Adjoint non-Abelian Coulomb gas at large $N$

G.W. Semenoff

*Department of Physics, University of British Columbia,*  
*6224 Agricultural Road, Vancouver, British Columbia, Canada V6T 1Z1*

and

K. Zarembo

*Steklov Mathematical Institute,*  
*Vavilov Street 42, GSP-1, 117966 Moscow, RF*

and

*Institute of Theoretical and Experimental Physics,*  
*B. Cheremushkinskaya 25, 117259 Moscow, RF*

**Abstract**

The non-Abelian analog of the classical Coulomb gas is discussed. The statistical mechanics of arrays of classical particles which transform under various representations of a non-Abelian gauge group and which interact through non-Abelian electric fields are considered. The problem is formulated on the lattice and, for the case of adjoint charges, it is solved in the large $N$ limit. The explicit solution exhibits a first order confinement-de-confinement phase transition with computable properties. In one dimension, the solution has a continuum limit which describes 1+1-dimensional quantum chromodynamics (QCD) with heavy adjoint matter.

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1E–mail: semenoff@physics.ubc.ca / semenoff@nbivms.nbi.dk
2E–mail: zarembo@class.mian.su
1 Introduction

The classical Coulomb gas is an important model in statistical mechanics. It is solvable in one dimension and in two dimensions it exhibits interesting critical phenomena. In this Paper, we shall formulate a non-Abelian generalization of the Coulomb gas. We consider the thermodynamic properties of arrays of non-dynamical quarks which transform under irreducible representations of the gauge group and which interact via non-Abelian electric fields. The main motivation is to study the confinement-deconfinement phase transition which is expected to occur in gauge theory at sufficiently high temperature or density.

The model which we formulate is most interesting in the case where the quarks transform under the adjoint representation of the gauge group. In that case, it can be solved explicitly in the large $N$ limit and exhibits a first order phase transition when the spatial dimension is $D=1,2,3$. When $D=1$, it has a continuum limit where it represents $(1+1)$-dimensional QCD with very heavy adjoint matter fields.

In the absence of quarks, i.e. the zero density limit, the model which we consider reduces to the strong coupling limit of lattice Yang-Mills theory. In that limit, only the color-electric interactions are retained. The phase structure of that system was studied in the large $N$ limit in refs.[1, 2]. There, it was found that in $D=2$ and $D=3$, there is a first order phase transition which occurs as the parameter $\gamma = e^2 N/2T$ is varied. The strong coupling, low temperature, confining phase occurs at $\gamma >> 1$ and the weak coupling, high temperature, de-confined phase at $\gamma << 1$. In the former case one can find an explicit solution of the lattice theory in the large $N$ limit. This phase is stable to small fluctuations for $\gamma$ greater than some critical value. In the small $\gamma$ limit, approximate techniques are required and a de-confined phase is found. There is a co-existence region where both phases are stable to small fluctuations, and are therefore separated by an energy barrier. In that region, there is a first order phase transition between them.

In $D = 2, 3$, the strong coupling theory is not renormalizable. The phase transition is first order, and there is no limit where the latent heat is small, i.e. of order $N^2$ times a quantity which remains finite as the lattice spacing is taken to zero. One could speculate that, if the strong coupling limit were corrected to include the magnetic term in the Hamiltonian, one would recover a renormalizable theory, manifest in the fact that there would be some limit in which the first order phase transition had finite latent heat in the continuum. There are good arguments for this scenario in the literature. As well, large $N$ techniques have been used to study the confinement-deconfinement transition in pure Yang-Mills theory [3].

In $D=1$, there is no magnetic term in the full Hamiltonian, and Yang-Mills theory can be solved exactly using a technique similar to the strong coupling approximation. Rather than being generated dynamically, the confining quark-antiquark potential appears at the tree level where the string tension is proportional to the coupling constant $e^2$. However, in $D=1$, pure Yang-Mills theory is trivial in that it has no propagating degrees of freedom.
It therefore does not exhibit a phase transition when the temperature is non-zero and it is in the confining phase for all values of $\gamma$.

In this paper, we consider the model where a gas of charged sources (quarks) in the adjoint representation of the gauge group are added to strongly coupled Yang-Mills theory. For $D = 2, 3$ there is still a first order phase transition which occurs on a critical line in the $\gamma - \lambda$ plane, where $\lambda \sim e^{-\mu/T}$ is the fugacity and $\mu$ is the chemical potential of the quarks. When $\gamma$ is greater than a certain critical value, the deconfinement transition can be induced by increasing the density, which is controlled by increasing $\lambda$. There is also a third order phase transition in the de-confined phase, similar to the Gross-Witten transition of $D=1$ lattice Yang-Mills theory.

In $D=1$, the coupling of sources to Yang-Mills theory makes the model non-trivial. The continuum version of this model has been studied in ref. 6. There is a first order phase transition between the confining and de-confined phases which can be obtained by increasing $\lambda$ for any value of $\gamma$. This phase transition originates in a percolation transition for electric flux lines. The quantum states are arrays of adjoint quarks on the line with non-dynamical strings of electric flux joining them. Each quark must have one line of flux entering and one line leaving it. For a fixed number of quarks, the model is explicitly solvable and has a finite dimensional Hilbert space, corresponding with the different ways of distributing electric flux so that the states are gauge invariant. In the large $N$ limit, the energy of a state is proportional to the total length of lines of electric flux plus the chemical potential times the number of quarks. At low temperature and density, the statistical sum is dominated by configurations which are a dilute gas of mesons – color neutral bound states of two or more adjoint quarks bound together by the appropriate number of strings of electric flux.

The property of confinement is characterized by measuring the energy necessary to insert a fundamental representation quark-antiquark pair into the system. Gauge invariance requires that the pair is connected by a single string of electric flux. At low temperature, the energy of this string is proportional to its length, leading to the linear, confining interaction. In a deconfined phase, for large separation, the energy would be a constant as the distance between the quark and antiquark is varied. This occurs when, at the deconfinement phase transition, the strings percolate in the one-dimensional space, and the addition of a flux string between a quark and an antiquark adds a small amount (or no) energy to the energy of the typical configuration.

In the next few subsections, we review some of the formalism which we require in the later sections - the Hamiltonian formalism of lattice Yang-Mills theory, the construction of the partition function in that formalism and some properties of states in the non-Abelian Coulomb gas.

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3 1+1 dimensional Yang-Mills theory on the sphere is known to have a third order Kazakov-Douglas phase transition. However, in the cylindrical geometry which is appropriate to finite temperature Yang-Mills theory on the open line, $R^1$, there is no third order phase transition.
1.1 Hamiltonian formulation of lattice Yang-Mills theory with external sources

We begin by reviewing the Hamiltonian formalism of Yang-Mills theory on the lattice. This is standard material which can be found in ref. [7], for example. In the Hamiltonian formalism, time is continuous and the space is approximated by a hypercubic lattice with sites \( x, y, \ldots \) and oriented links \( l \). The spatial gauge fields are operator valued unitary matrices, \( U_l \), and electric fields are operator valued Lie algebra elements, \( E_l \),

\[
U_l U_l^\dagger = 1 = U_l^\dagger U_l, \quad E_l^\dagger = E_l
\]

Both are associated with oriented links and obey the reflection conditions

\[
U_{-l} = U_l^\dagger, \quad E_{-l} = -U_l^\dagger E_l U_l
\]

The electric fields can be expanded in a basis

\[
E_l = \sum_A E_l^A T^A
\]

where \( T^A \) are hermitean generators in the fundamental representation of the Lie algebra

\[
\left[ T^A, T^B \right] = if^{ABC} T^C
\]

The operators have the algebra

\[
[U_l, U_{l'}] = 0, \quad \left[ E_l^A, E_{l'}^B \right] = if^{ABC} E_{l'}^C \delta_{ll'}, \quad \left[ E_l^A, U_{l'} \right] = T^A U_{l'} \delta_{ll'}
\]

The Hamiltonian is

\[
H = \sum_{l,A} \frac{e^2}{2} \left( E_l^A \right)^2 + \frac{1}{2} \frac{e^2}{2} \text{Tr} \left( \prod U + \prod U^\dagger \right)
\]

and the gauge constraint is

\[
\mathcal{G}^A(x) \equiv \sum_{l \in n(x)} E_l^A + \sum_{i=1}^K T_{R_i}^A \delta(x - x_i) \sim 0
\]

The first term in the Hamiltonian is the electric energy and the second term, which is summed over oriented elementary plaquettes \( \square \), is the magnetic energy. In \( \mathcal{G}^A(x) \), \( n(x) \) is the set of links one of whose endpoints is the site \( x \) and with orientation toward \( x \). Also, we have considered the system in the presence of an array of \( K \) classical quarks which transform under representations \( R_i \) and are situated at lattice sites \( x_i \).

The gauge constraint commutes with the Hamiltonian and generates the gauge transformation,

\[
U_l \to U_l^g \equiv g(x) U_l g^\dagger(y), \quad E_l \to E_l^g \equiv g(x) E_l g^\dagger(x)
\]
where $g(x)$ is a unitary matrix in the fundamental representation of the gauge group and $x$ and $y$ are the endpoints of the oriented link $l$: $\delta l = [y] - [x]$. It is useful to consider the Schrödinger representation where quantum states are functions of the gauge fields and the action of the electric field on these states is defined algebraically. In this representation, the gauge constraint implies that physical states transform as

$$\psi_{a_1 \ldots a_K}(U) = g^{R_1}_{a_1 b_1}(x_1) \ldots g^{R_K}_{a_K b_K}(x_K) \psi_{b_1 \ldots b_K}(U)$$

(1.9)

Since the Hamiltonian is gauge invariant, the eigenstates of the Hamiltonian carry a representation of the gauge group. The physical states are those which transform like (1.9).

In the continuum limit, the spatial gauge field $\bar{A}$ is obtained from $U_l = \mathcal{P} e^{i \int_\gamma \bar{A} d\gamma} \approx 1 + ia\bar{\ell} \cdot \bar{A} + \ldots$ where $a$ is the lattice spacing and $\bar{\ell}$ is a unit vector in the direction of the link $l$. Also, the continuum electric field operator is defined as $E^A = a^{D-1} \bar{\ell} \cdot \bar{E}^A$. In $D > 1$, the magnetic energy reduces to the magnetic field squared. The Hamiltonian (1.4), gauge constraint (1.7) and operator algebra (1.5) reduce to those of continuum Yang-Mills theory,

$$H = \int d^D x \left( \frac{e^2}{2} (\bar{E}^A)^2 + \frac{1}{2e^2} (\bar{B}^A)^2 \right)$$

(1.10)

$$G^A(x) = (\bar{D} \cdot \bar{E})^A + \sum_{i=1}^{K} T^A_{\overline{R}_i} \delta(x - x_i) \sim 0$$

(1.11)

and

$$[A_i(x), A_j(y)] = 0 , \quad [E_i(x), E_j(y)] = 0 , \quad [E^A_i(x), A^B_j(y)] = -i \delta^{AB} \delta_{ij} \delta(x - y)$$

(1.12)

respectively. On the lattice, energies are measured in units of the lattice spacing. It is only the very low-lying states of the dimensionless lattice Hamiltonian, with energy of order $a \cdot \epsilon$ which have finite energy, $\epsilon$, in the continuum limit, $a \to 0$.

### 1.2 Partition function

The partition function at temperature $T$ is the trace of the Gibbs distribution $e^{-H/T}$ over gauge invariant physical states. It is convenient to take this trace in the “coordinate” representation. The complete set of states is obtained by taking the direct product of the “position” eigenstates of the unitary matrices

$$|U\rangle = \prod_l |U_l\rangle$$

(1.13)

for each link $l$, together with some basis elements, $e_{a_1} \ldots e_{a_K}$, which carry the representations $R_1 \ldots R_K$ of the gauge group. The states $|U_l\rangle$ are normalized by

$$\langle U_l | U'_l \rangle = \delta(U_l, U'_l)$$

(1.14)
where $\delta(U_l, U'_l)$ is invariant $\delta$--function on the group manifold. The states (1.13) do not satisfy Gauss law constraint (1.7). The projection on the physical subspace can be realized by gauge transforming the states at one side of the trace and subsequently integrating over all gauge transforms. The result is

$$Z[x_i, R_i, T] = \int \prod_l [dU_l] \prod_x [dg(x)] \langle U | e^{-H/T} | U^g \rangle \ Tr g^{R_1}(x_1) \ldots Tr g^{R_K}(x_K)$$  \hspace{1cm} (1.15)$$

where $[dU_l]$ and $[dg(x)]$ are invariant Haar measures. We consider the case where all external quarks are in the adjoint representation. In the adjoint representation, the trace can be taken as the modulus squared trace of the fundamental representation group element,

$$Tr g^{\text{adj}}(x) = |Tr g(x)|^2 - 1$$  \hspace{1cm} (1.16)$$

We then multiply by the K'th power of the fugacity, $\lambda^K$, multiply by $1/K!$, sum over positions, $x_1, \ldots, x_K$ and sum over $K$. This produces the effective theory

$$Z[\lambda, T] = \int \prod_l [dU_l] \prod_x [dg(x)] e^{-S_{\text{eff}}[U, g]}$$  \hspace{1cm} (1.17)$$

where the effective action is

$$e^{-S_{\text{eff}}[U, g]} \equiv \langle U | e^{-H/T} | U^g \rangle e^{\sum_x \lambda^2 (|Tr g(x)|^2 - 1)}$$  \hspace{1cm} (1.18)$$

This effective action has gauge symmetry,

$$S_{\text{eff}}[hU h^\dagger, hgh^\dagger] = S_{\text{eff}}[U, g]$$  \hspace{1cm} (1.19)$$

Where $h$ is an element of the gauge group. It also has a global symmetry

$$S_{\text{eff}}[U, zg] = S_{\text{eff}}[U, g]$$  \hspace{1cm} (1.20)$$

where $z$ is a constant element of the center of the gauge group. This latter symmetry is related to confinement $[8] - [12]$. For the gauge group SU(N), the center is $Z_N$ and it is referred to as $Z_N$-symmetry. For U(N), the center is U(1).

When this symmetry is realized faithfully, the theory is confining. When it is spontaneously broken, the system is in the de-confined phase. An order parameter for this symmetry is the Polyakov loop operator$^4$ which is the trace of the group element,

$$P(x) \equiv Tr g(x)$$  \hspace{1cm} (1.21)$$

$^4$In the spacetime path integral formulation of finite temperature gauge theory $[13]$ $A_0(\tau, x)$ is a Lagrange multiplier field which enforces the gauge constraint. The Euclidean action is

$$S = \int_0^{1/T} d\tau \left( \sum_l E_l U_l \frac{d}{d\tau} U_l - H[U, E] + i \sum_x A_0^A(\tau, x) G^A(\tau, x) \right),$$

and the partition function is

$$Z = \int dA_0 dE [dU] e^{-S[A_0, E, U]}$$
where \( g(x) \) is in the fundamental representation of the gauge group. It is gauge invariant and, under (1.20), it transforms as

\[
P(x) \rightarrow z \, P(x)
\]

(1.22)

The quantity

\[
F(x, R, \lambda, T) = -T \ln \langle P(x) \rangle
\]

(1.23)

is the free energy which is necessary to insert a classical source in the fundamental representation into the system. In the confining phase, the expectation value vanishes and this free energy is infinite. In a de-confined phase the expectation value is non-zero and the free energy is finite.

The two-point correlators of Polyakov loops are related to the free energy of the quark–antiquark pair inserted in the vacuum. Assuming that the cluster property holds, the two–point correlator behaves at \( |x - y| \rightarrow \infty \) as

\[
\langle P(x) P^\dagger(y) \rangle \rightarrow \frac{\text{const}}{|x - y|^{(D-1)/2}} e^{-M_1|x-y|} + \ldots,
\]

(1.24)

where \( M_1 \) is the mass of the lowest excitation with appropriate quantum numbers. The two–point correlator goes to zero if the Polyakov loop expectation value vanishes and goes to a constant otherwise. In the deconfining phase, the quark–antiquark potential can be defined by subtracting the self–energies (1.23) from the logarithm of the correlator (1.24):

\[
V(x, y) = -T \ln \left( \langle P(x) P^\dagger(y) \rangle \right) + T \ln \langle P(x) \rangle + T \ln \langle P^\dagger(y) \rangle
\]

\[
\sim -\frac{\text{const}}{|x - y|^{(D-1)/2}} e^{-M_1|x-y|}.
\]

(1.25)

In the confining phase the self–energies are infinite, but the potential still can be defined. It grows linearly with distance:

\[
V(x, y) = -T \ln \left( \langle P(x) P^\dagger(y) \rangle \right) \sim \sigma |x - y|,
\]

(1.26)

where the string tension is

\[
\sigma = T M_1.
\]

(1.27)

For the models that we shall consider, the mass \( M_1 \) will be calculated exactly in Sec. 2 and 3.

with periodic boundary conditions in Euclidean time. In the strong coupling limit, this model can be reduced to (1.15) by integrating the time dependent modes explicitly. The dynamical variable \( g(x) \) is the holonomy group element for transport of a fundamental representation charge around the periodic Euclidean time,

\[
g(x) = \mathcal{P} \exp \left( i \int_0^{1/T} d\tau A_0(\tau, x) \right).
\]

The Polyakov loop operator is the trace of this holonomy element \( P(x) = \text{Tr} g(x) \).
1.3 Adjoint non-Abelian Coulomb gas

We shall treat the effective action (1.18) exactly in $D = 1$ where there is no magnetic term in the Hamiltonian. In $D > 1$, we consider the strong coupling limit where

$$e^2 \to \infty \ , \ T \to \infty \ , \ \gamma = e^2 N/2T \ \text{finite} \quad (1.28)$$

(We shall eventually also consider the limit where $N \to \infty$ with $\gamma$ finite.) Here $e^2$ and $T$ (as well as the lattice Hamiltonian $H$) are dimensionless quantities which will eventually get their engineering dimensions from factors of the lattice spacing. In this limit, the magnetic term in the Hamiltonian is ignored, so that the Hamiltonian is given by the electric term only and the Boltzmann weight (1.18) factorizes on the product of the heat kernels

$$K (g, h|\tau) = \langle h | e^{-\tau \Delta/N} | g \rangle \quad (1.29)$$

for the Laplacian

$$\Delta = \sum_A (E^A)^2 \quad (1.30)$$

on the group manifold with “time” $\gamma = e^2 N/2T$. In this limit, the magnetic interactions of the external quarks as well as the magnetic self-interactions of gluons are absent. It isolates the electric interactions and self-interactions.

In this strong coupling limit, the partition function has the form

$$Z = \int \prod_x [dg(x)] \prod_l [dU_l] \ \prod_l K \left(U_l, g U_l g^\dagger \gamma\right) e^{\sum_x \lambda (|Tr g(x)|^2 - 1)} \quad (1.31)$$

It is this model which we call the adjoint non-Abelian Coulomb gas.

1.3.1 Lattice string statistical mechanics

In strong coupling limit, the effective Hamiltonian is

$$H_0 = \sum_{l,A} e^2 \frac{1}{2} (E^A_l)^2 \quad (1.32)$$

The local moments of the electric field distribution are conserved,

$$\left[ E^A(x), H_0 \right] = 0 \quad (1.33)$$

and distribution of electric fields is frozen in time. The eigenstates of the Hamiltonian (1.32) can be constructed by acting with link variables $U_l$ on the vacuum. On a given link, we begin with the singlet state,

$$E^A_l |0\rangle = 0 \quad (1.34)$$

and the action of $E^A_l$ on excited states is defined combinatorially,

$$E^A_l (U_l)_{ab} |0\rangle = (T^A U_l)_{ab} |0\rangle \quad (1.35)$$
Figure 1: Graphical representation of the states $|\Psi_\pm\rangle$

$E_i^A (U_i)_{ab} (U_i)_{cd} |0\rangle = (T^A U_i)_{ab} (U_i)_{cd} |0\rangle + (U_i)_{ab} (T^A U_i)_{cd} |0\rangle$

and so on. The gauge invariant states contain closed loops of electric flux and lines of electric flux, appropriate numbers of which end at the quarks. Closed lines of flux are created by the traces of the spatial Wilson loop operators in different representations of the gauge group, for the closed curve $\Gamma$,

$W_R[\Gamma] = \text{Tr} \prod_{l \in \Gamma} U_i^R$

For a non-intersecting array of electric flux strings, the energy is proportional to the total length of all strings,

$E = \frac{e^2}{2} C_2 \sum_{\Gamma} L[\Gamma]$

Strings interact when they have common links. For example, we consider an array of strings where two intersecting strings share a common link. In this case, the eigenstates of the Hamiltonian are linear combinations of the uncrossed and crossed strings, fig. 1:

$|\Psi_\pm\rangle = (U_{ab} U_{cd} \pm U_{ad} U_{cb}) |0\rangle$

The contribution of the states $|\Psi_\pm\rangle$ to the energy of the string configuration is equal to $e^2 (N - 1)(N + 2)/N$ and $e^2 (N + 1)(N - 2)/N$, respectively. In the large $N$ limit, these states are degenerate and diagonalization of the Hamiltonian does not mix the crossed and uncrossed strings. Therefore, the interaction vanishes in the large $N$ limit. Thus, in the large $N$ limit, the statistical mechanics of the strong coupling Yang-Mills theory is equivalent to a statistical model of non-interacting lattice strings for which the partition sum can be written

$Z = \sum_{\Gamma} e^{-C_2 \gamma L[\Gamma]/2N} = \sum_{L} n(L) e^{-C_2 \gamma L/2N}$.

Such a system is expected to have a percolation transition when the entropy of configurations of strings overtakes the energy [14]. For large curves, the number of strings with a given length $L$ grows like

$n(L) \sim \text{const.} (2D - 1)^L$

The critical temperature is given by

$\gamma_{\text{crit.}} = 2 \ln(2D - 1)$

This is the result which was found in [1, 2] by the direct solution of the model.

When there are adjoint quarks present, quantum states still contain closed loops of electric flux. In addition, it is necessary that each quark absorbs and emits one line of flux. Increasing the density of quarks decreases the phase transition temperature somewhat.
1.3.2 Continuum limit

In the continuum limit, the trace which is taken to obtain the partition function can be taken over a complete set of eigenstates of the gauge field operator, twisted by the gauge transformation \[ |A\rangle = g(A + id)g^\dagger \]. Formally, we can consider the strong coupling limit and retain only the electric term in the Hamiltonian:

\[ \langle A | e^{-H/T} | A^g \rangle = \text{const. exp} \left( -\frac{T}{e^2} \int d^D x \ Tr (A - A^g)^2 \right) \]

(1.43)

where \( Dg = \nabla g + i [A, g] \). The partition function is

\[ Z = \int [dA] [dg] \ e^{-\int d^D x \left[ \frac{N}{2} Tr[Dg(x)]^2 - \lambda (|Trg(x)|^2 - 1) \right]} \]

(1.44)

The effective action is that of a gauged principal chiral model in \( D \) dimensions. The continuum treatment of the strong coupling limit, however, is valid only at \( D = 1 \), where the magnetic term in the Hamiltonian is absent from the beginning; this model is considered in the next section. At \( D > 1 \) the field theory defined by (1.44) is nonrenormalizable unless the kinetic term for the gauge fields is added.

The mean density of quarks is obtained from the partition function as

\[ \langle n \rangle = \frac{1}{V} \lambda \frac{\partial}{\partial \lambda} \ln Z \left[ \lambda, T \right] = \lambda \langle Trg^\dagger Trg - 1 \rangle \rightarrow \lambda \langle Trg \rangle^2 + O(N^0) \quad (\text{as } N \rightarrow \infty) \]

(1.45)

where we have assumed that the expectation value is translation invariant. The last equality follows from factorization of invariant correlators in the large \( N \) limit. From eq. (1.43) we see that, in the large \( N \) limit, the Coulomb gas in the confining phase where \( |\langle Tr g(x)\rangle| = |\langle P(x)\rangle| = 0 \) is dilute with density is of order one. This is a result of the fact that the number of species of glue-balls and hadrons does not grow in the large \( N \) limit. Above the deconfining temperature, where \( |\langle P(x)\rangle| \neq 0 \) there are \( N^2 - 1 \) gluon and \( N^2 \) quark degrees of freedom and the density is of order \( N^2 \). A similar consideration applies to the free energy. The free energy in the confining phase should grow like the number of degrees of freedom and be \( O(N^0) \) in the large \( N \) limit. In the de-confined phase the free energy should be of order \( N^2 \). For \( N \) infinite, this implies that a phase transition between the two phases would have infinite latent heat. However, one should consider the large \( N \) limit as analogous to the infinite volume limit in statistical mechanics where, strictly speaking, there is no phase transition in finite volume, but the analysis in infinite volume is to a good approximation accurate in the physical finite system too.

This will be particularly relevant to the case of \( D = 1 \), where, if \( N \) is finite, the dimensionality of the system is too low to have spontaneous symmetry breaking. It has
local interactions and is effectively a one-dimensional system. This implies that, when \( N \) is finite, it is always in the confining phase with unbroken center symmetry. In the infinite \( N \) limit, there can be a phase transition and a phase with broken symmetry. However, the latter limit should describe the physical behavior of the system accurately even when \( N \) is finite if the “finite size” corrections of order \( 1/N^2 \) are small.

Even this case is subtle, since the effective symmetry in the infinite \( N \) limit is \( U(1) \), a continuous symmetry and the effective dimension is two. The small \( \gamma \) phase can be ordered only because the action is non-local in index space, similar to an infinite-ranged spin model.

In Section 2 we shall analyze the large \( N \) limit of the one dimensional model directly in the continuum. We find an exact solution of the continuum model and show explicitly that it has a first order confinement-deconfinement phase transition which occurs at a critical line in the \( \gamma - \lambda \) plane. In Section 3 we discuss the strong coupling lattice model in any dimensions. We find an exact solution in the confined phase and an approximate solution of the deconfined phase. As on one dimension, there is a first order confinement-deconfinement transition. In Section 4, we discuss the results.

## 2 Large \( N \) limit of one–dimensional model

It is possible to analyze the large \( N \) limit of the \( D=1 \) model directly in the continuum limit. We consider the gauge group \( U(N) \) which has center \( U(1) \), or \( SU(N) \) with center \( \mathbb{Z}_N \) – there is no difference in the large \( N \) limit. In the gauged principal chiral model (1.44), since the space is an open line, we can choose the gauge \( A = 0 \). The partition function for the resulting model,

\[
Z = \int [Dg] e^{-\int dx \left[ \frac{N}{2} \text{Tr}(g'g') - \lambda \text{Tr}g \text{Tr}g\right]} \tag{2.1}
\]

is equivalent to that of unitary matrix quantum mechanics (where \( x \) is imaginary time) and can be solved in the large \( N \) limit by the methods of collective field theory \([16, 17]\). The method is essentially based on the relation between matrix quantum mechanics and nonrelativistic fermions \([18]\).

The dynamical variables in (2.1) are the phases \( \alpha_k(x) \) of the eigenvalues \( e^{i\alpha_k(x)} \) of the Polyakov loop variables \( g(x) \). They can be interpreted as coordinates of fermions which live on a circle. In the large \( N \) limit, the statistical integral in (2.1) is dominated by a single distribution of eigenvalues. We introduce the eigenvalue density

\[
\rho(\theta, x) = \frac{1}{N} \sum_{k=1}^{N} \delta(\theta - \alpha_k(x)) \tag{2.2}
\]

which is a periodic function which is normalized to unity on the interval \((-\pi, \pi)\). It has
the mode expansion
\[ \rho(\theta, x) = \frac{1}{2\pi} \left( 1 + \sum_{n \neq 0} c_n(x) e^{i\theta n} \right), \quad c_n^* = c_{-n} \] (2.3)

In a truly confining phase, all moments of the eigenvalue distribution vanish, \( \rho_{\text{conf}} = 1/2\pi. \) If any of the other moments are non-zero, there are Polyakov loop operators which have non-zero expectation values, if \( c_n(x) \neq 0 \) then \( \langle \text{Tr} \ g^n(x) \rangle \neq 0. \) Thus, all combinations of \( c_n(x) \neq 0 \) characterize all de-confined phases.

In the large \( N \) limit the eigenvalue density obeys a classical, saddle-point equation which can be deduced from canonical analysis of the collective field theory Hamiltonian [16], [19]:
\[ H = \int d\theta \left[ \frac{\gamma}{2} \rho(\theta) \frac{\partial \Pi}{\partial \theta}^2 + \frac{\pi^2 \gamma}{6} \rho^3(\theta) \right] - \lambda \left| \int d\theta \rho(\theta) e^{i\theta} \right|^2 - \frac{\gamma}{24} \] (2.4)
with subsequent Wick rotation to an imaginary time. Here \( \Pi(\theta, x) \) is the variable which is the canonical conjugate to \( \rho(\theta, x) \), so that the Poisson bracket is
\[ \{ \rho(\theta, x), \Pi(\theta', x) \} = \delta(\theta - \theta') \] (2.5)
(here \( x \) is the time variable) and \( v(\theta) = \partial \Pi / \partial \theta \) is the velocity of the Fermi fluid. The equations of motion, following from (2.4), read, after the change \( x \to ix, \, v \to -iv \), as follows:
\[ \frac{\partial \rho}{\partial x} + \gamma \frac{\partial}{\partial \theta} (\rho v) = 0, \] (2.6)
\[ \frac{\partial v}{\partial x} + \gamma v \frac{\partial v}{\partial \theta} - \frac{\pi^2 \gamma}{6} \rho \frac{\partial \rho}{\partial \theta} + 2\lambda \text{Im} \left( e^{-i\theta} c_1(x) \right) = 0, \] (2.7)
where
\[ c_1(x) = \int_{-\pi}^{\pi} d\theta \rho(\theta, x) e^{i\theta}. \] (2.8)
It is expected that the solution of these equations corresponding to the physical vacuum is a constant \( \rho_0(\theta) \).

One may expect that, at least at sufficiently low temperature or, equivalently, at sufficiently large \( \gamma \), the system is in the confining phase with unbroken center group symmetry. This symmetry acts on \( \rho(\theta, x) \) by translation \( \theta \to \theta + \theta_0 \), so the only symmetric solution is \( \rho_{\text{conf}}(\theta) = \frac{1}{2\pi} \). It always satisfies the equations of motion, as \( c_1(x), \) defined by eq. (2.8), is equal to zero when \( \rho = \rho_{\text{conf}}. \)

However, this solution, which is an exact solution of (2.6) and (2.7), is stable to small fluctuations only if \( \gamma \) is large enough. It becomes unstable for \( \gamma < \gamma_c(\lambda) \). To see this, it is necessary to analyze the spectrum of fluctuations in the strong coupling phase. Consider \( \rho(\theta, x) = \frac{1}{2\pi} (1 + \varphi(\theta, x)) \) and consider the equations (2.6) and (2.7) to linear order in \( \varphi \):
\[ \frac{\partial \varphi}{\partial x} + \gamma \frac{\partial v}{\partial \theta} = 0, \] (2.9)
\[
\frac{\partial v}{\partial x} - \frac{\gamma}{4} \frac{\partial \varphi}{\partial \theta} + 2\lambda \text{Im}(c_1 e^{-i\theta}) = 0. \tag{2.10}
\]

Differentiating eq. (2.9) with respect to \(x\) and eq. (2.10) with respect to \(\theta\) and subtracting the latter, multiplied by \(\gamma\), from the former, we obtain an equation for the density fluctuations:
\[
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\gamma^2}{4} \frac{\partial^2 \varphi}{\partial \theta^2} + 2\lambda \text{Re}(c_1 e^{-i\theta}) = 0. \tag{2.11}
\]

The eigenmodes are the Fourier harmonics of \(\varphi(\theta, x)\):
\[
\varphi(\theta, x) = \sum_{n \neq 0} c_n(x) e^{in\theta} \tag{2.12}
\]

where now \(c_n\) are infinitesimal:
\[
c''_n - \left[ \frac{\gamma^2 n^2}{4} - \lambda \gamma (\delta_{n,1} + \delta_{n,-1}) \right] c_n = 0, \quad n \neq 0. \tag{2.13}
\]

We obtain the following spectrum of excitations:
\[
M_n^2 = \gamma^2 n^2 \frac{\gamma}{4} - \lambda \gamma \delta_{n,1}, \quad n = 1, 2, \ldots \tag{2.14}
\]

At \(\gamma = \gamma_c(\lambda)\), where
\[
\gamma_c(\lambda) = 4\lambda, \tag{2.15}
\]

the lowest eigenvalue \((n = 1)\) goes to zero. For smaller \(\gamma\) this eigenvalue is negative and leads to the instability of the strong coupling solution where \(c_1\) is the first mode to become unstable. However, for reasons which will become clear once we consider the weak coupling phase, \(\gamma_c(\lambda)\) should not be identified with the point of the deconfining phase transition.

The contribution of fluctuations to the free energy is given by
\[
\delta F/V = \pi T \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left( \sum_n \ln \left( k^2 + \frac{\gamma^2 n^2}{4} \right) + \ln \left( \frac{k^2 + 4\gamma^2/4 - \lambda \gamma}{k^2 + 4\gamma^2/4} \right) \right) \tag{2.16}
\]

the singular part of which, near the critical line is \(\delta F/V \sim \frac{\pi^2 T}{2} \left( \sqrt{1 - 4\lambda/\gamma} - 1 \right)\).

The deconfining solution can be obtained by integration of eq. (2.7) at \(v = 0\). The density \(\rho_0(\theta)\) can always be chosen to be an even function of \(\theta\). Thus \(c_1\) is real, and one finds from eq. (2.7):
\[
\rho_0(\theta) = \frac{1}{\pi} \sqrt{\frac{2}{\gamma} \left( E + 2\lambda c_1 \cos \theta \right)}. \tag{2.17}
\]

The Fermi energy \(E\) and the constant \(c_1\) are to be determined from the normalization condition and eq. (2.8):
\[
\int_{-\theta_{\text{max}}}^{\theta_{\text{max}}} \frac{d\theta}{\pi} \sqrt{\frac{2}{\gamma} \left( E + 2\lambda c_1 \cos \theta \right)} = 1, \tag{2.18}
\]
Figure 2: The large $N$ phase diagram of the one-dimensional model. I – strong coupling (confining) phase, II – weak coupling (deconfining) phase; 1 – line on which the weak coupling phase terminates: $\gamma = \gamma_*(\lambda)$, 2 – line of the first-order phase transition: $\gamma = \gamma_0(\lambda)$, 3 – line of the instability of the strong coupling phase: $\gamma = \gamma_c(\lambda)$

\[
\int_{-\theta_{\text{max}}}^{\theta_{\text{max}}} \frac{d\theta}{\pi} \cos \theta \sqrt{\frac{2}{\gamma} (E + 2\lambda c_1 \cos \theta)} = c_1, \tag{2.19}
\]

\[
\theta_{\text{max}} = \pi - \arccos \frac{E}{2\lambda c_1}. \tag{2.20}
\]

It follows from these equations that $\theta_{\text{max}}$ tends to zero at $\gamma \to 0$ and grows with the increase of $\gamma$. Eventually it reaches $\pi$, where the weak coupling phase terminates, because the eigenvalue distribution begins to overlap with itself due to $2\pi$-periodicity. At the critical point $E_* = 2\lambda c_1*$, the integrals in (2.18) and (2.19) can be done explicitly, and we find that $c_1* = \frac{1}{3}$ and

\[
\gamma_*(\lambda) = \frac{128}{3\pi^2} \lambda = 4.323 \lambda. \tag{2.21}
\]

At this critical line, the weak coupling phase is unstable. It is there that the translation zero mode of the linearization of the equations becomes normalizable and the resulting fluctuations restore the translation invariance in $\theta$, i.e. the center symmetry. In the phase with density given by wcs this zero mode is not normalizable and therefore it is ineffective in restoring the symmetry.

We obtain the following picture of the deconfining phase transition (fig. 2). The weak and strong coupling phases can coexist, because $\gamma_c(\lambda) < \gamma_*(\lambda)$, although the region, where both phases are stable is very narrow, since $\gamma_c(\lambda)$ and $\gamma_*(\lambda)$ are numerically closed to each other. The phase transition is of the first order and takes place at some $\gamma_0(\lambda)$ between $\gamma_c(\lambda)$ and $\gamma_*(\lambda)$ (fig. 2). The line of phase transitions is defined as that line where the free energies of the both phases are equal to each other. Substituting $\rho_0(\theta)$ into eq.(2.4) one can find the free energy per unit volume, to leading order in the large $N$ limit:

\[
\frac{F}{VN^2} = \begin{cases} 
0, & \text{in confining phase,} \\
\frac{1}{3}E - \frac{1}{3} \lambda c_1^2 - \frac{\gamma}{\pi}, & \text{in deconfining phase.}
\end{cases} \tag{2.22}
\]

The equations determining the critical point can be solved numerically, the result is:

\[
\gamma_0(\lambda) = 4.219 \lambda. \tag{2.23}
\]
3 Strong coupling lattice model in any dimension

The model (1.31) is an unitary matrix analog of the Kazakov–Migdal model [20], which can be treated at large $N$ by the saddle point methods [21]–[24]. Similar methods have been applied to the model (1.31) with $\lambda = 0$ [1]–[3]. Here we generalize the consideration of [1] to the case of $\lambda \geq 0$.

To begin, we choose the gauge in which the Polyakov loops are diagonal: $g(x) = \text{diag}(e^{i\alpha_k(x)})$. Then, integrating over the gauge fields $U_i$ we obtain an effective action for the eigenvalues $\alpha_k(x)$:

$$S_{\text{eff}} = -\sum_x \left[ \lambda \sum_{kj} e^{i\alpha_k(x) - i\alpha_j(x)} + \sum_{k<j} \ln \sin^2 \frac{\alpha_k(x) - \alpha_j(x)}{2} \right] + \frac{1}{2} \sum_{\mu=-D}^D \ln I(\alpha(x), \alpha(x + \mu) | \gamma) \right],$$

(3.1)

where the second term comes from the Faddeev–Popov determinant and $I(\alpha, \alpha' | \tau)$ is a one–link integral:

$$I(\alpha, \alpha' | \tau) = \int DU K(e^{i\alpha}, U e^{i\alpha'} U^\dagger | \tau),$$

(3.2)

and we have used the invariance properties of the heat kernel. The link integral is calculable for any $N$ [25] and can be represented in the following form:

$$I(\alpha, \alpha' | \tau) = \text{const} \cdot \frac{\det_{kj} \vartheta \left( \frac{\alpha_k - \alpha_j}{2\pi} \right)}{J(\alpha) J(\alpha')}, \quad J(\alpha) = \prod_{i<j} \sin \frac{\alpha_i - \alpha_j}{2},$$

(3.3)

where $\vartheta(z | \tau)$ is the Riemann theta function.

In the large $N$ limit, since the effective action (3.1) is of order $N^2$ and there are $N$ degrees of freedom, the statistical sum is dominated by the configuration which minimizes the action, i.e. the solution of the classical equation of motion. In terms of the eigenvalue density (2.2) the equation of motion reads:

$$-2\lambda \text{Im} \left( c_1(x) e^{-i\theta} \right) - \varphi \int_{-\pi}^\pi d\theta' \rho(\theta', x) \cot \frac{\theta - \theta'}{2}$$

$$\quad = \frac{1}{N^2} \sum_{\mu=-D}^D \frac{\partial}{\partial \theta} \delta \rho(\theta, x) \ln I(\alpha(x), \alpha(x + \mu) | \gamma).$$

(3.4)

where

$$c_1(x) = \int_{-\pi}^\pi d\theta \rho(\theta, x) e^{i\theta}.$$  

(3.5)

The large $N$ limit of the one–link integral (3.2) can be considered analogously to its Hermitean–matrix counterpart along the lines of [26, 24]. The method again is based on the correspondence between matrix quantum mechanics – cf. eqs. (1.29) and (3.2) (the integration in (3.2) acts as a projection on the singlet states) – and the quantum
mechanics of the free nonrelativistic fermions – note that the numerator in eq. (3.3) has a form of the Slater determinant. In the large \( N \) limit the fermions behave semiclassically and the problem reduces to the equations of motion of the collective field theory:

\[
\frac{\partial \sigma}{\partial \tau} + \frac{\partial}{\partial \theta}(\sigma s) = 0, \tag{3.6}
\]

\[
\frac{\partial s}{\partial \tau} + s \frac{\partial s}{\partial \theta} - \pi^2 \sigma \frac{\partial \sigma}{\partial \theta} = 0, \tag{3.7}
\]

which should be solved on each link of the lattice with the following boundary conditions:

\[
\sigma(\theta, 0; x, \mu) = \rho(\theta, x), \tag{3.8}
\]

\[
\sigma(\theta, \gamma; x, \mu) = \rho(\theta, x + \mu). \tag{3.9}
\]

The right hand side of eq. (3.4) can be expressed through the solution of (3.6) – (3.9) as follows:

\[
-2\lambda \Im \left( c_1(x) e^{-i\theta} \right) + (D - 1) \varphi \int_{-\pi}^{\pi} d\theta' \rho(\theta', x) \cot \frac{\theta - \theta'}{2} = \sum_{\mu=-D}^{D} \left( s(\theta, 0; x, \mu) - s(\theta, \gamma; x - \mu, \mu) \right). \tag{3.10}
\]

The equations (3.5) – (3.10) completely determine the large \( N \) dynamics of the model under consideration. The vacuum should be identified with the \( x \)-independent solution. At \( D = 1 \) these equations have a continuum limit. One should introduce the lattice spacing \( a \), recover the canonical dimensions of the couplings, i.e. rescale \( \gamma \to a\gamma \) and \( \lambda \to a\lambda \), and take the limit \( a \to 0 \). As a result of this procedure, one obtains eqs. (2.6) – (2.8) of sec. 2 [1].

The vacuum solution in the large \( \gamma \) confining phase is dictated by the center group symmetry – \( \rho_0(\theta) = \frac{1}{2\pi} \). To find the spectrum of excitations we linearize the equations of motion around \( \rho_0(\theta) \):

\[
\rho(\theta, x) = \frac{1}{2\pi} + \frac{1}{2\pi} \sum_{n \neq 0} c_n(x) e^{in\theta}, \quad c_n^* = c_{-n}, \tag{3.11}
\]

\[
\sigma(\theta, \tau; x, \mu) = \frac{1}{2\pi} + \frac{1}{2\pi} \sum_{n \neq 0} \alpha_n(\tau; x, \mu) e^{in\theta}, \quad \alpha_n^* = \alpha_{-n}, \tag{3.12}
\]

\[
s(\theta, \tau; x, \mu) = \sum_{n \neq 0} \beta_n(\tau; x, \mu) e^{in\theta}, \quad \beta_n^* = \beta_{-n}. \tag{3.13}
\]

The solution of eqs. (3.6), (3.7) linearized in \( \alpha_n \) and \( \beta_n \) reads:

\[
\alpha_n(\tau; x, \mu) = \alpha_n^+(x, \mu) e^{\frac{in\tau}{2}} + \alpha_n^-(x, \mu) e^{-\frac{in\tau}{2}}, \tag{3.14}
\]

\[
\beta_n(\tau; x, \mu) = \frac{i}{2} \left[ \alpha_n^+(x, \mu) e^{\frac{in\tau}{2}} - \alpha_n^-(x, \mu) e^{-\frac{in\tau}{2}} \right]. \tag{3.15}
\]
Substituting these equations in the boundary conditions (3.8), (3.9) we express α_n and β_n in terms of c_n:

$$\alpha_n^\pm(x, \mu) = \mp e^{\mp \frac{n\gamma}{2}c_n(x) - c_n(x + \mu)} \frac{1}{2 \sinh \frac{n\gamma}{2}}. \quad (3.16)$$

The equality (3.10) then gives an equation for the Fourier coefficients of the eigenvalue density:

$$\sum_{\mu=1}^{D} \{c_n(x + \mu) - 2 \left[ \cosh \frac{n\gamma}{2} - \left( \frac{D-1}{D} + \delta_n \lambda \right) \sinh \frac{n\gamma}{2} \right] c_n(x) + c_n(x - \mu) \} = 0, \quad (3.17)$$

which leads to the following mass spectrum:

$$M_n^2 = 2D \left( \cosh \frac{n\gamma}{2} - 1 \right) - 2(D-1 + \delta_n D \lambda) \sinh \frac{n\gamma}{2}. \quad (3.18)$$

The strong coupling solution becomes unstable when $M_1^2$ turns to zero. Form eq. (3.18) one obtains:

$$\gamma_c(\lambda) = 2 \ln \frac{D + \sqrt{(D-1)^2 + \lambda D \left[2(D-1) + \lambda D\right]}}{1 - \lambda D}. \quad (3.19)$$

We have not been able to obtain an exact solution in the weak coupling phase, but an approximate one, valid in the limit $\gamma \to 0$, can be found. At small $\gamma$ one can expand $g(x) = e^{i\sqrt{\nu}(x)} \approx 1 + i\sqrt{\nu}\Phi(x)$, and the partition function (1.31) reduces to that of the Kazakov–Migdal model with the quadratic potential. From the point of view of the large N equations of motion this approximation corresponds to the expansion of the left hand side of eq. (3.10) in the powers of $\theta$. Actually, since at small $\gamma$ the eigenvalues are peaked about zero with the width of the distribution of order $\gamma$, we can expand the second term in eq. (3.10), retaining only the contributions of order $\gamma^{-1}$ and $\gamma^0$:

$$\varphi \int d\theta' \rho(\theta', x) \cot \frac{\theta - \theta'}{2} = 2 \varphi \int d\theta' \rho(\theta', x) \frac{\theta - \theta'}{\theta - \theta'} - \frac{1}{6} \theta - O(\gamma^2). \quad (3.20)$$

The first term in (3.10) is equal, with the same accuracy, to $2\lambda \theta$, as $c_1 = 1 + O(\gamma^2)$. In this approximation, we obtain the equations of motion for the Kazakov–Migdal model with the quadratic potential, as expected, and effective mass is equal to

$$m_{\text{eff}}^2 = 2D + \left[ 2\lambda - \frac{1}{6} (D-1) \right] \gamma. \quad (3.21)$$

The vacuum solution of the latter model is known [22]:

$$\rho_0(\theta) = \frac{1}{\pi} \sqrt{\frac{1}{4} - \mu^2 \theta^2}, \quad (3.22)$$

$$\mu = \frac{m_{\text{eff}}^2 (D-1) + D \sqrt{m_{\text{eff}}^4 - 4(2D-1)}}{(2D-1)\gamma}. \quad (3.23)$$
There are two reasons for which the weak coupling solution can terminate. First, it may become unstable due to the appearance of the massless excitation in the spectrum of fluctuations around it. For the solution \((3.22), (3.23)\) of the Kazakov–Migdal model it happens at \(m_{\text{eff}}^2 = 2\sqrt{2D - 1}\) \([23]\); for smaller values of \(m_{\text{eff}}^2\) this solution does not exist since \(\mu\) in \((3.23)\) becomes complex. However, the analyses of the \(\lambda = 0\) case shows \([3]\) that the weak coupling phase terminates before it reaches the point of instability, because the eigenvalue distribution hits \(\pi\) and begins to overlap with itself for smaller \(\gamma\). More precise criterion for a critical point is the condition that the width of the support of \(\sigma(\theta, \tau)\) exceeds \(2\pi\) at some intermediate \(\tau\) between 0 and \(\gamma\) \([3]\). For the solution \((3.22), (3.23)\), the last criterion is satisfied when \([3]\):

\[
\gamma_*(\lambda)^2 = \pi^4 - \frac{4\pi^2}{\mu(\gamma_*(\lambda), \lambda)}.
\]

(3.24)

The solution \((3.22), (3.23)\) being an approximate one, eq. \((3.24)\) gives only an estimate for the critical coupling. In fact, it should give an upper bound. The arguments for this are the following: All higher terms in the expansion of the cotangent in \((3.20)\) are negative, or, equivalently, the effective potential generated due to nonlinearity of the field \(g(x)\), is upside–down. Thus the neglected corrections can only strengthen the instability. These arguments are rigorous for \(\lambda = 0\). When \(\lambda > 0\), the first term in \((3.10)\) which is proportional to \(\lambda\) has an alternating Taylor expansion in powers of \(\theta\) when \(c_1\) is real. However, the first correction is also negative and we expect that \((3.24)\) gives an upper bound for the actual value of \(\gamma_*(\lambda)\) for all \(\lambda\).

The consideration above gives the following picture of the phase transition (fig. 3). In all cases the confining and deconfining phases can coexist and are separated by a first–order phase transition. The model also undergoes a third–order large–\(N\) phase transition in the deconfining phase. Actually, the line on which the weak coupling solution terminates, \(\gamma = \gamma_*(\lambda)\), obtained from the approximate equation \((3.24)\), crosses the line of the instability of the strong coupling phase \(\gamma = \gamma_c(\lambda)\) (fig. 3). As \((3.24)\) should give an upper bound for \(\gamma_*(\lambda)\), this crossing is not an artifact of the approximation done. So the weak coupling solution terminates before the transition to the strong coupling region and there exist two deconfining phases. The correlation length does not turn to zero at the critical line separating these phases. The phase transition is connected with the large–\(N\) critical behaviour of the link integral \((3.2)\). Such third–order phase transitions are typical for unitary matrix models \([5, 16, 17]\).

The phase diagram of the \(D = 1\) model is depicted on fig. 3a. One can verify that in the continuum limit \(\gamma \to 0, \lambda \to 0\) eq. \((3.13)\) reduces to eq. \((2.13)\) and eq. \((3.24)\) really gives an upper bound \((\gamma_*(\lambda) \approx \frac{\pi}{r^2} \lambda = 48.70 \lambda)\) for \((2.21)\).
4 Results and discussion

We have considered the thermodynamics of the gas of quarks in adjoint representation of the gauge group interacting via non-Abelian electric forces. In 1+1 dimension the model that we have considered is adjoint QCD in the limit where quarks are heavy. The fugacity parameter is proportional to the Boltzmann weight of the classical particle of mass $m - \lambda \propto e^{-m/T}$. The pre-exponential factor is not determined by classical theory, but it can be computed from a loop diagram:

$$\lambda = \sqrt{\frac{mT}{2\pi}} e^{-m/T}. \quad (4.25)$$

For the classical thermodynamics to be applicable, particle mass should be much larger than temperature – $m \gg T$. We also assume that $m^2 \gg e^2N$ and pair production is suppressed. The results of Sec. 3 indicate that there exists a region of parameters in which both of the conditions are satisfied and the model undergoes the deconfining phase transition. It is of the first order with critical line given approximately by the equation

$$\frac{e^2N}{2T_c} \approx 4.2 \sqrt{\frac{mT_c}{2\pi}} e^{-m/T_c}, \quad (4.26)$$

or, explicitly,

$$T_c = \frac{m}{\ln \frac{m^2}{e^2N} + O \left(\ln \ln \frac{m^2}{e^2N}\right)}. \quad (4.27)$$
The value of $T_c$ determines, by standard arguments, the asymptotics of the density of states in 1+1 dimensional QCD with heavy adjoint matter – $\rho(E) \propto e^{E/T_c}$ for large $E$. The spectrum and other properties of 1+1-dimensional QCD with adjoint matter fields have been investigated recently [27] - [31]. As emphasized by Kutasov [27], this model shares some features with string theories in that it exhibits an infinite number of asymptotically linear Regge trajectories and its density states increases exponentially at high energy. It should therefore exhibit a Hagedorn temperature, which could either be an upper limiting temperature, or the critical temperature of a phase transition. Kogan and Zhitnitsky [30] outlined features which the spectrum would have to possess in order that the behavior is a phase transition.

In the multidimensional case, we consider the limit of the lattice theory in which the coupling constant and the temperature measured in lattice units are large. In this limit the magnetic interactions can be neglected and the model also appears to be explicitly solvable. The deconfining phase transition for $D > 1$ takes place even in pure gluodynamics. For the model under consideration, this corresponds to $\lambda = 0$, the case which was considered previously [1] – [3]. The new feature of the model with adjoint matter is appearance of the additional third–order phase transition of Gross–Witten type for sufficiently large value of the fugacity parameter.

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