NON ABELIAN TENSOR SQUARE OF NON ABELIAN PRIME POWER GROUPS

PEYMAN NIROOMAND

Abstract. For every $p$-group of order $p^n$ with the derived subgroup of order $p^m$, Rocco in [7] has shown that the order of tensor square of $G$ is at most $p^{n(m-1)+2}$. In the present paper not only we improve his bound for non-abelian $p$-groups but also we describe the structure of all non-abelian $p$-groups when the bound is attained for a special case. Moreover, our results give as well an upper bound for the order of $\pi_3(SK(G,1))$.

1. Introduction and Preliminaries

The tensor square $G \otimes G$ of a group $G$ is a group generated by the symbols $g \otimes h$ subject to the relations

$$gg' \otimes h = (g' \otimes g)(g \otimes h)$$

and

$$g \otimes hh' = (g \otimes h)(h \otimes g')$$

for all $g, g', h, h' \in G$, where $g' = g^{-1}$. The non abelian tensor square is a special case of non abelian tensor product, which was introduced by R. Brown and J.-L. Loday in [3].

There exists a homomorphism of groups $\kappa : G \otimes G \to G'$ sending $g \otimes h$ to $[g, h] = ghg^{-1}h^{-1}$. The kernel of $\kappa$ is denoted by $J_2(G)$; its topological interest is in the formula $\pi_3(SK(G,1)) = J_2(G)$ (see [3]).

According to the formula $\pi_3(SK(G,1)) = J_2(G)$ computing the order of $G \otimes G$ has interests in topology in addition to its interpretation as a problem in the group theory.

Rocco in [7] and later Ellis in [4] have shown that the order of tensor square of $G$ is at most $p^{n(m-1)+2}$ for every $p$-group of order $p^n$ with the derived subgroup of order $p^m$.

The purpose of this paper is a further investigation on the order of tensor square of non abelian $p$-groups. We focus on non abelian $p$-groups because in abelian case the non abelian tensor square coincides with the usual abelian tensor square of abelian groups. To be precise, for a non abelian $p$-group of order $p^n$ and the derived subgroup of order $p^m$, we prove that $|G \otimes G| \leq p^{n(n-1)+2}$ and also we obtain the explicit structure of $G$ when $|G \otimes G| = p^{n(n-1)+2}$. It easily seen that the bound is less than of Rocco’s bound, unless that $G \cong Q_8$ or $G \cong E_1$, which causes two bounds to be equal. As a corollary by using the fact $\pi_3(SK(G,1)) = \text{Ker}(G \otimes G \to G')$, we can see that $|\pi_3(SK(G,1))| = |J_2(G)| \leq p^{n(n-1)+2}$.

Thorough the paper, $D_8$, $Q_8$ denote the dihedral and quaternion group of order 8, $E_1$ and $E_2$ denote the extra-special $p$-groups of order $p^3$ of exponent $p$ and $p^2$.

Key words and phrases. Tensor square, non abelian $p$-groups.
Mathematics Subject Classification 2010. 20D15.
respectively. Also \( C_{p^k} \) and \( \nabla(G) \) denote the direct product of \( k \) copies of the cyclic group of order \( p^k \) and the subgroup generated by \( g \otimes g \) for all \( g \) in \( G \), respectively.

2. Main Results

The aim of this section is finding an upper bound for the order tensor square of non abelian \( p \)-groups of order \( p^n \) in terms of the order of \( G' \). Also in the case for which \( |G'| = p \), the structure of groups is obtained when \( |G \otimes G| \) reaches the upper bound.

**Proposition 2.1.** [2, Proposition 9]. Given a central extension

\[
1 \rightarrow Z \rightarrow H \rightarrow G \rightarrow 1
\]

there is an exact sequence

\[
(Z \otimes H) \times (H \otimes Z) \xrightarrow{L} H \otimes H \rightarrow G \otimes G \rightarrow 1
\]

in which \( \text{Im} \ l \) is central.

**Proposition 2.2.** [2, Proposition 13, 14] The tensor square of \( D_8 \) and \( Q_8 \) is isomorphic to

\[
C_2^{(3)} \times C_4 \text{ and } C_2^{(2)} \times C_4^{(2)},
\]

respectively.

Recall that [2, 3] the order of tensor square of \( G \) is equal to \( |\nabla(G)||\mathcal{M}(G)||G'| \), where \( \mathcal{M}(G) \) is the Schur multiplier of \( G \).

Put \( G^{ab} = G/G' \). In analogy with the above proposition the following lemma is characterized the tensor square of extra-special \( p \)-groups of order \( p^3 \) \( (p \neq 2) \).

**Lemma 2.3.** The tensor square of \( E_1 \) and \( E_2 \) are isomorphic to \( C_p^{(6)} \) and \( C_p^{(4)} \), respectively.

**Proof.** It can be proved from [3, Theorem 2] that \( E_1 \otimes E_1 \) is elementary abelian. Now, by invoking [1, Proposition 2.2 (iii)], \( \nabla(E_1) \cong \nabla(E_1^{ab}) \) and hence \( |\nabla(E_1)| = p^3 \). On the other hand, [3, Theorem 3.3.6] implies that the Schur multiplier of \( E_1 \) is of order \( p^2 \), and so \( |E_1 \otimes E_1| = p^5 \).

In the case \( G = E_2 \) in a similar fashion, we can prove that \( E_2 \otimes E_2 \cong E_2^{ab} \).

**Lemma 2.4.** [6, Corollary 2.3] The tensor square of an extra-special \( p \)-group \( H \) of order \( p^{2m+1} \) is elementary abelian of order \( p^{4m^2} \), for \( m \geq 2 \).

**Proposition 2.5.** Let \( G \) be a \( p \)-group of order \( p^n \) and \( |G'| = p \). If one of the following conditions holds, then the order of tensor square is less than \( p^{(n-1)^2+2} \).

(i) \( G^{ab} \) is not elementary abelian;

(ii) \( G^{ab} \) is elementary abelian and \( Z(G) \) is not elementary abelian.

**Proof (i).** The proof is an upstanding result of Proposition [2, 2] while \( Z = G' \). Let \( G^{ab} = C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}} \) where \( \sum_{i=1}^{k} m_i = n - 1 \) and \( m_i \leq m_{i+1} \) for all \( i \).
1 \leq i \leq k - 1. Then
\[ |G \otimes G| \leq |G' \otimes G^{ab}\|G^{ab} \otimes G^{ab}| \]
\[ = \left| C_p \otimes C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}} \right| \times \left| C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}} \right| \]
\[ = \left( p^{m_k + \ldots + m_1 + 2(m_{k-1} + \ldots + m_1 + m_{k-2} + \ldots + m_1 + \ldots + m_1)} + k \right) \]
\[ \leq p^{n-1+2(n-3+n-4+\ldots+n-2k+3)+k} \]
\[ < p^{(n-1)^2+2}, \]
as required.

(ii). Since $G^{ab}$ is a vector space on $C_p$, let $H/G'$ be the complement of $Z(G)/G'$ in $G^{ab}$. Moreover $H$ is extra-special and $G = HZ(G)$. There is an epimorphism $H \times Z(G) \otimes H \times Z(G) \twoheadrightarrow G \otimes G$, so
\[ |G \otimes G| \leq |H \times Z(G) \otimes H \times Z(G)|. \]
Let $|Z(G)| = p^k$ and $|H| = p^{2m+1}$, we can suppose that $k \geq 2$ by using Proposition 2.12. Now the following two cases can be considered.
Case (i). First suppose that $m \geq 2$.
Let $Z(G) \cong C_{p^{k_1}} \times \ldots \times C_{p^{k_t}}$ and $\sum_{i=1}^t k_i = n - 2m$. Applying Lemma 2.13 and Proposition 11, we have
\[ |G \otimes G| \leq |H \otimes Z(G)|^2 |Z(G) \otimes Z(G)| \]
\[ = p^{4m^2} (C_p^{(m)} \otimes C_{p^{k_1}} \times \ldots \times C_{p^{k_t}}) \times (C_{p^{k_1}} \times \ldots \times C_{p^{k_t}}) \]
\[ = p^{4m^2} p^{2m} (2t-1)k_1 + (2t-3)k_2 + \ldots + k_t \]
\[ \leq p^{4m^2} p^{2mt} p^{n-2m+2(n-2m-2+\ldots+n-2m-t)} \]
\[ < p^{(n-1)^2+2}, \]
as required.

Case (ii). Without loss of generality, we can suppose that $Z(G) \cong C_{p^2}$. Now the result is obtained by using Proposition 2.14 and the fact that $|Iml| \geq p$.

\[ \square \]

**Theorem 2.6.** Let $G$ be a non abelian $p$-group of order $p^n$. If $|G'| = p$, then
\[ |G \otimes G| \leq p^{(n-1)^2+2}, \]
and the equality holds if and only if $G$ is isomorphic to $H \times E$, where $H \cong E_1$ or $H \cong Q_8$ and $E$ is an elementary abelian $p$-group.

**Proof.** One can assume that $G^{ab}$ and $Z(G)$ are elementary abelian and $|Z(G)| \geq p^2$ by Proposition 2.14. Let $E$ be the complement of $G'$ in $Z(G)$. Thus there exists an extra-special $p$-group $H$ of order $p^{2m+1}$ such that $G \cong H \times E$.

In the case $m \geq 2$, it is easily seen that $|G \otimes G| < p^{(n-1)^2+2}$. For $m = 1$,
\[ |G \otimes G| = |H \otimes H||E \otimes E||E \otimes H|^2 \]
where $|E \otimes E||E \otimes H|^2 = p^{(n-1)(n-3)}$. 

\[ \square \]
Now Proposition 2.2 and Lemma 2.3 imply that \(|G \otimes G| = p^{(n-1)^2+2}\) when \(H \cong Q_8\) or \(H\) has exponent \(p\).

**Theorem 2.7.** Let \(G\) be a non abelian \(p\)-group of order \(p^n\). If \(|G'| = p^m\), then

\[|G \otimes G| \leq p^{(n-1)(n-m)+2} .\]

**Proof.** We prove theorem by induction on \(m\). For \(m = 1\) the result is obtained by Theorem 2.6.

Let \(m \geq 2\) and \(K\) be a central subgroup of order \(p^m\) contained in \(G'\). Induction hypothesis and Proposition 2.2 yield

\[|G \otimes G| \leq |K \otimes G^{ab}| |G/K \otimes G/K| \leq p^{n-m}p(n-m)(n-2)+2 = p^{(n-1)(n-m)+2} .\]

**Corollary 2.8.** Let \(G\) be a non abelian \(p\)-group of order \(p^n\). If \(|G'| = p^m\), then

\[|\pi_3SK(G, 1)| \leq p^{n(n-m-1)+2} .\]

In particular when \(m = 1\), then

\[|\pi_3SK(G, 1)| \leq p^{n(n-2)+2} ,\]

and the equality holds if and only if \(G\) is isomorphic to \(H \times E\), in which \(H\) is extra-special of order \(p^3\) of exponent \(p\) or \(H \cong Q_8\) and \(E\) is an elementary abelian \(p\)-group.

**Corollary 2.9.** If the order of tensor square of \(G\) is equal to \(p^{(n-1)^2+2}\), then \(G \otimes G \cong C_p^{((n-1)^2+2)} (p \neq 2)\) or \(G \otimes G \cong C_4(2)^2 \times C_2^{((n-1)^2-2)}\).

REFERENCES

[1] R.D. Blyth, F. Fumagalli and M. Morigi. Some structural results on the non-abelian tensor square of groups. \textit{J. Group Theory.} \textbf{13}(1) (2010), 83–94.

[2] R. Brown, D.L. Johnson and E.F. Robertson. Some Computations of non Abelian tensor products of groups. \textit{J. Algebra.} \textbf{111} (1987), 177-202.

[3] R. Brown and J.-L Loday. Van Kampen theorems for diagrams of spaces. With an appendix by M. Zisman, \textit{Topology} \textbf{26} (1987), 311-335.

[4] G. Ellis. On the tensor square of a prime power group. \textit{Arch. Math.} \textbf{66} (1996), 467-469.

[5] G. Karpilovsky. \textit{The Schur multiplier} (London Math. Soc. Monogr. (N.S.) \textbf{2} 1987).

[6] P. Niroomand, M. R. Moghadam. Some properties on the specific subgroup of tensor square, to appear in \textit{Comm. Algebra.}

[7] N. R. Rocco. On a construction related to the nonabelian tensor square of a group. \textit{Bol. Soc. Brasil. Mat.} \textbf{22}(1) (1991), 63-79.

[8] N. R. Rocco. A presentation for crossed embeding of finite solvable groups. \textit{Comm. Algebra.} \textbf{22}(6) (1994), 1975-1998.

School of Mathematics and Computer Science, Damghan University, Damghan, Iran
E-mail address: niroomand@du.ac.ir, P. niroomand@yahoo.com