ON THE HOMOLOGY OF THE COMMUTATOR SUBGROUP OF
THE PURE BRAID GROUP

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Abstract. We study the homology of \([P_n, P_n]\), the commutator subgroup of
the pure braid group on \(n\) strands, and show that \(H_i([P_n, P_n])\) contains a
free abelian group of infinite rank for all \(1 \leq i \leq n - 2\). As a consequence
we determine the cohomological dimension of \([P_n, P_n]\): for \(n \geq 2\) we have
cd([P_n, P_n]) = n - 2.

1. Introduction

Let \(n \geq 2\) and denote by \(F_n\) the ordered configuration space of \(n\) points in the
complex plane:

\[ F_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \forall i \neq j\}. \]

The pure braid group on \(n\) strands is defined as \(P_n = \pi_1(F_n)\).

In [3] David Recio-Mitter and the author posed the question of determining the
cohomological dimension of \([P_n, P_n]\), the commutator subgroup of the pure braid
group, and conjectured that, for \(n \geq 2\),

\[ \text{cd}([P_n, P_n]) = n - 2. \]

In this work we prove this conjecture by computing a large part of the homology
of \([P_n, P_n]\); in particular we prove that \(H_*([P_n, P_n])\) contains a free abelian group
of infinite rank in all degrees \(1 \leq * \leq n - 2\) (see Theorem 6.1, Corollary 6.2 and
Theorem 8.1).

To the best of the author’s knowledge there is no result in the literature concerning
the homology of \([P_n, P_n]\) for large values of \(n\); on the contrary the homology of the
commutator subgroup of Artin’s full braid group [2] has been extensively studied
[12, 17, 6, 4], as well as the homology of Milnor fibers of discriminant fibrations
associated with other hyperplane arrangements in \(\mathbb{C}^n\) [7, 8, 5, 21].

Our strategy is the following. We consider the Salvetti complex \(\text{Sal}_n\) associated
with the \(n\)-th braid arrangement: the cell complex \(\text{Sal}_n\) is a classifying space for
\(P_n\), and it has a covering \(\text{Sal}^\log_n\) which is a classifying space for \([P_n, P_n]\).

The group \(P_{ab} \cong \mathbb{Z}^{\binom{n}{2}}\) acts on \(\text{Sal}^\log_n\) by deck transformations, and the action is cellular: hence
the associated cellular chain complex \(\text{Ch}^{\log}_{\bullet}\) is a chain complex of modules over the
commutative ring \(\mathbb{Z}[[P_{ab}]]\), and consequently the homology \(H_*([P_n, P_n])\) is also a
\(\mathbb{Z}[[P_{ab}]]\)-module.

We replace \(\text{Ch}^{\log}_{\bullet}\) with a homotopy equivalent subcomplex \(\text{Ch}^{\log}_{\bullet}\); the chain complex
\(\text{Ch}^{\log}_{\bullet}\) is only invariant for the action of a certain subgroup \(\mathbb{Z}^{\binom{n}{2}} - 1 \subset P_{ab}\), and we
restrict this action also in homology, i.e. we consider \( H_*(\{P_n, P_n\}) \) as a module over the commutative ring \( \mathbb{Z} \left[ \mathbb{Z}^{(2)} \right] \).

We define a filtration on \( \mathcal{C}_n \); the associated Leray spectral sequence, after localisation to the quotient field of \( \mathbb{Z} \left[ \mathbb{Z}^{(2)} \right] \), collapses on its first page: more precisely we have \( E^1_{p,q} = 0 \) for all \((p, q) \neq (n - 2, 0)\). This proves the statement for \( H_{n-2}(\{P_n, P_n\}) \) (see Theorem 6.1).

To prove the statement in lower degrees we consider the interaction between commutator subgroups of different pure braid groups. We present two proofs. The first proof is group-theoretic and rather simple. In the second proof we replace \( S\mathcal{A}^{\log}_n \) with a homotopy equivalent space \( \mathcal{C}^{\log}_2(n) \). The collection of spaces \( \left( \mathcal{C}^{\log}_2(n) \right)_{n \geq 0} \) forms a non-symmetric operad, and we use the composition maps to translate homology classes from one space to another (see Theorem 8.1).

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2. Preliminaries

We recall some classical constructions and results about configuration spaces and pure braid groups.

For all \( 1 \leq i \leq n + 1 \) there is a map \( \varphi_i : F_{n+1} \to F_n \), which forgets the \( i \)-th point of each configuration. This is a fiber bundle with fiber the punctured plane \( \mathbb{C} \setminus \{n \text{ points}\} \), called the Fadell-Neuwirth fibration (see [10]):

\[
\mathbb{C} \setminus \{n \text{ points}\} \to F_{n+1} \xrightarrow{\varphi_i} F_n.
\]

The space \( \mathbb{C} \setminus \{n \text{ points}\} \) is a classifying space for the free group on \( n \) generators \( \mathbb{Z}^* \), in particular it is an aspherical space. An induction argument shows that \( F_n \) is also aspherical, and therefore \( F_n \) is a classifying space for its fundamental group \( P_n \). We obtain a short exact sequence

\[
1 \to \mathbb{Z}^n \to P_{n+1} \to P_n \to 1.
\]

**Definition 2.1.** For all \( 1 \leq i < j \leq n \) there is a forgetful map \( \psi_{ij} : F_n \to F_2 \), which forgets all points of a configuration except the \( i \)-th and the \( j \)-th. This map of spaces induces a map, that we still call \( \psi_{ij} \), on fundamental groups:

\[
\psi_{ij} : P_n \to P_2 \simeq \mathbb{Z}.
\]

The collection of all these maps gives a homomorphism of groups \( \psi : P_n \to \mathbb{Z}^{(2)} \).

A classical result by Arnold [1] states that \( \psi \) is the abelianisation homomorphism, i.e. \( P_n^{ab} \simeq \mathbb{Z}^{(2)} \) along the map induced by \( \psi \). In this article we focus on the group \([P_n, P_n] = \ker \psi\), the commutator subgroup of the pure braid group.

3. Two classifying spaces for \([P_n, P_n]\)

We introduce two convenient models for the classifying space of \([P_n, P_n]\).
**Definition 3.1.** We define the space $F_n^{\text{log}}$. A point in $F_n^{\text{log}}$ is determined by a configuration $(z_1, \ldots, z_n) \in F_n$ together with a choice $w_{ij} \in \mathbb{C}$ of a logarithm of $(z_j - z_i)$ for all $i < j$:
\[
F_n^{\text{log}} = \left\{ \left( (z_i)_{1 \leq i \leq n}, (w_{ij})_{1 \leq i < j \leq n} \right) \mid z_j - z_i = e^{w_{ij}} \forall 1 \leq i < j \leq n \right\}.
\]
This space has a topology as subspace of $\mathbb{C}^n \times C\binom{n}{2}$.

There is a covering map $p: F_n^{\text{log}} \to F_n$, which forgets the numbers $w_{ij}$. The fiber is isomorphic to $\mathbb{Z}\binom{2}{n}$: to see this fix a point $((z_i), (\bar{w}_{ij}))$ lying over some point $(z_i) \in F_n$. Let $((z_i), (w_{ij}))$ be any other point lying over $(z_i)$: then there are integers $(k_{ij})_{1 \leq i < j \leq n}$ such that $w_{ij} - \bar{w}_{ij} = 2\pi \sqrt{-1}k_{ij}$ for all $1 \leq i < j \leq n$. Vice versa given integers $(k_{ij})_{1 \leq i < j \leq n}$ one can define a point $((z_i), (w_{ij}))$ in the fiber of $(z_i)$ by setting $w_{ij} = \bar{w}_{ij} + 2\pi \sqrt{-1}k_{ij}$ for all $1 \leq i < j \leq n$.

The last construction gives a free action of $\mathbb{Z}\binom{2}{n}$ on $F_n^{\text{log}}$: this is an action by deck transformations of $p$ and is transitive on fibers of $p$: therefore $\mathbb{Z}\binom{2}{n}$ is the whole group of deck transformations of $p$ and there is a short exact sequence
\[
1 \to \pi_1(F_n^{\text{log}}) \to \pi_1(F_n) \to \mathbb{Z}\binom{2}{n} \to 1.
\]
We can then conclude that $[P_n, P_n] \subseteq \pi_1(F_n^{\text{log}})$, because $[P_n, P_n]$ is contained in the kernel of any map from $P_n$ to an abelian group.

On the other hand the maps $\psi_{ij} : F_n \to F_2$ lift to maps $\psi_{ij}^{\text{log}} : F_n^{\text{log}} \to F_2^{\text{log}}$: the map $\psi_{ij}^{\text{log}}$ is defined by forgetting all data except $z_i, z_j$ and $w_{ij}$.

The space $F_2^{\text{log}}$ is contractible: this is a particular case of Lemma 3.7 and can be checked also directly. Therefore $\pi_1(F_2^{\text{log}})$ is a subgroup of $P_n$ contained in the kernel of all maps $\psi_{ij}$, i.e. $\pi_1(F_2^{\text{log}}) \subseteq [P_n, P_n]$. We obtain the following lemma.

**Lemma 3.2.** The space $F_n^{\text{log}}$ is a classifying space for the group $[P_n, P_n]$.

The action of $P_n^{ab}$ on $F_n^{\text{log}}$ induces an action of the ring $\mathbb{Z}[P_n^{ab}]$ on $H_*(F_n^{\text{log}})$, so our first attempt is to study $H_*(F_n^{\text{log}}) = H_*([P_n, P_n])$ as a module over this ring.

**Definition 3.3.** Let $R(n) = \mathbb{Z}[P_n^{ab}]$ be the ring of Laurent polynomials in $\binom{n}{2}$ variables $\mathbb{Z}[t_{ij}]_{1 \leq i < j \leq n}$. The variable $t_{ij}$ corresponds to the generator of $P_n^{ab}$ which is dual to the map $\psi_{ij} : P_n \to P_2$, i.e. for all $i < j$ and $k < l$ we have $\psi_{ij}(t_{kl}) = \delta_{ik} \delta_{jl}$.

The ring $R(n)$ is a domain and we call $\mathbb{K}(n)$ its quotient field.

The following lemma tells us that $H_*(F_n^{\text{log}})$ cannot be too large.

**Lemma 3.4.**
\[
H_*(F_n^{\text{log}}) \otimes_R \mathbb{K}(n) = 0.
\]

**Proof.** Consider the following homotopy $H : F_n^{\text{log}} \times [0, 2\pi] \to F_n^{\text{log}}$ of the space $F_n^{\text{log}}$ into itself. At time $0$, the map $H(\cdot, 0)$ is the identity of $F_n^{\text{log}}$; at time $\theta$ we rotate each configuration by an angle $\theta$ counterclockwise, adjusting logarithms:
\[
H \left( \left( (z_i), (w_{ij}) \right); \theta \right) = \left( e^{\theta \sqrt{-1}} z_i, (w_{ij} + \theta \sqrt{-1}) \right).
\]
At time $2\pi$, the map $H(\cdot; 2\pi)$ preserves all $z_i$’s and shifts all $w_{ij}$’s by $2\pi\sqrt{-1}$: this last map is precisely the map

$$\prod_{1 \leq i < j \leq n} t_{ij} : F_n^{\text{log}} \to F_n^{\text{log}},$$

i.e. the product of all deck transformations $t_{ij} : F_n^{\text{log}} \to F_n^{\text{log}}$.

Since $\prod_{1 \leq i < j \leq n} t_{ij}$ is homotopic to the identity of $F_n^{\text{log}}$, it induces the identity map on $H_*(F_n^{\text{log}})$.

Hence $H_*(F_n^{\text{log}})$, as a $R(n)$-module, is $[\prod_{1 \leq i < j \leq n} t_{ij} - 1]$-torsion, in particular its $\mathbb{K}(n)$-localisation vanishes. \hfill \Box

The proof of the previous lemma tells us that the variable $t_{12}$ acts on $H_*(([P_n, P_n]))$ as the product $\prod_{(i < j) \neq (1,2)} t_{ij}^{-1}$; therefore it seems convenient to replace $R(n)$ with a smaller ring, containing one variable less.

**Definition 3.5.** We call

$$\bar{R}(n) = \mathbb{Z}\left[\ell^{\pm 1}_{ij} | 1 \leq i < j \leq n, (i, j) \neq (1, 2)\right]$$

the ring of Laurent polynomials in $\binom{n}{2} - 1$ variables. $\bar{R}(n)$ is naturally a subring of $R(n)$ by identifying each $\ell_{ij}$ with the corresponding $t_{ij}$, and therefore each $R(n)$-module is also a $\bar{R}(n)$-module.

We can also identify $\bar{R}(n)$ as the quotient of $R(n)$ by the ideal generated by the element $[\prod_{1 \leq i < j \leq n} t_{ij}] - 1$. The composition of maps of rings $\bar{R}(n) \subset R(n) \to \bar{R}(n)$ is the identity of $\bar{R}(n)$.

The ring $\bar{R}(n)$ is a domain, and we call $\mathbb{K}(n)$ its quotient field.

We want now to study $H_*([P_n, P_n])$ as a $\bar{R}(n)$-module. We introduce our second model of a classifying space for $[P_n, P_n]$.

**Definition 3.6.** The space $\bar{F}_n^{\text{log}}$ is defined as the subspace of $F_n^{\text{log}}$ of configurations $((z_i), (w_{ij}))$ such that $z_2 = 1$, $z_1 = 0$ and $w_{12} = 0$.

The space $\bar{F}_n^{\text{log}}$ is not invariant under the action of the whole group $P_n^{ab}$ on $F_n^{\text{log}}$: the action of $t_{12}$ consists in shifting $w_{12}$ by $2\pi\sqrt{-1}$, and this is not allowed inside $\bar{F}_n^{\text{log}}$. The other generators $l_{ij}$ of $P_n^{ab}$ preserve $\bar{F}_n^{\text{log}}$: we conclude that $H_*(\bar{F}_n^{\text{log}})$ has a natural structure of $\bar{R}(n)$-module, and the inclusion map $\bar{F}_n^{\text{log}} \subset F_n^{\text{log}}$ induces a map of $\bar{R}(n)$-modules in homology.

**Lemma 3.7.** $\bar{F}_n^{\text{log}}$ is a deformation retract of $F_n^{\text{log}}$, and therefore it is also a classifying space for $[P_n, P_n]$.

**Proof.** We define a homotopy $H : F_n^{\text{log}} \times [0, 1] \to F_n^{\text{log}}$ starting with the identity of $F_n^{\text{log}}$ and ending with a retraction onto $\bar{F}_n^{\text{log}}$: the space $\bar{F}_n^{\text{log}}$ will be fixed pointwise throughout the homotopy.

Let $((z_i), (w_{ij})) \in F_n^{\text{log}}$. Then

$$H(((z_i), (w_{ij})); t) = \left((e^{-tw_{12}} \cdot (z_i - tw_{12})), (w_{ij} - tw_{12})\right).$$

\hfill \Box
4. Chain complexes

In this section let \( n \geq 2 \) be fixed. Our next aim is to describe explicitly a chain complex that computes the homology of \( F_n^{\log} \). We first recall the classical chain complex computing the homology of \( F_n \): it can be seen both as the dual of the reduced cochain complex of the one-point-compactification of \( F_n \), in the spirit of Fuchs [13], or as chain complex associated with the Salvetti complex [20] of the \( n \)-th braid arrangement.

**Definition 4.1.** An ordered partition of \( \{1, \ldots, n\} \) of degree \( 1 \leq k \leq n \) is a partition of \( \{1, \ldots, n\} \) into \( n-k \) non-empty subsets \( (\pi_1, \ldots, \pi_{n-k}) \), where each piece \( \pi_r \) is endowed with a total order.

For \( a, b \in \pi_r \), we write \( a \prec b \) if \( a \) precedes \( b \) in the order associated with \( \pi_r \), and we keep writing \( a \prec b \) if \( a \) is smaller than \( b \) as natural numbers.

We define the chain complex \( \mathcal{C}_n^* = \mathcal{C}_n^*(n) \). Let \( \mathcal{C}_n^k \) be the free abelian group (also called *cell*) for each ordered partition of \( \{1, \ldots, n\} \) of degree \( k \).

In order to describe the boundary maps of \( \mathcal{C}_n^* \), it is enough, for any two ordered partitions \( (\pi_r)_1 \leq \pi_{n-k} \) and \( (\pi'_r)_1 \leq \pi_{n-k+1} \) of degree \( k \) and \( k-1 \) respectively, to give a formula for the boundary index \( [\partial(\pi_r); (\pi'_r)] \), i.e. the coefficient of \( (\pi'_r) \) in \( \partial(\pi_r) \). There are two possibilities:

- \((\pi'_r)\) is obtained from \((\pi_r)\) by
  - splitting some piece \( \pi_l \) into two pieces \( \pi'_l \) and \( \pi'_{l+1} \), each having as total order the restriction of the total order on \( \pi_l \);
  - setting \( \pi'_l = \pi_r \) for \( r < l \) and \( \pi'_r = \pi_{r-1} \) for \( r > l+1 \), with the same total orders.

Then

\[
[\partial(\pi_r); (\pi'_r)] = (-1)^{i+\text{sgn}(\pi_l; \pi'_l)} = \pm 1,
\]

where, for an ordered set \((A, \prec)\) and a subset \( B \), we define \( \text{sgn}(A, B) \) as the parity of the number of couples \((a, b)\) of elements of \( A \) with \( b \prec a, b \in B \) and \( a \notin B \).

- \((\pi'_r)\) is not obtained from \((\pi_r)\) as before. Then \([\partial(\pi_r); (\pi'_r)] = 0\).

The chain complex \( \mathcal{C}_n^* \) is the cellular chain complex of the Salvetti complex \( \text{Sal}_n \); it is a finite cell complex contained in \( F_n \), onto which \( F_n \) deformation retracts [20]. Alternatively, in the spirit of Fuchs [13], one can consider the following stratification of \( F_n \). For every ordered partition \( (\pi_r)_1 \leq \pi_{n-k} \) of some degree \( k \), we consider the subspace \( e(\pi_r) \subset F_n \) consisting of all configurations \((z_1, \ldots, z_n)\) satisfying the following properties:

- there are exactly \( n-k \) vertical lines in \( \mathbb{C} \) passing through some of the \( n \) points;
- for \( 1 \leq r \leq n-k \), the \( r \)-th vertical line from left contains precisely the points \( z_i \) with \( i \in \pi_r \), and these points are assembled from the top to the bottom according to the total order \( \prec \).

In particular for configurations \((z_i) \in e(\pi_r)\) the following properites hold:

- for all \( i \neq j \), if \( i \in \pi_l \) and \( j \in \pi_{l'} \) with \( l < l' \), the point \( z_i \) lies on left of the point \( z_j \), i.e. \( \Re(z_i) < \Re(z_j) \);
- for all \( i \neq j \), if both \( i \) and \( j \) belong to the same piece \( \pi_l \) and if \( i < j \), then \( z_i \) lies above \( z_j \), or equivalently \( \Im(z_i) > \Im(z_j) \).

The one-point compactification \( F_n^+ \) of \( F_n \) has a CW structure given by the subspaces \( e(\pi_r) \) together with the point at infinity \( \infty \). The associated reduced cellular cochain
complex is precisely the one described in Definition 4.1. Note that each cell $e(\pi_r)$ is modeled on the interior of a product of simplices

$$\Delta^{n-k} \times \Delta^{[\pi_1]} \times \cdots \times \Delta^{[\pi_n-k]}.$$  

The local coordinates are the horizontal positions of the $n-k$ vertical lines and the vertical positions of the points $z_l$ on these lines. We regard $e(\pi_r)$ as a manifold of dimension $2n-k$; an orientation can be given by declaring a total order on the simplicial local coordinates, and we choose the lexicographic order associated with the product structure written above.

With this convention, the boundary index $[\partial e(\pi'_r)]: e(\pi_r)]$ in the reduced cellular chain complex of $F_n^+$ equals the formula for $[\partial(\pi_r)]: (\pi'_r)]$ in Definition 4.1.

The space $F_n$ is a $2n$-dimensional manifold and its stratification by the subspaces $e(\pi_r)$ gives rise to a Poincaré-dual cell complex, which is exactly the Salvetti complex $Sal_n$.

The space $Sal_n$ has a covering $Sal^{log}_n$ corresponding to the subgroup $[P_n,P_n]$ of its fundamental group $P_n$, and we can lift to $Sal^{log}_n$ the cell complex structure on $Sal_n$. The group of deck transformations $P^{log}_n$ acts freely on the cells of $Sal^{log}_n$; the associated chain complex is a chain complex of finitely generated, free $R(n)$-modules.

**Definition 4.2.** We define a chain complex $\mathcal{C}_k^{log}$. Let $\mathcal{C}_k^{log}$ be the free abelian group with one generator (called cell) for each choice of the following set of data:

- an ordered partition $((\pi_r)_{1 \leq r \leq n-k})$ of $\{1, \ldots, n\}$ of degree $k$;
- integers $W_{ij} \in \mathbb{Z}$ for all $1 \leq i < j \leq n$.

The boundary map has a similar formula as in Definition 4.1. Consider cells $(\pi_r,W_{ij})_{1 \leq r \leq n-k}$ and $(\pi'_r,W'_{ij})_{1 \leq r \leq n-k+1}$ in degrees $k$ and $k-1$ respectively.

- If $(\pi'_r,W'_{ij})$ cannot be obtained from $(\pi_r,W_{ij})$ as in the first case in Definition 4.1, splitting some $\pi_i$ into $\pi'_i$ and $\pi'_{i+1}$. Suppose that for all $i<j$ satisfying
  - $i,j \in \pi_i$;
  - $i < j$ in $\pi_i$;
  - $i \in \pi'_i$ and $j \in \pi'_{i+1}$

  we have $W'_{ij} = W_{ij} + 1$. Finally, suppose that for all other couples of indices $i<j$ we have $W'_{ij} = W_{ij}$. Then

  $$[\partial(\pi_r,W_{ij}): (\pi'_r,W'_{ij})] = (-1)^{l+\text{sgn}(\pi_i;\pi'_j)} = \pm 1.$$  

- If $(\pi'_r,W'_{ij})$ cannot be obtained from $(\pi_r,W_{ij})$ as before, then the boundary index is zero.

Similarly as before, we can stratify $F^{log}_n$ as follows: for all $(\pi_r,W_{ij})_{1 \leq r \leq n-k}$ as in Definition 4.2 consider the subspace $e(\pi_r,W_{ij})$ of $F^{log}_n$ determined by the following properties:

- $e(\pi_r,W_{ij})$ is a connected component of $p^{-1}(e(\pi_r))$, where $p: F^{log}_n \to F_n$ is the usual covering map;
- for all $i<j$, there exists a configuration $((z_l),(w_{ij})) \in e(\pi_r,W_{ij})$, depending on $i$ and $j$, such that one of the following four situations occurs, depending on the position of $i$ and $j$ in the ordered partition $((\pi_r))$:
  - $z_j = z_i + 1$ and $w_{ij} = 2\pi \sqrt{-1}(W_{ij})$, assuming $i \in \pi_l$ and $j \in \pi_{l'}$ for some $l<l'$;
z_j = z_i + \sqrt{-1} and \( w_{ij} = 2\pi \sqrt{-1}(W_{ij} + \frac{i}{j}) \), assuming \( i, j \in \pi_l \) for some \( l \), and \( j < i \);
- \( z_j = z_i - 1 \) and \( w_{ij} = 2\pi \sqrt{-1}(W_{ij} + \frac{j}{i}) \), assuming \( i \in \pi_l \) and \( j \in \pi_{l'} \) for some \( l > l' \);
- \( z_j = z_i - \sqrt{-1} \) and \( w_{ij} = 2\pi \sqrt{-1}(W_{ij} + \frac{i}{j}) \), assuming \( i, j \in \pi_l \) for some \( l \), and \( j < i \).

This stratification is the pull-back along \( p \) of the stratification on \( F_n \). We can add a point \( \infty \) to \( F_n^{\log} \) and obtain a space \((F_n^{\log})^+\) with a CW structure with the cells \( e(\pi_r, W_{ij}) \) together with the point \( \infty \).

The space \((F_n^{\log})^+\) is not the one-point compactification of \( F_n^{\log} \), but it is universal among topological spaces satisfying the following properties:

- \((F_n^{\log})^+\) is obtained from \( F_n^{\log} \) by adding one point \( \infty \);
- for every \( X \subset F_n^{\log} \) meeting finitely many strata \( e(\pi_r, (W_{ij})) \), the closure of \( X \) in \((F_n^{\log})^+\) is compact.

The genuine one-point compactification of \((F_n^{\log})\) would have a coarser topology than \((F_n^{\log})^+\), and in particular it would not have the topology of a CW complex.

The chain complex \( \mathcal{C}_h^{\log} \) coincides with the complex of reduced, compactly supported cochains of \((F_n^{\log})^+\); the formulas for the indices are the same because we lift the canonical orientations of cells \( e(\pi_r) \subset F_n \) to their preimages along \( p \). The manifold \( F_n^{\log} \) is stratified by the spaces \( e(\pi_r, W_{ij}) \) and there is a Poincaré dual cell complex, which is precisely the covering \( Sal_n^{\log} \) of the Salvetti complex \( Sal_n \).

Putting together all \( \mathbb{Z} \)-summands generated by cells \((\pi_r, W_{ij}) \in \mathcal{C}_h^{\log}\) for fixed \( \pi_r \) and varying \( W_{ij} \) we obtain one \( R(\pi_r) \)-summand of \( \mathcal{C}_h^{\log} \): the action of \( F_n^{\log} \) on this summand is analogous to the one discussed for the space \( F_n^{\log} \) (see the discussion preceding Lemma 3.2): multiplication times \( t_{kl} \) consists in shifting the number \( W_{kl} \) by 1, while keeping the other numbers \( W_{ij} \) as well as the ordered partition \( \pi_r \).

We note that \( \mathcal{C}_h^{\log} \) is a chain complex of finitely generated, free \( R(\pi_r) \)-modules; a \( R(\pi_r) \)-basis is given by those elements \((\pi_r, W_{ij}) \in \mathcal{C}_h^{\log}\) with \( W_{ij} = 0 \) for all \( i, j \); we call these basis elements \((\pi_r, 0) \in \mathcal{C}_h^{\log}\) to distinguish them from the elements \((\pi_r) \in \mathcal{C}_h\) generating \( \mathcal{C}_h \) over \( \mathbb{Z} \).

The differentials of \( \mathcal{C}_h^{\log} \) with respect to the basis of the elements \((\pi_r, 0) \) are expressed in a similar way as in Definition 4.2, but boundary indices are no longer always equal to 0 or \( \pm 1 \), rather they can take the form of a product of some variables \( \pm \) determined in the same way as in Definition 4.2. It is however still true that all boundary indices of \( \mathcal{C}_h^{\log} \) are either 0 or invertible elements of \( R(\pi_r) \).

There is a natural map \( \mathcal{C}_h^{\log} \to \mathcal{C}_h \) of chain complexes of abelian groups, mapping the generator \((\pi_r, W_{ij}) \) to the generator \((\pi_r)\): this map is induced by the covering map \( Sal_n^{\log} \to Sal_n \), which by construction is a cellular map.

**Definition 4.3.** The chain complex \( \mathcal{C}_h^{\log} \) contains a subcomplex \( \hat{\mathcal{C}}_h^{\log} \) of free abelian groups generated by cells \((\pi_r, W_{ij}) \) such that:
- there are indices \( l < l' \) with \( 1 \in \pi_l \) and \( 2 \in \pi_{l'} \);
- \( W_{12} = 0 \).
Note that $\mathcal{C}_n^\log$ is a subcomplex of abelian groups, and in particular is closed along boundary maps: if $(\pi_r, W_{ij})$ is a generator of $\mathcal{C}_n^\log$, then 1 and 2 already belong to different pieces of the partition $(\pi_r)$, so that $W_{12}$ cannot change along boundaries, according to Definition 4.2. The degrees of cells in $\mathcal{C}_n^\log$ range from 0 to $n - 2$, because there are always at least two pieces in the partition.

**Lemma 4.4.** The chain complex $\mathcal{C}_n^\log$ computes the homology of $\tilde{F}_n^\log$.

**Proof.** The space $\tilde{F}_n^\log$ can be also defined as follows. Let $F_{n-2}(\mathbb{C} \setminus \{0, 1\})$ be the subspace of $F_n$ of configurations $(z_1, \ldots, z_n)$ with $z_1 = 0$ and $z_2 = 1$. The space $F_{n-2}(\mathbb{C} \setminus \{0, 1\})$ is the ordered configuration space of $n-2$ points in the 2-punctured plane, so it is the fiber over $(z_1 = 0, z_2 = 1)$ of the bundle map $\psi_{12} : F_n \to F_2$ forgetting all points but the first two (see Definition 2.1).

The space $F_{n-2}(\mathbb{C} \setminus \{0, 1\})$ is aspherical, and its fundamental group is the kernel of the map induced by $\psi_{12}$ on fundamental groups; moreover $H_1(F_{n-2}(\mathbb{C} \setminus \{0, 1\})) \simeq \mathbb{Z}^{n-2}/\mathbb{Z}^2$, where the isomorphism is exhibited by the collection of maps $\psi_{ij}$ with $(i, j) \neq (1, 2)$.

The commutator subgroup of $\pi_1(F_{n-2}(\mathbb{C} \setminus \{0, 1\}))$ can be identified with $[F_n, F_n]$, and $\tilde{F}_n^\log$ is the covering of $F_{n-2}(\mathbb{C} \setminus \{0, 1\})$ corresponding to this group.

The space $F_{n-2}(\mathbb{C} \setminus \{0, 1\})$ is the complement in $\mathbb{C}^{n-2}$ of a hyperplane arrangement: using $z_3, \ldots, z_n$ as coordinates of $\mathbb{C}^{n-2}$ we are considering the following hyperplanes with real equations

- $z_i = 0$, for $3 \leq i \leq n$;
- $z_i = 1$, for $3 \leq i \leq n$;
- $z_i = z_j$, for $3 \leq i < j \leq n$.

Hence also $F_{n-2}(\mathbb{C} \setminus \{0, 1\})$ deformation retracts onto a Salvetti complex, that we call $\text{Sal}_n \subset F_{n-2}(\mathbb{C} \setminus \{0, 1\})$. Using the definition of the Salvetti complex [20] it is straightforward to check that the cellular chain complex $\mathcal{C}_n$ of $\text{Sal}_n$ is isomorphic to the subcomplex of $\mathcal{C}_n^\log$ generated by cells $(\pi_r)$ satisfying the first condition of Definition 4.3.

Another possibility is the following. For every ordered partition $(\pi_r)$ satisfying the first condition of Definition 4.3 we can consider the subspace $\tilde{e}(\pi_r) \subset F_{n-2}(\mathbb{C} \setminus \{0, 1\})$ containing configurations $(z_3, \ldots, z_n)$ such that the point $(0, 1, z_3, \ldots, z_n) \in F_n$ belongs to the subspace $e(\pi_r)$. The subspaces $\tilde{e}(\pi_r)$, together with the point at infinity $\infty$, give a CW structure of the one-point compactification $F_{n-2}(\mathbb{C} \setminus \{0, 1\})^+$ of the manifold $F_{n-2}(\mathbb{C} \setminus \{0, 1\})$. The reduced cellular cochain complex $\mathcal{C}_n$ of the space $F_{n-2}(\mathbb{C} \setminus \{0, 1\})^+$ is by construction isomorphic to the subcomplex of $\mathcal{C}_n$ generated by cells $(\pi_r)$ satisfying the first condition of Definition 4.3 up to a shift in dimension due to the fact that $F_n$ has (real) dimension $2n$, whereas $F_{n-2}(\mathbb{C} \setminus \{0, 1\})$ has dimension $2n - 4$. The Salvetti complex $\text{Sal}_n$ is the Poincaré dual of the cell decomposition of $F_{n-2}(\mathbb{C} \setminus \{0, 1\})^+$, and its cellular chain complex is also isomorphic to $\mathcal{C}_n$.

We can now restrict the covering $p : F_n^\log \to F_n$ first to a connected covering $p : F_n^\log \to F_{n-2}(\mathbb{C} \setminus \{0, 1\})$, and then to a connected covering $\tilde{\text{Sal}}_n^\log \to \text{Sal}_n$. Note that $F_n^\log$ is only one connected component of $p^{-1}(F_{n-2}(\mathbb{C} \setminus \{0, 1\})) \subset F_n^\log$: there is indeed one connected component for any fixed value of $w_{12} \in 2\pi \sqrt{-1}\mathbb{Z}$.
We pull back the cell structure on $\tilde{Sal}_n$ along $p$ to a cell structure on $\tilde{Sal}_n^{\log}$; thus the chain complex associated with $\tilde{Sal}_n^{\log}$ is precisely $\tilde{\mathcal{C}}h_\bullet^{\log}$. □

We define filtrations on the chain complexes that we have introduced.

**Definition 4.5.** For each generator $(\pi_r)_{1 \leq r \leq n-k}$ of $\mathcal{C}h_\bullet$ there is an index $l$ such that $1 \in \pi_l$: we denote $\iota(\pi_r) = l$.

We filter $\mathcal{C}h_\bullet$ in the following way: a generator $(\pi_r)_{1 \leq r \leq n-k}$ in some degree $k$ has *height* $p$, with $0 \leq p \leq n-1$, if there are exactly $p$ indices $i \in \pi_{l(e)}$ such that $i < 1$. Note that by Definition 4.4 the height can only decrease along boundaries in $\mathcal{C}h_\bullet$.

In the same way we can filter the chain complex $\mathcal{C}h^{\log}_\bullet$: a generator $(\pi_r, W_{ij})$ has the same height as the corresponding generator $(\pi_r)$ of $\mathcal{C}h_\bullet$. Note that we obtain a $P^\log_n$-invariant filtration on $\mathcal{C}h^{\log}_\bullet$: in other words $\mathcal{C}h^{\log}_\bullet$ becomes a filtered chain complex of $R(n)$-modules.

The chain complex $\tilde{\mathcal{C}}h_\bullet^{\log}$ has a natural action of the group $H_1(F_{n-2}(\mathbb{C} \setminus \{0, 1\})) \simeq \mathbb{Z}^{(n-2)-1}$; as we have already seen, the group $H_1(F_{n-2}(\mathbb{C} \setminus \{0, 1\}))$ can be identified with the kernel of the map $\psi_{12}: P^b_n \to \mathbb{Z}$, and is generated by elements $i_j$ for $1 \leq i < j \leq n$ with $(i, j) \neq (1, 2)$. Hence $\tilde{\mathcal{C}}h_\bullet^{\log}$ can be seen as a chain complex of free $\tilde{R}(n)$-modules.

**Definition 4.6.** We consider $\mathcal{C}h_\bullet^{\log}$ as a chain complex of free $\tilde{R}(n)$-modules and call $\Omega$ the basis containing those elements $(\pi_r, 0) \in \mathcal{C}h_\bullet^{\log}$ that lie in $\tilde{\mathcal{C}}h_\bullet^{\log}$.

The chain complex $\mathcal{C}h_\bullet^{\log}$ inherits a filtration from $\mathcal{C}h^{\log}_\bullet$, with heights $p$ ranging from 0 to $n-2$: this is a filtration in $\tilde{R}(n)$-modules.

We call $\mathcal{F}_p/\mathcal{F}_{p-1}[\mathcal{C}h_\bullet^{\log}]$ the $p$–th filtration stratum.

Note that $\Omega$ is a filtered basis for $\mathcal{C}h_\bullet^{\log}$.

### 5. Morse flows

In this section we simplify the complex $\mathcal{C}h_\bullet^{\log}$ to a chain complex with fewer generators; we use Forman’s discrete Morse theory, which was first introduced in [11]; see [14] or [15] for an introduction to discrete Morse theory. The Morse complex that we present has already appeared in a similar way in [9] and [16].

**Definition 5.1.** Recall from Definition 4.6 that $\Omega$ is a basis for $\tilde{\mathcal{C}}h_\bullet^{\log}$ as a chain complex of finitely generated, free $\tilde{R}(n)$-modules. For a cell $\epsilon = (\pi_r, 0) \in \Omega$, the index $\iota(\epsilon)$ was introduced in Definition 4.5. We define a matching $\mathcal{M}$ on $\Omega$:

- a cell $\epsilon = (\pi_r, 0)$ is critical if $\iota(\epsilon) = 1$ (i.e. $1 \in \pi_1$), and if 1 is the last element of $\pi_1$ according to $\prec$ (i.e. $i \prec 1$ for all $i \in \pi_1$ with $i \neq 1$);
- a cell $\epsilon = (\pi_r, 0)$ is collapsible if 1 is not the last element of $\pi_{l(e)}$. In this case the redundant partner of $\epsilon$ is $\epsilon' = (\pi'_r, 0)$, where $(\pi'_r)$ is obtained from $(\pi_r)$ by splitting $\pi_{l(e)}$ into $\pi_{l(e)}' = \{i \in \pi_1 | 1 < i\}$ and $\pi_{l(e)+1}' = \{i \in \pi_1 | i \leq 1\}$, as in Definition 4.2 with $l = \iota(\epsilon)$, and $\prec$ is restricted to the two pieces. Informally, we push all elements $i$ lying below 1 to the left. Note that $\iota(\epsilon') = \iota(\epsilon) + 1 \geq 2$. We write $\epsilon' \succ \epsilon$, meaning that the couple $(\epsilon', \epsilon)$ is in $\mathcal{M}$. 

To check that $M$ is acyclic on each filtration stratum $S_p/\mathcal{F}_{p-1}\mathcal{M}_{\log}$, hence it suffices to check that $M$ is acyclic on each filtration stratum $\mathfrak{F}_p/\mathfrak{F}_{p-1}\mathcal{M}_{\log}$. Let $\epsilon = (\pi_r,0) \not\mathcal{P} \epsilon' = (\pi'_r,0) \mathcal{P} \epsilon'' = (\pi''_r,0)$ be an alternating path of three distinct cells of degrees $k,k+1,k$, all having the same height $p$. This means that the redundant cell $\epsilon$ is matched with the collapsible cell $\epsilon'$, and that $[\partial \epsilon': \epsilon''] \neq 0$. Suppose also that $\epsilon''$ is redundant. Then both $\epsilon$ and $\epsilon''$ are obtained from $\epsilon'$ by splitting precisely the piece $\pi'_{i(\epsilon')}$, as in Definition 4.2, indeed 1 is not the last element in $\pi'_{i(\epsilon')}$, but it is the last element of both $\pi_{i(\epsilon)}$ and $\pi''_{i(\epsilon')}$.

Moreover there are exactly two ways to split $\pi'_{i(\epsilon')}$ in two pieces, so that the following conditions hold:

- 1 becomes the last element of its piece;
- the height $p$ doesn’t decrease, i.e. all elements preceding 1 in $\pi'_{i(\epsilon')}$ still belong to the same piece as 1 and precede 1.

The two pieces must be, in some order, $\left\{i \in \pi'_{i(\epsilon')} \mid i \leq 1\right\}$ and $\left\{i \in \pi'_{i(\epsilon')} \mid 1 \prec i\right\}$, and we can only choose which piece is split to the left and which to the right.

If $\left\{i \in \pi'_{i(\epsilon')} \mid 1 \prec i\right\}$ is split to the left, then we get the redundant partner of $\epsilon'$, that is, $\epsilon$; in the other case we must get $\epsilon''$.

We conclude that $i(\epsilon) = i(\epsilon') + 1$, and $i(\epsilon'') = i(\epsilon')$; in particular $i(\epsilon'') > i(\epsilon)$. This shows that the matching is acyclic on each stratum $p$, because the index $i$ strictly increases along alternating paths.

**Definition 5.2.** We call $\mathcal{M}_{\log} \mathcal{M}_{\log}$ the Morse complex associated with the acyclic matching $M$: it is a chain complex of finitely generated, free $\hat{R}(n)$-modules, with basis $\Omega^M$ given by $M$-critical cells in $\Omega$. The chain complex $\mathcal{M}_{\log} \mathcal{M}_{\log}$ is also a filtered chain complex of $\hat{R}(n)$-modules: the subcomplex $\mathfrak{F}_p \mathcal{M}_{\log}$ is generated by $M$-critical cells of height $\leq p$, and the $p$-th filtration stratum is denoted by $\mathfrak{F}_p/\mathfrak{F}_{p-1}\mathcal{M}_{\log}$.

We conclude this section by analysing more carefully the structure of the filtration strata.

**Definition 5.3.** Let $S$ be a subset of $\{2,\ldots,n\}$ containing 2. We denote by $R(S)$ the ring $\mathbb{Z}[t_{i,j}^{\pm 1}]_{i,j \in S}$. This is a domain and is naturally contained in $\hat{R}(n)$; its quotient field is denoted by $\mathbb{K}(S)$, and there is an inclusion $\mathbb{K}(S) \subset \hat{K}(n)$. In the particular case $S = \{2\}$ we have $R(S) = \mathbb{Z}$. 

- a cell $c = (c_0)_{c_1}$ is redundant if $i(c) \geq 2$ and 1 is the last element of $\pi_{i(c)}$ according to $\prec$. In this case the collapsible partner of $c$ is $c' = (c'_0)_{c'_1}$, where $(\pi'_{i(c)})$ is obtained from $(\pi_r)$ by concatenating $\pi_{i(c)}$ and $\pi_{i(c)-1}$ into $\pi'_{i(c)-1}$: on the new set $\pi'_{i(c)-1}$ the order $\prec$ is defined by extending $\prec$ on $\pi_{i(c)}$ and $\pi_{i(c)-1}$ with the rule $i \prec j$ for all $i \in \pi_{i(c)}$ and $j \in \pi_{i(c)-1}$. In particular $1 \prec j$ for all $j \in \pi_{i(c)-1}$. Informally, we push the column on left of 1 underneath 1. We write $c \not\mathcal{P} c'$. 

By Definition 4.2 if two cells $c \not\mathcal{P} c'$ are matched, then $[\partial c' : c]$ is invertible in $\hat{R}(n)$. To check that $M$ is acyclic, note first that $M$ is compatible the filtration of the chain complex $\mathcal{D}_{\log}$, hence it suffices to check that $M$ is acyclic on each filtration stratum $\mathfrak{F}_p/\mathfrak{F}_{p-1}\mathcal{D}_{\log}$.
Let $\mathcal{C}_h^S$ and $\mathcal{C}_h^{\log, S}$ be defined in analogy with Definitions 4.1 and 4.2 but using, instead of the set of indices $\{1, \ldots, n\}$, its subset $S$. In particular generators of $\mathcal{C}_h^S$ are given by ordered partitions $(\pi_r)_{1 \leq r \leq |S| - k}$ of $S$; generators of $\mathcal{C}_h^{\log, S}$ are given by an ordered partition of $S$ together with a choice of integers $(W_{ij})$ for all $i < j$ with $i, j \in S$.

Note that $\mathcal{C}_h^{\log, S}$ is a chain complex of finitely generated, free $R(S)$-modules, supported in degrees ranging from 0 to $|S| - 1$. In the particular case $S = \{2\}$ we have that $\mathcal{C}_h^{\log, S}$ consists of a copy of $\mathbb{Z}$ in degree 0.

**Lemma 5.4.** Let $0 \leq p \leq n - 2$; then there is an isomorphism of chain complexes of $R(n)$-modules

$$
\mathfrak{F}_p/\mathfrak{F}_{p-1}M\mathcal{C}_h^{\log, \omega} \cong \bigoplus_S \left( \bar{R}(n)^{p!} \otimes_{R(S)} \mathcal{C}_h^{\log, S} \right),
$$

where the sum is taken over all sets $S \subset \{2, \ldots, n\}$ with $|S| = n - p - 1$ and $2 \in S$.

This isomorphism shifts degrees by $-p$.

**Proof.** Recall that the differential in the chain complex $M(\mathcal{C}_h^{\omega})$ is defined as follows: for two $M$-critical cells $\mathfrak{c} = (\pi_r, 0)$ and $\mathfrak{c}' = (\pi_{r}', 0)$ in $\Omega^M$ the boundary index $[\partial \mathfrak{c}': \mathfrak{c}]$ is the sum of the weights of all alternating paths from $\mathfrak{c}'$ to $\mathfrak{c}$.

If $\mathfrak{c}$ and $\mathfrak{c}'$ have the same height $p$, then an alternating path $\mathfrak{c}' = \mathfrak{c}_0 \searrow \mathfrak{c}_1 \nearrow \cdots \nearrow \mathfrak{c}_t = \mathfrak{c}_0$ must contain only cells of height $p$. Since $\mathfrak{c}_0$ is critical, 1 is the last element of $\pi_1'$, and splitting in two pieces $\pi_1'$ would let the height $p$ of $\mathfrak{c}_0'$ decrease to a smaller height in $\mathfrak{c}_0$: hence $\mathfrak{c}_0$ is obtained from $\mathfrak{c}'$ by splitting some other piece $\pi_i'$ with $i \geq 2$, and therefore $e_0$ is already critical, hence $e_0 = e$.

Thus the differential in the chain complex $\mathfrak{F}_p/\mathfrak{F}_{p-1}M\mathcal{C}_h^{\log, \omega}$ is isomorphic to the differential obtained from Definition 4.2 by allowing only a splitting in two pieces of some part of the partition $\pi_1$ with $l \geq 2$.

In particular we can split our chain complex $\mathfrak{F}_p/\mathfrak{F}_{p-1}M\mathcal{C}_h^{\log, \omega}$ into many subcomplexes according to which $p$ elements, all different from 2, appear in $\pi_1$ and in which order $\prec$, provided that 1 is the last element of $\pi_1$.

To determine one of these subcomplexes we can equivalently choose a set $S \subset \{2, \ldots, n\}$ of $n - p - 1$ elements, with $2 \in S$, and declare that the other $p + 1$ elements $i \in \{1, \ldots, n\}$, including 1, are the elements of $\pi_1$. Moreover there are exactly $p!$ ways to order these $p + 1$ elements inside $\pi_1$, if we require 1 to be the last in the order: each of these possible choices of $\prec$ on $\pi_1$ gives rise to a different subcomplex.

Finally we note that each of these subcomplexes is isomorphic to the chain complex $\bar{R}(n) \otimes_{R(S)} \mathcal{C}_h^{\log, S}$, where the isomorphism is given by mapping the $M$-critical cell $(\pi_r, 0)_{1 \leq r \leq n - k}$ to the cell $1 \otimes (\pi_r, 0)_{2 \leq r \leq n - k}$; this map has degree $-p$. \hfill\Box

6. THE SPECTRAL SEQUENCE WITH COEFFICIENTS IN $\tilde{\mathbb{K}}(n)$

In this section we prove that $H_{n-2}([P_n, P_n]) \neq 0$. More precisely we prove the following theorem.

**Theorem 6.1.** For $n \geq 2$ the graded $\tilde{\mathbb{K}}(n)$-vector space

$$
\tilde{\mathbb{K}}(n) \otimes_{\bar{R}(n)} H_*([P_n, P_n])
$$

has dimension $(n - 2)!$ in degree $n - 2$ and vanishes in all other degrees.
This means, in particular, that \( H_{n-2}([P_n, P_n]) \) contains an embedded copy of \( \tilde{R}(n)^{(n-2)!} \), which for \( n \geq 3 \) is a free abelian group of infinite rank.

The following is an immediate consequence of Theorem 6.1.

**Corollary 6.2.** For \( n \geq 2 \) the cohomological dimension of \([P_n, P_n]\) is \( n - 2 \).

**Proof.** We have \( \text{cd}([P_n, P_n]) \geq n - 2 \) because \( H_{n-2}([P_n, P_n]) \neq 0 \). Moreover, as already seen in the proof of Lemma 4.4, the space \( \tilde{F}_n^{\log} \), deformation retracts onto the space \( \tilde{S}^n_{\log} \), which is a cell complex of dimension \( n - 2 \); hence \( \text{cd}([P_n, P_n]) \leq n - 2 \). \( \square \)

**Proof of Theorem 6.1.** We consider the filtered chain complex \( \mathcal{M} \tilde{c}_h^\log \). Since localisation is exact we can compute \( H_*([P_n, P_n]) \otimes \tilde{R}(n) \tilde{K}(n) \) as the homology of the chain complex \( \tilde{K}(n) \otimes \tilde{R}(n) \mathcal{M} \tilde{c}_h^\log \), which is a filtered chain complex of \( \tilde{K}(n) \)-vector spaces.

The first page of the associated Leray spectral sequence is

\[
E_1^{p,q} = H_{p+q} \left( \tilde{F}_{p}/\tilde{F}_{p-1} \left( \tilde{K}(n) \otimes \tilde{R}(n) \mathcal{M} \tilde{c}_h^\log \right) \right),
\]

and our aim is to show that the latter groups are all trivial, except for \( p = n - 2 \) and \( q = 0 \), where we have

\[
H_{n-2} \left( \tilde{F}_{n-2}/\tilde{F}_{n-3} \left( \tilde{K}(n) \otimes \tilde{R}(n) \mathcal{M} \tilde{c}_h^\log \right) \right) \simeq \tilde{K}(n)^{(n-2)!}.
\]

Once this statement is proved, Theorem 6.1 follows immediately because the spectral sequence collapses on its first page.

By Lemma 5.3 the chain complex \( \tilde{F}_{n-2}/\tilde{F}_{n-3} \left( \mathcal{M} \tilde{c}_h^\log \right) \) is isomorphic to the chain complex \( \tilde{R}(n)^{(n-2)!} \otimes \tilde{R}(2) \mathcal{C}_h^{\log,2} \). Since the ring \( \tilde{R}(2) \) is just \( \mathbb{Z} \), and since the chain complex \( \mathcal{C}_h^{\log,2} \) is just a copy of \( \mathbb{Z} \) in degree 0, we have that the filtration stratum \( \tilde{F}_{n-2}/\tilde{F}_{n-3} \left( \mathcal{M} \tilde{c}_h^\log \right) \) is concentrated in degree \( n - 2 \) and its homology is \( \tilde{R}(n)^{(n-2)!} \), also concentrated in degree \( n - 2 \). Tensoring with \( \tilde{K}(n) \) we have that \( E_{n-2,0} \simeq \tilde{K}(n)^{(n-2)!} \), and \( E_{n-2,q} = 0 \) for all \( q \neq 0 \).

We want now to show that the chain complex \( \tilde{F}_p/\tilde{F}_{p-1} \left( \tilde{K}(n) \otimes \tilde{R}(n) \mathcal{M} \tilde{c}_h^\log \right) \) is acyclic for all \( 0 \leq p \leq n - 3 \). By Lemma 5.4 it suffices to prove that, for any set \( S \subset \{ 2, \ldots, n \} \) containing 2, the chain complex

\[
\tilde{K}(n) \otimes \tilde{R}(n) \tilde{R}(n) \otimes \tilde{R}(S) \mathcal{C}_h^S
\]

is acyclic. We note that \( \tilde{K}(n) \) contains \( K(S) \), so we can equally consider

\[
\tilde{K}(n) \otimes K(S) K(S) \otimes \tilde{R}(S) \mathcal{C}_h^S
\]

and the latter is acyclic because \( K(S) \otimes \tilde{R}(S) \mathcal{C}_h^S \) is acyclic by Lemma 3.4 and extending the field \( K(S) \subset \tilde{K}(n) \) is exact. \( \square \)

We note that it was not necessary to localise \( \tilde{R}(n) \) with respect to all non-zero elements, i.e. passing from \( \tilde{R}(n) \) to its quotient field \( \tilde{K}(n) \).
**Definition 6.3.** Let $S$ be a finite subset of $\{2, \ldots, n\}$ containing 2. We call

$$\tau_S = \left( \prod_{i,j \in S; i < j} t_{ij} \right) - 1 \in \tilde{R}(n) \subset R(n).$$

Define also

$$\tau_n = \prod_S \tau_S \in \tilde{R}(n) \subset R(n)$$

where the product is extended over all subsets $S \subset \{2, \ldots, n\}$ containing 2.

Then the same argument of the proof of Lemma 5.4 tells us that, for all subsets $2 \in S \subset \{2, \ldots, n\}$ with $S \neq \{2\}$, we have

$$\tilde{R}(n) \left[ \tau_n^{-1} \right] \otimes_{R(S)} H_* \left( \mathfrak{C}h^\log_S \right) = 0.$$

Therefore we can repeat the proof of Theorem 6.1 to show that

$$\tilde{R}(n) \left[ \tau_n^{-1} \right] \otimes_{R(n)} H_*([P_n, P_n])$$

is concentrated in degree $n - 2$, where it is equal to $\tilde{R}(n) \left[ \tau_n^{-1} \right]^{(n-2)!}$.

### 7. A False Conjecture

In this section we disprove a natural conjecture that may arise from the arguments of the previous section.

Let $\mathcal{E}_{p,q}$ denote the spectral sequence associated to the filtered $\tilde{R}(n)$-chain complex $\mathfrak{C}h^\log$ (without localising at $\tilde{K}(n)$). Then any element $x$ in $\mathcal{E}^1_{n-2,0} = \tilde{R}(n)^{(n-2)!}$ which is divisible by the product

$$\prod_{i \geq 3} \tau_{(2,i)}$$

has a trivial $\mathcal{E}^1$-differential, because this product kills all homology groups in the column $\mathcal{E}_{n-3,0}$; indeed Lemma 5.4 relates this column to the homology of the chain complexes $\mathfrak{C}h^\log_{(2,i)}$ for $i \geq 3$; by Lemma 3.3 the homology of $\mathfrak{C}h^\log_{(2,i)}$ is killed by the factor $\tau_{(2,i)}$.

Suppose now that $x$ is also divisible by the product

$$\prod_{j > i \geq 3} \tau_{(2,i,j)}.$$

Then also the $\mathcal{E}^2$-differential of $x$ is zero, because this product kills all homology groups in the column $\mathcal{E}_{n-4,1}$, and therefore also the whole column $\mathcal{E}^2_{n-4,1}$.

Iterating this argument we see that if $x \in \mathcal{E}_{n-2,0}$ is divisible by $\tau_n$, then $x$ must survive to the limit of the spectral sequence.

One could then conjecture that $\mathcal{E}^\infty_{n-2,0} = \tau_n \mathcal{E}^1_{n-2,0}$; an immediate consequence would be that $H_{n-2}([P_n, P_n]) \cong \tilde{R}(n)^{(n-2)!}$ as $\tilde{R}(n)$-modules. This conjecture is however false for $n \geq 4$, as we shall see in the following discussion.

Let $\mathcal{F}$ denote the set $\{2, \ldots, n\}$: we prove that there are elements in $\mathcal{E}_{n-2,0}$ which are not multiple of $\tau_{\mathcal{F}}$ but survive in the spectral to the $\mathcal{E}^\infty$-page. Therefore the inclusion

$$\tau_n \mathcal{E}^1_{n-2,0} \simeq \tilde{R}(n)^{(n-2)!} \subset \mathcal{E}^\infty_{n-2,0} \cong H_{n-2}([P_n, P_n])$$

is not an equality.
Write $\tau_n$ as a product $\tau_n = \tau_{n}^R \cdot \tau_{n}^E$ in $\hat{R}(n)$, after Definition 6.3. To prove our claim it suffices to study the localisation of the spectral sequence $E_{p,q}^1$ at the element $\tau_n^E$, i.e. the spectral sequence $\tilde{E}_{p,q}^1$, associated to the filtered chain complex $\hat{R}(n)[\tau_n^{-1}] \otimes \mathcal{C}_h^*$. More precisely, it suffices to find an element $x$ in $\tilde{E}_{n-2,0}^1$ that is not a multiple of $\tau_{n}^S$ but survives to the limit $\tilde{E}_{\infty,n-2,0}^1$.

Using the same arguments as in the previous section, one can prove the following facts:

- the column $\tilde{E}_{n-2,q}^1$ is non trivial only for $q = 0$, and we have the equality $\tilde{E}_{n-2,0}^1 = \hat{R}(n)[\tau_n^{-1}]^\infty$.
- for all $1 \leq p \leq n - 3$ the columns $\tilde{E}_{p,q}^1$ are trivial.

This follows essentially from the fact that the page $\tilde{E}_{p,q}^1$ is the $\tau_n^E$-localisation of the page $E_{p,q}^1$. Lemma 5.4 tells us that the columns $1 \leq p \leq n - 3$ of the page $E_{p,q}^1$ are built out of the homology of the chain complexes $\mathcal{C}_h^{log,S}$, with $2 \leq S \subseteq \{2, \ldots, n\}$, $S \neq \{2\}$ and $S \neq \mathcal{S}$. Lemma 5.4 implies that $\tau_n^E$ kills the homology of all these chain complexes, hence $\tau_n^E$ also kills the page $\tilde{E}_{p,q}^1$ of the spectral sequence, except the columns $\tilde{E}_{n-2,0}^1$ and $\tilde{E}_{0,q}^1$.

In particular the groups $\tilde{E}_{n-2,q}^1$ and $\tilde{E}_{0,n-3}^1$ survive until the $(n - 2)$-th page, where we have a differential $\partial_{n-2} : \tilde{E}_{n-2,0}^1 \rightarrow \tilde{E}_{n-3,0}^1$.

Our next aim is to compute explicitly the group $\tilde{E}_{0,n-3}^1$. By definition it is equal to

$$H_{n-3} \left( \hat{R}(n)[\tau_n^{-1}] \otimes \hat{R}(n) \mathcal{S}_0 \mathcal{C}_h^* \right),$$

and using Lemma 5.4 we can compute the latter as

$$H_{n-3} \left( \hat{R}(n)[\tau_n^{-1}] \otimes \hat{R}(S) \mathcal{C}_h^{log,S} \right),$$

where $R(S) = \mathbb{Z}[t_{ij}]_{2 \leq i < j \leq n} \subset \hat{R}(n)$ was introduced in Definition 6.3. Note that both $\tau_n^E$ and $\tau_{n}^S$ are naturally elements of $R(S)$.

Theorem 6.1 implies that $\tau_n^E$ kills the homology of $\mathcal{C}_h^{log,S}$ in degrees $q \leq n - 4$ (recall that $S$ has $n - 1$ elements); this implies that also the column $\tilde{E}_{0,q}^1$ is trivial for $q \leq n - 4$.

Again by Theorem 6.1 there is an isomorphism of $\hat{R}(S)[\tau_n^{-1}]$-modules

$$H_{n-3} \left( R(S)[\tau_n^{-1}] \otimes R(S) \mathcal{C}_h^{log,S} \right) \cong \left( R(S) / (\tau_{n}^S) [\tau_n^{-1}] \right)^{(n-3)!},$$

where we quotient out the ideal generated by $\tau_{n}^S$ and we invert the element $\tau_n^E$. This isomorphism gives, after applying again Lemma 5.4 the following isomorphism of $\hat{R}(n)[\tau_n^{-1}]$-modules

$$\tilde{E}_{0,n-3}^1 \cong \left( \hat{R}(n) / (\tau_{n}^S) [\tau_n^{-1}] \right)^{(n-3)!}.$$

The $\tilde{E}_{n-2}$-differential takes the form of a map of $\hat{R}(n)[\tau_n^{-1}]$-modules

$$\partial_{n-2} : \hat{R}(n)[\tau_n^{-1}]^{(n-2)!} \rightarrow \left( \hat{R}(n) / (\tau_{n}^S) [\tau_n^{-1}] \right)^{(n-3)!},$$

and it is evident that in the kernel of this map there must be elements $x$ which are not multiple of $\tau_{n}^S$. Otherwise, after quotienting out all multiples of $\tau_{n}^S$ in the
source module, we would get an injective map of \( \hat{R}(n)/\langle \tau_2 \rangle [r_n^{-1}] \)-modules
\[
\left( \hat{R}(n)/\langle \tau_2 \rangle [r_n^{-1}] \right)^{(n-2)!} \hookrightarrow \left( \hat{R}(n)/\langle \tau_2 \rangle [r_n^{-1}] \right)^{(n-3)!},
\]
and this is a contradiction, because \( \hat{R}(n)/\langle \tau_2 \rangle [r_n^{-1}] \) is a domain and we have found an injective map from a free module of rank \((n-2)!\) to a free module of rank \((n-3)!\), with \((n-2)!\) strictly bigger than \((n-3)!\) because we assumed \(n \geq 4\).

8. Homology in lower degrees

In this section we prove non-triviality of \(H_*(\langle P_n, P_n \rangle)\) in all degrees \(* \leq n-2\). More precisely, we prove the following theorem.

**Theorem 8.1.** For all \(1 \leq * \leq n-2\) the group \(H_*(\langle P_n, P_n \rangle)\) contains a free abelian group of infinite rank.

We will first give a short, group-theoretic proof of Theorem 8.1; we will then construct a particular non-symmetric operad, called \(C^\log_2\), and reprove Theorem 8.1 in a more geometric way.

**Short proof of Theorem 8.1.** By Theorem 6.1 we know that \(H_{n-2}(\langle P_n, P_n \rangle)\) contains a free abelian group of infinite rank. In the following we fix \(3 \leq k \leq n-1\) and prove that \(H_{k-2}(\langle P_n, P_n \rangle)\) has the same property.

Consider the map \(\psi_k^n : F_n \to F_k\) that forgets the last \(n-k\) points of a configuration (compare with the maps \(\psi_{ij}\) from Definition 2.1):
\[
\psi_k^n(z_1, \ldots, z_n) = (z_1, \ldots, z_k) \in F_k.
\]
The map \(\psi_k^n\) is a fibration (see [10]) and there is a section \(\sigma_k^n : F_k \to F_n\) given by adjoining \(n-k\) points far on the right: formally we set \(M(z_1, \ldots, z_k) = \max_{i=1}^k |z_i|\) and then we define
\[
\sigma_k^n(z_1, \ldots, z_k) = (z_1, \ldots, z_k, M + 1, \ldots, M + n - k) \in F_n.
\]
We have induced maps on fundamental groups \(\psi_k^n : P_n \to P_k\) and \(\sigma_k^n : P_k \to P_n\); the composition \(\psi_k^n \circ \sigma_k^n : P_k \to P_k\) is the identity of \(P_k\).

The maps \(\psi_k^n\) and \(\sigma_k^n\) restrict to maps between commutator subgroups; in particular the composition \(\psi_k^n \circ \sigma_k^n : [P_k, P_k] \to [P_k, P_k]\) is the identity of \([P_k, P_k]\).

This implies that the induced map in homology
\[
(\sigma_k^n)_* : H_{k-2}(\langle P_k, P_k \rangle) \to H_{k-2}(\langle P_n, P_n \rangle)
\]
is injective, and again by Theorem 6.1 we know that \(H_{k-2}(\langle P_k, P_k \rangle)\) contains a free abelian group of infinite rank. \(\square\)

We turn now to the geometric proof of Theorem 8.1. We introduce a non-symmetric operad \(C^\log_2\), which is an analogue of May’s operad \(C_2\) of little squares (see [19]). Here non-symmetric is in the sense of [15]: there is no defined action of the symmetric group \(S_k\) on the space \(C^\log_2(k)\), but composition maps \(\gamma^\log\) are defined and satisfy the usual axioms of unity and associativity.

We define \(C_2(k)\) as the space of disjoint embeddings \(\iota_1, \ldots, \iota_k\) of \((0,1)^2\) into \((0,1)^2\) which are a composition of a dilation and a translation. Every \(\iota_i\) is called a little square.

In the definition given by May in [19], a dilation may shrink the square \((0,1)^2\) by different factors horizontally and vertically, but for us it will be convenient to
consider only \textit{conformal} dilations; in particular, using $z$ as complex coordinate of $(0, 1)^2$, each $i_t$ has the form $z \mapsto \lambda_t z + \zeta_t$ for some $\lambda_t > 0$ and $\zeta_t \in \mathbb{C}$. The number $\lambda_t$ is called the \textit{dilation factor} of $i_t$.

There is a map $g: \mathbb{C}_2(k) \to F_{k}$ given by taking the centres of the little squares:

$$g(t_1, \ldots, t_k) = \left( \frac{1}{2} \left( 1 + \sqrt{-1} \right) t_1, \ldots, \frac{1}{2} \left( 1 + \sqrt{-1} \right) t_k \right);$$

this map is a homotopy equivalence (see [19]).

Fix now a point $(t_1, \ldots, t_k) \in \mathbb{C}_2(k)$. We call $Q_i$ the image of $i_t$, which is an open square in $(0, 1)^2$. For every $i < j$, we can consider the open complex 2-manifold $Q_i \times Q_j$, with coordinates $z_i$ and $z_j$ corresponding to the restriction of the coordinate $z$ of $(0, 1)^2$ to $Q_i$ and $Q_j$ respectively.

Then $z_j - z_i$ is a holomorphic function on $Q_i \times Q_j$ with no zeroes, and since $Q_i \times Q_j$ is contractible we can determine a logarithm of this function, i.e. a function $w_{ij}: Q_i \times Q_j \to \mathbb{C}$ satisfying $e^{w_{ij}} = z_j - z_i$ on the whole $Q_i \times Q_j$. Different choices of the function $w_{ij}$ differ by a constant function, which is an integral multiple of $2\pi\sqrt{-1}$, i.e. has the form $2\pi\sqrt{-1}k_{ij}$ for some $k_{ij} \in \mathbb{Z}$.

\textbf{Definition 8.2.} We define the space $\mathbb{C}_2^{\log}(k)$ as the space of choices of the following set of data:

- disjoint embeddings $i_1, \ldots, i_k: (0, 1)^2 \to (0, 1)^2$ of the form $z \mapsto \lambda_i z + \zeta_i$ for some $\lambda_i > 0$ and $\zeta_i \in \mathbb{C}$; we call $Q_1, \ldots, Q_k \subset (0, 1)^2$ the images of these embeddings;
- for all $1 \leq i < j \leq k$, holomorphic functions $w_{ij}: Q_i \times Q_j \to \mathbb{C}$ satisfying $e^{w_{ij}} = z_j - z_i$, where $z_i$ and $z_j$ are the coordinates of $Q_i$ and $Q_j$ respectively.

A point in $\mathbb{C}_2^{\log}(k)$ is usually denoted as $(i_1)_{1 \leq i \leq k}, (w_{ij})_{1 \leq i < j \leq k}$.

The topology on $\mathbb{C}_2^{\log}(k)$ is defined as the weakest topology for which the following two maps are continuous.

The first is the map $p: \mathbb{C}_2^{\log}(k) \to \mathbb{C}_2(k)$ which forgets the functions $w_{ij}$ and remembers the embeddings $i_t$.

The second is the map $g: \mathbb{C}_2^{\log}(k) \to F_{k}^{\log}$ given by taking the centres of the little squares and evaluating the functions $w_{ij}$ at couples of centres:

$$g((i_1)_{1 \leq i \leq k}, (w_{ij})_{1 \leq i < j \leq k}) = ((c_1)_{1 \leq i \leq k}, (w_{ij})_{1 \leq i < j \leq k}),$$

where $c_i = i_t \left( \frac{1}{2} \left( 1 + \sqrt{-1} \right) \right)$ and $w_{ij} = w_{ij}(c_i, c_j)$.

The space $\mathbb{C}_2^{\log}(0)$ consists only of one point, whereas $p: \mathbb{C}_2^{\log}(1) \to \mathbb{C}_2(1)$ is a homeomorphism.

The map $g: \mathbb{C}_2^{\log}(k) \to F_{k}^{\log}$ is a homotopy equivalence, and the map $p: \mathbb{C}_2^{\log}(k) \to \mathbb{C}_2(k)$ is the pullback along $g$ of the covering map $p: F_{k}^{\log} \to F_k$, so that we have a pullback square

$$\begin{array}{ccc}
\mathbb{C}_2^{\log}(k) & \xrightarrow{g} & F_{k}^{\log} \\
p \downarrow & & \\
\mathbb{C}_2(k) & \xrightarrow{g} & F_k.
\end{array}$$

For all non-negative integers $k, n_1, \ldots, n_k$ recall that there is a composition map

$$\gamma: \mathbb{C}_2(k) \times \mathbb{C}_2(n_1) \times \cdots \times \mathbb{C}_2(n_k) \to \mathbb{C}_2(n_1 + \cdots + n_k).$$
For 1 \leq l \leq k let \((\iota_{l,i})_{1 \leq i \leq n_i} \in \mathcal{C}_2(n_i)\).

Here and in the following we identify the set \{((l, i) \mid 1 \leq l \leq k, 1 \leq i \leq n_i)\}, ordered lexicographically, with the ordered set \{1, \ldots, n_1 + \cdots + n_k\}. Then

\[(u_l)_{1 \leq l \leq k} \times (u_{1,i})_{i \leq n_1} \times \cdots \times (u_{k,i})_{i \leq n_k} \rightarrow (u_{l,i})_{l \leq n} \in \mathcal{C}_2(n_1 + \cdots + n_k),\]

where for all 1 \leq l \leq k and 1 \leq i \leq n_i we set \(u_{l,i} : = u_l \circ u_{l,i} : (0,1)^2 \rightarrow (0,1)^2\).

We now define composition maps making the collection of space \(\mathcal{C}_2^{\log}(k)\) for \(k \geq 0\) into a non-symmetric operad \(\mathcal{C}_2^{\log}\).

**Definition 8.3.** For all non-negative integers \(k, n_1, \ldots, n_k\) we define a map

\[\gamma_{\log} : \mathcal{C}_2^{\log}(k) \times \mathcal{C}_2^{\log}(n_1) \times \cdots \times \mathcal{C}_2^{\log}(n_k) \rightarrow \mathcal{C}_2^{\log}(n_1 + \cdots + n_k).\]

Let \(\left(\left(u_{l,i}\right)_{i \leq n_1}, \left(w_{l,i,j}\right)\right) \in \mathcal{C}_2^{\log}(k)\), and for all \(l \leq k\) let \(\left(\left(u_{l,i}\right)_{i \leq n_1}, \left(w_{l,i,j}\right)\right) \in \mathcal{C}_2^{\log}(n_1)\).

Then

\[\left(\left(u_{l,i}\right), \left(w_{l,i,j}\right)\right) \times \prod_{l=1}^{k} \left(\left(u_{l,i}\right), \left(w_{l,i,j}\right)\right) \rightarrow \prod_{l=1}^{k} \left(\left(u_{l,i}\right), \left(w_{l,i,j}\right)\right) \in \mathcal{C}_2^{\log}(n_1 + \cdots + n_k),\]

which is defined as follows. First, set \(u_{l,i} : = u_l \circ u_{l,i}\) just as in the definition of the composition map \(\gamma\) for the operad \(\mathcal{C}_2\).

For all \(i \leq k\), denote by \(Q_i\) the image of \(u_i\). For all \(l \leq k\) and \(i \leq n_i\), denote by \(Q_{l,i}\) and \(Q_{l,i}''\) the images of \(u_{l,i}\) and \(u_{l,i}''\) respectively. Clearly \(Q_{l,i}'' \subset Q_l\).

For \(1 \leq l < l' \leq k\), \(1 \leq i \leq n_i\) and \(1 \leq j \leq n_j\), define \(w_{l,i,j}''\) as the restriction of \(w_{l,i,j}\) on the product \(Q_{l,i}'' \times Q_{l,j}''\).

For \(1 \leq l \leq k\) and \(1 \leq i < j \leq n_i\), define \(w_{l,i,j}''\) on the product \(Q_{l,i}'' \times Q_{l,j}''\) as the composition

\[Q_{l,i}'' \times Q_{l,j}'' \xrightarrow{\iota_{l,i}^{-1} \times \iota_{l,j}^{-1}} Q_{l,i}'' \times Q_{l,j}'' \xrightarrow{(w_{l,i,j})_{l,i} + \log \lambda_l} \mathbb{C}\]

Here \(\lambda_l > 0\) is the dilation coefficient of the embedding \(u_l\), and \(\log \lambda_l\) denotes here the real logarithm.

The maps \(\gamma_{\log}\) satisfy the associativity and unity axioms of operads; nevertheless one cannot define a good action of the symmetric groups \(\mathfrak{S}_k\) on the spaces \(\mathcal{C}_2^{\log}(k)\); hence \(\mathcal{C}_2^{\log}\) is a non-symmetric operad and the collection of coverings \(\mathcal{C}_2^{\log}(k) \rightarrow \mathcal{C}_2(k)\) gives a map of non-symmetric operads \(\mathcal{C}_2^{\log} \rightarrow \mathcal{C}_2\).

Here a good action would be one making \(\mathcal{C}_2^{\log}\) into a symmetric operad, such that the map \(\mathcal{C}_2^{\log} \rightarrow \mathcal{C}_2\) is a map of operads. Suppose by contradiction that such an action existed; then we would have in particular a \(\mathfrak{S}_2\)-equivariant map \(\mathcal{C}_2^{\log}(2) \rightarrow \mathcal{C}_2(2)\), and as the action of \(\mathfrak{S}_2\) is free on \(\mathcal{C}_2(2)\), we would also have a free action of \(\mathfrak{S}_2\) on the space \(\mathcal{C}_2^{\log}(2)\). But this is not possible, because \(\mathcal{C}_2^{\log}(2)\) is a finite-dimensional and contractible CW-complex.

Let \(k \geq 0\) and \(n_1, \ldots, n_k \geq 1\). The map \(\gamma : \mathcal{C}_2(k) \times \prod_{l=1}^{k} \mathcal{C}_2(n_l) \rightarrow \mathcal{C}_2(n_1 + \cdots + n_k)\) admits a homotopy retraction

\[\rho : \mathcal{C}_2(n_1 + \cdots + n_k) \rightarrow \mathcal{C}_2(k) \times \prod_{l=1}^{k} \mathcal{C}_2(n_l).\]
For all \((u''_{i,k})_{i \leq k; i \leq n_i}\) we define

\[
(u''_{i,k})_{i \leq k; i \leq n_i} \overset{\gamma}{\longrightarrow} (u_i)_{i \leq k} \times \prod_{i=1}^{k} (u'_{i,i})_{i \leq n_i},
\]

where \(u_i = u''_{i,1}\) for all \(1 \leq i \leq k\), and \(u'_{i,i} = u''_{i,i}\) for all \(1 \leq l \leq k\) and \(1 \leq i \leq n_i\).

The composition \(\rho \circ \gamma\) is homotopic to the identity of \(C_2(k) \times \prod_{i=1}^{k} C_2(n_i)\).

To see this, note first that every little square \(\iota\) of little squares inflations of all little squares forms a homotopy between the identity of \(\text{Id}_n\).

The point \((\iota)_{i \leq k} \times \prod_{i=1}^{k} (\iota'_{i,i})_{i \leq n_i}\) is mapped by \(\rho \circ \gamma\) to the point

\[
(t_i \circ t'_i)_{i \leq k} \times \prod_{i=1}^{k} (t_i \circ t'_{i,i})_{i \leq n_i};
\]

using the inflation of \(t_{i,1}\) on the first factor and the inflation of \(t_i\) on the other factors we get the desired homotopy between \(\rho \circ \gamma\) and the identity of \(C_2(k) \times \prod_{i=1}^{k} C_2(n_i)\).

**Definition 8.4.** For all \(k \geq 0\) and \(n_1, \ldots, n_k \geq 1\) we define a map

\[
\rho^\log : C_2^{\log}(n_1 + \cdots + n_k) \rightarrow C_2^{\log}(k) \times C_2^{\log}(n_1) \times \cdots \times C_2^{\log}(n_k).
\]

Let \(\left((u''_{i,i}), (w''_{i,i,i,j})\right) \in C_2^{\log}(n_1 + \cdots + n_k)\). Then

\[
\left((u''_{i,i}), (w''_{i,i,i,j})\right) \overset{\rho^\log}{\longrightarrow} \left((u_i)_{i \leq k}, (w_{i,j})\right) \times \prod_{i=1}^{k} \left((u'_{i,i})_{i \leq n_i}, (w'_{i,i,j})\right),
\]

where

- \(u_i = u''_{i,1}\) for all \(1 \leq i \leq k\);
- \(w_{i,j} = w''_{i,(i,j)(i,j)}\) for all \(1 \leq i < j \leq k\);
- \(u'_{i,i} = u''_{i,i}\) for all \(1 \leq l \leq k\) and \(1 \leq i \leq n_i\);
- \(w'_{i,i,j} = w''_{i,(i,j)(i,j)}\) for all \(1 \leq l \leq k\) and \(1 \leq i < j \leq n_i\).

**Lemma 8.5.** The map \(\rho^\log\) is a homotopy retraction of \(\gamma^\log\), i.e. the composition \(\rho^\log \circ \gamma^\log\) is homotopic to the identity of \(C_2^{\log}(k) \times \prod_{i=1}^{k} C_2^{\log}(n_i)\).

**Proof.** Consider a point \(\left((u_i)_{i \leq k}, (w_{i,j})\right) \times \prod_{i=1}^{k} \left((u_{i,i})_{i \leq n_i}, (w'_{i,i,j} + \log \lambda_i)\right)\) and let \(Q_i\), \(Q'_{i,i}\) and \(Q''_{i,i}\) denote the images of the maps \(\iota_i\), \(\iota''_{i,i}\) and \(\iota \circ \iota''_{i,i}\) respectively. The image of this point along the map \(\rho^\log \circ \gamma^\log\) is the point

\[
\left((t_i \circ t''_{i,1})_{i \leq k}, (w_{i,j})\right) \times \prod_{i=1}^{k} \left((t_{i,i})_{i \leq n_i}, (w'_{i,i,j})\right),
\]

where
For all times $t \in [0,1]$ we consider the restriction of the functions $w_{ij}$ to the subspaces $Q''_{l,i} \times Q''_{j,i} \subset Q'_l \times Q'_j$.

- $\lambda_i$ is the dilation coefficient of $\iota_t$ and $\log \lambda_i$ is its real logarithm;
- the function $w'_{l,ij} + \log \lambda_t$ is transferred to $Q''_{l,i} \times Q''_{j,i}$ along the identification $\iota_t: Q'_{l,i} \times Q'_{j,i} \cong Q''_{l,i} \times Q''_{j,i}$.

We use the same description of the homotopy between $\iota_t \circ \iota'_t \to \iota_t$ and $\iota_t \circ \iota'_t \to \iota'_t$ that we have already considered in the discussion before Definition 8.4.

We note that the functions $w_{ij}$ are defined on the whole products $Q_l \times Q_j$, where the inflations of the first factor take place: we can use at each time these functions, restricted to the corresponding product of squares; thus we have defined the desired homotopy on the first factor.

For the second factor we write explicitly

$$\mathcal{H}_l: (0,1)^2 \times [0,1] \to (0,1)$$

for the inflation between $u$ and $\text{Id}_{(0,1)^2}$: $\mathcal{H}_l(z; t) = tz + (1-t)u_l(z)$.

For all times $t \in [0,1]$ let $Q''_{l,i,t}$ and $Q''_{l,j,t}$ denote the images of the maps $\mathcal{H}_l(\cdot; t)$ and $\mathcal{H}_l(\cdot; t)$ respectively. The map $\mathcal{H}_l(\cdot; t) \times \mathcal{H}_l(\cdot; t)$ identifies $Q''_{l,i,t} \times Q''_{l,j,t}$ with $Q'_{l,i,t} \times Q'_{l,j,t}$, and we can then push forward the function $w'_{l,ij} + \log (t + (1-t)\lambda_t)$.

This gives a homotopy between the function $w_{l,ij} + \log \lambda_i$ pushed forward to $Q''_{l,i} \times Q''_{l,j} = Q'_{l,i} \times Q'_{l,j} \times Q'_{l,i} \times Q'_{l,j}$ and the function $w_{ij}$ pushed forward (along the identity) to $Q'_{l,i} \times Q'_{l,j} = Q'_{l,i,0} \times Q'_{l,j,0}$.

Fix now $k \geq 2$ and $N \geq k + 2$. We want to prove that $H_{N-k-1}(C_2^\log(N))$ contains a free abelian group of infinite rank.

By Lemma 8.3 there is an inclusion of abelian groups

$$H_* \left( C_2^\log(k) \right) \otimes \bigotimes_{l=1}^k H_* \left( C_2^\log(n_l) \right) \subseteq H_* \left( C_2^\log(k) \times \prod_{l=1}^k C_2^\log(n_l) \right) \subseteq H_* \left( C_2^\log(N) \right),$$

For the first inclusion we have used the Künneth formula: this map is injective and, up to torsion (here in the sense of abelian groups), it is an isomorphism. If we knew that $H_*([P_n, P_n])$ was a free abelian group, this first inclusion would be indeed an isomorphism. The map $\gamma_2^\log$ is split-injective, since it admits a retraction $\rho_*^\log$.

We consider the following particular case: let $n_1 = N-k+1$ and $n_2 = \cdots = n_k = 1$. Then we have an inclusion

$$H_0 \left( C_2^\log(k) \right) \otimes H_{N-k-1} \left( C_2^\log(N-k+1) \right) \otimes H_0(\bar{C}_2^\log(1)) \otimes^{k-1} H_{N-k-1}(\bar{C}_2^\log(N)), $$

and we know by Theorem 8.1 that the first group has infinite rank. This proves that $H_{N-k-1}(\bar{C}_2^\log(N))$ has infinite rank, which is the statement of Theorem 8.1.

The last argument does not use efficiently the structure of non-symmetric operad $C_2^\log$, but only the underlying topological monoid structure. We have described the non-symmetric operad $C_2^\log$ in the hope that it may shed in future more light on the structure of $H_*([P_n, P_n])$. 


Computing the homology of $[P_n, P_n]$ as a $R(n)$-module seems a difficult task, in particular because $R(n)$ is not a principal ideal domain and we lack a good classification of finitely generated modules over $R(n)$. We can still observe that $H_\ast([P_n, P_n])$ is finitely generated over $R(n)$: indeed the chain complex $\mathcal{C}_\ast^{\operatorname{log}}$ is finitely generated over $R(n)$, and $R(n)$ is a noetherian ring.

Computing $H_\ast([P_n, P_n])$ directly as an abelian group seems not to be easy either. In Theorems 6.1 and 8.1 we have proved that $H_k([P_n, P_n])$ contains a free abelian group of infinite rank for all $1 \leq k \leq n - 2$; we conjecture that $H_\ast([P_n, P_n])$ is indeed a free abelian group, and that in particular it is torsion-free.

Our conjecture is related to a conjecture by Denham \cite{8} about the structure of the homology of the Milnor fibre of a complexified real arrangement; this conjecture was investigated also by Settepanella in \cite{21}. We observe that for $n = 3$ our conjecture is true because $[P_n, P_n]$ is a free group. It would also be interesting to study $H_\ast([P_n, P_n])$ as a representation, in the following way. Let $B_n$ denote Artin’s braid group on $n$ strands \cite{2}, and recall that there is a short exact sequence

$$1 \to P_n \to B_n \to \mathfrak{S}_n \to 1,$$

where $\mathfrak{S}_n$ denotes the $n$-th symmetric group. In particular $P_n$ is a normal subgroup of $B_n$; since $[P_n, P_n]$ is a characteristic subgroup of $P_n$, it is also a normal subgroup of $B_n$ and we have a short exact sequence

$$1 \to [P_n, P_n] \to B_n \to B_n/[P_n, P_n] \to 1.$$

Therefore the homology $H_\ast([P_n, P_n])$ is a representation of the group $B_n/[P_n, P_n]$, which fits into the following short exact sequence

$$1 \to P_n^{\operatorname{ab}} \cong \mathbb{Z}^{\left(\frac{n}{2}\right)} \to B_n/[P_n, P_n] \to \mathfrak{S}_n \to 1.$$

It would be interesting to compare the representation $H_\ast([P_n, P_n])$ of $B_n/[P_n, P_n]$ with the representation theories of the groups $P_n^{\operatorname{ab}}$ and $\mathfrak{S}_n$.

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