ON THE SCALING CRITICAL REGULARITY OF THE YANG-MILLS-HIGGS AND
THE YANG-MILLS-DIRAC SYSTEM IN THE LORENZ GAUGE

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ABSTRACT. In this paper, we study the local well-posedness of the (1 + 3)-dimensional Yang-Mills-Higgs (YMH) and the Yang-Mills-Dirac (YMD) system in the Lorenz gauge. Since there is some bilinear term in (YMH), which is a lack of null structure, one may obtain the well-posedness at most the energy space. However, we attain the scaling critical regularity of (YMH) by imposing the extra weighted regularity in the angular variables. In (YMD), for the coupled system to persist in time, it is required to impose the angular regularity on the Dirac spinor as much as the Yang-Mills gauge potential and curvature. We can then prove the scaling critical regularity of (YMD) using angular regularity instead of the null structure of the spinor field. In this manner, we present an approach to attack the scaling critical regularity of (YMH) and (YMD) simultaneously. This result is an application of our recent study on the Yang-Mills system in the Lorenz gauge [11].

1. INTRODUCTION

The Yang-Mills system and its coupled equations are interesting topics in that it is the non-abelian extension of the Maxwell-Dirac system, and it is a much more complicated system than the Dirac-Klein-Gordon system, and it is on the higher-dimensional space compared to the Chern-Simons gauge field.

This section is organised as follows. We start with the Yang-Mills system. Then we introduce its coupled equations; Higgs and Dirac fields. We note that two systems have gauge invariance property and then impose the Lorenz gauge to derive nonlinear wave equations. Then we state the main theorem on the scaling critical regularity of (YMH) and (YMD). Finally, we review the previous works on the related systems and then discuss the main scheme of the proof of our results.

1.1. Yang-Mills system. Let \( G \) be a compact Lie group and \( \mathfrak{g} \) its Lie algebra. We shall assume \( G = SU(n, \mathbb{C}) \), the group of unitary matrices of determinant one for simplicity. Then \( \mathfrak{g} = \mathfrak{su}(n, \mathbb{C}) \), the algebra of trace-free skew-Hermitian matrices. We define its inner product by

\[
\langle X, Y \rangle = -\text{Tr}(X \cdot Y^*),
\]

where Tr denotes the trace of a matrix and \( * \) is the complex conjugate. The matrix commutator is given by

\[
[X, Y] = X \cdot Y - Y \cdot X.
\]

For a \( \mathfrak{g} \)-valued 1-form \( A \) on the Minkowski space \( \mathbb{R}^{1+3} \) with the signature \((-,-,+,+))\), we denote by \( F = F[A] \) the associated curvature \( F = dA + [A, A] \). More explicitly, given 1-form \( A_\mu : \mathbb{R}^{1+3} \to \mathfrak{g} \), we define \( F_{\mu\nu} \) by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].
\]
We say that $A$ solves the Yang-Mills equation if it is a critical point of the Yang-Mills action \[17\]:

$$
\mathcal{L}_{YM} = \int_{\mathbb{R}^{1+3}} \langle F[A]^{\mu\nu}, F[A]_{\mu\nu} \rangle \, dx dt.
$$

Then the Euler-Lagrange equation, which we call the Yang-Mills equation, is of the form

\[1.2\]

$$
\mathcal{D}^\mu F_{\mu\nu} = 0,
$$

where $\mathcal{D}^\mu = \partial^\mu + [A^\mu, \cdot]$ denotes the covariant derivative associated with $A$. Here we note that $A$ is not necessarily unique. Indeed, we consider the following gauge transformation:

\[1.3\]

$$
A_\mu \to A'_\mu = U A_\mu U^{-1} - (\partial_\mu U) U^{-1},
$$

for sufficiently smooth function $U : \mathbb{R}^{1+3} \to G$. Let us denote $F' = F[A']$ and $\mathcal{D}'_\mu = \partial_\mu + [A'_\mu, \cdot]$. Then we see that

\[1.4\]

$$
F' = UF U^{-1}, \quad \mathcal{D}'_\mu F' = U(\mathcal{D}_\mu F)U^{-1}.
$$

Now this implies that

\[1.5\]

$$
D'^\mu F'^{\mu\nu} = \partial'^\mu F'^{\mu\nu} + [A'^\mu, F'^{\mu\nu}] = 0,
$$

which concludes that \[1.2\] is invariant under the gauge transformation \[1.3\]. We refer the readers to \[17\] for more geometric information of the Yang-Mills system.

1.2. **Yang-Mills-Higgs system: generalisation of Yang-Mills system I.** Now we are ready to introduce the Yang-Mills-Higgs system. It is the Yang-Mills equation coupled with the Higgs field

\[1.6\]

$$
\phi : \mathbb{R}^{1+3} \to g.
$$

To be precise, the Yang-Mills-Higgs action is the following integral \[20\]:

\[1.7\]

$$
\mathcal{L}_{YMH} = -\int_{\mathbb{R}^{1+3}} \left( \frac{1}{4} \langle F[A]_{\mu\nu}, F[A]^{\mu\nu} \rangle + \frac{1}{2} \langle \mathcal{D}^\mu \phi, \mathcal{D}_\mu \phi \rangle + V(\phi) \right) \, dx dt,
$$

where $V(\phi)$ is the Higgs potential. In this paper, we assume $V(\phi) = 0$ for simplicity.

Then the Euler-Lagrange equation gives the covariant form of the Yang-Mills-Higgs system:

\[1.8\]

$$
\mathcal{D}^\mu F_{\mu\nu} = -[\mathcal{D}_\nu \phi, \phi],
$$

$$
\mathcal{D}^\mu \mathcal{D}_\mu \phi = 0.
$$

We also note that \[1.8\] is invariant under the following gauge transformation: $\phi \to \phi' = U \phi U^{-1}$. In fact, it gives

$$
F' = UF U^{-1}, \quad \mathcal{D}'_\mu F' = U(\mathcal{D}_\mu F)U^{-1}, \quad \mathcal{D}'_\mu \phi' = U(\mathcal{D}_\mu \phi)U^{-1}.
$$

and hence we have

$$
\mathcal{D}'^\mu F'_{\mu\nu} = -[\mathcal{D}'_\nu \phi', \phi'], \quad \mathcal{D}'^\mu \mathcal{D}'_\mu \phi' = 0.
$$

Thus we conclude that \[1.8\] is invariant under the gauge transformation.
1.3. Yang-Mills-Dirac system: generalisation of Yang-Mills system II. Now we consider the Yang-Mills-Dirac system. It is the Yang-Mills equation coupled with the Dirac spinor field

\[ (1.14) \]

\[ \psi : \mathbb{R}^{1+3} \to \mathbb{C}^4. \]

More explicitly, a locally $SU(n, \mathbb{C})$ invariant Yang-Mills-Dirac action is the following integral \[22\]:

\[ (1.10) \]

\[ \mathcal{L}_{YMD} = -\int_{\mathbb{R}^{1+3}} \left( \frac{1}{4} (F[A]_{\mu\nu}, F[A]^{\mu\nu}) - \sum_{i,j=1}^{n} \overline{\psi}(i\delta_{ij}\gamma^\mu \partial^\mu + A^\mu_{ij} T^\mu_{ij} \gamma^\mu) \psi_j \right) dxdt, \]

where $\overline{\psi} = \psi^\dagger \gamma^0$ and $\psi^\dagger$ is the complex conjugate transpose of $\psi$. Here we write $A^\mu = A^\mu_{ij} T^\mu_{ij}$, where $T^a_{ij}$ are generators of $\mathfrak{su}(n, \mathbb{C})$. However, in view of mathematical analysis, the generators $T^a_{ij}$ play no crucial roles. Hence we assume $n = 1$ and then omit $T^a_{ij}$, for brevity. We leave its discussion to Appendix.

Then the Euler-Lagrange equation gives the covariant form of the Yang-Mills-Dirac system:

\[ (1.11) \]

\[ D^\mu F_{\mu\nu} = -\overline{\psi} \gamma_\nu \psi, \]

\[ (1.12) \]

\[ i\gamma^\mu \partial^\mu \psi = -A^\mu \gamma^\mu \psi. \]

Similarly, an easy computation shows that (1.11) and (1.12) are invariant under the gauge transformation $\psi \to \psi' = U \psi$. (Use $U^\dagger = U^{-1}$ in $G = SU(n, \mathbb{C})$.)

1.4. Yang-Mills-Higgs system as nonlinear wave equations. In the previous two sections, we see that the Yang-Mills-Higgs and the Yang-Mills-Dirac system have freedom of gauge choice. Now we are willing to derive the Yang-Mills-Higgs system as nonlinear wave equations. We first expand (1.8) to get

\[ \square A_\nu = \partial_\lambda \partial^\lambda A_\nu - [\partial^\mu A_\mu, A_\nu] - 2[A^\mu, \partial_\mu A_\nu] + [A^\mu, \partial_\nu A_\mu] + [\phi, \partial_\nu \phi] - [A^\mu, [A_\mu, A_\nu]] + [\phi, [A_\mu, \phi]], \]

\[ \square \phi = -[\partial^\mu A_\mu, \phi] - 2[A_\mu, \partial^\mu \phi] - [A^\mu, [A_\mu, \phi]]. \]

We impose the Lorenz gauge: $\partial^\mu A_\mu = 0$. Then we have

\[ (1.13) \]

\[ \square A_\nu = -2[A^\mu, \partial_\mu A_\nu] + [A^\mu, \partial_\nu A_\mu] + [\phi, \partial_\nu \phi] - 2[A^\mu, [A_\mu, A_\nu]] + [\phi, [A_\mu, \phi]], \]

and

\[ (1.14) \]

\[ \square \phi = -2[A_\mu, \partial^\mu \phi] - [A^\mu, [A_\mu, \phi]]. \]

We also have

\[ (1.15) \]

\[ \square F_{\mu\nu} = -[A^\lambda, F_{\mu\nu}] - \partial^\lambda[A_\lambda, F_{\mu\nu}] - [A^\lambda, [A_\lambda, F_{\mu\nu}]] - 2[F^\lambda_{\mu}, F_{\nu\lambda}] - 2[D_\mu \phi, D_\nu \phi] - [\phi, [F_{\mu\nu}, \phi]], \]

for any choice of gauge condition. (See \[24\] for the derivation of (1.15).) Then we expand the second, fourth, and fifth terms in (1.15) and impose the Lorenz gauge condition to get

\[ (1.16) \]

\[ \square F_{\mu\nu} = -2[A^\lambda, \partial_\lambda F_{\mu\nu}] + 2[\partial_\nu A^\lambda, \partial_\lambda A_\mu] - 2[\partial_\mu A^\lambda, \partial_\lambda A_\nu] + 2[\partial^\lambda A_\mu, \partial_\lambda A_\nu] \\ + 2[\partial_\mu A^\lambda, \partial_\lambda A_\nu] - 2[\partial_\mu \phi, \partial_\lambda \phi] - [A^\lambda, [A_\lambda, F_{\mu\nu}]] + 2[F^\lambda_{\mu}, [A^\lambda, A_\nu]] \\ - 2[F_{\nu\lambda}, [A^\lambda, A_\mu]] - [\partial_\lambda \phi, [A_\nu, \phi]] + 2[\partial_\nu \phi, [A_\mu, \phi]] - [\phi, [F_{\mu\nu}, \phi]] \\ - 2[[A^\lambda, A_\mu], [A_\lambda, A_\nu]] - 2[[A_\mu, \phi], [A_\nu, \phi]]. \]
1.5. Yang-Mills-Dirac system as nonlinear wave equations. Now we derive the Yang-Mills-Dirac system as nonlinear wave equations. We expand (1.11) to get

\[ \Box A_\nu = \partial_\nu \partial^\mu A_\mu - \partial^\rho A_\mu, A_\rho - 2[A_\mu, \partial_\nu A_\rho] + [A_\mu, \partial_\nu A_\rho] - \bar{\psi} \gamma_\nu \psi - [A_\mu, [A_\mu, A_\rho]]. \]

We impose the Lorenz gauge. Then we have

\[ \Box A_\nu = -2[A_\mu, \partial_\nu A_\rho] + [A_\mu, \partial_\nu A_\rho] - \bar{\psi} \gamma_\nu \psi - 2[A_\mu, [A_\mu, A_\rho]], \]

(1.17)

We also have (again, we refer to [24] for the derivation of (1.18))

\[ \Box F_{\mu\nu} = -[A^\lambda, F_{\lambda\mu\nu}] - \partial^\lambda [A_\lambda, F_{\mu\nu}] - [A^\lambda, [A_\lambda, F_{\mu\nu}]] - 2[F_{\lambda\mu}, F_{\nu\lambda}] \]

(1.18)

for any choice of gauge condition. Then we expand the first, second, and fourth terms in (1.18) and then impose the Lorenz gauge condition to get

\[ \Box F_{\mu\nu} = -2[A^\lambda, \partial_\mu F_{\lambda\nu}] + 2[\partial_\mu A^\lambda, \partial_\lambda A_\nu] - 2[\partial_\mu A^\lambda, \partial_\lambda A_\nu] + 2[\partial_\lambda A^\lambda, \partial_\lambda A_\nu] \]

\[ + 2[\partial_\mu A^\lambda, \partial_\nu A_\lambda] - [A^\lambda, [A_\lambda, F_{\mu\nu}]] - \partial_\mu (\bar{\psi} \gamma_\nu \psi) + \partial_\nu (\bar{\psi} \gamma_\mu \psi) \]

\[ + 2[F_{\lambda\nu}, [A_\lambda, A_\nu]] - 2[F_{\lambda\nu}, [A^\lambda, A_\mu]] - A_\mu (\bar{\psi} \gamma_\nu \psi) + A_\nu (\bar{\psi} \gamma_\mu \psi) \]

\[ - 2[[A^\lambda, A_\lambda], [A_\mu, A_\nu]]. \]

(1.19)

1.6. Main results. In this section, we state our result on the well-posedness of the Yang-Mills-Higgs and the Yang-Mills-Dirac system in the Lorenz gauge. We begin with initial data:

\[ A(0) = a \in \langle \Omega \rangle^{-\sigma} B^2_{2,1}, \quad \partial_t A(0) = \dot{a} \in \langle \Omega \rangle^{-\sigma} B^{-\frac{3}{2}}_{2,1}, \]

\[ \psi(0) = \psi_0 \in \langle \Omega \rangle^{-\sigma} B^0_{2,1}, \]

\[ \phi(0) = \phi_0 \in \langle \Omega \rangle^{-\sigma} B^2_{2,1}, \quad \partial_t \phi(0) = \dot{\phi} \in \langle \Omega \rangle^{-\sigma} B^{-\frac{3}{2}}_{2,1}, \]

\[ F(0) = f \in \langle \Omega \rangle^{-\sigma} B^2_{2,1}, \quad \partial_t F(0) = \dot{f} \in \langle \Omega \rangle^{-\sigma} B^{-\frac{3}{2}}_{2,1}, \]

(1.20)

where \( B^s_{2,1} \) is the usual inhomogeneous Besov space whose norm is defined by \( \| f \|_{B^s_{2,1}} = \sum_{n \geq 1} N^s \| P_{|\xi| \approx N} f \|_{L^2} \), and \( P_{|\xi| \approx N} \) is the Littlewood-Paley projection on the set \( \{ \xi \in \mathbb{R}^3 : |\xi| \approx N \} \). Here, we introduce the infinitesimal generators of the rotations on Euclidean space

\[ \Omega_{i,j} = x_i \partial_j - x_j \partial_i, \]

and the spherical Laplacian is defined by \( \Delta_{S^2} = \sum_{i < j} \Omega_{i,j}^2 \). We write \( \langle \Omega \rangle^s = (1 - \Delta_{S^2})^{-\frac{s}{2}} \).

Initial conditions for the Yang-Mills-Higgs. We note that the initial data \( f \) for \( F \) is completely determined by \( (a, \dot{a}) \) and \( (\phi_0, \phi_1) \). Indeed, we have

\[ \left\{ \begin{array}{l}
 f_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j], \\
 f_{0i} = \dot{a}_i - \partial_t a_i + [a_0, a_i], \\
 \dot{f}_{ij} = \partial_i \dot{a}_j - \partial_j \dot{a}_i + [\dot{a}_i, a_j] + [a_i, \dot{a}_j], \\
 \dot{f}_{0i} = \partial^i f_{ji} + [a^i, f_{ai}] + [\partial_t \phi_0, \phi_0] + [[a_i, \phi_0], \phi_0].
 \end{array} \right. \]

(1.21)

We also note the following constraints:

\[ \left\{ \begin{array}{l}
 \dot{a}_0 = \partial^i a_i, \\
 \partial^i f_{i0} + [a^i, f_{i0}] = -[\phi, \phi_0] - [[a_0, \phi_0], \phi_0].
 \end{array} \right. \]

(1.22)
Initial conditions for the Yang-Mills-Dirac. Similarly, we have the following initial condition for the Yang-Mills-Dirac system in the Lorenz gauge:

\[
\begin{align*}
  f_{ij} &= \partial_i a_j - \partial_j a_i + [a_i, a_j], \\
  f_{0i} &= \dot{a}_i - \partial_i a_0 + [a_0, a_i], \\
  \dot{f}_{ij} &= \partial_i \dot{a}_j - \partial_j \dot{a}_i + [\dot{a}_i, a_j] + [a_i, \dot{a}_j], \\
  \dot{f}_{0i} &= \partial^i f_{ji} + [a^\alpha, f_{\alpha i}] + \bar{\psi}_0 \gamma_i \psi_0,
\end{align*}
\]

and

\[
\begin{align*}
  \dot{a}_0 &= \partial^j a_i, \\
  \partial^i f_{j0} + [a^i, f_{0j}] &= -\bar{\psi}_0 \gamma_0 \psi_0.
\end{align*}
\]

We arrive at our main results:

**Theorem 1.1** (Well-posedness of Yang-Mills-Higgs system in the Lorenz gauge). Let \( \sigma \geq 1 \). Suppose that given initial data \((a, \dot{a}, \phi_0, \varphi_1, f, \dot{f})\) satisfy (1.20) and (1.24). Then there exists local existence time \( T \) which depends on the norm of initial data such that there exist solutions

\[
\begin{align*}
  A &\in C([-T, T]; (\Omega)^{-\sigma} B_{2,1}^{3/2}), \\
  \phi &\in C([-T, T]; (\Omega)^{-\sigma} B_{2,1}^{3/2}), \\
  F &\in C([-T, T]; (\Omega)^{-\sigma} B_{2,1}^{3/2}),
\end{align*}
\]

of (1.13), (1.14), and (1.15), respectively. The solutions \( A, \phi, \) and \( F \) have the regularity

\[
\phi \pm \frac{1}{i|\nabla|} \partial_0 \phi, \quad A \pm \frac{1}{i|\nabla|} \partial_0 A \in (\Omega)^{-\sigma} B_{\pm 1}^{3/2} (S_T), \quad F \pm \frac{1}{i|\nabla|} \partial_0 F \in (\Omega)^{-\sigma} B_{\pm 1}^{3/2} (S_T),
\]

where \( S_T = (-T, T) \times \mathbb{R}^3 \) is the restricted domain, and \( B_{\pm}^{s,b} \) is the Besov type \( X_{s,b}^b \) space.

**Theorem 1.2** (Well-posedness of Yang-Mills-Dirac system in the Lorenz gauge). Let \( \sigma \geq 1 \). Suppose that given initial data \((a, \dot{a}, \psi_0, \varphi_1, f, \dot{f})\) satisfy (1.20) and (1.24). Then there exists local existence time \( T \) which depends on the norm of initial data such that there exist solutions

\[
\begin{align*}
  A &\in C([-T, T]; (\Omega)^{-\sigma} B_{2,1}^{3/2}), \\
  \psi &\in C([-T, T]; (\Omega)^{-\sigma} B_{2,1}^{3/2}), \\
  F &\in C([-T, T]; (\Omega)^{-\sigma} B_{2,1}^{3/2}),
\end{align*}
\]

of (1.17), (1.18), and (1.19), respectively. The solutions \( A, \psi, \) and \( F \) have the regularity

\[
\Pi_{\pm} \psi \in (\Omega)^{-\sigma} B_{\pm 1}^{3/2} (S_T), \quad A \pm \frac{1}{i|\nabla|} \partial_0 A \in (\Omega)^{-\sigma} B_{\pm 1}^{3/2} (S_T), \quad F \pm \frac{1}{i|\nabla|} \partial_0 F \in (\Omega)^{-\sigma} B_{\pm 1}^{3/2} (S_T),
\]

where \( \Pi_{\pm} := \Pi_{\pm} (-i\nabla) \) is the Dirac projection operator.

Before discussion on the proof of Theorem 1.1 and Theorem 1.2 we shall mention a few selected works on the related systems. We first review the Maxwell-Dirac system, which reads

\[
-i \gamma^\mu \partial_\mu \psi + M \psi = A_\mu \gamma^\mu \psi,
\]

under the Lorenz gauge condition. Bouaviesa [8] studied the local existence of the system (1.20) and then in the Lorenz gauge, the authors of [8] revealed the system null structure, which depends on the structure of the whole system and proved the almost optimal well-posedness. The Dirac-Klein-Gordon system is interestingly
relevant to the Yang-Mills-Dirac system since it is also a nonlinear wave equation coupled with the Dirac field, which presents
\begin{equation}
-\imath \gamma^\mu \partial_\mu \psi + M \psi = \phi \psi,
\end{equation}
\begin{equation}
\Box \phi + m^2 \phi = \overline{\psi} \psi,
\end{equation}
for massive case. We refer the readers to [7, 2, 28, 4, 5] for this well-studied system (1.30). We also mention the Chern-Simons gauge field, which concerns the $(1+2)$ dimensional system. In particular, the Chern-Simons-Dirac system in the Lorenz gauge is given by
\begin{equation}
\imath \gamma^\mu \partial_\mu \psi = m \psi - A_\mu \gamma^\mu \psi,
\end{equation}
\begin{equation}
\Box A_\nu = -2 \partial^\mu (\epsilon_{\mu \nu \lambda} \overline{\psi} \gamma^\lambda \psi).
\end{equation}
For study on the well-posedness of (1.31), see [12, 18, 21] and references therein.

The Yang-Mills-Higgs and the Yang-Mills-Dirac system can be viewed as a generalisation of (1.29), (1.30), (1.31). The authors of [6] studied the existence of global solutions of the Yang-Mills, Higgs and Dirac field equations. Eardley and Moncrief [9], [10] studied local and global well-posedness of Yang-Mills-Higgs system in the temporal gauge ($A_0 = 0$) for initial data in $H^s \times H^{s-1}$ for $s \geq 2$. Then Keel [13] proved the global well-posedness of Yang-Mills-Higgs system in the Coulomb gauge ($\partial^\mu A_\mu = 0$) for finite energy initial data. We refer to [25, 14, 15, 16, 24, 27, 19, 11] for the study on the well-posedness of the Yang-Mills system in several gauges.

Recently, Tesfahun [26] was concerned with the Yang-Mills-Higgs system for initial data in energy space $H^1 \times L^2$ under the Lorenz gauge condition. However, the well-posedness below energy space is not known. Furthermore, the study on the well-posedness of the Yang-Mills-Dirac equations is hardly found.

To the best of our knowledge, Theorem 1.1 and Theorem 1.2 are the first results on the local well-posedness of the $(1+3)$-dimensional Yang-Mills-Higgs and the Yang-Mills-Dirac system in the Lorenz gauge for initial data below energy space.

1.7. **Strategy of the proof.** Coupling with the Higgs and Dirac field gives significantly different structures. In fact, compared to the Higgs fields, the Dirac spinor itself has the null structure [1]. However, our approach does not exploit its null structure and presents a scheme which deals with two systems simultaneously.

One should note that there is a bilinear form which has no null structure in the Yang-Mills-Higgs equations, (See the proof of (4.1).)
\begin{equation}
||[\phi_{\pm 1}, \partial_\nu \phi_{\pm 2}]||_{B^{-1,1}_{2,0,T}} \lesssim ||\phi_{\pm 1}||_{B^{0,1}_{2,1,T}} ||\phi_{\pm 2}||_{B^{0,1}_{2,1,T}}.
\end{equation}
Then we can prove the local well-posedness only for $H^s$, $s \geq 1$, using (2.6), (2.7). (See also Remark 4.1 and 26.) We attack this problem by imposing the extra weighted regularity in the angular variables. Indeed, we truly gain some regularity via spherical harmonic expansions:
\begin{equation}
||e^{\imath \nu i D_x} P^{K}_{N,L} H f ||_{L^2([R^{1+3}])} \lesssim N^{-\frac{L}{2}} L^\frac{1}{2} ||P^{K}_{N,L} H f ||_{L^2([R^{1+3}]),
\end{equation}
where $N$ is a spatial frequency, $L$ is a modulation, and $H_I$ is a spherical harmonic projection. (See Lemma 2.2) In addition, to avoid the summation problem of the angular regularity, one should obtain only $l_{\min}^{12}$ and hence consider the following summation:
\begin{equation}
\sum_{l_0, l_1, l_2} l_0^{10} l_1^{12} ||H_{l_0} u_1 || ||H_{l_2} u_2 || \lesssim ||(\Omega)^{\sigma} u_1 || ||(\Omega)^{\sigma} u_2 ||.
\end{equation}
Thus in the proof of bilinear estimates, we are concerned with the two cases: $l_1 \ll l_2$ and $l_2 \ll l_1$.

When we consider the higher-order terms, the problem becomes easier; we obtain even better estimates. We shall apply Hlder’s inequality and Bernstein’s inequality as a first step. Indeed, we have

$$
\|A_{\pm_1} \phi_{\pm_2} \partial \phi_{\pm_0}\|_{L^{2,0}_{t,x}} \lesssim \sum_{N,L,l} N_1^\frac{1}{2} L_0^{-\frac{1}{2}} \|P_{K_{N_0,l_0}} (A_{\pm_1} \phi_{N_1,L_1} \phi_{\pm_2} \phi_{N_2,L_2})\| (N_1^\frac{3}{2} L_0^\frac{3}{2} \|\phi_{N_0,L_0}\|),
$$

and hence it is no harm to get $l_1 l_2$ using Lemma 2.2 twice. In other words, we gain the regularity as $(N_1 N_2)^{-\frac{1}{2}}$, which allows to attain the sharp estimates. (See also Remark 5.1.) In our previous paper, we observe that even if we do not lose any regularity by duality, its treatment is much more cumbersome. We refer the readers to Appendix in [11].

As we mentioned early, we do not use the null structure of the Dirac spinor. Instead, we apply the angular regularity to attain the critical regularity for spinor fields. One may obtain the same result by using the null structure; however, the direct use of Lemma 2.2 gives simple proof. See the proof of (4.2) and Remark 4.2.

In summary, we would like to highlight the crucial role of the extra weighted regularity in the angular variables. Even if bilinear terms have no null structure, we can obtain the scaling critical regularity by recalling briefly the analysis on the sphere $S^2$. We state the key estimates in Section 4, which is a crucial part of the proof of our main results. In the end of Section 4, we give more description on the organisation of Section 5, which are devoted to the proof of the estimates of bilinear forms and higher-order terms, respectively. Finally, Appendix presents to justify our notation abuse in the Yang-Mills-Dirac system.

**Organisation.** This paper is organised as follows. In Section 2, we give some preliminaries containing Dirac gamma matrices, the decomposition of d’Alembertian and the Besov type $X^{s,b}$ space and recall briefly the analysis on the sphere $S^2$. We also introduce the sharp bilinear estimates of wave type localised in a thickened null cone. We state the key estimates in Section 4, which is a crucial part of the proof of our main results. In the end of Section 4, we give more description on the organisation of Section 5, which are devoted to the proof of the estimates of bilinear forms and higher-order terms, respectively. Finally, Appendix presents to justify our notation abuse in the Yang-Mills-Dirac system.

**Notations.** Since we only use $L^2_{t,x}$ norm, by $\|F\|$ we abbreviate $\|F\|_{L^2_{t,x}} := \sum_a \|F_a\|_{L^2_{t,x}}$ for $F = F_a T^a$, where $\{T^a\}_{a=1}^{n^2-1}$ is the set of infinitesimal generators of given Lie algebra $\mathfrak{g} = \mathfrak{su}(n,\mathbb{C})$. As usual different positive constants independent on dyadic numbers such as $N$ and $L$ are denoted by the same letter $C$, if not specified. $A \lesssim B$ and $A \gtrsim B$ means that $A \leq CB$ and $A \geq C^{-1}B$, respectively for some $C > 0$. $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

The spatial Fourier transform and space-time Fourier transform on $\mathbb{R}^3$ and $\mathbb{R}^{1+3}$ are defined by

$$
\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) \, dx, \quad \hat{u}(X) = \int_{\mathbb{R}^{1+3}} e^{-i(\tau + x \cdot \xi) t} u(t, x) \, dt \, dx,
$$

where $\tau \in \mathbb{R}, \xi \in \mathbb{R}^3$, and $X = (\tau, \xi) \in \mathbb{R}^{1+3}$. Also we denote $\mathcal{F}(u) = \hat{u}$. Then we define space-time Fourier projection operator $P_E$ by $P_E u(\tau, \xi) = \chi_E \hat{u}(\tau, \xi)$, for $E \subset \mathbb{R}^{1+3}$. We define spatial Fourier projection operator, similarly. For example, $P_{|\xi| \approx N}$ is the Littlewood-Paley projection on $|\xi| \approx N$.

Since we prefer to use the differential operator $|\nabla|$ rather than $-i\nabla$, for the sake of simplicity, we put $D := |\nabla|$ whose symbol is $|\xi|$.

For brevity, we denote the maximum, median, and minimum of $N_0$, $N_1$, $N_2$ by

$$
N_{\text{max}}^{012} = \max(N_0, N_1, N_2), \quad N_{\text{med}}^{012} = \text{med}(N_0, N_1, N_2), \quad N_{\text{min}}^{012} = \min(N_0, N_1, N_2).
$$
2. Preliminaries

2.1. Dirac projection operators. Let \( \eta_{\mu\nu} \) be the Minkowski metric on \( \mathbb{R}^{1+3} \) given by

\[
\eta = \text{diag}(-1, +1, +1, +1).
\]

We consider the Dirac matrices \( \gamma^\mu \in \mathbb{C}^{4 \times 4} \) given by

\[
\gamma^0 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \gamma^i = \begin{bmatrix}
0 & \sigma^i \\
-\sigma^i & 0
\end{bmatrix},
\]

with the Pauli matrices

\[
\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Now we introduce the Dirac projection operators

\[
\Pi_{\pm}(\xi) = \frac{1}{2} \begin{bmatrix} I_{4 \times 4} \pm \frac{1}{|\xi|} (\xi \gamma^0 \gamma^i) \end{bmatrix}
\]

and the associated Fourier multiplier \( \widehat{\Pi_{\pm}} f(\xi) = \Pi_{\pm}(\xi) \hat{f}(\xi) \). An easy computation shows that \( \Pi_+ \Pi_- = \Pi_- \Pi_+ = 0 \) and \( \Pi_\pm \Pi_\pm = \Pi_\pm \). Also, for any spinor field \( \psi \), we write \( \psi_{\pm} := \Pi_\pm \psi \) with \( \psi = \psi_+ + \psi_- \) and

\[
\Pi_{\pm}(i\gamma^0 \gamma^\mu \partial_\mu \psi) = -(-i\partial_t \pm D)\psi_{\pm}.
\]

Thus we can rewrite (1.12) by

\[
(-i\partial_t \pm D)\psi_{\pm} = -\Pi_{\pm}(A^\mu \gamma^0 \eta_{\mu\nu} \psi).
\]

2.2. Decomposition of d’Alembertian. In this section, we introduce the standard decomposition of d’Alembertian to rewrite (1.13) and (1.16) as first order system. We use the following transform:

\[
(A, \partial_t A, \phi, \partial_t \phi, F, \partial_t F) \rightarrow (A_+, A_-, \phi_+, \phi_-, F_+, F_-),
\]

where

\[
A_{\pm} = \frac{1}{2} \left( A \pm \frac{1}{iD} \partial_t A \right), \quad \phi_{\pm} = \frac{1}{2} \left( \phi \pm \frac{1}{iD} \partial_t \phi \right), \quad F_{\pm} = \frac{1}{2} \left( F \pm \frac{1}{iD} \partial_t F \right).
\]

Recall the system

\[
\begin{align*}
\Box A_\nu &= \mathbf{A}_\nu (A, \partial_t A, F, \partial_t F, \phi, \partial_t \phi), \\
\Box \phi &= \mathbf{\Phi} (A, \partial_t A, \phi, \partial_t \phi), \\
\Box F_{\mu\nu} &= \mathbf{\Gamma}_{\mu\nu} (A, \partial_t A, F, \partial_t F, \phi, \partial_t \phi),
\end{align*}
\]

where

\[
\mathbf{A}_{\nu} (A, \partial_t A, F, \partial_t F, \phi, \partial_t \phi) = -2[A^\mu, \partial_\mu A_\nu] + [A^\mu, \partial_\nu A_\mu] + [\phi, \partial_\nu \phi] - 2[A^\mu, [A_\mu, A_\nu]] + [\phi, [A_\nu, \phi]],
\]

\[
\mathbf{\Gamma}_{\mu\nu} (A, \partial_t A, F, \partial_t F, \phi, \partial_t \phi) = -2[A^\lambda, \partial_\lambda F_{\mu\nu}] + 2[\partial_\nu A^\lambda, \partial_\lambda A_\mu] - 2[\partial_\mu A^\lambda, \partial_\lambda A_\nu] + 2[\partial_\lambda A_\mu, \partial_\lambda A_\nu] + 2[\partial_\nu A_\lambda, [A^\lambda, A_\mu]] + 2[F_{\mu\nu}, [A^\lambda, A_\nu]] - 2[F_{\lambda\nu}, [A^\lambda, A_\mu]] - 2[[A^\lambda, A_\nu], [A_\lambda, A_\mu]] - 2[[A_\mu, \phi], [A_\nu, \phi]] + 2[\partial_\nu \phi, [A_\mu, \phi]] - 2[\partial_\mu \phi, [A_\nu, \phi]] - 2[[A^\mu, \phi], [A_\nu, \phi]] - 2[[A_\nu, \phi], [A_\mu, \phi]],
\]

\[
\mathbf{\Phi} (A, \partial_t A, \phi, \partial_t \phi) = -2[A_\mu, \partial^\mu \phi] - [A^\mu, [A_\mu, \phi]].
\]
Then our system (2.2) is rewritten as

\[
\begin{cases}
(i\partial_0 + D)A_{\nu,\pm} = \mp \frac{1}{2D} \Lambda_{\nu}(A_+, A_-, F_+, F_-, \phi_+, \phi_-), \\
(i\partial_0 + D)\phi_{\pm} = \mp \frac{1}{2D} \Phi(A_+, A_-, \phi_+, \phi_-), \\
(i\partial_0 + D)F_{\mu\nu,\pm} = \mp \frac{1}{2D} \Gamma_{\mu\nu}(A_+, A_-, F_+, F_-, \phi_+, \phi_-),
\end{cases}
\]

(2.3)

where

\[
\begin{align*}
\Lambda_{\nu}(A_+, A_-, F_+, F_-, \phi_+, \phi_-) &= \Lambda_{\nu}(A, \partial_0 A, F, \partial_0 F, \phi, \partial_0 \phi), \\
\Phi(A_+, A_-, \phi_+, \phi_-) &= \Phi(A, \partial_0 A, \phi, \partial_0 \phi), \\
\Gamma_{\mu\nu}(A_+, A_-, F_+, F_-, \phi_+, \phi_-) &= \Gamma_{\mu\nu}(A, \partial_0 A, F, \partial_0 F, \phi, \partial_0 \phi).
\end{align*}
\]

Also the initial data transforms to

\[
A_{\pm}(0) = a_{\pm} := \frac{1}{2} \left( a \pm \frac{1}{iD} \hat{a} \right) \in \langle \Omega \rangle^{-\sigma} B^\frac{1}{2},
\]

\[
\phi_{\pm}(0) = \phi_{0,\pm} := \frac{1}{2} \left( \phi_0 \pm \frac{1}{iD} \phi_1 \right) \in \langle \Omega \rangle^{-\sigma} B^\frac{1}{2},
\]

\[
F_{\pm}(0) = f_{\pm} := \frac{1}{2} \left( f \pm \frac{1}{iD} f \right) \in \langle \Omega \rangle^{-\sigma} B^\frac{1}{2}.
\]

Similarly, we can rewrite the nonlinear wave system of the Yang-Mills-Dirac equations as follows:

\[
\begin{cases}
(i\partial_0 + D)A_{\nu,\pm} = \mp \frac{1}{2D} \Lambda_{\nu}(A_+, A_-, F_+, F_-, \psi_+, \psi_-), \\
(i\partial_0 + D)\psi_{\pm} = \mp \frac{1}{2D} \Phi(A_+, A_-, \psi_+, \psi_-), \\
(i\partial_0 + D)F_{\mu\nu,\pm} = \mp \frac{1}{2D} \Gamma_{\mu\nu}(A_+, A_-, F_+, F_-, \phi_+, \phi_-).
\end{cases}
\]

(2.4)

2.3. Besov type $X^{s,b}$ spaces. We adapt the Besov type $X^{s,b}$ space. In this section, we also introduce linear estimate of the adapted function space.

We first define the function space as follows. For dyadic numbers $N \geq 1$ and $L$, we define the set

\[
K^\pm_{N,L} = \{ (\tau, \xi) \in \mathbb{R}^{1+3} : |\xi| \approx N, \quad |\tau \pm |\xi|| \approx L \},
\]

which is a thickened null cone. Then the Besov type $X^{s,b}$ space is given by

\[
B_{s,b} = \{ u \in L^2 : \| u \|_{B^s_{b}} = \sum_{N,L} \| P_{K^\pm_{N,L}} u \| < \infty \}.
\]

Note that we have assumed $N \geq 1$ to avoid the singularity at the origin in the spatial Fourier side because of $D^{-1}$ term in the right-hand-side of (2.3), (2.4).

Since we are only concerned with local time existence $T \leq 1$ throughout this paper, it is convenient to utilize our function space in the local time setting. Hence we introduce the restriction space. The time-slab which is the subset of $\mathbb{R}^{1+3}$ is given by

\[
S_T = (-T, T) \times \mathbb{R}^3.
\]

Then we define the restriction norm $B^s_{b}(S_T)$ for a function $u$ on a time slab $S_T$ by

\[
\| u \|_{B^s_{b}(S_T)} = \inf_{v = u \text{ on } S_T} \| v \|_{B^s_{b}}.
\]

Now we state the energy estimate lemma:
Lemma 2.1 (Lemma 2.1 of [11]). Let us consider the integral equation:
\[ v(t) = e^{±itD}f + \int_0^t e^{±i(t-t')D} F(t') \, dt' \]
with sufficiently smooth \( f \) and \( F \). If \( T \leq 1 \), then for any \( s \in \mathbb{R} \), we have
\[ \|v\|_{B^{±\frac{3}{4}}(S_T)} \lesssim T^{\frac{1}{4}} \|f\|_{B_{s}^{±\frac{3}{4}}} + \|F\|_{B_{s}^{−\frac{3}{4}}(S_T)}. \]

2.4. Analysis on the sphere. In this section, we introduce the linear estimates associated to the spherical harmonic projections which plays a crucial role in the proof of our main result. We refer to [11, Section 2.4] for the basic notations as preliminaries to avoid verbatim. Given dyadic number \( l \), we define the spherical harmonic projections \( H_l : f \mapsto H_l f \), which is an orthogonal projection in \( L^2(S^2) \). We also define the \( \sigma \) angular derivatives
\[ \langle \Omega \rangle^\sigma f = \sum_{l \text{dyadic}} l^\sigma H_l f. \]

Lemma 2.2 (Lemma 2.3 of [11]). Let \( N, L, \) and \( l \) be dyadic numbers. We have the following linear estimates:
\[ \|P_{K_{N,L}} H_l u\|_{L^2} \lesssim N^{-\frac{1}{2}} L^\frac{1}{2} l \|P_{K_{N,L}} H_l u\|_{L^2}. \]

From now on, since we often use the projection operators \( P_{K_{N,L}} H_l \) throughout this paper, we abbreviate it by \( P_{K_{N,L}}^l H_l \) for simplicity. We also write \( P_{K_{N,L}}^l u = u^l_{N,L,l} \) for brevity. Now given \( s \in \mathbb{R} \), \( b \geq \frac{3}{4} \) and \( \sigma \geq 1 \), we define the Banach space
\[ \|u\|_{B^{b,s}} = \sum_{N \geq 1} \sum_{L} \sum_{l} N^s L^b \|u^l_{N,L,l}\|_{L^2}. \]

2.5. Bilinear estimates. Given dyadic numbers \( N, L, \) we invoke that
\[ K_{N,L}^\pm = \{ (\tau, \xi) \in \mathbb{R}^{1+3} : |\xi| \approx N, |\tau \pm |\xi|| \approx L \}. \]

We introduce the key ingredient to handle multilinear estimates.

Theorem 2.3 (Theorem 1.1. of [23]). For all \( u_1, u_2 \in L^2(\mathbb{R}^{1+3}) \) such that \( \tilde{u}_j \) is supported in \( K_{N_j,L_j}^\pm \), then the estimates
\[ \|P_{K_{N_0,L_0}} (u_1 u_2)\| \leq C \|u_1\| \|u_2\| \]
holds with
\[ C \sim (N_{0j}^0 N_{1j}^1 L_1 L_2)^{\frac{1}{2}}, \]
\[ C \sim (N_{0j}^0 N_{1j}^1 L_0 L_j)^{\frac{1}{2}}, \quad j = 1, 2, \]
for any choice of signs \((\pm_0, \pm_1, \pm_2)\).

3. Proof of well-posedness

3.1. Proof of Theorem 1.1 We now arrive at the proof of our main results. We need to construct a solution \( (A_, \phi_+, F_+) \in B_{\pm,T}^{\frac{3}{4}+\frac{1}{2}+\sigma} \times B_{\pm,T}^{\frac{3}{4}+\frac{1}{2}+\sigma} \times B_{\pm,T}^{-\frac{1}{2}+\frac{1}{2}+\sigma} \) to the system (2.3). We first define the Duhamel
The key estimates are

\[ A_{\nu,\pm}(t) = A_{\nu,\pm}^a(t) \mp \int_0^t e^{\mp i(t-t')D} \frac{1}{2iD} \Lambda_\nu(A_+, A_-, F_+, F_-, \phi_+, \phi_-)(t') \, dt', \]

(3.1)

\[ F_{\mu\nu,\pm}(t) = F_{\mu\nu,\pm}^a(t) \mp \int_0^t e^{\mp i(t-t')D} \frac{1}{2iD} \tilde{\Gamma}_{\mu\nu}(A_+, A_-, F_+, F_-, \phi_+, \phi_-)(t') \, dt', \]

\[ \phi_{\pm}(t) = \phi_{\pm}^a(t) \mp \int_0^t e^{\mp i(t-t')D} \frac{1}{2iD} \Phi(A_+, A_-, \phi_+, \phi_-)(t') \, dt', \]

where the homogeneous part is given by

\[ \phi_{\alpha,\pm}^a = \frac{1}{2} e^{\mp iD} \left( \phi_\alpha(0, x) \mp \frac{1}{iD} \partial_x \phi_\alpha(0, x) \right). \]

To prove Theorem 1.2 we need to check the following nonlinear estimates:

\[ \| \Lambda_\nu(A_+, A_-, F_+, F_-, \phi_+, \phi_-) \|_{B_{2,1}^{\frac{1}{4}, \frac{1}{4}}(\mathbb{R}_+)} \lesssim \mathcal{G}^2(1 + \mathcal{G} + \mathcal{G}^2), \]

(3.2)

\[ \| \tilde{\Gamma}_{\mu\nu}(A_+, A_-, F_+, F_-, \phi_+, \phi_-) \|_{B_{2,1}^{\frac{1}{4}, \frac{1}{4}}(\mathbb{R}_+)} \lesssim \mathcal{G}^2(1 + \mathcal{G} + \mathcal{G}^2), \]

\[ \| \Phi(A_+, A_-, \phi_+, \phi_-) \|_{B_{2,1}^{\frac{1}{4}, \frac{1}{4}}(\mathbb{R}_+)} \lesssim \mathcal{G}^2(1 + \mathcal{G}), \]

where

\[ \mathcal{G} = \sum_{\pm} \left( \| A_\pm \|_{B_{2,1}^{\frac{1}{4}, \frac{1}{4}}} + \| \phi_\pm \|_{B_{2,1}^{\frac{1}{4}, \frac{1}{4}}} + \| F_\pm \|_{B_{2,1}^{\frac{1}{4}, \frac{1}{4}}} \right). \]

We also need the estimates of homogeneous part:

\[ \| \langle \Omega \rangle^\sigma A_{\nu,\pm}^a \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}} \lesssim \| \langle \Omega \rangle^\sigma a \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}} + \| \langle \Omega \rangle^\sigma \hat{a} \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}}, \]

\[ \| \langle \Omega \rangle^\sigma \phi_{\alpha,\pm}^a \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}} \lesssim \| \langle \Omega \rangle^\sigma \phi_\alpha \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}} + \| \langle \Omega \rangle^\sigma \hat{\phi}_\alpha \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}}, \]

\[ \| \langle \Omega \rangle^\sigma F_{\mu\nu,\pm}^a \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}} \lesssim \| \langle \Omega \rangle^\sigma f \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}} + \| \langle \Omega \rangle^\sigma \hat{f} \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}}, \]

which are obvious. If we had the nonlinear estimates (3.2), then the standard contraction argument would give the local well-posedness of the solution to 2.3.

3.2. Proof of Theorem 1.2. Similarly, to prove Theorem 1.2 we construct a solution \((A_\pm, \psi_\pm, F_\pm) \in B_{2,1}^{\frac{1}{4}, \frac{1}{4}} \times B_{2,1}^{\frac{1}{4}, \frac{1}{4}} \times B_{2,1}^{\frac{1}{4}, \frac{1}{4}}\) to the system (2.4). The Duhamel integrals are given by

\[ A_{\nu,\pm}(t) = A_{\nu,\pm}^a(t) \pm \int_0^t e^{i(t-t')D} \frac{1}{2iD} \Lambda_\nu(A_+, A_-, F_+, F_-, \psi_+, \psi_-)(t') \, dt', \]

(3.3)

\[ F_{\mu\nu,\pm}(t) = F_{\mu\nu,\pm}^a(t) \pm \int_0^t e^{i(t-t')D} \frac{1}{2iD} \tilde{\Gamma}_{\mu\nu}(A_+, A_-, F_+, F_-, \psi_+, \psi_-)(t') \, dt', \]

\[ \psi_{\pm}(t) = e^{iD} \psi_{0,\pm} + i \int_0^t e^{i(t-t')D} \tilde{\Psi}(A_+, A_-, \psi_+, \psi_-)(t') \, dt'. \]

The key estimates are

\[ \| \Lambda_\nu(A_+, A_-, F_+, F_-, \psi_+, \psi_-) \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}} \lesssim \mathcal{G}^2(1 + \mathcal{G} + \mathcal{G}^2), \]

(3.4)

\[ \| \tilde{\Gamma}_{\mu\nu}(A_+, A_-, F_+, F_-, \psi_+, \psi_-) \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}} \lesssim \mathcal{G}^2(1 + \mathcal{G} + \mathcal{G}^2), \]

\[ \| \tilde{\Psi}(A_+, A_-, \psi_+, \psi_-) \|_{B_{2,1}^{\frac{1}{2}, \frac{1}{2}}} \lesssim \mathcal{G}^2, \]
where
\[ \mathcal{E}' = \sum_{\pm} \left( \|A_{\pm}\|_{B^{\frac{1}{2},\infty}_x} + \|\psi_{\pm}\|_{B^{\frac{1}{2},\infty}_x} + \|F_{\pm}\|_{B^{\frac{1}{2},\infty}_x} \right). \]

Hence it is enough to consider the proof of (3.4). In the rest of the paper, we will focus on the proof of (3.2) and (3.4). We end this section with the organisation of the following sections.

**Organisation.** We shall introduce the outline of the rest of the paper briefly. Section 4 is devoted to the proof of bilinear estimates. There is a bilinear form without null structure, \([\phi, \partial_t \phi] \). We also treat the estimates of bilinear forms including spinor field \(\psi\) in this section. Compared to the Higgs field \(\phi\), one may use the null structure in the spinor field \(\psi\). However, we only use Lemma 2.2 and treat these bilinear forms simultaneously. In Section 5, we estimate the higher-order terms which appear in (2.3) and (2.4). We can treat these higher-order terms by Bernstein's inequality (5.1). In this manner, we even obtain better estimates than we required. In this paper, we do not give the explicit treatment of the bilinear forms of the terms \([A^\lambda, \partial_\lambda \varphi]\) and \([A^\lambda, \partial_\mu A_\lambda]\). For the estimates of these bilinear forms, we refer the readers to [11, Section 5.6]. Note that we are concerned with \(\mathfrak{su}(n, \mathbb{C})\)-valued fields; however, we remark that the estimates of such functions is reduced the nonlinear estimates of \(\mathbb{C}\)-valued functions. See Section 3.1 and Section 5.1 of [11] for more details.

### 4. Bilinear forms

In this section, we consider the following estimates:

\[ (4.1) \quad \|[[\phi_{\pm 1}, \partial_\nu \phi_{\pm 2}]]\|_{B^{\frac{1}{2},\infty}_x} \lesssim \|\phi_{\pm 1}\|_{B^{\frac{1}{2},\infty}_x} \|\phi_{\pm 2}\|_{B^{\frac{1}{2},\infty}_x}, \]

\[ (4.2) \quad \|\Pi_{\pm 0}(A^\mu_{\pm 1}) \gamma^\nu \psi_{\pm 2}\|_{B^{0,\frac{1}{2}}_x} \lesssim \|A_{\pm 1}\|_{B^{0,\frac{1}{2}}_x} \|\psi_{\pm 2}\|_{B^{0,\frac{1}{2}}_x}, \]

\[ (4.3) \quad \|\nabla^\nu \psi_{\pm 1} \gamma^\nu \psi_{\pm 2}\|_{B^{0,\frac{1}{2}}_x} \lesssim \|\psi_{\pm 1}\|_{B^{0,\frac{1}{2}}_x} \|\psi_{\pm 2}\|_{B^{0,\frac{1}{2}}_x}, \]

\[ (4.4) \quad \|\partial_\mu (\psi_{\pm 1} \gamma^\nu \psi_{\pm 2})\|_{B^{0,\frac{1}{2}}_x} \lesssim \|\psi_{\pm 1}\|_{B^{0,\frac{1}{2}}_x} \|\psi_{\pm 2}\|_{B^{0,\frac{1}{2}}_x}. \]

We prove only (4.1), (4.2), and (4.3). The treatment of (4.4) can be absorbed in the proof of (4.3).

#### 4.1. Proof of (4.1)

Recall the definition of \(\mathcal{B}^{a,b,\sigma}_x\). We write

\[ \|\phi_{\pm 1} \partial_\nu \phi_{\pm 2}\|_{B^{0,\frac{1}{2}}_x} = \sum_{N_0, L_0, l_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} l_0^{-\frac{1}{4}} \|P_{N_0}^{l_0} (\phi_{\pm 1} \partial_\nu \phi_{\pm 2})\| \]

\[ \lesssim \sum_{N, L, l} \sum_{N_0, L_0} \sum_{l_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} N_2^{-\frac{1}{2}} l_0^{-\frac{1}{4}} \|P_{N_0}^{l_0} (\phi_{N_1, L_1, l_1} \phi_{N_2, L_2, l_2})\| \]

\[ \lesssim \sum_{N, L, l} \sum_{N_0, L_0} \sum_{l_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} N_2^{-\frac{1}{4}} l_0^{-\frac{1}{4}} \|N_0 C_{N, L, l_0} (\phi_{N_1, L_1, l_1} \|\phi_{N_2, L_2, l_2})\| \]

Thus to prove the estimate (4.1), it is enough to show that

\[ \mathcal{I}^1(N, L) := \sum_{N_0, L_0} \sum_{N, L, l} \sum_{N_0, L_0} \sum_{l_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} N_2^{-\frac{1}{4}} l_0^{-\frac{1}{4}} \|N_0 C_{N, L, l_0} (\phi_{N_1, L_1, l_1} \|\phi_{N_2, L_2, l_2})\| \]

\[ \lesssim (N_1 N_2)^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{4}} \|\phi_{N_1, L_1, l_1}\| \|\phi_{N_2, L_2, l_2}\|. \]
4.1.1. Case 1: $L_0 \ll L_2$, $l_1 \ll l_2$. We use (2.7) with $j = 1$ and Lemma 2.2 to get

$$
I^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} N_2 (N^{12}_{\min} N^{01}_{\min} L_0 L_1)^{\frac{1}{2}} N_1^{\frac{1}{2}} L_1^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||
$$

$$
\lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} (N^{12}_{\min})^{\frac{1}{2}} N_1^{\frac{1}{2}} N_2 (N^{12}_{\min})^{\frac{1}{2}} L_0^\frac{1}{2} L_1^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||
$$

$$
\lesssim (N_1 N_2)^{\frac{1}{2}} L_1^\frac{1}{2} L_2^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||.
$$

4.1.2. Case 2: $L_0 \ll L_2$, $l_2 \ll l_1$. Similarly,

$$
I^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} N_2 (N^{12}_{\min} N^{12}_{\min} L_0 L_1)^{\frac{1}{2}} N_1^{\frac{1}{2}} L_1^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||
$$

$$
\lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} (N^{12}_{\min})^{\frac{1}{2}} N_1^{\frac{1}{2}} N_2 (N^{12}_{\min})^{\frac{1}{2}} L_0^\frac{1}{2} L_1^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||
$$

$$
\lesssim (N_1 N_2)^{\frac{1}{2}} L_1^\frac{1}{2} L_2^\frac{1}{2} l_2 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||.
$$

4.1.3. Case 3: $L_2 \ll L_0$, $l_1 \ll l_2$. Using (2.6) and Lemma 2.2 we get

$$
I^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} N_2 (N^{12}_{\min} N^{01}_{\min} L_0 L_1)^{\frac{1}{2}} N_1^{\frac{1}{2}} L_1^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||
$$

$$
\lesssim \sum_{N_0} N_0^{-\frac{1}{2}} (N^{12}_{\min})^{\frac{1}{2}} N_1^{\frac{1}{2}} N_2 (N^{01}_{\min})^{\frac{1}{2}} L_1^\frac{1}{2} L_2^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||
$$

$$
\lesssim (N_1 N_2)^{\frac{1}{2}} L_1^\frac{1}{2} L_2^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||.
$$

4.1.4. Case 4: $L_2 \ll L_0$, $l_2 \ll l_1$.

$$
I^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} N_2 (N^{12}_{\min} N^{12}_{\min} L_0 L_1)^{\frac{1}{2}} N_1^{\frac{1}{2}} L_1^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||
$$

$$
\lesssim \sum_{N_0} N_0^{-\frac{1}{2}} (N^{12}_{\min})^{\frac{1}{2}} N_1^{\frac{1}{2}} N_2 (N^{12}_{\min})^{\frac{1}{2}} L_1^\frac{1}{2} L_2^\frac{1}{2} l_1 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||
$$

$$
\lesssim (N_1 N_2)^{\frac{1}{2}} L_1^\frac{1}{2} L_2^\frac{1}{2} l_2 \||\phi_{N_1, L_1, l_1}^{\pm_1}||\phi_{N_2, L_2, l_2}^{\pm_2}||.
$$

This completes the proof of (4.1).

Remark 4.1. Here one should note that since $[\phi, \partial_r \phi]$ has no null structure, the direct use of bilinear estimates (2.6), (2.7) gives only well-posedness in $H^1 \times L^2$. Indeed, if initial data satisfy $(a, \phi_0) \in H^s \times H^s$, then (2.6) and (2.7) implies that $N^{1-s} N^2 \lesssim N^{2s}$ and hence $s \geq 1$. This shows the advantage of imposing additional angular regularity. We refer to [26] for the local well-posedness of (YMH) in the energy space.

4.2. Proof of (4.2). Similarly, we write

$$
|A_{\pm_1, \psi_{\pm_2}}|_{g^{\frac{1}{2} - \frac{1}{2}}} = \sum_{N_0, L_0, l_0} L_0^{-\frac{1}{2}} P_{L_0}^{l_0} \|P_{L_0}^{l_0} (A_{\pm_1, \psi_{\pm_2}})\|
$$

$$
\lesssim \sum_{N, L, l} L_0^{-\frac{1}{2}} P_{L_0}^{l_0} \|A_{N_1, L_1, l_1}^{\pm_1, \psi_{\pm_2}}\|
$$

$$
\lesssim \sum_{N, L, l} L_0^{-\frac{1}{2}} P_{L_0}^{l_0} C_{N, L}^{l_0} \|A_{N_1, L_1, l_1}^{\pm_1, \psi_{\pm_2}}\|.
$$
To prove the required estimate, we need to show that

\[ T^2(N, L) := \sum_{N_0, L_0} L_0^{-\frac{1}{2}} C_{N, L} \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2 \]

\[ \lesssim N_1^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2. \]

4.2.1. Case 1: \( L_0 \ll L_2, l_1 \ll l_2 \).

\[ T^2(N, L) \lesssim \sum_{N_0, L_0} L_0^{-\frac{1}{2}} (N_{min}^{012} N_{min}^{01} L_0 L_1)^{\frac{1}{2}} N_1^{-\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2 \]

\[ \lesssim \sum_{N_0} (N_{min}^{012})^{\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2 \]

\[ \lesssim N_1^{\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2. \]

4.2.2. Case 2: \( L_0 \ll L_2, l_2 \ll l_1 \).

\[ T^2(N, L) \lesssim \sum_{N_0, L_0} L_0^{-\frac{1}{2}} (N_{min}^{012} N_{min}^{01} L_0 L_1)^{\frac{1}{2}} N_2^{-\frac{1}{2}} L_2^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2 \]

\[ \lesssim \sum_{N_0} (N_{min}^{012})^{\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2 \]

\[ \lesssim N_1^{\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2. \]

4.2.3. Case 3: \( L_2 \ll L_0, l_1 \ll l_2 \).

\[ T^2(N, L) \lesssim \sum_{N_0, L_0} L_0^{-\frac{1}{2}} (N_{min}^{012} N_{min}^{01} L_0 L_2)^{\frac{1}{2}} N_1^{-\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2 \]

\[ \lesssim \sum_{N_0} (N_{min}^{012})^{\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2 \]

\[ \lesssim N_1^{\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2. \]

4.2.4. Case 4: \( L_2 \ll L_0, l_2 \ll l_1 \).

\[ T^2(N, L) \lesssim \sum_{N_0, L_0} L_0^{-\frac{1}{2}} (N_{min}^{012} N_{min}^{01} L_0 L_2)^{\frac{1}{2}} N_2^{-\frac{1}{2}} L_2^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2 \]

\[ \lesssim \sum_{N_0} (N_{min}^{012})^{\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2 \]

\[ \lesssim N_1^{\frac{1}{2}} L_1^2 \| A_{N_1, L_1, l_1} \|_2 \| \psi_{N_2, L_2, l_2} \|_2. \]

This completes the proof of (4.2).

Remark 4.2. In the proof of (4.2), we do not exploit the role of \( \Pi_{\pm 0} \). One can reveal null structure via duality and decomposition of \( A \). Indeed, we see that

\[ \| \Pi_{\pm 0} (A_{\mu, \pm 1} \gamma^\alpha \gamma^\mu \psi_{\pm 2}) \| \leq \| \Pi_{\pm 0} (A_{\mu, \pm 1} \Pi_{\pm 2} \gamma^\alpha \gamma^\mu \psi_{\pm 2}) \| + \| \Pi_{\pm 0} (A_{\mu, \pm 1} R_{\pm 2} \psi_{\pm 2}) \|, \]

where we used the commutator identity for the projection operators \( \Pi_{\pm} \):

\[ \gamma^\alpha \gamma^\mu \Pi_{\pm} = \Pi_{\pm} \gamma^\alpha \gamma^\mu \Pi_{\pm} - R_{\pm} \Pi_{\pm}, \]

and \( R_{\pm} = R_{\pm}^0 / m^2 \) and \( R_{\pm} = -1 \). For the first term, we use the following inequality via duality:

\[ |\mathcal{F}[\tilde{\psi}_{\pm 0} \Pi_{\pm 2} \gamma^\alpha \gamma^\mu \psi_{\pm 2}]| \lesssim \int_{\xi_1 = \xi_0 - \xi_2} \theta_{02} |\tilde{\psi}_{\pm 0}(\xi_0)| |\tilde{\psi}_{\pm 2}(\xi_2)| d\xi_0 d\xi_2. \]
For the second term, we write $A = A^{df} + A^{af}$ and then it becomes $Q$-type null form. Then the proof is also straightforward. We refer to [12, Section 3.2] for more details. However, we observe that without null structure we still obtain the required estimates via direct use of Lemma 2.2.

4.3. Proof of (4.3). We write

$$ \| \psi_{\pm 1} \psi_{\pm 2} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} = \sum_{N_0, L_0, l_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \| P_{\kappa_{N_0, L_0}} l_0 (\psi_{\pm 1} \psi_{\pm 2}) \| $$

$$ \lesssim \sum_{N, L, l} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \| P_{\kappa_{N_0, L_0}} l_0 (\psi_{\pm 1} \psi_{\pm 2}) \| $$

$$ \lesssim \sum_{N, L, l} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} l_0 \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0}. $$

Then it suffices to show that

$$ T^3(N, L) := \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0}. $$

4.3.1. Case 1: $L_0 \ll L_2, l_1 \ll l_2$.

$$ T^3(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} (N_0^{12} N_0^{12} L_0 L_1) \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0} $$

$$ \lesssim L_1^2 L_2^2 l_1 \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0}. $$

4.3.2. Case 2: $L_0 \ll L_2, l_2 \ll l_1$.

$$ T^3(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} (N_0^{12} N_0^{12} L_0 L_1) \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0} $$

$$ \lesssim L_1^2 L_2^2 l_2 \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0}. $$

4.3.3. Case 3: $L_2 \ll L_0, l_1 \ll l_2$.

$$ T^3(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} (N_0^{12} N_0^{12} L_0 L_2) \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0} $$

$$ \lesssim L_1^2 L_2^2 l_1 \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0}. $$

4.3.4. Case 4: $L_2 \ll L_0, l_2 \ll l_1$.

$$ T^3(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} (N_0^{12} N_0^{12} L_0 L_2) \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0} $$

$$ \lesssim L_1^2 L_2^2 l_2 \| \psi_{\pm 1} \| \| \psi_{\pm 2} \| _{N_0, L_0, l_0}. $$

5. Higher-order terms

To the end of the proof of our main theorem, we need to prove the following:

$$ \| [A^\mu, [A^\nu, A^\rho]] \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} \lesssim \| A_{\pm 1} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} \| A_{\pm 2} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} \| A_{\pm 0} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}}. $$

$$ \| \partial_\mu \phi [A^\nu, \phi] \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} \lesssim \| A_{\pm 1} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} \| A_{\pm 2} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} \| \phi_{\pm 0} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}}. $$

$$ \| \psi_{\pm 1} \gamma_\mu \psi_{\pm 2} A_{\pm 0} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} \lesssim \| A_{\pm 0} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} \| \psi_{\pm 1} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}} \| \psi_{\pm 2} \|_{B_{2,0}^{\frac{1}{2}, \frac{1}{2}, \sigma}}. $$


We only prove \([5.2]\) and \([5.3]\). Recall the Bernstein’s inequality:

\[
\|P_{|\xi|=N}f\|_{L^p(\mathbb{R}^4)} \lesssim \|P_{|\xi|=N}f\|_{L^q(\mathbb{R}^4)},
\]

for \(q < p \leq \infty\). We also have to consider several quartic terms appearing in \([111], [118]\); however, the treatment can be absorbed in the estimates of the bilinear forms \([\partial_\mu A^x, \partial_\nu A_\lambda]\). See \([11]\) Section 6 for details.

5.1. **Proof of** \([5.2]\). By Bernstein’s inequality \([5.4]\), we have

\[
\|A_{\pm1, \phi_{\pm2}} \partial \phi_{\pm0}\|_{B^{\frac{3}{4}, \frac{7}{4}}_{\infty}} = \sum_{N_0, L_0, l_0} N_0^{-\frac{3}{4}} L_0^{-\frac{1}{4}} \|P_{|\xi|=N_0}^l (A_{\pm1, \phi_{\pm2}} \partial \phi_{\pm0})\| \\
\lesssim \sum_{N_0, L_0, l_0} N_0 L_0^{\frac{1}{4}} \|P_{|\xi|=N_0}^l (A_{\pm1, \phi_{\pm2}})\| \|\phi_{\pm0}\|_{N_0, L_0, l_0} \\
\lesssim \sum_{N, L, l} N_0^{\frac{1}{2}} L_0^{-\frac{1}{2}} \|P_{|\xi|=N_0}^l (A_{\frac{N_1, L_1, l_1}{N_2, L_2, l_2}} \phi_{N_2, L_2, l_2})\| (N_0^2 L_0^2) \|\phi_{N_0, L_0, l_0}\| \\
\lesssim \sum_{N_0, L_0, l_0} \mathcal{H}^1(N, L) N_0^{\frac{1}{2}} L_0^{-\frac{1}{4}} \|\phi_{N_0, L_0, l_0}\|,
\]

where

\[
\mathcal{H}^1(N, L) := \sum_{N_1, L_1, l_1} \sum_{N_2, L_2, l_2} N_0^{\frac{1}{2}} L_0^{-\frac{1}{4}} C_{N, L}^{0, 12} (N_1 N_2)^{\frac{1}{2}} (L_1 L_2)^{\frac{3}{4}} l_1 l_2 \|A_{N_1, L_1, l_1}^{N_2, \frac{1}{2}} \phi_{N_2, L_2, l_2}\|.
\]

Hence it is enough to show that

\[
\mathcal{H}^1(N, L) \lesssim \sum_{N_1, L_1, l_1} \sum_{N_2, L_2, l_2} (N_1 N_2)^{\frac{1}{4}} (L_1 L_2)^{\frac{1}{4}} l_1 l_2 \|A_{N_1, L_1, l_1}^{N_2, \frac{1}{2}} \phi_{N_2, L_2, l_2}\|.
\]

We use Lemma \([2.2]\) to get

\[
\mathcal{H}^1(N, L) \lesssim \sum_{N_1, L_1, l_1} \sum_{N_2, L_2, l_2} N_0^{\frac{1}{2}} L_0^{-\frac{1}{4}} C_{N, L}^{0, 12} (N_1 N_2)^{\frac{1}{4}} (L_1 L_2)^{\frac{1}{4}} l_1 l_2 \|A_{N_1, L_1, l_1}^{N_2, \frac{1}{2}} \phi_{N_2, L_2, l_2}\|.
\]

If \(L_0 \ll L_2\), then \([2.7]\) gives

\[
\mathcal{H}^1(N, L) \lesssim \sum_{N_1, L_1, l_1} \sum_{N_2, L_2, l_2} N_0^{\frac{1}{2}} L_0^{-\frac{1}{4}} (N_1 N_2)^{\frac{1}{4}} (L_1 L_2)^{\frac{1}{4}} l_1 l_2 \|A_{N_1, L_1, l_1}^{N_2, \frac{1}{2}} \phi_{N_2, L_2, l_2}\|.
\]

For \(L_2 \ll L_0\), we use \([2.6]\) to get

\[
\mathcal{H}^1(N, L) \lesssim \sum_{N_1, L_1, l_1} \sum_{N_2, L_2, l_2} N_0^{\frac{1}{2}} L_0^{-\frac{1}{4}} (N_1 N_2)^{\frac{1}{4}} (L_1 L_2)^{\frac{1}{4}} l_1 l_2 \|A_{N_1, L_1, l_1}^{N_2, \frac{1}{2}} \phi_{N_2, L_2, l_2}\|.
\]

Thus we obtain even better estimates. This completes the proof of Theorem \([1.1]\).

**Remark 5.1.** One should observe that even if we lose some regularity by using Bernstein’s inequality in trilinear estimates, we truly gain much more regularity by applying Lemma \([2.2]\) twice. It is not harmful to the summation of angular regularity. In this manner, we can attain sharp estimates in the higher-order terms.
5.2. **Proof of (5.3).** We can prove it similarly. Indeed, we write

\[
\| \psi_{\pm}^1, \psi_{\pm}^2 A_{\pm} \|_{L^2}^2 \lesssim \sum_{N_0, L_0, l_0} N_0^{-\frac{1}{4}} L_0^{-\frac{1}{4}} l_0^2 \| P_{K_{N_0, L_0}}^{l_0} (\psi_{\pm}^1, \psi_{\pm}^2) \|_{L^2}^2 \]

\[
\lesssim \sum_{N_0, L_0} \mathcal{H}^2 (N, L) N_0^{\frac{1}{2}} L_0^{\frac{1}{2}} A_{N_0, L_0, l_0}^4 \| A_{N_0, L_0, l_0}^4 \|
\]

where

\[
\mathcal{H}^2 (N, L) := \sum_{N_1, L_1, l_1, N_2, L_2} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} C_{N, L}^{012} \| \psi_{N_1, L_1, l_1} \| \| \psi_{N_2, L_2, l_2} \|.
\]

Thus we are left to prove that

\[
\mathcal{H}^2 (N, L) \lesssim \sum_{N_1, L_1, l_1, N_2, L_2} (L_1 L_2)^{\frac{1}{2}} l_1 l_2 \| \psi_{N_1, L_1, l_1} \| \| \psi_{N_2, L_2, l_2} \|.
\]

Now we apply Lemma 2.2 to get

\[
\mathcal{H}^2 (N, L) \lesssim \sum_{N_1, L_1, l_1, N_2, L_2} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} C_{N, L}^{012} (N_1 N_2)^{-\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} l_1 l_2 \| \psi_{N_1, L_1, l_1} \| \| \psi_{N_2, L_2, l_2} \|.
\]

We only consider the case \( L_2 \ll L_0, l_2 \ll l_1 \). We have

\[
\mathcal{H}^2 (N, L) \lesssim \sum_{N_1, L_1, l_1, N_2, L_2} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} (N_1 N_2)^{-\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} l_1 l_2 \| \psi_{N_1, L_1, l_1} \| \| \psi_{N_2, L_2, l_2} \|
\]

This completes the proof of (5.2) and hence the proof of Theorem 1.2.

6. **Appendix**

6.1. **Quantum Field Theory.** The aim of this section is to justify our notation abuse in the Yang-Mills-Dirac system. In Section 1, we introduced a locally \( SU(n, \mathbb{C}) \) invariant Yang-Mills-Dirac action given by

\[
\mathcal{L}_{\text{YMD}} = - \int_{\mathbb{R}^{1+3}} \left( \frac{1}{4} \langle F[A]_{\mu \nu}, F[A]^\mu_{\nu} \rangle - \sum_{i,j=1}^n \bar{\psi}_i (i \delta_{ij} \gamma_{\mu} \partial^\mu + A^a_{ij} T^a_{ij} \gamma_{\mu}) \psi_j \right) \ dx \ dt.
\]

Here, \( \{ T^a \}_{a=1}^n \) is the set of generators. We also write explicitly the row and column indices \( i \) and \( j \) as in \( T^a_{ij} \). For \( T^a \), we have the following identity:

\[
[T^a, T^b] = i f^{abc} T^c,
\]

where \( f^{abc} \) is called structure coefficient. In particular, for \( G = SU(n, \mathbb{C}) \), we have

\[
\sum_a T^a_{ij} T^a_{kl} = \frac{1}{2} \left( \delta_{il} \delta_{kj} - \frac{1}{n} \delta_{ij} \delta_{kl} \right),
\]

known as Fierz identity. The Euler-Lagrange equation of (6.1) is given by

\[
\partial_{\mu} F_{\mu \nu}^a + \epsilon^{abc} A_{\mu}^b F_{\mu \nu}^c = -\bar{\psi}_i \gamma_{\mu} T^a_{ij} \psi_j,
\]

\[
i \gamma_{\mu} \partial_{\mu} \psi_i = -A_{\mu}^a \gamma_{\mu} T^a_{ij} \psi_j.
\]
Here, $F^a_{\mu}\nu$ is the $T^a$-component of $F_{\mu\nu}$. That is, $F_{\mu\nu} = F^a_{\mu\nu}T^a$. Now we are concerned with the wave equation for $F$:

$$\Box F_{\mu\nu} = -\{A^\lambda, F_{\mu\nu}\} - \partial^\lambda[A_{\lambda}, F_{\mu\nu}] - \{A^\lambda, [A_{\lambda}, F_{\mu\nu}]\} - D_\mu(\overline{\psi_i \gamma^\mu T_{ij}} \psi^j) + D_\nu(\overline{\psi_i \gamma^\mu T_{ij}} \psi^j) - 2\{F^\lambda, F_{\lambda\mu}\}.$$  

By definition of $D^\mu$, we write

$$D_\mu(\overline{\psi_i \gamma^\mu T_{ij}} \psi^j) = \partial_\mu(\overline{\psi_i \gamma^\mu T_{ij}} \psi^j) + [A_{\mu}, \overline{\psi_i \gamma^\mu T_{ij}} \psi^j]$$

$$= \partial_\mu(\overline{\psi_i \gamma^\mu T_{ij}} \psi^j) + [A_{\mu, kl} T_{kl}, \overline{\psi_i \gamma^\mu T_{ij}} \psi^j]$$

$$= \partial_\mu(\overline{\psi_i \gamma^\mu T_{ij}} \psi^j) + A_{\mu, ij} \overline{\psi_i \gamma^\mu T_{ij}} [T_{kl}, T_{ij}].$$

Thus we see that

$$\|D_\mu(\overline{\psi_i \gamma^\mu T_{ij}} \psi^j)\|_{X(A_{\mu}(n, \mathbb{C}))} = \|\partial_\mu(\overline{\psi_i \gamma^\mu T_{ij}} \psi^j)\|_{X(\mathbb{C})} |T_{ij}| + \|A_{\mu, ij} \overline{\psi_i \gamma^\mu T_{ij}} [T_{kl}, T_{ij}]|.$$

Hence we conclude that the estimates of $\mathfrak{su}(n, \mathbb{C})$-valued fields is reduced to the estimates of $\mathbb{C}$-valued fields; $T^a_{ij}$ plays no role in our analysis. We refer to [22, Chapter 25] for more discussion.

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