An experimental study into the bilinear oscillator close to grazing

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Abstract: A linear oscillator undergoing impact with a secondary support is studied experimentally. Smooth as well as nonsmooth bifurcations are observed. The bifurcations are explained with help from simulations based on mapping solutions between locally smooth subspaces. Experimental stability studies are also presented, justifying the normal form maps used to show the response close to grazing for systems with and without prestress of the secondary spring. The high degree of correspondence lends support to the modelling approach, and the highly complicated response justifies continued study of this system.

1. Introduction

Simple piecewise systems have been studied extensively in the past. While much work has been devoted to finding normal forms and classifying the possible types of bifurcations in such systems [1-4] and references therein, comparatively little has been devoted to experimental verification of these [5-8], and then only to a limited extent or with simple rigid impact assumptions. This study presents detailed analysis of a number of bifurcation scenarios close to grazing in terms of the nonlinear bimodal maps that result from solving the linear equations in each subspace. The normal form 3/2 map resulting from grazing is often either the cause of or a direct precursor to a smooth bifurcation. For this reason the 3/2 verses square root form of maps resulting from either a continuous or discontinuous vector field across the subspace border is examined experimentally, and found to be in good accordance with theoretical predictions.

2. Experimental Setup

A 1 kg mass is supported by parallel leaf springs to allow vertical displacement without rotation. An additional beam is placed to prevent large displacement. Contact is controlled via an adjustable bolt. The mass of the additional support is very small compared to the mass of the block, and the damping is very small. This allows modelling of the additional support as a massless spring, with contact starting and ending at the equilibrium position of the support. Harmonic excitation of the system via the base is then provided by an electromagnetic shaker. It is assumed that there is no coupling between oscillator and the shaker. For further design details see [9]. Measurement is conducted using an eddy current displacement probe to monitor the displacement of the mass, and accelerometers to measure the base and mass acceleration. 100 Hz low pass filtering is performed on the pre-amplified

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accelerometer signals. The time history is then plotted, and the Savitsky-Golay algorithm used for polynomial smoothing of the data. As a by product of this the first derivative is available, which enables plotting of the phase portrait. Bifurcations are monitored as a function of frequency by incrementing the frequency a small amount, allowing transients to decay, and then using the base acceleration signal to construct an appropriate Poincaré stroboscopic map. The system was driven from a nonimpacting to an impacting response, and care was taken to follow each attractor for as long as it remained stable, in order to capture bifurcation phenomena in the experimental system which could be compared to the model.

3. Modelling
Since we can make the simplifying assumption that the discontinuity surface is not motion or time dependent, and is located at $x - e = 0$, the nondimensionalised system of equations becomes,

$$\begin{align*}
x' &= v \\
v' &= \Gamma \sin(\omega \tau) - 2\xi v - x - \beta (x - e)H (x - e)
\end{align*}$$

(the last term becomes $(x - e + l)H (x - e)$ if a prestress of $l$ is introduced into the secondary spring).

This results in two linear equations which can be solved subject to initial conditions, and then maps $P_1$ and $P_2$ of these initial conditions between the two subspaces $X_1$ and $X_2$ produces the global response, see Fig. 1. A periodic solution is found either by iterating the map, or, if a particular solution is being sought via the Newton method (once the Jacobian has been determined). The stability was analysed by finding the eigenvalues of the global Jacobian, constructed by composition from the local Jacobians in each subspace. Further details and explicit formulae can be found in [9,10].

![Fig. 1](image)

Fig. 1 (a) Physical model of the oscillator. (b) Phase space is divided by the discontinuity boundaries $S_1$ and $S_2$. The locally valid maps $P_1$ and $P_2$ project the points from one boundary to another.

4. Stability Analysis
Once the flow has been reduced to a map, linear stability analysis of the map will in most cases be sufficient to determine the bifurcation scenario. For the perturbation about a fixed point of the stroboscopic map, $v^* = P(v^*)$, we can write the first order terms in the Taylor expansion,

$$u = (v - v^*)_{n+1} = \frac{\partial P}{\partial v}_{v=v^*} (v - v^*)_n + (O)^2$$

where the derivative matrix is the Jacobian. For a piecewise system this requires composition from the local Jacobians of the smooth subspaces. The eigenvalues of the global Jacobian give the stability of
the underlying flow being examined. Experimentally it is possible to estimate the global Jacobian matrix in the following way: First determine a stable solution by allowing transients to die out, then perturb the system slightly by gently tapping the oscillating mass. The above theory tells us that providing the perturbation remains small the linear map \( u_{n+1} = A u_n \) provides a good estimation of the evolution of the perturbed Poincarè map. In principle then for a 2 dimensional map 3 successive points would provide enough information to determine all the Jacobian coefficients. Between 7 and 10 points were used to allow a minimisation of the squares of the errors between data and the map to reduce the effect of noise. Further points were important to test the validity of the assumption that the Jacobian coefficients remain constant. The system was driven from nonimpacting to impacting motion by increasing either the amplitude or the frequency. The other parameters were set such that a stable period-1 solution existed after the grazing bifurcation. Existing theory has shown that close to grazing, local maps can be constructed on the discontinuity surface to represent flow in the high stiffness region, e.g. [3,11]. When the force across the boundary is discontinuous, the first order term has a power 1/2. This results in a square root singularity in the trace of the Jacobian. When the force is continuous with a discontinuous derivative, the first order term is 3/2 and the trace remains continuous. Experiments were conducted to determine the validity of these approximate local maps, the results of which are displayed in Fig. 2. Where impact occurs with the spring in a relaxed state, the determinant and trace are continuous. The determinant appears not to change value, and the trace may increase slightly. When the secondary spring is prestressed at the point of impact, the determinant does not change as the system is driven to the impacting state, but the trace jumps rapidly to a high value and then begins to return to the pre-impacting value. Experiments were also conducted for the system with a higher stiffness ratio (Fig. 3), which is of more interest for many engineering applications. The transition of the trace is much more noticeably abrupt due to the high stiffness, but still continuous. It is this change due to the grazing bifurcation that is found in many cases to be the precursor to a smooth bifurcation.

Fig. 2 The determinant remains continuous and does not change both for the prestressed (a) and unprestressed (b) systems. The trace however is different in each case, showing a large slope consistent with the square-root singularity predicted by the theory for the pre-stressed case. The grazing point is indicated by the red line.
The determinant is constant for the linear system, but the trace oscillates as a function of frequency (Part of which can be seen in the figure). Upon grazing there is a rapid, but continuous transition to new trace and determinant values, the stability of which depends on the frequency just before impact. Experimental results are given by points, simulation by the continuous line, and the frequency at which grazing occurs is indicated by the red line.

5. Bifurcation Analysis

The response after grazing obviously depends upon the eigenvalues at the grazing point. The forcing amplitude does not affect the eigenvalues of the linear system, but it changes the frequency at which the amplitude becomes large enough for grazing to occur. Different bifurcation diagrams have been constructed for various forcing amplitudes to reveal the various bifurcation scenarios which are possible when contact is initiated at different frequencies (i.e. different trace). For frequencies less than around 4 Hz or greater than 8 Hz the grazing bifurcation causes a stable impacting period-1 to be born. In between these values the impacting period 1 is unstable and the response therefore goes somewhere else. Each experimental bifurcation diagram presented lies within this region, and the simulations allow a full explanation of the bifurcation scenario. Figure 4 (a), for an amplitude of 0.25 mm, is not obtainable from simulation, as the predicted response after grazing is a stable period-1 orbit. It is suspected that the noise induces a chaotic response caused by successive loops in the phase portrait being perturbed from impacting to nonimpacting solutions. For the excitation amplitude equal to 0.38 mm, a quite different bifurcation scenario was observed, which is shown in Fig. 4(b). The grazing occurs at \( f = 7.95 \) Hz, where a period-4 response was recorded. As the excitation frequency increases, period-3 oscillations with one impact per period are obtained for \( f \in [8.00; 8.45] \) Hz. Numerical simulation reveals the nature of the atypical transitions from period-1 to period-4 to period-3 as resulting from a complex interplay between coexisting orbits and grazing-induced bifurcations. Simulation shows that a period-4 orbit coexists with the non-impacting period-1 orbit before it makes the first contact with the switching surface and then with impacting period-1 orbit after grazing for a narrow range of frequency. At about \( f = 7.95 \) Hz, there is a boundary crisis, and the state moves to the period-4 orbit. At a larger value of the parameter, another coexisting period-3 orbit (with one impact) is numerically found to be stable. As the parameter is increased, there is another boundary crisis, and the state jumps from period-4 to the period-3 orbit. As the parameter is further increased, another grazing event occurs around \( f \sim 8.45 \) Hz, and the period-3 orbit loses stability. After this event the state jumps to a coexisting period-2 orbit with two impacts at \( f \sim 8.50 \) Hz. Following this, the unstable period-1 becomes stable and is reached via a reverse period doubling. Shown in 5(a) chaotic behaviour is observed close to grazing at \( f = 7.55 \) Hz, which changes to period-2 oscillations as the frequency is increased. The sample trajectory of this period-2 response is presented on the phase plane for \( f = 7.85 \) Hz. Next a window of chaotic behaviour is obtained and a typical discrete-time phase portrait is given for \( f = 8.35 \) Hz.
Simulation reveals that this is caused by a smooth period doubling rapidly followed by a grazing event which produces a small scale chaotic attractor. This undergoes a crisis and blows up to a larger attractor, which then undergoes another crisis resulting in an attractor similar to that seen in the additional window of Fig. 5(a). A reverse period doubling cascade then follows ending in a period-1 response with one impact per period. A typical phase space trajectory for this condition is shown for $f = 8.55$ Hz. For the excitation amplitude equal to 0.53 mm (see Fig. 5(b)), grazing occurs at $f = 7.15$ Hz which turns the period-1 orbit unstable. But the system does not collapse, as there is another coexisting periodic orbit at that parameter value, and so the orbit discontinuously jumps from a non-impacting period-1 orbit to a period-2 orbit with two impacts per period. Simulation shows that this period-2 orbit coexists with the nonimpacting period-1 orbit before the parameter value $f = 7.15$ Hz, and there is a different period-2 orbit with one impact that begins to coexist with it after this value. At $f \sim 7.45$ Hz, there is a boundary crisis, due to which the system behaviour jumps from the period-2 orbit with two impacts to the coexisting period-2 orbit with one impact. As the parameter is further increased, one of the loops of the period-2 orbit approaches the switching manifold, and at $f \sim 8.45$ Hz another grazing event occurs. This results in a period-2 orbit with two impacts. Following this, there is a smooth reverse period doubling bifurcation giving rise to a period-1 orbit with one impact per cycle. To demonstrate these transitions the sequence of phase plots and Poincaré maps for $f = 8.45$ Hz, $f = 8.50$ Hz, $f = 8.55$ Hz, and $f = 8.60$ Hz are given. Representative phase space trajectories of different period two oscillations and the period-1 orbit for $f > 8.6$ Hz are shown for $f = 7.45$ Hz, $f = 8.00$ Hz and $f = 9.10$ Hz respectively in additional windows in Fig. 5(b). Two other examples of possible bifurcation scenarios are presented in Fig. 6 (a) and (b) for excitation amplitudes equal to 0.66 mm and 0.70 mm respectively. As can be seen from these figures chaotic behaviour is observed at grazing frequencies $f = 6.50$ Hz and $f = 6.25$ Hz and the corresponding Poincaré maps are presented in additional windows. After a series of bifurcations in both cases the system settles down to an impacting period-1 response. However, the intermediate bifurcations are different. For the smaller excitation amplitude of 0.66 mm the chaotic regime is followed by period-3 oscillations (with three impacts per period) shown for $f = 6.70$ Hz. It transits to a coexisting period-2 orbit through crisis at $f \sim 6.75$ Hz, and then on to a different period-2 orbit at $f = 7.9$ Hz. Finally at $f = 8.50$ Hz we observe a bifurcation to a period-1 response with one impact per period. For the larger amplitude of 0.70 mm, the narrow range of chaotic behaviour is followed by impacting period-1 oscillations shown for $f = 6.65$ Hz, which bifurcates into
period-2 response at $f = 8.05$ Hz. After two grazing events, it bifurcates back to a period-1 response with one impact at $f = 8.55$ Hz.

![Bifurcation diagrams](image)

**Fig. 5** Bifurcation diagrams obtained for the mass displacement under varying frequency at $\omega_n = 9.38$ Hz; $c = 1.3$ kgs$^{-1}$; $\beta = 29$; $g = 1.26$ mm and excitation amplitude equal to (a) 0.44 mm and (b) 0.53 mm. Additional windows demonstrate the trajectories on the phase plane and obtained for (a) $f = 7.85$ Hz, 8.35 Hz (Poincaré map) and 8.55 Hz; and (b) $f = 7.15$ Hz, 8.00 Hz and 9.10 Hz respectively.

![Bifurcation diagrams](image)

**Fig. 6** Bifurcation diagrams obtained for the mass displacement under varying frequency at $\omega_n = 9.38$ Hz; $c = 1.3$ kgs$^{-1}$; $\beta = 29$; $g = 1.26$ mm and excitation amplitude equal to (a) 0.66 mm and (b) 0.70 mm. Additional windows demonstrate the trajectories on the phase plane and obtained for (a) $f = 6.50$ Hz (Poincaré map), 6.70 Hz, 6.95 Hz and 7.55 Hz; and (b) $f = 6.25$ Hz (Poincaré map), 6.65 Hz, 8.35 Hz and 8.60 Hz respectively.

**6. Conclusion**
Different bifurcation scenarios have been shown for a number of values of the excitation amplitude, with the excitation frequency as the bifurcation parameter. The most typical recorded scenario was when a non-impacting periodic orbit bifurcates into an impacting one via grazing mechanism. In some cases the resulting orbit is stable, but in most cases it loses stability through grazing. Following such an event, the evolution of the attractor is governed by a complex interplay between smooth and non-smooth bifurcations. In some cases the occurrence of coexisting attractors was manifested through a discontinuous transition from one orbit to another through boundary crisis, which seems to be closely related to the grazing events. A new experimental technique to study the stability of periodic orbits in impact oscillators was explained. The Jacobian matrix of the Poincaré map, obtained with the experimental technique, was compared with that obtained from theoretical analysis. A good correspondence between them was observed for different stiffness ratios and for a system with prestress. It was concluded that the determinant remains invariant for both cases while the trace exhibits a slope singularity at this transition when the force is continuous, and is continuous with discontinuous slope when the force is discontinuous. Overall the correspondence between experiment and theoretical predictions are excellent and lend weight to various modelling assumptions used.

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