NODAL CURVES AND COMPONENTS OF THE HILBERT SCHEME OF CURVES IN $\mathbb{P}^r$ WITH THE EXPECTED NUMBER OF MODULI

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ABSTRACT. We study the existence of components with the expected number of moduli of the Hilbert scheme of integral nodal curves $C \subset \mathbb{P}^r$ with prescribed degree, arithmetic genus and number of singular points.

1. Introduction

For all integers $g, r, d$ set $\rho(g, r, d) := (r + 1)d - rg - r(r + 1)$. We work in the range $\rho(g, r, d) < 0$. Several authors studied the existence of irreducible components of the Hilbert scheme of smooth curves of $\mathbb{P}^r$ with the expected number of moduli (\cite{1, 4, 5, 1, 2, 7, 9, 8, 3, 15}). In the set-up of irreducible components, $\Gamma$, of $\text{Hilb}(\mathbb{P}^r)$ with the expected dimension and the expected number of moduli in the range $\rho(g, r, d) \leq 0$ it also says that for a general $C \in \Gamma$ the $g, r, d$ coming from the inclusion $C \subset \mathbb{P}^r$ is an isolated point of the set of all $g, r, d$'s on $C$ (see \cite{9}, Definition 1.1.2); it does not say that $C$ has only finitely many $g, r, d$'s. The result announced in \cite{4}, part 6) at page 338, is stronger for the following reason: it asks about the existence of an irreducible variety $T \subset M_g$ over which the relative $G_{r, d}$ is generically finite, but $T \not\subset V$ for any integral $V \subset M_g$ with $\dim(V) > \dim(T)$ and a general $C \in V$ has a $g, r, d$. As far as we know no proof of \cite{4}, part 6) at page 338, was published. Our tools give no informations on these two more general (and very interesting) problems.

For any nodal and connected curve $A \subset \mathbb{P}^r$ let $N_A$ denote its normal bundle in $\mathbb{P}^r$. If $h^1(A, N_A) = 0$, then $\text{Hilb}(\mathbb{P}^r)$ is smooth at $A$ and hence $\text{Hilb}(\mathbb{P}^r)$ has a unique irreducible component, $\Gamma$, and $\dim(\Gamma) = h^0(A, N_A) = (r + 1) \cdot \deg(A) - (r - 3)(p_a(A) - 1)$. In this case we say that $\Gamma$ has the expected dimension. For all integers $r, d, g, t$ such that $0 \leq t \leq g$ let $E(r, d, g, t)$ denote the subset of the Hilbert scheme of $\mathbb{P}^r$ parametrizing integral and non-degenerate curves $C \subset \mathbb{P}^r$ such that $\deg(C) = d, p_a(C) = g$ and $C$ has exactly $t$ ordinary nodes as its only singularities. Set $E'(r, d, g, t) := \{ C \in E(r, d, g, t) : h^1(C, O_C(2)) = 0 \}$. We will always take the reduced structure as the scheme structure for $E(r, d, g, t)$ and its open subset $E'(r, d, g, t)$. Let $\overline{E}(r, d, g, t)$ denote the closure of $E'(r, d, g, t)$ in the Hilbert scheme of $\mathbb{P}^r$. Let $\overline{M}_g(t)$ denote the set of all integral $Y \in \overline{M}_g$ with exactly $t$ nodes and $\overline{M}_g[t]$ its closure in $\overline{M}_g$. Notice that $\overline{M}_g(t)$ is integral and $\dim(\overline{M}_g(t)) = 3g - 3 - t$. For instance, $\overline{M}_g(g)$ parametrizes the set of all rational curves with exactly $g$ nodes.

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Notice that we always have a morphism \( u_{r,d,g,t} : E(r,d,g,t) \to \overline{M}_g \{ t \} \). Let \( \Gamma \) be any irreducible component of \( E'(r,d,g,t) \). We say that \( \Gamma \) has the expected number of moduli if \( \dim(u_{r,d,g,t}(\Gamma)) = \min\{ 3g - 3 - t, 3g - 3 + \rho(d,g,r) - t \} \).

**Conjecture 1.** Fix an integer \( r \geq 3 \). There is a function \( \Omega_r : \mathbb{N} \to \mathbb{N} \) such that \( \lim_{g \to +\infty} \Omega_r(g)/g = 0 \) and the following property holds: there is an integer \( g_0 \) such that for all integers \( g,t,d \) with \( g \geq g_0, 0 \leq t \leq g, -\rho(g,r,d) + t \leq 2g - \Omega_r(g) \) there is an irreducible component \( \Gamma \) of \( E'(r,d,g,t) \) with the expected dimension and the expected number of moduli.

**Conjecture 2.** Fix integers \( r \geq 3 \) and \( t \geq 0 \). There is a function \( \Omega_{r,t} : \mathbb{N} \to \mathbb{N} \) such that \( \lim_{g \to +\infty} \Omega_{r,t}(g)/g = 0 \) and the following property holds: there is an integer \( g_0 \geq t \) such that for all integers \( g,x,d \) with \( g \geq g_0, 0 \leq x \leq t, -\rho(g,r,d) + x \leq 3g - \Omega_{r,t}(g) \) there is an irreducible component \( \Gamma \) of \( E'(r,d,g,x) \) with the expected dimension and the expected number of moduli.

The case \( t = 0 \) of Conjecture 2 is the question raised in [8], after the statement of Theorem 1.2. This case was proved for \( r = 3 \) and \( t = 0 \) in [9].

In this paper we prove the following result.

**Theorem 1.** Fix integers \( r \geq 4 \) and \( t \geq 0 \). There is a function \( \psi_{r,t} : \mathbb{N} \to \mathbb{N} \) such that \( \lim_{g \to +\infty} \psi_{r,t}(g)/g = 0 \) and the following property holds: there is an integer \( g_0 \geq t \) such that for all integers \( g,x,d \) with \( g \geq g_0, 0 \leq x \leq t, -\rho(g,r,d) + x \leq 2g + 3 - \psi_{r,t}(g) \) there is an irreducible component \( \Gamma \) of \( E(r,d,g,x) \) with the expected number of moduli. If \( r \geq 7 \), then \( \Gamma \) is a component of \( E'(r,d,g,x) \).

Our proof of Theorem 1 is just a small modification of the proofs in [8]. We only write the modifications needed to get nodal irreducible curves.

We work over an algebraically closed field \( \mathbb{K} \) such that \( \text{char}(\mathbb{K}) = 0 \).

## 2. The proof

**Lemma 1.** Fix a set \( S \subset \mathbb{P}^m, m \geq 2 \), such that \( \overline{\nu}(S) \leq m + 3 \) and \( S \) is in linearly general position. Let \( \Gamma \) be the set of all rational normal curves of \( \mathbb{P}^m \) containing \( S \). Then \( \Gamma \) is a non-empty and irreducible variety of dimension \( (m + 3 - \overline{\nu}(S))(m - 1) \).

**Proof.** If \( \overline{\nu}(S) = m + 3 \), then the result is classical. Now assume \( \overline{\nu}(S) \leq m + 2 \). Fix a general \( A \subset \mathbb{P}^r \) such that \( \overline{\nu}(A) = s + 3 - \overline{\nu}(S) \). Since \( \overline{\nu}(A \cup S) = m + 3 \), we may apply the case just done to the set \( S \cup A \). We get the nonemptiness and the irreducibility of \( \Gamma \) and its dimension, because for a fixed rational normal curve \( D \subset \mathbb{P}^m \) such that \( D \supset S \) the set of all \( A \)'s contained in \( D \) has dimension \( s + 3 - \overline{\nu}(S) \). \( \square \)

**Lemma 2.** Let \( Y = A \cup B \subset \mathbb{P}^r \) be a nodal curve such that \( h^1(A,\mathcal{O}_A(2)) = 0 \). Set \( S := A \cap B \) and see it as an effective Cartier divisor of \( B \). Assume \( h^1(B,\mathcal{O}_B(2)(-S)) = 0 \). Then \( h^1(Y,\mathcal{O}_Y(2)) = 0 \).

**Proof.** Since \( h^1(B,\mathcal{O}_Y(2)(-S)) = 0 \), we have \( h^1(B,\mathcal{O}_B(2)) = 0 \) and the restriction map \( H^0(B,\mathcal{O}_B(2)) \to H^0(S,\mathcal{O}_S(2)) \) is surjective. Hence the lemma follows from the Mayer-Vietoris exact sequence

\[
0 \to \mathcal{O}_Y(2) \to \mathcal{O}_A(2) \oplus \mathcal{O}_B(2) \to \mathcal{O}_S(2) \to 0
\]

\( \square \)
Proof of Theorem 7. Until the last step we will only get a component of $E(r, d, g, x)$. In the statement of [8], Theorem 1.2, there is a function which goes as $2rg/(r + 3)$ if $r \notin \{4, 6\}$, like $(30/19)g$ if $r = 4$ and like $(3/2)g$ if $r = 6$. For simplicity we used the weaker upper bound $2rg/(r + 3)$ even for $r \in \{4, 6\}$. Fix any integer $x$ such that $0 \leq x \leq t$. By [8], Theorem 1.2, we may assume $x \neq 0$. Fix any pair $(u, v)$ such that the proof of [8], Theorem 1.2, gives the existence of an irreducible component $A_{u, v, r}$ of $\text{Hilb}(\mathbb{P}^r)$ with the expected number of moduli and whose general element $Y$ is a smooth curve of degree $u$ and genus $v$ with $h^1(Y, N_\gamma Y) = 0$. From the range of degrees and genera covered in the proof of [8], Theorem 1.2, it follows that to prove Theorem 1.2 gives that the proof of (ii) an irreducible component $B$ of $\text{Hilb}(\mathbb{P}^r)$ and a maximal subfamily $B_x \subset B$ of integral nodal curves with exactly $x$ nodes and with the following properties:

(i) $B$ has the expected dimension and the expected number of moduli and a general $Y \in B$ is a smooth curve of degree $u + xv$ and genus $v + x(r + 1)$ with $h^1(Y, N_\gamma Y) = 0$;

(ii) $B_x \subset B$ is a maximal subfamily of integral nodal curves with exactly $x$ nodes of $B$, i.e. $B_x \subset B$ and $B_x$ is an open subset of an irreducible component of $E'(r, u + rx, v + (r + 1)x, x)$;

(iii) $B_x$ has the expected number of moduli, i.e. $u_r, u + rx, v + (r + 1)x, x \in B_x$ is generically finite.

Fix $x(r + 2)$ general points of $Y$ and divide them into $x$ subsets $A_1, \ldots, A_x$ with $\sharp(A_i) = r + 2$ for all $i$. Let $D_i \subset \mathbb{P}^r$ be a general rational normal curve containing $A_i$. For general $A_1, \ldots, A_x$ and general $D_1, \ldots, D_x$ we get $D_i \cap D_j = \emptyset$ for all $i \neq j$, $D_i \cap Y = A_i$ for all $i$ and that the curve $W := Y \cup D_1 \cup \cdots \cup D_x$ is nodal. Notice that $W$ is connected, $\deg(W) = u + rx$ and $p_a(W) = v + (r + 1)x$. By [11], Lemma 2.3 (which uses [13] and [6], Theorem 4.1 and Remark 4.1.1), the curve $W$ is smoothable and $h^1(W, N_\gamma W) = 0$. To prove Theorem 1.2 we may also restrict the previous proof to pairs $(u, v)$ with the additional condition $\rho(v, r, u) \leq 0$. In this range the proof of [8], Theorem 1.2, gives $h^0(Y, T_{\mathbb{P}^r}|Y) = (r + 1)^2 - 1$. As in [13] (use of the multiplication map $\mu_0(D)$) or [8], property $(\gamma)$ at page 3489, to get that the unique irreducible component $\Gamma$ of $\text{Hilb}(\mathbb{P}^r)$ containing $Y \cup D_1$ has the right number of moduli and that $h^0(X, T_{\mathbb{P}^r}|X) = (r + 1)^2 - 1$ for a general element $X$ of it, it is sufficient to prove $h^0(D_1, (T_{\mathbb{P}^r}|D_1)(-A_1)) = 0$. This vanishing is true, because the vector bundle $T_{\mathbb{P}^r}|D_1$ is a direct sum of $r$ line bundles of degree $r + 1$ ([16], [10], [11]). Iterating $x - 1$ times the proof we get that the irreducible component of $\text{Hilb}(\mathbb{P}^r)$ containing $W$ has the right number of moduli. Varying the curve $Y$ in $B$, the sets $A_i$ and the rational normal curves $D_i$ we get an irreducible family $W$ of nodal curves of degree $u + rx$, arithmetic genus $v + x(r + 1)$, $x + 1$ irreducible components and with exactly $x(r + 2)$ nodes. Since $\dim(B) = (r + 1)u - (r - 3)(v - 1)$, we have $\dim(W) = (r + 1)u - (r - 3)(v - 1) + x(r + 2) + x(r - 1)$ (use the case $m = r$ of Lemma 1). Since $\dim(\Gamma) = (r + 1)(u + rx) - (r - 3)(v + (r + 1)x) = (r + 1)u - (r - 3)(v - 1) + (3r + 3)x$, $W$ has codimension $x(r + 2)$ in $\Gamma$. Fix any integer $y$ such that $0 \leq y \leq x(r + 2)$ and $S \subseteq A_1 \cup \cdots \cup A_x$ such that $\sharp(S) = y$. By [14], Theorem 6.3, there is a neighborhood $U$ of $W$ in $\text{Hilb}(\mathbb{P}^r)$ and a non-empty locally closed subset $U_S$ of $U$ consisting of curves with exactly $y$ nodes, each of them being a deformation of a different point of $S$, and $\dim(U_S) = \dim(U) - y$. Taking $y = x$ and $\sharp(S \cap A_i) = 1$ for all $i$ we get that any $E \in U_S$ is irreducible. To prove that $U_S$ has the expected number of moduli, it is sufficient to prove that $u_{g, r, d}|U_S$ is
generically finite. Since $W \in \overline{U_S}$, it is sufficient to prove that $\mathcal{O}_W(1)$ is an isolated $g_{\deg(W)}$ on $W$. Take a general $g_{\deg(W)}^r \cdot L$, of an irreducible component $\Lambda$ of the set of all $g_{\deg(W)}^r$'s on $W$ such that $\mathcal{O}_W(1) \in \Lambda$. Since $\mathcal{O}_W(1)$ is very ample and $h^0(W, \mathcal{O}_W(1)) = r + 1$, $L$ is very ample and $h^0(W, L) = r + 1$. Call $h_L : W \to \mathbb{P}^r$ the embedding induced by the complete linear system $[L]$. Since $B$ has the expected number of moduli, $Y$ is general in $B$ and $\rho(v, r, u) \leq 0$, $\mathcal{O}_Y(1)$ is an isolated $g_m^r$ on $Y$. Since $L|Y$ is near to $\mathcal{O}_Y(1)$, we have $L|Y \cong \mathcal{O}_Y(1)$. Since we may assume that $A_1 \cup \cdots \cup A_x$ is not sent into itself by a non-trivial automorphism of $Y$, we also see that $h_L(W)$ is the union of $Y$ and $x$ rational normal curves $C_1, \ldots, C_x$ with $C_i \cap Y = A_i$ for all $i$ and $C_i \cap C_j = \emptyset$ for all $i \neq j$. Since $\tau(A_i) \geq 4$, only finitely many automorphisms of $D_i$ send $A_i$ into itself. Hence we only have finitely many possible curves $C_i$, $i = 1, \ldots, x$. Hence $L = \mathcal{O}_W(1)$.

To check that the component we got is a component of $E'(r, d, g, x)$ we need to check that at each step here and in [8] we may apply Lemma 2. For the curves $D_i$, it is easy. In [8], page 3490, the author quoted [7], Sublemma 3.5; here $B$ is a smooth rational curve, $\deg(B) = r - 1$ and $\tau(S) = r + 2$. In [8], Claim (3.4) at page 3490, twice it is quoted [7], Claim 3.7; here $B$ is a smooth elliptic curve, $\deg(B) = r + 1$. In [7], Proposition 2.1, we have as $B$ a smooth curve of degree $d \geq p_a(B) + r$ with $\tau(S) = r + 4$; here $\deg(\mathcal{O}_B(2)(-S)) \geq 2 \cdot p_a(B) + 2r - r - 4 > 2 \cdot p_a(B)$. In [8] the reader will often meet as $B$ a canonically embedded curve (hence $B$ has genus $r + 1$ and $\mathcal{O}_B(1) \cong \omega_B$) with $\tau(S) = r + 6 + \epsilon$ with $\epsilon = 0$ if $r \notin \{4, 6\}$, $\epsilon = 2$ if $r = 4$ and $\epsilon = 1$ if $r = 6$; here we need $r \geq 7$ to use Lemma 2.

**Remark 1.** Fix $r \in \{3, 4, 6\}$. Using the cases $r \in \{4, 6\}$ of [8], Theorem 1.2, or [9] (case $r = 3$) we may take $3g$ (case $r = 3$) or $(30/19)g$ (case $r = 4$) and $(3/2)g$ (case $r = 6$) instead of $2rg/(r + 3)$ in the statement of Theorem 1.

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