A Voronoï–Oppenheim Summation Formula for Number Fields

Zhi Qi

Abstract. In this note, we establish a Voronoï–Oppenheim summation formula for divisor functions over an arbitrary number field.

1. Introduction

In 1904, Voronoï [Vor] introduced his famous summation formula for the classical divisor function \( \tau(p^n q^{w_p n q}) \), whose smoothed form (see [Tem, (1.5, 1.6)] and [IK, §4.5]) reads as follows:

\[
\sum_{n=1}^{\infty} \tau(n) w(n) = \int_0^{\infty} w(x) (\log x + 2\gamma)dx \\
+ \sum_{n=1}^{\infty} \tau(n) \int_0^{\infty} w(x) \left( 4K_0(4\pi \sqrt{n x}) - 2\pi Y_0(4\pi \sqrt{n x}) \right) dx,
\]

for \( w(x) \in C^\infty_c(0, \infty) \), in which \( \gamma \) is Euler’s constant.

In 1927, Oppenheim [Opp] extended Voronoï’s summation formula for \( \tau_s(n) = \sum_{ab=n} (a/b)^s = n^{-s} \sum_{d|n} d^{2s}, \quad (s \in \mathbb{C}) \), as follows:

\[
\sum_{n=1}^{\infty} \tau_s(n) w(n) = \int_0^{\infty} w(x) \left( \zeta(1-2s)x^{-s} + \zeta(1+2s)x^s \right) dx \\
+ \sum_{n=1}^{\infty} \tau_s(n) \int_0^{\infty} w(x) \left\{ 4 \cos(\pi s) K_{2s}(4\pi \sqrt{n x}) \\
- 2\pi \left( \cos(\pi s) Y_{2s}(4\pi \sqrt{n x}) + \sin(\pi s) J_{2s}(4\pi \sqrt{n x}) \right) \right\} dx.
\]

In this note, we generalize the Voronoï–Oppenheim formula to an arbitrary number field. Actually, our formula is even more general, with additive twists included (see [IK, §4.5])—this feature is usually important for applications.\(^1\)

\(^1\)Our motivation of writing this note was the application of Voronoï over an imaginary quadratic field in [LQ], where it was proven that at least 33% of central \( L \)-values for \( \text{PGL}_2(\mathbb{C}) \)-Maass forms are non-vanishing (here \( \mathbb{C} \) is the ring of integers in an imaginary quadratic field).
proof is inspired by the adelic approach to the Voronoi summation formula for cusp forms inCogdell [Cog] and Templier [Tem]. In our setting, Eisenstein series are used instead of cusp forms. For the archimedean vectors, we use the constructions in Beineke–Bump [BB] and extend their result on the Whittaker integral to complex places by a kernel formula for GL₂(ℂ) established in [QI4]. Recently, the ideas in [BB] were used in [BBT] and [BBB] to establish a Voronoi–Oppenheim formula over totally real number fields and the Gaussian field, but our adelic approach is conceptually simpler while our formula is more general.

**Notation and Definitions.** Let \( F \) be a number field. Let \( \mathfrak{O}, \mathfrak{D}, \) and \( \mathfrak{A} \) be its ring of integers, different ideal, and adele ring. Let \( N \) denote the norm for \( F \).

For each place \( v \) of \( F \), we denote by \( F_v \) the corresponding local field. When \( v \) is non-archimedean, let \( \mathfrak{p}_v \) be the corresponding prime ideal of \( \mathfrak{O} \) and let \( \text{ord}_v \) denote the additive valuation. Let \( \| \|_v \) denote the normalized modulus of \( F_v \). We have \( \| \|_v = | \| \) if \( F_v = \mathbb{R} \) and \( \| \|_v = | \|_2 \) if \( F_v = \mathbb{C} \), where \( | \| \) is the usual absolute value.

Let \( S_x \) or \( S_f \) denote the set of archimedean or non-archimedean places of \( F \), respectively. Write \( v \mid \infty \) and \( v \mid \infty \) as the abbreviation for \( v \in S_x \) and \( v \in S_f \), respectively. For a finite set of places \( S \), denote by \( \mathfrak{A}^S \), respectively \( F_S \), the subring of adeles with trivial component above \( S \), respectively above the complement of \( S \). For brevity, write \( \mathfrak{A}_f = \mathfrak{A}^{\infty} \) and \( F_\infty = F_{S_{\infty}} \). The modulus on \( F_\infty \) will be denoted by \( \| \|_{\overline{x}} \).

Let \( e(z) = \exp(2\pi iz) \). Fix the (non-trivial) standard additive character \( \psi = \otimes_v \psi_v \) on \( \mathfrak{A}/F \) as in [Lan, §XIV.1] such that \( \psi_v(x) = e(-x) \) if \( F_v = \mathbb{R} \), \( \psi_v(z) = e(-z + \bar{z}) \) if \( F_v = \mathbb{C} \), and that \( \psi_v \) has conductor \( \mathfrak{D}_v^{-1} \) for any non-archimedean \( F_v \). We split \( \psi = \psi_\infty \psi_f \) so that \( \psi_\infty(x) = e(-\text{Tr}_{F_\infty}(x)) \) \( (x \in F_\infty) \). For a finite set of places \( S \), define \( \psi_S = \prod_{v \in S} \psi_v \) as an additive character of \( F_S \).

We choose the Haar measure \( dx \) of \( F \) self-dual with respect to \( \psi_v \) as in [Lan, §XIV.1]: the Haar measure is the ordinary Lebesgue measure on the real line if \( F_v = \mathbb{R} \), and twice the ordinary Lebesgue measure on the complex plane if \( F_v = \mathbb{C} \). The measure \( dx \) on \( F_\infty \) is defined to be the product of \( dx_v \) for \( v \mid \infty \).

In general, we use Gothic letters \( a, b, \ldots \) to denote non-zero fractional ideals of \( F \), while we reserve \( n \) and \( \pi \) for non-zero integral ideals of \( F \). Let \( N(a) \) denote the norm of \( a \).

Let \( \zeta_F(s) \) be the Dedekind \( \zeta \) function for \( F \):

\[
\zeta_F(s) = \sum_{n \in \mathfrak{O}} \frac{1}{N(n)^s}, \quad \text{Re}(s) > 1.
\]

It is well-known that \( \zeta_F(s) \) is a meromorphic function on the complex plane with a simple pole at \( s = 1 \). Let \( \gamma^{(-1)}_F \) and \( \gamma^{(0)}_F \) respectively be the residue and the constant term of \( \zeta_F(s) \) at \( s = 1 \); namely,

\[
\zeta_F(s) = \frac{\gamma^{(-1)}_F}{s-1} + \gamma^{(0)}_F + O(|s-1|), \quad s \to 1.
\]

**Definition 1.1** (Bessel kernel). Let \( s \in \mathbb{C} \).

1. When \( F_v = \mathbb{R} \), for \( x \in \mathbb{R}_+ \) we define

\[
B_s(x) = \frac{\pi}{\sin(\pi s)} (J_{-2s}(4\pi \sqrt{x}) - J_{2s}(4\pi \sqrt{x})),
\]

where \( J_n \) is the Bessel function of the first kind of order \( n \).
Eisenstein series and using the local computations in his work for twisted divisor sums can be obtained in a similar fashion by considering certain ramified functions.

By letting \( \zeta = 0 \) and \( a = (1) \) in (1.6) (it is understood that if \( \zeta = 0 \) then \( S = 0 \), \( b = (1) \), and \( \psi\langle 0 \rangle = 1 \), we recover the Vorono"i–Oppenheim formula (1.2) when \( F = \mathbb{Q} \) as well as its generalization in (BBT) when \( F \) is totally real.

Let \( \tau(n) = \tau_0(n) \) be the (usual) divisor function for \( F \). The following Vorono"i summation formula is the formula (1.6) in the special case \( s = 0 \) (see (6.1)). When \( F = \mathbb{Q} \), this is the Vorono"i summation formula in (IK) §4.5.

---

2Edgar Assing informed the author that a general Vorono"i–Oppenheim summation formula for twisted divisor sums can be obtained in a similar fashion by considering certain ramified Eisenstein series and using the local computations in his work [Ass].
Corollary 1.4. Let $\zeta$, $a$, $b$, $S$ be as in Theorem 1.3. Let $w(x)$ and $\tilde{w}_0(y)$ be as in Definition 1.2. Define

\[
\tilde{w}_0(0; b) = \gamma_F^{-1} \tilde{w}_0'(0) + (2 \gamma_F^{(0)} - \gamma_F^{(-1)}) \log N(b\mathfrak{O}^{-1})) \tilde{w}_0(0),
\]

where the constants $\gamma_F^{(-1)}$ and $\gamma_F^{(0)}$ are defined as in (1.3), and $\tilde{w}_0(0)$ and $\tilde{w}_0'(0)$ are the integrals

\[
\tilde{w}_0(0) = \int_{F^\times_E} w(x) dx, \quad \tilde{w}_0'(0) = \int_{F^\times_E} w(x) \log \|x\|_{F^\times} dx.
\]

Then we have the identity

\[
\frac{1}{\sqrt{N(a)}} \sum_{\gamma \in (a\mathfrak{O})^{-1} \setminus \{0\}} \psi_a(\gamma) \tau(\gamma a \mathfrak{O}) \overline{w}(\gamma) = \sqrt{N(a)} \sqrt{N(b)} \tilde{w}_0(0; b)
\]

\[
+ \frac{1}{\sqrt{N(b)}} \sum_{\gamma \in (b\mathfrak{O})^{-1} \setminus \{0\}} \psi_S(\gamma) \tau(\gamma b \mathfrak{O}) \tilde{w}_0(\gamma).
\]

Remark 1.5. With some efforts, one may prove that the above formulae are valid for any $w : F^\times_E \to \mathbb{C}$ with compact support. Note that for such $w$ the integral transforms in Definition 1.2 are still well-defined.

Acknowledgements. The author thanks Edgar Assing and the referee for their helpful comments.

2. Review of Eisenstein Series

In this section, we recollect some basic facts on Eisenstein series. The reader is referred to [Bum §3.7] for more details.

Define the parabolic subgroup

\[
P = \left\{ \begin{pmatrix} x & r \\ y & \end{pmatrix} \right\} \subset \text{GL}_2.
\]

For $s \in \mathbb{C}$ let $\pi(s)$ be the representation of $\text{GL}_2(\mathbb{A})$ obtained by (normalized) parabolic induction of the following character of $P(\mathbb{A})$:

\[
\begin{pmatrix} x & r \\ y & \end{pmatrix} \mapsto \|x/y\|^{s+\frac{1}{2}}.
\]

To be precise, the space $V(s)$ of this representation consists of all smooth functions $\phi$ on $\text{GL}_2(\mathbb{A})$ that satisfy

\[
\phi(\begin{pmatrix} x & r \\ y & \end{pmatrix} g) = \|x/y\|^{s+\frac{1}{2}} \phi(g),
\]

on which the action of $\text{GL}_2(\mathbb{A})$ is by right translation.

For $\text{Re}(s) > \frac{1}{2}$ define the Eisenstein series $E(g; \phi)$ associated with $\phi \in V(s)$ by

\[
E(g; \phi) = \sum_{\gamma \in P(F) \setminus \text{GL}_2(F)} \phi(\gamma g), \quad g \in \text{GL}_2(\mathbb{A});
\]

the series is absolutely convergent. The Fourier–Whittaker expansion of $E(g; \phi)$ is given by

\[
E(g; \phi) = \phi(g) + M\phi(g) + \sum_{\gamma \in F^\times} W_\phi(a(\gamma)g),
\]
where
\[
M \phi(g) = \int_{\mathbb{H}} \phi(wn(r)g)dr, \quad \phi \in V(s),
\]
is the intertwining integral, and
\[
W \phi(g) = \int_{\mathbb{H}} \phi(wn(r)g)\overline{\psi(r)}dr, \quad \phi \in V(s),
\]
is the Whittaker integral, with
\[
M \phi = \int \phi(wn(r)g)\overline{\psi(r)}dr, \quad \phi \in V(s),
\]
is the intertwining integral, and
\[
W \phi = \int \phi(wn(r)g)\overline{\psi(r)}dr, \quad \phi \in V(s),
\]
is the intertwining integral, with
\[
a(x) = \begin{pmatrix} x & r \\ 1 & 1 \end{pmatrix}, \quad n(r) = \begin{pmatrix} 1 & r \\ 1 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
\]
It is known that the (global) integrals in (2.2) and (2.3) are converge for \(\Re(s) > \frac{1}{2}\) and have analytic continuation onto the whole complex plane, except for a simple pole at \(s = \frac{1}{2}\) that occurs in the case of \(M \phi(g)\). Moreover, we have \(M : V(s) \to V(-s)\), and \(W \phi(n(r)g) = \psi(r)W \phi(g)\).

### 3. An Adelic Identity

The starting point of our approach is the following identity, which is the analogue of [Theorem 3.1](tem) in the case of Eisenstein series.

**Lemma 3.1.** Let \(\zeta \in \mathbb{A}_f\) and \(\alpha \in \mathbb{A}_f^\times\). For \(\phi \in V(s)\) we have
\[
\phi(a(\alpha)) + M \phi(a(\alpha)) + \sum_{\gamma \in F^x} \psi_f(\gamma \zeta)W \phi(a(\alpha \gamma)) =
\]
\[
\phi(wn(\zeta)a(\alpha)) + M \phi(wn(\zeta)a(\alpha)) + \sum_{\gamma \in F^x} W \phi(a(\gamma)wn(\zeta)a(\alpha)).
\]

**Proof.** In view of (2.1), it is easily seen that the left-hand side is \(E(\mu(\zeta)a(\alpha); \phi)\) while the right-hand side is \(E(wn(\zeta)a(\alpha); \phi)\). Since \(E(g; \phi)\) is left \(GL_2(F)\)-invariant, the identity follows because \(w \in GL_2(F)\). Q.E.D.

We now assume that \(\phi \in V(s)\) is factorizable as \(\prod_v \phi_v\); it is clear that \(M \phi)\) and \(W \phi\) are also factorizable. More precisely, for each \(v\), \(\phi_v\) is a smooth function on \(GL_2(F_v)\) such that
\[
\phi_v(\begin{pmatrix} x & r \\ y & 1 \end{pmatrix}) = \|x/y\|_v^{s + \frac{1}{2}} \phi_v(g), \quad x, y \in F_v^\times, r \in F_v, g \in GL_2(F_v);
\]
by convention, let \(V_v(s)\) denote the space of such \(\phi_v\) with the above property. Put \(\phi_x = \prod_{v \mid x} \phi_v, M \phi_x = \prod_{v \mid x} M \phi_v,\) and \(W \phi_x = \prod_{v \mid x} W \phi_v\).

Choose \(\alpha \in \mathbb{A}_f^\times\) so that \(\text{ord}_v(a_v) = \text{ord}_v(\alpha)\) for every \(v \mid x\). Define
\[
S = \{v \mid x : \text{ord}_v(\zeta) < \text{ord}_v(\alpha)\}, \quad B = \prod_v \mathbb{P}^{\max\{\text{ord}_v(1/\alpha), \text{ord}_v(\alpha/\zeta^2)\}}.
\]
For every \(v \mid x\), choose \(\phi_v = \phi_{s,v}\) to be the canonical spherical vector in \(V_v(s)\) with \(\phi_{s,v}(k) = 1\) for all \(k \in GL_2(O_v)\); namely,
\[
\phi_{s,v}(\begin{pmatrix} x & r \\ y & 1 \end{pmatrix}) = \|x/y\|_v^{s + \frac{1}{2}}, \quad x, y \in F_v^\times, r \in F_v, k \in GL_2(O_v).
The local integrals $M_{\phi_v}$ and $W_{\phi_v}$ are very explicit in the spherical case (see for example [Bum] §4.6.4). Globally, if we define

$$c_s(0) = \frac{\zeta_F(2s)}{\sqrt{N(\mathcal{O})} \zeta_F(1 + 2s)}, \quad c_s(n) = \frac{\tau_s(n)/\sqrt{N(n)}}{N(\mathcal{D})^s \zeta_F(1 + 2s)},$$

then

$$\phi_f(a(\alpha)) = \frac{1}{N(\alpha)^{\frac{3}{2} + s}},$$

by definition, and

$$M_{\phi_f}(a(\alpha)) = \frac{c_s(0)}{N(\alpha)^{\frac{3}{2} - s}}, \quad W_{\phi_f}(a(\alpha^\gamma)) = c_s(\gamma a(\mathcal{D})),\quad \text{by Proposition 4.6.7 and Theorem 4.6.5 in [Bum].}$$

Consequently, the left-hand side of (3.1) is equal to

$$\phi_f(\gamma) = \frac{\phi_f(12)}{N(\alpha)^{\frac{3}{2} + s}} + c_s(0) M_{\phi_f}(12) \frac{N(\alpha)^{\frac{3}{2} - s}}{N(\alpha)^{\frac{3}{2} - s}} + \sum_{\gamma \in (\mathcal{A}_D)^{-1} \setminus \{0\}} \psi_f(\gamma) c_s(\gamma a(\mathcal{D})) W_{\phi_f}(a(\gamma)).$$

Next, we compute the right-hand side of (3.1). Keep in mind that $\phi_v$, $M_{\phi_v}$, and $W_{\phi_v}$ are right $\text{GL}_2(\mathcal{O}_v)$-invariant. Recall that $\mathcal{S}$ is defined in (3.2). When $v \in S_f \setminus S$ so that $\|\zeta/\alpha\|_v \leq 1$, we have the Iwasawa decomposition

$$a(\gamma) w_n(\zeta_v) a(\alpha_v) = \begin{pmatrix} \gamma & 1 \\ \alpha_v & \zeta_v/\alpha_v \end{pmatrix},$$

and hence

$$\phi_v(w_n(\zeta_v) a(\alpha_v)) = \|1/\alpha\|_v^{\frac{3}{2} + s}, \quad M_{\phi_v}(w_n(\zeta_v) a(\alpha_v)) = \|1/\alpha\|_v^{\frac{3}{2} - s} M_{\phi_v}(12),$$

$$W_{\phi_v}(a(\gamma) w_n(\zeta_v) a(\alpha_v)) = W_{\phi_v}(a(\alpha_v)).$$

When $v \in S$ so that $\|\zeta/\alpha\|_v > 1$, we have the Iwasawa decomposition

$$a(\gamma) w_n(\zeta_v) a(\alpha_v) = \begin{pmatrix} 1 & \gamma/\zeta_v \\ -\gamma/\zeta_v & 1 \end{pmatrix} \begin{pmatrix} \gamma \alpha_v/\zeta_v \\ \zeta_v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_v/\zeta_v & 1 \end{pmatrix},$$

and hence

$$\phi_v(w_n(\zeta_v) a(\alpha_v)) = \|\alpha/\zeta_v^2\|_v^{\frac{3}{2} + s}, \quad M_{\phi_v}(w_n(\zeta_v) a(\alpha_v)) = \|\alpha/\zeta_v^2\|_v^{\frac{3}{2} - s} M_{\phi_v}(12),$$

$$W_{\phi_v}(a(\gamma) w_n(\zeta_v) a(\alpha_v)) = \psi_v(-\gamma/\zeta_v) W_{\phi_v}(a(\gamma \alpha_v/\zeta_v^2)).$$

In view of the definition of $b$ in (3.2), it readily follows that

$$\phi_f(w_n(\zeta) a(\alpha)) = \frac{1}{N(b)^{\frac{3}{2} + s}},$$

and

$$M_{\phi_f}(w_n(\zeta) a(\alpha)) = \frac{c(0)}{N(b)^{\frac{3}{2} - s}}, \quad W_{\phi_f}(a(\gamma) w_n(\zeta) a(\alpha)) = \psi_s(-\gamma/\zeta) c_s(\gamma b(\mathcal{D})).$$

\footnote{A subtle issue is that the results in [Bum] §4.6 are proven for $\psi_v$ of conductor $\mathcal{O}_v$, but this may be easily addressed by re-scaling the character and the Haar measure.}
Therefore, the right-hand side of (3.1) is equal to
\[ \frac{\phi_x(w)}{N(b)^{\frac{3}{2}+s}} + c_s(0) \frac{M \phi_x(w)}{N(b)^{\frac{5}{2}-s}} + \sum_{\gamma \in (B_2)\cap \langle 0 \rangle} \psi_S(-\gamma/\zeta)c_s(\gamma bD)W_{\phi_x}(a(\gamma)w). \]

Lemma 3.1 says that (3.4) and (3.5) are equal to each other.

For each \( v \mid x \), we will make the choice of \( \phi_v \) later in 4.2.

4. Archimedean Kirillov Model

In this section, we will work exclusively on an archimedean local field \( F_v \). For simplicity, the place \( v \) will be suppressed from our notation. Accordingly, let \( F \) be either \( \mathbb{R} \) or \( \mathbb{C} \). Let \( \psi(x) = e(-\text{Tr}_F(x)) \), and \( dx \) be the corresponding self-dual Haar measure on \( F \). Let \( \| \cdot \| \) denote the standard modulus of \( F \). For \( s \in \mathbb{C} \), let \( V(s) \) be the space of smooth functions on \( \text{GL}_2(F) \) that satisfy
\[ \phi \left( \begin{pmatrix} x & r \\ y & 1 \end{pmatrix} g \right) = \|x/y\|^{\frac{s}{2}} \phi(g), \]
and let \( \pi(s) \) denote the representation of \( \text{GL}_2(F) \) that acts on \( V(s) \) by right translation.

For simplicity, we assume that \( 2s \notin \mathbb{Z} \setminus 2\mathbb{Z} \) or \( 2s \notin \mathbb{Z} \setminus \{0\} \) according as \( F \) is real or complex, so that \( \pi(s) \) is irreducible.

For \( \text{Re}(s) > 0 \), the Whittaker functional \( L \) on \( V(s) \) is defined by
\[ L(\phi) = \int_F \phi(wn(r))\overline{\psi(r)}dr, \]
in which the integral is convergent for \( \text{Re}(s) > 0 \) (see [God]). The Whittaker function \( W_\phi \) associated to \( \phi \in V(s) \) is
\[ W_\phi(g) = L(\pi(g)\phi) = \int_F \phi(wn(r)g)\overline{\psi(r)}dr. \]

By definition, the Kirillov model \( \mathcal{K}(\pi(s)) \) comprises all the functions \( W_\phi(a(x)) \) \( (x \in F^\times) \). It is known that \( C_c^\infty(F^\times) \subset \mathcal{K}(\pi(s)) \) (see [IT, Lemma 5.1]). Moreover, we define the intertwining operator \( M : V(s) \to V(-s) \) by the integral
\[ M \phi(g) = \int_F \phi(wn(r)g)dr. \]

Again, this integral is convergent for \( \text{Re}(s) > 0 \). It is known that both the Whittaker integral and the intertwining integral in (4.2) and (4.3) have meromorphic continuation to the entire \( s \) plane, but we will not need this fact since \( \text{Re}(s) > \frac{1}{2} \) will be assumed.

4.1. A Kernel Formula. We have the following kernel formula for the action of the Weyl element \( w \) on the Kirillov model \( \mathcal{K}(\pi(s)) \) as the Hankel integral transform with Bessel kernel \( B_s \).

**Proposition 4.1.** For \( W_\phi(a(x)) \in C_c^\infty(F^\times) \), we have
\[ W_\phi(a(y)w) = \int_{F^\times} W_\phi(a(x))B_s(xy)\sqrt{\|y/x\|}dx, \]
where \( B_s(x) \) is the Bessel kernel associated to \( \pi(s) \) as in Definition 1.1.
PROOF. For the formula (4.3) in our case of \( \pi(s) \), which is not unitary in general, we refer to Proposition 3.14, 3.17, Remark 17.6, and (18.1)–(18.4) in [Qi4]. Q.E.D.

The occurrence of Bessel functions in the representation theory of \( \text{SL}_2(\mathbb{R}) \) may be traced back to the books of Gel’fand, Graev, and Piatetski-Shapiro [GGPS], and Vilenkin [Vil]. We refer the reader to [CPS] §§6, 8], [BM1] Appendix 2], and [Qi4 §§17, 18] for the kernel formulae for unitary representations of \( \text{GL}_2(\mathbb{R}) \) and \( \text{GL}_2(\mathbb{C}) \) (see also [Mot, Bar, BM3, BBA]) for \( \text{SL}_2(\mathbb{R}) \) and \( \text{SL}_2(\mathbb{C}) \) (in special cases). For its applications in establishing the Kuznetsov formula and the Waldspurger formula, see [CPS, Qi1, BM1, CQ2, BM2, CQ1].

For any infinite dimensional admissible representation of \( \text{GL}_2(\mathbb{R}) \) or \( \text{GL}_2(\mathbb{C}) \), it follows from the Casselman–Wallach completion theorem (see [Cas, Wal1] Chapter 11 in [Wal2]) that, after dividing \( \sqrt{|x|} \), the Kirillov model \( \mathcal{X} \) is exactly the \( \mathcal{S}_{\text{sis}} \)-space as defined in [Qi4] (see also [MS]). It should be stressed that for the unitary case, the kernel formula is actually valid for all \( W_d(a(x)) \) in the Kirillov model. However, this is not necessarily true in general, and the action of \( w \) needs to be interpreted in terms of \( \text{GL}_2 \times \text{GL}_1 \) local functional equations.

Finally, we conclude this sub-section with some discussions on the various proofs of this kernel formula in the literature. The case of \( \text{GL}_2(\mathbb{R}) \) or \( \text{GL}_2(\mathbb{C}) \) is relatively easier, and there are three proofs in [CPS] §§8, [Mot], and [BM1] Appendix 2]. The methods of the latter two proofs were generalized to \( \text{SL}_2(\mathbb{C}) \) in [BM3] and [BBA]. However, certain conditions are required due to some convergence issues. In [BM3], an integral representation of the Bessel function is used but it is valid only for \( \text{Re}(s) < \frac{1}{2} \). In [BBA], it requires that \( \text{Re}(s) \neq 0 \) and hence the case of unitary principal series is excluded. The approach in [Qi4] is quite different and works without any condition. It is based on the sophisticated harmonic analysis for the Mellin transforms on \( \mathcal{S}_{\text{sis}} \)-spaces (see [Qi4 §1–3]). Also the ideas in [CPS] §§8 are followed and generalized in [Qi4 §17] to \( \text{GL}_n(\mathbb{R}) \) and \( \text{GL}_n(\mathbb{C}) \).

4.2. Choice of Archimedean Vectors. Let \( w \in C_c^\infty(\mathbb{F}^\times) \). We define the function \( \phi_{s,w} \) by

\[
\phi_{s,w}(g) = \begin{cases} 
\|x/y\|^{s+\frac{1}{2}} \int_F w(v)|v|^s \psi(\sqrt{v})dv, & \text{if } g = \begin{pmatrix} x & u \\ y & v \end{pmatrix}, \\ 0, & \text{if } g = \begin{pmatrix} x & u \\ y & v \end{pmatrix}.
\end{cases}
\]

LEMMA 4.2. We have \( \phi_{s,w} \in \mathcal{V}(s) \), that is, \( \phi_{s,w} \) is smooth and satisfies (4.1).

PROOF. It is clear that \( \phi_{s,w} \) satisfies (4.1). It follows that the smoothness of \( \phi_{s,w} \) is equivalent to the smoothness of its restriction to \( \text{SO}_2(\mathbb{R}) \) or \( \text{SU}_2(\mathbb{C}) \). For \( |a|^2 + |b|^2 = 1 \) (\( a, b \in \mathbb{R} \) or \( \mathbb{C} \)), with \( b \neq 0 \), if we let

\[
\begin{pmatrix} a & -b \\ b & \bar{a} \end{pmatrix} = \begin{pmatrix} x & u \\ y & v \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix},
\]

then \( x = 1/b, y = \bar{b}, u = a, \) and \( r = \bar{a}/b \). Therefore

\[
\phi_{s,w} \left( \begin{pmatrix} a & -b \\ b & \bar{a} \end{pmatrix} \right) = |b|^{-2s-1} \int_F w(v)|v|^s \psi(\sqrt{v})dv.
\]

\footnote{It should be noted that our parametrization is slightly different from theirs.}
The issue of smoothness is at the points where $b = 0$, but the Fourier transform here is a rapidly decreasing function of $\overline{a/b}$, so $\phi_{s,w}$ is smooth at these points. Q.E.D.

**Lemma 4.3.** Let $\Re(s) > 0$. We have

\begin{equation}
\phi_{s,w}(1_2) = 0, \quad \phi_{s,w}(w) = \int_F w(x)\|x\|^s dx,
\end{equation}

and

\begin{equation}
M\phi_{s,w}(1_2) = 0, \quad M\phi_{s,w}(w) = \frac{\gamma(2s)}{\gamma(1-2s)} \int_F w(x)\|x\|^{-s} dx,
\end{equation}

where

\begin{equation}
\gamma(s) = \begin{cases} 
\pi^{-s/2}\Gamma(s/2), & \text{if } F \text{ is real}, \\
(2\pi)^{-s}\Gamma(s), & \text{if } F \text{ is complex}.
\end{cases}
\end{equation}

The formulae in (4.6) follow immediately from the definitions in (4.5). By (4.3) and (4.5),

\begin{equation}
M\phi_{s,w}(1_2) = \int_F \int_F w(x)\|x\|^s \psi(rx) dx dr.
\end{equation}

By the Fourier inversion formula, this is the value of $w(x)\|x\|^s$ at $x = 0$. However, this function is compactly supported in $F \setminus \{0\}$, so $M\phi_{s,w}(1_2) = 0$. As for $M\phi_{s,w}(w)$, it follows from (4.3) that

\[ M\phi_{s,w}(w) = \int_F \phi_{s,w}(wn(r)w) dr, \]

while for $r \neq 0$ we have

\[ wn(r)w = \begin{pmatrix} 1/r & -1 \\ r & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1/r \end{pmatrix}, \]

so, on changing $r$ into $-1/r$, we obtain from (4.5) that

\[ M\phi_{s,w}(w) = \int_F \|r\|^{2s-1} \int_F w(x)\|x\|^s \psi(rx) dx dr. \]

Since $w(x)\|x\|^s$ is smooth and compactly supported, its Fourier transform is of Schwartz class, and hence the integral is convergent and analytic for all $\Re(s) > 0$.

Next, we formally change the order of integration. After this, the $r$-integral may be evaluated by Lemma 4.4 below, with $v = s$ or $2s$, then the formula for $M\phi_{s,w}(w)$ as in (4.7) follows.

**Lemma 4.4.** For $0 < \Re(v) < \frac{1}{2}$ we have

\begin{equation}
\int_0^\infty x^{2v-1}(e(-xy) + e(xy)) dx = \frac{\pi^{\frac{1}{2}-2v} \Gamma(v)}{y^{2v} \Gamma\left(\frac{1}{2} - v\right)},
\end{equation}

and

\begin{equation}
2 \int_0^\infty \int_0^{2\pi} x^{2v-1} e(-2xy \cos(\phi + \omega)) d\phi dx = \frac{(2\pi)^{1-2v} \Gamma(v)}{y^{2v} \Gamma(1 - v)},
\end{equation}

where $y \in (0, \infty)$ and $\omega \in [0, 2\pi)$; the integrals are convergent conditionally.
Proof of Lemma 4.4. By [GR] 3.761 9 and [Qi5] Lemma 4.4, the integrals in (4.9) and (4.10), respectively, are equal to \(2(2\pi y)^{-2}\Gamma(2\nu)\cos(\pi\nu)\) and \(2(2\pi y)^{-2}\Gamma(\nu)\sin(\pi\nu)\), and we arrive at the right-hand sides of (4.9) and (4.10) by the duplication and the reflection formulae for the gamma function. Q.E.D.

However, the change of the order of integration is not quite rigorous as the double integral does not converge absolutely. To justify this, we introduce an exponential factor \(\exp(-2\pi\epsilon|\tau|)\) or \(\exp(-4\pi\epsilon|\tau|)\) in the \(r\)-integral to ensure absolute convergence. To evaluate the \(r\)-integral, we use the following Lemma 4.5 instead of Lemma 4.4. Finally, we proceed to the limit as \(\epsilon \to 0\) to conclude the proof. In view of the duplication and the reflection formulae for the gamma function, (4.11) and (4.12) are the limiting forms of (4.9) and (4.10), respectively. For the complex case, note that

\[
\binom{2F_1}{\nu, \frac{1}{2} - \nu; 1; 1} = \frac{\sqrt{\pi}}{\Gamma(1 - \nu)\Gamma(\frac{1}{2} + \nu)},
\]

by the Gauss formula.

Lemma 4.5. Let \(\epsilon > 0\). For Re\((\nu) > 0\), we have

\[
\int_0^\infty x^{2\nu-1} \exp(-2\pi \epsilon x)(e(-xy) + e(xy))dx = \frac{2\Gamma(2\nu)\cos(2\nu\arctan(y/\epsilon))}{(2\pi)^{2\nu}(y^2 + \epsilon^2)^\nu},
\]

and

\[
2 \int_0^\infty \int_0^{2\pi} x^{2\nu-1} \exp(-4\pi \epsilon x)\cos(-2xy\cos(\phi + \omega))d\phi dx
\]

\[
= \frac{\Gamma(2\nu)}{(4\pi)^{2\nu-1}(y^2 + \epsilon^2)^\nu} \binom{2F_1}{\nu, \frac{1}{2} - \nu; 1; \frac{y^2}{y^2 + \epsilon^2}},
\]

where \(y \in (0, \infty)\) and \(\omega \in [0, 2\pi]\); the integrals are absolutely convergent.

Proof of Lemma 4.5. The formula (4.11) is a direct consequence of [GR] 3.944 6. As for (4.12), we first compute the \(\phi\)-integral by Bessel’s formula (see [Wat] 2.2 (1))

\[
J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(ix\cos\phi)d\phi,
\]

so that the integral in (4.12) turns into

\[
4\pi \int_0^\infty x^{2\nu-1} \exp(-4\pi \epsilon x)J_0(4\pi xy)dx,
\]

and this integral can be evaluated by [Wat] 13.2 (3)], giving the right-hand side of (4.12). Q.E.D.

Finally, for the Whittaker function associated to \(\phi_{s,w}\) we have the following lemma.

Lemma 4.6. Let \(W_{s,w} = W_\phi(with \phi = \phi_{s,w})\). We have

\[
W_{s,w}(a(x)) = \sqrt{x}w(x), \quad W_{s,w}(a(y)w) = \sqrt{y} \int_{\mathbb{R}^+} w(x)B_s(xy)dx.
\]
The formula for \( W_{s,w}(a(x)) \) is precisely the kernel formula in Proposition 4.1.

Q.E.D.

5. Proof in the Case \( \text{Re}(2s) > 1 \)

Assume that \( 2s \) is not an integer and that \( \text{Re}(2s) > 1 \). Let \( w \in \mathcal{C}_{\mathcal{E}}(F_{\mathcal{X}}^s) \), with \( w = \prod_{v \mid \mathfrak{F}} \phi_{s,w_v} \). We choose \( \phi_{s,x} \) to be the product \( \prod_{v \mid \mathfrak{F}} \phi_{s,w_v} \). In view of Lemma 4.3 and 4.6 if we change \( \zeta \) into \( -\zeta \), the sums in (3.4) and (3.5), respectively, equal to

\[
\frac{1}{N(a)^{s}N(\mathfrak{D})^{\frac{1}{2}+s}\zeta_F(1+2s)} \sum_{\gamma \in (\mathfrak{a}\mathfrak{D})^{-1}_{-\gamma}(0)} \psi_f(-\gamma \zeta) \tau_x(\gamma \mathfrak{a}\mathfrak{D}) w(\gamma),
\]

and

\[
\frac{\tilde{w}_s(0)}{N(b)^{\frac{1}{2}+s}} + \frac{\zeta_F(2s) \gamma_F(2s)}{N(\mathfrak{D})^{\frac{1}{2}}\zeta_F(1+2s) \gamma_F(1-2s) N(b)^{\frac{1}{2}-s}} \tilde{w}_{-s}(0) + \frac{1}{N(b)^{\frac{1}{2}}N(\mathfrak{D})^{\frac{1}{2}+s}\zeta_F(1+2s)} \sum_{\gamma \in (\mathfrak{b}\mathfrak{D})^{-1}_{-\gamma}(0)} \psi_S(\gamma/\zeta) \tau_x(\gamma \mathfrak{b}\mathfrak{D}) \tilde{w}_s(\gamma),
\]

where \( \gamma_F(s) \) is the product of the \( \gamma_v(s) \) defined as in (4.8). Recall the functional equation for \( \zeta_F \) (see [Lan, §XIV.8]):

\[
N(\mathfrak{D})^{s/2}\zeta_F(s) \gamma_F(s) = N(\mathfrak{D})^{1-s/2}\zeta_F(1-s) \gamma_F(1-s).
\]

Hence

\[
\frac{\zeta_F(2s) \gamma_F(2s)}{N(\mathfrak{D})^{\frac{1}{2}}\zeta_F(1+2s) \gamma_F(1-2s) N(b)^{\frac{1}{2}-s}} = \frac{N(\mathfrak{D})^{-s}\zeta_F(1-2s)}{N(\mathfrak{D})^{s}\zeta_F(1+2s)}.
\]

Note that \( \psi_f(-\gamma \zeta) = \psi_{\gamma \zeta}(\gamma \zeta) \) for \( \gamma, \zeta \in F \). Since (3.1) and (3.2) are equal to each other, we obtain (4.6) after multiplying them by \( N(\mathfrak{D})^{\frac{1}{2}+s}\zeta_F(1+2s) \).

6. Analytic Continuation

To complete the proof, we need to verify the validity of (4.6) for all values of \( s \in \mathbb{C} \) by the principle of analytic continuation. To this end, it suffices to verify that both sides of (4.6) are entire functions of \( s \).

Since \( w \) has compact support on \( F_{\mathcal{X}}^s \), while \( (\mathfrak{a}\mathfrak{D})^{-1} \) is a lattice in \( F_{\mathcal{X}} \), the left-hand side is a finite sum and hence gives rise to an entire function of \( s \). The function \( \zeta_F(s) \) is analytic except for a simple pole at \( s = 1 \), hence the first sum on the right is entire, and at \( s = 0 \) it takes value

\[
\sqrt{N(\mathfrak{D})} \int_{F_{\mathcal{X}}} w(x) \{ \gamma_F^{-1}(1) \log (\|x\|_{\mathcal{X}} N(\mathfrak{D})^s) + 2\gamma_f(0) \} dx,
\]
for \( \gamma_F^{(-1)} \) and \( \gamma_F^{(0)} \) defined as in \((1.3)\). Finally, the series on the right converges absolutely and uniformly on compact subsets by Lemma 6.1 and 6.2 below, with \( V = 1, c = \sigma + \varepsilon, d = 2, \) and \( A = \sigma + 2, \) so it converges to an entire function of \( s. \)

6.1. Averages of Divisor Functions. Actually, we can establish bounds, not just the convergence, for certain averages of \( \tau_s(n) . \) See \([Q12] \S 4\) and \([Q13] \S 4.3\) for their analogues in the cases of GL\(_2\) and GL\(_3\) cuspidal Fourier coefficients.

**Lemma 6.1.** Define \( N_v = 1 \) if \( F_v = \mathbb{R} \) and \( N_v = 2 \) if \( F_v = \mathbb{C} . \) For \( V \in \mathbb{R}_{+}^{\left| S_{x_{v}} \right|} \) and \( S \subset S_{X}, \) define \( N(V) = \prod_{v \mid x} V_{v}^{N_{v}}, \|V\|_{S} = \prod_{v \in S} V_{v}^{N_{v}} \), and

\[
(6.2) \quad F_X^{S}(V) = \{ x \in F_{X} : \|x\|_{v} > V_{v}^{N_{v}}, \text{if } v \in S, \|x\|_{v} \leq V_{v}^{N_{v}}, \text{if } v \in S_{x} \setminus S \}. 
\]

Let \( \sigma = |\text{Re}(s)| \) and \( 0 \leq c - \sigma < 1 \). Then for any \( 0 < \varepsilon < d - 1 \) we have

\[
(6.3) \quad \sum_{\gamma \in F_{x} \cap F_{x}^{S}(V)} \frac{1}{N_{\gamma}} \frac{|\tau_{s}(\gamma a)|}{\|a\|^{|d-c+\sigma|}} = O_{\varepsilon,c,d,\sigma,F}(N(a)^{1+\sigma+\varepsilon}N(V)^{1-c+\sigma+\varepsilon} / \|V\|_{S}^{d-c+\sigma}),
\]

with the implied constant uniformly bounded for \( \sigma \) in compact sets.

**Proof.** Firstly, by partial summation, we deduce from

\[
\sum_{N(n) \leq X} 1 = O_{F}(X)
\]

that

\[
(6.4) \quad \sum_{N(n) \leq X} \frac{|\tau_{s}(n)|}{N(n)^{c}} \leq \sum_{N(b) \leq X} \frac{1}{N(b)^{c+\sigma}} \sum_{N(a) \leq X/N(b)} \frac{1}{N(a)^{e-\sigma}} \leq F \frac{X^{1-\varepsilon+\sigma} \log X}{1-c+\sigma},
\]

for \( X \geq 2, \) provided that \( 0 \leq c - \sigma < 1. \) Next, we use \((6.4)\) as a substitute of \((4.3)\) in \([Q12]\) and apply his Lemma 4.1 to prove for any \( V \in \mathbb{R}_{+}^{\left| S_{x_{v}} \right|} \) (see also the proof of \([Q13] \) Lemma 4.10)

\[
(6.5) \quad \sum_{\gamma \in F_{x} \cap F_{x}^{S}(V)} \frac{1}{N_{\gamma}} \frac{|\tau_{s}(\gamma a)|}{\|a\|^{c}} = O_{\varepsilon,c,\sigma,F}(N(a)^{1+\sigma+\varepsilon}N(V)^{1-c-\sigma+\varepsilon}),
\]

which is an analogue of his Lemma 4.2. Finally, we proceed as in the proof of Lemma 4.3 in \([Q12]\) to derive \((6.3)\) from \((6.5)\). It is easy to verify the uniformity in \( \sigma \) at each step.

Q.E.D.

6.2. Estimates for the Hankel Transform. Finally, we have the following crude but uniform estimates for the Hankel transform. For brevity, we will suppress the place \( v \) from our notation.

**Lemma 6.2.** Let \( \sigma = |\text{Re}(s)|. \) For \( w(x) \in C_{c}^{\infty}(F_{x}) \) we have

\[
\int_{F_{x}} w(x)B_{s}(xy)dx \leq_{s,\varepsilon,A,w} \begin{cases} 1/\|y\|^{|\sigma+\varepsilon|}, & \text{if } \|y\| \leq 1, \\ 1/\|y\|^A, & \text{if } \|y\| > 1, \end{cases}
\]

for any \( \varepsilon > 0 \) and \( A \geq 0, \) with the implied constants uniformly bounded for \( s \) in compact sets.
PROOF. For fixed $s$ the estimates follow immediately from Theorem 3.12, 3.15 and Proposition 3.14, 3.17 in [Q14]. However, to prove the uniformity in $s$, we require uniform bounds and asymptotics for the Bessel kernel $B_s$.

Now fix $c > 1$ and let $|s| \leq c^4$.

Proceeding as in [Q12] §5.1, by estimating the Mellin–Barnes type integrals of certain gamma factors, for $|x| \leq c^4$ we deduce the bounds

$$B_s(x) \lesssim c_{c, \varepsilon} 1/|x|^{\sigma + \varepsilon}.$$  

It is critical that the integral contours therein can be chosen fixed for given $c$ and $\varepsilon$. Then the first uniform estimate follows directly.

Next, we invoke the formulae

$$B_s(x) = \pi i \left( e^{\pi i s} H_{2s}^{(1)}(4\pi \sqrt{x}) - e^{-\pi i s} H_{2s}^{(2)}(4\pi \sqrt{x}) \right),$$

$$B_s(-x) = 4 \cos(\pi s) K_{2s}(4\pi \sqrt{x}),$$

for $x \in \mathbb{R}^+$, and

$$B_s(z) = \pi^2 i \left( e^{2\pi i s} H_{2s}^{(1)}(4\pi \sqrt{z}) H_{2s}^{(1)}(4\pi \sqrt{\bar{z}}) - e^{-2\pi i s} H_{2s}^{(2)}(4\pi \sqrt{z}) H_{2s}^{(2)}(4\pi \sqrt{\bar{z}}) \right),$$

for $z \in \mathbb{C}^*$; see [Wat] (3.61 (1), (2)]. By [Olv] §7.13.1, Ex. 13.2], we deduce the uniform asymptotic formulae:

$$B_s(x) = \sum_{\pm} \frac{e(\pm(2\sqrt{x} + 1/8))}{x^{1/4}} \sum_{k=0}^{K-1} \frac{(\pm)^k A_k(s)}{x^{k/2}} + O_{c,K} \left( \frac{1}{x^{(2K+1)/4}} \right),$$

$$B_s(-x) = O_c \left( \frac{\exp(-4\pi \sqrt{x})}{x^{1/4}} \right),$$

for $x > c^4$, and

$$B_s(z) = \sum_{\pm} \frac{e(\pm 2\sqrt{z})}{|z|^{1/2}} \sum_{k=0}^{K-1} \frac{(\pm)^{k+l} A_k(s) A_l(s)}{z^{k/2} \bar{z}^{l/2}} + O_{c,K} \left( \frac{1}{|z|^{(K+1)/2}} \right),$$

for $|z| > c^4$, where $K$ is any non-negative integer, and the coefficient $A_k(s)$ is a certain polynomial in $s$ of degree $2k$. Then the second uniform estimate follows from repeated partial integration (we obtain Fourier integrals on letting $\sqrt{x}$ or $\sqrt{z}$ be the new variable) or directly from the exponential decay (in the real case). Q.E.D.

References

[Ass] E. Assing. On sup-norm bounds part II: GL(2) Eisenstein series. Forum Math., 31(4):971–1006, 2019.

[Bar] E. M. Baruch. The classical Hankel transform in the Kirillov model of the discrete series. Integral Transforms Spec. Funct., 24(5):339–356, 2013.

[BB] J. Beineke and D. Bump. Moments of the Riemann zeta function and Eisenstein series. I. J. Number Theory, 105(1):150–174, 2004.

[BBA] E. M. Baruch and O. Beit-Aharon. A kernel formula for the action of the Weyl element in the Kirillov model of SL(2, C). J. Number Theory, 146:23–40, 2015.

[BBB] D. Banerjee, E. M. Baruch, and D. Bump. Voronoi summation formula for Gaussian integers. Ramanujan J., 57:253–274, 2022.

[BBT] D. Banerjee, E. M. Baruch, and E. Tenetov. A Voronoi–Oppenheim summation formula for totally real number fields. J. Number Theory, 199:63–97, 2019.

[BM1] E. M. Baruch and Z. Mao. Bessel identities in the Waldspurger correspondence over the real numbers. Israel J. Math., 145:1–81, 2005.

[BM2] E. M. Baruch and Z. Mao. Central value of automorphic L-functions. Geom. Funct. Anal., 17(2):333–384, 2007.
[BM3] R. W. Bruggeman and Y. Motohashi. A note on the mean value of the zeta and \( L \)-functions. XIII. Proc. Japan Acad. Ser. A Math. Sci., 78(6):87–91, 2002.
[Bum] Daniel Bump. Automorphic Forms and Representations, Cambridge Studies in Advanced Mathematics, Vol. 55. Cambridge University Press, Cambridge, 1997.
[Cas] W. Casselman. Canonical extensions of Harish-Chandra modules to representations of \( G \). Canad. J. Math., 41(3):385–438, 1989.
[Cog] J. W. Cogdell. Bessel functions for \( GL_2 \). Indian J. Pure Appl. Math., 45(5):557–582, 2014.
[CPS] J. W. Cogdell and I. Piatetski-Shapiro. The Arithmetic and Spectral Analysis of Poincaré Series. Perspectives in Mathematics, Vol. 13. Academic Press, Inc., Boston, MA, 1990.
[CQ1] J. Chai and Z. Qi. On the Waldspurger formula and the metaplectic Ramanujan conjecture over number fields. J. Funct. Anal., 277(10):3757–3782, 2019.
[CQ2] J. Chai and Z. Qi. Bessel identities in the Waldspurger correspondence over the complex numbers. Israel J. Math., 235(1):439–463, 2020.
[GGPS] I. M. Gel’fand, M. I. Graev, and I. I. Pyatetskii-Shapiro. Representation Theory and Automorphic Functions. W. B. Saunders Co., Philadelphia, Pa.-London-Toronto, Ont., 1969. Translated from the Russian by K. A. Hirsch.
[God] R. Godement. Notes on Jacquet-Langlands’ Theory. The Institute for Advanced Study, Princeton, NJ, 1970.
[GR] I. S. Gradshteyn and I. M. Ryzhik. Table of Integrals, Series, and Products. Elsevier/Academic Press, Amsterdam, 7th edition, 2007.
[IK] H. Iwaniec and E. Kowalski. Analytic Number Theory, American Mathematical Society Colloquium Publications, Vol. 53. American Mathematical Society, Providence, RI, 2004.
[IT] A. Ichino and N. Templier. On the Voronoï formula for \( GL(n) \). Amer. J. Math., 135(1):65–101, 2013.
[Lan] S. Lang. Algebraic Number Theory. Graduate Texts in Mathematics, Vol. 110. Springer-Verlag, New York, 2nd edition, 1994.
[LQ] S.-C. Liu and Z. Qi. Moments of central \( L \)-values for Maass forms over imaginary quadratic fields. Trans. Amer. Math. Soc., 375(5):3381–3410, 2022.
[Mot] Y. Motohashi. A note on the mean value of the zeta and \( L \)-functions. XII. Proc. Japan Acad. Ser. A Math. Sci., 78(3):36–41, 2002.
[MS] S. D. Miller and W. Schmid. Distributions and analytic continuation of Dirichlet series. J. Funct. Anal., 214(1):155–220, 2004.
[Olv] F. W. J. Olver. Asymptotics and Special Functions. Academic Press, New York-London, 1974.
[Opp] A. Oppenheim. Some identities in the theory of numbers. Proc. London Math. Soc. (2), 26:295–350, 1927.
[Qi1] Z. Qi. On the Kuznetsov trace formula for \( PGL_2(\mathbb{C}) \). J. Funct. Anal., 272(8):3259–3280, 2017.
[Qi2] Z. Qi. Cancellation in the additive twists of Fourier coefficients for \( GL_2 \) and \( GL_3 \) over number fields. Amer. J. Math., 141(5):1317–1345, 2019.
[Qi3] Z. Qi. Subconvexity for \( L \)-functions on \( GL_3 \) over number fields. to appear in J. Eur. Math. Soc. (JEMS), 2020.
[Qi4] Z. Qi. Theory of fundamental Bessel functions of high rank. Mem. Amer. Math. Soc., 267(1303):vii+123, 2020.
[Qi5] Z. Qi. On the Fourier transform of regularized Bessel functions on complex numbers and Beyond Endoscopy over number fields. Int. Math. Res. Not. IMRN, (19):14445–14479, 2021.
[Tem] N. Templier. Voronoï summation for \( GL(2) \). Representation Theory, Automorphic Forms & Complex Geometry. A Tribute to Wilfried Schmid, pages 163–196. Int. Press, Somerville, MA, 2020.
[Vil] N. Ja. Vilenkin. Special Functions and the Theory of Group Representations. Translations of Mathematical Monographs, Vol. 22. American Mathematical Society, Providence, R.I., 1968.
[Vor] G. Voronoï. Sur une fonction transcendante et ses applications à la sommation de quelques séries. Ann. Sci. École Norm. Sup. (3), 21:207–267, 1904.
[Wal1] N. R. Wallach. Asymptotic expansions of generalized matrix entries of representations of real reductive groups. Lie Group Representations, I, Lecture Notes in Mathematics, Vol. 1024, pages 287–369. Springer, Berlin, 1983.
[Wal2] N. R. Wallach. *Real Reductive Groups. II*. Pure and Applied Mathematics, Vol. 132. Academic Press, Inc., Boston, MA, 1992.

[Wat] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1944.

School of Mathematical Sciences, Zhejiang University, Hangzhou, 310027, China

Email address: zhi.qi@zju.edu.cn