A NON-INJECTIVE VERSION OF WIGNER’S THEOREM

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Abstract. Let $H$ be a complex Hilbert space and let $\mathcal{F}_s(H)$ be the real vector space of all self-adjoint finite rank operators on $H$. We prove the following non-injective version of Wigner’s theorem: every linear operator on $\mathcal{F}_s(H)$ sending rank one projections to rank one projections (without any additional assumption) is either induced by a linear or conjugate-linear isometry or constant on the set of rank one projections.

1. Introduction

Wigner’s theorem plays an important role in mathematical foundations of quantum mechanics. Pure states of a quantum mechanical system are identified with rank one projections (see, for example, [21]) and Wigner’s theorem [22] characterizes all symmetries of the space of pure states as unitary and anti-unitary operators. We present a non-injective version of this result in terms of linear operators on the real vector space of self-adjoint finite rank operators which send rank one projections to rank one projections.

Let $H$ be a complex Hilbert space. For every natural $k < \dim H$ we denote by $\mathcal{P}_k(H)$ the set of all rank $k$ projections, i.e. bounded self-adjoint idempotent operators of rank $k$. Let $\mathcal{F}_s(H)$ be the real vector space of all self-adjoint finite rank operators on $H$. This vector space is spanned by $\mathcal{P}_k(H)$, see e.g. [10, Lemma 2.1.5].

Classical Wigner’s theorem says that every bijective transformation of $\mathcal{P}_1(H)$ preserving the angle between the images of any two projections, or equivalently, preserving the trace of the composition of any two projections, is induced by a unitary or anti-unitary operator. The first rigorous proof of this statement was given in [8], see also [20] for the case when $\dim H \geq 3$. By the non-bijective version of this result [2, 3, 4], arbitrary (not necessarily bijective) transformation of $\mathcal{P}_1(H)$ preserving the angles between the images of projections (it is clear that such a transformation is injective) is induced by a linear or conjugate-linear isometry.

Various analogues of Wigner’s theorem for $\mathcal{P}_k(H)$ can be found in [3, 6, 7, 9, 10, 11, 12, 13, 15, 17]. In particular, transformations of $\mathcal{P}_1(H)$ preserving principal angles between the images of any two projections and transformations preserving the trace of the composition of any two projections are determined in [9, 11] and [5], respectively. All such transformations are induced by linear or conjugate-linear isometries, except in the case $\dim H = 2k \geq 4$ when there is an additional class of transformations. The description of transformations preserving the trace of the composition given in [5] is based on the following fact from [9]: every transformation of $\mathcal{P}_k(H)$ preserving the trace of the composition of two projections can be extended to an injective linear operator on $\mathcal{F}_s(H)$. So, there is an intimate relation between

Key words and phrases. projection, self-adjoint operator of finite rank, Wigner’s theorem.
Wigner’s type theorems mentioned above and results concerning linear operators sending projections to projections \[1, 14, 16, 19\].

Consider a linear operator \(L\) on \(F_s(H)\) such that
\[
L(\mathcal{P}_k(H)) \subseteq \mathcal{P}_k(H)
\]
such that the restriction of \(L\) to \(\mathcal{P}_k(H)\) is injective. We also assume that \(\dim H \geq 3\). By \[14\], this operator is induced by a linear or conjugate-linear isometry if \(\dim H \neq 2k\). In the case when \(\dim H = 2k\), it can be also a composition of an operator induced by a linear or conjugate-linear isometry and an operator which sends any projection on a \(k\)-dimensional subspace \(X\) to the projection on the orthogonal complement \(X^\perp\). This statement is a small generalization of the result obtained in \[1\]. The main result of \[14\] concerns linear operators sending \(\mathcal{P}_k(H)\) to \(\mathcal{P}_m(H)\), as above, whose restrictions to \(\mathcal{P}_k(H)\) are injective.

In this paper, we determine all possibilities for a linear operator \(L\) on \(F_s(H)\) satisfying (1) for \(k = 1\) without any additional assumption. Such an operator is either induced by a linear or conjugate-linear isometry or its restriction to \(\mathcal{P}_1(H)\) is constant. We mention that this result could be easily obtained from \[18, \text{Theorem 2.1}\], as such a map \(L\) clearly preserves the adjacency relation on the set \(F_s(H)\). However, we will present an elementary approach by only using the Wigner’s theorem.

Some remarks concerning the case when \(k > 1\) will be given in the last section.

2. The main result

We investigate linear maps on \(F_s(H)\) preserving the set of projections of rank one. Our main result is the following.

**Theorem 1.** Let \(H\) be a complex Hilbert space, \(\dim H \geq 2\), and \(L: F_s(H) \to F_s(H)\) a linear map. Then we have
\[
L(\mathcal{P}_1(H)) \subseteq \mathcal{P}_1(H)
\]
if and only if either there exists \(P_0 \in \mathcal{P}_1(H)\) such that
\[
L(A) = (\text{tr}A)P_0, \quad A \in F_s(H)
\]
or there exists a linear or conjugate-linear isometry \(U: H \to H\) such that
\[
L(A) = UAU^*, \quad A \in F_s(H).
\]

3. Preliminaries

Denote by \(P_X\) the projection whose image is a closed subspace \(X \subseteq H\). Since \(P_X\) belongs to \(\mathcal{P}_k(H)\) if and only if \(X\) is \(k\)-dimensional, \(\mathcal{P}_k(H)\) will be identified with the Grassmannian \(G_k(H)\). For any subspace \(Z \subseteq H\), denote
\[
\langle Z \rangle_1 = \{X \in G_1(H) : X \subseteq Z\}.
\]
If \(\dim H \geq 2\), then \(G_1(H)\) is a projective space, whose projective lines are exactly sets of the form \(\langle S \rangle_1\), \(S \in G_2(H)\).

We will show that the maps \(f\), satisfying (2), behave nicely on projective lines in \(G_1(H)\). In order to do that, we will need the following concept, which is a modification of the concept, introduced in \[3\]. For any \(X, Y \in G_1(H)\) and \(t \in \left(\frac{1}{2}, \infty\right)\), define the set
\[
\chi_t(X, Y) = \{Z \in G_1(H) : t(P_X + P_Y) + (1 - 2t)P_Z \in \mathcal{P}_1(H)\}.
\]
The following lemma describes this set.

Lemma 1. Let $X, Y \in G_1(H)$ and $t \in \left(\frac{1}{2}, \infty\right)$. Then the following statements hold.

- $\chi_t(X, Y) \subset \langle X + Y \rangle_1$
- $\chi_t(X, Y) \neq \emptyset \iff \text{tr}(P_X P_Y) \geq (1 - \frac{1}{t})^2$
- If $X$ and $Y$ are orthogonal, then $\chi_t(X, Y) = \langle X + Y \rangle_1$.
- If $X \neq Y$ and $\text{tr}(P_X P_Y) > (1 - \frac{1}{t})^2$, then $\chi_t(X, Y)$ is homeomorphic to a circle.
- If $X = Y$ or $\text{tr}(P_X P_Y) = (1 - \frac{1}{t})^2 \neq 0$, then $\chi_t(X, Y)$ is a singleton.

Proof. It is easy to show that $\chi_t(X, X) = \{X\}$.

Assume now that $X \neq Y$ and denote $S = X + Y$ and $A = P_X + P_Y$. Then $A$ is a positive semidefinite operator with trace 2. Its kernel equals $X^\perp \cap Y^\perp$, so its range equals $S$. Therefore, if $Z \in \chi_t(X, Y)$, then $tA + (1 - 2t)P_Z$ is positive semidefinite, implying that $Z \in \langle S \rangle_1$.

Moreover, there exist $c \in [0, 1)$ and an orthonormal base $B$ of $S$, according to which we have the matrix representation

$$A|_S = \begin{bmatrix} 1 + c & 0 \\ 0 & 1 - c \end{bmatrix}.$$ 

Note that

$$\text{tr}(P_X P_Y) = \frac{1}{2t} \text{tr}(A^2 - A) = c^2.$$

If $Z$ is any element of $\langle S \rangle_1$, then, according to $B$,

$$P_Z|_S = \begin{bmatrix} s & w \\ \overline{w} & 1 - s \end{bmatrix}$$

for some $s \in [0, 1]$ and $w \in \mathbb{C}$, $|w| = \sqrt{s(1 - s)}$. Any such $Z$ belongs to $\chi_t(X, Y)$ if and only if

$$\det\left(t \begin{bmatrix} 1 + c & 0 \\ 0 & 1 - c \end{bmatrix} + (1 - 2t) \begin{bmatrix} s & w \\ \overline{w} & 1 - s \end{bmatrix}\right) = 0.$$

A straightforward calculation shows that the latter holds if and only if we have either $c = 0$ and $t = 1$ or $c \neq 0$ and $s$ equals

$$\frac{(1+c)(c(1+c)-1)}{2(2t-1)}.$$ 

Thus, if $X$ and $Y$ are orthogonal, then $\chi_t(X, Y)$ is non-empty if and only if $t = 1$ and in this case, it equals $\langle X + Y \rangle_1$. In the case when they are not orthogonal, $\chi_t(X, Y)$ is non-empty if and only if $3$ belongs to $[0, 1]$, which is equivalent to $c \geq 1 - \frac{1}{4}$. Next, if $3$ belongs to $\{0, 1\}$, which is equivalent to $c = 1 - \frac{1}{4}$, then $\chi_t(X, Y)$ is a singleton. Finally, if $3$ belongs to $(0, 1)$ and equals $s$, then any $Z \in \chi_t(X, Y)$ can be identified with an element $w$ of the circle with origin 0 and radius $\sqrt{s(1 - s)}$. \hfill $\square$

4. Proof of Theorem 1

Recall that $L$ is a linear map $F_s(H) \to F_s(H)$ satisfying $2$. Denote by $f$ the transformation $\hat{G}_1(H) \to \hat{G}_1(H)$, induced by $L$, i.e. $L(P_X) = P_{f(X)}$, $X \in \hat{G}_1(H)$.

Lemma 2. The following assertions are fulfilled:
(1) For any $t \in \mathbb{R} \setminus \{0, \frac{1}{2}\}$ and $X, Y \in \mathcal{G}_1(H)$ we have
\[ f(\chi_t(X, Y)) \subset \chi_t(f(X), f(Y)). \]
If $f(X) = f(Y)$, then $f$ is constant on $\chi_t(X, Y)$.

(2) $f$ transfers any projective line to a subset of a projective line.

Proof. (1) Easy verification.

(2) If $S \in \mathcal{G}_2(H)$ and $X, Y \in \langle S \rangle_1$ are orthogonal, then $\chi_1(X, Y)$ coincides with $\langle S \rangle_1$ by Lemma 1 and we have
\[ f(\langle S \rangle_1) \subset \chi_1(f(X), f(Y)) \subset \langle S' \rangle_1 \]
with $S' = f(X) + f(Y)$ if $f(X) \neq f(Y)$, otherwise we take any 2-dimensional subspace $S'$ containing $f(X) = f(Y)$.

\[ \square \]

**Lemma 3.** The restriction of $f$ to any projective line is either injective or constant.

Proof. Let $S \in \mathcal{G}_2(H)$. Suppose that the restriction of $f$ to $\langle S \rangle_1$ is not injective. Then there exist distinct $X, Y \in \langle S \rangle_1$ such that $f(X) = f(Y)$. For every $t \in \mathbb{R}$ define
\[ g(t) = \det((tP_X + P_Y) + (1 - 2t)P_{S \cap X^\perp})|S). \]
Then $g: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(\frac{1}{2}) > 0$. Let $A = P_X + P_Y - P_{S \cap X^\perp}$. Let $\langle \cdot, \cdot \rangle$ denote the scalar product on $H$. Since $\langle x, Ax \rangle > 0$ for $x \in X$ and $\langle Az, z \rangle \leq 0$ for $z \in S \cap X^\perp$, we have $g(1) \leq 0$. Therefore, there exists $t \in (\frac{1}{2}, 1]$ such that $g(t) = 0$. For such $t$ we have $S \cap X^\perp \subset \chi_t(X, Y)$. By Lemma 2, $f(S \cap X^\perp) = f(X) = f(Y)$. Another application of Lemma 2 yields that $f$ is constant on $\chi_1(X, S \cap X^\perp)$, which equals $\langle S \rangle_1$ by Lemma 1.

\[ \square \]

**Lemma 4.** The restriction of $f$ to any projective line is continuous.

Proof. Let $S \in \mathcal{G}_2(H)$. Then the linear span of $\{P_X \mid X \in \langle S \rangle_1\}$ is finite-dimensional, which implies that the restriction of $L$ to this linear span is bounded. Hence, the restriction of $f$ to $\langle S \rangle_1$ is continuous.

\[ \square \]

**Proof of Theorem 7.** The two examples in the conclusion of the theorem clearly satisfy (2). Assume now that (2) holds.

If $f$ is constant, then $\phi(A) = (\text{tr}A)P_0$, $A \in \mathcal{F}_2(H)$, for some $P_0 \in \mathcal{P}_1(H)$.

Assume now that $f$ is not constant. Then there exist $X, Y \in \mathcal{G}_1(H)$ such that $f(X) \neq f(Y)$. Denote $S = X + Y \in \mathcal{G}_2(H)$. We will first show that
\[ f(\langle S \rangle_1) = \langle f(X) + f(Y) \rangle_1. \]
By Lemma 2, Lemma 3 and Lemma 1, $f$ is an injective continuous map from $\langle S \rangle_1$ to $\langle f(X) + f(Y) \rangle_1$, which are both homeomorphic to the 2-dimensional sphere $S^2$.

Thus, $f$ induces an injective continuous map $\tilde{f}: S^2 \to S^2$. If $\tilde{f}$ was not surjective, then it would map into $S^2 \setminus \{p\}$ for some $p \in S^2$, which is homeomorphic to $\mathbb{R}^2$, but this would contradict the Borsuk–Ulam theorem. Therefore, we deduce (4).

We next assert that
\[ \text{tr}(P_{f(X)}P_{f(Y)}) = \text{tr}(P_XP_Y). \]
Assume first that $X$ and $Y$ are orthogonal. Lemma 1 implies that $\chi_1(X, Y) = \langle S \rangle_1$, so it follows from (4) that $\chi_1(f(X), f(Y)) = (f(X) + f(Y))_1$. Another application of Lemma 1 yields that $f(X)$ and $f(Y)$ are orthogonal, as desired. Suppose now
that $X$ and $Y$ are not orthogonal and denote $t = \frac{1}{1 + \sqrt{tr(P_X P_Y)}} \in (\frac{1}{2}, 1)$. By Lemma 1, $\chi_t(X, Y)$ is a singleton. We claim that

$$ (6) \quad f(\chi_t(X, Y)) = \chi_t(f(X), f(Y)). $$

Indeed, the left-hand side is contained in the right-hand side by Lemma 2. Let now $W \in \chi_t(f(X), f(Y))$. Then $W \in \langle f(X) + f(Y) \rangle_1$ by Lemma 1 so (4) yields that $W = f(W')$ for some $W' \in \langle S_1 \rangle$. Hence,

$$ (7) \quad t(P_f(X) + P_f(Y)) + (1 - 2t)P_f(W') = P_{W''} $$

for some $W'' \in G_1(H)$. Then we have $W'' \in \chi_{\frac{t}{2t-1}}(f(X), f(Y))$, hence another application of (4) implies that $W'' = f(W''')$ for some $W''' \in \langle S_1 \rangle$. Denote

$$ A = t(P_X + P_Y) + (1 - 2t)P_{W''} - P_{W'''}. $$

By (7), $L(A) = 0$. We assert that $A = 0$. Indeed, $A = aP_Z + bP_{S \cap Z^\perp}$ for some $Z \in \langle S_1 \rangle$ and $a, b \in \mathbb{R}$. Since $f|_{\langle S_1 \rangle}$ is injective, $f(Z) \neq f(S \cap Z^\perp)$, so $P_f(Z)$ and $P_f(S \cap Z^\perp)$ are linearly independent. Now $0 = L(A) = aP_f(Z) + bP_f(S \cap Z^\perp)$ implies that $a = b = 0$ and $A = 0$, which completes the proof of (6).

By (9), $\chi_t(f(X), f(Y))$ is a singleton. Another application of Lemma 1 yields that $t = \frac{1}{1 + \sqrt{tr(P_f(X), P_f(Y))}}$, so (5) holds.

We have shown that (5) holds whenever $X, Y \in G_1(H)$ are such that $f(X) \neq f(Y)$. By Lemma 2, the same holds for any pair from $\langle X + Y \rangle_1$.

We will next show $f$ is injective. If $\dim H = 2$, there is nothing more to do, so assume that $\dim H \geq 3$. Seeking a contradiction, suppose that there exist pairwise distinct $X, Y, Z \in G_1(H)$ such that $f(X) \neq f(Y)$ and $f(Z) = f(X)$. Denote $S = X + Y$ and let $Z' = (S + Z) \cap S^\perp$. By Lemma 3, $Z \not\in S$, thus $Z' \in G_1(H)$. Let $Y' \in \langle S_1 \rangle \setminus \{X\}$ be non-orthogonal to $X$. By the previous paragraph, $f(X)$ and $f(Y')$ are distinct and non-orthogonal. Since $Z'$ is orthogonal to $Y'$, $f(Z')$ is either equal or orthogonal to $f(Y')$, so $f(Z') \neq f(X)$. Because $Z'$ is orthogonal to $X$, $f(Z')$ is orthogonal to $f(X)$, which equals $f(Z)$. By the previous paragraph, $Z'$ is orthogonal to $Z$. Hence,

$$ \{0\} = (S + Z) \cap (S + Z)^\perp = Z' \cap Z^\perp = Z', $$

a contradiction. This contradiction shows that, since $f$ is not constant, it must be injective. Thus, (5) holds for all $X, Y \in G_1(H)$. The conclusion of the theorem now follows from Wigner’s theorem, see e.g. (4).

5. Final remarks

Consider a linear map $L$ on $F_*(H)$ satisfying

$$ L(\mathcal{P}_k(H)) \subset \mathcal{P}_k(H) $$

for a certain $k \in \mathbb{N}, k < \dim H$. As above, $L$ induces a transformation $f$ of $G_k(H)$ which is not necessarily injective. The general case can be reduced to the case when $\dim H \geq 2k$.

For subspaces $M$ and $N$ satisfying $\dim M < k < \dim N$ and $M \subset N$ we denote by $[M, N]_k$ the set of all $X \in G_k(H)$ such that $M \subset X \subset N$. For any $X, Y \in G_k(H)$ we have

$$ \chi_1(X, Y) = \{Z \in G_k(H) : P_X + P_Y - P_Z \in \mathcal{P}_k(H)\} \subset [X \cap Y, X + Y]_k $$
and the inverse inclusion holds if and only if $X, Y$ are compatible, i.e. there is an orthonormal basis of $H$ such that $X$ and $Y$ are spanned by subsets of this basis. If $X$ and $Y$ are orthogonal, then $\chi_1(X, Y) = \langle X + Y \rangle_k$ and

$$f((X + Y)_k) \subset \chi_1(f(X), f(Y)) \subset \langle f(X) + f(Y) \rangle_k.$$  

As in the proof of Lemma 4, we show that for any $(2k)$-dimensional subspace $S \subset H$ the restriction of $f$ to $\langle S \rangle_k$ is continuous. In the case when $k = 1$, the restriction of $f$ to any projective line is a continuous map to a projective line.

In the general case, a line of $G_k(H)$ is a subset of type $[M, N]_k$, where $M$ is a $(k-1)$-dimensional subspace contained a $(k+1)$-dimensional subspace $N$. This line can be identified with the line of $\langle M^\perp \rangle_1$ associated to the 2-dimensional subspace $N \cap M^\perp$. Two distinct $k$-dimensional subspaces are contained in a common line if and only if they are adjacent, i.e. their intersection is $(k-1)$-dimensional. If $X, Y \in G_k(H)$ are adjacent, then the line containing them is $[X \cap Y, X + Y]_k$. It was noted above that this line coincides with $\chi_1(X, Y)$ only in the case when $X$ and $Y$ are compatible. If $X$ and $Y$ are non-compatible, then $\chi_1(X, Y)$ is a subset of the line $[X \cap Y, X + Y]_k$ homeomorphic to a circle.

For every line there is a $(2k)$-dimensional subspace $S$ such that $\langle S \rangle_k$ contains this line, i.e. the restriction of $f$ to each line is continuous. Using analogous arguments as in the proof of Lemma 3, we establish that the restriction of $f$ to every line is either injective or constant; but we are not be able to show that $f$ sends lines to subsets of lines.

On the other hand, if $f$ is injective, then it is adjacency and orthogonality preserving (see [1, 5, 14] for the details). By [13], this immediately implies that $f$ is induced by a linear or conjugate-linear isometry if $\dim H > 2k$ and there is one other option for $f$ if $\dim H = 2k$.

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