REGULARITY OF SINGULAR SET IN OPTIMAL TRANSPORTATION

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Abstract. In this work, we establish a regularity theory for the optimal transport problem when the target is composed of two disjoint convex domains, denoted $\Omega^*_i$ for $i = 1, 2$. This is a fundamental model in which singularities arise. Even though the singular set does not exhibit any form of convexity a priori, we are able to prove its higher order regularity by developing novel methods, which also have many other interesting applications (see Remark 1.1). Notably, our results are achieved without requiring any convexity of the source domain $\Omega$. This aligns with Caffarelli’s celebrated regularity theory [2].

1. Introduction

Let $f, g \in L^1(\mathbb{R}^n)$ represent two probability densities concentrated on bounded open sets $\Omega, \Omega^* \subset \mathbb{R}^n$, respectively. Suppose there exists a positive constant $\lambda$ such that $\frac{1}{\lambda} < f, g < \lambda$ in $\Omega, \Omega^*$, respectively. Invoking Brenier’s theorem [1], we identify two globally Lipschitz convex functions, $u$ and $v$, both defined on $\mathbb{R}^n$, satisfying the following conditions:

\begin{equation}
(Du)^+_x f = g, \text{ with } Du(x) \in \overline{\Omega} \text{ for almost every } x \in \mathbb{R}^n; \tag{1.1}
\end{equation}

\begin{equation}
(Dv)^+_y g = f, \text{ with } Dv(y) \in \overline{\Omega} \text{ for almost every } y \in \mathbb{R}^n. \tag{1.2}
\end{equation}

From the regularity theory of Caffarelli [2], if $\Omega^*$ is convex, then the potential function $u$ belongs to $C^{1,\beta}_{\text{loc}}(\Omega)$ for some $\beta \in (0, 1)$ and is strictly convex inside $\Omega$.

If $\Omega^*$ consists of two disjoint convex domains, as seen in the example of [2] when mapping the unit disc onto two shifted half-discs, there appears a singular set $\mathcal{F} \subset \Omega$ on which $u$ is not differentiable. In this paper, considering the case when $\Omega^* = \Omega^*_1 \cup \Omega^*_2$, with $\Omega^*_1$ and $\Omega^*_2$ being bounded convex domains separated by a hyperplane $\mathcal{H}$, we will establish the regularity of the singular set $\mathcal{F}$.

From previous works [7, 16], one knows $\mathcal{F} \subset \Omega$ is a Lipschitz hypersurface, partitioning $\Omega$ into two subdomains $\Omega_i$, $i = 1, 2$, such that

\begin{equation}
(Du)^+_x (f\chi_{\Omega_i}) = g\chi_{\Omega^*_i} \text{ for } i = 1, 2. \tag{1.3}
\end{equation}

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Moreover, $F$ is characterised as $\partial \Omega_i \cap \Omega$ for $i = 1, 2$, which can be expressed as the graph of a certain Lipschitz function.

Define the functions $u_i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, 2$, as follows:

$$u_i(x) := \sup \{ L(x) : L \text{ is affine}, L \leq u \text{ in } \Omega_i, \text{ and } DL(x) \in \Omega_i^* \} \quad \forall x \in \mathbb{R}^n.$$  

From [7, 16], one has $F = \{ u_1 = u_2 \} \cap \Omega$. Hence, if $u_i$ are differentiable at $x \in F$, the unit normal vector to the hypersurface $F$ at $x$ satisfies

$$\nu_F(x) = \frac{Du_1(x) - Du_2(x)}{|Du_1(x) - Du_2(x)|}. $$

Under the additional assumption of strict convexity for $\Omega$, $\Omega_1^*$, and $\Omega_2^*$, the hypersurface $F$ has been proven to be locally $C^{1,\beta}$ regular for some $\beta \in (0, 1)$, see [7, 16]. We remark that the corresponding $C^{1,\beta}$ regularity of the free boundary in the optimal partial transport was obtained in [5, 15].

Notably, the domain convexity plays an important role in previous works. Indeed, if $\Omega$ is convex, one can invoke Caffarelli’s regularity theory [2] to derive the following key property:

$$\frac{1}{\lambda^2} \chi_{\Omega^*} \leq \det D^2v \leq \lambda^2 \chi_{\Omega^*}$$

interpreted in the Alexandrov sense. Thanks to (1.6), by using a localisation lemma one can apply the methods from [3] to obtain a quantitative strict convexity estimate for $v$ near $Du_i(x_0)$ for $x_0 \in F$, which in turn implies $C^{1,\beta}$ regularity of $u_i$, $i = 1, 2$. [7, 16]. However, the non-convexity of $\Omega$ renders (1.6) inapplicable, creating a significant obstacle in deriving the $C^{1,\beta}$ regularity as outlined above.

In this paper, we employ some new ideas and techniques to overcome these difficulties. Our first result is the $C^{1,\beta}$ regularity of $F$ without necessitating any convexity of $\Omega$.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $\Omega^* = \Omega_1^* \cup \Omega_2^*$ be a union of bounded convex domains $\Omega_i^*$ in $\mathbb{R}^n$, separated by a hyperplane $\mathcal{H}$. If $\frac{1}{\lambda} < f, g < \lambda$ within $\Omega$ and $\Omega^*$, respectively, for some positive constant $\lambda$, then the hypersurface $F$ is $C^{1,\beta}$ regular inside $\Omega$ for some $\beta \in (0, 1)$. Additionally, $u_i \in C^{1,\beta}_{\text{loc}}(\Omega \cap \overline{\Omega_i})$ for $i = 1, 2$.

A much more interesting and challenging problem concerns the higher order regularity of $F$, which is precisely solved in the main theorem of this paper:

**Theorem 1.2.** Under the assumptions of Theorem 1.1, and further assuming that $\Omega_i^*$ are $C^2$ and uniformly convex for $i = 1, 2$, and the densities $f \in C^\alpha_{\text{loc}}(\Omega)$ and $g \in C^\alpha(\Omega^*)$ for some $\alpha \in (0, 1)$, it follows that the singular set $F$ is $C^{2,\alpha}$ smooth.
Note that for each $i = 1, 2$, $u_i$ satisfies the second boundary value problem:

$$\begin{align*}
\det D^2 u_i &= \frac{f}{g(Du_i)} \quad \text{in } \Omega_i, \\
Du_i(\Omega_i) &= \Omega_i^*.
\end{align*}$$

From (1.5), it suffices to prove $u_i$ is $C^{2,\alpha}$ up to the singular set $F \subset \partial \Omega_i$. All previous related works require some sort of convexity of domains: In the study of global regularity of the optimal transport map, Caffarelli [4] requires the domains to be uniformly convex (see also [11, 18]). Recently, in [9] Chen-Liu-Wang relax this condition from uniform convexity to merely convexity. In the study of optimal partial transport [10], the semi-convexity of the free boundary plays a crucial role, which follows from the interior ball property [5, 12, 13].

The most striking point of this paper is that we manage to establish the higher order regularity of $u_i$ up to the singular set $F$, which does not exhibit any form of convexity a priori.

A crucial ingredient in the proof of Theorem 1.2 is the following obliqueness estimate:

**Proposition 1.1.** Under the hypotheses of Theorem 1.2, assume that $0 \in F$. Denote by $y_0 = Du_1(0) \in \partial \Omega_1^*$ and $\hat{y}_0 = Du_2(0) \in \partial \Omega_2^*$. Let $\nu, \nu^*, \hat{\nu}^*$ be the unit inner normals to $\Omega_1, \Omega_1^*, \Omega_2^*$ at $y_0, y_0, \hat{y}_0$, respectively. Then $\nu \cdot \nu^* > 0$ and $-\nu \cdot \hat{\nu}^* > 0$.

In previous works such as [4], [9] and [17], the obliqueness estimate was proved relying on the convexity of the domains. The exploration continued in [10], where the authors investigated the optimal partial transport problem and successfully established the obliqueness estimate at points on the free boundary. Notably, even though the free boundary is not convex in general, the interior ball property implies the semi-convexity of the free boundary, which enables the authors to make substantial progress in [10].

However, the interior ball property is not applicable to the problem considered in this paper. In fact, due to the absence of any variant of convexity of the singular set, all existing methods [4, 9, 10, 17] do not apply. Therefore, new observations and ideas are needed to establish the obliqueness estimate. In order to prove Proposition 1.1, it suffices to rule out the following three distinct scenarios:

- **Case I:** $\nu \cdot \nu^* = 0$, $\hat{\nu} \cdot \hat{\nu}^* = 0$, and $\nu, \nu^*, \hat{\nu}^*$ are non-coplanar.
- **Case II:** $\nu \cdot \nu^* = 0$, $\hat{\nu} \cdot \hat{\nu}^* = 0$, and $\nu, \nu^*, \hat{\nu}^*$ are coplanar.
- **Case III:** $\nu \cdot \nu^* = 0$, $\nu \cdot \hat{\nu}^* > 0$.

Although a detailed proof will be given in Section 4, here we would like to summarise some innovative ideas for each case.

For **Case I and III**: The approach is to show the splitting behaviour of the singular set and $\partial \Omega_1^*$ during a blow-up procedure. The strategy unfolds as follows:
(1) By a proper affine transformation, we can make the three normals \( \nu, \nu^*, \hat{\nu}^* \) mutually perpendicular to each other. By estimating the shape of sub-level sets of \( v \), we have a \((3 + \epsilon)\)-uniform convexity estimate of \( v \) at \( y_0 = Du_1(0) \) and \( \hat{y}_0 = Du_2(0) \). Utilising the duality between \( u \) and \( v \), we then obtain \( C^{1, \frac{1}{2} - \epsilon} \) estimates of \( u_1 \) and \( u_2 \) at 0. In the light of (1.5), we derive a \( C^{1, \frac{1}{2} - \epsilon} \) estimate of \( F \) at 0. Additionally, by combining the shape estimate on the sub-level set of \( v \) at \( y_0 \) with the tangential \( C^{1,1 - \epsilon} \) estimate of \( v \) at \( \hat{y}_0 \) in the \( \nu^* \)-direction, we obtain a rather unexpected “above the tangent” property, namely \( F \cap \text{span}\{\nu, \nu^*\} \) is above its tangent plane at 0.

(2) By applying the first blow-up, we can obtain that \( F \) becomes flat in \( n - 2 \) directions in the limit profile. Here, the term blow-up refers to the limit of a sequence of sets normalised by some affine transformations. The shape estimates for the sub-level set of \( v \) control these affine transformations, which surprisingly provide just enough information to confirm the flatness of \( F \) in \( n - 2 \) directions in the limit. This success is largely attributable to the \( C^{1, \frac{1}{2} - \epsilon} \) estimate of \( F \) at 0. The “above the tangent” property then guarantees that the limit domains are well-positioned.

(3) By applying the second blow-up, the limit of \( \Omega_1 \) becomes a cylindrical shape, and the limit of \( \partial \Omega^*_1 \) becomes the graph of a non-negative quadratic polynomial. Subsequently, after the secondary blow-up, we can employ techniques from [9, 10] to demonstrate the transformation of both \( \Omega_1 \) and \( \Omega^*_1 \) into cylindrical configurations, analogous to the scenarios depicted in [9].

For Case II: Observe that \( \nu^* \) is parallel to \( \hat{\nu}^* \). We can prove the width of the centred section of \( v \) in the \( \nu^* \)-direction is very small. On one hand, by combining this width estimate with (1.5) and the disjointness of \( \Omega^*_1 \) and \( \Omega^*_2 \), we can show that \( F \) becomes progressively flatter in the \( \nu^* \)-direction during the blow-up process. On the other hand, we establish a strict convexity estimate for the normalisation of \( u_1 \) that is independent of the blow-up scale, which however contradicts the flattening of \( F \).

Remark 1.1. The new ideas and methods developed in this paper are also useful for investigating many other interesting problems. In [8], we applied these methods to establish the obliqueness estimate at the intersection points of the free and fixed boundaries, which leads to the sharp global Sobolev regularity of the optimal map between active regions. More recently, in [6], we utilized these methods to investigate the partial regularity of free boundaries in optimal partial transportation between non-convex polygonal domains. In particular, we demonstrate that the free boundary is smooth except at a finite number of singular points. Additionally, in [6], we use these methods to provide a precise description of the singular set of the optimal map between non-convex polygonal domains, by showing that the singular set is a smooth curve away from a finite number of points.
The structure of this paper is organised as follows: Section 2 introduces important notations and definitions used for the subsequent discussions. Section 3 is dedicated to establishing the $C^{1,\beta}$ regularity of the singular set $F$. Section 4 focuses on the proof of the obliqueness estimate, a crucial step in our analysis. Finally, Section 5 outlines the proof of the $C^{2,\alpha}$ regularity of the singular set $F$.

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2. Preliminaries

Given a convex function $w : \mathbb{R}^n \to \mathbb{R}$, its subdifferential at $x \in \mathbb{R}^n$ is defined as:

$$\partial^- w(x) := \{ y \in \mathbb{R}^n : w(z) \geq w(x) + y \cdot (z - x) \text{ for all } z \in \mathbb{R}^n \}.$$  

(2.1)

For any subset $A \subset \mathbb{R}^n$, $\partial^- w(A) := \bigcup_{x \in A} \partial^- w(x)$.

Let $U \subset \mathbb{R}^n$ be an open set and $C$ be a positive constant. We say that a convex function $w$ satisfies the Monge-Ampère inequality:

$$\frac{1}{C} \chi_U \leq \det D^2 w < C \chi_U$$

in the Alexandrov sense if and only if for any Borel set $A$,

$$\frac{1}{C} |A \cap U| \leq |\partial^- w(A)| \leq C |A \cap U|,$$

(2.3)

where $| \cdot |$ denotes the $n$-dimensional volume.

Now, let us consider the functions and sets $u, v, u_i$ and $\Omega_i, \Omega^*_i$ for $i = 1, 2$ as delineated in the introduction. Specifically, we have the conditions (L.1)–(L.3) and $u_i$ satisfies (L.4). It is important to note that $\Omega$ can be expressed as the union $\Omega = \Omega_1 \cup \Omega_2 \cup F$, and within each $\Omega_i$, $i = 1, 2$, we have $u_i = u$. Additionally, we have the following dual relationships:

$$v(y) = \sup_{x \in \Omega} \{ y \cdot x - u(x) \} \text{ for all } y \in \mathbb{R}^n;$$

(2.4)

$$u_i(x) = \sup_{y \in \Omega^*_i} \{ y \cdot x - v(y) \} \text{ for all } x \in \mathbb{R}^n, \ i = 1, 2.$$  

(2.5)

Given that $\Omega^*_i$ are convex for $i = 1, 2$, Caffarelli’s regularity theory \cite{2} ensures that $u_i \in C^{1,\beta}_{\text{loc}}(\Omega_i)$ for some $\beta \in (0,1)$, and they satisfy the Monge-Ampère inequality in the sense of Alexandrov:

$$C^{-1} \chi_{\Omega_i} \leq \det D^2 u_i \leq C \chi_{\Omega_i} \text{ for } i = 1, 2.$$  

(2.6)

Lastly, we note that $F = \{ x \in \Omega : u_1(x) = u_2(x) \}$.

To proceed further, we introduce some useful definitions and notations. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a convex function whose graph does not contain any infinite straight line.
Definition 2.1. Given a point \( y_0 \in \mathbb{R}^n \) and a small positive constant \( h \), we define the centred sections of the convex function \( w \) at \( y_0 \) with height \( h \) as:

\[
S_h^c[w](y_0) := \{ y \in \mathbb{R}^n : w(y) < w(y_0) + (y - y_0) \cdot \bar{p} + h \},
\]

where \( \bar{p} \in \mathbb{R}^n \) is chosen such that the centre of mass of \( S_h^c[w](y_0) \) coincides with \( y_0 \). Additionally, we define the sub-level set of \( w \) at \( y_0 \) with height \( h \) as:

\[
S_h[w](y_0) := \{ y \in \mathbb{R}^n : w(y) < \ell_{y_0}(y) + h \},
\]

where \( \ell_{y_0} \) denotes an affine supporting function of \( w \) at \( y_0 \).

Remark 2.1. If \( w(0) = 0 \) and \( w \geq 0 \), by \cite{10} Remark 2.2, we have that

\[
w \leq Ch \quad \text{in} \quad S_h^c[v](0),
\]

where \( C \) is a constant depending only on \( n \).

In subsequent discussions, a constant will be referred to as universal if it depends solely on the parameters \( n, \Omega, \Omega^*, \lambda \), and \( \text{dist}(x_0, \partial \Omega) \). Additionally, we will employ the notation \( C_n \) to represent a positive constant dependent exclusively on the dimension \( n \), with the acknowledgment that its value may change across different contexts. The vectors \( e_i \), where \( i = 1, \ldots, n \), will be used to denote the standard basis vectors in the coordinate system.

3. \( C^{1,\beta} \) Regularity of \( \mathcal{F} \).

In this section, we continue to utilise the notations established in Section 2 for \( \Omega, \Omega^*, \Omega_i^* \) \((i = 1,2)\), \( u, v, u_i, \mathcal{F} \). Consider a specific point \( x_0 \in \mathcal{F} \subset \Omega \). Select sequences \( \{x_k\} \subset \Omega_1 \) and \( \{\hat{x}_k\} \subset \Omega_2 \), both converging to \( x_0 \) as \( k \to \infty \). By applying a translation of coordinates, we may assume \( x_0 = 0 \). Without loss of generality, and potentially passing to subsequences, we assume \( \lim_{k \to \infty} Du(x_k) = y_0 \in \Omega_1 \), and \( \lim_{k \to \infty} Du(\hat{x}_k) = \hat{y}_0 \in \Omega_2 \), respectively. This ensures that \( y_0, \hat{y}_0 \in \partial^- u(0) \). By (2.14), we deduce that \( 0 \in \partial^- v(y_0) \cap \partial^- v(\hat{y}_0) \).

By subtracting an appropriate constant, we may assume \( v(y_0) = v(\hat{y}_0) = 0 \) and \( v \geq 0 \). Define \( \Sigma = \{ y \in \mathbb{R}^n : v(y) = 0 \} \) and denote the set of extreme points of \( \Sigma \) as \( \text{ext}(\Sigma) \).

Recall Equation (1.2), especially \( Du(y) \in \Omega \) for almost every \( y \in \mathbb{R}^n \). Consider a section of \( v \), namely \( Z := \{ y \in \mathbb{R}^n : v(y) < \ell(y) \} \), with \( \ell \) being an affine function. Provided that \( Z \) is bounded and contains a point \( y \) such that \( 0 \in \partial^- v(y) \), since \( 0 \in \mathcal{F} \) is inside the interior of \( \Omega \), we can assert that \( \partial^- v(y) \cap \Omega \neq \emptyset \). Denoting the centre of mass of \( Z \) by \( z \), and employing John’s lemma, we identify an affine transformation \( L \) such that \( L(z) = 0 \) and \( B_1(0) \subset LZ \subset B_{C_n}(0) \), where \( C_n \) solely depends on the dimension \( n \).

From \cite{14} Inequality (4)], we deduce that:

\[
|\inf_Z (v - \ell)|^n \geq C_1 \left( \frac{1}{2} |Z| \right) \cap \Omega^* |Z|,
\]
where \((\frac{1}{2}Z)\) represents the dilation of \(Z\) with respect to \(z\), and \(C_1\) is a constant depending on \(n\) and \(\lambda\). From [14] Proposition 1, we have the following bound:

\[
|v(y) - \ell(y)|^n \leq C_2 |Z \cap \Omega^*| |Z| \text{ dist}(L(y), \partial(L(Z))) \quad \text{for } y \in Z,
\]

where \(C_2\) is a constant depending only on \(n\), \(\lambda\), and the diameter of \(\Omega\).

**Lemma 3.1.** The set \(\Sigma\) is both bounded and closed, and \(\text{ext}(\Sigma) \subset \partial \Omega^*\).

**Proof.** First, we prove that \(\Sigma\) is bounded and closed. Assume, to the contrary, that this is not the case. By convexity, \(\Sigma\) would then contain a half-line of the form \(\{q + te : t \geq 0\}\) for some \(q \in \Sigma\) and \(e \in \mathbb{S}^{n-1}\). For any \(y \in \mathbb{R}^n\) and \(x \in \partial^{-}v(y)\), by the convexity of \(v\) one has

\[
(x - 0) \cdot (y - q - te) \geq 0.
\]

As \(t\) tends to infinity, this implies that \(x \cdot e \leq 0\). Given the arbitrariness of \(y\), it follows that

\[
\partial^{-}v(\mathbb{R}^n) \subseteq \{x \cdot e \leq 0\}.
\]

Since we have \((Dv)^\sharp z = f\), it can be deduced that \(\Omega \subseteq \partial^{-}v(\mathbb{R}^n) \subseteq \{x \cdot e \leq 0\}\), which contradicts the fact that \(0 \in \Omega\). Thus, \(\Sigma\) must be bounded and closed.

Next, we prove that \(\text{ext}(\Sigma) \subseteq \partial \Omega^*\). Suppose to the contrary that there exists a point \(p \in \text{ext}(\Sigma)\) such that \(p\) is either in \(\Omega^*\) or in \(\mathbb{R}^n \setminus \overline{\Omega^*}\). If \(p \in \text{ext}(\Sigma) \cap \Omega^*\), then we have \(0 \in \partial^{-}v(p) \cap \Omega\). Referring to Proposition 2 and the subsequent discussion in [14], \(v\) is inferred to be \(C^1\) and strictly convex in a neighborhood of \(p\). Consequently, by duality, the function \(u\) must be differentiable at 0, leading to a contradiction. In the case where \(p \in \text{ext}(\Sigma) \cap (\mathbb{R}^n \setminus \overline{\Omega^*})\), the application of inequalities (3.1) and (3.2), along with the arguments presented in Section 3.1 of [14], also yields a contradiction. Hence, we conclude that \(\text{ext}(\Sigma) \subset \partial \Omega^*\). \(\square\)

Since exposed points are dense in the set of extreme points, we can choose points \(p \in \text{ext}(\Sigma) \cap \partial \Omega^*_1\) and \(\hat{p} \in \text{ext}(\Sigma) \cap \partial \Omega^*_2\) such that both are exposed points of \(\Sigma\). In this context, we establish the following localisation properties.

**Lemma 3.2.** Given any \(r_0 > 0\), there exists an \(h_0 > 0\) such that for all \(h \leq h_0\), the inclusion \(S_h^c[v](p) \subset B_{r_0}(p)\) holds.

**Proof.** Since \(p\) is an exposed point of \(\Sigma\), there exists a unit vector \(e \in \mathbb{S}^{n-1}\) such that

\[
\Sigma \subset \{y : (y - p) \cdot e \leq 0\},
\]

\[
\Sigma \cap \{y : (y - p) \cdot e = 0\} = \{p\}.
\]

Since \(S_h^c[v](p)\) is centred at \(p\), if \(S_h^c[v](p) \cap \{y : (y - p) \cdot e \geq 0\} \subset B_r(p)\), then it follows that \(S_h^c[v](p) \subset B_{C_{n,r}}(p)\).
Observing that $v > 0$ in $\mathbb{R}^n \setminus \Sigma$ and given any small constant $r_0 > 0$, the convexity and continuity of $v$ ensure the existence of a constant $\delta_0$ such that

\begin{equation}
(3.3) \quad v > \delta_0 \quad \text{in} \quad \{y : (y - p) \cdot e \geq 0\} \setminus B_{\frac{\delta_0}{C_n}}(p).
\end{equation}

By (2.9) we have $v \leq C_n h$ in $S^n_h[v](p)$. Setting $h_0 = \frac{\delta_0}{C_n}$, for all $h \leq h_0$, we then have $v \leq \delta_0$ in $S^n_h[v](p)$. From (3.3), it follows that

\[ S^n_h[v](p) \cap \{y : (y - p) \cdot e \geq 0\} \subset B_{\frac{\delta_0}{C_n}}(p) \quad \forall \ h \leq h_0. \]

Therefore, we conclude that $S^n_h[v](p) \subset B_{r_0}(p)$ whenever $h \leq h_0$. \qed

The next lemma establishes the localisation of centred sections near the point $p$.

**Lemma 3.3.** For any given $\frac{1}{2} > r_0 > 0$, let $h_0$ be the constant specified in Lemma 3.2. There exists a positive constant $\delta_0 > 0$ such that for all points $y \in \overline{\Omega}_1 \cap B_{h_0}(p)$ and for all $h \leq h_0$, the inclusion $S^n_{h_0}[v](y) \subset B_{2r_0}(p)$ holds.

**Proof.** Assume, for the sake of contradiction, that the statement is false. Then, there exist sequences $\{y_k\} \subset \overline{\Omega}_1$ converging to $p$ and $\{h_k\}$ with $h_k \leq h_0$, such that

\begin{equation}
(3.4) \quad S^n_{h_k}[v](y_k) \cap (\mathbb{R}^n \setminus B_{2r_0}(p)) \neq \emptyset.
\end{equation}

Given that $S^n_{h_k}[v](y_k)$ is balanced with respect to $y_k$, there exists a line segment $I_k \subset S^n_{h_k}[v](y_k)$ centred at $y_k$ satisfying $|I_k| \geq C_n r_0$. Without loss of generality, by taking a subsequence if necessary, we can assume that $h_k \to \bar{h} \leq h_0$ and $I_k \to I_{\infty}$ as $k \to \infty$, where $I_{\infty}$ is a segment centred at $p$ and satisfying

\begin{equation}
(3.5) \quad |I_{\infty}| \geq C_n r_0.
\end{equation}

First, we claim that $S^n_{h_k}[v](y_k)$ is uniformly bounded. From Lemma 3.1, $\Sigma$ is compact. Owing to the non-negativity and convexity of $v$, we deduce the existence of a large constant $R > 1$ satisfying

\[ v(y) > h_0 + \sup_{B_R(p)} v \quad \forall \ y \in \mathbb{R}^n \setminus B_R(p). \]

Suppose $S^n_{h_k}[v](y_k) = \{y \in \mathbb{R}^n : v(y) < \ell_k\}$, for some affine function $\ell_k$ that fulfils $\ell_k(y_k) = v(y_k) + h_k$. For any given unit vector $e \in \mathbb{S}^n$, we may assume $\ell_k$ is decreasing in the $e$-direction, otherwise consider $-e$ instead.

Since $y_k$ converges to $p$, we may assume that $y_k \in B_1(p)$ for all $k$. It can be verified that $y_k + 3Re \in \mathbb{R}^n \setminus B_R(p)$, leading to the inequality

\[ v(y_k + 3Re) > h_0 + \sup_{B_1(p)} v \geq \ell_k(y_k) \geq \ell_k(y_k + 3Re). \]
This implies \( y_k + 3\text{Re} \notin S^c_{h_k}[v](y_k) \). Since \( S^c_{h_k}[v](y_k) \) is centred at \( y_k \), we deduce \( y_k - 3C_n\text{Re} \notin S^c_{h_k}[v](y_k) \). Given that \( e \) is arbitrary, we conclude the claim that

\[
S^c_{h_k}[v](y_k) \subset B_{3C_nR}(y_k) \subset B_{3C_nR+1}(p).
\]  

Now, let’s consider a subsequence (without changing notation) such that \( S^c_{h_k}[v](y_k) \) converges to a limit convex set \( S_\infty \) in the Hausdorff distance, and \( h_k \to \bar{h} \leq h_0 \) as \( k \to \infty \). Write \( \ell_k = x_k \cdot (y - y_k) + h_k \) with \( x_k \in \mathbb{R}^n \), the global Lipschitz continuity of \( v \) ensures that \( \sup_k |x_k| \leq \|v\|_{\text{Lip}} < \infty \). Consequently, after possibly taking another subsequence, we can assert that \( x_k \to \bar{x} \) as \( k \to \infty \).

In the case when \( \bar{h} > 0 \), it follows that \( S_\infty = \{ y \in \mathbb{R}^n : v(y) < \bar{x} \cdot (y - p) + \bar{h} \} = S^c_{\bar{h}}(v)(p) \).

By (3.4), passing to limit we have

\[ S^c_{\bar{h}}[v](p) \cap (\mathbb{R}^n \setminus B_{2\rho_0}(p)) \neq \emptyset, \]

leading to a contradiction with Lemma 3.2.

In the case when \( \bar{h} = 0 \), we observe that \( 0 \leq v \leq \bar{x} \cdot (y - p) \) for all \( y \) in \( I_\infty \). Given that \( I_\infty \) is centred at \( p \), it follows that \( v \) must be identically zero on \( I_\infty \), implying that \( I_\infty \subseteq \Sigma \). This contradicts to the fact that \( p \) is an exposed point of \( \Sigma \).

Take \( \rho_0 \) small so that \( B_{2\rho_0}(p) \cap \Omega^*_2 = \emptyset \). Let \( h_0, \delta_0 \) be as in Lemma 3.2 and 3.3. Define \( \rho_0 := \frac{1}{2}\text{dist}(0, \partial \Omega) \). We will show the differentiability of \( v \) near \( p \) in the following lemma.

**Lemma 3.4.** There exists a constant \( \bar{r}_1 < \frac{1}{3}\delta_0 \), such that \( v \) is differentiable at all \( y \in \overline{\Omega^*_1} \cap B_{\bar{r}_1}(p) \).

**Proof.** First, we establish the differentiability of \( v \) at the point \( p \).

Consider \( n \) linearly independent unit vectors \( \hat{e}_i \) (may not be orthogonal), where \( i = 1, \ldots, n \). By convexity of \( v \), it suffices to show that \( v(p + t\hat{e}_i) \) is differentiable at \( t = 0 \) as a convex function of a single variable \( t \), for each \( i = 1, \ldots, n \).

Suppose to the contrary that \( v \) is not differentiable at \( p \). Then there exists a unit vector \( \hat{e} \), such that \( \{ p + te : t < 0 \} \cap \Omega^*_1 \neq \emptyset \) , and \( v(p + te) \) is not differentiable at \( t = 0 \). By subtracting an affine function \( \ell_1 \), we can assume the following behaviour:

\[
\begin{align*}
  v & \geq 0 & \text{on } \mathbb{R}^n, \\
  v(p + te) & = o(t) & \text{for } t < 0, \\
  v(p + te) & = at + o(t) & \text{for } t > 0,
\end{align*}
\]

where \( a > 0 \) is a constant. Note that the target of \( Dv \) is changed to \( \hat{\Omega} = \Omega - \{ D\ell_1 \} \), a translation of \( \Omega \). Given that \( \partial^- v(p) \cap \hat{\Omega} \) is non-empty, the estimates (3.1)–(3.2) still apply.
Define \( y_0 := p - t_0 e \) for \( t_0 \in (0, \frac{1}{2} \delta_0) \) sufficiently small. Let’s consider the centred section
\[
S_h^c[v](y_0) = \{ v < \ell \},
\]
where \( \ell \) is an affine function satisfying \( \ell(y_0) = v(y_0) + h \). Consider the intersection of boundary \( \partial S_h^c[v](y_0) \) with the line \( \{ p + te : t \in \mathbb{R} \} \), and let
\[
\partial S_h^c[v](y_0) \cap \{ p + te : t \in \mathbb{R} \} = \{ p - t_1(h)e, \ p + t_2(h)e \},
\]
where \( t_1(h) > t_0 \) and \( t_2(h) > -t_0 \). By [5, Lemma A.8], we have that \( S_h^c[v](y_0) \) varies continuously with respect to \( h \in (0, h_0) \). Hence, \( t_1(h) \) and \( t_2(h) \) depend continuously on \( h \). Since \( S_h^c[v](y_0) \) is centred at \( y_0 \), we have the following bounds:
\[
t_0 < t_1(h) < C_n(t_0 + t_2(h)),
\]
where \( C_n \) is a constant depending only on \( n \).

We first claim that \( t_2(h_0) \geq t_0 \), provided \( t_0 \) is chosen sufficiently small initially. Suppose, to the contrary, that \( t_2(h_0) < t_0 \). Then, by (3.7), we have
\[
\ell(p + t_2(h_0)e) = v(p + t_2(h_0)e) \leq 2at_0,
\]
provided \( t_0 \) is small enough. By (3.7) and (3.8), we obtain
\[
\ell(p - t_1(h_0)e) = v(p - t_1(h_0)e) = o(t_0).
\]
By (3.9), (3.10), and the balance of \( S_h^c[v](y_0) \) with respect to \( y_0 \), we have
\[
\ell(y_0) \leq C_n(2at_0 + o(t_0)),
\]
contradicting the fact that \( \ell(y_0) \geq \ell(y_0) - v(y_0) = h_0 \), provided \( t_0 \ll h_0 \) was chosen initially. Hence, the claim is proved.

We then claim that \( \liminf_{h \to 0} t_2(h) \leq 0 \), provided \( t_0 \) is chosen sufficiently small initially. Suppose, to the contrary, that there exists a constant \( a_0 \in (0, t_0) \) and a sequence \( \{ h_j \} \) converging to 0, such that \( t_2(h_j) \geq a_0 \) for all \( j \). Hence, from (3.7),
\[
v(p + t_2(h_j)e) \geq \frac{1}{2} a_0,
\]
provided \( t_0 \) is small enough. Moreover, since \( S_{h_j}^c[v](y_0) \) is centred at \( y_0 \) we have the segment
\[
I = \left\{ p + \theta e : -t_0 - \frac{1}{C_n} t_0 < \theta < a_0 \right\} \subset S_{h_j}^c[v](y_0) \quad \forall \ j = 1, 2, \ldots .
\]
Suppose \( S_{h_j}^c[v](y_0) = \{ y \in \mathbb{R}^n : v(y) < v(y_0) + x_j \cdot (y - y_0) + h_j \} \) for some vector \( x_j \in \mathbb{R}^n \). Since \( v \) is globally Lipschitz, \( \sup_j |x_j| \leq \| v \|_{Lip} < \infty \). Consequently, by taking a subsequence if necessary, we may assume \( x_j \to \bar{x} \) as \( j \to \infty \). In the limit, we thus obtain
\[
0 \leq v(y) \leq \bar{\ell}(y) \quad \forall y \in I,
\]
where \( \bar{\ell}(y) = v(y_0) + \bar{x} \cdot (y - y_0) \). Since \( y_0 \in I \), \( v(y_0) = \bar{\ell}(y_0) \) and \( v \leq \bar{\ell} \) on \( I \), the convexity of \( v \) implies that \( v \equiv \bar{\ell} \) on \( I \), namely \( v(p + te) \) is affine with respect to \( t \) for
Assume that \( \text{dist} \) (3.18) satisfying \( \ell(y_0) = v(y_0) + h' \).

Evaluating the affine function \( \ell \) at specific points yields:

\[
\ell(p - t_1 e) = v(p - t_1(h')e) = o(t_1) = o(t_0),
\]

\[
\ell(p + t_2(h')e) = v \left( p + \frac{1}{M} t_0 e \right) = \frac{a}{M} t_0 + o(t_0).
\]

The above equations imply that \( \ell \) is increasing along the \( e \) direction (provided \( t_0 \) is chosen sufficiently small at the beginning), leading to:

\[
\ell(p) - v(p) = \ell(p) \geq \ell(y_0) \geq \ell(y_0) - v(y_0) \approx \inf_{S_{h'}[v](y_0)} (v - \ell).
\]

Applying John’s lemma, we identify an affine transformation \( L \) such that \( L(y_0) = 0 \), and

\[
B_1(0) \subset L(S_{h'}[v](y_0)) \subset B_{C_n}(0).
\]

Since the affine transformation preserves the ratio of distances, we have

\[
\frac{\text{dist} \left( L(p), \partial L \left( S_{h'}[v](y_0) \right) \right)}{\text{diam}(L \left( S_{h'}[v](y_0) \right))} = \frac{\text{dist}(p, \partial S_{h'}[v](y_0))}{\text{diam}(S_{h'}[v](y_0))} \leq \frac{t_2(h')}{t_1(h')} \leq \frac{1}{M}.
\]

Hence, by (3.18)

\[
\text{dist} \left( L(p), \partial L \left( S_{h'}[v](y_0) \right) \right) \leq \frac{C_n}{M}.
\]

Since \( y_0 \in \Omega_1^* \cap B_{h_0}(p) \) and \( h' \leq h_0 \), Lemma 3.3 implies \( S_{h'}[v](y_0) \subset B_{2r_0}(p) \), ensuring that \( S_{h'}[v](y_0) \cap \Omega_2^* = \emptyset \). The convexity of \( \Omega_1^* \) leads to the volume comparison:

\[
\left| \left( \frac{1}{2} S_{h'}[v](y_0) \right) \right|_{y_0} \cap \Omega^* \geq C_n |S_{h'}[v](y_0) \cap \Omega^*|.
\]

Then by the estimate (3.11), we can obtain

\[
\inf_{S_{h'}[v](y_0)} (v - \ell)^n \geq C_1 C_n |S_{h'}[v](y_0) \cap \Omega^*||S_{h'}[v](y_0)|,
\]

where \( C_1 \) and \( C_n \) are constants independent of \( M \).
On the other hand, by estimates (3.22), (3.17) and (3.19), we can deduce
\[
\inf_{S_{\nu}([v](y_0))} (v - \ell) \leq |\ell(p) - v(p)| \leq \frac{C_2 C_n}{M} |S_{\nu}([v](y_0)) \cap \Omega^*| \|S_{\nu}([v](y_0))|,
\]
where $C_2$ is another constant independent of $M$. However, this contradicts the estimate (3.21), when the constant $M$ is sufficiently large.

Consequently, we conclude that the function $v$ is differentiable at the point $p$. By selecting a sufficiently small $\bar{r}_1 < \delta_0$, we further obtain:
\[
\partial^- v(B_{\bar{r}_1}(p)) \subset B_{\rho_0}(0) \subset \Omega.
\]
Last, for any point $y \in \overline{\Omega^*_1} \cap B_{\bar{r}_1}(p)$, we can employ the previous argumentation for establishing the differentiability of $v$ at $p$ to similarly show that $v$ is differentiable at $y$. \hfill \Box

**Remark 3.1.** Let $r_0, \delta_0, \rho_0$, and $\bar{r}_1$ be the constants given in preceding lemmas. According to (3.23), for any point $y \in \Omega^*_1 \cap B_{\bar{r}_1}(p)$, the set $\partial^- v(y) \cap \Omega$ is non-empty. By invoking Proposition 2 from [14] and the subsequent discussions, it follows that the function $v$ is continuously differentiable and exhibits strict convexity within $\Omega^*_1 \cap B_{\bar{r}_1}(p)$.

**Lemma 3.5** (Localisation). The function $v$ satisfies the Monge-Ampère inequality:
\[
\frac{1}{\lambda^2} \chi_{\Omega^*_1} \leq \det D^2 v \leq \lambda^2 \chi_{\Omega^*_1} \quad \text{in } B_{\bar{r}_1}(p)
\]
in the Alexandrov sense, as in (2.3).

**Proof.** It suffices to show $\det D^2 v$ has no mass in $B_{\bar{r}_1}(p) \setminus \Omega^*_1$, namely $|\partial^- v(B_{\bar{r}_1}(p) \setminus \Omega^*_1)| = 0$.

In fact, let $y \in B_{\bar{r}_1}(p) \setminus \Omega^*_1$, and $x \in \partial^- v(y)$. Owing to (3.23), it is guaranteed that $x$ belongs to $\Omega = \Omega_1 \cup \Omega_2 \cup F$. Recall that $u_i$ is $C^1$ and strictly convex inside $\Omega_i$ for $i = 1, 2$. If $x$ were to lie in either of $\Omega_i$, it would imply that $y \in \Omega^*_i$, being the gradient $Du_i(x)$. This, however, contradicts our initial assumption that $y \in B_{\bar{r}_1}(p) \setminus \Omega^*_1$. Consequently, it must be that $x$ resides in the set $F$.

This reasoning concludes that the subdifferential $\partial^- v$ maps $B_{\bar{r}_1}(p) \setminus \Omega^*_1$ entirely into $F$. Therefore, the Lebesgue measure of the image set satisfies $|\partial^- v(B_{\bar{r}_1}(p) \setminus \Omega^*_1)| = 0$. \hfill \Box

Utilising Lemmas 3.2 and 3.3, we are able to determine constants $\bar{r}_0 < \bar{r}_1$ and $h_0 > 0$ such that:
\[
S_{\nu}([v](y)) \subset B_{\bar{r}_1}(p) \quad \forall y \in B_{\bar{r}_0}(p) \cap \Omega^*_1, \quad \forall h \leq h_0.
\]
With these preparations, we are now ready to prove Theorem 1.1 as follows.

**Proof of Theorem 1.1.** By (3.21) and (3.25), we can invoke the proof of [4, Theorem 7.13] to obtain a quantitative uniform convexity of $v$ within $B_{\bar{r}_0}(p) \cap \Omega^*_1$. Specifically, this implies
that for any two points \( y, \tilde{y} \in B_{r_0}(p) \cap \overline{\Omega_1} \), one has the inequality
\[
|Dv(y) - Dv(\tilde{y})| \geq C|y - \tilde{y}|^a,
\]
where \( a > 2 \) and \( C \) is a positive constant.

Since \( u_1 = v^* \) in \( \Omega_1 \), where \( v^* \) is the Legendre transform of \( v \), we can apply the result of [5] Remark 7.10 to deduce that \( u_1 \in C^{1,\beta}(B_r(0) \cap \overline{\Omega_1}) \) for some \( \beta \in (0,1) \) and \( r > 0 \). By a similar argument, and a possibly smaller \( r \), we can ensure that \( u_2 \in C^{1,\beta}(B_r(0) \cap \overline{\Omega_2}) \).

Inferred from (1.5), we conclude that the singular set \( \mathcal{F} \) exhibits \( C^{1,\beta} \) regularity in \( B_r(0) \). Finally, a standard covering argument ensures the \( C^{1,\beta} \) regularity of \( \mathcal{F} \) inside \( \Omega \). \( \square \)

**Remark 3.2.** Since \( u_i \in C^{1,\beta}(B_r(0) \cap \overline{\Omega_i}) \) for \( i = 1,2 \), we have \( p = y_0 = Du_1(0) \) and \( \hat{p} = \hat{y}_0 = Du_2(0) \).

### 4. Obliqueness

In this section, we further assume that \( \Omega_i^* \) are \( C^2 \) and uniformly convex domains, for \( i = 1,2 \), and that the functions \( f, g \) are positive and belong to \( C^0_{\text{loc}}(\Omega) \) and \( C^0(\overline{\Omega}) \), respectively, for some \( \alpha \in (0,1) \). Without loss of generality, we can translate our coordinate system such that \( x_0 \in \mathcal{F} \) is at the origin, i.e., \( x_0 = 0 \).

By subtracting a constant, we may assume \( u_1(0) = u_2(0) = u(0) = 0 \). Denote \( y_{01} = y_0 = Du_1(0) \in \partial \Omega_1^* \) and \( y_{02} = \hat{y}_0 = Du_2(0) \in \partial \Omega_2^* \). As in (3.24) and (3.25), we identify constants \( \tilde{r}_1 > 0, \tilde{r}_0 > 0, \) and \( h_0 > 0 \) such that the following holds:
\[
  \det D^2 v = \tilde{g}_i \quad \text{in} \quad B_{\tilde{r}_1}(y_{0i}), \quad i = 1,2,
\]
interpreted in the Alexandrov sense, where \( \tilde{g}_i(y) = \frac{g(y)}{f(Dv(y))} \) for \( y \in \overline{\Omega_i^*} \cap B_{\tilde{r}_1}(y_{0i}) \), and \( \tilde{g}_i(y) = 0 \) for \( y \in B_{\tilde{r}_1}(y_{0i}) \) \( \setminus \overline{\Omega_i^*} \), with \( i = 1,2 \). Furthermore, for each \( i = 1,2 \), we have:
\[
  S^c_{h_i}[v](y) \subset B_{\tilde{r}_1}(y_{0i}) \quad \forall \ y \in B_{\tilde{r}_0}(y_{0i}) \cap \Omega_i^*, \quad \forall \ h \leq h_0.
\]

The subsequent analysis for \( v \) is in a neighbourhood of \( y_0 \), which nevertheless also applies to \( v \) near \( \hat{y}_0 \) analogously. Hereafter, let us write \( S_h^c[v] \) as \( S_h^c[v](y_0) \) for brevity. In light of (4.1) and (4.2), together with the convexity of \( \Omega_i^* \), we infer that the Monge-Ampère measure \( \det D^2 v \) satisfies a doubling property in \( S^c_{h_i}[v](y) \) for all \( y \in \overline{\Omega_i^*} \cap B_{\tilde{r}_1}(y_{0i}) \) and \( h < h_0 \). From [4] Lemma 2.2, the sections \( S^c_{h_i}[v] \) is of geometric decay. Specifically, given any \( 0 < s_1 < \bar{s} < 1 \), there exists a constant \( s_0 \in (0,1) \) independent of \( h \) such that \( \forall \ y, \tilde{y} \in \overline{\Omega_i^*} \cap B_{\tilde{r}_1}(y_{0i}) \) with \( y \in s_1S^c_{\bar{s}h_i}[v](\tilde{y}) \), one has
\[
  S^c_{sh_i}[v](y) \subset \bar{s}S^c_{h_i}[v](\tilde{y}) \quad \forall \ h \in (0,h_0), \quad \forall \ s \in (0,s_0).
\]
Moreover, the function \( v \) satisfies a strict convexity estimate of the form:
\[
  v(y) \geq v(y_0) + Dv(y_0) \cdot (y - y_0) + C|y - y_0|^{1+a} \quad \forall \ y \in \overline{\Omega_i^*} \text{ near } y_0,
\]
where \( a > 1 \) and \( C \) are constants.

Similar to that in the optimal partial transport \([10, \text{Lemma 2.6}]\), we have the uniform density property as follows:

**Lemma 4.1 (Uniform Density).** There exists a constant \( \delta > 0 \), depending only on \( \Omega, \Omega^*, \lambda, \) and \( \text{dist}(x_0, \partial \Omega) \), such that

\[
\frac{|S^c_h[v] \cap \Omega^*_1|}{|S^c_h[v]|} \geq \delta \quad \forall \, h \leq h_0.
\]

The proof of Lemma 4.1 aligns closely with that in \([4, \text{Theorem 3.1}]\) and \([10, \text{Lemma 2.6}]\), which is omitted here for brevity.

By a change of coordinates, assume the hyperplane \( H := \{ y \in \mathbb{R}^n : (y - y_0) \cdot e_1 = 0 \} \) tangentially touches \( \partial \Omega^*_1 \) at \( y_0 \). Following Lemma 4.1, and in line with that in \([10, \text{Corollary 2.7, Lemma 4.1}]\), we can deduce both a volume estimate and a tangential \( C^{1,1-\epsilon} \) estimate for the function \( v \).

**Corollary 4.1.** The following assertions hold true:

(i) Volume Estimate:

\[
|S_h[v](y_0) \cap \Omega^*_1| \approx |S^c_h[v](y_0) \cap \Omega^*_1| \approx |S^c_h[v](y_0)| \approx h^n. \tag{4.5}
\]

Moreover, for any affine transform \( A \), if either \( A(S^c_h[v](y_0)) \) or \( A(S_h[v](y_0) \cap \Omega^*_1) \) has a good shape, so is the other one.

(ii) Tangential \( C^{1,1-\epsilon} \) regularity of \( v \): \( \forall \, \epsilon > 0, \exists \) a constant \( C_\epsilon \) such that:

\[
B_{C_\epsilon h^{1+\epsilon}}(y_0) \cap H \subset S^c_h[v](y_0) \quad \forall \, h > 0 \text{ sufficiently small}. \tag{4.6}
\]

**Remark 4.1.** By the strict convexity of \( v \) (see (4.4)) and the fact that \( S^c_h[v] \) is centred at \( y_0 \), we have an equivalence relation between \( S_h[v] \) and \( S^c_h[v] \), namely \( \forall \, h > 0 \) small,

\[
S^c_{b^{-1}h}[v] \cap \Omega^*_1 \subset S_h[v] \cap \Omega^*_1 \subset S^c_h[v] \cap \Omega^*_1 \tag{4.7}
\]

where \( b > 1 \) is a constant independent of \( h \). The reader is referred to \([9, \text{Lemma 2.2}]\) for a proof.

Denote \( \nu = \nu(0) \) and \( \hat{\nu} = \hat{\nu}(0) \) as the unit inner normals to \( \partial \Omega_1 \) and \( \partial \Omega_2 \) at \( 0 \in \mathcal{F} \), respectively. Given that \( \mathcal{F} \) is \( C^{1,\beta} \) regular, one has \( \nu = -\hat{\nu} \). In a similar vein, let \( \nu^* = \nu^*(y_0) \) and \( \hat{\nu}^* = \hat{\nu}^*(y_0) \) denote the unit inner normals to \( \partial \Omega^*_1 \) and \( \partial \Omega^*_2 \) at \( y_0 \) and \( \hat{y}_0 \), respectively.

**Proposition 4.1.** The inner product of the normals, \( \nu \cdot \nu^* \) and \( \hat{\nu} \cdot \hat{\nu}^* \), are both strictly positive, i.e., \( \nu \cdot \nu^* > 0 \) and \( \hat{\nu} \cdot \hat{\nu}^* > 0 \).

By a change of coordinates, we can assume \( \nu = e_n \). Then from (1.5), it follows that

\[
e_n = \frac{y_0 - \hat{y}_0}{|y_0 - \hat{y}_0|}.
\]
By subtracting a linear function from \( u \), we can assume \( y_0 = le_n, \hat{y}_0 = -le_n \) for a constant \( l \geq \frac{1}{2} \text{dist}(\Omega_1^*, \Omega_2^*) > 0 \). The proof of Proposition 4.1 is by contradiction. Without loss of generality, suppose the obliqueness conditions are not satisfied at \( y_0 \in \partial \Omega_1^* \), namely \( e_n \cdot \nu^* = 0 \). By a rotation of coordinates, we may assume that \( \nu^* = e_1 \).

Given that \( F \in C^{1,\beta}, \) locally near \( 0, \) \( F \) can be expressed as

\[
(4.8) \quad F = \{ x \in \mathbb{R}^n : x_n = \rho(x_1, \ldots, x_{n-1}) \}
\]

for a function \( \rho \in C^{1,\beta} \) satisfying \( \rho(0) = 0 \) and \( D\rho(0) = 0 \).

Given that \( \partial \Omega_1^* \in C^2 \) is uniformly convex, locally near \( y_0, \) \( \partial \Omega_1^* \) can be described as

\[
(4.9) \quad \partial \Omega_1^* = \{ y \in \mathbb{R}^n : y_1 = \rho^*(y_2, \ldots, y_n - l) \}
\]

for a uniformly convex function \( \rho^* \in C^2 \) satisfying \( \rho^*(0) = 0 \) and \( D\rho^*(0) = 0 \).

Considering the convexity of \( v \) and the condition \( Dv(y_0) = 0 \), we may further adjust \( v \) by adding a constant to ensure \( v(y_0) = 0 \) and \( v \geq 0 \). To proceed, we divide our analysis into three distinct cases:

- **Case I:** \( \nu \cdot \nu^* = 0, \ \tilde{\nu} \cdot \tilde{\nu}^* = 0, \) and \( \nu, \nu^*, \tilde{\nu}^* \) are non-coplanar.
- **Case II:** \( \nu \cdot \nu^* = 0, \ \tilde{\nu} \cdot \tilde{\nu}^* = 0, \) and \( \nu, \nu^*, \tilde{\nu}^* \) are coplanar.
- **Case III:** \( \nu \cdot \nu^* = 0, \ \nu \cdot \tilde{\nu}^* > 0. \)

**4.1. Case I:** \( \nu \cdot \nu^* = 0, \ \tilde{\nu} \cdot \tilde{\nu}^* = 0, \) and \( \nu, \nu^*, \tilde{\nu}^* \) are non-coplanar. Since \( \nu, \nu^*, \) and \( \tilde{\nu}^* \) are non-coplanar, there exists an affine transformation \( A \) with \( \det A = 1, \) such that \( A\nu^* \cdot A\tilde{\nu}^* = 0 \). Note that \( u(A^{-1}x) \) is the potential function for the optimal transport from \( A\Omega \) to \( (A^t)^{-1}\Omega^* \).

- \( \frac{(A^t)^{-1}\nu}{|\nu|} \) is the unit inner normal of \( A\Omega_1 \) at 0.
- \( \frac{A\nu^*}{|\nu^*|} \) is the unit inner normal of \( (A^t)^{-1}\Omega_1^* \) at \( (A^t)^{-1}y_0 \).
- \( \frac{A\tilde{\nu}^*}{|\tilde{\nu}^*|} \) is the unit inner normal of \( (A^t)^{-1}\Omega_2^* \) at \( (A^t)^{-1}\hat{y}_0 \).

One can verify that \( \frac{(A^t)^{-1}\nu}{|\nu|}, \frac{A\nu^*}{|\nu^*|}, \) and \( \frac{A\tilde{\nu}^*}{|\tilde{\nu}^*|} \) are mutually perpendicular to each other.

Hence, under an appropriate affine transformation and a rotation of coordinates, we can assume that \( y_0 \) and \( \hat{y}_0 \) are as before, and \( F, \partial \Omega_1^* \) are as in (4.8) and (4.9) respectively, and locally near \( y_0 \) the boundary of \( \Omega_2^* \) is represented as

\[
\partial \Omega_2^* = \{ y \in \mathbb{R}^n : y_2 = \rho_2^*(y_1, y_3, \ldots, y_{n-1}, y_n + l) \}
\]

for a uniformly convex function \( \rho_2^* \in C^2 \) satisfying \( \rho_2^*(0) = 0 \) and \( D\rho_2^*(0) = 0 \).

Recall the sub-level set \( S_h[v] = S_h[v](y_0) = \{ v < h \} \) in (2.8). Define the width function

\[
d_e := \sup \{ |(y - y_0 \cdot e) : y \in S_h[v] \cap \Omega_1^* \} \quad \text{for} \ e \in \mathbb{S}^{n-1}.
\]
Let \( p_e \in \overline{S_h[v]} \cap \Omega_1^* \) be a point where \(|(p_e - y_0) \cdot e| = d_e\). When specifically considering the unit vector \( e_1 \), we shall denote \( p_e \) simply as \( p = (p_1, \ldots, p_n) \).

**Lemma 4.2.** For any \( \epsilon > 0 \) small, there exists a constant \( C_\epsilon \) such that
\[
d_{e_1} \leq C_\epsilon h^{\frac{2}{3} - \epsilon}.
\]
Moreover, for any unit vector \( e \in \text{span}\{e_2, \ldots, e_n\} \),
\[
d_e^2 \leq C d_{e_1}
\]
for some universal constant \( C \).

**Proof.** Let \( p \) be as above. Consider \( q \), the point of intersection between the segment \( \overline{op} \) and \( \partial \Omega_1^* \). Since both \( p \) and the origin are in \( \overline{S_h[v]} \), by convexity one has \( q \in \overline{S_h[v]} \). Thus,
\[q \in \overline{S_h[v]} \cap \partial \Omega_1^*.
\]
Recall that \( d_{e_1} = p_1 \). As the intersection \( S_h[v] \cap \Omega_1^* \) converges to \( y_0 \) as \( h \to 0 \), we deduce that \( \frac{q_0}{p_n} \geq \frac{1}{2} \) for sufficiently small \( h \). Consequently,
\[q_1 \geq \frac{q_n}{p_n} p_1 \geq \frac{1}{2} d_{e_1}.
\]
Define \( e_q := \frac{q - y_0}{|q - y_0|} \). Let \( D \subset \text{span}\{e_q, e_1\} \) be the planar region bounded by \( \partial \Omega_1^* \cap \text{span}\{e_q, e_1\} \) and the segment \( \overline{y_0q} \). Since \( \partial \Omega_1^* \in C^2 \) is uniformly convex, one has
\[\mathcal{H}^2(D) \geq C |q_1|^\frac{3}{2} \geq C d_{e_1}^{3/2}
\]
for a constant \( C \) independent of \( h \), where \( \mathcal{H}^2(\cdot) \) denotes the 2-dimensional Hausdorff measure.

Select \( \{\tilde{e}_2, \ldots, \tilde{e}_{n-1}\} \) to be an orthonormal basis for the orthogonal complement of \( \text{span}\{e_1, e_p\} \) in \( \mathbb{R}^n \). The tangential \( C^{1,1-\epsilon} \) estimate of \( v \) asserts that
\[y_0 + C_\epsilon h^{\frac{1}{2} + \epsilon} \tilde{e}_i \in S_{bh}^c[v] \quad \text{for} \quad i = 2, \ldots, n - 1,
\]
where \( b \) is the constant in \([4.7]\). Define \( G \) to be the convex hull of \( D \) and points \( \{y_0 + C_\epsilon h^{\frac{1}{2} + \epsilon} \tilde{e}_i, i = 2, \ldots, n - 1\} \). By convexity, it follows that \( G \subset S_{bh}^c[v] \).

Combining these results and the inclusion \( S_h[v] \cap \Omega_1^* \subset S_{bh}^c[v] \), we obtain:
\[C_\epsilon h^{\frac{1}{2} + (n-2)} \mathcal{H}^2(D) \leq |G| \leq |S_{bh}^c[v]| \approx h^\frac{2}{3},
\]
which in turn implies \( |d_{e_1}| \leq C_\epsilon h^{\frac{2}{3} - \epsilon} \).

Fix any unit vector \( e \in \text{span}\{e_2, \ldots, e_n\} \). The uniform convexity of \( \partial \Omega_1^* \) yields
\[d_e \geq (p_e - y_0) \cdot e \geq C_1 ((p_e - y_0) \cdot e)^2 = C_1 d_e^2
\]
for some universal constant \( C_1 > 0 \), which implies \( d_e^2 \leq C d_{e_1} \) for a universal constant \( C \). \( \square \)

**Corollary 4.2.** For all \( y \in \Omega_1^* \), the inequality \( v(y) \geq C_\epsilon |y - y_0|^{3+\epsilon} \) holds.
Lemma 4.3. The function
\[
(4.10)
\]
Recall that
Proof. \hspace{1cm}
\[
(4.13)
\]
\[
\nu
\]
which implies the desired estimate by integration. \hfill \Box

Lemma 4.4. There exists a small \( t_0 > 0 \) such that
\[
(4.15)
\]
\[
\rho(t, 0, \ldots, 0) > 0 \quad \text{whenever } 0 < |t| < t_0.
\]
Proof. Suppose to the contrary that for any small $t_0 > 0$, there exists a non-zero $t$ satisfying $|t| < t_0$ such that $te_1 \in \overline{\Omega}_1$. Define $q := Du_1(te_1) \in \overline{\Omega}_1^*$ and observe that $Dv(q) = te_1$.

If $t < 0$, the uniform convexity of $\Omega_1^*$ implies that

$$(Dv(q) - 0) : (q - y_0) < 0,$$

which contradicts the convexity of $v$. Consequently, one must have $t > 0$.

Now, set $h = v(q)$. Lemma 4.2 implies that $q_1 \leq C\epsilon h^{\frac{2}{3} - \epsilon}$. Since $Dv(q) = te_1$, the vector $e_1$ is the unit outward normal to the level set $S_h[v]$ at the point $q$. This leads to the inclusion

$$(4.16) S_h[v] \subseteq \left\{ y \in \mathbb{R}^n : y_1 \leq C\epsilon h^{\frac{2}{3} - \epsilon} \right\}.$$

On the other hand, by the tangential $C^{1,1-\epsilon}$ regularity of $v$ at $\hat{y}_0$, one has

$$p = \hat{y}_0 + C\epsilon h^{\frac{1}{3} + \epsilon} e_1 \in S_{\frac{1}{2}h^{-1}h}[v](\hat{y}_0) \subseteq S_h[v],$$

where the last inclusion follows from (4.7). Thus, $p \in S_h[v]$ with

$$p_1 = C\epsilon h^{\frac{1}{3} + \epsilon} \gg C\epsilon h^{\frac{2}{3} - \epsilon}$$

for $h > 0$ sufficiently small. However, this contradicts (4.16) when $h$ (and consequently $t$) is sufficiently small. Hence, the lemma is proved. $\square$

**First blow-up.** By a translation of coordinates, we may assume that $y_0 = 0$. Suppose $S_h[v] \approx E$ (namely $E \subset S_h[v] \subset C_n E$) for an ellipsoid $E$ centred at 0. Then, $E \cap \{y_1 = 0\}$ is an $(n - 1)$-dimensional ellipsoid aligned along the principal directions $e_2, \ldots, e_n$. Note that $\text{span}\{e_2, \ldots, e_n\} = \text{span}\{e_2, \ldots, e_n\}$.

The ellipsoid $E$ can be expressed as

$$(4.17) E = \left\{ y = y_1 e_1 + \sum_{i=2}^{n} y_i \hat{e}_i : \frac{y_1^2}{a_1^2} + \sum_{i=2}^{n} \frac{(y_i - k_i y_1)^2}{a_i^2} \leq 1 \right\}$$

for some constants $a_1, a_i, k_i$, where $i = 2, \ldots, n$. Applying Lemma 4.2 and the tangential $C^{1,1-\epsilon}$ estimate of $v$ at 0, we obtain the following bounds:

$$(4.18) 0 < a_1 \leq C\epsilon h^{\frac{2}{3} - \epsilon} \text{ and } C\epsilon h^{\frac{1}{3} + \epsilon} \leq a_i \leq C\epsilon h^{\frac{2}{3} - \epsilon} \text{ for } i = 2, \ldots, n.$$

Consider the affine transformations $T_1$ and $T_2$ defined as follows:

$$(4.19) T_1 : x = x_1 e_1 + \sum_{i=2}^{n} \bar{x}_i \hat{e}_i \mapsto z = x_1 e_1 + \sum_{i=2}^{n} (\bar{x}_i - k_i x_1) \hat{e}_i,$$

$$(4.20) T_2 : z = z_1 e_1 + \sum_{i=2}^{n} \tilde{z}_i \hat{e}_i \mapsto y = \frac{z_1}{a_1} e_1 + \sum_{i=2}^{n} \frac{\tilde{z}_i}{a_i} \hat{e}_i.$$

Setting $T = T_2 \circ T_1$, we have $T(E) = B_1(0)$, and thus $T(S_h^c[v]) \approx B_1(0)$. 
Define the rescaled function \(v_h(y) := \frac{1}{h} v(T^{-1}y)\). By \((4.1)\), one can verify that
\[
\text{det} D^2 v_h(y) = \frac{1}{h^n (\text{det} T)^2} \tilde{g}_1(T^{-1}y) \quad \forall y \in T(B_{r_1}(0)).
\] (4.21)

Note that \(h^n (\text{det} T)^2 \approx 1\), due to the relation \(T(S_h^c[v]) \approx B_1(0)\).

The gradient of this function transforms as
\[
D_{v_h}(x) = T^* D_{v}(T^{-1}x),
\]
where
\[
T^* = \left( T^t \right)^{-1} \circ \left( T^t \right)^{-1}.
\]
A direct computation reveals the inverses of the transpose s of \(T_1, T_2\) as
\[
(T_1^t)^{-1} : x = x_1 e_1 + \sum_{i=2}^n \bar{x}_i \bar{e}_i \mapsto z = \left( x_1 + \sum_{i=2}^n k_i \bar{x}_i \right) e_1 + \sum_{i=2}^n \bar{x}_i \bar{e}_i,
\]
\[
(T_2^t)^{-1} : z = z_1 e_1 + \sum_{i=2}^n \bar{z}_i \bar{e}_i \mapsto y = a_1 z_1 e_1 + \sum_{i=2}^n a_i \bar{z}_i \bar{e}_i.
\]

Let \(\Omega_{h_1}^v := T(\Omega_1^v)\). Given that the uniform density property (Lemma \(4.1\)) remains invariant under affine transformations, the centred sections of \(v_h\) satisfy
\[
\frac{|S_h^c[v_h] \cap \Omega_{h_1}^v|}{|S_h^c[v_h]|} \geq \delta \quad \forall \tilde{h} \leq 1.
\] (4.22)

Particularly, since \(S_1^c[v_h] = T S_1^c[v] \approx B_1(0)\), \((4.22)\) implies that
\[
\frac{|B_{C_n}(0) \cap \Omega_{h_1}^v|}{|B_1(0)|} \geq \delta.
\] (4.23)

Moreover, since \(0 \in \partial \Omega_{h_1}^v\) and \(\Omega_{h_1}^v\) is convex, by \((4.23)\) there exists an open convex cone
\[
C = \{ x \in \mathbb{R}^n : x \cdot e > c \}
\]
for some unit vector \(e \in S^{n-1}\), where \(c > 0\) is a constant independent of \(h\), such that
\[
C \cap B_1(0) \subset \Omega_{h_1}^v
\] (4.24)

for all sufficiently small \(h\).

We now proceed to establish the following estimates:

**Lemma 4.5.** There exist constants \(\beta \in (0, 1)\) and \(a > 1\) such that
\[
\frac{1}{C} |y|^{1+a} \leq v_h(y) \leq C |y|^{1+\beta} \quad \forall y \in B_{r_0}(0) \cap \Omega_{h_1}^v,
\] (4.25)

where \(C, r_0 > 0\) are constants independent of \(h\).

**Proof.** Note that \(v_h\) satisfies \(\frac{1}{C} \chi_{\Omega_{h_1}^v} \leq \text{det} D^2 v_h \leq C \chi_{\Omega_{h_1}^v}\) in \(B_{r_0}(0)\) for some universal constants \(C, r_0 > 0\). Observe also that the geometric decay property \((4.3)\) is invariant under rescaling and affine transformations. By the same argument of the proof of [10 Lemma 5.8]), one can obtain the desired estimates \((4.25)\). \(\square\)
Let \( T^* = \frac{1}{h}(T^t)^{-1} \). Denote by \( \Omega_{h1} \) the set \( T^* \Omega_1 \). By the above strict convexity estimate of \( v_h \) in (4.23), we have the inclusion
\[
B_r(0) \cap \Omega_{h1} \subset Dv_h(B_1(0) \cap \Omega_{h1}^*),
\]
for some small constant \( r > 0 \) independent of \( h \). Following a rescaling argument similar to the proof of [10 Lemma 5.10], we obtain the following inclusion:

**Corollary 4.3.** For any \( R > 0 \) large, there exists a constant \( M_R > 0 \) independent of \( h \) such that
\[
(4.26) \quad B_R(0) \cap \Omega_{h1} \subset Dv_h(B_{M_R}(0) \cap \Omega_{h1}^*) \quad \text{for } h > 0 \text{ small.}
\]

Passing to a subsequence \( \{h_j\} \to 0 \), we may assume that \( \{v_{h_j}\} \) converges to \( v_\infty \) locally uniformly for some convex function \( v_\infty \) defined on \( \mathbb{R}^n \). Additionally, up to a further subsequence, we may assume that \( T^* \Omega_1^* \) converges to some convex set \( \Omega_{1,\infty}^* \subset \{x_1 \geq 0\} \) locally uniformly in the Hausdorff distance. Let us denote by \( \Omega_{1,\infty} \) the interior of \( \partial^- v_\infty(\mathbb{R}^n) \). Since \( v_\infty \) is a convex function defined on the entire \( \mathbb{R}^n \), it is well-known that \( \Omega_{1,\infty} \) is a convex set.

By (4.21) and the continuity of \( f \) and \( g \) at 0, and passing to a subsequence, we have that
\[
(4.27) \quad \det D^2 v_\infty = c_0 \chi_{\Omega_{1,\infty}^*} \quad \text{on } \mathbb{R}^n,
\]
where \( c_0 \) is a positive constant. Define \( u_\infty(x) := \sup\{x \cdot y - v_\infty(y) : y \in \mathbb{R}^n\} \) for \( x \in \overline{\Omega_{1,\infty}} \).

Since \( \Omega_{1,\infty}^* \) is convex, (4.27) implies that the Monge-Ampère measure \( \det D^2 v_\infty \) is doubling for convex sets centred at points in \( \overline{\Omega_{1,\infty}} \). Consequently, by applying the same proof as in [10 Lemma 5.12], we establish the following properties for \( u_\infty \) and \( v_\infty \):

**Lemma 4.6.** The function \( v_\infty \) is \( C^1 \) and strictly convex in \( \overline{\Omega_{1,\infty}} \). Moreover, as a convex function defined on \( \mathbb{R}^n \), \( v_\infty \) is differentiable at every point \( y \in \overline{\Omega_{1,\infty}^*} \). The function \( u_\infty \) is \( C^1 \) and strictly convex in \( B_{r_0}(0) \cap \overline{\Omega_{1,\infty}} \) for some small \( r_0 > 0 \).

**Remark 4.3.** By the definition of \( \Omega_{1,\infty} \), one has
\[
Du_\infty(B_{r_0}(0) \cap \overline{\Omega_{1,\infty}}) \subset \overline{\Omega_{1,\infty}} \subset \{x_1 \geq 0\},
\]
which implies that \( u_\infty \) is nondecreasing in the \( e_1 \)-direction near 0.

Since \( v_\infty \) is differentiable at 0, one has
\[
\partial^- v_\infty(B_r(0)) \subset B_{r_0}(0) \cap \overline{\Omega_{1,\infty}}
\]
provided \( r \) is sufficiently small. We claim that: \( v_\infty \), as a convex function defined on \( \mathbb{R}^n \), is \( C^1 \) in \( B_r(0) \) for some \( r > 0 \) sufficiently small.

Indeed, suppose \( v_\infty \) is not differentiable at some \( y \in B_r(0) \), which would imply that \( \partial^- v_\infty(y) \) contains at least two distinct points \( x, \bar{x} \in B_{r_0}(0) \cap \overline{\Omega_{1,\infty}} \). Since \( \Omega_{1,\infty} \) is convex, and \( u_\infty \) is the Legendre transform of \( v_\infty \), it follows that \( u_\infty \) is affine along the segment connecting \( x \) and \( \bar{x} \), which contradicts the strict convexity of \( u_\infty \) in \( B_{r_0}(0) \cap \overline{\Omega_{1,\infty}} \).
Lemma 4.7. After an appropriate affine transformation, we obtain:

1. \( \Omega_{1,\infty} = \{x \in \mathbb{R}^n : x_1 > P(x)\} \), where \( P \) is a non-negative homogeneous quadratic polynomial satisfying \( P(0) = 0 \) and \( DP(0) = 0 \).

2. \( \Omega_{1,\infty} = \{x \in \mathbb{R}^n : x_n > \rho_\infty(x_1)\} \) for some convex function \( \rho_\infty \) when \( |x_1| \) is small. Moreover, \( \rho_\infty(0) = 0 \) and \( \rho_\infty \geq 0 \).

Proof. Fix any unit vector \( e \in \text{span}\{e_2, \ldots, e_{n-1}\} \). Let \( t_h \) be the positive number such that

\[
\left| \frac{1}{h}(T_{\frac{1}{h}}^h)^{-1}(t_h e) \right| = 1,
\]

yielding

\[
t_h \leq \frac{h}{\min_{2 \leq i \leq n} a_i} \leq C_{\epsilon} h^{\frac{1}{2} - \epsilon}.
\]

For any \( t \) satisfying \( |t| < h^{-\epsilon} t_h \leq C_{\epsilon} h^{\frac{1}{2} - 2\epsilon} \), define \( x^t := te = (0, x_2^t, \ldots, x_{n-1}^t, 0) \). It is straightforward to verify that

\[
(T^*(h^{-\epsilon} t_h e)) \geq \left| \frac{1}{h}(T_{\frac{1}{h}}^h)^{-1}(h^{-\epsilon} t_h e) \right| \to \infty \quad \text{as} \quad h \to 0.
\]

Now, let \( q^t := x^t + \rho(0, x_2^t, \ldots, x_{n-1}^t)e_n \in \mathcal{F} \). Let \( z_i := e_n \cdot \bar{e}_i \), then \( e_n = \sum_{i=2}^{n} z_i \bar{e}_i \).

Consequently, we have

\[
T^*(\rho(0, x_2^t, \ldots, x_{n-1}^t)e_n) = \frac{1}{h} \rho(0, x_2^t, \ldots, x_{n-1}^t) \left( a_1 \left( \sum_{i=2}^{n} a_i z_i \right) e_1 + \sum_{i=2}^{n} a_i z_i \bar{e}_i \right).
\]

Since \( |t| \leq C_{\epsilon} h^{\frac{1}{2} - 2\epsilon} \), by Lemma 4.3 we have \( |\rho(0, x_2^t, \ldots, x_{n-1}^t)| \leq C_{\epsilon} h^{\frac{1}{2} - \frac{1}{2}\epsilon} \). Given the above estimate for \( \rho \), \( |t| \leq C_{\epsilon} h^{\frac{1}{2} - 2\epsilon} \), \( |z_i| \leq 1 \), and \( |a_i| \leq C_{\epsilon} h^{\frac{1}{2} - \epsilon} \) for \( i = 2, \ldots, n \), we deduce that

\[
\left| \frac{1}{h} \rho(0, x_2^t, \ldots, x_{n-1}^t) \sum_{i=2}^{n} a_i z_i \bar{e}_i \right| \leq C_{\epsilon} h^{\frac{1}{2} - 10\epsilon}
\]

provided \( h \) is sufficiently small.

Let \( \tilde{d}_e := \sup\{|x \cdot e| : x \in S_h^e[v]\} \). By applying the uniform density property, we have a comparison between the widths of \( S_h^e[v] \) and \( S_h^e[v] \cap \Omega_1^e \) in the \( e \) direction, asserting their comparability. Furthermore, Remark 4.1 indicates that the width of \( S_h^e[v] \cap \Omega_1^e \) in the \( e \) direction is also comparable to that of \( S_h^e[v] \cap \Omega_1^e \) in the same direction. Synthesizing these comparative relationships with the estimates from Lemma 4.2 we are led to:

\[
\tilde{d}_e^2 \leq C a_1 \quad \forall \text{ unit vector } e \in \text{span}\{e_2, \ldots, e_n\}
\]

for a universal constant \( C \). Consequently, we obtain

\[
|k_i| \leq C \frac{a_1^{1/2}}{a_1} = C a_1^{-1/2}.
\]
Note that $|a_1| \leq C_\epsilon h^{2/3-\epsilon}$. Therefore, we have
\[
\left| \frac{1}{h^\epsilon} \rho(0, x'_2, \ldots, x'_{n-1}) a_1 \left( \sum_{i=2}^{n} k_i z_i \right) e_1 \right| \leq C_\epsilon \frac{1}{h} \left( C_\epsilon h^{1/2-2\epsilon} \right)^{3/2-\epsilon} \left( C_\epsilon h^{2/3-\epsilon} \right)^{1/2} \\
\leq C_\epsilon h^{1/12-10\epsilon}.
\]
Consequently,
\[
|T^*(q^t) - T^*(x^t)| \leq 2C_\epsilon h^{1/12-10\epsilon} \to 0 \quad \text{as} \quad h \to 0,
\]
provided that $\epsilon$ is initially chosen to be sufficiently small.

By taking a subsequence, if necessary, we may assume that $T^*(\text{span}\{e_2, \ldots, e_{n-1}\})$ converges to $H$, an $(n-2)$-dimensional subspace of $\mathbb{R}^n$. Consequently, from (4.28), (4.31), and Corollary 4.3, it follows that $H \subset \partial \Omega_{1,\infty}$. By convexity, we have that
\[
\Omega_{1,\infty} = \omega \times H
\]
for some two dimensional convex set $\omega$.

Now, we claim that
\[
e_1 \notin H.
\]
Suppose to the contrary that the $x_1$-axis is contained in $H \subset \overline{\Omega_{1,\infty}}$. From the definition of $u_\infty$, it follows that $u_\infty(0) = 0$ and $u_\infty \geq 0$ on $H$. On one hand, by Remark 4.3 we have $u_\infty(-te_1) \leq u_\infty(0) = 0$ for all small $t > 0$, and it follows that
\[
u_\infty(-te_1) = 0 \quad \text{for all sufficiently small} \quad t > 0.
\]
On the other hand, Lemma 4.4 ensures that $u_\infty$ is strictly convex in $B_r(0) \cap \overline{\Omega_{1,\infty}}$. This, however, contradicts (4.31).

Note that $\frac{T_{e_1}}{|T_{e_1}|}$ is the unit inner normal to $T^*\Omega_1$ at the origin. Without loss of generality, after passing to a subsequence of $h$, we may assume that $\frac{T_{e_1}}{|T_{e_1}|}$ converges to a unit vector $e_\infty$, orthogonal to both $H$ and $e_1$. By Lemma 4.4 the line $\{te_1 : t \in \mathbb{R}\}$ cannot intersect the interior of $\Omega_{1,\infty}$. Thus,
\[
\Omega_{1,\infty} \subset \{x : x \cdot e_\infty \geq 0\}.
\]
Recall that the boundary of $\Omega_1^*$ is given by $\partial \Omega_1^* = \{x : x_1 = \rho^*(x_2, \ldots, x_n)\}$, where $\rho^*$ is a $C^2$, uniformly convex function in a neighbourhood of the origin, satisfying $\rho^*(0) = 0$ and $D\rho^*(0) = 0$. Denote $x'' = (x_2, \ldots, x_n)$. In this setting, there exists a positive, homogeneous quadratic polynomial $P'$ such that
\[
\rho^*(x'') = P'(x'') + o(|P'(x'')|).
\]
We can extend $P'$ to $\mathbb{R}^n$ by defining $\tilde{P}(x_1,x''') = P'(x'')$. This leads to the following representation:

$$\partial \Omega_1^* = \left\{ x : \langle x, e_1 \rangle = \tilde{P}(x) + o(|\tilde{P}(x)|) \right\} \text{ near the origin.}$$

A straightforward computation yields

$$(4.36) \quad \partial(T\Omega_1^*) = \left\{ x : \langle x, e_1 \rangle = \tilde{P}_h(x) + o(\tilde{P}_h) \right\} \text{ near 0,}$$

where $\tilde{P}_h(x) = \frac{1}{|x|^{n-2}} \tilde{P}(T^{-1}x) \geq 0$, and

$$(4.37) \quad B_1(0) \cap T\Omega_1^* \subset \left\{ x : \langle x, e_1 \rangle \geq \frac{1}{2} \tilde{P}_h(x) \right\} \text{ for } h > 0 \text{ small.}$$

By applying the uniform density property of $v$ near the origin, and following a similar line of argument to the proof of [10, Lemma 5.9], we can deduce that the coefficients of $\tilde{P}_h$ are bounded by a constant that is independent of $h$. Consequently, upon passing to a subsequence if necessary, we may assume that $\tilde{P}_h$ converges to a non-negative homogeneous quadratic polynomial $P$, leading to

$$(4.38) \quad \Omega_{1,\infty}^* = \{ x \in \mathbb{R}^n : x_1 > P(x) \}.$$ 

Since $e_1 \cdot e_\infty = 0$, by a rotation of coordinates we may assume $e_\infty = e_n$. By [13,3], for a fixed unit vector $e \in H$, we can find a vector $\tilde{e} \in H^* := \text{span}\{e_2, \ldots, e_{n-2}\}$ such that $e$ is not orthogonal to $\tilde{e}$. Hence, there exists an affine transformation $A_1$ with $\det A_1 = 1$ such that $A_1(\text{span}\{e_n\}) = \text{span}\{e_n\}$, $A_1(\text{span}\{e_1, \ldots, e_{n-1}\}) = \text{span}\{e_1, \ldots, e_{n-1}\}$, and $A_1 e$ is parallel to $(A_1^*)^{-1}\tilde{e}$ (see [13] (3.2)). The unit inner normals of $A_1(\Omega_{1,\infty})$ and $(A_1^*)^{-1}\Omega_{1,\infty}^*$ at 0 are still orthogonal to each other. Denote $\bar{e}_2 = \frac{A_1 e}{|A_1 e|}$. Then, $A_1(\Omega_{1,\infty}) = \omega_1 \times H_1 \times \text{span}\{\bar{e}_2\}$, where $\omega_1$ is a two-dimensional convex subset and $H_1$ is an $(n-3)$-dimensional subspace in $\mathbb{R}^n$. It is easy to see that $(A_1^*)^{-1}H^* = H_1^* \times \text{span}\{\bar{e}_2\}$ for some $(n-3)$-dimensional subspace $H_1^*$ in $\mathbb{R}^n$.

Then we restrict ourselves to $H_1$ and $H_1^*$ in the $(n-2)$-space $\text{span}\{\bar{e}_2, e_n\}$. Similarly, as above, we can find unit vectors $e' \in H_1, e' \in H_1^*$ and an affine transformation $A_2$ such that $A_2(\text{span}\{\bar{e}_2, e_n\}) = \text{span}\{\bar{e}_2, e_n\}$, $A_2(\text{span}\{\bar{e}_2, e_n\}^\perp) = \text{span}\{\bar{e}_2, e_n\}^\perp$, and $A_2 e'$ is parallel to $(A_2^*)^{-1}e'$. Let $\tilde{e}_3 = \frac{A_2 e'}{|A_2 e'|}$. Repeating this process, after a sequence of affine transformations $A_i$, $i = 1, \ldots, n-2$, we have $A H = (A'^*)^{-1}H^*$, where $A = A_{n-2} \cdots A_1$.

Hence, by performing the affine transformation $A$ as above, and then by a rotation of coordinates, we may assume that $\Omega_{1,\infty}^*$ is as in the statement of Lemma 4.7 and that $\Omega_{1,\infty} = \omega \times \text{span}\{e_2, \ldots, e_{n-2}\}$ for some two-dimensional convex set $\omega \subset \text{span}\{e_1, e_n\}$ satisfying $0 \in \partial \omega$ and $\omega \subset \{x_n \geq 0\}$.

Finally, after an affine transformation of the form

$$\tilde{A} : \left\{ \begin{array}{ll}
    x_1 \to x_1 + k x_n & \text{for a constant } k \in \mathbb{R} \\
    x_i \to x_i & \text{for } i = 2, \ldots, n.
\end{array} \right.$$
we can transform \(\Omega_{1,\infty}\) to the position as in the statement of the lemma. Note that \(\tilde{A}\) makes \(\Omega_{1,\infty}\) slide along the \(x_1\)-direction, and at the same time \((\tilde{A}^t)^{-1}\) makes \(\Omega_{1,\infty}\) slide along the \(y_n\)-direction, while the \((n-2)\)-space \(\Span\{e_2, \ldots, e_{n-2}\}\) remains invariant. Hence, by choosing a proper constant \(k \in \mathbb{R}\), we may assume that \(\omega = \{(x_1, x_n) : x_n \geq \rho_\infty(x_1)\}\) for a convex function \(\rho_\infty\). Note that since \(P\) is a non-negative homogeneous quadratic polynomial, after the corresponding affine transformation \((\tilde{A}^t)^{-1}\), the set \(\Omega_{1,\infty}\) still satisfies \(\Omega_{1,\infty}^* = \{x \in \mathbb{R}^n : x_1 > P(x)\}\), but with a different non-negative homogeneous quadratic polynomial \(P\).

By the definition of \(V\), there exists a small \(r_1 \in (0, r_0)\) such that \(B_{r_1}(0) \cap \Omega_{1,\infty} \subset U\). It follows that \(\partial U\) is convex near 0.

By Lemma 4.7 in a neighbourhood of 0, we have
\[
U = \{x \in \mathbb{R}^n : x_n > \rho_\infty(x_1)\},
\]
where \(\rho_\infty\) is a convex function satisfying \(\rho_\infty(0) = 0\) and \(\rho_\infty \geq 0\), and
\[
V = \{y \in \mathbb{R}^n : y_1 > \bar{\rho}(y_2, \ldots, y_n)\},
\]
where \(\bar{\rho}\) is a smooth convex function satisfying \(\bar{\rho}(0) = 0\), \(D\bar{\rho}(0) = 0\).

Let \(\tilde{u} : \mathbb{R}^n \to \mathbb{R}\) be defined by
\[
\tilde{u}(x) := \sup\{x \cdot y - \bar{v}(y) : y \in V\} \quad \forall x \in \mathbb{R}^n.
\]
By the definition of \(u_\infty\) and Lemma 4.6, we have \(\tilde{u} = u_\infty\) on \(B_{r_1}(0) \cap \overline{\Omega_{1,\infty}} \subset U\).

Note also that
\[
\begin{align*}
\det D^2 \tilde{u} &= c_1 \chi_U \quad \text{in } \mathbb{R}^n, \\
D\tilde{u}(\mathbb{R}^n) &= \nabla
\end{align*}
\]
for some positive constant \(c_1\). Denote \(r_2 = \frac{1}{2} r_1\). Since \(\tilde{u}\) is strictly convex in \(B_{r_1}(0) \cap \overline{U} = B_{r_1}(0) \cap \overline{\Omega_{1,\infty}}\), there exists a constant \(h_0 > 0\) such that
\[
S^c_h[\tilde{u}](x) \cap U \subset B_{r_1}(0) \cap U \quad \forall x \in B_{r_2}(0) \cap \overline{U}\quad \text{and} \quad \forall \ h \leq h_0.
\]
It follows from (4.42) and (4.43) that the Monge-Ampère measure \(\det D^2 \tilde{u}\) is doubling for centred sections \(S^c_h[\tilde{u}](x)\), where \(h \leq h_0\) and \(x \in B_{r_2}(0) \cap \overline{U}\).

We proceed to summarise the properties of \(\tilde{u}\) and \(\tilde{v}\):

1. **Geometric decay.** By the obtained doubling property of the Monge-Ampère measures \(\det D^2 \tilde{u}\) and \(\det D^2 \tilde{v}\), from [Lemma 2.2] the centred sections \(S^c_h[\tilde{u}]\) and \(S^c_h[\tilde{v}]\) decay geometrically. Consequently, \(\tilde{u}\) (resp., \(\tilde{v}\)) is \(C^{1,\beta}\)-regular for some \(\beta \in (0, 1)\) and strictly convex in \(B_r(0) \cap \overline{U}\) (resp., \(B_r(0) \cap \overline{V}\)) for some positive \(r > 0\).
Applying [4, Corollary 2.2], we have the duality between $S_{\vhat}^c$ and $D\vhat(S_{\vhat}^c)$: if $E \subset S_{\vhat}^c = \{ \vhat < \ell \} \subset C_n E$ for some affine function $\ell$,

where $E$ is an ellipsoid centred at 0 with principal radii $\lambda_i \varepsilon_i$, $i = 1, \cdots, n$, then

$$E^* \subset D\vhat(S_{\vhat}^c) \subset C_n E^*,$$

where $E^*$ is an ellipsoid centred at $D\ell$ with principal radii $h \lambda_i \varepsilon_i$, $i = 1, \cdots, n$. A similar duality holds for $S_{\vhat}^c$ and $D\vhat(S_{\vhat}^c)$ as well.

By the proof of [9, Lemma 2.2], we have

$$S_{\vhat}^c - 1 h [\vhat] \cap V \subset S_{\vhat} [\vhat] \cap V \subset S_{\vhat}^c - b - 1 h [\vhat],$$

(4.44)

$$S_{\vhat}^c - 1 h [\vhat] \cap U \subset S_{\vhat} [\vhat] \cap U \subset S_{\vhat}^c [\vhat]$$

(4.45)

for some constant $b > 0$ independent of $h$.

(2) **Uniform density.** The affine invariance of the uniform density property ensures that

$$\frac{|S_{\vhat}^c [\vhat] \cap V|}{|S_{\vhat}^c [\vhat]|} \geq \delta$$

for a constant $\delta > 0$ independent of $h$. Indeed, it follows by taking limit of (4.22).

Similarly to (4.5), we have

$$|S_{\vhat} [\vhat] \cap V| \approx |S_{\vhat}^c [\vhat] \cap V| \approx |S_{\vhat}^c [\vhat]| \approx h^{\frac{n}{2}}.$$

(4.46)

To show the uniform density for $\tilde{u}$, we proceed as follows. Let $M > 0$ be a large constant to be determined. In the following, we always assume $h$ is small such that (4.43) holds. By performing an affine transformation, we may assume that

$$S_{\vhat}^c [\vhat] \approx B (\frac{h}{M})^{\frac{1}{2}} (0).$$

Using the duality between $S_{\vhat}^c [\vhat] \cap D\vhat(S_{\vhat}^c [\vhat])$, we find that

$$0 \in D\vhat \left( S_{\vhat}^c [\vhat] \right) \approx B (\frac{h}{M})^{\frac{1}{2}} (z)$$

for some $z \in U$. It follows that $D\vhat(S_{\vhat}^c [\vhat]) \approx (h/M)^{\frac{n}{2}}$.

For any $y \in S_{\vhat}^c [\vhat]$, from (4.11) we have

$$\tilde{u} (D\vhat(y)) = y \cdot D\vhat(y) - \vhat(y) \leq CM^{-1} h < b^{-1} h$$

by choosing $M > Cb$ to be a universal constant independent of $h$. Hence,

$$D\vhat \left( S_{\vhat}^c [\vhat] \right) \subset S_{\vhat} - b - 1 h [\vhat] \cap U \subset S_{\vhat}^c [\vhat] \cap U.$$

(4.47)

Combining the above estimates, we conclude that

$$|S_{\vhat}^c [\vhat] \cap U| \geq C_1 h^{\frac{n}{2}}$$

for some constant $C_1$ independent of $h$.
By the Alexandrov type estimate (see, for instance, [14 estimate (4)]), we obtain

$$h^n \geq C \left( \frac{1}{2} S_h^c[\tilde{u}] \right) \cap U \cdot |S_h^c[\tilde{u}]|$$

$$\geq C_2 |S_h^c[\tilde{u}] \cap U| \cdot |S_h^c[\tilde{u}]|,$$

where the second inequality follows from the convexity of $U$ near 0. Combining the above two estimates, we obtain the uniform density estimate for $\tilde{u}$:

$$\frac{|S_h^c[\tilde{u}] \cap U|}{|S_h^c[\tilde{u}]|} \geq \delta$$

for some $\delta > 0$, independent of $h$.

(3) **Conjugation between $S_h^c[\tilde{u}]$ and $S_h^c[\tilde{v}]$.** The uniform density property implies that $S_h^c[\tilde{u}]$ and $S_h^c[\tilde{v}]$ are conjugate for sufficiently small $h$. More precisely, if there exists an affine transformation $A$ with det $A = 1$ such that $AS_h^c[\tilde{v}] \approx B_{h^{1/2}}(0)$, then it follows that $(A^t)^{-1} S_h^c[\tilde{u}] \approx B_{h^{1/2}}(0)$.

To see why this is the case, let $A$ be the affine transformation such that det $A = 1$ and $AS_h^c[\tilde{v}] \approx B_{h^{1/2}}(0)$, and define

$$\bar{v}(y) := \tilde{v}(A^{-1}y), \quad \text{and} \quad \tilde{u}(x) := \tilde{u}(A^t x).$$

Then $D\bar{v}$ solves the optimal transport problem between the distributions of $AV$ and $(A^t)^{-1} U$. Note that now $S_h^c[\tilde{v}] \approx B_{h^{1/2}}(0)$. We aim to show $S_h^c[\tilde{u}] \approx B_{h^{1/2}}(0)$ as well.

Let $M$ be as in (4.48). By observation (b) in the proof of [4 Lemma 4.1], we have

$$\frac{M}{C_n} S_h^c[\tilde{v}] \subset S_h^c[\tilde{v}] \subset C_n M S_h^c[\tilde{v}].$$

Hence, for some constant $C_1$ independent of $h$, we have

$$B_{C_1^{-1} h^{1/2}}(0) \subset S_h^c[\tilde{v}] \subset B_{C_1 h^{1/2}}(0).$$

By the same argument leading to (4.47) and (4.48), we have that for some $z \in U$ and some constant $c_1 > 0$,

$$B_{c_1 h^{1/2}}(z) \subset D\bar{v} \left( S_h^c[\tilde{v}] \right) \subset S_h^c[\tilde{u}].$$

By the uniform density property, we have $|S_h^c[\tilde{u}]| \approx h^{n/2}$. From the fact that $S_h^c[\tilde{u}]$ is balanced with respect to the origin, it follows that

$$(A^t)^{-1} S_h^c[\tilde{u}] = S_h^c[\tilde{u}] \approx B_{h^{1/2}}(0).$$

Additionally, we can conclude that: for any $x \in S_h^c[\tilde{u}]$ and $y \in S_h^c[\tilde{v}]$, as $|(A^t)^{-1} x| \approx h^{1/2}$ and $|Ay| \approx h^{1/2}$, it follows that

$$|x \cdot y| = |((A^T)^{-1} x) \cdot (Ay)| \leq C h \quad \forall \ x \in S_h^c[\tilde{u}], \ \forall \ y \in S_h^c[\tilde{v}]$$

for some constant $C$ independent of $h$. 


Tangential $C^{1,1-\varepsilon}$ estimate. Owing to the flatness of $\partial U$ near the origin in the directions of $e_2, \ldots, e_{n-1}$, we invoke [4 Corollary 1.1] to deduce that $\tilde{u}$ is $C^{1,1}$ in these directions. Consequently, for any small $h > 0$, we have

$$C_1 h^{\frac{1}{2}} e \in S_h^c[\tilde{u}] \quad \forall e \in \text{span}\{e_2, \ldots, e_{n-1}\}$$

where $C_1 > 0$ is a constant independent of $h$. Then, by (4.49),

$$C_1 h^{\frac{1}{2}} e \cdot y \leq Ch \quad \forall y \in S_h^c[\tilde{v}].$$

Therefore,

$$\sup\{|y \cdot e| : y \in S_h^c[\tilde{v}]\} \leq Ch^{\frac{1}{2}} \quad \forall e \in \text{span}\{e_2, \ldots, e_{n-1}\}.$$  \hspace{1cm} (4.50)

Since $\partial V$ is smooth near the origin, we apply the result in [9, Lemma 3.1] to infer that $\tilde{v}$ is $C^{1,1-\varepsilon}$ tangentially for any $\varepsilon > 0$. More precisely,

$$B_{C_1 h^{\frac{1}{2}+\varepsilon}}(0) \cap \{x_1 = 0\} \subset S_h^c[\tilde{v}].$$  \hspace{1cm} (4.51)

Second blow-up. Suppose that $S_h^c[\tilde{v}]$ is comparable to an ellipsoid $E$ centred at the origin, which is given by (4.17). Thanks to estimates (4.50) and (4.51), we can improve the bounds in (4.18) such that

$$0 < a_1 \leq C_\varepsilon h^{\frac{1}{2}-\varepsilon},$$

$$c \varepsilon h^{\frac{1}{2}+\varepsilon} \leq a_i \leq C_\varepsilon h^{\frac{1}{2}-\varepsilon} \quad \text{for} \quad i = 2, \ldots, n - 1,$$

$$c \varepsilon h^{\frac{1}{2}+\varepsilon} \leq a_n \leq C_\varepsilon h^{\frac{1}{2}-\varepsilon}.$$  \hspace{1cm} (4.52)

Furthermore, the $C^{1,\beta}$ regularity of $\tilde{v}$ within $B_{r_0}(0) \cap V$ ensures that

$$B_{Ch^{1+\beta}}(0) \cap V \subset S_h^c[\tilde{v}].$$  \hspace{1cm} (4.53)

For the given $h > 0$ small, let $A_{h_1} = T_1$ defined in (4.19), $A_{h_2} = T_2$ defined in (4.20), and $A_h = A_{h_2} \circ A_{h_1}$ that maps $E$ onto the unit ball $B_1(0)$. Define $\tilde{v}_h : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{v}_h(x) := \frac{1}{h} \tilde{v}(A_h^{-1}x).$$

Take any point $x = (x_1, x'') \in \{x_n = 0\} \cap \partial V$ such that $|x''| \leq h^{\frac{1}{2}-5\varepsilon}$. From (4.50),

$$x_1 = \bar{\rho}(x''') \leq Ch^{1-10\varepsilon}.$$  \hspace{1cm} (4.54)

By (4.53), we deduce that $\|A_h\| \leq \frac{1}{h} h^{\frac{1}{2}} h^{-1+\beta}$, then it follows that

$$\text{dist} (A_h x, A_h \text{span}\{e_2, \ldots, e_{n-1}\}) \leq C_\varepsilon h^{1-10\varepsilon-\frac{1}{2}+\beta} \to 0 \quad \text{as} \quad h \to 0,$$

provided $\varepsilon$ is sufficiently small. Meanwhile, for $x = (x_1, x'') \in \{x_n = 0\} \cap \partial V$ with $|x''| = h^{\frac{1}{2}-5\varepsilon}$, we have

$$|A_h x| \geq C_\varepsilon h^{\frac{1}{2}-5\varepsilon+\frac{1}{2}+\varepsilon} = C_\varepsilon h^{-4\varepsilon} \to \infty \quad \text{as} \quad h \to 0,$$  \hspace{1cm} (4.55)
provided $\epsilon > 0$.

Thanks to (4.54) and (4.55), and by convexity of $V$, we can conclude that the sequence (possibly passing to a subsequence) $A_h(\{x_n = 0\} \cap \partial V)$ converges locally uniformly to an $(n-2)$-dimensional subspace $\tilde{H}^*$ of $\mathbb{R}^n$. Invoking convexity once more and further passing to a subsequence if necessary, we may assume that $A_h V$ converges locally uniformly to a smooth convex set represented by $\tilde{\Omega}^* = \omega^* \times \tilde{H}^*$, where $\omega^*$ is a two-dimensional convex set with smooth boundaries. The smoothness of $\omega^*$ can be inferred through an argument analogous to that of (4.35) to (4.38).

Invoking convexity once more and passing to a subsequence, we can assume that the set $(A^t_h)^{-1}(U \cap B_{r_1}(0))$ converges locally uniformly to a convex set $\tilde{\Omega}$ in the Hausdorff distance. According to (4.39), we find that

$$\{x \in \mathbb{R}^n : x_1 = x_n = 0\} \cap B_{r_1}(0) \subseteq U \cap B_{r_1}(0),$$

and by passing to a further subsequence if necessary, we may deduce that

$$(A^t_h)^{-1}(\{x \in \mathbb{R}^n : x_1 = x_n = 0\})$$

converges locally uniformly to an $(n-2)$-dimensional subspace $\tilde{H}$ of $\mathbb{R}^n$ in the Hausdorff distance. Consequently, $\tilde{H} \subseteq \partial \tilde{\Omega}$, and by convexity, it follows that $\tilde{\Omega} = \omega \times \tilde{H}$ for some two-dimensional convex set $\omega$. Furthermore, upon passing to another subsequence if necessary, we may assume that $(A^t_h)^{-1}e_1$ and $A_he_n$ converge to the vectors $\tilde{e}^*$ and $\tilde{e}$, respectively, such that $\tilde{e}^* \perp \tilde{e}$ and

$$\tilde{\Omega} \subseteq \{x \in \mathbb{R}^n : x \cdot \tilde{e} > 0\},$$
$$\tilde{\Omega}^* \subseteq \{x \in \mathbb{R}^n : x \cdot \tilde{e}^* > 0\}.$$

Similarly to the discussion after (4.38) in the proof of Lemma 4.7 (or see the proof of [10, Lemma 5.14]), by performing an additional affine transformation, we further have

(4.56) $$\tilde{\Omega} = \{x \in \mathbb{R}^n : x_n \geq \tilde{\rho}(x_1)\},$$
where $\tilde{\rho} \geq 0$ is a convex function defined in a neighbourhood of 0 with $\tilde{\rho}(0) = 0$, and

(4.57) $$\tilde{\Omega}^* = \{x \in \mathbb{R}^n : x_1 \geq \tilde{\rho}^*(x_n)\},$$
where $\tilde{\rho}^* \geq 0$ is a smooth convex function defined in a neighbourhood of 0 with $\tilde{\rho}^*(0) = 0$.

Additionally, we may assume that $\tilde{v}_h$ converges to $\tilde{v}_\infty$ locally uniformly, where $\tilde{v}_\infty$ satisfies $\tilde{v}_\infty(0) = 0$, $\tilde{v}_\infty \geq 0$, and

(4.58) $$\det D^2\tilde{v}_\infty = c_0 \chi_{\tilde{\Omega}^*} \quad \text{in } \mathbb{R}^n,$$
$$D\tilde{v}_\infty(\tilde{\Omega}^*) = \tilde{\Omega}$$
for some constant $c_0 > 0$. 

The above blow-up limits satisfy all the conditions required in [10] Proposition 5.1 (ii). Finally, a contradiction can be derived by following the argument presented in [10] Section 6.2, concluding that Case I cannot occur.

4.2. Case II: $\nu \cdot \nu^* = 0$, $\dot{\nu} \cdot \dot{\nu}^* = 0$, and $\nu, \nu^*, \dot{\nu}^*$ are coplanar. As before, we assume: $0 \in F$, $y_0 = Du_1(0) = le_n \in \partial \Omega^*_1$, and $\tilde{y}_0 = Du_2(0) = -le_n \in \partial \Omega^*_2$, where $l > 0$ is a universal constant; the unit normals $\nu = \nu(0) = e_n$, $\nu^* = \nu^*(y_0) = e_1$ and $\dot{\nu}^* = \dot{\nu}^*(\tilde{y}_0) = e_1$ or $-e_1$.

By subtracting a constant, we may assume $u_1(0) = 0$, $i = 1, 2$ and $v(y_0) = v(\tilde{y}_0) = 0$.

Define the quantities

$$d_h := \sup \{|y \cdot e_1| : y \in S_h[v] \cap \Omega^*_1\},$$

$$\hat{d}_h := \sup \{|y \cdot e_1| : y \in S_h[v] \cap \Omega^*_2\}.$$

We then have two possible scenarios: $d_h \geq \hat{d}_h$ or $d_h < \hat{d}_h$.

Consider a sequence $\{h_k\} \to 0$ such that either $d_k := d_{h_k} \geq \hat{d}_{h_k} := \hat{d}_k$ holds for all $k$, or the opposite inequality holds for all $k$. Assume without loss of generality that the former is true for all $k$. For any unit vector $e \in \text{span}\{e_2, \ldots, e_n\}$, define

$$d_{k,e} := \sup \{|(y - y_0) \cdot e| : y \in S_{h_k}[v] \cap \Omega^*_1\},$$

$$\hat{d}_{k,e} := \sup \{|(y - \tilde{y}_0) \cdot e| : y \in S_{h_k}[v] \cap \Omega^*_2\}.$$

By the proof of Lemma 4.2 we can obtain

$$d_{k,e}, \hat{d}_{k,e} \leq C \epsilon \chi_k \frac{1}{\eta} \quad \forall e \in \text{span}\{e_2, \ldots, e_n\}.$$  

Suppose the centred section $S_{h_k}^c[v](y_0)$ is comparable to $E + \{y_0\}$, where $E$ is an ellipsoid centred at the origin. Similarly as before, the ellipsoid $E$ can be expressed by (4.17), namely

$$E = \left\{ y = y_1 e_1 + \sum_{i=2}^{n} \bar{y}_i \bar{e}_i : \frac{y_1^2}{a_1^2} + \sum_{i=2}^{n} \frac{(\bar{y}_i - k_i y_1)^2}{a_i^2} \leq 1 \right\},$$

where $\bar{e}_2, \ldots, \bar{e}_n$ are the principal directions of $E \cap \{x_1 = 0\}$. Using Lemma 4.2 and the tangential $C^{1,1-\epsilon}$ estimate of $v$ at $y_0$, we have the same estimates as in (4.18) that

$$0 < a_1 < C \epsilon \chi_k \frac{2}{3} \epsilon, \quad C \chi_k \frac{1}{2} + \epsilon < a_i < C \epsilon \chi_k \frac{1}{3} - \epsilon \quad \text{for } i = 2, \ldots, n.$$

By a similar reasoning as in (4.30), it follows that

$$|k_i| \leq C a_1^{-\frac{1}{3}}, \quad i = 2, \ldots, n.$$

Finally, we remark that from the above assumption and definition,

$$\hat{d}_k \leq d_k \approx a_1.$$

For each $k = 1, 2, \ldots$, let $T_{k_1}$ be the transformation given in (4.59) and $T_{k_2}$ be the transformation given in (4.20), and $T_k = T_{k_2} \circ T_{k_1}$. Then $T_k(E) = B_1(0)$, and $T_k \left(S_{h_k}^c[v](y_0)\right) \approx$
Moreover, diam $(T)$ and the definitions of $\Omega^*_k$ in (4.66) it follows that
\begin{equation}
B_{\frac{1}{C}}(z_k) \cap \Omega^*_k \subseteq T_k(S_{h_k}[v] \cap \Omega^*_i) \subseteq B_C(z_k).
\end{equation}

To analyze the scaling behaviour, we define the rescaled functions
\[ v_k(\cdot) = \frac{1}{h_k} v(T^{-1}_k(\cdot)); \quad u_{ki}(\cdot) = \frac{1}{h_k} u_i(T^i_k(\cdot)), \quad i = 1, 2. \]

Denote $\Omega_{ki} = \frac{1}{h_k}(T^i_k)^{-1}\Omega_i$, for $i = 1, 2$, and $\mathcal{F}_k = \frac{1}{h_k}(T^k_k)^{-1}\mathcal{F}$. For any point $x \in \mathcal{F}_k$, the unit normal $v_k$ to $\mathcal{F}_k$ satisfies
\[ v_k(x) = \frac{Du_{k1}(x) - Du_{k2}(x)}{|Du_{k1}(x) - Du_{k2}(x)|}. \]

After a suitable rotation of coordinates that aligns $T_ke_n$ with the $e_n$-axis, we have
\begin{equation}
|z_k - l_ke_n|, \quad \text{where} \quad l_k \to \infty \quad \text{as} \quad k \to \infty.
\end{equation}

Moreover, at the points $z_k = Du_{k1}(0)$ and $-z_k = Du_{k2}(0)$, the unit inner normals to $\partial \Omega^*_{ki}$ for $k = 1, 2$ are parallel to $e_1$. By Lemma 3.4, we have that $v_k$ is continuously differentiable in $(B_1(z_k) \cap \Omega^*_{k1}) \cup (B_1(-z_k) \cap \Omega^*_{k2})$ for $k$ large.

Since $d_k \geq \hat{d}_k$ and $S_1[v_k] \cap \Omega^*_k$ is normalised in the sense of (4.63), we conclude
\begin{equation}
\sup\{|y \cdot e_1 : y \in S_1[v_k] \cap \Omega^*_k, \quad i = 1, 2\} \leq C \quad \forall k = 1, 2, \ldots,
\end{equation}
where $C > 0$ is a universal constant.

**Lemma 4.8.** The distance between $S_1[v_k] \cap \Omega^*_k$ and $S_1[v_k] \cap \Omega^*_{k2}$ satisfies:
\[ \frac{\text{dist}(S_1[v_k] \cap \Omega^*_k, \quad S_1[v_k] \cap \Omega^*_{k2})}{l_k} \to 2 \quad \text{as} \quad k \to \infty. \]

Moreover, diam $(S_1[v_k] \cap \Omega^*_k) \ll l_k$ and diam $(S_1[v_k] \cap \Omega^*_{k2}) \ll l_k$ for $k$ large.

**Proof.** By the definition of $T_k$ and estimates (4.60)–(4.61), the width of $T_k(S_{h_k} \cap \Omega^*_k)$ in the $e_i$-direction is upper bounded by $C_e h_k^{\frac{4}{3} - \epsilon}$ for $i = 2, \ldots, n$. Subsequently, by (4.60) again and the definitions of $T_{k2}$, we infer that the width of $S_1[v_k] \cap \Omega^*_k$ in the $e_i$-direction is bounded from above by:
\begin{equation}
\frac{C_e h_k^{\frac{4}{3} - \epsilon}}{\min\{a_i : i = 2, \ldots, n\}} \leq C_e h_k^{-\frac{1}{6} - 2\epsilon} \quad \text{for} \quad i = 2, \ldots, n.
\end{equation}

Additionally, from (4.65), the width of $S_1[v_k] \cap \Omega^*_k$ in the $e_1$ direction is upper bounded by a constant $C$ independent of $k$.

By the definition of $T_k$ and (4.60), one can see that $l_k \geq C_e h_k^{-\frac{1}{3} - \epsilon} \gg C_e h_k^{-\frac{1}{6} - 2\epsilon}$. Hence, diam $(S_1[v_k] \cap \Omega^*_{k1}) \ll l_k$ and diam $(S_1[v_k] \cap \Omega^*_{k2}) \ll l_k$ for $k$ large. Therefore, from the above discussions we can conclude that:
\[ \frac{\text{dist}(S_1[v_k] \cap \Omega^*_{k1}, S_1[v_k] \cap \Omega^*_{k2})}{l_k} = \frac{|z_k - (-z_k)|}{l_k} + o(1) \to 2 \quad \text{as} \quad k \to \infty. \]
Similar to (4.24), there exists an open cone $C_k$ with vertex $z_k$ and the size of opening independent of $k$, such that

\begin{equation}
B_1(z_k) \cap C_k \subset \Omega_{k1}^*.
\end{equation}

**Lemma 4.9.** There exist constants $\beta \in (0, 1)$, $a > 1$, $r_1 > 0$ and $C > 0$ (independent of $k$) such that for any $y \in B_{r_1}(z_k)$, we have

\begin{equation}
0 \leq v_k(y) \leq C|y - z_k|^{1+\beta},
\end{equation}

and for any $y \in B_{r_1}(z_k) \cap \overline{\Omega_{k1}^*}$,

\begin{equation}
v_k(y) \geq \frac{1}{C}|y - z_k|^{1+a}.
\end{equation}

**Proof.** By applying an argument similar to the proof of Lemma 4.5, we can obtain

\begin{equation}
\frac{1}{C}|y - z_k|^{1+a} \leq v_k(y) \leq C|y - z_k|^{1+\beta} \quad \forall \, y \in B_{r_1}(z_k) \cap \overline{\Omega_{k1}^*}
\end{equation}

with constants $\beta \in (0, 1)$, $a > 1$, $r_1 > 0$, and $C > 0$ independent of $k$.

Given that relation (4.7) remains invariant under affine transformations, the function $v_k$ preserves this relation with the same constant $b$ in (4.7). By (4.70), we deduce that

\begin{equation}
B_{c_1 \tilde{h}^{1+\beta}}(z_k) \cap \Omega_{k1}^* \subset S_{\tilde{h}}^c[v_k](z_k) \cap \Omega_{k1}^* \subset \Omega_{k1}^* \quad \forall \, \tilde{h} \in (0, \tilde{h}_0),
\end{equation}

where $\tilde{h}_0, c_1 = (C b)^{-\frac{1}{1+\beta}}$ are constants independent of $k$.

Combining inequalities (4.67) and (4.71), we infer that

\begin{equation}
B_{c_1 \tilde{h}^{1+\beta}}(z_k) \cap C_k \subset S_{\tilde{h}}^c[v_k](z_k) \quad \forall \, \tilde{h} \in (0, \tilde{h}_0).
\end{equation}

Taking into account that $S_{\tilde{h}}^c[v_k](z_k)$ is centred at $z_k$, we conclude that the “opposite” cone

\[ W_k := -\frac{1}{C_n} \left( B_{c_1 \tilde{h}^{1+\beta}}(z_k) \cap C_k \right) - \{z_k\} \subset S_{\tilde{h}}^c[v_k](z_k) \]

for some large constant $C_n$, dependent only on $n$. Thanks to the convexity of $S_{\tilde{h}}^c[v_k](z_k)$, it must contain the convex hull of

\[ (B_{c_1 \tilde{h}^{1+\beta}}(z_k) \cap C_k) \cup W_k. \]

Since the size of opening of the cone $C_k$ is independent of $k$, one can see that

\[ B_{\tilde{h}^{1+\beta}}(z_k) \subset \text{convex hull of } (B_{c_1 \tilde{h}^{1+\beta}}(z_k) \cap C_k) \cup W_k, \]

where $C$ is a constant independent of $k$. Consequently, we obtain

\begin{equation}
B_{\tilde{h}^{1+\beta}}(z_k) \subset S_{\tilde{h}}^c[v_k] \subset S_{\tilde{h}}^c[v_k]
\end{equation}

for $k$ large. The desired estimate (4.68) readily follows from (4.72).
Lemma 4.10. There exists a universal constant $N$ such that for $u_{k1} - l_k x_n$ and $u_{k2} + l_k x_n$, we have the inclusions

\begin{align}
S_{\frac{1}{N}} [u_{k1} - l_k x_n] &\subset Dv_k(S_1[v_k] \cap \Omega_{k1}^r), \\
S_{\frac{1}{N}} [u_{k2} + l_k x_n] &\subset Dv_k(S_1[v_k] \cap \Omega_{k2}^r),
\end{align}

respectively.

**Proof.** By the duality between $u_{k1}$ and $v_k$ (see (2.5)), we have

$$u_{k1}(x) = \sup \{ x \cdot y - v_k(y) : y \in \Omega_{k1}^r \}.$$ 

By the strict convexity estimate of $v_k$ (see (4.69)), we have

$$B_{r_2}(0) \cap \Omega_{k1}^r \subset Dv_k(S_1[v_k] \cap \Omega_{k1}^r)$$

for some constant $r_2 > 0$ independent of $k$. Hence, for any $x \in B_{r_2}(0) \cap \Omega_{k1}^r$,

$$u_{k1}(x) = \sup \{ x \cdot y - v_k(y) : y \in B_{r_1}(z_k) \cap \Omega_{k1}^r \}$$

$$= \sup \{ x \cdot y - v_k(y) : y \in B_{r_1}(z_k) \}$$

$$\geq \sup \{ x \cdot y - C|y - z_k|^{1+\beta} : y \in B_{r_1}(z_k) \}$$

$$= \sup \{ x \cdot (y - z_k) - C|y - z_k|^{1+\beta} : y \in B_{r_1}(z_k) \} + x \cdot z_k$$

$$\geq C_1 |x|^{1+a'} + l_k x_n$$

for $a' = \frac{1}{p} > 1$, where $C_1$ is a constant independent of $k$, the first inequality follows from (4.68), and the last inequality follows from a direct computation. It follows from the above estimate that

$$u_{k1}(x) - l_k x_n \geq C_1 |x|^{1+a'} \quad \forall x \in B_{r_2}(0) \cap \Omega_{k1}^r.$$

By (4.63), we have that

$$B_{\frac{1}{N}}(z_k) \cap \Omega_{k1}^r \subset S_1[v_k] \cap \Omega_{k1}^r.$$ 

Then, the desired inclusion (4.73) follows from (4.70) and (4.75).

The proof of (4.74) is analogous. Indeed, Lemma 4.9 also holds for $v_k$ near $\hat{z}_k := -z_k$ if we normalise $S_{h_k}[v](z_k) \cap \Omega_2^r$ instead of $S_{h_k}[v](\hat{z}_k) \cap \Omega_1^r$. It is important to note that both (4.73) and (4.74) are affine invariant. \(\Box\)
By (4.69) and the duality between $u_{k1}$ and $v_k$ (see (2.5)), we have
\[
    u_{k1}(x) = \sup\{x \cdot y - v_k(y) : y \in B_r(z_k) \cap \Omega_{k1}^r\}
\]
\[
    \leq \sup\{x \cdot y - C|y - z_k|^{1+a} : y \in B_r(z_k) \cap \Omega_{k1}^r\}
\]
\[
    \leq \sup\{x \cdot y - C|y - z_k|^{1+a} : y \in \mathbb{R}^n\}
\]
\[
    = \sup\{x \cdot (y - z_k) - C|y - z_k|^{1+a} : y \in \mathbb{R}^n\} + x \cdot z_k
\]
\[
    = C_1|x|^{1+\beta'} + l_k x_n
\]
for any $x \in B_r(0) \cap \Omega_{k1}$, where $\beta' = \frac{1}{a}$, and the last equality follows from the fact that the Legendre transform of $|x|^{1+a}$ is $c|x|^{1+\frac{a}{2}}$ for some constant $c > 0$. Hence, it follows that
\[
    u_{k1}(x) - l_k x_n \leq C_1|x|^{1+\beta'}
\]
for any $x \in B_r(0) \cap \Omega_{k1}$. By (4.75) and choosing $M$ sufficiently large, we may also ensure that
\[
    B_{\frac{1}{\sqrt{M}}}(0) \cap \Omega_{k1} \subset Du_k(S[v_k] \cap \Omega_{k1}^*),
\]
for $k$ sufficiently large.

Recall that $v_k(x)$ is the unit inner normal of $\Omega_{k1}$ at $x \in F_k \subset \partial \Omega_{k1}$. Let $G_k$ be a connected component of $\{x \in B_1(0) \cap F : v_k(x) \cdot e_n > 0\}$ such that $0 \in G_k$, and
\[
    G_k = \{x = (x', x_n) \in \mathbb{R}^n : x_n = \rho_k(x'), \ x' \in G_k'\}
\]
for some $C^{1,\beta}$ function $\rho_k$ satisfying $\rho_k(0) = 0$ and $D\rho_k(0) = 0$, where $G_k' \subset \mathbb{R}^{n-1}$ is the orthogonal projection of $G_k$ onto the plane $\{x_n = 0\}$. Note that $G_k'$ is a connected open set in $\mathbb{R}^{n-1}$ containing the origin.

If $p \in F_k \cap B_{\frac{1}{\sqrt{M}}}(0) \cap Du_k(S[v_k] \cap \Omega_{k2}^*)$, we have that $Du_{k2}(p) \in S[v_k] \cap \Omega_{k2}^*$. By (4.78), we have that $Du_{k1}(p) \in S[v_k] \cap \Omega_{k1}^*$. Since the width of $S[v_k] \cap \Omega_{k1}^*$ (for $i = 1, 2$) in the $e_1$ direction is bounded by a universal constant $C$, it follows that
\[
    |(Du_{k1}(p) - Du_{k2}(p)) \cdot e_1| \leq C.
\]
By Lemma 4.8 we have that $|Du_{k1}(p) - Du_{k2}(p)| \geq l_k$ provided $k$ is sufficiently large. Hence, it follows that
\[
    |\nu_k(p) \cdot e_1| = \frac{|Du_{k1}(p) - Du_{k2}(p)|}{|Du_{k1}(p) - Du_{k2}(p)|} \leq \frac{C}{l_k}
\]
for some constant $C$ independent of $k$, provided $k$ is sufficiently large. By Lemma 4.8 we have that $\text{diam}(S[v_k] \cap \Omega_{k1}^*) \ll l_k$ and $\text{diam}(S[v_k] \cap \Omega_{k2}^*) \ll l_k$ for sufficiently large $k$, which implies that
\[
    \nu_k(p) \cdot e_n > \frac{1}{2} \quad \forall p \in F_k \cap B_{\frac{1}{\sqrt{M}}}(0) \cap Du_k(S[v_k] \cap \Omega_{k2}^*).
From inequality (4.79), if we further assume that \( p \in G_k \), we can deduce that
\[
|D_1 \rho_k(p)| \leq \frac{C}{l_k}
\]
for sufficiently large \( k \).

Let
\[
p_t = (-t,0,\ldots,0,\rho_k(-t,0,\ldots,0))
\]
denote a point in \( G_k \cap B_1(0) \cap \text{span}\{e_1,e_n\} \). Now, define
\[
s_{k0} := \sup \left\{ s \mid -te_1 \in G'_k, \ p_t \in G_k \cap B_{\frac{1}{4M}}(0) \cap Dv_k(S_1[v_k] \cap \Omega_{k2}^*) \ \forall \ t \in [0,s] \right\}.
\]
We aim to establish a lower bound for \( s_{k0} \) in the following lemma.

**Lemma 4.11.** For sufficiently large \( k \), we have
\[
s_{k0} \geq \min \left\{ \frac{1}{4M}, \frac{1}{4NC} \right\},
\]
where \( C \) is a positive constant independent of \( k \).

**Proof.** Denote \( s_{k0} \) by \( s_0 \) for simplicity. Up to a subsequence, we may assume that \( p_t \) converges to a point \( p_{s0} \in B_{\frac{1}{4M}}(0) \cap F_k \) as \( t \to s_0 \). For any \( t \in (0,s_0) \), we have that
\[
p_t \in F_k \cap B_{\frac{1}{4M}}(0) \cap Dv_k(S_1[v_k] \cap \Omega_{k2}^*).
\]
Then, it follows from (4.80) that \( \nu_k(p_t) \cdot e_n \geq \frac{1}{2} \) for \( t \in (0,s_0) \) and \( k \) sufficiently large. By continuity, we have
\[
\nu_k(p_{s0}) \cdot e_n \geq \frac{1}{2},
\]
which implies that
\[
p_{s0} \in G_k.
\]
Suppose that (4.83) does not hold; then, \( s_0 < \frac{1}{4M} \). By estimate (4.81), we have
\[
|\rho_k(-t,0,\ldots,0)| \leq \frac{Ct}{l_k} \leq \frac{C}{4ML_k} \leq \frac{1}{4M}
\]
for \( t < s_0 \)
provided \( k \) is sufficiently large. Now, for \( t \in (0,s_0) \), we have that
\[
|p_t| = \sqrt{t^2 + |\rho_k(-t,0,\ldots,0)|^2} \leq \frac{1}{2\sqrt{2M}}.
\]
By continuity, this implies that
\[
p_{s0} \in B_{\frac{1}{2M}}(0)
\]
provided \( k \) is sufficiently large.

We claim that
\[
p_{s0} \notin Dv_k(S_1[v_k] \cap \Omega_{k2}^*)
\]
for \( k \) sufficiently large. Otherwise, by (4.84) and (4.85), for sufficiently large \( k \), we could extend \( t \) beyond \( s_0 \), namely,

\[
p_{s_0+\varepsilon} := (-s_0 - \varepsilon, 0, \ldots, 0, \rho_k(-s_0 - \varepsilon, 0, \ldots, 0)) \in G_k \cap B \frac{1}{2M}(0) \cap Dv_k \left( S_1[v_k] \cap \Omega_{k1}^* \right),
\]

which would be contradicting the definition of \( s_0 \).

Combining (4.86) and (4.74), we obtain

(4.87)

\[
u_{k2}(p_{s_0}) + l_k p_{s_0} \cdot e_n \geq \frac{1}{N}.
\]

On the other hand, since \( s_0 < \frac{1}{4N} \) by assumption, by (4.77) and (4.81) we obtain

(4.88)

\[
u_{k1}(p_{s_0}) \leq \frac{1}{4N} + l_k p_{s_0} \cdot e_n \leq \frac{1}{4N} + \frac{1}{4N} \leq \frac{1}{2N}.
\]

Because \( u_{k1} = u_{k2} \) on \( F_k \), by (4.88) we have

\[
u_{k2}(p_{s_0}) + l_k p_{s_0} \cdot e_n = u_{k1}(p_{s_0}) + l_k p_{s_0} \cdot e_n \leq \frac{1}{2N} + \frac{1}{4N} < \frac{1}{N},
\]

which contradicts (4.87).

Let \( t_0 := \min \left\{ \frac{1}{4M}, \frac{1}{4NC} \right\} \). By (4.81) and Lemma 4.11 we deduce that

\[
|\rho_k(-t_0 e_1)| = |p_{t_0} \cdot e_n| \leq \frac{C}{l_k} t_0 \to 0 \quad \text{as} \quad k \to \infty.
\]

Define

\[
\tilde{u}_k(x) := \sup \{ x \cdot y - v_k(y + z_k) : y + z_k \in S_1[v_k] \cap \Omega_{k1}^* \}.
\]

Since \((S_1[v_k] \cap \Omega_{k1}^*) \setminus \{x_k\} \subset B_C(0)\) for some constant \( C \) independent of \( k \), we have that \( \|\tilde{u}_k\|_{Lip} \leq C \). Note that \( \tilde{u}_k = u_{k1} - l_k x_n \) on \( B \frac{1}{2M}(0) \cap \Omega_{k1} \). With the assumption that \( \Omega_{k1}^* \subset \{x_1 \geq 0\} \), we have \( D_1 \tilde{u}_k \geq 0 \). It implies that \( \tilde{u}_k(-t_0 e_1) \leq \tilde{u}_k(0) = 0 \). Consequently,

\[
0 \leq u_{k1}(p_{t_0}) - l_k p_{t_0} \cdot e_n = \tilde{u}_k(p_{t_0}) \leq \tilde{u}_k(-t_0 e_1) + \|\tilde{u}_k\|_{Lip} p_{t_0} \cdot e_n \leq C p_{t_0} \cdot e_n \leq \frac{C}{l_k} t_0 \to 0 \quad \text{as} \quad k \to \infty,
\]

which contradicts (4.76) for sufficiently large \( k \).
4.3. **Case III** : $\nu \cdot \nu^* = 0$, $\nu \cdot \hat{\nu}^* > 0$. After an appropriate affine transformation, we can assume that $\nu$ is parallel to $\hat{\nu}^*$. Indeed, we only need to choose an affine transform $A$ such that $A\nu$ is parallel to $(A^t)^{-1}\hat{\nu}^*$, see [10] (3.2) for the existence of such a transformation. Note that after the transformation, $\frac{\nu}{\nu} = \frac{\hat{\nu}^*}{\hat{\nu}^*}$ as the inner unit normal of $A\Omega_1$ at 0 is still orthogonal to $\frac{A\nu}{|A\nu|}$ as the inner unit normal of $(A^t)^{-1}\Omega_1^*$ at $Ay_0$.

As in (4.8) and (4.9), we have that $0 \in F$, $y_0 = Du_1(0) = le_n \in \partial \Omega_1^*$, and $\hat{y}_0 = Du_2(0) = -le_n \in \partial \Omega_2^*$, where $l$ is a positive universal constant. Additionally, we have the unit normals $\nu = \nu(0) = e_n$, $\nu^* = \nu^*(y_0) = e_1$, and $\hat{\nu}^* = \hat{\nu}^*(\hat{y}_0) = -e_n$. By subtracting a constant, we may assume $u_i(0) = 0$ for $i = 1, 2$ and $v(y_0) = v(\hat{y}_0) = 0$.

In this case, Lemma 5.4 is applicable, leading to

$$u_2(x) - \hat{y}_0 \cdot e_n \leq C_\epsilon |x|^{2-\epsilon} \quad \forall x \in B_{r_0}(0).$$

Given that $v = u_2^*$ within $\Omega_2^*$, it follows that

$$v(y) \geq C_\epsilon |y - \hat{y}_0|^{2+\epsilon} \quad \forall y \in \Omega_2^*.$$

It is noteworthy that in Case I, the specific conditions $\hat{\nu} \cdot \hat{\nu}^* = 0$ and the non-coplanarity of $\nu, \nu^*, \hat{\nu}^*$ are exclusively used in the proofs of Lemmas 4.3 and 4.4.

The proof of Lemma 4.3 relies on the inequality

$$v(y) \geq C_\epsilon |y - \hat{y}_0|^{3+\epsilon}, \quad \forall y \in \Omega_2^*.$$

In the present scenario, this inequality remains valid since $|y - \hat{y}_0|^{2+\epsilon}$ dominates $|y - \hat{y}_0|^{3+\epsilon}$ when $y$ is close to $\hat{y}_0$.

By inclusions (4.6) and (4.7), we can find a point $p \in S_h[v]$ such that $p_1 \geq h^{2+\epsilon}$, creating a contradiction with (4.16). Hence, Lemma 4.4 is still valid in this particular scenario. As a result, the remainder of the proof from Case I can be replicated to eliminate the possibility of the current case.

5. **$C^{2,\alpha}$ estimates**

Let $\Omega, \Omega_1, \Omega_2, \Omega^*, \Omega_1^*, \Omega_2^*, F, f, g, u, u_1, u_2, v, \nu, \hat{\nu}, \nu^*$, and $\hat{\nu}^*$ be as defined in Section 3. Fix a point $x_0 \in F$ and denote $y_0 = Du_1(x_0)$. Thanks to Proposition 4.1, $\nu \cdot \nu^* > 0$, we can make a change of coordinates such that $x_0 = y_0 = 0$, and for some sufficiently small $r > 0$,

$$\Omega_1 \cap B_r(0) = \{x_n > \rho(x')\} \cap B_r(0) \quad \text{and} \quad \Omega_1^* \cap B_r(0) = \{x_n > \rho^*(x')\} \cap B_r(0),$$

where $\rho$ is a $C^{1,\beta}$ function satisfying $\rho(0) = 0$ and $D\rho(0) = 0$, and $\rho^*$ is a $C^2$ uniformly convex function with $\rho^*(0) = 0$ and $D\rho^*(0) = 0$. Here, $x' = (x_1, \ldots, x_{n-1})$.

By the proof of Lemmas 3.1 and 3.2 of [10], we can conclude that:
Lemma 5.1. For any \( \epsilon > 0 \) small, there exists a constant \( C_{\epsilon} \) such that:

\[
\begin{align*}
(5.1) 
& u_1(x) \geq C_{\epsilon}|x'|^{2+\epsilon} \quad \forall x \in \Omega_1 \cap B_r(0), \\
& u_1(tc_n) \leq C_{\epsilon}|t|^{2-\epsilon} \quad \forall t \text{ satisfying that } |t| \text{ is sufficiently small.}
\end{align*}
\]

In [10], by utilising \( \rho(x') \leq C|x'|^2 \) (derived from the interior ball property for optimal partial transport), a uniform density estimate for \( u_1 \) can then be derived. In the current situation, we only know that \( \mathcal{F} \) is \( C^{1,\beta} \) regular, namely \( \rho(x') \leq C|x'|^{1+\beta} \) for some \( \beta \in (0,1) \). Below, we modify the proof of [10, Lemma 3.3] to obtain the uniform density estimate under this weaker condition.

Lemma 5.2. For any \( h > 0 \) small, one has

\[
|S_h^c[u_1] \cap \Omega_1| \geq \delta_0,
\]

where \( \delta_0 > 0 \) is a constant independent of \( h \), and \( S_h^c[u_1] = S_h^c[u_1](0) \) denotes the centred section of \( u_1 \) with height \( h \).

Proof. Consider the intersections of \( \partial S_h^c[u_1] \) with the \( x_n \)-axis, denoted by \( z = se_n \) and \( \tilde{z} = -\tilde{s}e_n \), where \( s, \tilde{s} > 0 \). Since \( S_h^c[u_1] \) is centred at \( 0 \), one has \( s \approx \tilde{s} \). Consequently, we deduce that either \( u_1(z) \geq Ch \) or \( u_1(\tilde{z}) \geq Ch \) holds. Then, by Lemma 5.1, we have

\[
(5.2) \quad s \approx \tilde{s} \geq C_{\epsilon}h^{1+\epsilon}
\]

for any arbitrarily small \( \epsilon > 0 \).

Utilising (4.7) and Lemma 5.1, we derive that

\[
(5.3) \quad S_h^c[u_1] \cap \Omega_1 \subset S_{Ch}[u_1] \cap \Omega_1 \subset \left\{ x : |x'| < C_{\epsilon}h^{\frac{1}{2}-\epsilon} \right\}.
\]

Recall the bound \( |\rho(x')| \leq C|x'|^{1+\beta} \). For any point \( x \in S_h^c[u_1] \cap \{ x_n \geq C'h^\frac{1}{2}+\frac{1}{4}\beta \} \cap \overline{\Omega_1} \), inequality (5.3) ensures that \( |x'| < C_{\epsilon}h^{\frac{1}{2}-\epsilon} \), which, in turn, implies

\[
\rho(x') < C'h^{\left(\frac{1}{2}-\epsilon\right)(1+\beta)} \leq C'h^{\frac{1}{2}+\frac{1}{4}\beta} \leq x_n
\]

provided \( \epsilon \) is sufficiently small, where \( C' = 2CC_{\epsilon}^{1+\beta} \). Consequently, \( x \in \Omega_1 \). This leads us to conclude that

\[
(5.4) \quad S_h^c[u_1] \cap \{ x_n \geq C'h^\frac{1}{2}+\frac{1}{4}\beta \} \cap \overline{\Omega_1} \subset \Omega_1.
\]

Assume, for the sake of contradiction, that there exists a point

\[
x \in S_h^c[u_1] \cap \{ x_n \geq C'h^\frac{1}{2}+\frac{1}{4}\beta \} \setminus \Omega_1.
\]

Connecting \( x \) and \( z \) with a line segment, we observe that it intersects \( \partial \Omega_1 \) at some point \( y \). As both \( x \) and \( z \) belong to \( S_h^c[u_1] \cap \{ x_n \geq C'h^\frac{1}{2}+\frac{1}{4}\beta \} \), the convexity of \( u_1 \) implies \( y \in S_h^c[u_1] \cap \{ x_n \geq C'h^\frac{1}{2}+\frac{1}{4}\beta \} \cap \partial \Omega_1 \), contradicting (5.4). Therefore, we deduce that

\[
(5.5) \quad S_h^c[u_1] \cap \{ x_n \geq C'h^\frac{1}{2}+\frac{1}{4}\beta \} \subset \Omega_1.
\]
which implies that a “substantial” portion of $S^c_h[u_1]$ is contained within $\Omega_1$.

Invoking John’s Lemma, we identify an ellipsoid $E$ centred at the origin such that $E \subset S^c_h[u_1] \subset CE$. Due to (5.2), we have $s \gg C'h^\frac{1}{2}+\frac{1}{4}\beta$ for sufficiently small $h$. Considering the convexity of $S^c_h[u_1]$ and (5.5), we estimate

$$|S^c_h[u_1] \cap \Omega_1| \geq \left| S^c_h[u_1] \cap \{ x_n \geq C'h^\frac{1}{2}+\frac{1}{4}\beta \} \right|$$

$$\geq \left| E \cap \{ x_n \geq C'h^\frac{1}{2}+\frac{1}{4}\beta \} \right|$$

$$\geq \frac{c}{s} \frac{s-h^\frac{1}{2}+\frac{1}{4}\beta}{s} |E|$$

$$\geq \frac{1}{2} |S^c_h[u_1]|,$$

where $c > 0$ depends only on $n$. Consequently, we obtain

$$\frac{|S^c_h[u_1] \cap \Omega_1|}{|S^c_h[u_1]|} \geq \frac{c}{2}.$$

Once having the uniform density estimate, by the proof and techniques employed in [10, Lemma 3.4 and Corollary 3.2], we can obtain the subsequent estimates for $u_1$.

Lemma 5.3. For any $\epsilon > 0$ small, there exists a constant $C_\epsilon > 0$ such that:

1. Tangential $C^{1,1-\epsilon}$ estimate: $B_{C_\epsilon h^\frac{1}{2}+\epsilon}(0) \cap \{ x_n = 0 \} \subseteq S^c_h[u_1]$.
2. Almost $C^{1,1}$ estimate: $u_1(x) \leq C_\epsilon |x|^{2+\epsilon} \forall x \in B_{r_0}(0)$.
3. Strict convexity in $\Omega_1$: $u_1(x) \geq C_\epsilon |x|^{2+\epsilon} \forall x \in \Omega_1 \cap B_{r_0}(0)$.
4. Gradient bound: $|Du_1(x)| \leq C_\epsilon |x|^{1+\epsilon} \forall x \in B_{r_0}(0)$.

Here, $r_0 > 0$ is a suitably chosen small constant.

Thanks to Proposition 4.3, $\tilde{\nu} \cdot \tilde{\nu} > 0$, similarly we can establish analogous estimates for the function $u_2$. As before, by a change of coordinates we may assume $x_0 = \tilde{y}_0 = 0$ and

$$\Omega_2 \cap B_r(0) = \{ x_n > \tilde{\rho}(x') \} \cap B_r(0) \quad \text{and} \quad \Omega_2^* \cap B_r(0) = \{ x_n > \tilde{\rho}^*(x') \} \cap B_r(0),$$

where $\tilde{\rho}$ is a $C^{1,\beta}$ function satisfying $\tilde{\rho}(0) = 0$ and $D\tilde{\rho}(0) = 0$, and $\tilde{\rho}^*$ is a $C^2$ uniformly convex function with $\tilde{\rho}^*(0) = 0$ and $D\tilde{\rho}^*(0) = 0$.

Lemma 5.4. For any $\epsilon > 0$ small, there exists a constant $C_\epsilon > 0$ such that:

1. Tangential $C^{1,1-\epsilon}$ estimate: $B_{C_\epsilon h^\frac{1}{2}+\epsilon}(0) \cap \{ x_n = 0 \} \subseteq S^c_h[u_2]$.
2. Almost $C^{1,1}$ estimate: $u_2(x) \leq C_\epsilon |x|^{2+\epsilon} \forall x \in B_{r_0}(0)$.
3. Strict convexity in $\Omega_2$: $u_2(x) \geq C_\epsilon |x|^{2+\epsilon} \forall x \in \Omega_2 \cap B_{r_0}(0)$.
4. Gradient bound: $|Du_2(x)| \leq C_\epsilon |x|^{1+\epsilon} \forall x \in B_{r_0}(0)$.

Here, $r_0 > 0$ denotes a sufficiently small constant.
Proof of Theorem 1.2. Lemmas 5.3 and 5.4 imply that the functions $u_i$ for $i = 1, 2$ are $C^{1,1-\epsilon}$ along the singular set $\mathcal{F}$ for any sufficiently small $\epsilon > 0$. Subsequently, from (1.3), it follows that $\mathcal{F}$ itself possesses $C^{1,1-\epsilon}$ regularity.

Having established the $C^{1,1-\epsilon}$ regularity of both $u_i$ and $\mathcal{F}$, we are now in a position to apply the perturbation argument tailored for the optimal partial transport problem, as delineated in [10, Section 4]. This argument ensures that the regularity of $u_i$ can be further improved to $C^{2,\alpha}$ along $\mathcal{F}$. From (1.3) once more, we conclude that the singular set $\mathcal{F}$ is also of class $C^{2,\alpha}$.

□

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