Generalized Quantum Inverse Scattering

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ABSTRACT

A generalization of the quantum inverse scattering method is proposed replacing the quantum group $RLL$ commutation relations of Lax operators by reflection equation type $RLRL$ commutation relations. Under some natural assumptions the most general algebra of this type allowing to construct the necessary integrals of

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motion is found. It serves to describe Lax operators with completely non-ultralocal commutation relations. An example of this new formalism is an integrable model on monodromies of flat connections on a Riemann surface which is related to the XXZ quantum spin chain.
1. INTRODUCTION

The quantum inverse scattering method (QISM) was developed at the end of the '70s by the Leningrad School in order to handle quantum versions of classically integrable models in 1+1 dimensions [1]. The algebraic framework that characterizes the method was interpreted later as a deformation of Poisson-Lie groups [2] and led to the discovery of quantum groups (QG).

For our purposes the formulation on discretized space with \( N \) lattice points (time dependence is suppressed throughout) will be appropriate. Given a model on such a space it is sometimes possible to find so-called Lax operators satisfying a linear differential equation encoding the equations of motion which can be derived from the Hamiltonian of the model. Then the fundamental commutation relations of the Lax operators \( L^n(\lambda) \) at lattice sites \( n = 1, \ldots, N \) have the form

\[
R_{12}(\lambda - \mu)L^2_1(\lambda)L^2_2(\mu) = L^2_2(\mu)L^2_1(\lambda)R_{12}(\lambda - \mu) \\
L^n_1(\lambda)L^n_2(\mu) = L^n_2(\mu)L^n_1(\lambda), \quad m \neq n
\]  

(1.1)

where all quantities depend on spectral parameters \( \lambda, \mu \in \mathbb{C} \). Lax operators of different sites commute, this is referred to as ultralocality. Indices \( i = 1, 2 \) denote the auxiliary spaces \( V_i \) of dimension \( r \) on which the matrices \( L^n_i = L^n \otimes 1, L^n_2 = 1 \otimes L^n \) and \( R_{12} \) act nontrivially (we employ the usual quantum group terminology, see [3, 4] for example). The matrix \( R \) of dimension \( 2r \times 2r \) satisfies the Yang-Baxter equation (YBE)

\[
R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu).
\]  

(1.2)

As a consequence of ultralocality the product \( L^{n+1}(\lambda)L^n(\lambda) \) also satisfies the fundamental commutation relations (this is equivalent to existence of a coproduct for the algebra of Lax operators). By iteration the same holds for the monodromy
matrix

\[ T(\lambda) = L^N(\lambda)L^{N-1}(\lambda) \cdots L^1(\lambda), \quad (1.3) \]

and (1.1) implies that the operators \( t(\lambda) = Tr[T(\lambda)] \) are commuting for different values of spectral parameters

\[ [t(\lambda), t(\mu)] = 0. \quad (1.4) \]

Expanding \( t(\lambda) \) in powers of \( \lambda \) one obtains a set of mutually commuting operators and, moreover, they are conserved because it can be shown that the Hamiltonian is among them. This gives a systematic procedure to obtain all the integrals of motion necessary for integrability of the model.

In this paper we suggest a generalization of the quantum inverse scattering method based on reflection equation (RE) type algebras rather than the RLL algebra. This way we can describe Lax operators which are mutually non-commuting for any pair of lattice sites. The idea for such a generalized formalism relies on our experience with so-called extended RE algebras [5] (see also [6, 7]) and the explicit example of an integrable model that can be constructed out of them [8]. Motivated by Chern-Simons theory in [8] an integrable model was introduced on the moduli space of flat connections on Riemann surfaces which is related to the XXZ quantum spin chain. For motivation and to show that the subsequently developed formalism is not empty we outline the main points of that model in the following. We use the notation of [5] where part of the construction was also carried out in order to describe the braid group on a handlebody.

Single reflection equations without spectral parameters were investigated in [9, 10].

Extended RE algebras consist of \( N \) reflection equations for matrices \( K^n, n = 1, \ldots, N \) of size \( r \times r \) with operator valued entries belonging to some algebra \( \mathcal{A} \), and additional commutation relations between them

\[ RK^n_1 \tilde{R} K^n_2 = K^n_2 R K^n_1 \tilde{R}, \]

\[ RK^m_1 R^{-1} K^n_2 = K^n_2 R K^m_1 R^{-1}, \quad m > n \quad (1.5) \]
where $R = R_{12}$, $\tilde{R} = R_{21} \equiv PRP$ and $P$ is the permutation operator (we suppress indices where possible). Here the matrix $R$ of size $2r \times 2r$ is any invertible solution of the constant (spectral parameter independent) Yang-Baxter equation (1.2), but for the following example we restrict it to the $R$-matrix of $sl_q(2)$ given by

$$
R = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \omega & 1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}, \quad \omega = q - q^{-1}.
$$

One of the main properties of (1.5) is invariance w.r.t. QG coaction, i.e. the transformed $K^n_T = T^{(n)} K^n (T^{(n)})^{-1}$ are also solutions of (1.5) if $K^n_1 T^{(m)}_2 = T^{(m)}_2 K^n_1$ for all $m, n$, i.e. all elements of $K^n$ and $T^{(m)}$ are commuting and $T^{(n)}$ obeys the system of QG relations

$$
RT_1^{(n)} T_2^{(n)} = T_2^{(n)} T_1^{(n)} R, \\
RT_1^{(m)} T_2^{(n)} = T_2^{(n)} T_1^{(m)} R, \quad m > n.
$$

A further property of (1.5) is that $K^{n+1} K^n$ satisfies also the extended RE algebra (this can be interpreted as a braided coproduct for the extended RE algebra in the sense of [11]). In fact, the same holds for any strictly ordered product of $K$-matrices with values of indices decreasing from left to right, for the second equation of (1.5) it means that $m$ has to be greater than the largest index in the ordered product.

Now we describe a representation of (1.5) in terms of $sl_q(2)$ algebra generators $H, X^\pm$ which obey the relations

$$
[H, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = \omega^{-1}(q^H - q^{-H}).
$$

They can be conveniently rewritten in terms of three matrix equations

$$
\tilde{RL}_1^{\varepsilon_1} L_2^{\varepsilon_2} = L_2^{\varepsilon_2} L_1^{\varepsilon_1} \tilde{R}, \quad (\varepsilon_1, \varepsilon_2) \in \{(+, +), (+, -), (-, -)\}
$$
where the $L^\pm$ are triangular matrices expressed in terms of $sl_q(2)$ generators

\[
L^+ = \begin{pmatrix}
q^{H/2} & q^{-1/2}\omega X^- \\
0 & q^{-H/2}
\end{pmatrix},
L^- = \begin{pmatrix}
q^{-H/2} & 0 \\
-q^{1/2}\omega X^+ & q^{H/2}
\end{pmatrix}.
\tag{1.10}
\]

These matrices can be formally inverted by applying the so-called antipode map $S$ to the generators defined by $S(H) = -H$, $S(X^\pm) = -q^{\mp 1}X^\pm$. Define

\[
K^1 = S(L^-)L^+ = \begin{pmatrix}
q^H & q^{-1/2}\omega q^{H/2}X^- \\
q^{-1/2}\omega X^+ q^{H/2} & q^{-H} + q^{-1}\omega^2 X^+ X^-
\end{pmatrix},
\tag{1.11}
\]

it is straightforward to show with help of (1.9) that $K^1$ satisfies the first equation of (1.5). To represent the whole algebra (1.5) we define further

\[
L_i^\pm = 1 \otimes \cdots 1 \otimes L^\pm \otimes 1 \cdots 1,
1 \leq i \leq N
\tag{1.12}
\]

with $L^\pm$ inserted into the $i$-th position. The dot over the tensor product means matrix multiplication of these $2 \times 2$ matrices such that (1.12) again is a $2 \times 2$ matrix whose entries take value in the $N$-fold tensor product of the (universal enveloping) quantum algebra $sl_q(2)$. Operators contained in different spaces of the tensor product are commuting. Setting $K_i = S_i^- L_i^+$, we then have the following set of operators $K^n$ satisfying (1.5)

\[
K^n = S_1^- \cdots S_{n-1}^- K_n L_{n-1}^- \cdots L_1^-,
\tag{1.13}
\]

where we have put $S_i^- \equiv S(L_i^-)$ for brevity. More explicitly, the operators defined in (1.13) are written as $K^1 = K_1$, $K^2 = S_1^- K_2 L_1^-$, $\ldots$, $K^N = S_1^- \cdots S_{N-1}^- K_N L_{N-1}^- \cdots L_1^-$. In (1.13) we could have equivalently used $L^+$ instead of $L^-$, see [5].

These structures can be utilized to obtain an integrable model from monodromies of flat connections along fundamental cycles of a Riemann surface which
obey (1.5). Introduce the spectral parameter dependent ‘Lax operators’

\[ L^n(\lambda) = K^n + \lambda 1, \]  

(1.14)

as well as a spectral parameter dependent matrix \( S(\lambda, \mu) \) which satisfies the YBE

\[
S(\lambda, \mu) = \lambda \tilde{R}^{-1} - \mu R, \quad \tilde{S}(\lambda, \mu) = \lambda R^{-1} - \mu \tilde{R},
\]

(1.15)

then it can be proven with help of (1.5) that these quantities satisfy the system of equations

\[
S(\lambda, \mu)L^n_1(\lambda)\tilde{R}L^n_2(\mu) = L^n_2(\mu)RRL^n_1(\lambda)\tilde{S}(\lambda, \mu) \\
RRL^m_1(\lambda)R^{-1}L^n_2(\mu) = L^n_2(\mu)RRL^m_1(\lambda)R^{-1}, \quad m > n.
\]

(1.16)

In order to show this the Hecke relation \( PR - (PR)^{-1} = \omega I \) must be used which restricts the model to the fundamental representation of \( sl_q(2) \). Equations (1.16) are a spectral parameter dependent version of the extended RE algebra but some \( R \)-matrices remain constant. Comparing them to (1.1) we observe that they are completely non-ultralocal. From (1.5) they inherit the property that ordered products of Lax operators do also satisfy (1.16) such that the monodromy \( T(\lambda) \) defined as in (1.3) has the commutation relation

\[
S(\lambda, \mu)T_1(\lambda)\tilde{R}T_2(\mu) = T_2(\mu)RT_1(\lambda)\tilde{S}(\lambda, \mu).
\]

(1.17)

It can be proven that the quantum trace of \( T(\lambda) \)

\[
t(\lambda) = Tr_q[T(\lambda)] = Tr[MT(\lambda)], \quad M = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}
\]

(1.18)

commutes with itself for different values of spectral parameters as in (1.4), and thereby produces a family of conserved operators in involution. It is obvious
that the model can be generalized to $sl_q(3)$, etc. We refer to [8,12] for its relation to Chern-Simons theory. Here we only note that it is related to the $XXZ$ quantum spin chain, however by a nonunitary transformation. Define $H^n = L_n^- L_{n-1}^- \cdots L_1^-$, $H^0 = 1$, then the transformed $L^n(\lambda)$

$$H^n L^n(\lambda) S(H^{n-1}) = L^+_n + \lambda L^-_n,$$

is equivalent to the Lax operator of the $XXZ$ chain (for a recent review see [13]).

It seems to us that the quantum group invariant $n$-state vertex model on a torus constructed in [14] is the statistical mechanics analogue of the model above, and that (1.16) with all $R$-matrices spectral parameter dependent should hold for the monodromies defined there (before letting a certain parameter go to infinity). We will show in the next section how the model fits into the general framework for theories with mutually non-commuting Lax operators developed there.

2. PROPERTIES OF GENERALIZED QISM

The analogy of QISM based on fundamental commutation relations (1.1) with the integrable model associated to the spectral parameter dependent extended RE algebra (1.16) suggests to investigate the possibility to establish a general theory of integrable models whose Lax operators obey RE type algebras. The problem to be solved then is what does the most general form of such an algebra look like under the condition that a system of Hamiltonians in involution can be constructed? We will see that the answer is unique given some reasonable assumptions.

Generalized QISM will be based on the general spectral parameter dependent RE

$$R^1(\lambda, \mu)L^n_1(\lambda)R^2(\lambda, \mu)L^n_2(\mu) = L^n_2(\mu)R^3(\lambda, \mu)L^n_1(\lambda)R^4(\lambda, \mu),$$

with, as yet, four arbitrary $R$-matrices $R^i$, $i = 1, \ldots, 4$ which have to be subjected to some consistency conditions to be derived. It is not clear a priori what assumptions are possible such that a consistent formalism is guaranteed. Conventional
QISM was developed by generalizing common properties of a number of models, as they are scarce in our case we will copy properties of conventional QISM as far as possible and be guided by the example above to arrive at a set of natural and hopefully minimal assumptions.

We remark that a reflection equation of the type (2.1) was studied systematically for the first time in [15] where it emerged as a consistency condition for scattering off the endpoint of particles moving on a half-line, much like the YBE is a consistency condition for particle scattering on a line. But it appeared even earlier in [16] where it described the commutation relation of the monodromy of Lax operators for the non-abelian quantum Toda chain (two of the $R$-matrices there are constant like in (1.17)). It then reappeared in [17] where it was used to encode boundary conditions for non-periodic integrable models. The fact that there exist so-called non-ultralocal integrable models like the Toda chain where Lax operators of neighbouring sites do not commute, but commute for $|m-n| \geq 2$ (no commutation relations of those Lax operators are of RE type) and nevertheless have a monodromy obeying the RE led to a systematic study of quadratic algebras defined by the RE in [18]. This will save us some work because our monodromy matrix will also satisfy the RE which allows to take over several results of [18], the following can be viewed as a generalization of that work. Closest to our program comes [19] where some constant quadratic algebra was Yang-Baxterized to describe the non-ultralocal models mentioned above. This gave a RE type algebra part of whose $R$-matrices then were restricted to the identity in order to classify some known integrable models with nearest neighbour interactions. We are not aiming at these models but rather completely non-ultralocal models like the one discussed above and try to find the most general form compatible with integrability.

**General construction.** The first, natural assumption is simply to demand that the product $L^{n+1}(\lambda)L^n(\lambda)$ again satisfies RE (2.1). Upon insertion of this product into (2.1) it is easy to see that we need a relation interchanging $L^\tau_1(\lambda)$ and
$L_{2}^{n+1}(\mu)$ of the form

$$L_{1}^{n}(\lambda)R_{2}^{n}(\lambda, \mu)L_{2}^{n+1}(\mu) = R^{\alpha}(\lambda, \mu)L_{2}^{n+1}(\mu)R_{1}^{\beta}(\lambda, \mu)L_{1}^{n}(\lambda)R^{\gamma}(\lambda, \mu), \quad (2.2)$$

with three more $R$-matrices $R^{\alpha}, R^{\beta}, R^{\gamma}$ to be determined. Then we have to use (2.1) for both $L^{n}$ and $L^{n+1}$ to see what conditions on the unknown $R$-matrices arise. This analysis gets more involved by the fact that (2.1) can be written equivalently as

$$(\tilde{R}_{1}^{1}(\mu, \lambda))^{-1}L_{1}^{n}(\lambda)\tilde{R}_{2}^{3}(\mu, \lambda)L_{2}^{n}(\mu) = L_{2}^{n}(\mu)\tilde{R}_{2}^{1}(\mu, \lambda)L_{1}^{n}(\lambda)(\tilde{R}_{2}^{4}(\mu, \lambda))^{-1}, \quad (2.3)$$

where $\tilde{R}^{i}(\mu, \lambda) = P\tilde{R}^{i}(\lambda, \mu)P$ is obtained by interchanging $\lambda \leftrightarrow \mu$ and conjugating with the permutation operator. Hence, four cases must be considered and the $R$-matrices be fixed differently in each case. We do not go into details here and present the result which is surprisingly simple. There is only one general case depending on three $R$-matrices which we choose to be $R^{1}, R^{2}, R^{3}$ and all other possibilities are contained in this case given by

$$R^{1}(\lambda, \mu)L_{1}^{n}(\lambda)R^{2}(\lambda, \mu)L_{2}^{n}(\mu) = L_{2}^{n}(\mu)\tilde{R}^{2}(\mu, \lambda)L_{1}^{n}(\lambda)[\tilde{R}^{3}(\mu, \lambda)R^{1}(\lambda, \mu)(R^{3}(\lambda, \mu))^{-1}]$$

$$\tilde{R}^{2}(\mu, \lambda)L_{1}^{n+1}(\lambda)\tilde{R}^{3}(\mu, \lambda)L_{2}^{n}(\mu) = L_{2}^{n}(\mu)\tilde{R}^{2}(\mu, \lambda)L_{1}^{n+1}(\lambda)(\tilde{R}^{2}(\mu, \lambda))^{-1}. \quad (2.4)$$

As in the conventional formalism we demand that the monodromy satisfies the same equation as the individual Lax operator, this is our second assumption. For this it is clearly necessary that the product $L^{n+2}(\lambda)L^{n+1}(\lambda)L^{n}(\lambda)$ and any ordered product of Lax operators also satisfies the first equation of (2.4). We encounter a problem then because we would need equations of the type (2.2) for any pair of Lax operators introducing more and more unknown $R$-matrices. The analysis for arbitrary $N$ soon becomes so complicated that we did not attempt to find general rules which determine the $R$-matrices that remain free (moreover an integrable system where almost all Lax operators commute differently is hard to conceive). The way out is to impose that pairs of Lax operators separated by equal distances on the lattice
share the same commutation relation, this is our third assumption. It fixes the solution uniquely, we find that Lax operators which are not on neighbouring sites have all the same commutation relation depending only on $R^2(\lambda, \mu)$. We rewrite (2.4) in a more systematic way (renaming $R^2(\lambda, \mu) \rightarrow (R^3(\lambda, \mu))^{-1}$), together with the additional relation this comprises the most general system compatible with the three assumptions above

$$R^1(\lambda, \mu)L^n_1(\lambda)R^2(\lambda, \mu)L^n_2(\mu) = L^n_2(\mu)\tilde{R}^2(\mu, \lambda)L^n_1(\lambda)\left[(\tilde{R}^3(\mu, \lambda))^{-1}R^1(\lambda, \mu)R^3(\lambda, \mu)\right],$$

$$n = 1, \ldots, N$$

$$(R^2(\lambda, \mu))^{-1}L^n_1(\lambda)R^2(\lambda, \mu)L^{n+1}_2(\mu) = L^{n+1}(\mu)(R^3(\lambda, \mu))^{-1}L^n_1(\lambda)R^2(\lambda, \mu),$$

$$n = 1, \ldots, N - 1$$

$$(R^2(\lambda, \mu))^{-1}L^n_1(\lambda)R^2(\lambda, \mu)L^m_2(\mu) = L^m_2(\mu)(R^2(\lambda, \mu))^{-1}L^n_1(\lambda)R^2(\lambda, \mu),$$

$$1 \leq n < m \leq N, \ m \neq n + 1.$$  

(2.5)

This result corresponds to [19] although Yang-Baxterization of their constant algebra did not yield spectral parameter dependent $R^2, R^3$. If we rewrite these equations into one by attaching site labels to the $R$-matrices it is also reminiscent of an equation in [20], applied to the case of non-ultralocal lattice current algebras (with constant $R$-matrices).

**The monodromy.** By construction the monodromy $T(\lambda)$ defined as in (1.3) satisfies

$$R^1(\lambda, \mu)T_1(\lambda)R^2(\lambda, \mu)T_2(\mu) = T_2(\mu)\tilde{R}^2(\mu, \lambda)T_1(\lambda)\left[(\tilde{R}^3(\mu, \lambda))^{-1}R^1(\lambda, \mu)R^3(\lambda, \mu)\right],$$

(2.6)

and in addition we can define the partial monodromy $T^{k,l}(\lambda) = L^k(\lambda)L^{k-1}(\lambda)\cdots L^l(\lambda), \ k > l$ which has commutation relations given by

$$(R^2(\lambda, \mu))^{-1}T^{k,l}_1(\lambda)R^2(\lambda, \mu)T^{m,n}_2(\mu) = T^{m,n}_2(\mu)(R^2(\lambda, \mu))^{-1}T^{k,l}_1(\lambda)R^2(\lambda, \mu),$$

$$m > n > k > l.$$  

(2.7)

Here (2.7) gives two equations with $R^3_{(k,k+1)} = R^3$, and $R^3_{(k,n)} = R^2$ for $n \neq k + 1$. 

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Readers will have noticed already that above formulas contain fundamental commutation relations (1.1) and equation (1.3) for the monodromy of conventional (ultralocal) QISM as the special case $R^2 = R^3 = I$. However, non-ultralocal models with nearest neighbour interactions like the Toda chain which require $R^2 = I$ are not compatible with our assumption that the monodromy satisfies the same equation as the $L^n$. We restrict the discussion to completely non-ultralocal cases with $R^2 \neq I$, $R^3 \neq I$.

Unlike in the conventional case here it is not straightforward to obtain the set of commuting operators, the trace of (2.6) over auxiliary spaces does not factorize. The way out is to use a trick developed in [17] which was generalized to monodromy algebras of the type (2.6) with four arbitrary $R$-matrices in [18]. In our case the theorem states that a solution $M$ of the following RE

$$
((R^1(\lambda, \mu))^{t_1 t_2})^{-1} M_1(\lambda)((R^2(\lambda, \mu))^{t_1})^{-1} t_2 M_2(\mu) = \\
M_2(\mu)((R^2(\mu, \lambda))^{t_2})^{-1} t_1 M_1(\lambda)((R^4(\lambda, \mu))^{t_1 t_2})^{-1},
$$

(2.8)

which further must satisfy $T_1(\lambda) M_2(\mu) = M_2(\mu) T_1(\lambda)$, defines the desired operator

$$
t(\lambda) = Tr[M^t(\lambda) T(\lambda)]
$$

(2.9)

that obeys (1.4). Superscripts $t$ (resp. $t_1, t_2$) indicate transposition of the matrices (in auxiliary spaces), and we have put $R^4(\lambda, \mu) \equiv (\tilde{R}^3(\mu, \lambda))^{-1} R^1(\lambda, \mu) R^3(\lambda, \mu)$. From (2.8) we can read off the conditions such that $M = I$ is a solution, sufficient are

$$
((R^2(\lambda, \mu))^{t_1})^{-1} = ((R^2(\lambda, \mu))^{-1})^{t_1}, \quad ((R^2(\lambda, \mu))^{t_1 t_2})^{-1} = ((R^2(\lambda, \mu))^{-1})^{t_1 t_2}.
$$

(2.10)

**Consistency conditions.** Several conditions must be imposed on the $R$-matrices in (2.6) such that the monodromy matrix has a consistent quadratic algebra which can be used in the algebraic Bethe ansatz, for example. They were
studied in [18], in our case they read

\[ R_{12}^{1}(\lambda, \mu)R_{13}^{1}(\lambda, \nu)R_{23}^{1}(\mu, \nu) = R_{23}^{1}(\mu, \nu)R_{13}^{1}(\lambda, \nu)R_{12}^{1}(\lambda, \mu) \]

\[ R_{12}^{1}(\lambda, \mu)R_{31}^{2}(\nu, \lambda)R_{32}^{2}(\nu, \mu) = R_{32}^{2}(\nu, \mu)R_{31}^{2}(\nu, \lambda)R_{12}^{1}(\lambda, \mu) \]

\[ R_{12}^{4}(\lambda, \mu)R_{13}^{2}(\lambda, \nu)R_{23}^{2}(\mu, \nu) = R_{23}^{2}(\mu, \nu)R_{13}^{2}(\lambda, \nu)R_{12}^{4}(\lambda, \mu) \]

\[ R_{12}^{4}(\lambda, \mu)R_{13}^{4}(\lambda, \nu)R_{23}^{4}(\mu, \nu) = R_{23}^{4}(\mu, \nu)R_{13}^{4}(\lambda, \nu)R_{12}^{4}(\lambda, \mu) \].

They ensure also consistency of the first equation of (2.5) but as for the other two extra conditions have to be imposed (which guarantee that the braiding of the coproduct, in the sense of [11], embodied by them is indeed a braid group representation). If \( R^{3} = R^{2} \) then there is only one extra condition, namely the YBE for \( R^{2} \)

\[ R_{12}^{2}(\lambda, \mu)R_{13}^{2}(\lambda, \nu)R_{23}^{2}(\mu, \nu) = R_{23}^{2}(\mu, \nu)R_{13}^{2}(\lambda, \nu)R_{12}^{2}(\lambda, \mu). \]

In the general case the following YBE like conditions involving \( R^{3} \) have to be added

\[ R_{12}^{1}(\lambda, \mu)R_{31}^{3}(\nu, \lambda)R_{32}^{3}(\nu, \mu) = R_{32}^{3}(\nu, \mu)R_{31}^{3}(\nu, \lambda)R_{12}^{1}(\lambda, \mu) \]

\[ R_{12}^{4}(\lambda, \mu)R_{13}^{3}(\lambda, \nu)R_{23}^{3}(\mu, \nu) = R_{23}^{3}(\mu, \nu)R_{13}^{3}(\lambda, \nu)R_{12}^{4}(\lambda, \mu) \]

\[ R_{12}^{3}(\lambda, \mu)R_{13}^{2}(\lambda, \nu)R_{23}^{2}(\mu, \nu) = R_{23}^{2}(\mu, \nu)R_{13}^{2}(\lambda, \nu)R_{12}^{3}(\lambda, \mu) \]

\[ R_{12}^{2}(\lambda, \mu)R_{13}^{4}(\lambda, \nu)R_{23}^{4}(\mu, \nu) = R_{23}^{4}(\mu, \nu)R_{13}^{4}(\lambda, \nu)R_{12}^{2}(\lambda, \mu) \].

As can be expected they give no new conditions for \( R^{2} \) if \( R^{3} = R^{2} \). The example shows that not all three \( R \)-matrices neccessarily have to be spectral parameter dependent. Anyway, the case \( R^{3} \neq R^{2} \) in completely non-ultralocal models seems to be a less likely possibility.

It can then be seen how the example of the introduction fits into this formalism, namely \( R^{1}(\lambda, \mu) = S(\lambda, \mu) \), \( R^{2} = R^{3} = \tilde{R} \) (indeed \((\tilde{R}^{3})^{-1}R^{4}(\lambda, \mu)R^{3} = \tilde{S}(\lambda, \mu))\), and \( M = \text{diag}(q^{-1}, q) \) is a solution of (2.8) with this choice of \( R \)-matrices satisfying (2.11) and (2.12).
**Periodic case.** If we identify \( n + N \equiv n \) two more conditions have to be imposed on the \( R \)-matrices. The commutation relation between \( L^N \) and \( L^1 \) which are now on neighbouring sites is described by the third equation of (2.5) instead of the second one. This introduces an aperiodicity into the chain, in order to avoid this it is necessary to demand

\[
R^2(\lambda, \mu) = (\tilde{R}^2(\mu, \lambda))^{-1}, \quad R^3(\lambda, \mu) = R^2(\lambda, \mu). \tag{2.14}
\]

**The case \( R^2 = R^3 = \tilde{R}^1 \).** This particular choice of \( R^2, R^3 \) has two interesting properties. Namely, we can ask whether it is possible to introduce a QG comodule structure into (2.5) such that

\[
L_nT^1(\lambda) = T^1(n)\cdot L_n(\lambda)(T^1(n))^{-1}
\]

also satisfies (2.5). The answer is affirmative if \( R^2(\lambda, \mu) = R^3(\lambda, \mu) = \tilde{R}^1(\mu, \lambda) \) and \( T^1(n) \) obeys extended QG relations

\[
\begin{align*}
R^1(\lambda, \mu)T_1^{(n)}(\lambda)T_2^{(n)}(\mu) &= T_2^{(n)}(\mu)T_1^{(n)}(\lambda)R^1(\lambda, \mu), \\
R^1(\lambda, \mu)T_1^{(m)}(\lambda)T_2^{(n)}(\mu) &= T_2^{(n)}(\mu)T_1^{(m)}(\lambda)R^1(\lambda, \mu), \quad m > n
\end{align*}
\tag{2.15}
\]

and moreover \( L_1^n(\lambda)T_2^{(m)}(\mu) = T_2^{(m)}(\mu)L_1^n(\lambda) \) for all \( m, n \). This is of course the spectral parameter dependent counterpart of the QG comodule property of (1.5), but it is restricted to this special choice of \( R^2, R^3 \) and does not hold for (2.5) in general.

The second property in this case is that given certain ultralocal algebras they can be used to realize (2.5) in terms of their generators (the converse is not true of course). For example, an algebra with generators \( M^\pm \) and relations

\[
R(\lambda, \mu)M_{1}^{\varepsilon_1}(\lambda)M_{2}^{\varepsilon_2}(\mu) = M_{2}^{\varepsilon_2}(\mu)M_{1}^{\varepsilon_1}(\lambda)R(\lambda, \mu), \quad (\varepsilon_1, \varepsilon_2) \in \{(+, +), (+, -), (-, -)\}
\tag{2.16}
\]

allows to construct

\[
L^n(\lambda) = S_1^{-}(\lambda) \cdots S_{n-1}^{-}(\lambda)M_n(\lambda)M_{n-1}^{-}(\lambda) \cdots M_1^{-}(\lambda), \quad n = 1, \ldots, N \tag{2.17}
\]

where \( M_n(\lambda) = S_n^{-}(\lambda)M_n^+(\lambda), \ S_n^{-}(\lambda) \equiv (M_n^{-}(\lambda))^{-1} \) and the \( M_i^\pm(\lambda) \) are defined by a tensor product analogous to (1.12). Then \( L^n(\lambda) \) satisfies (2.5) with \( R(\lambda, \mu) \) \( R^1(\lambda, \mu)T_1^{(n)}(\lambda)T_2^{(n)}(\mu) = T_2^{(n)}(\mu)T_1^{(n)}(\lambda)R^1(\lambda, \mu), \quad 1 \leq m \leq n \)

\[
\begin{align*}
R^2(\lambda, \mu) &= (\tilde{R}^2(\mu, \lambda))^{-1}, \\
R^3(\lambda, \mu) &= R^2(\lambda, \mu)
\end{align*}
\tag{2.14}
\]

**The case \( R^2 = R^3 = \tilde{R}^1 \).** This particular choice of \( R^2, R^3 \) has two interesting properties. Namely, we can ask whether it is possible to introduce a QG comodule structure into (2.5) such that \( L_n^n(\lambda) = T(\lambda)\cdot L_n^n(\lambda)(T(n))^{-1} \) also satisfies (2.5). The answer is affirmative if \( R^2(\lambda, \mu) = R^3(\lambda, \mu) = \tilde{R}^1(\mu, \lambda) \) and \( T(\lambda) \) obeys extended QG relations

\[
\begin{align*}
R^1(\lambda, \mu)T_1^{(n)}(\lambda)T_2^{(n)}(\mu) &= T_2^{(n)}(\mu)T_1^{(n)}(\lambda)R^1(\lambda, \mu), \\
R^1(\lambda, \mu)T_1^{(m)}(\lambda)T_2^{(n)}(\mu) &= T_2^{(n)}(\mu)T_1^{(m)}(\lambda)R^1(\lambda, \mu), \quad m > n
\end{align*}
\tag{2.15}
\]

and moreover \( L_1^n(\lambda)T_2^{(m)}(\mu) = T_2^{(m)}(\mu)L_1^n(\lambda) \) for all \( m, n \). This is of course the spectral parameter dependent counterpart of the QG comodule property of (1.5), but it is restricted to this special choice of \( R^2, R^3 \) and does not hold for (2.5) in general.

The second property in this case is that given certain ultralocal algebras they can be used to realize (2.5) in terms of their generators (the converse is not true of course). For example, an algebra with generators \( M^\pm \) and relations

\[
R(\lambda, \mu)M_{1}^{\varepsilon_1}(\lambda)M_{2}^{\varepsilon_2}(\mu) = M_{2}^{\varepsilon_2}(\mu)M_{1}^{\varepsilon_1}(\lambda)R(\lambda, \mu), \quad (\varepsilon_1, \varepsilon_2) \in \{(+, +), (+, -), (-, -)\}
\tag{2.16}
\]

allows to construct

\[
L^n(\lambda) = S_1^{-}(\lambda) \cdots S_{n-1}^{-}(\lambda)M_n(\lambda)M_{n-1}^{-}(\lambda) \cdots M_1^{-}(\lambda), \quad n = 1, \ldots, N \tag{2.17}
\]

where \( M_n(\lambda) = S_n^{-}(\lambda)M_n^+(\lambda), \ S_n^{-}(\lambda) \equiv (M_n^{-}(\lambda))^{-1} \) and the \( M_i^\pm(\lambda) \) are defined by a tensor product analogous to (1.12). Then \( L^n(\lambda) \) satisfies (2.5) with \( R(\lambda, \mu) \) \( R^1(\lambda, \mu)T_1^{(n)}(\lambda)T_2^{(n)}(\mu) = T_2^{(n)}(\mu)T_1^{(n)}(\lambda)R^1(\lambda, \mu), \quad 1 \leq m \leq n \)
\( R^2(\lambda, \mu) = R^3(\lambda, \mu) = \tilde{R}^1(\mu, \lambda) \), and the monodromy matrix (1.3) which obeys (2.6) is found to be

\[
T(\lambda) = S_1^-(\lambda) \cdots S_N^{-}(\lambda) M_1^+ (\lambda) \cdots M_1^+(\lambda). 
\]

However, this monodromy can be equivalently obtained by observing that (2.16) implies the same commutation relations (2.16) for the monodromies \( T^\pm(\lambda) = M_N^\pm(\lambda) \cdots M_1^\pm(\lambda) \), and defining then \( T(\lambda) = S(T^-(\lambda))T^+(\lambda) \) we get (2.18). Whether there are any advantages in constructing (2.5) out of such a local algebra seems then questionable. An example where this construction works is the Yangian double [2] obeying (2.16) with \( R(\lambda - \mu) = (\lambda - \mu)I + hP \), and \( h \) is the deformation parameter. But for this case solutions of (2.8) seem not to exist.

**Classical limit.** Finally, we conclude this section by mentioning that (2.5) has a well defined semi-classical limit. Given the limit \( R(\lambda, \mu) = I + ihr(\lambda, \mu) + O(h^2) \), with a skew-symmetric classical \( r \)-matrix \( r_{21}(\mu, \lambda) = -r_{12}(\lambda, \mu) \), for all \( R \)-matrices of (2.5) in case of infinitesimal quantization parameter \( h \) and the correspondence \( \hbar [\cdot, \cdot] = \{ \cdot, \cdot \} \) then the following Poisson brackets are obtained from (2.5)

\[
\{ L_1^n(\lambda), L_2^m(\mu) \} = \begin{cases} 
 2(\lambda, \mu) L_1^n(\lambda) L_2^m(\mu) - L_2^m(\mu) L_1^n(\lambda) \left[ r_1(\lambda, \mu) + r_3(\lambda, \mu) - \tilde{r}_3(\mu, \lambda) \right] 
  & \text{if } m > n \\
 0 & \text{if } m = n 
\end{cases} 
\]

\[
\{ L_1^n(\lambda), L_2^m(\mu) \} = -r_2(\lambda, \mu) L_1^n(\lambda) L_2^m(\mu) - L_2^m(\mu) L_1^n(\lambda) r_2(\lambda, \mu) + L_1^n(\lambda) r_2(\lambda, \mu) L_2^m(\mu) 
\]

\[
+ L_2^m(\mu) r_3(n,m)(\lambda, \mu) L_1^n(\lambda), \quad m > n 
\]

where as in (2.7) we define \( r_3^{(n,n+1)} = r_3 \), and \( r_3^{(n,m)} = r_2 \) if \( m \neq n + 1 \). They reduce to the well known classical limit of (1.1) if we put \( r^2 = r^3 = I \).
3. CONCLUSIONS

We conclude with a few remarks about possible applications. It would be highly desirable to find more physical examples fitting this formalism. In [19] it was argued that supersymmetric integrable models exhibit completely non-ultralocal commutation relations. In these cases the commutation relations of the Lax operators contain additional factors accounting for the statistics of entries of the $L^n$. These $\pm$ signs can be conveniently accommodated by the constant matrices $R^2 = R^3$. In that sense the model of section 1 can be viewed as their $q$-generalization.

Further, it would be interesting to investigate the continuum limit of (2.5) with Lax operators $L(x; \lambda)$ obtained from

$$L^n(\lambda) = : \exp \left( \int_{x_n}^{x_{n+1}} L(x; \lambda) dx \right) : = 1 + \Delta L(x; \lambda) + O(\Delta^2), \quad \Delta = x_{n+1} - x_n$$

where the colons around the path ordered exponential indicate normal ordering which might be necessary. One could expect in the commutation relations of $L(x; \lambda)$ beside $\delta(x - y)$ (and possibly its derivative) also a nonlocal term like $\epsilon(x - y)$ from the condition $m > n$ in (2.5) which turns into $y > x$ in the continuum limit.

Finally, there is a vague connection to 2+1 dimensions as exemplified by the model of [8] related to Chern-Simons theory. The $L^n$ should then already be interpreted as monodromies along one space direction, however, not commuting ones in contrast to the situation in 2-dimensional statistical mechanics models.

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Note added. When this work was completed I found a new paper by L. Hlavaty where the same topic is discussed [21].
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