Finite-time Stabilization of Switched Positive Systems with Time-varying Delays and Actuator Saturation

Xuefei Wang¹, Jiwei Wen²*

¹School of Internet of Things Engineering, Jiangnan University, Wuxi, 214122, P.R. China
²Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), School of Internet of Things Engineering, Jiangnan University, Wuxi, 214122, P.R. China

*E-mail: wjw8143@aliyun.com

Abstract: In this paper, the finite-time stabilization for a class of switched positive systems with time-varying delays and actuator saturation under average dwell time switching is investigated. Through multiple copositive Lyapunov-Krasovskii functionals method, the sufficient conditions for finite-time bounded of continuous-time case are given, and the appropriate switching rules and state feedback controllers are presented together. Moreover, the convex hull technique is employed to deal with actuator saturation. Finally, an illustrative example is given to show the effectiveness of the proposed method.

1. Introduction

Positive systems are a kind of remarkable systems whose state variables and outputs are always confined in the non-negative orthant for any non-negative initial conditions. Recently, a kind of switched positive linear systems (SPLSs), which are composed of a family of positive subsystems and a switching signal, and the signal control the switching behavior between the subsystems, have been highlighted by many researchers. These systems are widely applied in formation flying control [1], network congestion control [2] and vehicle control [3].

SPLSs inherit many special properties of positive systems, which means that SPLSs combines the variable dynamic characteristics of switched systems and the unique non-negative characteristics of positive systems. As the non-negative characteristics of positive systems, using the classical quadratic Lyapunov function method to study the stability of positive systems will be conservative. As a powerful approach to deal with SPLSs, linear copositive Lyapunov functions (LCLFs) was used frequently in recent years, and it can also reflect the non-negative characteristics of the positive systems [4]. Naturally, the common linear copositive Lyapunov functions method [5] and the multiple linear copositive Lyapunov functions method [4] are used for stability analysis of SPLSs.

Actuator saturation widely appears in practical applications. For example, the input and output voltage and current of the system are limited to a range of certain standard, and the temperature and pressure in the chemical industry production process are limited. On the other hand, actuator can lead to poor performance of control systems and even lead to instability of the control systems. So, it’s necessary to research the actuator saturation phenomenon. In the literature [6], the stabilization and \( L_1 \)-gain performance of switched positive systems with actuator saturation was studied through convex hull technique. The \( L_2 \)-gain analysis of discrete-time switched systems with actuator saturation was
investigated by anti-windup design method in the literature [7].

On the other hand, when it comes to stability, most of the research is about Lyapunov stability or asymptotic stability, which are defined over an infinite time interval. In fact, this concept is totally different with finite-time stability. In many practical applications in some real systems, however, the system variables are required to be bounded in a finite-time interval very precisely. Hence, the concept of finite-time bounded (FTB) is proposed.

The exploration of finite-time control for switched positive systems with actuator saturation is very fair [8]. In the literature [9], through multiple LCLFs methods and convex hull technique, the stabilization of SPLSs with actuator saturation under both state-dependent and time-dependent switching are investigated. In the literature [8], finite-time \( H_{\infty} \) control for discrete-time switched singular time-delay systems subject to actuator saturation was investigated via static output feedback, and the uniformly FTB condition was given by LMIs. In fact, the switching action, coupled with the positivity constraint of state variables, together with the actuator saturation, make the behavior of SPLSs with actuator saturation very complicated. Which means the finite-time control of SPLSs with actuator saturation will be difficult and challenging.

Motivated by the above statement, this paper will focus on the study of finite-time control of SPLSs with both time-varying delays and actuator saturation. The main contributions of this paper are given in the following aspects: (1) Dealing with the FTB problem for switched positive system with both time-varying delays and actuator saturation is the first time; (2) Sufficient conditions of FTB for continuous-time switched positive systems with time-varying delays and actuator saturation are obtained by non-quadratic multiple copositive Lyapunov-Krasovskii functionals method; (3) Realized the joint design of switching laws and state feedback controller under the average dwell time switching to ensure that the closed-loop system is FTB and positive.

The rest of this paper is organized as follows. Some necessary statements and definitions are given in Section 2. In section 3, sufficient conditions for state feedback controllers and switching laws of FTB of switched positive systems with time-varying delays and actuator saturation are proposed. A numerical example is given in section 4. Section 5 is the conclusion of this paper.

2. Problem statements and preliminaries

Given the following switched system with actuator saturation and time-varying delays

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t - d(t)) + F_{\sigma(t)}\text{sat}(u(t)),
\]

\[x(t) = \varphi(t), t \in [-d_M, 0],\]

(1)

where \( x(t) \in \mathbb{R}^n \) denote the system state ; \( u(t) \in \mathbb{R}^m \) denote the control input; \( \sigma(t) : [0, \infty) \rightarrow S = \{1, 2, \cdots, s\} \) is a right continuous piecewise constant function of time, which is called switching signal, and it takes its values in the finite set \( S \), \( s \) means the number of subsystems; Subsystem \( A_i \) is activated at time \( t \) when \( \sigma(t) = i, i \in S \). \( A_i, B_i \) and \( F_i, i \in S \), are known constant matrices , and \( B_i \geq 0 \). \( d(t) \) is the time-varying delay, and \( 0 < d_m \leq d(t) \leq d_M, \dot{d}(t) \leq \rho < 1 \), in this paper, \( d_m, d_M \) and \( \rho \) are constants; \( \varphi(t) \) is a initial function defined in the time interval \([-d_M, 0]\). The sat : \( R^n \rightarrow R^n \) is the standard saturation function defined as sat(u) = [sat(u_1) sat(u_2) \cdots sat(u_m)]^T, where sat(u) = sign(u) min(|u_i|,1) . In this paper, we take the notation by using sat(\cdot) to denote both the scalar valued and the vector valued saturation functions.

In this paper, notation \( A > 0(A \geq 0) \) means all the elements of \( A \) are positive (non-negative).

**Definition 1** [10]: System (1) is said to be positive if for any switching signals \( \sigma(t) \) and any initial state \( \varphi(t) \geq 0, t \in [-d_M, 0] \), and its trajectory \( x(t) \geq 0 \) for all \( t \geq 0 \).

**Definition 2** [11]: A is called a Metzler matrix, if the off-diagonal entries of the matrix \( A \) are non-negative.

**Definition 3** [12]: For a switching signal \( \sigma(t) \) and each \( t_0 \geq t_1 \geq 0 \), let \( N_a(t_1, t_2) \) denote the number of
discontinuities of $\sigma(t)$ in the interval $(t_i, t_{i+1})$. We say that $\sigma(t)$ has an average dwell time $\tau_a$ if there exist two positive numbers $N_0$ (we call $N_0$ the chatter bound here) and $\tau_a$, such that

$$N_0(t_{i+1} - t_i) \leq N_0 + (t_{i+1} - t_i) / \tau_a, \quad \forall t_{i+1} \geq t_i \geq 0,$$

without loss of generality, in this paper we choose $N_0 = 0$.

**Definition 4** [13]: Given positive constants $c_1, c_2$ ($c_1 < c_2$), $T_f$ and any switching signals $\sigma(t)$, if

$$\|x(0)\| \leq c_1 \Rightarrow \|x(t)\| \leq c_2, \quad t \in [0, T_f]$$

then system (1) is said to be FTB with respect to $(c_1, c_2, T_f, \sigma(t))$, where $c_1 = \sup_{t \in [0, T_f]} \|x(t)\|$, $\|y\| = \sum_{i=1}^n |x_i|$, $x_i$ is the $i$th elements of vector $x$.

There are some important notations in the following.

For any vector $v$, we define $\varepsilon(v, 1)$ as

$$\varepsilon(v, 1) = \{x \in \mathbb{R}^n \mid x^T v \leq 1\}.$$

For any matrix $H \in \mathbb{R}^{n \times n}$, we define $L(H)$ as

$$L(H) = \{x \in \mathbb{R}^n \| h_i x \leq 1\},$$

where $h_i$ denotes the $i$th row of $H$, $i = 1, 2, \ldots, m$.

**Lemma 1** [14]: Given two matrices $K, H \in \mathbb{R}^{n \times n}$, $\forall x(t) \in \mathbb{R}^n$, if $x(t) \in L(H)$, then $\text{sat}(Kx) = c_0 \{D_i K x + D_i^H H x, z \in Q\}$, $Q = \{1, 2, \ldots, 2^m\}$, where $c_0$ denotes the convex hull. $D_i$ is $m \times m$ diagonal matrix with elements either 1 or 0 and $D_i^T = I - D_i$. There are $2^m$ possible such matrices, therefore

$$\text{sat}(Kx(t)) = \sum_{i=1}^{2^m} \eta_i (D_i K + D_i^T H) x(t), \quad \sum_{i=1}^{2^m} \eta_i = 1, \quad 0 \leq \eta_i \leq 1$$

$\eta_i$ are related to the system state.

Then, we are going to design a state feedback controller

$$u(t) = K_i x(t),$$

(2)

where $K_i \in \mathbb{R}^{n \times n}$ are gain matrix, $\forall i \in S$, such that the system (1) is not only switched positive system but also FTB.

Applying the state feedback controller (2) to system (1), and if $x(t) \in L(H_i)$, $\forall x(t) \in \mathbb{R}^n$, $\forall i \in S$, we can obtain the closed-loop system as

$$\dot{x}(t) = [A_{i\sigma(t)} + F_{i\sigma(t)} \sum_{z \in S} \eta_z (D_z K_{i\sigma(t)} + D_z^T H_{i\sigma(t)})] x(t) + B_{i\sigma(t)} x(t - d(t)), \quad x(t) = \varphi(t), t \in [-d_M, 0].$$

(3)

**Lemma 2** [14]: Switched system (3) is positive if and only if

$$A_{i\sigma(t)} + F_{i\sigma(t)} \sum_{z \in S} \eta_z (D_z K_{i\sigma(t)} + D_z^T H_{i\sigma(t)})$$

are Metzler matrices, $B_{i\sigma(t)} \succeq 0$, and initial state $\varphi(t) \succeq 0, t \in [-d_M, 0]$ hold for $\sigma(t) \in S$.

3. Main results

In this part, sufficient conditions for finite-time stabilization of continuous-time switched positive system (1) are presented.

**Theorem:** Consider the continuous-time switched system (1), given positive constants $c_1, c_2$ ($c_1 < c_2$), $T_f$, $\lambda$, $\mu \geq 1$, if there exist vectors $v_i \succ 0$ and $g_i \succ 0$, matrices $K_i$ and $H_i$, such that $\varepsilon(v_i, c_2, 1) \in L(H_i)$, $\forall (i \times j) \in S \times S$, and the following inequalities hold
\[
[A + F(T(D_k + D_h))]^T v_i - \lambda v_i + g_i \leq 0, \quad (4)
\]
\[
B_i^T v_i - (1 - \rho) e^{d_i t} g_i \leq 0, \quad (5)
\]
\[
\check{I}_e(A + F(T(D_k + D_h)))^T \geq 0, \quad g \neq l, \quad (6)
\]
\[
v_i \leq \mu v_i, \quad g_i \leq \mu g_i, \quad (7)
\]
where \( I_e = [0 \cdots 0 \ 0 \cdots 0], \quad I_j = [0 \cdots 0 \ 1 \cdots 0], \quad \forall z \in \{1, 2, \cdots, 2^n\}, \) then the continuous-time SPS (1) is FTB with respect to \((c_1, c_2, T, \sigma(t))\) for any switching signal with ADT

\[
\frac{T_j}{\ln \mu} \ln(\phi - \lambda T_j) = r^* \leq r_a, \quad (8)
\]
here, \( \phi = (W_1 + (e^{i\sigma} - 1)W_2 / \lambda)c_i / W_3, \quad c_i = \sup_{t \in [-d_i, 0]} \|x(t)\|, \quad W_1 = \max(v_i(j)), \quad W_2 = \max(g_i(j)), \quad W_3 = \min(v_i(j)), \ j \in \{1, 2, \cdots, n\}, \) and \( v_i(j) \) is the \( j \)-th element of \( v_i. \)

**Proof.** For \( t_0 = 0, \) we denote \( t_1 < t_2 < \cdots < t_3 < t_{n+1} < \cdots < t_n < t_{n+1} \) as the switching times on the interval \([0, T_j]\), which means the \( i \)-th subsystem is activated when \( t \in [t_i, t_{i+1}) \).

As \( e(v_i / c_2, 1) \subseteq L(H_1), \) for every \( x(t) \in e(v_i / c_2, 1), \) we get \( x(t) \in L(H_1). \) According to lemma 1, we can obtain

\[
\text{sat}(K, x(t)) = \sum_{i=1}^{2^n} \eta_i(D_kK_i + D_hH_i)x(t),
\]
then, it is easy to get the closed-loop system

\[
\dot{x}(t) = [A + F(T(D_k + D_h))]x(t) + B_i(x(t - d_i(t))), \quad (9)
\]

The inequality (6) implies that \( A + F(T(D_k + D_h)) \) is Metzler matrix. By lemma 2, system (9) is a switched positive system.

Establish the following multiple copositive Lyapunov-Krasovskii functionals for (9)

\[
V_{\alpha(t)}(t) = x^T(t)v_{\alpha(t)} + \int_{0}^{x(t)} e^{x(T-x)} x^T(s)g_{\alpha(t)}ds.
\]

When \( t \in [t_i, t_{i+1}), \) the derivative of \( V_{\alpha(t)}(t) \) with respect to \( t \) along the trajectory of the closed-loop system (9) leads to

\[
\dot{V}_{\alpha(t)} = \lambda V_{\alpha(t)} + x^T(t)[(A + F(T(D_k + D_h)))]^T v_i + g_i - \lambda v_i + x^T(t - d(t))[B_i^T v_i - (1 - \rho e^{d_i t}) g_i]
\]

\[
\leq \lambda V_{\alpha(t)} + x^T(t)[(A + F(T(D_k + D_h)))]^T v_i + g_i - \lambda v_i + x^T(t - d(t))[B_i^T v_i - (1 - \rho) e^{d_i t} g_i],
\]
then, from (4) and (5), one can obtain

\[
\dot{V}_{\alpha(t)} \leq \lambda V_{\alpha(t)}, \quad (11)
\]
integrating (11) from \( t_i \) to \( t, \) we obtain

\[
V_{\alpha(t)}(t) \leq e^{\lambda(x(t) - t)} \dot{V}_{\alpha(t)}(t), \quad (12)
\]
Then, by (7), we have the following inequality at the switching instant \( t_i \).

\[
V_{\alpha(t)}(t_i) \leq \mu V_{\alpha(t)}(t_i), \quad (13)
\]
Due to (12) and (13), one can obtain
\[ V(t) \leq e^{\lambda(t-t_0)}\|V_i(t_{N_t(0)})\| \leq \mu e^{\lambda(t-t_0)}\|V_i(t_{N_t(0)})\| \leq e^{\lambda(t-t_0)} e^{\lambda(t_{N_t(0)+1})} \|V_j(t_{N_t(0)+1})\| \leq \cdots \]

combining this and definition 3, one has

\[ V(t) \leq \mu^{i/e} e^{\lambda t} V(0). \]  

By (10), one can obtain

\[ V(0) = x^T(0)\sigma_{e(0)} + \int_{-d(0)}^{0} e^{-\lambda s} x^T(s)\sigma_{e(0)} ds \leq x^T(0)\left[ W_1 \cdots W_1 \right]^T + \int_{-d(0)}^{0} e^{-\lambda s} x^T(s)\left[ W_2 \cdots W_2 \right]^T ds \]

\[ \leq W_1 \|x(0)\| + W_2 \sup_{i\in[-d_u,d_u]} \|x(t)\| \int_{-d}^{0} e^{-\lambda s} ds \leq (W_1 + (e^{\lambda d_u} - 1)W_2 / \lambda) \|x(t)\| \]

\[ \leq (W_1 + (e^{\lambda d_u} - 1)W_2 / \lambda)c_1, \]  

from (10) one has

\[ W_2 \|x(t)\| \leq x^T(t)\|v_\sigma(t)\| \leq V_\sigma(t)(t). \]

It follows from (14)-(16) that

\[ \|x(t)\| \leq \mu^{i/e} e^{\lambda t} (W_1 + (e^{\lambda d_u} - 1)W_2 / \lambda)c_1 / W_2 \leq \mu^{i/e} e^{\lambda t} \phi, \]

by (8), we obtain

\[ \mu^{i/e} e^{\lambda t} \phi \leq c_2, \]

combining (17) and (18), we obtain

\[ \|x(t)\| \leq c_2. \]

The proof is completed.

**Remark:** The variables \( K_i, v_i \) in inequality (4) are mutually coupled, so it can’t be solved by LMI toolbox directly. For solving Theorem 1, the Algorithm 1 is giving to get appropriate controller gain matrices \( K_i \) and \( \text{ADT} \tau_i \).

**Algorithm**

**Step 1:** Given the system matrices \( A_i, B_i, F_i \), and proper positive constants \( c_1, c_2, T_f \).

**Step 2:** Choosing proper matrices \( H_i \), and proper positive constants \( \lambda, \mu \).

**Step 3:** Introducing variables \( \xi_i \in R^n \) which satisfy \( K_i^T D_i^T B_i^T v_i < \xi_i \), then, we obtain the following inequalities

\[ [A + F_i D_i H_i] v_i + \xi_i - \lambda v_i + g_i \leq 0, \]  

so, if (4a) holds, the inequalities (4) holds.

**Step 4:** Solving the LMIs of (4a), (5), (7) and \( e(v_i / c_i,1) \subseteq L(H_i) \). Then, we obtain the variables \( \xi_i, v_i \) and \( g_i \).

**Step 5:** By the inequalities (6) and \( K_i^T D_i^T B_i^T v_i < \xi_i \), we obtain the controller gain matrices \( K_i \). Next, we calculate \( \text{ADT} \tau_i \) by (8).

**Step 6:** If the inequalities are infeasible, increasing \( \lambda \) and \( \mu \) appropriately, then go to Step 3.

**4. Numerical example**

In this section, we are going to provide an example of the SPLS to verify the result we got in the section 3.

**Example:** Consider the continuous-time switched system (1) consisting of two subsystems, and the system parameters are given as

\[ A_i = \begin{bmatrix} -3.5 & 1.1 \\ 0.8 & -4.8 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.18 \end{bmatrix}, \quad F_i = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}, \]

\[ v \in R^3, \quad c_1 = 1, \quad c_2 = 2, \quad \lambda = 0.5, \quad \mu = 1.5. \]
\[
A_2 = \begin{bmatrix}
-3 & 0.2 \\
0.7 & -2
\end{bmatrix},
B_2 = \begin{bmatrix}
0.2 & 0 \\
0.1 & 0.2
\end{bmatrix},
F_2 = \begin{bmatrix}
0.6 & 0 \\
0.3 & 0
\end{bmatrix},
\varphi(t) = [0 \ 0]^T, d(t) = 0.1.
\]

Take \( c_1 = 2 \), \( c_2 = 20 \), \( T_f = 10 \), \( \mu = 1.05 \), \( \lambda = 0.1 \), \( H_1 = [-0.05 \ -0.1] \) and \( H_2 = [-0.1 \ -0.1] \), based on algorithm 1, we obtain
\[
\nu_1 = [0.6033 \ 1.0826]^T, \nu_2 = [0.6058 \ 1.0843]^T, \\
g_1 = [0.6981 \ 1.3304]^T, \\
g_2 = [0.6979 \ 1.3290]^T, \\
\xi_1 = [0.2139 \ 1.7709]^T, \\
\xi_2 = [0.1377 \ 0.3314]^T,
\]
\( W_1 = 1.0843, W_2 = 1.3304, W_3 = 0.6033 \) and the infimum of the ADT \( \tau^* = 0.8139 \). Then, we obtain the controller gain matrices \( K_1 = [-1.3101 \ 0.6699], K_2 = [-0.9726 \ 0.0110] \).

Furthermore, choosing \( \tau_a = 1 \), \( x(0) = [1.8 \ 0.2]^T \) and the initial subsystem \( \sigma(0) = 1 \). Figs. 1-2 give the simulation results which contain the switching signal and the state response of continuous-time switched positive system (1). It can be seen clearly that the closed-loop system (1) is positive and finite-time stable from Figs. 1-2, which means that the method of the Theorem is effective.

5. Conclusions
The finite-time control for switched positive systems with actuator saturation and time-varying delays under average dwell time switching has been studied in this paper. Together with convex hull technique for solving actuator saturation, the state feedback controller gain matrices and the ADT switching rules are designed through multiple copositive Lyapunov functionals method.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (Nos. 61722306).

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