Hybridized Ackermann’s Methods

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ABSTRACT Further elaborations on the modified Ackermann's method (MAM) for eigenvalue assignment are considered in this paper. Additional results concerning the incomplete eigenvalue assignment (IEA) are stated, verified, and commented. The advantages of IEA are pursued even further in this study beyond that mentioned in [1]. The study proposes two newly appended approaches based on MAM; named spectral and truncated methods. They are grounded on IEA, which fundamentally exemplify a hybridized approach to eigenvalue assignment. Necessary and sufficient conditions for stability of the truncated hybridized method are established, proved, and validated by examples. All results obtained apply equally well to identical eigenvalue assignment, complex eigenvalue assignment, as well as to uncontrollable systems. Besides, they lead to simplified state feedback matrix determination. Three numerical examples are fully worked out to substantiate the nature of the IEA and the two hybridized methods. Simulation and visualization using MATLAB demonstrate the flexibility of the proposed methods.

INDEX TERMS Ackermann’s method, characteristic polynomials, eigenvalue assignment, hybridized methods, incomplete assignment, MATLAB, uncontrollability.

I. INTRODUCTION

In a recent paper [1], the classical method of Ackermann [2-5] has been revisited, extended, and generalized; contributing an alternative proof and newly compact expositions. A modified Ackermann’s method (MAM) proposed in [1], enabled incomplete eigenvalue assignment (IEA) and a generalization to uncontrollable systems. We make a comeback in this paper with further elaborations, extensions, and systemization to MAM and IEA. Doing so, we are led to new additional forms of hybridized nature.

This paper acts as a continuation or as part II of an earlier one [1] entitled: Ackermann’s Method: Revisited, Extended, and Generalized to Uncontrollable Systems. Thus, the references needed in this paper are limited in number; mainly those listed in [1], together with additional relevant ones. It’s worth pointing out that IEA is different from partial eigenvalue assignment as treated in [6-8].

In essence, IEA enjoys the liberty of assigning an incomplete set of eigenvalues say, \( q \) where \( 1 < q < n \) without having to assign the remaining \( n-q \) eigenvalues.
In other words, we are left with \( n-q \) eigenvalues that are enforced. Their values are shown to be subject to certain limitations. They depend on the structure of the system and on the specific choice of an \( n \times (n-q) \) matrix \( N \) chosen to generate a square nonsingular modified controllability matrix (MCM).

The controllability matrix \( C \) is the backbone of the classical method. In [1], a modified form of \( C \) was proposed to enable IEA and to generalize the method to uncontrollable systems. The modified method determines a state feedback matrix which assigns say \( q \) eigenvalues explicitly and implicitly enforces the assignment of the remaining \( n-q \) eigenvalues. Limitations regarding those enforced eigenvalues were recognized and identified in [1]. Such limitations are followed in this paper. They are shown to be not as stringent, and can be relaxed in certain cases. This is discussed in Section IV and Section V.

We envisage incomplete assignment as a form of hybridization worth additional investigation. Two hybridized methods, termed spectral and truncated are the subject of our study shedding light on their nature, structure, and advantages. The entire \( n \) eigenvalues can be assigned through what we call the \( q \)-phase and through what we call the \( N \)-phase.

A particular matrix \( N^eAN \) arose when deriving the sum of the remaining \( n-q \) enforced eigenvalues encountered when applying IEA. The author in [1], went as far as showing that the trace of \( N^eAN \) equals the sum of the enforced eigenvalues. We pursue such assertion further and show that \( N^eAN \) encompasses the remaining \( n-q \) enforced eigenvalues as well.

In conjunction with the use of IEA as a mechanism of extending the method to uncontrollable systems, it can be justified on its own right in obtaining other methods beyond the assignment of a single eigenvalue as has been done in [1]. It enables the introduction and derivation of two new methods named: the spectral and the truncated methods. They are systematic and of hybridized nature. They are formally stated and proved in Section III and Section IV.

The spectral hybridized method relies on a particular choice of \( N \) based on certain candidate vectors obtained when solving a particular set of linear equations. An advantage of the spectral hybridized method is the ability to affect closed loop system structure beyond merely shifting the eigenvalues.

The truncated method is based on polynomials of order \( n-q \). It yields a systematic and simplified feedback controller of lower dimensionality. In fact, the enforced eigenvalues turn out to be the roots of sub-polynomials; those truncated from the open loop characteristic polynomial. A stable open loop system is shown to always result in a stabilized system. An unstable open loop system can sometimes result in a stabilized closed loop system conditionally. A necessary condition for stability when employing any order of truncated polynomial is a negative trace of \( A \). Moreover, this condition is also sufficient when employing a first order sub-polynomial. These facts are proved in Section V. The sub-polynomials are dictated by the coefficients in the last column of a transformed matrix \( C^{-1}AC \) which happen to be the coefficients of the open loop characteristic polynomial as proved in Section IV. A particular similarity transformation is therefore derived to facilitate our analysis.

The case of repeated and complex eigenvalue assignment pose no problem within the classical Ackermann’s method. The same applies with the hybridized methods. The case of uncontrollable eigenvalue assignment enabled by IEA requires no special treatment, and is shown to even simplify computations in terms of reduced complexity and reduced dimensionality. However, knowledge of the actual values of the uncontrollable eigenvalues as required by other methods [9-10] is not required by ours. Furthermore, as an added advantage, a simplified feedback matrix results. The number of terms and the highest power of \( A \) are now reduced adding to the numerical attractiveness of the proposed methods.

Finally, we recapitulate the following contributions brought about by our research.

- The IEA approach enabled the inception of two new assignment methods. Both accrediting hybridization to the modified Ackermann’s method.
- Establishing a necessary condition concerning stability when using the truncated method.
- Relaxing the condition concerning closed loop stability when the open loop system is unstable.
- Resolving a limitation posed in [1] regarding the inability of \( N^eAN \) to assign more than a single eigenvalue. Furthermore, it is shown that \( N^eAN \) even encompasses all of the remaining \( n-q \) eigenvalues.
- Exposing the role played by \( N^eAN \) when \( N=[A^0b\ A^1b\ \cdots\ A^n b] \) in simplifying and in reducing complexity of the controllers when using the truncated method.
Demonstrating the spectral method ability to provide more control over the entire spectrum beyond ordinary eigenvalue assignment.

Knowledge of the number and values of the uncontrollable eigenvalues is optional. Assignment can be carried out regardless. This is not the case with most methods of eigenvalue assignment.

II. PRELIMINARY SETTINGS FOR THE HYBRIDIZED METHODS

The system considered is a linear time invariant system.

\[ x = Ax + Bu \quad \text{ where } x \in \mathbb{R}^n, u \in \mathbb{R}^q \]  

Since matrix \( B \) has full rank \( 1 \) it is routinely replaced by \( b \). The state feedback controller used is

\[ u = -Kx \]  

where \( K \) is a \( 1 \times n \) row matrix yielding the closed loop system.

\[ x = (A-bK)x = A_Kx \]  

Assuming that \( \gamma_i : i = 1,2,\ldots,n \) are the coefficients of the closed loop characteristic polynomial (equation), the classical Ackermann’s formula \([2-5]\) calculates \( K \) as

\[ K = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}^{-1} \begin{bmatrix} A^n + \gamma_1 A^{n-1} + \gamma_2 A^{n-2} + \cdots + \gamma_n I_n \end{bmatrix} \]  

The method has been revisited in \([1]\), where the author derived an alternate compact form of (4) given by

\[ K = (A^{n-1}b)^g (A^n + \gamma_1 A^{n-1} + \gamma_2 A^{n-2} + \cdots + \gamma_n I_n) \]  

Where \((.)^g\) stands for a specialized left inverse of a matrix.

A modification to the controllability matrix in \([1]\) led to the introduction of what has been termed IEA, which explicitly assign \( q < n \) eigenvalues and implicitly assign \( n-q \) eigenvalues. The introduction of IEA facilitated additional enrichment to (5) culminating in a generalization of the method to uncontrollable systems.

The controllability matrix and later the modified controllability matrix(MCM) play central roles in our study.

The controllability matrix \( C \) is well known; given by

\[ C = \begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-2}b & A^{n-1}b \end{bmatrix} \]  

The following partitioning of the inverse of \( C \) is most convenient.

\[ C^{-1} = \begin{bmatrix} b^g \\
(\gamma_1)^g \\
(\gamma_2)^g \\
\vdots \\
(A^{n-1}b)^g \\
N^g \end{bmatrix} \]  

The fact that \( C^{-1}C = I_n \) necessitates

\[ (A^i b)^g (A^j b) = 0 \quad i \neq j \quad i,j = 0,1,2,\ldots,n-1 \]  

while \( CC^{-1} = I_n \) necessitates

\[ \sum_{i=0}^{n-1} (A^i b)(A^i b)^g = I_n \]  

Each row matrix with the superscript \( g \) is a unique generalized inverse of a column of the \( C \) matrix, \([11-13]\). The generalized inverses are unique in our case since they satisfy the additional conditions given in (8) and (9).

To cater for IEA and assignment of uncontrollable systems, the controllability matrix was modified in \([1]\) and justifiably given the name MCM. In our study, MCM is denoted by a matrix \( E \) given by

\[ E = \begin{bmatrix} b & Ab & \cdots & A^{q-1}b & \vdots & N \end{bmatrix} \]  

\[ = \begin{bmatrix} M & \vdots & N \end{bmatrix} \]  

Where \( N \) is \( n \times n-q \) matrix ensuring the nonsingularity of \( E \) together with other terms and conditions on its selection, which will be uncovered later. Consequently, the inverse of \( E \) assumes the following form shown in (11) together with the adjoining conditions listed below:

\[ E^{-1} = \begin{bmatrix} M_g \\
N_g \end{bmatrix} = \begin{bmatrix} b^g \\
\gamma_1^g \\
\gamma_2^g \\
\vdots \\
(A^{q-1}b)^g \\
N^g \end{bmatrix} \]  

and \( E^{-1}E = I_n \) necessitate

\[ (A^i b)^g (A^j b) = 0 \quad i \neq j \]

\[ = 1 \quad i = j \]

\[ i,j = 0,1,\ldots,q-1 \Rightarrow M_g M = I_q \]

\[ (A^i b)^g N = 0_{n \times q-a} \Rightarrow M_g N = 0_{q \times q-a} \]

\[ N^g (A^i b) = 0_{n-q \times 1} \Rightarrow N^g M = 0_{n-q \times q} \]

\[ N^g N = I_{n-q} \]
and \( E E^{-1} = I_n \) necessitate
\[
M^s M + NN^s = \left( \sum_{i=0}^{q-1} (A^s b)(A^s b)^s \right) + N N^s = I_n \tag{13}
\]
Conditions (12) and (13) will be referred to quite frequently. As proved in [1], such modification to \( C \) ending as \( E \) enabled a new matrix \( K_q \), which by construct, guarantees assignment of \( q \) eigenvalues, \( \{ \lambda_1, \lambda_2, \ldots, \lambda_{q-1}, \lambda_q \} \); being the roots of a monic polynomial of coefficients \( v_1, v_2, \ldots, v_{q-1}, v_q \), i.e.
\[
\psi_q(\lambda) = \prod_{i=1}^{q} (\lambda - \lambda_i) = \lambda^q + v_1 \lambda^{q-1} + \cdots + v_q \lambda + v_q \tag{14}
\]
As shown in [1],
\[
K_q = \begin{bmatrix} 0_{n \times (n-q)} & b & A^s b & \ldots & A^s b \end{bmatrix}^{-1} \times \begin{bmatrix} A^s + v_1 A^{s-1} + \cdots + v_q A + v_q I_n \end{bmatrix}
\]
\[
K_q = (A^s b)^s (A^s + v_1 A^{s-1} + \cdots + v_q A + v_q I_n)
\tag{15}
\]
Where the 1 in the first row matrix is positioned at the \( q \)th column. The remaining \( n-q \) eigenvalues are implicitly assigned and are dependent on \( N \). See example 1, where \( q = 1 \) and \( N \) is an arbitrary vector. Methods for selecting appropriate \( N \) is the main theme of this paper as stated in Section III and Section IV.

III. THE SPECTRAL METHOD

The classical Ackermann’s method offers no control over the system spectrum other than changing the eigenvalues. Association between eigenvalues and eigenvectors exist in the sense that knowledge of one leads to knowledge of the second. Therefore, one may think of the columns of \( N \) as a kind of eigenvectors resulting in assignment of the associated eigenvalue. In which case, we have a blending of two approaches to eigenvalue assignment; one explicit through the \( q \)-phase and another implicit through the \( N \)-phase. A type of hybridization we shall call the spectral method.

The spectral method is justifiable whenever more control on the structural properties of the closed loop system beyond eigenvalue assignment are desired, and whenever control over the entire spectrum beyond stabilizability is required as demonstrated in example 3.

It has the advantage of knowing the closed loop eigenvectors prior to the calculation of \( K \). When used with uncontrollable systems, the freedom in choosing the eigenvectors is even broadened resulting in diversified choices of \( K \). By and large, the spectral method provides additional freedom utilized in shaping the transient and the steady state response. See example 3.

To develop the spectral method we proceed as follows.

Referring to (3), using the following state transformation
\[
x = P \bar{x},
\]
\[
\hat{x} = P^{-1} A_k P \bar{x} = \bar{A}_k \bar{x}
\tag{16}
\]
Let,
\[
\begin{align*}
\bar{A}_k &= P^{-1} A_k P = M^s N^s \\
&= \begin{bmatrix} M^s A_k M & M^s A_k N \\
N^s A_k M & N^s A_k N \end{bmatrix}
\end{align*}
\tag{19}
\]

The essence of the spectral method is to assign the remaining \( n-q \) eigenvalues through a specific choice of \( N^s A_k N \), which will be shown later to equal \( N^s AN \). As proved in [1], the \( q \)-phase ensures assignment of \( q \) eigenvalues using (15). What remains now is to select an appropriate \( N \) matrix to ensure assignment of the rest \( n-q \) eigenvalues.

Utilizing properties of block diagonal matrices [14-18], the spectral method is proved if we manage to make \( \bar{A}_k \) as in (19) assume a form where either \( M^s A_k N \) or/and \( N^s A_k M \) are zero matrices. By this property, the eigenvalues of a matrix are separable specified independently by the two diagonal blocks of \( \bar{A}_k \) [14-18]. Note that due to controllability of the system, \( N^s A_k M \) can never be a zero matrix. Therefore, we are left with the only remaining alternative of making \( M^s A_k N \) a zero matrix, i.e., we aim to have \( \bar{A}_k \) assume the following block decomposition.
\[ \bar{A}_K = \begin{bmatrix} M^a A_k M & 0_{q^w - q} \\ N^a A_k M & N^a A_k N \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 1 \\ 0_{q^w - q} \end{bmatrix} = e_1 \] (20)

With such partitioning, \( N^a A_k N \) now dictates the remaining \( n-q \) eigenvalues. Besides, \( M^a A_k M \) dictates the \( q \) eigenvalues assigned explicitly using \( K \).

As proved in [1], by construct, \( K \) assigns \( q \) eigenvalues through the use of (15). Let \( \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \) be the remaining \( n-q \) eigenvalues yet to be assigned and let each vector \( v_i, i = q+1, q+2, \ldots, n \) be determined individually to satisfy

\[ (A-bK)v_i = \lambda_i v_i \] (21)

Or, for repeated eigenvalue assignment.

\[ (A-bK)v_{i+1} = \lambda_i v_{i+1} + v_i \] (22)

To facilitate solutions for \( v_i, i = q+1, q+2, \ldots, n \), each equation is rearranged as

\[ (A-\lambda_i I_n)v_i = b Kv_i \] (23)

Or, as

\[ (A-\lambda_i I_n)v_{i+1} = b Kv_{i+1} + v_i \] (24)

Since \( K \) is not yet finalized, we set \( K v_i = \zeta_i \), where generally, \( \zeta_i \) is any scalar that makes the equations consistent and results in nonzero solutions for \( v_i \). It is worth mentioning that to guarantee consistency of some specific systems, we set \( \zeta_1 = 0 \).

Once solutions \( v_i \) are obtained, \( N \) as in (18) is thus determined as.

Collectively, and based on (23) and (24), we now have

\[ (A-bK) \begin{bmatrix} v_{q+1} & v_{q+2} & \cdots & v_{n-1} & v_n \end{bmatrix} = N \Lambda_{n-q} \] (25)

Where \( \Lambda_{n-q} \) is a Jordan block specifying \( \lambda_1, \lambda_2, \ldots, \lambda_{n-q}, \lambda_n \). In a compact form,

\[ A_k N = N \Lambda_{n-q} \] (26)

Using the decompositions as in (10)-(12), together with \( E^{-1} E = I_n \) yields

\[ \begin{bmatrix} M^a M & M^a N \\ N^a M & N^a N \end{bmatrix} = I_n = \begin{bmatrix} I_q & 0_{q^w - q} \\ 0_{n-q} & I_{n-q} \end{bmatrix} \] (27)

Using (26) together with \( N^a N = I_{n-q} \), as in (12), we get

\[ N^a A_k N = \Lambda_{n-q} \] (28)

Besides, \( M^a N = 0_{q^w - q} \) as in (12) gives

\[ M^a A_k N = M^a (A-bK)N = M^a N \Lambda_{n-q} = 0_{q^w - q} \] (29)

Therefore, (19) is now

\[ \bar{A}_K = \begin{bmatrix} M^a A_k M & 0_{q^w - q} \\ N^a A_k M & N^a A_k N - \Lambda_{n-q} \end{bmatrix} \] (30)

Furthermore, using (12) where \( N^a M = 0_{n-q} \) yields

\[ N^a b = 0_{n-q} \] in particular, which results in

\[ N^a (A-bK)N = N^a AN - N^a A_b K N = N^a AN \] (31)

Hence,

\[ \bar{A}_K = \begin{bmatrix} M^a A_k M & 0_{q^w - q} \\ N^a A_k M & N^a AN - \Lambda_{n-q} \end{bmatrix} \] (32)

The remaining \( n-q \) eigenvalues are thus identified as those of \( \Lambda_{n-q} \), which are the same as those of \( N^a A_k N \), which equals \( N^a AN \). Furthermore, The \( q \) eigenvalues assigned using (15) are those of \( M^a A_k M \).

IV. THE TRUNCATED METHOD

In principle, other blends involving the \( q \)-phase and \( N \)-phase are possible. The difficulty is in selecting an \( N \) matrix that results in at least a stabilized closed loop system. Doing that by trial and error is fruitless and unjustifiable. However, if the open loop system is known to be stable, or exhibiting what may be called deferred instability, then another design method is conceivable. Once more, such approach is a type of hybridization we shall call the truncated method. Such property can be exploited to obtain simplified controllers, proves to be credible, can be systemized, and authenticated.
The truncated method is based on the IEA method where $q$ eigenvalues are explicitly specified and $n-q$ eigenvalues are the roots of a truncated polynomial extracted out of the open loop system characteristic polynomial.

The Truncated method is justifiable whenever simplicity of design is desired, stabilizability is tolerable, and an open loop system characteristic polynomial can result in closed loop stability. Furthermore, utilization of unstable open loop systems is possible to a certain extent. However, the truncated method cannot be used if the trace of $A$ is positive. Such case always results in an unstable closed loop system. All notions above are well demonstrated in example 2.

To develop the proposed method, we need to expose the system structure through a suitable similarity transformation $T$ followed by relevant stability study.

The similarity transformation $T$ is worked out as follows. Let $x = T \bar{x}$, giving $\bar{A} = T^{-1}AT$, and $\bar{b} = T^{-1}b$. (33)

Besides, referring to (4), and letting

$$P_k(A) = [A^n + \gamma_1 A^{n-1} + \gamma_2 A^{n-2} + \ldots + \gamma_n I_n]$$

$$K = [0 \ \cdots \ 0 \ 1] C^{-1} P_k(A)$$

Where $C$ is the controllability matrix as in (6), [4],[18].

Equivalently, using $\bar{A} = T^{-1}AT$,

$$K = [0 \ \cdots \ 0 \ 1] C^{-1} T T^{-1} P_k(A) T T^{-1}$$

$$K = [0 \ \cdots \ 0 \ 1] C^{-1} T P_k(T^{-1}AT) T^{-1}$$

$$K = [0 \ \cdots \ 0 \ 1] (T^{-1} C T) \bar{A} T$$

(36)

To get the most simple form for (36), let $T^{-1} C = I_n$; resulting in $T = C$, i.e., $T$ is the controllability matrix as in (6). In which case,

$$\bar{b} = C^{-1}b = e_1 = [1 \ 0 \ 0 \ \cdots \ 0]^T$$

And,

$$K = [0 \ \cdots \ 0 \ 1] P_k(\bar{A}) C^{-1} = \bar{K} C^{-1}$$

Hence,

$$\bar{K} = [0 \ \cdots \ 0 \ 1] P_k(\bar{A})$$

(39)

Basically, (39) is the last row of $P_k(\bar{A})$. To expose the structure of $\bar{A}$ let,

$$C = [b \ \cdots \ A^{n-1}b \ A^n b \ \cdots \ A^{n-1}b]$$

$$=[M \ | \ N] = T$$

$$\Rightarrow C^{-1} = T^{-1} = \begin{bmatrix} M^s & (A^{n-1}b)^s \\ N^s & (A^n b)^s \end{bmatrix}$$

(41)

Resulting in,

$$\bar{A} = T^{-1}AT = \begin{bmatrix} M^s AM & M^s AN \\ N^s AM & N^s AN \end{bmatrix}$$

(42)

If the open loop characteristic equation is given by,

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_{n-1} \lambda + a_n = 0$$

(43)

Invoking the Cayley-Hamilton theorem [4][18]

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_{n-1} A + a_n I_n = 0$$

(44)

It can be shown that (see Appendix).

$$\bar{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \ ; \ \bar{b} = \bar{e}_1$$

(45)

That is, $\bar{A}$ boils down to what is known by some authors as the Frobenius companion form [19].

In which case, since $\bar{b} = e_1$ and letting

$$\bar{K} = \begin{bmatrix} \bar{k}_1 & \bar{k}_2 & \cdots & \bar{k}_{n-1} & \bar{k}_n \end{bmatrix}$$

(46)

$$\bar{A}_K = \bar{A} - \bar{b}\bar{K}$$ exhibits the following form.
A linear time-invariant system is asymptotically stable if and only if matrix $A$ is Hurwitz. A more popular approach is that of Routh’s which is based on the coefficients of the characteristic polynomial. A system has all poles in the open left half plane if and only if all first-column elements of the Routh’s array have the same sign.

Alongside Routh’s method, [20],[21] give simple ratio checking inequalities that determine stability of a system. A necessary condition for stability is that all the coefficients of the characteristic polynomial have the same sign.

Given a system characteristic equation

$$\Delta_n(\lambda) = a_0s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n = 0 \quad (51)$$

Assume all $a_i$ have the same sign. This condition is necessary. Otherwise, the system is unstable since a change in sign in the characteristic polynomial renders the system unstable as established in control theory.

So, to establish stability, the following inequalities regarding the ratios of the coefficients of $\Delta_n(\lambda)$ should be satisfied [20],[21].

$$\frac{a_1}{a_0} > \frac{a_2}{a_1} > \cdots > \frac{a_{n-1}}{a_n} \quad (52)$$

The inequalities given in (52) are easier to check in comparison with the matrix-minor calculations required when applying the conventional Routh-Hurwitz’s methods.

Based on the inequalities given in (52), we now claim the following assertion:

If an open loop system is known to be stable and hence, satisfying (52), then any subsystem having a truncated characteristic equation of order $p$ where $p \leq n$ and,

$$\Delta_p(\lambda) = a_0s^p + a_1s^{p-1} + \cdots + a_{p-1}s + a_p = 0 \quad (53)$$

will be stable.

The coefficients $a_i$ are those of $\Delta_n(\lambda)$ as in (52).

The proof is straightforward. It follows from (52). Since the system is stable then (52) is satisfied.

Since fewer terms as dictated by (51) are involved, satisfaction of the inequalities in (52) is preserved. Hence, the subsystem as given by its characteristic polynomial as in (50) will be stable.
However, the roots of the subsystem given by (50) are generally different from those of the original open loop system given by (51).

Such assertion is needed when using the Truncated method, which centers on the involvement of truncated polynomials extracted out of the open loop characteristic polynomial.

In utilization of the above facts, to settle down on a proper truncated polynomial one must observe the following:

- If a system is open loop stable then every extracted truncated characteristic polynomial employed in the truncated method leads to a stabilized closed loop system.
- If the open loop system is unstable then stabilizability is still possible when employing particular truncated polynomial of certain orders. In this case, vigilance is needed to involve as many terms needed in the truncated polynomial as to ensure stability. That is, consider as many terms as the conditions in (52) permit.

When deciding to use the truncated method, if the trace of \( A \) is positive, the truncated method cannot be used at all as the open loop system will be unstable and the closed loop system will be unstable as well no matter what truncated polynomial is used. This is due to a change in sign in the first two highest powers of the truncated polynomial used. However, an open loop system can be unstable, but a stabilized closed loop system is possible as long as the trace of \( A \) is negative (hence, \( a_1 > 0 \)) and a truncated open loop characteristic polynomial results in a stabilized system according to (52). In this case, one can involve as many terms as those abiding (52) as demonstrated in example 2.

Conveniently, the coefficients \( a_i \) of the characteristic polynomial of \( A \) are given by the terms in the last column of any \( W^{-1}AW \) matrix where \( W \) has the form.

\[
W = \begin{bmatrix} w & A^2w & \cdots & A^{n-1}w \end{bmatrix}
\]

(54)

\( w \) can be any convenient column-vector ensuring a nonsingular \( W \). Obviously, it may be the system input matrix \( b \) whenever the system is controllable.

However, a more efficient method is to use the iterative Faddeeva-Leverrier algorithm [22-23]. It is an efficient method for finding the coefficients of the characteristic polynomials. Furthermore, if \( A \) is nonsingular, an additional advantage is that the inverse of \( A \) is readily obtainable at no extra computational cost.

The decomposition given in (42) can lead to another structural property of the closed loop system as represented in (42). That is, the sum of the \( q \) eigenvalues assigned through the \( q \)-phase are given by the trace of \( M^sA_M \) as proved below.

Let \( tr \) stands for trace of a matrix. Hence, using (13),

\[
tr(M^sA_M) = tr(MM^sA_M) = tr(MM^s(A-bK)) = tr(I_n - NN^s)(A-bK))
\]

\[
= tr(A-bK - NN^sA + N \lambda N^s b K)
\]

\[
= tr(A-bK - N^sA)
\]

\[
= tr((A-bK)) - tr(N^sAN)
\]

Referring to (32) and (41) in [1],

\[
tr(M^sA_M) = \sum_{i=1}^{q} \lambda_i + \sum_{j=q+1}^{n} \lambda_j - \sum_{j=q+1}^{n} \lambda_j
\]

\[
= \sum_{i=1}^{q} \lambda_i
\]

(55)

i.e., \( tr(M^sA_M) \) and hence, \( tr(M^s\bar{A}_M) \) give the sum of the \( q \) eigenvalues assigned explicitly by \( K \) or by \( \bar{K} \). Note that neither of the two matrices give the \( q \) eigenvalues except in certain special cases as when using the spectral method as verified in (32).

**VI. UNCONTROLLABLE SYSTEMS**

Dealing with uncontrollable systems using Ackermann's method was made possible in [1] owing to a modification to the controllability matrix. The problem of uncontrollability was resolved by replacing the columns causing a singular controllability matrix \( C \) by columns of another matrix \( \bar{N} \) to establish nonsingularity of the ensuing MCM.

As asserted in [1], there is no need to determine the uncontrollable eigenvalues (considered an advantage); only their number say \( v \) is needed in order to arrive at the dimension of \( \bar{N} \), which has to be \( v \). The columns of \( \bar{N} \) can be arbitrary; subject to ensuring a nonsingular MCM. Note that other available software methods like MATLAB place command [10] necessitates knowledge of the
uncontrollable eigenvalues, otherwise, an error message is issued.

The feedback matrix used is $K_u$ given by

$$K_u = (A^{\kappa b})^T [A^\kappa + \sigma_1 A^{\kappa-1} + \ldots + \sigma_{n-1} A + \sigma_n I_n]$$ (56)

Where $\kappa \leq n$ is the number of controllable eigenvalues. The remaining uncontrollable eigenvalues are thus inescapably re-assigned. Bear in mind that $(A^{\kappa b})^T$ is not unique. It depends on the $N$ used, which has to satisfy the conditions in (12).

The non-uniqueness of $N$ leads to a non-unique $K_u$. This is fully justified in control theory as any feedback matrix for uncontrollable systems is not unique even for single-input systems and even when using alternative approaches within the same method as in the IEA method. Depending on the choice of $N$, a different $K_u$ is obtained. This fact deserves the following warrant: you cannot validate the $K_u$ computed as in (56) using another different method such as MATLAB place command [10]. One better check validity of design by calculating the eigenvalues of $A - bK_u$.

The non-uniqueness of $K_u$ can be considered a design parameter fulfilling other system requirements such as minimizing energy, reducing certain norms, and modifying eigenvectors. Besides, it is possible to avoid feeding back certain states. This is made possible by settling on a $K_u$ with zero or almost zero coefficients.

Caution: The number of eigenvalues assigned by $K_u$ cannot exceed $n - \nu$. If tried inadvertently, the $K$ obtained will result in incorrect assignment of the controllable eigenvalues.

### VII. EXAMPLES

**Example 1**

An unstable uncontrollable system has the following system matrices with a nonzero initial condition.

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ (57)

It is required to assign $-8$ and inescapably the uncontrollable eigenvalue $-2$. The method used is IEA with different $N$ matrices, see (10).

Since the system is uncontrollable there are many candidate members for $N$ such as $N_1 = [1 \ 1]^T$, $N_2 = [2 \ -1]^T$, and $N_3 = [0 \ 1]^T$.

They are arbitrarily chosen, but ensuring a nonsingular MCM. Each $N_i$ results in $-8$ and $-2$ as closed loop eigenvalues. For each $N_i$, a $K_i$ is given by (15).

$$K_i = [1 \ 0][b \ N_i]^{-1} (A + 8I_2) = b^T (A + 8I_2)$$ (58)

Note that each $b^T$ associated with a $K_i$ is different. It satisfies $b^T b = 1$, but implicitly depends on $N_i$ through $b^T N_i = 0$ as follows from (12).

MATLAB was used to calculate $K$ for the three cases of $N$ getting.

$$K_1 = [-2 \ 5], K_2 = [13 \ 26]/7, K_3 = [7 \ 2]$$ (59)

To get and plot the time response, the initial function of MATLAB was used [4],[24]. The closed loop time response of $x_i(t)$ using (3) is plotted and is shown in Fig. 1. Each curve relates to that particular value of $K_i$ as in (59) calculated depending on the $N_i$ used.

![FIGURE 1. Time response of an uncontrollable system resulting from different feedback matrices $K_1$, $K_2$, and $K_3$; all assigning the same eigenvalue.](image-url)
therefore, one may opt for \( N_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \) when speed is his/her main concern.

In other words, \( N \) can be a design tuning parameter, not only in shifting the eigenvalues assigned, but also in shaping the response through their influence on the closed loop eigenvectors.

Example 2

An unstable controllable system has the following system matrices with a nonzero initial condition,

\[
A = \begin{bmatrix} -5.5 & 3 & 3 \\ -6 & 2.5 & 4 \\ 0 & 1 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

The truncated method is to be used. Referring to the appendix, using \( w = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \), the characteristic polynomial is given by the coefficients in the last column of \( W^{-1}AW \), where

\[
W^{-1}AW = \begin{bmatrix} 0 & 0 & 1.875 \\ 1 & 0 & -1.75 \\ 0 & 1 & -3.5 \end{bmatrix}
\]

Hence, the system has the following characteristic polynomial

\[
\Delta(\lambda) = \lambda^3 + 3.5\lambda^2 + 1.75\lambda - 1.875
\]

\( \Delta(\lambda) \) confirms the instability of the system due to the change in sign within the coefficients of \( \Delta(\lambda) \). It will be used to extract the sub-polynomials needed by the truncated method. Closed loop stability is still possible as stated in Section V and demonstrated below.

Such claim is validated by assigning a single eigenvalue, say \(-3\); leaving the remaining two eigenvalues dictated by a second order truncated polynomial.

Hence, for \( K = K_1 \), let \( N = \begin{bmatrix} A^2b & A^2b \end{bmatrix} \), and using format rat of MATLAB to get rational number representation for exact determination of \( K_1 \) using (15).

\[
K_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b & N \end{bmatrix}^{-1} (A + 3I_3) = b^T (A + 3I_3)
= \begin{bmatrix} -47/60 & 313/480 & 119/240 \end{bmatrix}
\]

\( K_1 \) results in assigning \(-3\) and two eigenvalues given by the roots of the truncated polynomial \( \Delta_1(\lambda) = \lambda^2 + 3.5\lambda + 1.75 \) which are \(-2.8956\) and \(-0.6044\).

An alternative approach is to assign two eigenvalues say \(-3\) and \(-3\); leaving the third one dictated by the only possibility of using a first order truncated polynomial.

For \( K = K_2, \ N = \begin{bmatrix} A^2b \end{bmatrix} \), using (15), one gets

\[
K_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b & Ab & N \end{bmatrix}^{-1} (A + 3I_3)^2 = (Ab)^T (A + 3I_3)^2
= \begin{bmatrix} -97/360 & 113/180 & 361/360 \end{bmatrix}
\]

\( K_2 \) results in assigning \(-3\), \(-3\) and \(-3.5\); being the root of the first order truncated polynomial \( \Delta_2(\lambda) = \lambda + 3.5 \).

MATLAB was used to calculate \( K \) for the two cases of \( N \). To get and plot the time response, the initial function of MATLAB was used. The closed loop time responses of \( x_1(t) \) are plotted and shown in Fig. 2. Each curve relates to that particular \( K \) calculated depending on the \( N \) used. No attempt to plot the unstable open loop system on the same axes as it will swamp the other two responses.

![FIGURE 2. Time response of a controllable system due to two feedback matrices \( K_1 \) and \( K_2 \) using the Truncated method.](image-url)

Note that two stable closed loop systems are possible even though the characteristic polynomial of the unstable open loop system was used as commented in Section IV and Section V. The simulations show \( K_2 \) results in a faster closed loop system than that resulting from the use of \( K_1 \) though exhibiting an appreciable undershoot.
The uncontrollable eigenvalue is \(-1\). Using the spectral method, let’s assign the following eigenvalues: \(-2 \pm i\), \(-4\), as well as the uncontrollable eigenvalue \(-1\).

To avoid complex number manipulation, we assign \(-4\) and \(-1\) through the \(N\)-phase by means of a suitable \(N\). This necessitates a determination of two vectors \(v_1\) and \(v_2\).

Since \(-4\) is not an eigenvalue of \(A\), \(A + 4I\) is nonsingular, and \(v_1\) will in this case be unique within a scalar multiplier \(\xi\), say \(\xi = 1\). Using (23),

\[
(A + 4I_4)v_1 = b \times 1 \Rightarrow v_1 = (A + 4I_4)^{-1}b
\]

\[
\Rightarrow v_1 = \begin{bmatrix} -3 & 17 & -1 & 2.6 \end{bmatrix}^T/8 \tag{66}
\]

For \(v_2\), when using (23) and since \(-1\) is an eigenvalue of \(A\), \(A + I\) is singular. Thus, we either let \(\xi = 0\), in which case,

\[
(A + I_4)v_2 = b \times 0
\]

\[
\Rightarrow v_2 = \text{null}(A + I_4) = \begin{bmatrix} 0 & r & 0 & 0 \end{bmatrix}^T \tag{67}
\]

Where \(r\) is an arbitrary scalar.

Or, let \(\xi\) be nonzero, say \(\xi = 1\). In which case,

\[
(A + I_4)v_2 = b \times \xi \Rightarrow (A + I_4)v_2 = b \tag{68}
\]

Recalling that \(A + I\) is singular and the system is consistent. A systemic non-unique solution for (68) can be obtained as follows.\([18],[25]\]

\[
(A + I_4)v_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 5 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 2 \end{bmatrix}, v_2 = b = \begin{bmatrix} -1 \\ 5 \\ 0 \\ 1 \end{bmatrix} \tag{69}
\]

\[
\begin{bmatrix} 0 & -1 & 0 \\ 2 & 5 & 0 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} v_{21} \\ v_{23} \\ v_{24} \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} v_{21} \\ v_{23} \\ v_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -0.5 \end{bmatrix} \tag{70}
\]

The remaining solution \(v_{22}\) is an arbitrary scalar, say \(v_{22} = r\). Thus,

\[
v_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ r \\ 1 \\ -0.5 \end{bmatrix} \tag{71}
\]

Using (15) with \(q = 2\) and \(N = [v_1 \ v_2]\) the spectral method gives \(K_2\) as

\[
K_2 = \frac{[0 \ 1 \ 0 \ 0][b \ Ab \ v_1 \ v_2]^{-1}(A^2 + 4A + 5I_4)}{(Ab)^r (A^2 + 4A + 5I_4)} \tag{72}
\]

Hence, according to (66) and (67), a first option for the feedback matrix \(K = K_1\) is

\[
K_1 = \begin{bmatrix} 29 & 0 & 19 & 50 \end{bmatrix}/3 \tag{73}
\]

According to (66) and (71) with \(r = 1\), a second option for the feedback matrix \(K = K_2\) is

\[
K_2 = \begin{bmatrix} 44 & 3 & 25 & 50 \end{bmatrix}/3 \tag{74}
\]

Since the system is uncontrollable a third \(v_3\) vector named \(v_{2r}\) may be opted, which can be totally arbitrary yielding a third \(K = K_3\). The justification is that: an uncontrollable eigenvalue is always reassigned irrespective of the \(K\) matrix used. Such fact renders any arbitrary \(v_{2r}\) a viable candidate. Strikingly, as the simulation shows in Fig. 3., the best time response is obtained when using \(K = K_3\) where, to five significant digits.

\[
K_3 = [0.7015 \ -1.7930 \ 2.7473 \ 16.667] \tag{75}
\]

We have arrived at this value of \(K_3\) after almost a hundred-run on MATLAB; randomizing the four
components of $v_{2r}$ at each run. On every run we judged $x_i(t)$ for small overshoot, small undershoot, and satisfactory settling time.

For the sake of verifying $K_3$ as in (74), $N = \begin{bmatrix} v_1 & v_{2r} \end{bmatrix}$. The random vector $v_{2r}$ used was.

$$v_{2r} = \begin{bmatrix} 0.2348 & 0.5286 & 0.0514 & 0.7569 \end{bmatrix}^T \quad (76)$$

The study has shown that uncontrollable systems require no unnecessary special approaches and that they can simplify design and calculations to the full. The three graded numerical examples worked-out and simulated clarify the designs presented.

Lastly, the classical Ackermann's method can only be used when the system is controllable. When stabilizability and ease of design is sought, the truncated hybridized method is recommended. The spectral hybridized method should be used whenever more control on the entire spectrum and the shape of response are required.

Future work may tackle the problem of adapting the classical Ackermann's method and the hybridized methods to the eigenvalue assignment of controllable and uncontrollable multi-input systems. Furthermore, exploration of other matrix configurations and forms may result in even more simplified forms concerning the truncated and the spectral methods.

**APPENDIX.**

Given $\begin{bmatrix} b & Ab & \cdots & A^{n-2}b & A^{n-1}b \end{bmatrix}$ as in (6) and,

$$C^{-1} = \begin{bmatrix} b^\delta \\ (Ab)^\delta \\ \vdots \\ (A^{n-2}b)^\delta \\ (A^{n-1}b)^\delta \end{bmatrix} \quad (A1)$$

$$C^{-1}AC = \begin{bmatrix} b^\delta Ab & \cdots & b^\delta A^{n-2}b & b^\delta A^{n-1}b \\ (Ab)^\delta Ab & \cdots & (Ab)^\delta A^{n-2}b & (Ab)^\delta A^{n-1}b \\ \vdots & \ddots & \vdots & \vdots \\ (A^{n-2}b)^\delta Ab & \cdots & (A^{n-2}b)^\delta A^{n-2}b & (A^{n-2}b)^\delta A^{n-1}b \end{bmatrix} \quad (A2)$$

Using (8),

$$C^{-1}AC = \begin{bmatrix} 0 & \cdots & 0 & b^\delta A^{n-1}b \\ 1 & \cdots & 0 & (Ab)^\delta A^{n-1}b \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & (A^{n-1}b)^\delta A^{n}b \end{bmatrix} \quad (A3)$$

To get each term $(A^i b)^\delta A^j b$, $i = 0, 1, \cdots, n-1$, in the last column, use the Cayley-Hamilton theorem applied to the characteristic equation

$$\Delta(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \quad (A4)$$

\[ \text{FIGURE 3. Time response of an uncontrollable fourth order system using the spectral method and three values for } K. \]
\( \Delta(A) = A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI_n = 0_{n,n} \) \hspace{1cm} (A5)

Repeatedly premultiplying \((A')^i\), \(i = 0, 1, \cdots, n-1\), postmultiplying it by \(b\) and observing \((8)\), one gets

\[
C^{-1} A C = \begin{bmatrix}
0 & \cdots & 0 & -a_n \\
1 & \cdots & 0 & -a_{n-1} \\
\vdots & & \ddots & \vdots \\
0 & \cdots & 1 & -a_1
\end{bmatrix}
\]

\hspace{1cm} (A6)

i.e., the coefficients of the characteristic polynomial are given by the negated last column of \( C^{-1} A C \).

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