On the Magnetohydrodynamics/Gravity Correspondence

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Abstract

The fluid/gravity correspondence relates solutions of the incompressible Navier-Stokes equation to metrics which solve the Einstein equations. In this paper we extend this duality to a new magnetohydrodynamics/gravity correspondence, which translates solutions of the equations of magnetohydrodynamics (describing charged fluids) into geometries that satisfy the Einstein-Maxwell equations. We present an explicit example of this new correspondence in the context of flat Minkowski space. We show that a perturbative deformation of the Rindler wedge satisfies the Einstein-Maxwell equations provided that the parameters appearing in the expansion, which we interpret as fluid fields, satisfy the magnetohydrodynamics equations. As a byproduct of our analysis we show that in four dimensions, the dual geometry is algebraically special Petrov type II.
1 Introduction

In the last fifteen years, the holographic viewpoint has become increasingly central to our understanding of black hole physics. An explicit duality between fluid dynamics and black hole geometries has been established by several groups [1, 2, 3, 4, 5]. Some constructions make use of the hydrodynamic expansion, while others rely on the near-horizon expansion or special algebraic properties of the metric. In each case, regardless of the details, the Navier-Stokes equation always captures the low-energy horizon dynamics. We may regard this as the universal aspect of the correspondence. However, some of its details do depend on the particular setting in which the correspondence is applied. For instance, the dual fluid may obey the relativistic Navier-Stokes equation [6, 5, 7], or it may be subject to particular forcing terms [1, 2].

When faced with the full complexity of the nonlinear Navier-Stokes equation, one may be tempted to start looking for a solution in the absence of external forces. However, there is a particular choice of forcing term arising from the Lorentz force which has been extensively studied in plasma physics. This suggests that it may be advantageous to establish a new version of the correspondence for the charged fluid, which would allow us to bring our knowledge of the magnetohydrodynamics (MHD) to bear on the problem of solving the Einstein-Maxwell equations. Previous attempts to establish such a correspondence [8] have considered background magnetic fields interacting with the fluid. Instead, we propose to examine dynamical magnetic fields (induced by the fluid’s motion) without background electromagnetic fields.

As such, we work in the Rindler wedge of flat Minkowski space and investigate perturbations of the geometry in the hydrodynamic limit, subject to some boundary conditions. Our perturbation, which we carry out to third order, is parametrized by fluid fields which satisfy the MHD equations. Interestingly, we find that in this setup, the conductivity $\sigma$ of the charged fluid is precisely the reciprocal of its fluid viscosity $\eta = 1/4\pi\sigma$. Similarly to [2], we show that the dual metric, after some suitable rescaling, depends on only one parameter, which is obtained from a combination of the derivative expansion parameter and
the distance between the metric horizon and the fluid surface. It is therefore possible to translate the
derivative expansion into the near-horizon expansion, and vice versa.

In some cases [2] the fluid/gravity duality involves metrics which are algebraically special. In four
dimensions, we show that the metric dual to MHD obeys Petrov type II conditions up to third order.

The main results of the paper are in section 5, where we formulate the Cauchy problem for the Einstein-
Maxwell equations, and describe one of its solutions in a hydrodynamic expansion. This analysis is
preceded by a short review of the Einstein-Maxwell theory (section 2) and of the MHD equations (section
3) and their scaling properties (section 4). In section 6, we elaborate on our solution by constructing it
order by order in perturbation theory. We supplement this presentation with sections 7 and 8, in which
we provide some additional details about the near-horizon expansion and offer some checks for the Petrov
type II. We conclude the paper with some thoughts, open questions and possible generalizations.

2 Einstein-Maxwell theory

The Einstein-Maxwell equations

\[ G_{\mu\nu} = 8\pi GT_{\mu\nu} \]  

describe gravity coupled to the electromagnetic stress tensor

\[ 4\pi T_{\mu\nu} = F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g_{\mu\nu}F^2, \]

where the gauge field itself solves the Maxwell equations

\[ \nabla_{\mu}F^{\mu\nu} = 0. \]

In the remainder of this paper, we work in units where \(8\pi G = 1\) and \(c = 1\). Rather than working directly
with the gauge field \(A_\mu\), it is more convenient to use the field strength \(F_{\mu\nu}\) and impose the Bianchi
constraint

\[ \nabla[\lambda F_{\mu\nu}] = 0. \]

These equations are well studied and several exact solutions are known. A famous example is y the
Reissner-Nordström solution, which describes a spherically-symmetric charged object in an asymptotically
flat 4-dimensional space. Other known solutions include planar charged objects obeying Anti-deSitter
asymptotics, as well as less familiar gravitational-wave-like solutions with both metric and gauge field
fluctuations. We will also be interested in the latter type of solution.

3 Magnetohydrodynamics

In 3 + 1 dimensions, the MHD equations with finite conductivity \(\sigma\) take the following form [9]:

\[ \nabla \cdot B = 0, \quad \nabla \cdot v = 0, \]

\[ \nabla \times E = -\partial_t B, \]

\[ \nabla \times B = 4\pi J = 4\pi \sigma(E + [v \times B]), \]

\[ \partial_t v + (v \cdot \nabla)v + \nabla P - \eta\nabla^2 v = [J \times B], \]

The equations can be partially solved for the electric field \(E\) and current \(J\). The remaining equations
then form a nonlinear system describing a fluid with velocity \(v\), subject to pressure \(P\) and a magnetic
field \(B\):

\[ \nabla \cdot B = 0, \quad \nabla \cdot v = 0, \]

\[ \partial_t B = \nabla \times [v \times B] + \frac{1}{4\pi \sigma} \nabla^2 B, \]

\[ \partial_t v + (v \cdot \nabla)v + \nabla P - \eta\nabla^2 v = \frac{1}{4\pi}(B \cdot \nabla)B - \frac{1}{8\pi} \nabla (B^2). \]
As in the case of the incompressible Navier-Stokes equation, there are sufficiently many equations to determine all the variables. It will prove useful to rewrite the system \((6)\) in terms of the electromagnetic fields \(f_{ij}, f_{\tau i}\), for \(i = 1, \ldots, 3\) (\(E_i = f_{i\tau}, B_i = \frac{1}{2} \epsilon_{ijk} f^{jk}\))

\[
\partial_k f_{ij} = 0, \quad \partial_i v^i = 0, \\
\partial_\tau f_{ij} = \partial_i f_{\tau j} - \partial_j f_{\tau i}, \quad f_{\tau i} = -\frac{1}{4\pi}\partial_j f_{ij} - v^k f_{ki}, \\
\partial_\tau v_i + v^j \partial_j v_i + \partial_i P - \eta \partial^2 v_i + \partial^j \pi_{ji} = 0, \quad \pi_{jk} = \frac{1}{4\pi} \left(f_{jl} f_{kl} - \frac{1}{4} f^2 \delta_{jk}\right).
\]

The MHD equations \((7)\) can be generalized to higher dimensions by assuming \(i = 1, \ldots, p\).

### 4 Scaling properties

The Navier-Stokes equation (equation \((7)\) with no electromagnetic field) is famous for its scaling property: simultaneous rescaling of the coordinates and fields

\[
v(x, \tau) \rightarrow \epsilon v(\epsilon x, \epsilon^2 \tau), \\
P(x, \tau) \rightarrow \epsilon^2 P(\epsilon x, \epsilon^2 \tau),
\]

leaves the equation invariant while preserving the viscosity \(\eta\). This scaling property is responsible for the universality of the NS equation in capturing the low energy dynamics of fluids. It may be extended to the MHD equations \((7)\) by requiring that the electromagnetic field obey the following scaling relation:

\[
f_{ij}(x, \tau) \rightarrow \epsilon f_{ij}(\epsilon x, \epsilon^2 \tau) \\
f_{\tau i}(x, \tau) \rightarrow \epsilon^2 f_{\tau i}(\epsilon x, \epsilon^2 \tau).
\]

The scaling properties of the MHD equations allow us to write an ansatz for the bulk gauge field:

\[
F_{ij} = \epsilon F_{ij}^0 + \epsilon^2 F_{ij}^1 + \ldots \\
F_{\tau i} = \epsilon^2 F_{\tau i}^0 + \epsilon^3 F_{\tau i}^1 + \ldots
\]

In order to ensure that the Bianchi identities hold at each expansion order independently (so that fields of different orders do not mix), the rest of the components should be chosen to be:

\[
F_{ir} = \epsilon^0 F_{ir}^0 + \epsilon F_{ir}^1 + \ldots \\
F_{\tau \tau} = \epsilon^1 F_{\tau \tau}^0 + \epsilon^2 F_{\tau \tau}^1 + \ldots
\]

### 5 MHD/gravity correspondence

We can now describe the MHD/gravity correspondence in the simplest possible setup. The starting point is the flat Minkowski metric in \((p+2)\)-dimensional space,

\[
ds^2 = -rd\tau^2 + 2d\tau dr + dx_i^2, \quad i = 1, \ldots, p,
\]

with no background electromagnetic field. The hyper surface \(\Sigma_c\) at fixed radius \(r = r_c\), whose induced metric is flat, is the background space in which the fluid theory evolves.\footnote{The Brown-York stress tensor is diagonal and can be trivially identified with the fluid stress tensor at rest.} We will now study a perturbative deformation of the metric and electromagnetic field which obeys the Einstein-Maxwell equations

\[
G_{\mu \nu} = 2G \left(F_{\mu \lambda} F_{\nu}^\lambda - \frac{1}{4} g_{\mu \nu} F^2\right), \\
\nabla_\mu F^{\mu \nu} = 0, \quad \nabla_{[\lambda} F_{\mu \nu]} = 0, \quad \mu = r, \tau, 1, \ldots, p,
\]

\(1\)
and the following boundary conditions:

Regularity at the horizon: both the field strength $F$ and the metric are regular at $r = 0$.

Dirichlet boundary conditions: the induced metric on $\Sigma_c$ is a flat Minkowski metric, and there is no induced charge nor current on $\Sigma_c$, i.e.

$$F^{\alpha\beta}(r_c) = F^{\gamma\tau}(r_c) = 0. \quad (15)$$

One of the solutions to the Cauchy problem is\footnote{Details of the derivation are provided in the next section.}

$$ds^2_{p+t} = -r dr^2 + 2 dr d\tau + dx_i dx^i - 2 \left(1 - \frac{r}{r_c}\right) v_i dx_i d\tau - 2 \frac{v_i}{r_c} dx_i dr$$

$$+ \left(1 - \frac{r}{r_c}\right) \left( (\nu^2 + 2P) d\tau^2 + \frac{v_i v_i}{r_c} dx_i dx^i \right) + \left(\frac{\nu^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr$$

$$- \frac{1}{16\pi\rho} \left(1 - \frac{r}{r_c}\right)^2 \frac{1}{2}\pi r_c \left(1 - \frac{r}{r_c}\right) \left( f_{ik} f_{jl} \delta^{kl} - \frac{1}{4} \delta_{ij} f^2 \right) dx^i dx^j \quad (16)$$

$$- \left(\frac{r^2 - r_c^2}{r_c}\right) \partial^2 v_i dx^i d\tau + O(e^3),$$

$$r_c F = \frac{1}{2} f_{ij} dx^i \wedge dx^j + f_{ij} dx^i \wedge d\tau - \partial_j f_{ij} dx^i \wedge dr + O(e^3).$$

This geometry, which is parametrized by the fluid fields $f_{ij}, v_i, P$ and $f_{\tau\tau}$ (which depend only on $x^i$ and $\tau$), will satisfy the Einstein-Maxwell equations to order $O(e^4)$ provided that the fluid fields satisfy the MHD equations,

$$\partial_\tau v_i + v^i \partial_j v_i + \partial_i \left( P - \frac{\nu^2 + 2P}{16\pi\rho} f^2 \right) - r_c \partial^2 v_i + \partial^i \pi_{ji} = 0, \quad \pi_{jk} = \frac{1}{4\pi} \left( f_{jil} f_{kl} - \frac{1}{4} f^2 \delta_{jk} \right),$$

$$\partial_i v^i = 0,$$

$$\partial_\tau f_{ij} = \partial_i f_{\tau j} - \partial_j f_{\tau i},$$

$$\partial_k f_{ij} = 0,$$

$$f_{\tau i} = -r_c \partial_j f_{ij} - v^k f_{ki}. \quad (17)$$

Interestingly, $\pi_{ij}$ is the lowest component of the electromagnetic energy momentum tensor on $\Sigma_c$ in the $\epsilon$-expansion. Moreover, the two diffusion constants which enter the MHD equations turn out to be equal:

$$\eta = \frac{1}{4\pi r_c} = r_c. \quad (18)$$

Perhaps this relation is not unexpected for such a simple background metric, since there are no dimensionless parameters for this ratio to depend on.

### 6 Solution

The Cauchy problem described in section 5 is generally hard to solve exactly. Nevertheless, due to the scaling properties of the MHD system, it is possible to construct a perturbative solution, as was done in [2]. The expansion assumes small perturbation size and slowly-varying spacetime dependence:

$$v_i \sim O(\epsilon), \quad \partial_i \sim O(\epsilon), \quad \partial_\tau \sim O(\epsilon^2), \quad P \sim O(\epsilon^2), \quad f_{ij} \sim O(\epsilon), \quad f_{\tau\tau} \sim O(\epsilon^2). \quad (19)$$

The problem may be simplified even further, as follows. At each given order in the expansion, we may divide the equations into two groups: constraint equations and propagating equations. The former depend only on the data from lower orders because of the extra spatial $\partial_i$ and time derivatives $\partial_\tau$ present, whereas the latter fix the radial dependence of the new metric components introduced at the same order. The
Navier-Stokes equation with magnetic forcing is a constraint equation which appears at third order $\mathcal{O}(\epsilon^3)$ so it can be written in terms of the metric solution at $\mathcal{O}(\epsilon^2)$ order.

In the remainder of this section, we will construct the geometry up to and including the $\mathcal{O}(\epsilon^2)$ order and describe the constraint equations at $\mathcal{O}(\epsilon^3)$ order.

### 6.1 Zeroth order

The background metric is flat Minkowski space, which solves Einstein’s equations with no source terms:

$$ds^2 = -rd\tau^2 + 2d\tau dr + dx_i dx^i.$$  \hfill (20)

At zeroth order $\mathcal{O}(\epsilon^0)$, there is one nontrivial Maxwell equation:

$$\partial_r (r F^0_{\tau i}) = 0.$$  \hfill (21)

The only solution that is regular at $r = 0$ is the trivial solution $F^0_{\tau i} = 0$.

To summarize, at zeroth order $\mathcal{O}(\epsilon^0)$, the solution is:

$$ds^2 = -rd\tau^2 + 2d\tau dr + dx_i dx^i + \mathcal{O}(\epsilon),$$  \hfill (22)

$$F = \mathcal{O}(\epsilon).$$

### 6.2 First order

Next, we wish to introduce a deformation of the metric parameterized by the fluid fields $v^i$ and $P$. The simplest way to do this is to use small Lorentz boosts, as was done in [10], resulting in

$$ds^2 = -rd\tau^2 + 2d\tau dr + dx_i dx^i - 2 \left( 1 - \frac{r}{r_c} \right) v_i dx^i d\tau - 2 \frac{v_i}{r_c} dx^i dr + \mathcal{O}(\epsilon^2).$$  \hfill (23)

There are no corrections to the electromagnetic field since the background field vanished in the first place.

At first order in $\epsilon$, the Maxwell equations are:

$$\partial_r F^0_{\tau r} = 0 \quad \implies \quad r_c F^0_{\tau r} = Q^0(x, \tau),$$

$$\partial_r (r F^1_{\tau i}) = 0 \quad \implies \quad F^1_{\tau i} = 0,$$  \hfill (24)

$$\partial_r F^0_{ij} = 0 \quad \implies \quad r_c F^0_{ij} = f_{ij}(x, \tau).$$

In the above, $Q^0$ can be interpreted as the charge density of the dual fluid. The only solution satisfies boundary condition (15) corresponds to $Q^0 = 0$.

In summary, the solution to first order $\mathcal{O}(\epsilon^1)$ is:

$$ds^2 = -rd\tau^2 + 2d\tau dr + dx_i dx^i - 2 \left( 1 - \frac{r}{r_c} \right) v_i dx^i d\tau - 2 \frac{v_i}{r_c} dx^i dr + \mathcal{O}(\epsilon^2),$$  \hfill (25)

$$r_c F = \frac{1}{2} f_{ij} dx^i \wedge dx^j + \mathcal{O}(\epsilon^2).$$

### 6.3 Second order

Note that the zeroth and first order solutions did not impose any constraints on the fluid fields $v^i$ and $f_{ij}$.

On the other hand, in order to solve the second order equations, we will need to impose some constraints on the fluid fields. In addition, we will have to introduce some extra fields such as $P(x, \tau)$ and $f_{\tau i}(x, \tau)$.
As aforementioned, at this order in the expansion, we must introduce a constraint equation. In this case, the equation at Σc has the form
\[ Q = 0 \]
which has the following solution:
\[ f_{ri} = -r_{c} \partial_{j} f_{ij} - v^{k} f_{ki} \]  
(28)

The equation above is one of the MHD equations (7) that we obtained by solving the Einstein-Maxwell equations at second order \( \mathcal{O}(\epsilon^{2}) \). Having obtained the field strength to this order, we can evaluate the stress tensor to second order as well:
\[ r_{c}^{2} F^{2} = f^{2} + \mathcal{O}(\epsilon^{3}), \quad 4 \pi r_{c}^{2} T_{ij} = f_{ti} f_{ij} - \frac{1}{4} f^{2} \delta_{ij}, \]
(29)

The nontrivial contributions to the stress tensor will backreact on the metric and produce additional terms of order \( \mathcal{O}(\epsilon^{2}) \) in \( g_{ij}^{(2)} \) and \( g_{\tau\tau}^{(2)} \). For example, the \((i, j)\) component of the Einstein equations will take the form
\[ R_{ij} = -\frac{1}{2} \partial_{r} \left( r \partial_{r} g_{ij}^{(2)} \right) = 8 \pi G \left( T_{ij} - \frac{1}{r} T_{\mu}^{\mu} \delta_{ij} \right). \]
(30)

Using the boundary condition on \( \Sigma_{c} \), we can write the solution in the form
\[ g_{ij}^{(2)} = \frac{1}{2 \pi r_{c}} \left( 1 - \frac{r}{r_{c}} \right) \left( f_{ti} f_{ij} - \frac{1}{2 \pi r_{c}} f^{2} \delta_{ij} \right), \]
\[ g_{\tau\tau}^{(2)} = -\frac{2 + p}{16 \pi p} \left( 1 - \frac{r}{r_{c}} \right)^{2} f^{2}. \]
(31)

As aforementioned, at this order in the expansion, we must introduce a constraint equation. In this case, it amounts to the requirement that the velocity field be divergence free:
\[ \partial_{r} v^{i} = 0. \]
(32)

To summarize, at second order \( \mathcal{O}(\epsilon^{2}) \), the solution is defined by:
\[ ds_{p+2}^{2} = -r d\tau^{2} + 2 d\tau dr + dx^{i} dx^{i} - 2 \left( 1 - \frac{r}{r_{c}} \right) v_{i} dx^{i} d\tau - 2 \frac{v_{i} v_{j}}{r_{c}} dx^{i} dr \]
\[ + \left( 1 - \frac{r}{r_{c}} \right) \left( v^{2} + 2 P \right) d\tau^{2} + \frac{v_{i} v_{j}}{r_{c}} dx^{i} dx^{j} + \left( \frac{v^{2}}{r_{c}} + \frac{2 P}{r_{c}} \right) d\tau dr \]
\[ + \frac{1}{2 \pi r_{c}} \left( 1 - \frac{r}{r_{c}} \right) \left( f_{ti} f_{ij} - \frac{1}{2 \pi r_{c}} f^{2} \delta_{ij} \right) dx^{i} dx^{j} - \frac{2 + p}{16 \pi p} \left( 1 - \frac{r}{r_{c}} \right)^{2} f^{2} d\tau^{2} + \mathcal{O}(\epsilon^{3}), \]
(33)

\[ r_{c} F = \frac{1}{2} f_{ij} dx^{i} \wedge dx^{j} + \frac{1}{2} f_{ij}^{1} dx^{i} \wedge dx^{j} + f_{\tau i} dx^{i} \wedge d\tau - \partial_{j} f_{ij} dx^{j} \wedge dr + \mathcal{O}(\epsilon^{3}), \]
\[ f_{ri} = -r_{c} \partial_{j} f_{ij} - v^{k} f_{ki}, \quad \partial_{[k} f_{ij]} = 0, \]
\[ \partial_{r} v^{i} = 0. \]
6.4 Third order

As in the second order case, at third order in the $\epsilon$-expansion we must once again introduce new fields and impose additional constraints on the ones that were introduced at lower orders.

To be more precise, the equations which are tangent to $\Sigma_c$ are constraint equations, while the remaining equations fix the radial dependence of the geometry at order $O(\epsilon^3)$ in terms of the fluid fields. We illustrate this point in the context of the Bianchi identity:

$$\partial_r F^1_{\tau r} = -\partial_\tau F^1_r,$$
$$\partial_r F^2_{ij} = \partial_i F^2_{rj} - \partial_j F^2_{ri},$$
$$\partial_\tau F^1_{ij} = 0,$$
$$\partial_r F^0_{ij} = \partial_i F^0_{rj} - \partial_j F^0_{ri}.$$

The equations in the last two lines are tangent to $\Sigma_c$ and therefore impose constraints on the fluid data $f_{ij}, f^1_{ij}, f_\tau$ from lower orders. On the other hand, the equations in the first two lines fix the $O(\epsilon^3)$ radial dependence of the field strength components in terms of the newly introduced fluid fields.

The Einstein-Maxwell constraint equations on $\Sigma_c$ have the following form:

$$n^\mu \nabla_\nu F^{\mu \nu} = 0,$$
$$n^\nu G_{\mu \nu} = 8\pi G n^\nu T_{\mu \nu},$$
$$n^\nu n^\mu G_{\mu \nu} = 8\pi G n^\nu n^\mu T_{\mu \nu},$$

where $n^\mu$ is a unit normal vector to $\Sigma_c$. At third order $O(\epsilon^3)$, the Maxwell constraint is

$$\partial_i (r F^1_{\tau i} + v^k F^0_{ki} + F^0_{r \tau}) + \partial_\tau F^0_{\tau r} = 0.$$  (36)

This condition is trivially satisfied due to our choice of boundary condition (15). The only nontrivial gravitational constraint at third order $O(\epsilon^3)$ is:

$$0 = n^\mu G_{\mu t} - 8\pi G n^\mu T_{\mu t} = \frac{1}{2r_c} \left[ \partial_r v_i + v^i \partial_\tau v_i + \partial_\tau P - r_c \partial^2 v_i + \frac{1}{4\pi} \partial_\tau \left( f_{jl} f^l_i - \frac{p}{2p} f^2 \delta_{ij} \right) \right].$$  (37)

It can be identified with the last of the MHD equations (7) after performing the redefinition

$$P - \left( \frac{p + 2}{16\pi p} \right) f^2 \rightarrow P.$$  (38)

Having established the solvability of the constraint equations at $O(\epsilon^3)$ order, the Cauchy theorem applied to the Einstein-Maxwell equations guarantees the existence of a solution for the entirety of the $O(\epsilon^3)$ equations. This full solution differs from (16) in that it contains additional fluid field terms at order $O(\epsilon^3)$. Finally, note that we may choose $f^1_{ij} = 0$, which trivially satisfies the Bianchi identity. This concludes the derivation of the duality proposed in section 5.

7 Near-horizon expansion

In this section, we will establish the equivalence of the hydrodynamic expansion for the metric (16) to the near-horizon expansion of the geometry. In order to achieve this, we begin by performing the coordinate redefinition from (2), namely:

$$x^i = \frac{r_c}{\epsilon} \hat{x}^i, \quad \tau = \frac{r_c}{\epsilon^2} \hat{\tau}, \quad r = r_c \hat{r},$$  (39)
In these new coordinates, the derivatives are no longer assumed to be small, i.e. $\partial_t = O(\epsilon^0), \partial_r = O(\epsilon^0)$, and the dual metric (40) takes the following form

$$\frac{\epsilon^2}{r_c^2} ds_{p+2}^2 = -\frac{\hat{r}}{\lambda} d\hat{r}^2 + 2d\hat{r}d\hat{r} + d\hat{x}_i d\hat{x}^i - 2(1 - \hat{r}) \hat{v}_i d\hat{x}^i d\hat{r} + (1 - \hat{r}) \left( \hat{v}^2 + 2\hat{P} \right) d\hat{r}^2 - \frac{1}{16\pi p} (1 - \hat{r})^2 \hat{f}^2 d\hat{r}^2$$

$$+ \lambda \left[ (1 - \hat{r}) \left( \hat{v}_i \hat{v}_j + \frac{1}{2\pi} \hat{f}_{ik} \hat{f}_{jl} \delta^{kl} - \frac{1}{4\pi p} \hat{f}^2 \delta_{ij} \right) d\hat{x}^i d\hat{x}^j - 2\hat{v}_i d\hat{x}^i d\hat{r} \right]$$

$$+ \left( \hat{v}^2 + 2\hat{P} \right) d\hat{r}d\hat{r} + (1 - \hat{r}^2) \hat{\omega}^2 \hat{v}_i d\hat{x}^i d\hat{r} + \ldots$$

where we introduced a new expansion parameter $\lambda = \frac{\epsilon^2}{r_c}$ as well as new fluid fields defined by

$$\hat{v}_i(\hat{x}, \hat{r}) = \frac{1}{\epsilon^2} v_i(\hat{x}(x), \hat{r}(\tau)), \quad \hat{P}(\hat{x}, \hat{r}) = \frac{1}{\epsilon^2} P(\hat{x}(x), \hat{r}(\tau)), \quad \hat{f}_{ij}(\hat{x}, \hat{r}) = \frac{1}{\epsilon^2} f_{ij}(\hat{x}(x), \hat{r}(\tau)), \quad \hat{f}_{i\tau}(\hat{x}, \hat{r}) = \frac{1}{\epsilon^2} f_{i\tau}(\hat{x}(x), \hat{r}(\tau)).$$

After a suitable rescaling, the geometry (40) will no longer depend on the two independent parameters $r_c$ and $\epsilon$, rather, it will be parameterized by the single parameter $\lambda$. Likewise, the $r_c$ dependence also drops out of the MHD equations, which become:

$$\partial_{\hat{r}} \hat{v}_i + \hat{v}^j \partial_{\hat{r}} \hat{v}_j + \partial_i \left( \hat{P} - \frac{P + 2}{16\pi p} \hat{f}^2 \right) - \hat{\omega}^2 \hat{v}_i + \frac{1}{4\pi} \hat{\omega}^2 \left( \hat{f}_{ij} \hat{f}_{kl} - \frac{1}{4} \hat{f}^2 \delta_{ij} \right) = 0,$$

$$\hat{f}_{i\tau} = -\hat{\omega} \hat{f}_{ij} - \hat{v}_k \hat{f}_{ki}.$$ 

The distance between the metric horizon at $r = 0$ and the cutoff surface at $r = r_c$ in the rescaled metric (40) behaves as $\frac{1}{\sqrt{r_c}}$, so should not be surprising that there are two ways to make $\lambda$ small: one way is to perform a hydrodynamic expansion in $\epsilon \ll 1$ on the fluid surface $\Sigma_c$ while keeping $r_c$ fixed; the other way consists of pushing the cutoff surface $\Sigma_c$ close to the horizon ($r_c \gg 1$) while removing the small derivative restriction on the fluid fields (so that $\epsilon$ can be arbitrarily large).

8 Petrov type

As in [2], we find that in four dimensions ($p = 2$), the geometry (16) is of algebraically special Petrov type II, meaning that there exists a null vector $k^\mu$ such that the Weyl tensor satisfies

$$W_{\mu\nu\rho\sigma}[k] k^\nu k^\rho = 0. \quad (43)$$

One may verify the existence of such a null vector by evaluating the invariant $I^3 - 27J^2$, which is a function of the metric. The details about $I$ and $J$ and their explicit value in terms of the metric components can be found in [11]. The lowest nontrivial components of $I$ and $J$ are typically of order $O(\epsilon^4)$ and $O(\epsilon^6)$, respectively. Hence we generally expect the invariant $I^3 - 27J^2$ to be of order $O(\epsilon^{12})$, while an explicit computation for the invariant of the metric (16) reveals it to be of order $O(\epsilon^{14})$.

9 Conclusion and open questions

The primary purpose of this work was to show that the fluid/gravity correspondence can be naturally extended to include electromagnetic fields, and to shed some light on this new facet of the duality.

We illustrated this new aspect of the correspondence in the simplest nontrivial background, namely the Rindler wedge of flat Minkowski space. In that context, we were able to obtain an explicit solution to the Einstein-Maxwell equations as a hydrodynamic expansion parameterized by the fluid fields with
polynomial bulk dependence. In the process, we discovered that the dual MHD equations have equal magnetic and fluid diffusion constants.

In light of the results in [4], which were cast in a similar framework to ours [2], we believe that the Cauchy problem from section 5 admits a solution at all orders in the hydrodynamic expansion. In the 4-dimensional case, we were able to perform a test of the algebraically special character of the geometry, which turned out to be of Petrov type II. It is very likely that this statement will continue to hold in higher dimensions, though in such cases there is no analogue to the invariant $I^3 - 27J^2$ which can be used to perform the check. Nevertheless, it should be possible to generalize our solution to other background geometries. It seems worth investigating the dimensionless ratio of the two diffusion constants, as it might be subject to certain restrictions in the case of MHD theories with gravity duals. In particular, it would be interesting to find a background corresponding to the infinitely conducting fluid $\sigma = \infty$, which serves as a good approximation to real world MHD problems.

In [2], the observation that the metric was of an algebraically special type strongly suggested the hypothesis that algebraically special metrics have fluid duals [12]. The fact that the metric (10) is algebraically special leads us to formulate a new conjecture: Petrov type I metrics which solve the Einstein-Maxwell equations with properly aligned electromagnetic field strength appear to be dual to MHD-like fluid equations on codimension-one hypersurfaces. In the limit when the mean curvature of the hypersurface is large, these fluid equations reduce to the usual MHD equations; some work in this direction was done in [8].

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