Triangular-Grid Billiards and Plabic Graphs

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Based on joint work with Pakawut (Pro) Jiradilok.

FPSAC
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The billiards permutation $\pi$ of this polygon $P$ is

$$(1 \ 3 \ 32 \ 26 \ 6 \ 30 \ 2 \ 33 \ 25 \ 12 \ 14 \ 9 \ 21 \ 19 \ 29 \ 28 \ 4 \ 31)(5 \ 24 \ 13 \ 10 \ 20 \ 27)(7 \ 22 \ 23 \ 15 \ 17)(8 \ 11 \ 18 \ 16).$$
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Triangular-Grid Billiards

The billiards permutation $\pi_P$ of this polygon $P$ is

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Main Theorem

Throughout this talk, \( P \) is a polygon in the triangular grid.

Theorem (D.–Jiradilok, 2023)

\[
\text{area}(P) \geq 6 \text{cyc}(P) - 6
\]

and

\[
\text{perim}(P) \geq 7\frac{2}{2} \text{cyc}(P) - 3\frac{2}{2}.
\]

Also,

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\text{area}(P) = 6 \text{cyc}(P) - 6
\]

if and only if \( P \) is a “tree of unit hexagons.”

Conjecture (D.–Jiradilok, 2023)

We have

\[
\text{perim}(P) \geq 4 \text{cyc}(P) - 2.
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**Theorem (D.–Jiradilok, 2023)**

We have $\text{area}(P) \geq 6 \cyc(P) - 6$ and $\text{perim}(P) \geq \frac{7}{2} \cyc(P) - \frac{3}{2}$. Also, $\text{area}(P) = 6 \cyc(P) - 6$ if and only if $P$ is a “tree of unit hexagons.”
Main Theorem

Throughout this talk, $P$ is a polygon in the triangular grid.

**Theorem (D.–Jiradilok, 2023)**

We have $\text{area}(P) \geq 6 \cdot \text{cyc}(P) - 6$ and $\text{perim}(P) \geq \frac{7}{2} \cdot \text{cyc}(P) - \frac{3}{2}$. Also, $\text{area}(P) = 6 \cdot \text{cyc}(P) - 6$ if and only if $P$ is a “tree of unit hexagons.”

**Conjecture (D.–Jiradilok, 2023)**

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A plabic graph is a planar graph embedded in a disc such that each vertex is colored either black or white.

The trip permutation of this plabic graph is the cycle $(1\ 3\ 5\ 2\ 4)$. 

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Triangular-Grid Billiards
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![Diagram of a plabic graph](image-url)
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![Plabic Graph Diagram](image-url)
Plabic Graphs from Grid Polygons

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Triangular-Grid Billiards
Membranes are certain triangulated surfaces in Euclidean space defined by Lam and Postnikov.
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\[ \phi : \{ \text{membranes} \} \to \{ \text{reduced plabic graphs} \}. \]

Lam and Postnikov showed that if \( G \) has \( n \) marked boundary points, then its essential dimension is at most \( n - 1 \).

Connected reduced plabic graphs with essential dimension 2 are exactly those coming from triangular-grid polygons.
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Reformulation of Main Theorem

Let $G$ be a connected reduced plabic graph with essential dimension 2. Suppose $G$ has $n$ marked boundary points and $v$ vertices, and let $c$ be the number of cycles in the trip permutation $\pi^G$.

Corollary (D.–Jiradilok, 2023)
We have $v \geq 6c - 6$ and $n \geq \frac{7}{2}c - \frac{3}{2}$.

Corollary (Honglin Zhu, last Friday++)
We have $n \geq 4c - 2$.

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We have

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Let $G$ be a connected reduced plabic graph with $n$ marked boundary points, $v$ vertices, and $c$ cycles in its trip permutation.

Problem (D.–Jiradilok, 2023)

Find inequalities relating $n$ and $v$ to $c$ when $G$ is taken from some "interesting" family of plabic graphs.
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Find inequalities relating $n$ and $v$ to $c$ when $G$ is taken from some “interesting” family of plabic graphs.
Problem (D.–Jiradilok, 2023)
Obtain analogues of our results for billiards systems in triangular-grid polygons with holes cut out.
Future Directions: Random Polygons

Question (D.–Jiradilok, 2023)

What can one say about $cyc(P)$ when $P$ is a large random triangular-grid polygon?
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Future Directions: Unitrajectorial Polygons

Question (D.–Jiradilok, 2023)

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What can one say about the triangular-grid polygons $P$ such that $\text{cyc}(P) = 1$?
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For $w \in P$ and $j \in \mathbb{Z}/(d + 1)\mathbb{Z}$, let
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\tau_j(w) = \begin{cases} 
  s_jw & \text{if } s_jw \in P \\
  w & \text{if } s_jw \not\in P.
\end{cases}
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$$\tau_j(w) = \begin{cases} s_j w & \text{if } s_j w \in P \\ w & \text{if } s_j w \notin P. \end{cases}$$

Start at an alcove in $P$ and apply the sequence

$$\tau_0, \tau_1, \tau_2, \ldots, \tau_d, \tau_0, \tau_1, \tau_2, \ldots, \tau_d, \tau_0, \tau_1, \tau_2, \ldots, \tau_d, \ldots.$$
Other Future Directions: Higher Dimensions

Problem (D.–Jiradilok, 2023)

Compare the number of trajectories in $P$ with $|P|$.
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THANKS!