Modular Invariants, Graphs and $\alpha$-Induction for
Nets of Subfactors I

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September 24, 2021

Abstract

We analyze the induction and restriction of sectors for nets of subfactors defined by Longo and Rehren. Picking a local subfactor we derive a formula which specifies the structure of the induced sectors in terms of the original DHR sectors of the smaller net and canonical endomorphisms. We also obtain a reciprocity formula for induction and restriction of sectors, and we prove a certain homomorphism property of the induction mapping.

Developing further some ideas of F. Xu we will apply this theory in a forthcoming paper to nets of subfactors arising from conformal field theory, in particular those coming from conformal embeddings or orbifold inclusions of $SU(n)$ WZW models. This will provide a better understanding of the labeling of modular invariants by certain graphs, in particular of the A-D-E classification of $SU(2)$ modular invariants.

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1 Introduction

Modular invariants associated to $SU(2)$ characters have been classified by [3], each being labeled by a graph, a Dynkin Diagram of type A-D-E. Similarly subfactors give rise to natural invariants, e.g. their principal graphs. Each $A, D_{\text{even}}, E_{\text{even}}$ is the principal graph (or fusion graph) of a subfactor of index less than four. Here we begin to look systematically at this relation between modular invariants, graphs and subfactors. Our treatment begins with the formulae for the extension ($\lambda \mapsto \alpha_\lambda$) and the restriction endomorphism ($\beta \mapsto \sigma_\beta$) for nets of subfactors $\mathcal{N} \subset \mathcal{M}$ defined by Longo and Rehren [19]. We derive several properties of these extension and restriction endomorphisms, including a reciprocity formula, and therefore we prefer the names $\alpha$-induced and $\sigma$-restricted endomorphisms.

We apply the procedure of $\alpha$-induction to several nets of subfactors arising from conformal field theory. We pay special attention to the current algebras of the $SU(n)_k$ WZW models. There we are dealing with nets of subfactors $\mathcal{N} \subset \mathcal{M}$ where the smaller net $\mathcal{N}$ is given in terms of representations of local loop groups of $SU(n)$. Firstly, we consider conformal embeddings of type $SU(n)_k \subset G_1$ with $G$ simple. In this case the enveloping net $\mathcal{M}$ is given by the local loop groups of $G$ in the level 1 vacuum representation. To such a conformal embedding corresponds a modular invariant. Secondly, we consider modular invariants of orbifold type. In this case we can construct the enveloping net $\mathcal{M}$ as an extension of $\mathcal{N}$ by simple currents; this crossed product construction is similar to the construction of the field algebra in [3]. Our treatment gives some new insights in the programme of labeling (block-diagonal) modular invariants by certain graphs initiated by Di Francesco and Zuber [3] (see also [4]). With $\lambda$ being the
localized endomorphisms associated to the positive energy representations of $LSU(n)$ at level $k$ we obtain a fusion algebra generated by the subsectors of the $\alpha$-induced endomorphisms $\alpha \lambda$. Graphs are obtained by drawing the fusion graphs of the $\alpha$-induced endomorphisms associated to the fundamental representation(s). They satisfy the axioms for graphs which Di Francesco and Zuber associate to modular invariants [5] (see also [21]), and for all our $(SU(2)$ and $SU(3))$ examples we reproduce in fact their graphs. For $SU(2)$ our theory yields in fact an explanation why the entries in the (non-trivial) block-diagonal modular invariants correspond to Coxeter exponents of the $D_{\text{even}}$, $E_6$ and $E_8$ Dynkin diagrams. We will also discuss the application of $\alpha$-induction to extended $U(1)$ theories from [2] and to the minimal models.

In [21], Xu defined a map $\lambda \mapsto a_{\lambda}$ by a similar, but different formula for the induced endomorphism. (In fact in his setting both $\lambda$ and $a_{\lambda}$ are endomorphisms of the same $\text{III}_1$-factor $M$.) He has already obtained the fusion graphs for the conformal inclusions involving $SU(n)$, however, we can also treat the orbifold inclusions of $SU(n)$. Our underlying framework is more general because it applies, for a given net of subfactors $\mathcal{N} \subset \mathcal{M}$ satisfying certain assumptions (which are fulfilled for many chiral conformal field theory models), to the whole class of localized, transportable endomorphisms of $\mathcal{N}$ whereas Xu restricts his analysis to the $LSU(n)$ setting. Moreover, we believe that our formalism is more appropriate as the nature of induction and restriction of sectors becomes more transparent, and we believe that our setting enables us to present simpler proofs.

This article is the first in a series of papers about modular invariants, graphs, and nets of subfactors. Here we develop the machinery of $\alpha$-induction in a general setting. In Section 2 we derive the braiding fusion equations that arise naturally from the notion of localized transportable endomorphisms of algebraic quantum field theory, and which play a crucial role in our analysis. In Section 3 we give the definition and prove several properties of $\alpha$-induction; we derive an important formula and the homomorphism property of $\alpha$-induction, and we also establish $\alpha \sigma$-reciprocity of $\alpha$-induction and $\sigma$-restriction. The game of $\alpha$-induction and $\sigma$-restriction of sectors generalizes the restriction and (Mackey) induction of group representations to nets of subfactors which are in general not governed by group symmetries. Nevertheless, as an illustration we briefly discuss the case of a net of subfactors arising from a subgroup of a finite group in Subsection 4.2. In a forthcoming paper we will present the above mentioned applications of this theory to several models of conformal field theory.
2 Preliminaries

In this section we review several facts about subfactors, sectors, algebraic quantum field theory and nets of subfactors, which we will need for our analysis.

2.1 Subfactors and sectors

We first briefly review some basic facts about subfactors and Longo’s theory of sectors. For a detailed treatment of these topics we refer to textbooks on operator algebras, e.g. [11].

A von Neumann algebra is a weakly closed subalgebra $M \subset \mathcal{B}(\mathcal{H})$ of the algebra of bounded operators on some Hilbert space $\mathcal{H}$. It is called a factor if its center is trivial, $M' \cap M = \mathbb{C}1$. A factor is called infinite if there is an isometry $v \in M$ with range projection $vv^* \neq 1$, and purely infinite or type III if $M_p = pMp$ is infinite for every non-zero projection $p \in M$.

An inclusion $N \subset M$ of factors with common unit is called a subfactor. A subfactor is called irreducible if the relative commutant is trivial, $N' \cap M = \mathbb{C}1$, and it is called infinite if $N$ and $M$ are infinite factors. Let $N \subset M$ be an infinite subfactor on a separable Hilbert space $\mathcal{H}$. Then there is a vector $\Phi \in \mathcal{H}$ which is cyclic and separating for both $M$ and $N$. Let $J_M$ and $J_N$ be the modular conjugations of $M$ and $N$ with respect to $\Phi$. Then the endomorphism

$$\gamma = \text{Ad}(J_NJ_M)|_M$$

of $M$ satisfies $\gamma(M) \subset N$ and is called a canonical endomorphism from $M$ into $N$. It is unique up to conjugation by a unitary in $N$. The restriction $\theta = \gamma|_N$ is called a dual canonical endomorphism. If the Kosaki index [13] is finite, $[M : N] < \infty$, then there are isometries $v \in M$ and $w \in N$ such that

$$vm = \gamma(m)v, \quad m \in M,$$

$$wn = \theta(n)w, \quad n \in N,$$

$$w^*v = [M : N]^{-1/2} 1 = w^*\gamma(v)$$

Then $E^M_N(m) = w^*\gamma(m)w$, $m \in M$, is a conditional expectation from $M$ onto $N$ and the identity

$$m = [M : N] \cdot E^M_N(mv^*)v, \quad m \in M,$$

holds [13]. This means in particular that every $m \in M$ can be written as $m = nv$ for some $n \in N$, i.e. $M = Nv$. 

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For any unital *-algebra $M$ we denote by $\text{End}(M)$ the set of unital *-endomorphisms of $M$. For $\lambda, \mu \in \text{End}(M)$ we define the intertwiner space
\[
\text{Hom}_M(\lambda, \mu) = \{ t \in M : t\lambda(m) = \mu(m)t, \quad m \in M \}
\]
and
\[
\langle \lambda, \mu \rangle_M = \dim \text{Hom}_M(\lambda, \mu).
\]
We have $\langle \lambda, \mu \rangle_M = \langle \mu, \lambda \rangle_M$. Now let $M$ be a type III factor. An endomorphism $\lambda \in \text{End}(M)$ is called irreducible if $\lambda(M)' \cap M = \mathbb{C}1$. Endomorphisms $\lambda, \mu \in \text{End}(M)$ are called (inner) equivalent if there is a unitary $u \in M$ such that $\lambda = \text{Ad}(u) \circ \mu$. The quotient of $\text{End}(M)$ by inner equivalence is called the set of sectors of $M$ and denoted by $\text{Sect}(M)$, and the equivalence class of $\lambda \in \text{End}(M)$ is denoted by $[\lambda]_M$. However, we often drop the suffix and write $[\lambda]$ for $[\lambda]_M$ as long as it is clear which factor is meant. There is a natural product of sectors coming from the composition of endomorphisms. Explicitly, $[\lambda] \times [\mu] = [\lambda \circ \mu]$. There is also an addition of sectors. Let $\lambda_i \in \text{End}(M), i = 1, 2, \ldots, n$. Since $M$ is infinite we can take a set of isometries $t_i \in M, i = 1, 2, \ldots, n$, satisfying the relations of the Cuntz algebra $\mathcal{O}_n$,
\[
t_i^* t_j = \delta_{i,j} 1, \quad \sum_{i=1}^n t_i^* t_i = 1.
\]
Define $\lambda \in \text{End}(M)$ by
\[
\lambda(m) = \sum_{i=1}^n t_i \lambda_i(m) t_i^*, \quad m \in M.
\]
Then $[\lambda]$ does not depend on the choice of the set of isometries and hence we can define the sum
\[
\bigoplus_{i=1}^n [\lambda_i] = [\lambda].
\]
Each $[\lambda_i]$ is called a subsector of $[\lambda]$. With the operations $\times$ and $\oplus$ that fulfill associativity and distributivity, $\text{Sect}(M)$ becomes a unital semi-ring, and the unit is given by the identity (or trivial) sector $[\text{id}]$.

For $\lambda \in \text{End}(M)$ irreducible $\bar{\lambda} \in \text{End}(M)$ is called conjugate if $[\lambda \circ \bar{\lambda}]$ and $[\bar{\lambda} \circ \lambda]$ both contain the identity sector once. The conjugate is unique up to inner equivalence. For general $\lambda$ let $\gamma_\lambda$ be the canonical endomorphism of $M$ into $\lambda(M)$. Then a conjugate is given by $\bar{\lambda} = \lambda^{-1} \circ \gamma_\lambda$. $[\bar{\lambda}]$ is called the conjugate sector, and the map $[\lambda] \mapsto [\bar{\lambda}]$ preserves sums (if $[\lambda] = [\lambda_1] \oplus [\lambda_2]$ then $[\bar{\lambda}] = [\bar{\lambda}_1] \oplus [\bar{\lambda}_2]$) and reverses products (if $[\lambda] = [\mu] \times [\nu]$ then $[\bar{\lambda}] = [\bar{\nu}] \times [\bar{\mu}]$).
Furthermore, for an automorphism \( \alpha \in \Aut(M) \) we have \([\alpha^{-1}] = [\overline{\alpha}]\). The number \( d_\lambda = [M : \lambda(M)]^{1/2} \) is called the statistical dimension of \( \lambda \). Then \( d_\lambda = d_\chi \) if \([\lambda] = [\chi']\), and \( d_\lambda = d_\overline{\lambda} \). For \( \lambda_1, \lambda_2 \in \End(M) \) such that \([\lambda] = [\lambda_1] \oplus [\lambda_2] \) we have

\[
\langle \lambda, \mu \rangle_M = \langle \lambda_1, \mu \rangle_M + \langle \lambda_2, \mu \rangle_M.
\]

If \( \lambda, \mu, \nu, \lambda, \mu \in \End(M) \) have finite statistical dimension and \( \lambda, \mu \) are conjugates of \( \lambda \) and \( \mu \), respectively, then we have \( \langle \lambda, \mu \rangle_M = \langle \overline{\lambda}, \overline{\mu} \rangle_M \).

2.2 Statistics operators in algebraic quantum field theory

Let us briefly review some facts about the algebraic framework of quantum field theory \([7, 8, 9, 10, 14]\). As all our later applications are chiral theories we present the whole setting with the unit circle \( S^1 \) as the underlying “space-time” from the beginning. Since we will make explicit use of several well-known results and in order to make this article more self-contained we prefer to present the proofs which are simple and instructive, but compare also \([12, 13]\). Fix a point \( z \in S^1 \) on the circle and set

\[
\mathcal{J}_z = \{ I \subset S^1 \text{ non-void open interval, } z \notin \overline{I} \},
\]

where \( \overline{I} \) denotes the closure of \( I \). A Haag-Kastler net on the punctured circle \( \mathcal{A} = \{ A(I), I \in \mathcal{J}_z \} \) is a family of von Neumann algebras acting on a Hilbert space \( \mathcal{H}_0 \) such that isotony holds, i.e. \( I \subset J \) implies \( A(I) \subset A(J) \), and we also have locality, i.e. \( I_1 \cap I_2 = \emptyset \) implies \( A(I_1) \subset A(I_2') \). For subsets \( R \subset S^1 \) (which may touch or contain the “point at infinity” \( z \)) we define

\[
\mathcal{C}_\mathcal{A}^{(0)}(R) = \bigcup_{J \in \mathcal{J}_z, J \subset R} A(J), \quad \mathcal{C}_\mathcal{A}(R) = \mathcal{C}_\mathcal{A}^{(0)}(R) = \mathcal{A}^{\text{closed}}(R).
\]

As usual, we denote the \( C^* \)-algebra of the whole circle by the same symbol as the net itself, \( \mathcal{A} = \mathcal{C}_\mathcal{A}(S^1) \). An endomorphism \( \lambda \in \End(\mathcal{A}) \) is called localized in an interval \( I \in \mathcal{J}_z \) if \( \lambda(a) = a \) for all \( a \in \mathcal{C}_\mathcal{A}(I') \), where \( I' \) denotes the interior of the complement of \( I \). A localized endomorphism \( \lambda \) is called transportable if for all \( J \in \mathcal{J}_z \) there are unitaries \( U_{\lambda, J} \in \mathcal{A} \), called charge transporters, such that \( \lambda = \Ad(U_{\lambda, J}) \circ \lambda \) is localized in \( J \). By \( \Delta_\mathcal{A}(I) \)
we denote the set of localized transportable ("DHR") endomorphisms of \( \mathcal{A} \) localized in \( I \in \mathcal{J}_z \).

Let us now assume Haag duality (on the punctured circle),

\[
\mathcal{A}(I) = \mathcal{C}_\mathcal{A}(I')', \quad I \in \mathcal{J}_z.
\]

Note that then an endomorphism \( \lambda \in \Delta_{\mathcal{A}}(I) \) leaves any local algebra \( A(K) \) with \( K \in \mathcal{J}_z, I \subset K \), invariant since \( a'\lambda(a) = \lambda(a'a) = \lambda(aa') = \lambda(a)a' \) for any \( a \in A(K) \) and \( a' \in \mathcal{C}_\mathcal{A}(K') \), hence \( \lambda(a) \in A(K) \) by Haag duality.

**Lemma 2.1** Let \( I_1, I_2 \in \mathcal{J}_z \) such that \( I_1 \cap I_2 = \emptyset \) and let \( \lambda_i \in \Delta_{\mathcal{A}}(I_i), \)

\( i = 1, 2 \). Then \( \lambda_1 \) and \( \lambda_2 \) commute, \( \lambda_1 \circ \lambda_2 = \lambda_2 \circ \lambda_1 \).

**Proof.** Take \( I \in \mathcal{J}_z \) arbitrary. Then choose intervals \( J_1, J_2 \in \mathcal{J}_z \) such that \( J_i \cap I = \emptyset, i = 1, 2, \) and that there are also intervals \( K_1, K_2 \in \mathcal{J}_z, K_i \supset I_i \cup J_i, i = 1, 2, \) and \( K_1 \cap K_2 = \emptyset \). By transportability there are unitaries \( U_i \equiv U_{\lambda_i; I_i; J_i} \), such that \( \tilde{\lambda}_i = \text{Ad}(U_i) \circ \lambda_i \in \Delta_{\mathcal{A}}(I_i), i = 1, 2 \). Then \( U_i \in A(K_i) \) by Haag duality, hence \( U_1U_2 = U_2U_1 \) and \( \lambda_1(U_2) = U_2 \) and \( \tilde{\lambda}_2(U_1) = U_1 \). Then for any \( a \in A(I) \) we have \( \tilde{\lambda}_i(a) = a, i = 1, 2, \) and thus

\[
\lambda_1 \circ \lambda_2(a) = \text{Ad}(U_1^*) \circ \tilde{\lambda}_1 \circ \text{Ad}(U_2^*) \circ \tilde{\lambda}_2(a)
\]

\[
= \text{Ad}(U_1^* \tilde{\lambda}_1(U_2^*)) \circ \tilde{\lambda}_1 \circ \tilde{\lambda}_2(a)
\]

\[
= U_1^* U_2^* a U_2 U_1 = U_2 U_1 a U_1 U_2
\]

\[
= \text{Ad}(U_2^* \tilde{\lambda}_2(U_1^*)) \circ \tilde{\lambda}_2 \circ \tilde{\lambda}_1(a)
\]

\[
= \text{Ad}(U_2^*) \circ \tilde{\lambda}_2 \circ \text{Ad}(U_1^*) \circ \tilde{\lambda}_1(a)
\]

\[
= \lambda_2 \circ \lambda_1(a).
\]

Since \( I \) was arbitrary it follows \( \lambda_1 \circ \lambda_2(a) = \lambda_2 \circ \lambda_1(a) \) for any \( a \in \mathcal{A} \). Q.E.D.

Now assume that \( \lambda, \mu \) are localized in the same interval \( I \in \mathcal{J}_z, \lambda, \mu \in \Delta_{\mathcal{A}}(I) \). Then they will in general not commute, however, they are intertwined by a unitary operator which will be discussed in the following. Choose \( I_1, I_2 \in \mathcal{J}_z \) such that \( I_1 \cap I_2 = \emptyset \). Then there are unitaries \( U_1 \equiv U_{\lambda; I_1} \) and \( U_2 \equiv U_{\mu; I_2} \) such that \( \lambda_1 = \text{Ad}(U_1) \circ \lambda \in \Delta_{\mathcal{A}}(I_1) \) and \( \mu_2 = \text{Ad}(U_2) \circ \mu \in \Delta_{\mathcal{A}}(I_2) \). We set

\[
\epsilon_{U_1, U_2}^{I_1, I_2}(\lambda, \mu) = \mu(U_1^*) U_2 U_1 \lambda(U_2).
\]

This operator has remarkable invariance properties. Let

\[
\mathcal{J}_z^{2 \text{dis}} = \{(I_1, I_2) \in \mathcal{J}_z \times \mathcal{J}_z, I_1 \cap I_2 = \emptyset \}.
\]
For disjoint intervals $I_1, I_2 \in \mathcal{J}_z$ denote $I_2 > I_1$ (respectively $I_2 < I_1$) if $I_1$ lies clockwise (respectively counter-clockwise) to $I_2$ relative to the point $z$. Let
\[
\mathcal{J}_z^{2,+} = \{(I_1, I_2) \in \mathcal{J}_z \times \mathcal{J}_z, I_2 > I_1\},
\]
\[
\mathcal{J}_z^{2,-} = \{(I_1, I_2) \in \mathcal{J}_z \times \mathcal{J}_z, I_2 < I_1\}.
\]
Then clearly $\mathcal{J}_z^{2,\text{dis}} = \mathcal{J}_z^{2,+} \cup \mathcal{J}_z^{2,-}$.

**Lemma 2.2** The operators $\epsilon_{U_1, U_2}(\lambda, \mu)$ do not depend on the special choice of $U_1$ and $U_2$, moreover, varying $I_1$ and $I_2$, $\epsilon_{U_1, U_2}(\lambda, \mu)$ remains constant on $\mathcal{J}_z^{2,+}$ and $\mathcal{J}_z^{2,-}$.

**Proof.** First replace $U_1$ by $\tilde{U}_1$ such that $\tilde{\lambda}_1 = \text{Ad}(\tilde{U}_1) \circ \lambda \in \Delta_A(I_1)$ as well. Then with $V_1 = \tilde{U}_1 U_1^*$ we have $\tilde{\lambda}_1 = \text{Ad}(V_1) \circ \lambda_1$ and hence $V_1 \in A(I_1)$ by Haag duality. Then we have
\[
\epsilon_{U_1, U_2}(\lambda, \mu) = \mu(U_1^*) U_2^* V_1 U_1 \lambda(U_2)
\]
\[
= \mu(U_1^*) V_1^* U_2^* V_1 U_1 \lambda(U_2)
\]
\[
= \mu(U_1^*) U_2^* \mu_2(V_1^*) V_1 U_1 \lambda(U_2)
\]
\[
= \epsilon_{U_1, U_2}(\lambda, \mu),
\]
since $\mu_2(V_1) = V_1$ by $I_1 \cap I_2 = \emptyset$. In the same way we can replace $U_2$ by some $\tilde{U}_2$ such that $\tilde{\mu}_2 = \text{Ad}(\tilde{U}_2) \circ \mu \in \Delta_A(I_2)$. In the next step we replace $U_1$ by some $\tilde{U}_1$ such that $\tilde{I}_1 \cap I_1 \neq \emptyset$ but still $\tilde{I}_1 \cap I_2 = \emptyset$. We can now assume that our chosen $\tilde{U}_1$ is such that $\tilde{\lambda}_1 \in \Delta_A(\tilde{I}_1 \cap I_1)$, and hence we can use the same $\tilde{U}_1$ for the new interval $\tilde{I}_1$. In the same way we can replace $I_2$ by $\tilde{I}_2$. As long as $\tilde{I}_1 \cap \tilde{I}_2 = \emptyset$ we have the freedom to vary $\tilde{U}_1$ and $\tilde{U}_2$, and so on. Now assume that we have $I_2 > I_1$ for our initial intervals. By iteration of the above arguments we can reach any pair of intervals in $\mathcal{J}_z^{2,+}$, and similarly in $\mathcal{J}_z^{2,-}$ if $I_1 < I_2$, the lemma is proven. \[Q.E.D.\]

We conclude that for any $\lambda, \mu \in \Delta_A(I)$ there are only two operators $\epsilon(\lambda, \mu) = \epsilon_{U_1, U_2}(\lambda, \mu)$, where $(I_1, I_2) \in \mathcal{J}_z^{2,\pm}$, but $\epsilon^+(\lambda, \mu)$ and $\epsilon^-(\lambda, \mu)$ may be different in general. We now have even the choice to set $I_1 = I$ and $U_1 = 1$. We choose intervals $I_{\pm} \in \mathcal{J}_z$ such that $I_+ > I$ and $I_- < I$. If $U_{\mu, \pm} \equiv U_{\mu, I, I_{\pm}}$ are unitaries such that $\mu_{\pm} = \text{Ad}(U_{\mu, \pm}) \circ \mu \in \Delta_A(I_{\pm})$ then we find by putting $I_2 = I_+$ or $I_2 = I_-$
\[
\epsilon^\pm(\lambda, \mu) = U_{\mu, \pm}^* \lambda(U_{\mu, \pm}).
\]
The $\epsilon^\pm(\lambda, \mu)$'s are usually called statistics operators. Choose $K_+, K_- \in \mathcal{J}_z$ such that $I \cup I_\pm \subset K_\pm$ and $I_\pm \cap K_\mp = \emptyset$. Note that $U_{\mu, \pm} \in A(K_\pm)$ by Haag duality.

**Lemma 2.3** For $\lambda, \mu, \nu \in \Delta_A(I)$ we have

$$
\begin{align*}
\epsilon^\pm(\lambda, \mu) \cdot \lambda \circ \mu(a) &= \mu \circ \lambda(a) \cdot \epsilon^\pm(\lambda, \mu), \quad a \in A, \quad (3) \\
\epsilon^\pm(\lambda, \mu) &\in A(I), \quad (4) \\
\epsilon^+(\lambda, \mu) &= (\epsilon^-(\mu, \lambda))^*, \quad (5) \\
\epsilon^\pm(\lambda \circ \mu, \nu) &= \epsilon^\pm(\lambda, \nu) \lambda(\epsilon^\pm(\mu, \nu)), \quad (6) \\
\epsilon^\pm(\lambda, \mu \circ \nu) &= \mu(\epsilon^\pm(\lambda, \nu)) \epsilon^\pm(\lambda, \mu). \quad (7)
\end{align*}
$$

**Proof.** Ad Eq. (3): For $a \in A$ we compute

$$
\begin{align*}
\epsilon^\pm(\lambda, \mu) \cdot \lambda \circ \mu(a) &= U_{\mu, \pm}^* \lambda(U_{\mu, \pm}) \cdot \lambda \circ \mu(a) \\
&= U_{\mu, \pm}^* \lambda(U_{\mu, \pm}(a)) \\
&= U_{\mu, \pm}^* \lambda(\mu \circ \lambda(a)) \\
&= U_{\mu, \pm}^* \cdot \mu \circ \lambda(a) \cdot \lambda(U_{\mu, \pm}) \\
&= \mu \circ \lambda(a) \cdot U_{\mu, \pm}^* \lambda(U_{\mu, \pm}) \\
&= \mu \circ \lambda(a) \cdot \epsilon^\pm(\lambda, \mu).
\end{align*}
$$

Ad Eq. (4): For $a \in C_A(I')$ Eq. (3) reads $\epsilon^\pm(\lambda, \mu) a = a \epsilon^\pm(\lambda, \mu)$, i.e. $\epsilon^\pm(\lambda, \mu) \in C_A(I')' = A(I)$.

Ad Eq. (5): From $U_{\mu, \pm} \in A(K_\pm)$ it follows $\lambda_-(U_{\mu, +}) = U_{\mu, +}$ and $\mu_+(U_{\lambda, -}) = U_{\lambda, -}$. Hence

$$
\begin{align*}
\epsilon^+(\lambda, \mu) &= U_{\mu, +}^* \lambda(U_{\mu, +}) \\
&= U_{\mu, +}^* U_{\lambda, -}^* U_{\lambda, -} U_{\mu, +} \lambda(U_{\mu, +}) \\
&= U_{\mu, +}^* U_{\lambda, -}^* \lambda_-(U_{\mu, +}) U_{\lambda, -} \\
&= U_{\mu, +}^* U_{\lambda, -}^* U_{\mu, +} U_{\lambda, -} \\
&= U_{\mu, +}^* U_{\lambda, -}^* (U_{\lambda, -}^* U_{\mu, +}) U_{\lambda, -} \\
&= U_{\mu, +}^* U_{\mu, +} \mu(U_{\lambda, -}^*) U_{\lambda, -} \\
&= \mu(U_{\lambda, -}^*) U_{\lambda, -} \\
&= (\epsilon^-(\mu, \lambda))^*.
\end{align*}
$$
Ad Eq. (6): Clearly \((\lambda \circ \mu)_{\pm} \in \Delta_A(I_{\pm})\) where
\[(\lambda \circ \mu)_{\pm} = \lambda_{\pm} \circ \mu_{\pm} = \Ad(U_{\lambda \circ \mu, \pm}) \circ \lambda \circ \mu, \quad U_{\lambda \circ \mu, \pm} = U_{\lambda_{\pm}, \pm}(U_{\mu, \pm}).\]
Hence
\[
\epsilon^\pm(\lambda \circ \mu, \nu) = U^*_{\nu, \pm} \cdot \lambda \circ \mu(U_{\nu, \pm}) \\
= U^*_{\nu, \pm} \lambda(U_{\nu, \pm}) \lambda(U^*_{\nu, \pm}) \cdot \lambda \circ \mu(U_{\nu, \pm}) \\
= \epsilon^\pm(\lambda, \nu) \lambda(\epsilon^\pm(\mu, \nu)).
\]
Ad Eq. (7): This follows now easily from Eqs. (5) and (6). Q.E.D.

Note that Eq. (5) nicely reflects the invariance properties of \(\epsilon^\pm(\lambda, \mu)\) as stated in Lemma 2.2.

2.3 The braiding fusion equations

We will now describe how the naturality and braiding fusion equations (BFEs) arise in the algebraic framework. The content of this subsection is not essentially new (e.g. versions of these equations have already been given in [13]), however, as we will make explicit use of the different versions of the BFE we again present the proofs. Moreover, in view of our applications we want to formulate the BFEs for local intertwiners and therefore we have to require strong additivity of the underlying Haag-Kastler net. Strong additivity (or “irrelevance of points”) means that \(A(I) = A(I_1) \lor A(I_2)\) whenever intervals \(I_1\) and \(I_2\) are obtained by removing one single point from the interval \(I \in \mathcal{J}_z\). This requirement basically ensures the equivalence of local and global intertwiners. In the following we will often consider elements of the set \(\Delta_A(I)\) as elements of \(\text{End}(A(K))\) for \(I, K \in \mathcal{J}_z\) such that \(I \subset K\) which is possible since elements of \(\Delta_A(I)\) leave \(A(K)\) invariant.

**Lemma 2.4** Suppose that \(A\) is strongly additive. Then for \(\lambda, \mu \in \Delta_A(I_o), I_o \in \mathcal{J}_z\), we have
\[
\text{Hom}_A(\lambda, \mu) = \text{Hom}_{A(I_o)}(\lambda, \mu).
\]

*Proof.* We first show “\(\subset\)”. Assume \(T \in \text{Hom}_A(\lambda, \mu)\). Then clearly \(T \lambda(a) = \mu(a)T\) for all \(a \in A(I_o)\). Moreover, as \(T a = T \lambda(a) = \mu(a)T = aT\) for all \(a \in \mathcal{C}_A(I'_o)\) we find \(T \in \mathcal{C}_A(I'_o)' = A(I_o)\), proving “\(\subset\)”.

Next, we show “\(\supset\)”. Assume \(T_o \in \text{Hom}_{A(I_o)}(\lambda, \mu)\). It suffices to show \(T_o \lambda(a) = \mu(a)T_o\) for all \(a \in A(I)\) and all \(I \in \mathcal{J}_z\) such that \(I_o \subset I\) \((I_o \neq I)\) because then \(T_o \in \text{Hom}_A(\lambda, \mu)\) by norm continuity. First assume that \(I_o\)
and $I$ have one boundary point in common, i.e. $I$ extends $I_0$ one one side. Then $I_1 = I \cap I'_0$ is an interval in $J_z$ and $A(I) = A(I_0) \lor A(I_1)$ by strong additivity. We have $T_0 \lambda(a) = \mu(a)T_0$ for all $a \in A(I_0)$ by assumption and also $T_0 \lambda(a) = T_0a = \mu(a)T_0$ for all $a \in A(I_1)$ since $T_0 \in A(I_0)$. Hence $T_0$ intertwines $\lambda$ and $\mu$ on the subalgebra of $A(I)$ which is algebraically generated by $A(I_0)$ and $A(I_1)$ and is weakly dense by strong additivity. As endomorphisms in $\Delta A(I_0)$ are weakly continuous on any $A(I)$, $I_0 \subset I$, it follows $T_0 \lambda(a) = \mu(a)T_0$ for all $a \in A(I)$. If $I$ has no common boundary point with $I_0$ we just have to repeat the procedure to extend the interval also on the other side.

Q.E.D.

Now we are ready to prove the naturality equations for local intertwiners.

**Proposition 2.5** For $\lambda, \mu, \rho \in \Delta A(I_0)$, $I_0 \in J_z$, and $T \in \text{Hom}_{A(I_0)}(\lambda, \mu)$ we have the naturality equations

\[
\rho(T) \, \epsilon^\pm(\lambda, \rho) = \epsilon^\pm(\mu, \rho) \, T, \quad (9)
\]

\[
T \, \epsilon^\pm(\rho, \lambda) = \epsilon^\pm(\rho, \mu) \, \rho(T). \quad (10)
\]

**Proof.** Choose intervals $I_+, I_- \in J_z$ such that $I_- < I_0 < I_+$. We take unitaries $U_{\rho, \pm} \in \mathcal{A}$ such that $\rho_\pm = \text{Ad}(U_{\rho, \pm}) \circ \rho$ are localized in $I_\pm$. Then $T\lambda(U_{\rho, \pm}) = \mu(U_{\rho, \pm})T$ by Lemma 2.4. Moreover, $\rho_\pm(T) = T$ as $T \in A(I_0)$. We can now compute

\[
\rho(T) \, \epsilon^\pm(\lambda, \rho) = \rho(T) \, U_{\rho, \pm}^* \lambda(U_{\rho, \pm}) \\
= U_{\rho, \pm}^* \, \rho_\pm(T) \, \lambda(U_{\rho, \pm}) \\
= U_{\rho, \pm}^* \, U_{\rho, \pm} \lambda(U_{\rho, \pm}) \\
= U_{\rho, \pm}^* \, \mu(U_{\rho, \pm})T \\
= \epsilon^\pm(\mu, \rho) \, T,
\]

and Eq. (11) is obtained just by applying Eq. (9) to $T^* \in \text{Hom}_{A(I_0)}(\mu, \lambda)$ and using Eq. (8). Q.E.D.

By use of Eqs. (8) and (7) we obtain immediately the following

**Corollary 2.6** For $\lambda, \mu, \nu, \rho \in \Delta A(I_0)$, $I_0 \in J_z$, and $S \in \text{Hom}_{A(I_0)}(\lambda \circ \mu, \nu)$ we have the BFEs

\[
\rho(S) \, \epsilon^\pm(\lambda, \rho) \lambda(\epsilon^\pm(\mu, \rho)) = \epsilon^\pm(\nu, \rho) \, S, \quad (11)
\]

\[
S \, \lambda(\epsilon^\pm(\rho, \mu)) \epsilon^\pm(\rho, \lambda) = \epsilon^\pm(\rho, \nu) \, \rho(S). \quad (12)
\]
By Lemma 2.3, Eqs. (3) and (4), we find \( \epsilon^\pm(\lambda, \mu) \in \text{Hom}_{A(I)}(\lambda \circ \mu, \mu \circ \lambda) \). Using Eq. (11) and also Eq. (6) we obtain the Yang-Baxter equation (YBE).

**Corollary 2.7** For \( \lambda, \mu, \nu \in \Delta_A(I) \) we have the YBE

\[
\nu(\epsilon^\pm(\lambda, \mu)) \epsilon^\pm(\lambda, \nu) \lambda(\epsilon^\pm(\mu, \nu)) = \epsilon^\pm(\mu, \nu) \mu(\epsilon^\pm(\lambda, \nu)) \epsilon^\pm(\lambda, \mu) .
\]

We remark that the YBE is also true without the assumption of strong additivity because the statistics operators are global intertwiners.

Assume we have a Haag-Kastler net \( \mathcal{N} = \{N(I), I \in \mathcal{J}_z\} \) of von Neumann algebras acting on a Hilbert space \( \mathcal{H} \). If (the \( C^* \)-algebra) \( \mathcal{N} \) leaves a subspace \( \mathcal{H}_0 \subset \mathcal{H} \) invariant and the corresponding subrepresentation \( \pi_0 \) of the defining representation of \( \mathcal{N} \) is faithful, we denote by \( A = \{A(I), I \in \mathcal{J}_z\} \) the isomorphic net given by

\[
A(I) = \pi_0(N(I)), \quad I \in \mathcal{J}_z .
\]

Then strong additivity of the net \( \mathcal{N} \) is equivalent to strong additivity of the net \( A \). If the net \( A \) is Haag dual the we say that \( \mathcal{N} \) has a faithful Haag dual subrepresentation. In that case one checks that \( N(I) = C_N(I)^\prime \cap \mathcal{N} \) for \( I \in \mathcal{J}_z \).

Let \( \Delta_N(I) \) denote the set of transportable endomorphisms of \( \mathcal{N} \) localized in \( I \in \mathcal{J}_z \), i.e. for \( \lambda \in \Delta_N(I) \) and any \( J \in \mathcal{J}_z \) there are unitary charge transporters \( u_{\lambda;I,J} \in \mathcal{N} \) such that \( \hat{\lambda} = \text{Ad}(u_{\lambda;I,J}) \circ \lambda \) is localized in \( J \). Then \( U_{\lambda_0;I,J} = \pi_0(u_{\lambda;I,J}) \) is a charge transporter of

\[
\lambda_0 = \pi_0 \circ \lambda \circ \pi_0^{-1} \in \Delta_A(I) .
\]

Note that, if \( \mathcal{N} \) has a Haag dual subrepresentation, elements of \( \Delta_N(I) \) leave \( N(K) \) invariant whenever \( K \in \mathcal{J}_z \) contains \( I \), so that elements of \( \Delta_N(I) \) can also be considered as elements of \( \text{End}(N(K)) \).

Now choose again \( I_0, I_\pm \in \mathcal{J}_z \) such that \( I_- < I_0 < I_+ \). For \( \lambda, \mu \in \Delta_N(I_0) \) we set \( u_{\mu,\pm} = u_{\mu;I_0,I_\pm} \), and

\[
\epsilon^\pm(\lambda, \mu) = u_{\mu,\pm}^* \lambda(u_{\mu,\pm})
\]

so that

\[
\epsilon^\pm(\lambda_0, \mu_0) = \pi_0(\epsilon^\pm(\lambda, \mu)) .
\]

We call the \( \epsilon^\pm(\lambda, \mu) \)'s statistics operators as well.
Now assume that $\mathcal{N}$ is strongly additive and let $\lambda, \mu, \rho \in \Delta_N(I_0)$ and $t \in \text{Hom}_{N(I_0)}(\lambda, \mu)$. Then $T = \pi_0(t) \in \text{Hom}_{N(I_0)}(\lambda_0, \mu_0)$. This way we obtain $\text{Hom}_N(\lambda, \mu) = \text{Hom}_{N(I_0)}(\lambda, \mu)$ from Lemma 2.4, and we have the naturality equations

$$
\rho_0(T) \epsilon^\pm(\lambda_0, \rho_0) = \epsilon^\pm(\mu_0, \rho_0) T,
\quad T \epsilon^\pm(\rho_0, \lambda_0) = \epsilon^\pm(\rho_0, \mu_0) \rho_0(T).
$$

Applying $\pi_0^{-1}$ to this and Lemma 2.3 we arrive at

**Corollary 2.8** Assume that $\mathcal{N}$ has a faithful Haag dual subrepresentation. Then we have for $\lambda, \mu \in \Delta_N(I_0)$, $I_0 \in \mathcal{F}_z$,

$$
\epsilon^\pm(\lambda, \mu) \cdot \lambda \circ \mu(n) = \mu \circ \lambda(n) \cdot \epsilon^\pm(\lambda, \mu), \quad n \in \mathcal{N}, \quad (14)
$$

$$
\epsilon^\pm(\lambda, \mu) \in N(I_0), \quad (15)
$$

$$
\epsilon^\pm(\lambda, \mu) = (\epsilon^-(\mu, \lambda))^*, \quad (16)
$$

$$
\epsilon^\pm(\lambda \circ \mu, \nu) = \epsilon^\pm(\lambda, \nu) \lambda(\epsilon^\pm(\mu, \nu)), \quad (17)
$$

$$
\epsilon^\pm(\lambda, \mu \circ \nu) = \mu(\epsilon^\pm(\lambda, \nu)) \epsilon^\pm(\lambda, \mu). \quad (18)
$$

If in addition $\mathcal{N}$ is strongly additive and also $\nu, \rho \in \Delta_N(I_0)$, then for $t \in \text{Hom}_{N(I_0)}(\lambda, \mu)$ we have the naturality equations

$$
\rho(t) \epsilon^\pm(\lambda, \rho) = \epsilon^\pm(\mu, \rho) t, \quad (19)
$$

$$
t \epsilon^\pm(\rho, \lambda) = \epsilon^\pm(\rho, \mu) \rho(t), \quad (20)
$$

for $s \in \text{Hom}_{N(I_0)}(\lambda \circ \mu, \nu)$ we have the BFEs

$$
\rho(s) \epsilon^\pm(\lambda, \rho) \lambda(\epsilon^\pm(\mu, \rho)) = \epsilon^\pm(\nu, \rho) s, \quad (21)
$$

$$
s \lambda(\epsilon^\pm(\rho, \mu)) \epsilon^\pm(\rho, \lambda) = \epsilon^\pm(\rho, \nu) \rho(s), \quad (22)
$$

and the YBE

$$
\nu(\epsilon^\pm(\lambda, \mu)) \epsilon^\pm(\lambda, \nu) \lambda(\epsilon^\pm(\mu, \nu)) = \epsilon^\pm(\mu, \nu) \mu(\epsilon^\pm(\lambda, \nu)) \epsilon^\pm(\lambda, \mu). \quad (23)
$$

**2.4 Nets of subfactors**

A net of von Neumann algebras (or even factors) over a partially ordered index set $\mathcal{J}$ is an assignment $\mathcal{M} : \mathcal{J} \ni i \mapsto M_i$ of von Neumann algebras (or factors) on a Hilbert space $\mathcal{H}$ such that we have isotony, $M_i \subset M_j$ whenever $i \leq j$. A net of subfactors consists of two nets of factors $\mathcal{N}$ and $\mathcal{M}$ such that we have subfactors $N_i \subset M_i$ for all $i \in \mathcal{J}$. We simply write $\mathcal{N} \subset \mathcal{M}$.
A net of subfactors is called standard if there is a vector \( \Omega \in \mathcal{H} \) that is cyclic and separating for every \( M_i \) on \( \mathcal{H} \) and \( N_i \) on a subspace \( \mathcal{H}_0 \subset \mathcal{H} \). Note that the projection \( e_N \in \mathcal{B}(\mathcal{H}) \) onto \( \mathcal{H}_0 \) is the Jones projection for each inclusion \( N_i \subset M_i \) for a standard net of subfactors. If there is also an assignment \( E : J \ni i \mapsto E_i \) of faithful normal conditional expectations from \( M_i \) onto \( N_i \) such that \( E_i = E_j|_{M_i} \) for \( i \leq j \) then we say that \( \mathcal{N} \subset \mathcal{M} \) has a faithful normal conditional expectation. \( E \) is called standard if it preserves the vector state \( \omega = \langle \Omega, \cdot \Omega \rangle \). If the index set \( J \) is directed we simply say \( \mathcal{N} \subset \mathcal{M} \) is a directed net and we can form the \( C^* \)-algebras \( \bigcup_{i \in J} N_i \) and \( \bigcup_{i \in J} M_i \) and denote it, by abuse of notation, by the same symbols as used for the nets, \( \mathcal{N} \) and \( \mathcal{M} \), respectively.

In [19] the following is proven

**Proposition 2.9** Let \( \mathcal{N} \subset \mathcal{M} \) be a directed standard net of subfactors with a standard conditional expectation. For every \( i \in J \) there is an endomorphism \( \gamma \) of the \( C^* \)-algebras \( \mathcal{M} \) into \( \mathcal{N} \) such that \( \gamma|_{M_j} \) is a canonical endomorphism of \( M_j \) into \( N_j \) whenever \( i \leq j \). Furthermore, \( \gamma \) acts trivially on \( M_i \cap N \). As \( i \in J \) varies to any \( i' \in J \) the corresponding \( \gamma \) and \( \gamma' \) are inner equivalent by a unitary in \( N_k \) provided \( i, i' \leq k \).

Since (for a fixed \( i \in J \)) \( \gamma \) is a canonical endomorphism of \( M_j \) into \( N_j \) whenever \( i \leq j \) there is a restriction of \( \gamma \) to \( \mathcal{N} \) that we denote by \( \theta \),

\[
\theta = \gamma|_{\mathcal{N}} \in \text{End}(\mathcal{N}).
\]  

**Proposition 2.10** Let \( \mathcal{N} \subset \mathcal{M} \) be a directed standard net of subfactors with a standard conditional expectation. Let \( \gamma \in \text{End}(\mathcal{M}) \) be associated with some \( i \in J \) and \( \theta \in \text{End}(\mathcal{N}) \) its restriction as above. Then we have unitary equivalences

\[
\pi^0 \simeq \pi_0 \circ \gamma \quad \text{and} \quad \pi^0|_{\mathcal{N}} \simeq \pi_0 \circ \theta
\]

where \( \pi^0 \) is the defining representation of \( \mathcal{M} \) on \( \mathcal{H} \) and \( \pi_0 \) the ensuing representation of \( \mathcal{N} \) on \( \mathcal{H}_0 = \overline{\mathcal{N}\Omega} \).

It is also proven in [19] that the Kosaki index is constant in a directed standard net of subfactors with a standard conditional expectation. Moreover, for such nets the following is shown in [19]. Pick \( \gamma \) and \( \theta \) for some \( i \in J \) as above. Then there is an isometry \( w \in N_i \) satisfying \( wn = \theta(n)w \) for all \( n \in \mathcal{N} \) and inducing the conditional expectation \( E \) by \( E(m) = w^*\gamma(m)w \) for \( m \in \mathcal{M} \). If in addition the index is finite, \( [M : N] \equiv [M_i : N_i] < \infty \), then
there is also an isometry \( v \in M \) satisfying \( vm = \gamma(m)v \) for all \( m \in M \) and \( w^*v = [M : N]^{-1/2}1 = w^*\gamma(v) \). Then clearly \( E(vv^*) = [M : N]^{-1}1 \), and we have also \( M_j = N_jv \) whenever \( i \leq j \), and finally \( M = Nv \).

A directed standard net of subfactors with a standard conditional expectation is called a quantum field theoretical net of subfactors if the index set \( J \) admits a causal structure and we have \( N_i \subset M_j' \) if \( i \) and \( j \) are causally disjoint. For our purposes we choose the directed set \( J = J_z \) and assume that we have a given quantum field theoretical net of subfactors \( N \subset M \).

We denote by \( A \) the net (and the C*-algebra)

\[
A(I) = \pi_0(N(I)), \quad I \in J_z.
\]

As we are dealing with factors, \( \pi_0 \) is automatically faithful. We assume that \( A \) satisfies Haag duality, i.e. \( N \) has a faithful Haag dual subrepresentation.

Fix an interval \( I_o \in J_z \) and take the endomorphism \( \gamma \) of Prop. 2.9. First note that Proposition 2.10 tells us that \( \theta \in \Delta_N^0(I_o) \). Let us consider the situation that \( \pi_0 \) decomposes into a finite number of representations of \( N \) as follows,

\[
\pi_0 \circ \theta \simeq \bigoplus_{\ell=0}^n m_\ell \pi_\ell
\]

where \( \pi_\ell, \ell = 0, 1, \ldots, n, \) are irreducible, mutually disjoint representations of \( N \) and \( m_\ell \) are multiplicities. Assume that \( \pi_\ell \) are such that we can write

\[
\pi_0 \circ \theta \simeq \bigoplus_{\ell=0}^n m_\ell \cdot \pi_0 \circ \lambda_\ell
\]

with \( \lambda_\ell \in \Delta_N^0(I_o) \). Then this means that we have isometries \( T_{\ell,r} \in \mathcal{B}(H_0), \ell = 0, 1, \ldots, n, r = 1, 2, \ldots, m_\ell, \) such that

\[
T_{\ell,r}T_{\ell',r'} = \delta_{\ell,\ell'}\delta_{r,r'}1, \quad \sum_{\ell=0}^n \sum_{r=1}^{m_\ell} T_{\ell,r}T_{\ell',r'}^* = 1,
\]

and

\[
\pi_0 \circ \theta(n) = \sum_{\ell=0}^n \sum_{r=1}^{m_\ell} T_{\ell,r} \cdot \pi_0 \circ \lambda_\ell(n) \cdot T_{\ell',r'}^*, \quad n \in N.
\]

As \( \theta, \lambda_\ell \in \Delta_N^0(I_o) \) it follows

\[
a = \sum_{\ell=0}^n \sum_{r=1}^{m_\ell} T_{\ell,r}aT_{\ell',r'}^*, \quad a \in \mathcal{C}_\mathcal{A}(I_o'),
\]
hence $T_{\ell,r} \in \mathcal{C}_A(I_0) = A(I_0)$. Thus we can define $t_{\ell,r} = \pi_0^{-1}(T_{\ell,r}) \in N(I_0)$, and we find in particular

$$\theta(n) = \sum_{\ell=0}^n \sum_{r=1}^{m_\ell} t_{\ell,r}(n) t_{\ell',r'}^* , \quad n \in N(I_0),$$

and this is in terms of sectors of $N(I_0)$

$$[\theta] = \bigoplus_{\ell=0}^n m_\ell [\lambda_\ell]. \quad (27)$$

3 α-induction for nets of subfactors

From now on we assume that we have a given quantum field theoretical net of subfactors $N \subset M$ over the index set $J_z$, i.e. $N(I_1) \subset M(I_2)'$ if $I_1 \cap I_2 = \emptyset$. This implies locality of the net $N$ but we even assume the net $M$ to be local, and we also assume the net $A = \{ A(I) = \pi_0(N(I)) , \ I \in J_z \}$ to satisfy Haag duality. We also assume the net $N$ (or equivalently the net $A$) to be strongly additive. Moreover, we require the net $N \subset M$ to be of finite index, $[M : N] < \infty$. We fix an arbitrary interval $I_0 \in J_z$ and take the corresponding endomorphism $\gamma$ of Proposition 2.9.

3.1 Definition of α-induction

In the following we set $\varepsilon(\lambda, \mu) = \varepsilon^+(\lambda, \mu)$ for any $\lambda, \mu \in \Delta_N(I_0)$. As usual, we denote by $v \in M(I_0)$ and $w \in N(I_0)$ the isometries which intertwine $\gamma \in \text{End}(M)$ and its restriction $\theta \in \Delta_N(I_0)$, respectively, and satisfy $w^*v = [M : N]^{-1/2}1 = w^*\gamma(v)$.

Lemma 3.1 For $\lambda \in \Delta_N(I_0)$ we have

$$\text{Ad}(\varepsilon(\lambda, \theta)) \circ \lambda \circ \gamma(v) = \theta(\varepsilon(\lambda, \theta)^*)\gamma(v). \quad (28)$$

Proof. By the intertwining property of $v$ we find $\gamma(v)^* \in \text{Hom}_{N(I_0)}(\theta^2, \theta)$. Hence we can apply the BFE, Eq. (22), and obtain

$$\varepsilon(\lambda, \theta) \cdot \lambda \circ \gamma(v)^* = \gamma(v)^*\theta(\varepsilon(\lambda, \theta))\varepsilon(\lambda, \theta),$$

hence

$$\varepsilon(\lambda, \theta) \cdot \lambda \circ \gamma(v) \cdot \varepsilon(\lambda, \theta)^* = \theta(\varepsilon(\lambda, \theta)^*)\gamma(v).$$
If \( I \in \mathcal{J}_z \) contains \( I_0 \) then for \( n \in N(I) \) we have \( \text{Ad}(\varepsilon(\lambda, \theta)) \circ \lambda \circ \gamma(n) = \text{Ad}(\varepsilon(\lambda, \theta)) \circ \lambda \circ \theta(n) = \theta \circ \lambda(n) \in \theta(N(I)) \subset \gamma(M(I)) \), and note that then also \( \theta(\varepsilon(\lambda, \theta)^* \gamma(v)) = \gamma(M(I)) \). Since each \( m \in M(I) \) can be written as \( m = nv \) for some \( n \in N(I) \) we find

**Corollary 3.2** For any \( I \in \mathcal{J}_z \) such that \( I_0 \subset I \) we have

\[
\text{Ad}(\varepsilon(\lambda, \theta)) \circ \lambda \circ \gamma(M(I)) \subset \gamma(M(I)).
\] (29)

Now we are ready to define \( \alpha \)-induction — just by the formula (3.10) for the extended endomorphism in Proposition 3.9 in [19]. However, we have shown that this endomorphism leaves each algebra \( M(I) \) with \( I \in \mathcal{J}_z \) such that \( I_0 \subset I \) invariant.

**Definition 3.3** For \( \lambda \in \Delta_N(I_0) \) we define the \( \alpha \)-induced endomorphism \( \alpha_\lambda \in \text{End}(\mathcal{M}) \) by

\[
\alpha_\lambda = \gamma^{-1} \circ \text{Ad}(\varepsilon(\lambda, \theta)) \circ \lambda \circ \gamma.
\] (30)

Thanks to Corollary 3.2, \( \alpha_\lambda \) is well defined and can also be considered as an element of \( \text{End}(M(I)) \) as long as \( I \in \mathcal{J}_z \) contains \( I_0 \). The definition of \( \alpha \)-induction is such that \( \alpha_\lambda \) is an extension of \( \lambda \), i.e. we have \( \alpha_\lambda(n) = \lambda(n) \) obviously for \( n \in \mathcal{N} \).

### 3.2 The main formula for \( \alpha \)-induction

Choose \( I_+ \in \mathcal{J}_z \) such that \( I_0 < I_+ \) and denote by \( \gamma_+ \) a (canonical) endomorphism associated to \( I_+ \) as in Proposition 2.9 and let \( \theta_+ \) its restriction to \( \mathcal{N} \). Then the unitary \( u = [M : N] \cdot E(v_+ v^*) \in \mathcal{N} \) intertwines \( \gamma \) and \( \gamma_+ \) and relates isometries \( v \) and \( v_+ \in M(I_+) \) by \( v_+ = uv \) [19]. The proof of the following lemma from [19] makes use of locality of the net \( \mathcal{M} \).

**Lemma 3.4** We have

\[
\varepsilon(\theta, \theta)v^2 = \varepsilon(\theta, \theta)^* v^2 = v^2, \quad \varepsilon(\theta, \theta)\gamma(v) = \varepsilon(\theta, \theta)^* \gamma(v) = \gamma(v).
\] (31)
Proof. By the intertwining property of \( u \) we have in particular \( \theta_+ = \text{Ad}(u) \circ \theta \). Therefore \( u = u_{\theta,+} \) is a charge transporter for \( \theta \) and we can write \( \varepsilon(\theta, \theta) = u^* \theta(u) \). By locality of \( M \) we find \( v_+ v = v v_+ \), i.e. \( v v = \theta(v) v \), hence \( \varepsilon(\theta, \theta) v^2 \equiv u^* \theta(u) v^2 = v^2 \). Since \( v^2 = \gamma(v) v \) we obtain \( \varepsilon(\theta, \theta) \gamma(v) v^* = \gamma(v) v^* \) by right multiplication with \( v^* \). Application of the conditional expectation yields \( \varepsilon(\theta, \theta) \gamma(v) = \gamma(v) \) since \( E(v v^*) = w^* \gamma(v v^*) w = [M : N]^{-1} \mathbf{1} \). Multiplying the obtained relations by \( \varepsilon(\theta, \theta)^* \) from the left yields the full statement. Q.E.D.

Later we will use the following important

**Lemma 3.5** Let \( t \in M(I_\circ) \) such that \( t^\lambda(n) = \mu(n) t \) for all \( n \in N(I_\circ) \) and some \( \lambda, \mu \in \Delta_{\chi}(I_\circ) \). Then \( t \in \text{Hom}_{M(I_\circ)}(\alpha_\lambda, \alpha_\mu) \).

**Proof.** As \( \alpha_\lambda, \alpha_\mu \) restrict, respectively, to \( \lambda, \mu \) on \( N(I_\circ) \) it suffices to show \( t \alpha_\lambda(v) = \alpha_\mu(v) t \). Let \( s = \gamma(t) \). Then clearly \( s \in \text{Hom}_{N(I_\circ)}(\theta \circ \lambda, \theta \circ \mu) \). By the BFE, Eq. (21), we obtain

\[
\varepsilon(\theta \circ \mu, \theta) s = \varepsilon(\theta, \theta) \theta(\varepsilon(\lambda, \theta)).
\]

Since \( \varepsilon(\theta \circ \mu, \theta) = \varepsilon(\theta, \theta) \theta(\varepsilon(\mu, \theta)) \) we find

\[
s \theta(\varepsilon(\lambda, \theta)^*) = \theta(\varepsilon(\mu, \theta)^*) \varepsilon(\theta, \theta)^* \varepsilon(s, \theta, \theta).
\]

So let us compute

\[
s \cdot \text{Ad}(\varepsilon(\lambda, \theta)) \circ \lambda \circ \gamma(v) = s \theta(\varepsilon(\lambda, \theta)^*) \gamma(v)
\]

\[
= \theta(\varepsilon(\mu, \theta)^*) \varepsilon(\theta, \theta)^* \varepsilon(s, \theta, \theta) \gamma(v)
\]

\[
= \theta(\varepsilon(\mu, \theta)^*) \varepsilon(\theta, \theta)^* \varepsilon(s, \theta, \theta) \gamma(v)
\]

\[
= \theta(\varepsilon(\mu, \theta)^*) \gamma(v) s
\]

\[
= \theta(\varepsilon(\mu, \theta)^*) \gamma(v) s
\]

\[
= \text{Ad}(\varepsilon(\mu, \theta)) \circ \mu \circ \gamma(v) \cdot s,
\]

where we repeatedly used Lemmata 3.3, 3.4 and also that \( \theta(s) \gamma(v) = \gamma(t) \gamma(v) = \gamma(v) \gamma(t) = \gamma(v) s \). Thanks to Corollary 3.2 we can now apply \( \gamma^{-1} \) and obtain \( t \alpha_\lambda(v) = \alpha_\mu(v) t \). Q.E.D.

Note that we obtained Lemma 3.3 just by the following ingredients: Haag duality and strong additivity of the net \( A \), implying existence of statistics operators and the BFEs for local intertwiners of endomorphisms in \( \Delta_{\chi}(I_\circ) \), and locality of the net \( M \), implying Lemma 3.4, and of course, finiteness.
of the index guaranteeing the existence of the isometry \( v \). Now consider the following special situation \( \lambda = \mu = \text{id} \) in Lemma 3.5. First note that \( \alpha_{\text{id}} = \text{id} \) by the definition of \( \alpha \)-induction. Then for each \( t \in \text{Hom}_{M(I_0)}(\text{id}, \text{id}) = M(I_0)' \cap M(I_0) \), i.e.

\[ N(I_0)' \cap M(I_0) \subset M(I_0)' \cap M(I_0) = \mathbb{C} 1, \]

and \( I_0 \in \mathcal{J}_z \) was arbitrary. Somewhat surprisingly, we gained

**Corollary 3.6** Let \( N \subset M \) be a directed quantum field theoretical net of subfactors over \( \mathcal{J}_z \) with finite index. If \( N \) is strongly additive and has a Haag dual subrepresentation and \( M \) satisfies locality, then \( N \subset M \) is a net of irreducible subfactors.

Another immediate consequence of Lemma 3.5 is the following

**Corollary 3.7** If \( [\lambda] = [\mu] \) for some \( \lambda, \mu \in \Delta_N(I_0) \), then \( [\alpha_\lambda] = [\alpha_\mu] \).

(Here and in the following we use the sector brackets for sectors of either \( N(I_0) \) or \( M(I_0) \).)

**Lemma 3.8** If \( n \in N \) then \( nv = 0 \) implies \( n = 0 \). Similarly, for \( m \in M \), \( w^* \gamma(m) = 0 \) implies \( m = 0 \).

**Proof.** This follows from the identities \( n = [M : N]^{1/2} w^* \gamma(v) = [M : N]^{1/2} w^* \gamma(nv), n \in N, \) and \( m = [M : N]^{1/2} w^* \gamma(m) = [M : N]^{1/2} w^* \gamma(m)v, m \in M \).

We are now ready to prove the main formula for \( \alpha \)-induction given in the following

**Theorem 3.9** For \( \lambda, \mu \in \Delta_N(I_0) \) we have

\[ \langle \alpha_\lambda, \alpha_\mu \rangle_{M(I_0)} = \langle \theta \circ \lambda, \mu \rangle_{N(I_0)}. \]  

**Proof.** We first show “\( \leq \)”. Let \( t \in \text{Hom}_{M(I_0)}(\alpha_\lambda, \alpha_\mu) \). We show that \( r = w^* \gamma(t) \in \text{Hom}_{M(I_0)}(\theta \circ \lambda, \mu). \) Clearly, \( r \in N(I_0) \). By assumption, we have \( t \alpha_\lambda(m) = \alpha_\mu(m)t \) for all \( m \in M(I_0) \). Restriction to \( N(I_0) \) and application of \( \gamma \) yields \( \gamma(t) \in \text{Hom}_{M(I_0)}(\theta \circ \lambda, \theta \circ \mu) \). It follows for all \( n \in N(I_0) \)

\[ r \cdot \theta \circ \lambda(n) = w^* \cdot \gamma(t) \cdot \theta \circ \lambda(n) = w^* \cdot \theta \circ \mu(n) \cdot \gamma(t) = \mu(n) r \]
since $w^*\theta(n) = nw^*$. By Lemma 3.8 the map $t \mapsto r = w^*\gamma(t)$ is injective, thus “$\leq$” is proven.

We now turn to “$\geq$”. Suppose $r \in \text{Hom}_{N(I)}(\theta \circ \lambda, \mu)$ is given. We show that $t = rv \in \text{Hom}_{M(I)}(\alpha \lambda, \alpha \mu)$. Clearly, $t = rv \in M(I)$, and we have for all $n \in N(I)$

$$t\lambda(n) = rv\lambda(n) = r \cdot \theta \circ \lambda(n) \cdot v = \mu(n) rv = \mu(n) t.$$ 

Hence, by Lemma 3.5, we have $t \in \text{Hom}_{M(I)}(\alpha \lambda, \alpha \mu)$. By Lemma 3.8, the map $r \mapsto t = rv$ is injective; the proof is complete. Q.E.D.

3.3 Homomorphism property of $\alpha$-induction

As $\alpha \lambda$ restricts to $\lambda$ on $N(I)$ which is of finite index in $M(I)$, we find $d_{\alpha \lambda} = d_\lambda$. This is an immediate consequence of the multiplicativity of the minimal index \[17\]: Consider the chain of inclusions $\alpha \lambda(N(I)) \subset N(I) \subset M(I)$. Choose $\eta \in \text{End}(M(I))$ such that $\eta(M(I)) = N(I)$. Then $[M(I) : N(I)] = d_\lambda^2$ and $[M(I) : \alpha \lambda(N(I))] = d_{\alpha \lambda}^2$, hence $[N(I) : \alpha \lambda(N(I))] = d_{\alpha \lambda}^2$ but $[N(I) : \lambda(N(I))] = d_\lambda^2$, thus indeed $d_{\alpha \lambda} = d_\lambda$. However, there are more properties.

Lemma 3.10 For any $\lambda, \mu \in \Delta_N(I)$ we have $\alpha_{\lambda \mu} = \alpha \lambda \circ \alpha \mu$.

Proof. We compute

$$\alpha_{\lambda \mu} = \gamma^{-1} \circ \text{Ad}(\varepsilon(\lambda \circ \mu, \theta)) \circ \lambda \circ \mu \circ \gamma$$

$$= \gamma^{-1} \circ \text{Ad}(\varepsilon(\lambda, \theta) \lambda(\varepsilon(\mu, \theta))) \circ \lambda \circ \mu \circ \gamma$$

$$= \gamma^{-1} \circ \text{Ad}(\varepsilon(\lambda, \theta)) \circ \lambda \circ \text{Ad}(\varepsilon(\mu, \theta)) \circ \mu \circ \gamma$$

$$= \alpha \lambda \circ \alpha \mu,$$

where we used Eq. \[3\]. Q.E.D.

As $\varepsilon(\lambda, \mu) \in \text{Hom}_{N(I)}(\lambda \circ \mu, \mu \circ \lambda)$ we obtain from Lemma 3.5 that $\varepsilon(\lambda, \mu)\alpha_{\lambda \mu}(m) = \alpha_{\mu \lambda}(m)\varepsilon(\lambda, \mu)$ for all $m \in M(I)$, in particular for $m = v$. Since $M = Nv$ we obtain from Lemma 3.10 the following

Corollary 3.11 For $\lambda, \mu \in \Delta_N(I)$ we have

$$\alpha \mu \circ \alpha \lambda = \text{Ad}(\varepsilon(\lambda, \mu)) \circ \alpha \lambda \circ \alpha \mu.$$

(33)
As $\alpha_\lambda$ restricts to $\lambda$ on $\mathcal{N}$ we clearly have $\alpha_\lambda(\varepsilon(\mu, \nu)) = \lambda(\varepsilon(\mu, \nu))$ for $\lambda, \mu, \nu \in \Delta_N(I_o)$. Therefore, by rewriting the YBE, Eq. (23), and recalling that $\varepsilon(\lambda, \lambda) \in \alpha_2^2(M(I_o))' \cap M(I_o)$ by Corollary 3.11, we arrive at

**Corollary 3.12** For $\lambda, \mu, \nu \in \Delta_N(I_o)$ we have the YBE

$$\alpha_\nu(\varepsilon(\lambda, \mu)) \varepsilon(\lambda, \nu) \alpha_\lambda(\varepsilon(\mu, \nu)) = \varepsilon(\mu, \nu) \alpha_\mu(\varepsilon(\lambda, \nu)) \varepsilon(\lambda, \mu), \quad (34)$$

in particular, the endomorphisms $\alpha_\lambda$ are braided endomorphisms, i.e. setting $\sigma_i = \alpha_i^{-1}(\varepsilon(\lambda, \lambda))$, $i = 1, 2, 3, \ldots$, yields a representation of the braid group $B_\infty$.

Next we show that $\alpha$-induction preserves also sums of sectors.

**Lemma 3.13** Let $\lambda, \lambda_1, \lambda_2 \in \Delta_N(I_o)$ such that $[\lambda] = [\lambda_1] \oplus [\lambda_2]$. Then $[\alpha_\lambda] = [\alpha_{\lambda_1}] \oplus [\alpha_{\lambda_2}].$

**Proof.** As $[\lambda] = [\lambda_1] \oplus [\lambda_2]$ we have isometries $y_1, y_2 \in N(I_o)$ fulfilling the relations of $O_2$, $y_i^\dagger y_j = \delta_{i,j} 1$, $\sum_{i=1}^2 y_i y_i^* = 1$, and

$$\lambda(n) = \sum_{i=1}^2 y_i \lambda_i(n) y_i^*, \quad n \in N(I_o).$$

We now choose an interval $I_+ \in J_z$ such that $I_o < I_+$. Note that $y_i \in \text{Hom}_{N(I_o)}(\lambda_i, \lambda) = \text{Hom}_{\Delta_N}(\lambda_i, \lambda)$. Choose a charge transporter $u_{\theta,+} \in \mathcal{N}$ such that $\theta_+ = \text{Ad}(u_{\theta,+}) \circ \theta \in \Delta_N(I_+)$. Then we have

$$\varepsilon(\lambda, \theta) = u_{\theta,+}^\dagger \lambda(u_{\theta,+}) = \sum_{i=1}^2 u_{\theta,+}^\dagger_\lambda(u_{\theta,+}) y_i y_i^* = \sum_{i=1}^2 u_{\theta,+} y_i \lambda_i(u_{\theta,+}) y_i^*.$$

Since $y_i \in N(I_o)$ we also find $\theta_+(y_i) = y_i$, $i = 1, 2$, and thus we compute for $n \in N(I_o)$

$$\text{Ad}(\varepsilon(\lambda, \theta)) \circ \lambda(n) = \sum_{i=1}^2 u_{\theta,+}^\dagger_\lambda y_i \lambda_i(u_{\theta,+} n u_{\theta,+}^\dagger) y_i^* u_{\theta,+}$$

$$= \sum_{i=1}^2 u_{\theta,+}^\dagger_\lambda(y_i) \lambda_i(u_{\theta,+} n u_{\theta,+}^\dagger) \theta_+(y_i^*) u_{\theta,+}$$

$$= \sum_{i=1}^2 \theta(y_i) u_{\theta,+}^\dagger \lambda_i(u_{\theta,+} n u_{\theta,+}^\dagger) u_{\theta,+} \theta(y_i^*)$$

$$= \sum_{i=1}^2 \theta(y_i) \cdot \text{Ad}(\varepsilon(\lambda_i, \theta)) \circ \lambda_i(n) \cdot \theta(y_i^*).$$

Specializing to $n = \gamma(m)$, $m \in M(I_o)$, and applying $\gamma^{-1}$ yields

$$\alpha_\lambda(m) = \sum_{i=1}^2 y_i \alpha_\lambda_i(m) y_i^*, \quad m \in M(I_o),$$

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the lemma is proven. \[Q.E.D.\]

For sectors with finite statistical dimension we can show that \(\alpha\)-induction preserves also sector conjugation.

**Lemma 3.14** If \(\overline{\lambda} \in \Delta_N(I_o)\) is a conjugate to \(\lambda \in \Delta_N(I_o)\), \(d_\lambda < \infty\), then \(\alpha_{\overline{\lambda}}\) is a conjugate to \(\alpha_\lambda\), i.e. \([\alpha_{\overline{\lambda}}] = [\overline{\alpha_{\lambda}}]\).

**Proof.** Using Lemma 3.10, Theorem 3.9 and Eq. (1) we get

\[
\langle \alpha_\lambda, \alpha_\lambda \rangle_{M(I_o)} = \langle \theta \circ \lambda, \lambda \rangle_{N(I_o)} = \langle \theta \circ \lambda \circ \overline{\lambda}, \text{id}_{N(I_o)} \rangle_{N(I_o)}
= \langle \alpha_{\lambda \circ \overline{\lambda}}, \text{id}_{M(I_o)} \rangle_{M(I_o)}
= \langle \alpha_\lambda \circ \alpha_{\overline{\lambda}}, \text{id}_{M(I_o)} \rangle_{M(I_o)}
= \langle \alpha_\lambda, \overline{\alpha_{\lambda}} \rangle_{M(I_o)}.
\]

Replacing \(\lambda\) by \(\overline{\lambda}\) yields \(\langle \alpha_{\overline{\lambda}}, \alpha_{\overline{\lambda}} \rangle_{M(I_o)} = \langle \alpha_{\overline{\lambda}}, \alpha_{\overline{\lambda}} \rangle_{M(I_o)}\) whereas conjugation yields \(\langle \overline{\alpha_{\lambda}}, \overline{\alpha_{\lambda}} \rangle_{M(I_o)} = \langle \overline{\alpha_{\lambda}}, \overline{\alpha_{\lambda}} \rangle_{M(I_o)}\). Thus we found

\[
\langle \alpha_{\overline{\lambda}}, \alpha_{\overline{\lambda}} \rangle_{M(I_o)} = \langle \alpha_{\overline{\lambda}}, \alpha_{\overline{\lambda}} \rangle_{M(I_o)} = \langle \overline{\alpha_{\lambda}}, \overline{\alpha_{\lambda}} \rangle_{M(I_o)},
\]

and because we assumed finite statistical dimensions, these expressions are finite. Then this implies the statement. \[Q.E.D.\]

Next we want to discuss certain commutativity rules between sectors arising from \(\alpha\)-induction.

**Lemma 3.15** Let \(\lambda, \mu, \rho \in \Delta_N(I_o)\) and \(r \in M(I_o)\) such that \(r(\lambda(n)) = \mu(n)r\) for all \(n \in N(I_o)\). Then we have \(r\varepsilon(\rho, \lambda) = \varepsilon(\rho, \mu)\alpha_{\rho}(r)\).

**Proof.** Note that \(s = \gamma(r) \in \text{Hom}_{N(I_o)}(\theta \circ \lambda, \theta \circ \mu)\). Thus the BFE, Eq. (22), yields

\[
s \theta(\varepsilon(\rho, \lambda))\varepsilon(\rho, \theta) = \varepsilon(\rho, \theta \circ \mu)\rho(s),
\]

hence we obtain by using Eq. (18)

\[
s \theta(\varepsilon(\rho, \lambda)) = \theta(\varepsilon(\rho, \mu)) \varepsilon(\rho, \theta)\rho(s)\varepsilon(\rho, \theta)^*,
\]

and applying \(\gamma^{-1}\) yields the statement. \[Q.E.D.\]

**Proposition 3.16** Let \(\lambda, \mu \in \Delta_N(I_o)\) and \(\beta \in \text{End}(M(I_o))\) such that \([\beta]\) is a subsector of \([\alpha_\mu]\). Then \([\alpha_\lambda \circ \beta] = [\beta \circ \alpha_\lambda]\).
Proof. By assumption, there is an isometry $t \in M(I_0)$, $t^*t = 1$, such that
\[ t\beta(m) = \alpha_\mu(m)t, \quad m \in M(I_0). \]
Then $u = t^*\varepsilon(\lambda, \mu)\alpha_\lambda(t) \in \text{Hom}_{M(I_0)}(\alpha_\lambda \circ \beta, \beta \circ \alpha_\lambda)$ as we have for all $m \in M(I_0)$
\[
\begin{align*}
    t^*\varepsilon(\lambda, \mu)\alpha_\lambda(t) \cdot \alpha_\lambda \circ \beta(m) &= t^*\varepsilon(\lambda, \mu) \cdot \alpha_\lambda \circ \alpha_\mu(m) \cdot \alpha_\lambda(t) \\
    &= t^* \cdot \alpha_\mu \circ \alpha_\lambda(m) \cdot \varepsilon(\lambda, \mu)\alpha_\lambda(t) \\
    &= \beta \circ \alpha_\lambda(m) \cdot t^*\varepsilon(\lambda, \mu)\alpha_\lambda(t),
\end{align*}
\]
where we used Corollary 3.11. All we have to show is that $u$ is unitary. Note that $tt^* \in \text{Hom}_{M(I_0)}(\alpha_\mu, \alpha_\lambda)$ and hence in particular $tt^* \in \mu(N(I_0))'$ and $M(I_0)$ as $\alpha_\mu$ restricts to $\mu$ on $N(I_0)$. Then Lemma 3.15 yields $tt^*\varepsilon(\lambda, \mu) = \varepsilon(\lambda, \mu)\alpha_\lambda(tt^*)$. Therefore
\[
    u^*u = \alpha_\lambda(t^*)\varepsilon(\lambda, \mu)^*tt^*\varepsilon(\lambda, \mu)\alpha_\lambda(t) = \alpha_\lambda(t^*tt^*) = 1,
\]
and
\[
    uu^* = t^*\varepsilon(\lambda, \mu)\alpha_\lambda(tt^*)\varepsilon(\lambda, \mu)^*t = t^*tt^* = 1,
\]
the proof is complete. Q.E.D.

3.4 $\sigma$-restriction and $\alpha\sigma$-reciprocity

In [19] there is also defined a restriction for endomorphisms. In our context, we will call that $\sigma$-restriction.

Definition 3.17 For $\beta \in \text{End}(\mathcal{M})$ the $\sigma$-restricted endomorphism $\sigma_\beta \in \text{End}(\mathcal{N})$ is defined by
\[
    \sigma_\beta = \gamma \circ \beta|_{\mathcal{N}}. \tag{35}
\]

If $\beta \in \text{End}(\mathcal{M})$ leaves $M(I)$ invariant for $I \in \mathcal{J}_z$, $I_0 \subset I$, then clearly $\sigma_\beta$ leaves $N(I)$ invariant. Moreover, the formula $\sigma_{\beta}(n) = \gamma \circ \beta(n)$, $n \in N(I)$, defines also a map from $\text{End}(M(I))$ to $\text{End}(N(I))$. For $\lambda \in \Delta_\mathcal{N}(I_0)$ we obviously have $\sigma_{\alpha_\lambda} = \theta \circ \lambda$ so that in particular $[\lambda]$ is a subsector of $[\sigma_{\alpha_\lambda}]$. It is natural to ask whether $[\beta]$ is a subsector of $[\alpha_{\sigma_\beta}]$. For localized, transportable $\beta$ we are going to prove an even stronger result which is a sort of Frobenius reciprocity for $\alpha$-induction and $\sigma$-restriction. For this we need some more preparation.

Clearly, if $\beta$ is localized in $I_0$ then so is $\sigma_\beta$ as for $n \in C_N(I_0')$ we find $\sigma_{\beta}(n) = \gamma \circ \beta(n) = \gamma(n) = \theta(n) = n$ since $\theta$ is localized in $I_0$. Now suppose that $\beta$ is also transportable: For each $I_1 \in \mathcal{J}_z$ we have unitary charge transporters $Q_{\beta;I_0,I_1} \in \mathcal{M}$ such that $\beta_{I_1} = \text{Ad}(Q_{\beta;I_0,I_1}) \circ \beta$ is localized in $I_1$.
Lemma 3.18 If $\beta \in \Delta_M(I_o)$ then $\sigma_\beta \in \Delta_N(I_o)$. Namely, for any $I_1 \in \mathcal{J}_z$ we have $\sigma_{\beta,I_1} = \text{Ad}(u_{\sigma_\beta;I_o,I_1}) \circ \sigma_\beta \in \Delta_N(I_1)$ with
\[ u_{\sigma_\beta;I_o,I_1} = u_{\theta;I_o,I_1} \gamma(Q_{\beta;I_o,I_1}) . \tag{36} \]

Proof. We have to show that $\sigma_{\beta,I_1} = \text{Ad}(u_{\sigma_\beta;I_o,I_1}) \circ \sigma_\beta$ is localized in $I_1$. Now for $n \in \mathcal{C}_N(I'_1)$ we have
\[
\begin{align*}
\sigma_{\beta,I_1}(n) &= u_{\theta;I_o,I_1} \gamma(Q_{\beta;I_o,I_1}) \cdot \gamma \circ \beta(n) \cdot \gamma(Q_{\beta;I_o,I_1})^* u_{\theta;I_o,I_1}^* \\
&= u_{\theta;I_o,I_1} \cdot \gamma \circ \beta_1(n) \cdot u_{\theta;I_o,I_1}^* \\
&= u_{\theta;I_o,I_1} \gamma(n) u_{\theta;I_o,I_1}^* \\
&= u_{\theta;I_o,I_1} \theta(n) u_{\theta;I_o,I_1}^* \\
&= \theta_1(n) = n,
\end{align*}
\]
since $\theta_1 = \text{Ad}(u_{\theta;I_o,I_1}) \circ \theta$ is localized in $I_1$. Q.E.D.

For some interval $I_- \in \mathcal{J}_z$ such that $I_- < I_o$ we set $Q_{\beta,-} = Q_{\beta;I_o,I_-}$.

Lemma 3.19 For $\beta \in \Delta_M(I_o)$ we have
\[ \varepsilon(\sigma_\beta, \theta) = \gamma^2(Q_{\beta,-})^* \varepsilon(\theta, \theta) \gamma(Q_{\beta,-}) . \tag{37} \]

Proof. We compute
\[
\begin{align*}
\varepsilon(\sigma_\beta, \theta) &= \varepsilon^-(\theta, \sigma_\beta)^* = \theta(u_{\sigma_\beta;I_o,I_-})^* u_{\sigma_\beta;I_o,I_-} \\
&= \theta(\gamma(Q_{\beta,-})^* u_{\theta,-})^* u_{\theta,-} \gamma(Q_{\beta,-}) \\
&= \gamma^2(Q_{\beta,-}) \theta(u_{\theta,-})^* u_{\theta,-} \gamma(Q_{\beta,-}) \\
&= \gamma^2(Q_{\beta,-})^* \varepsilon^-(\theta, \theta)^* \gamma(Q_{\beta,-}) \\
&= \gamma^2(Q_{\beta,-})^* \varepsilon(\theta, \theta) \gamma(Q_{\beta,-}) ,
\end{align*}
\]
where we used Eq. (16). Q.E.D.

For $I \in \mathcal{J}_z$ let $\Delta_M^{(0)}(I)$ denote the set of transportable endomorphisms localized in $I$ which leave $M(K)$ invariant for any $K \in \mathcal{J}_z$ with $I \subset K$. Note that $\lambda(M(K)) \subset M(K)$ for $\lambda \in \Delta_M(I)$ is automatically satisfied if $\mathcal{M}$ is Haag dual, i.e. $\Delta_M^{(0)}(I) = \Delta_M(I)$ in this case. However, in order to
be as general as possible we do not assume Haag duality of $\mathcal{M}$ (although it is satisfied in the applications we have in mind) but we do need invariance of local algebras as we often consider elements of $\Delta_{\mathcal{M}}^{(0)}(I)$ as elements of $\text{End}(M(K))$ for $I \subset K$.

**Lemma 3.20** Let $t \in M(I_0)$ such that $t\lambda(n) = \beta(n)t$ for all $n \in N(I_0)$ and some $\lambda \in \Delta_N(I_0)$ and $\beta \in \Delta_{\mathcal{M}}^{(0)}(I_0)$. Then $t \in \text{Hom}_{M(I_0)}(\alpha_\lambda, \beta)$.

**Proof.** As $\alpha_\lambda(n) = \lambda(n)$ for all $n \in N(I_0)$ it suffices to show $t\alpha_\lambda(v) = \beta(v)t$. Let $s = \gamma(t)$. Then clearly $s \in \text{Hom}_{N(I_0)}(\theta \circ \lambda, \sigma_\beta)$. By the BFE, Eq. (21), we obtain
\[
s \theta(\epsilon(\lambda, \theta)^*) = \epsilon(\sigma_\beta, \theta)^* \theta(s) \epsilon(\theta, \theta).
\]
So let us compute
\[
s \cdot \text{Ad}(\epsilon(\lambda, \theta)) \circ \lambda \circ \gamma(v) = s \theta(\epsilon(\lambda, \theta)^*) \gamma(v)
\]
\[
= \epsilon(\sigma_\beta, \theta)^* \theta(s) \epsilon(\theta, \theta) \gamma(v)
\]
\[
= \epsilon(\sigma_\beta, \theta)^* \theta(s) \gamma(v)
\]
\[
= \epsilon(\sigma_\beta, \theta)^* \gamma(v)s
\]
\[
= \gamma(Q_{\beta, -})^* \epsilon(\theta, \theta)^* \gamma^2(Q_{\beta, -}) \gamma(v)s
\]
\[
= \gamma(Q_{\beta, -})^* \epsilon(\theta, \theta)^* \gamma(v) \gamma(Q_{\beta, -})s
\]
\[
= \gamma(Q_{\beta, -})^* \gamma(v) \gamma(Q_{\beta, -})s
\]
\[
= \gamma(Q_{\beta, -} v Q_{\beta, -})s
\]
\[
= \gamma(Q_{\beta, -} \beta_I(v) Q_{\beta, -})s
\]
\[
= \gamma \circ \beta(v) \cdot s,
\]
where we repeatedly used Lemmata 3.1, 3.4 and 3.19. Applying $\gamma^{-1}$ yields $t\alpha_\lambda(v) = \beta(v)t$.

Q.E.D.

Now we are ready to prove the reciprocity theorem.

**Theorem 3.21** For $\lambda \in \Delta_N(I_0)$ and $\beta \in \Delta_{\mathcal{M}}^{(0)}(I_0)$ we have $\alpha\sigma$-reciprocity,
\[
\langle \alpha_\lambda, \beta \rangle_{M(I_0)} = \langle \lambda, \sigma_\beta \rangle_{N(I_0)}.
\]

**Proof.** We first show “$\leq$”. Let $t \in \text{Hom}_{M(I_0)}(\alpha_\lambda, \beta)$. We show that $r = \gamma(t)w \in \text{Hom}_{N(I_0)}(\lambda, \sigma_\beta)$. Clearly, $r \in N(I_0)$. By assumption, we have

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$t_{\alpha}(m) = \beta(m)t$ for all $m \in M(I_0)$. Restriction to $N(I_0)$ and application of $\gamma$ yields $\gamma(t) \in \text{Hom}_{N(I_0)}(\theta \circ \lambda, \sigma_\beta)$. It follows for all $n \in N(I_0)$

\begin{equation}
    r \lambda(n) = \gamma(t)w \lambda(n) = \gamma(t) \cdot \theta \circ \lambda(n) \cdot w = \sigma_\beta(n) \gamma(t)w = \sigma_\beta(n)r.
\end{equation}

By Lemma 3.8 the map $t \mapsto r = \gamma(t)w$ is injective, thus “$\leq$” is proven.

We now turn to “$\geq$”. Suppose $r \in \text{Hom}_{N(I_0)}(\lambda, \sigma_\beta)$ is given. We show that $t = v^*r \in \text{Hom}_{M(I_0)}(\alpha_\lambda, \beta)$. Clearly, $t = v^*r \in M(I_0)$, and we have for all $n \in N(I_0)$

\begin{equation}
    t \lambda(n) = v^*r \lambda(n) = v^* \sigma_\beta(n)r = v^* \cdot \gamma \circ \beta(n) \cdot r = \beta(n) v^*r = \beta(n)t.
\end{equation}

Hence, by Lemma 3.20, we have $t \in \text{Hom}_{M(I_0)}(\alpha_\lambda, \beta)$. It follows again from Lemma 3.8 that the map $r \mapsto t = v^*r$ is injective; the proof is complete. Q.E.D.

It follows from the proof that we have $v^* \in \text{Hom}_{M(I_0)}(\alpha_{\sigma_\beta}, \beta)$ since $1 \in \text{Hom}_{N(I_0)}(\sigma_\beta, \sigma_\beta)$. (Recall $\sigma_\beta \in \Delta_N(I_0)$ by Lemma 3.18.) We conclude that $[\beta]$ is a subsector of $[\alpha_{\sigma_\beta}]$.

Remark. Note that Theorem 3.21 is not a generalization of Theorem 3.9 since we assumed in particular that $\beta$ is localized. However, $\alpha_\mu$ is in general not localized; it is localized if and only if the monodromy $\varepsilon(\mu, \theta)\varepsilon(\theta, \mu)$ is trivial (Prop. 3.9 in [19]).

Note that $\sigma$-restriction does not preserve sector products, i.e. $[\sigma_{\beta_1 \circ \beta_2}]$ is in general different from $[\sigma_{\beta_1 \circ \beta_2}]$, e.g. for $\beta_1 = \beta_2 = \text{id}$. However, we add the following

**Lemma 3.22** Let $\beta, \beta_1, \beta_2 \in \text{End}(M(I_0))$. If $[\beta] = [\beta_1] \oplus [\beta_2]$ then $[\sigma_\beta] = [\sigma_{\beta_1}] \oplus [\sigma_{\beta_2}]$. If $[\beta_1] = [\beta_2]$ then $[\sigma_{\beta_1}] = [\sigma_{\beta_2}]$.

**Proof.** If $[\beta] = [\beta_1] \oplus [\beta_2]$ then there are isometries $t_1, t_2 \in M(I_0)$ satisfying the relations of $O_2$ and $\beta(m) = \sum_{i=1}^{2} t_i \beta_i(m) t_i^*$ for $m \in M(I_0)$. Then $s_i = \gamma(t_i)$ satisfy the relations of $O_2$ as well and

\begin{equation}
    \sigma_\beta(n) = \gamma \circ \beta(n) = \sum_{i=1}^{2} s_i \gamma \circ \beta_i(n) \cdot s_i^* = \sum_{i=1}^{2} s_i \sigma_{\beta_i}(n) s_i^*, \quad n \in N(I_0).
\end{equation}

If $[\beta_1] = [\beta_2]$ then $\beta_2 = \text{Ad}(u) \circ \beta_1$ with some unitary $u \in M(I_0)$. Then clearly $\sigma_{\beta_2} = \text{Ad}(\gamma(u)) \circ \sigma_{\beta_1}$, and $\gamma(u) \in N(I_0)$ is unitary. Q.E.D.

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3.5 The inverse braiding

We have used the statistics operators \( \varepsilon(\lambda, \theta) \equiv \varepsilon^+(\lambda, \theta) \) for the definition of the \( \alpha \)-induced endomorphism \( \alpha_\lambda \equiv \alpha_\lambda^+ \). Of course, all our results we derived hold similarly for the endomorphisms \( \alpha_\lambda^- \), analogously defined by use of \( \varepsilon^-(\lambda, \theta) \). However, \( \alpha_\lambda \) and \( \alpha_\lambda^- \) are in general not the same. In this subsection we investigate several relations between \( \alpha_\lambda \) and \( \alpha_\lambda^- \). The following Proposition is instructive.

**Proposition 3.23** For \( \lambda \in \Delta_N'(I_o) \) the following are equivalent:

1. \( [\alpha_\lambda] = [\alpha_\lambda^-] \),
2. \( \alpha_\lambda = \alpha_\lambda^- \),
3. The monodromy is trivial: \( \varepsilon(\lambda, \theta)\varepsilon(\theta, \lambda) = 1 \).

**Proof.** If \( [\alpha_\lambda] = [\alpha_\lambda^-] \) then there is a unitary \( u \in \text{Hom}_{M(I_o)}(\alpha_\lambda, \alpha_\lambda^-) \), i.e. \( u\alpha_\lambda(m) = \alpha_\lambda^-(m)u \) for all \( m \in M(I_o) \). Restriction yields \( u\lambda(n) = \lambda(n)u \) for all \( n \in N(I_o) \). By Lemma 3.5 we find \( u \in \text{Hom}_{M(I_o)}(\alpha_\lambda, \alpha_\lambda^-) \), in particular \( u\alpha_\lambda(v) = \alpha_\lambda(v)u \), hence \( \alpha_\lambda(v)u = \alpha_\lambda^-(v)u \), thus \( \alpha_\lambda(v) = \alpha_\lambda^-(v) \). But \( \alpha_\lambda(n) = \lambda(n) = \alpha_\lambda^-(n) \) for all \( n \in \mathcal{N} \), therefore \( \alpha_\lambda(m) = \alpha_\lambda^-(m) \) for any \( m \in \mathcal{M} \), proving \( \alpha_\lambda = \alpha_\lambda^- \). Now by Lemma 3.4 we have \( \alpha_\lambda(v) = \varepsilon(\lambda, \theta)^*v \), and similarly \( \alpha_\lambda^-(v) = \varepsilon^-(\lambda, \theta)^*v = \varepsilon(\theta, \lambda)v \). Therefore \( \alpha_\lambda(v) = \alpha_\lambda^-(v) \) implies \( \varepsilon(\lambda, \theta)\varepsilon(\theta, \lambda)v = v \) and hence \( \varepsilon(\lambda, \theta)\varepsilon(\theta, \lambda) = 1 \) by Lemma 3.8. Now if the monodromy is trivial then \( \varepsilon(\lambda, \theta) = \varepsilon^-(\lambda, \theta) \), and this trivially leads to \( [\alpha_\lambda] = [\alpha_\lambda^-] \).

Q.E.D.

Nevertheless we have the following

**Lemma 3.24** For \( \lambda, \mu \in \Delta_N'(I_o) \) we have

\[
\text{Ad}(\varepsilon(\lambda, \mu)) \circ \alpha_\lambda \circ \alpha_\mu^- = \alpha_\mu^- \circ \alpha_\lambda. \tag{39}
\]

**Proof.** As \( \alpha_\lambda \) and \( \alpha_\mu^- \) restrict to \( \lambda \) and \( \mu \), respectively, on \( \mathcal{N} \) it suffices to show

\[
\varepsilon(\lambda, \mu) \cdot \alpha_\lambda \circ \alpha_\mu^-(v) = \alpha_\mu^- \circ \alpha_\lambda(v) \cdot \varepsilon(\lambda, \mu).
\]

Recall \( \alpha_\lambda(v) = \varepsilon(\lambda, \theta)^*v \) by Lemma 3.4 and similarly \( \alpha_\mu^-(v) = \varepsilon^-(\mu, \theta)^*v = \varepsilon(\theta, \mu)v \). The YBE, Eq. (23), can be written as

\[
\varepsilon(\lambda, \mu)\lambda\varepsilon(\theta, \mu)\varepsilon(\lambda, \theta)^* = \mu(\varepsilon(\lambda, \theta)^*)\varepsilon(\theta, \mu)\theta(\varepsilon(\lambda, \mu)).
\]

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Now we compute
\[\varepsilon(\lambda, \mu) \cdot \alpha_\lambda \circ \alpha^-_\mu(v) = \varepsilon(\lambda, \mu) \alpha_\lambda(\varepsilon(\theta, \mu)v) = \varepsilon(\lambda, \mu) \lambda(\varepsilon(\theta, \mu)) \varepsilon(\lambda, \theta)^* v = \mu(\varepsilon(\lambda, \theta)^*) \varepsilon(\theta, \mu) \theta(\varepsilon(\lambda, \mu)) v = \mu(\varepsilon(\lambda, \theta)^*) \varepsilon(\theta, \mu) v \varepsilon(\lambda, \mu) = \alpha^-_\mu(\varepsilon(\lambda, \theta)^*) \varepsilon(\lambda, \mu) = \alpha^-_\mu \circ \alpha_\lambda(v) \cdot \varepsilon(\lambda, \mu),\]
proving the lemma. Q.E.D.

The following lemma establishes a sort of naturality equations for the \(\alpha\)-induced endomorphisms.

**Lemma 3.25** Let \(\lambda, \mu, \rho \in \Delta_N(I)\). For an \(r \in M(I)\) such that \(r \lambda(n) = \mu(n)r\) for all \(n \in N(I)\) we have
\[\begin{align*}
\alpha^\pm_\rho(r) \varepsilon^\mp(\lambda, \rho) &= \varepsilon^\mp(\mu, \rho) r, & (40) \\
r \varepsilon^\pm(\rho, \lambda) &= \varepsilon^\pm(\rho, \mu) \alpha^\pm_\rho(r). & (41)
\end{align*}\]

**Proof.** Completely analogous to Lemma 3.15 we also obtain \(r \varepsilon^-(\rho, \lambda) = \varepsilon^-(\rho, \mu) \alpha^-_\rho(r)\), establishing Eq. (41). Now note that \(r^* \mu(n) = \lambda(n)r^*\) for all \(n \in N(I)\), therefore we can apply Eq. (41) yielding Eq. (40) by use of Eq. (16).
Q.E.D.

We are now ready to prove the following

**Proposition 3.26** Let \(\lambda, \mu \in \Delta_N(I)\) and \(\beta, \delta \in \text{End}(M(I))\) such that \([\beta]\) and \([\delta]\) are subsectors of \([\alpha_\lambda]\) and \([\alpha^-_\mu]\), respectively. Then \([\beta \circ \delta] = [\delta \circ \beta]\).

**Proof.** By assumption, there are isometries \(t, s \in M(I)\), \(t^* t = s^* s = 1\), such that
\[t \beta(m) = \alpha_\lambda(m) t, \quad s \delta(m) = \alpha^-_\mu(m) s, \quad m \in M(I)\].
Then \( u = s^* \alpha^-_\mu (t^*) \varepsilon(\lambda, \mu) \alpha_\lambda(s) t \in \text{Hom}_{M(I_o)}(\beta \circ \delta, \delta \circ \beta) \) as we have for all \( m \in M(I_o) \)
\[
s^* \alpha^-_\mu (t^*) \varepsilon(\lambda, \mu) \alpha_\lambda(s) t \cdot \beta \circ \delta(m) = s^* \alpha^-_\mu (t^*) \varepsilon(\lambda, \mu) \alpha_\lambda(s) \cdot \alpha_\lambda \circ \delta(m) \cdot t
= s^* \alpha^-_\mu (t^*) \varepsilon(\lambda, \mu) \cdot \alpha_\lambda \circ \alpha^-_\mu (m) \cdot \alpha_\lambda(s) t
= s^* \alpha^-_\mu (t^*) \cdot \alpha^-_\mu \circ \alpha_\lambda(m) \cdot \varepsilon(\lambda, \mu) \alpha_\lambda(s) t
= s^* \cdot \alpha^-_\mu \circ \beta(m) \cdot \alpha^-_\mu (t^*) \varepsilon(\lambda, \mu) \alpha_\lambda(s) t
= \delta \circ \beta \cdot s^* \alpha^-_\mu (t^*) \varepsilon(\lambda, \mu) \alpha_\lambda(s) t,
\]
where we used Lemma 3.24. All we have to show is that \( u \) is unitary. Note that \( tt^* \in \text{Hom}_{M(I_o)}(\alpha_\lambda, \alpha_\lambda) \) and \( ss^* \in \text{Hom}_{M(I_o)}(\alpha^-_\mu, \alpha^-_\mu) \) and hence in particular \( tt^* \in \lambda(N(I_o))' \cap M(I_o) \) and \( ss^* \in \mu(N(I_o))' \cap M(I_o) \) as \( \alpha_\lambda \) and \( \alpha^-_\mu \) restrict to \( \lambda \) and \( \mu \), respectively, on \( N(I_o) \). Then Lemma 3.25 yields \( \alpha^-_\mu (tt^*) \varepsilon(\lambda, \mu) = \varepsilon(\lambda, \mu) tt^* \) by Eq. (40) and \( ss^* \varepsilon(\lambda, \mu) = \varepsilon(\lambda, \mu) \alpha_\lambda(ss^*) \) by Eq. (11). Therefore
\[
u^* u = t^* \alpha_\lambda(s^*) \varepsilon(\lambda, \mu)^* \alpha^-_\mu(t) ss^* \alpha^-_\mu(t^*) \varepsilon(\lambda, \mu) \alpha_\lambda(s) t
= t^* \alpha_\lambda(s^*) \varepsilon(\lambda, \mu)^* ss^* \alpha^-_\mu(tt^*) \varepsilon(\lambda, \mu) \alpha_\lambda(s) t
= t^* \alpha_\lambda(s^*) \varepsilon(\lambda, \mu)^* ss^* \varepsilon(\lambda, \mu) tt^* \alpha_\lambda(s) t
= t^* \alpha_\lambda(ss^*) tt^* t = 1,
\]
and
\[
u u^* = s^* \alpha^-_\mu(t^*) \varepsilon(\lambda, \mu) \lambda(s) tt^* \alpha_\lambda(s^*) \varepsilon(\lambda, \mu)^* \alpha^-_\mu(t) s
= s^* \alpha^-_\mu(t^*) \varepsilon(\lambda, \mu) tt^* \alpha_\lambda(ss^*) \varepsilon(\lambda, \mu)^* \alpha^-_\mu(t) s
= s^* \alpha^-_\mu(t^* tt^*) ss^* \alpha^-_\mu(t) s = 1,
\]
the proof is complete. \( Q.E.D. \)

Recall that \( [\beta] \) is a subsector of \( [\alpha_{\sigma \beta}] \) for any \( \beta \in \Delta^{(0)}_{\lambda}(I_o) \), and in the same way it is a subsector of \( [\alpha^-_{\sigma \beta}] \). From Proposition 3.26 we obtain immediately

**Corollary 3.27** For any \( \beta \in \Delta^{(0)}_{\lambda}(I_o) \) and any \( \delta \in \text{End}(M(I_o)) \) such that \( [\delta] \) is a subsector of some \( [\alpha_{\mu}], \mu \in \Delta_N(I_o), \) we have \( [\beta \circ \delta] = [\delta \circ \beta] \).

### 4 Miscellanea

#### 4.1 The results in terms of sector algebras

We now want to present our results in the language of sector algebras. We need some preparation.
Definition 4.1 Let $V$ be a (real or complex), finite dimensional, unital, associative algebra (with addition $\oplus$ and multiplication $\times$) together with a basis $V = \{v_0, v_1, v_2, \ldots, v_{d-1}\}$ (in the linear space sense) such that 1 $\in V$, say $v_0 = 1$. Let $N^k_{i,j}$ be the structure constants, defined by

$$v_i \times v_j = \bigoplus_{k=0}^{d-1} N^k_{i,j} v_k.$$ (42)

If

1. (Conjugation) there is an involutive permutation $i \mapsto \bar{i}$, $\bar{i} = i$, satisfying $N^0_{i,j} = \delta_{i,j}$ and $N^k_{i,j} = N^k_{j,i}$ (so that it extends to an anti-automorphism of $V$),

2. (Positive Integrality) the structure constants are non-negative integers, $N^k_{i,j} \in \mathbb{N}_0$,

then $(V,V)$ (or simply $V$) is called a sector algebra. If $V$ is commutative, $N^k_{i,j} = N^k_{j,i}$, then it is called a fusion algebra.

Now let $M$ be an infinite factor and $V = \{[\lambda_0], [\lambda_1], [\lambda_2], \ldots, [\lambda_{d-1}]\}$ be a finite set of irreducible sectors with finite statistical dimension, which contains the trivial sector, say $[\lambda_0] = [\text{id}]$, and is closed under sector conjugation and the sector product. The latter means that the irreducible decomposition of each product $[\lambda_i] \times [\lambda_j]$ is a sum of elements in $V$ (possibly with some multiplicities). We simply call such a set a sector basis. We can consider a sector basis as the basis of an algebra $V$ where the summation $\oplus$ and multiplication $\times$ comes from the sum and product of sectors in the obvious sense.

By the properties of addition and multiplication of sectors, $V$ is indeed a sector algebra, and the structure constants are given by $N^k_{i,j} = \langle \lambda_i \circ \lambda_j, \lambda_k \rangle_M$, where $\lambda_i$ denote representative endomorphisms of the sector $[\lambda_i]$.

Now suppose that we have a net of subfactors $\mathcal{N} \subset M$ as described at the beginning of Section 3. We denote by $[\Delta]\mathcal{N}(I_0) \subset \text{Sect}(N(I_0))$ the set of DHR sectors, i.e. the quotient of $\Delta_N(I_0)$ by inner equivalence in $N(I_0)$ (and similarly $[\Delta]_0(0) M(I_0) \subset \text{Sect}(M(I_0))$ as the quotient of $\Delta^0_M(I_0)$ by inner equivalence in $M(I_0)$). Suppose we have a given sector basis $W \subset [\Delta]\mathcal{N}(I_0)$. Because of the commutativity of sectors in $[\Delta]\mathcal{N}(I_0)$, the associated sector algebra $W$ is indeed a fusion algebra. As $\alpha$-induction preserves unitary equivalence by Corollary 3.7, the map $\lambda \mapsto \alpha_\lambda$ extends to a map $[\alpha] : [\lambda] \mapsto [\alpha_\lambda]$, from $W$ to $\text{Sect}(M(I_0))$. Now let $\mathcal{V}$ denote the set of all irreducible subsectors $[\beta] \in \text{Sect}(M(I_0))$ of every $[\alpha_\lambda]$, $[\lambda] \in W$. Since $\alpha$-induction
preserves the sector product and conjugation, $V$ must be a sector basis and we denote by $V$ the associated sector algebra. However, $V$ is not necessarily commutative. We now summarize the results of Subsection 3.3, Lemmata 3.10, 3.13, 3.14 and Proposition 3.16, in the following

**Theorem 4.2** Let $W \subset [\Delta]_N^I$ be a sector basis and $W$ the associated fusion algebra, and let $V \subset \text{Sect}(M(I_o))$ be the corresponding sector basis obtained by $\alpha$-induction and $V$ the associated sector algebra. Then $\alpha$-induction extends to a homomorphism $[\alpha]: W \to V$, preserving conjugates and statistical dimensions. Each $[\alpha \lambda], [\lambda] \in W$, commutes with each $[\beta] \in V$. If $[\alpha]$ is surjective i.e. each element in $V$ can be written as a linear combination of $[\alpha \lambda]$'s, $[\lambda_i] \in W$, then the sector algebra $V$ is a fusion algebra.

Now we turn to the discussion of $\sigma$-restriction in terms of sectors. By Lemma 3.22, the map $\beta \mapsto [\sigma \beta]$ extends to a map from $\text{Sect}(M(I_o))$ to $\text{Sect}(N(I_o))$. We can therefore summarize the results of Theorem 3.21 and Corollary 3.27 as follows.

**Theorem 4.3** Let $T \subset [\Delta]_M^{(0)}(I_o)$ be a sector basis and $T$ the associated fusion algebra. Let also $W \subset [\Delta]_N^I$ be a sector basis with associated fusion algebra $W$, and $V, V$ obtained by $\alpha$-induction as above. If all elements of $T$ are mapped to elements in $W$ by $\sigma$-restriction, then $T \subset V$ and $T \subset V$ is a (sector) subalgebra. Moreover, any element of $T$ commutes with every element of $V$.

### 4.2 The subgroup net of subfactors

Although we postpone all our (conformal field theory) applications to the forthcoming paper [1], let us briefly discuss a simple example here. Consider a situation as in the DHR theory [7], i.e. we have a net $\mathcal{F}$ of local field algebras $F(I)$, $I \in \mathcal{J}$, that are type III-factors, and we have a compact gauge group $G$ acting outerly on each $F(I)$, and this action is implemented on the Hilbert space $\mathcal{H}$ by a unitary representation $U$. The net $\mathcal{N}$ of observable algebras is then given by the fixed point algebras $\mathcal{N}(I) = F(I)^G$. (There are also some more physically motivated assumptions, e.g. certain space-time transformation properties and that observables and fields associated to relatively spacelike regions commute.) Now supppose that we are dealing with a finite gauge group, and that $H \subset G$ is a subgroup. We define another net $\mathcal{M}$ by taking the fixed point algebras with respect to the subgroup, $M(I) = F(I)^H$. Then we clearly obtain a net of subfactors $\mathcal{N} \subset M \subset \mathcal{M}$ of finite index. (The index is in fact $[G : H]$.) Under the standard assumptions of
the DHR theory \[7\] the Hilbert space $\mathcal{H}$ decomposes with respect to the action of $\mathcal{N}$ as

$$\mathcal{H} = \bigoplus_{\pi \in \hat{G}} \mathcal{H}_\pi \otimes \mathbb{C}^{d_\pi}. \quad (43)$$

Here $\pi \in \hat{G}$ are the irreducible representations of $G$ of dimension $d_\pi$, and $\mathcal{H}_\pi$ are pairwise inequivalent representation spaces of $\mathcal{N}$, the superselection sectors. The gauge group $G$ acts on the multiplicity spaces $\mathbb{C}^{d_\pi}$ by the representation $\pi$, i.e.

$$U(g) = \bigoplus_{\pi \in \hat{G}} 1_{\mathcal{H}_\pi} \otimes \pi(g), \quad g \in G. \quad (44)$$

With respect to $\mathcal{M}$ we have another decomposition

$$\mathcal{H} = \bigoplus_{\rho \in \hat{H}} \mathcal{H}_\rho \otimes \mathbb{C}^{d_\rho}, \quad (45)$$

where now $\rho \in \hat{H}$ label the irreducible representations (of dimension $d_\rho$) of the subgroup $H$. Since $N(I) = F(I) \cap U(G)'$ and $M(I) = F(I) \cap U(H)'$ it is not hard to see that the decompositions of Eq. (43) and Eq. (45) are related by

$$\mathcal{H}_\rho = \bigoplus_{\pi \in \hat{G}} \mathcal{H}_\pi \otimes \mathbb{C}^{n_\pi^\rho}, \quad (46)$$

where $n_\pi^\rho$ are the induction-restriction coefficients

$$n_\pi^\rho = \langle \rho, \text{res}_H^G \pi \rangle_{\mathbb{Z}[\hat{H}]} = \langle \text{ind}_H^G \rho, \pi \rangle_{\mathbb{Z}[\hat{G}]}. \quad (40)$$

Now let us assume that our requirements of Haag duality and strong additivity for the net $\mathcal{N}$ and locality of the net $\mathcal{M}$ are fulfilled. Let $\lambda_\pi \in \Delta_{\mathcal{N}}(I_0)$ and $\beta_\rho \in \Delta_{\mathcal{M}}^{(0)}(I_0)$, $I_0 \in \mathcal{J}_2$, denote localized endomorphisms corresponding to the superselection sectors $\mathcal{H}_\pi$, $\pi \in \hat{G}$, and $\mathcal{H}_\rho$, $\rho \in \hat{H}$, so that they obey in particular the fusion rules of $\hat{G}$ and $\hat{H}$, respectively, and their statistical dimensions coincide with the dimensions of the corresponding group representations. We learn from Proposition 2.10 (see also [19]) that $\sigma$-restriction corresponds to the restriction of representations of the net $\mathcal{M}$ to the net $\mathcal{N}$, i.e. $\pi^0 \circ \beta_\rho|_{\mathcal{N}} \simeq \pi^0_0 \circ \sigma_{\beta_\rho}$. This restriction can be read off from Eq. (16), hence we conclude for $\rho \in \hat{H}$

$$[\sigma_{\beta_\rho}] = \bigoplus_{\pi \in \hat{G}} n_\pi^\rho [\lambda_\pi]. \quad (41)$$
From $\alpha\sigma$-reciprocity, Theorem 3.21:

$$\langle \alpha_{\lambda}, \beta_{\rho} \rangle_M(I_\circ) = \langle \lambda, \sigma_{\beta_{\rho}} \rangle_N(I_\circ) = \pi_{\rho},$$

we conclude (recall $d_{\alpha_{\lambda}} = d_{\lambda}$)

$$[\alpha_{\lambda_{\pi}}] = \bigoplus_{\rho \in H} n_{\rho}^{\pi} \cdot [\beta_{\rho}].$$

In other words, for this particular example of the subgroup net of subfactors, 
$\sigma$-restriction corresponds to the induction, $\alpha$-induction corresponds to the 
restriction of group representations, and $\alpha\sigma$-reciprocity reflects Frobenius 
reciprocity.

### 4.3 Remarks

In view of our later applications to chiral conformal field theories \[1\] we 
have presented the theory for nets of subfactors indexed by the set $\mathcal{J}_z$, i.e. 
with the punctured circle $S^1 \setminus \{z\}$ as the underlying “space-time”, and we 
also required strong additivity of $\mathcal{N}$ or, equivalently, of $\mathcal{A}$. Note that for 
chiral Conformal field theories strong additivity is equivalent to the already 
assumed Haag duality (on the punctured circle). For the general case we 
assumed strong additivity so that local intertwiners (of localized endomorphisms) 
extend to global ones and therefore satisfy the naturality equations 
and BFEs. One may however drop the strong additivity assumption and 
work with global intertwiners from the beginning. The invariance of local 
algebras $M(I)$, $I \in \mathcal{J}_z$, $I_\circ \subset I$, under the action of $\alpha_{\lambda}$ is also true without the 
strong additivity assumption because $v$ itself is a global intertwiner. Moreover, many of our results possess global analogues, e.g. Theorem 3.9 then 
reads $\langle \alpha_{\lambda}, \alpha_{\mu} \rangle_M = \langle \theta \circ \lambda, \mu \rangle_N$ for $\lambda, \mu \in \Delta_N(I_\circ)$ or Theorem 3.21 becomes 
$\langle \alpha_{\lambda}, \beta \rangle_M = \langle \lambda, \sigma_{\beta} \rangle_N$, $\beta \in \Delta_N(I_\circ)$. However, we cannot obtain Corollary 3.6 
without the strong additivity assumption and we need the local formulation for the results concerning the subsectors of the $[\alpha_{\lambda}]$'s. But global 
analogues of the results not depending on the strong additivity can also be 
generalized to other space-times like the $D$-dimensional Minkowski space $\mathbb{M}^D$ with $D = 2, 3, 4, \ldots$ (as long as we have transportable endomorphisms). 
One just has to replace intervals $I$ by double cones $\mathcal{O}$ and to substitute 
“disjoint”, $I_1 \cap I_2 = \emptyset$, by “causally disjoint”, i.e. “relatively spacelike”, 
$\mathcal{O}_1 \subset \mathcal{O}_2$. But notice that for $\mathbb{M}^D$ with $D > 2$ the spacelike complement of any double cone is connected, and this implies that we only have one 
statistics operator. There are no longer two different braidings and hence
we have $\alpha_\lambda \equiv \alpha_\lambda^+ = \alpha_\lambda^-$, and in particular all induced endomorphisms $\alpha_\lambda$ are localized.

Acknowledgement

We are grateful to K.-H. Rehren for several useful comments on an earlier version of the manuscript. This project is supported by the EU TMR Network in Non-Commutative Geometry.

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