Phase transition dynamics in a strong first-order quark-hadron transition

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We investigate the dynamics of a strong first-order quark-hadron transition driven by cubic interaction via homogeneous bubble nucleation in the Friedberg-Lee model. The one-loop effective thermodynamics potential of the Friedberg-Lee model and the critical bubble profiles have been calculated at different temperatures and chemical potentials. By taking the temperature and the chemical potential as the variables, the evolutions of the surface tension, the critical radius of the bubble and the shift in the coarse-grained free energy in the presence of a nucleation bubble are obtained and the limit on the reliability of the thin-wall approximation is also addressed accordingly. Our results are compared to those obtained for a weak first-order quark-hadron phase transition, especially the spinodal decomposition is relevant.

I. INTRODUCTION

At sufficiently high temperatures and densities, one expects that normal nuclear matter undergoes a phase transition to quark-gluon plasma (QGP), where quarks and gluons become deconfined and essentially chiral. This is a topic of great interest related to the physics of heavy-ion collisions at ultrarelativistic energies as well as to the astrophysics of neutron stars\textsuperscript{1,2}. Quantum chromodynamics (QCD) as a theory of strong interaction is applicable to determine the properties of strongly interacting matter at high temperatures and densities, however, because of the phenomenon of asymptotic freedom, the nature of the quark-hadron phase transition nevertheless remains an open question, especially when quark chemical potentials are involved in the practical calculations\textsuperscript{2}. Therefore, we still lack the capabilities to describe the low-energy nonperturbative phenomena in the framework of QCD theory and have to resort to effective models to study the nontrivial structure of the QCD vacuum, such as the Nambu-Jona-Lasinio (NJL)\textsuperscript{3,4}, the linear sigma model (LSM)\textsuperscript{5}, or their modernized versions, the Polyakov Nambu-Jona-Lasinio model (PNJL)\textsuperscript{6} and the Polyakov Quark Meson Model (PQM)\textsuperscript{7}.

The nature of the QCD phase diagram in the temperature and chemical potential plane has been intensively studied in past decades. Most effective models usually predict a smooth crossover transition at low chemical potential and non-zero temperature, while at high density and low temperature, there is a first-order phase transition for the chiral and the deconfinement transitions. At the endpoint of the first-order phase boundary, there should exist a so-called QCD critical endpoint (CEP)\textsuperscript{8}. How to find and identify the CEP in experiment is the main goals of the beam energy scan (BES) program at Relativistic Heavy-Ion Collider (RHIC)\textsuperscript{9} and the Super-Proton Synchrotron (SPS) facilities\textsuperscript{10}. On the theoretical side, although a recent study on the QCD phase diagram based on chiral effective models shows that the biggest part of the QCD phase diagram is crossover transition rather than true first-order one if the quark and meson fluctuations are included via the functional renormalization group\textsuperscript{11}, the possibility of a first-order phase transition is not ruled out from both the experimental and the theoretical point of view. In reality, most descriptions of the equation of state (EoS) of neutron stars with a quark core are undertaken in a hybrid equation of state with a hadron phase connected to a quark phase through a first-order phase transition\textsuperscript{12,13}. Moreover, the properties of hybrid stars with a strong first-order phase transition and their relevance to gravitational wave observations will place constraints on the tidal deformabilities of compact stars, allowing one to probe EoS for matter at extreme circumstances\textsuperscript{14}. Besides the quark-hadron phase transition, the first-order phase transition shall also play important roles in the evolution history of the early universe, such as its possible roles in electroweak baryogenesis and dark matter\textsuperscript{15,16}. Recently, a strong first-order phase transition is also taken as a potential source of gravitational waves (GW) which could be measured by future detectors\textsuperscript{17,18}. Especially, the approved Laser Interferometer Space Antenna (LISA) project makes it extremely important to directly probe the electroweak phase transition dynamics and their corresponding GW signals\textsuperscript{19}.

In a first-order phase transition, the initial metastable (or false) vacuum decays to the stable vacuum through the nucleation of bubbles larger than a critical size, and the nucleation rate of critical bubbles can be calculated from

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the microphysics using semiclassical methods in Euclidean thermal field theory\cite{22, 23}. Within this framework, an effective thermodynamic potential in the form of a Landau function with cubic interaction is an important and useful theoretical tool\cite{1}. According to the mean-field theory of phase transitions, the free energy density of the system can be expanded in terms of the parameter near the critical point, we can make a general consideration without going into much detail about the underlying dynamics\cite{24}. Therefore, at least in the level of the mean-field approximation, the thermodynamical potential of the effective models can be parametrized in the form of a Landau expansion around the equilibrium phase with all terms up to a = 4 in the region of the first-order phase transition. This scenario has been adopted to describe the dynamical mechanism of bubble nucleation in a strong first-order cosmological electroweak phase transition\cite{25} and in a weak first-order quark-hadron phase transition\cite{26, 27}. The benefit of this kind of parameterization is to simplify the effective potential and then make it greatly convenient to solve the equation of motion of the critical bubble profile both in numerical method and in analytical method.

For a weak first-order quark-hadron phase transition, when temperatures is slightly less than the critical temperature $T_c$, the thermodynamic potential exhibits a local minimum aside from the global minimum, as the temperature decreases to some specific value $T_{sp}$, the local minimum gradually disappears and ends at a point of inflection known as spinodal instability. Hence, the effective potential has no potential barrier for $T < T_{sp}$, and the shift in the coarse-grained free energy due to the appearance of the critical bubble monotonously decreases with the decreasing of the temperature and should eventually become zero at some specific temperature as shown in Refs.\cite{26, 27}. Since the weak first-order quark-hadron phase transition has been intensively investigated in the framework of the linear sigma model coupled to quarks\cite{26, 28, 29} and the hybrid model by combining EoS obtained within lattice QCD for the quark phase with that of gas of resonances in hadron phase\cite{24}. In this work, a strong first-order quark-hadron phase transition induced by an effective potential with a zero-temperature potential barrier is to be considered in the context of the Friedberg-Lee model\cite{30}. The model consists of quark fields interacting with a phenomenological scalar field which is introduced to describe the complicated nonperturbative features of the QCD vacuum, and it has been very successful in describing the static properties of isolated hadrons at low energy. Recently, the model has been also extended to finite temperatures and densities to study the deconfinement phase transition in Refs.\cite{31–35}, in which most of works focus on the thermodynamic effective potential and phase diagram. Because the strong first-order phase transition gains more and more attentions both in the astrophysics of neutron stars and cosmological phase transitions in the early universe, especially when GWs are relevant, our current study will be concentrated on the dynamics of a first-order phase transition via bubble nucleation, and the general discussions presented in this work can be applied to study the dynamics of the first-order phase transitions in various fields driven by cubic interaction, especially beyond the thin-wall approximation.

The paper is organized as follows. In the following section we briefly describe the Friedberg-Lee model and its effective potential at finite temperatures and densities. In Sec. II, we give detailed description of homogeneous nucleation and the methods used for both numerical and analytic computations of the critical bubble profiles. Our results and discussions are presented in Sec. IV, whereas in the last section we give summary.

## II. MODEL FORMULATION

We start with the Lagrangian of the Friedberg-Lee model for a phenomenological scalar field $\sigma$ interacted with the spin-$\frac{1}{2}$ quark fields $\psi$ of the form\cite{30},

$$\mathcal{L} = \overline{\psi}(i\partial - g\sigma)\psi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - U(\sigma),$$

where the potential, which exhibits a typically first-order phase transition, is parametrized in the form of a Landau expansion with all the terms up to quartic term as

$$U(\sigma) = \frac{1}{2!}a\sigma^2 + \frac{1}{3!}b\sigma^3 + \frac{1}{4!}c\sigma^4.$$  \hspace{1cm} (2)

The model parameters $a$, $b$ and $c$ are well chosen as $b^2 > 3ac$ in order to ensure that $U(\sigma)$ has a local minimum at $\sigma = 0$ and an global minimum at a relative larger value of the $\sigma$ field

$$\sigma_v = \frac{3|b|}{2c} \left[ 1 + \left( 1 - \frac{8ac}{3b^2} \right)^{\frac{3}{2}} \right].$$  \hspace{1cm} (3)

Usually, the global minimum at $\sigma = \sigma_v$ is interpreted as the physical or true vacuum, whereas the local minimum at $\sigma = 0$ represents a metastable vacuum where the condensates vanishes and quarks have zero rest mass. The difference

$$\sigma_v = \frac{3|b|}{2c} \left[ 1 + \left( 1 - \frac{8ac}{3b^2} \right)^{\frac{3}{2}} \right].$$  \hspace{1cm} (3)
FIG. 1: The one-loop effective potential $V_{\text{eff}}$ as a function of $\sigma$ at $T = 0$ MeV, $T = 80$ MeV and $T = 119.8$ MeV when fixing the chemical potential $\mu$ at 0 MeV. For our choice of parameters, the two minima appear as degenerate at $T_C \simeq 119.8$ MeV, which is usually defined as the critical temperature.

in the potential values of the two vacuum states is defined as the bag constant $\varepsilon$. For convenience, in the following discussions, we’d like to take $U(0) = 0$, therefore, the bag constant $\varepsilon$ can be expressed as

$$\varepsilon = \frac{a}{2!} \sigma^2 + \frac{b}{3!} \sigma^3 + \frac{c}{4!} \sigma^4. \tag{4}$$

There is a wide range of the model parameters $a$, $b$, $c$ and $g$ adopted in Refs. [32, 33, 43] in order to confront the basic properties of nucleon in vacuum. However, for the problem we discuss here, different sets of values will show similar physical results, hereafter we just take one set of parameters $a = 17.70\, fm^{-2}$, $b = -1457.4\, fm^{-1}$, $c = 20000$ and $g = 12.16$, which has been widely used in the literatures.

A convenient framework of studying phase transitions is the thermal field theory. Within this framework, the finite temperature effective potential is an important and useful theoretical tool. Keeping only contributions to one-loop order, the effective potential of the Friedberg-Lee model can be computed exactly in closed form following the steps presented in Ref. [37]

$$V_{\text{eff}}(\sigma; T, \mu) = U(\sigma) + V_B(\sigma; T) + V_F(\sigma; T, \mu), \tag{5}$$

where $U(\sigma)$ is the classical potential of the Lagrangian. $V_B(\sigma; T)$ is the finite temperature contribution from boson one-loop diagrams, and $V_F(\sigma; T, \mu)$ is the finite temperature and density contribution from fermion one-loop diagrams [34, 37]. These contribute the following terms in the effective potential

$$V_B(\sigma; T) = T \int \frac{d^3 \vec{p}}{(2\pi)^3} \ln \left( 1 - e^{-E_{\sigma}/T} \right), \tag{6}$$

$$V_F(\sigma; \beta, \mu) = -2N_f N_c T \int \frac{d^3 \vec{p}}{(2\pi)^3} \left[ \ln \left( 1 + e^{-(E_{\sigma} - \mu)/T} \right) + \ln \left( 1 + e^{-(E_{\sigma} + \mu)/T} \right) \right], \tag{7}$$

in which $N_f = 2$, $N_c = 3$. $E_{\sigma} = \sqrt{\vec{p}^2 + m_{\sigma}^2}$ and $E_q = \sqrt{\vec{p}^2 + m_q^2}$ are energies for the $\sigma$ mesons and quarks, in which the constituent quark (antiquark) mass $m_q$ is defined as $m_q = g\sigma$, while the effective mass of scalar meson field is set by $m_{\sigma}^2 = a + b\sigma + \frac{c}{2}\sigma^2$. To ensure the effective masses of $\sigma$ fields are always positive, in this work we prefer to fix $m_{\sigma}$ by taking its value at the physical vacuum state.

In the absence of the chemical potential, the one-loop effective potential at different temperatures has been plotted in Fig.1. The shape of the potential shows that a first-order phase transition takes place as it exhibits two degenerate minima at a certain temperature $T_c \simeq 119.8$ MeV, which is usually defined as the critical temperature. Normally, apart
FIG. 2: The one-loop effective potential $V_{\text{eff}}$ as a function of $\sigma$ at $\mu = 0$ MeV, $\mu = 150$ MeV and $\mu = 256.4$ MeV when fixing the temperature $T$ at 50 MeV. For our choice of parameters, the critical chemical potential is set at $\mu_C \simeq 256.4$ MeV when two minima are equal.

from this critical temperature, there exists another particular temperature which is widely discussed in a first-order phase transition as one of the minima of the potential disappears when the temperature is at a higher temperature. Between these two particular temperatures, metastable states exist and lie close to $\sigma_v$, and the system can exhibit supercooling or superheating. On the contrary, with the temperature decreasing, as soon as the temperature is across the critical temperature, the metastable vacuum and physical vacuum will get flipped, and the metastable states now are centred around the origin $\sigma = 0$ since the value of potential in the minimum nearby $\sigma_v$ becomes less than that of the minimum at $\sigma = 0$. Then, it is possible to define the bag constant as the difference between the effective potential at the metastable vacuum state and the effective potential at the physical vacuum state,

$$\varepsilon(T) = V_{\text{eff}}(0; T) - V_{\text{eff}}(\sigma_v; T).$$

(8)

It is easy to check that the bag constant $\varepsilon$ will decrease with the increasing of the temperature, and when $T = T_c$ the two vacuum are equal, the bag constant is zero.

As we known the nucleon is treated as a soliton bag in the Friedberg-Lee model, and there are two dynamical mechanisms to support the confinement of the quarks inside the nucleons. One is the effective mass of the quark in the physical vacuum which is very large outside the nucleon and makes it energetically unfavourable to exist there. The other is the bag constant. Since at the critical temperature $T = T_c$, the physical vacuum will be located at $\sigma = 0$, the quarks have zero rest mass and only kinetic energy. Meanwhile as the two minima of the effective potential degenerate, the bag constant becomes zero. Therefore, for the moment, there is no more dynamical mechanism to be provided to confine the quarks, the deconfinement phase transition is believed to occur.

When fixing temperature at $T = 50$ MeV, we plot the one-loop effective potential $V_{\text{eff}}$ as a function of $\sigma$ at various chemical potentials $\mu = 0$ MeV, $\mu = 150$ MeV and $\mu = 256.4$ MeV in Fig.2 From this figure, we can find that the shapes of the potentials show similar behaviors when compared with the results presented in Fig.1 For $\mu = 256.4$ MeV, the values of the effective potentials at the two vacuums are equal. In this moment, this certain chemical potential is defined as the critical chemical potential $\mu_c = 256.4$ MeV for the first-order phase transition. With the decreasing of the chemical potential from $\mu_c$, the global minimum of the potential jumps from the position of $\sigma = 0$ to that of $\sigma_v$, and the state at $\sigma_v$ is often referred to the physical vacuum. Whereas the local minimum around $\sigma = 0$ becomes a metastable state and it is set as a metastable or false vacuum. The coexistence of the physical vacuum and the false vacuum shows the coexistence of the deconfinement and confinement phases when $\mu \leq \mu_c$, this is the typical character of the first-order phase transition. The temperature and the chemical potential dependence of the bag constant is defined as the difference between the values of the effective potential at the false vacuum and at the physical vacuum as usual,

$$\varepsilon(T, \mu) = V_{\text{eff}}(0; T, \mu) - V_{\text{eff}}(\sigma_v; T, \mu).$$

(9)
It is easy to check that the bag constant $\varepsilon$ will decrease with the increasing of the chemical potential, and when $\mu = \mu_c$ the two vacuums are equal, the bag constant is zero.

## III. HOMOGENEOUS THERMAL NUCLEATION

For the first-order phase transition, when the temperature or chemical potential is around its critical value, the effective potential exhibits degenerate minima which are separated by a barrier. As the temperature or chemical potential is lowered, the local minimum at $\sigma \geq 0$ becomes the false vacuum, while the global minimum of the effective potential at $\sigma \approx \sigma_v$ is taken as the stable or physical vacuum. The false vacuum would be stable classically, but quantum mechanically it is only a metastable state and can decay via the nucleation of bubbles larger than a critical size. Technically, this decay may be triggered by either quantum or thermal fluctuations, depending on what kind of physics we are interested in. In this work, we will be mostly concerned with the regime in which thermal fluctuations are much larger than quantum fluctuations.

The dynamics of a first-order phase transition can be described by the mechanism of bubble nucleation of the stable vacuum inside the false vacuum, and the appearance of a bubble of the stable vacuum inside a metastable vacuum is believed to be a natural consequence of the thermal and quantum fluctuations of any thermodynamical systems closely interrelated with a first-order phase transition. For $T < T_c$ or $\mu < \mu_c$, bubbles of the stable vacuums created by thermal fluctuations may grow or shrink inside the false vacuum depending on its energy budget with regard to homogeneous false vacuum. Because the bulk free energy density of the false vacuum is higher than that of the stable vacuum, the phase conversion from the false vacuum to the stable vacuum makes the bulk free energy of the whole system lower. However, the appearance of a spherical bubble means there is an interface which is needed in order to separate the stable vacuum from the exterior of the false vacuum, and the creation of such an interface represents an energy cost. Therefore, the mechanism of phase conversion from the metastable phase $\sigma \approx 0$ to the stable phase $\sigma \simeq \sigma_v$ proceeds by a competition between the free energy gain from the phase transition of the bulk and the energy cost from the formation of an interface. Note that the free energy shift due to the appearance of a spherical bubble of the stable vacuum is proportional to $-R^3$, where $R$ is the bubble radius, and the surface tension of the interface between two phase is proportional to $+R^2$. For the nucleation of small bubbles, the energy cost is higher than the energy gain, small bubbles tends to shrink. On the contrary, a bubble with a sufficiently large radius represents a large bulk energy gain, the energy gain in the system is going to be higher than the surface energy cost in creating the bubble. As a consequence, these large bubbles tend to expand even more and to coalesce completely, completing the phase conversion. Therefore, only bubbles of a very large radius play a decisive role in the theory of dynamics of a first-order phase transition.

In the theory of the bubble nucleation, a scalar field $\sigma$ is treated as the order parameter and a coarse-grained free energy functional of the system is defined as

$$F(\sigma) = \int dr^3 \left[ \frac{1}{2} (\nabla \sigma)^2 + V_{\text{eff}}(\sigma; T, \mu) \right].$$

The critical bubble configuration is an extremum of the coarse-grained free energy functional $F(\sigma)$ with respect to the scalar field $\sigma$, so that the equation of motion to be solved is now the nonlinear ordinary differential equation,

$$\frac{d^2\sigma(r)}{dr^2} + \frac{2}{r} \frac{d\sigma(r)}{dr} = \frac{\partial V_{\text{eff}}(\sigma; T, \mu)}{\partial \sigma},$$

with boundary conditions $\lim_{r \to \infty} \sigma(r) = 0$ and $\frac{d\sigma(0)}{dr} = 0$. The solution for this equation of motion is a saddle point solution $\sigma_b$, which connect the two minima of the effective potential.

Once the solution $\sigma_b$ is found, the shift in the coarse-grained free energy due to the formation of a nucleation bubble can be calculated as

$$\Delta F_b = 4\pi \int r^2 dr \left[ \frac{1}{2} \left( \frac{d\sigma_b}{dr} \right)^2 + V_{\text{eff}}(\sigma_b; T, \mu) \right].$$

It should be pointed out that here and after that $\Delta F_b(0; T, \mu)$ is well normalized to be zero for simplicity. The nucleation rate per unit volume is expressed as

$$\Gamma = \mathcal{P} \exp \left[ -\frac{\Delta F_b}{T} \right],$$

with $\mathcal{P}$ being the quantum correction factor.
where the pre-exponential factor $P$ corresponds to the probability for a critical bubble-like field fluctuation $\sigma$ to be generated and grow. Evaluation of the pre-exponential factor is a nontrivial matter. A rough estimate of their ratio can be obtained by dimensional arguments and we could approximate $P$ by $T^4$ for simplicity \[26\]. The surface tension of the nucleation bubble interface between the false vacuum and the stable vacuum is then defined as

$$\Sigma = \int dr \left[ \frac{1}{2} \left( \frac{d\sigma}{dr} \right)^2 + V_{\text{eff}}(\sigma_b; T, \mu) \right]. \quad (14)$$

For the critical bubble, since there exists an energy competition between the free energy gain and the surface energy cost, any bubbles smaller than the critical bubble will contract and disappear, while any larger bubble will continue to grow and drive the phase transition. The critical bubble with a critical radius $R_c$ can be estimated as follows:

$$\Delta F_b = 4\pi \int r^2 dr \left[ \frac{1}{2} \left( \frac{d\sigma}{dr} \right)^2 + V_{\text{eff}}(\sigma_b; T, \mu) \right] = -\frac{4}{3} \pi r^3 \varepsilon + 4\pi r^2 \Sigma, \quad (15)$$

where $\varepsilon$ is the bag constants in Eqs. (8) and (9). This is essentially a three-dimensional bubble of true vacuum surrounded by a bubble wall, and the last equation (15) represents the Euclidean action relative to that of the false vacuum. The critical radius $R_c$ is determined by minimization of the coarse-grained free energy $\Delta F_b$ with respect to $r$, which in turn requires that

$$R_c = \frac{2\Sigma}{\varepsilon}. \quad (16)$$

Hence, only bubbles that have a size equal to or larger than the critical radius $R_c$ are energetically favourable and would play an important role in the dynamical seed of the phase conversion.

For a generic effective potential $V_{\text{eff}}$, the equation of motion (11) with some certain boundary conditions usually cannot be solved analytically. However, when the system is very close to the critical coexistence line, e.g. $T \sim T_c$ or $\mu \sim \mu_c$, the problem can be essentially simplified. In such a situation, the difference between the effective potential at the false vacuum and the effective potential at physical vacuum is much smaller than the height of the barrier separated these two vacuum, because of the competition between the free energy gain and the surface energy cost, the critical radius of the bubbles becomes much greater than the wall thickness, the second term in the equation of motion (11) can be neglected. Then the so-called thin-wall approximation is applicable and the equation of motion (11) reduces to the equation for a typical one-dimensional soliton

$$\frac{d^2\sigma(r)}{dr^2} = \frac{dV_{\text{eff}}}{d\sigma}. \quad (17)$$

This static field equation implies that

$$\frac{d\sigma(r)}{dr} = \pm \sqrt{2V_{\text{eff}}}. \quad (18)$$

Integrating Eq. (18) yields

$$r = \int_\sigma^{\sigma_v} \frac{d\sigma}{\sqrt{2V_{\text{eff}}}}. \quad (19)$$

In the case of an arbitrary potential $V_{\text{eff}}$ with two or more degenerate global minima as in the limit $\varepsilon \to 0$, the profile of the critical bubble can be estimated as follows. For a smoothly varying potential $V_{\text{eff}}$, the integral on the right-hand side diverges as $\sigma(r)$ approaches any of the global minima. Hence, as $r$ ranges from 0 to $\infty$, $\sigma(r)$ must vary monotonically from one global minimum of $V_{\text{eff}}$ at $\sigma = \sigma_v$ to an adjacent global minimum at $\sigma = 0$. In this case, the approximate solution for the bubble with the critical size is then given by

$$\sigma(r) = \begin{cases} \sigma_v & 0 < r < R_c - \Delta R, \\ \sigma_{\text{wall}}(r) & R_c - \Delta R < r < R_c + \Delta R, \\ 0 & r > R_c + \Delta R, \end{cases} \quad (20)$$

which describes a three-dimensional bubble of radius $R_c$ of the stable vacuum at $\sigma = \sigma_v$ surrounded by a thin wall inside the metastable vacuum at $\sigma = 0$. $\sigma_{\text{wall}}(r)$ is a solution to Eq. (19) and describes the bubble wall separating the two vacua.
Furthermore, in the absence of the bag constant $\varepsilon$, the one-dimensional energy or the surface tension of the bubble is
\[
\Sigma_{tw} = \int_0^\infty dr \left[ \frac{1}{2} \left( \frac{d\sigma_b}{dr} \right)^2 + V_{\text{eff}} \right] = \int_0^{\sigma_c} d\sigma \sqrt{2V_{\text{eff}}}. \tag{21}
\]
From the equations (19) and (21), a saddle point field configuration $\sigma(r)$ and the surface tension can be directly obtained by using the effective potential $V_{\text{eff}}$ without solving the equation of motion in Eq. (11), which is usually difficult to be solved both in the numerical method and in the analytical method. This is the main advantage of the thin-wall approximation approach. Since the thin-wall approximation is so widely adopted in literatures [26–29, 38–41], in what follows, we focus our study on the exact numerical computations and establish limits on the reliability of the thin-wall approximation.

**IV. RESULTS AND DISCUSSION**

![FIG. 3: Left panel: Critical bubble profiles for different temperatures and zero chemical potential. From left to right, the curves correspond to $T = 0, 70, 109, 118, 119, 119.8$ MeV. Right panel: Critical bubble profiles for different chemical potentials when fixing the temperature $T$ at 50 MeV. From left to right, the curves correspond to $\mu = 0, 200, 234, 253, 254, 265.4$ MeV.](image)

In what follows, we numerically solve the equation of motion in Eq. (11) with some proper boundary conditions, $\sigma \to 0$ as $r \to \infty$ and $\sigma_{(0)} = 0$. The exact numerical solutions by taking the temperatures as $T = 0, 70, 109, 118, 119, 119.8$ MeV in the absence of chemical potential are plotted in the left panel of Fig. 3. One can see that with the temperature decreasing from $T = T_c$, all curves approach to zero when the radius $r$ is large, whereas $\sigma(r)$ at the center of the bubble is changed dramatically. For the temperature is sufficiently close to the critical temperature at $T_c = 119.8$ MeV, the $\sigma$ field at the center of the bubble only slightly deviates from its stable vacuum value at $\sigma = \sigma_c$, however, for $T \leq 109$ MeV the $\sigma$ field at the center of the bubble is visibly different from its stable vacuum value. This means that the thin-wall approximation is expected to be valid and applicable only when the temperature is very close to the critical temperature $T_c$. Any further extension of the thin-wall approximation to lower temperatures deviation from $T_c$ should be checked very carefully.

Similar discussion can be applied to the second case, when the temperature is fixed, the critical bubble profiles at different chemical potentials are illustrated in the right panel of Fig. 3, where the chemical potentials are taken as $\mu = 0, 200, 230, 253, 254$ and 265.4 MeV for a fixed temperature $T = 50$ MeV. The evolution of the $\sigma(r)$ for different chemical potentials tells that the radius of the critical bubble should increase as well with the chemical potential increasing, and the nontrivial behavior of the $\sigma(r)$ in the center of the bubble can also be interpreted as a limit to the applicability of the thin-wall approximation. From the right panel of Fig. 3 since the $\sigma(0)$ reaches its maximum when $\mu \simeq 234$ MeV, this specific value is taken as the lower limit for the validity of the thin-wall approximation.

Once the bubble profiles have been solved, The surface tension of the nucleation bubble interface between the false vacuum and the stable vacuum as a function of the temperature is shown in the left panel of Fig. 4 in the case of zero chemical potential. From the left panel of Fig. 4 we can see that the surface tension exhibits an interesting behavior. With the increasing of the temperature, the surface tension $\Sigma(T)$ does grow up steadily, when $T > 60$ MeV, the...
surface tension will increase quickly and reach to a maximum value at $\Sigma(T) \simeq 7.38 \text{ MeV/fm}^2$ when $T \simeq 109 \text{ MeV}$, after that its value decreases from this top value. This nontrivial behavior of $\Sigma(T)$ at $T \simeq 109 \text{ MeV}$ can be analyzed by the evolution of the bubble profile with the temperature. From the left panel of Fig. 3 as $T$ goes down from its critical temperature $T_c$, the $\sigma(r)$ field nearby the center of the bubble will depart from its thin-wall approximate solution $\sigma_c$ in Eq. (20) gradually, when $T \simeq 109 \text{ MeV}$, the $\sigma(r)$ field reaches its maximal value before it starts to decrease. This implies that the turning point of the surface tension could be treated as a landmark for the breaking down of the thin-wall approximation since the point is the right moment when the $\sigma(r)$ deviates from the thin-wall approximation value mostly in the beginning. Furthermore, the nonmonotonic behavior of the surface tension can also be revealed by the evolution of the critical bubble profiles shown in the left panel of Fig. 3 since the $\sigma(r)$ becomes more and more steep with $T$ goes down from $T_c$ to $T = 109 \text{ MeV}$. As a consequence, the surface tension increases. On the contrary, for $T < 109 \text{ MeV}$, the $\sigma(r)$ is progressively softened, in the mean time the value of the second term $V_{\text{eff}}(\sigma_0; T, \mu)$ in Eq. (11) becomes more and more larger, therefore the surface tension decreases as the temperature falls down. It is interesting to note that in the case of a weak first-order phase transition, as long as the temperature is under a spinodal temperature $T_{\text{sp}}$, a small barrier between the two minima in the potential will disappear, and there is only one minimum left in the effective potential. According to a standard criterion to guarantee the existence of the stable bounce or soliton solutions, it is indispensable for the potential of the order parameter fields, e.g. $\sigma$ field in this work, to exhibit three distinct extrema \[2, 14\]. So that we can only have a trivial solution to the equation of motion \[11\] as $\sigma(r) = 0$ if $T < T_{\text{sp}}$, and the surface tension should approach to the zero when $T \rightarrow T_{\text{sp}}$ as displayed in Fig. 4 in Ref. \[27\].

For the second case presented in the right panel of Fig. 3, $\Sigma(\mu)$ shows a non-monotonically decreasing function of $\mu$ too. With the increase of the chemical potential, $\Sigma(\mu)$ goes up accordingly until it reaches the top of its values, in which the chemical potential is around 234 MeV. In general, the turning point of the surface tension is still treated as a generous limit to the applicability of the thin-wall approximation.

The critical radius of the bubble as a function of temperature and chemical potential are displayed in Fig. 6. As mentioned above, any bubble smaller than the critical bubble will shrink and rapidly disappear, and any larger bubble will grow and drive the phase conversion. For this reason, bubbles with radii larger than $R_c$ will have a decisive role and can be taken as the dynamical seed of the first-order phase conversion. From Fig. 5, it gives a signal of a swelling of the critical bubble with the increase of temperature and chemical potential. By taking the temperature and the chemical potential as the active variables, $R_c$ increases only slightly with the increase of the variables as variables are small. However as the variables are close to their critical values, e.g. $T_c$ and $\mu_c$, $R_c$ sharply grows and diverges. The divergent behaviors of the $R_c$ at $T = T_c$ and $\mu = \mu_c$ are in agreement with the definition of the critical radius in Eq. (16), note that $\varepsilon \rightarrow 0$ as $T \rightarrow T_c$. For the numerical subtlety, besides the nontrivial numerical solutions presented in Fig. 3, the equation of motion in Eq. (11) can always possess two trivial solutions: $\sigma(r) = \sigma_c$ and $\sigma(r) = 0$. For the proper boundary conditions in our work, we choose the solution $\sigma(r) = \sigma_c$, this means that the critical bubble has an infinite radius when $T = T_c$, or $\mu = \mu_c$, or there is no exact nontrivial solution for the equation of motion when the temperature is exactly located in $T = T_c$, or the chemical potential is exactly equal to $\mu_c$.

The shift in the coarse-grained free energy due to the activation of a nucleation bubble $\Delta F_b$ can be calculated.
FIG. 5: Left panel: Radius of the critical bubble as a function of temperature $T$ when $T \leq T_c$ at zero chemical potential. Right panel: Radius of the critical bubble as a function of chemical potential $\mu$ when $\mu \leq \mu_c$ for fixing the temperature at 50 MeV.

FIG. 6: Left panel: The bubble activation free energy shift $\Delta F_b/T$ as a function of temperature $T$ for $T \leq T_c$ at zero chemical potential. Right panel: The bubble activation free energy shift $\Delta F_b/T$ as a function of chemical potential $\mu$ for $\mu \leq \mu_c$ when fixing the temperature at 50 MeV.

directly from Eq.(12) or in a more convenient way by using the formula in Eq.(15) when $r = R_c$,

$$\Delta F_b = -\frac{4}{3} \pi R_c^3 \epsilon + 4\pi R_c^2 \Sigma,$$

$$= \frac{16\pi \Sigma^3}{3} \epsilon^2.$$

\hspace{1cm} (22)

In this work, we concentrate on the relatively violent behavior of the exponential factor in Eq.(13), which is an essential ingredient for the nucleation rate per unit volume $\Gamma$, whereas the pre-exponential factor $P$ is crudely chosen as $T^4$. To show the shift in the coarse-grained free energy due to the appearance of the critical bubble and its crucial role played in the nucleation rate for the first-order phase transition, $\Delta F_b/T$ as functions of the temperature $T$ and the chemical potential $\mu$ are plotted in Fig.6. In the absence of the chemical potential, the $\Delta F_b/T$ decreases with the increase of the temperature and touches down some minimum point, then it will rise very quickly and diverge nearby the critical temperature $T_c$, just as expected in Eq.(22). For $T \approx 114.5$ MeV, $\Delta F_b/T \approx 1$, then $\Gamma$ will be strongly suppressed by the exponential factor, and the system is likely to stay in the metastable vacuum for a relatively long time. On the contrary, for $T < 114.5$ MeV, the unstable vacuum tends to decay very quickly to the true vacuum, the nonmonotonic behavior of the $\Delta F_b/T$ as functions of the temperature $T$ is also reported in a recent work in a strong cosmological first-order phase transition[20], where the bounce action firstly decreases to a minimum value and then
increases again with decreasing of the temperature.

In the case of fixing the temperature at 50 MeV, the bubble activation free energy shift $\Delta F_b/T$ as a function of chemical potential $\mu$ for $\mu \leq \mu_c$, is addressed in the right panel in Fig.6. In this case, the $\Delta F_b/T$ shows a monotonic behavior and grows up rapidly with the increase of the chemical potential. When the chemical potential approaches to the critical value $\mu_c$, the $\Delta F_b/T$ tends to rise up dramatically and diverge again due to the fact that $\epsilon$ is zero when $\mu = \mu_c$. For $\mu$ is about 231 MeV, $\Delta F_b/T \approx 1$, so that the system is likely to remain in the metastable vacuum as long as the chemical potential is larger than 231 MeV.

V. SUMMARY

In the present paper we have investigated a dynamics of a strong first-order phase transition via homogeneous bubble nucleation within the Friedberg-Lee model at finite temperature and chemical potential. After obtaining the effective thermodynamical potential, a saddle point solution of the equation of motion has been numerically calculated and the exact bubble profiles evolving with the temperatures and chemical potentials have been obtained. The differences between the energy density of the false vacuum and that of the physical vacuum or the bag constant functions are precisely defined for finite temperatures and densities. The critical temperature $T_c$ for zero chemical potential is around 119.8 MeV when the two minima of the effective potential are equal to each other as the parameters chosen in this work. On the contrary, if we treat the chemical potential as a variable, a critical chemical potential is to be set up at $\mu \approx 256.4$ MeV as temperature is fixed at 50 MeV.

Surface tensions as a function of temperature and as a function of chemical potential show similar behaviors. They will firstly increase to a maximum value and then decrease again with the decreases of the temperature and chemical potential accordingly. The inflection point of the surface tension can be taken as a limit on the reliability of the thin-wall approximation due to the fact that the critical bubble profiles obtained from the exact numerical method will deviate from that of values based on the thin-wall approximation more and more dramatically since then. Moreover, in the case of our study in the Friedberg-Lee model, the classical potential is possessed of three distinct extrema and its two minima are separated by a barrier. According to a standard criterion to guarantee the existence of the stable bounce or soliton solutions [42–44], no matter how small the barrier will be, we can always have a nontrivial bounce solution for the equation of motion of the critical bubble profiles. This indicates that as the temperature or the chemical potential goes to zero, the surface tension $\Sigma$ shall approach to some small value rather than zero as long as the barrier exists there. However, for the case of a weak first-order phase transition, since the local minimum of the effective potential gradually disappears as the system cools down, for $T < T_{sp}$, there is no nontrivial stable bounce solution for the equation of motion, the surface tension $\Sigma$ should subsequently become zero [27]. This is an apparent difference between the zero-temperature effective potential with and without a barrier. Furthermore, because of its important role in heavy-ion collision and in astrophysics, the surface tension has attracted much attention recently. Our calculations give a rather low value, for example the surface tension is about 7.38 MeV/fm² for zero chemical potential and 7.73 MeV/fm² for fixing the temperature at $T = 50$ MeV when the system is at the critical point. Besides the model adopted in this work, most effective models, such as the MIT bag model [45], the quark-meson model [28, 29, 11, 16], NJL model [47, 48], three-flavor PQM model [39], the nucleon-meson model [10], also predict small values and $\Sigma \leq 30$ MeV/fm².

This is unlike the evolution of the surface tension as a function of temperature and chemical potential, the radius of the critical bubble as function of temperature or as a function of chemical potential exhibits a monotonic property with the increase of temperature or the increase of chemical potential. In both cases, $R_c$ starts from a small value and then increases slightly with the increase of the variable, when the system is close to its critical point, it sharply grows and disappears. While for a weak first-order quark-hadron phase transition, for $T < T_{sp}$, since there only exists a trivial bounce solution for the equation of motion as $\sigma(r) = 0$, $R_c \to 0$ as $T \to T_{sp}$.

The shift in the coarse-grained free energy due to the presentation of a nucleation bubble over the temperature $\Delta F_b/T$ show a very interesting behavior with the system warming up. When the temperature rises up, $\Delta F_b/T$ firstly decreases to a minimum value and then increases rapidly. As the temperature is close to the critical temperature $T_c$, it will quickly go across the unity 1 and become divergent. In comparison with the works based on a weak first-order quark-hadron phase transition [20, 25], we find the $\Delta F_b/T$ as a function of temperature shows similar behavior when the temperature is nearby the critical temperature $T_c$, however, when the temperature is apart from $T_c$, our results present a non-monotonic behavior with the decrease of the temperature, whereas for the case of a weak first-order phase transition with the spinodal instability, the $\Delta F_b/T$ will drop monotonically to zero very soon as $T \to T_{sp}$. This is another apparent difference between the zero-temperature effective potential with and without a barrier.
Acknowledgments

We thank Jinshuang Jin for valuable comments and discussions. This work is supported in part by National Natural Science Foundation of China (NSFC) under No.11675048.

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