More eigenvalue problems of Nordhaus-Gaddum type

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Abstract

Let $G$ be a graph of order $n$ and let $\mu_1(G) \geq \cdots \geq \mu_n(G)$ be the eigenvalues of its adjacency matrix. This note studies eigenvalue problems of Nordhaus-Gaddum type. Let $\overline{G}$ be the complement of a graph $G$. It is shown that if $s \geq 2$ and $n \geq 15(s - 1)$, then

$$|\mu_s(G)| + |\mu_s(\overline{G})| \leq n/\sqrt{2(s - 1)} - 1.$$  

Also if $s \geq 1$ and $n \geq 4s$, then

$$|\mu_{n-s+1}(G)| + |\mu_{n-s+1}(\overline{G})| \leq n/\sqrt{2s} + 1.$$  

If $s = 2^k + 1$ for some integer $k$, these bounds are asymptotically tight. These results settle infinitely many cases of a general open problem.

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1 Introduction

Let $\overline{G}$ denote the complement of a graph $G$. A Nordhaus-Gaddum problem is of the type:

Given a graph parameter $p(G)$, determine

$$\max \{ p(G) + p(\overline{G}) : v(G) = n \} \quad \text{or} \quad \min \{ p(G) + p(\overline{G}) : v(G) = n \}.$$  

Since first introduced by Nordhaus and Gaddum in [9], such problems have been studied for a huge variety of graph parameters; see [2] for a recent comprehensive survey. The Nordhaus-Gaddum problems attract attention because they help to get deeper insights in extremal graph questions. Also, these problems are the closest analog to Ramsey problems for non-discrete parameters $p(G)$.

In this note we shall be interested in the case when $p(G)$ is a spectral graph parameter; thus, given a graph $G$ of order $n$, let us index the eigenvalues of the adjacency matrix of $G$ as $\mu_1(G) \geq \cdots \geq \mu_n(G)$ and set $\mu(G) = \mu_1(G)$.

The first known spectral Nordhaus-Gaddum results belong to Nosal [10], and to Amin and Hakimi [1], who showed that for every graph $G$ of order $n$,

$$n - 1 \leq \mu(G) + \mu(\overline{G}) < \sqrt{2(n - 1)}. \quad (1)$$  

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The lower bound in (1) is best possible and is attained if and only if $G$ is a regular graph; however the upper bound can be improved significantly. A minor improvement has been shown in [8], but an essentially best possible bound has been found only recently, by Csikvari [4] and Terpai [11] who showed that $\mu(G) + \mu(\overline{G}) \leq 4n/3 - 1$.

A similar problem for other eigenvalues has been proposed in [8]:

Given $s$ and $n$, find or estimate the functions

$$f_s(n) = \max_{v(G)=n} |\mu_s(G)| + |\mu_s(\overline{G})| \quad \text{and} \quad f_{n-s}(n) = \max_{v(G)=n} |\mu_{n-s+1}(G)| + |\mu_{n-s+1}(\overline{G})|.$$ 

Several bounds have been proved in [8]; among these is a tight bound for $f_2(n)$:

$$n^{\sqrt{2}} - 3 < f_2(n) < \frac{n}{\sqrt{2}}.$$ 

The problem of finding $f_s(n)$ for $s \neq 2$ has remained largely open for some time, and has been recently reiterated in [2]. In this paper we make further progress along these lines and settle asymptotically an infinite number of cases. In addition we extend the study to even more general spectral parameters.

Our first statement is about a function similar to $f_s(n)$.

**Theorem 1** If $s \geq 2$, $n \geq 3s - 2$, and $G$ is a graph of order $n$, then

$$\sum_{i=2}^{s} (\mu_i^2(G) + \mu_i^2(\overline{G})) < \frac{n^2}{4}.$$ 

Applying the AM-QM inequality to (2), we obtain another Nordhaus-Gaddum result.

**Corollary 2** If $s \geq 2$, $n \geq 3s - 2$, and $G$ is a graph of order $n$, then

$$\sum_{i=2}^{s} (|\mu_i(G)| + |\mu_i(\overline{G})|) < n\sqrt{(s-1)/2}.$$ 

However, we were not able to deduce the following natural statement directly from Theorem 1 so we shall provide a separate proof for it.

**Theorem 3** If $s \geq 2$, $n \geq 3s - 2$, and $G$ is a graph of order $n$, then

$$\mu_s^2(G) + \mu_s^2(\overline{G}) < \frac{n^2}{4(s-1)}.$$ 

Applying the AM-QM inequality to the left side of (3), one immediately sees that

$$|\mu_s(G)| + |\mu_s(\overline{G})| < \frac{n}{\sqrt{2(s-1)}},$$

which is a new bound on $f_s(n)$. However, we can do better, using a trick that was pioneered by the first author in [7], and has been applied on numerous occasions since then. We thus get the following bound.

**Theorem 4** If $s \geq 2$, $n \geq 15(s-1)$, and $G$ is a graph of order $n$, then

$$|\mu_s(G)| + |\mu_s(\overline{G})| \leq \frac{n}{\sqrt{2(s-1)}} - 1.$$ 

It turns out that the last inequality is asymptotically tight, at least for some values of $s$, as shown in Theorem 9 below.

Finding $f_{n-s}(n)$ turns to be slightly different. We begin with an analog of Theorem 1.
Theorem 5 If \( s \geq 1, n > 2s, \) and \( G \) is a graph of order \( n \), then
\[
\sum_{i=1}^{s} \left( \mu_{n-i+1}^2(G) + \mu_{n-i+1}^2(\overline{G}) \right) \leq \left( \frac{n}{2} + s \right)^2.
\] (4)

From (4) we easily obtain another Nordhaus-Gaddum result, similar to Corollary 2.

Corollary 6 If \( s \geq 1, n > 2s, \) and \( G \) is a graph of order \( n \), then
\[
\sum_{i=1}^{s} \left( |\mu_{n-i+1}(G)| + |\mu_{n-i+1}(\overline{G})| \right) \leq \left( \frac{n}{2} + s \right) \sqrt{2s}.
\]

We also can deduce the following corollary, whose short proof is in Section 2.

Corollary 7 If \( s \geq 1, n > 4s, \) and \( G \) is a graph of order \( n \), then
\[
\mu_{n-s+1}^2(G) + \mu_{n-s+1}^2(\overline{G}) \leq \frac{1}{s} \left( \frac{n}{s} + 2s \right)^2.
\] (5)

Note that the right side of (5) includes low order terms. Such terms may be reduced but not removed completely, at least for some values of \( s \): e.g., if \( s = 1 \), taking the complete balanced bipartite graph \( K_{n/2,n/2} \), we see that
\[
\mu_{n-s+1}^2(K_{n/2,n/2}) + \mu_{n-s+1}^2(\overline{K_{n/2,n/2}}) = \frac{n^2}{4} + 1.
\]

We shall use Corollary 7 to obtain a new bound on \( f_{n-s}(n) \) as well.

Theorem 8 If \( s \geq 1, n \geq 4s, \) and \( G \) is a graph of order \( n \), then
\[
|\mu_{n-s+1}(G)| + |\mu_{n-s+1}(\overline{G})| \leq \frac{n}{\sqrt{2s}} + 1.
\]

All above bounds are essentially best possible whenever \( s = 2^k + 1 \) and \( n \) is sufficiently large.

Theorem 9 Let \( s = 2^k - 1 + 1 \) for some integer \( k \geq 1 \). There exists infinitely many graphs \( G \) such that if \( 2 \leq i \leq s \), then
\[
\mu_i(G) \geq \frac{v(G)}{2 \sqrt{2(s-1)}} - 1, \quad \mu_{n-i+2}(G) \leq -\frac{v(G)}{2 \sqrt{2(s-1)}},
\]
\[
\mu_i(\overline{G}) \geq \frac{v(G)}{2 \sqrt{2(s-1)}} - 1, \quad \mu_{n-i+2}(\overline{G}) \leq -\frac{v(G)}{2 \sqrt{2(s-1)}}.
\]

2 Proofs

For graph notation and concepts undefined here, we refer the reader to [3]. In particular, if \( G \) is a graph, we write \( v(G) \) for the number of vertices of \( G \). For short we set \( \mu_i := \mu_i(G) \), \( \mu := \mu(G) \), \( \overline{\mu}_i := \mu_i(\overline{G}) \), and \( \overline{\mu} := \mu(\overline{G}) \).

2.1 Some useful observations

Lemma 10 Let \( G \) be a graph of order \( n \). If \( X \subset \{2, \ldots, n\} \), then
\[
\sum_{i \in X} \mu_i^2(G) \leq n^2/4.
\]
Proof Indeed, if $A$ is the adjacency matrix of $G$, then

$$\mu^2 + \sum_{i \in X} \mu_i^2 \leq \sum_{i=1}^{n} \mu_i^2 = \text{tr} (A^2) = 2e(G).$$

Hence, in view of $\mu \geq 2e(G)/n$, we find that

$$\sum_{i \in X} \mu_i^2 \leq 2e(G) - \mu^2 \leq 2e(G) - (2e(G)/n)^2 \leq n^2/4,$$

completing the proof.

Lemma 11 Let $n \geq s \geq 2$, and let $G$ be a graph of order $n$. If $\mu_s \leq 0$, then

$$|\mu_s| \leq \frac{n}{2\sqrt{n-s+1}}.$$

Proof Indeed, since $\mu_n(G) \leq \cdots \leq \mu_s(G) \leq 0$, we see that

$$\sum_{i=s}^{n} \mu_i^2 \geq \sum_{i=s}^{n} \mu_i^2 = (n-s+1) \mu_s^2,$$

and the assertion follows by Lemma 10.

Theorem 12 Let $k \geq 0$ and $n \geq 4^k$. If $G$ is a graph of order $n$, then either

$$\mu_{n-k+1}(G) \leq -1 \quad \text{and} \quad \mu_{n-k+1}(\overline{G}) \leq 0$$

or

$$\mu_{n-k+1}(\overline{G}) \leq -1 \quad \text{and} \quad \mu_{n-k+1}(G) \leq 0.$$

Proof A classical bound of Ramsey theory implies that every graph of order at least $4^k$ contains either a complete graph on $k+1$ vertices or an independent set on $k+1$ vertices. Suppose that $G$ contains a complete graph on $k+1$ vertices, and so $\overline{G}$ contains an independent set on $k+1$ vertices. For an induced subgraph $H$ of graph $G$, the Cauchy interlacing theorem implies that

$$\mu_{m-i}(H) \geq \mu_{m-i}(G)$$

for all $i = 0, \ldots, v(H) - 1$; therefore,

$$\mu_{n-k+1}(G) \leq \mu_2(K_{k+1}) = -1 \quad \text{and} \quad \mu_{n-k+1}(\overline{G}) \leq \mu_2(\overline{K_{k+1}}) = 0$$

as claimed.

Using Weyl’s inequalities ([M], p. 181), we come up with the following pair of useful bounds:

Lemma 13 (Weyl) If $G$ is a graph of order $n$ and $2 \leq k \leq n$, then

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \leq -1,$$  \hspace{1cm} (6)

$$\mu_k(G) + \mu_{n-k+1}(\overline{G}) \geq -1.$$  \hspace{1cm} (7)
2.2 Blow-ups of graphs

For any graph $G$ and integer $t \geq 1$, write $G^{(t)}$ for the graph obtained by replacing each vertex $u$ of $G$ by a set $V_u$ of $t$ independent vertices and every edge $\{u, v\}$ of $G$ by a complete bipartite graph with parts $V_u$ and $V_v$. Usually $G^{(t)}$ is called a blow-up of $G$. Blow-up graphs have a very useful algebraic relation to $G$: thus, if $A$ is the adjacency matrix of $G$, then the adjacency matrix $A(G^{(t)})$ of $G^{(t)}$ is given by

$$A(G^{(t)}) = A \otimes J_t$$

where $\otimes$ is the Kronecker product and $J_t$ is the all ones matrix of order $t$. This observation yields the following fact.

Proposition 14 The eigenvalues of $G^{(t)}$ are $t \mu_1(G), \ldots, t \mu_n(G)$, together with $n (t - 1)$ additional 0’s.

We also want to find the eigenvalues of the complements of graph blow-ups. Given a graph $G$ and an integer $t > 0$, set $G^{(t)} = \overline{G^{(t)}}$, i.e., $G^{(t)}$ is obtained from $G^{(t)}$ by joining all vertices within $V_u$ for every vertex $u$ of $G$. We easily can check the following fact.

Proposition 15 The eigenvalues of $G^{(t)}$ are $t \mu_1(G) + t - 1, \ldots, t \mu_n(G) + t - 1$, together with $n (t - 1)$ additional $(-1)$’s.

2.3 Proofs

Proof of Theorem 1 Let $2 \leq i \leq s$. First, we shall show that

$$\mu_i^2 \leq \mu_{s+i-1}^2 + \overline{\rho}_{n-i+2}^2. \quad (8)$$

Indeed, if $\mu_i \leq 0$, then (6) implies that $\overline{\rho}_{n-i+2} \leq 0$, and so

$$\mu_i^2 < (\mu_i + 1)^2 \leq \overline{\rho}_{n-i+2}^2.$$

On the other hand, if $\mu_i > 0$, then $\mu_{s+i-1} \leq \mu_i < 0$ implies that $\mu_i^2 \leq \mu_{s+i-1}^2$. So (8) is always true. Further, we obviously have

$$\mu^2 + \sum_{i=2}^{n} \mu_i^2 + \overline{\rho}^2 + \sum_{i=2}^{n} \overline{\rho}_i^2 = 2e(G) + 2e(\overline{G}) = n(n - 1).$$

Note that Weyl’s inequality (5) implies that

$$2 \sum_{i=2}^{s} \mu_i^2 \leq \sum_{i=2}^{s} \mu_i^2 + \sum_{i=s+1}^{2s-1} \mu_i^2 + \sum_{i=n-s+2}^{n} \overline{\rho}_i^2, \quad (9)$$

and by symmetry,

$$2 \sum_{i=2}^{s} \overline{\rho}_i^2 \leq \sum_{i=2}^{s} \overline{\rho}_i^2 + \sum_{i=s+1}^{2s-1} \mu_i^2 + \sum_{i=n-s+2}^{n} \mu_i^2. \quad (10)$$

Further, the condition $n \geq 3s - 2$, implies that $n - s + 2 > 2s - 1$ and so

$$\{s + 1, \ldots, 2s - 1\} \cap \{n - s + 2, \ldots, n\} = \emptyset.$$

Therefore, adding (9) and (10) together with $\mu^2$ and $\overline{\rho}^2$, we see that

$$\mu^2 + 2 \sum_{i=2}^{s} \mu_i^2 + \overline{\rho}^2 + 2 \sum_{i=2}^{s} \overline{\rho}_i^2 \leq \sum_{i=1}^{n} \mu_i^2 + \sum_{i=1}^{n} \overline{\rho}_i^2 \leq n(n - 1).$$
Finally, using (11) we find that $\mu^2 + \mu^2 \geq (\mu + \mu)^2 / 2 \geq (n-1)^2 / 2$, and so 
$$
\sum_{i=2}^{s} \mu_i^2 + \sum_{i=2}^{s} \mu_i^2 \leq \frac{1}{2} n (n-1) - \frac{1}{4} (n-1)^2 = \frac{n^2 - 1}{4} < \frac{n^2}{4},
$$
completing the proof of Theorem 1.

\[\square\]

**Proof of Theorem 3** If $\mu_s(G) \geq 0$ and $\mu_s(G) \geq 0$, then

$$
\sum_{i=2}^{s} (\mu_i^2 + \mu_i^2) \geq (s-1) (\mu_s^2 + \mu_s^2),
$$
and inequality (3) follows by Theorem 1.

Next, if $\mu_s < 0$ and $\mu_s < 0$, then Lemma 11 implies that $\mu_s^2 + \mu_s^2 \leq n^2 / 2 (n-s+1) < n^2 / 4 (s-1)$, so (3) follows in this case as well.

Finally, assume that $\mu_s < 0$ and $\mu_s < 0$. Then $\mu_{2s-1} \leq \cdots \leq \mu_{s+1} \leq \mu_s < 0$ and so,

$$
(s-1) \mu_s^2 \leq \mu_{s+1}^2 + \cdots + \mu_{2s-1}^2.
$$

(11)

Since $\mu_2 \geq \cdots \geq \mu_s \geq 0$, inequality (3) implies that $\mu_{n-i+2} \leq 0$, and so

$$
\mu_i^2 \leq (\mu_i + 1)^2 \leq \mu_{n-i+2}^2
$$

for every $i = 2, \ldots, s$. Therefore

$$
(s-1) \mu_s^2 \leq \mu_{n}^2 + \cdots + \mu_{n-s+2}^2.
$$

(12)

Since the condition $n \geq 3s-2$, implies that $n-s+2 > 2s-1$, we see that

$$
\{s+1, \ldots, 2s-1\} \cap \{n-s+2, \ldots, n\} = \emptyset.
$$

Hence, setting $X := \{s+1, \ldots, 2s-1\} \cup \{n-s+2, \ldots, n\}$, inequalities (11), (12) and Lemma 11 imply that

$$
(s-1) (\mu_s^2 + \mu_s^2) \leq \sum_{i=s+1}^{2s-1} \mu_i^2 + \sum_{i=n-s+2}^{n} \mu_i^2 = \sum_{i \in X} \mu_i^2 \leq \frac{n^2}{4},
$$

completing the proof of Theorem 3.

\[\square\]

**Proof of Theorem 4** Assume that $s \geq 2$, $n \geq 15 (s-1)$, and $G$ is a graph of order $n$. As mentioned above, using the AM - QM inequality and Theorem 3 we always have

$$
|\mu_s(G)| + |\mu_s(G)| \leq \frac{n}{\sqrt{2 (s-1)}},
$$

(13)

To the end of the proof we shall show that we can add a $-1$ to the right side of this inequality. Thus, let $G$ be a graph of order $n$ with

$$
|\mu_s(G)| + |\mu_s(G)| = f_s(n),
$$

and assume for a contradiction that

$$
f_s(n) > \frac{n}{\sqrt{2 (s-1)}} - 1,
$$

(14)
Our first aim is to show that $\mu_s > 0$ and $\overline{\mu}_s > 0$. Indeed, if both $\mu_s$ and $\overline{\mu}_s$ are non-positive, then Lemma 11 implies that

$$|\mu_s(G)| + |\mu_s(G)| < \frac{n}{\sqrt{n - s + 1}} \leq \frac{n}{\sqrt{2(s - 1)}} - 1.$$ 

Now, let $\mu_s > 0$ and $\overline{\mu}_s \leq 0$. Then, Lemmas 10 and 11 imply that

$$|\mu_s| \leq \frac{n}{2\sqrt{s - 1}} \quad \text{and} \quad |\overline{\mu}_s| \leq \frac{n}{2\sqrt{n - s + 1}},$$

and so, in view of $n \geq 15(s - 1)$, we see that

$$\frac{n}{2\sqrt{s - 1}} + \frac{n}{2\sqrt{n - s + 1}} < \frac{n}{\sqrt{2(s - 1)}} - 1,$$

contradicting (13). Therefore $\mu_s > 0$ and $\overline{\mu}_s > 0$.

Now, let $t$ be a positive integer and set $H := G^{(t)}$. Since $\mu_s > 0$ and $\overline{\mu}_s > 0$, Propositions 14 and 15 imply that

$$\mu_s(H) = t\mu_s \quad \text{and} \quad \overline{\mu}_s(H) = \mu_s(G^{[t]}) = t\overline{\mu}_s + t - 1,$$

and therefore

$$|\mu_s(H)| + |\mu_s(H)| = tf_s(n) + t - 1.$$

On the other hand, (13) implies that

$$|\mu_s(H)| + |\mu_s(H)| \leq \frac{tn}{\sqrt{2(s - 1)}},$$

hence

$$f_s(n) \leq \frac{n}{\sqrt{2(s - 1)}} - \frac{t - 1}{t}.$$

Now, letting $t$ tend to $\infty$, we obtain a contradiction to (14), and thus complete the proof of Theorem 4.

**Proof of Theorem 5** We start with the obvious fact

$$\sum_{i=2}^{n} (\mu_i + \overline{\mu}_i) = -\mu - \overline{\mu}. \quad (15)$$

For $i = 2, \ldots, n$, set $w_i := \mu_i + \overline{\mu}_{n-i+2}$. Rearranging (16) and using (10), we find that

$$\sum_{i=2}^{s+1} (w_i + w_{n-i+2}) = -\mu - \overline{\mu} - \sum_{i=s+2}^{n-s} w_i \geq -\mu - \overline{\mu} + n - 2s - 1.$$ 

On the other hand,

$$\mu_i^2 = (w_i - w_{n-i+2})^2 = w_i^2 - 2w_i\overline{\mu}_{n-i+2} + \overline{\mu}_{n-i+2}^2 > \overline{\mu}_{n-i+2}^2 - 2w_i\overline{\mu}_{n-i+2}.$$ 

Since $w_i < 0$ and $\overline{\mu}_{n-i+2} \geq -n/2$ (by Lemma 11), we see that

$$\mu_i^2 \geq \overline{\mu}_{n-i+2}^2 + nw_i.$$
Therefore,
\[
\sum_{i=2}^{s+1} (\mu_i^2 + \overline{\mu}_i^2) \geq \sum_{i=2}^{s+1} (\mu_{n-i+2}^2 + \overline{\mu}_{n-i+2}^2) + n \sum_{i=2}^{s+1} (w_i + w_{n-i+2})
\]
\[
= \sum_{i=1}^{s} (\mu_{n-i+1}^2 + \overline{\mu}_{n-i+1}^2) + n \sum_{i=2}^{s+1} (w_i + w_{n-i+2})
\]
\[
\geq \sum_{i=1}^{s} (\mu_{n-i+1}^2 + \overline{\mu}_{n-i+1}^2) + n (n - 2s - 1 - \mu - \overline{\mu}).
\]

Using that \( n > 2s \), we see that
\[
\mu^2 + \overline{\mu}^2 + \sum_{i=2}^{s+1} (\mu_i^2 + \overline{\mu}_i^2) + \sum_{i=1}^{s} (\mu_{n-i+1}^2 + \overline{\mu}_{n-i+1}^2) \leq n (n - 1),
\]
and so,
\[
\mu^2 + \overline{\mu}^2 + 2 \sum_{i=1}^{s} (\mu_{n-i+1}^2 + \overline{\mu}_{n-i+1}^2) + n (n - 2s - 1 - \mu - \overline{\mu}) \leq n (n - 1).
\]
Rearranging this inequality, we find that
\[
2 \sum_{i=1}^{s} (\mu_{n-i+1}^2 + \overline{\mu}_{n-i+1}^2) \leq 2sn + n (\mu + \overline{\mu}) - \mu^2 - \overline{\mu}^2
\]
\[
\leq 2sn + n (\mu + \overline{\mu}) - (\mu + \overline{\mu})^2 / 2
\]
\[
\leq 2sn + n^2 / 2,
\]
completing the proof of Theorem 5.

Proof of Corollary 7 Since Theorem 12 implies that \( \mu_{n-s+1} \leq 0 \) and \( \overline{\mu}_{n-s+1} \leq 0 \), we get
\[
\sum_{i=1}^{s} (\mu_{n-i+1}^2 + \overline{\mu}_{n-i+1}^2) \geq s (\mu_{n-s+1}^2 + \overline{\mu}_{n-s+1}^2),
\]
and the assertion follows by Theorem 5.

Proof of Theorem 8 Assume that \( s \geq 1, n \geq 4^s \), and \( G \) is a graph of order \( n \) with
\[
|\mu_{n-s+1}(G)| + |\mu_{n-s+1}(\overline{G})| = f_{n-s}(n).
\]
To begin with, using the AM - QM inequality and Corollary 7, we see that
\[
f_{n-s}(n) \leq \frac{n}{\sqrt{2s}} + \sqrt{2s}.
\]
Note that this inequality is almost what we need, as the main term is the correct one, but the constant term is larger than desired. Thus, to the end of the proof we shall show that we can make the additive term equal to 1. We shall use the same techniques as in the proof of Theorem 4.

Assume for a contradiction that
\[
f_{n-s}(n) > \frac{n}{\sqrt{2s}} + 1, \quad (16)
\]

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First, Theorem 12 implies that $\overline{\mu_{n-s+1}} \leq 0$, and so Lemma 11 implies that $|\overline{\mu_{n-s+1}}| \leq n/(2\sqrt{s})$. Hence, 

$$|\mu_{n-s+1}| > \frac{n}{\sqrt{2s}} + 1 - |\overline{\mu_{n-s+1}}| > \frac{n}{\sqrt{2s}} + 1 - \frac{n}{2\sqrt{s}} > 1,$$

and, by symmetry, $|\overline{\mu_{n-s+1}}| > 1$. That is to say, $\mu_{n-s+1} < -1$ and $\overline{\mu_{n-s+1}} < -1$.

Let $t$ be a positive integer and set $H := G^{(t)}$. Since $\mu_{n-s+1} < -1$ and $\overline{\mu_{n-s+1}} < -1$, Propositions 14 and 15 imply that $\mu_{tn-s+1} = t\mu_{n-s+1}$ and $\mu_{tn-s+1}(H) = t\mu_{n-s+1}(G^{(t)}) = t\overline{\mu_{n-s+1}} + t - 1$,

and therefore

$$|\mu_{tn-s+1}(H)| + |\mu_{tn-s+1}(\overline{H})| = tf_{n-s}(n) - t + 1.$$

On the other hand, Corollary 7 implies that

$$|\mu_{tn-s+1}(H)| + |\mu_{tn-s+1}(\overline{H})| \leq \frac{tn}{\sqrt{2s}} + \sqrt{2s};$$

hence

$$f_{n-s}(n) \leq \frac{n}{\sqrt{2s}} + \frac{\sqrt{2s}}{t} + \frac{t - 1}{t}.$$

Now, letting $t$ tend to $\infty$, we obtain a contradiction to (16), and thus complete the proof of Theorem 8.

\[ \square \]

3 Lower bounds on $f_s(n)$ and $f_{n-s}(n)$

Define an infinite sequence of square (0,1) matrices $A_1, A_2, \ldots$ as follows. Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and for every $k = 1, 2, \ldots$ set

$$A_{k+1} = \frac{1}{2}((2A_k - J_{2^k}) \otimes B + J_{2^{k+1}}).$$

First note that $A_{k+1}$ is a $(0,1)$ symmetric matrix of order $2^{k+1}$.

To give some properties of the eigenvalues of the matrices $A_{k+1}$ we first point out a fact without a proof.

**Lemma 16** Let $M$ be a symmetric real matrix of order $n$ with all row-sums equal to $r$, and $r, \mu_2(M), \ldots, \mu_n(M)$ be the eigenvalues of $M$. If $a$ and $b$ are real numbers, then the eigenvalues of the matrix $aM + bJ_n$ are

$$ar + bn, a\mu_2(M), \ldots, a\mu_n(M).$$

In the following lemma and its proof we shall use $a^{[b]}$ to indicate an eigenvalue $a$ of multiplicity $b$.

**Lemma 17** If $k \geq 1$, then the row-sums of $A_{k+1}$ are equal to $2^k$ and its spectrum is

$$2^k, \left(\frac{2^k}{2}\right)^{[2^k-1]} , \left(\frac{2^k}{2}\right)^{[-2^{k-1}]}.$$
Proof We shall prove the lemma by induction on \( k \). If \( k = 1 \), we see that

\[
A_2 = \frac{1}{2} \left( (2A_1 - J_2) \otimes B + J_4 \right) = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}.
\]

The row-sums of \( A_2 \) are equal to 2, and its eigenvalues are 2, \( \sqrt{2} \), 0 and \(-\sqrt{2}\). Assume that the statement holds for \( A_k \); in particular, the row-sums of \( A_k \) are equal to \( 2^{k-1} \). Hence, the row-sums of both \( 2A_k - J_{2k} \) and \( (2A_k - J_{2k}) \otimes B \) are zero, and so the row-sums of \( A_{k+1} \) are \( 2^k \), proving the first part of the statement.

Further, the spectrum of \( A_k \) is

\[
2^{k-1}, \left( 2^{(k-1)/2} \right)^{[2^{k-2}]}, 0^{[2^{k-1}-1]}, \left( -2^{(k-1)/2} \right)^{[2^{k-2}]}, \text{ and so by Lemma 16 the spectrum of } 2A_k - J_{2k} \text{ is}
\]

\[
\left( 2^{(k+1)/2} \right)^{[2^{k-2}]}, 0^{[2^{k-1}]}, \left( -2^{(k+1)/2} \right)^{[2^{k-2}]}, \text{ and the same bound holds also for } A_{k+1} \text{ is}
\]

\[
2^k, \left( 2^{k/2} \right)^{[2^{k-1}]}, 0^{[2^{k-1}]}, \left( -2^{k/2} \right)^{[2^{k-1}]},
\]

completing the induction step and the proof of Lemma 17. 

If \( P \) is a Hermitian matrix of size \( n \), we index the eigenvalues of \( P \) as \( \mu_1 (P) \geq \cdots \geq \mu_n (P) \). Observe the following special case of Weyl’s inequalities that we shall need in the proof of Theorem 9.

If \( P \) and \( Q \) are Hermitian matrices of size \( n \), then for each \( 1 \leq s \leq n \)

\[
\mu_s (P - Q) \leq \mu_s (P) - \mu_s (Q) \leq \mu_1 (P - Q).
\]

Proof of Theorem 9 Let \( A_{k+1} (t) \) be matrix obtained from \( A_{k+1} \otimes J_t \) by zeroing all diagonal entries. Clearly \( A_{k+1} (t) \) is the adjacency matrix of a graph \( G \) of order \( n = 2^{k+1} t \). For \( s = 2^{k-1} + 1 \), and for each \( i \leq s \) we have

\[
\mu_i (G) \geq \mu_s (A_{k+1} \otimes J_t) - \mu_1 (A_{k+1} \otimes J_t - A_{k+1} (t)) = 2^{k/2} t - 1 = \frac{v (G)}{2\sqrt{2} (s - 1)} - 1.
\]

Next, it is not hard to see that the adjacency matrix of \( \overline{G} \) is obtained from \( (J_{2k+1} - A_{k+1}) \otimes J_t \) by zeroing all diagonal entries. Since \( A_{k+1} \) and \( J_{2k+1} - A_{k+1} \) have the same spectrum, we also find that

\[
\mu_i (\overline{G}) \geq \mu_s ((J_{2k+1} - A_{k+1}) \otimes J_t) - 1 = 2^{k/2} t - 1 = \frac{v (G)}{2\sqrt{2} (s - 1)} - 1.
\]

Finally, we have

\[
\mu_{n-i+2} (G) \leq \mu_{n-s+2} (A_{k+1} \otimes J_t) - \mu_n (A_{k+1} \otimes J_t - A_{k+1} (t)) = -2^{k/2} t = -\frac{v (G)}{2\sqrt{2} (s - 1)},
\]

and the same bound holds also for \( \mu_{n-i+2} (\overline{G}) \). The proof of Theorem 9 is completed.

\[\square\]
4 Concluding remarks

We would like to emphasize the decisive role of Weyl’s inequalities in our proofs. It turns out that they offer almost unlimited possibilities for variations. The upper bounds on $f_s(n)$ determined in Corollary 4 and Theorem 5 seem asymptotically tight for every $s$ and $n$ tending to infinity. However, if would be hard to disprove such conjecture if it turns out to be false. Thus, we raise the following problem.

**Problem 18** For which values of $s$ it is true that

$$
\lim_{n \to \infty} \frac{1}{n} f_s(n) = \frac{1}{\sqrt{2(s-1)}} ?
$$

(17)

For which values of $s$ it is true that

$$
\lim_{n \to \infty} \frac{1}{n} f_{n-s}(n) = \frac{1}{\sqrt{2s}} ?
$$

(18)

Further, for those $s$ for which the answer to of the above problem is positive, we can ask subtler and more definite questions.

**Problem 19** If $s$ is such that equality holds in (17), is it true that

$$
\lim_{n \to \infty} \left( f_s(n) - \frac{n}{\sqrt{2(s-1)}} \right) = -1 ?
$$

If $s$ is such that equality holds in (18), is it true that

$$
\lim_{n \to \infty} \left( \frac{1}{n} f_{n-s}(n) - \frac{1}{\sqrt{2s}} \right) = 1 ?
$$

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