ON \([L]\)-HOMOTOPY GROUPS

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Abstract. The paper is devoted to investigation of some properties of \([L]\)-homotopy groups. It is proved, in particular, that for any finite CW-complex \(L\), satisfying double inequality \(\left[S^n\right] < [L] \leq [S^{n+1}]\), \(\pi_n^{[L]}(S^n) = \mathbb{Z}\). Here \([L]\) denotes extension type of complex \(L\) and \(\pi_n^{[L]}(X)\) denotes \(n\)-th \([L]\)-homotopy group of \(X\).

1. Introduction

A new approach to dimension theory, based on notions of extension types of complexes and extension dimension leads to appearance of \([L]\)-homotopy theory which, in turn, allows to introduce \([L]\)-homotopy groups (see [1]). Perhaps the most natural problem related to \([L]\)-homotopy groups is a problem of computation. It is necessary to point out that \([L]\)-homotopy groups may differ from usual homotopy groups even for complexes.

More specifically the problem of computation can be stated as follows: describe \([L]\)-homotopy groups of a space \(X\) in terms of usual homotopy groups of \(X\) and homotopy properties of complex \(L\).

The first step on this way is apparently computation of \(n\)-th \([L]\)-homotopy group of \(S^n\) for complex whose extension type lies between extension types of \(S^n\) and \(S^{n+1}\).

In what follows we, in particular, perform this step.

2. Preliminaries

Follow [1], we introduce notions of extension types of complexes, extension dimension, \([L]\)-homotopy, \([L]\)-homotopy groups and other related notions.

We also state Dranishnikov’s theorem, characterizing extension properties of complex [2].

All spaces are polish, all complexes are countable finitely-dominated CW complexes.

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For spaces $X$ and $L$, the notation $L \in \text{AE}(X)$ means, that every map $f : A \to L$, defined on a closed subspace $A$ of $X$, admits an extension $\bar{f}$ over $X$.

Let $L$ and $K$ be complexes. We say (see [1]) that $L \leq K$ if for each space $X$ from $L \in \text{AE}(X)$ follows $K \in \text{AE}(X)$. Equivalence classes of complexes with respect to this relation are called extension types. By $[L]$ we denote extension type of $L$.

**Definition 2.1.** ([1]). The extension dimension of a space $X$ is extension type $\text{ed}(X)$ such that $\text{ed}(X) = \min\{[L] : L \in \text{AE}(X)\}$.

Observe, that if $[L] \leq [S^n]$ and $\text{ed}(X) \leq [L]$, then $\dim X \leq n$.

Now we can give the following

**Definition 2.2.** [1] We say that a space $X$ is an absolute (neighbourhood) extensor modulo $L$ (shortly $X$ is ANE($[L]$)) and write $X \in \text{ANE}([L])$ if $X \in \text{ANE}(Y)$ for each space $Y$ with $\text{ed}(X) \leq [L]$.

Definition of $[L]$-homotopy and $[L]$-homotopy equivalence [1] are essential for our consideration:

**Definition 2.3.** Two maps $f_0, f_1 : X \to Y$ are said to be $[L]$-homotopic (notation: $f_0 \simeq f_1$) if for any map $h : Z \to X \times [0,1]$, where $Z$ is a space with $\text{ed}(Z) \leq [L]$, the composition $(f_0 \oplus f_1)h|_{h^{-1}(X \times \{0,1\})} : h^{-1}(X \times \{0,1\}) \to Y$ admits an extension $H : Z \to Y$.

**Definition 2.4.** A map $f : X \to Y$ is said to be $[L]$-homotopy equivalence if there is a map $g : Y \to X$ such that the compositions $gf$ and $fg$ are $[L]$-homotopic to $\text{id}_X$ and $\text{id}_Y$ respectively.

Let us observe (see [1]) that ANE([L])-spaces have the following $[L]$-homotopy extension property.

**Proposition 2.1.** Let $[L]$ be a finitely dominated complex and $X$ be a Polish ANE([L])-space. Suppose that $A$ is closed in a space $B$ with $\text{ed}(B) \leq [L]$. If maps $f, g : A \to X$ are $[L]$-homotopic and $f$ admits an extension $F : B \to X$ then $g$ also admits an extension $G : B \to X$, and it may be assumed that $F$ is $[L]$-homotopic to $G$.

To provide an important example of $[L]$-homotopy equivalence we need to introduce the class of approximately $[L]$-soft maps.

**Definition 2.5.** [1] A map $f : X \to Y$ is said to be approximately $[L]$-soft, if for each space $Z$ with $\text{ed}(Z) \leq [L]$, for each closed subset $A \subset Z$, for an open cover $\mathcal{U} \in \text{cov}(Y)$, and for any two maps $g : A \to X$ and $h : Z \to Y$ such that $fg = h|_A$ there is a map $k : Z \to X$ satisfying condition $k|_A = g$ and the composition $fk$ is $\mathcal{U}$-close to $h$. 
Proposition 2.2. [1] Let $f : X \to Y$ be a map between ANE([L])-compacta and $\text{ed}(Y) \leq [L]$. If $f$ is approximately $[L]$-soft then $f$ is a $[L]$-homotopy equivalence.

In order to define $[L]$-homotopy groups it is necessary to consider an $n$-th $[L]$-sphere $S^n_{[L]}$, namely, an $[L]$-dimensional ANE([L]) - compactum admitting an approximately $[L]$-soft map onto $S^n$. It can be shown that all possible choices of an $[L]$-sphere $S^n_{[L]}$ are $[L]$-homotopy equivalent. This remark, coupled with the following proposition, allows us to consider for every finite complex $L$, every $n \geq 1$ and for any space $X$, the set $\pi_n^{[L]}(X) = [S^n_{[L]}, X]_{[L]}$ endowed with natural group structure (see [1] for details).

Theorem 2.3. [1] Let $L$ be a finitely dominated complex and $X$ be a finite polyhedron or a compact Hilbert cube manifold. Then there exist a $[L]$-universal ANE([L]) compactum $\mu^L_X$ with $\text{ed}(\mu^L_X) = [L]$ and an $[L]$-invertible and approximately $[L]$-soft map $f^L_X : \mu^L_X \to X$.

The following theorem is essential for our consideration.

Theorem 2.4. Let $L$ be simply-connected CW-complex, $X$ be finite-dimensional compactum. Then $L \in \text{AE}(X)$ iff $c - \dim_{H_i(L)} X \leq i$ for any $i$.

From the proof of Theorem 2.4 one can conclude that the following theorem also holds:

Theorem 2.5. Let $L$ be a CW-complex (not necessary simply-connected). Then for any finite-dimensional compactum $X$ from $L \in \text{AE}(X)$ follows that $c - \dim_{H_i(L)} X \leq i$ for any $i$.

3. Cohomological properties of $L$

In this section we will investigate some cohomological properties of complexes $L$ satisfying condition $[L] \leq S^n$ for some $n$. To establish these properties let us first formulate the following

Proposition 3.1. [4] Let $(X, A)$ be a topological pair, such that $H_q(X, A)$ is finitely generated for any $q$. Then free submodules of $H^q(X, A)$ and $H_q(X, A)$ are isomorphic and torsion submodules of $H^q(X, A)$ and $H_{q-1}(X, A)$ are isomorphic.

Now we use Theorem 2.5 to obtain the following lemma.

Lemma 3.2. Let $L$ be finite CW complex such that $[L] \leq [S^{n+1}]$ and $n$ is minimal with this property. Then for any $q \leq n$ $H_q(L)$ is torsion group.
Proof. Suppose that there exists \( q \leq n \) such that \( H^q(L) = \mathbb{Z} \oplus G \). To get a contradiction let us show that \( |L| \leq |S^n| \). Consider \( X \) such that \( L \in \text{AE}(X) \). Observe, that \( X \) is finite-dimensional since \( |L| \leq |S^{n+1}| \) by our assumption. Denote \( H = H_q(L) \). By Theorem 2.5 we have \( c - \dim H X \leq q \). From the other hand, universal coefficients formula implies that \( H^{q+1}(X, A) \approx H^{q+1}(X, A) \otimes H \oplus \text{Tor}(H^{q+2}(X, A), H) \). Hence, \( H^{q+1}(X, A) \otimes H = \{0\} \). Observe, however, that by our assumption we have \( H^{q+1}(X, A) \otimes H = H^{q+1}(X, A) \oplus (H^{q+1}(X, A) \otimes G) \). Therefore, \( H^{q+1}(X, A) = 0 \).

From the last fact we conclude that \( c - \dim X \leq q \) and therefore since \( X \) is finite-dimensional, \( \dim X \leq q \) which implies \( S^q \in \text{AE}(X) \). \( \square \)

From this lemma and Proposition 3.1 we obtain

**Corollary 3.3.** In the same assumptions \( H^q(L) \) is torsion group for any \( q \leq n \).

The following fact is essential for construction of compacts with some specific properties which we are going to construct further.

**Lemma 3.4.** Let \( L \) be as in previous lemma. For any \( m \) there exists \( p \geq m \) such that \( H^q(L; \mathbb{Z}_p) = \{0\} \) for any \( q \leq n \).

Proof. From Corollary 3.3 we can conclude that \( H^q(L) = \bigoplus_{i=1}^{l_k} \mathbb{Z}_{m_{qi}} \) for any \( q \leq n \). Additionally, let \( \text{Tor } H^{n+1}(L) = \bigoplus_{i=1}^{l_{n+1}} \mathbb{Z}_{m_{(n+1)i}} \).

For any \( m \) consider \( p \geq m \) such that \( (p, m_{ki}) \) for every \( k = 1 \ldots n+1 \) and \( i = 1 \ldots l_k \). Universal coefficients formula implies that \( H^q(L; \mathbb{Z}_p) = \{0\} \) for every \( k \leq n \). \( \square \)

Finally let us proof the following

**Lemma 3.5.** Let \( X \) be a metrizable compactum, \( A \) be a closed subset of \( X \). Consider a map \( f : A \to S^n \). If there exists extension \( \bar{f} : X \to S^n \) then for any \( k \) we have \( \delta_k : f^*(\zeta) = 0 \) in group \( H^{n+1}(X, A; \mathbb{Z}_k) \), where \( \zeta \) is generator in \( H^{n}(S^n, \mathbb{Z}_k) \).

Proof. Let \( \bar{f} \) be an extension of \( f \). Commutativity of the following diagram implies assertion of lemma:
4. SOME PROPERTIES OF $[L]$-HOMOTOPY GROUPS

In this section we will investigate some properties of $[L]$-homotopy groups.

From this point and up to the end of the text we consider finite complex $L$ such that $[S^n] < [L] \leq [S^{n+1}]$ for some fixed $n$.

Remark 4.1. Let us observe that for such complexes $S^n_{[L]}$ is $[L]$-homotopic equivalent to $S^n$ (see Proposition 2.2). Therefore for any $X$ $\pi_n^{[L]}(X)$ is isomorphic to $G = \pi_n(S^n)/N([L])$ where $N([L])$ denotes the relation of $[L]$-homotopic equivalence between elements of $\pi_n(S^n)$.

From this observation one can easily obtain the following fact.

**Proposition 4.1.** For $\pi_n^{[L]}(S^n)$ there are three variants: $\pi_n^{[L]}(S^n) = \mathbb{Z}$, $\pi_n^{[L]}(S^n) = \mathbb{Z}_m$ for some integer $m$ or this group is trivial.

Let us characterize the hypothetical equality $\pi_n^{[L]}(S^n) = \mathbb{Z}_m$ in terms of extensions of maps.

**Proposition 4.2.** If $\pi_n^{[L]}(S^n) = \mathbb{Z}_m$ then for any $X$ such that $\text{ed}(X) \leq [L]$, for any closed subset $A$ of $X$ and for any map $f : A \to S^n$, there exists extension $\bar{h} : X \to S^m$ of composition $\bar{h} = z_m f$, where $z_m : S^n \to S^n$ is a map having degree $m$.

**Proof.** Suppose, that $\pi_n^{[L]}(S^n) = \mathbb{Z}_m$. Then from Remark 4.1 and since $[z_m] = m[\text{id}_{S^n}] = [*]$ (where $[f]$ denotes homotopic class of $f$) we conclude that $z_m : S^n \to S^n$ is $[L]$-homotopic to constant map. Let us show that $h = z_m f : A \to S^n$ is also $[L]$-homotopic to constant map. This fact will prove our statement. Indeed, by our assumption $\text{ed}(X) \leq [L]$ and $S^n \in ANE$ and therefore we can apply Proposition 2.1.

Consider $Z$ such that $\text{ed}(Z) \leq [L]$ and a map $H : Z \to A \times I$, where $I = [0, 1]$. Pick a point $s \in S^n$. Let $f_0 = z_m f$, $f_1 \equiv s$ -- constant map considered as $f_i : A \times \{i\} \to S^n$, $i = 0, 1$.

Define $F : A \times I \to S^n \times I$ as follows: $F(a, t) = (f(a), t)$ for each $a \in A$ and $t \in I$. Let $f'_0 \equiv z_m$ and $f'_1 \equiv s$ considered as $f'_i : S^n \times \{i\} \to S^n$, $i = 0, 1$. 

\[
\begin{align*}
H^n(A; \mathbb{Z}_k) & \xrightarrow{\delta_{X,A}} H^{n+1}(X, A; \mathbb{Z}_k) \\
\xrightarrow{\delta_{z_n,S^n}} & H^{n+1}(S^n, S^n; \mathbb{Z}_k) = \{0\}
\end{align*}
\]
Consider a composition $G = FH : Z \to S^n \times I$. By our assumption $f_0$ is $[L]$-homotopic to $f_1$. Therefore a map $g : G^{-1}(S^n \times \{0\} \cup S^n \times \{1\}) \to S^n$, defined as $g|_{G^{-1}(S^n \times \{i\})} = f_i|L$ for $i = 0, 1$, can be extended over $Z$. From the other hand we have $G^{-1}(S^n \times \{i\}) \equiv H^{-1}(A \times \{i\})$ and $g|_{G^{-1}(S^n \times \{i\})} = f_i|LH = f_i$ for $i = 0, 1$. This remark completes the proof. □

Now consider a special case of complex having a form $S^n < L = K_s \vee K \leq S^{n+1}$, where $K_s$ is a complex obtained by attaching to $S^n$ a $(n + 1)$-dimensional cell using a map of degree $s$.

**Proposition 4.3.** Let $[\alpha] \in \pi_n(X)$ be an element of order $s$. Then $\alpha$ is $[L]$-homotopy to constant map.

**Proof.** Observe that similar to proof of Proposition 4.2 it is enough to show that for every $Z$ with $\text{ed}(Z) \leq [L]$, for every closed subspace $A$ of $Z$ and for any map $f : Z \to S^n$ a composition $\alpha f : A \to X$ can be extended over $Z$.

Let $g : S^n \to K_s^{(n)}$ be an embedding (by $M^{(n)}$ we denote $n$-dimensional skeleton of complex $M$) and $r : L \to K_s$ be a retraction.

Since $\text{ed}(Z) \leq [L]$, a composition $gf$ has an extension $F : Z \to L$. Let $F' = rF$ and $\alpha'$ be a map $\alpha$ considered as a map $\alpha' : K_s^{(n)} \to X$. Observe that $\alpha'F'$ is a necessary extension of $\alpha f$. □

5. **Computation of $\pi_n^{[L]}(S^n)$**

In this section we will prove that $\pi_n^{[L]}(S^n) = Z$.

Suppose the opposite, i.e. $\pi_n^{[L]}(S^n) = Z_m$ (we use Proposition 4.1; the same arguments can be used to prove that $\pi_n^{[L]}(S^n)$ is non-trivial).

To get a contradiction we need to obtain a compact with special extension properties. We will use a construction of [3]

Let us recall the following definition.

**Definition 5.1.** [3] Inverse sequence $S = \{X_i, p_i^{i+1} : i \in \omega\}$ consisting of metrizable compacta is said to be $L$-resolvable if for any $i$, $A \subseteq X_i$-closed subspace of $X_i$ and any map $f : A \to L$ there exists $k \leq i$ such that composition $f p_k^i : (p_k^i)^{-1}A \to L$ can be extended over $X_k$.

The following lemma (see [3]) expresses an important property of $[L]$-resolvable inverse sequences.

**Lemma 5.1.** Suppose that $L$ is a countable complex and that $X$ is a compactum such that $X = \lim S$ where $S = (X_i, \lambda_i), q_i^{i+1}$ is a $L$-resolvable inverse system of compact polyhedra $X_i$ with triangulations $\lambda_i$ such that $\text{mesh}\{\lambda_i\} \to 0$. Then $L \in \text{AE}(X)$
Let us recall that in [3] inverse sequence \( S = \{(X_i, \tau_i), p_i^{i+1}\} \) was constructed such that \( X_i \) is compact polyhedron with fixed triangulation \( \tau_i, X_0 = S^{n+1}, \) mesh \( \tau_i \to 0, S \) is \([L]\)-resolvable and for any \( x \in X_i \) we have \( (p_i^{i+1})^{-1} x \simeq L \) or *.

It is easy to see that using the same construction one can obtain inverse sequence \( S = \{(X_i, \tau_i), p_i^{i+1}\} \) having the same properties with exeption of \( X_0 = D^{n+1} \) where \( D^{n+1} \) is \( n+1 \)-dimensional disk.

Let \( X = \lim S \). Observe, that \( \text{ed}(X) \leq [L] \). Let \( p_0 : X \to D^{n+1} \) be a limit projection.

Pick \( p \geq m + 1 \) which Lemma 3.4 provides us with. By Vietoris-Begle theorem (see [4]) and our choice of \( p \), for every \( i \) and every \( X'_i \subseteq X_i \), a homomorphism \( (p_i^{i+1})^* : H^k(X'_i; Z_p) \to H^k((p_i^{i+1})^{-1} X'_i; Z_p) \) is isomorphism for \( k \leq n \) and monomorphism for \( k = n+1 \).

Therefore for each \( D' \subseteq X_0 = D^{n+1} \) homomorphism \( p_0^* : H^k(D'; Z_p) \to H^k((p_0)^{-1} D'; Z_p) \) is isomorphism for \( k \leq n \) and monomorphism for \( k = n+1 \). In particular, \( H^n(X; Z_p) = \{0\} \) since \( X_0 = D^{n+1} \) has trivial cohomology groups.

Let \( A = (p_0)^{-1} S^n \) and \( \zeta \in H^n(S^n; Z_p) \approx Z_p \) be a generator.

Since \( p_0^* : H^n(S^n; Z_p) \to H^n(A; Z_p) \) is isomorphism, \( p_0^*(\zeta) \) is generator in \( H^n(A, Z_p) \approx Z_p \). In particular, \( p_0^*(\zeta) \) is element of order \( p \).

From exact sequence of pair \((X, A)\)

\[
\ldots \to H^n(X; Z_p) = \{0\} \overset{i_X,A}{\longrightarrow} H^n(A; Z_p) \overset{\delta^*_{X,A}}{\longrightarrow} H^{n+1}(X, A; Z_p) \to \ldots
\]

we conclude that \( \delta^*_{X,A} \) is monomorphism and hence \( \delta^*_{X,A}(p_0^*(\zeta)) \in H^{n+1}(X, A; Z_p) \) is element of order \( p \).

Consider now a composition \( h = z_m p_0 \). By our assumption this map can be extended over \( X \) (see Proposition 4.2). This fact coupled with Lemma 3.5 implies that \( \delta^*_{X,A}(h^*(\zeta)) = 0 \) in \( H^{n+1}(X, A; Z_p) \). But \( \delta^*_{X,A}(h^*(\zeta)) = m \delta^*_{X,A}(p_0^*(\zeta)) \). We arrive to a contradiction which shows that

**Theorem 5.2.** Let \( L \) be a complex such that \( [S^n] < [L] \leq [S^{n+1}] \). Then \( \pi_n^{[L]}(S^n) \approx Z \).

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