Three Prior Selection Estimate Reliability Function of Exponential Distribution

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Abstract. In this paper, we studied the function of reliability for exponential distribution using traditional methods of estimation (Mle, Jackknife). As well as studying Bayes’ methods as only prior and Double priors with a new suggestion used three Prior by using distributions Exponential and Gamma and Chi-Square. An experimental study was conducted to compare these methods and to demonstrate the efficiency of the methods proposed in practice by relying on the generated observations from Exponential distribution. We compare between these methods by using Mean square error (MSE). We did study was performed by using simulation with different values parameter (θ) and different sample sizes (N=10, 30, 50, 100). With the writing of the simulation program by MATLAB 2015a.

1. Introduction
The exponential distribution is often concerned with the amount of time until some specific event occurs. Exponential distributions are commonly used in calculations of product reliability, or the length of time a product lasts [1]. This distribution plays an important role in the development of the theory, that is, any new theory developed can be easily illustrated by the exponential distribution due to its mathematical tractability; Barlow and Proschan [2] and Leemis [3]. Krishna and Kumar [4] have studied the reliability estimation based on the progressive type-II censored sample under the classical setup. They studied Patel, R. M. & Patel, A. C. (2017) A comparison of double informative priors assumed for the parameter of the exponential lifetime model is considered. Three different sets of double priors are included, and the results are compared with a fourth single prior. The data is Type II censored and Bayes estimators for the parameter and reliability are carried out under a squared error loss function in the cases of the four different sets of prior distributions[5]. Haq and Dey (2011) considered comparison and selection of a suitable prior using Bayesian methodology [6]. Haq, A., & Aslam, M. (2009) studied comparison of double informative priors which are assumed for an unknown parameter of the Poisson distribution. The idea is that some time for a single true unknown parameter, different prior information is available; usually, we use one informative prior to before incorporate that prior knowledge and ignoring the other information. So to include two different kinds of information in the analysis, two different priors have been selected for a single unknown parameter of the Poisson distribution. We have assumed three double priors: Gamma-Chi-square distribution, Gamma-Exponential distribution, Chi-square-Exponential distribution and one as usual prior: Gamma distribution for the unknown parameter of the Poisson model [7].
The aim of this study is to compare the reliability function of the exponential distribution using traditional, representative methods of estimation (Maximum Likelihood and Jackknife). In addition, also, using Bayesian methods based on double prior information with a new proposal that uses three prior information. It will be as follow only prior by using Gamma distribution and double Priors by using distributions (Gamma and Chi-Square) Three Priors by using distributions (Exponential, Gamma, Chi-Square).

2. Theoretical Study

2.1.1. Exponential Distribution
The notation $T \sim \text{Exponential}(\theta)$ means that the random variable $T$ has an Exponential distribution with shape parameter $\theta$. The probability density function ($T > 0$) and cumulative distribution function (cdf) and reliability function at time $t$, is given respectively by

$$f(t; \theta) = \theta e^{-\theta t}, \quad 0 < t < \infty, \quad \theta > 0$$

$$F(t; \theta) = 1 - e^{-\theta t}, \quad \theta > 0$$

$$R(t; \theta) = e^{-\theta t}, \quad 0 < t < \infty, \quad \theta > 0$$

2.1.2. Maximum Likelihood Estimation (MLE)
This way is one of the most important methods of appreciation aims to make possible a function at the end of maximizes. All that is done to which write down the likelihood function $L(t; \theta)$, and then find the value $\hat{\theta}$ of $\theta$ which maximizes $L(t; \theta)$. The log-likelihood function based on the random sample $t_1, t_2, \ldots, t_n$, is given by:

$$L = \prod_{i=1}^{n} f(t; \theta) = L(t_1, \ldots, t_n, \theta) = \theta^n e^{-\theta \sum_{i=1}^{n} t_i}$$

$$\ln L(t_1, \ldots, t_n, \theta) = n \ln \theta - \theta \sum_{i=1}^{n} t_i$$

And by taking the partial derivative of $\ln L(x_1, \ldots, x_n, \theta)$ we get

$$\frac{d \ln L(t_1, \ldots, t_n, \theta)}{d\theta} = n - \sum_{i=1}^{n} t_i$$

solving of the equations $\frac{d \ln L(t_1, \ldots, t_n, \theta)}{d\theta} = 0$ yields the maximum likelihood

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} t_i}$$

Depending on property invariant by the Maximum Likelihood Estimation, then the estimator Maximum Likelihood to reliability function will be as follows

$$R_{MLE}(t) = e^{-t \hat{\theta}_{MLE}}$$

2.1.3. Jackknife Method
This method is one of the methods used to improve the estimators’ values, and depend on the value of the deleted one of the values of views. Let $(t_i)$ and re-estimate Values estimators based on residual values that number $(n-1)$ and are re-style on this all the values of the sample and sequentially. If the $\hat{\theta}_{MLE}(j)$ represents the Maximum Likelihood Estimator resulting from the application of Maximum
Likelihood Estimator of all data except the value \((t_i)\), then estimator Jackknife to Maximum Likelihood Estimator for the parameter \((\theta)\) \([8]\). It is calculated by the following equation.
\[
\hat{\theta}_{j i a k n i f e - m l e} = n\hat{\theta}_{m l e} - (n - 1) \frac{\sum_{j=1}^{n} \hat{\theta}_{M L E}(j)}{n}
\]  
(7)
And the estimator reliability function will take the following formula
\[
\hat{R}_{j a k n i f e - M L E}(t) = e^{-t(\hat{\theta}_{j a k n i f e - M L E})}
\]  
(8)

3. Methods of Bayesian Estimation
In this method, we will rely on the posterior distribution of parameter \(\theta\) of exponential distribution under different prior distributions estimating. we have considered the following three different double priors and single and three priors. It will be as follows:

- Only Prior by Using Gamma distribution
- Double Priors by Using Gamma and Chi-Square Distributions
- Three Priors by using distributions (Gamma, Exponential, Chi-Square)

3.1.1. Gamma \(G(a_1, b_1)\) distribution as Only Prior (BAY1)
In this method, we will use one prior is a Gamma \(G(a_1, b_1)\) distribution for parameter \(\theta\), Its density function of Gamma is given by
\[
P_{1}(\theta) = \frac{\theta^{a_1-1} b_1 e^{-\theta b_1}}{\Gamma(a_1)} , \quad \theta > 0 ; \quad a_1 \text{ and } b_1 > 0
\]  
(9)
And by using equation (4), (9). Then density joint function of \(T\) and \(\theta\) described in will be as follows
\[
f^\ast(t_1, t_2, ..., t_m, \theta) = L(t_1, ..., t_m | \theta) P_1(\theta) = \frac{\theta^n e^{-\theta \sum_{i=1}^{n} t_i} \frac{\theta^{a_1-1} b_1^{a_1} e^{-\theta b_1}}{\Gamma(a_1)}}{r}
\]  
(10)
which is a Gamma distribution \(G(y_1, \delta_1)\) with parameters \(y_1 = a_1 - 1, \ \delta_1 = \sum_{i=1}^{n} t_i + b_1\)
\[
\therefore f^\ast(t_1, t_2, ..., t_m, \theta) = \frac{\theta^{y_1} b_1^{a_1} e^{-\delta_1 \theta}}{\Gamma(a_1)}
\]  
(11)
Hence from (11) we find marginal density function of \(T\) is given by
\[
f^\ast(t_1, t_2, ... , t_m) = \int_{0}^{\infty} f^\ast(T, \theta) d\theta = \int_{0}^{\infty} \theta^{y_1} b_1^{a_1} e^{-\delta_1 \theta} \frac{\theta^{y_1} b_1^{a_1} e^{-\delta_1 \theta}}{\Gamma(a_1)} d\theta
\]  
Hence \(f^\ast(t_1, t_2, ... , t_m) = \frac{b_1^{a_1} r^{y_1+1}}{\Gamma(a_1)} \frac{1}{\delta_1^{y_1+1}}
\]  
(12)
By using (11) and (12), then density function of the posterior distribution of \(\theta\) is given by
\[
\mathcal{A}_1^\ast(\theta | t_1, ..., t_m) = \frac{f^\ast(t_1, ..., t_m, \theta)}{f^\ast(t_1, ..., t_m)} = \frac{\theta^{y_1} b_1^{a_1} e^{-\delta_1 \theta}}{\Gamma(a_1) \delta_1^{y_1+1}}
\]  
(13)
By using the quadratic loss function \(c(\hat{\theta} - \theta)^2\). Then Bayes' estimator will be the estimator that minimizes the posterior risk given by
\[
Risk(\theta) = E[c(\hat{\theta} - \theta)] = \int_{0}^{\infty} (\hat{\theta} - \theta)^2 \mathcal{A}_1^\ast(\theta | t_1) \ d\theta , \ \text{Let} \ \frac{\partial Risk(\theta)}{\partial \theta} = 0
\]
Then \( \hat{\theta}_{BAY1} = \int_0^\infty \theta \mathcal{A}_1^*(\theta | t_i) \ d\theta = \int_0^\infty \frac{\theta^{n+\gamma_1+1} \delta_1^{n+\gamma_1+1} e^{-\theta \delta_1}}{r(n+\gamma_1+1)} \ d\theta \)

\[ \therefore \hat{\theta}_{BAY1} = \frac{n+\gamma_1+1}{\delta_1} \] (14)

And by using of (3) and (13), the Bayes estimator for the reliability function \( R(t) \) is given by

\[ \hat{R}_{BAY1}(t) = \int_0^\infty e^{-t \theta} \mathcal{A}_1^*(\theta | t_i) \ d\theta = \int_0^\infty \frac{\theta^{n+\delta_1} \delta_1^{n+\gamma_1+1} e^{-\theta (t+\delta_1)}}{r(n+\gamma_1+1)} \ d\theta \]

by using transformation \( u = \theta (t + \delta_1) \Rightarrow \theta = \frac{u}{t + \delta_1} \Rightarrow d\theta = \frac{du}{t + \delta_1} \)

\[ \therefore \hat{R}_{BAY1}(t) = \left( \frac{\delta_1}{t + \delta_1} \right)^{n+\gamma_1+1} = \left( \frac{\sum_{i=1}^n t_i + b_2}{t + \sum_{i=1}^n t_i + b_1} \right)^{n+\delta_1} \] (15)

3.1.2. **Double Priors by Using Gamma and Chi-square Distributions (BAY 2)**

We will here use two density functions for two different distributions, the first prior distribution for Gamma Distributions of \( \theta \) with a parameter with hyperparameter \( a_2 \) and \( b_2 \), where it possesses (pdf)

\[ P_2(\theta) = \frac{\theta^{a_2-1} b_2 a_2 e^{-\theta b_2}}{r(a_2)} , \theta > 0 ; a_2, b_2 > 0 \] (16)

And second Prior Distributions of \( \theta \) is density functions pdf for Chi-square Distributions with hyperparameters \( a_3 \), where it possesses (pdf)

\[ P_3(\theta) = \frac{\theta^{a_3-1} e^{-\theta a_3}}{2^a_3 r(a_3/2)} , \theta > 0 \text{ and } a_2 > 0 \] (17)

By combining the two previous functions (16) and (17), We will can define Double prior define \( \theta \) as follows

\[ P_{T1}(\theta) \propto P_2(\theta) \times P_3(\theta) = \theta^{3a_2/2 - 2} e^{-\theta(b_2 + \frac{1}{2})} \] (18)

And by using equation (4),(18), Then density joint function of \( T \) and \( \theta \) described in will be as follows

\[ f^*(t_1, t_2, ..., t_m, \theta) = L(t_1, t_2, ..., t_m | \theta) P_{T1}(\theta) \]

\[ = \theta^n e^{-\theta \sum_{i=1}^n t_i} \times \theta^{3a_2/2 - 2} e^{-\theta(b_2 + \frac{1}{2})} \] (19)

Where \( \gamma_2 = \frac{3a_2}{2} - 2 \) and \( \delta_2 = \sum_{i=1}^n t_i + b_2 + \frac{1}{2} \)

\[ \therefore f^*(t_1, t_2, ..., t_m, \theta) = \theta^{n+\gamma_2} e^{-\theta \delta_2} \] (20)

Hence from (20), we find marginal density function of \( T \) is given by

\[ f^*(t_1, t_2, ..., t_m) = \int_0^\infty \theta^{n+\gamma_2} e^{-\theta \delta_2} d\theta = \frac{r(n+\gamma_2+1)}{\delta_2^{n+\gamma_2+1}} \] (21)

By using (20) and (21), then pdf of the posterior distribution of \( \theta \) is given by
By using the same previous quadratic loss function. Then

\[ \tilde{\theta}_{BAY2} = \frac{n + y_2 + 1}{\delta_2} \]  

(23)

Table 1. Mean squared error for reliability function where \( \theta = 1.0 \)

| N  | bi | ai | t  | MLE  | Jacknife | BAY1 | BAY2 | BAY3 | BEST |
|----|----|----|----|------|----------|------|------|------|------|
| 10 | -1 | -1 | 0.5000 | 0.0109 | 0.0118 | 0.0144 | 0.1412 | 0.0413 | MLE   |
|    |    |    | 0.7500 | 0.0130 | 0.0155 | 0.0143 | 0.0848 | 0.0375 | MLE   |
|    |    |    | 1.0000 | 0.0115 | 0.0116 | 0.0136 | 0.0508 | 0.0293 | MLE   |
|    |    |    | 1.2500 | 0.0109 | 0.0161 | 0.0121 | 0.0304 | 0.0212 | MLE   |
|    |    |    | 1.5000 | 0.0090 | 0.0148 | 0.0104 | 0.0182 | 0.0148 | MLE   |
|    |    |    | 1.7500 | 0.0072 | 0.0131 | 0.0088 | 0.0109 | 0.0101 | MLE   |
|    |    |    | 2.0000 | 0.0056 | 0.0114 | 0.0074 | 0.0065 | 0.0068 | MLE   |
| 20 | -1 | -1 | 0.5000 | 0.0053 | 0.0054 | 0.0057 | 0.1376 | 0.0112 | MLE   |
|    |    |    | 0.7500 | 0.0068 | 0.0070 | 0.0071 | 0.0819 | 0.0125 | MLE   |
|    |    |    | 1.0000 | 0.0070 | 0.0075 | 0.0073 | 0.0487 | 0.0113 | MLE   |
|    |    |    | 1.2500 | 0.0063 | 0.0071 | 0.0067 | 0.0283 | 0.0092 | MLE   |
|    |    |    | 1.5000 | 0.0054 | 0.0063 | 0.0058 | 0.0167 | 0.0070 | MLE   |
|    |    |    | 1.7500 | 0.0044 | 0.0053 | 0.0049 | 0.0099 | 0.0052 | MLE   |
|    |    |    | 2.0000 | 0.0035 | 0.0043 | 0.0040 | 0.0058 | 0.0038 | MLE   |
| 50 | -1 | -1 | 0.5000 | 0.00201 | 0.00209 | 0.00207 | 0.1202 | 0.0028 | MLE   |
|    |    |    | 0.7500 | 0.0026 | 0.0027 | 0.0027 | 0.0691 | 0.0036 | MLE   |
|    |    |    | 1.0000 | 0.00282 | 0.0029 | 0.00285 | 0.0396 | 0.0034 | MLE   |
|    |    |    | 1.2500 | 0.00263 | 0.0027 | 0.00269 | 0.0227 | 0.0030 | MLE   |
|    |    |    | 1.5000 | 0.0022 | 0.0024 | 0.0023 | 0.0130 | 0.0025 | MLE   |
|    |    |    | 1.7500 | 0.0018 | 0.0020 | 0.0019 | 0.0075 | 0.0020 | MLE   |
|    |    |    | 2.0000 | 0.00152 | 0.0016 | 0.0016 | 0.0044 | 0.00157 | MLE   |
| 100 | -1 | -1 | 0.5000 | 0.00091 | 0.00090 | 0.00092 | 0.0966 | 0.00100 | Jacknife |
|     |    |    | 0.7500 | 0.001227 | 0.001225 | 0.001230 | 0.0525 | 0.001413 | Jacknife |
|     |    |    | 1.0000 | 0.001310 | 0.001319 | 0.001323 | 0.0287 | 0.001478 | MLE   |
|     |    |    | 1.2500 | 0.001232 | 0.00125 | 0.00124 | 0.0160 | 0.001344 | MLE   |
|     |    |    | 1.5000 | 0.001072 | 0.001094 | 0.001083 | 0.0096 | 0.00114 | MLE   |
|     |    |    | 1.7500 | 0.00088 | 0.00091 | 0.00090 | 0.0063 | 0.000920 | MLE   |
|     |    |    | 2.0000 | 0.00069 | 0.000726 | 0.000720 | 0.0047 | 0.000712 | MLE   |
And by using of (3) and (22), the Bayes estimator for the reliability function \( R(t) \) is given by

\[
\hat{R}_{\text{BAYZ}}(t) = \int_0^\infty e^{-t \theta} A_2^*(\theta | t_i) \, d\theta = \int_0^\infty \frac{\theta^{n+\gamma_2} \delta_2^{n+\gamma_2+1} e^{-\theta(t+\delta_2)}}{\Gamma(n+\gamma_2+1)} \, d\theta
\]

\[= \hat{R}_{\text{BAYZ}}(t) = \left( \frac{\delta_2}{t+\delta_2} \right)^{n+\gamma_2+1} = \left( \frac{\sum_{i=1}^n t_i + b_2 + \frac{1}{2}}{\sum_{i=1}^n t_i + t + b_2 + \frac{1}{2}} \right)^{n+\frac{3\delta_2}{2} - 1} \tag{24}\]

3.1.3 Suggested method using Gamma and Chi-square and Exponential Distributions as Three Priors

In this method, we suggested using three different distributions are suggested as priors. Distributions of \( \theta \) with hyperparameter \( a_4, b_4 \), Where it has (pdf) respectively as it comes

The first Prior distribution of \( \theta \) to be exponential with hyperparameter \( a_4 \)

\[P_4(\theta) = a_4 \, e^{-\theta \, a_4}, \, \theta > 0 ; a_4 > 0 \tag{25}\]

The second Prior Distributions of \( \theta \) is density functions pdf for Gamma Distributions with hyperparameters \( a_4, b_4 \), where it possesses (pdf)

\[P_5(\theta) = \frac{\theta^{a_4-1} \, b_3^{a_4} \, e^{-\theta \, b_3}}{\Gamma(a_4)}, \, \theta > 0 ; a_4 \text{ and } b_3 > 0 \tag{26}\]

The three Prior Distributions of \( \theta \) is density functions pdf for Chi-square Distributions with hyperparameters \( a_4 \), where it possesses (pdf)

\[P_6(\theta) = \frac{\theta^{a_4-1} \, e^{-\theta \, \frac{a_4}{2}}}{2^{a_4} \, \Gamma \left( \frac{a_4}{2} \right)}, \, \theta > 0 \text{ and } a_4 > 0 \tag{27}\]

We will can define Three prior for \( \theta \) can define as follows

\[P_{T_2}(\theta) \propto P_4(\theta) * P_5(\theta) * P_6(\theta) = \theta^{\frac{a_4}{2}} \, e^{-\theta(a_4 + b_3 - \frac{1}{2})} \tag{28}\]

By using (4)(28).Then density joint function of \( T \) and \( \theta \) described in will be as follows

\[f^*(t_1, t_2, ..., t_m, \theta) = L(t_1, t_2, ..., t_m | \theta) \, P_{T_2}(\theta) = \theta^{n+\gamma_3} \, e^{-\theta \, \delta_3} \tag{29}\]

Where \( \gamma_3 = \frac{a_4}{2} \) and \( \delta_3 = \sum_{i=1}^n t_i + a_4 + b_3 - \frac{1}{2} \)

Hence from (23), we find marginal density function of \( T \) is given by

\[f^*(t_1, t_2, ..., t_m) = \frac{\Gamma(n+\gamma_3+1)}{\delta_3^{n+\gamma_3+1}} \tag{30}\]

By using (29) and (30) , then pdf of the posterior distribution of \( \theta \) is given by

\[A_3^*(\theta | t) = \frac{f^*(t_1, ..., t_m, \theta_1)}{f^*(t_1, ..., t_m)} = \frac{\theta^{n+\gamma_3} \, \delta_3^{n+\gamma_3+1} \, e^{-\theta \delta_3}}{\Gamma(n+\gamma_3+1)}, \, \theta > 0 \tag{31}\]

By using the quadratic loss function \( c(\hat{\theta} - \theta)^2 \), Then

\[\hat{\theta}_{\text{BAY3}} = \frac{n+\frac{a_4}{2}+1}{\delta_3} \tag{32}\]
And by using of (3) and (31), the Bayes estimator for the reliability function $R(t)$ is given by

$$
\hat{R}_{\text{BAF}}(t) = \int_0^\infty e^{-t \theta} \mathcal{A}_3^*(\theta | t_i) \ d\theta = \left( \frac{\sum_{i=1}^n t_i + a_1 + b_3 - \frac{1}{2}}{t + \sum_{i=1}^n t_i + a_1 + b_3 - \frac{1}{2}} \right)^{n + \frac{a_2}{2} + 1}
$$

(33)

4. Practical Aspect (Simulation)

Using the simulation method gives us a good impression of the nature of the data and also reduces time, effort and costs. Where the data was created in theory without obtaining it scientifically, knowing that data generation, in theory, does not contradict the accuracy of the results. , And a summary of this method we will extract in the following stages

i. The initial values for parameter $\theta$ and choose different sample sizes

This step is important upon which later steps depend. We assumed the initial values for parameter $(\theta = 1.5, 1)$ and sample sizes $(10, 25, 50, 100)$. Choosing these values is important because changing parameter values and different sample sizes (small - medium - large) will give us an idea of the estimates and their pattern of behaviour. And we take the value parameter $\delta_i$ and $\gamma_i$. Where hyperparameters $a_i = 0.5, -1$ and $b_i = -1$

ii. Step of data generation

In this step, the generation of exponential distribution data using the inverse method is as follows $F(t_i; \theta) = U_i = 1 - e^{-t_i \theta} \Rightarrow t_i = \frac{\ln(U_i - 1)}{\theta}$

$F(t_i; \theta) =$ The cumulative distribution

$U_i =$ Uniformly distributed random variable (0,1)

iii. Measure comparison

$$
MSE(\hat{R}(t_i)) = \frac{\sum_{i=1}^l (R(t_i) - \hat{R}(t_i))^2}{l}, \text{ Where } R=1000 \text{ is the number of replications. The tables below show the results of the estimation using the simulation. Program the simulation written by using (Mathlab-2015 b )}
$$
Table 2. Mean squared error for reliability function where $\theta = 1.5$

| N | bi | ai | T  | MLE   | Jackknife | BAY1   | BAY2   | BAY3   | BEST   |
|---|----|----|----|-------|-----------|--------|--------|--------|--------|
| 10 | -1 | 0.5 | 0.5000 | 0.0129 | 0.0170 | 0.0226 | 0.28531 | 0.0267 | MLE    |
|    |    |     | 0.7500 | 0.0124 | 0.0194 | 0.0171 | 0.13423 | 0.0195 | MLE    |
|    |    |     | 1.0000 | 0.0099 | 0.0185 | 0.0121 | 0.06308 | 0.0112 | MLE    |
|    |    |     | 1.2500 | 0.0072 | 0.0167 | 0.0072 | 0.02960 | 0.0071 | MLE    |
|    |    |     | 1.5000 | 0.0051 | 0.0148 | 0.0045 | 0.01387 | 0.0042 | BAY3   |
|    |    |     | 1.7500 | 0.0034 | 0.0135 | 0.0029 | 0.00649 | 0.0025 | BAY3   |
|    |    |     | 2.0000 | 0.0023 | 0.0130 | 0.0019 | 0.00141 | 0.0015 | BAY3   |
| 20 | -1 | 0.5 | 0.5000 | 0.0059 | 0.0064 | 0.0082 | 0.28417 | 0.0093 | MLE    |
|    |    |     | 0.7500 | 0.0060 | 0.0069 | 0.0073 | 0.13671 | 0.0081 | MLE    |
|    |    |     | 1.0000 | 0.0049 | 0.0060 | 0.0053 | 0.02650 | 0.0057 | MLE    |
|    |    |     | 1.2500 | 0.0036 | 0.0047 | 0.0037 | 0.02915 | 0.0035 | BAY3   |
|    |    |     | 1.5000 | 0.0025 | 0.0034 | 0.0024 | 0.01356 | 0.0023 | BAY3   |
|    |    |     | 1.7500 | 0.0017 | 0.0024 | 0.0015 | 0.00630 | 0.0014 | BAY3   |
|    |    |     | 2.0000 | 0.0011 | 0.0016 | 0.0010 | 0.00291 | 0.0009 | BAY3   |
| 50 | -1 | 0.5 | 0.5000 | 0.002408 | 0.002390 | 0.002818 | 0.1301 | 0.001804 | BAY3 |
|    |    |     | 0.7500 | 0.002489 | 0.002531 | 0.002755 | 0.0599 | 0.003038 | MLE |
|    |    |     | 1.0000 | 0.002052 | 0.002137 | 0.002166 | 0.0274 | 0.002922 | MLE |
|    |    |     | 1.2500 | 0.001500 | 0.001599 | 0.001525 | 0.0125 | 0.002257 | MLE |
|    |    |     | 1.5000 | 0.001019 | 0.001112 | 0.001007 | 0.0057 | 0.001558 | BAY1 |
|    |    |     | 1.7500 | 0.000666 | 0.00073 | 0.00063 | 0.0026 | 0.001009 | BAY1 |
|    |    |     | 2.0000 | 0.00041 | 0.00047 | 0.00039 | 0.0012 | 0.000625 | BAY1 |
| 100 | -1 | 0.5 | 0.5000 | 0.001299 | 0.001298 | 0.001407 | 0.1285 | 0.00082 | BAY3 |
|    |    |     | 0.7500 | 0.001355 | 0.001370 | 0.001426 | 0.0566 | 0.00146 | MLE |
|    |    |     | 1.0000 | 0.001123 | 0.001148 | 0.001154 | 0.0253 | 0.00147 | MLE |
|    |    |     | 1.2500 | 0.000822 | 0.000851 | 0.000829 | 0.0113 | 0.00117 | MLE |
|    |    |     | 1.5000 | 0.000557 | 0.000583 | 0.000554 | 0.0051 | 0.00083 | BAY1 |
|    |    |     | 1.7500 | 0.000359 | 0.000376 | 0.000353 | 0.0023 | 0.00055 | BAY1 |
|    |    |     | 2.0000 | 0.00022 | 0.000233 | 0.000218 | 0.0011 | 0.000213 | BAY3 |
Figure 1. Reliability function for different sample sizes and for different parameter values.

5. Conclusions
The observation of tables 1 and 2 shows the following:
• The results Table show method maximum likelihood estimation (MLE) is best over all other methods, especially when the parameter's default value $\theta = 1.0$.
• Bayes method that is used Using Gamma and Chi-square and Exponential Distributions as Three Priors (BAY3) is the best Especially in small samples $N = 10, 20$ and at $\theta = 1.5$ and especially in times ($t = 1.25, 1.50, 1.75, 2.0$).
• We noted decreasing values of (MES) with an increasing sample size of all cases this corresponds to the statistical theory.

Acknowledgment
We recommend extending the study to include other distributions with two parameters by using traditional and other Bayesian methods and applying other estimation methods and comparing them with the methods used in the study.

References
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