On discriminants, Tjurina modifications and the geometry of determinantal singularities

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Abstract

We describe a method for computing discriminants for a large class of families of isolated determinantal singularities – more precisely, for sub-families of $G$-versal families. The approach intrinsically provides a decomposition of the discriminant into two parts and allows the computation of the determinantal and the non-determinantal loci of the family without extra effort; only the latter manifests itself in the Tjurina transform. This knowledge is then applied to the case of Cohen-Macaulay codimension 2 singularities putting several known, but previously unexplained observations into context and explicitly constructing a counterexample to Wahl’s conjecture on the relation of Milnor and Tjurina numbers for surface singularities.

1 Introduction

Isolated hypersurface and complete intersection singularities are well studied objects and there are many classical results about different aspects such as topology, deformation behaviour, invariants, classification and even metric properties (see any textbook on singularities, e.g. [21], [18], [20]). Beyond complete intersections, however, knowledge is rather scarce and unexpected phenomena arise. In this article, we focus on the class of determinantal singularities to pass beyond ICIS, as the properties already differ significantly, but classical results on determinantal varieties and free resolutions provide strong tools to treat this case. Recently, significant progress has been made for this class, e.g. in [26], [2], [24], [8], [17]. In [15] the use of Tjurina modifications made it possible to relate a given determinantal singularity to an often singular variety, which happens to be an ICIS under rather mild conditions. This could e.g. be exploited in [15] and [32] to determine the topology of the Milnor fibre of an isolated

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Cohen-Macaulay codimension 2 singularity (ICMC2 singularity for short). But it is also obvious from those results that the Tjurina transform is blind to certain other properties of an ICMC2 singularity. In this article, we explain which properties of the singularity manifest themselves in the Tjurina transform and which do not by studying the discriminant and a natural decomposition thereof. The general approach to determining the discriminant of a given family of varieties $V(I_t)$ is based on the Jacobian criterion. It involves the elimination of the original variables from the ideal generated by $I_t$ and the ideal of minors of appropriate size of the relative Jacobian matrix of $I_t$. However, the complexity of this approach, which originates from the sensitivity of Gröbner basis computations to the number of occurring variables, makes it impractical for many examples. It is hence important to understand the structure of the discriminant theoretically and to be able to decompose it appropriately by a priori arguments. Making use of Hironaka’s smoothness criterion [19], the structure of the perturbed matrix can be used to split the problem into two smaller problems, one dealing with the locus of determinantal singularities, the other one closely related to the Tjurina transform as it describes the locus above which there are singularities adjacent to an $A_1$ singularity.

In section 2, we first recall known facts about determinantal singularities and then proceed to revisit Hironaka’s smoothness criterion. In the following section 3 we consider the discriminant of versal families of determinantal singularities of type $(m, n, t)$ starting with the simplest case $(2, k, 2)$, then passing on to maximal minors of matrices of arbitrary size and finally to smaller minors. As a sideeffect, we also obtain a quite explicit formulation of the Tjurina transform in the non-maximal case. The above mentioned decomposition of the discriminant into the two parts also gives rise to the surprising behaviour of some aspects of ICMC2 singularities, as we are seeing an interplay of influences related to properties of the generic determinantal singularity and to the Tjurina transform. With these two contributions in mind, it is possible to predict some properties of the singular locus of the Tjurina transform, to explain observations of [8] about ICMC2 3-folds and prove the easy direction of Wahl’s conjecture [31] on the relation between Milnor and Tjurina number for ICMC2 surface singularities. The knowledge from this proof then leads to the construction of a class of counter examples for the converse direction of the conjecture. These applications to the ICMC2 case are discussed in the final section.

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were computed in Singular [9].

2 Basic Facts on EIDS

Before focussing on the discriminant, we shall first recall the definition of properties of the class of singularities on which we focus in the following sections: essentially isolated determinantal singularities (EIDS) as first introduced by Ebeling and Gusein-Zade in [10]. Although certain classes of such singularities had been studied before (e.g. [13], [14] and [5]), simultaneous studies of certain properties of EIDS of all matrix sizes and types only appear recently e.g. in [2] and [24]. In this section, we cover well-known facts about EIDS to give the reader the background knowledge for the subsequent considerations on the discriminant and the Tjurina transform.

Definition 2.1 Let $M_{m,n}$ denote the set of all $m \times n$-matrices with entries in $\mathbb{C}$ and let $1 \leq t \leq \min\{m, n\}$. Then

$$M^t_{m,n} := \{ A \in M_{m,n} \mid \text{rk}(A) < t \}$$

is the generic determinantal variety.

Remark 2.2 $M^t_{m,n}$ can be understood as the variety in $M_{m,n} \cong \mathbb{C}^{mn}$ which is the vanishing locus of the ideal of $t$-minors of the matrix

$$
\begin{pmatrix}
x_1 & \cdots & x_n \\
\vdots & & \vdots \\
x_{(m-1)n+1} & \cdots & x_{mn}
\end{pmatrix} \in \text{Mat}(m,n,\mathbb{C}[x_1,\ldots,x_{mn}]).
$$

$M^t_{m,n}$ has codimension $(n-t+1)(m-t+1)$ in $M_{m,n}$ and $\text{Sing}(M^t_{m,n}) = M^{t-1}_{m,n}$. Moreover, it is known that the sets $M^i_{m,n} \setminus M^{i-1}_{m,n}$ for $1 \leq i \leq t$ form a Whitney stratification for $M^t_{m,n}$.

Remark 2.3 By a result of Ma [23] extending several previous results for maximal, for submaximal and for 2-minors (see e.g. [9], [1]), the first syzygy module of the ideal of $M^t_{m,n}$ is generated by linear relations.

As we are interested in minors of matrices with arbitrary power series as entries, the generic determinantal varieties are not the objects of our primary focus. They are only a tool to formulate the following definition:

Definition 2.4 Let $F : \mathbb{C}^k \rightarrow M_{m,n} \cong \mathbb{C}^{mn}$ be a polynomial map. Then $X = F^{-1}(M^t_{m,n})$ is a determinantal variety of type $(m,n,t)$, if $\text{codim}(X) = (m-t+1)(n-t+1)$ in $\mathbb{C}^k$.

1Passing to the convergent power series ring, we can also define and use these notions in the setting of analytic space germs. Analogously all of the subsequent notions can be carried over to space germs and the analytic setting.
Remark 2.5 By a well-known result of Eagon and Hochster [11], the condition on the codimension ensures that the local ring of a germ of a determinantal variety is Cohen-Macaulay. The same condition on the codimension also implies that all syzygies arise from the linear ones of the generic matrix (by the extension of scalars via the map of local rings induced by $F$). As a consequence, the entries of the syzygy matrix are linear combinations of the components $F_{i,j}$ of the map $F$.

Definition 2.6 ([10]) A germ $(X, 0) \subset (\mathbb{C}^k, 0)$ of a determinantal variety of type $(m, n, t)$ is called an EIDS at $0$, if the corresponding $F$ is transverse to all strata $M^t_{m,n} \setminus M^{t-1}_{m,n}$ of $M^t_{m,n}$ outside the origin.

Remark 2.7 For an EIDS $(X, 0)$ defined by $F^{-1}(M^t_{m,n})$ the singular locus is precisely the preimage of the singular locus of $M^t_{m,n}$, i.e. it is $F^{-1}(M^t_{m,n})$. Thus $(X, 0)$ has an isolated singularity at the origin iff $k \leq (m - t + 2)(n - t + 2)$; moreover, a smoothing of it exists if and only if the previous inequality is strict.

As invertible row and column operations applied to a matrix do not change the ideal of its minors, this holds for any matrix of an EIDS. Moreover, two singularities should be considered equivalent, if one arises from the other by means of a coordinate change. These observations explain the structure of the group which describes the most suitable equivalence relation for determinantal singularities:

Definition 2.8 Let $G$ denote the group $(Gl_m(\mathbb{C}\{x\}) \times Gl_n(\mathbb{C}\{x\})) \rtimes \text{Aut}(\mathbb{C}\{x\})$. Two determinantal singularities $(X_1, 0), (X_2, 0) \subset (\mathbb{C}^k, 0)$ of the same type $(m, n, t)$ and defined by $F_1$ and $F_2$ respectively are called $G$-equivalent, if there is a tuple $(R, L, \Phi) \in G$ such that $F_1 = L^{-1}(\Phi^* F_2) R$.

Recall that a map germ is $s$-determined, if any other map which coincides with it in all terms up to degree $s$ is equivalent to it. If we want to stress the existence of such an $s$ without specifying the value of $s$, the map is referred to as finitely determined.

Theorem 2.9 [25] An EIDS $(X, 0)$ corresponding to a map $F$, defined by the ideal of minors of the corresponding matrix (which we also denote by $F$ by abuse of notation), is finitely $G$-determined if and only if it has finite $G$-Tjurina number

$$\tau_G = \dim_{\mathbb{C}}(\text{Mat}(m,n; \mathbb{C}\{x\})/(J_F + J_{op}))$$

where $J_F$ denotes the submodule generated by the $k$ matrices, each holding the partial derivatives of the entries of $F$ w.r.t. one of the variables, and

$$J_{op} = \langle AF + FB | A \in \text{Mat}(m,m; \mathbb{C}\{x\}), \text{Mat}(n,n; \mathbb{C}\{x\}) \rangle$$

Remark 2.10 The group $G$ is a subgroup of the group $K$ of Mather and coincides with it e.g. for complete intersections and Cohen-Macaulay codimension 2 singularities. It also appears as the subgroup $K_V$ of $K$ in the literature, where
\( V \) is the generic determinantal singularity of the appropriate type (defined by \( M_{m,n} \)). In the cases, where \( G \)-equivalence coincides with analytic equivalence, \( \tau_G \) is precisely the usual Tjurina number. In general, however, \( G \) is a proper subgroup of \( K \) as it respects the underlying matrix size. Thus there can e.g. be no element of \( G \) leading from one representation of Pinkham’s famous example \([27]\) to the other, i.e. leading from a determinantal singularity of type \((2,4,2)\) to one of type \((3,3,2)\) (with the additional constraint of the matrix to be symmetric) or vice versa. It is important to observe that a restriction to \( G \)-equivalence does not fix the minimal size of the matrix, it only fixes some size, as any determinantal singularity of type \((m,n,t)\) can easily be considered as one of type \((m+1,n+1,t+1)\) by simply adding an extra line and an extra column of which all entries are zero except the one where the row and column meet, which should then be chosen to be 1. If, on the other hand, a \( G \)-equivalence class of \((m+1) \times (n+1)\)-matrices contains a matrix of this particular structure, the class will be referred to as essentially of type \((m,n,t)\).

Keeping in mind, that the chosen equivalence can only be used to compare determinantal singularities of compatible types, we shall restrict our considerations to families for which each fibre is a determinantal singularity of appropriate type. A similar restriction has already been used by Schaps in \([28]\), where she considered the notion of an \( M \)-deformation (or more generally an \( f \)-deformation) by deforming the entries of a given matrix \( M \). There she gives a criterion when an \( M \)-deformation is versal and provides examples of situations, in which it is not. However, the \( M \)-deformation of Schaps does not even provide a versal unfolding of the \( F \) as her example 1 shows, which she attributes to D. S. Rim without further reference:

**Example 2.11** Consider the determinantal singularity given by the 2-minors of the matrix

\[
\begin{pmatrix}
x_1 & \alpha x_2 & \beta x_3 & \gamma x_4 \\
x_1 & x_2 & x_3 & x_4
\end{pmatrix}
\sim\begin{pmatrix}
x_1 & 0 & sx_3 & tx_4 \\
0 & x_2 & x_3 & x_4
\end{pmatrix},
\]

where \( \alpha, \beta, \gamma \in \mathbb{C} \) sufficiently general (according to Schaps) or more precisely \((s:t) \in \mathbb{P}^1 \setminus \{(0:1),(1:0),(1:1)\}\), which immediately allows us to pass to the affine chart \( t \neq 0 \) writing \( S \) for \( \frac{sx_3}{tx_4} \). Note that two such matrices corresponding to different points in \( \mathbb{P}^1 \setminus \{(0:1),(1:0),(1:1)\} \) lead to non-\( G \)-equivalent matrices, as this would alter the cross-ratio of the four points \((0:1),(1:0),(1:1),(s:t)\) in \( \mathbb{P}^1 \). The corresponding space germs, on the other hand, are isomorphic in a trivial way, as a change of \( e \) only manifests itself by multiplying some of the minors by invertible constants.

\( \tau_G \) is 5 in this example and a versal unfolding of the corresponding morphism \( F \) is given by

\[
\begin{pmatrix}
x_1 & a & Sx_3 + b & x_4 + c + ex_4 \\
d & x_2 & x_3 & x_4
\end{pmatrix},
\]

where Schaps only accepts the deformation parameters \( a, b, c, d \), but refuses \( e \), as it alters the original matrix. An interesting property of this family is that
changes to the parameter \( e \) only have an effect on the \( K \)-equivalence class of the corresponding space germ, if at least one of the other four parameters is non-zero.

Choosing a monomial \( C \)-basis \( m_1, \ldots, m_\tau \) of the Mat\( (m, n; \mathbb{C}\{x\})(J_f + J_{op}) \), it is then easy to write down a semiuniversal unfolding of the morphism \( F \), by simply perturbing the corresponding matrix as follows:

\[
F_{t_1, \ldots, t_\tau} = F + \sum_{i=1}^\tau t_i m_i.
\]

The corresponding family of space germs is also versal for determinantal deformations of determinantal singularities\(^3\) in the following sense: Any family of space germs with given determinantal singularity in the zero-fibre and only determinantal singularities of appropriate type as fibres can be induced from the family of space germs described by \( F_{t_1, \ldots, t_\tau} \). To make notation a bit shorter, we call such a family of space germs \( \mathcal{G} \)-versal.

But the relation between the base of a \( \mathcal{G} \)-versal family and a versal family can be quite subtle and has not yet been studied in generality. In Pinkham’s example, we only see one of the components of the base of the versal family as the base of the \( \mathcal{G} \)-versal family; in Rim’s example cited above the relation is significantly less obvious:

**Example 2.12 (2.11 continued)** A \( \mathcal{G} \)-versal family with base \((\mathbb{C}^5, 0)\) has already been constructed above.

A versal family of space germs with the given special fibre is

\[
I_X = \langle (S - 1)x_3x_4 + Ax_4 + Dx_3, x_2x_4 - Bx_2, x_1x_4 + Cx_4 - CG, \\
x_2x_3 + Ex_3 + Hx_2, x_1x_3 - Fx_3 - \frac{1}{S}FH, x_1x_2 + \frac{1}{S}EF \rangle
\]

over a base \((V, 0)\) which is the germ of the cone of the Segre embedding of \( \mathbb{P}^3 \times \mathbb{P}^1 \) in \( \mathbb{P}^7 \). More precisely,

\[
I_V = \langle SAB + AE - (S - 1)BH, SAC + SAF - (S - 1)FH, SBC - EF, \\
SBD + DE + (S - 1)EG, CD + DF + (S - 1)CG, \\
SAG - DH - (S - 1)GH \rangle
\]

(see example 3.4 in the Thèse of Buchweitz \(^7\) for the construction). The relation between these seemingly completely unrelated base spaces becomes apparent,

\(^2\)Following tradition in the standard basis community we also refer to a module element, of which the only non-zero entry is a monomial, as a monomial.

\(^3\)For simplicity of notation, we can consider a non-singular codimension \( k \) germ as determinantal singularity essentially of type \((1, k, 1)\).
as soon as we consider them as the images of \((V(B), 0)\) under the projections to the first and second factor of \((\mathbb{C}^5, 0) \times (\mathbb{C}^8, 0)\), where

\[
B = ((S - 1)b - (S - 1 - e)A, a - (1 + e)B, (1 + e)d + C,
(S - 1)c + (S - 1 - e)D, a + E, Sd - F, e + (1 + e)G, b - H).
\]

(Note that the coefficients \(S, S - 1, S - 1 - e\) and \(1 + e\) are all units in the local ring).

The non-trivial comparison illustrated above also makes the question of semi-universality difficult to answer in generality; in some cases as, e.g. the Hilbert-Burch case, it is obvious, in others it may boil down to a case by case check for different matrix structures. In view of the possibility of semiuniversality in further cases, we continue with the considerations in full generality and leave this question to be answered at the time of application to specific matrix sizes (or even matrices).

At this point, it is important to observe that the \(G\)-versal families obtained by the above construction are indeed flat, as any relation lifts to a relation of the family by the second part of remark \(2.5\).

Having restricted our interest to \(G\)-versal deformations, we can now state the object, which we want to determine algorithmically in the subsequent section: the locus in the base \(\mathbb{C}^{\tau_G}\) of the \(G\)-versal family, above which the fibres possess singularities, i.e. the \(G\)-discriminant locus of the family.

For later use, we also need one further construction concerning determinantal singularities: a Tjurina modification as introduced in \(30\) and used e.g. in \(29\) and \(15\). This construction relies on the fact that the rows of an \(m \times n\) matrix \(A\) representing a point of \(M_{m,n}^t\) span a \((n-t+1)\)-dimensional subspace in \(\mathbb{C}^n\). This gives rise to a rational map \(\tilde{P} : M_{m,n}^t \rightarrow \text{Grass}(n - t + 1, n)\), of which we can resolve indeterminacies to obtain a map \(\hat{P} : W = \Gamma_{\tilde{P}}(M_{m,n}^t \setminus M_{m,n}^{t-1}) \subset \mathbb{C}^{m \cdot n} \times \text{Grass}(n - t + 1, n)\). Combining this with a map \(F\) defining a determinantal variety \(X\) of type \((m, n, t)\) as in definition \(2.3\), we obtain the following commutative diagram:

\[
\begin{align*}
Y = X \times_{M_{m,n}^t} W & \xrightarrow{\tilde{F}} W \\
X & \xrightarrow{F} M_{m,n}^t \\
\end{align*}
\]

If the dimension of \(X\) is large enough to allow the exceptional locus to be a proper subset of \(Y\), this is indeed a modification. Explicit equations for \(Y\) are given in \(15\), in the simplest case, \(t = n \leq m\), the equations are given by

\[
F \cdot \mathbf{s} = 0,
\]

where \(\mathbf{s} = (s_1, \ldots, s_n)\) denotes the tuple of variables of \(\text{Grass}(n - 1, n) = \text{Grass}(1, n) = \mathbb{P}^{n-1}\).
Analogously the columns can be used for the same construction, as they span a \((m - t + 1)\)-dimensional subspace in \(\mathbb{C}^m\).

3 The discriminant for EIDS

To study the discriminant, we need to detect the singularities of the fibres. As already mentioned in the introduction, we shall decompose the discriminant and compute the contributions separately. To this end, we shall exploit the smoothness criterion of Hironaka in a similar way as in [3], but with a slightly more involved train of thought. For readers’ convenience, we postpone the general case and work out the key ideas in the smallest non-trivial case first: 2-minors of \(2 \times (2 + k)\) matrices.

**Definition 3.1** [19] Let \((X, 0) \subset (\mathbb{C}^n, 0)\) be a germ with defining ideal \(I_{X,0}\) generated by \(f_1, \ldots, f_s \in \mathbb{C}\{x\} := \mathbb{C}\{x_1, \ldots, x_n\}\), and assume that these power series form a standard basis of the ideal with respect to some local degree ordering. Assume further that the power series \(f_i\) are sorted by increasing order. The tuple \(\nu^* \in \mathbb{N}^s\) then denotes the sequence of orders of the \(f_i\).

The tuple \(\nu^*\) detects singularities, as the following lemma states, which is implicitly already present in Hironaka’s work:

**Lemma 3.2** The germ \((X, 0) \subset (\mathbb{C}^n, 0)\) is singular at \(p\) if and only if

\[
\nu^*(X, 0) >_{\text{lex}} (1, \ldots, 1)\]

with respect to the lexicographical ordering \(>_{\text{lex}}\).

Of course, the above definition and lemma also make sense for the germ at any other point \(p\) on \(X\), as can be seen by moving the point \(p\) to 0 by a coordinate transformation and then passing to the germ.

Based on these considerations, we now want to decide whether for a given \(2 \times (2 + k)\)-matrix \(M\), defining a determinantal variety \(X\) of type \((2, 2 + k, 2)\), \(\nu^*(X, p) >_{\text{lex}} (1, \ldots, 1)\) at some point \(p\). If \(\nu^*(X, p)\) starts with 1 as first entry, the singularity needs to be essentially of type \((1, 1 + k, 1)\) at \(p\), as \(M\) has to contain one entry which is of order zero and hence a unit in the local ring at \(p\). In other words: if all entries of \(M\) are of order at least 1, the first entry of \(\nu^*(X, p)\) cannot be lower than 2. We hence know that \(X\) is singular at \(p\), if and only if one of the following two alternatives holds:

(A) the ideal \(I_A\) generated by the entries of \(M\) has order at least 1 at \(p\),

(B) \(X\) is essentially of type \((1, 1 + k, 1)\) and singular.
In case (A), it suffices to determine where the ideal of 1-minors of \( M \) is of order at least 1 to describe this contribution to the singular locus.

The second case is significantly more subtle: Such a unit might be sitting at any position in the matrix, which implies a priori that the respective contributions need to be computed for each matrix entry. We know, however, that in case (B) the matrix is of rank precisely 1 at \( p \), i.e., all column vectors are collinear and hence correspond to a point in \( \mathbb{P}^1 \). At such points, the Tjurina transformation is an isomorphism, which allows us to pass to the Tjurina transform, determine its singular locus outside \( V(I_A) \) and take the closure thereof as the contribution (B).

Considering a larger matrix size, say a singularity of type \((m, n, m)\) with \( n \geq m \), and the corresponding maximal minors, we can proceed analogously, but may a priori encounter \( m \) cases corresponding to the singularity being essentially of type \((m - i, n - i, m - i)\) with \( 0 \leq i < m \). As larger minors also vanish, whenever all minors of a smaller size vanish, it suffices to consider the vanishing locus of the \((m - 1) \times (m - 1)\) minors to determine contribution (A).

For contribution (B), we need to assume that the rank of the matrix is precisely \( m - 1 \). Hence its columns span a hyperplane in \( \mathbb{C}^m \) and we can again make use of the fact that the Tjurina transform is an isomorphism at such points. So we can simply determine the singular locus of the Tjurina transform outside \( V(I_A) \) and take the closure thereof as we did for contribution (B) in the previous case.

To give a concise overview of the necessary computations, this is summarized in algorithm 1. There the input is restricted to polynomials for purely practical reasons: it should consist of finitely many terms.

**Algorithm 1** Discriminant for EIDS of type \((m, n, m)\) (sequential)

**Require:** \( M \subset \text{Mat}(m, n; \mathbb{C}[x_1, \ldots, x_r]) \), \( m \leq n \) defining EIDS at 0

**Ensure:** ideals \( I_A \), \( I_B \) describing the discriminant of the versal family of the given EIDS as follows:

- \( I_A \) describes contribution (A)
- \( I_B \) describes contribution (B)
- \( I_A \cap I_B \) describes the discriminant

1: matrix \( N := \text{versal}(G(M)) \)
2: ideal \( I_A := (\text{minors of } N) \)
3: \( I_A = \text{eliminate}(I_A; x_1, \ldots, x_r) \)
4: ideal \( I_{Tj} := (s_1, \ldots, s_m) \cdot N \), with generators denoted as \( f_1, \ldots, f_n \)
5: ideal \( I_B := I_{Tj} + \text{minor} \left( \frac{\partial f_i}{\partial x_j} \right)_{i,j} n - m + 1 \) where \( \mathbf{v} = (\mathbf{x}, \mathbf{s}) \)
6: \( I_B = (I_B : \langle \mathbf{x} \rangle)^\infty \)
7: \( I_B = \text{eliminate}(I_B; x_1, \ldots, x_r, s_1, \ldots, s_n) \)
8: \( I_B = (I_B : I_A^\infty) \)
9: return \((I_A, I_B)\)

9
The algorithm requires a saturation in step 6 to remove any contribution of the irrelevant ideal. As this can be a significant bottleneck, a parallel approach can be helpful: replace step 6 by running step 7 in all charts \( D(s_i) \) of the projective space and intersect the resulting ideals \( I_{B,i} \) to obtain \( I_B \).

**Remark 3.3** In lines 3 and 7 of \( \mathbb{I} \), we use elimination which means that we endow the resulting complex space with the annihilator structure (cf. [18], Def. 1.45), which is not compatible with base change. We might as well have chosen the Fitting structure relying on resultant methods instead of elimination, as this is compatible with base change.

The choice of elimination over resultants is mostly based on the purely practical fact that the implementation of elimination in Singular is significantly more refined than the one of resultants.

**Example 3.4**

(a) Consider the determinantal singularity defined by the 2-minors of a matrix of the form

\[
M = \begin{pmatrix}
x_1 & \cdots & x_{r-1} & x_r \\
x_{r+1} & \cdots & x_{2r-1} & f(x_1, \ldots, x_{r-1}) 
\end{pmatrix}.
\]

The \( \mathcal{G} \)-versal family with this special fiber can be written as

\[
M_1 = \begin{pmatrix}
x_1 + a_1 & \cdots & x_{r-1} + a_{r-1} & x_r \\
x_{r+1} & \cdots & x_{2r-2} & F(x_1, \ldots, x_{r-1}, a_r, \ldots, a_s) + a_0 
\end{pmatrix},
\]

where \( F(x_1, \ldots, x_{r-1}, a_r, \ldots, a_s) \) corresponds to a versal deformation with section of \( f(x) \). For determining the contributions to the discriminant, we now apply our algorithm and obtain:

\[
I_A = \langle x_1 + a_1, \ldots, x_{r-1} + a_{r-1}, x_r, \ldots, x_{2r-2}, F(a_1, \ldots, a_s) + a_0 \rangle \cap \mathbb{C}[a]
\]

\[
I_B = \text{discriminant of } F(x_1, \ldots, x_{r-1}, a_r, \ldots, a_s) + a_0
\]

The locus, above which we see determinantal singularities, is the smooth hypersurface \( V(F(a_1, \ldots, a_s) + a_0) \) and the remaining part of the discriminant is precisely the discriminant of the versal family with section in the right hand lower entry.

(b) As the next example, we consider 3 families of ICMC2 singularities:

\[
M_1 = \begin{pmatrix}
x & y \\
w & x
\end{pmatrix} \quad M_2 = \begin{pmatrix}
x & y \\
w & u
\end{pmatrix} \quad M_3 = \begin{pmatrix}
x & y \\
w & x + g(u,v)
\end{pmatrix},
\]

where \( f(v) = v^k \) for some \( k \in \mathbb{N} \) and \( (g(u,v), h(u,v)) \) describes a fat point in the plane. Then, \( M_1 \) is a 3-fold in \( (\mathbb{C}^5, 0) \) and the other two are 4-folds.
in \((\mathbb{C}^6, 0)\). Direct computation yields the following versal families:

\[
M_{1; a, b} = \left( x \quad w \quad x + \sum_{i=0}^{k-1} a_i v^i \quad y + v^k + \sum_{i=0}^{k-2} b_i v^i \right)
\]

\[
M_{2; b} = \left( x \quad y \quad w \quad u \quad x + v^k + \sum_{i=0}^{k-2} b_i v^i \right)
\]

\[
M_{3; t} = \left( x \quad y \quad w \quad u \quad x + G(u, v, t) \quad y + H(u, v, t) \right),
\]

with suitably chosen \(G(u, v, t)\) and \(H(u, v, t)\) (cf. \([14]\)). The Tjurina transforms for the three cases are:

\[
I_{T_{j, 1}} = \langle sx + tw, sy + t(x + \sum_{i=0}^{k-1} a_i v^i), sz + t(y + v^k + \sum_{i=0}^{k-2} b_i v^i) \rangle
\]

\[
I_{T_{j, 2}} = \langle sx + tw, sy + tu, sz + t(x + v^k + \sum_{i=0}^{k-2} b_i v^i) \rangle
\]

\[
I_{T_{j, 3}} = \langle sx + tw, sy + t(x + G(u, v)), sz + t(y + H(u, v)) \rangle
\]

Passing to the two affine charts of \(\mathbb{P}^1\), we immediately see that all the \(V(I_{T_{j, i}})\) are non-singular. Hence, \(I_B = \langle 1 \rangle\) and \(I_A\) describes the whole discriminant in these cases.

(c) To illustrate contributions (A) and (B) not only in the extremal cases shown above, we give two surface and two 3-fold examples from the list of simple ICMC2 singularities \([14]\), which for the surface case coincides with Tjurina’s list of rational triple point singularities in \([30]\).

As first example of a surface singularity, we consider Tjurina’s \(A_{0,1,2}\) singularity. Its versal family is:

\[
\begin{pmatrix}
 x_3 \\
 x_3^4 + x_4 a_1 + a_2 \\
 x_2 + a_3 \\
 x_1
\end{pmatrix}
\]

for which the two contributions to the discriminant are:

\[
I_A = \langle a_2^3 + a_5, a_4^3 + a_1 a_4 - a_2 \rangle
\]

\[
I_B = \langle a_5 (4a_1^3 + 27a_2^2) \rangle.
\]

So contribution (A) is a 5-dimensional smooth subvariety of the base and contribution (B) consists of two hypersurfaces, a smooth one and a cylinder over a plane cusp.

As next example, we consider Tjurina’s \(D_0\) singularity:

\[
\begin{pmatrix}
 x_3 + a_1 \\
 x_1 + x_3 a_3 + x_4 a_2 + a_4 \\
 x_2 + a_5 \\
 x_1
\end{pmatrix}
\]

for which a direct computation yields

\[
I_A = \langle a_1 a_3 + a_2 a_6 - a_4, a_1^2 + a_5 a_6 + a_7 \rangle.
\]
which again happens to be a smooth subvariety of codimension 2. Contribution (B) is an irreducible hypersurface of degree 16, of which we do not give the explicit equations here.

The first 3-fold example has the versal family

\[
\begin{pmatrix}
  x_5^2 + x_4x_5 + x_4a_1 + a_2 & x_2 + x_5a_3 + a_4 & x_1 \\
  x_1 + x_5a_2 + a_3 & x_3^2 + x_2x_5 + x_4a_6 + a_7 & x_3 \\
  x_3 & x_4 & x_5^3 + x_2x_5 + a_7
\end{pmatrix}.
\]

Here contribution (A) is an irreducible hypersurface of degree 7, whereas contribution (B) is the hypersurface defined by

\[I_B = \langle a_7(a_1^3 - a_2) \rangle.\]

The versal family in the final example is:

\[
\begin{pmatrix}
  x_3 + a_1 & x_2 + x_5a_4 + a_5 \\
  x_1 + x_5a_2 + a_3 & x_3^2 + x_2x_5 + x_4a_6 + a_7 & x_4
\end{pmatrix}.
\]

In this case, contribution (A) is an irreducible hypersurface of degree 5 and contribution (B) is an irreducible hypersurface of degree 14.

Up to now, we had restricted our considerations to ideals of maximal minors to allow a clearer exposition of the material. For considering non-maximal minors, we first observe that the \(\mathbb{P}^m - 1\) which was used in case (B) above is just a manifestation of a Grassmannian in the simplest case, hyperplanes in \(\mathbb{C}^m\). Passing to non-maximal minors, however, the Grassmannian has more structure which we need to recall before continuing with our study of the discriminant.

Classically the Grassmannian describing the set of \(r\)-dimensional linear subspaces of an \(n\)-dimensional vector space \(V\) or equivalently of \((r-1)\)-dimensional linear subspaces \(\mathbb{P}^{r-1} \subset \mathbb{P}^{n-1}\) can be embedded into projective space by the Plücker embedding:

\[
\text{Grass}(r, n) \longrightarrow \mathbb{P} \left( \bigwedge^r V \right) \cong \mathbb{P}^{\binom{n}{r} - 1}
\]

\[
\text{span}(v_1, \ldots, v_r) \longrightarrow v_1 \wedge \cdots \wedge v_r.
\]

The image of this embedding is closed; the equations of the image are quadratic in the variables of \(\mathbb{P}^{\binom{n}{r} - 1}\); we denote the variables as \(x_{i_1, \ldots, i_r}\) for any given sequence of indices \(1 \leq i_1 < i_2 < \cdots < i_r \leq n\). Purely for convenience of notation, we extend this to any subset of \(\{1, \ldots, n\}\) with \(r\) elements. To this end, we set \(x_{i_1, \ldots, i_r} = 0\), if two elements of the index coincide and postulate that permutations of indices change the sign by the sign of the permutation. Then each Plücker relation is of the form

\[
\sum_{j=0}^{r} (-1)^j x_{i_1, \ldots, i_{r-1}, k_j} \cdot x_{k_0, \ldots, \hat{k}_j, \ldots, k_r} = 0
\]
where \( i_1, \ldots, i_{r-1} \) and \( k_0, \ldots, k_r \) are subsets of \( \{1, \ldots, n\} \), i.e. the ideal of the image of the Plücker embedding is generated by quadratic polynomials. At this point it is important to stress that a Plücker coordinate \( x_{i_1, \ldots, i_r} \) can be interpreted as the \( r \)-minor of the matrix with columns \( v_1, \ldots, v_r \) involving the rows \( i_1, \ldots, i_r \).

Now we are ready to consider the general case of an EIDS of any type \((m, n, t)\) with \( 1 < t \leq m \leq n \). The case (A) does not present additional difficulties here, as the ideal \( I_A \) can be computed directly as \((t-1)\)-minors of the given matrix \( M \). As we have seen before, case (B) comprises all singular points at which the matrix \( M \) has rank precisely \( t-1 \). Appending \( t-1 \) columns, whose entries are \( n(t-1) \) new variables, and imposing the condition that at least one \((t-1)\)-minor of this part does not vanish, allows us to restrict to this part of \( X \) by taking the \( t \) minors of the new matrix. With our previous considerations about the Grassmannian, this can also be more conveniently expressed by introducing Plücker coordinates instead of the additional columns and then leads to a set of equations of the form

\[
\sum_{i=1}^{t} (-1)^{i} y_{i_1, \ldots, i_t} m_{i_t,j} = 0
\]

for all strictly increasing \( t \)-tuples \( \{i_1, \ldots, i_t\} \subset \{1, \ldots, m\} \) and for all \( j \in \{1, \ldots, n\} \), where \( m_{i_t,j} \) denotes the entry of \( M \) at the position \((l,j)\). We now consider the ideal generated by these polynomials and by the generators of the image of the Plücker embedding. After saturating out the irrelevant ideal, the new ideal describes the part of \( X \) which is relevant for contribution (B). We can then compute the singular locus thereof, saturate out the maximal ideal in the \( y_{i_1, \ldots, i_t} \) and then eliminate the original variables \( x \) and all variables \( y_{i_1, \ldots, i_t} \) as before to obtain \( I_B \). For practical purposes, a parallel approach using a covering of the Grassmannian with affine charts should again be the choice in implementations due to the extremely high number of variables and the particularly simple structure of the ideal of the Grassmannian in each chart.

Comparing the construction above with the general construction of the Tjurina transform in [15], we see that the use of the Grassmannian in both settings is the same and that \( I_B \) captures precisely the singular locus of the Tjurina transform as before. Therefore, we have decomposed the discriminant of a determinantal singularity in the following way:

**Proposition 3.5** Let \((X, 0) \subset \mathbb{C}^N\) be a determinantal singularity of type \((m, n, t)\), \( m > n \), defined by \( F^{-1}(M_{m,n}^t) \) and let \( X \) be its \( \mathcal{G} \)-versal family. Further assume that \( \dim(X) \geq m \). Then the discriminant of \( X \) decomposes naturally into two contributions:

(A) points in the base space, above which there are determinantal singularities

(B) points in the base space, above which there are singularities leading to singular points in the Tjurina transform.
The condition on the dimension of $X$ in the preceding proposition ensures that the exceptional locus of the Tjurina modification is a lower-dimensional closed subset of the Tjurina transform. In the above decomposition, the contribution related to the Tjurina transform may be empty in some cases, whereas the other one always contains at least the origin.

4 Applications to the ICMC2 case

The above considerations not only yield a decomposition of the discriminant. They show that determinantal singularities $(F^{-1}(M_{m,n}^t), 0)$ possess in general two kinds of contributions to the singular locus: The structural contribution arising from $F^{-1}(M_{m,n}^t)$, to which the Tjurina transform is partly blind, and a contribution arising from the map $F$ itself, which manifests itself in the Tjurina transform. The well-studied special case of ICMC2 singularities, in which $G$-versality and versality are known to coincide, provides a good setting to consider this in more detail and illustrate the consequences.

Lemma 4.1 Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an ICMC2 singularity, i.e. of type $(t, t+1, t)$, with generic linear entries.

1. It has a smooth Tjurina transform, if and only if $N \geq 2t$.

2. The singular locus of the Tjurina transform is a determinantal variety of type $(t+1, N, t+1)$ of dimension $2t - N - 1$ in $\mathbb{P}^{t-1}$ for $t+1 \leq N < 2t$.

Proof: In the case of generic linear entries in the matrix $M$ of $X$, the ideal of the Tjurina transform $Y$ is generated by bi-homogeneous polynomials of bidegree, $(1,1)$. Therefore the Jacobian matrix of it is of the form

\[
\begin{pmatrix}
A & M^T
\end{pmatrix},
\]

where the first columns hold the derivatives w.r.t. the original variables and the remaining ones the derivatives w.r.t. the variables of the $\mathbb{P}^{t-1}$. Then $A$ is a $(t+1) \times N$ matrix with homogeneous entries of degree 1, which only involve the variables of the $\mathbb{P}^{t-1}$ and which are generic, because the entries of $M$ were generic. As the singular locus of $X$ is just the origin, the Tjurina modification is an isomorphism outside the origin and we therefore only need to evaluate the Jacobian criterion above $V(x)$. This causes the last $t$ columns of the Jacobian matrix and all generators of the ideal of the Tjurina transform to vanish. Hence the singular locus of the Tjurina transform is precisely the vanishing locus of the maximal minors of $A$.

For the first claim, it suffices to observe that the codimension of the singular locus of $Y$ is $N - t$ in $0 \times \mathbb{P}^{t-1}$, which needs to exceed $t - 1$ for $Y$ to be smooth, i.e. we obtain the condition $N > 2t - 1$. These arguments also prove the second claim. \qed
As the matrix describing the singular locus of the Tjurina transform is a square matrix for \( N = t + 1 \), we immediately get the following corollary:

**Corollary 4.2** The singular locus of the Tjurina transform of an ICMC2 singularity \((X, 0) \subset (\mathbb{C}^{t+1}, 0)\) of type \((t, t + 1, t)\) with generic linear entries is a hypersurface of degree \( t + 1 \) in \( \mathbb{P}^{t-1} \).

For the other extreme of \( N = 2t - 1 \), i.e. for isolated singular points in the Tjurina transform, it is also possible to determine the number of points as it coincides with the value of the only non-zero term in the Hilbert Polynomial, the constant term, in this case. This polynomial itself can be obtained from a graded free resolution (given by the Eagon-Northcott complex for determinantal varieties of type \((m, n, m)\)). We only give an example of such a computation:

**Corollary 4.3** The singular locus of the Tjurina transform of an ICMC2 singularity \((X, 0) \subset (\mathbb{C}^5, 0)\) of type \((3, 4, 3)\) with generic linear entries consists of 10 points in general position in \( \mathbb{P}^2 \).

**Proof:** It is well-known that the Hilbert polynomial can be read off from the Betti diagram of a minimal free resolution (see e.g. [15] or [12]). Here the situation is particularly simple: the choice of \( N = 5 \) and \( t = 3 \) leads to a singular locus \( \Sigma \) of the Tjurina transform which only consists of points in \( \mathbb{P}^2 \) and can be described by the vanishing of the 4-minors of a \( 5 \times 4 \) matrix with generic linear entries. In particular, this is the Hilbert-Burch case for which von Bothmer, Busé and Fu give an even more explicit formula in [4]: Given the minimal graded free resolution

\[
0 \to \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbb{P}^2}(-l_i) \to \bigoplus_{i=1}^{t+1} \mathcal{O}_{\mathbb{P}^2}(-k_i) \to I_\Sigma \to 0,
\]

the Hilbert polynomial and hence the number of points is

\[
\frac{\sum_{i=1}^{t} l_i^2 - \sum_{i=1}^{t+1} k_i^2}{2}.
\]

In our setting, all \( l_i \) are have the value \(-5\) and all \( k_i \) are \(-4\) which yields \( \frac{1}{2}(4 \cdot 25 - 5 \cdot 16) = 10 \) points.

\[\square\]

As the Tjurina transform can be non-singular, there are cases in which the contribution \((B)\) of the discriminant of the versal family is empty as e.g. for the generic ICMC2 of type \((2, 3, 2)\) in \((\mathbb{C}^{k}, 0)\) for \( k \geq 4 \). But there are, of course, many non-generic matrices with singular Tjurina transform even in these dimensions. The smoothness of the Tjurina transform is actually a statement about the adjacencies of an EIDS, as the following lemmata show:

**Lemma 4.4** Let \((X, 0) \subset (\mathbb{C}^{k}, 0)\), be an EIDS for which contribution \((B)\) to the discriminant of the versal family is not empty, then \((X, 0)\) is adjacent to an \( A_1 \) singularity and has a smoothing passing through \( A_1 \) singularities.
Proof: If \( p \) is a point in the base of the versal deformation belonging to contribution (B), then the Tjurina transform of the fibre above this point is singular and has only ICIS singularities, which are themselves adjacent to an \( A_1 \). Moreover, as contribution (B) is non-empty, it contains by construction an open set which does not meet contribution (A). Above this open set, there are no fibres with determinantal singularities, and the Tjurina modification is already an isomorphism for these fibres. Hence the original singularity is also adjacent to an \( A_1 \) and possesses a smoothing which passes through \( A_1 \) singularities.

\[ \square \]

Remark 4.5 There are smoothable EIDS for which no smoothing passes through an \( A_1 \) singularity as can be seen from the results in [15] and [32]. For surface singularities of type \((2, 3, 2)\) in \((\mathbb{C}^4, 0)\) this is precisely the determinantal singularity with generic linear entries. For 3-fold singularities of type \((2, 3, 2)\) in \((\mathbb{C}^5, 0)\) these are precisely the singularities with \(b_3 - b_2 = -1\). As we always have \(b_2 = 1\), this difference implies \(b_3 = 0\), whence the Tjurina transform is smooth, no adjacency to an \( A_1 \) is possible and the contribution (B) to the discriminant is empty. However, these singularities are smoothable through a different mechanism: They pass through the EIDS with generic linear entries of the appropriate ambient dimension. For the latter, any non-trivial deformation is a smoothing. In dimensions, in which the determinantal singularity with generic linear entries is a rigid EIDS, the contribution (B) will always be empty, as the terminal object in the adjacency diagramm is a rigid determinantal singularity which causes contribution (A) to be the whole base of the versal family. Therefore, we cannot decide in general whether a singularity is adjacent to an \( A_1 \) based solely on the fact that contribution (B) is empty. But even in this case, it can make sense to consider the locus above which there are ICIS singularities, by omitting the final saturation by \( I_A \) in the computation of contribution (B).

These last observations also indicate that passing to the Tjurina transform provides valuable information about the original singularity, but this information also relies on knowledge about the contribution (A). The cases of surfaces in \((\mathbb{C}^4, 0)\) and 3-folds in \((\mathbb{C}^5, 0)\) show how different the behaviour can be. To illustrate this, we first discuss Wahl’s conjecture about the relation between Milnor and Tjurina number in the surface case: we reprove the easier direction that quasihomogeneity implies \(\mu = \tau - 1\) for the special situation that there are only isolated singularities in the Tjurina transform. This already indicates, where it might be possible to find counterexamples for the other direction, i.e. non-quasihomogeneous codimension 2 surface singularities satisfying \(\mu = \tau - 1\), and we pursue this thought to construct a whole class of counterexamples. Contrasting the rather controlled situation of surfaces, we then give an explanation of the observations of Damon and Pike [8] in the 3-fold case, relying on the same mechanism, but with very different outcome.

Lemma 4.6 Let \((X, 0) \subset (\mathbb{C}^4, 0)\) be a quasihomogeneous isolated determinantal singularity of type \((2, 3, 2)\) with at most isolated singularities in the Tjurina
transform. Then

\[ \mu = \tau - 1. \]

**Proof:** In this proof we denote the Tjurina transform of \( X \) by \( Y \) and we assume that the presentation matrix of \( X \) is chosen to have quasihomogeneous entries and respect row and column weights as in [13]. This implies that \( Y \) is quasihomogeneous w.r.t. the same weights.

From [13], we know that

\[ \mu(X) = 1 + \sum_{p \in \text{Sing}(Y)} \mu(Y, p), \]

i.e. it differs from the sum over the Milnor numbers of the singularities \((Y, p)\) of the Tjurina transform by 1. The singularities of the Tjurina transform are at most ICIS singularities and hence satisfy \( \mu(Y, p) \geq \tau(Y, p) \) with equality precisely in the case of quasihomogeneous singularities [22]. So it remains to establish the relation

\[ \tau(X) = \sum_{p \in \text{Sing}(Y)} \tau(Y, p) + 2 \]

to prove the claim. However, after a few preliminary considerations this leads to a Gröbner basis computation which we will sketch for a general matrix of the given properties in the rest of the proof.

To this end, we first recall from [13] that

\[ T^1(X) \cong N' = H^1(Y_0, T_{Y_0}) \oplus \bigoplus_{p \in \text{Sing}(Y)} T^1(Y, p) \]

implying for the corresponding dimensions

\[ \tau(X) = \dim_{\mathbb{C}} H^1(Y, TY) + \sum_{p \in \text{Sing}(Y)} \tau(Y, p). \]

As all of these \( \mathbb{C}\{x\}\)-modules are finite dimensional \( \mathbb{C} \)-vector spaces, this also induces an isomorphism of \( \mathbb{C} \)-vector spaces which can be expressed in terms of a monomial basis of \( T^1_X \). To complete the proof, we therefore need to identify those basis elements in \( T^1_X \) which do not contribute to \( \bigoplus_{p \in \text{Sing}(Y)} T^3_{Y,p} \).

To keep the presentation of the rest of the proof as simple as possible, we denote the variables by \( x, y, z, w \) and denote the tuple of these four variables by \( \mathbf{x} \) in the following. Since the Tjurina transform only contains isolated singularities, \( X \) can be expressed in terms of a matrix

\[ A = \begin{pmatrix} x & y & z \\ a & b & c \end{pmatrix} \]
according to [15], where \( a, b, c \in \mathfrak{m} \subset \mathbb{C}\{x\} \) and no term of \( a \) is divisible by \( x \).

The Tjurina transform is described by the ideal \( I_{Tj} = (sx + ta, sy + tb, sz + tc) \) with Jacobian matrix

\[
\begin{pmatrix}
s & ta_y & ta_z & ta_w & x & a \\
tb_x & s + tb_y & tb_z & tb_w & y & b \\
tc_x & tc_y & s + tc_z & tc_w & z & c \\
\end{pmatrix},
\]

where a subscript stands for the partial derivative by the respective variable.

At \( t = 0 \) there is a 3-minor \( s^3 \), whence \((0, 0, 0, 0) \times (1 : 0) \) cannot be a singular point of the Tjurina transform. Therefore it suffices to consider the chart \( D(t) \).

We know from [15] that

\[
N' = ((\mathbb{C}[s, t]{\{x\}})^3 / K)_{(1)}
\]

where the subscript \((1)\) denotes the degree 1 part in \( s \) and \( t \) and the module \( K \) is generated by the columns of the Jacobian matrix of above and \( I_{Tj} \cdot \mathbb{C}\{x\}^3 \). As we are interested in \( K_{(1)} \) and possibly slices of higher degree, but not in \( K_{(0)} \), we now replace the columns 5 and 6 of the above matrix by their multiples with \( s \) and \( t \). To fix a numbering of the generators of \( K_{(1)} \), we keep the resulting numbering of the columns: starting with the partial derivatives by the \( x, y, z \) and \( w \) and continuing with the \( s \) and \( t \) multiples we just introduced, the first eight generators are the columns of the following matrix:

\[
\begin{pmatrix}
s & ta_y & ta_z & ta_w & sx & sa & tx & ta \\
tb_x & s + tb_y & tb_z & tb_w & sy & sb & ty & tb \\
tc_x & tc_y & s + tc_z & tc_w & sz & sc & tz & tc \\
\end{pmatrix}.
\]

The last 9 generators are then ordered as in the following matrix:

\[
\begin{pmatrix}
sx + ta & sy + tb & sz + tc & 0 & 0 \\
0 & 0 & 0 & sx + ta & \ldots & 0 \\
0 & 0 & 0 & 0 & sz + tc \\
\end{pmatrix}.
\]

We also know that the sum over the \( T^1(Y, p) \), which are all sitting above the origin of \( \mathbb{C}^5 \), can be computed as the \((s)-local, but s-global) Tjurina module in the respective chart of \( \mathbb{P}^1 \) by considering the module

\[
T = (\mathbb{C}[s]{\{x\}})^3 / \overline{K}
\]

obtained by dehomogenizing the previous module w.r.t. the variable \( t \). Our task will now be a comparison of \( K_{(1)} \) and \( \overline{K} \) and of the respective quotients by means of the corresponding leading ideals, which we obtain from a standard basis computation.

By the assumption that the entries of the original matrix are contained in \( \mathfrak{m} \), only the first 4 generators of \( \overline{K} \) and \( \overline{K} \) can possibly contain \( s \)-degree zero entries. Using a mixed ordering which first compares w.r.t. a global ordering
This only leaves potential differences between standard bases of $K(1)$ and $\overline{K}$ in s-polynomials arising from $C_4$ and $C_i$ for $1 \leq i \leq 3$.

In $K(1)$, we can see no contributing s-polynomial which arises between $C_i$
and $C_4$ with $1 \leq i \leq 3$, because its leading monomial would be in $(s, t)$-degree 2. In $\mathbb{K}$ on the other hand, such an $s$-polynomial is relevant, as $C_4$ is non-zero (due to the need for a pure power in $w$ to appear to allow finite dimension), is of $s$-degree zero and has lower $w$-order in each entry than the corresponding entry of the second row of $M$. Considering this more closely, we see that $sC_4$ and $s^2C_4$ can both contribute to relevant $s$-polynomials with $C_1, C_2, C_3$. On the other hand, using the linear combinations of $C_1, C_2, C_3$ indicated by the columns of the right adjoint of the $3 \times 3$ square matrix with columns $C_1, C_2, C_3$, the minor of the Jacobian matrix corresponding to $C_1, C_2, C_3$ reduces to zero in all entries. Hence $s^3C_4$ does not provide any new contribution but reduces to lower degree terms in $s$. Therefore we obtain precisely two $(s, t)$-degrees for which the standard basis of $\mathbb{K}$ contains elements not necessarily appearing in the one of $\mathbb{K}(1)$. This implies that their leading monomials can be part of the computed monomial basis of $N'$, but reduce to zero in $T = (\mathbb{C}[x]/(x^3/K)$. If they are in the monomial basis of $N'$, they contribute to $\dim \mathbb{C}H^1(Y, T_Y)$.

Showing that no $x$-multiple of two leading monomials of the still remaining $s$-polynomials is non-zero in $N'$, we obtain $\dim \mathbb{C}H^1(Y, T_Y) \leq 2$ which in turn proves one side of the inequality in the original claim. Again, we simply state the relations:

$$x \cdot \text{NF}(\text{spoly}(sC_4, \{C_1, C_2, C_3\})) = a_w tC_9 + b_w tC_{12} + c_w tC_{15} - atC_4$$
$$y \cdot \text{NF}(\text{spoly}(sC_4, \{C_1, C_2, C_3\})) = a_w tC_{10} + b_w tC_{13} + c_w tC_{16} - btC_4$$
$$z \cdot \text{NF}(\text{spoly}(sC_4, \{C_1, C_2, C_3\})) = a_w tC_{11} + b_w tC_{14} + c_w tC_{17} - ctC_4$$

Of course, such relations continue to hold after multiplication with $s$ and subsequent reduction by $C_1, C_2, C_3$, which now leaves only one case to be considered:

$$w \cdot \text{NF}(\text{spoly}(sC_4, \{C_1, C_2, C_3\})),$$

but by the Euler relation this expression reduces to zero, as we are considering the quasihomogeneous case.

For proving the equality part of the statement, it suffices to establish

$$\tau(X) = \sum_{p \in \text{Sing}(Y)} \tau(Y, p) + 2.$$ 

Thus we need to show that $\dim \mathbb{C}H^1(Y, T_Y)$ is at least 2 for all quasihomogeneous ICMC2 surface singularities of type $(2, 3, 2)$ which have at most isolated singularities in their Tjurina transform. This certainly holds for the simplest such singularity given by the matrix

$$\begin{pmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_1 & x_2 
\end{pmatrix},$$

as an explicit Gröbner basis computation for $K$ provides the leading monomials $(0, t^2, 0)$ and $(0, 0, t^2)$ arising from the $s$-polynomials of the pairs $(sC_4, C_1)$ and
As we know that all ICMC2 surface singularities of type (2, 3, 2) are adjacent to this singularity (see [14]), the principle of conservation of number then ensures the upper semicontinuity of \( \dim \mathbb{C} H^1(Y, T_Y) \) concluding the proof.

\[ \square \]

**Remark 4.7** The Milnor number of a semiquasihomogeneous ICIS is known to coincide with the Milnor number of its quasihomogeneous initial part, the Tjurina number of a semiquasihomogeneous ICIS is bounded from above by the Tjurina number of its quasihomogeneous initial part. Therefore the preceding lemma also establishes the inequality

\[ \mu \geq \tau - 1 \]

for semiquasihomogeneous isolated determinantal surface singularities of type (2, 3, 2) with at most isolated singularities in the Tjurina transform.

**Remark 4.8** The considerations in the proof also show that \( \dim \mathbb{C} H^1(Y, T_Y) \) can be computed explicitly in the non-quasihomogeneous case: Considering the module \( K \) defined above, we first observe that already \( N' = (((\mathbb{C}[s,t]\{x\})^3/K)_{(1)} \) has to be a finite dimensional vector space due to finite determinacy of the given singularity. Then the desired dimension of \( H^1(Y, T_Y) \) is precisely \( \dim \mathbb{C}(N') - \dim \mathbb{C}(T) \) where \( T = (\mathbb{C}[s]\{x\})^3/K. \) Note that the value of \( \dim \mathbb{C} H^1(Y, T_Y) \) can exceed 2, as the following example shows:

\[
\begin{pmatrix}
 x_1 \\
 x_2 + 2x_3^2 - x_3^3 \\
 -x_1^2 + x_3^2 \\
 -x_2^2 + x_1^2 + x_2^3 \\
 x_3
\end{pmatrix}
\]

In this example, \( \tau(X, 0) = 34 \) and \( \sum_{p \in \text{Sing}(Y)} \tau(Y, p) = 31 \) which yields a difference of \( \dim \mathbb{C} H^1(Y, T_Y) = 3. \) However, \( \mu(Y) = 39 \) and hence this does not provide a counterexample to Wahl’s conjecture.

**Remark 4.9** Looking at the above proof more closely, we can even determine the basis elements contributing to \( N'/T. \) Considering a standard basis of \( K \) w.r.t. the ordering chosen in the proof, they are precisely the monomials in the basis of \( N' \) which are divisible by monomials in the leading module \( L(K) \) arising from reduction of elements of the form

\[
s^j x_i^i \begin{pmatrix} t a_{ix_i} \\ t b_{ix_i} \\ t c_{ix_i} \end{pmatrix}
\]

where \( j \in \{1, 2\} \) as we know from the proof and \( i \in \mathbb{N} \) takes all values which are sufficiently low not to push the whole element beyond the determinacy bound.

We now construct a counterexample to Wahl’s conjecture, i.e. a non-quasihomogeneous determinantal surface singularity for which \( \mu = \tau - 1 \) holds. More precisely, we try to salvage as much of the situation of lemma [4.6] as we can: We
search for an isolated surface singularity \((X, 0)\), which gives rise to more than one isolated singularity in the Tjurina transform \(Y\). If we can choose all of these singularities as quasihomogeneous, but w.r.t. different weights for the respective singularities, and if we can furthermore ensure that \(\dim_{\mathbb{C}} H^1(Y, T_Y) = 2\), this is precisely the desired counterexample. All of these constraints are e.g. satisfied for the singularity \((X, 0)\) defined by the maximal minors of the matrix

\[
\begin{pmatrix}
z + x & y & x^k + w^2 \\
w^l & z & y
\end{pmatrix},
\]

where the value of \(k, l \in \mathbb{N}\) is at least 3. Its Tjurina transform \(Y\) has two quasihomogeneous singularities, an \(A_{l-1}\) at \((0 : 0 : 1)\) and a \(D_{k+1}\) at \((0 : 1 : 0)\):

\[
A_{l-1} : \langle w^l + tx + tz, z + ty, y + tx^k + tw^2 \rangle \sim_{\mathbb{C}} \langle w^l + tx + t^3 x^k + t^3 w^2, y, z \rangle
\]

with monomial basis of the Tjurina algebra

\[
(1, 0, 0), (w, 0, 0), \ldots, (w^{l-2}, 0, 0)
\]

and

\[
D_{k+1} : \langle sw^l + z + x, sz + y, sy + x^k + w^2 \rangle \sim_{\mathbb{C}} \langle z, y, s^2 x + x^k + w^2 + s^3 w^l \rangle
\]

with monomial basis

\[
(0, 0, 1), (0, 0, s), (0, 0, s^2), (0, 0, x), \ldots, (0, 0, x^{k-2}).
\]

As both of these are quasihomogeneous, the Tjurina and Milnor number coincide for each of the two singularities and we have a total Milnor number of \(Y\) of \(k + l\) which implies that \(\mu(X, 0) = k + l + 1\). On the other hand, it is an easy computation to see that a \(\mathbb{C}\)-vector space basis of \(T^1(X)\) is given by the monomials:

\[
\begin{pmatrix}0 & 0 & 1 \\0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix}0 & 0 & x^{k-1} \\0 & 0 & 0 \end{pmatrix}, \begin{pmatrix}0 & 0 & 0 \\0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix}0 & 0 & 0 \\0 & 0 & 1 \end{pmatrix}, \begin{pmatrix}0 & 0 & 0 \\0 & 0 & 0 \end{pmatrix}
\]

Hence, the Tjurina number is \(k + l + 2\) and we can even discern the three contributions in the monomial basis: the first \(k - 1\) basis elements and the last 2 correspond directly to a basis of the \(T^1\) of the \(D_{k+1}\) singularity, the \((k + 1)\)-st to the \((k + l - 1)\)-th element to a basis of the \(T^1\) of the \(A_{l-1}\) singularity leaving precisely two elements for \(H^1(Y, T_Y)\).

To see that the determinantal singularity is not quasihomogeneous, we consider two hyperplane sections: with \(V(w)\) and with \(V(x)\). Both lead to quasihomogeneous space curve singularities, but w.r.t. different weights:

\[
\begin{pmatrix}z + x & y & x^k \\
0 & z & x
\end{pmatrix}
\]

of Type \(A_k \lor L\) and

\[\text{More precisely, the last 2 elements correspond to those basis elements involving } s.\]
\[ \left( \frac{z}{w^l}, \frac{y}{z}, \frac{w^2}{y} \right) \text{ of Type } E_{2l+6}(2) \text{ (for } l \not\equiv 2 \mod 3) \text{ or } J_{l+4,0} \left( \frac{l+4}{6} \right) \]

which are quasihomogeneous w.r.t. weights \((2, k+1, 2, -)\) in the first case and 
\((-l+4, 2l+2, 3)\) in the second. By an easy calculation, we see that it is impossible to choose weights for \(y\) and \(z\) satisfying both conditions at the same time for \(k, l \geq 3\).

We now come back to 3-folds. The results of [15] and in this article also allow a more geometric interpretation for the new and surprising phenomena observed in [8] for the simple ICMC2 3-fold singularities from [14]. Following Damon and Pike, we now consider the difference between the Euler characteristic of the Milnor fibre \(b_3 - b_2\) and the Tjurina number \(\tau_X\) using their invariant \(\gamma := \tau - (b_3 - b_2)\):

**Observation 4.10 (8)**

a) \(\gamma \geq 2\) for all simple ICMC2-singularities of dimension 3 and increases in value as we move higher in the classification.

b) \(b_3 - b_2 \geq -1\), with equality for the generic linear section and one infinite family.

c) \(b_3 - b_2\) is constant for certain infinite families with values \(-1\) (one family), \(0\) (two families), and \(1\) (two families).

d) \(\gamma\) is constant in all other considered infinite families in the table of simple singularities with only one exception where both \(b_3 - b_2\) and \(\gamma\) increase with \(\tau\).

e) For singularities of the form \(\left( \frac{x}{w}, \frac{y}{v}, \frac{g(x, y)}{z} \right)\) with \(g\) a simple hypersurface singularity, \(\gamma = 3\) and \(b_3 - b_2 = \mu(g) - 1\).

To explain these observations, which differ greatly from the rather rigid structure observed in the surface case, we use again the Tjurina modification. In all cases in question, the Tjurina transform has at most quasihomogeneous hypersurface singularities, whence we know that its Milnor and Tjurina numbers coincide. From [15], we know that \(b_2 = 1\) and \(b_3\) coincides with the Tjurina number of the Tjurina transform. So observation a) simply states that \(\tau_X - \tau_Y \geq 1\). In particular, we have \(\tau_Y = 0\) and \(\tau_X = 1\) for the generic linear section, the \(A_1^+\) singularity, implying that this singularity is not adjacent to an \(A_1\) singularity. As all other singularities in the list are adjacent to it, this explains the lower bound for the difference and hence observation a).

The first part of observation b) follows immediately from the fact that \(b_2 = 1\). The second part is concerned with the family

\(\left( \frac{x}{w}, \frac{y}{x}, \frac{z}{y + v^k} \right)\).
The Tjurina transform of this family is smooth, which implies that \( b_3 = 0 \). The Tjurina number of \( X \), however, increases in the family, \( \tau_X = 2k - 1 \), and is closely related to the maximal number of \( A_0^+ \)-singularities which can appear in a fibre of the versal family. This maximal number is achieved e.g. by the perturbation

\[
\begin{pmatrix}
x & y & z \\
w & x & y + v^k + \alpha
\end{pmatrix}
\]

for any \( 0 \neq \alpha \in \mathbb{C} \), where we see precisely \( k \) such singularities. Observation c) then simply states that similar behaviour with constant topological type of the Tjurina transform also occurs for other families which have singularities in their Tjurina transform.

Observation d), on the other hand, singles out families in which the increase of Tjurina number originates from the increase in Milnor/Tjurina number of the Tjurina transform and the maximal number of \( A_0^+ \)-singularities appearing in a fibre of the versal family does not change. In the last considered family, where \( b_3 - b_2 \) and \( \gamma \) increase with \( \tau_X \), we see a first example of increasing contributions to both the Tjurina transform and the purely determinantal part.

The only part of the last observation, which still remains to be explained, is the statement \( \gamma = 3 \). As already observed in [4], \( T_X^1 \) is isomorphic to \( T_X^{\text{section}} \) of the plane curve singularity defined by the right hand lower entry. Hence \( \tau_X - \tau_Y \) is the difference arising from deformations with section as opposed to usual deformations for the respective plane curve. In all cases in question, this difference is precisely 2 giving rise to \( \gamma = (\tau_X - \tau_Y) + b_2 = 3 \).

**Remark 4.11** Although this article explains many of the recent surprising observations about ICMC2 singularities, this is merely a glimpse into the new phenomena we are seeing in determinantal singularities. Extending existing tools to the determinantal setting and combining methods from the theory of syzygies, from classical singularity theory and topology, we seem to have reached a point now, where we can start thinking about a more systematic study of general determinantal singularities.

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