Approximation of Random Diffusion Equation by Nonlocal Diffusion Equation in Free Boundary Problems of One Space Dimension

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Abstract. We show how the Stefan type free boundary problem with random diffusion in one space dimension can be approximated by the corresponding free boundary problem with nonlocal diffusion. The approximation problem is a slightly modified version of the nonlocal diffusion problem with free boundaries considered in [4, 8]. The proof relies on the introduction of several auxiliary free boundary problems and constructions of delicate upper and lower solutions for these problems. As usual, the approximation is achieved by choosing the kernel function in the nonlocal diffusion term of the form \( J_\epsilon(x) = \frac{1}{\epsilon} J(\frac{x}{\epsilon}) \) for small \( \epsilon > 0 \), where \( J(x) \) has compact support. We also give an estimate of the error term of the approximation by some positive power of \( \epsilon \).

Key words: Free boundary, random diffusion, nonlocal diffusion, approximation.

MSC2010 subject classifications: 35K20, 35R35, 35R09.

1. Introduction

Free boundary problems of the form

\[
\begin{align*}
\frac{v_t}{v(t,g(t))} &= dv_{xx} + f(t,x,v), & t > 0, & x \in (g(t), h(t)), \\
v(t,g(t)) &= v(t,h(t)) = 0, & t > 0, \\
g'(t) &= -\mu v_x(t,g(t)), & t > 0, \\
h'(t) &= -\mu v_x(t,h(t)), & t > 0, \\
g(0) &= -h_0, & h(0) = h_0, & v(0,x) = v_0(x), & x \in [-h_0, h_0]
\end{align*}
\]

have been widely studied in recent years, after the work [15], where a logistic type nonlinear term \( f = f(v) \) was considered, and the initial function \( v_0 \) was assumed to satisfy \( v_0 \in C^2([−h_0, h_0]), v_0(±h_0) = 0 \) and \( v_0 > 0 \) in \((-h_0, h_0)\). For continuous initial function \( v_0 \) and general \( f = f(t,x,v) \), the well-posedness of (1.1) was proved in [10]. We refer to [11, 12, 14, 16, 19, 21, 26, 27, 29, 31, 34, 36] and the references therein for a sample of the recent works on (1.1). See also [5, 20, 23, 32] for some related earlier works.

If \( f \equiv 0 \) in (1.1), then the problem reduces to the well known one-phase Stefan equation [9, 33], which was proposed by Josef Stefan in 1890 to describe the melting of ice in contact with water, and was extensively studied in the past half century; see, for example, [2, 3, 21, 22, 25, 28].

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More recently, the following nonlocal version of \([1.1]\) was proposed and investigated in \([4]\) (see \([8]\) for the case \(f \equiv 0\)):

\[
\begin{align*}
\begin{cases}
u_t = \delta \left[ \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - u(t,x) \right] + f(t,x,u), & t > 0, \ x \in (g(t), h(t)), \\
u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\
g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{\infty} J(x-y)u(t,x)dydx, & t > 0, \\
h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)u(t,x)dydx, & t > 0, \\
g(0) = -h_0, \ h(0) = h_0, \ u(0,x) = u_0(x), & x \in [-h_0, h_0],
\end{cases}
\end{align*}
\tag{1.2}
\]

In both \([1.1]\) and \([1.2]\), \(\mu\) and \(h_0\) are given positive numbers, and for their respective well-posedness, the usual basic assumptions are:

- The initial functions \(u_0, v_0\) belong to \(I_0\), where
  \[
  I_0 := \{ \phi \in C([-h_0, h_0]) : \phi(\pm h_0) = 0, \ \phi(x) > 0 \text{ in } (-h_0, h_0) \};
  \]

- The function \(f: \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}\) satisfies
  \(f_1\): \(f \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+), \ f(t,x,0) \equiv 0, \ f(t,x,u)\) is locally Lipschitz in \(u \in \mathbb{R}^+\), uniformly in \((t,x) \in \mathbb{R}^+ \times \mathbb{R}\),

- \(f_2\): There exists \(K > 0\) such that \(f(t,x,u) \leq 0\) for \(u > K\) and \((t,x) \in \mathbb{R}^+ \times \mathbb{R}\);

- The kernel function \(J: \mathbb{R} \to \mathbb{R}\) in \([1.2]\) is continuous, nonnegative and satisfies
  \(J\): \(J(0) > 0, \int_{\mathbb{R}} J(x)dx = 1, \ J\) is even.

In \((f_1)\), the requirement \(f(t,x,0) \equiv 0\) can be relaxed to \(f(t,x,0) \geq 0\). Assumption \((f_2)\) is a simple sufficient condition to guarantee that the positive solution stays bounded and hence is defined for all \(t > 0\). For local existence it is not needed.

For logistic type \(f(t,x,u) = f(u)\) (also known as Fisher-KPP type), it was shown in \([4]\) that the long-time behaviour of \([1.2]\), similar to \([1.1]\), is governed by a spreading-vanishing dichotomy. However, when spreading happens, it was proved in \([13]\) that the spreading speed of \([1.2]\) could be finite or infinite, depending on the properties of the kernel function \(J\) in \([1.2]\); this is very different from \([1.1]\), where the spreading speed is always finite whenever spreading happens (\([11,12,14,16,18,26,30]\)).

For the corresponding fixed boundary problems of \([1.1]\) and \([1.2]\), it is well-known \([11,16,7,37]\) that, over any finite time interval \([0,T]\), the unique solution \(v\) of the local diffusion problem is the limit of the unique solution of the nonlocal problem as \(\epsilon \to 0\), when the kernel function \(J\) in the nonlocal problem is replaced by

\[
\tilde{J}_\epsilon(x) = C J_\epsilon(x) := C \frac{1}{\epsilon} J \left( \frac{x}{\epsilon} \right)
\]

with a suitable positive constant \(C\), provided that \(J\) has compact support, \(f\) and the common initial function are all smooth enough.

For example, if \(J\) satisfies \((J)\) with supporting set contained in \([-1,1]\), and \(\tilde{J}_\epsilon, J_\epsilon\) are defined as above with

\[
C = C_* := \left[ \frac{1}{2} \int_{\mathbb{R}} J(z)z^2dz \right]^{-1} = \left[ \int_0^1 J(z)z^2dz \right]^{-1},
\tag{1.3}
\]
and $F(t, x, u)$ is $C^1$ in $t$, $C^3$ in $(x, u)$, and $u_0 \in C^3([a, b])$, then it follows from Theorem A of [37] that the unique solution $u_\epsilon$ of the nonlocal diffusion problem

$$
\begin{cases}
  u_t = \frac{C}{\epsilon^2} \left[ \int_a^b J(x-y)u(t,y)dy - u(t,x) \right] + F(t, x, u), & x \in [a, b], \ t > 0, \\
  u(0, x) = u_0(x), & x \in [a, b]
\end{cases}
$$

converges to the unique solution $u$ of the corresponding random diffusion problem

$$
\begin{cases}
  u_t = u_{xx} + F(t, x, u), & x \in [a, b], \ t > 0, \\
  u = 0, & x \in \{a, b\}, \ t > 0, \\
  u(0, x) = u_0(x), & x \in [a, b],
\end{cases}
$$

in the following sense: For any $T \in (0, \infty)$,

$$\lim_{\epsilon \to 0} \|u_\epsilon - u\|_{C([0,T] \times [a, b])} = 0.
$$

If $F \equiv 0$ and $u_0 \in C^{2+\alpha}([a, b])$, $0 < \alpha < 1$, then it follows from Theorem 1.1 of [6] that

$$\|u_\epsilon - u\|_{C([0,T] \times [a, b])} \leq C\epsilon^\alpha
$$

for some $C > 0$ and all small $\epsilon > 0$.

In this paper, we examine whether similar results hold between the free boundary problems (1.1) and (1.2). We show that (1.1) is the limiting problem of a slightly modified version of (1.4). The modification occurs in the free boundary equations

$$
\begin{align*}
  g'(t) &= -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx, \\
  h'(t) &= \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)u(t,x)dydx.
\end{align*}
$$

In [4], the equations in (1.4) are obtained from the assumption that the changing population range $[g(t), h(t)]$ of the species with population density $u(t, x)$ expands at each of its end point $(x = g(t)$ and $x = h(t)$) with a rate proportional to the population flux across that end point.

If we assume instead that these rates are proportional to the population flux across the end points of a slightly reduced region of the population range, say $[g(t) + \delta, h(t) - \delta]$ for some small $\delta > 0$, then (1.4) should be changed accordingly to

$$
\begin{align*}
  g'(t) &= -\mu \int_{g(t)+\delta}^{h(t)-\delta} \int_{-\infty}^{g(t)+\delta} J(x-y)u(t,x)dydx, \\
  h'(t) &= \mu \int_{g(t)+\delta}^{h(t)-\delta} \int_{h(t)-\delta}^{\infty} J(x-y)u(t,x)dydx.
\end{align*}
$$

So in the context of population spreading as explained in [4], the expansion of the population range governed by (1.5) is also meaningful.

---

1Note that this problem is equivalent to

$$
\begin{cases}
  u_t = \int_a^b \frac{J(x-y)}{\epsilon^2} \left[ u(t,y) - u(t,x) \right]dy + F(t, x, u), & x \in [a, b], \ t > 0, \\
  u = 0, & x \in \mathbb{R} \setminus [a, b], \ t > 0, \\
  u(0, x) = u_0(x), & x \in [a, b].
\end{cases}
$$

2See Remark 1.4 below on the possible necessity of the variation from (1.2).
The modified (1.2) then has the form

\[
\begin{aligned}
&u_t = d \int_{g(t)}^{h(t)} J(x - y)u(t, y)dy - du(t, x) + f(t, x, u), \quad t > 0, \quad x \in (g(t), h(t)), \\
u(t, g(t)) = u(t, h(t)) = 0, & \quad t > 0, \\
g'(t) = -\mu \int_{g(t)+\delta}^{h(t)-\delta} \int_{-\infty}^{\infty} J(x - y)u(t, x)dydx, & \quad t > 0, \\
h'(t) = \mu \int_{g(t)+\delta}^{h(t)-\delta} \int_{h(t)-\delta}^{\infty} J(x - y)u(t, x)dydx, & \quad t > 0, \\
g(0) = -h_0, \quad h(0) = h_0, & \quad u(0, x) = u_0(x), \quad x \in [-h_0, h_0].
\end{aligned}
\]

We are now ready to state the main results of this paper.

(1.6)

We are now able to describe the nonlocal approximation problem of (1.1). Suppose that

\[
\begin{aligned}
&J \text{ and in what follows, spt}(J) \subset [-1, 1], \\
&J_\epsilon(x) := \frac{1}{\epsilon}J\left(\frac{x}{\epsilon}\right),
\end{aligned}
\]

and some extra smoothness conditions on \( f \) and \( v_0 \) (to be specified below) are satisfied. (Here and in what follows, spt(\( J \)) stands for the supporting set of \( J \).) Then we will show that the following problem, with \( 0 < \epsilon \ll 1 \), is an approximation of (1.1):

\[
\begin{aligned}
&u_t = d \int_{g(t)}^{h(t)} J_\epsilon(x - y)u(t, y)dy - du(t, x) + f(t, x, u), \quad t > 0, \quad x \in (g(t), h(t)), \\
u(t, g(t)) = u(t, h(t)) = 0, & \quad t > 0, \\
g'(t) = -\mu \int_{g(t)+\sqrt{\epsilon}}^{h(t)-\sqrt{\epsilon}} \int_{-\infty}^{\infty} J_\epsilon(x - y)u(t, x)dydx, & \quad t > 0, \\
h'(t) = \mu \int_{g(t)+\sqrt{\epsilon}}^{h(t)-\sqrt{\epsilon}} \int_{h(t)-\sqrt{\epsilon}}^{\infty} J_\epsilon(x - y)u(t, x)dydx, & \quad t > 0, \\
g(0) = h(0) = h_0, & \quad u(0, x) = v_0(x), \quad x \in [-h_0, h_0],
\end{aligned}
\]

where \( C_\epsilon \) is given by (1.3) and

\[
(1.9) \quad C_0 := \left[ \int_{-1}^{1} J(y)dydx \right]^{-1} = \left[ \int_{0}^{1} J(y)dydx \right]^{-1} = \left[ \int_{0}^{1} J(y)dy \right]^{-1} < C_\epsilon.
\]

Let us note that, from (1.7) we have \( J_\epsilon(x) = 0 \) for \( |x| \geq \epsilon \), and hence, for \( 0 < \epsilon \ll 1 \),

\[
\begin{aligned}
&\int_{g(t)+\sqrt{\epsilon}}^{h(t)-\sqrt{\epsilon}} \int_{-\infty}^{\infty} J_\epsilon(x - y)u(t, x)dydx = \int_{-\epsilon}^{\epsilon} \int_{0}^{1} J_\epsilon(x - y)u(t, g(t) + \sqrt{\epsilon} + x)dydx, \\
&\int_{g(t)+\sqrt{\epsilon}}^{h(t)-\sqrt{\epsilon}} \int_{h(t)-\sqrt{\epsilon}}^{\infty} J_\epsilon(x - y)u(t, x)dydx = \int_{-\epsilon}^{\epsilon} \int_{0}^{1} J_\epsilon(x - y)u(t, h(t) - \sqrt{\epsilon} - x)dydx.
\end{aligned}
\]

Therefore in (1.8), for \( 0 < \epsilon \ll 1 \), we may rewrite

\[
(1.10) \quad \begin{aligned}
g'(t) &= -\mu \frac{C_0}{\epsilon^{1/2}} \int_{-\epsilon}^{\epsilon} \int_{0}^{1} J_\epsilon(x - y)u(t, g(t) + \sqrt{\epsilon} + x)dydx, \\
h'(t) &= \mu \frac{C_0}{\epsilon^{1/2}} \int_{-\epsilon}^{\epsilon} \int_{0}^{1} J_\epsilon(x - y)u(t, h(t) - \sqrt{\epsilon} - x)dydx.
\end{aligned}
\]

The extra smoothness conditions on \( f \) and \( v_0 \) mentioned above are: There exists some \( \alpha \in (0, 1) \) such that

\[
(1.11) \quad f \in C^{\alpha, \alpha}(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+),
\]

\[
(1.12) \quad v_0 \in C^{2+\alpha}([-h_0, h_0]), \quad v_0(\pm h_0) = 0 < |v_0'(\pm h_0)|, \quad v_0(x) > 0 \text{ in } (-h_0, h_0).
\]

We are now ready to state the main results of this paper.
Theorem 1.1. Suppose \((f_1), (f_2), (f_3), (J)\) and \((1.7)\) hold, and \(v_0\) satisfies \((1.11)\). Then for every small \(\epsilon > 0\), problem \((1.8)\) has a unique positive solution, denoted by \((u_\epsilon, g_\epsilon, h_\epsilon)\). Moreover, if \((v, g, h)\) is the unique positive solution of \((1.1)\) and if we define \(v(t, x) = 0\) for \(x \in \mathbb{R} \setminus (g(t), h(t))\) and \(u_\epsilon(t, x) = 0\) for \(x \in \mathbb{R} \setminus (g_\epsilon(t), h_\epsilon(t))\), then, for any \(T \in (0, \infty)\),

\[
\begin{align*}
\lim_{\epsilon \to 0} & \sup_{t \in [0, T]} \|v_\epsilon(t, \cdot) - v(t, \cdot)\|_{L^\infty(\mathbb{R})} = 0, \\
\lim_{\epsilon \to 0} & \|g_\epsilon - g\|_{L^\infty([0, T])} = 0, \quad \lim_{\epsilon \to 0} \|h_\epsilon - h\|_{L^\infty([0, T])} = 0.
\end{align*}
\]

If we further raise the smoothness requirements on \(v_0\) and \(f\), namely assuming additionally

\[(1.12)\quad v_0 \in C^{3+\alpha}([-h_0, h_0]),
\]

\[(f_4)\quad f \in C^{1+\alpha, 1+\alpha, 1+\alpha}(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+),
\]

then we can obtain an error estimate as follows.

Theorem 1.2. Under the assumptions of Theorem 1.1, if additionally \((f_4)\) and \((1.12)\) are satisfied, then for any \(T > 0\) and any \(\gamma \in (0, \min\{\alpha, \frac{1}{2}\})\), there exists \(0 < \epsilon_* \ll 1\) such that for every \(\epsilon \in (0, \epsilon_*)\),

\[
\begin{align*}
\sup_{t \in [0, T]} \|u_\epsilon(t, \cdot) - v(t, \cdot)\|_{L^\infty(\mathbb{R})} & \leq \epsilon^\gamma, \\
\sup_{t \in [0, T]} |g_\epsilon(t) - g(t)| & \leq \epsilon^\gamma, \quad \sup_{t \in [0, T]} |h_\epsilon(t) - h(t)| \leq \epsilon^\gamma.
\end{align*}
\]

Remark 1.3. Theorem 1.1 still holds if in \((1.8)\), the free boundary conditions are changed to, for an arbitrary \(\beta \in (0, 1)\),

\[
\begin{align*}
g'(t) &= -\mu \frac{C_0}{\epsilon^{1+\beta}} \int_{g(t)+\epsilon^\beta}^{h(t)-\epsilon^\beta} \int_{-\infty}^{g(t)+\epsilon^\beta} J_\epsilon(x-y)u(t, x)dydx, \quad t > 0, \\
h'(t) &= \mu \frac{C_0}{\epsilon^{1+\beta}} \int_{h(t)+\epsilon^\beta}^{h(t)+\epsilon^\beta} \int_{h(t)-\epsilon^\beta}^{\infty} J_\epsilon(x-y)u(t, x)dydx, \quad t > 0,
\end{align*}
\]

or equivalently, in \((1.10)\) the equations are changed to

\[
\begin{align*}
g'(t) &= -\mu \frac{C_0}{\epsilon^{1+\beta}} \int_{0}^{t} J_\epsilon(x-y)u(t, g(t)+\epsilon^\beta + x)dydx, \\
h'(t) &= \mu \frac{C_0}{\epsilon^{1+\beta}} \int_{0}^{t} J_\epsilon(x-y)u(t, h(t) - \epsilon^\beta - x)dydx.
\end{align*}
\]

In such a case, Theorem 1.2 still holds if \(\gamma \in (0, \min\{\alpha, 1-\beta\})\). Only minor changes are needed in the proofs; for example, for such a case, in \((2.4)\), \(\gamma_1\) should belong to \((\gamma, \min\{\alpha, 1-\beta\})\).

Remark 1.4. We believe that the modification of \((1.2)\) to \((1.6)\) is necessary in order to obtain an approximation problem of \((1.1)\) such as \((1.8)\). Some analysis leading us to this conjecture is given in Section 5.

The rest of the paper is organised as follows. In Section 2, we collect some preparatory results for the proof of the main results, and also explain the strategy of the proof (near the end of the section). Sections 3 and 4 consist of the proofs of Theorems 1.1 and 1.2 respectively, based on the construction of delicate upper and lower solutions, following the strategy set in Section 2. In Section 5, we discuss the conjecture in Remark 1.4 through some detailed calculations.

### 2. Preparations

In this section, we prepare some results to be used in the proof of Theorems 1.1 and 1.2 in Sections 3 and 4. These preparatory results can be proved by simple variations of existing methods and techniques.
Theorem 2.1. Suppose (J), (f₁) and (f₂) hold, u₀ ∈ I₀ and 0 ≤ δ ≪ h₀. Then problem (1.6) has a unique positive solution defined for all t > 0. In particular, for 0 < ε ≪ 1, problem (1.8) admits a unique positive solution (uᵦ, gᵦ, hᵦ) defined for all t > 0.

Proof. In [4, Theorem 2.1], existence and uniqueness for problem (1.2) is proved by using the contraction mapping theorem several times. If the third and fourth equations of (1.2) are replaced by (1.6), the proof in [4] can be carried over with only minor and obvious changes. We leave the details to the interested reader.

Theorem 2.2. Suppose (f₁) and (f₂) hold, and u₀ ∈ I₀. (i) Assume that T ∈ (0, ∞), 0 ≤ δ ≪ h₀, and the kernel function J satisfies (J). If (ν, g, h) ∈ C(Ḋ) × C([0, T]) × C([0, T]) with D = {(t, x) : t ∈ (0, T}, x ∈ (g(t), h(t))} satisfies

\[
\begin{align*}
\nu_t & \geq d \int_{\bar{g}(t)}^{\bar{h}(t)} J(x - y) \nu(t, y) dy - d\bar{u}(t, x) + f(t, x, \nu), \quad t \in (0, T], \quad x \in (g(t), h(t)), \\
\bar{u}(t, g(t)) & \geq 0, \quad \nu(t, h(t)) \geq 0, \quad \bar{u}(t, \bar{h}(t)) \geq 0, \\
\bar{g}(t) & \leq -\mu \int_{\bar{g}(t)}^{\bar{h}(t)} \int_{-\infty}^{\infty} J(x - y) \bar{u}(t, x) dy dx, \quad t \in (0, T], \\
\bar{h}(t) & \geq \mu \int_{\bar{g}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y) \bar{u}(t, x) dy dx, \quad t \in (0, T], \\
\nu(0, x) & \geq u₀(x), \quad x \in [-h₀, h₀] \subset [g(0), h(0)],
\end{align*}
\]

then

\[
\begin{align*}
\bar{g}(t, x) & \leq g(t, x), \quad h(t, x) \leq h(t, x) \quad \text{for } t \in [0, T], \\
u(t, x) & \leq \nu(t, x) \quad \text{for } t \in (0, T], \quad x \in [\bar{g}(t), \bar{h}(t)],
\end{align*}
\]

where (u, g, h) is the unique positive solution of (1.6).

(ii) Assume T ∈ (0, ∞), g, h, P₁, P₂ ∈ C¹([0, T]), and ν ∈ C¹₂(Ω) satisfies

\[
\begin{align*}
\nu_t & \geq \mu \nu_{xx} + f(t, x, \nu), \quad t \in (0, T], \quad x \in (\bar{g}(t), \bar{h}(t)), \\
\bar{v}(t, g(t)) & = 0, \quad \bar{v}(t, h(t)) = 0, \quad t \in (0, T], \\
\bar{g}(t) & \leq -\mu \nu_{xx}(t, \bar{g}(t)) + P₁(t), \quad t \in (0, T], \\
\bar{h}(t) & \geq -\mu \nu_{xx}(t, \bar{h}(t)) + P₂(t), \quad t \in (0, T], \\
\nu(0, x) & \geq v₀(x), \quad x \in [-h₀, h₀] \subset [g(0), h(0)].
\end{align*}
\]

If (v, g, h) ∈ C¹₂(Ω) × C¹([0, T]) × C¹([0, T]) with Ω = {(t, x) : t ∈ (0, T], x ∈ (g(t), h(t))} satisfies (2.2) with all the inequalities replaced by equalities, then

\[
\begin{align*}
\bar{g}(t) & \leq g(t), \quad h(t) \leq h(t) \quad \text{for } t \in (0, T], \\
v(t, x) & \leq \nu(t, x) \quad \text{for } (t, x) \in [0, T] \times [\bar{g}(t), \bar{h}(t)].
\end{align*}
\]

Proof. For conclusion (i), if δ = 0, then it follows directly from [4, Lemma 3.1]. When δ > 0, one can similarly prove it since the proof of [4, Lemma 3.1] is not affected.

The comparison principle in part (ii) can be proved by following the proof of [15, Lemma 3.5], because the extra terms P₁ and P₂ in the inequalities for \( \bar{g}' \) and \( \bar{h}' \) do not affect the argument there.

The triple (ν, g, h) may be called an upper solution. We can define a lower solution by reversing all the inequality signs and obtain analogous comparison results.

Next, for \( i = 1, 2, T₀ > 0 \) and

\[
\gamma_i \in (\gamma, \min\{\alpha, \frac{1}{2}\})
\]
with \( \gamma \) given by Theorem 1.2 we consider the following perturbation problems of (1.1),

\[
\begin{align*}
\partial_t v_i &= d \partial_{xx} v_i + f(t, x, v_i) + A_i \epsilon^{\gamma_1}, & t \in (0, T_0], & x \in (g_i(t), h_i(t)), \\
v_i(t, g_i(t)) &= v_i(t, h_i(t)) = 0, & t \in (0, T_0], \\
g_i'(t) &= -\mu \partial_x v_i(t, g_i(t)) - B_i \epsilon^{\gamma_1}, & t \in (0, T_0], \\
h_i'(t) &= -\mu \partial_x v_i(t, h_i(t)) + B_i \epsilon^{\gamma_1}, & t \in (0, T_0], \\
v_i(0, x) &= v_0(x), & x \in [-h_0, h_0],
\end{align*}
\]

where \( A_1 = B_1 = 1, A_2 = 0 \) and \( B_2 = -2 \). We require \( A_i \geq 0 \) to guarantee the solution is nonnegative and well-defined. Solutions of (2.4) will be used to construct upper and lower solutions of (1.8). We have the following results.

**Theorem 2.3.** (i) Suppose (f), (g) and (h) hold, and \( v_0 \) satisfies (1.11). Then for any \( T_0 \in (0, +\infty) \), there exists \( \epsilon_0 > 0 \) small depending on \( T_0 \) such that for any \( \epsilon \in [0, \epsilon_0] \), problem (2.4) has a unique positive solution \( (v_ie, g_ie, h_ie) \). Moreover, there exists \( M_1 > 0 \) depending on \( T_0 \), \( u_0 \) and \( \alpha \in (0, 1) \), such that, for every \( \epsilon \in [0, \epsilon_0] \) and \( i = 1, 2, \)

\[
\begin{align*}
\|v_ie\|_{C^{1+\alpha/2, \alpha}([T_0])}, & \quad \|g_ie\|_{C^{1+\alpha/2}([0, T_0])}, & \quad \|h_ie\|_{C^{1+\alpha/2}([0, T_0])} \leq M_1,
\end{align*}
\]

where \( \Omega_{ie} := \{ (t, x) : t \in (0, T_0], x \in [g_ie(t), h_ie(t)] \}. \)

(ii) If in addition, (g) and (1.12) are satisfied, then there exists \( M_2 > 0 \) depending on \( T_0 \) and \( u_0 \) such that, for \( \epsilon \in [0, \epsilon_0] \) and \( i = 1, 2, \)

\[
\begin{align*}
\|v_ie\|_{C^{2+\alpha, \alpha}([T_0])}, & \quad \|g_ie\|_{C^{2+\alpha/2}([0, T_0])}, & \quad \|h_ie\|_{C^{2+\alpha/2}([0, T_0])} \leq M_2.
\end{align*}
\]

Let us note that if \( \epsilon = 0 \), then (2.4) reduces to (1.1). The relationship between (1.1) and (2.4) with \( \epsilon > 0 \) is given in the following result.

**Theorem 2.4.** Under the assumptions of Theorem 2.3 part (i), the unique solution \((v, g, h)\) of (1.1) satisfies, for \( \epsilon \in (0, \epsilon_0) \),

\[
\begin{align*}
[g_2(t), h_2(t)] & \subset [g(t), h(t)] \subset [g_1(t), h_1(t)]], & t \in [0, T_0], \\
v_2(t, x) & \leq v(t, x) \leq v_1(t, x), & t \in [0, T_0], x \in \mathbb{R},
\end{align*}
\]

where we have assumed \( v(t, x) = 0 \) for \( x \in \mathbb{R} \setminus (g(t), h(t)) \) and \( v_ie(t, x) = 0 \) for \( x \in \mathbb{R} \setminus (g_ie(t), h_ie(t)) \). Moreover,

\[
\begin{align*}
\lim_{\epsilon \to 0} \sup_{t \in [0, T_0]} \|v_ie(t, \cdot) - v(t, \cdot)\|_{L^\infty(\mathbb{R})} = 0,
\end{align*}
\]

\[
\begin{align*}
\lim_{\epsilon \to 0} \|g_ie - g\|_{L^\infty([0, T_0])} = 0, \quad \lim_{\epsilon \to 0} \|h_ie - h\|_{L^\infty([0, T_0])} = 0.
\end{align*}
\]

**Proof of Theorem 2.3.** This follows from simple variations of existing techniques. So unless necessary, we will be brief and leave the details to the interested reader.

Let the assumptions in part (i) be satisfied. For the local existence and uniqueness result to (2.4), we will follow the proof of [15] Theorem 2.1 with some minor modifications. Since the use of the Sobolev embedding theorem there requires some corrections, we provide the necessary details in this part of the proof. An alternative correction can be found in [35].

Firstly, as in [15], for small \( t > 0 \) we straighten the free boundary of (2.4) by the transformation \((t, y) \to (t, x)\), where

\[
x = \Gamma(t, y) := y + \zeta(y)[h(t) - h_0] + \zeta(-y)[g(t) + h_0], \quad y \in \mathbb{R},
\]

where \((g(t), h(t))\) stands for \((g_i(t), h_i(t))\) with \( i = 1 \) or \( 2 \), and \( \zeta \in C^\infty(\mathbb{R}) \) satisfies

\[
\zeta(y) = 1 \text{ if } |y - h_0| < \frac{h_0}{4}, \quad \zeta(y) = 0 \text{ if } |y - h_0| > \frac{h_0}{2}, \quad |\zeta'(y)| < \frac{6}{h_0} \text{ in } \mathbb{R}.
\]
Then for small $t > 0$, say $t \in [0, S]$ such that $|g(t) - h_0|, |h(t) - h_0| \leq \frac{h_0}{10}$, the interval $[g(t), h(t)]$ in the $x$-axis is changed to $[-h_0, h_0]$ in the $y$-axis, and with

$$w(t, y) := v_i(t, \Gamma(t, y)),$$

problem (2.4) for $t \in [0, S]$ is changed to

$$\begin{cases}
\partial_t w = \tilde{A}(t, y) w_y + \tilde{B}(t, y) w_f + f(t, \Gamma(t, y), w) + A_i \epsilon \gamma, & t \in (0, S), y \in (-h_0, h_0), \\
w(t, -h_0) = w(t, h_0) = 0, & t \in (0, S),
\end{cases}$$

(2.9)

where

$$\begin{cases}
\tilde{A}(t, y) = A(g(t), h(t), y), \\
\tilde{B}(t, y) = B(g(t), h(t), y) + g'(t) C_1(g(t), h(t), y) + h'(t) C_2(g(t), h(t), y),
\end{cases}$$

and $A(\xi, \eta, y), B(\xi, \eta, y), C_1(\xi, \eta, y), C_2(\xi, \eta, y)$ are $C^\infty$ functions in $[h_0 - \frac{h_0}{10}, h_0 + \frac{h_0}{10}] \times \mathbb{R}$, with $d/4 \leq A(\xi, \eta, y) \leq 16d$ in this range.

Denote

$$g_1 := -\mu v'_0(-h_0), h_1 := -\mu v'_0(h_0), S := \frac{h_0}{16(1 + |g_1| + h_1)},$$

and for $T \in (0, S]$ define $\Delta_T := [0, T] \times [-h_0, h_0]$.

$$\begin{align*}
\mathcal{D}_T &:= \{ w \in C(\Delta_T) : w(0, y) = u_0(y), \|w - u_0\|_{C(\Delta_T)} \leq 1 \}, \\
\mathcal{D}_{1T} &:= \{ g \in C^1([0, T]) : g(0) = -h_0, g'(0) = g_1, \|g' - g_1\|_{C([0, T])} \leq 1 \}, \\
\mathcal{D}_{2T} &:= \{ h \in C^1([0, T]) : h(0) = h_0, h'(0) = h_1, \|h' - h_1\|_{C([0, T])} \leq 1 \}.
\end{align*}$$

It is easily seen that $\mathcal{D}_T := \mathcal{D}_T \times \mathcal{D}_{1T} \times \mathcal{D}_{2T}$ is a complete metric space with the metric

$$d((w, g, h), (\tilde{w}, \tilde{g}, \tilde{h})) := \|w - \tilde{w}\|_{C(\Delta_T)} + \|g' - \tilde{g}'\|_{C([0, T])} + \|h' - \tilde{h}'\|_{C([0, T])}.$$  

Given $(w, g, h) \in \mathcal{D}_T$ with $T \in (0, S]$, we extend $(w, g, h)$ to $t > T$ by defining

$$(w(t, y), g(t), h(t)) = (w(T, y), g(T), h(T))$$

for $t > T$, $y \in [-h_0, h_0]$, and we extend the associated $\tilde{A}(t, y)$ and $\tilde{B}(t, y)$ similarly. For simplicity the extended functions are still denoted by themselves.

Fix $T \in (0, S]$ and for $(w, g, h) \in \mathcal{D}_T$ we consider the following initial boundary value problem

$$\begin{cases}

\partial_t \overline{w} - \tilde{A}(t, y) \overline{w}_y - \tilde{B}(t, y) \overline{w}_f = f(t, \Gamma(t, y), w) + A_i \epsilon \gamma, & t \in (0, S), y \in (-h_0, h_0), \\
w(t, -h_0) = w(t, h_0) = 0, & t \in (0, S),
\end{cases}$$

(2.10)

where the above extensions of $(w, g, h)$ and $\tilde{A}, \tilde{B}$ are assumed for $t > T$.

We note that the modulus of continuity of $\tilde{A}$, and the $L^\infty$ bound of $\tilde{B}$ are independent of the choice of $g$ and $h$ above, and $d/4 \leq \tilde{A} \leq 16d$ always holds. Hence, by standard $L^p$ theory, there exists $C_1 > 0$ depending only on $p, \Delta_S$, $\|v_0\|_{C^2([-h_0, h_0])}$, and $A_i \epsilon \gamma$, such that

$$\|\overline{w}\|_{W^{1,2}_p(\Delta_S)} \leq C_1.$$  

By the Sobolev imbedding theorem, for any $\sigma \in (0, 1)$, there exists $K_1 > 0$ and $p > 1$ depending on $\sigma, h_0$ and $S$ such that

$$\|\overline{w}\|_{C^{(1+\sigma)/2, 1+\sigma}(\Delta_S)} \leq K_1 \|\overline{w}\|_{W^{1,2}_p(\Delta_S)} \leq C_2 := K_1 C_1.$$
It follows that

\[ \|\overline{w}\|_{C^{(1+\sigma)/2,1+\sigma}(\Delta_T)} \leq C_2. \]

Define, for \( t \in [0, S] \),

\[
\begin{align*}
\overline{h}(t) &= h_0 - \int_0^t \mu \overline{w}_y(t, h_0)d\tau + B_i \epsilon t, \\
\overline{g}(t) &= -h_0 - \int_0^t \mu \overline{w}_y(t, -h_0)d\tau - B_i \epsilon t.
\end{align*}
\]

Then clearly

\[ \|\overline{g}\|_{C^{\sigma/2}([0,S])}, \|\overline{h}\|_{C^{\sigma/2}([0,S])} \leq C_3 := \mu C_2 + B_i \epsilon. \]

We now define \( \tilde{F} : \mathcal{D}_T \to C(\Delta_S) \times C^1([0, S]) \) and \( F : \mathcal{D}_T \to C(\Delta_T) \times C^1([0, T]) \) by

\[ \tilde{F}(w, g, h) = (\overline{w}, \overline{g}, \overline{h}), \quad F(w, g, h) = (\overline{w}, \overline{g}, \overline{h})|_{\{t \in [0, T]\}}. \]

Then the same reasoning as in [15] shows that \( F \) maps \( \mathcal{D}_T \) into itself if \( T \leq S_0 \in (0, S) \) for some \( S_0 = S_0(C_2, C_3, \sigma) \) small enough.

We show next that by shrinking \( T \) further if necessary, \( F : \mathcal{D}_T \to \mathcal{D}_T \) is a contraction mapping. Let \((w_j, g_j, h_j) \in \mathcal{D}_T \) for \( j = 1, 2 \) and denote \((\overline{w}_j, \overline{g}_j, \overline{h}_j) = \tilde{F}(w_j, g_j, h_j)\). We assume that \((g_j, h_j)\) are extended to \( t > T \) as before. We denote the associated \( \tilde{A}(t, y) \) and \( \tilde{B}(t, y) \) by \( \tilde{A}_j(t, y) \) and \( \tilde{B}_j(t, y) \) and assume that they are also extended to \( t > T \) as before.

Setting \( \tilde{U} = \overline{w}_1 - \overline{w}_2 \), we obtain

\[
\begin{align*}
U(t, \pm h_0) &= 0, \quad t \in [0, S], \quad y \in (-h_0, h_0), \\
U(0, y) &= 0, \quad y \in [-h_0, h_0].
\end{align*}
\]

Applying the \( L^p \) estimate we obtain, for some \( C_4 \) depending only on \( \Delta_S \) and \( p > 1 \),

\[ \|U\|_{W^{1,2}_p(\Delta_S)} \leq C_4(\|w_1 - w_2\|_{C(\Delta_T)} + \|g_1 - g_2\|_{C^1([0, T])} + \|h_1 - h_2\|_{C^1([0, T])}), \]

since the right hand side of the first equation for \( U \) above has its \( L^p(\Delta_S) \) norm controlled by \( \|w_1 - w_2\|_{C(\Delta_T)} + \|g_1 - g_2\|_{C^1([0, T])} + \|h_1 - h_2\|_{C^1([0, T])} \) due to our extension of the functions, and the required conditions on \( \tilde{A}_2, \tilde{B}_3 \) for the \( L^p \) estimate are not affected by the choice of \( g_2 \) and \( h_2 \), similar to the situation for \((2.10)\).

We may now use the Sobolev embedding theorem to deduce, as before

\[ \|U\|_{C^{(1+\sigma)/2,1+\sigma}(\Delta_S)} \leq C_5(\|w_1 - w_2\|_{C(\Delta_T)} + \|g_1 - g_2\|_{C^1([0, T])} + \|h_1 - h_2\|_{C^1([0, T])}), \]

with \( C_5 \) depending only on \( \Delta_S, \sigma \) and \( C_4 \). Therefore, for every \( T \in (0, S_0] \),

\[ \|\overline{w}_1 - \overline{w}_2\|_{C^{(1+\sigma)/2,1+\sigma}(\Delta_T)} \leq C_5(\|w_1 - w_2\|_{C(\Delta_T)} + \|g_1 - g_2\|_{C^1([0, T])} + \|h_1 - h_2\|_{C^1([0, T])}). \]

Using this estimate, we can follow the argument in [15] to deduce that there exists \( S_1 \in (0, S_0] \) depending on \( C_5, \sigma \) and \( \mu \) such that \( F \) is a contraction mapping on \( \mathcal{D}_T \) for any \( T \in (0, S_1] \). This guarantees a unique fixed point of \( F \), which is a positive solution of \((2.3)\) for \( t \in (0, S_1) \).

As in [15], for any given \( T_0 > 0 \), by repeating the above process finitely many times, the positive solution of \((2.3)\) can be extended to \( t \in [0, T_0) \), except that for the case \( i = 2 \), some further explanation is needed, since the extra term \( 2\epsilon^{\alpha/2} \) in the equations of \( h'(t) \) and \( g'(t) \) may cause \( h(t) - g(t) \) to decrease in \( t \), and the above process requires this quantity to be bounded from below by \( h_0 \). However, it is easy to show that this lower bound can be guaranteed over a finite time interval \([0, T_0]\) if \( \epsilon > 0 \) is small enough, say \( \epsilon \in (0, \epsilon_0) \).

We now consider the estimates in \((2.5)\). Since the solution over \( t \in [0, T_0] \) can be obtained by repeating the local existence proof finitely many times, it is enough to see how the estimates can be obtained over \( t \in [0, S_1] \) in the above arguments. With the regularity for \( w, g \) and \( h \)
obtained above through the use of $L^p$ theory and Sobolev embedding theorem, the estimates for $g$ and $h$ in (2.5) already hold. Moreover, in view of the assumptions (f$_3$) and (1.11), we see that all the conditions are satisfied to apply the Schauder estimate to (2.10) to obtain a $C^{1+\alpha/2,2+\alpha}$ bound for $w$, which yields a $C^{1+\alpha/2,2+\alpha}$ bound for $v$. This proves (2.5), and the proof of part (i) is complete.

It remains to prove (2.6) in part (ii). We first take $\sigma \in (0,1)$ in the above arguments so that $\sigma \geq (1 + \alpha)/2$. Then $w \in C^{(1+\alpha)/2,1+\alpha}$ and $g,h \in C^{1+\sigma}$ by using the $L^p$ theory and Sobolev embedding theorem in part (i). From these facts, (f$_4$) and (1.12), we see that $\tilde{A}, \tilde{B} \in C^{(1+\alpha)/2,1+\alpha}$, and the right hand side of the first equation in (2.10) belongs to $C^{(1+\alpha)/2,1+\alpha}$. Hence we can apply the Schauder estimate to (2.10) to obtain a $C^{2+\alpha/2}$ bound for $w$, which yields a $C^{2+\alpha/2}$ bound for $g$ and $h$ through (2.11), and then a $C^{3+\alpha/2,2+\alpha}$ bound for $v$. This proves (2.6).

Proof of Theorem 2.4: The validity of (2.8) follows from the continuous dependence of the solution of (2.4) with respect to the parameter $\epsilon \in [0, \epsilon_0]$, and (2.7) follows from the comparison principle.

Remark 2.5. The convergences in (2.8) actually hold under stronger norms. For example, combining (2.8) with (2.5), we immediately see that, for $i = 1, 2$,

$$\lim_{\epsilon \to 0} ||g_{ie} - g||_{C^1([0,T_0])} = 0, \quad \lim_{\epsilon \to 0} ||h_{ie} - h||_{C^1([0,T_0])} = 0.$$ 

Since $g'(t) < 0 < h'(t)$ for $t \geq 0$ (here the assumption $v_0'(\pm h_0) \neq 0$ is used), the above identities imply that, for $i = 1, 2$,

$$(2.12) \quad g_{ie}'(t) < 0 < h_{ie}'(t) \quad \text{for all} \quad t \in [0,T_0] \quad \text{and all small} \quad \epsilon > 0.$$ 

Strategy: We are now in a position to briefly describe the strategy of the proof of the main results. In the next section, we will modify $v_{ie}$ to obtain $v_{ie}^* = v_{ie} + O(\epsilon^n)$ for $i = 1, 2$ and $0 < \epsilon \ll 1$ such that $(v_{ie}^*, g_{ie}, h_{ie})$ is an upper solution of (1.8) and $(v_{ie}^*, g_{ie}, h_{ie})$ is a lower solution of (1.8). Hence the unique solution $(u_{ie}, g_{ie}, h_{ie})$ of (1.8) satisfies

$$v_{ie}^* \geq v_{ie} \geq v_{ie}^*, \quad g_{ie} \leq g_{ie} \leq g_{ie}, \quad h_{ie} \geq h_{ie} \geq h_{ie}.$$ 

From these inequalities and (2.8), we immediately obtain the desired conclusions in Theorem 1.1.

The proof of Theorem 1.2 follows the same strategy, but with $(v_{ie}, g_{ie}, h_{ie})$ replaced by an upper solution $(V_{ie}, G_{ie}, H_{ie})$ of (2.4) with $i = 1$, obtained by modifying the solution $(v_{ie}, g_{ie}, h_{ie})$ of (2.4) with $i = 2$, so that $|V_{ie} - v_{ie}| + |G_{ie} - g_{ie}| + |H_{ie} - h_{ie}|$ is bounded by $C\epsilon^n$ for some $C > 0$.

Notations: We conclude this section by observing that (f$_1$) implies, for any $k > 0$ there exists $L_0 = L_0(k) > 0$ such that

$$(2.13) \quad |f(t,x,u_1) - f(t,x,u_2)| \leq L_0|u_1 - u_2| \quad \text{for} \quad u_1, u_2 \in [0,k], (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$ 

And (f$_3$) implies, for any $k > 0$ there exists $L = L(k) > 0$ such that

$$(2.14) \quad |f(t_1,x_1,u_1) - f(t_1,x_1,u_2)| \leq L(|t_1 - t_2| + |x_1 - x_2| + |u_1 - u_2|)$$

for $u_1, u_2 \in [0,k], t_1, t_2 \in \mathbb{R}^+$ and $x_1, x_2 \in \mathbb{R}$.  

3. Proof of Theorem 1.1

Throughout this section we assume that (f$_1$), (f$_2$), (f$_3$), (J), (1.7) and (1.11) hold. Then, from Section 2, we know that for any $T_0 \in (0, +\infty)$, there exists $\epsilon_0 > 0$ small depending on $T_0$ such that for every $\epsilon \in (0, \epsilon_0]$ and $i = 1, 2$, problem (2.4) has a unique positive solution $(v_{ie}, g_{ie}, h_{ie})$.  

For $0 < \epsilon \ll 1$, let $(u_\epsilon, g_\epsilon, h_\epsilon)$ be the unique solution of (1.8), which we know from Section 2 is defined for all $t > 0$.

Define

$$m_\epsilon(t, x) = m_\epsilon(t, x; g_{2\epsilon}, h_{2\epsilon}) := \begin{cases} 1 - \left[ \frac{x}{g_{2\epsilon}(t)} \right]^2, & t \in [0, T_0], \ x \in [g_{2\epsilon}(t), 0], \\ 1 - \left[ \frac{x}{h_{2\epsilon}(t)} \right]^2, & t \in [0, T_0], \ x \in [0, h_{2\epsilon}(t)]. \end{cases}$$

(3.1)

Clearly $m_\epsilon$ is a $C^1$ function and $0 \leq m_\epsilon(t, x) \leq 1$. Moreover, $\partial_x m_\epsilon(t, x)$ is Lipschitz continuous in $x$.

As we will see below, Theorem 1.1 follows easily from Proposition 3.1. Before doing that, let us see how Theorem 1.1 follows easily from Proposition 3.1

Proposition 3.1. There exist positive constants $\tilde{M}_1$ and $\epsilon_1 \in (0, \epsilon_0]$ such that for any $\epsilon \in (0, \epsilon_1]$, we have

$$\left\{ \begin{array}{l} [g_{2\epsilon}(t), h_{2\epsilon}(t)] \subset [g_1(t), h_1(t)] \subset [g_1(t), h_1(t)], \quad t \in [0, T_0], \\ u_\epsilon(t, x) \geq v_{2\epsilon}(t, x) - \tilde{M}_1 \epsilon^{\gamma_1}, \quad t \in [0, T_0], \ x \in [g_{2\epsilon}(t), h_{2\epsilon}(t)], \\ u_\epsilon(t, x) \leq v_{1\epsilon}(t, x) + 3M_1 \epsilon, \quad t \in [0, T_0], \ x \in [g_\epsilon(t), h_\epsilon(t)], \end{array} \right.$$  

(3.2)

where $M_1$ is given in (2.5).

We will use a series of lemmas to prove Proposition 3.1. Before doing that, let us see how Theorem 1.1 follows easily from Proposition 3.1.

Proof of Theorem 1.1 (assuming Proposition 3.1): Combining (3.2) and (2.8), we immediately obtain the desired conclusion in Theorem 1.1 with $T = T_0$. \qed

We now set to prove Proposition 3.1.

Lemma 3.2. If $L_1 \geq L$ with $L = L(M_1 + 1)$ given by (2.14) and $M_1$ given by (2.5), and

$$\tilde{v}_{2\epsilon}(t, x) := v_{2\epsilon}(t, x) - \epsilon^{\gamma_1} Ke^{L_1t} m_\epsilon(t, x) \quad \text{with} \quad K := \frac{h_0}{2\mu e^{2L_1T_0}},$$

then there exists $\epsilon_1 \in (0, \epsilon_0]$ such that for all $\epsilon \in (0, \epsilon_1]$,

$$\tilde{v}_{2\epsilon}(t, x) > 0 \quad \text{for} \quad t \in (0, T_0], \ x \in (g_{2\epsilon}(t), h_{2\epsilon}(t)),$$

and $(\tilde{v}_2, g_2, h_2) = (\tilde{v}_{2\epsilon}, g_{2\epsilon}, h_{2\epsilon})$ satisfies

$$\begin{cases} \begin{aligned} \partial_t \tilde{v}_2 &\leq d\partial_{xx} \tilde{v}_2 + f(t, x, \tilde{v}_2) - \tilde{A}_2 \epsilon^{\gamma_1}, & t \in (0, T_0], \ x \in (g_2(t), h_2(t)) \setminus \{0\}, \\ \tilde{v}_2(t, g_2(t)) = \tilde{v}_2(t, h_2(t)) = 0, & t \in (0, T_0], \end{aligned} \\ g_2'(t) \geq -\mu \partial_x \tilde{v}_2(t, g_2(t)) + \epsilon^{\gamma_1}, \quad t \in (0, T_0], \\ h_2'(t) \leq -\mu \partial_x \tilde{v}_2(t, h_2(t)) - \epsilon^{\gamma_1}, \quad t \in (0, T_0], \\ \tilde{v}_2(0, x) \leq \tilde{v}_0(x), \quad x \in [-h_0, h_0], \end{cases}$$

(3.3)

where $\tilde{A}_2 := \frac{2dK}{M_1^2}$.

Proof. From the definition of $m_\epsilon$ and $\tilde{v}_2$, we have $\tilde{v}_2(t, g_2(t)) = \tilde{v}_2(t, h_2(t)) = 0$, and $\tilde{v}_2(0, x) \leq \tilde{v}_0(x)$. By our assumptions on $u_0$, the definition of $m_\epsilon(t, x)$, and the Hopf boundary lemma applied to $\partial_x v_{2\epsilon}(t, x)$ with $x \in \partial(g_{2\epsilon}(t), h_{2\epsilon}(t))$, we immediately see that for all small $\epsilon > 0$ (depending on $L_1$ and $T_0$),

$$\tilde{v}_{2\epsilon}(t, x) > 0 \quad \text{for} \quad t \in [0, T_0], \ x \in (g_{2\epsilon}(t), h_{2\epsilon}(t)).$$

For

$$\tilde{m} = \tilde{m}_\epsilon := \epsilon^{\gamma_1} Ke^{L_1t} m_\epsilon(t, x),$$
a simple computation gives,

\[
\begin{aligned}
\partial_t \tilde{m} &= L_1 \tilde{m} + \epsilon^\gamma_1 Ke^{L_1 t} 2x^2 h^{-3}_2 h'_2 \geq L_1 \tilde{m}, \\
\partial_{xx} \tilde{m} &= -2\epsilon^\gamma_1 Ke^{L_1 t} h^{-2}_2 \leq 0, \\
\partial_t \tilde{m} &= L_1 \tilde{m} + \epsilon^\gamma_2 Ke^{L_1 t} 2x^2 g^2_2 g'_2 = L_1 \tilde{m}, \\
\partial_{xx} \tilde{m} &= -2\epsilon^\gamma_2 Ke^{L_1 t} g^{-2}_2 \leq 0.
\end{aligned}
\]  

(3.4)

Then by (2.13) and (2.4) we have, provided that \(L_1 \geq L_0\),

\[
\begin{aligned}
\partial_t \tilde{v}_2 &= \partial \partial_{xx} \tilde{v}_2 + f(t, x, \tilde{v}_2) - \partial \partial_x \tilde{m} + \partial \partial_{xx} \tilde{m} + f(t, x, v_2) - f(t, x, \tilde{v}_2) \\
&\leq \partial \partial_{xx} \tilde{v}_2 + f(t, x, \tilde{v}_2) - L_1 \tilde{m} + \partial \partial_{xx} \tilde{m} + L_0 |\tilde{v}_2 - v_2| \\
&\leq \partial \partial_{xx} \tilde{v}_2 + f(t, x, \tilde{v}_2) - \tilde{d}d \partial_{xx} \tilde{m} \\
&\leq \partial \partial_{xx} \tilde{v}_2 + f(t, x, \tilde{v}_2) - 2dKM_1^2 \epsilon^\gamma_1 \text{ for } t \in (0, T_0), \ x \in (g_2(t), h_2(t)) \ \setminus \ \{0\},
\end{aligned}
\]

where we have used \(|g_2(t)|, h_2(t) \leq M_1 \) and (2.12).

Next we verify the third and fourth inequalities in (3.3). Applying the fourth equation of (2.4) and \(K = \frac{h_0}{2e\epsilon^{\gamma_1} M_1}\), we deduce

\[
h'^2_2(t) = -\mu \partial_x v_2(t, h_2) - 2\epsilon^\gamma_1 = -\mu[\partial_x \tilde{v}_2(t, h_2) + \partial_x \tilde{m}(t, h_2)] - 2\epsilon^\gamma_1
\]

\[
= -\mu \partial_x \tilde{v}_2(t, h_2) + 2\mu Ke^{L_1 t} h^{-1}_2 \epsilon^\gamma_1 - 2\epsilon^\gamma_1
\]

\[
\leq -\mu \partial_x \tilde{v}_2(t, h_2) + 2\mu Ke^{L_1 t} h^{-1}_0 \epsilon^\gamma_1 - 2\epsilon^\gamma_1
\]

\[
= -\mu \partial_x \tilde{v}_2(t, h_2) - \epsilon^\gamma_1 \text{ for } t \in (0, T_0],
\]

and analogously,

\[
g'_2(t) \geq -\mu \partial_x w(t, h_2) - \epsilon^\gamma_1 \quad \text{ for } t \in (0, T_0].
\]

The proof is finished. \(\square\)

Since \(v_1 = v_\iota \in C^{1+\alpha/2,2+\alpha}(\Omega_i)\) for \(i = 1, 2\) and \(\epsilon \in [0, \epsilon_0]\), there exist \(\tilde{v}_i = \tilde{v}_\iota \in C^{0,2+\alpha}(D)\) for \(i = 1, 2\) and \(D = [0, T_0] \times \mathbb{R}\) such that \(\tilde{v}_i = v_i\) in \(\Omega_i\). Moreover, in view of (2.5), we may further require, for \(i = 1, 2\) and \(\epsilon \in [0, \epsilon_0]\),

\[
\begin{aligned}
\sup_{0 \leq t \leq T_0} ||\tilde{v}_i(t, \cdot)||_{C^{2+\alpha}([g_2(t) - \epsilon, g_2(t) + \epsilon])} &< 2M_1.
\end{aligned}
\]

We now define, for \(\epsilon \in (0, \epsilon_0]\) and \((t, x) \in D = [0, T_0] \times \mathbb{R}\),

\[
\begin{aligned}
\tilde{v}_1^*(t, x) &= v_1^*(t, x) := \tilde{v}_1(t, x) + 3M_1 \epsilon, \\
\tilde{v}_2^*(t, x) &= v_2^*(t, x) := \tilde{v}_2(t, x) - \tilde{m}_\epsilon(t, x).
\end{aligned}
\]

Since \(v_2' = v_2(t, h_2(t)) \neq 0\) and, by the Hopf boundary Lemma and the assumptions on \(u_0, \partial_x v_2(t, h_2(t)) > 0\) and \(\partial_x v_2(t, h_2(t)) < 0\) for \(t \in [0, T_0]\), for all sufficiently small \(\epsilon > 0\), say \(\epsilon \in (0, \epsilon_2) \subset (0, \epsilon_1]\), we have \(\partial_x \tilde{v}_2(x, t) > 0\) for \(x \in [g_2(t) - \epsilon, g_2(t)]\) and \(\partial_x \tilde{v}_2(x, t) < 0\) for \(x \in [h_2(t), h_2(t) + \epsilon]\), which immediately leads to

\[
\begin{aligned}
\tilde{v}_2^*(t, x) < 0 \quad \text{ for } t \in [0, T_0], \ \epsilon \in (0, \epsilon_2], \ x \in (g_2(t) - \epsilon, g_1(t)) \cup (h_2(t), h_2(t) + \epsilon).
\end{aligned}
\]

On the other hand, due to (3.5), by shrinking \(\epsilon_2\) if necessary, we have

\[
\begin{aligned}
v^*_1(t, x) > 0 \quad \text{ for } t \in [0, T_0], \ \epsilon \in (0, \epsilon_2], \ x \in (g_1(t) - \epsilon, g_1(t)) \cup (h_1(t), h_1(t) + \epsilon).
\end{aligned}
\]

Set

\[
\begin{aligned}
\mathcal{L}_\epsilon[v_1^*](t, x) &:= \frac{C_\epsilon}{\epsilon^2} \left[ \int_{g_1(t)}^{h_1(t)} J_\epsilon(x, y) v_1^*(t, y) dy - v_1^*(t, x) \right], \\
\tilde{\mathcal{L}}_\epsilon[v_1^*](t, x) &:= \frac{C_\epsilon}{\epsilon^2} \left[ \int_{g_1(t)}^{h_1(t) + \epsilon} J_\epsilon(x, y) v_1^*(t, y) dy - v_1^*(t, x) \right].
\end{aligned}
\]
It follows from (3.6) and (3.7) that, for \( t \in [0, T_0] \) and \( \epsilon \in (0, \epsilon_2) \),

\[
L_\epsilon^1[v_1^*](t, x) < \widetilde{L}_\epsilon^1[v_1^*](t, x), \quad L_\epsilon^2[v_2^*](t, x) > \widetilde{L}_\epsilon^2[v_2^*](t, x).
\]

We show next that \((v_1^*, g_1, h_1)\) (resp. \((v_2^*, g_2, h_2)\)) is an upper (resp. a lower) solution of (1.8).

**Lemma 3.3.** For all small \( \epsilon > 0 \), we have

\[
\begin{aligned}
\partial_t v_1^* &\geq dL_\epsilon^1[v_1^*] + f(t, x, v_1^*) \quad \text{for } t \in (0, T_0), \ x \in (g_1(t), h_1(t)), \\
\partial_x v_2^* &\leq dL_\epsilon^2[v_2^*] + f(t, x, v_2^*) \quad \text{for } t \in (0, T_0), \ x \in (g_2(t), h_2(t)).
\end{aligned}
\]

**Proof.** By definition,

\[
\begin{aligned}
v_1^*(t, x) &= v_1(t, x) + 3M_1 \epsilon, \quad t \in [0, T_0], \ x \in [g_1(t), h_1(t)], \\
v_2^*(t, x) &= v_2(t, x) - \tilde{m}_\epsilon(t, x) = \tilde{v}_2(t, x), \quad t \in [0, T_0], \ x \in [g_2(t), h_2(t)],
\end{aligned}
\]

and it follows from (2.4) and (3.3) that

\[
\begin{aligned}
\partial_t v_1^* &= \partial_t v_1 = d\partial_{xx} v_1 + f(t, x, v_1) + \epsilon \gamma_1 = d\partial_{xx} v_1^* + f(t, x, v_1^*) + \epsilon \gamma_1 \\
&= dL_\epsilon^1[v_1^*] + f(t, x, v_1^*) + d\left( \partial_{xx} v_1^* - \widetilde{L}_\epsilon^1[v_1^*] \right) \\
&\quad + f(t, x, v_1) - f(t, x, v_1^*) + \epsilon \gamma_1 \quad \text{for } t \in (0, T_0), \ x \in (g_1(t), h_1(t))
\end{aligned}
\]

and

\[
\begin{aligned}
\partial_t v_2^* &= \partial_t \tilde{v}_2 \leq d\partial_{xx} \tilde{v}_2 + f(t, x, \tilde{v}_2) - \tilde{A}_2 \epsilon \gamma_1 = d\partial_{xx} v_2^* + f(t, x, v_2^*) - \tilde{A}_2 \epsilon \gamma_1 \\
&= dL_\epsilon^2[v_2^*] + f(t, x, v_2^*) \\
&\quad + d\left( \partial_{xx} v_2^* - \widetilde{L}_\epsilon^2[v_2^*] \right) - \tilde{A}_2 \epsilon \gamma_1 \quad \text{for } t \in (0, T_0), \ x \in (g_2(t), h_2(t)) \setminus [-\epsilon, \epsilon].
\end{aligned}
\]

Since \( \partial_{xx} \tilde{M}_\epsilon(t, x) \) and hence \( \partial_{xx} v_2^*(t, x) \) does not exist at \( x = 0 \), we estimate \( v_2^*(t, x) \) differently for \( x \in [-\epsilon, \epsilon] \). As the kernel function in the operator \( \widetilde{L}_\epsilon^2 \) is \( J_\epsilon \) whose support is contained in \([-\epsilon, \epsilon]\) with \( 0 < \epsilon \ll 1 \), we have, for \( x \in [-\epsilon, \epsilon] \),

\[
\widetilde{L}_\epsilon^2[v_2^*](t, x) = \widetilde{L}_\epsilon^2[\tilde{v}_2](t, x) = \widetilde{L}_\epsilon^2[v_2](t, x) - \tilde{L}_\epsilon^2[\tilde{m}_\epsilon](t, x),
\]

and so \( v_2^* \) satisfies, for \( (t, x) \in (0, T_0) \times [-\epsilon, \epsilon] \),

\[
\begin{aligned}
\partial_t v_2^* &= \partial_t \tilde{v}_2 - \partial_t \tilde{m}_\epsilon = d\partial_{xx} v_2 + f(t, x, v_2) - \partial_t \tilde{m}_\epsilon \\
&= dL_\epsilon^2[v_2^*] + f(t, x, v_2^*) + d\left( \partial_{xx} v_2^* - \widetilde{L}_\epsilon^2[v_2^*] \right) \\
&\quad + \left( \partial_{xx} \tilde{m}_\epsilon - \partial_t \tilde{m}_\epsilon \right) + f(t, x, v_2) - f(t, x, v_2^*).
\end{aligned}
\]

We now prove (3.9) in several steps.

**Step 1:** We show that, for \( t \in (0, T_0) \), \( x \in (g_1(t), h_1(t)) \) and all small \( \epsilon > 0 \),

\[
\left| \widetilde{L}_\epsilon^1[v_1^*] - \partial_{xx} v_1^* \right| \leq 3M_1 \epsilon \alpha.
\]

Recall \( \text{spt}(J_\epsilon) \subset [-\epsilon, \epsilon] \). By Taylor expansion, we obtain, for such \((t, x)\),

\[
\left| \widetilde{L}_\epsilon^1[v_1^*] - \partial_{xx} v_1^* \right| = \left| \frac{C_{\epsilon}}{\epsilon^2} \left[ \int_{g_1(t)-\epsilon}^{h_1(t)+\epsilon} \frac{1}{\epsilon} J_{\epsilon}(x-y) v_1^*(t, y) dy - v_1^*(t, x) \right] - \partial_{xx} v_1^* \right|
\]

\[
= \left| \frac{C_{\epsilon}}{\epsilon^2} \left[ \int_{x-\epsilon}^{x+\epsilon} \frac{1}{\epsilon} J_{\epsilon}(x-y) v_1^*(t, y) dy - v_1^*(t, x) \right] - \partial_{xx} v_1^* \right|
\]

\[
= \left| \frac{C_{\epsilon}}{\epsilon^2} \int_{-1}^{1} J(z) [v_1^*(t, x+\epsilon z) - v_1^*(t, x)] dz - \partial_{xx} v_1^* \right|
\]
calculations as in Steps 1 and 2, we obtain for (t, x) ∈ [−ε, ε] and all small ε > 0,
\[ \left| L_2^2[v_2] - \partial_{xx}v_2 \right| \leq 3M_1\epsilon^\alpha, \]

where \( M_1 \) is given by Lemma 3.2.

Similarly to Steps 1 and 2, for all small \( \epsilon > 0 \), we have
\[ \left| L_2^2[\tilde{m}_\epsilon] \right| \leq 2\epsilon L_1 h_0^{-2}. \]

Step 2: We show that, for \( t \in (0, T_0), x \in (g_2(t), h_2(t)) \setminus [-\epsilon, \epsilon] \) and all small \( \epsilon > 0 \),
\[ \left| \tilde{L}_2^1 [v_1^*] - \partial_{xx} v_1^* \right| \]

where \( \delta(t, y) \) lies between 0 and \( \epsilon z \). Due to the symmetry of \( J \) and the choice of \( C_* \), we have
\[ \frac{C_*}{\epsilon^2} \int_{-1}^1 J(z) \left[ \epsilon z \partial_x v_1^*(t, x) + \frac{1}{2} (\epsilon z)^2 \partial_{xx} v_1^*(t, x) \right] dz - \partial_{xx} v_1^* = 0. \]

Thus
\[ \left| \tilde{L}_2^1 [v_1^*] - \partial_{xx} v_1^* \right| \leq \frac{C_*}{\epsilon^2} \int_{-1}^1 J(z) \frac{1}{2} (\epsilon z)^2 \left| \partial_{xx} v_1^*(t, x) - \partial_{xx} v_1^*(t, x + \delta_1(t, z)) \right| dz \]

where \( \delta_1(t, y) \) lies between 0 and \( \epsilon z \). Due to the symmetry of \( J \) and the choice of \( C_* \), we have
\[ \frac{C_*}{\epsilon^2} \int_{-1}^1 J(z) \left[ \epsilon z \partial_x v_1^*(t, x) + \frac{1}{2} (\epsilon z)^2 \partial_{xx} v_1^*(t, x) \right] dz - \partial_{xx} v_1^* = 0. \]

where we have used the estimates in (3.5) with \( \tilde{v}_1 \) replaced by \( v_1^* \). This concludes Step 1.

Step 3: We show that, for \( t \in (0, T_0), x \in [-\epsilon, \epsilon] \) and all small \( \epsilon > 0 \),
\[ \left| L_2^2[v_2] - \partial_{xx} v_2 \right| \leq 3M_1\epsilon^\alpha, \]

where \( K = \frac{h_0}{2\mu_0 h_1} \) is given by Lemma 3.2.

Similarly to Steps 1 and 2, for all small \( \epsilon > 0 \), we have
\[ \left| L_2^2[\tilde{m}_\epsilon] \right| \leq 4\epsilon^{\gamma_1} Ke^{L_1 t} h_0^{-2}, \]

where \( L_1 \) is given by Lemma 3.2.

It remains to prove the second inequality. Using the mean value theorem, by similar calculations as in Steps 1 and 2, we obtain for \( (t, x) \in (0, T_0) \times [-\epsilon, \epsilon] \),
\[ \left| L_2^2[\tilde{m}_\epsilon] \right| = \frac{C_*}{\epsilon^2} \int_{g_2(t)-\epsilon}^{g_2(t)+\epsilon} \frac{1}{\epsilon} J \left( \frac{x - y}{\epsilon} \right) [\tilde{m}_\epsilon(t, y) - \tilde{m}_\epsilon(t, x)] dy \]

where \( \delta(t, y) \) lies between \( x \) and \( y \), and we have used
\[ \int_{x-\epsilon}^{x+\epsilon} \frac{1}{\epsilon} J \left( \frac{x - y}{\epsilon} \right) (y-x) dy = 0. \]

From the definition of \( \tilde{m}_\epsilon \) and (3.34), we see \( \partial_x \tilde{m}_\epsilon \) is Lipschitz continuous, and
\[ \left| \partial_x \tilde{m}_\epsilon(t, \delta(t, y)) - \partial_x \tilde{m}_\epsilon(t, x) \right| \leq 2\epsilon^{\gamma_1} Ke^{L_1 t} h_0^{-2} |\delta(t, y) - y| \leq 2\epsilon^{\gamma_1} Ke^{L_1 t} h_0^{-2} |x - y|. \]
Hence, we can apply the definition of $C_\epsilon$ to deduce
\[ |\tilde{L}_\epsilon^2[\tilde{m}_\epsilon]| \leq 2\epsilon^{\gamma_1} Ke^{L_1^2 t} h_0^{-2} C_\epsilon \int_0^{x+h_\epsilon} \frac{1}{\epsilon} J\left(\frac{x-y}{\epsilon}\right) (y-x)^2 dy \]
\[ = 2\epsilon^{\gamma_1} Ke^{L_1^2 t} h_0^{-2} C_\epsilon \int_{-1}^1 J(z) z^2 dz = 4\epsilon^{\gamma_1} Ke^{L_1^2 t} h_0^{-2} \quad \text{for } (t, x) \in (0, T_0] \times [-\epsilon, \epsilon]. \]

This completes Step 3.

**Step 4:** We show that for $t \in (0, T_0]$, $x \in [-\epsilon, \epsilon]$ and all small $\epsilon > 0$,
\[ \partial_t v_2^* \leq d\tilde{L}_\epsilon^2[v_2^*] + f(t, x, v_2^*). \]

By the identity just before Step 1, it suffices to show, for $t \in (0, T_0]$, $x \in [-\epsilon, \epsilon]$ and all small $\epsilon > 0$,
\[ d\left(\partial_{xx} v_2 - \tilde{L}_\epsilon^2[v_2]\right) + \left(d\tilde{L}_\epsilon^2[\hat{m}_\epsilon] - \partial_t \hat{m}_\epsilon\right) + f(t, x, v_2) - f(t, x, v_2^*) \leq 0. \]

By (2.13), (3.4) and the estimates in Step 3, we have, for $t \in (0, T_0]$, $x \in [-\epsilon, \epsilon]$ and all small $\epsilon > 0$,
\[ d\left(\partial_{xx} v_2 - \tilde{L}_\epsilon^2[v_2]\right) + \left(d\tilde{L}_\epsilon^2[\hat{m}_\epsilon] - \partial_t \hat{m}_\epsilon\right) + f(t, x, v_2) - f(t, x, v_2^*) \leq 3dM_1 \epsilon^\alpha + 4dKe^{L_1^2 t} h_0^{-2} \epsilon^{\gamma_1} - L_1 \hat{m}_\epsilon + L_0 v_2^* - v_2 \]
\[ = -(L_1 - L_0) \hat{m}_\epsilon + 3dM_1 \epsilon^\alpha + 4dKe^{L_1^2 t} h_0^{-2} \epsilon^{\gamma_1} \]
\[ \leq -\left((L_1 - L_0) Ke^{L_1^2 t} h_0^{-2}\right) \epsilon^{\gamma_1} + 3dM_1 \epsilon^\alpha \]
\[ \leq -\left[\frac{1}{2} (L_1 - L_0) - 4d h_0^{-2}\right] Ke^{L_1^2 t} \epsilon^{\gamma_1} + 3dM_1 \epsilon^\alpha \]
\[ \leq -\left[\frac{1}{2} (L_1 - L_0) - 4d h_0^{-2}\right] Ke^{L_1^2 t} \epsilon^{\gamma_1} + 3dM_1 \epsilon^\alpha < 0, \]

provided that we first choose $L_1 > L_0$ such that $\frac{1}{2} (L_1 - L_0) - 4d h_0^{-2} > 0$ and then choose $\epsilon > 0$ sufficiently small. Step 4 is now completed.

**Step 5:** We show that for $t \in (0, T_0]$, $x \in (g_2(t), h_2(t)) \setminus [-\epsilon, \epsilon]$ and all small $\epsilon > 0$,
\[ \partial_t v_2^* \leq d\tilde{L}_\epsilon^2[v_2^*] + f(t, x, v_2^*). \]

Using our earlier calculation and the estimate in Step 2, we obtain, for $t \in (0, T_0]$, $x \in (g_2(t), h_2(t)) \setminus [-\epsilon, \epsilon]$ and all small $\epsilon > 0$,
\[ \partial_t v_2^* - d\tilde{L}_\epsilon^2[v_2^*] - f(t, x, v_2^*) \]
\[ \leq d\left(\partial_{xx} v_2^* - \tilde{L}_\epsilon^2[v_2^*]\right) - \tilde{A}_2 \epsilon^{\gamma_1} \]
\[ \leq 3dM_1 \epsilon^\alpha - \tilde{A}_2 \epsilon^{\gamma_1} < 0. \]

This completes Step 5.

**Step 6:** We show that for $t \in (0, T_0]$, $x \in (g_1(t), h_1(t))$ and all small $\epsilon > 0$,
\[ \partial_t v_1^* \geq d\tilde{L}_\epsilon^1[v_1^*] + f(t, x, v_1^*). \]

From (2.13), we have
\[ |f(t, x, v_1) - f(t, x, v_1^*)| \leq L_0 |v_1 - v_1^*| = 3L_0 M_1 \epsilon. \]
Combining this with our earlier calculations and the estimate in Step 1, we obtain, for 
\( t \in (0, T_0], \ x \in (g_1(t), h_1(t)) \) and all small \( \epsilon > 0 \),
\[
\partial_t v_1^* - d\mathcal{L}_0^1[v_1^*] - f(t, x, v_1^*) \\
= d \left( \partial_{xx} v_1^* - \mathcal{L}_0^1[v_1^*] \right) + f(t, x, v_1) - f(t, x, v_1^*) + \epsilon \gamma \leq -3dM_1\epsilon + \epsilon \gamma > 0.
\]

This completes Step 6.
Clearly (3.9) follows directly from (3.8) and the inequalities proved in Steps 4, 5 and 6. \( \square \)

**Lemma 3.4.** For all small \( \epsilon > 0 \), we have

\[
\begin{aligned}
& h_1'(t) \geq \frac{\mu C_0}{\epsilon^{3/2}} \int_{g_1(t)+\sqrt{\epsilon}}^{h_1(t)-\sqrt{\epsilon}} \int_{h_1(t)-\sqrt{\epsilon}}^{\infty} J_\epsilon(x-y) v_1^*(t,x) dy dx, \\
& h_2'(t) \leq \frac{\mu C_0}{\epsilon^{3/2}} \int_{g_2(t)+\sqrt{\epsilon}}^{h_2(t)-\sqrt{\epsilon}} \int_{h_2(t)-\sqrt{\epsilon}}^{\infty} J_\epsilon(x-y) v_2^*(t,x) dy dx.
\end{aligned}
\]

**Proof.** Let us first recall \( \text{spt}(J) \subset [-1,1], \ J_\epsilon(\xi) = \frac{1}{\epsilon} J(\xi) \) and \( \text{spt}(J_\epsilon) \subset [-\epsilon, \epsilon] \). By (1.10) we have
\[
\begin{aligned}
\frac{\mu C_0}{\epsilon^{3/2}} \int_{g_1(t)+\sqrt{\epsilon}}^{h_1(t)-\sqrt{\epsilon}} \int_{h_1(t)-\sqrt{\epsilon}}^{\infty} J_\epsilon(x-y) v_1^*(t,x) dy dx &= \frac{\mu C_0}{\epsilon^{3/2}} \int_{-\epsilon}^{\epsilon} \int_{0}^{w} J_\epsilon(x-y) v_1^*(t,h_1(t) - \sqrt{\epsilon} + x) dy dx \\
&= \frac{\mu C_0}{\epsilon} \int_{-\epsilon}^{\epsilon} \int_{0}^{w} J(z) v_1^*(t,h_1(t) - \epsilon w - \sqrt{\epsilon}) dz dw \\
&= \frac{\mu C_0}{\epsilon} \int_{-\epsilon}^{\epsilon} \int_{-1}^{1} J(z) v_1(t,h_1(t) - \epsilon w - \sqrt{\epsilon}) dz dw + 3\mu_1 C_0 M_1 \sqrt{\epsilon} \int_{-1}^{1} \int_{-1}^{w} J(z) dz dw \\
&= \frac{\mu C_0}{\epsilon} \int_{-\epsilon}^{\epsilon} \int_{-1}^{1} J(z) \left[ (\epsilon w - \sqrt{\epsilon}) \partial_x v_1(t,h_1(t)) + \frac{(\sqrt{\epsilon} w + 1)^2}{2} \partial_{xx} v_1(t,h_1 + \delta_1(t,w)) \right] dz dw \\
&\quad + 3\mu_1 C_0 M_1 \sqrt{\epsilon} \int_{-1}^{1} \int_{-1}^{w} J(z) dz dw \\
&= -\mu \partial_x v_1(t,h_1(t)) + \mu C_0 \sqrt{\epsilon} \int_{-1}^{1} \int_{-1}^{w} J(z)[w] \partial_x v_1(t,h_1(t)) dz dw \\
&\quad + \mu C_0 \sqrt{\epsilon} \int_{-1}^{1} \int_{-1}^{w} J(z) \left( \frac{\sqrt{\epsilon} w + 1}{2} \right) \partial_{xx} v_1(t,h_1 + \delta_1(t,w)) dz dw \\
&\quad + 3\mu_1 C_0 M_1 \sqrt{\epsilon} \int_{-1}^{1} \int_{-1}^{w} J(z) dz dw,
\end{aligned}
\]

where \( \epsilon w - \sqrt{\epsilon} \leq \delta_1(t,w) \leq 0 \) for \( -1 \leq w \leq 0 \).

Thus, making use of (2.35) and \( \int_{-1}^{1} J(z) dz = \frac{1}{2} \), we get
\[
\begin{aligned}
& \frac{\mu C_0}{\epsilon^{3/2}} \int_{g_1(t)+\sqrt{\epsilon}}^{h_1(t)-\sqrt{\epsilon}} \int_{h_1(t)-\sqrt{\epsilon}}^{\infty} J_\epsilon(x-y) v_1^*(t,x) dy dx \\
&\leq -\mu \partial_x v_1(t,h_1(t)) + \mu C_0 M_1 \sqrt{\epsilon} \int_{-1}^{1} \int_{-1}^{w} J(z)[w] + \frac{1}{2} (\sqrt{\epsilon} w + 1)^2 dz dw \\
&\quad + 3\mu_1 C_0 M_1 \sqrt{\epsilon} \int_{-1}^{1} \int_{-1}^{w} J(z) dz dw
\end{aligned}
\]
\[
\leq -\mu \partial_x v_1(t, h_1(t)) + \mu C_0 M_1 \sqrt{\epsilon} \int_{-1}^{0} \int_{-1}^{w} J(z)(1 + 2) \, dz \, dw
\]
\[
+ 3\mu C_0 M_1 \sqrt{\epsilon} \int_{-1}^{0} \int_{-1}^{w} J(z) \, dz \, dw
\]
\[
\leq -\mu \partial_x v_1(t, h_1(t)) + 6\mu C_0 M_1 \sqrt{\epsilon} \int_{-1}^{0} \int_{-1}^{0} J(z) \, dz \, dw
\]
\[
= -\mu \partial_x v_1(t, h_1(t)) + 3\mu C_0 M_1 \sqrt{\epsilon}
\]
\[
\leq -\mu \partial_x v_1(t, h_1(t)) + \epsilon \gamma = h'_1(t) \quad \text{for all small } \epsilon > 0.
\]

By similar calculations and the fourth inequality of (3.3), we deduce
\[
\mu \frac{C_0}{\epsilon^{3/2}} \int_{g_2(t)+\sqrt{\epsilon}}^{h_2(t)-\sqrt{\epsilon}} \int_{g_2(t)-\sqrt{\epsilon}}^{g_1(t)+\sqrt{\epsilon}} J_\epsilon(x-y)v_2^*(t, x) \, dy \, dx
\]
\[
\geq -\mu \partial_x \hat{v}_2(t, h_2(t)) - 3\mu C_0 M_1 \sqrt{\epsilon} \int_{-1}^{0} \int_{-1}^{0} J(z) \, dz \, dw
\]
\[
\geq -\mu \partial_x \hat{v}_2(t, h_2(t)) - 3\mu C_0 M_1 \sqrt{\epsilon}
\]
\[
\geq -\mu \partial_x \hat{v}_2(t, h_2(t)) - \epsilon \gamma \geq h'_2(t) \quad \text{for all small } \epsilon > 0.
\]

Therefore, (3.11) holds. \qed

Analogously we can prove

Lemma 3.5. For all small \(\epsilon > 0\),

\[
g_1'(t) \leq -\mu \frac{C_0}{\epsilon^{3/2}} \int_{g_2(t)+\sqrt{\epsilon}}^{h_1(t)-\sqrt{\epsilon}} \int_{g_2(t)-\sqrt{\epsilon}}^{g_1(t)+\sqrt{\epsilon}} J_\epsilon(x-y)v_1^*(t, x) \, dy \, dx,
\]

\[
g_2'(t) \geq -\mu \frac{C_0}{\epsilon^{3/2}} \int_{h_2(t)-\sqrt{\epsilon}}^{g_2(t)+\sqrt{\epsilon}} \int_{h_2(t)+\sqrt{\epsilon}}^{h_1(t)+\sqrt{\epsilon}} J_\epsilon(x-y)v_2^*(t, x) \, dy \, dx.
\]

We are now ready to complete the proof of Proposition 3.1. We note that

\[
v_1^*(t, x) = 3M_1 \epsilon > 0, \quad t \in (0, T_0], \quad x \in \{g_1(t), h_1(t)\},
\]

\[
v_2^*(t, x) = v_2(x) - \tilde{m}_\epsilon(t, x) = 0, \quad t \in (0, T_0], \quad x \in \{g_2(t), h_2(t)\},
\]

and

\[
v_1^*(0, x) = v_1(0, x) + 3M_1 \epsilon = v_0(x) + 3M_1 \epsilon \geq v_0(x) = u_\epsilon(0, x), \quad x \in [-h_0, h_0],
\]

\[
v_2^*(0, x) = v_2(0, x) - m_1(0, x) \leq v_0(x) = u_\epsilon(0, x), \quad x \in [-h_0, h_0].
\]

Hence, in view of the inequalities proved in the previous three lemmas, we can conclude that \((v_1^*, g_1, h_1)\) (resp. \((v_2^*, g_2, h_2)\)) is an upper (resp. a lower) solution of (1.8). Now the comparison principle in Theorem 2.2 (i), combined with (3.10), yields the desired conclusions in (3.2) with

\[
\tilde{M}_1 := Ke^{L_1T_0} = \frac{h_0}{2\mu e^{L_1T_0}}.
\]

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. A crucial step in the proof is to construct an upper solution \((V_{1\epsilon}, G_{1\epsilon}, H_{1\epsilon})\) of (2.4) with \(i = 1\) by modifying the solution \((v_{2\epsilon}, g_{2\epsilon}, h_{2\epsilon})\) of (2.4) with \(i = 2\), so that \(|V_{1\epsilon} - v_{2\epsilon}| + |G_{1\epsilon} - g_{2\epsilon}| + |H_{1\epsilon} - h_{2\epsilon}|\) is bounded by \(C\epsilon \gamma\) for some \(C > 0\).

To construct \((V_{1\epsilon}, G_{1\epsilon}, H_{1\epsilon})\), for positive constants \(\xi_1\) and \(\xi_2\), we define

\[
\phi(t) = \phi_\epsilon(t) := (1 + \xi_2 \epsilon^{3/4} t^\gamma) t,
\]
where $\gamma_1$ is given by (2.3). Then for sufficiently large $\xi_1$, $\xi_2$, $\xi_3$ and all small $\epsilon > 0$, define

$$
G_{1\epsilon}(t) := g_{2\epsilon}(\phi(t)), \quad H_{1\epsilon}(t) := h_{2\epsilon}(\phi(t)) \quad \text{for } 0 < t < \tilde{T},
$$

$$
V_{1\epsilon}(t, x) = v_{2\epsilon}(\phi(t), x) + \epsilon^{\gamma_1}M_\epsilon(t, x) \quad \text{for } t \in [0, \tilde{T}], \; x \in [G_{1\epsilon}(t), H_{1\epsilon}(t)],
$$

where $(v_{2\epsilon}, g_{2\epsilon}, h_{2\epsilon})$ is the solution of (2.4) with $i = 2$ and $T_0 = 2T > 0$, $\tilde{T} > 0$ is uniquely determined by $\phi(\tilde{T}) = 2T$, and

$$
M_\epsilon(t, x) := (\xi_1 t + \xi_3)\epsilon^{\gamma_1} m_\epsilon(t, x; G_{1\epsilon}, H_{1\epsilon}) = (\xi_1 t + \xi_3)\epsilon^{\gamma_1} m_\epsilon(\phi(t), x; g_{2\epsilon}, h_{2\epsilon}),
$$

with $m_\epsilon(t, x; G_{1\epsilon}, H_{1\epsilon})$ given by (3.1) with obvious modifications. Clearly, $M_\epsilon \in C^{1+\alpha/2,1}(\bar{\Omega}_\epsilon)$, where $\Omega_\epsilon = \{(t, x) : t \in (0, \tilde{T}], \; x \in (G_{1\epsilon}, H_{1\epsilon})\}$.

**Proposition 4.1.** Suppose $J$, (f1), (f2) and (f4) hold, and $u_0$ satisfies (1.12). Then there exist $\xi_1$, $\xi_2$, $\xi_3$ large and $\epsilon^* > 0$ small such that when $\epsilon \in (0, \epsilon^*)$, the above defined triple $(V_{1\epsilon}, G_{1\epsilon}, H_{1\epsilon})$ is a weak upper solution of (2.4) with $i = 1$ and $T_0 = \tilde{T}$.

Before giving the proof of Proposition 4.1, let us see how it is used to prove Theorem 1.2.

**Proof of Theorem 1.2** (assuming Proposition 4.1): It follows from Proposition 3.1 and Proposition 4.1 that

$$
\begin{cases}
[g_{2\epsilon}(t), h_{2\epsilon}(t)] \subset [g_\epsilon(t), h_\epsilon(t)] \subset [g_{1\epsilon}(t), h_{1\epsilon}(t)], & \text{for } t \in [0, 2T], \\
u_\epsilon(t, x) \leq v_2(t, x) - \bar{M}_1 \epsilon^{\gamma_1}, & \text{for } t \in [0, 2T], \; x \in [g_{2\epsilon}(t), h_{2\epsilon}(t)], \\
u_\epsilon(t, x) \leq v_{1\epsilon}(t, x) + 3M_1 \epsilon, & \text{for } t \in [0, 2T], \; x \in [g_\epsilon(t), h_\epsilon(t)],
\end{cases}
$$

and

$$
\begin{cases}
[g_{1\epsilon}(t), h_{1\epsilon}(t)] \subset [G_{1\epsilon}(t), H_{1\epsilon}(t)], & \text{for } t \in [0, \tilde{T}], \\
v_{1\epsilon}(t, x) \leq V_{1\epsilon}(t, x), & \text{for } t \in [0, \tilde{T}], \; x \in [g_{1\epsilon}(t), h_{1\epsilon}(t)].
\end{cases}
$$

**Claim 1.** For sufficiently small $\epsilon > 0$, we have $T < \tilde{T} < 2T$ and

$$
|h_\epsilon(t) - h(t)|, \; |g_\epsilon(t) - g(t)| \leq K\epsilon^{\gamma_1} \approx \epsilon^3, \quad t \in [0, T],
$$

where $K = M_2 G_{2\epsilon} T \epsilon^{\gamma_1}$, and $(v, g, h)$ is the solution of (1.1).

Recall that $\tilde{T}$ satisfies $\tilde{T} + \xi_2 \tilde{T} \epsilon^{\gamma_1} T \epsilon^{\gamma_1} = 2T$. Then clearly $\tilde{T} < 2T$, and for small $\epsilon > 0,

$$
\tilde{T} - T = 2T - \xi_2 \tilde{T} \epsilon^{\gamma_1} T \epsilon^{\gamma_1} = T - \xi_2 2T \epsilon^{\gamma_1} 2T \epsilon^{\gamma_1} > 0.
$$

Taking advantages of (2.6), (2.7), (4.1) and (4.2), we deduce for $t \in [0, T]$,

$$
|h_\epsilon(t) - h(t)| \leq h_1(t) - h_2(t) = h_{1\epsilon}(t) - H_{1\epsilon}(t) + h_{2\epsilon}(\phi(t)) - h_{2\epsilon}(t)
\leq h_{2\epsilon}(\phi(t)) - h_{2\epsilon}(t) \leq ||h_{2\epsilon}'||_{C([0,2T])} ||\phi(t) - t||
\leq ||h_{2\epsilon}'||_{C([0,2T])} \xi_2 T \epsilon^{\gamma_1} T \epsilon^{\gamma_1} \leq M_2 \xi_2 T \epsilon^{\gamma_1} T \epsilon^{\gamma_1} = K\epsilon^{\gamma_1} \approx \epsilon^3 \quad \text{if } 0 < \epsilon \ll 1,
$$

and similarly $|g_\epsilon(t) - g(t)| \leq K\epsilon^{\gamma_1}$ for such $\epsilon$. Hence (4.3) holds.

Moreover, the above calculations also imply that for $t \in [0, T]$ and $i = 1, 2$,

$$
|h_{\epsilon i}(t) - h(t)|, |g_{\epsilon i}(t) - g(t)| \leq K\epsilon^{\gamma_1} \quad \text{if } 0 < \epsilon \ll 1,
$$

and so, for such $t$, $\epsilon$ and $i = 1, 2$,

$$
[g(t) + K\epsilon^{\gamma_1}, h(t) - K\epsilon^{\gamma_1}] \subset [g_\epsilon(t), h_\epsilon(t)] \cap [g_{\epsilon i}(t), h_{\epsilon i}(t)].
$$

**Claim 2.** For sufficiently small $\epsilon > 0$, the following estimate holds:

$$
|u_\epsilon(t, x) - v(t, x)| < \epsilon^3 \quad \text{for } t \in [0, T], \; x \in [g(t) + K\epsilon^{\gamma_1}, h(t) - K\epsilon^{\gamma_1}].
$$

---

3This is a classical upper solution except on the line $\{(t, 0) : t > 0\}$ where $\partial_t V_\epsilon$ exists and is continuous. Therefore the usual comparison principle holds as for the case of classical upper solutions.
By (2.7) and (4.1), for all small $\epsilon > 0$ and $t \in [0, T]$, $x \in [g_{2\epsilon}(t), h_{2\epsilon}(t)]$,

$$v(t, x) \in [v_{2\epsilon}(t, x), v_{1\epsilon}(t, x)], \quad u_\epsilon(t, x) \in [v_{2\epsilon}(t, x) - M_1\epsilon^{\gamma_1}, v_{1\epsilon}(t, x) + 3M_1\epsilon].$$

Hence, by (4.4), for all small $\epsilon > 0$ and $t \in [0, T]$, $x \in [g(t) + K\epsilon^{\gamma_1}, h(t) - K\epsilon^{\gamma_1}]$, we have

$$v(t, x), \quad u_\epsilon(t, x) \in [v_{2\epsilon}(t, x) - M_1\epsilon^{\gamma_1}, v_{1\epsilon}(t, x) + 3M_1\epsilon].$$

We may now make use of (2.6), (4.2) and (4.4) to conclude that, for all small $\epsilon > 0$,

$$|u_\epsilon(t, x) - v(t, x)| \leq v_{1\epsilon}(t, x) - v_{2\epsilon}(t, x) + M_1\epsilon^{\gamma_1} + 3M_1\epsilon$$(4.5)

$$\leq v_{1\epsilon}(t, x) - v_{2\epsilon}(t, x) + M_1\epsilon^{\gamma_1} + 3M_1\epsilon$$

$$= v_{2\epsilon}(\phi(t), x) - v_{2\epsilon}(t, x) + M_\epsilon(t, x)^{\gamma_1} + M_1\epsilon^{\gamma_1} + 3M_1\epsilon$$

$$\leq \frac{1}{2} K\epsilon^{\gamma_1} + ||\partial_t v_{2\epsilon}||_{L^\infty} |\phi(t) - t|$$

$$\leq \frac{1}{2} K\epsilon^{\gamma_1} + M_2\epsilon T^{\gamma_1} \epsilon^{\gamma_1}$$

$$\leq \epsilon^{\gamma_1} \quad \text{for} \quad t \in [0, T], \quad x \in [g(t) + K\epsilon^{\gamma_1}, h(t) - K\epsilon^{\gamma_1}]$$

Claim 3. For sufficiently small $\epsilon > 0$, we have

$$|u_\epsilon(t, x) - v(t, x)| < \epsilon^{\gamma_1} \quad \text{for} \quad t \in [0, T], \quad x \in \mathbb{R},$$

where $v(t, x) = 0$ for $x \in \mathbb{R} \setminus (g(t), h(t))$ and $u_\epsilon(t, x) = 0$ for $x \in \mathbb{R} \setminus (g_{1\epsilon}(t), h_{1\epsilon}(t))$.

From (2.7) and (4.1), we see that $[g(t), h(t)] \cup [g_{1\epsilon}(t), h_{1\epsilon}(t)] \subset [g_{2\epsilon}(t), h_{1\epsilon}(t)]$. Hence in view of Claim 2, we just need to consider the estimate of $|u_\epsilon(t, x) - v(t, x)|$ for $x \in [g_{2\epsilon}(t), h_{1\epsilon}(t)] \setminus [g(t) + K\epsilon^{\gamma_1}, h(t) - K\epsilon^{\gamma_1}]$.

Clearly (4.5) holds also for $x \in [g_{2\epsilon}(t), h_{1\epsilon}(t)]$ since by our convention $v_{2\epsilon}(t, x) = 0$ for $x \in \mathbb{R} \setminus (g_{2\epsilon}(t), h_{2\epsilon}(t))$. Hence for $x \in [g_{2\epsilon}(t), h_{1\epsilon}(t)] \setminus [g(t) + K\epsilon^{\gamma_1}, h(t) - K\epsilon^{\gamma_1}]$, $t \in [0, T]$ and $0 < \epsilon \ll 1$, we have

$$|u_\epsilon(t, x) - v(t, x)| \leq 2 \epsilon^{\gamma_1}$$

Moreover, taking advantages of (2.6) and $h(t) - K\epsilon^{\gamma_1} \leq h_{2\epsilon} \leq h_{1\epsilon}$, we deduce for such $t$, $\epsilon$ and $x \in [h(t) - K\epsilon^{\gamma_1}, h_{1\epsilon}(t)]$,

$$|u_\epsilon(t, x) - v(t, x)| \leq |v_{1\epsilon}(t, x) - v_{1\epsilon}(t, h_{1\epsilon}(t))| - |v_{2\epsilon}(t, x) - v_{2\epsilon}(t, h_{2\epsilon}(t))| + \epsilon^{\gamma_1}$$

$$\leq ||\partial_x v_{1\epsilon}(t, \cdot)||_{L^\infty} |x - h_{1\epsilon}(t)| + ||\partial_x v_{2\epsilon}(t, \cdot)||_{L^\infty} |x - h_{2\epsilon}(t)| + \epsilon^{\gamma_1}$$

$$\leq 4 M_2 K\epsilon^{\gamma_1} + \epsilon^{\gamma_1}$$

$$\leq \epsilon^{\gamma_1}.$$
(We note that $|\partial_x v_2(t, x)| \leq M_2$ is the only place in the proof of Theorem 1.2 where (2.5) is not enough, and (2.6) has to be used.)

Due to $v_2(t, g_2(t)) = v_2(t, h_2(t)) = 0$ and $v_2(s, g_2(t)), v_2(s, h_2(t)) > 0$ for $s > t$, we also have

$$
\partial_t v_2(t, x) \geq 0 \text{ for } t \in [0, T_0], \ x = g_2(t) \text{ or } h_2(t).
$$

Define

$$
P(t, x, k) := km(t, x) + \partial_t v_2(t, x).
$$

Then for any $k \geq (M_2)^2$,

$$
P(t, g_2(t), k), P(t, h_2(t), k) \geq 0 \quad \text{for } t \in [0, T_0],
$$

$$
\partial_x P(t, x, k) \geq \frac{k}{M_2} - M_2 \geq 0 \quad \text{for } t \in [0, T_0], \ x \in [g_2(t), \frac{1}{2} g_2(t)],
$$

$$
\partial_x P(t, x, k) \leq \frac{-k}{M_2} + M_2 \leq 0 \quad \text{for } t \in [0, T_0], \ x \in [\frac{1}{2} h_2(t), h_2(t)],
$$

which imply

$$
P(t, x, k) \geq 0 \text{ for } t \in [0, T_0], \ x \in \left[ g_2(t), \frac{1}{2} g_2(t) \right] \cup \left[ \frac{1}{2} h_2(t), h_2(t) \right].
$$

On the other hand, for $(t, x) \in [0, T_0] \times \left[ \frac{1}{2} g_2(t), \frac{1}{2} h_2(t) \right]$ we have $m(t, x) \geq \frac{4}{3}$ and hence

$$
P(t, x, k) \geq \frac{3}{4} k - M_2 \geq 0 \text{ for } t \in [0, T_0], \ x \in \left[ \frac{1}{2} g_2(t), \frac{1}{2} h_2(t) \right],
$$

provided that $k \geq 4M_2/3$. Therefore, (4.6) holds for $k_1 \geq \max\{M_2^2, 4M_2/3\}, \ \epsilon \in (0, \epsilon_0]$.

(ii) By Remark 2.5 we have

$$
\lim_{\epsilon \to 0} \|g_2 - g\|_{C^1([0, T_0])} = \lim_{\epsilon \to 0} \|h_2 - h\|_{C^1([0, T_0])} = 0.
$$

By the assumption $|u_0'(\pm h_0)| > 0$ and $g'(t) = -\mu v_x(t, g(t)), h'(t) = -\mu v_x(t, h(t))$, we obtain $|g'(t), h'(t) > 0$ for all $t \in [0, T_0]$. Hence, $C := \min_{\epsilon \in [0, T_0]} \{-g'(t), h'(t)\} > 0$ and there exists $\epsilon_0 \in (0, \epsilon_0]$ such that

$$
g_2'(t) < \frac{g'(t)}{2} < 0, \quad 0 < \frac{h'(t)}{2} < h_2'(t) \quad \text{for all } t \in [0, T_0], \ \epsilon \in [0, \epsilon_0].
$$

Therefore, if $k_2 := C/(2\mu)$, then for $t \in [0, T_0]$ and $\epsilon \in [0, \epsilon_0]$,

$$
\partial_x v_2(t, g_2(t)) = \frac{-g_2(t)}{\mu} \geq \frac{-g'(t)}{2\mu} \geq k_2,
$$

$$
- \partial_x v_2(t, h_2(t)) = \frac{h_2'(t)}{\mu} \geq \frac{h'(t)}{2\mu} \geq k_2.
$$

This proves (4.7).
Proof of Proposition 4.1. By (2.1), \( \tilde{v}(t, x) = \tilde{v}_\varepsilon(t, x) := v_{2\varepsilon}(\phi(t), x) \) satisfies
\[
\begin{cases}
\tilde{v}_t = d\tilde{v}_{xx} + \frac{\phi'(t)-1}{\phi(t)} \tilde{v}_t + f(\phi(t), x, \tilde{v}), & t \in (0, \bar{T}], x \in (G_{1\varepsilon}(t), H_{1\varepsilon}(t)), \\
\tilde{v}(t, G_{1\varepsilon}(t)) = \tilde{v}(t, H_{1\varepsilon}(t)) = 0, & t \in (0, \bar{T}],
\end{cases}
\]
(4.8)
\[
\begin{align*}
G_{1\varepsilon}(t) &= \phi'(t)[-\mu \tilde{v}_x(t, G_{1\varepsilon}(t)) + 2\varepsilon \gamma_1], & t \in (0, \bar{T}], \\
H_{1\varepsilon}(t) &= \phi'(t)[-\mu \tilde{v}_x(t, H_{1\varepsilon}(t)) - 2\varepsilon \gamma_1], & t \in (0, \bar{T}], \\
\tilde{v}(0, x) &= v_0(x), & x \in [g_0, h_0].
\end{align*}
\]
From the first equation of (4.8) and (2.14), we obtain
\[
(V_{1\varepsilon})_t = \partial_t \tilde{v} + \varepsilon \gamma_1 \partial_t M_\varepsilon = d\tilde{v}_{xx} + \frac{\phi'(t)-1}{\phi(t)} \tilde{v}_t + f(\phi(t), x, \tilde{v}) + \varepsilon \gamma_1 \partial_t M_\varepsilon
\]
\[
= d(V_{1\varepsilon})_{xx} + f(t, x, V_{1\varepsilon}) + \frac{\phi'(t)-1}{\phi(t)} \tilde{v}_t
\]
\[
+ \varepsilon \gamma_1 \partial_t M_\varepsilon - d \varepsilon \gamma_1 \partial_{xx} M_\varepsilon + f(\phi(t), x, \tilde{v}) - f(t, x, V_{1\varepsilon})
\]
\[
\geq d(V_{1\varepsilon})_{xx} + f(t, x, V_{1\varepsilon}) + \frac{\phi'(t)-1}{\phi(t)} \tilde{v}_t
\]
\[
+ \varepsilon \gamma_1 \partial_t M_\varepsilon - d \varepsilon \gamma_1 \partial_{xx} M_\varepsilon - L(\varepsilon \gamma_1 M_\varepsilon + |\phi(t) - t|)
\]
\[
=: d(V_{1\varepsilon})_{xx} + f(t, x, V_{1\varepsilon}) + E(t, x) \text{ for } t \in (0, \bar{T}], x \in (G_{1\varepsilon}(t), 0) \cup (0, H_{1\varepsilon}(t)).
\]
A simple computation gives
\[
\begin{cases}
\partial_t M_\varepsilon \geq \xi_1(\xi_1 t + \xi_3) \varepsilon \gamma_1 t \left[ 1 - \frac{x^2}{H_{1\varepsilon}(t)} \right] = \xi_1 M_\varepsilon & \text{for } x > 0,
\partial_{xx} M_\varepsilon = -2(\xi_1 t + \xi_3) \varepsilon \gamma_1 t \left[ \frac{1}{H_{1\varepsilon}(t)} \right] < 0 & \text{for } x > 0,
\partial_t M_\varepsilon \geq \xi_1(\xi_1 t + \xi_3) \varepsilon \gamma_1 t \left[ 1 - \frac{x^2}{G_{1\varepsilon}(t)} \right] = \xi_1 M_\varepsilon & \text{for } x < 0,
\partial_{xx} M_\varepsilon = -2(\xi_1 t + \xi_3) \varepsilon \gamma_1 t \left[ \frac{1}{G_{1\varepsilon}(t)} \right] < 0 & \text{for } x < 0.
\end{cases}
\]
(4.9)
Claim 1. We can choose \( \xi_1, \xi_2 \) and \( \xi_3 \) such that
\[
E(t, x) \geq \varepsilon \gamma_1 \text{ for } t \in (0, \bar{T}], x \in (G_{1\varepsilon}(t), 0) \cup (0, H_{1\varepsilon}(t)) \text{ and } 0 < \varepsilon \ll 1.
\]
In the following, we just verify \( E(t, x) \geq \varepsilon \gamma_1 \) for \( t \in (0, \bar{T}], x \in (0, H_{1\varepsilon}(t)) \) since the proof for \( t \in (0, \bar{T}], x \in (G_{1\varepsilon}(t), 0) \) is similar.

Since \( m_\varepsilon(t, x; G_{1\varepsilon}(t), H_{1\varepsilon}(t)) = m_\varepsilon(\phi(t), x; g_{2\varepsilon}, h_{2\varepsilon}), \) from (4.6) we deduce
\[
\partial_t v_{2\varepsilon}(\phi, x) \geq -k_1 m_\varepsilon(t, x; G_{1\varepsilon}, H_{1\varepsilon})
\]
where \( k_1 \) is given by Lemma 4.2 and so, for \( 0 < \varepsilon \ll 1 \) and \( t \in (0, \bar{T}], x \in [G_{1\varepsilon}(t), H_{1\varepsilon}(t)], \) we have
\[
\tilde{v}_t(t, x) = \phi' \partial_t v_{2\varepsilon}(\phi, x) \geq -2k_1 m_\varepsilon(t, x; G_{1\varepsilon}, H_{1\varepsilon}),
\]
where we have used
\[
0 < \xi_2 \varepsilon \gamma_1 < \phi'(t) - 1 = \xi_2(1 + \xi_1 t)e^{\varepsilon t}e^{\gamma_1 t} < 1 \text{ for small } \varepsilon > 0.
\]
Making use of (4.9), (4.10) and (4.11), we obtain, for \( (t, x) \in (0, \bar{T}] \times (0, H_{1\varepsilon}(t)) \) and \( 0 < \varepsilon \ll 1, \)
\[
E = \frac{\phi'(t)-1}{\phi(t)} \tilde{v}_t + \varepsilon \gamma_1 \partial_t M_\varepsilon - d \varepsilon \gamma_1 \partial_{xx} M_\varepsilon - L(\varepsilon \gamma_1 M_\varepsilon + |\phi(t) - t|)
\]
\[
\geq -2k_1[\phi'(t) - 1]m_\varepsilon + \xi_1 M_\varepsilon e^{\gamma_1 t} + 2d(\xi_1 t + \xi_3) \frac{e^{\xi_1 t}e^{\gamma_1 t}}{H_{1\varepsilon}^2(t)} e^{\gamma_1 t} - LM_\varepsilon e^{\gamma_1 t} - L|\phi(t) - t|
\]
\[
\geq -2k_1 \xi_2 M_\varepsilon e^{\gamma_1 t} + \xi_1 M_\varepsilon e^{\gamma_1 t} + 2d(\xi_1 t + \xi_3) \frac{e^{\xi_1 t}e^{\gamma_1 t}}{H_{1\varepsilon}^2(t)} e^{\gamma_1 t} - LM_\varepsilon e^{\gamma_1 t} - L\xi_2 e^{\xi_1 t}e^{\gamma_1 t}
\]
\((\xi_1 - 2k_1\xi_2 - L)M_\epsilon \gamma_1 + \left[ 2d(\xi_1 t + \xi_3)\frac{e^{\xi_1 t}}{H_{1e}(t)} - L\xi_2 e^{\xi_1 t} \right] \gamma_1.\)

Since \(H_{1e}(\tilde{T}) = h_{2e}(T_0) \leq M_2\), we have

\[
2d(\xi_1 t + \xi_3)\frac{e^{\xi_1 t}}{H_{1e}(t)} - L\xi_2 e^{\xi_1 t} \geq \frac{2d}{M_2} \xi_3 + \left( \frac{2d}{M_2} \xi_1 - L\xi_2 \right) t e^{\xi_1 t} > 1
\]

provided that

\[
\xi_1 > \frac{LM_2^2}{2d}\xi_2, \quad \xi_3 > \frac{M_2^2}{2d}.
\]

Therefore, for \((t, x) \in (0, \tilde{T}) \times (0, H_{1e}(t))\) and \(0 < \epsilon \ll 1, E(t, x) > \epsilon_1\)

provided that

\[
(4.12) \quad \xi_1 > \max \left\{ 2k_1\xi_2 + L_e, \frac{LM_2^2}{2d}\xi_2 \right\}, \quad \xi_3 > \frac{M_2^2}{2d}.
\]

Claim 1 is thus proved, and hence, for such \(\epsilon, \xi_1, \xi_2\) and \(\xi_3\), we have

\( (V_{1e})_t \geq d(V_{1e})_x + f(t, x, V_{1e}) + \epsilon_1 \) for \( t \in (0, \tilde{T}), x \in (G_{1e}(t), 0) \cup (0, H_{1e}(t)). \)

Next, we deal with the estimates of \(G'_{1e}\) and \(H'_{1e}\). From the forth equation of \(4.8\), \(4.11\) and \(V_{1e} = \tilde{v} + \epsilon_1 M_e, \) we obtain

\[
H'_{1e}(t) = \phi'(t)[-\mu\tilde{v}_x(t, H_{1e}(t)) - 2\epsilon_1] \geq -\mu\phi'(t)\tilde{v}_x(t, H_{1e}(t)) - 4\epsilon_1
\]

\[
= -\mu\tilde{v}_x(t, H_{1e}(t)) - \mu[\phi'(t) - 1]\tilde{v}_x(t, H_{1e}(t)) - 4\epsilon_1
\]

\[
= -\mu(V_{1e})_x(t, H_{1e}(t)) - \epsilon_1 \partial_x M_e(t, H_{1e}(t)) - \mu[\phi'(t) - 1]\tilde{v}_x(t, H_{1e}(t)) - 4\epsilon_1
\]

\[
= -\mu(V_{1e})_x(t, H_{1e}(t)) - \mu[\phi'(t) - 1]\tilde{v}_x(t, H_{1e}(t)) + \mu \partial_x M_e(t, H_{1e}(t)) e^{\gamma_1} - 4\epsilon_1
\]

\[
\therefore -\mu(V_{1e})_x(t, H_{1e}(t)) + E_1(t).
\]

**Claim 2.** We can choose \(\xi_1, \xi_2\) and \(\xi_3\) satisfying (4.12) such that

\( E_1(t) \geq \epsilon_1 \) for \( t \in (0, \tilde{T}) \) and \( 0 < \epsilon \ll 1. \)

With \(k_2\) determined by Lemma 4.2 by (4.7), we have

\[
(4.13) \quad -\tilde{v}_x(t, H_{1e}(t)) \geq k_2 \quad \text{for} \quad 0 \leq t \leq \tilde{T}.
\]

Then applying (4.13) and \(H_{1e}(t) \geq h_0\), we deduce, for \(0 \leq t \leq \tilde{T}\) and \(0 < \epsilon \ll 1, \)

\[
E_1 = -\mu[\phi'(t) - 1]\tilde{v}_x(t, H_{1e}(t)) + \mu \partial_x M_e(t, H_{1e}(t)) e^{\gamma_1} - 4\epsilon_1
\]

\[
\geq \mu[\phi'(t) - 1]k_2 - 2\mu(\xi_1 t + \xi_3)e^{\xi_1 t}H_{1e}^{-1}e^{\gamma_1} - 4\epsilon_1
\]

\[
= \xi_2(1 + \xi_1 t)e^{\xi_1 t}k_2 - 2\mu(\xi_1 t + \xi_3)e^{\xi_1 t}H_{1e}^{-1}e^{\gamma_1} - 4\epsilon_1
\]

\[
= (\xi_2 k_2 - 2\xi_3 H_{1e}^{-1})\mu e^{\xi_1 t}e^{\gamma_1} + (\xi_2 - 2H_{1e}^{-1})\mu \xi_1 t e^{\xi_1 t}e^{\gamma_1} - 4\epsilon_1
\]

\[
\geq (\xi_2 k_2 - 2\xi_3 h_0^{-1})\mu e^{\gamma_1} + (\xi_2 - 2h_0^{-1})\mu \xi_1 t e^{\gamma_1} - 4\epsilon_1
\]

\[
\geq 5\epsilon_1 - 4\epsilon_1 = \epsilon_1,
\]

provided that

\[
(4.14) \quad \xi_2 \geq \max \left\{ \frac{2}{h_0}, \frac{5}{\mu k_2} + \frac{2}{h_0 k_2}\xi_3 \right\}.
\]

This proves Claim 2 and hence, for \(\xi_1, \xi_2, \xi_3\) satisfying (4.12) and (4.14), and \(0 < \epsilon \ll 1, \)

\(H'_{1e}(t) \geq -\mu(V_{1e})_x(t, H_{1e}(t)) + \epsilon_1, \quad t \in [0, \tilde{T}].\)
Analogously, for such $\xi_1, \xi_2, \xi_3$ and $0 < \epsilon \ll 1$,
$$G_1^{\epsilon}(t) \leq -\mu(V_1^{\epsilon})_x(t, G_1^{\epsilon}(t)) - \epsilon \gamma, \quad t \in [0, \tilde{T}].$$

Furthermore, from the definition of $M_\epsilon$,
$$V_1^{\epsilon}(t, G_1^{\epsilon}(t)) = V_1^{\epsilon}(t, H_1^{\epsilon}(t)) = 0 \quad \text{for } t \in [0, \tilde{T}],$$
$$V_1^{\epsilon}(0, x) = \bar{v}(0, x) + \epsilon \gamma M(0, x) \geq u_0(x) \quad \text{for } x \in [-h_0, h_0].$$
Since $M_\epsilon(t, x)$ and hence $V_1^{\epsilon}(t, x)$ is $C^1$ at $x = 0$, we may now conclude that $(V_1^{\epsilon}, G_1^{\epsilon}, H_1^{\epsilon})$ is a weak upper solution of (2.4) with $i = 1.$

$$\Box$$

5. About Remark 1.4

Here we provide some analysis which leads us to believe the modification of (1.2) is needed for the approximation of (1.1). Without modifying (1.2), the natural candidate for the approximation problem of (1.1) is the following one:

$$\left\{ \begin{array}{l}
\frac{\partial u_\epsilon}{\partial t} = \frac{\partial^2 u_\epsilon}{\partial y^2} + f(t, x, u_\epsilon),
\quad t > 0, \ x \in (g(t), h(t)),
\end{array} \right.$$

$$u(t, g(t)) = u(t, h(t)) = 0, \quad t > 0,$$

$$g'(t) = -\mu C_1 \epsilon^2 \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_\epsilon(x - y)u(t, x)dydx, \quad t > 0,$$

$$h'(t) = \mu C_1 \epsilon^2 \int_{h(t)}^{g(t)} \int_{h(t)}^{\infty} J_\epsilon(x - y)u(t, x)dydx, \quad t > 0,$$

$$-g(0) = h(0) = h_0, \ u(0, x) = v_0(x), \quad x \in [-h_0, h_0]$$

for some $C_1 > 0$, where $C_* = \left[ \int_0^1 J(y)g(y)^2 dy \right]^{-1}$ as in (1.3).

**Proposition 5.1.** Suppose the conditions of Theorem 1.4 hold, $(v, g, h)$ is the solution of (1.1), and $(u_\epsilon, g_\epsilon, h_\epsilon)$ is the solution of (5.1). If $(u_\epsilon, g_\epsilon, h_\epsilon) \to (v, g, h)$ as $\epsilon \to 0$ uniformly for $x \in \mathbb{R}$ and $t \in [0, T]$ for every $T > 0$, then

$$C_1 = C_*.$$

**Proof.** A simple calculation gives

$$\frac{\partial}{\partial t} \left[ \int_{g_\epsilon(t)}^{h_\epsilon(t)} u_\epsilon(t, x)dx \right] = h_\epsilon'(t)u_\epsilon(t, h_\epsilon(t)) - g_\epsilon'(t)u_\epsilon(t, g_\epsilon(t)) + \int_{g_\epsilon(t)}^{h_\epsilon(t)} \frac{\partial}{\partial t} u_\epsilon(t, x)dx$$

$$= \int_{g_\epsilon(t)}^{h_\epsilon(t)} \frac{\partial}{\partial t} u_\epsilon(t, x)dx$$

and

$$\int_{g_\epsilon(t)}^{h_\epsilon(t)} \left[ \int_{g_\epsilon(t)}^{h_\epsilon(t)} J_\epsilon(x - y)u_\epsilon(t, y)dy - u_\epsilon(t, x) \right] dx$$

$$= \int_{g_\epsilon(t)}^{h_\epsilon(t)} \int_{g_\epsilon(t)}^{h_\epsilon(t)} J_\epsilon(x - y)[u_\epsilon(t, y) - u_\epsilon(t, x)]dydx$$

$$- \int_{g_\epsilon(t)}^{h_\epsilon(t)} \int_{\mathbb{R} \setminus [g_\epsilon(t), h_\epsilon(t)]} J_\epsilon(x - y)u_\epsilon(t, x)dydx$$

$$= - \int_{g_\epsilon(t)}^{h_\epsilon(t)} \int_{h_\epsilon(t)}^{\infty} J_\epsilon(x - y)u_\epsilon(t, x)dydx - \int_{g_\epsilon(t)}^{h_\epsilon(t)} \int_{-\infty}^{g_\epsilon(t)} J_\epsilon(x - y)u_\epsilon(t, x)dydx$$
Integrating the first equation in (5.1) over \( \{(x, s) : x \in (g_\epsilon(s), h_\epsilon(s)), s \in (0, t)\} \), we thus obtain
\[
\int_{g_\epsilon(t)}^{h_\epsilon(t)} u_\epsilon(t, x)dx - \int_{g_0}^{h_\epsilon(s)} v_0(x)dx = \int_0^t \int_{g_\epsilon(s)}^{h_\epsilon(s)} \partial_t u_\epsilon(s, x)dxds
= -\frac{dC_\epsilon}{\mu C_1}[h_\epsilon(t) - g_\epsilon(t) - 2h_0] + \int_0^t \int_{g_\epsilon(s)}^{h_\epsilon(s)} f(s, x, u_\epsilon)dxds.
\]
Letting \( \epsilon \to 0 \) we deduce
\[
(5.2) \quad \int_{g(t)}^{h(t)} v(t, x)dx - \int_{g_0}^{h_\epsilon(s)} v_0(x)dx = -\frac{dC_*}{\mu C_1}[h(t) - g(t) - 2h_0] + \int_0^t \int_{g(s)}^{h(s)} f(s, x, v)dxds.
\]
On the other hand, from (1.1) we have
\[
\frac{\partial}{\partial t} \left[ \int_{g(t)}^{h(t)} u(t, x)dx \right] = h'(t)v(t, h(t)) - g'(t)v(t, g(t)) + \int_{g(t)}^{h(t)} \partial_x v(t, x)dx
= \int_{g(t)}^{h(t)} \partial_t v(t, x)dx
\]
and
\[
\int_{g(t)}^{h(t)} v_{xx}dx = v_x(t, h(t)) - v_x(t, g(t)) = \frac{1}{\mu}[h'(t) - g'(t)].
\]
So similarly we have
\[
\int_{g(t)}^{h(t)} v(t, x)dx - \int_{g_0}^{h_\epsilon(s)} v_0(x)dx = -\frac{d}{\mu}[h(t) - g(t) - 2h_0] + \int_0^t \int_{g(s)}^{h(s)} f(s, x, v)dxds.
\]
Comparing this identity with (5.2), we immediately obtain \( C_1 = C_* \). \( \square \)

Next we examine the asymptotic limit of the solution \((u_\epsilon, h_\epsilon, g_\epsilon)\) of (5.1) with \( C_1 = C_* \), as \( \epsilon \to 0 \).

**Proposition 5.2.** Suppose the conditions of Theorem 1.1 hold, and \((u_\epsilon, g_\epsilon, h_\epsilon)\) is the solution of (5.1) with \( C_1 = C_* \). Then there exists \( \mu > 0 \) such that
\[
\liminf_{\epsilon \to 0} u_\epsilon(x, t) \geq v(x, t), \quad \liminf_{\epsilon \to 0} h_\epsilon(t) \geq h(t), \quad \limsup_{\epsilon \to 0} g_\epsilon(t) \leq g(t)
\]
uniformly for \( x \in \mathbb{R}, t \in [0, T] \) with every \( T > 0 \), where \((v(t, x), g(t), h(t))\) denotes the unique solution of (1.1) with \( \mu = \mu_\ast \).

Here, we assume that \( u_\epsilon \) and \( v \) are extended by 0 outside their supporting sets.

**Proof.** For \( 0 < \epsilon \ll 1 \), let \((u_\epsilon, g_\epsilon, h_\epsilon)\) be the unique solution of (5.1) with \( \mu = \mu_\ast \) and \( i = 2 \). The value of \( \mu \) will be determined later.

For an arbitrarily given \( T_0 > 0 \), fix \( T = T_0 \). By Lemma 3.2, the function
\[
\hat{v}_\epsilon(t, x) := v_\epsilon(t, x) - \epsilon^{\gamma_1} Ke^{L_1 t}m_\epsilon(t, x)
\]
satisfies (3.3), where \( m_\epsilon \) is given by (3.1) with \((\mu, g_\epsilon, h_\epsilon)\) in place of \((\mu, g_\epsilon, h_\epsilon)\), and the same change is understood in (3.3). We assume that \( \tilde{v} \) has been extended to a \( C^{0,2+\alpha}([0, T_0] \times \mathbb{R}) \) function.
Let \( M(x) \) and \( N(x) \) be smooth nonnegative functions over \([0,1]\) vanishing at \( x \in \{0,1\} \).

For convenience of notation, we define \( M(x) = N(x) = 0 \) for \( x \notin [0,1] \). Then define, for \( 0 < \epsilon \ll 1, t \in (0, T_0] \) and \( x \in \{g_\epsilon(t), h_\epsilon(t)\} \),

\[
v_\epsilon(t, x) := \hat{v}_\epsilon(t, x) + \epsilon M\left(\frac{h_\epsilon(t) - x}{\epsilon}\right) \frac{h_\epsilon'(t)}{\mu} - \epsilon N\left(\frac{x - g_\epsilon(t)}{\epsilon}\right) \frac{g_\epsilon'(t)}{\mu};
\]

and so we have, by (3.3),

\[
\mu \frac{C_\star}{\epsilon^2} \int_{g_\epsilon(t)}^{h_\epsilon(t)} \int_{h_\epsilon(t)}^\infty J_\epsilon(x - y) v_\epsilon(t, x) dy dx
\]

\[
= \mu \frac{C_\star}{\epsilon^2} \int_{-\epsilon}^{\epsilon} \int_0^1 J_\epsilon(x - y) v_\epsilon(t, h_\epsilon(t) + x) dy dx
\]

\[
= \mu \frac{C_\star}{\epsilon} \int_{-1}^{1} \int_0^1 J(x - y) v_\epsilon(t, h_\epsilon(t) + x) dy dx
\]

\[
= \mu \frac{C_\star}{\epsilon} \int_0^1 \int_0^1 J(z) v_\epsilon(t, h_\epsilon(t) - ew) dz dw
\]

\[
= \mu \frac{C_\star}{\epsilon} \int_0^1 \int_0^1 J(z) \left[ - (v_\epsilon)_x(t, h_\epsilon(t)) ew + O(\epsilon^2 w^2) + \frac{h_\epsilon'(t)}{\mu} \epsilon M(w) \right] dz dw
\]

\[
\geq \mu \frac{h_\epsilon'(t) C_\star}{\mu} \int_0^1 \int_0^1 J(z) [w + M(w)] dz dw
\]

\[
= \mu \frac{h_\epsilon'(t) C_\star}{\mu} \int_0^1 \int_0^z J(z) \int_0^w [w + M(w)] dz dw
\]

\[
= \mu \frac{h_\epsilon'(t) C_\star}{\mu} \left[ \frac{1}{2} + C_\star \int_0^1 J(z) \int_0^z M(w) dw dz \right]
\]

provided that

\[
\mu \leq \mu \left[ \frac{1}{2} + C_\star \int_0^1 J(z) \int_0^z M(w) dw dz \right].
\]

Analogously,

\[
\mu \frac{C_\star}{\epsilon^2} \int_{g_\epsilon(t)}^{h_\epsilon(t)} \int_{-\infty}^{g_\epsilon(t)} J_\epsilon(x - y) v_\epsilon(t, x) dy dx
\]

\[
\geq \mu \frac{|g_\epsilon'(t)| C_\star}{\mu} \int_0^1 \int_{w}^1 J(z) [w + N(w)] dz dw
\]

\[
= \mu \frac{|g_\epsilon'(t)| C_\star}{\mu} \left[ \frac{1}{2} + C_\star \int_0^1 J(z) \int_0^z N(w) dw dz \right]
\]

provided that

\[
\mu \leq \mu \left[ \frac{1}{2} + C_\star \int_0^1 J(z) \int_0^z N(w) dw dz \right].
\]
For $0 < \epsilon \ll 1$, $t \in (0, T_0]$ and $x \in \left( g_e(t), h_e(t) \right]$, we further have
\begin{align}
\frac{d C_s}{\epsilon^2} & \left[ \int_{g_e(t)}^{h_e(t)} J_e(x - y)v_e(t, y)dy - v_e(t, x) \right] \\
& = \frac{d C_s}{\epsilon^2} \left[ \int_{g_e(t) - \epsilon}^{h_e(t) + \epsilon} J_e(x - y)\hat{v}_e(t, y)dy - \hat{v}_e(t, x) \right] - L_e(t, x) - R_e(t, x),
\end{align}
(5.3)
where
\[ L_e(t, x) := \frac{d C_s}{\epsilon^2} \int_{g_e(t) - \epsilon}^{h_e(t)} J_e(x - y)\hat{v}_e(t, y)dy \]
\[ + \epsilon \frac{g_e'(t)}{\mu} \left[ \int_{g_e(t)}^{h_e(t) + \epsilon} J_e(x - y)N \left( y - \frac{g_e(t)}{\epsilon} \right)dy - N \left( \frac{x - g_e(t)}{\epsilon} \right) \right], \]
\[ R_e(t, x) := \frac{d C_s}{\epsilon^2} \int_{h_e(t)}^{h_e(t) + \epsilon} J_e(x - y)\hat{v}_e(t, y)dy \]
\[ - \epsilon \frac{h_e'(t)}{\mu} \left[ \int_{h_e(t)}^{h_e(t) + \epsilon} J_e(x - y)M \left( \frac{h_e(t) - y}{\epsilon} \right)dy - M \left( \frac{h_e(t) - x}{\epsilon} \right) \right]. \]

From spt$(J_e) \subset [-\epsilon, \epsilon]$ and spt$(M)$, spt$(N) \subset [0, 1]$, we see immediately that
\[ L_e(t, x) = 0 \text{ for } x \notin \left[ g_e(t), h_e(t) + 2\epsilon \right], \]
\[ R_e(t, x) = 0 \text{ for } x \notin \left[ h_e(t) - 2\epsilon, h_e(t) \right], \]
and due to $(\hat{v}_e)_x(t, h_e(t)) < 0 < (\hat{v}_e)_x(t, g_e(t))$, that
\[ L_e(t, x) \leq 0 \text{ for } x \in \left[ g_e(t) + \epsilon, h_e(t) + 2\epsilon \right], \]
\[ R_e(t, x) \leq 0 \text{ for } x \in \left[ h_e(t) - 2\epsilon, h_e(t) - \epsilon \right]. \]

For $x \in \left[ h_e(t) - \epsilon, h_e(t) \right]$, letting
\[ z = \frac{y - h_e(t)}{\epsilon} \text{ and } w = \frac{h_e(t) - x}{\epsilon} \in [0, 1], \]
we obtain
\begin{align*}
R_e(t, x) &= \frac{d C_s}{\epsilon^2} \left\{ \int_0^1 J(w + z)\hat{v}_e(t, h_e + \epsilon z)dz - \epsilon \frac{h_e'(t)}{\mu} \left[ \int_{-1}^{0} J(w + z)M(-z)dz - M(w) \right] \right\} \\
& = \frac{d C_s}{\epsilon^2} \left\{ \int_0^1 J(w + z) \left[ (\hat{v}_e)_x(t, h_e)\epsilon z + O(\epsilon^2 z^2) \right]dz - \epsilon \frac{h_e'(t)}{\mu} \left[ \int_{-1}^{0} J(w + z)M(-z)dz - M(w) \right] \right\} \\
& \leq \frac{d C_s h_e'(t)}{\epsilon \mu} \left\{ - \int_0^1 J(w + z)dz - \int_0^1 J(w - z)M(z)dz + M(w) \right\} \\
& \quad - \frac{d C_s h_e'(t)}{\mu} \int_0^1 J(w + z)zdz \left[ \epsilon^{\gamma - 1} + O(1) \right] \\
& \leq \frac{d C_s h_e'(t)}{\epsilon \mu} \left\{ - \int_0^1 \left[ J(w + z)z + J(w - z)M(z) \right]dz + M(w) \right\} \\
& \quad - \epsilon^{\gamma - 1} \frac{d C_s}{2 \mu} \int_0^1 J(w + z)zdz.
\end{align*}
We now choose $M$ such that
\begin{equation}
\int_0^1 \left[ J(w - z)M(z) + J(w + z)z \right]dz \geq M(w) \text{ for } w \in [0, 1].
\end{equation}
This can be easily achieved. Indeed, for \( w \in [0, 1] \), write
\[
F(w) := \int_0^1 J(w + z)dz.
\]
Then
\[
F(w) = \int_w^{1+w} J(\xi)(\xi - w)d\xi = \int_0^1 J(\xi)(\xi - w)d\xi,
\]
\[
F(1) = 0, \quad F(0) = \int_0^1 J(\xi)d\xi \in (0, \frac{1}{2})
\]
and
\[
F'(w) = -\int_w^1 J(\xi)d\xi \leq 0, \quad F'(0) = -\frac{1}{2}, \quad F'(1) = 0, \quad F''(w) = J(w) \geq 0 \text{ for } w \in [0, 1].
\]
It follows that
\[
F(0) - \frac{1}{2}w \leq F(w) < \frac{1}{2}(1 - w) \text{ for } w \in [0, 1).
\]
Therefore we can find \( M \) satisfying, apart from the earlier requirements, that
\[
M(w) \leq F(w) \text{ in } [0, 1], \quad M(w) = F(w) \text{ for } w \in [F(0), 1].
\]
Fix such an \( M \); then clearly (5.4) holds and we have
\[
R_{\epsilon}(t, x) \leq -\epsilon^{\gamma_1-1} \frac{dC_*}{2\mu} F(w)
\]
for \( 0 < \epsilon \ll 1 \) and \( w = \frac{h_c(t) - x}{\epsilon} \in [0, 1] \).

Similarly,
\[
L_{\epsilon}(t, x) \leq dC_*|g_\epsilon'(t)|\left\{ -\int_{[0, 1]} J(w + z) + J(w - z)N(z)dz + N(w) \right\}
\]
\[
-\epsilon^{\gamma_1-1} \frac{dC_*}{2\mu} \int_0^1 J(w + z)dz.
\]
So if we take \( N(w) = M(w) \), then
\[
L_{\epsilon}(t, x) \leq -\epsilon^{\gamma_1-1} \frac{dC_*}{2\mu} F(w)
\]
for \( 0 < \epsilon \ll 1 \) and \( w = \frac{x - g_\epsilon(t)}{\epsilon} \in [0, 1] \).

With \( M = N \) defined as above, we now take
\[
\mu := \mu \left[ \frac{1}{2} + C_* \int_0^1 J(z) \int_0^z M(w)dwdz \right].
\]
Then
\[
\begin{cases}
\mu \frac{C_*}{\epsilon^2} \int_{\frac{h_c(t)}{\epsilon}}^{\frac{h_c(t)}{\epsilon} + \epsilon} \int_{-\infty}^\infty J_\epsilon(x - y)v_\epsilon(t, x)dydx \geq h_\epsilon'(t), \\
\mu \frac{C_*}{\epsilon^2} \int_{\frac{h_c(t)}{\epsilon}}^{\frac{h_c(t)}{\epsilon} + \epsilon} \int_{-\infty}^\infty J_\epsilon(x - y)v_\epsilon(t, x)dydx \geq g_\epsilon'(t),
\end{cases}
\]
for \( 0 < \epsilon \ll 1 \) and \( t \in [0, T_0] \).

By Steps 4 and 5 in the proof of Lemma 3.3, there exists \( c_0 > 0 \) such that
\[
\partial_t \hat{v}_\epsilon \leq d \frac{C_*}{\epsilon^2} \int_{\frac{g_\epsilon(t)}{\epsilon} - \epsilon}^{\frac{g_\epsilon(t)}{\epsilon} + \epsilon} J_\epsilon(x - y)\hat{v}_\epsilon(t, y)dy - \hat{v}_\epsilon(t, x) + f(t, x, \hat{v}_\epsilon) - c_0 \epsilon^{\gamma_1}
\]
for \( 0 < \epsilon \ll 1 \), \( t \in [0, T_0] \) and \( x \in [g_\epsilon(t), h_\epsilon(t)] \).
We show next that

\begin{equation}
\frac{\partial v_\epsilon}{\partial t} \leq \frac{C_*}{2} \left[ \int_{g_0(t)}^{h_\sigma(t)} J_\epsilon(x-y)v_\epsilon(t,y) \, dy - v_\epsilon(t,x) \right] + f(t, x, v_\epsilon)
\end{equation}

for $0 < \epsilon \ll 1$, $t \in [0, T_0]$ and $x \in [g_\epsilon(t), h_\epsilon(t)]$.

When $x \in [\hat{g}_\epsilon(t) + \epsilon, h_\epsilon(t) - \epsilon]$, we have $v_\epsilon(t,x) = \hat{v}_\epsilon(t,x)$, and by (5.3), (5.5), we obtain

\begin{align*}
\frac{dC_*}{\epsilon_2} \left[ \int_{g_0(t)}^{h_\sigma(t)} J_\epsilon(x-y)v_\epsilon(t,y) \, dy - v_\epsilon(t,x) \right] \\
\geq \frac{dC_*}{\epsilon^2} \left[ \int_{g_0(t)}^{h_\sigma(t)+\epsilon} J_\epsilon(x-y)v_\epsilon(t,y) \, dy - \hat{v}_\epsilon(t,x) \right] \\
\geq \partial v_\epsilon - f(t, x, v_\epsilon),
\end{align*}

as we wanted.

For $x \in [\hat{g}_\epsilon(t), \hat{g}_\epsilon(t) + \epsilon]$, by (5.3) and (5.5), we obtain

\begin{align*}
\frac{dC_*}{\epsilon^2} \left[ \int_{g_0(t)}^{h_\sigma(t)} J_\epsilon(x-y)v_\epsilon(t,y) \, dy - v_\epsilon(t,x) \right] \\
\geq \partial v_\epsilon - f(t, x, \hat{v}_\epsilon) + c_0 \epsilon^{\gamma_1} + \epsilon^{\gamma_1-1} \frac{dC_*}{2\mu} F\left( \frac{x - g_\epsilon(t)}{\epsilon} \right),
\end{align*}

and by the Lipschitz continuity of $f$ we also have

\[-f(t, x, \hat{v}_\epsilon) \geq -f(t, x, v_\epsilon) - O(\epsilon).

Moreover, due to our choice of $M$,

\begin{align*}
\partial_t \hat{u}_\epsilon &= \partial_t v_\epsilon + \epsilon M(\frac{x - g_\epsilon(t)}{\epsilon}) \frac{g_\epsilon''(t)}{\mu} - M'(\frac{x - g_\epsilon(t)}{\epsilon}) \frac{[g_\epsilon(t)]^2}{\mu} \\
&\geq \begin{cases} 
\partial_t v_\epsilon - O(\epsilon) & \text{for } x \in [g_\epsilon(t) + \epsilon F(0), g_\epsilon(t) + \epsilon], \\
\partial_t v_\epsilon - O(1) & \text{for } x \in [g_\epsilon(t), g_\epsilon(t) + \epsilon F(0)],
\end{cases}
\end{align*}

and

\[\epsilon^{\gamma_1-1} \frac{dC_*}{2\mu} F\left( \frac{x - g_\epsilon(t)}{\epsilon} \right) \geq c_1 \epsilon^{\gamma_1-1} \text{ for } x \in [g_\epsilon(t), g_\epsilon(t) + \epsilon F(0)],
\]

where

\[c_1 := \frac{dC_*}{2\mu} \min_{w \in [0, F(0)]} F(w) > 0.
\]

Substituting these estimates to (5.7), we obtain

\[\frac{dC_*}{\epsilon^2} \left[ \int_{g_0(t)}^{h_\sigma(t)} J_\epsilon(x-y)v_\epsilon(t,y) \, dy - v_\epsilon(t,x) \right] \geq \partial v_\epsilon - f(t, x, v_\epsilon)
\]

for $0 < \epsilon \ll 1$, $t \in [0, T_0]$ and $x \in [g_\epsilon(t), g_\epsilon(t) + \epsilon]$. Thus (5.6) holds in this range of the variables. For $x \in [h_\epsilon(t) - \epsilon, h_\epsilon(t)]$, the proof is parallel and we omit the details. Therefore (5.6) holds for $0 < \epsilon \ll 1$, $t \in [0, T_0]$ and $x \in [g_\epsilon(t), h_\epsilon(t)]$, as desired.

Since $0 < v_\epsilon(0, x) \leq v_0(x)$ for $0 < \epsilon \ll 1$ and $x \in (h_0, h_0)$, we may now use the comparison principle to conclude that

\begin{align*}
(5.8) \quad u_\epsilon(t, x) \geq v_\epsilon(t, x), \quad h_\epsilon(t) \geq h_\epsilon(t), \quad g_\epsilon(t) \leq g_\epsilon(t) \quad \text{for } t \in [0, T_0], \quad x \in [g_\epsilon(t), h_\epsilon(t)].
\end{align*}

By Theorem 2.4, we have $(v, \hat{g}_\epsilon, h_\epsilon) \to (v, \hat{g}, \hat{h})$ uniformly as $\epsilon \to 0$, and hence $v_\epsilon \to v$ uniformly as $\epsilon \to 0$. The required estimates now follow directly by letting $\epsilon \to 0$ in (5.8).

\[\square\]
Remark 5.3. (i) Note that if for some kernel function \( J \) we can find a function \( M \) satisfying (5.3) and
\[
\int_0^1 J(z) \int_0^z M(w) dw dz > \frac{1}{2} C_*^{-1} = \frac{1}{2} \int_0^1 J(z) z^2 dz,
\]
then from the above proof we see that \( \underline{\mu} > \mu \), and therefore the conclusion in Proposition 5.2 implies that \((u_\epsilon, g_\epsilon, h_\epsilon)\) cannot converge to \((v, g, h)\), the unique solution of (1.1), uniformly as \( \epsilon \to 0 \), since \( \underline{\mu} > \mu \) implies \( \underline{\nu} > v \). However, we have not been able to find such a pair \((J, M)\) so far, though we suspect such a pair exists.

(ii) It is also possible to obtain an estimate for \((u_\epsilon, g_\epsilon, h_\epsilon)\) in the form
\[
\limsup_{\epsilon \to 0} u_\epsilon(t, x) \leq \overline{u}(t, x), \quad \limsup_{\epsilon \to 0} h_\epsilon(t) \leq \overline{h}(t), \quad \liminf_{\epsilon \to 0} g_\epsilon(t) \geq \underline{g}(t)
\]
uniformly for \( x \in \mathbb{R}, \ t \in [0, T] \) with every \( T > 0 \), where \((\overline{u}(t, x), \overline{g}(t), \overline{h}(t))\) denotes the unique solution of (1.1) with \( \mu = \underline{\mu} \).

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