_j),
\]

where \( \Delta_n(z) = \prod_{1 \leq j < k \leq n} (z_k - z_j) \) for the nonnegative measures \( d\mu_1 \) and \( d\mu_2 \).

Firstly, we consider the recurrence relations of these two-parameter polynomials with general weights, which is of essential importance to clarify the properties of...
For the two-parameter Cauchy bi-orthogonal polynomials \( \{ P_n(x^{\theta_1}) \}_{n=0}^{\infty} \), they have the following \( (k_1 + k_2 + 2) \)-term recurrence relation

\[
x (P_{n+1}(x^{\theta_1}) + a_n P_n(x^{\theta_1})) = \sum_{\alpha = n-k_2}^{n+k_1+1} \eta_{n,\alpha} P_{\alpha}(x^{\theta_1}), \tag{2.4}
\]

where

\[
a_n = -\int_{R_+} P_{n+1}(x^{\theta_1}) d\mu_1(x), \quad \eta_{n,\alpha} = \frac{\langle x(P_{n+1}(x^{\theta_1}) + a_n P_n(x^{\theta_1}), Q_{\alpha}(y^{\theta_2}) \rangle}{\langle P_{\alpha}(x^{\theta_1}), Q_{\alpha}(y^{\theta_2}) \rangle}.
\]

**Proof** The proof is based on the definition of \( a_n \) and the orthogonality of these polynomials. By making use of the formula for \( a_n \), one can find

\[
\langle x(P_{n+1}(x^{\theta_1}) + a_n P_n(x^{\theta_1})), Q_{m}(y^{\theta_2}) \rangle = -(P_{n+1}(x^{\theta_1}) + a_n P_n(x^{\theta_1}), y Q_{m}(y^{\theta_2})).
\]

Furthermore, since \( y Q_{m}(y^{\theta_2}) \in \text{span}\{Q_0(y^{\theta_2}), \ldots, Q_{m+k_2}(y^{\theta_2})\} \), we know the above equation equals zero if \( m + k_2 < n \) according to the orthogonality. Noting that

\[
x (P_{n+1}(x^{\theta_1}) + a_n P_n(x^{\theta_1})) = \sum_{\beta = 0}^{n+k_1+1} \eta_{n,\beta} P_{\beta}(x^{\theta_1})
\]

and

\[
\langle x(P_{n+1}(x^{\theta_1}) + a_n P_n(x^{\theta_1})), Q_{m}(y^{\theta_2}) \rangle = 0 \quad \text{if} \quad m < n - k_2,
\]

we know \( \eta_{n,\beta} = 0 \) if \( \beta < n - k_2 \) and the coefficients of \( \{\eta_{n,\alpha}\}_{\alpha = n-k_2}^{n+k_1+1} \) can be computed from the orthogonality. \( \square \)

**Remark 2.2** One can show that \( a_n = -(\sigma_n \tau_n)/(\tau_n + \sigma_n) \) (see Proposition 3.4 in our paper) and it was proved (Bertola et al. 2010, Proposition 3.1) that both \( \sigma_n \) and \( \tau_n \) are strictly positive. Thus, here \( a_n \) is well-defined.

**Corollary 2.3** There is a dual recurrence relation for the polynomials \( \{ Q_n(y^{\theta_2}) \}_{n=0}^{\infty} \)

\[
y (Q_{n+1}(y^{\theta_2}) + \hat{a}_n Q_n(y^{\theta_2})) = \sum_{\alpha = n-k_1}^{n+k_2+1} \hat{\eta}_{n,\alpha} Q_{\alpha}(y^{\theta_2}), \tag{2.5}
\]

where

\[
\hat{a}_n = -\int_{R_+} Q_{n+1}(y^{\theta_2}) d\mu_2(y), \quad \hat{\eta}_{n,\alpha} = \frac{\langle P_{\alpha}(x^{\theta_1}), y(Q_{n+1}(y^{\theta_2}) + \hat{a}_n Q_n(y^{\theta_2})) \rangle}{\langle P_{\alpha}(x^{\theta_1}), Q_{\alpha}(y^{\theta_2}) \rangle}.
\]
Therefore, the recurrence relations (2.4) and (2.5) are in fact the spectral problems (1.3) if we write the spectral problem in matrix form and introduce the shift operator. Moreover, the constraint in $a_{k+1}a_{k+2} \neq 0$ in (1.3) is again equal to the constraint the $\tau_n \neq 0$, which we have discussed before.

2.2 Special Case: Two-Parameter Cauchy–Laguerre Bi-Orthogonal Polynomials

To claim the importance of these polynomials, in this subsection we show that with Laguerre weights, these polynomials are related to bi-orthogonal Jacobi polynomials in Borodin (1999) and Madhekar and Thakare (1982), which could be used in the further studies about the hard edge scaling limit of the two-parameter Cauchy–Laguerre matrix model. Moreover, the partition shown in this subsection can be viewed as a special case of what we would discuss in Sect. 3.

Let’s consider the Laguerre weights $d\mu_1(x) = x^a e^{-x} dx$ and $d\mu_2(y) = y^b e^{-y} dy$. In this case, the moments in (2.1) could be written as

$$I_{j,k} := m_{(j-1)\theta_1,(k-1)\theta_2} = \int_{\mathbb{R}^2_+} \frac{x^{a+(j-1)\theta_1} y^{b+(k-1)\theta_2}}{x + y} e^{-(x+y)} dxdy.$$  

These moments are two-parameter generalisations of Cauchy bi-orthogonal polynomials which are slightly different from the one-parameter generalisation considered in Forrester and Li (2019)—unlike the procedure to connect the one-parameter Cauchy bi-orthogonal polynomials with the standard Jacobi polynomials, we’d like to evaluate the moments and connect them with Jacobi bi-orthogonal polynomials (Madhekar and Thakare 1982). Consider an evolution of the moments

$$J_{j,k}(s) = \int_{\mathbb{R}^2_+} \frac{x^{a+(j-1)\theta_1} y^{b+(k-1)\theta_2}}{x + y} e^{-s(x+y)} dxdy,  \tag{2.6}$$

and by making the use of transformations on variables $x = \tilde{x}/s$ and $y = \tilde{y}/s$, one can show

$$J_{j,k}(s) = s^{-(1+a+b+\theta_1(j-1)+\theta_2(k-1))} I_{j,k}.$$  

Therefore,

$$\frac{d}{ds} J_{j,k}(s) = -s^{-(2+a+b+\theta_1(j-1)+\theta_2(k-1))} (1 + a + b + \theta_1(j - 1) + \theta_2(k - 1)) I_{j,k}.$$  

On the other hand, the derivative of $s$ can directly lead us to

$$\frac{d}{ds} J_{j,k}(s) = \int_{\mathbb{R}^2_+} x^{a+\theta_1(j-1)} y^{b+\theta_2(k-1)} e^{-s(x+y)} dx dy$$

$$\quad = \int_{\mathbb{R}_+} x^{a+\theta_1(j-1)} e^{-sx} dx \int_{\mathbb{R}_+} y^{b+\theta_2(k-1)} e^{-sy} dy$$

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\[ I_{j,k} = \frac{\Gamma(1 + a + \theta_1(j - 1))\Gamma(1 + b + \theta_2(k - 1))}{1 + a + b + \theta_1(j - 1) + \theta_2(k - 1)}. \]

The equivalence of these two expressions gives rise to

\[ \frac{1}{\Gamma(1 + a + \theta_1(j - 1))\Gamma(1 + b + \theta_2(k - 1))} \]

If we consider two systems of functions (Borodin 1999, Proposition 3.3)

\[ \xi_n(x^{\theta_1}) = \sum_{i=0}^{n} (-1)^i \frac{(1 + a + b + \theta_1i)/\theta_2)_{a} x^{\theta_1i}}{i!(n-i)!} := \sum_{i=0}^{n} c_{n,i} x^{\theta_1i}, \]

\[ \psi_n(x^{\theta_2}) = \sum_{i=0}^{n} (-1)^i \frac{(1 + a + b + \theta_2i)/\theta_1)_{a} x^{\theta_2i}}{i!(n-i)!} := \sum_{i=0}^{n} d_{n,i} x^{\theta_2i}, \]

where \((a)_m = a(a+1) \cdots (a+m-1)\) stands for the Pochhammer symbol, then one can check that \(\{\xi_n, \psi_n\}_{n=0}^{\infty}\) are bi-orthogonal in \(L^2([0, 1], x^{a+b} dx)\). In other words,

\[ \int_0^1 \xi_n(x^{\theta_1})\psi_m(x^{\theta_2})x^{a+b} dx = \tilde{h}_n \delta_{n,m}, \quad \tilde{h}_n = \frac{1}{1 + a + b + (\theta_1 + \theta_2)n}. \]

The moments of the above bi-orthogonal polynomials are given by

\[ \int_0^1 x^{\theta_1(j-1)}x^{\theta_2(k-1)}x^{a+b} dx = \frac{1}{1 + a + b + \theta_1(j - 1) + \theta_2(k - 1)}. \]

Furthermore, if we set

\[ \hat{P}_n(x^{\theta_1}) = \sum_{l=0}^{n} \frac{c_{n,l}}{\Gamma(1 + a + \theta_1l)} x^{\theta_1l}, \quad \hat{Q}_n(y^{\theta_2}) = \sum_{l=0}^{n} \frac{d_{n,l}}{\Gamma(1 + b + \theta_2l)} y^{\theta_2l}, \]

then it is found that

\[ \int_{\mathbb{R}_+^2} e^{-(x+y)} \frac{P_m(x^{\theta_1})}{x+y} \frac{Q_n(y^{\theta_2})}{x^{a+b}} dxdy = \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{c_{m,j}d_{n,k}}{\Gamma(1 + a + \theta_1j)\Gamma(1 + b + \theta_2k)} I_{j+1,k+1} \]

\[ = \sum_{j=0}^{m} \sum_{k=0}^{n} c_{m,j}d_{n,k} I_{j+1,k+1} = \int_0^1 \xi_m(x^{\theta_1})\psi_n(x^{\theta_2})x^{a+b} dx = \tilde{h}_n \delta_{n,m}. \]

Therefore, the polynomials defined in (2.8) are Cauchy bi-orthogonal polynomials with Laguerre weight. To make it monic, let’s take

\[ P_n(x^{\theta_1}) = \frac{\Gamma(1 + a + \theta_1n)}{c_{n,n}} \hat{P}_n(x^{\theta_1}) \]
and the density of the first matrix is related to
\[
\rho(\gamma) = \frac{1}{\Gamma((1+a+b+\gamma)/\theta)} \hat{P}_n(x^{\theta_1}),
\]
\[
Q_n(y^{\theta_2}) = \frac{\Gamma((1+b+\gamma)/\theta)}{d_{n,n}} \hat{Q}_n(y^{\theta_2})
\]
\[
= \frac{\Gamma((1+\gamma)/\theta)}{\Gamma((1+a+b+\gamma)/\theta)} \hat{Q}_n(y^{\theta_2}),
\]
and thus \( h_n \) in orthogonal relation (2.2) is
\[
\frac{\Gamma((n+1)) \Gamma(1+a+\gamma) \Gamma(1+b+\gamma)}{\Gamma((1+a+b+\gamma)/\theta) \Gamma((1+\gamma)/\theta)} \hat{Q}_n(y^{\theta_2}).
\]
Moreover, the two-parameter Cauchy bi-orthogonal polynomials can be written as contour integral expression
\[
\hat{P}_n(x^{\theta_1}) = \int_{\gamma} du \frac{\Gamma((1+a+b-\gamma)/\theta)}{2\pi i \Gamma((1+n+u) \Gamma((1+a-\gamma)/\theta))} x^{-\gamma u},
\]
where \( \gamma \) is a contour encloses \( \{0, -1, \cdots, -n\} \) and \( \hat{Q}_n(y^{\theta_2}) = \hat{P}_n(x^{\theta_1})|_{a \leftrightarrow b, x \leftrightarrow y, \theta_1 \leftrightarrow \theta_2} \). The Mellin–Barnes integral shows that the bi-orthogonal polynomials are still Meijer G-functions of the variable \( x^{\theta_j} \). These polynomials can be expressed in terms of Fox H-functions as well and in the special symmetric case, the expressions degenerate to Forrester and Li (2019, Eqs. (2.8a)–(2.8b)).

We want to demonstrate how the \( \theta \) parameter affects the density. To make it simpler, we consider \( \theta_1 = \theta_2 = \theta \) and the corresponding theoretical results about the density was given in Forrester and Li (2019). The matrix model considered here is of size \( N \times N \). To be precise, we’re interested in the following \( (r, s) \)-points correlation functions (c.f. Bertola et al. 2009, Definition 3.2)
\[
\rho^{(r,s)}(x_1, \cdots, x_r; y_1, \cdots, y_s) = \det \begin{bmatrix}
K_{01}(x_i, x_j)_{1 \leq i, j \leq r} & K_{00}(x_i, y_j)_{1 \leq i, j \leq s}
K_{11}(y_i, x_j)_{1 \leq i, s, 1 \leq j \leq r} & K_{10}(y_i, y_j)_{1 \leq i, j \leq s}
\end{bmatrix},
\]
and the density of the first matrix is related to \( K_{01} \) considered here; the others can be similarly analysed. In particular, \( K_{01} \) can be written in terms of the bi-orthogonal polynomials \( \{\hat{P}_n, \hat{Q}_n\} \) by
\[
K_{01}(x, x') = x^a e^{-x} \sum_{k=0}^{N-1} \frac{1}{h_k} \hat{P}_k(x^{\theta}) C[\hat{Q}_k](x'),
\]
where \( C[\hat{Q}_k] \) means the following Cauchy transform of \( \hat{Q}_k(y^{\theta}) \)
\[
C[\hat{Q}_k](x') = \int_{\mathbb{R}^+} \frac{\hat{Q}_k(y^{\theta})}{x' + y} y^b e^{-y} dy.
\]
Moreover, these polynomials are of the forms

\[ \hat{P}_n(x^\theta) = \sum_{l=0}^{n} \frac{c_{n,l}}{\Gamma(1 + a + \theta l)} x^{\theta l}, \quad \hat{Q}_n(y^\theta) = \sum_{l=0}^{n} \frac{c_{n,l}}{\Gamma(1 + b + \theta l)} y^{\theta l}, \]

with

\[ c_{n,l} = (-1)^l \frac{\Gamma(\alpha + l + n + 1)}{\Gamma(l + 1)\Gamma(n - l + 1)\Gamma(\alpha + n + 1)}, \quad \alpha = \frac{1 + a + b}{\theta} - 1. \]

By taking into explicit formula, the Cauchy transform (2.9) can be computed as

\[ C[\hat{Q}_k](x') = e^{x'} \sum_{l=0}^{k} c_{n,l}x'^{b+\theta l}\Gamma(-b - \theta l; x') \]

with \( \Gamma(a; x) \) the incomplete gamma function

\[ \Gamma(a; x) = \int_{x}^{\infty} t^{a-1}e^{-t}dt. \]

Therefore, to show how the parameter \( \theta \) influences the density, firstly, we need the following proposition.

**Proposition 2.4** The density is given by

\[ \rho(x) = \lim_{N \to \infty} N^{-2/\theta} K_{01}(\frac{x}{N^{2/\theta}}, \frac{x}{N^{2/\theta}}). \]

It was proven in Forrester and Li (2019, Proposition 2.17) that the scaling limit of the above right-hand side is independent of \( N \) and can be written in terms of Fox H-function. In Fig. 1, we can see that when \( \theta = 1 \), the scaling limit converges to the kernel slower than the one shown in Bertola et al. (2014, Fig. 1), it is because the scheme shown here is at the convergence rate \( O(1/N) \) and the one in Bertola et al. is at \( O(1/N^2) \). We remark that the optimal convergence rate can be obtained by making use of asymptotic analysis of ratio of gamma functions (Forrester and Li) and some rigorous analysis will be discussed in the future.

In Fig. 2, we find that the different impacts on \( \theta \) when \( N = 16 \). One can see that the amplitude is affected by the parameter.

### 3 Time Evolutions and Integrable Lattices

In this section, let’s consider such time evolutions in the measures that

\[ I_{j,k}(t) = \langle x^{\theta_1(j-1)}, y^{\theta_2(k-1)} \rangle = \int_{\mathbb{R}^2_+} \frac{x^{\theta_1(j-1)}y^{\theta_2(k-1)}}{x + y} d\mu_1(x; t)d\mu_2(y; t), \quad (3.1) \]
Fig. 1 The exact density verse the asymptotic formula

Fig. 2 Density with different $\theta$-values for $N = 16$
This kind of assumption is made to decouple the double integral into single integrals, like the rank 1 shift condition discussed in Bertola et al. (2010), Li and Yu, which is useful to compute in terms of x and y separately (c.f. Eq. (2.7)). Specifically, one of the special solutions of \(d\mu_1\) and \(d\mu_2\) is \(d\mu_1(x; t) = e^{tx}d\mu_1(x)\) and \(d\mu_2(y; t) = e^{ty}d\mu_2(y)\) and if we take \(d\mu_1(x; t) = e^{tx}dx\) and \(d\mu_2(y; t) = e^{ty}dy\), then the moments \(I_{j,k}(t)\) are the same with \(J_{j,k}(-t)\) defined in (2.6). However, the measures \(d\mu_1(x; t)\) and \(d\mu_2(y; t)\) we consider are arbitrary and own finite moments, and thus, the partition function of the two-parameter Cauchy–Laguerre matrix model is a special tau function we discuss below.

From the time-dependent moments/inner product (3.1), we can similarly define a family of time-dependent two-parameter Cauchy bi-orthogonal polynomials \(\{P_n(x^{\theta_1}; t)\}_{n=0}^\infty\) and \(\{Q_n(y^{\theta_2}; t)\}_{n=0}^\infty\) via the orthogonal relation

\[
\langle P_n(x^{\theta_1}; t), Q_m(y^{\theta_2}; t) \rangle = h_n(t)\delta_{n,m} \quad \text{with} \quad h_n = \frac{\tau_{n+1}(t)}{\tau_n(t)}.
\]

(3.2)

Note that \(\{P_n(x^{\theta_1}; t)\}_{n=0}^\infty\) (resp. \(\{Q_n(y^{\theta_2}; t)\}_{n=0}^\infty\)) also admit \((k_1 + k_2 + 2)\)-term recurrence relations following Proposition 2.1 with the coefficients \(a_n, \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n\) and \(\hat{\eta}_n, \alpha\) time-dependent. To derive the corresponding integrable system, we need the following derivative formula for the two-parameter Cauchy bi-orthogonal polynomials.

**Proposition 3.1** We have the time evolution equation

\[
\partial_t \left( P_{n+1}(x^{\theta_1}; t) + a_n P_n(x^{\theta_1}; t) \right) = \frac{\partial_t (a_n h_n)}{h_n} P_n(x^{\theta_1}; t).
\]

(3.3)

**Proof** Since the polynomials are monic and \(\{P_k(x^{\theta_1}; t)\}_{k=0}^n\) expand a basis of polynomial at order \(n\), we have

\[
\partial_t \left( P_{n+1}(x^{\theta_1}; t) + a_n P_n(x^{\theta_1}; t) \right) = \sum_{k=0}^n \xi_{n,k}(t) P_k(x^{\theta_1}; t).
\]

Moreover, from the orthogonal relation (3.2), we know

\[
\partial_t \left( \langle P_{n+1}(x^{\theta_1}; t) + a_n P_n(x^{\theta_1}; t), Q_m(y^{\theta_2}; t) \rangle \right) = \partial_t h_{n+1} \delta_{n+1,m} + \partial_t (a_n h_n) \delta_{n,m}.
\]

(3.4)

The left-hand side can be equivalently expressed as

\[
\langle \partial_t (P_{n+1}(x^{\theta_1}; t) + a_n P_n(x^{\theta_1}; t)), Q_m(y^{\theta_2}; t) \rangle
\]

\[
+ \langle P_{n+1}(x^{\theta_1}; t) + a_n P_n(x^{\theta_1}; t), \partial_t Q_m(y^{\theta_2}; t) \rangle,
\]

whose second term is zero if \(m < n\). Therefore, one can find

\[
\langle \partial_t (P_{n+1}(x^{\theta_1}; t) + a_n P_n(x^{\theta_1}; t)), Q_m(y^{\theta_2}; t) \rangle = 0 \quad \text{if} \quad m < n,
\]
to conclude that $\xi_{n,m} = 0$ if $m < n$. Moreover, if $m = n$, one can find $\xi_{n,n} = \partial_t (a_n h_n)/h_n$ directly from (3.4), and thus complete the proof. □

Similar to Li and Li (2019, Proposition 3.2), this proposition gives us a time evolution equation for the eigenfunction. However, unlike the method demonstrated therein, it is insufficient to derive an integrable system from Eqs. (3.3) and (3.4) only. The coefficients of $x^{\theta_1 n}$ in $P_{n+1}(x^{\theta_1}; t)$ could hardly be expressed as derivatives of tau functions and the equation cannot be easily closed. We need a novel method to derive the integrable lattice. The basic idea is to use the recurrence relation (2.4) and time evolution equation (3.3) and make the use of compatibility condition to formulate the integrable equation.

From Eq. (3.3) and by the use of recurrence relation with $e_n = \partial_t (a_n h_n)/h_n$, we can write down

$$x \left[ \partial_t (P_{n+1} + a_n P_n) + \frac{e_n}{e_{n-1}} a_{n-1} \partial_t (P_n + a_{n-1} P_{n-1}) \right] = e_n \sum_{\alpha = n-k_2-1}^{n+k_1} \eta_{n-1,\alpha} P_\alpha. \tag{3.5}$$

Moreover, if we denote $f_n := e_n a_{n-1}/e_{n-1}$, then the left-hand side of the above equation can also be written as

$$\partial_t [x (P_{n+1} + a_n P_n)] + f_n \partial_t [x (P_n + a_{n-1} P_{n-1})]$$

$$= \partial_t \left( \sum_{\alpha = n-k_2}^{n+k_1+1} \eta_{n,\alpha} P_\alpha \right) + f_n \partial_t \left( \sum_{\alpha = n-k_2}^{n+k_1} \eta_{n-1,\alpha} P_\alpha \right)$$

$$= \sum_{\alpha = n-k_2}^{n+k_1} \left( \partial_t \eta_{n,\alpha} + f_n \partial_t \eta_{n-1,\alpha} \right) P_\alpha + f_n \partial_t \eta_{n-1,n-k_2-1} P_{n-k_2-1}$$

$$= \partial_t P_{n+1} + \sum_{\alpha = n-k_2}^{n+k_1} (\eta_{n,\alpha} + f_n \eta_{n-1,\alpha}) \partial_t P_\alpha + f_n \eta_{n-1,n-k_2-1} \partial_t P_{n-k_2-1}. \tag{3.6}$$

To see the compatibility condition of Eqs. (3.5) and (3.6), we express the underlined term as a linear combination of basis $\{ P_\alpha (x^{\theta_1}; t) \}_{\alpha = 0}^{n+k_1}$. Notice that there exist indefinite parameters $\xi_{n-k_2}, \cdots, \xi_{n+k_1}$ (where we assume $\xi_{n+k_1+1} = 1$) such that

$$\sum_{\alpha = n-k_2}^{n+k_1+1} \xi_{\alpha} \partial_t (P_\alpha + a_{\alpha-1} P_{\alpha-1}) = \sum_{\alpha = n-k_2}^{n+k_1+1} (\xi_{\alpha} \partial_t a_{\alpha-1}) P_{\alpha-1}$$

$$+ \partial_t P_{n+1} + \sum_{\alpha = n-k_2}^{n+k_1} (\xi_{\alpha} + \xi_{\alpha+1} a_{\alpha}) \partial_t P_\alpha + \xi_{n-k_2} a_{n-k_2-1} \partial_t P_{n-k_2-1}.$$
Proposition 3.2

\[
\begin{aligned}
\xi_n, n, \alpha & = \partial_t \eta_{n, \alpha} + f_n \partial_t \eta_{n-1, \alpha} + \xi_{\alpha+1} (e_\alpha - \partial_t a_\alpha), \quad \alpha = n - k_2, \ldots, n + k_1, \\
\xi_n, n, n-k_2 - 1 & = f_n \partial_t \eta_{n-1, n-k_2 - 1} + \xi_{n-k_2} (e_{n-k_2 - 1} - \partial_t a_{n-k_2 - 1}), \\
0 & = f_n \eta_{n-1, n-k_2 - 1} - \xi_{n-k_2} a_{n-k_2 - 1}, \\
\end{aligned}
\]

(3.7)

where \(\{\xi_{\alpha}, \alpha = n - k_2, \ldots, n + k_1\}\) satisfy the linear system \(\xi_{\alpha} + \xi_{\alpha+1} a_\alpha = \eta_{n, \alpha} + f_n \eta_{n-1, \alpha}\) with \(\xi_{n+k_1 + 1} = 1\).

**Proof** By using Eq. (3.3), we know

\[
\sum_{\alpha = n-k_2}^{n+k_1+1} \xi_\alpha \partial_t (P_\alpha + a_{\alpha-1} P_{\alpha-1}) = \sum_{\alpha = n-k_2}^{n+k_1+1} \xi_\alpha e_{\alpha-1} P_{\alpha-1}.
\]

Therefore, a combination of the above two equations leads us to

\[
\partial_t P_{n+k_1+1} + \sum_{\alpha = n-k_2}^{n+k_1} (\xi_\alpha + \xi_{\alpha+1} a_\alpha) \partial_t P_\alpha + \xi_{n-k_2} a_{n-k_2-1} \partial_t P_{n-k_2-1} = \sum_{\alpha = n-k_2}^{n+k_1+1} \xi_\alpha (e_{\alpha-1} - \partial_t a_{\alpha-1}) P_{\alpha-1}.
\]

Moreover, if we assume that \(\xi_{n-k_2}, \ldots, \xi_{n+k_1}\) satisfy

\[
\xi_\alpha + \xi_{\alpha+1} a_\alpha = \eta_{n, \alpha} + f_n \eta_{n-1, \alpha}, \quad \text{for} \ \alpha = n - k_2, \ldots, n + k_1,
\]

then the underlined term in (3.6) can be expressed as

\[
\sum_{\alpha = n-k_2}^{n+k_1+1} \xi_\alpha (e_{\alpha-1} - \partial_t a_{\alpha-1}) P_{\alpha-1} + (f_n \eta_{n-1, n-k_2 - 1} - \xi_{n-k_2} a_{n-k_2 - 1}) \partial_t P_{n-k_2 - 1}.
\]

Hence, we can rewrite Eq. (3.6) as

\[
\partial_t [x (P_{n+1} + a_n P_n)] + f_n \partial_t [x (P_n + a_{n-1} P_{n-1})]
\]

\[
= \sum_{\alpha = n-k_2}^{n+k_1} \left( \partial_t \eta_{n, \alpha} + f_n \partial_t \eta_{n-1, \alpha} + \xi_{\alpha+1} (e_\alpha - \partial_t a_\alpha) \right) P_\alpha
\]

\[
+ (f_n \partial_t \eta_{n-1, n-k_2 - 1} + \xi_{n-k_2} (e_{n-k_2 - 1} - \partial_t a_{n-k_2 - 1})) P_{n-k_2 - 1}
\]

\[
+ (f_n \eta_{n-1, n-k_2 - 1} - \xi_{n-k_2} a_{n-k_2 - 1}) \partial_t P_{n-k_2 - 1}.
\]

(3.8)
where the last term is a linear combination of $\{P_{\alpha}(x^{\theta_1}; t)\}^{n-k_2-2}_{\alpha=0}$. Therefore, according to the independent of the basis, one can finally arrive at the compatibility condition (3.7).

The equations for $P_n(x^{\theta_1}; t)$ are not closed in this case. One should consider dual equations for $Q_n(y^{\theta_2}; t)$. By making the use of the spectral problem (2.5) and time evolution

$$
\partial_t \left( Q_{n+1}(y^{\theta_2}; t) + \hat{a}_n Q_n(y^{\theta_2}; t) \right) = \frac{\partial_t (\hat{h}_n h_n)}{\hat{h}_n} := \hat{e}_n Q_n(y^{\theta_2}; t),
$$

one can get a dual equation for (3.7) by changing $a \mapsto \hat{a}, e \mapsto \hat{e}, f \mapsto \hat{f}, \xi \mapsto \hat{\xi}$ and $\eta \mapsto \hat{\eta}$. After combining the equations held by $P_n(x^{\theta_1}; t)_{n=0}^\infty$ and $\{Q_n(y^{\theta_2}; t)\}_{n=0}^\infty$, the lattice equations are closed and an example is illustrated in the following subsection.

**Remark 3.3** Even in the case $\theta_1 = \theta_2 = 1$, if $d \mu_1 \neq d \mu_2$ (i.e. the moments are not symmetric), the time-dependent partition function of the original Cauchy two-matrix model doesn’t satisfy the CKP hierarchy. However, it is interesting to see, by the use of the Andréief formula, the partition function can be written as

$$
\tau_n = \frac{1}{(n!)^2} \int_{\mathbb{R}_+^n \times \mathbb{R}_+^n} \det \left[ \frac{1}{x_j + y_k} \right]^{n}_{j,k=1} \Delta_n(x) \Delta_n(y) \prod_{j=1}^{n} d\mu_1(x_j; t) d\mu_2(y_j; t)
$$

and the moments $I_{j,k}$ are defined by

$$
I_{j,k} = \int_{-\infty}^{t} \phi_j(t) \psi_k(t) dt, \quad \text{where } \phi_j(t) = \int_{\mathbb{R}_+} x^{j-1} d\mu_1(x; t) \text{ and } \psi_k(t) = \int_{\mathbb{R}_+} y^{k-1} d\mu_2(y; t).
$$

The determinant with this kind of moment is usually called as the Gram determinant in the soliton theory; please see Hirota (2004, §2) for details. Moreover, if the time-dependent measures are taken as $d\mu_i(x; t) = \sum_{k=1}^{\infty} \exp(t_{2k+1} x^{2k+1}) d\mu_i(x)$ with $i = 1, 2$, then the Gram determinant satisfies the KP hierarchy with odd flows, as shown in Hirota (2004, §3.2). Therefore, we can tell that even though the asymmetric moments are not related to the CKP hierarchy any more, there is another integrable hierarchy behind this random matrix model, which is a special case of KP hierarchy with odd flows only. Based on these facts, we would like to demonstrate the first nontrivial asymmetric case and show its exact solvability.

---

1 One can follow the procedure exhibited in Wang et al. (2010) and take the Gram determinant into the first equation of CKP equation (i.e. Wang et al. 2010, Eq. (3)). It is not difficult to see the asymmetric Gram determinant doesn’t satisfy the bilinear equation of CKP equation.
3.1 The First Nontrivial Asymmetric tau Function and Integrable Lattice

In this part, we’d like to consider the first nontrivial asymmetric tau function, i.e. \( \theta_1 = \theta_2 = 1 \) case with \( d\mu_1 \neq d\mu_2 \). In this setting, we can define the moments

\[
m_{i,j} = \langle x^i, y^j \rangle = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{x^i y^j}{x+y} d\mu_1(x; t) d\mu_2(y; t)
\]

and the corresponding \( \tau \)-function

\[
\tau_n = \det(m_{j,k})_{j,k=0}^{n-1}.
\]

Similar to (2.3), one can define the asymmetric Cauchy bi-orthogonal polynomials \( \{P_n(x; t)\}_{n=0}^{\infty} \) and \( \{Q_n(y; t)\}_{n=0}^{\infty} \) such that

\[
\langle P_n(x; t), Q_m(y; t) \rangle = h_n \delta_{n,m} \text{ with } h_n = \tau_{n+1}/\tau_n.
\]

Furthermore, these polynomials admit the following properties.

**Proposition 3.4** \( P_n(x; t) \) and \( Q_n(y; t) \) satisfy the recurrence relations

\[
x(P_{n+1}(x; t) + a_n P_n(x; t)) = P_{n+2}(x; t) + b_n P_{n+1}(x; t) + c_n P_n(x; t) + d_n P_{n-1}(x; t),
\]

\[
y(Q_{n+1}(y; t) + \hat{a}_n Q_n(y; t)) = Q_{n+2}(y; t) + \hat{b}_n Q_{n+1}(y; t) + \hat{c}_n Q_n(y; t) + \hat{d}_n Q_{n-1}(y; t),
\]

where the coefficients are given by

\[
a_n = \hat{\hat{C}}_n, \quad b_n = \hat{\hat{C}}_n + B_{n+1}, \quad c_n = -A_{n+1} - \hat{\hat{C}}_n \hat{\hat{B}}_n, \quad d_n = -A_n \hat{\hat{C}}_n,
\]

\[
\hat{a}_n = C_n, \quad \hat{b}_n = C_n + \hat{B}_{n+1}, \quad \hat{c}_n = -A_{n+1} - C_n \hat{B}_n, \quad \hat{d}_n = -A_n C_n,
\]

with

\[
A_n = \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2}, \quad B_n = \frac{\xi_{n+1}}{\tau_{n+1}} - \frac{\xi_n}{\tau_n}, \quad \hat{\hat{B}}_n = \frac{\hat{\hat{\xi}}_{n+1}}{\tau_{n+1}} - \frac{\hat{\hat{\xi}}_n}{\tau_n}, \quad C_n = -\frac{\sigma_{n+1} \tau_n}{\sigma_n \tau_{n+1}}, \quad \hat{\hat{C}}_n = -\frac{\hat{\hat{\sigma}}_{n+1} \tau_n}{\hat{\hat{\sigma}}_n \tau_{n+1}}.
\]

and the functions \( \sigma \) and \( \xi \) (resp. \( \hat{\sigma} \) and \( \hat{\xi} \)) have the determinant expressions

\[
\sigma_n = \det \left( \frac{m_{i,j}}{\phi_j} \right)_{i=0,\ldots,n-1, \ j=0,\ldots,n}, \quad \xi_n = \det \left( \frac{m_{i,j}}{\hat{\hat{\phi}}_j} \right)_{i=0,\ldots,n-2, \ j=0,\ldots,n-1},
\]

\[
\hat{\sigma}_n = \det \left( \frac{m_{i,j} \hat{\hat{\phi}}_j}{\phi_j} \right)_{i=0,\ldots,n, \ j=0,\ldots,n-1}, \quad \hat{\xi}_n = \det \left( \frac{m_{i,j}}{\hat{\hat{\phi}}_j} \right)_{i=0,\ldots,n-1, \ j=0,\ldots,n-2, \ n}.
\]

The single moments \( \phi_j \) and \( \hat{\phi}_j \) are defined by \( \hat{\hat{\phi}}_j = \int_{\mathbb{R}_+} x^j d\mu_1(x; t) \) and \( \phi_j = \int_{\mathbb{R}_+} y^j d\mu_2(y; t) \).

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This proposition is a direct consequence of Proposition 2.1 and we omit its proof here. Moreover, it follows from the derivative formula in Proposition 3.1 and we have the following proposition.

**Proposition 3.5** $P_n(x; t)$ and $Q_n(y; t)$ evolve as

\[
\begin{align*}
\partial_t P_{n+1}(x; t) + \hat{C}_n \partial_t P_n(x; t) &= \hat{C}_n (B_n + \hat{B}_n) P_n(x; t), \\
\partial_t Q_{n+1}(y; t) + C_n \partial_t Q_n(y; t) &= C_n (B_n + \hat{B}_n) Q_n(y; t).
\end{align*}
\]

**Proof** Compared with Proposition 3.1, we are left to prove that

\[
\partial_t \log h_n = B_n + \hat{B}_n,
\]

which is equivalent to

\[
\partial_t \tau_n = \xi_n + \hat{\xi}_n.
\]

This can be immediately achieved by following Lemma 3.2 in Chang et al. (2018). □

Following the procedure demonstrated above, one can derive the following nonlinear lattice system from the compatibility condition

\[
\begin{align*}
\partial_t A_n &= A_n (B_n + \hat{B}_n - B_{n-1} - \hat{B}_{n-1}), \\
\partial_t B_n &= (B_{n-1} + \hat{B}_{n-1}) \hat{C}_{n-1} - (B_n + \hat{B}_n) \hat{C}_n, \\
\partial_t \hat{B}_n &= (B_{n-1} + \hat{B}_{n-1}) C_{n-1} - (B_n + \hat{B}_n) C_n, \\
\partial_t C_n &= C_n \left( C_n \frac{A_n}{C_{n-1}} - \hat{B}_n - C_{n+1} + \frac{A_{n+1}}{C_n} + \hat{B}_{n+1} \right), \\
\partial_t \hat{C}_n &= \hat{C}_n \left( \hat{C}_n \frac{A_n}{\hat{C}_{n-1}} - B_n - \hat{C}_{n+1} + \frac{A_{n+1}}{\hat{C}_n} + B_{n+1} \right).
\end{align*}
\]

We need to emphasise that although the recurrence relations (3.10) seem to admit eight coefficients, in fact, they do have only five independent coefficients expressed by $A_n$, $B_n$, $\hat{B}_n$, $C_n$ and $\hat{C}_n$. It means that the equations held by these recurrence coefficients should be only five equations rather than eight equations; the other three are automatically compatible. To summarise, we have the following proposition.

**Theorem 3.6** The system (3.13a)–(3.13e) admit the matrix integrals solution

\[
\begin{align*}
A_n &= \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2}, \quad B_n = \frac{\xi_{n+1}}{\tau_{n+1}} - \frac{\xi_n}{\tau_n}, \quad \hat{B}_n = \frac{\hat{\xi}_{n+1}}{\tau_{n+1}} - \frac{\hat{\xi}_n}{\tau_n}, \quad C_n = - \frac{\sigma_{n+1} \tau_n}{\sigma_n \tau_{n+1}}, \quad \hat{C}_n = - \frac{\hat{\sigma}_{n+1} \tau_n}{\hat{\sigma}_n \tau_{n+1}}.
\end{align*}
\]

where

\[
\tau_n = \frac{1}{(n!)^2} \int_{\mathbb{R}_+^n \times \mathbb{R}_+^n} \Delta_n^2(x) \Delta_n^2(y) \prod_{j,k=1}^n \mu(x_j + y_k) \prod_{j=1}^n d\mu_1(x_j; t) d\mu_2(y_j; t);
\]
Proposition 3.7

The tau functions satisfy the following bilinear form

\[ D_i \tau_{n+1} \cdot \tau_n = \sigma_n \hat{\sigma}_n, \quad (3.14a) \]

\[ D_i \xi_n \cdot \tau_n = \hat{\sigma}_n \sigma_{n-1}, \quad (3.14b) \]
\[ D_i \dot{\xi}_n \cdot \tau_n = \sigma_n \dot{\sigma}_{n-1}, \quad (3.14c) \]
\[ D_i \xi_{n+1} \cdot \tau_n + D_i \tau_{n+1} \cdot \dot{\xi}_n = \sigma_n \partial_i \dot{\sigma}_n, \quad (3.14d) \]
\[ D_i \dot{\xi}_{n+1} \cdot \tau_n + D_i \tau_{n+1} \cdot \dot{\xi}_n = \dot{\sigma}_n \partial_i \sigma_n, \quad (3.14e) \]

**Proof** First, by following Lemma 3.2 in Chang et al. (2018), we have the derivative formulae as follows:

\[ \partial_i \tau_n = \xi_n + \dot{\xi}_n = -\tau_n, \quad \partial_i \xi_n = -\dot{\xi}_n, \quad \partial_i \dot{\xi}_n = -\ddot{\xi}_n, \quad \partial_i \sigma_n = \alpha_n + \beta_n, \quad \partial_i \dot{\sigma}_n = \dot{\alpha}_n + \dot{\beta}_n, \]

where

\[
\alpha_n = \det \begin{pmatrix} m_{i,j} \phi_j \end{pmatrix}_{i=0, \ldots, n-1, j=0, \ldots, n-1}, \quad \dot{\alpha}_n = \det \begin{pmatrix} m_{i,j} \dot{\phi}_i \end{pmatrix}_{i=0, \ldots, n-1, j=0, \ldots, n-1},
\]
\[
\beta_n = \det \begin{pmatrix} m_{i,j} \phi_j \end{pmatrix}_{i=0, \ldots, n-2, j=0, \ldots, n}, \quad \dot{\beta}_n = \det \begin{pmatrix} m_{i,j} \dot{\phi}_i \end{pmatrix}_{i=0, \ldots, n-2, j=0, \ldots, n},
\]
\[
\tau_n = \det \begin{pmatrix} m_{i,j} \dot{\phi}_i \phi_j \end{pmatrix}_{i=0, \ldots, n-1, j=0, \ldots, n}, \quad \dot{\tau}_n = \det \begin{pmatrix} m_{i,j} \dot{\phi}_i \phi_j \end{pmatrix}_{i=0, \ldots, n-1, j=0, \ldots, n},
\]
\[
\xi_n = \det \begin{pmatrix} m_{i,j} \dot{\phi}_i \phi_j \end{pmatrix}_{i=0, \ldots, n-2, j=0, \ldots, n-2}, \quad \ddot{\xi}_n = \det \begin{pmatrix} m_{i,j} \dot{\phi}_i \phi_j \end{pmatrix}_{i=0, \ldots, n-2, j=0, \ldots, n-2}.
\]

The remaining part can be completed by use of the Jacobi identity (Hirota 2004), which reads

\[
\mathcal{D} \mathcal{D} \begin{pmatrix} i_1 & i_2 \\ j_1 & j_2 \end{pmatrix} = \mathcal{D} \left( \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} \right) \mathcal{D} \left( \begin{pmatrix} i_2 \\ j_2 \end{pmatrix} \right) - \mathcal{D} \left( \begin{pmatrix} i_1 \\ j_2 \end{pmatrix} \right) \mathcal{D} \left( \begin{pmatrix} i_2 \\ j_1 \end{pmatrix} \right). \quad (3.15)
\]

Here, \( \mathcal{D} \) is the determinant of an arbitrary square matrix. \( \mathcal{D} \left( \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} \right) \) with \( i_1 < i_2 < \cdots < i_k, \ j_1 < j_2 < \cdots < j_k \) denotes the determinant of the matrix obtained from \( \mathcal{D} \) by removing the rows with indices \( i_1, i_2, \ldots, i_k \) and the columns with indices \( j_1, j_2, \ldots, j_k \).

The first three relations (3.14a)–(3.14c) can be derived by taking

\[
\mathcal{D}_1 = \det \begin{pmatrix} m_{i,j} \dot{\phi}_i \phi_j \end{pmatrix}_{i=0, \ldots, n, j=0, \ldots, n}, \quad i_1 = j_1 = n + 1, \ i_2 = j_2 = n + 2,
\]
\[
\mathcal{D}_2 = \det \begin{pmatrix} m_{i,j} \dot{\phi}_i \phi_j \end{pmatrix}_{i=0, \ldots, n, j=0, \ldots, n-1}, \quad i_1 = n, \ i_2 = n + 1, \ j_1 = n + 1, \ j_2 = n + 2,
\]
\[
\mathcal{D}_3 = \det \begin{pmatrix} m_{i,j} \dot{\phi}_i \phi_j \end{pmatrix}_{i=0, \ldots, n-1, j=0, \ldots, n}, \quad i_1 = n + 1, \ i_2 = n + 2, \ j_1 = n, \ j_2 = n + 1,
\]
respectively. The validity of (3.14d) can be confirmed by combining two relations

\[ \tau_n \partial_t \xi_{n+1} = \xi_{n+1} \partial_t \tau_n + \sigma_n \hat{\sigma}_n, \]
\[ \hat{\xi}_n \partial_t \tau_{n+1} = \tau_{n+1} \partial_t \hat{\xi}_n + \sigma_n \hat{\beta}_n, \]

which are consequences of applying the Jacobi identity to

\[ D_{4,1} = \det \begin{pmatrix} m_{i,j} \hat{\phi}_i \\ \phi_j \\ 0 \end{pmatrix} \begin{pmatrix} i = 0, \ldots, n-1, n+1 \\ j = 0, \ldots, n \end{pmatrix}, \quad i_1 = j_1 = n + 1, \ i_2 = j_2 = n + 2, \]
\[ D_{4,2} = \det \begin{pmatrix} m_{i,j} \hat{\phi}_i \\ \phi_j \\ 0 \end{pmatrix} \begin{pmatrix} i = 0, \ldots, n \\ j = 0, \ldots, n \end{pmatrix}, \quad i_1 = n + 1, \ i_2 = n + 2, \ j_1 = n, \ j_2 = n + 2, \]

respectively. Similarly, by combining two relations

\[ \tau_n \partial_t \hat{\xi}_{n+1} = \hat{\xi}_{n+1} \partial_t \tau_n + \hat{\sigma}_n \xi_n, \]
\[ \xi_n \partial_t \tau_{n+1} = \tau_{n+1} \partial_t \hat{\xi}_n + \hat{\sigma}_n \beta_n, \]

obtained by applying the Jacobi identity to

\[ D_{5,1} = \det \begin{pmatrix} m_{i,j} \hat{\phi}_i \\ \phi_j \\ 0 \end{pmatrix} \begin{pmatrix} i = 0, \ldots, n \\ j = 0, \ldots, n-1, n+1 \end{pmatrix}, \quad i_1 = j_1 = n + 1, \ i_2 = j_2 = n + 2, \]
\[ D_{5,2} = \det \begin{pmatrix} m_{i,j} \hat{\phi}_i \\ \phi_j \\ 0 \end{pmatrix} \begin{pmatrix} i = 0, \ldots, n \\ j = 0, \ldots, n \end{pmatrix}, \quad i_1 = n, \ i_2 = n + 2, \ j_1 = n + 1, \ j_2 = n + 2, \]

respectively, one can prove (3.14e). \qed

The bilinear form (3.14a)–(3.14e) has many intriguing properties. Firstly, when we consider the moments are symmetric, i.e. \( \phi_i = \hat{\phi}_i \) and \( m_{i,j} = m_{j,i} \), then this lattice equation can be degenerated to the C-Toda lattice, which was introduced in Chang et al. (2018) and Li and Li (2019). The reason is that in the symmetric case, \( \xi_n = \hat{\xi}_n, \sigma_n = \hat{\sigma}_n \) and \( 2\xi_n \) can be regarded as the derivative of \( \tau_n \). Therefore, Eqs. (3.14d) and (3.14e) are the derivatives of (3.14b) and (3.14c), which are naturally valid. At the same time, the nonlinear system can be degenerated as well. The nonlinear system (3.13a) will reduce to the nonlinear C-Toda lattice in Chang et al. (2018, eq (3.17)) by setting \( B_n = \hat{B}_n, \ C_n = \hat{C}_n = \sqrt{B_{n+1} A_{n+1}} / B_n \) in the symmetric case.

Secondly, this bilinear form can be iterated. Starting with the initial values \( \tau_0 = \hat{\tau}_0 = 1, \ \sigma_0 = \hat{\sigma}_0 = 0 \) and one can get \( \tau_1 \) from Eq. (3.14a), \( \hat{\xi}_1 \) from Eqs. (3.14d) and (3.14e) and \( \sigma_1, \hat{\sigma}_1 \) from Eqs. (3.14b) and (3.14c). By iterating these equations repeatedly, one can get the general expressions for these variables (i.e. \( \tau_n = \det(m_{i,j})_{i,j=0}^{n-1} \) and \( \sigma_n, \hat{\sigma}_n, \hat{\xi}_n \) and \( \hat{\xi}_n \) have expressions in (3.12)), which means that the lattice equation is recursively solvable.
4 Conclusion and Discussion

In this article, we focus on the two-parameter generalisation of the Cauchy two-matrix model, its average characteristic polynomials and corresponding integrable lattice. We show that the asymmetric Gram determinant plays an important role in the tau function theory of the integrable lattice (3.14a)–(3.14e) and therefore, in the general integrable lattice (3.7) as well as its dual lattice equation. However, it is not the end of the story. First of all, there should be some further discussions about the \( \theta \)-deformed Cauchy two-matrix model. First of all, it seems that under the mapping \( x^{\theta_1} \mapsto x \) and \( y^{\theta_2} \mapsto y \), then the jpdf (1.5) can be equivalently written as

\[
\det \left[ \frac{1}{x_{i}^{k_1} + y_{j}^{k_2}} \right]_{i,j=1}^{N} \Delta_{N}(x) \Delta_{N}(y) \prod_{j=1}^{N} d\mu_{1}(x_{j}^{k_1})d\mu_{2}(y_{j}^{k_2}), \quad k_i = 1/\theta_i.
\]

It suggests us that the jpdf should be appeared in an analogue of the following matrix integral (Harnad and Yu Orlov 2006)

\[
\int_{U(N)} \frac{dU}{\det(X + UYU^\dagger)^{N}} = \frac{\det(1/(x_{i} + y_{j}))_{i,j=1}^{N}}{\Delta_{N}(x) \Delta_{N}(y)},
\]

and it is of interest in our further study since this observation also plays an important role in integrable system. For example, if we consider an antisymmetric Hermitian \( N \times N \) matrix, then it has the jpdf \( \Delta_{N}^{2}(x^{2}) \prod_{j=1}^{N} d\mu(x_{j}^{2}) \) when \( N \) is even (Forrester 2010, Exer. 1.3.5), and the time-dependent partition function related to this jpdf is the Lotka–Volterra lattice. However, if we map \( x^{2} \) to \( x \), the time-dependent partition function is the celebrated Toda lattice, whose Bäcklund transformation is the former one. Therefore, the change of variables in jpdf in random matrix theory should have some extra explanations in integrable systems and we put it into our future studies.

Moreover, in the work Bertola et al. (2009) and Forrester and Li (2019), it has been shown that with \( d\mu_{1}(x) = x^{a}e^{-x}dx \) and \( d\mu_{2}(y) = y^{a+1}e^{-y}dy \), the \( \theta \)-deformed Cauchy two-matrix model is related to the \( \theta \)-deformed) Bures ensemble, and the partition function of the latter can be regarded as the \( \tau \)-function of the B-Toda lattice (Chang et al. 2018, Li and Yu). It implies that the asymmetric lattice equation (3.14a)–(3.14e) can be degenerated to the B-Toda lattice as well, which has a four-term recurrence relation of the form (1.1) with proper coefficients (c.f. Chang et al. 2018, Equation 3.43). Therefore, it is of interest to understand the mechanism of the symmetric/skew-symmetric reduction and whether there is any \( \theta \)-deformed Toda lattice of BKP type. The mechanism of reduction would reveal some subgroups of the affine Weyl group \( \tilde{W}^{(k,k+1)}(A_{1}) \), and it is interesting to consider how to make reductions on the discrete eigenfunctions. Regarding the \( \theta \)-deformed B-Toda lattice, since the tau function of B-type is of Pfaffian form, one needs to consider more Pfaffian techniques to involve in the tau function expression. Besides, as there is an equation connected Sawada–Kotera equation and Kaup–Kupershmidt equation, we’d like to explore the discrete version to connect B-Toda and C-Toda lattices.
As we showed, the tau function of the lattice (3.14a)–(3.14e) is related to the partition function of the generalised Cauchy two-matrix model. The Cauchy two-matrix model, and corresponding Meijer G-function, recently has been studied in the theories of Hurwitz number and topological recursion (Bertola and Harnad 2019). It is still open for us to know whether the rationally weighted Hurwitz number is related to this generalised $\theta$-deformed integrable hierarchy.

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Appendix A

This appendix is devoted to a statement about how the superpotential (1.3) relates to the extended affine group $\tilde{W}^{(k,k+1)}(A_\ell)$. To this end, firstly, we give some basic concepts about the extended affine Weyl group.

Let $V$ be an $\ell$-dimensional Euclidean space with inner product $(\cdot, \cdot)$ and $\mathcal{R}$ be an irreducible root system defined in $V$ with simple roots $\alpha_1, \cdots, \alpha_\ell$ and coroots $\alpha_1^\vee, \cdots, \alpha_\ell^\vee$. Then, the Weyl group $W(\mathcal{R})$ can be generated by the reflection

$$x \mapsto x - (\alpha_j^\vee, x)\alpha_j, \quad \forall x \in V, \quad j = 1, \cdots, \ell.$$ 

The affine Weyl group $W_a(\mathcal{R})$ which acts on the Euclidean space $V$ can be realised as the semi-product of $W(\mathcal{R})$ by the lattice of coroots via the map

$$x \mapsto w(x) + \sum_{j=1}^{\ell} m_j \alpha_j^\vee, \quad w \in W(\mathcal{R}), \quad m_j \in \mathbb{Z}.$$ 

Dubrovin and Zhang then proposed the extended affine Weyl group $\tilde{W}^{(k)}(A_\ell)$ acting on $V \oplus \mathbb{R}$ in the study of Frobenius manifold. It is generated by the transformations

$$(x, x_{\ell+1}) \mapsto (w(x) + \sum_{j=1}^{\ell} m_j \alpha_j^\vee, x_{\ell+1}), \quad w \in W(\mathcal{R}), \quad m_j \in \mathbb{Z},$$

$$(x, x_{\ell+1}) \mapsto (x + \gamma w_k, x_{\ell+1} - \gamma), \quad 1 \leq k \leq \ell$$

with $w_k$ being the fundamental weights defined by the relations $(w_k, \alpha_j^\vee) = \delta_{k,j}$ for $k, j = 1, \cdots, \ell$ and $\gamma$ is a constant related the root system. With $\mathcal{R}$ chosen as $A_\ell$, a construction of Frobenius manifold structure on the orbit of $\tilde{W}^{(k)}(A_\ell)$ was given in Dubrovin and Zhang (1998). It was shown that $\tilde{W}^{(k)}(A_\ell)$ describes the monodromy.
of roots of trigonometric polynomials admits the form
\[
\lambda(\phi) = e^{ik\phi} + a_1 e^{i(k-1)\phi} + \cdots + a_{\ell+1} e^{i(k-\ell-1)\phi}, \quad a_{\ell+1} \neq 0.
\]
If one sets \(e^{i\phi}\) as the shift operator \(\Lambda\), then the above polynomials will become
\[
L = \Lambda^k + a_1 \Lambda^{k-1} + \cdots + a_{\ell+1} \Lambda^{k-\ell-1}, \quad a_{\ell+1} \neq 0
\]
which is regarded as the spectral operator of the bigraded Toda hierarchy (c.f. Eq. (1.2)).

In the recent work (Zuo 2020), the author studied another extended affine Weyl group \(\tilde{W}(k, k+1)(A_\ell)\) acting on the space \(V \oplus \mathbb{R}^2\), generated by the transformation
\[
(x, x_{\ell+1}, x_{\ell+2}) \mapsto (w(x) + \sum_{j=1}^{\ell} m_j \alpha_j^\vee, x_{\ell+1}, x_{\ell+2}), \quad w \in W(A_\ell), \quad m_j \in \mathbb{Z},
\]
\[
(x, x_{\ell+1}, x_{\ell+2}) \mapsto (x + \gamma w_k, x_{\ell+1} - \gamma, x_{\ell+2}), \quad (x, x_{\ell+1}, x_{\ell+2})
\]
\[
\mapsto (x + \gamma w_k, x_{\ell+1} - \gamma)
\]
for \(1 \leq k \leq \ell - 1\). It was proven by Zuo (2020) that the orbit space of the extended affine Weyl group is locally isomorphic to a simple Hurwitz space \(\mathbb{M}_{k, \ell-k+1, 1}\) which contains a natural structure of Frobenius manifold. Moreover, this space consists of trigonometric Laurent series admitting the form
\[
\lambda(\phi) = (e^{i\phi} - a_{\ell+2})^{-1}(e^{i(k+1)\phi} + a_1 e^{i(k\phi)} + \cdots + a_{\ell+1} e^{i(k-\ell)\phi}), \quad a_{\ell+1} a_{\ell+2} \neq 0.
\]
Similarly, by setting \(e^{i\phi} = \Lambda\), one gets the spectral operator (1.3)
\[
L = (\Lambda - a_{\ell+2})^{-1}(\Lambda^{k+1} + a_1 \Lambda^k + \cdots + a_{\ell+1} \Lambda^{k-\ell}), \quad 1 \leq k \leq \ell - 1, \quad a_{\ell+1} a_{\ell+2} \neq 0.
\]

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