ON MIXED PLANE CURVES OF POLAR DEGREE 1

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ABSTRACT. Let \( f(z, \bar{z}) \) be a mixed strongly polar homogeneous polynomial of 3 variables \( z = (z_1, z_2, z_3) \). It defines a Riemann surface \( V := \{ [z] \in \mathbb{P}^2 | f(z, \bar{z}) = 0 \} \) in the complex projective space \( \mathbb{P}^2 \). We will show that for an arbitrary given \( g \geq 0 \), there exists a mixed polar homogeneous polynomial with polar degree 1 which defines a projective surface of genus \( g \). For the construction, we introduce a new type of weighted homogeneous polynomials which we call polar weighted homogeneous polynomials of twisted join type.

1. Introduction

Let \( f(z, \bar{z}) \) be a strongly polar homogeneous mixed polynomial of \( n \)-variables \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) with polar degree \( q \) and radial degree \( d \). Recall that a strongly polar homogeneous polynomial \( f(z, \bar{z}) \) satisfies the equality ([6]):

\[
(1) \quad f((t, \rho) \circ z, (t, \rho) \circ \bar{z}) = t^d \rho^q f(z, \bar{z}), \quad (t, \rho) \in \mathbb{R}^+ \times S^1.
\]

Here \((t, \rho) \circ z\) is defined by the usual action \((t, \rho) \circ z = (t \rho z_1, \ldots, t \rho z_n)\). Let \( \tilde{V} \) be the mixed affine hypersurface

\[
\tilde{V} = f^{-1}(0) = \{ z \in \mathbb{C}^n | f(z, \bar{z}) = 0 \}.
\]

We assume that \( \tilde{V} \) has an isolated singularity at the origin. Let \( f : \mathbb{C}^n \setminus \tilde{V} \to \mathbb{C}^* \) be the global Milnor fibration defined by \( f \) and let \( F \) be the fiber. Namely \( F \) is the hypersurface \( f^{-1}(1) \subset \mathbb{C}^n \). The monodromy map \( h : F \to F \) is defined by

\[
h(z) = (\eta z_1, \ldots, \eta z_n), \quad \eta = \exp\left(\frac{2\pi i}{q}\right).
\]

We consider the smooth projective hypersurface \( V \) defined by

\[
V = \{ [z] \in \mathbb{P}^{n-1} | f(z, \bar{z}) = 0 \}.
\]

By (1), if \( z \in f^{-1}(0) \) and \( z' \) is in the same \( \mathbb{R}^+ \times S^1 \) orbit of \( z \), then \( z' \in f^{-1}(0) \). Thus the hypersurface \( V = \{ [z] \in \mathbb{P}^{n-1} | f(z) = 0 \} \) is well-defined. Consider the quotient map \( \pi : \mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1} \) and its restriction to the Milnor fiber \( \pi : F \to \mathbb{P}^{n-1} \setminus V \). This is a \( q \)-cyclic covering map. In the previous paper, we have shown that

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**Theorem 1.** (Theorem 11, [6]) The embedding degree of $V$ is equal to the polar degree $q$.

First we observe that

**Proposition 2.** The Euler characteristics satisfy the following equalities.

1. $\chi(F) = q \chi(\mathbb{P}^{n-1} \setminus V)$.
2. $\chi(\mathbb{P}^{n-1} \setminus V) = n - \chi(V)$ and $\chi(V) = n - \chi(F)/q$.
3. The following sequence is exact.

$$1 \rightarrow \pi_1(F) \xrightarrow{\pi_{\sharp}} \pi_1(\mathbb{P}^{n-1} \setminus V) \rightarrow \mathbb{Z}/q\mathbb{Z} \rightarrow 1.$$  

**Corollary 3.** If $q = 1$, the projection $\pi : F \rightarrow \mathbb{P}^{n-1} \setminus V$ is a diffeomorphism.

**Corollary 4.** Assume that $n = 3$. Then the genus $g(V)$ of $V$ is given by the formula:

$$g(V) = \frac{1}{2} \left( \frac{\chi(F)}{q} - 1 \right)$$

The monodromy map $h : F \rightarrow F$ gives free $\mathbb{Z}/q\mathbb{Z}$ action on $F$. Thus using the periodic monodromy argument in [3], we get

**Proposition 5.** The zeta function of the monodromy $h : F \rightarrow F$ is given by

$$\zeta(t) = (1 - t^q)^{-\chi(F)/q}.$$  

In particular, if $q = 1$, $h = \text{id}_F$ and $\zeta(t) = (1 - t)^{-\chi(F)}$.

1.1. **Projective mixed Curves.** Let $C$ be a smooth $C^\infty$ surface embedded in $\mathbb{P}^2$ and let $g$ be the genus of $C$ and let $q$ be the embedding degree of $C$. It is known that the following inequality is satisfied.

$$g \geq \frac{(q - 1)(q - 2)}{2}.$$  

This was first conjectured by R. Thom and it has been proved by many people. For example see Kronheimer-Mrowka, [2]. We are interested to present $C$ as a mixed algebraic curve in the smallest embedding degree $q$ of a Riemann surface of a given genus $g$ as a mixed algebraic curve. (So we are not interested in the embedding with $q = 0$.) In our previous paper, we have used the join type construction starting from a strongly polar homogeneous polynomial of two variables $f(z_1, z_2, \bar{z}_1, \bar{z}_2)$ of polar degree $q$ and radial degree $q + 2r$ and we considered

$$g(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) = f(z_1, z_2, \bar{z}_1, \bar{z}_2) + z_3^{q+r} \bar{z}_3^r.$$  

Using such a polynomial, we have shown that there exists a mixed curve of a given genus $g$ with the embedding degree 2 ([6]). Note that if degree $q = 1$, the join theorem ([1]) says that the Euler number of the Milnor fiber of $g$ is 1 (i.e., the Milnor number is 0) and thus we only get genus 0. Thus to get a mixed curve of polar degree 1 and the genus arbitrary large, we have to find another type of polynomials. This is the reason we introduce polar weighted
homogeneous polynomials of twisted join type (See §3). For example, in the above setting, we consider the polynomial:

\[ g'(z_1, z_2, \bar{z}_1, \bar{z}_2, \bar{z}_3) = f(z_1, z_2, \bar{z}_1, \bar{z}_2) + \bar{z}_2 z_3^{q+r} \bar{z}_3^{-1}. \]

Using polynomials of this type, we will show that there exists a mixed surface with the polar degree \( q = 1 \) for any \( q \) (Theorem 9, Corollary 10).

This paper is a continuation of our previous papers [7, 5, 6] and we use the same notations as those we have used previously.

2. Mixed projective curves

Let \( M(q + 2r, q; n) \) be the space of strongly polar homogeneous polynomials of \( n \)-variables \( z_1, \ldots, z_n \) with polar degree \( q \) and radial degree \( q + 2r \).

2.1. Important mixed affine curves. We consider the following mixed strongly polar homogeneous polynomial of two variables:

\[ h_{q,r,j}(w, \bar{w}) = (w_1^{q+j} \bar{w}_1^j + w_2^{q+j} \bar{w}_2^j) (w_1^{r-j} - \alpha w_2^{r-j}) (\bar{w}_1^{r-j} - \beta \bar{w}_2^{r-j}), \quad r \geq j \geq 0 \]

with \( \alpha, \beta \in \mathbb{C}^* \)-generic. This polynomial plays a key role for the construction. Note that \( h_{q,r,j} \) is a strongly polar homogeneous polynomial with radial degree \( q + 2r \) and polar degree \( q \) respectively i.e., \( h_{q,r,j} \in M(q + 2r, q; 2) \).

Then the Milnor fiber \( H_{q,r,j} := h_{q,r,j}^{-1}(1) \) of \( h_{q,r,j} \) is connected. The Euler characteristic of \( \chi(H_{q,r,j}) \) (where \( H_{q,r,j}^* = H_{q,r,j} \cap \mathbb{C}^* \)) is given by

\[ \chi(H_{q,r,j}) = -r_{q,r,j} \times q \quad \text{and} \quad \chi(H_{q,r,j}) = -r_{q,r,j} q + 2q \]

where \( r_{q,r,j} \) is the link component number of the mixed curve \( C = h_{q,r,j}^{-1}(0) \).

Note that the link component number \( r_{q,r,j} \) is given by \( r_{q,r,j} = q + 2(r - j) \) by Lemma 64, [7]. Thus

**Proposition 6.**

\[ \chi(H_{q,r,j}) = -q ((q - 2) + 2(r - j)) \]

2.2. Join type polynomials. We consider the following strongly polar homogeneous polynomial of join type.

\[ f_{q,r,j}(z, \bar{z}) = h_{q,r,j}(w, \bar{w}) + z_3^{q+r} \bar{z}_3^r, \quad w = (z_1, z_2) \]

The the Milnor fiber \( F_{q,r,j} = f_{q,r,j}^{-1}(1) \) of \( f_{q,r,j} \) is connected. By the Join theorem (Cisneros-Molina [1]), \( F_{q,r,j} \) is a simply connected 2-dimensional CW-complex so that

\[ \chi(F_{q,r,j}) = -(q - 1) \chi(H_{q,r,j}) + q = q(q - 1)(q - 2) + 2q(q - 1)(r - j) + q. \]

Let \( C_{q,r,j} \) be the projective curve defined by \( \{ f_{q,r,j}(z, \bar{z}) = 0 \} \) in \( \mathbb{P}^2 \). By Corollary 4, the genus \( g(C_{q,r,j}) \) of \( C_{q,r,j} \) is given by

\[ g(C_{q,r,j}) = \frac{(q - 1)(q - 2)}{2} + (q - 1)(r - j) \geq \frac{(q - 1)(q - 2)}{2}. \]
For \( q = 2 \), we get

\[
g(C_{2,r,j}) = (r - j) \geq 0.
\]

Thus this shows that for arbitrary \( g \geq 0 \), the mixed curve \( C_{2,g+j,j} \) is a curve of genus \( g \) and the embedding degree 2. Note that \( g(C_{1,r,j}) = 0 \). Thus \( q = 1 \) gives only rational curves. Therefore to get a mixed curve with the embedding degree 1, the join type polynomials can not be used.

3. Twisted join type polynomial

In this section, we introduce a new class of mixed polar weighted polynomials which we use to construct curves with embedded degree 1. Let \( f(z, \bar{z}) \) be a polar weighted homogeneous polynomial of \( n \)-variables \( z = (z_1, \ldots, z_n) \).

Let \( Q = t(q_1, \ldots, q_n) \), \( P = t(p_1, \ldots, p_n) \) be the radial and polar weight respectively and let \( d, q \) be the radial and polar degree respectively. For simplicity, we call that \( Q' = t(q_1/d, \ldots, q_n/d) \) and \( P' = t(p_1/q, \ldots, p_n/q) \) the normalized radial weights and the normalized polar weights respectively.

Consider the mixed polynomial of \((n + 1)\)-variables:

\[
g(z, \bar{z}, w, \bar{w}) = f(z, \bar{z}) + \bar{z}_n w^a \bar{w}^b, \quad a > b.
\]

Consider the rational numbers \( q_{n+1}, p_{n+1} \) satisfying

\[
\frac{q_n}{d} + (a + b) q_{n+1} = 1, \quad -\frac{p_n}{q} + (a - b) p_{n+1} = 1.
\]

We assume that \( q_n < d \) so that \( q_{n+1}, p_{n+1} \) are positive rational numbers. The polynomial \( g \) is a polar weighted homogeneous polynomial with the normalized radial and polar weights \( \tilde{Q}' = t(q_1/d, \ldots, q_n/d, q_{n+1}) \) and \( \tilde{P}' = t(p_1/q, \ldots, p_n/q, p_{n+1}) \) respectively. The radial and polar degree of \( g \) are given by \( \text{lcm} (d, \text{denom}(q_{n+1})) \) and \( \text{lcm} (q, \text{denom}(p_{n+1})) \) where \( \text{denom}(x) \) is the denominator of \( x \in \mathbb{Q} \). We call \( g \) a twisted join of \( f(z, \bar{z}) \) and \( \bar{z}_n w^a \bar{w}^b \).

We say that \( g \) is a polar weighted homogeneous polynomial of twisted join type. A twisted join type polynomial behaves differently than the simple join type, as we will see below.

We recall that \( f(z, \bar{z}) \) is called to be 1-convenient if the restriction of \( f \) to each coordinate hyperplane \( f_i := f|_{(z_i = 0)} \) is non-trivial for \( i = 1, \ldots, n \) ([4])

**Lemma 7.** Assume that \( n \geq 2 \) and \( f \) is 1-convenient. Then

\[
\phi : \pi_1((\mathbb{C}^*)^n \setminus F^*_f) \cong \mathbb{Z}^n \times \mathbb{Z}
\]

is an isomorphism where \( \phi \) is the canonical mapping \( \phi : (\mathbb{C}^*)^n \setminus F^*_f \to (\mathbb{C}^*)^n \times (\mathbb{C} \setminus \{1\}) \) defined by \( \phi(z) = (z, f(z, \bar{z})) \) and \( F^*_f := f^{-1}(1) \cap (\mathbb{C}^*)^n \).

**Proof.** Let us use the notations:

\[
D_\delta := \{ \eta \in \mathbb{C} ||\eta|| \leq \delta \}, \quad S_\delta(1) = \{ \eta \in \mathbb{C} ||\eta - 1|| = \delta \}.
\]

Denote by \( \hat{f} \) the restriction of \( f \) to \( (\mathbb{C}^*)^n \). The fact that the mapping \( \hat{f} : (\mathbb{C}^*)^n \setminus f^{-1}(0) \to \mathbb{C}^* \) is a fibration and the inclusion \( D_{1-\varepsilon} \cup S_\varepsilon^1 \hookrightarrow \mathbb{C} \setminus \{1\} \)
is a deformation retract implies the following inclusion is also a deformation retract:
\[ \iota: \hat{f}^{-1}(D_{1-\epsilon}) \cup \hat{f}^{-1}(S_\epsilon(1)) \subset (\mathbb{C}^*)^n \setminus F_f^*, \quad 0 < \epsilon \ll 1. \]
On the other hand, \( \hat{f}^{-1}(S_\epsilon(1)) \cong f^{-1}(1 - \epsilon) \times S_1^1(1) \cong F_f^* \times S_1^1 \) and \( \pi_1(\hat{f}^{-1}(S_\epsilon(1))) \cong \pi_1(F_f^*) \times \mathbb{Z} \). The 1-convenience of \( f \) implies the homomorphism \( \iota_*: \pi_1(F_f^*) \to \pi_1((\mathbb{C}^*)^n) \) is surjective. Moreover \( \hat{f}^{-1}(D_{1-\epsilon}) \) is homotopic to \((\mathbb{C}^*)^n\), as \( D_{1-\epsilon} \hookrightarrow \mathbb{C} \) is a deformation retract. Thus the assertion follows from the van Kampen lemma, applied to the decomposition
\[
\hat{f}^{-1}(D_{1-\epsilon}) \cup \hat{f}^{-1}(S_\epsilon(1)) = \hat{f}^{-1}(D_{1-\epsilon}) \cup \hat{f}^{-1}(S_\epsilon(1)),
\]
\[
\hat{f}^{-1}(D_{1-\epsilon}) \cap \hat{f}^{-1}(S_\epsilon(1)) = \hat{f}^{-1}(1 - \epsilon) \cong F_f^*.
\]
\[ \square \]

Put \( F_{f,n} := f_n^{-1}(1) = F_f \cap \{ z_n = 0 \} \subset \mathbb{C}^{n-1} \) with \( f_n := f|_{\mathbb{C}^n \setminus \{ z_n = 0 \}} \).

**Theorem 8.** Assume that \( n \geq 2 \) and \( f \) is 1-convenient and \( g(z, \bar{z}, w, \bar{w}) \) is a twisted join polynomial as above. Then

1. the Milnor fiber of \( g \), \( F_g = g^{-1}(1) \), is simply connected.
2. The Euler characteristic of \( F_g \) is given by the formula:
\[
\chi(F_g) = -(a - b - 1)\chi(F_f) + (a - b)\chi(F_{f,n}).
\]

**Proof.** Consider \( F_g^* := F_g \cap (\mathbb{C}^*)^{n+1} \) and the projection map \( \pi: F_g^* \to (\mathbb{C}^*)^n \) defined by \((z, w) \mapsto z\). Then the image of \( F_g^* \) by \( \pi \) is \((\mathbb{C}^*)^n \setminus F_f^* \) and \( \pi: F_g^* \to (\mathbb{C}^*)^n \setminus F_f^* \) gives an \((a - b)\)-cyclic covering. In fact the fiber \( \pi^{-1}(z) \) is given as
\[
\pi^{-1}(z) = \{(z, w) | w^a\bar{w}^b = \frac{1 - f(z, \bar{z})}{z_n}\}
\]
Therefore
\[
\pi_1((\mathbb{C}^*)^n \setminus F_f^*)/\pi_2(\pi_1(F_g^*)) \cong \mathbb{Z}/(a - b)\mathbb{Z}.
\]
By Lemma 7, we see that \( \pi_1((\mathbb{C}^*)^n \setminus F_f^*) \cong \mathbb{Z}^{n+1} \) and any subgroup of \( \mathbb{Z}^{n+1} \) with a finite index is a free abelian group of the same rank \( n + 1 \). Therefore \( \pi_1(F_g^*) \cong \mathbb{Z}^{n+1} \). Note that \( g(z, \bar{z}, w, \bar{w}) \) is 1-convenient. Thus taking normal slice of each smooth divisor \( z_i = 0 \) in \( F_g \), we see that
\[
\iota_*: \pi_1(F_g^*) \to \pi_1((\mathbb{C}^*)^{n+1})
\]
is surjective. Consider the inclusion map \( \iota: F_g^* \to (\mathbb{C}^*)^{n+1} \). If \( \iota_* \) is not injective, \( \pi_1((\mathbb{C}^*)^{n+1}) \cong \pi_1(F_g^*)/\text{Ker} \iota_* \) can not be a free abelian group of rank \( n + 1 \). Thus \( \iota_*: \pi_1(F_g^*) \to \pi_1((\mathbb{C}^*)^{n+1}) \) is an isomorphism.

For the proof of the assertion (2), we apply the additivity of the Euler characteristic to the union \( F_g = F_g^{\{n\}} \cup F_{g,n} \) where \( F_g^{\{n\}} := F_g \cap \{ z_n \neq 0 \} \) and \( F_{g,n} := F_g \cap \{ z_n = 0 \} \). Note that \( F_{g,n} \cong F_{f,n} \times \mathbb{C} \). Put \( \mathbb{C}^{\{n\}} = \mathbb{C}^n \cap \{ z_n \neq 0 \} \) and \( F_f^{\{n\}} = F_f \cap \{ z_n \neq 0 \} \). In the following, we consider the projection
\[ \pi_n : \mathbb{C}^{n+1} \to \mathbb{C}^n \text{ defined by } \pi_n(z, w) = z. \]  
Note that \( \pi_n^{-1}(F_f) = F_f \times \mathbb{C} \) and \( F_g^{*}(n) \cap \pi_n^{-1}(F_f) = \{(z, 0) \mid z \in F_f^{*}(n)\} \).

\[
\chi(F_g^{*}(n)) = \chi(F_g^{*}(n) \setminus \pi_n^{-1}(F_f)) + \chi(F_g^{*}(n) \cap \pi_n^{-1}(F_f)) \\
= (a - b)\chi(\mathbb{C}^{*}(n) \setminus \pi_n^{*}(n)) + \chi(F_f^{*}(n)) \\
= -(a - b - 1)\chi(F_f^{*}(n)) \\
\chi(F_{g_n}) = \chi(F_{f_n} \times \mathbb{C}) = \chi(F_{f_n}).
\]

The last equality follows from \( F_{g_n} = F_{f_n} \times \mathbb{C} \). To complete the proof, we use the additivity of the Euler characteristic which gives the equality

\[
\chi(F_f) = \chi(F_f^{*}(n)) + \chi(F_{f_n}).
\]

\[ \square \]

3.1. Construction of a family of mixed curves with polar degree \( q \).

Now we are ready to construct a key family of mixed curves with embedding degree \( q \). Recall the polynomial:

\[
h_{q,r,j}(w, \bar{w}) := (z_1^{q+j}z_2^{q+j} - z_1^r - \alpha z_2^r - \beta z_2^r), \quad w = (z_1, z_2),
\]

\( h_{q,r,j}(w, \bar{w}) \) is 1-convenient strongly polar homogeneous polynomial with the radial degree \( q + r \) and the polar degree \( q \) respectively. The constants \( \alpha, \beta \) are generic. For this, it suffices to assume that \( |\alpha|, |\beta| \neq 0,1 \) and \( |\alpha| \neq |\beta| \).

Consider the twisted join polynomial of 3 variables \( z_1, z_2, z_3 \):

\[
s_{q,r,j}(z, \bar{z}) = h_{q,r,j}(w, \bar{w}) + \bar{z}_2 z_3^{q+r} \bar{z}_3^{r-1}, \quad z = (z_1, z_2, z_3).
\]

Let \( F_{q,r,j} = s_{q,r,j}^{-1}(1) \subset \mathbb{C}^3 \) be the Milnor fiber and let \( S_{q,r,j} \subset \mathbb{P}^2 \) be the corresponding mixed projective curve:

\[
S_{q,r,j} = \{[z] \in \mathbb{P}^2 \mid s_{q,r,j}(z, \bar{z}) = 0\}.
\]

Note that \( S_{q,r,j} \) is a smooth mixed curve. The following describes the topology of \( F_{q,r,j} \) and \( S_{q,r,j} \).

**Theorem 9.**  
1. The Euler characteristic of the Milnor fiber \( F_{q,r,j} \) is given by:
   \[
   \chi(F_{q,r,j}) = q(q^2 - q + 1 + 2(r - j)).
   \]
2. The genus of \( S_{q,r,j} \) is given by:
   \[
   g(S_{q,r,j}) = \frac{q(q - 1)}{2} + (r - j)
   \]

**Proof.** Let \( H_{q,r,j} = h_{q,r,j}^{-1}(1) \). Then by Proposition 6,

\[
\chi(H_{q,r,j}) = -q(q - 2 + 2(r - j)) \\
\chi(H_{q,r,j} \cap \{z_2 = 0\}) = q
\]

and the assertion follows from Theorem 8. \( \square \)
3.2. Mixed curves with polar degree 1. We consider the case $q = 1, j = 0$:

\[
\begin{cases}
  h(w, \bar{w}) := (z_1 + z_2)(z_1^r - \alpha z_2^r)(\bar{z}_1^r - \beta \bar{z}_2^r) \\
  f_r(z, \bar{z}) := h(w, \bar{w}) + \bar{z}_2 z_3^{r+1} \bar{z}_3^{r-1} \\
  S_r := \{ [z] \in \mathbb{P}^2 \mid f_r(z, \bar{z}) = 0 \}.
\end{cases}
\]

Corollary 10. Let $S_r$ be the mixed curve as above. Then the degree of $S_r$ is 1 and the genus of $S_r$ is $r$.

Proof. Let $F_r = f_r^{-1}(1)$ be the Milnor fiber of $f_r$. By Theorem 8, we have $\chi(F_r) = 2r + 1$. Thus by Corollary 4, the assertion follows immediately. □

Remark 11. $h(w, \bar{w})$ can be replaced by $(z_1^{r+1} - z_2^{r+1})(\bar{z}_1 - \beta \bar{z}_2^r)$ without changing the topology.

4. Further embeddings of smooth curves

Consider a smooth curve $C \subset \mathbb{P}^2$ with genus $g$. If $C$ is a complex algebraic curve of degree $q$, they are related by the Plücker formula $g = \frac{(q-1)(q-2)}{2}$. In particular, $q$ is the positive integer root of $x^2 - 3x + 2 = 0$. Thus for a given $g \geq 1$, $q$ is unique if it exists. In this section, we consider this problem in the category of mixed projective curves. Consider the family of mixed curves

\[ S_{q,r,1} : h_{q,r,1}(w, \bar{w}) + \bar{z}_2 z_3^{q+r} \bar{z}_3^{r-1} \]

We have shown that the genus $g$ is given as follows.

\[ g = \frac{q(q-1)}{2} + r - 1. \]

Assume that $g$ is fixed and we consider the possible degree $q$. We can solve as

\[ r = g - \frac{q(q-1)}{2} + 1. \]

This shows that

Theorem 12. For a given $g > 0$ and $q$ which satisfies the inequality

\[ g \geq \frac{q(q-1)}{2}, \]

the mixed curve $S_{q,r,1}$ with $r = g - \frac{q(q-1)}{2} + 1$ has genus $g$ and degree $q$.

Remark 13. Assume that

\[ \left(\#\right) \quad \frac{q(q-1)}{2} \geq g \geq \frac{(q-1)(q-2)}{2}. \]

For the construction of a curve with \{g, q\} satisfying (\#), we can not use the surface $S_{q,r,1}$. If $g - \frac{(q-1)(q-2)}{2} \equiv 0 \mod q - 1$, we can use the mixed curve $C_{q,r,1}$. If $g \not\equiv \frac{(q-1)(q-2)}{2} \mod q - 1$, we do not know if such an embedding exists.
Let us consider mixed polar weighted homogeneous polynomials of \( n \) variables with polar degree 1. They have the following strong property:

**Theorem 14.** Let \( f(z, \bar{z}) \) be a polar weighted homogeneous polynomial of degree 1 of radial weight \((q_1, \ldots, q_n; d)\) and polar weight \((p_1, \ldots, p_n; 1)\). Then the Milnor fibration \( \varphi = f/|f| : S^{2n-1} \setminus K \to S^1 \) with \( K = f^{-1}(0) \cap S^{2n-1} \) is trivial. In fact, the explicit diffeomorphism is given using the one-parameter family of diffeomorphisms of the monodromy flows \( h_\theta : F \to F_\theta \) with \( \theta \in \mathbb{R} \) and \( F_\theta := \varphi^{-1}(\exp(i\theta)) \) and

\[
h_\theta(z) = \exp(i\theta) \circ z
\]

where \( \rho \circ z = (\rho^1 z_1, \ldots, \rho^n z_n) \) and \( \rho \in S^1 \). Note that \( h_{2\pi} = \text{id} \). The trivialization of the fibration is given by the diffeomorphism \( \psi : F \times S^1 \to S^{2n-1} \setminus K \) which is defined by

\[
\psi(z, \exp(i\theta)) = h_\theta(z)
\]

Observe that the trivialization is not an extension of the trivialization of the normal bundle of \( K \) in \( S^{2n-1} \).

**Corollary 15.** Let \( f(w), \ w = (z_1, z_2) \) be a polar weighted homogeneous polynomial with polar degree 1. Then the link \( K := f^{-1}(0) \cap S^3 \) is trivially fibered over the circle. Thus we have

\[
\pi_1(S^3 \setminus K) \cong \mathbb{Z} \times \pi_1(F)
\]

where \( F \) is the Milnor fiber.

Let \( f(z, \bar{z}) \) be a polar weighted homogeneous polynomial of \( n \) variables. On the topology of the hypersurface \( F = f^{-1}(1) \), we propose the following basic question.

*Is the homological (or homotopical) dimension of \( F \) is \( n-1 \) under a certain condition (say mixed non-degeneracy)?*

We say that \( f(z, \bar{z}) \) satisfies the **homological dimension property** if the assertion is satisfied for \( F = f^{-1}(1) \). There are several cases in which the assertion is true.

1. **Simplicial type:** Assume that \( f(z, \bar{z}) \) is a simplicial type polar weighted homogeneous polynomial. Then the homological dimension of \( F \) is at most \( n-1 \). This follows from Theorem 10, [4].

2. **(Join type)** Assume that \( f(z, \bar{z}) = h(w, \bar{w}) + k(u, \bar{u}) \) where \( w = (w_1, \ldots, w_m), u = (u_1, \ldots, u_\ell) \) and \( z = (w, u) \). Assume that \( h(w, \bar{w}), k(u, \bar{u}) \) are polar weighted homogeneous polynomials which satisfies the homological dimension property. Then \( f \) also satisfies the property. This follows from the Join theorem by Cisneros Molino [1].
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