A Note on Target Q-Learning for Solving Finite MDPs with a Generative Oracle

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Abstract

Q-learning with function approximation could diverge in the off-policy setting and the target network is a powerful technique to address this issue. In this manuscript, we examine the sample complexity of the associated target Q-learning algorithm in the tabular case with a generative oracle. We point out a misleading claim in [Lee and He, 2020] and establish a tight analysis. In particular, we demonstrate that the sample complexity of the target Q-learning algorithm in [Lee and He, 2020] is \( \tilde{O}(|S|^2|A|^2(1-\gamma)^{-5}\varepsilon^{-2}) \). Furthermore, we show that this sample complexity is improved to \( \tilde{O}(|S||A|(1-\gamma)^{-5}\varepsilon^{-2}) \) if we can sequentially update all state-action pairs and \( \tilde{O}(|S||A|(1-\gamma)^{-4}\varepsilon^{-2}) \) if \( \gamma \) is further in \((1/2, 1)\). Compared with the vanilla Q-learning, our results conclude that the introduction of a periodically-frozen target Q-function does not sacrifice the sample complexity.

1 Introduction

Q-learning is one of the most simple yet popular algorithms in the reinforcement learning (RL) community [Sutton and Barto, 2018]. However, Q-learning suffers the divergence issue when (linear) function approximation is applied [Baird, 1995, Tsitsiklis and Van Roy, 1997]. To address this instability issue, a technique called target network is proposed in the famous DQN algorithm [Mnih et al., 2015]. In particular, DQN implements a duplication of the main Q-network (i.e., the so-called target network), which is further used to generate the bootstrap signal for updates. One important feature is that the target network is fixed over intervals. Unlike Q-learning, the learning targets do not change during an interval for DQN. In [Mnih et al., 2015, Table 3], it is reported that the target network contributes a lot to the superior performance of DQN.

Since then, it has been an active area of research to theoretically understand the target network technique [Lee and He, 2019, 2020, Fan et al., 2020, Zhang et al., 2021, Agarwal et al., 2022, Chen et al., 2022] and design variants based on this technique [Lillicrap et al., 2016, Fujimoto et al., 2018, Haarnoja et al., 2018, Carvalho et al., 2020]. In this manuscript, we take a “sanity check”: we examine the sample complexity of target Q-learning in the tabular case with a generative oracle. We want to know whether target Q-learning sacrifices the sample complexity as it periodically freezes the target Q-function, which is believed to “may ultimately slow down training” in [Piché et al., 2021].

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First, we revisit the target Q-learning algorithm and analysis in [Lee and He, 2020]. In particular, once the target Q-function is fixed, this algorithm randomly picks up a state-action pair to perform the stochastic gradient descent (SGD) update. To avoid the confusion with algorithms introduced later, we call this algorithm StoTQ-learning (stochastic target Q-learning). In particular, Lee and He [2020] showed that the sample complexity of StoTQ-learning is \( \tilde{O}(|S|^2|A|^3(1- \gamma)^{-4} \varepsilon^{-2} \log^{-1}(1/\gamma)) \), where \(|S|\) is the number of states, \(|A|\) is the number of actions, \(\gamma \in (0, 1)\) is the discount factor, and \(\varepsilon \in (0, 1/(1-\gamma))\) is the error between the obtained Q-function and the optimal Q-function with respect to the \(\ell_{\infty}\)-norm. We point out that [Lee and He, 2020] made a mis-claim that the dependence on the effective horizon is \(\tilde{O}(1/(1-\gamma)^4)\) as they ignored \(1/\log(1/\gamma) = \tilde{O}(1/(1-\gamma)^2)\). In other word, the correct dependence on the effective horizon is \(\tilde{O}(1/(1-\gamma)^5)\). As one can see, this sample complexity suffers a poor dependence on the problem size \(|S| \times |A|\). To this end, we refine the analysis in [Lee and He, 2020] and builds a tighter upper bound on the variance of the SGD update. Consequently, we show that StoTQ-learning enjoys a sample complexity \(\tilde{O}(|S|^2|A|^2(1- \gamma)^{-5} \varepsilon^{-2})\), in which the dependence on \(1/(1-\gamma)\) is same with phased Q-learning [Kearns and Singh, 1999] and Q-learning [Wainwright, 2019].

Second, we demonstrate that the dependence on the problem size can be improved to \(\tilde{O}(|S||A|)\) if we sequentially update state-action pairs for target Q-learning. We call such an algorithm SeqTQ-learning (sequential target Q-learning). Technically, SeqTQ-learning ensures that all state-action pairs can be updated after one “epoch”, which cannot be achieved by StoTQ-learning since StoTQ-learning randomly picks up a state-action pair to update during an “epoch”. In particular, the proposed modification is similar to the “random shuffling” technique in the deep-learning community, which is shown to reduce the variance compared with the original SGD update for finite-sum optimization (see [Mishchenko et al., 2020] and references therein).

Finally, we conclude that if \(\gamma \in (1/2, 1)\), the sample complexity of SeqTQ-learning is improved to \(\tilde{O}(|S||A|(1- \gamma)^{-4} \varepsilon^{-2})\), which is identical with the sharp sample complexity of Q-learning in [Li et al., 2021]. This good result builds on the tight analysis in [Li et al., 2021, Agarwal et al., 2022]. Therefore, we conclude that compared with the vanilla Q-learning, the introduction of a periodically-frozen target Q-function does not sacrifice the statistical accuracy in the tabular case with a generative oracle.

2 Preliminary

An infinite-horizon Markov Decision Process (MDP) [Puterman, 2014] can be describe by a tuple \(\mathcal{M} = \langle S, A, P, r, \gamma, d_0 \rangle\). Here \(S\) and \(A\) are the state and the action space, respectively. We assume that both \(S\) and \(A\) are finite. Here \(P(s'|s, a)\) specifies the transition probability of the next state \(s'\) based on current state \(s\) and current action \(a\). The quality of each action \(a\) on state \(s\) is judged by the reward function \(r(s, a)\). Without loss of generality, we assume that \(r(s, a) \in [0, 1]\) for all \((s, a) \in S \times A\) throughout. Finally, \(\gamma \in [0, 1]\) is a discount factor, weighting the importance of future returns, and \(d_0\) specifies the initial state distribution.

From the view of the agent, it maintains a policy \(\pi\) to select actions based on \(\pi(a|s)\). The quality of a policy \(\pi\) is measured by the state-action value function \(Q^\pi(s, a) := \mathbb{E}[\sum_{t=0}^\infty \gamma^t r(s_t, a_t) \mid s_0 = s, a_0 = a]\), i.e., the cumulative discounted rewards starting from \((s, a)\). According to the theory of MDP [Puterman, 2014], there exists an optimal policy \(\pi^*\) such that its state-action value function is optimal, i.e., \(Q^{\pi^*}(s, a) = \max_{\pi} Q^\pi(s, a)\) for all \((s, a) \in S \times A\).
For simplicity, let $Q^*$ denote the optimal state-action value function, which further satisfies the Bellman equation:

$$Q^*(s, a) = r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, a)} \left[ \max_{a'} Q^*(s', a') \right], \quad \forall (s, a) \in S \times A.$$ 

Let us define the Bellman operator $T : \mathbb{R}^{|S||A|} \to \mathbb{R}^{|S||A|},$

$$TQ(s, a) = r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, a)} \left[ \max_{a'} Q(s', a') \right].$$

It is obvious that $Q^*$ is the unique fixed point of $T$. Furthermore, $T$ is $\gamma$-contractive with respect to the $\ell_\infty$-norm:

$$\forall Q_1, Q_2, \quad \| TQ_1 - TQ_2 \|_\infty \leq \gamma \| Q_1 - Q_2 \|_\infty.$$ 

As a result, we can perform the fixed point iteration to solve $Q^*$, which is known as the value iteration algorithm [Puterman, 2014]. However, if the transition function $P$ is unknown and we have access to the sample $(s, a, r, s')$, we can define the empirical Bellman operator $T$ for a state-action value function $Q$:

$$\hat{T}Q(s, a) = r(s, a) + \gamma \max_{a'} Q(s', a'), \quad s' \sim P(\cdot|s, a).$$

With the noisy estimate $\hat{T}$, we can implement the stochastic approximation and such an algorithm is called Q-learning [Watkins and Dayan, 1992]. Without loss of generality, we assume the reward function is known.

### 3 Algorithms and Main Results

In this section, we investigate the sample complexity of two target Q-learning algorithms with a generative oracle (see Oracle 1). In particular, the generative oracle provides a simple way of studying the sample complexity by allowing i.i.d. samples. Nevertheless, results under the setting of i.i.d. samples can be extended to the Markovian case by the coupling arguments (see for example [Nagaraj et al., 2020, Agarwal et al., 2022]).

**Oracle 1** (Generative Oracle). Given a state-action pair $(s, a)$, the oracle returns the next state $s'$ by independently sampling from the transition function $P(\cdot|s, a)$.

#### 3.1 Stochastic Target Q-learning

First, we focus on the algorithm proposed in [Lee and He, 2020] (see Algorithm 1), which is re-named after Stochastic Target Q-learning (StoTQ-learning) for ease of presentation. In particular, we consider the simplified version where $(s, a)$ is uniformly sampled from $S \times A$ in Line 5 of Algorithm 1. For the update rule in Line 6, it can be viewed as one-step stochastic gradient descent of the following optimization problem:

$$\min_{Q \in \mathbb{R}^{|S||A|}} L(Q; Q_k) := \frac{1}{2|S||A|} \sum_{(s, a)} (TQ_k(s, a) - Q(s, a))^2,$$  

Specifically, the randomness comes from the sample index $(s, a)$ and the label noise in $TQ_k(s, a)$ because we use the empirical Bellman update $r(s, a) + \gamma \max_{a'} Q_k(s', a')$. From this viewpoint, it is reasonable that $Q_{k,T}$ is close to $TQ_k$ as long as the step size is properly designed and the iteration number $T$ is sufficiently large. Consequently, StoTQ-learning generates a sequence $\{Q_k\}$, which performs the approximate Bellman update as the phased Q-learning algorithm (a.k.a. sampling-based value iteration) [Kearns and Singh, 1999]. This connection is clear in the following error bound.

**Lemma 1** (Proposition 1 of [Lee and He, 2020]). For each outer iteration $k$, suppose that the optimization error of the inner loop satisfies that $\mathbb{E}\|Q_k - TQ_{k-1}\|_2^2 \leq \varepsilon_{\text{opt}}$ for all $k \leq K$. Then, we have that

$$\mathbb{E}\|Q_K - Q^*\|_\infty \leq \sqrt{\frac{\varepsilon_{\text{opt}}}{1 - \gamma}} + \gamma^K \mathbb{E}\|Q_0 - Q^*\|_\infty.$$ 

Lemma 1 claims that to control the final error $\varepsilon$, it is essential to ensure the optimization error is small for each inner loop. Since the learning targets are generated by a fixed variable $Q_k$ in the inner loop, SGD is stable for the optimization problem (1) (see [Bottou et al., 2018] and references therein). As a consequence, we expect that $\varepsilon_{\text{opt}}$ is well-controlled. In terms of the analysis, the key is to upper bound the variance of stochastic gradients. Let $\nabla L(Q; Q_k)$ be the true gradient and $\nabla L(Q; Q_k)$ be the stochastic gradient.
Algorithm 1 Stochastic Target Q-learning (StoTQ-learning)

Input: outer loop iteration number $K$, inner loop iteration number $T$, initialization $Q_0$, and step-sizes $\{\eta_t\}$.
1: for iteration $k = 0, 1, \cdots, K - 1$ do
2: \hspace{1em} set $Q_{k,0} = Q_k$
3: for iteration $t = 0, 1, \cdots, T - 1$ do
4: \hspace{2em} randomly pick up $(s, a)$, calculate $r(s, a)$, and obtain $s'$ by the generative oracle.
5: \hspace{2em} update $Q_{k,t+1}(s, a) = Q_{k,t}(s, a) + \eta_t (r(s, a) + \gamma \max_{a'} Q_k(s', a') - Q_{k,t}(s, a))$.
6: end for
7: \hspace{1em} set $Q_{k+1} = Q_k$.
8: end for

Output: $Q_K$

Lemma 2 (Lemma 7 of [Lee and He, 2020]). For any $Q \in \mathbb{R}^{|S||A|}$, we have
\[
\mathbb{E} \left[ \left\| \nabla L(Q; Q_k) \right\|_2^2 \right] \leq 8|S||A| \left\| \nabla L(Q; Q_k) \right\|_2^2 + 12\gamma^2 |S||A| \left\| Q_k - Q^* \right\|_\infty^2 + \frac{18|S||A|}{(1 - \gamma)^2}.
\]

Based on Lemma 2, Lee and He [2020] proved the sample complexity $\tilde{O}(|S|^3|A|^3(1 - \gamma)^{-5}\varepsilon^{-2})$ for Algorithm 1, in which the dependence on the problem size $|S||A|$ is inferior to algorithms like Phased Q-learning and Q-learning (see Table 1). In this manuscript, we point out that the proof of Lemma 2 can be improved to obtain a tighter upper bound and a better sample complexity.

Lemma 3 (Refined Version of Lemma 2). For any $Q \in \mathbb{R}^{|S||A|}$, we have
\[
\mathbb{E} \left[ \left\| \nabla L(Q; Q_k) \right\|_2^2 \right] \leq |S||A| \left\| \nabla L(Q; Q_k) \right\|_2^2 + 6\gamma^2 \left\| Q_k - Q^* \right\|_\infty + \frac{3\gamma^2}{(1 - \gamma)^2}.
\]

As one can see, the upper bound in Lemma 3 is better than that in Lemma 2 in terms of the dependence on $|S||A|$ on the last two terms. With Lemma 3, we arrive at a better sample complexity.

Theorem 1 (Sample Complexity of Algorithm 1). For any tabular MDP with a generative oracle, consider Algorithm 1 with the following parameters:
\[
Q_0 = 0, \quad K = \mathcal{O} \left( \frac{1}{1 - \gamma} \log \left( \frac{1}{(1 - \gamma)\varepsilon} \right) \right), \quad T = \mathcal{O} \left( \frac{|S|^2|A|^2}{(1 - \gamma)^2\varepsilon^2} \right), \quad \eta_t = \frac{\eta}{\lambda + t},
\]
where $\eta = 2|S||A|$ and $\lambda = 13/2 \cdot \gamma^2 |S||A|$. Then, we have that $\mathbb{E}[\|Q_K - Q^*\|_\infty] \leq \varepsilon$. Accordingly, the number of required samples is
\[
KT = \tilde{O} \left( \frac{|S|^2|A|^2}{(1 - \gamma)^2\varepsilon^2} \right).
\]

Compared with the lower bound $\Omega(|S||A|(1 - \gamma)^{-3}\varepsilon^{-2})$ [Azar et al., 2013], the sample complexity shown in Theorem 1 is sub-optimal in the dependence on the problem size $|S||A|$ and effective horizon $1/(1 - \gamma)$. In the following parts, we discuss how to improve the orders.

3.2 Sequential Target Q-learning

To overcome the sample barrier of StoTQ-learning, a simple yet effective approach is to sequentially update all state-action pairs (see Algorithm 2). This ensures that the optimality gap with respect to the $\ell_\infty$-norm is reduced after $|S||A|$ iterations, which is consistent with $\gamma$-contraction of the Bellman operator. In contrast, the uniform sampling strategy in StoTQ-learning is designed to minimize the optimality gap with respect to the $\ell_2$-norm. In fact, the translation between $\ell_2$-norm and $\ell_\infty$-norm results in the poor dependence on the problem size $|S||A|$ for StoTQ-learning.

Theorem 2 (Sample Complexity of Algorithm 2). For any tabular MDP with a generative oracle, consider Algorithm 2 with the following parameters:
\[
Q_0 = 0, \quad K = \mathcal{O} \left( \frac{1}{1 - \gamma} \log \left( \frac{1}{(1 - \gamma)\varepsilon} \right) \right), \quad T = \mathcal{O} \left( \frac{1}{(1 - \gamma)^4\varepsilon^2} \log (|S||A|) \right), \quad \eta_t = \frac{1}{t + 2}.
\]

Then, we have that $\mathbb{E}[\|Q_K - Q^*\|_\infty] \leq \varepsilon$. Accordingly, the number of required samples is
\[
K \cdot T \cdot |S||A| = \tilde{O} \left( \frac{|S||A|}{(1 - \gamma)^3\varepsilon^2} \right).
\]
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Algorithm 2 Sequential Target Q-learning (SeqTQ-learning)

**Input:** outer loop iteration number $K$, inner loop iteration number $T$, initialization $Q_0$, and step-sizes $\{\eta_t\}$.

1: for iteration $k = 0, 1, \cdots, K - 1$ do
2: \hspace{1em} set $Q_{k,0} = Q_k$
3: for iteration $t = 0, 1, \cdots, T - 1$ do
4: \hspace{2em} for each state-action pair $(s,a) \in S \times A$ do
5: \hspace{3em} calculate $r(s,a)$ and obtain $s'$ by the generative oracle.
6: \hspace{3em} update $Q_{k,t+1}(s,a) = Q_{k,t}(s,a) + \eta_t (r(s,a) + \gamma \max_a Q_k(s',a') - Q_{k,t}(s,a))$
7: \hspace{2em} end for
8: Set $Q_{k+1} = Q_{k,T}$
9: end for
10: end for

**Output:** $Q_K$.

**Remark 1.** We note that SeqTQ-learning uses more conservative step-sizes than Q-learning. Specifically, it is a common choice that Q-learning uses the step-size $\eta_t = 1/(1 + (1 - \gamma)(t + 1))$ [Wainwright, 2019, Li et al., 2021]. We explain the difference here. For each inner loop, the update rule of SeqTQ-learning is

\[
\text{SeqTQ-learning} : \quad Q_{k,t+1} = (1 - \eta_t)Q_{k,t} + \eta_t \tilde{T}_t Q_k.
\]

Define the error term $\Delta_{k,t} := Q_{k,t+1} - TQ_k$. Then, we have that

\[
\text{SeqTQ-learning} : \quad \Delta_{k,t+1} = (1 - \eta_t)\Delta_{k,t} + \eta_t(\tilde{T}_t Q_k - T Q_k).
\] (2)

On the other hand, the update rule of Q-learning is

\[
\text{Q-learning} : \quad Q_{t+1} = (1 - \eta_t)Q_t + \eta_t \tilde{T}_t Q_t.
\]

Define the error term $\Delta_t = Q_t - TQ^*$. Then, we have that $\Delta_{t+1} = (1 - \eta_t)\Delta_t + \eta_t(\tilde{T}_t Q_t - T Q^*) = (1 - \eta_t)\Delta_t + \eta_t(\tilde{T}_t Q_t - \tilde{T}_t Q^*) + \eta_t(\tilde{T}_t Q^* - T Q^*)$. By the $\gamma$-contraction of the empirical Bellman operator $\tilde{T}_t$, we obtain

\[
\text{Q-learning} : \quad \Delta_{t+1} \leq (1 - \eta_t)\Delta_t + \gamma \eta_t \|\Delta_t\|_\infty 1 + \eta_t(\tilde{T}_t Q^* - T Q^*),
\] (3)

where $\leq$ holds elementwise and $1$ is the vector filled with $1$. We note that the variances of the noise terms in (2) and (3) have the same order. Furthermore, we see that the contraction coefficient does not rely on $(1 - \gamma)$ in SeqTQ-learning, which explains the step-size design of SeqTQ-learning.

Finally, we remark that the independence on effective horizon can be further improved to $1/(1 - \gamma)^4$ in the regime of $\gamma \in (1/2, 1)$. This improvement is based on the sharp analysis in [Li et al., 2021, Agarwal et al., 2022].

**Theorem 3** (Tight Sample Complexity of Algorithm 2 when $\gamma > 1/2$). For any tabular MDP with a generative oracle and $\gamma \in (1/2, 1)$, consider Algorithm 2 with the following parameters:

\[
Q_0 = 0, \quad K = \mathcal{O} \left( \frac{1}{1 - \gamma} \log^2 \left( \frac{1}{1 - \gamma} \right) \right), \quad T = \mathcal{O} \left( \frac{1}{(1 - \gamma)^3 \varepsilon^2} \log \left( \frac{K|S||A|}{\delta} \right) \right), \quad \eta_t = \frac{1}{t + 2},
\]

where $\delta \in (0, 1)$ is the failure probability. Then, with probability at least $1 - \delta$, we have that $\|Q_K - Q^*\|_\infty \leq \varepsilon$. Accordingly, the number of required samples is

\[
K \cdot T \cdot |S||A| = \tilde{O} \left( \frac{|S||A|}{(1 - \gamma)^4 \varepsilon^2} \right).
\]

**Remark 2.** We note that the sample complexity of SeqTQ-learning in Theorem 3 has the same order with the vanilla Q-learning [Li et al., 2021] under the same setting. Compared with the lower bound in [Azar et al., 2013], the sample complexity in Theorem 3 is still sub-optimal in the dependence on $1/(1 - \gamma)$. To further overcome the hurdle, the variance reduction scheme for sampling-based value iteration in [Sidford et al., 2018a,b] should be considered. Since the inner loop of the target Q-learning is an online version of the sampling-based value iteration\(^3\), it is likely that the sample complexity of target Q-learning with variance reduction can match the lower bound.

\(^3\)Given a target Q-function $Q_k$ to evaluate, sampling-based value iteration performs the batched update $1/T \sum_{t=1}^T \tilde{T}_t Q_k$ with $T$ i.i.d. samples in each iteration, while target Q-learning performs the online update by taking a small gradient step in each iteration.
4 Conclusion

In this manuscript, we establish the tight sample complexity for target Q-learning in the tabular setting with a generative oracle, which provides a sanity check. In particular, we conclude that compared with the vanilla Q-learning, the introduction of a periodically-frozen target Q-function does not sacrifice the sample complexity. We hope our results could provide insights for future research.

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A Proofs of Main Results

In the following proofs, we often use $c$ to denote an absolute constant, which may change in different lines.

A.1 Proof of Theorem 1

We prove Theorem 1 by following the analysis in [Lee and He, 2020]. In particular, we obtain a stronger convergence result by Lemma 3. To make the notations consistent with [Lee and He, 2020], we consider the population loss is defined by

$$L(Q; Q_k) = \frac{1}{2} \sum_{(s,a)} d(s,a)(T Q_k(s,a) - Q(s,a))^2,$$

where $d(s,a) > 0$ assigns sampling probability for each state-action pair in Line 4 of Algorithm 1. Specifically, we consider $d(s,a) = 1/(|S||A|)$ in Theorem 1, which yields the tightest sample complexity among all sampling distributions. To facilitate later analysis, let $D \in \mathbb{R}^{|S||A| \times |S||A|}$ be the diagonal matrix of $d$. In addition, define the weighted norm by $\|x\|_{2,D} = \sqrt{x^T D x}$. When the context is clear, we simply write $\|x\|_{2,D}$ by $\|x\|_D$.

**Lemma 4** (Gradient Lipschitz Continuity and Strong Convexity; Lemma 6 of [Lee and He, 2020]). The objective function $L(Q; Q_k)$ in (4) is $\mu$-strongly convex with $\mu = \min_{(s,a)} d(s,a)$ and $\beta$-gradient Lipschitz continuous with $\beta = \max_{(s,a)} d(s,a)$.

Based on Lemma 4, we arrive at the following convergence result.

**Proposition 1** (Inner Loop Convergence of Algorithm 1). Considering Algorithm 1, let us set $\eta_t = \eta/(\lambda + t)$ with $\eta = 2|S||A|$ and $\lambda = 13|S||A|\gamma^2/2$. Then, for all $t \geq 0$ and $k \geq 0$, we have that

$$\mathbb{E} \left[ L(Q_{k,t}; Q_k) \right] = \frac{1}{2} \mathbb{E} \left[ \|Q_{k,t} - T Q_k\|_{2,D}^2 \right] \leq \frac{104\beta \gamma^2}{\mu^2(1-\gamma)^2} \frac{1}{\lambda + t}.$$

**Proof.** By Lemma 4, we know that $L(Q_{k,t}; Q_k)$ is a $\beta$-smooth and $\mu$-strongly convex function. Following the typical analysis of SGD on a $\beta$-smooth and $\mu$-strongly convex function, we have that

$$L(Q_{k,t+1}; Q_k) \leq L(Q_{k,t}; Q_k) - \eta_t \langle \nabla L(Q_{k,t+1}; Q_k), \nabla L(Q_{k,t}; Q_k) \rangle + \frac{\beta \eta_t^2}{2} \left\| \nabla L(Q_{k,t}; Q_k) \right\|_2^2.$$

By taking the expectation over the randomness in the stochastic gradient, we obtain that

$$\mathbb{E} \left[ L(Q_{k,t+1}; Q_k) \right] \leq L(Q_{k,t}; Q_k) - \eta_t \mathbb{E} \left[ \left\| \nabla L(Q_{k,t}; Q_k) \right\|_2^2 \right] + \frac{\beta \eta_t^2}{2} \mathbb{E} \left[ \left\| \nabla L(Q_{k,t}; Q_k) \right\|_2^2 \right].$$

As a corollary of Lemma 7 and Lemma 10, we have that

$$\mathbb{E} \left[ \left\| \nabla L(Q_{k,t}; Q_k) \right\|_2^2 \right] \leq \left\| \nabla L(Q_{k,t}; Q_k) \right\|_{2,D^{-1}}^2 + 6\gamma^2 \left\| Q_k - Q^* \right\|_\infty + \frac{3\gamma^2}{(1-\gamma)^2} \leq \left\| \nabla L(Q_{k,t}; Q_k) \right\|_{2,D^{-1}}^2 + \frac{51\gamma^2}{(1-\gamma)^2} \leq \frac{1}{\mu} \left\| \nabla L(Q_{k,t}; Q_k) \right\|_2^2 + \frac{51\gamma^2}{(1-\gamma)^2}.$$

Thus, we know that

$$\mathbb{E} \left[ L(Q_{k,t+1}; Q_k) \right] \leq L(Q_{k,t}; Q_k) - (\eta_t - \frac{\beta \eta_t^2}{2\mu}) \left\| \nabla L(Q_{k,t}; Q_k) \right\|_2^2 + \frac{\beta \eta_t^2}{2} \frac{51\gamma^2}{(1-\gamma)^2} \leq L(Q_{k,t}; Q_k) - (\eta_t - \frac{\beta \eta_t^2}{2\mu}) 2\mu L(Q_{k,t}; Q_k) + \frac{\beta \eta_t^2}{2} \frac{51\gamma^2}{(1-\gamma)^2} \leq (1 - \mu \eta_t) L(Q_{k,t}; Q_k) + \frac{\beta \eta_t^2}{2} \frac{51\gamma^2}{(1-\gamma)^2}.$$
where (1) is because the strong convexity implies that \(\|\nabla L(Q_{k,t}; Q_k)\|^2 \geq 2\mu L(Q_{k,t}; Q_k)\), and the last inequality holds when \(0 < \eta_t \leq \mu/\beta\). Taking the expectation over the randomness before iteration \((k, t)\), we have that
\[
\mathbb{E}[L(Q_{k,t+1}, Q_k)] \leq (1 - \mu\eta_t)\mathbb{E}[L(Q_{k,t}; Q_k)] + \frac{26\beta^2\eta_t^2}{(1 - \gamma)^2}.
\]
By choosing the diminishing step size \(\eta_t = \eta/(\lambda + t)\) satisfying \(\mu\eta > 1\), we obtain that
\[
\mathbb{E}[L(Q_{k,t}; Q_k)] \leq \frac{\nu}{\lambda + t}, \quad \forall t \geq 0, k \geq 1,
\]
where \(\nu = \max\{\lambda\mathbb{E}[L(Q_{k,0}; Q_k)], \eta^2C\}\) and \(C = (26\beta^2\gamma^2)/((\mu\eta - 1)(1 - \gamma)^2)\). Compared with the result in [Lee and He, 2020], \(C\) is improved by a factor of \(|S||A|\).

For the initial distance, we have that
\[
\mathbb{E}[L(Q_{k,0}; Q_k)] = \mathbb{E}[L(Q_k; Q_k)] = \frac{1}{2}\mathbb{E}[\|Q_k - TQ_k\|_2^2] \leq 2\mathbb{E}\left[\|Q_k - Q^*\|_2^2\right] \leq \frac{16}{(1 - \gamma)^2}.
\]
Thus, by choosing \(\eta = 2/\mu\) and \(\lambda = (13\beta^2)/2\mu^2\), we know that \(\nu = (104\beta\gamma^2)/(\mu^2(1 - \gamma)^2)\).

**Proof of Theorem 1.** According to Lemma 1, if we have
\[
\frac{\sqrt{\varepsilon_{\text{opt}}}}{1 - \gamma} \leq \frac{\varepsilon}{2},
\]
then we can sure that \(\mathbb{E}[\|Q_k - Q^*\|_\infty] \leq \varepsilon\). If we initialize \(Q_0 = 0\), the second condition is satisfied when
\[
K = \frac{1}{1 - \gamma} \log\left(\frac{4}{(1 - \gamma)\varepsilon}\right).
\]
For the first condition, we can ensure that \(\varepsilon_{\text{opt}} = (1 - \gamma)^2\varepsilon^2/4\). Notice that for all \(k \geq 1\), we have
\[
\mathbb{E}\left[\|Q_k - TQ_{k-1}\|_2^2\right] \leq \frac{2}{\mu}\mathbb{E}[L(Q_k; Q_{k-1})].
\]
Hence, it suffices to set that
\[
\frac{2}{\mu} \cdot \frac{104\beta\gamma^2}{\mu^2(1 - \gamma)^2} \cdot \frac{1}{\lambda + T} \leq \frac{(1 - \gamma)^2\varepsilon^2}{4} \implies T \geq \frac{832\beta\gamma^2}{\mu^3(1 - \gamma)^4\varepsilon^2}.
\]
The conditions in (5) and (6) give the desired sample complexity.

**A.2 Proof of Theorem 2**

**Lemma 5.** For Algorithm 2, assume \(\eta_t \in (0, 1)\) for all \(t \geq 0\). In addition, suppose that \(\|Q_0 - Q^*\|_\infty \leq 1/(1 - \gamma)\). Then, we have that
\[
\|Q_k\|_\infty \leq \frac{1}{1 - \gamma}, \quad \|Q_{k,t}\|_\infty \leq \frac{1}{1 - \gamma} \quad \forall k \geq 0, t \geq 0.
\]

**Proof.** The proof is done by a simple induction and details are therefore omitted.

**Lemma 6.** Assume that we have that \(\mathbb{E}[\|Q_k - TQ_{k-1}\|_\infty] \leq \varepsilon_{\text{opt}}\) for all \(k \leq K\). Then, we have that
\[
\mathbb{E}[\|Q_K - Q^*\|_\infty] \leq \frac{\varepsilon_{\text{opt}}}{1 - \gamma} + \gamma^K\mathbb{E}[\|Q_0 - Q^*\|_\infty].
\]
Proof.

\[ E[\|Q_K - Q^*\|_\infty] \leq E[\|Q_K - TQ_{K-1} + TQ_{K-1} - Q^*\|_\infty] \]
\[ \leq E[\|Q_K - TQ_{K-1}\|_\infty] + E[\|TQ_{K-1} - Q^*\|_\infty] \]
\[ \leq \varepsilon_{\text{opt}} + \gamma E[\|Q_{K-1} - Q^*\|_\infty] \]
\[ \leq \frac{\varepsilon_{\text{opt}}}{1 - \gamma} + \gamma^K E[\|Q_0 - Q^*\|_\infty]. \]

\[ \square \]

Proof of Theorem 2. Let us write down the update rule

\[ Q_{k,t+1} = (1 - \eta_t)Q_{k,t} + \eta_t \hat{T}_t Q_k, \]
where \( \hat{T}_t \) is the empirical Bellman operator associated with iteration \( t \). Define the error term \( \Delta_{k,t} := Q_{k,t} - TQ_k \). Then, we have that

\[ \Delta_{k,t+1} = (1 - \eta_t)\Delta_{k,t} + \eta_t (\hat{T}_t Q_k - TQ_k) \]
\[ = \prod_{i=0}^t (1 - \eta_i) \Delta_{k,0} + \sum_{i=0}^t \eta_i \left( \prod_{j=i+1}^t (1 - \eta_j) \right) (\hat{T}_i Q_k - TQ_k). \]

Let us consider the step-size \( \eta_t = 1/(2 + t) \), which satisfies the condition that \( (1 - \eta_t) \leq \eta_t / \eta_{t-1} \). Accordingly,

\[ \|\Delta_{k,t+1}\|_\infty \leq \eta_t \|\Delta_{k,0}\|_\infty + \left\| \sum_{i=0}^t \eta_i \left( \prod_{j=i+1}^t (1 - \eta_j) \right) (\hat{T}_i Q_k - TQ_k) \right\|_\infty. \]

We see that the noise term \( \{E_t : E_t = \hat{T}_t Q_k - TQ_k\} \) are i.i.d. random variables with zero-mean. Furthermore, each element of \( E_t \) is upper bounded by \( \|Q_k\|_{\text{span}} \) and and its variance is upper bounded by \( \|\sigma^2(Q_k)\|_\infty \):

\[ \|Q_k\|_{\text{span}} = \max_{(s,a)} Q_k(s,a) - \min_{(s,a)} Q_k(s,a), \]
\[ \sigma^2(Q_k)(s,a) = \gamma^2 \mathbb{E}_{a'} \left[ \left( \max_{s'} Q_k(s',a') - \mathbb{E}_{a'} \left[ \max_{a' \in A} Q_k(s',a') \right] \right)^2 \right], \]
\[ \|\sigma(Q_k)\|_\infty = \sqrt{\max_{(s,a)} \sigma^2(Q_k)(s,a)}. \]

Define \( P_t \) by the following recursion:

\[ P_{t+1} = (1 - \eta_t)P_t + \eta_t \left( \hat{T}_t Q_k - TQ_k \right) \quad \text{with} \quad P_0 = 0. \]

This is a stationary auto-regressive process. By [Wainwright, 2019, Lemma 3], we should have that

\[ E[\|P_{t+1}\|_\infty] \leq c \left\{ \sqrt{\eta_t} \|\sigma(Q_k)\|_\infty \sqrt{\log (2|\mathcal{S}||\mathcal{A}|)} + \eta_t \|Q_k\|_{\text{span}} \log (2|\mathcal{S}||\mathcal{A}|) \right\}, \]
where \( c > 0 \) is an absolute constant. As a result, we have that

\[ E[\|\Delta_{k,t+1}\|_\infty] \leq c \sqrt{\eta_t} \left( \|\Delta_{k,0}\|_\infty + \|\sigma(Q_k)\|_\infty \sqrt{\log (2|\mathcal{S}||\mathcal{A}|)} + \|Q_k\|_{\text{span}} \log (2|\mathcal{S}||\mathcal{A}|) \right). \]

By Lemma 5, we have that

\[ \|\Delta_{k,0}\|_\infty \leq 2 \|Q_k - Q^*\|_\infty \leq \frac{2}{1 - \gamma}, \]
\[ \|\sigma(Q_k)\|_\infty \leq \frac{1}{1 - \gamma}, \]
\[ \|Q_k\|_{\text{span}} \leq 2 \|Q_k\|_\infty \leq \frac{2}{1 - \gamma}. \]
Consequently, we obtain that
\[
\mathbb{E} \left[ \| \Delta_{k,T} \|_\infty \right] \leq \frac{c}{1-\gamma} \sqrt{\frac{\log (4|S||A|)}{T}}.
\]
According to Lemma 6, it suffices to consider that
\[
\varepsilon_{\text{opt}} = \frac{(1-\gamma)\varepsilon}{2}, \quad K = \frac{1}{1-\gamma} \log \left( \frac{4}{(1-\gamma)\varepsilon} \right).
\]
This further implies that
\[
\frac{c}{1-\gamma} \sqrt{\frac{\log (4|S||A|)}{T}} \leq \frac{(1-\gamma)\varepsilon}{2} \implies T \geq \frac{c \log (4|S||A|)}{(1-\gamma)^2 \varepsilon^2}.
\]
Hence, the total sample complexity is
\[
T \cdot K \cdot |S||A| = \tilde{O} \left( \frac{|S||A|}{(1-\gamma)^5 \varepsilon^2} \right).
\]

A.3 Proof of Theorem 3

Proof of Theorem 3: Following the same steps in the proof of Theorem 2, we have that
\[
\Delta_{k,t+1} = \eta_t \Delta_{k,0} + \eta_t \sum_{i=0}^{t} \left( \hat{T}_i Q_k - T Q_k \right).
\]
For our purpose, let us define \( \mathcal{F}_t \) be the sigma-algebra of all state-action-reward pairs generated before iteration \( t \). Then, for all \((s, a) \in S \times A\), we have that
\[
\mathbb{E} \left[ \hat{T}_t Q_k(s, a) - T Q_k(s, a) \mid \mathcal{F}_t \right] = 0, \quad \forall t \geq 0,
\]
Furthermore, for all \((s, a) \in S \times A\) and \( t \geq 0 \), we have that
\[
\mathbb{E} \left[ \left( \hat{T}_t Q_k(s, a) - T Q_k(s, a) \right)^2 \mid \mathcal{F}_t \right] = \sigma^2(Q_k)(s, a) = \gamma^2 \text{Var}_P(Q_k)(s, a),
\]
where \( \text{Var}_P(Q_k)(s, a) = \mathbb{E}_s'[(\max_{a'} Q_k(s', a') - \mathbb{E}_{s'}[\max_{a'} Q_k(s', a')])^2] \). Consider the sum of conditional variances:
\[
W_{t+1}(s, a) := \sum_{i=0}^{t} \mathbb{E} \left[ \left( \hat{T}_i Q_k(s, a) - T Q_k(s, a) \right)^2 \mid \mathcal{F}_t \right] = (t+1)\gamma^2 \text{Var}_P(Q_k)(s, a).
\]
According to Lemma 5, we have that \( \|Q_k\|_\infty \leq 1/(1-\gamma) \). Now, we can apply Lemma 11 with \( R = 1/(1-\gamma) \) and \( \sigma^2 = (t+1)\gamma^2 \text{Var}_P(Q_k)(s, a) \) and \( K = 1 \) to obtain that for any \((s, a) \in S \times A\), with probability \( 1 - \delta \), we have
\[
\left| \eta_t \sum_{i=0}^{t} \left( \hat{T}_i Q_k(s, a) - T Q_k(s, a) \right) \right| \leq c\eta_t \left( (t+1)\gamma^2 \text{Var}_P(Q_k)(s, a) \log \left( \frac{2}{\delta} \right) + \frac{1}{1-\gamma} \log \left( \frac{2}{\delta} \right) \right).
\]
In summary, we have that with probability \( 1 - \delta/K \), we have that
\[
Q_{k+1} = T Q_k + E_k,
\]
where \( E_k \) is an error term satisfying that
\[
\|E_k(s, a)\| \leq \eta_{T-1} \|Q_k - T Q_k\|_\infty + c\eta_{T-1} \sqrt{Tgamma^2 \text{Var}_P(Q_k)(s, a) \log \left( \frac{2K|S||A|}{\delta} \right)} + c\eta_{T-1} \frac{1}{1-\gamma} \log \left( \frac{2K|S||A|}{\delta} \right).
\]
\[
\leq c \frac{1}{T + 1} \frac{1}{1 - \gamma} + c \frac{1}{T + 1} \sqrt{T \gamma^2 \text{Var}_P(Q_k)(s, a) \log \left( \frac{2K|S||A|}{\delta} \right)} \\
+ c \frac{1}{T + 1} \frac{1}{1 - \gamma} \log \left( \frac{2K|S||A|}{\delta} \right)
\]
\[
\leq \alpha_T + c \frac{1}{T + 1} \frac{1}{1 - \gamma} \sqrt{T \gamma^2 \text{Var}_P(Q_k)(s, a) \log \left( \frac{2K|S||A|}{\delta} \right)},
\]
where
\[
\alpha_T = c \frac{1}{T(1 - \gamma)} + c \frac{1}{T(1 - \gamma)} \log \left( \frac{2K|S||A|}{\delta} \right).
\]
Now, we see that the recursion in (7) has the same form with that in [Agarwal et al., 2022]. Following the same steps in [Agarwal et al., 2022], when \( T \geq 1 / \log(K|S||A|/\delta) \), with probability at least 1 - \( \delta \), we have
\[
\|\Delta_K\|_\infty \leq c \left[ \frac{1}{T(1 - \gamma)^3} \log \left( \frac{K|S||A|}{\delta} \right) + \alpha_T + \gamma L \frac{1}{1 - \gamma} + \sqrt{\frac{1}{T(1 - \gamma)^3} \log \left( \frac{K|S||A|}{\delta} \right)} \right],
\]
where \( L = cK / \log(1/(1 - \gamma)) \). Thus, by
\[
K = \tilde{O} \left( \frac{1}{1 - \gamma} \right), \quad T = \tilde{O} \left( \frac{1}{(1 - \gamma)^3} \right),
\]
with probability 1 - \( \delta \), we obtain that \( \|Q_K - Q^*\|_\infty \leq \epsilon \).

### B Technical Lemmas

#### Lemma 7 (Upper Bound of Stochastic Gradient Variance)

**Lemma 7 (Upper Bound of Stochastic Gradient Variance).** *In iteration \( k \) and timestep \( t \),*
\[
\mathbb{E} \left[ \left\| \nabla L(Q_{k,t}; Q_k) \right\|_2^2 \right] \leq \|\nabla L(Q_{k,t}; Q_k)\|_{2,D-1}^2 + 6\gamma^2 \|Q_k - Q^*\|_\infty + \frac{3\gamma^2}{(1 - \gamma)^2}.
\]

**Proof.** Conditioned on \( Q_k, Q_{k,t} \), we have
\[
\mathbb{E} \left[ \left\| \nabla L(Q_{k,t}; Q_k) \right\|_2^2 \right] = \mathbb{E} \left[ \sum_{(s,a) \in S \times A} \mathbb{I}\{(S, A) = (s, a)\}^2 \left( Q_{k,t}(s,a) - r(s,a) - \gamma \max_a Q_k(S', a') \right)^2 \right]
\]
\[
= \sum_{(s,a) \in S \times A} \mathbb{E} \left[ \mathbb{I}\{(S, A) = (s, a)\}^2 \left( Q_{k,t}(s,a) - r(s,a) - \gamma \max_a Q_k(S', a') \right)^2 \right]
\]
\[
\overset{(1)}{=} \sum_{(s,a) \in S \times A} \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{I}\{(S, A) = (s, a)\}^2 \left( Q_{k,t}(s,a) - r(s,a) - \gamma \max_a Q_k(S', a') \right)^2 \mid S, A \right] \right]
\]
\[
\overset{(2)}{=} \sum_{(s,a) \in S \times A} \mathbb{E} \left[ \mathbb{I}\{(S, A) = (s, a)\}^2 \mathbb{E} \left[ \left( Q_{k,t}(s,a) - r(s,a) - \gamma \max_a Q_k(S', a') \right)^2 \mid S, A \right] \right]
\]
\[
\overset{(3)}{=} \sum_{(s,a) \in S \times A} \mathbb{E} \left[ \mathbb{I}\{(S, A) = (s, a)\}^2 \mathbb{E} \left[ \left( Q_{k,t}(s,a) - r(s,a) - \gamma \max_a Q_k(S', a') \right)^2 \mid s, a \right] \right].
\]

Equality (1) follows the Tower property, equality (2) follows that \( \mathbb{I}\{(S, A) = (s, a)\} \) is determined by \( S, A \) and equality (3) holds because of the indicator function. We first consider the term \( \mathbb{E}_{S' \sim P(\cdot|s, a)}[(Q_{k,t}(s,a) - r(s,a) - \gamma \max_a Q_k(S', a'))^2] \). With \( \mathbb{E}[X^2] = (\mathbb{E}[X])^2 + \text{Var}[X] \), we obtain
\[
\mathbb{E}_{S' \sim P(\cdot|s, a)} \left[ (Q_{k,t}(s,a) - r(s,a) - \gamma \max_a Q_k(S', a'))^2 \right]
\]
\[
= (Q_{k,t}(s,a) - \mathcal{T}Q_k(s,a))^2 + \text{Var}\left[Q_{k,t}(s,a) - r(s,a) - \gamma \max_a Q_k(S', a')\right]
\]
\[
= (Q_{k,t}(s,a) - \mathcal{T}Q_k(s,a))^2 + \text{Var}\left[\hat{\mathcal{T}}Q_k(s,a)\right]
\]
\[
\leq (Q_{k,t}(s,a) - \mathcal{T}Q_k(s,a))^2 + 6\gamma^2 \|Q_k - Q^*\|_\infty^2 + \frac{3\gamma^2}{(1 - \gamma)^2}
\]  
(Lemma 9).

Then we have that
\[
\mathbb{E}\left[\|\nabla L(Q_{k,t}; Q_k)\|_2^2\right]
\]
\[
\leq \sum_{(s,a)\in S \times A} \mathbb{E}\left[\mathbb{I}\{(S,A) = (s,a)\}^2 \left((Q_{k,t}(s,a) - \mathcal{T}Q_k(s,a))^2 + 6\gamma^2 \|Q_k - Q^*\|_\infty^2 + \frac{3\gamma^2}{(1 - \gamma)^2}\right)\right]
\]
\[
= \sum_{(s,a)\in S \times A} \left((Q_{k,t}(s,a) - \mathcal{T}Q_k(s,a))^2 + 6\gamma^2 \|Q_k - Q^*\|_\infty^2 + \frac{3\gamma^2}{(1 - \gamma)^2}\right) \mathbb{I}\{(S,A) = (s,a)\}^2
\]
\[
= \|\nabla L(Q_{k,t}; Q_k)\|_{2,D-1}^2 + 6\gamma^2 \|Q_k - Q^*\|_\infty^2 + \frac{3\gamma^2}{(1 - \gamma)^2}.
\]

\[\square\]

**Lemma 8 (Initial Distance).** Before the inner loop starts, we have that
\[
\|\mathcal{T}Q_k - Q_{k,0}\|_D^2 \leq 4 \|Q_k - Q^*\|_\infty^2.
\]

**Proof.** We have that
\[
\|\mathcal{T}Q_k - Q_{k,0}\|_D^2 = \|\mathcal{T}Q_k - Q_k\|_D^2
\]
\[
= \|\mathcal{T}Q_k - \mathcal{T}Q^* + \mathcal{T}Q^* - Q_k\|_D^2
\]
\[
= \sum_{(s,a)} d(s,a) \left((\mathcal{T}Q_k(s,a) - \mathcal{T}Q^*(s,a) + \mathcal{T}Q^*(s,a) - Q_k(s,a))^2\right)
\]
\[
\leq \sum_{(s,a)} d(s,a) \left[2(\mathcal{T}(Q_k)(s,a) - \mathcal{T}Q^*(s,a))^2 + 2(\mathcal{T}Q^*(s,a) - Q_k(s,a))^2\right]
\]
\[
\leq \sum_{(s,a)} d(s,a) \left[2\gamma^2 \|Q_k - Q^*\|_\infty^2 + 2\|Q_k - Q^*\|_\infty^2\right]
\]
\[
\leq 4 \|Q_k - Q^*\|_\infty^2.
\]

\[\square\]

**Lemma 9 (Variance of \(\hat{\mathcal{T}}(Q_k)\)).** For each \(k\), we have that
\[
\text{Var}[\hat{\mathcal{T}} Q_k(s,a)] \leq 6\gamma^2 \|Q_k - Q^*\|_\infty^2 + \frac{3\gamma^2}{(1 - \gamma)^2}.
\]

**Proof.**
\[
\text{Var}[\hat{\mathcal{T}} Q_k(s,a)]
\]
\begin{align*}
= & \mathbb{E}\left[\left(\hat{T}Q_k(s, a) - TQ_k(s, a)\right)^2\right] \\
= & \mathbb{E}\left[\left(\hat{T}Q_k(s, a) - TQ_k(s, a) - \hat{T}Q^*(s, a) + \hat{T}Q^*(s, a) - TQ^*(s, a) + TQ^*(s, a)\right)^2\right] \\
\leq & 3\mathbb{E}\left[\left(\hat{T}Q_k(s, a) - \hat{T}Q^*(s, a)\right)^2\right] + 3\mathbb{E}\left[(TQ_k(s, a) - TQ^*(s, a))^2\right] + 3\mathbb{E}\left[(\hat{T}Q^*(s, a) - TQ^*(s, a))^2\right] \\
\leq & 3\gamma^2 \|Q_k - Q^*\|_\infty^2 + 3\gamma^2 \|Q_k - Q^*\|_\infty^2 + 3\frac{\gamma^2}{(1 - \gamma)^2} \\
= & 6\gamma^2 \|Q_k - Q^*\|_\infty^2 + \frac{3\gamma^2}{(1 - \gamma)^2}.
\end{align*}

Lemma 10 (Boundedness of Estimate; Lemma 8 of [Lee and He, 2020]). Suppose that \(\mathbb{E}[\|Q_i - TQ_{i-1}\|_2^2] \leq \varepsilon_{\text{opt}}, \forall i \leq k\) and \(\varepsilon_{\text{opt}} \leq (1 - \gamma)^2\). Then, we have that

\[
\mathbb{E}\left[\|Q_k - Q^*\|_\infty^2\right] \leq \frac{8}{(1 - \gamma)^2}.
\]

Lemma 11 (Freedman’s Inequality). Suppose that \(Y_n = \sum_{k=1}^n X_k \in \mathbb{R}\), where \(\{X_k\}\) is a real-valued scalar sequence obeying

\[
|X_k| \leq R, \quad \text{and} \quad \mathbb{E}[X_k \mid \{X_j\}_{j<k}] = 0 \quad \forall k \geq 1.
\]

Define

\[
W_n := \sum_{k=1}^n \mathbb{E}_{k-1}[X_k^2],
\]

where the expectation \(\mathbb{E}_{k-1}\) is conditional on \(\{X_j\}_{j<k}\). Then, for any given \(\sigma^2 \geq 0\), we have that

\[
\mathbb{P}\left(|Y_n| \geq \tau \text{ and } W_n \leq \sigma^2\right) \leq 2\exp\left(-\frac{\tau^2}{\sigma^2 + R\tau/3}\right).
\]

In addition, if \(W_n \leq \sigma^2\) almost surely, for any positive integer \(K \geq 1\), we have that

\[
\mathbb{P}\left(|Y_n| \leq \sqrt{8 \max\left\{W_n, \frac{\sigma^2}{2K}\right\} \log \frac{2K}{\delta} + \frac{4}{3} R \log \frac{2K}{\delta}}\right) \geq 1 - \delta.
\]