Option pricing under path-dependent stock models

Kiseop Lee∗, Seongje Lim† and Hyungbin Park‡

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Abstract

This paper studies how to price and hedge options under stock models given as a path-dependent SDE solution. When the path-dependent SDE coefficients have Fréchet derivatives, an option price is differentiable with respect to time and the path, and is given as a solution to the path-dependent PDE. This can be regarded as a path-dependent version of the Feynman-Kac formula. As a byproduct, we obtain the differentiability of path-dependent SDE solutions and the SDE representation of their derivatives. In addition, we provide formulas for Greeks with path-dependent coefficient perturbations. A stock model having coefficients with time integration forms of paths is covered as an example.

1 Introduction

1.1 Overview

This paper aims to investigate option prices under a stock model given as a solution to an SDE with path-dependent coefficients having Fréchet derivatives. We show that an option price is differentiable with respect to time and path, and is given as a solution to a path-dependent PDE. Using this result, PDE representations and hedging portfolios of options under path-dependent stock models are investigated. In addition, we derive option Greeks with path-dependent coefficient perturbations.

One of main purposes of this paper is to provide a PDE representation of option prices under path-dependent stock models. This is a path-dependent version of the Feynman-Kac formula and can be regarded as extended results of Peng and Wang (2016). We consider a market model in which the logarithm of stock price follows a path-dependent SDE. The stock price process $S = (S_t)_{t \geq 0}$ is given as $S_t = S_0 e^{X_t}$, where

$$dX_t = b_t(X) \, dt + \sigma_t(X) \, dW_t, \quad X_0 = 0,$$

for a non-anticipative functional $b, \sigma$ under the risk-neutral measure. A path-dependent option price is $v_t(X_t) = e^{-r(T-t)} u_T(X_t)$ where $u_t$ is defined as

$$u_t(X_t) := e^{-r(T-t)} \mathbb{E} \left[ g(X_T) | \mathcal{F}_t \right], \quad 0 \leq t \leq T,$$

∗Department of Statistics, Purdue University, West Lafayette, IN, United States, kiseop@purdue.edu
†Department of Mathematical Sciences, Seoul National University, 1, Gwanak-ro, Gwanak-gu, Seoul, Republic of Korea, tjdwpdla@snu.ac.kr
‡Department of Mathematical Sciences, Seoul National University, 1, Gwanak-ro, Gwanak-gu, Seoul, Republic of Korea, hyungbin@snu.ac.kr, hyungbin2015@gmail.com
for a constant short rate $r$ and a non-anticipative functional $g$. We show that $u$ has first- and second- order vertical derivatives and a horizontal derivative, denoted as $D_x u$, $D_{xx} u$ and $D_t u$, respectively, and $u$ satisfies a path-dependent PDE:

\begin{align*}
D_t u_t(\gamma_t) + b_t(\gamma_t) D_x u_t(\gamma_t) + \frac{1}{2} \sigma^2_t(\gamma_t) D_{xx} u_t(\gamma_t) &= 0, & \gamma_t &\in D([0, t], \mathbb{R}^d), \\
u_T(\gamma_T) &= g(\gamma_T), & \gamma_T &\in D([0, T], \mathbb{R}^d).
\end{align*}

(1.1)

The converse also holds true. The precise statement is given in Theorem 4.4.

Peng and Wang (2016) studied a nonlinear Feynman-Kac formula for non Markovian BS-DEs as in (1.1) with a standard Brownian motion $X_t = W_t$ (i.e. $b \equiv 0, \sigma \equiv 1$). It is not straightforward to extend their results to an Ito process $X$ having path dependent drift and volatility terms. The reason is as follows. As a main technique, they used the property that a Brownian motion with initial value $x_0$ is the parallel translation of a standard Brownian motion by $x_0$. More precisely, the Brownian increment is given as $W^{\gamma} + \delta t - W^{\gamma} = \delta t$ where $W^{\gamma}(s) = \gamma(s) 1_{[0, t]}(s) + (\gamma(t) + W(s) - W(t)) 1_{(t, T]}(u)$ with a standard Brownian motion $W$. However, we cannot expect that this property holds for a path-dependent SDE solution, thus it cannot be extended directly to a path-dependent SDE solution. This paper is an extension of the above paper to an Ito process $X$ having path dependent drift and volatility. To overcome the technique in Peng and Wang, we adopt Fréchet derivative to estimate the term as $X^{\gamma+\delta t} - X^{\gamma}$.

As another application, we conduct a sensitivity analysis of option prices with respect to the coefficient perturbations of stock. For the perturbed coefficients $b^\epsilon$ and $\sigma^\epsilon$ with the perturbation parameter $\epsilon$, let $X^\epsilon$ be a solution to

$$dX^\epsilon_t = b^\epsilon_t(X^\epsilon) dt + \sigma^\epsilon_t(X^\epsilon) dW_t, \quad X^\epsilon_0 = 0.$$ 

At time 0, the perturbed option price is

$$v^\epsilon_0(X_0) = e^{-r^\epsilon T} \mathbb{E} [g(X^\epsilon_T)].$$

for the perturbed short rate $r^\epsilon$. We show that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} (v^\epsilon_0(X_0) - v_0(X_0)) = - u_0(X_0) e^{-r^\epsilon T \hat{r} T} - e^{-r^\epsilon T} \mathbb{E} \left[ \int_0^T D_x u_s(X_s) \hat{b}_s(X) + D_{xx} u_s(X_s) \hat{\sigma}_s(X) \sigma_s(X) ds \right]$$

where $\hat{b}, \hat{\sigma}, \hat{r}$ are the partial derivatives of $b, \sigma, r$, respectively, of $\epsilon$ evaluated at $\epsilon = 0$. Precise assumptions regarding perturbations and the results are given in Section 1.2.

The rest of this paper is structured as follows. The related literature is reviewed in Section 1.2. We explain the basic concepts of functional Itô calculus in Section 2. In Section 3 we investigate the differentiability with respect to the initial path of path-dependent SDEs and state the main result of this paper. In Section 4 our main results are applied to option pricing theory for an exponential path-dependent stock price model. We show that the option price is a $C^{1,2}$ solution to a path-dependent partial differential equation and provide sensitivity formulas. Section 5 presents an example for a stock price model. Finally, the last section summarizes the paper. The proofs of the main results are presented in the appendices.

### 1.2 Literature review

Recently, path-dependent SDEs have been actively researched. Dupire (2019) extended Itô calculus to functionals of the current path of a process. To obtain an Itô formula, they expressed the differential of the functional in terms of partial derivatives. Moreover, they developed an
extension of the Feynman-Kac formula to the functional case and an explicit expression of the integrand in the martingale representation theorem. Cont and Fournié (2010a) derived a change of variable formula for non-anticipative functionals. Their results led to functional extensions of the Itô formula for a large class of stochastic processes. Cont and Fournié (2010b) showed that the functional derivative admits a suitable extension to the space of square-integrable martingales. This extension defines a weak derivative viewed as a non-anticipative lifting of the Malliavin derivative. These results led to a constructive martingale representation formula for Itô processes.

The Feynman-Kac formula has been extended to non-Markovian cases. Peng (2010) studied a type of path-dependent quasi-linear parabolic PDEs. These PDEs are formulated through a classical BSDE in which the terminal values and generators are general functions of Brownian motion paths. As a result, they established the nonlinear Feynman-Kac formula for a general non-Markovian BSDE. Ekren, Keller, Touzi, and Zhang (2014) proposed a notion of viscosity solutions for path-dependent semi-linear parabolic PDEs. This can be viewed as viscosity solutions of non-Markovian backward SDEs. They proved the existence, uniqueness, stability, and comparison principle for the viscosity solutions. Bouchard, Loeper, and Tan (2021) introduced a notion of approximate viscosity solution for nonlinear path-dependent PDEs including the Hamilton-Jacobi-Bellman type equations. They investigated the existence, stability, comparison principle and regularity of the solution to the PDEs.

Estimating Greeks is an important topic in mathematical finance. Fournié, Lasry, Lebuchoux, Lions, and Touzi (1999) presents a probabilistic method for the numerical computations of Greeks in finance. They applied the integration-by-parts formula developed in the Malliavin calculus to exotic European options in the framework of the Black and Scholes model. Their method was compared to the Monte Carlo finite difference approach and turned out to be efficient in the case of discontinuous payoff functionals.

2 Functional Itô calculus

In this section, we introduce basic notions of functional Itô calculus. We fix a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and time maturity $T \in \mathbb{R}_+$. Let $D([0, T], \mathbb{R}^d)$ be a collection of all functions with càdlàg paths from $[0, t]$ to $\mathbb{R}^d$. We also define $\bar{D} = \bigcup_{t \in [0, T]} D([0, t], \mathbb{R}^d)$. We define $\mathcal{D}^d$ as a collection of all $d$-dimensional stochastic processes with càdlàg paths. (i.e., $X \in \mathcal{D}^d$ means that $X$ is a stochastic process defined on $[0, T]$ and $X(\omega) \in D([0, T], \mathbb{R}^d)$ for $\omega \in \Omega$).

We write $\eta_t(\cdot) = \eta(\cdot \wedge t) \in D([0, T], \mathbb{R}^d)$ to denote the stopped path of $\eta$ at time $t \in [0, T]$ for $\eta \in D([0, T], \mathbb{R}^d)$. We denote the value of path $\gamma_t \in D([0, t], \mathbb{R}^d)$ at time $s \in [0, t]$ as $\gamma_t(s) \in \mathbb{R}^d$.

For each $t \in [0, T]$, let $\gamma_t \in D([0, t], \mathbb{R}^d)$. We introduce several notations

\[
\|\gamma_t\|_s := \sup_{u \in [0, s]} \|\gamma(u)\| := \sup_{u \in [0, s]} \max_{i=1, \ldots, d} |(\gamma(u)_1, \gamma(u)_2, \ldots, \gamma(u)_d)|,
\]

\[
\|\gamma_t\|_{s_0, s_1} := \sup_{u \in [s_0, s_1]} \|\gamma(u)\|, \quad \|\gamma_t\| := \|\gamma_t\|_t,
\]

for $s, s_0, s_1 \in [0, t]$. The map $\|\cdot\| = \|\cdot\|_t$ is regarded as a norm in $D([0, t], \mathbb{R}^d)$. We use the same notation in $\mathcal{D}^d$,

\[
\|X\|_s := \sup_{u \in [0, s]} \|X(u)\|, \quad \|X\|_{s_0, s_1} := \sup_{u \in [s_0, s_1]} \|X(u)\|, \quad \|X\| := \|X\|_T. \tag{2.1}
\]

Let $F : [0, T] \times D([0, t], \mathbb{R}^d) \to \mathbb{R}$. We say that $F$ is a non-anticipative functional if for any $\eta, \gamma \in D([0, t], \mathbb{R}^d)$ satisfying $\eta^s = \gamma^s$, the equality $F(s, \eta) = F(s, \gamma)$ holds. This means that the value of $F(s, \gamma)$ depends only on the $[0, s]$ part of path $\gamma$. The non-anticipative functional $F$ can be regarded as an operator $\bar{F}$ from $\bar{D}^d$ to $\mathbb{R}$ by defining $\bar{F}(\gamma) = F(t, \gamma^T)$ for $\gamma \in D([0, t], \mathbb{R}^d)$. 

3
where \( \gamma^T(s) = \gamma(s)\mathbf{1}_{[0,0)}(s) + \gamma(t)\mathbf{1}_{[t,T]}(s) \). We can denote \( F_t(\eta) = F(t,\eta) \in \mathbb{R} \) for a non-anticipative functional \( F \) if there is no ambiguity. Next, we introduce the notions of horizontal and vertical derivatives, which indicate the sensitivity of a functional to each perturbation of a stopped path \((t,\eta)\).

**Definition 2.1.** Let \( F : [0,T] \times D([0,T],[\mathbb{R}^d]) \to \mathbb{R} \) be a non-anticipative functional.

(i) We say that \( F \) is left continuous if

\[
F(t,\cdot) : (D([0,T],[\mathbb{R}^d]), \| \cdot \|_T) \to \mathbb{R}
\]

is continuous,

(ii) for any \((t,\eta) \in [0,T] \times D([0,T],[\mathbb{R}^d])\) and \( \epsilon > 0 \), there exists \( \delta > 0 \) if \((t',\eta') \in [0,T] \times D([0,T],[\mathbb{R}^d])\) satisfy \( t' < t \) and \( \| \eta - \eta' \|_T + |t - t'| < \delta \), then \( |F(t,\eta) - F(t',\eta')| < \epsilon \).

(iii) \( F \) is said to be horizontally differentiable at \((t,\eta) \in [0,T] \times D([0,T],[\mathbb{R}^d])\) if the limit

\[
D_tF(t,\eta) = \lim_{\delta \to 0^+} \frac{|F(t+\delta,\eta') - F(t,\eta')|}{\delta}
\]

exists. We call \( D_tF(t,\eta) \) the horizontal derivative \( D_tF(t,\eta) \) of \( F \) at \((t,\eta)\).

(iv) \( F \) is said to be vertically differentiable at \((t,\eta) \in [0,T] \times D([0,T],[\mathbb{R}^d])\) if the limit

\[
\partial_i F(t,\eta) = \lim_{\delta \to 0^+} \frac{|F(t,\eta^i + \delta e_i\mathbf{1}_{[t,T]}(\xi) - F(t,\eta^i)|}{\delta}
\]

exists for \( i = 1, \ldots, d \) where \( e_i \) is \( i \)-th basis.

We call \( D_xF(t,\eta) := (\partial_1 F(t,\eta), \ldots, \partial_d F(t,\eta)) \in \mathbb{R}^{1 \times d} \) the vertical derivative \( D_xF(t,\eta) \) of \( F \) at \((t,\eta)\).

If \( F \) is horizontally differentiable at all \((t,\eta)\), the map \( DF : (t,\eta) \mapsto DF(t,\eta) \) is a non-anticipative functional. The same holds for the vertical derivative. Similarly, we can define the second-order vertical derivative \( D_{xx}F = D_x(D_xF) \) of \( F \).

**Definition 2.2.** We say \( F \) is a regular functional if there exist \( D_tF, D_xF, D_{xx}F \) for all \((t,\eta) \in [0,T] \times D([0,T],[\mathbb{R}^d])\), and we have

(i) For each \( t \in [0,T] \), \( D_tF(t,\cdot) : (D([0,T],[\mathbb{R}^d]), \| \cdot \|_T) \to \mathbb{R} \) is continuous,

(ii) \( D_xF, D_{xx}F \) are left continuous,

(iii) and \( D_tF, D_xF, D_{xx}F \) are boundedness-preserving.

From Bally, Caramellino, and Cont (2016), we get the continuous version of the functional Itô formula.

**Theorem 2.1.** (Functional Itô formula : continuous case) Let \( X \) be a continuous \( d \)-dimensional semimartingale and \( F \) be a regular functional. For any \( t \in [0,T] \),

\[
F(t,X_t) - F(0,X_0) = \int_0^t D_tF(u,X_u) \, du + \int_0^t D_xF(u,X_u) \, dX(u) + \frac{1}{2} \int_0^t \text{tr}(D_{xx}F(u,X_u) \, d[X](u)) \quad \text{a.s.}
\]
3 Differentiability of stochastic flow

In this section, we investigate the differentiability with respect to the initial path of a path-dependent SDE. The result is the path-dependent version of Theorem V.39 of Protter (2005). Protter (2005) deals with the differentiability with respect to an initial data point of an SDE when the coefficients of the SDE are a function depending on a finite point. The condition of this theorem is that the coefficient function of the SDE is differentiable. Therefore, we introduce the Fréchet derivative of a non-anticipative functional corresponding to the differentiability of the function.

Let $V$ and $W$ be two normed vector spaces. We denote the family of linear bounded operators from $V$ to $W$ with the operator norm by $L(V, W)$. The Fréchet derivative of an operator between two normed vector spaces can be defined.

**Definition 3.1.** Let $V$, $W$ be vector spaces with norm $\| \cdot \|_V$ and $\| \cdot \|_W$.

(i) We say that an operator $F : V \to W$ has a Fréchet derivative at $v \in V$ if there exists a bounded linear operator $DF(v) : V \to W$ such that

$$
\lim_{\|u\|_V \to 0} \frac{\|F(v + u) - F(v) - DF(v)(u)\|_W}{\|u\|_V} = 0.
$$

(ii) Suppose that an operator $F : V \to W$ has a Fréchet derivative for all $v \in V$. We say that $F$ has a second-order Fréchet derivative at $v \in V$ if $DF : V \to L(V, W)$ has a Fréchet derivative at $v \in V$. This means that there exists a bounded linear operator $D^2F(v) : V \to L(V, W)$ such that

$$
\lim_{\|u\|_V \to 0} \frac{\|DF(v + u) - DF(v) - D^2F(v)(u)\|_{L(V, W)}}{\|u\|_V} = 0.
$$

An operator $F : V \to W$ is said to have a Fréchet derivative if $F$ has a Fréchet derivative for all $v \in V$. We say that the operator $F$ has a second Fréchet derivative as in a similar way. For simplicity, we may sometimes denote $D^2F(v)(w)$ as $D^2F(v, w)$.

Before applying the above definitions to non-anticipative functionals, we define some notions following Protter (2005). Fix a maturity $T \in \mathbb{R}_+$. A non-anticipative functional $F : [0, T] \times D([0, T], \mathbb{R}^d) \to \mathbb{R}^N$ is functional Lipschitz continuous if for each $t \in [0, T]$, there exists a constant $K(t)$ such that

$$
\|F_t(\eta) - F_t(\delta)\| \leq K(t) \|\eta - \delta\|_t
$$

for any $\eta, \delta \in D([0, T], \mathbb{R}^d)$. There are some conditions for a non-anticipative functional $F$ to apply the Fréchet derivative.

**Assumption 3.1.** Let $F : [0, T] \times D([0, T], \mathbb{R}^d) \to \mathbb{R}$ be a non-anticipative functional. The functional $F$ satisfies the following:

(i) for each $\eta \in D([0, T], \mathbb{R}^d)$, the map $F(\eta) : [0, T] \to \mathbb{R}^N$ is a càdlàg path,

(ii) there exists an increasing sequence of open sets $A_k$ such that $\bigcup_k A_k = D([0, T], \mathbb{R}^d)$ and $F$ is functional Lipschitz continuous on each $A_k$.

The first condition means that we can regard the functional $F : [0, T] \times D([0, T], \mathbb{R}^d) \to \mathbb{R}^N$ as the operator $F : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^N)$. The second is a necessary condition for the existence of the Fréchet derivative and the solution to the SDE. We refer to satisfying the second condition as a local functional Lipschitz continuity.
Let $Z$ be a $N$-dimensional semimartingale with $Z_0 = 0$ and $F : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^{d \times N})$ be a non-anticipative functional satisfying Assumption 3.1 and having the Fréchet derivative $DF$. Consider a solution $(X^x, DX^x)$ to the $N$-dimensional SDE with a non-anticipative functional coefficient

$X^x_t = x + \int_0^t F_s-(X^x) \, dZ_s,$  \hspace{1cm} (3.1)

$DX^x_t = 1 + \int_0^t DF_s-(X^x)(DX^x) \, dZ_s,$ \hspace{1cm} (3.2)

for $x \in \mathbb{R}^d$. Since $F$ and $DF$ are locally functional Lipschitz continuous, we know that the solutions of (3.1) and (3.2) uniquely exist.

The following theorems deal with the differentiability of a solution to an SDE with a non-anticipative functional coefficient. Under the good condition of the coefficient $F$, a solution to the SDE can be differentiable, and the differential is written as a solution to an SDE. This means we can apply the property of SDE to the differential. We can regard this proposition as a non-anticipative functional version of Theorem V.39 of Protter (2005).

**Proposition 3.1.** Let $F : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^{d \times N})$ be a non-anticipative functional with Assumption 3.1. Assume that $F$ has the Fréchet derivative $DF$ which is locally Lipschitz continuous in the operator norm, i.e., for any $\eta \in D([0, T], \mathbb{R}^d)$, there exist constant $\epsilon, K(\eta) > 0$ such that if $\|\eta - \tilde{\eta}\| < \epsilon$ then

$$\|DF(\eta)(\delta) - DF(\tilde{\eta})(\delta)\| \leq K(\eta) \|\eta - \tilde{\eta}\| \|\delta\|,$$ \hspace{1cm} (3.3)

holds for all $\delta \in D([0, T], \mathbb{R}^d)$. Suppose that $X$ is a solution to SDE (3.1). Then, the following conditions are satisfied.

(i) For almost everywhere $\omega \in \Omega$, there exists a function $X^x(t, \omega)$ that is continuous differentiable in $\mathbb{R}^d$ with respect to $x$.

(ii) Let $DX^x(t, \omega) = \frac{\partial X^x}{\partial x}(t, \omega)$. Then, $DX^x$ is the solution to SDE (3.2).

**Proposition 3.2.** Let $F : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^{d \times N})$ be a non-anticipative functional with Assumption 3.1. Assume that $F$ has the second-order Fréchet derivative $D^2F$, and that $DF$ and $D^2F$ are locally Lipschitz continuous in the operator norm, i.e., for each $\eta \in D([0, T], \mathbb{R}^d)$, there exist constants $\epsilon > 0, K(\eta)$ such that if $\|\eta - \tilde{\eta}\| \leq \epsilon$ then

$$\|DF(\eta)(\gamma) - DF(\tilde{\eta})(\gamma)\| \leq K(\eta) \|\eta - \tilde{\eta}\| \|\gamma\|,$$

$$\|D^2F(\eta, \gamma)(\delta) - D^2F(\tilde{\eta}, \gamma)(\delta)\| \leq K(\eta) \|\eta - \tilde{\eta}\| \|\gamma\| \|\delta\|,$$ \hspace{1cm} (3.4)

for all $\gamma, \delta \in D([0, T], \mathbb{R}^d)$. Also, let $D^2X^x$ be a solution to the SDE for each $x \in \mathbb{R}^d$,

$$D^2X^x_t = \int_0^t D^2F_s-(X^x, DX^x)(DX^x) \, dZ_s + DF_s-(X^x)(DX^x) \, dZ_s,$$ \hspace{1cm} (3.5)

Then, $D^2X^x$ exists and is unique. Moreover, we obtain the following.

(i) For almost everywhere $\omega \in \Omega$, there exists a function $DX^x(t, \omega)$ that is continuous differentiable in $\mathbb{R}^d$ with respect to $x$.

(ii) The differential $\frac{\partial DX^x}{\partial x}(t, \omega)$ of $DX^x$ is the solution to SDE (3.3).

The above theorems state when the initial value of the SDE is a point. We can easily extend the theorems to cases in which the initial value of SDE is a path. Let $(\Omega, \mathcal{Q}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ be the filtered probability space, where $\mathcal{F}_t$ is a sigma-algebra generated by an $N$-dimensional Wiener
process $W_s$ for $0 \leq s \leq t$. Suppose that the stochastic process $X^{\gamma_t}$ follows an SDE for each $t \in [0, T]$ and $\gamma_t \in D([0, t], \mathbb{R}^d)$,

$$
X^{\gamma_t}(s) = \gamma_t(t) + \int_t^s b_r(X^{\gamma_t}) \, dr + \int_t^s \sigma_r(X^{\gamma_t}) \, dW_r, \quad t \leq s \leq T,
$$

$$
X^{\gamma_t}(s) = \gamma_t(s), \quad 0 \leq s \leq t,
$$

where $b$ and $\sigma$ are non-anticipative functionals. The proofs of the following theorems are given at the end of Appendix [A].

**Theorem 3.3.** Let $b : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^d)$, $\sigma : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^{d \times N})$ be a non-anticipative functional with Assumption [3.7]. Assume that $b, \sigma$ have the Fréchet derivatives $Db, D\sigma$ satisfying (3.3). Let $X^{\gamma_t}$ be a solution to SDE (3.6) and $D_{\delta_t}X^{\gamma_t}$ be a solution to the SDE

$$
D_{\delta_t}X^{\gamma_t}(s) = \delta_t(t) + \int_t^s Db_r(X^{\gamma_t})(D_{\delta_t}X^{\gamma_t}) \, dr + \int_t^s D\sigma_r(X^{\gamma_t})(D_{\delta_t}X^{\gamma_t}) \, dW_r, \quad t \leq s \leq T,
$$

$$
D_{\delta_t}X^{\gamma_t}(s) = \delta_t(s), \quad 0 \leq s < t.
$$

for each $\delta_t \in D([0, t], \mathbb{R}^d)$. Then, for $(s, \omega) \in [0, T] \times \Omega$ with $\omega$ not in the exceptional set, we have

$$
\lim_{h \to 0} \left| \frac{X^{\gamma_t+h\delta_t}(s, \omega) - X^{\gamma_t}(s, \omega) - D_{\delta_t}X^{\gamma_t}(s, \omega)}{h} \right| = 0,
$$

for $h \in \mathbb{R}$.

**Theorem 3.4.** Let $b : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^d)$, $\sigma : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^{d \times N})$ be a non-anticipative functional with Assumption [3.7]. Assume that $b, \sigma$ have Fréchet derivatives up to order 2 as $Db, D\sigma, D^2b, D^2\sigma$ satisfying (3.3). Let $X^{\gamma_t}, DX^{\gamma_t}$ be a solution to SDEs (3.6), (3.7) and $D^2X^{\gamma_t}$ be a solution to the SDE

$$
D_{\delta_t}D_{\delta_t}X^{\gamma_t}(s) := \int_s^t D^2b_r(X^{\gamma_t}, D_{\delta_t}X^{\gamma_t})(D_{\delta_t}X^{\gamma_t}) \, dr + \int_s^t D^2\sigma_r(X^{\gamma_t}, D_{\delta_t}X^{\gamma_t})(D_{\delta_t}X^{\gamma_t}) \, dW_r \quad t \leq s \leq T,
$$

$$
D_{\delta_t}D_{\delta_t}X^{\gamma_t}(s) = 0, \quad 0 \leq s < t.
$$

Then, for $(s, \omega) \in [0, T] \times \Omega$ with $\omega$ not in the exceptional set, the equality

$$
\lim_{|h| \to 0} \left| \frac{D_{\delta_t}X^{\gamma_t+h\delta_t}(s, \omega) - D_{\delta_t}X^{\gamma_t}(s, \omega) - h \cdot D_{\delta_t}D_{\delta_t}X^{\gamma_t}(s)}{h} \right| = 0
$$

holds.

The results of the above theorems indicate that the process $X^{\gamma_t}$ is differentiable in the direction $\delta_t$ and the derivative is represented as a solution to SDE (3.7). We summarize the assumption of coefficients $b, \sigma$ in Theorem 3.3.

**Assumption 3.2.** The non-anticipative functionals $b : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^d)$, $\sigma : D([0, T], \mathbb{R}^d) \to D([0, T], \mathbb{R}^{d \times N})$ satisfy the following:

(i) Assumption [3.7]

(ii) $b, \sigma$ have Fréchet derivatives up to order 2 in sense of Definition 3.1 and they satisfy locally functional Lipschitz continuity (3.4).
4 Path-dependent option pricing

Through this section, we set up the exponential path-dependent stock price model and apply our results to option pricing. Under certain conditions regarding the coefficients and payoff functional, the option price is a $C^{1,2}$ solution to a path-dependent partial differential equation (PPDE). Using this, we obtain sensitivity formulas of option prices for changes of underlying model coefficients.

4.1 Path-dependent SDEs and PDEs

We set up the exponential path-dependent stock price model in which the logarithm of stock price follows a general path-dependent SDE. Since this setting is constructed under the risk-neutral model, an option price is represented by a conditional expectation. Next, we introduce a backward stochastic differential equation (BSDE) to which a solution is an option price. Finally, we state a theorem that implies that the option price operator is the solution to a PPDE and a solution to the PPDE is the option price.

Let us denote a logarithm of $d$-dimensional stock price by $X_t$. Let $r \in \mathbb{R}_+$ be the deterministic short rate term and $q : D([0,T],\mathbb{R}^d) \to D([0,T],\mathbb{R}^d)$ be the non-anticipative dividend functional. To start under the risk-neutral setting, we define a logarithm of stock price for each $i = 1, \cdots, d$ as

$$dX^{(i)}_t = (r - q^{(i)}(X) - \frac{1}{2}\sigma^{(i)}_t(X)\sigma^{(i)}_t(X)^\top)dt + \sigma^{(i)}_t(X)dW_t, \quad X^{(i)}_0 = 0,$$

where $W_t$ is $N$-dimensional Brownian motion and $\sigma^{(i)} : D([0,T],\mathbb{R}^d) \to D([0,T],\mathbb{R}^N)$ is a non-anticipative functional. We define a non-anticipative functional $\sigma : D([0,T],\mathbb{R}^d) \to D([0,T],\mathbb{R}^{d \times N})$ such that its $i$-th row is $\sigma^i$ for all $i = 1, \cdots, d$. Then, the $d$-dimensional process $X$ can be written as

$$dX_t = (r \mathbb{1}^d - q_t(X) - \frac{1}{2}|\sigma_t(X)|^2)dt + \sigma_t(X)dW_t, \quad X_0 = 0^d,$$

where $0^d$, $\mathbb{1}^d$ are zero and a unit vector, respectively, in $\mathbb{R}^d$, and $|\sigma|^2 : D([0,T],\mathbb{R}^d) \to D([0,T],\mathbb{R}^d)$ is a non-anticipative functional, the $i$-th coordinate of which is $\sigma^{(i)}_t(X)\sigma^{(i)}_t(X)^\top$ for $i = 1, \cdots, d$.

The $d$-dimensional stock price is represented by $S_t = (S^{(1)}_t, \cdots, S^{(d)}_t) := (S^{(1)}_0 e^{X_t^{(1)}}, \cdots, S^{(d)}_0 e^{X_t^{(d)}})$, where $X = (X^{(1)}, \cdots, X^{(d)})$ and $S^{(i)}_0 > 0$ for all $i = 1, \cdots, d$. We obtain the equation from the multidimensional Itô formula for all $i = 1, \cdots, d$

$$dS^{(i)}_t = S^{(i)}_t dX^{(i)}_t + \frac{1}{2} S^{(i)}_t d[X^{(i)}]_t = S^{(i)}_t (r - q^{(i)}(X))dt + S^{(i)}_t \sigma^{(i)}_t(X)dW_t,$$

where $\sigma^{(i)}$ is the $i$-th row of $\sigma$. Thus, the above probability space is a risk-neutral measure space.

Let $H : D([0,T],\mathbb{R}^d) \to \mathbb{R}$ be a payoff non-anticipative functional. Then, the option price at time $t$ is

$$e^{-r(T-t)}\mathbb{E}[H(S_0 e^{X_T})|\mathcal{F}_t] = e^{-r(T-t)}\mathbb{E}[g(X_T)|\mathcal{F}_t],$$

where $g : D([0,T],\mathbb{R}^d) \to \mathbb{R}$ with $g^{(i)}(\eta) = H^{(i)} \circ \exp(S_0 \eta)$ for all $i = 1, \cdots, d$.

Now, we introduce a BSDE to which a solution is the above option price (4.2). Suppose that the stochastic process $X^{\gamma_t}$ follows SDE (3.5). For a more general setting, the non-anticipative functional $b$ is used instead of $r \mathbb{1}^d - q_t - \frac{1}{2}|\sigma_t|^2$. Note that we set $b = r \mathbb{1}^d - q_t - \frac{1}{2}|\sigma_t|^2$ and $\gamma_t = X_t$ to ensure that the solution to SDE (4.2) corresponds to the solution to (4.1).
In the remainder of this section, we assume that the coefficients $b, \sigma$ satisfy Assumption \ref{assumption_i}. Then, directional derivatives $D_bX^\gamma_\cdot$ and $D_{\sigma \sigma}X^\gamma_\cdot$ of $X^\gamma_\cdot$ exist for $\delta_t, \delta'_t \in D([0, T], \mathbb{R}^d)$. In particular, we focus on the vertical derivative in Definition \ref{def_vertical_derivative}. Let $e_i$ be the $i$-th standard basis for $\mathbb{R}^d$. For a path $\tilde{e}_i := e_i I_{[0, t]} \in D([0, t], \mathbb{R}^d)$, we denote $\partial_i X^\gamma_\cdot$ as a vertical derivative $D_{\tilde{e}_i}X^\gamma_\cdot$.

In addition, we denote $D_x F(t, \eta)$ as $(\partial_t F(t, \eta), \cdots, \partial_1 F(t, \eta)) \in \mathbb{R}^{1 \times d}$.

Let $S^2(0, T; \mathbb{R})$ be the set of all $(\mathcal{F}_t)$-adapted processes $Y$ with $\mathbb{E}[\sup_{u \in [0, T]} |Y(u)|^2] < \infty$ and $\mathcal{H}^2(0, T; \mathbb{R})$ be the set of all $(\mathcal{F}_t)$-adapted processes $Z$ with $\mathbb{E}[\int_0^T |Z(u)|^2 du] < \infty$. In a similar way, we can define the spaces $S^p$ and $\mathcal{H}^p$ for $p \geq 2$. Suppose that $(Y^\gamma, Z^\gamma)$ is a solution to BSDE (4.3):  

$$Y^\gamma(s) = g(X^\gamma_T) - \int_s^T Z^\gamma(r) \, dW_r, \quad t \leq s \leq T,$$  

for $X^\gamma$ in (3.3). From the martingale representation theorem, BSDE (4.3) has a unique solution $(Y^\gamma, Z^\gamma)_{0 \leq t \leq T}$ in $S^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R})$ for a given $\gamma \in D([0, T], \mathbb{R}^d)$. Already, we know that $Y^\gamma(u) = \mathbb{E}[g(X^\gamma_T)|\mathcal{F}_u]$ for $t \leq u \leq T$.

In addition, we assume that $g : D([0, T], \mathbb{R}^d) \to \mathbb{R}$ has Fréchet derivatives up to order 2.

**Assumption 4.1.** The payoff functional $g : D([0, T], \mathbb{R}^d) \to \mathbb{R}$ satisfies the following conditions:

(i) for each $\eta \in D([0, T], \mathbb{R}^d)$, there exists a constant $\delta = \delta(\eta) > 0$ such that if $\|\tilde{\eta} - \eta\| < \delta$, then $\|g(\eta) - g(\tilde{\eta})\| \leq C(1 + \|\eta\| + \|\tilde{\eta}\|)\|\eta - \tilde{\eta}\|$ for some $k = k(\eta) \geq 1$,

(ii) $g$ has Fréchet derivatives up to order 2 in sense of Definition \ref{def_order_2_derivatives} and they satisfy local Lipschitz continuity (4.3). More precisely, for each $\eta \in D([0, T], \mathbb{R}^d)$, there exist constant $\epsilon > 0$, $K(\eta)$ such that if $\|\eta - \tilde{\eta}\| \leq \epsilon$ then

$$|Dg(\eta)(\gamma) - Dg(\tilde{\eta})(\gamma)| \leq K(\eta) \|\eta - \tilde{\eta}\| \|\gamma\|,$$

$$|D^2g(\eta, \gamma)(\delta) - D^2g(\tilde{\eta}, \gamma)(\delta)| \leq K(\eta) \|\eta - \tilde{\eta}\| \|\gamma\| \|\delta\|,$$

for all $\gamma, \delta \in D([0, T], \mathbb{R}^d)$.

Note that we assume the Lipschitz continuity of the coefficients $b, \sigma$ and local Lipschitz continuity of the payoff functional $g$. The difference is followed by the existence of solutions to the SDE (3.6) and BSDE (4.3). To guarantee the existence and uniqueness of solutions to the SDE (3.6) for the entire interval $[0, T]$, we need the coefficients $b, \sigma$ to be globally Lipschitz continuous. There may be a blowup in finite time if the coefficients $b$ and $\sigma$ are only locally Lipschitz continuous, not globally. Conversely, we already know that solution to BSDE (4.3) uniquely exists from the martingale representation theorem. Thus, the first condition of Assumption \ref{assumption_i} is only required for a technical reason.

Now, we introduce the result of this subsection. This result implies that the conditional expectation is the solution to a PPDE. Conversely, it also states that a solution to the PPDE is the conditional expectation. This means that we can estimate the conditional expectation as solving the PPDE. Recall that $\hat{D}^d = \bigcup_{t \in [0, T]} D([0, t], \mathbb{R}^d)$. The proof of this theorem is in Appendix B.2.

**Theorem 4.1.** Under Assumptions 3.2 and 4.1, we define a non-anticipative functional $u : \hat{D}^d \to \mathbb{R}$ as

$$u_t(\gamma_t) := Y^\gamma(t) = \mathbb{E} \left[ g(X^\gamma_T) | \mathcal{F}_t \right]$$  

for $\gamma_t \in D([0, T], \mathbb{R}^d)$, where $Y^\gamma$ is a solution to BSDE (4.3). Then $u$ has first and second-order vertical derivatives and the horizontal derivative, denoted as $D_x u$, $D_{xx} u$, and $D_t u$, respectively. Moreover, $u$ satisfies a PPDE

$$D_t u_t(\gamma_t) + \langle b_t(\gamma_t), D_x u_t(\gamma_t) \rangle + \frac{1}{2} \text{tr} \left( \sigma_t(\gamma_t) D_{xx} u_t(\gamma_t) \sigma_t^	op(\gamma_t) \right) = 0, \quad \text{for } \gamma_t \in D([0, T], \mathbb{R}^d),$$

$$u_T(\gamma_T) = g(\gamma_T), \quad \text{for } \gamma_T \in D([0, T], \mathbb{R}^d).$$  

(4.5)
Conversely, suppose that the regular functional $u$ is a solution to PPDE (4.5). Then,

$$u_t(\gamma_t) = Y^{\gamma_t}(t), \quad \text{for} \quad \gamma_t \in D([0, t], \mathbb{R}^d),$$

where $Y^{\gamma_t}$ is a solution to BSDE (4.3).

Now, we set the non-anticipative functional $b_t = r1^d - q_t - \frac{1}{2}|\sigma_t|^2$ as in (4.1) to examine the option price. Then, the PPDE (4.5) is changed to

$$D_t u_t(\gamma_t) + \langle r1^d - q_t(\gamma_t), D_x u_t(\gamma_t) \rangle + \frac{1}{2} \text{tr} \left( \sigma_t(\gamma_t) \cdot (D_{xx} u_t(\gamma_t) - \text{diag}_d(D_x u_t(\gamma_t))\sigma_t^\top(\gamma_t)) \right) = 0, \quad \text{for} \quad \gamma_t \in D([0, t], \mathbb{R}^d),$$

$$u_T(\gamma_T) = g(\gamma_T) \quad \text{for} \quad \gamma_T \in D([0, T], \mathbb{R}^d),$$

(4.6)

where $\text{diag}_d(D_x u_t)$ is a $d \times d$ diagonal matrix whose $i$-th coordinate is $\partial_i u_t$ for each $i = 1, \ldots, d$.

Let $u$ be the solution to the above PPDE (4.6) and define a non-anticipative functional $v$ as $v_t(\gamma_t) = u_t(\gamma_t) \exp(-r(T-t))$. Then, the option price is $v_t(X_t)$, since $v_t(X_t) = u_t(X_t) \exp(-r(T-t)) = \mathbb{E}[g(X_T)|F_t] \exp(-r(T-t))$. Moreover, we can obtain the hedging portfolio of option. This construction is motivated by Theorem 5.1 in [Fournier (2010)]. Recall that the 1-dimensional adapted process $C_t$ is a self-financing portfolio with initial value $x$ and $d$-dimensional position process $a_t = (a_t^{(1)}, \ldots, a_t^{(d)})$ under the $d$-dimensional stock model $S_t$ if $C_t$ satisfies

$$dC_t = \frac{C_t - \langle a_t, S_t \rangle}{R_t} \, dR_t + \sum_{i=1}^d a_t^{(i)} S_t^{(i)} q_t^{(i)}(X_t) \, dt + \sum_{i=1}^d a_t^{(i)} \, dS_t^{(i)}$$

$$C_0 = x,$$

where $R_t = \exp(rt)$ denotes bond price with short rate interest $r$ (i.e., $dR_t = rR_t \, dt$).

**Proposition 4.2.** Let $u$ be the solution to the above PPDE (4.6) and define a non-anticipative functional $v$ as $v_t(\gamma_t) = u_t(\gamma_t) e^{-r(T-t)}$. Then, $v_t(X_t)$ is the self-financing portfolio with the initial value $\mathbb{E}[u_T(X_T)] e^{-rT}$ and the $d$-dimensional position process whose $i$-th coordinate is $\partial_i u_t(X_t)/S_t^{(i)} e^{-r(T-t)}$.

### 4.2 Greeks

For simplicity, we set the dimension $N = d = 1$. We continue using all settings in Section 4.1 except for the dimension. This section studies the sensitivity of option prices for changes in short rate $r$ and volatility $\sigma$. Since we have $b_t = r - q_t - \frac{1}{2}|\sigma_t|^2$ to examine option prices, the changes in short rate and volatility can be expressed by the changes in $b$ and $\sigma$, which are the coefficients of SDE (4.1). In this process, we define families of coefficients under certain assumptions. For each coefficient, we compare the value of the portfolio obtained in Proposition 4.2. Using this, we obtain the sensitivity formula. This approach is inspired by Chapter 5 in [Fournier (2010)].

First, we consider families of coefficients $(r^\epsilon)_{\epsilon \in I}, (q^\epsilon)_{\epsilon \in I}$ and $(\sigma^\epsilon)_{\epsilon \in I}$ where $I = (-1, 1)$. Here, $\epsilon$ is the perturbation parameter. For convenience, we define $b_t^\epsilon := r^\epsilon - q_t^\epsilon - \frac{1}{2}(\sigma_t^\epsilon)^2$ for each $\epsilon \in I$, the perturbed stock price is $S_t^\epsilon = S_0 e^{X^\epsilon}$ where its logarithm $X^\epsilon$ is a solution to the SDE

$$dX_t^\epsilon = b_t^\epsilon(X^\epsilon) \, dt + \sigma_t^\epsilon(X^\epsilon) \, dW_t, \quad X_0^\epsilon = 0.$$  

(4.7)

**Assumption 4.2.** The family of coefficients $b^\epsilon$ and $\sigma^\epsilon$ satisfy the following conditions:
(i) For \( \psi = b, \sigma, q \), there exist differentials \( \psi : [0, T] \times D([0, T], \mathbb{R}) \rightarrow \mathbb{R} \) satisfying the following: for each \( \eta_0 \in D([0, T], \mathbb{R}) \), there is a \( \delta = \delta(\eta_0) > 0 \) such that for all \( \eta \in D([0, T], \mathbb{R}) \) with \( \| \eta - \eta_0 \| < \delta \),
\[
|\psi_t^\varepsilon(\eta) - \psi_t(\eta) - \varepsilon \psi_t(\eta)| \leq \varepsilon \phi(\eta_0, \varepsilon) \| \eta \|^k_t,
\]
where \( t \in [0, T] \), \( k > 0 \) is independent of \( \eta \) and \( \phi : D([0, T], \mathbb{R}) \times I \rightarrow \mathbb{R}^+ \) is continuous with respect to \( \varepsilon \) with \( \phi(\cdot, 0) = 0 \).

(ii) The families \( b^\varepsilon, \sigma^\varepsilon \) satisfy Assumption 3.2. Moreover, the Lipschitz continuity constant is independent of \( \varepsilon \). More precisely, for each \( \eta \in D([0, T], \mathbb{R}^d) \), there exist constant \( \kappa > 0 \), \( K(\eta) > 0 \) such that if \( \| \eta - \tilde{\eta} \| \leq \kappa \) then
\[
\| D\phi^\varepsilon(\eta)(\gamma) - D\phi^\varepsilon(\tilde{\eta})(\gamma) \| \leq K(\eta) \| \eta - \tilde{\eta} \| \| \gamma \|, \\
\| D^2\phi^\varepsilon(\eta)(\gamma, \delta) - D^2\phi^\varepsilon(\tilde{\eta}, \gamma)(\delta) \| \leq K(\eta) \| \eta - \tilde{\eta} \| \| \gamma \| \| \delta \|,
\]
for all \( \gamma, \delta \in D([0, T], \mathbb{R}^d) \) where \( \phi^\varepsilon = b^\varepsilon, \sigma^\varepsilon \).

(iii) The differentials \( b, \sigma \) satisfy locally functional Lipschitz continuity, which is the second condition of Assumption 3.1.

The first condition indicates the local existence of differentials of the family of coefficients. The second condition means that we can apply Theorem 4.1 to the stock model in (4.7) for each \( \varepsilon \in I \). The third condition is needed for a technical reason to prove Theorem 4.2.

Let \( u \) be a solution to PPDE (4.6). As shown in the paragraph before Proposition 4.2 the option price at time 0 is \( v_0(X_0) = u_0(X_0)e^{-rT} \) and the perturbed one is \( v_0(X_0) = \mathbb{E}[g(X_T^\varepsilon)]e^{-rT} \) for perturbed short rate term \( r^\varepsilon \). From the terminal condition of the PPDE (4.6), we know \( g(X_T^\varepsilon) = u_T(X_T^\varepsilon) \). Therefore, the sensitivity of option prices is written as
\[
\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (v_0^\varepsilon(X_0) - v_0(X_0)) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{E}[u_T(X_T^\varepsilon)e^{-r^\varepsilon T} - u_T(X_T)e^{-rT}].
\]
To make the term \( u_T(X_T^\varepsilon) \), apply the portfolio in Proposition 4.2 to the perturbed stock dynamic \( S^\varepsilon \) in (4.7). The next proposition states the value of the portfolio at time \( t \).

**Proposition 4.3.** For each \( \varepsilon \in I \), define the stochastic process \( G^\varepsilon \) as
\[
G^\varepsilon(t) = u_t(X_t^\varepsilon) + \int_0^t D_xu_s(X_s^\varepsilon)(b_s(X^\varepsilon) - b^\varepsilon_s(X^\varepsilon)) + \frac{1}{2} D_{xx}u_s(X_s^\varepsilon)(\sigma_s^2(X^\varepsilon) - (\sigma^\varepsilon_s)^2(X^\varepsilon)) ds,
\]
and \( K^\varepsilon(t) = G^\varepsilon(t)e^{-r^\varepsilon(T-t)} \). Then, \( G^\varepsilon \) is a martingale and \( K^\varepsilon \) is a self-financing portfolio under the perturbed stock model \( S^\varepsilon_t \) in (4.7) with initial value \( \mathbb{E}[u_T(X_T^\varepsilon)]e^{-r^\varepsilon T} \) and the position
\[
\frac{D_xu(X^\varepsilon)}{S^\varepsilon_t} e^{-r^\varepsilon T}.
\]

Using the above arguments, we can obtain the following formula, which gives the sensitivity of option price when the short rate \( r \) and the volatility \( \sigma \) are perturbed. For the last equality in the theorem, we used \( \hat{r} = \hat{b} + \hat{\sigma}_1 \sigma_1 + \hat{q}_1 \) from \( \hat{b}_t^\varepsilon := r^\varepsilon - q_t^\varepsilon - \frac{1}{2}(\hat{\sigma}_1^\varepsilon)^2 \),

**Theorem 4.4.** Under Assumptions 3.2, 4.1 and 4.2, the sensitivity of the option price (4.9) is
\[
\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (v_0^\varepsilon(X_0) - v_0(X_0)) = -u_0(X_0)e^{-rT}\hat{r}T - \mathbb{E}\left[ \int_0^T D_xu_s(X_s)(\hat{b}_s(X) + D_{xx}u_s(X_s)\hat{\sigma}_s(X)\sigma_s(X)) ds \right] e^{-rT}
\]
\[
\text{(4.10)}
\]
\[
= -u_0(X_0)e^{-rT}\hat{r}T - \mathbb{E}\left[ \int_0^T (\hat{r} - \hat{q}_s(X))D_xu_s(X_s) + (D_{xx}u_s(X_s) - D_xu_s(X_s))\hat{\sigma}_s(X)\sigma_s(X) ds \right] e^{-rT}.
\]
Remark 4.1. We introduce Fournie (2010)’s work with respect to the sensitivity analysis using our notation. First, we sketch a model and an option price. The stock model is
\[ dS_t = r_t S_t \, dt + \sigma_t(S) \, dW_t \]
where the deterministic short rate is \( r \) and volatility \( \sigma \) is a non-anticipative functional satisfying Assumption 3.2. Let \( u \) be a locally regular functional in Definition 2.2 and \( g \) be an option.
Assume that for all \( t < T \), \( u_t \) satisfies
\[
D_t u_t (\gamma_t) + r_t D_x u_t (\gamma_t) \gamma_t (t) + \frac{1}{2} D_{xx} u_t (\gamma_t) \gamma_t^2 (t) \sigma_t^2 (\gamma_t) = r_t u_t (\gamma_t)
\]
for \( \gamma_t \in D([0, t], \mathbb{R}^d) \) and \( \gamma_T \in D([0, T], \mathbb{R}^d) \). Then, \( u_t (S_t) \) is a self-financing portfolio with the initial value \( u_0 (S_0) \) and position \( D_x u_t (S_t) \).

Now, we turn to sensitivity. Let \( (\sigma^\epsilon)_{\epsilon \in (0, \infty)} \) be families of functionals satisfying the first condition of the Assumption 3.2 for each \( \epsilon > 0 \). We denote the perturbed stock model \( S^\epsilon \) as the unique solution to
\[ dS^\epsilon_t = r_t S^\epsilon_t \, dt + \sigma^\epsilon_t (S) \, dW_t. \]
Under some additional conditions as an existence of differential \( \hat{\sigma} \), we have
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}[g(S^\epsilon_T) - g(S_T)]e^{-rT} = \mathbb{E} \left[ \int_0^T \hat{\sigma}_t (S) \sigma_t (S) S^2 (t) D_{xx} u_t (S_t) e^{-rt} \right].
\]

5 Example

5.1 Coefficients with time integration
In this subsection, we consider the no-dividend stock price model with a time integration coefficient as setting \( \sigma_t (\eta) = \alpha \int_0^t f (t, u) \eta (u) \, du \) where \( \alpha : \mathbb{R} \to \mathbb{R} \) is a function and \( f (t, \cdot) : [0, t] \to \mathbb{R} \) for each \( t \in [0, T] \). The dynamics of the logarithm of stock price follows the SDE
\[
dX_t = r - \frac{1}{2} \alpha^2 \left( \int_0^t f (t, u) X_u \, du \right) \, dt + \alpha \left( \int_0^t f (t, u) X_u \, du \right) \, dW_u. \tag{5.1}
\]
To ensure that the coefficient \( \sigma \) is well-defined, we may assume that for each \( t \in [0, T] \), the function \( f (t, \cdot) : [0, t] \to \mathbb{R} \) is in \( L^1 \), i.e., \( \| f (t, \cdot) \|_{L^1 [0, t]} = \int_0^t |f (t, u)| \, du < \infty \). Including this condition, there is a sufficient condition for the existence and uniqueness of solution (5.1) and Assumption 3.2.

Proposition 5.1. Assume that for each \( t \in [0, T] \), the norm \( \| f (t, \cdot) \|_{L^1 [0, t]} = \int_0^t |f (t, u)| \, du \) is finite. If \( \alpha \) and \( \alpha^2 \) are Lipschitz continuous in \( \mathbb{R} \), then a solution to the SDE (5.1) uniquely exists. Moreover, if the function \( \alpha \) in (5.1) satisfies
(i) \( \alpha \) is in \( C^2 (\mathbb{R}) \), i.e., is continuously differentiable up to order 2,
(ii) for each \( \eta \in D([0, T], \mathbb{R}) \), a path \( (\int_0^t f (t, u) \eta (u) \, du)_{t \in [0, T]} \) is càdlàg with respect to \( t \),
then Assumption 3.2 holds.

Example 5.1. There are two examples of the function \( f \). The first is the Dirac delta function.
This means that we define an integral \( \int_0^t f (t, u) \eta (u) \, du = \eta (k (t)) \) for some \( k (t) \in [0, t] \). Although \( f \) is not actually a function, the proof of Proposition 5.1 holds. Using this, we can understand that the model (5.1) is a generalization of the Markov process.
The sensitivity of an option price is calculated using the formula
\[ \epsilon \]
From the chain rule, we obtain the differential of \( \sigma \) of coefficients \( b \), partial derivative of the option price with respect to the short rate term change. We follow the notation in Section 4.2. First, we discuss the rho value, which is the change. We study two examples in this subsection.

Example 5.3. Assume that for each \( u \in [0, T] \), there exists a differential \( \dot{f} \) of \( f \) such that
\[ \left| f(t + \epsilon, u) - f(t, u) - \epsilon \dot{f}(t, u) \right| \to 0, \]
as \( \epsilon \to 0 \) for all \( t \in [u, T] \). We define the family of the volatility \( \sigma_t^\epsilon(\eta) = \alpha\left(\int_0^t f(t, u + \epsilon)\eta(u)\,du\right) \). From the chain rule, we obtain the differential of \( \sigma \) as
\[ \dot{\sigma}_t(\eta) = \alpha \left( \int_0^t f(t, u)\eta(u)\,du \right) \int_0^t \dot{f}(t, u)\,du. \]
The sensitivity of an option price is calculated using the formula 4.10.

Remark 5.1. The typical example of \( \alpha \) satisfying both Lipschitz continuity and differentiability is bounded and \( C^2(\mathbb{R}) \). This includes the sine and cosine functions. If we already know that a solution to the SDE 5.1 exists, we can drop the condition of Lipschitz continuity on \( \mathbb{R} \). Moreover, we can reduce a domain of \( \alpha \) to \( \{ \int_0^t f(t, u)\eta(u)\,du \mid t \in [0, T], \omega \in \Omega \} \), which may be a subset of \( \mathbb{R} \).

5.2 Greeks

In this section, we study the sensitivity of an option price as the short rate \( r \) and volatility \( \sigma \) change. We follow the notation in Section 4.2. First, we discuss the rho value, which is the partial derivative of the option price with respect to the short rate term \( r \). Applying the families of coefficients \( b^\epsilon_t = b_t + \epsilon \) and \( \sigma^\epsilon_t = \sigma_t \), we obtain \( \sigma = 0 \) and \( \dot{r} = 1 \). Hence, the sensitivity of the option price 4.9 is actually the value rho. From Theorem 4.11, we obtain
\[ \rho = \frac{\partial v_0(X_0)}{\partial r} = -u_0(X_0)\epsilon e^{-rT}T - E\left[ \int_0^T D_x u_s(X_s)\,ds \right] e^{-rT}. \]

Second, we consider the case in which the volatility changes and the short rate does not. As we study in Section 4.2, the sensitivity differs from how we define the family of volatility \( \sigma^\epsilon \). We study two examples in this subsection.

Example 5.2. Define the family of volatility \( \sigma^\epsilon_t(\eta) = \sigma_t(\eta + \epsilon \mathbf{1}_{[0, T]}). \) The differential \( \dot{\sigma} \) is written as
\[ \dot{\sigma} = \alpha \left( \int_0^t f(t, u)\eta(u)\,du \right) \int_0^t \dot{f}(t, u)\,du, \]
because the inequality 4.8 holds from the Fréchet derivative of \( \sigma \). To make the differential of the short rate zero, we define the family of \( b^\epsilon \) by \( b^\epsilon_t = r - (\sigma^\epsilon_t)^2/2 \). It is easily seen that the families of coefficients \( b^\epsilon \), \( \sigma^\epsilon \) satisfy all conditions of Assumption 4.2. Similarly, we consider the family of the volatility \( \sigma^\epsilon_t(\eta) = \alpha\left(\int_0^t f(t, u + \epsilon)\eta(u)\,du\right) \). Then, differential \( \dot{\sigma} \) is written as
\[ \dot{\sigma} = \alpha \left( \int_0^t f(t, u)\eta(u)\,du \right) \int_0^t \eta(u)\,du, \]
and the sensitivity formula can be calculated using the formula 4.10.

Example 5.3. Assume that for each \( u \in [0, T] \), there exists a differential \( \dot{f} \) of \( f \) such that
\[ \left| f(t + \epsilon, u) - f(t, u) - \epsilon \dot{f}(t, u) \right| \to 0, \]
as \( \epsilon \to 0 \) for all \( t \in [u, T] \). We define the family of the volatility \( \sigma^\epsilon_t(\eta) = \alpha\left(\int_0^t f(t + \epsilon, u)\eta(u)\,du\right) \). From the chain rule, we obtain the differential of \( \sigma \) as
\[ \dot{\sigma}_t(\eta) = \alpha \left( \int_0^t f(t, u)\eta(u)\,du \right) \int_0^t \dot{f}(t, u)\,du. \]
6 Conclusion

We studied the differentiability of solutions to path-dependent SDEs. Given an SDE with path-dependent coefficients having Fréchet derivatives, for each \( \gamma_t, \delta_t \in D([0, t], \mathbb{R}^d) \), we proved that the SDE solution \( X^{\gamma_t} \) is differentiable with respect to the initial path \( \gamma_t \) in the direction of \( \delta_t \), and that the directional derivative \( D_{\delta_t} X^{\gamma_t} \) is given as a solution to

\[
D_{\delta_t} X^{\gamma_t}(s) = \delta_t(t) + \int_t^s D_b r(X^{\gamma_t})(D_{\delta_t} X^{\gamma_t}) \, dr + \int_t^s D \sigma r(X^{\gamma_t})(D_{\delta_t} X^{\gamma_t}) \, dW_t \quad t \leq s \leq T, \\
D_{\delta_t} X^{\gamma_t}(s) = \delta_t(s) \quad 0 \leq s < t,
\]

as given in (3.7).

PDE representations and sensitivities of option prices under stock models described in Section 4.1 are studied as applications. Under Assumptions 3.2 and 4.1, an option price \( \alpha(X_t) \) is given in (3.7). For the families of coefficients \((b_\epsilon, \sigma_\epsilon)\) for each \( \epsilon \in \mathbb{I} \), we proved that

\[
D_t u_\epsilon(\gamma_t) + \langle r \mathbb{1}^d - q(\gamma_t), D_x u_\epsilon(\gamma_t) \rangle \\
+ \frac{1}{2} \text{tr} \left( \sigma(\gamma_t) \cdot ((D_{xx} u_\epsilon(\gamma_t) - \text{diag}(D_x u_\epsilon(\gamma_t)))\sigma(\gamma_t)) \right) = 0 \quad \text{for} \quad \gamma_t \in D([0, t], \mathbb{R}^d),
\]

for \( \gamma_T \in D([0, T], \mathbb{R}^d) \).

Conversely, a regular solution \( u \) of the above PDE (4.5) satisfies

\[ u(\gamma_t) = Y^{\gamma_t}(t), \quad \text{for} \quad \gamma_t \in D([0, t], \mathbb{R}^d), \]

where \( Y^{\gamma_t} \) is a solution to BSDE (4.3).

Furthermore, we introduced formulas for Greeks of option prices for small changes of the short rate \( r \) and volatility \( \sigma \). For the families of coefficients \((b', \sigma') \) for \( \epsilon \in \mathbb{I} \), we considered the perturbed stock model \( S^\epsilon = S_0 e^{X^\epsilon} \) where

\[
dX^\epsilon_t = b^\epsilon(X^\epsilon) \, dt + \sigma^\epsilon(X^\epsilon) \, dW_t, \quad X_0^\epsilon = 0.
\]

Then, the perturbed option price is

\[
u_0^\epsilon(X_0) = e^{-rT} \mathbb{E} [g(X_T^\epsilon)],
\]

and under Assumption 4.2 the Greek value is expressed as

\[-u_0(X_0) e^{-rT} \iota T - \mathbb{E} \left[ \int_0^T D_x u_\epsilon(X_s) \dot{b}_s(X) + D_{xx} u_\epsilon(X_s) \sigma_s(X) \sigma_s(X) \, ds \right] e^{-rT}.
\]

As an example, we studied the stock model having coefficients with time integration form. The logarithm of the stock price is a solution to SDE (5.1),

\[
dX_t = r - \frac{1}{2} \alpha^2 \left( \int_0^t f(t, u) X_u \, du \right) \\
+ \alpha \left( \int_0^t f(t, u) X_u \, du \right) \, dW_u,
\]

where \( \alpha : \mathbb{R} \to \mathbb{R} \) is a smooth function and \( f(t, \cdot) : [0, t] \to \mathbb{R} \) is integrable for each \( t \in [0, T] \). Proposition 5.1 provided conditions on \( \alpha \) and \( f \) to satisfy Assumption 3.2. We then applied the Greek formulas in (4.10) to several types of perturbations.

A Fréchet derivative

In this appendix, we prove Propositions 5.1 and 5.2 as well as Theorems 3.3 and 5.3. For simplicity, we only deal with the case of \( N = d = 1 \). The flow of proofs follows that of Theorem V.39 in Protter (2005). First, we define the necessary terms.
Recall that \( D([0,T], \mathbb{R}) \) is the collection of functions with càdlàg paths from \([0,T]\) to \(\mathbb{R}\) and \(\mathbb{D}\) is the space of adapted right continuous processes with left limit (RCLL) processes for fixed maturity \(T \in \mathbb{R}_+\). For a process \( H \in \mathbb{D}\), we use notations in (2.1) and define \(S^p\) norms in \(\mathbb{D}\) for \(p \geq 2\)

\[
\|H\|_{S^p} = \left( \sup_{t \in [0,T]} |H_t|^p \right)^{1/p}.
\]

We define the \(H^p\)-norm of semimartingale \(Z\) with \(Z_0 = 0\) as Émery norm. See Chapter 5 in Protter (2005) for detail. Note that from Theorem V.2 in Protter (2005), \(H^p\) norm is stronger than \(S^p\) norm.

The following proposition says that the Fréchet derivative of non-anticipative functional at \(t\) depends only on the path on \([0,t]\). It is understood that the Fréchet derivative of a non-anticipative functional is a non-anticipative functional. Recall that we write \(\eta^t \in D([0,T], \mathbb{R})\) as the stopped path of \(\eta\) at time \(t \in [0,T]\).

**Proposition A.1.** Let \(F : D([0,T], \mathbb{R}) \to D([0,T], \mathbb{R})\) be a non-anticipative functional with a Fréchet derivative \(\eta, \tilde{\eta}, \gamma, \tilde{\gamma} \in D([0,T], \mathbb{R})\). If \(\eta^t = \tilde{\eta}^t\) and \(\gamma^t = \tilde{\gamma}^t\) for some \(t \in [0,T]\), then \(DF_t(\gamma)(\eta) = DF_t(\gamma)(\tilde{\eta})\).

**Proof.** Let \(\delta \in \mathbb{R}_+\) be the scale of \(\eta\). Apply Definition 3.1 to \(u = \delta \eta\) and \(v = \gamma\). It follows that

\[
0 = \lim_{\delta \to 0} \|F(\gamma + \delta \eta) - F(\gamma) - DF(\gamma)(\delta \eta)\| = \lim_{\delta \to 0} \frac{|F(\gamma + \delta \eta) - F(\gamma) - DF(\gamma)(\delta \eta)|}{\|\delta \eta\|}
\]

\[
= \lim_{\delta \to 0} \left| \frac{F(\tilde{\gamma} + \delta \tilde{\eta}) - F(\tilde{\gamma})}{\delta \|\tilde{\eta}\|} - DF_t(\gamma) \left( \frac{\eta}{\|\tilde{\eta}\|} \right) \right| \|\tilde{\eta}\|.
\]

since \(DF(\gamma)(\cdot)\) is linear operator in Definition 3.1. By applying Definition 3.1 to \(u = \delta \tilde{\eta}\) and \(v = \tilde{\gamma}\), we can obtain what we desire. \(\square\)

Let \(DF\) be the Fréchet derivative of non-anticipative functional \(F\). For fixed \(t \in [0,T]\) and the adapted processes \(X\) and \(Y\), the value \(DF_t(X)(Y)\) depends only on the paths of \(X\) and \(Y\) on \([0,t]\). Note that the stopped path \(X^t(\cdot) := X(t \wedge \cdot)\) is \(\mathcal{F}_t\)-measurable. From the above proposition, we have \(DF_t(X)(Y) = DF_t(X^t)(Y^t)\) and \(DF_t(X^t)(Y^t)\) is \(\mathcal{F}_t\)-measurable. Thus, as a stochastic process, the operator \(DF(X)(Y) : [0,T] \times \Omega \to \mathbb{R}\) is adapted.

Let us repeat the setup of Proposition 3.1. Let \(Z\) be a semimartingale with \(Z_0 = 0\) and \(F : D([0,T], \mathbb{R}) \to D([0,T], \mathbb{R})\) be locally functional Lipschitz continuous with Fréchet derivative \(DF\). Consider a solution \((X, DX)\) to the SDE with non-anticipative functional coefficient

\[
X_t^x = x + \int_0^t F_{s-}(X^s) \, dZ_s,
\]

\[
DX_t^x = 1 + \int_0^t DF_{s-}(X^s)(DX^s) \, dZ_s.
\]

Because \(F\) and \(DF\) are locally functional Lipschitz continuous, we know that the solutions to (A.1) and (A.2) uniquely exist.

The following lemma is the restatement of the Lemma V.2 in Protter (2005). It says that the solution to SDE (A.1) uniquely exists in \(S^p\) and its \(S^p\) norm is estimated independently of the coefficient of the SDE.

**Lemma A.2.** Let \(1 \leq p \leq \infty\), let \(J \in S^p\), let \(F\) be functional Lipschitz continuous satisfying \(F(0) = 0\). Then, the equation

\[
X_t = J_t + \int_0^t F_{s-}(X) \, dZ_s
\]

has a solution in \(S^p\). It is unique, and moreover, \(\|X\|_{S^p} \leq C(p, Z)\|J\|_{S^p}\), where \(C(p, Z)\) is a constant depending only on \(p\) and \(Z\).
Note that we can also apply the lemma to the solution to the SDE (A.2) since the operator $DF(\eta)(\cdot)$ for $\eta \in D([0, T], \mathbb{R})$ is linear.

For the continuity of $X^x$ in (A.1), we apply Theorem V.37 in Protter (2005). Using continuity, we can extend Theorem V.16 in Protter (2005). Let $\pi = \{0 = T_0 \leq T_1 \leq \cdots T_k \leq T\}$ denote a finite sequence of finite stopping time. The sequence $\pi$ is called a random partition. We say a sequence of random partition $\pi_n = \{0 = T^n_0 \leq T^n_1 \leq \cdots T^n_{k_n} \leq T\}$ tends to the identity if

(i) $\lim_n \sup_k T^n_k = T$ a.s and

(ii) $\|\pi_n\| := \sup_k |T^n_{k+1} - T^n_k|$ converges to 0 a.s.

For a stochastic process $Y$, we define approximations of $Y$ as

$$Y^\pi \equiv Y_0 1_{\{0\}} + \sum_{j=0}^k Y_{T_j 1_{\{T_j, T_{j+1}\}}} \quad Y^{\pi+} \equiv \sum_{j=0}^k Y_{T_j 1_{\{T_j, T_{j+1}\}}} + Y_T 1_{\{T\}}.$$  

(A.3)

If $Y$ is adapted and càdlàg (i.e. $Y \in \mathbb{D}$), then $(Y^{\pi}(s))_{s \geq 0}$ is left continuous with right limits and adapted and $(Y^{\pi+}(s))_{s \geq 0}$ is right continuous.

The next lemma is an extension of Theorem V.16 in Protter (2005) to functional Lipschitz continuity. Using this lemma, we can approximate solutions to the SDEs (A.1) and (A.2) using solutions to SDEs with function coefficients.

**Lemma A.3.** Suppose that $F$ is functional Lipschitz continuous. Let $X$ be a solution to SDE (A.1) and $X^{(\pi)}$ be a solution to the following SDE:

$$X^{(\pi)}_t = x + \int_0^t F_s((X^{(\pi)})^{\pi+}) \, dZ_s$$

for a random partition $\pi$. If $\pi_n$ is a sequence of random partitions tending to the identity, then $X^{(\pi_n)}$ tends to $X$ in $S^p$.

**Idea of proof.** Note that $X(\cdot, \omega) : [0, T] \to \mathbb{R}$ is uniformly continuous for a.e. $\omega \in \Omega$. For any small $\epsilon > 0$, there exists large $n$ such that $|X^{(\pi_n)}_t(\omega) - X_u(\omega)| < \epsilon$ for all $u \in [0, t]$. We can conclude $\|F(X^{(\pi+)}(\omega)) - F(X(\omega))\|_t < C\epsilon$ for some constant $C$ from Lipschitz continuity of $F$. The remaining part is same as in Theorem V.16 in Protter (2005).

Now, we prove Proposition 3.1. The construction of the proof follows that of Theorem V.39 of Protter (2005) with no explosion time.

**Proof of Proposition 3.1.** Step 1. As in Step 1 of Theorem V.39 of Protter (2005), we can assume that $DF$ is globally Lipschitz continuous. This also means that we may assume that the constant $K(\eta)$ in (3.3) is independent of the choice $\eta$. Note that from Lipschitz continuity of $DF$, we have the linear growth condition for $DF$, i.e., for all $\eta, \gamma \in D([0, T], \mathbb{R})$, there exist constant $K > 0$ such that

$$\|DF(\eta)(\gamma)\| \leq K(1 + \|\eta\|) \|\gamma\|. \quad \text{(A.4)}$$

Step 2. We show that $DX^x$ is continuous with respect to $x$. Let $V(t, \omega) \equiv DX^x(t, \omega) - DX^y(t, \omega)$ for $t \in [0, T], \omega \in \Omega$. Then, we have

$$V(t) = \int_0^t DF_{s-}(X^x)(V) \, dZ_s + \int_0^t J_s \, dZ_s$$

where $J_s = [DF_s(X^x) - DF_s(X^y)](DX^y)$. From the fact that the $H^p$ norm is stronger than the $S^p$ norm and Emery’s inequality, we obtain $\|\int_0^T J_s \, dZ_s\|_{S^p} \leq \|J\|_{S^p} \|Z\|_{H^\infty}$. We can estimate $\|J\|_{S^p}$ as

$$\|J\|_{S^p} \leq KE \left[ \sup_{u \in [0, T]} \|X^x - X^y\|_u \|DX^y\|_u \right]^{\frac{1}{2}} \leq KE \left[ \|X^x - X^y\|^{\frac{1}{p}} \|DX^y\|^{\frac{1}{p}} \right]^{\frac{1}{2}} \leq CKE \left[ \|X^x - X^y\|^{2p} \right]^{\frac{1}{2p}} \|DX^y\|_{S^{2p}} = CK \|X^x - X^y\|_{S^{2p}} \|DX^y\|_{S^{2p}}.$$
We use the Lipschitz continuity of $DF$ in the 2nd inequality and Hölder’s inequality in the 4th inequality. The constant may be different for each inequality. Applying Lemma A.2 with $DF(\cdot)(0) = 0$, we obtain

$$
\|V\|_{S^p} \leq C \left\| \int_0^T J_{s-} \, dZ_s \right\|_{S^p} \leq C \|X^x - X^y\|_{S^{2p}} \|DX^y\|_{S^{2p}} \|Z\|_{H^\infty}. \tag{A.5}
$$

The Lemma A.2 indicates that $\|DX^y\|_{S^{2p}}$ is bounded in $S^{2p}$. By the proof of Theorem V.37 in Protter (2005), we have that $\|X^x - X^y\|_{S^p} \leq C |x - y|^{p}$, which means that $X^x$ is Lipschitz continuous in $S^p$ with respect to $x$. Now, from (A.5) and Lemma A.2 we can assert that $DX^x$ is continuous with respect to $x$ using Kolmogorov’s continuity theorem.

Step 3. We verify $\frac{\partial X^x}{\partial x}(t, \omega) = DX^x(t, \omega)$ for $(t, \omega) \in [0, T] \times \Omega$. We fix $\omega \in \Omega$. From definitions (A.3), (A.6), we have the following equalities for $t \in [T_i, T_{i+1}], i = 0, \cdots, k_n$, and $s \in [0, t]$,

$$
\int_0^t F_s(\hat{X}^\pi)^y \, dZ_s = \sum_{j=0}^{i-1} F_{T_j}(\hat{X}^\pi)^y \cdot (Z_{T_{j+1}} - Z_{T_j}) + F_{T_i}(\hat{X}^\pi)^y \cdot (Z_t - Z_{T_i}).
$$

Hence, $\hat{X}, D\hat{X}$ can be described as

$$
\hat{X}_t = \hat{X}_{T_i} + F_{T_i}(\hat{X}^\pi)^y \cdot (Z_t - Z_{T_i})
$$

$$
D\hat{X}_t = D\hat{X}_{T_i} + DF_{T_i}(\hat{X}^\pi)^y \cdot (Z_t - Z_{T_i}) \tag{A.7}
$$

for $t \in [T_i, T_{i+1}]$.

We use induction to show $\frac{\partial X^x}{\partial x}(t, \omega) = D\hat{X}(t, \omega)$. If $t \in [T_0, T_1] = [0, T_1]$, it is easily seen to show $\frac{\partial X^x}{\partial x} = D\hat{X}_t$ from Proposition A.1.

Now, we prove the induction step. Suppose that $\frac{\partial X^x}{\partial x} = D\hat{X}_t$ for $t \in [0, T_j]$. Let $t \in (T_j, T_{j+1}]$. From (A.7), we obtain that $\frac{\partial X^x}{\partial x} = D\hat{X}_t + \frac{\partial F_{T_j}(\hat{X}^\pi)^y}{\partial x}(Z_t - Z_{T_j})$. We need to show $\frac{\partial F_{T_j}(\hat{X}^\pi)^y}{\partial x} = DF_{T_j}(\hat{X}^\pi)^y$. For small $\delta > 0$,

$$
\left| \frac{F_{T_j}(\hat{X}^{x+\delta} - F_{T_j}(\hat{X}^x)}{\delta} - DF_{T_j}(\hat{X}^\pi)^y \right| \leq AB + C
$$
where $A, B, C$ are defined as
\[
A = \frac{|F_{T_l}((\hat{X}^x+\delta,\pi^+)) - F_{T_l}((\hat{X}^x,\pi^+)) - D F_{T_l}((\hat{X}^x,\pi^+))(\hat{X}^x+\delta,\pi^+ - \hat{X}^x,\pi^+)|}{\|X^x+\delta,\pi^+ - X^x,\pi^+\|_{T_l}}
\]
\[
B = \frac{\|\hat{X}^x+\delta,\pi^+ - \hat{X}^x,\pi^+\|_{T_l}}{\delta}
\]
\[
C = \left|\frac{D F_{T_l}((\hat{X}^x,\pi^+))(\hat{X}^x+\delta,\pi^+ - \hat{X}^x,\pi^+)}{\delta} - D F_{T_l}((\hat{X}^x,\pi^+))(D \hat{X}^x)\right|.
\]

Since $\frac{\partial \hat{X}^x}{\partial x} = D \hat{X}_t$, $A$ converges to 0 as $\delta \to 0$ from the definition of the Fréchet derivative. In addition, since $D F(\eta)(\cdot)$ is a continuous linear operator for each $\eta \in D([0, T, \mathbb{R}])$, we obtain $C \to 0$ as $\delta \to 0$. Now, we derive that $B$ is bounded uniformly in $\delta$. For any $\epsilon > 0$, we can choose $\xi > 0$ such that if $0 < \delta < \xi$ then
\[
\frac{|F_{T_l}((\hat{X}^x+\delta,\pi^+)) - F_{T_l}((\hat{X}^x,\pi^+)) - D F_{T_l}((\hat{X}^x,\pi^+))(D \hat{X}^x)|}{\delta} < \epsilon,
\]
for all $i = 0, \ldots, j - 1$. Thus, we obtain
\[
B = \sup_{\eta \in [0, T_l]} \frac{|\hat{X}^x_{u+\delta,\pi^+} - \hat{X}^x_{u,\pi^+}|}{\delta} \leq \left(\|D F_{T_l}((\hat{X}^x,\pi^+))(D \hat{X}^x)\|_{T_l} + \epsilon\right) \times \sup_{i=0, \ldots, j-1} \sup_{u \in [T_i, T_{i+1}]} |Z_u - Z_{T_l}|.
\]
We have already fixed $\omega \in \Omega$, and the $|Z_u - Z_{T_l}|$ part is bounded by $2 \sup_{u \in (0, T_l]} |Z_u|$. Thus, we have $B$ is uniformly bounded in $\delta$. Consequently, we obtain $\frac{\partial \hat{X}^x}{\partial x} = D \hat{X}_t$ for all $t \in [0, T]$ from the induction argument.

Step 4. The remaining part is same as Theorem V.39 of Protter (2005). From Step 1, we can consider $(X, DX)$ as a distribution called a generalized function. In distribution theory, we use the theorem that if $g$ is a continuous function and $Dg$ is a continuous derivative in the distribution sense, then $Dg$ is the derivative of $g$ in classical sense. Therefore, we can conclude $\frac{\partial X}{\partial x}(t, \omega) = DX(t, \omega)$ for $\omega$ not in exceptional set.

\begin{remark} \label{rem:A.1}
The Fréchet derivative includes the notion of the usual derivative. If a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x$, then the Fréchet derivative of $f$ at $x \in \mathbb{R}$ is represented by $Df(x) : \mathbb{R} \to \mathbb{R}$ as $DF(x)(h) = f'(x) \cdot h$.

The chain rule also holds for the Fréchet derivative and we use it for the normed space $D([0, T], \mathbb{R})$ or $\mathbb{R}$. Note that there exists the Fréchet derivative of $b(X^x)$ for an operator $b : D([0, T], \mathbb{R}) \to D([0, T], \mathbb{R})$ which is represented by $Db(X^x)(DX^x)$.

We omit the proof of Proposition \ref{prop:A.2} since it is the generalize of Proposition \ref{prop:A.1} to the second-order Fréchet derivative.
\end{remark}

\textbf{Proofs of Theorems \ref{thm:A.3} and \ref{thm:A.4}}. Basically, we follow the proofs of Propositions \ref{prop:A.1} and \ref{prop:A.2}. Here, we only describe differences between the proofs. By switching the initial data from a point to a path, we extend Lemma \ref{lem:A.3} to the case of an SDE \ref{eq:A.4}. For a fixed initial path $\gamma_t$ and derivative direction $\delta_t$, we take a random partition $\pi = \{0 = T_0 \leq T_1 \leq \cdots \leq T_n = T\}$ including the discontinuity points of $\gamma_t$ and $\delta_t$. In the interval of the partition, there is no difference from Lemma \ref{lem:A.3} as the paths $\gamma_t$ and $\delta_t$ are continuous in the interval. The discontinuity points at partition $T_0, \ldots, T_n$ can be covered well from the right continuous version $X^{\pi^+}$ and $DX^{\pi^+}$ because the paths $\gamma_t$ and $\delta_t$ are càdlàg. Thus, we have $X^{(\pi_n)}, DX^{(\pi_n)}$ converges to $X, DX$ in $S^p$. \hfill \Box
B Non-anticipative PDE

In this section, we prove Theorem 4.1. The entire proof follows that of Peng and Wang (2016). Applying the traditional BSDE theory to the case of a path-dependent BSDE, we obtain the vertical differentiability of the solution to the BSDE. For the time derivative, we approximate the given path-dependent BSDE to the case of a classic function coefficient BSDE and use the result of Peng and Wang (2016). In this process, we determine that the non-anticipative functional $u$ in (4.1) is the solution to the path-dependent PDE (4.5).

B.1 Vertical derivative

In this part, we show that the operator $u_t(\gamma_t) = Y^{\gamma_t}(t)$ has a second-order vertical derivative. For this, we show that the $S^p$ norm and $H^p$ norm of solutions to an SDE or BSDE depend only on the initial path, applying the arguments in Zhang (2017) and Peng and Wang (2016). After that, we estimate the $S^p$ norm of a difference of $Y^{\gamma_t}$ when the initial path changes vertically. This argument is deeply involved in Peng and Wang (2016). Then, we have the differentiability of $u$ using Kolmogorov’s continuity theorem.

Before stating the theorems and lemmas to be proved, we mention an important thing in proofs. As in the proof of Proposition 3.1, we can assume that $b, \sigma$ and $g$ are globally Lipschitz continuous. It means that the inequality (A.4) holds for $DF = Db, D\sigma$. Moreover, the linear growth for $Dg$ holds, i.e., for all $\eta, \gamma \in D([0,T], \mathbb{R})$, there exists $K > 0$ such that $|Dg(\eta)(\gamma)| \leq K(1 + ||\eta||)||\gamma||$. Finally, the constant $C$ in proofs may be different in each line.

Theorem B.1. For any $p \geq 2$, the following inequality holds,

$$ \mathbb{E} \left[ \sup_{s \in [t,T]} |Y^{\gamma}(s)|^p + \left| \int_t^T |Z^{\gamma}(s)|^2 \, ds \right|^\frac{p}{2} \right] \leq C \mathbb{E} ||g(X^{\gamma})||^p. $$

The proof of this theorem is same as Lemma 3.4 in Peng and Wang (2016). We now study the continuity and differentiability of $(Y^{\gamma}, Z^{\gamma})$ with respect to the initial path $\gamma_t$. For $\gamma_t \in D([0,t], \mathbb{R}^d), \gamma_{\bar{t}} \in D([0, \bar{t}], \mathbb{R})$, we define $\Delta_h Y^{\gamma_t} = \frac{1}{h}(Y^{\gamma_{t+h}} - Y^{\gamma_t}), \Delta_h Z^{\gamma_t} = \frac{1}{h}(Z^{\gamma_{t+h}} - Z^{\gamma_t})$ where $\gamma^{h}_{t} \in D([0,t], \mathbb{R}^d)$ is defined as $\gamma^{h}_{t}(s) = \gamma_{t}(s)1_{[0, t]}(s) + (\gamma_{t}(t) + h)1_{[t, t]}(s)$ for $s \in [0, t]$.

Lemma B.2. For any $p \geq 2$, there exist some constants $C_p$ and $q$ dependent only on $C, T, k, p$ such that for any $t, \bar{t} \in [0, T], h, \bar{h} \in \mathbb{R} - \{0\}, \gamma_t \in D([0,t], \mathbb{R}^d), \gamma_{\bar{t}} \in D([0, \bar{t}], \mathbb{R})$,

(i) \[ \mathbb{E} \left[ \sup_{u \in [t \land \bar{t}, T]} |Y^{\gamma_t}(u) - Y^{\gamma_{\bar{t}}}(u)|^p \right] \leq C_p (1 + ||\gamma_t||^q + ||\gamma_{\bar{t}}||^q)(||\gamma_t - \gamma_{\bar{t}}||\land + ||t - \bar{t}||^p). \]

(ii) \[ \mathbb{E} \left[ \left| \int_{t \land \bar{t}}^T (Z^{\gamma_t}(u) - Z^{\gamma_{\bar{t}}}(u))^2 \, du \right|^\frac{p}{2} \right] \leq C_p (1 + ||\gamma_t||^q + ||\gamma_{\bar{t}}||^q)(||\gamma_t - \gamma_{\bar{t}}||\land + ||t - \bar{t}||^p). \]

(iii) \[ \mathbb{E} \left[ \sup_{u \in [t \land \bar{t}, T]} |\Delta_h Y^{\gamma_t}(u) - \Delta_h Y^{\gamma_{\bar{t}}}(u)|^p \right] \leq C_p (1 + ||\gamma_t||^q + ||\gamma_{\bar{t}}||^q)(||\gamma_t - \gamma_{\bar{t}}||\land + ||t - \bar{t}||^p). \]

(iv) \[ \mathbb{E} \left[ \left| \int_{t \land \bar{t}}^T (\Delta_h Z^{\gamma_t}(u) - \Delta_h Z^{\gamma_{\bar{t}}}(u))^2 \, du \right|^\frac{p}{2} \right] \leq C_p (1 + ||\gamma_t||^q + ||\gamma_{\bar{t}}||^q)(||\gamma_t - \gamma_{\bar{t}}||\land + ||t - \bar{t}||^p). \]
Before we prove the above lemma, we need to estimate the case of \( X^\gamma \). The proof of this lemma is from standard argument in SDE theory.

**Lemma B.3.** For any \( p \geq 2 \), there exists some constant \( C_p \) dependent only on \( C, T, k, p \) such that for any \( t, \bar{t} \in [0, T] \), \( \gamma_t \in D([0, t], \mathbb{R}) \), \( \bar{\gamma}_t \in D([0, \bar{t}], \mathbb{R}) \),

\[
E \left[ \sup_{s \in [0,T]} \left| X^{\gamma_t}(s) - X^{\bar{\gamma}_t}(s) \right|^p \right] \leq C_p \left( 1 + \|\gamma_t\|_p^p + \|\bar{\gamma}_t\|_p^p \right) \left( \|\gamma_t - \bar{\gamma}_t\|_{L^\infty}^p + |t - \bar{t}|^p \right).
\]

Now, we prove Lemma [B.2]. In this proof, we use the Fréchet derivative of \( X^{\gamma_t} \) introduced in the propositions in Section 3. More specifically, let \( Dg \) be the Frechet derivative of \( X^{\gamma_t} \). From Remark [A.1] we obtain the differential of \( g(X^{\gamma_t}) \),

\[
\frac{\partial g(X^{\gamma_t})}{\partial x} = Dg(X^{\gamma_t})(DX^{\bar{\gamma}_t}).
\]

**Proof of Lemma [B.3]** The proof follows that of Theorem 3.7 in Peng and Wang [2016]. Without loss of generality, we assume \( t \leq \bar{t} \). From Theorem [B.1] Assumption [4.1] and Lemma [B.3] we obtain the inequalities

\[
E \left[ \sup_{u \in [t,T]} \left| Y^{\gamma_t}(u) - Y^{\bar{\gamma}_t}(u) \right|^p \right] \leq CE \left[ \left( 1 + \|X^{\gamma_t}\| + \|X^{\bar{\gamma}_t}\| \right)^{2kp} \right] \frac{1}{2} E \left[ \left\| X^{\gamma_t} - X^{\bar{\gamma}_t} \right\|^{2p} \right]^{\frac{1}{2}} \leq C \left( \|\gamma_t - \bar{\gamma}_t\|_{L^\infty} + |t - \bar{t}|^p \right) \left( 1 + \|\gamma_t\| + \|\bar{\gamma}_t\| \right)^{kp}.
\]

The proof of \((ii)\) in the lemma is same as in Theorem [B.1] regarding the control of \( Z \).

To show \((iii)\), let us consider \((\Delta_h Y^{\gamma_t} - \Delta_h Y^{\bar{\gamma}_t}, \Delta_h Z^{\gamma_t} - \Delta_h Z^{\bar{\gamma}_t})\) as a solution to the BSDE

\[
Y(r) = \frac{g(X^{\gamma_t}_r)}{h} - \frac{g(X^{\bar{\gamma}_t}_r)}{h} - \int_r^T Z(r) \, dW_r, \quad \bar{t} \leq r \leq T.
\]

Applying [B.1], we estimate \((iii)\) in the lemma as above. We obtain that

\[
E \left[ \sup_{s \in [\bar{t},T]} \left| \Delta_h Y^{\gamma_t}(s) - \Delta_h Y^{\bar{\gamma}_t}(s) \right|^p \right] \leq CE \left[ \left\| \frac{g(X^{\gamma_t}_r)}{h} - \frac{g(X^{\gamma_t}_r)}{h} \right\| + \left\| \frac{g(X^{\bar{\gamma}_t}_r)}{h} - \frac{g(X^{\bar{\gamma}_t}_r)}{h} \right\| \right]
\]

\[
= CE \left[ \int_0^1 Dg(X^{\gamma_t}_r)(DX^{\bar{\gamma}_t}_r) - Dg(X^{\bar{\gamma}_t}_r)(DX^{\bar{\gamma}_t}_r) \, d\theta \right].
\]

We use the fundamental theorem of calculus to estimate the second equality. These terms can be estimated using Lipschitz continuity of \( Dg \).

The proof of the last term is similar to the proof of Theorem [B.1].

The next step is to apply Kolmogorov’s continuity theorem. Note that this theorem is the version of Lemma 3.8 in Peng and Wang [2016].

**Theorem B.4.** For each \( \gamma_t \in D([0,t], \mathbb{R}) \), \( \{Y^{\gamma_t}(s) : s \in [0,T], x \in \mathbb{R} \} \) has a version which is a.e. in \( C^{0,2}([0,T] \times \mathbb{R}) \).

**Proof.** Fix \( k, \bar{k} \in \mathbb{R} \). From the second inequality in Lemma [B.2] we obtain that \( (Y^{\gamma_t}, Z^{\gamma_t}) \) is continuous with respect to \( x \) in \( S^2 \) and \( M^2 \) from Kolmogorov’s continuity theorem. Let \((D_x Y^{\gamma_t}, D_x Z^{\gamma_t})\) be a solution to the BSDE

\[
D_x Y^{\gamma_t}(s) = Dg(X^{\gamma_t})(DX^{\gamma_t}) - \int_s^T D_x Z^{\gamma_t}(u) \, du.
\]
As in the proof of Lemma \[\text{B.2}\] we have
\[
\mathbb{E}\left[\frac{g(X_t^h) - g(X_t^0)}{h} - Dg(X_t^0)(DX_t^0)|^p\right] \leq C(1 + \|\gamma_t^h\|^q)\|\gamma_t^h - \gamma_t\|^p.
\]

It also means that \(\Delta_h Y^\tau\) converges to \(D\gamma Y^\tau\) as \(h \to 0\) in \(S^p\) space. Using the \((ii)\) in Lemma \[\text{B.2}\] it is easy to show the continuity of \(D_x Y^\tau\) with respect to \(x\).

Similarly, let \((D_{xx} Y^\tau, D_{xx} Z^\tau)\) be a solution to the BSDE

\[
D_{xx} Y^\tau(s) = D^2 g(X^\tau, DX^\tau)(DX^\tau) + Dg(X^\tau)(D^2 X^\tau) - \int_s^T D_{xx} Z_{\gamma}^\tau dw_u,
\]

where \(D^2 X^\tau = \frac{\partial^2 DX^\tau}{\partial h^2}|_{h=0}\). The same proof works for \(D_{xx} Y^\tau\).

\[\square\]

### B.2 Time derivative

In this subsection, we prove the horizontal differentiability of \(u\) in \[\text{[4.4]}\]. In addition, the operator \(u\) is the solution to the path-dependent PDE. The entire idea follows the flow of Theorems 3.10 and 4.5 in Peng and Wang (2016). Recall that the operator \(u : \tilde{D} \to \mathbb{R}\) is defined as \(u_t(\gamma_t) = Y^\tau(t)\), where \(\tilde{D} = \bigcup_{t \in [0,T]} D([0,t], \mathbb{R})\).

**Theorem B.5.** The operator \(u\) in \[\text{[4.4]}\] has the horizontal derivative \(D_t u : \tilde{D} \to \mathbb{R}\) and \(u\) satisfies a path-dependent PDE:

\[
\begin{align*}
D_t u_t(\gamma_t) + b_t(\gamma_t) D_x u_t(\gamma_t) + \frac{1}{2} \sigma_t^2(\gamma_t) D_{xx} u_t(\gamma_t) &= 0, \quad \text{for} \quad \gamma_t \in D([0,t], \mathbb{R}), \\
 u_T(\gamma_T) &= g(\gamma_T), \quad \text{for} \quad \gamma_T \in D([0,T], \mathbb{R}).
\end{align*}
\]

(B.3)

We denote a differential operator \(L u = b D_x u + \frac{1}{2} \sigma^2 D_{xx} u\). First, we give the main ideas of the proof. For the initial path \(\gamma_t \in D([0,t], \mathbb{R})\), define the horizontally extended path \(\gamma_t,t+\delta \in D([0,t+\delta], \mathbb{R})\) as \(\gamma_{t+\delta}(s) = \gamma_t(s) 1_{[0,t]} + \gamma_t(t) 1_{(t,t+\delta]}\) for small \(\delta > 0\). The main idea is to prove each of the following equalities

\[
\begin{align*}
 u_{t+\delta}(\gamma_{t,t+\delta}) - u_t(\gamma_t) &= \lim_{n \to \infty} u^{(n)}(\gamma_{t,t+\delta}) - u_t^{(n)}(\gamma_t) \\
 &= \lim_{n \to \infty} \int_t^{t+\delta} \partial_t \tilde{u}^{(n)}(s, \gamma_t(t_0), \cdots, \gamma_t(t_j)) ds \\
 &= \lim_{n \to \infty} \int_t^{t+\delta} -\tilde{L}^{(n)}(s, \gamma_t(t_0), \cdots, \gamma_t(t_j)) \tilde{u}^{(n)}(s, \gamma_t(t_0), \cdots, \gamma_t(t_j)) ds \\
 &= -\int_t^{t+\delta} L(\gamma_{t,s}) u_s(\gamma_{t,v}) ds,
\end{align*}
\]

for all small \(\delta > 0\) and some suitable \(u^{(n)}, \tilde{u}^{(n)}\) and \(\tilde{L}^{(n)}\). After that, we obtain that

\[
D_t u_t(\gamma_t) := \lim_{\delta \to 0} \frac{u_{t+\delta}(\gamma_{t,t+\delta}) - u_t(\gamma_t)}{\delta} = -Lu_t(\gamma_t).
\]

(B.5)

This means that \(u\) has a horizontal derivative, and also that \(u\) is a solution to the path-dependent PDE

\[
D_t u_t(\gamma_t) + Lu_t(\gamma_t) = 0, \quad \text{for} \quad \gamma_t \in D([0,t], \mathbb{R}).
\]

First, we introduce \(X^{n,\tau}\) and \((Y^{n,\tau}, Z^{n,\tau})\) by approximating the coefficient function \(b, \sigma\) and define the operator \(u^{(n)}\) in \[\text{[B.4]}\]. The approximation may be considered as the detailed version in Proposition \[\text{3.1}\].
Fix the time $t \in [0, T]$ and initial path $\gamma_t \in D([0, t], \mathbb{R})$. Let $n \in \mathbb{N}$ be a number sufficiently larger than the number of discontinuity points of $\gamma_t$. Then, we can define a sequence of partitions $\pi_n = \{0 = t_0 \leq t_1 \leq \cdots \leq t_n = T\}$ tending to the identity (see the paragraph after Lemma A.2), which covers all discontinuity points of $\gamma_t$ and the time $t$. We also define linear operators $\varphi^n_L, \varphi^n_R : \tilde{D} \to \tilde{D}$ as for $s \in [0, T]$:

$$
(\varphi^n_L)_v(\eta) = \sum_{j=0}^{n-1} \eta(t_j \wedge s) \mathbb{1}_{[t_j \wedge s, t_{j+1} \wedge s)}(v) + \eta(T \wedge s) \mathbb{1}_{\{T \wedge s\}}(v),
$$

$$
(\varphi^n_R)_v(\eta) = \sum_{j=0}^{n-1} \eta(t_{j+1} \wedge s) \mathbb{1}_{[t_j \wedge s, t_{j+1} \wedge s)}(v) + \eta(T \wedge s) \mathbb{1}_{\{T \wedge s\}}(v),
$$

for $\eta \in D([0, s], \mathbb{R})$ and $v \in [0, s]$ where $\tilde{D} = \bigcup_{t \in [0, T]} D([0, t], \mathbb{R})$ defined in Section 2.1. The operators $\varphi^n_L$ and $\varphi^n_R$ convert the path to a piecewise constant path. They help us to approximate a non-anticipative functional by some good function.

For all sufficiently large $n \in \mathbb{N}$ and a path $\tilde{\gamma}_t \in D([0, \tilde{t}], \mathbb{R})$, we define the process $X^{n, \tilde{\gamma}_t}$ as a solution to the SDE

$$
X^{n, \tilde{\gamma}_t}(s) = \tilde{\gamma}_t(s) + \int_t^s (b \circ \varphi^n_L)_r(X^{n, \tilde{\gamma}_t}) \, dr + \int_t^s (\sigma \circ \varphi^n_L)_r(X^{n, \tilde{\gamma}_t}) \, dW_r, \quad s \in [\tilde{t}, T],
$$

$$
X^{n, \tilde{\gamma}_t}(s) = \tilde{\gamma}_t(s), \quad s \in [0, \tilde{t}].
$$

Similarly, we define $(Y^{n, \tilde{\gamma}_t}, Z^{n, \tilde{\gamma}_t})$ as a solution to the BSDE

$$
Y^{n, \tilde{\gamma}_t}(s) = g \circ \varphi^n_R(X^{n, \tilde{\gamma}_t}) - \int_s^T Z^{n, \tilde{\gamma}_t}(r) \, dW_r.
$$

Then, we can consider the derivative $DX^{n, \tilde{\gamma}_t}$ and $D_x Y^{n, \tilde{\gamma}_t}$ as in (3.1) and (3.2). We set $u^{(n)}(\tilde{\gamma}_t) = Y^{n, \tilde{\gamma}_t}(\tilde{t})$ and $D_x u^{(n)}(\tilde{\gamma}_t) = D_x Y^{n, \tilde{\gamma}_t}(\tilde{t})$. Then, $u^{(n)}, D_x u^{(n)} : \tilde{D} \to \mathbb{R}$ is the approximation of $u, D_x u$.

**Lemma B.6.** Fix $\tilde{\gamma}_t \in D([0, \tilde{t}], \mathbb{R})$. If the discontinuity points of $\tilde{\gamma}_t$ are included in those of $\gamma_t$, we have that $u^{(n)}(\tilde{\gamma}_t)$ and $D_x u^{(n)}(\tilde{\gamma}_t)$ converge to $u(\tilde{\gamma}_t)$ and $D_x (\tilde{\gamma}_t)$ as the mesh $n \to \infty$.

**Proof.** First, we need to show $X^{n, \gamma_t}$ and $DX^{n, \gamma_t}$ converge to $X^{\gamma_t}$ and $DX^{\gamma_t}$. From the simple extension of Theorem 3.2.2 and Theorem 3.2.4 in [Zhang (2017)] to the case of path-dependent, it is enough to show

$$
\mathbb{E}\left[\left\| \int_t^T F^{(1)}(X^{\gamma_t})(DX^{\gamma_t}) - F^{(2)}(X^{\gamma_t})(DX^{\gamma_t}) \, dr \right\|^p \right] \to 0,
$$

where $F^{(1)} = b, \sigma, Db, D\sigma$ and $F^{(2)} = b \circ \varphi^n_L, \sigma \circ \varphi^n_L, D[b \circ \varphi^n_L], D[\sigma \circ \varphi^n_L]$.

Since $\varphi^n_L$ is a linear operator and $\|\varphi^n_L(\eta)\| = \|\varphi^n_L(\eta)\| \leq 1$ from the definition, we obtain the Fréchet derivative of $F \circ \varphi$ as $D[F \circ \varphi^n_L](\eta)(\delta) = DF(\varphi^n_L(\eta))(\varphi^n_L(\delta))$. Using Lipschitz continuity, linear growth and the dominant convergence theorem, we have $X^{n, \gamma_t}, DX^{n, \gamma_t} \to X^{\gamma_t}, DX^{\gamma_t}$ in $S^p$ space.

Since $u^{(n)}(\tilde{\gamma}_t), D_x u^{(n)}(\tilde{\gamma}_t), u_\tilde{t}(\tilde{\gamma}_t), D_x u_\tilde{t}(\tilde{\gamma}_t)$ are all solution to some BSDE, we can use the argument in Lemma B.1. Thus, showing that

$$
\mathbb{E}[\|D[g \circ \varphi^n_R(X^{n, \gamma_t})(DX^{n, \gamma_t}) - Dg(X^{\gamma_t})(DX^{\gamma_t})]^p\|, \mathbb{E}[\|g(X^{\gamma_t}) - g \circ \varphi^n_R(X^{n, \gamma_t})]^p\|]
$$

converges to 0 as $n \to \infty$ completes the proof. Note that this is easy to check from Lipschitz continuity and linear growth of $g, Dg$. $\square$
The above Lemma [B.6] can be extended to the case of $D_{xx}u^{(n)}(\tilde{\gamma}_t) \rightarrow D_{xx}u(\tilde{\gamma}_t)$. The next step is to find the family of functions $(u_k)_{k=0,1,\ldots,n}$ satisfying

$$Y^{n,\tilde{\gamma}_t}(\tilde{t}) = u^{(n)}(\tilde{\gamma}_t) = u_k(\tilde{t}, \tilde{\gamma}_t(t_0), \ldots, \tilde{\gamma}_t(t_k), \tilde{\gamma}_t(\tilde{t}))$$

for $\tilde{\gamma}_t \in D([0, \tilde{t}], \mathbb{R})$ with $\tilde{t} \in [t_k, t_{k+1}]$. In this process, we determine that $u_k$ is actually a solution to some BSDE and is a smooth solution to some PDE with a differential operator. Later, we approximate the differential operator to what we desire.

To do this, we need to regard $X^{n,\tilde{\gamma}_t}$ as a solution to a function coefficient SDE. First, we fix $s \in [t_k, t_{k+1})$ for some $k = 0, \ldots, n-1$. For each path $\eta \in D([0, T], \mathbb{R})$, the value $b_s(\varphi^n_L(\eta))$ depends on only the k points of a path $\eta$, since the operator $b$ is non-antipredictive functional. Thus, we can define $b_k^{(n)} : [t_k, t_{k+1}) \times \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ for each $k = 0, \ldots, n-1$,

$$b_k^{(n)}(s, \eta(t_0), \eta(t_1), \ldots, \eta(t_k)) = (b \circ \varphi^n_L)_s(\eta) \quad (B.6)$$

Similarly, we define $\sigma_k^{(n)}$ for each $k = 0, \ldots, n-1$. Then, the dynamics of $X^{n,\tilde{\gamma}_t}$ is represented using the notation $b_k^{(n)}$, $\sigma_k^{(n)}$ on each time interval $s \in [t_k, t_{k+1})$ by

$$X^{n,\tilde{\gamma}_t}(s) = X^{n,\tilde{\gamma}_t}(t_k) + \int_{t_k}^{s} b_k^{(n)}(r, X^{n,\tilde{\gamma}_t}(t_0), \ldots, X^{n,\tilde{\gamma}_t}(t_k)) \, dr + \int_{t_k}^{s} \sigma_k^{(n)}(r, X^{n,\tilde{\gamma}_t}(t_0), \ldots, X^{n,\tilde{\gamma}_t}(t_k)) \, dW_r. \quad (B.7)$$

In the same way, we define $g^{(n)}$ by the function representation of $g \circ \varphi^n_R$ as

$$g^{(n)}(\eta(t_0), \ldots, \eta(t_n)) = g \circ \varphi^n_R(\eta).$$

Then, the dynamics of $(Y^{n,\tilde{\gamma}_t}, Z^{n,\tilde{\gamma}_t})$ satisfies the following for each time interval $s \in [t_k, t_{k+1})$:

$$Y^{n,\tilde{\gamma}_t}(s) = Y^{n,\tilde{\gamma}_t}(t_k) - \int_{t_k}^{s} Z^{n,\tilde{\gamma}_t}(r) \, dW_r, \quad \text{for} \quad s \in [t_k, t_{k+1}), \quad (B.8)$$

Finally, we define the differential operator $L_k^{(n)} = b_k^{(n)} \frac{\partial}{\partial x_k} + \frac{1}{2} \sigma_k^{(n)} \frac{\partial^2}{\partial x_k \partial x_k}$, where $x_k$ is the last coordinate for each $k = 0, \ldots, n-1$.

Now, we find a PDE from the last interval which the solution to BSDE $Y^{n,\tilde{\gamma}_t}(\tilde{t})$ satisfies. Let consider for $[t_{n-1}, t_n]$ in (B.7) and (B.8). If $X^{n,\tilde{\gamma}_t}(t_0), \ldots, X^{n,\tilde{\gamma}_t}(t_{n-1})$ are known, we can regard (B.7) and (B.8) as standard SDE and BSDE with function coefficients. Thus, we can apply the result in [Pardoux and Peng (1992)]. For parameters $x_0, \ldots, x_{n-1} \in \mathbb{R}$, let $u_{n-1} : [t_{n-1}, t_n] \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be a solution to the PDE

$$\partial_t u_{n-1}(s, x_0, \ldots, x_{n-1}, y) + L_{n-1}^{(n)}(s, x_0, \ldots, x_{n-1}) u_{n-1}(s, x_0, \ldots, x_{n-1}, y) = 0,$$

$$u_{n-1}(T, x_0, \ldots, x_{n-1}, y) = g^{(n)}(x_0, \ldots, x_{n-1}, y),$$

for $s \in [t_{n-1}, t_n]$ and $y \in \mathbb{R}$. The parameter determine coefficients of PDE and the solution at $(t_{n-1}, y)$ means that the initial point of SDE is $y$ at time $t_{n-1}$. Thus, we have from Theorem 3.2 in [Pardoux and Peng (1992)]

$$u_{n-1}(t_{n-1}, X^{n,\tilde{\gamma}_t}(t_0), \ldots, X^{n,\tilde{\gamma}_t}(t_{n-1}), X^{n,\tilde{\gamma}_t}(t_{n-1})) = Y^{n,\tilde{\gamma}_t}(t_{n-1}).$$

Note that this term becomes a terminal condition for the next interval $[t_{n-2}, t_{n-1}]$. Moreover, for a path $\tilde{\gamma}_t$ with $\tilde{t} \in [t_{n-1}, t_n]$, the following holds,

$$Y^{n,\tilde{\gamma}_t}(\tilde{t}) = u_{n-1}(\tilde{t}, X^{n,\tilde{\gamma}_t}(t_0), \ldots, X^{n,\tilde{\gamma}_t}(t_{n-1}), X^{n,\tilde{\gamma}_t}(\tilde{t})) = u_{n-1}(\tilde{t}, \tilde{\gamma}_t(t_0), \ldots, \tilde{\gamma}_t(t_{n-1}), \tilde{\gamma}_t(\tilde{t})). \quad (B.9)$$
Now, we expand the above argument to the entire interval. For $k = 0, \ldots, n - 2$, we inductively define $u_k(s, x_0, \ldots, x_k, y) : [t_k, t_{k+1}] \times \mathbb{R}^{k+1} \times \mathbb{R}$ by a solution to the following PDE with parameters $x_0, \ldots, x_k \in \mathbb{R}$:

$$\partial_t u_k(s, x_0, \ldots, x_k, y) + L_k^{(n)}(s, x_0, \ldots, x_k)u_k(t, x_0, \ldots, x_k, y) = 0,$$

$$u_k(t_{k+1}, x_0, \ldots, x_{k-1}, y) = u_{k+1}(t_{k+1}, x_0, \ldots, x_{k-1}, y).$$

(B.10)

For a path $\tilde{\gamma}_t$ with $t \in [t_k, t_{k+1}]$, we have an equation as in equation (B.9),

$$u^{(n)}(\tilde{\gamma}_t) = Y^{n,\tilde{\gamma}_t}(t) = u_k(\tilde{\gamma}_t(t_0), \ldots, \tilde{\gamma}_t(t_k), \tilde{\gamma}_t(t)).$$

Under the above setting, we define a function $\tilde{u}^{(n)}$ and a differential operator $\tilde{L}^{(n)}$ that are used in (B.11). Because the sequence of partitions $\pi^{(n)}$ covers the time $t$, there exists $j \in \{0, 1, \ldots, n - 1\}$ such that $t = t_j$. We define a function $\tilde{u}^{(n)} : [t, t + \delta] \times \mathbb{R}^{j+1} \rightarrow \mathbb{R}$ for small $\delta > 0$ and a differential operator as

$$\tilde{u}^{(n)}(v, \eta(t_0), \ldots, \eta(t_j)) = \sum_{i=j}^{n-1} u_i(v, \eta(t_0), \ldots, \eta(t_j), \eta(t), \ldots, \eta(t)) \mathbb{1}_{[t_i, t_{i+1}]}(v)$$

$$\tilde{L}^{(n)}(v, \eta(t_0), \ldots, \eta(t_j)) = \sum_{i=j}^{n-1} L_i^{(n)}(v, \eta(t_0), \ldots, \eta(t_j), \eta(t), \ldots, \eta(t)) \mathbb{1}_{[t_i, t_{i+1}]}(v).$$

(B.11)

Note that the function $\tilde{u}^{(n)}$ is differentiable with respect to the time except at points in the partition $\sigma$. This definition is used to represent $u^{(n)}(\gamma_{t,t+\delta}) - u^{(n)}(\gamma_t)$ as a single term.

**Lemma B.7.** With a defined $\tilde{u}^{(n)}$ and a differential operator $\tilde{L}^{(n)}$ in (B.11), the following holds:

$$u^{(n)}(\gamma_{t,t+\delta}) - u^{(n)}(\gamma_t) = \int_t^{t+\delta} \partial_t \tilde{u}^{(n)}(v, \gamma_t(t_0), \ldots, \gamma_t(t_j)) dv$$

The integral of $\partial_t \tilde{u}^{(n)}$ is defined except at points in the partition.

**Proof.** We use the differentiability of $u_k$ with respect to $y$ in (B.11) to obtain $u_k(s, x_0, \ldots, x_k, y) - u_k(t_k, x_0, \ldots, x_k, y) = \int_k^s \partial_t u_k(v, x_0, \ldots, x_k, y) dv$ for each $k = 0, \ldots, n - 1$ and $s \in [t_k, t_{k+1}]$. Suppose $t_j = t$ and $t + \delta \in [t_k, t_{k+1})$ for some $j, k$. Since the function $u_i$ for $j \leq i \leq k-1$ satisfies the terminal condition of PDE (B.10), we have

$$u^{(n)}(\gamma_{t,t+\delta}) - u^{(n)}(\gamma_t) = u_k(t + \delta, \gamma_t(t_0), \ldots, \gamma_t(t_j), \gamma_t(t), \ldots, \gamma_t(t_k)) - u_k(t, \gamma_t(t_0), \ldots, \gamma_t(t_j), \gamma_t(t), \ldots, \gamma_t(t_k))$$

$$= \int_{t_k}^{t+\delta} \partial_t u_k(v, \gamma_t(t_0), \ldots, \gamma_t(t_j), \gamma_t(t), \ldots, \gamma_t(t_k)) dv$$

$$+ \sum_{i=j}^{k-1} \int_{t_j}^{t_{i+1}} \partial_t u_i(v, \gamma_t(t_0), \ldots, \gamma_t(t_j), \gamma_t(t), \ldots, \gamma_t(t_k)) dv$$

$$= \int_{t_j}^{t+\delta} \partial_t \tilde{u}^{(n)}(v, \gamma_t(t_0), \ldots, \gamma_t(t_j)) dv$$

The last equation is what we want. \(\square\)

Now, we can proceed to the last equality in (B.4).
Lemma B.8. Let \( \tilde{f}^{(n)} \), \( \tilde{u}^{(n)} \) be as defined in the previous lemma. Suppose that all partition cover the discontinuous points of \( \gamma_t \) and \( t \). For fixed \( \delta > 0 \), we have

\[
\lim_{n \to \infty} \int_{t}^{t+\delta} \tilde{f}^{(n)}(v, \gamma_t(t_0), \ldots, \gamma_t(t_{j(n)})) \tilde{u}^{(n)}(v, \gamma_t(t_0), \ldots, \gamma_t(t_{j(n)})) \, dv = -\int_{t}^{t+\delta} L(\gamma_{t,v}) u_v(\gamma_{t,v}) \, dv,
\]

where \( t_{j(n)} \in \pi^{(n)} \) satisfy \( t_{j(n)} = t \) for each \( n \in \mathbb{N} \).

**Proof.** Let \( t_k(n) \in \pi^{(n)} \) satisfy \( v \in [t_k(n), t_{k(n)+1}) \). From the definition of \( \gamma_{t,v} \), we have

\[
u_{k(n)}(v, \gamma_{t}(t_0), \ldots, \gamma_{t}(t_{j(n)}), \gamma_{t}(t), \ldots, \gamma_{t}(t)) = \nu_{k(n)}(v, \gamma_{t,v}(t_0), \ldots, \gamma_{t,v}(t_k(n)), \gamma_{t,v}(v)) = Y^{n,\gamma_{t,v}}(v) = \nu^{(n)}(\gamma_{t,v}).
\]

Then, we can understand the partial derivatives as a vertical derivative of a solution \( Y^{n,\gamma_{t,v}}(v) \), since it holds that

\[
\frac{\partial \nu_{k(n)}(v, \gamma_{t}(t_0), \ldots, \gamma_{t}(t_{j(n)}), \gamma_{t}(t), \ldots, \gamma_{t}(t))}{\partial x_{k(n)}} = \lim_{h \to 0} \frac{\nu^{(n)}(\gamma_{t,v}^h) - \nu^{(n)}(\gamma_{t,v})}{h} = D_x \nu^{(n)}(\gamma_{t,v}).
\]

Using (B.11) and (B.6) we have

\[
L(\gamma_{t,v}) u_v(\gamma_{t,v}) = -b_v(\varphi^2_v(\gamma_{t,v})) D_x u^{(n)}(\gamma_{t,v}) - \frac{1}{2}(\sigma_v)^2(\varphi^2_v(\gamma_{t,v})) D_{xx} u^{(n)}(\gamma_{t,v})
\]

for \( v \in [t, t+\delta] \). Note that the \( b_v(\varphi^2_v(\gamma_{t,v})) \) converges to \( b_v(\gamma_{t,v}) \) in a sup-norm as \( n \to \infty \). The case of \( \sigma \) is same. From Lemma B.6 and linear growth of \( b \) and \( \sigma \), the dominating convergence theorem gives what we want. \( \square \)

Now, we are ready to prove Theorem B.5.

**Proof of Theorem B.5.** To conclude (B.5) using the fundamental theorem of calculus from equations (B.4), we only need to show that \( L u_s(\gamma_{t,s}) \) is continuous with respect to \( s > t \). Since the non-anticipative functionals \( b, \sigma \) are functional Lipschitz continuous, it is sufficient to show the continuity of \( D_x u_s(\gamma_{t,s}) \) and \( D_{xx} u_s(\gamma_{t,s}) \). We give the proof only in the case in which \( D_x u_s(\gamma_{t,s}) \); the other cases are similar. For \( s > t \), we have

\[
|D_x u_s(\gamma_{t,s}) - D_x u_t(\gamma_t)| \leq |D_x Y^{\gamma_{t,s}}(s) - D_x Y^{\gamma_{t}}(s)| + |D_x Y^{\gamma_{t,s}}(s) - D_x Y^{\gamma_{t}}(t)|.
\]

Because \( D_x Y^{\gamma_{t}} \) is the solution to BSDE (B.2), and \( D_x Y^{\gamma_{t}}(v) \) is continuous for \( v \geq t \) almost everywhere. Thus, the second term goes to 0 as \( s \to t^+ \). The first term can be estimated as in the argument in (iii) Lemma B.2 from the functional Lipschitz continuity of \( Dg \). \( \square \)

We have proven the existence of the vertical and horizontal derivatives of non-anticipative functional \( u \). Now, we describe the proof of Theorem 4.1.

**Proof of Theorem 4.1.** From the argument in Section 2 recall that we may regard the non-anticipative function \( F \) as an operator \( \tilde{F} \) from \( \tilde{D} \) to \( \mathbb{R} \). Conversely, the operator \( \tilde{F} \) from \( \mathbb{R} \) to \( \mathbb{R} \) can be regard as a non-anticipative function by \( F(t, \gamma) = \tilde{F}(\gamma^t) \). Therefore, we can regard \( u \) defined in (4.4) as a non-anticipative functional.

The existences of \( D_x u, D_{xx} u, D_t u \) are derived from Theorems B.3 and B.5. We now turn to the converse direction. The operator \( u : \tilde{D} \to \mathbb{R} \) corresponds to the non-anticipative functional \( u : [0, T] \times D([0, T], \mathbb{R}) \to \mathbb{R} \) as \( \tilde{u}(t, \gamma) = u(\gamma) \) for \( \gamma \in D([0, t], \mathbb{R}) \). From the functional Itô
Applying the functional Itô formula Theorem 2.1 to Proof of Proposition 4.3.

Using the above argument, we prove the sensitivity formula of Theorem 4.4. We also assume that families of coefficient \( b \) and \( \sigma \) satisfy Assumption 4.2. Finally, we prove Proposition 4.2, which states the self-financing portfolio of options.

Proof of Proposition 4.2 We apply the Itô formula to \( v(X_t) \). Since \( u_t(X_t) \) is a solution to PPDE (4.10), we have

\[
du_t(X_t) = D_x u_t(X_t) \sigma_t(x) \, dW_t = -D_x u_t(X_t)(r - q_t(X)) \, dt + \frac{D_x u_t(X_t)}{S_t} \, dS_t
\]

using the definition of \( S_t \). Thus, we obtain that

\[
dv_t(X_t) = rv_t(X_t) - D_x u_t(X_t)(r - q_t(X)) e^{-r(T-t)} + \frac{D_x u_t(X_t)}{S_t} e^{-r(T-t)} \, dS_t.
\]

This means that \( v_t(X_t) \) is a self-financing portfolio with initial value \( u_0(X_0) e^{-rT} \) with position \( \frac{D_x u_0(X_0)}{S_0} e^{-r(T-t)} \). Finally, using the martingale property of \( u_t(X_t) \), we obtain \( \mathbb{E}[u_T(X_T)] = u_0(X_0) \).

C Greeks

In this section, we prove the propositions and theorem in Section 4.2. We follow the setting in Section 4.2. Under the risk-neutral setting, we define the stock model \( S_t \) and logarithm \( X_t \) described in Section 4.1. We consider families of coefficients \( (r^\epsilon)_{\epsilon \in I}, (q^\epsilon)_{\epsilon \in I} \) and \( (\sigma^\epsilon)_{\epsilon \in I} \) where \( I = (-1, 1) \) and \( \epsilon \) is the perturbation parameter. For convenience, we define \( b^\epsilon_t := r^\epsilon - q^\epsilon_t - \frac{1}{2}(\sigma^\epsilon_t)^2 \).

First, we mention that \( X_t \) and \( X^\epsilon \) are actually close in \( S^p \) space.

Lemma C.1. For \( \epsilon \) sufficiently close to 0 and \( p \geq 2 \), we have

\[
\mathbb{E}
\left[
\sup_{u \in [0,T]} |X(u) - X^\epsilon(u)|^p
\right] \leq C \epsilon^p.
\]

The proof is the same as in Theorem B.3 of Fournie (2010). Now, let us prove Proposition 4.2.

Proof of Proposition 4.3 Applying the functional Itô formula Theorem 2.1 to \( u_t(X^\epsilon_t) \), we obtain

\[
du_t(X^\epsilon_t) = (D_x u_t(X^\epsilon_t)(b^\epsilon_t(X^\epsilon_t) - b_t(X^\epsilon_t))) + \frac{1}{2}((\sigma^\epsilon_t)^2(X^\epsilon_t) - \sigma_t^2(X^\epsilon_t)) D_x u_t(X^\epsilon_t) \, dt + D_x u_t(X^\epsilon_t) \sigma_t^\epsilon(X^\epsilon_t) \, dW_t.
\]

We use the fact that \( u \) is the solution to PPDE (4.5) in the third equality. Therefore, we obtain

\[
dG^\epsilon(t) = D_x u_t(X^\epsilon_t) \sigma_t^\epsilon(X^\epsilon_t) \, dW_t = -(r^\epsilon - q^\epsilon_t(X^\epsilon_t)) D_x u_t(X^\epsilon_t) \, dt + \frac{D_x u_t(X^\epsilon_t)}{S_t^\epsilon} \, dS_t^\epsilon.
\]

Therefore, as in the proof of Proposition 4.2 we obtain what we want.

Now, for each \( \epsilon > 0 \), define \( A_t = G^\epsilon(t) - u_t(X^\epsilon_t) \) Then, for each \( \epsilon > 0 \), we get equalities from the martingale property of \( u_t(X_t) \) and \( G^\epsilon \)

\[
v^\epsilon_0(X_0) - v_0(X_0) = \mathbb{E}[u_T(X^\epsilon_T) e^{-rT} - u_T(X_T) e^{-rT}]
\]

\[
= \mathbb{E}[G^\epsilon(T) - A_T] e^{-rT} - u_0(X_0) e^{-rT} = -\mathbb{E}[A_T] e^{-rT} + u_0(X_0)(e^{-rT} - e^{-rT}).
\]

Using the above argument, we prove the sensitivity formula of Theorem 4.4.
Proof of Theorem 4.4. We have shown that
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} (v_0(X_0) - v_0(X_0)) = \lim_{\epsilon \to 0} \left( u_0(X_0) \frac{e^{-rT} - e^{-rT}}{\epsilon} - \mathbb{E}[\frac{\epsilon}{A_T} e^{-rT}] \right).
\]
Note that for \( \psi = b, \sigma, q \), we have \( \psi^\epsilon(\eta) \to \psi(\eta) \) and \( \frac{\psi^\epsilon(\eta) - \psi(\eta)}{\epsilon} \to \dot{\psi}(\eta) \) as \( \epsilon \to 0 \) for each \( \eta \in D([0, T], \mathbb{R}) \). Then, \( \dot{r} \) comes from the formula of \( r \) and \( r^\epsilon \) as \( r = b t + \frac{1}{2} \sigma^2 t + q t \), \( r^\epsilon = b^\epsilon_t + \frac{1}{2} (\sigma^\epsilon_t)^2 + q^\epsilon t \).
Thus, we obtain that \( \dot{r} = b_\epsilon + \sigma_\epsilon r + q_\epsilon \).

For the term \( \mathbb{E}[A_T/\epsilon] \), we sequentially use the triangle inequality and apply the Lebesgue dominating convergence theorem to each term. In the process, we use Theorem B.1, Lemma C.1 and the Lipschitz continuity of \( b \) and \( \sigma \).

The fractional part in \( \mathbb{E}[A_T/\epsilon] \) can be regarded as differential by showing \( \mathbb{E}[\int_0^T |D_x u_s(X_s^\epsilon) (b_s^\epsilon(x^\epsilon) - b_s(x^\epsilon))|/\epsilon - \dot{b}_s(x^\epsilon)| ds \) goes to 0. Then, we can omit \( \epsilon \) of \( X \) as \( \mathbb{E}[\int_0^T |D_x u_s(X_s^\epsilon) (b_s^\epsilon(x^\epsilon) - \dot{b}_s(x)| ds \) and \( \mathbb{E}[\int_0^T |(D_x u_s(X_s^\epsilon) - D_x u_s(x_s)) \dot{b}_s(x)| ds \) converges to 0. Consequently, we obtain
\[
\mathbb{E}\left[ \int_0^T |D_x u_s(X_s^\epsilon) \left( \frac{b_s^\epsilon(x^\epsilon) - b_s(x^\epsilon)}{\epsilon} - D_x u_s(x_s) \dot{b}_s(x) \right) | ds \right] \to 0, \quad \text{as} \quad \epsilon \to 0.
\]
Similarly, we can estimate the remaining term of \( \sigma \). Therefore, we conclude that
\[
\lim_{\epsilon \to 0} \mathbb{E}\left[ \frac{A_T}{\epsilon} \right] e^{-rT} = \mathbb{E} \left[ \int_0^T D_x u_s(x_s) \dot{b}_s(x) + D_{xx} u_s(x_s) \sigma_\epsilon(x) \sigma_\epsilon(x) ds \right] e^{-rT}.
\]

\[\square\]

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