Positive lyapunov exponents for quasiperiodic Szegő cocycles

Zhenghe Zhang

Department of Mathematics, Northwestern University, 2033 Sheridan Road Evanston, IL 60208-2730, USA

E-mail: zhenghe@math.northwestern.edu

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Abstract

In this paper, we first obtain a formula of averaged Lyapunov exponents for ergodic Szegő cocycles via the Herman–Avila–Bochi formula. Then using acceleration, we construct a class of analytic quasiperiodic Szegő cocycles with uniformly positive Lyapunov exponents. Finally, a simple application of the main theorem in Young (1997 Ergod. Theory Dyn. Syst. 25 483–504) allows us to estimate the Lebesgue measure of support of the measure associated with certain class of $C^1$ quasiperiodic 2-sided Verblunsky coefficients. Using the same method, we also recover the Sorets and Spencer (1991 Commun. Math. Phys. 142 543–66) results for Schrödinger cocycles with nonconstant real analytic potentials and obtain some nonuniform hyperbolicity results for arbitrarily fixed Brjuno frequency and for certain $C^1$ potentials.

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1. Introduction

In this paper, we study the Lyapunov exponents for two special families of $SL(2, \mathbb{C})$ cocycles: Szegő and Schrödinger cocycles. In particular, we are interested in how to produce positive Lyapunov exponents. We first introduce the $SL(2, \mathbb{C})$ cocycles and define the associated Lyapunov exponents.

1.1. $SL(2, \mathbb{C})$ cocycles and Lyapunov exponents

Let $(X, \mu)$ be a probability space and $T : X \rightarrow X$ be a $\mu$-preserving transformation. Let $A : X \rightarrow SL(2, \mathbb{C})$ be a measurable function satisfying the integrability condition

$$\int_X \ln \| A(x) \| \, d\mu < \infty.$$
Then, we can use \((T, A)\) to define a dynamical system:
\[
(T, A) : X \times \mathbb{C}^2 \to X \times \mathbb{C}^2, \ (x, w) \mapsto (T(x), A(x)w).
\]

\(A\) is the so-called cocycle map. The \(n\)th iteration of dynamics will be denoted by \((T, A)^n = (T^n, A_n)\), thus
\[
A_n(x) = A(T^{n-1}(x)) \cdots A(x), \quad n \geq 1, \quad A_0 = Id.
\]

If furthermore \(T\) is invertible, then
\[
A_{-n}(x) = A_n(T^{-n}(x))^{-1}, \quad n \geq 1.
\]

One of the most important objects in understanding dynamics of \(SL(2, \mathbb{C})\) cocycles is the Lyapunov exponent, which is denoted by \(L(T, A)\) and given by
\[
\lim_{n \to \infty} \frac{1}{n} \int_X \log \|A_n(x)\| \, d\mu = \inf_n \frac{1}{n} \int_X \log \|A_n(x)\| \, d\mu \geq 0.
\]

The limit exists and is equal to the infimum since \(\int_X \log \|A_n(x)\| \, d\mu\) is a subadditive sequence. If in addition \(T\) is \(\mu\)-ergodic, then by Kingman’s subadditive ergodic theorem we also have
\[
L(T, A) = \lim_{n \to \infty} \frac{1}{n} \int_X \log \|A_n(x)\|, \quad \text{for } \mu \text{ almost every } x.
\]

### 1.2. Positive Lyapunov exponents for Schrödinger cocycles

The Schrödinger cocycle map \(A(E - \lambda v) : X \to SL(2, \mathbb{R})\) is given by
\[
A(E - \lambda v)(x) = \begin{pmatrix}
E - \lambda v(x) & -1 \\
1 & 0
\end{pmatrix},
\]
where \(v : X \to \mathbb{R}\) is the potential function (we assume \(v \in L^\infty(X)\), \(E \in \mathbb{R}\) is the energy and \(\lambda\) is the coupling constant. Schrödinger cocycles arises from the one-dimensional discrete Schrödinger operator \(H\) on \(l^2(\mathbb{Z})\). We fix the potential \(v\). For \(u \in l^2(\mathbb{Z})\), the Schrödinger operator is given by
\[
(H_{\lambda, x} u)_n = u_{n+1} + u_{n-1} + \lambda v(T^n(x))u_n.
\]

Let \(u \in \mathbb{C}^2\) be a solution of equation \(H_{\lambda, x} u = Eu\) (note \(u\) is not necessary in \(l^2(\mathbb{Z})\)); then the relation between cocycle and operator is
\[
A(E - \lambda v)(T^n(x)) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}.
\]

Let \(\Sigma_{\lambda, x}\) be the spectrum of \(H_{\lambda, x}\). That it,
\[
\Sigma_{\lambda, x} = \{E \in \mathbb{C} : H_{\lambda, x} - E \text{ is not invertible}\}.
\]

The positivity of Lyapunov exponents for Schrödinger cocycles is intensely studied since the Lyapunov exponent is very important in understanding the spectrum of the Schrödinger operators. For potential functions belong to different regularity classes, the mechanisms which lead to positivity of Lyapunov exponents are very different. We list some of the results which are closely related to this paper.
1.2.1. Continuous potentials. We assume $X$ is a compact metric space, $T$ is a homeomorphism which is also $\mu$-ergodic and $\mu$ is nonatomic. Then [AD] (see [AD, theorem 1]) shows that there is residual subset $G$ of $C(X, \mathbb{R})$ such that for every $v \in G$, $L(T, A(Ev)) > 0$ for almost every $E$. What lies behind this result is actually the monotonicity of the family $(T, A(Ev))$ with respect to the energy $E$, from which one can deduce the so-called Kotani theory (see [Ko2, AK] or [Sim1]). Then one can perform the following steps:

1. $\int_{-N}^{N} L(T, A(E_{-v})) dE : (L^1(X) \cap B(L^\infty(X)), \| \cdot \|_1) \to \mathbb{R}$ is continuous (see [AD, lemma 1]). That is, after some suitable integration, Lyapunov exponent is a continuous function in the $L^1$ sense;

2. the set of simple functions with nonperiodic sequences along the orbit of $T$ is dense in $L^\infty(X)$ (see [AD, lemma 2]);

3. for $v$ take finitely many values which are nonperiodic, $L(T, A(Ev)) > 0$ for almost every $E$ (see [Ko1]).

(1) and (2) reduce the proof of theorem 1 in [AD] to (3), which is due to the Kotani theory and nondeterminism of nonperiodic sequences. One can also add a coupling constant since it does not affect the nondeterminism (see [AD, theorem 2]). Note the positivity of Lyapunov exponents in this case holds only for a full measure set of energy. So in this case, it is basically the randomness of potential functions which leads to the positivity of Lyapunov exponents.

1.2.2. Real-analytic potentials. To consider higher regularity potentials, we restrict to the case $(X, \mu, T) = \left( \mathbb{R}^d/\mathbb{Z}^d, \text{Leb}, R_\alpha \right)$ and $v \in C^r(\mathbb{R}^d/\mathbb{Z}^d, \mathbb{R})$, $r \in \mathbb{Z}^+ \cup \{ \infty, \omega \}$ (here $\mathbb{Z}^+$ is the set of positive integers and $C^\infty$ means analyticity), where $R_\alpha : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d/\mathbb{Z}^d$ is the translation $R_\alpha(x) = x + \alpha$. In this case $x$ is the so-called phase and $\alpha$ is the frequency. For $R_\alpha$ ergodic, $v$ is the so-called quasiperiodic potential, which is the most intensively studied.

The first breakthrough, which perhaps is also the most famous one, is [H], where in the case $r = \omega$ and $d = 1$ Herman (among other things) shows that for these $v$ which are nonconstant trigonometric polynomials, one has $L(\alpha, A(E-\lambda v)) > 0$ for all $E$ and all $\alpha$, provided that $\lambda$ is large, depending only on $v$ (this in fact also holds for $d > 1$). Herman’s result was later generalized by Sorets and Spencer [S-S] to all nonconstant real-analytic potential functions. Bourgain [Bo] generalizes this result to the case $d > 1$ (for higher dimensional Diophantine frequencies, it is first proved in [BoG]). For some one-dimensional strong Diophantine frequencies, Klein generalizes this result to some Gevrey potentials, see [Kl].

What lies behind these series of results is basically the analyticity of the potential functions, which implies the subharmonicity of Lyapunov exponents. This allows one to move the phase $x$ to a complex plane to get around the small divisor problems. Note here largeness of couplings is needed. But one has that the positivity of Lyapunov exponents holds for all $E$ in these cases, not just a full measure subset.

Inspired by a new notion, the acceleration of Lyapunov exponents, which was first introduced in [A2], we will give a different proof of [S-S]’s result, see theorem A’. Our approach can be applied to more general $SL(2, \mathbb{C})$ cocycles. The dynamics reasons which lead to positive Lyapunov exponents are also clearer in our approach.

In [E], Eliasson also shows that if $v$ is Gevrey and satisfies a generic transversality condition (which is also a generalization of nonconstant real-analytic functions), and if $\alpha$ is Diophantine, then for large $\lambda$, the spectrum is a pure point. By the Kotani theory (see [Ko2]), $L(\alpha, A(E-\lambda v)) > 0$ for almost every $E$. He also obtains that the measure of the spectrum grows as $\lambda$ goes to $\infty$. 

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In these cases, it is basically the analyticity of potential functions and largeness of couplings which lead to positivity of Lyapunov exponents.

1.2.3. Smooth potentials. For $r \in \mathbb{Z}^+ \cup \{\infty\}$, it is more subtle to produce positive Lyapunov exponents, because in these cases Kotani theory cannot easily be used to produce positive Lyapunov exponents and there is no subharmonicity. It seems that a complicated induction and arithmetic properties on frequencies (such as Diophantine or Brjuno conditions) are necessary to take care of the small divisor problems in these cases.

Early works can be found in [FSW, Sin], where the authors used multi-scale analysis and a very special shape of graph of potential functions is needed. Recently, [Bj, C] obtained some results for some general smooth potentials. In [Bj], Bjerklöv’s approach is close in spirit to Benedicks–Carleson’s approach for the Hénon map (see [BC]). In [C], Chan uses multi-scale analysis; he also obtained positive Lyapunov exponents for all $E$ and most frequencies for some typical $C^1$ potentials via a variation method. Both of their results need to eliminate frequencies.

We will prove some similar results with [Bj, C], see theorem $B’$. Our proof is a simple application of a theorem in [Y], but note that the method in [Y] is also close in spirit to Benedicks–Carleson’s approach.

The main advantage of our approach is that we can, for a certain class of $C^1$ potentials, fix the arbitrary Brjuno frequency to produce positive Lyapunov exponents. Thus in our approach, it is clearer how the geometric properties of potential functions can affect the estimate of Lyapunov exponents. We can also reobtain Eliasson’s result on the estimate of measure of the spectrum for analytic potentials, see corollary 7.

In fact, our approach implies that, if one is allowed to eliminate frequencies, it is enough to assume $v$ is $C^1$ to obtain some corresponding results (see remark 14). We also have a very precise description of the ‘critical sets’ for large couplings (see remark 14).

The other advantage of our approach is again that it is more general. We will also deal with analytic and smooth cases in an unified form, in which obstructions to produce positive Lyapunov exponents are clearer.

For a fixed frequency, positivity of Lyapunov exponents for all $E$ with smooth potentials is more subtle, because one needs to deal with the appearance and disappearance of ‘critical points’ in the induction taking care of the small divisor problems. We are currently working on this aspect.

1.3. Positive Lyapunov exponents for Szegő cocycles

The Szegő cocycle map $A^{(E, f)} : X \to SU(1, 1)$ is given by

$$A^{(E, f)}(x) = (1 - |f(x)|^2)^{-1/2} \begin{pmatrix} \sqrt{E} & -f(x) \\ -f(x) \sqrt{E} & 1/\sqrt{E} \end{pmatrix},$$

where $E \in \partial \mathbb{D}$, $\mathbb{D}$ is the open unit disc in $\mathbb{C}$, and $f : X \to \mathbb{D}$ is a measurable function satisfying

$$\int_X \ln(1 - |f|) \, d\mu > -\infty.$$
that is, $Q^* SU(1, 1)Q = SL(2, \mathbb{R})$. Szegő cocycles arises naturally in the orthogonal polynomials on unit circle in the following way. For a polynomial $Q_n$ of degree $n$, we first define $Q^*_n$, the reversed polynomial, by

$$Q^*_n(E) = E^n Q_n(1/E).$$

Now start with $\varphi_0 = \varphi_0^* \equiv 1$, we can define a sequence of polynomials in $E$ as

$$\left( \begin{array}{c} \varphi_{n+1} \\ \varphi^*_{n+1} \end{array} \right) = (E A(E, f)(T^n(x))) \left( \begin{array}{c} \varphi_n \\ \varphi^*_n \end{array} \right), \quad n \geq 0.$$

We can use $\varphi_n$ to define a sequence of probability measure $d\mu_n$ on $\partial D$ as

$$d\mu_n = \frac{d\theta}{2\pi |\varphi_n(e^{i\theta})|^2}.$$

Then as $n \to \infty$, $d\mu_n$ converges weakly to a nontrivial probability measure $d\mu$ on $\partial D$ (here trivial means that $\mu$ is supported on a finite set). Then $\{\varphi_n\}_{n \in \mathbb{N}}$ is nothing other than the orthonormal set of the Hilbert space $\mathcal{H} = L^2(\partial D, d\mu)$, which one obtains by applying the Gram–Schmidt procedure to $1, E, E^2, \ldots$. In other words,

$$\varphi_n = \frac{P_n[E^n]}{\|P_n[E^n]\|}, \quad P_n = \text{projection onto } \{1, \ldots, E^{n-1}\}^\perp$$

for $n \in \mathbb{N}$, where $\| \cdot \|$ is the $\mathcal{H}$ norm.

Here the terms in the sequence $\{f(T^n(x))\}_{n \geq 0}$ appearing in $A(E, f)(f(T^n(x)))$ are called Verblunsky coefficients and $\mu$ is the associated measure. The correspondence between $\Pi_0^n \mathbb{D}$ and the set of nontrivial probability measures on $\partial D$ is actually one to one (this is exactly Verblunsky’s theorem). One can see [Sim2, chapter 1] for detailed descriptions of the above discussion.

While the Lyapunov exponent is important in understanding the relation between the Verblunsky coefficients and the associated measure $\mu$, it has not been very intensively studied as in the Schrödinger case. In this paper, inspired by the Schrödinger case, we will give some similar results for positivity of Lyapunov exponents for Szegő cocycles.

First we will show that $A(E, f)$ is actually another typical monotonic family of $SL(2, \mathbb{R})$ cocycles, which is of the form $BR\theta$, where $B$ is a $SL(2, \mathbb{R})$-valued cocycle map and

$$R\theta = \begin{pmatrix} \cos 2\pi \theta & -\sin 2\pi \theta \\ \sin 2\pi \theta & \cos 2\pi \theta \end{pmatrix} \in SO(2, \mathbb{R}).$$

This allows us to apply the Herman–Avila–Bochi formula to obtain a formula of averaged Lyapunov exponent for ergodic Szegő cocyles, see proposition 1. Then as in the Schrödinger case, we can easily draw the conclusion that for a generic set $\mathcal{G} \subset C(X, \mathbb{D})$, if $f \in \mathcal{G}$, then $L(T, A(E, f)) > 0$ for almost every $E$. This takes care of continuous Verblunsky coefficients.

For $r = o$, by the method we used to recover the result of [S-S], we can prove for a certain class of analytic quasiperiodic verblunsky coefficients, $L(T, A(E, f)) > 0$ for all $E \in \partial D$, see theorem A and corollary 4. This answers a question proposed in [Sim3], section 10.16 and [DK], section 3.

For a certain class of smooth quasiperiodic Verblunsky coefficients, we also obtain some results as we do in the Schrödinger case, see theorem B.

We will prove theorems A and A’, and theorems B and B’ in unified ways.
2. Statement of main results

2.1. A formula of averaged Lyapunov exponent for ergodic Szegő cocycles

We first consider the $SL(2, \mathbb{R})$-valued cocycle map $A : X \to SL(2, \mathbb{R})$ and assume $T$ is $\mu$-ergodic. In this case, we have the following Herman–Avila–Bochi formula ([AB, theorem 12]):

$$\int_{\mathbb{R}/\mathbb{Z}} L(T, A \circ R_\theta) \, d\theta = \int_X \ln \frac{\|A(x)\| + \|A(x)\|^{-1}}{2} \, d\mu.$$

The Herman–Avila–Bochi formula is first obtained as an inequality ([H, sections 6.2]):

$$\int_{\mathbb{R}/\mathbb{Z}} L(T, A \circ R_\theta) \, d\theta \geq \int_X \ln \frac{\|A(x)\| + \|A(x)\|^{-1}}{2} \, d\mu.$$

Since the right-hand side is typically positive (unless $A(\mathbb{R}/\mathbb{Z}) \subseteq SO(2, \mathbb{R})$), this gives a lower bound for average Lyapunov exponent in the family $(T, A \circ R_\theta)$. Now in Szegő cocycle family $A(E, f)$, we let $E = e^{2\pi i \theta}, \theta \in \mathbb{R}/\mathbb{Z}$. Then applying the Herman–Avila–Bochi formula to the Szegő cocycles, we obtain the following

**Proposition 1.** For $(T, A(E, f))$ as above, we have

$$\int_{\mathbb{R}/\mathbb{Z}} L(T, A(E, f)) \, d\theta = -\frac{1}{2} \int_X \ln(1 - |f(x)|^2) \, d\mu.$$

**Proof.** We first write $A(E, f)(x)$ as

$$(1 - |f(x)|^2)^{-1/2} \begin{pmatrix} 1 & -f(x) \\ -f(x)^* & 1 \end{pmatrix} \begin{pmatrix} \sqrt{E} & 0 \\ 0 & \sqrt{E} \end{pmatrix}.$$

Now we conjugate the above matrices to $Q^* A(E, f) Q \in SL(2, \mathbb{R})$, which is

$$(1 - |f(x)|^2)^{-1/2} Q^* \begin{pmatrix} 1 & -f(x) \\ -f(x)^* & 1 \end{pmatrix} QR_{-\frac{\theta}{2}} = B(x) R_{-\frac{\theta}{2}}.$$

It is easily calculated that

$$\|B(x)\| + \|B(x)\|^{-1} = \sqrt{\text{tr}(B^*B) + 2} = \frac{2}{\sqrt{1 - |f(x)|^2}},$$

where tr stands for trace. Now by the Herman–Avila–Bochi formula, we have

$$\int_{\mathbb{R}/\mathbb{Z}} L(T, A(E, f)) \, d\theta = \int_{\mathbb{R}/\mathbb{Z}} L(T, B R_{-\frac{\theta}{2}}) \, d\theta = \int_{\mathbb{R}/\mathbb{Z}} L(T, B R_\theta) \, d\theta = \int_X \ln \frac{\|B(x)\| + \|B(x)\|^{-1}}{2} \, d\mu = -\frac{1}{2} \int_X \ln(1 - |f(x)|^2) \, d\mu. \quad \square$$

Thus if $|f(x)| > 0$ on a positive measure set of $x,$ we can obtain $L(T, A(E, f)) > 0$ for a positive measure set of $E \in \partial \mathbb{D}$. As we said in section 1.3, this computation shows that the typical monotonic family of cocycles, $(f, A R_\theta)$ (monotonic with respect to $\theta$) arise naturally as Szegő cocycles. Then, as in section 1.2.1, we can obtain the following conclusion. Assuming further that $X$ is a compact metric space, $T$ a homeomorphism and $\mu$ nonatomic, there is a
generic set $G \subset C(X, \mathbb{D})$ (the space of continuous functions on $X$ taking value in $\mathbb{D}$) such that for every $f \in G$, we have $L(T, A^{E,f}) > 0$ for Lebesgue almost every $E \in \partial \mathbb{D}$.

As in [AD, remark 4.3], to draw the above conclusion we only need the following replacement in step (1) of section 1.2.1:

$$v \mapsto \int_{-N}^{N} L(T, A^{E,v}) \, dE \quad \text{by} \quad f \mapsto \int_{X} \ln(1 - |f|^2) \, d\mu,$$

where the corresponding continuity conclusion is immediate by the bounded convergence theorem.

As explained in [Sim2, theorem 12.6.1], an immediate consequence is that, for every $f \in G$ and almost every $x$ and Lebesgue almost every $\eta \in \partial \mathbb{D}$, the Aleksandrov measures $d\mu_{x,\eta}$ are a pure point. Here the family of Aleksandrov measures $d\mu_{x,\eta}$ are the measures associated with the Verblunsky coefficients $\{\eta f(T^n(x))\}_{n \geq 0}$, $\eta \in \partial \mathbb{D}$.

2.2. $C^r$ quasiperiodic Szegő and Schrödinger cocycles

From now on, we will focus on the case that $X = \mathbb{R}/\mathbb{Z}$, $T = R\alpha$ and $\mu$ is the unique probability Haar measure $d\mu_x$. We will consider only $C^r$ Szegő and Schrödinger cocycles for $r \in \mathbb{Z}^+ \cup \{\infty, \omega\}$. We shall use the notation $(\alpha, A)$ instead of $(T\alpha, A)$.

We first introduce the definition of uniform hyperbolicity, which plays a central role in this paper. We consider the Riemann surface $\mathbb{C}P^1$, which is given by the projection

$$\pi : (\mathbb{C}^2)^* = (\mathbb{C}^2 \setminus \{0\}) \to \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}, \quad \pi(z_0, z_1) = \frac{z_0}{z_1}, \quad z_1 \neq 0; \quad \pi(z_0, 0) = \infty.$$

Then the following commutative diagram induces the Möbius transformation for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$

$$\begin{array}{ccc}
(C^2)^* & \xrightarrow{A} & (C^2)^* \\
\pi \downarrow & & \pi \\
\mathbb{C}P^1 & \xrightarrow{A} & \mathbb{C}P^1
\end{array}$$

In other words, $A \cdot z = \frac{az+b}{cz+d}$. Now we are ready to give the following definition

**Definition.** We say $(\alpha, A) \in C^r(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))$ is uniformly hyperbolic if there are two continuous functions $u, s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}P^1$ such that

1. $u, s$ are $(\alpha, A)$-invariant, which means that $A(x) \cdot u(x) = u(x + \alpha)$ and $A(x) \cdot s(x) = s(x + \alpha)$;

2. there exists $C > 0, \lambda > 1$ such that $\|A^{n}(x)v\|, \|A^{n}(x)w\| \leq C\lambda^{-n}$ for every $n \geq 1$ and all unit vectors $v \in u(x), w \in s(x)$.

Here $u$ is called the unstable direction and $s$ is the stable direction of $(\alpha, A)$.

We denote the set of uniformly hyperbolic systems by $UH$. It is clear that $L(\alpha, A) > 0$ for $(\alpha, A) \in UH$. If $L(\alpha, A) > 0$ and $(\alpha, A) \notin UH$, we say that $(\alpha, A)$ is nonuniformly hyperbolic. We will denote the set of nonuniformly hyperbolic systems by $NH$.

We now state our main results.
2.2.1. $C'$ quasiperiodic Szegő cocycles. We have already introduced Szegő cocycles in section 1. Recall that the cocycle map is given by

$$A^{(E,f)}(x) = (1 - |f(x)|^2)^{-1/2} \begin{pmatrix} \sqrt{E} & -f(x) \\ f(x)\sqrt{E} & 1 \end{pmatrix}. $$

In section 1.3 we introduced a way to build the relation between the 1-sided Verblunsky coefficient $\{f(x + na)\}_{a \geq 0}$ and its associated measure $d\mu_{\alpha,x}$. Here we need to consider 2-sided Verblunsky coefficients $\{f(x + na)\}_{a \in \mathbb{Z}}$. We introduce another way to build this relation in the following. In fact, there is an unitary operator $C_{\alpha,x} : L^2(\mathbb{N}) \to L^2(\mathbb{N})$, the CMV operator, associated with $\{f(x + na)\}_{a \geq 0}$. Then $d\mu_{\alpha,x}$ is the spectral measure for $C_{\alpha,x}$ and $\delta_0 = (1, 0, 0, \ldots) \in L^2(\mathbb{N})$. Thus the spectrum of $C_{\alpha,x}$ is the support of the measure $d\mu_{\alpha,x}$. See [Sim2, theorem 4.2.8] for a detailed description.

Now for the 2-sided Verblunsky coefficient $\{f(x + na)\}_{a \in \mathbb{Z}}$, there is also an unitary operator $E_{\alpha,x} : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$, the induced extended CMV operator associated with it. Let $\Sigma_{\alpha,x}$ be the spectrum of $E_{\alpha,x}$. The definition of the extended CMV matrix can be found in [Sim3, sections 10.5.34 and 10.5.35]. Then, we have for irrational frequencies, the following dynamics description of $\Sigma_{\alpha,x}$.

**Fact A.** $\Sigma_{\alpha,x} = \Sigma_{\alpha,0} = \{E : (\alpha, A^{(E,f)}) \text{ is not uniformly hyperbolic}\}^1$.

For simplicity, let $\Sigma_{\alpha} = \Sigma_{\alpha,0}$ in this case. One can see [Sim2] and [Sim3] for more detailed discussion.

From now on, we will fix the function $f$ in the Szegő cocycle families to be of the form $f(x) = \lambda v(x)$ with $\lambda \in (0, 1)$ as the coupling constant, and $v(x) = e^{2\pi i h(x)}$, where $h \in C'(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$. In this case, for irrational frequencies, we denote the spectrum by $\Sigma_{\lambda,a}$.

**Remark 2.** Every function $h \in C'(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ can be written as $h(x) = kx + \theta(x)$ with $\theta \in C'(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $k$ is the degree. We assume $h$ is written in this form.

Our first theorem is concerned with the analytic case $\theta \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. This means that there exists a $\delta > 0$ such that $\theta$ can naturally be extended to a holomorphic function on $\Omega_\delta = \{z = x + iy \in \mathbb{C}/\mathbb{Z} : |y| < \delta\}$.

Then $A^{(E,\lambda,t)}$ can be extended naturally to a holomorphic map from $\Omega_\delta$ to $SL(2, \mathbb{C})$ as

$$A^{(E,\lambda,t)}(z) = (1 - \lambda^2)^{-1/2} \begin{pmatrix} \sqrt{E} & -\lambda \sqrt{E}v(z) \\ -\lambda\sqrt{E}v(z) & 1 \end{pmatrix}. $$

We will use $C^\infty(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))$ to denote the set of $A \in C^\infty(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))$ that can be holomorphically extended to $\Omega_\delta$. We also denote by $C^\infty_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ the set of real-analytic functions that can be holomorphically extended to $\Omega_\delta$.

We assume that $\theta \in C^\infty_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ is nonconstant. Then there is largest positive integer $q = q(\theta)$ such that $\theta(x + \frac{1}{q}) = \theta(x)$. Let $\mathbb{R}/\mathbb{Z} \times \partial \mathbb{D}$ be the set of $(\alpha, E)$ and $\pi_1 : \mathbb{R}/\mathbb{Z} \times \partial \mathbb{D} \to \mathbb{R}/\mathbb{Z}$ be the projection to the first component. Then the first main theorem of this paper is as follows:

**Theorem A.** Let $\theta \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be nonconstant, $k \in \mathbb{Z}$ and $v(x) = e^{2\pi i kx + \theta(x)}$. Then there exists a finite set $\mathcal{F} = \mathcal{F}(\theta, k) \subset \mathbb{R}/\mathbb{Z} \times \partial \mathbb{D}$ with $\pi_1(\mathcal{F}) \subset \{\frac{p}{q(\theta)} : p = 0, 1, \ldots, q(\theta) - 1\}$.

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1 The author is grateful to David Damanik for pointing this out.
with the following property. For any compact set \( C \subset (\mathbb{R}/\mathbb{Z} \times \partial \mathbb{D}) \setminus \mathcal{F} \), there exists a constant \( c_0 = c_0(\theta, k, C) \in \mathbb{R} \) such that

\[
L(\alpha, A(E,\lambda v)) \geq -\frac{1}{2} \ln(1 - \lambda) + c_0
\]

for all \((\alpha, E; \lambda) \in C \times (0, 1)\). Moreover, we have

\[
L(\alpha, A(E,\lambda v)) = 0,
\]

for all \((\alpha, E; \lambda) \in \mathcal{F} \times (0, 1)\).

**Remark 3.** Note that we obviously have \( L(0, A(-1,\lambda v)) = 0 \) for all \( v \), since it is always the case that \( \text{tr}(A(-1,\lambda v)) = 0 \). Thus, the finite set \( \mathcal{F} \) always contains \((0, -1)\). The other elements of \( \mathcal{F} \) are similarly selected using a trace criterion (determined by \( \theta \) and \( k \)).

**Theorem A** easily implies the following corollary:

**Corollary 4.** Let \( \theta, k, v \) and \( \mathcal{F} \) be as in theorem A. Then \( \forall \alpha \) with \( d(\alpha, \pi_1(\mathcal{F})) > 0 \), there is a constant \( c \in (0, 1) \) such that we have

\[
L(\alpha, A(E,\lambda v)) > 0
\]

for all \((E, \lambda) \in \partial \mathbb{D} \times (c, 1)\).

Thus the uniform positivity of Lyapunov exponents with respect to \( E \) is established.

Before corollary 4, the only almost periodic Szegő cocycle with uniformly positive Lyapunov exponents (uniform in \( E \)) was known in [DK], where the base dynamics is the ergodic translation on \( \mathbb{R}/\mathbb{Z} \times \mathbb{Z}_2 \). Again, as in the end of section 2.1, an immediate consequence of corollary 4 is that, for any irrational \( \alpha \) and for Lebesgue almost every \( x \) and Lebesgue almost every \( \eta \in \partial \mathbb{D} \), the Aleksandrov measures \( d\mu_{x,\eta} \) (here measures are for 1-sided Verblunsky coefficients) are a pure point for \( \lambda \) sufficiently close to 1.

We now turn to the question of estimating the Lebesgue measure of \( \Sigma_{\lambda,\alpha} \) associated with some 2-sided quasiperiodic Verblunsky coefficients. We will only assume that \( \theta \) is \( C^1 \), but we have to put some additional conditions on it. We first introduce the notion of the Brjuno number. For \( \alpha \) irrational, let \( p_n/q_n \) be its \( n \)th continued fraction approximant. Then \( \alpha \) is called a Brjuno number if

\[
\sum_{n=1}^{\infty} \frac{\ln q_{n+1}}{q_n} < \infty.
\]

Note this is a full measure condition since it contains all the Diophantine numbers.

We first let \( E = e^{2\pi i t}, t \in \mathbb{R}/\mathbb{Z} \). To make the dependence on parameters clearer, we write \( A(t,\lambda v) \) instead of \( A(E,\lambda v) \). We need two additional assumptions on \( \theta(x) \):

1. For each irrational \( \alpha \), \( \theta'(x + \alpha) - \theta'(x) = 0 \) exactly at two points in \( \mathbb{R}/\mathbb{Z} \), which are the unique maximum and unique minimum of \( \theta(x + \alpha) - \theta(x) \) (if we allow \( \theta \) to be \( C^2 \), then an easier but stronger condition is that \( \partial_2 \theta(x) = 0 \) exactly at two points in \( \mathbb{R}/\mathbb{Z} \));
2. \( \text{Leb}(\theta(\mathbb{R}/\mathbb{Z})) \leq \frac{1}{2} \).

An example that satisfies these conditions is \( \theta(x) = \frac{1}{2} \cos(2\pi x) \).

Now for any \( \epsilon > 0 \), let

\[
\Delta_\epsilon(\lambda, \alpha) = \{ t : (\alpha, A^{(t,\lambda v)}) \in NH \text{ and } L(\alpha, A^{(t,\lambda v)}) > (1 - \epsilon) \ln \lambda \}
\]

and

\[
\Gamma_\epsilon(\lambda) = \{(\alpha, t) : t \in \Delta_\epsilon(\lambda, \alpha)\}.
\]

Then our next main theorem is as follows:
Theorem B. For any Brjuno $\alpha$, there is a connected interval $I_\epsilon \subset \mathbb{R}/\mathbb{Z}$ of $t$ such that for any $\epsilon > 0$,
\[
\lim_{\lambda \to 1} \text{Leb}(I_\epsilon \cap \Delta_\epsilon(\lambda, \alpha)) = \text{Leb}(I_\epsilon).
\]
Furthermore, let $\mathcal{K} = \bigcup I_\epsilon$ be the compact domain in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$; then
\[
\lim_{\lambda \to 1} \text{Leb}(\mathcal{K} \cap \Gamma_\epsilon(\lambda)) = \text{Leb}(\mathcal{K}).
\]
Now by fact A, we have $\text{Leb}(\Sigma_1) \geq \text{Leb}(I_\epsilon \cap \Delta(\lambda, \alpha))$. Combined with theorem B, we obtain for Brjuno $\alpha$:
\[
\lim_{\lambda \to 1} \text{Leb}(\Sigma_1) = \text{Leb}(I_\epsilon).
\]
The equality holds because we will see in the proof of theorem B that for every $t/\epsilon \in I_\epsilon$,
\[
(\alpha, A(t,\lambda v)) \in \mathcal{UH} \text{ for } \lambda \text{ sufficiently close to } 1.
\]

2.3. C' quasiperiodic Schrödinger cocycles

Now we turn to the Schrödinger case. Recall the Schrödinger cocycle map is given by
\[
A(E-\lambda v)(x) = \begin{pmatrix} E - \lambda v(x) & -1 \\ 1 & 0 \end{pmatrix}.
\]
We denote the corresponding Schrödinger operator by $H_{\lambda,\alpha,x}$. Let $\Sigma_{1,\alpha,x}$ be the set of spectrum of the operator $H_{\lambda,\alpha,x}$; then similar to the Szegö case, for $\alpha$ irrational, the following basic fact (see [JM]) is well known:

**Fact A'.** $\Sigma_{1,\alpha,x} = \Sigma_{1,\alpha,0} = \{ E : (\alpha, A(E-\lambda v)) \text{ is not uniformly hyperbolic} \}$.

Let $\Sigma_{1,\alpha} = \Sigma_{1,\alpha,x}$. Again we are first concerned with the analytic case. Let $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be nonconstant. Then we have the following analogue of theorem A for Schrödinger cocycles:

**Theorem A'.** Let $v \in C^\omega_{\text{c}}(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be nonconstant. Then there exists a constant $c_0 = c_0(v)$ such that
\[
L(\alpha, A(E-\lambda v)) \geq \ln \lambda + c_0
\]
for all $(\alpha, E, \lambda) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \times (0, \infty)$.

**Remark 5.** It is interesting to note the difference between the Szegö and Schrödinger cases. In the Szegö case we remove a finite set and consider any compact set in its complement while in the Schrödinger case we need to remove nothing. Since the proof of both cases are basically the same from our method, one can easily see where this difference arises.

Similarly, to estimate the measure of the spectrum of the associated Schrödinger operators, we only need to assume $v$ is $C^1$, but we need some additional conditions. After a translation and scaling, we can, without loss of generality, assume $v(\mathbb{R}/\mathbb{Z}) = [0, 1]$. We further assume that:

1. $v$ has finitely many critical points;
2. for each $t \in [0, 1]$ outside of a finite set, $v'(x)$ takes different values for different $x \in v^{-1}(t)$. 
Let \( t = \frac{E}{\lambda} \in [0, 1] \) and \( \mathbb{R}/\mathbb{Z} \times [0, 1] \) be the set of \((\alpha, t)\); let \( \Delta_\epsilon(\lambda, \alpha) \) and \( \Gamma_\epsilon(\lambda) \) as in the Szegő case. Then our theorem is:

**Theorem B’**. Fixing arbitrary \( \epsilon > 0 \). Then for each Brjuno \( \alpha \),

\[
\lim_{\lambda \to \infty} \text{Leb}(\{0, 1\} \cap \Delta_\epsilon(\lambda, \alpha)) = 1
\]

and

\[
\lim_{\lambda \to \infty} \text{Leb}(\mathbb{R}/\mathbb{Z} \times [0, 1] \cap \Gamma_\epsilon(\lambda)) = 1.
\]

**Remark 6.** Again the corresponding estimate of \( \text{Leb}(\Sigma_{\lambda, \alpha}) \) is immediate by Fact A’. Here it is different from the Szegő case since \( t \) is related to \( E \) in a different way: we know for our \( v \), \( \Sigma_{\lambda, \alpha} \subset [−2, \lambda + 2] \); thus we obtain for Brjuno \( \alpha \),

\[
\lim_{\lambda \to \infty} \text{Leb}(\Sigma_{\lambda, \alpha}) = 1.
\]

It might seem that the conditions we put on \( v \) in theorem B’ are not very natural, but in fact we have the following corollary. First note that

\[
\Sigma_{\lambda, \alpha} \subset \left[ \lambda \inf_{x \in \mathbb{R}/\mathbb{Z}} v(x) − 2, \lambda \sup_{x \in \mathbb{R}/\mathbb{Z}} v(x) + 2 \right].
\]

Then we also have:

**Corollary 7.** Let the frequency \( \alpha \in \mathbb{R}/\mathbb{Z} \) be a Brjuno number and the potential function \( v \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \) be real analytic. Then we have

\[
\lim_{\lambda \to \infty} \frac{\text{Leb}(\Sigma_{\lambda, \alpha})}{\lambda(\sup v − \inf v) + 4} = 1.
\]

**Proof.** Again after a translation and scaling we can assume that \( v(\mathbb{R}/\mathbb{Z}) = [0, 1] \). Now if \( v \) is constant, then by Fact A’

\[
\Sigma_{\lambda, \alpha} = \{ E : |\text{tr}(A^{(E−\lambda v)})| \leq 2 \} = [\lambda v − 2, \lambda v + 2].
\]

Next we assume that the least positive period of \( v \) is 1. Then by analyticity, \( \{ x \in \mathbb{R}/\mathbb{Z} : v'(x) = 0 \} \) is a finite set. This is condition (1) for \( v \) in theorem B’. We assume that \( v \) violates condition (2). Then we can assume there are two sequences \( \{ x_n \}_{n \geq 1} \) and \( \{ y_n \}_{n \geq 1} \) such that

\[
v(x_n) = v(y_n),
\]

\[
\lim_{n \to \infty} x_n = x_0 \neq y_0 = \lim_{n \to \infty} y_n.
\]

Fix a sufficiently small number \( \gamma > 0 \); let \( B(x_0, \gamma) \) and \( B(y_0, \gamma) \) be neighbourhoods around \( x_0 \) and \( y_0 \) with radius \( \gamma \). Then we consider the analytic curves \( (x, v(x))_{x \in B(x_0, \gamma)} \) and \( (x, v(x))_{x \in B(y_0, \gamma)} \). What we want to show is that these two pieces of analytic curves coincide after a translation. Thus we are allowed to add some linear function \( k x \) so that we can assume that \( v'(x_0) = v'(y_0) \neq 0 \). Then by the inverse function theorem, there is an analytic function \( f : v(B(x_0, \gamma)) \to B(x_0, \gamma) \) such that \( f[v(x)] = x, x \in B(x_0, \gamma) \). Now we consider the analytic function \( g(x) = f[v(x) − x_0 + y_0], x \in B(x_0, \gamma) \). By assumption we have

\[
g'(x_0 − y_0 + y_n) = f'(v(y_n))v'(y_n) = f'(v(x_n))v'(x_n) = 1,
\]

where \( N \) is some large positive integer. Since \( g(x_0) = f[v(y_0)] = x_0 \), we have \( g(x) = f[v(x) − x_0 + y_0] = x \). By analyticity, we must have that \( v(x − x_0 + y_0) = v(x), x \in B(x_0, \gamma) \) and hence \( v(x − x_0 + y_0) = v(x), x \in \mathbb{R}/\mathbb{Z} \). Since \( x_0 − y_0 \in \mathbb{R}/\mathbb{Z} \) is not zero, this contradicts with the assumption that \( v(x) \) is of period 1. Thus theorem B’ implies the corollary 7 for this kind of \( v \).
Finally, if the least positive period of \( v \) is \( \frac{1}{n}, \ n > 1 \), then we can instead consider the dynamical system \((n\alpha, A(E^{-\lambda v})^{\frac{1}{n}})\). Indeed, we have the following facts:

1. \( L(\alpha, A(E^{-\lambda v})) = L(n\alpha, A(E^{-\lambda v})^{\frac{1}{n}}) \);
2. \((\alpha, A(E^{-\lambda v})) \in UH \Leftrightarrow (n\alpha, A(E^{-\lambda v})^{\frac{1}{n}}) \in UH \);
3. \( \alpha \) is Brjuno \( \Rightarrow n\alpha \) is Brjuno.

These facts obviously reduce the this case to the case that \( v \) is of period 1 and hence prove the corollary.

For the proof of these facts, (1) is straightforward. (2) follows from the fact that \( u \) is the unstable direction of \((\alpha, A(E^{-\lambda v}))\) if and only if \( u(E^{-\lambda v}) \) is the unstable direction \((n\alpha, A(E^{-\lambda v})^{\frac{1}{n}})\).

Same holds for the stable direction \( s \). Finally if \( \alpha \) is Brjuno, then for large \( s \) the \( s \)th continued fraction approximant \( q_s(n\alpha) \) of \( n\alpha \) is some \( c_s(n)q_l(\alpha) \), where \( q_l(\alpha) \) is the \( l \)th continued fraction approximant of \( \alpha \). Then it is not very difficult to see that \( c_{s+1}(n\alpha) = c_s(n)q_{l+1}(\alpha) \). Here \( \{c_s(n)\}_{s \geq 1} \) is a sequence of constants depending only on \( n \) and there is a constant \( c > 0 \) such that \( c^{-1} < c_s(n) < c, \ s \geq 1 \). This obviously implies that

\[
\sum_{s=1}^{\infty} \frac{\ln q_{s+1}(n\alpha)}{q_s(n\alpha)} < \infty,
\]

that is \( n\alpha \) is a Brjuno number. This completes the proof of corollary 7. \( \square \)

The same estimate in corollary 7 is obtained in [E] (see [E, theorem]) for a class of Gevrey potentials and Diophantine frequencies, where the class of potentials also includes all real-analytic functions.

2.4. Outline of the remainder of the paper

In section 3 we introduce acceleration for analytic \( SL(2, \mathbb{C}) \) cocycles: we state a main theorem about it and prove a weaker version which will be sufficient for this paper. Then we construct a class of uniformly hyperbolic systems. In section 4, based on section 3, we prove theorems A and A'. In section 5, we first state the main theorem in [Y] in a slightly different form, which makes the application easier. Then we use it to prove theorems B and B'. Finally in section 6 we end with some discussion.

3. Acceleration and uniform hyperbolicity

In this section we give a brief introduction of acceleration and some related results. Then we construct a class of uniformly hyperbolic systems.

3.1. Quantization of acceleration

The main tool we are going to use is the acceleration of quasiperiodic analytic \( SL(2, \mathbb{C}) \) cocycles, which was first introduced in [A2], where lots of important results for it have been proved. We will only introduce what we need here. If \( A \in C^0(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C})) \), then for each \( y \in (-\delta, \delta) \) we can define \( A_y \in C^0(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C})) \) by \( A_y(x) = A(x + iy) \). Then the acceleration is defined by

\[
\omega(\alpha, A) = \lim_{y \to 0^+} \frac{1}{2\pi y} (L(\alpha, A_y) - L(\alpha, A)),
\]
where the existence of the limit is guaranteed by the fact that the Lyapunov exponent $L(\alpha, A_y)$ is a convex function of $y$. Indeed, setting $l(y) = L(\alpha, A_y)$, we have
\[
l(y) = \inf_k \frac{1}{2^n} \int_{\mathbb{R}/\mathbb{Z}} \ln \| A_{2^n} (x + iy) \| \, dx.
\]
Thus if we complexify $y$, then $l(y)$ is the limit of a decreasing sequence of subharmonic functions which implies it is also subharmonic; furthermore the change of variable $x \mapsto x + \Im y$ in the integral shows that $l(y)$ is independent of $\Im y$. These together imply that $l(y)$ is convex in $\Re y$. Note also that this fact holds for all $\alpha$. While we will not use it explicitly, the following result underlies the philosophy of the proof of theorems A and A’.

**Theorem 8 (Acceleration is quantized).** The acceleration of analytic $\text{SL}(2, \mathbb{C})$ cocycles with irrational frequency is always an integer.

Note that one immediate consequence of theorem 8 is that $l(y)$ is piecewise linear for an irrational frequency. The proof of theorem 8 uses the continuity of Lyapunov exponents on
\[(\mathbb{R}/\mathbb{Z}) \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{C}))\],
which is proved in [JKS] (see [JKS, corollary 2]). But what we need here is the analogous result in the $U\mathcal{H}$ case, where the frequency can also be rational. As remarked in [A2], the proof in this case is much easier. We formulate and prove it.

**Theorem 9.** The acceleration $\omega(\alpha, A)$ is integer-valued and is $C^\infty$ on

$U\mathcal{H}_\omega = U\mathcal{H} \cap [\Re \times C^\omega(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{C}))]$.

Thus $\omega(\alpha, A)$ is constant on any connected component of $U\mathcal{H}_\omega$.

**Proof.** For $(\alpha, A) \in U\mathcal{H}_\omega$, we have that the unstable and stable directions $u, s \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{C}P^1)$ (see [A2, lemma 10]). Then we can let $B : \mathbb{R}/\mathbb{Z} \to \text{SL}(2, \mathbb{C})$ be analytic with column vectors in $u(x)$ and $s(x)$. Then $B$ diagonalizes $A$ as
\[
B(x + \alpha)^{-1} A(x) B(x) = \begin{pmatrix} r(x) & 0 \\ 0 & r(x)^{-1} \end{pmatrix}.
\]
Then it is easy to see that
\[
L(\alpha, A) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_{\mathbb{R}/\mathbb{Z}} \Re \ln r(x + j\alpha) \, dx = \int_{\mathbb{R}/\mathbb{Z}} \Re \ln r(x) \, dx.
\]
Note that $r : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is also analytic. By openness of $U\mathcal{H}$ and analyticity, we have that the corresponding upper-left entry of the diagonalized cocycle for $A_y(x)$ can be chosen to be $r(x + yi)$. Thus
\[
\omega(\alpha, A) = \lim_{y \to 0^+} \frac{L(\alpha, A_y) - L(\alpha, A)}{2\pi y} = \Re \int_{\mathbb{R}/\mathbb{Z}} \ln r(x + yi) - \ln r(x) \, dx = \Im \frac{1}{2\pi} \int_{\mathbb{R}/\mathbb{Z}} \ln r^{-1}(x)
\]
is the winding number of $r^{-1}$ about the origin and hence an integer.
We also have that $L(\alpha, A)$ is $C^\infty$ on $U\mathcal{H}_w$ (see [A2, section 1.3] for a detailed discussion). Thus
\[
\omega(\alpha, A) = \frac{1}{2\pi} \frac{\partial}{\partial y} L(\alpha, A),
\]
is also $C^\infty$ on $U\mathcal{H}_w$. □

It is interesting to note that using theorem 9 alone, we will give a self-contained proof of theorems $A$ and $A'$ in our paper.

3.2. A class of uniformly hyperbolic systems

The main tool we are going to use in this section is the polar decomposition of $SL(2, \mathbb{C})$ matrices, which will enable us to construct a class of uniformly hyperbolic systems. For $A \in SL(2, \mathbb{C})$ it is a standard result that we can decompose it as $A = U_1 \sqrt{A^* A}$, where $U_1 \in SU(2)$ and $\sqrt{A^* A}$ is a positive Hermitian matrix. We can further decompose $\sqrt{A^* A}$ as $\sqrt{A^* A} = U_2 \Lambda U_2^*$, where column vectors of $U_2$ are eigenvectors of $A^* A$ (thus $U_2$ can be chosen such that $U_2 \in SU(2)$) and $\Lambda = \text{diag}(\|A\|, \|A\|^{-1})$. Thus $A = U_1 U_2 \Lambda U_2^*$. By this decomposition procedure and after some fixed choice of $U_2$, we can consider $U_1$, $U_2$ and $\Lambda$ as maps from $SL(2, \mathbb{C})$ to $SL(2, \mathbb{C})$ so that for each $A \in SL(2, \mathbb{C})$

\[
A = U_1(A)U_2(A)\Lambda(A)U_2^*(A).
\]

Then we have the following claim:

**Lemma 10.** For some suitable choices of column vectors of $U_2(A)$, we have $U_1, U_2, \Lambda : SL(2, \mathbb{C}) \setminus SU(2) \to SL(2, \mathbb{C})$ are all $C^\infty$ maps. Here $C^\infty$ is in the sense that all these maps are between real manifolds.

**Proof.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \setminus SU(2)$. First note that by the above decomposition procedure we know that $\|A\|^2, \|A\|^{-2}$ are two eigenvalues of $A^* A$, thus
\[
\text{tr}(A^* A) = |a|^2 + |b|^2 + |c|^2 + |d|^2 = \|A\|^2 + \|A\|^{-2} > 2
\]
and
\[
\|A\|^2 = \frac{1}{2} (\text{tr}(A^* A) + \sqrt{\text{tr}(A^* A)^2 - 4}).
\]
Thus $\|A\|^2$ and hence $\|A\|$ are $C^\infty$ on $SL(2, \mathbb{C}) \setminus SU(2)$. This proves that $\Lambda \in C^\infty(SL(2, \mathbb{C}) \setminus SU(2), SL(2, \mathbb{C})).$

Let $U_2'$ be a matrix that diagonalizes $A^* A$. Then the column vectors are solutions of the equations
\[
(A^* A - \|A\|^2 I_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \text{and} \quad (A^* A - \|A\|^{-2} I_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,
\]
where $I_2$ is the $2 \times 2$ identity matrix. Thus we can choose $U_2'$ to be
\[
\begin{pmatrix} \tilde{a}b + \tilde{c}d, \\ \|A\|^2 - |a|^2 - |c|^2, \quad \|A\|^{-2} - |a|^2 - |c|^2 \end{pmatrix},
\]
which has a nonzero determinant since $A \notin SU(2)$. Then we can choose $U_2$ as
\[
U_2(A) = \frac{1}{\sqrt{\det(U_2'(A))}} U_2'(A).
\]
This shows that
\[ U_2 \in C^\infty(SL(2, \mathbb{C}) \setminus SU(2), SL(2, \mathbb{C})). \]
Finally since \( U_1(A) = AU_2(A) \Lambda^{-1}(A)U_2^*(A) \), we have
\[ U_1 \in C^\infty(SL(2, \mathbb{C}) \setminus SU(2), SL(2, \mathbb{C})). \]

Now we consider \((\alpha, A)\), where \( \alpha \in \mathbb{R}/\mathbb{Z} \) and \( A \in C^r(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C})) \) with \( r \in \mathbb{N} \cup \{\infty, \omega\} \). We assume further that \( A(x) \) is \( SU(2) \) free. As in lemma 10, we can decompose \( A(x) \) as \( A(x) = U_1(x)U_2(x)\Lambda(x)U_2^*(x) \). Then by lemma 10, \( U_1(x), U_2(x) \) and \( \Lambda(x) \) are \( C^r, 0 \leq r \leq \infty \) and \( C^\omega, r = \omega \) in \( x \in \mathbb{R}/\mathbb{Z} \). Let \( U_3(x) = U_1(x - \alpha)U_2(x - \alpha) \in SU(2) \). Then we have
\[ U_3(x + \alpha)^*A(x)U_3(x) = \Lambda(x)U(x), \]
where \( U(x) = U_2(x)^*U_1(x - \alpha)U_2(x - \alpha) \in SU(2) \). This is equivalent to
\[ (0, U_3)^{-1}(\alpha, A)(0, U_3) = (\alpha, \Lambda U). \]
Thus, we can instead consider the dynamical system \((\alpha, \Lambda U)\). Note that \( \alpha \) has already been involved when we transform \((\alpha, A)\) to \((\alpha, \Lambda U)\).

Now let \( U(x) = \left( \begin{smallmatrix} c(x) & -\overline{d(x)} \\ d(x) & c(x) \end{smallmatrix} \right) \), so \( |c(x)|^2 + |d(x)|^2 = 1 \), then we have the following lemma.

**Lemma 11.** Let \((\alpha, A)\) be a \( SU(2) \) free system with the equivalent system \((\alpha, \Lambda U)\), where \( U \) is as above; assume there exists a \( 0 < \gamma < 1 \) such that \( \inf_{x \in \mathbb{R}/\mathbb{Z}} |c(x)| \geq \gamma \); let
\[ \rho = \frac{1}{\gamma} + \sqrt{\frac{1}{\gamma^2} - 1} > 1. \]
If
\[ \inf_{x \in \mathbb{R}/\mathbb{Z}} \|A(x)\| = \lambda > \rho, \]
then \((\alpha, A) \in \mathcal{UH}\). Moreover, we have that
\[ L(\alpha, A) \geq \ln \lambda - \ln 2 \rho \]
for all \( \lambda \in (\rho, \infty) \).

**Proof.** The idea is to consider the projectivized dynamics \((\alpha, \Lambda U \cdot) : \mathbb{R}/\mathbb{Z} \times \mathbb{CP}^1 \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{CP}^1 \) and construct an invariant cone field. Here again \( \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \) and \( \Lambda U \cdot \) acts on \( \mathbb{CP}^1 \) as Möbius transformations, which we introduced at the beginning of section 2.2.

Let \( \lambda_1, \lambda_2 \) be arbitrary numbers satisfying \( \lambda > \lambda_2 > \lambda_1 > \rho \); let \( B(\infty, r) = \{ z \in \mathbb{CP}^1, |z| > r \} \) and \( B(0, r) = \{ z \in \mathbb{CP}^1, |z| < r \} \) for \( r > 0 \). Then we have the following facts:

1. \( (U(x) \cdot B(\infty, \lambda_1)) \cap B(0, \frac{1}{\lambda_1}) = \emptyset \) and
2. \( \Lambda(x) \cdot \tilde{B}(\infty, \frac{1}{\lambda_2}) \subset \Lambda(x) \cdot B(\infty, \frac{1}{\lambda_2}) \subset B(\infty, \lambda) \)

for all \( x \in \mathbb{R}/\mathbb{Z} \). Given this, we have
\[ (\Lambda U) \cdot B(\infty, \lambda_1) \subset \Lambda \cdot B \left( 0, \frac{1}{\lambda_1} \right)^c = \Lambda \cdot \tilde{B} \left( \infty, \frac{1}{\lambda_1} \right) \]
\[ \subset \Lambda \cdot B \left( \infty, \frac{1}{\lambda_2} \right) \subset B(\infty, \lambda) \subset B(\infty, \lambda_1). \]

Namely, \( B(\infty, \lambda_1) \) is an invariant cone field for \((\alpha, \Lambda U)\) which is also uniformly contracted into a sub-disc. Then \((\alpha, \Lambda U)\) and \((\alpha, A)\) is uniformly hyperbolic. Here we use the following
two facts:
(1) $\mathcal{U}$ is conjugate invariant. Indeed, if $(\alpha, A) \in \mathcal{U}$ and $u, s$ are the two associated invariant sections, then for arbitrary $B \in C^0(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))$, $B(x)^{-1} \cdot u(x)$ and $B(x)^{-1} \cdot s(x)$ are the two invariant sections of $B(x + \alpha)^{-1} A(x) B(x)$;

(2) $(\alpha, A) \in \mathcal{U}$ is equivalent to the existence of an invariant conefield (see [A1], section 2.1).

For the proof of (1) and (2), (2) is obvious. For (1), just note that when $|c| \geq \gamma$, $|d| = \sqrt{1 - |c|^2} \leq \sqrt{1 - \gamma^2}$ and let $t = \frac{|d|}{|c|} > \frac{\gamma}{\sqrt{1 - \gamma^2}}$, then

$$|U \cdot (re^{i\theta})| \geq \frac{tr - 1}{t + r} > \frac{1}{r}$$

for all $r > \rho$.

In fact, if we consider the function $g(t, r) = \frac{tr - 1}{t + r}$, it is symmetrical in $t$ and $r$ and increasing in both variables, and for $t = \frac{\gamma}{\sqrt{1 - \gamma^2}}$, $g(t, r) = \frac{1}{r}$ exactly at $r = \rho$.

For the estimate of the Lyapunov exponent, note that for $z \in B(\infty, \lambda_1)$, $|U \cdot z| > \frac{1}{\lambda_1}$ by the above argument. Thus for arbitrarily fixed $\lambda_1 \in (\rho, \lambda)$,

$$\frac{\|\Lambda(U(z), 1)^T\|}{\sqrt{|z|^2 + 1}} \geq \frac{\|\Lambda(u(z), 1)^T\|}{\sqrt{|z|^2 + 1}} \geq \sqrt{\frac{\lambda - 2 + \left(\frac{\lambda}{\lambda_1}\right)^2}{1 + \lambda_1^{-2}}} \geq \frac{\lambda}{2\lambda_1}.$$

Thus we have that for $z \in B(\infty, \lambda_1)$

$$L(\alpha, A) \geq \lim_{n \to \infty} \frac{1}{n} \ln \|\Lambda(U(z)) u(z)^T\| \geq \ln \lambda - \ln 2 \lambda_1,$$

where the first inequality follows from the definition of Lyapunov exponents and invariance of $B(\infty, \lambda_1)$. Now since the above inequality holds for all $\lambda_1 > \rho$, we obtain

$$L(\alpha, A) \geq \ln \lambda - \ln 2 \rho.$$

We continue to use the notation $A(x), U(x), U_1(x), U_2(x), \Lambda(x), c(x), d(x)$ in the remainder of this paper.

4. Uniformly positive Lyapunov exponents

In this section we prove theorems A and $A'$ in an unified way. Recall that the family of Szegö cocycles are given by $(\alpha, A^{(E, \lambda, \nu)})$, $0 < \lambda < 1$, where

$$A^{(E, \lambda, \nu)}(z) = (1 - \lambda z)^{-1/2} \begin{pmatrix} \sqrt{E} & -\frac{1}{\sqrt{E}} \nu(z) \\ -\lambda \sqrt{E} v(z) & 1 \end{pmatrix};$$

the family of Schrödinger cocycles are given by $(\alpha, A^{(\lambda(t - \nu(z)))})$, $0 < \lambda < \infty$, where

$$A^{(\lambda(t - \nu(z)))}(z) = \begin{pmatrix} \lambda (t - \nu(z))^{-1} & -1 \\ 1 & 0 \end{pmatrix}.$$
is the strip to where the analytic cocycle maps can be extended. Thus they share the parameters 
\( (z; \alpha, t; \lambda) \). For simplicity, we write the cocycle maps as \( A^{(\lambda)} \) in both cases. Then the unified strategy is as follows:

1. We prove the theorem step by step as introduced in the beginning of 4.1. Proof of main theorem: theorem A

2. By this strategy, it is clear that how we can actually construct a certain class of parametrized Lyapunov Exponents for Szegő cocycles.

3. Thus \( A^{(\lambda)} \) are \( SU(2) \) free for \( \lambda \) close to 1 in the Szegő case and for \( \lambda \) sufficiently large in the Schrödinger case. We also compute the upper-left entry of \( A(\lambda, r) \) explicitly.

4. Using theorem 9 to find an unique \( SU(2) \) free for \( \lambda \) close to 1 in the Szegő case and for \( \lambda \) sufficiently large in the Schrödinger case. Then we can fix in each case a compact region \( K \subset \mathbb{R}/\mathbb{Z} \times I \) of \( (\alpha, t) \), \( I \subset \mathbb{R} \) is a compact interval, with the following property. For each \( (\alpha, t) \in K \), the algebraic set \( \{ z \in \Omega_0 : g(z; \alpha, t) = 0 \} \) is finite.

5. We obtain uniformly positive Lyapunov exponents.

6. Using 9 to find an unique acceleration \( n_j \) on each \( O_j \times [\lambda_j, \infty) \).

By this strategy, it is clear that how we can actually construct a certain class of parametrized analytic \( SL(2, \mathbb{C}) \) cocycles with uniformly positive Lyapunov exponents.

4.1. Proof of main theorem: theorem A

Now we are ready to prove theorem A. We do it step by step as introduced in the beginning of this section.

**Step 1.** We start with the polar decomposition of \( A = A^{(E, \nu)} \), \( E = e^{2\pi i u} \). For simplicity, let \( z = x + yi \) and \( v(z) = \rho(z) e^{2\pi i b(z)} = \rho e^{2\pi i h} \), where both \( r \) and \( h \) are real valued function. Let \( a = a(\lambda, r) = r(\lambda - r^{-1}) + \sqrt{4 + (\lambda(\lambda - r^{-1}))} \), then obviously both \( r \) and \( a \) are uniformly bounded away from 0 and 0 in any compact subregion of \( \Omega_d \). A direct computation shows

\[
\|A(z)\| = \sqrt{\frac{2 + \lambda^2 (r^2 + r^{-2}) + \lambda (r + r^{-1}) \sqrt{4 + (\lambda(\lambda - r^{-1}))}}{2(1 - \lambda^2)}},
\]

which is uniformly of size \( \sqrt{\frac{1}{1 - \lambda^2}} \). In particular, if \( y = 0 \), then \( r = 1 \) and \( \|A(x)\| = \sqrt{\frac{1}{1 - \lambda^2}} \) for all \( x \in \mathbb{R}/\mathbb{Z} \). Also we obtain

\[
U_2(z) = \frac{1}{a^2 + 4} \begin{pmatrix} a & \frac{2}{F} e^{-2\pi i h} \\ -2 E e^{2\pi i h} & a \end{pmatrix}.
\]

Since it is easy to see that

\[
U(z) = U_2(z)^* A(z - \alpha) U_2(z - \alpha) A(z - \alpha)^{-1},
\]
we obtain the upper-left coefficient of \( U \) is
\[
c(z; \alpha, E; \lambda) = \frac{c_1}{\sqrt{E}} \left\{ a(z - \alpha) a(z) E + 4 e^{2\pi i [h(z) - h(z')]}
\right. \\
+ 2 E \lambda a(z) \left. \frac{r(z - \alpha) + 2 \lambda r(z - \alpha) a(z - \alpha) e^{2\pi i [h(z) - h(z')]}}{r(z - \alpha)} \right\},
\]
where
\[
c_1 = \frac{\|A(z - \alpha)\|^{-1}}{\sqrt{(a(z)^2 + 4)(a(z - \alpha)^2 + 4)(1 - \lambda^2)}}
\]
is uniformly bounded away from \( \infty \) and 0 for all \( z \) in any compact subregion of \( \Omega_\lambda \) and all \( \lambda \in [0, 1] \).

**Step 2.** If \( \lambda = 1 \), we have \( a(1, r) = 2r \), and thus
\[
c(z; \alpha, E; \lambda) = 4 e^{2\pi i h(z)} [r(z - \alpha) + r^{-1}(z - \alpha)] [E v(z) + v(z - \alpha)]
\]
where again \( |c_2|, |c_2^{-1}| \) are uniformly bounded. Thus, we can reduce the analysis of uniform positivity of \( C \) to that of \( g(z; \alpha, E) \) with \( g(z; \alpha, E) = E v(z) + v(z - \alpha) \). Then, we obtain
\[
E v(z) + v(z - \alpha) = 0 \iff \theta(z) = \theta(z - \alpha) = \frac{1}{2} - \frac{t}{k \alpha + m},
\]
where \( m \) is any integer; since we can choose \( \delta \) such that \( \theta(z) \) is uniformly bounded on \( \Omega_\lambda \), we only need to care about finitely many such \( m \). Now since \( q \) is the largest positive integer such that \( \theta(z + \frac{1}{q}) = \theta(z) \) and \( \theta \) is a nonconstant real-analytic function, these together imply that:

1. \( \theta(z) = \theta(z - \alpha) = \frac{1}{2} - t - k \alpha + m, \forall z \in \Omega_\lambda \), only if \( (\alpha, t) = (\frac{p}{q}, \frac{1}{2} + m - k \alpha + m) \), where \( p = 0, 1, \ldots, q - 1 \). Obviously, such pair \( (\alpha, t) \) is finite, where \( (\alpha, e^{2\pi i \nu}) \) is nothing other than our \( \mathcal{F} \) in the main theorem. We also denote \( \mathcal{F} \) as such pairs in \( (\alpha, t) \).
2. \( \theta(z) = \theta(z - \alpha) = \frac{1}{2} - t - k \alpha + m, \) has at most finite solutions in \( \Omega_\lambda \) otherwise.

**Step 3.** Now let \( C \) be as in theorem A. Then each \( (\alpha_j, e^{2\pi i \nu_j}) \in C \) satisfying condition (2). So for any \( (\alpha_j, e^{2\pi i \nu_j}) \in C \), we can find some height \( y_j \) such that
\[
|c(x + iy_j; \alpha_j, t_j; 1)|
\]
is bounded away from zero for all \( x \in \mathbb{R} / \mathbb{Z} \). Then for each \( (\alpha_j, e^{2\pi i \nu_j}) \in C \), we can find some connected open set \( \mathcal{O}_j \) satisfying \( (\alpha_j, e^{2\pi i \nu_j}) \in \mathcal{O}_j \subset (\mathbb{R} / \mathbb{Z} \times \partial \mathbb{D}) \setminus \mathcal{F} \) and some large \( \lambda_j > 0 \) such that
\[
|c(x + iy_j; \alpha, t; \lambda) |
\]
is bounded away from zero for all \( (x, \alpha, e^{2\pi i \nu_j}, \lambda) \in \mathbb{R} / \mathbb{Z} \times \mathcal{O}_j \times [\lambda_j, 1] \). Here we use the straightforward fact that for fixed \( (y_j, \alpha_j, t_j) \), as \( (\alpha, t, \lambda) \rightarrow (\alpha_j, t_j, 1) \) \( c(x + iy_j; \alpha, t; \lambda) \rightarrow c(x + iy_j; \alpha_j, t_j; 1) \) in \( C^0(\mathbb{R} / \mathbb{Z}, \mathbb{R}) \) as a function of \( x \). On the other hand, we know that \( \|A(x)\| \) is uniformly large and of size \( \sqrt{1 - \lambda} \).

**Step 4.** Now by lemma 11, without loss of generality, we can assume \( \lambda_j \) is such that there is a constant \( \eta_j = \eta_j(\lambda_j, \mathcal{O}_j) \) and for each \( (\alpha, E, \lambda) \in \mathcal{O}_j \times [\lambda_j, 1] \), the following two things
where

\(\|B^n\|_1 = \lambda^n\),

for all \(n \geq 1\). This completes the proof of main theorem.
4.2. Recovery of the Schrödinger case: proof of theorem A′

Without loss of generality, we assume $|v(z)| \leq 1$ on $\Omega$. Before we perform our common steps in section 4.1, we first use a simple trick to avoid that $A^{(E-\lambda v)}$ can always touch $SU(2)$ for $t = \frac{E}{\lambda} \in v(\mathbb{R}/\mathbb{Z})$, which leads to the discontinuity of the polar decomposition (this simple trick is also crucial in the proof of theorem B′, where uniform largeness of $\|A^{(E-\lambda v)}\|$ is required).

We instead consider $\hat{A} = TA^{(E-\lambda v)}T^{-1}$, where
\[
T = \begin{pmatrix}
\sqrt{\lambda}^{-1} & 0 \\
0 & \sqrt{\lambda}
\end{pmatrix}.
\]

This obviously does not change the dynamics. Now we carry out our steps.

**Step 1.** Now we instead consider the polar decomposition of
\[
\hat{A}^{(E-\lambda v)}(z) = \begin{pmatrix}
\lambda(t - v(z)) & -\lambda^{-1} \\
\lambda & 0
\end{pmatrix}.
\]

Let $r(z, t) = t - v(z)$; then $r(z, t)$ is uniformly bounded on $\Omega \times [-2, 2]$. If we set
\[
a = a(z, t, \lambda) = |r|^2 + 1 + \frac{1}{\lambda^4} + \left( |r|^2 + 1 + \frac{1}{\lambda^4} \right) - \frac{4}{\lambda^4},
\]
then obviously $a$ and $a^{-1}$ are uniformly bounded for all $(z, t, \lambda) \in \Omega \times [-2, 2] \times [\lambda_0, \infty)$, where $\lambda_0$ is any large positive number. Then a direct computation shows that $\|\hat{A}\| = \lambda \sqrt{T}$.

Thus, $\|\hat{A}\|$ is uniformly of size $\lambda$ as $\lambda \to \infty$. We also have
\[
\hat{\mathcal{U}}_2 = \frac{1}{\sqrt{(a - \frac{2}{\lambda^4})^2 + \frac{4}{\lambda^4} |r(z)|^2}} \begin{pmatrix}
a - \frac{2}{\lambda^4} & \frac{2r(z)}{\lambda^2} \\
\frac{2r(z)}{\lambda^2} & a - \frac{2}{\lambda^4}
\end{pmatrix}.
\]

For simplicity let $f(z, t, \lambda) = 1/\sqrt{(a - \frac{2}{\lambda^4})^2 + \frac{4}{\lambda^4} |r(z)|^2}$. Then we obtain that the corresponding upper-left element of $\hat{\mathcal{U}}$ is
\[
\hat{c}(z; \alpha, t; \lambda) = c_4 \left\{ r(z - \alpha) - \frac{2r(z)}{\lambda^2 a(z)} + \frac{2r(z - \alpha)}{\lambda^4 a(z)} - \frac{4r(z)}{\lambda^6 a(z - \alpha)a(z)} \right\},
\]
where
\[
c_4 = \sqrt{\frac{2}{a(z - \alpha)}} f(z - \alpha) f(z) a(z - \alpha) a(z)
\]
satisfies that $c_4$ and $c_4^{-1}$ are uniformly bounded for all $(z, \alpha, t, \lambda) \in \Omega \times \mathbb{R}/\mathbb{Z} \times [-2, 2] \times [\lambda_0, \infty)$.

**Step 2.** Let $g(z; \alpha, t; \lambda) = \frac{1}{\sqrt{T}} \hat{c}(z; \alpha, t; \lambda)$; then we can reduce the analysis of uniform positivity of $|\hat{c}(z; \alpha, t; \lambda)|$ to that of $|g(z; \alpha, t; \lambda)|$. Note that
\[
g(z; \alpha, t; \infty) = t - v(z - \alpha).
\]

**Step 3.** Now by the analyticity and nonconstancy of $v$, for each $t \in [-2, 2]$, we can pick some height $y_t$ such that
\[
| t - v(x + iy_t - \alpha) |
\]
is bounded away from zero for all \((x, \alpha) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\). Then for fixed \(y_t\), there is a small open interval \(I_t\) around \(t\) and a large \(\lambda_t > 0\) such that

\[
|g(x + iy_t; \alpha, s; t)|
\]
is bounded away from zero for all \((x, \alpha, s, \lambda) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times I_t \times \{\lambda_t, \infty\}\). Here we use the obvious fact that for fixed \(t\) and \(y_t\), as \((s, \lambda) \to (t, \infty)\), \(g(x + iy_t; \alpha, s; \lambda) \to g(x + iy_t; \alpha, t; \infty)\) in \(C^0(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}, \mathbb{R})\) as a function of \((x, \alpha)\).

Now by the compactness of \([-2, 2]\) we can find finitely many \(t\), say \(t_1, \ldots, t_k\), such that:

1. \([-2, 2] \subset \bigcup_j I_{t_j}\), and
2. \(|g(x + iy_{t_j}; \alpha, s; \lambda)|\) is bounded away from zero uniformly for all \((x, s, \lambda, \alpha) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times I_t \times [\lambda_t, \infty)\).

This implies \(|\hat{c}(x + iy_t; \alpha, s; \lambda)|\) are uniformly bounded away from zero for all \((x, \alpha, s, \lambda) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times I_t \times \Omega_0\), and \(|\hat{A}(x + iy_t)|\) is uniformly of size \(O(\lambda)\).

**Step 4–Step 6** are the same as those of section 4.1. This addresses the case \((t, \lambda) \in [-2, 2] \times [\lambda_0, \infty)\).

On the other hand, condition (2) above, concerning the estimates of \(|g(x + iy_\lambda; \alpha, t; \lambda)|\), is automatically satisfied for all \((t, \lambda) \in (\mathbb{R} \setminus [-2, 2]) \times [\lambda_0, \infty)\) for large \(\lambda_0 > 0\), because \(|\epsilon(z)| < 1\) on \(\Omega_0\). Since \(|\hat{A}(x + iy_t)|\) is uniformly of size \(O(\lambda)\), we can apply lemma 11 to these parameters simultaneously, which completes the proof of theorem \(A\).

Note all the necessary estimates in the Schrödinger case are for all \(\alpha \in \mathbb{R}/\mathbb{Z}\), which illustrates the difference between the Szegő and Schrödinger cases.

5. Nonuniform hyperbolicity

In this section, we prove theorems \(B\) and \(B'\) in an unified way. The main result we are going to use is the main theorem in [Y]. We first state it and give some discussion to make application easier.

5.1. Young’s theorem for nonuniformly hyperbolic \(SL(2, \mathbb{R})\) cocycles

Now let \(A(\cdot, t) \in C^r(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{R}))\), \(t \in [0, 1]\) and \(r \geq 1\), be a one parameter family of cocycle maps which is \(C^1\) in \((x, t)\); let \((\alpha, A(\cdot, t)U(\cdot, t))\) be the corresponding equivalent systems and \(c(x, t)\) is the upper-left element of \(U(x, t)\); let

\[
B(x, t, \lambda) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \quad A(x, t) = \begin{pmatrix} b(x, t, \lambda) & 0 \\ 0 & b(x, t, \lambda)^{-1} \end{pmatrix}
\]

and

\[
\Delta_c(\lambda, \alpha) = \{ t : (\alpha, A_t) \in \mathcal{N}\mathcal{U}\mathcal{H} \text{ with } L(\alpha, A_t) > (1 - \epsilon) \ln \lambda \}.
\]

Then the following theorem is in [Y] (see [Y, theorem 2]):

**Theorem 12.** Fixing arbitrary \(\epsilon > 0\). Let \(\alpha\) be a Brjuno number and \(A(\cdot, t)\) as above; assuming \(|b(x, t, \lambda)|^{\frac{1}{2}}\), \(|b(x, t, \lambda)|^{\alpha}\) are uniformly bounded for all \((x, t, \lambda) \in \mathbb{R}/\mathbb{Z} \times [0, 1] \times [\lambda_0, \infty)\) for some \(\lambda_0\) large. If \(c(x, t)\) is such that for each \(t\)

1. \(\{ x : c(x, t) = 0 \} \neq \emptyset\) and is finite;
2. \(x \mapsto c(x, t)\) is transversal to \(\{ x = 0 \}\);
3. \(\frac{\partial \gamma}{\partial t} / \frac{\partial \gamma}{\partial x}\) takes different values at different zeros of \(c(x, t)\).

Then \(\text{Leb}(\Delta_c(\lambda, \alpha)) \to 1\) as \(\lambda \to \infty\).
Conditions (1)–(3) are not exactly these in [Y], but it is not difficult to see the equivalence. Indeed, in [Y] the author identifies \( \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\} \) with \( \mathbb{R}/(\pi \mathbb{Z}) \), where the Möbius transformation of \( BU \) on \( \mathbb{RP}^1 \) is conjugated to be \( \widehat{BU} : \mathbb{R}/(\pi \mathbb{Z}) \to \mathbb{R}/(\pi \mathbb{Z}) \) via the commutative diagram

\[
\begin{array}{ccc}
\mathbb{R}/(\pi \mathbb{Z}) & \xrightarrow{BU} & \mathbb{R}/(\pi \mathbb{Z}) \\
\downarrow \cot & & \downarrow \cot \\
\mathbb{R} \cup \{\infty\} & \xrightarrow{BU} & \mathbb{R} \cup \{\infty\}
\end{array}
\]

Then she considers the function \( \beta(x,t) = (\widehat{BU}(x,t))^{-1}\left(\frac{\pi}{2}\right) \); then conditions (1)–(3) are stated in terms of \( \beta(x,t) \) in her main theorem. If we let \( c(x,t) = \cos(\varphi(x,t)) \), then the relation between \( \beta(x,t) \) and \( c(x,t) \) is

\[
\beta(x,t) = \frac{\pi}{2} - \varphi(x,t) + m\pi.
\]

Then it is easy to see the equivalence of these conditions in \( c(x,t) \) and in \( \beta(x,t) \). In fact, the equivalent conditions can be stated using any of the following functions:

\[
\beta(x,t), \quad \tan(\beta(x,t)) = \cot(\varphi(x,t)), \quad c(x,t) = \cos(\varphi(x,t)) \text{ and } \varphi(x,t) = \frac{\pi}{2} + m\pi.
\]

Furthermore, \( c(x,t) = \cos(\varphi(x,t)) \) can also be replaced by \( f(x,t)c(x,t) \), where \( f(x,t) \) satisfying that \( |f(x,t,\lambda)|, \ |\frac{\partial f(x,t,\lambda)}{\partial x}| \) is uniformly bounded for all \( (x,t,\lambda) \in \mathbb{R}/\mathbb{Z} \times [0,1] \times [\lambda_0, \infty) \).

Note condition (2) in theorem 12 does not imply the finiteness of \( \{x : c(x,t) = 0\} \) since we have functions such as \( x^3 \cos\left(\frac{1}{x}\right) \). There is no particular reason we should use \( t \in [0,1] \). In fact, \( [0,1] \) can be replaced by any connected interval and the result still holds (this also implies that conditions (1)–(3) can be violated at finitely many \( t \)).

5.2. Proof of theorem B

We need to consider \( \hat{A} = Q^*A^{(\alpha,\lambda)}Q \in SL(2,\mathbb{R}) \). By the definition of \( \hat{A} \) and the fact that \( Q \in U(2) \), we have \( \|\hat{A}(x)\| = \sqrt{1 + \chi} \) (this implies the corresponding \( b(x,t) \) in theorem 12 is constantly 1, which obviously satisfies the conditions) and the corresponding \( \hat{U}_2(x) = Q^*U_2(x)P(x) \in SO(2) \), where

\[
P = \begin{pmatrix}
\frac{e^{\frac{\pi}{2}}}{\sqrt{Ev(x)}} & 0 \\
0 & \frac{e^{-\frac{\pi}{2}}}{\sqrt{Ev(x)e^{-\frac{\pi}{2}}}}
\end{pmatrix}.
\]

Let \( (\alpha, \Lambda \hat{U}) \) be the corresponding equivalent system with \( \hat{U} \in SO(2) \). Then a direct computation shows that the corresponding \( \hat{c}(x,t) \) of \( \hat{U}(x) \) is

\[
\hat{c}(x; \alpha, t) = \cos \pi[\theta(x) - \theta(x - \alpha) + k\alpha + t].
\]

Hence by the discussion following theorem 12, we can instead consider

\[
\varphi(x,t,\alpha) = \theta(x) - \theta(x - \alpha) + k\alpha + t + \frac{m}{2}.
\]

Let \( f(x) = \theta(x - \alpha) - \theta(x) - \frac{1}{2} - k\alpha \). We set for each Brjuno \( \alpha, I_\alpha = f(\mathbb{R}/\mathbb{Z}) \cap \mathbb{R}/\mathbb{Z} \). Then we can show \( \varphi(x,t,\alpha) \) satisfies conditions (1)–(3) in theorem 12 for each \( t \in I_\alpha \). Indeed, (1) is obviously satisfied for each \( t \in I_\alpha \) by our choice of \( \theta(x) \). For (2), we note for each
irrational $\alpha$, $\frac{d\theta}{dx}(x, t, \alpha) = \theta'(x) - \theta'(x - \alpha) = 0$ has only two solutions which are independent of $t$. Thus, except finitely many $t$, (2) is satisfied for each $t \in I_\alpha$. For (3), we note $\frac{d\theta}{dx} \equiv 1$ is nonzero and independent of $x$; thus we only need that $\frac{d\theta}{dx}$ takes different values for different zeros of $\varphi(x, t)$, which again is equivalent to that $\theta'(x) - \theta'(x - \alpha)$ takes on distinct values. But we know for irrational $\alpha$, $\theta'(x) - \theta'(x - \alpha) = 0$ exactly at two points; furthermore, we also assumed $\text{Leb}[\theta(\mathbb{R}/\mathbb{Z})] \leq \frac{1}{\pi}$, which implies $\text{Leb}[(\theta(\cdot) - \theta(\cdot - \alpha)](\mathbb{R}/\mathbb{Z})) \leq 1$. These together imply that $\theta'(x) - \theta'(x - \alpha)$ takes different values at level sets of $\theta(x) - \theta(x - \alpha)$.

Now set $\mathcal{K} = \bigcup_{\lambda \in \mathbb{I}_\alpha} \text{Leb}(I_\alpha \cap \Delta_\delta(\lambda, \alpha))$; then theorem 12 implies for Lebesgue almost every $\alpha$, $\text{Leb}(I_\alpha) \to \text{Leb}(I_\alpha)$ as $\lambda \to 1$. Hence, bounded Convergence theorem implies

$$\text{Leb}(\mathcal{K} \cap \Gamma_\varepsilon(\lambda)) = \int_{\mathbb{R}/\mathbb{Z}} s(\alpha, \lambda) d\alpha \to \int_{\mathbb{R}/\mathbb{Z}} \text{Leb}(I_\lambda) d\alpha = \text{Leb}(\mathcal{K})$$

as $\lambda \to 1$. This completes the proof.

Remark 13. By the proof of the main theorem, it is easy to see that any region in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ which is away from $\mathcal{K}$ with a positive distance is in $\mathcal{U}_\alpha$ as $\lambda \to 1$. On the other hand, obviously we have $\text{Leb}(I_\lambda) \to 0$ as $\alpha \to 0$. If we take $v(x) = e^{i\pi \cos(2\pi x)}$, then $\text{Leb}(I_\lambda) \to 1$ as $\alpha \to \frac{1}{2}$. Thus $\text{Leb}(\Sigma) \to 1$ as $\lambda \to 1$ and Brjuno $\alpha \to \frac{1}{2}$. Namely, we have constructed some analytic quasiperiodic 2-sided Verblunsky coefficients, of which the associated $\mu_x$ satisfies that $\text{supp}(\mu_x)$ can be arbitrarily close to full measure.

5.3. Proof of theorem B'

For the proof of theorem B' we continue to use the polar decomposition of proof of theorem A' in section 4.2. Note here we assume $v(\mathbb{R}/\mathbb{Z}) = [0, 1]$ and we restrict to $y = 0$.

Recall we have $\|A\| = \lambda \sqrt{2}$, where for $t = \frac{E}{2}$ and $r(x) = t - v(x)$,

$$a = a(x, t, \lambda) = r(x)^2 + 1 + \frac{1}{\lambda^2} + \sqrt{r(x)^2 + 1 + \frac{1}{\lambda^2}} - \frac{4}{\lambda^2}$$

satisfies that $a$ and $a^{-1}$ are uniformly bounded for all $(x, t, \lambda) \in \mathbb{R}/\mathbb{Z} \times [0, 1] \times [\lambda_0, \infty]$. Thus the corresponding $b(x, t, \lambda) = \sqrt{\frac{4}{\lambda}}$ satisfies all the conditions in theorem 12. We also have

$$\hat{c}(x; t; \lambda) = c_4 \left\{ \frac{2r(x)}{\lambda^2 a(x)} + \frac{2r(x - \alpha)}{\lambda^2 a(x)} - \frac{4r(x)}{\lambda^6 a(x - \alpha)} \right\},$$

where

$$c_4 = \sqrt{\frac{2}{a(x - \alpha)}} f(x - \alpha) f(x) a(x - \alpha) a(x)$$

and $f(x, t, \lambda) = 1/\sqrt{(a - \frac{2}{\pi})^2 + 4r(x)^2}$. Hence, we have

$$\hat{c}(x; t; \infty) = \frac{t - v(x - \alpha)}{\sqrt{(t - v(x - \alpha))^2 + 1}}.$$

Furthermore it is not difficult to see that for any fixed $\alpha$,

$$\hat{c}(x; \alpha, t; \lambda) \to \hat{c}(\alpha; \alpha, t; \infty)$$

in $C^1(\mathbb{R}/\mathbb{Z} \times [0, 1], \mathbb{R})$ as $\lambda \to \infty$.

Indeed, it is easy to see this reduces to the convergence of $a(x, t, \lambda)$ to $a(x, t, \infty)$ in $C^1$ topology, which is immediate.
Now by the discussion following theorem 12, conditions (1)–(3) for \( \hat{c}(x, t) \) in theorem 12 can be reduced to these for \( t - v(x - \alpha) \) for sufficiently large \( \lambda \), which are immediate by our assumption on \( v(x) \) in theorem B'. Indeed, outside of a finite set of \( t \) containing critical values of \( v \), assumption (1) implies conditions (1)–(2) of theorem 15 and assumption (2) is the same as condition (3).

By exactly the same proof of theorem B, it follows that
\[
\text{Leb}(\mathbb{R}/\mathbb{Z} \times [0, 1] \cap \Gamma'_t(\lambda)) \to 1 \quad \text{as} \quad \lambda \to \infty.
\]
This completes the proof of theorem B'.

**Remark 14.** In fact, the proof of theorem 12 in [Y] contains more information: for each \( \alpha \) Brjuno and \( t \in \Delta_v(\lambda, \alpha) \), there are finitely many \( x_i \in \mathbb{R}/\mathbb{Z} \), say \( i = 1, \cdots, l \), along whose orbits there are sequences \( \{\|A_n^{(E, \lambda)}(x_i)\|_{n \in \mathbb{Z}} \} \) (for some \( w \in \mathbb{R}^2 \setminus \{0\} \)) which decay exponentially as \( n \to \pm \infty \). Namely, let \( E \in \lambda \Delta_v(\lambda, \alpha) \) and \( x_i \in \mathbb{R}/\mathbb{Z} \) be critical points of \( (\alpha, A^{(E, \lambda)}) \); then \( E \) is an eigenvalue of operator \( H_{\alpha, \lambda, x_i} \) with an exponential decay eigenfunction. This is the so-called Anderson Localization. \( \{x_1, \ldots, x_l\} \) is the so-called ‘critical set’, the existence of which leads to nonuniform hyperbolicity. Another fact that we can obtain from the proof of theorem 12 is that in our Schrödinger cocycle setting, both the ‘initial critical set’ (zeros of \( \hat{c}(x; \alpha, t) \) and ‘critical set’ converge to zeros of \( t - v(x - \alpha) \) as \( \lambda \to \infty \). This gives an explicit description of a number of ‘critical points’. We can relax our conditions on \( v(x) \) to obtain weak results. Namely any \( C^1 \) potential function \( v \) with an interval \( I \) in its image satisfying conditions (1)–(3) can be our choice (for example, we can relax \( v(x) \) to have an unique minimum or maximum and pick a small interval \( I \) of \( t \) near this minimal or maximal value). This method obviously allows us to produce cocycles \( (\alpha, A^{(E, \lambda)}) \) which have \( 2n \) ‘critical points’, where \( n \geq 1 \) can be any natural number.

There is another theorem in [Y] (see [Y, theorem 1]) taking the frequency \( \alpha \) as a parameter, the proof of which is basically the same as that of theorem 12. Combining this theorem with the Sard theorem and our decomposition procedure, it is easy to see that if we fix an arbitrary \( C^1 \) potential \( v \) and arbitrary \( \epsilon > 0 \), then for almost every \( E \in \lambda v(\mathbb{R}/\mathbb{Z}) \),
\[
\lim_{\lambda \to \infty} \text{Leb}(\alpha: (\alpha, A^{(E, -\lambda \epsilon)}) \in \mathcal{NUH} \quad \text{and} \quad L(\alpha, A^{(E, -\lambda \epsilon)}) > (1 - \epsilon) \ln \lambda) = 1.\quad (2)
\]

6. Discussion

In the Schrödinger case, under the condition \( |v| \leq 1 \) on \( \Omega_d \), a little bit more computation shows that for any fixed \( \lambda > 1 \), \( (\alpha, A^{(E, -\lambda \epsilon)}) \) satisfies all the conditions in lemma 11 for all \( E \in \mathbb{R} \setminus [-3 - \lambda, 3 + \lambda] \) (in the case of theorem B', this also implies that \( E \in \mathbb{R}: L(E) > (1 - \epsilon) \ln \lambda \) tends to be a full measure set as \( \lambda \to \infty \). The uniform hyperbolicity result implied by lemma 11 is nothing new, because \( \mathbb{R} \setminus [-2 - \lambda, 2 + \lambda] \) is in the resolvent set and we have the basic fact related uniform hyperbolicity and resolvent set. The new fact is that by lemma 11, for \( E \) and \( \lambda \) in the case above, \( (\alpha, A^{(E, -\lambda \epsilon)}) \) admits an invariant cone field such that for each vector in this cone, it is expanded under forward iteration on each step. This is uniform hyperbolicity in some strong sense and is not true for \( E \) in the spectral gap.

More concretely, as in the proof of theorem 9, \( (\alpha, A^{(E, -\lambda \epsilon)}) \in \mathcal{UH} \) can be analytically conjugated to a diagonal system
\[
\left( \begin{array}{cc}
\alpha, & 0 \\
0, & r(x)^{-1}
\end{array} \right)
\]

The author is grateful to Anton Gorodetski and Vadim Kaloshin for showing me their notes, where they pointed this out.
via its stable and unstable direction (where we assume \( r(x) \) corresponds to unstable direction). Then for \((E, \lambda)\) satisfying conditions in lemma 11, we have \(|r(x)| \geq \epsilon > 1\) for all \(x \in \mathbb{R}/\mathbb{Z}\). We emphasize here that the invariant cone fields are for the original system \((\alpha, A(E-\lambda v)).\) Because for \(\alpha \) and \(v \in C^\omega_{\mathbb{C}}(\mathbb{R}/\mathbb{Z}, \mathbb{R})\) satisfying \(\delta > \beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}\) (where \(q_n\) is the continued fraction approximants of \(\alpha\)), we can always analytically conjugate the diagonal system to the constant system

\[
\begin{pmatrix}
\alpha, & e^{L(\alpha, A(E-\lambda v))}
\end{pmatrix}
\]

by solving a simple cohomological equation:

\[
\ln |b(x + \alpha)| - \ln |b(x)| = \ln |r(x)| - L(\alpha, A(E-\lambda v))
\]

Note that in both cases, for large couplings, one can always choose a suitable height \(y\) such that \((\alpha, Ay)\) is uniformly hyperbolic. This fact reflects another important theorem in [A2].

Recall that for irrational \(\alpha, y \mapsto L(\alpha, A_y)\) is a piecewise affine function in \(y\), thus it is natural to give the following definition.

**Definition.** We say that \((\alpha, A) \in (\mathbb{R}/\mathbb{Z}) \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))\) is regular if \(L(\alpha, Ay)\) is affine for \(y\) in a neighbourhood of 0.

Then the theorem in [A2] (see [A2, theorem 6]) is:

**Theorem 15.** Assume that \(L(\alpha, A) > 0\). Then \((\alpha, A)\) is regular if and only if \((\alpha, A)\) is uniformly hyperbolic.

Thus if \(a\) priori we know \(L(\alpha, A) > 0\), then for each nonzero height \(y\) sufficiently close to 0, \((\alpha, Ay)\) is always uniformly hyperbolic.

In both cases, the essential obstruction is that \(c(x; \alpha, t)\) oscillates around 0, which forces us to consider the real-analytic case and choose suitable heights to obtain uniformly positive Lyapunov exponents.

On the other hand, we know for some \(t\) and \(\{x : c(x; \alpha, t) = 0\} \neq \emptyset\), we can still have uniformly hyperbolic systems, for example, the \(E\) in the spectral gap of Almost Mathieu operators \((v(x) = 2 \cos(2\pi x))\) with \(\lambda > 2\) can be our choice (see [AJ, main theorem]). Thus, it will be very interesting to understand that, under the condition \(\{x : c(x, t) = 0\} \neq \emptyset\) and with uniformly positive Lyapunov exponents, how can one go between uniform hyperbolicity and nonuniform hyperbolicity when \(t\) varies.

The proof of theorem 12 implies that in the case of theorem \(B'\), it is exactly these \(t\), near which resonance occurs or near a critical value of \(v\), that have been excluded. Here, for example, since the induction step starts at the continued fractional approximant \(q_N\) for some large \(N\), resonance at the initial step means that there exist some \(x_0\) and \(1 \leq k < q_N\) such that

\[
t = v(x_0)\text{ and } |v(x_0) - v(x_0 + k\alpha)| \ll \frac{1}{q_N^2}.
\]

Thus the natural next step is to study these \(t\) and perform the following possible generalization: for a fixed Diophantine frequency, put some additional conditions on \(v\), like higher but finite regularity (for example, \(C^3\)) and nondegeneracy of critical points (i.e. \(v''(x_0) \neq 0\) where \(v'(x_0) = 0\)) to obtain the positive Lyapunov exponents for all \(E\) for sufficiently large couplings.
(the difference between this result and that in [C] is the following: here one tries to fix frequency and potential while in [C] the author eliminates frequencies and varies potentials). As we stated in section 1.2.3, a new induction step is needed to take care of the appearance and disappearance of ‘critical points’ near resonance. One can even try to prove Anderson Localization (AL) for almost every phase or try to produce counterexamples such that AL does not hold. Similar problems are also proposed in [Kl].

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