A LITTLEWOOD-PALEY TYPE THEOREM FOR BERGMAN SPACES

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Abstract. In this paper, we prove that the original Littlewood-Paley g-functions can be used to characterize Bergman spaces as well.

1. Introduction

Let \( D \) be the unit disk in the complex plane \( \mathbb{C} \) with \( \mathbb{T} := \partial D \) being the unit circle. Recall that for \( 0 < p < \infty \), the Hardy space \( \mathcal{H}^p \) on \( D \) is defined as the set of all analytic functions \( f \) on \( D \) satisfying

\[
\| f \|_{\mathcal{H}^p} := \sup_{0 < r < 1} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} < \infty.
\]

It is classical that for any \( f \in \mathcal{H}^p \), almost everywhere on \( \mathbb{T} \) there exist radial limits \( \lim_{r \to 1^-} f(re^{i\theta}) \), denoted by \( f(e^{i\theta}) \), and there holds the relation \( \| f \|_{\mathcal{H}^p} = \| f \|_{L^p(\mathbb{T})} \). We refer to [9] for theory of classical Hardy spaces.

Suppose \( f \in \mathcal{H}^p \) and \( f = \sum_n a_n z^n \) is the power series of \( f \). Consider the following two quantities

\[
(1.1) \quad d(f)(z) = \left( \sum_{n=0}^{\infty} |\Delta_n(f)(z)|^2 \right)^{\frac{1}{2}}
\]

where \( \Delta_0(f)(z) = a_0 \) and \( \Delta_n(f)(z) = \sum_{2n-1 \leq k < 2n} a_k z^k \) for \( n \geq 1 \), and

\[
(1.2) \quad g(f)(z) = \left( \int_0^1 \left( 1 - r^2 \right) |f'(r z)|^2 dr \right)^{\frac{1}{2}},
\]

for \( z \in \mathbb{D} \cup \mathbb{T} \). Two results on Hardy spaces, essentially due to Littlewood and Paley [6], assert that

\[
(1.3) \quad \| f \|_{L^p(\mathbb{T})} \approx \| d(f) \|_{L^p(\mathbb{T})} \quad \text{and} \quad \| f - f(0) \|_{L^p(\mathbb{T})} \approx \| g(f) \|_{L^p(\mathbb{T})},
\]

the constants involved being only dependent on \( p \) with \( 0 < p < \infty \). The two equivalent relations (1.3) can be considered to be the beginning of the Littlewood-Paley theory.

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The main purpose of this paper is to prove that these two equivalent relations (1.3) hold true as well in the case of $L^p(D)$ replacing $L^p(T)$, characterizing the so-called Bergman spaces. Recall that for $0 < p < \infty$, the Bergman space $A^p$ consists of functions $f$ analytic in $D$ with

$$\|f\|_{A^p} = \left( \int_D |f(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty$$

where $dA(z) = dx dy/\pi$ with $z = x + iy$ in $D$. Note that for $1 \leq p < \infty$, $A^p$ is a Banach space under the norm $\|f\|_{A^p}$. If $0 < p < 1$, the space $A^p$ is a quasi-Banach space with $p$-norm $\|f\|_{A^p}^p$.

**Notation.** For two nonnegative (possibly infinite) quantities $X$ and $Y$, by $X \asymp Y$ we mean that there exists a positive constant $C$ depending only on $p$ such that $X \leq CY$, and by $X \approx Y$ that $X \asymp Y$ and $Y \asymp X$.

2. **Main results**

We state our main results as Theorems 2.1 and 2.2.

**Theorem 2.1.** Let $0 < p < \infty$. There are two constants $A_p$ and $B_p$ depending only on $p$ such that

$$A_p \|f\|_{A^p} \leq \|d(f)\|_{L^p(D)} \leq B_p \|f\|_{A^p}$$

for any $f \in A^p$.

This characterization of those functions in $A^p$ is a straightforward consequence of the first equivalent relation in (1.3), but one of the important features of this characterization is that linear operators obtained by multipliers $m_k$ (of the coefficients $a_k$) that vary boundedly on the dyadic blocks $\triangle_n$ preserve the class $A^p$. More generally, this yields a Marcinkiewicz multiplier theorem for Bergman spaces stating that, for any $0 < p < \infty$ there exists a constant $C_p$ depending only on $p$ such that

$$\left\| \sum_{k=0}^{\infty} m_k a_k z^k \right\|_{A^p} \leq C_p \left( \sup_k |m_k| + \sup_{n \geq 0} \sum_{2^n \leq k < 2^n + 1} |m_{k+1} - m_k| \right) \|f\|_{A^p}.$$

The proof of this inequality can be obtained as in the case of Hardy spaces (see for example [9], Theorem XV.4.14).

**Proof of Theorem 2.1.** Let $0 < p < \infty$. Denote by $f_r(z) = f(rz)$ for $0 < r < 1$ and $z \in D$. By the first equivalent relation in (1.3), one has

$$\int_D |f(z)|^p dv(z) = 2 \int_0^1 rdr \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi}$$

$$\approx 2 \int_0^1 rdr \int_0^{2\pi} |d(f_r)(e^{i\theta})|^p \frac{d\theta}{2\pi}$$

$$= \|d(f)\|_{L^p(D)}^p.$$

This completes the proof of (2.1). \qed
Theorem 2.2. Let $0 < p < \infty$. There are two constants $\alpha_p$ and $\beta_p$ depending only on $p$ such that

\begin{equation}
\alpha_p \|f\|_{A^p} \leq \|g(f)\|_{L^p(D)} \leq \beta_p \|f\|_{A^p}
\end{equation}

for any $f \in A^p$ with $f(0) = 0$. Consequently,

\begin{equation}
\|f\|_{A^p} \approx |f(0)| + \|g(f)\|_{L^p(D)} \quad \text{for } 1 \leq p < \infty,
\end{equation}

and

\begin{equation}
\|f\|_{A^p}^p \approx |f(0)|^p + \|g(f)\|_{L^p(D)}^p \quad \text{for } 0 < p < 1.
\end{equation}

We will deduce this theorem from some classical results, essentially due to Littlewood and Paley, Marcinkiewicz and Zygmund, and a theorem of Coifman and Rochberg [3] on atomic decomposition for Bergman spaces (see Lemma 2.1 below). The proof is thus considerably elementary.

Lemma 2.1. (cf. [7], Theorem 8.3.1) Let $0 < p \leq 1$. There exists a sequence $\{a_k\}$ in $D$ and a constant $C$ such that $A^p$ consists exactly of functions of the form

\begin{equation}
f(z) = \sum_{k=1}^{\infty} c_k \frac{1 - |a_k|^2}{(1 - \bar{a}_k z)^{2/p+1}}, \quad z \in D,
\end{equation}

where $\{c_k\}$ belongs to the sequence space $\ell^p$ and the series converges in the quasi-norm topology of $A^p$, and

\[ C^{-1} \left( \sum_k |c_k|^p \right)^{1/p} \leq \|f\|_{A^p} \leq C \left( \sum_k |c_k|^p \right)^{1/p}. \]

Proof of Theorem 2.2. We begin with the first inequality in (2.2). Denote by $f_r(z) = f(rz)$ for $0 < r < 1$ and $z \in D$. Then by the second equivalent relation in (1.3) we have for any $0 < p < \infty$,

\[
\int_D |f(z) - f(0)|^p dv(z) = 2 \int_0^1 \|f_r - f_r(0)\|_{H^p}^p r dr \approx \int_0^1 r dr \int_0^{2\pi} \left( \int_0^1 (1 - s^2)|f_r'(se^{i\theta})|^2 ds \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \\
\leq \int_0^1 r dr \int_0^{2\pi} \left( \int_0^1 (1 - s^2)|f'(rse^{i\theta})|^2 ds \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \\
\approx \|g(f)\|_{L^p(D)}^p.
\]

This proves the first inequality in (2.2).

To prove the second inequality in (2.2) for the case $0 < p \leq 1$, we will adopt Lemma 2.1. To this end, we write

\[ f_k(z) = \frac{1 - |a_k|^2}{(1 - z\bar{a}_k)^{2/p+1}}. \]
An immediate computation yields that
\[ |f'_k(rz)|^2 = \frac{(2/p + 1)^2|\bar{a}_k|^2(1 - |a_k|^2)^2}{|1 - rz\bar{a}_k|^{2(2/p + 2)}} \]

Also, it is easy to check that
\[ |1 - tz| \leq (1 - t) + |1 - z| \leq 3|1 - tz|, \quad 0 < t \leq 1, \ \forall z \in \mathbb{D}. \]

Then
\[ g(f_k)(z) = |a_k|(2/p + 1)(1 - |a_k|^2) \left( \int_0^1 \frac{(1 - r^2)dr}{|1 - rz\bar{a}_k|^{2(2/p + 2)}} \right)^{\frac{1}{2}} \]
\[ \lesssim (1 - |a_k|^2) \left( \int_0^1 \frac{dr}{[(1 - r) + |1 - z\bar{a}_k|]^{2(2/p + 1) + 1}} \right)^{\frac{1}{2}} \]
\[ \lesssim (1 - |a_k|^2) \frac{1}{|1 - z\bar{a}_k|^{2/p + 1}}. \]

Hence, for \( f = \sum_k c_kf_k \) with \( \sum_k |c_k|^p < \infty \) we have
\[ \int_\mathbb{D} |g(f)(z)|^p dv(z) \leq \sum_{k=1}^\infty |c_k|^p \int_\mathbb{D} |g(f_k)(z)|^p dv(z) \]
\[ \lesssim \sum_{k=1}^\infty |c_k|^p (1 - |a_k|^2)^p \int_\mathbb{D} \frac{1}{|1 - z\bar{a}_k|^{2+p}} dv(z) \]
\[ \lesssim \sum_{k=1}^\infty |c_k|^p, \]
where the last inequality is obtained by the fact that
\[ \int_\mathbb{D} \frac{1}{|1 - z\bar{w}|^{2+p}} dv(z) \approx \frac{1}{(1 - |w|^2)^p} \quad \text{as } |w| \to 1^- , \]
for \( p > 0 \) (see Theorem 1.7 in [5]). By Lemma 2.1, we conclude the second inequality in (2.2) for the case \( 0 < p \leq 1 \).

Finally, let \( 1 < p < \infty \). If \( f \in A^p \), then \( f \) has the integral representation
\[ f(z) = \int_\mathbb{D} \frac{f(w)dv(w)}{(1 - zw)^2}, \quad \forall z \in \mathbb{D}. \]

An immediate computation yields that
\[ |f'(rz)| \lesssim (1 - |rz|^2)^{-\frac{1}{2}} \int_\mathbb{D} \frac{|f(w)|dv(w)}{|1 - rz\bar{w}|^{\frac{3}{2}}}. \]
Then,

\[
g(f)^2(z) \lesssim \int_0^1 \frac{1-r^2}{1-|rz|^2} \left| \int_{D} \frac{|f(w)|dv(w)}{|1-rzw|^2} \right|^2 dr
\]

\[
\lesssim \int_{D \times D} |f(w)f(\xi)|dv(w)dv(\xi) \int_0^1 \frac{dr}{|1-rzw|^2|1-rz\bar{\xi}|^2}
\]

\[
\lesssim \int_{D \times D} |f(w)f(\xi)|dv(w)dv(\xi)
\]

\[
\times \left( \int_0^1 \frac{dr}{|[1-z\bar{w}|+(1-r)]^{\frac{5}{2}}|[1-z\xi|+(1-r)]^2} \right)^{\frac{1}{2}}
\]

\[
\lesssim \int_{D \times D} \frac{|f(w)f(\xi)|}{|1-z\bar{w}|^2|1-z\xi|^2}dv(w)dv(\xi)
\]

\[
= \left( \int_{D} \frac{|f(w)|}{|1-z\bar{w}|^2}dv(w) \right)^2.
\]

However, the mapping

\[
f \mapsto \int_{D} \frac{f(w)}{|1-z\bar{w}|^2}dv(w)
\]

is bounded on $L^p(D)$ for $1 < p < \infty$ (e.g. Theorem 1.9 in [5]). Therefore, we conclude the second inequality in (2.2) for the case $1 < p < \infty$.  

\[\square\]

Remark 2.1. (1) Since 1930’s the Littlewood-Paley theory was developed considerably and mainly carried out by E. M. Stein [8], widening its applicability both in the classical setting involving $\mathbb{R}^n$ (even when $n = 1$) and in abstract situations involving, among other things, Lie groups, symmetric spaces, diffusion semigroups and martingales. We consult [4] and references therein for more recent information.

(2) Some real-variable characterizations of Bergman spaces involving maximal and area integral functions in terms of the Bergman metric, have been obtained recently by the present authors [1, 2].

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