Regular conditional distributions for semimartingale SDEs

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Abstract

In this paper we prove existence of the regular conditional distribution of strong solutions to a large class of semimartingale-driven stochastic differential equations. For proving this result, we show for the first time that solutions of these stochastic differential equations can be written as measurable functions of their driving processes into the space of all càdlàg functions equipped with the Borel algebra generated by all open sets with respect to the Skorohod metric. As a corollary, the two theorems prove a Markov property, which is for example relevant in computational stochastics.

Keywords: semimartingale-driven stochastic differential equations, function representation in the Skorohod space, regular conditional distribution, Markov property

Mathematics Subject Classification (2020): 60H10

1 Introduction

The existence of the regular conditional distribution of strong solutions of stochastic differential equations (SDEs) is well-known for equations that are driven by processes with continuous sample paths. In particular, it follows from [1, p. 396, Theorem 12.9] together with [2, p. 185, Theorem 8.37], since for continuous processes the space of all continuous functions equipped with the sup-norm is a Polish space. This is however not true for processes that involve jumps, since the space of all càdlàg functions with the sup-norm is not separable.

We prove for the first time the existence of the regular conditional distribution of strong solutions of a large class of semimartingale-driven SDEs that do involve jumps. This kind of result is needed, for example, in computational stochastics as it implies the Markov property. For an example where the classical result for the Brownian case has been applied, see [3]. The class of SDEs we consider is very general and therefore covers a wide range of applications. In particular, we consider SDEs of the form

\[
X_t = H_t + \int_0^t g(s, G, X) \, dY_s, \quad t \in [0, T]
\]  

(1)

where \(m, d, r \in \mathbb{N}\), \(T \in (0, \infty)\), \(Y\) is an \(\mathbb{R}^m\)-valued càdlàg semimartingale, \(H\) is an \(\mathbb{R}^d\)-valued càdlàg adapted process, and \(G\) is an \(\mathbb{R}^r\)-valued càdlàg adapted process on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})\) that satisfies the usual conditions. Let for all \(n \in \mathbb{N}\), \(\mathbb{D}_n = \{f : [0, T] \to \mathbb{R}^n : f \text{ is càdlàg}\}\). For the functions \(f, g : [0, T] \times \mathbb{D}_r \times \mathbb{D}_d \to L(\mathbb{R}^m, \mathbb{R}^d)\) we assume that the mapping \(t \mapsto f(t, \zeta, \gamma)\) is càdlàg for all \((\zeta, \gamma) \in \mathbb{D}_r \times \mathbb{D}_d\) and that \(g\) is defined using \(f\) for all \(t \in (0, T)\) as

\[
g(t, \zeta, \gamma) = f(t-, \zeta, \gamma)
\]  

(2)

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and \( g(0, \zeta, \gamma) = f(0, \zeta, \gamma) \).

For proving the existence of the regular conditional distribution, we show for the first time that \( X \) can be written as a measurable function of \( (H, G, Y) \) into the space \( \mathbb{D}_d \) equipped with the Borel algebra generated by all open sets with respect to the Skorohod metric, which is a Polish space.

A seminal result concerning the functional representation of SDEs is [11], where it is proven that the solution of an SDE driven by time and a Brownian motion can be expressed as a function of the initial value and the Brownian motion. The drawback of this representation is that the function is dependent on the distribution of the solution at time \( 0 \). This has been solved in [3], where a universal representation for such SDEs has been proven. In [5] and [6], for the solution of general semimartingale SDEs, a functional representation has been proven and in [4, p. 396, Theorem 12.9] this result was generalized to the setting which we also employ in the current paper. However, in [4] the Skorohod function space was equipped with the topology of uniform convergence on compact subsets, see [4, p. 35 f. and Equation (12.3.10)]. Under this topology, however, the space of all càdlàg functions is not separable. But equipped with the Skorohod metric, which was originally introduced in [10], the space is a Polish space. Hence, the current paper resolves this issue.

The proof of our result requires major changes in the technique known from [4]. The key novel step in the first part of the proof is showing Skorohod measurability of the approximating sequence; in the second part of the proof the key lies in proving that the limit of the sequence of stochastic processes in the Skorohod sense is indistinguishable from the solution of SDE (1). Our functional representation is of interest on its own and – most importantly – enables us to prove the existence of the regular conditional distribution of \( X \).

As an example for an application of both results, we finally prove a Markov property.

### 2 Notations and Preliminaries

We equip this space with the Skorohod topology, which is defined in the following, cf. [2, p. 125 ff.]. First, we define the set

\[
\Lambda = \{ \lambda : [0, T] \to [0, T] : \lambda \text{ is strictly increasing and continuous with } \lambda(0) = 0 \text{ and } \lambda(T) = T \}
\]

and then we set for all \( \lambda \in \Lambda \),

\[
\| \lambda \|_\Lambda = \sup_{0 \leq s < t \leq T} \left| \log \left( \frac{\lambda(t) - \lambda(s)}{t - s} \right) \right|.
\]

Using this we define the Skorohod metric \( d^S \) for all \( x, y \in \mathbb{D}_n \) as

\[
d^S(x, y) = \inf_{\lambda \in \Lambda} \max \left\{ \| \lambda \|_\Lambda, \sup_{0 \leq t \leq T} \| x(t) - y(\lambda(t)) \| \right\}.
\]

The space \((\mathbb{D}_n, d^S)\) is separable and complete, see [2, p. 128, Theorem 12.2]. We define the Skorohod topology as the topology induced by the open subsets of \( \mathbb{D}_n \) under \( d^S \). Then the space \( \mathbb{D}_n \) with this topology is a Polish space.

Further, denote by \( \mathcal{B}(\mathbb{D}_n) \) the Borel-\( \sigma \)-field generated by all open sets in \((\mathbb{D}_n, d^S)\). This \( \sigma \)-field is the same as the one generated by all coordinate mappings, see [4, p. 36]. In particular, this implies Borel measurability of the following mappings

\[
\mathbb{D}_n \ni f \mapsto f(t),
\]

\[
\mathbb{D}_n \ni f \mapsto f(t^-),
\]
for all $t \in [0, T]$, see [3]. The space $D_n$ can also be equipped with the sup-norm $\| \cdot \|_\infty$. By $d$ we denote the corresponding metric. It is well known that for all $x, y \in D_n$ it holds that $d^S(x, y) \leq d(x, y)$. Let $L(\mathbb{R}^m, \mathbb{R}^d)$ be the space of all linear bounded operators from $\mathbb{R}^m$ to $\mathbb{R}^d$ equipped with the operator norm and let $B(L(\mathbb{R}^m, \mathbb{R}^d))$ be the induced Borel $\sigma$-field.

Before we state our assumptions we define for all $n \in \mathbb{N}$, $x \in D_n$, and $t \in [0, T]$ the function $x_t : [0, T] \to \mathbb{R}^n$, $x_t(s) = x(s \land t)$.

\textbf{Assumption 2.1.} For the function $f$ we assume:

(i) $f$ is measurable with respect to $B([0, T]) \otimes B(\mathbb{D}_r) \otimes B(\mathbb{D}_d)$,

(ii) $f(t, \zeta, \gamma) = f(t, \zeta^1, \gamma^1)$ for all $\zeta \in \mathbb{D}_r$, $\gamma \in \mathbb{D}_d$, and $t \in [0, T]$,

(iii) there exists a function $C : [0, T] \times \mathbb{D}_r \to (0, \infty)$, which is measurable with respect to $B([0, T]) \otimes B(\mathbb{D}_r)$ and has the property that $t \mapsto C(t, \zeta)$ is càdlàg, such that for all $\zeta \in \mathbb{D}_r$, $\gamma, \gamma_1, \gamma_2 \in \mathbb{D}_d$ and $t \in [0, T]$ we have

$$
\|f(t, \zeta, \gamma)\| \leq C(t, \zeta) \left(1 + \sup_{0 \leq s \leq t} \|\gamma(s)\|\right)
$$

$$
\|f(t, \zeta, \gamma_1) - f(t, \zeta, \gamma_2)\| \leq C(t, \zeta) \left(\sup_{0 \leq s \leq t} \|\gamma_1(s) - \gamma_2(s)\|\right).
$$

\textbf{Lemma 2.1.} If $f$ fulfills Assumption 2.1 (i), then $g$ defined by equation (2) is $B([0, T]) \otimes B(\mathbb{D}_r) \otimes B(\mathbb{D}_d)$-measurable.

\textbf{Proof.} Define for all $n \in \mathbb{N}$ the function $f_n : [0, T] \times \mathbb{D}_r \times \mathbb{D}_d \to L(\mathbb{R}^m, \mathbb{R}^d)$ through

$$
f_n(t, \zeta, \gamma) = f\left(\left\lfloor \frac{nt}{n} \right\rfloor, \zeta, \gamma\right) \mathbf{1}_{t \in [0, T]} + f(0, \zeta, \gamma) \mathbf{1}_{\{0\}}.
$$

Then for all $n \in \mathbb{N}$ the function $f_n$ is $B([0, T]) \otimes B(\mathbb{D}_r) \otimes B(\mathbb{D}_d)$-measurable as a composition of measurable functions. Together with the fact that $g = \lim_{n \to \infty} f_n$ this implies that $g$ is $B([0, T]) \otimes B(\mathbb{D}_r) \otimes B(\mathbb{D}_d)$-measurable. \hfill $\square$

\section{Representation of the SDE solution as Skorohod measurable function of the driving processes}

The goal of this section is to prove that the solution of SDE (1) can be expressed as a Skorohod measurable function of its initial value, the process $G$, and the semimartingale $Y$. Note that in [4] it is proven that the solution can be expressed as a measurable function with respect to the Borel-$\sigma$-field generated by the sup-norm. This is, however, not sufficient for the existence of a regular conditional distribution, which we finally aim at, because the normed space induced by the sup-norm on the càdlàg functions is not separable. We will prove the following theorem.

\textbf{Theorem 3.1.} Let Assumption 2.1 hold. Then there exists a Skorohod measurable function $\Psi : \mathbb{D}_d \times \mathbb{D}_r \times \mathbb{D}_m \to \mathbb{D}_d$ such that for all $\mathbb{R}^d$-valued càdlàg processes $H$, all $\mathbb{R}^r$-valued càdlàg processes $G$, and all $\mathbb{R}^m$-valued càdlàg semimartingales $Y$,

$$X = \Psi(H, G, Y)$$

is the unique solution of

$$X_t = H_t + \int_0^t g(s, G, X) \, dY_s, \quad t \in [0, T].$$
For the convenience of the reader, we split the proof of this theorem in two parts. In the first part we construct a function $\Psi$ and prove that it is Skorohod measurable. In the second part we show that $X = \Psi(H, G, Y)$ solves the SDE.

For the construction of the function $\Psi$ in the first part we operate similar as in [4, p. 394], but with the additional requirement that the function needs to be measurable with respect to the Borel-$\sigma$-algebra generated by the Skorohod topology. Hence, our proof will differ at some crucial points from the one of [4, p. 394].

**Proof of Theorem 3.1, part 1.** We start by defining the mapping

$$
Ψ(0): D_d \times D_r \times D_m \to D_d, \quad Ψ(0)(\gamma, \zeta, \eta) = γ.
$$

Since for an open set $O \in B(D_d)$ it holds that $(Ψ(0))^{-1}(O) = O \times D_r \times D_m$ is again an open set, we know that $Ψ(0)$ is continuous and hence also measurable with respect to $(B(D_d) \otimes B(D_r) \otimes B(D_m))/B(D_d)$.

Now we define $(Ψ(n))_{n \in \mathbb{N}}$ and prove its measurability inductively: assume that $Ψ(n-1): D_d \times D_r \times D_m \to D_d$ is already defined and proven to be measurable. Consider the function

$$
Γ(n-1): [0, T) \times D_d \times D_r \times D_m \to L(\mathbb{R}^m, \mathbb{R}^d), \quad Γ(n-1)(t, γ, ζ, η) = f(t, ζ, Ψ(n-1)(γ, ζ, η)).
$$

This function is measurable as a concatenation of measurable functions. Based on $Γ(n-1)$ a sequence of time points $(t_j^{(n)})_{j \in \mathbb{N}} \subset [0, T]$ can be defined iteratively by $t_0^{(n)} = 0$ and

$$
t_{j+1}^{(n)} = \inf \{s \in [t_j^{(n)}, T]: \|Γ(n-1)(s, γ, ζ, η) - Γ(n-1)(t_j^{(n)}, γ, ζ, η)\| ≥ 2^{-n} \text{ or } \|Γ(n-1)(s, γ, ζ, η) - Γ(n-1)(t_j^{(n)}, γ, ζ, η)\| ≥ 2^{-n} \},
$$

where we set $\inf \emptyset = T$. This sequence of time points we write as functions by defining

$$
t_{j+1}^{(n)}: [0, T] \times D_d \times D_r \times D_m \to [0, T], \quad (t, γ, ζ, η) \mapsto t_{j+1}^{(n)}.
$$

We now prove that these functions are measurable. Let $n, j \in \mathbb{N}$ and $c \in [0, T)$. Then

$$(t_{j+1}^{(n)})^{-1}([0, c]) = \{ (t_j^{(n)}, γ, ζ, η) \in [0, T] \times D_d \times D_r \times D_m : \inf \{ s \in [t_j^{(n)}, T]: \|Γ(n-1)(s, γ, ζ, η) - Γ(n-1)(t_j^{(n)}, γ, ζ, η)\| ≥ 2^{-n} \text{ or } \|Γ(n-1)(s, γ, ζ, η) - Γ(n-1)(t_j^{(n)}, γ, ζ, η)\| ≥ 2^{-n} \} ≤ c \}$$

$$= \{ (t_j^{(n)}, γ, ζ, η) \in [0, T] \times D_d \times D_r \times D_m : \exists s \in [t_j^{(n)}, c]: \|Γ(n-1)(s, γ, ζ, η) - Γ(n-1)(t_j^{(n)}, γ, ζ, η)\| ≥ 2^{-n} \text{ or } \forall m \in \mathbb{N}, m ≥ 2^{n+1} \exists s \in [t_j^{(n)}, c]: \|Γ(n-1)(s, γ, ζ, η) - Γ(n-1)(t_j^{(n)}, γ, ζ, η)\| ≥ 2^{-n} - 1/m \}.$$
Since $Q$ is dense in $\mathbb{R}$,
\[
( t^{(n)}_{j+1} )^{-1}( [0, c] ) \\
= \bigcup_{q \in Q} \{ ( t^{(n)}_{j}, \gamma, \zeta, \eta ) : \| \Gamma^{(n-1)}( 0, t^{(n)}_{j}, \gamma, \zeta, \eta ) - \Gamma^{(n-1)}( t^{(n)}_{j}, \gamma, \zeta, \eta ) \| \geq 2^{-n} \text{ or } \forall m \in \mathbb{N}, m \geq 2^{n+1} \exists s \in [q, c] : \| \Gamma^{(n-1)}( s, \gamma, \zeta, \eta ) - \Gamma^{(n-1)}( t^{(n)}_{j}, \gamma, \zeta, \eta ) \| \geq 2^{-n} - 1/m \}
\]
and $q \geq t^{(n)}_{j}$
\[
= \bigcup_{q \in Q} \bigg( \bigcup_{s \in [q, c] \cap Q} \{ ( t^{(n)}_{j}, \gamma, \zeta, \eta ) \} \bigg)
\]
\[
\bigcup \bigg( \bigcup_{m=2^{n+1}}^{\infty} \bigcup_{s \in [q, c] \cap Q} \{ ( t^{(n)}_{j}, \gamma, \zeta, \eta ) \} \bigg)
\]
\[
= \bigcup_{q \in Q} \bigg( \bigcup_{s \in [q, c] \cap Q} \{ ( t^{(n)}_{j}, \gamma, \zeta, \eta ) \} \bigg)
\]
\[
= \bigcup_{q \in Q} \bigg( \bigcup_{m=2^{n+1}}^{\infty} \bigcup_{s \in [q, c] \cap Q} \{ ( t^{(n)}_{j}, \gamma, \zeta, \eta ) \} \bigg)
\]
\[
\bigcup \bigg( \bigcup_{m=2^{n+1}}^{\infty} \bigcup_{s \in [q, c] \cap Q} \{ ( t^{(n)}_{j}, \gamma, \zeta, \eta ) \} \bigg)
\]
As $\Gamma^{(n-1)} : [0, T] \times \mathbb{D}_d \times \mathbb{D}_r \times \mathbb{D}_m \rightarrow L(\mathbb{R}^m, \mathbb{R}^d)$ is measurable, $\Gamma^{(n-1)}$ is also measurable for fixed time $s \in [0, T]$ as function from $\mathbb{D}_d \times \mathbb{D}_r \times \mathbb{D}_m$ to $L(\mathbb{R}^m, \mathbb{R}^d)$. As a consequence also the function
\[
( t^{(n)}_{j}, \gamma, \zeta, \eta ) \mapsto \| \Gamma^{(n-1)}( s, \gamma, \zeta, \eta ) - \Gamma^{(n-1)}( t^{(n)}_{j}, \gamma, \zeta, \eta ) \| \geq 2^{-n} - 1/m
\]
is measurable for all $s \in [0, T]$. Hence, $t^{(n)}_{j+1}$ is measurable as its preimage is a countable union of measurable sets, see (5). Recalling $t^{(n)}_{0} = 0$, we can prove that $t^{(n)}_{1}$ is measurable as a function
from $\mathbb{D}_d \times \mathbb{D}_r \times \mathbb{D}_m$ to $[0,T]$ in a similar way. Now we interpret $t_j^{(n)}$ as a function from $\mathbb{D}_d \times \mathbb{D}_r \times \mathbb{D}_m$ to $[0,T]$ by plugging in the functions for $t_j^{(n)}, \ldots, t_1^{(n)}$ recursively. This function is measurable as a concatenation of measurable functions.

As a next step, following [3, p. 395], we define $\Psi^{(n)}: \mathbb{D}_d \times \mathbb{D}_r \times \mathbb{D}_m \to \mathbb{D}_d$ by

$$\Psi^{(n)}(\gamma, \zeta, \eta)(s) = \gamma(s) + \sum_{j=0}^{\infty} \Gamma^{(n-1)}(t_j^{(n)}, \gamma, \zeta, \eta)(\eta_j^{(n+1)}(s) - \eta_j^{(n)}(s)).$$  \hfill (6)

In [3, p. 395] instead of the interval $[0,T]$ the unbounded interval $[0,\infty)$ is considered. In our case (6) simplifies to a finite sum. To prove the measurability of $\Psi^{(n)}$, we first show that the function $h: [0,T] \times \mathbb{D}_m \to \mathbb{D}_m$ defined by $h(t,\eta) = \eta'$ is measurable. For this, we use [1, p. 153, Lemma 4.51], which states that it is sufficient to prove that for fixed $t \in [0,T]$ the mapping $\eta \mapsto \eta'$ is measurable and for fixed $\eta \in \mathbb{D}_m$ the mapping $t \mapsto \eta'$ is continuous in $t \in [0,T]$.

First, we prove that for fixed $t \in [0,T]$ the mapping $\eta \mapsto \eta'$ is measurable. For this, we apply [3, Theorem 4], which claims that it is enough to show that the operator $T_i: \mathbb{D}_m \to \mathbb{D}_m$, $T_i\eta = \eta'$ is a linear bounded operator, if we consider $\mathbb{D}_m$ equipped with the sup-norm instead of the Skorohod topology. The linearity is easy to check and the boundedness follows from the fact that

$$\|\eta'\|_\infty = \sup_{s \in [0,T]} \|\eta(t \land s)\| = \sup_{s \in [0,T]} \|\eta(s)\| \leq \sup_{s \in [0,T]} \|\eta(s)\| = \|\eta\|_\infty.$$  \hfill (7)

Next, we prove that for fixed $\eta \in \mathbb{D}_m$, the mapping $t \mapsto \eta'$ is continuous in $t \in [0,T]$. Let $t \in [0,T]$ and let $(t_k)_{k \in \mathbb{N}} \subset [0,T]$ be a monotone sequence that converges to $t$. Without loss of generality let $(t_k)_{k \in \mathbb{N}}$ be monotonically increasing. Then with $\lambda = \text{Id}$ we have for all $k \in \mathbb{N}$,

$$d^S(\eta', \eta'^k) \leq \sup_{s \in [0,T]} \|\eta(t \land s) - \eta(t_k \land s)\| = \sup_{s \in [t_k, t]} \|\eta(s) - \eta(t_k)\|.$$  

Now assume that $\sup_{s \in [t_k, t]} \|\eta(s) - \eta(t_k)\|$ is not convergent to 0 as $k \to \infty$, i.e. there exist $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ there exists $s_k \in (t_k, t)$ such that $\|\eta(s_k) - \eta(t_k)\| > \varepsilon$. But $\lim_{k \to \infty} \eta(s_k) = \eta(\cdot)$ and $\lim_{k \to \infty} \eta(t_k) = \eta(\cdot)$, which forms a contradiction. Hence, we have proven the continuity. This together with (7) implies the measurability of $h$

Next, observe that in [3, $\Gamma^{(n-1)}(t_j^{(n)}, \gamma, \zeta, \eta) \in L(\mathbb{R}^m, \mathbb{R}^d)$. This can also be considered a bounded linear operator from $\mathbb{D}_m$ to $\mathbb{D}_d$, which is applied for each time point to the càdlàg function evaluated at that point. Here $\mathbb{D}_m$ and $\mathbb{D}_d$ are equipped with the sup-norm. Then using [3, p. 387, Theorem 4], we get the measurability of this mapping in the Skorohod topology. Consequently, $\Psi^{(n)}$ is measurable as a concatenation and sum of measurable functions.

Now we define the function $\Psi: \mathbb{D}_d \times \mathbb{D}_r \times \mathbb{D}_m \to \mathbb{D}_d$ by

$$\Psi(\gamma, \zeta, \eta) = \begin{cases} \lim_{n \to \infty} \Psi^{(n)}(\gamma, \zeta, \eta), & \text{if the limit exists in the Skorohod topology,} \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathbb{D}_d$ is complete with respect to the Skorohod topology, we see that $\lim_{n \to \infty} \Psi^{(n)}(\gamma, \zeta, \eta)$ exists if and only if $(\Psi^{(n)}(\gamma, \zeta, \eta))_{n \in \mathbb{N}}$ is a Cauchy sequence. Hence, the set of all $(\gamma, \zeta, \eta)$ for which the above limit exists is given by

$$S := \left\{ (\gamma, \zeta, \eta) \in \mathbb{D}_d \times \mathbb{D}_r \times \mathbb{D}_m : \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} : \forall m, n \geq n_0 : d^S(\Psi^{(m)}(\gamma, \zeta, \eta), \Psi^{(n)}(\gamma, \zeta, \eta)) < \varepsilon \right\} = \bigcap_{k \in \mathbb{N}} \bigcup_{n_0} \bigcap_{m, n \geq n_0} \left\{ (\gamma, \zeta, \eta) \in \mathbb{D}_d \times \mathbb{D}_r \times \mathbb{D}_m : d^S(\Psi^{(m)}(\gamma, \zeta, \eta), \Psi^{(n)}(\gamma, \zeta, \eta)) < \frac{1}{k} \right\}.$$
Hence, $\Psi^{(n)}1_{S}$ is measurable, because $S$ is measurable and the function $\Psi = \lim_{n \to \infty} \Psi^{(n)}1_{S}$ is measurable as the limit of measurable functions. This closes part 1 of the proof of Theorem 3.1. □

To prove that the above constructed Skorohod measurable function is indeed the function we are looking for in Theorem 3.1 we need the following definitions and notations.

**Definition 3.2** (cf. [3, p. 368, Definition 11.4]). An adapted increasing process $V$ is said to be a dominating process for a real-valued semimartingale $\xi$, if there exists a decomposition $\xi = M + A$, with $M$ a locally square integrable martingale with $M_0 = 0$, $A$ a process with finite variation paths such that the process $B$ defined by

$$B_t = V_t - 2\sqrt{2}[M, M]_t + \langle M, M \rangle_t^{1/2} - \sqrt{2}|A|_t, \quad t \in [0, T]$$

is an increasing process with $B_0 \geq 0$.

**Remark 3.3.** It is known that for each semimartingale $\xi$ there exists a dominating process $V$, see [3, p. 368, Theorem 11.5]. Furthermore, for each stopping time $\nu: \Omega \to [0, T]$ it holds that

$$\mathbb{E}[\sup_{0 \leq t < \nu} |\xi_t|^2] \leq \mathbb{E}[V^2],$$

see [3, p. 369, Theorem 11.7].

We also need the following theorem.

**Theorem 3.4** (cf. [3, p. 373, Theorem 11.13]). Let $\xi$ be a semimartingale and let $P$ be a locally bounded predictable process. Let $V$ be a dominating process for $\xi$. Then for any stopping time $\nu: \Omega \to [0, T]$ it holds

$$\mathbb{E}\left[\sup_{0 \leq t < \nu} \left|\int_0^t P_s \, d\xi_s \right|^2\right] \leq \mathbb{E}[\theta_\nu^2(P, V)],$$

where $\theta_\nu(P, V)$ is for all $t \in [0, T]$ defined by

$$\theta_t(P, V) = \left(\int_0^t |P_s|^2 \, dV_s^2\right)^{1/2} + \int_0^t |P_s| \, dV_s.$$

Further,

$$\mathbb{E}\left[\sup_{0 \leq t < \nu} \left|\int_0^t P_s \, d\xi_s \right|^2\right] \leq 4\mathbb{E}\left[\left(\sup_{0 \leq s < \nu} |P_s|^2\right) V_{\nu-}\right].$$

We continue with the second part of the proof of Theorem 3.1 where we show that $\Psi$ constructed in part 1 of the proof has the property that $X = \Psi(H, G, Y)$ is the solution of SDE $\Pi$.

**Proof of Theorem 3.1** part 2. To prove the desired property we proceed similar as in the first part of the proof, but instead of defining a sequence of functions approximating $\Psi$, we define a sequence of càdlàg adapted stochastic processes approximating the solution of SDE $\Pi$. First, we set $Z^{(0)} = H$. Next, we define $Z^{(n)}$ inductively as follows: assume that $Z^{(0)}, \ldots, Z^{(n-1)}$ are already defined. We define a sequence $(\tau^{(n)}_j)_{j \in \mathbb{N}}$ by $\tau^{(n)}_0 = 0$ and for $j \geq 1$ recursively by

$$\tau^{(n)}_{j+1} = \inf \left\{ s \in [\tau^{(n)}_j, T] : \|f(s, G, Z^{(n-1)}) - f(\tau^{(n)}_j, G, Z^{(n-1)})\| \geq 2^{-n} \right\}.$$
This is a sequence of stopping times, since \( s \mapsto \| f(s, G, Z^{(n-1)}) - f(\tau^{(n)}_j, G, Z^{(n-1)}) \| \) is a càdlàg adapted process. Further, it holds that \( \lim_{j \to \infty} \tau^{(n)}_j = T \). Now we are able to define \( Z^{(n)}_0 = H_0 \) and for all \( j \in \mathbb{N}_0, \tau^{(n)}_j < t \leq \tau^{(n)}_{j+1} \) by

\[
Z^{(n)}_t = H_t + f(\tau^{(n)}_j, G, Z^{(n-1)})(Y_t - Y_{\tau^{(n)}_j}).
\]

Equivalent for all \( t \in [0, T] \),

\[
Z^{(n)}_t = H_t + \sum_{j=0}^{\infty} f(\tau^{(n)}_j, G, Z^{(n-1)}) \left( Y^{(n)}_{\tau^{(n)}_{j+1}} - Y^{(n)}_t \right).
\] (8)

Comparing equation (8) and (6) we obtain that for almost all \( \omega \in \Omega \) it holds

\[
Z^{(n)}_t(\omega) = \Psi^{(n)}(H(\omega), G(\omega), Y(\omega)).
\]

In a next step we explore the convergence of the sequence \( (Z^{(n)}_t)_{n \in \mathbb{N}} \). For this we follow [4, p. 396] and define for all \( n \in \mathbb{N} \) the sequences \( (S^{(n)}_t)_{n \in \mathbb{N}} \) and \( (R^{(n)}_t)_{n \in \mathbb{N}} \) through

\[
S^{(n)}_t = \sum_{j=0}^{\infty} f(\tau^{(n)}_j, G, Z^{(n-1)}) \mathbb{1}_{[\tau^{(n)}_j, \tau^{(n)}_{j+1})}(t), \quad t \in [0, T],
\] (9)

\[
R^{(n)}_t = H_t + \int_0^t f(s, G, Z^{(n-1)}) \, dY_s, \quad t \in [0, T].
\]

It holds that

\[
Z^{(n)}_t = H_t + \int_0^t S^{(n)}_{s-} \, dY_s,
\] (10)

here we need \( s- \) in the integrand, because the half open interval \( [\tau^{(n)}_j, \tau^{(n)}_{j+1}) \) needs to have the form \( (\tau^{(n)}_j, \tau^{(n)}_{j+1}] \) to deliver the correct integral. Furthermore,

\[
\| S^{(n)}_t - f(t, G, Z^{(n-1)}) \| \leq 2^{-n}.
\] (11)

Since \( Y \) is an \( \mathbb{R}^m \)-valued semimartingale, we know that each component of \( Y \) is a semimartingale and hence admits a dominating process by Remark 3.3. Summing up these dominating processes we obtain a common dominating process, which we denote by \( V \). Now we define the process \( U \) by

\[
U_t = V_t + V_t^2 + \sup_{0 \leq s \leq t} \| H_s \| + C(t, G), \quad t \in [0, T].
\]

Here \( C(t, G(\omega)) \) is as in Assumption 2.4 (iii). Further define the sequence of stopping times \( (\nu_j)_{j \in \mathbb{N}} \) for all \( j \in \mathbb{N} \) by

\[
\nu_j = \inf \left\{ \{ t \in [0, T] : U_t \geq j \text{ or } U_{t-} \geq j \} \cup \{ T \} \right\}.
\] (12)

The sequence defined in (12) is indeed a sequence of stopping times, because it is the minimum of two first hitting times and \( U \) as well as \( U_- \) are progressively measurable. Further, it holds that \( \lim_{j \to \infty} \nu_j = T \). Next, combining (9) and (10) we get

\[
E \left[ \sup_{0 \leq t \leq \nu_j} \| R^{(n)}_t - Z^{(n)}_t \| \right] = E \left[ \sup_{0 \leq t \leq \nu_j} \| H_t + \int_0^t f(s, G, Z^{(n-1)}) \, dY_s - H_t - \int_0^t S^{(n)}_{s-} \, dY_s \| \right] = E \left[ \sup_{0 \leq t \leq \nu_j} \| \int_0^t (f(s, G, Z^{(n-1)}) - S^{(n)}_{s-}) \, dY_s \| \right].
\] (13)
To continue this calculation, recall the following standard notation. For a matrix $A$ we denote by $A_{ij}$ the entry of $A$ in the $i$-th row and the $j$-th column and for a vector $b$ we denote by $b_j$ the $j$-th entry of the vector. Hence, for $i \in \{1, \ldots, d\}$ the $i$-th entry of the vector

$$
\int_0^t (f(s-, G, Z^{(n-1)}) - S_{s-}^{(n)}) \, dY_s
$$

is given by

$$
\left(\int_0^t (f(s-, G, Z^{(n-1)}) - S_{s-}^{(n)}) \, dY_s\right)_i = \sum_{k=1}^m \int_0^t (f(s-, G, Z^{(n-1)}) - S_{s-}^{(n)})_{ik} \, d(Y_s)_k. \quad (14)
$$

Using (14) and applying the Cauchy-Schwarz inequality to (13) we obtain that

$$
\mathbb{E}\left[ \sup_{0 \leq t < \nu} \left\| \int_0^t (f(s-, G, Z^{(n-1)}) - S_{s-}^{(n)}) \, dY_s \right\|^2 \right]
\leq \mathbb{E}\left[ \sup_{0 \leq t < \nu} \sum_{i=1}^d m \sum_{k=1}^m \left| \int_0^t (f(s-, G, Z^{(n-1)}) - S_{s-}^{(n)})_{ik} \, d(Y_s)_k \right|^2 \right]
\leq m \sum_{i=1}^m \sup_{0 \leq t < \nu} \left| \int_0^t (f(s-, G, Z^{(n-1)}) - S_{s-}^{(n)})_{ik} \, d(Y_s)_k \right|^2
$$

(15)

We combine this with the second statement of Theorem 3.4 and use the definition of the stopping time $\nu_j$ in (12) as well as the estimate in (11) to obtain that

$$
\sum_{i=1}^d m \sum_{k=1}^m \mathbb{E} \left[ \sup_{0 \leq t < \nu_j} \left| \int_0^t (f(s-, G, Z^{(n-1)}) - S_{s-}^{(n)})_{ik} \, d(Y_s)_k \right|^2 \right]
\leq 4 \sum_{i=1}^d m \sum_{k=1}^m \mathbb{E} \left[ \sup_{0 \leq s < \nu_j} \left| (f(s-, G, Z^{(n-1)}) - S_{s-}^{(n)})_{ik} \right|^2 V_{\nu_j} \right]
\leq 4 \sum_{i=1}^d m m \sum_{k=1}^m \mathbb{E} \left[ \sup_{0 \leq s < \nu_j} \left| (f(s-, G, Z^{(n-1)}) - S_{s-}^{(n)})_{ik} \right|^2 \right]
\leq 4 \sum_{i=1}^d m m \sum_{k=1}^m 2^{-2n} j^2 = 4dm^2 2^{-2n} j^2. \quad (16)
$$

Combining (13), (15), and (16) we conclude that

$$
\mathbb{E} \left[ \sup_{0 \leq t < \nu_j} \| R_t^{(n)} - Z_t^{(n)} \|^2 \right] \leq 4dm^2 2^{-2n} j^2.
$$

Next, we define for all $n \in \mathbb{N}_0$, $t \in [0, T]$,

$$
A_t^{(n)} = \sup_{0 \leq s \leq t} \| Z_s^{(n+1)} - Z_s^{(n)} \|.
$$
For any stopping time \( \tau \leq \nu_j \) and for all \( n \in \mathbb{N} \) we estimate the second moment of \( A_{\tau-}^{(n)} \) by

\[
\mathbb{E}\left[ (A_{\tau-}^{(n)})^2 \right] = \mathbb{E}\left[ \sup_{0 \leq t < \tau} \| Z_t^{(n+1)} - Z_t^{(n)} \|^2 \right] \\
\leq \mathbb{E}\left[ \sup_{0 \leq t < \tau} \left( \| Z_t^{(n+1)} - R_t^{(n+1)} \| + \| R_t^{(n+1)} - Z_t^{(n)} \| + \| R_t^{(n+1)} - R_t^{(n)} \| \right)^2 \right] \\
\leq 3\mathbb{E}\left[ \sup_{0 \leq t < \tau} \| R_t^{(n+1)} - Z_t^{(n+1)} \|^2 \right] + 3\mathbb{E}\left[ \sup_{0 \leq t < \tau} \| R_t^{(n+1)} - Z_t^{(n)} \|^2 \right] \\
+ 3\mathbb{E}\left[ \sup_{0 \leq t < \tau} \| R_t^{(n+1)} - R_t^{(n)} \|^2 \right] \\
\leq 12dm^2j^22^{-2n}(1 + 2^{-2}) + 3\mathbb{E}\left[ \sup_{0 \leq t < \tau} \| R_t^{(n+1)} - R_t^{(n)} \|^2 \right]. 
\]

Further,

\[
\mathbb{E}\left[ \sup_{0 \leq t < \tau} \| R_t^{(n+1)} - R_t^{(n)} \|^2 \right] \\
= \mathbb{E}\left[ \sup_{0 \leq t < \tau} \left| \int_0^t (f(s-, G, Z^{(n)}) - f(s-, G, Z^{(n-1)})) dY_s \right|^2 \right] \\
\leq \mathbb{E}\left[ \sup_{0 \leq t < \tau} \sum_{i=1}^d \sum_{k=1}^m \left| \int_0^t (f(s-, G, Z^{(n)}) - f(s-, G, Z^{(n-1)}))_{ik} dY_s(k) \right|^2 \right] \\
\leq \sum_{i=1}^d \sum_{k=1}^m \mathbb{E}\left[ \sup_{0 \leq t < \tau} \left( \int_0^t (f(s-, G, Z^{(n)}) - f(s-, G, Z^{(n-1)}))_{ik} dY_s(k) \right)^2 \right]. 
\]

Next, we apply the first statement of Theorem 3.4 to estimate for all \( i \in \{1, \ldots, d\} \), \( k \in \{1, \ldots, m\} \),

\[
\mathbb{E}\left[ \sup_{0 \leq t < \tau} \left| \int_0^t (f(s-, G, Z^{(n)}) - f(s-, G, Z^{(n-1)}))_{ik} dY_s(k) \right|^2 \right] \\
\leq \mathbb{E}\left[ \theta^2_{ik} (f(s-, G, Z^{(n)}) - f(s-, G, Z^{(n-1)}))_{ik}, V_s \right] \\
\leq \mathbb{E}\left[ \left( \int_0^{\tau-} \left| (f(s-, G, Z^{(n)}) - f(s-, G, Z^{(n-1)}))_{ik} \right|^2 dV_s^2 \right)^{1/2} \right. \\
\left. + \int_0^{\tau-} \left| (f(s-, G, Z^{(n)}) - f(s-, G, Z^{(n-1)}))_{ik} \right| dV_s \right]^2 \\
\leq \mathbb{E}\left[ \left( \int_0^{\tau-} C(s-, G) \sup_{0 \leq u < s} \| Z_u^{(n)} - Z_u^{(n-1)} \|^2 dV_s^2 \right)^{1/2} \right. \\
\left. + \int_0^{\tau-} C(s-, G) \sup_{0 \leq u < s} \| Z_u^{(n)} - Z_u^{(n-1)} \| dV_s \right] \\
\leq j^2 \mathbb{E}\left[ \left( \int_0^{\tau-} \sup_{0 \leq u < s} \| Z_u^{(n)} - Z_u^{(n-1)} \|^2 dV_s^2 \right)^{1/2} \right. \\
\left. + \int_0^{\tau-} \sup_{0 \leq u < s} \| Z_u^{(n)} - Z_u^{(n-1)} \| dV_s \right] \\
= j^2 \mathbb{E}\left[ \left( \int_0^{\tau-} (A_{\tau-}^{(n-1)})^2 dV_s^2 \right)^{1/2} + \int_0^{\tau-} A_{\tau-}^{(n-1)} dV_s \right]^2 \\
\leq 2j^2 \mathbb{E}\left[ \int_0^{\tau-} (A_{\tau-}^{(n-1)})^2 dV_s^2 + \left( \int_0^{\tau-} A_{\tau-}^{(n-1)} dV_s \right)^2 \right]. 
\]

Here we used additionally Assumption 2.1(iii) the definition of \( \nu_j \), and the definition of \( A_{\tau-}^{(n-1)} \).
In the next step we apply Jensen’s inequality and obtain
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t \left( f(s, G, Z^{(n)}) - f(s, G, Z^{(n-1)}) \right)_{ik} d(Y_s)_k \right|^2 \right]
\]
\[
\leq 2 j^2 \mathbb{E} \left[ \int_0^\tau (A_{s-}^{(n-1)})^2 dV_s^2 + V_{s-} \int_0^\tau (A_{s-}^{(n-1)})^2 dV_s \right]
\leq 2 j^3 \mathbb{E} \left[ \int_0^\tau (A_{s-}^{(n-1)})^2 d(V_s^2 + V_s) \right].
\] (19)

Plugging (19) and (18) into (17) yields
\[
\mathbb{E} \left[ (A_{\tau-}^{(n)})^2 \right]
\leq 15dm^2 j^2 2^{-2n} + 3 \sum_{i=1}^d \sum_{k=1}^m 2j^3 \mathbb{E} \left[ \int_0^\tau (A_{s-}^{(n-1)})^2 d(V_s^2 + V_s) \right] \hspace{1cm} (20)
\]
\[
= 15dm^2 j^2 2^{-2n} + 6dm^2 j^3 \mathbb{E} \left[ \int_0^\tau (A_{s-}^{(n-1)})^2 d(V_s^2 + V_s) \right].
\]

Using similar considerations as in (15) and the second statement of Theorem 3.4 we get
\[
\mathbb{E} \left[ (A_{\tau-}^{(0)})^2 \right] = \mathbb{E} \left[ \left( \sup_{0 \leq t \leq \tau} \| Z^{(1)}_t - Z^{(0)}_t \| \right)^2 \right] = \mathbb{E} \left[ \left( \sup_{0 \leq t \leq \tau} \| H_t + \int_0^t S^{(1)}_s dY_s - H_t \| \right)^2 \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \sum_{i=1}^d \sum_{k=1}^m (S^{(1)}_{i-})_{ik} d(Y_s)_k \right]^2 \leq \sum_{i=1}^d \sum_{k=1}^m \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \| (S^{(1)}_{i-})_{ik} \|^2 V_{s-}^2 \right] \leq 4 dm^2 j^2 \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \| S^{(1)}_t \|^2 \right].
\] (21)

Using (9) and Assumption 2.1 (iii) we obtain that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \| S^{(1)}_t \|^2 \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \left\| \sum_{j=0}^{\infty} f(t_j^{(1)}, G, Z^{(0)}) \mathbf{1}_{(t_j^{(1)}, t_j^{(1)+1})}(t) \right\|^2 \right]
\leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \sum_{j=0}^{\infty} \| f(t_j^{(1)}, G, H) \|^2 \mathbf{1}_{(t_j^{(1)}, t_j^{(1)+1})}(t) \right]
\leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \sum_{j=0}^{\infty} \| C(t_j^{(1)}, G) \|^2 \left( 1 + \sup_{0 \leq s \leq t_j^{(1)+1}} \| H_s \| \right)^2 \mathbf{1}_{(t_j^{(1)}, t_j^{(1)+1})}(t) \right]
\leq \mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \| C(t, G) \|^2 \sup_{0 \leq t \leq \tau} (1 + \| H_t \|)^2 \right]
\leq j^2 (1 + j)^2.
\] (22)

Plugging (22) into (21) we get
\[
\mathbb{E} \left[ (A_{\tau-}^{(0)})^2 \right] \leq 4 dm^2 j^4 (1 + j)^2.
\] (23)
Next, we define for all \( t \in [0, T] \),
\[
B_t = \sum_{n=0}^{\infty} 2^n (A_t^{(n)})^2.
\]

For some stopping time \( \tau \leq \nu_j \), (20) and (23) assure
\[
\mathbb{E}[B_{\tau-}] = \mathbb{E} \left[ \sum_{n=0}^{\infty} 2^n (A_{\tau-}^{(n)})^2 \right] = \sum_{n=0}^{\infty} 2^n \mathbb{E}[(A_{\tau-}^{(n)})^2]
\]
\[
\leq \mathbb{E}[(A_{\tau-}^{(0)})^2] + \sum_{n=1}^{\infty} \left( 2^n 15d^2 j^2 2^{-2n} + 2^n 6dm^2 j^3 \mathbb{E} \left[ \int_0^{\tau-} (A_s^{(n-1)})^2 \, d(V_s^2 + V_s) \right] \right)
\]
\[
\leq 4dm^2 j^4 (1 + j)^2 + 15dm^2 j^2 \sum_{n=1}^{\infty} 2^{-n} + 12dm^2 j^3 \mathbb{E} \left[ \int_0^{\tau-} (A_s^{(n-1)})^2 \, d(V_s^2 + V_s) \right]
\]
\[
= 4dm^2 j^4 (1 + j)^2 + 15dm^2 j^2 + 12dm^2 j^3 \mathbb{E} \left[ \int_0^{\tau-} B_s \, d(V_s^2 + V_s) \right].
\]

Now we apply Gronwall’s inequality, see [4, Theorem 12.1], to conclude that there exists a constant \( c \in (0, \infty) \), which only depends on \( j \), such that
\[
\mathbb{E}[B_{\nu_j-}] \leq c.
\]

Hence, for all \( j \in \mathbb{N} \) we have proven that
\[
\sum_{n=0}^{\infty} 2^n \mathbb{E}[(A_{\nu_j-}^{(n)})^2] < \infty.
\]

This implies that for large \( n \in \mathbb{N} \), \( \mathbb{E}[(A_{\nu_j-}^{(n)})^2] < 2^{-n} \). Therefore, it holds that
\[
\sum_{n=0}^{\infty} \left( \mathbb{E}[(A_{\nu_j-}^{(n)})^2] \right)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \left( \mathbb{E} \left[ \sup_{0 \leq s < \nu_j} \| Z_s^{(n+1)} - Z_s^{(n)} \|^2 \right] \right)^{\frac{1}{2}}
\]
\[
= \sum_{n=0}^{\infty} \left\| \sup_{0 \leq s < \nu_j} Z_s^{(n+1)} - Z_s^{(n)} \right\|_2 < \infty.
\]

Hence, we obtain
\[
\left\| \sum_{n=0}^{\infty} \sup_{0 \leq s < \nu_j} Z_s^{(n+1)} - Z_s^{(n)} \right\|_2 < \infty. \tag{24}
\]

This implies that
\[
N = \bigcup_{j=1}^{\infty} \left\{ \omega \in \Omega : \sum_{n=0}^{\infty} \sup_{0 \leq s < \nu_j} \| Z_s^{(n+1)} - Z_s^{(n)} \| = \infty \right\}
\]
is a \( \mathbb{P} \)-null set. Hence, for all \( \omega \in \Omega \setminus N \) the sequence \( (Z_s^{(n)}(\omega))_{n \in \mathbb{N}} \) converges uniformly on \( [0, \nu_j(\omega)) \) for all \( j \in \mathbb{N} \). Therefore, the process \( \tilde{Z} \) defined by
\[
\tilde{Z}_t(\omega) = \begin{cases} 
\lim_{n \to \infty} Z_t^{(n)}(\omega), & \omega \in \Omega \setminus N, \\
0, & \omega \in N
\end{cases}
\]
is well-defined. Since \( \mathbb{P}(N) = 0 \) and because of the definition of \( \nu_j \) we have that \( Z^{(n)} \) converges uniformly on \( [0, T] \) to \( \tilde{Z} \) almost surely. Hence, \( Z^{(n)} \) also converges almost surely to \( \tilde{Z} \) in the Skorohod metric, since by the definition of the Skorohod metric it follows that every convergent
Hence, recalling the definition of $Z^{(n)}$, we observe that $Z$ and $\tilde{Z}$ differ only on the subset of a nullset and are hence indistinguishable. We get that $Z^{(n)} = \Psi^{(n)}(H, G, Y)$ converges almost surely to $Z$ in the Skorohod topology. Hence, $Z = \Psi(H, G, Y)$ a.s.

Furthermore, it follows from (23) that

$$\sup_{k \geq 1} \sum_{j=n+1}^{\infty} \sum_{0 \leq s \leq \nu_j} \|Z(j) - Z(j-1)\|_{2}^2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$  \hspace{1cm} (25)

The definition of $\tilde{Z}$ and (25) yield

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sup_{0 \leq s \leq \nu_j} \|\tilde{Z}_s - Z^{(n)}_s\|^2 \right] = 0.$$ \hspace{1cm} (26)

Further, we get by Assumption 2.1(iii) and the definitions of $U$ and $\nu_j$ that

$$\sup_{0 \leq s \leq \nu_j} \|f(s, G, \tilde{Z}) - f(s, G, Z^{(n)})\| \leq \sup_{0 \leq s \leq \nu_j} C(s, G)\|\tilde{Z} - Z^{(n)}\| \leq j \sup_{0 \leq s \leq \nu_j} \|\tilde{Z}_s - Z^{(n)}_s\|.$$ \hspace{1cm} (27)

The second statement of Theorem 3.4 together with (27) and (26) yields

$$\mathbb{E} \left[ \sup_{0 \leq s \leq \nu_j} \left\|H_t + \int_0^t f(s, G, \tilde{Z}) \, dY_s - H_t - \int_0^t f(s, G, Z^{(n)}) \, dY_s \right\|^2 \right]$$

$$= \mathbb{E} \left[ \sup_{0 \leq s \leq \nu_j} \left\| \int_0^t \left( f(s, G, \tilde{Z}) - f(s, G, Z^{(n)}) \right) \, dY_s \right\|^2 \right]$$

$$\leq \mathbb{E} \left[ \sup_{0 \leq s \leq \nu_j} \sum_{i=1}^d \sum_{k=1}^m \left\| \int_0^t \left( f(s, G, \tilde{Z}) - f(s, G, Z^{(n)}) \right)_{ik} \, d(Y_s)_k \right\|^2 \right]$$

$$\leq \sum_{i=1}^d \sum_{k=1}^m \mathbb{E} \left[ \sup_{0 \leq s \leq \nu_j} \left\| \int_0^t \left( f(s, G, \tilde{Z}) - f(s, G, Z^{(n)}) \right)_{ik} \, d(Y_s)_k \right\|^2 \right]$$

$$\leq 4dm^2 j \mathbb{E} \left[ \sup_{0 \leq s \leq \nu_j} \|f(s, G, \tilde{Z}) - f(s, G, Z^{(n)})\|^2 \nu_{ij} \right]$$

$$\leq 4dm^2 j \mathbb{E} \left[ \sup_{0 \leq s \leq \nu_j} \|\tilde{Z}_s - Z^{(n)}_s\|^2 \right] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, recalling the definition of $Z^{(n)}$, we observe that $\tilde{Z}$ is a solution of SDE (1). Since $Z$ and $\tilde{Z}$ are indistinguishable, we get that $Z = \Psi(H, G, Y)$ is a solution of SDE (1). This closes the proof of Theorem 3.1. \hfill \square
4 Existence of the regular conditional distribution

In this section we prove the existence of the regular conditional distribution for solutions of SDEs. This proof is based on Theorem 3.1. As a corollary we obtain a Markov property. For the proof we need the following result.

Theorem 4.1 ([3, p. 185, Theorem 8.37]). Let \( \mathcal{A} \subset \mathcal{F} \) be a sub-\( \sigma \)-algebra. Let \( Z \) be a random variable with values in a Polish space \((E, \mathcal{E})\). Then there exists a regular conditional distribution \( \kappa_{Z,\mathcal{A}} \) of \( Z \) given \( \mathcal{A} \).

Theorem 4.2. Let Assumption 2.1 hold. Let \( t_0 \in [0,T] \). Then the regular conditional distribution of \((X_s)_{s \in [t_0,T]} \) given \( X_{t_0} \) exists, i.e. there exists a stochastic kernel \( \kappa_{(X_s)_{s \in [t_0,T]}},\sigma(X_{t_0}) \) from \((\Omega, \mathcal{F})\) to \((\mathbb{D}_d, \mathcal{B}(\mathbb{D}_d))\) such that for all \( A \in \mathcal{F} \) and all \( B \in \mathcal{B}(\mathbb{D}_d) \) it holds that

\[
\int_A \mathbb{1}_B((X_s)_{s \in [t_0,T]}) \, d\mathbb{P} = \int_A \kappa_{(X_s)_{s \in [t_0,T]}},\sigma(X_{t_0})(\cdot, B) \, d\mathbb{P}.
\]

Proof. First, we prove that for all \( n \in \mathbb{N} \), any \( \mathbb{R}^n \)-valued càdlàg adapted stochastic process \( Z = (Z_t)_{t \in [0,T]} \) is also a random variable with values in \((\mathbb{D}_n, \mathcal{B}(\mathbb{D}_n))\). Recall that \( \mathcal{B}(\mathbb{D}_n) \) is generated by all coordinate mappings \( \pi_t \) with \( t \in [0,T] \), which are defined for all \( \gamma \in \mathbb{D}_n \) through

\[
\pi_t(\gamma) = \gamma(t).
\]

Let \( A \in \mathcal{B}(\mathbb{R}^n) \) and \( t \in [0,T] \). Then it holds that

\[
Z^{-1}(\pi_t^{-1}(A)) = Z^{-1}(\{ \gamma \in \mathbb{D}_n : \gamma(t) \in A \}) = \{ \omega \in \Omega : Z(\omega) \in \{ \gamma \in \mathbb{D}_n : \gamma(t) \in A \} \} = \{ \omega \in \Omega : Z_t(\omega) \in A \} \in \mathcal{F}.
\]

The measurability of the stochastic processes \( H, G, \) and \( Y \) implies that the mapping

\[
\omega \mapsto X(\omega) = \Psi(H(\omega), G(\omega), Y(\omega))
\]

is a measurable mapping with values in \((\mathbb{D}_d, \mathcal{B}(\mathbb{D}_d))\) as a concatenation of measurable mappings. Hence for \( t_0 \in [0,T] \) obviously also the mapping

\[
\omega \mapsto \Psi(H(\omega), G(\omega), Y(\omega)) \mathbb{1}_{s \in [t_0,T]}
\]

is measurable. Next we choose \( \mathcal{A} \) in Theorem 4.1 to be \( \sigma(X_{t_0}) \) and set \( Z = \Psi(H,G,Y) \mathbb{1}_{s \in [t_0,T]} \). Note that \((\mathbb{D}_d, \mathcal{B}(\mathbb{D}_d))\) is a Polish space. With this Theorem 4.1 ensures the existence of a regular conditional distribution \( \kappa_{\Psi(H,G,Y) \mathbb{1}_{s \in [t_0,T]},\sigma(X_{t_0})} \). Hence, \( \mathbb{P}(X_s \mathbb{1}_{s \in [t_0,T]} | X_{t_0} = x) \) exists and is given for all \( B \in \mathcal{B}(\mathbb{D}_d) \) by

\[
\mathbb{P}(X_s \mathbb{1}_{s \in [t_0,T]} | X_{t_0} = x)(B) = \kappa_{\Psi(H,G,Y) \mathbb{1}_{s \in [t_0,T]},\sigma(X_{t_0})}(X_{t_0}^{-1}(x), B).
\]

This proves the claim.

In the end we give an example how this result can be applied for proving Markov properties for solutions of semimartingale SDEs. In the special case of a classical Brownian motion-driven SDE such a result has been proven in [8] using similar arguments and referring to the functional representation proven in [3].
Corollary 4.3. Let $\hat{X}^x_t$ be the strong solution of the semimartingale SDE
\[
\hat{X}^x_t = x + \int_0^t \hat{f}(\hat{X}_{s-}) \, d\hat{Y}_s, \quad t \in [0, T],
\]  
where $x \in \mathbb{R}^d$, $\hat{Y}$ is an $\mathbb{R}^m$-valued càdlàg semimartingale, and $\hat{f}: \mathbb{R}^d \to L(\mathbb{R}^m, \mathbb{R}^d)$ is a function which is measurable with respect to $\mathcal{B}(\mathbb{R}^d)$ and such that there exists a constant $\hat{C} \in (0, \infty)$ such that for all $x, y \in \mathbb{R}^d$,
\[
\|\hat{f}(x)\| \leq \hat{C} \left(1 + \|x\|\right),
\]
\[
\|\hat{f}(x) - \hat{f}(y)\| \leq \hat{C} \left(\|x - y\|\right).
\]
Furthermore, let $t \in [0, T]$ and assume that for $\hat{Y}$ it holds that $(\hat{Y}_{t+s} - \hat{Y}_t)_{s \in [0, T-t]}$ has the same distribution as $(\hat{Y}_s)_{s \in [0, T-t]}$. Then it holds that
\[
\mathbb{P}(\hat{X}^x_t)_{s \in [0, T]} | X^x_t = y = \mathbb{P}(\hat{X}^y_t)_{s \in [0, T-t]}.
\]

Proof. SDE (28) is a special case of SDE (11) with $H \equiv x$, $G \equiv 0$, and $f(t-\zeta, \gamma) = \hat{f}(\gamma_{t-})$, and fulfills Assumptions [21]. Hence, we may apply Theorem 3.1 to SDE (11) with the time interval $[0, T-t]$ and get the existence of a measurable function $\Psi$ such that $X = \Psi(H, G, Y)$ for all $\mathbb{R}^d$-valued càdlàg processes $H$, all $\mathbb{R}^r$-valued càdlàg processes $G$, and all $\mathbb{R}^m$-valued càdlàg semimartingales $Y$. This implies that for each initial value $y \in \mathbb{R}^d$ it holds that $(\hat{X}^y_t)_{s \in [0, T-t]} = \Psi(y, 0, (\hat{Y}_s)_{s \in [0, T-t]}) = \Psi(y, (\hat{Y}_s)_{s \in [0, T-t]}).

Furthermore, $(\hat{X}^x_{t+s}, (\hat{Y}_{t+s} - \hat{Y}_t)_{s \in [0, T-t]})$, since
\[
\hat{X}^x_{t+s} = \hat{X}^x_t + \int_t^{t+s} \hat{f}(\hat{X}^x_{u-}) \, d\hat{Y}_u = \hat{X}^x_t + \int_0^s \hat{f}(\hat{X}^x_{u+t-}) \, d\hat{Y}_{u+t}
\]
\[
= \hat{X}^x_t + \int_0^s \hat{f}(\hat{X}^x_{u+t-}) \, d(\hat{Y}_{u+t} - \hat{Y}_t).
\]

By Theorem 1.2 the regular conditional distribution $\mathbb{P}((\hat{X}^x_t)_{s \in [0, T]} | X^x_t = y)$ exists. Hence,
\[
\mathbb{P}(\hat{X}^x_t)_{s \in [0, T]} | X^x_t = y = \mathbb{P}(\hat{X}^x_t)_{s \in [0, T]} | (\hat{Y}_{t+s} - \hat{Y}_t)_{s \in [0, T-t]} = \mathbb{P}(\hat{X}^y_t)_{s \in [0, T-t)} = \mathbb{P}(\hat{X}^y_t)_{t \in [0, T-t]},
\]
which proves the claim. \square

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