Some Results on Inner Quasidiagonal $C^*$-algebras

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Abstract In the current article, we prove the crossed product $C^*$-algebra by a Rokhlin action of finite group on a strongly quasidiagonal $C^*$-algebra is strongly quasidiagonal again. We also show that a just-infinite $C^*$-algebra is quasidiagonal if and only if it is inner quasidiagonal. Finally, we compute the topological free entropy dimension in just-infinite $C^*$-algebras.

Keywords Inner quasidiagonal $C^*$-algebras, crossed product $C^*$-algebras, strongly quasidiagonal $C^*$-algebras, just-infinite $C^*$-algebras, topological free entropy dimension

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1 Introduction

To distinguish the class of NF algebras and the class of strong NF algebras, Blackadar and Kirchberg introduced the concept of inner quasidiagonal $C^*$-algebras in [3]. From its definition, it is apparent that the class of inner quasidiagonal $C^*$-algebras is a subclass of quasidiagonal $C^*$-algebras. Many basic properties of inner quasidiagonal $C^*$-algebras have been discussed in [3] and [4]. It was also shown that a separable $C^*$-algebra is a strong NF algebra if and only if it is nuclear and inner quasidiagonal. Therefore the class of all strong NF algebras is strictly contained in the class of nuclear and quasidiagonal $C^*$-algebras (i.e., NF algebras). Examples of separable nuclear $C^*$-algebras which are quasidiagonal but not inner quasidiagonal were given in the same article. And we also know that all separable simple quasidiagonal $C^*$-algebras are inner quasidiagonal, all strongly quasidiagonal $C^*$-algebras are inner quasidiagonal. Recall that a $C^*$-algebra is called strongly quasidiagonal if it is separable and all its representations are quasidiagonal.

Since not all $C^*$-subalgebras of inner quasidiagonal $C^*$-algebras are inner, the crossed product $C^*$-algebra by an action of a finite group on an inner quasidiagonal $C^*$-algebra may not be inner quasidiagonal again. In [22], it was shown that the crossed product $C^*$-algebra by a Rokhlin action of a finite group on a unital inner quasidiagonal $C^*$-algebra is inner again. So

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it is natural to ask whether the same conclusion still holds for non-unital inner quasidiagonal $C^*$-algebras. In the current paper, we will prove that the crossed product $C^*$-algebra by a Rokhlin action of finite group on a strongly quasidiagonal $C^*$-algebra (may not be unital) is strongly quasidiagonal again.

Just-infinite $C^*$-algebras were first introduced by Grigorchuk, Musat and Rørdam, in [9] as an analogous notion of the well-established notions of just-infinite groups and just-infinite (abstract) algebras. A $C^*$-algebra is called just-infinite if it is infinite-dimensional and all its proper quotient are finite-dimensional. The necessary and sufficient conditions for a $C^*$-algebra to be just-infinite were also given in the same article when the algebras are separable. By using the characterizations of the just-infinite $C^*$-algebras given in [9], we will prove that every separable just-infinite $C^*$-algebra is quasidiagonal if and only if it is inner quasidiagonal.

At last, we analyze the topological free entropy dimension in just-infinite $C^*$-algebras. The notion of topological free entropy dimension $\delta_{\text{top}}$ of $n$-tuples of elements in a unital $C^*$-algebra was introduced by Voiculescu in [21], where some basic properties of free entropy dimension are discussed. For more information about $\delta_{\text{top}}$, we refer the readers to [12, 14, 15].

The organization of the paper is as follows. In Section 2, we first recall some definitions and fix some notation. Then we prove that the crossed product $C^*$-algebra by a Rokhlin action of finite group on a strongly quasidiagonal $C^*$-algebra is strongly quasidiagonal again. At last, we prove that every just-infinite $C^*$-algebra is quasidiagonal if and only if it is inner quasidiagonal. In Section 3, we compute the topological free entropy dimension $\delta_{\text{top}}$ in just-infinite $C^*$-algebras.

2 Inner Quasidiagonal $C^*$-algebras

2.1 Definitions and Preliminaries

The concept of inner quasidiagonal $C^*$-algebras was given by Blackadar and Kirchberg in [3], which is obtained by slightly modifying the Voiculescu’s characterization of quasidiagonal $C^*$-algebras.

Definition 2.1 ([3]) A $C^*$-algebra $A$ is inner quasidiagonal if, for every $x_1, \ldots, x_n$ in $A$ and $\varepsilon > 0$, there is a representation $\pi$ of $A$ on a Hilbert space $\mathcal{H}$, and a finite-rank projection $P \in \pi(A)''$ such that $\|P\pi(x_i) - \pi(x_i)P\| < \varepsilon, \|P\pi(x_i)P\| > \|x_i\| - \varepsilon$ for $1 \leq i \leq n$.

The next result gives us a characterization of inner quasidiagonal $C^*$-algebras.

Theorem 2.2 ([4]) A separable $C^*$-algebra is inner quasidiagonal if and only if it has a separating family of quasidiagonal irreducible representations.

Now let us recall the concept of completely positive maps. A map $\varphi$ from $C^*$-algebra $A$ to a $C^*$-algebra $B$ is said to be completely positive if $\varphi_n : M_n(A) \to M_n(B)$, defined by

$$\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})],$$

is positive for every $n$. We use c.p. to abbreviate “completely positive”, u.c.p. for “unital completely positive” and c.c.p. for “contractive completely positive”.

Definition 2.3 Let $A$ and $B$ be $C^*$-algebras and $\varphi : A \to B$ be a c.c.p. map. The $C^*$-subalgebra

$$A_\varphi = \{a \in A : \varphi(a^*a) = \varphi(a)^*\varphi(a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a)^*\}$$
is called the multiplicative domain of $\varphi$. It is well-known that $A_{\varphi}$ is the largest subalgebra of $A$ on which $\varphi$ restricts to a $*$-homomorphism.

Suppose $\pi : B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H})$ is the canonical mapping onto the Calkin algebra.

**Lemma 2.4** (Stinespring) Let $A$ be a unital $C^*$-algebra and $\varphi : A \to B(\mathcal{H})$ be a c.p. map. Then, there exist a Hilbert space $\hat{\mathcal{H}}$, a $*$-representation $\pi_\varphi : A \to B(\hat{\mathcal{H}})$ and an operator $V : \mathcal{H} \to \hat{\mathcal{H}}$ such that

$$\varphi(a) = V^*\pi_\varphi(a)V$$

for every $a \in A$. In particular, $\|\varphi\| = \|V^*V\| = \|\varphi(1)\|$. We call the triplet $(\pi_\varphi, \hat{\mathcal{H}}, V)$ in the preceding lemma a Stinespring dilation of $\varphi$. When $\varphi$ is unital, $V^*V = \varphi(I) = I$, and hence $V$ is an isometry. So in this case we may assume that $\mathcal{H} \subseteq \hat{\mathcal{H}}$ and $V$ is a projection $P$ on $\mathcal{H}$ such that $\varphi(a) = P\pi_\varphi(a)|_\mathcal{H}$.

So the next result gives us another characterization of inner quasidiagonal $C^*$-algebras.

**Theorem 2.5** ([6]) A unital $C^*$-algebra $A$ is inner quasidiagonal if and only if there is a sequence of u.c.p. maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $\|a\| = \lim \|\varphi_n(a)\|$ and $\text{dist}(a, A_{\varphi_n}) \to 0$ for all $a \in A$ where $A_{\varphi_n}$ is the multiplicative domain of $\varphi_n$.

Recall that a faithful representation of a $C^*$-algebra $A$ is called essential if $\pi(A)$ contains no nonzero finite rank operators.

**Proposition 2.6** (1.7, [19]) Let $\pi : A \to B(\mathcal{H})$ be a faithful essential representation. Then $A$ is quasidiagonal if and only if $\pi(A)$ is a quasidiagonal set of operators.

A $C^*$-algebra $A$ is called antiliminal if $A$ contains no nonzero abelian elements, i.e., if every nonzero hereditary $C^*$-subalgebra of $A$ is noncommutative. Recall that a $C^*$-algebra is called primitive if it admits a faithful irreducible representation. It is said to be prime if, whenever $\mathcal{I}$ and $\mathcal{J}$ are closed two-sided ideals in $A$ such that $\mathcal{I} \cap \mathcal{J} = 0$, then either $\mathcal{I} = 0$, or $\mathcal{J} = 0$. It is well-known that a separable $C^*$-algebra is prime if and only if it is primitive. Combining Theorem 2.2, Proposition 2.6 and IV.1.1.7 in [1], we can quickly get the following lemma.

**Lemma 2.7** ([3, Corollary 2.6]) Every separable antiliminal quasidiagonal prime $C^*$-algebra is inner. Every separable simple quasidiagonal $C^*$-algebra is inner quasidiagonal.

**Remark 2.8** In [3], it has been shown that an arbitrary inductive limit (with injective connecting maps) of inner quasidiagonal $C^*$-algebras is inner quasidiagonal. Every residually finite-dimensional $C^*$-algebra is inner quasidiagonal.

### 2.2 Crossed Product $C^*$-algebras by Actions of Finite Groups

Even though we can regard a crossed product $C^*$-algebra by an action of a finite group on an inner quasidiagonal $C^*$-algebra as a subalgebra of the tensor product of this inner quasidiagonal algebra and a matrix algebra, we can not tell that this crossed product $C^*$-algebra is inner quasidiagonal again since a subalgebra of an inner quasidiagonal algebra may not be inner quasidiagonal again.

**Remark 2.9** Let $A \subseteq B(\mathcal{H})$ be a separable inner quasidiagonal $C^*$-algebra and $\alpha : G \to \text{Aut}(A)$ be an action of finite group $G$ on $A$. Suppose $\pi : A \to B(\mathcal{K})$ is a $*$-homomorphism and $\{e_{g,a}\}$ is the family of canonical matrix units of $B(l^2(G))$. Then we define an action
\[ \beta : G \to \text{Aut}(\pi(A)) \]
by setting
\[ \beta_g(\pi(a)) = \pi(\alpha_g(a)). \]

Therefore it is not hard to verify that the mapping \( \rho : A \rtimes_\alpha G \to \pi(A) \rtimes_\beta G \) by letting

\[
\rho(I_\mathcal{H} \otimes \lambda_g) = I_K \otimes \lambda_g \quad \text{and} \quad \rho \left( \sum_{g \in G} \alpha_g^{-1}(a) \otimes e_{g,g} \right) = \sum_{g \in G} \beta_g^{-1}(\pi(a)) \otimes e_{g,g}
\]
is a \( \ast \)-homomorphism again.

**Theorem 2.10** Let \( A \) be a unital separable \( C^* \)-algebra. If there are a sequence of u.c.p maps \( \varphi_n : A \to \mathcal{M}_k(\mathbb{C}) \) such that \( \|a\| = \lim \|\varphi_n(a)\| \) and \( d(a,A_{\varphi_n}) \to 0 \) for all \( a \in A \) and an automorphic representation \( \alpha : G \to \alpha_g \) of finite group \( G \) on \( A \) satisfying that \( A_{\varphi_n} \) is \( \alpha_g \) invariant (i.e., \( \alpha_g|_{A_{\varphi_n}} \) is an automorphism of \( A_{\varphi_n} \)) for each \( n \) and \( g \in G \), then \( A \rtimes_\alpha G \) is inner quasidiagonal again.

**Proof** By Lemma 2.4, for each \( \varphi_n \), there are a Hilbert space \( \mathcal{H}_n \) containing a copy of \( \mathbb{C}^{k_n} \) and a representation \( \pi_n : A \to \mathcal{B}(\mathcal{H}_n) \) such that \( \varphi_n(a) = P_n \pi_n(a) P_n \) for \( a \in A \) where \( P_n \) is the finite-rank projection from \( \mathcal{H}_n \) onto \( \mathbb{C}^{k_n} \). Then there is a \( \ast \)-homomorphism

\[
\rho_n : A \rtimes_\alpha G \to \pi_n(A) \rtimes_\beta_n G,
\]

where \( \beta_n \) and \( \rho_n \) are defined in the way introduced by Remark 2.9. Note \( P_n \otimes I_{|G|} \) is a finite-rank projection where \( I_{|G|} \) is the identity in \( \mathcal{M}_{|G|}(\mathbb{C}) \) and then the map \( \hat{\varphi}_n \) defined by the form

\[
\hat{\varphi}_n(c) = (P_n \otimes I_{|G|}) \rho_n(c)(P_n \otimes I_{|G|}) \quad \text{for every} \quad c \in A \rtimes_\alpha G
\]
is a u.c.p. map. It is obvious that \( I_A \otimes \lambda_g \in (A \rtimes_\alpha G) \hat{\varphi}_n \), where \( (A \rtimes_\alpha G) \hat{\varphi}_n \) is the multiplicative domain of \( \hat{\varphi}_n \). For \( a,b \in A_{\varphi_n} \), we have \( \alpha_g(a) \) and \( \alpha_g(b) \) are both in \( A_{\varphi_n} \) by the fact that \( \alpha_g|_{A_{\varphi_n}} \) is an automorphism of \( A_{\varphi_n} \) for each \( n \) and \( g \in G \), then

\[
\hat{\varphi}_n \left( \sum_{g \in G} \alpha_g^{-1}(a) \otimes e_{g,g} \cdot \sum_{g \in G} \alpha_g^{-1}(b) \otimes e_{g,g} \right)
\]

\[
= (P_n \otimes I_{|G|}) \rho_n \left( \sum_{g \in G} \alpha_g^{-1}(ab) \otimes e_{g,g} \right) (P_n \otimes I_{|G|})
\]

\[
= \sum_{g \in G} P_n \beta_g^{-1}(\pi(ab)) P_n \otimes e_{g,g}
\]

\[
= \sum_{g \in G} P_n \pi(\alpha_g^{-1}(ab)) P_n \otimes e_{g,g}
\]

\[
= \left( \sum_{g \in G} P_n(\beta_g^{-1}(\pi(a))) P_n \otimes e_{g,g} \right) \left( \sum_{g \in G} P_n(\beta_g^{-1}(\pi(a))) P_n \otimes e_{g,g} \right)
\]

\[
= \hat{\varphi}_n \left( \sum_{g \in G} \alpha_g^{-1}(a) \otimes e_{g,g} \right) \hat{\varphi}_n \left( \sum_{g \in G} \alpha_g^{-1}(b) \otimes e_{g,g} \right)
\]

and it is easy to check

\[
\hat{\varphi}_n \left( \sum_{g \in G} \alpha_g^{-1}(a) \otimes e_{g,g} \right) \cdot (I_A \otimes \lambda_g) = \hat{\varphi}_n \left( \sum_{g \in G} \alpha_g^{-1}(a) \otimes e_{g,g} \right) \hat{\varphi}_n(I_A \otimes \lambda_g).
\]

Therefore

\[
(A \rtimes_\alpha G) \hat{\varphi}_n \supseteq A_{\varphi_n} \rtimes_\alpha G \quad \text{for every} \quad g \in G. \tag{2.1}
\]
Since $(I \otimes \lambda_g)(I \otimes \lambda_h) = I \otimes \lambda_{gh}$ and
\[
(I \otimes \lambda_h) \left( \sum_{g \in G} \alpha_g^{-1}(a) \otimes e_{g,g} \right) (I \otimes \lambda_h^n) = \sum_{g \in G} \alpha_g^{-1}(\alpha_h(a)) \otimes e_{g,g},
\]
we only need to prove $\text{dist}(c, (A \rtimes G)\hat{\varphi}_n) \to 0$ and $\lim \sup \|\varphi_n(c)\| = \|c\|$ when $c$ is in the form
\[
c = \left( \sum_{g \in G} \alpha_g^{-1}(a) \otimes e_{g,g} \right) (I \otimes \lambda_h)
\]
without loss of generality. Then by Theorem 2.5, $A \rtimes G$ is inner quasidiagonal.

Now for $c = (\sum_{g \in G} \alpha_g^{-1}(a) \otimes e_{g,g})(I \otimes \lambda_h)$, we have $\text{dist}(a, A_{\varphi_n}) \to 0$ for all $a \in A$ from the assumption. Then there are $a_n \in A_{\varphi_n}$ for $n \in \mathbb{N}$ such that $\|a - a_n\| \to 0$. Note
\[
\left( \sum_{g \in G} \alpha_g^{-1}(a_n) \otimes e_{g,g} \right) (I \otimes \lambda_h) \in (A \rtimes G)\hat{\varphi}_n
\]
by (2.1). Then
\[
\|c - \left( \sum_{g \in G} \alpha_g^{-1}(a_n) \otimes e_{g,g} \right) (I \otimes \lambda_h)\|
= \left\| \left( \sum_{g \in G} \alpha_g^{-1}(a - a_n) \otimes e_{g,g} \right) (I \otimes \lambda_h) \right\|
\leq \|a - a_n\| \to 0.
\]
Therefore $\text{dist}(c, (A \rtimes G)\hat{\varphi}_n) \to 0$. By $\|a\| = \lim \sup \|\varphi_n(a)\|$ in the assumption, we also have
\[
\lim \sup \|\varphi_n(c)\| = \lim \sup \left\| (P_n \otimes I_{[G]}) \left( \sum_{g \in G} \pi(\alpha_g^{-1}(a)) \otimes e_{g,g} \right) (P_n \otimes I_{[G]})\hat{\varphi}_n (I \otimes \lambda_h) \right\|
= \lim \sup \left\| \sum_{g \in G} \varphi_n(\alpha_g^{-1}(a)) \otimes e_{g,g} \right\|
= \left\| \sum_{g \in G} (\alpha_g^{-1}(a)) \otimes e_{g,g} \right\|
= \left\| \sum_{g \in G} (\alpha_g^{-1}(a)) \otimes e_{g,g} (I \otimes \lambda_h) \right\|
= \|c\|.
\]
Hence by Lemma 2.5, $A \rtimes G$ is inner quasidiagonal again. \qed

In [22], Rokhlin actions have been considered for unital inner quasidiagonal $C^*$-algebras. In this section, we are going to consider the Rokhlin actions on strongly quasidiagonal $C^*$-algebras where $C^*$-algebras may not be unital. Note every strongly quasidiagonal $C^*$-algebra is inner quasidiagonal.

**Lemma 2.11** Let $A$ be a strongly quasidiagonal $C^*$-algebra and $B \subseteq A$ a hereditary $C^*$-subalgebra. Then $B$ is strongly quasidiagonal again.

**Proof** By Proposition 2.8 in [3], we only need to show $\pi(B)$ is a quasidiagonal set of operators for any irreducible representation $\pi$ of $B$. 

Some Results on Inner Quasidiagonal C*-algebras

Let \( \pi : \mathcal{B} \to \mathcal{B}(\mathcal{H}) \) be an irreducible representation of \( \mathcal{B} \). Then we can extend \( \pi \) to an irreducible representation \( \tilde{\pi} : \mathcal{A} \to \mathcal{B}(\tilde{\mathcal{H}}) \) with \( \mathcal{H} \subseteq \tilde{\mathcal{H}} \) and \( \tilde{\pi}(x)|_{\mathcal{H}} = \pi(x) \) for all \( x \in \mathcal{B} \) by Proposition 2.10.2 in [8]. From the fact that \( \mathcal{A} \) is strongly quasidiagonal, we know \( \tilde{\pi}(\mathcal{A}) \) is a quasidiagonal set of operators and then \( \tilde{\pi}(\mathcal{B}) \) is a quasidiagonal set of operators too. Since \( \mathcal{B} \) is hereditary, by II.6.1.9 in [1], \( \tilde{\pi}(\mathcal{B})|_{\tilde{\mathcal{H}}} \) is irreducible where \( \tilde{\mathcal{H}} \subseteq \tilde{\mathcal{H}} \) is the essential subspace (see II.6.1.5 in [1]) of \( \tilde{\pi}(\mathcal{B}) \). Therefore

\[
\tilde{\pi}(\mathcal{B}) = \tilde{\pi}(\mathcal{B})|_{\tilde{\mathcal{H}}} \oplus 0|_{\tilde{\mathcal{H}}^\perp}.
\]

Since the projection \( p_{\mathcal{H}} \) on \( \mathcal{H} \) is in \( \tilde{\pi}(\mathcal{B})' \), we also have

\[
\tilde{\pi}(\mathcal{B}) = \pi(\mathcal{B}) \oplus \tilde{\pi}(\mathcal{B})|_{\mathcal{H}^\perp}.
\]

Since \( \mathcal{H} \subseteq \tilde{\mathcal{H}} \) by the fact that \( \tilde{\mathcal{H}} \) is an essential subspace of \( \tilde{\pi}(\mathcal{B}) \) and \( \tilde{\pi}(\mathcal{B})|_{\tilde{\mathcal{H}}} \) is irreducible, we have \( \tilde{\mathcal{H}} = \mathcal{H} \) and \( \tilde{\pi}(\mathcal{B})|_{\mathcal{H}^\perp} = 0|_{\mathcal{H}^\perp} \). It implies that

\[
\tilde{\pi}(\mathcal{B}) = \pi(\mathcal{B}) \oplus 0|_{\mathcal{H}^\perp}.
\]

So if \( \mathcal{H}^\perp = \{0\} \), then \( \tilde{\pi}(\mathcal{B}) = \pi(\mathcal{B}) \) is a quasidiagonal set of operators. Otherwise, by [5, Lemma 15.5], we know that \( \pi(\mathcal{B}) \) is a quasidiagonal set of operators on \( \mathcal{H} \). Hence \( \mathcal{B} \) is a strongly quasidiagonal C*-algebra.

It is natural to ask whether a similar conclusion holds for inner quasidiagonal C*-algebras.

**Proposition 2.12** Let \( \mathcal{A} \) be a separable C*-algebra. If \( \mathcal{A} \) is inner quasidiagonal and \( \mathcal{B} \subseteq \mathcal{A} \) is a hereditary C*-subalgebra, then \( \mathcal{B} \) is inner quasidiagonal again.

**Proof** By Theorem 2.2, we only need to prove there is a separating family of irreducible quasi-diagonal representations of \( \mathcal{B} \).

Let \( \{\tilde{\pi}_n\} \) be a separating family of irreducible quasi-diagonal representations of \( \mathcal{A} \) on \( \tilde{\mathcal{H}}_n \). Then \( \tilde{\pi}_n(\mathcal{B}) \subseteq \tilde{\pi}_n(\mathcal{A}) \) is a quasidiagonal set of operators in \( \mathcal{B}(\tilde{\mathcal{H}}_n) \). By II.6.1.9 in [1], \( \tilde{\pi}_n(\mathcal{B})|_{\mathcal{H}_n} \) is irreducible where \( \mathcal{H}_n \subseteq \tilde{\mathcal{H}}_n \) is the essential subspace of \( \tilde{\pi}_n(\mathcal{B}) \). Therefore \( \tilde{\pi}_n(\mathcal{B}) = \tilde{\pi}_n(\mathcal{B})|_{\mathcal{H}_n} \oplus 0|_{\mathcal{H}_n^\perp} \). So by Lemma 3.10 in [5], we have \( \tilde{\pi}_n(\mathcal{B})|_{\mathcal{H}_n} \) is a quasidiagonal set of operators in \( \mathcal{B}(\mathcal{H}_n) \). It implies that \( \tilde{\pi}_n|_{\mathcal{H}_n} \) is a separating family of irreducible quasidiagonal representation of \( \mathcal{B} \). Therefore \( \mathcal{B} \) is inner quasidiagonal again. \( \Box \)

A closed two-sided ideal \( \mathcal{I} \) in a C*-algebra \( \mathcal{A} \) is said to be primitive if \( \mathcal{I} \neq \mathcal{A} \) and \( \mathcal{I} \) is the kernel of an irreducible representation of \( \mathcal{A} \) on some Hilbert space. The primitive ideal space, \( \text{prim}(\mathcal{A}) \), is the set of all primitive ideals in \( \mathcal{A} \). For more information about \( \text{prim}(\mathcal{A}) \), we refer the readers to [1].

**Lemma 2.13** Let \( \mathcal{A} \) be a separable C*-algebra. If \( \mathcal{A} \) is strongly quasidiagonal, then \( \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \) is strongly quasidiagonal again for any positive integer \( n \).

**Proof** By IV.3.4.25 in [1], we know that every primitive ideal in \( \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \) is in the form \( \mathcal{I} \otimes \mathcal{M}_n(\mathbb{C}) \) where \( \mathcal{I} \) is a primitive ideal of \( \mathcal{A} \). For an arbitrary irreducible representation \( \pi \) of \( \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \), there is a representation \( \pi' : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) such that \( \ker(\pi') = \mathcal{I} \) for a primitive ideal \( \mathcal{I} \). So we extend \( \pi' \) to \( \tilde{\pi} : \tilde{\mathcal{A}} \to \mathcal{B}(\mathcal{H}) \) by letting

\[
\tilde{\pi}(a + \lambda I_{\tilde{\mathcal{A}}}) = \pi'(a) + \lambda I_{\mathcal{B}(\mathcal{H})},
\]

where \( \tilde{\mathcal{A}} \) is the unitization of \( \mathcal{A} \). Therefore \( \ker(\tilde{\pi} \otimes \text{id}) = \mathcal{I} \otimes \mathcal{M}_n(\mathbb{C}) \) where \( \text{id} \) is the identity representation of \( \mathcal{M}_n(\mathbb{C}) \). Since \( \mathcal{A} \) is strongly quasidiagonal
quasidiagonal if and only if $\tilde{A}$ is strongly quasidiagonal, we have $\tilde{\pi}(\tilde{A})$ is a quasidiagonal set of operators in $B(\mathcal{H})$.

Now we extend $\pi$ to a unital representation $\overline{\pi}$ on $\tilde{A} \otimes M_n(\mathbb{C})$. Hence $\overline{\pi}$ is irreducible too and

$$\ker(\overline{\pi}) = \ker(\pi) = I \otimes M_n(\mathbb{C}).$$

Since

$$\ker(\pi \otimes \text{id}) = \ker(\overline{\pi}),$$

$\overline{\pi} \otimes \text{id}$ and $\overline{\pi}$ are both irreducible, then by Corollary 4 in [10] we know that $\overline{\pi}$ is quasidiagonal precisely when $\pi \otimes \text{id}$ is. By Lemma 14 in [10], $\pi \otimes \text{id}$ is quasidiagonal since $A$ is strongly quasidiagonal. It follows that $\overline{\pi}$ is quasidiagonal. Note $\pi(A \otimes M_n(\mathbb{C})) \subseteq \pi(\tilde{A} \otimes M_n(\mathbb{C}))$, therefore $\pi(A \otimes M_n(\mathbb{C}))$ is a quasidiagonal set of operators. It follows that $A \otimes M_n(\mathbb{C})$ is strongly quasidiagonal by Proposition 2.8 in [3].

For showing our next result, we need the next concept.

**Definition 2.14** ([17, Definition 3]) Let $S$ be a class of $C^*$-algebras. A $C^*$-algebra $A$ is called a local $S$-algebra if for every finite subset $F \subseteq A$ and $\forall \varepsilon > 0$ there exist a $C^*$-algebra $B$ in $S$ and a $*$-homomorphism $\varphi : B \to A$ such that $\text{dist}(x, \varphi(B)) < \varepsilon$ for all $x \in F$. When $A$ is unital and the $C^*$-algebras in $S$ and the $*$-homomorphism are unital, the $C^*$-algebra $A$ is called unital local $S$-algebra.

**Lemma 2.15** If $S$ is a class of strongly quasidiagonal $C^*$-algebras, then each local $S$-algebra is strongly quasidiagonal again.

**Proof** Let $A$ be a local $S$-algebra. Then for every finite set $F \subseteq A$ and $\varepsilon > 0$, there are a strongly quasidiagonal $C^*$-algebra $B$ and a $*$-homomorphism $\varphi : B \to A$ such that $d(a, \varphi(B)) < \varepsilon$ for every $a \in F$. Hence we can find a finite subset $D \subseteq \varphi(B)$ such that for $a \in F$ there exists $d \in D$ such that $\|a - d\| < \varepsilon$. Assume $\pi : A \to B(\mathcal{H})$ is a representation of $A$, then $\pi \circ \varphi : B \to B(\mathcal{H})$ is a representation of $B$. Therefore $\pi(\varphi(B))$ is a quasidiagonal set of operators on $\mathcal{H}$ since $B$ is strongly quasidiagonal. It implies that there are a finite-rank projection $p \in B(\mathcal{H})$ and a finite subset $X \subset \mathcal{H}$ such that $\|p\pi(d) - \pi(d)p\| < \varepsilon$ and $\|p(x) - x\| < \varepsilon$ for every $d \in D \subseteq \varphi(B)$ and $x \in X$. Now it is easy to calculate that for $a \in F$

$$\|p\pi(a) - \pi(a)p\| < \|p\pi(a) - p\pi(d)\| + \|p\pi(d) - \pi(d)p\| + \|\pi(d)p - \pi(a)p\| < 3\varepsilon.$$ 

These imply that $\pi(A)$ is a quasidiagonal set of operators. Since $\pi$ is arbitrary, we know $A$ is a strongly quasidiagonal $C^*$-algebra.

Now we need to introduce the Rokhlin action on $C^*$-algebras.

**Definition 2.16** ([17, Definition 2]) Let $A$ be a $C^*$-algebra and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the Rokhlin property if for any $\varepsilon > 0$ and any finite subset $F \subseteq A$ there exist mutually orthogonal positive contractions $(r_g)_{g \in G} \subseteq A$ such that

1. $\|\alpha_g(r_h) - r_{gh}\| < \varepsilon$ for all $g, h \in G$;
2. $\|r_ga - ar_g\| < \varepsilon$ for all $a \in F$ and $g \in G$;
3. $\|(\sum_{g \in G} r_g)a - a\| < \varepsilon$ for all $a \in F$.

For showing our main result in this subsection, we need the following result.
Theorem 2.17 ([17, Theorem 2]) Let $A$ be a $C^*$-algebra and $G$ be a finite group. Let $\alpha : G \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property. Then for any finite subset $\mathcal{F} \subseteq A \rtimes_\alpha G$ and any $\varepsilon > 0$ there exist a positive element $a \in A$ and a $*$-homomorphism $\varphi : M_{|G|}(aAa) \rightarrow A \rtimes_\alpha G$ such that $\text{dist}(x, \text{Im}(\varphi)) < \varepsilon$ for all $x \in \mathcal{F}$.

The $C^*$-algebra $A$ in Theorem 2.17 may not be unital. The next result is our main result in this subsection.

Theorem 2.18 Let $A$ be a strongly quasidiagonal $C^*$-algebra and $\alpha$ a Rokhlin action on $A$ by a finite group $G$. Then the crossed product $A \rtimes_\alpha G$ is a strongly quasidiagonal $C^*$-algebra again.

Proof By Lemma 2.11 and Lemma 2.13, $M_{|G|}(aAa) \cong aAa \otimes M_{|G|}$ is strongly quasidiagonal and then $A \rtimes_\alpha G$ is a locally $S$-algebra where $S$ is a class of strongly quasidiagonal $C^*$-algebras by Theorem 2.17. Hence by Lemma 2.15, $A \rtimes_\alpha G$ is strongly quasidiagonal.

2.3 Just-infinite $C^*$-algebras

In this subsection, we will see the relationship between just-infinite $C^*$-algebras and inner quasidiagonal $C^*$-algebras. A $C^*$-algebra is called just-infinite if it is infinite-dimensional and all its proper quotients are finite-dimensional.

For each $n \in \{0, 1, 2, \ldots, \infty\}$, the $T_0$-space $Y_n$ is defined to be the disjoint union $Y_n = \{0\} \cup Y'_n$, where $Y'_n = \{1, 2, \ldots, n\}$ is a set with $n$ elements if $n$ is finite, and $Y'_n = \mathbb{N}$ has countably infinitely many elements if $n = \infty$. Equip $Y_n$ with the topology for which the closed subsets of $Y_n$ are precisely the following set: $\emptyset, Y_n$ and all finite subsets of $Y'_n$. So prim($A$) and a just-infinite $C^*$-algebra $A$ are characterized by using $Y_n$ in the following theorem.

Theorem 2.19 ([9, Theorem 3.10]) Let $A$ be separable $C^*$-algebra. Then $A$ is just-infinite if and only if prim($A$) is homeomorphic to $Y_n$, for some $n \in \{0, 1, 2, \ldots, \infty\}$, and each non-faithful irreducible representation of $A$ is finite dimensional (If $n = 0$, we must also require that $A$ is infinite dimensional; this is automatic when $n \geq 1$). Moreover:

(a) prim($A$) = $Y_0$ if and only if $A$ is simple. Every infinite dimensional simple $C^*$-algebra is just-infinite.

(b) prim($A$) = $Y_n$, for some integer $n \geq 1$, and $A$ is just-infinite, if and only if $A$ contains a simple non-zero essential infinite dimensional ideal $I_0$ such that $A/I_0$ is finite dimensional.

In this case, $n$ is equal to the number of simple summand of $A/I_0$.

(c) The following conditions are equivalent:

(i) $A$ is just-infinite and prim($A$) = $Y_\infty$.

(ii) $A$ is just-infinite and RFD.

(iii) prim($A$) is an infinite set, all of its infinite subsets are dense, and $A/I$ is finite dimensional, for each non-zero $I \in \text{prim}(A)$.

(iv) prim($A$) is an infinite set, the direct sum representation $\bigoplus_i \pi_i$ is faithful for each infinite family $\{\pi_i\}_{i}$ of pairwise inequivalent irreducible representations of $A$, and each non-faithful irreducible representation of $A$ is finite dimensional.

Lemma 2.20 ([9, Lemma 3.2]) Every just-infinite $C^*$-algebra is prime.

It implies that every non-zero ideal $J$ in just-infinite $C^*$-algebras $A$ is essential, i.e., $I \cap J$
≠ \{0\} for every non-zero closed ideal \( I \) in \( \mathcal{A} \).

Now we are going to show that a just-finite \( C^* \)-algebra is quasidiagonal if and only if it is inner quasidiagonal.

**Theorem 2.21** Let \( \mathcal{A} \) be a separable \( C^* \)-algebra. If \( \mathcal{A} \) is just-infinite, then \( \mathcal{A} \) is quasidiagonal if and only if it is inner quasidiagonal.

**Proof** Since every inner quasidiagonal \( C^* \)-algebra is quasidiagonal, then one direction of the proof is clear. Now we only need to prove that a just-infinite \( C^* \)-algebra \( \mathcal{A} \) is inner quasidiagonal as \( \mathcal{A} \) is quasidiagonal.

Since \( \mathcal{A} \) is separable, \( \mathcal{A} \) is just-infinite if and only if it is in the three cases listed in Theorem 2.19. In the case (\( \alpha \)), we already know \( \mathcal{A} \) is inner quasidiagonal by Lemma 2.7. In the case (\( \gamma \)), it is obvious that \( \mathcal{A} \) is inner quasidiagonal by Remark 2.8. Hence we only need to verify that \( \mathcal{A} \) is inner quasidiagonal in the case (\( \beta \)).

By Lemma 2.20, we know \( \mathcal{A} \) is prime and then every ideal in \( \mathcal{A} \) is essential. Now we consider two subcases. In the first subcase, we suppose \( \mathcal{A} \) is antiliminal. Then \( \mathcal{A} \) is inner quasidiagonal by Lemma 2.7. In the second subcase, we suppose \( \mathcal{A} \) is not antiliminal. Since \( \mathcal{A} \) has a faithful irreducible representation \( \pi \) and there is an element \( x \in \mathcal{A} \) such that \( \text{rank}(\pi(x)) \leq 1 \) by IV.1.1.7 in [1], \( \mathcal{A} \) has an essential ideal isomorphic to \( \mathbb{K} \). So we may assume \( \mathbb{K} \subseteq \mathcal{A} \). Note \( \mathcal{I}_0 \) in Theorem 2.19 is essential, so we know that \( \mathbb{K} \cap \mathcal{I}_0 \neq \{0\} \). By the fact that \( \mathcal{I}_0 \) is simple, we have

\[ \mathbb{K} \cap \mathcal{I}_0 = \mathbb{K} = \mathcal{I}_0. \]

It follows that \( \mathcal{A} \) is an extension of two AF-algebras, it is itself an AF-algebra. By Remark 2.8, \( \mathcal{A} \) is inner quasidiagonal. \( \square \)

**Remark 2.22** A \( C^* \)-algebra is called FDI if it has no infinite-dimensional irreducible representation ([7]). By Theorem 2.21 every just-infinite quasidiagonal \( C^* \)-algebra is inner quasidiagonal in the separable case. Meanwhile, it is easy to see that every FDI \( C^* \)-algebra is inner quasidiagonal, and we also know that no FDI \( C^* \)-algebra is just-infinite by the fact that just-infinite \( C^* \)-algebras are prime. Therefore in the separable case the set of just-infinite quasidiagonal \( C^* \)-algebras has no intersection with the set of FDI \( C^* \)-algebra even though they are both inner quasidiagonal.

### 3 Topological Free Entropy Dimension in Just-infinite \( C^* \)-algebras

In this section, we will first recall the notion of topological free entropy dimension \( \delta_{\text{top}}(x_1, \ldots, x_n) \) for \( n \)-tuple \( \vec{x} = (x_1, \ldots, x_n) \) in a unital \( C^* \)-algebra. After that, we are going to analyze the topological free entropy dimension in just-infinite \( C^* \)-algebras.

#### 3.1 Definition of \( \delta_{\text{top}} \) for \( n \)-tuple

A separable \( C^* \)-algebra \( \mathcal{A} \) is called MF if it can be embedded into

\[ \prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C}) \]

for a sequence of positive integers \( \{n_k\}_{k=1}^\infty \). The concept of MF algebras was first introduced by Blackadar and Kirchberg in [2]. The topological free entropy dimension is well-defined for the \( n \)-tuple \( (x_1, \ldots, x_n) \) if the unital \( C^* \)-algebra generated by \( \{x_1, \ldots, x_n\} \) is an MF \( C^* \)-algebra.
Definition 3.1  The topological free entropy dimension of \( x_1, \ldots, x_n \) is defined by

\[
\delta_{\text{top}}(x_1, \ldots, x_n) = \limsup_{\omega \to 0^+} \inf_{\varepsilon > 0} \limsup_{k \to \infty} \frac{\log(\nu_\infty(\Gamma^{(\text{top})}(x_1, \ldots, x_n; k, \varepsilon, Q_1, \ldots, Q_r), \omega))}{-k^2 \log \omega}.
\]

For the exact definition of topological free entropy dimension, we refer the readers to \([14, 21]\) or \([15]\). One of main questions about topological free entropy dimension is whether it is independent of the generators of unital MF \( C^* \)-algebras. In \([15]\), we can find the following results.

Theorem 3.2 (\([15, \text{Theorem 4.5}]\))  If \( A = C^*(x_1, \ldots, x_n) \) is MF-nuclear, then

\[
\delta_{\text{top}}(x_1, \ldots, x_n) \leq 1.
\]

Lemma 3.3 (\([15, \text{Theorem 5.3}]\))  Suppose \( A = C^*(x_1, \ldots, x_n) \) is a unital MF-algebra and either

(i) \( A \) has no finite-dimensional representations, or

(ii) \( A \) has infinitely many non-unitarily-equivalent finite-dimensional irreducible representation.

Then \( \delta_{\text{top}}(x_1, \ldots, x_n) \geq 1 \).

By Lemma 3.3, we can quickly get the following corollary.

Corollary 3.4  Let \( A = C^*(x_1, \ldots, x_n) \) be a simple infinite-dimensional MF algebra. Then \( \delta_{\text{top}}(x_1, \ldots, x_n) \geq 1 \).

Remark 3.5  In the von Neumann version of free entropy dimension, Voiculescu in \([20]\) proved that if \( x = x^* \) is an element of a von Neumann algebra with faithful trace \( \tau \), then

\[
\delta_0(x) = 1 - \sum_{t \text{ is an eigenvalue of } x} \tau(P_t)^2,
\]

where \( P_t \) is the orthogonal projection onto \( \ker(x - t) \). If \( x \) has no eigenvalues, then \( \delta_0(x) = 1 \).

Suppose \( \tau \) is a tracial state of \( A \) and \( \pi_\tau \) is the GNS construction induced by \( \tau \) with cyclic vector \( e \). Then \( \tau : \pi(A)^\prime\prime \to \mathbb{C} \) is the faithful tracial state defined by \( \tau(T) = \langle Te, e \rangle \). So we can find the next lemma in \([15]\).

Lemma 3.6  Suppose \( A = C^*(x_1, \ldots, x_n) \) is a unital MF-aglebra and \( \tau \in T_{\text{MF}}(A) \). Suppose \( b = b^* \in \pi_\tau(A)^\prime\prime \). Then

\[
\delta_{\text{top}}(x_1, \ldots, x_n) > \delta_0(b, \tau).
\]

Definition 3.7 (\([15]\))  Suppose \( A = C^*(x_1, \ldots, x_n) \) is an MF \( C^* \)-algebra. A tracial state \( \tau \) on \( A \) is an MF-trace if there are a sequence \( \{m_k\} \) of positive integers and sequences \( \{A_{1k}\}, \ldots, \{A_{nk}\} \) with \( A_{1k}, \ldots, A_{nk} \in M_{m_k}(\mathbb{C}) \) such that, for every \( * \)-polynomial \( Q \),

(1) \( \lim_{k \to \infty} ||Q(A_{1k}, \ldots, A_{nk})|| = ||Q(x_1, \ldots, x_n)|| \), and

(2) \( \lim_{k \to \infty} \tau_{m_k}(Q(A_{1k}, \ldots, A_{nk})) = \tau(Q(x_1, \ldots, x_n)) \).

We let \( T_S(A) \) denote the set of all tracial states on \( A \) and \( T_{\text{MF}}(A) \) denote the set of all MF-traces on \( A \). The MF-ideal of \( A \) is defined as

\[
J_{\text{MF}}(A) = \{ a \in A : \forall \tau \in T_{\text{MF}}(A), \tau(a^*a) = 0 \}.
\]
It has been shown that \( J_{\text{MF}}(\mathcal{A}) \) is a nonempty weak*-compact convex set in [15]. Furthermore, we let \( \mathcal{S} \) denote the class of MF \( C^* \)-algebras \( \mathcal{A} \) for which every trace is an MF-trace, i.e.,
\[
T \mathcal{S}(\mathcal{A}) = T_{\text{MF}}(\mathcal{A}) \text{ and } W
\]
denote the class of all MF algebras \( \mathcal{A} \) such that \( J_{\text{MF}}(\mathcal{A}) = \{0\} \). For more details about \( \mathcal{S} \) and \( W \), we refer the readers to [15].

**Theorem 3.8** ([15, Theorem 3.6]) Suppose \( \mathcal{A} = C^*(x_1, \ldots, x_n) \) is an MF-algebra and \( \mathcal{A}/J_{\text{MF}}(\mathcal{A}) \) has dimensional \( d < \infty \). Then \( \delta_{\text{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{d} \).

The previous theorem tells us that the topological free entropy dimension of \( \mathcal{A} \) is independent of the generators of \( \mathcal{A} \) if \( \mathcal{A}/J_{\text{MF}}(\mathcal{A}) \) has dimensional \( d < \infty \).

### 3.2 \( \delta_{\text{top}} \) in Just-infinite \( C^* \)-algebras

The following two results are easy consequences of the properties of just-infinite \( C^* \)-algebras and \( \delta_{\text{top}} \).

**Proposition 3.9** Let \( \mathcal{A} = C^*(x_1, \ldots, x_n) \) be a unital just-infinite MF \( C^* \)-algebra. If \( \mathcal{A} \not\in \mathcal{W} \), then \( \delta_{\text{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim(J_{\text{MF}}(\mathcal{A}))} \).

**Proof** If \( \mathcal{A} \not\in \mathcal{W} \), then \( J_{\text{MF}}(\mathcal{A}) \neq 0 \). Since \( \mathcal{A} \) is just-infinite, we have \( \mathcal{A}/J_{\text{MF}}(\mathcal{A}) \) is finite-dimensional. Then \( \delta_{\text{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim(J_{\text{MF}}(\mathcal{A}))} \) by Theorem 3.8.

**Proposition 3.10** Let \( \mathcal{A} = C^*(x_1, \ldots, x_n) \) be a unital just-infinite MF \( C^* \)-algebra. If \( \mathcal{A} \) is of type \( (\alpha) \) or type \( (\gamma) \) in Theorem 2.19, then \( \delta_{\text{top}}(x_1, \ldots, x_n) \geq 1 \).

**Proof** Suppose \( \mathcal{A} \) is of type \( (\alpha) \), then \( \mathcal{A} \) is simple and infinite-dimensional. By Corollary 3.4, \( \delta_{\text{top}}(x_1, \ldots, x_n) \geq 1 \). Suppose \( \mathcal{A} \) is of type \( (\gamma) \), then \( \mathcal{A} \) has infinite many non-unitarily-equivalent finite dimensional irreducible representations, so \( \delta_{\text{top}}(x_1, \ldots, x_n) \geq 1 \) by Lemma 3.3.

**Remark 3.11** Note if \( \mathcal{A} \) is of type \( (\alpha) \) or type \( (\gamma) \), then \( \mathcal{A} \in \mathcal{W} \).

**Theorem 3.12** Let \( \mathcal{A} = C^*(x_1, \ldots, x_n) \) be a unital MF \( C^* \)-algebra. If \( \mathcal{A} \) is just-infinite and \( \mathcal{A} \in \mathcal{W} \cap \mathcal{S} \), then \( \delta_{\text{top}}(x_1, \ldots, x_n) \geq 1 \).

**Proof** By Remark 3.11 and Proposition 3.10, we know that \( \delta_{\text{top}}(x_1, \ldots, x_n) \geq 1 \) as \( \mathcal{A} \) is of type \( (\alpha) \) or \( (\gamma) \). So we only need to consider the case in which \( \mathcal{A} \) is type \( (\beta) \) in Theorem 2.19.

Assume \( \mathcal{A} \) is not antiliminal. Since \( \mathcal{A} \) is prime, \( \mathcal{A} \) has a faithful irreducible representation \( \pi \) and there is an element \( x \in \mathcal{A} \) such that \( \text{rank}(\pi(x)) \leq 1 \) by IV.1.1.7 in [1]. It follows that \( \mathcal{A} \) has an essential ideal isomorphic to \( K \). So we may assume \( K \subset \mathcal{A} \). Note \( I_0 \) in Theorem 2.19 is essential, so we know that \( K \cap I_0 \neq \{0\} \). By the fact that \( I_0 \) is simple, we have
\[
K \cap I_0 = K = I_0.
\]
Since \( \mathcal{A}/I_0 \) is finite-dimensional and every tracial state vanishes on \( K \), we have
\[
K = J_{\text{MF}}(\mathcal{A}) \neq 0.
\]
This contradicts to the fact that \( \mathcal{A} \in \mathcal{W} \). Then \( \mathcal{A} \) must be antiliminal. Let \( \tau \) be a factor tracial state on \( \mathcal{A} \). Then
\[
\ker \pi_\tau \supset I_0 \quad \text{or} \quad \ker \pi_\tau = 0 \quad (3.1)
\]
by the fact that \( I_0 \) is essential and simple where \( \pi_\tau \) is a representation induced by \( \tau \). Note \( \mathcal{A} \in \mathcal{W} \cap \mathcal{S} \), then every factor tracial state is an MF tracial state and
\[
J_{\text{MF}}(\mathcal{A}) = 0. \quad (3.2)
\]
In [11], it was shown that the set of factor tracial states is the set of extreme points of $TS(A)$. It follows that there is at least one factor tracial state $\tau$ such that $\pi_\tau$ is faithful by (3.1) and (3.2). Then for such $\tau$, $\pi_\tau(A)$ is antiliminal too. If $\pi_\tau(A)''$ is type I factor with tracial state, then $\pi_\tau(A) = \pi_\tau(A)'' \cong M_n(\mathbb{C})$ for an integer $n$. This contradicts the fact that $\pi_\tau(A)$ is antiliminal. Hence $\pi_\tau(A)''$ is type II$_1$ factor. It implies that there is an element $a \in \pi_\tau(A)''$ which has no eigenvalues, so $\delta_0(a) = 1$ by Remark 3.5. Now by Lemma 3.6,

$$\delta_{top}(x_1, \ldots, x_n) \geq 1.$$ 

Combining the preceding results and Theorem 3.2, we have the following two corollaries.

**Corollary 3.13**  Let $A = C^*(x_1, \ldots, x_n)$ be a unital $C^*$-algebra. If $A$ is just-infinite MF-nuclear and $A \in \mathcal{W} \cap \mathcal{S}$, then $\delta_{top}(x_1, \ldots, x_n) = 1$.

**Corollary 3.14**  Let $A = C^*(x_1, \ldots, x_n)$ be a unital $C^*$-algebra. If $A$ is just-infinite MF-nuclear and is of type ($\alpha$) or type ($\gamma$), then $\delta_{top}(x_1, \ldots, x_n) = 1$.

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